Local Deterministic Transformations of Three-Qubit Pure States

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The properties of deterministic LOCC transformations of three qubit pure states are studied. We show that the set of states in the GHZ class breaks into an infinite number of disjoint classes under this transformation. These classes are characterized by the value of a quantity that is invariant under these transformations, and is defined in terms of the coefficients of a particular canonical form in which only states in the GHZ class can be expressed. This invariant also imposes a strong constraint on any POVM that is part of a deterministic protocol. We also consider a transformation generated by a local 2-outcome POVM and study under what conditions it is deterministic, i.e., both outcomes belong to the same orbit. We prove that for real states it is always possible to find such a POVM and we discuss analytical and numerical evidence that suggests that this result also holds for complex states. We study the transformation generated in the space of orbits when one or more parties apply several deterministic POVMs in succession and use these results to give a complete characterization of the real states that can be obtained from the GHZ state with probability 1.

I. INTRODUCTION

A very important part of the study of the entangled states of spatially separated systems, is the study of the transformations that are possible when using only local operations and classical communication (LOCC), since it allows us to classify entangled states and it can be used as one way of quantifying this resource. Two states that are related by local unitary transformations are considered equivalent as far as entanglement is concerned, since both states can be obtained from each other and local operations cannot increase entanglement. The action of the group of local unitaries breaks the space of states into orbits [1]. Then, to transform a pure state into another state in a different orbit by local operations, we need to allow each party to apply a local generalized measurement, i.e., a POVM, on her part of the state.

For bipartite pure states, the problem of deterministically transforming a state into another has been solved by Nielsen [5], who gave necessary and sufficient conditions for a given transformation to be achievable with probability 1. Later Vidal [6] extended this result by calculating the maximal probability of success of any LOCC transformation of bipartite pure states. For more than two parties, this problem is still unsolved. The bipartite case seems to be very special due to the existence of the Schmidt decomposition. Any pure bipartite state can be transformed by applying local unitaries into a state of the form

$$|\psi\rangle = \sum_{i} \lambda_i |ii\rangle,$$

where the $\lambda_i$ are positive real numbers, $|ii\rangle = |i\rangle_A \otimes |i\rangle_B$ and $\{|i\rangle\}$ are orthonormal vectors on each subsystem. This greatly simplifies the analysis of LOCC transformations: it gives a canonical expression for states in a given orbit, and allows the reduction of an arbitrary LOCC protocol to a protocol in which one party applies local unitaries and local POVMs, and the other party only has to apply a local unitary, conditional on the results obtained by the first party [6]. For multipartite states with three parties or more there is no known reduction of LOCC protocols.

For a system of three qubits, several Schmidt-like decompositions have been proposed [5-7], all based on the idea of using local unitaries to get rid of as many coefficients as possible. One interesting property that emerges from these decompositions is that in general it is not possible to make all the coefficients real. In particular there are states that have at least one coefficient that is complex for any local basis, and this has as a consequence that these states are not locally unitarily equivalent to their conjugates (the states obtained by taking the complex conjugate of the coefficients). This contrasts with the bipartite case in which, since the Schmidt decomposition has only real coefficients, every state is in the same orbit as its conjugate.

A POVM applied to a state has, in general, outcomes that belong to different orbits. However, a protocol that transforms a state into another with probability 1, has to include at least one POVM for which all outcomes are in the same orbit. For instance, this has to be the case for the last POVM of the protocol: if its outcomes are not in the same orbit, then the protocol has not achieved the transformation with probability 1. We will call a POVM with this property a deterministic POVM, because we can use such a POVM and suitable local unitaries, to obtain any state in the orbit of the outcomes with probability 1, attaining a deterministic transformation. Since any local POVM can be replaced by a sequence of 2-outcome POVMs, it is then interesting to study the case of a deterministic 2-outcome POVM.

In this paper we will study some properties of deterministic LOCC protocols and deterministic POVMs applied to 3-qubit pure states. We will only be interested in transformations between states that have genuine tri-
partite entanglement (i.e., all three reduced density matrices have rank 2), since other cases can be reduced to the bipartite case. In Section II, we prove that a certain function of the states is invariant under any deterministic LOCC protocol and show that this imposes a constraint on the local POVMs that can be a part of a deterministic transformation. We also show that the set of states in the GHZ class breaks into an infinite number of disjoint subclasses under this particular type of transformation. In Section III, we study the particular case of a 2-outcome deterministic POVM, and discuss what are the conditions for its existence. We prove that such a POVM can always be found for real states, and present some evidence that the same situation holds for complex states. In Section IV we analyze the transformation in the space of orbits. In Section V we study the case of the GHZ state and give a complete characterization of all the states with real coefficients that can be obtained deterministically from it. Finally, the conclusions are presented in Section VI.

II. GENERAL PROPERTIES OF LOCC TRANSFORMATIONS OF 3-QUBIT STATES

Pure states of three qubits with 3-particle entanglement are divided in two inequivalent classes: the GHZ class and the W class. They have the property that any local POVM applied to a state in a given class, can only have as outcomes states in the same class. In particular, states in the W class can always be transformed by local unitary operations, into a state with real coefficients. In this paper we will call a state “real” if it is locally unitarily equivalent (LUeq) to a state with real coefficients. States in the GHZ class can be either real or complex.

Any state in the GHZ class is LUeq to a state of the form

\[
|\psi\rangle = \mu|000\rangle + \nu e^{i\gamma}|\varphi_A\rangle|\varphi_B\rangle|\varphi_C\rangle,
\]

where \(\mu \geq \nu > 0\) are real numbers, \(\gamma \in [0, 2\pi]\) and \(|\varphi_X\rangle = \cos \delta_X |0\rangle + \sin \delta_X |1\rangle\) with \(\delta_X \in (0, \frac{\pi}{2})\) and \(X = A, B, C\). We will assume that the state \(|\psi\rangle\) is normalized, so only five of the six parameters in (2) are independent. If we write \(|\psi\rangle = |\mu\rangle + |\nu\rangle\) where \(|\mu\rangle\) and \(|\nu\rangle\) correspond to the first and second term in (2) respectively, we can construct the invariant

\[
\Omega(|\psi\rangle) = \langle\mu|\nu\rangle = \nu e^{i\gamma} \cos \delta_A \cos \delta_B \cos \delta_C.
\]

If \(\mu = \nu\), the sign of the phase \(\gamma\) is not well defined, since in this case there is an ambiguity with respect to which product state in (2) is \(|\mu\rangle\), and hence we can interchange \(|\mu\rangle\) and \(|\nu\rangle\) by local unitaries, and transform the state into its conjugate, which changes the sign of \(\gamma\). As shown in (2) this means that the state is real, although we need to use complex coefficients if we want to write it in the particular form given by (2). Aside from this ambiguity, this decomposition is unique. If \(\mu > \nu\) then the state \(|\psi\rangle\) is complex if and only if \(\text{Im}(\Omega(|\psi\rangle)) \neq 0\). If \(\text{Im}(\Omega(|\psi\rangle)) = 0\), then either \(\gamma\) is equal to \(0\) or \(\pi\) (and in both cases all the coefficients are real, so the state is real), or \(\delta_X = \frac{\pi}{2}\) for some \(X\). If this is the case, we can get rid of the phase by applying the local unitary

\[
U = \begin{pmatrix} 1 & 0 \\ 0 & e^{-i\gamma} \end{pmatrix},
\]
to party \(X\), which makes all the coefficients real.

Let \(\{A_i\}, i = 1, \ldots, n\) represent a local POVM applied by Alice. If we apply it to a state \(|\psi\rangle\) we can write the normalized outcomes as \(|\phi_j\rangle = q_j^{-\frac{1}{2}} A_i \otimes 1 \otimes 1 |\psi\rangle\), where \(q_j = \langle\psi|A_i^\dagger A_i \otimes 1 \otimes 1 |\psi\rangle\) is the probability of outcome \(i\). Let’s consider the case in which none of the operators \(A_i\) corresponds to a projective measurement (i.e., they all have rank 2). If we apply this POVM to a state with genuine tripartite entanglement, all the outcomes will still have 3-particle entanglement. To understand why this is true, suppose that there is an operator \(A_j\) of the POVM such that its corresponding outcome \(|\phi_j\rangle\) has no 3-particle entanglement. Then \(|\phi_j\rangle\) has to be the product of a pure state of one of the parties, let’s call it \(X\), and a pure state (possibly entangled) of the remaining two parties, so party \(X\) is completely disentangled from the other two. Since we are assuming that \(A_j\) is invertible (it is a rank two, 2 by 2 matrix), we can construct a local POVM with operators \(\{A_j^{-1} \sqrt{1 - (A_j^{-1})^\dagger A_j^{-1}}\}\) that when applied to \(|\phi_j\rangle\) has at least one outcome that has 3-particle entanglement (the one corresponding to \((A_j^{-1})^\dagger A_j^{-1}\) also has rank two). Then we would have a protocol that with finite probability and only applying local operations, allows us to create entanglement between party \(X\) and the other two, starting from a state in which party \(X\) was disentangled, and this is clearly not possible.

Let’s consider a state \(|\psi\rangle\) in the GHZ class and let Alice apply a local POVM to it. Then all the outcomes \(|\phi_i\rangle\) have to be in the GHZ class too, so we know that we can apply local unitaries to them such that

\[
(U_{A(i)} \otimes U_{B(i)} \otimes U_{C(i)})|\phi_i\rangle = |\mu_i\rangle + |\nu_i\rangle,
\]

where

\[
|\mu_i\rangle = \mu_i |000\rangle,
|\nu_i\rangle = \nu_i e^{i\gamma} |\varphi_{A(i)}\rangle |\varphi_{B(i)}\rangle |\varphi_{C(i)}\rangle.
\]

Since \(|\mu_i\rangle\), \(|\nu_i\rangle\) and \(|\mu\rangle\), \(|\nu\rangle\) are product states, and the action of the POVM and any local unitaries is still local, for every outcome \(i\) we must have either

\[
\sqrt{q_i}|\mu_i\rangle = (U_{A(i)} \otimes U_{B(i)} \otimes U_{C(i)}) (A_i \otimes 1 \otimes 1)|\mu\rangle,
\]

\[
\sqrt{q_i}|\nu_i\rangle = (U_{A(i)} \otimes U_{B(i)} \otimes U_{C(i)}) (A_i \otimes 1 \otimes 1)|\nu\rangle,
\]
or
\[\sqrt{q_i} |\mu_i\rangle = (U_{A(i)} \otimes U_{B(i)} \otimes U_{C(i)})(|A_i \otimes 1 \otimes 1\rangle |\nu\rangle) \]
\[\sqrt{q_i} |\nu_i\rangle = (U_{A(i)} \otimes U_{B(i)} \otimes U_{C(i)})(|A_i \otimes 1 \otimes 1\rangle |\mu\rangle). \quad (8)\]

To decide which one is the case, we note that decomposition (2) requires that \(\mu_i \geq \nu_i\), and \(\mu_i, \nu_i\) are the norms of the states \(|\mu_i\rangle\) and \(|\nu_i\rangle\) respectively. Then, if \(\langle \mu | A_i^\dagger A_i \otimes 1 \otimes 1 |\nu\rangle \geq \langle \nu | A_i^\dagger A_i \otimes 1 \otimes 1 |\mu\rangle\) (which is equivalent to \(\mu_i |\mu_i\rangle \geq \nu_i |\nu_i\rangle\)), we have that (8) must hold. Otherwise, (8) holds. Using \(\sum_i A_i^\dagger A_i = 1\), we can then write

\[\text{Re}(\Omega(|\psi\rangle)) = \langle \mu |\nu\rangle + \langle \nu |\mu\rangle = \sum_i q_i \text{Re}(\Omega(|\phi_i\rangle)). \quad (9)\]

This result is due to Vidal [9]. It puts a strong constraint on deterministic LOCC protocols, as we show in the following theorem.

**Theorem 1** Let \(|\psi\rangle\) and \(|\xi\rangle\) be two states in the GHZ class and assume there is a LOCC protocol that transforms \(|\psi\rangle\) into \(|\xi\rangle\) with probability 1. Then,

\[\text{Re}(\Omega(|\psi\rangle)) = \text{Re}(\Omega(|\xi\rangle)), \quad (10)\]

i.e., the quantity \(\text{Re}(\Omega)\) is invariant under deterministic LOCC transformations. Furthermore, it must be invariant for every local POVM in the protocol, that is, if the POVM is applied to a state \(|\chi\rangle\) and has outcomes \(|\phi_i\rangle\), then

\[\text{Re}(\Omega(|\chi\rangle)) = \text{Re}(\Omega(|\phi_i\rangle)), \quad (11)\]

for all \(i\).

**Proof:** The most general LOCC protocol is a sequence of local unitaries, local POVMs and classical communication between all the parties. Local unitaries cannot change \(\text{Re}(\Omega)\) because \(\Omega(|\psi\rangle)\) is an invariant of the orbit. Thus, it can only be changed by applying POVMs. Consider the first POVM of the protocol, that takes the state \(|\psi\rangle\) into one of its possible outcomes \(|\phi_i\rangle\), each occurring with probability \(q_i\). Then, according to equation (2) (and because \(q_i > 0\), either all outcomes \(|\phi_i\rangle\) satisfy \(\text{Re}(\Omega(|\phi_i\rangle)) = \text{Re}(\Omega(|\psi\rangle))\) or there are at least two outcomes \(|\phi_1\rangle\) and \(|\phi_2\rangle\) that satisfy \(\text{Re}(\Omega(|\phi_1\rangle)) < \text{Re}(\Omega(|\phi_2\rangle)) < \text{Re}(\Omega(|\psi\rangle))\). It is easy to see that in the latter case, at any stage in the protocol, we will have two outcomes \(|\phi_j\rangle\) and \(|\phi_k\rangle\) that satisfy \(\text{Re}(\Omega(|\phi_1\rangle)) < \text{Re}(\Omega(|\phi_2\rangle)) < \text{Re}(\Omega(|\phi_3\rangle))\). This will be true in particular for the last stage of the protocol. But that would mean that \(|\phi_j\rangle\) and \(|\phi_k\rangle\) are in different orbits (because \(\Omega\) is invariant under local unitaries), and that contradicts the fact that the protocol is deterministic. Thus, the only possibility is that all the outcomes of the first POVM have the same value of \(\text{Re}(\Omega)\). We can apply exactly the same reasoning to all the POVMs in the protocol, and then conclude that all the final outcomes satisfy \(\text{Re}(\Omega(|\phi_i\rangle)) = \text{Re}(\Omega(|\psi\rangle))\). Since this is a deterministic protocol that transforms \(|\psi\rangle\) into \(|\xi\rangle\), then all these outcomes should be in the same orbit as \(|\xi\rangle\), and so we have \(\text{Re}(\Omega(|\xi\rangle)) = \text{Re}(\Omega(|\phi_i\rangle)) = \text{Re}(\Omega(|\psi\rangle))\).

This theorem tells us that under deterministic LOCC transformations the class of GHZ states breaks into an infinite number of subclasses that are labeled by the real part of the complex invariant \(\Omega\). Two states in different subclasses cannot be transformed one into the other with probability 1 by means of local operations and classical communication. From equation (3) and from the range of the parameters, we see that the set of these subclasses is isomorphic to the open segment \((-\frac{1}{2}, \frac{1}{2})\). The subclass that contains the GHZ state \(|\text{GHZ}\rangle = \frac{1}{\sqrt{2}}(|000\rangle + |111\rangle)\), corresponds to the center of the segment, and it is defined by \(\text{Re}(\Omega) = 0\). Note that all subclasses contain both real and complex states.

This result gives a broad description of how a state can be transformed in the space of orbits with probability 1. Tighter constraints can be obtained from studying the behavior of the entanglement monotones (10), which usually introduce some necessary conditions that must be satisfied in order for a transformation to be possible to be implemented locally. To find sufficient conditions we have to be able to show that a protocol exists that accomplishes the transformation. A first step in that direction is to study deterministic POVMs.

**III. DETERMINISTIC 2-OUTCOME POVM**

In this section we will study under what conditions a 2-outcome POVM is a deterministic POVM (i.e., both outcomes are in the same orbit). A general 3-qubit state can be written

\[|\psi\rangle = \frac{1}{\sqrt{2}} \sum_{ijk} t_{ijk} |ijk\rangle. \quad (12)\]

Following [8], we can define matrices \(T_0\) and \(T_1\), where

\[(T_i)_{jk} = t_{ijk}. \quad (13)\]

The group of Local Unitary (LU) transformations of three qubits is locally isomorphic (i.e., has the same Lie algebra) to \(U(1) \times [SU(2)]^3\). Under a LU transformation performed only by Bob and Charlie with matrices \(U_B\) and \(U_C\), the matrices \(T\) transform according to

\[T_i \rightarrow U_B T_0 U_C, \quad (14)\]

while if the transformation is performed by Alice, we have

\[T_0 \rightarrow u_{00}^A T_0 + u_{01}^A T_1, \quad T_1 \rightarrow u_{10}^A T_0 + u_{11}^A T_1, \quad (15)\]

where \(u_{ij}^A\) are the matrix elements of \(U_A\).
We know that the orbits of 3-qubit states can be parametrized with 5 continuous invariants plus a discrete invariant, since in general a 3-qubit state is not LUeq to its complex conjugate. There are many ways of choosing these invariants. In this paper we will use the following set

\[ I_1 = \sum_{ijkmpq} t_{ikj} t_{mij} t_{mpq} t_{pqk} = tr \rho_A^2 \]

\[ I_2 = \sum_{ijkmpq} t_{ikj} t_{mij} t_{mpq} t_{pqk} = tr \rho_B^2 \]

\[ I_3 = \sum_{ijkmpq} t_{ikj} t_{mij} t_{mpq} t_{pqk} = tr \rho_C^2 \]

\[ I_4 = \sum_{ijklmnopqrst} t_{ijklmnopqrst} \epsilon_{i1} \epsilon_{j2} \epsilon_{k3} \epsilon_{l4} \epsilon_{m5} \epsilon_{n6} \epsilon_{o7} \epsilon_{p8} \epsilon_{q9} \epsilon_{r10} \epsilon_{s11} \epsilon_{t12} | \]

\[ I_5 = \sum_{ijklmnopqrst} t_{ijklmnopqrst} n_{pqrst} n_{pqmn} n_{pqmk} n_{pqkn} \]

(16)

where \( \epsilon_{ij} \) is the antisymmetric symbol and all the indices are summed from 0 to 1. \( I_4 \) is the 3-tangle introduced in [11]. As shown in [12] these 5 invariants are algebraically independent. However, since they are all real and invariant under complex conjugation of the coefficients \( t_{ijkl} \), they cannot distinguish between a state and its conjugate. To fix this ambiguity we use the complex invariant [13]

\[ I_6 = \sum_{ijkl} t_{i,j,k} t_{i,j,k} t_{i,j,k} t_{i,j,k} t_{i,j,k} t_{i,j,k} \]

(17)

where again all indices are summed from 0 to 1. To completely specify an orbit we need the value of \( I_1 \) through \( I_5 \) plus the sign of the imaginary part of \( I_6 \). It is worth noting that \( I_1 - I_2, I_1 - I_3, I_2 - I_3 \) and \( I_4 \) are decreasing entanglement monotones, while \( I_5 \) is not an entanglement monotone [13].

We will consider the case of a 2-outcome POVM applied by Alice on a pure state \( |\psi\rangle \) of three qubits. The most general POVM is given by the operators \( A_0 \) and \( A_1 \), where

\[ A_0 = V_0 \left( \begin{array}{cc} \sqrt{x} & 0 \\ 0 & \sqrt{y} \end{array} \right) U \]

\[ A_1 = V_1 \left( \begin{array}{cc} \sqrt{1-x} & 0 \\ 0 & \sqrt{1-y} \end{array} \right) U, \]

(18)

where \( V_0, V_1 \) and \( U \) are unitary matrices [16], and \( 0 \leq x, y \leq 1 \). It is easy to see that they satisfy \( A_i^* A_0 + A_0^* A_1 = 1 \), where 1 is the identity matrix. When we apply this POVM to a state \( |\psi\rangle \), we obtain two outcomes \( |\phi_0\rangle \) and \( |\phi_1\rangle \) given by

\[ |\phi_i\rangle = \frac{1}{\sqrt{q_i}} (A_i \otimes 1 \otimes 1 |\psi\rangle) \quad i = 0, 1, \]

(19)

where \( q_i \) is the probability of outcome \( i \). From [13] and [14] we can see that the action of this POVM on \( |\psi\rangle \) is equivalent to applying a unitary transformation first given by \( U \), applying a diagonal and real POVM and finally applying a unitary \( V_i \) conditional on the outcome of the POVM. This last local unitary cannot change the order of the orbit \( |\phi_i\rangle \). Since we are considering two states in the same orbit to be equivalent, we can take this unitary to be the identity without loss of generality.

Let us consider first the case in which \( U = 1 \). Then both elements of the POVM reduce to real and diagonal matrices

\[ E_0 = \left( \begin{array}{cc} \sqrt{x} & 0 \\ 0 & \sqrt{y} \end{array} \right), \quad E_1 = \left( \begin{array}{cc} \sqrt{1-x} & 0 \\ 0 & \sqrt{1-y} \end{array} \right) \].

(20)

From now on, we will take \( 0 < x, y < 1 \), since when \( x \) or \( y \) are equal to zero or one, the POVM becomes a projective measurement, which destroys three particle entanglement. We can write explicit expressions for both outcomes of the POVM

\[ |\phi_0\rangle = \frac{1}{\sqrt{q_0}} \sum_k (\sqrt{x} t_{0jk} |0jk\rangle + \sqrt{y} t_{1jk}|1jk\rangle) \]

\[ |\phi_1\rangle = \frac{1}{\sqrt{q_1}} \sum_k (\sqrt{1-x} t_{0jk} |0jk\rangle + \sqrt{1-y} t_{1jk}|1jk\rangle). \]

(21)

Now we calculate the invariants \( I_1 \) through \( I_5 \) for \( |\phi_0\rangle \) as a function of \( x \) and \( y \)

\[ I_1(x, y) = \frac{x^2 a^2 + 2 x y T r[T_0 T_1^\dagger]}{(a + b y)^2} + y^2 b^2 \]

\[ I_2(x, y) = \frac{x^2 F_0 + 2 x y T r[T_0 T_1^\dagger] T r[T_1^\dagger T_0^\dagger]}{(a + b y)^2} + y^2 F_1 \]

\[ I_3(x, y) = \frac{x^2 F_0 + 2 x y T r[T_0 T_1^\dagger] T r[T_1^\dagger T_0^\dagger]}{(a + b y)^2} + y^2 F_1 \]

\[ I_4(x, y) = \frac{x y I_4(|\psi\rangle)}{(a + b y)^2} \]

\[ I_5(x, y) = \frac{x^3 G_0 + 3 x^2 y G_0 + 3 x y^2 G_10 + y^3 G_11}{(a + b y)^3} \]

(22)

where the matrices \( T_i \) are as defined in [13], \( a = T r[T_0 T_0^\dagger], b = T r[T_1 T_1^\dagger], a + b = 1 \) for a normalized \( |\psi\rangle \), \( F_i = T r[(T_i T_i^\dagger)^2] \) and \( G_{ij} = T r[T_i T_j T_i^\dagger T_j^\dagger] \).

The invariants for \( |\phi_1\rangle \) are obtained from (22) by replacing \( x \) by \( 1-x \) and \( y \) by \( 1-y \). For the two outcomes to be in the same orbit, we need the five invariants to take the same values for both states, i.e.,

\[ I_i(x, y) = I_i(1-x, 1-y) \quad i = 1, \ldots, 5. \]

(23)

If these conditions are satisfied, then either \( |\phi_0\rangle \) is LUeq to \( |\phi_1\rangle \), or \( |\phi_0\rangle \) is LUeq to \( |\phi_1\rangle^* \). To determine which one is the case, we need to calculate the sign of the imaginary part of the complex invariant \( I_6 \). For now, let us concentrate on the equations in (23). These equations
have a common solution with $0 < x, y < 1$ if and only if the following conditions are satisfied (see appendix)
\[
a^2 \text{Tr}[|T_1 T_0^\dagger|^2] = b^2 \text{Tr}[|T_0 T_0^\dagger|^2], \quad a \text{Tr}[T_0 T_0^\dagger T_1^\dagger T_1] = b \text{Tr}[T_0 T_1^\dagger T_0^\dagger T_1^\dagger T_0^\dagger T_0]
\]
\[a^2 x(1 - x) = b^2 y(1 - y).
\] (26)

Furthermore, the solution satisfies $I_5(\{\phi_1\}) < I_5(\{\psi\})$. This is worth noting because $I_5$ is not an entanglement monotone, but behaves monotonically under this particular class of POVMs. Equations (24) and (25) are real valued polynomial constraints on the coefficients of the state, and in general are not satisfied for an arbitrary state. From (24) and (25), we can see that these constraints are invariant under LU transformations applied by Bob and Charlie, while they are not invariant under local unitaries by Alice. Equation (25) is a constraint on the parameters of the POVM that depends on the state we are transforming.

Now let $U$ be any unitary matrix, so our POVM takes the form $\{E_0 U, E_1 U\}$, with $E_0, E_1$ given by (20). This is equivalent to applying a local unitary $U$ to Alice’s part of the state, followed by a diagonal POVM, and we know the conditions that need to be satisfied in this last stage. So we can reduce the problem to finding a local unitary performed by Alice that would transform the original state $\psi$ into a state that satisfies (24) and (25). Then we can choose a POVM that satisfies (25), where now $a$ and $b$ are calculated using the coefficients of the transformed state $U \otimes 1 \otimes 1 |\psi\rangle$. We will consider the cases of real and complex states separately.

### A. Real States

To characterize the orbit of a real state $|\psi\rangle$ we only need four parameters instead of the five needed for an arbitrary state. First, note that, by our definition, any real state can be transformed by means of local unitary transformations, into a state with only real coefficients. Of the (at most) eight coefficients of this state, only seven are independent if we are considering a normalized state, and we can get rid of three more by applying local real unitary (orthogonal) transformations on each of the 3 qubits. Since $I_i, i = 1, \ldots, 4$ are algebraically independent, we can use this set to parametrize the orbits of real states. This greatly simplifies our analysis because, as seen in the appendix, (24) is enough to assure that $I_i, i = 1, \ldots, 4$ have the same values for both outcomes of our POVM. So, given a real state, we need to find a $U$ such that $|\psi\rangle = U \otimes 1 \otimes 1 |\psi\rangle$ satisfies (24). Let

\[U(\alpha) = \begin{pmatrix}
\cos \alpha & \sin \alpha \\
-\sin \alpha & \cos \alpha
\end{pmatrix}.
\] (27)

In terms of the matrices $T_i$, this transformation can be written

\[T_0' = \cos \alpha T_0 + \sin \alpha T_1 \quad T_1' = -\sin \alpha T_0 + \cos \alpha T_1.
\] (28)

If we plug this into (24), take out a common factor $\cos^8 \alpha$, introduce the variable $z = \tan \alpha$ and move all terms to one side, we can write (24) as polynomial $p_1(z)$ of degree 8 with real coefficients, of the form

\[p_1(z) = A(1 - z^8) + B(z + z^7) + C(z^2 - z^6) + D(z^2 + z^5) = 0.
\]

If $z_0$ is a real root of $p_1$, then $U(\alpha_0)$, with $\alpha_0 = \arctan(z_0)$ is the unitary matrix we are looking for. Now it’s easy to check that $p_1(1) = -p_1(-1)$, so $p_1$ has at least one real root in $[-1, 1]$, which means that we can always find a unitary $U$, such that $|\psi\rangle = U \otimes 1 \otimes 1 |\psi\rangle$ satisfies (24). Now we can apply to $|\psi\rangle$ a diagonal POVM that satisfies (24), and we are certain that both outcomes have the same values of the four invariants $I_i, i = 1, \ldots, 4$. But in the case of real states this is enough to completely specify the orbit, because since $|\psi\rangle$ is real, so is $|\psi\rangle$ because $U$ was chosen to be real, and the outcomes of the POVM, $|\phi_0\rangle$ and $|\phi_1\rangle$, are also real, because the POVM itself is real. In this case we don’t have to worry about the value of the complex invariant. Finally, since $|\phi_0\rangle$ and $|\phi_1\rangle$ are in the same orbit, we can apply local unitaries to transform them into any state in the same orbit. So the net result of this protocol is to transform any state in the orbit of $|\psi\rangle$ into any state in the orbit of $|\phi_0\rangle$, with probability 1. The results presented so far show that for any real state, there is some set of orbits that can be reached deterministically from that state, although we haven’t yet characterized this set. We will discuss this problem in Section IV.

### B. Complex states

The analysis of the complex states turns out to be more complicated, because now we need to find $U$ such that $|\psi\rangle$ satisfies both (24) and (25). We can write any unitary $U$ as

\[e^{i\phi} \begin{pmatrix} e^{i\beta} & 0 \\ 0 & e^{-i\beta} \end{pmatrix} \begin{pmatrix} \cos \alpha & \sin \alpha \\ -\sin \alpha & \cos \alpha \end{pmatrix} \begin{pmatrix} e^{i\zeta} & 0 \\ 0 & e^{-i\zeta} \end{pmatrix}.
\] (30)

The phase and the matrix on the left will commute with the diagonal matrices of the POVM, so their action is equivalent to applying a local unitary to the outcomes of the POVM. But we know that this action will not change the orbit of the outcome state, so we can fix them to be the identity. So $U$ will take the state $|\psi\rangle$ with matrices $T_0$ and $T_1$ to a state $|\psi\rangle$ with matrices

\[T_0' = \cos \alpha e^{i\zeta} T_0 + \sin \alpha e^{-i\zeta} T_1 \quad T_1' = -\sin \alpha e^{i\zeta} T_0 + \cos \alpha e^{-i\zeta} T_1.
\] (31)

We can substitute (31) into the homogeneous form of (24) and (25), again divide by $\cos^8 \alpha$ and introduce the
variable \( z = \tan \alpha \), so both conditions are expressed as polynomials in \( z \) equal to zero, with real coefficients, of the form

\[
p_i(z) = A_i(1 - z^6) + B_i(z + z^7) + C_i(z^2 - z^6) + D_i(z^3 + z^5) = 0 \quad i = 1, 2,
\]

with the coefficients given by

\[
\begin{align*}
A_i &= a_{0i} \\
B_i &= b_{1i} \cos(2\zeta) + b_{2i} \sin(2\zeta) \\
C_i &= c_{0i} + c_{1i} \cos(4\zeta) + c_{2i} \sin(4\zeta) \\
D_i &= d_{1i} \cos(2\zeta) + d_{2i} \sin(2\zeta) + d_{3i} \cos(6\zeta) + d_{4i} \sin(6\zeta),
\end{align*}
\]

where \( a_{0i}, b_{1i}, c_{ji}, d_{ji} \) are real valued polynomials on the coefficients of \(|\psi\rangle\), whose exact form can be computed from regrouping the terms obtained after substituting \( \frac{1}{\pi} \) into \( (\frac{\pi}{2}) \) and \( (\frac{\pi}{2}) \).

Finding a local unitary performed by Alice on \(|\psi\rangle\) that would yield a state that satisfies \( (\frac{\pi}{2}) \) and \( (\frac{\pi}{2}) \) is equivalent to finding values \( z \) and \( \zeta \) (which parametrize the unitary) such that both polynomials \( p_1 \) and \( p_2 \) vanish. We can think of \( \zeta \) as a parameter for these polynomials, and what we are looking for is a value of \( \zeta \) such that \( p_1 \) and \( p_2 \) have a common real root.

The polynomials \( p_i \) have certain useful symmetries. First of all, because their coefficients are real, complex roots appear in conjugate pairs. Also, because of the First of all, because their coefficients are real, complex roots appear in conjugate pairs.

We can now use the results of the previous section to give a characterization of the states that can be obtained from \(|\psi\rangle\) by applying a 2-outcome deterministic POVM. Let us assume that the state \(|\psi\rangle\) is a gate state. We will also assume that \( a < b \) (if it’s not, we apply a bit flip on Alice’s qubit, which interchanges the matrices \( T_0 \) and \( T_1 \), and hence \( a \) and \( b \)). We can use the invariants evaluated for \(|\phi_0\rangle\) (given by \( (\frac{\pi}{2}) \)) to characterize the orbit of the outcomes. These equations are homogeneous of degree zero in \( x \) and \( y \), so we can write them in terms of only one parameter \( \lambda = \frac{x}{y} \).

\[
I_i(\lambda) = \alpha_i + \beta_i \left( \frac{\lambda}{(a + b\lambda)^2} \right) \quad i = 1, \ldots, 4
\]

where \( r_k \) and \( s_k \) are polynomials on the coefficients of \(|\psi\rangle\).

We can see that this resultant vanishes several times in \([0, 2\pi]\), which is the range of \( \zeta \), and this is useful because the resultant of two polynomials vanishes if and only if they have a common factor. This falls short of saying that we can find \( \zeta \) such that \( g_1 \) and \( g_2 \) have a common real root, because there is in principle the possibility that the common factor is a polynomial of degree 2 irreducible over the real numbers, so \( g_1 \) and \( g_2 \) have a common root but it is complex. However, after checking this procedure with many randomly generated states, we found that the common factor always corresponds to a real root.

Let’s assume that in fact, we can always find a value \( \zeta_0 \) such that \( p_1 \) and \( p_2 \) have a common real root \( z_0 \). Then we know that if we apply \( U(\alpha_0, \zeta_0) \otimes 1 \otimes 1 \) (where \( \alpha_0 = \arctan(z_0) \)) to \(|\psi\rangle\), we obtain a state \(|\psi'\rangle\) that satisfies \( (\frac{\pi}{2}) \) and \( (\frac{\pi}{2}) \). Then, we can choose a POVM that satisfies \( (\frac{\pi}{2}) \), and we can be sure that both outcomes of this POVM, when applied to \(|\psi'\rangle\), will have the same values of \( I_i, i = 1, \ldots, 5 \). However, as we pointed out before, this is still not enough to say that both outcomes are in the same orbit. There’s still the possibility that they are in orbits that are conjugate to each other, since we are dealing with complex states, which are not LUeq to their conjugates. To decide which one is the case, we can calculate the sign of the imaginary part of \( I_6 \) for both outcomes. Unfortunately, the expression of \( I_6 \) for both outcomes is too complicated and it’s not possible to extract the sign of the imaginary part analytically for an arbitrary state, although it is very easy to compute it numerically for a given state. We analysed randomly generated states, and found that we can always find a value of \( \zeta \) for which both outcomes are indeed in the same orbit (although there are other values of \( \zeta \) for which the outcomes are in conjugate orbits). We will refer to states with this property as gate states, since we can use them as a gate to leave one orbit and move to another with probability 1.

**IV. THE TRANSFORMATION IN THE SPACE OF ORBITS**

We can now use the results of the previous section to give a characterization of the states that can be obtained from \(|\psi\rangle\) by applying a 2-outcome deterministic POVM. Let us assume that the state \(|\psi\rangle\) is a gate state. We will also assume that \( a < b \) (if it’s not, we apply a bit flip on Alice’s qubit, which interchanges the matrices \( T_0 \) and \( T_1 \), and hence \( a \) and \( b \)). We can use the invariants evaluated for \(|\phi_0\rangle\) (given by \( (\frac{\pi}{2}) \)) to characterize the orbit of the outcomes. These equations are homogeneous of degree zero in \( x \) and \( y \), so we can write them in terms of only one parameter \( \lambda = \frac{x}{y} \).
\[ I_5(\lambda) = \alpha_5 + \frac{\lambda(\beta_5 + \gamma_5 \lambda)}{(a + b \lambda)^3}, \]  
\[ \alpha_1 = 1, \quad \alpha_2 = \alpha_3 = \frac{\text{Tr}[(T_0 T_0^\dagger)^2]}{a^2}, \quad \alpha_4 = 0 \]

where

\[ \alpha_5 = \frac{\text{Tr}[(T_0 T_0^\dagger)^3]}{a^3} \]
\[ \beta_1 = 2(\text{Tr}[T_0 T_0^\dagger] \text{Tr}[T_1 T_1^\dagger] - ab) \]
\[ \beta_2 = 2(\text{Tr}[T_0 T_0^\dagger] - b \text{Tr}[(T_0 T_0^\dagger)^2]) \]
\[ \beta_3 = 2(\text{Tr}[T_0 T_0^\dagger] - b \text{Tr}[(T_0 T_0^\dagger)^2]) \]
\[ \beta_4 = I_4(\psi) \]
\[ \beta_5 = 3(\text{Tr}[T_0 T_0^\dagger] - b \text{Tr}[(T_0 T_0^\dagger)^3]) \]
\[ \gamma_5 = 3(\text{Tr}[T_1 T_1^\dagger] - b^2 \text{Tr}[(T_0 T_0^\dagger)^3]) \]

The range of \( \lambda \) is \([1, +\infty)\) (when \( a < b \)), with \( \lambda = 1 \) corresponding to no transformation \((E_0 \propto 1)\), so we have \( I_5(\lambda = 1) = I_5(|\psi\rangle) \), and \( \lambda = +\infty \) corresponds to a projective measurement \((y = 1, x = 0)\), that destroys any 3-particle entanglement. From (36) we can see that the set of orbits we can reach from \(|\psi\rangle\) can be described as a one parameter family \(\{I_5(\lambda)\}\) that corresponds to a curve in the space of orbits, that starts at state \(|\psi\rangle\) and ends on a state that has no tripartite entanglement.

It is possible for some orbits to have more than one gate state. The values of the coefficients \([37]\) will be in general different for different gate states. Since these coefficients determine the curve \(\{I_5(\lambda)\}\), we will be able to transform to different sets of orbits depending on which gate state we use. We can also reach a different family of orbits if we let Bob or Charlie apply a deterministic POVM instead of Alice. This is because the matrices \(T_i\), are different for different parties, and so will give in general different gate states.

If we fix the sign of the imaginary part of \(I_6\), we can use the invariants \(\{I_i, i = 1, \ldots, 5\}\) as coordinates for the orbits. All the previous results can be summarized in the following picture. Every point in this space (which represents the orbit of some state \(|\psi\rangle\)), is the starting point of a finite number of curves, each representing a set of orbits that can be obtained from \(|\psi\rangle\) with probability 1 with a local 2-outcome POVM.

More orbits can be reached if several rounds of deterministic POVMs are allowed. The general protocol will be something like this: (i) starting with the state \(|\psi\rangle\), Alice applies a local unitary to transform it into a gate state; (ii) she applies a POVM on her part of the system, that satisfies \([38]\); (iii) according to the outcome she obtains, she communicates to Bob and Charlie the state \(|\psi'\rangle\) they are sharing after the measurement; (iv) they decide which one will apply the next POVM and repeat the same steps, now starting with the state \(|\psi'\rangle\). A simplified pictorial representation of this transformation is given in Figure 1.

The transformation occurs in the 5-dimensional space defined by the invariants \(I_i\), but for simplicity, we represent only two of them \((I_4\text{ and }I_5\)\). We start with a gate state \(|\psi\rangle\) and we apply a deterministic 2-outcome POVM \((\text{with some parameter }\lambda_0)\), that transforms it into state \(|\psi'\rangle\). The line connecting \(|\psi\rangle\) and \(|\psi'\rangle\) represents all the orbits that can be reached from \(|\psi\rangle\) by applying a POVM with parameter \(\lambda\) between 1 and \(\lambda_0\). The dotted lines originating at \(|\psi\rangle\) represent the set of orbits that can be reached from the same orbit, but using a gate state different from \(|\psi\rangle\) (which is in the same orbit as \(|\psi\rangle\) so it’s represented by the same point in the plot). In the actual space of orbits, these curves extend until they reach an orbit that represents a state with no 3-particle entanglement, that corresponds to the point where the POVM becomes a projective measurement \((\text{i.e., }\lambda = +\infty)\). For clarity, we are only plotting the beginning of these curves. After deterministically transforming \(|\psi\rangle\) into \(|\psi'\rangle\), the parties can choose again from several gate states to apply the next POVM. This will determine which party will apply this POVM, because in general, a state is a gate state only for a particular party. In the figure, the full line represents a POVM that transforms \(|\psi''\rangle\) into \(|\psi'''\rangle\), while again, the dotted lines correspond to other possible deterministic transformations that can be applied to \(|\psi'\rangle\). By applying many deterministic POVMs with different parameters, we can reach many different orbits.
V. TRANSFORMATION OF THE STATE

\(|\text{GHZ}\rangle = \frac{1}{\sqrt{2}}(|000\rangle + |111\rangle)\)

As an example of the use of 2-outcome deterministic POVMs, we will now study the particular case of the \(|\text{GHZ}\rangle\) state. As it was mentioned in Section II, this state belongs to the subclass of states that satisfy \(\text{Re}(\Omega) = 0\). We will show that it can be transformed with probability 1, into any real state in that subclass.

First, we need to identify the real states that satisfy \(\text{Re}(\Omega) = 0\). Then clearly we must have that either \(\Omega(|\psi\rangle)\) is zero or pure imaginary. In the former case, this means that \(\langle \mu | \nu \rangle = 0\), and then decomposition \([2]\) takes one of the following forms:

\[
\begin{align*}
\mu |000\rangle + \nu |1\rangle |\varphi\rangle |\varphi'\rangle & \\
\mu |000\rangle + \nu |\varphi\rangle |1\rangle |\varphi'\rangle & \\
\mu |000\rangle + \nu |\varphi\rangle |\varphi'\rangle |1\rangle.
\end{align*}
\]

(38)

If \(\Omega(|\psi\rangle)\) is pure imaginary, then the only case in which \(|\psi\rangle\) is actually a real state is the case in which \(\mu = \nu\), as discussed in Section II. In this case, the state takes the form

\[
\frac{1}{\sqrt{2}}(|000\rangle \pm i|\varphi\rangle|\varphi'\rangle|\varphi''\rangle),
\]

(39)

where none of the states in the second term can be equal to \(|0\rangle\) or \(|1\rangle\) (otherwise it could be transformed into a real state by a local unitary), and we obtain \(\mu = \frac{1}{\sqrt{2}}\) by imposing normalization of the state. The two states in \([36]\) (corresponding to the two possible signs of the second term) are \(\text{LU}eq\) to each other.

Since the GHZ state is symmetric under a permutation of the parties, it is clear that if we find a protocol that transforms it into the first state in \([35]\), then we can also transform it into the other two. In this section we will use the results of Section III to explicitly construct protocols that transform the GHZ state into the state

\[
|\phi\rangle = \mu |000\rangle + \nu |1\rangle |\varphi\rangle |\varphi'\rangle,
\]

(40)

or the state

\[
\frac{1}{\sqrt{2}}(|000\rangle + i|\varphi''\rangle |\varphi\rangle |\varphi'\rangle),
\]

(41)

for all allowed values of \(\mu, \nu, |\varphi\rangle, |\varphi'\rangle\) and \(|\varphi''\rangle\). These protocols will be divided into three steps. First, Charlie applies a local deterministic POVM that transforms \(|\text{GHZ}\rangle\) into \(\frac{1}{\sqrt{2}}(|000\rangle + |111\rangle |\varphi'\rangle)\). Then, Bob applies another local POVM that takes the state to \(\frac{1}{\sqrt{2}}((000) + |1\rangle |\varphi\rangle |\varphi'\rangle)\). Finally, Alice applies the last POVM, which she can choose to take the state to \(\mu |000\rangle + \nu |1\rangle |\varphi\rangle |\varphi'\rangle\) or \(\frac{1}{\sqrt{2}}((000) + i|\varphi''\rangle |\varphi\rangle |\varphi'\rangle)\).

Step 1. The \(T_i\) matrices for the GHZ state are given by

\[
T_0 = \left(\begin{array}{cc}
\frac{1}{\sqrt{2}} & 0 \\
0 & 0
\end{array}\right), \quad T_1 = \left(\begin{array}{cc}
0 & 0 \\
0 & \frac{1}{\sqrt{2}}
\end{array}\right),
\]

(42)

and they have the same form for all parties. If Charlie applies a local unitary \(U\) on its qubit, where

\[
U = \frac{\sqrt{2}}{2} \left(\begin{array}{cc}
1 & 1 \\
-1 & 1
\end{array}\right),
\]

(43)

the \(T_i\) matrices for the state \(|\psi'\rangle = 1 \otimes 1 \otimes U|\text{GHZ}\rangle\) are

\[
T'_0 = \frac{1}{2} \left(\begin{array}{cc}
1 & 0 \\
0 & 1
\end{array}\right), \quad T'_1 = \frac{1}{2} \left(\begin{array}{cc}
-1 & 0 \\
0 & 1
\end{array}\right).
\]

(44)

It is very easy to see that these matrices satisfy equation \([24]\), so the state \(|\psi'\rangle\) is a gate state. Thus, Charlie can apply a deterministic POVM to it. In particular, this state satisfies \(b' = a' = Tr[T'_0 T_0] = \frac{1}{2}\), so according to equation \([39]\) we have \(y = 1 - x\), so Charlie can apply a deterministic POVM of the form

\[
E_0 = \left(\begin{array}{cc}
\sqrt{x} & 0 \\
0 & \frac{1}{\sqrt{1-x}}
\end{array}\right), \quad E_1 = \left(\begin{array}{cc}
\sqrt{1-x} & 0 \\
0 & \sqrt{x}
\end{array}\right),
\]

(45)

where \(x \in [\frac{1}{2}, 1)\). The normalized state corresponding to the outcome zero is

\[
|\phi_0\rangle = \frac{E_0 U |0\rangle}{\langle 0 | U^\dagger E_0 U |0\rangle^{\frac{1}{2}}} = \frac{E_0 U |1\rangle}{\langle 1 | U^\dagger E_0 U |1\rangle^{\frac{1}{2}}},
\]

(47)

are normalized states. A straightforward calculation shows that \(\langle 0 | U^\dagger E_0 U |0\rangle = \langle 1 | U^\dagger E_0 U |1\rangle = \frac{1}{2}\), so we can write

\[
|\phi_0\rangle = \frac{1}{\sqrt{2}}(|000\rangle |0'\rangle + |111\rangle |1'\rangle).
\]

(48)

This state can be taken to the canonical form \([3]\) by letting Charlie apply a local (real) unitary on his qubit, that takes the state \(|0'\rangle\) into \(|0\rangle\), and \(|1'\rangle\) into \(|\varphi'\rangle = \cos \delta |0\rangle + \sin \delta |1\rangle\). Thus, \(|0|\varphi'\rangle = |0'\rangle |1'\rangle\) and then we have

\[
\cos \delta' = \langle 0' | 1' \rangle = 2 \langle 0 | U^\dagger E_0 U |0\rangle = 2x - 1.
\]

(49)

We can see that for any \(\delta' \in (0, \frac{\pi}{2}\)], we can find \(x \in [\frac{1}{2}, 1)\) that satisfies this equation. This means that we can
transform \(|GHZ\rangle\) into \(\frac{1}{\sqrt{2}}(|000⟩ + |11⟩|ϕ′⟩)\) with probability 1, for any \(|ϕ′⟩\).

**Step 2.** In this step Bob applies a deterministic POVM to transform the state \(|ϕ⟩ = \frac{1}{\sqrt{2}}(|000⟩ + |11⟩|ϕ⟩)\) into \(\frac{1}{\sqrt{2}}(|000⟩ + |1⟩|ϕ⟩|ϕ′⟩)\). The \(T_i\) matrices for \(|ϕ⟩\) from Bob’s point of view, are given by

\[
T_0 = \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 \end{pmatrix}, \quad T_1 = \begin{pmatrix} \frac{1}{\sqrt{2}} \cos δ' & 0 \\ \frac{1}{\sqrt{2}} \sin δ' \end{pmatrix}.
\]

(50)

First, Bob applies the local unitary \(U\) from (43) to his qubit, obtaining the state \(|ϕ′⟩ = 1 \otimes U \otimes 1|ϕ⟩\), characterized by matrices \(T_i'\) given by

\[
T'_0 = \frac{1}{2} \begin{pmatrix} 1 & 0 \\ \cos δ' & \sin δ' \end{pmatrix}, \quad T'_1 = \frac{1}{2} \begin{pmatrix} -1 & 0 \\ \cos δ' & \sin δ' \end{pmatrix}.
\]

(51)

Again, it is easy to show that \(T'_i\) satisfy (24), so \(|ϕ′⟩\) is a gate state. We also have that \(a' = Tr[T'_0T'_0] = \frac{1}{2}\), so Bob can apply the POVM of equation (45) to his qubit and obtain two outcomes in the same orbit. We can apply the same analysis we did in Step 1 to the outcome \(|χ₀⟩\) of Bob’s POVM, and show that

\[
|χ₀⟩ = \frac{1}{\sqrt{2}}(|0⟩|0′⟩|0⟩ + |1⟩|1′⟩|ϕ′⟩)
\]

(52)

where the normalized states \(|0′⟩\) and \(|1′⟩\) are also given by (43). It should be clear from Step 1 that, again, we can choose \(x\) and a suitable local unitary on Bob’s qubit to transform this state into

\[
|χ⟩ = \frac{1}{\sqrt{2}}(|000⟩ + |1⟩|ϕ⟩|ϕ′⟩)
\]

(53)

for any \(|ϕ⟩ = \cos δ|0⟩ + \sin δ|1⟩\), with \(δ \in (0, \frac{π}{2})\).

**Step 3.** Now Alice has to choose between two local POVMs depending on whether she wants to obtain (40) or (41). Consider first the case in which she wants to transform \(|χ⟩\) into \(μ|000⟩ + ν|1|ϕ′⟩\). The \(T_i\) matrices for \(|χ⟩\) from Alice’s point of view are

\[
T₀ = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad T₁ = \frac{1}{\sqrt{2}} \begin{pmatrix} \cos δ \cos δ' & \cos δ \sin δ' \\ \sin δ \cos δ' & \sin δ \sin δ' \end{pmatrix}.
\]

(54)

These matrices already satisfy equation (24) and since \(a = Tr[T₀T₀] = \frac{1}{2}\), Alice can apply the deterministic POVM given by (43). The state corresponding to outcome zero is

\[
|ξ⟩ = \sqrt{x}|000⟩ + \sqrt{1-x}|1|ϕ′⟩ = μ|000⟩ + ν|1|ϕ′⟩,
\]

(55)

where we set \(μ = \sqrt{x}\) and \(ν = \sqrt{1-x}\). Since \(x \in [\frac{1}{2}, 1]\), we have \(μ ≥ ν\). The state in (55) is the same as in (40).

Consider now the case in which Alice wants to obtain \(\frac{1}{\sqrt{2}}(|000⟩ + i|ϕ⟩|ϕ⟩|ϕ′⟩)\) from \(|χ⟩\). In this case we can construct the appropriate POVM \(\{A₀, A₁\}\) by inspection. If \(|ϕ⟩ = \cos δ'|0⟩ + \sin δ'|1⟩\), we define

\[
A₀ = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & i \cos δ'' \\ 0 & i \sin δ'' \end{pmatrix}, \quad A₁ = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -i \cos δ'' \\ 0 & -i \sin δ'' \end{pmatrix}.
\]

(56)

It is easy to verify that they satisfy \(A₀ + A₀ + A₁ + A₁ = 1\), and that the probabilities of both outcomes are equal to \(\frac{1}{2}\). The normalized state that corresponds to outcome zero is

\[
\frac{1}{\sqrt{2}}(|000⟩ + i|ϕ⟩|ϕ⟩|ϕ′⟩),
\]

(57)

while the one corresponding to outcome 1 is just the complex conjugate of (57). But we know that these two states are actually in the same orbit, so we can transform outcome 1 into (57) by local unitaries, so we obtain (57) with probability 1. The state in (57) is the same as in (41). This concludes the protocol.

Note that all three steps involve only local unitaries and deterministic POVMs, so these protocols allow Alice, Bob and Charlie to transform the GHZ state into any other real state that belongs to the subclass defined by \(\text{Re}(Ω) = 0\) with probability 1, using only local operations and classical communication. This is then a complete characterization of the real states that can be obtained from the GHZ state, since by Theorem 1 we know that we cannot reach real states that belong to a different subclass. It is interesting to note that it does not seem to be that easy to find a deterministic protocol to transform the GHZ state into any complex state in the same subclass. Whether this is actually possible is still an open question.

**VI. SUMMARY AND CONCLUSIONS**

In this paper, we studied the properties of deterministic LOCC transformations of 3-qubit pure states with tripartite entanglement. We showed that the set of states in the GHZ class breaks into an infinite number of disjoint subclasses, characterized by the real part of a complex function \(Ω(|ϕ⟩)\). Two states that belong to different subclasses cannot be transformed one into the other with probability one, by means of local operations and classical communication. This quantity is not only invariant under deterministic transformations, but it also must be conserved by any local POVM that is part of a deterministic protocol. This imposes a strong constraint on the POVMs that can be used for deterministically transforming a given state.

It is interesting to point out that the invariance of \(\text{Re}(Ω)\) under deterministic LOCC transformations (and its invariance under any local POVM that is part of such
a transformation), follows from the invariance of \( \Omega \) under local unitaries and the very particular form of equation (4). In the language of entanglement monotones, we can say that \( \text{Re}(\Omega) \) is both an increasing and decreasing entanglement monotone. Any function of the states that is invariant under local unitaries and satisfies an equation like (4) for an arbitrary local POVM, will be invariant under deterministic LOCC protocols, and hence will break the set of states into inequivalent classes that will be labeled by that function. This will be true even in the multipartite case, so identifying quantities with these properties could be very useful in the study of deterministic transformations of entanglement.

We also discussed the case of a deterministic 2-outcome POVM. We showed that for this POVM to exist, both the state and the parameters of the POVM have to satisfy certain polynomial conditions. In particular the coefficients of the state have to satisfy two polynomial constraints. To be able to apply a deterministic POVM to a given state, we need to find a local unitary that will transform our original state into another state that satisfies the two constraints. For real states, the problem actually simplifies and only one constraint has to be satisfied. In this case, it was proven in general that the necessary local unitary could be found, allowing us to apply a local 2-outcome POVM that would send the state to some other orbit with probability 1. For complex states we found some analytical evidence that the unitary could be found, but a rigorous proof of this fact is still an open problem. However, it is important to stress that of all random numerical examples analysed, the algorithm discussed in Section II never failed to find a gate state for some other orbit with probability 1. For complex states we found some analytical evidence that the unitary could be found, allowing us to apply a POVM and which POVM they choose. Although it is difficult in general to study this procedure analytically, in order to characterize the set of states that can be obtained from \( |\psi\rangle \) (except for states with high symmetry like the GHZ state), a numerical analysis is easy to implement, and can be used to study general properties of this set, that could help us to have a better understanding of deterministic transformations.

Finally, we combined the two main results of this paper to give a complete characterization of the real states that can be obtained from the GHZ state with probability 1. First we used the results of Section II to characterize the subclass of states that could in principle be obtained deterministically from it, and then we constructed an explicit protocol that allows the three parties to transform the GHZ state into any real state in that subclass. Finding a protocol to transform it to a complex state in the same subclass does not seem to be as easy, and thus whether this transformation is possible or not is still an open question.

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APPENDIX A: SOLUTION OF

\[ I_i(x, y) = I_i(1 - x, 1 - y) \]

We want to know under which conditions does (23) have a nontrivial solution (i.e., \( x \neq y \) and \( x, y \neq 0, 1 \)). We will consider only states that have 3-particle entanglement, which means that \( a, b \neq 0, 1 \). First, let us note that we can write \( I_1(x, y) \) as

\[ I_1(x, y) = 1 + \frac{2xy(Tr[T_0 T_1^i T_0^j] - ab)}{(ax + by)^2}, \tag{A1} \]

where \( (Tr[T_0 T_1^i T_0^j] - ab) \neq 0 \) if \( |\psi\rangle \) has 3-particle entanglement. Then \( I_1(x, y) = I_1(1 - x, 1 - y) \) has a solution if and only if

\[ \frac{xy}{(ax + by)^2} = \frac{(1 - x)(1 - y)}{(a(1 - x) + b(1 - y))^2}, \tag{A2} \]

which is the same as

\[ a^2 x(1 - x) = b^2 y(1 - y). \tag{A3} \]

This also implies that \( I_4(x, y) = I_4(1 - x, 1 - y) \). Both \( I_2 \) and \( I_3 \) have the form

\[ I_i(x, y) = \frac{F_0 x^2 + F_1 y^2 + 2C_i xy}{(ax + by)^2} \quad i = 2, 3. \tag{A4} \]

We can use (A2) to write \( I_i(x, y) = I_i(1 - x, 1 - y), \) where

\[ \frac{F_0 + F_1 z^2}{(a + bz)^2} = \frac{F_0 + F_1 w^2}{(a + bw)^2}, \tag{A5} \]

where we introduced the variables \( z = \frac{y}{x} \) and \( w = \frac{(1 - y)}{(1 - x)} \). From (A3) we see that these variables are not independent, and satisfy the condition \( zw = (\frac{x}{y})^2 \). Furthermore, both \( z \) and \( w \) are positive, since \( x \) and \( y \) are between 0 and 1. If we expand (A3) and use the relationship between \( z \) and \( w \), we have

\[ (F_0 b^2 - F_1 a^2)(z (\frac{a^2 + b^2}{a^2} + 2 \frac{a}{b}) = 0, \tag{A6} \]

and since \( z \) has to be positive (and \( a \) and \( b \) are positive), we have the condition
which is equation (24).

To study the equation $I_5(x, y) = I_5(1 - x, 1 - y)$ we can assume that both (A7) and (A3) are satisfied, since we are looking for a simultaneous solution of (23). Let $\mu = I_5(x, y)$. Introducing $z = \frac{x}{y}$ and using (22) we can write

$$G_{00} + 3G_{01}z + 3G_{10}z^2 + G_{11}z^3 = \mu(a + bz)^3,$$  \hspace{1cm} (A8)

where $G_{ij} = Tr[T_i T_j T_k T_l T]^i _j T]^k _l]$, and we can expand this into

$$(G_{00} - \mu a^3) + 3(G_{01} - \mu a^2b) + 3(G_{10} - \mu ab^2)z^2 + (G_{11} - \mu b^3) = 0.$$  \hspace{1cm} (A10)

A root of this cubic polynomial represents an operator of a POVM for which the value of $I_5$ for the outcome of that operator is $\mu$. We are looking for two operators whose outcomes have the same value of $I_5$, but that also satisfy equation (A3). That is the same as finding two roots $z_0$ and $z_1$ of (A9), that satisfy the condition

$$z_0 z_1 = \frac{a^2}{b^2}.$$ \hspace{1cm} (A11)

Let $z_2$ be the third root of (A9). From elementary algebra we know that the product of the three roots is equal to minus the quotient of the independent and the cubic coefficients, so we can write

$$z_0 z_1 z_2 = \frac{G_{00} - \mu a^3}{G_{11} - \mu b^3} = -\frac{a^3 G_{00}}{b^3} - \mu.$$ \hspace{1cm} (A12)

Using (A3) and the Cayley-Hamilton theorem, it can be shown that

$$\frac{G_{00}}{a^3} = \frac{G_{11}}{b^3},$$ \hspace{1cm} (A13)

so (A12) reduces to

$$z_0 z_1 z_2 = -\frac{a^3}{b^3}.$$  \hspace{1cm} (A14)

If we want (A11) to be satisfied we need $z_2 = -\frac{a}{b}$. If we plug this into (A9), we find that $z_2$ is actually a root if and only if

$$b G_{01} = a F_{10},$$ \hspace{1cm} (A15)

which is equation (25). There is one more detail we need to check. We need $z_0 = \frac{a}{b}$ and $z_1 = \frac{1-\frac{a}{x}}{1-\frac{a}{y}}$ to be positive numbers, because $x$ and $y$ are between 0 and 1, and only one of them should be greater than 1 (which can be seen from their explicit form in terms of $x$ and $y$). We know that the other root $z_2 = -\frac{a}{b}$ is negative, so the condition for only one of them to be greater than 1 can be written

$$(z_0 - 1)(z_1 - 1)(z_2 - 1) > 0.$$ \hspace{1cm} (A16)

Expanding this inequality we get

$$(z_0 z_1 - (z_0 z_1 + z_0 z_2 + z_1 z_2) + (z_0 + z_1 + z_2) - 1 > 0.$$ \hspace{1cm} (A17)

All the symmetric polynomials on the roots of a polynomial equation can be written in terms of the coefficients of that polynomial, so we can rewrite this inequality as

$$-\frac{(G_{00} - \mu a^3)}{(G_{11} - \mu b^3)} - 3\frac{(G_{01} - \mu a^2b)}{(G_{11} - \mu b^3)} - 3\frac{(G_{10} - \mu ab^2)}{(G_{11} - \mu b^3)} - 1 > 0.$$ \hspace{1cm} (A18)

Expanding this and using $a + b = 1$ we get

$$G_{00} + 3 G_{01} + 3 G_{10} + G_{11} > \mu.$$ \hspace{1cm} (A19)

But the left hand side is just the value of $I_5$ for the state $|\psi\rangle$, while $\mu$ is the value of $I_5$ for the transformed state $|\phi_0\rangle$ (or $|\phi_1\rangle$). So this condition is telling us that under a deterministic 2-outcome POVM, $I_5$ behaves monotonically, even though it is not an entanglement monotone in general.

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[16] This decomposition can be understood in the following way: $A_i$ are positive semidefinite operators, and by performing a singular value decomposition we know that there are unitary matrices $V_i$ and $U_i$ such that $V_i A_i U_i T$ are positive semidefinite diagonal matrices. The matrices $U_i$ can be used to diagonalize the hermitian matrices $A_i A_i^T$. But since we have the constraint that $A_i A_i^T A_i = I$, and $I$ is already a diagonal matrix, it is easy to see that we can take $U_i = U_i T$.