Efficient Strategy Computation in Zero-Sum Asymmetric Repeated Games

Lichun Li and Jeff S. Shamma

Abstract

Zero-sum asymmetric games model decision making scenarios involving two competing players who have different information about the game being played. A particular case is that of nested information, where one (informed) player has superior information over the other (uninformed) player. This paper considers the case of nested information in repeated zero-sum games and studies the computation of strategies for both the informed and uninformed players for finite-horizon and discounted infinite-horizon nested information games. For finite-horizon settings, we exploit that for both players, the security strategy, i.e., Nash equilibrium, and also the opponent's corresponding best response depend only on the informed player's history of actions. Using this property, we refine the sequence form, and formulate an LP computation of player strategies that is linear in the size of the uninformed player's action set. For the infinite-horizon discounted game, we construct LP formulations to compute the approximated security strategies for both players, and provide a bound on the performance difference between the suboptimal strategies and the security strategies. Finally, we illustrate the results on a network interdiction game between an informed system administrator and uniformed intruder.

I. INTRODUCTION

There are many competitive settings in which players have asymmetric information about the underlying state of the game. A historical example [1] is the gradual disarmament procedure between the United States and the Soviet Union, where each country has better understanding of its own armament situation than the other. Similar information asymmetry is also found between competitive companies. For example, in air transportation systems, airline companies competing for earliest available time slots in flow constrained areas have a very rough understanding about the routing preferences of other companies [2], [3].

The particular setting of interest in this paper is where there is nested information and repeated interactions. Nested information is where one (informed) player has more information than the other (uninformed) player. In particular, there is an underlying state of the world that is initially selected, once and for all, at random. The informed player knows the realized state, whereas the uninformed player only knows the prior distribution. Repeated interactions means that players play over stages and can make observations about past play. Here we assume the case of full monitoring, in which players observe the actions taken by their opponents.

Dealing with information asymmetry is subtle in repeated games for the informed player. The tension is that by exploiting the superior information, the informed player also consequentially reveals this information. Indeed, there are classical extreme examples [4] where the optimal policy of the informed player ignores the state realization (non-revelation). In other cases, the optimal policy of the informed player is to immediately fully utilize the state realization in a manner that exposes the realized state (full revelation). A middle ground is partial revelation, in which the informed player selects actions based on probability distribution that depends on the realized state. This approach forces the uninformed player to make inferences about the realized state based on the actions of the informed player.

There has been extensive prior work analyzing the structure of such asymmetric information repeated zero-sum games [5], [6], [4]. In particular, it has been shown that the security strategy (i.e. Nash equilibrium) for the informed player only depends on past actions of the informed player. The informed player can model the uninformed player as making Bayesian updates with an evolving belief state, which are the posterior probabilities of the realized state based on observations. These belief states are a sufficient statistic for the informed player. That is, the informed player’s security strategy is shown to be a function of this belief state. Likewise, the security strategy for the uninformed player only depends on past actions of the informed player (and not on his own actions), which was shown to be also true in asymmetric vector payoff games [7]. Moreover, from the dual game of the asymmetric repeated game, it was shown that the security strategy for the uninformed player in the asymmetric repeated game is the same as the security strategy for the uninformed player in the dual game with a ‘special’ initial vector payoff which, however, was not explicitly given. It was also shown that the uninformed player’s security strategy depends on a vector payoff of the same size of the state set.

The main contribution of this paper is an LP formulation for the explicit computation (i.e., beyond a characterization) of player strategies for both the informed and uninformed player. In the finite-horizon case, the one can enumerate all strategies.
of both players and thereby embed the game in a large but finite matrix game and apply classical methods based on LPs (e.g., [8] and references therein). This approach quickly becomes intractable as a function of the game parameters (horizon length, number of states, and number of actions). Prior work, notably [9], [10], showed that the security strategies of such an extensive form game is the solution of an LP whose size is linear in the size of the game tree, and hence polynomial in the size of the uninformed player’s action set. This approach required a transformation of the game into a so-called sequence form. Prior work studied an explicit LP formulation for asymmetric information games for one-stage games [11].

Based on the idea of realization plan in sequence form and the fact that both the security strategy and the corresponding best responses depend on informed player’s past actions, this paper refines the sequence form and proposes explicit LP formulations to compute both players’ security strategies. For finite-horizon problems with fixed horizon length, the sizes of these LP constructions are polynomial with respect to the size of the informed player’s action set, and in particular, only linear in the size of the uninformed player’s action set as compared with prior work that had polynomial dependence. The computational complexity reduction is realized by exploiting the aforementioned property of strategy dependence on only the informed player’s actions. Therefore, the branches associated with the uninformed player’s actions in the game tree are cut, which significantly reduces the size of the game tree and hence the computational complexity of the security strategies.

Computing the security strategies in discounted infinite-horizon asymmetric repeated games is non-convex [12], [13]. Therefore, we present approximated security strategies for both players, provide LP formulations to compute the suboptimal strategies, and analyze the bound on the performance difference between the approximated security strategies and the security strategies. The informed player’s suboptimal strategy is stationary, and depends only on the aforementioned belief state of the uninformed player. For the uninformed player, as mentioned before, his security strategy is the same as the one in the dual game with a special initial vector payoff. We first find out that the special initial vector payoff is indeed the uninformed player’s regret with respect to every possible system state, and then provide an LP formulation to approximate the special initial vector payoff. With this approximated special initial vector payoff, using the same technique as for the informed player, we construct an LP to compute an approximated security strategy for the uninformed player in the dual game, which serves as the approximated security strategy for the uninformed player in the asymmetric repeated game. Since the security strategy for the uninformed player in the dual game is stationary, and depends on the regret only, our approximated security strategy for the uninformed player in the asymmetric repeated game depends only on the regret, too. Since the size of the regret is the same as the size of the underlying state set, the approximated security strategy for the uninformed player significantly reduces the required memory as compared to the information set.

The remainder of this paper is organized as follows. Section II presents the main results for finite stage games. Section III discusses discounted infinite-horizon games. Section IV illustrates the results on a network interdiction game. Finally, Section V presents some future work.

II. FUTURE STAGE ASYMMETRIC REPEATED GAMES

Notation. Let \( \mathbb{R}^n, \mathbb{R}_0^+ \) and \( \mathbb{Z}^+ \) denote \( n \)-dimensional real space, non-negative real numbers, and positive integers, respectively. For a finite set \( K \), \( |K| \) denotes its cardinality, and \( \Delta(K) \) indicates the set of probability distributions over \( K \). The vectors \( 1 \) and \( 0 \) are appropriately dimensioned column vectors with all elements being \( 1 \) and \( 0 \), respectively. For \( v(0), v(1), v(2), ... \) a sequence of real numbers, we adopt the convention that \( \sum_{t=1}^t v(t) = 0 \) and \( \prod_{t=1}^0 v(t) = 1 \). The supreme norm of a function \( f : D \to \mathbb{R} \) is defined as \( \|f\|_{\sup} = \sup_{x \in D} |f(x)| \), where \( D \) is a non-empty set.

A. Setup

A two-player zero-sum asymmetric repeated game is specified by a five-tuple \((K, A, B, M, p_0)\), where

- \( K \) is a non-empty finite set, called the state set, the elements of which are called states.
- \( A \) and \( B \) are non-empty finite sets, called player 1 and 2’s action sets, respectively.
- \( M : K \times A \times B \to \mathbb{R} \) is the one-stage payoff function. \( M^k \) indicates the payoff matrix given state \( k \in K \). The matrix element \( M^k_{a,b} \), also denoted as \( M(k, a, b) \), is the payoff given state \( k \in K \), player 1’s action \( a \in A \), and player 2’s action \( b \in B \). The notation \( M^k_{a,:) \) indicates the row vector payoff given state \( k \) and player 1’s action \( a \in A \).
- \( p_0 \in \Delta(K) \) is the initial probability on \( K \). We assume that \( p_0^k > 0 \) for any \( k \in K \).

A \( T \)-stage asymmetric repeated game is played as follows. Let \( a_t, b_t \) denote the actions of player 1 and player 2 for stages \( t \in \{1, 2, ..., T\} \), respectively. At stage \( t = 1 \), a state \( k \) is chosen once and for all according to the probability distribution \( p_0 \), and communicated to player 1 only. Player 1 and 2 are called the informed and the uninformed player, respectively. Each player chooses his action independently, and the pair \((a_t, b_t)\) is observed by both players. At stage \( t = 2 \), both players again simultaneously choose their actions, and these are observable by both players. The payoff of player 1 (and penalty to player 2) at stage \( t \) is \( M^k_{a_t,b_t} \). The process is repeated for the remaining \( t = 2, 3, ..., T \). These payoffs/penalties are not observed by player 2.

More formally, we will use the concept of behavioral strategies. For any stage \( t = 1, \ldots, T \), the histories of player 1 and 2’s actions prior to time \( t \) are denoted by \( h^A_t = \{a_1, \ldots, a_{t-1}\} \) and \( h^B_t = \{b_1, \ldots, b_{t-1}\} \), respectively. For \( t = 1 \), the null histories are denoted \( h^A_1 = h^B_1 = 0 \). The corresponding set of possible action sequences are denoted by \( H^A_t = A^{t-1} \) and
A behavioral strategy for player 1 is a collection of mappings \( \sigma = (\sigma_t)_{t=1}^T \), where each \( \sigma_t \) is a map from \( K \times H^A_t \times H^B_t \) to \( \Delta(A) \). Similarly, but taking into account the lack of information on the state \( k \in K \), a behavioral strategy for player 2 is a collection of mappings \( \tau = (\tau_t)_{t=1}^T \), where \( \tau_t \) is a map from \( H^A_t \times H^B_t \) to \( \Delta(B) \). Denote by \( \Sigma \) and \( \mathcal{T} \) the sets of behavioral strategies of player 1 and 2, respectively. The values \( \sigma_t^a(k, h^A_t, h^B_t) \) for \( a \in A \) and \( \tau_t^b(h^A_t, h^B_t) \) for \( b \in B \) denote the probabilities of playing \( a \) and \( b \) at stage \( t \), respectively, given the histories \( h^A_t \in H^A_t \) and \( h^B_t \in H^B_t \), and realized state, \( k \in K \).

Play proceeds as follows. As previously stated, at stage \( t = 1 \), a state \( k \) is chosen once and for all according to the probability distribution \( p_0 \). The action \( a_1 \) is a randomized outcome according to the behavioral strategy distribution \( \sigma_1(k, \emptyset, \emptyset) \in \Delta(A) \), and the action \( b_1 \) is a randomized outcome according to the behavioral strategy distribution \( \tau_1(0, \emptyset) \). At stage \( t = 2, ..., T \), the action \( a_t \) is a randomized outcome according to the behavioral strategy distribution \( \sigma_t(k, h^A_t, h^B_t) \in \Delta(A) \), and the action \( b_t \) is a randomized outcome according to the behavioral strategy distribution \( \tau_t(h^A_t, h^B_t) \), where we assume that these outcomes are conditionally independent given \( h^A_t \) and \( h^B_t \).

A triple \((p_0, \sigma, \tau)\) induces a probability distribution \( P_{p_0,\sigma,\tau} \) on the set \( \Omega = K \times (A \times B)^T \) of plays. Let \( E_{p_0,\sigma,\tau} [\cdot] \) denote the corresponding expectation. The payoff with initial probability \( p_0 \) and strategies \( \sigma \) and \( \tau \) of the \( T \)-stage asymmetric information repeated game is defined as

\[
\gamma_T(p_0, \sigma, \tau) = E_{p_0,\sigma,\tau} \left[ \sum_{t=1}^{T} M(k, a_t, b_t) \right].
\]

The \( T \)-stage game \( \Gamma_T(p_0) \) is defined as the two-player zero-sum asymmetric repeated game equipped with initial distribution \( p_0 \), strategy spaces \( \Sigma \) and \( \mathcal{T} \), and payoff function \( \gamma_T(p_0, \sigma, \tau) \). In game \( \Gamma_T(p_0) \), the informed player seeks to maximize the payoff \( \gamma_T(p_0, \sigma, \tau) \), while the uninformed player seeks to minimize it.

For the \( T \)-stage game \( \Gamma_T(p_0) \), the security level \( V_T(p_0) \) of the informed player is defined as

\[
V_T(p_0) = \max_{\sigma \in \Sigma} \min_{\tau \in \mathcal{T}} \gamma_T(p_0, \sigma, \tau),
\]

and the strategy \( \sigma^* \in \Sigma \) which achieves the security level is called the security strategy of the informed player. Similarly, the security level \( \bar{V}_T(p_0) \) of the uninformed player is defined as

\[
\bar{V}_T(p_0) = \min_{\tau \in \mathcal{T}} \max_{\sigma \in \Sigma} \gamma_T(p_0, \sigma, \tau),
\]

and the strategy \( \tau^* \in \mathcal{T} \) which achieves the security level is called the security strategy of the uninformed player. When \( V_T(p_0) = \bar{V}_T(p_0) \), we say game \( \Gamma_T(p_0) \) has a value, i.e. there exists a Nash equilibrium. Since the game \( \Gamma_T(p_0) \) is a finite game, the game value always exists, and is denoted by \( V_T(p_0) \).

### B. \( H^B \) independent strategies

A fundamental difference between a repeated asymmetric game and a one-shot asymmetric game is that in the repeated asymmetric game, the uninformed player can learn the system state from the informed player’s actions. Indeed, the uninformed player’s belief about the system state plays an important role for both players to make decisions. Since only the informed player’s actions are directly related to the system state, the uninformed player’s history action sequence doesn’t provide extra information about the system state given the informed player’s history action sequence. Therefore, it is not surprising to see that given informed player’s history action sequence, both players’ security strategies are independent of the uninformed player’s history action sequence. Let’s define an \( H^B \) independent behavioral strategy of player 1 as a collection of mappings \( \bar{\sigma} = (\bar{\sigma}_t)_{t=1}^T \) where each \( \bar{\sigma}_t \) is a map from \( K \times H^A_t \) to \( \Delta(A) \). Similarly, an \( H^B \) independent behavioral strategy of player 2 is a collection of mappings \( \bar{\tau} = (\bar{\tau}_t)_{t=1}^T \) where \( \bar{\tau}_t \) is a map from \( H^A_t \) to \( \Delta(B) \). Denote by \( \bar{\Sigma} \) and \( \bar{\mathcal{T}} \) the sets of \( H^B \) independent behavior strategies of player 1 and 2. Clearly, \( \bar{\Sigma} \) and \( \bar{\mathcal{T}} \) are subsets of \( \Sigma \) and \( \mathcal{T} \), respectively.

**Proposition 1** (**14, 5, 6**). Consider a two-player zero-sum \( T \)-stage asymmetric repeated game \( \Gamma_T(p_0) \). Each player has a security strategy in game \( \Gamma_T(p_0) \) that is independent of player 2’s history action sequence, i.e.

\[
\max_{\sigma \in \Sigma} \min_{\tau \in \mathcal{T}} \gamma_T(p_0, \sigma, \tau) = \max_{\bar{\sigma} \in \bar{\Sigma}} \min_{\bar{\tau} \in \bar{\mathcal{T}}} \gamma_T(p_0, \bar{\sigma}, \bar{\tau})
\]

\[
\min_{\tau \in \mathcal{T}} \max_{\sigma \in \Sigma} \gamma_T(p_0, \sigma, \tau) = \min_{\bar{\tau} \in \bar{\mathcal{T}}} \max_{\bar{\sigma} \in \bar{\Sigma}} \gamma_T(p_0, \bar{\sigma}, \bar{\tau}).
\]

If one player’s behavior strategy is independent of the uninformed player’s history action sequence, then the other player’s best response to the \( H^B \) independent strategy is independent of the uninformed player’s history action sequence, too.

**Proposition 2**. Consider a two-player zero-sum \( T \)-stage asymmetric repeated game \( \Gamma_T(p_0) \). For any \( \bar{\sigma} \in \bar{\Sigma} \), and any \( \bar{\tau} \in \bar{\mathcal{T}} \),

\[
\min_{\bar{\tau} \in \bar{\mathcal{T}}} \gamma_T(p_0, \bar{\sigma}, \bar{\tau}) = \min_{\bar{\tau} \in \bar{\mathcal{T}}} \gamma_T(p_0, \bar{\sigma}, \bar{\tau})
\]

\[
\max_{\bar{\sigma} \in \bar{\Sigma}} \gamma_T(p_0, \bar{\sigma}, \bar{\tau}) = \max_{\bar{\sigma} \in \bar{\Sigma}} \gamma_T(p_0, \bar{\sigma}, \bar{\tau}).
\]
Proof. Since $T \in T$, we have $\min_{\tau \in T} \gamma_T(p_0, \bar{\sigma}, \bar{\tau}) \leq \min_{\tau \in T} \gamma_T(p_0, \bar{\sigma}, \bar{\tau})$. Meanwhile, for any $\tau \in T$, we can design $\bar{\pi}_t^0(h^A_t) = \sum_{h^B_t \in H^B_t} \bar{\pi}_t^0(h^A_t, h^B_t)$ for all $t = 1, \ldots, T$, such that $\gamma_T(p_0, \bar{\sigma}, \bar{\tau}) = \gamma_T(p_0, \bar{\sigma}, \bar{\tau})$. Hence, we have $\min_{\tau \in T} \gamma_T(p_0, \bar{\sigma}, \bar{\tau}) \geq \min_{\tau \in T} \gamma_T(p_0, \bar{\sigma}, \bar{\tau})$. Therefore, equation (1) is shown.

Similarly, $\Sigma \in \Sigma$ implies that $\max_{\sigma \in \Sigma} \gamma_T(p_0, \sigma, \bar{\tau}) \geq \max_{\sigma \in \Sigma} \gamma_T(p_0, \sigma, \bar{\tau})$. Meanwhile, for any $\sigma \in \Sigma$, we can design $\bar{\sigma}_t^\sigma(k, h^A_t) = \frac{\sum_{h^B_t \in H^B_t} \bar{\sigma}_t^\sigma(k, h^A_t, h^B_t) \prod_{t=1}^{\tau} \Sigma_{h^A_t}((h^A_t))}{\sum_{h^B_t \in H^B_t} \prod_{t=1}^{\tau} \Sigma_{h^B_t}((h^B_t))}$ for all $t = 1, \ldots, T$, such that $\gamma_T(p_0, \sigma, \bar{\tau}) = \gamma_T(p_0, \sigma, \bar{\tau})$, which implies that $\max_{\sigma \in \Sigma} \gamma_T(p_0, \sigma, \bar{\tau}) \leq \max_{\sigma \in \Sigma} \gamma_T(p_0, \sigma, \bar{\tau})$. Therefore, equation (2) is shown. \hfill \Box

Proposition 1 and 2 imply that when computing players’ security strategies, we can ignore the uninformed player’s history action sequence, which greatly reduces the number of both players’ information sets in the extensive game tree, and hence reduces the computational complexity of the security strategies.

C. LP formulations of security strategies

A $T$-stage asymmetric information repeated game, as a finite game, can always be expressed as a finite extensive game tree[13]. Assuming perfect recall, we can use sequence form to construct a linear program to compute the security strategy. Moreover, the size of the linear program is linear in the size of the game tree, and hence polynomial in the size of the uninformed player’s action set [10]. In our case, the analysis in subsection 4.E indicates that both players can ignore the uninformed player’s history action sequence when making decisions. In other words, the uninformed player can forget what he did before, which violates the perfect recall assumption in the sequence form. Here, we will adopt the realization plan in the sequence form, and take advantage of the $H^B$ independent strategies to develop LP formulations with reduced computational complexity to compute the $H^B$ independent security strategies.

As in the sequence form, we define the realization plan $q_t(h^A_t; k)$ of the informed player’s history action sequence $h^A_t$ given state $k$ at stage $t$ as

$$q_t(h^A_t; k) = \prod_{s=1}^{t-1} \bar{\sigma}_s^{a_s}(k, h^A_s),$$

(3)

where $a_s$ and $h^A_s$ are the informed player’s action and history action sequence at stage $s$ in the history action sequence $h^A_t$, denoted by $a_s, h^A_s \in h^A_t$. Therefore, the realization plan $q_t$ satisfy the following constraints:

$$q_t(h^A_t; k) = 1, \quad \forall k \in K,$$

(4)

$$\sum_{a_t \in A} q_{t+1}(h^A_t, a_t; k) = q_t(h^A_t; k), \quad \forall k \in K, h^A_t \in H^A_t, \forall t = 1, \ldots, T,$$

(5)

$$q_t(h^A_t; k) \geq 0, \quad \forall k \in K, h^A_t \in H^A_t, \forall t = 2, \ldots, T + 1,$$

(6)

where $(h^A_t, a_t)$ indicates concatenation. A realization plan of the informed player is a collection of the informed player’s realization plans $q = (q_t)_{t=1}^T$ at all stages. Indeed, the realization plan $q_t(h^A_t; k)$ is the conditional probability $P[h^A_t | k]$. The set of realization plans of the informed player is denoted by $Q$, including all properly dimensioned real vectors satisfying equation (4\textcircled{6}).

A very important difference between a one-shot game and a repeated game is that the uninformed player can learn the system state from the informed player’s history actions. The informed player can characterize his revelation of information by the posterior probability $P[k | h^A_t]$, which is also called the belief state of player 2. Let $p_t \in \Delta(K)$ denote the posterior probability over the system state $k \in K$ at stage $t$ given $h^A_t$, i.e. $p_t(h^A_t) = P[k | h^A_t]$. The belief state $p_{t+1}$ at stage $t + 1$ can be computed recursively as a function of $p_t$, the informed player’s strategy $x^k_t = \bar{\sigma}_t(k, h^A_t)$, and the informed player’s realized action $a_t$ based on the Bayesian law. Therefore, we have

$$p_t(h^A_t) = \pi(p_t, x_t, a_t) = \frac{p_t(h^A_t)x^k_t(a_t)}{\bar{x}_{p_t, x_t}(a_t)}, \text{ with } p_1 = p \text{ in game } \Gamma_T(p)$$

(7)

where $x^k_t(a_t) = \bar{\sigma}_t^\sigma(k, h^A_t)$, and $\bar{x}_{p_t, x_t}(a_t) = \sum_{k \in K} p_t(h^A_t)x^k_t(a_t)$. The variable $\bar{x}$ can be seen as the weighted average of $x_t$. Based on the belief state, the value function $V_T(p)$ satisfies a backward recursive equation which is similar to the Bellman’s equation [11], [3].

$$V_t(p) = \max_{x \in \Delta(A)} \min_{y \in \Delta(B)} \sum_{k \in K} p_t^k x^k T M_k y + \sum_{a_t \in A} \bar{x}_{p_t, x_t}(a_t) V_{t-1}(\pi(p, x, a_t)).$$

(8)

Based on the realization plan $q_t$ and the backward recursive formula [3], we construct a linear program to compute the security strategy for the informed player.
Theorem 3. Consider a two-player zero-sum $T$-stage asymmetric repeated game $\Gamma_T(p)$. The game value $V_T(p)$ satisfies

$$V_T(p) = \max_{q, \ell \in Q, L} \sum_{t=1}^{T} \sum_{h^A_t \in H^A_t} \ell_{h^A_t}$$

s.t. \( \sum_{k \in H, a \in A} p^k q_{t+1} ((h^A_t, a); k) M_{a, z} \geq \ell_{h^A_t} 1^T, \forall t = 1, \ldots, T, \forall h^A_t \in H^A_t. \) \hspace{1cm} (9)

where $Q$ is a set including all properly dimensioned real vectors satisfying (10). $L$ is a properly dimensioned real space, and $(h^A_t, a)$ indicates concatenation. The informed player’s security strategy is linear in $(\bar{h}^A_t, a)$, $t = 1, \ldots, T$, $\forall h^A_t \in H^A_t$. \hspace{1cm} (10)

Proof. By the duality theorem, it is easy to see that equation (9-10) is true for $T = 1$. Let’s assume that $V_{t-1}(p)$ satisfies (9-10) for all $t = 2, \ldots$. According to Lemma III.1 of [16], we have $\bar{x}_{p, a}(1)V_{t-1}(p, a) = V_{t-1}(\bar{x}_{p, a}^T, p, a)).$ Therefore, the second term of (9) satisfies

$$\sum_{a_1 \in A} \bar{x}_{p, a}(1)V_{t-1}(p, a, a_1) = \sum_{a_1 \in A} \max_{q, \ell, a, h^A_t} \sum_{t=1}^{t-1} \sum_{h^A_t \in H^A_t} \ell_{h^A_t}$$

s.t. \( \sum_{k \in K} p^k q_{t+1} ((h^A_t, a); k) \geq \ell_{h^A_t} 1^T, \forall s = 1, \ldots, t, s \in A. \hspace{1cm} (11)

Let $s = s' + 1$ and $h^A_{s'} = (a_1, h^A_{s'}).$ we have

$$\sum_{a_1 \in A} \bar{x}_{p, a}(1)V_{t-1}(p, a, a_1) = \max_{q, \ell, a, h^A_t} \sum_{t=1}^{t-1} \sum_{h^A_t \in H^A_t} \ell_{h^A_t}$$

s.t. \( p^k q_{t+1} ((h^A_t, a); k) \geq \ell_{h^A_t} 1^T, \forall s = 2, \ldots, t, h^A_s \in H^A_s. \)

By the duality theorem, it is easy to verify that

$$\min_{y \in \Delta(B)} \sum_{k \in K} p^k x^k T M^k y = \max_{\ell \in R} \ell_{h^A_t}$$

s.t. \( \sum_{k \in K} p^k x^k T M^k \geq \ell_{h^A_t} 1^T. \)

According to equation (9), and with the fact that $x^k(a_1) = q_2(a_1; k)$, we show that equation (9-10) still holds for $V_t(p)$ for $t = 2, \ldots$. Once we get the optimal solution $q^*$, according to (3), the security strategy of the informed player can be computed according to (11).

Our LP formulation of informed player’s security strategy has its size linear in the size of the state set and the size of uninformed player’s action set, polynomial in the size of informed player’s action set, and exponential in time horizon. Let’s first analyze the variable size. Variable $\bar{x}$ consists of $(q_t)_{t=1}^{T-1}$, where $q_t$ is of size $|K| \times |H^A_t| = |K| \times |A|^{t-1}$, and hence $q$ consists of $|K| |1 + A| + \cdots + |A| T = O(|K| A|^{T-1})$ scalars. Variable $\ell$ consists of $|(1 + |A|)^{t-1} = O(|A| T)$ scalars. In all, we see that the LP formulation has $O(|K| A|^{T-1})$ scalar variables. Next, let’s take a look at the constraint size. Constraint (10) includes $|K|$ equations. Constraint (11) includes $|K| |1 + A| + \cdots + |A| T = O(|K| A|^{T-1})$ equations. Constraint (12) includes $|K| |1 + A| + \cdots + |A| T = O(|K| A|^{T-1})$ equations. In all, there are $O(|K| + |B| A|^{T-1})$ equations. Therefore, the size of the LP formulation to compute the informed player’s security strategy is linear in $|K|$ and $|B|$, polynomial in $|A|$, and exponential in $T$.

Next, let’s take a look at the uninformed player’s security strategy. Define the conditional expected total payoff $u(\tilde{\tau}; k, h^A_{T+1})$ given uninformed player’s strategy $\tilde{\tau} \in \tilde{T}$, state $k \in K$, and informed player’s history action sequence $h^A_{T+1} \in H^A_{T+1}$ as

$$u(\tilde{\tau}; k, h^A_{T+1}) = \mathbb{E}_T \left[ \sum_{t=1}^{T} M(k, a_t, b_t) h^A_{T+1} \right]. \hspace{1cm} (12)$$

It is easy to show that

$$u(\tilde{\tau}; k, h^A_{T+1}) = \sum_{t=1}^{T} M_{a_t} y_{h^A_{t+1}} \hspace{1cm} (13)$$

where $y_{h^A_{t+1}} = \tilde{\tau}(h^A_{t+1})$, and $a_t, h^A_t \in H^A_{T+1}$. We notice that $u(\tilde{\tau}; k, h^A_{T+1})$ is a linear function of $\tilde{\tau}$, or in other words, $y$. 
Theorem 4. Consider a two-player zero-sum $T$-stage asymmetric repeated game $\Gamma_T(p)$. The game value $V_T(p)$ satisfies
\[
V_T(p) = \min_{y \in Y, \ell \in \mathbb{R}^{|K|}} p^T \ell \\
\text{s.t. } u(y; k, :) \leq \ell^k 1, \quad \forall k \in K_
1^T y_{h_t^A} = 1, \quad \forall h_t^A \in H_t^A, \forall t = 1, \ldots, T, \quad \forall h_t^A \in H_t^A, \forall t = 1, \ldots, T, \quad \forall h_t^A \in H_t^A, \forall t = 1, \ldots, T,
\]
where $Y$ is a properly dimensioned real space, and $u(y; k, :) = [H_{t+1}^A]$ dimensional column vector whose element is $u(y; k, h_{t+1}^A)$, a linear function of $y$ satisfying equation (13). The uninformed player’s security strategy $\nu^k(\bar{\tau}^*; h_t^A)$ is $y^*(y^t, h_t^A)$.

Proof. Let’s define $\nu_T^k(\bar{\tau}) = \max_{\sigma(k) \in \Sigma(k)} E_{\sigma, \bar{\tau}} \left[ \sum_{t=1}^T M_{a_t, b_t} |k| \right]$, where $\sigma(k)$ indicates the informed player’s $H^B$ independent behavior strategy given the system state $k \in K$ and $\Sigma(k)$ is the corresponding set including all possible $\sigma(k)$. We have
\[
\nu_T^k(\bar{\tau}) = \max_{\sigma(k) \in \Sigma(k)} \sum_{h_{t+1}^A \in H_{t+1}^A} P [h_{t+1}^A |k] u(\bar{\tau}; k, h_{t+1}^A) = \max_{q_{T+1}(h_{T+1}^A)} \sum_{h_{T+1}^A \in H_{T+1}^A} q_{T+1}(h_{T+1}^A; k) u(\bar{\tau}; k, h_{T+1}^A).
\]
According to the duality theorem, we have
\[
\mu_T^k(\bar{\tau}) = \min_{\ell \in \mathbb{R}} \ell^k \text{s.t. } u(\bar{\tau}; k, :) \leq \ell^k 1.
\]
The game value $V_T(p)$ satisfies
\[
V_T(p) = \min_{\bar{\tau} \in \mathbb{R}^T} \sum_{k \in K} p^k \mu_T^k(\bar{\tau}) = \min_{y \in Y, \ell \in \mathbb{R}^{|K|}} \sum_{k \in K} p^k \ell^k \text{s.t. } u(y; k, :) \leq \ell^k 1, \forall k \in K.
\]

The LP formulation of the uninformed player’s security strategy has its size linear in the size of the state set and his own action set, polynomial in the size of the informed player’s action set, and exponential in time horizon. We first analyze the variable size. Variable $y$ includes $(y_t)_{t=1}^T$, where $y_t$ is of size $|B||A^{t-1}|$, and hence $y$ has $|B|(1 + |A| + \cdots + |A|^{T-1}) = O(|B||A|^T)$ scalar variables. Variable $\ell$ is of size $|K|$. In all, the variable size is in the order of $|B||A|^T + |K|$. We then study the constraint size. Constraint (16) consists of $(1 + |A| + \cdots + |A|^{T-1}) = O(|A|^T)$ equations. Constraint (17) consists of $|B|(1 + |A| + \cdots + |A|^{T-1}) = O(|B||A|^T)$ equations. Constraint (18) consists of $|A|^T$ equations. In all, the constraint size is of order $O((|B| + |K| + 1)|A|^T)$. Therefore, the size of the LP formulation to compute the uninformed player’s security strategy is linear in $|K|$ and $|B|$, polynomial in $|A|$, and exponential in $T$.

III. $\lambda$-DISCOUNTED ASYMMETRIC REPEATED GAMES

In finite-stage asymmetric information repeated games, the security strategies of the players depends on the informed player’s history actions. As the time horizon gets long, players need a large amount of memories to record the history actions. Since the horizon of a $\lambda$-discounted asymmetric repeated game is infinite, it is necessary for players to find fixed-sized sufficient statistics for decision making. After figuring out the fixed-sized sufficient statistics, we find that players’ security strategies in $\lambda$-discounted asymmetric repeated game are still hard to compute, and hence approximated security strategies with guaranteed performance are provided. This section talks about the sufficient statistics and the approximated security strategies player by player.

A. Setup

A two-player zero-sum $\lambda$-discounted asymmetric repeated game is specified by the same five-tuple $(K, A, B, M, p_0)$ and played in the same way as described in the two-player zero-sum $T$-stage asymmetric repeated game. The payoff of player $1$ at stage $t$ is $\lambda(1 - \lambda)^{t-1} M(k, a_t, b_t)$ for some $\lambda \in (0, 1)$, and the game is played for infinite horizon. The payoff of the $\lambda$-discounted asymmetric repeated game with initial probability $p_0$ and strategies $\sigma$ and $\tau$ is defined as
\[
\gamma_{\lambda}(p_0, \sigma, \tau) = E_{p_0, \sigma, \tau} \left( \sum_{t=1}^{\infty} \lambda(1 - \lambda)^{t-1} M(k, a_t, b_t) \right).
\]
The $\lambda$-discounted game $\Gamma_{\lambda}(p_0)$ is defined as a two-player zero-sum asymmetric repeated game equipped with initial distribution $p_0$, strategy spaces $\Sigma$ and $\mathcal{T}$, and payoff function $\gamma_{\lambda}(p_0, \sigma, \tau)$. The security strategies $\sigma^*$ and $\tau^*$, and security levels $V^\lambda_{\lambda}(p_0)$ and $V^\lambda_{\lambda}(p_0)$ are defined in the same way as in $T$-stage game in Section [11] for player 1 and 2, respectively. Since $\gamma_{\lambda}(p_0, \sigma, \tau)$ is bilinear over $\sigma$ and $\tau$, $\Gamma_{\lambda}(p_0)$ has a value $V_{\lambda}(p_0)$ according to Sion’s minimax Theorem, i.e. $V_{\lambda}(p_0) = V^\lambda_{\lambda}(p_0) = V^\lambda_{\lambda}(p_0)$ [6].

B. The informed player

1) The informed player’s security strategy: The belief state $p_t$ in (7) plays an important role in decision making of the informed player. Indeed, in a $\lambda$-discounted asymmetric repeated game $\Gamma_{\lambda}(p)$, the belief state $p_t$ is sufficient statistics of the informed player.

**Proposition 5 ([8]).** Consider a two-player zero-sum $\lambda$-discounted asymmetric repeated game $\Gamma_{\lambda}(p)$. The game value $V_{\lambda}(p)$ satisfies

$$V_{\lambda}(p) = \max_{x \in \Delta(A)_{\lambda}} \min_{y \in \Delta(B)} \left( \lambda \sum_{k \in K} p^k x^k T M^k y + (1 - \lambda) T_{p,x}(V_{\lambda}) \right),$$  \hspace{1cm} (20)

where

$$T_{p,x}(V_{\lambda}) = \sum_{a \in A} \bar x_{p,x} V_{\lambda}(\pi(p,x,a)).$$  \hspace{1cm} (21)

Moreover, the informed player has a security strategy that depends only on the belief state $p_t$ at each stage $t$, and is independent of the uninformed player’s history action sequence.

First of all, Proposition [5] points out that the informed player’s security strategy is independent of the uninformed player’s history action, just as what it is in $T$-stage game. Following the same steps, we can show that the uninformed player’s best response to an $H^B$ independent strategy is also $H^B$ independent. Second, Proposition [5] provides the sufficient statistics $p_t$ of the informed player. So the informed player only needs to record $p_t \in \Delta(K)$ instead of all of his own history actions. Finally, given the belief state $p_t$, Proposition [5] gives a Bellman-like equation (20) to compute the informed player’s security strategy. Unfortunately, computing the value $V_{\lambda}(p)$ and the informed player’s corresponding security strategy $\sigma^*$ is non-convex [12], [13]. Therefore, we need to find an approximated security strategy that is easy to compute, and has some performance guarantee.

2) The informed player’s approximated security strategy: One way to approximate the strategy is to approximate the game value $V_{\lambda}(p)$ first, and then compute the security strategy based on the approximated game value. Here, we will use the game value $V_{\lambda,T}(p)$ of a $\lambda$-discounted $T$-stage asymmetric repeated game $\Gamma_{\lambda,T}(p)$ to approximate the game value $V_{\lambda}(p)$. A $\lambda$-discounted $T$-stage asymmetric repeated game $\Gamma_{\lambda,T}(p)$ is the same as the $\lambda$-discounted asymmetric repeated game $\Gamma_{\lambda}(p)$ except that the game is only played for $T$ stages. Following the standard arguments as in the proof of Proposition [5], we see that the game value $V_{\lambda,T+1}(p)$ of the $\lambda$-discounted $T$-stage game $\Gamma_{\lambda,T+1}(p)$ satisfies the recursive formula as below.

$$V_{\lambda,T+1}(p) = \max_{x \in \Delta(A)_{\lambda}} \min_{y \in \Delta(B)} \left( \lambda \sum_{k \in K} p^k x^k T M^k y + (1 - \lambda) T_{p,x}(V_{\lambda,T}) \right), \text{ with } V_{\lambda,0}(p) \equiv 0.$$  \hspace{1cm} (22)

Before we go ahead to provide the approximated security strategy based on this approximated game value, we are interested in how good the approximated game value is, and how fast it converges to the real game value. To this purpose, we define an operator $F_x$ as

$$F^V_x(p) = \min_{y \in \Delta(B)} \left\{ \lambda \sum_{k \in K} p^k x^k T M^k y + (1 - \lambda) T_{p,x}(V) \right\}.$$  \hspace{1cm} (23)

It’s clear that $V_{\lambda}(p) = \max_{x \in \Delta(A)_{\lambda}} F^V_x(p)$, and $V_{\lambda,T+1}(p) = \max_{x \in \Delta(A)_{\lambda}} F^V_{\lambda,T}(p)$. The operator $F_x$ is actually a contraction mapping.

**Lemma 6.** Let $\mathcal{V}$ be the set of functions mapping from $\Delta(K)$ to $\mathbb{R}$. Given any $x \in \Delta(A)_{\lambda}$ and $\lambda \in (0, 1)$, the operator $F_x : \mathcal{V} \to \mathcal{V}$ defined in (22) is a contraction mapping with contraction constant $1 - \lambda$, i.e.

$$\|F^V_{x1} - F^V_{x2}\|_{\sup} \leq (1 - \lambda) \|V_1 - V_2\|_{\sup}, \forall V_1, V_2 \in \mathcal{V}.$$ 

**Proof.** Since the second term of mapping $F_x$ in equation (23) is irrelevant to $y$, we have

$$F^V_{x1}(p) = \min_{y \in \Delta(B)} \left\{ \lambda \sum_{k \in K} p^k x^k T M^k y + (1 - \lambda) T_{p,x}(V_1) \right\},$$

$$F^V_{x2}(p) = \min_{y \in \Delta(B)} \left\{ \lambda \sum_{k \in K} p^k x^k T M^k y + (1 - \lambda) T_{p,x}(V_2) \right\}.$$
Proof. From equation (20) and (22), we have
\[
T \quad \text{Theorem 7. Given}
\]
\[\bar{\sigma}_H(t) \quad \text{is the solution to}
\]
\[
\lambda \sum_{k \in K} p_k^T M_k y + (1 - \lambda) T_{p,x} (\bar{\sigma}(t)),
\]
where \( \bar{\sigma}(t) \) is a \([A] \times [K]\) matrix whose \( k \)th column is \( \bar{\sigma}_H(k, p) \). Clearly, \( \bar{\sigma}_H(t) \) can be also seen as player 1’s security strategy at stage 1 in the \( \lambda \)-discounted \( T + 1 \)-stage asymmetric repeated game \( \Gamma_{\lambda, T+1}(p) \). Following the same steps as in Theorem 3 we can construct a linear program to compute the approximated game value \( V_{\lambda, T+1}(p) \) and the corresponding approximated security strategy \( \bar{\sigma}_H(k, p) \).

Theorem 8. Consider a two-player zero-sum \( \lambda \)-discounted asymmetric game \( \Gamma_{\lambda}(p) \). The approximated game value \( V_{\lambda, T+1}(p) \) satisfies
\[
V_{\lambda, T+1}(p) = \max_{q \in \mathbb{Q}} \sum_{t=1}^{T+1} \sum_{h_t^A \in H_t^A} \lambda (1 - \lambda)^{t-1} \ell_{h_t^A}^A
\]
\[
\sum_{k \in K, a \in A} q_{t+1}(k, (h_t^A, a)) M_{k}^A \geq \ell_{h_t^A}^A 1^T, \forall t = 1, 2, \ldots, T + 1, h_t^A \in H_t^A.
\]
where \( q \in \mathbb{Q} \) is a set including all properly dimensioned real vectors satisfying (41). \( L \) is a properly dimensioned real space, and \((h_t^A, a)\) corresponds to concatenation. The approximated security strategy
\[
\bar{\sigma}_H^A(k, p) = q_k^*(a; k), \forall a \in A.
\]
3) The performance analysis of the informed player’s approximated security strategy: Now that we can compute the informed player’s approximated security strategy, the next question is which performance this strategy can guarantee. To this purpose, we first define the security level $J^{\bar{\sigma},T}(p)$ guaranteed by the approximated security strategy $\bar{\sigma}_{\lambda,T}$ as

$$J^{\bar{\sigma},T}(p) = \min_{\bar{\sigma}} \gamma_{\lambda}(p, \bar{\sigma}_{\lambda,T}, T).$$

Since $\bar{\sigma}_{\lambda,T}$ is a stationary strategy, according to the standard procedure of dynamic programming, its security level $J^{\bar{\sigma},T}$ has the following property.

**Lemma 9.** Let $\bar{\sigma} \in \bar{\Sigma}$ be the informed player’s stationary strategy that depends only on the belief state $p_{\tau}$ besides the state $k \in K$. The security level $J^{\bar{\sigma}}$ of $\bar{\sigma}$ satisfies $J^{\bar{\sigma}}(p) = \text{F}^{\bar{J}^{\bar{\sigma}}}_{\bar{\sigma}(\cdot; p)}(p)$.

**Proof.** Since player 1’s strategy is fixed to be $\bar{\sigma}$, the discounted game $\Gamma_{\lambda}$ becomes a discounted optimization problem, and hence satisfies Bellman’s principle, i.e.

$$J^{\bar{\sigma}}(p) = \min_{y \in \Delta(M)} \left( \lambda \sum_{k \in K} p^{k} \bar{\sigma}(k, p)^{T} M^{k} y + (1 - \lambda) \sum_{a \in A} \bar{\bar{\sigma}}_{p, \bar{\sigma}(\cdot; p)}(a) J^{\bar{\sigma}}(\bar{\pi}(p, \bar{\sigma}(\cdot; p), a)) \right)$$

$$= \min_{y \in \Delta(M)} \left( \lambda \sum_{k \in K} p^{k} \bar{\sigma}(k, p)^{T} M^{k} y + (1 - \lambda) \bar{T}_{p, \bar{\sigma}(\cdot; p)}(J^{\bar{\sigma}}) \right)$$

$$= \text{F}^{\bar{J}^{\bar{\sigma}}}_{\bar{\sigma}(\cdot; p)}(p). \quad \square$$

Now, we are ready to show that the difference between the suboptimal strategy’s security level $J^{\bar{\sigma},T}$ and the game value is bounded from above, which is stated in the following theorem.

**Theorem 10.** The suboptimal policy $\bar{\sigma}_{\lambda,T}$ defined in equation (28) guarantees a security level $J^{\bar{\sigma},T}$ satisfying

$$\|V_{\lambda} - J^{\bar{\sigma},T}\|_{\text{sup}} \leq \frac{2(1 - \lambda)}{\lambda} \|V_{\lambda} - V_{\lambda,T}\|_{\text{sup}}.$$ (30)

**Proof.** Lemma 9 indicates that

$$|V_{\lambda}(p) - J^{\bar{\sigma},T}(p)| \leq |V_{\lambda}(p) - V_{\lambda,T+1}(p)| + |V_{\lambda,T+1}(p) - J^{\bar{\sigma},T}(p)|$$

$$= |V_{\lambda}(p) - V_{\lambda,T+1}(p)| + |\text{F}^{V_{\lambda,T}}_{\bar{\sigma}_{\lambda,T}(\cdot; p)}(p) - \text{F}^{\bar{J}^{\bar{\sigma},T}}_{\bar{\sigma}_{\lambda,T}(\cdot; p)}(p)|.$$ 

Take the supreme norm on both sides, and use Lemma 6 and Theorem 7. We have

$$\|V_{\lambda} - J^{\bar{\sigma},T}\|_{\text{sup}} \leq (1 - \lambda) \left( \|V_{\lambda} - V_{\lambda,T}\|_{\text{sup}} + \|V_{\lambda,T} - J^{\bar{\sigma},T}\|_{\text{sup}} \right)$$

$$\leq (1 - \lambda) \left( \|V_{\lambda} - V_{\lambda,T}\|_{\text{sup}} + \|V_{\lambda,T} - V_{\lambda}\|_{\text{sup}} + \|V_{\lambda} - J^{\bar{\sigma},T}\|_{\text{sup}} \right),$$

which implies equation (30). \quare

We would like to provide an algorithm to conclude this subsection about the informed player’s approximated security strategy.

**Algorithm 11.** The informed player’s algorithm in $\lambda$-discounted asymmetric repeated game

(i) **Initialization**

- Read payoff matrices $M$, initial probability $p_0$, and system state $k$.
- Set $T$.
- Let $t = 1$, and $p_1 = p_0$.

(ii) **Compute the informed player’s approximated security strategy $\bar{\sigma}_{\lambda,T}$ based on** (28) **where $q^*_{p}$ is the optimal solution of LP (26, 27) with $p = p_0$.

(iii) **Choose an action $a \in A$ according to the probability $\bar{\sigma}_{\lambda,T}(k, p_t)$, and announce it publicly.**

(iv) **Update the belief state $p_{t+1}$ according to** (7).

(v) **Update $t = t + 1$ and go to step (ii).**
C. The uninformed player’s approximated security strategy

Because of the lack of access to the informed player’s strategy, the belief state $p_t$ is not available to the uninformed player, and hence cannot serve as the uninformed player’s sufficient statistics. De Meyer first introduced the dual game of an asymmetric repeated game in [5], and pointed out that the uninformed player’s security strategy in the dual game with a special initial regret is also the uninformed player’s security strategy in the ‘primal’ game. One applausible property of the uninformed player’s security strategy in the dual game is that the security strategy depends only on a fixed-sized sufficient statistics that is fully available to the uninformed player. The questions are what is the ‘special’ initial regret, and how to compute the corresponding security strategy in the dual game. To answer these questions, we first introduce the dual game of an asymmetric repeated game.

1) The uninformed player’s security strategy and the dual game: Given a $\lambda$-discounted asymmetric repeated game $\Gamma_\lambda(p)$, which is also called the primal game hereafter, its dual game $\bar{\Gamma}_\lambda(w)$ is defined with respect to $p$, where $w \in \mathbb{R}^{|K|}$ is called the initial regret. The dual game is played the same way as in the primal game, except that the system state $k \in K$ is chosen by player 1 (informed player) instead of the nature. In the dual game, Player 2 (uninformed player) is still not informed of the system state. Let $p$ be player 1’s strategy to choose the system state, player 1’s payoff or player 2’s penalty in the dual game $\bar{\Gamma}_\lambda(w)$ is defined as

$$\tilde{\gamma}_\lambda(w, p, \sigma, \tau) = E_{p, \sigma, \tau} \left[ w_k + \sum_{t=1}^{\infty} \lambda(1-\lambda)^{t-1} M(k, a_t, b_t) \right].$$

(31)

The $\lambda$-discounted asymmetric repeated dual game $\bar{\Gamma}_\lambda(w)$ has a game value denoted by $\bar{V}_\lambda(w)$ satisfying [5]

$$\bar{V}_\lambda(w) = \min_{\tau \in T} \max_{p \in \Delta(K), \sigma \in \Sigma} \tilde{\gamma}_\lambda(w, p, \sigma, \tau) = \max_{p \in \Delta(K), \sigma \in \Sigma} \min_{\tau \in T} \tilde{\gamma}_\lambda(w, p, \sigma, \tau).$$

(32)

The game value of the dual game and the game value of the primal game are related in the following way [5], [6]. Following the same steps as in Proposition 2, we can show that the informed player’s independent strategies for both players in the rest of this subsection.

Let’s define the anti-discounted regret $w^*_k$ at stage $t$ with respect to state $k$ given informed player’s history action sequence $h^A_t$ as

$$w^*_k(h^A_t) = \frac{E \left[ \sum_{s=1}^{t-1} \lambda (1-\lambda)^{s-1} M^k_{a_s, b_s} \left| k, h^A_t \right. \right]}{1-\lambda}, \forall k \in K.$$

The anti-discounted regret $w^*_k(h^A_t)$ can be computed recursively as

$$w^*_{t+1}(h^A_t, a_t) = \frac{w^*_k(h^A_t) + \lambda M^k_{a_t, h^A_t}}{1-\lambda}, \forall k \in K, \text{with } w_1 = w.$$  

(35)

The anti-discounted regret $w_t$ is indeed the sufficient statistics for the uninformed player in the dual game.

Proposition 12. [5], [6] The value $\bar{V}_\lambda(w)$ of the $\lambda$-discounted dual asymmetric repeated game $\bar{\Gamma}_\lambda(w)$ satisfies

$$\bar{V}_\lambda(w) = \min_{y \in \Delta(B)} \max_{a \in A} (1-\lambda) \bar{V}_\lambda \left( \frac{w + \lambda M_y a}{1-\lambda} \right).$$

where $M_y$ is a $|K| \times |B|$ matrix whose $k$th row is $M^k_{a,y}$. Moreover, the uninformed player has a security strategy that depends only at each stage $t$ on $w_t$.

Meanwhile, it was also shown in [5], [6] that the security strategy of the uninformed player in the dual game $\bar{\Gamma}_\lambda(w^*)$ with some special initial regret $w^*$ is also the security strategy for the uninformed player in the primal game $\Gamma_\lambda(p)$.

Proposition 13. (Corollary 2.10 and 3.25 in [5]) Consider a $\lambda$-discounted asymmetric repeated game $\Gamma_\lambda(p)$ and its dual game $\bar{\Gamma}_\lambda(w)$. Let $w^*$ be the optimal solution to the optimal problem on the right hand side of equation (34). The security strategy for the uninformed player in the dual game $\bar{\Gamma}_\lambda(w^*)$ is also the security strategy for the uninformed player in the primal game $\Gamma_\lambda(p)$.

Proposition 13 indicates that given an initial probability in the primal game, there exists an initial regret in the dual game such that the security strategies of the uninformed player in the primal and the dual games are the same. Therefore, when playing the primal game $\Gamma_\lambda(p)$, the uninformed player can find out the corresponding initial regret $w^*$ in the dual game first, and then play the dual game instead.
2) The special initial regret $w^*$ and its approximation: The next question is what the special initial regret $w^*$ is. Mathematically speaking, $w^*$ is the optimal solution to the problem $\min_{w \in \mathbb{R}^{|K|}} \{ V_\lambda(w) - p^T w \}$. We are also curious about the physical meaning of $w^*$, i.e. what exactly $w^*$ stands for in the primal game $\Gamma_\lambda(p)$. To this purpose, let’s first define the uninformed player’s worst case regret $\mu_\lambda \in \mathbb{R}^{|K|}$ of strategy $\tau \in T$ in the primal game as

$$
\mu_\lambda^k(\tau) = - \max_{\sigma(k) \in \Sigma(k)} \mathbb{E}_{p,\sigma,\tau} \left[ \sum_{t=1}^{\infty} \lambda(1 - \lambda)^{t-1} M^k_{a_t,b_t} | k \right],
$$

where $\sigma(k)$ indicates informed player’s behavior strategy if the system state is $k \in K$, and $\Sigma(k)$ is the corresponding set including all $\sigma(k)$.

The special initial regret $w^*$ is actually the uninformed player’s worst case regret of his security strategy.

**Theorem 14.** Consider a two-player zero-sum $\lambda$-discounted asymmetric repeated game $\Gamma_\lambda(p)$. Let $\tau^*$ be the uninformed player’s security strategy in $\Gamma_\lambda(p)$. An optimal solution $w^*$ to the optimal problem $\min_{w \in \mathbb{R}^{|K|}} \{ V_\lambda(w) - p^T w \}$ is $w^* = \mu_\lambda(\tau^*)$, i.e.

$$
\min_{w \in \mathbb{R}^{|K|}} \{ V_\lambda(w) - p^T w \} = \tilde{V}_\lambda(\mu_\lambda(\tau^*)) - p^T \mu_\lambda(\tau^*). \tag{38}
$$

**Proof.** Equation (34) shows that the left hand side of (38) equals to $V_\lambda(p)$. We will show that the right hand side of (38) equals to $V_\lambda(p)$, too.

First, we have

$$
V_\lambda(p) = \max_{\sigma \in \Sigma} \gamma_\lambda(p, \sigma, \tau^*) = -p^T \mu_\lambda(\tau^*). \tag{39}
$$

Next, we show that

$$
\tilde{V}_\lambda(\mu_\lambda(\tau^*)) = 0. \tag{40}
$$

Equation (33) implies that $\tilde{V}_\lambda(\mu_\lambda(\tau^*)) \geq V_\lambda(p) + p^T \mu_\lambda(\tau^*) = 0$. Meanwhile, for any $p' \in \Delta(K)$, we have

$$
V_\lambda(p') = \min_{\tau \in T} \max_{\sigma \in \Sigma} \gamma_\lambda(p', \sigma, \tau) \leq \max_{\sigma \in \Sigma} \gamma_\lambda(p, \sigma, \tau^*) = -p^T \mu_\lambda(\tau^*). \tag{41}
$$

Notice here that $\tau^*$ is the uninformed player’s security strategy in $\Gamma_\lambda(p)$, and is not necessarily the uninformed player’s security strategy in $\Gamma_\lambda(p')$. Equation (41) implies that $V_\lambda(p) + p^T \mu_\lambda(\tau^*) \leq 0$. Therefore, equation (40) is true. Together with (39), we show that the right hand side of (38) equals to $V_\lambda(p)$, which completes the proof. \[\square\]

The uninformed player’s worst case regret $w^*$ of this security strategy can be seen as the dual variable of the initial probability $p$. The production of the two variables recovers the opposite of the game value (see equation (39)). While the informed player’s security strategy depends only on $p$ and its Bayesian update $p_t$, the uninformed player can fully rely on $w^*$ and its anti-discounted update $\bar{w}_t$ to generate his security strategy. Moreover, the belief state $p_t$ is fully available to the informed player, while the anti-discounted regret $w_t$ is fully available to the uninformed player.

Theorem [14] characterizes the physical meaning of the special initial regret $w^*$. The next question is how to compute it. Unfortunately, computing $w^*$ is difficult, since it relies on the security strategy for the uninformed player and the game value in the primal game, which is non-convex [13]. Therefore, we propose to approximate $w^*$ based on the $\lambda$-discounted $T$-stage asymmetric repeated game $\Gamma_\lambda,T(p)$, a truncated version of the primal game $\Gamma_\lambda(p)$. Let $\bar{\tau}^* \in \bar{T}$ be the security strategy for the uninformed player in $\Gamma_\lambda,T(p)$. The approximation $w^*$ of the special initial regret is $\mu_{\lambda,T}(\bar{\tau}^*)$ which is defined as

$$
\mu_{\lambda,T}(\bar{\tau}^*) = - \max_{\sigma(k) \in \Sigma(k)} \mathbb{E}_{p,\sigma,\bar{\tau}^*} \left[ \sum_{t=1}^{T} \lambda(1 - \lambda)^{t-1} M^k_{a_t,b_t} | k \right].
$$

Similarly to the $T$-stage game, in the $\lambda$-discounted $T$-stage game, we define the conditional expected total payoff $u_{\lambda,T}(\bar{\tau}; k, h_{T+1}^A)$ given uninformed player’s strategy $\bar{\tau} \in \bar{T}$, state $k \in K$ and informed player’s history action sequence $h_{T+1}^A \in H_{T+1}^A$ as

$$
u_{\lambda,T}(\bar{\tau}; k, h_{T+1}^A) = \mathbb{E}_{\bar{\tau}} \left[ \sum_{t=1}^{T} \lambda(1 - \lambda)^{t-1} M^k_{a_t,b_t} | k, h_{T+1}^A \right], \tag{42}
$$

which satisfies

$$
u_{\lambda,T}(\bar{\tau}; k, h_{T+1}^A) = \sum_{t=1}^{T} \lambda(1 - \lambda)^{t-1} M^k_{a_t,b_t}. \tag{43}
$$

Following the same steps as in Theorem [4] we can construct an LP formulation to compute $V_{\lambda,T}(p)$ and $\mu_{\lambda,T}(\bar{\tau}^*)$. 


Theorem 15. Consider a \( \lambda \)-discounted asymmetric repeated game \( \Gamma_\lambda(p) \). The approximated game value \( V_{\lambda,T}(p) \) satisfies
\[
V_{\lambda,T}(p) = \min_{y \in Y, t \in [\mathcal{R}]} \sum_{k \in K} p^k l^k \\
\text{s.t. } u_{\lambda,T}(y,k,:) \leq \ell^k \mathbf{1}, \quad \forall k \in K, \\
y^T h_{\lambda} = 1, \quad \forall h^A \in H^A, \forall t = 1, \ldots, T, \\
y_{\lambda} \geq 0, \quad \forall h^A \in H^A, \forall t = 1, \ldots, T,
\]
where \( Y \) is a properly dimensioned real space, and \( u_{\lambda}(y,k,:) \) is a \(|H^A|\) dimensional column vector whose element is \( u_{\lambda}(y,k,:) \), a linear function of \( y \) satisfying equation (42). The approximated regret \( w^* \) is \(-\ell^*\).

3) The Uninformed player’s approximated security strategy: Now that the approximated initial regret \( w^* \) for the dual game \( \bar{\Gamma}_\lambda(w^*) \) is computed, the next step is to compute the uninformed player’s security strategy in the dual game, which is again non-convex [13]. Similar to what we do in approximating the informed player’s security strategy, we use the game value of a \( \lambda \)-discounted \( T \)-stage dual game \( \bar{\Gamma}_{\lambda,T}(w^*) \) to approximate \( \bar{V}_\lambda(w^*) \), and derive the uninformed player’s approximated security strategy based on the approximated game value.

A \( \lambda \)-discounted \( T \)-stage asymmetric repeated dual game \( \bar{\Gamma}_{\lambda,T}(w) \) is played the same way as a \( \lambda \)-discounted asymmetric repeated dual game \( \bar{\Gamma}_\lambda(w) \) except that \( \bar{\Gamma}_{\lambda,T}(w) \) is only played for \( T \)-stages. Since \( \bar{\Gamma}_{\lambda,T}(w) \) is a finite game, it has a value denoted by \( \tilde{V}_{\lambda,T}(w) \), i.e.
\[
\tilde{V}_{\lambda,T}(w) = \min_{\tau \in T} \max_{p \in \Delta(K)} E_{\tilde{p},\tilde{\sigma},\tau} \left[ w + \sum_{t=1}^{T} \lambda(1-\lambda)^{t-1} M_{\text{a},\text{b}} \right] = \max_{p \in \Delta(K)} \min_{\tau \in T} \left[ \sum_{t=1}^{T} \lambda(1-\lambda)^{t-1} M_{\text{a},\text{b}} \right]
\]
Following the same steps as in the proof of Proposition 3.23 in [6], we derive that the game value \( \tilde{V}_{\lambda,T+1}(w) \) of dual game \( \bar{\Gamma}_{\lambda,T+1}(w) \) satisfies the following recursive formula.
\[
\tilde{V}_{\lambda,T+1}(w) = \min_{y \in \Delta(B) \in \mathcal{A}} \max(1-\lambda) \tilde{V}_{\lambda,T}(w + \lambda M_{\text{a},y}) \left( \frac{w + \lambda M_{\text{a},y}}{1-\lambda} \right)
\]
Moreover, since \( \bar{\Gamma}_{\lambda,T}(w) \) is a dual game of \( \Gamma_{\lambda,T}(p) \), their game values have the following relations.
\[
\tilde{V}_{\lambda,T}(w) = \max_{p \in \Delta(K)} \{ V_{\lambda,T}(p) + p^T w \}, \quad (50)
\]
Based on the relations between the game values of the \( \lambda \)-discounted game \( \Gamma_\lambda(p) \), the \( \lambda \)-discounted \( T \)-stage games \( \Gamma_{\lambda,T}(p) \) and their duals, we have the following lemma.

Lemma 16. Consider a two-player zero-sum \( \lambda \)-discounted asymmetric repeated game \( \Gamma_\lambda(p) \) and its dual game \( \bar{\Gamma}_\lambda(w) \), and a two-player zero-sum \( \lambda \)-discounted \( T \)-stage asymmetric repeated game \( \Gamma_{\lambda,T}(p) \) and its dual game \( \bar{\Gamma}_{\lambda,T}(w) \). Their game values satisfy
\[
\| V_\lambda - \bar{V}_{\lambda,T} \|_{\sup} = \| \tilde{V}_\lambda - \tilde{\bar{V}}_{\lambda,T} \|_{\sup}. \quad (52)
\]
Proof. First, we show \( \| V_\lambda - \bar{V}_{\lambda,T} \|_{\sup} \leq \| \tilde{V}_\lambda - \tilde{\bar{V}}_{\lambda,T} \|_{\sup} \). According to equation (49) and (51), we have
\[
\| V_\lambda(p) - V_{\lambda,T}(p) \| = \min_{w \in [\mathcal{R}]} \{ \tilde{V}_\lambda(w) - p^T w \} - \min_{w \in [\mathcal{R}]} \{ \tilde{V}_{\lambda,T}(w) - p^T w \}.
\]
Let \( w^* \) and \( w^\ast \) be the optimal solution to the problem \( \min_{w \in [\mathcal{R}]} \{ \tilde{V}_\lambda(w) - p^T w \} \) and \( \min_{w \in [\mathcal{R}]} \{ \tilde{V}_{\lambda,T}(w) - p^T w \} \), respectively. If \( \min_{w \in [\mathcal{R}]} \{ \tilde{V}_\lambda(w) - p^T w \} \geq \min_{w \in [\mathcal{R}]} \{ \tilde{V}_{\lambda,T}(w) - p^T w \} \), then we have \( \| V_\lambda(p) - V_{\lambda,T}(p) \| \leq \| \tilde{V}_\lambda(w^\ast) - \tilde{V}_{\lambda,T}(w^\ast) \| \). Otherwise, we have \( \| V_\lambda(p) - V_{\lambda,T}(p) \| \leq \| \tilde{V}_\lambda(w^* - \tilde{V}_{\lambda,T}(w^* \ast) \| \). Therefore, for any \( p \in \Delta(K) \), \( V_\lambda(p) - V_{\lambda,T}(p) \leq \| V_\lambda - \tilde{V}_{\lambda,T} \|_{\sup} \), which implies that \( \| V_\lambda - V_{\lambda,T} \|_{\sup} \leq \| V_\lambda - \tilde{V}_{\lambda,T} \|_{\sup} \), which results in \( \| V_\lambda - V_{\lambda,T} \|_{\sup} \leq \| \tilde{V}_\lambda - \tilde{V}_{\lambda,T} \|_{\sup} \).

Following the same steps, based on equation (34) and (50), we derive that \( \| \tilde{V}_\lambda - \tilde{V}_{\lambda,T} \|_{\sup} \leq \| V_\lambda - V_{\lambda,T} \|_{\sup} \). Therefore, equation (52) is shown.

Before we draw the uninformed player’s approximated security strategy based on the approximated game value \( \tilde{V}_{\lambda,T}(w^\ast) \), we are interested in how far away the approximated game value is from the real game value. To this purpose, we define an operator \( \tilde{\mathbf{F}} \) as
\[
\tilde{\mathbf{F}}_y (w) = (1-\lambda) \max_{a \in \mathcal{A}} \tilde{V}_\lambda \left( \frac{w + \lambda M_{\text{a},y}}{1-\lambda} \right),
\]
where \( y \in \Delta(B) \), \( w \in [\mathcal{R}] \), and \( \tilde{V} : [\mathcal{R}] \to \mathbb{R} \). With the same technique as in Lemma 6 we can show that \( \tilde{\mathbf{F}} \) is also a contraction mapping.
Lemma 17. Given any $y \in \Delta(B)$ and $\lambda \in (0, 1)$, the operator $\tilde{F}_y$ defined as in (53) is a contraction mapping with contraction constant $1 - \lambda$, i.e.,

\[ \|\tilde{F}_{y_1} - \tilde{F}_{y_2}\|_{\sup} \leq (1 - \lambda)\|\tilde{V}_1 - \tilde{V}_2\|_{\sup}, \tag{54} \]

where $\tilde{V}_{1.2} : \mathbb{R}^{|K|} \to \mathbb{R}$.

Lemma [17] further implies that the approximated value $\tilde{V}_{\lambda,T}$ converges to the real game value $\tilde{V}_\lambda$ exponentially fast with respect to $T$. The proof is similar to the proof of Theorem [7].

Theorem 18. Consider the game value $\tilde{V}_\lambda(w)$ of a $\lambda$-discounted asymmetric repeated dual game $\tilde{\Gamma}_\lambda(w)$ and the game value $\tilde{V}_{\lambda,T}(w)$ of a $\lambda$-discounted $T$-stage asymmetric repeated dual game $\tilde{\Gamma}_{\lambda,T}(w)$. The game value $\tilde{V}_{\lambda,T}$ converges to $\tilde{V}_\lambda$ exponentially fast with respect to the time horizon $T$ with convergence rate $1 - \lambda$, i.e.,

\[ \|\tilde{V}_\lambda - \tilde{V}_{\lambda,T} \|_{\sup} \leq (1 - \lambda)\|\tilde{V}_\lambda - \tilde{V}_{\lambda,T} \|_{\sup}. \tag{55} \]

Applying the approximated game value $\tilde{V}_{\lambda,T}$ in equation (36), we derive the uninformed player’s approximated security strategy $\tilde{\sigma}_{\lambda,T}(w_t)$ in dual game $\tilde{\Gamma}_\lambda(w^*)$ as

\[ \tilde{\sigma}_{\lambda,T}(w_t) = \arg \min_{y \in \Delta(B)} \max_{a \in A} (1 - \lambda)\tilde{V}_{\lambda,T} \left( \frac{w_t + \lambda M_\lambda y}{1 - \lambda} \right), \tag{56} \]

where $w_t$ is updated according to (33). Comparing equation (56) and (49), we see that the approximated security strategy $\tilde{\sigma}_{\lambda,T}(w_t)$ can be seen as the uninformed player’s security strategy at stage 1 in a $\lambda$-discounted $T + 1$-stage dual game $\tilde{\Gamma}_{\lambda,T+1}(w_t)$. Similar to the LP formulation computing the game value of $\Gamma_{\lambda,T}(p)$, we construct an LP formulation to compute the game value of $\tilde{\Gamma}_{\lambda,T+1}(w)$ and the uninformed player’s approximated security strategy $\tilde{\sigma}_{\lambda,T}$.

Theorem 19. Consider a two-player zero-sum $\lambda$-discounted $T + 1$-stage dual game $\tilde{\Gamma}_{\lambda,T+1}(w)$. Its game value $\tilde{V}_{\lambda,T+1}(w)$ satisfies

\[ \tilde{V}_{\lambda,T+1}(w) = \min_{y \in Y, t \in [T+1], \ell \in \mathbb{R}} L \]

\[ s.t. w + \ell \leq L \mathbf{1} \]

\[ u_{\lambda,T+1}(y; k, :) \leq \ell^k \mathbf{1}, \quad \forall k \in K, \tag{59} \]

\[ \mathbf{1}^T \mathbf{y}_{\ell t}^A = 1, \quad \forall h_t^A \in H_t^A, \forall t = 1, \ldots, T + 1, \tag{60} \]

\[ \mathbf{y}_{\ell t}^A \geq 0, \quad \forall h_t^A \in H_t^A, \forall t = 1, \ldots, T + 1, \tag{61} \]

where $Y$ is a properly dimensioned real space, and $u_{\lambda,T+1}(y; k, :)$ is a $|H_{\ell t+2}|$ dimensional column vector whose element is $u_{\lambda,T+1}(y; k, h_{\ell t+2}^A)$, a linear function of $y$ satisfying equation (43).

Moreover, suppose in dual game $\tilde{\Gamma}_\lambda(w_0)$, at stage $t$, the anti-discounted regret $w_t = w$. The uninformed player’s approximated security strategy $\tilde{\sigma}_{\lambda,T}(w)$ is $y_{\ell t}^A$.

Proof. According to equation (48), we have

\[ \tilde{V}_{\lambda,T+1}(w) = \min_{\tilde{\sigma} \in T} \max_{p \in \Delta(K)} \sum_{k \in K} p^k (w^k + \mu_{\lambda,T+1}(\tilde{\sigma})). \]

Similar to how we derive equation (18), we have

\[ \mu_{\lambda,T+1}(\tilde{\sigma}) = \min_{\ell^k \in \mathbb{R}} \ell^k \]

\[ s.t. u_{\lambda,T+1}(\tilde{\sigma}; k, :) \leq \ell^k \mathbf{1}. \]

Therefore, we have

\[ \tilde{V}_{\lambda,T+1}(w) = \min_{\tilde{\sigma} \in T} \max_{p \in \Delta(K)} \min_{\ell^k \in \mathbb{R}} \ell^k \sum_{k \in K} p^k (w^k + \ell^k) \]

\[ s.t. u_{\lambda,T+1}(\tilde{\sigma}; k, :) \leq \ell^k \mathbf{1}, \forall k \in K. \]

Since $\sum_{k \in K} p^k (w^k + \ell^k)$ is bilinear in $p$ and $\ell$, according to Sion’s minimax theorem, we have

\[ \tilde{V}_{\lambda,T+1}(w) = \min_{\tilde{\sigma} \in T} \max_{p \in \Delta(K)} \sum_{k \in K} p^k (w^k + \ell^k) \]

\[ s.t. u_{\lambda,T+1}(\tilde{\sigma}; k, :) \leq \ell^k \mathbf{1}, \forall k \in K. \]
According to the duality theorem, given any $\bar{\tau} \in \bar{T}$ and $\ell \in \mathbb{R}^{[K]}$, we have

$$\max_{p \in \Delta(K)} \sum_{k \in K} p^k (w^k + \ell^k)$$

$$\text{s.t.} u_{\lambda,T+1}(\bar{\tau}; k, :) \leq \ell^k \mathbf{1}, \forall k \in K$$

$$= \min_{L \in \mathbb{R}} \bar{L}$$

$$\text{s.t.} w + \ell \leq \bar{L} \mathbf{1},$$

$$u_{\lambda,T+1}(\bar{\tau}; k, :) \leq \ell^k \mathbf{1}, \forall k \in K,$$

which completes the proof.

Now, we know how to compute the approximated special initial regret $w^*$ and the uninformed player’s approximated security strategy $\bar{\tau}_{\lambda,T}$ in the dual game $\bar{\Gamma}_{\lambda}(w^*)$. This approximated security strategy $\bar{\tau}_{\lambda,T}$ is also the uninformed player’s approximated security strategy in the primal game $\Gamma_{\lambda}(p)$. Let’s conclude this subsection with the uninformed player’s algorithm in the $\lambda$-discounted asymmetric repeated game $\Gamma_{\lambda}(p)$.

**Algorithm 20.** The uninformed player’s approximated security strategy in $\lambda$-discounted asymmetric repeated game $\Gamma_{\lambda}(p_0)$

(i) **Initialization**

- Read payoff matrices $M$ and initial probability $p_0$.
- Set $T$.
- Solve the LP problem (45-46) with $p = p_0$, and let $w^* = -\ell^*$. Let $t = 1$ and $w_1 = w^*$.

(ii) Solve the LP problem (57-61) with $w = w_t$, and the uninformed player’s approximated security strategy $\bar{\tau}(w_t)$ is $y^*_k\lambda$.

(iii) Choose an action $b \in B$ according to the probability $\bar{\tau}_{\lambda,T}(w_t)$, and announce it publicly.

(iv) Read the informed player’s action, and update the anti-discounted regret $w_{t+1}$ according to (62).

(v) Update $t = t + 1$ and go to step (ii).

4) The performance difference between the suboptimal strategy and the security strategy: With the uninformed player’s approximated security strategy $\bar{\tau}_{\lambda,T}$, we are interested in the worst case cost guaranteed by this strategy, which is also called the security level of $\bar{\tau}_{\lambda,T}$. Given an uninformed player’s strategy $\tau \in \bar{T}$, the security level $J^\tau(p)$ in game $\Gamma_{\lambda}(p)$ is defined as

$$J^\tau(p) = \max_{\sigma \in \Sigma} \gamma_{\lambda}(p, \sigma, \tau).$$

Since the uninformed player’s approximated security strategy is derived from his approximated security strategy in the dual game, the security levels of the approximated security strategy in the primal and dual games are highly related. Hence, we would also like to define the security level $\bar{J}^\tau(w)$ of $\tau \in \bar{T}$ in the dual game $\bar{\Gamma}_{\lambda}(w)$ as

$$\bar{J}^\tau(w) = \max_{p \in \Delta(K)} \max_{\sigma \in \Sigma} \bar{\gamma}_{\lambda}(w, p, \sigma, \tau).$$

Following the same steps as in the proof of (33,34) in [5], we can show that $J^\tau(p)$ and $\bar{J}^\tau(w)$ have the following relations.

$$\bar{J}^\tau(w) = \max_{p \in \Delta(K)} \{ J^\tau(p) + p^T w \},$$

$$J^\tau(p) = \min_{w \in \mathbb{R}^{[K]}} \{ \bar{J}^\tau(w) - p^T w \}.$$  

Meanwhile, we also notice that in dual game $\bar{\Gamma}_{\lambda}(w)$, the security level $\bar{J}^\tau$ of a stationary strategy $\tau$ that depends only on $w_t$ satisfies $\bar{J}^\tau(w) = \bar{F}^\tau_{\tau(w)}(w)$.

**Lemma 21.** Let $\tau \in \bar{T}$ be the uninformed player’s stationary strategy that depends only on the anti-discounted regret $w_t$. The security level $\bar{J}^\tau$ of $\tau$ in a $\lambda$-discounted asymmetric information repeated game $\bar{\Gamma}_{\lambda}(w)$ satisfies $\bar{J}^\tau(w) = \bar{F}^\tau_{\tau(w)}(w)$, where $\bar{F}_{\tau(w)}$ is defined in (53).
Theorem 22. Consider a two-player zero-sum \( \lambda \)-discounted asymmetric information repeated game \( \Gamma_\lambda(p) \) and the uninformed player’s approximated security strategy \( \bar{\tau}_{\lambda,T} \) defined in (56). The security level \( J_{\lambda,T}(p) \) of \( \bar{\tau}_{\lambda,T} \) in game \( \Gamma_\lambda(p) \) satisfies
\[
||J_{\lambda,T} - V_\lambda||_{\text{sup}} \leq \frac{2(1 - \lambda)}{\lambda} ||V_\lambda - \bar{V}_{\lambda,T}||_{\text{sup}}
\]

Proof. According to equation (65) and (64), we have \( |J_{\lambda,T}(p) - V_\lambda(p)| = |\min_{w \in \mathbb{R}_+^{|K|}} \{ J_{\lambda,T}(w) - p^T w \} - \min_{w \in \mathbb{R}_+^{|K|}} \{ \bar{V}_\lambda(w) - p^T w \}| \). Let \( w^* \) be the solution to the optimal problem \( \min_{w \in \mathbb{R}_+^{|K|}} \{ V_\lambda(w) - p^T w \} \). Since \( J_{\lambda,T}(p) \geq V_\lambda(p) \), we have
\[
|J_{\lambda,T}(p) - V_\lambda(p)| \leq |J_{\lambda,T}(w^*) - \bar{V}(w^*)| \leq ||J_{\lambda,T} - \bar{V}||_{\text{sup}}, \forall p \in \Delta(K).
\]
Following the same steps as in the proof of Theorem 10, we can show that \( ||J_{\lambda,T} - \bar{V}||_{\text{sup}} \leq \frac{2(1 - \lambda)}{\lambda} ||V_\lambda - \bar{V}_{\lambda,T}||_{\text{sup}} \). Therefore, we have \( ||J_{\lambda,T} - V_\lambda||_{\text{sup}} \leq \frac{2(1 - \lambda)}{\lambda} ||V_\lambda - \bar{V}_{\lambda,T}||_{\text{sup}} \). According to Equation (52), equation (66) is proved.

IV. CASE STUDY: A NETWORK INTERDICTIO PROBLEM

This section uses game theoretic tools to study a network interdiction problem developed from [17], and provides security strategies and approximated security strategies for both players (attacker and network) in finite-horizon game and discounted game, respectively.

Consider a network with a source node and a sink node. There are two channels from the source node to the sink node. One of them has high capacity of 3, and the other one has low capacity of 1. Only the network knows which channel has high capacity. The network needs to choose a channel to use at each stage to maximize the throughput over a certain horizon. Meanwhile, the attacker will either block one channel with cost 1 or observe the usage of channels with cost 0 to minimize the throughput over the same horizon. Notice that the attacker can only detect whether a channel is in use, but cannot measure the capacity of a channel. Our objective is to design security or approximated security strategies for both players.

The network interdiction problem is modeled as an asymmetric repeated game with the network to be the informed player and the attacker to be the uninformed player. The network’s action is to either use channel 1 (1) or use channel 2 (2), and the attacker’s action is to observe (o), block channel 1 (1), or block channel 2 (2). The payoff matrices are provided as in Table 1.

The initial probability that channel 1 has high capacity is 0.5.

We first compute the security strategies and security levels for both the network and the attacker in a 3-stage asymmetric game according to Theorem 3 and 4, respectively. The linear program used to compute the network’s security strategy has 65 constraints and 35 variables, while the attacker’s LP formulation has 44 constrains and 23 variables. The security level of the network is 6.57 which meets the security level of the attacker.

| \( k \) | \( 1 \) | \( 2 \) | \( \text{o} \) |
|-----|---|---|-----|
| 1   | 1 | 2 | o   |
| 2   | 2 | 1 | 1   |

PAPER MATRIX \( M^k \) IF CHANNEL \( k \) HAS HIGH CAPACITY
The security strategy of the network is given in Table II. Consider the case in which channel 1 has high capacity. At stage 1, the network uses the high capacity channel with probability 0.64 instead of 1, because if the network reveals the high capacity channel at stage 1, the attacker will block the high capacity channel for the next two stages. At stage 2, if channel 1 was used at stage 1, then the network thinks that the attacker may guess that channel 1 has high capacity, and hence the network reduces its probability of using channel 1 to 0.56. Otherwise, the probability of using channel 1 is increased to 0.8. At the final stage, unless channel 1 is continuously used, the network will use high capacity channel for sure.

The security strategy of the attacker is shown in Table III. Notice that because the cost of blocking a channel is low compared with the gain of blocking the high capacity channel, the attacker prefers blocking channels to observing channels. Therefore, for many cases, the attacker launches attacks instead of observing channels unless he is almost sure which channel has high capacity. In this case, because the loss of blocking low capacity channel is higher than the loss of observing channels (see Table I), the attacker would prefer observing channels to blocking low capacity channel. At stage 1, since the initial probability over the states is [0.5 0.5], the attacker will block either channel with equal probability. At stage 2, the attacker will increase the probability of blocking channel 1 by 0.04 if channel 1 is used at stage 1. Otherwise, the probability of blocking channel 1 is decreased by 0.04. At stage 3, if one channel was used continuously, the attacker’s realized loss in the case that this channel has high capacity is already high, so his strategy focuses more on minimizing the payoff if the continuously used channel has high capacity, as if he is playing only a single game.

The security strategies of both players are, then, used in the 3-stage network interdiction game. We ran the 3-stage game for 5000 times, and the average total payoff of the network was 6.58 which was approximately the game value 6.57 computed according to Theorem 2 and 3.

Next, we compute the approximated security strategies for both players in a 0.7-discounted asymmetric repeated game. According to Theorem 8, the network computes his approximated security strategy based on the approximated game value \( V_{w_t} \). The game values from the discounted 1-stage game to the discounted 4-stage game are presented in the left plot of Figure 1. We see that the approximated game value converges, and that the more unsure the attacker is about the high capacity channel, the higher throughput the network can get, and the highest approximated game value is 2.24 when the initial probability is [0.5 0.5]. The approximated security strategy is given in the right plot of Figure 1. For both cases, the probability of using channel 1 is lower if the network thinks that the attacker has stronger belief that channel 1 has higher capacity. Meanwhile, compared to the case in which channel 2 has high capacity (green dots), it is more possible for the network to use channel 1 if channel 1 has high capacity (blue crosses).

To compute the attacker’s approximated security strategy, we first need to compute the approximated special initial regret \( w^* \) according to Theorem 8 which is \([-2.24; -2.24]\) in this case. At each stage, the attacker computes his approximated security strategy based on the anti-discounted regret \( w_t \) with \( w_t = w^* \). We assume that the anti-discounted regret \( w_t^\lambda \) if channel 2 has high capacity is 2, and use the approximated value \( \tilde{V}_{w_t^\lambda} \) to compute the approximated security strategy. The approximated
game values $\tilde{V}_{\lambda,T}$ where $T$ varies from 1 to 4 are presented in the left plot of Figure 2. We see that the approximated game value converges over $T$, and increases with respect to $w^1_i$. The attacker’s approximated security strategy is shown in the right plot of Figure 2. When $w^1_i$ is relatively low compared with $w^2_i$, the attacker will block channel 2 with higher probability to balance the payoffs of both cases, as if he believes that it is more possible for channel 2 to have high capacity. Contrarily, when $w^1_i$ is larger than $w^2_i$, the attacker will block channel 1 with higher probability to balance the payoffs of both cases, as if he believes that it is more possible for channel 1 to have high capacity.

The approximated security strategies of both players are, then, used in a 0.7-discounted network interdiction game. Before running the game, we first anticipate the payoff of the game. From equation (24), we have that
\[
\left| \lambda V^*_{\lambda,T}(p) \right| \leq (1 - \frac{2(1-\lambda)^{T+1}}{\lambda}) \left| \lambda V^*_{\lambda,T} \right| + 1.96.
\]
Together with equation (30) and (24), the network can guarantee a payoff $|J^*_{\lambda,T}(p)| \leq (1 + \frac{2(1-\lambda)^{T+1}}{\lambda}) \left| \lambda V^*_{\lambda,T} \right| + 2.59$. Therefore, we anticipate that the payoff is in the interval $[1.96, 2.59]$. When running the game, we stopped at stage 10 since the sum of the payoff after stage 10 is less than $10^{-7}$. The 10-stage 0.7-discounted game was ran for 100 times, and the average payoff is 2.35 which is within our anticipated interval, and demonstrates our main results.

V. Future Work

This paper studies asymmetric repeated games in which one player has superior information about the game over the other, and provides LP formulations to compute both player’s security strategies in finite-horizon games and approximated security strategies in discounted games. In the future, we will generalize these results to the case in which one player has superior knowledge of one part of the information, while the other player has superior knowledge of the other part.

REFERENCES

[1] R. J. Aumann and M. Maschler, Repeated games with incomplete information. MIT press, 1995.
[2] L. Crucio, J.-P. Clarke, and L. Weigang, “Trajectory option set planning optimization under uncertainty in ctos,” in Intelligent Transportation Systems (ITSC), 2015 IEEE 18th International Conference on. IEEE, 2015, pp. 2084–2089.
[3] L. Li, J.-P. Clarke, E. Feron, and J. Shamma, “Robust trajectory option set planning in ctop based on bayesian game modell,” in 2017 American Control Conference (ACC). AACC, 2017.
[4] S. Zamir, “Repeated games of incomplete information: Zero-sum,” Handbook of Game Theory, vol. 1, pp. 109–154, 1992.
[5] B. De Meyer, “Repeated games and partial differential equations,” Mathematics of Operations Research, vol. 21, no. 1, pp. 209–236, 1996.
[6] S. Sorin, A First Course on Zero-Sum Repeated Games. Springer Science & Business Media, 2002, vol. 37.
[7] V. Kamble, Games with vector payoffs: a dynamic programming approach. PhD thesis, 2015.
[8] I. Adler, “The equivalence of linear programs and zero-sum games,” International Journal of Game Theory, vol. 42, no. 1, pp. 165–177, 2012.
[9] D. Koller, N. Megiddo, and B. Von Stengel, “Efficient computation of equilibria for extensive two-person games,” Games and Economic Behavior, vol. 14, no. 2, pp. 247–259, 1996.
[10] B. Von Stengel, “Efficient computation of behavior strategies,” Games and Economic Behavior, vol. 14, no. 2, pp. 220–246, 1996.
[11] J.-P. Ponsard and S. Sorin, “The lp formulation of finite zero-sum games with incomplete information,” International Journal of Game Theory, vol. 9, no. 2, pp. 99–105, 1980.
[12] A. Gilpin and T. Sandholm, “Solving two-person zero-sum repeated games of incomplete information,” in Proceedings of the 7th International Joint Conference on Autonomous Agents and Multiagent Systems-Volume 2. International Foundation for Autonomous Agents and Multiagent Systems, 2008, pp. 903–910.
[13] T. Sandholm, “The state of solving large incomplete-information games, and application to poker,” AI Magazine, vol. 31, no. 4, pp. 13–22, 2010.
[14] S. Zamir, “On the relation between finitely and infinitely repeated games with incomplete information,” International Journal of Game Theory, vol. 1, no. 1, pp. 179–198, 1971.
[15] D. M. Kreps and R. Wilson, “Sequential equilibria,” Econometrica: Journal of the Econometric Society, pp. 863–894, 1982.
[16] L. Li and J. Shamma, “LP formulation of asymmetric zero-sum stochastic games,” in Decision and Control (CDC), 2014 IEEE 53rd Annual Conference on. IEEE, 2014, pp. 1930–1935.
[17] J. Zheng and D. A. Castanon, “Dynamic network interdiction games with imperfect information and deception,” in Decision and Control (CDC), 2012 IEEE 51st Annual Conference on. IEEE, 2012, pp. 7758–7763.