Gauge and Gravitational Couplings from Modular Orbits in Orbifold Compactifications

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ABSTRACT

We discuss the appearance of modular functions at the one-loop gauge and gravitational couplings in (0,2) non-decomposable N=1 four dimensional orbifold compactifications of the heterotic string. We define the limits for the existence of states causing singularities in the moduli space in the perturbative regime for a generic vacuum of the heterotic string. The "proof" provides evidence for the explanation of the stringy Higgs effect.
1 Introduction

The purpose of this paper is to examine the appearance of one-loop threshold corrections in gauge and gravitational couplings, in four dimensional non-decomposable orbifolds of the heterotic string. In $4D$ $N = 1$ orbifold compactifications the process of integrating out massive string modes, causes the perturbative one-loop threshold corrections\(^1\), to receive non-zero corrections in the form of automorphic functions of the target space modular group. The one-loop threshold corrections can be calculated either by calculation of string amplitudes or by the sum over modular orbits. The latter technique will be used in this work.

At special points in the moduli space, previously massive states become massless, and contribute to gauge symmetry enhancement. The net result of the appearance of massless states in the running gauge coupling constants appears in the form of a dominant logarithmic term. In section two we will discuss the logarithmic term effect and suggest that its appearance, due to the nature of the underlying modular integration, sets specific limits in the mass of the previously massive states that becoming massless at the enhanced symmetry point.

In addition, in this paper, we are particularly interested in the calculation of one-loop threshold effects in non-decomposable orbifolds, using the technique of summing over modular orbits, that arise after integrating out the moduli dependent contributions of the heavy string modes. The last technique have been used in a variey of contexts, such as, the calculation of target space free energies of toroidal compactifications in \([\text{1}]\) and of Calabi-Yau compactification models \([\text{2}]\), in addition to the calculation of threshold effects to gauge and gravitationalcouplings in $N = 1$ $4D$ decomposable orbifold compactifications \([\text{3}]\) and the calculation of target space free energies and $\mu$-term contributions in $N = 1$ $4D$ non-decomposable orbifold compactifications in \([\text{4}]\). In section three we will discuss the appearance of automorphic functions of $\Gamma_o(3)_{T,U}$ via the calculation of modular orbits of target space free energies and thus the threshold corrections, generalizing to non-decomposable orbifolds the discussion in \([\text{3}]\) for decomposable ones. In sections four and five we will complete the picture by extending the calculation of one-loop threshold effects to gauge and gravitational couplings respectively, using the sum over modular orbits (SMO), to $N = 1$ $4D$ non-decomposable orbifolds. We will exhibit the application of SMO by examining a $Z_6$ $N = 1$ non-decomposable orbifold which exhibits a $\Gamma^o(3)_T \times \Gamma^o(3)_U$ target space duality group in one of its two dimensional untwisted subspaces. The gauge embedding in the gauge degrees of freedom will not be specified, apart for its $T^2$ torus subspace part, and

\(^1\)which receive non-zero moduli dependent corrections from the $N = 2$ unrotated complex planes
kept generic in order for the threshold effects to be dependent only on the Wilson line context of its two dimensional subspace.

2 Massless singularity limit

In general, if one wants to describe globally the moduli space and not just the small field deformations of an effective theory around a specific vacuum solution, one has to take into account the number of massive states that become massless at a generic point in moduli space. This is a necessary, since the full duality group $SO(22,6;\mathbb{Z})_T$ mixes massless with massive modes. It happens because there are transformations of $O(6,22,\mathbb{Z})$ acting as automorphisms of the Lorentzian lattice metric of $\Gamma^{(6,22)} = \Gamma^{(6,6)} \oplus \Gamma^{(0,16)}$ that transform massless states into massive states.

Let us consider now the $T_2$ torus, coming from the decomposition of the $T_6$ orbifold into the form $T_2 \oplus T_4$. At the large radius limit of the $T^2$ it was noticed that in the presence of states that become massless at a point in moduli space e.g, when the $T \rightarrow U$, the threshold corrections to the gauge coupling constants receive the most dominant logarithmic contribution in the form,

$$\Delta_a(T, \bar{T}) \approx b'_a \int_T \frac{d^2 \tau}{\tau_2} e^{-M^2(T)\tau_2} \approx -b'_a \log M^2(T),$$

(1)

where $b'_a$ is the contribution to the $\beta$-function from the states that become massless at the point $T = U$. Strictly speaking the situation is slightly different. We will argue that if we want to include in the string effective field theory large field deformations and to describe the string Higgs effect and not only small field fluctuations, eqn.(1) must be modified. We will see that massive states which become massless at specific points in the moduli space do so, only if the values of the untwisted moduli dependent masses are between certain limits. In this point was not emphasized and it was presented in a way that the appearance of the singularity in eqn. (1) had a general validity for generic values of the mass parameter.

We introduce the function Exponential Integral $E_1(z)$

$$E_1(z) = \int_z^\infty \frac{e^{-t}}{t} dt \quad (|\text{arg } z| < \pi),$$

(2)

with the expansion

$$E_1(z) = -\gamma - \ln z - \sum_{n=1}^{\infty} \frac{(-)^n z^n}{nn!}.$$  

(3)
It can be checked that for values of the parameter $|z| > 1$, the $lnz$ term is not the most dominant, while for $0 < |z| < 1$ it is.

In the latter case $|z| > 1$, the $lnz$ term is not the most dominant, while for $0 < |z| < 1$ it is. The $E_1(z)$ term is approximated as

$$E_1(z) = -ln(z) + a_0 + a_1z + a_2z^2 + a_4z^3 + a_4z^4 + a_5z^5 + \epsilon(z).$$  \hspace{1cm} (5)

Take now the form of eqn.(1) explicitly

$$\triangle(z, \bar{z}) \approx b_a' \int_{|\tau_1|<1/2}^{\infty} d\tau_1 \int_{\sqrt{1-\tau_1}} e^{-M^2(T)\tau_2}. \hspace{1cm} (6)$$

Then by using eqn.(2) in eqn.(6), we can see that the $-b_a' \ln M^2(T)$ indeed arise. Notice now, that the limits of the integration variable $\tau_1$ in the world-sheet integral in eqn.(6) are between $-1/2$ and $1/2$. Then especially for the value $|1/2|$ the lower limit in the integration variable $\tau_2$ takes its lowest value e.g $(1-\tau_2^{1/2}) = (1-(1/2)^{1/2}) = \sqrt{3}/2$. Use now eqn.(3). Rescaling the $\tau_2$ variable in the integral, and using the condition $0 < z < 1$ we get the necessary condition for the logarithmic behaviour to be dominant$^3$

$$0 < M^2(T) < \frac{4}{\sqrt{3}a'}.$$ \hspace{1cm} (7)

This means that the dominant behaviour of the threshold corrections appears in the form of a logarithmic singularity, only when the moduli scalars satisfy the above limit.

We know that for particular values of the moduli scalars, the low energy effective theory appears to have singularities, which are due to the appearance of charged massless states in the physical spectrum. At this stage, the contribution of the mass to the low energy gauge coupling parameters is given by $[11]$

$$M^2 \to -n_H|T - p|^2,$$ \hspace{1cm} (8)

where the $n_H$ represents the number of states $\phi_H$ which become massless at the point $p$.

The parameter $(a'\sqrt{3/4})M^2$ must always be between the limits zero and one in order that the dominant contribution of the physical singularity to $\triangle$ to be in the ”mild”

$^2$ The ”Exponential Integral” $E_1(x)$ for $0 \leq x \leq 1$ is $E_1(x) = -ln(x) + a_0 + a_1x + a_2x + a_3x + a_4x + a_5x + \epsilon(x), |\epsilon(x)| \leq 2 \times 10^{-7}$, with the numerical constants $a_i$ to be given by

$$a_0 = -5.77 \hspace{0.5cm} a_1 = 0.99 \hspace{0.5cm} a_2 = -0.25 \hspace{0.5cm} a_3 = -0.55 \hspace{0.5cm} a_4 = -0.009 \hspace{0.5cm} a_5 = 0.00107$$ \hspace{1cm} (4)

$^3$Restoring units in the Regge slope parameter $a'$. 

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logarithmic form \((8)\). Therefore, the complete picture of the threshold effects, when the asymptotic behaviour of the threshold corrections is involved, reads

\[
\frac{1}{g^2_a(\mu)} = \frac{k_a}{g^2_{\text{string}}} + \frac{b_a}{16\pi^2} \ln \frac{M^2_{\text{string}}}{\mu^2} - \Theta(-M^2 + \frac{4}{\sqrt{3}a^\prime})b^\prime_a \log M^2(T),
\]

where \(\Theta\) is the step function. Threshold effect dependence on the \(\Theta\) function, takes place in Yang-Mills theories, via the decoupling theorem \([13]\). The contribution of the various thresholds decouples from the full theory, and the net effect is the appearance of mass suppressed corrections to the physical quantities. Their direct effect on the low energy effective theory is the appearance of the automorphic functions of the moduli dependent masses, after the integration of the massive modes.

So far, we have seen that the theory can always approach the enhanced symmetry point behaviour from a general massive point on the moduli space under specific conditions. For "large" values of the moduli masses the enhanced symmetry point can be approached if its mass is inside the limit \((7)\). Remember that at the point \(T = p\) eqn.\((9)\) breaks down, since at this point perturbation theory is not valid any longer.

### 3 Target space automorphic functions from string compactifications

Before looking at the appearance of automorphic functions in the one-loop gauge and gravitational couplings of 4D orbifold compactifications, using the sum over modular orbits, we need some background on the mass operator moduli dependence in orbifolds. For orbifold compactifications, where the underlying internal torus does not decompose into a \(T_6 = T_2 \oplus T_4\), the \(Z_2\) twist associated with the reflection \(-I_2\) does not put any additional constraints on the moduli \(U\) and \(T\). As a consequence the moduli space of the untwisted subspace is the same as in toroidal compactifications and orbifold sectors which have the lattice twist acting as a \(Z_2\), give non-zero threshold one-loop corrections to the gauge coupling constants in \(N = 1\) supersymmetric orbifold compactifications.

In the study of the untwisted moduli space, we will assume initially that under the action of the internal twist there is a sublattice of the Narain lattice \(\Gamma_{22,6}\) in the form \(\Gamma_{22,6} \supset \Gamma_2 \oplus \Gamma_4\) with the twist acting as \(-I_2\) on \(\Gamma_2\). In the general case, we assume that there is always a sublattice\(^4\) \(\Gamma_{q+2,2} \oplus \Gamma_{r+4,4} \subset \Gamma_{16,6}\), where the twist acts as \(-I_{q+4}\), on

\(^4\)this does not correspond to a decomposition of the Narain lattice as \(\Gamma_{22,6} = \Gamma_{q+2,2} \oplus \ldots\) since the gauge lattice \(\Gamma_{16}\) is an Euclidean even self-dual lattice. So the only way for it to factorize as \(\Gamma_{16} = \Gamma_q \oplus \Gamma_r\), with \(q + r = 16\), is when \(q = r = 8\).
\[ \Gamma_{q+2,2} \text{ and with eigenvalues different than } -I \text{ on } \Gamma_{r+4,4}. \] In this case, the mass formula for the untwisted subspace \( \Gamma_{q+2,2} \) depends on the factorised form \( P^2_R = v^T \phi \phi^T v \), with \( v^T \) taking values as a row vector, namely as

\[ v^T = (a^1, \ldots, a^q; n^1, n^2; m^1, m^2). \quad (10) \]

The quantities in the parenthesis represent the lattice coordinates of the untwisted sublattice \( \Gamma_{q+2,2} \), with \( a^1, \ldots, a^q \) the Wilson line quantum numbers and \( n^1, n^2, m^1, m^2 \) the winding and momentum quantum numbers of the two dimensional subspaces.

Let us consider first the generic case of an orbifold where the internal torus factorizes into the orthogonal sum \( T_6 = T_2 \oplus T_4 \) with the \( Z_2 \) twist acting on the 2-dimensional torus lattice. We will be interested in the mass formula of the untwisted subspace associated with the \( T_2 \) torus lattice. We consider as before that there is a sublattice of the Euclidean self-dual lattice \( \Gamma_{22,6} \) as \( \Gamma_{q+2,2} \oplus \Gamma_{20-q;4} \subset \Gamma_{22,6} \). In this case, the momentum operator factorises into the orthogonal components of the sublattices with \((p_L; P_R) \subset \Gamma_{q+2,2}\) and \((P_L; P_R) \subset \Gamma_{20-q;4}\). As a result the mass operator factorises into the form

\[ \frac{\alpha'}{2} M^2 = p^2_R + P^2_R + 2N_R. \quad (11) \]

On the other hand, the spin operator \( S \) for the \( \Gamma_{q+2,2} \) sublattice changes as

\[ p^2_L - p^2_R = 2(N_R + 1 - N_L) + P^2_R - P^2_L = 2n^T m + q^T C q, \quad (12) \]

where \( C \) is the Cartran metric operator for the invariant directions of the sublattice \( \Gamma_q \) of the \( \Gamma_{16} \) even self-dual lattice. In eqn’s (11,12), we discussed the level matching condition in the case of a \( T_6 \) orbifold admitting an orthogonal decomposition.

Let us now consider the gauge symmetry enhancement of the \( Z_6-II-b \) orbifold. This orbifold is defined on the torus lattice \( SU(6) \times SU(2) \) and the twist in the complex basis is defined as \( \Theta = \exp((2, -3, 1) \frac{2\pi i}{6}) \). This orbifold is non-decomposable in the sense that the action of the lattice twist does not decompose into the orthogonal sum \( T_6 = T_2 \oplus T_4 \) with the fixed plane lying in \( T_2 \). The orbifold twists \( \Theta^2 \) and \( \Theta^4 \), leave the third and complex plane unrotated. The lattice in which the twists \( \Theta^2 \) and \( \Theta^4 \) act as a lattice automorphism is the \( SO(8) \). In addition there is a fixed plane which lies in the \( SU(3) \) lattice and is associated with the \( \Theta^3 \) twist.

Consider now the k-twisted sector of a six-dimensional orbifold of the the heterotic string associated with a twist \( \theta^k \). The twisted sector quantum numbers have to satisfy

\[ Q^k n = n, \quad Q^k m = m, \quad M^k l = l, \quad (13) \]

\(^5\) The information about the nature of singularities will then used in the calculation of the modular orbits.
where $Q$ defines the action of the twist on the internal lattice and $M$ defines the action of the gauge twist on the $E_8 \times E_8$ lattice.

In the $Z_6 - II - b$ orbifold, for the $N = 2$ sector associated to the $\Theta^2$ twist, $n^T m = m_1 n^1 + 3 m_2 n^2$, and

$$m^2 = \sum_{m_1, m_2 n^1, n^2} \left| \frac{1}{Y} - T U' n^2 + i T n^1 - i U' m_1 + 3 m_2 |_{U' - U + 2} = M/(Y/2), \quad (14)$$

with $Y = (T + \bar{T})(U + \bar{U})$. The quantity $Y$ is associated with the Kähler potential, $K = -\log Y$. The target space duality group is found to be $\Gamma^0(3)_{T} \times \Gamma^0(3)_{U'}$, where $U' = U + 2$. Mixing of the equations (11, 12) gives us the following equation

$$p_L^2 - \frac{\alpha'}{2} M^2 = 2(1 - N_L - \frac{1}{2} P_L^2) = 2n^T m + q^T C q. \quad (15)$$

The previous equation gives us a number of different orbits invariant under $SO(q + 2, 2; Z)$ transformations:

a) the untwisted orbit with $2n^T m + q^T C q = 2$. In this orbit, $N_L = 0, \ P_L^2 = 0$. In particular, when $M^2 = 0$, this orbit is associated with the string Higgs effect. The string Higgs effect appears as a special solution of the (15) at the point where $p_L^2 = 2$, where additional massless particles may appear.

b) the untwisted orbit where $2n^T m + q^T C q = 0$ where, $2N_L + P_L^2 = 2$. This is the orbit relevant to the calculation of threshold corrections to the gauge couplings, without taking into account the enhanced gauge symmetry points.

c) The massive untwisted orbit with $2N_L + P_L^2 \geq 4$. Now always $M^2 \geq 0$. This orbit will be of no use to our attempt of exhibiting the singular behaviour of threshold corrections. Let us now consider, for the orbifold $Z_6 - II - b$, the modular orbit associated with the string Higgs effect. We are looking for points in the moduli space where singularities associated with the additional massless particles appear and have as a result gauge group enhancement. This point correspond to $T = U$ with $m_2 = n_2 = 0$ and $m_1 = n_1 = \pm 1$. At this point the gauge symmetry is enhanced to $SU(2) \times U(1)$. In particular, the left moving momentum for the two dimensional untwisted subspace yields

$$p_L^2 = \frac{1}{2 T_2 U_2} | T U n_2 - \bar{T} n_1 - i U' m_1 + 3 m_2 |^2 = 2, \quad (16)$$

while

$$p_R^2 = \frac{1}{2 T_2 U_2'} | - T U' n^2 + i T n^1 - i U' m_1 + 3 m_2 |^2 = 0. \quad (17)$$

At the fixed point of the modular group $\Gamma^0(3), \frac{\sqrt{3}}{2}(1 + i \sqrt{3})$, there are no additional massless states, so there is no further enhancement of the gauge symmetry.
We will now use eqn.(14) to calculate the stringy one-loop threshold corrections to the gauge coupling constants coming from the integration of the massive compactification modes with \((m, m', n, n') \neq (0, 0, 0, 0)\). The total contribution to the threshold corrections, coming from modular orbits and associated with the presence of massless particles, is connected to the existence of the following\(^6\) orbits\(\cite{2,3,4}\),

\[
\Delta_0 = \sum_{2n^+m+q^T Cq=2} \log \mathcal{M}|_{\text{reg}} \\
\Delta_1 = \sum_{2n^+m+q^T Cq=0} \log \mathcal{M}|_{\text{reg}}.
\]  

(18)

In the previous expressions, a regularization procedure is assumed that takes place, which renders the final expressions finite, as infinite sums are included in their definitions. Moreover, we demand that the regularization procedure for \(e^\Delta\) has to respect both modular invariance and holomorphicity. The regularization is responsible for the subtraction of a moduli independent quantity from the infinite sum e.g \(\sum_{n,m \in \text{orbit}} \log \mathcal{M}\). The regularization procedure for the case of a decomposable orbifold, where the threshold corrections are invariant under the \(SL(2, \mathbb{Z})\), were discussed in \(\cite{2}\). The general case of the regularization procedure for the case of non-decomposable orbifolds, where the threshold corrections are invariant under subgroups of \(SL(2, \mathbb{Z})\), was discussed in \(\cite{4}\).

Let us consider first the orbit relevant for the string Higgs effect. This orbit is associated with the quantity \(2n^T m + q^T Cq = 2\), where \(n^T m = m_1 n^1 + 3m_2 n^2\). The total contribution from the previously mentioned orbit yields:

\[
\Delta_0 \propto \sum_{n^T m = 0,q^2 = 1} \sum_{n^T m + q^T Cq = 2} \log \mathcal{M} = \sum_{n^T m = 0,q^2 = 1} \log \mathcal{M} + \sum_{n^T m = 1,q^2 = 0} \log \mathcal{M} + \sum_{n^T m = -1,q^2 = 2} \log \mathcal{M} + \ldots
\]  

(19)

We must notice here that we have written the sum \(\sum_{n^T m + q^T Cq = 2}\) over the states associated with the \(SO(4, 2)\) invariant orbit \(2n^T m + q^T Cq = 2\) in terms of a sum over \(\Gamma^0(3)\) invariant orbits \(n^T m = \text{constant}\). We will be first considering the contribution from the orbit \(2n^T m + q^T Cq = 0\). Note that we are working in analogy with calculations associated with topological free energy considerations \(\cite{1,2,3,4}\). From the second equation in eqn.(18), considering in general the \(SO(4, 2)\) coset, we get for example that

\[
\Delta_1 \propto \sum_{n^T m + q^2 = 0} \log \mathcal{M} = \sum_{n^T m = 0,q^2 = 0} \log \mathcal{M} + \sum_{n^T m = -1,q^2 = 1} \log \mathcal{M} + \ldots
\]  

(20)

Consider in the beginning the term \(\sum_{n^T m = 0,q^2 = 0} \log \mathcal{M}\). We are summing up initially the orbit with \(n^T m = 0; (n, m) \neq (0, 0)\),

\[
\mathcal{M} = 3m_2 - im_1 U' + in^1 T + n^2 (-U' T + B C) + q \text{ dependent terms}.
\]  

\(^6\) we calculate only \(\sum \log \mathcal{M}\) since \(\sum \log \mathcal{M}^\dagger\) is its complex conjugate,
We calculate the sum over the modular orbit \( n^T m + q^2 = 0 \). As in (3) we calculate initially the sum over massive compactification states with \( q_1 = q_2 = 0 \) and \( (n, m) \neq (0, 0) \). Namely, the orbit

\[
\sum_{n^T m = 0, \ q = 0} \log M = \sum_{(n, m) \neq (0, 0)} \log(3m_2 - im_1 U' + in_1 T + n_2(-U'T)) + BC \sum_{(n, m) \neq (0, 0)} \frac{n_2}{3m_2 - im_1 U' + in_1 T - n_2 U'T} + \mathcal{O}((BC)^2). \tag{22}
\]

The sum in relation (22) is topological (it excludes oscillator excitations) and is subject to the constraint \( 3m_2n_2 + m_1n_1 = 0 \). Its solution receives contributions from the following sets of integers:

\[
m_2 = r_1r_2, \quad n_2 = s_1s_2, \quad m_1 = -3r_2s_1, \quad n_1 = r_1s_2, \tag{23}
\]

\[
m_2 = r_1r_2, \quad n_2 = s_1s_2, \quad m_1 = -r_2s_1, \quad n_1 = 3r_1s_2, \tag{24}
\]

and

\[
\sum_{n^T m = 0, \ q = 0} \log M = \log\left[\left(\eta^{-2}(T)\frac{1}{3}\eta^{-2}\left(U'\frac{1}{3}\right)\right)\left(1 - 4 BC (\partial_T \log \eta(T)) \times \left(\partial'_U \log \eta(U')\right)\right)\right] + \mathcal{O}((BC)^2).
\tag{25}
\]

The previous expression is associated with the non-perturbative [2, 3, 4] gaugino generated superpotential \( W \), which comes by direct integration of the string massive orbifold modes. The contribution of this term could give rise to a direct Higgs mass in the effective action and represents a particular solution to the \( \mu \) term problem. These issues are discussed in [4]. The threshold contribution of (25) to the modular orbit \( \Delta_1 \) of eqn. (18) is obtained by substituting (25) in (20) yielding

\[
\Delta_1 \propto \log\left[\left(\eta^{-2}(T)\frac{1}{3}\eta^{-2}\left(U'\frac{1}{3}\right)\right)\left(1 - 4 BC (\partial_T \log \eta(T)) \times \left(\partial'_U \log \eta(U')\right)\right)\right] + \ldots.
\tag{26}
\]

The previous discussion was restricted to small values of the Wilson lines where our \((0, 2)\) orbifold goes into a \((2, 2)\). We turn now our discussion to the contribution from the first equation in (33) which is relevant to the stringy Higgs effect. Take for example the expansion (19). Let’s examine the first orbit corresponding to \( \Delta_{0,0} = \sum_{n^T m = 1, q = 0} \log M \). This
orbit is the one for which some of the previously massive states, now become massless. At these points $\Delta_{0,0}$ have to exhibit the logarithmic singularity. In principle we could predict, in the simplest case when the Wilson lines have been switched off that $\Delta_{0,0}$ may be given by

$$\Delta_{0,0} = \sum_{n^2 m=1} \log(TU'n^2 + Tn^1 - U'm_1 + 3m_2) = \log\{(\omega(T) - \omega(U'))^\xi \times \{\eta(T)^{-2} \eta(U')^{-2} + \eta(T) \eta(U')^{-2}\} + \ldots, \tag{27}$$

where $\omega(T)$ is the hauptmodul $[14]$ for the subgroup $\Gamma^o(3)_T$, namely $\omega = [\eta(T/3)/\eta(T)]^{12}$. The behaviour of $\Delta_0$ term reflects the fact that at the points with $T = U$, generally previously massive states becoming massless, while the eta-terms are needed for consistency under modular transformations. Finally, the integers $\chi$, $\zeta$ have to be calculated from a string loop calculation or by directly performing the sum. Note that for the R.H.S of (27) there is no known way of directly performing the sum.

After this parenthesis, we continue our discussion by turning on, Wilson lines. When we turn the Wilson lines on, for the $SO(4,2)$ orbit of the relevant untwisted two dimensional subspace, $\Delta_{0,0}$ becomes

$$\Delta_{0,0} = \sum_{n^2 m=1} \log\{3m_2 - im_1 U + in_1 T - n_2(UT - BC)\}. \tag{28}$$

The sum after using an ansatz, similar to [3], and keeping only lowest order terms satisfy

$$\Delta_{0,0} = \log(\omega(T) - \omega(U) - BC X(T, U))^{\xi} + \log\{\eta(T)^{-2} \eta(U')^{-2} + \eta(T) \eta(U')^{-2} - BC Y(T, U)\} + \ldots \tag{29}$$

The functions $X(T, U), Y(T, U)$, may be calculated by the demand of duality invariance. Let us first discuss the calculation of $X(T, U)$. Demanding duality invariance of the first term in (27), under $\Gamma^o(3)_U$ modular transformations, we get that $X(T, U)$ has to obey - to the lowest non-trivial order in B C - the transformation

$$X(T, U) \Gamma^o(3)_U \rightarrow (i\gamma U + \delta)^2 X(T, U) - i\gamma(i\gamma U + \delta) (\partial_T \omega(T)). \tag{30}$$

In (30) we have used the fact that under the $\Gamma^o(3)_U$ target space duality transformations

$$U \rightarrow \frac{\alpha U - i\beta}{i\gamma U + \delta}, \quad T \rightarrow T - i\gamma \frac{BC}{i\gamma U + \delta}, \quad \alpha \delta - \beta \gamma = 1,$$

$$B \rightarrow \frac{B}{i\gamma U + \delta}, \quad C \rightarrow \frac{C}{i\gamma U + \delta}, \quad \beta = 0 \mod 3, \tag{31}$$

7in the following we will be using the variable U instead of U'.
which leave the tree level Kähler potential

\[ K = -\log[(T + \bar{T})(U + \bar{U}) - (\bar{B} + C)(B + \bar{C})] \]  

(32)

invariant \[ \Gamma_0^\gamma \], the following transformation is valid

\[ \omega(T) - \omega(U) \xrightarrow{\Gamma_0^\gamma} \omega(T) - \omega(U) - i\gamma \frac{BC}{i\gamma U + \delta} (\partial_T \omega(T)). \]  

(33)

In a similar way invariance under \( \Gamma_0^\gamma(T) \) transformations

\[ T \xrightarrow{\Gamma_0^\gamma(T)} \alpha T - i\beta, \quad U \xrightarrow{\Gamma_0^\gamma(T)} U - i\gamma \frac{BC}{i\gamma U + \delta}, \quad \alpha \delta - \beta \gamma = 1, \]  

(34)

which leave \[ (32) \] invariant, \( X(T, U) \) has to transform as

\[ X(T, U) \xrightarrow{\Gamma_0^\gamma(T)} (i\gamma T + \delta)^2 X(T, U) + i\gamma (i\gamma T + \delta) (\partial_U \omega(U)), \]  

(35)

up to the lowest order in \( BC \). So far we have described the properties of \( X(T, U) \) under modular transformations. The final form of our function, which has to respect the proper modular transformations, and to reveal the presence of physical singularities in the quantum moduli space reads

\[ X(T, U) = -3\partial_U \{\log \eta^2(T)\} \omega'(T) + \partial_T \{\log \eta^2(U)\} \omega'(U) + \]  

\[ \beta \{\omega(T) - \omega(U)\} \{\eta^4(T)\eta^4(U)\} + O((BC)^2), \]  

(36)

where \( \beta \) is a constant which may be decided from a loop calculation. Lets us now try to determine the \( Y \)-term in

\[ D = \log \left( \eta(T)^{-2} \eta(U)^{-2} + \eta(T)^{-2} \eta(U)^{-2} - BC \mathcal{Y}(T, U) \right) \]  

(37)

of \[ (29) \]. It should transform with modular weight -1 under \( \Gamma_0^\gamma(U) \) transformations. In this case we find that \( \mathcal{Y} \) has to transform, up to order \( BC \) as

\[ \mathcal{Y}(T, U) \xrightarrow{\Gamma_0^\gamma(U)} (i\gamma U + \delta) \mathcal{Y}(T, U) - i\gamma \{\eta^{-2}(U)(\partial_T \eta^{-2}(T)) + (\partial_T \eta^{-2}(T)) \eta^{-2}(U)\}. \]  

(38)

On the other hand, if we demand that it transforms with modular weight -1 under \( \Gamma_0^\gamma(T) \) we get that, up to lowest order in \( BC \),

\[ \mathcal{Y}(T, U) \xrightarrow{\Gamma_0^\gamma(T)} (i\gamma T + \delta) \mathcal{Y}(T, U) - i\gamma \{\partial_T \eta^{-2}(U) \eta^{-2}(T) + (\partial_U \eta^{-2}(U)) \eta^{-2}(U)\}. \]  

(39)
The modular properties (38), (39) and the presence of the physical singularities in our moduli space fix the function \( \mathcal{Y}(T, U) \) up to order \((BC)^2\) as

\[
\mathcal{Y}(T, U) = \{ \eta^{-2}(T) \eta^{-2}(U/3)(\partial_T \eta^2(T))(\partial_U \eta^2(U)) + \\
\eta^{-2}(T) \eta^{-2}(U/3)(\partial_T \eta^2(T))(\partial_U \eta^2(U)) \} + \rho[ (\eta^2(T) \eta^2(U/3)) + \eta^2(U/3)\eta^2]\]

(40)

where \( \rho \) may be decided from a loop calculation. It follows now, from (29) that

\[ e^{\Delta_{0,0}} \propto \left[ (\omega(T) - \omega(U))^2 \eta(T)^{-2} \eta(U)^{-2} - BCY[\omega(T) - \omega(U)]^\xi - \right. \\
\left. -\xi (\omega(T) - \omega(U))^{\xi-1} BCG \{ \eta(T)^{-2} \eta(U)^{-2} + \eta(T)^{-2} \eta(U)^{-2} \} + O((BC)^2) \right]. \]

(41)

We must notice here that the expression for \( e^{\Delta_{0,0}} \) transforms with modular weight \(-1\) under the \( \Gamma_0(3) \) modular transformations (31, 34). This is natural since from the relations [2],

\[ Z = e^{-F_{\text{fermionic}}} = -\det\left( \frac{M^\dagger}{Y^\dagger} \frac{M}{Y} \right) = -\frac{|W|^2}{Y}, \]

\[ F_{\text{fermionic}} = \sum_{(n,m)\neq(0,0)} \log \det\left( \frac{M^\dagger}{Y^\dagger} \frac{M}{Y} \right), \]

(42)

where \( F_{\text{fermionic}} \) the fermionic free energy, the quantity \( e^{\Delta_{0,0}} \) is identified with \( W \), the superpotential.

4 Threshold corrections to gauge couplings

We will now analyze the threshold corrections to the gauge couplings, due to the integration of massive modes, in the case of \( N = 1 \) symmetric \((2,2)\) non-decomposable orbifold compactifications of the heterotic string. When considering an effective locally supersymmetric theory, we have to distinguish between the kind of renormalized physical couplings involved in the theory. These are the cut-off dependent Wilsonian gauge couplings and the moduli and momentum dependent effective gauge couplings (EGC). Let us consider contributions to the EGC from the \((2,2)\) symmetric non-decomposable \( Z_6 - II - b \) orbifold considered in the previous section. We want to examine the EGC when the embedding in the gauge degrees of freedom is such that the gauge group in the "observable" sector gets broken to a subgroup by turning on Wilson line moduli fields B, C on the untwisted subspace of the non-decomposable orbifold. We consider a general embedding in the gauge
degrees of freedom such that the gauge group, in the "hidden" sector remains unbroken, namely $E_8'$. The contributions to the EGC receive contributions from all the $N = 2$ sectors of the nondecomposable orbifold. Here for simplicity reasons we will consider only the contribution to the thresholds of the EGC from the $N = 2 G^2$ sector that were examined in the previous section. We examine first the contributions to the EGC from the unbroken $E_8'$ gauge group. In this case the threshold corrections $\Delta_{E_8'}$ receive contributions from the untwisted $N = 2$ orbit, $2 n^T m + q^T C q = 0$, of (23) yielding

$$\Delta_{E_8'} = c(E_8') \log \left( 9 |\eta(T)|^4 |\eta(U)|^4_3 |1 - BC (\partial_T \log \eta^2(T))(\partial_U \log \eta^2(U))^{-2} \right) +$$

$$+ c(E_8') \log \left( 9 |\eta(U)|^4 |\eta(T)|^4_3 |1 - BC (\partial_T \log \eta^2(T))(\partial_U \log \eta^2(U))^{-2} \right).$$

(43)

The full threshold corrections to the EGC receive an additional contribution from the massless modes, due to Kähler and sigma model anomalies equal to

$$\Delta_{\text{massive}} = C_a K - 2 \sum r T_a(r) \log \text{det} g_r,$$

(44)

where $C_a = -C(G_a) + \sum r T_a(r)$, $C_a$ the quadratic Casimir of the gauge group $G_a$, $K$ is the Kähler potential of the $N = 2$ unrotated subspace, the sum is over the chiral matter superfields transforming in a representation $r$ of $G_a$ and $g_r$ is the $\sigma$-model metric of the massless sector that the matter fields in the representation $r$ belong. The equation for the EGC associated to the $E_8'$ for a scale $p^2 \ll M_{E_8}$, after taking into account (44) and the contribution from the massive states that have been integrated out, namely eqn. (13), becomes

$$\frac{1}{g_{E_8'}} = \frac{S + \bar{S}}{2} + \frac{b_{E_8'}}{16 \pi^2} \log \frac{M^2_{\text{string}}}{p^2} - \tilde{a}_{E_8'} \log ((T + \bar{T})(B + \bar{B}) - (\bar{B} + C)(\bar{C} + B))$$

$$+ c(E_8') \log \left( 9 |\eta(T)|^4 |\eta(U)|^4_3 |1 - BC (\partial_T \log \eta^2(T))(\partial_U \log \eta^2(U))^{-2} \right) +$$

$$+ c(E_8') \log \left( 9 |\eta(U)|^4 |\eta(T)|^4_3 |1 - BC (\partial_T \log \eta^2(T))(\partial_U \log \eta^2(U))^{-2} \right),$$

(45)

where $b_{E_8'} = -3 c(E_8')$ and

$$\tilde{a}_{E_8'} = C_{E_8'} = -c(E_8').$$

(46)

In the following we will consider that, in the running of EGC at the "observable" sector, beyond the high energy string scale there is an additional scale, e.g $M_1$, for which supersymmetry remains unbroken and the gauge group $G$, sitting at the high energy scale, gets spontaneously broken at a subgroup. By inspection of (27) we can realize that below the string scale $M_{\text{string}}$ there is an additional scale given by $M_1 = |\omega(T) - \omega(U)| M_{\text{string}}$. 

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This is exactly the scale corresponding to gauge symmetry enhancement to $SU(2) \times U(1)$. Let us now try to calculate the running gauge coupling for the two additional massless vector multiplets\footnote{In the case of $N = 1$ four dimensional compactifications of heterotic string vacua, the moduli of the invariant subspace belongs to vector multiplets.} present in the spectrum at the point $T = U$. Note that the running of the gauge couplings for points different than $T = U$, between the scales $M_I$ and $M_{\text{string}}$, is given by

$$\frac{1}{g^2(M_I^2)} = \frac{1}{g^2(M_{\text{string}}^2)} + \frac{b_a}{16\pi^2} \log \frac{M_{\text{string}}^2}{M_I^2} + \Delta_{\text{massive}}, \quad (47)$$

where $\Delta_{\text{massive}}$ is given in (44). Here, $b_a = -3c(G_a) + \sum_C T_a(c) - 3 \sum_V T_a(r_V)$, with the first sum runs over the chiral matter superfields transforming under a representation $r_C$ of the gauge group with $T_a(c) = Tr_{r_C}(T_a^2)$, the second sum runs over light vector multiplet representations $r_V$, and $T_a$ denotes a generator of the gauge group. In the case that the gauge coupling of the vector multiplets is in the region $p^2 \ll M_I$, we get

$$\frac{1}{g_{U(1)}^2(p^2)} = \frac{1}{g^2(M_I^2)} + \frac{\tilde{b}_a}{16\pi^2} \log \frac{M_I^2}{p^2} + \Delta, \quad (48)$$

where

$$\Delta = -\frac{a_{U(1)}}{16\pi^2} \{ \log \left( 9|\eta(T)\eta(U)|^4 \right) + \log \left( 9|\eta(U)\eta(U)|^4 \right) \}. \quad (49)$$

Here,

$$a_{U(1)} = -c(U(1)) + \sum_C T_{U(1)}(1 + 2n_{U(1)}), \quad (50)$$

where $n_{U(1)}$ the modular weights of the light chiral superfields. Note that the moduli metric of the untwisted $N = 2$ plane, from (44), $g_r = ((T + \bar{T})(U + \bar{U}))^{nC}$, with $n_C$ the modular weight of the light chiral superfields. Let us now apply [47, 48, 49, 50] to the running gauge coupling belonging to the 2 additional vector multiplets present in the spectrum above the threshold scale $M_I$, for the $Z_6 - II - b$ orbifold,

$$\frac{1}{g_{U(1)}^2(p^2)} = \frac{s + \bar{s}}{2} + \frac{\tilde{b}_{U(1)}}{16\pi^2} \log \frac{M_{\text{string}}^2}{p^2} + \frac{\tilde{b}_{U(1)} - b_{U(1)}}{16\pi^2} \log (\omega(T) - \omega(U))^2 - \frac{a_{U(1)}}{16\pi^2} \{ \log ((T + \bar{T})(U + \bar{U})|\eta(U)|^4) + \log ((T + \bar{T})(U + \bar{U})|\eta(U)|^4) \}. \quad (51)$$

Here, $\tilde{b}_{U(1)} = 0$, since $c_{U(1)} = 0$ and there are no hypermultiplets charged under the $U(1)$. In the same way, $a_{U(1)} = 0$, since the gauge group under the additional threshold scale $M_I$ is abelian. The coefficient $b_{U(1)}$ equals $\tilde{b}_{U(1)} + 2b_{\text{vec}}$, where $b_{\text{vec}}$ the contribution from the $\beta$-function coefficients of the $N = 2$ vector multiplets which are massless above the threshold scale and 2 counts their multiplicity. The additional threshold scale beyond the traditional string tree level unification scale is the one associated with the term $\omega(T) - \omega(U)$. The threshold scale is associated with the enhancement of the abelian part of the gauge group to $SU(2)$. 
5 Threshold corrections to gravitational couplings

Let us now discuss contributions to the running gravitational couplings in (2, 2) symmetric $Z_N$ orbifold constructions of the heterotic string.

For (0, 2) $Z_N$ orbifolds the effective low energy action of the heterotic string is

$$\mathcal{L} = \frac{1}{2} \mathcal{R} + \frac{1}{4} g_{grav} \mathcal{C} + \frac{1}{4} S_R (GB) + \frac{1}{4} S_I R_{abcd} R^{abcd}, \quad (52)$$

where $S_R \equiv (S + \bar{S})$, $S_I \equiv 2 Im(S)$. We have used the conventional choice for the gravitational couplings is $1/g_{grav} \equiv S_R$, while $GB$ is the Gauss-Bonnet combination

$$4 \ (GB) = \mathcal{C}^2 - 2 \mathcal{R}_{ab}^2 + \frac{2}{3} \mathcal{R}^2 \quad (53)$$

and $\mathcal{C}$ the Weyl tensor $C_{abcd}$. When the above relation is written in the form

$$\mathcal{L} \propto \Delta^{grav}(T, \bar{T})(\mathcal{R}_{ab}^2 - 4 \mathcal{R}_{ab}^2 + \mathcal{R}^2) + \Theta^{grav}(T, \bar{T}) \epsilon_{abcd} \mathcal{R}_{ab} \mathcal{R}_{cd}, \quad (54)$$

where $(\Theta^{grav}(T, \bar{T})$ the CP-odd part of GB, then the one-loop corrections $\Delta^{grav}$ to the gravitational action in $N = 1$ decomposable orbifolds, in the absence of Green-Schwarz mechanism, give

$$\Delta^{grav} \propto b_{grav}^{N=2} \log \big( (T + \bar{T}) |\eta(iT)|^4 \big),$$

where $b_{grav}^{N=2}$ is the gravitational $\beta$-function coefficient that receives non-zero contributions from the $N = 2$ sectors.

The corrections to the gravitational couplings considered up to now in the literature, are concerned with the decomposable orbifolds. We will complete the discussion of corrections to the running gravitational couplings by examining non-decomposable orbifolds. For the latter orbifolds the threshold corrections are expressed in terms of automorphic functions belonging to subgroups of the inhomogeneous modular group $PSL(2, \mathbb{Z}) = SL(2, \mathbb{Z})/\pm 1$.

We focus our attention to the case of $Z_6 - II - b$ orbifold. We consider the case of vanishing Wilson lines in the $\Theta^2$ sector. In the presence of the threshold $p^2 \ll M_i^2 \ll M_{\text{string}}^2$, we get

$$\frac{1}{g_{grav}^2(M_i^2)} = \frac{1}{g_{grav}^2(M_{\text{string}}^2)} + \frac{b_{grav}}{16\pi^2} \log \frac{M_i^2}{M_{\text{string}}^2} - \frac{a_{grav}}{16\pi^2} \log \left( \eta^4(T) \eta^4 \left( \frac{U}{3} \right) \frac{9}{9} \right) - \frac{a_{grav}'}{16\pi^2} \log \left( 9 \eta^4 \left( \frac{T}{3} \right) \eta^4(U) \right), \quad (55)$$

and

$$\frac{1}{g_{grav}^2(p^2)} = \frac{1}{g_{grav}^2(M_i^2)} + \frac{b_{grav}}{16\pi^2} \log \frac{M_i^2}{p^2} - \frac{a_{grav}}{16\pi^2} \log \left( (T + \bar{T})(U + \bar{U}) \right). \quad (56)$$
Here \( \frac{1}{g_{\text{grav}}^2 (M_{\text{string}}^2)} = \frac{S+S}{2} \) the treel level coupling, \( \tilde{b}_{\text{grav}} \), \( b_{\text{grav}} \) the \( \beta \)-function coefficients for the range \( p^2 \ll M_I^2 \), \( M_I^2 \ll p^2 \ll M_{\text{string}} \) respectively. Note that in (53, 54) we neglected the contributions for the \( Z_6 \) \( I \) \( b \) orbifold that are coming from the other \( N = 2 \) sectors. For all our study and conclusions regarding (55, 56) we have considered that our orbifold has only one \( N = 2 \) sector, the \( \Theta^2 \) sector. If we want to consider the full \( Z_6 \) \( I \) \( b \) orbifold, we should add the holomorphic contributions from the other \( N = 2 \) sector in addition to the contributions to the \( \beta \)-function coefficients of the fixed plane lying in the \( SU(3) \) lattice, invariant under the \( \Theta^3 \) twist, for which the contributions to the gravitational running couplings transform under \( PSL(2, Z) \). The \( \tilde{a}_{\text{grav}} \) comes from non-holomorphic contributions from Kähler and \( \sigma \)-model anomalies and is given by \( \tilde{a}_{\text{grav}} = \frac{1}{24} (21 + 1 - \dim G + \gamma_M + \sum C(1 + 2nC)) \), where \( \gamma_M \) is the contribution from modulinos. The \( \tilde{a}_{\text{grav}} \) has been calculated in the absence of continuous Wilson lines [16] as coefficients of the Gauss-Bonnet term in the gravitational action and represents the contribution of the completely rotated \( N = 2 \) plane. In that case \( \tilde{a}_{\text{grav}} = \hat{b}_{\text{grav}}^{N=2} \). The coefficient \( a'_{\text{grav}} \) has also been calculated in [16] and equals \( \hat{b}_{\text{grav}}^{N=2} \). Moreover, because of the contribution of the additional vector multiplet which become massless above the enhancement scale \( M_I \), \( b_{\text{grav}} - \hat{b}_{\text{grav}} = \gamma^C_{\text{grav}} + \gamma^V_{\text{grav}} \), with \( \gamma^C_{\text{grav}}, \gamma^V_{\text{grav}} \) the contributions to the gravitational \( \beta \)-function arising from the decomposition of the additional \( N = 2 \) vector multiplet in terms of its \( N = 1 \) multiplets. That happens because any \( N = 2 \) vector multiplet, can be decomposed into a \( N = 1 \) vector multiplet and a \( N = 1 \) chiral multiplet. Substituting (53) into (54) we get

\[
-\frac{a'_{\text{grav}}}{16\pi^2} \log((T + \bar{T})(U + \bar{U})\eta^4(T)\eta^4(U)^3) - \frac{a'_{\text{grav}}}{16\pi^2} \log((T + \bar{T})(U + \bar{U})\eta^4(U)\eta^4(\frac{T}{3})^3),
\]

which is invariant under \( \Gamma_0(3)_{T,U} \) transformations.

Orbifolds, where the target space modular groups belong to a subgroup of the modular group may be found from further compactifying six-dimensional F-theory compactifications on a general Calabi-Yau 3-fold with an \( F_n \) base where the order of the Mordell-Weyl group may be three [17, 18].

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