ABSTRACT. We prove non-trivial bounds for bilinear forms with hyper-Kloosterman sums with characters modulo a prime $q$, which, for both variables of length $M$, are non-trivial as soon as $M \geq q^{3/8+\delta}$ for any $\delta > 0$. This range, which matches Burgess’s range, is identical with the best results previously known only for simpler exponentials of monomials. The proof combines refinements of the analytic tools from our previous paper and new geometric methods. The key geometric idea is a comparison statement that shows that even when the “sum-product” sheaves that appear in the analysis fail to be irreducible, their decomposition reflects that of the “input” sheaves, except for parameters in a high-codimension subset. This property is proved by a subtle interplay between étale cohomology in its algebraic and diophantine incarnations. We prove a first application concerning the first moment of a family of $L$-functions of degree 3.

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1. Introduction

1.1. Presentation of the results. Let $k \geq 1$ be an integer. For a prime $q$ and a tuple $\chi = (\chi_1, \ldots, \chi_k)$ of $k$ Dirichlet characters modulo $q$ (each of which might be trivial), the $(k - 1)$-dimensional generalized hyper-Kloosterman sums associated to $\chi$ are the exponential sums defined

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for \( x \in F_q^\times \) by

\[
\text{Kl}_k(x; \chi, q) = \frac{1}{q^{k-1}} \sum_{y_1, \ldots, y_k \in F_q^\times, y_1 \cdots y_k = x} \chi_1(y_1) \cdots \chi_k(y_k) e\left( \frac{y_1 + \cdots + y_k}{q} \right).
\]

These sums were introduced by Deligne in the case \( \chi_i = 1 \) [SGA4\footnote{SGA4}2] (corresponding to hyper-Kloosterman sums), and then generalized by Katz in [Kat88, Ch. 4]. The most special case is the classical Kloosterman sums \( \text{Kl}_2(x; q) = \text{Kl}_2(x; (1, 1), q) \). As an application of the Riemann Hypothesis over finite fields, Deligne and Katz established the highly non-trivial pointwise bounds

\[
| \text{Kl}_k(x; \chi, q) | \leq k.
\]

Many deep properties of these sums were studied in great depth by Katz in [Kat88] and [Kat90]. Among other things, Katz proved equidistribution statements that describe precisely the distribution of generalized hyper-Kloosterman sums inside \( \mathbb{C} \), at least for most possible choices of \( \chi \).

In [KMS17], motivated by various applications to the analytic theory of automorphic forms and associated \( L \)-functions (especially our work with Blomer, Milićević and Fouvry [BFK\footnote{BFK}^+17, BFK\footnote{BFK}^+]\)), we considered the problem of bounding non-trivially hyper-Kloosterman sums “on average”, namely to estimate general bilinear sums

\[
B(K, \alpha, \beta) := \sum_{m \leq M, n \leq N} \alpha_m \beta_n K(mn; q)
\]
in the case

\[
K(x; q) = \text{Kl}_k(x; (1, \ldots, 1), q),
\]

where \( K \geq 2 \) and \( \alpha = (\alpha_m)_{m \leq M}, \beta = (\beta_n)_{n \leq N} \) are arbitrary complex numbers. The goal in such problems is to achieve a non-trivial bound when the ranges of the summation variables \( M \) and \( N \) are as small as possible. In that first paper, we were able to get significantly below the so-called Polyá-Vinogradov (or Fourier-theoretic) range, which is essentially \( M = N = q^{1/2} \). Precisely, we obtained a bound for the bilinear form with hyper-Kloosterman sums which is non-trivial as long as \( MN \geq q^{7/8+\delta} \) for some \( \delta > 0 \).

The proof involved analytic number theoretic techniques developed in the work of Fouvry and Michel [FM98] (rooted in earlier works of Karatsuba-Vinogradov and Friedlander-Iwaniec) and \( \ell \)-adic cohomology methods having the Riemann Hypothesis of Deligne and the books of Katz [Kat88, Kat90] as starting points.

The earlier paper [FM98] of Fouvry and Michel had considered the problem of bounding bilinear forms as above but with \( K(x) = e(f(x)/q) \) for some fixed rational function \( f \) with coefficients in \( \mathbb{Q} \). In particular, for \( f = X^k + a \), where \( k \in \mathbb{Z}_{\geq 0} \) and \( a \neq 0 \), it was possible to bound the corresponding bilinear sum non-trivially as long as \( MN \geq q^{3/4+\delta} \) for some \( \delta > 0 \). The key tool there was a variant of Katz’s equidistribution results.

The purpose of the present paper is twofold: (1) to generalize the results of [KMS17] to generalized Kloosterman sums, which are likely to be useful in analytic number theory; (2) to reach the same limit of validity \( MN \geq q^{3/4+\delta} \) for generalized hyper-Kloosterman sums.

A special case of our main result, Theorem 4.1, is the following:

**Theorem 1.1.** Assume that \( \chi \) has Property NIO of Definition 2.1. For any \( \delta > 0 \) there exists \( \eta > 0 \) such that for any integer \( k \geq 2 \), any prime number \( q \), and any integers \( M, N \geq 1 \) such that

\[
M + N \geq q^{\delta}, \quad MN \geq q^{3/4+\delta}
\]

we have

\[
\sum_{m \leq M} \sum_{n \leq N} \alpha_m \beta_n \text{Kl}_k(amn; \chi, q) \ll \|\alpha\|_2 \|\beta\|_2 (MN)^{1/2-\eta}
\]
for any \( a \in \mathbb{F}^\times_q \) and for arbitrary families of complex numbers \( \alpha = (\alpha_m)_{m \leq M} \) and \( \beta = (\beta_n)_{n \leq N} \). The implied constant depends only on \( \delta \) and \( k \).

Property NIO (short for “Not Induced or Orthogonal”) is an elementary combinatorial property that we define below in Section 2; it is easy to check, and it is “generically” satisfied in some sense. For instance, the case \( \chi = (1, \ldots, 1) \) corresponding to hyper-Kloosterman sums themselves has NIO, and so does \( (1, \ldots, 1, \chi) \) if \( k \) is odd.

The exponent \( 3/4 \) seem to be a recurring barrier in other related problems: first and foremost in the work of Burgess [Bur62] and more recently (see [FKM15a, FKM14]) when dealing with sums of the shape

\[
\sum_{\substack{p \leq N \text{ prime}}} K(p; q)
\]

where \( p \) ranges over prime numbers, or

\[
\sum_{n \leq N} \lambda_f(n)K(n; q)
\]

where \( K \) is a general trace function modulo \( q \) and \( (\lambda_f(n))_{n \leq N} \) are the Hecke eigenvalues of a fixed Hecke eigenform \( f \) (cuspial or Eisenstein). We cannot pass this barrier in full generality yet but we can at least offer some progress for special bilinear forms, where one of the variable is smooth, i.e., for

\[
B(K, \alpha, 1_N) := \sum_{m \leq M, n \leq N} \alpha_m K(mn; q).
\]

A special case of Theorem 4.2 is:

**Theorem 1.2.** Assume that \( \chi \) has NIO. For any \( \delta > 0 \) there exists \( \eta > 0 \) such that for \( k \geq 2 \) an integer, \( q \) a prime and \( M, N \geq 1 \) some integers satisfying

\[
M + N \geq q^\delta, \quad MN^2 \geq q^{1+\delta}
\]

we have

\[
\sum_{m \leq M, n \leq N} \alpha_m K_l_k(amn; \chi, q) \ll \|\alpha\|_2 (MN^2)^{1/2-\eta}
\]

for any \( a \in \mathbb{F}^\times_q \) and for any tuple of complex numbers \( \alpha = (\alpha_m)_{m \leq M} \), where the implicit constant depends on \( \delta \) and \( k \).

In particular, for \( M = N \), we obtain a non-trivial bound as long as

\[
M = N \geq q^{1/3+\delta}
\]

for some \( \delta > 0 \). If we denote by \( d_2(n) \) the classical divisor function, we deduce the following result:

**Corollary 1.3.** Assume that \( \chi \) has NIO. For any \( \delta > 0 \), there exists \( \eta > 0 \) such that for any integer \( k \geq 2 \), any prime number \( q \), and any \( N \geq q^{2/3+\delta} \), we have

\[
\sum_{n \leq N} d_2(n) K_l_k(an; \chi, q) \ll Nq^{-\eta},
\]

for any \( a \in \mathbb{F}^\times_q \) where the implicit constant depends on \( \delta \) and \( k \).

**Remark 1.4.** It is of interest to generalize results like Theorem 1.1 to other functions \( K \) modulo \( q \). We suspect that the methods of this paper could potentially be applicable to trace functions \( K \) satisfying suitable “big monodromy” assumptions, and the following property: \( K \) belongs to a family \( K_a \) parameterized by non-trivial additive characters \( x \mapsto e(ax/p) \) of \( \mathbb{F}_q \), and this family
satisfies a relation of the type \( K_{a^\mu}(x) = K(a^\nu x) \) for some fixed non-zero integers \( \mu \) and \( \nu \). For instance, this holds for the generalized Kloosterman sums with \( \mu = 1, \nu = k \) when defining
\[
K_a(x) = \frac{1}{q^{k-1}} \sum_{y_1, \ldots, y_k \in \mathbb{F}_q^\times \atop y_1 \cdots y_k = x} \chi_1(y_1) \cdots \chi_k(y_k) e\left(\frac{a(y_1 + \cdots + y_k)}{q}\right).
\]

1.2. Applications to moments of \( L \)-functions. As for our previous paper [KMS17], Theorems 1.1 and 1.2 have applications to the evaluation of moments of \( L \)-functions index by Dirichlet characters modulo \( q \). As a simple illustration, we will prove in Section 3 the following result, which generalizes some recent work of Zacharias [Zac17]:

**Theorem 1.5.** Let \( f \) be a primitive holomorphic cusp form of level 1. For \( q \) prime, let \( \xi \) be a non-trivial Dirichlet character modulo \( q \). There exist an absolute constant \( \delta > 0 \) such that
\[
\frac{1}{q-1} \sum_{\chi \pmod{q}} L(f \otimes \chi, 1/2)L(\xi \chi, 1/2) = 1 + O_f(q^{-\delta}).
\]

**Remark 1.6.** Zacharias established this asymptotic for \( \xi = 1 \) using amongst other ingredients the bounds from [KMS17] for \( K(x) = K_l(x; (1,1,1), q) \); he evaluated more generally a mollified version of this average, enabling him to establish that, for \( q \) large, there is a positive proportion of \( \chi \pmod{q} \) such that \( L(f \otimes \chi, 1/2) \) and \( L(\chi, 1/2) \) are both non-vanishing. Most likely a similar result may be established in our case.

As in [BFK+17, KMS17, BFK+17], we also expect that our results will prove useful to estimate other averages of certain \( L \)-functions of degree 3 and 4. For instance, we may consider:

- The twisted first moment
\[
\frac{1}{q-1} \sum_{\chi \pmod{q}} L(f \otimes \chi, 1/2)L(\xi \chi, 1/2)\prod_i \xi^{k_i}
\]
where \( \xi = (\xi_i)_i \) a tuple of characters of modulus \( q \) (possibly trivial) and \( k = (k_i)_i \) is a family of integers;
- The shifted second moment
\[
\frac{1}{q-1} \sum_{\chi \pmod{q}} L(f \otimes \chi, 1/2)L(f \otimes \xi \chi, 1/2).
\]

1.3. Principle of the proof. We denote \( K(x) = K_{l_k}(ax; \chi, q) \) for a fixed \( k \)-tuple \( \chi \) with Property NIO and a fixed \( a \in \mathbb{F}_q^\times \).

As in our previous work (and [FIS5, FM98]), the proof starts with an application of the \(+ab\)-shifting trick of Karatsuba and Vinogradov. Let us recall that the shifting trick builds on the almost invariance of an interval under sufficiently small translations. The interval to be shifted here is that of the almost invariance of an interval under sufficiently small translations. The interval to be shifted is by product \(+ab\) with \( (a, b) \in [A, 2A] \times [B, 2B] \) for \( A, B \) suitable parameters (such that \( AB = N \)). As \( K(mn; q) \) depends only on the congruence class of \( mn \pmod{q} \) the replacement of \( n \leftrightarrow n + ab \) leads to the following transformations
\[
mn \pmod{q} \leftrightarrow m(n + ab) = an(\overline{a}n + b) = s(r + b) \pmod{q},
\]
\[
(m_1 n, m_2 n) \pmod{q} \leftrightarrow (an_1(\overline{a}n_1 + b), an_2(\overline{a}n_2 + b)) = (s_1(r + b), s_2(r + b)) \pmod{q}
\]
with \( (r, s), (r, s_1, s_2) \) taking values in \( \mathbb{F}_q \times \mathbb{F}_q^\times \) or \( \mathbb{F}_q \times (\mathbb{F}_q^\times - \Delta(\mathbb{F}_q^\times 2)) \). Under suitable assumptions on \( A, M, N \) one then show that the above maps are essentially injective (ie. have fibers bounded in size by \( q^{o(1)} \)). However, these maps are far from being surjective, so performing such a change of variable will result in a loss. This can be tamed by an application of the Hölder inequality with
a sufficiently large exponent, which we denote by $2l$ in the sequel. This process leads then to the problem of bounding sums of the shape

$$
\sum_{b \in B} |\Sigma_I(K, b)|, \quad \sum_{b \in B} |\Sigma_{II}(K, b)|,
$$

where $B$ denotes the set of $2l$-uples of integers $b = (b_1, \ldots, b_{2l}) \in [B, 2B]^{2l}$ and

$$
\Sigma_I(K, b) = \sum_{r \in F_q} \sum_{s \in F_q^*} K(sr, sb),
$$

$$
\Sigma_{II}(K, b) = \sum_{r \in F_q} \sum_{s_1, s_2 \in F_q^*} K(s_1 r, s_1 b)K(s_2 r, s_2 b)
$$

where

$$
(1.1) \quad K(r, b) = \prod_{i=1}^l K(r + b_i)K(r + b_{i+l}).
$$

The next step will be to use methods from $\ell$-adic cohomology to bound the individual sums $\Sigma_I(K, b)$ and $\Sigma_{II}(K, b)$ with square-root cancellation, namely to derive

$$
\Sigma_I(K, b) \ll q,
$$

$$
\Sigma_{II}(K, b) \ll q^{3/2}.
$$

In fact, these bounds do not always hold, but it will be enough to prove them outside a sufficiently small subset $B^{diag}$ of “diagonal” tuples $b$. In [KMS17], we were able to implement this strategy by proving, when $l = 2$, that the set $B^{diag}(\bmod q)$ is contained in the set of $F_q$-point of a proper algebraic subvariety $V^{diag} \subset A_{F_q}^{2l}$, i.e., a variety with codimension $\geq 1$. However, as was already noticed in [FM98], the use of a larger exponent $2l$ in Hölder’s inequality could improve the ranges $M, N$ for which $B(K, \alpha, \beta)$ is bounded non-trivially (in a way similar as Burgess proof), provided this came with a sufficiently good lower bound on the codimension of the variety $V^{diag}$.

It is the purpose of the present paper to implement fully this strategy for any exponent $l \geq 1$. Precisely, we will prove the required estimates with a diagonal subvariety such that

$$
(1.2) \quad \text{codim}(V^{diag}) \geq \frac{l - 1}{2}.
$$

The outcome is that by taking $l$ very large, we obtain non-trivial estimates of $B(K, \alpha, \beta)$ and $B(K, \alpha, 1_N)$ in the ranges defined by

$$
MN \geq q^{3/4+\delta} \quad \text{and} \quad MN^2 \geq q^{2/3+\delta}
$$

for any $\delta > 0$.

**Remark 1.7.** (1) It would be reasonable to expect that the correct codimension is

$$
\text{codim}(V^{diag}) \geq l + o(l)
$$

as $l \to +\infty$, which would indeed be best possible (it is easy to see that the codimension is $\leq l$). A lower bound of this quality was established in [FM98] in case $K(x) = e((x^k + a)/q)$ already mentioned. Although bound (1.2) only goes half of the way to this expectation, it is nevertheless sufficient for our purpose, and it seems that even the full lower bound would not help in improving the exponents $3/4$ and $2/3$ in Theorems 1.1 and 1.2.

(2) Readers who have some familiarity with either [FM98] or [KMS17] will have noticed that we have made a compromise here: the new variables $s$ and $(s_1, s_2)$ belong to the subsets $[A, 2AM]$ and $[A, 2AM]^2 - \Delta([A, 2AM]^2)$ of the larger sets $F_q^\times$ or $F_q^{\times 2} - \Delta(F_q^{\times 2})$, so that we lose something
by “forgetting” this fact by positiviy. It is certainly possible to compensate this loss using the completion method, introducing additional twists by additive characters in the $s$-variable, and handling them by arguments similar to those of [KMS17, §4.5]. However, when $l$ is very large, the improvement in the final bounds is very small (because of (1.2)), more importantly the final limiting exponents $3/4$ and $2/3$ are not improved. So we have chosen to avoid the completion step, in order to simplify an already complex argument. It should be noted however that, for small values of $l$, the completion step is worth pursuing, and that is was crucial in [KMS17] to obtain non-trivial bounds for $l = 2$.

We now sketch briefly the proof of (1.2) in the case of general bilinear forms (the special bilinear forms are easier to handle, and the diagonal variety in the general case also works in the special case). Setting

$$R(r, b) = \sum_{s \in \mathbb{F}_q} K(sr, sb)$$

we have

$$\Sigma_{II}(K, b) = \sum_{r \in \mathbb{F}_q} |R(r, b)|^2 - \sum_{s \in \mathbb{F}_q^*} \sum_{r \in \mathbb{F}_q} |K(sr, sb)|^2,$$

and our aim is to show that the two main terms of the above difference compensate exactly, except for some diagonal $b$, whose cardinality we aim to control as precisely as possible.

For this, we first interpret the functions

$$(r, b) \rightarrow K(r, b), \ R(r, b)$$

as trace functions of $\ell$-adic sheaves $K$ and $R$ on $\mathbb{A} \times \mathbb{A}^{2l}$, which are pointwise pure of weight 0 and mixed of weight $\leq 1$ respectively. This exploits the general formalism of Kloosterman sheaves and $\ell$-adic sheaves. The functions

$$(r, b) \rightarrow |K(r, b)|^2, \ |R(r, b)|^2$$

are the trace functions of the endomorphisms sheaves $\text{End}(K)$ and $\text{End}(R)$. By means of the Grothendieck–Lefschetz trace formula and of Deligne’s most general form of the Riemann Hypothesis over finite fields [Del80], the bound

$$\sum_{r \in \mathbb{F}_q} |R(r, b)|^2 - \sum_{s \in \mathbb{F}_q^*} \sum_{r \in \mathbb{F}_q} |K(sr, sb)|^2 \ll q^{3/2}$$

for a given $b$ amounts to the fact that the specialized sheaves $K_b$ and $R_b$ each have decompositions into geometric irreducible components whose dimensions precisely match.

Our argument in [KMS17] is to show that both these specializations are geometrically irreducible when $b$ lies outside of a hypersurface in $\mathbb{A}^{2l}$. However, it is hopeless to expect to reduce the codimension in the same manner, because one can see that the sheaves will cease to be irreducible for instance on the hypersurface $b_1 = b_{l+1}$. (The argument does apply if the input sheaf is of rank 1, instead of a Kloosterman sheaf of rank $k \geq 2$; this corresponds to the situation in [FM98]).

What we must do then is to prove that the matching between the dimensions of the irreducible components does continue to hold on subvarieties $X$ such as that hypersurface, and this as long as their dimension of $X$ is large enough. The proof of this is intricate, and combines geometric steps as well as a number of diophantine computations that exploit the criteria for irreducibility of algebraic varieties or of sheaves based on point-counting formulas and the Riemann Hypothesis.

**Notation.** For any prime number $\ell$, we fix an isomorphism $\iota : \mathbb{Q}_\ell \rightarrow \mathbb{C}$. Let $q$ be a prime number. Given an algebraic variety $X_{\mathbb{F}_q}$, a prime $\ell \neq q$ and a constructible $\mathbb{Q}_\ell$-sheaf $\mathcal{F}$ on $X$, we denote by $t_{\mathcal{F}} : X(\mathbb{F}_q) \longrightarrow \mathbb{C}$ its trace function, defined by

$$t_{\mathcal{F}}(x) = \iota(\text{Tr}(\mathcal{F}|_{x} | \mathcal{F}_x)), \quad x \in X(\mathbb{F}_q).$$
where \( \mathcal{F}_x \) denotes the stalk of \( \mathcal{F} \) at \( x \). More generally, for any finite extension \( \mathbf{F}_{q^d}/\mathbf{F}_q \), we denote by \( t_{\mathcal{F}}(\cdot; \mathbf{F}_{q^d}) \) the trace function of \( \mathcal{F} \) over \( \mathbf{F}_{q^d} \), namely

\[
t_{\mathcal{F}}(x; \mathbf{F}_{q^d}) = \nu(\text{Tr}(\mathbf{F}_x \mathbf{F}_{q^d} | \mathcal{F}_x)).
\]

An \( \ell \)-adic sheaf will always means a \( \overline{\mathbf{Q}}_\ell \)-sheaf. For standard facts in \( \ell \)-adic cohomology (such as proper base change, cohomological dimension, etc), we refer to the books of Fu [Fu11] and Milne [Mi80], and to the notes of Deligne [SGA4 \( \frac{1}{2} \)].

We will usually omit writing down \( \nu \). In any expression where some element \( z \) of \( \overline{\mathbf{Q}}_\ell \) has to be interpreted as a complex number, we mean to consider \( \nu(z) \).

We denote by \( \mathcal{F} \) the dual of a constructible sheaf \( \mathcal{F} \); if \( \mathcal{F} \) is a middle-extension sheaf, we will use the same notation for the middle-extension dual.

Let \( \psi \) (resp. \( \chi \)) be a non-trivial additive (resp. multiplicative) character of \( \mathbf{F}_q \). We denote by \( \mathcal{L}_\psi \) (resp. \( \mathcal{L}_\chi \)) the associated Artin-Schreier (resp. Kummer) sheaf on \( \mathbf{A}^1_{\mathbf{F}_q} \) (resp. on \( (\mathbf{G}_m)_{\mathbf{F}_q} \)), as well (by abuse of notation) as their middle extension to \( \mathbf{P}^1_{\mathbf{F}_q} \). The trace functions of the latter are given by

\[
t_{\psi}(x; \mathbf{F}_{q^d}) = \psi(\text{Tr}_{\mathbf{F}_{q^d}/\mathbf{F}_q}(x)) \quad \text{if} \quad x \in \mathbf{F}_{q^d}, \quad t_{\psi}(x; \mathbf{F}_{q^d}) = 0,
\]

\[
t_{\chi}(x; \mathbf{F}_{q^d}) = \chi(\text{N}_{\mathbf{F}_{q^d}/\mathbf{F}_q}(x)) \quad \text{if} \quad x \in \mathbf{F}_{q^d}, \quad t_{\chi}(x; \mathbf{F}_{q^d}) = t_{\chi}(x; \mathbf{F}_q) = 0.
\]

For the trivial additive or multiplicative character, the trace function of the middle-extension is the constant function \( 1 \).

Given \( \lambda \in \mathbf{F}_{q^d} \), we denote by \( \mathcal{L}_{\psi, \lambda} \) the Artin-Schreier sheaf of the character of \( \mathbf{F}_{q^d} \) defined by \( x \mapsto \psi(\text{Tr}_{\mathbf{F}_{q^d}/\mathbf{F}_q}(\lambda x)) \).

If \( X_{\mathbf{F}_q} \) is an algebraic variety, \( \psi \) (resp. \( \chi \)) is an \( \ell \)-adic additive character of \( \mathbf{F}_q \) (resp. \( \ell \)-adic multiplicative character) and \( f : X \to \mathbf{A}^1 \) (resp. \( g : X \to \mathbf{G}_m \)) is a morphism, we denote by either \( \mathcal{L}_{\psi(f)} \) or \( \mathcal{L}_{\psi}(f) \) (resp. by \( \mathcal{L}_{\chi(g)} \) or \( \mathcal{L}_{\chi}(g) \)) the pullback \( f^* \mathcal{L}_\psi \) of the Artin-Schreier sheaf associated to \( \psi \) (resp. the pullback \( g^* \mathcal{L}_\chi \) of the Kummer sheaf). These are lisse sheaves on \( X \) with trace functions \( x \mapsto \psi(f(x)) \) and \( x \mapsto \chi(g(x)) \), respectively. The meaning of the notation \( \mathcal{L}_\psi(f) \), which we use when putting \( f \) as a subscript would be typographically unwieldy, will always be unambiguous, and no confusion with Tate twists will arise.

Given a variety \( X_{/\mathbf{F}_q} \), an integer \( k \geq 1 \) and a function \( c \) on \( X \), we denote by \( \mathcal{L}_\psi(cs^{1/k}) \) the sheaf on \( X \times \mathbf{A}^1 \) (with coordinates \( (x, s) \)) given by \( \alpha_\ast \mathcal{L}_{\psi(c(x))} \), where \( \alpha \) is the covering map \( (x, s, t) \mapsto (x, s) \) on the \( k \)-fold cover

\[
\{(x, s, t) \in X \times \mathbf{A}^1 \times \mathbf{A}^1 \mid t^k = s\}.
\]

Given a field extension \( L_{/\mathbf{F}_p} \), and elements \( \alpha \in L^\times \) and \( \beta \in L \), we denote by \([\cdot \alpha],[\cdot \alpha] \) the scaling map \( x \mapsto \alpha x \) on \( \mathbf{A}_L^1 \), and by \([+\beta],[+\beta] \) the additive translation \( x \mapsto x + \beta \). For a sheaf \( \mathcal{F} \), we denote by \([\cdot \alpha] \ast \mathcal{F} \) (resp. \([+\alpha] \ast \mathcal{F} \)) the respective pull-back operation.

We will usually not indicate base points in étale fundamental groups; whenever this occurs, it will be clear that the properties under consideration are independent of the choice of a base point.

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## 2. Preliminaries

We begin by defining Property NIO, and a useful variant called CGM (for “Connected Geometric Monodromy”). These are motivated by results of Katz (see [Kat90, Cor. 8.9.2, Th. 8.8.1–8.8.2]).
Definition 2.1. Let $A$ be a finite cyclic group and $\chi = (\chi_1, \ldots, \chi_k)$ a tuple of characters of $A$. Let $\Lambda = \chi_1 \cdots \chi_k$.

1. The tuple $\chi$ is **Kummer-induced** if there exists a divisor $d$ of $k$, $d \neq 1$, and a tuple $(\xi_1, \ldots, \xi_{k/d})$ of characters of $A$ such that the $\chi$’s are all the characters with $\chi^d = \xi_j$ for some $j$, with multiplicity.

2. The tuple $\chi$ is **self-dual** if there is a character $\xi$ such that the set of characters $\chi \in \chi$, with multiplicity, is stable under $\chi \mapsto \xi \chi^{-1}$. The character $\xi$ is called a “dualizing character”.

3. A self-dual tuple $\chi$ is **alternating** if $k$ is even and $\Lambda = \xi^{k/2}$, and otherwise, it is **symmetric**.

4. A tuple $\chi$ has Property NIO if it is not Kummer-induced and, if $k$ is even, if it is not self-dual symmetric.

5. A tuple $\chi$ has Property CGM if it is not Kummer-induced, and $\chi_1 \cdots \chi_k = 1$, and one of the following conditions holds:
   - $k$ is odd,
   - $\chi$ is not self-dual,
   - $k$ is even, $\chi$ is self-dual and alternating, and the dualizing character $\xi$ is trivial.

Example 2.2. We consider Dirichlet characters modulo $q$ in these examples.

1. Consider the case $k = 2$ and $q$ odd, $\chi = (\chi_1, \chi_2)$. Denote by $\chi(2)$ the non-trivial real character of $\mathbb{F}_q^\times$. Then $\chi$ is:
   - Kummer-induced if and only if $\chi_2 = \chi_1 \chi(2)$.
   - If not Kummer-induced, always self-dual alternating, taking $\xi = \chi_1 \chi_2$ as dualizing character.

   In particular, for $\chi = (1, \chi_2)$, the alternating case is $\chi_2 = 1$, corresponding to the “classical” Kloosterman sum, and the non self-dual case is $\chi_2^2 = 1$. The Kummer-induced tuple $\chi = (1, \chi(2))$ corresponds to Salié sums.

2. If $k$ is odd, then $\chi$ has NIO if and only if it is not Kummer-induced. In particular, this is the case if $\chi_1 = \cdots = \chi_{k-1} = 1$.

3. If $\chi_1 = \cdots = \chi_k = 1$, then $\chi$ has NIO.

In the next section, we will need the following useful lemma which bounds the number of integral points in a box that satisfy a system of polynomial equations modulo $q$. We include a proof, since we do not know a convenient reference.

Lemma 2.3. Let $k \geq 1$ be an integer and let $A > 0$. Let $X_A \subset \mathbb{A}^k_Z$ be an algebraic variety of dimension $d \geq 0$ given by the vanishing of $\leq A$ polynomials of degree $\leq A$. Let $p$ be a prime number and $0 < B < p/2$ an integer. Then

\[ |\{ x = (x_1, \ldots, x_k) \in \mathbb{F}_p^k \mid x \in X(\mathbb{F}_p) \text{ and } B \leq x_i \leq 2B \text{ for } 1 \leq i \leq k \} \} \leq B^d \]

where the implied constant depends only on $k$ and $A$, and the notation $B \leq x_i \leq 2B$ means that the unique integer between 1 and $p-1$ congruent to $x_i$ modulo $p$ belongs to the interval $[B, 2B]$.

Proof. We proceed by induction on the dimension $d$. For $d = 0$, the statement is elementary since $X$ is then finite. We assume that it holds for dimension $\leq d - 1$, and proceed now by induction on $k \geq d$. If $k = d$, then $X = \mathbb{A}^d$, and the result is clear. We suppose that the statement holds for $X \subset \mathbb{A}^l$ with $l < k$. We denote by $f : X \to \mathbb{A}^1$ the projection $x \mapsto x_1$. Now for a given prime $p > 2B$, we have

\[ |\{ x = (x_1, \ldots, x_k) \in \mathbb{F}_p^k \mid x \in X(\mathbb{F}_p) \text{ and } B \leq x_i \leq 2B \text{ for } 1 \leq i \leq k \} | \leq \sum_{B \leq x \leq 2B} |f^{-1}(x)|. \]

Let $U$ be the set of $x \in \mathbb{A}^1$ such that $\dim(f^{-1}(x)) = d - 1$. This is an open dense subset of $\mathbb{A}^1$ and the cardinality of its complement is bounded in terms of $k$ and $A$ only. For $x \in U(\mathbb{F}_p)$, we have
|f^{-1}(x)| \ll B^{d-1} by the induction step from dimension \(d-1\). For \(x \notin U(F_p)\), we have |f^{-1}(x)| \ll B^d by the induction step from \(k-1\) to \(k\). Hence
\[
\sum_{B \leq x \leq 2B} |f^{-1}(x)| \ll B \cdot B^{d-1} + B^d,
\]
as desired. \(\square\)

3. An Application to Moments of \(L\)-functions

In this section, we will prove Theorem 1.5, which we recall is a variation of a recent result of Zacharias [Zac17].

Let \(f\) be a primitive cusp form of level 1, trivial nebentypus and weight \(k_f\), with Hecke eigenvalues \(\lambda_f(n)\). For Dirichlet characters \(\chi\) and \(\xi\) modulo \(q\), we consider the \(L\)-function
\[
L((f \oplus \xi) \otimes \chi, s) = L(f \otimes \chi, s)L(\chi, s)
\]
of degree 3. Note that for \(\text{Re}(s) > 1\), we have the Dirichlet series expansion
\[
L((f \oplus \xi) \otimes \chi, s) = \sum_{n \geq 1} \chi(n)(\lambda_f \cdot \xi)(n)n^{-s}.
\]
We wish to evaluate the average
\[
\mathcal{M} = \frac{1}{q-1} \sum_{\chi \pmod{q}} L((f \oplus \xi) \otimes \chi, 1/2),
\]
proving that \(\mathcal{M} = 1 + O(q^{-\alpha})\) for some \(\alpha > 0\).

The proof is every similar to [Zac17, §6.2], which correspond to the case \(\xi = 1\), so we will only sketch certain steps.

We assume for simplicity \(\xi\) is even (ie. \(\xi(-1) = 1\)), and we will only evaluate the even moment
\[
\mathcal{M}^+ = \frac{2}{q-1} \sum_{\chi \pmod{q}} L((f \oplus \xi) \otimes \chi, 1/2)
\]
where \(\sum^+\) restricts the sum to even primitive characters modulo \(q\). We will prove that \(\mathcal{M}^+ = \frac{1}{2} + O(q^{-\alpha})\) for some \(\alpha > 0\). The sum over odd characters satisfies the same asymptotics, hence this implies Theorem 1.5.

Define \(\Gamma_R(s) = \pi^{-s/2}\Gamma(s/2)\) and let
\[
L_\infty(f, s) = \Gamma_R\left(s + \frac{k - 1}{2}\right)\Gamma_R\left(s + \frac{k + 1}{2}\right), \quad L_\infty(\chi \xi, s) = \Gamma_R(s),
\]
be the archimedean \(L\)-factors of \(L(f, s)\) and \((\chi \xi, s)\) respectively. Further, let
\[
\varepsilon((f \oplus \xi) \otimes \chi) = \varepsilon(f)\varepsilon_\chi^2\varepsilon_\xi
\]
where \(\varepsilon_\eta\) denotes the normalized Gauss sum of a Dirichlet character. Define then the completed \(L\)-function
\[
\Lambda((f \oplus \xi) \otimes \chi, s) = q^{3s/2}L_\infty(s)L((f \oplus \xi) \otimes \chi, s).
\]
For \(\xi\) and \(\chi\) even, we then have the functional equation
\[
\Lambda((f \oplus \xi) \otimes \chi, s) = \varepsilon((f \oplus \xi) \otimes \chi)\Lambda(f \otimes \chi \otimes \overline{\chi}, 1 - s).
\]

Let \(0 < \alpha < 1/4\) be a parameter to be fixed later. For \(\chi\) even, non-trivial and not equal to \(\xi^{-1}\), we apply the approximate functional equation to the \(L\)-function \(L((f \oplus \xi) \otimes \chi, s)\), in an unbalanced form ([IK04, Th. 5.3] with \(q\) replaced by the conductor \(q^3\) and \(X = q^{1/2-2\alpha}\)). After adding the
contribution of the character $\chi^{-1}$, which is $\ll q^{-1/5+\varepsilon}$ for any $\varepsilon > 0$, this gives $M^+ = M_1 + M_2$, where

$$M_1 = \frac{2}{q-1} \sum_{\chi \pmod{q}}^+ \sum_{\chi(\overline{n})}^{\chi(\overline{n})} \frac{\chi(n)(\lambda_f \ast \xi)(n)}{n^{1/2}} V\left(\frac{n}{q^{2-2\alpha}}\right),$$

$$M_2 = \frac{2}{q-1} \sum_{\chi \pmod{q}}^+ \varepsilon((f \otimes \xi) \otimes \chi) \sum_{\chi(\overline{n})}^{\chi(\overline{n})} \frac{\chi(n)(\lambda_f \ast \xi)(n)}{n^{1/2}} V\left(\frac{n}{q^{1+2\alpha}}\right),$$

where the function $V$ is defined by

$$V(y) = \frac{1}{2i\pi} \int_{(1)} \frac{L_x\left(\frac{1}{2} + s\right)}{L_x\left(\frac{1}{2}\right)} G(s)y^{-s} ds, \quad G(s) = \exp(s^2)$$

for $y > 0$. Shifting the $s$-contour to the right if $y \geq 1$ or to Re$(s) = -1/2$ if $y \leq 1$, we deduce that

$$y^i V^{(i)}(y) \ll_A f (1 + y)^{-A}$$

for any $A > 0$ and $i \geq 0$, and

$$V(y) = 1 + O(y^{1/2}) \quad \text{for} \quad y \leq 1.$$}

It follows from the first of these bounds that, for any $\kappa > 0$, the contribution to both sums of the integers $n \geq q^{3/2+\kappa}$ is $\ll_A, f, \kappa q^{-A}$ for any $A \geq 0$.

We first bound $M_1$. We add to $M_1$ the contribution of the trivial character, up to an error term bounded by $O(q^{-1/5})$, and perform the summation over the even characters $\chi$. We obtain

$$M_1 = \sum_{n \equiv \pm 1 \pmod{q}} (\lambda_f \ast \xi)(n) \frac{\chi(n)}{n^{1/2}} V\left(\frac{n}{q^{2-2\alpha}}\right) + O(q^{-1/5}) = V\left(\frac{1}{q^{2-2\alpha}}\right) + O(q^{-\alpha+\varepsilon}) = 1 + O(q^{-\alpha+\varepsilon}),$$

for any $\varepsilon > 0$, where the first term $V(q^{-2+2\alpha})$ is the contribution of the trivial solution $n = 1$ of the congruence $n \equiv \pm 1 \pmod{q}$.

Now we consider $M_2$. We add to $M_2$ the contribution of the trivial character, up to an error of size $\ll q^{\varepsilon+\frac{1}{2}+\alpha-1} \ll q^{\varepsilon+\alpha-1/2}$, for any $\varepsilon > 0$. We then perform the summation over $\chi$ even. We have

$$\frac{1}{q-1} \sum_{\chi \pmod{q}}^+ \varepsilon((f \otimes \xi) \otimes \chi) \chi(n) = \frac{\varepsilon(f)}{q-1} \sum_{\chi \pmod{q}}^+ \varepsilon \xi \chi \chi(n) = \frac{\varepsilon(f)}{q^{1/2}} \left(\text{Kl}_3(n; \xi, q) + \text{Kl}_3(-n; \xi, q)\right),$$

where we abbreviate

$$\text{Kl}_3(\pm n; \xi, q) = \text{Kl}_3(\pm n; (1, 1, \xi), q).$$

Hence we have

$$M_2 = \frac{\varepsilon(f)}{q^{1/2}} \sum_n (\lambda_f \ast \xi)(n) \frac{\chi(n)}{n^{1/2}} (\text{Kl}_3(n; \xi, q) + \text{Kl}_3(-n; \xi, q)) V\left(\frac{n}{q^{1+2\alpha}}\right) + O(q^{-1/5}).$$

We open the Dirichlet convolution

$$(\lambda_f \ast \xi)(n) = \sum_{ab=n} \lambda_f(a) \xi(b).$$

By standard techniques (dyadic subdivisions, inverse Mellin transform to separate the variables), we establish that $M_2$ is, up to a factor $\ll q^2$ for any $\varepsilon > 0$, bounded by the sum of $\ll (\log q)^2$ bilinear sums of the type

$$M_2(M, N) = \frac{1}{(qMN)^{1/2}} \sum_{m, n} \lambda_f(m) \xi(n) \text{Kl}_3(\pm mn; \xi, q) V\left(\frac{m}{M}\right) W\left(\frac{n}{N}\right)$$
where
\[ 1 \leq MN \leq q^{1+2\alpha}, \]
a = 1 or \(-1\), and \(V\) and \(W\) are smooth functions, compactly supported in \([1, 2]\), such that
\[ x^i V^i(x), \ x^i W^i(x) \ll_{f, \varepsilon} q^{i\varepsilon} \]
for any \(\varepsilon > 0\) and \(i \geq 0\).

We set \(M = q^\mu\) and \(N = q^\nu\). The trivial bound is
\[ M_2(M, N) \ll q^\varepsilon \left( \frac{MN}{q} \right)^{1/2} = q^{(\mu + \nu)/2 - 1/2 + \varepsilon} \]
for any \(\varepsilon > 0\), which is \(\ll q^{-\alpha + \varepsilon}\) if \(\mu + \nu \leq 1 - 2\alpha\). Now assume that
\[ 1 - 2\alpha \leq \mu + \nu \leq 1 + 2\alpha. \]
Estimating the sum over \(n\) by the Polyá-Vinogradov technique (completion), summing trivially over the \(m\) variable, we obtain
\[ M_2(M, N) \ll q^\varepsilon \left( \frac{MN}{N} \right)^{1/2} \ll q^{1/2 - \nu + \alpha + \varepsilon} \]
for any \(\varepsilon > 0\). This bound is \(\ll q^{-\alpha + \varepsilon}\) if \(\nu \geq \frac{1}{2} + 2\alpha\). We then assume that
\[ \nu \leq \frac{1}{2} + 2\alpha. \]

If \(\nu\) is small, so that \(\mu\) is large, we apply [FKM15a, Th. 1.2] to the sum over \(m\), summing trivially over \(n\). We get
\[ M_2(M, N) \ll Nq^{-1/8 + \alpha + \varepsilon} = q^{-1/8 + \nu + \alpha + \varepsilon} \]
for any \(\varepsilon > 0\). Again, this is \(\ll q^{-\alpha + \varepsilon}\) provided \(\nu \leq \frac{1}{8} - 2\alpha\). Now assume that
\[ \frac{1}{8} - 2\alpha \leq \nu \leq \frac{1}{2} + 2\alpha. \]
Then \(\frac{1}{2} - 4\alpha \leq \mu \leq \frac{7}{8} + 4\alpha\). The general bilinear form estimate in [FKM14, Thm 1.17] gives
\[ M_2(M, N) \ll q^{\varepsilon + \alpha} \min(N^{-1} + M^{-1}q^{1/2}, M^{-1} + N^{-1}q^{1/2})^{1/2} \]
which is \(\ll q^{-\alpha + \varepsilon}\) provided \(\alpha \leq 1/32\) and
\[ \max(\mu, \nu) \geq \frac{1}{2} + 2\alpha. \]
We finally consider the case when \(\alpha \leq 1/32\) and
\[ \frac{1}{2} - 4\alpha \leq \mu, \ \nu \leq \frac{1}{2} + 2\alpha. \]

In this situation, we can then apply Theorem 1.1 for the triple \(\chi = (1, 1, \xi)\), which has Property NIO for any \(\xi\) by Example 2.2 (2). We obtain the bound
\[ M_2(M, N) \ll q^\varepsilon \left( \frac{MN}{q} \right)^{1/2} (MN)^{-\eta} \ll q^{2\alpha + \varepsilon}(MN)^{-\eta} \ll q^{2\alpha - 3\eta/4 + \varepsilon} \]
for any \(\varepsilon > 0\), where \(\eta > 0\) is the saving exponent in Theorem 1.1 when the parameter \(\delta\) there is
\[ \delta = \frac{1}{4} - 8\alpha. \]
Hence, for \(\alpha > 0\) fixed and small enough, we obtain
\[ M_2(M, N) \ll q^{-\eta' + \varepsilon} \]
for some fixed \(\eta' > 0\) and any \(\varepsilon > 0\), where the implied constant depends on \(\varepsilon\) and \(f\).
4. Reduction to complete exponential sums

In this section, we will state the general forms of Theorems 1.1 and 1.2, and reduce their proofs
to certain bounds for families of exponential sums over finite fields. In fact, we begin with slightly
more general bilinear sums.

Let \( q \) be a prime number, and let \( K : \mathbb{F}_q \to \mathbb{C} \) be any function. Let \( M, N \) be integers such that
\( 1 \leq M, N \leq q - 1 \). Let \( M \) be a subset of the positive integers \( m \leq q - 1 \) of cardinality \( M \). We set
\( M^+ = \max_{m \in M} m \). Let finally \( N = \{ n \mid 1 \leq n < N \} \).

Given tuples of complex numbers \( \alpha = (\alpha_m)_{m \in M} \) and \( \beta = (\beta_n)_{n \in N} \), we set
\[
B(K, \alpha, \beta) = \sum_{m \in M} \sum_{n \in N} \alpha_m \beta_n K(mn).
\]

We will prove the following:

**Theorem 4.1.** Fix an integer \( k \geq 2 \). Let \( q \) be a prime and let \( a \in \mathbb{F}_q^\times \). Let \( \chi \) be a \( k \)-tuple of
Dirichlet characters modulo \( q \). Suppose that \( \chi \) has Property NIO, and define \( K(x) = K_{l_k}(ax; \chi, q) \).
With notations as above, for any integer \( l \geq 2 \) and any \( \varepsilon > 0 \), we have
\[
B(K, \alpha, \beta) \ll q^\varepsilon \|\alpha\|_1 \|\beta\|_1 (MN)^{1/2} \left( \frac{1}{M} + \left( \frac{q^{3/2} + q^l}{MN} \right)^{1/2} \right)^{1/2},
\]
where the implied constant depends only on \( (k, l, \varepsilon) \), provided one of the two following two conditions
holds:
\[
q^{3/2} \leq N < \frac{1}{2} q^{1 - \frac{3}{4} l}, \quad q^{3/2} \leq N, \quad NM^+ < \frac{1}{2} q^{1 - \frac{3}{4} l}.
\]

Remark. This bound is non-trivial only for \( l \) large enough, precisely for \( l \geq 9 \). As we will explain,
this limitation results from our simplifying choice of not applying the completion method to detect
that an auxiliary variable belongs to some interval in \( \mathbb{F}_q \).

In the special case of “type I” sums, we obtain

**Theorem 4.2.** With the same notation and assumption as in Theorem 4.1, especially assuming
that \( \chi \) has NIO, and with the additional condition that \( \alpha_n = 1 \) for \( n \in N \), for any integer \( l \geq 1 \) and
any \( \varepsilon > 0 \), we have
\[
B(K, \alpha, 1) \ll q^\varepsilon \|\alpha\|_1^{1-1/7} \|\alpha\|_2^{1/7} M^{1/2} N \left( \frac{q^{3/2} + q^l}{MN^2} \right)^{1/2} q^{1 + 1/2l},
\]
where the implied constant depends on \( (k, l, \varepsilon) \), provided one of the following two conditions holds:
\[
q^{1/2} \leq N \leq \frac{1}{2} q^{1/2 + 1/2l}, \quad q^{1/2} \leq N, \quad NM^+ \leq \frac{1}{2} q^{1 + 1/2l}.
\]

Remark. As \( l \) gets large, this bound is non-trivial if
\[
M^+ N \leq q, \quad MN^2 \geq q^{1+\delta}
\]
for some \( \delta > 0 \). In particular for \( M = M^+ = N \), this is non trivial if
\[
N \geq q^{1/3+\delta}.
\]
4.1. The type II bilinear sum. We now start the proof of the reduction step for Theorem 4.1. Applying Cauchy’s inequality, we obtain
\[
|B(K, \alpha, \beta)| \leq \|\beta\| \left( \sum_n \left| \sum_m \alpha_m K(mn) \right|^2 \right)^{1/2} \ll \|\alpha\|^2 N + S^+)^{1/2}
\]
where
\[
S^+ = \sum_{m_1 + m_2} \alpha_{m_1} \alpha_{m_2} \sum_n K(m_1 n) K(m_2 n).
\]

We now use the +ab-shift trick of Karatsuba-Vinogradov as in [FM98, KMS17]. For this we introduce two integer parameters \( A, B \geq 1 \) such that \( AB \leq N \). Using the notation \( a \sim A \) for \( A \leq a < 2A \), we then have
\[
S^+ = \frac{1}{AB} \sum_{a \sim A, b \sim B} \sum_{n + ab \equiv n} \alpha_{m_1} \alpha_{m_2} \sum_{n + abN} K(m_1 (n + ab)) K(m_2 (n + ab)).
\]

Using the fact that \( N \) is an interval, we deduce as in [FM98, KMS17] that
\[
S^+ \ll \frac{\log q}{AB} \sum_{a, m_1, m_2, n} \left| \alpha_{m_1} \alpha_{m_2} \right| \left| \sum_{b \sim B} K(m_1 (n + ab)) K(m_2 (n + ab)) e(bt) \right|
\]
for some \( t \in \mathbb{R} \) and \( n \) varying over an interval of length \( \ll N + AB \). For \( (r, s_1, s_2) \in (\mathbb{F}_q^\times)^3 \) set
\[
\nu(r, s_1, s_2) = \sum_{a, m_1, m_2, n} \left| \alpha_{m_1} \alpha_{m_2} \right|
\]
so that
\[
S^+ \ll \frac{\log q}{AB} \sum_{r, s_1, s_2} \nu(r, s_1, s_2) \left| \sum_{b \sim B} K(s_1 (r + b)) K(s_2 (r + b)) e(bt) \right|
\]
(by the change of variable \( r = \alpha \cdot n, s_i = a \cdot m_i, i = 1, 2 \)). We have
\[
\sum_{r, s_1, s_2} \nu(r, s_1, s_2) = \sum_{a, n, m_1 + m_2} \left| \alpha_{m_1} \alpha_{m_2} \right| \leq AN\|\alpha\|^2 \leq AMN\|\alpha\|^2
\]
and
\[
\sum_{r, s_1, s_2} \left( \sum_{a, n, m_1 + m_2} \left| \alpha_{m_1} \right| \left| \alpha_{m_2} \right| \right)^2 = \sum_{a, n, m_1 + m_2} \left| \alpha_{m_1} \right| \left| \alpha_{m_2} \right| \sum_{a', n', m_1' + m_2'} \left| \alpha_{m_1'} \right| \left| \alpha_{m_2'} \right|.
\]

Now assume that
\[
2AN < q.
\]
Then the equation \( \alpha n' \equiv \alpha n \pmod{q} \) is equivalent to \( an' \equiv a'n \pmod{q} \), which is equivalent to \( an' \equiv a'n \). Therefore if we fix \( a \) and \( n' \), the integers \( a' \) and \( n \) are determined up to \( q^{o(1)} \) values.

Suppose that \( a, a', n, n' \) are so chosen. For \( i = 1, 2 \), we then have
\[
\sum_{m_i, m_i'} \left| \alpha_{m_i} \right| \left| \alpha_{m_i'} \right| \leq \sum_{m_i, m_i'} \left| \alpha_{m_i} \right|^2 + \sum_{m_i, m_i'} \left| \alpha_{m_i'} \right|^2 \ll \|\alpha\|^2.
\]
Indeed, since \( M \) is a subset of \([1, q - 1]\), once \( m_i \) (resp. \( m_i' \)) is given, the congruence \( am_i \equiv a'm_i' \pmod{q} \) uniquely determines \( m_i' \) (resp. \( m_i \)). Therefore
\[
\sum_{r, s_1, s_2} \nu(r, s_1, s_2)^2 \ll q^{o(1)} AN\|\alpha\|^4.
\]
Alternatively, if we assume instead of (4.1) that
\[ 2AM^+ < q, \]
then the same reasoning with the equation \( am_1 \equiv a'm'_1 \pmod{q} \) also leads to (4.2).

Fix an integer \( l \geq 2 \). We apply Hölder’s inequality in the following form:
\[
\sum_{r,s_1,s_2} \left| \sum_{b \sim B} \cdots \right| \left( \sum_{r,s_1,s_2} \nu \sum_{b \sim B} \cdots \right)^{1/2} \leq q \beta (AN)^{1-\frac{1}{p}} M^{1-\frac{1}{q}} \left( \sum_{b \in \mathcal{B}} |\Sigma_{II}(K, b)| \right)^{1/2l},
\]
where \( \mathcal{B} = [B, 2B]^{2l} \), and
\[
\Sigma_{II}(K, b) = \sum_{r \in \mathcal{F}_q} \sum_{s_1, s_2 = \mathcal{F}_q^{s_1+s_2}} K(s_1 r, s_1 b) \overline{K}(s_2 r, s_2 b)
\]
is the exponential sum defined in (1.1), where
\[
K(r, b) = \prod_{i=1}^{l} K(r + b_i) K(r + b_{i+1}^l).
\]
We observe at this point that the sum \( \Sigma_{II}(K, b) \) is independent of the parameter \( a \) such that \( K(x) = Kl_k(ax; \chi, q) \), by changing the variables \( s_1 \) and \( s_2 \) to \( as_1 \) and \( as_2 \) respectively.

We will estimate these sums in different ways depending on the position of \( b \). Precisely:

**Theorem 4.3.** There exist affine varieties
\[ \mathcal{V}^\Delta \subset \mathcal{W} \subset \mathbb{A}_{\mathbb{Z}}^{2l} \]
defined over \( \mathbb{Z} \) such that
\[ \text{codim}(\mathcal{V}^\Delta) = l, \quad \text{codim}(\mathcal{W}) \geq \frac{l - 1}{2}, \]
which have the following property: for any prime \( q \) large enough, depending only on \( k \), for any tuple \( \chi \) of characters of \( \mathcal{F}_q^\times \) with Property NIO, for any \( a \in \mathcal{F}_q^\times \), and for all \( b \in \mathcal{F}_q^{2l} \), with
\[
K(x) = Kl_k(ax; \chi, q),
\]
we have
\[
\Sigma_{II}(K, b) \ll q^3 \text{ if } b \in \mathcal{V}^\Delta(\mathcal{F}_q)
\]
(4.4)
\[
\Sigma_{II}(K, b) \ll q^2 \text{ if } b \in (\mathcal{W} - \mathcal{V}^\Delta)(\mathcal{F}_q)
\]
(4.5)
\[
\Sigma_{II}(K, b) \ll q^{3/2} \text{ if } b \not\in \mathcal{W}(\mathcal{F}_q).
\]
(4.6)
In all cases, the implied constant depends only on \( k \).

We emphasize that the varieties \( \mathcal{V}^\Delta \) and \( \mathcal{W} \) are independent of the tuple of characters.

We will apply these estimates for the parameters \( b \) belonging to the box \([B, 2B]^{2l}\), and for this we use Lemma 2.3.
Let \( B^V \) (resp. \( B^W \)) be the set of \( b \in B \) such that \( b \in V_\Delta(F_q) \) (resp. \( b \in W(F_q) \)). Since the subvarieties \( V_\Delta \) and \( W \) are defined over \( \mathbb{Z} \), it follows from Lemma 2.3 that
\[
\sum_b |\Sigma_{11}(K, b)| \ll q^3|B^V| + q^2|B^W| + q^{3/2}B^{2l}
\]
(4.7)
\[
\ll q^3B^{2l-\operatorname{codim}(V_\Delta)} + q^2B^{2l-\operatorname{codim}(W)} + q^{3/2}B^{2l}.
\]
We also choose \( A \) such that \( A \) and that either of (4.1) or (4.3) hold. Writing \( \operatorname{codim}(W) = \gamma l \), we deduce that
\[
|B(K, \alpha, \beta)| \leq \|eta\|\|\alpha\|^2N + S^\perp)^{1/2}
\]
where
\[
S^\perp \ll \frac{q^\varepsilon}{AB}\|\alpha\|^2(AN)^{1-\frac{1}{M}}M^{1-\frac{1}{2}}(q^2B^{2-\gamma l}) + q^{3/2}B^{2l})^{1/2}.
\]
Hence
\[
|B(K, \alpha, \beta)| \ll q^\varepsilon\|\alpha\|\|\beta\|(MN)^{1/2}\left(\frac{1}{M} + \left(\frac{q^2B^{-\gamma l}}{AM^2N} + \frac{q^{3/2}}{AM^2N}\right)^{1/2}\right)^{1/2}
\]
(4.8)
\[
\ll q^\varepsilon\|\alpha\|\|\beta\|(MN)^{1/2}\left(\frac{1}{M} + \left(\frac{q^{2-\gamma l}}{AM^2N} + \frac{q^{3/2}}{(MN)^2}\right)^{1/2}\right)^{1/2}.
\]
This holds under the condition that
\[
A = Nq^{-\frac{3}{2}} \geq 1
\]
and that either of (4.1) or (4.3) hold.

In particular, since \( \gamma \geq 1/3 \), the second term on the right-hand side of (4.8) is smaller than the third. This implies Theorem 4.1. Theorem 1.1 follows by choosing \( l \) large enough depending on \( \delta \).

4.2. Bounding type I sums. We turn now to Theorem 4.2, and consider the special bilinear form
\[
B(K, \alpha, 1_N) = \sum_{m \in M} \sum_{n \in N} \alpha_m K(mn).
\]
Given \( l \geq 2 \), a trivial bound is
\[
B(K, \alpha, 1_N) \leq \|\alpha\|^{1-\frac{1}{M}}\|\alpha\|^\frac{3}{2}M^\frac{M}{2}N.
\]
Proceeding as before, we get
\[
B(K, \alpha, 1_N) = \frac{1}{AB} \sum_{a \sim A} \sum_{b \sim B} \sum_{m \in M} \sum_{n \in N} \alpha_m \sum_{ab \in N} K(m(n + ab))
\]
\[
\ll \varepsilon q^\varepsilon \sum_{r \in F_q \times \mathbb{F}_q^*} \nu(r, s) \left| \sum_{b \sim B} \eta_b K(s(r + b)) \right|
\]
with
\[
\nu(r, s) = \sum_{a \sim A} \sum_{m \in M} \sum_{n \in N} \left| \alpha_m \right|
\]
\[
\sum_{am = s, m \equiv r (\text{mod} q)} \sum_{n \in N} \left| \alpha_n \right|
\]
\[
\ll \varepsilon \left(\frac{q}{AB}\right)^{1/2} M^{1/2} + \left(\frac{q^2}{AM^2N}\right)^{1/2} \left(\frac{q^{3/2}}{AM^2N}\right)^{1/2}.
\]

This holds under the condition that
\[
A = Nq^{-\frac{3}{2}} \geq 1
\]
and that either of (4.1) or (4.3) hold.
and $|\eta_b| \leq 1$. We have
\[
\sum_{r,s} \nu(r, s) \ll AN \sum_{m \in M} |\alpha_m|.
\]
We also have
\[
\sum_{r,s} \nu(r, s)^2 = \sum_{a,m,n,a',m',n'} |\alpha_m||\alpha_{m'}|.
\]
Assuming that
\[
(4.9) \quad 2AN \text{ or } 2A.M^+ < q
\]
we show by the same reasoning as above that
\[
\sum_{r,s} \nu(r, s)^2 \ll \sum_{a,m} |\alpha_m|^2 \sum_{n,a',m',n'} 1 \ll q^\epsilon AN \sum_{m} |\alpha_m|^2,
\]
We next apply Hölder’s inequality in the form
\[
\sum_{r,s} \nu(r, s) \left| \sum_{B < b \leq 2B} \eta_b K(s(r + b)) \right| \\
\leq \left( \sum_{r,s} \nu(r, s) \right)^{1 - \frac{1}{2l}} \left( \sum_{r,s} \nu(r, s)^2 \right)^{\frac{1}{2l}} \left( \sum_{B < b \leq 2B} \eta_b K(s(r + b)) \right)^{2l} \left( \sum_{r,s} \nu(r, s)^2 \right)^{\frac{1}{2l}} \\
\ll q^\epsilon (AN)^{1 - \frac{1}{2l}} \|\alpha\|_1^{1 - \frac{1}{2l}} \|\alpha\|_2^{\frac{1}{2l}} \left( \sum_{B < b \leq 2B} \eta_b K(s(r + b)) \right)^{2l} \left( \sum_{r,s} \nu(r, s)^2 \right)^{\frac{1}{2l}}.
\]
Expanding the $2l$-th power, we have
\[
\sum_{r,s} \sum_{B < b \leq 2B} \left| \sum_{B < b \leq 2B} \eta_b K(s(r + b)) \right|^{2l} \leq \sum_{B \in \mathcal{B}} |\Sigma_l(K, b)|
\]
with
\[
(4.10) \quad \Sigma_l(K, b) = \sum_{r \in \mathbb{F}_q} \sum_{s \in \mathbb{F}_q^\times} K(sr, sb) = \sum_{r \in \mathbb{F}_q} R(r, b).
\]
Note that $\Sigma_l(K, b)$ is independent of the choice of $a \in \mathbb{F}_q^\times$ such that $K(x) = Kl_k(ax; \chi, q)$. We have reached the bound
\[
(4.11) \quad B(K, \alpha, 1_N) \ll q^\epsilon \|\alpha\|_1^{1 - \frac{1}{2l}} \|\alpha\|_2^{\frac{1}{2l}} M \frac{N}{AB^{2l}} \sum_{b \in \mathcal{B}} |\Sigma_l(K, b)|^{\frac{1}{2l}}.
\]
As before, we can prove different bounds on $\Sigma_l(K, b)$ depending on the position of $b$.

**Theorem 4.4.** Let $\mathcal{V}^\Delta$ and $\mathcal{W}$ be the affine varieties on Theorem 4.3. For any prime $q$ large enough, depending only on $k$, for any tuple $\chi$ with Property NIO, for any $a \in \mathbb{F}_q^\times$ and for all $b \in \mathbb{F}_q^{2l}$, with
\[
K(x) = Kl_k(ax; \chi, q),
\]
we have
\[
(4.12) \quad \Sigma_l(K, b) \ll q^2 \text{ if } b \in \mathcal{V}^\Delta(\mathbb{F}_q)
\]
\[
(4.13) \quad \Sigma_l(K, b) \ll q^{l/2} \text{ if } b \in (\mathcal{W} - \mathcal{V}^\Delta)(\mathbb{F}_q)
\]
\[
(4.14) \quad \Sigma_l(K, b) \ll q \text{ if } b \notin \mathcal{W}(\mathbb{F}_q).
\]
In all cases, the implied constant depends only on $k$. 


Using the same notation \( \text{codim}(W) = \gamma l \) as before, we have therefore
\[
\sum_{b \in \mathcal{B}} \left| \Sigma_f(K, b) \right| \ll \| B \| q^2 + \| B \|^{3/2} q^3 + | B | q
\]
\[
\ll B^l q^2 + B^{(2-\gamma)l} q^{3/2} + B^{2l} q,
\]
by Lemma 2.3. Choosing
\[
B = q^{1/l}
\]
to equate the first and third terms above and
\[
A = N/B = Nq^{-1/l}
\]
we obtain from (4.11) the estimate
\[
B(K, \alpha, 1_N) \ll_{k,c} q^{l} \| \alpha \|_1^{-1/2} \| \alpha \|_2^{1/2} M N \left( \frac{(MN)^{-1}}{AB^2} (q B^{2l} + q^{1/2} B^{(3-\gamma)l}) \right)^{1/2l}
\]
\[
\ll_{k,c} q^{l} \| \alpha \|_1^{-1/2} \| \alpha \|_2^{1/2} M N \left( \frac{q^{1+\gamma}}{MN^2} + \frac{q^{-\gamma+1}}{MN^2} \right)^{1/2l},
\]
assuming that (4.9) holds and that \( A \geq 1 \). Since \( \gamma \geq 1/2 \) (by Theorem 4.3), the second term on the right-hand side of the last inequality is smaller than the first. Together with (4.9), this leads to Theorem 4.2, and Theorem 1.2 follows by letting \( l \) get large.

5. Algebraic preliminaries

We collect in this section some definitions and statements of algebraic geometry that we will use later. Most are standard, but we include some proofs for completeness and by lack of a convenient reference.

Let \( C_{F_q} \) be a smooth and geometrically connected curve with smooth projective model \( S \). The conductor of a constructible \( \ell \)-adic sheaf \( \mathcal{F} \) on \( C \) is defined by
\[
c(\mathcal{F}) = g(S) + \text{rank}(\mathcal{F}) + |\text{Sing}(\mathcal{F})| + \sum_{x \in \text{Sing}(\mathcal{F})} \text{Swan}_x(\mathcal{F}) + \dim H^0_c(C_{F_q}, \mathcal{F}),
\]
where \( g(S) \) is the genus of \( S \), \( \text{Sing}(\mathcal{F}) \) is the set of points of \( S \) where the middle-extension of \( \mathcal{F} \) is not lisse and \( \text{Swan}_x(\mathcal{F}) \) is the Swan conductor at \( x \).

Let \( C_{F_q} \) be a curve (not necessarily smooth or irreducible). Let \( (C_i)_{i \in I} \) be the geometrically irreducible components of \( C_{F_q} \) and \( \pi_i : \tilde{C}_i \to C_i \) their canonical desingularization. We define the conductor of a constructible \( \ell \)-adic sheaf \( \mathcal{F} \) on \( C_{F_q} \) by
\[
c(\mathcal{F}) = \sum_{i \in I} c(\pi_i^*(\mathcal{F}|C_i)) + \sum_{x \in C_{\text{sing}}} m_x(C),
\]
where \( C_{\text{sing}} \) is the singular set of \( C \) and \( m_x(C) \) the multiplicity of \( x \) as a singularity of \( C \).

If \( C_{F_q} \) is a curve, \( f \) is a function on \( C \) and \( \mathcal{F} \) an \( \ell \)-adic sheaf on \( C \), then
\[
c(\mathcal{F} \otimes \mathcal{L}_{f(x)}) \ll c(\mathcal{L}_{f(x)})^2 c(\mathcal{F})^2,
\]
where the implied constant is absolute.

We will use the following version of Deligne’s Riemann Hypothesis over finite fields [Del80].

**Proposition 5.1.** Let \( F_q \) be a finite field with \( q \) elements and let \( C \) be a curve over \( F_q \). Let \( \mathcal{F} \) and \( S \) be constructible \( \ell \)-adic sheaves on \( C \) which are mixed of weights \( \leq 0 \) and pointwise pure of weight
0 on a dense open subset. Suppose that the restriction of $\mathcal{F} \otimes \mathcal{G}^\circ$ to any geometrically irreducible component of $C$ has no trivial summand. We then have
\[
\sum_{x \in C(F_q)} t_{\mathcal{F}}(x; F_q) t_{\mathcal{G}}(x; F_q) \ll \sqrt{q}
\]
where the implied constant depend only on the conductors of $\mathcal{F}$ and of $\mathcal{G}$.

**Proof.** If $C$ is smooth and geometrically connected, and $\mathcal{F}$ and $\mathcal{G}$ are geometrically irreducible middle-extensions, this is deduced from Deligne’s results in [FKM13, Lemma 3.5]; the extension to general $\mathcal{F}$ and $\mathcal{G}$ satisfying our assumptions is immediate. For a general smooth curve, one need only apply the bound to each component separately.

For a general curve, observe that the difference between the sum over $C$ and the sum over a desingularization of $C$ is the sum over singular points of $t_{\mathcal{F}}(x; F_q) t_{\mathcal{G}}(x; F_q)$ minus the sum over points of the desingularization lying over singular points of $t_{\mathcal{F}}(x; F_q) t_{\mathcal{G}}(x; F_q)$. Since the size of both those sets of points may be bounded in terms of the sum of the multiplicities of singular points, and the value of $t_{\mathcal{F}}(x; F_q) t_{\mathcal{G}}(x; F_q)$ at those points may be bounded in terms of the conductors, this contribution is also bounded in terms of the conductors. \(\square\)

We will also use a criterion for a sheaf to be lisse that might be well-known but for which we do not know of a suitable reference.

**Lemma 5.2.** Let $\text{Spec}(O)$ be an open dense subset of the spectrum of the ring of integers in a number field and $U \to \text{Spec}(O)$ a reduced scheme of finite type. Let $\ell$ be a prime number invertible in $O$. Let $r \geq 1$ be an integer and let $\mathcal{F}$ be a constructible $\ell$-adic sheaf on $U$.

Assume that:

(1) For any finite-field valued point $\text{Spec}(k) \to \text{Spec}(O)$, the sheaf $\mathcal{F}_k$ on $U_k$ is lisse of rank $r$.

(2) For any finite-field valued point $\text{Spec}(k) \to \text{Spec}(O)$, any generic point $\eta$ of $U_k$, and any $s \in \Gamma(\text{Spec}(O^\text{et}_k), \mathcal{F})$, if $s$ is non-zero at the special point of the étale local ring $O^\text{et}_\eta$, then it is non-zero at the generic point.

Then $\mathcal{F}$ is lisse on $U$.

**Proof.** Let $x \in U_k \subset U$ and let $s$ be a non-zero section of $\mathcal{F}$ over the étale local ring $O^\text{et}_x$ at $x$. Since (the pullback of) $\mathcal{F}$ is lisse on $O^\text{et}_{x,k}$ by Assumption (1), the generic point of $O^\text{et}_{x,k}$ belongs to the support of $s$. Hence (the pullback of) $s$ is non-zero at the special point of $O^\text{et}_\eta$, which maps to the generic point $O^\text{et}_{x,k}$ (for some generic point $\eta$ of $U_k$). By Assumption (2), we deduce that the generic point of $O^\text{et}_\eta$ belongs to the support of (the pullback of) $s$. Since this generic point maps to the generic point of $O^\text{et}_x$, this means that the support of $s$ contains the generic point of $O^\text{et}_x$, hence because the support of $s$ is closed, it is the whole $\text{Spec}(O^\text{et}_x)$.

Now let $(s_1, \ldots, s_r)$ be a basis of the stalk $\mathcal{F}_x = \Gamma(O^\text{et}_x, \mathcal{F})$. These sections define a morphism
\[
\mathcal{Q}^\ell \to \mathcal{F}_{O^\text{et}_x}
\]
whose induced map on stalks is, by the above, injective. By Assumption (1) and the fact that the rank of the stalk of a constructible $\ell$-adic sheaf is a constructible function, the rank of the stalk of $\mathcal{F}$ at every point is $r$. Hence both stalks have the same dimension, thus the induced map on stalks is an isomorphism. This means that $\mathcal{F}$ is locally constant at $x$, and we conclude that $\mathcal{F}$ is lisse. \(\square\)

### 6. Generalized Kloosterman sheaves

In this section, we summarize the basic properties of the generalized Kloosterman sheaves whose trace functions are the sums $\text{Kl}_k(x; X, q)$. These were defined by Katz in [Kat88, Th. 4.1.1], building on Deligne’s work [SGA4², Sommes trig., Th. 7.8]. They are special cases of the hypergeometric sheaves defined by Katz in [Kat90, 8.2.1].
Throughout this section, we fix a prime number $p$, a prime number $\ell \neq p$, and we consider a finite field $\mathbb{F}_q$ of characteristic $p$ with $q$ elements and a non-trivial $\ell$-adic additive character $\psi$ of $\mathbb{F}_q$. We fix an integer $k \geq 2$ coprime to $q$, and a tuple $\chi = (\chi_1, \ldots, \chi_k)$ of $\ell$-adic characters of $\mathbb{F}_q^\times$. We denote by $\Lambda(\chi)$ (or $\Lambda$ if $\chi$ is understood) the product $\chi_1 \cdots \chi_k$.

**Proposition 6.1** (Generalized Kloosterman sheaves). There exists a constructible $\overline{\mathbb{Q}}_\ell$-sheaf $\mathcal{K}\ell = \mathcal{K}\ell_{k,\psi}(\chi)$ on $\mathbb{P}^1_{\overline{\mathbb{F}}_q}$, called a generalized Kloosterman sheaf, with the following properties:

1. For any $d \geq 1$ and any $x \in G_m(\mathbb{F}_q^d)$, we have
   
   $t_{\mathcal{K}\ell}(x; \mathbb{F}_q^d) = K_l_k(x; \chi, \mathbb{F}_q^d) \equiv \frac{(-1)^{k-1}}{q^{d(k-1)/2}} \sum_{x_1 \cdots x_k = x} \chi_1(N_{\mathbb{F}_q^d/\mathbb{F}_q}x_1) \cdots \chi_k(N_{\mathbb{F}_q^d/\mathbb{F}_q}x_k) \psi \left( \text{Tr}_{\mathbb{F}_q^d/\mathbb{F}_q}(x_1 + \cdots + x_k) \right)$.

2. The sheaf $\mathcal{K}\ell_{k,\psi}(\chi)$ is lisse of rank $k$ on $G_m$.

3. On $G_m$, the sheaf $\mathcal{K}\ell_{k,\psi}(\chi)$ is geometrically irreducible and pure of weight 0.

4. The sheaf $\mathcal{K}\ell_{k,\psi}(\chi)$ is tamely ramified at 0, and its $I(0)$-decomposition is

$$\bigoplus_{\chi \in \chi} \mathcal{L}_\chi \otimes J(n_\chi),$$

where $J(n)$ is a unipotent Jordan block of size $n$, and $n_\chi$ is the multiplicity of $\chi$ in $\chi$.

5. The sheaf $\mathcal{K}\ell_{k,\psi}(\chi)$ is wildly ramified at $\infty$, with a single break equal to $1/k$, and with Swan conductor equal to 1.

6. The stalks of $\mathcal{K}\ell_{k,\psi}(\chi)$ at 0 and $\infty$ both vanish.

7. If $\gamma \in \text{PGL}_2(\overline{\mathbb{F}}_q)$ is non-trivial, there does not exist a rank 1 sheaf $\mathcal{L}$ such that we have a geometric isomorphism

$$\gamma^* \mathcal{K}\ell_{k,\psi}(\chi) \simeq \mathcal{K}\ell_{k,\psi}(\chi) \otimes \mathcal{L}$$

over a dense open set.

8. The conductor of $\mathcal{K}\ell_{k,\psi}(\chi)$ is $k + 3$.

**Proof.** Let $j : G_m \to \mathbb{P}^1$ be the open inclusion. We define

$$\mathcal{K}\ell_{k,\psi}(\chi) = j_! \mathbb{K}l(\psi; \chi; 1, \ldots, 1) \left( \frac{n-1}{2} \right),$$

where the sheaf on the right-hand side is the lisse sheaf on $G_m$ defined by Katz in [Kat88, 4.1.1]. We also have a formula in terms of hypergeometric sheaves, namely

$$\mathcal{K}\ell_{k,\psi}(\chi) = j_! \mathcal{H}l(1, \psi; \chi, \emptyset) \left( \frac{n-1}{2} \right)$$

(see [Kat90, 8.4.3]). Assertions (1) and (2) are, respectively, assertions (2) and (1) of [Kat88, 4.1.1]. Assertion (3) results from the identification with hypergeometric sheaves and [Kat90, Th. 8.4.2 (1), (4)].

Assertions (4) and (5) are given in [Kat90, Th. 8.4.2 (6)]. Assertion (6) is clear from the definition as an extension by zero of a sheaf on $G_m$.

Finally, (7) is a special case of [FKM15b, Prop. 3.6 (2)], and (8) follows from the definition of the conductor and the previous statements. \qed

All parts of Definition 2.1, including the definition of Property CGM and Property NIO, make sense for tuples of $\ell$-adic characters of $\mathbb{F}_q^\times$. When we wish to emphasize the base finite field, we will speak of Property CGM or NIO over $\mathbb{F}_q$. The names CGM and NIO are justified by the following theorem of Katz.
Lemma 6.4. Assume that $\chi$ is not Kummer induced. Let $G$ be the geometric monodromy group of $\mathcal{K}_k,\psi(\chi)$. We then have $G^0 = G^{0,\text{der}}$, the derived group. Moreover

(1) If $k$ is odd, then $G^0 = G^{0,\text{der}} = \text{SL}_k$.
(2) If $k$ is even, then $G^0 = G^{0,\text{der}}$ is either
   - $\text{SO}_k$ if $\chi$ is self-dual and symmetric.
   - $\text{Sp}_k$ if $\chi$ is self-dual and alternating.
   - $\text{SL}_k$ if $\chi$ is not self-dual.

Finally, if $\chi$ has CGM, then $G = G^0$ is either $\text{SL}_k$ or $\text{Sp}_k$.

Proof. The claims about $G^0$ are proved by Katz in [Kat90, Th. 8.11.3 and Corollary 8.11.2.1].

To evaluate $G$, note that when $G^0 = \text{SL}_k$, $G$ is contained in $\text{GL}_k$. To show $G = G^0$, it suffices to show the determinant is trivial. But the determinant character is $\chi$ where $\chi$ is actually self-dual and not just self-dual up to a twist. This follows from [Kat90, Theorem 8.8.1]. Reviewing Definition 2.1, we obtain the desired statements.

The need to sometimes increase the base field is justified by the following lemma that will allow us to work with tuples satisfying the weaker CGM Property.

Lemma 6.3. Assume that $\chi$ has NIO. Then there exists an $\ell$-adic character $\chi_0$, possibly over a finite extension $F_{q^r}$ of $F_q$, such that the tuple $\chi_0\chi$ has CGM over $F_{q^r}$.

Proof. If $k$ is even and $\chi$ is self-dual alternating, take $\chi_0$ to be the inverse of a square root of the duality character. Otherwise, take $\chi_0$ to be the inverse of a $k$-th root of $\Lambda$.

For convenience, we will most often simply denote $\mathcal{K}_k = \mathcal{K}_k,\psi(\chi)$ since we assume that $\psi$ and $\chi$ are fixed.

The next lemma computes precisely the local monodromy of $\mathcal{K}_k,\psi(\chi)$ at $\infty$.

Lemma 6.4. Assume $p > k > 2$. Denote by $\widehat{\psi}$ the additive character $x \mapsto \psi(kx)$ of $F_q$. Then, as representations of the inertia group $I(\infty)$ at $\infty$, there exists an isomorphism

$$\mathcal{K}_k,\psi(\chi) \simeq [x \mapsto x^k]_*(L_{\chi_{(2)}} \otimes L_{\psi_{(2)}}),$$

where $\chi_{(2)}$ is the unique non-trivial character of order 2 of $F_q^\times$.

Proof. This follows from a more precise result of L. Fu [Fu11, Prop. 0.8] (who describes the local representations of the decomposition group).

7. Sheaves and statement of the target theorem

As in the previous section, we fix a prime number $p$, a prime number $\ell \neq p$, and we consider a finite field $F_q$ of characteristic $p$ with $q$ elements and a non-trivial $\ell$-adic additive character $\psi$ of $F_q$. We assume that $p > 2k + 1$.

Let $\chi$ be a $k$-tuple of $\ell$-adic characters of $F_q^\times$. We define

$$\mathcal{F} = \mathcal{K}_k,\psi(\chi),$$
a a constructible $\ell$-adic sheaf on $A_{F_q}$. In this section we impose no further conditions on $\chi$.

Fix $l \geq 2$. For $1 \leq i \leq 2l$, let $f_i = s(r + b_i)$ on $A^{2+2l}$ with coordinates $(r, s, b)$. 20
We now define the “sum-product” sheaf
\[ \mathcal{K}(\chi) = \bigotimes_{1 \leq i \leq l} f_i^* \mathcal{F} \otimes f_{i+1}^* \mathcal{F}' \]
on \mathbb{A}^{2+2l}_q.

Let \( V/\mathbb{Z} \) be the open subset of \( \mathbb{A}^{2+2l}_\mathbb{Z} \) where \( s(r + b_i) \neq 0 \) for \( i \), so that \( \mathcal{K} \) is lisse on \( V_{\mathbb{F}_q} \) for all \( q \). Let \( \pi: \mathbb{A}^{2+2l} \to \mathbb{A}^{1+2l} \) be the projection \( (r, s, b) \mapsto (r, b) \) (defined over \( \mathbb{Z} \)). We define
\[ \mathcal{R}(\chi) = R^1 \pi_! \mathcal{K}(\chi), \]
a constructible \( \ell \)-adic sheaf on \( \mathbb{A}^{1+2l}_{\mathbb{F}_q} \).

We will most often drop the dependency on \( \chi \) in these notation and write \( \mathcal{K} = \mathcal{K}(\chi) \) and \( \mathcal{R} = \mathcal{R}(\chi) \).

We define the diagonal variety \( \mathcal{V}^\Delta \) by the condition
\[ \mathcal{V}^\Delta = \{ b \in \mathbb{A}^{2l} \mid \text{for all } i, \text{there exists } j \neq i \text{ such that } b_i = b_j \}. \]
Note that \( \mathcal{V}^\Delta \) does not depend on the tuple of characters considered.

**Lemma 7.1.** Outside \( \mathcal{V}^\Delta \), we have \( R^0 \pi_! \mathcal{K} = R^2 \pi_! \mathcal{K} = 0 \).

**Proof.** This is very similar to [KMS17, Lemma 4.1 (2)]. By the proper base change theorem, the stalk of \( R^2 \pi_! \mathcal{K} \) at \( x = (r, b) \in \mathbb{A}^{1+2l} \) is
\[ H^i_c(\mathbb{A}_{\mathbb{F}_q}^{1+l}, \mathcal{F}) \]
where \( s \) is the coordinate on \( \mathbb{A}^1 \). This cohomology group vanishes for \( i = 0 \) and any \( x \), and it vanishes for \( i = 2 \) and \( x \notin \mathcal{V}^\Delta \) by [FKM15b, Theorem 1.5]. \( \square \)

We now compute the local monodromy at infinity of the sheaf \( \mathcal{K} \). For any additive character \( \psi \), we denote by \( \tilde{\psi} \) the character \( x \mapsto \psi(kx) \).

**Lemma 7.2.** (1) Let \( r \in \mathbb{F}_q \) and \( b \in \mathbb{F}_q^{2l} \) be such that \( r + b_i \neq 0 \) for all \( i \). Let \( (r + b_i)^{1/k} \) be a fixed \( k \)-th root of \( r + b_i \) in \( \mathbb{F}_q \). Define signs \( \varepsilon_i = 1 \) for \( 1 \leq i \leq l \) and \( \varepsilon_i = -1 \) for \( l + 1 \leq i \leq 2l \).

The local monodromy at \( s = \infty \) of \( \mathcal{K}_{r, b} \) is isomorphic to the local monodromy at \( s = \infty \) of the sheaf
\[ \bigoplus_{(\zeta_2, \ldots, \zeta_{2l}) \in \mu_{k}^{2l-1}} \mathcal{L}_{\tilde{\psi}}^{\zeta}(r, b_1)^{1/k} \sum_{i=2}^{2l} \varepsilon_i \zeta_i (r, b_i)^{1/k}s^{1/k}. \]
where \( \mu_k \) is the group of \( k \)-th roots of unity in \( \mathbb{F}_q \).

(2) Let \( K \) be a field of characteristic \( p \nmid k \), and let \( r \in K \) and \( b \in K^{2l} \) be such that \( r + b_i \neq 0 \) for all \( i \). Assume that \( K \) contains all \( k \)-th roots \( (1 + b_i/r)^{1/k} \) of \( 1 + b_i/r \) for all \( i \). Let \( \psi \) be a non-trivial \( \ell \)-adic additive character and let \( \chi \) be a \( k \)-tuple of multiplicative characters of a finite subfield of \( K \). The local monodromy at \( t = \infty \) of the lisse sheaf
\[ \tilde{\mathcal{K}} = \bigotimes_{1 \leq i \leq l} \mathcal{K} \ell_{k, \psi}(\chi)(t(1 + b_i/r)) \otimes \mathcal{K} \ell_{k, \psi}(\chi)(t(1 + b_i+t/r))^{1/k} \]
on \mathbb{G}_{m, K} \) is isomorphic to the local monodromy at \( s = \infty \) of the sheaf
\[ \bigoplus_{(\zeta_2, \ldots, \zeta_{2l}) \in \mu_{k}^{2l-1}} \mathcal{L}_{\tilde{\psi}}^{\zeta}(t(1+b_1/r)^{1/k} \sum_{i=2}^{2l} \varepsilon_i \zeta_i (t(1+b_i/r)^{1/k}). \]
Proof. Since Lemma 6.4 has the same form as [KMS17, Lemma 4.9], up to the additional factor \( \mathcal{L}_\Lambda \), the first assertion may be proved exactly like [KMS17, Lemma 4.16 (1)] (with \( \lambda = 0 \) there), replacing throughout the tensor product

\[
\bigotimes_{i=1}^{2} [(x + b_i)]^* \mathcal{K} \ell_{k, \psi}(\chi) \otimes [(x + b_{i+2})]^* \mathcal{K} \ell_{k, \psi}(\chi)'^* 
\]

by

\[
\bigotimes_{i=1}^{l} [(x + b_i)]^* \mathcal{K} \ell_{k, \psi}(\chi) \otimes [(x + b_{i+2})]^* \mathcal{K} \ell_{k, \psi}(\chi)'^* 
\]

(note that the factors involving \( \Lambda \) cancel-out at the end). The second statement is proved in the same manner. \( \square \)

Let \( \tilde{Z} \subset \mathbb{A}_Z^{1+2l} \) be the image of

\[
(7.1) \quad \tilde{Z} = \{(r, b, x) \in \mathbb{A}_Z^{1+4l} \mid x_i^k = r + b_i \text{ for } 1 \leq i \leq 2k, \quad \sum_{i=1}^{l} x_i = \sum_{i=l+1}^{2l} x_i \} \subset \mathbb{A}_Z^{1+4l}
\]

under the projection onto \((r, b)\). Let

\[
Z = \tilde{Z} \cup \bigcup_{1 \leq i \leq 2l} \{r = -b_i\}.
\]

Let \( U \) be the complement of \( Z \). We emphasize that \( \tilde{Z}, Z \) and \( U \) are defined over \( Z \), and independent of \( \chi \).

Lemma 7.3. The subscheme \( \tilde{Z} \) of \( \mathbb{A}_Z^{2l} \) is closed and irreducible, and \( \mathcal{R} \) is lisse on \( U_{\mathbb{F}_q} \).

Proof. This is analogue to [KMS17, Lemma 4.26, (1) and (2)], so we will be brief.\(^1\) The projection \((r, b, x) \mapsto (r, b)\) from the subscheme

\[
\mathcal{Z'} = \{(r, b, x) \in \mathbb{A}_Z^{1+4l} \mid x_i^k = r + b_i \text{ for } 1 \leq i \leq 2k\}
\]

to \( \mathbb{A}_Z^{1+2l} \) is finite, since the domain is defined by adjoining the coordinates \((x_1, \ldots, x_{2l})\) to \( \mathbb{A}_Z^{1+2l} \), and each satisfies a monic polynomial equation. Thus the closed subscheme \( \tilde{Z} \) defined by (7.1) is also finite over \( \mathbb{A}_Z^{1+2l} \), and its image \( \tilde{Z} \) is closed. Moreover, the subscheme (7.1) is the divisor in \( \mathcal{Z'} \) given by the equation

\[
\sum_{i=1}^{l} x_i = \sum_{i=l+1}^{2l} x_i.
\]

In particular, this subscheme, and consequently its projection \( \tilde{Z} \), is irreducible.

To prove that \( \mathcal{R} \) is lisse on \( U_{\mathbb{F}_q} \), we use Deligne’s semicontinuity theorem [Lau81]. The sheaf \( \mathcal{K} \) is lisse on the complement of the divisors given by the equations \( r = -b_i \) and \( s = 0 \) in \( \mathbb{A}_Z^{2+2l} \). We compactify the \( s \)-coordinate by \( \mathbb{P}^1 \) and work on

\[
X = (\mathbb{A}^1 \times \mathbb{P}^1 \times \mathbb{A}^{2l}) \cap \{(r, s, b) \mid (r, b) \in U\}.
\]

By extending by 0, we view \( \mathcal{K} \) as a sheaf on \( X \) which is lisse on the complement in \( X \) of the divisors \( s = 0 \) and \( s = \infty \) (because \( U \) is contained in the complement of the divisors \( r = -b_i \) and thus \( X \) is as well). Let

\[
\pi^{(2)} : X \to U
\]

\(^1\)To avoid confusion, note that what is called \( Z \) in KMS is not the analogue of what is called \( Z \) here.
denote the projection \((r, s, b) \mapsto (r, b)\). Then \(\pi^{(2)}\) is proper and smooth of relative dimension 1 and \(R\pi^{(2)} = \mathcal{R}|U = R^{1}\pi^{(2)}_s\mathcal{K}\).

Since the restrictions of \(\mathcal{K}\) to the divisors \(s = \infty\) and \(s = 0\) are zero, this sheaf is the extension by zero from the complement of those divisors to the whole space of a lisse sheaf. Deligne’s semicontinuity theorem [Lau81, Corollary 2.1.2] implies that the sheaf \(\mathcal{K}\) is lisse on \(U\) if the Swan conductor is constant on each of these two divisors. By Proposition 6.1, the generalized Kloosterman sheaf has tame ramification on \(s = 0\), hence any tensor product of generalized Kloosterman sheaves (such as \(\mathcal{K}\)) has tame ramification, hence Swan conductor 0, on \(s = 0\). On the other hand, Lemma 7.2 gives a formula for the local monodromy representation of \(\mathcal{K}\) as \(s = \infty\) as a sum of pushforward of representations from the tame covering \(x \mapsto x^k\). Since the Swan conductor is additive and since the Swan conductor is invariant under pushforward by a tame covering (see, e.g., [Kat88, 1.13.2]), it follows that

\[
\text{Swan}_x(\mathcal{K}_r, b) = \sum_{\zeta_2, \ldots, \zeta_{2l} \in \mu_k} \text{Swan}_x\left(\mathcal{L}_\psi\left(\left((r + b_1)^{1/k} + \sum_{i=2}^{2l} \varepsilon_i \zeta_i (r + b_i)^{1/k}\right)s^{1/k}\right)\right) = k^{2l-1}
\]

by definition of \(U\), since the Swan conductor of \(\mathcal{L}_\psi(at)\) is 1 for \(a \neq 0\).

**Lemma 7.4.** The subscheme \(Z\) is a hypersurface in \(A^{1+2l}_\mathbb{Z}\). It is defined by the vanishing of a polynomial \(P\) in \(\mathbb{Z}[r, b_1, \ldots, b_{2l}]\) such that, for any fixed \(b \notin \mathcal{Y}_\Delta\), the polynomial \(P_b = P(\cdot, b)\) of the variable \(r\) is not zero.

**Proof.** First we check that \(\tilde{Z}\) is a hypersurface in \(A^{1+2l}_\mathbb{Z}\). It is the projection of the closed subscheme

\[
\tilde{Z} = \left\{(r, b, x) \in A^{1+4l} | x_i^k = r + b_i \text{ for } 1 \leq i \leq 2l, \sum_{i=1}^{l} x_i = \sum_{i=l+1}^{2l} x_i \right\} \subset A^{1+4l}.
\]

This closed subscheme is pure of dimension \(2l\), since the first \(2l\) equations let us eliminate the variables \(b_i\) and the last equation is nontrivial. The projection \(\tilde{Z} \to \tilde{Z}\) is finite (as already observed in the proof of the previous lemma) and hence \(\tilde{Z}\) is a closed subscheme of \(A^{2l+1}\) that is pure of dimension \(2l\), i.e., a hypersurface. Since \(Z\) is the union of \(\tilde{Z}\) and the hyperplanes with equation \(r + b_i = 0\), it is also a hypersurface.

Let \(P \in \mathbb{Z}[r, b]\) be a polynomial whose vanishing set is \(\tilde{Z}\). Suppose \(b\) is such that \(P_b\) is the zero polynomial in the variable \(r\), i.e., such that the projection \(\tilde{Z}_b \to A^1\) given by \((r, x) \mapsto r\) is surjective.

The scheme \(C \subset A^{1+2l}\) given by the equations

\[
x_i^k = r + b_i \quad 1 \leq i \leq 2k
\]

is a curve and the projection \(C \to A^1\) given by \((r, x) \mapsto r\) is finite. The fiber \(\tilde{Z}_b\) is the intersection of \(C\) and the hyperplane

\[
\sum_{i=1}^{l} x_i = \sum_{i=l+1}^{2l} x_i,
\]

so that \(P_b = 0\) if and only if the function

\[
F = \sum_{i=1}^{l} x_i - \sum_{i=l+1}^{2l} x_i
\]

vanishes on an irreducible component of \(C\).

If we assume that \(b \notin \mathcal{Y}_\Delta\) then by definition there exists some \(i\) such that \(b_i \neq b_j\) for all \(j \neq i\). Locally on \(A^1\) with coordinate \(r\) near the point \(r = -b_i\), the covering maps \(x_i^k = r + b_j\) for \(j \neq i\)
are étale, so the functions \( x_j \) (on the curve \( C \)) “belong” to the étale local ring \( R \) of \( \mathbb{A}^1 \) at \(-b_i\). The function \( x_1 \), however, does not belong to \( R \), hence the function \( F \) is non-zero in an algebraic closure of the fraction field of \( R \), which is also an algebraic closure of the function field of any irreducible component of \( C \). This concludes the proof.

\section*{Definition 7.5}  
The sheaf \( \mathcal{R}^* \) on \( U_{F_q} \) is the maximal quotient of the sheaf \( \mathcal{R}|U_{F_q} \) that is pure of weight 1 (see [Del80]).

Define \( f: U \rightarrow \mathbb{A}^2 \) over \( \mathbb{Z} \) by \((r, b) \mapsto b\).

Below, by \( \text{End}_{V_b}(\mathcal{G}) \), where \( \mathcal{G} \) is a lisse sheaf on \( V_{F_q}b \), we mean the \( \pi_1(V_{F_q}b \times \overline{F_q})\)-homomorphisms, etc.

Let \( b \in \mathbb{A}_{F_q}^2 \) and let \( \kappa(b) \) be the residue field of \( b \). Since \( \mathcal{R}_b = R^1\pi_1\mathcal{K}_b \) by the proper base change theorem, there exists a natural \( \text{Gal}(\kappa(b)/\kappa(b))\)-equivariant morphism

\[ \text{End}_{V_b}(\mathcal{K}_b) \longrightarrow \text{End}_{U_b}(\mathcal{R}_b^*). \]

Since every \( V_b \)-endomorphism of \( \mathcal{K}_b \) preserves the weight filtration, the image of this morphism is contained in the subring of endomorphisms of \( \mathcal{R}_b \) that preserve the weight filtration, and hence we have an induced morphism

\[ \theta_b: \text{End}_{V_b}(\mathcal{K}_b) \longrightarrow \text{End}_{U_b}(\mathcal{R}_b^*), \]

which by construction is still Frobenius-equivariant.

In the next definition, we already describe the subvariety \( \mathcal{W} \) of Theorem 4.3; in particular, we see that it is independent of the tuple of characters \( \chi \), since this is the case for \( X_\infty \) and \( Z \). The difficulty will be to prove that it satisfies the required properties.

\section*{Definition 7.6}  
We denote \( X_\infty = \mathbb{A}^2 \setminus \mathcal{V}^\Delta \), and for any integer \( j \geq 0 \), we let

\[ X_j = \{ b \in X_\infty \mid |Z_b| \leq j \}. \]

We define \( \mathcal{W} \) to be the union of \( \mathcal{V}^\Delta \) and of all irreducible components of all \( X_j \) of dimension strictly less than \((3l+1)/2\).

By definition, we therefore have the codimension bound

\[ \text{codim}(\mathcal{W}) \geq \frac{l-1}{2}. \]

Our main geometric goal will be to prove the following result:

\section*{Theorem 7.7}  
Assume that \( \chi \) has NIO. If \( p \) is large enough, depending only on \( k \) and \( l \), then the natural morphism \( \theta_b \) is an isomorphism for all \( b \in \mathbb{A}^2(F_q) \setminus \mathcal{W}(F_q) \). Furthermore, each geometrically irreducible component of \( \mathcal{R}_b^* \) has rank greater than one.

The basic strategy to be used is as follows:

\begin{enumerate}
\item We show that for \( q \) large enough and for \( b \in \mathbb{A}^2(F_q) \) outside an explicit subscheme \( W_1 \) of codimension \( l - 1 \), the natural morphism \( \theta_b \) is injective. This reduces the target statement to a proof that the dimensions \( \text{End}_{V_b}(\mathcal{K}_b) \) and \( \text{End}_{U_b}(\mathcal{R}_b^*) \) are equal.
\item We show that, when these dimensions agree for the generic point of an irreducible component of a stratum, this implies the corresponding statement on the whole irreducible component.
\item Finally, we prove the target theorem at the generic point of an irreducible component of a stratum with dimension \( > (3l+1)/2 \).
\end{enumerate}

The most difficult part is the last one. This we prove by showing the strata can be covered by the vanishing sets of equations of a certain type in products of curves. Using this description, and a variant of Katz’s Diophantine criterion for irreducibility, we show that the dimension of the space of endomorphisms of \( \mathcal{K} \) is equal to the space of endomorphisms of \( \mathcal{R} \) that are invariant under the
Galois group of the function field of this cover. Finally, by a vanishing cycles argument, we show that the Galois group in fact acts trivially.

**Remark 7.8.** We have defined $U$, the stratification $X_j$, and $W$ as objects over the integers rather than over a finite field $\mathbf{F}_q$. This is used in a few different places: first, when comparing the generic point and the special point of a stratum, we use a tameness property of the sheaf $\mathcal{R}$, which we verify by showing that the sheaf is defined over the integers. Second, when describing the defining equations of the strata, at one point we make a large characteristic assumption. Third, we need the set $W$ to be uniform in $q$ to allow us to apply Lemma 2.3.

8. Integrality

We fix an integer $n \geq 1$ and an integer $k \geq 2$. Let $\ell$ be a prime number. We denote in this section $S = \text{Spec}(\mathbf{Z}[\mu_n, 1/n\ell])$. For any $\ell$-adic character $\chi$ of $\mathbf{Z}_\ell$, we have an associated lisse $\ell$-adic sheaf $\mathcal{L}_\chi$ over $S$ defined by Kummer theory. If $\mathbf{F}_q$ is a residue field of $S$ of characteristic $p \nmid n\ell$, so that $q \equiv 1 \mod n$, then there is a natural isomorphism between the group of $\ell$-adic characters $\chi$ of $\mathbf{Z}_\ell$ and the group of $\ell$-adic characters $\chi$ of $\mathbf{F}_q$ of order dividing $n$ of $\mathbf{F}_q^\times$, such that $\chi(x) = \chi^\ell(x)$, where $\xi$ is the $n$-th root of unity in $\mathbf{Z}[\mu_n, 1/n\ell]$ mapping to $x^{(q-1)/n}$. We then have a natural isomorphism $\mathcal{L}_{\chi|_{\mathbf{F}_q}} = \mathcal{L}_\chi$ of $\ell$-adic sheaves.

**Proposition 8.1.** Let $\bar{\chi}$ be a $k$-tuple of characters of $\mathbf{Z}_\ell$. There exists an $\ell$-adic sheaf $\mathcal{R}^\text{univ}(\bar{\chi})$ on $\mathbf{A}^{1+2l}_{\mathbb{S}}$, lisse on $\mathbb{U}_S$, with the following property: for any prime $p \nmid \ell n$, for any finite field $\mathbf{F}_q$ of characteristic $p$ which is a residue field of a prime ideal in $\mathbf{Z}[\mu_n, 1/n\ell]$, for any non-trivial additive character $\psi$ of $\mathbf{F}_q$, we have

$$\mathcal{R}^\text{univ}(\bar{\chi})|_{\mathbf{A}_{\mathbf{F}_q}^{1+2l}} = \mathcal{R}(\chi)$$

where $\chi$ is the $k$-tuple of $\ell$-adic characters of $\mathbf{F}_q^\times$ corresponding to $\bar{\chi}$.

**Proof.** We will first construct a sheaf $\mathcal{R}^\text{univ}(\bar{\chi})$ over $S$ with the desired specialization property, and we will then check that the sheaf thus defined is lisse on $\mathbb{U}_S$. The existence statement is a fairly straightforward generalization of [KMS17, Lemma 4.27], but we give full details since the precise construction is needed to check the lisseness assertion.

Let $X_1 \subset \mathbb{G}_{m}^{k+1}$ be the subscheme over $S$ with equation

$$x_1 \ldots x_k = t$$

and let

$$f_1 : X_1 \longrightarrow \mathbf{A}^1$$

be the projection $(x_1, \ldots, x_k, t) \mapsto t$. Let $X_2$ be the subscheme of $\mathbb{G}_{m}^{2lk} \times \mathbf{A}^{2+2l}$ over $S$ defined by the equations

$$\prod_{j=1}^{k} x_{i,j} = s(r + b_i), \quad 1 \leq i \leq 2l,$$

and let $f_2 : X_2 \longrightarrow \mathbf{A}^{1+2l}$ be the projection

$$f_2(x_{1,1}, \ldots, x_{2l,k}, r, s, b) = (r, b).$$

Let further $X \subset X_2$ be the closed subscheme over $S$ defined by the equation $x_{1,1} = 1$. The morphism

$$\mathbb{G}_{m} \times X \longrightarrow X_2$$

defined by

$$(t, x_{1,1}, \ldots, x_{2l,k}, r, s, b) \mapsto (tx_{1,1}, \ldots, tx_{2l,k}, r, t^ks, b)$$

is
is an isomorphism, with inverse given by
\[(x_{1,1}, \ldots, x_{2l,k}, r, s, b) \mapsto \left( x_{1,1}, \frac{x_{1,2}}{x_{1,1}}, \ldots, \frac{x_{2l,k}}{x_{1,1}}, r, s, \frac{b}{x_{1,1}} \right) \].

Let now \( p \nmid n \ell \) be a prime and \( \mathbb{F}_q \) a finite field of characteristic \( p \) that is a residue field of a prime ideal in \( S \). Let \( \psi \) be a non-trivial additive character of \( \mathbb{F}_q \). We have an isomorphism
\[
\mathcal{K}(\ell, \psi, \chi) \left( \frac{1}{2} \right) [1 - k] \simeq Rf_{\ell\psi}(x_1 + \cdots + x_k) \otimes \bigotimes_{i=1}^k \mathcal{L}_{\chi_i}(x_i)
\]
of sheaves on \( \mathbb{A}^1_{\mathbb{F}_q} \). By definition and Lemma 7.1, it follows that
\[
\mathcal{R}(\chi) = R^{2(k-1)+1}f_{\ell\psi} \left( \mathcal{L}_{\psi} \left( \sum_{j=1}^k \left( \sum_{i=1}^l x_{i,j} - \sum_{i=1}^l x_{l+i,j} \right) \right) \otimes \bigotimes_{j=1}^l \mathcal{L}_{\chi_j}(x_{i,j}/x_{l+i,j}) \right).
\]

We now translate this by “transport of structure” to \( G_m \times X \simeq X_2 \). First, we have \( f_2 = f \circ p_2 \) where \( p_2 \) is the projection \( G_m \times X \to X \). Next, let \( f : X \to \mathbb{A}^{1+2l} \) be the projection onto \((r, b)\), and let \( g : X \to \mathbb{A}^1 \) be defined by
\[
g(x_{1,1}, \ldots, x_{2l,k}, r, s, b) = \sum_{j=1}^k \left( \sum_{i=1}^l x_{i,j} - \sum_{i=1}^l x_{l+i,j} \right).
\]

Let \( g' \) be the function
\[g' = \sum_{j=1}^k \left( \sum_{i=1}^l x_{i,j} - \sum_{i=1}^l x_{l+i,j} \right)
\]
on \( X_2 \). Then \( g' \) corresponds to \( tg \) under the isomorphism \( X_2 \simeq G_m \times X \). Moreover, the sheaves \( \mathcal{L}_{\chi_j}(x_{i,j}/x_{l+i,j}) \) are transported to \( \mathcal{L}_{\chi_j}(x_{i,j}/x_{l+i,j}) \) under this isomorphism (since both variables involved are multiplied by \( t \)). We conclude that
\[
\mathcal{R}(\chi)[-2l(k-1)-1] \simeq R(f \circ p_2) \left( \mathcal{L}_{\psi} (tg) \otimes \bigotimes_{j=1}^l \mathcal{L}_{\chi_j}(x_{i,j}/x_{l+i,j}) \right)
\]
on \( \mathbb{A}^{1+2l}_{\mathbb{F}_q} \).

We can now apply the strategy of [KMS17, Lemma 4.23]. By the projection formula, we have
\[
R_{p_2} \left( \mathcal{L}_{\psi}(tg) \otimes \bigotimes_{j=1}^l \mathcal{L}_{\chi_j}(x_{i,j}/x_{l+i,j}) \right) = \left( \bigotimes_{j=1}^l \mathcal{L}_{\chi_j}(x_{i,j}/x_{l+i,j}) \right) \otimes R_{p_2} \mathcal{L}_{\psi}(tg)
\]
and \( R_{p_2} \mathcal{L}_{\psi}(tg) \) is the pullback along \( g \) of the Fourier transform of the extension by zero of the constant sheaf on \( G_m \mathbb{F}_q \), which is \( (Ru^* \mathcal{O}_q[-1])_{\mathbb{F}_q} \) for \( u : G_m \to \mathbb{A}^1 \) the inclusion.

We then define the sheaf
\[
\mathcal{R}^{\text{univ}}(\bar{\chi}) = R^{2l(k-1)}f_1 \left( g^*(Ru^* \mathcal{O}_q) \otimes \bigotimes_{j=1}^l \mathcal{L}_{\chi_j}(x_{i,j}/x_{l+i,j}) \right)
\]
over \( S \). The preceeding computation gives an isomorphism \( \mathcal{R}^{\text{univ}}(\bar{\chi})_{\mathbb{F}_q} \simeq \mathcal{R}(\chi) \) over \( \mathbb{F}_q \).

Furthermore, since the complex
\[
Rf_1 \left( g^*(Ru^* \mathcal{O}_q) \otimes \bigotimes_{j=1}^l \mathcal{L}_{\chi_j}(x_{i,j}/x_{l+i,j}) \right),
\]
is supported in degree $2l(k - 1)$ over $U_{\mathbb{F}_q}$ for all $\mathbb{F}_q$, the corresponding complex

$$RF_i\left(g^*(Ru_*\overline{Q}_\ell) \otimes \bigotimes_{j=1}^k \bigotimes_{i=1}^l \mathcal{L}_{\chi_j}^i(x_{i,j}/x_{l+i,j})\right)$$

is supported in a single degree on $S$.

We will now check that $\mathcal{R}^\text{univ}(\chi)$ is lisse on $U_S$. By the specialization property and Lemma 7.3, we know that $\mathcal{R}^\text{univ}(\chi)$ is lisse on $U_{\mathbb{F}_q}$ for any residue field $\mathbb{F}_q$ of characteristic $p \nmid \ell n$, and that it has constant rank. Because it is a constructible sheaf, its rank is a constructible function, and hence it has the same rank everywhere on $U_S$.

Write $\mathcal{R}^\text{univ} = \mathcal{R}^\text{univ}(\chi)$ for simplicity. We show that $\mathcal{R}^\text{univ}$ is lisse on $U_S$ by contradiction. By the criterion in Lemma 5.2, if $\mathcal{R}^\text{univ}$ is not lisse on $U_S$, then there exists a finite-field-valued point (say over $\mathbb{F}_q$) and a section of $\mathcal{R}^\text{univ}$ over the étale local ring $\mathcal{O}^\ell_{\eta}$ for some generic point $\eta$ of $U_{\mathbb{F}_q}$ which is non-zero at the special point, but zero at the generic point. If we denote by $i$ the inclusion of $\eta$ in $\text{Spec}(\mathcal{O}^\ell_{\eta})$, then such a section corresponds to a morphism $i_*\overline{Q}_\ell \to \mathcal{R}^\text{univ}$ over this local ring that is non-trivial at the generic point. Because

$$\mathcal{R}^\text{univ} = R^{2l(k-1)}f_i\left(g^*(Ru_*\overline{Q}_\ell) \otimes \bigotimes_{j=1}^k \bigotimes_{i=1}^l \mathcal{L}_{\chi_j}^i(x_{i,j}/x_{l+i,j})\right)$$

and the complex

$$RF_i\left(g^*(Ru_*\overline{Q}_\ell) \otimes \bigotimes_{j=1}^k \bigotimes_{i=1}^l \mathcal{L}_{\chi_j}^i(x_{i,j}/x_{l+i,j})\right)$$

is supported in a single degree, we obtain a nontrivial map.

$$Ri_*\overline{Q}_\ell[-2l(k-1)] \to RF_i\left(g^*(Ru_*\overline{Q}_\ell) \otimes \bigotimes_{j=1}^k \bigotimes_{i=1}^l \mathcal{L}_{\chi_j}^i(x_{i,j}/x_{l+i,j})\right). \quad (8.1)$$

We then apply the Verdier duality functor, taking our base scheme $S = \text{Spec}(\mathcal{O}^\ell_{\eta})$. In this case our dualizing complex is $\overline{Q}_\ell$ and we set $D(F) = \text{Hom}(F, \overline{Q}_\ell)$. Later, we will apply also apply Verdier duality on schemes of finite type over $S$ (see, e.g., [Fu11, Ch. 8, Ch. 10.1] for the $\ell$-adic formalism of Verdier duality in this setting). As usual, for a scheme of finite type over $S$ with structural morphism $\varpi$, we set $D(F) = \text{Hom}(F, F^\varpi\overline{Q}_\ell)$. Dualizing the morphism (8.1), we obtain a morphism

$$D RF_i\left(g^*(Ru_*\overline{Q}_\ell) \otimes \bigotimes_{j=1}^k \bigotimes_{i=1}^l \mathcal{L}_{\chi_j}^i(x_{i,j}/x_{l+i,j})\right) \to D Ri_*\overline{Q}_\ell[2l(k - 1)], \quad (8.2)$$

that is also nontrivial, since by double-duality its dual is (8.1).

We have

$$D Ri_*\overline{Q}_\ell = Ri_! D \overline{Q}_\ell = Ri_! i_* \overline{Q}_\ell = Ri_! i_*[\overline{Q}_\ell[-2]] = Ri_* \overline{Q}_\ell[-2],$$

where the last two equalities follow respectively from the fact that $i$ is the inclusion of a smooth divisor of codimension $1$ and the fact that $i$ is proper. The left-hand side of (8.2) is

$$RF_\ast D\left(g^*(Ru_*\overline{Q}_\ell) \otimes \bigotimes_{j=1}^k \bigotimes_{i=1}^l \mathcal{L}_{\chi_j}^i(x_{i,j}/x_{l+i,j})\right) = RF_\ast D(g^*(Ru_*\overline{Q}_\ell)) \otimes \bigotimes_{j=1}^k \bigotimes_{i=1}^l \mathcal{L}_{\chi_j}^{i-1}(x_{i,j}/x_{l+i,j}),$$

since duality is local, and therefore commutes with twisting with a locally constant sheaf. Hence the existence of a non-trivial morphism (8.2) would lead to a morphism

$$i^*RF_\ast D g^*(Ru_*\overline{Q}_\ell) \otimes \bigotimes_{j=1}^k \bigotimes_{i=1}^l \mathcal{L}_{\chi_j}^{i-1}(x_{i,j}/x_{l+i,j}) \to \overline{Q}_\ell[2l(k - 1) + 2]$$

27
that is nontrivial at $\eta$. Finally, this would force the stalk of the sheaf

$$i^*Rf_* D g^*(Ru_*Q_\ell) \otimes \bigotimes_{j=1}^{k} \otimes_{i=1}^{l} \mathcal{L}_{\chi_j^{-1}}(x_{i,j}/x_{l+i,j})$$

in degree $-2l(k-1) - 2$ to be nontrivial at the generic point of $\mathbb{A}^{2l+1}$. We will now prove that this last property fails.

Away from the vanishing set of $g$, the sheaf $g^*(Ru_*Q_\ell)$ is the constant sheaf $Q_\ell$, so its dual is $Q_\ell[2(2l(k-1))]$, where $2l(k-1)$ is the relative dimension of $X$.

On the other hand, we claim that the morphism $g$ is smooth in a Zariski-open neighborhood of the vanishing set of $g$. To check this, because $g' = gt$, it suffices to check that $g'$ is smooth in a neighborhood of its vanishing set. Examining just the contribution $\sum_{j=1}^{k} x_{i,j}$ to $g'$, observe that the only equation defining $X_2$ involving $(x_{i,1}, \ldots, x_{i,k})$ is of the form $\prod_{j=1}^{k} x_{i,j} = \alpha$, so the derivative of this contribution in a transverse direction is nonzero, and $g'$ is smooth, unless $x_{i,1} = x_{i,2} = \cdots = x_{i,k}$. In this case, all the $x_i$ are equal to some $k$-th root of $s(r + b_i)$, and thus

$$g' = \sum_{i=1}^{l} (s(r + b_i))^{1/k} - \sum_{i=l+1}^{2l} (s(r + b_i))^{1/k}$$

which is non-zero when $(r, b) \in U$.

Since $g$ is smooth in a neighborhood of the vanishing locus of $g$, the sheaf $D g^*(Ru_*Q_\ell) = g^! D(Ru_*Q_\ell)$ is there a shift (and Tate twist) of $g^* D(Ru_*Q_\ell)$, which is a shift (and Tate twist) of $g^* Ru_*Q_\ell$, and thus vanishes on the zero-set of $g$. We conclude that $D g^*(Ru_*Q_\ell)$ is everywhere supported in degree $-4l(k-1)$.

Finally, we observe that $f$ is an affine morphism from a scheme of dimension $2l(k-1)$. By results of Gabber (see [ILJ14, XV, Theorem 1.1.2]), the support of the sheaf

$$R^d f_* D \left( g^*(Ru_*Q_\ell) \otimes \bigotimes_{j=1}^{k} \otimes_{i=1}^{l} \mathcal{L}_{\chi_j^{-1}}(x_{i,j}/x_{l+i,j}) \right)$$

has dimension $2l(k-1) - d - 4l(k-1)$ relative to $S$. Hence, its stalk in degree $2 - 2l(k-1)$ has support of dimension

$$2l(k-1) + 2l(k-1) - 2 - 4l(k-1) = -2$$

and therefore vanishes at the generic point of the special fiber, which has dimension $-1$ (relative to Spec($\mathcal{O}_q^d$)). This is the desired contradiction.

\[ \square \]

9. Injectivity

Let

$$W_1 = W^\Delta \cup \{ b \in \mathbb{A}^{2l} \mid \text{at most two coordinates of } b \text{ have multiplicity 1} \}.$$ 

This is a closed subvariety of codimension $l - 1$ of $\mathbb{A}^{2l}_Z$. The goal of this section is to prove the following injectivity statement for $\theta_b$:

**Theorem 9.1.** Let $p > 2k + 1$ be a prime and let $F_q$ be a finite field of characteristic $p$ with $q$ elements. Let $\chi$ be a $k$-tuple of $\ell$-adic characters of $F_q^\times$ with Property CGM.

For $p$ large enough, depending only on $(k,l)$ and for $b \in \mathbb{A}^{2l}(F_q)$ outside $W_1(F_q)$, the natural morphism

$$\theta_b : \text{End}_{\mathcal{O}_q^d}(\mathcal{X}_b) \longrightarrow \text{End}_{\mathcal{O}_q^d}(\mathcal{R}_b^e)$$

is injective.
We begin with a lemma. First, we observe that for any \( b \), and any geometrically irreducible component \( \mathcal{H} \) of \( \mathcal{K}_b \), we can meaningfully speak of the weight one part of \( R^1\pi_!\mathcal{H} \), since \( \mathcal{H} \) is defined over a finite field extension of \( \mathbb{F}_q \).

**Lemma 9.2.** For any \( b \in \mathbb{A}^2(\mathbb{F}_q) \), the morphism \( \theta_b \) is injective if, and only if, for any geometrically irreducible component \( \mathcal{H} \) of \( \mathcal{K}_b \), the weight one part of \( R^1\pi_!\mathcal{H} \) is non-zero.

**Proof.** Since \( \mathcal{K}_b \) is pointwise pure, hence geometrically semisimple, it is geometrically isomorphic to a direct sum

\[
\bigoplus_{i \in I} \mathcal{F}_i^{\otimes n_i}
\]

for some geometrically irreducible sheaves \( \mathcal{F}_i \) and some integers \( n_i \geq 1 \). Then

\[
R^1\pi_!\mathcal{K}_b \cong \bigoplus_{i \in I} (R^1\pi_!\mathcal{F}_i)^{\otimes n_i},
\]

and the maximal weight one quotient of \( R^1\pi_!\mathcal{K}_b \) is also the corresponding direct sum of the maximal weight one quotients \( (R^1\pi_!\mathcal{F}_i)^{w=1} \) of \( R^1\pi_!\mathcal{F}_i \), with multiplicity \( n_i \). If one of these quotients vanishes, then any \( u \in \text{End}_{\mathcal{K}_b}(\mathcal{K}_b) \) that is non-zero only on the corresponding summand \( \mathcal{F}_i \) satisfies \( \theta_b(u) = 0 \).

Conversely, suppose that all the quotients \( (R^1\pi_!\mathcal{F}_i)^{w=1} \) are non-zero. By Schur’s Lemma, the endomorphism algebra \( \text{End}_{\mathcal{K}_b}(\mathcal{K}_b^*) \) is isomorphic to a product of matrix algebras \( M_{n_i}(\overline{\mathbb{Q}}_l) \). For each \( i \), \( \theta_b \) maps an endomorphism \( u \) to the endomorphism of \( (R^1\pi_!\mathcal{F}_i)^{w=1} \), represented by a block matrix with diagonal scalar matrices in each block, whose entries are the coefficients of the matrix in \( M_{n_i}(\overline{\mathbb{Q}}_l) \) corresponding to \( u \). Since the blocks have non-zero size, such a matrix is zero if and only if \( u \) is zero. \( \square \)

Let \( G \) be the geometric monodromy group of \( \mathcal{K}_\ell_k,\psi(\chi) \). Let \( b \in \mathbb{A}^2(\mathbb{F}_q) \). We denote by \( B \subset \mathbb{A}^1 \) the set of values \( \{b_i\} \). For any family \( \varrho = (\varrho_x)_{x \in B} \) of irreducible representations of \( G \), we denote by \( \mathcal{K}_\varrho \) the sheaf

\[
\mathcal{K}_\varrho = \bigotimes_{x \in B} \varrho_x(\mathcal{K}_\ell_k,\psi(\chi))(s(r + x)).
\]

on \( \mathbb{A}^2 \) with coordinates \((r, s)\).

**Lemma 9.3.** Assume that \( \chi \) has CGM. Any geometrically irreducible component \( \mathcal{H} \) of \( \mathcal{K}_b \) is isomorphic to \( \mathcal{K}_\varrho \) for some family \( \varrho = (\varrho_x)_{x \in B} \) such that, for all \( x \in B \), the representation \( \varrho_x \) is an irreducible summand of the representation \( \text{Std}^{\otimes n_1} \otimes (\text{Std}^{\vee})^{\otimes n_2} \), where

\[
(9.1) \quad n_1 = \sum_{1 \leq i \leq l} 1, \quad n_2 = \sum_{l+1 \leq i \leq 2l} 1.
\]

**Proof.** Write

\[
\mathcal{K}_b = \bigotimes_{x \in B} \mathcal{K}_\ell_k,\psi(\chi)(s(r + x))^{\otimes n_1} \otimes (\mathcal{K}_\ell_k,\psi(\chi)(s(r + x))^{\vee})^{\otimes n_2}.
\]

By the Goursat–Kolchin–Ribet criterion (see \cite{Kat90} or \cite{FKM15b}), which may be applied since the sheaf \( \mathcal{K}_\ell_k,\psi(\chi) \) has geometric monodromy group \( \text{SL}_k \) or \( \text{Sp}_k \) by Theorem 6.2, the sheaf

\[
\bigoplus_{x \in B} \mathcal{K}_\ell_k,\psi(\chi)(s(r + x))
\]

has geometric monodromy group \( G^{[B]} \), so that its irreducible components correspond exactly to the tuples \( \varrho \). \( \square \)

**Lemma 9.4.** Let \( b \) be a point in \( \mathbb{A}^2 - V_\Delta \). Let \( \mathcal{K}_\varrho \) be an irreducible component of \( \mathcal{K}_b \). Then the rank of \( R^1\pi_!\mathcal{K}_\varrho \) on the dense open set where \( P_b(r) \neq 0 \) is equal to the rank of \( \mathcal{K}_\varrho \) divided by \( k \).
Proof. Note that the set where $P_b$ doesn’t vanish is indeed a dense open subset by Lemma 7.4.

Let $r$ be such that $P_b(r) \neq 0$. Then by proper base change, the stalk of $R^1\pi_1(H_b)$ at $r$ is equal to $H^1_c(G_m,\mathcal{F}_q, \mathcal{K}_{\mathbb{F},r})$.

Because $P_b(r) \neq 0$, Lemma 7.2 shows that the local monodromy representation at $\infty$ of $\mathcal{K}_{b,r}$ is isomorphic to a sum of sheaves of the form $\mathcal{L}_\psi(\alpha \cdot s^{1/k})$ for nonzero $\alpha$. Each sheaf $\mathcal{L}_\psi(\alpha \cdot s^{1/k})$ has all breaks $1/k$ at $\infty$, so the same is true for $\mathcal{K}_{b,r}$.

The sheaf $\mathcal{K}_{\mathbb{F},r}$ is a summand of $\mathcal{K}_{b,r}$, hence it also lisse on $G_m$, namely ramified at 0, and has all breaks $1/k$ at $\infty$. Moreover, it also satisfies

$$H^0_c(G_m,\mathcal{F}_q, \mathcal{K}_{\mathbb{F}}) = H^2_c(G_m,\mathcal{F}_q, \mathcal{K}_{\mathbb{F}}) = 0,$$

and therefore the Euler-Poincaré characteristic formula for a lisse sheaf on $G_m$ implies that

$$\dim H^1_c(G_m,\mathcal{F}_q, \mathcal{K}_{\mathbb{F}}) \leq \chi(G_m,\mathcal{F}_q, \mathcal{K}_{\mathbb{F}}) = \text{Swan}(\mathcal{K}_{\mathbb{F}}) + \text{Swan}_c(\mathcal{K}_{\mathbb{F}}) = \frac{1}{k} \text{rk}(\mathcal{K}_{\mathbb{F}}).$$

In the next lemmas, we fix a point $b$ in $\mathbb{A}^d - V(\Delta)$, and an index $i$ such that $b_i \neq b_j$ for $j \neq i$.

We denote $\epsilon = -1$ if $1 \leq i \leq l$, and $\epsilon = 1$ if $l + 1 \leq i \leq 2l$. For any character $\chi$, we denote $n_\chi$ the multiplicity of $\chi$ in $\chi$, which is 0 if $\chi \notin \chi$.

For an irreducible component

$$\mathcal{K}_{\mathbb{F}} = \bigotimes_{x \in B} \mathcal{L}_\psi(\mathcal{K}_x,\mathcal{K}(\chi))(s(r + x))$$

of $\mathcal{K}_b$ (all are of this type by Lemma 9.3), we denote

$$M_\mathbb{F} = \bigotimes_{x \in B} \mathcal{L}_\psi(\mathcal{K}_x,\mathcal{K}(\chi))(s(r + x)).$$

Since $M_\mathbb{F}$ is tamely ramified at 0, its local monodromy representation at $s = 0$ can be expressed as a sum of Jordan blocks, which we write

$$\bigoplus_\eta \mathcal{L}_\eta \otimes J(m_\eta)$$

where $\eta$ runs over a finite set of characters.

**Lemma 9.5.** With notation as above, the rank of the weight one part of $R^1\pi_1(\mathcal{K}_{\mathbb{F}})$ on the nonempty open set where $P_b(r) \neq 0$ is equal to

$$\sum_\eta \max(m_\eta - n_\eta, 0).$$

**Proof.** Because $b_i$ occurs with multiplicity one in $B$, the representation $\varrho_{b_i}$ is necessarily the standard representation if $i \leq l$ or its dual if $i > l$ (see (9.1)), and in any case has rank $k$. This implies that

$$\text{rk}(\mathcal{K}_{\mathbb{F}}) = k \text{rk}(M_\mathbb{F})$$

and hence by Lemma 9.4, we have

$$\text{rk}(R^1\pi_1(\mathcal{K}_{\mathbb{F}})) = \text{rk}(M_\mathbb{F}) = \sum_\eta m_\eta,$$

so that it suffices to show that the weight < 1 part of $R^1\pi_1(\mathcal{K}_{\mathbb{F}})$ has the rank

$$\sum_\eta \min(m_\eta, n_\eta).$$

To prove this, observe that the weight < 1 part is the sum over the singularities of the sheaf of the local monodromy invariants (see, e.g., [KMS17, Lemma 4.22(2)]). Because $\mathcal{K}_{\mathbb{F},r}$ is a summand
of $K_{b,r}$ which by Lemma 7.2 has no nontrivial local monodromy invariants at $\infty$, $H_{b,r}$ has no nontrivial local monodromy invariants at $\infty$.

If $i \leq l$, then the local monodromy representation at 0 is given by

$$H_{b,r} = \mathcal{M}_b \otimes \mathcal{K}_{k,\psi}(\chi)(s(r + b_i)) = \left( \bigoplus_{\eta} \mathcal{L}_\eta \otimes J(m_\eta) \right) \otimes \left( \bigoplus_{\lambda \in \chi} \mathcal{L}_\chi \otimes J(n_\chi) \right)$$

$$= \bigoplus_{\eta, \lambda \in \chi} \mathcal{L}_{\eta \chi} \otimes J(m_\eta) \otimes J(n_\chi).$$

The dimension of the invariant subspace of $L_p$ is $\min(9,i)$, case it is $\min(m_\eta, n_\chi)$, hence the result follows in that case.

If $l + 1 \leq i \leq 2l$, the same calculation applies, except that $\mathcal{L}_{\chi^{-1}}$ appears instead of $\mathcal{L}_\chi$.

The next lemma continues with the same notation.

**Lemma 9.6.** Assume that $\chi$ has CGM. Then the rank of the weight one part of $R^1\pi_1 H_{b,r}$ is at least two.

**Proof.** By the previous lemma, it is enough to prove that

$$\sum_{\eta} \max(m_\eta - n_\eta, 0) \geq 2.$$ (9.3)

Since $b \notin W_1$, there are at least three elements of $B$ that occur with multiplicity one, say $b_i, b_j$ and $b_j'$.

Let $\delta = 1$ if $j \leq l$ and $\delta = -1$ if $j > l$, so that $\rho_{b_i}$ is the standard representation if $\delta = 1$ and the dual representation if $\delta = -1$.

Let

$$M'_b = \bigotimes_{x \in B, x \neq b_i, b_j} \rho_x(\mathcal{K}_{k,\psi}(\chi))(s(r + x))$$

so that

$$M_b = M'_b \otimes \mathcal{K}_{k,\psi}(\chi)(s(r + b_j))$$

if $\delta = 1$ and

$$M_b = M'_b \otimes \mathcal{K}_{k,\psi}(\chi)(s(r + b_j))'$$

if $\delta = -1$.

Let $\mathcal{L}_\theta \otimes J(r)$ be a Jordan block in the local monodromy representation of $M'_b$ at $s = 0$. We estimate the contribution from this factor in the local monodromy representation (9.2) of $M_b$.

This contribution contains a direct sum

$$\bigoplus_{\chi \in \chi} \mathcal{L}_{\chi}\otimes J(n_\chi + r - 1).$$ (9.4)

If the character $\theta$ is nontrivial, then the tuple of characters $\theta' \chi'^{\delta'}$ cannot be equal to $\chi$, up to permutation because this would contradict the CGM assumption. Hence, there exists a character $\chi$ such that $n_\chi > n_{\chi^{\delta'}}$, and therefore the Jordan blocks (9.4) include a character $\eta = \chi^{\delta}\theta$ with $m_\eta > n_\eta$. Hence these blocks have a contribution

$$\geq \min(n_\chi + r - 1 - n_{\chi^{\delta'}} + r - 1) \geq r$$

to the sum on the left-hand side of (9.3).

On the other hand, if $\theta$ is trivial, then the character $\chi$ with $n_\chi$ maximal contributes

$$\geq \min(n_\chi + r - 1 - n_\chi, 0) = r - 1.$$

In particular, we obtain (9.3) except if the local monodromy of $M'_b$ at zero consists of at most one unipotent Jordan block of rank two, or of at most one nontrivial character of rank one, plus
a sum of any number of trivial representations. This conditions means that local monodromy representation of \( M_\alpha \) at zero is either trivial or is a pseudoreflection (unipotent or not).

In the first case, we have a sheaf with trivial local monodromy at 0 that is expressed as a tensor product. Then all the tensor factors must have scalar local monodromy at 0. This is impossible here, since one of the tensor factors is \( \mathcal{K}_{k,\psi}(\chi)(s(r+b_j)) \) or its dual, and the local monodromy of this sheaf is not scalar (because \( k \geq 2 \)).

If the local monodromy representation is a pseudoreflection, then when it is expressed as a tensor product, all but one of the tensor factors must be one-dimensional, and the remaining factor must have local monodromy that is given by a pseudoreflection times a scalar. Again, because one of the tensor factors is \( \mathcal{K}_{k,\psi}(\chi)(s(r+b_j)) \) or its dual, this must be the special factor, and this can only happen when \( k = 2 \) by Proposition 6.1. All the remaining tensor factors are one-dimensional. But since the geometric monodromy group is \( SL_2 \) in that case (because \( \chi \) has CGM), and the only one-dimensional representation of \( SL_2 \) is the trivial representation, and this only appears in even tensor powers of the standard representation, we conclude that all remaining factors must have even multiplicity. This is a contradiction, since we have three factors with multiplicity one, and the sum of the multiplicities is \( 2l \), which is even.

□

Now Theorem 9.1 follows immediately from Lemma 9.2 and Lemma 9.6.

10. Specialization statement

We continue with the previous notation. Recall that \( X_\infty = A^{2l} - \mathcal{V}^\Delta \) and that \( X_j \) is defined in Definition 7.6. We recall that we have the projection \( f: U \to A^{2l} \).

Lemma 10.1. For each \( j \), the subvariety \( X_j \) is closed in \( X_\infty \).

For each irreducible component \( X \) of \( X_j \) that intersects the characteristic zero part, the morphism

\[
f: Z \cap f^{-1}(X - X \cap X_{j-1}) \to X - X \cap X_{j-1}
\]

is finite étale.

Proof. These claims follow from Lemma 7.4. Indeed, \( Z \) is the solution set of a family of nonzero polynomials in one variable indexed by points of \( X_\infty = A^{2l} - \mathcal{V}^\Delta \). The set \( X_j \) is constructible, so to show it is closed it suffices to show that it is closed under specialization. The polynomial factorizes completely over any geometric generic point into one distinct factor for each root, raised to some power, and each factor has at most one root over the special point, so the number of roots over the special point is at most the number of roots over the generic point, as desired.

To check that \( Z \cap f^{-1}(X - X \cap X_{j-1}) \) is finite étale over \( X - X \cap X_{j-1} \), we consider the polynomial \( P(r) \) over the étale local ring of a point of \( X - X \cap X_{j-1} \), which is an integral strict Henselian local ring, and use the fact that the polynomial has the same number of roots over the special point and over the geometric point. By the previous discussion each linear factor over the geometric generic point must admit a root over the residue field, which means the polynomial is monic. Because it is monic, and the ring is strict henselian, we can factor it into a product of irreducible factors, each with exactly one root in the residue field. Over the generic point each such factor will have only one root in the residue field, hence have only one root in the fraction field. Therefore, because the generic point has characteristic zero, so all polynomials are separable, each such factor is a power of \( (x - \alpha) \) where \( \alpha \) is its unique root, so the polynomial is a product of linear factors, with at most one distinct linear factor with each possible root in the residue field, hence its vanishing set is the disjoint union of the vanishing sets of these linear factors and thus is finite étale.

□

Fix \( j \geq 0 \). Let \( X \subset X_j \subset A^{2l} \) be an irreducible component of \( X_j \) over \( \mathbb{Z} \) which intersects the characteristic zero part. We consider a finite field \( \mathbb{F}_q \) of characteristic \( p > 2k + 1 \) such that \( X_{\mathbb{F}_q} \) is irreducible and nonempty.
Lemma 10.2. Let $\chi$ be a $k$-tuple of characters of $\mathbb{F}_q^\times$. The sheaf $\mathcal{R}^*(U \cap f^{-1}(X_{\mathbb{F}_q} - X_{\mathbb{F}_q} \cap X_{j-1}))$ is tamely ramified around the divisor $Z \cup \{x\}$.

Proof. Let $n$ be the lcm of the orders of the characters $\chi_i$. By the remarks before Proposition 8.1, there exists a tuple $\tilde{\chi}$ of characters of $\mu_n$ such that $\chi$ is associated to this tuple. Let $\mathcal{R}^\text{univ}(\tilde{\chi})$ be the sheaf over $\mathbb{Z}[\mu_n, 1/(n\ell)]$ given by Proposition 8.1. This sheaf $\mathcal{R}^\text{univ}(\tilde{\chi})$ is lisse on the open set $U \cap f^{-1}(X_{j} - X \cap X_{j-1})$, whose complement is the étale divisor $Z \cup \{x\}$. Hence, by Abyankhar’s Lemma [SGA1, Exposé XIII, §5], the sheaf $\mathcal{R}^\text{univ}(\tilde{\chi})$ is tamely ramified, and hence so is

$$\mathcal{R}^\text{univ}(\tilde{\chi})|_{\mathbb{A}^{2l}|_{\mathbb{F}_q}} = \mathcal{R}(\chi),$$

and also $\mathcal{R}^*(\chi)$.

\[\Box\]

Proposition 10.3. Let $\eta$ be the generic point of $X_{\mathbb{F}_q}$, and let $\bar{\eta}$ be a geometric generic point over $\eta$. Let $\chi$ be a $k$-tuple of characters of $\mathbb{F}_q^\times$ with Property CGM. Suppose that

$$\dim \text{End}_{U_\eta}(\mathcal{R}^*_\eta) = \dim \text{End}_{V_\bar{\eta}}(\mathcal{X}_\bar{\eta}).$$

Let $b \in X(\mathbb{F}_q)$ such that $b \notin X_{j-1}$ and $b \notin W_1$. Then $\theta_b$ is an isomorphism.

Proof. Consider the sheaf $\mathcal{E} = R^2 f_!(\mathcal{R}^* \otimes \mathcal{R}^{*, \nu})$

on $\mathbb{A}^{2l}|_{\mathbb{F}_q}$. We claim that

(a) The restriction of $\mathcal{E}$ to $X_j - X_{j-1}$ is lisse.

(b) We have an isomorphism $\mathcal{E}_\eta \simeq \text{End}_{U_\eta}(\mathcal{R}^*_\eta)(-1)$.

(c) We have an isomorphism $\mathcal{E}_b \simeq \text{End}_{U_b}(\mathcal{R}^*_b)(-1)$.

Moreover, let $g: V \to \mathbb{A}^{2l}$ be the map $(r, s, b) \mapsto b$ over $\mathbb{Z}$ and

$$\tilde{\mathcal{E}} = R^4 g_!(\mathcal{K} \otimes \mathcal{K}^{\nu})$$

on $\mathbb{A}^{2l}|_{\mathbb{F}_q}$. We claim that

(a') The restriction of $\tilde{\mathcal{E}}$ to $X_j - X_{j-1}$ is lisse.

(b') We have an isomorphism $\tilde{\mathcal{E}}_\eta \simeq \text{End}_{V_\eta}(\mathcal{X}_\eta)(-1)$.

(c') We have an isomorphism $\tilde{\mathcal{E}}_b \simeq \text{End}_{V_b}(\mathcal{X}_b)(-1)$.

Assuming these facts, we have

$$\dim \text{End}_{U_b}(\mathcal{R}^*_b) = \dim \mathcal{E}_b = \dim \tilde{\mathcal{E}}_b = \dim \text{End}_{U_\eta}(\mathcal{R}^*_\eta),$$

with the identities following from respectively (c), (a), (b), the assumption, (b'), (a'), and (c'). (In particular, when we apply assumption (a) and (a'), we use the fact that $b$ is a specialization of $\bar{\eta}$, hence they lie on the same connected component of $X_j - X_{j-1}$, and so any lisse sheaf on $X_j - X_{j-1}$ has equal ranks at these two points.)

We now prove the claims. The assertions (b)/(b') and (c)/(c') follow from the proper base change theorem, Poincaré duality, and semisimplicity.

Assertion (a) is a consequence of Deligne’s semicontinuity theorem and the tameness of $\mathcal{R}^*$. Specifically, by Lemma 10.1, we know that $U$, over $X_{\mathbb{F}_q} - (X_{\mathbb{F}_q} \cap X_{j-1})$, is the complement of a finite étale divisor inside a morphism smooth and proper of relative dimension one, and $\mathcal{R}^* \otimes \mathcal{R}^{*, \nu}$
is a lisse sheaf on it. By Lemma 10.2, the Swan conductor of $R^s \otimes R^s$ at this divisor vanishes, and so by Deligne’s semicontinuity theorem [Lau81, Corollary 2.1.2] the cohomology sheaf is lisse.

Assertion (a): Let $Y = X_{\mathbf{F}_q} - (X_{\mathbf{F}_q} \cap X_j)$. Then $\mathcal{K} \otimes \mathcal{K}^\nu$ is lisse on $V \times \mathbf{A}^2 Y$. Let $(\mathcal{K} \otimes \mathcal{K}^\nu)^{\pi_1(V \times \mathbf{A}^2 Y)}$ be its (geometric) monodromy invariants. Then there is a natural map

$$(\mathcal{K} \otimes \mathcal{K}^\nu)^{\pi_1(V \times \mathbf{A}^2 Y)} \to \mathcal{K} \otimes \mathcal{K}^\nu$$

over $V \times \mathbf{A}^2 Y$, where we interpret $(\mathcal{K} \otimes \mathcal{K}^\nu)^{\pi_1(V \times \mathbf{A}^2 Y)}$ as a constant sheaf. This induces by functoriality a map

$$R^4g\_! (\mathcal{K} \otimes \mathcal{K}^\nu)^{\pi_1(V \times \mathbf{A}^2 Y)} \to R^4g\_! \mathcal{K} \otimes \mathcal{K}^\nu$$

over $Y$. Because $V$ is an open subset of $\mathbf{A}^{2+2}$ whose fibers under $g$ are all nonempty, the top cohomology of a constant sheaf along $g$ is a constant sheaf, so this gives a map

$$(\mathcal{K} \otimes \mathcal{K}^\nu)^{\pi_1(V \times \mathbf{A}^2 Y)} \to R^4g\_! \mathcal{K} \otimes \mathcal{K}^\nu.$$

We claim that this last map is an isomorphism. It is sufficient to check this on the stalk at each point $b$. To do this, first check that the monodromy group of $\mathcal{K} \otimes \mathcal{K}^\nu$ over $V \times \mathbf{A}^2 Y$ is equal to the monodromy of the same sheaf on $V_b$. This can be done using Goursat-Kolchin-Ribet, since $\chi$ has CGM and $p > 2k + 1$. We also use the fact that, because $Z$ is finite etale over $Y$, and $Z$ includes $\{-b_1, \ldots, -b_{2g}\}$, no $b_i, b_j$ that are distinct generically on the $Y$ stratum can become equal at any point of $Y$.

Next observe that this map is simply the natural map from the monodromy invariants of $\mathcal{K} \otimes \mathcal{K}^\nu$ to the monodromy coinvariants of $\mathcal{K} \otimes \mathcal{K}^\nu$. Because the monodromy is semisimple, it is an isomorphism.

11. Preliminaries for the proof of the generic statement

This section uses independent notation from the rest of the paper. In particular, we will use the letter $k$ to denote finite fields.

We will use the following variant of the Diophantine Criterion for irreducibility of Katz (compare [Kat05, p. 25] and [KMS17, Lemma 4.14]).

Lemma 11.1. Let $w$ be an integer. Let $X$ be a geometrically irreducible separated scheme of finite type over a finite field $k$, and let $U$ be a normal open dense subset of $X$. Let $\ell$ be a prime different from the characteristic of $k$. Let $\mathcal{F}$ be an $\ell$-adic sheaf on $X$, mixed of weights $\leq w$ on $X$, and lisse and pure of weight $w$ on $U$. We have then

$$\dim \operatorname{End}^{\pi_1(U \times \mathbf{F}_q)}(\mathcal{F}|U) = \limsup_{\nu \to +\infty} \frac{1}{|k|^{|\nu(\dim(X_{\mathbf{F}_q}) + w)|}} \sum_{x \in X(k_{\nu})} |t_\mathcal{F}(x; k_{\nu})|^2,$$

where $k_{\nu}$ is the extension of $k$ of degree $\nu$ in a fixed algebraic closure.

In particular, if the right-hand side of the formula above is equal to 1, then $\mathcal{F}|U$ is geometrically irreducible.

Proof. Let $n = \dim(X_{\mathbf{F}_q})$. Up to performing a Tate twist on $\mathcal{F}$, we may assume that $w = 0$. For any $x \in X(k_{\nu})$ we have then

$$|t_\mathcal{F}(x; k_{\nu})|^2 \leq \operatorname{rk}(\mathcal{F})^2$$

hence by trivial counting we get

$$\frac{1}{|k|^{|\nu|}} \sum_{x \in X(k_{\nu})} |t_\mathcal{F}(x; k_{\nu})|^2 = \frac{1}{|k|^{|\nu|}} \sum_{x \in U(k_{\nu})} |t_\mathcal{F}(x; k_{\nu})|^2 + \frac{1}{|k|^{|\nu|}} \sum_{x \in (X \setminus U)(k_{\nu})} |t_\mathcal{F}(x; k_{\nu})|^2$$

$$= \frac{1}{|k|^{|\nu|}} \sum_{x \in U(k_{\nu})} |t_\mathcal{F}(x; k_{\nu})|^2 + O(|k|^{-\nu}).$$
This shows that we may restrict the sum on the right-hand side of (11.1) to \( U(k_\nu) \).

Since \( \mathcal{F} \) and its dual \( \mathcal{F}^\vee \) are lisse and pointwise pure of weight 0 on \( U \), the sheaf \( \text{End}(\mathcal{F}) = \mathcal{F} \otimes \mathcal{F}^\vee \) is also lisse and pointwise pure of weight 0 on \( U \). Moreover, for all \( x \in U(k_\nu) \), we have

\[
t_{\text{End}(\mathcal{F})}(x; k_\nu) = |t_\mathcal{F}(x; k_\nu)|^2.
\]

By the Grothendieck–Lefschetz trace formula, we have

\[
\frac{1}{|k|^{|\nu|}} \sum_{x \in U(k_\nu)} |t_\mathcal{F}(x; k_\nu)|^2 = \frac{1}{|k|^{|\nu|}} \text{Tr}(\text{Fr}_{k_\nu} | H_c^{2n}(U \times \overline{\mathbb{F}}_q, \text{End}(\mathcal{F}))) + \frac{1}{|k|^{|\nu|}} \sum_{i=0}^{2|\nu|-1} (-1)^i \text{Tr}(\text{Fr}_{k_\nu} | H_c^{1}(U \times \overline{\mathbb{F}}_q, \text{End}(\mathcal{F}))).
\]

By Deligne’s Riemann Hypothesis \([\text{Del80}]\), all eigenvalues of the Frobenius of \( k_\nu \) acting on the cohomology group \( H_c(U \times \overline{\mathbb{F}}_q, \text{End}(\mathcal{F})) \) have modulus \( \leq |k|^{1/2} \), and therefore

\[
|\text{Tr}(\text{Fr}_{k_\nu} | H_c^{1}(U \times \overline{\mathbb{F}}_q, \text{End}(\mathcal{F})))| \leq \text{dim}(H_c^{1}(U \times \overline{\mathbb{F}}_q, \text{End}(\mathcal{F})))|k|^{-\nu/2},
\]

so that we derive

\[
\frac{1}{|k|^{|\nu|}} \sum_{x \in U(k_\nu)} |t_\mathcal{F}(x; k_\nu)|^2 = \frac{1}{|k|^{|\nu|}} \text{Tr}(\text{Fr}_{k_\nu} | H_c^{2n}(U \times \overline{\mathbb{F}}_q, \text{End}(\mathcal{F}))) + O(|k|^{-\nu/2}).
\]

On the other hand, we have a Frobenius-equivariant isomorphism

\[
H_c^{2n}(U \times \overline{\mathbb{F}}_q, \text{End}(\mathcal{F})) \simeq \text{End}(\mathcal{F})_{\pi_1(U \times \overline{\mathbb{F}}_q)}(\overline{\mathbb{F}}_q)(-n).
\]

The eigenvalues of Frobenius on \( \text{End}(\mathcal{F})_{\pi_1(U \times \overline{\mathbb{F}}_q)}(\overline{\mathbb{F}}_q)(-n) \) have modulus \( q^n \). Therefore

\[
|k|^{-\nu} \text{Tr}(\text{Fr}_{k_\nu} | H_c^{2n}(U \times \overline{\mathbb{F}}_q, \text{End}(\mathcal{F})))
\]

is the sum of the \( \nu \)-th power of \( \text{dim} H_c^{2n}(U \times \overline{\mathbb{F}}_q, \text{End}(\mathcal{F})) \) complex numbers, each of modulus 1, and by a standard lemma, we have therefore

\[
\limsup_{\nu \to +\infty} \frac{1}{|k|^{|\nu|}} \text{Tr}(\text{Fr}_{k_\nu} | H_c^{2n}(U \times \overline{\mathbb{F}}_q, \text{End}(\mathcal{F}))) = \text{dim} H_c^{2n}(U \times \overline{\mathbb{F}}_q, \text{End}(\mathcal{F})) = \text{dim} \text{End}_{\pi_1(U \times \overline{\mathbb{F}}_q)}(\mathcal{F}),
\]

by the geometric semi-simplicity of \( \mathcal{F}|U \).

This result, combined with the injectivity statement, reduces the desired isomorphism to a bound on exponential sums, where \( b \) are summed over a stratum of the stratification. The technique we will use to obtain cancellation is a form of separation of variables, where we essentially obtain cancellation in the sum over each individual variable \( b_i \).

We now describe a general geometric form of the type of separation of variables that we will use.

- Let \( m \) and \( N \) be natural numbers. Let \( S \) be a finite set.
- Let \( \mathcal{O}_K \) be the ring of integers of a number field, and \( B \) a separated scheme of finite type over \( \mathcal{O}_K[1/n] \).
- Let \( C_i \) for \( i \in S \) be curves over \( B \). Let \( A \) be a smooth geometrically irreducible curve over \( \mathbb{Z}[1/N] \). We will use \( s \) as a variable for points of \( A \) and \( x_i \) for points of \( C_i \).
- We denote \( \mathcal{C} = C_1 \times_B \cdots \times_B C_n \). We view functions on \( C_i \) as functions on \( \mathcal{C} \) by composing with the \( i \)-th projection.
- For \( 1 \leq j \leq m \), let \( f_j = (f_{i,j})_{1 \leq i \leq n} \in \Gamma \) be a tuple of functions on the curves \( C_i \), and let \( g_j \) be a function on \( B \).
- Let \( Y \subseteq \mathcal{C} \) be the common zero locus of the \( m \) functions
  \[
  \Sigma_j := g_j + \sum_{i \in S} f_{i,j} \in \Gamma(\mathcal{C}, \mathcal{O}_\mathcal{C}), \quad j = 1, \ldots, m.
  \]

- Let \( \pi : Y \times A \to Y \) be the obvious projection, and \( g_i : Y \times A \to C_i \times A \) the obvious morphisms.

- Let \( \ell \) be a prime number dividing \( N \). For \( i \in S \), and \( q \) some prime ideal of \( \mathcal{O}_K \) coprime to \( N \), we assume given a lisse \( \ell \)-adic sheaf \( \mathcal{F}_i \), pointwise pure of weight 0, on \( C_i \times A_{\mathbb{F}_q} \). We denote by \( (\varrho, x, s) \mapsto t_i(\varrho, x, s; k) \) the trace function of \( \mathcal{F}_i \) over some finite extension \( k/\mathbb{F}_q \).

- For \( s \in A(k) \) and \( \varrho \in B(k) \) we set
  \[
  \mathcal{F}_{i,\varrho,s} := \mathcal{F}_i|_{C_i \times B(\varrho) \times \{s\}}
  \]
  the sheaf on \( C_i \times k \) obtained by restricting to the fiber of \( \varrho \) and “freezing” the \( s \)-variable. We assume that for any \( q \), any \( k/\mathbb{F}_q \) and any point \( s \in A(k) \) the conductor of \( \mathcal{F}_{i,\varrho,s} \) is bounded by some constant \( C \geq 1 \).

  - For \( q \) some prime of \( \mathcal{O}_K \) coprime with \( n \), we are given a lisse \( \ell \)-adic sheaf \( \mathcal{G} \), pointwise pure of weight 0, on \( B \times A_{\mathbb{F}_q} \). We denote by \( (\varrho, s) \mapsto t_*(\varrho, s; k) \) its trace function.

We make the following “twist-independence” assumption:

\((\text{TI})\). For all \( i \), for all \( \varrho \in B \) and for all \( s_1 \neq s_2 \) in \( A \), the lisse sheaf \( \mathcal{F}_{i,\varrho,q_1} \otimes \mathcal{F}_{i,\varrho,q_2} \) on each geometrically irreducible component of \( C_{i,\varrho} \) has no geometrically irreducible component that is of rank 1.

The implicit constants associated with the symbols \( O(\cdots) \) or \( \ll \) are assumed to depend on \( \mathcal{C}, A \), the maps \( (f_j)_{j=1,\ldots,m} \), and the conductors of the sheaves involved.

The main estimate on exponential sums we will need is the following

**Proposition 11.2.** Assume that Assumption (TI) holds. We have

\[
(11.2) \quad \sum_{(\varrho, x) \in Y(k)} \sum_{s \in A(k)} |t_*(\varrho, s; k)|^2 \prod_{i=1}^n |t_i(\varrho, x_i, s; k)|^2 = \sum_{(\varrho, x) \in Y(k)} \sum_{s \in A(k)} |t_*(\varrho, s, k)|^2 \prod_{i=1}^n |t_i(\varrho, x_i, s; k)|^2 + O\left(|k|^{|\dim B|+|S|/2+2}\right).
\]

**Remark.** One can often show (by fiberining by curves) that as \( |k| \to \infty \) the first term on the righthand side of \((11.2)\) satisfies

\[
\sum_{(\varrho, x) \in Y(k)} \sum_{s \in A(k)} |t_*(\varrho, s, k)|^2 \prod_{i=1}^n |t_i(\varrho, b_i, s; k)|^2 \gg |k|^{|\dim(Y \times A)|_{\mathbb{F}_q}}
\]

while the error term is

\[
\ll |k|^{(n-m+1)-1/2} \ll |k|^{|\dim(Y \times A)|_{\mathbb{F}_q}-1/2}
\]

as soon as

\[
m \leq \frac{|S| - 3}{2}.
\]

**Example 11.3.** Take \( B \) a point, \( C_i = A = \mathbb{G}_m \), \( \mathcal{F}_i = [(b_i,s) \mapsto b_i s^* \mathcal{K}_\ell] \) on \( \mathbb{G}_m^2 \), and \( \mathcal{G} = \mathcal{O}_\ell \).

Define \( f_{i,j}(b_i) = b_i^j \) and \( Y \) be the subvariety of \( \mathbb{G}_m^2 \) defined by the equations

\[
\sum b_i = \cdots = \sum b_i^m = 0.
\]
One has \( \dim V_{\mathbf{F}_q} = n - m \) for \( q \) large enough. Then (TI) is satisfied and Proposition 11.2 states that

\[
\sum_{b_1,\ldots,b_n \in \mathbf{F}_q^n} \left| \sum_{s \in \mathbf{F}_q^*} \prod_{i=1}^n \mathsf{Kl}_2(b_i s; q) \right|^2 = \sum_{b_1,\ldots,b_n \in \mathbf{F}_q^n} \left| \sum_{s \in \mathbf{F}_q^*} \prod_{i=1}^n \mathsf{Kl}_2(b_i s; q) \right|^2 + O(q^{(n-m+1)/2}),
\]

provided \( m \leq (n-3)/2 \).

**Proof.** We will omit the indication of the finite field, which is always \( k \), in the notation for trace functions. Opening the square, we have

\[
(11.3) \quad \sum_{(q, x) \in Y(k)} \left| \sum_{s \in A(k)} t_s(q, s) \prod_{i=1}^n t_i(q, x, s) \right|^2 = \sum_{(q, x) \in A(k)} \sum_{s \in A(k)} |t_s(q, s)|^2 \prod_{i=1}^n |t_i(q, x, s)|^2 + \sum_{s_1, s_2 \in A(k)} \sum_{(q, x) \in Y(k)} t_s(q, s_1) \overline{t_s(q, s_2)} \prod_{i=1}^n t_i(q, x, s_1) \overline{t_i(q, x, s_2)}.
\]

We detect the condition \((q, x) \in Y(k)\) through additive characters. Thus, let \( \psi \) a non-trivial character of \( k \). For \( x = (x_i)_{i \in S} \in \mathcal{C}(k) \), we have

\[
\delta_{(q,x) \in Y(k)} = \prod_{j=1}^m \frac{1}{|k|^m} \sum_{\lambda_j \in k} \psi(\lambda_j \sum_j (q, x)) = \frac{1}{|k|^m} \sum_{\lambda \in k^m} \psi(g_{\lambda}(q)) + \sum_{j=1}^m \sum_{i \in S} \psi(\lambda_j f_{i,j}(x_i))
\]

where \( \lambda = (\lambda_j)_{j \leq m} \), and

\[
g_{\lambda}(q) = \sum_{j=1}^m \lambda_j g_j(q), \quad f_{i,\lambda}(x_i) = \sum_{j=1}^m \lambda_j f_{i,j}(x_i).
\]

Thus the second sum on the right-hand side of (11.3) is equal to

\[
\frac{1}{|k|^m} \sum_{s_1, s_2 \in A(k)} \sum_{\lambda \in k^m} \sum_{(q, x) \in \mathcal{C}(k)} \psi(\lambda \psi(q)) t_s(q, s_1) \overline{t_s(q, s_2)} \prod_{i \in S} t_i(q, x, s_1) \overline{t_i(q, x, s_2)} \psi(f_{i,\lambda}(x_i))
\]

\[
= \frac{1}{|k|^m} \sum_{s_1, s_2 \in A(k)} \sum_{\lambda \in k^m} \sum_{q \in B(k)} \psi(\lambda \psi(q)) t_s(q, s_1) \overline{t_s(q, s_2)} \times \prod_{i \in S} \left( \sum_{x_i \in C_{i,\psi}(k)} t_i(q, x_i, s_1) \overline{t_i(q, x_i, s_2)} \psi(f_{i,\lambda}(x_i)) \right).
\]

For \( s_1 \neq s_2 \), it follows from the twist-independence assumption and the Riemann Hypothesis (Proposition 5.1 and (5.1)) that for each \( i \in S \), we have

\[
\sum_{x_i \in C_{i,\psi}(k)} t_i(q, x_i, s_1) \overline{t_i(q, x_i, s_2)} \psi(f_{i,\lambda}(x_i)) \ll |k|^{1/2}
\]

and \( t_s(q, s_1) t_s(q, s_2) \ll 1 \) for all \( q \in B(k) \). Hence the sum above is \( \ll |k|^{\dim B + |S|/2 + 2} \), which concludes the proof. \( \square \)
12. Parameterization of strata

The goal of this section is to give a convenient parameterization of the irreducible components of the strata of the stratification \((X_j)\) (Definition 7.6).

Let \(j\) be an integer with \(X_j\) non-empty. Let \(X \subset X_j \subset \mathbb{A}^{2l}\) be an irreducible component of \(X_j\) over \(\mathbb{Z}\) which intersects the characteristic zero part. Let \(\overline{\eta}\) be a geometric generic point of \(X\).

We will show that \(X\) is the projection of a space defined by equations of a certain explicit type; more precisely, these will be exactly of the type that can be handled using Lemma 11.2, allowing us to evaluate the sums that appear in Lemma 11.1. To describe these equations and to perform an inductive process, where we express better and better approximations of \(X\) as the image of such space, we need to package certain data, which we do using the following definitions.

Definition 12.1. A perspective datum \(\Pi\) on \(X\) is a tuple

\[
\Pi = (m, S, B, (C_i), (b_i), (f_{i,j}), (g_j))
\]

where

- \(m \geq 0\) is an integer.
- \(S \subseteq \{1, \ldots, 2l\}\).
- \(B\) is a separated scheme of finite type over \(\mathbb{Q}\).
- \((C_i)_{i \in S}\) is a family of relative curves over \(B\).
- \((b_i)_{i \in S}\) is a family of functions \(b_i : B \to \mathbb{A}^1\) if \(i \notin S\) and \(b_i : C_i \to \mathbb{A}^1\) if \(i \in S\), such that if \(i \in S\), the function \(b_i\) is not constant on any irreducible component of any geometric fiber of \(C_i \to B\).
- \((f_{i,j})_{1 \leq j \leq m}\) is a family of functions \(f_{i,j} : C_i \to \mathbb{A}^1\).
- \((g_j)_{1 \leq j \leq m}\) is a family of functions \(g_j : B \to \mathbb{A}^1\).

To simplify the notation, we will sometimes write \(\Pi \cdot m, \ldots, \Pi \cdot (g_j)\) for the corresponding data.

Let \(\Pi\) be a perspective datum over \(X\). We denote \(\mathcal{C}_\Pi\) the fiber product over \(B\) of the curves \(C_i\) for \(i \in S\), and \(\gamma_\Pi\) the subvariety of \(\mathcal{C}_\Pi\) defined as the zero locus of the functions

\[
g_j + \sum_{i \in S} f_{i,j}
\]

for \(1 \leq j \leq m\), where we extend the functions \(f_{i,j}\) and the functions \(g_j\) by pullback to \(\mathcal{C}_\Pi\).

A perspective over \(X\) is a triple \((\Pi, Y, \gamma)\) where

- \(\Pi\) is a perspective datum on \(X\),
- \(Y\) is an irreducible component of \(\gamma_\Pi\)
- \(\gamma\) is a geometric point of \(Y\),

such that the functions morphism \(g : \gamma_\Pi \to \mathbb{A}^{2l}\) defined by \((b_1, \ldots, b_{2l})\) induces a quasi-finite morphism

\[
Y - g^{-1}(\mathbb{V}^\Delta) \to \mathbb{A}^{2l} - \mathbb{V}^\Delta
\]

which maps \(\gamma\) to \(\overline{\eta}\).

The goal of this section will be to construct a perspective on \(X\) where \(Y\) is irreducible and the image of the map \(Y \to \mathbb{A}^{2l}\) is \(X\). More precisely, the main result is the following:

Theorem 12.2. There exists a perspective \((\Pi, Y, \gamma)\) on \(X\) such that \(Y\) is irreducible, \(\gamma\) is a geometric generic point of \(Y\), and

\[
2l - |\Pi \cdot S| + 2\Pi \cdot m \leq 4(2l - \dim(X)).
\]

The reader is encouraged to first finish reading the proof of the main theorems of this paper, assuming that this statement holds, since this will illustrate how the perspective data is exploited in the final steps.
The basic strategy is the following:

1. We start with a perspective with \( S \) as large as possible, \( m \) as small as possible, but \( \bar{\gamma} \) potentially a quite special point of \( Y \) (Lemma 12.3). We plan to reduce \( \dim Y \) while keeping the growth of \( m \) and the loss of \( |S| \) controlled by a step-by-step induction.
2. At each step, we find some equations that are satisfied at \( \bar{\gamma} \) but not at the generic point of \( Y \) (Lemmas 12.4 and 12.5).
3. We construct a new perspective by adding these new equations (which may require also adjoining some new variables to \( B \) and \( C_i \)), lowering \( \dim Y \) (Lemma 12.6). However, the solution set in \( Y \) of these new equations might not contain any irreducible components of the solution set in \( Y_\Pi \) of the new equations, since they may instead be absorbed into other irreducible components of \( Y_\Pi \). To deal with this, we must assume \( Y_\Pi = Y \).
4. We can ensure that this condition holds by a Diophantine argument, which requires increasing \( |S| \) (Lemma 12.9). This requires certain irreducibility assumptions on \( B \) and on the curves \( C_i \), which we ensure in Lemma 12.10 by a direct construction.
5. Finally, we prove 12.2 by showing that an induction involving all these steps terminates in a suitable perspective.

We begin by exhibiting trivial examples of perspectives that will be used to start the induction process (or to terminate it in a trivial case).

**Lemma 12.3.** (1) The tuple
\[
\Pi_0 = (0, \{1, \ldots, 2l\}, \text{Spec}(\overline{Q}), (\mathbb{A}^1)_{1 \leq i \leq 2l}, (\text{Id}_{\mathbb{A}^1})_{1 \leq i \leq 2l}, \emptyset, \emptyset)
\]
is a perspective datum, and \( (\Pi_0, X, \bar{\eta}) \) is a perspective.

(2) The tuple
\[
\Pi_1 = (0, \emptyset, X, \emptyset, (b_i | X), \emptyset, \emptyset)
\]
is a perspective datum and \( (\Pi_1, X, \bar{\eta}) \) is a perspective.

**Proof.** This is an elementary check. In (1), we have \( \Psi_{\Pi_0} = \mathbb{A}^{2l} \), and the morphism \( X \to \mathbb{A}^{2l} \) is quasi-finite, while in (2) we have \( \mathcal{E}_{\Pi_1} = X \), with the same conclusion. \( \square \)

In the next three lemmas, we begin the proof of the second step by studying how the roots of the polynomial \( P_b \), which are the \( r \)-coordinates of the points in the fiber \( Z_b \), can change under specialization.

Let \( F \) be an algebraically closed field. Let \( r_0, b_1, \ldots, b_{2l} \) be elements of \( F \). Formally, the polynomial \( P_b \in F[r] \) is the product
\[
P_b = \prod_{i=1}^{2l} (r + b_i) \prod_{(\zeta_i) \in \mathbb{F}_k^2} \left( \sum_{i=1}^{2l} \zeta_i (r + b_i)^{1/k} \right).
\]
This expansion makes sense unambiguously in an algebraic closure \( K \) of the complete local field \( F((r-r_0)) \), provided we fix a choice of \( k \)-th roots of \( r + b_i \) in \( K \). In particular, the order of vanishing of \( P_b \) at \( r_0 \) is the sum of the valuation of the factors, where the valuation on \( F((r-r_0)) \) is extended uniquely to \( K \).

For \( 1 \leq i \leq 2l \), fix \( k \)-th roots \( (r_0 + b_i)^{1/k} \) of \( r + b_i \) in \( F \) consistent with the choice of \( (r + b_i)^{1/k} \) in \( K \). Then the multiplicity of the factor
\[
\sum_{i=1}^{2l} \zeta_i (r + b_i)^{1/k}
\]
at $r_0$ is
\[
\begin{cases}
0 & \text{if } \sum_{i=1}^{2l} \zeta_i (r_0 + b_i)^{1/k} \neq 0, \\
1/k & \text{if } \sum_{i=1}^{2l} \zeta_i (r_0 + b_i)^{1/k} = 0 \text{ but } \sum_{1 \leq i < 2l, r_0 + b_i = 0} \zeta_i \neq 0,
\end{cases}
\]
and otherwise it is equal to the multiplicity of the formal power series
\[
\sum_{1 \leq i < 2l, r_0 + b_i \neq 0} \zeta_i (r + b_i)^{1/k} \in F[[r]] \subset K
\]
at $r_0$, when one chooses the branch of $(r + b_i)^{1/k}$ with constant coefficient $(r_0 + b_i)^{1/k}$.

Lemma 12.4. Let $R$ be a local integral domain with algebraically closed residue field $F$, and let $K$ be an algebraic closure of the fraction field of $R$. Let $b_1, \ldots, b_{2l}$ be elements of $R$, and $b \in F^{2l}$ their reductions modulo the maximal ideal. Let $r_0$ be some root of $P_b \in F[r]$. Assume that there exist at least two roots of $P_b$ in $K$ that reduce to $r_0$. For $1 \leq i \leq 2k$, fix a $k$-th root of $r_0 + b_i$ in $F$.

Consider an algebraic closure $\bar{K}$ of $K((u))$. For $\zeta \in \mu_k^{2l}$, let $n(\zeta) \geq 0$ be the multiplicity of
\[
\sum_{i=1}^{2l} \zeta_i (r + b_i)^{1/k}
\]
at $r_0$, as defined above.

There is no solution $(u_0, v_1, \ldots, v_{2l}) \in R^{1+2l}$ of the system of equations
\[
\begin{align*}
(12.1) & \quad v_i^k = u_0 + b_i \\
(12.2) & \quad u + b_i = 0, \text{ for all } i \text{ such that } r_0 + b_i = 0 \\
(12.3) & \quad \sum_{i=1}^{2l} \zeta_i v_i = 0, \text{ for all } \zeta \text{ such that } \sum_{i=1}^{2l} \zeta_i (r_0 + b_i)^{1/k} = 0 \in F \\
(12.4) & \quad \sum_{1 \leq i < 2l, r_0 + b_i \neq 0} \zeta_i v_i^{1-kt} = 0, \text{ if } n(\zeta) \geq 2 \text{ and } 0 \leq t \leq n(\zeta) - 1.
\end{align*}
\]

Proof. Suppose that there exists a solution $u_0 \in R$. We estimate from below the multiplicity of $u_0$ as a root of $P_b$. For each factor of $P_b$, the valuation at $u_0$ is at least the valuation of the corresponding factor of $P_b$ at $r_0$, hence by summing, the order of vanishing of $P_b$ at $u_0$ is at least the order of vanishing of $P_b$ at $r_0$. But this contradicts the assumption that there exist two roots of $P_b$ reducing to $r_0$. \qed

Lemma 12.5. Let $R$ be a local integral domain with algebraically closed residue field $F$ containing a primitive $k$-th root of unity. Let $b_1, \ldots, b_{2l}$ be elements of $R$ and $b$ the reduction of $b$ modulo the maximal ideal. Assume that $\deg(P_b) < \deg(P_b)$.

1. If $b \notin \mathbb{V}^\Delta$, then for any $\zeta = (\zeta_i) \in \mu_k^{2l}$ there exists an integer $n_\zeta \geq 0$ such that
\[
\sum_{i=1}^{2l} \zeta_i b_i^{n_\zeta} = 0 \in F.
\]

2. There exists some $\zeta = (\zeta_i) \in \mu_k^{2l}$ and some integer $\nu$ with $0 \leq \nu \leq n_\zeta - 1$ such that
\[
\sum_{i=1}^{2l} \zeta_i b_i^{\nu} = 0 \in R.
\]
Proof. Writing
\[ \sum_{i=1}^{2l} \zeta_i(r + b_i)^{1/k} = r^{1/k} \sum_{i=1}^{2l} \zeta_i (1 + b_i/r)^{1/k} = r^{1/k} \sum_{t=0}^{\infty} \left( \prod_{j=0}^{t-1} \frac{1/k - j}{1+j} \right) \left( \sum_{i=1}^{2l} \zeta_i b_i^t \right) \frac{1}{r^t} \]
for \( (\zeta_i) \in \mu_k^{2l} \), we first see that if (1) fails, then the left-hand side is identically 0, which implies that \( b \in \mathcal{V}^\Delta \). Then we obtain
\[ \deg(P_b) = 2l + k^{2l-1} - \sum_{(\zeta_i) \in \mu_k^{2l}} m_{\zeta} \]
where \( m_{\zeta} \geq 0 \) is the largest integer such that
\[ \sum_{i=1}^{2l} \zeta_i b_i^t = 0 \]
for \( 0 \leq t \leq m_{\zeta} \). If condition (2) does not hold, we therefore deduce that \( \deg(P_b) \leq \deg(P_b) \), which contradicts the assumption. \( \square \)

The next lemma is one of the key ingredients of the proof of Theorem 12.2.

Lemma 12.6. Let \( \Pi \) be a perspective datum on \( X \) and \( (\Pi, Y, \gamma) \) a perspective. If \( y_\Pi \) is irreducible, so that \( Y = y_\Pi \), and \( \gamma \) is not a geometric generic point of \( y_\Pi \), then there exists a perspective \((\Pi', Y', \gamma') \) with

\[ \Pi' \cdot S = \Pi \cdot S, \quad \dim(\Pi' \cdot B) \leq \dim(\Pi \cdot B) + 1 \quad \dim(Y') < \dim(Y). \]

Proof. Let \( \bar{\alpha} \) be a geometric generic point of \( Y \), and \( \bar{\beta} \) its image in \( A^{2l} \). By definition of a perspective, the fiber of \( Y \to A^{2l} \) over \( \bar{\eta} \) is finite, and since it contains \( \bar{\gamma} \), it cannot contain the point\(^2 \bar{\alpha} \) that specializes to \( \bar{\gamma} \). Hence \( \bar{\beta} \neq \bar{\eta} \), and since \( \bar{\alpha} \) specializes to \( \bar{\gamma} \), it follows that \( \bar{\beta} \) specializes to \( \bar{\eta} \). In particular, we deduce that \( \bar{\beta} \notin \mathcal{V}^\Delta \).

By definition, \( \bar{\gamma} \) is a geometric generic point of \( X \subset X_j \). If \( \bar{\beta} \) was a point of \( X_j \), it would follow that they are equal, which is not the case. Hence the fiber of \( f: Z \to A^{2l} - \mathcal{V}^\Delta \) over \( \bar{\beta} \) has \( j+1 \) points, whereas the fiber over \( \bar{\eta} \) has \( j \) points.

Consider now the local ring \( R \) of the closure of \( \bar{\beta} \) at the point \( \bar{\eta} \). It has algebraically closed residue field. The polynomial \( P_\beta \in R[r] \) has \( j+1 \) roots, and the specialization \( P_\eta \) has \( j \) roots. So either there exist two roots of \( P_\eta \) that have the same image in the residue field, or \( \deg(P_\beta) > \deg(P_\eta) \).

Case 1 (two roots coincide).

Let \( r_0 \) be the common reduction of at least two roots of \( P_\beta \). We will apply Lemma 12.4 to \( R \) and to this \( r_0 \). We define the multiplicity \( n(\zeta) \) for \( \zeta \in \mu_k^{2l} \) as in that lemma.

We consider the covering \( \tilde{B} \to B \times A^1 \), with coordinate \( u \) on \( A^1 \), obtained by adjoining \( k \)-th roots \( v_i \) of \( u + b_i \) for all \( i \notin S \). We then define \( B' \) as the complement in \( \tilde{B} \) of the zero locus of \( u + b_i \) for all \( i \notin S \) such that \( r_0 + b_i \neq 0 \). For \( i \notin S \), the functions \( b_i \) define functions \( B' \to A^1 \) by composing with the projection \( B' \to B \).

For \( i \in S \), we consider the curve \( \tilde{C}_i \to \tilde{B} \) obtained from the base change of \( C_i \times A^1 \to B \times A^1 \) to \( B' \) by adjoining a \( k \)-th root \( v_i \) of \( u + b_i \), so we have a diagram

\[
\begin{array}{ccc}
C_i & \to & C_i \times_B \tilde{B} \\
\downarrow & & \downarrow \\
B & \leftarrow & \tilde{B}
\end{array}
\]

\(^2\)To be precise, the image of this geometric point. We will usually not make the distinction between \( \bar{\gamma} \) and its image, when no confusion can arise.
If \( r_0 + b_i \neq 0 \), we define \( C'_i \) as the complement in \( C_i \) of the zero locus of \( u + b_i \), and otherwise we define \( C'_i = C_i \). In all cases, the morphism \( C'_i \to C_i \) allows us to define a function \( b_i : C'_i \to \mathbb{A}^1 \).

The fibers of this function over a geometric point of \( B' \) project to geometric fibers of \( C_i \to B \), hence irreducible components project to irreducible components, and so hence \( b_i \) is not constant on any irreducible component of any geometric fiber, since \( \Pi \) is a perspective datum.

We next define the scheme \( C' \to B' \) as the fiber product for \( i \in S \) of the curves \( C'_i \) over \( B' \).

There exists a lift \( \tilde{\gamma}' \) of \( \tilde{\gamma} \) in \( C' \) such that \( u(\tilde{\gamma}') = r_0 \) (indeed, we can lift \( \tilde{\gamma} \) to the fiber product of the \( \tilde{C}_i \) over \( \tilde{B} \), and the resulting point lies in \( C' \) since \( r_0 + b_i = 0 \) if \( u + b_i = 0 \)). We fix such a lift. This choice defines canonical \( k \)-th roots of \( u(\tilde{\gamma}') + b_i(\tilde{\gamma}') = r_0 + b_i \), and we will use these later.

The functions \( g_j, 1 \leq j \leq m \) and \( f_{i,j} \) of the perspective datum \( \Pi \) extend to \( B' \) and \( C'_i \), respectively, by composing with the projections \( B' \to B \) and \( C'_i \to C_i \). We will now add additional functions (corresponding to a change of the value of the parameter \( m \)).

Precisely, let \( m' = m + m_1 + m_2 + m_3 \), where \( m_1 \) (resp. \( m_2, m_3 \)) is the number of equations (12.2) in Lemma 12.4 (resp. number of equations (12.3) or (12.4)). We define the additional functions \( g_j \) and \( f_{i,j} \) for \( m + 1 \leq j \leq m' \), making a one-to-one correspondence between the values of \( j \) and the equations of those three types.

If \( j \) corresponds to an equation (12.2), i.e., to an integer \( i \) with \( 1 \leq i \leq 2I \) such that \( r_0 + b_i = 0 \), then we define

\[
\begin{align*}
  f_{i,j} &= u + b_i & \text{if } i' \in S & \text{if } i' = i \\
  f_{i,j} &= 0 & \text{if } i' \in S & \text{if } i' \neq i \\
  g_j &= 0,
\end{align*}
\]

if \( i \in S \), and otherwise we define

\[
\begin{align*}
  f_{i,j} &= 0 & \text{if } i' \in S \\
  g_j &= u + b_i.
\end{align*}
\]

If \( j \) corresponds to an equation (12.3), i.e., to some \( \zeta \in \mu^{2l}_k \) such that

\[
\sum_{i=1}^{2l} \zeta (r_0 + b_i)^{1/k} = 0
\]

we define

\[
\begin{align*}
  f_{i,j} &= \zeta_i v_i & \text{for } i \in S \\
  g_j &= \sum_{i \notin S} \zeta_i v_i.
\end{align*}
\]

Finally, if \( j \) corresponds to an equation (12.4), i.e., to \( \zeta \in \mu^{2l}_k \) and \( t \) such that \( n(\zeta) \geq 2 \) and \( 0 \leq t \leq n(\zeta) - 1 \), then we define

\[
\begin{align*}
  f_{i,j} &= \zeta_i v_1^{1-kt} & \text{if } i \in S & \text{and } r_0 + b_i \neq 0 \\
  g_j &= \sum_{i \notin S} \zeta_i v_1^{1-kt}.
\end{align*}
\]

(note that by the definition of \( C'_i \), the function \( v_i \) is non-vanishing). We now have defined the perspective datum

\[
\Pi' = (m', S, B', (C'_i)_{i \in S}, (b_i), (f_{i,j})_{i \in S, 1 \leq j \leq m'}, (g_j)_{1 \leq j \leq m'}).
\]

The associated variety, i.e., the vanishing locus \( Y' \) of

\[
\frac{g_j + \sum_{i \in S} f_{i,j}}{42}
\]
for $1 \leq j \leq m'$, contains $\bar{\gamma}'$ by construction (see Lemma 12.4 again). Let $Y'$ be an irreducible component of $Y'$ containing $\bar{\gamma}'$. We claim that $(\Pi', Y', \bar{\gamma}')$ is the required perspective.

First, for $y \in Y$, the points of the fiber of $Y' \to Y$ over $y$ are determined by the value of the function $u$ on $Y'$, whose values lie in the set of roots of the polynomial $P_{\Pi(Y)}$. In particular, the fiber is finite, and hence $Y'$ is quasi-finite over $Y$. It follows on the one hand that $Y'$ has dimension $\leq \dim(Y)$, and on the other hand that $Y'$ is quasi-finite over $A^{2l} \setminus \Gamma$. So $(\Pi', Y', \bar{\gamma}')$ is a perspective.

We have $\dim(B') \leq \dim(B) + 1$. It remains therefore to check that $\dim(Y') < \dim(Y)$. We have already observed that $\dim(Y') \leq \dim(Y)$. Suppose the dimensions were equal. Then, since $Y' \to Y$ is quasi-finite, the geometric generic point $\bar{\gamma}'$ would map to $\bar{\alpha}$ in $Y$, and therefore to $\bar{\beta}$ in $A^{2l}$. By applying finally Lemma 12.4, we obtain a contradiction: since two roots of $P_{\beta}$ reduce to the same root of $P_{\Pi}$, there cannot be solutions in $R$ of the system of equations (12.2), (12.3), (12.4), whereas this is exactly what we obtain from the fact that $\bar{\beta}$ is the image of $\bar{\gamma}'$.

**Case 2** (the degree drops).

We now consider instead Lemma 12.5, and define integers $n_\zeta$ for $\zeta \in \mu^2_k$ as the least integer $\geq 0$ such that
\[
\sum_{i=1}^{2l} \zeta_i b_i^{n_\zeta} = 0
\]
at $\eta$ (this exists by statement (1) in the lemma). We define $m' = m + m_1$, where $m_1$ is the number of pairs $(\zeta, \nu)$ with $\zeta \in \mu^2_k$ and $0 \leq \nu \leq n_\zeta$. For $m + 1 \leq j \leq m'$, corresponding in one-to-one fashion to $(\zeta, \nu)$, we define
\[
\begin{align*}
f_{i,j} &= \zeta_i b_i^{n_\zeta} \quad \text{for } i \in S \\
g_j &= \sum_{i \notin S} \zeta_i b_i^{n_\zeta}.
\end{align*}
\]
Then $\Pi' = (m', S, B, (C_i), (b_i), (f_{i,j})_{1 \leq i \leq m', 1 \leq j \leq m'})$ is a perspective datum (since the $b_i$ have not changed, the non-constancy condition is also unchanged). The point $\bar{\gamma}$ belongs to the associated variety $Y' \subset Y'_{\Pi} \subset C_{\Pi}$ (by definition of $n_\zeta$), so $(\Pi', Y', \bar{\gamma})$ is a perspective, where $Y'$ is the irreducible component of $Y'$ containing $\bar{\gamma}$. By Lemma 12.5, on the other hand, $\bar{\alpha}$ does not lie in $Y'$, so all its irreducible components, including $Y'$, have dimension $< \dim(Y_{\Pi}) = \dim(Y)$.

In the next lemma, we produce from a a perspective another one with a specific value of the parameter $m$.

**Lemma 12.7.** Let $(\Pi, Y, \bar{\gamma})$ be a perspective on $X$. There exists a perspective $(\Pi', Y', \bar{\gamma}')$ such that
\[
\begin{align*}
\Pi' \cdot S &= \Pi \cdot S, \\
\Pi' \cdot B &= \Pi \cdot B, \\
\Pi' \cdot (C_i) &= \Pi \cdot (C_i), \\
\Pi' \cdot (b_i) &= \Pi \cdot (b_i),
\end{align*}
\]
$\Pi' \cdot m = \dim(\Pi \cdot B) + |\Pi \cdot S| - \dim(Y)$

$Y'$ is isomorphic to $Y$, $Y_{\Pi} \subset Y'_{\Pi}$ as $B$-schemes.

**Proof.** Let $m' = \dim(\Pi \cdot B) + |\Pi \cdot S| - \dim(Y)$. It is the codimension of $Y$ in $C_{\Pi}$. Let $X$ be the subspace of $\Gamma(C_{\Pi}, 0)$ generated by the functions
\[
h_j = g_j + \sum_{1 \leq i \leq m} f_{i,j}
\]
for $1 \leq j \leq m$. We claim that for any integer $\nu$ with $0 \leq \nu \leq m'$, there exist $(\varphi_1, \ldots, \varphi_\nu)$ in $X$ such that all irreducible components of the zero locus $V(\varphi_1, \ldots, \varphi_\nu)$ in $C_{\Pi}$ that contain $Y$ have codimension $\nu$ in $C_{\Pi}$.

We prove this by induction on $\nu$. The statement is true for $\nu = 0$. Assume that $\nu \leq m'$ and that the property holds for $\nu - 1$ and the functions $(\varphi_1, \ldots, \varphi_{\nu-1})$. Let $W$ be an irreducible component of the zero locus $V(\varphi_1, \ldots, \varphi_{\nu-1})$. It has codimension $\nu - 1 < m' = \codim(Y)$ in $C_{\Pi}$ so $Y$ is a
proper closed irreducible subset of $W$. Hence there exists $j$ such that $h_j$ does not vanish identically on $W$, and in particular the set of $\varphi \in X$ such that $\varphi$ does not vanish on $W$ is a non-empty Zariski-open subset of $X$. Taking intersection of these open sets, there exists $\varphi_\nu \in X$ such that $\varphi_\nu$ is non-vanishing on all irreducible components $W$ containing $Y$. It follows that $(\varphi_1, \ldots, \varphi_\nu)$ satisfy the induction assumption.

For $\nu = m'$, this means that all irreducible components of $V(\varphi_1, \ldots, \varphi_{m'})$ containing $Y$ have codimension $m' = \text{codim}(Y)$ in $C_\Pi$. Hence $Y$ is one of the irreducible components of $V(\varphi_1, \ldots, \varphi_{m'})$.

For $1 \leq \nu \leq m'$, write

$$\varphi_\nu = \sum_{1 \leq j \leq m} \alpha_{\nu,j} h_j.$$  

We define

$$g'_\nu = \sum_{1 \leq j \leq m} \alpha_{\nu,j} g_j, \quad f'_{i,\nu} = \sum_{1 \leq j \leq m} \alpha_{\nu,j} f_{i,j},$$

for $i \in S$ and $1 \leq \nu \leq m'$ so that

$$g'_\nu + \sum_{i \in S} f'_{i,\nu} = \varphi_\nu.$$  

Then

$$\Pi' = (m', S, B, (C_i)_{i \in S}, (b_i), (f'_{i,j})_{1 \leq j \leq m'}, (g'_j)_{1 \leq j \leq m'})$$

is a perspective datum on $X$; by construction $Y$ is an irreducible component of $\mathcal{Y}_\Pi'$ and $\mathcal{Y}_\Pi \subset \mathcal{Y}_\Pi'$ as $B$-schemes, so $(\Pi', Y, \tilde{\gamma})$ is a perspective with the desired properties.

In the next lemma, we have a single perspective, so we don’t use the selector notation.

**Lemma 12.8.** Let $(\Pi, Y, \tilde{\gamma})$ be a perspective on $X$. For any $T \subset S$ and $b \in B$, we put

$$\tilde{\Gamma}_{T,b} = \prod_{i \in T} \Gamma(C_i, \mathcal{O}_{C_i,b}), \quad \Gamma_{T,b} = \prod_{i \in T} (\Gamma(C_i, \mathcal{O}_{C_i,b})/\kappa_b),$$

where the $\kappa_b$ is the residue field at $b$. The spaces $\tilde{\Gamma}_{T,b}$ and $\Gamma_{T,b}$ are $\kappa_b$-vector spaces. For $1 \leq j \leq m$, we denote $f_{T,j,b} = (f_{i,j})_{i \in T} \in \tilde{\Gamma}_{T,b}$.

Assume that $S$ is not empty, that $B$ is irreducible, and that the generic fiber of $C_i \to B$ is geometrically irreducible for all $i \in S$.

One of the following properties holds:

(a) The scheme $\mathcal{Y}_\Pi$ has a unique geometrically irreducible component whose projection to $B$ is dominant.

(b) There exists a proper subset $T \subset S$ such that the images of $(f_{T,1,b}, \ldots, f_{T,m,b})$ span a subspace of $\Gamma_{T,\eta}$ of dimension $\leq m - (|S| - |T|)/2$, for $\eta$ the generic point of $B$.

**Proof.** There exists a number field and an open dense subset $\emptyset$ of its ring of integers in a number field such that the perspective datum is defined over $\emptyset$. We fix one model of $\Pi$ over $\emptyset$, and we will use the same notation for its components as for the original objects over $Q$. We assume that property (b) does not hold and we will show that (a) holds. We will do this by studying fibers of $\mathcal{Y}_\Pi \to B$ over finite-valued field points of a suitable dense open subset of $B$, using the point-counting criterion for irreducibility over finite fields.

For $b \in B$, the condition that the all curves $C_{i,b}$ are geometrically irreducible is a constructible condition. So is the condition $(f_{T,1,b}, \ldots, f_{T,m,b})$ generate a subspace of $\Gamma_{T,b}$ of dimension $> m - (|S| - |T|)/2$ for all proper subsets $T$ of $S$.

By assumption, including the negation of (b), these properties both hold at the generic point, hence we can find a dense open subset $B^o$ where both properties hold.
Let $\text{Spec}(\kappa) \to \text{Spec}(\emptyset)$ be a finite-field valued point of $\text{Spec}(\emptyset)$. Fix $b \in B^\circ(\kappa)$. Let $\psi$ be a fixed non-trivial additive character of $\kappa$. We denote $V = \mathcal{I}_{\Pi, b, \kappa}$. We compute $|V(\kappa)|$ using additive characters (as in the proof of Proposition 11.2). For $\lambda \in \kappa^m$ and $x \in \mathcal{C}_\Pi(\kappa)$, we denote

$$f_\lambda(x) = \sum_{j=1}^m \lambda_j \sum_{i \in S} f_{i,j}(x).$$

and

$$\xi(\lambda) = \psi\left(\sum_{j=1}^m \lambda_j g_j(b)\right).$$

We have

$$|V(\kappa)| = \frac{1}{|\kappa|^m} \sum_{x \in \mathcal{C}_\Pi(\kappa)} \prod_{\lambda \in \kappa} \psi\left(\lambda \left(\sum_{i \in S} f_{i,j}(x)\right)\right)$$

$$= \frac{1}{|\kappa|^m} \sum_{x \in \mathcal{C}_\Pi(\kappa)} \prod_{\lambda \in \kappa} \psi(\lambda g_j(b)) \psi\left(\lambda \sum_{i \in S} f_{i,j}(x)\right)$$

$$= \frac{1}{|\kappa|^m} \sum_{\lambda \in \kappa^m} \xi(\lambda) E(b; \lambda),$$

where

$$E(b; \lambda) = \sum_{x \in \mathcal{C}_\Pi(\kappa)} \psi(f_\lambda(x)).$$

By definition of $\mathcal{C}_\Pi$ as a fiber product, we have the separation of variable formula

$$E(b; \lambda) = \prod_{i \in S} \sum_{x \in \mathcal{C}_{i,b}(\kappa)} \psi\left(\sum_{j=1}^m \lambda_j f_{i,j}(x)\right).$$

Let

$$S_\lambda = \left\{i \in S \mid \sum_{j=1}^m \lambda_j f_{i,j} \text{ is constant on } C_{i,b}\right\} \subset S.$$

Applying the Weil bound for the exponential sums over $C_{i,b}(\kappa)$ (assuming the characteristic is larger than the degree of the functions $f_{i,j}$), it follows that

$$E(b; \lambda) \ll |\kappa|^{\left|S_\lambda\right| + (|S| - |S_\lambda|)/2} = |\kappa|^{(|S| + |S_\lambda|)/2}.$$

We now split the expression for $|V(\kappa)|$ above according to the value of $S_\lambda$, and isolate the term corresponding to $S_\lambda = S$ from the others. This gives $|V(\kappa)| = N_1 + N_2$, where

$$N_1 = \frac{1}{|\kappa|^m} \sum_{\lambda \in \kappa^m} \xi(\lambda) E(b; \lambda), \quad N_2 = \frac{1}{|\kappa|^m} \sum_{\lambda \in \kappa^m} \xi(\lambda) E(b; \lambda).$$

Taking $T = S - \{i\}$ for a fixed $i \in S$ in the defining property of $B^\circ$, we observe that the tuple $(f_{T,1,b}, \ldots, f_{T,m,b})$ generates a subspace of $\Gamma_{T,b}$ of dimension $> m - (|S| - |T|)/2 > m - 1/2$, hence are linearly independent in $\Gamma_{T,b}$, and thus are linearly independent in $\Gamma_{S,b}$. The condition $S_\lambda = S$ arises then only when $\lambda = 0$. Hence

$$N_1 = \frac{1}{|\kappa|^m} \prod_{i \in S} |C_{i,b}(\kappa)|.$$
Since $C_{i,b}$ is a geometrically irreducible curve (by the choice of $B^\circ$), we have $|C_{i,b}(\kappa)| = |\kappa| + O(|\kappa|^{1/2})$ for all $i$. Hence

$$N_1 = |\kappa|^{|S|-m}(1 + O(|\kappa|^{-1/2}))^{|S|} + O(|\kappa|^{-m+|S|-1/2}) = |\kappa|^{|S|-m} + O(|\kappa|^{|S|-m-1/2}).$$

On the other hand, we have

$$N_2 \ll \frac{1}{|\kappa|^{m}} \sum_{T \subset S, T \neq S} |\kappa|^{n(T)}|\kappa|^{(|S|+|T|)/2}$$

where $n(T)$ is the dimension of the $\kappa$-vector subspace of $\kappa^m$ whose elements are all $\lambda$ such that $S_{\lambda} \subset T$. We have $n(T) = \ker(\varphi_T)$, where $\varphi_T : \kappa^m \to \Gamma_{T,b,\kappa}/\kappa$ is the linear map

$$\lambda \mapsto \sum_{j=1}^{m} \lambda_j f_{T,j} \pmod{\kappa}.$$ 

Since $T$ is a proper subset of $S$, by the definition of $B^\circ$, we must have $\dim \operatorname{Im}(\varphi_T) > m - \frac{|S|-|T|}{2}$, so that $n(T) < (|S| - |T|)/2$, which implies $n(T) \leq (|S| - |T|)/2 - 1/2$, so we derive

$$N_2 \ll |\kappa|^{-m+(|S|-|T|)/2+(|S|+|T|)/2-1/2} = |\kappa|^{|S|-m-1/2}.$$

We conclude that

$$|V(\kappa)| = |\kappa|^{|S|-m} + O(|\kappa|^{|S|-m-1/2}).$$

Applying this to finite extensions of $\kappa$ and applying the Lang-Weil estimates, we conclude that $V$ is geometrically irreducible.

Recalling that $V$ was the fiber of $Y_{\Pi}$ over an arbitrary point $b \in B^\circ(\kappa)$, we see that all the fibers of $Y_{\Pi}$ over finite-field valued points of $B^\circ$ with sufficiently large characteristic are geometrically irreducible, so all the fibers of $Y_{\Pi}$ over points of $B^\circ$ are geometrically irreducible. Therefore $Y_{\Pi}$ has a unique geometrically irreducible component that is dominant over $B$, concluding the proof that condition (a) holds.

**Lemma 12.9.** Let $(\Pi,Y,\gamma)$ be a perspective on $X$ defined over an open subscheme $\operatorname{Spec}(\mathcal{O})$ of the ring of integers in a number field. Assume that $S$ is not empty, that $B$ is geometrically irreducible, that each $C_i$ is irreducible and that the generic fiber of $C_i \to B$ is geometrically irreducible for all $i \in S$.

If $Y_{\Pi}$ is reducible and all irreducible components of $Y_{\Pi}$ are dominant over $B$, then there exists a perspective $(\Pi',Y',\gamma')$ on $X$ such that $\dim Y' = \dim Y$ and

$$1 \leq |\Pi \cdot S| - |\Pi' \cdot S| \leq 2(|\Pi \cdot m - \Pi' \cdot m|).$$

**Proof.** We apply Lemma 12.8 to $(\Pi,Y,\gamma)$, and use the same notation. Since $Y_{\Pi}$ is reducible and all its irreducible components are dominant over $B$, there are at least two irreducible components that are dominant over $B$. By Lemma 12.8, we conclude that there exists a proper subset $T \subset \Pi \cdot S$ such that the span of $(f_{T,1,\eta}, \ldots, f_{T,m,\eta})$ in $\Gamma_T$ has dimension $\leq m - (|S| - |T|)/2$.

For $\lambda \in \ker(\varphi_T)$ and $i \in T$, $\sum_{j=1}^{m} \lambda_j f_{i,j}$ is equal to an element of $\kappa_{\eta}$ and hence a rational function on $B$. Let $B^\ast$ be an open subset of $B$ on which all these functions are defined. Because $C_i$ is irreducible, $\sum_{j=1}^{m} \lambda_j f_{i,j}$ is equal to this function on $B^\ast$ not just at the generic point, but everywhere.

Let $m'$ be the dimension of the span $X$ of $(f_{T,1,\eta}, \ldots, f_{T,m,\eta})$ in $\Gamma_T$. We have then

$$1 \leq |\Pi \cdot S| - |T| \leq 2(|\Pi \cdot m - m'|).$$
Let $\tilde{\mathcal{C}}$ be the fibre product of $C_i$ for $i \in S - T$ with $B^*$ over $B$. We have an evaluation map

$$\varphi_T : A^m \to \Gamma_T$$

sending $(\lambda_i)_{i \in T}$ to

$$\sum_{j=1}^{m} \lambda_j f_{T,j}.$$ We define $B' \subset \tilde{\mathcal{C}}$ to be the common zero locus of the functions

$$\sum_{j=1}^{m} \lambda_j \left( g_j + \sum_{i \in S} f_{i,j} \right)$$

for all $\lambda$ in $\ker(\varphi_T)$. These expressions are indeed well-defined functions on $\tilde{\mathcal{C}}$ because, as we saw earlier

$$\sum_{j=1}^{m} \lambda_j f_{i,j}$$

is equal to a function on $B^*$ for $i \in T$ if $\lambda \in \ker(\varphi_T)$.

Furthermore, we choose $f'_{i,j}$ in $X$ for $i \in T$ and $1 \leq j \leq m'$ so that $f'_{i,j} = \sum_{\nu=1}^{m} \beta_{j,\nu} f_{i,\nu}$ for $(\beta_{j,\nu})_{1 \leq j \leq m'}$ a set of elements of $A^m$ that span its image $X$ under $\varphi_T$. Define

$$g'_j = \sum_{\nu=1}^{m} \beta_{\nu,j} \left( g_\nu + \sum_{i \in S - T} f_{i,\nu} \right).$$

Then the tuple

$$\Pi' = (m', T, B', (C_i \times_B B')_{i \in T}, (b'_i)_{i \in T}, (f'_{i,j}), (g'_j))$$

is a perspective datum on $X$, where $b'_i$ is the extension of $b_i$ to $C'_i = C_i \times_B B'$ by pullback for $i \in T$, the projection $B' \to A^1$ if $i \in S - T$, and the projection $B' \to B \to A^1$ otherwise.

By construction, the fiber product $\mathcal{C}_{\Pi'}$ is a locally closed subset of contained in $\mathcal{C}_{\Pi}$. The subscheme $\mathcal{Y}_{\Pi'}$ is an open subset of $\mathcal{Y}_{\Pi}$, because it has the same set of defining equations after restricting to an open subset $B^*$ of $B$. Because the irreducible component $Y$ was dominant over $B$, its restriction to this open subset has the same dimension, and because $\bar{\gamma}$ was dominant over $B$, it remains in this open subset as well. Hence $(\Pi', Y', \bar{\gamma})$ is the desired perspective on $X$.

The last preparatory lemma constructs a perspective where the base $B$ satisfies the assumptions of the last lemma.

**Lemma 12.10.** Let $(\Pi, Y, \bar{\gamma})$ be a perspective on $X$. Then there exists a perspective $(\Pi', Y', \bar{\gamma}')$ such that

$$\Pi' \cdot m = \Pi \cdot m, \quad \Pi' \cdot S = \Pi \cdot S,$$

and $\dim(Y') = \dim(Y)$, and moreover

(a) $\Pi' \cdot B$ is irreducible.

(b) For all $i \in S$, the curve $\Pi' \cdot C_i$ are irreducible and the fiber of $\Pi' \cdot C_i$ over the geometric generic point of $\Pi' \cdot B$ is irreducible.

(c) All irreducible components of $\mathcal{Y}_{\Pi'}$ are dominant over $B$, as is $\bar{\gamma}$.

The strategy of the proof is to make several modifications to the given perspective datum to ensure that these three conditions hold. We will first replace $B$ by an irreducible scheme, ensuring condition (a). We then pass to a finite cover of $B$ over which generic geometrically irreducible components of $C_i$ are defined and choose one for each $i$, ensuring condition (b). Finally we remove a closed subset from $B$, containing all the irreducible components that are not dominant over $B$, ensuring condition (c).
Proof. Let $A \subset C_\Pi$ be an irreducible component containing $Y$. Let $B_0$ be the schematic closure of the image of $\gamma$ under the projection $Y \to B$. It is closed and irreducible. Let $\beta$ be its generic point. Let $\beta' \to \beta$ be a finite extension such that all irreducible components of the generic fibers of the curves $C_i$ for $i \in S$ are defined over $\beta'$. Let then $B' \to B_0$ be a finite flat morphism whose generic fiber is $\beta' \to \beta$ (we can construct such a morphism by taking a generator of the field extension $\beta'/\beta$, and multiplying it by a regular function on $B_0$ so that its minimal polynomial $P$ becomes monic; then the cover $B'$ of $B_0$ obtained by adjoining a root of $P$ has the required property).

Fix a lift $\hat{\gamma}$ of $\gamma$ to $Y \times_B B'$. Let $Y'$ be an irreducible component of $Y \times_B B'$ containing $\hat{\gamma}$, and let $\mathcal{A}'$ be an irreducible component of $C_\Pi \times_B B'$ containing $Y'$. Because $\hat{\gamma}$ maps to the generic point $\beta$ of $B_0$, $\hat{\gamma}'$ must map to the generic point $\beta'$ of $B'$ (the only point lying in the fiber), and so $Y' \to B'$ and $\mathcal{A}' \to B'$ are dominant maps. Because $\mathcal{A}'$ is an irreducible component of $C_\Pi \times_B B'$, and maps dominantly to $B'$, it follows that $\mathcal{A}'_{\beta'}$ is an irreducible component of the pullback $\mathcal{C}_{\beta'}$ of the product of the curves $C_i$ to $\beta'$. Hence there are irreducible components $\mathcal{C}_{i,\beta'}$ of $C_{i,\beta'}$ for $i \in S$, such that $\mathcal{A}'_{\beta'}$ is contained in the product of the $\mathcal{C}_{i,\beta'}$. Let $C_i'$ be the closure of $\mathcal{C}_{i,\beta'}$. This is an irreducible curve over $B'$. We can pullback the functions $b_i$, $g_j$ and $f_{i,j}$ to $B'$ and $C_i'$, respectively. We have then constructed a perspective datum

$$\Pi' = (m,S,B', (C_i'), (b_i'), (f_{i,j}'), (g_j')).$$

The irreducible component $Y'$ is contained in $\mathcal{C}_\Pi'$, hence in $Y_{\Pi'}$. Since the morphism $Y_{\Pi'} \to Y_\Pi$ is finite, it is an irreducible component of $Y_\Pi$. It contains $\hat{\gamma}'$ and so maps dominantly onto $B'$.

Let $B''$ be the complement in $B'$ of the closure of the images of all irreducible components of $Y_{\Pi'}$ that are not dominant over $B'$. We can pullback the data $C_i', b_i', g_j', f_{i,j}, Y''$ further to $B''$. This defines a perspective datum

$$\Pi'' = (m,S,B'', (C_i''), (b_i''), (f_{i,j}''), (g_j''))$$

and a perspective $(\Pi'', Y'', \hat{\gamma}')$.

By construction, $B'$ and $B''$ are geometrically irreducible. Since the curves $C_i''$ are generically irreducible, and their geometric generic fibers are defined over $\beta''$, they are generically geometrically irreducible. Because $\hat{\gamma}'$ maps dominantly to $B$, $\hat{\gamma}'$ maps dominantly to $B'$ and $B''$. Finally, all irreducible components of $Y_{\Pi''}$ map dominantly to $B''$ by construction. □

We can now conclude this section.

Proof of Theorem 12.2. Consider the set $\mathcal{P}$ of perspectives $(\Pi, Y, \hat{\gamma})$ on $X$ such that

$$(12.5) \quad 2 \dim(\Pi \cdot B) + 2 \dim(Y) + |\Pi \cdot S| \leqslant 6l.$$

This set is nonempty by Lemma 12.3 (1), hence it contains some element where

$$\dim(Y) + |\Pi \cdot S|$$

is minimal.

Using Lemma 12.10, we obtain a perspective $(\Pi, Y, \hat{\gamma}) \in \mathcal{P}$ such that $\Pi \cdot B$ is geometrically irreducible, the curves $\Pi \cdot C_i$ are irreducible, the geometric generic fibers of $\Pi \cdot C_i$ are irreducible, and all irreducible components of $Y_{\Pi}$ as well as $\hat{\gamma}$ are dominant over $B$. By Lemma 12.7, we may assume that

$$\Pi \cdot m = \dim(\Pi \cdot B) + |\Pi \cdot S| - \dim(Y)$$

(note that the last condition in Lemma 12.7 implies that all irreducible components of $Y_{\Pi'}$ as well as $\hat{\gamma}'$ are dominant over $B$ for the new perspective given by that lemma with input $(\Pi, Y, \hat{\gamma})$.)

We will then see that, except in a trivial case, a perspective with these properties satisfies the desired conclusion that $\Pi \cdot Y$ is irreducible, $\hat{\gamma}$ is the generic point of $Y$, and

$$2l - |\Pi \cdot S| + 2\Pi \cdot m \leqslant 4(2l - \dim(X)).$$
First, if $Y$ is irreducible and $\bar{\gamma}$ is the generic point of $Y$, then because $Y$ is quasi-finite over $\mathbb{A}^{2l}$, we have $\dim(Y) = \dim(X)$, hence
\[
2l - |\Pi \cdot S| + 2|\Pi \cdot m| = 2 \dim(\Pi \cdot B) + 2l + |\Pi \cdot S| - 2 \dim(Y) \leq 8l - 4 \dim(Y) = 4(2l - \dim(X)).
\]

Next assume that $Y$ is irreducible and $\bar{\gamma}$ is not the generic point of $Y$. Then Lemma 12.6 provides a perspective $(\Pi', Y', \bar{\gamma}')$ with
\[
|\Pi' \cdot S| = |\Pi \cdot S|, \quad |\Pi' \cdot B| \leq |\Pi \cdot B| + 1, \quad \dim(Y') < \dim(Y)
\]
so
\[
2 \dim(\Pi' \cdot B) + \dim(Y') + |\Pi' \cdot S| \leq 6l
\]
but satisfying
\[
\dim(Y') + |\Pi' \cdot S| < \dim(Y) + |\Pi \cdot S|,
\]
which contradicts the minimality of $\Pi$.

Suppose now that $Y$ is reducible and $\Pi \cdot S$ is nonempty. Then Lemma 12.9 provides a perspective $(\Pi', Y', \bar{\gamma}')$ which satisfies $|\Pi' \cdot S| < |\Pi \cdot S|$, and moreover
\[
\dim(Y') = \dim(Y') \geq \dim(\Pi' \cdot B) + |\Pi' \cdot S| - \Pi' \cdot m
\]
\[
\geq \dim(\Pi' \cdot B) - \Pi \cdot m + \frac{1}{2}(|\Pi' \cdot S| + |\Pi \cdot S|)
\]
\[
= \dim(\Pi' \cdot B) - \dim(\Pi \cdot B) + \frac{1}{2}(|\Pi' \cdot S| - |\Pi \cdot S|) + \dim(Y)
\]

hence
\[
2 \dim(\Pi' \cdot B) - 2 \dim(\Pi \cdot B) + \leq |\Pi \cdot S| - |\Pi' \cdot S|,
\]
which because of (12.5) implies
\[
2 \dim(\Pi' \cdot B) + 2 \dim(Y') + |\Pi' \cdot S| \leq 6l.
\]
On the other hand, we have
\[
\dim(Y') + |\Pi' \cdot S| < \dim(Y) + |\Pi \cdot S|,
\]
again contradicting the assumption of minimality.

Finally, the remaining case when $\Pi \cdot S$ is empty is trivial: in that case, $Y$ is a closed subscheme of $\Pi \cdot B$ so that
\[
4 \dim(X) \leq 4 \dim(Y) \leq 2 \dim(\Pi \cdot B) + 2 \dim(Y) \leq 6l
\]
and we may simply take the trivial perspective $(\Pi_1, X, \bar{\eta})$ of Lemma 12.3 (2), for which
\[
2l - |\Pi_1 \cdot S| + 2|\Pi_1 \cdot m| = 2l \leq 4(2l - \dim(X)).
\]

\[\square\]

13. The generic statement

We continue with the previous notation. Fix $j \geq 0$. Let $X \subset X_j \subset \mathbb{A}^{2l} - \mathcal{V}^\Delta$ be an irreducible component of $X_j$ over $\mathbb{Z}$ which intersects the characteristic zero part. Let $\overline{X}$ be the closure of $X$ in $\mathbb{A}^{2l}$.

Fix a perspective $(\Pi, Y, \bar{\gamma})$ on $X$ such that $Y_{\Pi}$ is irreducible, $\overline{\gamma}$ is a geometric generic point of $Y$, and $2l - |S| + 2m \leq 4 \codim_{\mathbb{A}^{2l}}(X)$, which exists by Theorem 12.2. By definition, all of the perspective data is defined over $\mathbb{Q}$. However, by standard finiteness arguments, everything is necessarily defined over a finitely generated subring of $\mathbb{Q}$, i.e. over a ring $\mathcal{O}_K[1/N]$, where $\mathcal{O}_K$ is the ring of integers of a number field $K$ and $N \geq 1$ is some integer. We will use the same notation $Y, C_l, b_l$, etc. to refer to the objects over this ring. Since, by assumption, $Y_{\Pi, \mathbb{Q}}$ is irreducible, and equal to $Y_{\mathbb{Q}}$, we deduce that $Y_{\Pi}$ is geometrically irreducible and equal to $Y$. 49
Because the geometric generic point of $Y$ is a lift of the geometric generic point of $X$, the image of $Y$ in $\mathbb{A}^M$ is a dense subset of $\overline{X}$. For all but finitely many prime ideals $r$ of $\mathcal{O}_K[1/N]$, with residue field denoted $\mathbb{F}_q$, the variety $Y_{\mathbb{F}_q}$ is irreducible and nonempty, $X_{\mathbb{F}_q}$ is irreducible and nonempty, and the map $Y_{\mathbb{F}_q} \to \overline{X}_{\mathbb{F}_q}$ is dominant. In the remainder of this section, we only consider finite fields $\mathbb{F}_q$ arising in this manner, and we also always assume that the characteristic of $\mathbb{F}_q$ is $> 2k + 1$.

Lemma 13.1. Assume that $\chi$ has CGM. If $p$ is large enough with respect to $(k, l, X)$ and $\dim(X_{\mathbb{Q}}) \geq (3l + 1)/2$, then we have

$$\begin{align*}
\sum_{y \in Y(\mathbb{F}_q)} \sum_{r \in \mathbb{F}_q^\times} \left| \sum_{s \in \mathbb{F}_q^\times} \prod_{i=1}^l Kl_k(r(s + b_i(y)); \chi, q) Kl_k(r(s + b_{i+l}(y)); \chi, q) \right|^2 \leq \\
\sum_{y \in Y(\mathbb{F}_q)} \sum_{r \in \mathbb{F}_q^\times} \left| \prod_{i=1}^l Kl_k(r(s + b_i(y)); \chi, q) Kl_k(r(s + b_{i+l}(y)); \chi, q) \right|^2 + O(q^{\dim(X_{\mathbb{Q}}) + 3/2}),
\end{align*}$$

where the implied constant depends only on $(\Pi, k, l)$.

Proof. We first fix $r \in \mathbb{F}_q^\times$. We apply Proposition 11.2 with data $(m, B, S, (C_i))$ coming from the perspective datum $\Pi$, $A = \mathbb{G}_m$, and the sheaf $\mathcal{F}_i = [(b_i, s) \mapsto s(r + b_i)]*\mathcal{K}_{k, \psi}(\chi)$.

Assumption (TI) holds by a Goursat-Kolchin-Ribet argument (see [Kat90] and [FKM15b]). Indeed, each irreducible component of $C_{i, \theta}$ is a geometrically irreducible curve on which $b_i$ is a nonconstant function. The sheaf $\mathcal{F}_{i, \theta, s_1} \otimes \mathcal{F}_{i, \theta, s_2}$ is the pullback along $b_i$ of the sheaf

$$\mathcal{G} = [b_i \mapsto (s_1(r + b_i))]*\mathcal{K}_{k, \psi}(\chi) \otimes [b_i \mapsto (s_1(r + b_i))]*\mathcal{K}_{k, \psi}(\chi)'.$$

The monodromy group after pulling back along the map $b_i$ is a finite index subgroup, so it suffices to show that no finite-index subgroup of the geometric monodromy group of $\mathcal{G}$ admits an one-dimensional irreducible component. However, by Goursat’s lemma, the geometric monodromy group of $\mathcal{G}$ is a product of two copies of the monodromy group of $\mathcal{K}_{k, \psi}(\chi)$, acting by the tensor product of the standard representation with its dual. This group is connected, so has no proper finite-index subgroups, and does not admit a one-dimensional representation, which proves the claim.

The conductor of all the sheaves $\mathcal{F}_{i, \theta, s_1}$ which are pullbacks of (shifted and translated) generalized Kloosterman sheaves are bounded by constants depending only on $\Pi$.

Applying Proposition 11.2 we obtain

$$\sum_{y \in Y(\mathbb{F}_q)} \left| \sum_{s \in \mathbb{F}_q^\times} \prod_{i=1}^l Kl_k(r(s + b_i(y)); \chi, q) Kl_k(r(s + b_{i+l}(y)); \chi, q) \right|^2 = \\
\sum_{y \in Y(\mathbb{F}_q)} \left| \prod_{i=1}^l Kl_k(r(s + b_i(y)); \chi, q) Kl_k(r(s + b_{i+l}(y)); \chi, q) \right|^2 + O(q^{\dim(B_Q + |S|/2 + 2)},$$

where the implied constant depends only on $(\Pi, k, l)$.

Summing over $r$, we get the formula (13.1), except that the error term is $O(q^{\dim(B_Q + |S|/2 + 3)})$. However, since $X$ is the vanishing set of $m$ equations in a fiber product of $|S|$ curves over $B$, we have

$$\dim(X_{\mathbb{Q}}) \geq \dim(B_Q + |S| - m = \dim(B_Q + \frac{|S|}{2} + l - (l + m - \frac{|S|}{2}) \geq \dim(B_Q + \frac{|S|}{2} + l - 2(l - \dim(X)) \geq \dim(B_Q + \frac{|S|}{2} + 1/2),$$

50
where the last two inequalities hold by the assumption on the perspective and the assumption on \( \dim X \), respectively.

Let \( \eta \) be the generic point of \( X_{\overline{F}_q} \) and let \( \overline{\eta} \) be a geometric generic point over \( \eta \). Let \( \eta' \) be the the generic point of \( Y_{\overline{F}_q} \). We fix a \( k \)-tuple \( \chi \) of characters of \( \overline{F}_q^\times \).

**Lemma 13.2.** Assume that \( \chi \) has CGM. We have

\[
(13.2) \quad \dim \text{End}_{\eta'}(\mathcal{K}_{\eta'} \times \overline{F}_q) = \dim \text{End}_{\eta'}(\mathcal{R}_{\eta'}^{\ast} \times \overline{F}_q).
\]

**Proof.** Let \( Y^\circ \) be the smooth locus of \( Y \). The endomorphisms \( \text{End}_{\eta'}(\mathcal{K}_{\eta'} \times \overline{F}_q) \) are the same as the endomorphisms of the pullback of \( \mathcal{K} \) to \( Y_{\overline{F}_q}^\circ \times \overline{A}_{\mathbb{Q}} \), because the monodromy representations of both sheaves are the same (as they are normal, with the same generic point). We calculate the endomorphisms by applying Lemma 11.1 as the lim-sup of

\[
q^{-\dim X - 2} \sum_{y \in Y^\circ(\overline{F}_q)} \sum_{r \in \overline{F}_q^\times} \prod_{i=1}^l K_l(r(s + b_i(y)); \chi, q) K_l(r(s + b_{i+1}(y)); \chi, q)^2.
\]

We do the same for \( \text{End}_{\eta'}(\mathcal{R}_{\eta'}^{\ast} \times \overline{F}_q) \), obtaining the lim-sup of

\[
q^{-\dim X - 2} \sum_{y \in Y^\circ(\overline{F}_q)} \sum_{r \in \overline{F}_q^\times} \prod_{i=1}^l K_l(r(s + b_i(y)); \chi, q) K_l(r(s + b_{i+1}(y)); \chi, q)^2.
\]

By Lemma 13.1, these two quantities are equal up to \( O(q^{-1/2}) \), and therefore their limsup are equal.

In the remainder of this section, we will prove an analogous statement with \( \overline{\eta} \) instead of \( \eta' \). The method is to prove that

\[
(13.3) \quad \dim \text{End}_{\overline{\eta}}(\mathcal{K}_{\overline{\eta}}) = \dim \text{End}_{\eta'}(\mathcal{K}_{\eta'} \times \overline{F}_q)
\]

and

\[
(13.4) \quad \dim \text{End}_{\eta'}(\mathcal{R}_{\eta'}^{\ast} \times \overline{F}_q) = \dim \text{End}_{\overline{\eta}}(\mathcal{R}_{\overline{\eta}}^{\ast}).
\]

We will prove (13.3) immediately. The formula (13.4) is more difficult, and its proof will use vanishing cycles.

**Proposition 13.3.** Assume that \( \chi \) has CGM. For any extension \( \eta' \) of \( \eta \) we have

\[
\dim \text{End}_{\eta'}(\mathcal{K}_{\eta'}) = \dim \text{End}_{\eta'}(\mathcal{K}_{\eta'} \times \overline{F}_q).
\]

**Proof.** Let \( G \) be the geometric monodromy group of \( \mathcal{K} \), and let \( B \) be the set of distinct values of \( b_1, \ldots, b_{2l} \) at \( \eta \). Then certainly the arithmetic monodromy group of \( \mathcal{K}_{\eta} \times \overline{F}_q \) is contained in \( G^{[B]} \). By Goursat-Kolchin-Ribet, the geometric monodromy group of \( \mathcal{K}_{\eta} \times \overline{F}_q \) is \( G^{[B]} \), so the arithmetic and geometric monodromy groups are equal. Therefore \( \text{Gal}(\overline{F}/\eta \times \overline{F}_q) \) acts trivially on \( \text{End}_{\overline{\eta}}(\mathcal{K}_{\overline{\eta}} \times \overline{F}_q) \) as this action factors through the quotient of the arithmetic monodromy group by the geometric monodromy group. It follows that \( \text{Gal}(\overline{F}/\eta' \times \overline{F}_q) \) acts trivially and so \( \text{End}_{\eta'}(\mathcal{K}_{\eta'} \times \overline{F}_q) \), which is the space of invariants of that action, is equal to the whole space.

In order to prove (13.4), we first introduce some notation. We write \( \overline{\eta} = \eta \times \overline{F}_q \). We consider the projective line \( P^1_{\overline{\eta}} \) with coordinate \( r \). We denote by \( \mathcal{O}^{et} \) the étale local ring of \( P^1_{\overline{\eta}} \) at \( \infty \) and by \( K \) its field of fractions. We will often identify \( K \) (resp. a separable closure \( K^{sep} \) of \( K \)) with the corresponding spectra.

What follows is the key lemma.
**Lemma 13.4.** With assumptions as above, the action of $\text{Gal}(K^{\text{sep}}/K)$ on $\mathcal{R}^*_{K^{\text{sep}}}$ is unipotent.

Note that to make sense of this action, we use the fact that the image of the natural morphism $\text{Spec}(K) \to \mathbf{A}^{1+2l}$ has image in $U$, which follows from Lemma 7.4.

**Proof.** We denote by $\sigma$ the special point of $\text{Spec}(\mathcal{O}^{\text{et}})$. We consider the projective line $P^1_{\mathcal{O}^{\text{et}}}$, with coordinate $t$, and denote by $j$ (resp. by $g$) the open immersion $G_{m, \mathcal{O}^{\text{et}}} \to P^1_{\mathcal{O}^{\text{et}}}$ (resp. the open immersion $A^1_{\mathcal{O}^{\text{et}}} \to P^1_{\mathcal{O}^{\text{et}}}$).

We consider the lisse sheaf $\widehat{\mathcal{K}} = \bigotimes_{1 \leq i \leq l} \mathcal{K}_{\ell, \psi}(\chi)(t(1+b_i/r)) \otimes \mathcal{K}_{\ell, \psi}(\chi)(t(1+b_{i+l}/r))^v$ on $G_{m, \mathcal{O}^{\text{et}}}$.

By the change of variable $t = rs$ and the proper base change theorem, the $\text{Gal}(K^{\text{sep}}/K)$-action on $\mathcal{R}_{K^{\text{sep}}}$ is isomorphic to the action on $H^1(P^1_{K^{\text{sep}}}, j^!\widehat{\mathcal{K}})$. Since $\mathcal{R}^*_{K^{\text{sep}}}$ is a quotient of $\mathcal{R}_{K^{\text{sep}}}$, the lemma will follow if we prove that the action of $\text{Gal}(K^{\text{sep}}/K)$ on $H^1(P^1_{K^{\text{sep}}}, j^!\widehat{\mathcal{K}})$ is unipotent.

By the long exact sequence for vanishing cycles, we have a long exact sequence

$$
\cdots \to H^i(P^1_{\sigma}, j^!\widehat{\mathcal{K}}) \to H^i(P^1_{K^{\text{sep}}}, j^!\widehat{\mathcal{K}}) \to H^i(P^1_{\sigma}, R\Phi j^!\widehat{\mathcal{K}}) \to \cdots
$$

(13.5)

For each $i$, we have an isomorphism

$$
H^i(P^1_{\sigma}, j^!\widehat{\mathcal{K}}) = H^i(P^1_{\sigma}, j^! \left( \mathcal{K}_{\ell, \psi}(\chi)^\otimes \otimes (\mathcal{K}_{\ell, \psi}(\chi)^v)^\otimes \right)),
$$

hence the $\text{Gal}(K^{\text{sep}}/K)$-action on these spaces is trivial.

On the other hand, the vanishing cycle complex $R\Phi j^!\widehat{\mathcal{K}}$ is zero away from the point at $\infty$ of $P^1_{\sigma}$ (local acyclicity of smooth morphisms and lisseness of $j^!\widehat{\mathcal{K}}$) and is zero at 0 (because of tame ramification and Deligne’s semicontinuity theorem).

We therefore only need to understand $R\Phi j^!\widehat{\mathcal{K}}$ at $t = \infty$. By the second part of Lemma 7.2, the local monodromy at infinity of $j^!\widehat{\mathcal{K}}$ is isomorphic to that of a direct sum of sheaves of the form

$$
\mathcal{L}_\psi \left( (t(1+b_1/r))^{1/k} + \sum_{i=2}^{2k} \varepsilon_i \zeta_i (t(1+b_i/r))^{1/k} \right).
$$

Since $(1+b_i/r)^{1/k}$ belongs to the étale local ring $\mathcal{O}^{\text{et}}$, this is isomorphic to the local monodromy of a direct sum of sheaves of the form $\mathcal{L}_\psi(\gamma(r)^{1/k})$. We have $\mathcal{L}_\psi(\gamma(r)^{1/k}) = \varpi_* \mathcal{L}_\psi(\gamma(r)u)$ where $\varpi$ is the finite covering $u \mapsto u^k$. We compute the local monodromy at $\infty$ of this sheaf. This is a standard computation. We use the long exact sequence

$$
\cdots \to H^i(P^1_{\sigma}, g_!\mathcal{G}) \to H^i(P^1_{K^{\text{sep}}}, g_!\mathcal{G}) \to H^i(P^1_{\sigma}, R\Phi g_!\mathcal{G}) \to \cdots
$$

and distinguish three cases:

1. If $\gamma(r) = 0$ in $\mathcal{O}^{\text{et}}$, then $\mathcal{G}$ is tamely ramified at $\infty$, so the vanishing cycles vanish.
2. If $\gamma(r) \neq 0$ in $\mathcal{O}^{\text{et}}$ but $\gamma(r) = 0$ at the special point, then all $H^i$’s with coefficients in $g_!\mathcal{G}$ in the above exact sequence vanish except

$$
H^2(P^1_{\sigma}, g_!\mathcal{G}),
$$

which is one-dimensional with a trivial action of $\text{Gal}(K^{\text{sep}}/K)$; this implies that the action on $H^1(P^1_{\sigma}, R\Phi g_!\mathcal{G})$ is trivial.
3. If $\gamma(r) \neq 0$ at $\sigma$, then all cohomology groups in the sequence vanish by properties of the Artin-Schreier sheaves.
In any of the three cases, by local acyclicity of smooth morphisms we see that $R\Phi_{g}\mathcal{G}$ vanishes outside the point at $\infty$, so knowing that $H^i(\mathbb{P}^1_k, R\Phi_{g}\mathcal{G})$ has trivial Galois action implies that the Galois action on the stalk at $\infty$ vanishes.

Since the vanishing cycle functor is additive and commutes with finite pushforward, we conclude that $\text{Gal}(K^{\text{sep}}/K)$ acts trivially on $H^i(\mathbb{P}^1_{\overline{K}}, R\Phi_{g}\mathcal{X})$ for all $i$, hence by the exact sequence (13.5), this group acts unipotently on $H^i(\mathbb{P}^1_{\overline{K^{\text{sep}}}}, j_{\overline{\mathcal{X}}})$, as desired.

**Proposition 13.5.** Assume that $\chi$ has CGM. We have

$$\dim \text{End}_{U_{\eta'}}(\mathcal{R}^*_{\eta'} \times \overline{\mathbb{F}}_q) = \dim \text{End}_{U_{\eta}}(\mathcal{R}^*_{\eta}).$$

*Proof.* We first note that we have an inclusion

$$\text{End}_{U_{\eta}}(\mathcal{R}^*_{\eta}) \subset \mathcal{R}^*_{\eta} \otimes (\mathcal{R}^*_{\eta})' \uparrow.$$

Moreover, we have a commutative triangle

$$\begin{array}{ccc}
\text{Gal}(K^{\text{sep}}/K) & \longrightarrow & \pi_1(U_{\eta}) \\
\alpha \downarrow & & \downarrow \\
\text{Gal}(\overline{\eta}/\eta) & & \\
\end{array}$$

where $\alpha$ is surjective because $K$ does not contain a finite extension of $\overline{\eta}$.

The fundamental group $\pi_1(U_{\eta})$ acts on $\mathcal{R}^*_{\eta} \otimes (\mathcal{R}^*_{\eta})'$ and the Galois group $\text{Gal}(\overline{\eta}/\eta)$ acts on $\text{End}_{U_{\eta}}(\mathcal{R}^*_{\eta})$, and these actions are compatible with the inclusion above.

By Lemma 13.4, the action of $\text{Gal}(K^{\text{sep}}/K)$ on $\mathcal{R}^*_{\eta} \otimes (\mathcal{R}^*_{\eta})'$ is unipotent, hence the action of $\text{Gal}(\overline{\eta}/\eta)$ on $\text{End}_{U_{\eta}}(\mathcal{R}^*_{\eta})$ is also unipotent since $\alpha$ is surjective. But we know, by purity, that this action is semisimple, and it follows that the action $\text{Gal}(\overline{\eta}/\eta)$ on $\text{End}_{U_{\eta}}(\mathcal{R}^*_{\eta})$ is in fact trivial. In particular, we have

$$\dim \text{End}_{U_{\eta'}}(\mathcal{R}^*_{\eta'} \times \overline{\mathbb{F}}_q) = \dim \text{End}_{U_{\eta}}(\mathcal{R}^*_{\eta}).$$

*□*

Finally, we can deduce:

**Theorem 13.6.** Let $X$ be an irreducible component of $X$, which intersects the characteristic zero part. Assume that $p$ is a prime sufficiently large with respect to $(k, l, X)$. Let $\mathbb{F}_q$ be a finite field of characteristic $p$, and let $\overline{\eta}$ be the geometric generic point of $X_{\overline{\mathbb{F}}_q}$. Suppose that $X$ has dimension at least $(3l + 1)/2$. Let $\chi$ be a $k$-tuple of characters of $\mathbb{F}_q^\times$ with Property CGM. Then we have

$$\dim \text{End}_{U_{\eta}}(\mathcal{G}_{\eta}) = \dim \text{End}_{U_{\eta}}(\mathcal{G}^*_{\eta}).$$

*Proof.* Since the assertion is geometric, we may replace $\mathbb{F}_q$ by a finite extension that is a residue field of the base $\mathcal{O}_K[1/N]$ of the “spread-out” perspective. The equality then follows, when the characteristic of $\mathbb{F}_q$ is sufficiently large in terms of $(k, l, X)$, by combining Proposition 13.3, Lemma 13.2 and Proposition 13.5. *□*

14. **Conclusion of the Proof**

We recall that we want to prove Theorem 7.7, which we restate for convenience:

**Theorem 14.1.** Assume that $\chi$ has NIO. If $p$ is large enough, depending only on $k, l$, then for any $b \in A^2(\mathbb{A}_q) - W(\mathbb{F}_q)$, the natural morphism $\theta_b$ is an isomorphism.

Furthermore, each irreducible component of $\mathcal{R}^*_{\eta}$ has rank greater than one.
Proof. Since $\chi$ has NIO, by Lemma 6.3 there exists a character $\xi$, possibly over a finite extension $F_q^r$ of $F_q$, such that $\chi' = \xi\chi$ has CGM over $F_q^r$. Consider $\chi$ as a tuple of characters of $F_q^r$. Then $\mathcal{K}_{\ell, \psi}(\chi') = \mathcal{L}_\xi \otimes \mathcal{K}_{\ell, \psi}(\chi)$, and it follows that the auxiliary sheaves $\mathcal{K}$ and $\mathcal{R}^*$ for $\chi$ are obtained from those associated to $\chi'$ by twisting by a rank 1 sheaf $\mathcal{L}_\xi ((r+b_1) \ldots (r+b_l)(r+b_{l+1}) \ldots (r+b_{2l})^{-1})$. Then the corresponding endomorphism rings (and the morphism $\theta_b$) are the same for $\chi$ and $\chi'$. Up to renaming the field, this implies that we may as well assume that $\chi$ has CGM over $F_q$.

Let $b \in A^2(F_q) - W(F_q)$ be a point. Let $j$ be the minimum $j$ such that $b \in X_j$. Let $X$ be an irreducible component of $\hat{j}$ containing $X_j$. By taking $q$ sufficiently large, we may assume that $X$ intersects the characteristic zero part. As the set of irreducible components is finite and depends only on $k, l$, the minimum value for $q$ depends only on $k, l$.

If the dimension of $X$ is less than $(3l + 1)/2$, then $b \in X \subseteq W$.

Otherwise, let $\eta$ be the generic point of $X$. Then by Theorem 13.6, taking $q$ sufficiently large,

$$\dim \text{End}_{U_b}(\mathcal{K}_\eta) = \dim \text{End}_{U_b}(\mathcal{R}^*_b).$$

Because $U_1$ has dimension $\leq l + 1$, and $\dim X \geq (3l + 1)/2 > l + 1$ as $l > 1$, $\eta$ is not contained in $U_1$. By Lemma 10.1, $Z$ is finite étale over $X_j - X_{j-1}$. Because the $b_i$ are sections of $Z$, and $b$ is a specialization of $\eta$ inside $X_j - X_{j-1}$, any two of the $b_i$ which are unequal over $\eta$ must remain unequal over $b$, so $b \notin U_1$.

So by Theorem 9.1, the natural map

$$\theta_b : \text{End}_{U_b}(\mathcal{K}_b) \rightarrow \text{End}_{U_b}(\mathcal{R}^*_b)$$

is an isomorphism, hence by Proposition 10.3, $\theta_b$ is an isomorphism.

Each irreducible component of $\mathcal{R}^*_b$ is the image of an idempotent element of $\text{End}_{U_b}(\mathcal{R}^*_b)$, which because $\theta_b$ is an isomorphism is induced by an idempotent element of $\text{End}_{U_b}(\mathcal{K}_b)$, and thus is equal to the weight one part of the cohomology of the image of that idempotent element of $\text{End}_{U_b}(\mathcal{K}_b)$. In other words, it is the weight one part of the cohomology of an irreducible component of $\mathcal{K}_b$. Hence by Lemma 9.6, its rank is at least two.

We finally can conclude the proof by showing how Theorem 14.1 allows us to give the estimates for complete sums used in the proof of our main theorems. In both cases, we use the fact (as remarked before the statements of Theorem 4.3 and 4.4) that we may assume that the function $K$ is $K_{l_k}(x; \chi, q)$. By Lemma 7.1 and the Grothendieck–Lefschetz trace formula, for any $b \notin \mathcal{V}^\Delta$, the function $R$ is equal to minus the trace function of the sheaf $\mathcal{R}$, if the additive character $\psi$ is chosen so that $\psi(x) = e(x/q)$ for $x \in F_q$.

**Proof of Theorem 4.3.** We have defined $\mathcal{V}^\Delta$ and $W$, and they satisfy the codimension bounds stated in the theorem (see (7.2)).

We need to estimate the complete sums

$$\Sigma_{II}(b) = \sum_{r \in F_q} |R(r, b)|^2 - \sum_{s \in F_q} \sum_{r \in F_q} |K(sr, sb)|^2$$

for $b \in F_q^{2l}$. Since $K_{l_k}$ is bounded, we have $\Sigma_{II}(b) \ll q^3$ for all $b$, which is the trivial bound (4.4).

If $b \in W(F_q)$ and $b \notin \mathcal{V}^\Delta(b)$, then we obtain $\Sigma_{II}(b) \ll q^2$ by estimating the two terms in $\Sigma_{II}$ separately, and using the Riemann Hypothesis together with the fact that the $\mathcal{R}$-sheaf is mixed of weights $\leq 1$ on $A^{2l} - \mathcal{V}^\Delta$, and the $\mathcal{K}$-sheaf is pure of weight 0. This proves (4.5).

Now assume that $b \notin W(F_q)$. By Theorem 14.1, the Frobenius-equivariant map

$$\theta_b : \text{End}_{U_b}(\mathcal{K}_b) \rightarrow \text{End}_{U_b}(\mathcal{R}^*_b)$$

...
is an isomorphism. In particular the Frobenius automorphism of $F_q$ has the same trace on both spaces. The trace on $\End_U(b)$ is, by the Grothendieck–Lefschetz trace formula, equal to

$$\sum_{r \in F_q} |R(r, b)|^2 + O(q^{3/2})$$

where the error term arises from the contribution of the $H^1$-cohomology and of the weight < 1 part of $R$. Similarly, the trace of Frobenius on $\End_V(b)$ is equal to

$$\sum_{s \in F_q^*} \sum_{r \in F_q} |K(s, r, b)|^2 + O(q^{3/2})$$

where the error term arises from the contribution of the $H^1$-cohomology. Comparing, we obtain (4.6).

It remains to observe that, in all these estimates, the implied constant depends only on the sum of the Betti numbers of the relevant sheaves. These are estimated in the usual way by reducing to expressions as exponential sums and applying the Betti number bounds of Bombieri–Katz (see [Kat01, Th. 12] and [KMS17, Prop. 4.24] for the analogue argument in our previous paper). □

Proof of Theorem 4.4. We recall that we need to estimate

$$\Sigma_f(b) = \sum_{r \in F_q} R(r, b)$$

(see (4.10)). Since $K_k$ is bounded, we have $\Sigma_f(b) \ll q^2$ for all $b$, which is the trivial bound (4.12).

If $b \in W(F_q)$ and $b \notin \mathcal{V}(b)$, then we obtain $\Sigma_f(b) \ll q^{3/2}$ because the $R$-sheaf is of weights $\leq 1$ on $A^2 - \mathcal{V}$ and has no geometrically trivial irreducible component (by Theorem 14.1 it doesn’t even have rank 1 components), proving (4.13).

Finally, if $b \notin W(F_q)$, then we obtain $\Sigma_f(b) \ll q$ straightforwardly from Deligne’s Riemann Hypothesis, since $R^*$ is of weight 1 and has no geometrically trivial irreducible component (by Theorem 14.1 it doesn’t even have rank 1 components), proving (4.14).

Again, the implied constants in these estimates depend only on the sum of the Betti numbers of the relevant sheaves, and are estimated in by reducing to expressions as exponential sums and applying the Betti number bounds of Bombieri–Katz [Kat01]. □

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