1. The Lattice Effective Action

The Euclidean Schrödinger functional in Yang-Mills theories without matter fields is:

\[ Z[A^{(f)}, A^{(i)}] = \langle A^{(f)} \rangle \exp(-H T) \mathcal{P}[A^{(i)}] \]  

(1)

where \( \mathcal{P} \) projects onto the physical states and \( H \) is the Hamiltonian of the gauge system.

On the lattice the Schrödinger functional reads:

\[ Z[U^{(f)}, U^{(i)}] = \int \mathcal{D}U \exp(-S), \]  

(2)

where \( S \) is the Wilson action and we impose the boundary conditions:

\[ U(x)|_{x_4=0} = U^{(i)}, \quad U(x)|_{x_4=T} = U^{(f)}. \]  

(3)

Note that the Schrödinger functional is invariant under arbitrary lattice gauge transformations of the boundary links. If we consider:

\[ U(x)|_{x_4=0} = U(x)|_{x_4=T} = U^{\text{ext}}(0, \vec{x}). \]  

(4)

where the lattice links are related to the continuum gauge fields by the well-known relation (\( P \) is the path-ordering operator):

\[ U_{\mu}^{\text{ext}}(x) = P \exp \left\{ +i a g \int_{0}^{1} dt A_{\mu}^{\text{ext}}(x + a t \hat{\mu}) \right\}, \]  

(5)

then we define \( \mathcal{H} \) the lattice effective action for the background field \( A_{\mu}^{\text{ext}}(\vec{x}) \) by means of the lattice Schrödinger functional Eq. (2) as:

\[ \Gamma[A^{\text{ext}}] = -\frac{1}{T} \ln \left\{ \frac{Z[U^{\text{ext}}]}{Z(0)} \right\}. \]  

(6)

In Eq. (6) \( T \) is the extension in Euclidean time, \( Z[U^{\text{ext}}] = Z[U^{\text{ext}}, U^{\text{ext}}] \) and \( Z(0) \) is the lattice Schrödinger functional without external background field \( (U^{\text{ext}}_1 = 1) \).

Our effective action is by definition gauge invariant and can be used for a non-perturbative investigation of the properties of the quantum vacuum. In the case of background fields which give rise to a constant field strength \( \Gamma[A^{\text{ext}}] \) is proportional to the spatial volume \( V \). Thus we are interested in the density of the effective action:

\[ \varepsilon[A^{\text{ext}}] = -\frac{1}{\Omega} \ln \left[ \frac{Z[U^{\text{ext}}]}{Z(0)} \right], \quad \Omega = V \cdot T. \]  

(7)

Our proposal for the gauge invariant effective action on the lattice has been successfully checked in the U(1) lattice gauge theory [13].

Let us consider the \( SU(2) \) lattice gauge theory in a constant Abelian background magnetic field:

\[ A_{\mu}^{\text{ext}}(\vec{x}) = \vec{A}^{\text{ext}}(\vec{x}) \delta_{\mu,3}, \quad A_{k}^{\text{ext}}(\vec{x}) = \delta_{k,2} x_1 H. \]  

(8)

On the lattice:

\[ U_2^{\text{ext}}(x) = \cos \left( \frac{agHx_1}{2} \right) + i \sigma^3 \sin \left( \frac{agHx_1}{2} \right), \]  

\[ U_1^{\text{ext}}(x) = U_3^{\text{ext}}(x) = U_4^{\text{ext}}(x) = 1. \]  

(9)

The periodic boundary conditions imply the quantization of the external magnetic field

\[ \frac{a^2gH}{2} = \frac{2\pi}{L_1} n^{\text{ext}} \]  

(10)
where $n^{\text{ext}}$ is an integer, $L_1$ the lattice extension in the $x_1$ direction (in lattice units).

We perform numerical simulations by using the standard Wilson action. The links belonging to the time slice $x_4 = 0$ are frozen to the configuration Eq. (2). We also impose that the constraints Eq. (2) apply to links at the spatial boundaries (in the continuum this condition amounts to the usual requirement that the fluctuations over the background field vanish at the infinity). In order to evaluate the density of the effective action Eq. (6) we are faced with the problem of computing a partition function. To avoid this problem we consider the derivative of $\epsilon[\bar{A}^{\text{ext}}]$ with respect to $\beta$:

$$
\epsilon'[\bar{A}^{\text{ext}}] = \left\langle \frac{1}{\Omega} \sum_{x,\mu>\nu} \frac{1}{2} \text{Tr} U_{\mu\nu}(x) \right\rangle_0 - \left\langle \frac{1}{\Omega} \sum_{x,\mu>\nu} \frac{1}{2} \text{Tr} U_{\mu\nu}(x) \right\rangle_{A^{\text{ext}}}.
$$

(11)

2. The Unstable Modes

Let us briefly discuss the origin of the Nielsen-Olesen unstable modes $\mathcal{H}$. To evaluate the continuum effective action one writes:

$$
A(x) = A^{\text{ext}}(x) + \eta(x)
$$

(12)

where $\eta(x)$ is the quantum fluctuation on the background field. In the one-loop approximation one retains in the action only the terms quadratic in $\eta(x)$. After performing the Gaussian functional integration in the background gauge one obtains [5]:

$$
\epsilon(H) = \frac{1}{2} H^2 + \frac{11}{48 \pi^2} g^2 H^2 \ln(\frac{gH}{\Lambda^2}) + \mathcal{O}(g^2 H^2).
$$

(13)

However, it was pointed out [4] that the quadratic action is affected by the presence of negative eigenvalues which give rise to the unstable modes. It turns out that the stabilization of the unstable modes induces a negative classical-like term [6] which cancels the classical energy [7]:

$$
\epsilon(H) = \frac{1}{48 \pi^2} g^2 H^2 \ln(\frac{\Lambda^2}{gH}) + \mathcal{O}(g^2 H^2)
$$

(14)

in the thermodynamic limit $V \to \infty$.

On the lattice we must evaluate the effective action in the weak coupling region. In the one-loop approximation it turns out that even the lattice quadratic action displays the unstable modes. Discarding some irrelevant operators we find an approximate formula for the unstable modes eigenvalues:

$$
\lambda_u = (1 - \cos p_4) + (1 - \cos p_3) - \sin(\frac{gH}{2}),
$$

(15)

where $p_\mu = \frac{2\pi}{L_\mu} n_\mu$. In our simulations we fix $L_3 = L_4 = 32$ and vary $L_1 = L_2 = L$ with $n^{\text{ext}} = 1$. Inserting these values into Eq. (15) we find $\lambda_u \leq 0$ when $L \geq L_{\text{crit}} \approx 10$. So that we can switch on and off the unstable modes on the lattice [8].

In Figs. 1 and 2 we show the derivative of the density of the effective action (in units of $\epsilon^{\text{ext}} = 1 - \cos(\frac{2\pi}{L} n^{\text{ext}})$) for different lattice sizes. We see that in the weak coupling region $\epsilon'[\bar{A}^{\text{ext}}]$ tends to the derivative of the external action if $L \leq 8$. On the other hand, if $L > 8$ the derivative of the density of the effective action decreases monotonously by increasing the lattice volume. This peculiar behaviour is a truly dynamic effect due to the unstable modes. Indeed Fig. 3, where we contrast the $U(1)$ and $SU(2)$ case in the weak coupling region ($\beta = 3$ for $U(1)$ and $\beta = 5$ for $SU(2)$), shows that in the $U(1)$ theory $\epsilon'[\bar{A}^{\text{ext}}] \approx 1$. Moreover, according to Eq. (14) we find...
that in the weak coupling region ($\beta = 5$):

$$\frac{\varepsilon'[\vec{A}^\text{ext}]}{\varepsilon'^\text{ext}} = \frac{a}{L_{\text{eff}}^\alpha}; \quad \alpha = 1.47(7),$$

(16)

where $L_{\text{eff}}$ is the effective linear size of the lattice [2].

Remarkably, it turns out that also the peak value of $\varepsilon'[\vec{A}^\text{ext}]$ tends towards zero with the same law as implied by Eq. (16):

$$\left.\frac{\varepsilon'[\vec{A}^\text{ext}]}{\varepsilon'^\text{ext}}\right|_{\text{peak}} = \frac{a'}{L_{\text{eff}}^\alpha'}; \quad \alpha' = 1.5(1).$$

(17)

Equations (16) and (17) tell us that $\varepsilon'[\vec{A}^\text{ext}]$ tends uniformly towards zero in the thermodynamic limit.

3. Conclusions

Our numerical results are suggesting that $\varepsilon'[\vec{A}^\text{ext}] \to 0$ when $L_{\text{eff}} \to \infty$ in the whole $\beta$-region. Thus in the continuum limit $L_{\text{eff}} \to \infty$, $\beta \to \infty$ the confining vacuum screens completely the external chromomagnetic Abelian field. In other words, the continuum vacuum behaves as an Abelian magnetic condensate medium in accordance with the dual superconductivity scenario. Moreover the magnetic condensate dynamics seems to be closely related to the presence of the Nielsen-Olesen unstable modes. Thus, our approach to a gauge-invariant effective action on the lattice opens the door towards the understanding of the dynamics of color confinement.

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