COACTIONS AND SKEW PRODUCTS OF TOPOLOGICAL GRAPHS

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Abstract. We show that the $C^*$-algebra of a skew-product topological graph $E \times \kappa G$ is isomorphic to the crossed product of $C^*(E)$ by a coaction of the locally compact group $G$.

1. Introduction

In [4, Theorem 2.4] we proved that if $E$ is a directed graph and $\kappa$ is a function from the edges of $E$ to a discrete group $G$, then the graph algebra $C^*(E \times \kappa G)$ of the skew-product graph is a crossed product of $C^*(E)$ by a coaction of $G$. This was later generalized to homogeneous spaces $G/H$ in [2, Theorem 3.4], and to higher-rank graphs in [9, Theorem 7.1]. In this paper we generalize the result to topological graphs and locally compact groups. More precisely, we prove in Theorem 3.1 that if $\kappa: E \to G$ is a continuous function (that is, a cocycle), then there exists a coaction $\varepsilon$ of $G$ on $C^*(E)$ such that

$$C^*(E \times \kappa G) \cong C^*(E) \times_{\varepsilon} G.$$ 

We give two distinct approaches to the coaction: in Section 3 we obtain the coaction indirectly, via an application of Landstad duality, and in Section 4 we construct the coaction directly, applying techniques developed in [6]. We thank Iain Raeburn for helpful conversations concerning this direct approach.

In Section 2 we record our conventions for topological graphs, $C^*$-correspondences, skew products, multiplier modules, and functoriality of Cuntz-Pimsner algebras. In an appendix we develop a few tools that we need for dealing with certain bimodule multipliers in terms of function spaces.

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2. Preliminaries

In general, we refer to [11] (see also [8]) for topological graphs, and to [11, 3] (see also [7]) for $C^*$-correspondences, except we make a few minor, self-explanatory modifications. Thus, a topological graph $E$ comprises locally compact Hausdorff spaces $E^1, E^0$ and maps $s, r: E^1 \to E^0$ with $s$ a local homeomorphism and $r$ continuous. Let $A = C_0(E^0)$, and let $X = X(E)$ be the associated $A$-correspondence, which is the completion of $C_c(E^1)$ with operations defined for $f \in A$ and $\xi, \eta \in C_c(E^1)$ by

\[
\begin{align*}
    f \cdot \xi(e) &= f(r(e))\xi(e) \\
    \xi \cdot f(e) &= \xi(e)f(s(e)) \\
    \langle \xi, \eta \rangle(v) &= \sum_{s(e)=v} \overline{\xi(e)}\eta(e).
\end{align*}
\]

Throughout this paper we will also write $A' = C_0(E^1)$, so that $X$ can be regarded as an $A' - A$ correspondence as well as an $A$-correspondence. Recall from [8] that the left $A'$-module multiplication is nondegenerate in the sense that $A' \cdot X = X$, and is determined by the homomorphism $\pi_E: A' \to \mathcal{L}(X)$ given by $(\pi_E(f)\xi)(e) = f(e)\xi(e)$ for $f \in A'$ and $\xi \in C_c(E^1)$, and the (nondegenerate) left $A$-module multiplication $\varphi_A: A \to \mathcal{L}(X)$ is then given by $\varphi_A(f) = \pi_E(\mathbb{1} \circ r)$ for $f \in A$.

We denote by $(k_X, k_A): (X, A) \to C^*(E) = \mathcal{O}_X$ the universal Cuntz-Pimsner covariant representation, and for any Cuntz-Pimsner covariant representation $(\psi, \pi)$ of $(X, A)$ in a $C^*$-algebra $B$ we denote by $\psi^{(1)}: K(X) \to B$ the associated homomorphism\footnote{and here we use the notation of [1]: Raeburn would write $(\psi, \pi)^{(1)}$} determined by $\psi^{(1)}(\theta_{\xi, \eta}) = \psi(\xi)\psi(\eta)^*$, and by $\psi \times \pi: C^*(E) \to B$ the unique homomorphism satisfying

\[
(\psi \times \pi) \circ k_X = \psi \quad \text{and} \quad (\psi \times \pi) \circ k_A = \pi.
\]

Note that in [3], correspondences were called right-Hilbert bimodules, and nondegeneracy was built into the definition. All our correspondences will in fact be nondegenerate, so we can freely apply the results from [3].

For skew products of topological graphs, we use a slight variation of the definition in [1]: the main difference is that we use the same notational conventions as those in [11] for skew products of discrete directed graphs. Thus, a cocycle of a locally compact group $G$ on a
topological graph $E$ is a continuous map $\kappa: E^1 \to G$, and the skew product is the topological graph $E \times _\kappa G$ with

$$(E \times _\kappa G)^i = E^i \times G \quad (i = 0, 1),$$

$$r(e, t) = (r(e), \kappa(e)t), \quad \text{and} \quad s(e, t) = (s(e), t).$$

Our conventions for multipliers of correspondences are taken primarily from [3, Chapter 1], but also see [7]. If $(\pi, \psi, \tau): (A, X, B) \to (M(C), M(Y), M(D))$ is a correspondence homomorphism, then there is a unique homomorphism $\psi^{(1)}: \mathcal{K}(X) \to M(\mathcal{K}(Y)) = \mathcal{L}(Y)$ such that $\psi^{(1)}(\theta _{\xi, \eta}) = \psi(\xi)\psi(\eta)^*$ for $\xi, \eta \in X$. (For this result in the stated level of generality, in particular with no nondegeneracy assumption on $(\psi, \pi)$, see [7, Lemma 2.1].) If $(\psi, \pi)$ happens to be nondegenerate, then so is $\psi^{(1)}$, and hence $\psi^{(1)}$ extends uniquely to a homomorphism $\psi^{(1)}: \mathcal{L}(X) \to \mathcal{L}(Y)$.

A correspondence homomorphism $(\psi, \pi): (X, A) \to (M(Y), M(B))$ is defined in [7] to be Cuntz-Pimsner covariant if

(i) $\psi(X) \subset M_B(Y)$,

(ii) $\pi: A \to M(B)$ is nondegenerate,

(iii) $\pi(J_X) \subset M(B; J_Y)$, and

(iv) the diagram

$$
\begin{array}{ccc}
J_X & \xrightarrow{\pi|} & M(B; J_Y) \\
\varphi _A| \downarrow & & \downarrow \varphi _B| \\
\mathcal{K}(X) & \xrightarrow{\psi^{(1)}} & M_B(\mathcal{K}(Y))
\end{array}
$$

commutes,

where, for an ideal $I$ of a $C^*$-algebra $C$,

$$M(C; I) := \{ m \in M(C) : mC \cup Cm \subset I \}.$$

By [7, Corollary 3.6], for each Cuntz-Pimsner covariant homomorphism $(\psi, \pi)$, there is a unique homomorphism $O_{\psi, \pi}$ making the diagram

$$
\begin{array}{ccc}
(X, A) & \xrightarrow{(\psi, \pi)} & (M_B(Y), M(B)) \\
(\kappa_X, \kappa_A) \downarrow & & \downarrow (\kappa_Y, \kappa_B) \\
O_X & \xrightarrow{O_{\psi, \pi}} & M_B(O_Y)
\end{array}
$$

commute. Moreover, $O_{\psi, \pi}$ is nondegenerate, and is injective if $\pi$ is.

Our conventions for coactions on correspondences mainly follow [3], but see also [6].
3. INDIRECT APPROACH

In this section we apply Landstad duality to give an indirect approach to the following result:

**Theorem 3.1.** If $\kappa: E^1 \to G$ is a cocycle on a topological graph $E$, then there is a coaction $\varepsilon$ of $G$ on $C^*(E)$ such that

$$C^*(E \times_\kappa G) \cong C^*(E) \times_\varepsilon G.$$ 

Throughout the rest of this paper, in addition to $A = C_0(E^0)$ and $X = X(E)$, we will also use the following abbreviations:

- $F = E \times_\kappa G$;
- $Y = X(E \times_\kappa G)$;
- $B = C_0((E \times_\kappa G)^0)$.

**Proof.** To apply Landstad duality [10, Theorem 3.3] (stated in more modern form in [5, Theorem 4.1]), we need the following ingredients:

- an action $\alpha: G \to \text{Aut } C^*(F)$, an $rt - \alpha$ equivariant nondegenerate homomorphism $\mu: C_0(G) \to M(C^*(F))$ (where “rt” is action of $G$ on $C_0(G)$ by right translation), and an injective nondegenerate homomorphism $\Pi: C^*(E) \to M(C^*(F))$ whose image coincides with Rieffel’s generalized fixed-point algebra $C^*(F)^\alpha$. Note that in [5], $C^*(F)^\alpha$ would be written as $\text{Fix}(C^*(F), \alpha, \mu)$.

Since $G$ acts on the right of the skew-product topological graph $F$ via right translation in the second coordinate, by [11, Proposition 5.4 and discussion preceding Remark 5.3] we have an action $\beta = (\beta^1, \beta^0): G \to \text{Aut } Y$ such that

$$\beta^1_t(\xi)(e, r) = \xi(e, rt) \quad \text{for } \xi \in Y,$$

$$\beta^0_t(g)(v, r) = g(v, rt) \quad \text{for } g \in B,$$

which in turn gives an action on $C^*(F)$ such that

$$\alpha_t \circ k_Y = k_Y \circ \beta^1_t$$

$$\alpha_t \circ k_B = k_B \circ \beta^0_t.$$ 

Since $F^0 = E^0 \times G$, we have

$$B = C_0(F^0) = C_0(E^0) \otimes C_0(G) = A \otimes C_0(G),$$

so we can define a nondegenerate homomorphism $\mu: C_0(G) \to M(C^*(F))$ by

$$\mu(g) = k_B(1_{M(A)} \otimes g),$$

and then it is routine to verify that $\mu$ is $rt - \alpha$ equivariant.
Finally, since the action of $G$ on $F$ is free and proper, the proof of \cite[Theorem 5.6]{[1]} constructs an isomorphism

$$\Pi: C^*(E) \overset{\cong}{\to} C^*(F)^\alpha,$$

and then the result follows from Landstad duality. \hfill \qed

4. A direct approach to the coaction

As in Section 3, we suppose we are given a cocycle $\kappa: E^1 \to G$ of a locally compact group $G$ on a topological graph $E$, and we continue to write $A = C_0(E^0), A' = C_0(E^1), X = X(E), F = E \times_{\kappa} G, Y = X(F),$ and $B = C_0(F^0)$.

Recall that the canonical embedding $G \subseteq M(C^*(G))$ is identified with a unitary element $w_G$ of $M(C_0(G) \otimes C^*(G))$. Similarly, we may identify $\kappa$ with a unitary element of

$$M^\beta(C^*(G)) = M(A' \otimes C^*(G)),$$

where $M^\beta(C^*(G))$ denotes the multiplier algebra $M(C^*(G))$ with the strict topology.

Define a nondegenerate homomorphism $\kappa^*: C_0(G) \to M(A')$ by $\kappa^*(f) = f \circ \kappa$, and a nondegenerate homomorphism $\nu: C_0(G) \to \mathcal{L}(X)$ by

$$\nu = \pi_E \circ \kappa^*,$$

where $\pi_E: A' \to \mathcal{L}(X)$ is the homomorphism given on $C_c(E^1)$ by pointwise multiplication.

**Proposition 4.1.** With the above notation, there is a coaction $(\sigma, \text{id}_A \otimes 1)$ of $G$ on $(X, A)$ defined by

$$\sigma(\xi) = v \cdot (\xi \otimes 1),$$

where

$$v = \nu \otimes \text{id}(w_G) \in \mathcal{L}(X \otimes C^*(G)),$$

and moreover there is a coaction $\zeta$ of $G$ on $C^*(E)$ such that

$$\zeta \circ k_X = k_X \otimes \text{id} \circ \sigma \quad \zeta \circ k_A = k_A \otimes 1.$$

**Proof.** This follows from \cite[Corollaries 3.4–3.5]{[6]}, because $\nu: C_0(G) \to \mathcal{L}(X)$ commutes with $\varphi_A$. \hfill \qed

It will be convenient for us to find an equivalent expression for the coaction $\sigma$. Note that we may regard $X$ as an $A' - A$ correspondence,
and hence $X \otimes C^*(G)$ as an $(A' \otimes C^*(G)) - (A \otimes C^*(G))$ correspondence. Thus we can write
\[ \sigma(\xi) = \kappa^* \otimes \text{id}(w_G) \cdot (\xi \otimes 1). \]
However, we can go further: by construction the unitary element $\kappa^* \otimes \text{id}(w_G)$ of $M(A' \otimes C^*(G))$ coincides with the function in $C_b(E_1, M^\beta(C^*(G)))$ whose value at an edge $e$ is $\kappa^* \otimes \text{id}(w_G)(e) = w_G(\kappa(e)) = \kappa(e)$; thus we can write
\[ \sigma(\xi) = \kappa \cdot (\xi \otimes 1). \]

In Theorem 3.1 we used Landstad duality to show that $C^*(F)$ is isomorphic to the crossed product of $C^*(E)$ by a coaction $\varepsilon$ of $G$; on the other hand, in Proposition 4.1 we directly constructed a coaction $\zeta$ of $G$ on $C^*(E)$. To show that also $C^*(E) \times^\zeta G \cong C^*(F)$, we now show that in fact the coactions $\varepsilon$ and $\zeta$ coincide. Since the mechanism behind Landstad duality is that $\varepsilon$ is pulled back along $\Pi^{-1}$ from the inner coaction $\delta^\mu$ on $C^*(F)$, this is accomplished by the following:

**Proposition 4.2.** Let $\zeta$ be the coaction on $C^*(E)$ from Proposition 4.1 and let $\Pi: C^*(E) \to M(C^*(F))$ and $\mu: C_0(G) \to M(C^*(F))$ be as in the proof of Theorem 3.1. Then $\Pi$ is $\zeta - \delta^\mu$ equivariant, and hence $\zeta$ coincides with the coaction $\varepsilon$ from Theorem 3.1.

**Proof.** It is equivalent to show that $(\Pi, \mu): (C^*(E), C_0(G)) \to M(C^*(F))$ is a covariant representation for the coaction $\zeta$, and for this we will apply [6, Corollary 4.3].

We will need to know how the homomorphism $\Pi$ from [1] can be described using the techniques of [7]: [1] Proof of Theorem 5.6 constructs a correspondence homomorphism
\[ (\psi, \pi): (X, A) \to (M_B(Y), M(B)), \]
although the notation in [1] is substantially different. In the terminology of [7, Definition 3.1], [1] Proof of Theorem 5.6 shows that $(\psi, \pi)$ is Cuntz-Pimsner covariant, so that by [7, Corollary 3.6] there is a nondegenerate homomorphism $O_{\psi, \pi}$ making the diagram
\[
\begin{array}{ccc}
(X, A) & \xrightarrow{(\psi, \pi)} & (M_B(Y), M(B)) \\
\downarrow (k_X, k_A) & & \downarrow (k_Y, k_B) \\
C^*(E) & \xrightarrow{\sigma_{\psi, \pi}} & M_B(C^*(F))
\end{array}
\]

\[3\]The roles of $E, X, A$ and $F, Y, B$ are interchanged, and what we call $(\psi, \pi)$ here was written as $(\mu, \nu)$ in [1].
commute; the homomorphism \( \Pi \) from [1] coincides with \( \mathcal{O}_{\psi,\pi} \).

Thus, by [6, Corollary 4.3] it suffices to show that

\[
(\psi, \pi, \mu): (X, A, C_0(G)) \to (M_B(Y), M(B))
\]

is covariant for \((\sigma, \text{id}_A \otimes 1)\), in the sense of [6, Definition 2.9]. Thus we must show that

(i) \((\pi, \mu)\) is covariant for \((A, \text{id}_A \otimes 1)\), and

(ii) \(\psi \otimes \text{id} \circ \sigma(\xi) = \mu \otimes \text{id}(w_G) \cdot (\psi(\xi) \otimes 1) \cdot (\mu \otimes \text{id}(w_G)^*\) for all \(\xi \in X\).

Condition (i) is immediate because \(\pi\) and \(\mu\) commute. Next, we rewrite (ii) in an equivalent form:

\[
(ii)' \psi \otimes \text{id}(\sigma(\xi)) \cdot \mu \otimes \text{id}(w_G) = (\mu \otimes \text{id}(w_G) \cdot (\psi(\xi) \otimes 1))\]

To proceed further, notice that the maps \(\psi\), \(\pi\), and \(\mu\) from [1] take a particularly simple form in our present context:

- \(\psi = \text{id}_X \otimes 1_{M(C_0(G))}\);
- \(\pi = \text{id}_A \otimes 1_{M(C_0(G))}\);
- \(\mu = 1_{M(A)} \otimes \text{id}_{C_0(G)}\).

(We should explain our notation in the above expression for \(\psi\): it follows from the definitions that, as a Hilbert \((A \otimes C_0(G))\)-module, \(Y\) coincides with the external tensor product \(X \otimes C_0(G)\) (where \(C_0(G)\) is regarded as a Hilbert module over itself in the canonical way). One just has to keep in mind that \(Y\) does not coincide with \(X \otimes C_0(G)\) as a \(B\)-correspondence — the left \(B\)-module multiplication is twisted by the cocycle \(\kappa\).) Thus, for \(\xi \in X\) we can write:

- \(\psi(\xi) = \xi \otimes 1\);
- \(\overline{\psi \otimes \text{id}(\sigma(\xi))} = \sigma(\xi)_{13} = \kappa_{13} \cdot (\xi \otimes 1 \otimes 1)\);
- \(\mu \otimes \text{id}(w_G) = 1 \otimes w_G\).

Since both sides of (ii)' are adjointable Hilbert-module maps from \(B \otimes C^*(G)\) to \(Y \otimes C^*(G)\), and \(A \otimes C_c(G)\) is dense in \(B\), it suffices to check that the two sides of (ii)' take equal values on elementary tensors of the form \(f \otimes g \otimes a\), with \(f \in A, g \in C_c(G), a \in C^*(G)\). Evaluating the right-hand side of (ii)' gives

\[
(1 \otimes w_G) \cdot (f \otimes g \otimes a) = (1 \otimes w_G) \cdot (\xi \otimes 1 \otimes 1) \cdot (f \otimes g \otimes a) = (1 \otimes w_G) \cdot (\xi \cdot f \otimes g \otimes a).
\]

Now we must use the function-space techniques from Appendix A. We have \(1 \otimes w_G \in C_b(F^0, M^3(C^*(G)))\), with value \(t\) at \((v, t) \in F^0\)
\[ \xi \cdot f \otimes g \otimes a \in C_c(F^1, C^*(G)), \] so by Corollary A.3 we can evaluate the last quantity in (4.1) at \((e, t) \in F^1\), giving
\[
\begin{align*}
(1 \otimes w_G)(r(e, t))(\xi \cdot f \otimes g \otimes a)(e, t) &= (1 \otimes w_G)(r(e), \kappa(e)t)(\xi \cdot f)(e)g(t)a \\
&= \kappa(e)t \xi(e)f(s(e))g(t)a \\
&= \xi(e)f(s(e))g(t)\kappa(e)ta.
\end{align*}
\]

We proceed similarly with the left-hand side of (ii)′:
\[
(\kappa_{13} \cdot (\xi \otimes 1 \otimes 1) \cdot (1 \otimes w_G)) \cdot (f \otimes g \otimes a)
\]
\[
= \kappa_{13} \cdot (\xi \cdot f \otimes w_G(g \otimes a))
\]
Now, \(\kappa_{13} \in C_b(F^1, M^\beta(C^*(G)))\), with value \(\kappa(e)\) at \((e, t)\), and \(\xi \cdot f \otimes w_G(g \otimes a) \in C_c(F^1, C^*(G))\) because \(\xi \cdot f \in C_c(E^1)\) and \(w_G(g \otimes a) \in C_c(G, C^*(G))\), so by Corollary A.3 we can evaluate the right-hand side of (4.2) at \((e, t) \in F^1\), giving
\[
\begin{align*}
\left(\kappa_{13} \cdot (\xi \cdot f \otimes w_G(g \otimes a))\right)(e, t) &= \kappa_{13}(e, t)(\xi \cdot f \otimes w_G(g \otimes a))(e, t) \\
&= \kappa(e)(\xi \cdot f(e)(w_G(g \otimes a))(t) \\
&= \kappa(e)\xi(e)f(s(e))w_G(t)(g \otimes a)(t) \\
&= \kappa(e)\xi(e)f(s(e))tg(t)a \\
&= \xi(e)f(s(e))g(t)\kappa(e)ta.
\end{align*}
\]

Therefore we have verified (ii)′, and this finishes the proof. \( \square \)

**Appendix A. Functions and multipliers**

In Section 4, we need to compute with bimodule multipliers in terms of functions. If \(T\) is a locally compact Hausdorff space and \(C\) is a \(C^*\)-algebra, we will use without comment the following identifications (see, e.g., [12] or [3 Appendix C]):

- \(C_0(T, C) = C_0(T) \otimes C\);
- \(M(C_0(T) \otimes C) = C_b(T, M^\beta(C))\),

where we write \(M^\beta(C)\) to denote \(M(C)\) with the strict topology. Note that since the action of \(C_b(T, M^\beta(C))\) by multipliers on \(C_0(T, C)\) is via pointwise multiplication, it preserves \(C_c(T, C)\).

We will need to use functions as multipliers on certain \(C^*\)-correspondences; since this theory is not easily available in the literature, we give details for all the results we need. However, we
make no attempt to construct a general theory — rather, we do only enough to establish Corollary \ref{cor:genskew} which we need in Section \ref{sec:skew}

For a topological graph $E$, we write (as in the rest of the paper) $A = C_0(E^0)$, $X = X(E)$, and $A' = C_0(E^1)$. We will regard $X \otimes C$ both as an $(A' \otimes C) - (A \otimes C)$ correspondence and as an $(A \otimes C)$-correspondence.

The following lemma is routine:

\textbf{Lemma A.1.} $C_c(E^1, C)$ embeds densely in the $(A' \otimes C) - (A \otimes C)$ correspondence $X \otimes C$ in the following way: if $\xi, \eta \in C_c(E^1, C) \subset X \otimes C$, $f \in C_c(E^1, C) \subset A' \otimes C$, and $g \in C_c(E^0, C) \subset A \otimes C$, then $f \cdot \xi$ and $\xi \cdot g$ are the elements of $C_c(E^1, C)$ given by

\begin{align}
(A.1) & \quad (f \cdot \xi)(e) = f(e)\xi(e) \\
(A.2) & \quad (\xi \cdot g)(e) = \xi(e) \cdot g(s(e)),
\end{align}

and $(\xi, \eta)$ is the element of $C_c(E^0, C) \subset A \otimes C$ given by

\begin{equation}
(A.3) \quad \langle \xi, \eta \rangle(v) = \sum_{s(e)=v} \xi(e)^*\eta(e).
\end{equation}

Moreover, $g \cdot \xi$ is the element of $C_c(E^1, C)$ given by

\begin{equation}
(A.4) \quad (g \cdot \xi)(e) = g(r(e))\xi(e).
\end{equation}

\textbf{Proof.} First of all, \ref{eq:prehilbert} \ref{eq:prehilbert} \ref{eq:prehilbert} make $C_c(E^1, C)$ into a pre-Hilbert $C_c(E^0, C)$-module (where the latter is regarded as a dense $*$-subalgebra of $C_0(E^0, C) = A \otimes C$). The only non-obvious property of pre-Hilbert modules is that \ref{eq:prehilbert} does give an element of $C_c(E^0, C)$, but this can be proved by an argument similar to those used in \cite[Lemma 1.5]{Buss:2012}.

Observe that the Hilbert-module norm on $C_c(E^1, C)$ is given by

\begin{equation}
(A.5) \quad \|\xi\| = \sup_{v \in E^0} \left\|\sum_{s(e)=v} \xi(e)^*\xi(e)\right\|^{1/2},
\end{equation}

which is larger than the uniform norm. In particular, for $e \in E^1$ the evaluation map $\xi \mapsto \xi(e)$ from $C_c(E^1, C)$ to $C$ is bounded from the Hilbert-module norm to the norm of $C$.

Computing with elementary tensors of the form $\xi \otimes c$ for $\xi \in C_c(E^1)$ and $c \in C$, it is now routine to verify that the completion of the pre-Hilbert module $C_c(E^1, C)$ is isomorphic to the external tensor product $X \otimes C$ of the Hilbert $A$-module $X$ and the Hilbert $C$-module $C$.

Now regarding $X \otimes C$ as an $(A' \otimes C) - (A \otimes C)$ correspondence, \ref{eq:prehilbert} is obviously true on elementary tensors, hence for $f, \xi \in C_c(E^1) \otimes C$, and therefore as stated by density of $C_c(E^1) \otimes C$ in $C_c(E^1, C)$ and by continuity of evaluation. Finally, \ref{eq:prehilbert} follows from \ref{eq:prehilbert}. \hfill \box
Lemma A.2. Let \( K \subset E^1 \) be compact. On the subspace
\[
C_K(E^1, C) := \{ \xi \in C_c(E^1, C) : \text{supp} \xi \subset K \}
\]
of \( X \otimes C \), the Hilbert-module norm and the uniform norm are equivalent. Consequently, \( C_K(E^1, C) \) is norm-closed in \( X \otimes C \).

Proof. By (A.5), the uniform norm on \( C_K(E^1, C) \) is smaller than the Hilbert-module norm from \( X \otimes C \). Thus it suffices to show that the Hilbert-module norm is bounded above by a multiple of the uniform norm. Let \( \xi \in C_K(E^1, C) \). Using compactness of \( K \) and local homeomorphism of \( s \), it is easy to verify that the cardinalities of the intersections \( K \cap s^{-1}(v) \) for \( v \in E^0 \) are bounded above by some nonnegative integer \( d \). Then for any \( v \in E^0 \) we have
\[
\left\| \sum_{s(e)=v} \xi(e)^* \xi(e) \right\| \leq \sum_{s(e)=v} \|\xi(e)\|^2 \leq d\|\xi\|^2_u,
\]
where \( \|\xi\|^2_u \) denotes the uniform norm of \( \xi \), and the result follows. \( \square \)

Since \( X \otimes C \) is a nondegenerate \((A' \otimes C) - (A \otimes C)\) correspondence, the left module action of \( A' \otimes C \) extends canonically to the multiplier algebra \( M(A' \otimes C) = C_b(E^1, M^\beta(C)) \) (and similarly for the left module action of \( A \otimes C \)). The following corollary allows us to compute this extended left module action on generators:

Corollary A.3. If \( m \in C_b(E^1, M^\beta(C)) \) and \( \xi \in C_c(E^1, C) \subset X \otimes C \), then the element \( m \cdot \xi \) of \( X \otimes C \) lies in \( C_c(E^1, C) \), and
\[
(m \cdot \xi)(e) = m(e)\xi(e) \quad \text{for} \ e \in E^1.
\]
If \( n \in C_b(E^0, M^\beta(C)) \) then both \( n \cdot \xi \) and \( \xi \cdot n \) lie in \( C_c(E^1, C) \), and
\[
(n \cdot \xi)(e) = n(r(e))\xi(e);
\]
\[
(\xi \cdot n)(e) = \xi(e)n(s(e)).
\]

Proof. Choose a net \( \{m_i\} \) in \( C_c(E^1, C) \) converging strictly to \( m \) in \( M(C_0(E^1, C)) \). The conclusion holds for each \( m_i \cdot \xi \), by Lemma A.1. Let \( K = \text{supp} \xi \), a compact subset of \( E^1 \). Then \( m_i \cdot \xi \in C_K(E^1, C) \) for all \( i \). Since \( m_i \cdot \xi \to m \cdot \xi \) in the norm of \( X \otimes C \), we have \( m \cdot \xi \in C_K(E^1, C) \), by Lemma A.2.

For each \( e \in E^1 \), by norm-continuity of evaluation on \( C_c(E^1, C) \) we have
\[
(m \cdot \xi)(e) = \lim_i (m_i \cdot \xi)(e) = \lim_i m_i(e)\xi(e).
\]
Moreover, for any \( a \in C \), we can choose \( f \in C_0(E^1, C) \) such that \( f(e) = a \) and compute:
\[
m_i(e)a = m_i(e)f(e) = (m_if)(e) \to (m_f)(e) = m(e)a.
\]
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Thus evaluation is strictly continuous on \( C_b(E^1, M^\beta(C)) \); in particular,
\[
\lim_i m_i(e)\xi(e) = m(e)\xi(e),
\]
which establishes (A.6).

The statement for \( n \cdot \xi \) follows by composing with the range map \( r: E^1 \to E^0 \), and the statement for \( \xi \cdot n \) is proved similarly to the above argument for \( m \cdot \xi \).

\[\square\]

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