COMPATIBILITY CONDITION BETWEEN RING AND CORING.

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Abstract. We introduce the notion of bi-monoid in general monoidal category generalizing by this the notion of bialgebra. In the case of bimodules over a noncommutative algebra, we obtain a compatibility condition between ring and coring whenever both structures admit the same underlying bimodule.

Introduction

Let $R$ be an associative algebra over a commutative ring with identity $k$, and consider its category of unital and $k$-central bimodule $R\mathcal{M}_R$ as a monoidal category with multiplication the two variables functor defined by the tensor product $- \otimes_R -$ and with identity object the regular bimodule $R_R R$. A compatibility condition between a ring structure (monoid in $R\mathcal{M}_R$, see below) and coring structure (comonoid in $R\mathcal{M}_R$, see below), is a well known problem in noncommutative algebra. Even though, the object is the same (i.e., the underlying $R$-bimodules structures coincide), there is no obvious way in which a tensor product of bimodules can be equipped with a monoid structure. In this direction M. E. Sweedler [10, §5] introduced the $\times_A$-product, and few years later M. Takeuchi gave in [11] the notion of $\times_A$-bialgebra with noncommutative basering. This notion can be seen as an approach to a compatibility condition problem by taking, in appropriate way, a comonoid in $R\mathcal{M}_R$ and a monoid in $R\otimes_k R\mathcal{M}_{R\otimes_k R}$ ($R^e$ is the opposite ring of $R$).

In this note we give another approach to the compatibility condition problem by considering a comonoid and monoid in the same monoidal category. The basic ideas behind our approach are the notions of wreath and cowreath recently introduced by S. Lack and R. Street in [6], which are a generalization of distributive law due to J. Beck [1]. In the bimodules category, wreath and cowreath lead in a formal way to endow certain tensor product within a structure of ring and of coring. A double distributive law is then a wreath and cowreath induced, respectively, by a ring and coring taking the same underlying bimodule. In this way we arrive to the compatibility condition by assuming the existence of a double distributive law for a bimodule which admits a structures of both ring and coring.

In Section 1 we review the Eilenberg-Moore categories attached to a monoid (resp. comonoid) in a strict monoidal category. We give, as in the case of bimodules category [5], a simplest and equivalent definition for a wreath (resp. cowreath) over monoid (resp. comonoid), see Proposition 1.1 (resp. Proposition 1.4). In particular we show that the wreath (resp. cowreath) product satisfies a universal property, Proposition 1.8 (resp. Proposition 1.11). In Section 2 we use the notion of double distributive law (Definition 2.1) in order to prove an equivalent compatibility conditions for an object with a structures of both monoid and comonoid, Proposition 2.2. An object satisfying the equivalent conditions of such proposition is called a bi-monoid. In Section 3 presents an application of results stated in previous sections to the bimodules category. In particular, we give an example which shows that the class of bialgebras is "strictly contained" in the class of bi-monoid in the monoidal category of $k$-modules.

Notations and Basic Notions: Given any Hom-set category $\mathcal{C}$, the notation $X \in \mathcal{A}$ means that $X$ is an object of $\mathcal{C}$. The identity morphism of $X$ will be denoted by $X$. The set of all morphisms $f : X \to X'$ in $\mathcal{C}$, is denoted by $\text{Hom}_{\mathcal{C}}(X, X')$. Let $\mathcal{M}$ be a strict monoidal category with multiplication $- \otimes -$ and identity object $I$. Recall from
[\[ \text{Proof.} \]

Lemma 1.1. We have the following useful lemma defined as follows:

This is the right Eilenberg-Moore monoidal category associated to the comonoid \( \Delta : C \to C \otimes C \) (comultiplication), \( \varepsilon : C \to 1 \) (counit) such that \( (\varepsilon \otimes \varepsilon) \circ \Delta = C = (C \otimes \varepsilon) \circ \Delta \) and \( (\Delta \otimes \varepsilon) \circ \Delta = (\varepsilon \otimes \varepsilon) \circ \Delta \).

A morphisms of comonoids \( \phi : (C, \Delta, \varepsilon) \to (D, \Delta', \varepsilon') \) is a morphism \( \phi : C \to D \) in \( \mathcal{M} \) such that \( \varepsilon' \circ \phi = \varepsilon \) (counit property) and \( \Delta' \circ \phi = (\phi \otimes \phi) \circ \Delta \) (coassociativity property). Dually, a monoid in \( \mathcal{M} \) is a three-tuple \( (\Lambda, \mu, \eta) \) consisting of an object \( \Lambda \in \mathcal{M} \) and two morphisms \( \mu : \Lambda \otimes \Lambda \to \Lambda \) (multiplication), \( \eta : 1 \to \Lambda \) (unit) such that \( \mu \circ (\eta \otimes \Lambda) = \Lambda = \mu \circ (\Lambda \otimes \eta) \) and \( \mu \circ (\Lambda \otimes \mu) = \mu \circ (\mu \otimes \Lambda) \). A morphism of monoid \( \psi : (\Lambda, \mu, \eta) \to (\Lambda', \mu', \eta') \) is a morphism \( \psi : \Lambda \to \Lambda' \) such that \( \psi \circ \eta = \eta' \) and \( \psi \circ \mu = \mu' \circ (\psi \otimes \psi) \).

A right \( C \)-comodule is a pair \((X, \rho_X)\) consisting of an object \( X \in \mathcal{M} \) and a morphism \( \rho_X : X \to X \otimes C \) (a right \( C \)-coaction) such that \((X \otimes \varepsilon) \circ \rho_X = X\) and \((\rho_X \otimes C) \circ \rho_X = (X \otimes \Delta) \circ \rho_X\). A morphism of right \( C \)-comodules \( f : (X, \rho_X) \to (X', \rho_{X'}) \) is a morphism \( f : X \to X' \) in the category \( \mathcal{M} \) such that \( \rho_{X'} \circ f = (f \otimes C) \circ \rho_X \).

Left \( C \)-comodules and their morphisms are similarly defined; we use the Greek letter \( \lambda^- \) to denote their left \( C \)-coactions. A \( C \)-bicomodule is a three-tuple \((X, \rho_X, \lambda_X)\) where \((X, \rho_X)\) is a right \( C \)-comodule and \((X, \lambda_X)\) is a left \( C \)-comodule such that \((\lambda_X \otimes C) \circ \rho_X = (C \otimes \rho_X) \circ \lambda_X\). A morphism of \( C \)-bicomodules is a morphism of left and of right \( C \)-comodules. We use the notation \( \text{Hom}_{C-C}(\cdot, \cdot) \) for the sets of all \( C \)-bicomodules morphisms. Dually, a right \( \Lambda \)-module is a pair \((P, \tau_P)\) consisting of an object \( P \in \mathcal{M} \) and a morphisms \( \tau_P : P \otimes \Lambda \to P \) (a right \( \Lambda \)-action) such that \( \tau_P \circ (P \otimes \eta) = P \) and \( \tau_P \circ (P \otimes \mu) = \tau_P \circ (\tau_P \otimes \Lambda) \). A morphism of right \( \Lambda \)-modules \( g : (P, \tau_P) \to (P', \tau_{P'}) \) is a morphism \( g : P \to P' \) in \( \mathcal{M} \) such that \( g \circ \tau_P = \tau_{P'} \circ (g \otimes \Lambda) \).

A \( \Lambda \)-module and their morphisms are similarly defined and we use the letter \( I^- \) to denote a left \( \Lambda \)-actions. An \( \Lambda \)-bimodule is a three-tuple \((P, \tau_P, \iota_P)\) where \((P, \tau_P)\) is a right \( \Lambda \)-module and \((P, \iota_P)\) is a left \( \Lambda \)-module such that \( \tau_P \circ (1_P \otimes \Lambda) = 1_P \circ (\Lambda \otimes \tau_P) \). A morphisms of \( \Lambda \)-bimodules is a morphism of left and of right \( \Lambda \)-modules. We denote by \( \text{Hom}_{\Lambda-\Lambda}(\cdot, \cdot) \) the sets of all \( \Lambda \)-bimodules morphisms. For a details on comodules corings, definitions and basic properties of bicomodules over corings, the reader is referred to monograph [2].

1. Review on (right) Eilenberg-Moore monoidal categories.

Let \( \mathcal{M} \) denote a strict monoidal category with multiplication \(- \otimes -\) and identity object \( I \). In this section we review the Eilenberg-Moore monoidal categories [6] associated to a monoid and to a comonoid both defined in \( \mathcal{M} \). We start by considering a comonoid \( C \) in \( \mathcal{M} \) with a structure morphisms \( \Delta : C \to C \otimes C \) and \( \varepsilon : C \to I \).

1.1. The monoidal category \( \mathcal{R}_C^\varepsilon \).

This is the right Eilenberg-Moore monoidal category associated to the comonoid \( C \) (see [6] for a general context) defined as follows:

Objects of \( \mathcal{R}_C^\varepsilon \): Are pairs \((X, \tau)\) consisting of an object \( X \in \mathcal{M} \) and a morphism \( \tau : C \otimes X \to X \otimes C \) such that

\[
\begin{align*}
(X \otimes \Delta) \circ \tau & = (\tau \otimes C) \circ (C \otimes \tau) \circ (\Delta \otimes X) \\
(X \otimes \varepsilon) \circ \tau & = \varepsilon \circ X.
\end{align*}
\]

We have the following useful lemma:

Lemma 1.1. For every object \( X \in \mathcal{M} \), the following conditions are equivalent:

(i) \( C \otimes X \) is a \( C \)-bicomodule with a left \( C \)-coaction \( \lambda_{C \otimes X} = \Delta \otimes X \);
(ii) there is a morphism \( \tau : C \otimes X \to X \otimes C \) satisfying equalities (1.1) and (1.2).

Proof. (ii) \(\Rightarrow\) (i). Take the right \( C \)-coaction \( \rho_{C \otimes X} = (C \otimes \tau) \circ (\Delta \otimes X) \).

(i) \(\Rightarrow\) (ii). Take \( \tau = (\varepsilon \otimes X \otimes C) \circ \rho_{C \otimes X} \), where \( \rho_{C \otimes X} \) is the given right coaction. \(\square\)

In this way the morphisms in \( \mathcal{R}_C^\varepsilon \) are defined in their unreduced form as follows.
Morphisms in $\mathcal{E}_C$:

$$\text{Hom}_{\mathcal{E}_C} \left( (X, \xi), (X', \xi') \right) = \text{Hom}_{\mathcal{C}-\mathcal{C}} \left( C \otimes X, C \otimes X' \right),$$

where $C \otimes X$ and $C \otimes X'$ are endowed with the structure of $C$-bicomodule defined in Lemma 1.2. That is a morphism $\alpha : (X, \xi) \to (X', \xi)$ in $\mathcal{E}_C$ is morphism $\alpha : C \otimes X \to C \otimes X'$ in $\mathcal{M}$ satisfying

\begin{equation}
(\Delta \otimes X') \circ \alpha = (C \otimes \alpha) \circ (\Delta \otimes X),
\end{equation}

\begin{equation}
(C \otimes \xi') \circ (\Delta \otimes X) \circ \alpha = (\alpha \otimes C) \circ (C \otimes \xi) \circ (\Delta \otimes X).
\end{equation}

The multiplications of $\mathcal{E}_C$: Let $\alpha : (X, \xi) \to (X', \xi')$ and $\beta : (Y, \eta) \to (Y', \eta')$ two morphisms in $\mathcal{E}_C$. One can easily proves that

\begin{align*}
(X, \xi) \otimes_C (Y, \eta) &:= \left( X \otimes Y, (X \otimes \eta) \circ (\xi \otimes Y) \right) \\
(X', \xi') \otimes_C (Y', \eta') &:= \left( X' \otimes Y', (X' \otimes \eta') \circ (\xi' \otimes Y') \right)
\end{align*}

are also an objects of the category $\mathcal{E}_C$, which defines the horizontal multiplication. The vertical one is defined as a composed morphism:

\begin{align*}
(\alpha \otimes_C \beta) &:= (C \otimes X' \otimes \epsilon \otimes Y') \circ (C \otimes X' \otimes \beta) \circ (C \otimes \xi' \otimes Y') \circ (C \otimes \alpha \otimes Y) \circ (\Delta \otimes X \otimes Y) \\
&= (C \otimes X' \otimes \epsilon \otimes Y') \circ (\alpha \otimes \beta) \circ (C \otimes \xi \otimes Y) \circ (\Delta \otimes X \otimes Y).
\end{align*}

Lastly, the identity object of the multiplication $- \otimes_C -$ is given by the pair $(I, C)$ (here $C$ denotes the identity morphism of $C$ in $\mathcal{M}$).

1.2. The monoidal category $\mathcal{E}_C$.

This is the left Eilenberg-Moore category associated to $C$.

Objects of $\mathcal{E}_C$: Are pairs $(p, P)$ consisting of an object $P \in \mathcal{M}$ and a morphism $p : P \otimes C \to C \otimes P$ such that

\begin{align*}
(\Delta \otimes P) \circ p &= C \otimes p \circ (p \otimes \Delta) \\
(\epsilon \otimes P) \circ p &= P \otimes \epsilon.
\end{align*}

As before one can easily checks the following lemma

**Lemma 1.2.** For every object $P \in \mathcal{M}$, the following conditions are equivalent

(i) $P \otimes C$ is a $C$-bicomodule with a right $C$-coaction $\rho^{P \otimes C} = P \otimes \Delta$;

(ii) there is a morphism $p : P \otimes C \to C \otimes P$ satisfying equalities (1.5) and (1.6).

In this way the morphisms in $\mathcal{E}_C$ are defined in their unreduced form as follows

Morphisms in $\mathcal{E}_C$:

$$\text{Hom}_{\mathcal{E}_C} \left( (p, P), (p', P') \right) = \text{Hom}_{\mathcal{C}-\mathcal{C}} \left( P \otimes C, P' \otimes C \right),$$

where $P \otimes C$ and $P' \otimes C$ are endowed with the structure of $C$-bicomodule defined in Lemma 1.2. That is a morphism $\gamma : (p, P) \to (p', P')$ in $\mathcal{E}_C$ is morphism $\gamma : P \otimes C \to P' \otimes C$ in $\mathcal{M}$ satisfying

\begin{align*}
(P' \otimes \Delta) \circ \gamma &= (\gamma \otimes C) \circ (P \otimes \Delta), \\
(p' \otimes C) \circ (P' \otimes \Delta) \circ \gamma &= (C \otimes \gamma) \circ (p \otimes C) \circ (P \otimes \Delta).
\end{align*}

The multiplications of $\mathcal{E}_C$: Let $\gamma : (p, P) \to (p', P')$ and $\sigma : (q, Q) \to (q', Q')$ two morphisms in $\mathcal{E}_C$. One can easily proves that

\begin{align*}
(p, P) \otimes_C (q, Q) &:= \left( (p \otimes Q) \circ (P \otimes q), P \otimes Q \right) \\
(p', P') \otimes_C (q', Q') &:= \left( (p' \otimes Q') \circ (P' \otimes q'), P' \otimes Q' \right)
\end{align*}
are also an objects of the category \( \mathcal{L}^c_C \), which leads to the horizontal multiplication. The vertical one is defined as the composed morphism:

\[
(\gamma \otimes^c \sigma) := (P' \otimes \varepsilon \otimes Q' \otimes C) \circ (\gamma \otimes Q' \otimes C) \circ (P \otimes q' \otimes C) \circ (P \otimes \sigma \otimes C) \circ (P \otimes Q \otimes \Delta)
\]

Lastly, the identity object of the multiplication \(- \otimes_C -\) is given by the pair \((\mathcal{C}, I)\) (here \(\mathcal{C}\) denotes the identity morphism of \(\mathcal{C}\) in \(\mathcal{M}\)).

1.3. Cowreath and their products. \(\mathcal{C}\) still denotes a comonoid in \(\mathcal{M}\), \(\mathcal{R}^c_C\) and \(\mathcal{L}^c_C\) are the monoidal categories defined, respectively, in subsections [1.1] and [1.2]. The notion of wreath was introduced in [6] in the general context of 2-categories, in the monoidal case they are defined as follows:

**Definition 1.3.** Let \(\mathcal{C}\) be a comonoid in a strict monoidal category \(\mathcal{M}\). A **right cowreath over \(\mathcal{C}\)** (or **right \(\mathcal{C}\)-cowreath**) is a comonoid in the monoidal category \(\mathcal{R}^c_C\). A **right wreath over \(\mathcal{C}\)** (or **right \(\mathcal{C}\)-wreath**) is a monoid in the monoidal category \(\mathcal{R}^c_C\). The left versions of these definitions are obtained in the monoidal category \(\mathcal{L}^c_C\).

The following gives, in terms of the multiplication of \(\mathcal{M}\), a simplest and equivalent definition of cowreath.

**Proposition 1.4.** Let \(\mathcal{C}\) be a comonoid in a strict monoidal category \(\mathcal{M}\) and \((\mathcal{R}, \tau)\) an object of the category \(\mathcal{R}^c_C\). The following statements are equivalent

(i) \((\mathcal{R}, \tau)\) is a right \(\mathcal{C}\)-cowreath;

(ii) There is a \(\mathcal{C}\)-bicomodules morphisms \(\xi : \mathcal{C} \otimes \mathcal{R} \rightarrow \mathcal{C}\) and \(\delta : \mathcal{C} \otimes \mathcal{R} \rightarrow \mathcal{C} \otimes \mathcal{R} \otimes \mathcal{R}\) converting the following diagrams commutative

\[
\begin{array}{ccc}
\mathcal{C} \otimes \mathcal{R} & \xrightarrow{\delta} & \mathcal{C} \otimes \mathcal{R} \otimes \mathcal{R} \\
\downarrow{\xi} & & \downarrow{\tau} \\
\mathcal{C} \otimes \mathcal{R} & & \mathcal{C} \otimes \mathcal{R} \\
\end{array}
\quad \quad \quad
\begin{array}{ccc}
\mathcal{C} \otimes \mathcal{R} & \xrightarrow{\delta} & \mathcal{C} \otimes \mathcal{R} \otimes \mathcal{R} \\
\downarrow{\delta} & & \downarrow{\delta \otimes \mathcal{R}} \\
\mathcal{C} \otimes \mathcal{R} \otimes \mathcal{R} & & \mathcal{C} \otimes \mathcal{R} \otimes \mathcal{R} \\
\end{array}
\]

**Proof.** Analogue to that of [5] Proposition 2.2]. □

The cowreath products was introduced in [6] for comonads in a general 2-categories. In the particular case of strict monoidal categories this product is expressed by the following

**Proposition 1.5.** Let \(\mathcal{C}\) be a comonoid in \(\mathcal{M}\), and \((\mathcal{R}, \tau)\) a right \(\mathcal{C}\)-cowreath with structure morphisms \(\xi : \mathcal{C} \otimes \mathcal{R} \rightarrow \mathcal{C}\) and \(\delta : \mathcal{C} \otimes \mathcal{R} \rightarrow \mathcal{C} \otimes \mathcal{R} \otimes \mathcal{R}\). The object \(\mathcal{C} \otimes \mathcal{R}\) admits a structure of comonoid with comultiplication and counit given by

\[
\begin{array}{ccc}
\mathcal{C} \otimes \mathcal{R} & \xrightarrow{\Delta \otimes \mathcal{R}} & \mathcal{C} \otimes \mathcal{C} \otimes \mathcal{R} \\
\xrightarrow{\mathcal{C} \otimes \delta} & & \xrightarrow{\mathcal{C} \otimes \tau} \\
\mathcal{C} \otimes \mathcal{C} \otimes \mathcal{R} & & \mathcal{C} \otimes \mathcal{R} \otimes \mathcal{C} \otimes \mathcal{R},
\end{array}
\]

Moreover, with this comonoid structure the morphism \(\xi : \mathcal{C} \otimes \mathcal{R} \rightarrow \mathcal{C}\) becomes a morphism of comonoid.

**Proof.** Straightforward. □
The comonoid $C \otimes R$ of the previous Proposition is referred as the cowreath product of $C$ by $R$.

Remark 1.6. Notice that the object $R$ occurring in Proposition [1.4] need not be a comonoid. However, if $R$ is itself a comonoid with a structure morphisms $\Delta': R \to R \otimes R$, $\epsilon': R \to I$ such that the pair $(R, C)$ belongs to the left monoidal category $\mathcal{M}_L$, then the morphisms $\xi := C \otimes \epsilon'$ and $\delta := C \otimes \Delta'$ endow $(R, \tau)$ with a structure of right $C$-cowreath while $\xi' := \epsilon \otimes R$, and $\delta' := \Delta \otimes R$ gave to $(C, R)$ a structure of left $R$-cowreath. Furthermore, by Proposition [1.5], the morphisms $\xi : C \otimes R \to C$ and $\xi' : C \otimes R \to R$ are in fact a comonoids morphisms.

In this way the morphism $r : C \otimes R \to R \otimes C$ should satisfies the following equalities

\begin{align}
(1.9) & \quad (R \otimes \Delta) \circ r = (R \otimes C) \circ (\Delta \otimes R) \\
(1.10) & \quad (R \otimes \epsilon) \circ r = \epsilon \otimes R \\
(1.11) & \quad (\Delta' \otimes C) \circ r = (R \otimes r) \circ (R \otimes \Delta') \\
(1.12) & \quad (\epsilon' \otimes C) \circ r = C \otimes \epsilon'.
\end{align}

A morphism satisfying the four previous equalities is called a comonoid distributive law from $C$ to $R$, see $\Pi$ for the original definition.

A cowreath product satisfies a universal property in the following sense

**Proposition 1.7.** Let $(C, \Delta, \epsilon)$ be a comonoid in a strict monoidal category $\mathcal{M}$, and $(R, \tau)$ a $C$-cowreath with a structure morphisms $\xi : C \otimes R \to C$ and $\delta : C \otimes R \to C \otimes R \otimes R$.

Let $(D, \Delta', \epsilon')$ be a comonoid with a comonoid morphism $\alpha : D \to C$ and with morphism $\beta : D \to R$ satisfying

\begin{align}
(1.13) & \quad \xi \circ (C \otimes \beta) = C \otimes \epsilon' \\
(1.14) & \quad \delta \circ (C \otimes \beta) = (C \otimes \beta \otimes \beta) \circ (C \otimes \Delta')
\end{align}

Assume that $\alpha$ and $\beta$ satisfy the equality

\begin{equation}
(1.15) \quad r \circ (\alpha \otimes \beta) \circ \Delta' = (\beta \otimes \alpha) \circ \Delta'
\end{equation}

then there exists a unique comonoid morphism $\gamma : D \to C \otimes R$ such that $\xi \circ \gamma = \alpha$ and $(\epsilon \otimes R) \circ \gamma = \beta$.

**Proof.** If there exists such a morphism, then it should be unique by the following computations

\begin{equation}
(\alpha \otimes \beta) \circ \Delta' = \left( \left( \xi \circ \gamma \right) \otimes \left( (\epsilon \otimes R) \circ \gamma \right) \right) \circ \Delta' = \left( \xi \otimes (\epsilon \otimes R) \otimes (\gamma \otimes \gamma) \right) \circ \Delta' = \left( \xi \otimes \epsilon \otimes R \otimes \epsilon \otimes R \right) \circ (C \otimes \delta) \circ (\Delta \otimes R) \circ \gamma = \left( \xi \otimes R \right) \circ (C \otimes \epsilon \otimes R \otimes R) \circ (\Delta \otimes R) \circ \delta \circ \gamma = (\xi \otimes R) \circ \delta \circ \gamma = \gamma.
\end{equation}

Since by hypothesis $\xi \circ (\alpha \otimes \beta) \circ \Delta' = \alpha$ and $(\epsilon \otimes R) \circ (\alpha \otimes \beta) \circ \Delta' = \beta$, it suffice to show that $(\alpha \otimes \beta) \circ \Delta'$ is a comonoid morphism. The counitary property comes out as

\begin{equation}
\varepsilon \circ (\alpha \otimes \beta) \circ \Delta' = \left( \varepsilon \circ \xi \circ (C \otimes \beta) \circ (\alpha \otimes D) \circ \Delta' \right) = \left( \varepsilon \circ (C \otimes \epsilon') \circ (\alpha \otimes D) \circ \Delta' \right) = \varepsilon \circ \left( (D \otimes \epsilon') \circ \Delta' \right) = \epsilon'.
\end{equation}
Now the coassociativity property is obtained by the following computations

\[
\Delta \circ (\alpha \otimes \beta) \circ \Delta' = (C \otimes r) \circ (C \otimes \delta) \circ (\Delta \otimes R) \circ (\alpha \otimes \beta) \circ \Delta'
\]

\[
= (C \otimes r) \circ (\Delta \otimes R \otimes R) \circ \delta \circ (\alpha \otimes \beta) \circ \Delta'
\]

\[
= (C \otimes r) \circ (\Delta \otimes R \otimes R) \circ \delta \circ (C \otimes \Delta') \circ (\alpha \otimes D) \circ \Delta'
\]

\[
= (C \otimes r) \circ (\Delta \otimes R \otimes R) \circ (\alpha \otimes R \otimes R) \circ (D \otimes \beta \otimes \beta) \circ (D \otimes \Delta') \circ \Delta'
\]

\[
= (\alpha \otimes R \otimes C \otimes R) \circ (D \otimes (r \circ (\alpha \otimes \beta)) \otimes R) \circ (\Delta' \otimes D \otimes R) \circ (D \otimes D \otimes \beta) \circ (\Delta' \otimes D) \circ \Delta'
\]

\[
= (\alpha \otimes R \otimes C \otimes R) \circ (D \otimes (r \circ (\alpha \otimes \beta)) \otimes R) \circ (\Delta' \otimes D \otimes R) \circ (D \otimes \beta) \circ \Delta'
\]

\[
= (\alpha \otimes R \otimes C \otimes R) \circ (D \otimes (r \circ (\alpha \otimes \beta)) \otimes R) \circ (\Delta' \otimes D \otimes R) \circ (D \otimes \beta) \circ \Delta'
\]

In what follows we announce the analogue notion for a given monoid in a strict monoidal category. So consider a monoid \((A, \mu, \eta)\) in \(M\). We start by defining the right and left Eilenberg-Moore monoidal categories attached to \(A\).

### 1.4. The monoidal category \(R^\circ_A\)

This is the right Eilenberg-Moore monoidal category associated to the monoid \(A\) (see [3]), and defined as follows

**Objects of \(R^\circ_A\):** Are pairs \((U, u)\) consisting of an object \(U \in M\) and a morphism \(u : A \otimes U \to U \otimes A\) such that

\[
\begin{align*}
\text{(1.16)} & \quad u \circ (\mu \otimes U) = (U \otimes \mu) \circ (u \otimes A) \circ (A \otimes u) \\
\text{(1.17)} & \quad u \circ (\eta \otimes U) = U \otimes \eta.
\end{align*}
\]

We have the following lemma

**Lemma 1.8.** For every object \(U \in M\), the following conditions are equivalent

(i) \(U \otimes A\) is an \(A\)-bimodule with a right \(A\)-action \(r_{U \otimes A} = U \otimes \mu\);

(ii) there is a morphism \(u : A \otimes U \to U \otimes A\) satisfying equalities (1.16) and (1.17).

**Proof.** (ii) \(\Rightarrow\) (i). Take the left \(A\)-action \(l_{U \otimes A} = (U \otimes \mu) \circ (u \otimes A)\).

(i) \(\Rightarrow\) (ii). Take \(u = l_{U \otimes A} \circ (A \otimes U \otimes \eta)\), where \(l_{U \otimes A}\) is the given left action. \(\square\)

In this way the morphisms in \(R^\circ_A\) are defined in their unreduced form as follows

**Morphisms in \(R^\circ_A\):**

\[
\text{Hom}_{R^\circ_A} \left( (U, u), (U', u') \right) := \text{Hom}_{A \otimes A} \left( U \otimes A, U' \otimes A \right),
\]

where \(U \otimes A\) and \(U' \otimes A\) are endowed with the structure of \(A\)-bimodule defined in Lemma 1.8. That is any morphism \(\nu : (U, u) \to (U', u')\) in \(R^\circ_A\) is a morphism \(\nu : U \otimes A \to U' \otimes A\) satisfying

\[
\begin{align*}
\text{(1.18)} & \quad \nu \circ (U \otimes \mu) = (U' \otimes \mu) \circ (\nu \otimes A), \\
\text{(1.19)} & \quad \nu \circ (U \otimes \mu) \circ (u \otimes A) = (U' \otimes \mu) \circ (u' \otimes A) \circ (A \otimes \nu).
\end{align*}
\]
The multiplications of \( R^a_A \): Let \( \nu : (U, u) \to (U', u') \) and \( \upsilon : (V, v) \to (V', v') \) two morphisms in \( R^a_A \). One can easily proves that
\[
(U, u) \otimes_A (V, v) := (U \otimes V, (U \otimes v) \circ (u \otimes V))
\]
are also an objects of the category \( R^a_A \), which gives the horizontal multiplication. The vertical one is defined by the composition:
\[
(\nu \otimes_A \upsilon) := (U' \otimes V' \otimes \mu) \circ (U' \otimes v \otimes \Delta_A) \circ (U \otimes \nu \otimes \Delta_A) \circ (U \otimes \eta \otimes V \otimes \Delta_A)
\]
Lastly, the identity object of the multiplication \(- \otimes_A -\) is given by the pair \((I, \Delta_A)\) (here \( \Delta_A \) denotes the identity morphism of \( A \) in \( M \)).

1.5. The monoidal category \( L^a_A \).

This is the left Eilenberg-Moore category associated to \( A \), and defined as follows

Objects of \( L^a_A \): Are pairs \((m, M)\) consisting of an object \( M \in M \) and a morphism \( m : M \otimes A \to A \otimes M \) such that
\[
\tag{1.20}
m \circ (M \otimes \mu) = (\mu \otimes M) \circ (A \otimes m) \circ (m \otimes A)
\]
\[
\tag{1.21}
m \circ (M \otimes \eta) = \eta \otimes M.
\]
As before one can easily check the following lemma

**Lemma 1.9.** For every object \( M \in M \), the following conditions are equivalent

(i) \( A \otimes M \) is an \( A \)-bimodule with a left \( A \)-action \( I_{A \otimes M} = \mu \otimes M \);
(ii) there is a morphism \( m : M \otimes A \to A \otimes M \) satisfying equalities \[1.20\] and \[1.21\].

In this way the morphisms in \( L^a_A \) are defined in their unreduced form as follows

Morphisms in \( L^a_A \):
\[
\text{Hom}_{L^a_A} \left( (m, M), (m', M') \right) := \text{Hom}_{A\otimes A} \left( A \otimes M, A \otimes M' \right),
\]
where \( A \otimes M \) and \( A \otimes M' \) are endowed with the structure of \( A \)-bimodule defined in Lemma 1.9. That is amorphism \( \theta : (m, M) \to (m', M') \) in \( L^a_A \) is a morphism \( \theta : A \otimes M \to A \otimes M' \) satisfying
\[
\tag{1.22}
\theta \circ (\mu \otimes M) = (\mu \otimes M') \circ (A \otimes \theta),
\]
\[
\tag{1.23}
\theta \circ (\mu \otimes M) \circ (A \otimes m) = (\mu \otimes M') \circ (A \otimes m') \circ (\theta \otimes A).
\]

The multiplications of \( L^a_A \): Let \( \theta : (m, M) \to (m', M') \) and \( \vartheta : (n, N) \to (n', N') \) two morphisms in \( L^a_A \). One can easily prove that
\[
(m, M) \otimes_A (n, N) := (m \otimes N) \circ (M \otimes n), M \otimes N
\]
\[
(m', M') \otimes_A (n', N') := (m' \otimes N') \circ (M' \otimes n'), M' \otimes N'
\]
are also an objects of the category \( L^a_A \), which leads to the horizontal multiplication. The vertical one is defined as the composed morphism:
\[
(\theta \otimes_A \vartheta) := (\mu \otimes M' \otimes N') \circ (A \otimes \theta \otimes N') \circ (A \otimes m \otimes N') \circ (A \otimes M \otimes \vartheta) \circ (A \otimes M \otimes \eta \otimes N)
\]
\[
= (\mu \otimes M' \otimes N') \circ (A \otimes m' \otimes N') \circ (\theta \otimes \vartheta) \circ (A \otimes M \otimes \eta \otimes N).
\]
Lastly, the identity object of the multiplication \(- \otimes_A -\) is given by the pair \((A, \mathbb{1})\).
1.6. Wreaths and their products. \( A \) still denotes a monoid in \( \mathcal{M} \), \( \mathcal{R}^a_A \) and \( \mathcal{L}_A^a \) are the monoidal categories defined, respectively, in subsections [48] and [49]. As in the comonoid case, we have

**Definition 1.10.** Let \( A \) be a monoid in a strict monoidal category \( \mathcal{M} \). A right wreath over \( A \) (or right \( A \)-wreath) is a monoid in the monoidal category \( \mathcal{R}^a_A \). A right cowreath over \( A \) (or right \( A \)-cowreath) is a comonoid in the monoidal category \( \mathcal{L}_A^a \). The left versions of these notions are defined in the monoidal category \( \mathcal{L}_C^a \).

The following gives, in terms of the multiplication of \( \mathcal{M} \), a simplest and equivalent definition of wreath.

**Proposition 1.11.** Let \( A \) be a monoid in a strict monoidal category \( \mathcal{M} \) and \((\mathcal{T}, t)\) an object of the category \( \mathcal{R}^a_A \). The following statements are equivalent

(i) \((\mathcal{T}, t)\) is a right \( A \)-wreath;

(ii) There is an \( A \)-bimodules morphisms \( \zeta : A \to \mathcal{T} \otimes A \) and \( \nu : \mathcal{T} \otimes \mathcal{T} \otimes A \to \mathcal{T} \otimes A \) rendering the following diagrams commutative

\[
\begin{array}{ccc}
\mathcal{T} \otimes \mathcal{T} \otimes A & \xrightarrow{\nu} & \mathcal{T} \otimes A \\
\mathcal{T} \otimes A & \xrightarrow{\zeta} & \mathcal{T} \otimes A \\
\mathcal{T} \otimes A & \xrightarrow{\zeta \otimes \mathcal{T}} & \mathcal{T} \otimes \mathcal{T} \otimes A \\
\mathcal{T} \otimes \mathcal{T} \otimes \mathcal{T} \otimes A & \xrightarrow{\nu \otimes \mathcal{T}} & \mathcal{T} \otimes \mathcal{T} \otimes \mathcal{T} \otimes A \\
\mathcal{T} \otimes \mathcal{T} \otimes \mathcal{T} \otimes A & \xrightarrow{\zeta \otimes \nu} & \mathcal{T} \otimes \mathcal{T} \otimes A \\
\end{array}
\]

Proof. Is left to the reader. \( \square \)

The wreath products is expressed by the following

**Proposition 1.12.** Let \((A, \mu, \eta)\) be a monoid in \( \mathcal{M} \), and \((\mathcal{T}, t)\) a right \( A \)-wreath with structure morphisms \( \zeta : A \to \mathcal{T} \otimes A \) and \( \nu : \mathcal{T} \otimes \mathcal{T} \otimes A \to \mathcal{T} \otimes A \). The object \( \mathcal{T} \otimes A \) admits a structure of monoid with multiplication and unit given by

\[
\begin{align*}
\mu' : \mathcal{T} \otimes A \otimes \mathcal{T} \otimes A & \to \mathcal{T} \otimes A, \\
\eta' : 1 & \to A, \\
\zeta & \to \mathcal{T} \otimes A.
\end{align*}
\]

Moreover, with this monoid structure the morphism \( \zeta : A \to \mathcal{T} \otimes A \) becomes a morphism of monoides.

Proof. Straightforward. \( \square \)

The monoid \( \mathcal{T} \otimes A \) of the previous Proposition is referred as the wreath product of \( A \) by \( T \).

**Remark 1.13.** Notice here also that the object \( T \) occurring in Proposition 1.11 need not be a monoid. However, if \( T \) is it self a monoid with a structure morphisms \( \mu' : \mathcal{T} \otimes T \to \mathcal{T} \), \( \eta' : 1 \to \mathcal{T} \) such that the pair \((t, A)\) belongs to the left monoidal category \( \mathcal{L}^a_T \), then the morphisms \( \zeta := \eta' \otimes A \) and \( \nu := \mu' \otimes A \) endow \((T, t)\) with a structure of right \( A \)-wreath while \( \zeta' := \eta \otimes T \), and \( \nu' := \mu \otimes A \) gave to \((t, A)\) a structure of left \( T \)-wreath. Furthermore, by Proposition 1.12 the morphisms \( \zeta : A \to \mathcal{T} \otimes A \) and \( \zeta' : \mathcal{T} \to \mathcal{T} \otimes A \) are in fact a monoids morphisms.
In this way the morphism \( t : T \otimes A \rightarrow A \otimes T \) should satisfies the following equalities

\[
\begin{align*}
(1.24) & \quad t \circ (\mu \otimes T) = (T \otimes \mu) \circ (t \otimes A) \circ (A \otimes t) \\
(1.25) & \quad t \circ (\eta \otimes T) = T \otimes \eta \\
(1.26) & \quad t \circ (A \otimes \mu') = (\mu' \otimes A) \circ (T \otimes t) \circ (t \otimes T) \\
(1.27) & \quad t \circ (A \otimes \eta') = \eta' \otimes A.
\end{align*}
\]

A morphism satisfying the four previous equalities is called a monoid distributive law from \( A \) to \( T \), see also \[ \text{II} \] for the original definition.

The universal property of wreath products is expressed as follows

**Proposition 1.14.** Let \((\mathcal{M}, \mu, \eta)\) be a monoid in a strict monoidal category \( \mathcal{M} \), and \((T, t)\) a right \( \mathcal{A} \)-wreath with a structure morphisms \( \zeta : \mathcal{A} \rightarrow T \otimes \mathcal{A} \), \( \nu : T \otimes T \otimes \mathcal{A} \rightarrow T \otimes \mathcal{A} \).

Let \((L, \mu', \eta')\) be a monoid with a monoid morphism \( \varphi : \mathcal{A} \rightarrow L \) and with morphism \( \psi : T \rightarrow L \) satisfying

\[
\begin{align*}
(1.28) & \quad (\psi \otimes \mathcal{A}) \circ \zeta = \eta' \otimes \mathcal{A} \\
(1.29) & \quad (\psi \otimes \mathcal{A}) \circ \nu = (\mu' \otimes \mathcal{A}) \circ (\psi \otimes \psi \otimes \mathcal{A})
\end{align*}
\]

Assume that \( \varphi \) and \( \psi \) satisfy the equality

\[
(1.30) \quad \mu' \circ (\varphi \otimes \psi) = \mu' \circ (\psi \otimes \alpha) \circ t
\]

then there exists a unique monoid morphism \( \phi : T \otimes \mathcal{A} \rightarrow L \) such that \( \phi \circ \zeta = \varphi \) and \( \phi \circ (T \otimes \eta) = \psi \).

2. **Bi-monoid in general monoidal category.**

The letter \( \mathcal{M} \) sill denotes a strict monoidal category with multiplication \(- \otimes -\) and identity object \( \mathbb{1} \). Consider \( \mathcal{B} \) an object of \( \mathcal{M} \) such that \((\mathcal{B}, \Delta, \varepsilon)\) is a comonoid in \( \mathcal{M} \) and \((\mathcal{B}, \mu, \eta)\) is also a monoid in \( \mathcal{M} \). Assume that there is a morphism \( h : \mathcal{B} \otimes \mathcal{B} \rightarrow \mathcal{B} \otimes \mathcal{B} \) which satisfies the following equalities:

\[
\begin{align*}
(2.1) & \quad h \circ (\eta \otimes \mathcal{B}) = \mathcal{B} \otimes \eta \\
(2.2) & \quad h \circ (\mu \otimes \mathcal{B}) = (\mathcal{B} \otimes \mu) \circ (h \otimes \mathcal{B}) \circ (\mathcal{B} \otimes h) \\
(2.3) & \quad h \circ (\mathcal{B} \otimes \eta) = \eta \otimes \mathcal{B} \\
(2.4) & \quad h \circ (\mathcal{B} \otimes \mu) = (\mu \otimes \mathcal{B}) \circ (h \otimes \mathcal{B}) \circ (\mathcal{B} \otimes h) \\
(2.5) & \quad (\mathcal{B} \otimes \varepsilon) \circ h = \varepsilon \otimes \mathcal{B} \\
(2.6) & \quad (\mathcal{B} \otimes \Delta) \circ h = (h \otimes \mathcal{B}) \circ (\mathcal{B} \otimes h) \circ (\Delta \otimes \mathcal{B}) \\
(2.7) & \quad (\varepsilon \otimes \mathcal{B}) \circ h = \mathcal{B} \otimes \varepsilon \\
(2.8) & \quad (\Delta \otimes \mathcal{B}) \circ h = (\mathcal{B} \otimes h) \circ (h \otimes \mathcal{B}) \circ (\mathcal{B} \otimes \Delta).
\end{align*}
\]

Observe that the equalities \( (2.1)-(2.2) \) means that \((\mathcal{B}, h) \in \mathcal{R}_{\mathcal{B}}^a\) and \( (2.2)-(2.3) \) means that \((h, \mathcal{B}) \in \mathcal{L}_{\mathcal{B}}^a\) while \((\mathcal{B}, h) \in \mathcal{R}_{\mathcal{B}}^b\) by equalities \( (2.1)-(2.2) \) and \((h, \mathcal{B}) \in \mathcal{L}_{\mathcal{B}}^b\) by equalities \( (2.2)-(2.3) \). Moreover \( (2.1)-(2.4) \) say that \( h \) is a monoid distributive law from \( \mathcal{B} \) to \( \mathcal{B} \), and \( (2.5)-(2.8) \) say that \( h \) is a comonoid distributive law form \( \mathcal{B} \) to \( \mathcal{B} \).

**Definition 2.1.** The morphism \( h : \mathcal{B} \otimes \mathcal{B} \rightarrow \mathcal{B} \otimes \mathcal{B} \) satisfying equalities \( (2.1)-(2.8) \), is called a double distributive law between the monoid \((\mathcal{B}, \mu, \eta)\) and the comonoid \((\mathcal{B}, \Delta, \varepsilon)\).

Notice that if \( h : \mathcal{B} \otimes \mathcal{B} \rightarrow \mathcal{B} \otimes \mathcal{B} \) is a double distributive law, then \( \left( \mathcal{B} \otimes \mathcal{B}, (\mathcal{B} \otimes h \otimes \mathcal{B}) \circ (\Delta \otimes \Delta), \varepsilon \circ (\mathcal{B} \otimes \varepsilon) \right) \) is by Remark \( \text{II} \) and Proposition \( \text{I.6} \) a comonoid in \( \mathcal{M} \), and \( \left( \mathcal{B} \otimes \mathcal{B}, (\mu \otimes \mu) \circ (\mathcal{B} \otimes h \otimes \mathcal{B}), (\eta \otimes \eta) \circ \eta \right) \) is by Remark \( \text{I.3} \) and Proposition \( \text{I.12} \) a monoid in \( \mathcal{M} \). Using the both structures we can now enounce our main result
Proposition 2.2. Let $\mathcal{M}$ be a strict monoidal category with multiplication $- \otimes -$ and identity object $I$. Consider a 6-tuple $(B, \Delta, \varepsilon, \mu, h)$ where $(B, \Delta, \varepsilon)$ is a comonoid in $\mathcal{M}$, $(B, \mu)$ is a monoid in $\mathcal{M}$ and $h : B \otimes B \rightarrow B \otimes B$ is a double distributive law between them (i.e., satisfies equalities (2.1)-(2.8)). The following statements are equivalent

(i) $\Delta$ and $\varepsilon$ are morphisms of monoids;
(ii) $\mu$ and $\eta$ are morphisms of comonoids;
(iii) $\Delta$, $\varepsilon$, $\mu$ and $\eta$ satisfy:

\begin{enumerate}[(a)]
  \item $\Delta \circ \eta = \eta \otimes \eta$
  \item $(\mu \otimes \mu) \circ (B \otimes h \otimes B) \circ (\Delta \otimes \Delta) = \Delta \circ \mu$
  \item $\varepsilon \circ \eta = I$
  \item $\varepsilon \circ \mu = \varepsilon \otimes \varepsilon$.
\end{enumerate}

Proof. It is clear from definitions that $\Delta$ is monoid morphism if and only if the equalities (iii)(a) $-$ (b) are satisfied, and that $\varepsilon$ is a monoid morphism if and only if the equalities (iii)(c) $-$ (d) are verified. This shows that $(iii) \iff (i)$. On the other hand (iii)(b) $-$ (d) is equivalent to say that $\mu$ is a comonoid morphisms, and (iii)(a) $-$ (c) is equivalent to say that $\eta$ is a comonoid morphism. This leads to the equivalence $(iii) \iff (ii)$. \qed

Definition 2.3. Let $\mathcal{M}$ be a strict monoidal category with multiplication $- \otimes -$ and identity object $I$. A bi-monoid is a 6-tuple $(B, \Delta, \varepsilon, \mu, \eta, h)$ satisfying the equivalent conditions of Proposition 2.2.

Remark 2.4. The results stated in Sections 1 and 2 can be extended to the case of not necessary strict monoidal category by using the multiplicative equivalence between any monoidal category and a strict one, see [4, Corollary 1.4].

3. Some applications.

In what follows $k$ denotes a commutative ring with 1, and $\mathcal{M}_k$ its category of modules.

3.1. Compatibility condition between ring and coring. Let $R$ be $k$-algebra, all bimodules are assumed to central $k$-bimodules, and their category will be denoted by $R.\mathcal{M}_k$. This is a monoidal category with multiplication the two variables functor $- \otimes -$ the tensor product over $R$, and with identity object the regular $R$-bimodule $RR$. Let $(C, \Delta, \varepsilon)$ be an $R$-coring \cite{9} (i.e., a comonoid in $R.\mathcal{M}_R$), we use Sweedler’s notation for the comultiplication, that is $\Delta(c) = c_{(1)} \otimes_R c_{(2)}$, for every $c \in C$ (summation is understood). Given an $R$-bilinear morphism $h : C \otimes_R C \rightarrow C \otimes_R C$, we denote the image of $x \otimes y \in C \otimes_R C$ by $h(x \otimes y) = y^h \otimes x^h$ (summation is understood).

Now taking into account Remark 2.4 we can announce the compatibility condition in the monoidal category $R.\mathcal{M}_R$.

Corollary 3.1. Let $R$ be a $k$-algebra, and $\iota : R \rightarrow C$ is a rings extension. Consider $C$ as an $R$-bimodule by restricting $\iota$. Assume that this $R$-bimodule admits a structure of an $R$-coring with comultiplication and counit, respectively, $\Delta$ and $\varepsilon$. If $h : C \otimes_RC \rightarrow C \otimes_RC$ is a double distributive law (i.e., an $R$-bilinear map satisfying (2.1)-(2.8)), then the 6-tuple $(C, \Delta, \varepsilon, \mu, 1_C, h)$ is a bi-monoid in the monoidal category $R.\mathcal{M}_R$ (Definition 2.3) if and only if

\begin{enumerate}[(a)]
  \item $\Delta(1_C) = 1_C \otimes_R 1_C$.
  \item $\Delta(xy) = x_{(1)}y_{(1)}^h \otimes x_{(1)}y_{(2)}^h$, for every $x, y \in C$.
  \item $\varepsilon(1_C) = 1_R$
  \item $\varepsilon(xy) = \varepsilon(x) \varepsilon(y)$, for every $x, y \in C$.
\end{enumerate}

In particular $R$ is a trivial bi-monoid in $R.\mathcal{M}_R$. 

3.2. Algebras and coalgebras.

Example 3.2. Let $(B, \Delta, \epsilon)$ be a $k$-coalgebra and $(B, \mu, 1_B)$ a $k$-algebra. Denote by $\tau : B \otimes_k B \to B \otimes_k B$ the usual flip map i.e., $\tau(x \otimes y) = y \otimes x$, for all $x, y \in B$. One can easily check that $\tau$ satisfies all equalities \[2.1\]–\[2.4\] and \[2.5\]–\[2.8\] with respect to $\Delta$, $\epsilon$, $\mu$ and $1_B$. That is, in our terminology, $\tau$ is a double distributive law. Therefore, by Proposition \[2.2\] $(B, \Delta, \epsilon, \mu, 1_B, \tau)$ is a bi-monoid in the monoidal category $\mathcal{M}_k$ if and only if $B$ is a bialgebra in the usual sense $[5]$. 

Example 3.3. Let $L$ be a $k$-module and set $B := k \oplus L$. An element belonging to $B$ will be denoted by a pair $(k, x)$ where $k \in k$ and $x \in L$. We consider $B$ as a $k$-algebra with multiplication and unit defined by 

$$\mu((k, x) \otimes (l, y)) = (k, x)(l, y) = (kl, ky + ly), \quad 1_B = (1, 0), \quad \forall (k, x), (l, y) \in B,$$

and also as a $k$-coalgebra with comultiplication and counit defined by 

$$\Delta(k, x) = (k, x) \otimes (1, 0) + (1, 0) \otimes (k, x), \quad \epsilon(k, x) = (k, 0), \quad \forall (k, x) \in B.$$ 

It is well known that $B$ is not a $k$-bialgebra, since $\Delta$ is not a multiplicative map. Now consider the $k$-linear map 

$$h : B \otimes_k B \to B \otimes_k B$$ 

$$\sum (k_i, x_i) \otimes (l_i, y_i) \mapsto \sum \left( (l_i, y_i) \otimes (k_i, 0) + (l_i, 0) \otimes (x_i) - (0, x_i) \otimes (0, y_i) \right)$$ 

Let $(k, x), (l, y), (l', y') \in B$, we claim that $h$ is a double distributive law. First, we have 

$$h((1, 0) \otimes (l, y)) = (l, y) \otimes (1, 0)$$ 

$$h((k, x) \otimes (1, 0)) = (1, 0) \otimes (k, 0) + (1, 0) \otimes (0, x)$$ 

$$= (1, 0) \otimes (k, x)$$ 

that is $h$ satisfies \[2.1\] and \[2.3\]. On the other hand 

$$(B \otimes \epsilon) \circ h((k, x) \otimes (l, y)) = B \otimes \epsilon \left( (l, y) \otimes (k, 0) + (l, 0) \otimes (0, x) - (0, x) \otimes (0, y) \right)$$ 

$$= (l, y)k = \epsilon \otimes B \left( (k, x) \otimes (l, y) \right),$$ 

$$( \epsilon \otimes B ) \circ h((k, x) \otimes (l, y)) = \epsilon \otimes B \left( (l, y) \otimes (k, 0) + (l, 0) \otimes (0, x) - (0, x) \otimes (0, y) \right)$$ 

$$= l(k, 0) + l(0, x) = l(k, x) = B \otimes \epsilon \left( (k, x) \otimes (l, y) \right).$$ 

This implies that $h$ satisfies equalities \[2.5\] and \[2.7\]. The equalities \[2.2\] and \[2.4\] are obtained from the following two computations: 

$$(B \otimes \mu) \circ (h \otimes B) \circ (B \otimes h) \left( (k, x) \otimes (l', y') \otimes (l, y) \right)$$ 

$$= (B \otimes \mu) \circ (h \otimes B) \left( (k, x) \otimes (l, y) \otimes (l', 0) + (k, x) \otimes (l, 0) \otimes (0, y') - (k, x) \otimes (0, y') \otimes (0, y) \right)$$ 

$$= (B \otimes \mu) \left[ \left( (l, y) \otimes (k, 0) + (l, 0) \otimes (0, x) - (0, x) \otimes (0, y) \right) \otimes (l', 0) + \left( (l, 0) \otimes (k, 0) + (l, 0) \otimes (0, x) \right) \otimes (0, y') \right.$$ 

$$- \left( 0, y' \otimes (k, 0) - (0, x) \otimes (0, y') \right) \otimes (0, y) \right]$$ 

$$= (l, y) \otimes (kl', 0) + (l, 0) \otimes (0, l'y) - (0, x) \otimes (0, l'y) + (l, 0) \otimes (0, ky') - (0, y') \otimes (0, ky)$$ 

$$= (l, y) \otimes (kl', 0) + (l, 0) \otimes (0, l'x + ky') - (0, l'x + ky') \otimes (0, y)$$ 

$$= h \left( (kl', ky' + l'x) \otimes (l, y) \right),$$
(\mu \otimes B) \circ (B \otimes h) \circ (h \otimes B) \left( (k, x) \otimes (l', y') \otimes (l, y) \right) \\
= (\mu \otimes B) \circ (B \otimes h) \left( (l', y') \otimes (k, 0) \otimes (l, y) + (l', 0) \otimes (0, x) \otimes (l, y) - (0, x) \otimes (0, y') \otimes (l, y) \right) \\
= (\mu \otimes B) \left[ (l', y') \otimes (l, y) \otimes (k, 0) + (l', 0) \otimes (l, 0) \otimes (0, y) - (l', 0) \otimes (0, x) \otimes (0, y) \right] \\
\quad - (0, x) \otimes (l, 0) \otimes (0, y') + (0, x) \otimes (0, y') \otimes (0, y) \\
= (l' l', l' y + l y') \otimes (k, 0) + (l' l', 0) \otimes (0, x) - (0, l' l') \otimes (0, y) - (0, x) \otimes (0, y') \\
= (l' l', l' y + l y') \otimes (k, 0) + (l' l', 0) \otimes (0, x) - (0, x) \otimes (0, l' y + l y') \\
= h \left( (k, x) \otimes (l' l', l' y + l y') \right). \\

Equalities (2.6) and (2.8), are obtained as follows:

\begin{align*}
(h \otimes B) \circ (B \otimes h) \circ (\Delta \otimes B) \left( (k, x) \otimes (l, y) \right) &= (h \otimes B) \circ (B \otimes h) \left( (k, x) \otimes (1, 0) \otimes (l, y) + (1, 0) \otimes (0, x) \otimes (l, y) \right) \\
&= (h \otimes B) \left( (k, x) \otimes (l, y) \otimes (1, 0) + (1, 0) \otimes (l, 0) \otimes (0, y) - (1, 0) \otimes (0, x) \otimes (0, y) \right) \\
&= (l, y) \otimes (k, 0) \otimes (1, 0) + (l, 0) \otimes (0, x) \otimes (1, 0) - (0, x) \otimes (0, y) \otimes (1, 0) + (l, 0) \otimes (1, 0) \otimes (0, x) - (0, x) \otimes (1, 0) \otimes (0, y) \\
&= (B \otimes \Delta) \circ h \left( (k, x) \otimes (l, y) \right), \\

(B \otimes h) \circ (h \otimes B) \circ (B \otimes \Delta) \left( (k, x) \otimes (l, y) \right) &= (B \otimes h) \circ (h \otimes B) \left( (k, x) \otimes (l, y) \otimes (1, 0) + (k, x) \otimes (1, 0) \otimes (0, y) \right) \\
&= (B \otimes h) \left( (l, y) \otimes (k, 0) \otimes (1, 0) + (l, 0) \otimes (0, x) \otimes (1, 0) - (0, x) \otimes (0, y) \otimes (1, 0) + (1, 0) \otimes (k, x) \otimes (0, y) \right) \\
&= (l, y) \otimes (1, 0) \otimes (k, 0) + (l, 0) \otimes (1, 0) \otimes (0, x) - (0, x) \otimes (1, 0) \otimes (0, y) + (1, 0) \otimes (0, y) \otimes (k, 0) - (1, 0) \otimes (0, x) \otimes (0, y) \\
&= (\Delta \otimes B) \circ h \left( (k, x) \otimes (l, y) \right).
\end{align*}

This finishes the proof of the claim. Next we show that the 6-tuple \((B, \Delta, \varepsilon, \mu, 1_B, h)\) is by Proposition 2.2, a bi-monoid in the monoidal category \(\mathcal{M}_k\). The equalities (2.2)(iii)(a), (2.2)(iii)(c), and (2.2)(iii)(d) are easily checked, and (2.2)(iii)(b) is derived from the following computation:

\begin{align*}
(\mu \otimes \mu) \circ (B \otimes h \otimes B) \circ (\Delta \otimes \Delta) \left( (k, x) \otimes (l, y) \right) \\
= (\mu \otimes \mu) \circ (B \otimes h \otimes B) \left( (k, x) \otimes (1, 0) \otimes (l, y) \otimes (1, 0) + (k, x) \otimes (1, 0) \otimes (1, 0) \otimes (0, y) \\
\quad + (1, 0) \otimes (0, x) \otimes (l, y) \otimes (1, 0) + (1, 0) \otimes (0, x) \otimes (1, 0) \otimes (0, y) \right) \\
= (\mu \otimes \mu) \left( (k, x) \otimes (l, y) \otimes (1, 0) \otimes (1, 0) + (k, x) \otimes (1, 0) \otimes (1, 0) \otimes (0, y) \\
\quad + (1, 0) \otimes \left( (l, 0) \otimes (0, x) - (0, x) \otimes (0, y) \right) \otimes (1, 0) + (1, 0) \otimes (1, 0) \otimes (0, x) \otimes (0, y) \right) \\
\text{and so} \\
(\mu \otimes \mu) \circ (B \otimes h \otimes B) \circ (\Delta \otimes \Delta) \left( (k, x) \otimes (l, y) \right) &= (kl, ky + lx) \otimes (1, 0) + (k, 0) \otimes (0, y) + (l, 0) \otimes (0, x) \\
&= (kl, ky + lx) \otimes (1, 0) + (1, 0) \otimes (0, ky + lx) \\
&= \Delta \left( (kl, ky + lx) \right) = \Delta \circ \mu \left( (k, x) \otimes (l, y) \right).
\end{align*}

Remark 3.4. Take \(\mathcal{M}\) the monoidal category \(\mathcal{M}_k\), let \(A\) be a k-algebra and \(C\) a k-coalgebra. If in Proposition 1.12 the right \(A\)-wreath is induced by a k-algebra \(T\), then the wreath product \(A \otimes_k T\) is the well known smash product \(A \mathcal{U}_k T\) as was proved in [3 Theorem 2.5] (see also the references cited there). Dually, if in Proposition 1.15 the right \(C\)-cowreath is induced by a k-coalgebra \(\mathcal{R}\), then the cowreath product \(C \otimes_k \mathcal{R}\) is the well known smash coproduct \(C \mathcal{R} \times \mathcal{R}\) [3 Theorem 3.4]. In this way, the universal properties of smash product and smash coproduct stated, respectively, in [3 Proposition 2.12] and [3 Proposition 3.8], are in fact a particular cases of Proposition 1.14 and Proposition 1.17.
On the other hand, the factorization problem [3, Theorem 4.5] between two \( k \)-bialgebras can be re-formulated using double distributive law. Explicitly, given two \( k \)-bialgebras \( A \) and \( C \) together with two \( k \)-linear maps \( c : A \otimes C \to C \otimes A \) and \( a : C \otimes A \to A \otimes C \) such that \( (C, c) \in \mathcal{R}_A^a \), \( (a, C) \in \mathcal{L}_A^c \), and that \( (A, a) \in \mathcal{R}_C^c \), \( (c, A) \in \mathcal{L}_C^a \). So as above \( A \otimes C \) is a \( k \)-coalgebra and \( k \)-algebra which is not necessary a \( k \)-bialgebra (this is the factorization problem). Using Corollary 3.1 one can give as in [3, Theorem 4.5] a necessary and sufficient conditions for a double distributive law \( \mathcal{h} : (A \otimes C) \otimes (A \otimes C) \to (A \otimes C) \otimes (A \otimes C) \) in order to get a structure of bi-monoid on \( A \otimes C \) in the monoidal category \( \mathcal{M}_k \).

**References**

[1] J. Beck, *Distributive laws.* In Seminar on Triples and Categorical Homology Theory. Lecture Notes in Math., vol. 80, Springer-Verlag, 1969, pp. 119–140.

[2] T. Brzeziński and R. Wisbauer, *Corings and Comodules.* Cambridge University Press, LMS 309, (2003).

[3] S. Caenepeel, B. Ion, G. Militaru and S. Zhu, *The factorization problem and the smash biproduct of algebras and coalgebras,* Algebras Represent. Theory, 3 (2002), 19–42.

[4] A. Joyal and R. Street, *Braided tensor categories,* Adv. Math. 102 (1993), 29–78.

[5] L. El Kaoutit, *Extended distributive law: Co-wreath over co-rings,* arXiv:math.RA/0612818 (2006).

[6] S. Lack and R. Street, *The formal theory of comonads II,* J. Pure Appl. Algebra 175 (2002), 243-265.

[7] S. Mac Lane, *Categories for the Working Mathematician.* Vol. 5 of Graduate Texts of Mathematics. Springer-Verlag New York, (1998) second edition.

[8] M. E. Sweedler, *Hopf algebras,* Benjamin, New York, 1969.

[9] M. E. Sweedler, *The predual theorem to the Jacobson-Bourbaki theorem,* Tran. Amer. Math. Soc. 213 (1975), 391–406.

[10] M. E. Sweedler, *Groups of Simple Algebras,* I. H. E. S. Publ., n°44 (1975), 79–189.

[11] M. Takeuchi, *Groups of algebras over \( A \otimes \mathcal{T} \),* J. Math. Soc. Japan 29 (1977), 459–492.

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