STRICHARTZ TRANSFORMS WITH RIESZ POTENTIALS AND
SEMYANISTYI INTEGRALS

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Abstract. In this paper, we study the general orthogonal Radon transform $R_{p,q}^k$ first studied by R.S. Strichartz in [19]. An sharp existence condition of $R_{p,q}^k f$ on $L^p$-spaces will be given. Then we devote to the relation formulas connecting Strichartz transform $R_{p,q}^k$ and Semyanisty integrals. We prove the corresponding Fuglede type formulas, through which a number of explicit inversion formulas for $R_{p,q}^k f$ will be given. Different from the “inclusion” Radon transform and “Gonzalez” type orthogonal transform, Strichartz transform is more complicated. Our conclusions generalize the corresponding results of the two particular cases above.

1. Introduction

Problems of integral geometry related to Grassmann manifolds constitute a core of different branches of mathematics and arise in many applications, which can be found in the book by Helgason [7], Gonzalez[3] and Rubin[12], containing many references on this subject. Here we concern the orthogonal Radon transform on affine Grassmannians first studied by R.S. Strichartz in [19]. A background information about this kind of transforms can be found in [1, 2, 4, 5, 9, 10, 11, 13, 14, 17] etc.. Let $G(n,j)$ and $G(n,k)$ be a pair of Grassmannian bundles of $j$-dimensional and $k$-dimensional affine planes in $\mathbb{R}^n$. For $p \geq 0$, $q \geq 0$, $l \geq 0$, that satisfying $p + q = j$ and $p + l = k$, we call $j$-plane $\tau \in G(n,j)$ and $k$-plane $\zeta \in G(n,k)$ incident if $\tau$ intersect $\zeta$ orthogonally in a $p$-plane. Let $\hat{\tau}$ be the set of all $j$-planes incident to $\zeta$ and $\hat{\zeta}$ the set of $k$-planes incident to $\tau$. Then for a good function $f$ on $G(n,j)$, the orthogonal Radon transform $R_{p,q}^j f$ is a function on $G(n,k)$, where the value $R_{p,q}^j f(\zeta)$, $\zeta \in G(n,k)$ is defined as an integral of $f$ over the set of all $j$-planes incident to $\zeta$. Similarly, for a good function $g$ on $G(n,k)$, the corresponding Radon transform $R_{p,l}^j g(\tau)$ is a function on $G(n,j)$ that integrates $g$ over the set of all $k$-planes incident to $\tau$. Formly,

$$R_{p,q}^j f(\zeta) = \int_{\hat{\zeta}} f(\tau) d\tau, \quad R_{p,l}^j g(\tau) = \int_{\hat{\tau}} f(\zeta) d\zeta,$$

where $d\tau$ and $d\zeta$ are the invariant measure, see (2.17), (2.18) for precise meaning.

There have been numerous publications devoted to its two particular cases. When $q = 0$, our transform reduces to the case of “inclusion” transform $R_{j,k}$ studied by Gonzalez and
Kakehi [5], which takes functions on the Grassmannian of $j$-dimensional affine planes in $\mathbb{R}^n$ to functions on a similar manifold of $k$-dimensional planes by integration over the set of all $j$-planes that contained in a given $k$-plane. Under the condition $k - j$ is even, they studied the range characterization and inversion problems, using the Lie algebra language on only smooth rapidly decreasing functions. A sharp existence condition for this transform was given by B. Rubin [11], where Rubin inverted these transforms in the framework of Lebesgues spaces for arbitrary $k - j > 0$ provided that these operators are injective. But the method of stereographic projection makes all formulas more complicated because of inevitable weight factors. The inversion formulas for continuous functions and general $L^p$ functions were obtained by us directly through part Radon transform [13], which is a quite different approach.

The case of $p = 0$ reduces to the orthogonal transform $R^k_j$ studied by Gonzalez [1, 2], which integrates a function on $G(n, j)$ over all the $j$-planes orthogonal to a $k$-plane with only one point intersection. (In the following statements, we call it “Gonzalez” transform.) These publications deal with the inversion formulas of the transforms corresponding to smooth functions, where Gonzalez also studied the intertwining relations connecting $R^k_{ij}$ and invariant differential operators. The general intertwining relationship between these transforms and fractional integral operators were studied by the author and Rubin (see [14, 15, 16]). Our approach extends to the case of “inclusion” Radon transform talked above.

In the present paper, we will investigate the general Strichartz type Radon transforms in which the two corresponding orthogonal planes have a $p$-plane intersection, under the assumption that

$$p > 0, \quad q > 0, \quad l > 0, \quad p + q + l < n.$$  

For this general case, Strichartz [19] mainly focused on its harmonic properties. In [4], Gonzalez studied the intertwining relations between Strichartz transforms and some invariant differential operators, where only smooth functions with compact support are considered. Here, our main concerns are (a) sharp conditions for $f$ under which $R^k_{p,q}f$ exists in the Lebesgue sense, (b) functional relations connecting $R^k_{p,q}$ with Riesz potentials, Radon-John $k$-plane transforms, Semyanistyi integrals and (c) explicit inversion formulas for arbitrary dimensions $j$ and $k$.

Our paper is organized as follows. Section 2 contains some basic notations about fractional integral operators and Radon-John transforms, “inclusion” Radon transforms, “Gonzalez” transforms and Strichartz transforms, etc.. Some known necessary intertwining relations connecting them will be introduced. In section 3, we mainly study Strichartz transforms on radial functions, through which a sharp existence condition for this transform will be given in section 4. In the last section, we devote to the intertwining relations connecting Strichartz transforms, Semyanistyi integrals and potential integral operators. We prove the Fuglede type theorems, through which some inversion formulas can be obtained when function $f$ belongs to the range of the $j$-plane transform.

2. Preliminaries

2.1. Riesz Potentials and Erdélyi–Kober Fractional Integrals. In this part, we introduce the basic knowledge about the Riesz potential operators first, which can be
regard as a negative power of the minus-Laplacian
\[(I_n^\alpha)^{-1} \sim (-\Delta_n)^{-\alpha/2}, \quad \Delta_n = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \ldots + \frac{\partial^2}{\partial x_n^2}. \quad (2.1)\]

Explicitly, for a good function \(f\) on \(\mathbb{R}^n\), the Riesz potential of a positive \(\alpha\) is defined by
\[
(I_n^\alpha f)(x) = \frac{1}{\gamma_n(\alpha)} \int_{\mathbb{R}^n} \frac{f(y) dy}{|x-y|^{n-\alpha}}, \quad \gamma_n(\alpha) = \frac{2^{\alpha} \pi^{n/2} \Gamma(\alpha/2)}{\Gamma((n-\alpha)/2)}; \quad (2.2)
\]
\[
\alpha \neq n, n+2, n+4, \ldots
\]
If \(f \in L^p(\mathbb{R}^n), 1 \leq p < n/\alpha\), then \((I_n^\alpha f)(x) < \infty\) for almost all \(x\), and the bounds for \(p\) are sharp; see [12, Chapter III]. For the left inverse of \(I_n^\alpha\), denoted by \(D_n^\alpha\), numerous investigations are devoted to this question; see [Rul, SKM] and references therein.

**Theorem 2.1.** [12, Theorem 3.41] Suppose \(\varphi = I_n^\alpha f\), where \(f \in L^p(\mathbb{R}^n), 1 \leq p < n/\alpha\). If \(\mu\) is a radial finite complex Borel measure on \(\mathbb{R}^n\), satisfying
\[
\int_{|x|>1} |x|^\beta d|\mu|(x) < \infty, \quad \text{for some } \beta > \alpha; \quad (2.3)
\]
\[
\int_{\mathbb{R}^n} |x|^j d\mu(x) = 0, \quad \text{for } |j| = 0, 2, 4, \ldots 2\lfloor \alpha/2 \rfloor; \quad (2.4)
\]
\[
d_{\mu}(\alpha) = \frac{\pi^{n/2-\alpha} \varsigma(\alpha)}{\sigma_{n-1} \Gamma((n+\alpha)/2)} \neq 0 \quad (2.5)
\]
where
\[
\varsigma(\alpha) = \int_0^\infty t^{-\alpha/2-1} dt \int_{\mathbb{R}^n} e^{-t|y|^2} d\mu(y)
\]
\[
= \begin{cases} 
\Gamma(-\alpha/2) \int_{\mathbb{R}^n} |y|^\alpha d\mu(y) & \text{if } \alpha \neq 2, 4, 6, \ldots, \\
\frac{2(-1)^{\alpha/2+1}}{(\alpha/2)!} \int_{\mathbb{R}^n} |y|^\alpha \log|y| d\mu(y) & \text{if } \alpha = 2, 4, 6, \ldots,
\end{cases}
\]
then
\[
(D_n^\alpha \varphi)(x) \equiv \frac{1}{d_{\mu}(\alpha)} \int_0^\infty \frac{(\varphi * \mu_t)(x)}{t^{1+\alpha}} dt = \lim_{\varepsilon \to 0} \frac{1}{d_{\mu}(\alpha)} \int_0^\varepsilon \frac{(\varphi * \mu_t)(x)}{t^{1+\alpha}} dt; \quad (2.6)
\]
represents the left inverse of Riesz potential \(I_n^\alpha f\), that is, \(D_n^\alpha I_n^\alpha f = f\). The limit in (2.6) exists in the \(L^p\) norm and in the almost everywhere sense. If \(f \in C_0(\mathbb{R}^n) \cap L^p(\mathbb{R}^n)\), this limit is uniform on \(\mathbb{R}^n\).

The inversion formula (2.6) is non-local. If \(\alpha\) is an even integer, then the local inversion formula \((-\Delta_n)^{\alpha/2} I_n^\alpha f = f\) is available under additional smoothness assumptions for \(f\).
We also work with Erdélyi-Kober type fractional integrals of positive order $\alpha$, which arise in integral geometry and many other applications of Fractional Calculus. Detailed information can be found in [12, Section 2]. The well known two Erdélyi-Kober type fractional integrals have the following expressions,

\begin{align*}
(I_{+}^{\alpha}f)(t) &= \frac{2}{\Gamma(\alpha)} \int_{0}^{t} (t^2 - r^2)^{\alpha-1} f(r) \, r \, dr, \\
(I_{-}^{\alpha}f)(t) &= \frac{2}{\Gamma(\alpha)} \int_{t}^{\infty} (r^2 - t^2)^{\alpha-1} f(r) \, r \, dr.
\end{align*}

For these operators, we have the following existence theorems.

**Lemma 2.2.** [12, Lemma 2.42] 

(i) The integral $(I_{+}^{\alpha}f)(t)$ is absolutely convergent for almost all $t > 0$ whenever $r \to rf(r)$ is a locally integrable function on $\mathbb{R}_+$. 

(ii) If

\[ \int_{a}^{\infty} |f(r)| r^{2\alpha-1} \, dr < \infty, \quad a > 0, \]

then $(I_{-}^{\alpha}f)(t)$ is finite for almost all $t > a$. If $f$ is non-negative, locally integrable on $[a, \infty)$, and (2.9) fails, then $(I_{-}^{\alpha}f)(t) = \infty$ for every $t \geq a$.

Fractional derivatives of the Erdélyi–Kober type are defined as the left inverses $D_{\pm}^{\alpha} = (I_{\pm}^{\alpha})^{-1}$ and have different analytic expressions. For example, if $\alpha = m + \alpha_0$, $m = \lfloor \alpha \rfloor$, $0 \leq \alpha_0 < 1$, then, formally,

\[ D_{\pm}^{\alpha} \varphi = (\pm D)^{m+1} I_{\pm,2}^{1-\alpha_0} \varphi, \quad D = \frac{1}{2t} \frac{d}{dt}. \]

More precisely, the following statements hold.

**Theorem 2.3.** Let $\varphi = I_{+}^{\alpha}f$, where $rf(r)$ is locally integrable on $\mathbb{R}_+$. Then $f(t) = (D_{+}^{\alpha} \varphi)(t)$ for almost all $t \in \mathbb{R}_+$, as in (2.10).

**Theorem 2.4.** [12, Theorem 2.44] If $f$ satisfies (2.9) for every $a > 0$ and $\varphi = I_{-}^{\alpha}f$, then $f(t) = (D_{-}^{\alpha} \varphi)(t)$ for almost all $t \in \mathbb{R}_+$, where $D_{-}^{\alpha} \varphi$ can be represented as follows.

(i) If $\alpha = m$ is an integer, then

\[ D_{-}^{\alpha} \varphi = (-D)^{m} \varphi, \quad D = \frac{1}{2t} \frac{d}{dt}. \]

(ii) If $\alpha = m + \alpha_0$, $m = \lfloor \alpha \rfloor$, $0 < \alpha_0 < 1$, then

\[ D_{-}^{\alpha} \varphi = t^{2(1-\alpha)}(-D)^{m+1} t^{2\alpha} \psi, \quad \psi = I_{-,2}^{1-\alpha_0} t^{-2m-2} \varphi. \]
In particular, for \( \alpha = k/2, k \text{ odd} \),
\[
\mathcal{D}^{k/2}_D \varphi = t ( -D )^{k+1/2} t^{1/2} I^{1/2} t^{-k-1} \varphi.
\] (2.13)

These fractional integrals and their derivatives possess the semi-group property
\[
\mathcal{D}^\alpha_n \mathcal{D}^\beta_n = \mathcal{D}^{\alpha+\beta}_n, \quad I^\alpha_n I^\beta_n = I^{\alpha+\beta}_n;
\]
\[
\mathcal{D}^{\alpha,2}_\pm \mathcal{D}^{\beta,2}_\pm = \mathcal{D}^{\alpha+\beta,2}_\pm, \quad I^{\alpha,2,2}_\pm = I^{\alpha+\beta,2}_\pm, \tag{2.14}
\]
in suitable classes of functions that guarantee the existence of the corresponding expressions.

2.2. Radon transforms on Grassmanians. Let \( G(n, j) \) and \( G_{n,j} \) be the affine Grassmann manifold of all non-oriented \( j \)-planes \( \tau \) and the ordinary Grassmann manifold of \( j \)-dimensional subspaces \( \xi \) of \( \mathbb{R}^n \), respectively. Similarly, \( G(n, k) \) and \( G_{n,k} \) denote the set of \( k \)-planes \( \zeta \) and the set of \( k \)-subspaces \( \eta \). Then every \( j \)-plane \( \tau \) can be parameterized by the pair \((\xi, u)\), where \( \xi \in G_{n,j} \) and \( u \in \xi^\perp \). Similarly, we write \( \zeta = (\eta, v) \in G(n, k) \) where \( \eta \in G_{n,k} \) and \( v \in \eta^\perp \). For any integer \( m \) and subspace \( X \) in \( \mathbb{R}^n \), denote by \( G_m(X) \) and \( G(m, X) \) the sets of \( m \)-dimensional subspaces and \( m \)-dimensional planes in \( X \), respectively. If \( P \) and \( Q \) are two orthogonal subspaces with no other intersection except for the original point, let \([P, Q]\) denote the smallest subspace that contains both \( P \) and \( Q \). For incident planes \( \zeta = (\eta, v) \in G(n, k) \) and \( \tau = (\xi, u) \in G(n, j) \), denote \( P = \xi \cap \eta \) and \( Q = \eta^\perp \cap \xi \). Then \( P \) is a \( p \)-dimensional subspace, \( Q \) a \( q \)-dimensional subspace and \( R \) a \( l \)-dimensional subspace, where \( p + q = j \) and \( p + l = k \). Obviously, \( \xi = [P, Q], \eta = [P, R] \). Then for \( \zeta = (\eta, v) \in G(n, k) \) and \( \tau = (\xi, u) \in G(n, j) \), we have
\[
\hat{\zeta} = \{ [P, Q] + v + u : P \in G_p(\eta), Q \in G_q(\eta^\perp), \text{and } u \in P^\perp \cap \eta \},
\] (2.15)
\[
\hat{\tau} = \{ [P, R] + u + v : P \in G_p(\xi), R \in G_l(\xi^\perp), \text{and } v \in P^\perp \cap \xi \},
\] (2.16)
where \( p \geq 0, q \geq 0, l \geq 0 \) satisfying \( p + q = j \), \( p + l = k \), and \( G_q(\eta^\perp) \) represents the \( q \)-dimensional subspace in \( \eta^\perp \). The manifolds \( G(n, j) \) and \( G(n, k) \) will be endowed with \( d\tau = d\xi du \) and \( d\zeta = d\eta dv \), where \( d\xi \) and \( d\eta \) are the corresponding \( O(n) \)-invariant measure on \( G_{n,j} \) and \( G_{n,k} \) with total mass 1 and \( du, dv \) the usual Lebesgue measure on subspace \( \xi^\perp \) and \( \eta^\perp \), respectively. Then the orthogonal Radon transform \( R^k_{p,q} \) in (1.1) can be rewritten in the following form,
\[
R^k_{p,q} f(\zeta) = \int_{G_q(\eta^\perp)} d\eta^\perp Q \int_{G_p(\eta)} d\eta P \int_{P^\perp \cap \eta} f([P,Q] + u + v)du.
\] (2.17)

where \( d\eta^\perp Q \) and \( d\eta P \) denote the corresponding invariant measure on \( G_q(\eta^\perp) \) and \( G_p(\eta) \) with total mass 1. Similarly, we can define the corresponding dual transform. For a function \( g = g(\zeta) \) on \( G(n, k) \), the dual transform of \( R^l_{p,q} g \) is a function on \( G(n, j) \), defined by
\[
R^l_{p,q} g(\xi, u) = \int_{G_l(\xi^\perp)} d\xi^\perp L \int_{G_p(\xi)} d\xi P \int_{P^\perp \cap \xi} g([P, L] + u + v)dv.
\] (2.18)
where \( d_{\xi}L \) and \( d_{\xi}P \) denote the corresponding invariant measure on \( G_{l}(\xi^{\perp}) \) and \( G_{p}(\xi) \) with total mass 1, respectively.

When \( p = 0 \), in the meantime \( q = j \) and \( l = k \), transforms (2.17) and (2.18) reduce to the Gonzalez type orthogonal Radon transform,

\[
(R_{j}^{k}f)(\eta, v) = \int_{G_{j}(\eta^{\perp})} d_{\eta^{\perp}} \xi \int_{\eta} f(\xi + u + v)du ,
\]

and its dual transform

\[
(R_{j}^{k}f)(\xi, u) = \int_{G_{k}(\xi^{\perp})} d_{\xi^{\perp}} \eta \int_{\xi} f(\eta + u + v)dv ,
\]

which studied in [1, 2]; If moreover \( j = 0 \), we get the \( k \)-plane transform and its dual:

\[
(R_{k}f)(\eta, v) = \int_{\eta} f(u + v)du , \quad \zeta = (\eta, v) \in G(n, k), \quad 1 \leq k \leq n - 1;
\]

\[
(R_{k}^{*}\varphi)(x) = \int_{O(n)} \varphi(\eta_{0}x) d\rho ,
\]

where \( \eta_{0} \) is an arbitrary fixed \( k \)-plane through the origin and \( d\rho \) the invariant measure with mass 1. The dual \( k \)-plane transform \( R_{k}^{*} \) averages a function \( \varphi \) on \( G(n, k) \) over all \( k \)-planes passing through a fixed point \( x \in \mathbb{R}^{n} \). When \( \varphi = \varphi(\xi, u) \) is a good radial function on \( G(n, k) \), \( \varphi(\eta, v) = \varphi_{0}(|v|) \), then

\[
R_{k}^{*}\varphi(x) = \frac{\sigma_{k-1} \sigma_{n-k-1}}{\sigma_{n-1} r^{n-2}} \int_{0}^{r} \varphi_{0}(s)(r^{2} - s^{2})^{k/2 - 1} s^{n-k-1} ds , \quad r = |x| .
\]

If \( p = j \), in the meantime \( q = 0 \), our transform reduces to the “inclusion” Radon transform

\[
(R_{j,k}f)(\eta, v) = \int_{G_{j}(\eta)} d_{\eta^{\perp}} \xi \int_{\xi^{\perp} \cap \eta} f(\xi + u + v)du ,
\]

and its dual

\[
(R_{j,k}^{*}f)(\xi, u) = \int_{\xi \subset \eta} f(\eta + u) d\xi \eta ,
\]

that integrates function \( f \) on \( G(n, j) \) over all the \( k \)-planes containing \( j \)-plane \( (\xi, u) \). When \( f \) is a radial function \( f(\xi, u) = f_{0}(|u|) \), then \( R_{j,k}f(\eta, v) \) is also radial. Explicitly,

\[
R_{j,k}f(\eta, v) = \sigma_{k-j-1} \int_{s}^{\infty} f_{0}(r)(r^{2} - s^{2})^{(k-j)/2 - 1} r dr , \quad s = |v| ,
\]

see [11] for detailed informations.
2.3. Intertwining relations and Fuglede equality. In this part, we recall the known relations between fractional integrals and the Radon transforms on Grassmannians. The following theorem is due to Rubin [10], that Radon-John $j$-plane transforms and their duals interwine Riesz Potentials on the source space and target space.

**Theorem 2.5.** [10, Section 3] For functions $f$ on $\mathbb{R}^n$, $\psi$ on $G(n,k)$,

\[ R_k I_{n-k}^\alpha f = I_{n-k}^\alpha R_k f, \quad R_k I_{n-k}^\alpha \psi = I_{n-k}^\alpha R_k \psi. \quad (2.27) \]

Here $I_{n-k}^\alpha$ stands for the Riesz potential on the $(n-k)$-dimensional fiber of the Grassmannian bundle $G(n,k)$. As above, it is assumed that either side of the corresponding equality exists in the Lebesgue sense.

A similar statement for the orthogonal Radon transform $R_j^k$ of the Gonzalez type was proved by the author and Rubin [14];

**Theorem 2.6.** [14, section 4] If $0 < \alpha < n - k - j$, then

\[ (I_{n-k}^\alpha R_j^k f)(\zeta) = (R_j^k I_{n-k}^\alpha f)(\zeta), \quad \zeta \in G(n,k), \quad (2.28) \]

provided that either side of this equality exists in the Lebesgue sense.

The case of $j = 0$ agrees with the equality (2.27). When $j + k = n - 1$, $m = 1$, the Laplacian form of (2.28) was proved by Gonzalez under the assumption that $f$ is infinitely differentiable and compactly supported; cf. [2, Lemma 3.3].

At last, we introduce the Fuglede’s formula, which plays an important role in the construction of original function $f$ from its $k$-plane transform $R_k f$.

**Theorem 2.7.** [6] For any $1 \leq k \leq n - 1$,

\[ R_k^* R_k f = c_{k,n} I_n^k f, \quad c_{k,n} = \frac{2^k \pi^{k/2} \Gamma(n/2)}{\Gamma((n-k)/2)}, \quad (2.29) \]

provided that either side of this equality exists in the Lebesgue sense.

Similar statements have been proved for “Gonzalez” transform and “inclusion” transform.

\[ R_k^* R_j^k R_j^h = R_j^* R_j^k R_k h = c I_{n-k}^{j\!+\!k} h, \quad c = \frac{2^{j+k} \pi^{(j+k)/2} \Gamma(n/2)}{\Gamma((n-j-k)/2)}; \quad (2.30) \]

\[ R_k^* R_{j,k}^h = R_k^* R_{j,k} R_k h = c_{k,n} I_n^k h, \quad c_{k,n} = \frac{2^k \pi^{k/2} \Gamma(n/2)}{\Gamma((n-k)/2)}, \quad (2.31) \]

both under the conditions that the Riesz potentials on the right-hand side exist in the Lebesgue sense. In this paper, we will generalize them to Strichartz type transforms, see Theorem 5.11.

In the following statement, we assume the integer numbers $p, q, l$ satisfying

\[ p > 0, \quad q > 0, \quad l > 0 \quad \text{and} \quad p + q + l < n. \quad (2.32) \]
3. Radon transforms of Radial functions

We recall that a function $f$ on $G(n,j)$ is radial, if there is a function $f_0$ on $\mathbb{R}^+$, such that $f(\tau) = f_0(|\tau|)$. If radial function $f$ is good enough, then, by (2.17), we can write

$$(R_{p,q}^k f)(\eta, v) = \int_{G_\eta(\eta^\perp)} F(Q + v) dQ, \quad (3.1)$$

where

$$F(Q + v) = \int_{G_\eta(\eta^\perp)} d\eta P \int_{P^\perp \cap \eta} f_0(|u + P_{Q^\perp}v|) du, \quad (3.2)$$

and $P_{Q^\perp}v$ denotes the orthogonal projection of $v$ onto $Q^\perp$. For fixed $Q$, the expression of $F(Q + v)$ can be seen as a $p$-plane to $k$-plane inclusion transform restricted in subspace $Q^\perp$. Explicitly,

$$F(Q + v) = R_{p,k} f_0(\eta, P_{Q^\perp}v), \quad (3.3)$$

where $R_{p,k}$ denotes the $p$-plane to $k$-plane inclusion transform in $Q^\perp$ and function $f_0$ is a radial function on $G(p, Q^\perp)$ satisfying $f_0(P, \omega) = f_0(|\omega|)$. Using [11, Lemma 2.3], $F(Q + v)$ is also a radial function on $G(q, \eta^\perp)$, we write $F(Q + v) = F_0(|P_{Q^\perp}v|)$. According to the formula (2.26),

$$F_0(s) = \sigma_{t-1} \int_{s}^{\infty} f_0(r) r^{-l/2-1} dr. \quad (3.4)$$

From (3.1), for fixed $\eta$, the function $R_{p,q}^k f_0(\eta, \cdot)$ is a dual $q$-plane transform restricted in subspace $\eta^\perp$. Then from (2.24), $R_{p,q}^k f_0(\eta, \cdot)$ is also a radial function with the expression

$$R_{p,q}^k f(\eta, v) = \frac{\sigma_{t-1} \sigma_{n-k-1} \sigma_{n-k-2}}{\sigma_{n-k-1} t^{n-k-2}} \int_{0}^{t} F_0(s) t^{l/2-1} r^{n-k-2} ds, \quad t = |v|. \quad (3.5)$$

Combining (3.4) and (3.5), using (2.7) and (2.8), we have the following theorem.

**Theorem 3.1.** If $f(\tau) \equiv f_0(|\tau|)$ satisfies the conditions

$$\int_{0}^{a} |f_0(t)| t^{n-j-1} dt < \infty \quad \text{and} \quad \int_{a}^{\infty} |f_0(t)| t^{l-1} dt < \infty \quad (3.6)$$

for some $a > 0$, then

$$(R_{p,q}^k f)(\zeta) = (I_{j,k} f_0)(|\zeta|), \quad (3.7)$$

where

$$(I_{j,k} f_0)(s) = \frac{c_1}{s^{n-k-2}} \int_{0}^{s} (s^2 - r^2)^{l/2-1} r^{n-k-2} dr \int_{r}^{\infty} f_0(t) (t^2 - r^2)^{l/2-1} dt$$

$$= \frac{c_1}{s^{n-k-2}} (I_{j+1,2}^{f_0} (t^{l/2-1} r^{l/2-1} f_0)(s), \quad \ell = n - k - q \geq 1, \quad (3.8)$$
\[ c_1 = \frac{\sigma_{l-1}\sigma_{q-1}\sigma_{\ell-1}}{\sigma_{n-k-1}}, \quad \bar{c}_1 = \frac{\pi^{l/2} \Gamma((n-k)/2)}{\Gamma(\ell/2)}. \] (3.9)

Moreover,
\[ \int_{\alpha}^{\beta} |(I_{j,k}f_0)(s)| \, ds < \infty \quad \text{for all} \quad 0 < \alpha < \beta < \infty. \] (3.10)

**Proof.** We just need to show that the assumptions in (3.6) imply (3.10). Then \((I_{j,k}f_0)(s) < \infty\) for almost all \(s > 0\), and therefore (3.7) is meaningful. The proof of (3.6) is very similar to the proof in [13, Lemma 3.1]. To keep the paper complete and readable, we give its proof here. It suffices to assume \(f_0 \geq 0\). Let \(\ell = n - q - k \geq 1\) and the letter \(c\) stands for a constant that can be different at each occurrence. Then through (3.5), for any \(0 < \alpha < \beta < \infty\),
\[
\int_{\alpha}^{\beta} (I_{j,k}f_0)(s) \, ds \leq c \int_{\alpha}^{\beta} ds \int_{0}^{s} (s^2 - r^2)^{q/2-1} r^{\ell-1} F_0(r) \, dr \\
\leq c \int_{\alpha}^{\beta} ds \int_{0}^{s} (s-r)^{q/2-1} r^{\ell-1} F_0(r) \, dr \\
= c \int_{0}^{\beta} r^{\ell-1} F_0(r) \, dr \int_{\alpha}^{\beta} (s-r)^{q/2-1} ds + c \int_{\alpha}^{\beta} r^{\ell-1} F_0(r) \, dr \int_{r}^{\beta} (s-r)^{q/2-1} ds \\
\leq c \int_{0}^{\beta} r^{\ell-1} (\beta - r)^{q/2} F_0(r) \, dr - c \int_{0}^{\alpha} r^{\ell-1} (\alpha - r)^{q/2} F_0(r) \, dr.
\]

These integrals have the same form, so we just need to show that
\[
I(\alpha) \equiv \int_{0}^{\alpha} r^{\ell-1} (\alpha - r)^{q/2} F_0(r) \, dr < \infty \quad \forall \alpha > 0.
\]

Indeed,
\[
I(\alpha) \leq \int_{0}^{\alpha} r^{\ell-1} (\alpha - r)^{q/2} dr \int_{r}^{\infty} f_0(t) (t-r)^{l/2-1} t^{l/2} dt \\
\leq c \int_{0}^{\alpha} f_0(t) t^{l/2+\ell-1} dt \int_{r}^{t} (t-r)^{l/2-1} dr + c \int_{\alpha}^{\infty} f_0(t) t^{l/2} dt \int_{0}^{\alpha} (t-r)^{l/2-1} dr \\
= c \int_{0}^{\alpha} f_0(t) t^{l+\ell-1} dt + c \int_{\alpha}^{\infty} f_0(t) t^{l/2} (t^{l/2} - (t - \alpha)^{l/2}) dt,
\]

where...
So

\[ I(\alpha) \leq c \int_0^\alpha f_0(t) t^{n-j-1} dt + c \int^\infty_\alpha f_0(t) t^{l-1} dt < \infty, \]

The last expression is finite by (3.6). \(\square\)

**Remark 3.2.**

1. If the first integral in (3.6) is not finite, function \(f(\tau) = \frac{1}{|\tau|^{n-j}}\) only satisfying the second integral condition in (3.6) gives an example that its Strichartz transform \(R^k_{p,q} f(\eta, v) = \infty\) for almost every point \((\eta, v) \in G(n, k)\).

2. Through Lemma 2.2, to guarantee the existence of the corresponding integrals \((I_{j,k}f_0)(s)\), the finiteness of the second integral in (3.6) is essentially necessary.

The following analogue of Theorem 3.1 for the dual transform \(R^j_{p,l}g\) follows from Theorem 3.1 by the symmetry.

**Theorem 3.3.** If \(g(\zeta) \equiv g_0(|\zeta|)\) satisfies the conditions

\[
\int_0^a |g_0(s)| s^{n-k-1} ds < \infty \quad \text{and} \quad \int_0^\infty |g_0(s)| s^{q-1} ds < \infty, \tag{3.11}
\]

for some \(a > 0\), then

\[
(R^j_{p,l}g)(\tau) = (I_{k,j}g_0)(|\tau|), \tag{3.12}
\]

where

\[
(I_{k,j}g_0)(t) = \frac{c_2}{t^{n-j-2}} \int_0^t (t^2 - r^2)^{l/2-1} r^{\ell-1} dr \int_0^\infty g_0(s) (s^2 - r^2)^{q/2-1} s ds
\]

\[
= \frac{\tilde{c}_2}{t^{n-j-2}} (I_{l+2}^1 r^\ell - I_{l-2}^q g_0)(t), \quad \ell = n - j - k \geq 1, \tag{3.13}
\]

\[
c_2 = \frac{\sigma_{l-1} \sigma_{q-1} \sigma_{l-1}}{\sigma_{n-j-1}}, \quad \tilde{c}_2 = \frac{\pi^{q/2} \Gamma((n-j)/2)}{\Gamma(\ell/2)}. \tag{3.14}
\]

Moreover,

\[
\int_\alpha^\beta |(I_{k,j}g_0)(t)| dt < \infty \quad \text{for all} \quad 0 < \alpha < \beta < \infty. \tag{3.15}
\]

**Remark 3.4.** A similar statement as in Remark 3.2 explains the importance of the two integral conditions in (3.11).
Example 3.5. The following formulas can be easily obtained from (3.8) and (3.13),
(i) If \( f(\tau) = |\tau|^{-\lambda}, l < \lambda < n - j \), then \( (R_{p,q}^k f)(\zeta) = c_1 |\zeta|^{l-\lambda} \);
(ii) If \( g(\zeta) = |\zeta|^{-\lambda}, q < \lambda < n - k \), then \( (R_{p,t}^g)(\tau) = c_2 |\tau|^{q-\lambda} \);
(iii) If \( f(\tau) = (1 + |\tau|^2)^{-\frac{n-k}{2}}, \) then \( (R_{p,q}^k f)(\zeta) = c_3 (1 + |\zeta|^2)^{-\frac{n-k-q}{2}} \);
(iv) If \( g(\zeta) = (1 + |\zeta|^2)^{-\frac{n-j}{2}}, \) then \( (R_{p,t}^g)(\tau) = c_4 (1 + |\tau|^2)^{-\frac{n-j-l}{2}} \),
where
\[
\begin{align*}
c_1 &= \frac{\pi^{l/2} \Gamma((\lambda - l)/2) \Gamma((-\lambda + n - j)/2) \Gamma((n - k)/2)}{\Gamma(\lambda/2) \Gamma((-\lambda + n - q)/2) \Gamma((n - k - q)/2)}, \\
c_2 &= \frac{\pi^{q/2} \Gamma((\lambda - q)/2) \Gamma((-\lambda + n - k)/2) \Gamma((n - j)/2)}{\Gamma(\lambda/2) \Gamma((-\lambda + n - l)/2) \Gamma((n - j - l)/2)}, \\
c_3 &= \frac{\pi^{l/2} \Gamma((n - k)/2)}{\Gamma((n - p)/2)}, \quad c_4 = \frac{\pi^{q/2} \Gamma((n - j)/2)}{\Gamma((n - p)/2)}.
\end{align*}
\]

Theorem 3.6. Suppose \( p + q + l < n \) and \( p + q = j, p + l = k \), and let \( f(\tau) \equiv f_0(|\tau|) \) be a radial function on \( G(n, j) \) satisfying the same integral conditions (3.6). Then function \( f_0 \) can be recovered from the Radon transform \( (R_{p,q}^k f)(\zeta) \equiv (I_{j,k} f_0)(|\zeta|) \) by the formula
\[
f_0(t) = \tilde{c}_1^{-1} (D_{-2}^{q/2} r^{2-\ell} D_{+2}^{q/2} s^{n-k-2} I_{j,k} f_0)(t), \quad (3.16)
\]
where \( \tilde{c}_1 = \pi^{l/2} \Gamma((n - k)/2) / \Gamma(\ell/2) \) and the Erdélyi–Kober fractional derivatives \( D_{-2}^{q/2} \) and \( D_{+2}^{q/2} \) are defined by (2.10)-(2.13).

Proof. By (3.8),
\[
(I_{j,k} f_0)(s) = \tilde{c}_1 s^{2+k-n} (D_{+2}^{q/2} r^{\ell-2} D_{-2}^{q/2} f_0)(s). \quad (3.17)
\]
Using the finiteness of the integrals in (3.6), through a simple calculation, we can prove \( r^{\ell-1} I_{-2}^{q/2} f_0 \in L^1_{loc}(\mathbb{R}_+) \). It follows that the conditions of Theorems 2.3 and 2.4 are satisfied and both fractional integrals in (3.17) can be inverted. This gives (3.16). \( \square \)

Interchanging \( j \) and \( k \), the reader can easily arrive the following similar statement for the dual transform \( R_{p,t}^g \).

Theorem 3.7. Suppose \( p + q + l < n \) and \( p + q = j, p + l = k \), and let \( g(\zeta) \equiv g_0(|\zeta|) \) be a radial function on \( G(n, k) \) satisfying the same integral conditions (3.11). Then function \( g_0 \) can be recovered from the Radon transform \( (R_{p,t}^q g)(\tau) \equiv (I_{k,j} g_0)(|\tau|) \) by the formula
\[
g_0(t) = \tilde{c}_2^{-1} (D_{-2}^{q/2} r^{2-\ell} D_{+2}^{q/2} s^{n-j-2} I_{k,j} g_0)(t), \quad (3.18)
\]
where \( \tilde{c}_2 = \pi^{q/2} \Gamma((n - j)/2) / \Gamma(\ell/2) \) and the Erdélyi–Kober fractional derivatives \( D_{-2}^{q/2} \) and \( D_{+2}^{q/2} \) are defined by (2.10)-(2.13).
4. Existence theorem

4.1. Duality. In this section, we prove the dual relationship between $R^k_{p,q}$ and $R^l_{p,l}$ first. Using the idea of double fibration (cf. [7], P57, or [14], Lemma 3.4), we can prove the following identity, which combined the dual relationship in [14, Lemma 3.4] and [11, Lemma 2.1].

**Theorem 4.1.** Suppose $p + q + l < n$ and $p + q = j, p + l = k$, for $f = f(\xi, u)$ on $G(n, j)$ and $g = g(\eta, v)$ on $G(n, k)$,

$$< R^k_{p,q}f, g > = < f, R^l_{p,l}g >,$$

the equality holds if any side of the integral exists in the Lebesgue sense.

**Proof.** Let

\[
R^k = Re_1 + Re_2 + \cdots + Re_k, \quad R^{n-k} = Re_{k+1} + Re_{k+2} + \cdots + Re_n,
\]

\[
R^p = Re_1 + Re_2 + \cdots + Re_p, \quad R^q = Re_{n-q+1} + Re_{n-q+2} + \cdots + Re_n,
\]

\[
R^l = Re_{p+1} + Re_{p+2} + \cdots + Re_k, \quad R^{n-p} = Re_{p+1} + Re_{p+2} + \cdots + Re_n.
\]

Then $R^k = [R^p, R^l]$, $R^l = [R^p, R^q]$. We begin with the following integral,

\[
\int_{O(n)} d\gamma \int_{R^{n-p}} f_\gamma([R^p, R^q] + u)g_\gamma([R^p, R^l] + u)du,
\]

(4.2)

where $f_\gamma(\tau) = f(\gamma \tau)$, $g_\gamma(\zeta) = g(\gamma \zeta)$, for any $\gamma \in O(n)$. Notice that $R^{n-p} = R^{n-k} \oplus R^l$, the upper integral equals

\[
\int_{O(n)} d\gamma \int_{O(k)} d\rho_1 \int_{O(n-k)} d\rho_2 \int_{R^{n-k}} d\nu \int_{R^l} f_{\gamma \rho_1 \rho_2}([R^p, R^q] + u + v)g_{\gamma \rho_1 \rho_2}([R^p, R^l] + v)du
\]

where $O(k), O(n-k)$ are the stabilizers of $R^k, R^{n-k}$ in $O(n)$, and $d\rho_1, d\rho_2$ the corresponding invariant probability measures, respectively. Since rotation $\rho_1 \rho_2$ leaves subspaces $R^k$ and $R^{n-k}$ invariant, so the last integral equals

\[
\int_{O(n)} \int_{R^{n-k}} g_\gamma(R^k + v)dv \int_{O(k)} d\rho_1 \int_{O(n-k)} d\rho_2 \int_{R^l} f_\gamma([\rho_1 R^p, \rho_2 R^q] + \rho_1 u + v)du
\]

\[
= \int_{O(n)} \int_{R^{n-k}} g_\gamma(R^k + v)R^k_{p,q}f_\gamma(R^k + v)dv = 1.s. \text{ of (4.1)},
\]

where the last equality follows from the expression (2.17).

Similarly, we can also prove

(4.2) = r.s. of (4.1)

Then we finish the proof of the dual equality. \qed

Combining Example 3.5 and Theorem 4.1, we have the following formulas.
Corollary 4.2.

\[
\int_{G(n,j)} \frac{R^j_{p,q} g(\tau)}{|\tau|^\lambda} d\tau = c_1 \int_{G(n,k)} \frac{g(\zeta)}{|\zeta|^{\lambda-l}} d\zeta, \quad l < \lambda < n - j; \tag{4.3}
\]

\[
\int_{G(n,k)} \frac{R^k_{p,q} f(\zeta)}{|\zeta|^\lambda} d\zeta = c_2 \int_{G(n,j)} \frac{f(\tau)}{|\tau|^{\lambda-q}} d\tau, \quad q < \lambda < n - k; \tag{4.4}
\]

\[
\int_{G(n,j)} \frac{R^j_{p,l} g(\tau)}{(1 + |\tau|^2)(n-p)/2} d\tau = c_3 \int_{G(n,k)} \frac{g(\zeta)}{(1 + |\zeta|^2)(n-k-q)/2} d\zeta; \tag{4.5}
\]

\[
\int_{G(n,k)} \frac{R^k_{p,q} f(\zeta)}{(1 + |\zeta|^2)(n-p)/2} d\zeta = c_4 \int_{G(n,j)} \frac{f(\tau)}{(1 + |\tau|^2)(n-j-l)/2} d\tau, \tag{4.6}
\]

where \(c_1, c_2, c_3\) and \(c_4\) are the same numbers as in Example 3.5.

Then we come to the following existence theorem.

Theorem 4.3.

(i) If \(f \in L^s(G(n,j)), 1 \leq s < (n - j)/l\), then \(R^k_{p,q} f(\zeta)\) exists for almost all \(\zeta \in G(n,k)\).

(ii) If \(g \in L^r(G(n,k)), 1 \leq r < (n - k)/q\), then \((R^j_{p,l} g)(\tau)\) exists for almost all \(\tau \in G(n,j)\).

The bounds \(s < (n - j)/l\) and \(r < (n - k)/q\) in these statements are sharp.

Proof. (i) follows from (4.6) if we apply Hölder’s inequality to the right-hand side. If \(s \geq (n - j)/l\), the function

\[
f(\tau) = (2 + |\tau|)^{(j-n)/s}(\log(2 + |\tau|))^{-1}
\]

which belongs to \(L^s(G(n,j))\) gives a counter-example, because it does not meet the second integral condition in (3.6). The result for \((R^j_{p,l} g)(\tau)\) follows if we interchange \(j\) and \(k\), \(q\) and \(l\). \(\square\)

5. Strichartz transforms, Riesz Potentials and Semyanistyi integrals

In this part, we devote to the interesting formulas that connect Strichartz Radon transforms \(R^k_{p,q}\), Semyanistyi type integrals and the Riesz Potentials. Fuglede type formulas about \(R^k_{p,q}\) will also be considered, through which we can reconstruct function \(f\) from \(R^k_{p,q} f\), when \(f\) belongs to the range of Radon-John \(j\)-plane transforms. These formulas generalize the corresponding conclusions in [10] and [14].
5.1. Strichartz Transforms with Riesz Potentials.

**Theorem 5.1.** If $0 < \alpha < n - k - q$, then
\[ R_{p,q} I_{n-j}^{\alpha} f(\eta, v) = I_{n-j}^{\alpha} R_{p,q} f(\eta, v), \quad (\eta, v) \in G(n, k), \tag{5.1} \]
provided that either side of this equality exists in the Lebesgue sense.

**Proof.** Through the expression (2.17) and (2.2), the left hand of (5.1)
\[ R_{p,q} I_{n-j}^{\alpha} f(\eta, v) = \int G_p(\eta^+) \int G_p(\eta) \int d\eta P \int \frac{I_{n-j}^{\alpha} f([P, Q] + u + v)du}{|x|^{n-j-\alpha}} \]
\[ = \frac{1}{\gamma_{n-j}^{\alpha}} \int G_p(\eta^+) \int G_p(\eta) \int d\eta P \int \frac{f([P, Q] + u + v - x)dx}{|x|^{n-j-\alpha}}. \]
Note that $Q^\perp = \eta \oplus (Q^\perp \cap \eta^\perp)$, then the last integral equals
\[ \frac{1}{\gamma_{n-j}^{\alpha}} \int G_p(\eta^+) \int G_p(\eta) \int d\eta P \int \frac{f([P, Q] + u + v - x)}{|x + y|^{n-j-\alpha}}dy \]
\[ = c_1 \int G_p(\eta^+) \int G_p(\eta) \int d\eta P \int \frac{f([P, Q] + u + v - x)}{|x|^{n-k-q-\alpha}}dy \tag{5.2} \]
where
\[ c_1 = \frac{\sigma_{l-1}}{\gamma_{n-j}^{\alpha}} \int_0^\infty \frac{x^{l-1}}{(1 + x^2)^{(n-j-\alpha)/2}}dx = \frac{\Gamma((n - j - l - \alpha)/2)}{2\alpha \pi^{(n-j-\alpha)/2} \Gamma(\alpha)}. \]

Similarly, we calculate the right side of (5.1),
\[ I_{n-k}^{\alpha} R_{p,q} f(\eta, v) = \int G_p(\eta^+) \int G_p(\eta) \int d\eta P \int \frac{f([P, Q] + u + v - w)}{|w|^{n-k-\alpha}}dw \]
\[ = \frac{1}{\gamma_{n-k}^{\alpha}} \int G_p(\eta^+) \int G_p(\eta) \int d\eta P \int \frac{f([P, Q] + u + v - w)du}{|x|^{n-k-\alpha}} \]
Note that $\eta^\perp = Q \oplus (Q^\perp \cap \eta^\perp)$, we interchange the integrals order and get
\[ \frac{1}{\gamma_{n-k}^{\alpha}} \int G_p(\eta^+) \int G_p(\eta) \int d\eta P \int \frac{f([P, Q] + u + v - x)}{|x|^{n-k-\alpha}}dx \]
\[ = c_2 \int G_p(\eta^+) \int G_p(\eta) \int d\eta P \int \frac{f([P, Q] + u + v - x)}{|x|^{n-k-q-\alpha}}dx \tag{5.3} \]
where
\[ c_2 = \frac{\sigma_{q-1}}{\gamma_{n-k}^{\alpha}} \int_0^\infty \frac{x^{q-1}}{(1 + x^2)^{(n-k-\alpha)/2}}dx = \frac{\Gamma((n - k - q - \alpha)/2)}{2\alpha \pi^{(n-k-\alpha)/2} \Gamma(\alpha)}. \]
Comparing the two expressions (5.2) and (5.3), we get the equality (5.1).  \( \square \)
The differential form of (5.1) was first studied by Gonzalez under the assumption that $f$ is infinitely differentiable and compactly supported; cf. [4, Section 4].

At the case of $p = 0$, that is, “Gonzalez” case, equality (5.1) coincides with the conclusion in Theorem 2.6. At another case $q = 0$, we get the intertwining formula for the “inclusion” Radon transforms.

Corollary 5.3. For $f = f(\xi, u)$ a function on $G(n, j)$,

$$R_{j,k}^\alpha I^\alpha_{n-j} f(\eta, v) = I^\alpha_{n-k} R_{j,k} f(\eta, v), \quad (\eta, v) \in G(n, k), \quad (5.4)$$

provided that either side of this equality exists in the Lebesgue sense.

5.2. Strichartz transforms with Semyanistyi type integrals. In this part, we devote to the relationship between Strichartz transforms and Semyanistyi type integrals with the following expressions:

$$ (P_k^\alpha f)(\zeta) = \frac{1}{\gamma_{n-k}(\alpha)} \int_{\mathbb{R}^n} f(x) |x - \zeta|^{\alpha + k - n} dx, $$

$$ (P_k^{\alpha*} \varphi)(x) = \frac{1}{\gamma_{n-k}(\alpha)} \int_{G(n,k)} \varphi(\zeta) |x - \zeta|^{\alpha + k - n} dx, \quad (5.5) $$

where $\gamma_{n-k}(\alpha) = \frac{2^{\alpha(n-k)/2} \Gamma(\alpha/2)}{\Gamma((n-k-\alpha)/2)}$, $\Re \alpha > 0$, $\alpha + k - n \neq 0, 2, 4, ...$. We recall some known formulas about the Semyanistyi integrals, which connect them with the $k$-plane transforms and Riesz potentials,

$$ P_k^\alpha f = I^\alpha_{n-k} R_k f, \quad P_k^{\alpha*} \varphi = R_k^* I^\alpha_{n-k} \varphi, \quad (5.6) $$

where $f$ is a function on $\mathbb{R}^n$, $g$ a function on $G(n, k)$ and the equalities hold if and only if either side of the formulas exists in the Lebesgue sense. Combined with Theorem 2.7,

$$ P_k^{\alpha*} R_k f = c_{k,n} I^{k+\alpha}_n f, \quad (5.7) $$

$$ R_k^* P_k^\alpha f = c_{k,n} I^{k+\alpha}_n f, \quad (5.8) $$

where $c_{k,n} = (2\pi)^k \sigma_{n-k-1}/\sigma_{n-1}$, see [10, Section 3] for more detailed information. Next, we will generalize these formulas to Strichartz type transform.

The following theorem connects Strichartz transforms and the dual transforms of Semyanistyi type integrals.

Theorem 5.4. For a function $f = f(\tau)$ on $G(n, j)$,

$$ P_k^{\alpha*} R_{p,q}^k f = c P_j^{(\alpha+l)*} f , \quad c = \frac{2^l \pi^{l/2} \Gamma((n-j)/2)}{\Gamma((n-j-l)/2)}, \quad (5.9) $$

provided that either side of this equality exists in the Lebesgue sense.
Proof. Through Theorem 5.1 and the second equality in (5.6), to prove Theorem 5.4, we just need to prove for a good function \( f(\tau) \) on \( G(n,j) \), the following equality hold,

\[
R^* R_{p,q} f = R_{p,q} P^j f, \quad c = \frac{2^l \pi^{l/2} \Gamma((n-j)/2)}{\Gamma((n-j-l)/2)}.
\] (5.10)

In fact, through the definition of \( R_{p,q} \) (2.17) and \( R^* \) (2.22),

\[
R^* R_{p,q} f(x) = \int_{O(n)} R_{p,q} f(\gamma R^k + x) d\gamma
\]

\[
= \int_{O(n)} d\gamma \int_{O(k)} d\rho_1 \int_{O(n-k)} d\rho_2 \int_{\mathbb{R}^l} f(\gamma(\rho_1 \mathbb{R}^p, \rho_2 \mathbb{R}^q) + \rho_1 u + x)du
\]

\[
= \int_{O(n)} d\gamma \int_{\mathbb{R}^l} f(\gamma(\mathbb{R}^j + u) + x)du = c \int_{G_{n,j}} \int_{\xi} \frac{f(\xi + u + x)}{|u|^{n-j-l}} du,
\]

where \( c = \frac{2^l \pi^{l/2} \Gamma((n-j)/2)}{\Gamma((n-j-l)/2)} \). Owing to the second formula in (5.6), equality (5.10) is proved. Then (5.9) follows. \( \square \)

Remark 5.5.

1. In the case of \( p = q = 0 \), in the meantime \( j = 0 \) and \( l = k \), formula (5.9) coincides with the formula (5.7).

2. From the proof of Theorem 5.4, we have the following intertwining relation, which connects Strichartz transforms with the dual \( k \)-plane transforms and the dual Semyanistyi type integrals.

**Corollary 5.6.** For a function \( f = f(\tau) \) on \( G(n,j) \),

\[
R^* R_{p,q} f = c P^j f, \quad c = \frac{2^l \pi^{l/2} \Gamma((n-j)/2)}{\Gamma((n-j-l)/2)}.
\] (5.11)

provided that either side of this equality exists in the Lebesgue sense.

The following theorem connecting Strichartz transforms with the Semyanistyi type integrals.

**Theorem 5.7.** For a suitable function \( f = f(x) \) on \( \mathbb{R}^n \),

\[
R_{p,q} P_{\alpha}^j f(\eta, v) = c P_{\alpha}^j f(\eta, v), \quad c = \frac{2^q \pi^q \Gamma((n-k)/2)}{\Gamma((n-k-q)/2)}.
\] (5.12)

provided that either side of this equality exists in the Lebesgue sense.

**Proof.** Using Theorem (2.7), through the definition of Semyanistyi integral (5.5), to prove this theorem we just need to prove the following equality holds for a good function \( f \) on
\[ R^k_{p,q} R_j f(\eta, v) = c P^q_k f(\eta, v), \quad c = \frac{2^q \pi^q \Gamma((n - k)/2)}{\Gamma((n - k - q)/2)}. \] (5.13)

In fact,
\[
R^k_{p,q} R_j f(\eta, v) = \int_{G_q(\eta)} d_{\eta} Q \int_{G_p(\eta)} d_{\eta} P \int_{P_{\eta} \cap Q} R_j f([P, Q] + u + v) du
\]
\[
= \int_{G_q(\eta)} d_{\eta} Q \int_{G_p(\eta)} d_{\eta} P \int_{P_{\eta} \cap Q} du \int_{P} dx \int Q f(x + y + u + v) dy
\]
\[
= \frac{\sigma_{q-1}}{\sigma_{n-k-1}} \int_{\eta} du \int_{\eta} f(y + u + v) |y|^{k+q-n} dy
\]
\[
= \frac{\sigma_{q-1}}{\sigma_{n-k-1}} \int_{\mathbb{R}^n} f(x) |x - \tau|^{k+q-n} dx = c P^q_k f(\eta, v).
\]

where \( c = \frac{2^q \pi^q \Gamma((n - k)/2)}{\Gamma((n - k - q)/2)}. \) Then we finish the proof. \( \square \)

The proof of Theorem 5.7 implies the following corollary.

**Corollary 5.8.** For a suitable function \( f = f(x) \) on \( \mathbb{R}^n \),
\[ R^k_{p,q} R_j f(\eta, v) = c P^q_k f(\eta, v), \quad c = \frac{2^q \pi^q \Gamma((n - k)/2)}{\Gamma((n - k - q)/2)}, \] (5.14)
provided that either side of this equality exists in the Lebesgue sense.

**Remark 5.9.**

1. In the case of \( q = 0 \), Strichartz transform \( R^k_{p,q} \) reduces to “inclusion” transform \( R_{j,k} \). Then the formula (5.14) reads
\[ R_{j,k} R_j f(\eta, v) = R_k f(\eta, v). \] (5.15)

2. In the case of \( p = 0 \), Strichartz transform \( R^k_{p,q} \) reduces to “Gonzalez” transform \( R^k_j \). Then the formula (5.14) reads
\[ R^k_j R_j f(\eta, v) = c P^j_k f(\eta, v), \quad \text{where} \quad c = \frac{2^j \pi^j \Gamma((n - k)/2)}{\Gamma((n - k - j)/2)}. \] (5.16)

Analogue statements still hold for the transform \( R_{p,l}^j \), by interchanging the index \( j \) and \( k \), \( q \) and \( l \). In particular, we have the following conclusions:
1. For a suitable function \( g = g(\zeta) \) on \( G(n, k) \),
   \[
P^\alpha_j R_{p,l}^g = cP_k^{(\alpha+j+l)}g, \quad R^j_{p,l} R_{p,l}^g = cP_k^{l}g,
   \]
   where \( c = \frac{2^{\pi n/2} \Gamma((n-k)/2)}{\Gamma((n-k-q)/2)} \).

2. For a suitable function \( f = f(x) \) on \( \mathbb{R}^n \),
   \[
   R_j^p P_k^\alpha f(\xi, u) = cP_j^{l+\alpha} f(\xi, u), \quad R_j^l R_k f(\xi, u) = cP_l^i f(\xi, u).
   \]
   where \( c = \frac{2^{\pi n/2} \Gamma((n-j)/2)}{\Gamma((n-j-l)/2)} \).

5.3. Fuglede Type Equalities and Inversion Formulas.

**Theorem 5.10.** For a good function \( f = f(x) \) on \( \mathbb{R}^n \), we have the following two formulas,

(i) \[
P_k^{\beta_j} R_{p,q} P_j^\alpha f(x) = cI_n^{\alpha+\beta+j+l} f(x) = cI_n^{\alpha+\beta+k+q} f(x),
   \]
(ii) \[
P_j^\alpha R_{p,l}^j P_k^{\beta_j} f(x) = cI_n^{\alpha+\beta+j+l} f(x) = cI_n^{\alpha+\beta+k+q} f(x),
   \]
   where \( c = \frac{2^{\pi n/2} \Gamma((n-j)/2)}{\Gamma((n-j-l)/2)} \), both equalities hold provided that the Riesz potential \( I_n^{\alpha+\beta+k+q} f \) exists in the Lebesgue sense.

From the relationship between Semyanistyi integrals and Radon-John transforms, we just need to prove the following theorem.

**Theorem 5.11.** For a good function \( f = f(x) \) on \( \mathbb{R}^n \),

(i) \[
R_k^p R_{p,q} R_j f(x) = cI_n^{j+l} f(x) = cI_n^{k+q} f(x),
   \]
(ii) \[
R_j^p R_{p,l} R_k f(x) = cI_n^{j+l} f(x) = cI_n^{k+q} f(x),
   \]
   where \( c = \frac{2^{\pi n/2} \Gamma((n-j-l)/2)}{\Gamma((n-j+l)/2)} \), both equalities hold provided that the Riesz potential \( I_n^{k+q} f \) exists in the Lebesgue sense.

This Theorem is obvious through the Lemma 5.6, Theorem 2.7 and Theorem 2.5. It can also be obtained through Lemma 5.8 and Theorem 2.7.

**Remark 5.12.**

1. When \( q = 0 \), in the meantime \( j = p \), our formula (5.21) coincides with equality (2.31) of the inclusion transform case.

2. When \( p = 0 \), in the meantime \( j = q \), our formula (5.21) coincides with equality (2.30) of the Gonzalez transform case.

Theorem 5.11 enables us to reconstruct \( f \) from \( R_{p,q}^k f \) provided that \( f \) belongs to the range of the \( j \)-plane transform.
Theorem 5.13. Let \( f = R_j h, \ h \in L^s(\mathbb{R}^n) \). If \( 1 \leq s < n/(j + l) \), then

\[
f = c^{-1} R_j \mathbb{D}^{j+l}_n R^*_k R^k_{p,q} f, \quad c = \frac{2^{j+l} \pi^{(j+l)/2} \Gamma(n/2)}{\Gamma((n - j - l)/2)} \tag{5.23}
\]

where \( \mathbb{D}^{j+l}_n \) is the Riesz fractional derivative (2.6). More generally, if \( 0 < \alpha < n - j - l \) and \( 1 \leq s < n/(j + l + \alpha) \), then

\[
f = c^{-1} R_j \mathbb{D}^{j+l+\alpha}_n R^*_k \mathcal{I}^\alpha_n R^k_{p,q} f \tag{5.24}
\]

with the same constant \( c \).

Proof. By (5.21),

\[
c^{-1} R_j \mathbb{D}^{j+l}_n R^*_k R^k_{p,q} f = c^{-1} R_j \mathbb{D}^{j+l}_n R^*_k R^k_{p,q} R_j h = R_j \mathbb{D}^{j+l}_n \mathcal{I}^j_n h = R_j h = f.
\]

Further, combining (5.21) with the semigroup property of Riesz potentials, we obtain

\[
c^{-1} \mathcal{I}^\alpha_n R^*_k R^k_{p,q} f = c^{-1} \mathcal{I}^\alpha_n R^*_k R^k_{p,q} R_j h = \mathcal{I}^\alpha_n \mathcal{I}^j_n h = \mathcal{I}^{j+l+\alpha}_n h.
\]

However, by (2.27), \( \mathcal{I}^\alpha_n R^*_k R^k_{p,q} f = R^*_k \mathcal{I}^\alpha_n R^k_{p,q} f \). Hence

\[
\mathcal{I}^{j+l+\alpha}_n h = c^{-1} R^*_k \mathcal{I}^\alpha_n R^k_{p,q} f,
\]

and therefore \( h = c^{-1} \mathbb{D}^{j+l+\alpha}_n R^*_k \mathcal{I}^\alpha_n R^k_{p,q} f \). Applying \( R_j \) to both sides, we obtain (5.24). \( \square \)

At the case \( p = 0 \) and \( q = 0 \), these inversion formulas coincide with the Theorem 5.1 and Theorem 5.2 in [14], respectively. Interchanging the index \( j, k \) and \( q, l \), we have the following inversion formulas for transform \( R^j_{p,l} \):

Theorem 5.14. Let \( g = R_k h, \ h \in L^t(\mathbb{R}^n) \). If \( 1 \leq t < n/(k + q) \), then

\[
g = c^{-1} R_k \mathbb{D}^{k+q}_n R^*_j R^j_{p,q} g, \quad c = \frac{2^{k+q} \pi^{(k+q)/2} \Gamma(n/2)}{\Gamma((n - k - q)/2)} \tag{5.25}
\]

where \( \mathbb{D}^{k+q}_n \) is the Riesz fractional derivative (2.6). More generally, if \( 0 < \alpha < n - k - q \) and \( 1 \leq t < n/(k + q + \alpha) \), then

\[
g = c^{-1} R_k \mathbb{D}^{k+q+\alpha}_n R^*_j \mathcal{I}^\alpha_n R^j_{p,q} g \tag{5.26}
\]

with the same constant \( c \).

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