CLASSICAL LOGIC AND INTUITIONISTIC LOGIC:
EQUIVALENT FORMULATIONS IN NATURAL DEDUCTION,
GÖDEL-KOLMOGOROV-GLIVENKO TRANSLATION

by

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Abstract. — This report first shows the equivalence between several formulations of classical logic in intuitionistic logic (tertium non datur, reductio ad absurdum, Pierce’s law). Then it establishes the correctness of the Gödel-Kolmogorov translation, whose restriction to the propositional case is due to Glivenko. This translation maps a formula $F$ of first order logic to a formula $F^\sim\sim$ in such a way that $F$ is provable in classical logic if and only if $F^\sim\sim$ is provable in intuitionistic logic. All formal proofs are presented in natural deduction.

These questions are well-known proof theoretical facts, but in textbooks, they are often ignored or left to the reader. Because of the combinatorial difficulty of some of the needed formal proofs, we hope that this report may be useful, in particular to students and colleagues from other areas.

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1. Foreword and references

The results in here are not new, but it is hard to tell where they are properly written down and published. They are often left as exercises to the reader. These are exercises on the combinatorics of proofs, but some of the cases are not that easy for people who are new to proof theory, especially students (though we encourage everyone to try these proofs themselves).

Natural deduction is a tree-like framework for formal proofs which is naturally intuitionistic. This tree-like formulation was introduced in [10], where NJ [2, 3] is reformulated in terms of pseudo-trees. More modern references include [4, 1].

The translation of a formula $F$ into a formula $F^{\land\land\land}$ which is intuitionistically provable if and only if $F$ is classically provable is due to [5] (in French) for the propositional case and to [8] (in Russian, summary in French, in English in [13]) and [6, 7] (in German) for the first order case.

The equivalence of the various formulations of classical logic in intuitionistic logic (tertium non datur, reductio ad absurdum, Pierce law) can be found here and there e.g. in [11, 12, 9].

2. Natural deduction rules

Let us recall the natural deduction rules that we use throughout this report.

A proof is a tree plus additional information:
- the nodes are formulæ
- the root is the conclusion of the proof
- the leaves are the hypothesis which can be:
  - cancelled or discharged (if so, they are between square brackets)
  - free (nothing particular)
- every branch (unary, binary or ternary) is labelled by a rule name.
- some branches (named $\rightarrow_e, \lor_e, \exists_e$) include an index which also appears on the hypotheses which are cancelled during the application of the rule.

If the multiset of free hypotheses of a proof $d$ is $\Gamma$ and the conclusion of $d$ is $C$, then $d$ is a proof of $\Gamma \vdash C$ that is a proof of $C$ under the (conjunction of the) assumptions $\Gamma$.

If a rule says that $H$ is cancellable in $d$ then any number of free occurrences of $H$ can be cancelled (one also says discharged). The cancelled hypotheses and the rule receive a fresh new index that encodes this fact (as this information is not recoverable from the proof tree, this is why natural deductions are more than trees).
### Classical Logic and Intuitionistic Logic

| Introduction rules | Elimination rules |
|--------------------|-------------------|
| **Implication**    |                   |
| \[ A \Gamma [A] \Delta [A] \] \[ \vdots \] \[ B \] \rightarrow_{\alpha} \[ A \rightarrow B \] | \[ \vdots \] | \[ A \rightarrow B \rightarrow_{\epsilon} \] |
|                   | \[ A \text{ cancellable.} \] |                   |
| **Conjunction**    |                   |
| \[ \frac{A \quad B}{A \wedge B} \] \[ \wedge_{i} \] | \[ \frac{A \wedge B}{A} \] \[ \wedge_{1} \] | \[ \frac{A \wedge B}{B} \] \[ \wedge_{2} \] |
| **Falsum**         |                   |
| \[ \frac{\bot}{C} \] \[ \perp \] |                   |
|                   | \[ \text{ex falso quod libet sequitur} \] |
| **Disjunction**    |                   |
| \[ \frac{A}{A \vee B} \] \[ \vee_{1} \] | \[ \frac{B}{C} \] \[ \vee_{2} \] | \[ \frac{A \vee B}{C} \] \[ \vee_{\alpha} \] |
|                   | \[ A \text{ cancellable in } d_{1}, B \text{ in } d_{2}. \] |

### Introduction and Elimination Rules for Quantifiers

| Introduction rules | Elimination rules |
|--------------------|-------------------|
| **Existential quantifier** |                   |
| \[ \frac{A(t)}{\exists x A(x)} \] \[ \exists_{i} \] | \[ \Theta \] | \[ \frac{A(u) \Gamma [A(x)] \Delta [A(x)]}{C} \] \[ \exists_{\alpha} \] |
|                   | \[ \text{Afterwards, no free } x \text{ in } C \text{ nor in any free hypothesis.} \] |
| **Universal quantifier** |                   |
| \[ \frac{A(x)}{\forall x A(x)} \] \[ \forall_{i} \] | \[ \frac{\forall x A(x)}{A(t)} \] \[ \forall_{\epsilon} \] |
|                   | \[ No \text{ free } x \text{ in } \Gamma. \] |
Natural deduction is “naturally intuitionistic”: formulæ like \( \neg \neg A \to A \) or \( (\neg X) \lor X \) are not provable. It is equivalent to other formulations of intuitionistic logic like the sequent calculus with many hypothesis and one conclusion.

There are no rules for negation \( \neg X \), which is treated as a shorthand for \( X \to \bot \):
\[
\neg X \equiv_{\text{def}} X \to \bot
\]

For convenience, we will sometimes write the negation rules as follows.
\[
A \Gamma\![A] \alpha, \Delta\![A] \alpha \vdash \neg A \neg \neg A \bot
\]

The reader can easily verify that given \( \neg A \equiv_{\text{def}} A \to \bot \), these are just instances of \( \to_e \) and \( \to_i \) where the subformula \( B \) of the \( \to \) rules is \( \bot \).

3. Three formulations of classical logic

To obtain a natural deduction calculus for classical logic, one has to add a family of proper axioms (i.e. axioms other than \( A \), that is \( A \vdash A \), which unfortunately complicates normalisation and the proof of the subformula property).

Tertium Non Datur
\[
\frac{\text{tna}}{A \lor \neg A}
\]

for all formula \( A \)

Reductio Ad Absurdum
\[
\frac{\text{raa}}{(\neg A) \to A}
\]

for all formula \( A \)

Pierce law
\[
\frac{\text{Pierce}}{((P \to Q) \to P) \to P}
\]

for all formulæ \( P \) and \( Q \)

RAA can also be expressed as rule cancelling several occurrences of \( \neg A \).
\[
\frac{A \Gamma\![\neg A] \alpha, \Delta\![\neg A] \alpha \vdash \bot \text{ raa' } \alpha}{\bot}
\]

\( \text{raa' }\) is clearly equivalent to the axiom \( \text{raa} \) given above: using the invertible rule \( \to_i \), one obtains \( \neg \neg A \) and then obtains \( A \) by \( \text{raa} \) above, as follows.
\[
\frac{\text{raa}}{(\neg A) \to \bot \text{ raa'} \alpha}{A}
\]
Given \( raa' \), \( raa \) becomes derivable as follows.

\[
\frac{[\neg\neg A]^2 [\neg A]^1}{\vdash A \quad raa' (1)}
\frac{\neg\neg A \rightarrow A \rightarrow_i (2)}{
eg\neg A \rightarrow A \rightarrow_i (2)}
\]

4. Equivalence of the three formulations of classical logic in intuitionistic logic

4.1. Reductio ad Absurdum entails Tertium Non Datur. —

\[
\frac{[\neg (a \lor \neg a)]^3}{[a]^2 \lor \neg a \rightarrow_i \gamma_e}
\frac{\neg (a \lor \neg a) \rightarrow_i \gamma_e}{[\neg a]^1 \lor \neg a \rightarrow_i \gamma_e}
\frac{\bot \rightarrow e}{\neg \neg (a \lor \neg a) \rightarrow \gamma_e}
\frac{\neg (a \lor \neg a) \rightarrow \gamma_e}{\neg (a \lor \neg a) \rightarrow (a \lor \neg a) \rightarrow \gamma_e}
\]

4.2. Tertium Non Datur entails Pierce law. —

\[
\frac{\neg p \rightarrow p \rightarrow_i \gamma_e}{\neg p \rightarrow p \rightarrow_i \gamma_e}
\frac{[p]^3}{(p \rightarrow q) \rightarrow p \rightarrow_i \gamma_e}
\frac{\bot \rightarrow e}{p \rightarrow q \rightarrow_i \gamma_e}
\frac{[(p \rightarrow q) \rightarrow p]^2 \rightarrow \gamma_e^2}{((p \rightarrow q) \rightarrow p) \rightarrow \gamma_e^2}
\frac{p \rightarrow q \rightarrow_i \gamma_e}{((p \rightarrow q) \rightarrow p) \rightarrow \gamma_e^3}
\]

4.3. Pierce law entails Reductio ad Absurdum. —

\[
\frac{\neg\neg P^2 \rightarrow \neg P^1}{Pierce \ avec \ Q = \bot}
\frac{\neg P \rightarrow P \rightarrow_i \gamma_e}{\neg P \rightarrow P \rightarrow_i \gamma_e}
\frac{\bot \rightarrow e}{\neg P \rightarrow P \rightarrow_i \gamma_e}
\frac{P \rightarrow P \rightarrow_i \gamma_e}{\neg P \rightarrow P \rightarrow_i \gamma_e}
\]

5. The Gödel-Kolmogorov translation

The not-not translation \( F^{\neg\neg} \) of a formula \( F \) is inductively defined as follows:
\[ \begin{align*}
\bot \lnot\lnot &= \bot \\
\top \lnot\lnot &= \lnot\top \\
A \lnot\lnot &= \lnot\lnot A \\
(A \land B) \lnot\lnot &= A \lnot\lnot \land B \lnot\lnot \\
(A \rightarrow B) \lnot\lnot &= A \lnot\lnot \rightarrow B \lnot\lnot \\
(\forall x. A) \lnot\lnot &= \forall x. A \lnot\lnot \\
(A \lor B) \lnot\lnot &= \lnot\lnot (A \lnot\lnot \lor B \lnot\lnot) \\
(\exists x. A) \lnot\lnot &= \lnot\lnot\exists x. A \lnot\lnot \end{align*} \]

\( \text{NJ} \) stands for plain natural deduction, which is intuitionistic.

\( \text{NK} \) stands for classical natural deduction, that is \( \text{NJ} \) enriched by one of the families of axioms given above (or all of them, since they are equivalent): tertium non datur, reductio ad absurdum or Pierce law.

Since \( \bot \lnot\lnot = \bot \) and \( \lnot A = (A \rightarrow \bot) \), the definition of the not not translation of an implicative formula yields the following remark:

**Remark 1.** \( \lnot\lnot A \equiv \lnot(A \lnot\lnot) \).

**Proposition 2.** \( \lnot\lnot\lnot A \vdash_{\text{NJ}} \lnot A \)

**Proof.** Here is the natural deduction proof of it:

\[ \begin{array}{c}
\frac{[A]^2 \quad [\lnot A]^1}{\lnot\lnot\lnot A} \quad \gamma_c \\
\frac{\lnot\lnot\lnot A \quad \lnot\lnot A}{\lnot A} \quad \gamma_c (1) \\
\frac{\lnot A}{\lnot\lnot A} \quad \gamma_c (2) \\
\end{array} \]

**Lemma 3.** \( \lnot\lnot F \lnot\lnot \vdash_{\text{NJ}} F \lnot\lnot \)

**Proof.** We proceed by induction on \( F \).

1. If \( F = \bot \) one has to show that \( \lnot\lnot\bot \vdash \bot \):

\[ \begin{array}{c}
\frac{[\bot]^1}{\lnot\lnot\bot} \quad \gamma_l (1) \\
\frac{\bot}{\lnot\lnot\bot} \quad \gamma_c \\
\end{array} \]

2. If \( F = a \) we have to show \( \lnot\lnot\lnot a \vdash \lnot\lnot a \) which is a consequence of Proposition 2 with \( A = \lnot a \).

3. If \( F = X \lor Y \) we have to show that \( \lnot\lnot\lnot(X \lnot\lnot \lor Y \lnot\lnot) \vdash \lnot\lnot(X \lnot\lnot \lor Y \lnot\lnot) \), which is a consequence of Proposition 2 with \( A = \lnot(X \lnot\lnot \lor Y \lnot\lnot) \).

4. If \( F = \exists x P \) one has to show that \( \lnot\lnot\lnot(\exists x P \lnot\lnot) \vdash \lnot\lnot(\exists x P) \) which is a consequence of Proposition 2 with \( A = \lnot(\exists x P \lnot\lnot) \).
5. If \( F = A \rightarrow B \), one has to show that \( \neg(\neg A \rightarrow \neg B) \vdash (\neg A \rightarrow \neg B) \). The induction hypothesis (IH) makes sure that \( \neg \neg B \vdash B \):

\[
\frac{[\neg A] \land [B \rightarrow \neg B]}{\neg(\neg A \rightarrow \neg B)} \quad \gamma(1)
\]

\[
\frac{\frac{\frac{[\neg A] \land [B \rightarrow \neg B]}{\neg(\neg A \rightarrow \neg B)}}{\neg \neg B \vdash \neg(\neg A \rightarrow \neg B) \quad \gamma(2)}}{\vdash \neg(\neg A \rightarrow \neg B) \quad \gamma(3)}
\]

6. If \( F = A \land B \), one has to show that \( \neg(\neg (A \land B)) \vdash (A \land B) \) and because of the induction hypothesis (IH) we can assume that \( \neg \neg A \vdash A \) and \( \neg \neg B \vdash B \):

\[
\frac{[\neg \neg A \land \neg \neg B]}{\neg \neg (A \land B)} \quad \gamma(1)
\]

\[
\frac{\frac{\frac{[\neg \neg A \land \neg \neg B]}{\neg \neg (A \land B)}}{\neg \neg A \vdash \neg \neg (A \land B) \quad \gamma(2)}}{\vdash \neg \neg (A \land B) \quad \gamma(3)}
\]

\[
\frac{\frac{\frac{[\neg \neg A \land \neg \neg B]}{\neg \neg (A \land B)}}{\neg \neg B \vdash \neg \neg (A \land B) \quad \gamma(4)}}{\vdash \neg \neg B \quad \gamma(5)}
\]

From those two proofs, both with the single undischarged hypothesis \( \neg(\neg (A \land B)) \), one easily gets \( A \land B \) by the rule \( \land_i \).

7. If \( F = \forall x A \) one has to show that \( \neg(\neg \forall x A) \vdash (\forall x A) \) and the induction hypothesis (IH) guarantees that \( \neg \neg A \vdash A \).
\[
\frac{[\forall x. A^{-}]^1}{\forall e \quad \neg A^{-}} \quad \neg e
\]
\[
\frac{\neg \forall x. A^{-}}{\forall x. A^{-}} \quad \neg i(1) \quad \neg e
\]
\[
\frac{\neg \exists e}{\neg A^{-}} \quad \neg i(2)
\]
\[
\frac{IH}{A^{-}} \quad \forall_i
\]

Notice that the hypothesis \([\neg A^{-}]^2\) is cancelled before the \(\forall_i\) rule is applied.

\[
\Box
\]

**Theorem 4.** — \(\Gamma^{-} \vdash_{\text{NJ}} F^{-}\) if and only if \(\Gamma \vdash_{\text{NK}} F\).

5.1. **If \(\Gamma^{-} \vdash_{\text{NJ}} F^{-}\) then \(\Gamma \vdash_{\text{NK}} F\).** — If \(\Gamma^{-} \vdash_{\text{NJ}} F^{-}\) is provable in NJ, then it is also provable in NK: indeed the rules of NJ are rules of NK.

Since in NK it possible to add and delete double negations, for every formula \(A\), both \(A \vdash_{\text{NK}} A^{-}\) and \(A^{-} \vdash_{\text{NK}} A\) hold, as an easy induction on the formula shows. Thus, a proof in NK can be constructed.

\[
\begin{array}{cccc}
A_1^{-} & A_n^{-} \\
\vdash_{\text{NK}} & \vdash_{\text{NK}} \\
A_1^{-} & \ldots & A_n^{-} & F^{-} \\
\vdash_{\text{NJ}} & \vdash_{\text{NK}} & \vdash_{\text{NK}} & \vdash_{\text{NK}} \\
F^{-} & \vdash_{\text{NK}} & F^{-} & \vdash_{\text{NK}} & \vdash_{\text{NK}} & \vdash_{\text{NK}} & F
\end{array}
\]

5.2. **If \(\Gamma \vdash_{\text{NK}} F\) then \(\Gamma^{-} \vdash_{\text{NJ}} F^{-}\).** — We proceed by induction on the height of the proof in NK. Observe that the obtained NJ proof has the same occurrences of free variables.

5.2.1. **The height of the proof is 0 and the proof is an axiom \(A \vdash_{\text{NK}} A\).** —

\[
\begin{array}{c}
A^{-} & \vdash_{\text{NK}} \\
A
\end{array}
\]

5.2.2. **The height of the proof is 0 and it is an application tertium non datur \(\vdash_{\text{NK}} A \lor \neg A\).** — Remember that \((\neg A)^{-} = \neg (A^{-})\).
5.2.3. The height of the proof is 0 and it comes from reductio ad absurdum \( \vdash^{NK} (\neg\neg A) \rightarrow A \). — We have to show that \( \vdash (\neg\neg A) \rightarrow A \) but since \( \neg A = (A \rightarrow \bot) \) we know that \( (\neg\neg A) \rightarrow = \neg\neg A \). We therefore have to show that \( \vdash \neg\neg (A \neg\neg) \rightarrow A \neg\neg \), but this true by Lemma 3.

5.2.4. The proof ends with \( \bot_e \). — We apply the induction hypothesis (IH) to the proof without this last rule, using the fact that \( \bot \neg\neg = \bot \).

5.2.5. The proof ends with \( \rightarrow_e \). — The induction hypothesis (IH) can be applied to the two proofs obtained by suppressing this last rule.

5.2.6. The last rule is \( \rightarrow_i \). — The induction hypothesis (IH) can be applied to the proof obtained by suppressing this last rule.

5.2.7. The last rule is \( \land_e \). — The induction hypothesis (IH) can be applied to the proof obtained by suppressing this last rule.

5.2.8. The last rule is \( \land_i \). — The induction hypothesis (IH) can be applied to the two proofs obtained by suppressing this last rule.
5.2.9. The last rule is $\lor_i$. — The induction hypothesis (IH) can be applied to the three proofs obtained by suppressing this last rule. We use Lemma 3.

$$
\frac{
\Delta \quad [A] \quad \Theta \quad [B] \quad \lor_i
}{\forall_e}
$$

5.2.10. The last rule is $\lor_i$. — The induction hypothesis (IH) can be applied to the proof obtained by suppressing this last rule.

$$
\frac{
\Delta \quad [A] \quad \Theta \quad [B] \quad \lor_i
}{\forall_e}
$$

5.2.11. The last rule is $\forall_e$. — The induction hypothesis (IH) can be applied to the proof obtained by suppressing this last rule.

$$
\frac{
\forall x. A \quad \forall i
}{A[x := t]} \quad \forall_e
\quad \quad \frac{
\forall x. A \quad \forall i
}{A[x := t]} \quad \forall_e
$$
5.2.13. The last rule is $\exists$. — The induction hypothesis (IH) can be applied to the two proofs obtained by suppressing this last rule. We use Lemma 3.

References

[1] D. van Dalen – Logic and structure, fifth ed., Universitext, Springer-Verlag, 2013.
[2] G. Gentzen – “Untersuchungen über das logische Schließen I”, Mathematische Zeitschrift 39 (1934), p. 176–210, Traduction Française de R. Feys et J. Ladrère: Recherches sur la déduction logique, Presses Universitaires de France, Paris, 1955.
[3] _______ – “Untersuchungen über das logische Schließen II”, Mathematische Zeitschrift 39 (1934), p. 405–431, Traduction française de J. Ladrère et R. Feys: Recherches sur la déduction logique, Presses Universitaires de France, Paris, 1955.
[4] J.-Y. Girard, Y. Lafont & P. Taylor – Proofs and types, Cambridge Tracts in Theoretical Computer Science, no. 7, Cambridge University Press, 1988.
[5] V. Glivenko – “Sur quelques points de la logique de M. Brouwer”, Bulletin de la Société Mathematique de Belgique 15 (1929), p. 183–188.
[6] K. Gödel – “Eine interpretation des intuitionistischen aussagenkalküls”, Erg. Math. Kolloqu. 4 (1933), p. 39–40 (German).
[7] _______ – “Zur intuitionistischen arithmetik und zahlentheorie”, Ergebnisse eines mathematischen Kolloquiums 4 (1933), p. 34–38 (German).
[8] A. Kolmogorov – “Sur le principe de tertium non datur”, Mathematicheskii Sbornik 32 (1925), no. 4, p. 646–667.

[9] G. Mints – A short introduction to intuitionistic logic, University Series in Mathematics, Springer, 2000.

[10] D. Prawitz – Natural Deduction, a Proof-theoretical Study, Acta universitatis stockholmiensis — Stockholm studies in philosophy, no. 3, Almqvist and Wiksell, Stockholm, 1965.

[11] H. Rasiowa & R. Sikorski – The mathematics of metamathematics, Monografie matematyczne, vol. 41, Polish Scientific Publishers, 1963.

[12] A. Troelstra & D. van Dalen – Constructivism in mathematics (vol. 1), Studies in Logic and the foundations of mathematics, vol. 121, North-Holland, 1988.

[13] J. van Heijenoort (ed.) – From frege to gödel. a source book in mathematical logic, 1879–1931., Cambridge, MA: Harvard University Press, 1967 (English).