Numerical Simulation of Acoustic Scattering by a Cylinder Based on the Enhanced Optimized Scheme

Siqi Yuan¹, Ruixuan Ma¹²*, Conghai Wu¹ and Shuhai Zhang¹

¹ State Key Laboratory of Aerodynamics, China Aerodynamics Research and Development Center, MianYang, SiChuan 621000, China
² Key Laboratory of Aerodynamics Noise Control, China Aerodynamics Research and Development Center, MianYang, SiChuan, 621000, China
Email: maruixuan@cardc.cn

Abstract. The enhanced optimized scheme we developed in the early work is employed to simulate the scattering of acoustic waves from a two-dimensional cylinder by solving the Euler equations. The numerical results of a benchmark problem are found to be in excellent agreement with the exact solution. Our numerical results show that when acoustic waves propagate through a cylinder, the acoustic scattering results in a spatial redistribution of the acoustic energy as well as an alteration of the phase of the waves. The directivities of the scattered fields change significantly for the different length ratios of acoustic wavelength to the radius of the cylinder.

Keywords. Numerical simulation, acoustic scattering, enhanced optimized scheme

1. Introduction
In recent years, many studies have been carried out on the acoustic scattering characteristics [1, 2] of regular configurations, such as cylinders, spheres and ellipsoids. Among them, the most classic Biot theory [3] that introduces a displacement potential function established the equation of scattered acoustic waves, and it gave a theoretical solution about infinite cylinders, elliptical cylinders, spheres, and ellipsoids.

Numerical simulation is one of the main methods on aeroacoustics. Due to the unsteady, multi-scale, and slow attenuation characteristics of aeroacoustics, the numerical methods are required to have high-resolution, low-dispersion, and low-dissipation features. In traditional computations, the numerical scheme with the high-order accuracy on a given stencil is preferred. This kind of schemes has high accuracy for smooth problems, but has relatively low resolution in the high wave number region, which is not conducive to compute aerodynamic noise. For these problems, the most used difference methods are high-order compact scheme [4] and high-order optimized scheme such as DRP scheme [5].

The high-order compact scheme is widely used due to its good spectral properties, but it needs to solve equations in every time step, which reduces computational efficiency and is not good at parallel computation. The optimized difference scheme [6] is a good method to solve the problems of aerodynamic noise. They optimize the scheme within a certain wave number range by sacrificing accuracy order to improve the resolution properties.

The enhanced optimized scheme [7] is a weighted combination of the order-optimized scheme and a resolution-optimized scheme on the seven-point symmetric stencil. With the well-designed weight, the resolution is improved without loss of convergence order.
In this paper, the enhanced optimized scheme is employed to study the acoustic scattering by a cylinder. In section 2, the problem and the numerical method are shown. In section 3, the effects of ratios of the wavelength of the incident wave to the radius of the cylinder are analysed. The last section is the conclusion.

2. Methodology

2.1. Exact Solution

In this paper, the problem of scattering of plane waves by a cylinder is considered. The radius of the cylinder is \( a \), and the span of the cylinder is parallel to the \( z \) axis. The cross section of the cylinder is the \( xoy \) plane, and the incident wave is along the \( x \) direction. It is noticed that the normal component of the velocity field is vanished at the boundary of the cylinder. The scattered wave acoustic pressure \( p_s \) and radial velocity \( v_{sr} \) are

\[
\begin{align*}
    p_s &= p_e \left[ -B_m H_m^{(1)} \left( \frac{2\pi}{\lambda} r \right) - 2 \sum_{m=1}^{\infty} (-j)^{m+1} B_m H_m^{(1)} \left( \frac{2\pi}{\lambda} r \right) \cos(m\theta) \right] \\
    v_{sr} &= -\frac{1}{j \frac{2\pi}{\lambda} \rho_0 c_0} \frac{\partial p_s}{\partial r} \\
    &= \frac{p_0}{\rho_0 c_0} \left[ j B_0 H_0^{(1)} \left( \frac{2\pi}{\lambda} r \right) - \sum_{m=1}^{\infty} (-j)^m B_m \left[ H_{m+1}^{(1)} \left( \frac{2\pi}{\lambda} r \right) - H_{m-1}^{(1)} \left( \frac{2\pi}{\lambda} r \right) \right] \cos(m\theta) \right]
\end{align*}
\]

(1)

where \( B_m \) can be determined according to the specific problem, and \( H_m^{(1)} \) is the first kind of \( m \)-th order Hankel function, \( \rho_0 \) and \( c_0 \) are the density and wave velocity of the plane incident wave. The total acoustic field contains both of the incident acoustic field and the scattered acoustic field. Here, we have \( H_0^{(1)}(x) = -H_0^{(1)}(x) \).

The intensity of the scattered acoustic wave can be approximated by a relatively simple form when \( \frac{\lambda}{a} \) is very large or very small. In the case of \( \frac{\lambda}{a} \ll 2\pi \), this intensity under high frequency scattering is approximated as

\[
I_{sr} = \frac{a}{2r} \sin \left( \frac{\theta}{2} \right) + \frac{1}{2\pi kr} \cot^2 \theta \sin^2 \left( \frac{2\pi}{\lambda} a \sin \theta \right).
\]

(2)

When \( \frac{\lambda}{a} \gg 2\pi \), the approximation can be reduced to:

\[
I_{sr} = \frac{\pi a}{8r} \left( \frac{2\pi}{\lambda} a \right)^3 (1 - 2\cos \theta)^2.
\]

(3)

2.2. Enhanced Optimized Scheme

In this subsection, we introduce the enhanced optimized scheme proposed in [7]. It is a nonlinear weighted combination of the order-optimized scheme and a resolution-optimized scheme. The two sub-schemes are the order-optimized scheme, and a resolution-optimized scheme where the wavenumber error is vanished at a given scaled wavenumber. The enhanced optimized scheme adopts seven-point symmetrical stencil and can be expressed as
\[
\hat{f}_0' = (1 - \lambda) \hat{f}^S + \lambda \hat{f}^O
\]  

(4)

In this equation, \(\hat{f}^S\) is the 6-th order scheme which is the order-optimized one, and \(\hat{f}^O\) is a 4-th order optimized scheme which is exact for scaled wavenumber \(\kappa_0 = \frac{\pi}{3}\). The nonlinear weight \(\lambda_j = \min\{1, \gamma_j\}, \gamma_j = \sqrt{IW_j \cdot a + IW_j \cdot b}\), where \(IW\) is the wave number indicator [8]

\[
IW_j = \left( \frac{f_{j-2} - 4f_{j-4} + 6f_j - 4f_{j+4} + f_{j+6}}{f_{j-4} - 2f_j + f_{j+4}} \right)^2 + \left( -f_{j-2} + 2f_{j-4} - 2f_{j+4} + f_{j+6} \right)^2 + \varepsilon
\]

(5)

According to Taylor expansion, the truncation error of a q-th-order difference scheme can be written as

\[
T_h = \hat{f}^{(n)} - f'(x) = M f^{(q+1)}(x) h^q + O\left(f^{(q+2)}(x) h^{q+1}\right)
\]

(6)

where \(M\) is the constant corresponding to the scheme. We let the scaled wavenumber \(\kappa = kh\), and the effective wave number is \(\bar{\kappa} = \kappa h\). The wavenumber error is \(\bar{\kappa} - \kappa\), which is used to measure the resolution properties of the scheme. The wave number error and the truncation error of Eq.(4) have the following relationship

\[
\bar{\kappa} - \kappa = (1 - \lambda_j)(\bar{\kappa}^S - \kappa) + \lambda_j(\bar{\kappa}^O - \kappa) = -C_6(1 - \lambda_j)\kappa^5 + C_4\lambda_j\kappa^3 + O(\kappa^5)
\]

(7)

We minimize the wavenumber error at zero wavenumber and ensure that the wavenumber error is vanished at \(\kappa_0 = \frac{\pi}{3}\), we can obtain \(a = 0.776433285797188\) and \(b = 0.223566714202812\).

3. Results and Discussions

The exact solution of the acoustic scattering by a cylinder can be obtained by equations (1), (2), (3). The numerical simulation is performed by computing the Euler equations in polar coordinates. The time discretization adopts the fourth-order Runge-Kutta method [9]. The spatial discretization adopts the enhanced optimized scheme we develop, according to the equations (4), (5), (6). Figure 1 shows that the comparison between the numerical simulation results and the exact solution when the plane wave incident on a solid cylinder. Figure 1(a) presents the root-mean-square pressure of the scattered field obtained by the boundary element method in case of \(\lambda / a = 2\pi / 5\) [10] and figure 1(b) presents the root-mean-square pressure of the scattered field obtained by the Enhanced optimized scheme in case of \(\lambda / a = 1\). It is easy to see that the result from the boundary element method has a clear difference from the accurate solution. This difference reaches its maximum behind the cylinder. The results obtained by the enhanced optimized scheme agree well with the exact solution, which remains well behind the cylinder. These indicate that the enhanced scheme can resolve the characters of acoustic scattering better than the traditional acoustic methods.
(a) the boundary element method, $\lambda/a = 2\pi/5$   
(b) the enhanced optimized scheme, $\lambda/a = 1$

**Figure 1.** Comparison of root-mean-square pressure of the scattered field between the numerical results and the theory studies.

Figure 2 shows the root-mean-square pressure of the scattered fields at different ratios of $\lambda/a$ obtained by using the enhanced optimized scheme. We can see that the directivities of the scattered fields change significantly for the different length ratios of $\lambda/a$. The directivity is symmetric across the direction of the incident wave. With the increase of $\lambda/a$, the strength of the scattered field gradually weakens. With the decrease of $\lambda/a$, the scattered fields show more irregularities, and more and more secondary lobes are visible behind the cylinder. When $\lambda/a \leq 2$, the strength of the scattered waves are comparable to the incident waves and focused closer to the direction of the incident wave. When $2 \leq \lambda/a \leq 2^2$, the amount of front and back scattering is roughly equivalent. When $\lambda/a \geq 2^2$, the amplitude of the scattered acoustic pressure in front of the cylinder is higher, which is dominant in the whole acoustic scattering.

**Figure 2.** Root-mean-square pressure of the scattered fields at different ratios.
Figure 3 shows acoustic characters of the total and scattered fields with the length ratio $\lambda/a=2\pi$. In this case, the scattering of the plane wave is relatively mild and there is only a slight alteration of the wavefronts. The cylindrical spreading of the scattered waves is observed away from the cylinder. As shown in figure 3(a), the total acoustic field is stronger in front of the cylinder, and weaker behind the cylinder after the plane acoustic wave passes over the cylinder. Figure 3(b) shows that the scattering phenomenon behind the cylinder is more obvious. Figure 3(c) and 3(d) illustrate the instantaneous total acoustic pressure and scattered acoustic pressure. It can be seen that when acoustic waves propagate through a cylinder, the acoustic scattering results in a spatial redistribution of the acoustic energy as well as an alteration of the phase of the waves. Figure 3(d) describes that the scattered acoustic waves radiate from the cylinder, which displays as a typical cylindrical wave.

![Figure 3](image_url)

**Figure 3.** Snapshots of acoustic characters on acoustic scattering by a cylinder with the length ratio $\lambda/a=2\pi$. (a) root-mean-square pressure of the total pressure, (b) root-mean-square pressure of the scattered pressure, (c) the instantaneous total pressure, and (d) the instantaneous scattered pressure.

4. Conclusion
In this paper, the scattering of plane acoustic waves by a cylinder at wide length ratios has been studied by solving the Euler equations with the enhanced optimized scheme. Numerical errors associated with the enhanced optimized scheme are made much smaller than those associated with the conventional acoustic boundary element method. The scattering of plane acoustic waves by a cylinder at a wide range of the length-scale ratio $\lambda/a$ has been investigated by solving the Euler equations in polar coordinates. It can be concluded that the total acoustic wave scattering decreases as the length ratio increases. The present study provides a new method for numerical simulation of acoustic scattering, and also gives discussions on the length scale related to the problem of acoustic scattering by a cylinder, which is of benefit to acoustic measurements in practical engineering applications.
Acknowledgements
The authors gratefully acknowledge the support for this research provided by National Natural Science Foundation of China under research grant no. 11732016, Sichuan Science and Technology Program under research grant no. 2018JZ0076 and National Numerical Windtunnel project.

References
[1] Tam C K W 2006 Recent advances in computational aeroacoustics Fluid Dynamics Research 38 (9) 591–615.
[2] Ma R X, Wang Y M, Zhang S H, Wu C H and Wang X N 2021 Numerical investigation of scale effect on acoustic scattering by vortex Acta Physica Sinica 70 (10) 190-199.
[3] Biot M A 1962 Mechanics of deformation and acoustic propagation in porous media Journal of Applied Physics 33 (4) 1483-1498.
[4] Lele S K 1992 Compact difference schemes with spectral-like resolution Journal of Computational Physics 103 (1) 16-42.
[5] Tam C and Webb J C 1993 Dispersion-relation-preserving finite difference schemes for computational acoustics Journal of Computational Physics 107 (2) 262-281.
[6] Wu C H, Wang Y M and Zhang S H 2019 A hybrid optimized difference scheme Physics of Gases 4 (6) 22-28.
[7] Wu C H, Ma R X, Wang Y M and Zhang S H 2020 Application of enhanced optimized difference schemes in acoustic scattering Acta Aerodynamica Sinica 38 (6) 1120-1128.
[8] Wu C H 2012 Research on High Order Accurate and High Resolution Difference Methods of Fluid Dynamics (Nanjing: Nanjing University of Aeronautics and Astronautics).
[9] Jiang G S and Shu C W 1996 Efficient implementation of weighted ENO schemes Journal of Computational Physics 126 (1) 202-22.
[10] Ma J, Wang H B, Gao S, Liu X Z, Liu J H and He A J 2018 The boundary element method to solve the acoustic scattering characteristics of a rigid cylinder Journal of Nanjing University (Natural Sciences) 54 (05) 875-886.