Truncated Wigner approximation as non-positive Kraus map

A B Klimov, I Sainz and J L Romero

Dept. de Física, Universidad de Guadalajara, 44420 Guadalajara, Mexico

E-mail: klimov.andrei@gmail.com

Received 15 September 2019, revised 23 March 2020
Accepted for publication 27 April 2020
Published 7 May 2020

Abstract
We show that the Truncated Wigner Approximation developed in the flat phase-space is mapped into a Lindblad-type evolution with an indefinite metric in the space of linear operators. As a result, the classically evolved Wigner function corresponds to a non-positive operator \( \hat{R}(t) \), which does not describe a physical state. The rate of appearance of negative eigenvalues of \( \hat{R}(t) \) can be efficiently estimated. The short-time dynamics of the Kerr and second harmonic generation Hamiltonians are discussed.

Keywords: Phase space, Wigner function, Kraus map, Truncated Wigner Approximation

(Some figures may appear in colour only in the online journal)

1. Introduction

The Liouvillian, or Truncated Wigner Approximation (TWA) is one of the most popular semiclassical approximations, which has been widely used in numerous applications (see [1] for a recent review). Developed in the framework of the phase-space approach [2, 3], the TWA allows us to employ classical intuition in order to describe the initial stage of the quantum evolution.

According to the general ideas of the phase-space mapping, every operator \( \hat{f} \) acting in the Hilbert space of a quantum system is in one-to-one correspondence with its symbol \( W_f(\Omega) \), defined in the classical phase-space \( \mathcal{M} \), \( \Omega \in \mathcal{M} \),

\[
\hat{f} \Leftrightarrow W_f(\Omega)
\]  

This procedure allows the computation of average values of any \( \hat{f} \) by convoluting \( W_f(\Omega) \) with the symbol of the density matrix \( W_\rho(\Omega) \) (the Wigner function). The von Neumann evolution equation for the density matrix is mapped into the Moyal equation [4],

\[
\partial_t W_\rho(\Omega) = i \{ W_\rho(\Omega), W_H(\Omega) \}_M,
\]  

where \( W_H \) is the symbol of the Hamiltonian and \( \{ , \}_M \) denote the Moyal brackets. Equation (2) in general contains higher order derivatives on the phase-space coordinates, which makes its solution a difficult task. Nevertheless, equation (2) admits an expansion in powers of a semiclassical parameter \( \epsilon \ll 1 \), and acquires the Liouvillian form,

\[
\partial_t W_\rho(\Omega) \approx \epsilon \{ W_\rho(\Omega), W_H(\Omega) \}_P,
\]  

leading to order in \( \epsilon \), where \( \{ , \}_P \) is the Poisson bracket on \( \mathcal{M} \). The solution of equation (3), that approximates the exact Wigner function \( W_\rho(\Omega|\rho) \) as a ‘classically evolve’,

\[
W_\rho(\Omega|\rho) \approx W_\rho(\Omega|\rho(t)),
\]  

where \( \Omega^cl(t) \) denotes classical trajectories, is known as the Truncated Wigner Approximation (TWA). This approximation describes the propagation of every point of the initial distribution along the corresponding classical trajectory. The evolution of average values are computed by integrating symbols of operators with \( W_\rho(\Omega|\rho(t)) \),

\[
\langle \hat{f}(t) \rangle \approx \int d\Omega W_f(\Omega) W_\rho(\Omega|\rho(t)).
\]  

Initially applied to the analysis of semiclassical dynamics of quantum systems in the flat \( p - q \) phase space, [5, 6] (see also [1, 7] and references therein), TWA was extended to quantum systems with SU(2) [8, 9] and SU(3) symmetries [9, 10]. TWA leads to exact results only for harmonic quantum dynamics. In the flat \( p - q \) phase-space a harmonic evolution is governed by at most quadratic (in the phase-space variables) Hamiltonians, and leads to a symplectic deformation of the initial distribution (interestingly, a similar behavior is also observed for some dissipative scenarios [11]). For quantum systems with a semi-simple...
dynamic symmetry group the harmonic evolution is generated by Hamiltonians linear in the group generators. In this case the initial distribution is rigidly displaced (i.e. without distortion) in the corresponding phase-space as a consequence of the covariance of phase-space distributions under group transformations.

Although formally speaking, TWA fails from the very beginning in case of non-linear evolution [12] (since the classical dynamics preserves the phase-space area), it describes relatively well a short-time non-linear dynamics of smooth and localized distributions (representing the so-called semiclassical states). In the simplest case of quantum systems with the Heisenberg-Weyl symmetry, it was noted [13] by using the Hudson theorem that the operator $\hat{R}(t)$ corresponding to the inverse map of $W_p(\Omega^2(-t))$, in general, does not describe a physical state.

In the present paper we will analyze the evolution equation for $\hat{R}(t)$ and show that it has an indefinite Lindblad form, i.e. contains positive and negative Lindblad-like terms. In other words, an anharmonic classical dynamics, if viewed from the quantum point of view, corresponds to a very specific non-unitary quantum evolution which results in a non-positivity of $\hat{R}(t)$. We discuss the algebraic structure of this equation and the physical implications of its non-positivity. We show that negative eigenvalues of the operator $\hat{R}(t)$ appear already at $t = 0^+$ with the rate that depends both on the initial state and the degree of non-linearity of the Hamiltonian.

Here we will focus only on the evolution in the flat phase-space, although a similar approach can be applied to quantum systems with higher symmetries.

2. Operator form of the Liouvillean evolution equation

2.1. General considerations

In case of the Heisenberg-Weyl symmetry the map from the Hilbert space to the flat phase-space is defined by the kernel

$$\hat{\omega}(\alpha) = \hat{D}(\alpha)(-1)^{\hat{a}^\dagger \hat{a}} \hat{D}(\alpha),$$

$$\text{Tr} \hat{\omega}(\alpha) = 1, \int d^2\alpha \hat{\omega}(\alpha) = \delta^2(\alpha),$$

(5)
in such a way that\(^1\)

$$W_f(\alpha) = \text{Tr}(\hat{f} \hat{\omega}(\alpha)),\quad \text{and the inverse transformation has the form}$$

$$\hat{f} = \int d^2\alpha \hat{\omega}(\alpha) W_f(\alpha),$$

(7)

where $\hat{D}(\alpha) = \exp(\alpha \hat{a}^\dagger - \alpha^* \hat{a})$, is the displacement operator and $d^2\alpha = d\alpha d\alpha^*$. The Wigner function automatically satisfies the normalization condition,

$$\int d^2\alpha \ W_p(\alpha) = 1,$$

(8)
corresponding to $\text{Tr} \hat{\rho} = 1$. The purity of the state, $P = \text{Tr} \hat{\rho}^2$, is expressed in terms of the Wigner function as

$$P = \int d^2\alpha \ W_p^2(\alpha).$$

(9)

The TWA evolution equation (3) takes the form

$$i\partial_t W_p = \partial_\alpha W_p \partial_\alpha^* W_p - \partial_\alpha^* W_p \partial_\alpha W_p = i [W_H, W_p],$$

(10)

leading to the Liouvillian evolution of the Wigner function

$$W_p(\alpha) \to W_p(\alpha|t) \approx W_p(\alpha(-t)),$$

(11)

where the canonical transformation $\alpha \to \alpha(t)$, generated by $W_H(\alpha)$, is described by the classical Hamilton equations

$$\dot{\alpha}(t) = -i[\alpha, W_H(\alpha)].$$

Equation (11) contradicts Hudson’s theorem [14] in case of anharmonic dynamics (when $W_H(\alpha)$ contains greater than quadratic powers of $\alpha$ and $\alpha^*$). This is clearly seen in the example of evolution of the Wigner function of the initial coherent state, $|\alpha_0\rangle$,

$$W_p(\alpha) = 2 \exp(-|\alpha - \alpha_0|^2),$$

(12)

which evolves according to (11) into a positive, but non-Gaussian function, thus not corresponding to any physical quantum state [13].

The inverse map of the evolved Wigner function $W_p(\alpha(-t))$, leads to the following time-dependent Hermitian operator

$$\hat{R}(t) = \int d^2\alpha \hat{\omega}(\alpha) W_p(\alpha(-t)),$$

$$\hat{R}(0) = \hat{\rho}.$$\hspace{1cm}(13)
The average values calculation according to equation (4) is equivalent to tracing $\hat{R}(t)$ with the corresponding operator

$$\langle \hat{f}(t) \rangle \approx \text{Tr}(\hat{R}(t)\hat{f}).$$

Since the classical evolution is reduced to the canonical transformation of the phase-space coordinates the function $W_p(\alpha(-t))$ satisfies the normalization condition (8) and the purity (9) ‘conservation’

$$P(t) = \int d^2\alpha \ W_p^2(\alpha(-t)) = \text{Tr} \hat{R}^2(t) = P(0).$$

(14)

Thus, for the initial pure state, the operator (13) fulfills the conditions

$$\text{Tr} \hat{R}(t) = 1,$$

$$\text{Tr} \hat{R}^2(t) = 1,$$

(15)

(16)

which, however, does not mean that $\hat{R}^2(t)$ is equal to $\hat{R}(t)$, except in the case when the Hamiltonian is a quadratic function of $\alpha$ and $\alpha^*$, as shown in appendix A. Moreover, the trace-class operator $\hat{R}(t)$ does not describe a physical state (except for the harmonic evolution), which is reflected in
appearance of negative eigenvalues for $t = 0^+$ as shown below.

2.2. Evolution equation for $^\hat{R}(t)$

It is instructive to analyze the non-positivity of $^\hat{R}(t)$ operator on the level of the evolution equation in the Hilbert space. Let us consider a symmetrized $n + m$ degree Hamiltonian

$$\check{H}_{nm} = [a^\dagger n a^n + a^n a^\dagger n]_{\text{sym}}, \quad n \geq m,$$  \hspace{1cm} (17)

where $[\ldots]_{\text{sym}}$ means the full normalized symmetrization of the monomial $a^\dagger n a^n$, see appendix C. The (symmetrized) monomial Hamiltonian describes a variety of physical processes [5]. In addition, an arbitrary Hamiltonian on $a$ and $a^\dagger$ can be represented as a series on $\check{H}_{nm}$.

The symbol of (17) is

$$W_H(\alpha, a^\dagger a) = a^\alpha a^n + a^n a^\alpha a.$$

and the Liouville equation (10) takes the form

$$i\partial_t W_H = a^{\alpha m - 1}a^\alpha n W_H - \alpha a^n a^\alpha W_H + \alpha a^n a^\alpha W_H,$$  \hspace{1cm} (18)

It can be shown (see appendix B) that for the generic Hamiltonian (17) the equation for $^\hat{R}(t)$ defined in (13) can be reduced to the Lindblad-type [15] form

$$\partial_t ^\hat{R} = i[ ^\hat{R}, \check{H}_{eff}] + L(^\hat{R}),$$  \hspace{1cm} (19)

where the effective Hamiltonian is

$$\check{H}_{eff} = \frac{n + m}{2n + m}(a^\dagger n a^n + a^n a^\dagger n + a^n a^\dagger n + a^\dagger n a^n),$$  \hspace{1cm} (20)

and the $L$ has the following structure

$$L = G_{nm}^{(0)} + G_{mn}^{(1)} + G_{mn}^{(2)} + G_{mn}^{(3)},$$  \hspace{1cm} (21)

where

$$G_{nm}^{(0)} = \sum_{j=1}^{n-m-1} C_j^{nm} \left[ L_{0j}^{nm} - L_{j0}^{nm} - \check{L}_{nm}^{nm} \right],$$  \hspace{1cm} (22)

$$G_{mn}^{(1)} = \frac{n}{2n+m+1} \sum_{j=0}^{\lfloor n/2 \rfloor} C_j^{mn} \left[ L_{mj}^{m} + L_{j0}^{mn} - L_{nj}^{mn} \right],$$  \hspace{1cm} (23)

$$G_{mn}^{(2)} = \frac{1}{2n+m+1} \sum_{k=1}^{m} \sum_{j=0}^{\lfloor n/2 \rfloor} C_j^{mn} C_{n-k}^{mn} (n - 2k) \times \left[ L_{m-j}^{mn} + L_{m-j-n-k}^{mn} - L_{mj}^{mn} \right],$$  \hspace{1cm} (24)

where operators $L_{jk}^{pq}$ and $\check{L}_{jk}^{pq}$ defining the Lindblad superoperators

$$L = 2L \otimes L^\dagger - L^\dagger L \otimes i - i \otimes L^\dagger L,$$  \hspace{1cm} (25)

$$\check{L} = 2L \otimes L^\dagger - L^\dagger L \otimes i - i \otimes L^\dagger L,$$  \hspace{1cm} (26)

are

$$L_{jk}^{mn} = a^\dagger i a^k - i a^p a^q - i a^q a^p - i a^k a^p, \quad \check{L}_{jk}^{mn} = (L_{jk}^{mn})^*,$$

$p, q = m, n; \ j = 0 \ldots p; \ k = 0, \ldots, q$,

and $C_m$ are the binomial coefficients.

The following observations about the structure of equation (19)–(24) can be made:

1. $L(^\hat{R}) = 0$ and $\check{H}_{eff} = \check{H}_{nm}$ in case of harmonic evolution, i.e., when the Hamiltonian is a quadratic form on quadratic function of $a$ and $a^\dagger$.

2. The evolution of $^\hat{R}(t)$ is not unitary for $n + m > 2$, in the sense that $^\hat{R}(t) \neq ^\hat{R}(t)^*$ for $t > 0$ (see appendix B). This, nevertheless, does not mean that a pure state decoheres into a mixed state as it evolves according to equation (19). It is discussed below that this specific non-unitarity along with the condition (14) leads to the appearance of negative eigenvalues of the operator $^\hat{R}(t)$.

3. The Hamiltonian ‘sector’ of the evolution decreases for higher non-linearities. In addition, $\check{H}_{eff} \sim \check{H}_{nm}$ only for $m = 1$ and $\check{H}_{eff} \sim \check{H}_{22} + \check{H}_{22}^*$ in the particular case $n = m = 2$ (see appendix C).

4. The number of positive and negative coefficients of the Lindblad operators, $\check{L}_{jk}^{mn, \check{L}_{jk}^{mn}, \check{L}_{jk}^{mn}, \check{L}_{jk}^{mn}, \check{L}_{jk}^{mn}, \check{L}_{jk}^{mn}, \check{L}_{jk}^{mn}, \check{L}_{jk}^{mn}, \check{L}_{jk}^{mn}, \check{L}_{jk}^{mn}, \check{L}_{jk}^{mn}, \check{L}_{jk}^{mn}$, is the same. The evolution of $^\hat{R}(t)$ is induced by the Kraus map [16] $\hat{\rho} \rightarrow S(\hat{\rho})$ of the form

$$S = \check{K}_0 \otimes \check{K}_0^\dagger + \sum_j (\check{K}_j \otimes \check{K}_j^\dagger - \check{K}_j \otimes \check{K}_j^\dagger),$$  \hspace{1cm} (27)

$$\check{K}_0 \check{K}_0^\dagger + \sum_j (\check{K}_j \check{K}_j^\dagger - \check{K}_j \check{K}_j^\dagger) = I.$$  \hspace{1cm} (28)

Thus, the map generated by the classical dynamics is not completely positive [17]. It will be shown below that such a map is actually non-positive.

5. In general, the non-Hamiltonian part of the evolution equation (19) has the structure:

$$L(^\hat{R}) = i(F(^\hat{R}) - F^\dagger(^\hat{R})),$$  \hspace{1cm} (29)

where

$$F = \sum_j \hat{A}_j \otimes \hat{B}_j,$$

is a (non-Lindbladian) map, which leads to the conservation conditions (15), (16) and, as a consequence, to the following overlap relation

$$Tr(L(^\hat{R}(t))R(t)) = 0.$$  \hspace{1cm} (30)

6. For an initial pure state it follows from (15)–(16) that

$$\sum_k \lambda_k(t) = 1, \quad \sum_k \lambda_k^2(t) = 1,$$

and thus

$$\sum_k \lambda_k(t)(1 - \lambda_k(t)) = 0.$$  \hspace{1cm} (31)
where \( \{ \lambda_k(t), k = 1, 2, \ldots \} \) are eigenvalues of the \( \hat{R}(t) \) operator \cite{18}
\[
\hat{R}(t) = \sum_k \lambda_k(t) \langle k(t) \rangle \langle k(t) \rangle, \quad \hat{R}(0) = |\psi_0\rangle \langle \psi_0|.
\] (31)

According to equation (A5) the rank of the operator \( \hat{R}(t) \) is not preserved by the equation of motion (19). Thus, the initial (first rank) density matrix evolves into the form (31) in the anharmonic case, since \( |\lambda_k(t) > 0| < 1 \), unless the initial states is an eigenstate of the Hamiltonian (17),
\[
[\hat{R}(0), \hat{H}] = 0.
\] (32)

It immediately follows from the relation (30) that at least one negative eigenvalue of \( \hat{R}(t) \) appears at \( t = 0^+ \) for an initial pure state if the condition (32) is not fulfilled. The rates of appearance of negative eigenvalues depend both on the Hamiltonian and the initial state.

The upper bound of negative eigenvalues can be estimated by using the min-max theorem in the subspace orthogonal to the initial state \( |\psi_0\rangle \), i.e. finding a state \( |\phi\rangle \), \( \langle \psi_0|\phi\rangle = 0 \), such that
\[
\lambda_{\min} \leq \min \langle \phi | \hat{R}(t) | \phi \rangle < 0.
\] (33)

For short times, when
\[
\hat{R}(t) \approx \hat{R}_0 + it[\hat{R}_0, \hat{H}_{\text{eff}}] + t \mathcal{L} \hat{R}_0,
\] (34)
the condition equation (33) is reduced to
\[
\lambda_{\min} \leq t \min \langle \phi | \mathcal{L} \hat{R}_0 | \phi \rangle < 0.
\] (35)

It is worth noting that the evolution equation in the form (19)–(24) greatly simplifies the estimation of the self-correlation function
\[
G(t) = \langle \psi(0)|\hat{R}(t)|\psi(0)\rangle,
\] (36)
and the fidelity
\[
\mathcal{F}(t) = \langle \psi(t)|\hat{R}(t)|\psi(t)\rangle,
\] (37)
where \( |\psi(t)\rangle \) is the exact state vector. The deviation of \( \mathcal{F}(t) \) from the identity, in particular, in the beginning of evolution, quantifies the quality of TWA.

3. Examples

In this section we consider two representative examples of anharmonic evolution.

3.1. Kerr evolution

Kerr dynamics is generated by the following (symmetrized) fourth-order Hamiltonian
\[
\hat{H}_{\text{Kerr}} = \{2\hat{a}^{\dagger 2}\hat{a}^2\}_{\text{sym}}.
\]

The effective Hamiltonian (20) is
\[
\hat{H}_{\text{eff}} = \frac{1}{2}(\hat{a}^{\dagger 2}\hat{a}^{2} + \hat{a}^{2}\hat{a}^{\dagger 2}) = \frac{1}{2}\hat{H}_K + \frac{1}{4}\mathcal{H},
\] (38)
and the non-unitary part of the evolution equation (21)–(24) takes the form
\[
\mathcal{L}_{\text{Kerr}} = \frac{1}{4}(\hat{L}_{10}^{22} + \hat{L}_{12}^{22} - \hat{L}_{12}^{12} - \hat{L}_{10}^{22}),
\] (39)
where the operators defining (25)–(26) are
\[
L_{10}^{22} = \hat{a}^{\dagger} - i\hat{a}\hat{a}^{\dagger 2}, \quad L_{12}^{22} = \hat{a}^{\dagger} + i\hat{a}\hat{a}^{\dagger 2},
\]
\[
L_{12}^{12} = i(\hat{a} - \hat{a}^{\dagger}\hat{a}^{\dagger 2}), \quad L_{10}^{12} = -i(\hat{a} + \hat{a}^{\dagger}\hat{a}^{\dagger 2}).
\]

It is instructive to represent (39) in form (28) where
\[
F(\hat{R}) = \frac{1}{2}(\hat{a}\hat{R}\hat{a}^{\dagger 2} + \hat{a}^{\dagger}\hat{R}\hat{a}^{\dagger 2})
\] (40)
The Kraus map \( \hat{R}(t) \rightarrow \hat{R}(t + i\delta t) \) has the form (27) with
\[
\tilde{K}_{+1} = \frac{1}{4}\sqrt{\delta t} L_{10}^{22}, \quad \tilde{K}_{+2} = \frac{1}{4}\sqrt{\delta t} L_{12}^{22},
\]
\[
\tilde{K}_{-1} = \frac{1}{4}\sqrt{\delta t} L_{10}^{12}, \quad \tilde{K}_{-2} = \frac{1}{4}\sqrt{\delta t} L_{12}^{12},
\]
\[
\hat{K}_0 = \hat{I} - i\delta t\hat{H}_{\text{eff}}
\]
where \( H_{\text{eff}} \) is defined in (38).

In what follows we analyze evolution of negative eigenvalues of \( \hat{R}(t) \).

(a) Fock states \( |n\rangle \) do not evolve under action of Kerr Hamiltonian, \( [|n\rangle \langle n|, H_{\text{Kerr}}] = 0 \). It is straightforward to see that \( [|n\rangle \langle n|, H_{\text{eff}}] = 0 \) and \( \mathcal{L}_{\text{Kerr}}(|n\rangle \langle n|) = 0 \).

(b) For the initial state
\[
|\psi_0\rangle = \frac{|0\rangle + \alpha|1\rangle}{\sqrt{1 + |\alpha|^2}}, \quad |\alpha| \ll 1,
\] (41)
which approximately describes a low excited coherent state, the short time solution (34) has the form
\[
\hat{R}(t) \approx \frac{1}{1 + |\alpha|^2}(|0\rangle \langle 0| + |\alpha|^2|1\rangle \langle 1|)
\]
\[
+ \alpha(1 - 2i\delta t)|1\rangle \langle 0| + \alpha^2(1 + 2i\delta t)|0\rangle \langle 1|
\]
\[
- \frac{it}{\sqrt{2}}(\alpha|2\rangle \langle 1| - \alpha^2|1\rangle \langle 2|) + O(\delta t^2),
\] (42)
where the second line describes the action of the Lindblad-like operator \( \mathcal{L}_{\text{Kerr}} \), satisfying equation (29). It is easy to see that for short times there are three non-zero eigenvalues: \( \lambda_1 = 1 - O(\delta t^2) \),
\[
\lambda_+ = \pm \frac{|\alpha|^2}{\sqrt{2}(1 + |\alpha|^2)^{3/2}} + O(\delta t^2),
\] (43)
and the negative one appears with the rate \( \lambda_- \sim |\alpha|/\sqrt{2} \). Interestingly, the same result can be obtained by optimizing the min-max solution in the orthogonal to the state (41) subspace. For instance,
considering a sample state
\[ |\phi\rangle = \frac{\alpha^0|0\rangle + |1\rangle + \beta|2\rangle}{\sqrt{1 + |\alpha|^2 + |\beta|^2}}, \quad \langle \phi | \psi_0 \rangle = 0, \]
we immediately obtain that \( \min_{\beta} \langle \phi | \vec{R}(t) | \phi \rangle = \lambda_\min \).

(c) For the initial coherent state \( |\alpha\rangle \) we make use of the minmax theorem (35) and for the sample state
\[ |\phi\rangle = \frac{|\beta\rangle - \langle \alpha | \beta \rangle |\alpha\rangle}{\sqrt{1 + |\langle \beta | \alpha \rangle|^2}}, \quad \langle \phi | \psi_0 \rangle = 0, \quad (44) \]
one obtains the following upper bound for the negative eigenvalue of \( \vec{R}(t) \) for \( |\alpha| \gg 1 \)
\[ \lambda_\min \leq \min_{\beta} \langle \phi | \vec{R}(t) | \phi \rangle \approx \frac{|\alpha|^2 e^{-|\alpha|^2 t}}{\sqrt{2|\alpha|^2 t}}. \]
The fidelity (37) also is deviated very slowly from the unity for the initial coherent state \( |\alpha\rangle \) in the scale of classical period of oscillations \( T_{cl} \sim |\alpha|^2 \)
\[ F(t) \sim 1 - \frac{3}{2}|\alpha|^2 t^2 = 1 - \frac{3}{2} |\alpha|^2 \left( \frac{t}{T_{cl}} \right)^2, \quad (45) \]
while the self-correlation function (36) exhibits a fast change of the initial state at short times,
\[ G(t) \sim 1 - 4 |\alpha|^2 t^2 = 1 - 4 |\alpha|^2 \left( \frac{t}{T_{cl}} \right)^2. \]

3.2. Second harmonic generation

The Hamiltonian describing the effect of second harmonic generation (and down conversion) is of third order
\[ \tilde{H}_{SG} = [\hat{a}^2 \hat{a} + \hat{a} \hat{a}^2]_{\text{sym}}. \]
The effective Hamiltonian and the set of Lindblad operators are
\[ \tilde{H}_{\text{eff}} = \frac{3}{8} (\hat{a}^3 \hat{a} + \hat{a}^2 \hat{a}^2 + \hat{a} \hat{a}^2 \hat{a} + \hat{a}^2 \hat{a} \hat{a}) = \frac{3}{4} \tilde{H}_{SG}, \quad (46) \]
\[ \mathcal{L} = \frac{1}{16} (\tilde{L}_0^{21} + \tilde{L}_0^{12} - \tilde{L}_0^{21} - \tilde{L}_0^{21}) + \frac{1}{8} (\tilde{L}_1^{21} + \tilde{L}_1^{21} - \tilde{L}_1^{21} - \tilde{L}_1^{21}), \quad (47) \]
where
\[ \tilde{L}_0^{21} = \hat{a} - i \hat{a}^2, \quad \tilde{L}_0^{12} = \hat{a} + i \hat{a}^2, \]
\[ \tilde{L}_0^{12} = i (\hat{a}^+ - \hat{a}^+ \hat{a}^2), \quad \tilde{L}_0^{21} = -i (\hat{a}^+ + \hat{a}^+ \hat{a}^2), \]
\[ \tilde{L}_1^{21} = \hat{a}^3 - i \hat{a}^3 \hat{a}, \quad \tilde{L}_1^{21} = \hat{a}^3 + i \hat{a}^3 \hat{a}, \]
\[ \tilde{L}_1^{21} = i (\hat{a}^+ - i \hat{a}^+ \hat{a}), \quad \tilde{L}_1^{21} = -i (\hat{a}^+ + i \hat{a}^+ \hat{a}). \]
In the representation (28) the operator (47) has the form
\[ F(\vec{R}) = \frac{1}{8} (2 \hat{a}^2 \hat{R} \hat{a}^2 + 2 \hat{a} \hat{R} \hat{a} \hat{a} \hat{a}^+ + \hat{a}^2 \hat{R} + \hat{R} \hat{a}^2 \hat{a} + \hat{a} \hat{R} \hat{a} \hat{R}^+ + \hat{R} \hat{a} \hat{a} \hat{R}^+ + \hat{R} \hat{a} \hat{a} \hat{R}). \]
The construction of the Kraus operators is similar to the previous example.
It is straightforward to find that for the vacuum initial state \( |\psi_0\rangle = |0\rangle \),
the short-time expansion for \( \vec{R}(t) \) has the form
\[ \vec{R}(t) \approx |0\rangle \langle 0| + \frac{it}{2}(|1\rangle \langle 1| - |1\rangle \langle 0|) + \frac{\sqrt{2} t}{4} (|1\rangle \langle 2| - |2\rangle \langle 1|) + O(t^2), \]
leading to the following negative eigenvalue
\[ \lambda_- = -\frac{t}{2 \sqrt{2}} + O(t^2). \quad (48) \]
For the coherent state \( |\alpha\rangle \) we proceed as in the previous example of Kerr Hamiltonian, minimizing the average value of (34) over the sample states (44) obtaining for \( |\alpha| \gg 1 \)
\[ \lambda_- \leq -\frac{|\alpha|^2 e^{-|\alpha|^2 t}}{2 \sqrt{2} |\alpha|^2 t}. \]
The fidelity (37) behaves very differently for the initial coherent and number states. In particular, for a number state \( |N\rangle \) one obtains
\[ 1 - F(t) \sim \frac{1}{8} (10 N^3 + 6 N^2 + 10 N + 3) t^2, \]
while for coherent states \( |\alpha\rangle \),
\[ 1 - F(t) \sim \frac{3}{8} t^2. \]
This confirms the intuition that TWA works significantly better for coherent states than for number states with the same average energy.

In conclusion we have shown that the Truncated Wigner Approximation (10) developed in phase-space corresponds to a Lindblad-type (non-unitary) evolution with indefinite metric in the space of linear operators, except for the case of linear evolution (governed by quadratic Hamiltonians). As a result, the inverse image of the classically evolved Wigner function is a non-positive operator \( \vec{R}(t) \) satisfying the relations (15)–(16). In other words, the classical dynamics generates a Kraus-like map containing both positive and negative terms that transform an initial density matrix into an operator not corresponding to a physical state. The positivity of the operator corresponding to the classically evolved Wigner function is broken for \( t = 0^+ \) in case of anharmonic evolution. Observe, that the unitarity is lost by the approximation
\[ i [\vec{w}(\alpha), \vec{H}] \to [\vec{w}(\alpha), W_{\vec{H}}(\alpha)]_p, \]
which should be considered as a weak asymptotic limit on the states with \( \vec{n} = T \vec{r} (\hat{a}^\dagger \hat{a}) \to \infty \) (which basically corresponds to considering only a zero-order term of singularly perturbed phase-space evolution equation).
It is interesting to contrast the quantum and TWA dynamics of an initial coherent state, described in the phase-space by the Wigner function equation (12):

(a) the quantum and TWA harmonic evolutions preserve the positivity of the Wigner function and positive definiteness of its inverse matrix of the evolved state;
(b) during an anharmonic TWA evolution the positivity of the Wigner function is maintained, but the operator counterpart of $W_p(\alpha(-t))$ is not a positively defined operator; the quantum anharmonic evolution in phase-space leads to a non-positive Wigner function, the negativity of which can be used for the detection of non-classicality [19].

**Acknowledgments**

This work is partially supported by the Grant 254127 of CONACyT (Mexico).

**Appendix A**

Here we show that the Liouvillian evolution in phase-space does not correspond to quantum unitary dynamics.

The classical phase-space evolution of the Wigner function can be symbolically described as

$$W_p(t) = e^{\i t \{H, \cdot\}} W_p(0),$$

where $W_p(\alpha, \alpha^*)$ is the symbol of the Hamiltonian and the flat space Poisson brackets has the form

$$\{\cdot\} = i(\partial_{\alpha} \otimes \partial_{\alpha^*} - \partial_{\alpha^*} \otimes \partial_{\alpha}),$$

the symbol $\otimes$ indicates the order of application of the derivatives. The TWA evolution of the square of the density operator can be determined in terms of the star-product as follows

$$W^*(t) = e^{-\i t \{H, \cdot\}} (W_p(t) W_p(t)) = W_p(t)^* W_p(t),$$

where $\cdot$ denotes the star-product operator.

Since $\{W_p$, $\cdot\}$ is a first-order differential operator we can write

$$e^{\i t \{W_p, \cdot\}} (W_p(0)^* W_p(0)) = \hat{S}_t(W_p(\alpha(-t)) W_p(\alpha(-t))),$$

(A1)

where

$$W_p(\alpha(-t)) = e^{\i t \{W_p, \cdot\}} W_p(0),$$

is the classically evolved Wigner function (11) and

$$\hat{S}_t = \exp \left\{ \frac{1}{2} e^{\i t \{W_p, \cdot\}} (\partial_{\alpha} \otimes \partial_{\alpha^*} - \partial_{\alpha^*} \otimes \partial_{\alpha}) e^{-\i t \{W_p, \cdot\}} \right\}.$$  

(A2)

is the transformed star-product operator. For quadratic in $\alpha$ and $\alpha^*$ symbols $W_p(\alpha, \alpha^*)$ the operator (A2) is invariant under transformation generated by $W_p$. This leads to the well known result,

$$W_p(\alpha(-t))^* W_p(\alpha(-t)) = W_p(\alpha(-t)),$$

(A3)

i.e. an initial pure state evolves into a pure state under action of quadratic Hamiltonians.

In general case, by taking into account that

$$e^{\i t \{H, \cdot\}} \partial_{\alpha} e^{-\i t \{H, \cdot\}} = \partial_{\alpha} - \i t (\partial_{\alpha} W_p) \partial_{\alpha^*} \partial_{\alpha} + \partial_{\alpha^*} W_p \partial_{\alpha} + O(t^2),$$

$$e^{\i t \{H, \cdot\}} \partial_{\alpha^*} e^{-\i t \{H, \cdot\}} = \partial_{\alpha^*} + \i t (\partial_{\alpha^*} W_p) \partial_{\alpha} \partial_{\alpha^*} \partial_{\alpha} + O(t^2),$$

we arrive at the following deformation of the star-product operator at short times,

$$\hat{S} = \exp \left[ -\i \frac{t}{2} \{\cdot\} + \i \frac{t}{2} \hat{d} \right],$$

being

$$\hat{d} = \partial_{\alpha} \otimes \{\partial_{\alpha^*} W_p, \cdot\} - \{\partial_{\alpha^*} W_p, \cdot\} \otimes \partial_{\alpha} + \{\partial_{\alpha} W_p, \cdot\} \otimes \partial_{\alpha^*} - \partial_{\alpha^*} \otimes \{\partial_{\alpha^*} W_p, \cdot\},$$

(A4)

the first-order defect operator. It is easy to see that for non-harmonic Hamiltonians the defect becomes non-trivial.

For instance, in the case of Kerr evolution, $W_p \sim |\alpha|^4$,

$$\hat{d} \sim \partial_{\alpha} \otimes \alpha^2 \partial_{\alpha^*} \partial_{\alpha^*} - \alpha^2 \partial_{\alpha^*} \partial_{\alpha} + c.c.$$

$$+ 2 |\alpha|^2 \partial_{\alpha} \otimes \partial_{\alpha^*} - \partial_{\alpha^*} \otimes 2 |\alpha|^2 \partial_{\alpha^*} + c.c.$$

Thus, in general the relation (A3) is not fulfilled, so that

$$\hat{R}(t) \neq \hat{R}^2(t),$$

(A5)

i.e. the initial pure state becomes mixed except when

(a) $\{W_p, W_p(0)\} = 0$, i.e. the initial state is an eigenstate of the Hamiltonian;
(b) The Hamiltonian is a quadratic function in $\alpha$ and $\alpha^*$.

The above result can be also obtained directly from the evolution equation (19), by observing that the dynamics of $\hat{R}^2(t)$ is described by

$$\partial_t \hat{R}^2 = i [\hat{R}^2, \hat{H}_{\text{eff}}] + \hat{R} \mathcal{L}(\hat{R}) + \mathcal{L}(\hat{R}) \hat{R},$$

and for an initial pure state $\hat{R}^2(0) = \hat{R}(0) = |\psi\rangle \langle \psi|$. However, due to the structure of the non-Hamiltonian part (28), $\mathcal{L}(\hat{R}) = i (F(\hat{R}) - F^*(\hat{R}))$,

$$\mathcal{L}(\hat{R}) = \hat{R} \mathcal{L}(\hat{R}) + \mathcal{L}(\hat{R}) \hat{R},$$

(A6)

unless $F(\hat{R}) = F^*(\hat{R})$ or $\mathcal{L}(\hat{R}) = 0$. Thus, the evolution of $\hat{R}(t)$ and $\hat{R}^2(t)$ is completely different.
Appendix B

In this appendix we outline the derivation of equation (19)–(24).

Applying the first-order differential operator in the right-hand side of (18) to the kernel (5) and making use the correspondence rules

\[ \hat{a}^\dagger \hat{w}(\alpha) - \hat{w}(\alpha) \hat{a}^\dagger, \]

\[ \alpha \hat{w}(\alpha) = \frac{\hat{w}(\alpha) + \hat{w}(\alpha) \hat{a}}{2}, \]

and their generalizations

\[ \alpha^k \hat{w}(\alpha) = \frac{1}{2} \left( \frac{\hat{w}(\alpha)}{k} \hat{a}^{k-1} \hat{a} \hat{w}(\alpha) \right), \]

\[ \alpha^k \hat{w}(\alpha) = \frac{1}{k} \left( \frac{\hat{w}(\alpha)}{k} \hat{a}^{k-1} \hat{a} \hat{w}(\alpha) \right), \]

we arrive at equation (19)–(24) after the normal re-ordering

\[ \hat{a}^k \hat{a}^l = \sum_{p=0}^{\min(k,l)} \frac{k! l!}{p!(k-p)!(l-p)!} \hat{a}^{l-p} \hat{a}^{k-p}. \]

Appendix C

Here we give explicit expressions of the Hamiltonians (17) and (20) in the normal ordered form,

\[ H_{\text{eff}} = \frac{n + m}{2n + m} \left( 2a^\dagger a a^n + \sum_{k=1}^m C_m^k n! \alpha^{m-k} a^{n-k} \right) + \frac{2a^\dagger a a^n + \sum_{k=1}^m C_m^k n! \alpha^{m-k} a^{n-k}}{2}, \]

and

\[ H_{mn} = a^\dagger a a^n + \sum_{k=1}^m C_m^k n! \left( \frac{1}{2} \right)^k \alpha^{m-k} a^{n-k} + a^\dagger a a^n + \sum_{k=1}^m C_m^k n! \left( \frac{1}{2} \right)^k \alpha^{m-k} a^{n-k}, \]

here \( n \geq m \). One can observe that \( H_{\text{eff}} \sim H_{mn} \) only for \( m = 1 \) and \( H_{\text{eff}} = \frac{1}{2} H_{mn} + \frac{1}{2} \) in the particular case \( n = m = 2 \).

ORCID iDs

A B Klimov & https://orcid.org/0000-0001-8493-721X
I Sainz & https://orcid.org/0000-0002-6671-0262

References

[1] Polkovnikov A 2011 Ann. Phys. 325 1790
[2] Zachos C K et al 2005 Quantum Mechanics in Phase Space (Singapore: World Scientific)
[3] Berry M V 1977 Phil. Trans. R. Soc. 287 237
[4] Littlejohn R G 1986 Phys. Rep. 138 193
[5] Klimov A B 1986 Phys. Rep. 138 193
[6] Lefevre-Seguin M and Seguin J C 1996 J Chem. Phys. 105 368
[7] Emary E C M and Hühne R 2012 J. Phys. A: Math. Theor. 45 204012
[8] Aberg J and Zyczkowski K 2004 J. Phys. A: Math. Theor. 37 7213
[9] Emary E C M and Hühne R 2012 J. Phys. A: Math. Theor. 45 204012
[10] Bracken A and Wood J 2006 J. Phys. A: Math. Theor. 39 657
[11] Bracken A and Wood J 2006 J. Phys. A: Math. Theor. 39 657
[12] Bracken A and Wood J 2006 J. Phys. A: Math. Theor. 39 657
[13] Bracken A and Wood J 2006 J. Phys. A: Math. Theor. 39 657
[14] Bracken A and Wood J 2006 J. Phys. A: Math. Theor. 39 657
[15] Bracken A and Wood J 2006 J. Phys. A: Math. Theor. 39 657
[16] Bracken A and Wood J 2006 J. Phys. A: Math. Theor. 39 657
[17] Bracken A and Wood J 2006 J. Phys. A: Math. Theor. 39 657
[18] Bracken A and Wood J 2006 J. Phys. A: Math. Theor. 39 657
[19] Bracken A and Wood J 2006 J. Phys. A: Math. Theor. 39 657