Supersymmetric partners of the harmonic oscillator with an infinite potential barrier

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Abstract
Supersymmetry transformations of first- and second-order are used to generate Hamiltonians with known spectra departing from the harmonic oscillator with an infinite potential barrier. Also studied is the way in which the eigenfunctions of the initial Hamiltonian are transformed. The first- and certain second-order supersymmetric partners of the initial Hamiltonian possess third-order differential ladder operators. Since systems with this kind of operators are linked with the Painlevé IV equation, several solutions of this nonlinear second-order differential equation will be simply found.

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(Some figures may appear in colour only in the online journal)

1. Introduction

Supersymmetric quantum mechanics (SUSY QM) has proven to be an exceptional technology for generating quantum mechanical potentials with known spectra [1–15]. In this method the spectrum of an initial Hamiltonian is modified, by creating or deleting levels, in order to implement the so-called spectral design [8, 10, 14]. An important fact is that the intertwining operator technique and the factorization method are procedures which are equivalent to SUSY QM [10].

It is well known that the harmonic oscillator Hamiltonian has an equidistant energy spectrum, which is due to its intrinsic algebraic structure known as the Heisenberg–Weyl algebra. On the other hand, the polynomial Heisenberg algebras are deformations of the oscillator algebra, where the differential ladder operators are of order \( m + 1 \) and the commutator relation

\[
\frac{d}{dx} \, A(x) = \frac{A(x)}{x} 
\]
between them is a polynomial of order $m$ in the Hamiltonian. Due to this algebraic structure, the Hamiltonian spectrum turns out to be the juxtaposition of several equidistant energy ladders.

Let us note that systems described by second-order polynomial Heisenberg algebras (for $m = 2$) are connected to the Painlevé IV (PIV) equation [16–25]. Conversely, if a system characterized by third-order differential ladder operators and their extremal states are found, thus one can find solutions to the PIV equation in a simple way.

The supersymmetric partners of the harmonic oscillator Hamiltonian provide explicit realizations of the polynomial Heisenberg algebras [7, 25–27]. In particular, for first-order SUSY the involved supercharges are linear in the momentum and such Hamiltonians have third-order ladder operators [1], i.e., they fulfill the commutation relations associated to a second-order polynomial Heisenberg algebra, thus they will lead to solutions to the PIV equation [7].

On the other hand, Hamiltonians obtained from the harmonic oscillator through second-order SUSY with supercharges which are quadratic in the momentum possess, in general, fifth-order differential ladder operators. However, it is possible to identify a subfamily of these Hamiltonians which, in addition to have these fifth-order operators, possesses also third-order ones and, consequently, lead to new solutions of the PIV equation [12]. The same property can be found for a subset of $k$th-order SUSY partner Hamiltonians of the oscillator, which have the two kinds of differential ladder operators, those of order $2k + 1$ and third-order ones, the last leading also to solutions to the PIV equation.

Using the previously mentioned technique, plenty of non-singular explicit solutions to the PIV equation, either real or complex, have been derived [12, 28, 29] (for other methods to generate solutions to the PIV equation the reader can seek [21, 30]). It would be important to address a systematic analysis of the corresponding singular solutions. In this paper we will start this study by allowing the existence of one fixed singularity. Our treatment will be based on the harmonic oscillator with an infinite potential barrier [4], which for simplicity will be placed at the origin. We will be mainly interested in transformations which reproduce again the singularity present in the initial potential, i.e., PIV solutions having singularities for $x = 0$.

We shall describe also the induced spectral modifications and the second-order polynomial Heisenberg algebra characterizing the new Hamiltonians, which will naturally lead to new solutions to the PIV equation.

In order to achieve our goals, in section 2 we will review briefly the SUSY QM and the way in which Hamiltonians which are intertwined with the harmonic oscillator realize the second-order polynomial Heisenberg algebras, connecting them later with the PIV equation and some of its solutions. In section 3 we will study the harmonic oscillator with and infinite potential barrier at the origin, and we will apply to it the first and second-order SUSY techniques. In section 4 we will obtain several solutions to the PIV equation, either non-singular or with a singularity at $x = 0$, by using the extremal states of the SUSY partners of the harmonic oscillator with an infinite potential barrier. Finally, in section 5 we will emphasize the original results obtained in this paper as well as our conclusions.

### 2. Supersymmetric quantum mechanics

SUSY QM describes systems characterized by a supersymmetric Hamiltonian $H_{ss}$ and two supercharges $Q_1, Q_2$, all of them Hermitian operators satisfying the following supersymmetry algebra with two generators [31–43]:

$$[H_{ss}, Q_i] = 0, \quad \{Q_i, Q_j\} = \delta_{ij}H_{ss}, \quad i, j = 1, 2,$$  \hspace{1cm} (1)
where \([F, G] = FG - GF\) and \([F, G] = FG + GF\) are the commutator and anticommutator of the operators \(F\) and \(G\) respectively.

The simplest realizations of such an algebra arise from the intertwining operator technique as follows \([43]\). Let us suppose that a pair of Hamiltonians

\[
H = -\frac{1}{2} \frac{d^2}{dx^2} + V(x), \quad \tilde{H} = -\frac{1}{2} \frac{d^2}{dx^2} + \tilde{V}(x),
\]

where \(V(x)\) and \(\tilde{V}(x)\) are real potentials, obey the intertwining relations

\[
\tilde{H} A^+ = A^+ H \iff HA = A \tilde{H},
\]

with \(A^+\) and \(A\) being differential intertwining operators of order \(k\) (we are using units such that \(\hbar = m = 1\)). These operators satisfy

\[
AA^+ = \prod_{i=1}^{k} (H - \epsilon_i), \quad A^+ A = \prod_{i=1}^{k} (\tilde{H} - \epsilon_i), \quad \epsilon_i \in \mathbb{R}.
\]

The requirement \(\epsilon_i \in \mathbb{R}, i = 1, \ldots, k\) is taken mainly by two reasons: (i) \(\tilde{V}(x)\) should be real; (ii) also we will look for real solutions to the PIV equation. In fact, if we just would require that \(V(x)\) be real, without worrying about the PIV solution, (4) could include pairs of complex conjugate factorization energies \(\epsilon_j, \bar{\epsilon}_j\), leading to a real SUSY partner potential \([14, 36, 43]\). Moreover, this transformation can be decomposed into first- and second-order ones, the second-order transformations involving \(\epsilon_j\) and \(\bar{\epsilon}_j\) in an irreducible way, i.e., they also can be produced by two first-order transformations but the intermediate potential would be complex \([36]\). For the purposes of this paper it is enough to assume that \(\epsilon_i \in \mathbb{R}, i = 1, \ldots, k\), and the same will be done further on for any other factorization of this kind.

In order to realize the standard supersymmetry algebra of (1) let us choose

\[
Q_1 = \frac{Q^+ + Q^-}{\sqrt{2}}, \quad Q_2 = \frac{Q^+ - Q^-}{i\sqrt{2}}, \quad Q^+ = \begin{pmatrix} 0 & A^+ \\ A & 0 \end{pmatrix}, \quad Q^- = \begin{pmatrix} 0 & 0 \\ A^+ & 0 \end{pmatrix},
\]

so that

\[
H_{ss} = \{Q^-, Q^+\} = \begin{pmatrix} A^+ A & 0 \\ 0 & AA^+ \end{pmatrix}.
\]

Since \(A^+\) and \(A\) are the previous \(k\)th-order differential intertwining operators, this representation is known as \(k\)-SUSY QM. Hence, the supersymmetric Hamiltonian \(H_{ss}\) turns out to be

\[
H_{ss} = (H_d - \epsilon_1) \cdots (H_d - \epsilon_k),
\]

where

\[
H_d - \epsilon_i = \begin{pmatrix} \tilde{H} - \epsilon_i & 0 \\ 0 & H - \epsilon_i \end{pmatrix}, \quad i = 1, \ldots, k.
\]

### 2.1. 1-SUSY

Let the operators \(A^+\) and \(A\) be of first order \([2, 6, 14, 31, 32, 43]\), i.e.,

\[
A^+ = \frac{1}{\sqrt{2}} \left[ -\frac{d}{dx} + \alpha(x) \right], \quad A = \frac{1}{\sqrt{2}} \left[ \frac{d}{dx} + \alpha(x) \right],
\]

where \(\alpha(x)\) is a real function of \(x\). By plugging these expressions in the intertwining relations (3) one gets that \(\alpha\) must satisfy

\[
\alpha' + \alpha^2 = 2(V - \epsilon).
\]
Moreover, the substitution $\alpha = [\ln(u)]' = u'/u$ transforms this Riccati equation for $\alpha$ into a stationary Schrödinger one for $u$,

$$-\frac{1}{2}u'' + Vu = Hu = \epsilon u,$$

where $\epsilon$ is a real constant called factorization energy. Besides (10), it is obtained the following expression for the potential $\tilde{V}(x)$:

$$\tilde{V} = V - \alpha' = V - [\ln(u)]''.$$  \hfill (12)

Hence, if we choose a nodeless seed solution $u$ of the stationary Schrödinger equation (also called transformation function) associated to a given factorization energy $\epsilon$, then the intertwining operators $A^+$, $A$, and the new Hamiltonian $\tilde{H}$ become completely determined. Moreover, departing from the normalized eigenfunctions $\psi_n(x)$ of $\tilde{H}$ associated to the eigenvalues $E_n$, the corresponding ones $\phi_n(x)$ of $\tilde{H}$ are typically found through

$$\phi_n(x) = \frac{A^+\psi_n(x)}{\sqrt{E_n - \epsilon}}.$$  \hfill (13)

An additional eigenfunction $\phi_\epsilon(x)$ of $\tilde{H}$, associated to the eigenvalue $\epsilon$, could exist, which obeys

$$A\phi_\epsilon(x) = 0 \Rightarrow \phi_\epsilon(x) \propto \exp\left[-\int \alpha(x) \, dx\right] \propto 1/u(x).$$  \hfill (14)

Since $\tilde{H}\phi_\epsilon(x) = \epsilon\phi_\epsilon(x)$, then if $\phi_\epsilon(x)$ satisfies the given boundary conditions it turns out that $\epsilon$ must be incorporated to the set of eigenvalues of $\tilde{H}$.

### 2.2. 2-SUSY

Let us suppose now that the intertwining operators $A^+$ and $A$ are of second order [35–43], i.e.,

$$A^+ = \frac{1}{2} \left[ \frac{d^2}{dx^2} - \eta(x) \frac{d}{dx} + \gamma(x) \right],$$

$$A = \frac{1}{2} \left[ \frac{d^2}{dx^2} + \eta(x) \frac{d}{dx} + \eta'(x) + \gamma(x) \right].$$  \hfill (15)

A similar treatment as for 1-SUSY leads now to the following nonlinear second-order differential equation for $\eta$:

$$\frac{\eta\eta''}{2} - \left(\frac{\eta'}{4}\right)^2 + \eta^2 \left(\eta' + \frac{\eta^2}{4} = 2V + \epsilon_1 + \epsilon_2\right) + (\epsilon_1 - \epsilon_2)^2 = 0.$$  \hfill (16)

Its solutions, in terms of either two solutions $\alpha_{1,2}$ of the Riccati equation associated to $\epsilon_{1,2}$ or the corresponding Schrödinger ones $u_{1,2}$ for $\epsilon_1 \neq \epsilon_2$, read

$$\eta = -\frac{2(\epsilon_1 - \epsilon_2)}{\alpha_{1} - \alpha_{2}} = [\ln W(u_1, u_2)]',$$

where $W(u_1, u_2) = u_1u_2' - u_1'u_2$ is the Wronskian of $u_1$ and $u_2$. Two additional expressions arise from the intertwining relations (3):

$$\gamma = \frac{\eta'}{2} + \frac{\eta^2}{2} - 2V + \epsilon_1 + \epsilon_2,$$

$$\tilde{V} = V - \eta' = V - [\ln W(u_1, u_2)]''.$$  \hfill (19)

Hence, if we choose now two seed solutions $u_{1,2}$ of the stationary Schrödinger equation associated to $\epsilon_{1,2}$ such that $W(u_1, u_2)$ is nodeless inside the domain of $V(x)$, it turns out that $A^+$, $A$, and $\tilde{H}$ become once again completely determined. Moreover, the eigenfunctions
Two extra eigenfunctions $\phi_{\epsilon_1,\epsilon_2}(x)$ of $\tilde{H}$, associated to the eigenvalues $\epsilon_{1,2}$, could exist, which obey 

$$A \phi_{\epsilon_1,\epsilon_2}(x) = 0 \Rightarrow \phi_{\epsilon_1}(x) \propto \frac{u_2}{W(u_1,u_2)}, \quad \phi_{\epsilon_2}(x) \propto \frac{u_1}{W(u_1,u_2)}.$$  

(21)

Since $\tilde{H}\phi_{\epsilon_1,\epsilon_2}(x) = \epsilon_{1,2}\phi_{\epsilon_1,\epsilon_2}(x)$, then if the two $\phi_{\epsilon_1,\epsilon_2}(x)$ satisfy the given boundary conditions it turns out that $\epsilon_{1,2}$ must be included in the spectrum of $\tilde{H}$.

2.3. Polynomial Heisenberg algebras

The polynomial Heisenberg algebras are deformations of the harmonic oscillator algebra, which are characterized by two standard commutation relations

$$[\mathbb{H}, L^\pm] = \pm L^\pm,$$  

(22)

plus one defining the deformation

$$[L^-, L^+] = N(\mathbb{H} + 1) - N(\mathbb{H}) = P_m(\mathbb{H}),$$  

(23)

where the analogue to the number operator, $N(\mathbb{H}) \equiv L^+L^-$, is a polynomial of degree $m + 1$ in the Hamiltonian $\mathbb{H}$ so that $P_m(\mathbb{H})$ becomes of degree $m$ [27]. Note that $N(\mathbb{H})$ admits the following factorization:

$$N(\mathbb{H}) = \prod_{i=1}^{m+1} (\mathbb{H} - \epsilon_i).$$  

(24)

Let us realize now the polynomial Heisenberg algebras of (22)–(24) by using the intertwining operator technique of section 2. In order to do that, let us express first the commutation relation which involves $\mathbb{H}$ and $L^+$ in the standard intertwining form:

$$(\mathbb{H} - 1)L^+ = L^+\mathbb{H}.$$  

(25)

By comparing this with (3) it is natural to make $H = \mathbb{H}$, $\tilde{H} = \mathbb{H} - 1$, $A^+ = L^+$, $A = L^-$, $k = m + 1$ and $\epsilon_i = \epsilon_i - 1$. Thus, (4) automatically leads to the commutation relation of (23). In this way it is obtained a realization of the polynomial Heisenberg algebras of (22)–(24) in terms of differential operators of finite order. There is, however, an important difference that must be stressed: while in the first part of section 2 it was assumed that $H$ is known in order to generate $\tilde{H}$, now the potential $V(x)$ associated to $\mathbb{H}$ has to be determined from the very algebraic treatment.

In the realization just built $L^+$ and $L^-$ are differential ladder operators of order $m + 1$. Let us consider now the functions $\phi(x)$ which belong to the kernel of $L^-,$

$$L^-\phi = 0 \Rightarrow N(\mathbb{H})\phi = L^+L^-\phi = \prod_{i=1}^{m+1} (\mathbb{H} - \epsilon_i)\phi = 0.$$  

(26)

Since this kernel is invariant under $\mathbb{H}$, we can choose as the linearly independent functions $\phi$ generating this subspace the solutions of the stationary Schrödinger equation for $\mathbb{H}$ associated to $\epsilon_i$:

$$\mathbb{H}\phi_{\epsilon_i} = \epsilon_i\phi_{\epsilon_i}.$$  

(27)
Departing from each of these extremal states $\phi_1$, it can be constructed a ladder of eigenfunctions of $H$ associated to the eigenvalues $\epsilon_i$, $n = 0, 1, 2, \ldots$ so that the system described by $H$ will have at most $m + 1$ ladders with eigenfunctions built up by the repeated action of $L^\pm$ onto such extremal states.

By taking $m = 0, 1$, and looking for the more general systems ruled by the corresponding polynomial Heisenberg algebras, we will arrive to the harmonic oscillator and effective ‘radial’ oscillator potentials (which have ladder operators of first and second orders respectively). On the other hand, for $m = 2$ (third-order ladder operators which will be specifically denoted by $L^{\pm}$) anticipating the reduced operators obtained from the theorem in section 2.4) it turns out that the corresponding potential becomes determined by a function which satisfies the PIV equation [7, 24] (see also [34]). In order to see this explicitly, let us assume that

$$\mathbb{H} = -\frac{1}{2} \frac{d^2}{dx^2} + \mathcal{V}(x),$$

and $L^+ = L_1^+ L_2^+$, $L^- = L_2^- L_1^-$, where

$$I_1^+ = \frac{1}{\sqrt{2}} \left[ - \frac{d}{dx} + f(x) \right],$$

$$I_2^+ = \frac{1}{2} \left[ \frac{d^2}{dx^2} + g(x) \frac{d}{dx} + h(x) \right].$$

The previous factorized expressions for $L^\pm$ are useful since it is employed an auxiliar Hamiltonian $H_a = -\frac{1}{2} \frac{d^2}{dx^2} + V_a(x)$ which is intertwined with $H$ as follows:

$$(\mathbb{H} - 1) I_1^+ = I_1^+ H_a, \quad H_a I_2^+ = I_2^+ \mathbb{H}.$$  

(30)

By using then the formulae obtained for 1-SUSY and 2-SUSY and after several calculations we arrive to the following final results:

$$f = x + g,$$

$$h = \frac{g'}{2} - \frac{g^2}{2} - 2xg + x^2 + \epsilon_2 + \epsilon_3 - 2\epsilon_1 - 1,$$

$$\mathcal{V} = \frac{x^2}{2} - \frac{g'}{2} + \frac{g^2}{2} + xg + \epsilon_1 - \frac{1}{2},$$

(31)-(33)

where $g(x)$ satisfies the PIV equation,

$$g'' = \frac{(g')^2}{2g} + \frac{3}{2} g^3 + 4xg^2 + 2(x^2 - a)g + \frac{b}{g},$$

(34)

with parameters $a = \epsilon_2 + \epsilon_3 - 2\epsilon_1 - 1$, $b = -2(\epsilon_2 - \epsilon_3)^2$.

Let us recall that $\epsilon_i$, $i = 1, 2, 3$ are the three roots involved in (24) for $m = 2$, which at the same time coincide with the energies for the three extremal $\phi_i$, of $H$, i.e.,

$$I^{\pm} \phi_i = 0 = N \phi_i = I^{\pm} I^{\pm} \phi_i, \quad i = 1, 2, 3.$$  

(35)

Since $L^- = I_2^+ I_1^-$, where $I_2^+ = (I_2^+)^\dagger$ and $I_1^- = (I_1^-)^\dagger$ arise from (29), one of the extremal states, denoted $\phi_1$, can be easily obtained

$$I_1^- \phi_1 = \frac{1}{\sqrt{2}} \left[ \frac{d}{dx} + f(x) \right] \phi_1 = 0 \quad \Rightarrow \quad \phi_1 = c \exp \left( -\frac{x^2}{2} - \int g \, dx \right).$$

(36)

The other two extremal states are found in a more complicated way; however, their analytic expressions can be obtained explicitly [7]:
Furthermore, the eigenfunctions $\phi_{\epsilon j}$ take this form with

$$
\epsilon = \epsilon_2 + \epsilon_3 - 2\epsilon_1 - 1, \quad b = -2(\epsilon_2 - \epsilon_3)^2, \quad \epsilon_{1,2,3} \in \mathbb{R},
$$

a system obeying a second-order polynomial Heisenberg algebra, characterized by the potential of (33), can be constructed.

On the other hand, if we find a system ruled by second-order polynomial Heisenberg algebras, in particular its extremal states, then we can build solutions to the PIV equation as long as the extremal state is not identically null. In order to see this, let us rewrite the expression for the extremal state $\phi_{\epsilon j}$ of (36) in the form

$$
g(x) = -x - [\ln \phi_{\epsilon j}]',
$$
i.e., a solution $g(x)$ to the PIV equation in terms of the extremal state $\phi_{\epsilon j}$ of $\mathbb{H}$ has been found.

### 2.4. Harmonic oscillator

Let us apply now the $k$-SUSY technique to the harmonic oscillator Hamiltonian

$$
\begin{align*}
H &= -\frac{1}{2} \frac{d^2}{dx^2} + \frac{x^2}{2}.
\end{align*}
\tag{40}
$$

If the transformation is of order $k \geq 1$, it is possible to create $k$ new levels below the ground state energy $E_0 = 1/2$ of the oscillator (let us suppose that this happens), at the positions defined by the factorization energies $\epsilon_j, \ j = 1, \ldots, k$ involved in (4) [26, 43]. The eigenfunctions $\phi_{\epsilon j}(x)$ of the new Hamiltonian $\tilde{H}$, associated to the eigenvalues $E_{\epsilon j} = n + 1/2$ of the initial Hamiltonian $H$, are given by a generalization of (13) and (20):

$$
\phi_{\epsilon j}(x) = \frac{A^+ \psi_{\epsilon j}(x)}{\sqrt{(E_{\epsilon j} - \epsilon_1) \cdots (E_{\epsilon j} - \epsilon_k)}}.
\tag{41}
$$

Furthermore, the eigenfunctions $\phi_{\epsilon j}$ associated to the new levels $\epsilon_j$ can be written as

$$
\phi_{\epsilon j} \propto \frac{W(u_1, \ldots, u_{j-1}, u_{j+1}, \ldots, u_k)}{W(u_1, \ldots, u_k)}, \quad j = 1, \ldots, k,
\tag{42}
$$

where $W(u_1, \ldots, u_k)$ is the Wronskian of the $k$ seed solutions $u_j, \ j = 1, \ldots, k$ used to implement the transformation, which satisfy

$$
Hu_j = \epsilon_j u_j.
\tag{43}
$$

Up to a constant factor, the general solution to this equation with $V(x) = \frac{x^2}{2}$ and $\epsilon$ arbitrary is given by

$$
\begin{align*}
u(x) &= e^{-x^2/2} \left[ _1F_1 \left( \frac{1 - \epsilon}{4}; \frac{1}{2}; x^2 \right) + 2i \frac{\Gamma(\frac{3-\epsilon}{4})}{\Gamma(\frac{1-\epsilon}{4})} x \ _1F_1 \left( \frac{3 - \epsilon}{4}; \frac{3}{2}; x^2 \right) \right].
\tag{44}
\end{align*}
$$

Thus, each $u_j$ takes this form with $\epsilon$ substituted by $\epsilon_j$ and $v$ by $v_j$. For this transformation not to be singular $W(u_1, \ldots, u_k)$ must not have zeros in the real axis. For simplicity let us assume from now on that $\epsilon_k < \epsilon_{k-1} < \cdots < \epsilon_j < E_0 = 1/2$. With this ordering $W(u_1, \ldots, u_k)$ will not have zeros if $|v_j| < 1$ for $j$ odd and $|v_j| > 1$ for $j$ even, and thus the new potential

$$
\tilde{V}(x) = \frac{x^2}{2} - [\ln W(u_1, \ldots, u_k)]''
\tag{45}
$$

will not have singularities.
It is important to notice that the Hamiltonian $\hat{H}$ has well defined ladder operators. In fact, the harmonic oscillator has first-order ladder operators $a^+$ and $a^-$. Let us define now two ladder operators for the system described by $\hat{H}$ [1, 26]:

$$L^+ = A^+ a^+ A, \quad L^- = A^+ a^- A. \quad (46)$$

While $A^+$ and $A$ are differential operators of order $k$, $L^+$ and $L^-$ are of order $2k + 1$. Due to the intertwining relations (3) and the defining commutation relations of the ladder operators $a^+$ and $a^-$, $[\hat{H}, a^\pm] = \pm a^\pm$, it turns out that $[\hat{H}, L^\pm] = \pm L^\pm$, i.e., $L^+$ and $L^-$ are $(2k + 1)$th-order ladder operators for $\hat{H}$.

If $k = 1$ the ladder operators $L^\pm$ are of third order and $[\hat{H}, L^+, L^-]$ directly generate a second-order polynomial Heisenberg algebra. On the other hand, since $A^+$ and $A$ are of second order if $k = 2$, then $L^\pm$ will be of fifth order in such a case. It is important to know under which circumstances $L^\pm$ can be ‘reduced’ to third-order ladder operators. The answer is contained in the following theorem [12]: if the seed solutions $u_1(x)$ and $u_2(x)$ are such that $u_2 = a^- u_1$ and $\epsilon_2 = \epsilon_1 - 1$, then $L^\pm$ can be factorized as

$$L^+ = (\hat{H} - \epsilon_1) l^+, \quad L^- = l^- (\hat{H} - \epsilon_1), \quad (47)$$

where $l^+$ and $l^-$ are third-order differential ladder operators of $\hat{H}$, such that $[\hat{H}, l^\pm] = \pm l^\pm$, which also satisfy

$$l^+ l^- = (\hat{H} - \epsilon_2)(\hat{H} - \epsilon_1 - 1)(\hat{H} - 1/2). \quad (48)$$

Once the Hamiltonian having third-order differential ladder operators is identified, it is straightforward to generate then the solutions to the PIV equation through its extremal states (see (39)). Using this technique, plenty of non-singular explicit solutions to the PIV equation, either real or complex, have been derived. It would be interesting to explore systematically the corresponding singular solutions. In this paper we will start this study by allowing the existence of one fixed singularity at the origin. Since our treatment is based on the harmonic oscillator with an infinite potential barrier at $x = 0$, it is natural to start first by studying the associated problem of eigenvalues and then its corresponding SUSY partners.

3. Harmonic oscillator with an infinite potential barrier at the origin

We are interested in studying the Hamiltonian $H_0 = -\frac{1}{2} \frac{d^2}{dx^2} + V_0(x)$ with

$$V_0(x) = \begin{cases} \frac{x^2}{2} & \text{if } x > 0 \\ \infty & \text{if } x \leq 0. \end{cases}$$

The eigenvalues of $H_0$ take the form $E_n = 2n + \frac{1}{2}$ with corresponding eigenfunctions

$$\psi_n(x) = C_n x e^{-x^2/2} _1F_1\left(-n; \frac{1}{2}; x^2\right), \quad (49)$$

with $n \in \mathbb{N}$ and $C_n = \frac{2 (2n+1)!}{2^{2n+1} n!} \left[\frac{2}{2n+1}\right]^\frac{1}{2}$ being normalization constants [44]. These eigenfunctions $\psi_n(x)$ correspond to the odd eigenfunctions of the standard harmonic oscillator normalized in the domain $(0, \infty)$, which are the ones that satisfy the boundary conditions at $x = 0$ and in the limit $x \to \infty$. A plot of the potential corresponding to the harmonic oscillator with an infinite potential barrier along with the first three eigenfunctions can be found in figure 1.

It will be required further ahead the even eigenfunctions of the standard harmonic oscillator but now normalized in the domain $(0, \infty)$,

$$\chi_n(x) = B_n e^{-x^2/2} _1F_1\left(-n; \frac{1}{2}; x^2\right), \quad (50)$$
Figure 1. The potential $V_0$ and its first three eigenfunctions.

which are associated to $E_n = 2n + \frac{1}{2}$, where $n \in \mathbb{N}$ and $B_n = \frac{(2n)!}{2^n \sqrt{2^n n!}}$ are their normalization constants [44]. Although they satisfy $H_0 \chi_n = E_n \chi_n$, they do not obey the boundary condition at $x = 0$, and thus they are not eigenfunctions of $H_0$. We will say then that the $\chi_n$ are non-physical eigenfunctions (NPE) of $H_0$ associated to $E_n$.

3.1. 1-SUSY

Let us suppose now that $H_0$ is intertwined with another Hamiltonian $H_1 = -\frac{1}{2} \frac{d^2}{dx^2} + V_1$ as in (3), where $H_0$ is identified with the initial Hamiltonian $H$ and $H_1$ with the final one $\tilde{H}$ and the intertwining operators $A^+, A$ are given by (9). Consequently, the pair of Hamiltonians $H_0$ and $H_1$ can be factorized in the following way (see (4)):

$$H_0 = AA^+ + \epsilon, \quad H_1 = A^+ A + \epsilon,$$

(51)

where the factorization energy $\epsilon$ is supposed to be real. In addition, the transformation function $u(x)$ must satisfy the stationary Schrödinger equation:

$$-\frac{1}{2} u'' + V_0 u = \epsilon u.$$

(52)

For $x > 0$, this equation has a general solution given by

$$u(x) = e^{-x^2/2} \left[ b_1 _1 F_1 \left( \frac{1 - 2\epsilon}{4}; \frac{1}{2}; x^2 \right) + b_2 _1 F_1 \left( \frac{3 - 2\epsilon}{4}; \frac{3}{2}; x^2 \right) \right],$$

(53)

$b_1, b_2$ being real constants [3]. Since the bound states of $H_0$ vanish at $x = 0$ and for $x \to \infty$, then the same boundary conditions will be required for the eigenfunctions of the new Hamiltonian $H_1$. This implies that the transformation function of (53) must have a well defined parity in order that the bound states of $H_1$ vanish at $x = 0$; hence, two different cases can be identified.

3.2. Odd transformation function

Let us choose in the first place an odd transformation function by taking $b_1 = 0$ and $b_2 = 1$ in (53) so that

$$u(x) = xe^{-x^2/2} _1 F_1 \left( \frac{3 - 2\epsilon}{4}; \frac{3}{2}; x^2 \right).$$

(54)
0
2
4
6
8
10
12
14
16
18
20
x

1
2
3
4
5
6
7
Figure 2. The potential $V_1(x)$ as a function of $x$ and the factorization energy $\epsilon$ for odd transformation functions.

The substitution of this expression in (12) leads us immediately to the new potential

$$V_1 = V_0 + \frac{1}{x^2} + 1 - \left\{ \ln \left[ \psi_n \left( \frac{3 - 2 \epsilon}{4}; \frac{5}{2}; x^2 \right) \right] \right\}', \quad x > 0. \tag{55}$$

This potential contains a term of the form $\frac{1}{x^2}$, which is singular at $x = 0$ and by itself induces in a natural way the vanishing boundary condition for the eigenfunctions of $H_1$ at $x = 0$. Transformations with $\epsilon > \frac{3}{2}$ are not allowed since they generate additional singularities for $x > 0$, and thus they modify the domain $(0, \infty)$ of the initial potential. Figure 2 shows the dependence of $V_1(x)$ on $x$ and on the parameter $\epsilon$ for an intertwining that uses an odd transformation function.

As was shown in (13), an eigenfunction $\psi_n(x)$ of $H_0$ associated to the eigenvalue $E_n$ typically transforms into an eigenfunction $\phi_n(x)$ of $H_1$ associated to $E_n$, i.e.,

$$\phi_n(x) \propto A^+ \psi_n(x) \propto -\psi_n' + \frac{u'}{u} \psi_n \propto \frac{W[u, \psi_n]}{u}. \tag{56}$$

In our case this becomes true (see however the next subsection), and when substituting the expressions for $u(x)$ and $\psi_n(x)$ we obtain explicitly the eigenfunctions $\phi_n(x)$ (which satisfy the equation $H_1 \phi_n = E_n \phi_n$ and the boundary conditions $\phi_n(0) = \phi_n(\infty) = 0$):

$$\phi_n(x) = -D_n x^2 e^{-x^2/2} \left\{ \frac{4n}{3} \psi_n \left( \frac{3 - 2 \epsilon}{4}; \frac{5}{2}; x^2 \right) + \left( 1 - \frac{2}{3} \epsilon \right) \frac{1}{u} \left[ \psi_n \left( \frac{3 - 2 \epsilon}{4}; \frac{5}{2}; x^2 \right) \frac{1}{u} \psi_n \left( -n; \frac{3}{2}; x^2 \right) \right] \right\}. \tag{57}$$

with $D_n = \frac{C_n}{\sqrt{2^{3n/2} n!}}$ being normalization constants. The corresponding energies $E_n = 2n + \frac{3}{2}$, $n = 0, 1, 2, \ldots$ thus belong to the spectrum of $H_1$. Some eigenfunctions $\phi_n(x)$ along with their corresponding potential have been drawn in figure 3.

The even eigenfunctions of the standard harmonic oscillator $\chi_n(x)$, which are NPE of $H_0$, transform into NPE $\psi_n(x)$ of $H_1$ which diverge at $x = 0$, as can be seen from the following
explicit expressions which were calculated by using the right-hand side of (56) with \( \psi_n \) substituted by \( \chi_n \):

\[
\begin{align*}
\varphi_n(x) & \propto \frac{1}{x} e^{-x^2/2} \left\{ 4n x^2 \, _1F_1 \left( 1 - n; \frac{3}{2}; x^2 \right) \\
& \quad + \left[ \left( 1 - \frac{2}{3} \epsilon \right) x^2 \, _1F_1 \left( \frac{2-2\epsilon}{4}; \frac{3}{2}; x^2 \right) - 1 \right] \, _1F_1 \left( \frac{-n}{2}; \frac{1}{2}; x^2 \right) \right\}.
\end{align*}
\]

Thus, the energies \( E_n = 2n + \frac{1}{2}, \ n = 0, 1, 2, \ldots \) do not belong to the spectrum of \( H_1 \). In addition, since \( \varphi_{\epsilon}(x) \propto 1/u(x) \) diverges also for \( x = 0 \), then \( \epsilon \) belongs neither to the spectrum of \( H_1 \).

The limit case \( \epsilon \to \frac{1}{2} \) is worth of attention, since for this factorization energy the ground state energy level of \( H_0 \) is erased from the spectrum of the new Hamiltonian \( H_1 \). Although the new spectrum is equivalent to the old one through a finite displacement in the energy, the form of the new potential, however, is drastically different from the initial one due to the singular term \( 1/x^2 \) (see (55)).

### 3.3. Even transformation function

Let us choose now the even solution of (52) as transformation function,

\[
u(x) = e^{-x^2/2} \, _1F_1 \left( \frac{1-2\epsilon}{4}; \frac{1}{2}; x^2 \right).
\]

Using once again (12), the potential \( V_1(x) \) turns out to be

\[
V_1(x) = V_0(x) + 1 - \left\{ \ln \left[ \, _1F_1 \left( \frac{1-2\epsilon}{4}; \frac{1}{2}; x^2 \right) \right] \right\}^n.
\]

This potential has also a singularity at \( x = 0 \), since \( V_0(x) \) includes the infinite potential barrier. Transformations with \( \epsilon > \frac{1}{2} \) are not allowed, due to they generate additional singularities for \( x > 0 \) and, thus, they modify the domain of definition of the initial potential. Figure 4 shows the dependence of \( V_1(x) \) on \( x \) and on the parameter \( \epsilon \) for an intertwining that uses an even transformation function.
Let us calculate now the eigenfunctions of $H_1$ from the corresponding ones of $H_0$ using (56). For the odd eigenfunctions $\psi_n(x)$ we obtain

$$
\phi_n(x) \propto e^{-\frac{x^2}{2}} \left\{ 4n x^2 \left[ 1 - n; \frac{5}{2}; x^2 \right] + 3 \left[ (1 - 2\epsilon) x^2 \left[ 1 - 2\epsilon; \frac{3}{2}; x^2 \right] - 1 \right] \left[ -n; \frac{3}{2}; x^2 \right] \right\},
$$

(61)

with $D_n = \frac{B_n}{\sqrt{2(E_n - \epsilon)}}$ being normalization constants. Some of them are plotted in figure 5 along with the corresponding potential. Note that $\phi_n(x)$ satisfy the boundary conditions, while $\psi_n(x)$ do not. As in the previous case, the function $\phi_n(x) \propto 1/u(x)$ does not obey the boundary condition at $x = 0$ and thus the associated factorization energy $\epsilon$ does not belong to the spectrum of $H_1$. In conclusion, the spectrum of $H_1$ is composed by the levels

$$
E_n = \frac{2n + \frac{1}{2}}{2}, \quad n = 0, 1, \ldots
$$

Once again, there is a notorious limit $\epsilon \to \frac{1}{2}$, since in this case the otherwise ground state energy level $E_0 = \frac{1}{2}$ is erased from the spectrum of the new Hamiltonian $H_1$. Notice that in this case the new potential and its spectrum become the same as the initial ones (up to a finite displacement in the energy).

3.4. 2-SUSY

Let us suppose that $H_0$ is intertwined with a different Hamiltonian $H_2$ as in (3), $H_2$ being identified now with $\tilde{H}$ and the intertwining operators $A^+$. $A$ with the second-order ones of
According to subsection 2.2, the transformation functions \( u_1(x) \) and \( u_2(x) \) must satisfy the stationary Schrödinger equation, whose general solution for \( x > 0 \) is the one of (53). From (19) we can see that the new potential can be written as

\[
V_2 = V_0 - \eta' = V_0 - \left[ \ln W(u_1, u_2) \right]''.
\]

In addition, the eigenfunctions \( \psi_n(x) \) of \( H_0 \) typically transform into eigenfunctions \( \phi_n(x) \) of \( H_2 \) through the action of the intertwining operator \( A^\dagger \) as follows:

\[
\phi_n(x) = \frac{A^\dagger \psi_n(x)}{\sqrt{(E_n - \epsilon_1) (E_n - \epsilon_2)}} - \frac{[\ln W(u_1, u_2)]'' \psi_n(x)}{2} + \frac{[\ln W(u_1, u_2)]'''}{2} - 2E_n + \epsilon_1 + \epsilon_2
\]

As it was seen previously, for a given \( \epsilon \) there are two solutions with opposite parity: the odd solution \( u(x) = x e^{-\frac{x}{2}} \frac{\Gamma(\frac{3-2\epsilon}{4}; \frac{3}{2}; x^2)}{\frac{\Gamma(\frac{3-2\epsilon}{4}; \frac{3}{2}; x^2)}} \) and the even one \( u(x) = e^{-\frac{x}{2}} \frac{\Gamma(\frac{3-2\epsilon}{4}; \frac{3}{2}; x^2)}{\frac{\Gamma(\frac{3-2\epsilon}{4}; \frac{3}{2}; x^2)}} \). We are going to choose \( u_1(x) \) and \( u_2(x) \) as parity definite solutions for the ordering \( \epsilon_1 > \epsilon_2 \).

Note that there exist four different parity combinations leading to four different kinds of second-order transformations which are explored below.

### 3.5. Odd–odd transformation functions

Let us choose \( u_1(x) = x e^{-\frac{x}{2}} \frac{\Gamma(\frac{3-2\epsilon}{4}; \frac{3}{2}; x^2)}{\frac{\Gamma(\frac{3-2\epsilon}{4}; \frac{3}{2}; x^2)}} \) and \( u_2(x) = x e^{-\frac{x}{2}} \frac{\Gamma(\frac{3-2\epsilon}{4}; \frac{3}{2}; x^2)}{\frac{\Gamma(\frac{3-2\epsilon}{4}; \frac{3}{2}; x^2)}} \). Since \( u_1(x) = F(x)G(x) \) and \( u_2(x) = F(x)H(x) \) with \( F(x) = x e^{-x^2/2}, G(x) = \frac{\Gamma(\frac{3-2\epsilon}{4}; \frac{3}{2}; x^2)}{\frac{\Gamma(\frac{3-2\epsilon}{4}; \frac{3}{2}; x^2)}} \), \( H(x) = \frac{\Gamma(\frac{3-2\epsilon}{4}; \frac{3}{2}; x^2)}{\frac{\Gamma(\frac{3-2\epsilon}{4}; \frac{3}{2}; x^2)}} \), then the Wronskian \( W(u_1, u_2) \) can be expressed as

\[
W(u_1, u_2) = W(FG, FH) = F^2 W(G, H) = x^3 e^{-x^2} w(x),
\]

where \( w(x) \equiv \frac{w(G,H)}{x} \) turns out to be a continuous function without zeros in \( x \geq 0 \). In this way we have separated the singularity at \( x = 0 \) induced by the transformation on the new potential (e.g. [8]), i.e.,

\[
V_2(x) = V_0 - \left[ \ln W(u_1, u_2) \right]'' = \frac{x^2}{2} + \frac{3}{x^2} + 2 - \left[ \ln w(x) \right]' \quad \text{for} \quad x \geq 0.
\]
Let us take now $u \in \mathbb{R}$ and associated to $\epsilon$ for arriving to $H$. On the other hand, in the domain $x \geq E_j$, the two consecutive levels $E_j$ are discarded since they modify the domain of definition of the initial potential and thus its corresponding spectral problem.

It is worth noticing that this choice of $u_1(x)$ and $u_2(x)$ produces a non-singular transformation for $x > 0$ as long as the factorization energies satisfy $\epsilon_2 < \epsilon_1 \leq \frac{3}{2} = E_0$ or $E_j = \frac{3}{2} \leq \epsilon_2 < \epsilon_1 \leq \frac{3}{2} + \frac{3j}{2} = E_{j+1}$. As in the previous section, singular transformations are discarded since they modify the domain of definition of the initial potential and thus its corresponding spectral problem.

There are several limit cases through which we can delete either one or two levels of $H_0$ for arriving to $H_2$. For instance, the initial ground state energy $E_0$ can be deleted by making $\epsilon_1 = E_0$, $\epsilon_2 < E_0$ since now the solution of the stationary Schrödinger equation for $H_2$ associated to $\epsilon_1 = E_0$ does not satisfy the boundary conditions and thus $E_0 \notin \text{Sp}(H_2)$. On the other hand, in the domain $E_j \leq \epsilon_2 < \epsilon_1 \leq E_{j+1}$ we can delete either $E_j$ or $E_{j+1}$, by taking $\epsilon_2 = E_j$ with $E_j < \epsilon_1 < E_{j+1}$ in the first case or $\epsilon_1 = E_{j+1}$ and $E_j < \epsilon_2 < E_{j+1}$ in the second. Moreover, the two consecutive levels $E_j, E_{j+1}$ can be deleted by choosing $\epsilon_2 = E_j$ and $\epsilon_1 = E_{j+1}$.

In figure 6 we can see an example of the new potential $V_2$ and several of its eigenfunctions $\phi_n(x)$ for $\epsilon_2 < \epsilon_1 < 3/2$.

### 3.6. Even–even transformation functions

Let us take now $u_1 = e^{-\frac{x^2}{4}} \frac{1}{\sqrt{\pi}} F_1\left(\frac{1}{4},\frac{1}{2};x^2\right)$ and $u_2 = e^{-\frac{x^2}{8}} \frac{1}{\sqrt{\pi}} F_1\left(\frac{1}{4},\frac{1}{2};x^2\right)$. We obtain that $W(u_1, u_2) = x e^{-x^2/2}$ is a continuous function without zeros for $x \geq 0$ with $F = \frac{1}{\sqrt{\pi}} F_1\left(\frac{1}{4},\frac{1}{2};x^2\right)$ and $G = \frac{1}{\sqrt{\pi}} F_1\left(\frac{1}{4},\frac{1}{2};x^2\right)$. Hence

$$V_2(x) = \frac{x^2}{2} + \frac{1}{x^2} + 2 - [\ln w(x)]'' \quad \text{for} \quad x \geq 0. \quad (67)$$

![Figure 6. The potential $V_2$ and its first three eigenfunctions obtained from two odd seed solutions with factorization energies $\epsilon_1 = \frac{5}{8}$ and $\epsilon_2 = \frac{1}{8}$.](image-url)
Note that the eigenfunctions $\psi_n$ of $H_0$ are mapped here into NPE $\psi_n(x) = \frac{A^+ \phi_n(x)}{\sqrt{|E_n - (x - x_n)T_{2n+1}|}}$ of $H_2$ that do not satisfy the boundary conditions and then the energies $E_n$ are not in the spectrum of $H_2$. Meanwhile, the even NPE $\chi_n(x)$ of $H_0$, that do not satisfy the boundary conditions, transform into the correct eigenfunctions $\phi_n(x) = \frac{A^+ \chi_n(x)}{\sqrt{|E_n - (x - x_n)T_{2n+1}|}}$ of $H_2$, which do satisfy the boundary conditions and thus, the corresponding eigenvalues $E_n$ belong to the spectrum of $H_2$. As in the previous case, the NPE $\phi_{1,2}$ of $H_2$ associated to $\epsilon_{1,2}$ diverge at $x = 0$ and thus $\epsilon_{1,2} \notin \text{Sp}(H_2)$.

For this parity combination of $u_1$ and $u_2$ the transformation is non-singular for $x > 0$ as long as the factorization energies satisfy $\epsilon_2 < \epsilon_1 \leq \frac{1}{2} = \mathcal{E}_0$ or $\mathcal{E}_j = \frac{1 + j^2}{2} < \epsilon_2 < \epsilon_1 \leq \frac{1 + (j+1)^2}{2} = \mathcal{E}_{j+1}$. Similarly to the previous section, singular transformations with singularities at $x > 0$ are not allowed due to they change the initial spectral problem.

The limit cases for which one or two neighbor levels $\mathcal{E}_j$ disappear from $\text{Sp}(H_2)$ work similarly as in the previous case. Thus, by taking $\epsilon_1 = \mathcal{E}_0$, $\epsilon_2 < \mathcal{E}_0$ it turns out that $\mathcal{E}_0 \notin \text{Sp}(H_2)$. On the other hand, if we make either $\epsilon_2 = \mathcal{E}_j$ with $\mathcal{E}_j < \epsilon_1 < \mathcal{E}_{j+1}$ or $\epsilon_1 = \mathcal{E}_{j+1}$ with $\mathcal{E}_j < \epsilon_2 < \mathcal{E}_{j+1}$, it turns out that either $\mathcal{E}_j \notin \text{Sp}(H_2)$ or $\mathcal{E}_{j+1} \notin \text{Sp}(H_2)$ respectively. In addition, if $\epsilon_2 = \mathcal{E}_j$ and $\epsilon_1 = \mathcal{E}_{j+1}$ then both $\mathcal{E}_j, \mathcal{E}_{j+1} \notin \text{Sp}(H_2)$.

In figure 7 one can find some examples of the eigenfunctions $\phi_n(x)$ along with the corresponding potential $V_2$ for $\epsilon_2 < \epsilon_1 < \frac{1}{2}$.

### 3.7. Odd–even transformation functions

Let $u_1 = xe^{-x^2} \, _1F_1\left(\frac{1-2\epsilon_1}{4}; \frac{3}{2}; x^2\right)$ and $u_2 = e^{-x^2} \, _1F_1\left(\frac{1-2\epsilon_2}{4}; \frac{3}{2}; x^2\right)$ with $\epsilon_2 < \epsilon_1$. Since again $u_1(x) = F(x)G(x)$, $u_2(x) = F(x)H(x)$ with $F(x) = e^{-x^2/2}$, $G(x) = x \, _1F_1\left(\frac{1-2\epsilon_1}{4}; \frac{3}{2}; x^2\right)$ and $H(x) = x \, _1F_1\left(\frac{1-2\epsilon_2}{4}; \frac{3}{2}; x^2\right)$, it turns out that now the Wronskian becomes $W(u_1, u_2) = e^{-x^2}W(G, H)$, where $W(G, H)$ is a continuous function without zeros for $x \geq 0$. Thus, the new potential can be written as

$$V_2(x) = \frac{x^2}{2} + 2 - \left[\ln W(G, H)\right]^\gamma \quad \text{for} \quad x \geq 0. \quad (68)$$

Let us note that the eigenfunctions of $H_2$ are found here by acting the intertwining operator $A^+$ onto those $\psi_n$ of $H_0$, $\phi_n(x) = \frac{A^+ \psi_n(x)}{\sqrt{|E_n - (x - x_n)T_{2n+1}|}}$, since they satisfy the boundary conditions so that their corresponding eigenvalues $E_n$ belong to the spectrum of $H_2$. Meanwhile, the
Finally let us take 3.8. Even–odd transformation functions with even NPE $\chi_{E}$ of $H_0$, which do not satisfy the boundary conditions, transform into NPE $\varphi_{\alpha}(x) = \sqrt{\frac{A^{3} \psi_{\alpha}(x)}{(x^{2} - e_{\alpha}^{1}x^{2})}}$ of $H_2$ that do not satisfy the boundary conditions and, consequently, the energies $E_{n}$ do not belong to the spectrum of $H_2$.

For this choice of $u_{1}(x)$ and $u_{2}(x)$ the transformation is found to be non-singular for $x > 0$ as long as the factorization energies satisfy $\frac{T_{1} + T_{2}}{2} = \frac{e_{2} - e_{1}}{2} < e_{1} < \frac{2T_{1} - T_{2}}{2} = E_{j}$. As in the previous section, singular transformations with singularities at $x > 0$ are once again discarded.

Now we need to know if either $\phi_{\epsilon_{1}}, \phi_{\epsilon_{2}}$, or both in (21) satisfy the boundary conditions to become also eigenfunctions of $H_{2}$. For $E_{j} < e_{2} < e_{1} < E_{j}$ it turns out that $\phi_{\epsilon_{1}}$ satisfies the boundary conditions while $\phi_{\epsilon_{2}}$ does not. This implies that $e_{2} \in \text{Sp}(H_{2})$ and $e_{1} \notin \text{Sp}(H_{2})$, i.e., through the second-order SUSY transformation it can be created a new level at the position $e_{2}$. This is a surprising result since by means of the first-order SUSY transformation we produced potentials which were just isospectral to the initial one. In addition, for $e_{1} = E_{j}$ with $E_{j} < e_{2} < E_{j}$ the same result is obtained, but now it implies that $e_{1} = E_{j} \notin \text{Sp}(H_{2})$ and $e_{2} \in \text{Sp}(H_{2})$. Thus, by employing the second-order SUSY transformation we have deleted the level $E_{j}$ and at the same time we have created a new one at $e_{2}$, so we have effectively ‘moved down’ $E_{j}$ to its new position $e_{2}$. For $e_{2} = E_{j}$ and $E_{j} < e_{1} < E_{j}$ neither $\phi_{\epsilon_{1}}$ nor $\phi_{\epsilon_{2}}$ satisfy the boundary conditions so that $e_{1,2} \notin \text{Sp}(H_{2})$. Finally, for $e_{1} = E_{j}$ and $e_{2} = E_{j}$ the same happens, i.e., we have deleted the level $E_{j}$ in order to produce $H_{2}$.

In figure 8 one can find an example of the potential $V_{2}$ along with some of its eigenfunctions $\phi_{\alpha}(x)$ for $E_{1} < e_{2} < e_{1} < E_{1}$.

### 3.8. Even–odd transformation functions

Finally let us take $u_{1} = e^{-x^{2}}_{1} F_{1}(\frac{1}{2}, 1; x^{2})$ and $u_{2} = x e^{-x^{2}}_{1} F_{1}(\frac{1}{2}, 1; x^{2})$ with $e_{2} < e_{1}$. A similar calculation as in the previous section leads to a $V_{2}(x)$ having the same form of (68), where now $G(x) = x F_{1}(\frac{1}{2}, 1; x^{2})$ and $H(x) = x F_{1}(\frac{1}{2}, 1; x^{2})$. Once again, the eigenfunctions $\phi_{\alpha}$ of $H_{2}$ are obtained from those of $H_{0}$ through $\phi_{\alpha}(x) = \frac{A^{3} \psi_{\alpha}}{\sqrt{(x^{2} - e_{\alpha}^{1}x^{2})(x^{2} - e_{\alpha}^{2})}}$ which satisfy the boundary conditions so that the eigenvalues $E_{n}$ belong to the spectrum of $H_{2}$. On the other hand, the even NPE $\chi_{E}$ of $H_{0}$ which do not obey the boundary conditions of the original problem, transform into NPE $\varphi_{\alpha}(x) = \frac{A^{3} \psi_{\alpha}}{\sqrt{(x^{2} - e_{\alpha}^{1}x^{2})(x^{2} - e_{\alpha}^{2})}}$ of $H_{2}$ which do not satisfy

![Figure 8](image-url)
Figure 9. The potential $V_2$ and its first four eigenfunctions obtained from even and odd seed solutions with factorization energies $\epsilon_1 = -\frac{5}{4}$ and $\epsilon_2 = -\frac{7}{4}$.

neither the boundary conditions. Thus, their corresponding energies $\mathcal{E}_n$ are not contained in the spectrum of $H_2$.

Note that for this choice of $u_1(x)$ and $u_2(x)$ the transformation is non-singular as long as the factorization energies satisfy $\epsilon_2 < \epsilon_1 \leq \frac{1}{2}$, i.e., a new level has been created at $\epsilon_1$. For $\epsilon_1 = \mathcal{E}_0$ and $\epsilon_2 < \mathcal{E}_0$ it is obtained that $\epsilon_{1,2} \notin \text{Sp}(H_2)$, namely, there is no additional level in $\text{Sp}(H_2)$. On the other hand, for $\epsilon_2 = \mathcal{E}_j$ and $\mathcal{E}_j < \epsilon_1 < \mathcal{E}_{j+1}$ once again $\epsilon_1 \in \text{Sp}(H_2)$ and $\epsilon_2 = \mathcal{E}_j \notin \text{Sp}(H_2)$, i.e., through the second-order SUSY transformation the level $\mathcal{E}_j$ has been ‘moved up’ to the position $\epsilon_1$. For $\epsilon_1 = \mathcal{E}_{j+1}$ and $\mathcal{E}_j < \epsilon_2 < \mathcal{E}_{j+1}$ neither $\phi_{\epsilon_1}$ nor $\phi_{\epsilon_2}$ satisfy the boundary conditions so that $\epsilon_{1,2} \notin \text{Sp}(H_2)$. Finally, for $\epsilon_1 = \mathcal{E}_{j+1}$ and $\epsilon_2 = \mathcal{E}_j$ the same happens, which implies that the level $\mathcal{E}_j$ has been deleted.

Figure 9 shows a potential $V_2$ and some of its eigenfunctions $\phi_n(x)$ for $\epsilon_2 < \epsilon_1 < \frac{1}{2}$.

4. Solutions to the Painlevé IV equation

In section 2 it was stated that it is possible to find solutions $g(x)$ to the PIV equation through

$$g(x) = -x - [\ln \phi_{\epsilon_1}]'$$

where $\phi_{\epsilon_1}$ is an extremal state for a system having third-order ladder operators $l^+$ and $l^-$, which satisfy (35). Moreover, if we know the explicit form of the three extremal states (and their associated eigenvalues), we can identify each one of these with $\phi_{\epsilon_1}$ and thus three solutions to the PIV equation can be generated, associated to different parameters $a, b$. Since Hamiltonians generated from the harmonic oscillator with an infinite potential barrier at the origin through supersymmetric techniques can have third-order ladder operators, hence solutions to the PIV equation can be straightforwardly obtained, as detailed ahead.
4.1. 1-SUSY

Recall that for a first-order supersymmetry transformation the analogue to the number operator is given by

\[ N \equiv L^+ L^- = A^+ a^+ A A^+ a^- A = (H_1 - \epsilon) (H_1 - 1 - \epsilon) \left( H_1 - \frac{1}{2} \right). \]  

(70)

In addition, there are three extremal states \( \phi_1, \phi_2 \) and \( \phi_3 \) with eigenvalues \( \epsilon_1, \epsilon_2 \) and \( \epsilon_3 \) respectively which satisfy

\[ N \phi_i = L^+ L^- \phi_i = 0, \quad i = 1, 2, 3. \]  

(71)

Explicit expressions for such extremal states are well known, and we can label them firstly in the way

\[ \phi_1 \propto \frac{1}{u(x)}, \quad \phi_2 \propto A^+ a^+ u(x), \quad \phi_3 \propto A^+ \chi_0, \]  

(72)

where \( \{ \epsilon_1 = \epsilon, \epsilon_2 = \epsilon + 1, \epsilon_3 = \frac{1}{2} \} \). Moreover, the cyclic permutations of the indices of \( \{ \epsilon_1, \epsilon_2, \epsilon_3 \} \) lead immediately to additional solutions of the PIV equation with parameters determined by three different choices: \( \{ \epsilon_1 = \epsilon, \epsilon_2 = \epsilon + 1, \epsilon_3 = \frac{1}{2} \}, \{ \epsilon_1 = \epsilon + 1, \epsilon_2 = \frac{1}{2}, \epsilon_3 = \epsilon \}, \{ \epsilon_1 = \frac{1}{2}, \epsilon_2 = \epsilon, \epsilon_3 = \epsilon + 1 \} \).

It is worth noticing that the solutions to PIV depend on our selection of the transformation function \( u(x) \), for which there are two different choices (for a fixed \( \epsilon \)).

4.2. Odd transformation function

For \( u(x) = x e^{-x^2/2} _1F_1 \left( \frac{1-2\epsilon}{4}; \frac{3}{2}; x^2 \right) \) we obtain the following extremal states:

\[ \phi_1 \propto \frac{e^{x^2/2}}{x} _1F_1 \left( \frac{1-2\epsilon}{4}; \frac{3}{2}; x^2 \right), \]  

(73)

\[ \phi_2 \propto u(\ln u)^\nu - 1, \]  

(74)

\[ \phi_3 \propto e^{-x^2/2} \left[ \left( 1 - \frac{2}{3} \epsilon \right) x _1F_1 \left( \frac{7-2\epsilon}{4}; \frac{5}{2}; x^2 \right) + \frac{1}{x} \right]. \]  

(75)

These expressions and the cyclic permutations of \( \{ \epsilon_1, \epsilon_2, \epsilon_3 \} \) lead to the following three solutions \( g_i(x) = -x - [\ln \phi_i]' \) of the PIV equation

\[ g_1 = \frac{1}{x} - 2x + \left( 1 - \frac{2}{3} \epsilon \right) x _1F_1 \left( \frac{7-2\epsilon}{4}; \frac{5}{2}; x^2 \right), \]  

(76)

\[ g_2 = -g_1 - 2x - 2 \left[ \frac{x + (2\epsilon - x^2)(g_1 + x) + (g_1 + x)^3}{x^2 - 2\epsilon - 1 - (g_1 + x)^2} \right], \]  

(77)

\[ g_3 = -\frac{g_1 + 2}{g_1 + 2x} = \frac{g_1^2 + 2xg_1 + 2\epsilon - 1}{g_1 + 2x}. \]  

(78)

Note that \( g_1 \) solves the PIV equation with parameters \( a_1 = -\epsilon + \frac{1}{2} \) and \( b_1 = -2 \left( \epsilon + \frac{1}{2} \right)^2 \) while \( g_2 \) and \( g_3 \) do it for \( a_2 = -\epsilon - \frac{5}{2} \) and \( b_2 = -2 \left( \epsilon - \frac{1}{2} \right)^2 \) and \( a_3 = 2\epsilon - 1 \) and \( b_3 = -2 \) respectively. Since they involve the confluent hypergeometric function, it is said that these belong to the confluent hypergeometric function hierarchy of solutions to PIV.
In addition, for some particular values of the factorization energy $\epsilon$ they reduce to well known special functions, some examples of which are reported in the following table (here and in the following section $F(x) = \frac{1}{2}\sqrt{\pi} e^{-x^2} \text{erfi}(x)$ will represent the Dawson function):

| $\epsilon$ | $\frac{1}{2}$ | $\frac{1}{2}$ | $\frac{3}{2}$ |
|------------|---------------|---------------|---------------|
| $g_1(x)$   | $2e^{-x^2}$  | $2e^{-x^2}$  | $2e^{-x^2}$  |
| $g_2(x)$   | $\frac{2e^{-x^2}}{\sqrt{\pi}\text{erf}(x)} + \frac{1}{\sqrt{\pi}} e^{x^2}\text{erf}(x) + x$ | $-\frac{1}{x} F(x) - 2x$ | $1 - 2x^2$ |
| $g_3(x)$   | $\frac{2e^{-x^2}}{\sqrt{\pi}\text{erf}(x)} + \frac{1}{\sqrt{\pi}} e^{x^2}\text{erf}(x) + x$ | $-\frac{1}{x} F(x) - 2x$ | $\frac{1}{x}$ |

### 4.3. Even transformation function

For $u(x) = e^{-x^2/2} \text{F}_1\left(\frac{1-2\epsilon}{4}; \frac{1}{2}; x^2\right)$ we obtain

$$\phi_1 \propto e^{x^2/2} \text{F}_1\left(\frac{1-2\epsilon}{4}; \frac{1}{2}; x^2\right),$$

$$\phi_2 \propto u[(\ln u)^\nu - 1],$$

$$\phi_3 \propto (1 - 2\epsilon) x e^{-x^2/2} \frac{1}{\text{F}_1\left(\frac{1-2\epsilon}{4}; \frac{1}{2}; x^2\right)}.$$

These states and their cyclic permutations of $\{\epsilon_1, \epsilon_2, \epsilon_3\}$ lead to the following three solutions of the PIV equation:

$$g_1 = -2x + (1 - 2\epsilon) x \frac{1}{\text{F}_1\left(\frac{1-2\epsilon}{4}; \frac{1}{2}; x^2\right)},$$

$$g_2 = -g_1 - 2x - 2 \left[\frac{x + (2\epsilon - x^2) (g_1 + x) + (g_1 + x)^3}{x^2 - 2\epsilon - 1 - (g_1 + x)^2}\right],$$

$$g_3 = \frac{g_1^2 + 2xg_1 + 2\epsilon - 1}{g_1 + 2x},$$

which once again belong to the confluent hypergeometric function hierarchy of solutions to the PIV equation with parameters given by the expressions found in the previous subsection. For some particular values of $\epsilon$ we get the solutions of the following table:

| $\epsilon$ | $\frac{1}{2}$ | $\frac{1}{2}$ | $\frac{3}{2}$ |
|------------|---------------|---------------|---------------|
| $g_1(x)$   | $0$           | $-2x$         | $2[(1 - 2x^2)F(x) + x] - 2x^2 F(x) - 1$ |
| $g_2(x)$   | undetermined  | $0$           | $\frac{[(2x^2 - 1)F(x) - x][2(x - F(x))F(x) - 1]}{(2xF(x) - 1)(F(x)(2x^2F(x) + F(x) - 3x) + 1)}$ |
| $g_3(x)$   | $\frac{1}{x}$ | undetermined  | $\frac{2F(x)}{2xF(x) - 1} - \frac{1}{F(x)}$ |
4.4. 2-SUSY

Recall now that, for the second-order SUSY partner Hamiltonians generated from the harmonic oscillator by using as transformation function \( u_1 \) and \( u_2 = a^- u_1 \) with \( \epsilon_2 = \epsilon_1 - 1 \), there are third-order ladder operators \( l^+ \) and \( l^- \) such that the analogue to the number operator factorizes as

\[
l^+ l^- = (H_2 - \epsilon_1 + 1)(H_2 - \epsilon_1 - 1)(H_2 - 1/2).
\]  

Therefore, there are three extremal states \( \phi_1, \phi_2 \) and \( \phi_3 \) with eigenvalues chosen as \( \epsilon_1 = \epsilon_1 - 1, \epsilon_2 = \epsilon_1 + 1 \) and \( \epsilon_3 = \frac{1}{2} \), respectively, which satisfy

\[
N\phi_i = l^+ l^- \phi_i = 0, \quad i = 1, 2, 3.
\]  

Their explicit expressions are given by:

\[
\phi_1 \propto \frac{u_1}{W[u_1, u_2]}, \quad \phi_2 \propto A^+ a^+ u_1, \quad \phi_3 \propto A^+ \chi_0.
\]  

Using the connection formula \( u_2(x) = a^- u_1(x) \) with \( \epsilon_2 = \epsilon_1 - 1 \), these states can be expressed in terms of just one transformation function \( u_1(x) \equiv u(x) \) with \( \epsilon_1 = \epsilon \) as follows:

\[
\phi_1 \propto \frac{u}{W[u, u]} = \frac{1}{u [x^2 + 1 - 2\epsilon - (\frac{u'}{u})^2]},
\]  

\[
\phi_2 \propto 2u' - \eta u
\]  

\[
\phi_3 \propto e^{-\frac{\sqrt{2}}{2} \left[ \left( x + \frac{u'}{u} \right) \eta + 2\epsilon - 1 \right]},
\]  

where

\[
\eta = \frac{2(x + \frac{u'}{u})}{x^2 + 1 - 2\epsilon - (\frac{u'}{u})^2}.
\]  

Once again, we can choose any permutation of the indices of \( \{\epsilon_1, \epsilon_2, \epsilon_3\} \) in order to identify \( \phi_1 \) with any of the three extremal states of the system departing from the choice of \( (87) \). Hence we will obtain the following three different solutions of the PIV equation:

\[
g_1 = -x - \alpha + 2 \left[ \frac{x + \alpha}{x^2 + 1 - 2\epsilon - \alpha^2} \right]
\]  

\[
g_2 = g_1 + \frac{2\alpha^2 - 2\epsilon^2 + 2(2\epsilon + 1)}{\alpha - g_1 - x},
\]  

\[
g_3 = \frac{(x + \alpha)g_1^2 + [2\epsilon - 1 + (x + \alpha)^2] g_1 + (2\epsilon - 3)(x + \alpha)}{(x + \alpha)^2 + (x + \alpha)g_1 + 2\epsilon - 1}.
\]

Here we should remember that \( \alpha = \frac{\sqrt{2}}{u} \).

Note that \( g_1 \) solves the PIV equation with parameters \( a_1 = -\epsilon + \frac{5}{2} \) and \( b_1 = -2 \left( \epsilon + \frac{1}{2} \right)^2 \) while \( g_2 \) and \( g_3 \) do it for \( a_2 = -\epsilon - \frac{7}{2} \) and \( b_2 = -2 \left( \epsilon - \frac{3}{2} \right)^2 \) and \( a_3 = 2(\epsilon - 1) \) and \( b_3 = -8 \) respectively. Also, the solutions to PIV depend on our selection of the transformation function \( u(x) \), which leads once again to two possible options.
4.5. Odd transformation function

Taking \( u = x e^{-x^2/2} F_1 \left( \frac{3}{2}; \frac{1}{2}; x^2 \right) \) we obtain the following particular solutions \( g_i = -x - [\ln \phi_i]' \) of the PIV equation corresponding to different factorization energies \( \epsilon \):

| \( \epsilon \)   | \( g_1(x) \)                                                                                     | \( g_2(x) \)                                                                                     | \( g_3(x) \)                                                                                     |
|----------------|------------------------------------------------------------------------------------------------|------------------------------------------------------------------------------------------------|------------------------------------------------------------------------------------------------|
| \( -\frac{3}{2} \) | \( \frac{1+2x^2}{2x^2-x} \)                                                                    | \( -\frac{1}{3} - 2x \)                                                                          | \( \frac{4x(3-4x^2+4x^4)}{(1-x^2)(1+3x^2)} \)                                                    |
| \( \frac{1}{2} \)  | \( \frac{9-48x^4+32x^6-16x^8}{4x(243+855x^2-459x^4+168x^6-120x^8+112x^{10}-48x^{12})} \) | \( \frac{4x(243+855x^2-459x^4+168x^6-120x^8+112x^{10}-48x^{12})}{(1-x^2)(1+3x^2)} \) | \( \frac{4x(3-4x^2+4x^4)}{(1-x^2)(1+3x^2)} \)                                                    |
| \( \frac{3}{2} \)  | \( \frac{9-48x^4+32x^6-16x^8}{4x(243+855x^2-459x^4+168x^6-120x^8+112x^{10}-48x^{12})} \) | \( \frac{4x(243+855x^2-459x^4+168x^6-120x^8+112x^{10}-48x^{12})}{(1-x^2)(1+3x^2)} \) | \( \frac{4x(3-4x^2+4x^4)}{(1-x^2)(1+3x^2)} \)                                                    |

4.6. Even transformation function

For \( u = e^{-x^2/2} F_1 \left( \frac{1}{2}; \frac{1}{2}; x^2 \right) \) we get solutions \( g_i(x) = -x - [\ln \phi_i]' \) of the PIV equation corresponding to distinct factorization energies \( \epsilon \):

| \( \epsilon \)   | \( g_1(x) \)                                                                                     | \( g_2(x) \)                                                                                     | \( g_3(x) \)                                                                                     |
|----------------|------------------------------------------------------------------------------------------------|------------------------------------------------------------------------------------------------|------------------------------------------------------------------------------------------------|
| \( -\frac{5}{2} \) | \( \frac{4x(3-4x^2+4x^4)}{(1-x^2)(1+3x^2)} \)                                                    | \( \frac{3+9x^2+2x^4}{3-4x^2} \)                                                                   | \( \frac{4x}{1-2x^2} \)                                                                       |
| \( -\frac{1}{2} \) | 0                                                                                               | \( \frac{45-39x^2+14x^4-52x^6+40x^8}{(-45x+109x^3+30x^5-12x^7-8x^9)} \)                      | \( \frac{4x}{1+2x^2} \)                                                                       |
| \( \frac{5}{2} \)  | \( \frac{4x(3-4x^2+4x^4)}{(1-x^2)(1+3x^2)} \)                                                    | \( \frac{3+9x^2+2x^4}{3-4x^2} \)                                                                   | \( \frac{4x}{1-2x^2} \)                                                                       |

5. Conclusions

By applying the first-order SUSY QM to the harmonic oscillator with an infinite potential barrier at the origin the supersymmetric partner Hamiltonians, which are isospectral to the initial one, have been generated. On the other hand, the second-order transformations enlarge the spectral design possibilities for generating new Hamiltonians with a prescribed spectrum, since now one can either erase a selected level, or two consecutive ones. We can also add a new level to the original spectrum almost everywhere, the only restricted energies which cannot be produced are the ones corresponding to the even eigenstates of the harmonic oscillator.

When using a first-order differential intertwining operator to implement the technique, two choices appear for the transformation function \( u(x) \) (related to the parity).

If \( u(x) \) is odd there will not be singularities in the generated potential for \( x > 0 \) as long as the factorization energy satisfies that \( \epsilon \leq \frac{3}{2} = E_0 \). Besides, the eigenfunctions of the harmonic oscillator, which represent the bound states of the original system, transform into eigenfunctions of the new Hamiltonian.

On the other hand, if \( u(x) \) is even, there will not be singularities in the new potentials for \( x > 0 \) as long as the factorization energy satisfies that \( \epsilon \leq \frac{3}{2} = E_0 \). Moreover, this choice becomes peculiar, in the sense that eigenfunctions of the initial Hamiltonian \( H_0 \) are mapped into NPE of the new Hamiltonian \( H_1 \), while the NPE of \( H_0 \) transform now into the correct eigenfunctions of \( H_1 \).

When using a second-order differential intertwining operator, four parity combinations for the two transformation functions \( u_1(x) \) and \( u_2(x) \) will appear.

If both \( u_1(x) \) and \( u_2(x) \) are taken to be odd, no singularities will appear in the transformed potential for \( x > 0 \) if the factorization energies are chosen as \( \epsilon_1 < \epsilon_3 \leq \frac{3}{2} = E_0 \) or \( E_j = \frac{3+4j}{2} \leq \epsilon_2 < \epsilon_1 \leq \frac{3+4j+1}{2} = E_{j+1} \). If \( u_1(x) \) is odd and \( u_2(x) \) is even no singularities
will appear in the transformed potential for $x > 0$ as long as the factorization energies obey that $E_j = \frac{1 + 4j^2}{2} \leq \epsilon_1 < \epsilon_1 \leq 3 + 4j^2 = E_j$. When taking $u_1(x)$ even and $u_2(x)$ odd, there will be no extra singularities in the transformed potential as long as the factorization energies satisfy that $\epsilon_2 < \epsilon_1 < \epsilon_1 \leq \frac{1}{2} = E_0$ or $\epsilon_j = \frac{1 + 4j^2}{2} \leq \epsilon_2 < \epsilon_1 \leq 5 + 4j^2 = E_{j+1}$. Moreover, for these three cases it turns out that the eigenfunctions of the initial Hamiltonian transform into eigenfunctions of the new Hamiltonian $H_2$.

When both $u_1(x)$ and $u_2(x)$ are even, there will not be singularities in the new potential for $x > 0$ if the factorization energies obey that $\epsilon_2 < \epsilon_1 \leq \frac{1}{2} = E_0$ or $\epsilon_j = \frac{1 + 4j^2}{2} \leq \epsilon_2 < \epsilon_1 \leq \frac{1}{2} = E_{j+1}$. In addition, eigenfunctions of the initial Hamiltonian transform into NPE of $H_2$ while the NPE of $H_0$ transform into the correct eigenfunctions of $H_2$.

Finally, a direct and simple procedure to obtain explicit solutions to the Painlevé IV equation was implemented by using the extremal states for some families of supersymmetric partners of the harmonic oscillator with and infinite potential barrier at the origin. Let us note that some rational PIV solutions derived here coincide with several ones contained in tables 26.1 and 26.2 of [30]. A further study of the hierarchies of PIV solutions which can be generated by applying the SUSY techniques to this truncated harmonic oscillator is still required (see however [12, 28, 29, 45]).

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References

[1] Mielnik B 1984 J. Math. Phys. 25 3387
[2] Cooper F, Khare A and Sukhatme U 1995 Phys. Rep. 251 267
[3] Junker G and Roy P 1998 Ann. Phys. 270 155
[4] Márquez I F, Negro J and Nieto L M 1998 J. Phys. A: Math. Gen. 31 4115
[5] Quesne C and Vansteenkiste N 1999 Helv. Phys. Acta 72 71
[6] Mielnik B and Rosas-Ortiz O 2004 J. Phys. A: Math. Gen. 37 10007
[7] Carballo J M, Fernández D J, Negro J and Nieto L M 2004 J. Phys. A: Math. Gen. 37 10349
[8] Contreras-Astorga A and Fernández D J 2008 J. Phys. A: Math. Theor. 41 475303
[9] Marquette I 2009 J. Math. Phys. 50 095202
[10] Fernández D J 2010 AIP Conf. Proc. 1287 3
[11] Quesne C 2011 Mod. Phys. Lett. A 26 1843
[12] Bermúdez D and Fernández D J 2011 SIGMA 7 025
[13] Marquette I 2012 J. Math. Phys. 53 012901
[14] Andrianov A A and Ioffe M V 2012 J. Phys. A: Math. Theor. 45 503001
[15] Gómez-Ullate D, Grandati Y and Milson R 2013 Rational extensions of the quantum harmonic oscillator and exceptional Hermite polynomials arXiv:1306.5143 [math-ph]
[16] Shabat A 1992 Inverse Problems 8 303
[17] Veselov A P and Shabat A B 1993 Funct. Anal. Appl. 27 81
[18] Adler V E 1994 Physica D 73 335
[19] Dubov S Y, Eleonskii V M and Kulagin N E 1994 Chaos 4 47
[20] Eleonskii V M, Korolev V G and Kulagin N E 1994 Chaos 4 583
[21] Bassom A P, Clarkson P A and Hicks A C 1995 Stud. Appl. Math. 95 1
[22] Andrianov A A, Ioffe M V and Nishnianidze N D 1995 Phys. Lett. A 201 103
[23] Sukhatme U P, Rasinaru C and Khare A 1997 Phys. Lett. A 234 401
[24] Andrianov A A, Cannata F, Ioffe M and Nishnianidze D 2000 Phys. Lett. A 266 341
[25] Mateo J and Negro J 2008 J. Phys. A: Math. Theor. 41 045204
[26] Fernández D J and Hussin V 1999 J. Phys. A: Math. Gen. 32 3603
[27] Fernández A J, Negro J and Nieto L M 2004 Phys. Lett. A 324 139
[28] Bermúdez D and Fernández D J 2011 Phys. Lett. A 375 2974
[29] Bermúdez D 2012 SIGMA 8 069
[30] Gromak V I, Laine I and Shimomura S 2002 Painleve Differential Equations in the Complex Plane (Berlin: de Gruyter)
[31] Witten E 1981 Nucl. Phys. B 185 513
[32] Witten E 1982 Nucl. Phys. B 202 253
[33] Aoyama H, Sato M and Tanaka T 2001 Nucl. Phys. B 619 105
[34] Ioffe M V and Nishnianidze D N 2004 Phys. Lett. A 327 425
[35] Andrianov A A, Ioffe M V and Spiridonov V P 1993 Phys. Lett. A 174 273
[36] Andrianov A A, Ioffe M V, Cannata F and Dedonder J P 1995 Int. J. Mod. Phys. A 10 2683
[37] Bagrov V G and Samsonov B F 1997 Phys. Part. Nucl. 28 374
[38] Fernández D J, Glasser M L and Nieto L M 1998 Phys. Lett. A 240 15
[39] Fernández D J, Hussin V and Mielnik B 1998 Phys. Lett. A 244 309
[40] Samsonov B F 1999 Phys. Lett. A 263 274
[41] Mielnik B, Nieto L M and Rosas-Ortiz O 2000 Phys. Lett. A 269 70
[42] Cariñena J F, Ramos A and Fernández D J 2001 Ann. Phys. 292 42
[43] Fernández D J and Fernández-García N 2005 AIP Conf. Proc. 744 236
[44] Flügge S 1971 Practical Quantum Mechanics I (Berlin: Springer) p 68
[45] Bermúdez D and Fernández D J 2013 Supersymmetric quantum mechanics and Painleve equations arXiv:1311.0647 [math-ph]