Nonlocal Andreev Reflections as Source of Spin Exchange and Kondo Screening

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We report on a novel exchange mechanism, mediating the Kondo screening, in a correlated double quantum dot structure in the proximity of superconductor, stemming from nonlocal Andreev reflection processes. We estimate its strength perturbatively and corroborate analytical predictions with accurate numerical renormalization group calculations using an effective model for the superconductor-proximized nanostructure. We demonstrate that superconductor-induced exchange leads to a two-stage Kondo screening. We determine the dependence of the Kondo temperature on the coupling to superconductor and predict a characteristic modification of conventional low-temperature transport behavior, which can be used to experimentally distinguish this phenomenon from other Kondo effects.

Introduction.— The exchange interactions control the magnetic order and properties of a vast number of materials \cite{1} and lead to many fascinating phenomena, such as various types of the Kondo effect \cite{2–4}. Double quantum dots (DQDs), and in general multi-impurity systems, constitute a convenient and controllable playground, where nearly as much different exchange mechanisms compete with each other to shape the ground state of the system. Local exchange between the spin of a quantum dot (QD) and the spin of conduction band electrons gives rise to the Kondo effect \cite{2, 5}. Direct exchange arriving with an additional side-coupled QD may destroy it or lead to the two-stage Kondo screening \cite{4, 6–10}. In a geometry where the two QDs contact the same lead, conduction band electrons mediate the RKKY exchange \cite{11–13}. The RKKY interaction competes with the Kondo effect and leads to the quantum phase transition of a still debated nature \cite{14–22}. Moreover, in DQDs coupled in series also superexchange can alter the Kondo physics significantly \cite{23, 24}.

Recently, hybrid quantum devices, in which the interplay between various magnetic correlations with superconductivity (SC) plays an important role, have become an important direction of research \cite{25, 26}. In particular, chains of magnetic atoms on SC surface have proven to contain self-organized Majorana quasi-particles and exotic spin textures \cite{27–30}, while hybrid DQD structures have been used to split the Cooper pairs coherently into two entangled electrons propagating to separated normal leads \cite{31–35}. The latter is possible due to non-local (crossed) Andreev reflections (CARs), in which each electron of a Cooper pair tunnels into different QD, and subsequently to attached lead. As we show in this letter, such processes give rise to an exchange mechanism, that we henceforth call CAR exchange, which can greatly modify the low-temperature transport behavior of correlated hybrid nanostructures.

The proximity of SC induces pairing in QDs \cite{36, 37} and tends to suppress the Kondo effect if the superconducting energy gap $2\Delta$ becomes larger than the relevant Kondo temperature $T_K$ \cite{38–45}. Moreover, the strength of SC pairing can greatly affect the Kondo physics in the sub-gap transport regime: For QDs attached to SC and normal contacts, it can enhance the Kondo effect \cite{46–48}, while for DQD-based Cooper pair splitters, it tends to suppress both the SU(2) and SU(4) Kondo effects \cite{49}. Here, we show that the superconducting proximity effect, which gives rise to CAR exchange, can be the sole cause of the Kondo screening. We demonstrate and discuss such screening in a setup comprising T-shaped DQD with normal and superconducting contacts, see Fig. 1(a). We note that despite quite generic character of CAR exchange, and its presence in systems containing at least two QDs coupled close to each other to the same SC contact, to best of our knowledge CAR-induced screening has not been identified in previous studies \cite{31–35, 44, 49–53}. In the system proposed here [Fig. 1(a)], its presence is evident. Moreover, its magnitude can be directly related to the relevant energy scales, such as the Kondo temperature, which provides a fingerprint for quantitative experimental verification of our predictions.

Model.— The schematic of the considered system is depicted in Fig. 1(a). It contains two QDs attached to a common SC lead. Only one of them (QD1) is directly attached to the left (L) and right (R) normal leads, while the other dot (QD2) remains coupled only through QD1. The SC is modeled by the BCS Hamiltonian, $H_S = \sum_{k\sigma} \xi_k a_{k\sigma}^\dagger a_{k\sigma} - \Delta \sum_k (a_{k\uparrow}^\dagger a_{-k\downarrow}^\dagger + a_{-k\uparrow} a_{k\downarrow})$, with energy dispersion $\xi_k$, energy gap $2\Delta > 0$ and $a_{k\sigma}$ annihilation operator of electron possessing spin $\sigma$ and momentum $k$. The coupling between SC and QDs is described by the hopping Hamiltonian $H_{\text{TS}} = \sum_{k\sigma} \nu_{k\sigma} (d_{k\sigma}^\dagger a_{k\sigma} + \text{h.c.})$, with $d_{k\sigma}^\dagger$ creating a spin-$\sigma$ electron at QD1. The matrix element $\nu_{k\sigma}$ and the normalized density of states of SC in normal state, $\rho_S$, contribute to the coupling of QD1 to SC electrode as $\Gamma_{SI} = \pi \rho_S |\nu_{S1}|^2$. We focus on the sub-gap regime, therefore, we integrate out SC degrees of freedom lying outside the energy gap \cite{36}. This gives rise to the following effective Hamiltonian,
Figure 1. (a) Schematic of the considered system. Left/right (L/R) lead is coupled to the first quantum dot (QD1), while superconductor is attached to both QD1 and QD2. (b)-(d) illustrate an example of direct spin exchange: spin-up electron from the initial state (b) hops to the other QD (c) and spin-down electron hops back (d). Note, that the final state is in fact the same singlet state, only with opposite sign. (e)-(g) show an example of process contributing to crossed Andreev reflection (CAR) exchange. A Cooper pair from SC approaches DQD (e) and two singlets of the same charge are formed (f), before the Cooper pair is re-emitted (g). (h)-(j) present an example of RKKY process: an electron scattered off one QD (h) mediates the spin exchange towards the other (i), before it is finally scattered off there, too (j).

\[ H_{\text{eff}} = H_{\text{SDQD}} + H_L + H_R + H_T, \]

\[ H_{\text{SDQD}} = \sum_{i\sigma} \varepsilon_i n_i\sigma + \sum_i U n_i^1 n_i^\dagger + U'(n_1 - 1)(n_2 - 1) + \sum_{i\sigma} t(d_i^\dagger \sigma d_{i'}^\dagger \sigma + h.c.) + J\bar{S}_1 \bar{S}_2 + \sum_{i\sigma} \left[ \Gamma_{S_i}(d_i^\dagger \sigma + h.c.) + \Gamma_{S_X}(d_i^\dagger \sigma + h.c.) \right] \]

is the Hamiltonian of the SC-proximized DQD [49, 54], with QD1 energy level \( \varepsilon_i \), inter-site (intra-site) Coulomb interactions \( U' (U) \), inter-dot hopping \( t \), and CAR coupling \( \Gamma_{S_X} \). \( n_i^\dagger \sigma d_{i'}^\dagger \sigma \) denotes the electron number operator at QD1, \( n_i = n_i^\dagger + n_i^\dagger \), and \( i \equiv 3 - i \). Our model is strictly valid in the regime where \( \Delta \) is the largest energy scale. Nevertheless, all discussed phenomena are present in a full model for energies smaller than SC gap. The presence of out-gap states shall result mainly in additional broadening of DQD energy levels, changing the relevant Kondo temperatures. We note that the procedure of integrating out out-gap states neglects the SC-mediated RKKY interaction. To compensate for this, we explicitly include the Heisenberg term \( J\bar{S}_1 \bar{S}_2 \) in \( H_{\text{SDQD}} \), with \( \bar{S}_i \) denoting the spin operator of QD1 and a Heisenberg coupling \( J \) substituting the genuine RKKY exchange.

The normal leads are treated as a reservoirs of non-interacting electrons, \( H_r = \sum_{\mathbf{k}\sigma} \varepsilon_{\mathbf{k}\sigma} c_{\mathbf{r}\mathbf{k}\sigma}^\dagger c_{\mathbf{r}\mathbf{k}\sigma} \), where \( c_{\mathbf{r}\mathbf{k}\sigma} \) annihilates an electron of spin \( \sigma \) and momentum \( \mathbf{k} \) in lead \( r \) (\( r = L, R \)) with the corresponding energy \( \varepsilon_{\mathbf{r}\mathbf{k}\sigma} \). The tunneling Hamiltonian reads, \( H_T = \sum_{\mathbf{r}\mathbf{k}\sigma} V_r (d_{\mathbf{i}\sigma}^\dagger c_{\mathbf{r}\mathbf{k}\sigma} + h.c.) \), giving rise to coupling between lead \( r \) and QD1 of strength \( \Gamma_r = \pi \rho_r v_r^2 \), with \( \rho_r \) the normalized density of states of lead \( r \) and \( v_r \) the local hopping matrix element, assumed momentum-independent. We consider a wide-band limit, assuming constant \( \Gamma_r = \Gamma / 2 \) within the cutoff \( \pm D = \pm 2U \) around the Fermi level.

For thorough analysis of the CAR exchange mechanism and its consequences for transport, we determine the linear conductance between the two normal leads from

\[ G = \frac{2e^2}{h} \pi T \left[ -\frac{\partial f_T}{\partial \omega} \right] A(\omega) d\omega, \]

where \( f_T \) the Fermi function at temperature \( T \), while \( A(\omega) \) denotes the normalized local spectral density of QD1 [55]. Henceforth, we assume a maximal CAR coupling, \( \Gamma_{S_X} = \sqrt{\Gamma_{S_1} \Gamma_{S_2}} [49, 54], \) \( \Gamma_{S_1} = \Gamma_{S_2} = \Gamma_S \) and consider DQD tuned to the particle-hole symmetry point, \( \varepsilon_1 = \varepsilon_2 = -U/2. \) However, these assumptions are not crucial for the results presented here, as discussed in Supplemental Material [56].

Estimation of relevant energy scales.— Since we analyze a relatively complex system, let us build up the understanding of its behavior starting from the case of a QD between two normal-metallic leads, which can be obtained in our model by setting \( t = \Gamma_S = J = U' = 0. \) Then, the conductance as a function of temperature, \( G(T) \), grows below the Kondo temperature \( T_K \) and reaches maximum for \( T \to 0, \) \( G(T = 0) = G_{\text{max}}. \) At particle-hole symmetry point, the unitary transmission is achieved, \( G_{\text{max}} = G_0 = 2e^2/h; \) see short-dashed line in Fig. 2. An experimentally relevant definition of \( T_K \) is that at \( T = T_K \) \( G(T) = G_{\text{max}}/2. \) \( T_K \) is exponentially small in the local exchange \( J_0 = 8\Gamma/(\pi U) \), and is approximated by \( T_K \approx D \exp[-1/(\rho J_0)] \) [5].

The presence of a second side-coupled QD, \( t > 0 \), significantly enriches the physics of the system by introducing direct exchange between QDs, see Fig. 1(b-d). In general, effective inter-dot exchange can be defined as energy difference between the triplet and singlet states of isolated DQD, \( J_{\text{eff}} = E_S - E_{\text{sc}}. \) Unless \( U \) becomes very large, superexchange can be neglected [23] and \( J_{\text{eff}} \) is determined by direct exchange, \( J_{\text{eff}} \approx 4t^2/(U - U') > 0. \) When the hopping \( t \) is tuned small [31], one can expect \( J_{\text{eff}} \approx T_K \), which implies the two-stage Kondo screening [4, 6]. Then, for \( T < T_K \), the local spectral density of QD1 serves as a band of width \( \sim T_K \) for QD2. The spin of an electron occupying QD2 experiences the Kondo screening below the associated Kondo temperature

\[ T^* = a T_K \exp(-b T_K / J_{\text{eff}}) \]

with \( a \) and \( b \) constants of order of unity [4, 6]. This is reflected in conductance, which drops to 0 with lowering \( T \), maintaining characteristic Fermi-liquid \( G \sim T^2 \) dependence [6]; see the curves indicated with crosses and full squares in Fig. 2(a). Similarly to \( T_K \), experimentally relevant definition of \( T^* \) is that \( G(T = T^*) = G_{\text{max}}/2. \)
Even at the particle-hole symmetry point $G_{\text{max}} < G_0$, because the single-QD strong-coupling fixed point is unstable in the presence of QD2 and $G(T)$ does not achieve $G_0$ exactly, before it starts to decrease.

The proximity of SC gives rise to two further exchange mechanisms that determine the system’s behavior. First of all, the well-known RKKY interaction appears, $J \sim \Gamma_S^2$ [11–13]. Moreover, the CAR exchange emerges as a consequence of finite $\Gamma_S$. It can be understood on the basis of perturbation theory as follows. DQD in the inter-dot singlet state may absorb and re-emit a Cooper pair approaching from SC; see Fig. 1(e)-(g). As a second-order process, it reduces the energy of the singlet, which is a ground state of isolated DQD. A similar process is not possible in the triplet state due to spin conservation. Therefore, the singlet-triplet energy splitting $J_{\text{eff}}$ is increased (or generated for $t = J = 0$). More precisely, the leading (2nd-order in $t$ and $\Gamma_S$) terms in the total exchange are

$$J_{\text{eff}} \approx J + \frac{4t^2 J}{U - U' + \frac{3}{4} J} + \frac{4\Gamma_S^2}{U + U' + \frac{3}{4} J}.$$  \hspace{1cm} (4)

Using this estimation, one can predict $T^*$ for finite $\Gamma_S$, $t$ and $J$ with Eq. (3). Apparently, from three contributions corresponding to: (i) RKKY interaction, (ii) direct exchange and (iii) CAR exchange, only the first may bear a negative (ferromagnetic) sign. The two other always have an anti-ferromagnetic nature. More accurate expression for $J_{\text{eff}}$ is derived in Supplemental Material [56] [see Eq. (S-5)] by the Hamiltonian down-folding procedure. The relevant terms differ by factors important only for large $\Gamma_S/U$.

It is noteworthy that inter-dot Coulomb interactions decrease the energy of intermediate states contributing to direct exchange [Fig. 1(e)], while increasing the energy of intermediate states causing the CAR exchange [Fig. 1(f)]. This results in different dependence of corresponding terms in Eq. (4) on $U'$. As can be seen in Figs. 2(a) and 2(b), it has a significant effect on the actual values of $T^*$.

**CAR exchange and Kondo effect.**— To verify Eqs. (3)-(4) we calculate $G$ using accurate full density matrix numerical renormalization group (NRG) technique [57–60]. We compare $U' = 0$ case with experimentally relevant value $U' = U/10$ [61]. While for two close adatoms on SC surface RKKY interactions may lead to prominent consequences [28], these should vanish rapidly with the inter-impurity distance [62]. Therefore, we first neglect RKKY coupling and analyze its consequences afterwards.

The main results are presented in Fig. 2(a), showing the temperature dependence of $G$ for different circumstances. For reference, results for $\Gamma_S = 0$ are shown, exhibiting the two-stage Kondo effect caused by direct exchange mechanism. As can be seen in Fig. 2(b), an excellent agreement of $T^*$ found from NRG calculations and Eq. (3) is obtained with $a = 0.42$ and $b = 1.51$, the same for both $U' = 0$ and $U' = U/10$. Note, however, that $J_{\text{eff}}$ is different in these cases, cf. Eq. (4), and $U'$ leads to increase of $T^*$.

Furthermore, for $t = 0$ and $\Gamma_S > 0$ the two-stage Kondo effect caused solely by the CAR exchange is present; see Fig. 2(a). Experimentally, this situation corresponds to a distance between the two QDs smaller than the superconducting coherence length, but large enough for the exponentially suppressed direct hopping to be negligible. While intuitively one could expect pairing to compete with any kind of magnetic ordering, the Kondo screening induced by CAR exchange is a beautiful example of a superconductivity in fact leading to magnetic order, namely the formation of the Kondo singlet. This CAR-exchange-mediated Kondo screening is one of our main findings. For such screening Eq. (3) is still fulfilled with very similar parameters, $a = 0.37$ ($a = 0.35$) and $b = 1.51$ ($b = 1.50$) for $U' = 0$ ($U' = U/10$), correspondingly; see Fig. 2(b). Moreover, as follows from Eq. (4), $U'$ reduces CAR exchange, and therefore diminishes $T^*$. For the same values of $J_{\text{eff}},$ the dependence of $G(T)$ for $t = 0$ and $\Gamma_S > 0$ is hardly different from the one for $\Gamma_S = 0$ and $t > 0$ for $T > T^*$, compare the curves denoted by crosses and full circles in Fig. 2(a). However, $G(T)$ saturates at residual value $G_{\text{res}}$ as $T \to 0$ only for finite $\Gamma_S$, which at
particle-hole symmetry makes $G_{\min}$ the hallmark of SC proximity and following CAR exchange processes. From numerical results, one can estimate it as

$$G_{\min} = \frac{e^2}{h} \cdot \frac{\Gamma_S^2}{U^2}, \tag{5}$$

with $c \approx 2.25$, barely depending on $U'$ and getting smaller for $t > 0$. This is illustrated in Fig. 2(c), where the dotted line corresponds to Eq. (5) with $c = 2.25$.

Lastly, in Fig. 2(a) we also present the curves obtained for $t = \Gamma_S$ chosen such that for $U' = 0$ the total exchange $J_{\text{eff}}$ according to Eq. (4) remains the same as in the cases considered earlier. This is to illustrate what happens when both (direct and CAR) exchange interactions are present. Fig. 2(a) clearly shows that $T^*$ remains practically unaltered. On the contrary, $G_{\min}$ decreases for larger $t$ below the estimation given by Eq. (5), as can be seen in Fig. 2(c). Finally, $U'$ is of limited significance in this case, because the enhancement of direct exchange is compensated by decrease of CAR exchange.

While analyzing the results concerning $G_{\min}(\Gamma_S)$ plotted in Fig. 2(c) one needs to keep in mind that $G_{\min}$ is obtained at deeply cryogenic conditions. To illustrate this better, $G(\Gamma_S)$ obtained for $t = 0$ and $T = 10^{-12}U$ is plotted with solid line in Fig. 3. Clearly, for weak $\Gamma_S$ the system exhibits rather conventional (single-stage) Kondo effect with $G = G_{\text{max}} \approx 2e^2/h$, while QD2 is effectively decoupled ($G_{\text{max}} < 2e^2/h$ in the proximity of SC lead [47]). Only for larger values of $\Gamma_S$ the CAR exchange is strong enough, such that $T^* < T$ and the dependence $G(\Gamma_S)$ continuously approaches the $T = 0$ limit estimated by Eq. (5) and presented in Fig. 2(c).

**CAR-RKKY competition.**— Let us now discuss the effects introduced by RKKY interaction. We choose $t = 0$ for the sake of simplicity and analyze a wide range of $\Gamma_S$, starting from the case of anti-ferromagnetic RKKY interaction ($J > 0$). Large $J > 0$ leads to the formation of a molecular singlet in the nanostructure. This suppresses conductance, unless $\Gamma_S$ becomes of the order of $U/2$, when the excited states of DQD are all close to the ground state. This is illustrated by double-dotted line in Fig. 3. Smaller $J > 0$ causes less dramatic consequences, namely it just increases $J_{\text{eff}}$ according to Eq. (4), leading to enhancement of $T^*$, cf. Eq. (3). This is presented with dot-dashed line in Fig. 3.

The situation changes qualitatively for ferromagnetic RKKY coupling, $J < 0$. Then, RKKY exchange and CAR exchange have opposite signs and compete with each other. Depending on their magnitudes and temperature, one of the following scenarios may happen: For $J_{\text{eff}} > 0$, i.e. large enough $\Gamma_S$, and $T < T^*$, the system is in the singlet state due to the two-stage Kondo screening of DQD spins. $G(T = 0)$ is reduced to $G_{\min}$, which tends to increase for large negative $J$; see dashed lines in Fig. 3. In the inset to Fig. 3, the spectral density of QD1 representative for this regime is plotted as curve indicated by triangle. It corresponds to a point on the $J = -0.1U$ curve in the main plot, also indicated by triangle. The dip in $A(\omega)$ has width of order of $T^*$.

For finite $T$, there is always a range of sufficiently small $|J_{\text{eff}}|$, where QD2 becomes effectively decoupled, and, provided $T < T_K$, $G$ reaches $G_{\text{max}}$ due to conventional Kondo effect at QD1. This is the case for sufficiently small $\Gamma_S$ for $J = 0$ or $J = -0.01U$, and in the narrow range of $\Gamma_S$ around the point indicated by a circle in Fig. 3 for $J = -0.1U$ (for $J = 0.005U$, the considered $T$ is still below $T^*$). The conventional Kondo effect manifests itself with a characteristic peak in $A(\omega)$, as illustrated in the inset in Fig. 3 with line denoted by circle.

Finally, large enough $J_{\text{eff}} < 0$ and low $T$ give rise to an effective ferromagnetic coupling of DQDs spins into triplet state. Consequently, the underscreened Kondo effect occurs [3, 63] for weak $\Gamma_S$ and, e.g., $J = -0.1U$; see the point indicated by square in Fig. 3. This leads to $G = G_{\text{max}}$ and a peak in $A(\omega)$, whose shape is significantly different from the Kondo peak, cf. the curve denoted by square in the inset in Fig. 3.

**Conclusions.**— The CAR exchange mechanism is present in any system comprising at least two QDs or magnetic impurities coupled to the same superconducting contact in a way allowing for crossed Andreev reflections. In the proposed setup it leads to the two-stage Kondo screening even in the absence of other exchange mechanisms, characterized by residual low-temperature conductance at particle-hole symmetric case. We showed that the competition between CAR exchange and RKKY interaction may result in completely different Kondo screening scenarios. Our results bring further insight into the low-temperature behavior of hybrid coupled quantum dot systems, which hopefully could be verified with the present-day experimental techniques. Moreover, we believe they may also be relevant for heavy-fermion superconductors, where in principle non-local pairing giving rise to similar exchange mechanism is also possible.

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Supplemental Material for:

Nonlocal Andreev Reflections as Source of Spin Exchange and Kondo Screening

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CONTENTS

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In the following Supplemental Material we demonstrate that the perturbations expected in the experimental reality, yet neglected in the main article for the sake of simplicity, indeed do not change the results qualitatively. Within the Supplemental Material we add a prefix "S-") to references, e.g., Eq. (S-1) and Fig. S-1, whereas regular numbers, e.g., Eq. (1) and Fig. 1, refer to the main article.

In Sec. I we determine an expression for the effective inter-dot exchange interaction \( J_{\text{eff}} \) (defined in the main article) with the aid of Hamiltonian down-folding method. It is more precise than the perturbative result in Eq. (4), yet reduces to it when expanded to the second order in \( \Gamma_S/U \). In Sec. II we analyze consequences of the detuning of quantum dots energy levels from the exact particle-hole symmetry (PHS) point. This point is an important issue, since the exact PHS is hardly achievable in experimental setups. However, we show that in a reasonably wide range of detunings around the PHS point, the results of the main article are valid. In Sec. III we demonstrate that the asymmetry between the couplings of respective quantum dots to the superconductor, \( \Gamma_{S1} \neq \Gamma_{S2} \), does not lead to qualitative changes of the transport properties of the system. Finally, in Sec. IV we investigate the role of reduced efficiency of CAR processes, modeled within the Hamiltonian (1) by setting \( \Gamma_{SX} < \sqrt{\Gamma_{S1}\Gamma_{S2}} \), and show that it does not lead to qualitative changes.

I. MORE PRECISE ESTIMATION OF THE EFFECTIVE EXCHANGE

We first briefly describe the Hamiltonian down-folding method, then we apply it to the model considered in the main paper.

A. General formulation of down-folding method

Say we have the full Hilbert space of states divided into two sections, A and B. We think of A as being most important and of B as some addition, typically a set of high-energy states (in terms of the non-interacting part of the Hamiltonian). We can structure the secular equation for \( H \) as follows,

\[
\begin{bmatrix}
H_A & H_{AB}^\dagger \\
H_{AB} & H_B
\end{bmatrix}
\begin{bmatrix}
\psi_A \\
\psi_B
\end{bmatrix}
= E
\begin{bmatrix}
\psi_A \\
\psi_B
\end{bmatrix}.
\] (S-1)
Treating it as a set of linear equations one can calculate components ψ_B as functions of matrix elements of H, elements of ψ_A and the unknown eigenvalue E, ψ_B = (E − H_B)^{-1}H_{AB}ψ_A. Thus, we obtain the equation

\[ [H_A + H_{AB}^t(E - H_B)^{-1}H_{AB}] \psi_A = E \psi_A, \]  

(S-2)

where the term in the square bracket can be called the effective Hamiltonian of the states A, H_{eff}^{A}(E). Note, that as long as E on the left-hand-side is not approximated, this expression is exact, nonlinear equation for eigenvalues E of the original, full Hamiltonian and the projections of the full eigenstates (ψ_A ψ_B)^T onto the space A. In particular, if part A has a finite basis, one obtains the characteristic equation for H eigenvalues by subsequently eliminating one state after another. In practice, however, one is usually interested in eliminating states of high energies and determination of low-energy spectrum of the Hamiltonian, E ≪ H_B. Then, one can put E ≈ 0. In more general case, E ≈ ⟨H_A⟩ can be used, where ⟨H_A⟩ is some kind of estimation of the relevant energy. The procedure is valid, as long as the energy dependence of H_{eff}^{A}(E) does not influence its spectrum strongly, in particular for small interaction H_{AB}.

B. Application to the model under consideration

The spin singlet S = 0 subspace of the effective Hamiltonian for the superconductor-proximized double quantum dot H_{SDQD} can be written in the form

\[ H_{SDQD}^{S=0} = \begin{pmatrix}
-U - \frac{1}{2}J + \delta_1 + \delta_2 & t\sqrt{2} & t\sqrt{2} & \Gamma_{S\chi X} \sqrt{2} & -\Gamma_{S\chi X} \sqrt{2} \\
\frac{t\sqrt{2}}{2} & 2\delta_1 - U' & 0 & \Gamma_{S1} & \Gamma_{S2} \\
\frac{t\sqrt{2}}{2} & 0 & 2\delta_2 - U' & \Gamma_{S2} & \Gamma_{S1} \\
\Gamma_{S\chi X} \sqrt{2} & \Gamma_{S1} & \Gamma_{S2} & U' & 0 \\
-\Gamma_{S\chi X} \sqrt{2} & \Gamma_{S2} & \Gamma_{S1} & 0 & 2(\delta_1 + \delta_2 + U')
\end{pmatrix}, \]  

(S-3)

with δ_i = ε_i + U/2; see Eq. (1). The order of basis states is as follows: |1⟩ = 2^{-1/2}(|↑↓⟩ − |↓↑⟩), |2⟩ = |00⟩, |3⟩ = |02⟩, |4⟩ = |00⟩, |5⟩ = |22⟩, where |χ_1χ_2⟩ denotes such state, that QD is in a state |χ_1⟩, and the possible states are χ_1 = 0 (empty QD), χ_1 = 2 (doubly occupied QD), or χ_1 = σ (QD occupied by a single electron of spin σ; σ = ↑ or σ = ↓).

From Eq. (S-3) one can clearly see that in the regime of U dominating over other energy scales there is one state possessing energy of the order of −U, and other states have energies regular in the limit U → ∞. Therefore, the first approximation to E_{GS} is the energy of that state, E_{GS}^{0} = −U − \frac{1}{2}J + \delta_1 + \delta_2. The down-folding procedure shall be used to correct this estimation. This correction is crucial, since otherwise we get an obvious yet crude result J_{eff} = J.

Let us first examine Γ_{S1} = Γ_{S2} = Γ_{S\chi} = Γ_S = 0 case. Then, the charge is conserved, so the states |4⟩ and |5⟩ are decoupled. Taking |1⟩ to subspace A, while keeping |2⟩ and |3⟩ in subspace B one gets the estimation of E_{GS}, which for E substituted by E_{GS}^{0} leads to

\[ J_{eff}^{Γ_S=0} = J + 4\frac{t^2}{U - U'}\left[1 - \frac{(\delta_1 - \delta_2)^2}{(U - U')^2}\right]^{-1}, \]  

(S-4)

in agreement with Refs. [1, 2].

For finite Γ_{S1}, Γ_{S2} and Γ_{S\chi} the situation is more complicated, because there are four states to be eliminated. We assign them to subspace B and obtain cumbersome expressions, resulting from inverting explicitly non-diagonal 4 × 4 matrix. Therefore, we limit ourselves to the case of particle-hole symmetry, δ_1 = δ_2 = 0. Then, keeping only |1⟩ in subspace A and the remaining 4 states in B, and using E → E_{GS} substitution one gets feasible result,

\[ J_{eff} = J + \frac{4\Gamma_{S\chi X}^2}{U + U' + \frac{3}{4}J} \left[1 - \frac{4\pi^2 \Gamma_S^2}{(U + \frac{3}{4}J)^2 - U'^2}\right]^{-1} + \frac{4t^2}{U - U' + \frac{3}{4}J} \left[1 - \frac{4\Gamma_S^2}{(U + \frac{3}{4}J)^2 - U'^2}\right]^{-1}, \]  

(S-5)

where x = (Γ_{S1} − Γ_{S2})/(Γ_{S1} + Γ_{S2}) and Γ_S = (Γ_{S1} + Γ_{S2})/2. Expanding Eq. (S-5) in powers of Γ_{S\chi}, Γ_S and t, we obtain in the leading order

\[ J_{eff} ≈ J + \frac{4\Gamma_{S\chi X}^2}{U + U' + \frac{3}{4}J} + \frac{4t^2}{U - U' + \frac{3}{4}J}, \]  

(S-6)

identical to Eq. (4), yet extended to Γ_{S1} ≠ Γ_{S2} and Γ_{S\chi} < \sqrt{Γ_{S1}Γ_{S2}}; see Sec. IV for discussion of the importance of the latter.
II. EFFECTS OF DETUNING FROM THE PARTICLE-HOLE SYMMETRY POINT

At PHS $G_{\text{min}} = G(T=0) = 0$ in the absence of superconducting lead, making $G_{\text{min}} > 0$ a hallmark of SC-induced two-stage Kondo effect. However, outside of PHS point $G_{\text{min}} > 0$ even in the case of the two-stage Kondo effect caused by the direct exchange. Exact PHS conditions are hardly possible in real systems, and the fine-tuning of the QD energy levels to PHS point is limited to some finite accuracy. Therefore, there may appear a question, if the result obtained at PHS is of any importance for the realistic setups. As we show below — it is, in a reasonable range of detunings $\delta_i = \varepsilon_i + U/2$.

In Fig. S-1(a) we present the $G(T)$ dependence in and outside the PHS, corresponding to parameters of Fig. 2(a) in the main article, with only $U' = U/10$ taken into account. Clearly, for considered small values of $\delta_1 = \delta_2 = \delta$, $G_{\text{min}} < 10^{-3}e^2/h$ for direct exchange only, while $G_{\text{min}}$ in the presence of a superconductor is significantly increased and close to the PHS value. Furthermore, for $|\delta_1| \sim |\delta_2| \sim \delta$, the residual conductance caused by the lack of PHS $G_{\text{min}} \approx e^2/h \cdot (\delta/U)^2$, which is a rapidly decreasing function in the vicinity of PHS point, as illustrated in Fig. S-1(b) with solid lines. Evidently, in the regime $|\delta_1| < 0.01U$ the residual conductance caused by SC is orders of magnitude larger, leading to the plateau in $G_{\text{min}}(\delta_1)$ dependence, visible in Fig. S-1(b). Taking into account that the realistic values of $U$ in the semiconductor quantum dots are rather large, this condition seems to be realizable by fine-tuning of QD gate voltages.

Lastly, let us point out that while in the presence of only one exchange mechanism, CAR or direct, $G_{\text{min}}(\delta_1)$ dependencies depicted in Fig. S-1(b) are symmetrical with respect to sign change of $\delta_1$, for both exchange mechanisms the dependence is non-symmetric. This behavior shall be explained elsewhere.
FIG. S-2. (a) Linear conductance between the normal leads, $G$, as a function of temperature, $T$, for parameters corresponding to Fig. 2(a) with $\xi = U/20$ and $U' = U/10$, for different values of asymmetry coefficient $x$ [see Eq. (S-7)] in the presence of CAR exchange only, or both CAR and direct exchange. (b) The second-stage Kondo temperature $T^*$ normalized by $T_K$ as a function of $x$, calculated with the aid of NRG (points) and a fit to Eq. (3) (lines) with $J_0$ from Eq. (4). (c) The zero-temperature conductance $G_{\text{min}}$ as a function of QD1 coupling to SC lead, $\Gamma_{S1}$, compiled from data obtained at different circumstances (as indicated in the legend) for different $x$. Dotted line corresponds to Eq. (S-9) with $c = 2.25$.

III. EFFECTS OF ASYMMETRY OF COUPLINGS TO SUPERCONDUCTOR

Similarly to PHS, the ideal symmetry in the coupling between respective QDs and SC lead is hardly possible in experimental reality. As shown below, it does not introduce any qualitatively new features. On the other hand, it decreases the second Kondo temperature, which is already small, therefore, quantitative estimation of this decrease may be important for potential experimental approaches. To analyze the effects of $\Gamma_{S1} \neq \Gamma_{S2}$, we introduce the asymmetry parameter $x$ and extend the definition of $\Gamma_S$,

$$x = \frac{\Gamma_{S1} - \Gamma_{S2}}{\Gamma_S}, \quad \Gamma_S = \frac{\Gamma_{S1} + \Gamma_{S2}}{2}. \tag{S-7}$$

Note, that even for a fixed $\Gamma_S$, the actual CAR coupling $\Gamma_{SX} = \Gamma_S \sqrt{1 - x^2}$ decreases with increasing $|x|$, which is a main mechanism leading to a decrease of $T^*$ outside the $x = 0$ point visible in Fig. S-2(a) and (b). To illustrate this, the curves corresponding to both exchange mechanisms were calculated using $x$-dependent $t = \Gamma_{SX}$ instead of $t = \xi/\sqrt{2}$. Therefore, $\xi$ was generalized for $x \neq 0$ by setting $\xi = \sqrt{t^2(1-x^2)^{-1} + \Gamma_{S2}^2}$. Clearly, in Fig. S-2(b) the curves for different exchange mechanisms are very similar and differ mainly by a constant factor, resulting from different influence of $U'$; see discussion of Fig. 2 in the main article. The magnitude of $T^*$ changes is quite large, exceeding an order of magnitude for $x = \pm 0.5$ and $\xi = U/20$. Moreover, $T^* \to 0$ for $x \to \pm 1$. Consequently, for strongly asymmetric devices one cannot hope to observe the second stage of Kondo screening; see for example the curves for $x = \pm 0.8$ in Fig. S-2(a), where $T^*$ goes beyond the considered range of $T$ (all 4 curves lie on top of each other).

A careful observer can note that the $T^*(x)$ dependency is not symmetrical; note for example different $T^*$ for $x = \pm 0.5$ in Fig. S-2(a). This is caused by the dependence of the first Kondo temperature $T_K$ on $\Gamma_{S1}$ [3, 4],

$$T_K(\Gamma_{S1}) = T_K(\Gamma_{S1} = 0) \cdot \exp \left( \frac{\pi \Gamma_{S1}^2}{2 U} \right). \tag{S-8}$$

As $T_K$ grows for increasing $\Gamma_{S1}$ (or $x$), $T^*$ decreases according to Eq. (3). Note, that in the main article $T_K$ is defined...
in the absence of SC. Its $\Gamma_S$ dependence can be accounted for by small changes in the coefficients $a$ and $b$ in Eq. (3), as long as $x$ is kept constant.

To close the discussion of $T^*(x)$ dependence let us point out, that in Eq. (S-5) there appears a correction to Eq. (4) for $x \neq 0$. However, it is very small due to additional factor $\Gamma_S^2/U^2$ in the leading order. Its influence on curves plotted in Fig. S-2(b) is hardly visible.

In turn, let us examine the $x$ dependence of the $T = 0$ conductance $G_{\min}$. As can be seen in Fig. S-2(a), it monotonically increases with $x$, as it crosses $x = 0$ point. In fact, Eq. (5) can be generalized to

$$G_{\min} = \frac{e^2}{h} \cdot c \cdot \frac{\Gamma_{S1}^2}{U^2},$$

with $c \approx 2.25$ (indicated by a dotted line in Fig. S-2(c)). The values of $G_{\min}$ obtained from all analyzed $G(T)$ dependencies for different $x$ have been compiled in Fig. S-2(c). It is evident, that Eq. (S-9) is approximately fulfilled for all the considered cases.

Finally, it seems noteworthy that the normal-lead coupling asymmetry, $\Gamma_L \neq \Gamma_R$, is irrelevant for the results except for a constant factor diminishing the conductance $G$ [5].

**IV. THE ROLE OF CAR EFFICIENCY**

In the main article we assumed $\Gamma_{Sx} = \sqrt{\Gamma_{S1}\Gamma_{S2}}$, which is valid when the two quantum dots are much closer to each other then the coherence length in the superconductor. This does not have to be the case in the real setup, yet relaxing this assumption does not introduce qualitative changes.

To quantitatively analyze the consequences of less effective Andreev coupling we define the CAR efficiency as $C = \Gamma_{Sx}/\sqrt{\Gamma_{S1}\Gamma_{S2}}$ and analyze $C < 1$ in the wide range of $\Gamma_{S1} = \Gamma_{S2} = \Gamma_S$ and other parameters corresponding to Fig. 3. The results are presented in Fig. S-3.

Clearly, decreasing $C$ from $C = 1$ causes diminishing of $\Gamma_{Sx}$, and consequently of CAR exchange. For a change as small as $C = 0.9$, the consequences reduce to some shift of the conventional Kondo regime, compare Fig. S-3(a) with Fig. 3. Stronger suppression of CAR may, however, increase the SC coupling necessary to observe the second stage of Kondo screening caused by CAR outside the experimentally achievable range, see Fig. S-3(b). Moreover, the reduced $T^*$ leads to narrowing of the related local spectral density dip, while the increased critical $\Gamma_S$ necessary for

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**FIG. S-3.** Linear conductance between the normal leads $G$ as a function of coupling to SC lead, $\Gamma_S$, for indicated values of RKKY exchange $J$ and the efficiency of CAR processes reduced by factor (a) $C = 0.9$ and (b) $C = 0.5$. Other parameters as in Fig. 3. Inset: QD1 local spectral density $A(\omega)$ as a function of energy $\omega$ for points on $J = -0.1U$ curve, indicated with corresponding symbols.
observation of the second stage of screening leads to the shallowing of the dip. This is visible especially in the inset in Fig. S-3(b).

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