Mechanism for Surface Waves in Vibrated Granular Material

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March 2, 1999

Abstract

We use molecular dynamics simulations to study the formation of surface waves in vertically vibrated granular material. We find that horizontal movements of particles, which are essential for the formation of the waves, consist of two distinct processes. First, the movements sharply increase while the particles are colliding with a bottom plate, where the duration of the collisions is very short compared to the period of the vibration. Next, the movements gradually decrease between the collisions, during which the particles move through the material. We also find that the horizontal velocity field after the collisions is strongly correlated to the surface profile before the collisions.

PACS Number: 85.70.F; 47.20; 03.20.+i; 03.40.Kf

Granular material under vibration has been a constant source of interesting phenomena among which surface waves are particularly interesting [1, 2, 3, 4, 5]. Particles in a vertically vibrating box display a variety of steady state patterns—stripes, squares, hexagons, and localized excitations called “oscillons”, and even a richer set of transient patterns [6, 7, 8, 9, 10, 11]. The fact that such diverse patterns can exist in the seemingly simple system has attracted a lot of efforts on the study of the mechanism for the formation of the surface waves [12, 13, 14, 15, 17, 16, 18, 19].

It is generally accepted that horizontal movements of particles play an important role in the formation of the surface waves. The present theories can be roughly divided into two groups according to how such movements are generated and maintained. In the first group of theories, it is argued that horizontal movements are generated while particles are colliding with the bottom of a box [12, 13, 14]. The duration of the collisions is very short compared to the period of the vibration. It is also argued that between the collisions with the bottom, horizontal movements gradually decrease while the particles move through the medium. Using these assumptions, many of the patterns observed in the experiments can be reproduced. In the other group of theories, it is argued that horizontal movements depend only on continuum variables such as height and/or density fields [17, 14, 16, 18]. Since these fields change smoothly over time, changes in the movements are also gradu-
ual. In particular, changes in the movements during the collisions (with the bottom) are not particularly different from those during other phases of the vibration. These theories can also reproduce many of the experimental patterns, typically by coupling density and height fields.

The underlying assumptions of all of these theories sound plausible. Also, it is very difficult to determine which of the theories describes the experiments better. All of them reproduce many of the experimental patterns, and have their own strong and weak points. Also, one should note that such theories cannot be validated just because they reproduce the observed patterns. Quantitative agreements (e.g., of dispersion relation) are necessary for the validation. In order to check the validity of the theories, it is thus necessary to obtain detailed information on the system, and compare it with the predictions of the theories. Unfortunately, such information is usually difficult to obtain by experiments.

In this paper, we use molecular dynamics (MD) simulation method, which provides detailed information on the motion of individual particles as well as time averaged fields. We study the system in two dimensions. We focus on horizontal movements of particles, which are essential to the formation of the surface waves. We find that the time evolution of average horizontal speed \(U\) is made up of two separate processes. First, there are sharp increases in \(U\) during the short periods that the pile is colliding with the bottom plate. The other process is gradual decay of \(U\) between such collisions, which results from interparticle collisions. The time evolution of \(U\) strongly supports the theories in the first group \([12, 13, 14]\). The present results do not imply that a continuum description is not possible for the system, it implies that the interpretation of the continuum variables has to be modified.

We then study the processes in more detail. We find that the horizontal velocity field \(V_x(x)\) after collisions (with the bottom) shows a strong correlation with \(\partial_x h(x)\) before the collisions, where \(h(x)\) is the center of mass field. Such correlation is assumed in the theories of \([12, 14]\). The second process can be characterized by temporal decay of \(U\). We find that the decay time is very small when there is no surface wave, and it is comparable to the period of the vibration when surface waves are present. We also study the parameter dependence of the decay time and maximum horizontal speed.

The simulations are done in two dimensions with disk shaped particles, using a form of interaction due to Cundall and Strack \([21, 22]\). Particles interact only by contact, and the force between two such particles \(i\) and \(j\) is the following. Let the coordinate of the center of particle \(i\) (\(j\)) be \(\vec{R}_i\) (\(\vec{R}_j\)), and \(\vec{r} = \vec{R}_i - \vec{R}_j\). The normal component \(F_{n_{i \rightarrow j}}\) of the force acting on particle \(i\) from particle \(j\) is

\[
F_{n_{i \rightarrow j}} = k_n(a_i + a_j - |\vec{r}|) - \gamma m_e(\vec{v} \cdot \hat{n}),
\]

where \(a_i\) (\(a_j\)) is the radius of particle \(i\) (\(j\)), \(\hat{n} = \vec{r}/r\), and \(\vec{v} = d\vec{r}/dt\). Here, \(k_n\) is the elastic constant, \(\gamma\) the friction coefficient, and \(m_e\) is the effective mass, \(m_i m_j/(m_i + m_j)\). The shear component \(F_{s_{i \rightarrow j}}\) is given by

\[
F_{s_{i \rightarrow j}} = -\text{sign}(\delta s) \min(k_s|\delta s|, \mu|F_{n_{i \rightarrow j}}|),
\]

where \(\mu\) is the friction coefficient, \(\delta s\) the total shear displacement during a contact, and \(k_s\) is the elastic constant of a virtual tangential spring. The shear force applies a torque to the particles, which then rotate.

Particles can also interact with walls. The force and torque on particle \(i\), in contact with a wall, are given by \([3, 4]\) with \(a_i = 0\) and \(m_e = m_i\). Also, the system is in a vertical gravitational field \(\vec{g}\). The interaction parameters used in this study are fixed as follows, unless otherwise specified: \(k_n = k_s = 10^7, \gamma = 10^4\) and \(\mu = 0.2\). And, the timestep for integration is \(5 \times 10^{-7}\). In order to avoid artifacts of a monodisperse system (e.g., hexagonal packing), we choose the radius of the particles from a Gaussian distribution with the mean 0.1 and width 0.02. The density of the particles is 5. Throughout this paper, CGS units are implied.

We put particles on a horizontal plate which oscillates sinusoidally along the vertical direction with given amplitude \(A\) and frequency \(f\). Let the width of the plate be \(W\). We apply a periodic boundary condition in the horizontal direction. We start the
simulation by inserting the particles at random positions above the plate. We let them fall by gravity and wait while they lose energy by collisions. We wait for $10^6$ iterations for the particles to relax, and during this period we keep the plate fixed. The typical velocity at the end of the relaxation is of order $10^{-2}$. After the relaxation, we vibrate the plate, and start to take measurements.

The coefficient of restitution between the particles $e_{pp}$, determined from the above interaction parameters, is 0.21, and the coefficient between the particles and the plate $e_{pw}$ is $8.0 \times 10^{-2}$. The particles are thus strongly inelastic. We have studied the motion of a single particle for several values of $A$ with $f = 10$, and find good agreements with the predictions by Mehta and Luck [22]. Also, the present model was shown to reproduce the dispersion relation from experiment [23].

We then study the center of mass motion of the particles. As the depth of the pile increases, its motion can be different from that of a single particle [24]. For the parameters with which the motion of a single particle is periodic with period $T = 1/f$, the motion of the pile can be subharmonic at large depth. Although studying the effect of the subharmonic motion on the surface waves can be interesting, here we limit ourselves to the simpler case of no such motion. The minimum depth at which the subharmonic motion occurs depends on the interaction parameters, and is an increasing function of $k_n$. We thus use a rather large value of $k_n (10^7)$, so the subharmonic motion is absent in all cases studied here. As a check, we study the motion of the particles in a narrow tube of $W = 1$ (five particle width) for several $\Gamma$ with fixed $f$. Here, $\Gamma$ is dimensionless peak acceleration $4\pi Af^2/g$. We find that the pile behaves like a single particle for at least 10 particle depth. In particular, a bifurcation of the motion occurs at $\Gamma \simeq 3.7$, which then terminates at $\Gamma \simeq 4.4$, which agree well with the results of a single inelastic particle [25].

We proceed to study surface waves. We choose $W = 16$ and $N = 800$. Thus the system is, on average, 80 particle wide and 10 particle deep. We fix the frequency $f = 10$, and study the waves for several values of $\Gamma$. We find that $f/2$ waves start to appear for $\Gamma \sim 2.5$, then disappear when $\Gamma$ becomes about 4. When $\Gamma$ further increases, $f/4$ waves start to appear for $\Gamma \sim 5.5$. The features of this “phase diagram” agree well with those of the experiments [26].

Inspection of the motion of the particles shows that horizontal movements of the particles play a crucial role in maintaining the waves—an observation which was made in many previous studies on the problem (e.g., [1]). We focus on the horizontal movements which we characterize by average horizontal speed $U(t)$, defined as

$$U(t) = \frac{1}{N} \sum_{i=1}^{N} |v_i^x(t)|,$$

where $v_i^x$ is the horizontal velocity of particle $i$. Even if there are active horizontal movements, the average horizontal velocity can be small since the movements can occur in both positive and negative $x$ directions. We thus use $U(t)$ instead of the average horizontal velocity. We normalize $U(t)$ with $2\pi Af$, the maximum velocity of the bottom plate. In Fig. 1, we show normalized $U(t)$ for $\Gamma = 2$ and 3 for 10 vibration periods. Here, $f = 10$, $W = 16$, and $N = 800$. The $\Gamma = 3$ curve has been offset for clarity.

![Figure 1](image_url)

**Figure 1**: Dimensionless average horizontal speed $U(t)$ for first 10 vibration cycles. Here, $W = 16$, $N = 800$, $f = 10$ and $\Gamma = 2$ and 3. The $\Gamma = 3$ curve has been offset for clarity.

As shown in the figure, there clearly exist two separate processes: sharp increase in $U$ within short time intervals and gradual decay of $U$ during other phases
of the vibration. We also measure the time series of the total pressure on the plate \( p(t) \), which consists of sharp peaks occurring when the pile collides with the plate. We find that the locations of the peaks in \( U(t) \) coincide with those in \( p(t) \). Thus, the horizontal movements are being fed by collisions of the pile with the plate, and are being lost while the pile is not in contact with the plate.

In order to maintain the surface waves, the particles have to travel distance \( \lambda \) within the period of the surface waves, where \( \lambda \) is their wavelength. When \( U(t) \) decays slowly, we expect that the particles can travel long enough distance to maintain the waves. Indeed, we find that the surface waves are present only when \( U(t) \) decays slowly. To be more precise, the waves are observed when the decay time \( \tau \), which will be defined later, is comparable to the period of the vibration.

These observations strongly support the theories \([12, 13, 14]\) which argue that the horizontal motion is being supplied by collisions with the bottom plate, and is being dissipated by interparticle collisions. The short time intervals during which the pile is colliding with the plate play a crucial role in maintaining the surface waves.

In Fig. 2, we show time evolution of the horizontal velocity field \( V_x(x,t) \), which is defined as the average horizontal velocity of the particles whose centers are in \([x, x + dx]\). Here, \( \Gamma = 3 \), \( f = 10 \), and other parameters are the same as in Fig. 1. We also normalize \( V_x(x,t) \) by \( 2\pi Af \). In the figure, sharp changes in \( V_x(x, t) \) occur at certain values of \( t \), which we find to be identical to the positions of the peaks in \( U(t) \). Thus, sudden increases in \( V_x(x, t) \) occur when the pile collides with the plate. The shape of \( V_x(x, t) \) after such collisions does not change much until next collisions, but the overall magnitude of \( V_x(x, t) \) decreases.

The profile of \( V_x \) just after the collisions is important for the dynamics of the surface waves. Since the motion of the particles is deterministic, it is possible that the profile is determined from certain field just before the collisions. We check the dependency of \( V_x \) on several fields. Specifically, we calculate the correlation coefficient \( r \) between \( V_x \) after the collisions and the following fields just before the collisions: \( x \) and \( y \) velocity fields \( V_x \) and \( V_y \), their spatial derivative \( \partial_x V_x \) and \( \partial_y V_y \), the number density \( m \) and its derivative \( \partial_x m \), and the center of mass field \( h \) and its derivative \( \partial_x h \). Here, \( m(x)dx \ (h(x)) \) is defined to be the total number (the center of mass) of the particles whose centers are in \([x, x + dx]\). Since the determination of the collision times becomes difficult for large \( \Gamma \), we study only for \( \Gamma \leq 4 \). We average \( r \) over 10 collisions. We find that \( r \) is large (\( \sim -0.8 \)) for \( \partial_x m, \partial_x h \) and \( V_x \), and there is essentially no correlation with the other fields. The large value of \( r \) for \( \partial_x h \) suggests

\[ V_x(x, t_a) \propto -\partial_x h(x, t_b), \tag{4} \]

where \( t_a \) and \( t_b \) is the time just after and before the collisions, respectively. This very equation was assumed in the theories of Cerda et al. \([12]\) and Eggers and Riecke \([4]\), where they proposed the form motivated from the flow of granular material on an inclined plane. The strong correlation occurs for all \( \Gamma \) studied here, including the \( \Gamma = 2 \) case where no surface wave is observed.

We find that the spatial variation of the packing fraction is not significant, so \( m \) is roughly proportional to \( h \). We thus expect that the correlation for
\( \partial_{\tau} m \) is also large. What is strange is the strong correlation between \( V_z \) before and after the collisions, which states that the particles reverse the direction of their horizontal movements while the pile is colliding with the plate. We do not understand the origin of the reversal of the motion.

We also measure the magnitude of the horizontal movements for several values of \( \Gamma \). To be more specific, we measure the peak values of \( U(t) \) like the ones shown in Fig. 1, and averaged them over all the collisions. One might expect that the average is proportional to the collision velocity of particles on the plate. Even though both display minimum around \( \Gamma = 4.6 \), we find that there is no strong correlation between them, and the average seems to show complicated dependence on \( \Gamma \). Further study is necessary to quantify and understand the dependence.

Next, we study the decay process, where horizontal movements of the particles gradually decrease between the collisions with the plate. We quantify the process as follows. We start from a time series of \( U(t) \) like the ones shown in Fig. 1. We then translate the positions of the peaks so that they all coincide. We then normalize the peak values of \( U(t) \) to unity, and average \( U(t) \) over all the peaks. The resulting quantity, defined as \( U_a(t) \), can be used to characterize the decay process. In Fig. 3, we show \( U_a(t) \) for \( \Gamma = 2, 3 \) and 4, where other parameters remain unchanged. We then define decay time \( \tau \) as the “half-life”—the time at which \( U_a(t) \) becomes 1/2. The half-life for \( \Gamma = 2 \) is about 0.0045, which is much smaller than the period of the vibration (\( T = 0.1 \)). On the other hand, \( \tau \) is close to 0.08 for \( \Gamma = 3 \) and 4. For all \( \Gamma \) we have studied, we find that the surface waves are present if and only if \( \tau \) is comparable to the period of the vibration. The need for large \( \tau \) for the formation of the surface waves is evident as previously discussed. The fact that the dynamics consists of two separate processes does not seem to depend on small variations of interaction parameters.

The quantitative understanding of the decay process requires that one should be able to predict the dependence of \( \tau \) on parameters such as \( \Gamma \). However, it seems that such dependences are rather complex. For example, we find that \( \tau \) does not depend on \( \Gamma \) in a simple fashion: \( \tau \) is not a monotonic function of \( \Gamma \), and a small increase in \( \Gamma \) can result in a ten-fold increase or decrease in \( \tau \). At the particle level, the decay process occurs by the collisions between the particles, and the resulting changes in the horizontal movements. In order to make quantitative predictions, one thus has to understand both the collision frequency and the resulting momentum changes. Unfortunately, both of these quantities are poorly understood. More studies are again needed in order to understand the decay process.

The time evolution of horizontal movements of the particles can also be studied using auto-correlation function \( c(t) \) defined as \( \langle v_x(0)v_x(t) \rangle \), where the average is taken over the particles. We find that \( c(t) \) also displays two basic processes: sharp change during the collisions with the plate and slow decay between them.

In sum, we show that horizontal movements of the particles, which are essential to the formation of the surface waves, consist of two separate processes: sharp increase of the movements by the collisions with the bottom plate, and slow decrease of the movements due to interparticle collisions, which strongly support the theories of [12, 13, 14]. We also find that the horizontal velocity field \( V_x(x) \) after the collisions with the plate is strongly correlated to \( \partial_x h(x) \) just before the collisions. Here, we are mainly interested

Figure 3: The time evolution of \( U_a(t) \) for \( \Gamma = 2, 3 \) and 4. The “half-life” \( \tau \) is small for \( \Gamma = 2 \), but becomes comparable to the period of the vibration for \( \Gamma = 3 \) and 4.
in qualitative features of the mechanism for the formation of the surface waves. Their quantitative understanding, such as the parameter dependence of \( \tau \), will be subject of future works.

After the present work has been completed, we became aware of the work of Kim et al \[27\], where similar results were obtained.

I thank H. K. Pak, S. Kim, K. Kim and S.-O. Jeong for useful discussions. This work is supported in part by the Department of Energy under grant DE-FG02-93-ER14327, Korea Science and Engineering Foundation through the Brain-Pool program, and SNU-CTP.

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