Are IR Renormalons a Good Probe for the Strong Interaction Domain?

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Abstract

We study the origin of non-analyticity in $\alpha_s$ of a short-distance QCD observable to demonstrate that the infrared renormalons, the same-sign factorial growth of the perturbative expansion, is a universal phenomenon that originates entirely from the small coupling domain. In particular, both the position and the nature of the singularity of the Borel transform of the perturbative series prove to be independent of whether the running coupling $\alpha(k^2)$ becomes singular at some finite scale (“Landau pole”), or stays finite down to $k^2=0$. We argue that getting hold of the infrared renormalons per se can help next to nothing in quantifying non-perturbative effects.
1 Introduction

Last years showed revival of interest to the problem of asymptotic behaviour of the perturbative (PT) expansion in QCD [1]. Considering the Borel transform of PT series has been suggested [2] as technical means of improving convergence, and demonstrated that the PT expansion by itself prompts about nontrivial dynamics at low momentum scales via infrared (IR) renormalons.

In reality, one does not expect QCD observables to have proper analyticity in $\alpha_s$ in the vicinity of $\alpha_s = 0$, which is necessary to justify the Borel resummation as a mathematically well defined operation. The original arguments rely on the nontrivial analytic properties of QCD observables with respect to the external momentum scale parameter $Q^2$. These properties are driven by causality and unitarity constraints, supplemented with the pattern of the hadron spectrum. On the other hand, the IR renormalons themselves have a simpler origin related merely to the nature of the perturbative expansion. In the present paper we address these features of the PT expansion to illustrate that the IR renormalons are not a reflection of the actual non-perturbative (NP) dynamics. Rather they are an artefact of an attempt to describe physics occurring at quite different scales by means of one and the same expansion parameter. The salient conclusions we draw from our analysis are:

- IR renormalons persist even if the coupling stays finite at arbitrarily small momentum scales when no apparent uncertainty associated with the Landau singularity shows up.
- The position and the nature of the IR renormalon singularity in the Borel image of a generic hard QCD observable are determined by the first two coefficients of the $\beta$-function.
- The modification of the Borel integration prescription one can design to represent the true answer, depends heavily on the details of strong interaction dynamics as well as on the particular observable. Therefore, such a possibility seems to carry no practical value as a perturbative algorithm.
- The $Q^2$-dependence of a short distance observable, inferred from the Borel resummation, typically yields an incorrect image of the actual magnitude of condensate effects, when considered at intermediate $Q^2$.

We also briefly address the OPE-motivated procedure to illustrate that it is free from the IR renormalon problems.

In this paper we outline the main framework and the results of the analysis supplying minimal illustrations when necessary. A more complete discussion with better elaborated qualitative and quantitative considerations and deeper physical arguments about the relevance of the analysis undertaken, will be presented in a forthcoming publication [3].

2 Toy model

To study the origin of the IR renormalons and their relation to NP dynamics we concentrate on a simplified model peeled off inessential details. To this end, we address the problem of the PT expansion of the integral

$$I(\alpha) = \int_0^{Q^2} \frac{dk^2}{k^2} \left( \frac{k^2}{Q^2} \right)^p \alpha(k^2)$$

as a model for a short-distance dominated observable. Integrals of this type naturally emerge, for example, when one calculates the one-gluon correction to a collinear safe QCD observable using the running coupling in the integrand. Correctness of this substitution can be unambiguously proven in an Abelian theory; in the QCD context such a doing is often motivated by the “naive non-Abelization” or by the “extended BLM prescription” for fixing the relevant hardness scale in...
It is easy to see that the observable \((2.1)\) satisfies the inhomogeneous renormalization group (RG) equation

\[
\beta(\alpha) \frac{dI(\alpha)}{d\alpha} + p I(\alpha) = \alpha , \tag{2.2}
\]

where

\[
\frac{d\alpha(k^2)}{d \log k^2} = \beta \left( \alpha(k^2) \right). \tag{2.3}
\]

Hereafter, to simplify notation, we absorb the factor \(\frac{\beta_0}{4\pi}\) into the coupling and denote by \(\alpha\) (without an argument) its value at the external hard scale of the problem,

\[
\alpha \equiv \frac{\beta_0}{4\pi} \alpha_s(Q^2).
\]

In this notation the PT expansion of the QCD \(\beta\)-function is

\[
\beta(\alpha) = -\alpha^2 - \gamma \alpha^3 + \text{higher terms} , \quad \gamma = \frac{\beta_1}{\beta_0^2} > 0 . \tag{2.4}
\]

In this Section we concentrate on the properties of \(I\) as a function of \(\alpha\) (its perturbative expansion, Borel representation, analyticity in \(\alpha\)). The analytic properties of \(I(\alpha)\) stem from the dependence of \(\alpha(k^2)\) on \(\alpha\) at \(k^2 < Q^2\), which dependence is determined by the RG trajectories

\[
\int_\alpha^{\alpha(k^2)} \frac{d\alpha'}{-\beta(\alpha')} = \ln \frac{Q^2}{k^2} \equiv t. \tag{2.5}
\]

In what follows we shall refer to \(t\) as “time”.

It is clear that the integral \((2.1)\) is sensible only if \(\alpha(k^2)\) does not develop a Landau singularity at finite positive \(k^2\) (at finite time), which implies that \(\beta(\alpha)/\alpha\) has a zero on the positive real axis. (The first such zero will be generically denoted by \(\bar{\alpha}, \ 0 < \bar{\alpha} \leq +\infty\).) Arguments in favour of an “infrared finite” coupling as the only reasonable expansion parameter for QCD observables, at least in the present context, will be given in a more detailed publication [3].

Our aim is to compare the “exact” expression \((2.1)\) with the results one obtains using the Borel resummation tricks.

### 2.1 Analyticity in \(\alpha\): perturbative and physical “phases”

It is straightforward to solve the RG equation \((2.2)\). First, one finds the solution of the homogeneous equation

\[
\beta(\alpha) \frac{dX(\alpha)}{d\alpha} + p X(\alpha) = 0 \ , \quad X(\alpha) = (f(\alpha))^p ; \tag{2.6a}
\]

\[
f(\alpha) = \exp \left\{ -\int_\alpha^{\alpha(k^2)} \frac{d\alpha'}{\beta(\alpha')} \right\} . \tag{2.6b}
\]

Function \(f(\alpha)\) is nothing but the RG-invariant expression for a pure power of the momentum scale. For small \(\alpha\), for example, one invokes the expansion \((2.4)\) to obtain, up to an overall factor,

\[
f(\alpha) = \exp \left\{ -\frac{1}{\alpha} \right\} \alpha^{-\gamma} (1 + \mathcal{O}(\alpha)) = \frac{\text{const}}{Q^2} . \tag{2.7}
\]
In terms of \( f(\alpha) \) one can write the solution of the inhomogeneous equation in the form

\[
I(\alpha) = f^p(\alpha) \int_{\alpha}^{\alpha_\infty} \frac{d\alpha'}{\beta(\alpha')} \frac{\alpha'}{f^p(\alpha')}.
\] (2.8)

Here \( \alpha_\infty \) is the asymptotic value of the running coupling at \( t=\infty \) along the trajectory (2.5) that starts from a given \( \alpha \) at \( t=0 \). We shall call such point(s) attractive. In the physical phase, that is for real positive (small) initial \( \alpha \) values, \( \alpha_\infty = \bar{\alpha} \):

\[
I^{\text{PH}}(\alpha) = \int_{\alpha}^{\bar{\alpha}} \frac{d\alpha'}{\beta(\alpha')} \exp \left\{ -p \int_{\alpha'}^{\bar{\alpha}} \frac{d\alpha''}{\beta(\alpha'')} \right\}.
\] (2.9)

From explicit expressions (2.8), (2.7) it is clear that an analytic continuation of \( I(\alpha) \), starting from some small positive \( \alpha \), can fail – for small \( \alpha \) – only due to non-analyticity in \( \alpha \) of the lower limit of the integral, that is \( \alpha_\infty(\alpha) \). Let us demonstrate that it does fail; the faster, the smaller initial value of \( \alpha \) is taken.

To this end let us look at the RG trajectories \( \alpha(t) \) for complex initial values of \( \alpha \). We start with the case when \(|\alpha| \) is small but its phase is finite. In this case the value of \(|\alpha(t)| \) stays uniformly small along the whole trajectory, \( 0 \leq t \leq \infty \), so that it suffices to approximate the full \( \beta \)-function by its one-loop expression to find

\[
\frac{1}{\alpha(t)} = \left( \frac{1}{\alpha} - t \right) \left[ 1 + \mathcal{O}(\alpha(t) \ln \alpha(t)) \right] \approx \frac{1}{\alpha} - t.
\] (2.10)

In this approximation trajectories lie on small circles (either in the upper or in the lower half-plane) that touch the real axis at \( \alpha = 0 \). The radius of a circle is related to the phase of the initial \( \alpha \) by

\[
r = \frac{1}{2} |\text{Im} \alpha|^{-1} = \frac{1}{2} \frac{|\alpha|^2}{|\text{Im} \alpha|} \ll 1.
\] (2.11)

It is important to realize that the existence of the family of such trajectories having the common limiting point \( \alpha_\infty \equiv \bar{\alpha}_0 = -0 \) is a universal, purely perturbative feature of the asymptotically free theory with \( \beta \propto -\alpha^2 \). In QCD the double-zero of the \( \beta \)-function acts as a repulsive point for trajectories with \( \alpha > 0 \) and as an attractive one for \( \alpha < 0 \). The domain in the complex \( \alpha \)-plane covered by these trajectories will be called the “perturbative phase”. Within this domain \( I(\alpha) \) is an analytic function but explicitly different from the physical answer (2.3). So, we define

\[
I^{\text{PT}}(\alpha) = \int_{-0}^{\alpha} \frac{d\alpha'}{\beta(\alpha')} \exp \left\{ -p \int_{\alpha'}^{\alpha} \frac{d\alpha''}{\beta(\alpha'')} \right\}.
\] (2.12)

With the decrease of the phase of \( \alpha \), the radius (2.11) increases and trajectories start to distort, being affected by the higher terms in the \( \beta \)-function. At some critical point \( (r \sim 1, \text{ in general}) \) a bifurcation occurs and the trajectories switch to another attractive point, different from \( -0 \). Such a jump results in a singularity (non-analyticity) of \( I(\alpha) \). It is easy to see from (2.11) that this happens for almost real initial \( \alpha \) values:

\[
\frac{\text{Im } \alpha}{\text{Re } \alpha} = \frac{c |\alpha|^2}{\text{Re } \alpha} \approx c |\alpha| \approx c \text{ Re } \alpha \ll 1.
\] (2.13)

In the general case, the topological structure of the analyticity domains and, correspondingly, the number of jumps \( I(\alpha) \) experiences while approaching the real axis, can be quite complicated, reflecting the structure of the attractive zeroes of the \( \beta \)-function in the complex \( \alpha \)-plane. For the
Figure 1: Examples of RG trajectories (dashed) and separatrices (solid) for the polynomial $\beta$-function (2.14a) with $\alpha_1 = \infty$, $\bar{\alpha} = 1$ (left) and for a singular $\beta$ (2.14b) with $\gamma = 0$, $h = 1$ (right).

sake of simplicity we shall restrict our wording to the scenario when the very first bifurcation ("phase transition" from the perturbative domain) leads us directly to the physical phase; the generalization is straightforward.

To illustrate this phenomenon one can consider two simplest examples with the two attractive points, $-0$ and $\bar{\alpha}$, with both examples respecting the first two PT terms of the QCD $\beta$-function (2.4). In the first model with a polynomial $\beta$ the physical coupling freezes at a finite value $\bar{\alpha}$:

$$\beta(\alpha) = -\alpha^2(1 + \alpha/\alpha_1)(1 - \alpha/\bar{\alpha}) , \quad \alpha_1^{-1} - \bar{\alpha}^{-1} = \gamma.$$  \hspace{1cm} (2.14a)

The second model employing a rational $\beta$-function possesses a pair of complex conjugated poles and yields $\alpha(k^2)$ increasing logarithmically with $k^2 \to 0$, that is $\bar{\alpha} = \infty$:

$$\beta(\alpha) = -\frac{\alpha^2}{(1 - \frac{1}{2}\gamma\alpha)^2 + h^2\alpha^2}.$$  \hspace{1cm} (2.14b)

In Fig.1 examples of trajectories are shown together with the characteristic lines – separatrices – which border PT and PH domains. Crossing a separatrix in $\alpha$ causes non-analyticity in $I(\alpha)$. It is worth reminding that the fact that the separatrices emerge from the origin $+0$ along the circular arcs, is of the most general nature and does not depend on the details of the model chosen for illustration. Thus, we arrive at the main conclusion of this Section, that $I(\alpha)$ (for a rather trivial, purely perturbative, reason) cannot be analytic in any sector with a finite opening angle $\theta_0$ covering the positive real direction in the $\alpha$-plane. As a result, the usual Borel representation of the physical quantity $I^{PH}(\alpha)$ does not exist. Indeed, if the integral

$$\tilde{I}(\alpha) = \int_0^\infty \, du \, B(u) \, \exp\left\{-\frac{u}{\alpha}\right\}$$

existed for some $\alpha = \alpha_0$, it would define the function analytic for all $\alpha$ with $\text{Re}\, \alpha^{-1} > \alpha_0^{-1}$ which is the interior of the circle $|\alpha - \frac{1}{2}\alpha_0| = \frac{1}{2}\alpha_0$.

This non-analyticity in $I(\alpha)$ at arbitrarily small positive $\alpha$ manifests itself in a singularity of the Borel image on the real positive $u$-axis, which is equivalent to the same-sign factorial asymptote of the PT coefficients. In the following we show that the Borel resummation actually yields the perturbative function (2.12), $\tilde{I}(\alpha) = I^{PT}(\alpha)$. 

4
2.2 Borel transform for the PT phase

Following the above line of reasoning, one concludes that the analyticity domain in the perturbative phase, contrary to the physical phase, is broad enough to allow one to represent $I^{PT}(\alpha)$ in terms of the Borel integral. Such a representation, however, involves imaginary values of the Borel parameter $u$. Namely, for sufficiently small $\alpha$ in the upper half-plane ($\text{Im} \alpha > 0$) one can write

$$I^{PT}(\alpha) = \int_0^{i\infty} du \, B(u) \exp\left\{ -\frac{u}{\alpha} \right\}. \tag{2.15a}$$

The inverse relation gives the Borel transform $B(u)$ as a regular function (for imaginary $u$),

$$B(u) = \int_{-\infty-i/2}^{-\infty+i/2} \frac{dz}{2\pi i} \, I^{PT}(z^{-1}) \exp\{uz\}. \tag{2.15b}$$

The integration line in $z = 1/\alpha$ shown in (2.15b) corresponds to a (small) circle in the upper half-plane of $\alpha$, namely, $|\alpha - ir| = r$.

In terms of the PT coefficients, one substitutes the (asymptotic) PT expansion for $I$,

$$I^{PT} = \sum_{n=1}^{\infty} a_n \alpha^n, \tag{2.16}$$

into (2.15b). Closing the contour by the (multiple) pole at $z = 0$ ($\text{Re} \, uz < 0$ for $\text{Im} \, z > 0$), one obtains the standard improved series for the function $B(u)$

$$B(u) = \sum_{n=0}^{\infty} \frac{a_{n+1}}{n!} u^n. \tag{2.17}$$

Thus, for imaginary values of $u$ this series determines the regular Borel image $B(u)$.

2.3 “Condensate” contribution

The IR renormalon problem arises when one attempts to reconstruct the observable $I(\alpha)$ for real $\alpha$ by means of the perturbative series (2.16) processed via the Borel machinery (2.17) and the integral representation (2.15a). The latter now runs, however, along the real positive $u$-axis. Then $B(u)$ shows up a singularity (a pole, for the one-loop $\beta$-function, or a cut in general) at $u = u_0$ (in our normalization for the coupling, $u_0 = p$). As a result, the answer for $I$ becomes ambiguous and, generally, complex, depending on the way one choses to pass by the singular point on the integration line.

The PT coefficients $a_n$ do not care whether the expansion parameter $\alpha$ has been chosen real or imaginary. Therefore, we come to conclude that the IR renormalon is nothing but the outcome of an unjustified attempt to force the perturbative answer $I^{PT}$ (2.12) analytically continued outside its native phase, to represent the true $I^{PH}$. Now we are in a position to cure the wound. To do so we observe that the physical answer as given by (2.9) can be reconstructed as a sum of two contributions: the PT piece, $I^{PT}(\alpha)$, analytically continued to real $\alpha$ values plus the “condensate” (or “confinement”) contribution originating from the discontinuity due to crossing the separatrix on the way to the real axis. Indeed, let us split the original integral along the physical trajectory in (2.9), running from $\bar{\alpha}$ to $\alpha$ into two pieces, $-\infty \to \alpha$ and $\bar{\alpha} \to -0$, to write

$$I^{PH}(\alpha) = I^{PT}(\alpha) + H(\alpha), \tag{2.18}$$
where
\[ H(\alpha) \equiv -\int_0^\alpha \frac{d\alpha'}{\beta(\alpha')} \exp \left\{ -p \int_0^{\alpha'} \frac{d\alpha''}{\beta(\alpha'')} \right\} = -f^p(\alpha) \int_0^\alpha \frac{d\alpha'}{\beta(\alpha')} \frac{\alpha'}{f^p(\alpha')} \alpha' \]
(2.19)
The “condensate” contribution \( H \) satisfies the homogeneous RG equation (2.6). Therefore, turning from the \( \alpha \) representation to the momentum dependence, one observes a pure power
\[ H(\alpha) = C f^p(\alpha) = \frac{C_\Lambda}{Q^{2p}} \]
(2.20)
with a complex constant \( C \) (or, equivalently, a dimensionful constant \( C_\Lambda \)) to be determined from (2.19). For example, for two models (2.14) one has
\[ f(\alpha) = \alpha^{-\gamma} \exp \left\{ -\frac{1}{\alpha} \left( 1 + \frac{\alpha}{\alpha_1} \right) \frac{\pi}{\alpha_1^{\alpha_1+\gamma}} (1 - \frac{\alpha}{\alpha_1})^{-\frac{\alpha}{\alpha_1^{\alpha_1+\gamma}}} \right\} \]
(2.21a)
and
\[ f(\alpha) = \alpha^{-\gamma} \exp \left\{ -\frac{1}{\alpha} + \left( \frac{\gamma^2}{4} + h^2 \right) \alpha \right\} \]
(2.21b)
The “condensate” constant \( C \) is expressed then, respectively, in terms of the hypergeometric or the Bessel function, depending on \( p \) and the parameters of the \( \beta \)-function [3].

From (2.20) the reason becomes clear why do we refer to \( H \) as the condensate contribution (without quotation marks, from now on). Let us draw the reader’s attention to the fact that \( I^{PH} \) and \( I^{PT} \) have identical PT expansions, and \( H \) has obviously none.

3 Rescuing Borel representation: mission impossible

A question arises, whether the Borel-type integral representation for the physical answer can be rescued. At least within our simplified approach in which the NP dynamics has been embodied into the behaviour of the \( \beta \)-function at \( \alpha \sim 1 \), the answer to this question is: in principle, yes. However, as a more extensive analysis shows, the Borel construction one is looking for proves to be not universal, depending essentially on the NP features of the theory. In other words, getting hold of the best possible PT information appears to be insufficient for this purpose, calling for genuinely new, non-perturbative, input. The problem is mathematically more involved, so that in the present note we give only a brief sketch of the procedure.

One starts by explicitly constructing the Borel image of \( I^{PT} \). This can be done by translating the RG equation (2.2) into an equation for \( B(u) \). To this end one writes (2.15b) in the form
\[ B(u) = \frac{1}{u-p} \int_C \frac{dz}{2\pi i} e^{pz} I^{PT}(z^{-1}) \frac{d}{dz} e^{(u-p)z} \]
(3.1)
and, upon integrating by parts, makes use of (2.2) to arrive at the following compact symbolic equation
\[ uB(u) + \phi \left( \frac{1}{\partial_u} \right) \{ (p-u)B(u) - 1 \} = 0 ; \quad \partial_u \equiv \frac{d}{du} \]
(3.2a)
The function \( \phi \) here quantifies the deviation of the \( \beta \)-function from its one-loop expression and is given by
\[ \phi(\alpha) = \frac{\alpha^2}{\beta(\alpha) + \alpha^2} \]
(3.2b)
For the case of the pure one-loop \( \beta(\alpha) = -\alpha^2 \), one has \( 1/\phi \equiv 0 \), and one gets a simple pole solution

\[
B(u) = \frac{1}{p-u} = \frac{1}{p} \left( 1 - \frac{u}{u_0} \right)^{-1}, \quad u_0 = p \left( \frac{4\pi p}{\beta_0} \right) \text{ in the standard normalization}. \tag{3.3}
\]

Given a (rational) \( \beta \)-function, it is straightforward to reduce (3.2a) to the differential equation (with proper initial conditions) for \( B(u) \). It is important to emphasize that the master (integral) equation (3.2), as well as its differential counterpart, is an equation with the only singular point

\[
x \equiv 1 - \frac{u}{u_0} = 0,
\]

which, in general, results in a branch point in the solution, \( B(u) \), at \( u = u_0 \). Near the singularity

\[
B(u) = x^{-1}F(x) \approx c_0 \left( 1 - \frac{u}{u_0} \right)^{-1-p\gamma}, \tag{3.4}
\]

where \( \gamma \) is given by the first two loops of the \( \beta \)-function, see (2.4). Thus, one observes that both the position and the nature of the Borel singularity have purely perturbative origin. It is this universal singularity (3.4) that governs the same-sign factorial growth of the PT series.

For example, within the models (2.14) one arrives at a second order differential equation. In the first case (2.14a) it is a homogeneous confluent hypergeometric equation for the function \( F(x) = xB \), with the initial conditions \( F(1) = 1, F'(1) = 0 \). For the second model (2.14b) \( B(u) \) is given by the solution of the inhomogeneous Bessel equation (see [3] for details).

Having obtained \( B(u) \), one then has to construct the Borel integral

\[
I^\text{PT}(\alpha) = \int_0^\infty du \, B(u) \exp \left\{ \frac{u}{\alpha} \right\}, \tag{3.5}
\]

passing the singular point, say, from above and to add the condensate contribution due to crossing the upper half-plane separatrix. From the Borel plane point of view, the latter contribution, \( H(\alpha) \), as a solution of the homogeneous RG equation (2.6), can be written (up to a factor) as an integral of the same function \( B(u) \) along a quasi-closed contour, namely, around the cut \( u_0 < u < \infty \). Thus, the full answer can be represented as a sum of two integrals, one from zero to plus infinity (3.5) and the second embracing the cut (3.4). The imaginary parts of these two contributions clearly cancel, as together they constitute the physical answer \( I^\text{PH} \).

How much of the real stuff remains? To address this problem we restrict ourselves to a polynomial \( \beta \)-function of power \( n+2 \). In this case one has the \( n \)-th order differential equation to determine \( B(u) \) (generalized confluent hypergeometric equation), with the initial conditions

\[
pB(0) = 1, \quad \left( \frac{d}{du} \right)^k (p-u)B(u) \bigg|_{u=0} = 0, \quad k = 1 \ldots n-1.
\]

Let us examine the large \( u \) asymptote of the solution. For \( u \to \infty \) equation (3.2a) reduces to

\[
0 = \left[ 1 - \phi(\partial_u^{-1}) \right] uB(u) \quad \Rightarrow \quad \beta(\partial_u^{-1})uB(u) = 0,
\]

which implies

\[
uB(u) \simeq d_0 \exp \left\{ \frac{u}{\bar{\alpha}} \right\} + \sum_i d_i \exp \left\{ \frac{u}{\bar{\alpha}_i} \right\}. \tag{3.6}
\]

Here \( \bar{\alpha}_i \neq 0 \) is a set of zeroes of the \( \beta \)-function, other than the relevant one, the physical fixed point \( \bar{\alpha} \). This particular term should cancel, however, in the combination of the ordinary (PT) and
the contour (NP) integrals (the discontinuity of $B(u)$ is a solution of the homogeneous equation as well). Otherwise, the answer would be singular at $\alpha = \bar{\alpha}$, which is not the case, since in the vicinity of the fixed point it is analytic:

$$I_{\text{PH}}^{\alpha} = \frac{\bar{\alpha}}{p} + \mathcal{O}(\alpha - \bar{\alpha}).$$

Moreover, for quite a while $I(\alpha)$ stays regular above the fixed point, $\alpha > \bar{\alpha}$, before the RG trajectories betray $\bar{\alpha}$ for another, higher attractive point, i.e. another (unphysical) NP phase occurs in which the coupling is never small (no asymptotic freedom, that is).

From this consideration one concludes that if $\bar{\alpha}$ were the only zero, the condensate contribution (contour integral) would have to cancel the PT integrand above the singular point $u_0$ completely. This is simply because there would be no other terms in the sum (3.6) to maintain the asymptote.

This explains a miracle of the finite Borel representation one obtains within the model for the two-loop $\beta$-function with an (anti-QCD) negative $\gamma$. One can get this model from (2.14a) setting $\alpha_1 = \infty$ ($\gamma = -\bar{\alpha}^{-1}$). The master equation (3.2) for the Borel image then becomes the first order differential equation,

$$\phi(\partial_u) = \frac{1}{-\gamma} \partial_u = \bar{\alpha} \partial_u; \quad \bar{\alpha} \frac{d}{du} [(p - u)B(u)] + uB(u) = 0, \quad B(0) = 1/p. \quad (3.7a)$$

Its solution,

$$B(u) = \frac{1}{p} \left( 1 - \frac{u}{p} \right)^{-1-p\gamma} \exp \left\{ \frac{u}{\bar{\alpha}} \right\} \equiv \frac{1}{p} \left( 1 - \frac{u}{u_0} \right)^{-1+p/\bar{\alpha}} \exp \left\{ \frac{u}{\bar{\alpha}} \right\}, \quad (3.7b)$$

reveals explicitly the general features, namely, the nature of the singularity at $u_0$ and the expected exponential behaviour at infinity.

To evaluate the condensate contribution one first puts $1/\alpha_1 = 0$ in (2.21a) to fix

$$f(\alpha) = \left( \frac{\bar{\alpha} - \alpha}{\alpha} \right)^{-1/\bar{\alpha}} \exp \left\{ -\frac{1}{\alpha} \right\}. \quad (3.8)$$

Then, one performs integration in (2.19) to obtain

$$C = -\int_{-\infty-i0}^{\bar{\alpha}} \frac{d\alpha'}{\beta(\alpha')} \frac{\alpha'}{f^p(\alpha')} = \int_{-\infty-i0}^{\bar{\alpha}} \frac{d\alpha'}{\alpha'(1-\alpha'/\bar{\alpha})} \left( \frac{\bar{\alpha} - \alpha'}{\alpha'} \right)^{\gamma_p} \exp \left\{ \frac{p}{\alpha'} \right\}$$

$$= -\int_{-\infty-i0}^{1} dz \left( z - 1 \right)^{-1+\gamma_p} \exp \left\{ \gamma_p z \right\} = \left( \frac{e}{\gamma_p} \right)^{\gamma_p} \Gamma(\gamma_p) e^{-i\pi\gamma_p}; \quad \gamma_p = -p\gamma = \frac{p}{\bar{\alpha}} > 0. \quad (3.8)$$

Finally,

$$H(\alpha) = \left( \frac{e}{\gamma_p} \right)^{\gamma_p} \Gamma(\gamma_p) e^{-i\pi\gamma_p} \cdot \left( \frac{\alpha}{\bar{\alpha} - \alpha} \right)^{\gamma_p} \exp \left\{ -\frac{p}{\alpha} \right\}. \quad (3.9)$$

This expression, upon inspection, differs only by sign from the part of the ordinary (PT) Borel integral from $u_0$ to $\infty$,

$$H(\alpha) = -\int_{u_0}^{\infty+i0} du B(u) \exp \left\{ -\frac{u}{\alpha} \right\}. \quad (3.9)$$

As a result, for the physical answer in this model the finite-support Borel-type representation holds,

$$I_{\text{PH}}^{\alpha} = \int_{0}^{u_0} du B(u) \exp \left\{ -\frac{u}{\alpha} \right\}. \quad (3.10)$$
This observation has been independently made in \[5\].

We now try to examine on this explicit example the standard routine employed in the renormalon analysis. Namely, it is accustomed to take the PV of the standard Borel integrals which amounts, for this model, to taking the value

\[
I_B(\alpha) \equiv I^{\text{PH}}(\alpha) - \text{Re} H(\alpha) = I^{\text{PH}}(\alpha) - f^p(\alpha) \left( \frac{e}{\gamma_p} \right)^{\gamma_p} \Gamma(\gamma_p) \cos \pi \gamma_p .
\]  

(3.11)

At the same time, the uncertainty associated with the Borel resummation procedure is usually estimated as

\[
\delta I(\alpha) = \frac{1}{2\pi} \left| \int_C du B(u) e^{-u/\alpha} \right| ,
\]  

(3.12)

with the contour running around the cut, which in our case yields

\[
\delta I(\alpha) = f^p(\alpha) \left( \frac{e}{\gamma_p} \right)^{\gamma_p} \Gamma(\gamma_p) \frac{1}{\pi} |\sin \pi \gamma_p| .
\]  

(3.13)

Both the error of the PV Borel summation \(\Delta_B = I^{\text{PH}} - I_B\) and the estimated uncertainty \(\delta I\) have the same correct power behavior \((\Lambda^2/Q^2)^p\). However, their relative magnitude, in general, mismatches:

\[
\frac{\Delta_B}{\delta I} = \frac{\pi}{|\tan \pi \gamma_p|} ,
\]  

(3.14)

so that the actual error can become much larger than the estimated one, which happens, for example, when \(\gamma_p\) is numerically small\[\text{[6]}\]. The origin of such a mismatch is readily understood \[\text{[3]}\].

It is worthwhile to mention another interesting feature of an interplay between PT and NP contributions. Namely, when \(\gamma_p\) happens to be a positive integer, the singularity (3.4) disappears and \(B(u)\) remains analytic in the entire complex \(u\)-plane. There is no ambiguity in choosing the path of integration, and the Borel summation yields unambiguous – but incorrect – result! What is more intriguing from the viewpoint of the naive PT analysis is that the factorial growth of the coefficients in the expansion of \(I(\alpha)\) disappears:

\[
a_n \sim n^{\gamma_p-1} (\bar{\alpha})^{-n} .
\]  

(3.15)

Resummation of the “subleading” \(1/n\) corrections to the asymptote of \(a_n\) kills factorials altogether. Analyzing mere perturbative series one would not infer any deficiency of the expansion, although the presence of the “NP condensate” has been demonstrated explicitly.

For illustration, in the simplest case \(\gamma_p=1\) one has

\[
I(\alpha) = \frac{\alpha \bar{\alpha}}{p(\bar{\alpha} - \alpha)} \left( 1 - \exp \left\{ 1 - \frac{\bar{\alpha}}{\alpha} \right\} \right) .
\]  

(3.16)

The first term is what one gets by the PT summation whereas the second one is the NP contribution. The former is merely a geometric series in \(\alpha\), showing no indication of NP effects. The latter term, on the contrary, has literally no PT expansion. In general, however, the situation is different and such an apparent splitting is absent, so that PT and NP pieces cannot be explicitly separated. It is just the genuine case of intrinsically mixed PT and NP contributions when the PT series normally send the message, via IR renormalons, about the presence of the non-perturbative effects.

\[\text{[1]}\] We parenthetically note that it is actually the case for the limit \(\beta_0 \to \infty\) (or \(n_f \to -\infty\)), viz., one gets \(\gamma \sim 1/n_f\); if one assumes that the higher order terms in the \(\beta\)-function can be in turn obtained by only the leading in \(1/n_f\) contributions, though literally one gets positive \(\gamma\).
Notice, that for a generic $\beta$-function ($n>1$) an integer $\gamma$ does not mean analyticity: $u_0$ becomes a logarithmic branch point of $B(u)$.

A curious observation is that for a given (polynomial) $\beta$-function one can construct a polynomial $P_n(\alpha(k^2))$ to replace $\alpha(k^2)$ in the definition of the "observable" \cite{2}, such that $B(u)$ becomes analytic and, thus, the PT expansion factorial-free \cite{3}. However, such a construction proves to be non-universal with respect to $p$.

In conclusion, let us mention that an attractive scenario of the finite-support Borel representation yielding $I^{PH}(\alpha)$ analytic everywhere but at $\alpha = 0$, is a peculiar property of the oversimplified two-loop model. Elsewhere such a possibility may accidentally occur only at specific, contrived values of the parameters. However, this would hold, once again, for one particular value of $p$, while for observables with the canonical dimension other than $2p$ the contribution from $u > u_0$ gets resurrected. Therefore we consider such an accident carrying no physical significance.

4 Toy OPE for the Toy model

For short distance Euclidean observables the Wilson OPE is known to provide a proper framework for describing power suppressed effects. It automatically cures, once and forever, the IR renormalon trouble for the price of modifying perturbative coefficients \cite{3}. The latter are changed only a little for describing power suppressed effects. It automatically cures, once and forever, the IR renormalon \cite{3} such that the series converge for $Q^2/\mu^2$ if $\alpha$ is chosen above $\alpha_{sh}$, and diverge (for sufficiently small $\alpha$, that is, large $\log(Q^2/\mu^2)$, sic!) if $\mu < \mu_{sh}$. We denote the value of the coupling corresponding to this scale by $\alpha_{sh} = \alpha(\mu_{sh}) < \bar{\alpha}$. Leaving aside tiny details concerning perverted $\beta$-functions \cite{3}, one can determine this OPE-borderline value solving the equation

$$
\text{Re} \int_{\alpha_{sh}}^{\infty} \frac{d\alpha'}{-\beta(\alpha')} = 0,
$$

where the integration contour travels to infinity along a separatrix. Such an equation always has a unique solution between zero and $\bar{\alpha}$. In a general case when the structure of zeroes is rich and there is a set of separatrices, one should take the minimal solution. (Singularities of the $\beta$-function, if any, should also be considered, with the position of the singularity replacing $\infty$ in the integral in
For example, in the two-loop model (3.7) one obtains $\alpha_{sh} \simeq 0.782 \bar{\alpha}$. For the model (2.14b) with $\gamma = 0$, discussed in [7], $\alpha_{sh} = h^{-1}$.

Borel non-summability in $I$ within the OPE lies in the second, large distance, piece $I_{ld}^{\mu}$, which, of course, cannot be even dreamed to be calculated in terms of $\alpha$ expansion. This contribution is explicitly associated with the particular, physical, phase.

We note that an attempt to do the OPE without an IR cutoff, i.e. putting $\mu \rightarrow 0$ (or, equivalently, using Dimensional Regularization to “renormalize” the power convergent integrals like $I$) means calculating $I_{PT}(\alpha)$ instead of $I_{ld}^{\mu}(\alpha)$, and leaving only $H(\alpha)$ in $I_{ld}^{\mu}(\alpha)$. In other words, it is an attempt to entirely subtract “perturbative corrections” from $I_{ld}$. It is just this action that generates the IR renormalons in the coefficient functions making them uncalculable. As long as the exact analytic expression for $I(\alpha)$ is known, such a routine seemingly does not pose particular problems. However, in practical applications it looks rather dangerous when multiloop corrections are incorporated. Indeed, consider for example the $Q^2$-dependence of the observable. Translating the behaviour in $\alpha(Q^2)$ into the $Q^2$-dependence one finds a smooth behaviour of $I$ at small $Q^2$:

$$ I(\alpha(Q^2)) \simeq \frac{\bar{\alpha}}{p} + \frac{1}{p + \rho} \left( \frac{Q^2}{\Lambda^2_{QCD}} \right)^\rho, \quad \rho \equiv \frac{d\beta(\alpha)}{d\alpha} \bigg|_{\alpha = \bar{\alpha}} > 0; \quad \text{for } Q < \Lambda_{QCD}. \quad (4.4) $$

On the contrary, according to (2.20), both the “perturbative” and “purely non-perturbative” parts, taken separately, are singular:

$$ I_{PT}(\alpha(Q^2)) \sim H(\alpha(Q^2)) \propto \left( \frac{\Lambda^2_{QCD}}{Q^2} \right)^p. \quad (4.5) $$

Therefore, the naive Borel resummation typically yields a strongly corrupted picture of the actual scale of long distance phenomena. We will return to this important point in [3].

5 Conclusions

We argued that the IR renormalons have a transparent mathematical origin and are associated with employing the same short distance coupling as an expansion parameter for describing physics at both large and small scales. They do not actually show what happens at low momentum scales but only send a message that something may happen there. Inspired by this idea [9] we studied a simplified model for hard QCD observables in which all the details of strong dynamics are embodied into an IR non-trivial $\beta$-function.

We explicitly showed that, contrary to a rather popular believe, IR renormalons are not directly related to the Landau singularity in the running coupling. Even if it freezes at a small value, the same-sign factorial growth of the PT coefficients remains intact. At the same time, there are curious examples when no IR renormalon is around (the PT series converges!) but the PT expansion is as deficient as in a general case.

The position and the nature of the IR renormalon singularities are determined by the first two coefficients of the $\beta$-function, i.e. are governed by the deep PT domain. The true origin of the IR renormalons is rooted in a non-trivial phase structure of QCD.

We examined how well can renormalons represent, in general, the actual long distance effects and found them unsatisfactory in many important respects including the absolute magnitude, an estimate of incompleteness of the purely PT approximation and the steepness of the $Q^2$-dependence in the transition regime from short distances to the strong interaction domain.
As far as the ultraviolet renormalons are concerned, in the framework of our model they appear in the observables represented by an integral

\[ U(\alpha) = \int_{Q^2}^{\infty} \frac{dk^2}{k^2} \left( \frac{Q^2}{k^2} \right)^p \alpha(k^2). \] (5.1)

Straightforward application of the analysis outlined in the present paper shows that \( U \) is analytic at small \( \alpha \) except for only a narrow beak around the real negative axis. Therefore, its Borel resummation yields the correct result. This being a positive statement, we are not sure of how heavily does the conclusion rely on particular features of the oversimplified model. Therefore we refrain from definite claims of its applicability to actual QCD.

When this paper was in writing we learned about the preprint by G. Grunberg [5] where the finite support Borel representation has been constructed for the two-loop (“non-QCD”) \( \beta \)-function, in agreement with our finding.

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