ABSTRACT. We introduce two new distances for zigzag persistence modules. The first uses Auslander-Reiten quiver theory, and the second is an extension of the classical interleaving distance. Both are defined over completely general orientations of the $A_n$ quiver. We compare the first distance to the block distance introduced by M. Botnan and M. Lesnick and obtain the full set of sharp Lipschitz bounds between the two (as bottleneck distances) over pure zigzag orientations. The final portion of the paper presents sharp Lipschitz bounds necessary for the extended interleaving distance to dominate the distance that is created from the Auslander-Reiten quiver. These bounds are obtained for general orientations of the $A_n$ quiver.

1.0.1 INTRODUCTION

Both classical 1-D persistent homology and zigzag persistent homology use data structures that fall under the same quiver theoretic notion: they are both orientations of Dynkin quivers of type $A_n$, which are written throughout as $A_n$ where $n$ is the number of vertices of the quiver.

Quiver theory treats all orientations of $A_n$ equally regarding the result that any representation of such a quiver (i.e., any persistence module over the underlying poset) decomposes into interval representations [Gab72], the collection of which in turn form a barcode—a stable topological invariant of the representation/persistence module (or the data set that generated it).

In this paper we propose two new distances on persistence modules over $A_n$-type quivers. We will spend the rest of the paper constructing them, laying out their properties and advantages, and proving stability results between these distances and some of those already in use in persistent homology literature.

We primarily focus our attention on the comparison of distances via their induced bottleneck distances (Definition 1.2.6): distances that first associate a pair of modules to their barcodes (collections of interval summands), and then pair up the elements of the barcodes in some “closest” manner.

Here we briefly introduce and summarize these two new distances on zigzag persistence modules and relay some of their most overt properties.

- $A_n$-modules as multisets of vertices of the Auslander-Reiten quiver.

The AR distance (section 2) can be applied to persistence modules over any orientation of $A_n$ and is a bottleneck distance by construction. When some notion of ‘endpoint parity’ between a pair of interval modules agrees, their distance is simply sum of difference between endpoints (an $\ell^1$-type distance when considering intervals to be coordinate pairs, as is commonly seen in persistence diagrams). The distance behaves differently when parity does not agree. Over pure zigzag orientations, this distance’s change in behavior relative to endpoint parity is a feature shared by the block distance [MBB18], which is reviewed in subsection 2.5.2 and compared in full with the AR distance in section 3.
The properties of the AR distance are strongly influenced by the algebra of the underlying quiver. For instance, in pure zigzag orientations, interval modules of [sink, sink] endpoint parity are close to projective simple modules, and those with [source, source] endpoint parity are close to injective simples modules (in this situation “closeness” is relative to support size). In general, when a pair of intervals has non-matching endpoint parity, the poset structure influences their distance to a much greater degree than similarity in supports.

- \(A_n\)-modules as persistence modules over a suspended poset.

The weighted interleaving distance (section 4) considers an arbitrary orientation of \(A_n\) as a series of connected ‘valleys’ (maximal upward posets of the form [source, \(\infty\)]), and then measures the distance between two modules by the depth of the valleys on which the intervals must be isomorphic. On all shallower valleys they are free to differ.

The general construction was pursued in our previous paper [MM17] for the purpose of applying interleaving distance to finite posets without inevitably encountering an excessive number of module pairs whose interleaving distance was infinite.

1.0.2 Contributions

A summary of our contributions are as follows:

- We provide full and sharp Lipschitz bounds between our AR distance and the block distance, with the latter treated as its own induced bottleneck distance (Theorem 3.4.6).

- Included as part of the elucidation of the AR distance is as a topic of potentially independent interest: we provide an explicit formulation of the Auslander-Reiten quiver for any orientation of \(A_n\) in Section 2.2. While this formulation follows from the Knitting Algorithm (see [Sch14] for details on the Knitting Algorithm and other methods of calculating the Auslander-Reiten quiver for orientations of \(A_n\)), our formulation provides full information about the Auslander-Reiten quiver without any iterative construction.

- We provide sharp bounds for the weighted interleaving distance to dominate the AR distance. (Theorem 4.1.2.)

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1.1 Preliminaries

Notation 1.1.1. Throughout, we say that a distance on a set \(X\) is a function \(d : X \times X \to [0, \infty]\) such that

1. \(d(x, x) = 0\) for all \(x \in X\),
2. \(d(x, y) = d(y, x)\) for all \(x, y \in X\), and
3. \(d(x, y) \leq d(x, z) + d(z, y)\) for any \(x, y, z \in X\).
That is, from the standard definition of ‘metric’ we surrender identification between points with $d(x, y) = 0$ and allow distances to take on infinite values.

**Definition 1.1.2.** A generalized persistence module (GPM) $F$ over a poset $P$ in a category $\mathcal{D}$ is a functor $F : P \to \mathcal{D}$. That is, $F$ is an assignment

- $x \to F(x)$ for all $x \in P$,
- $(x \leq y) \to F(x \leq y) \in \text{Hom}_\mathcal{D}(F(x), F(y))$ for all $x \leq y$ in $P$

such that, for any $x \leq y \leq z$, the inequalities are sent to morphisms satisfying $F(y \leq z) \circ F(x \leq y) = F(x \leq z)$.

The category of such functors is denoted $\mathcal{D}^P$, where morphisms in this category are given by natural transformations of functors.

A persistence module is a GPM with values in the category of finite dimensional vector spaces, and is the object of primary interest in this document.

### 1.1.1 Quivers

In this paper we will frequently view our underlying structures as both posets and as quivers. We would like to work with familiar persistent homology structures while applying quiver-theoretic machinery. The following is a short, formal definition of quivers, as well as an explanation for why they can be thought of as equivalent to posets in our setting.

**Definition 1.1.3.** A quiver is a quadruple $(Q_0, Q_1, h, t)$ where

- $Q_0$ is some finite set called the vertex set,
- $Q_1$ is a collection of arrows between vertices,
- $h : Q_1 \to Q_0$ is a map that sends each arrow to its destination (head), and
- $t : Q_1 \to Q_0$ is a map that sends each arrow to its source (tail).

A representation $V$ of a quiver $Q$ is

- a vector space $V(i)$ assigned to every vertex, and
- a linear map $V(a) : V(ta) \to V(ha)$ assigned to every arrow.

The space of finite-dimensional representations of a quiver $Q$, denoted $\text{rep}(Q)$, is a category with morphisms given pointwise, $f = \{f_i\}_{i \in Q_0} : V \to W$, such that they satisfy commutative squares $f(ha)V(a) = W(a)f(ta)$ for all arrows $a \in Q_1$.

Quivers may, in general, have closed loops or multiple arrows between the same pair of vertices. These features may prevent such quivers from being posets under the relation

$$\{x \leq y \text{ if and only if there exists a path } x \to y\}.$$ 

However, the converse—that posets always give rise to quivers in a canonical way—is true.

**Definition 1.1.4.** For a poset $P$, the Hasse quiver $Q(P)$ is the quiver given by:

- $Q_0 = P$ as a set of vertices.
- There exists an arrow $i \to j$ whenever $i \leq j$ in $P$, and there is no $k$ (distinct from $i$ and $j$) such that $i \leq k \leq j$. 

3
Under certain restrictions, quivers do give rise to posets in a fashion that inverts the Hasse construction. When there is such a bi-directional correspondence, as seen in the following proposition, the space of representations of the quiver is equivalent to the space of persistence modules over the poset.

**Proposition 1.1.5.** Let $Q$ be a quiver such that:
- $Q$ has no cycles (including stationary loops),
- for any two $i, j \in Q_0$, there exists at most one arrow between $i$ and $j$.

Then $Q$ is the Hasse quiver of some poset $P$. Furthermore, suppose $Q$ also satisfies:
- For $i, j \in Q_0$, there is at most one path from $i \rightarrow \ldots \rightarrow j$.

Then, the category of finite-dimensional representations of the quiver $Q$ is equivalent to the category of functors from the poset category $P$ to the category of finite-dimensional vector spaces. I.e.,
\[ \text{rep}(Q) \cong \text{vect}^P \]
as categories.

**Remark 1.1.6.** The extra condition above (at most one path $i \rightarrow \ldots \rightarrow j$) is necessary for the equivalence of categories for the following reason. Any GPM over a poset, by virtue of being a functor from a thin category (cardinality of any $\text{Hom}$-space is at most 1), has the property that the morphisms given by composition along any two parallel paths are equal; see Definition 1.1.2. Contrast this with Definition 1.1.3 in which there are no parallel-path commutativity conditions on a quiver representation.

(If one wished to obtain equivalence between the two categories while allowing for the existence of parallel paths, this would require the use of bound quivers: quivers with commutativity relations, for a general reference see [Sch14]. Such pursuits are not within the scope of this document.)

By virtue of this equivalence, from here onward we will denote quivers/posets by $P$, rather than $Q$.

The quiver of interest in this paper is $P = A_n$, the ‘straight line’ quiver, with arbitrary orientations for its arrows. It satisfies all the conditions of Proposition 1.1.5.

**Definition 1.1.7.** For $n \in \mathbb{N}$, an $A_n$-type quiver is any quiver with vertex set $\{1, \ldots, n\}$ whose arrow set consists of exactly one of $i \rightarrow i + 1$ or $i \leftarrow i + 1$ for every $i$.

The corresponding poset (whose Hasse quiver returns the original quiver) is given by
\[ 1 \sim \ldots \sim n \]
where each $\sim$ corresponds to $<$ (for quiver arrows of the form $\rightarrow$) or $>$ (for quiver arrows of the form $\leftarrow$).

$A_n$ will be said to be equioriented if all arrows face the same way.

$A_n$ will be said to have pure zigzag orientation if arrows alternate (i.e., each vertex is either a source or sink).

The following definition is a fundamental one to persistent homology.
Definition 1.1.8. For an orientation of $\mathbb{A}_n$, define the interval persistence module (indecomposable quiver representation) $[x, y]$ to be the one

$$[x, y](i) = \begin{cases} K & \text{if } 1 \leq x \leq i \leq y \leq n \\ 0 & \text{otherwise} \end{cases}$$

where $K$ is some base field. The internal morphisms of $[x, y]$ are $1_K$ when possible, and 0 otherwise.

From context it should always be clear when we mean the indecomposable $[x, y]$ or the $\mathbb{Z}$-interval $[x, y]$.

Lastly, we will often abbreviate interval persistence modules of the form $[x, x]$ as $[x]$.

For $P = \mathbb{A}_n$, as it turns out, every $\mathbb{A}_n$-representation — equivalently, every $P$ persistence module — is isomorphic to a direct sum of interval persistence modules. The original result cited below is quiver theoretic in origin, but this result has since been proved independently for pointwise finite dimensional persistence modules over $\mathbb{R}$ [Cra12].

Proposition 1.1.9 ([Gab72]). Representations / persistence modules over any $P = \mathbb{A}_n$ decompose into interval persistence modules. This decomposition is unique up to ordering and isomorphism of summands.

Furthermore, interval persistence modules are precisely the indecomposable persistence modules (up to isomorphism) of $P$.

For a very efficient exposition of the definitions and features of additive categories, categorical products and coproducts, and categories possessing unique decomposability properties, the authors recommend the paper [Kra15].

Notation 1.1.10. Throughout, by indecomposable representation of $P = \mathbb{A}_n$ we mean the unique representative of the isomorphism class that is precisely an interval representation.

1.1.2 The Auslander-Reiten Quiver

The following is a crucial piece of quiver theoretic machinery that renders possible the development of this paper’s first distance.

Definition 1.1.11. Given a quiver $P$, its Auslander-Reiten (AR) quiver is a new quiver in which:

- the vertex set is the collection of isomorphism classes of indecomposable representations of $P$,
- an arrow exists from one vertex to another whenever there exists an irreducible morphism between the corresponding $P$-indecomposables.

When $P = \mathbb{A}_n$, there are finitely many indecomposable representations up to isomorphism, and representatives of the distinct isomorphism classes can be chosen to be precisely the collection of interval representations of $P$ (Proposition 1.1.9). That is, the Auslander-Reiten quiver of some $P = \mathbb{A}_n$ has vertex set consisting of the interval representations of $P$.

See any of [ASS06, Sch14, Kra08, HD17] for general introductions to Auslander-Reiten theory.

What is important to note for now is that, when $P = \mathbb{A}_n$, its Auslander-Reiten quiver has a finite vertex set, unique arrows, no closed loops, and is a connected graph ([ARS97] VI Thm 1.4). The nature of the Auslander-Reiten quiver of any $P = \mathbb{A}_n$ will be discussed in detail in Subsection 2.2.
1.2 **Classic Persistent Homology Distances**

We now define two fundamental distances to persistent homology.

### 1.2.1 Interleaving Distance

The interleaving distance is a distance on generalized persistence modules with values in any category $\mathcal{D}$ over any poset $P$ (Definition 1.1.2). We offer the following definitions in their full generality, though in the remainder of the paper they will be applied only to persistence modules (GPMs with values in $\text{vect}$) over very specific posets.

We first define translations, which are used to ‘shift’ GPMs within a poset and are how the size of an interleaving is measured.

**Definition 1.2.1.** A translation $\Lambda$ on a poset $P$ is a map $\Lambda : P \to P$ such that

- $x \leq \Lambda x$ for all $x \in P$,
- if $x \leq y$ in $P$, then $\Lambda x \leq \Lambda y$.

The height of a translation is

$$h(\Lambda) = \max_{x \in P} \{ d(x, \Lambda x) \}$$

where $d$ is some distance on $P$.

The collection of translations over a poset $P$ form a monoid with left action on any $\mathcal{D}^P$, given by the pointwise statement

$$F\Lambda(x) = F(\Lambda x) \text{ for all } x \in P.$$ 

In brief, before the full definition below, an interleaving between two GPMs is a translation $\Lambda$ and a pair of morphisms from each GPM to a $\Lambda$-shift of the other such that certain commutativity conditions are fulfilled.

**Definition 1.2.2.** An interleaving between two GPMs $F,G$ in $\mathcal{D}^P$ is a translation $\Lambda$ on $P$ and a pair of morphisms (natural transformations) $\phi : F \to G\Lambda$, $\psi : G \to F\Lambda$ such that the following diagram commutes:

```
F  \rightarrow F\Lambda  \rightarrow F\Lambda\Lambda
|   \phi      \downarrow \phi\Lambda |
|   \downarrow \psi     \downarrow   |
G  \rightarrow G\Lambda  \rightarrow G\Lambda\Lambda
```

Alternatively, we say that $F,G$ are $\Lambda$-interleaved.

The interleaving distance between $F$ and $G$ is

$$D_{IL}(F,G) = \inf \{ \epsilon : F, G \text{ have a } \Lambda \text{-interleaving with } h(\Lambda) = \epsilon \}$$

The translations $\phi$ and $\psi$ are sometimes referred to as “approximate isomorphisms”, and the interleaving distance can be thought of as the shift distance by which there fails to be a true isomorphism between the persistence modules.
Remark 1.2.3. The above definition is not quite the traditional one seen most often in the literature (see [BdS13]). In many definitions there are two translations Λ and Γ (one to shift $F$, and the other to shift $G$), and the height of the interleaving is the height of the larger translation. In the posets we are interested in, the values of the interleaving distance do not change when allowing for two distinct translations rather than using the same translation twice. So, for the sake of simplicity, and without altering the distance, we have reduced Definition 1.2.2 to a statement involving only a single translation Λ.

The collection of translations on a poset $P$ is itself a poset under the partial order given by the relation

$$\Lambda \leq \Gamma \text{ if } \Lambda(x) \leq \Gamma(x) \text{ for all } x \in P.$$ 

There is rarely a unique translation of a given height, though occasionally it is easier to assume that we are using a full translation of some height.

Remark 1.2.4. By a full translation of height $\epsilon$, we will mean a maximal element in the poset of translations that has height $\epsilon$. In the case $P = \mathbb{Z}, \mathbb{R}$, there is always a unique full translation of height $\epsilon$: the translation $\Lambda_{\epsilon}(x) = x + \epsilon$ for all $x \in \mathbb{R}$. In posets that are not totally ordered, there may be multiple distinct full translations of certain heights.

By the next result, any $\epsilon$-interleaving can always be taken as using a full translation of height $\epsilon$.

Proposition 1.2.5. Let $\Lambda, \Lambda'$ be two translations over some poset $P$ such that $\Lambda' \geq \Lambda$, and let $F, G$ be two GPMs in $D^P$ for some $D$. If $F, G$ are $\Lambda$-interleaved, then $M, N$ are $\Lambda'$-interleaved.

### 1.2.2 Bottleneck Distances

We first define the general notion of a bottleneck distance (Definition 1.2.6), then present the classic bottleneck distance (Example 1.2.8), and lastly put forward the meaning of a general distance’s induced bottleneck distance (Remark 1.2.9).

A bottleneck distance (also a Wasserstein metric — see [BSS18]) acts on pairs of multisets of some set $\Sigma$. It requires

- a distance $d$ on $\Sigma$, and
- a function $W : \Sigma \to [0, \infty)$

such that

$$|W(f) - W(g)| \leq d(f, g),$$

(Δ-ineq)

for all $f, g \in \Sigma$.

Let $\Sigma$ be some set, and $F, G$ two multisets (subsets with multiplicities of elements) of $\Sigma$. A matching between $F$ and $G$ is a bijection

$$x : F' \leftrightarrow G'$$

where $F' \subset F, G' \subset G$.

The height of a matching $x : F \leftrightarrow G$ is

$$h(x) = \max \{ \max_{f \in F} \{d(f, x(f))\}, \max_{f \notin F'} \{W(f)\}, \max_{g \notin G'} \{W(g)\} \}.$$ 

That is, take the maximum over all distances (using $d$) between paired elements, as well as the maxima over all of the ‘widths’ (using $W$) of the unpaired elements of $F$ and $G$. 

Definition 1.2.6. Given a set $\Sigma$, and any functions $d$ and $W$ as above holding to the $\triangle$-ineq relationship, the bottleneck distance generated by $d$ and $W$ between two multisets $F, G$ of $\Sigma$ is

$$D(F, G) = \min\{h(x) : x \text{ is a matching between } F \text{ and } G\}.$$ 

The following connects bottleneck distances to persistence modules. From [Cra12], this can be generalized to $\mathbb{R}$ persistence modules.

Definition 1.2.7. For $\mathbb{A}_n$, let $\Sigma$ denote the set of (isomorphism classes of) indecomposable persistence modules: i.e., its intervals.

For a persistence module $M$ over $\mathbb{A}_n$, define its barcode to be the multiset of $\Sigma$ containing exactly the summands in its decomposition (with existence and uniqueness guaranteed by Proposition 1.1.9):

$$B(M) = \{[x_i, y_i]_{i \in I}, \text{ where } M = \bigoplus_{i \in I} [x_i, y_i]\}.$$ 

Example 1.2.8. The ‘classical’ bottleneck distance on persistence modules over $\mathbb{R}$) is the one given by

- $d(f, g) = D_{IL}(\{f\}, \{g\})$
- $W(f) = D_{IL}(\{f\}, \emptyset),$

where $D_{IL}$ is the interleaving distance of Definition 1.2.2.

Let $f = [x_1, y_1], g = [x_2, y_2]$ be indecomposable/interval persistence modules over $\mathbb{R}$. Unpacking the definition of interleaving distance yields the equations:

- $d(f, g) = \max\{|x_1 - x_2|, |y_1 - y_2|\}$ and
- $W(f) = I(f, 0) = 1/2(y - x).$

This is precisely the ($\ell^\infty$ or $\infty$-Wasserstein) bottleneck distance that is most commonly used to measure distance between persistence diagrams in persistent homology literature. □

Remark 1.2.9. Let $C$ be any Krull-Schmidt category [Kra15] and $D$ any distance on the collection of objects in the category. Then there is a unique or canonical bottleneck distance induced by $D$, that being the one in which any two objects $X, Y$ become associated to the multisets corresponding to their Krull-Schmidt decompositions

$$X = X_1 \oplus \ldots \oplus X_m, \quad Y = Y_1 \oplus \ldots \oplus Y_n,$$

and the bottleneck distance between those multisets is given by

- $d(X_i, Y_j) = D(X_i, Y_j)$ and
- $W(X_i) = D(X_i, 0).$

1.2.3 Comparison of Bottleneck Distances

As one of the goals of this paper is finding minimal Lipschitz bounds between bottleneck distances, we discuss the relationship between comparing bottleneck distances directly, and comparing their component $d$’s and $W$’s.

For two bottleneck distances $D_1 = \{d_1, W_1\}$ and $D_2 = \{d_2, W_2\}$, while the inequality $D_1 \leq D_2$ implies $W_1 \leq W_2$, it does not necessitate that $d_1 \leq d_2$. This has the potential to be a frustrating obstacle to comparing different bottleneck distances.
To remedy this, we define a canonical $d$ and $W$ for a given bottleneck distance $D$ that will allow for a more natural means of comparison.

**Definition 1.2.10** (Minimal generators). For a bottleneck distance $D = \{d, W\}$ on multisets of some set $\Sigma$, define $\bar{d}(\sigma, \tau) = D(\{\sigma\}, \{\tau\})$. Then $\bar{d} \leq d$, and the pairs $\{d, W\}$ and $\{\bar{d}, W\}$ both generate the same $D$. Call $\{\bar{d}, W\}$ the minimal generators of the bottleneck distance $D$.

A bottleneck distance $D$ fully recovers its minimal generators. Specifically:

- $\bar{d}(\sigma, \tau) = D(\{\sigma\}, \{\tau\})$, and
- $W(\sigma) = D(\{\sigma\}, \emptyset)$.

We now get the desired comparison statement:

**Proposition 1.2.11.** For two bottleneck distances $D_1 = \{d_1, W_1\}$ and $D_2 = \{d_2, W_2\}$, $D_1 \leq D_2$ if and only if $\bar{d}_1 \leq \bar{d}_2$ and $W_1 \leq W_2$.

**Proof.** The forward implication is immediate from the above statements about the recovery of $\bar{d}, W$ from $D$. The reverse implication is immediate from the definition of $D$ (as is the stronger statement: $d_1 \leq d_2$ and $W_1 \leq W_2 \implies D_1 \leq D_2$). □

**Notation 1.2.12.** From this point onward, we allow $D(\{\sigma\}, \{\tau\})$ to be shortened to $D(\sigma, \tau)$ for bottleneck distances.

## 2 AR-BOTTLENECK DISTANCE

This bottleneck distance uses the graph-structure of some original quiver $Q$’s corresponding Auslander-Reiten quiver as a means of measuring the distance between indecomposable persistence modules.

### 2.1 DEFINITIONS

Let $Q = A_n$. Let $Q'$ be the AR quiver of $Q$. For indecomposables $\sigma, \tau$ of $Q$, let $p = p_0 \ldots p_l$ denote an unoriented path in $Q'$ from $\sigma$ to $\tau$. The tail and head of a path are those of the first and last vertex, respectively: $tp = t p_l = \sigma$, and $hp = h p_0 = \tau$.

**Definition 2.1.1.** Define the AR distance between two indecomposables to be

$$\delta_{\text{AR}}(\sigma, \tau) = \min_{p: \sigma \to \tau} \left\{ \sum_{i=1}^{l-1} |\dim(Q'(hp_i)) - \dim(Q'(tp_i))| \right\},$$

where dimension of an indecomposable $M$ of $Q$ (equivalently, a vertex of $Q'$) is

$$\dim(M) = \sum_{i \in Q_0} \dim_K M(i), \text{ i.e., } \dim([x, y]) = y - x + 1.$$

That is, $\delta_{\text{AR}}(\sigma, \tau)$ is the dimension-weighted path-length between $\sigma$ and $\tau$, minimized over all possible paths.

**Example 2.1.2.** See Figure 1. Consider the interval modules $[2, 3], [3, 6]$.

- **Figure 1a:** $\delta_{\text{AR}}([2, 3], [3, 6]) = 4$.
- **Figure 1b:** $\delta_{\text{AR}}([2, 3], [3, 6]) = 8$. 

(A) Equi-orientation of $A_8$ and its corresponding AR quiver.

(B) Orientation of $A_8$ and its corresponding AR quiver.

(C) Zigzag orientation of $A_8$ and its corresponding AR quiver.

**Figure 1.** Three orientations of $A_8$ and their AR quivers, where edges of weight more than 1 are drawn with double lines and labeled by the difference in dimensions between the two indecomposables that they connect.
• Figure 1c: $\delta_{AR}([2, 3], [3, 6]) = 10$.

**Definition 2.1.3.** Define the AR bottleneck distance $D_{AR}$ on the space of indecomposable representations of $Q$ to be the bottleneck distance induced by:

- $d_{AR}(\sigma, \tau) = \delta_{AR}(\sigma, \tau)$,
- $W_{AR}(\sigma) = \min_{t \in Q_0} \{\delta_{AR}(\sigma, [t])\} + 1$.

We can immediately check that $D_{AR}$ is indeed a bottleneck distance.

**Proposition 2.1.4.** $D_{AR}$ satisfies $\Delta$-ineq.

**Proof.** Simply note that for any $\sigma, \tau$, and any simple $[t]$, by the graph-distance definition of $\delta_{AR}$ it is immediate that

$$d_{AR}(\sigma, [t]) \leq d_{AR}(\sigma, \tau) + d_{AR}(\tau, [t]),$$

and so, minimizing over $[t]$ with respect to $W_{AR}(\tau)$,

$$W_{AR}(\sigma) \leq d_{AR}(\sigma, [t]) + 1 \leq d_{AR}(\sigma, \tau) + W_{AR}(\tau).$$

Combining with the symmetric statement (swapping $\sigma$ and $\tau$) we get the full statement of the equation $\Delta$-ineq. □

**Remark 2.1.5.** The reason for the $+1$ in the definition of $W_{AR}$ above is simply that there are no zero representations in the AR quiver. As in [EH14], we account for the distance to zero being distance to a simple indecomposable, plus one additional traversal (of dimension-weight 1).

Put another way, we attach a zero representation to every simple indecomposable in the AR quiver (see Figure 1a). For $Q = A_n$, let $Q'$ denote the AR quiver of $Q$ supplemented with the vertices 0; for all vertices $i$ of $Q$, and with extra edges $[i] \rightarrow 0_i$. Then we may alternatively define $W_{AR}(\sigma) = \min_{i \in Q_0} \{\delta_{AR}(\sigma, 0_i)\}$.

**Example 2.1.6.** See Figure 1. Consider the interval modules $[2, 3], [3, 6]$.

- Figure 1a: $D_{AR}([2, 3], [3, 6]) = 4$.
- Figure 1b: $D_{AR}([2, 3], [3, 6]) = 4$.
- Figure 1c: $D_{AR}([2, 3], [3, 6]) = 8$.

## 2.2 AR Quiver Construction Algorithm

From here we present an algorithm for determining the shape of the Auslander-Reiten quiver for any quiver of the form $Q = A_n$. This algorithm arises as a consequence of the Knitting Algorithm (see [Sch14] Chapter 3.1.1), but has been streamlined to the specific case of $Q = A_n$, and is able to elucidate the full structure of such AR quivers without the sequential construction method that the Knitting Algorithm and other similar methods require.

We maintain the convention of many quiver theoretic publications, in which the AR quiver is drawn with arrows always directed left to right, with the leftmost indecomposables being simple projectives and the rightmost indecomposables being simple injectives. Vertical orientation is arbitrary, but will be fixed under the following method. Key to this structural result about AR quivers for arbitrary orientations of any $A_n$ is the fact that the indecomposables fit into a diagonal grid with axes for the left and right endpoints of the intervals. The algorithm instructs the formation of these axes, which subsequently induce the entire shape of the AR quiver.
Notation 2.2.1. There are two separate and obvious orderings on the vertices of any orientation of $A_n$, the first being the ordering of the vertices according to their labeling as a subset of $\mathbb{Z}$, and the second being the ordering given by the poset relation $\leq_P$. The following discussions are carried out in the language of the vertices as a subset of $\mathbb{Z}$. So, by all comparative words (increasing, decreasing, greater, lesser) we will mean relative to the inherited $\mathbb{Z}$-ordering of the vertices from left to right in the poset.

Algorithm 2.2.2. The construction of the left and right $(x \text{ and } y)$ axes of the AR quiver for some $Q = A_n$ are as follows.

- For the $x$-axis (south west to north east), list the vertices in the following order:
  Take all vertices of $A_n$ that are in some segment of the form $(\text{min}, \text{next max})$, and list them on the axis in reverse $\leq_Z$ order. Then, take all remaining vertices and list them in forward $\leq_Z$ order.
  Note that the values of this $x$-axis always increase away from $x = 1$.

- For the $y$-axis (north west to south east), list the vertices in the following order:
  Take all vertices of $A_n$ that are in some segment of the form $(\text{max}, \text{next min})$, and list them on the axis in forward $\leq_Z$ order. Then, take all remaining vertices and list them in reverse $\leq_Z$ order.
  Note that the values of this $y$-axis always increase toward $y = n$.

Example 2.2.3. In Figure 2, we represent an orientation of $A_n$ with an implied arbitrary density of vertices along the edges. Segments of the poset are taken and rearranged to form the $x$ and $y$ axes according to the algorithm.

Notation 2.2.4. From the separation made by the diagonals $x = 1$ and $y = n$, we label the corresponding regions of the AR quiver by the four cardinal compass directions.

$\mathcal{E}_Q \subset \Sigma_Q$ is the collection of all interval modules $[x, y]$ where the vertex $x$ is contained in some $Q$ interval of the form $(\text{sink, next source})$, and $y$ is in some $(\text{source, next sink})$ (and $x \neq 1, y \neq n$).

$\mathcal{W}_Q \subset \Sigma_Q$ is the collection of all interval modules $[x, y]$ where the vertex $x$ is contained in some $(\text{source, next sink})$, and $y$ is in some $(\text{sink, next source})$ ($x \neq 1, y \neq n$).

$\mathcal{S}_Q \subset \Sigma_Q$ is the collection of all interval modules $[x, y]$ where the vertex $x$ is contained in some $(\text{sink, next sink})$, and $y$ is in some $(\text{sink, next source})$ ($x \neq 1, y \neq n$).

$\mathcal{N}_Q \subset \Sigma_Q$ is the collection of all interval modules $[x, y]$ where the vertex $x$ is contained in some $(\text{sink, next source})$, and $y$ is in some $(\text{sink, next source})$ ($x \neq 1, y \neq n$).

Let $\bar{E}$ (similarly $\bar{W}$, $\bar{S}$, $\bar{N}$) denote the original region along with all diagonal modules (those with either $x = 1$ or $y = n$) that are adjacent to it in the AR quiver. In addition, in all four cases, let this set also include the module $[1, n]$.

Remark 2.2.5. Within each of the regions $\bar{E}$, $\bar{W}$, $\bar{S}$, and $\bar{N}$, the $x$ and $y$ coordinate axes are monotone (Figure 3).

The following is a direct consequence of Algorithm 2.2.2 (and Remark 2.2.5).

Proposition 2.2.6. Formula for $\delta_{AR}$.

Let $\sigma = [x_1, y_1]$ and $\tau = [x_2, y_2]$ be indecomposables over $Q$. Then the graph distance $\delta_{AR}(\sigma, \tau)$ of Definition 2.1.3 is given by

$$\delta_{AR}(\sigma, \tau) = \delta^x(x_1, x_2) + \delta^y(y_1, y_2),$$

where

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Figure 2. An orientation of $\mathbb{A}_n$ and the subsequent arrangements of the $x$ and $y$ axes.

Figure 3. AR quiver for an orientation on $\mathbb{A}_n$. The purple arrows denote the direction of decreasing dimension of indecomposables (always away from the diagonals $x = 1$ and $y = n$).
where
\[
\delta^x(\sigma, \tau) = \begin{cases} 
|x_1 - x_2| & \text{if } \sigma, \tau \in \bar{\mathcal{E}} \cup \bar{\mathcal{N}} \\
1 + x_1 + x_2 - 1 & \text{otherwise.}
\end{cases}
\]

and
\[
\delta^y(\sigma, \tau) = \begin{cases} 
|y_1 - y_2| & \text{if } \sigma, \tau \in \bar{\mathcal{W}} \cup \bar{\mathcal{N}} \\
-n + y_1 + n - y_2 & \text{otherwise.}
\end{cases}
\]

Proposition 2.2.6 follows immediately from the monotonicity of the two axes in each of the four regions of the AR quiver.

### 2.3 Distance to Zero in $D_{\text{AR}}$

The dimension of an indecomposable is a lower bound for its $W_{\text{AR}}$ value. The following characterizes precisely when this is achieved.

**Proposition 2.3.1.** For any indecomposable $\sigma = [x, y]$, $W_{\text{AR}}(\sigma) \geq y - x + 1$. Furthermore, $W_{\text{AR}}(\sigma) = y - x + 1$ if and only if there is a path of decreasing dimension from $\sigma$ to a simple indecomposable in the AR quiver.

**Proof.** The first statement is immediate from the dimension-weighting of the edges in the definition of $\delta_{\text{AR}}$ (Definition 2.1.1) and the induced distance $D_{\text{AR}}$ (Definition 2.1.3).

Let $\sigma = [x, y]$ be an indecomposable with decreasing path to some simple $[t]$. Then necessarily $x \leq t \leq y$, and the existence of a decreasing path guarantees that $[x, y]$ and $[t]$ are in the same compass region. Hence, $\delta_{\text{AR}}([x, y], [t]) = t - x + y - t = y - x$, and so $W_{\text{AR}}(\sigma) = y - x + 1 = \dim(\sigma)$.

The converse also follows from the definitions cited above. If there is not a path of decreasing dimension, then any path of minimal weight from $[x, y]$ to $[t]$ must be of the form
\[
[x, y] \rightarrow \ldots \rightarrow [x_1, y_1] \rightarrow [x_2, y_2] \rightarrow \ldots \rightarrow [t]
\]
where $[x, y] \supset [x_1, y_1] \subset [x_2, y_2] \supset [t]$. Then, $\delta_{\text{AR}}([x, y], [t]) \geq t - x + y - t + (x_1 - x_2) + (y_2 - y_1)$ where at least one of the parenthetical terms is strictly positive.

**Corollary 2.3.2.** For any indecomposable $[x, y] \in \bar{\mathcal{E}} \cup \bar{\mathcal{W}}$,
\[
W_{\text{AR}}(\sigma) = \dim(\sigma).
\]

**Proof.** Note that the projective simple and injective simple indecomposable modules form (respectively) the outer corners of the east and west regions, and it is immediate from the shape of the AR quiver (Algorithm 2.2.2) that there are decreasing paths from any module in $\bar{\mathcal{E}}$ or $\bar{\mathcal{W}}$ to one of these.

For any indecomposable in the north and south regions, from Figure 2 we see that there exists a path of decreasing dimension to the flat north or south boundary, but these boundaries are not comprised of exclusively simple indecomposables. This complicates the situation for $W_{\text{AR}}(\sigma)$ when $\sigma \in \mathcal{N} \cup \mathcal{S}$. 


\[ \text{Figure 4. From Example 2.3.4. These are the truncated views of an AR quiver highlighting the structures of its north and south boundaries. The indecomposables with } W_{\text{AR}} > \dim \text{ are outlined.} \]

**Definition 2.3.3.** The north boundary is the collection of indecomposables that comprise the very top of the AR quiver. As a consequence of Algorithm 2.2.2 (see also Notation 2.2.4) this is exactly the set 

\[ B_N = \{ N_i = [\text{source}, \text{next sink}] \} \cup \{ [s] : s \notin \bigcup_i N_i \} \subset \mathcal{N}. \]

The intervals are listed left to right on the boundary of the AR quiver in increasing order of their endpoints (as a subset of \( \mathbb{Z} \)). This is the construction pictured above: the north boundary is all red intervals and blue simples listed in sequence according to \( \leq_{\mathbb{Z}} \).

The south boundary is 

\[ B_S = \{ S_j = [\text{sink}, \text{next source}] \} \cup \{ [s] : s \notin \bigcup_j S_j \} \subset \mathcal{S}. \]

These are listed left to right in the AR quiver in decreasing order (as a subset of \( \mathbb{Z} \)).

**Example 2.3.4.** Consider the orientation of \( Q = \mathbb{A}_{10} \) and its north and south boundaries as seen in Figure 4.

The red intervals are the starting points for finding intervals with \( W_{\text{AR}} > \dim \). Do note first that by Corollary 2.3.2 the red intervals \([1, 2]\) and \([9, 10]\) in fact satisfy \( W_{\text{AR}} = \dim \) as they are in \( \mathcal{W} \) and \( \mathcal{E} \) respectively.

The boundary intervals contained strictly within \( \mathcal{N} \) or \( \mathcal{S} \) are of potential concern. Any non-simple such indecomposables have \( W_{\text{AR}} > \dim \). This is immediate by observing that all paths leading away from these indecomposables are paths of *increasing* dimension, violating the condition of Proposition 2.3.1. These are still not the only intervals with \( W_{\text{AR}} > \dim \), however. In this example, we see that the full collection of such intervals is

- North: \([4, 5]\), \([8, 9]\).
- South: \([5, 8]\), \([2, 4]\), \([2, 8]\).

The southern collection of intervals manifest the final feature of interest: since the boundary intervals \([5, 8]\) and \([2, 4]\) are adjacent, the interval \([2, 8]\) caught above them also has no decreasing path to a simple indecomposable. \( \square \)

The preceding discussion motivates the following classification.
Definition 2.3.5. For an orientation $Q$ of $A_n$, define $hull(Q)$ to be the union of the following sets:

$$H_N = \{[source, sink] \subset [2, n - 1] : any subintervals of the form [sink, next source] have length one\},$$
and

$$H_S = \{[sink, source] \subset [2, n - 1] : any subintervals of the form [source, next sink] have length one\}.$$

Each set vacuously includes the intervals with no interior subintervals of opposite orientation.

Call $H_N \subset N$ the north hull and $H_S \subset S$ the south hull.

To conclude this section we will provide explicit formulas for $W_{AR}(\sigma)$ when $\sigma \in hull(Q)$, resulting in upper and lower bounds on $W_{AR}$ (Proposition 2.3.8).

Lemma 2.3.6. Values of $W_{AR}$ for $hull(P)$.

Let $Q$ be some orientation of $A_n$ with non-empty hull. Suppose $[x, y] \in H_N$ (symmetrically, $[x, y] \in H_S$). Define $[x, y]_\bullet$ to be the largest interval containing $[x, y]$ that is also in $H_N$. Let $e = x_\bullet - 1$ and $E = y_\bullet + 1$. Then $W_{AR}([x, y])$ is attained by passing through one of the simples $[e]$ or $[E]$. That is,

$$W_{AR}([x, y]) = \min\{\delta_{AR}([x, y], [e]), \delta_{AR}([x, y], [E])\} + 1.$$

Moreover, the precise distances to these indecomposables are given by

$$\delta_{AR}([x, y], [e]) = \begin{cases} 
  x + y - 2 & \text{if } e > 1 \text{ and is the leftmost sink} \\
  x + y - 2e & \text{otherwise}
\end{cases}$$

and

$$\delta_{AR}([x, y], [E]) = \begin{cases} 
  2E - (x + y) & \text{if } E < n \text{ and is the rightmost source} \\
  2n - (x + y) & \text{otherwise}.
\end{cases}$$

Proof. Let $[x, y] \in H_N$, meaning that $x$ is a source and $y$ is a sink. Let $1 \leq t \leq n$.

Case $t < x_\bullet$:
We proceed by possible regions in which $[t]$ may lie and give the corresponding $\delta_{AR}$.

$$\begin{align*}
[t] \in \tilde{N} : & \quad \delta_{AR}([x, y], [t]) = x + y - 2t & \text{(low-N)} \\
[t] \in \tilde{E} \setminus \tilde{N} : & \quad \delta_{AR}([x, y], [t]) = 2(n - t) - (y - x) & \text{(low-E)} \\
[t] \in \tilde{W} \setminus \tilde{N} : & \quad \delta_{AR}([x, y], [t]) = x + y - 2 & \text{(low-W)} \\
[t] \in \tilde{S} : & \quad \delta_{AR}([x, y], [t]) = 2(n - 1) - (y - x) & \text{(low-S)}
\end{align*}$$

Case $y_\bullet < t$:

$$\begin{align*}
[t] \in \tilde{N} : & \quad \delta_{AR}([x, y], [t]) = 2t - x - y & \text{(high-N)} \\
[t] \in \tilde{E} \setminus \tilde{N} : & \quad \delta_{AR}([x, y], [t]) = 2n - x - y & \text{(high-E)} \\
[t] \in \tilde{W} \setminus \tilde{N} : & \quad \delta_{AR}([x, y], [t]) = 2(t - 1) - (y - x) & \text{(high-W)} \\
[t] \in \tilde{S} : & \quad \delta_{AR}([x, y], [t]) = 2(n - 1) - (y - x) & \text{(high-S)}
\end{align*}$$

Case $x_\bullet \leq t \leq y_\bullet$:

The only possibilities are

1. $[t]$ is some source with $x_\bullet \leq t < y_\bullet$, so $[t]$ is in the east region. That is,

$$\delta_{AR}([x, y], [t]) = |x - t| + 2n - y - t.$$

Clearly, this value is minimized by all sources $x \leq m < y_\bullet$. Choosing any of these gives us

$$\begin{align*}
\delta_{AR}([x, y], [t]) = \delta_{AR}([x, y], [x]) = 2n - x - y. & \quad \text{(mid-E)}
\end{align*}$$
(2) \( [t] \) is some sink with \( x_\bullet < t \leq y_\bullet \), so \( [t] \) is in the west region. That is,
\[
\delta_{AR}([x,y],[t]) = x - 1 + t - 1 + |y - t|.
\]
Clearly, this value is minimized by all sinks \( x_\bullet < m \leq y \). Choosing any of these gives us
\[
\delta_{AR}([x,y],[t]) = \delta_{AR}([x,y],[y]) = x + y - 2. \tag{mid-W}
\]

(3) \( [t] \) is anything else, in which case it is interior to a segment of the form \([\text{source, next sink}]\), and thus lies on the south boundary. That is,
\[
\delta_{AR}([x,y],[t]) = 2(n - 1) - (y - x). \tag{mid-S}
\]

We exclude various equations from consideration.

- It is easy to check that (low-N) \( \leq \) (low-W) \( \leq \) (high-W) \( \leq \) (high-S). As \( x_\bullet \) is a source and \( x_\bullet \geq 2 \), there always exists some sink \( t < x_\bullet \) (and thus \( [t] \in W_\bullet \)), so we need never use the biggest two equations.

- Similarly, (high-N) \( \leq \) (high-E) \( \leq \) (low-E) \( \leq \) (low-S). As \( y_\bullet \) is a sink and \( y_\bullet \leq n - 1 \), there always exists some source \( t > y_\bullet \) (and thus \( [t] \in E_\bullet \)), so we need never use the biggest two equations.

- All mid-type equations are unnecessary for consideration as well. Simply note that (mid-E) = (high-E), (mid-W) = (low-W), and (mid-S) = (low,high-S).

From this, we can conclude that no matter the poset orientation, the only candidates for minimizing \( \delta_{AR}([x,y],[t]) \) are (low-N), (low-W), (high-N), and (high-E).

The only time that there is no (low-N) candidate is if \( e \) is the leftmost sink and \( e \neq 1 \). But in this case, \( e = x_\bullet - 1 \) is a candidate for (low-W). Conversely, if there is any (low-N) candidate, then \( e = x_\bullet - 1 \) is also a candidate, and minimizes the equation.

The symmetric statements are true of (high-N) and (high-E), which are minimized by substituting \( E \).

The statement of the lemma follows. \( \square \)

**Lemma 2.3.7.** If \([x,y] \in (\mathcal{N} \cup \mathcal{S}) \setminus \text{hull}(P)\), then \( W_{AR}([x,y]) = \dim([x,y]) \).

**Proof.** If \([x,y] \not\in \text{hull}(P)\), then there exists \( t \in [x,y] \) such that \([t] \) is on the boundary of the same region in which \([x,y] \) lies. Then \( \delta_{AR}([x,y],[t]) = y - t + x - t \), and so \( W_{AR}([x,y]) = y - x + 1 = \dim([x,y]) \). \( \square \)

The subsequent proposition follows from Lemmas 2.3.6, 2.3.7 and Corollary 2.3.2:

**Proposition 2.3.8.** All intervals \( \sigma \) have the property that
\[
\dim(\sigma) \leq W_{AR}(\sigma) \leq n.
\]
The set \( \text{hull}(P) \) is precisely the collection of intervals \( \sigma \) such that \( W_{AR}(\sigma) > \dim(\sigma) \). Furthermore, the diameter \( W_{AR} = n \) is always attained by the indecomposable \([1,n]\).

And, as \( D_{AR}(\sigma,\tau) \leq \max\{W_{AR}(\sigma),W_{AR}(\tau)\} \) for all pairs \( \sigma,\tau \), we get the following corollary.

**Corollary 2.3.9.** For any \( P = \mathbb{A}_n \), \( D_{AR} \leq n \).
2.4 Behavior of $D_{\text{AR}}$ on Pure Zigzag Orientations

Recall that in Definition 1.1.7 we say $P = A_n$ has pure zigzag orientation if the directions of any two adjacent arrows are opposite; alternatively, if every vertex is a source (minimal) or a sink (maximal).

As zigzag is an orientation that is often of particular independent interest, we will here espouse some properties of $D_{\text{AR}}$ specifically for the zigzag setting.

The Auslander-Reiten quiver of a zigzag orientation of $A_{11}$ is shown in Figure 5.
**Notation 2.4.1.** There are slight differences in the AR quiver based on the original orientation starting and ending at a max or min. This results in four zigzag orientation types, which we label as follows for convenience:

- In (uu) orientation, 1 and n are sinks,
- In (ud) orientation, 1 is a sink and n is source,
- In (du) orientation, 1 is a source and n is a sink,
- In (dd) orientation, 1 and n are sources.

**Remark 2.4.2 (Hull of zigzag orientation).** From Definition 2.3.5, we immediately see that an \( A_n \) quiver with zigzag orientation has \( H_N = \{ \text{min, max} \} \subset [2, n-1] \) and \( H_S = \{ \text{max, min} \} \subset [2, n-1] \). That is, hull(\( P \)) is precisely the entire north and south regions of AR quiver (which excludes the diagonals).

As an immediate consequence of Lemma 2.3.6, we have the following.

**Corollary 2.4.3** (to Lemma 2.3.6). For a zigzag orientation of some \( A_n \) quiver, any \( \sigma = [x, y] \) in hull(\( P \)) has

\[
W_{AR}(\sigma) = \min\{x + y - 1, 2n - x - y + 1\}.
\]

**Example 2.4.4.** For \( D_{AR} \) over zigzag orientations, there will be intervals of small dimension and large \( W_{AR} \) value. Consider \( A_{100} \) with either (ud) or (du) zigzag orientation. In either case, \( \sigma = [50, 51] \) has a dimension of 2, but by Corollary 2.4.3, \( W_{AR}(\sigma) = 100 = \text{diam}(W_{AR}) \) (Proposition 2.3.8).

**Example 2.4.5.** To extend the previous example to any zigzag orientation of \( A_n \), consider:

- If \( n \) is even, the indecomposable \( [n/2, n/2 + 1] \) has dimension 2 and \( W_{AR} \) value of \( n \),
- If \( n \) is odd, the indecomposable \( [n-1, n-1 + 1] \) has dimension 2 and \( W_{AR} \) value of \( n - 1 \).

**Remark 2.4.6.** Note that any orientation less than “pure” zigzag (Figure 6a) will possess reduced dimension-to-\( W_{AR} \) disparities.

For example, consider a poset with zigzag orientation everywhere save for the middle of the poset, in which there is a consecutive pair of rightward (or leftward) edges \( \rightarrow \rightarrow \) (Figure 6b). This splits the entire north region from one giant hull into two hulls by introducing the simple \([10]\) in the middle of the north boundary, providing a path of decreasing dimension to a simple for many modules formerly in the hull.

For another example, if \( A_n \) has orientation \( \cdots \rightarrow \leftrightarrow \rightarrow \leftrightarrow \rightarrow \cdots \) where the zigzag feature switches every other vertex (Figure 6c), then it turns out that the difference \( W_{AR}(\sigma) - \text{dim}(\sigma) \in \{0, 2\} \) for all indecomposables \( \sigma \) due to a high distribution of simples over the north boundary.

The last orientation in this example proves to be a worthwhile course of investigation for zigzag persistence, and is the focus of Section 3.3.

### 2.5 \( D_{BL} \) and \( D_{AR} \): Features and Stability

In this section we discuss the block distance \( D_{BL} \) of [MBB18] and explore the differences and similarities between \( D_{BL} \) and the Auslander-Reiten quiver distance \( D_{AR} \).

There is one rather cumbersome notational concern to be overcome when considering these two distances: for quiver theoretic purposes we have labeled our vertices in sequential order on the
Figure 6. Three orientations of $A_{21}$ and the north boundaries of their AR quivers. Not only do the “problem” regions shrink, but the disparities between $W_{AR}$ and dimension of the problem indecomposables also shrink.
zigzag quiver itself, while recent literature considers zigzag intervals as indexed over a particular poset denoted $\mathbb{Z}Z$, which then corresponds to some persistence module in $\mathbb{R}^2$. The disparity of notation and structure will be addressed with care when it comes time to consider the distances side by side (Definition 2.5.8), but is worth bearing in mind throughout. As such we will take to the following convention:

**Notation 2.5.1.** For a zigzag interval $I$, denote by $I_A$ the interval as viewed over a $\mathbb{Z}$-labeled $A_n$ quiver, and by $I_{\mathbb{Z}Z}$ a corresponding interval over the poset $\mathbb{Z}Z$ (Definition 2.5.2).

Of a final note is that there is no canonical association of vertices in $A_n$ with points in $\mathbb{Z}Z$. Throughout, we refuse to declare any point at which $A_n$ and $\mathbb{Z}Z$ are "fused". The reader is encouraged to keep this in mind during the subsequent material, and to be convinced that this lack of choice is of no consequence to the work provided. This is in fact ideal when taking into account that we will eventually consider extending to limits of zigzag quivers with unbounded length (Section 3.4).

2.5.1 **Posets**

**Definition 2.5.2.** Let $\mathbb{Z}Z$ be the poset consisting of all points $\{(i, i), (i, i - 1) \in \mathbb{Z}^2\}_{i \in \mathbb{Z}}$ and having the subposet order inherited from $\mathbb{Z}^{\text{op}} \times \mathbb{Z}$. Generally, an interval of this poset is written as $\langle i, j \rangle$, which denotes one of $(i, j), [i, j), (i, j], or [i, j]$.

An interval $\langle i, j \rangle$ in $\mathbb{Z}Z$ is the convex set

$$\langle i, j \rangle = \{(x, y) : i \sim x, y \sim j\},$$

where the $\sim$ represent either $\leq$ or $<$ depending on the respectively closed or open endpoints of $\langle i, j \rangle$.

An interval representation of $\mathbb{Z}Z$ is written $\langle i, j \rangle_{\mathbb{Z}Z}$. For any point $(x, y) \in \mathbb{Z}Z$,

$$\langle i, j \rangle_{\mathbb{Z}Z}(x, y) = \begin{cases} K & \text{if } (x, y) \in \langle i, j \rangle \\ 0 & \text{otherwise.} \end{cases}$$

The internal maps of $\langle i, j \rangle_{\mathbb{Z}Z}$ are $1_K$ where possible, and $0$ otherwise.

**Definition 2.5.3.** Let $U \subset \mathbb{R}^{\text{op}} \times \mathbb{R}$ be the subposet consisting of all points $(x, y) \in \mathbb{R}^2$ such that $x \leq y$. We will have it inherit the ordering of $\mathbb{R}^{\text{op}} \times \mathbb{R}$: that $(x, y) \leq (w, z)$ if and only if $x \geq w$ and $y \leq z$.

The connection between $\mathbb{Z}Z$ and $U$ as subposets of $\mathbb{R}^{\text{op}} \times \mathbb{R}$ is shown in Figure 7.

**Definition 2.5.4** (See [MBB18] sections 2.5 and 3 for original details). For a point $u \in U$, define $\mathbb{Z}Z[\leq u]$ to be the subposet of $\mathbb{Z}Z$ consisting of all the points of $\mathbb{Z}Z$ that are $\leq u$ when considering both $U$ and $\mathbb{Z}Z$ as subposets of $\mathbb{R}^{\text{op}} \times \mathbb{R}$ (see Figure 8).

For a zigzag persistence module $M_{\mathbb{Z}Z}$, define $M|_{\mathbb{Z}Z[\leq u]}$ to be the restriction of $M$ to the subposet $\mathbb{Z}Z[\leq u]$. Then define the colimit functor $\tilde{E} : \text{vect}^{\mathbb{Z}Z} \rightarrow \text{vect}^{\mathbb{R}^{\text{op}} \times \mathbb{R}}$ by:

$$\tilde{E}(M)(u) = \lim_{\longrightarrow} M|_{\mathbb{Z}Z[\leq u]},$$

the colimit of the diagram given by the $[\leq u]$ restriction, for every $u \in U \subset \mathbb{R}^{\text{op}} \times \mathbb{R}$.

Under $\tilde{E}$, interval $\mathbb{Z}Z$ modules are sent to the following block modules. (See figure 9.)
**Figure 7.** A visualization of the connection between the posets $\mathbb{Z}\mathbb{Z}$ and $\mathbb{U}$ as sub-posets of $\mathbb{R}^{\text{op}} \times \mathbb{R}$. The arrows denote the diagonal increasing vector under $\leq_{\mathbb{R}^{\text{op}} \times \mathbb{R}}$.

**Figure 8.** The restriction of the poset $\mathbb{Z}\mathbb{Z}$ under a point $u \in \mathbb{U}$.

**Figure 9.** The various types of $\mathbb{Z}\mathbb{Z}$ interval modules and their corresponding $\mathbb{U}$ modules under $\tilde{E}$.

\[
\begin{align*}
\tilde{E}((i, j)_{\mathbb{Z}\mathbb{Z}}) &= (i, j)_{\text{BL}} = \{(x, y) \in \mathbb{U} : i < x, y < j\} \\
\tilde{E}([i, j)_{\mathbb{Z}\mathbb{Z}}) &= [i, j)_{\text{BL}} = \{(x, y) \in \mathbb{U} : i \leq y < j\} \\
\tilde{E}((i, j]_{\mathbb{Z}\mathbb{Z}}) &= (i, j]_{\text{BL}} = \{(x, y) \in \mathbb{U} : i < x \leq j\} \\
\tilde{E}([i, j]_{\mathbb{Z}\mathbb{Z}}) &= [i, j]_{\text{BL}} = \{(x, y) \in \mathbb{U} : x \leq i, j \leq y\}
\end{align*}
\]
2.5.2 Block Module Distance

**Definition 2.5.5.** Let $D_U$ denote the interleaving distance on $U$.

Let $\epsilon = (-\epsilon, \epsilon) \in \mathbb{R}_{\geq 0} \times \mathbb{R}$ be the “increasing” vector of length $\epsilon$ for $U$. For a $U$ persistence module $M$, define the new $U$ persistence module $M(\epsilon)(u) = M(u + \epsilon)$. Similarly, for a morphism of persistence modules $\phi$, define $\phi(\epsilon)(u) = \phi(u + \epsilon)$.

For any $M, \epsilon$, let $1_{M, M(\epsilon)}$ be the morphism that takes the value $1_K$ on $u \in \text{supp}(M) \cap \text{supp}(M(\epsilon))$, and is zero otherwise. (It is simple to check that the $K$-span of this morphism gives precisely $\text{Hom}(M, M(\epsilon))$.)

Two $U$ persistence modules are said to be $\epsilon$-interleaved if there exist morphisms $\phi : M \to N(\epsilon)$ and $\psi : N \to M(\epsilon)$ such that

- $\psi(\epsilon) \circ \phi = 1_{M, M(2\epsilon)}$, and
- $\phi(\epsilon) \circ \psi = 1_{N, N(2\epsilon)}$.

For two $U$ persistence modules $M, N$,

$$D_U(M, N) = \inf\{ \epsilon : M, N \text{ are } \epsilon\text{-interleaved} \}.$$ (In full generality, this definition would make use of arbitrary $U$-translations, but implicit in this distance is the use of an $\ell^\infty$ norm, in which case we may as well default to the diagonal vector of length $\epsilon$ to define the translation at all points. This aligns with the earlier notion of a full translation of a given height.)

**Definition 2.5.6.** From Definitions 2.5.4 and 2.5.5, define the block distance to be the composition

$$D_{BL}(M_{ZZ}, N_{ZZ}) := (D_U \circ \tilde{E})(M_{ZZ}, N_{ZZ}) = D_U(\tilde{E}(M_{ZZ}), \tilde{E}(N_{ZZ})).$$

**Proposition 2.5.7 ([MBB18], Lemma 3.1).** The bottleneck distance induced by $D_{BL}$ can be generated by the following $W_{BL}$ and $d_{BL}$.

- $W_{BL}((i, j)_{ZZ}) = 1/4(j - i)$.
- $W_{BL}([i, j]_{ZZ}) = \infty$.
- $W_{BL}([i, j]_{ZZ}) = 1/2(j - i)$.
- $W_{BL}((i, j)_{ZZ}) = 1/2(j - i)$.

If $(i_1, j_1)_{ZZ}$ and $(i_2, j_2)_{ZZ}$ are two zigzag/block modules of the same endpoint parity, then

- $d_{BL}((i_1, j_1)_{ZZ}, (i_2, j_2)_{ZZ}) = \max\{|i_1 - i_2|, |j_1 - j_2|\}$

Otherwise, define $d_{BL} = \max\{W_{BL}((i_1, j_1)_{ZZ}), W_{BL}((i_2, j_2)_{ZZ})\}$, the max of the $W$-values.

The above result on interval modules is obtained from the more general definition, in which the projection of $ZZ$ interval modules to BL interval modules is by left Kan extension via colimit. See the original work [MBB18] for more detail.

2.5.3 Intervals of Zigzag $A_n$ as Intervals of $ZZ$

Finally, in order to make comparisons between $D_{AR}$ and $D_{BL}$, we need to be able to relate $A_n$ modules to $ZZ$ modules before embedding via $E$.

**Definition 2.5.8.** For some $P = A_n(z)$ define the functor $Z : \text{vect}^P \to \text{vect}^{ZZ}$ by how it acts on the following indecomposables. For any $x \in P$, there is some associated $(i, i) \in ZZ$ (the positioning in which $P$ is “fused” to $ZZ$ is fixed ahead of time and is entirely arbitrary).
• \( \mathcal{Z}([x + 1, x + 2k - 1]_A) = (i, i + k)_{\mathbb{Z}} \).
• \( \mathcal{Z}([x, x + 2k]_A) = [i, i + k]_{\mathbb{Z}} \).
• \( \mathcal{Z}([x, x + 2k - 1]_A) = [i, i + k]_{\mathbb{Z}} \).
• \( \mathcal{Z}([x + 1, x + 2k]_A) = (i, i + k)_{\mathbb{Z}} \).

**Definition 2.5.9.** Let \( P = A_n(z) \) and let \( Z \) be the \( \mathbb{Z} \)-interval (not module) given by \( \mathcal{Z}([1, n]_A) \). Define \( \Sigma_{\mathbb{Z}}(P) \) to be the subcategory of \( \text{vect}_{\mathbb{Z}} \) given by all modules with support contained in the \( \mathbb{Z} \)-interval \( Z = \mathcal{Z}([1, n]_A) \).

**Proposition 2.5.10.** The functor (natural transformation)
\[
\mathcal{Z} : \text{vect}^P \to \Sigma_{\mathbb{Z}}(P)
\]
is an equivalence of categories (natural equivalence).

**Proof.** The inverse of \( \mathcal{Z} \) is given by the reverse statements of Definition 2.5.8. \( \square \)

**Definition 2.5.11.** For a \( \mathbb{Z} \) module \( I_{\mathbb{Z}} \), define dimension \( \dim(I_{\mathbb{Z}}) = \sum_{i \in \mathbb{Z}} \dim_K(I_{\mathbb{Z}}(i)) \) to be the sum of the dimensions of the vector spaces of \( I_{\mathbb{Z}} \).

**Notation 2.5.12.** In any setting where we have fixed some \( P = A_n(z) \) and some \( \mathcal{Z} : \text{vect}^P \to \Sigma_{\mathbb{Z}}(P) \) (“some” only because this is technically dependent on our consistently hand-waved choice of \( A \leftrightarrow \mathbb{Z} \) anchor), we will drop the equivalence \( \mathcal{Z} \) altogether and simply denote by \( \sigma_A \) and \( \sigma_{\mathbb{Z}} \) the same module viewed as a member of either of the two equivalent categories.

Also, despite the disparity in labeling between \( \mathbb{A} \) and \( \mathbb{Z} \) modules (Definition 2.5.8), the dimension of \( \sigma \) is the same in both contexts:
\[
\dim_A(\sigma_A) = \dim_{\mathbb{Z}}(\mathcal{Z}(\sigma_A) = \sigma_{\mathbb{Z}}).
\]
For this reason, we will simply write \( \dim \) with no need for subscripting based on the category.

### 3 Stability Between \( D_{BL} \) and \( D_{AR} \) over Pure Zigzag

Algebraic stability results usually refer to obtaining bounds between some distance and its induced bottleneck distance (Remark 1.2.9). The following are two important examples of stability that have been paraphrased into this paper’s vocabulary.

The first stability result is in fact an isometry.

**Theorem.** For \( \text{vect} \)-valued persistence modules over \( \mathbb{R} \), the interleaving distance and its induced bottleneck distance are isometric.

That is, the interleaving distance can be taken to be diagonal over the indecomposable summands without any loss of sharpness.

The fact that a distance is a lower bound on its own induced bottleneck distance is trivial. The non-trivial direction for the above result is seen originally in [CSEH07]. It was then algebraically presented and proved in [CCSG+09] (Theorem 4.4). The categorically focused “induced matching” version of the result appears in [BL13] (Theorem 3.5), which is emphasized even further in the entirety of [BL16] (particularly Theorems 1.4, 1.7).

The following is the initial stability result for the block distance.
Theorem ([MBB18] Proposition 2.12 and Theorem 3.3). For vect-valued persistence modules over $\mathbb{Z}$ embedded via $E$ as block $\sqcup$ persistence modules, $D_{BL}$ and its induced bottleneck distance $\hat{D}_{BL}$ satisfy

$$D_{BL} \leq \hat{D}_{BL} \leq \frac{5}{2}D_{BL}.$$ 

As the block distance separates by $(\cdot, \cdot)_{\mathbb{Z}}$ type, the result above is proved independently for each of the four cases. In three of these cases the above statement is tight with the constant of $5/2$. In [Bak16] it is shown that for the case of $(\cdot, \cdot)_{\mathbb{Z}}$ modules, the block distance and its induced bottleneck distance are isometric (i.e., the $5/2$ can be replaced with 1).

These theorems are immensely important results for the topic at hand, but do not reflect the sort of stability theorem that we will provide for $D_{AR}$. As it has been defined, $D_{AR}$ is foundationally a bottleneck distance in the first place, and thus is its own induced bottleneck distance. As such, any algebraic stability result of the type discussed here would be trivial for $D_{AR}$. Instead, we examine comparative stability of the kind $D_{AR} \leq A \cdot D_{BL}$ and $D_{BL} \leq B \cdot D_{AR}$ over pure zigzag orientations.

The following is our final result for this section: full minimal Lipschitz constants comparing a modification of $D_{AR}$ to $D_{BL}$ over the four kinds of $\mathbb{Z}$ modules (this echoes the piecewise stability results of the block distance [MBB18], as this modified $D_{AR}$ also shares the trait that it “separates” modules by $(\cdot, \cdot)_{\mathbb{Z}}$ type).

**Theorem** (Theorem 3.4.6). The following are the minimal Lipschitz constants comparing $D_{BL}$ with the modification $D^{2,\infty}_{AR}$ of $D_{AR}$ over some poset $P = \mathbb{A}_n(z)$ of pure zigzag orientation.

- If $\sigma_{\mathbb{Z}}, \tau_{\mathbb{Z}} \in (\cdot, \cdot)_{\mathbb{Z}}$, then $2D_{BL} \leq D^{2,\infty}_{AR} \leq 16D_{BL}$.
- If $\sigma_{\mathbb{Z}}, \tau_{\mathbb{Z}} \in [\cdot, \cdot]_{\mathbb{Z}}$, then $2D_{BL} \leq D^{2,\infty}_{AR} \leq 4D_{BL}$ (if $D_{BL} < \infty$).
- If $\sigma_{\mathbb{Z}}, \tau_{\mathbb{Z}} \in [\cdot, \cdot]_{\mathbb{Z}}$, then $2D_{BL} \leq D^{2,\infty}_{AR} \leq 8D_{BL}$.
- If $\sigma_{\mathbb{Z}}, \tau_{\mathbb{Z}} \in (\cdot, \cdot)_{\mathbb{Z}}$, then $2D_{BL} \leq D^{2,\infty}_{AR} \leq 8D_{BL}$.

### 3.1 Partitioning of Intervals and Modifications of $D_{AR}$

Throughout, we compare $D_{BL}$ with the original $D_{AR}$ and then two further modifications of it. $D^{r}_{AR}$ is a modification of $D_{AR}$ that acts by projecting into a poset refinement of pure zigzag (called $r$-zigzag) in order to avoid a large hull, all while preserving the structure of the projected modules over sources and sinks. $D^{r,\infty}_{AR}$ is a further modification that views original zigzag modules over $r$-zigzag posets of unbounded length. This perspective both compares more favorably with $D_{BL}$ and may be of independent interest to anyone who does not wish to be limited to bounded zigzag posets in the first place.

The remainder of this section chronicles Lipschitz stability between $D_{BL}$ and original $D_{AR}$ and the fact that in both directions the minimal Lipschitz constants involve $n$ itself (the length of $P = \mathbb{A}_n$). The first modification $D^{r}_{AR}$ removes one of these dependencies, while the second modification to $D^{r,\infty}_{AR}$ removes the other.

The most persistent discrepancy (the one removed by the $D^{r,\infty}_{AR}$ modification) is discussed in the following remark.

**Remark 3.1.1** (Partitions of $\Sigma_P$: $\mathbb{Z}$ vs. compass). We require a brief discussion of the connection between the subsets $E, W, S, N$ of $\Sigma_P$ and the subsets $(\cdot, \cdot)_{\mathbb{Z}}, [\cdot, \cdot]_{\mathbb{Z}}, (\cdot, \cdot)_{\mathbb{Z}}, (\cdot, \cdot)_{\mathbb{Z}}$ of $\Sigma_{\mathbb{Z}}(P)$ under the
functor $\mathcal{Z}$ (Definition 2.5.8). When trying to pair the compass regions precisely to the partitions by $\mathcal{Z}$-type, the inconvenience becomes that the diagonals (of the AR quiver) belong to different members of the $\mathcal{Z}$-partition depending on the orientation of $A_n$.

Define the sets:

- $D_{nw} = \{[1, \cdot] \in \Sigma_P : [1, \cdot] \text{ is northwest of } [1, n]\}$.
- $D_{ne} = \{[, n] \in \Sigma_P : [, n] \text{ is northeast of } [1, n]\}$.
- $D_{se} = \{[1, \cdot] \in \Sigma_P : [1, \cdot] \text{ is southeast of } [1, n]\}$.
- $D_{sw} = \{[, n] \in \Sigma_P : [, n] \text{ is southwest of } [1, n]\}$.

Supplement when necessary with the bar notation from Notation 2.2.4, i.e.,

$$\bar{N} = N \cup D_{nw} \cup D_{ne} \cup \{[1, n]\}.$$ 

See Table 1.

This leads to Lemmas 3.2.2 and 3.3.7, which introduce $n$-dependence in $D_{BL} \leq A \cdot D_{AR}, D_{AR}^r$ Lipschitz constants. This is resolved at last when comparing with the modification $D_{AR}^\infty$, as seen in Proposition 3.4.4.

Finally, we introduce a notational convention for use in Tables 2 and 3.

**Notation 3.1.2.** For the remainder of the work on stability, we invoke the following notational conventions for the sake of filling out Tables 2 and 3 with greater readability.

Let $\sigma = [x_1, y_1]_A$ and $\tau = [x_2, y_2]_A$. We will denote the by the following values various quantities originating in Proposition 2.2.6:

- $\text{LH}^{\text{diff}}(\sigma, \tau) = |x_1 - x_2|$, the left hand support difference of the modules,
- $\text{RH}^{\text{diff}}(\sigma, \tau) = |y_1 - y_2|$, the right hand support difference of the modules,
- $\text{LH}^{\text{comp}}(\sigma, \tau) = x_1 - 1 + x_2 - 1$, the left hand support complements of the modules, also allowing for the notation $\text{LH}^{\text{comp}}(\sigma) = x_1 - 1$,
- $\text{RH}^{\text{comp}}(\sigma, \tau) = n - y_1 + n - y_2$, the right hand support complements of the modules, also allowing for the notation $\text{RH}^{\text{comp}}(\sigma) = n - y_1$. 

| $A_n^{uu}(z)$ | $\mathcal{E}$ | $\mathcal{W}$ | $S \cup D_{se}$ | $N \cup D_{ne}$ |
|----------------|-------------|--------------|----------------|-----------------|
| $A_n^{ld}(z)$ | $\mathcal{E} \cup D_{ne}$ | $\mathcal{W} \cup D_{nw}$ | $\tilde{S}$ | $N$ |
| $A_n^{dl}(z)$ | $\mathcal{E} \cup D_{se}$ | $\mathcal{W} \cup D_{sw}$ | $S$ | $\tilde{N}$ |
| $A_n^{dd}(z)$ | $\mathcal{E}$ | $\mathcal{W}$ | $S \cup D_{sw}$ | $N \cup D_{nw}$ |

**Table 1.** Equality of partitions by compass regions of the AR quiver and by endpoint type in $\mathcal{Z}$, dependent on orientation of $A_n$. 


Table 2. In both tables, recall that the difference between $\dim_{ZZ}$ and $\dim_A$ (which is the difference between $j - i$ and $y - x$) is given by Definition 2.5.8, and is in all cases essentially a factor of 2 (with $\dim_A$ being the larger one).

### 3.2 Unmodified Stability

**Proposition 3.2.1** (Unmodified Right-Hand Stability). Over pure zigzag orientation,

$$D_{AR} \leq 2n \cdot D_{BL}$$

is the minimal Lipschitz constant satisfying the above inequality.

**Proof.** Necessity is obtained from Example 2.4.5. A module of the form $[x, x + 1]_A$ can have $W_{AR} = n, n - 1$, and this corresponds to some module of the form $[i, i + 1]_{ZZ}$ or $(i, i + 1)_{ZZ}$, both of which have $W_{BL} = 1/2$. Sufficiency follows from Corollary 2.3.9. □

In the other direction, we must address the misalignment issues brought to attention in Remark 3.1.1.

**Lemma 3.2.2** (Partitioning Non-alignment (see Remark 3.1.1)). Let $P = A_n(z)$ be a poset of pure zigzag orientation. Then if $D_{BL} < \infty$,

$$D_{BL} \leq n/4 \cdot D_{AR}$$

where $n/4$ is a lower bound for the Lipschitz constant in the inequality above.
Proof. No matter the orientation of \( P \), one of \( \sigma = [1, n]_A, \tau_1 = [2, n]_A \) is in some \( \{ \cdot, \cdot \}_{ZZ} \) and the other is in the associated \( \{ \cdot, \cdot \}_{ZZ} \). Similarly, one of \( \sigma = [1, n]_A, \tau_2 = [1, n - 1]_A \) is in some \( \{ \cdot, \cdot \}_{ZZ} \) and the other is in the associated \( \{ \cdot, \cdot \}_{ZZ} \). That is to say, \( D_{BL}(\sigma, \tau_i) = \max\{W_{BL}(\sigma), W_{BL}(\tau_i)\} \approx n/4 \) or \( \infty \) (for \( i = 1, 2 \). See Table 2b).

However, both pairs have a \( D_{AR} \) distance of 1 (recall that all \( D_{AR} \) distances from a diagonal to an adjacent region are of the form \( LH^{\text{diff}} + RH^{\text{diff}} \)). \( \square \)

We are now prepared to state this stability result.

**Proposition 3.2.3** (Unmodified Left-Hand Stability). Over pure zigzag orientation, so long as \( D_{BL} < \infty \),

\[
D_{BL} \leq n/4 \cdot D_{AR}
\]

where \( n/4 \) is the minimal Lipschitz constant satisfying the above inequality across all pairs of indecomposables.

**Proof.** Necessity is given by Lemma 3.2.2. Sufficiency follows below.

Sufficiency follows from Tables 2 and 3, with special concern being given to the final column of Table 3. The most extreme comparison from this column (suppose \( uu \) orientation for ease of notation) are the pair of modules \( \sigma_A = [1, n - 1]_A \) and \( \tau_A = [2, n]_A \), which correspond to \( \sigma_{ZZ} = [i, i + (n - 1)/2]_{ZZ} \) and \( \tau_{ZZ} = (i, i + (n - 1)/2]_{ZZ} \) for some \( i \in \mathbb{Z} \). But though \( n \)-dependent, these only require a Lipschitz constant of \( n/8 \), and thus \( n/4 \) remains permissible. \( \square \)

### 3.3 Stability with \( r \)-zigzag

It seems to the authors that the AR distance’s tendency to have hulls in pure zigzag orientations such that intervals with small supports have \( W \)-values at or near the entire diameter of \( D_{AR} \) is undesirable under quite a few perspectives (namely, for finding Lipschitz bounds with other more “well-behaved” distances). See Example 2.4.5 and its subsequent discussion Remark 2.4.6 for motivation, from which we have already seen in Proposition 3.2.1 that that any relationship \( D_{AR} \leq A \cdot D_{BL} \) requires a constant that scales with \( n \).

**Definition 3.3.1.** Let \( P = \mathbb{A}_n(z) \) be some pure zigzag orientation and \( r \in \mathbb{Z}_{\geq 2} \). Define \( P^r = \mathbb{A}_n(z, r) \) to be the following poset. Let \( P^r \) have sources and sinks collectively labeled \( 1, 2, \ldots, (n-1), n \), alternating from source to sink in the same sequence as the vertices \( 1, 2, \ldots, n-1, n \) of \( P \). For each \( 1 \leq i \leq n-1 \), add \( r - 1 \) vertices between \( i \) and \( (i+1)r \), such that the segment \([i, (i+1)r] \) is totally ordered.

\[
\mathbb{A}_{11}(z) = \mathbb{A}_{11}(z, 1) \quad \mathbb{A}_{11}(z) = \mathbb{A}_{11}(z, 2) \quad \mathbb{A}_{11}(z) = \mathbb{A}_{11}(z, 3)
\]

Let \( R \) be the embedding from \( \Sigma_P \to \Sigma_{P^r} \) (the collections of isomorphism classes of indecomposable representations over each poset) given by \( R([x, y]) = [x, yr] \). (We note that \( R \) clearly depends on the original \( P \) and the choice of \( r \), but we will simply write \( R \) in all cases and leave the dependence on \( P, r \) clear by context.)

Finally, define \( D_{AR}^r \) on the set of indecomposable representations of \( P \) by

\[
D_{AR}^r(\sigma, \tau) = D_{AR}(R(\sigma), R(\tau)).
\]

where the right hand \( D_{AR} \) is the AR distance over \( P^r \).
The endpoint conversion from $\Lambda_n(z, r)$ intervals to ZZ intervals is similar to that of Definition 2.5.8, but has the labeling disparities increased by a factor of $R$.

**Remark 3.3.2.** For some module $[x, y]$ over a pure zigzag orientation $\Lambda_n(z)$ and some $r \in \mathbb{Z}_{>0}$,

$$\dim([x_r, y_{r-1}]) = r \cdot [\dim([x, y]) - 1] + 1 = r \cdot (y - x) + 1.$$ 

The following result is immediate from Definition 2.3.5.

**Remark 3.3.3.** Let $P = \Lambda_n(z)$ have pure zigzag orientation and $P^r$ be its $r$-zigzag refinement. As $\hull(\Lambda_n(z)) = \{(x_r, (x + 1)_r) \mid 1 \leq x < n\}$, it follows that $\{R([x, x + 1])) \}_{1 \leq x < n} = \hull(P^r)$.

**Corollary 3.3.4** (to Lemma 2.3.6). For any indecomposable $\sigma$ over pure zigzag orientation, if $\sigma = [x, x+1]$ then

$$W^r_{\AR}(\sigma) = \dim(\sigma) + 2 = r + 3,$$

and otherwise

$$W^r_{\AR}(\sigma) = \dim(\sigma).$$

**Example 3.3.5.** The following is a visualization of the module embedding $R$ from $P = \Lambda_6^{du}$ to its 3-zigzag refinement $P^3$. 

---

**Table 3.** Table of $d$-values over any poset of pure zigzag orientation, partitioned by ZZ interval type. For sources of individual formulas see: row 1, Prop 2.5.7; row 2, Prop 2.2.6 and Notation 3.1.2; row 3, Remark 3.3.2 and Example 3.3.5; row 4, Proposition 3.4.4.
Though unlabeled for clarity, interval modules maintain the same relative position across the two AR quivers under $R$. Shape, location, and relative distance between indecomposables are essentially unchanged. However, along the north and south boundaries, it is immediate that the gray dots in this area are simples, removing the presence of pure zigzag’s large hulls.

**Proposition 3.3.6** ($r$-zigzag Right-Hand Stability). Over an $r$-zigzag orientation $P = \mathbb{A}_n(z, r)$ with $r \geq 2$,

$$D_{AR} \leq 8r \cdot D_{BL}.$$  

Compare with Proposition 3.2.1 in which the large hull of unmodified $W_{AR}$ caused $n$-dependence in the inequality.

**Proof.** Necessity comes from the first column of Table 2b. Sufficiency of the remaining columns for $W$-values is easy to check.

First column $d$-values in Table 3 require only a constant of $2r$. We only show the sufficiency of $8r$ when comparing fourth column intervals from Table 3.

Suppose then that $\sigma, \tau$ are two indecomposables with opposite parity of both left and right endpoints. $d_{AR}^r(\sigma, \tau)$ becomes large (and $d_{BL}$ becomes small) when $\sigma, \tau$ have small supports and are positioned centrally within the poset. However, if the supports are too small $d_{AR}^r$ will revert to max $W_{AR}$ values, which we already know are stable.

The largest value of $d_{AR}^r(\sigma, \tau)$ such that $d_{AR}^r < \max W_{AR}^r$'s is with $\sigma$ and $\tau$ both having supports as close as possible to $Z([n/6, 5n/6]) = [n/6, 5n/6]$, while still possessing opposite parity on left and right endpoints. In such a situation, $d_{AR}^r(\sigma_\alpha, \tau_\alpha) \approx W_{AR}^r(\sigma_\alpha) \approx W_{AR}^r(\tau_\alpha) \approx r \cdot (2n/3)$. But then, $d_{AR}^r \approx 2r \cdot d_{BL}$, and so $8r$ remains permissible. □

Considering the opposite inequality, we encounter a repeat of the partition misalignments.

**Lemma 3.3.7** (Partitioning Non-alignment for $r$-zigzag). Let $P = \mathbb{A}_n(z)$ be a poset of pure zigzag orientation and $P^r$ be its $r$-zigzag extension. Then if $D_{BL} < \infty$,

$$D_{BL} \leq \frac{n}{4r} \cdot D_{AR}^r,$$

where $n/4r$ is a lower bound for the Lipschitz constant in the inequality above.

**Proof.** The proof follows identically to that of Lemma 3.2.2, where the example modules $\sigma, \tau_1, \tau_2$ are all viewed through the functor $R_P$. □

**Proposition 3.3.8** ($r$-zigzag Left-Hand Stability). Over pure zigzag orientation, so long as $D_{BL} < \infty$,

$$D_{BL} \leq \frac{n}{4r} \cdot D_{AR}^r$$
where \( n/4r \) is the minimal Lipschitz constant satisfying the above inequality.

**Proof.** Necessity follows from Lemma 3.3.7.

Sufficiency parallels the proof of Proposition 3.2.3 using Remark 3.3.2. (In the event that it is of interest to the reader, outside of the misalignment cases handled by Lemma 3.3.7, the smaller weight \( n/8r \) suffices for all remaining cases. This is a further mirroring of the proof of Proposition 3.2.3.)

By projecting from pure zigzag into an \( r \)-zigzag poset and removing the hull, we have successfully eliminated the \( n \) dependence of one side of our inequalities. The final modification at last removes the other.

### 3.4 Stability with Poset Limits

The following is a further modification of \( D_{AR}^r \) that assumes the representation category of some original \( P = A_n(z) \) or \( P^r = A_n(z, r) \) is embedded into a poset of similar structure that is lengthened on either end.

There are two advantages to this modification. 1) This modification obtains stability with \( D_{BL} \) in a way that does not depend on the original length \( n \) of the poset. 2) This modifies \( D_{AR} \) over pure zigzag orientations (via first modifying to \( D_{AR}^r \)) in such a way that one may consider the modules over a zigzag poset of unbounded length, which may be of independent interest to many.

**Definition 3.4.1.** Let \( P = A_n \) and \( P' = A_m \) be two orientations of \( A \)-type quivers of any lengths. Assign the labelling \( P = \{ 1 \sim 2 \sim \ldots \sim n \} \) and \( P' = \{ 1' \sim 2' \sim \ldots \sim m' \} \). Then define

\[ P \land P' \]

to be the poset obtained from joining the \( P \)-vertex \( n \) with the \( P' \)-vertex \( 1' \), along with the original \( \leq, \leq' \) relationships and any added inequalities induced by the association of \( n \) with \( 1' \).

**Definition 3.4.2.** Let \( P = A_n(z) \) be some pure zigzag orientation. Let \( P^r = A_n(z, r) \) be its \( r \)-zigzag refinement (Definition 3.3.1). For \( f \in \mathbb{Z}_{\geq 1} \), define the poset \( P^{r,f} = A_n(z, r \pm f) \) as follows.

First define the poset \( U_r = \{ u \geq \ldots \geq (1 + r)_u \leq \ldots \leq (1 + 2r)_u \} \) and \( D_r = \{ d \leq \ldots \leq (1 + r)_d \geq \ldots \geq (1 + 2r)_d \} (= \text{the opposite poset of } U_r) \). Define \( P^{r,1} \) to be

- \( U_r \land P^r \land U_r \) if \( P = A_n^{uu} \),
- \( U_r \land P^r \land D_r \) if \( P = A_n^{ud} \),
- \( D_r \land P^r \land U_r \) if \( P = A_n^{du} \),
- \( D_r \land P^r \land D_r \) if \( P = A_n^{dl} \).

Below is an example of \( P^{3,1} \) for \( P = A_6^{du} \).

\[ \text{Diagram} \]

Define \( P^{r,f} \) inductively (i.e., the number of wedges on both sides of appropriately chosen \( U_r \) or \( D_r \) is equal to \( f \)). In this way, the \( r \)-zigzag structure and sink/source orientation of the left and right endpoints remain unchanged from \( P^r \) to \( P^{r,f} \).
Let $F : \Sigma_{P^r} \to \Sigma_{P^r,f}$ be the functor $F((x, y)_{P^r}) = (x; y)_{P^r,f}$. That is, the supports of interval modules remain fixed within $P^r$ considered as a subposet of $P^{r,f}$.

**Definition 3.4.3.** For $\sigma, \tau$ over some pure-zigzag orientation $P = \mathbb{A}_n(z)$, define

$$D^{r,f}_{\text{AR}}(\sigma, \tau) = D_{\text{AR}}(F \circ R(\sigma), F \circ R(\tau)).$$

Define

$$D^{r,\infty}_{\text{AR}}(\sigma, \tau) = \lim_{f \to \infty} D^{r,f}_{\text{AR}}(\sigma, \tau).$$

Again, take note that in the following proposition the separation into pieces of the AR quiver of $P^r$ when embedded by $F_P$ align precisely with the $\langle \cdot, \cdot \rangle$ partitioning of the AR quiver.

**Proposition 3.4.4 ($D^{r,\infty}_{\text{AR}}$ Separates by ZZ-type).** For $P = \mathbb{A}_n(z)$, $D^{r,\infty}_{\text{AR}}$ separates modules by ZZ region. That is, the image of the functor $F : \Sigma_{P^r} \to \Sigma_{P^r,f}$ consists of the four connected components

$$F(\{(\cdot, \cdot)_{ZZ, P^r}\}) \subset \{(\cdot, \cdot)_{ZZ, P^r,f}\},$$

$$F(\{[\cdot, \cdot]_{ZZ, P^r}\}) \subset \{[\cdot, \cdot]_{ZZ, P^r,f}\},$$

$$F(\{[\cdot, \cdot]_{ZZ, P^r}\}) \subset \{(\cdot, \cdot)_{ZZ, P^r,f}\},$$

$$F(\{(\cdot, \cdot)_{ZZ, P^r}\}) \subset \{[\cdot, \cdot]_{ZZ, P^r,f}\}.$$

Moreover, $D^{r,\infty}_{\text{AR}}$ is the bottleneck distance given by:

- $d^{r,\infty}_{\text{AR}}(\sigma_h, \tau_h) = |x_1 - x_2| + |y_1 - y_2|$ if $\sigma, \tau$ are in the same $\langle \cdot, \cdot \rangle_{ZZ}$ region, and $d^{r,\infty}_{\text{AR}}(\sigma_h, \tau_h) = \infty$ otherwise.

- $W^{r,\infty}_{\text{AR}}(\sigma) = y_1 - x_1 + 3$ if $\sigma = [x_1, x_1 + r]$ where $x$ is a sink or source vertex, and $W^{r,\infty}_{\text{AR}}(\sigma) = y_1 - x_1 + 1$ otherwise. That is, $W^{r,\infty}_{\text{AR}}(\sigma) = W^r_{\text{AR}}(\sigma)$.

**Figure 10.** Again, the thicker dots represent indecomposables from the AR quiver of $P = \mathbb{A}_6^n$ under the 3-zigzag embedding functor $R$. Depicted here is the embedding $F$ of modules of the AR quiver of $P^3$ into that of the extension by $D_3$ on the left and $U_3$ on the right.

**Proof.** As we have seen, from $P = \mathbb{A}_n(z)$ to $P^r = \mathbb{A}_n(z, r)$, the AR quiver becomes refined by a factor of $r$ along both axes while the relative positions of the embedded modules from $P$ remain
the same (Example 3.3.5). This separation and the fact that $W_{AR}^{r,\infty}$ remains completely unchanged from $W_{AR}^r$ can be checked individually from the four possible orientations of $P^r$ in Figures 11 and 12.

In all four images, when wedging with $U_r$ or $D_r$, the new axis contains the $A$’s in sequence, the $B$’s in sequence, but separates the two sub-axes by the $C$’s. Wedges on the left side of the poset are added to the middle of the $x$-axis and to the ends of the $y$-axis. Wedges on the right side of the poset are added to the ends of the $x$-axis and to the middle of the $y$-axis.

Compare these case by case with the partitions in Table 1 in Remark 3.1.1.

![Figure 11](image)

(A) When $P$ is of $d^*$ orientation, the original $x = 1$ (contained in $B_1$ in the image) is grouped with the other $B_i$’s, which are all open left endpoints.

(B) For $u^*$ orientations, the original $x = 1$ is grouped with the closed endpoints when the original axis becomes split by the wedges.

![Figure 12](image)

(A) For $\ast u$ orientations, the original axis value $y = n$ is a closed right endpoint, and is grouped with the other closed endpoints.

(B) Finally, the original axis value $y = n$ is open, and is grouped in the new axis with the other open endpoints.

Remark 3.4.5. While $d_{AR}^{r,\infty}$ may attain infinite values, the final bottleneck distance $D_{AR}^{r,\infty}$ does not, by virtue of the fact that $W_{AR}^{r,\infty} = W_{AR}^r$ is always bounded by the length of the original $r$-zigzag orientation (Corollary 2.3.9).

The following theorem is our concluding result on comparisons of $D_{AR}$ with $D_{BL}$.

Theorem 3.4.6 (Sharp $D_{AR}^{r,\infty}$ vs. $D_{BL}$ Lipschitz Constants). Let $P = A_n(z)$ be of pure zigzag orientation. The following are the four stability results between $D_{BL}$ and $D_{AR}^{r,\infty}$ partitioned by ZZ-type (as neither distance directly compares modules from different regions of the partition).

- If $\sigma_{ZZ}, \tau_{ZZ} \in (-\cdot, \cdot)_{ZZ}$, then $r \cdot D_{BL} \leq D_{AR}^{r,\infty} \leq 8r \cdot D_{BL}$.
- If $\sigma_{ZZ}, \tau_{ZZ} \in [\cdot, \cdot]_{ZZ}$, then $r \cdot D_{BL} \leq D_{AR}^{r,\infty} \leq 2r \cdot D_{BL}$ (if $D_{BL} < \infty$).
• If $\sigma_{ZZ}, \tau_{ZZ} \in \left[\cdot, \cdot\right]_{ZZ}$, then $r \cdot D_{BL} \leq D_{AR}^{r, \infty} \leq 4r \cdot D_{BL}$.

• If $\sigma_{ZZ}, \tau_{ZZ} \in \left(\cdot, \cdot\right]_{ZZ}$, then $r \cdot D_{BL} \leq D_{AR}^{r, \infty} \leq 4r \cdot D_{BL}$.

Proof. All left hand inequalities $r \cdot D_{BL} \leq D_{AR}^{r, \infty}$ are necessary by the first column of Table 3, and sufficiency is easy to see by examination of Table 2 (columns two three and four of Table 3 simply revert to problems of comparing values in Table 2).

For the right hand inequalities, a Lipschitz constant of $D_{AR}^{r, \infty} \leq 2r \cdot D_{BL}$ is permissible when considering only Table 3. However, the different $W_{BL}$ behaviors in Table 2 force some of the values to be larger. □

As seen in the initial statement of the proof at the beginning of this section, one may as well choose the minimal zigzag extension of $r = 2$ if there is no contextual motivation for selecting a larger value.

4 Weighted Interleaving Distance

As briefly discussed in the introduction, the weighted interleaving distance on some orientation of $A_n$ measures similarity between two interval modules by the depth or shallowness on the ‘wells’ over which their supports differ (Figure 13).

Definition 4.0.1. For a general orientation $P = A_n$, enumerate the poset’s source vertices from left to right as $m_1, \ldots, m_p$. Define $V_i$ to be the maximal sub-poset given by all elements comparable to $m_i$.

$$V_i = \{x \in P : x \geq m_i\}.$$ Label the left and right sinks of $V_i$ (if they exist) as $1_i$ and $n_i$ respectively:

$$\{1_i \leftarrow \ldots \leftarrow m_i \rightarrow \ldots \rightarrow n_i\}.$$

Let $[V_i]$ denote the interval representation $[1_i, n_i]$.

Lastly, as independent posets, the wedge of $V_i$ and $V_{i+1}$ is the poset in which $n_i$ is identified with $1_{i+1}$:

$$V_i \wedge V_{i+1} = \{1_i \leftarrow \ldots \leftarrow m_i \rightarrow \ldots \rightarrow n_i = 1_{i+1} \leftarrow \ldots \leftarrow m_{i+1} \rightarrow \ldots \rightarrow n_{i+1}\}.$$ as in Definition 3.4.1.

Remark 4.0.2. Any orientation $P = A_n$ can be uniquely expressed as a wedge of $V_i$’s

$$P = V_1 \wedge V_2 \wedge \ldots \wedge V_l,$$

where $V_1$ and $V_l$ may be equioriented segments.

Moving forward, we will view representations of an orientation of $A_n$ as persistence modules over a one-vertex refinement of the original poset.

Definition 4.0.3. For a poset $P$, let $\tilde{P}$ be the poset $P \cup \{\infty\}$ with the relation $x \leq y$ if and only if either $x, y \in P$ with $x \leq_P y$, or $y = \infty$.

We call $\tilde{P}$ the poset $P$ suspended at infinity.

Translations on $P = A_n$ can be viewed as a wedge of translations on each individual $V_i$. 

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**Proposition 4.0.4.** Any translation $\Lambda$ on $P = V_1 \wedge \ldots \wedge V_p$ can be fully described by how it acts on the individual $V_i$. Similarly, any collection $\{\Lambda_i \in \text{Trans}(V_i)\}_{1 \leq i \leq p}$ determines a translation on $P$.

Similarly, any translation $\Lambda$ on $\check{P}$ where $P = V_1 \wedge \ldots \wedge V_p$ can be fully described by how it acts on the individual $\check{V}_i$. However, in reverse we must add the extra condition that pairs of translations for adjacent $V_i$ agree at the points of overlap. That is, any collection

$$\{\Lambda_i \in \text{Trans}(\check{V}_i)\}_{1 \leq i \leq p} : \Lambda_i(n_i) = \Lambda_{i+1}(1_{i+1}) \text{ for all } 1 \leq i < p$$

determines a translation on $\check{P}$.

We now define an interleaving-type distance using the poset suspended at $\infty$.

**Definition 4.0.5.** For a poset $P$ and a pair $(a, b) \in \mathbb{N} \times \mathbb{N}$, define the weighted height of a translation $\Lambda$ over $\check{P}$ to be

$$\tilde{h}(\Lambda) = \max_{x \in P} \delta^{(a,b)}(x, \Lambda x),$$

where $\delta^{(a,b)}(x, y)$ is the directed graph distance between $x$ and $y$, with edges of $P$ counted with weight $a$, and added edges of $\check{P}$ counted with weight $b$.

At a weight of $(1, 1)$, this is the directed graph distance induced by the poset structure. However, as we want to make the movement of former maximals possible without entirely losing track of the significance of that operation, we have the ability to feather the “penalty” of moving these former maximal to $\infty$ with the weight $b$ (or rather, the weight of $b$ relative to $a$).

**Definition 4.0.6.** For a poset $P$ and a pair $(a, b) \in \mathbb{N} \times \mathbb{N}$, we define the weighted interleaving distance $D^{(a,b)}_I$ to be the interleaving distance (Definition 1.2.2) on the set of representations of $P$, but with translations taken over $\check{P}$, using the height function given in Definition 4.0.5.

Throughout, the notation $D^{(a,b)}_I$ will be reduced to $D_I$.

We introduce the following notation for future convenience.

**Notation 4.0.7.** For a given $V_i$, let $T_i = \min\{m_i - 1, n_i - m_i\}$ be the length of the short side, and $S_i = \max\{m_i - 1, n_i - m_i\}$ be the length of the long side. Define $T_i = 0$ if $V_i$ is equioriented.

Define $T := \max_{1 \leq i \leq p} T_i$.

Define $S := \max_{1 \leq i \leq p} S_i$.

**Proposition 4.0.8 (Classification of Translations on $\check{P}$).** Let $P = V_1 \wedge V_2 \wedge \ldots \wedge V_p$ be an orientation of $\mathcal{A}_n$, and assume $a \leq b$. Let $\Lambda$ be a translation on $\check{P}$. The collection of full translations (Remark 1.2.4) are described below by how they act on each individual $\check{V}_i$ (Remark 4.0.4).

- If $\tilde{h}(\Lambda) < a$, then each $\Lambda_i$ (and so $\Lambda$ itself) is the trivial translation.
- If $a \leq \tilde{h}(\Lambda) < b$, then the sources and sinks of each $V_i$ are fixed by $\Lambda_i$. All other vertices move upwards by $k$ vertices, where $ak \leq \tilde{h}(\Lambda) < a(k + 1)$, or to their unique comparable sink, if that is closer than $k$ vertices.
- If $b \leq \tilde{h}(\Lambda) < a(T_i - 1) + b$, then $\Lambda$ can be described in the same way as above, save that now the sinks are sent to $\infty$. Also, if $ak + b \leq \tilde{h}(\Lambda) < a(k + 1) + b$, then any vertices (other than unique source) that are within $k$ vertices from their corresponding sink are also sent to $\infty$. 
Figure 13. An orientation of $A_n$ with two pairs of intervals. For $b \gg a$, the red intervals are much closer under $D_I^{(a,b)}$ than the blue intervals.

- If $a(T_i - 1) + b \leq \tilde{h}(\Lambda)$, then the entire shorter leg (sans the source) can be sent to $\infty$ by the translation. Each vertex of the longer leg (including the source), is sent up the longer side as far as the translation permits (including being sent to $\infty$).

Barring extreme differences between the length of the two sides combined with small values of $b$, $\Lambda^2$ will almost always send every vertex of $V_i$ to $\infty$.

To summarize, if $I$ and $J$ are two arbitrary persistence modules over $P$:

- $D_I(I, J) < b$ guarantees that $I$ and $J$ are isomorphic on every fixed point.
- $D_I(I, J) = ak + b$ for some $k \geq 0$ guarantees that they $I$ and $J$ are isomorphic on minimal vertices of any $V_i$ in which $T_i - 1 \geq ak$.

Example 4.0.9. With all definitions in place, we can make an easy example to convey what $D_I$ measures, and what it ignores.

In Figure 13, the red intervals are much closer to each other in the weighted interleaving distance than the blue intervals are.

In particular, $D_I$(red modules) requires a translation of height sufficient to annihilate all the shallow $V_i$'s, but not the large one. However, $D_I$(blue modules) immediately requires moving the minimal at the bottom of the deepest $V_i$, already demanding a larger translation than anything involved in interleaving the red modules.

4.1 Stability of $D_I$ over $D_{AR}$ as Bottleneck Distances

Remark 4.1.1. We will compare $D_I$ and $D_{AR}$ as bottleneck distances. From here onward, let $D_I$ denote the bottleneck distance induced by the weighted interleaving distance.

The focus of this section is the minimization of weights $(a, b)$ (under lexicographic $\mathbb{N} \times \mathbb{N}$ ordering) such that $D_{AR} \leq D_I^{(a,b)}$ (again, as bottleneck distances).

The weighted interleaving distance measures different features than the other distances in this paper, and was adopted as one of our directions of investigation due to its ability to preserve an interleaving-like approach to finite posets that is not immediately stalled by sink/source vertices, which must remain fixed under the ordinary interleaving distance.
The authors previously proved an algebraic stability result using this distance for “branch”-type posets [MM17]. While not supplying an algebraic stability result for arbitrary $\mathbb{A}_n$ quivers, we do compare $D_I$ (its induced bottleneck distance) against $D_{AR}$.

Instead of single-variable Lipschitz stability results, we state $D_I$ stability against another distance in terms of the two-parameter weight used to define it: $(a, b) \in \mathbb{N} \times \mathbb{N}$, where we consider $\mathbb{N} \times \mathbb{N}$ to be ordered lexicographically (Definition 4.0.6). This ordering is to prioritize first minimizing the weight attached to the original poset structure, and afterwards the weight that determines distances to $\infty$.

**Theorem 4.1.2.** Let $P = V_1 \wedge \ldots \wedge V_l$ be some orientation of $\mathbb{A}_n$. Let $T, S$ be as in Notation 4.0.7. The classification of stable weights for $D_I \geq D_{AR}$ as bottleneck distances is:

- for equioriented, see Corollary 4.2.4,
- for $S < 3$, see Proposition 4.3.1,
- for $S \geq 3$ and $T = 1$, see Proposition 4.3.1,
- other non-shallow posets are not addressed (see Definition 4.5.3), and remaining posets are split by centrality (see Definition 4.6.2),
- for shallow and central orientations, see Proposition 4.6.4,
- for shallow and non-central orientations, see Corollary 4.10.1.

### 4.2 Stable Values of $a$

**Proposition 4.2.1.** Let $P = V_1 \wedge \ldots \wedge V_p$.

- If $S \geq 3$, the minimal permissible weight is of the form $(2, b)$.
- If $S = 2$, the minimal permissible weight is of the form $(1, b)$ if there is only a single equioriented segment of two consecutive edges, and is of the form $(2, b)$ otherwise.
- If $S = 1$, the minimal permissible weight is of the form $(1, b)$.

**Proof.** *Necessity:* For $S \geq 3$ consider the following diagram.

For $S = 2$ consider the following. For any two equioriented segments with $i - 2 \leftrightarrow i - 1 \leftrightarrow i$ left of $j \leftrightarrow j + 1 \leftrightarrow j + 2$, the intervals $\sigma = [i - 1, i + 2]$ and $\tau = [i, j]$ are interleaved by a translation height $a$ while having an AR distance of 2. Two of the four possible configurations appear below, the remaining two being those with segments of $\nearrow \nearrow$ and $\searrow \searrow$ arrangements.
That is, the only $S = 2$ type of poset permitting $a = 1$ is pure zigzag with a single pair of consecutive edges with the same orientation.

**Sufficiency:** One need only consider $W$-values of interval modules containing no maximals orimals, and $d$-values of pairs of interval modules whose supports share precisely the same fixed points. This is easy to check in all cases.

\[\text{Corollary 4.2.2. For any poset } P \text{ with } S \geq 3 \text{ and weight } (2, b) \text{ with } b > 2, \text{ if } D_I(\sigma, \tau) < b \text{ then } D_AR(\sigma, \tau) \leq D_I(\sigma, \tau).\]

\[\text{Corollary 4.2.3. For any orientation } Q \text{ of } \mathbb{A}_n \text{ and the appropriate choice of } a = 1, 2, \text{ the weight } (a, n) \text{ is always permissible. I.e., } b = n \text{ is an upper bound for the value of } b \text{ in the minimal permissible weight.}\]

Proof. This follows immediately from Propositions 4.2.1 and 2.3.8. □

Lastly, a much less general corollary is the resulting minimal stable weights for equi-oriented $\mathbb{A}_n$.

\[\text{Corollary 4.2.4. Let } Q \text{ be an equi-orientation of } \mathbb{A}_n. \text{ Then } (2, 1) \text{ is the minimal stable weight.}\]

### 4.3 Stability when $S < 3$

**Proposition 4.3.1.** If $S < 3$, then stability is minimally obtained by $(a, b)$ where $a = 1$ or $a = 2$ according to Proposition 4.2.1, and $b = n$.

Proof. **Necessity:** $W_I([1, n]) = b$ and $W_AR([1, n]) = n$. **Sufficiency:** Due to Proposition 4.2.1, one needs only check $W, d$-values involving modules of the form $[x, n]$. □

### 4.4 Short and Long Escape

As translations split over wedges (Proposition 4.0.4), we now examine the translations required for realizing $W_AR([V_i])$ of any wedged component, where $P = V_1 \land \ldots \land V_l$.

**Definition 4.4.1.** Let $V = \{1 \geq \ldots \geq m \leq \ldots \leq n\}$. Assume that the left side is strictly shorter than the right. That is, according to Notation 4.0.7,

\[m - 1 = T < S = n - m.\]

We first construct the most basic translation $\Lambda$ such that $\phi, \psi = 0$ form a $\Lambda$-interleaving of $[1, n]$ and 0. This is the translation given by:

- $\Lambda x = \infty$ for all $x$ in $[1, m)$. Note that the distance in the weighted poset from $m - 1$ to $\infty$ is

\[\epsilon(b) = 2(T - 1) + b.\] (4.1)
For all \( x \in [m, n] \), \( \Lambda x \) moves up the right hand side (possibly to \( \infty \)) by a distance of \( \mathcal{E}(b) \), where

\[
\mathcal{E}(b) = \begin{cases} 
  b & \text{if } b \geq 2S \\
  1/2(2S + b) & \text{if } 2S + b \equiv 0 \mod 4 \\
  1/2(2S + b) + 1 & \text{if } 2S + b \equiv 2 \mod 4 \\
  \lceil 1/2(2S + b) \rceil & \text{if } 2S + b \equiv 1, 3 \mod 4
\end{cases}
\]  

(4.2)

In short, by the translation property that \( x \leq y \) demands \( \Delta x \leq \Delta y \), moving the minimal up one side requires that the entire other side by sent to \( \infty \) by \( \Lambda \). However, the side up which the minimal is moved is relaxed, and may take two \( \Lambda \)-applications in order to send all vertices to \( \infty \), by the properties of interleavings.

Define \( \epsilon(b) \) to be the short escape, and \( \mathcal{E}(b) \) to be the long escape. This construction is of minimal height, being \( h(\Lambda) = \max\{\epsilon(b), \mathcal{E}(b)\} \), such that \( \Lambda^2(x) = \infty \) for any \( x \in V \).

Replace the prototype translation and define \( \Lambda^b \) to be the maximal translation on \( V \) of height \( \max\{\epsilon(b), \mathcal{E}(b)\} \). This translation is unique (unless it is a symmetric \( V \), in which case choose the left side be considered the ‘short’ side).

**Proposition 4.4.2.** The translation \( \Lambda^b \) is of minimal height such that \( \phi, \psi = 0 \) form a \( \Lambda^b \)-interleaving of \([V]\) and 0.

I.e., \( \Lambda^b \) realizes \( W_I([V]) \) and no translation of smaller height does.

Proposition 4.4.2 pairs extremely well with the following. (Recall that we are now considering \( D_I \) to always refer to its induced bottleneck distance as per Remark 4.1.1.)

**Corollary 4.4.3.** [to Proposition 4.0.4] Let \( P = V_1 \wedge \ldots \wedge V_p \). The induced bottleneck distance \( D_I \) (by slight abuse of notation) and its generating functions \( W, d \) all split over wedges.

\[
\begin{align*}
  W_I(I) &= \max_{1 \leq i \leq p} \{W_I(I|V_i)\}, \\
  d_I(I, J) &= \max_{1 \leq i \leq p} \{d_I(I|V_i, J|V_i)\}, \\
  D_I(I, J) &= \max_{1 \leq i \leq p} \{D_I(I|V_i, J|V_i)\}
\end{align*}
\]

4.5 **Shallow Posets**

**Example 4.5.1.** Using Proposition 4.4.2 and Corollary 4.4.3, let us examine a powerful constraint for stability: \( W \)-values for the indecomposable \([1, n]\).

If we solve simultaneously for the conditions that (a) the largest long escape exceeds the largest short escape (i.e., \( W_I([1, n]) \) is determined by some long escape) and (b) stability of the form \( D_{AR} \leq D_I \), we get the two bounds

\[
b \geq 2n - 2S \quad \text{and} \quad b < 2S - 4T + 2.
\]

Combining inequalities, we see that such a \( b \) can only exist if (even with some permissive rounding),

\[
2(S - T) + 1 > n.
\]
Remark 4.5.2. One immediately sees from the equation above that the situation in which $W_1([1, n])$ is determined by some long escape value is incredibly specific, as it requires at the very least that the poset have one $V_i$ with longer side constituting more than half of the entire poset (using $T \geq 2$):

$$2S > n + 1.$$ 

As long escape dictates $W_1([1, n])$ only in this extreme case, we henceforward will only consider the complementary situation.

Definition 4.5.3. An orientation of $\mathcal{A}_n$ written $P = V_1 \land \ldots \land V_t$ that has $S \geq 3$ (Proposition 4.2.1 above) is shallow if $T \geq 2$ (to keep the hull small) and $2S \leq n$ (to ensure all $W_i$’s are determined by short escape).

Remark 4.5.4. Indeed, in a shallow poset short escape values are used for any $W_1([V_i])$ (and so, by Corollary 4.4.3, all $W_1$-values). To see that $W_1([V_i]) = \epsilon_l(b)$ for all $1 \leq l \leq p$, simply note that $\epsilon_l(b) = b < 2(T_l - 1) + b = \epsilon_l(b)$.

With only this, we can immediately get the stability statement for $W$-values out of the way.

Proposition 4.5.5. For a shallow poset and any weight $(2, b)$ with $b \geq n - T$,

$$W_{AR}(\sigma) \leq W_1(\sigma)$$

for any indecomposable $\sigma$.

Proof. If $\text{supp}(\sigma)$ contains no sink or source we are done by Corollary 4.2.2.

If $\sigma \in \text{Hull}(Q)$, then by the $T \geq 2$ tenet for shallow, for any $[x, y] \in \text{Hull}(Q)$ either $[x, y] \subset (1, m_t]$ or $[x, y] \subset [m_t, n]$. In particular, the corresponding $[e]$ and $[E]$ of Lemma 2.3.6 obey $e < E \leq m_t + 1$ or $m_t - 1 \leq e \leq E$. The formulas of Lemma 2.3.6 are all $\leq n - T \leq b \leq W_1([x, y])$. (It is possible for one equation to reach $n - T + 1$, but in this case $m_t \in [x, y]$ and $W_1([x, y]) \geq 2 + n - T$ as $T \geq 2$).

If $\sigma \notin \text{Hull}(Q)$ then $W_{AR}(\sigma) = \text{dim}(\sigma)$. If $m_t \notin \text{supp}(\sigma)$, then either $[1, m_t]$ or $[m_t, n]$ are disjoint from $\text{supp}(\sigma)$ (each of which has length at least $T$), guaranteeing that the dimension $\text{dim}(\sigma) \leq n - T$. Otherwise, $m_t \in \text{supp}(\sigma)$, and $W_1(\sigma) = W_1([1, n]) = 2(T - 1) + b \geq n + T - 2 \geq n = \text{diam}(W_{AR})$. 

\[\square\]

4.6 Stability for Shallow and Central

Lemma 4.6.1. If any of the following are true about a pair of intervals $\sigma, \tau$ over the shallow poset $P = V_1 \land \ldots \land V_p$, then any weight $(a, b)$ with $a = 2$ and $b \geq n - T$ is stable.

(1) $m_t \in \text{one of supp}(\sigma), \text{supp}(\tau)$, but not the other.

(2) $\text{dim}(\sigma) \leq b$.

(3) $[V_i] \subset \text{supp}(\sigma)$.

Throughout, assume the intervals are always labeled such that $W_{AR}(\sigma) \geq W_{AR}(\tau)$.

Proof of Lemma 4.6.1. (1) Any interleaving translation must move $m_t$, and so has height $D_1(\sigma, \tau) \geq 2(T - 1) + b \geq 2T - 2 + n - T = n + T - 2 \geq n = \text{diam}(W_{AR}) = \text{diam}(D_{AR})$.

(2) In the proof of (1) we saw that $d_{AR}(\sigma) \leq n - T \leq b$ when $\sigma \in \text{Hull}$. So for any $\sigma$, if $\text{dim}(\sigma) \leq b$, then $W_{AR}(\sigma) \leq b$. But then $D_{AR}(\sigma, \tau) = \min\{d_{AR}(\sigma, \tau), W_{AR}(\sigma)\}$ (by the running assumption of $W_{AR}(\sigma) \geq W_{AR}(\tau)$), and so $D_{AR}(\sigma, \tau) \leq b$. By Corollary 4.2.2, the pair is stable.
By (2), we may assume that \( \sigma = [x, y] \) where \( y - x \geq b \geq n - T \). As \( 1 + T \leq m_t \leq n - T \), it is immediate that \( m_t \in \text{supp}(\sigma) \). Hence, by (1), \( m_t \in \text{supp}(\tau) \) also.

Assume now that, in addition, all of \([V_t] \subset \text{supp}(\sigma)\). We will show by cases on the equation for \( d_{AR} \) that this must also yield stability. First note the following inequalities generated by the interleaving condition: as \( [V_t] \subset \text{supp}(\sigma) \), the endpoints of \( \tau = [x_2, y_2] \) are restricted by
\[
x_2 \leq 1_t + 1 + \frac{D_I - b}{2} \\
y_2 \geq n_t - 1 - \frac{D_I - b}{2}
\]
where \( D_I := D_I(\sigma, \tau) \).

Stability can now be checked across all possible cases of \( \delta^x, \delta^y \). We show only one of them here.
\[
D_{AR}(\sigma, \tau) \leq d_{AR}(\sigma, \tau) = |x_1 - x_2| + |y_1 - y_2| \\
\leq 1_t + 1 + \frac{D_I - b}{2} - 1 + n - \left( n_t - 1 - \frac{D_I - b}{2} \right) \\
\leq D_I - (n - T) + n - (n_t - 1_t) + 1 \\
\leq D_I + T - (S + T) + 1 \\
\leq D_I
\]

Recall the meanings of \( T, S \) from Notation 4.0.7.

**Definition 4.6.2.** We say a poset \( P = V_1 \wedge \ldots \wedge V_p \) is central if there is some \( V_t \) with \( T_t = T \) positioned in such a way that \( [V_t] \subset [T, n - T + 1] \).

**Proposition 4.6.3.** A shallow poset is central if and only if every pair of intervals fulfill at least one of the conditions of Lemma 4.6.1.

**Corollary 4.6.4.** If \( P \) is a shallow and central poset, then every pair of indecomposable modules \( \sigma, \tau \) satisfies the inequality
\[ D_{AR}(\sigma, \tau) \leq D_{1}(\sigma, \tau) \]
for any weight \((2, b)\) with \( b \geq n - T \).

### 4.7 Stability for Shallow and non-Central

**Proposition 4.7.1.** Suppose \( P = A_n \) is a shallow and non-central poset: suppose without loss of generality that \( 1_t < T \). Consider a weight \((2, b)\) with \( b \geq n - T \).

Any pair of indecomposables \( \sigma = [x_1, y_1], \tau = [x_2, y_2] \) is stable under this weight unless \( x_1, x_2 \in (1_t, m_t) \) and \( \delta^y = n - y_1 + n - y_2 \).

Proposition 4.7.1 follows from the subsequent lemma.

**Lemma 4.7.2.** If any of the following are true about a pair of intervals \( \sigma, \tau \) over a shallow poset \( P = V_1 \wedge \ldots \wedge V_p \), then any weight \((a, b)\) with \( a = 2 \) and \( b \geq n - T \) is stable.

Throughout, assume the intervals are always labeled such that \( W_{AR}(\sigma) \geq W_{AR}(\tau) \).

1. \( \sigma, \tau \) are in the same region of the AR quiver.
(2) \( \sigma, \tau \) are in opposite regions of the AR quiver (a north-south or east-west pair).

(3) \( 1_t \notin \text{supp}(\sigma) \) and \( x_2 \leq 1_t \) (symmetrically, \( n_t \notin \text{supp}(\sigma) \) and \( x_2 \geq n_t \)).

Proof. (1) From Lemma 4.6.1 we may assume \([V_t] \notin \text{supp}(\sigma)\). As \( m_t \in \text{supp}(\sigma) \), it follows that either \( 1_t \) or \( n_t \) is in \( \text{supp}(\sigma) \). Suppose then, without loss of generality, that \( 1_t \notin \text{supp}(\sigma) \): that is, \( x_1 \in (1_t, m_t] \).

We may assume that \( m_t \in \text{supp}(\tau) \). If \( x_2 \leq 1_t \), then the bound on \( |x_1 - x_2| + |y_1 - y_2| \) proceeds identically to the similar equation in the proof of Lemma 4.6.1 (3). Otherwise, \( x_2 \in (1_t, m_t] \). Then,

\[
D_{AR}(\sigma, \tau) \leq d_{AR}(\sigma, \tau) = |x_1 - x_2| + |y_1 - y_2| \\
\leq m_t - 1_t + n - \left( n_t - 1 - \frac{D_I - b}{2} \right) \\
< D_I - (n - T) + n - (n_t - m_t) - 1_t + 1 \\
\leq D_I
\]

(2) Again by Lemma 4.6.1, assume without loss of generality that \( x_1 \in (1_t, m_t] \). Then it must be that \( x_2 \leq 1_t \) in order to have \( \delta x = x_1 - 1 + x_2 - 1 \). But in such a situation, the bound on \( x_1 - 1 + x_2 - 1 + n - y_1 + n - y_2 \) proceeds identically to the similar equation in the proof of Lemma 4.6.1 (3).

(3) Using Lemma 4.6.1 and this lemma’s (1) and (2), we may assume without loss of generality that \( 1_t \notin \text{supp}(\sigma) \), and either

- \( d_{AR}(\sigma, \tau) = x_1 - 1 + x_2 - 1 + |y_1 - y_2| \) or
- \( d_{AR}(\sigma, \tau) = |x_1 - x_2| + n - y_1 + n - y_2. \)

However, given the assumption \( 1_t \notin \text{supp}(\sigma) \), the first equation above also yields stability. If \( d_{AR}(\sigma, \tau) \) is the first equation, then \( x_2 \leq 1_t \), and so:

\[
D_{AR}(\sigma, \tau) \leq d_{AR}(\sigma, \tau) = x_1 - 1 + x_2 - 1 + |y_1 - y_2| \\
\leq 1_t + 1 + \frac{D_I - b}{2} - 1 + 1_t - 1 + n - \left( n_t - 1 - \frac{D_I - b}{2} \right) \\
\leq D_I - b + n + 2 \cdot 1_t - n_t \\
\leq D_I + 1_t + T - (n_t - 1_t) \\
< D_I + 2T - (S + T)
\]

Assume the second equation, and assume that \( x_2 \leq 1_t \). However, one can immediately see from the bound on the similar equation in the proof of Lemma 4.6.1 (3) that this assumption results in stability as well. \(\square\)

This result allows us to narrow down a maximally anti-stable candidate pair for any shallow non-central poset.
4.8 Maximally Anti-Stable Pairs

The structure of this section is as follows.

Suppose \( P \) is a shallow but non-central orientation of \( \mathbb{A}_n \). Without loss of generality suppose that \( 1_t < T \). We have already shown by Lemmas 4.6.1 (3) and 4.7.2 that any anti-stable pair \( \sigma = [x_1, y_1], \tau = [x_2, y_2] \) has the property that \( x_1, x_2 \in (1_t, m_t) \) and \( y_1, y_2 \geq m_t \) are of opposite orientation from each other.

This means that \( \delta_{AR}(\sigma, \tau) = |x_1 - x_2| + n - y_1 + n - y_2 \) for any anti-stable pair. We measure anti-stability by the size of the difference \( D_{AR} - D_I \), and show that starting from any anti-stable pair, we can reduce down to one of two canonical anti-stable pairs that between them maximize anti-stability.

First, choosing \( x_1, x_2 \) as far apart as possible increases \( D_{AR} \) while having no effect on \( D_I \). But \( y_1 \) has a lower bound dependent on \( x_1 \)’s position (while \( y_2 \) does not depend on \( x_2 \)), so to maximize later freedom we choose \( x_1 = 1_t + 1 \) and \( x_2 = m_t \).

Then, \( y_2 \) has two \( d_{AR} \)-minimizing possibilities based on the orientation of \( y_1 \). Lastly, \( y_1 \) can be shifted left to further minimize \( d_{AR} \). This leftward shifting of \( y_1 \) potentially alters the interleaving distance between \( \sigma \) and \( \tau \), but as long as \( y_1 \) is chosen such that \( \dim(\sigma) > b \) [Lemma 4.6.1 (2)] it causes a strict increase in anti-stability of the pair.

**Definition 4.8.1.** For any \( y > n_t \), define \( k(y) = \max_{t < j \leq i} \{ T_j \} \) where \( y \in [m_i, m_{i+1}) \).

For and vertex \( y \) right of \( V_t \), the value \( k(y) \) returns the length of the longest shortest edge of the \( V_t \)’s contained between \( V_t \) and \( y \). This value determines the interleaving distance between two modules containing \( m_i \), one of whose right endpoints is \( y \), and the other of which is contained between \( m_i \) and \( m_{i+1} \).

As \( D_{AR}(\sigma, \tau) \leq W_{AR}(\sigma) \), if \( W_{AR}(\sigma) \leq D_I(\sigma, \tau) \) then we are done. It suffices to assume throughout that \( W_{AR}(\sigma) > D_I(\sigma, \tau) \), and to then show that \( d_{AR}(\sigma, \tau) \leq D_I \). The assumption \( W_{AR}(\sigma) > D_I(\sigma, \tau) \) amounts to the inequality

\[
y_1 - x_1 + 1 > 2(k(y_1) - 1) + b.
\]

This is clear from Lemma 4.7.2 plus the foreknowledge that we will be adjusting all other vertices such that the defining feature of \( D_I(\sigma, \tau) \) will be \( W_I \) of the \( V_p \)’s between \( n_t \) and \( y_1 \), as these are in the support of \( \sigma \) and outside the support of \( \tau \).

More conveniently, we will replace \( x_1 = 1_t + 1 \) and write the above inequality as

\[
y_1 > 2k(y_1) - 2 + b + 1_t.
\]

**Definition 4.8.2.** For a weight \( (2, b) \) and vertex \( y > n_t \), consider the statement

\[ \Theta(y) : y > 2k(y) - 2 + b + 1_t. \]

Define

\[ y_u(b) = \min \{ y : \Theta(y) \text{ holds and } y \text{ is upward oriented} \} \]

and

\[ y_d(b) = \min \{ y : \Theta(y) \text{ holds and } y \text{ is downward oriented} \} \]

where we will simply write \( y_u \) and \( y_d \) when context makes clear the value of \( b \).
Corollary 4.8.3. If there is any pair that violates stability for the weight \((2, n - T)\), then at least one of the pairs

\[
(\sigma_u = [1_t + 1, y_u], \tau_u = [m_t, n_t]) \text{ or } (\sigma_d = [1_t + 1, y_d], \tau_d = [m_t, n_t - k(y_d)])
\]

also violates stability for that weight and is maximally anti-stable out of all pairs of intervals over the poset (that is, the value of \(R = D_{AR} - D_t\) is positive and maximal for the correct pair).

In the event that there is any anti-stable pair for the poset, call the pair above with the greater anti-stability the maximal anti-stable pair for the poset. If both pairs are just as anti-stable, choose \((\sigma_u, \tau_u)\).

Proof. This follows from Propositions 4.9.1 and 4.9.2. \(\square\)

4.9 Maximally Anti-stable Pairs

Let \(P\) be a shallow and non-central orientation of \(\mathcal{A}_n\).

Suppose there exists a pair \(\check{\sigma} = [x'_1, y'_1], \check{\tau} = [x'_2, y'_2]\) with \(W_{AR}(\check{\sigma}) \geq W_{AR}(\check{\tau})\) such that \((\check{\sigma}, \check{\tau})\) is an anti-stable pair for any weight \((2, b)\) with \(b \geq n - T\).

Proposition 4.9.1. If \((\check{\sigma}, \check{\tau})\) is an anti-stable pair, then \(\check{\sigma} = [1_t + 1, y'_1], \check{\tau} = [m_t, y'_2]\) also comprise an anti-stable pair. Furthermore, \(R(\check{\sigma}, \check{\tau}) \geq R(\check{\sigma}, \check{\tau})\) and \(W_{AR}(\check{\sigma}) \geq W_{AR}(\check{\tau})\).

Proof. It is immediate that this choice of \(x_1, x_2\) maximize the value of \(\delta_{AR}(\sigma, \tau)\). The opposite assignment would do the same, however, \(y_1\) (which maximizes \(\delta_{AR}\) by being small) has an \(x_1\)-dependent lower bound, while \(y_2\) has no \(x_2\)-dependency. For this reason the precise assignment of \(x_1, x_2\) in the proposition is ideal going forward. \(\square\)

Suppose there exists a pair \(\check{\sigma} = [1_t + 1, y'_1], \check{\tau} = [m_t, y'_2]\) with \(W_{AR}(\check{\sigma}) \geq W_{AR}(\check{\tau})\) such that \((\check{\sigma}, \check{\tau})\) is an anti-stable pair for any weight \((2, b)\) with \(b \geq n - T\).

Proposition 4.9.2. If \((\check{\sigma}, \check{\tau})\) is an anti-stable pair, then \(\check{\sigma} = [1_t + 1, y'_1], \tau = [m_t, y'_2]\) also comprise an anti-stable pair, where \(y_2 = n_t\) or \(y_2 = n_t - k(y'_1)\); whichever has opposite \(y\)-orientation from \(y'_1\). Furthermore, \(R(\check{\sigma}, \tau) \geq R(\check{\sigma}, \check{\tau})\) and \(\check{\sigma}\) has larger dimension than \(\tau\).

Proof. (1) Suppose \(y'_1 \in [\max, \text{next min}]\). Then \(\tau = [m_t, n_t - k(y'_1)]\) and \(\check{\tau} = [m_t, y'_2]\), with \(y_2 \geq n_t - k(y'_1)\) and having orientation \(y_2 \in [\min, \text{next max}]\).

If \(n_t - k(y'_1) < y_2 < n_t\), then

\[
D_1(\check{\sigma}, \tau) = D_1(\check{\sigma}, \check{\tau})
\]

but

\[
D_{AR}(\check{\sigma}, \tau) - D_{AR}(\check{\sigma}, \check{\tau}) = y_2 - (n_t - k(y'_1)) \geq 0,
\]

and so

\[
R(\check{\sigma}, \tau) \geq R(\check{\sigma}, \check{\tau}).
\]

Otherwise, \(y_2 \in [m_p, n_p]\) for some \(p \geq t + 1\). From \(\tau\) to \(\check{\tau}\), the right endpoint increases, and so the value of \(D_1\) may decrease. Specifically, if \(D_1(\check{\sigma}, \tau)\) was determined by a particularly large 2-V that is then included in the larger support of \(\check{\tau}\), it will not be taken into account for that interleaving distance, and we will have a non-zero value for

\[
D_1(\check{\sigma}, \tau) - D_1(\check{\sigma}, \check{\tau}) = 2 \left( \max_{m_t < m_t \leq y'_1} \{T_i\} - \max_{y_2 < m_t \leq y'_1} \{T_i\} \right). \]
Let \( T_j = \max_{m_t < m_i \leq y'_1} \{ T_i \} \). Then the difference above is at most \( 2(T_j - 1) \). If we can show that the difference between the \( D_{AR}'s \) is larger than this, we will have shown a net increase in \( R(\hat{\sigma}, \tau) \) over \( R(\hat{\sigma}, \hat{\tau}) \).

\[
D_{AR}(\hat{\sigma}, \tau) - D_{AR}(\hat{\tau}, \hat{\tau}) = y_2 - (n_t - k(y'_1)) \geq m_j - n_t + k(y'_1),
\]
as the drop in \( D_j \)'s was assumed to have happened by \( y_2 \) exceeding the value of \( m_j \) (and so \( n_j \) by orientation conditions). As \( k(y'_1) = T_j - 1 \), the difference in \( D_{AR}'s \) becomes

\[
m_j - n_t + k(y'_1) \geq T_j + T_j - 1 = 2(T_j) - 1.
\]

This is precisely what was desired, and so we have the inequality for \( R \)-values.

(2) Suppose next that \( y'_1 \in [\min, \text{next max}] \). Let \( y_2 > n_t \) of orientation \([\max, \text{next min}] \).

If \( n_t < y_2 < m_{t+1} \), then

\[
D_I(\hat{\sigma}, [m_t, n_t]) = D_I(\hat{\sigma}, [m_t, y_2])
\]

and

\[
D_{AR}(\hat{\sigma}, [m_t, n_t]) > D_{AR}(\hat{\sigma}, [m_t, y_2]),
\]

so \( R \) strictly increases from choosing the left endpoint of \( \tau \) to be \( n_t \).

Otherwise, by the requirement of orientation, \( n_{t+1} \leq y_2 \). Then

\[
D_I(\hat{\sigma}, [m_t, n_t]) - D_I(\hat{\sigma}, [m_t, y_2]) = 2 \left( \max_{m_t < m_i \leq y'_1} \{ T_i \} - \max_{y_2 < m_i \leq y'_1} \{ T_i \} \right).
\]

The above difference is bounded above by \( 2(T_j) - 1 \), where \( T_j := \max_{y_2 < m_i \leq y'_1} \{ T_i \} \).

At the same time,

\[
D_{AR}(\hat{\sigma}, [m_t, n_t]) - D_{AR}(\hat{\sigma}, [m_t, y_2]) = y_2 - n_t,
\]

where \( y_2 \geq n_j \). But, \( y_2 - m_j \geq 2T_j \), and so \( y_2 - n_t > 2T_j \).

Combined, we see that \( R(\hat{\sigma}, [m_t, n_t]) > R(\hat{\sigma}, [m_t, y_2]) \) for any choice of \( y_2 > n_t \).

\[
\Box
\]

4.10 Permissibility of \( n - T/2 \)

**Corollary 4.10.1.** Let \( P = V_1 \land V_2 \land \ldots \land V_p \) be a shallow and non-central orientation of \( k_n \). The minimal weight such that \( D_{AR} \leq D_I \) is \((2, b)\) where \( b \) is bounded above by

\[
b \leq n - T/2 - 1.
\]

**Proof.** The minimal pair is always stable for \( b \geq n - T/2 - 1 \) : Of the two possible minimals pairs of Corollary 4.8.3 we will only show the proof of \( \sigma_d = [1_t + 1, y_d] \) and \( \tau_d = [m_t, n_t - k(y_d)] \). (The proof for \( \sigma_u, \tau_u \) is incredibly similar, and a slightly less restrictive inequality.) Recall that \( y_d \) is minimal such that \( x_1 + D_I = 1_t + n - T/2 + 2k(y_d) \leq y_d \) (Definition 4.8.2).
So $D_I = b + 2(k(y_d) - 1)$ and $\delta_{AR}(\sigma, \tau) = m_t - 1_t - 1 + n - (n_t - k(y_d)) + n - y_d$. Comparing, we get

$$\delta_{AR}(\sigma_d, \tau_d) \leq D_I(\sigma, \tau) \text{ if }$$

$$m_t - 1_t - 1 + n - n_t + k(y_d) + n - y_d \leq b + 2k(y_d) - 1 \text{ if }$$

$$m_t - 1_t - 1 + 2n - n_t + k(y_d) - (1_t + b + 2k(y_d)) \leq b + 2k(y_d) - 1 \text{ if }$$

$$m_t - 1 - 2 \cdot 1_t + 2n - n_t - k(y_d) - 2b \leq 2k(y_d) - 2 \text{ if }$$

$$m_t - 1 - 2 \cdot 1_t + 2n - n_t - 2n + T \leq 2k(y_d) - 2 \text{ if }$$

$$m_t - n_t + T - 2 \cdot 1_t + 3 \leq 3k(y_d) \text{ if }$$

$$-2 \cdot 1_t + 3 \leq 3k(y_d)$$

the last statement of which is true due to the left being $\leq 1$ and the right being $\geq 3$.

□

**Example 4.10.2.** (See Figure 14.) We show a sample poset in which the minimal stable value equals the upper bound $b = n - T/2 - 1$.

For $T > 1$, let $1_t = 1$, $m_t = T + 1$, $n_t = 2T + 1$, $n = 1 + 4T$. Let the region from $n_t$ to $n$ consist of $V_p$’s with $T_p = 1$ and of orientation such that $y_u$ is forced to be (even just slightly) larger than the minimization given by 4.8.2.
Then \((\sigma_d, \tau_d)\) form the minimal pair, and we can explicitly check that \(b = n - T/2 - 1\) is permissible while no smaller weight will be:

\[
\delta(\sigma_d, \tau_d) \leq D_I(\sigma_d, \tau_d) \text{ iff } T - 1 + n - 2T + n - (2 + b) \leq b \text{ iff } 2n - T - 3 \leq 2b \text{ iff } n - T/2 - 1 \leq b \text{ if } T \text{ is even, or } n - T/2 - 2 \leq b \text{ if } T \text{ is odd.}
\]

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