REGULARITY AND WEAK COMPARISON PRINCIPLES FOR DOUBLE PHASE QUASILINEAR ELLIPTIC EQUATIONS

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Abstract. We consider the Euler equation of functionals involving a term of the form
\[
\int_\Omega \left( |\nabla u|^p + a(x)|\nabla u|^q \right) \, dx,
\]
with \(1 < p < q < p + 1\) and \(a(x) \geq 0\). We prove weak comparison principle and summability results for the second derivatives of solutions.

1. Introduction. For \(n \geq 2\) we consider a bounded and smooth domain \(\Omega \subset \mathbb{R}^n\) and we are interested in the study of properties of positive solutions of the equation:
\[
\begin{cases}
-\text{div}(p|\nabla u|^{p-2}\nabla u + qa(x)|\nabla u|^{q-2}\nabla u) = f(u) & \text{in } \Omega \\
u = 0 & \text{on } \partial \Omega,
\end{cases}
\]
where:
- \(p, q \in \mathbb{R}\) are such that \(0 < p < q < p + 1\)
- \(a(x) \in C^1(\Omega)\) satisfies: \(a(x) \geq 0 \ \forall x \in \Omega\)
- \(f\) is a strictly positive and locally Lipschitz continuous function.
\(\Delta_p u = \text{div}(p|\nabla u|^{p-2}\nabla u)\) is the usual \(p\)-Laplace operator. Equation (1) is related to the Euler equation of functionals involving a term of the form
\[
F(u) = \int_\Omega \left( |\nabla u|^p + a(x)|\nabla u|^q \right) \, dx,
\]
which were introduced by Zhikov [29, 30] in order to study the behavior of anisotropic materials. Properties of minimizers for these variational integrals were firstly studied by Marcellini ([16, 17]) and recently by several authors (see for instance [1, 3, 4, 5, 6, 7] and the references quoted there). In particular in [4] \(C^{1,\alpha}\) regularity is established. As pointed out in [4, 5, 12, 29, 30], condition \(p < q < p + 1\) excludes the Lavrentiev phenomenon.

In [9, 10] were introduced several useful tools to handle equations involving the \(p\)-Laplace operator. Then these strategies were widely developed (see for instance [2, 13, 15, 18, 20, 23, 24, 25, 26] and the reference therein) in order to achieve properties of equations (and systems of equations) involving the \(p\)-Laplace operator in several frameworks (in bounded and unbounded domains, with lower order terms...

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or singular data). In this paper, by means of the linearized equation of (1), we give some bounds on the hessian of the solutions, which will allow us to achieve an estimate of the inverse of the gradient of the solutions (see Section 3). Using these tools, we are able to give some regularity results and to recover a weighted Poincaré inequality, useful to obtain a weak comparison principle (see Section 4). We recall that a weak comparison principle for the classical p-Laplace equation \((-\Delta_p u = f(u))\) was proved for \(p \in (1, 2)\) in [8] and for \(p \in [2, +\infty)\) in [9].

2. Notation and preliminary results. For \(x \in \mathbb{R}^n, |x|\) stands for the euclidian norm. \((\cdot, \cdot)\) is the euclidian scalar product. If \(A = (a_{ij}) \in \mathbb{R}^{n \times n}\) is a matrix, we set \(|A| = \sqrt{\sum_{i,j=1}^{n} a_{ij}^2}\). For \(r > 0\) and \(x_0 \in \mathbb{R}^n\), \(B_r(x_0)\) denotes the open ball of radius \(r\) and centered at \(x_0\). Given a function \(u : \mathbb{R}^n \to \mathbb{R}\), we denote by \(u_i\) the partial derivative of \(u\) with respect to the \(i\)-th variable. \(c\) and \(C\) will be positive constants, which can vary from line to line.

For \(m > 1\), we will use the standard notation \(W^{1,m}(\Omega)\) for the usual Sobolev spaces. In order to well define the so called linearized equation of (1), we need some properties about the weighted Sobolev spaces (for more details about them see for instance [19, 21, 28]). We recall here their definition and some basic tools.

For \(m \geq 1\) and \(\mu \in L^1(\Omega)\) the weighted Sobolev space \(W^{1,m}_\mu(\Omega)\) (with respect to the weight \(\mu\)) is defined as the set of functions \(v \in L^m(\Omega)\) which are bounded with respect to the norm:

\[
\|v\| = \|v\|_{L^m(\Omega)} + \|\nabla v\|_{L^\mu_m(\Omega)},
\]

where \(\nabla v\) is the distributional derivative of \(v\), \(\|v\|_{L^m(\Omega)} = \left(\int_{\Omega} |v|^m \right)^{\frac{1}{m}}\) and \(\|\nabla v\|_{L^\mu_m(\Omega)} = \left(\int_{\Omega} |\nabla v|^\mu \mu \right)^{\frac{1}{m}}\). According to [19], equivalently it is possible to define such a space as the completion of \(C^\infty_c(\Omega)\) with respect to the norm defined in (3).

As for the usual Sobolev spaces, the space \(W^{1,1}_\mu(\Omega)\) is defined as the closure of \(C^\infty_c(\Omega)\) in \(W^{1,1}_\mu(\Omega)\). We set \(H^1_\mu(\Omega) = W^{1,2}_\mu(\Omega)\) and \(H^{1,2}_\mu(\Omega) = W^{1,2}_{0,\mu}(\Omega)\), which are the Hilbert spaces where the linearized operator associated to equation (1) is defined. Notice that, for \(p > 2\), the space \(W^{1,p}(\Omega)\) is continuously embedded in \(H^{1,2}_\mu(\Omega)\) if \(\mu = |\nabla u|^{p-2}\).

3. Main estimates. In this section we state and prove our main results, starting with some local estimates of the second derivatives of solutions of equation (1).

We remark that the results in this section hold also if in the righthand side of equation (1) the function \(f\) depends on \(x\) (instead of \(u\)). Actually, the same proofs work with minor modifications. For \(u \in W^{1,p}(\Omega)\), the weak form of (1) is:

\[
\int_{\Omega} |\nabla u|^{p-2} \langle \nabla u, \nabla \psi \rangle + \int_{\Omega} a(x)|\nabla u|^{q-2} \langle \nabla u, \nabla \psi \rangle = \int_{\Omega} f(u)\psi, \quad \forall \psi \in C^\infty_c(\Omega). \quad (4)
\]

Our main step is to compute the so called linearized equation of (1).

We set \(Z = \{x \in \Omega : \nabla u(x) = 0\}\) and we consider \(\tilde{\psi} \in C^\infty_c(\Omega \setminus Z)\). For any \(i = 1, \ldots, n\), we take \(\psi = \nabla \tilde{\psi}_i\) in (4) and, integrating by parts, we get:

\[
\int_{\Omega} |\nabla u|^{p-2} \langle \nabla u_i, \nabla \tilde{\psi} \rangle + (p-2) \int_{\Omega} |\nabla u|^{p-4} \langle \nabla u_i, \nabla u \rangle \langle \nabla u, \nabla \tilde{\psi} \rangle + \int_{\Omega} a(x) \left[ |\nabla u|^{q-2} \langle \nabla u_i, \nabla \tilde{\psi} \rangle + (q-2) \int_{\Omega} |\nabla u|^{q-4} \langle \nabla u_i, \nabla u \rangle \langle \nabla u, \nabla \tilde{\psi} \rangle \right] +
\]
and thought extended equal to 0 inside $Z$.

Proof. (Hessian estimate) To give an estimate of the inverse of the weight $f$ of distributional second derivatives of $u$ we achieve the suitable regularity to ensure that these derivatives coincide with the distributional second derivatives of $u$ in the whole $\Omega$. We remark that, in order to achieve the estimate for the Hessian, it is required the local Lipschitz regularity of solutions, ensured if we take for example minima of functional in (2), which are $C^{1,\alpha}$ for some $\alpha \in (0, 1)$, as proved in [4]. Moreover, we remark that the strict positivity of $f$ does not need to prove the following proposition, while it is required in order to give an estimate of the inverse of the weight.

**Proposition 1** (Hessian estimate). Let $u \in W^{1,\infty}_{\text{loc}}(\Omega)$ be a solution of (4). Let $x_0 \in \Omega$ and let $r > 0$ be such that $B_r(x_0) \subset \Omega$. For $\beta \in [0, 1)$ and $\gamma < N - 2$ ($\gamma = 0$ if $n = 2$), there holds:

$$\sup_{y \in \Omega} \int_{B_r(x_0)} \frac{|\nabla u|^{p-2-\beta} |D^2 u|^2}{|x-y|^\gamma} \, dx \leq C,$$

where $C$ depends on $x_0, r, \beta, \gamma, p, N, ||u||_{W^{1,\infty}(\Omega)}, f$.

**Proof.** Let $G_\eta : \mathbb{R} \to \mathbb{R}$ be defined as:

$$G_\eta(t) = \begin{cases} t & \text{if } |t| \geq 2\eta \\ 2 \left[ t - \eta \frac{t}{|t|} \right] & \text{if } \eta < |t| < 2\eta \\ 0 & \text{if } |t| \leq \eta. \end{cases}$$

Fix $\beta \in [0, 1)$ and $\gamma < N - 2$ (or $\gamma = 0$ if $N = 2$) and set:

$$T_\varepsilon(t) = \frac{G_\varepsilon(t)}{|t|^\beta} \quad \text{and} \quad K_\delta(t) = \frac{G_\delta(t)}{|t|^\gamma + 1}.$$

Let $\varphi$ be a cut-off function such that

$$\varphi \in C_c^\infty(B_{2r}(x_0)) \quad \varphi \equiv 1 \quad \text{in } B_r(x_0) \quad \text{and} \quad |D\varphi| \leq \frac{2}{r},$$

with $2r < \text{dist}(x_0, \partial\Omega)$.

We take

$$\tilde{\psi}(x) = T_\varepsilon(u_i(x)K_\delta(|x-y|)\varphi(x)^2$$

as test function in (5) and we get:

$$\int_\Omega \nabla u|^{p-4} \left[ |\nabla u|^2 |\nabla u_i|^2 + (p-2)(\nabla u_i, \nabla u)^2 \right] T_\varepsilon(u_i)K_\delta(|x-y|)\varphi^2 +$$

$$+ \int_\Omega |\nabla u|^{p-4} \left[ |\nabla u|^2 |\nabla u_i|^2 + (p-2)(\nabla u_i, \nabla u)^2 \right] \nabla K_\delta(|x-y|) T_\varepsilon(u_i)\varphi^2 +$$

$$+ 2 \int_\Omega |\nabla u|^{p-4} \left[ |\nabla u|^2 |\nabla u_i|^2 + (p-2)(\nabla u_i, \nabla u)^2 \right] \nabla \varphi T_\varepsilon(u_i)K_\delta(|x-y|) \varphi^2 +$$

$$+ \int_\Omega a(x)|\nabla u|^{q-4} \left[ |\nabla u|^2 |\nabla u_i|^2 + (q-2)(\nabla u_i, \nabla u)^2 \right] T_\varepsilon(u_i)K_\delta(|x-y|) \varphi^2 +$$

$$+ \int_\Omega a(x)|\nabla u|^{q-4} \left[ |\nabla u|^2 |\nabla u_i|^2 + (q-2)(\nabla u_i, \nabla u)^2 \right] \nabla K_\delta(|x-y|) T_\varepsilon(u_i)\varphi^2.$$
and hence, recalling that the functions $a$ and $T_\varepsilon'$ are positive, we get:
\[
\begin{align*}
\int_\Omega |\nabla u|^{p-4} \left(|\nabla u|^2|\nabla u_i|^2 + (p-2)(\nabla u_i, \nabla u)^2 \right) T_\varepsilon'(u_i) K_\delta(|x-y|) \varphi^2 + (10) \\
+ \int_\Omega a(x)|\nabla u|^{q-4} \left(|\nabla u|^2|\nabla u_i|^2 + (q-2)(\nabla u_i, \nabla u)^2 \right) T_\varepsilon'(u_i) K_\delta(|x-y|) \varphi^2 \\
\geq \min\{1, p-1\} \int_\Omega |\nabla u|^{p-2}|\nabla u_i|^2 T_\varepsilon'(u_i) K_\delta(|x-y|) \varphi^2 \\
+ \min\{1, q-1\} \int_\Omega a(x)|\nabla u|^{q-2}|\nabla u_i|^2 T_\varepsilon'(u_i) K_\delta(|x-y|) \varphi^2 \\
\geq \min\{1, p-1\} \int_\Omega |\nabla u|^{p-2}|\nabla u_i|^2 T_\varepsilon'(u_i) K_\delta(|x-y|) \varphi^2.
\end{align*}
\]

By (9) and (10) we infer:
\[
\begin{align*}
\int_\Omega |\nabla u|^{p-2}|\nabla u_i|^2 T_\varepsilon'(u_i) K_\delta(|x-y|) \varphi^2 &\leq (11) \\
\leq c \int_\Omega |\nabla u|^{p-2}|\nabla u_i||\nabla K_\delta(|x-y|)||T_\varepsilon'(u_i)|| \varphi^2 \\
+ c \int_\Omega |\nabla u|^{p-2}|\nabla u_i||\nabla \varphi||T_\varepsilon'(u_i)||K_\delta(|x-y|) \varphi \\
+ c \int_\Omega |a(x)||\nabla u|^{q-2}|\nabla u_i||\nabla K_\delta(|x-y|)||T_\varepsilon'(u_i)|| \varphi^2 \\
+ c \int_\Omega |a(x)||\nabla u|^{q-2}|\nabla u_i||\nabla \varphi||T_\varepsilon'(u_i)||K_\delta(|x-y|) \varphi \\
+ c \int_\Omega |a_i(x)||\nabla u|^{q-2}|\nabla u||\nabla K_\delta(|x-y|)||T_\varepsilon'(u_i)|| \varphi^2 \\
+ c \int_\Omega |a_i(x)||\nabla u|^{q-2}|\nabla u||\nabla \varphi||T_\varepsilon'(u_i)||K_\delta(|x-y|) \varphi \\
+ c \int_\Omega |f'(u)||u_i||T_\varepsilon'(u_i)||K_\delta(|x-y|) \varphi^2.
\end{align*}
\]
Recalling that $a \in C^1(\Omega)$, $p < q$, $\nabla u$ is locally bounded and $|T_\varepsilon(t)| \leq |t|^{1-\beta}$ (with $\beta \in [0,1)$), we have:

$$\int_\Omega |\nabla u|^{p-2} |\nabla u_i|^2 T'_\varepsilon(u_i) K_\delta(|x-y|) \varphi^2 \leq$$

$$\leq c \int_\Omega |\nabla u|^{p-2} |\nabla u_i| |\nabla K_\delta(|x-y|)| |T_\varepsilon(u_i)| \varphi^2$$

$$+ c \int_\Omega |\nabla u|^{p-2} |\nabla u_i| |\nabla \varphi| |T_\varepsilon(u_i)| K_\delta(|x-y|) \varphi$$

$$+ c \int_\Omega |\nabla u|^{p-2} |\nabla u_i||T'_\varepsilon(u_i)| K_\delta(|x-y|) \varphi^2$$

$$+ c \int_\Omega |\nabla u|^{p-2} |\nabla u_i| |\nabla K_\delta(|x-y|)| \varphi^2$$

$$+ c \int_\Omega |\nabla \varphi| K_\delta(|x-y|) \varphi$$

$$+ c \int_\Omega |f'(u)| |u_i||T_\varepsilon(u_i)| K_\delta(|x-y|) \varphi^2.$$

Since $\gamma < n - 2$ and $\Omega$ is bounded, $\int_\Omega \frac{1}{|x-y|^2} \ dx$ is uniformly bounded and hence, for fixed $\varepsilon > 0$ we send $\delta$ to 0 and by dominate convergence we get:

$$\int_\Omega |\nabla u|^{p-2} \frac{|\nabla u_i|^2 T'_\varepsilon(u_i) \varphi^2}{|x-y|^\gamma} \leq$$

$$\leq c \int_\Omega \frac{|\nabla u|^{p-2} |\nabla u_i| |T_\varepsilon(u_i)| \varphi^2}{|x-y|^{\gamma+1}}$$

$$+ c \int_\Omega \frac{|\nabla u|^{p-2} |\nabla u_i| |\nabla \varphi| |T_\varepsilon(u_i)| \varphi}{|x-y|^\gamma}$$

$$+ c \int_\Omega \frac{|\nabla u|^{p-2} |\nabla u_i||T'_\varepsilon(u_i)| \varphi^2}{|x-y|^\gamma}$$

$$+ c \int_\Omega \frac{\varphi^2}{|x-y|^{\gamma+1}}$$

$$+ c \int_\Omega \frac{|\nabla \varphi| \varphi}{|x-y|^\gamma}$$

$$+ c \int_\Omega \frac{|f'(u)| |u_i||T_\varepsilon(u_i)| \varphi^2}{|x-y|^\gamma}.$$

We remark that the first term at right-hand side of (13) is equal to 0 when $n = 2$, because $\nabla K_\delta \to 0$ if $\gamma = 0$.

Using Young inequality ($ab \leq \theta a^2 + \frac{1}{4\theta} b^2 \forall a, b \in \mathbb{R} \forall \theta > 0$), we get

$$\int_\Omega \frac{|\nabla u|^{p-2} |\nabla u_i| |T_\varepsilon(u_i)| \varphi^2}{|x-y|^{\gamma+1}} \leq \theta \int_\Omega \frac{|\nabla u|^{p-2} |\nabla u_i|^2 G_\varepsilon(u_i)}{|x-y|^\gamma |u_i|^\beta \psi + c}$$

and

$$\int_\Omega \frac{|\nabla u|^{p-2} |\nabla u_i||T'_\varepsilon(u_i)| \varphi^2}{|x-y|^\gamma} \leq \theta \int_\Omega \frac{|\nabla u|^{p-2} |\nabla u_i|^2 G_\varepsilon(u_i)}{|x-y|^\gamma |u_i|^\beta \psi + c},$$
where we can choose $\vartheta > 0$ as we want. Recalling that $|\nabla \varphi| \leq \frac{2}{\gamma}$, we also have:

$$
\int_\Omega \frac{|\nabla u|^{p-2}|\nabla u_i||\nabla \varphi| |T_\varphi(u_i)||\varphi|}{|x-y|^\gamma} \leq \vartheta \int_\Omega \frac{|\nabla u|^{p-2}|\nabla u_i|^2}{u_i} G_z(u_i) \psi + c. 
$$

(16)

Recalling that $f$ is lipschitz continuous, we have:

$$
\int_\Omega \frac{\varphi^2}{|x-y|^\gamma+1} + \int_\Omega \frac{|\nabla \varphi|^2}{|x-y|^\gamma} + \int_\Omega \frac{|f_i||T_\varphi(u_i)||\varphi^2}{|x-y|^\gamma} \leq c \int_\Omega \frac{|f'(u)||u_i||T_\varphi(u_i)||\varphi^2}{|x-y|^\gamma} \leq C.
$$

(17)

Since $T'_\varphi(t) = \frac{1}{|t|^p} \left[ G'_z(t) - \beta \frac{G_z(t)}{t} \right]$, by estimates (13)-(17) we get:

$$
\int_\Omega \frac{|\nabla u|^{p-2}|\nabla u_i|^2}{|u_i|^2|x-y|^\gamma} \left( G'_z(u_i) - (\beta + \vartheta) \frac{G_z(u_i)}{u_i} \right) \psi^2 \leq c.
$$

(18)

We choose $\vartheta$ such that $\beta + \vartheta < 1$ and hence $G'_z(u_i) - (\beta + \vartheta) \frac{G_z(u_i)}{u_i}$ is positive. Recalling the definition of $G_z$, it follows that

$$
G'_z(t) - (\beta + \vartheta) \frac{G_z(t)}{t} \to 1 - (\beta + \theta)
$$

as $\varepsilon \to 0$ and hence by Fatou’s Lemma we get:

$$
\int_{\Omega \setminus \{u_i = 0\}} \frac{|\nabla u|^{p-2}|\nabla u_i|^2}{|u_i|^2|x-y|^\gamma} \psi^2 \leq c.
$$

(19)

Since $|u_i| \leq |\nabla u|$, it follows:

$$
\int_{\Omega \setminus \{u_i = 0\}} \frac{|\nabla u|^{p-2-\beta}|\nabla u_i|^2}{|x-y|^\gamma} \psi^2 \leq c,
$$

and finally, since $\nabla u_i = 0$ a.e. in $\{u_i = 0\} \cap (\Omega \setminus Z)$, we conclude:

$$
\int_{\Omega \setminus Z} \frac{|\nabla u|^{p-2-\beta}|\nabla u_i|^2}{|x-y|^\gamma} \psi^2 \leq c, 
$$

(20)

where $c$ does not depend on $y$. Recalling the definition of $\varphi$, (20) implies

$$
\sup_{y \in \Omega} \int_{B_r(x_0) \setminus Z} \frac{|\nabla u|^{p-2-\beta}|D^2 u|^2}{|x-y|^\gamma} \leq c
$$

and by standard arguments (see for instance [9, 25, 26]) we get the thesis.

Thanks to the above estimate on the Hessian of $u$ we are able to prove the following proposition, for which the positivity of $f$ will be needed. We first prove an estimate of $\frac{1}{|\nabla u|}$ on balls and then we conclude by means of a covering argument.

**Proposition 2** (Estimate of the inverse of the weight). Let $u \in W^{1,\infty}_{\text{loc}}(\Omega)$ be a positive solution of (4). Fix $\gamma \in [0, p - 1)$ and $\gamma < N - 2$ ($\gamma = 0$ if $N = 2$). Then, for any $\overline{\Omega}' \subset \subset \Omega$ there exists $c$ such that

$$
\sup_{y \in \Omega} \int_{\overline{\Omega}'} \frac{1}{|\nabla u|^\gamma |x-y|^\gamma} \, dx \leq c
$$

(21)

where $c = c(\Omega', \gamma, \tau, n, p, q, \|u\|_{W^{1, \infty}}, a, f)$.

Moreover, there holds: $|Z \cap \overline{\Omega}'| = 0$. 


Proof. As above, for \( x_0 \in \Omega \) and \( r > 0 \) such that \( B_{2r}(x_0) \subset \Omega \), we consider a cut-off function \( \varphi \) satisfying (8) and, for \( \beta \in [0,1) \) and \( \tau = p - 2 + \beta < p - 1 \), we take
\[
\psi = \frac{1}{|\nabla u|^r + \varepsilon} K_\delta(|x-y|) \varphi^2
\]
as test function in (4). After setting \( \lambda = \inf_{x \in B_{2r}(x_0)} f(x) \) (we recall that the assumptions on \( f \) ensure \( \lambda > 0 \)), we have
\[
\lambda \int_{B_{2r}(x_0)} \frac{1}{|\nabla u|^r + \varepsilon} K_\delta(|x-y|) \varphi^2 \leq \int_{B_{2r}(x_0)} f \frac{1}{|\nabla u|^r + \varepsilon} K_\delta(|x-y|) \varphi^2 = (22) \]
\[
= - \int_{B_{2r}(x_0)} \frac{|\nabla u|^{r-2}}{|(\nabla u|^{r+\varepsilon})^2} |\nabla u|^{p-2} \nabla u \cdot D^2 u \nabla u K_\delta(|x-y|) \varphi^2
+ \int_{B_{2r}(x_0)} \frac{|\nabla u|^{p-2} \nabla u \cdot \nabla K_\delta(|x-y|)}{|\nabla u|^r + \varepsilon} \varphi^2
+ 2 \int_{B_{2r}(x_0)} \frac{|\nabla u|^{q-2} \nabla u \cdot \nabla \psi K_\delta(|x-y|)}{|\nabla u|^r + \varepsilon} \varphi.
\]
Recalling that \( p < q \) and that \( \nabla u \) is bounded, by (22) we infer:
\[
\int_{B_{2r}(x_0)} \frac{1}{|\nabla u|^r + \varepsilon} K_\delta(|x-y|) \varphi^2 \leq \int_{B_{2r}(x_0)} \frac{1}{|\nabla u|^r + \varepsilon} |D^2 u| K_\delta(|x-y|) \psi^2 +
+ \int_{B_{2r}(x_0)} \frac{|\nabla u|^{p-1} \nabla K_\delta(|x-y|)}{|\nabla u|^r + \varepsilon} \varphi^2
+ \frac{2}{\lambda} \int_{B_{2r}(x_0)} \frac{|\nabla u|^{p-1} |\nabla \psi|}{|\nabla u|^r + \varepsilon} K_\delta(|x-y|) \varphi.
\]
As above, using dominate convergence theorem, we send \( \delta \) to 0 getting:
\[
\int_{B_{2r}(x_0)} \frac{1}{|\nabla u|^r + \varepsilon} \frac{1}{|x-y|^\gamma} \psi^2 \leq \int_{B_{2r}(x_0)} \frac{|\nabla u|^r}{|(\nabla u|^r + \varepsilon)^2} |\nabla u|^{p-2} |D^2 u| |x-y|^\gamma \psi^2 +
+ \int_{B_{2r}(x_0)} \frac{|\nabla u|^{p-1} \nabla \psi}{|\nabla u|^r + \varepsilon} |x-y|^\gamma + \varphi^2
+ \frac{2}{\lambda} \int_{B_{2r}(x_0)} \frac{|\nabla u|^{p-1} |\nabla \psi|}{|\nabla u|^r + \varepsilon} |x-y|^\gamma \varphi.
\]
By definition of \( \tau \), using Proposition 1, for \( \theta > 0 \), we get:
\[
\tau \int_{B_{2r}(x_0)} \frac{|\nabla u|^r}{|(\nabla u|^r + \varepsilon)^2} |\nabla u|^{p-2} |D^2 u| |x-y|^\gamma \psi^2 = (23)
\]
Proof.
If $4870 \text{GIUSEPPE RIEY}$ is bounded, by (29) (with $\gamma < n$) we ensure that

$$\int_{\Omega} |\nabla u|^p - 2 - |D^2 u|^2 - c_\theta \leq 0, \quad \theta \int_{B_{2r}(x_0)} \left( |\nabla u|^{p-1} + \varepsilon \right) |x-y|^{-\gamma} \varphi^2 + \frac{1}{\theta} \int_{B_{2r}(x_0)} |\nabla u|^{p-2-\beta} |D^2 u|^2 - c \varphi^2.$$

Since $\tau < p - 1$, then $\frac{|\nabla u|^{p-1}}{|\nabla u|^{p-1} + \varepsilon}$ is bounded and hence there holds:

$$J_2 \leq c \int_{B_{2r}(x_0)} \frac{1}{|x-y|^{\gamma+1}} \psi^2 \leq c. \quad (26)$$

By properties of $\psi$ we get:

$$J_3 \leq \frac{c}{\rho} \int_{B_{2r}(x_0)} \frac{1}{|x-y|^\gamma} \psi^2 \leq c. \quad (27)$$

We choose $\theta$ small enough and by (25),(26),(27) we get:

$$\int_{B_{2r}(x_0)} \left( |\nabla u|^{p-1} + \varepsilon \right) |x-y|^{-\gamma} \varphi^2 + \frac{1}{\theta} \int_{B_{2r}(x_0)} |\nabla u|^{p-2-\beta} |D^2 u|^2 \leq c \quad (28)$$

and the thesis follows using the Fatou’s Lemma as $\varepsilon$ tends to 0.

We remark that, as $\varepsilon \to 0$, $\frac{1}{|\nabla u|^{p-1} + \varepsilon} |x-y|^{-\gamma} \psi^2 \to +\infty$ in $Z \cap B_{2r}(x_0)$ and hence we infer that $|Z \cap B_{2r}(x_0)| = 0.$

If we consider solutions which are $C^1$ up to the boundary, we can apply the results in [22] to infer that the Hopf’s Lemma holds for this kind of operator and therefore we ensure that $Z \cap \partial \Omega = \emptyset$ and hence we can extend to the whole $\Omega$ the local results in Propositions 1 and 2, as stated in the two following corollaries.

**Corollary 1.** Let $u \in C^1(\overline{\Omega})$ be a weak solution of (1). Then for $\beta \in [0,1)$ and $\gamma < n - 2$ ($\gamma = 0$ if $n = 2$), there exists $C > 0$ such that

$$\sup_{y \in \Omega} \int_{\partial \Omega} |\nabla u|^{p-2-\beta} |u_{ij}|^2 \frac{|x-y|^{-\gamma}}{|x-y|^{\gamma}} \, dx \leq C \quad (29)$$

and

$$\sup_{y \in \Omega} \int_{\partial \Omega} |\nabla u|^{p-2-\beta} |D^2 u|^2 \frac{|x-y|^{-\gamma}}{|x-y|^{\gamma}} \, dx \leq C. \quad (30)$$

**Corollary 2.** Let $u \in C^1(\overline{\Omega})$ be a weak solution of (1). Fix $\tau \in [0,p - 1)$ and $\gamma < n - 2$ ($\gamma = 0$ if $n = 2$). Then there exists $C > 0$ such that

$$\sup_{y \in \Omega} \int_{\partial \Omega} \frac{dx}{|\nabla u|^{\gamma} |x-y|^\gamma} \leq C. \quad (31)$$

Moreover, there holds: $|Z| = 0.$

4. Regularity and weak comparison principle.

**Theorem 4.1** (Regularity). Let $u$ be a $C^1(\overline{\Omega})$ solution to (4). There hold:

1. if $p \in (1,3)$, then $u \in W^{2,2}(\Omega)$
2. if $p \in [3, +\infty)$, then $u \in W^{2,s}(\Omega)$ with $s \leq \frac{p-1}{p-2}$.

**Proof.** If $p \in (1,3)$, we choose $\beta < 1$ such that $p - 2 - \beta < 0$ and, recalling that $\nabla u$ is bounded, by (29) (with $\gamma = 0$) we infer:

$$\int_{\Omega} |u_{ij}|^2 \, dx \leq \sup_{y \in \Omega} |\nabla u|^{\beta+2-p} \int_{\Omega} |\nabla u|^{p-2-\beta} |u_{ij}|^2 \, dx \leq C.$$
If $p \in [3, +\infty)$, we have $p - 2 - \beta > 0$ and, since $\beta < 1$, taking $s < \frac{p - 1}{p - 2}$, we have $(p - 2 - \beta) \frac{s}{2} < p - 1$ and, using (29) and (31) (with $\gamma = 0$), we can conclude

$$
\int_{\Omega} |u_{ij}|^s \, dx = \int_{\Omega} |u_{ij}|^s |\nabla u|^{(p-2-\beta) \frac{s}{2}} \cdot \frac{1}{|\nabla u|^{(p-2-\beta) \frac{s}{2}}} \, dx \leq \left( \int_{\Omega} |u_{ij}|^2 |\nabla u|^{p-2-\beta} \, dx \right)^{\frac{s}{2}} \left( \int_{\Omega} \frac{1}{|\nabla u|^{(p-2-\beta) \frac{s}{2}}} \, dx \right)^{\frac{2-s}{2}} \leq C.
$$

\[\square\]

We recall that, for $u \in W_0^{1,1}(\Omega)$, we have

$$
u(x) = c \int_{\Omega} \frac{(x_i - y_i)u_j(y)}{|x - y|^n} \, dy$$

for suitable $c > 0$ (see [14, Lemma 7.14]) and hence (see [14, Lemma 7.16])

$$|\nu(x)| \leq C \int_{\Omega} \frac{|\nabla u(y)|}{|x - y|^{n-1}} \, dy.$$ 

Therefore, using potential estimates in [14, Lemma 7.12] and thanks to estimates in Corollaries 1 and 2, we can apply Theorem 3.2 in [9] getting the following theorem.

**Theorem 4.2** (Weighted Poincaré inequality). Let $u \in C^1(\overline{\Omega})$ be a solution of (4) with $p \geq 2$. Taking $\mu = |\nabla u|^{p-2}$, there holds:

$$||v||_{L^2(\Omega)} \leq c(\Omega)||\nabla u||_{L^2(\Omega)} \quad \forall \, v \in H^1_{0,\mu}(\Omega)$$

(32)

with $c(\Omega) \to 0$ as $|\Omega| \to 0$.

Theorem 4.2 is a crucial tool to prove the following weak comparison principle in small domain.

**Theorem 4.3** (Weak comparison principle). Let $u, v \in C^1(\overline{\Omega})$ be positive solutions of (4) and let $D \subseteq \Omega$ be such that $u \leq v$ on $\partial D$. Then there exists $\delta > 0$ such that, if $|D| < \delta$, there holds $u \leq v$ in $D$.

**Proof.** We recall that (see [8, Lemma 2.1]) for every $p > 1$ there exists $c > 0$ such that

$$\langle |\xi|^{p-2} \xi - |\eta|^{p-2} \eta, \xi - \eta \rangle \geq c (|\xi| + |\eta|)^{p-2} |\xi - \eta|^2$$

(33)

for every $\xi, \eta \in \mathbb{R}^n$. By (4) written for $u$ and $v$ with $\psi = (u - v)^+ \chi_D$ (being $\chi_D$ the characteristic function of $D$), we get:

$$
\int_D \langle |\nabla u|^{p-2} \nabla u - |\nabla v|^{p-2} \nabla v, \nabla (u - v)^+ \rangle + \int_D \langle a(x)|\nabla u|^{p-2} \nabla u - |\nabla v|^{p-2} \nabla v, \nabla (u - v)^+ \rangle = \int_D \frac{f(u) - f(v)}{u - v} [(u - v)^+]^2
$$

(34)
By (33) and (34) and recalling that \( a(x) \) is positive and that \( f \) is locally Lipshitz, we infer:
\[
c \int_D (|\nabla u| + |\nabla v|)^{p-2} |\nabla (u-v)^+|^2 \leq c \int_D (|\nabla u| + |\nabla v|)^{p-2} |\nabla (u-v)^+|^2 + c \int_D a(x)(|\nabla u| + |\nabla v|)^{p-2} |\nabla (u-v)^+|^2 \leq C \int_D [(u-v)^+]^2.
\]
If \( p \in (1,2) \), using classical Poincaré inequality, by (35) and recalling that \( \nabla u \) and \( \nabla v \) are bounded, we get
\[
\int_D (|\nabla u| + |\nabla v|)^{p-2} |\nabla (u-v)^+|^2 \leq C(|D|) \int_D |\nabla (u-v)^+|^2 = C(|D|) \int_D ((|\nabla u| + |\nabla v|)^2 - p(|\nabla u| + |\nabla v|)^{p-2} |\nabla (u-v)^+|^2 = cC(|D|) \int_D (|\nabla u| + |\nabla v|)^{p-2} |\nabla (u-v)^+|^2,
\]
where \( C(|D|) \to 0 \) as \( |D| \to 0 \) and hence the thesis follows choosing \( |D| \) small enough.

If \( p \in [2, +\infty) \), using Theorem 4.2, by (35) we achieve the thesis.  

REFERENCES

[1] P. Baroni, M. Colombo and G. Mingione, Regularity for general functionals with double phase, Calc. Var. Partial Differential Equations, 57 (2018), Art. 62, 48 pp.
[2] D. Castorina, G. Riey and B. Sciunzi, Hopf Lemma and Regularity Results for Quasilinear Anisotropic Elliptic Equations, Calc. Var. Partial Differential Equations, To appear.
[3] M. Colombo and G. Mingione, Bounded minimisers of double phase variational integrals, Arch. Raton. Mech. Anal., 215 (2015), 443–496.
[4] M. Colombo and G. Mingione, Bounded minimisers of double phase variational integrals, Arch. Raton. Mech. Anal., 218 (2015), 219–273.
[5] M. Colombo and G. Mingione, Calderón-Zygmund estimates and non-uniformly elliptic operators, J. Funct. Anal., 270 (2016), 1416–1478.
[6] G. Cupini, F. Leonetti and E. Mascolo, Existence of weak solutions for elliptic systems with \( p, q \)-growth conditions, Ann. Acad. Sci. Fenn. Ser A I Math., 40 (2015), 645–658.
[7] G. Cupini, P. Marcellini and E. Mascolo, Existence for elliptic equations under \( p, q \)-growth, Adv. Differential Equations, 19 (2014), 693–724.
[8] L. Damascelli, Comparison theorems for some quasilinear degenerate elliptic operators and applications to symmetry and monotonicity results, Ann. Inst. H. Poincaré. Anal. non linéaire, 15 (1998), 493–516.
[9] L. Damascelli and B. Sciunzi, Regularity, monotonicity and symmetry of positive solutions of \( m \)-Laplace equations, J. Differential Equations, 206 (2004), 483–515.
[10] L. Damascelli and B. Sciunzi, Harnack inequalities, maximum and comparison principles, and regularity of positive solutions of \( m \)-Laplace equations, Calc. Var. Partial Differential Equations, 25 (2006), 139–159.
[11] E. Di Benedetto, \( C^{1+\alpha} \) local regularity of weak solutions of degenerate elliptic equations, Nonlinear Anal., 7 (1983), 827–850.
[12] L. Esposito, F. Leonetti and G. Mingione, Sharp regularity for functionals with \( (p, q) \) growth, J. Differential Equations, 204 (2004), 5–55.
[13] A. Farina, L. Montoro, G. Riey and B. Sciunzi, Monotonicity of solutions to quasilinear problems with a first-order term in half-spaces, Ann. Inst. H. Poincaré. Anal. Non Linéaire, 32 (2015), 1–22.
[14] D. Gilbarg and N. S. Trudinger, *Elliptic Partial Differential Equations of Second Order*, Springer, 1983.
[15] T. Leonori, A. Porretta and G. Riey, Comparison principles for p-Laplace equations with lower order terms, *Ann. Mat. Pura Appl.*, 196 (2017), 877–903.
[16] P. Marcellini, Regularity of minimizers of integrals of the calculus of variations with non standard growth conditions, *Arch. Ratton. Mech. Anal.*, 105 (1989), 267–284.
[17] P. Marcellini, Regularity and existence of solutions of elliptic equations with p, q-growth conditions, *J. Differential Equations*, 90 (1991), 1–30.
[18] C. Mercuri, G. Riey and B. Sciunzi, A regularity result for the p-Laplacian near uniform ellipticity, *SIAM J. Math. Anal.*, 48 (2016), 2059–2075.
[19] N. G. Meyers and J. Serrin, $H=W$, *Proc. Nat. Acad. Sci. U.S.A.*, 51 (1964), 1055–1056.
[20] G. Montoro, G. Riey and B. Sciunzi, Qualitative properties of positive solutions to systems of quasilinear elliptic equations, *Adv. Differential Equations*, 20 (2015), 717–740.
[21] M. K. V. Murthy and G. Stampacchia, Boundary value problems for some degenerate-elliptic operators, *Ann. Mat. Pura Appl.*, 80 (1968), 1–122.
[22] P. Pucci and J. Serrin, *The Maximum Principle*, Birkhäuser, Basel, 2007.
[23] G. Riey, Boundary regularity for quasi-linear elliptic equations with lower order term, *Electron. J. Differential Equations*, 283 (2017), 1–9.
[24] G. Riey and B. Sciunzi, A note on the boundary regularity of solutions to quasilinear elliptic equations, *ESAIM Control Optim. Calc. Var.*, 24 (2018), 849–858.
[25] B. Sciunzi, Some results on the qualitative properties of positive solutions of quasilinear elliptic equations, *NoDEA Nonlinear Differential Equations Appl.*, 14 (2007), 315–334.
[26] B. Sciunzi, Regularity and comparison principles for p-Laplace equations with vanishing source term, *Commun. Contemp. Math.*, 16 (2014), 1450013, 20pp.
[27] P. Tolksdorf, Regularity for a more general class of quasilinear elliptic equations, *J. Differential Equations*, 51 (1984), 126–150.
[28] N. S. Trudinger, Linear elliptic operators with measurable coefficients, *Ann. Scuola Norm. Sup. Pisa*, 27 (1973), 265–308.
[29] V. V. Zhykov, Averaging of functional of the calculus of variations and elasticity theory, *Izv. Akad. Nauk. SSSR Ser. Mat.*, 50 (1986), 675–710.
[30] V. V. Zhykov, S. M. Kozlov and O. A. Oleinik, *Homogenization of Differential Operators and Integral Functionals*, Springer-Verlag, Berlin, 1994.

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