Universal fluctuations in radial growth models belonging to the KPZ universality class

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Abstract – We investigate the radius distributions (RD) of surfaces obtained with large-scale simulations of radial clusters that belong to the KPZ universality class. For all investigated models, the RDs are given by the Tracy-Widom distribution of the Gaussian unitary ensemble, in agreement with the conjecture of the KPZ universality class for curved surfaces. The quantitative agreement was also confirmed by two-point correlation functions asymptotically given by the covariance of the Airy $^2$ process. Our simulation results fill a lacking gap of the conjecture that had been recently verified analytically and experimentally.

Growth phenomena remain a topic of great interest in nonequilibrium Statistical Physics, mainly because of the self-similarity and universality emerging from dynamical local processes of different systems. In this context, one of the most important examples is the Kardar-Parisi-Zhang (KPZ) universality class introduced by equation [1]:

$$\frac{\partial h}{\partial t} = \nu \nabla^2 h + \lambda |\nabla h|^2 + \eta,$$

where $\eta$ is a Gaussian noise. This universality class was observed in a large number of models [2,3] and a few experimental systems [4–6].

Former works on surface dynamics were mainly concerned with the scaling properties of the surface fluctuations by means of scaling exponents [2,3,7]. However, there is a number of other universal quantities that are also suitable for determining universality in surfaces [6]. Examples include the stationary distributions of the global interface width and the extremal height [8,9], and the height distributions during the transient regime [10,11] that precedes saturation of the interface width.

The distributions during the transient (growth) regime were computed exactly for some models in the KPZ class, in $1+1$ dimension [10–13]. Among the most relevant, the height distribution (HD) of the single-step model [2], determined analytically by Johansson [10], has the Tracy-Widom distribution of the largest eigenvalue of the Gaussian unitary ensemble (GUE) of the random matrix theory [14] as a limit solution. Prähofer and Spohn [11,15] also obtained analytically the scaling form of the KPZ universality class and a Tracy-Widom distribution for the HD of the polynuclear growth model (PNG). Furthermore, they showed that the HD for the growth from flat substrates is given by the Gaussian orthogonal ensemble (GOE), while growth from a single seed (radial growth) leads to a GUE distribution for the radii [11,15]. Recently, Sasamoto and Spohn [16] found a solution of the one-dimensional KPZ equation with an initial condition that induces the growth of curved surfaces and the GUE distribution was confirmed as the limit for the radius fluctuations. Subsequently, a numerical evaluation of this solution corroborated these analytical results [17]. Prähofer and Spohn [18] also obtained an analytical solution for the limiting process describing the surface fluctuations in the PNG model as the so-called Airy $^2$ process.

Experimentally, the GOE and GUE distributions were obtained in a few experiments exhibiting KPZ scaling. An evidence of the KPZ exponents was found in the slow combustion of paper sheets [4]. The burning fronts evolve from a flat initial condition and the obtained HD has a reasonable agreement with GOE. A recent experiment on electroconvection of turbulent liquid crystal films [5] allowed to investigate an isotropic radial growth with high accuracy. It was shown that this system belongs to the
KPZ universality class exhibiting radius distribution (RD) in excellent agreement with GUE, including the cumulants from second to fourth order. However, the mean is shifted and tends to the GUE value as a power law $t^{-1/3}$. The same result was found by Sasamoto and Spohn [16] in a solution of the KPZ equation in 1+1 dimensions with an initial edge condition indicating a universal behaviour of the exponent 1/3. However, Ferrari and Frings [13] have recently shown that the scaling law featuring this approach to GUE mean is not universal. Indeed, they have shown that the mean shift decays as $t^{-1/3}$ for the totally asymmetric simple exclusion process (ASEP) whereas no correction (up order $O(t^{-2/3})$) is possible for the weakly ASEP, both models belonging to the KPZ class. The liquid crystal film setup was also used to induce a front growth from a flat surface, and a good agreement with GOE was obtained for the HDs [6].

The theoretical and experimental evidences above mentioned strongly suggest that the GUE distribution is a universal feature of one-dimensional growth with radial symmetry in the KPZ class. However, this conjecture is based on a limited number of models with exact results [11,13,15,16] and a single experimental work [5]. Numerical confirmation of the GUE distributions in non-solvable model is, up to this moment, missing. In order to fill this gap, we investigate the RD of large radial clusters (larger than $3 \times 10^6$ particles) generated with different versions of the Eden model [19–21]. We show that the RDs exhibit very good agreement with GUE distribution. We also observed that the cumulants of the distribution converge to the GUE values, as previously observed in other systems [5,16]. The two-point correlation function also converges to the Airy$_2$ process, as predicted by the conjecture [18].

The Eden model [19] consists in adding new particles in the empty neighbourhood of a growing cluster. If the growth starts with a single particle, the model yields asymptotically spherical clusters with a self-affine surface exhibiting the scaling exponents of the KPZ universality class [21]. We simulated off-lattice clusters in two-dimensions with the usual algorithm [21,22]: a particle in the active (growing) zone is selected at random and a new particle is added in a random position chosen in the empty neighbourhood of the selected particle. The procedure is repeated while the cluster does not reach $N$ particles. With suitable optimizations [21,22], we were able to grow clusters with up to $3 \times 10^7$ particles. Since we randomly pick up a particle from a constantly updated list containing $N_s$ surface sites, the time step is simply $\Delta t = 1/N_s$. A total number of up to $10^3$ off-lattice clusters were grown in order to perform statistical averages.

We also have simulated Eden models on a square lattice using an algorithm proposed in ref. [20] that removes the lattice anisotropy effects. The method consists in

\[ P(R) = \frac{1}{\sigma_R} G\left( \frac{R - \langle R \rangle}{\sigma_R} \right), \]  

where $\sigma_R^2$ is the variance of the RD and $G(x)$ is a normalized scaling function. This scaling form reduces finite size corrections, what has improved data collapses in other analyses, as interface width distributions [8], for example. In fig. 1, we compare the rescaled RDs for the three Eden models with the rescaled GUE distribution. An excellent collapse of all curves upon a single curve $G(x)$ was obtained. Similar results hold for different

![Fig. 1: (Color online) Radius distributions of on- and off-lattice Eden models rescaled to a null mean and a unitary variance. The mean radii of the aggregates are approximately 2500 for the off-lattice model and $3.2 \times 10^4$ for on-lattice models. The solid line is the rescaled GUE distribution. In this plot, $R^* \equiv (R - \langle R \rangle)/\sigma_R$.](48003-p2)
growth times. This data collapse confirms that, a part of corrections to scaling in the cumulants described below, the RDs of the all investigated models agree with the GUE distribution, as conjectured by Prähöfer and Spohn \cite{11}.

In radial growth belonging to the KPZ universality class, the radii are stochastic variables evolving in time as \cite{11,15}

\[ R(t) \simeq \lambda t + \left( A^2 \lambda t / 2 \right)^{1/3} \chi_{\text{GUE}}, \tag{3} \]

where \( \lambda \) and \( A \) are two non-universal (model dependent) parameters. The random variable \( \chi_{\text{GUE}} \) is distributed according to the GUE Tracy-Widom distribution \cite{14}. Therefore, \( \lambda \) is the asymptotic radial growth rate obtained from \( \lambda \simeq \partial R / \partial t + a_n t^{-2/3} \) in the limit \( t \to \infty \) \cite{23}. In fig. 2(a), we show the average radius growth rate against a power of time. The extrapolated asymptotic values are \( \lambda \simeq 1.1843(2) \) for off-lattice, \( \lambda \simeq 0.2639(2) \) for Eden A, and \( \lambda \simeq 0.4807(2) \) for Eden B, where the uncertainties obtained in the regressions are shown in parenthesis.

The parameter \( A \) was estimated in two independent ways. We can use the local squared surface roughness, in a window of size \( \epsilon \), defined as

\[ w^2(\epsilon, t) = \langle [R(x, t)]^2 \rangle_\epsilon - \langle R(x, t) \rangle_\epsilon^2, \tag{4} \]

where \( \langle \ldots \rangle_\epsilon \) denotes an average within several windows in the interface. Alternatively, we can estimate the \( A \) value using the height-height correlation function

\[ c(\epsilon, t) = \langle [R(x+\epsilon, t) - R(x, t)]^2 \rangle_\epsilon. \tag{5} \]

For long times, theoretical arguments predict that \( w^2 \simeq A \epsilon / 6 \) and \( c \simeq A \epsilon \) \cite{23}. Curves for \( 6w^2 / \epsilon \) as functions of \( \epsilon \) are shown in fig. 2(b). Well-defined plateaus are observed for both on-lattice models, except for short scales, when a small deviation is observed. Since off-lattice simulations are much smaller, the plateau is not so evident as in the on-lattice case, but the data also tend to a constant value. For the sake of comparison, we measured the local roughness exponent, defined as \( w(\epsilon) \sim \epsilon^\alpha \), and found \( \alpha \approx 0.43 \) for our largest off-lattice simulations. This value is considerably smaller than the expected exponent of the KPZ class \( \alpha = 1/2 \), confirming the presence of strong finite-time effects in the exponents. Roughness exponent for the on-lattice models are very close to the value \( \alpha = 1/2 \). The estimates of parameter \( A \) are represented by dashed lines in figs. 2(b) and (c). The estimates using local interfaces width are: \( A \simeq 1.46 \) for off-lattice, \( A \simeq 1.32 \) for Eden A, and \( A \simeq 0.71 \) for Eden B models. The correlation function estimates are slightly larger: \( A \approx 1.55 \) for off-lattice and A models, and \( A \approx 0.84 \) for Eden B.

In agreement to eq. (3), the quantity

\[ q = \frac{R - \lambda t}{(A^2 \lambda t / 2)^{1/3}} \tag{6} \]

is a random variable given by a GUE distribution. The RDs shown in fig. 3 were obtained with the values \( \lambda = 1.1842 \) and \( A = 1.45 \) for off-lattice simulations, \( \lambda = 0.263887 \) and \( A = 1.43 \) for Eden A, and \( \lambda = 0.4806 \) and \( A = 0.805 \) for Eden B. As predicted by the radial KPZ conjecture, a very good collapse for different models (and different times) is observed. The agreement with the GUE distribution is noticeable, except by a shift in \( q \), that vanishes as \( t \to \infty \). The shift is more evident for Eden B as can also be seen in fig. 4. Our results confirm the limiting scaling form conjectured by Prähöfer and Spohn \cite{11,15}, as previously observed in theoretical \cite{11,12,15,16} and experimental \cite{5} systems.

In order to quantify the agreement between RDs and GUE distributions, we investigate the \( n \)-th-order cumulants of the probability distribution \( P(q) \) denoted by \( \kappa_n \). The differences between the cumulants obtained for off-lattice simulations and the GUE values are shown, as a function of time, in fig. 4(a). As expected, all cumulants

Fig. 2: (Color online) Determination of the parameters related to radius evolution given by eq. (3). (a) Average radius growth rate \( d(R)/dt \) against \( t^{-2/3} \) for off-lattice (main plot) and on-lattice (insets) Eden models. Dashed lines are extrapolations to \( t \to \infty \) used to determine \( \lambda \). (b) Rescaled interface width for a scale \( \epsilon \). The horizontal lines are estimates of the parameter \( A \). Cluster average radii are 32000 for on-lattice and 2700 for off-lattice models. (c) The equivalent analysis of (b) for the correlation function.
converge to the GUE values. As observed experimentally by Takeuchi and Sano [5], the first moment decreases towards the GUE value while the higher-order cumulants increases towards GUE values. The same happens for Eden A, as can be seen in fig. 4(b). Differently, the mean for Eden B increases (more slowly than the others) towards GUE and the higher-order cumulants converges more quickly to the theoretical values. This negative amplitude of the difference between simulation and GUE mean was also observed in a solution of the KPZ equation [16]. Indeed, the law describing the convergence of the mean have recently attracted great interest [5,13,16]. Our off-lattice and Eden B simulations are very well described by the $t^{-1/3}$ law previously reported [5,16] while the simulation of Eden A are only consistent with this approach since a long power law regime was not observed.

A further evidence of the agreement between Eden growth and radial KPZ conjecture is yielded by the two-point correlation function given by $C_2(\epsilon, t) = (R(x+\epsilon,t)R(x,t)) - \langle R \rangle^2$, that, in agreement with the conjecture [18], scales at long times as $C_2(\epsilon, t) \simeq (A^2 \lambda t/2)^{2/3}g_2(u)$ with $u = (\lambda \epsilon^2)/(A^2 \lambda t)_2^{2/3}$, where $g_2(u)$ is the covariance of the Airy$_2$ process [24]. In fig. 5, we show the correlation function $C_2$ for different models. A very good agreement between the scaled $C_2$ for all Eden models and $g_2$ function is obtained, showing that the models are well described by the Airy$_2$ process. It is worth mentioning that scaled $C_2$ approaches $g_2$ for long times and that this approach is slower for on-lattice models.

In summary, the KPZ universality class in radial growth has been the subject of recent analytical [16] and experimental [5] investigations that agreed with the conjecture proposed by Prähofer and Spohn [11,15], where interface fluctuations in systems belonging to the KPZ universality class are described by well-known universal distributions. However, a computational verification in non-analytically solvable growth models was lacking until the present work. In the present letter, we have investigated the radius distributions in clusters obtained with Eden growth models [20,21] starting from a single particle. The radius distributions obtained for all models exhibit an excellent agreement with the scaling ansatz given by eq. (3), that associates the radius fluctuations with the
Tracy-Widom [14] distribution of the Gaussian Unitary Ensemble. The cumulants of order $n \geq 1$ associated to RD converge to the corresponding GUE cumulants for all investigated models. A finite-time correction of order $t^{-1/3}$ in the first moment was clearly observed for off-lattice and Eden B simulations, in agreement with other systems [5,16]. Finally, a correlation function in accordance with the so-called Airy$_2$ process yields a further strong evidence that the radius fluctuations in all investigated growth models are in agreement with the KPZ conjecture.

As a final remark, notice that the the small exponent $\alpha \approx 0.43$ obtained for the largest off-lattice simulations shows that scaling exponents undergo strong finite-time effects. Therefore, the RD analysis may be more reliable than scaling exponents to determine the universality class of a system. In particular, the scaling form given by eq. (2) is simpler than the analysis with eq. (3), since fit procedures are not required in the first approach and it does not have finite-time corrections.

In conclusion, the Pr"ahofer-Spohn conjecture is now fully verified in the three general branches of Statistical Physics: experimental, theoretical, and computational.

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Additional remark: After acceptance, we have become aware that an analytical solution of the KPZ equation in a curved surface was independently obtained in ref. [25], besides the above-mentioned results given in ref. [16].

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