A DYNAMICAL APPROACH TO NON-UNIFORM DENSITY THEOREMS
FOR COHERENT SYSTEMS

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Abstract. We introduce a notion of covolume for point sets in locally compact groups that simultaneously generalizes the covolume of a lattice and the reciprocal of the Beurling density for amenable, unimodular groups. This notion of covolume arises naturally from the transverse measure theory of the associated hull dynamical system of a point set. Using groupoid techniques, we prove a density theorem for coherent frames over unimodular groups using this new notion of covolume that generalizes both the previous theorems for uniform sampling in general unimodular groups and those for non-uniform sampling in compactly generated groups of polynomial growth. This density theorem also covers important new examples, in particular model sets arising from cut-and-project schemes.

1. Introduction

Let $G$ be a unimodular second-countable locally compact (sclc) group and let $(\pi, H_\pi)$ be a projective discrete series, i.e., a projective, square-integrable, irreducible, unitary representation of $G$. This article considers the reproducing properties of coherent systems arising from $\pi$, that is, sequences in $H_\pi$ of the form

$$\pi(\Lambda)\eta = (\pi(\lambda)\eta)_{\lambda \in \Lambda}$$

where $\Lambda$ is a discrete subset of $G$ and $\eta \in H_\pi$. For the complete recovery of any vector $\xi \in H_\pi$ from the coefficients $((\xi, \pi(\lambda)\eta))_{\lambda \in \Lambda}$, intuition suggests that the point set $\Lambda$ should be “sufficiently dense” in $G$. This intuition is formalized by density theorems, which roughly state that in order for $\pi(\Lambda)\eta$ to be a stable spanning sequence for $H_\pi$ (in the present paper, a frame), the density of $\Lambda$ should be no smaller than an associated critical density $d_\pi$, and for $\pi(\Lambda)\eta$ to be a stable linearly independent sequence (in the present paper, a Riesz sequence), the density should be no larger than $d_\pi$.

One of the challenges in proving a density theorem is to provide a suitable notion of density for the point sets under consideration. The existing density theorems for coherent systems fall into two categories. The first category is that of uniform sampling, i.e., from lattices $\Gamma$ in the underlying group $G$. In this case, the density of $\Gamma$ is given unambiguously by the reciprocal $1/\text{covol}(\Gamma)$ of lattice covolume, and an elementary proof of the density theorem for frames and Riesz sequences can be given by periodizing over the associated homogeneous space $G/\Gamma$ of cosets [40]. Similar techniques have been used in the specific setting of Gabor frames, notably in Janssen’s classroom proof of the density theorem [27]. The density theorem for lattices can also be deduced from the Atiyah–Schmid formula for the von Neumann dimension of $H_\pi$ as a Hilbert module over the (twisted) group von Neumann algebra of $\Gamma$, cf. [1, 4, 19].

The second category is that of non-uniform sampling, i.e., sampling from more general point sets. In this case the lack of a group structure on the point set is an obstacle and there is no clear candidate for the density of a point set in a general unimodular group. For compactly generated groups of polynomial growth however, Führ–Gröchenig–Haimi–Klotz–Romero [21] have shown that Beurling densities [8] defined in terms of a word metric give the decisive density theorems for coherent systems.
The main aim of this article is to unify these two categories of density theorems into one general theorem which also includes important new examples. One of the main innovations is a notion of covolume for arbitrary (relatively separated) point sets that simultaneously generalizes lattice covolume and the reciprocal of Beurling density for amenable, unimodular groups (possibly of exponential growth). For any relatively separated point set $\Lambda$ we introduce two (extended real-valued) numbers $\text{covol}^-\!(\Lambda)$ and $\text{covol}^+\!(\Lambda)$ which we term the lower and upper covolume, respectively. For these notions of covolume, the following theorem is proved:

**Theorem 1.1.** Let $\Lambda$ be a relatively separated set in a unimodular lcsc group $G$ and let $(\pi, \mathcal{H}_\pi)$ be a projective discrete series representation of $G$ with formal dimension $d_\pi > 0$. Then the following statements hold:

(i) If the coherent system $\pi(\Lambda)\eta = (\pi(\lambda)\eta)_{\lambda \in \Lambda}$ is a frame for $\mathcal{H}_\pi$, i.e., there exist $A, B > 0$ such that

$$A \|\xi\|^2 \leq \sum_{\lambda \in \Lambda} \|\xi, \pi(\lambda)\eta\|^2 \leq B \|\xi\|^2, \quad \xi \in \mathcal{H}_\pi,$$

then

$$d_\pi \text{covol}^+\!(\Lambda) \leq 1.$$

(ii) If $\pi(\Lambda)\eta$ is a Riesz sequence where $\eta \in \mathcal{H}_\pi$, i.e., there exist $A, B > 0$ such that

$$A \|\epsilon\|_2^2 \leq \left\| \sum_{\lambda \in \Lambda} c_\lambda \pi(\lambda)\eta \right\|_2^2 \leq B \|\epsilon\|_2^2, \quad \epsilon \in \ell^2(\Gamma),$$

then

$$d_\pi \text{covol}^-\!(\Lambda) \geq 1.$$

(The numbers $d_\pi \text{covol}^+\!(\Lambda)$ and $d_\pi \text{covol}^-\!(\Lambda)$ do not depend on the normalization of Haar measure on $G$.)

We provide some details on the definition of the lower and upper covolume of a point set $\Lambda$. Our starting point is that if $\Gamma$ is a lattice, then there exists a unique invariant probability measure $\mu$ on the homogeneous space $G/\Gamma$, and if $\mu$ is renormalized such that Weil’s integration formula holds, then $\text{covol}(\Gamma) = \mu(G/\Gamma)$. For general point sets $\Lambda$, there is a dynamical system that serves as a substitute for the homogeneous space in the lattice case, the so-called punctured hull $\Omega^\times(\Lambda)$ on which $G$ acts, see Section 3.3 for precise definitions. Invariant measures on hulls have received much attention in the theory of tilings and mathematical quasicrystals in $\mathbb{R}^d$ and other abelian groups $\mathbb{Z}^d, \mathbb{Z}^d$. More recently, aspects of this theory have been generalized the non-abelian setting $\mathbb{R}^d, \mathbb{Z}^d$. With the lattice case in mind, we define the covolume of $\Lambda$ with respect to such a finite invariant measure $\mu$ on $\Omega^\times(\Lambda)$ to be

$$\text{covol}_\mu(\Lambda) = \mu(\Omega^\times(\Lambda)),$$

where $\mu$ is normalized such that a natural generalization of Weil’s formula holds, cf. (8). Underlying this definition is the theory of transverse measures $\mathbb{M}_{\mathbb{R}^d}$, which to every invariant measure $\mu$ on $\Omega^\times(\Lambda)$ associates a transverse measure $\mu_0$ on a canonical transversal $\Omega_0(\Lambda)$. Assuming no normalization of $\mu$, the covolume $\text{covol}_\mu(\Lambda)$ can be instead defined as $\mu(\Omega^\times(\Lambda))/\mu_0(\Omega_0(\Lambda))$, see Section 4.1 for details.

Contrary to the lattice case however, the punctured hull of a general point set may admit several invariant probability measures (or, for point sets in non-amenable groups, possibly none at all). Hence, we introduce the lower and upper covolume of $\Lambda$ defined as

$$\text{covol}_\mu^-\!(\Lambda) = \inf_{\mu} \text{covol}_\mu(\Lambda), \quad \text{covol}_\mu^+\!(\Lambda) = \sup_{\mu} \text{covol}_\mu(\Lambda),$$
where the infimum and supremum are taken over all finite invariant measures on \( \Omega^x(\Lambda) \), and it is understood that \( \text{covol}^{-}(\Lambda) = \infty \) (resp. \( \text{covol}^{+}(\Lambda) = -\infty \)) if no such measures exist (in this case the statements of Theorem 1.1 are vacuously true).

1.1. Model sets. Of particular interest are point sets for which the associated punctured hull is uniquely ergodic, i.e., admits a unique invariant probability measure \( \mu \), in which case \( \text{covol}^{-}(\Lambda) = \text{covol}^{+}(\Lambda) = \text{covol}(\mu) \). The dynamical property of linear repetivity is known to be a sufficient condition for unique ergodicity, see [3, 34]. Another important class of uniquely ergodic point sets come from the cut-and-project schemes introduced by Meyer in [35] and their generalization beyond abelian groups [10]. A model set in \( G \) is a point set of the form \( \Lambda = p_G(\Gamma \cap (G \times W)) \) where \( \Gamma \) is a lattice in a product group \( G \times H \) for some locally compact group \( H \), \( W \) is a compact subset of \( H \), and \( p_G \) denotes the projection from \( G \times H \) onto \( G \). Under certain regularity assumptions on \( W \), Björklund–Hartnick–Pogorzelski [10] have proved that model sets in general lcsc groups are uniquely ergodic, and it can be derived from their work that the covolume of such a regular model set is given by the natural formula

\[
\text{covol}^{-}(\Lambda) = \text{covol}^{+}(\Lambda) = \frac{\text{covol}(\Gamma)}{m_H(W)}. \tag{1}
\]

where \( \text{covol}(\Gamma) \) is the covolume of \( \Gamma \) in \( G \times H \) and \( m_H \) denotes the Haar measure on \( H \) (cf. also Section 4.2). Hence, for regular model sets, Theorem 1.1 takes a particularly simple form.

1.2. Beurling densities. In ergodic theory, asymptotic frequencies of point sets in amenable groups are measured via Banach densities, which are defined in terms certain averaging sequences called (strong) Følner sequences, cf. Section 4.3. These densities are also applied in harmonic analysis and frame theory where they go by the name of Beurling densities. For Delone sets, that is, point sets which are both separated and relatively dense, we prove the following relation between covolumes and Beurling densities.

**Theorem 1.2.** Let \( \Lambda \) be a Delone set in an amenable, unimodular lcsc group. Denote by \( D^{-}(\Lambda) \) and \( D^{+}(\Lambda) \) respectively the lower and upper Beurling density of \( \Lambda \) defined with respect to a strong Følner sequence in \( G \). Then

\[
D^{-}(\Lambda) = \frac{1}{\text{covol}^{+}(\Lambda)}, \quad D^{+}(\Lambda) = \frac{1}{\text{covol}^{-}(\Lambda)}. \tag{1}
\]

Note that the above theorem implies in particular that any Delone set \( \Lambda \) for which \( \Omega^x(\Lambda) \) is uniquely ergodic has uniform density. In particular, it gives a new proof that a regular, cocompact model set has uniform Beurling density described by equation (1). For merely relatively separated point sets the relation between covolumes and Beurling densities is described in Lemma 4.8 and Theorem 4.10. For point sets that are not relatively dense, \( D^{-}(\Lambda) = 0 \), while \( \text{covol}^{+}(\Lambda) \) may still be finite (e.g. if \( \Lambda \) is a non-cocompact lattice).

As mentioned already, the decisive density theorems for non-uniform sampling are in terms of Beurling densities with respect to balls in groups of polynomial growth [21, Theorem 5.3]. By a result of Breuillard [12], groups of polynomial growth are characterized by the fact that balls with respect to an adapted metric form Følner sequences. This raises the natural question of whether the non-uniform density theorem for coherent systems in [21] generalizes to arbitrary amenable, unimodular groups when Beurling density is instead defined with respect to a strong Følner sequence. For Delone sets this now follows as an immediate corollary of Theorem 1.1 and Theorem 1.2. More generally, we prove the following:

**Theorem 1.3.** Let \( G \) be an amenable unimodular lcsc group, let \( \Lambda \subseteq G \) and let \( (\pi, \mathcal{H}_\pi) \) be a projective discrete series representation of \( G \).
(i) If \( \pi(\Lambda)\eta \) is a frame for \( \mathcal{H}_\pi \) for some \( \eta \in \mathcal{B}_\pi \), then
\[
D^-(\Lambda) \geq d_\pi.
\]

(ii) If \( \pi(\Lambda)\eta \) is a Riesz sequence for \( \mathcal{H}_\pi \) for some \( \eta \in \mathcal{H}_\pi \), then
\[
D^+(\Lambda) \leq d_\pi.
\]

We give some more details on our techniques for proving non-uniform density theorems. For this, fix a \( \sigma \)-projective discrete series \( \pi \) of a unimodular SLc group \( G \) and a relatively separated point set \( \Lambda \) in \( G \). Given a vector \( \eta \in \mathcal{H}_\pi \), we are interested in the structure of the coherent system \( \pi(\Lambda)\eta = (\pi(\lambda)\eta)_{\lambda \in \Lambda} \).

When \( \Gamma \) is a lattice in \( G \), the existence of coherent frames and Riesz sequences has a clear connection to the representation theory of \( \Gamma \) as an abstract discrete group. In the language of [22, 25], a vector \( \eta \in \mathcal{H}_\pi \) is a frame vector (resp. Riesz vector) for \( \pi|\Gamma \) if \( \pi(\Gamma)\eta \) is a frame (resp. Riesz sequence). The existence of a frame (resp. Riesz) vector for \( \pi|\Gamma \) being a subrepresentation (resp. superrepresentation) of the \( \sigma \)-projective left regular representation of \( \Gamma \), cf. [4, 19, 40]. A crucial part of this observation is the fact that for a lattice, the frame operator \( S \) of \( \pi(\Gamma)\eta \) intertwines \( \pi|\Gamma \); hence whenever \( \eta \) is a frame vector for \( \pi|\Gamma \) then \( S^{-1/2}\eta \) is a Parseval frame vector for \( \pi|\Gamma \), i.e., with frame bounds \( A = B = 1 \) (and similarly the existence of a Riesz vector implies the existence of an orthonormal vector).

For a non-uniform point set \( \Lambda \), there is no longer a group representation theoretic characterization of the existence of coherent frames and Riesz sequences over \( \Lambda \). The associated frame operator of a general coherent frame no longer commutes with \( \pi(\lambda) \) for \( \lambda \in \Lambda \), and consequently the associated Parseval frame is not coherent. We remedy these facts in the present paper by looking at a class of sequences more general than coherent systems over \( \Lambda \), namely \( \Lambda \)-indexed sequences of the form \( (\pi(\lambda)\eta_{\lambda^{-1}\Lambda})_{\lambda \in \Lambda} \) where \( (\eta_P)_{P \in \Omega_0(\Lambda)} \) is a collection of vectors in \( \mathcal{H}_\pi \) indexed by the transversal \( \Omega_0(\Lambda) \) of \( \Lambda \). It turns out that such frames and Riesz sequences are captured exactly by the notion of frame and Riesz vectors for an associated groupoid representation. These notions, which we introduce in Section 2.6, behave formally similar to frame and Riesz vectors for group representations, and have analogous representation theoretic characterizations. In particular, the existence of a frame (resp. Riesz) vector for a groupoid representation implies the existence of a Parseval frame (resp. orthonormal) vector. The groupoid in question is given by the restriction of the transformation groupoid \( G \ltimes \Omega^\times(\Lambda) \) to the transversal \( \Omega_0(\Lambda) \), which we denote by \( \mathcal{G}(\Lambda) \). For point sets in \( G = \mathbb{R}^d \), these groupoids as well as the closely related tiling groupoids have been studied extensively in the setting of operator algebras, noncommutative geometry and mathematical physics, see e.g. [2, 6, 13, 14, 28, 30]. The idea that coherent frames over non-uniform point sets can be viewed as “fibered” over the unit space of a groupoid already appeared in Kreisel’s work [31, 32], from which the authors of the present paper have drawn inspiration.

For our proof of Theorem 1.1 we prove in fact a density theorem for frame and Riesz vectors for the corresponding groupoid representation of \( \mathcal{G}(\Lambda) \), which are more general than coherent systems over \( \Lambda \), cf. Corollary 5.7. The proof technique is based on first establishing a relationship between frame bounds and covolume via integration of the frame inequalities, and then passing from a frame (resp. Riesz) vector to the associated Parseval (resp. orthonormal) vector, much like the technique of Romero–van Velthoven in [40]. One of the key technical results is that frames and Riesz sequences are stable under limits of point sets in the Chabauty–Fell topology, see Section 5.2 which is the topology inducing the weak limits in the sense of Beurling [8]. These results are adaptations of weak limit results on Gabor frames [26, 38] to the general coherent setting.
**Remark 1.4.** The set $B_\pi$ in [Theorem 1.1] consists of those vectors $\eta \in \mathcal{H}_\pi$ such that
\[
\int_G \sup_{y \in xQ} ||\langle \xi, \pi(y)\eta \rangle||^2 \, dx < \infty, \quad \xi \in \mathcal{H}_\pi,
\]
for some (equivalently all) compact neighborhood $Q$ of the identity of $G$. These vectors were termed admissible analyzing vectors in [21]. A more restrictive assumption than $\eta \in B_\pi$ is present in [21, Theorem 5.3]. The assumption that $\eta \in B_\pi$ is needed to show that the frame property of $\pi(\Lambda)\eta$ extends to $\pi(P)\eta$ for all $P \in \Omega^\infty(\Lambda)$. For IN groups the assumption that $\eta \in B_\pi$ can be omitted as we show in Appendix A.

1.3. Structure of the paper. The paper is organized as follows: Section 2 introduces the abstract notion of frame and Riesz vectors for a general groupoid representation, Section 3 defines the groupoid of a relatively separated point set in a lcsc group. Section 4 defines the lower and upper covolumes associated to a relatively separated point set and describes the relation to Beurling densities in the amenable case. In Section 5 the notions of frame and Riesz vectors of groupoid representations are applied to prove Theorem 1.1.

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2. Frames and Riesz vectors for groupoid representations

In this section we recall necessary notions for groupoids, their (projective) representations as well as frame theory. Then in Section 2.6 we introduce the notions of frame vectors and Riesz vectors for groupoid representations.

2.1. Groupoids. In this subsection we briefly introduce the terminology needed for groupoids, see e.g. [39] for a comprehensive reference.

A groupoid is a set $G$ together with a distinguished subset $G^{(0)}$, maps $s, r : G \to G^{(0)}$, a multiplication map $(\alpha, \beta) \mapsto \alpha \beta$ from $G^{(2)} := \{ (\alpha, \beta) \in G \times G : s(\alpha) = r(\beta) \}$ to $G$ and an inversion map $\alpha \mapsto \alpha^{-1}$ from $G \to G$ such that the following axioms are satisfied:

\begin{enumerate}
  \item $r(\alpha) = s(\alpha) = \alpha$ for all $\alpha \in G^{(0)}$;
  \item $r(\alpha)\alpha = \alpha = \alpha s(\alpha)$ for all $\alpha \in G$;
  \item $r(\alpha^{-1}) = s(\alpha)$ and $s(\alpha^{-1}) = r(\alpha)$ for all $\alpha \in G$;
  \item $\alpha^{-1}\alpha = s(\alpha)$ and $\alpha \alpha^{-1} = r(\alpha)$ for all $\alpha \in G$;
  \item $r(\alpha \beta) = r(\alpha)$ and $s(\alpha \beta) = s(\beta)$ for all $(\alpha, \beta) \in G^{(2)}$; and
  \item $(\alpha \beta)\lambda = \alpha(\beta \lambda)$ for all $(\alpha, \beta, \lambda) \in G^{(2)}$.
\end{enumerate}

The elements of $G^{(0)}$ are referred to as units, and $s$ and $r$ are called the source and range map, respectively. Given $\omega \in G^{(0)}$, we set
\[G_\omega = \{ \alpha \in G : s(\alpha) = \omega \}, \quad G^\omega = \{ \alpha \in G : r(\alpha) = \omega \}.
\]

A locally compact groupoid is a groupoid $G$ equipped with locally compact topology for which the maps $s$, $r$, multiplication and inversion are continuous with respect to the relative topologies of $G^{(0)}$ in $G$ and $G^{(2)}$ in the product $G \times G$. We also assume locally compact groupoids to be Hausdorff, which implies that $G^{(0)}$ is closed in $G$.

A locally compact groupoid $G$ is called étale if the source and range maps are local homeomorphisms when viewed as maps $s, r : G \to G$. For étale groupoids, the unit space $G^{(0)}$ is open.
in \( G \) and the fibers \( G_\omega \) and \( G^\omega \) are discrete subsets of \( G \) for each \( \omega \in G^{(0)} \). An open bisection is a subset \( U \subseteq G \) such that the restrictions of \( s \) and \( r \) to \( U \) are injective. An étale groupoid always admits a basis consisting of open bisections.

A measurable groupoid is a groupoid \( G \) which is also a measurable space such that \( G^{(0)} \) is a measurable set and the maps \( s, r, (\alpha, \beta) \mapsto \alpha\beta \) and \( \alpha \mapsto \alpha^{-1} \) are measurable (here we equip \( G \times G \) with the \( \sigma \)-algebra generated by products of measurable sets in \( G \)). If \( G \) is standard Borel as a measurable space, then \( G \) is called a Borel groupoid, and if the fiber \( G_\omega \) (equivalently \( G^\omega \)) is countable for each \( \omega \in G^{(0)} \), then \( G \) is called fibrewise countable.

An important class of groupoids that will be relevant in the present paper arise from group actions. Let \( G \) be a locally compact group with identity element \( e \) and let \( \Omega \) be a left \( G \)-space, i.e., a topological space equipped with a continuous left action of \( G \). Then the transformation groupoid \( G = G \times \Omega \) is the locally compact groupoid whose underlying set is \( G \times \Omega \) and whose structure is defined as follows: the unit space is equal to \( \Omega \), which we embed into \( G \) via \( \omega \mapsto (e, \omega) \) for \( \omega \in \Omega \). The source and range maps are given by \( s(x, \omega) = x^{-1}\omega \) and \( r(x, \omega) = \omega \) for \( (x, \omega) \in G \), groupoid multiplication is given by \( (x, \omega)(y, x^{-1}\omega) = (xy, \omega) \) and groupoid inversion is given by \( (x, \omega)^{-1} = (x^{-1}, x^{-1}\omega) \).

2.2. Abstract transversals. Let \( G \) be a locally compact groupoid and let \( \Omega_0 \) be a closed subset of the unit space \( G^{(0)} \) of \( G \). Then the set
\[
G|_{\Omega_0} = \{ \alpha \in G : s(\alpha), r(\alpha) \in \Omega_0 \}
\]
is a groupoid with unit space equal to \( \Omega_0 \). The source and range maps, groupoid multiplication and groupoid inversion are given by exactly the same formulas as those of \( G \). We will be interested in situations where \( \Omega_0 \) satisfies the following additional properties:

**Definition 2.1.** Let \( G \) be a locally compact groupoid with unit space \( \Omega \). An abstract transversal for \( G \) is a closed subset \( \Omega_0 \subseteq \Omega \) such that the following conditions hold:

(i) \( \Omega_0 \) meets every orbit of \( \Omega \) as a left \( G \)-space, i.e., for every \( \omega \in \Omega \) the intersection \( \Omega_0 \cap \{ r(\alpha) : \alpha \in G_\omega \} \) is nonempty.

(ii) With \( Z = \{ \alpha \in G : s(\alpha) \in \Omega_0 \} \),

the restrictions \( s : Z \to \Omega \) and \( r : Z \to \Omega_0 \) are open maps for the relative topologies on \( Z \) and \( \Omega_0 \).

The reason to consider transversals is that the space \( Z \) in [Definition 2.1] establishes a groupoid equivalence between \( G \) and the restriction groupoid \( G|_{\Omega_0} \), cf. [30] or [41, Chapter 9.4].

2.3. General frame theory. Let \( \mathcal{H} \) be a Hilbert space. A family \( (e_j)_{j \in J} \) of vectors in \( \mathcal{H} \) is called a frame if there exist \( 0 < A \leq B < \infty \) such that
\[
A \| \xi \|^2 \leq \sum_{j \in J} |\langle \xi, e_j \rangle|^2 \leq B \| \xi \|^2 \quad \text{for all } \xi \in \mathcal{H}.
\]
The numbers \( A \) and \( B \) are called lower and upper frame bounds for \( (e_j) \) respectively. A frame for which one can choose \( A = B = 1 \) is called a Parseval frame. If only an upper frame bound exists but not necessarily a lower frame bound, \( (e_j) \) is called a Bessel sequence.

To a Bessel sequence \( (e_j)_{j \in J} \) one can associate the analysis operator \( C : \mathcal{H} \to \ell^2(J) \), which is a bounded linear operator given by
\[
C \xi = (\langle \xi, e_j \rangle)_{j \in J} \quad \text{for } \xi \in \mathcal{H}.
\]
Hilbert spaces over $\Omega$.

The synthesis operator $D : \ell^2(J) \rightarrow \mathcal{H}$ is given by

$$D((c_j)_{j \in J}) = \sum_{j \in J} c_j e_j$$

for $(c_j)_{j \in J} \in \ell^2(J)$.

The operator $S = C^*C = DD^* : \mathcal{H} \rightarrow \mathcal{H}$ given by

$$S\xi = \sum_{j \in J} (\xi, e_j)e_j$$

for $\xi \in \mathcal{H}$ is called the frame operator associated to $(e_j)_{j \in J}$. A Bessel sequence $(e_j)_{j \in J}$ is a frame with frame bounds $0 < A \leq B < \infty$ if and only if its associated frame operator $S$ is invertible, with $AI \leq S \leq BI$.

A family $(e_j)_{j \in J}$ in $\mathcal{H}$ is called a Riesz sequence if there exists $A, B > 0$ such that

$$A\|c\|^2 \leq \left\| \sum_{j \in J} c_j e_j \right\|^2 \leq B\|c\|^2$$

for all $c = (c_j)_{j \in J} \in \ell^2(J)$.

The numbers $A$ and $B$ are called lower and upper Riesz bounds for $(e_j)_{j \in J}$, respectively. Note that an upper Riesz bound for $(e_j)_{j \in J}$ is precisely the same as an upper Bessel bound for $(e_j)_{j \in J}$, and that a Riesz sequence with Riesz bounds $A = B = 1$ is exactly an orthonormal sequence.

For a detailed account on frames, see [16].

2.4. Measurable fields of Hilbert spaces. Let $\Omega$ be a measurable space. A field of Hilbert spaces over $\Omega$ is simply a collection $(\mathcal{H}_\omega)_{\omega \in \Omega}$ of Hilbert spaces indexed by $\Omega$. We write $\langle \cdot, \cdot \rangle_\omega$ for the inner product of $\mathcal{H}_\omega$ and $\| \cdot \|_\omega$ for its norm. An element $\xi$ of the product $\prod_{\omega \in \Omega} \mathcal{H}_\omega$ is called a section of $(\mathcal{H}_\omega)_{\omega \in \Omega}$ and we denote by $\xi_\omega$ its value at $\omega \in \Omega$. The field $(\mathcal{H}_\omega)_{\omega \in \Omega}$ is called measurable if it is equipped with a linear subspace $V$ of $\prod_{\omega \in \Omega} \mathcal{H}_\omega$ such that the following properties are satisfied: There exists a countable family $(\eta^i)_{i=1}^\infty$ in $V$ such that

(i) $\{\eta^i_\omega : i \in \mathbb{N}\}$ is dense in $\mathcal{H}_\omega$ for every $\omega \in \Omega$, and

(ii) an element $\xi \in \prod_{\omega \in \Omega} \mathcal{H}_\omega$ is in $V$ if and only if $\omega \mapsto \langle \xi_\omega, \eta^i_\omega \rangle_\omega$ is measurable for every $i \in \mathbb{N}$.

The elements of the set $V$ are referred to as measurable sections.

Given measurable fields of Hilbert spaces $(\mathcal{H}_\omega)_{\omega \in \Omega}$ and $(\mathcal{K}_\omega)_{\omega \in \Omega}$ over $\Omega$, a measurable field of bounded linear operators from $(\mathcal{H}_\omega)_{\omega \in \Omega}$ to $(\mathcal{K}_\omega)_{\omega \in \Omega}$ is a collection $(T_\omega)_{\omega \in \Omega}$ where $T_\omega$ is a bounded linear operator from $\mathcal{H}_\omega$ to $\mathcal{K}_\omega$ for each $\omega \in \Omega$, such that whenever $\xi$ is a measurable section of $(\mathcal{H}_\omega)_{\omega \in \Omega}$, then $T_\omega \xi \in \prod_{\omega \in \Omega} \mathcal{H}_\omega$ given by $(T_\omega \xi)_\omega = T_\omega \xi_\omega$ is a measurable section of $(\mathcal{K}_\omega)_{\omega \in \Omega}$. Furthermore, $(T_\omega)_{\omega \in \Omega}$ is called bounded if the map $\Omega \rightarrow [0, \infty)$ given by $\omega \mapsto \|T_\omega\|_{\mathcal{H}_\omega \rightarrow \mathcal{K}_\omega}$ is bounded.

2.5. Twisted groupoid representations. Let $\mathcal{G}$ be a measurable groupoid with unit space $\Omega = \mathcal{G}^{(0)}$. A continuous map $\sigma : \mathcal{G}^{(2)} \rightarrow \mathbb{T}$ is called a 2-cocycle on $\mathcal{G}$ if

(i) $\sigma(\alpha, \beta)\sigma(\alpha\beta, \lambda) = \sigma(\alpha, \beta\lambda)\sigma(\beta, \lambda)$ for all $\alpha, \beta, \lambda \in \mathcal{G}$ with $s(\alpha) = r(\beta)$ and $s(\beta) = r(\lambda)$;

(ii) $\sigma(\alpha, s(\alpha)) = \sigma(r(\alpha), \alpha) = 1$ for all $\alpha \in \mathcal{G}$.

A $\sigma$-projective (weakly measurable) unitary representation $\pi$ of $\mathcal{G}$ on a field $(\mathcal{H}_\omega)_{\omega \in \Omega}$ of Hilbert spaces over $\Omega$ is given by a unitary map $\pi(\alpha) : \mathcal{H}_{s(\alpha)} \rightarrow \mathcal{H}_{r(\alpha)}$ for every $\alpha \in \mathcal{G}$ such that the following properties are satisfied:

(i) For every pair of measurable sections $\xi$ and $\eta$ of $(\mathcal{H}_\omega)_{\omega \in \Omega}$, the map $\mathcal{G} \rightarrow \mathbb{C}$ given by $\alpha \mapsto \langle \pi(\alpha)\xi_{s(\alpha)}, \eta_{r(\alpha)} \rangle$ is measurable; and

(ii) $\pi(\alpha)\pi(\beta) = \sigma(\alpha, \beta)\pi(\alpha\beta)$ whenever $s(\alpha) = r(\beta)$. 


Given two $\sigma$-projective unitary representations $\pi$ and $\rho$ of $G$ on measurable fields $(H_\omega)_{\omega \in \Omega}$ and $(K_\omega)_{\omega \in \Omega}$ respectively, a measurable field $(T_\omega)_{\omega \in \Omega}$ of bounded linear operators from $(H_\omega)_\omega$ to $(K_\omega)_\omega$ is said to intertwine $\pi$ and $\rho$ if
\[
T_{r(\omega)}(\alpha) = \pi(\alpha)T_{s(\omega)}(\alpha) \quad \text{for all } \alpha \in G.
\]
If $T$ has the same domain and codomain and $T_{r(\omega)}(\alpha) = \pi(\alpha)T_{s(\omega)}(\alpha)$ for all $\alpha \in G$, we simply say that $T$ intertwines $\pi$.

There is a canonical $\sigma$-projective unitary representation associated to a fibrewise countable measurable groupoid $G$, namely its $\sigma$-projective left regular representation $\lambda$, defined as follows: Consider the field of Hilbert spaces $(L^2(G^\omega))_{\omega \in \Omega}$. Note that an element $\xi = (\xi_\omega)_{\omega} \in \prod_{\omega \in \Omega} L^2(G^\omega)$ defines a function $G \to \mathbb{C}$ by $\alpha \mapsto \xi_{r(\omega)}(\alpha)$. We declare the measurable sections of the field $(L^2(G^\omega))_{\omega \in \Omega}$ to be those $\xi$ for which the associated function on $G$ is measurable. This gives $(L^2(G^\omega))_{\omega \in \Omega}$, the structure of a measurable field of Hilbert spaces over $\Omega$.

We now define the $\sigma$-projective left regular representation $\lambda^\sigma$ of $G$ on this field by defining $\lambda^\sigma(\alpha): L^2(G^\sigma(\alpha)) \to L^2(G^\tau(\alpha))$ via
\[
\lambda^\sigma(\alpha)f(\beta) = \sigma(\alpha, \alpha^{-1}\beta)f(\alpha^{-1}\beta) \quad \text{for } f \in L^2(G^\sigma(\alpha)) \text{ and } \beta \in G^\tau(\alpha).
\]

2.6. Frame and Riesz vectors. Throughout this subsection we fix a fibrewise countable measurable groupoid $G$ with measurable 2-cocycle $\sigma$ and a $\sigma$-projective unitary representation $\pi$ of $G$ on a measurable field $(H_\omega)_{\omega \in \Omega(0)}$ of Hilbert spaces.

Given a measurable section $\eta = (\eta_\omega)_\omega$ of $(H_\omega)_\omega$, we can for each $\omega \in \Omega(0)$ consider the $G^\omega$-indexed family
\[
\pi(G^\omega)\eta := (\pi(\alpha)\eta_{s(\alpha)})_{\alpha \in G^\omega}
\]
of vectors in $H_\omega$. In this context we make the following definition:

**Definition 2.2.**
(i) We say that $\eta$ is a frame vector for $\pi$ if there exist $0 < A \leq B < \infty$ such that
\[
A\|\xi\|^2 \leq \sum_{\alpha \in G^\omega} |\langle \xi_\omega, \pi(\alpha)\eta_{s(\alpha)} \rangle|^2 \leq B\|\xi\|^2 \quad \text{for all } \omega \in \Omega(0) \text{ and } \xi \in H_\omega.
\]
In other words, each family $\pi(G^\omega)\eta$ is a frame for $H_\omega$, with frame bounds $0 < A \leq B < \infty$ independent of $\omega$.

(ii) We say that $\eta$ is a Riesz vector for $\pi$ if there exist $A, B, > 0$ such that
\[
A\|c\|^2 \leq \sum_{\alpha \in G^\omega} c_{\alpha}\pi(\alpha)\eta_{s(\alpha)}\| \leq B\|c\|^2 \quad \text{for all } \omega \in \Omega(0) \text{ and all } c = (c_{\alpha})_{\alpha \in G^\omega} \in L^2(G^\omega).
\]
In other words, each family $\pi(G^\omega)\eta$ is a Riesz sequence for $H_\omega$, with Riesz bounds $0 < A \leq B < \infty$ independent of $\omega$.

If only the upper frame bounds exist in the definition of a frame vector, we call it a Bessel vector for $\pi$. We make analogous definitions for Parseval frame vectors and orthonormal vectors.

**Proposition 2.3.** Suppose $\eta$ is a Bessel vector for $\pi$ and denote by $C_\omega$ the analysis operator of $\pi(G^\omega)\eta$ for ($\mu$-almost) each $\omega \in \Omega(0)$. Then $C = (C_\omega)_{\omega \in \Omega(0)}$ is a bounded measurable field of bounded linear operators from $(H_\omega)_{\omega \in \Omega(0)}$ to $(L^2(G^\omega))_{\omega \in \Omega(0)}$ that intertwines $\pi$ and $\lambda^\sigma$.

**Proof.** Suppose that $\eta$ is a Bessel vector for $\pi$. If $\xi$ is a measurable section for $(H_\omega)_{\omega \in \Omega(0)}$, then measurability of $C\eta \in \prod_{\omega \in \Omega(0)} L^2(G^\omega)$ means exactly that the map $G \to \mathbb{C}$ given by $\alpha \mapsto (\xi_{r(\omega)}, \pi(\alpha)\eta_{s(\alpha)})$ is measurable, which follows from the definition of a weakly measurable groupoid representation. Hence $C$ defines a measurable field of bounded linear operators,
which is bounded because of the uniform Bessel bound of the families \( \pi(G^\omega)\eta \), \( \omega \in G^{(0)} \). To prove the intertwining relation, note that for \( \alpha, \beta \in G \) with \( r(\alpha) = r(\beta) \) we have
\[
\sigma(\alpha, \alpha^{-1}) = \sigma(\alpha, \alpha^{-1})\sigma(\alpha^{-1}, \beta) = \sigma(\alpha, \alpha^{-1}\beta)\sigma(\alpha^{-1}, \beta).
\]
Hence, letting \( \alpha \in G \), \( \xi \in \mathcal{H}_{s(\alpha)} \) and \( \beta \in G^{r(\alpha)} \), we get
\[
C_{r(\alpha)}\pi(\alpha)\xi(\beta) = \langle \pi(\alpha)\xi, \pi(\beta)\eta_{s(\beta)}\rangle_{r(\alpha)}
= \langle \xi, \pi(\alpha)^*\pi(\beta)\eta_{s(\beta)}\rangle_{r(\alpha)}
= \langle \xi, \sigma(\alpha, \alpha^{-1})\sigma(\alpha^{-1}, \beta)\pi(\alpha^{-1}\beta)\eta_{s(\beta)}\rangle_{r(\alpha)}
= \sigma(\alpha, \alpha^{-1}\beta)\langle \xi, \pi(\alpha^{-1}\beta)\eta_{s(\beta)}\rangle_{r(\alpha)}
= \lambda^\sigma(\alpha)C_{s(\alpha)}\xi(\beta).
\]

**Corollary 2.4.** The following statements hold:

(i) If \( \pi \) admits a frame vector, then \( \pi \) admits a Parseval frame vector.

(ii) If \( \pi \) admits a Riesz vector, then \( \pi \) admits an orthonormal vector.

**Proof.** (i): Let \( \eta \) be a frame vector for \( \pi \) and denote by \( S_\omega \) the frame operator of \( \pi(G^\omega)\eta \) for \( \omega \in G^{(0)} \). Then \( S_\omega = C_\omega^*C_\omega \) where \( C_\omega \) is the analysis operator of \( \pi(G^\omega)\eta \), hence from **Proposition 2.3** we get that \( S = (S_{\omega})_\omega \) is a bounded measurable field of bounded linear operators on \( (\mathcal{H}_\omega)_\omega \) that intertwines \( \pi \) in the sense that
\[
S_{r(\alpha)}\pi(\alpha) = \pi(\alpha)S_{s(\alpha)} \quad \alpha \in G.
\]
Taking inverses above, one sees that the field \( S^{-1} = (S_{\omega}^{-1})_\omega \) is also a bounded measurable field of bounded linear operators that intertwines \( \pi \). Using this relation repeatedly, one obtains \( p(S_{r(\alpha)}^{-1})\pi(\alpha) = \pi(\alpha)p(S_{r(\alpha)}^{-1}) \) for every polynomial \( p(x) \) in one indeterminate \( x \). Letting \( 0 < A \leq B < \infty \) be the frame bounds of \( \eta \) we have that \( AI \leq S_\omega \leq BI \) for all \( \omega \in \Omega \). Writing the square-root function as a uniform limit of polynomials \( p_n \) on the interval \([A, B]\), we get from the continuous functional calculus that
\[
||S_{r(\alpha)}^{-1/2}\pi(\alpha) - \pi(\alpha)S_{s(\alpha)}^{1/2}|| = \lim_n ||p_n(S_{r(\alpha)})\pi(\alpha) - \pi(\alpha)p_n(S_{s(\alpha)})|| = 0.
\]
It follows that the field \( S^{-1/2} = (S_{\omega}^{-1/2})_\omega \) intertwines \( \pi \). Hence the canonical Parseval frame associated to each \( \pi(G^\omega)\eta \) is of the form
\[
S_{\omega}^{-1/2}\pi(G^\omega)\eta = \pi(G^\omega)S^{-1/2}\eta.
\]
This shows that \( S^{-1/2}\eta \) is a Parseval frame vector for \( \pi \).

(ii): Let \( \eta \) be a Riesz vector associated to \( \pi \). For each \( \omega \in \Omega \) set \( K_\omega = \overline{\text{span}}(\pi(G^\omega)\eta) \). We then get a representation \( \pi|K_\omega \) by restricting each \( \pi(\alpha) \) to \( K_{s(\alpha)} \), and \( \eta \) becomes both a Riesz vector and a frame vector for \( \pi|K_\omega \), so we can form the associated frame operators \( S_\omega \) on \( K_\omega \). Arguing similarly as in (ii) it now follows that the field \( S^{-1/2} = (S_{\omega}^{-1/2})_\omega \) intertwines \( \pi|K_\omega \), hence \( S^{-1/2}\eta \) is an orthonormal vector for \( \pi|K_\omega \), hence also for \( \pi \). □

3. Point sets in locally compact groups

Throughout this section we fix a unimodular lcs group \( G \) with identity element \( e \). We fix a Haar measure \( m \) on \( G \) and denote integration with respect to this measure by \( \int_G \ldots \, dx \).
3.1. The Chabauty–Fell topology. Denote by \( C(G) \) the set of closed subsets of \( G \). The Chabauty–Fell topology on \( C(G) \) is given by the subbasis consisting of the sets
\[
\mathcal{O}_K = \{ C \in C(G) : C \cap K = \emptyset \},
\]
\[
\mathcal{O}^V = \{ C \in C(G) : C \cap V \neq \emptyset \},
\]
where \( K \) ranges over all compact subsets of \( G \) and \( V \) ranges over all open subsets of \( G \). Alternatively, it is determined by an open neighborhood basis at each \( C \in C(G) \) consisting of the sets
\[
\mathcal{V}_{K,V}(C) = \{ D \in C(G) : D \cap K \subseteq CV, C \cap K \subseteq DV \}
\]
where \( K \) runs through all compact subsets of \( G \) and \( V \) runs through all open neighborhoods of the identity, cf. [9, Proposition A.1]. For the rest of the paper, all topological considerations will be with respect to the Chabauty–Fell topology. The space \( C(G) \) is compact. Observe that since \( G \) is second-countable, also \( C(G) \) is second countable, so its topology is described by convergence of sequences.

The following lemma describes convergence of sequences in \( C(G) \) and will occasionally be useful, cf. [6, Proposition E.12].

**Lemma 3.1.** Let \( P_n, P \in C(G) \) for each \( n \in \mathbb{N} \). Then \( P_n \to P \) if and only if both of the following statements hold:

(i) Whenever \( x \in P \) then there exist \( x_n \in P_n \) such that \( x_n \to x \).

(ii) Whenever \( (n_k)_{k \in \mathbb{N}} \) is a subsequence of \( \mathbb{N} \) and \( x_{n_k} \in P_{n_k} \) with \( x_{n_k} \to x \in G \), then \( x \in P \).

3.2. Point sets. Let \( \Lambda \) be a subset of \( G \). Given \( S \subseteq G \) and \( \ell \in \mathbb{N} \), we say that \( \Lambda \) is

(i) \( \ell \)-relatively \( S \)-separated if \( |\Lambda \cap xS| \leq \ell \) for all \( x \in G \);

(ii) \( S \)-dense if \( |\Lambda \cap xS| \geq 1 \) for all \( x \in G \).

For a fixed \( \ell \in \mathbb{N} \) we say that \( \Lambda \) is \( \ell \)-relatively separated if there exists a nonempty open set \( U \subseteq G \) (which can always be taken a precompact, symmetric neighborhood of the identity) such that \( \Lambda \) is \( \ell \)-relatively \( U \)-separated. A set is relatively separated \(^1\) if it is \( \ell \)-relatively separated for some \( N \in \mathbb{N} \). Equivalently
\[
\text{rel}_U(\Lambda) := \sup_{x \in G} |\Lambda \cap xU| < \infty
\]
for some (equivalently) all nonempty open sets \( U \subseteq G \). We denote by
\[
\text{rel}(\Lambda) := \inf_U \sup_{x \in G} |\Lambda \cap xU|,
\]
the minimal number \( \ell = \text{rel}(\Lambda) \) such that \( \Lambda \) is \( \ell \)-relatively separated. If \( \Lambda \) is 1-relatively \( U \)-separated (resp. 1-relatively separated) we simply say that \( \Lambda \) is \( U \)-separated (resp. separated \(^2\)).

On the other hand, if \( \Lambda \) is \( K \)-dense for some compact set \( K \), then \( \Lambda \) is called relatively dense. If \( \Lambda \) is both separated and relatively dense, then \( \Lambda \) is called a Delone set.

**Proposition 3.2.** The following statements hold:

(i) For any nonempty open set \( U \subseteq G \) and \( \ell \in \mathbb{N} \) the set of \( \ell \)-relatively \( U \)-separated sets is closed in \( C(G) \).

(ii) For any compact set \( K \subseteq G \) the set of all \( K \)-dense sets is closed in \( C(G) \).

\(^1\) The term uniformly locally finite is also used in the literature.

\(^2\) The term uniformly discrete is also used in the literature.
Proof. (i) Let $P$ be in the closure of the set of all $\ell$-relatively $U$-separated subsets of $C(G)$. Let $x \in G$ and suppose for a contradiction that there exist $\ell + 1$ distinct elements $y_1, \ldots, y_{\ell+1} \in P \cap xu$. Pick a pairwise disjoint collection $(V_i)_{i=1}^{\ell+1}$ of open sets such that $y_i \in V_i$ and $V_i \subseteq xu$ for each $1 \leq i \leq \ell + 1$. Then $P \cap V_i = \emptyset$ for each $i$, and thus there exists a set $\Lambda \subseteq C(G)$ with $	ext{rel}_U(\Lambda) \leq \ell$ such that $\Lambda \subseteq \bigcap V_i \cap \bigcap V_i \cap \bigcap V_i \cap \bigcap V_i$. Thus we can find $\ell + 1$ elements $z_1, \ldots, z_{\ell+1} \in G$ with $z_i \in \Lambda \cap V_i \subseteq \Lambda \cap xu$ for each $i$, which are distinct since $(V_i)_i$ are pairwise disjoint. But this contradicts the fact that $\Lambda$ is $\ell$-relatively $U$-separated, and we can conclude that $P$ is $\ell$-relatively $U$-separated.

(ii) Let $P$ be in the closure of the set of all $K$-dense subsets of $G$. Assume for a contradiction that $P$ is not $K$-dense. Then there exists $x \in G$ such that $P \cap xK = \emptyset$, so $P \in O_{xK}$. By assumption there exists a $K$-dense set $Q$ with $Q \cap xK = \emptyset$. This is a contradiction, so $P$ must be $K$-dense.

The following proposition describes convergence of relatively separated sets.

**Proposition 3.3.** Let $(P_n)_{n \in \mathbb{N}}$ be a sequence of $\ell$-relatively $U$-separated sets that converges to a set $P$ in $C(G)$. Let $K \subseteq G$ be a compact subset of $G$ such that $P \cap \partial K = \emptyset$. Write $P \cap K = \{x^1, \ldots, x^k\}$. Then there exists an $N \in \mathbb{N}$ such that when $n \geq N$ the set $P_n \cap K$ can be partitioned into $k$ sets

$$P_n \cap K = M_1^n \cup \cdots \cup M_k^n$$

of cardinalities satisfying $1 \leq |M_j^n| \leq \ell$ such that $M_j^n \to \{x^j\}$ as $n \to \infty$ for each $1 \leq j \leq k$, that is $x_n \to x^j$ for every sequence of elements $x_n \in M_j^n$, $n \in \mathbb{N}$.

**Proof.** By Proposition 3.2 we have that $P$ is $\ell$-relatively $U$-separated. In particular $P \cap K$ is finite since $K$ is compact, so that one can indeed write $P \cap K = \{x^1, \ldots, x^k\}$ for some $k \in \mathbb{N}$.

Since $P$ is discrete, we can find an open neighborhood $V$ of the identity of $G$ such that $P \cap x^jV = \{x_j\}$ for each $1 \leq j \leq k$. We may assume that $V \subseteq U$, and since each $x^j$ lies in the interior of $K$ we may also assume that $x^jV \subseteq K$ for each $1 \leq j \leq k$. Set $M_j^n = P_n \cap x^jV$ for each $n \in \mathbb{N}$ and $1 \leq j \leq k$. Since $P_n \to P$, there exists by Lemma 3.1 an $N_1 \in \mathbb{N}$ such that when $n \geq N_1$ we can find elements $x_n \in P_n \cap x^jV$ for each $1 \leq j \leq k$. In particular, $|M_j^n| \geq 1$ for every $n \geq N_1$ and $1 \leq j \leq k$.

Set $K' = K \setminus (P \cap K)V$ and note that $P \cap K = \emptyset$, i.e., $P \in O_{K'}$. Since $P_n \to P$, there exists $N_2 \in \mathbb{N}$ such that when $n \geq N_2$ we have $P_n \in O_{K'}$. Hence $P_n \cap K = (P \cap x^jV) \cup \bigcup_{j=1}^k x^jV$ for $n \geq N_2$. Since the sets $\{x^jV\}_{j=1}^k$ are pairwise disjoint, every element of $P_n \cap K$ belongs to exactly one of the sets $M_j^n$ for $n \geq N_2$. This shows that we have a partition of $P_n \cap K$ into $M_1^n \cup \cdots \cup M_k^n$ for $n \geq N_2$ and furthermore $|M_j^n| \leq |P_n \cap x^jU| \leq \ell$. If $n \geq N := \max\{N_1, N_2\}$ then we get $1 \leq |M_j^n| \leq \ell$ for $n \geq N$. Finally, since $P_n \to P$ it follows that $M_j^n \to \{x^j\}$ for each $1 \leq j \leq k$. Equivalently, we have $x_n \to x^j$ for every sequence of elements $x_n \in M_j^n$, $n \in \mathbb{N}$.

A useful corollary of Proposition 3.2 is the following one.

**Corollary 3.4.** Let $U$ be a symmetric identity neighborhood in $G$ and let $\Lambda \subseteq G$ be an $\ell$-relatively $U$-separated set. For every compact set $K \subseteq G$ we have that

$$|P \cap K| \leq \ell \frac{m(KU)}{m(U)}, \quad \text{for all } P \in \Omega(\Lambda).$$
Proof. Let $P \in \Omega(\Lambda)$. Since $U$ is symmetric we have that $x \in yU$ if and only if $y \in xU$ for any $x, y \in G$. Using this and Proposition 3.2 we obtain

$$\sum_{y \in P \cap K} 1_{yU}(x) = \sum_{y \in P \cap K} 1_{xU}(y) = |P \cap K \cap xU| \leq |P \cap xU| \leq \ell.$$ 

Note also that if $x \in yU$ for some $y \in K$ then $x \in KU$. Consequently

$$m(U)|P \cap K| = \int_{G} \sum_{y \in P \cap K} 1_{yU}(x) \, dx = \int_{KU} \sum_{y \in P \cap K} 1_{yU}(x) \, dx \leq m(KU) \cdot \ell,$$

and the conclusion follows.

3.3. The hull of a point set. There is a continuous left $G$-action on $\mathcal{C}(G)$ given by

$$xC = \{xy : y \in C\} \text{ for } x \in G \text{ and } C \in \mathcal{C}(G).$$

The closure of the orbit of a set $\Lambda \in \mathcal{C}(G)$ under this action is called the hull of $\Lambda$ and is denoted by

$$\Omega(\Lambda) = \overline{\{x\Lambda : x \in G\}} \subseteq \mathcal{C}(G).$$

Since $\mathcal{C}(G)$ is compact, it follows that $\Omega(\Lambda)$ is a compact space. The punctured hull of $\Lambda$ is the locally compact space

$$\Omega^x(\Lambda) = \Omega(\Lambda) \setminus \{\emptyset\}. $$

Note that $\Lambda$ is relatively dense if and only if $\emptyset \notin \Omega(\Lambda)$, so that $\Omega^x(\Lambda) = \Omega(\Lambda)$ is compact in this case.

The transversal or discrete hull of $\Lambda$ is

$$\Omega_0(\Lambda) = \{P \in \Omega(\Lambda) : e \in P\} \subseteq \Omega^x(\Lambda).$$

Note that the transversal is compact since it is closed in $\Omega(\Lambda)$.

3.4. The groupoid of a point set. Fix a relatively separated set $\Lambda \subseteq G$. Since $\Omega(\Lambda)$ is a left $G$-space, we can consider the transformation groupoid $G \ltimes \Omega(\Lambda)$ as considered in Section 2.1.

Definition 3.5. Let $\Lambda$ be any closed subset of $G$. The groupoid of $\Lambda$, denoted by $G(\Lambda)$, is the restriction of the transformation groupoid $G \ltimes \Omega^x(\Lambda)$ to the transversal $\Omega_0(\Lambda)$ of $\Lambda$. That is,

$$G(\Lambda) = \{(x, P) \in G \times \Omega^x(\Lambda) : x^{-1}P, P \in \Omega_0(\Lambda)\} = \{(x, P) \in G \times \Omega_0(\Lambda) : x \in P\},$$

with source and range maps given by $s(x, P) = x^{-1}P$ and $r(x, P) = P$ for $(x, P) \in G(\Lambda)$, multiplication given by $(x, P)(y, x^{-1}P) = (xy, P)$ and inversion given by $(x, P)^{-1} = (x^{-1}, xP)$.

Proposition 3.6. For a closed subset $\Lambda \subseteq G$, the discrete hull $\Omega_0(\Lambda)$ is an abstract transversal for the transformation groupoid $G \ltimes \Omega^x(\Lambda)$ in the sense of Definition 2.1.

Proof. First we prove that every orbit of $\Omega(\Lambda)$ as a left $G \ltimes \Omega(\Lambda)$-space meets $\Omega_0(\Lambda)$. The orbit of a $P \in \Omega(\Lambda)$ is given by the set $\{xP : x \in G\}$. Picking any $x \in P^{-1}$, one has $e \in xP$, hence $xP \in \Omega_0(\Lambda)$.

Next, we prove that $s : Z(\Lambda) \rightarrow \Omega_0(\Lambda)$ and $r : Z(\Lambda) \rightarrow \Omega(\Lambda)$ are open maps. It suffices to check that the image of $(V \times W) \cap Z(\Lambda)$ under $s$ is open in $\Omega_0(\Lambda)$ and the corresponding image under $r$ is open in $\Omega(\Lambda)$ for every open $V$ in $G$ and open $W$ in $\Omega(\Lambda)$. For $s$, note that

$$s((V \times W) \cap Z(\Lambda)) = \{x^{-1}P : x \in V, P \in W, x \in P\} = \left( \bigcup_{x \in V} x^{-1}W \right) \cap \Omega_0(\Lambda),$$

which is open in $\Omega_0(\Lambda)$ by definition of the subspace topology and the continuity of the action of $G$ on $\Omega(\Lambda)$. Further,

$$r((V \times W) \cap Z(\Lambda)) = \{P \in W : x \in P \text{ for some } x \in V\} = W \cap O^V$$
is open in $\Omega(\Lambda)$. This finishes the proof.

\begin{lemma}
Let $U$ and $V$ be open sets in $G$ with $V^{-1}V \subseteq U$. If $\Lambda$ is $U$-discrete, then the map $V \times \Omega(\Lambda) \rightarrow \Omega(\Lambda)$ given by $(x,P) \mapsto xP$ is a homeomorphism onto its image.
\end{lemma}

\begin{proof}
Consider $V\Omega(\Lambda) = \{xP : x \in V, P \in \Omega(\Lambda)\}$. Because of the established injectivity, there exists for every $P \in V\Omega(\Lambda)$ a unique $xP \in V$ such that $P = xPQ$ for some $Q \in \Omega(\Lambda)$. This defines a continuous map $V\Omega(\Lambda) \rightarrow V$. Indeed, let $V' \subseteq V$ be an open neighborhood of $xP$. Then $V'\Omega(\Lambda)$ is an open neighborhood of $P$ since $P = xP(x^{-1}P) \in V'\Omega(\Lambda)$. Moreover, if $Q \in V'\Omega(\Lambda)$, then we can write $Q = xQ'$ for some $x \in V'$ and $Q' \in \Omega(\Lambda)$. But then $xQ = x$ by uniqueness of $x_Q$, hence $xQ \in V'$.

Having established continuity of $P \mapsto xP$, it now follows that the map $V\Omega(\Lambda) \rightarrow V \times \Omega(\Lambda)$ given by $P \mapsto (xP, x^{-1}P)$ is a continuous inverse of the map in the lemma. This finishes the proof.
\end{proof}

\begin{proposition}
Let $\Lambda \subseteq G$ be $U$-discrete with $U$ a symmetric, open neighborhood of the identity. Then as $x$ ranges over $G$, $V \subseteq G$ ranges over all symmetric, open neighborhoods of the identity with $V^2 \subseteq U$ and $W$ ranges over open sets in $\Omega(\Lambda)$, the sets

\[ U_{x,V,W} = ((xV \cap W) \times W) \cap G(\Lambda) \]

form a basis for the topology of $G(\Lambda)$ consisting of open bisections of $G(\Lambda)$. Further, the range and source maps of $G(\Lambda)$ are open, so that $G(\Lambda)$ is an étale groupoid.
\end{proposition}

\begin{proof}
For $x$ and $V$ as in the proposition, set $V_x = xV \cap Vx$. Since $G(\Lambda)$ is equipped with the subspace topology inherited from $G \times \Omega(\Lambda)$ and the sets $V_x \times W$ form a basis for the topology of the latter, it follows that the sets $(V_x \times W) \cap G(\Lambda)$ form a basis for the topology on $G(\Lambda)$.

Let $x$, $V$ and $W$ be as in the proposition. Since $W$ is open in $\Omega(\Lambda)$, we can find an open set $W'$ in $\Omega(\Lambda)$ such that $W = \Omega(\Lambda) \cap W'$. First we consider the restriction of $s$ to $U_{x,V,W}$. Since $V_x^{-1}V_x \subseteq (xV)^{-1}(xV) = V^2 \subseteq U$, we have from \ref{Lemma 3.7} that the map $V_x \times W' \rightarrow \Omega(\Lambda)$ given by $(x,P) \mapsto xP$ is a homeomorphism onto its image. Since inversion in $G$ is a homeomorphism, it follows that $V_x \times W' \rightarrow \Omega(\Lambda)$, $(x,P) \mapsto x^{-1}P$ is also a homeomorphism onto its image. Restricting this map to the subset $U_{x,V,W}$ of $V_x \times W'$, we obtain another homeomorphism onto its image. But this map is exactly $s$, which shows that the restriction of $s$ to $U_{x,V,W}$ is a homeomorphism onto its image. In particular, we conclude that $s$ is a local homeomorphism. Since $r = sc^{-1}$ is follows that also $r$ is a local homeomorphism and hence $G(\Lambda)$ is étale.

Next we consider the restriction of $r$ to $U_{x,V,W}$ and have to prove its injectivity. If $r(y, P) = r(y', P')$ for $(y, P), (y', P') \in U_{x,V,W}$, then $P = P'$ by definition of $r$. Hence $y, y' \in P \cap xU$, so $y = y'$ by $U$-discreteness of $P$. This shows injectivity of the restriction and hence that $U_{x,V,W}$ is an open bisection.
\end{proof}

4. Invariant measures and covolumes

In this section we continue to assume that $G$ is a unimodular ssc group.

4.1. Transverse measures. We will be concerned with the dynamics of the left $G$-action on the hulls of relatively separated sets $\Lambda \subseteq G$, in particular finite $G$-invariant Radon measures on $\Omega(\Lambda)$ which we will just refer to as finite invariant measures. We let $C_G(\Omega(\Lambda))$ denote the set of $G$-invariant probability measures on $\Omega(\Lambda)$.

The following proposition will be a key result for us. Its proof can be found in \ref{[33, Proposition 4.3]}, but for the convenience of the reader we present an argument specialized to the present setting.
Proposition 4.1. Let $\Lambda \subseteq G$ be relatively separated. Then for every non-zero finite invariant measure $\mu$ on $\Omega^x(\Lambda)$ there exists a unique measure $\mu_0$ on $\Omega_0(\Lambda)$ such that

$$
\int_{\Omega^x(\Lambda)} \sum_{x \in P} F(x, x^{-1}P) \, d\mu(P) = \int_{\Omega_0(\Lambda)} \int_G F(x, Q) \, dx \, d\mu_0(Q)
$$

(7)

for all $F \in C_c(G \times \Omega_0(\Lambda))$. The measure is called the transverse measure associated to $\mu$ and is finite and non-zero.

Proof. We consider the closed subspace $Z$ of $G \times \Omega^x(\Lambda)$ defined as

$$Z = \{(x, P) \in G \times \Omega^x(\Lambda) : x \in P\}.$$ 

There is a left action of $G$ on $Z$ given by $y \cdot (x, P) = (yx, yP)$ for $y \in G$ and $(x, P) \in Z$. We consider $Z$ as fibered over $\Omega^x(\Lambda)$ in the following way: If $\pi: Z \to \Omega^x(\Lambda)$ is given by $\pi(x, P) = P$ then $Z = \bigcup_{P \in \Omega^x(\Lambda)} \pi^{-1}(P)$. We equip each fiber $\pi^{-1}(P) = P \times \{P\}$ with the counting measure. Together with the measure $\mu$ on $\Omega^x(\Lambda)$, we obtain a Borel measure $\nu$ on $Z$ given by

$$\nu(S) = \int_{\Omega(\Lambda)} |S \cap \pi^{-1}(P)| \, d\mu(P), \quad S \subseteq Z \text{ Borel.}$$

Integration of functions $f \in C_c(Z)$ with respect to this measure is given by

$$\int_Z f(z) \, d\nu(z) = \int_{\Omega(\Lambda)} \sum_{x \in P} f(x, P) \, d\mu(P).$$

Note that $\nu$ is $G$-invariant: If $y \in G$ and $S \subseteq Z$ is Borel, then

$$\nu(yS) = \int_{\Omega(\Lambda)} |yS \cap \pi^{-1}(P)| \, d\mu(P)$$

$$= \int_{\Omega(\Lambda)} |S \cap \pi^{-1}(yP)| \, d\mu(P)$$

$$= \int_{\Omega(\Lambda)} |S \cap \pi^{-1}(P)| \, d\mu(P),$$

where we used the $G$-invariance of $\mu$ in the last equality.

Now equip $G \times \Omega_0(\Lambda)$ with the left $G$-action given by $y \cdot (x, Q) = (yx, yQ)$ for $y \in G$ and $(x, Q) \in G \times \Omega_0(\Lambda)$. The map $h: Z \to G \times \Omega_0(\Lambda)$ given by $h(x, P) = (x, x^{-1}P)$ is a $G$-invariant homeomorphism, which means that the pushforward $h_*\nu$ is a $G$-invariant measure on $G \times \Omega_0(\Lambda)$. For each fixed Borel set $T \subseteq \Omega_0(\Lambda)$, the map $S \mapsto h_*\nu(S \times T)$ defines an invariant measure on $G$, so there exist positive constants $\mu_0(T)$ such that

$$(h_*\nu)(S \times T) = \mu_0(T).$$

One then checks that $\mu_0$ defines a measure on $\Omega_0(\Lambda)$. At the level of integrals, the above formula becomes (7).

Uniqueness is straightforward. 

Note that we do not necessarily assume non-zero finite invariant measures $\mu$ on $\Omega^x(\Lambda)$ to be probability measures, i.e., $\mu(\Omega^x(\Lambda)) = 1$. Indeed, another natural normalization of $\mu$ is to instead assume that the associated transverse measure $\mu_0$ on $\Omega_0(\Lambda)$ is a probability measure. For $f \in C_c(G)$, the transverse measure formula gives

$$\mu_0(\Omega_0(\Lambda)) \int_G f(x) \, dx = \int_{\Omega_0(\Lambda)} \int_G f(x) \, dx \, d\mu_0(Q) = \int_{\Omega^x(\Lambda)} \sum_{x \in P} f(x) \, d\mu(P).$$
Hence, if $\mu$ is normalized such that $\mu_0(\Omega_0(\Lambda)) = 1$, the following generalization of Weil’s formula holds:

$$\int_G f(x) \, dx = \int_{\Omega_0^G(\Lambda)} \sum_{x \in P} f(x) \, d\mu(P), \quad \text{for all } f \in C_c(G).$$  \hfill (8)

**Definition 4.2.** Let $\Lambda \subseteq G$ be a relatively separated set and let $\mu$ be a non-zero finite invariant measure on $\Omega^G(\Lambda)$. The **covolume** of $\Lambda$ with respect to $\mu$ is the number

$$\text{covol}\mu(\Lambda) = \frac{\mu(\Omega(\Lambda))}{\mu_0(\Omega_0(\Lambda))}.$$  

Furthermore, we define the **lower** and **upper covolume** of $\Lambda$ to be the extended real-valued numbers given respectively by

$$\text{covol}_-(\Lambda) = \inf_{\mu \in \mathcal{P}_G(\Omega^G(\Lambda))} \text{covol}_\mu(\Lambda), \quad \text{covol}_+(\Lambda) = \sup_{\mu \in \mathcal{P}_G(\Omega^G(\Lambda))} \text{covol}_\mu(\Lambda),$$

where it is understood that $\text{covol}_-(\Lambda) = \infty$ (resp. $\text{covol}_+(\Lambda) = -\infty$) if $\mathcal{P}_G(\Omega^G(\Lambda)) = \emptyset$.

**Remark 4.3.** Note that for any $c > 0$ and any non-zero finite invariant measure $\mu$ on $\Omega^G(\Lambda)$, we have $\text{covol}_\mu(\Lambda) = c \mu(\Lambda)$, so we may assume no normalization in the infima/supremums in the definition of upper and lower covolume, or we may assume the normalization from Weil’s formula \[5\].

**Remark 4.4.** The covolume of a point set $\Lambda$ with respect to a finite invariant measure $\mu$ does depend on the choice of Haar measure $m$ on $G$. Indeed, if one rescales the Haar measure by some $c > 0$, then by \[7\] the transverse measure rescales by $c^{-1}$, so that the covolume rescales by $c$.

**Proposition 4.5.** For any Borel set $S \subseteq G$ with $0 < m(S) < \infty$ and non-zero finite invariant measure $\mu$ on $\Omega^G(\Lambda)$, we have that

$$\frac{1}{\text{covol}_\mu(\Lambda)} = \frac{1}{\mu(\Omega^G(\Lambda))} \int_{\Omega^G(\Lambda)} \frac{|P \cap S|}{m(S)} \, d\mu(P).$$

In particular, the following statements hold:

(i) If $\Lambda$ is $K$-dense for some compact set $K \subseteq G$, then $\text{covol}_\mu(\Lambda) \leq m(K)$.

(ii) If $\Lambda$ is $\ell$-relatively $U$-separated for some $\ell \in \mathbb{N}$ and open set $U \subseteq G$, then $\text{covol}_\mu(\Lambda) \geq m(U)/\ell$.

**Proof.** The transverse measure formula \[7\] gives

$$\mu_0(\Omega_0(\Lambda)) m(S) = \int_{\Omega_0(\Lambda)} \int_{G} 1_S(x) \, dx \, d\mu_0(Q) = \int_{\Omega^G(\Lambda)} \sum_{x \in P} 1_S(x) \, d\mu(P) = \int_{\Omega^G(\Lambda)} |P \cap S| \, d\mu(P).$$

Hence

$$\frac{1}{\text{covol}_\mu(\Lambda)} = \frac{\mu_0(\Omega_0(\Lambda))}{\mu(\Omega^G(\Lambda))} = \frac{1}{\mu(\Omega^G(\Lambda))} \int_{\Omega^G(\Lambda)} \frac{|P \cap S|}{m(S)} \, d\mu(P).$$  \hfill (9)

In particular, if $\Lambda$ is $K$-dense, so that $|P \cap K| \geq 1$ for all $P \in \Omega(\Lambda)$ by \[Proposition 3.2\] then $\text{covol}_\mu(\Lambda) \leq m(K)$. On the other hand, if $\Lambda$ is $\ell$-relatively $U$-separated, so that $|P \cap U| \leq \ell$ for all $P \in \Omega(\Lambda)$ by \[Proposition 3.2\] then $\text{covol}_\mu(\Lambda) \geq m(U)/\ell$.  \[\square\]
4.2. Model sets. In this subsection we consider regular model sets in $G$ and show that their covolume is given by the formula (1).

A cut-and-project scheme is a triple $(G, H, \Gamma)$ where $G$ and $H$ are locally compact groups and $\Gamma$ is a lattice in $G \times H$ such that the projection map $p_G : G \times H \to G$ is injective when restricted to $\Gamma$ and the image of $\Gamma$ under the projection map $p_H : G \times H \to H$ is dense in $H$. If $W$ is a compact subset of $H$, a set of the form

$$\Lambda = p_G(\Gamma \cap (G \times W))$$

is called a weak model set in $G$. The set $W$ is called the window. If $W$ is Jordan-measurable with dense interior, aperiodic and $\Gamma$-regular (cf. [10, Section 2.2] for definitions) then $\Lambda$ is called a regular model set. We will fix a regular model set coming from a cut-and-project scheme $(G, H, \Gamma)$ and window $W \subseteq H$ for the remainder of the section. We also assume that $G$ and $H$ are unimodular and second-countable. Denote by $m$ the Haar measure on $G$ and by $m_H$ the Haar measure on $H$. We set $Y = (G \times H)/\Gamma$, the homogeneous space associated to $\Gamma$. It is a left $G$-space under multiplication in the first coordinate.

By [10], there exists a unique $G$-invariant probability measure $\mu$ on $\Omega(\Lambda)$. The following theorem, cf. [10, Theorem 3.1] and [10, Lemma 4.12], relates the dynamics of $\Omega(\Lambda)$ to the dynamics of $Y$.

**Proposition 4.6.** There exists a unique $G$-equivariant Borel map $\beta : \Omega(\Lambda) \to Y$ with closed graph such that $\beta(\Lambda) = \Gamma$. Moreover, the following statements hold:

(i) If $Y^{ns} = \{(x, y)\Gamma : y^{-1}W \text{ is } \Gamma\text{-regular}\}$ and $\Omega^{\times}(\Lambda)^{ns} = \beta^{-1}(Y^{ns})$, then

$$\beta|_{\Omega^{\times}(\Lambda)^{ns}} : \Omega^{\times}(\Lambda)^{ns} \to Y^{ns}$$

is a bijection. Furthermore, $\Omega(\Lambda)^{ns}$ is $G$-invariant and $\mu$-conull in $\Omega(\Lambda)$.

(ii) For $P \in \Omega^{\times}(\Lambda)^{ns}$ it holds that

$$\sum_{x \in P} f(x) = \sum_{(x, y) \in \beta(P)} f(x) 1_W(y), \ f \in C_c(G).$$

The following proposition is essentially [10, Proposition 4.13], but we include a proof here which uses the present notation for completeness.

**Proposition 4.7.** The covolume of $\Lambda$ with respect to $\mu$ is given by

$$\text{covol}_\mu(\Lambda) = \frac{\text{covol}(\Gamma)}{m(W)}.$$ 

In particular

$$\text{covol}_-(\Lambda) = \text{covol}_+(\Lambda) = \frac{\text{covol}(\Gamma)}{m_H(W)}.$$ 

**Proof.** Let $\mu$ (resp. $\nu$) be the unique $G$-invariant (resp. $G \times H$-invariant) probability measure on $\Omega^{\times}(\Lambda)$ (resp. $Y$). From Proposition 4.6 it then follows that

$$\int_{\Omega^{\times}(\Lambda)} g(\beta(P)) \, d\mu(P) = \int_Y g(Q) \, d\nu(Q), \ g \in C_c(Y).$$

Because of the normalizations of $\mu$ and $\nu$, we also have that $\text{covol}_\mu(\Lambda) = 1/\mu_0(\Omega_0(\Lambda))$ and $\text{covol}_\nu(\Gamma) = 1/\nu_0(\Omega_0(\Gamma))$. 

Take any \( f \in C_c(G) \). Upon integrating the equality in Proposition 4.6 (ii) with respect to \( \mu \) and using the above formula as well as the transverse measure formula (7) one obtains

\[
\text{covol}_\mu(\Lambda) = \int_G f(x) \, dx = \int_{\Omega^\times(\Lambda)} \sum_{x \in P} f(x) \, d\mu(P)
\]

\[
= \int_{\Omega^\times(\Lambda)} \sum_{(x,y) \in \beta(P)} f(x) \chi_W(y) \, d\mu(P)
\]

\[
= \int_{\Omega^\times(\Lambda)} \sum_{(x,y) \in Q} f(x) \chi_W(y) \, d\nu(Q)
\]

\[
= \int_{\Omega^\times(\Lambda)} \sum_{(x,y) \in Q} f(x) \chi_W(y) \, d\mu(P) = m_H(W) \int_G f(x) \, dx.
\]

Hence

\[
\text{covol}_\mu(\Lambda) = \frac{\text{covol}(\Gamma)}{m_H(W)}.
\]

In particular, since \( \mu \) is the unique invariant probability measure on \( \Omega^\times(\Gamma) \), it follows that

\[
\text{covol}_+(\Lambda) = \text{covol}_\mu(\Lambda).
\]

4.3. Relation to Beurling density. In this subsection we assume that \( G \) is amenable. For such groups, the lower and upper Beurling density of a set \( \Lambda \subseteq G \) are defined respectively as

\[
D^-(\Lambda) = \sup_K \inf_{x \in G} \frac{|\Lambda \cap xK|}{m(K)}, \quad D^+(\Lambda) = \inf_K \sup_{x \in G} \frac{|\Lambda \cap xK|}{m(K)},
\]

where the infimum and supremum are taken over all compact sets \( K \subseteq G \) of positive measure.

A sequence \((F_n)_{n \in \mathbb{N}}\) of compact subsets of \( G \) is called a (right) strong Følner net if for every compact set \( K \subseteq G \), one has

\[
\lim_{n \to \infty} \frac{m(F_n K \cap F_n^c K)}{m(F_n)} = 0.
\]

The lower and upper Beurling densities may alternatively be described in terms of a strong Følner net as follows, cf. [37]:

\[
D^-(\Lambda) = \lim_n \inf_{x \in G} \frac{|\Lambda \cap x F_n|}{m(F_n)}, \quad D^+(\Lambda) = \lim_n \sup_{x \in G} \frac{|\Lambda \cap x F_n|}{m(F_n)}.
\]

Note that for compactly generated groups of polynomial growth, balls defined with respect to a corresponding word metric form a Følner net, cf. [15], so that Beurling densities can be defined with respect to balls as was done in e.g. [21].

It will be of interest of us to consider the following quantities, which we term the lower and upper hull Beurling density of \( \Lambda \) respectively:

\[
D^--(\Lambda) := \sup_K \inf_{P \in \Omega^\times(\Lambda)} \frac{|P \cap K|}{m(K)}, \quad D^{++}(\Lambda) := \inf_K \sup_{P \in \Omega^\times(\Lambda)} \frac{|P \cap K|}{m(K)},
\]

where the supremum and infimum are as before taken over all compact sets \( K \subseteq G \) of positive measure. Note that it does not matter if we take suprema/infima over \( \Omega(\Lambda) \) or \( \Omega^\times(\Lambda) \) in the above definition. The following lemma describes the precise relationship between Beurling densities and hull Beurling densities.

Lemma 4.8. Let \( \Lambda \subseteq G \) be \( \ell \)-relatively separated. Then

\[
D^--(\Lambda) \leq D^-(\Lambda) \leq \ell \cdot D^--(\Lambda), \quad D^+(\Lambda) = D^{++}(\Lambda).
\]

In particular, if \( \Lambda \) is separated then \( D^--(\Lambda) = D^-(\Lambda) \).
However, \( \lim_{U \to \infty} \Omega(\Lambda) \). This shows that \( \inf_{x \in G} |x \Lambda \cap K| \leq \ell \cdot \inf_{P \in \Omega^X(\Lambda)} |P \cap K| \), hence \( D^{-}(\Lambda) \leq \ell \cdot D^{--}(\Lambda) \).

For the upper densities, let \( \epsilon > 0 \) and pick a symmetric, precompact neighborhood \( V \) of the identity such that the sets \( \{x_j V\}_{j=1}^{k} \) are pairwise disjoint. Take \( K \) and \( P \) as before. Since \( P \in \Omega^X(\Lambda) \) there exists \( x \in G \) such that \( P \cap K \subseteq x \Lambda V \). Hence for each \( 1 \leq j \leq k \) there exists \( \lambda_j \in \Lambda \) such that \( x \lambda_j \in x_j V \subseteq (P \cap K)V \). In particular, \( x \lambda_i \neq x \lambda_j \) for \( i \neq j \). This shows that \( |P \cap K| \leq |x \Lambda \cap K V| \). Now Corollary 3.4 gives
\[
|x \Lambda \cap K V| - |x \Lambda \cap K| = |x \Lambda \cap (K V \setminus K)| \leq \ell \cdot \frac{m((K V \setminus K)U)}{m(U)} \leq \ell \cdot \frac{m(K L \cap K^{c} L)}{m(U)}
\]
where \( L = V \cup \overline{U} \). Combining the two inequalities we have
\[
\inf_{P \in \Omega^X(\Lambda)} \frac{|P \cap K|}{m(K)} \leq \inf_{x \in G} \frac{|x \Lambda \cap K V|}{m(K)} + \frac{\ell}{m(U)} \cdot \frac{m(K L \cap K^{c} L)}{m(K)}.
\]
Since \( G \) is amenable, taking infimums over all compact \( K \subseteq G \) of positive measure, Proposition 5.4 (iii) gives
\[
D^{++}(\Lambda) \leq D^{+}(\Lambda) + \frac{\ell}{m(U)} \cdot \inf_{K} \frac{m(K L \cap K^{c} L)}{m(K)} = D^{+}(\Lambda).
\]  

The following example shows that it need not be the case that \( D^{-}(\Lambda) = D^{--}(\Lambda) \) for point sets that are not separated.

**Example 4.9.** Let \( G = \mathbb{R} \) and let \( \Lambda = \mathbb{Z} \cup \{k + 1/k : k \in \mathbb{Z} \setminus \{0\}\} \). Then \( D^{-}(\Lambda) = 2 \). However, \( \mathbb{Z} \in \Omega^X(\Lambda) \), so
\[
D^{--}(\Lambda) = \sup_{K} \inf_{P \in \Omega^X(\Lambda)} \frac{|P \cap K|}{m(K)} \leq \sup_{K} \frac{|Z \cap K|}{m(K)} = 1.
\]

The following theorem establishes the exact relationship between hull Beurling densities and covolumes for relatively separated and relatively dense sets.

**Theorem 4.10.** Let \( \Lambda \subseteq G \) be a relatively dense and relatively separated set. Then
\[
D^{-}(\Lambda) = \frac{1}{\text{covol}_+ (\Lambda)}.
\]
If \( \Lambda \) is separated, then also
\[
D^{++}(\Lambda) = \frac{1}{\text{covol}_- (\Lambda)}.
\]

**Proof.** Since \( \Lambda \) is assumed relatively dense, \( \Omega^X(\Lambda) = \Omega(\Lambda) \) is a compact space. Throughout the proof, we assume without loss of generality that all non-zero finite invariant measures on \( \Omega(\Lambda) \) are probability measures. We let \( \Lambda \) be \( \ell \)-relatively \( U \)-separated for a precompact, symmetric, open neighborhood \( U \) of the identity.
Step 1: Bounding the reciprocal of covolume. First, note that for any invariant probability measure $\mu$ on $\Omega(\Lambda)$ and any compact set $K \subseteq G$ of positive measure we have that
\[
\inf_{P \in \Omega(\Lambda)} \frac{|P \cap K|}{m(K)} \leq \int_{\Omega(\Lambda)} \frac{|P \cap K|}{m(K)} d\mu(P) \leq \sup_{P \in \Omega(\Lambda)} \frac{|P \cap K|}{m(K)}.
\]
By Proposition 4.5 the middle term above equals $1/\text{covol}_\mu(\Lambda)$. Taking infima and suprema over all compact sets $K \subseteq G$ of positive measure in the first double inequality and using Lemma 4.8 we arrive at
\[
D^-(\Lambda) = D^--(\Lambda) \leq \frac{1}{\text{covol}_\mu(\Lambda)} \leq D^{++}(\Lambda) = D^+(\Lambda).
\]
The rest of the proof is spent constructing invariant probability measures $\mu$ such that $1/\text{covol}_\mu(\Lambda)$ attains both $D^{--}(\Lambda)$ and $D^{++}(\Lambda)$.

Step 2: Følner sequence estimate. Choose a strong Følner sequence $(F_n)_{n \in \mathbb{N}}$ for $G$ and define for each $n \in \mathbb{N}$ and $P \in \Omega(\Lambda)$ a probability measure $\mu_{n,P}$ on $\Omega(\Lambda)$ via
\[
\int_{\Omega(\Lambda)} f d\mu_{n,P} = \frac{1}{m(F_n)} \int_{F_n} f(x^{-1}P) dx
\]
for nonnegative Borel functions $f$ on $\Omega(\Lambda)$. In other words, $\mu_{n,P}$ is the convolution of the measure $1/m(F_n) \cdot 1_{F_n^{-1}} \cdot m$ on $G$ by the Dirac measure $\delta_P$ on $\Omega(\Lambda)$.

Fix $P \in \Omega(\Lambda)$. For $x \in F_n$ we claim that
\[
|P \cap xU| = \sum_{y \in P \cap F_n U} 1_{F_n \cap yU}(x).
\]
Indeed, if $y \in P \cap xU$ then $y \in P \cap F_nU$ and $x^{-1}y \in U$ which implies $y^{-1}x \in U^{-1} = U$, hence $x \in yU$. Thus $1_{F_n \cap yU}(x) = 1$. On the other hand, if $1_{F_n \cap yU}(x) = 1$ for some $y \in P \cap F_nU$ then $x \in yU$ so $y \in xU$, hence $y \in P \cap xU$.

Using (10), we obtain
\[
\int_{\Omega(\Lambda)} |P' \cap U| d\mu_{n,P}(P') = \frac{1}{m(F_n)} \int_{F_n} |P \cap xU| dx
\]
\[
= \frac{1}{m(F_n)} \int_{F_n} \sum_{y \in P \cap F_n U} \int_G 1_{F_n \cap yU}(x) dx
\]
\[
\leq \frac{1}{m(F_n)} \int_{F_n} \sum_{y \in P \cap F_n U} \int_G 1_{yU}(x) dx
\]
\[
= m(U) \frac{|P \cap F_n U|}{m(F_n)}.
\]
On the other hand, using the fact that $1_{F_n \cap yU}(x) = 1_{yU}(x)$ for $x \in F_n \setminus F_n^cU$ we get
\[
\int_{\Omega(\Lambda)} |P' \cap U| d\mu_{n,P}(P') = \frac{1}{m(F_n)} \sum_{y \in P \cap F_n U} \int_G 1_{F_n \cap yU}(x) dx
\]
\[
\geq \frac{1}{m(F_n)} \sum_{y \in P \cap (F_n \setminus F_n^cU)} \int_G 1_{yU}(x) dx
\]
\[
= m(U) \frac{|P \cap (F_n \setminus F_n^cU)|}{m(F_n)}.
\]
Combining these two inequalities, we arrive at
\[
\frac{|P \cap (F_n \setminus F_n^cU)|}{m(F_n)} \leq \frac{1}{m(U)} \int_{\Omega(\Lambda)} |P' \cap U| d\mu_{n,P}(P') \leq \frac{|P \cap F_n K|}{m(F_n)},
\]
where $K = \mathcal{U}$. We now compare the left and right hand side above. Using Corollary 3.4 we have that
\[
|P \cap (F_n K \cap F_n^c K)| \leq C \cdot m((F_n K \cap F_n^c K) U) \leq C \cdot m(F_n K^2 \cap F_n^c K^2)
\]
where $C = \ell/m(U)$. Hence
\[
0 \leq \frac{|P \cap F_n K|}{m(F_n)} - \frac{|P \cap (F_n \setminus F_n^c U)|}{m(F_n)} = \frac{|P \cap (F_n K \setminus (F_n \setminus F_n^c U))|}{m(F_n)} \leq \frac{C \cdot m(F_n K^2 \cap F_n^c K^2)}{m(F_n)}.
\]

By definition of a strong Følner sequence the final expression above goes to zero as $n$ tends to infinity. Since it is independent of $P$ and the term $|P \cap F_n|/m(F_n)$ is also bounded between the same two expressions, we have that
\[
\lim_{n \to \infty} \int_{\Omega(\Lambda)} \left| \frac{|P' \cap U|}{m(U)} d\mu_{n,P}(P') - \frac{|P \cap F_n|}{m(F_n)} \right| = 0 \text{ uniformly in } P. \tag{11}
\]

**Step 3:** The lower density. For each $n \in \mathbb{N}$ let $P_n \in \Omega(\Lambda)$ be such that $|P_n \cap F_n|/m(F_n) = \min_{P \in \Omega(\Lambda)} |P \cap F_n|/m(F_n) = \inf_{P \in \Omega(\Lambda)} |P \cap F_n|/m(F_n)$. It then follows that
\[
D^-(\Lambda) = \lim_{n \to \infty} \frac{|P_n \cap F_n|}{m(F_n)}.
\]

By the uniform convergence in (11) we have that
\[
\lim_{n \to \infty} \int_{\Omega(\Lambda)} \frac{|P' \cap U|}{m(U)} d\mu_{n,P}(P') = D^-(\Lambda). \tag{12}
\]

Passing to a subsequence, assume by the Banach–Alaoglu theorem that $(\mu_{n,P_n})_n$ converges in the weak* topology to a measure $\mu$ in the unit ball of the dual space of $C(\Omega(\Lambda))$, which is necessarily a probability measure due to the compactness of $\Omega(\Lambda)$. We claim that $\mu$ is $G$-invariant. Indeed, letting $f \in C(\Omega(\Lambda))$ and letting $y \in G$ we obtain
\[
\left| \int_{\Omega(\Lambda)} f(P) d\mu_{n,P_n}(P) - \int_{\Omega(\Lambda)} f(y^{-1}P) d\mu_{n,P_n}(P) \right| = \frac{1}{m(F_n)} \int_{F_n} \left| f(x^{-1}P_n) - f(y^{-1}x^{-1}P_n) \right| dx \leq \frac{1}{m(F_n)} \int_{F_n \Delta F_n} f(x^{-1}P_n) dx \leq \sup_{P \in \Omega(\Lambda)} |f(P)| \cdot \frac{m(F_n \Delta F_n y)}{m(F_n)}.
\]

Since $(F_n)_n$ is a Følner sequence, the last term above tends to 0 as $k \to \infty$, which shows that $\mu$ is $G$-invariant.

Since the function $\Omega(\Lambda) \to [0, \infty)$, $P' \mapsto |P' \cap U|$ is lower semi-continuous and bounded below, the weak* convergence $\mu_{n,P_n} \to \mu$ gives that
\[
\frac{1}{\text{covol}_\mu(\Lambda)} = \int_{\Omega(\Lambda)} \left| \frac{|P' \cap U|}{m(U)} d\mu(P') \right| \leq \liminf_{n \to \infty} \int_{\Omega(\Lambda)} \left| \frac{|P' \cap U|}{m(U)} d\mu_{n,P_n}(P') \right| = D^-(\Lambda).
\]
This shows that the lower bound $D^-\Lambda$ in Step 1 is attained by the reciprocal of the covolume of $\mu$. We conclude that $D^-\Lambda = 1 / \text{covol}_+(\Lambda)$.

**Step 4: The upper density.** Assume that $\Lambda$ is separated. As in Step 3 we can write

$$D^+\Lambda = \lim_{n \to \infty} \frac{|Q_n \cap F_n|}{m(F_n)}$$

for $Q_n \in \Omega(\Lambda)$, and assume that $(\mu_n, Q_n)$ converges in the weak* topology to an invariant probability measure $\nu$ on $\Omega(\Lambda)$. Since $\Lambda$ is separated, $P \mapsto |P \cap U|$ is continuous, so in this case

$$\frac{1}{\text{covol}_\nu(\Lambda)} = \int_{\Omega(\Lambda)} \frac{|P' \cap U|}{m(U)} \, d\nu(P') = \lim_{n \to \infty} \int_{\Omega(\Lambda)} \frac{|P' \cap U|}{m(U)} \, d\mu_{n, Q_n}(P') = D^+\Lambda.$$

Combined with Step 1 this shows that $D^+\Lambda = 1 / \text{covol}_-(\Lambda)$.

5. **Coherent systems**

We continue to assume that $G$ is a unimodular lcsc group.

5.1. **Discrete series representations.** A measurable 2-cocycle on $G$ is a Borel measurable function $\sigma : G \times G \to \mathbb{T}$ satisfying the following properties:

(i) $\sigma(x, y)\sigma(xy, z) = \sigma(x, yz)\sigma(y, z)$ for all $x, y, z \in G$; and

(ii) $\sigma(e, e) = 1$.

A $\sigma$-projective unitary representation $\pi$ of $G$ on a Hilbert space $\mathcal{H}_\pi$ is a map $\pi : G \to \mathcal{U}(\mathcal{H}_\pi)$, where $\mathcal{U}(\mathcal{H}_\pi)$ denotes the unitary operators on $\mathcal{H}_\pi$, such that

$$\pi(x)\pi(y) = \sigma(x, y)\pi(xy) \quad \text{for all } x, y \in G.$$

We also require that $\pi$ is strongly continuous, that is, $x \mapsto \pi(x)\xi$ is continuous for every $\xi \in \mathcal{H}_\pi$.

A $\sigma$-projective, irreducible, unitary representation $\pi$ of $G$ is called discrete series if there exists a non-zero $\xi \in \mathcal{H}_\pi$ such that

$$\int_G |\langle \xi, \pi(x)\xi \rangle|^2 \, dx < \infty.$$

If $\pi$ is discrete series, then in fact the above integral is finite for all $\xi \in \mathcal{H}_\pi$, and moreover one has the orthogonality relations

$$\int_G \langle \xi, \pi(x)\eta \rangle \langle \xi', \pi(x)\eta' \rangle \, dx = d_\pi^{-1} \langle \xi, \xi' \rangle \langle \eta, \eta' \rangle \quad \text{for all } \xi, \xi', \eta, \eta' \in \mathcal{H}_\pi,$$

where the number $d_\pi$ is called the formal dimension of $\pi$. It depends on the chosen Haar measure on $G$; if the Haar measure is scaled by a factor of $c > 0$, then the formal dimension is scaled by $c^{-1}$.

Given $\xi, \eta \in \mathcal{H}_\pi$, we will denote by $C_\eta \xi : G \to \mathbb{C}$ the associated matrix coefficient given by

$$C_\eta \xi(x) = \langle \xi, \pi(x)\eta \rangle, \quad x \in G.$$
5.2. **Frames and Riesz sequences over limits of translates.** This subsection concerns the stability of the frame and Riesz sequence property for coherent systems over limits of point sets in the Chabauty–Fell topology. Throughout, we fix a projective discrete series representation $(\pi, H_{n})$ of $G$.

The following proof is an adaptation of [17, Proposition 3.1].

**Proposition 5.1.** Let $\Lambda \subseteq G$ be a set and suppose that $\pi(\Lambda)\eta$ is a Bessel sequence for some non-zero $\eta \in H_{n}$. Then $\Lambda$ is relatively separated.

**Proof.** Pick $\xi \in H_{n}$ with $\|\xi\| = 1$. Since $\eta$ is non-zero, the function $C_{\eta}\xi$ is continuous and non-zero on $G$, so there exists a compact set $K \subseteq G$ with nonempty interior and a constant $c > 0$ such that $|C_{\eta}\xi(x)| \geq c$ for all $x \in K$.

Suppose that $\Lambda$ is not relatively separated and let $\ell \in \mathbb{N}$ be arbitrary. Then there exists $y \in G$ such that $|\Lambda \cap yK| \geq \ell$. Hence

$$\sum_{\lambda \in \Lambda} |\langle\pi(y)\xi, \pi(\lambda)\eta\rangle|^{2} \geq \sum_{\lambda \in \Lambda \cap yK} |\langle\pi(y)\xi, \pi(\lambda)\eta\rangle|^{2} = \sum_{\lambda \in \Lambda \cap yK} |C_{\eta}\xi(y^{-1}\lambda)|^{2} \geq \ell \|\xi\|^{2}. $$

Since $\ell$ was arbitrary, this shows that $\pi(\Lambda)\eta$ is not a Bessel sequence.

**Proposition 5.2.** Let $\Lambda \subseteq G$ be a set and suppose that $\pi(\Lambda)\eta$ is a Riesz sequence with Riesz bounds $0 < A \leq B < \infty$ where $\eta \in H_{n}$. Then $\Lambda$ is separated and $\pi(P)\eta$ is a Riesz sequence with Riesz bounds $0 < A \leq B < \infty$ for all $P \in \Omega(\Lambda)$.

**Proof.** Suppose that $\pi(\Lambda)\eta$ is a Riesz sequence with bounds $0 < A \leq B < \infty$.

First, assume for a contradiction that $\Lambda$ is not separated. Then using the strong continuity of $\pi$, we can find $\lambda, \lambda' \in \Lambda$ such that $\|\pi(\lambda)\eta - \pi(\lambda')\eta\|^{2} < A$. Applying the definition of Riesz sequence to the sequence $c \in \ell^{2}(\Lambda)$ given by $c_{\lambda} = 1$, $c_{\lambda'} = -1$ and zero otherwise, we obtain

$$2A = A\|c\|^{2} \leq \|\pi(\lambda)\eta - \pi(\lambda')\eta\|^{2} < A. $$

This is a contradiction, so we are forced to conclude that $\Lambda$ is $U$-separated for some open neighborhood $U$ of the identity.

Next, let $P \in \Omega(\Lambda)$, say $P = \lim_{n} P_{n}$ in the Chabauty–Fell topology, where each $P_{n}$ is a left translate of $\Lambda$. Let $c \in \ell^{2}(P)$. Without loss of generality we can assume that $c$ has finite support, say $\text{supp}(c) = \{x_{1}, \ldots, x_{k}\}$. By Lemma 3.4, we can find $x_{j}^{(n)} \in P_{n}$, $1 \leq j \leq k$, such that $x_{j}^{(n)} \to x_{j}$ as $n \to \infty$. Define now elements $c_{n} \in \ell^{2}(P_{n})$ by

$$c_{n}(x) = \begin{cases} c(x_{j}), & \text{if } n \geq N \text{ and } x = x_{j}^{(n)} \text{ for some } 1 \leq j \leq k, \\ 0, & \text{otherwise}. \end{cases} $$

Then for $n \geq N$ we get

$$\left\| \sum_{x \in P} c(x)\pi(x)\eta \right\| - \left\| \sum_{x \in P_{n}} c_{n}(x)\pi(x)\eta \right\| \leq \left\| \sum_{x \in P} c(x)\pi(x)\eta - \sum_{x \in P} c_{n}(x)\pi(x)\eta \right\| \\
= \left\| \sum_{j=1}^{k} c(x_{j})\pi(x_{j})\eta - \sum_{j=1}^{k} c(x_{j})\pi(x_{j}^{(n)})\eta \right\| \\
\leq \sum_{j=1}^{k} |c(x_{j})|\|\pi(x_{j})\eta - \pi(x_{j}^{(n)})\eta\|. $$

Given $\epsilon > 0$, we can find $n \geq N$ such that the last expression in the above inequalities is less than $\epsilon$ by using the strong continuity of $\pi$ at each $x_{j}$, $1 \leq j \leq k$. Since $(\pi(x)\eta)_{x \in P_{n}}$ is a Riesz
sequence with bounds $0 < A \leq B < \infty$ and $\|c\|_2 = \|c_n\|_2$, we obtain form the above estimate that

$$A\|c\|_2^2 - \epsilon \leq \left\| \sum_{x \in P} c_x \pi(x)\eta \right\|_2^2 \leq B\|c\|_2^2 + \epsilon.$$ 

Since $\epsilon > 0$ was arbitrary, this finishes the proof. \hfill \Box

Fix a compact neighborhood $Q$ of the identity. Define the local maximum function of $f \in L^2(G)$ to be

$$f^2(x) = \sup_{t \in Q} |f(t)|, \quad x \in G.$$ 

We define $W^2(G)$ to be following subspace of $L^2(G)$:

$$W^2(G) = \{ f \in L^2(G) : f^2 \in L^2(G) \}.$$ 

Note that $W^2(G)$ is an example of a Wiener amalgam space which usually goes by the name of $W(L^2, L^\infty)(G)$ in the literature, cf. [20]. The choice of a different $Q$ yields an equivalent norm. The main property of $W^2(G)$ we need is the following lemma, which can be deduced from [24, Lemma 1] or proved using similar techniques as in [23, Theorem 11.1.4].

**Lemma 5.3.** Let $f \in W^2(G)$ be continuous. Then for every $\epsilon > 0$ and every $\delta > 0$, there exists a compact set $K \subseteq G$ such that for every relatively separated set $\Lambda \subseteq G$ with $\text{rel}(\Lambda) \leq \delta$ we have

$$\left( \sum_{\lambda \in \Lambda \setminus K} |f(\lambda)|^2 \right)^{1/2} < \epsilon.$$ 

Set now

$$B_\pi = \{ \eta \in \mathcal{H}_\pi : C_\eta \xi \in W^2(G) \text{ for all } \xi \in \mathcal{H}_\pi \}.$$ 

The elements of this space were termed admissible analyzing vectors in [24]. We show in the appendix that they are dense in $\mathcal{H}_\pi$.

**Theorem 5.4.** Let $\Lambda \subseteq G$ be a set. If $\pi(\Lambda)\eta$ is a frame for $\mathcal{H}_\pi$ where $\eta \in B_\pi$, then $\Lambda$ must be relatively separated and relatively dense. Furthermore, if $0 < A \leq B < \infty$ are frame bounds for $\pi(\Lambda)\eta$ and $\Lambda$ is $\ell$-relatively separated, then $\pi(\Lambda)\eta$ is a frame with frame bounds $0 < A/\ell \leq B < \infty$ for every $P \in \Omega^\times(\Lambda)$.

**Proof.** Let $\pi(\Lambda)\eta$ be a frame with bounds $0 < A \leq B < \infty$ where $\eta \in B_\pi$. First, suppose for a contradiction that $\Lambda$ is not relatively dense. Let $\xi \in \mathcal{H}_\pi$ with $\|\xi\| = 1$ and $\epsilon > 0$. By Lemma 5.3 there exists a compact set $K \subseteq G$ such that

$$\sum_{x \in P \setminus K} |\langle \xi, \pi(x)\eta \rangle|^2 < \frac{A}{2}$$ 

for all $P \in \Omega^\times(\Lambda)$. By assumption there exists $y \in G$ such that $|y^{-1}\Lambda \cap K| = |\Lambda \cap yK| = 0$. But then

$$A = A\|\pi(y)\xi\|_2^2 \leq \sum_{x \in \Lambda} |\langle \xi, \pi(y)^* \pi(x)\eta \rangle|^2 = \sum_{x \in y^{-1}\Lambda} |\langle \xi, \pi(x)\eta \rangle|^2 = \sum_{x \in y^{-1}\Lambda \setminus K} |\langle \xi, \pi(x)\eta \rangle|^2 < \frac{A}{2}$$ 

Since we have reached a contradiction, $\Lambda$ must be relatively dense.

Let now $P \in \Omega^\times(\Lambda)$, say $P = \lim_n P_n$ where each $P_n$ is a left translate of $\Lambda$. By Proposition 5.1 $\Lambda$ must be $\ell$-relatively $U$-separated for some $\ell \in \mathbb{N}$ and open neighborhood $U$ of the identity. We then have that $\text{rel}_U(P_n), \text{rel}_U(P) \leq \ell$ by Proposition 3.2.
Let $\epsilon > 0$ and $\xi \in \mathcal{H}_\pi$. Since $\eta \in \mathcal{B}_\pi$, we get $C_\rho \xi \in W^2(G)$. From Lemma 5.3, we can then find a compact set $K \subseteq G$ such that

$$\sum_{x \in P' \setminus K} |\langle \xi, \pi(x)\eta \rangle|^2 < \frac{\epsilon}{4}$$

for all relatively separated sets $P' \subseteq G$ with $\text{rel}(P') < \ell$. In particular, we have that

$$\left| \sum_{x \in P' \setminus K} |\langle \xi, \pi(x)\eta \rangle|^2 - \sum_{x \in P_n \setminus K} |\langle \xi, \pi(x)\eta \rangle|^2 \right| < \frac{\epsilon}{2} \quad (14)$$

for all $n \in \mathbb{N}$.

Let $P \cap K = \{x_1, \ldots, x_k\}$. For the purposes of the estimate (14) we may assume that $P \cap \partial K = \emptyset$. Indeed, since $P$ is discrete we can find an open neighborhood $V$ of the identity such that $P \cap x_j V = \{x_j\}$ for each $1 \leq j \leq k$. Setting $K' := K \cup x_1 V \cup \cdots \cup x_k V$, we have that $P \cap K' = P \cap K$, $P \cap \partial K' = \emptyset$ and $K' \subseteq K'$, so that (14) holds with $K$ replaced by $K'$. We therefore assume that $P \cap \partial K = \emptyset$ for the rest of the proof, so that each $x_j$ lies in the interior of $K$.

Appealing to Proposition 3.3, we can find $N \in \mathbb{N}$ such that when $n \geq N$, $P_n \cap K$ is a disjoint union of sets $M_n^1, \ldots, M_n^k$ with $1 \leq |M_n^j| \leq \ell$ and $M_n^j \to \{x_j\}$ as $n \to \infty$. Together with the strong continuity of $\pi$, these properties imply that

$$\lim_{n \to \infty} \sum_{x \in M_n^j} |\langle \xi, \pi(x)\eta \rangle|^2 = |M_n^j| |\langle \xi, \pi(x)\eta \rangle|^2, \quad 1 \leq j \leq k.$$ 

Define a function $m: P \to \mathbb{N}$ by $m(x) = |M_n^j|$ if $x = x^j$ and $m(x) = 1$ otherwise. Using the above, there exists an $N' \geq N$ such that

$$\left| \sum_{x \in P' \setminus K} m(x)|\langle \xi, \pi(x)\eta \rangle|^2 - \sum_{x \in P_n \setminus K} |\langle \xi, \pi(x)\eta \rangle|^2 \right| < \frac{\epsilon}{2} \quad (15)$$

when $n \geq N'$.

Combining (14) and (15), we arrive at

$$\sum_{x \in P} m(x)|\langle \xi, \pi(x)\eta \rangle|^2 - \sum_{x \in P_n} |\langle \xi, \pi(x)\eta \rangle|^2 < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon, \quad n \geq N'.$$

Using that $\pi(P_n)\eta$ is a frame with bounds $0 < A \leq B < \infty$ for every $n \in \mathbb{N}$, we obtain

$$A \|\xi\|^2 - \epsilon \leq \sum_{x \in P} m(x)|\langle \xi, \pi(x)\eta \rangle|^2 \leq B \|\xi\|^2 + \epsilon.$$ 

Since $\epsilon > 0$ was arbitrary, this gives

$$\frac{A}{\ell} \|\xi\|^2 \leq \sum_{x \in P} \frac{m(x)}{\ell}|\langle \xi, \pi(x)\eta \rangle|^2 \leq \sum_{x \in P} \frac{m(x)}{\ell}|\langle \xi, \pi(x)\eta \rangle|^2 \leq \sum_{x \in P} \frac{m(x)}{\ell}|\langle \xi, \pi(x)\eta \rangle|^2 \leq B \|\xi\|^2.$$ 

Hence $\pi(P)\eta$ is a frame with bounds $0 < A/\ell \leq B < \infty$. \hfill \Box

### 5.3. Necessary conditions for frames and Riesz sequences

In this subsection we fix a $\sigma$-projective discrete series representation $\pi$ of $G$.

Let $\Lambda$ be a relatively separated subset of $G$ and let $\mathcal{G}(\Lambda)$ be the corresponding groupoid as constructed in Section 3.4. First we show how the 2-cocycle $\sigma$ on $G$ gives rise to a 2-cocycle $\sigma_\Lambda$ on $\mathcal{G}(\Lambda)$ in the sense of Section 2.3. Namely, for any $(x, P), (y, Q) \in \mathcal{G}(\Lambda)^{(2)}$, set

$$\sigma_\Lambda((y, Q), (x, P)) = \sigma(y, x). \quad (16)$$

The 2-cocycle identities for $\sigma_\Lambda$ reduce easily to the 2-cocycle identities for $\sigma$. 

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We will now construct a \( \sigma_\Lambda \)-projective groupoid representation \( \pi_\Lambda \) of \( G(\Lambda) \). As our measurable field of Hilbert spaces \( (H_Q)_{Q \in \Omega_0(\Lambda)} \) over the unit space \( \Omega_0(\Lambda) \) of \( G(\Lambda) \), we simply take \( H_Q = H_\pi \) for every \( Q \in \Omega_0(\Lambda) \). Hence \( \prod_{Q \in \Omega_0(\Lambda)} H_Q \) can be identified with the set of functions \( \Omega_0(\Lambda) \to H_\pi \), and we declare the measurable sections to be those functions \( \xi : \Omega_0(\Lambda) \to H_\pi \) for which \( Q \mapsto \langle \xi(Q), \eta \rangle \) is measurable for every \( \eta \in H_\pi \).

We define \( \pi_\Lambda \) as follows: Given \( (x, P) \in \Omega_0(\Lambda) \), we set
\[
\pi_\Lambda(x, P)\xi = \pi(x)\xi \in H_P = H_\pi, \quad \xi \in H_{x^{-1}P} = H_\pi.
\]
The verification that this defines a \( \sigma_\Lambda \)-projective unitary representation of \( G(\Lambda) \) on \( (H_Q)_{Q \in \Omega_0(\Lambda)} \) reduces straightforwardly to the fact that \( \pi \) is a \( \sigma \)-projective unitary representation of \( G \) on \( H_\pi \).

The following proposition characterizes frame and Riesz vectors for \( \pi_\Lambda \) in the sense of Section 2.6.

**Proposition 5.5.** Let \( \Lambda \) be a relatively separated subset of \( G \), let \( \mu \) be a finite invariant measure on \( \Omega^\times(\Lambda) \) and let \( \eta = (\eta_Q)_{Q \in \Omega_0(\Lambda)} \) be a measurable section of \( (H_Q)_Q \). Then the following statements are equivalent:

(i) \( \eta \) is a frame vector for \( \pi_\Lambda \).

(ii) The families \( (\pi(x)\eta_{x^{-1}Q})_{x \in Q} \) are frames for \( Q \in \Omega_0(\Lambda) \), with frame bounds uniform in \( Q \).

(iii) The families \( (\pi(x)\eta_{x^{-1}P})_{x \in P} \) are frames for \( Q \in \Omega^\times(\Lambda) \), with frame bounds uniform in \( P \).

Analogous equivalences hold with frame vector replaced by Riesz vector, Parseval frame vector or orthonormal vector.

**Proof.** By definition, \( \eta \) is a frame vector for \( \pi_\Lambda \) if each of the families
\[
\pi_\Lambda(G^Q)\eta = (\pi_\Lambda(\alpha)\eta_{b(\alpha)})_{\alpha \in G^Q} = (\pi_\Lambda(x, x^{-1}Q)\eta_{x^{-1}Q})_{x \in Q} = (\pi(x)\eta_{x^{-1}Q})_{x \in Q}
\]
is a frame for \( H_\pi \) with frame bounds uniform in \( Q \). This shows that (i) and (ii) are equivalent by definition.

To show that (ii) implies (iii), assume that \( \eta \) is a frame vector for \( \pi_\Lambda \). Let \( P \in \Omega^\times(\Lambda) \) and pick some \( y \in P \). Then \( y^{-1}P \in \Omega_0(\Lambda) \) so that
\[
(\pi(x)\eta_{x^{-1}y^{-1}P})_{x \in y^{-1}P} = (\pi(y)^*\pi(x)\eta_{x^{-1}P})_{x \in P}
\]
is a frame. Since \( (\pi(x)\eta_{x^{-1}P})_{x \in P} \) is an image of this frame under the unitary operator \( \pi(y)^* \), it follows that the latter is also a frame with the same frame bounds. Finally, the implication from (iii) to (ii) is trivial since \( \Omega_0(\Lambda) \subseteq \Omega^\times(\Lambda) \). \( \square \)

**Theorem 5.6.** Let \( \Lambda \subseteq G \) be a relatively separated set and let \( \mu \) be a finite invariant measure on \( \Omega^\times(\Lambda) \). If \( \eta = (\eta_Q)_{Q \in \Omega_0(\Lambda)} \) is a frame vector for \( \pi_\Lambda \) with frame bounds \( 0 < A \leq B < \infty \), then
\[
A \leq \frac{\mu_0(\Omega_0(\Lambda))^{-1} \int_{\Omega_0(\Lambda)} \|\eta_Q\|^2 d\mu_0(Q)}{d\pi \text{covol}_\mu(\Lambda)} \leq B.
\]
If \( \eta \) is merely a Bessel vector for \( \pi_\Lambda \) with Bessel bound \( B \), then the upper bound above holds.

**Proof.** Let \( \eta \) be a frame vector for \( \pi_\Lambda \) with frame bounds \( 0 < A \leq B < \infty \). By Proposition 5.5 the following double inequality holds for all \( P \in \Omega^\times(\Lambda) \) and all \( \xi \in H \):
\[
A\|\xi\|^2 \leq \sum_{x \in P} |\langle \xi, \pi(x)\eta_{x^{-1}P} \rangle|^2 \leq B\|\xi\|^2.
\]
Integrating over $\Omega^x(\Lambda)$ with respect to $\mu$, we obtain

$$A\mu(\Omega^x(\Lambda))\|\xi\|^2 \leq \int_{\Omega^x(\Lambda)} \sum_{x \in P} |\langle \xi, \pi(x)\eta_{x-1}p \rangle|^2 \, d\mu(P) \leq B\mu(\Omega^x(\Lambda))\|\xi\|^2.$$  

We consider the middle term of the above double inequality. Using the transverse measure formula (7) and the orthogonality relations (14), we obtain

$$\int_{\Omega^x(\Lambda)} \sum_{x \in P} |\langle \xi, \pi(x)\eta_{x-1}p \rangle|^2 \, d\mu(p) = \int_{\Omega_0(\Lambda)} \int_{G} |\langle \xi, \pi(x)\eta_Q \rangle|^2 \, d\mu_0(Q)$$

$$= \int_{\Omega_0(\Lambda)} d_x^{-1}\|\xi\|^2 \|\eta\|^2 \, d\mu_0(Q)$$

$$= d_x^{-1}\|\xi\|^2 \int_{\Omega_0(\Lambda)} \|\eta\|^2 \, d\mu_0(Q).$$

Cancelling $\|\xi\|^2$ and dividing the double inequality by $\mu_0(\Omega_0(\Lambda))$, the desired conclusion follows. The statement for Bessel vectors follows by the same argument. 

**Corollary 5.7.** Let $\Lambda \subseteq G$ be a relatively separated set. Then the following hold:

(i) If there exists a frame vector for $\pi_\Lambda$ then

$$d_\pi \text{covol}_+ (\Lambda) \leq 1.$$  

(ii) If there exists a Riesz vector for $\pi_\Lambda$ then

$$d_\pi \text{covol}_- (\Lambda) \geq 1.$$  

**Proof.** First, if there exist no finite invariant measures on $\Omega^x(\Lambda)$ then $\text{covol}_- (\Lambda) = \infty$ while $\text{covol}_+ (\Lambda) = -\infty$, hence the statements are vacuously true. Let therefore $\mu$ be a finite invariant measure on $\Omega^x(\Lambda)$.

(i) By Corollary 2.4 we may take a frame vector $\eta$ for $\pi_\Lambda$ to be Parseval. It then follows from Theorem 5.6 that $\mu_0(\Omega_0(\Lambda))^{-1} \int_{\Omega_0(\Lambda)} \|\eta_Q\|^2 \, d\mu_0(Q) = d_\pi \text{covol}_\mu (\Lambda)$. Since each $Q \in \Omega_0(\Lambda)$ contains $e$, the vector $\eta_Q = \pi(e)\eta_{x-1}Q$ is an element of the Parseval frame $(\pi(x)\eta_{x-1}Q)_{x \in Q}$. Since vectors in Parseval frames have norm at most 1, it follows that

$$d_\pi \text{covol}_\mu (\Lambda) = \frac{1}{\mu_0(\Omega_0(\Lambda))} \int_{\Omega_0(\Lambda)} \|\eta_Q\|^2 \, d\mu_0(Q) \leq 1.$$  

As $\mu$ was arbitrary, we conclude that $d_\pi \text{covol}_+ (\Lambda) \leq 1$.

(ii) By Corollary 2.4 we may take a Riesz vector $\eta$ for $\pi_\Lambda$ to be orthonormal, in particular a Bessel vector for $\pi_\Lambda$ with Bessel bound $B = 1$. It then follows from Theorem 5.6 that $\mu_0(\Omega_0(\Lambda))^{-1} \int_{\Omega_0(\Lambda)} \|\eta_Q\|^2 \, d\mu_0(Q) \leq d_\pi \text{covol}_\mu (\Lambda)$. Similarly as in (ii), each vector $\eta_Q$ is an element of an orthonormal sequence, hence $d_\pi \text{covol}_\mu (\Lambda) \geq 1$. As $\mu$ was arbitrary, we conclude that $d_\pi \text{covol}_- (\Lambda) \geq 1$. 

**Proof of Theorem 1.1:** (i): Suppose that $\pi(\Lambda)\eta$ is a frame for $\mathcal{H}_\pi$ with $\eta \in B_\pi$. By Theorem 5.3 $\pi(P)\eta$ is a frame for every $P \in \Omega^x(\Lambda)$ with frame bounds uniform in $P$. By Proposition 5.5 the constant section $(\eta)_{P \in \Omega_0(\Lambda)}$ is a frame vector for $\pi_\Lambda$. By Corollary 5.7 it follows that $d_\pi \text{covol}(\Lambda) \leq 1$.

(ii): Suppose that $\pi(\Lambda)\eta$ is a Riesz sequence for $\mathcal{H}_\pi$ for some $\eta \in \mathcal{H}_\pi$. By Proposition 5.2 $\pi(P)\eta$ is a Riesz sequence for every $P \in \Omega^x(\Lambda)$ with Riesz bounds uniform in $P$. By Proposition 5.5 the constant section $(\eta)_{P \in \Omega_0(\Lambda)}$ is a Riesz vector for $\pi_\Lambda$. By Corollary 5.7 it follows that $d_\pi \text{covol}(\Lambda) \geq 1$. 


Proof of Theorem 1.3. (i) If $\pi(\Lambda)\eta$ is a frame with $\eta \in B_\pi$, then $\Lambda$ is relatively separated and relatively dense by Theorem 5.3 and $d_\pi \text{covol}^+(\Lambda) \leq 1$ by Theorem 1.1. Since $D^-(\Lambda) = 1/\text{covol}^+(\Lambda)$ by Theorem 4.10 we conclude that $D^-(\Lambda) \geq D^-(\Lambda) \geq d_\pi$.

(ii) If $\pi(\Lambda)\eta$ is a Riesz sequence then $\Lambda$ is separated by Proposition 5.2 and $d_\pi \text{covol}^-(\Lambda) \geq 1$ by Theorem 1.1. Since $D^+(\Lambda) = 1/\text{covol}^-(\Lambda)$ by Theorem 4.10 and $D^+(\Lambda) = D^{++}(\Lambda)$ by Lemma 4.8 we conclude that $D^+(\Lambda) \leq d_\pi$. □

Appendix A. Admissible analyzing vectors

Let $\pi$ be a projective discrete series representation of a unimodular lcsc group $G$. Recall that

$$B_\pi = \{\eta \in H_\pi : C_\eta \xi \in W^2(G) \text{ for all } \xi \in H_\pi\}.$$  

The authors thank Jordy Timo van Velthoven for pointing out the necessity of the condition $\eta \in B_\pi$ in their approach for proving Theorem 1.1 and for providing them with the following proof:

Lemma A.1. The space $B_\pi$ is dense in $H_\pi$.

Proof. Let $Q$ be a compact neighborhood of the identity. First, note that if $f \in L^2(G)$ and $g \in C_c(G)$, then $f *_\pi g \in W^2(G)$. Indeed,

$$\begin{align*}
(f *_\pi g)^2(x) &= \sup_{t \in \pi} \left| \int_G \sigma(y, y^{-1}t) f(y) g(y^{-1}t) \, dy \right| \\
&\leq \sup_{t \in \pi} \int_G |f(y)||g(y^{-1}t)| \, dy \\
&\leq \int_G |f(y)| \sup_{t \in y^{-1}xQ} |g(t)| \, dy \\
&= (|f| * |g|^2)(x).
\end{align*}$$

Hence, using Young’s convolution inequality, we get

$$\|f *_\pi g^2\|_2 \leq \|f\|_2 \|g^2\|_1 < \infty.$$  

Picking any $\xi \in H_\pi$ such that $C_\xi : H_\pi \to L^2(G)$ is an isometry, we may assume that $H_\pi$ is a closed subspace of $L^2(G)$ invariant under the $\pi$-twisted left regular representation $\lambda^\pi$. The matrix coefficient $C_\eta \xi$ is then given by the twisted convolution $\xi *_\pi \eta$. Denoting by $P$ the orthogonal projection of $L^2(G)$ onto $H_\pi$, we have that $PC_c(G)$ is dense in $H_\pi$. Since $P$ commutes with $\lambda^\pi(x)$ for every $x \in G$, we get for $\xi \in H_\pi$ and $\eta \in PC_c(G)$, say $\eta = Pf$ for $f \in C_c(G)$, that $\xi *_\pi \eta = \xi *_\pi Pf = P\xi *_\pi f = \xi *_\pi f$. It then follows from the first claim that $\xi *_\pi \eta \in W^2(G)$. This shows that $PC_c(G)$ is a subset of $B_\pi$. Since the former is dense in $H_\pi$, this finishes the proof.

□

The density of $B_\pi$ in $H_\pi$ was also claimed in [23].

Proposition A.2. Suppose that $G$ is an IN group, i.e., there exists a compact neighborhood $Q$ of the identity such that $xQ = Qx$ for all $x \in G$. Then if $\pi(\Lambda)\eta$ is a frame where $\Lambda \subseteq G$ and where $\eta \in H_\pi$ is a general vector, then $\pi(P)\eta$ is a frame for all $P \in \Omega^\pi(\Lambda)$. Hence for projective discrete series of IN groups, the assumption that $\eta \in B_\pi$ can be omitted from Theorem 1.1.
**Proof.** Let $Q$ be a compact, symmetric neighborhood of the identity such that $xQ = Qx$ for all $x \in G$. Assuming that $\pi(\Lambda)\eta$ is a frame for $\eta \in \mathcal{H}_\pi$, we let $\xi \in \mathcal{B}_\pi$ so that $C\eta \xi \in W^2(G)$.

Note then that if we consider the local maximum function with respect to $Q$, we get

$$
(C\eta \xi)(x) = \sup_{t \in xQ} |\langle \xi, \pi(t)\eta \rangle| = \sup_{t \in xQ} |\langle \eta, \pi(t)^{-1}\xi \rangle| = \sup_{t \in x^{-1}Q} |\langle \eta, \pi(t)\xi \rangle|,
$$

where we used the invariance of $Q$ in the last step. Hence, using the unimodularity of $G$, we get

$$
\|C\eta \xi\|_{W^2(G)}^2 = \int_G \sup_{t \in x^{-1}Q} |\langle \eta, \pi(t)\xi \rangle| \, dx = \int_G \sup_{t \in xQ} |\langle \eta, \pi(t)\xi \rangle| \, dx = \|C\eta \xi\|_{W^2(G)}^2 < \infty.
$$

We can then proceed as in Theorem 5.4 showing that one obtains the frame property of $\pi(P)\eta$, $P \in \Omega^\times(\Lambda)$, for all $\xi \in \mathcal{B}_\pi$, and then by density of $\mathcal{B}_\pi$ in $\mathcal{H}_\pi$ conclude that $\pi(P)\eta$ is a frame.

The technique in the above proof for avoiding the assumption $\eta \in \mathcal{B}_\pi$ is also implicitly used in the Gabor frame case in [26, Lemma 5.19].

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