Interacting fermion dynamics in Majorana phase-space

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The problem of fermion dynamics is studied using the Q-function for fermions. This is a probabilistic phase-space representation, which we express using Majorana operators, so that the phase-space variable is a real antisymmetric matrix. We consider a general interaction Hamiltonian with four Majorana operators and arbitrary properties. Our model includes the Majorana Hubbard and Fermi Hubbard Hamiltonians, as well as general quantum field theories of interacting fermions. Using the Majorana Q-function we derive a generalized Fokker-Planck equation, with results for the drift and diffusion terms. The diffusion term is proved to be traceless, which gives a dynamical interpretation as a forwards-backwards stochastic process. This approach leads to a model of quantum measurement in terms of an ontology with real vacuum fluctuations.

I. INTRODUCTION

Fermionic physics is universal: all stable massive elementary particles are fermions. Here we analyze the nonlinear dynamics of a probabilistic phase-space of fermions [1, 2]. Recent developments in quantum measurement theory [3] have led to a theory of bosonic quantum dynamics as stochastic processes in phase-space, propagating in both time directions. We obtain the time-evolution equation for interacting fermions in Majorana phase-space, and show that it is also generalized Fokker-Planck equation with a traceless diffusion matrix [4]. This demonstrates that retrocausal physics [5, 6] of fermions is equivalent to quantum mechanics.

Such phase-space methods are potentially relevant to other developments in quantum computing [7], and to the control acquired in ultra-cold Fermi systems, which allows studies of strongly interacting fermions [8, 9]. Experiments in this area include superfluidity [10, 11], the crossover from Cooper pairs to the Bose-Einstein condensate (BEC) region [12–15], and the Hubbard model [16–18], which has been realized in optical lattices [19–21]. This exhibits phase-transitions [22, 23], transport properties [24, 25], and anti-ferromagnetism [26, 27]. It can be used as a quantum simulator [28, 29].

Since interactions play an important role in Fermi systems, it is important to develop first-principles theoretical methods to investigate the corresponding dynamics, without mean-field approximations. Some of the different theoretical methods that have been used to study strongly interacting fermions include Monte-Carlo methods, which have a sign problem [30], and Grassmann phase-space approaches [31, 32], although these can become exponentially complex.

An alternative approach is via Gaussian phase-space representations [33, 34] and a fermionic P-function [35–41]. Such methods have been used to investigate the Fermi-Hubbard model [35–37], but can lead to sampling errors [42, 43]. The technique used here is the generalized Q-function [44] defined in terms of Gaussian operators. Rather than using Fermi ladder operators, we use Majorana operators. This corresponds to the Majorana phase-space [2], which has been used to study the dynamics of shock waves [45] and information-related quantities like the Renyi entropy, purity and fidelity [46]. Here we extend this to fermion interactions. Interactions with bosonic fields have been treated elsewhere [47].

Majorana fermions and their related group structures have been heavily investigated in their own right in recent years [48, 49]. One of their features is that they are their own antiparticle [50] and are found in topological superconductors [51, 52]. They have a possible role in quantum computation [53–55], as well as more generally in condensed matter physics [56, 57], due to their relationship with the general DIII symmetry class [58]. One major topic of study is to incorporate interactions [59]. There are several Majorana models that include interactions, including the Majorana Hubbard model on square lattices [58, 59, 60], honeycomb lattices [61, 62], triangular lattices [63, 64], and vortex lattices [65].

The study of interactions for Majorana fermions in condensed matter physics [66] leads to a new classification of topological phases in one dimension [67]. We treat the most general interactions of four operators, using a Majorana Q-function [2]. This includes the Fermi Hubbard and other lattice models of quartic fermion interactions [50, 68]. In this paper, we obtain the generalized Fokker-Planck equation that describes Q-function dynamics. Our approach uses a phase-space of real antisymmetric matrices, which is a fundamental concept in group theory [69]. This is related to variational theories of Gaussian states [70, 71], except that Q-functions do not require a variational approximation.

This paper is organized as follows: Section II gives a summary of Majorana Q-functions and notation. In Section III we discuss the Hamiltonian and dynamical evolution. Sec. IV gives a derivation of the Fokker Planck equation, and properties of the diffusion term. In Sec. V we discuss the relation of the drift to the phase space of pure states. A summary is given in Sec. VI.
II. GAUSSIAN MAJORANA Q-FUNCTION

Phase-space methods have been used to study bosonic fields [73, 81] with considerable success in comparisons to experiment [82, 83]. One can analogously define fermionic phase-space representations using Grassmann variables [77, 84–86]. However these non-commuting variables have an exponential complexity.

Fermionic phase-space representations have also been introduced over complex phase-spaces, including the P-function [88–90] and the Q-function [46]. These use ordered Gaussian fermionic operators as a basis, together with a complex phase-space.

Here we treat a third approach, a representation which uses as a phase-space variable a real anti-symmetric matrix [2]. The advantage of this approach is that it corresponds mathematically to a well-defined compact homogeneous space. This representation is the Majorana Q-function. We start by giving a brief summary of its properties.

A. Gaussian operator definition

We consider a general $M$-mode lattice of quantum fermionic modes described by $M$ fermionic annihilation and creation operators $\hat{a}, \hat{a}^\dagger$. We denote $M$-dimensional vectors and matrices with a bold notation, and $2M$ dimensional vectors and matrices with an underline, so that an extended operator $\hat{u}$ is defined as

$$\hat{u} = \begin{pmatrix} \hat{a}_1 & \cdots & \hat{a}_M \end{pmatrix}^T. \quad (2.1)$$

One can obtain $2M$-dimensional Majorana operators from this extended vector of fermionic creation and annihilation operators [32] by the action of a matrix [85],

$$U_0 = \begin{bmatrix} 1 & 1 \\ -iI & iI \end{bmatrix}, \quad \text{so that:}$$

$$\hat{S} = U_0 \hat{u}. \quad (2.2)$$

With this relation, $\hat{S}_1 = \hat{a}_1 + \hat{a}_1^\dagger$, and $\hat{S}_{M+1} = i \left(\hat{a}_1^\dagger - \hat{a}_1\right)$. The resulting real Majorana operator $\hat{S}$ is a $2M$-dimensional Hermitian Fermi operator which obeys the following anti-commutation relation, for $i, j = 1, \ldots 2M$:

$$\{\hat{S}_i, \hat{S}_j\} = 2\delta_{ij}. \quad (2.3)$$

The Gaussian Majorana operator can be defined in several ways, either ordered or unordered. To obtain operator differential identities, we choose a normal ordering approach. The phase-space variable is defined for general Gaussian operators using an antisymmetric complex matrix, $\underline{x}$, in one of the irreducible bounded symmetric domains of group theory, defined [74, 80, 88] so that

$$\underline{xx}^\dagger \leq I. \quad (2.4)$$

Our definition of a Gaussian basis gives an exponential of a quadratic in the Majorana operators as:

$$\hat{\Lambda}(\underline{x}) = N(\underline{x}) \exp \left[ -i \hat{\gamma}^T \left[ \frac{1}{2} i + (\hat{\gamma} + i\hat{\gamma}^\dagger)^{-1}\right] \hat{\gamma} / 2 \right]. \quad (2.5)$$

Here $N(\underline{x})$ ensures that the Gaussian operator is normalized so that $\text{Tr} \left[ \hat{\Lambda}(\underline{x}) \right] = 1$, and we define $\hat{\gamma} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$, which is a matrix square root of $-I$.

From now on, we treat the case where the Gaussian operator is Hermitian and positive definite, so that $\underline{x}$ is a real anti-symmetric matrix, with fermionic pure states occurring at the boundary where $x = I$. Gaussian states on the boundary have also been classified as a fundamental symmetry class in the physics literature corresponding to certain condensed matter devices [89].

B. Q-function definition

Q-function phase-space representations are positive probability distributions that can provide powerful simulation methods. These include, for example, a recent 60 qubit simulation of mesoscopic multipartite Bell violations [91] using GHZ states in ion traps. A Majorana Q-function can be defined for any fermionic quantum density matrix, $\hat{\rho}$. This is defined as a distribution over the antisymmetric matrices:

$$Q(\underline{x}) = \text{Tr} \left[ \hat{\rho} \hat{\Lambda}^N(\underline{x}) \right], \quad (2.6)$$

where we introduce $\hat{\Lambda}^N(\underline{x})$ as a rescaling of the unit trace Gaussian $\hat{\Lambda}(\underline{x})$, such that:

$$\hat{\Lambda}^N(\underline{x}) = \frac{1}{N} \hat{\Lambda}(\underline{x}) S(\underline{x}^2). \quad (2.7)$$

The function $S(\underline{x}^2)$ is an arbitrary even function of $\underline{x}$, and the normalization $N$ is defined so that the following resolution of identity holds,

$$\hat{1} = \int d\underline{x} \hat{\Lambda}^N(\underline{x}), \quad (2.8)$$

where the antisymmetric real matrix integration measure $\underline{dx}$ is given by

$$d\underline{x} = \prod_{1 \leq j < k \leq 2M} dx_{ij}. \quad (2.9)$$

As a result, since $\text{Tr} \left[ \hat{\rho} \right] = 1$, the probability distribution is normalized to unity

$$\int d\underline{x} Q(\underline{x}) = 1. \quad (2.10)$$
Any fermionic observable can be calculated using the fermionic \( Q \)-function, together with the appropriate identities. To give an example, the expectation value of the Majorana two-fermion correlation function,

\[
\hat{X}_{\mu \nu} \equiv \frac{i}{2} [\gamma_\mu, \gamma_\nu]
\]  

(2.11)
is given by \[2\]:

\[
\langle \hat{X} \rangle = (4M - 1) \int \mathcal{D}Q(x) \, d\mu\nu
\]  

(2.12)

Explicit results obtained here will use the limit of \( S(x^2) = 1 \) for simplicity. Other choices are also possible, including the pure states with \( x^2 = -1 \), which are divided into two parity classes \[48\], each belonging to the DIII symmetric space of Cartan \[65\].

C. Notation and derivatives

In this section we give a summary of the derivatives that will be used, as well as the compact notation that is introduced. The phase-space variable of the Majorana \( Q \)-function is a real antisymmetric matrix. Derivatives with respect to \( x \) are defined to take account of this constraint, i.e. so that \( x_{ab} \equiv -x_{ba} \) \[2\], so that:

\[
\frac{\partial x_{ab}}{\partial x_{cd}} = \delta_{ac}\delta_{bd} - \delta_{ad}\delta_{bc}.
\]  

(2.13)

We note that antisymmetric derivatives \( dx/\partial x \) are defined here so that \((\partial/\partial x)_{ij} \equiv \partial_{ji} = -\partial_{ij}\). The differential identities given below are given in terms of the matrices \( x^\pm = x \pm iI \). As a result,

\[
\frac{\partial x^+_a}{\partial x_a} = \frac{\partial x_{ab}}{\partial x_{cd}} = \frac{\partial x^-_{ab}}{\partial x_{cd}},
\]  

(2.14)
since \( \delta_{ab} \) is a constant. We also define:

\[
\partial_{\alpha\beta} \equiv \frac{\partial}{\partial x_{\alpha\beta}}.
\]  

(2.15)

Where appropriate, we will use a shorthand form with a restricted index range that only includes the independent parameters, where \( \alpha \equiv (\alpha, \beta) \) with \( 1 \leq \alpha < \beta \leq 2M \), similarly \( \mu = (\mu, \nu) \), and hence, for \( \alpha < \beta \)

\[
\partial_{\alpha} \equiv \partial_{\alpha\beta} \equiv \frac{\partial}{\partial x_{\alpha\beta}}.
\]  

(2.16)

Derivatives of products of anti-symmetric matrices can be obtained by using the product rule, for example:

\[
\partial_{cd} (x_{ab}x_{ef}) = x_{ab} (\partial_{ce} \delta_{df} - \delta_{cf} \delta_{de}) + x_{ef} (\delta_{ac} \delta_{bd} - \delta_{ad} \delta_{bc}).
\]  

(2.17)

The indices of \( x^+ \) and \( x^- \) are related through the following expression:

\[
x^+_{ab} = -x^-_{ba}. \tag{2.18}
\]

Throughout the paper we use a four-index notation for the products of the form \( x^+ x^- \), which is:

\[
X^{\alpha\beta} \equiv x^{\alpha+} x^{\beta-},
\]

(2.19)

From Eq. (2.18), one obtains:

\[
X^{\beta\alpha} \equiv X^{\alpha\beta}. \tag{2.20}
\]

Real and imaginary parts are given by

\[
\Im X^{\alpha\beta} = \frac{1}{2i} \left( X^{\alpha\beta} - X^{\beta\alpha} \right)
\]

(2.21)

Derivatives of the variable \( X^{\alpha\beta}_{ij} \) are calculated using the product rules and the definition of the derivative given in Eq. (2.13), for example:

\[
\partial_{\mu\nu} X^{\alpha\beta}_{ij} = \partial_{\mu} X^{\alpha}_{i} \partial_{\nu} X^{\beta}_{j}
\]

(2.22)

\[
= x^+_{ij} \left( \delta_{\alpha\mu} \delta_{\nu\beta} - \delta_{\alpha\beta} \delta_{\mu\nu} \right) + x^+_{ij} \left( \delta_{\nu\beta} \delta_{\mu\nu} - \delta_{\beta\mu} \delta_{\mu\nu} \right).
\]

Some useful derivatives used in the calculations are:

\[
(\partial_{\alpha} x^+_{kl}) \, X^{\alpha}_{kl} = X^k_{ij} - X^l_{ij},
\]

\[
(\partial_{\alpha} x^-_{kl}) \, X^{\alpha}_{kl} = X^{kl}_{ij} - X^{kl}_{ij},
\]

\[
\partial_{\alpha} X^{\alpha}_{ij} = 0,
\]

\[
\partial_{\alpha} \Im X^{\alpha}_{ij} = -2x_{ij} (2M - 1). \tag{2.23}
\]

D. Majorana differential identities

The utility of the normally ordered approach is that straightforward differential identities exist for all fermionic observables. Observables are even polynomials in Majorana operators, and their identities can be obtained from the quadratic results given here. These are essential in order to obtain dynamical equations of motion and observables using phase-space representations.

For the Majorana Gaussian operator, quadratic differential identities were derived in \[2\], and are given below.

- Left product:

\[
\hat{\gamma}^{\alpha T} \hat{\Lambda} = i \left[ x^- \frac{dx}{dx^+} x^+ - \hat{\Lambda} x^+ \right]. \tag{2.24}
\]
Here $\hat{X}^{\pm} = \hat{X} \pm i\mathcal{L}^{-1}$. These identities will be used below to obtain the time evolution equation for the Q-function, and hence the corresponding Fokker-Planck equation.

III. TIME EVOLUTION

A. Model Hamiltonian

To obtain a formalism for the time evolution of the Majorana Q-function with interacting fermions, we now consider a general Hamiltonian with a non-interacting linear term and an interaction term. The interaction Hamiltonian describes a four-Majorana interaction. Depending on the parameters this Hamiltonian may correspond to the Majorana Hubbard model [50, 66], or to a generic four-fermion quantum field theory using a lattice discretization in space.

The Hamiltonian of the model is given by:

$$\hat{H} = \hat{H}_0 + \hat{H}_{\text{int}},$$

$$= i\hbar \sum_{i,j} t_{ij} \gamma_i \gamma_j + \frac{\hbar}{2} \sum_{i,j,k,l} g_{ijkl} \gamma_i \gamma_j \gamma_k \gamma_l.$$  

(3.1)

Due to the antisymmetry of fermion operator commutators, and with no loss of generality, we can impose the condition that $t_{ij}$ and $g_{ijkl}$ are elements of second and fourth order antisymmetric tensors respectively. This implies that $t_{ij} = -t_{ji}$ and

$$g_{ijkl} = \begin{cases} 
+g_{\sigma(ijkl)} & \text{if } (i,j,k,l) \text{ is an even permutation} \\
-g_{\sigma(ijkl)} & \text{if } (i,j,k,l) \text{ is an odd permutation} \\
0 & \text{otherwise.} 
\end{cases}$$

(3.2)

Since the Hamiltonian is Hermitian, it follows that the coefficients $t, g$ are all real. This follows since, from hermiticity and antisymmetry,

$$t_{ij} = -t_{ji}^*, \quad g_{ijkl} = g_{iklj}^*.$$  

(3.3)

B. Dynamical evolution

The time evolution equation for the density operator is given by:

$$i\hbar \frac{\partial}{\partial t} \rho = [\hat{H}, \hat{\rho}].$$

(3.4)

Therefore, the time evolution equation for the Majorana Q-function obtained from the definition of the Q-function given in Eq. (2.6) is:

$$\frac{dQ(x)}{dt} = \frac{1}{i\hbar} \text{Tr} \left[ [\hat{\Lambda}, \hat{\rho}] \hat{N}(x) \right].$$

(3.5)

Using the cyclic identities of the trace, we get:

$$\frac{dQ(x)}{dt} = \frac{1}{i\hbar} \text{Tr} \left[ \hat{\Lambda} \hat{N} \dot{\hat{H}} - \dot{\hat{H}} \hat{\Lambda} \hat{\rho} \right].$$

(3.6)

From the Hamiltonian given in Eq. (3.1), the time evolution equation in Eq. (3.6) can be written as:

$$\frac{dQ(x)}{dt} = \frac{S^2}{N} \text{Tr} \left\{ t_{ij} \rho \left[ \hat{\Lambda} (x) \gamma_i \gamma_j - \gamma_j \gamma_i \hat{\Lambda} (x) \right] \ight. + \left. \frac{g_{ij}}{2} \rho \left[ \hat{\Lambda} (x) \gamma_i \gamma_j \gamma_k \gamma_l - \gamma_i \gamma_j \gamma_k \gamma_l \hat{\Lambda} (x) \right] \right\}.$$  

(3.7)

Here we have defined $g_{ijkl} = g_{ij}$, and $i = (i,j,k,l)$. We follow the Einstein summation convention and thereby avoid summation signs throughout the paper. Repeated indices $i, j, \alpha, \beta, \mu, \nu$ are summed over $1, \ldots, 2M$. Using the differential identities of Section III and the procedure in the Appendix A, we obtain:

$$\frac{dQ(x)}{dt} = -it_{ij} \left( X_{ij}^{\alpha\beta} - X_{ij}^{\alpha\beta*} \right) \partial_{\alpha\beta} Q$$

$$+ \frac{i}{2} g_{ij} \left\{ \left( X_{ij}^{\alpha\beta} X_{kl}^{\mu\nu} - X_{ij}^{\alpha\beta*} X_{kl}^{\mu\nu*} \right) \partial_{\alpha\beta} \partial_{\mu\nu} Q \ight. + \left. \left( X_{ij}^{+} X_{ij}^{-} - X_{ij}^{\alpha\beta*} X_{ij}^{\alpha\beta} \right) \partial_{\mu\nu} Q \ight\}. $$

(3.8)

Here we have used the definitions given in Eqs. (2.19) as well as the following properties:

$$X_{ij}^{\alpha\beta*} = X_{ij}^{\alpha\beta}$$

$$X_{ij}^{\alpha\beta} = X_{ij}^{\beta\alpha}$$

$$X_{ij}^{\alpha\beta} = X_{ji}^{\alpha\beta}. $$

(3.9)

Since $X_{ij}^{\alpha\beta} - X_{ij}^{\alpha\beta*} = 2i\Im X_{ij}^{\alpha\beta}$, (see Appendix A), we can rewrite the time evolution equation as:

$$\frac{dQ(x)}{dt} = \left\{ \frac{i}{2} g_{ij} \left( X_{ij}^{\alpha\beta} X_{kl}^{\mu\nu} - X_{ij}^{\alpha\beta*} X_{kl}^{\mu\nu*} \right) \partial_{\mu\nu} Q \ight. + \left. 2\Im X_{ij}^{\alpha\beta} (t_{ij} - 3g_{ij} x_{kl}^+) \right\} \partial_{\alpha\beta} Q. $$

(3.10)
We note that Eq. (3.10) sums over all possible values for the indices marked by Greek labels. Following the procedure described in Appendix B, we arrive at the time evolution equation for the Majorana Q-function in terms of implicit Einstein summation over only independent phase-space variables, whose indices are denoted \( \alpha = (\alpha, \beta) \) with \( \alpha < \beta \). These are regarded as a vector,

\[
x^\alpha \equiv x_{\alpha \beta},
\]

\[
\partial_\alpha \equiv \frac{\partial}{\partial x^\alpha}.
\] (3.11)

After making this restriction, and using the antisymmetry of \( x_{\alpha \beta} \), we find that:

\[
\frac{dQ}{dt} = 4 \left[ 3X^\alpha_{ij}(t_{ij} - 3g_i x^i_{kl}) \partial_\alpha Q - 2g_i (\Re X^\mu_{ij}) (3X^\nu_{kl}) \partial_\alpha \partial_\mu Q \right].
\] (3.12)

Bold repeated indices \( \alpha, \mu \) are summed over independent variables with \( \alpha < \beta \) and \( \mu < \nu \). Here the implicit Einstein summation corresponds to \( \sum_\alpha \) and \( \sum_\mu \) which denote \( \sum_{\alpha < \beta} \) and \( \sum_{\nu < \mu} \) respectively. The partial differential equation above can therefore be written in the form:

\[
\frac{dQ}{dt} = \left[ -A^\alpha + 2 \frac{D^\alpha}{\mu} \partial_\mu \right] \partial_\alpha Q,
\] (3.13)

where

\[
D^\alpha_{\mu \nu} = -16g_i \Re X^\alpha_{ij} \Im X^\nu_{kl},
\]

\[
A^\alpha = 4 \Im X^\alpha_{ij}(3g_i x^i_{kl} - t_{ij}).
\] (3.14)

In this equation, we have denoted the first order coefficients as \( A^\alpha \). In the next section we see that these terms are related to the diffusion and drift terms of a generalized Fokker-Planck equation, which has an explicitly probability conserving form.

IV. GENERALIZED FOKKER-PLANCK EQUATION

In this section we will express the partial differential equation given in Eq. (3.13) in the form of a generalized Fokker-Planck equation. For this purpose, we use the product rule for the second order derivative, which allows us to write Eq. (3.13) as,

\[
\frac{dQ}{dt} = \frac{1}{2} \left[ \partial_\alpha \partial_\mu \left( D^{\alpha \mu} Q \right) - \left( \partial_\alpha \partial_\mu D^{\alpha \mu} \right) Q \right] - \left( A^\alpha + \left( \partial_\mu D^{\alpha \mu} \right) \right) \partial_\alpha Q,
\] (4.1)

where we use the result that,

\[
\partial_\alpha Q \partial_\mu D^{\alpha \mu} + \partial_\mu Q \partial_\alpha D^{\alpha \mu} = 2 \partial_\alpha Q \partial_\mu D^{\alpha \mu}.
\] (4.2)

We have exchanged the dummy indices \( \alpha \leftrightarrow \mu \) in the second term of the left hand side of the above equation to get the desired result. On defining:

\[
A^\alpha = \bar{A}^\alpha + \partial_\mu D^{\alpha \mu},
\] (4.3)

we obtain

\[
\frac{dQ}{dt} = \frac{1}{2} \left( \partial_\alpha \partial_\mu D^{\alpha \mu} - \left[ \partial_\alpha \partial_\mu D^{\alpha \mu} \right] Q \right) - \left( \partial_\alpha A^\alpha - \left[ \partial_\alpha A^\alpha \right] Q \right),
\] (4.4)

where \( A^\alpha \) is the drift term, while \( D^{\alpha \mu} \) is the diffusion term. Since \( \partial_\alpha \partial_\mu D^{\alpha \mu} = 0 = \partial_\alpha A^\alpha \), where the proof is given in Appendix C, the generalized Fokker Planck equation for the Majorana Q-function simplifies. It has an explicitly probability conserving, but not positive-definite, form [91, 92]:

\[
\frac{dQ}{dt} = - \partial_\alpha \left[ -A^\alpha + \frac{1}{2} \partial_\mu D^{\alpha \mu} \right] Q.
\] (4.5)

A. Traceless diffusion

We now investigate whether the diffusion matrix \( D^{\alpha \mu} \) is traceless, as is the case for other Q-function generalized Fokker-Planck equations. Such equations implement a diffusive ‘baker map’ transformation, in which the time-evolution results in mixing, but without the diffusive growth in entropy of a traditional Fokker-Planck equation [94, 95]. We will first show that the diffusion term given above includes both positive and negative-definite forms.

This term can be written explicitly as:

\[
D^{\alpha \mu} = -16g_i \Re X^\alpha_{ij} \Im X^\mu_{kl} = -8g_i \Re X^\alpha_{ij} \Im X^\mu_{kl} + \Im X^\alpha_{ij} \Re X^\mu_{kl}.
\] (4.6)

We define the following matrices, that depend only of pairs of indices \( (\alpha, \beta) \) or \( (\mu, \nu) \) as:

\[
B^\alpha_{(\pm) i} = \Re X^\alpha_{ij} \pm \Im X^\alpha_{kl},
\]

\[
D^{\alpha \mu}_{(\pm) i j} \equiv B^\alpha_{(\pm) i} B^\mu_{(\pm) j},
\] (4.7)

so that the diffusion term is:

\[
D^{\alpha \mu} = 4g_i \left( B^\alpha_{(-) j} B^\mu_{(-) i} - B^\alpha_{(+) j} B^\mu_{(+) i} \right).
\] (4.8)

The matrix is symmetric, and for each set of four indices \( i \), it is expressed as a difference of two real terms, each one being an outer product of identical vectors. Since any matrix of the form \( D^{\alpha \mu} = B^\alpha B^\mu \) is positive definite, it follows that each of these matrix product terms is individually positive definite. Therefore, it is explicitly shown that the Majorana Q-function diffusion matrix can always be expressed as the sum of multiple positive-definite and negative-definite terms.

The diffusion matrix of a standard Fokker Planck equation is symmetric and positive definite [91, 92].
the case of the Q-function this is not the case. Non-
positive diffusion matrices for the generalized Fokker-
Planck equations of Q-functions have been investigated
for bosonic and spin systems [47, 49]. This can be inter-
preted as a forward-backward stochastic process [4, 49],
i.e. a diffusion that takes place in the forward and back-
ward directions of time simultaneously.

In terms of the phase-space variables $x$, the diffusion
matrix is given explicitly as:

$$D^{\alpha\mu} = -4g_i \left[ (x_{i\alpha} x_{j\beta} - x_{i\alpha} \delta_{j\beta} + \delta_{i\alpha} x_{j\beta} + \delta_{i\alpha} \delta_{j\beta})
\times (x_{k\mu} x_{l\nu} - x_{k\mu} \delta_{l\nu} + \delta_{k\mu} x_{l\nu} + \delta_{k\mu} \delta_{l\nu})
- (x_{i\alpha} x_{j\beta} + x_{i\alpha} \delta_{j\beta} - \delta_{i\alpha} x_{j\beta} + \delta_{i\alpha} \delta_{j\beta})
\times (x_{k\mu} x_{l\nu} + x_{k\mu} \delta_{l\nu} - \delta_{k\mu} x_{l\nu} + \delta_{k\mu} \delta_{l\nu}) \right].$$

(4.9)

We have shown that the diffusion matrix of the Fokker-
Planck equation for the Majorana Q-function can be
expressed as a sum of a positive- and negative-definite
terms. We will now show that the diffusion matrix is
completely traceless. In order to prove this, we consider
the form of the diffusion equation given in Eq. (4.9),
which can also be expressed as:

$$D^{\alpha\mu} = -8g_i \left[ (x_{i\alpha} x_{j\beta} x_{k\gamma} x_{l\delta}) + (x_{i\alpha} x_{j\beta} x_{k\gamma} x_{l\delta})
- (x_{i\alpha} x_{j\beta} x_{k\gamma} x_{l\delta}) - (x_{i\alpha} x_{j\beta} x_{k\gamma} x_{l\delta}) \right].$$

(4.10)

To prove the traceless property, it is necessary to show
that $\sum_\alpha D^{\alpha\alpha} = 0$. We will show this through a proof
that every diagonal element of this matrix in this basis
is zero. Each diagonal element has the form:

$$D^{\alpha\alpha} = -4g_i \left[ (x_{i\alpha} x_{j\beta} x_{k\gamma} x_{l\delta}) + (x_{i\alpha} x_{j\beta} x_{k\gamma} x_{l\delta})
- (x_{i\alpha} x_{j\beta} x_{k\gamma} x_{l\delta}) - (x_{i\alpha} x_{j\beta} x_{k\gamma} x_{l\delta}) \right].$$

(4.11)

After expanding, we find that the imaginary terms are
either cubic or linear in $x$, so that:

$$D^{\alpha\alpha} = 4g_i \left[ (x_{i\alpha} x_{j\beta} x_{k\gamma} x_{l\delta}) + (x_{i\alpha} x_{j\beta} x_{k\gamma} x_{l\delta})
- (x_{i\alpha} x_{j\beta} x_{k\gamma} x_{l\delta}) - (x_{i\alpha} x_{j\beta} x_{k\gamma} x_{l\delta}) \right].$$

(4.12)

Inspecting these terms, we see that all the terms have
similar behavior, and in each case:

1. Cubic terms like $g_i x_{i\alpha} x_{j\beta} x_{k\gamma}$ cancel similar terms
in the sum with $i$ and $k$ swapped, since this is
an odd permutation which changes the sign of $g_i$.
These terms also cancel since $x_{ij} = -x_{ji}$.

2. Linear terms like $g_i x_{i\alpha} \delta_{j\beta} x_{k\gamma} \delta_{l\delta}$ vanish, since $g_i = 0$ if $l = j$.

In summary, the diffusion matrix in the $x$ variables has
the property that all the diagonal elements are zero, and
consequently it is traceless. Following the discussion of
such traceless equations given elsewhere [49], it is always
possible to make an orthogonal transformation which di-
agonalizes the diffusion and leaves the trace invariant, so
that it obeys a forwards-backwards stochastic equation.

V. DRIFT TERM AND PHASE-SPACE DOMAIN

The phase-space variables of the Majorana Q-function
are real antisymmetric matrices, which define a bounded
homogeneous phase-space with $M(M - 1)$ dimensions.
Inclusion of the boundary is given by

$$I + \bar{x}^2 \geq 0.$$

(5.1)

Physical states are characterized by the above condition,
which corresponds to Hermitian, positive density matrix.
Gaussian pure fermionic states are restricted to the
surface of the homogeneous space. It is possible that the
generalized Fokker-Planck solutions of Hamiltonian
evolution, even though non-Gaussian, may also be confined
to the surface of the homogeneous space. In this section,
we verify this conjecture for the drift term of the
generalized Fokker-Planck equation.

The condition that defines the surface of the real
subspace of the complex homogeneous space is given by:

$$I + \bar{x}^2 = 0,$$

(5.2)

which can also be written in the form:

$$x_{\alpha\eta} x_{\eta\beta} = -\delta_{\alpha\beta}.$$

(5.3)

On differentiating the above equation we obtain:

$$\frac{\partial x_{\alpha\eta}}{\partial t} x_{\eta\beta} + x_{\alpha\eta} \frac{\partial x_{\eta\beta}}{\partial t} = -\frac{\partial}{\partial t} \delta_{\alpha\beta} = 0.$$

(5.4)

We now wish to relate this condition with the drift term,
$A^{(\alpha\eta)}$ and the surface of the homogeneous space. On
considering the generic drift equation,

$$\frac{\partial x_{\alpha}}{\partial t} = A^\alpha,$$

(5.5)

Eq. (5.4) can be written as:

$$A^{(\alpha\eta)} x_{\eta\beta} + x_{\alpha\eta} A^{(\eta\beta)} = 0.$$

(5.6)

Using Eq. (3.13) as well as Eq. (3.14) the expression for
the drift term is given by:

$$A^{(\alpha\eta)} = -\frac{4g_i}{\hbar} 3 x_{\alpha\beta}^{\alpha\beta} \left[ 6x_{kl}^+ - \frac{t_{ij}}{g_i} + 8(3 - 2M) x_{kl} \right].$$

(5.7)

On substituting this expression in the left hand side of
Eq. (5.6) we obtain:

$$x_{\alpha\eta} A_{\eta\beta} + A_{\alpha\eta} x_{\eta\beta} =$$

$$-\frac{4g_i}{\hbar} \left[ 3 X_{ij}^{\eta\beta} \left( -x_{\alpha\eta} t_{ij} + 2(15 - 8M) x_{\alpha\eta} x_{kl}^+ \right) + 3 X_{ij}^{\alpha\eta} \left( -t_{ij} x_{\eta\beta} + 2(15 - 8M) x_{kl}^+ x_{\eta\beta} \right) \right]$$

$$= \frac{4g_i}{\hbar} \left[ 2(15 - 8M) \left( -\delta_{\beta j} x_{\eta\beta} + \delta_{i\alpha} x_{ij}^+ x_{\eta\beta} \right) x_{kl}^+ - t_{ij} \left( -\delta_{\beta j} x_{\eta\beta} + \delta_{i\alpha} x_{ij}^+ x_{\eta\beta} \right) \right].$$

(5.8)
Here we have used that,
\[ \Delta X_{ij}^{\eta_j} x_{\alpha \eta} = - (x_{i\eta} \delta_{j\alpha} x_{\alpha \eta} - x_{j\eta} x_{\alpha \eta}) . \]

We notice that the expression of Eq. (5.8) contains terms of the form \( x_{i\eta} x_{\alpha \eta} \). If we consider the condition for the pure states on the boundary of the homogeneous space, given in Eq. (5.3) we get:

\[
x_{\alpha \eta} A_{i\beta}^{\eta_j} + A_{i\eta}^{\eta_j} x_{\alpha \beta} =
\frac{4 g_i}{h} [2 (15 - 8 M) (\delta_{j\beta} \delta_{i\alpha} - \delta_{i\alpha} \delta_{j\beta}) x_{kl}^+ - \frac{1}{g_i} (\delta_{j\beta} \delta_{i\alpha} - \delta_{i\alpha} \delta_{j\beta})] = 0.
\] (5.9)

In summary, the drift term maintains the ‘surface’ condition that corresponds to a Gaussian pure state. It is a tangent vector in the space.

VI. SUMMARY

We have considered a completely general four-fermion interaction Hamiltonian that contains four Majorana operators. For this model, we have derived the time evolution equation for the Majorana Q-function phase-space representation. This type of interaction Hamiltonian can be used to describe the Majorana-Hubbard and Fermi-Hubbard models, as well as more general Hamiltonians in quantum field theory. In order to perform the calculations we have used the symmetry properties of the Hamiltonian. We have derived a generalized Fokker-Planck type equation, whose diffusion term is not positive definite. Instead we show it has a zero trace: the diffusion term can be expressed as a sum of positive definite and negative definite terms. This is consistent with a forward-backward stochastic evolution, as found previously for the evolution of bosonic and spin Q-functions. Such evolution has a probabilistic action and path integral \[ \int \mathcal{D}x \] compatible with an ontological interpretation \[ \int \mathcal{D}x \] as an objective field without requiring observers \[ \int \mathcal{D}x \].

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Appendix A: Dynamics of Majorana Q-functions

In this section we give details of the calculations used to obtain the time evolution of the Majorana Q-function given in Eq. (5.8). All repeated indices are summed over their full range of definition, where \( i, j, k, l = 1, \ldots, M \) and \( \mu, \nu, \alpha, \beta = 1, \ldots, 2M \). The bold notation \( \mu = (\mu, \nu) \), \( \alpha = (\alpha, \beta) \) indicates ordered indices summed with \( \mu < \nu, \alpha < \beta = 1, \ldots, 2M \).

First, consider the time evolution equation of the Majorana Q-function, Eq. (5.7), which is:

\[
\frac{d Q(x)}{dt} = S^2 \frac{N}{N} \text{Tr} \left\{ t_{ij} \hat{J} \left\{ x_{ij}^\alpha \gamma_i^j \right\} \gamma_j^i \right\}.
\] (A1)

In calculating the time evolution equation, products of the form \( \gamma_i^j \) and \( \gamma_j^k \gamma_k^i \) must be evaluated. The differentiable identities given in Eqs. (2.24) and (2.25) are used to obtain:

\[
\gamma_i^j \gamma_j^k \gamma_k^i = \left( -x_{ij}^\mu \partial_{\alpha \beta} \partial_{\mu \nu} - \partial_{\mu \nu} \partial_{\alpha \beta} x_{ij}^\mu \right) X_{kl}^\alpha^\beta^* + \left( \partial_{\alpha \beta} x_{ij}^+ \partial_{\mu \nu} \partial_{\alpha \beta} x_{ij}^+ \right) X_{kl}^\alpha^\beta^* - x_{ij}^\mu^* x_{kl}^+ \partial_{\mu \nu} \partial_{\alpha \beta} - \hat{J} X_{ij}^j.
\] (A2)

\[
\hat{J} \gamma_i^j \gamma_j^k \gamma_k^i = X_{ij}^\gamma^\alpha^\beta\left( -x_{kl}^\mu \partial_{\alpha \beta} \partial_{\mu \nu} - \partial_{\mu \nu} \partial_{\alpha \beta} x_{ij}^\mu \right) + \left( \partial_{\alpha \beta} x_{ij}^+ \partial_{\mu \nu} \partial_{\alpha \beta} x_{ij}^+ \right) - x_{ij}^\mu^* X_{kl}^+ \partial_{\mu \nu} \partial_{\alpha \beta} + \hat{J} X_{ij}^j.
\] (A3)

Here, the definitions given in Eq. (2.19) are utilized so that \( X_{ij}^\alpha^\beta = x_{ij}^\alpha x_{ij}^\beta \), and \( X_{ij}^\alpha^\beta = x_{ij}^\alpha x_{ij}^\beta \). The derivative of the Gaussian basis and the derivative of the Gaussian operator are related through the chain rule as:

\[
\frac{1}{N} S \left( \left[ x \right] \right)^2 \frac{d \hat{J} (x)}{dx} = \frac{d \hat{J} (x)}{dx} - \ln \left( \frac{1}{N} S \left( \left[ x \right] \right)^2 \right) \hat{J} (x).
\] (A4)

We now take the limit \( S \to 1 \), so that

\[
\frac{1}{N} S \left( \left[ x \right] \right)^2 \frac{d \hat{J} (x)}{dx} = \frac{d \hat{J} (x)}{dx},
\] (A5)

and \( \hat{J} (x) = \frac{1}{N} \hat{J} (x) \). Using the differential identities given in Eqs. (2.24), (2.25), (A2) and (A3) gives:

\[
\frac{d Q(x)}{dt} = 2 t_{ij} 3 X_{ij}^\alpha \partial_{\alpha \beta} Q + i \frac{g_i}{2} \left[ x_{ij}^+ X_{ij}^\alpha - x_{ij}^+ x_{ij}^\alpha \right] \partial_{\alpha \beta} + \left( X_{ij}^\alpha \partial_{\alpha \beta} X_{ij}^\mu - \partial_{\alpha \beta} x_{ij}^\mu X_{ij}^\alpha - X_{ij}^\mu X_{ij}^\alpha \right) \partial_{\alpha \beta} \partial_{\mu \nu} + x_{ij}^\mu \partial_{\nu \mu} Q + x_{ij}^\alpha \partial_{\alpha \beta} x_{ij}^+ X_{ij}^\alpha \partial_{\alpha \beta} \partial_{\mu \nu} Q.
\] (A6)

Here we have defined \( X_{ij}^\alpha - X_{ij}^\alpha = 2 i \partial_{\alpha \beta} X_{ij}^\alpha \). Next, the expressions given in Eq. (2.23) lead to:

\[
\partial_{\alpha \beta} x_{ij}^+ X_{ij}^\alpha - \partial_{\alpha \beta} x_{ij}^+ X_{ij}^\alpha = 0.
\]

Since the indices \( \alpha, \beta, \mu \) and \( \nu \) are dummy indices we can interchange \( \alpha \to \beta \) and \( \mu \to \nu \) in the terms that multiply.
\[ \partial_{\mu\nu}Q. \text{ Therefore we obtain:} \]

\[ \frac{dQ(x)}{dt} = 2t_{ij} \sum_{ij} X_{ij}^{\alpha\beta} \partial_{\alpha\beta} Q + \]

\[ + \frac{i}{2} g_i \left( \sum_{ij} X_{ij}^{\alpha\beta} X_{ij}^{\mu\nu} \right) \partial_{\alpha\beta} \partial_{\mu\nu} + \]

\[ + \sum_{ij} \left( X_{ij}^{\alpha\beta} \partial_{\mu\nu} X_{ij}^{\alpha\beta} - X_{ij}^{\mu\nu} \partial_{\alpha\beta} X_{ij}^{\alpha\beta} \right) \right] \partial_{\mu\nu} Q. \]  

\[ \text{(A7)} \]

To simplify the results further, we swap \( i \leftrightarrow k \) and \( j \leftrightarrow l \) in the terms \( x_{\alpha\beta} X_{ij}^{\alpha\beta} x_{ij} \) and \( g_{ijkl} X_{ij}^{\mu\nu} \partial_{\mu\nu} X_{ij}^{\alpha\beta} \). There is no sign change in \( g_i \) since it is an even permutation, so we obtain:

\[ \frac{dQ(x)}{dt} = 2t_{ij} \sum_{ij} X_{ij}^{\alpha\beta} \partial_{\alpha\beta} Q + \]

\[ + \frac{i}{2} g_i \left( \sum_{ij} X_{ij}^{\alpha\beta} X_{ij}^{\mu\nu} \right) \partial_{\alpha\beta} \partial_{\mu\nu} + \]

\[ + \sum_{ij} \left( X_{ij}^{\mu\nu} \partial_{\alpha\beta} X_{ij}^{\alpha\beta} - X_{ij}^{\alpha\beta} \partial_{\mu\nu} X_{ij}^{\alpha\beta} \right) \right] \partial_{\mu\nu} Q. \]  

\[ \text{(A8)} \]

Using the derivative properties described in Section [1], as well as the swapping of indices and the permutation properties of \( g_i \), we get:

\[ g_i \left( X_{ij}^{\mu\nu} \partial_{\mu\nu} X_{ij}^{\alpha\beta} - X_{ij}^{\alpha\beta} \partial_{\mu\nu} X_{ij}^{\alpha\beta} \right) = 8i g_i x_{ij}^{\alpha\beta} \sum_{ij} X_{ij}^{\alpha\beta}. \]

Therefore we finally obtain the simplified form of the time evolution equation of the Majorana \( Q \)-function:

\[ \frac{dQ(x)}{dt} = i \left[ \frac{g_i}{2} \left( X_{ij}^{\alpha\beta} X_{ij}^{\mu\nu} - X_{ij}^{\alpha\beta} X_{ij}^{\mu\nu} \right) \partial_{\alpha\beta} \partial_{\mu\nu} + \right. \]

\[ \left. + 2i \sum_{ij} X_{ij}^{\mu\nu} \left( \sum_{ij} X_{ij}^{\alpha\beta} x_{ij}^{\mu\nu} \right) \right] Q. \]  

\[ \text{(A9)} \]

This equation corresponds to Eq. [3.10].

**Appendix B: Identities with independent variables**

Since the \( x \) matrices are real antisymmetric matrices, \( x_{\alpha\beta} \) and \( x_{\beta\alpha} \) are not independent. However, Eq. \[ A9 \] includes all possible values for the indices labeled by Greek letters. This implies, for example, for the term involving second order derivatives, that:

\[ g_i X_{ij}^{\alpha\beta} X_{kl}^{\mu\nu} \partial_{\alpha\beta} \partial_{\mu\nu} Q = \sum_{\alpha < \beta < \mu < \nu} g_i X_{ij}^{\alpha\beta} X_{kl}^{\mu\nu} \partial_{\alpha\beta} \partial_{\mu\nu} Q + \]

\[ + \sum_{\alpha > \beta < \mu < \nu} g_i X_{ij}^{\alpha\beta} X_{kl}^{\mu\nu} \partial_{\alpha\beta} \partial_{\mu\nu} Q + \]

\[ + \sum_{\alpha < \beta < \mu > \nu} g_i X_{ij}^{\alpha\beta} X_{kl}^{\mu\nu} \partial_{\alpha\beta} \partial_{\mu\nu} Q + \]

\[ + \sum_{\alpha > \beta < \mu < \nu} g_i X_{ij}^{\alpha\beta} X_{kl}^{\mu\nu} \partial_{\alpha\beta} \partial_{\mu\nu} Q. \]  

\[ \text{(B1)} \]

In order to write Eq. [3.10], only considering independent variables, the following steps will be used. In the second term of the above expression one can swap the dummy indices \( \alpha \leftrightarrow \beta \) and \( \mu \leftrightarrow \nu \). We can rewrite the derivative term by using: \( x_{\beta\alpha} = -x_{\alpha\beta} \) and \( x_{\mu\nu} = -x_{\nu\mu} \). Latin indices can be swapped as \( i \leftrightarrow j \) and \( j \leftrightarrow l \). This swapping does not change the sign of \( g_i \), since it is an even permutation, so the second term becomes \( \sum_{\alpha < \beta < \mu < \nu} g_i X_{ij}^{\alpha\beta} X_{kl}^{\mu\nu} \partial_{\alpha\beta} \partial_{\mu\nu} Q \). Since \( X_{ij}^{\alpha\beta} = X_{ij}^{\beta\alpha} \), we notice that, after this swapping, the second term and the first term of the above expression are the same.

Following an analogous procedure to that described above, we can perform the corresponding swapping in the third and fourth term, showing that all four terms are equivalent. Hence, the above expression is written using independent variables, as:

\[ g_i X_{ij}^{\alpha\beta} X_{kl}^{\mu\nu} \partial_{\alpha\beta} \partial_{\mu\nu} Q = 4 \sum_{\alpha < \beta < \mu < \nu} g_i X_{ij}^{\alpha\beta} X_{kl}^{\mu\nu} \partial_{\alpha\beta} \partial_{\mu\nu} Q + \]

\[ 4g_i x_{ij}^{\alpha\beta} \sum_{ij} X_{ij}^{\alpha\beta} \partial_{\alpha\beta} Q. \]  

\[ \text{(B2)} \]

In the last line of the above equation we use the notation that bold repeated indices \( \alpha, \mu \) are summed over independent variables with \( \alpha < \beta \) and \( \mu < \nu \). We will also make a restricted summation of the coefficients of \( \partial_{\alpha\beta} Q \), again in order to only include independent variables. Following an analogous procedure, this gives an additional overall factor of 2. We will explain the procedure for the term that corresponds to the linear Hamiltonian, which is \( 2i \sum_{ij} X_{ij}^{\alpha\beta} t_{ij} \partial_{\alpha\beta} Q \). In this case,

\[ 2i \sum_{ij} X_{ij}^{\alpha\beta} t_{ij} \partial_{\alpha\beta} Q = t_{ij} \sum_{\alpha < \beta} 2i \sum_{\alpha < \beta} X_{ij}^{\alpha\beta} \partial_{\alpha\beta} Q + \]

\[ + t_{ij} \sum_{\beta < \alpha} 2i \sum_{\beta < \alpha} X_{ij}^{\alpha\beta} \partial_{\alpha\beta} Q. \]  

\[ \text{(B3)} \]

Next, swapping the Latin indices \( i \leftrightarrow j \), gives \(-2i \sum_{ij} X_{ij}^{\alpha\beta} t_{ij} \partial_{\alpha\beta} Q \). Since \( t_{ij} = -t_{ji} \) and \( X_{ji}^{\alpha\beta} = X_{ij}^{\alpha\beta} \), the second term is identical to the first term of the above expressions. Thus, in terms of independent variables the term corresponding to the linear Hamiltonian is:

\[ 2i \sum_{\alpha < \beta} X_{ij}^{\alpha\beta} t_{ij} \partial_{\alpha\beta} Q = 4it_{ij} \sum_{ij} X_{ij}^{\alpha\beta} \partial_{\alpha\beta} Q. \]  

\[ \text{(B4)} \]

Following this method for the other terms, and using implicit Einstein summation over \( \alpha \) and \( \mu \) in terms of only independent variables with \( \alpha < \beta \) and \( \mu < \nu \), Eq. \[ A9 \] becomes:

\[ \frac{dQ(x)}{dt} = [4 \sum_{ij} X_{ij}^{\alpha\beta} (t_{ij} - 3g_i x_{ij}^{\mu\nu}) \partial_{\alpha\beta} Q + \]

\[ + 2g_i (X_{ij}^{\alpha\beta} X_{kl}^{\mu\nu} - X_{kl}^{\alpha\beta} X_{ij}^{\mu\nu}) \partial_{\alpha\beta} Q]. \]  

\[ \text{(B5)} \]

This equation corresponds to Eq. [3.12].

On performing exchanges of indices, the following symmetry properties were used:
Symmetry of $x$ matrices: $x_{\alpha\beta} = -x_{\beta\alpha}$.

Symmetry of $g_i$: One can always exchange the Latin indices $i, j, k$ and $l$ while taking into account the number of permutations between indices.

We now simplify the term $g_i \left( X_{ij}^{\alpha\beta} X_{kl}^{\mu\nu} - X_{ij}^{\alpha\beta} X_{kl}^{\alpha\beta}\ast \right) \partial_\alpha \partial_\mu Q$. To do this, the symmetry properties above are used, together with:

- Swapping on restricted summations: One can always exchange $\alpha$ with $\mu$ and $\beta$ with $\nu$ simultaneously without any sign change. The second order derivative of the $Q$-function does not change, as $\partial_\alpha \partial_\mu Q = \partial_\mu \partial_\alpha Q$.

Using these symmetry properties, we obtain $g_i X_{ij}^{\alpha\beta} X_{kl}^{\mu\nu} = g_i \left( X_{ij}^{\ast \alpha\beta} X_{kl}^{\mu\nu} \right)$, and also that:

$$X_{ij}^{\alpha\beta} X_{kl}^{\mu\nu} - X_{ij}^{\alpha\beta} X_{kl}^{\alpha\beta}\ast = X_{ij}^{\ast \alpha\beta} X_{kl}^{\mu\nu} - \left( X_{ij}^{\alpha\beta} X_{kl}^{\mu\nu} \right)^\ast$$

$$= 2i \Im \left( X_{ij}^{\ast \alpha\beta} X_{kl}^{\mu\nu} \right)$$

$$= 2i \left( \Re X_{ij}^{\alpha\beta} \Im X_{kl}^{\mu\nu} + \Im X_{ij}^{\alpha\beta} \Re X_{kl}^{\mu\nu} \right).$$

(12)

Therefore, we see that:

$$2ig_i \left( X_{ij}^{\alpha\beta} X_{kl}^{\mu\nu} - X_{ij}^{\alpha\beta}\ast X_{kl}^{\mu\nu}\ast \right)$$

$$= -4g_i \left( \Re X_{ij}^{\alpha\beta} \Im X_{kl}^{\mu\nu} + \Im X_{ij}^{\alpha\beta} \Re X_{kl}^{\mu\nu} \right).$$

(13)

We now exchange $i \leftrightarrow k$ and $j \leftrightarrow l$, in the term $g_i \Re X_{ij}^{\alpha\beta} \Re X_{kl}^{\mu\nu}$, there is no sign change in $g_i$, since it is an even permutation, then $g_i \Re X_{ij}^{\alpha\beta} \Re X_{kl}^{\mu\nu} = g_i \Re X_{kl}^{\alpha\beta} \Re X_{ij}^{\mu\nu}$.

Next we swap the dummy indices $\alpha \leftrightarrow \mu$ and $\beta \leftrightarrow \nu$, obtaining $\Re X_{ij}^{\alpha\beta} \Re X_{kl}^{\mu\nu} = \Re X_{ij}^{\mu\nu} \Re X_{kl}^{\alpha\beta}$. Therefore we get:

$$2ig_i \left( X_{ij}^{\alpha\beta} X_{kl}^{\mu\nu} - X_{ij}^{\alpha\beta}\ast X_{kl}^{\mu\nu}\ast \right) = -8g_i \Re X_{ij}^{\alpha\beta} \Im X_{kl}^{\mu\nu}. \tag{14}$$

In terms of the phase-space variables $x_{\alpha\beta}$, the term $\Re X_{ij}^{\alpha\beta} \Im X_{kl}^{\mu\nu}$ is:

$$\Re X_{ij}^{\alpha\beta} \Im X_{kl}^{\mu\nu} = (x_{\alpha\beta} x_{\beta\alpha} + \delta_{\alpha\beta} \delta_{\alpha\beta} ) ( -x_{\alpha\beta} \delta_{\alpha\beta} + \delta_{\alpha\beta} x_{\alpha\beta} ). \tag{15}$$

This concludes the derivation of the generalized Fokker-Planck equation given in the main text.

**Appendix C: Derivatives**

In this section we give details of the calculations of the first order derivatives of the diffusion and drift term, as well as the second order derivatives of the diffusion term of the Fokker-Planck equation for the Majorana $Q$-function.
Using the product rule of the derivative we prove
\[ \frac{\partial^2 X_{ij}^{\alpha\beta}}{\partial x_{\alpha\beta}} = 0. \]
Implementation of this gives,
\[ \partial_\alpha \left( x_{kl} \overline{X}_{ij}^{\alpha\beta} \right) = (\delta_{k\alpha} \delta_{\beta j} - \delta_{k\beta} \delta_{\alpha i}) \overline{X}_{ij}^{\alpha\beta} \]
\[ = (3X_{ij}^{kl} - 3X_{ij}^{lk}) \]
\[ = 0. \]  
(C6)
This last result follows since we are considering that \( i \neq j \neq k \neq l \), therefore:
\[ \overline{3X}_{ij}^{kl} = -(x_{ik} \delta_{\beta j} - \delta_{ik} x_{ij}) = 0. \]  
(C7)
All other permutations of indices \( i, j, k \) and \( l \) for \( X_{ij}^{kl} \), gives the same result. Therefore we have proved that:
\[ \partial_\alpha \partial_\mu D^{\alpha\mu} = - \frac{32}{\hbar} (3 - 2M) \frac{\partial}{\partial x_{\alpha\beta}} x_{kl} \overline{3X}_{ij}^{\alpha\beta} \]
\[ = 0. \]  
(C8)

2. Derivatives of the drift term

In order to calculate the first order derivative of the drift term \( A^\alpha = A^{(\alpha\beta)} \), we consider the expression given in Eq. (4.3) as well as Eq. (8.14). So
\[ \partial_\alpha A^\alpha = \partial_\alpha \overline{A}^\alpha + \partial_\alpha \partial_\mu D^{\alpha\mu}. \]  
(C9)
As the second order derivative of the diffusion term is zero from Eq. (C6), the second term of the above expression is zero. Utilizing the result Eq. (2.23), one can obtain that:
\[ \partial_\alpha \overline{A}^\alpha = 4\partial_\alpha \left( \overline{3X}_{ij}^{\alpha\beta} \left( 3g_{ij} x_{kl}^{+} - t_{ij} \right) \right) \]
\[ = 12\partial_\alpha \left( \overline{3X}_{ij}^{\alpha\beta} g_{ij} x_{kl}^{+} \right) - 4\partial_\alpha \left( \overline{3X}_{ij}^{\alpha\beta} t_{ij} \right). \]  
(C10)
From Eq. (C6) and Eq. (2.23) one know that
\[ \partial_\alpha \left( x_{kl}^{+} \overline{3X}_{ij}^{\alpha\beta} \right) = 0 \]  and
\[ \partial_\alpha \overline{3X}_{ij}^{\alpha\beta} = 0, \]  so one can obtain that the first order derivative of the drift term is zero.
\[ \frac{\partial A^{(\alpha\beta)}}{\partial x_{\alpha\beta}} = 0. \]  
(C11)
This gives the result, in a more compact form, that
\[ \partial_\alpha A^\alpha = 0. \]
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