ARTIN–SCHREIER AND CYCLOTOMIC EXTENSIONS

JULIO CESAR SALAS–TORRES, MARTHA RZEDOWSKI–CALDERÓN, AND GABRIEL VILLA–SAL VADOR

Abstract. From class field theory we have that any Artin–Schreier extension of a rational congruence function field is contained in the composite of cyclotomic function fields and a constant field extension. In this paper we prove this result without using class field theory.

1. Introduction

Let $K$ be a function field over its constant field $k$ of characteristic $p > 0$. The extension $L/K$ is a cyclic extension of degree $p$ if and only if there exists $z \in L$ such that $L = K(z)$, where

$$z^p - z = a$$

for some $a \in K$ with $a \notin \wp(K) := \{b^p - b \mid b \in K\}$. Such extension $L/K$ is called an Artin–Schreier extension.

Let $K = k(T)$ be a field of rational functions where $k$ is a perfect field of characteristic $p > 0$ and let $L/K$ be a cyclic extension of degree $p$. Then $L = K(y)$, where $y$ satisfies an equation of the form

$$y^p - y = s(T),$$

with $s(T) \notin \wp(K)$ and the divisor of $s(T)$ given by

$$(s(T))_K = \frac{\mathcal{C}}{\mathcal{P}_1^{\alpha_1} \cdots \mathcal{P}_r^{\alpha_r}},$$

with $r \geq 0$, $\mathcal{P}_i$ a prime divisor in $K$, $\alpha_i \in \mathbb{N}$, $(\alpha_i, p) = 1$, $\mathcal{C}$ an integral divisor relatively prime to $\mathcal{P}_i$ for $i \in \{1, ..., r\}$. The prime divisor $\mathcal{P}_\infty$ may or may not be included in this set of divisors. This equation is known as the Artin–Schreier equation in its normal form.

The ramified primes in $L/K$ are precisely $\mathcal{P}_1, ..., \mathcal{P}_r$ and are totally and wildly ramified (see [1]). The different of $L/K$ is

$$\mathfrak{D}_{L/K} = \prod_{i=1}^{r} p_i^{(\alpha_i+1)(p-1)},$$

where $p_i$ is the divisor in $L$ that divides $\mathcal{P}_i$, for all $i \in \{1, ..., r\}$ (see [4, page 172]).

Let $k = \mathbb{F}_q(T)$ be a congruence rational function field, where $\mathbb{F}_q$ is the finite field of $q = p^s$ elements and $p$ is a prime number. Let $R_T = \mathbb{F}_q[T]$ be the ring
of polynomials. For \( N \in \mathbb{R}_T \setminus \{0\} \), \( \Lambda_N \) denotes the \( N \)-torsion of the Carlitz module. Finally, \( k(\Lambda_N) \) denotes the \( N \)-th cyclotomic function field (see [2]). We have \( G_N := \text{Gal}(k(\Lambda_N)/k) \cong (R_T/(N))^* \) and \( \Phi(N) := |G_N| \). For an irreducible polynomial \( P \in R_T \) and \( \alpha \in \mathbb{N} \) it holds \( \Phi(P^\alpha) = q^{(\alpha-1)d(q^d-1)} \) where \( d = \deg P \).

We know that the fact that any Artin–Schreier extension of a rational congruence function field is contained in the composite of cyclotomic extensions and a constant field extension is a consequence of the Kronecker–Weber Theorem for function fields proved by David Hayes using the Reciprocity Law of class field theory ([2]).

The purpose of this paper is to give a combinatorial proof that any Artin–Schreier extension of a rational congruence function field is contained in the composite of cyclotomic function fields and a constant field extension that are explicitly described.

The proof is given as follows. First, we consider Artin–Schreier extensions with only one prime ramifying. We compute the number of such extensions having their conductor a divisor of a given power of the ramified prime. Next, we prove that this number equals the number of Artin–Schreier extensions contained in the composite of the cyclotomic function field generated by the same power of the ramified prime and a constant extension of degree \( p \). The general case follows immediately from this case using partial fractions decomposition.

2. The result

The following proposition provides different manners that an Artin–Schreier extension can be generated.

**Proposition 2.1.** Let \( K \) be a field of characteristic \( p > 0 \) and let \( L_1 = K(y) \) and \( L_2 = K(z) \) be cyclic extensions of degree \( p \) over \( K \) given by

\[
y^p - y = a_1 \in K \quad \text{and} \quad z^p - z = a_2 \in K.
\]

Then the following statements are equivalent:

(a) \( L_1 = L_2 \).

(b) \( z = jy + b \) for some \( 1 \leq j \leq p - 1 \) and \( b \in K \).

(c) \( a_2 = ja_1 + (b^p - b) \) for some \( 1 \leq j \leq p - 1 \) and \( b \in K \).

**Proof.** See [4, page 171]. \( \square \)

We now consider Artin–Schreier extensions where precisely one prime divisor ramifies. First we treat the case when the divisor corresponds to a monic irreducible polynomial \( P \).

**Proposition 2.2.** Let \( K = \mathbb{F}_q(T) \), \( q = p^t \), \( P \) be a monic irreducible polynomial in \( \mathbb{F}_q[T] \), \( d = \deg P \), \( L_1 = K(y) \), where

\[
y^p - y = \frac{f(T)}{p^\alpha},
\]

\( f(T) \in \mathbb{F}_q[T], \deg f(T) \leq d\) \( a, (f(T), P) = 1, \alpha \in \mathbb{N} \), \( (\alpha, p) = 1 \), \( \alpha_0 := \left\lfloor \frac{\alpha}{p} \right\rfloor \), the integer part of \( \frac{\alpha}{p} \), and \( L_2 = K(z) \), where

\[
z^p - z = \frac{g(T)}{p^\alpha},
\]

where \( g(T) \) is a polynomial in \( \mathbb{F}_q[T] \) of degree \( d \).
contradiction. Similarly, we have equivalent:

**Corollary 2.3.**

Let \( h(T) = \frac{P}{\alpha} \) with \( h(T) \in \mathbb{F}_q[T] \), where either \( h(T) = 0 \) or \( \deg h(T) \leq d\alpha_0 \).

**Proof.** The equivalences follow from Proposition 2.1. We verify the conditions to be met by \( c \).

Let \( Q \) be a monic irreducible polynomial in \( \mathbb{F}_q[T] \). We have \( \nu_Q(c) \geq 0 \) since if \( \nu_Q(c) < 0 \) we would have

\[
0 \leq \nu_Q \left( \frac{g(T)}{\alpha} \right) = \nu_Q \left( j \frac{f(T)}{\alpha} + c^p - c \right)
= \min \left\{ \nu_Q \left( \frac{f(T)}{\alpha} \right), \nu_Q(c) \right\} = \nu_Q(c) < 0,
\]

which is a contradiction. Similarly, we have \( \nu_p(c) \geq 0 \). Finally, we have \( c = 0 \) or \( \nu_p(c) \leq 0 \) since \( \nu_p(c) > 0 \) implies \( 0 \geq \deg(c) = \sum_{Q \in P_K} \nu_Q(c) > 0 \) and this is a contradiction.

Therefore we consider the following possibilities:

(a) \( c = 0 \).

(b) \( c \neq 0 \), \( \nu_p(c) = 0 \) which implies \( c \in \mathbb{F}_q^* \).

(c) \( \nu_p(c) < 0 \), let \( \gamma := -\nu_p(c) \), then \( \nu_p(c^p - c) = \min\{p\nu_p(c), \nu_p(c)\} = -p\gamma \neq -\alpha \), because \((\alpha, p) = 1\). Thus

\[
-\alpha = \nu_p \left( \frac{g(T)}{\alpha} \right) = \nu_p \left( j \frac{f(T)}{\alpha} + c^p - c \right)
= \min \left\{ \nu_p \left( \frac{f(T)}{\alpha} \right), \nu_p(c^p - c) \right\} = \min \{-\alpha, -p\gamma\}.
\]

This implies \( -\alpha < -p\gamma \), so that \( p\gamma < \alpha \). Therefore \( \gamma \leq \alpha_0 = \left[ \frac{\alpha}{p} \right] \), then \( c = \frac{h_1(T)}{P^{\gamma}} \) with \( h_1(T) \in \mathbb{F}_q[T] \), \( (h_1(T), P) = 1 \), \( \deg h_1(T) \leq d\gamma \) and \( \gamma \leq \alpha_0 \), which implies \( c = \frac{h_1(T)P^{\alpha_0 - \gamma}}{P^{\alpha_0}} \).

In the first two cases we put \( h(T) = cP^{\alpha_0} \) and in the third \( h(T) = h_1(T)P^{\alpha_0 - \gamma} \).

We conclude that \( c = \frac{h(T)}{P^{\alpha_0}} \) with either \( h(T) = 0 \) or \( \deg h(T) \leq d\alpha_0 \). \( \square \)

**Corollary 2.3.** Let \( K, q, P \) and \( d \) be as in Proposition 2.2. Let \( L = K(y) \), be an Artin–Schreier extension given in normal form, that is

\[
y^p - y = \frac{f(T)}{P^{\alpha_0}}.
\]
Lemma 2.5. Let $d = \deg f, \alpha$ given in normal form $\alpha$ where $1 \leq c \leq d$ obtain the number of possible equations just note that $h > 0$. Proof. By Proposition 2.2 the number of possibilities for $\alpha$ in different cyclic extensions $L$ is $\alpha$ such that $L = K(z)$ is also given in normal form

$$z^p - z = \frac{g(T)}{p^\alpha}.$$  

Further, these elements $z$ are given by $z = y + c$, where $1 \leq j \leq p - 1$ and $c = \frac{h(T)}{p^\alpha}$, with $h(T) \in \mathbb{F}_q[T], \deg h(T) \leq d \alpha$. Moreover, there are

$$\frac{p - 1}{p} q^{(d \alpha + 1)}$$

different equations of the form (2.1) with $g(T) \in \mathbb{F}_q[T], \deg g(T) \leq d \alpha, (g(T), P) = 1$, so that $z$ is as above and $\frac{g(T)}{p^\alpha} = j \frac{f(T)}{p^\alpha} + c^p - c$.

Proof. By Proposition 2.2 the number of possibilities for $j$ is $p - 1$ and the number of possible $h(T)$ is $q^{(d \alpha + 1)}$, thus the number of possible $z$ is $(p - 1)q^{(d \alpha + 1)}$. To obtain the number of possible equations just note that $c^p - c = b^p - b$ if and only if $(c - b)^p = c - b$ if and only if $c - b \in \mathbb{F}_p$.

Corollary 2.4. Let $K, q, P$ and $d$ be as above. Then the number of different Artin–Schreier extensions $L = K(y)$ where

$$y^p - y = \frac{f(T)}{p^\alpha}$$

with $f(T) \in \mathbb{F}_q[T], \deg f(T) \leq d \alpha, (f(T), P) = 1, \alpha \in \mathbb{N}, (\alpha, p) = 1$, is

$$N_\alpha := \frac{p - 1}{p} q^{(\alpha - \alpha_0)},$$

where $\alpha_0 = \left\lceil \frac{\alpha}{p} \right\rceil$.

Proof. By the division algorithm we have $f(T) = a P^\alpha + h(T)$, where $a \in \mathbb{F}_q, h(T) \in \mathbb{F}_q[T]$, with either $\deg h(T) \leq d \alpha - 1$ and $(h(T), P) = 1$ or $h(T) = 0$. Then $y^p - y = a + \frac{h(T)}{P^\alpha}$. The number of equations of this type is

$$q \Phi(P^\alpha) = q \cdot q^{(\alpha - 1)d}(q^d - 1).$$

By Corollary 2.3 we obtain that there are

$$N_\alpha = \frac{p - 1}{p} q^{(\alpha - 1)d}(q^d - 1)$$

different cyclic extensions $L$ of degree $p$ over $K$ where $P$ is the only prime ramified and the power $\alpha$ appears in the Artin–Schreier equation in its normal form. \hfill \Box

Lemma 2.5. Let $K = \mathbb{F}_q(T), q = p^t$, $P$ a monic irreducible polynomial in $\mathbb{F}_q[T], d = \deg P$ and $\alpha \in \mathbb{N}, (\alpha, p) = 1$. 


(a) Let $F = K(y)$ where

$$y^p - y = \frac{f(T)}{P^\alpha}$$

with $f(T) \in \mathbb{F}_q[T]$, $\deg f(T) \leq \alpha d$ and $(f(T), P) = 1$.

Assume $F \subseteq K(\Lambda_M)$ for some $M \in \mathbb{R}_T \setminus \{0\}$. Then $F \subseteq K(\Lambda_{p^{\alpha+1}})$ but $F \not\subseteq K(\Lambda_{p^\alpha})$.

(b) Conversely, if $F/K$ is an Artin–Schreier extension such that $F \subseteq K(\Lambda_{p^{\alpha+1}})$ but $F \not\subseteq K(\Lambda_{p^\alpha})$, then $F = K(y)$, where

$$y^p - y = \frac{f(T)}{P^\alpha}$$

where $f(T) \in \mathbb{F}_q[T]$, $\deg f(T) \leq \alpha d$ and $(f(T), P) = 1$.

Proof. (a) We have that $\mathcal{P}$, the prime divisor associated to the polynomial $P$, is the only ramified prime divisor in $F/K$ and the different of $F/K$ is

$$\mathcal{D}_{F/K} = p^{\alpha+1}(p-1),$$

where $p$ is the only prime divisor in $F$ that divides $\mathcal{P}$. Then the discriminant of $F/K$ is

$$\delta_{F/K} = \mathcal{P}^{(\alpha+1)(p-1)}.$$

Since $F$ is contained in a cyclotomic field and the extension $F/K$ is cyclic, $F$ is the field associated to some character $\Theta$ of order $p$ and conductor $\mathfrak{f}_\Theta$ (see [4, Chapter 12]).

By the conductor-discriminant formula (see [3]) we have

$$\delta_{F/K} = \prod_{\varphi \in \langle \Theta \rangle} \mathfrak{f}_\varphi = \mathfrak{f}_\Theta^p \mathfrak{f}_\Theta^1 \cdots \mathfrak{f}_\Theta^{p-1} = \mathfrak{f}_\Theta^{p^\alpha - 1}.$$

Then $\mathcal{P}^{(\alpha+1)(p-1)} = \mathfrak{f}_\Theta^{p^\alpha - 1}$, so that $\mathfrak{f}_\Theta = \mathcal{P}^{(\alpha+1)}$. Hence we conclude that $F \subseteq K(\Lambda_{p^{\alpha+1}})$ but $F \not\subseteq K(\Lambda_{p^\alpha})$ (see [4, Chapter 12]).

(b) Since $F \subseteq K(\Lambda_{p^{\alpha+1}})$, $P$ is the only ramified prime in $F/K$. Then $F = K(y)$ where $y^p - y = \frac{f(T)}{P^\alpha}$, where $f(T) \in \mathbb{F}_q[T]$, $\deg f(T) \leq \beta d$, $(f(T), P) = 1$. It follows from (a) that $\beta = \alpha$. \hfill $\square$

Proposition 2.6. Let $K = \mathbb{F}_q(T)$, $q = p^t$ and $P$ be a monic irreducible polynomial in $\mathbb{F}_q[T]$, $d = \deg P$, $\beta \in \mathbb{N}$, $\beta \geq 2$. Then the number of Artin–Schreier extensions contained in the cyclotomic field $K(\Lambda_{p^\beta})$ is

$$\mathcal{N}_\beta := \frac{q^{(\beta - \left\lceil \frac{\beta}{p} \right\rceil)d} - 1}{p - 1},$$

where $\left\lceil \frac{\beta}{p} \right\rceil$ denotes the ceiling of $\frac{\beta}{p}$, namely the minimum integer greater than or equal to $\frac{\beta}{p}$.

Proof. Since the lattice of subgroups of an abelian group is symmetric, we have by Galois theory $\mathcal{N}_\beta$ equals the number of subgroups of order $p$ of the Galois group...
$\left(R_T/P^\beta\right)^*$ of the extension $K(\Lambda_{p^0})/K$. Let $r_p$ be the number of elements of order $p$ in $\left(R_T/P^\beta\right)^*$. We consider the exact sequence

$$
1 \rightarrow D_{p^0, P} \rightarrow \left(R_T/P^\beta\right)^* \rightarrow \left(R_T/P\right)^* \rightarrow 1.
$$

Therefore

$$
\left(R_T/P^\beta\right)^* \cong D_{p^0, P} \times \left(R_T/P\right)^*.
$$

We have $D_{p^0, P} = \{A \mod P^\beta | A \equiv 1 \mod P\} = \{A \mod P^\beta | A = 1 + h(T)P, h(T) \in \mathbb{F}_q[T] \text{ and } \deg h(T) \leq (\beta - 1)d - 1\}$. Observe that $|D_{p^0, P}| = q^{(\beta - 1)d}$. We also note that $A \mod P^\beta \equiv 1 \mod P^\beta$ if and only if $h(T) = 0$. Then $A \mod P^\beta$ is of order $p$ if and only if $h(T) \neq 0$ and $(1 + h(T)P)^p = 1 + h(T)P^p \equiv 1 \mod P^\beta$ if and only if $P^\beta | h(T)P^p$. We have two cases:

(a) If $\beta \leq p$, it holds for all $A \mod (\left(R_T/P^\beta\right)^*) \in D_{p^0, P}$ and so $r_p = q^{(\beta - 1)d} - 1$.

(b) If $\beta > p$, we have $P^\beta g(T) = h(T)P^p$, for some $g(T) \in \mathbb{F}_q[T]$. Therefore $P^{\beta - p}g(T) = h(T)^p$. So $h(T) = P^\gamma h_1(T)$ for some $\gamma \in \mathbb{N}$ and $h_1(T) \in \mathbb{F}_q[T]$ with $(h_1(T), P) = 1$. We have $P^{\beta - p}g(T) = P^\gamma P^p h_1(T)^p$. Then $\gamma p \geq \beta - p$,

$$\gamma \geq \frac{\beta}{p} - 1, \gamma + 1 \geq \frac{\beta}{p}.$$ 

Thus, $\gamma + 1 \geq \left\lceil \frac{\beta}{p} \right\rceil$. Therefore $h(T) = P^{\left\lceil \frac{\beta}{p} \right\rceil - 1} h_2(T)$, where

$$
\deg h_2(T) = \deg h(T) - \left(\left\lceil \frac{\beta}{p} \right\rceil - 1\right) d \leq (\beta - 1)d - 1 - \left(\left\lceil \frac{\beta}{p} \right\rceil - 1\right) d
$$

$$= \left(\beta - \left\lceil \frac{\beta}{p} \right\rceil\right) d - 1.
$$

It follows that $r_p = q^{\left(\beta - \left\lceil \frac{\beta}{p} \right\rceil\right) d - 1}$ is the number of elements of order $p$ of $\left(R_T/P^\beta\right)^*$.

Then, in any case, the number of subgroups of order $p$ of $\left(R_T/P^\beta\right)^*$ is

$$
N_\beta = \frac{r_p}{p - 1} = q^{\left(\beta - \left\lceil \frac{\beta}{p} \right\rceil\right) d - 1} / (p - 1).
$$

\[\square\]

**Corollary 2.7.** Let $K = \mathbb{F}_q(T)$, $q = p^t$ and $P$ be a monic irreducible polynomial in $\mathbb{F}_q[T]$, $d = \deg P$, $\alpha \in \mathbb{N}$, $(\alpha, p) = 1$, $\alpha_0 = \left\lceil \frac{\alpha}{p} \right\rceil$. Then the number of Artin–Schreier extensions contained in $K(\Lambda_{p^{\alpha_0}})$ but not in $K(\Lambda_{p^{\alpha_0 - 1}})$ is

$$
\Phi(P^{\alpha_0 - \alpha_0}) = \frac{\Phi(P^{\alpha_0 - \alpha_0})}{p - 1}.
$$
Proof. By Proposition 2.6 we have that number of Artin–Schreier extensions contained in $K(\Lambda_{p^\alpha})$ but not in $K(\Lambda_p)$ is equal to

$$N_{\alpha+1} - N_\alpha = \frac{q^{(\alpha+1-\frac{\alpha+1}{p})d} - q^{\alpha-\frac{\alpha}{p}d}}{p-1} - \frac{q^{(\alpha-\frac{\alpha}{p})d} - q^{\alpha-\frac{\alpha}{p}d}}{p-1} = \frac{q^{\alpha-\frac{\alpha}{p}d}q^{(1-\frac{\alpha+1}{p})d} - q^{\alpha-\frac{\alpha}{p}d}q^{(1-\frac{\alpha}{p})d}}{p-1}$$

$$= \Phi(P^{\alpha-\alpha_0}) = \frac{\Phi(P^{\alpha-\alpha_0})}{p-1},$$

because when $p \nmid \alpha$, we have $\left\lceil \frac{\alpha+1}{p} \right\rceil = \left\lceil \frac{\alpha}{p} \right\rceil$ and $\left\lfloor \frac{\alpha}{p} \right\rfloor = \left\lfloor \frac{\alpha}{p} \right\rfloor + 1$. □

Corollary 2.8. Let $K = \mathbb{F}_q(T)$, $q = p^t$ and $P$ be a monic irreducible polynomial in $\mathbb{F}_q[T]$, $d = \deg P$, $\alpha \in \mathbb{N}$, $(\alpha, p) = 1$, $\alpha_0 = \left\lfloor \frac{\alpha}{p} \right\rfloor$. Consider the Artin–Schreier extensions $F = K(y)$ of $K$ where

$$y^p - y = \frac{f(T)}{P^\alpha}$$

with $f(T) \in \mathbb{F}_q[T]$, $\deg f(T) \leq ad$ and $(f(T), P) = 1$. Then there are at least

$$N_\alpha = \frac{p}{p-1}\Phi(P^{\alpha-\alpha_0})$$

extensions $F$ of the type described in (2.2) contained in $K(\Lambda_{p^\alpha+1})\mathbb{F}_{q^p}$.

Proof. Consider the following diagram.

```
| K(\Lambda_{p^\alpha+1}) | F |
|-------------------------|---|
|                         | p |
|                         | K |
```

We have $\mathbb{F}_{q^p} = \mathbb{F}_q(\xi)$, where $\xi^p - \xi = \rho$ with $\rho \in \mathbb{F}_q \setminus \wp(\mathbb{F}_q)$.

By Corollary 2.7 there are $\frac{\Phi(P^{\alpha-\alpha_0})}{p-1}$ Artin–Schreier extensions $F/K$ contained in $K(\Lambda_{p^\alpha+1})$ but not in $K(\Lambda_p)$. By Lemma 2.5 such $F$ is of type (2.2). From one such $F$ we obtain $p$ fields $F_i = K(y_i)$, where $y_i = y + i\xi$ for $0 \leq i \leq p-1$, of degree $p$ over $K$ and contained in $K(\Lambda_{p^\alpha+1})\mathbb{F}_{q^p}$. We will verify that these fields are of the type required, that they are different and also that if $E, F \subseteq K(\Lambda_{p^\alpha+1})$ are of the type required and $E \neq F$, then $E_i \neq F_j$, for all $i, j \in \{0, \ldots, p-1\}$. With the above we conclude that at least $\frac{p}{p-1}\Phi(P^{\alpha-\alpha_0})$ extensions $F$ of the type required are contained in the composite of the cyclotomic extension and the constant extension.
We have
\[(y + i\xi)^p - (y + i\xi) = y^p - y + iP^p - i\xi = y^p - y + i(\xi^p - \xi)\]
\[= f(T) + ip = f(T) + ip\frac{P^\alpha}{P^\alpha} = g_i(T)\]
with \(g_i(T) \in \mathbb{F}_q[T]\), \(\deg g_i(T) \leq \alpha\) for all \(i \in \{0, ..., p - 1\}\).

Then \(F_i = K(y + i\xi)\) is an extension of the type required. Suppose now that \(0 \leq i, j \leq p - 1\), \(i \neq j\) and \(F_i = F_j\). Then \(y + i\xi, y + j\xi \in F_i = F_j\), therefore \(\xi \in F_i = F_j\), so that \(y \in F_i = F_j = F\). Thus \(\xi \in F_i = F_j = F\), which is a contradiction because the constant field of \(F\) is \(\mathbb{F}_q\).

Finally, assume that \(F, E\) are extensions of the type required, contained in \(K(\Lambda_{p^{o+1}})\) with \(F \neq E\) and \(E_i = F_j\) for some \(i, j \in \{0, ..., p - 1\}\).

Say \(F = K(y)\) with \(y^p - y = f(T)\), \(f(T) \in \mathbb{F}_q[T]\), \(\deg f(T) \leq \alpha\) and \(E = K(z)\) with \(z^p - z = g(T)\), \(g(T) \in \mathbb{F}_q[T], \deg g(T) = \alpha\).

We have \(i \neq 0\) and \(j \neq 0\), since otherwise \(\xi \in K(\Lambda_{p^{o+1}})\), which is a contradiction because the constant field of \(K(\Lambda_{p^{o+1}})\) is \(\mathbb{F}_q\). Let \(l = i^{-1}j\). We have \(l(y + j\xi) = l(y + i\xi), z + j\xi \in F_i = F_j\). Then \(z - ly \in F_i = F_j\). But \(z - ly \notin K\), for \(F \neq E\). Therefore \(\mathbb{F}_{q^p} \subseteq F_i = F_j = K(z - ly) \subseteq K(\Lambda_{p^{o+1}})\), which is a contradiction. \(\square\)

Next lemma is the main step to our main result. We prove that any cyclic extension of degree \(p\) over \(K\) in which \(P\) is the only ramified prime divisor and it appears to the power \(\alpha\) in the Artin-Schreier equation in its normal form is contained in the composite of a cyclotomic field and a constant extension.

**Lemma 2.9.** Let \(p\) be a prime number, \(q = p^i\) and \(P\) be a monic irreducible polynomial in \(\mathbb{F}_q[T]\), \(d = \deg P\). Let \(K = \mathbb{F}_q(T)\) and \(F = K(y)\), where
\[y^p - y = f(T)\]
with \(f(T) \in \mathbb{F}_q[T], \deg f(T) \leq \alpha d, (f(T), P) = 1, \alpha \in \mathbb{N}, \text{ and } (\alpha, p) = 1\). Then
\[F \subseteq K(\Lambda_{p^{o+1}})\mathbb{F}_{q^p} .\]

**Proof.** By Corollaries 2.4 and 2.8 we have that the number of cyclic extensions of degree \(p\) over \(K\) in which \(P\) is the only ramified prime and it appears to the power \(\alpha\) in the Artin-Schreier equation in its normal form must be equal to the number of such extensions contained in the composite of the cyclotomic and the constant extensions mentioned above. Therefore we conclude that any such extension \(F\) is contained in the composite \(K(\Lambda_{p^{o+1}})\mathbb{F}_{q^p}\). \(\square\)

In the following result are considered Artin–Schreier extensions in which the infinite prime divisor is the only ramified prime divisor.

**Lemma 2.10.** Let \(p\) be a prime number and \(q = p^i\). Let \(K = \mathbb{F}_q(T)\) and \(F = K(y)\), where \(y^p - y = f(T) \in \mathbb{F}_q[T], \deg f(T) = \alpha, \alpha \in \mathbb{N}, \text{ and } (\alpha, p) = 1\). Then
\[F \subseteq K(\Lambda_{p^{o+1}})\mathbb{F}_{q^p} .\]

**Proof.** It follows from Lemma 2.9 if we observe that \(K(T) = K\left(\frac{1}{T}\right)\). \(\square\)
We note that to simplify the statement of the main result in the abstract and in the introduction, we have included the field $K(\Lambda_{\frac{1}{\alpha+1}})$ among cyclotomic fields.

Finally, we present our main result.

**Theorem 2.11.** Let $K = F_q(T)$, where $q = p^l$ and $p$ is a prime number. Let $F/K$ be an Artin–Schreier extension, namely $F = K(y)$, where

$$y^p - y = s(T),$$

with $s(T) \in K$, $s(T) \notin \wp(K)$,

$$(s(T))_K = \frac{\mathfrak{c}}{P_1^{\alpha_1} \cdots P_r^{\alpha_r}},$$

where $P_i$ is a prime divisor in $K$, $\alpha_i \in \mathbb{N}$, $(\alpha_i, p) = 1$, $\mathfrak{c}$ is an integral divisor relatively prime to $P_i$ for $i \in \{1, \ldots, r\}$.

(a) If the prime divisor $P_\infty$ is not ramified in $F/K$, that is, if $P_\infty$ is not a factor of $\prod_{i=1}^{r} P_i^{\alpha_i}$, then $F \subseteq K(\Lambda_{\prod_{i=1}^{r} P_i^{\alpha_i+1}})F_q^p$.

(b) If $P_\infty$ is ramified in $F/K$, that is, if $P_\infty$ is a factor of $\prod_{i=1}^{r} P_i^{\alpha_i}$, say $P_\infty = P_1$, then $F \subseteq K(\Lambda_{\frac{1}{\alpha+1}})K(\Lambda_{\prod_{i=2}^{r} P_i^{\alpha_i+1}})F_q^p$.

**Proof.** The divisor $P_i$ corresponds to the polynomial $P_i$, for $i \in \{1, \ldots, r\}$ in case (a) and for $i \in \{2, \ldots, r\}$ in case (b).

(a) By the partial fractions method we have:

$$s(T) = \frac{f(T)}{P_1^{\alpha_1} \cdots P_r^{\alpha_r}} = \frac{f_1(T)}{P_1^{\alpha_1}} + \cdots + \frac{f_r(T)}{P_r^{\alpha_r}},$$

where $f_i(T) \in F_q[T]$ and $\deg f_i(T) \leq \alpha_i \deg P_i$ for all $i \in \{1, \ldots, r\}$. Consider $F_i = K(y_i)$, where $y_i^p - y_i = \frac{f_i(T)}{P_i^{\alpha_i}}$ for $i \in \{1, \ldots, r\}$. By Lemma 2.9, it follows that $F_i \subseteq K(\Lambda_{P_i^{\alpha_i+1}})F_q^p$ for $i \in \{1, \ldots, r\}$. We note that we can put $y = y_1 + \cdots + y_r$ since $y^p - y = y_1^p - y_1 + \cdots + y_r^p - y_r = \frac{f(T)}{P_1^{\alpha_1} \cdots P_r^{\alpha_r}}$. Then $F = K(y) \subseteq K(\Lambda_{P_1^{\alpha_1+1}}) \cdots K(\Lambda_{P_r^{\alpha_r+1}})F_q^p = K(\Lambda_{P_1^{\alpha_1+1} \cdots P_r^{\alpha_r+1}})F_q^p$.

(b) In this case $P_\infty = P_1$ and we have

$$y^p - y = s(T) = \frac{g(T)}{P_2^{\alpha_2} \cdots P_r^{\alpha_r}},$$

with $\deg \left(\frac{g(T)}{P_2^{\alpha_2} \cdots P_r^{\alpha_r}}\right) = \alpha_1 \in \mathbb{N}$, $g(T) \in F_q[T]$ and $P_2, \ldots, P_r$ monic irreducible polynomials. By the division algorithm

$$g(T) = (P_2^{\alpha_2} \cdots P_r^{\alpha_r})h(T) + l(T),$$
where \( h(T), l(T) \in \mathbb{F}_q[T] \), \( \deg h(T) = \alpha_1 \) and either \( \deg l(T) < \sum_{i=2}^r \alpha_i \deg P_i \) or \( l(T) = 0 \). Then
\[
y^p - y = h(T) + \frac{l(T)}{\prod_{i=2}^r P_i^{\alpha_i}}.
\]
We consider \( F_1 = K(y_1) \), where \( y^p_1 - y_1 = h(T) \). As in case (a), by the partial fractions method we have:
\[
\frac{l(T)}{\prod_{i=2}^r P_i^{\alpha_i}} = \frac{l_2(T)}{P_2^{\alpha_2}} + \cdots + \frac{l_r(T)}{P_r^{\alpha_r}},
\]
where \( l_i(T) \in \mathbb{F}_q[T] \) and \( \deg l_i(T) \leq \alpha_i d_i \) for all \( i \in \{2, \ldots, r\} \). From Lemmas 2.9 and 2.10 we have
\[
F_1 \subseteq K(\Lambda_{\frac{1}{\alpha_1+1}})\mathbb{F}_q^p,
F_i \subseteq K(\Lambda_{\frac{1}{\alpha_i+1}})\mathbb{F}_q^p \text{ for all } i \in \{2, \ldots, r\}.
\]
We have \( y = y_1 + \cdots + y_r \) since \( y^p - y = y_1^p - y_1 + \cdots + y_r^p - y_r = h(T) + \frac{l(T)}{\prod_{i=2}^r P_i^{\alpha_i}} \).

We conclude
\[
F = K(y) \subseteq K(\Lambda_{\frac{1}{\alpha_1+1}})K(\Lambda_{\frac{1}{\alpha_2+1}})\cdots K(\Lambda_{\frac{1}{\alpha_r+1}})\mathbb{F}_q^p.
\]

\[
= K(\Lambda_{\frac{1}{\alpha_1+1}})K(\Lambda_{\frac{1}{\alpha_2+1}})\cdots K(\Lambda_{\frac{1}{\alpha_r+1}})\mathbb{F}_q^p.
\]

\[\square\]

References
[1] Helmut Hasse, Theorie der relativ–zyklischen algebraischen Funktionenkörper, insbesondere bei endlichen Konstantenkörper, J. Reine Angew. Math. 172 (1934), 37–54.
[2] David Hayes, Explicit Class Field Theory for Rational Function Fields, Trans. Amer. Math. Soc. 189 (1974), 77–91.
[3] Martha Rzedowski–Calderón and Gabriel Villa–Salvador, ConductorDiscriminant Formula for Global Function Fields, International Journal of Algebra 5 (2011), 1557–1565.
[4] Gabriel Villa–Salvador, Topics in the theory of algebraic function fields, Birkhäuser, Boston, 2006.