Timeless path integral for relativistic quantum mechanics

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Received 8 November 2012, in final form 22 April 2013
Published 14 May 2013
Online at stacks.iop.org/CQG/30/125004

Abstract
Starting from the canonical formalism of relativistic (timeless) quantum mechanics, the formulation of a timeless path integral is rigorously derived. The transition amplitude is reformulated as the sum, or functional integral, over all possible paths in the constraint surface specified by the (relativistic) Hamiltonian constraint, and each path contributes with a phase identical to the classical action divided by \( \hbar \). The timeless path integral manifests the timeless feature as it is completely independent of the parametrization for paths. For the special case that the Hamiltonian constraint is a quadratic polynomial in momenta, the transition amplitude admits the timeless Feynman’s path integral over the (relativistic) configuration space. Meanwhile, the difference between relativistic quantum mechanics and conventional nonrelativistic (with time) quantum mechanics is elaborated on in light of the timeless path integral.

PACS numbers: 03.65.Ca, 03.65.Db, 03.65.Ta, 04.60.−m

1. Introduction

The idea that quantum mechanics can be well defined even if the notion of time is absent has been proposed [1, 2] and developed in a number of different strategies [3–7]. The motivation for formulating quantum mechanics in timeless description comes from the research on quantum gravity, as in the quantum theory of general relativity, the spacetime background is not fixed and generally it is not possible to make sense of quantum variables ‘at a moment of time’. This is closely related to the ‘problem of time’ in quantum gravity [8].

In particular, a comprehensive formulation for the relativistic (timeless) quantum mechanics and its probabilistic interpretation are presented in chapter 5 of [9]1. The formulation is based on the canonical (Hilbert spaces and self-adjoint operators) formalism and we wonder

1 The adjective ‘relativistic’ connotes relational correlations between physical variables in the timeless description. It should not be confused with the adjective for the theory of (special) relativity.
whether it also admits the covariant (sum-over-histories) formalism. In the conventional nonrelativistic (with time) quantum mechanics, the transition amplitudes are the matrix elements of the unitary evolution generated by the Hamiltonian and can be reformulated as the sum over histories, called the path integral. In the relativistic quantum mechanics, however, the concept of time evolution is not well defined at the fundamental level; therefore, conceptual issues and technical subtleties arise when one tries to derive the timeless path integral from the canonical formalism. Various aspects of sum-over-histories approaches to relativistic quantum mechanics have been considered for a variety of models [10–18]. Particularly, the method of path integral quantization elucidates the timelessness of the reparametrization-invariant quantum theory as the result of a superposition of clocks via Jacobi’s principle [15] and the existence of composition laws in relativistic quantum mechanics via path decomposition expansion [16]. However, rigorous derivation to the sum-over-histories formalism from a well-formulated canonical formalism is still lacking in relativistic quantum mechanics, and many important questions remain unclear such as what operator ordering has to be taken to yield a sensible path integral and what exactly the measure of the path integral is.

The aim of this paper is not to formulate a new sum-over-histories approach from a new perspective but instead to rigorously derive the timeless path integral for relativistic quantum mechanics, starting from the canonical formulation specifically described in [9]. The main difficulty lies in the fact that the ordinary time-slicing process of path integrals cannot be directly carried over as there is no privileged observable to be treated as time, and thus new techniques have to be devised. It turns out that, nevertheless, the transition amplitude can be reformulated as the sum, or functional integral, over all possible paths on the constraint surface \( \Sigma \) specified by the (relativistic) Hamiltonian constraint \( H(q^a, p_a) = 0 \) for the configuration variables \( q^a \) and their conjugate momenta \( p_a \), and each path contributes with a phase identical to the classical action divided by \( \hbar \). Unlike the conventional path integral in which each path is parametrized by the time variable \( t \), the timeless path integral is completely independent of the parametrization for paths, manifesting the timeless feature. Furthermore, for the special case that the Hamiltonian constraint is a quadratic polynomial in \( p_a \), the timeless path integral over \( \Sigma \) reduces to the timeless Feynman’s path integral over the (relativistic) configuration space.

The timeless path integral for relativistic quantum mechanics is appealing both conceptually and technically. Conceptually, the timeless path integral offers an alternative interpretation of relativistic quantum fluctuations and is more intuitive than the canonical

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2 For more references in the general area of ‘timeless’ quantum theories, also see references [5] and [10–15] cited in [17] and references [11–21] cited in [18].

3 It should be noted that the path integral derived in this paper is fundamentally different from that in [11]. While this paper gives the timeless path integral for relativistic quantum mechanics, what in [11] is the path integral for nonrelativistic quantum mechanics recast into a parametrized theory, which manifests reparametrization invariance in an apparently timeless fashion. The underlying mechanics considered in [11] is still the conventional (nonrelativistic) quantum mechanics, as opposed to the timeless (relativistic) quantum mechanics in this paper. The path integral in [11], although cast in a timeless fashion, is equivalent to the ordinary path integral of conventional quantum mechanics and thus no difficulty arises with regard to time slicing. When the system is strictly deparametrizable, the transition amplitude obtained from the timeless path integral coincides with that from the ordinary path integral, but their interpretations of probability and physics are profoundly different. The difference is subtle but crucial as discussed in sections 3.2 and 4.2. In the same spirit of [11], it is suggested in [12] that, following the paradigm of the relationship between Jacobi’s and Hamilton’s action principles, one can correspond the Wheeler–DeWitt equation to a time-independent Schrödinger equation and derive a time-dependent Wheeler–DeWitt equation of the Schrödinger type with the 4-volume of spacetime playing the role of physical time. (For this aspect of the problem of time, also see [15] for the much simpler case of nonrelativistic particles moving through a space-dependent potential.) The resulting quantum theory of gravity (if can be constructed consistently) is again fundamentally different from that by directly quantizing the Wheeler–DeWitt equation in the manner of timeless quantum mechanics specifically formulated in [9]. (Also see appendix of [13] for the path integral for Jacobi’s action based on the ideas of [12].)
formalism for many aspects. It can give a new point of view about how the conventional quantum mechanics with time emerges within a certain approximation and thus may help to resolve the problem of time. Technically, the timeless path integral provides tractable tools to compute (at least numerically or approximately) the transition amplitudes which otherwise remain formal in the canonical formalism, as various approximation methods in path integral approaches have been widely exploited and can be readily adapted to the timeless context.

In the research of loop quantum gravity (LQG), the sum-over-histories formulation is an active research area that goes under the name ‘spin foam models’ (SFMs) (see [9] and references therein for LQG and SFMs). In particular, over the past years, SFMs in relation to the kinematics of LQG have been clearly established [19–22]. However, the Hamiltonian dynamics of LQG is far from fully understood, and although well motivated, SFMs have not been systematically derived from any well-established theories of canonical quantum gravity. Meanwhile, loop quantum cosmology (LQC) has recently been cast in a sum-over-histories formulation, providing strong support for the general paradigm underlying SFMs [23, 24]. In this paper, the timeless path integral is systematically derived from the canonical formalism of relativistic quantum mechanics, and we hope it will shed new light on the issues of the interplay between LQG/LQC and SFMs.

This paper is organized as follows. We begin with a review on the classical theory of relativistic mechanics in section 2 and then a review on the quantum theory of relativistic mechanics in section 3. The main topic is presented in section 4, where the timeless path integral is derived and investigated in detail. Conclusions are summarized and discussed in section 5. Additionally, the stationary phase approximation for the timeless path integral is included in appendix A and, in order to compare with the timeless path integral, we re-derive the path integral for conventional quantum mechanics in appendix B.

2. Classical theory of relativistic mechanics

The conventional formulation of classical mechanics treats the time \( t \) on a special footing and therefore is not broad enough for general-relativistic systems, which treat time on the equal footing as other variables. To include general-relativistic systems, we need a more general formulation with a new conceptual scheme. A timeless formulation for relativistic classical mechanics is proposed for this purpose and described in detail in chapter 3 of [9], excerpts from which are presented in this section with some new materials added to give a review and define notations.

2.1. Hamiltonian formalism

Let \( C \) be the relativistic configuration space coordinatized by \( q^a \) for \( a = 1, 2, \ldots, d \), with \( q^a \) being the partial observables and \( d \) being the dimension of \( C \). In nonrelativistic mechanics, one of the partial observables can be singled out and treated specially as the time \( t \), i.e. \( q^t = (t, q^i) \), but this separation is generally not possible for general-relativistic systems. An observation yields a complete set of \( q^a \), which is called an event. In nonrelativistic mechanics, an observation is a reading of the time \( t \) together with other readings \( q^i \).

Consider the cotangent space \( \Omega = T^\ast C \) coordinatized by \( q^a \) and their momenta \( p_a \). The space \( \Omega \) carries a natural 1-form \( \tilde{\theta} = p_a dq^a \). Once the kinematics (i.e. the space \( C \) of the

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4 Readers who are familiar with the partial observable approach to timeless quantum mechanics as detailed in chapters 3 and 5 of [9] may skip sections 2 and 3 and come back whenever necessary. To avoid any confusion with other timeless notions of quantum theories (recall footnote 5), it is advised to still read section 3.2.
partial observables \( q^a \) is known, the dynamics is fully determined by giving a constraint surface \( \Sigma \) in the space \( \Omega \). The constraint surface \( \Sigma \) is specified by \( H = 0 \) with a function \( H : \Omega \to \mathbb{R}^k \). Denote by \( \tilde{\gamma} \) an unparametrized curve in \( \Omega \) (observables and momenta) and by \( \gamma \) its projection to \( C \) (observables only). The physical motion is determined by the function \( H \) via the following.

Variational principle. A curve \( \gamma \) in \( C \) is a physical motion connecting the events \( q_1^a \) and \( q_2^a \) if \( \tilde{\gamma} \) extremizes the action

\[
S[\tilde{\gamma}] = \int_{\tilde{\gamma}} p_a \, dq^a
\]

in the class of the curves \( \tilde{\gamma} \) which satisfy

\[
H(q^a, p_a) = 0,
\]

(i.e. \( \tilde{\gamma} \in \Sigma \)) and whose projection \( \gamma \) to \( C \) connects \( q_1^a \) and \( q_2^a \).

If \( k = 1 \), then \( H \) is a scalar function and called the Hamiltonian constraint. If \( k > 1 \), then there is gauge invariance and \( H \) is called the relativistic Hamiltonian. The pair \((C, H)\) describes a relativistic dynamical system. All (relativistic and nonrelativistic) Hamiltonian systems can be formulated in this timeless formalism.

By parametrizing the curve \( \tilde{\gamma} \) with a parameter \( \tau \), the action (2.1) reads

\[
S[q^a, p_a, N] = \int d\tau \left( p_a(\tau) \frac{dq^a(\tau)}{d\tau} - N_i(\tau) H_j(q^a, p_a) \right),
\]

where the constraint (2.2) has been implemented with the Lagrange multipliers \( N_i(\tau) \). Varying this action with respect to \( N_i(\tau) \), \( p_a(\tau) \) and \( q^a(\tau) \) yields the constraint equation(s) (2.2) together with the Hamilton equations

\[
\frac{dq^a}{d\tau} = N_i(\tau) \frac{\partial H^i(q^a, p_a)}{\partial p_a},
\]

\[
\frac{dp_a}{d\tau} = -N_i(\tau) \frac{\partial H^i(q^a, p_a)}{\partial q^a}.
\]

For \( k > 1 \), a motion is determined by a \( k \)-dimensional surfaces in \( C \) and different choices of the \( k \) arbitrary functions \( N_i(\tau) \) determine different curves and parametrizations on the single surface that defines a motion. For \( k = 1 \), a motion is a one-dimensional curve in \( C \) and different choices of \( N(\tau) \) correspond to different parametrizations for the same curve. Different solutions of \( q^a(\tau) \) and \( p_a(\tau) \) for different choices of \( N_i(\tau) \) are the gauge-equivalent representations of the same motion and different choices of \( N_i(\tau) \) have no physical significance.

Along the solution curve, the change rate of \( H \) with respect to \( \tau \) is given by

\[
\frac{dH^i}{d\tau} = \frac{dq^a}{d\tau} \frac{\partial H^i}{\partial q^a} + \frac{dp_a}{d\tau} \frac{\partial H^i}{\partial p_a} = N_i \frac{\partial H^i}{\partial p_a} \frac{\partial H^i}{\partial q^a} - N_j \frac{\partial H^i}{\partial q^a} \frac{\partial H^j}{\partial p_a} = N_i \{ H^i, H^j \}.
\]

To be consistent, the physical motion should remain on the constraint surface \( \Sigma \). That is, \( dH/d\tau \) has to vanish along the curve. Therefore, we must have the condition

\[
\{ H^i, H^j \} \big|_\Sigma = 0,
\]

abbreviated as \( \{ H^i, H^j \} \approx 0 \)
for all \(i\) and \(j\). A function \(F(q^i, p_i)\) defined in a neighborhood of \(\Sigma\) is called weakly zero if \(F|_\Sigma = 0\) (abbreviated as \(F \approx 0\)) and called strongly zero if

\[
F|_\Sigma = 0 \quad \text{and} \quad \left(\frac{\partial F}{\partial q^i}, \frac{\partial F}{\partial p_i}\right)|_{\Sigma} = 0,
\]

abbreviated as \(F \simeq 0\). (2.7)

It can be proven that \(F \approx 0\) implies \(F \simeq f_i H^i\) for some functions \(f_i(q^i, p_i)\). Consequently, we have

\[
\{H^i, H^j\} \simeq f_i^j(q^i, p_i)H^i.
\]

The condition (2.6) ensures all constraints \(H^i\) to be first class. (See [25] for more about constrained systems and the concept of first-class constraints.)

### 2.2. Nonrelativistic mechanics as a special case

The conventional nonrelativistic mechanics can also be formulated in the timeless framework as a special case. For the nonrelativistic systems, the relativistic configuration space has the structure \(\mathcal{C} = \mathbb{R} \times C_0\), where \(C_0\) is the conventional nonrelativistic configuration space; i.e., \(q^i = (t, q^i)\) as one of the partial observables is identified as the time \(t\). Correspondingly, the momenta read \(p_i = \{(p_t, p_i)\}, \) with \(p_t\) being the conjugate momentum of \(t\) and \(p_i\) being the conjugate momenta of \(q^i\). The Hamiltonian constraint is given by

\[
H(t, q^i, p_t, p_i) = p_t + H_0(q^i, p_i; t),
\]

where \(H_0(q^i, p_i; t)\) is the conventional nonrelativistic Hamiltonian function. Given the Hamiltonian constraint in the form of (2.9), the Hamilton equations (2.4) lead to

\[
\frac{d\tau}{d\tau} = N(\tau), \quad \frac{dp_i}{d\tau} = -N(\tau)\frac{\partial H_0}{\partial t}, \quad \frac{dq^i}{d\tau} = N(\tau)\frac{\partial H_0}{\partial p_i} \quad \frac{dp_i}{d\tau} = -N(\tau)\frac{\partial H_0}{\partial q^i},
\]

which reads

\[
\frac{dp_i}{d\tau} = -\frac{\partial H_0}{\partial p_i}, \quad \frac{dq^i}{d\tau} = \frac{\partial H_0}{\partial p_i}, \quad \frac{dp_i}{d\tau} = -\frac{\partial H_0}{\partial q^i},
\]

if particularly we use \(\tau\) to parametrize the curve of solutions. Furthermore, the constraint (2.2) dictates \(p_t = -H_0\). Thus, the momentum \(p_t\) is the negative of energy and it is a constant of motion if \(H_0 = H_0(q^i, p_i)\) has no explicit dependence on \(t\). The equations in (2.12) are precisely the conventional Hamilton equations for nonrelativistic mechanics.

The Hamilton equations in (2.12) form a system of first-order ordinary differential equations. Given the initial condition \(q^i(t_0) = q^i_0\) and \(p_i(t_0) = p_i_0\) at the time \(t_0\), the existence and uniqueness theorem for ordinary differential equations states that there exists a solution of (2.12) given by \(q^i = q^i(t)\) and \(p_i = p_i(t)\) for \(t \in \mathbb{R}\), and furthermore the solution is unique\(^5\).

As a consequence, \(q^i\) and \(p_i\) evolve as functions of \(t\), and a physical motion is an open curve in \(\mathcal{C} = \mathbb{R} \times C_0\), along which the observable \(t\) is monotonic.

\(^5\) In order to apply the existence and uniqueness theorem, we assume \(\partial H_0/\partial q^i\), \(\partial H_0/\partial p_i\), \(\partial^2 H_0/\partial q^i\), \(\partial^2 H_0/\partial p_i\) and \(\partial^2 H_0/\partial q^i\partial p_j\) all continuous.
A dynamical system in which a particular partial observable can be singled out as \( t \) such that the Hamiltonian is separated as in the form of (2.9) is called deparametrizable. For deparametrizable systems, the change of \( t \) is in accord with the ordinary notion of time, which does not turn around but grows monotonically along the physical motion. Generically, however, relativistic systems might be non-deparametrizable—no preferred observable can serve as the time such that other variables are described as functions of time along the physical motion. The classical theory predicts the physical motion as an unparametrized curve, which gives correlations between physical variables, not the way physical variables evolve with respect to a preferred time variable. In the following subsection, we will introduce the timeless double pendulum as an example to illustrate the timeless feature.

### 2.3. Example: timeless double pendulum

Let us now introduce a genuinely timeless system as a simple model to illustrate the mechanics without time. This model was first introduced in [3, 4] and used repeatedly as an example in [9].

Consider a mechanical system with two partial observables, \( a \) and \( b \), whose dynamics is specified by the relativistic Hamiltonian

\[
H(a, b, p_a, p_b) = \frac{1}{2} \left( p_a^2 + p_b^2 + a^2 + b^2 - 2E \right)
\]

with a given constant \( E \). The relativistic configuration space is \( \mathcal{C} = \mathbb{R}^2 \) coordinatized by \( a \) and \( b \), and the cotangent space \( \Omega = T^*\mathcal{C} \) is coordinatized by \( (a, b, p_a, p_b) \). The constraint surface \( \Sigma \) is specified by \( H = 0 \); it is a three-dimensional sphere of radius \( \sqrt{2E} \) in \( \Omega \).

In the \( N(\tau) = 1 \) gauge, the Hamilton equations (2.4) give

\[
\frac{da}{d\tau} = p_a, \quad \frac{db}{d\tau} = p_b, \quad \frac{dp_a}{d\tau} = -a, \quad \frac{dp_b}{d\tau} = -b,
\]

and the Hamiltonian constraint (2.2) gives

\[
a^2 + b^2 + p_a^2 + p_b^2 = 2E.
\]

The general solution is given by

\[
a(\tau) = A_a \sin(\tau), \quad b(\tau) = A_b \sin(\tau + \beta),
\]

where \( A_a = \sqrt{2E} \sin \alpha \) and \( A_b = \sqrt{2E} \cos \alpha \), and \( \alpha \) and \( \beta \) are constants.

Therefore, physical motions are closed curves (ellipses) in \( \mathcal{C} = \mathbb{R}^2 \). (Choosing different gauges for \( N \) yields the same curve with different parametrizations.) This system is non-deparametrizable and does not admit a conventional Hamiltonian formulation, because, as discussed in section 2.2, physical motions in \( \mathcal{C} = \mathbb{R} \times \mathbb{C}_0 \) for a nonrelativistic system are monotonic in \( t \) and thus cannot be closed curves.

### 2.4. Lagrangian formalism

Consider the special case that the relativistic Hamiltonian is given in the form

\[
H(q^a, p_a) = \sum_a \alpha_a p_a^2 + \sum_a \beta_a p_a q^a + \sum_a \gamma_a p_a q^a + V(q^a),
\]

where \( \alpha_a, \beta_a \) and \( \gamma_a \) are constant coefficients, and \( V(q^a) \) is the potential which depends only on \( q^a \). This form is quite generic and many examples of interest belong to this category such as the relativistic particle (free or subject to an external potential), the timeless double pendulum, and so on. It is usually convenient to choose \( (a, \beta_a, \gamma_a) \) in such a way that \( V(q^a) \) depends only on \( p_a \) and \( q^a \) separately. In this subsection, the repeated index \( a \) is not summed unless \( \sum_a \) is explicitly used.

\[\text{\[2.17\]}\]

6 In this subsection, the repeated index \( a \) is not summed unless \( \sum_a \) is explicitly used.
pendulum (harmonic or anharmonic) and the nonrelativistic system as described by (2.9) with
\[ H_0 = \sum_i p_i^2 / 2m_i + V(q^i, t). \] The Hamilton equations (2.4) yields
\[
\frac{dq^a}{d\tau} = N \left( 2\alpha_a p_a + \beta_a q^a + \gamma_a \right), \tag{2.18a}
\]
\[
\frac{dp_a}{d\tau} = -N \left( \beta_a p_a + \frac{\partial V}{\partial q^a} \right). \tag{2.18b}
\]
Equation (2.18a) gives the relation between the momenta \( p_a \) and the ‘velocities’ \( \dot{q}^a := dq^a / d\tau \), through which the inverse Legendre transform recasts the action (2.3) in terms of the Lagrangian function
\[
S[q^a, \dot{q}^a, N; \tau] = \int d\tau L(q^a, \dot{q}^a, N)
= \int d\tau \left( \sum_a \frac{N}{4\alpha_a} \left[ \frac{\dot{q}^a}{N} - \beta_a q^a - \gamma_a \right]^2 - NV(q^a) \right). \tag{2.19}
\]
Variation with respect to \( N \) yields
\[
\frac{\delta S}{\delta N} \equiv \frac{\partial L}{\partial N} = 0 \Rightarrow 0 = \sum_a \frac{1}{4\alpha_a} \left[ \frac{\dot{q}^a}{N} - \beta_a q^a - \gamma_a \right]^2 - \sum_a \frac{\dot{q}^a}{2\alpha_a N} \left[ \frac{\dot{q}^a}{N} - \beta_a q^a - \gamma_a \right] - V = -\left( \sum_a \alpha_a p_a^2 + \sum_a \beta_a p_a q^a + \sum_a \gamma_a p_a + V \right) = -H, \tag{2.20}
\]
which is precisely the Hamiltonian constraint (2.2). On the other hand, variation with respect to \( q^a \) gives the equation of motion as a second-order differential equation
\[
\frac{\delta S}{\delta q^a} \equiv \frac{\partial L}{\partial q^a} - \frac{d}{d\tau} \frac{\partial L}{\partial \dot{q}^a} = 0
\Rightarrow \frac{d}{N d\tau} \left( \frac{d q^a}{d\tau} \right) = \beta_a^2 q^a + \beta_a \gamma_a - 2\alpha_a \frac{\partial V}{\partial q^a}, \tag{2.21}
\]
which is equivalent to (2.18).

3. Quantum theory of relativistic mechanics

The timeless formulation for relativistic classical mechanics is reviewed in section 2. Based on the Hamiltonian framework of the classical theory, the quantum theory of relativistic mechanics can be formulated in canonical formalism. Unlike the conventional quantum theory, relativistic quantum mechanics does not describe evolution in time, but correlations between observables.

In section 3.1, we stipulate a general scheme for relativistic quantum mechanics, which is excerpted from chapter 5 of [9]. In section 3.2, we comment on the difference between relativistic quantum mechanics and conventional quantum mechanics when the system is deparametrizable. In section 3.3, as excerpted from chapter 5 of [9] again, we take the timeless double pendulum as a simple example to illuminate the timeless formalism. Issues on the physical Hilbert space are detailed in section 3.4 and the physical interpretations of quantum measurements and collapse are discussed in section 3.5.
3.1. General scheme

Let \( \mathcal{C} \) be the relativistic configuration space for the classical theory as described in the section 2.1. The corresponding quantum theory can be formulated timelessly in the following scheme.

Kinematical states. Let \( \mathcal{S} \subset \mathcal{K} \subset \mathcal{S}' \) be the Gelfand triple defined over \( \mathcal{C} \) with the measure \( dq_1 dq_2 \cdots dq_7 \). The kinematical states of a system are represented by vectors \( |\psi\rangle \in \mathcal{K} \), and \( \mathcal{K} \) is called the kinematical Hilbert space.

Partial observables. A partial observable is represented by a self-adjoint operator in \( \mathcal{K} \). The simultaneous eigenstates \( |s\rangle \) of a complete set of commuting partial observables are called quantum events. In particular, \( \hat{q}^i \) and \( \hat{p}_a \) are partial observables acting respectively as the multiplicative and differential operators on \( \psi(q^a) \); i.e., \( \hat{q}^i \psi(q^a) = q^a \psi(q^a) \) and \( \hat{p}_a \psi(q^a) = -i\hbar \frac{\partial}{\partial q^a} \psi(q^a) \). Their eigenstates \( |q^a\rangle \) (defined as \( \hat{q}^i |q^a\rangle = q^a |q^a\rangle \)) and \( |p_a\rangle \) (defined as \( \hat{p}_a |p_a\rangle = p_a |p_a\rangle \)) are both quantum events.

Dynamics. Dynamics is defined by a self-adjoint operator \( \hat{H} \) in \( \mathcal{K} \), called relativistic Hamiltonian. The operator from \( \mathcal{S} \) to \( \mathcal{S}' \) schematically defined as

\[
\hat{P} = \int \text{d} \tau \ e^{-i\tau \hat{H}}
\]  

(3.1)

is called the ‘projector’. The matrix elements

\[
W(s, s') := \langle s | \hat{P} | s' \rangle
\]  

(3.2)

are called transition amplitudes, which encode entire physics of the dynamics.

Physical states. A physical state is a solution of the quantum Hamiltonian constraint equation

\[
\hat{H} |\psi\rangle = 0,
\]  

(3.3)

which is the quantum counterpart of (2.2). Given an arbitrary kinematical state \( |\psi_a\rangle \in \mathcal{S} \), we can associate an element \( (\Psi_{\psi_a} | \in \mathcal{S}' \), defined by its (linear) action on arbitrary states \( |\psi_{\mu}\rangle \in \mathcal{S} \), as

\[
(\Psi_{\psi_a} | \psi_{\mu}\rangle = \int \text{d} \tau \langle e^{i\tau \hat{H}} \psi_a | \psi_{\mu}\rangle \equiv \langle \psi_a | \hat{P} | \psi_{\mu}\rangle,
\]  

(3.4)

such that \( (\Psi_{\psi_a} | \) is a physical state, namely a solution to (3.3). The solution space is endowed with the Hermitian inner product

\[
(\Psi_{\psi_a} | \Psi_{\psi_{\beta}}) := \langle \Psi_{\psi_a} | \Psi_{\psi_{\beta}} \rangle,
\]  

(3.5)

which is called the physical inner product. The Cauchy completion of the solution space with respect to the physical inner product \((\cdot | \cdot)\) is called the physical Hilbert space and denoted by \( \mathcal{H} \).

Measurements and collapse. If the measurement corresponding to a partial observable \( \hat{A} \) is performed, then the outcome takes the value of one of the eigenvalues of \( \hat{A} \) if the spectrum of \( \hat{A} \) is discrete, or in a small spectral region (with uncertainty) if the spectrum is continuous. Measuring a complete set of partial observables \( \hat{A}_i \) simultaneously is called a complete measurement at an ‘instance’\(^9\), the outcome of which gives rise to a kinematical state \( |\psi_a\rangle \) (which is a simultaneous eigenstate of \( \hat{A}_i \) if the spectra of \( \hat{A}_i \) are discrete). The physical state is said to be collapsed to \( |\Psi_{\psi_a}\rangle \) by the complete measurement.

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7 That is, \( \mathcal{S} \) is the space of the smooth functions \( f(q^a) \) on \( \mathcal{C} \) with fast decrease, \( \mathcal{K} = L^2(\mathcal{C}, dq^a) \) is a Hilbert space and \( \mathcal{S}' \) is formed by the tempered distributions over \( \mathcal{C} \).

8 The integration range depends on the system. It is over a compact space if the spectrum of \( \hat{H} \) is discrete and over a noncompact space if the spectrum is continuous. The operator \( \hat{P} \) is a projector in the precise sense only if zero is a part of the discrete spectrum of \( \hat{H} \).

9 In the timeless language, a complete measurement is said to be conducted at some ‘instance’, not at some ‘instant’. 
Prediction in terms of probability. If at one instance a complete measurement yields $|\psi_\alpha\rangle$, then the probability that at another instance another complete measurement yields $|\psi_\beta\rangle$ is given by

$$P_{\beta\alpha} = \left| \frac{W[\psi_\beta, \psi_\alpha]}{\sqrt{W[\psi_\beta, \psi_\beta] \sqrt{W[\psi_\alpha, \psi_\alpha]}}} \right|^2,$$

where

$$W[\psi_\beta, \psi_\alpha] := \langle \psi_\beta | \hat{P} | \psi_\alpha \rangle = \int ds \int ds' W(s, s') \psi_\beta(s) \psi_\alpha(s').$$

In particular, if the quantum events $s$ make up a discrete spectrum, then the probability of the quantum event $s$ given the quantum event $s'$ is

$$P_{ss'} = \left| \frac{W(s, s')}{\sqrt{W(s, s)}} \right|^2.$$

If the spectrum is continuous, then the probability of a quantum event in a small spectral region $R$ given a quantum event in a small spectral region $R'$ is

$$P_{RR'} = \left| \frac{W(R, R')}{\sqrt{W(R, R)} \sqrt{W(R', R')}} \right|^2,$$

where

$$W(R, R') := \int_R ds \int_{R'} ds' W(s, s').$$

The general scheme stipulated above gives a sound (axiomatic) framework for relativistic quantum mechanics. However, it is far from complete and remains provisional as many aspects need to be further clarified. One obvious problem is what precisely the small region $R$ in (3.10) should be associated with when a complete measurement is conducted in the case of a continuous spectrum. Additionally, although simultaneous exact measurements of non-commuting (partial) observables are impossible, may simultaneous inaccurate measurements of them (known as ‘joint measurements’) still be possible? If yes, given outcomes with inaccuracies, what exact kinematical state $|\psi_\alpha\rangle$ does the joint measurement yield? Furthermore, what are correct treatments for the measurements which are performed not at an instance but over a short continuous duration or repeatedly performed at successive instances\(^\text{10}\)?

3.2. Remarks on deparametrizable systems

It should be emphasized that, unlike the classical theory, the relativistic quantum mechanics formulated in section 3.1 is not equivalent to the conventional quantum theory, even if the system is deparametrizable. In conventional quantum mechanics, the time $t$ is treated as a parameter and not quantized as an operator. Thus, the measurement of $t$ is presumed to have zero uncertainty ($\Delta t = 0$). In relativistic quantum mechanics, by contrast, $t$ is on the same footing as other observables $q_i$ and the measurement of $t$ will yield nonzero $\Delta t$.

\(^{10}\)These problems already exist in the orthodox formulation of conventional quantum mechanics. A lot of research studies have been devoted to these issues and vigorous debates remain unsettled. Among them, see [26] for the simultaneous measurement of a pair of conjugate observables and [27] for the ‘joint measurement problem’. We hope these issues would gain more insight in light of relativistic quantum mechanics.
If a system is deparametrizable and particularly \( H_0 \) in (2.9) is not explicitly dependent on \( t \), then we have \( \hat{H} = \hat{p}_1 + \hat{H}_0(q^i, \hat{p}_1) \) and the projector \( \hat{P} \) can be cast as

\[
\hat{P} := \int dt \, e^{-i\hat{H}t} = \int dt \, dp_1 \, dE \, e^{-i\hat{H}t} \langle p_1, E \rangle \langle p_1, E \rangle
\]

\[
= \int dt \, dp_1 \, dE \, e^{-i(p_1 + E)t} \langle p_1, E \rangle \langle p_1, E \rangle
\]

\[
\propto \int dp_1 \, dE \, \delta (p_1 + E) \langle p_1, E \rangle \langle p_1, E \rangle = \int dE \langle -E, E \rangle \langle -E, E \rangle,
\]

(3.11)

where \( \langle p_1, E \rangle \equiv | p_1 \rangle \otimes | E \rangle \) are the simultaneous eigenstates of \( \hat{p}_1 \) and \( \hat{H}_0 \) with eigenvalues \( p_1 \) and \( E \) (note that \( \hat{p}_1 \) and \( \hat{H}_0 \) commute). Consequently, the transition amplitude for relativistic quantum mechanics is given by

\[
W(q^a, q^b) \equiv W(t, q^i, t', q^i) := \langle q^b | \hat{P} | q^a \rangle
\]

\[
\propto \int dE (t, q^i) \langle -E, E \rangle \langle -E, E \rangle = \int dE \, e^{-iE(t-t')} f_E (q^i) f_E (q^a),
\]

(3.12)

where \( f_E (q^i) := \langle q^i | E \rangle \) is the eigenfunction of \( \hat{H}_0 \) in \( q^i \)-representation. On the other hand, the transition amplitude for conventional quantum mechanics is given by

\[
G(q^i, t; q^i, t') := \langle q^b | e^{-iH_0(t-t')} | q^a \rangle
\]

\[
= \int dE dE' \langle q^i | E \rangle \langle E' | q^i \rangle = \int dE \, e^{-iE(t-t')} f_E (q^i) f_E (q^a),
\]

(3.13)

which happens to be identical to \( W(t, q^i, t', q^i) \) in (3.12). In fact, even if \( H_0(q^i, p_i; t) \) depends on \( t \) explicitly, as long as \( \langle \hat{H}_0(t_1), \hat{H}_0(t_2) \rangle = 0 \) for all \( t_1, t_2 \), \( W(t, q^i, t', q^i) \) and \( G(t, q^i, t', q^i) \) are identical to each other (up to an irrelevant normalization factor for \( W \)) as will be proven in light of the path integral in section 4.2. A system is said to be strictly deparametrizable if \( \langle \hat{H}_0(t_1), \hat{H}_0(t_2) \rangle = 0 \). For strictly deparametrizable systems, in a sense, relativistic quantum mechanics and conventional quantum mechanics are different at the level of kinematics (observation and measurement) but identical at the level of dynamics (transition amplitudes).

In conventional quantum mechanics, given a (non-relativistic) quantum event in a small spectral \( R_0 \) measured at time \( t' \) with small time inaccuracy \( \Delta t' \), the (averaged) probability of the quantum event in a small spectral \( R_0 \) measured at time \( t \) with small time inaccuracy \( \Delta t \) can be prescribed as

\[
\mathcal{P}_{R_0, t\pm \Delta t; R_0', t'\pm \Delta t'} = \frac{1}{\Delta t \Delta t'} \int_{t-\Delta t}^{t+\Delta t} \int_{t'-\Delta t'}^{t'+\Delta t'} dt \, dE \, dE' \, G(q^i, t; q^i, t') \]

\[
= \frac{1}{m^2 \Delta x^2} \int_{R_0} d^2q \int_{R_0'} d^2q' \, G(q^i, t; q^i, t') \]

(3.14)

which is different from (3.9) with \( R = [t - \Delta t, t + \Delta t] \times R_0 \) and \( R' = [t' - \Delta t', t' + \Delta t'] \times R_0' \) even though \( W(t, q^i, t', q^i) \propto \Delta t \Delta t' \), as amplitudes are summed over time \( \Delta t' \) in (3.9), while they are summed over time \( \Delta t \) in (3.14). Only if \( \Delta t \) and \( \Delta t' \) are small enough, the interference in time can be neglected. Particularly, for a simple harmonic oscillator governed by the relativistic Hamiltonian \( H = p_i + H_0 = p_i + p_i^2 / 2m + m^2 \omega^2 / 2 \), it was shown in [5, 6] that, if \( \Delta t \ll m \Delta x^2 / \hbar \), we can ignore the temporal resolution \( \Delta t \) and idealize the measurement of \( t \) as instantaneous, and the conventional nonrelativistic quantum theory is recovered as a good approximation of the relativistic quantum mechanics.

On the other hand, for non-strictly deparametrizable systems, i.e. \( \langle \hat{H}_0(t_1), \hat{H}_0(t_2) \rangle \neq 0 \), relativistic quantum mechanics and conventional quantum mechanics are different both for

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11 For deparametrizable systems, \( \langle \hat{H}_0(q^i, \hat{p}_1, t_1), \hat{H}_0(q^i, \hat{p}_1, t_2) \rangle = 0 \) if and only if \( \langle \hat{H}_0(q^i, \hat{p}_1, t_1), \hat{H}_0(q^i, \hat{p}_1, t_2) \rangle = 0 \).
kinematics and dynamics. For many situations of interest, nevertheless, $G(t, q^1, t', q^{1'})$ can be regarded as a reasonable approximation of $W(t, q^1, t', q^{1'})$ as will be discussed in section 4.2.

3.3. Example: timeless double pendulum

Take the timeless double pendulum introduced in section 2.3 as an example. The kinematical Hilbert space is $\mathcal{K} = L^2(\mathbb{R}^2, dadb)$, and the quantum Hamiltonian equation reads

$$\hat{H}\psi(a, b) = \frac{1}{2} \left(-\hbar^2 \frac{\partial^2}{\partial a^2} - \hbar^2 \frac{\partial^2}{\partial b^2} + a^2 + b^2 - 2E\right) \psi(a, b) = 0. \quad (3.15)$$

Since $\hat{H} = \hat{H}_a + \hat{H}_b = E$, where $\hat{H}_a$ (resp. $\hat{H}_b$) is the nonrelativistic Hamiltonian for a simple harmonic oscillator in the variable $a$ (resp. $b$), this equation can be easily solved by using the basis that diagonalizes $\hat{H}_a$ and $\hat{H}_b$. Let

$$\psi_n(a) \equiv \langle a|n \rangle = \frac{1}{\sqrt{n!}} H_n(a) e^{-\alpha^2/2\hbar} \quad (3.16)$$

be the normalized $n$th eigenfunction for the harmonic oscillator with the eigenvalue $E_n = \hbar(n + 1/2)$, where $H_n(a)$ is the $n$th Hermite polynomial. Clearly, the function

$$\psi_{n_a, n_b}(a, b) := \psi_{n_a}(a)\psi_{n_b}(b) \equiv \langle a, b|n_a, n_b \rangle \quad (3.17)$$

solves (3.15) if

$$\hbar(n_a + n_b + 1) = E, \quad (3.18)$$

which implies that the quantum theory exists only if $E = \hbar(N + 1)$ with $N \in \mathbb{Z}^+ \cup \{0\}$.

Consequently, for a given $N$, the general solution of (3.15) is given by

$$\Psi(a, b) = \sum_{n=0}^{N} c_n \psi_n(a)\psi_{N-n}(b), \quad (3.19)$$

and thus the physical Hilbert space $\mathcal{H}$ is an $(N + 1)$-dimensional proper subspace of $\mathcal{K}$ spanned by an orthonormal basis $\{|n, N - n\}_{n=0,\ldots,N}$.

The projector $\hat{P} : \mathcal{S} \rightarrow \mathcal{H}$ is a true projector as $\mathcal{H}$ is a proper subspace of $\mathcal{K}$ for the case that the spectrum of $\hat{H}$ is discrete. Obviously, $\hat{P}$ is given by

$$\hat{P} = \sum_{n=0}^{N} |n, N - n\rangle \langle n, N - n|, \quad (3.20)$$

which can be obtained (up to an irrelevant overall factor) from (3.1):

$$\int_0^{2\pi/h} d\tau e^{-i\tau\hat{H}} \propto \frac{1}{2\pi} \int_0^{2\pi} d\tau \sum_{n_a, n_b} |n_a, n_b\rangle e^{-i\tau(n_a + n_b + 1 - E)} \langle n_a, n_b| \quad (3.21)$$

Here, the integration range is so chosen because $\exp(-i\tau\hat{H})$ is periodic in $\tau$ with period $2\pi/h$ if $E = \hbar(N + 1)$.

The transition amplitudes are given by

$$W(a, b, a', b') := \langle a, b|\hat{P}|a', b'\rangle = \sum_{n=0}^{N} \langle a, b|n, N - n\rangle \langle n, N - n|a', b'\rangle$$

$$= \sum_{n=0}^{N} e^{-a^2 + b^2 + a'^2 + b'^2}/(2\hbar \cdot n!(N - n)!} H_n(a)H_{N-n}(b)H_n(a')H_{N-n}(b'), \quad (3.22)$$
which is the probability density of measuring \((a, b)\), given \((a', b')\) measured at another instance. Furthermore, the probability of the quantum event \((n_a, n_b)\) given the quantum event \((n'_a, n'_b)\) is

\[
W(\psi_{n_a, n_b}, \psi_{n'_a, n'_b}) := \langle n_a, n_b|\hat{P}|n'_a, n'_b\rangle = \delta_{n_a, n'_a+2n_b} \delta_{n_b, n'_b}.
\] (3.23)

### 3.4. More on the physical Hilbert space

The operator \(\hat{P} : \mathcal{S} \rightarrow \mathcal{S}'\) maps an arbitrary element of \(\mathcal{S}\) to its dual space \(\mathcal{S}'\). If zero is in the continuous spectrum of \(\hat{H}\), then \(\hat{P}\) maps \(\mathcal{S}\) to a larger space \(\mathcal{S}'\) and thus is not really a projector. In this case, the physical state \((\Psi_{\psi_\alpha}|\) mapped from \(|\psi_\alpha\rangle\) is a tempered distribution. \(\hat{P}\) becomes a true projector only if zero is a part of the discrete spectrum of \(\hat{H}\) such as in the timeless double pendulum.

The construction in (3.1) is a special case for the group averaging procedure [28, 29], the idea of which is to averaging over all states along the gauge flow (generated by the constraint operator) to yield the physical solution which satisfies the constraint equation. In the special case, let \(|E\rangle\) be the eigenstate of \(\hat{H}\) with eigenvalue \(E\); then schematically we have

\[
\hat{H}|\psi_\alpha\rangle = \int d\tau \hat{H} e^{-ir\hat{H}}|\psi_\alpha\rangle = \int d\tau \int dE \hat{H} e^{-ir\hat{H}}|E\rangle\langle E|\psi_\alpha\rangle
\]

\[
= \int d\tau \int dE e^{-irE}\langle\langle E|\psi_\alpha\rangle \propto \int dE \delta(E)\langle\langle E|\psi_\alpha\rangle = 0,
\] (3.24)

thus showing that \(\hat{P}\) maps an arbitrary kinematical state \(|\psi_\alpha\rangle\) to a physical state which satisfies the constraint equation (3.3). Furthermore, it can be easily shown that \((\Psi_{\psi_\alpha}|\psi_\beta\rangle = (\Psi_{\psi_\beta}|\psi_\alpha\rangle\) if \((\Psi_{\psi_\alpha}| = (\Psi_{\psi_\beta}|\), and therefore the physical inner product in (3.5) is well defined.

If there are multiple constraints, then we have to solve the multiple constraint equations simultaneously:

\[
\hat{H}_i |\psi_\alpha\rangle = 0, \quad \text{for} \; i = 1, \ldots, k.
\] (3.25)

In the simplest case that \([\hat{H}_i, \hat{H}_j] = 0\) for all \(i, j\), the projector can be easily constructed via

\[
\hat{P} = \int d\tau_1 \cdots \int d\tau_k e^{-ir\hat{H}}
\] (3.26)

as a direct extension of (3.1). In general, however, \(\hat{H}_i\) do not commute, as classically the Poisson brackets \([\hat{H}_i, \hat{H}_j]\) vanish only weakly (see (2.6) and (2.8)).

In the case that \(\hat{H}_i\) do not commute but form a closed Lie algebra, i.e.,

\[
[\hat{H}_i, \hat{H}_j] = f^{ij}_{\;k} \hat{H}_k
\] (3.27)

with \(f^{ij}_{\;k}\) being constants, the exponentials of \(\hat{H}_i\) form a Lie group \(G\) and the physical state can be obtained by group averaging:

\[
|\Psi_\psi\rangle = \int_G d\mu(\hat{U}) \hat{U}|\psi_\alpha\rangle,
\] (3.28)

where \(d\mu\) is the Haar measure. It follows

\[
\hat{U}^*|\Psi_\psi\rangle = \int_G d\mu(\hat{U}) \hat{U}^* \hat{U}|\psi_\alpha\rangle = \int_G d\mu(\hat{U}^{-1}) \hat{U}^* \hat{U}^\prime|\psi_\alpha\rangle
\]

\[
= \int_G d\mu(\hat{U}^\prime) \hat{U}^\prime|\psi_\alpha\rangle = |\Psi_\psi\rangle
\] (3.29)

for any \(\hat{U}^\prime \in G\). The fact that \(|\Psi_\psi\rangle\) is invariant under any \(\hat{U}^\prime \in G\) implies that it is annihilated by the generators of \(G\), namely, \(\hat{H}_i|\Psi_{\psi}\rangle = 0\). Furthermore, the physical inner product in (3.5)
is again well defined. (See [29] for more details and subtleties.) The averaging in (3.26) is
indeed a special case of (3.28).

Generically, $f^{ij}$ are functions of $q^a$ and $p_a$ in (2.8), and, correspondingly, $\hat{H}$ do not form
a closed Lie algebra in the kinematical space $K$. In this case, it is much more difficult to obtain
the physical solutions and construct the quantum theory which is free of quantum anomalies
(see [30] for the issues of anomalies).

In section 4, we will focus only on the case with a single Hamiltonian constraint, as
our original motivation is to derive the timeless path integral in the absence of time. The
trivial cases with multiple constraints are presented in section 4.4, while nontrivial cases are
discussed in section 5.

3.5. Remarks on measurements and collapse

Imagine that a quantum system is measured by Alice and Bob at two different instances,
yielding two outcomes corresponding to $|\psi_\alpha\rangle$ and $|\psi_\beta\rangle$, respectively. From the viewpoint of
Alice, the physical state is collapsed to $|\Psi_\alpha\rangle$ by her measurement and Bob’s measurement
affirms her prediction. Bob, on the other hand, regards the physical state to be collapsed to
$|\Psi_\beta\rangle$ by his measurement and predicts what Alice can measure. The striking puzzle arises:
Who, Alice or Bob, causes the physical state to collapse in the first place?

In the timeless framework, it turns out to be an invalid question to ask who collapses
the physical state first, since we cannot make any sense of time. The seemingly puzzle is
analogous to the Einstein–Podolsky–Rosen (EPR) paradox, in which a pair of entangled
particles are measured separately by Alice and Bob. In the context of special relativity, if
the two measurements are conducted at two spacetime events which are spacelike separated,
the time ordering of the two events can flip under a Lorentz boost and thus has no physical
significance. Alice and Bob can both claim that the entangled state is collapsed by her/his
measurement and thus have different knowledge about what the physical state should be, yet
the predictions by Alice and Bob are consistent with each other. In our case, the measurement
at an instance is analogous to the measurement on a single particle of the EPR pair; the
kinematical state is analogous to the (local) state of a single particle; and the physical state
is analogous to the (global) entangled state of the EPR pair. A complete knowledge (usually
from measurement) about the local state will collapse the global state at once through the
entanglement, which is analogous to the dynamics (or say, transition amplitudes) in our case.
Consistency also holds in our case as $\langle \Psi_\beta | \psi_\alpha \rangle = \langle \Psi_\alpha | \psi_\beta \rangle$. (See [31] for more on the EPR
paradox in the relational interpretation of quantum mechanics and also section 5.6 of [9] for
more on the philosophical issues.)

If the system is deparametrizable, then one can make sense of time ordering with respect
to the preferred time variable, say, Alice performs her measurement before Bob. For Alice, the
physical state is collapsed by her measurement and then she can predict Bob’s measurement.
What happens from the viewpoint of Bob? It turns out that, for Bob, the physical state is
collapsed by his measurement and he can retrodict Alice’s measurement. Alice’s prediction
and Bob’s retrodiction again are consistent with each other. (See [6] for more on consistency
of prediction and retrodiction.) Analogously, in the EPR paradox, even if two measurements
are conducted at two causally related events (and thus the time ordering of the two events
cannot be flipped), Alice’s prediction and Bob’s retrodiction lead to no inconsistency12.

12 At first thought, it seems doubtful that (conventional) quantum mechanics can be used for retrodiction as well as for
prediction, since this would imply that collapse by a measurement at present can affect states not only in the future but
also in the past. It turns out this commits no violation against causality and in fact is exactly what happens in Wheeler’s
delayed-choice Gedanken experiment [32], which has been confirmed by several experimental implementations [33].
The perplexity is closely related to the ‘consistent histories’ interpretation of quantum mechanics [34].
As a side remark, exploiting further the close analogy between the EPR pair and the timeless formalism of relativistic quantum mechanics, one might be able to conceive an analogue of Bell’s inequality, which would help to elaborate on the interpretational and conceptual issues of relativistic quantum mechanics at the level of thought experiments.

4. Timeless path integral

The canonical formalism for relativistic quantum mechanics is described in section 3. All information about quantum dynamics is encoded by the transition amplitudes (3.2). In particular, by choosing \(|s| = |q^0|\) and \(|s'| = |q^N|\), all physics can be obtained from the following transition amplitudes:

\[
W(q^a, q'^a) = \langle q^a|\hat{P}|q'^a\rangle \sim \int d\tau \langle q^a|e^{-i\hat{H}\tau}|q'^a\rangle.
\] (4.1)

From now on, we will use the notation \(\sim\) to denote the equality up to an overall constant factor which has no physical significance, as any overall constant is canceled out in the numerator and denominator in (3.8).

As a special case of group averaging, the integration range of \(\tau\) is taken to be a compact interval if \(\exp(-i\tau\hat{H})\) forms a compact Lie group \(U(1)\) (timeless double pendulum is an example) and it is taken to be \((-\infty, \infty)\) if the group of \(\exp(-i\tau\hat{H})\) is noncompact. For the case that \(\exp(-i\tau\hat{H})\) gives \(U(1)\), we can unwrap \(U(1)\) to its covering space \(\mathbb{R}\) and correspondingly integrate \(\tau\) over \((-\infty, \infty)\). The unwrapping only gives rise to an overall multiplicative factor (which is divergent if not properly regulated). Therefore, in any case, up to an irrelevant overall factor, transition amplitudes can be computed by

\[
W(q^a, q'^a) \sim \int_{-\infty}^{\infty} d\tau \langle q^a|e^{-i\hat{H}\tau}|q'^a\rangle, \tag{4.2}
\]

where \(\langle q^a|e^{-i\hat{H}\tau}|q'^a\rangle\) can be thought as the transition amplitude for a kinematical state \(|q^a\rangle\) to ‘evolve’ to the state \(|q'^a\rangle\) by the ‘parameter time’ \(\tau\). Equation (4.2) sums over \(\langle q^a|e^{-i\hat{H}\tau}|q'^a\rangle\) for all possible values of \(\tau\), suggesting that \(W(q^a, q'^a)\) is intrinsically timeless as the parameter time \(\tau\) has no essential physical significance.

Rigorously, the integration should be regularized via

\[
W(q^a, q'^a) \sim \lim_{M \to \infty} \frac{\int_{-M}^{M} d\tau \langle q^a|e^{-i\hat{H}\tau}|q'^a\rangle}{\int_{-M}^{M} d\tau}, \tag{4.3}
\]

as a cut-off \(M\) is introduced to regulate the integral and the irrelevant overall factor to be finite. As we will see, the variable \(\tau\) corresponds to the parametrization of curves in the path integral and integrating over all \(\tau\) indicates that the parametrization of curves has no physical significance. The above regularization scheme is physically well justified, as it cuts off the curves in the path integral which are ‘too wild’ (noncompact curves), given the endpoints \(q^N\) and \(q'^N\) fixed.

In the following, starting from (4.2), we will first derive the timeless path integral for the case of a single Hamiltonian constraint and then investigate it in more detail. In the end, we will study the path integral with multiple relativistic constraints which commute mutually.

4.1. General structure

For a given \(\tau\), let us introduce a parametrization sequence: \(\tau_0 = 0, \tau_1, \tau_2, \ldots, \tau_{N-1}, \tau_N = \tau\) with \(\tau_n \in \mathbb{R}\), and define \(\Delta \tau_n := \tau_n - \tau_{n-1}\). The conditions on the endpoints (\(\tau_0 = 0\) and \(\tau_N = \tau\)) correspond to \(\sum_{n=1}^{N} \Delta \tau_n = \tau\). The mesh of the parameter sequence is defined to be
max_{\tau=0,1,\ldots,n} \{\|\Delta\tau_n\|}. The parameter sequence is said to be fine enough if its mesh is smaller than a given small number \(\epsilon\).\(^{13}\)

As \(\tau\) is fixed now, identifying \(q^a = q^a_N\) and \(q^\mu = q^\mu_0\), and using \(\sum_{n=1}^N \Delta\tau_n = \tau\), we can rewrite \(\langle q^a_1|e^{-i\hat{H}\tau}\hat{H}(q^\mu)\rangle\) as

\[
\langle q^a_1|e^{-i\hat{H}\tau}\hat{H}(q^\mu)\rangle = \left(\prod_{n=1}^{N-1} \int dq^a_n \langle q^a_n|e^{-i\Delta\tau_n\hat{H}}\langle q^\mu_{n-1}|e^{-i\Delta\tau_n\hat{H}}|q^\mu_0\rangle\langle q^\mu_{n-1}|e^{-i\Delta\tau_n\hat{H}}|q^\mu_{n-2}\rangle \cdots \langle q^\mu_1|e^{-i\Delta\tau_1\hat{H}}|q^\mu_0\rangle\right)\langle q^a_1|e^{-i\Delta\tau_N\hat{H}}|q^a_0\rangle.
\]  

(4.4)

where we have inserted \(N - 1\) times the completeness relation

\[
\int dq \langle q|q\rangle := \int dq_1 \ldots dq_d |q_1,\ldots,q_d\rangle \langle q_1,\ldots,q_d|.
\]  

(4.5)

For a given arbitrary small number \(\epsilon\), by increasing \(N\), we can always make the parameter sequence fine enough such that mesh[\(\tau_n\] \(\leqslant |\tau|/N \leqslant \epsilon\).\(^{14}\) Consequently, we can approximate each \(\langle q^a_1|e^{-i\Delta\tau_1\hat{H}}|q^a_0\rangle\) to the first order in \(\epsilon\) as

\[
\langle q^a_1|e^{-i\Delta\tau_1\hat{H}}|q^a_0\rangle = \langle q^a_1|1 - i\Delta\tau_1\hat{H}(\hat{q}^a, \hat{\rho}_a)|q^a_0\rangle + O(\epsilon^2).
\]  

(4.6)

For the generic case that the Hamiltonian operator \(\hat{H}\) is a polynomial of \(\hat{q}^a\) and \(\hat{\rho}_a\) is a Weyl ordered, with the use of the completeness relation for the momenta

\[
\int \frac{d^d p}{(2\pi \hbar)^d} |p_a\rangle \langle p_a| := \int \frac{dp_1 \cdots dp_d}{(2\pi \hbar)^d} |p_1,\ldots,p_d\rangle \langle p_1,\ldots,p_d|,\]

(4.7)

it can be shown that

\[
\langle q^a|\hat{H}(\hat{q}^a, \hat{\rho}_a)|q^\mu\rangle = \int \frac{d^d p}{(2\pi \hbar)^d} \exp \left[\frac{1}{\hbar} p_a(q^a - q^\mu)\right] \hat{H} \left(\frac{q^a + q^\mu}{2}, p_a\right).
\]  

(4.8)

(See exercise 11.2 in [35] for the proof.) Applying (4.8) to (4.6), we have

\[
\langle q^a_{n+1}|e^{-i\Delta\tau_{n+1}\hat{H}}|q^a_n\rangle = \int \frac{d^d p_n}{(2\pi \hbar)^d} \exp \left[p_n(q^a_{n+1} - q^a_n)/\hbar\right] \left[1 - i\Delta\tau_{n+1}\hat{H} \left(\frac{q^a_{n+1} + q^a_n}{2}, p_n\right)\right] + O(\epsilon^2)
\]

\[
= \int \frac{d^d p_n}{(2\pi \hbar)^d} \exp \left[p_n(q^a_{n+1} - q^a_n)/\hbar\right] e^{-i\Delta\tau_{n+1}H(\bar{q}, \bar{\rho})} + O(\epsilon^2),
\]  

(4.9)

where we define \(\bar{q}^a_n := (q^a_{n+1} + q^a_n)/2\) and \(\Delta q^a_n := q^a_{n+1} - q^a_n\).

\(^{13}\) Let \(X\) be a topological space and \(s \in [0, 1]\). A continuous map \(\gamma : s \mapsto \gamma(s) \in X\) is called a path with an initial point \(\gamma(0) = x_0\) and an end point \(\gamma(1) = x_1\). The image of \(\gamma\) is called a curve, which can be reparametrized with respect to a new variable \(\tau\) as \(\gamma : \tau \mapsto \gamma(\tau)\) by introducing an arbitrary continuous function \(\tau : s \mapsto \tau(s) \in \mathbb{R}\).

The parametrization sequence \(\tau_0 = 0, \tau_1, \tau_2, \ldots, \tau_N = \tau\) can be viewed as a discrete approximation for the reparametrization function \(\tau(s)\) with \(\tau(s = 0) = 0\) and \(\tau(s = 1) = \tau\) if we identify \(\tau_n = \tau(n/N)\). For the case that \(\tau(s)\) is injective, the parametrization sequence is ordered (i.e., \(0 = \tau_0 < \tau_1 < \cdots < \tau_N = \tau\) and \(\Delta\tau_n > 0\) if \(\tau > 0\)) and called a partition of the interval \([0, \tau]\), which is used to define the Riemann integral as the continuous limit: \(\int_0^\tau f(\tau')d\tau' = \lim_{\text{mesh} \to 0} \sum_{n=0}^{N-1} f(\tau_n)\Delta\tau_n\). In the timeless formulation of relativistic mechanics, a dynamical solution is an unparametrized curve in \(\Omega\) and its parametrization has no physical significance. In order to exploit the timeless feature, we should keep the parametrizations generic and not restrict ourselves to injective ones. Correspondingly, the partition is generalized to a parametrization sequence and the Riemann integral is generalized to the Riemann–Stieltjes integral as \(\int_0^\tau f(\tau')d\tau' = \lim_{\text{mesh} \to 0} \sum_{n=0}^{N-1} f(\tau_n)\Delta\tau_n\), which is well defined even if \(\tau(s)\) is not injective.

\(^{14}\) More rigorously, for a given \(\epsilon\), the large number \(N\) should be chosen to satisfy mesh[\(\tau_n\] \(\leqslant |\tau|/N \leqslant M/N \leqslant \epsilon\), where \(M\) is the cut-off regulator defined in (4.3), so that the \(O(\epsilon^2)\) term in (4.9) can be dropped for any value of \(|\tau|\). In the end, we have to integrate (4.10) over all possible values of \(\tau\) to obtain \(W(q^a, q^\mu)\), and the regularization is essential to keep the \(O(\epsilon^2)\) terms under control for the arbitrary values of \(\tau\).
Making the parametrization sequence finer and finer (by decreasing \( \epsilon \) or equivalently by increasing \( N \)) and at the end going to the limit \( \epsilon \to 0 \) or \( N \to \infty \), we can cast (4.4) as

\[
(q^\mu |e^{-i\hat{H}q^\mu_0}) = \lim_{N\to\infty} \left( \prod_{n=1}^{N-1} \int d^dp_n \int_0^\infty \frac{d^f p_n}{(2\pi \hbar)^d} \right) \exp \left( -i \sum_{n=0}^{N-1} p_{na} \Delta q_n^a \right) \times \exp \left( \frac{N-1}{\hbar} \sum_{n=0}^{N-1} \Delta \tau_{n+1} H(\bar{q}_n^a, p_{na}) \right)
\]

In the limit \( N \to \infty \), the points \( q_n \) and \( p_n \) can be viewed as the sampled points of a continuous curve in \( \Omega = T^* C \) given by \( \bar{y}(\tau') = (q^\mu(\tau'), p_n(\tau')) \), which is parametrized by \( \tau' \) and with the endpoints projected to \( C \) fixed by \( q^\mu(\tau' = 0) = q^\mu_0 \) and \( q^\mu(\tau' = \tau) = q^\mu \). That is, \( q_n \) and \( p_n \) are the sampled points of \( \bar{y} \) as \( q_n = q^\mu(\tau_n) \) and \( p_{na} = p_n(\tau_n) \). In the treatment of the functional integral, it is customary to introduce the special notations for path integrals:

\[
\prod_{n=1}^{N-1} \int d^dp_n \to \int Dp_n
\]

Meanwhile, in the continuous limit \( (N \to \infty) \), the finite sums appearing in the exponents in (4.10) also converge to the integrals:

\[
i \sum_{n=0}^{N-1} p_{na} \Delta q_n^a \to \int \frac{d^f p_n}{(2\pi \hbar)^d} = \int Dp_n
\]

\[
-i \sum_{n=0}^{N-1} \Delta \tau_{n+1} H(\bar{q}_n^a, p_{na}) \to -i \int \bar{y} H(q^\mu(\tau'), p_n(\tau')) d\tau'.
\]

Note that the continuous limit above is defined via the Riemann–Stieltjes integral as an extension of the Riemann integral (see footnote 15). With the new notations, (4.10) can be written in a concise form

\[
(q^\mu |e^{-i\hat{H}q^\mu_0}) = \int Dq^\mu \int Dp_n \exp \left( \frac{i}{\hbar} \int \bar{y} p_n d\tau' \right) \exp \left( -i \int \bar{y} H(q^\mu(\tau'), p_n(\tau')) d\tau' \right).
\]

It is remarkable to note that (up to the factor \( i/\hbar \)) the continuous limit in (4.12) is simply the line integral of the 1-form \( \hat{\theta} = p_n d\tau' \) over the curve \( \bar{y} \), identical to (2.1), and is independent of the parametrization of \( \tau \). On the other hand, the integral in (4.13) depends on the parametrization of \( \tau \). Thus, to compute \( W(q^\mu, q^\mu_0) \) in (4.2), the integration over \( \tau \) only hits the second exponential in (4.14) and the first exponential simply factors out. The integration of the second exponential over \( \tau \) yields

\[
\int_{-\infty}^{\infty} d\tau' \exp \left( -i \int \bar{y} H(q^\mu(\tau'), p_n(\tau')) d\tau' \right) = \int_{-\infty}^{\infty} d\tau' \exp \left( -i \int \bar{y} H(q^\mu(\bar{\tau}), p_n(\bar{\tau})) d\bar{\tau} \right) = \delta \left( \int \bar{y} H(q^\mu(\bar{\tau}), p_n(\bar{\tau})) d\bar{\tau} \right)
\]

(4.15)
where we have rescaled the parametrization $\tau'$ to $\tilde{\tau} = \tau'/\tau$ so that the endpoints now read $q^{\alpha} = q^{\alpha}(\tilde{\tau} = 0)$ and $q^{a} = q^{a}(\tilde{\tau} = 1)$\textsuperscript{15}. The appearance of the Dirac delta function indicates that only the paths which satisfy $\int_{\tilde{\tau}_{\mathrm{f}}}^{\tilde{\tau}_{\mathrm{i}}} H(\tilde{\tau}) \, d\tilde{\tau} = 0$ will contribute to the path integral for $W(q^{a}, q^{w})$. The condition $\int_{\tilde{\tau}_{\mathrm{f}}}^{\tilde{\tau}_{\mathrm{i}}} H(\tilde{\tau}) \, d\tilde{\tau} = 0$ is, however, still not geometrical, since $\tilde{\tau}$ can be further reparametrized to $\tilde{\tau} = \tilde{\tau}'(\tilde{\tau})$ to yield $\int_{\tilde{\tau}'_{\mathrm{f}}}^{\tilde{\tau}'_{\mathrm{i}}} H(\tilde{\tau}') \, d\tilde{\tau}' \neq 0$ even with the initial and final values fixed, i.e., $\tilde{\tau}'(\tilde{\tau} = 0) = 0$ and $\tilde{\tau}'(\tilde{\tau} = 1) = 1$. On the other hand, $W(q^{a}, q^{w})$ cast in (4.2) has no dependence on the parametrization whatsoever, which implies that, in the continuous limit, the contribution of a path $\tilde{\gamma}$ satisfying the condition $\int_{\tilde{\tau}_{\mathrm{f}}}^{\tilde{\tau}_{\mathrm{i}}} H(\tilde{\tau}) \, d\tilde{\tau} = 0$ for a specific (rescaled) parametrization $\tilde{\tau}$ is somehow exactly canceled by that of another path satisfying the same condition. In the end, only the paths restricted to the constraint surface (i.e., $\tilde{\gamma} \in \Sigma$, or equivalently $H(\tilde{\tau}') = 0$ for all $\tilde{\tau}'$ along the path) contribute to the path integral for $W(q^{a}, q^{w})$. The constraint $\tilde{\gamma} \in \Sigma$ is now geometrical.

How the aforementioned cancellation takes place is obscure. To elucidate this point, we exploit the fact that $W(q^{a}, q^{w})$ is independent of the parametrization and play the trick by averaging over all possible parametrizations. That is, up to an overall factor of no physical significance, we can recast $W(q^{a}, q^{w})$ by summing over different parametrizations as follows:

\[
W(q^{a}, q^{w}) \sim \int d\tau \int [D\Delta \tau]_{\sum_{n=1}^{N} \Delta \tau_{n} = \tau} \langle q^{a} | e^{i\tau\hat{H}} | q^{w} \rangle
\]

\[
\sim \int d\tau \int [D\Delta \tau]_{\sum_{n=1}^{N} \Delta \tau_{n} = \tau} \int Dq^{a} \int Dp_{a} \exp \left( \frac{i}{\hbar} \int_{\tilde{\tau}_{\mathrm{i}}}^{\tilde{\tau}_{\mathrm{f}}} p_{a} \, dq^{a} \right)
\times \exp \left( -i \sum_{n=0}^{N-1} \Delta \tau_{n+1} H(q^{a}_{n}, p_{a}) \right),
\]

where the notation $\int [D\Delta \tau]_{\sum_{n=1}^{N} \Delta \tau_{n} = \tau}$ is a shorthand for

\[
\int_{-\tau/N}^{\tau/N} d\Delta \tau_{1} \int_{-\tau/N}^{\tau/N} d\Delta \tau_{2} \cdots \int_{-\tau/N}^{\tau/N} d\Delta \tau_{N} \rightarrow \int [D\Delta \tau]_{\sum_{n=1}^{N} \Delta \tau_{n} = \tau},
\]

which sums over all fine enough (namely, mesh[$\tau_i$] $\leqslant |\tau|/N$) parametrization sequences for a given $\tau$. It is easy to show that

\[
\int_{-\infty}^{\infty} d\tau \int_{-\tau/N}^{\tau/N} d\Delta \tau_{1} \int_{-\tau/N}^{\tau/N} d\Delta \tau_{2} \cdots \int_{-\tau/N}^{\tau/N} d\Delta \tau_{N} = \prod_{n=0}^{N-1} \int_{-\infty}^{\infty} d\Delta \tau_{n+1},
\]

when the cut-off regulator $M$ is removed (also see footnote 17). Consequently, for a given arbitrary parametrization $\tau'$, renaming the varying $\Delta \tau_{n}$ as $\Delta \tau_{n} = \hbar^{-1} N_{n} \Delta \tau_{n}'$, we can rewrite (4.16) as

\[
W(q^{a}, q^{w}) \sim \int Dq^{a} \int Dp_{a} \int D\eta \exp \left( \frac{i}{\hbar} \int p_{a} \, dq^{a} \right) \exp \left( -i \sum_{n=0}^{N-1} \Delta \tau_{n+1} N_{n+1} H(q^{a}_{n}, p_{a}) \right),
\]

where we introduce the notation

\[
\prod_{n=0}^{N-1} \int_{-\infty}^{\infty} dN_{n+1} \rightarrow \int D\eta.
\]

\textsuperscript{15} In the expression of (4.15), we have removed the cut-off regulator (i.e., the limit $M \rightarrow \infty$ has been taken). More rigorously, we have removed the regulator before the limit $N \rightarrow \infty$ is taken. The Dirac delta function in (4.15) would have been a nascent delta function if the regulator had not been removed.
Again, in the continuous limit, the finite sum converges to the Riemann–Stieltjes integral:

$$-\frac{i}{\hbar} \sum_{n=0}^{N-1} \Delta \tau_{n+1} N_{n+1} H(\tilde{q}_n, p_{n\alpha}) \rightarrow -\frac{i}{\hbar} \int N(\tau') H(q'(\tau'), p_{\alpha}(\tau')) \, d\tau', \quad (4.21)$$

and (4.19) can be neatly written as the path integral

$$W(q^a, q^{a'}) \sim \int Dq^a \int Dp_{\alpha} \int DN \exp \left[ \frac{i}{\hbar} \left( \int_{\tilde{\gamma}} p_{\alpha} \, dq^a - \int_{\tilde{\gamma}} N(\tau') H \, d\tau' \right) \right] \equiv \int Dq^a \int Dp_{\alpha} \int DN \exp \left[ \frac{i}{\hbar} \int_{\tilde{\gamma}} \left( p_{\alpha} \frac{dq^a}{d\tau'} - N(\tau') H \right) \, d\tau' \right]. \quad (4.22)$$

Integration over \( N \) can be carried out to obtain the delta functional

$$\int DN \exp \left( \frac{i}{\hbar} \int N(\tau') H \, d\tau' \right) \sim \delta[H] \equiv \lim_{N \to \infty} \prod_{n=0}^{N-1} \delta(H(\tilde{q}_n, p_{n\alpha})), \quad (4.23)$$

and thus the path integral (4.22) can be written in an alternative form as

$$W(q^a, q^{a'}) \sim \int Dq^a \int Dp_{\alpha} \delta[H] \exp \left[ \frac{i}{\hbar} \int_{\tilde{\gamma}} p_{\alpha} \, dq^a \right], \quad (4.24)$$

where the insertion of the delta functional \( \delta[H] \) confines the path to be in the constraint surface (i.e. \( \tilde{\gamma} \in \Sigma \)). Note that the phase in the exponent in (4.24) is identical to the classical action defined in (2.1) (divided by \( \hbar \)) and that in (4.22) is identical to the classical action in (2.3) with \( k = 1 \). Therefore, each path in \( \Sigma \) contributes with a phase, which is the classical action divided by \( \hbar \).

The path integral formalism is intuitively appealing. It gives us an intuitive picture about the transition amplitudes: \( W(q^a, q^{a'}) \) is described as the sum, with the weight \( \exp(iS/\hbar) \) (where \( S \) is the classical action of \( \tilde{\gamma} \)), over all arbitrary paths \( \tilde{\gamma} \) which are restricted to \( \Sigma \) and whose projection \( \gamma \) to \( C \) connects \( q^a \) and \( q^{a'} \). None of \( q^a \) is restricted to be monotonic along the paths, and in this sense the formulation is called the timeless path integral. The parameterization for the paths has no physical significance as can be seen in the expression of (4.24), which is completely geometrical and independent of parametrizations. On the other hand, the continuum notation of (4.22) is really a schematic for the discretized version

$$W(q^a, q^{a'}) \sim \lim_{N \to \infty} \prod_{n=0}^{N-1} \int dq_n^a \prod_{n=0}^{N-1} \int \frac{dp_{n\alpha}}{(2\pi\hbar)^{d}} \prod_{n=0}^{N-1} \int dN_{n+1}$$

$$\times \exp \left( -\frac{i}{\hbar} \sum_{n=0}^{N-1} \Delta \tau_{n+1} N_{n+1} H(\tilde{q}_n, p_{n\alpha}) \right) \quad (4.25a)$$

$$\sim \lim_{N \to \infty} \prod_{n=0}^{N-1} \int dq_n^a \prod_{n=0}^{N-1} \int \frac{dp_{n\alpha}}{(2\pi\hbar)^{d}} \prod_{n=0}^{N-1} \int dN_{n+1} \exp \left( -\frac{i}{\hbar} \sum_{n=0}^{N-1} N_{n+1} H(\tilde{q}_n, p_{n\alpha}) \right), \quad (4.25b)$$

where \( \Delta \tau^a_n \) in (4.25a) is absorbed to \( N_n \) in (4.25b) and this only results in an irrelevant overall factor. The expression of (4.25b) is explicitly independent of parametrizations.16

The contributing paths in the path integral can be very “wild”—not necessarily smooth or even continuous. This calls into question whether the path integral can achieve convergence.

16 Perhaps, more appropriately, the ‘timeless path integral’ should be renamed ‘timeless ‘curve’ integral’, as in the rigorous terminology, a curve is defined as the unparametrized image of a path, which is specified by a parameter. However, we keep the name of ‘path integral’ to conform to the conventional nomenclature.
We do not attempt to present a rigorous derivation here but refer to [36] for the legitimacy issues and subtleties of the path integral.

Each path in $\Sigma$ contributes with a different phase, and the contributions from the paths essentially cancel one another through destructive interference until we come near the stationary solution. As a result, most contributions come from the paths close to the stationary solution. The stationary solution can be obtained by taking the functional variations on (4.21) with respect to $N$, $p_n$ and $q^a$, which yield the classical Hamiltonian constraint (2.2) and the Hamilton equations (2.4); that is, the stationary solution coincides with the classical solution. Provided that the action for the classical solution is much greater than $\hbar$, i.e. $S[\gamma] \gg \hbar$, the stationary phase approximation (see appendix A) yields

$$W(q^a, q'^a) \approx \sum_i \xi_i e^{i S[\gamma_i]}, \quad (4.26)$$

where $\gamma_i$ are the classical solutions which connect $q^a$ and $q'^a$ and $\xi_i$ are the weight factors for each classical solution $\gamma_i$, which are proportional to the decoherence width of phases of the nearby trajectories around $\gamma_i$. By expanding the paths around the classical solutions, various semiclassical approximation methods in (conventional) path integral approaches can be easily carried over to the timeless path integral.

### 4.2. Deparametrizable systems as a special case

If the Hamiltonian happens to be deparametrizable, then the classical Hamiltonian is in the form of (2.9), and the path integral (4.24) reads

$$W(q^a, q'^a) \sim \int Dq \int d\tau \int Dp_\tau \delta[p_\tau + H_0] \exp \left[ \frac{i}{\hbar} \int \left( \frac{d\gamma}{dt} - H_0 \right) dt \right] \left( \prod_{n=1}^{N-1} \frac{dq_n}{\Delta t_n} \right) \left( \prod_{n=0}^{N-1} \frac{dq_n}{(2\pi \hbar)^{d-1}} \right) \left( \prod_{n=0}^{N-1} \Delta t_n \right). \quad (4.27)$$

If the system is strictly deparametrizable, i.e., $[\hat{H}_0(t_1), \hat{H}_0(t_2)] = 0$, then the transition amplitude for conventional quantum mechanics, denoted as $G(q', t; q, t')$, is given by (B.6). Note that, for given arbitrary $t_1, t_2, \ldots, t_N$, the integrand for $\int D\tau$ in (4.27) is formally identical to $G(q', t; q', t')$ given in (B.6) (with $t_n$ replaced by $t_n$), and thus we have

$$W(q^a, q'^a) \sim \int D\tau G(q', t; q', t') \sim G(q', t; q', t'), \quad (4.28)$$

where $\int D\tau$ simply factors out as an irrelevant overall factor. This implies that $W(t, q', t'; q^a)$ and $G(t, q'; t', q'^a)$ are identical to each other (up to an irrelevant normalization factor for $W$) for strictly deparametrizable systems as commented in section 3.2.

For non-strictly deparametrizable systems, i.e., $[\hat{H}_0(t_1), \hat{H}_0(t_2)] \neq 0$, on the other hand, we do not have (4.28), because while (4.27) sums over all possible paths $\gamma(\tau) = (q^a(\tau), p_n(\tau)) = \gamma_i$.

17 Generally, there could be multiple classical solutions connecting $q^a$ and $q'^a$ (as in the case of the timeless double pendulum), especially when the system is not deparametrizable.

18 There could be possible experimental tests of the weight factors in mesoscopic phenomena or optical/electron diffractions/interferences.
which can move forward and backward in $t$, (B.6) now sums over only the paths $\tilde{p}_0(t) = (q^a(t), p^a(t))$ which are monotonic in $t$ as the time-ordered condition (B.9) has to be imposed for the systems in which $[\hat{H}_0(q^a, \hat{p}^a; t_1), \hat{H}_0(q^a, \hat{p}^a; t_2)] \neq 0$. The difference is profound and shows that relativistic quantum mechanics and conventional quantum mechanics are different both at the level of kinematics and at the level of dynamics if the system is non-strictly deparametrizable, as already commented in the end of section 3.2.

However, for most situations, provided that the action for the classical solution is much greater than $\hbar$, we have the good approximation (4.26) and only the paths in the vicinity of the classical solution are important. Meanwhile, as discussed in section 2.2, the classical solution are important. Therefore, the conventional path integral, although not equivalent to, is a good approximation for a deparametrizable system is always monotonic in $t$. Thus, for non-strictly deparametrizable systems, it is a good approximation in (4.27) to sum over only the paths which are not too deviated from the classical solution and are monotonic in $t$. In this approximation, (4.27) reduces to the conventional path integral (B.6) as $\int Dt$ factors out as an irrelevant overall factor. Therefore, the conventional path integral, although not equivalent to, is a good approximation for the timeless path integral for non-strictly deparametrizable systems. Further research is needed to investigate when the approximation remains good and when it fails. This issue is closely related to the composition laws of relativistic quantum mechanics studied in [16] and the main idea of [18] that the complication with the quantum Zeno effect in the conventional path integral should be avoided by ‘softening’ the restriction on paths in a manner which gives rise to coarse-graining in timescale (also see [17] for the issues of probability distributions in the context of the decoherent histories approach to quantum theory).

4.3. Timeless Feynman’s path integral

Consider the special case that the classical Hamiltonian is given in the form of (2.17) and the Hamiltonian operator is Weyl ordered. As the Hamiltonian is a quadratic polynomial in $p^a$, the path integral over $Dp_a$ in (4.22) can be integrated out. That is, in the expression:

$$W(q^a, q'^a) \sim \int Dq^a \int DN \prod_{n=0}^{N-1} \frac{d^ep_n}{(2\pi \hbar)^d} \times \exp \left[ \frac{i}{\hbar} \sum_{n=0}^{N-1} \left( \sum_a p_{na} \Delta q^a_{n+1} - \Delta t'_{n+1} N_{n+1} H(q^a_n, p_{na}) \right) \right],$$

(4.29)

the integration over each $p_{na}$ can be explicitly carried out:

$$\int_{-\infty}^{\infty} dp_{na} \exp \left( \frac{i}{\hbar} \left[ \sum_a p_{na} \Delta q^a_{n+1} - \Delta t'_{n+1} N_{n+1} \left( \alpha_a p^2_{na} + \beta_a p_{na} \bar{q}^a_{na} + \gamma_a p_{na} \right) \right] \right) \propto \frac{1}{\sqrt{N_{n+1}}} \exp \left( \frac{i}{\hbar} \frac{\Delta t'_{n+1} N_{n+1}}{4\alpha_a} \left( \frac{\Delta q^a_{n+1}}{\Delta t'_{n+1} N_{n+1}} - \beta_a \bar{q}^a_{n+1} - \gamma_a \right)^2 \right)$$

(4.30)

by the Gaussian integral $\int_{-\infty}^{\infty} dx e^{-ax^2 + bx} = (\pi/a)^{1/2} e^{b^2/4a}$. Noting that $dN_{n+1} / \sqrt{N_{n+1}} = 2 d\sqrt{N_{n+1}}$ and introducing the shorthand notation

$$\prod_{n=0}^{N-1} \int_{-\infty}^{\infty} d\sqrt{N_{n+1}} \rightarrow \int D\sqrt{N},$$

(4.31)

In this subsection, the repeated index $a$ is not summed unless $\sum_a$ is explicitly used.
we then have
\[ W(q^\alpha, q^{\alpha\prime}) \sim \int Dq^\alpha \int D\sqrt{N} \times \exp \left[ \frac{1}{\hbar} \sum_{n=0}^{N-1} \left( \sum_{a} \frac{N_{a+1}}{4\alpha_a} \left[ \frac{\Delta q^\alpha_n}{\Delta \tau^\prime_{n+1} N_{a+1}} - \beta_a q^\alpha_n - \gamma_a \right]^2 - N_{a+1} V(q^\alpha_0) \right) \Delta \tau^\prime_{n+1} \right], \] (4.32)

which written in the continuous form reads
\[ W(q^\alpha, q^{\alpha\prime}) \sim \int Dq^\alpha \int D\sqrt{N} \exp \left( \frac{i}{\hbar} \int_0^\infty d\tau' \left( \sum_{a} \frac{N}{4\alpha_a} \left[ \frac{q^\alpha'}{N} - \beta_a q^\alpha - \gamma_a \right]^2 - N V(q^\alpha) \right) \right), \] (4.33)

where the ‘velocity’ \( \dot{q}^\alpha := dq^\alpha/d\tau' \) is the continuous limit of \( \Delta q^\alpha_n/\Delta \tau^\prime_{n+1} \).

Therefore, in the special case that the Hamiltonian is a quadratic polynomial in \( p_a \), the transition amplitude admits a path integral formalism over the configuration space, whereby the functional integration over \( N \) is modified as \( \int D\sqrt{N} \). This is called the configuration space path integral or Feynman’s path integral. The configuration space path integral (4.33) sums over all arbitrary paths \( \gamma \in \mathcal{C} \) whose endpoints are fixed at \( q^\alpha \) and \( q^{\alpha\prime} \), and each path contributes with a phase, which is identical to the Lagrangian function as given in (2.19) (divided by \( \hbar \)). The functional variations on (4.33) with respect to \( \sqrt{N} \) and \( q^\alpha \) yield the classical Hamiltonian constraint and equation of motion as in (2.20) and (2.21)\(^{20}\). This shows again that the stationary solution is the classical solution and thus (4.26) is a good approximation.

4.4. Timeless path integral with multiple constraints

If there are multiple constraints and the constraint operators \( \hat{H}^i \) commute, then the projector is given by (3.26), and (4.2) can be directly generalized as\(^{21}\)
\[ W(q^\alpha, q^{\alpha\prime}) \sim \int_{-\infty}^{\infty} d\tau^1 \cdots \int_{-\infty}^{\infty} d\tau^k \langle q^\alpha | e^{-i \sum_i \tau^i \hat{H}^i} | q^{\alpha\prime} \rangle. \] (4.34)

If each \( \hat{H}^i \) is a polynomial of \( q^\alpha \) and \( \hat{p}_a \) and Weyl ordered, then the linear sum \( \hat{H} = \sum_i \tau^i \hat{H}^i \) is also a polynomial and Weyl ordered. Thus, by replacing \( \tau \) with 1 and \( H \) with \( \hat{H}^i \) in (4.14), it can be shown
\[ \langle q^\alpha | e^{-i \sum_i \tau^i \hat{H}^i} | q^{\alpha\prime} \rangle = \int Dq^\alpha \int Dp_a \exp \left( \frac{i}{\hbar} \int_\gamma p_a dq^\alpha \right) \exp \left( -i \sum_i \int_\gamma \tau^i \hat{H}^i(q^\alpha(\bar{\tau}), p_a(\bar{\tau})) d\bar{\tau} \right), \] (4.35)

where \( \bar{\tau} \) is a parameter for the curve \( \bar{\gamma} \) with \( q^\alpha = q^\alpha(\bar{\tau} = 0) \) and \( q^{\alpha\prime} = q^{\alpha}(\bar{\tau} = 1) \). Redefining \( \tau^i \Delta \bar{\tau}^n \) as \( \Delta \tau^\prime_{n+1} \), we then have
\[ \langle q^\alpha | e^{-i \sum_i \tau^i \hat{H}^i} | q^{\alpha\prime} \rangle = \int Dq^\alpha \int Dp_a \exp \left( \frac{i}{\hbar} \int_\gamma p_a dq^\alpha \right) \exp \left( -i \sum_i \int_\gamma H^i(q^\alpha(\tau^i), p_a(\tau^i)) d\tau^i \right), \] (4.36)

As in the case with a single constraint, the first exponential in (4.36) is independent of parametrizations for the curve \( \bar{\gamma} \), and for the second exponential we can play the same

\(^{20}\) Note that \( 8W/\sqrt{N} = 2\sqrt{N} \delta W/\delta N \).

\(^{21}\) In this subsection, the repeated index \( i \) is not summed unless \( \sum_i \) is explicitly used.
trick by summing over different parametrizations to get rid of the seemingly dependence on parametrizations. Following the same steps in section 4.1, for each $i$, we have

$$\int \mathrm{d} \tau \int \left[ D \Delta \tau \right]_{\sum \Delta \tau = \tau'} \langle q' | e^{i\tau' \hat{H}} | q' \rangle$$

$$= \int Dq' \int Dp_a \int D\Sigma \exp \left[ \frac{i}{\hbar} \int \dot{q}' \left( p_a \frac{dq_a}{d\tau'} - N_i(\tau') \hat{H} \right) d\tau' \right]$$

(4.37)

for a given arbitrary parametrization $\tau'$. After summed over $\left[ D \Delta \tau \right]_{\sum \Delta \tau = \tau'}$ for each $i$, (4.34) yields

$$W(q^a, q'^a) \sim \int Dq' \int Dp_a \prod_{j=1}^k \int D\Sigma \exp \left[ \frac{i}{\hbar} \int \dot{q}' \left( p_a \frac{dq_a}{d\tau'} - \sum_{i=1}^k N_i \hat{H} \right) d\tau' \right]$$

(4.39a)

$$\sim \int Dq' \int Dp_a \prod_{j=1}^k \delta[H'] \exp \left[ \frac{i}{\hbar} \int \dot{q}' \left( p_a dq_a \right) \right]$$

(4.39b)

which is the direct generalization of (4.22) and (4.24). In the path integral, each path in $\Sigma$ contributes with a phase, which is the classical action given in (2.3) divided by $\hbar$. Functional variation on (4.39a) with respect to $N_i$, $q^a$ and $p_a$ again yields the classical equations (2.2) and (2.4).

5. Summary and discussion

Starting from the canonical formulation in [9], the timeless path integral for relativistic quantum mechanics is rigorously derived. Given in (4.24), the transition amplitude is formulated as the path integral over all possible paths on the constraint surface $\Sigma$ (through the confinement by the delta functional $\delta[H]$), and each path contributes with a phase identical to the classical action $\int \dot{q} dq$ divided by $\hbar$. The alternative expression is given in (4.22), which is the functional integral over all possible paths in the cotangent space $\Omega = T^*C$ as well as over the Lagrange multiplier $N$. The timeless path integral manifests the timeless feature of relativistic quantum mechanics, as the parametrization for paths has no physical significance. For the special case that the Hamiltonian constraint $H(q^a, p_a)$ is a quadratic polynomial in $p_a$, the transition amplitude admits the timeless Feynman’s path integral over the paths in the configuration space $C$, as given in (4.33).

The formulation of timeless path integral is intuitively appealing and advantageous in many respects as it generalizes the action principle of relativistic classical mechanics by replacing the classical notion of a single trajectory with a sum over all possible paths. It is easy to see that the classical solution contributes most to the transition amplitude and thus (4.26) is a good approximation for generic cases since the stationary solution is identical to the classical one. Various approximation methods developed in (conventional) path integral approaches can be readily adapted to the timeless description. Furthermore, the timeless path integral offers a new perspective to see how the conventional quantum mechanics emerges from relativistic quantum mechanics within a certain approximation (as discussed in section 4.2) and may provide new insight into the problem of time. Specifically, for strictly deparametrizable systems, relativistic quantum mechanics and conventional quantum mechanics are different at the level of kinematics but identical at the level of dynamics; for non-strictly deparametrizable systems, on the other hand, they are different for both kinematics and dynamics.

The formulation of timeless path integral can be directly extended for the dynamical systems with multiple constraints as given in (4.39), if the constraint operators $\hat{H}$ commute.
For the case that $\hat{H}$ do not commute but form a closed Lie algebra, the projector is no longer given by (3.26) but we have to invoke (3.28) to obtain the physical state, which leads to

$$W(q^a, q^a) \sim \int d\mu(\theta^i)(q^a|e^{-i\theta^i \hat{H}}|q^a),$$

(5.1)

where $\theta^i$ are the coordinates of the Lie group $G$ generated by $\hat{H}$. Starting from (5.1) and following the similar techniques used in this paper, one expects to obtain the timeless path integral, but the measure of the functional integral $\prod_{n=1}^{f} \int DN_i$ appearing in (4.39a) would have to be nontrivially modified, as $\theta^i$ play the same role of $\tau^i$ in (4.34) but now the nontrivial Haar measure $d\mu$ is involved and the nontrivial topology of $G$ has to be taken into account. Consequently, one also has to deal with the issues concerning global restrictions on the Lagrange multipliers that ensure compatibility with the boundary conditions of the path integral (see [11, 13, 15]). The BRST methods used in appendix of [13] to derive the path integral for Jacobi’s action could be adopted to handle the gauge redundancy by the multiple constraints. By elucidating the procedure of group averaging in light of BRST techniques, we wish to explicitly formulate the timeless path integral with multiple constraints in the future. For the case that $\hat{H}$ do not form a closed Lie algebra, it is not clear how to construct the quantum theory which is free of quantum anomalies even in the canonical formalism. The timeless path integral may instead provide a new conceptual framework to start with for constructing the quantum theory.

Throughout this paper, we have focused on simple mechanical systems, but not field theories. In section 3.3 of [9], the canonical treatment of classical field theories which maintains clear meaning in a general-relativistic context is presented as a direct generalization of the theories. In section 3.3 of [9], the canonical treatment of classical field theories which maintains clear meaning in a general-relativistic context is presented as a direct generalization of the timeless path integral treatments of timeless theories in which the notion of time slicing does not fully apply and any time slicing through spin foams is of no physical significance. The timeless path integral developed in this paper makes no reference whatsoever to time slicing and any time slicing through spin foams is of no physical significance. The timeless path integral contains the quantum field theory described in [9]. We leave it for the future research.

In SFMs, the transition amplitude between two spin networks (i.e. quantum states of gravitational fields) is given by the sum (with appropriate weights) over all spin foams whose boundary consists of the given spin networks. As spin foams are two complexes with colored faces and edges, the formulation of sum-over-spin-foams amplitudes is completely combinatorial and any time slicing through spin foams is of no physical significance. The timeless path integral developed in this paper makes no reference whatsoever to time slicing and thus bears a close resemblance to the formulation of SFMs, as compared to other path integral treatments of timeless theories in which the notion of time slicing does not fully disappear at the fundamental level (recall footnote 5). Therefore, as the timeless path integral in

\[ W(s, s') = \sum_{\sigma} w(\Gamma) \prod_{j} \dim(j) \prod_{e} A_{\gamma}(j_{e}, i_{e}) \prod_{v} A_{\gamma}(j_{v}, i_{v}), \]

where a spin foam $\sigma = (\Gamma, j_{f}, i_{e})$ is given by a two-complex $\Gamma$ with a half-integer $j_{f}$ associated with each face $f$ and an intertwiner $i_{e}$ associated with each edge $e$; the boundary of $\sigma$ is given by the spin networks $s$ and $s'$; and $A_{\gamma}$ and $A_{\gamma}$ are the amplitudes associated with each edge $e$ and each vertex $v$. Compared to the timeless path integral (4.24), the functional integral over all paths $\int Dq^a Dp_a$ analogous to the discrete sum over all spin foams $\sum_{\sigma}$ (more elaborately, $q^a$ corresponds to $\Gamma$ and $p_a$ to the coloring ($j_{f}, i_e$); the kinematic weight factor $\exp(i \mu \int_{\gamma} p_a dq^a \theta)$ for each path $\gamma = (q^a, p_a)$ is analogous to the weight factor $w(\Gamma) \prod_{j} \dim(j) \prod_{e} A_{\gamma}(j_{e}, i_{e})$ for each spin foam $\sigma = (\Gamma, j_{f}, i_{e})$; and the Hamiltonian constraint functional $\delta[H]$) $\approx \lim_{N \to \infty} \prod_{\mu=1}^{N} \delta(H(q^a, p_a))$ is analogous to the product of vertex amplitudes $\prod_{v} A_{\gamma}(j_{v}, i_{v})$, as the Hamiltonian of LQG acting on the nodes of spin networks gives rise to the vertices of spin foams at which edges branch. Note that the set $\{H(q^a, p_a)\}$ indexed by $n$ is not ordered in any sense of time slicing, and neither is the set $\{A_{\gamma}(j_{e}, i_{e})\}$ indexed by $v$. The resemblance is marked except for the striking difference between the functional integral over continuous variables in (4.24) and the discrete sum over discrete variables in SFMs.
this paper is systematically derived from the well-defined canonical formulation of relativistic quantum mechanics, we expect it to provide new insight into the issues of the connection between LQG/LQC and SFMs. Extending the timeless path integral to field theories will make the resemblance to SFMs even stronger.

Acknowledgments

The author would like to thank Biao Huang for helpful discussions and anonymous reviewers of the previous manuscripts for bringing related works to his notice. This work was supported in part by the grant no. 10675019 from the NSFC and the grants nos 20080440017 and 200902062 from the China Postdoctoral Science Foundation. The major revision to the original manuscript owes much to the valuable comments elicited by Professor Wei-Tou Ni.

Appendix A. Stationary phase approximation

The timeless phase integral (4.22) can be recast as

\[ W(q^a, q^a') \sim \int Dq^a \int Dp_a \int DN \exp \left( \frac{i}{\hbar} S[\tilde{\gamma}, N] \right), \]  

(A.1)

where \( S[\tilde{\gamma}, N] \equiv S[q^a, p_a, N] \) is the action given by (2.3) for the path \( \tilde{\gamma} \). For the situations where the action for the classical solution \( \tilde{\gamma}_{cl} \) is much greater than \( \hbar \), i.e. \( S[\tilde{\gamma}_{cl}] \gg \hbar \), the stationary phase approximation (à la method of steepest descent) can be applied. The main idea of the stationary phase method relies on the cancelation of the rapidly varying phases (as \( S[\tilde{\gamma}_{cl}] / \hbar \) behaves as a huge number) of the paths considerably deviated from the stationary solution, which is identical to the classical solution \( \tilde{\gamma}_{cl} \). Essentially, only the paths in the vicinity of the classical solution contribute in (A.1).

First, we functionally expand the action \( S[\tilde{\gamma}, N] \) in a Taylor series about the classical solution \( \tilde{\gamma}_{cl} \):

\[ S[\tilde{\gamma}, N] = S[\tilde{\gamma}_{cl}] + \frac{1}{2!} \int d\tau d\tau' \frac{\delta^2 S}{\delta q^a(\tau) \delta q^b(\tau')} \left. \right|_{\tilde{\gamma}_{cl}} (\eta^A(\tau) - \eta^A_{cl}(\tau)) (\eta^B(\tau') - \eta^B_{cl}(\tau')) + \cdots, \]  

(A.2)

where \( \eta^A = (q^a, p_a) \) and \( \eta^A_{cl} = (q^a_{cl}, p_{acl}) \) is the classical solution. Note that the first-order functional derivatives vanish as the classical solution coincides with the stationary solution, and the second-order derivatives explicitly read

\[
\left. \frac{\delta^2 S}{\delta q^a(\tau) \delta q^b(\tau')} \right|_{\tilde{\gamma}_{cl}} = -N(\tau) \left. \frac{\partial^2 H}{\partial q^a(\tau) \partial q^b(\tau')} \right|_{\tilde{\gamma}_{cl}} \delta(\tau - \tau'),
\]  

(A.3a)

\[
\left. \frac{\delta^2 S}{\delta q^a(\tau) \delta p_b(\tau')} \right|_{\tilde{\gamma}_{cl}} = -N(\tau) \left. \frac{\partial^2 H}{\partial q^a(\tau) \partial p_b(\tau')} \right|_{\tilde{\gamma}_{cl}} \delta(\tau - \tau') - \delta^b_a \frac{d}{d\tau} \delta(\tau - \tau'),
\]  

(A.3b)

\[
\left. \frac{\delta^2 S}{\delta p_a(\tau) \delta p_b(\tau')} \right|_{\tilde{\gamma}_{cl}} = -N(\tau) \left. \frac{\partial^2 H}{\partial p_a(\tau) \partial p_b(\tau')} \right|_{\tilde{\gamma}_{cl}} \delta(\tau - \tau').
\]  

(A.3c)

\(^{23}\) Note that the classical solution satisfies \( H = 0 \) and thus \( S[\tilde{\gamma}_{cl}] \) is independent of the gauge choice of \( N \).
Taking (A.3) into (A.2) yields

\[ S[\tilde{\gamma}, N] \approx S[\tilde{\gamma}_c] + \int d\tau \frac{d(q^a - q^a_c)}{d\tau}(p_a - p_{\text{act}}) \]

\[-\frac{1}{2} \int d\tau N(\tau) \frac{\partial^2 H}{\partial \eta^A(\tau)\partial \eta^B(\tau)} \bigg|_{\tilde{\gamma}_c} \left( \eta^A(\tau) - \eta^A_c(\tau) \right) \left( \eta^B(\tau) - \eta^B_c(\tau) \right) \]  

(A.4a)

\[ = S[\tilde{\gamma}_c] + \int d\tau \left( \delta p_a(\tau) \frac{d\delta q^a(\tau)}{d\tau} - N(\tau)\tilde{H}(\delta q^a, \delta p_a) \right). \]  

(A.4b)

where the perturbation variables \( \delta \eta^A \) are defined as \( \delta \eta^A := \eta^A - \eta^A_c \equiv (q^a - q^a_c, p_b - p_{\text{act}}) = : (\delta q^a, \delta p_b) \)

with \( \delta q^a = 0 \) at the endpoints of \( \delta \tilde{\gamma} := \tilde{\gamma} - \tilde{\gamma}_c \), and \( \tilde{H}(\delta q^a, \delta p_a) \) is a function of \( \delta q^a \) and \( \delta p_a \) defined as

\[ \tilde{H}(\delta q^a, \delta p_a) := \frac{1}{2} \frac{\partial^2 H}{\partial \eta^A \partial \eta^B} \bigg|_{\tilde{\gamma}_c} \delta \eta^A \delta \eta^B. \]  

(A.5)

Next, substituting (A.4) into (A.1) and ignoring the higher order contributions, we obtain the stationary phase approximation:

\[ W(q^a, q^{a\prime}) \approx \xi_{\tilde{\gamma}_c} e^{\frac{i}{\hbar}S[\tilde{\gamma}_c]} \]

(A.7)

where the weight factor \( \xi_{\tilde{\gamma}_c} \) is given by

\[ \xi_{\tilde{\gamma}_c} = \int \mathcal{D}q^a \int \mathcal{D}p_a \int \mathcal{D}N \exp \left[ \frac{i}{\hbar} \int_{\tilde{\gamma}_c} \delta p_a \frac{d\delta q^a}{d\tau} - N(\tau)\tilde{H}(\delta q^a, \delta p_a) \right]. \]  

(A.8)

Note that (A.8) takes the same form of (4.22) except that \( \eta^A \) is replaced by \( \delta \eta^A \) and \( H \) is replaced by \( \tilde{H} \), implying that \( \xi_{\tilde{\gamma}_c} \) is independent of the parametrization of \( \tau \) as it should be. If there is only one single classical solution connecting \( q^{a\prime} \) and \( q^a \), then the overall weight factor has no physical significance. On the other hand, if there are multiple classical solutions, (A.7) is generalized as (4.26), where the relative weights of \( \xi_{\tilde{\gamma}_c} \) are of physical importance and can be understood as the decoherence width of phases of the paths close to the classical solution \( \tilde{\gamma}_c \).

Equations (A.7) and (A.8) can be obtained less formally but more quickly by directly expanding the integrand in (2.3):

\[ p_a \frac{d\delta q^a}{d\tau} = NH(q^a, p_a) = (p_{\text{act}} + \delta p_a) \frac{d(q^a_c + \delta q^a)}{d\tau} - NH(q^a_c, p_{\text{act}}) - N \frac{\partial H}{\partial q^a} \bigg|_{\tilde{\gamma}_c} \delta q^a \]

\[-N \frac{\partial H}{\partial p_a} \bigg|_{\tilde{\gamma}_c} \delta p_a - N \frac{1}{2} \frac{\partial^2 H}{\partial \eta^A \partial \eta^B} \bigg|_{\tilde{\gamma}_c} \delta \eta^A \delta \eta^B + \ldots \]  

(A.9a)

\[ = \left( p_{\text{act}} \frac{d\delta q^a_c}{d\tau} - NH(q^a_c, p_{\text{act}}) \right) + \left( \delta p_a \frac{d\delta q^a}{d\tau} - N \tilde{H}(\delta q^a, \delta p_a) \right) + \ldots, \]  

(A.9b)

where from equations (A.9a) to (A.9b) we have applied (2.4) and integration by parts to eliminate the linear terms in \( \delta \eta^A \). As \( H(q^a_c, p_{\text{act}}) = 0 \), taking (A.9b) into (A.1) immediately gives (A.7) with (A.8).

24 Carrying out \( \int \mathcal{D}N \) in (A.8) gives rise to a delta functional \( \delta[\tilde{H}] \). However, it should be noted that \( \tilde{H}(\delta \eta^A) \approx 0 \) is not the constraint for \( \delta \eta^A \). Instead, given \( H(\eta^A_c) = 0 \), the Hamiltonian constraint \( H(\eta^A) = 0 \) leads to \( \frac{d\delta q^a}{d\tau} \bigg|_{\tilde{\gamma}_c} \delta \eta^A + \tilde{H}(\delta \eta^A) \approx 0 \) for \( \delta \eta^A \). The linear terms in \( \delta \eta^A \) are eliminated in (A.9) and thus do not contribute to (A.7) and (A.8).
Appendix B. Path integral with time

In order to compare with the timeless path integral for relativistic quantum mechanics, in this appendix, we re-derive the path integral formalism (with time) for conventional quantum mechanics but include more generality: the (nonrelativistic) Hamiltonian \( H_0 \) is allowed to have explicit dependence on time, i.e., \( \hat{H}_0 = \hat{H}_0(q^i, \dot{p}_i; t) \), with \( t \) being the time parameter. For a given Hamiltonian operator \( H_0 \), the (nonrelativistic) transition amplitude is defined as

\[
G(q^i, t; q'^i, t') := \langle q'^i | \hat{U}(t - t') | q^i \rangle = \langle q^i | \mathcal{T} e^{-\frac{i}{\hbar} \int_{t'}^{t} dt \hat{H}_0(\tau)} | q'^i \rangle,
\]

where \( \hat{U}(t - t') \) is the evolution operator from \( t' \) to \( t \) and \( \mathcal{T} \) represents time ordering.

B.1. First case

We first consider the systems in which \( \{ \hat{H}_0(q^i, \dot{p}_i; t_0), \hat{H}_0(q^i, \dot{p}_i; t_2) \} = 0 \) for all \( t_1, t_2 \). For these systems, the time ordering \( \mathcal{T} \) in (B.1) is superfluous and thus

\[
G(q^i, t; q'^i, t') = \langle q^i | e^{-\frac{i}{\hbar} \int_{t'}^{t} dt \hat{H}_0(\tau)} | q'^i \rangle
\]

(B.2)

For given \( t' \) and \( t \), let us introduce a parametrization sequence: \( t_0 = t' \), \( t_1 \), \( t_2 \), \ldots, \( t_{N-1} \), \( t_N = t \) with \( t_n \in \mathbb{R} \), and define \( \Delta t_n := t_n - t_{n-1} \). The conditions on the endpoints \( (t_0 = t' \) and \( t_N = t \) correspond to \( \sum_{n=0}^{N} \Delta t_n = t - t' \)).

For a given arbitrary small number \( \epsilon \), by increasing \( N \), we can always make the parameter sequence fine enough, i.e., \( \text{mesh} \{ t_n \} \leq |\tau|/N \leq \epsilon \), such that

\[
-i \int_{t'}^{t} d\tau \hat{H}_0(\tau) = -i \lim_{N \to \infty} \sum_{n=0}^{N-1} \Delta t_n \hat{H}_0(t_n).
\]

Consequently, we can recast the transition amplitude as

\[
G(q^i, t; q'^i, t') = \langle q^i | e^{-\frac{i}{\hbar} \int_{t'}^{t} dt \hat{H}_0(\tau)} | q'^i \rangle
\]

(B.3a)

\[
= \left( \prod_{n=1}^{N-1} \int d^d q_n \right) \langle q_N | e^{-\frac{i}{\hbar} \Delta t_N \hat{H}_0(t_{N-1})} | q_{N-1} \rangle \langle q_{N-1} | e^{-\frac{i}{\hbar} \Delta t_{N-1} \hat{H}_0(t_{N-2})} | q_{N-2} \rangle \ldots \langle q_1 | e^{-\frac{i}{\hbar} \Delta t_1 \hat{H}_0(t_0)} | q_0 \rangle + \mathcal{O}(\epsilon^2)
\]

(B.3b)

where we have identified \( q' = q_N \) and \( q'' = q_0 \) and in (B.3b) inserted \( N - 1 \) times the completeness relation

\[
\int dq^d |q^d\rangle \langle q^d| := \int dq^d \ldots dq^{d-1} |q^d, \ldots, q^{d-1}\rangle \langle q^d, \ldots, q^{d-1}|.
\]

(B.4)

Disregarding difference of \( \mathcal{O}(\epsilon^2) \), (B.3) is formally identical to (4.4) except that \( \hat{H} \) is now replaced by \( \hat{H}_0(t_n)/\hbar \) and \( q^d_n \) replaced by \( \vec{q}_n \). Therefore, taking the formal replacements

\[
d \to d - 1, \quad q_n^d \to q_n^d, \quad p_n^d \to p_n^d, \quad \vec{q}_n \to \vec{q}_n, \quad \hat{H}(\vec{q}_n, \vec{p}_n) \to \hat{H}_0(\vec{q}_n, \vec{p}_n; t)/\hbar, \quad \hat{H}(\vec{q}_n^d, \vec{p}_n^d; t)/\hbar, \quad t' \to t, \quad \tau = t \}
\]

(B.5)

and assuming that the Hamiltonian operator \( \hat{H}_0(\vec{q}_n, \vec{p}_n; t) \) is a polynomial of \( \vec{q}_n \) and \( \vec{p}_n \) (while the parameter \( t \) is viewed as coefficients) and is Weyl ordered, all the derivations from (4.4)

25 In the literature, the parametrization sequence is normally chosen to be uniform, i.e., \( \Delta t_n = (t - t')/N \). Here, we purposely keep it generic (non-uniform and even unordered) to be compared with the timeless path integral (see footnote 15).

26
to (4.14) can be carried over in the obvious way. Therefore, parallel to (4.10) and (4.14), the
(nonrelativistic) transition amplitude is given by
\[
G(q^i, t'; q^i, t) = \lim_{N \to \infty} \left( \prod_{n=0}^{N-1} \int dq_n \right) \left( \prod_{n=0}^{N-1} \int dp_n \right) \exp \left( i \sum_{n=0}^{N-1} P_n \Delta q_n \right) \times \exp \left( -i \hbar \sum_{n=0}^{N-1} \Delta \tau_n \hat{H}_0(\vec{q}_n^i, p_n; \tau_n) \right)
\]
\[
\equiv \int Dq' \int Dp' \exp \left( i \int \hat{p}_0 dq' \right) \exp \left( -i \hbar \int \hat{H}_0(q'(\tau), p'(\tau); \tau) d\tau \right). \tag{B.6a}
\]

where \( \vec{y}_0 \) are arbitrary paths in the cotangent space \( \Omega_0 = T^* C_0 \) which is given by
\( \vec{y}_0(\tau) = (q'(\tau), p'(\tau)) \) and with the endpoints projected to the nonrelativistic configuration
space \( C_0 \) fixed by \( q'(\tau = t') = q^i \) and \( q'(\tau = t) = q' \).

It should be remarked that (B.6a) is given for a fixed parametrization sequence \( \tau_0 = t', \tau_1, \tau_2, \ldots, \tau_{N-1}, \tau_N = t \), which is not necessarily uniform or ordered. If we choose different
parametrization sequences (fine enough), they nevertheless yield the same transition amplitude.
No further sum over different parametrization sequences is necessary, as opposed to \( \int D\Delta \tau \)
in (4.16), which results in \( \int DN \) in (4.19), for timeless path integral. If the parametrization
sequence is not an ordered partition of the interval \([t', t]\), the paths \( \vec{y}_0(\tau) \) are envisioned turning
back and forth in time within some periods. This still gives the same transition amplitude as
that by an ordered partition, because in (B.3a) any array segment turning back and forth in
time gives rise to
\[
e^{-\frac{i}{\hbar} \Delta \tau_n \hat{H}_0(\tau_n) - \frac{i}{\hbar} \Delta \tau_{n-1} \hat{H}_0(\tau_{n-1}) - \cdots - \frac{i}{\hbar} \Delta \tau_1 \hat{H}_0(\tau_1) \cdots} = 1 + O(\epsilon^2), \tag{B.7}
\]
whenever \( \tau_n \approx \tau_{n+1} \equiv \bar{\tau} + O(\epsilon) \).

B.2. Second case

Secondly, let us consider the systems in which \( [\hat{H}_0(\vec{q}^i, \hat{p}_i; t_1), \hat{H}_0(\vec{q}^i, \hat{p}_i; t_2)] \neq 0 \). For these
systems, the time ordering \( T \) in (B.1) is not superfluous but we have
\[
Te^{-\frac{i}{\hbar} \int_{t'}^{t} \hat{H}_0(\tau) d\tau} = 1 + \left( -\frac{i}{\hbar} \right) \int_{t'}^{t} \hat{H}_0(\tau_1) d\tau_1 + \int_{t'}^{t} \hat{H}_0(\tau_2) d\tau_2 + \cdots \]
\[
\equiv 1 + \left( -\frac{i}{\hbar} \right) \int_{t'}^{t} \hat{H}_0(\tau_1) d\tau_1 + \left( -\frac{i}{\hbar} \right)^2 \int_{t'}^{t} \hat{H}_0(\tau_2) d\tau_2 + \cdots \tag{B.8a}
\]
\[
= \lim_{N \to \infty} \left( 1 + \left( -\frac{i}{\hbar} \right) \sum_{n=0}^{N-1} \Delta \tau_n \hat{H}_0(\tau_n) + \left( -\frac{i}{\hbar} \right)^2 \sum_{n=0}^{N-1} \sum_{n=0}^{N-1} \Delta \tau_n \Delta \tau_{n+1} \hat{H}_0(\tau_n) \hat{H}_0(\tau_{n+1}) \right)
\]
\[
= \lim_{N \to \infty} \left( 1 - \frac{i}{\hbar} \Delta \tau_0 \hat{H}_0(\tau_0) \right) \left( 1 - \frac{i}{\hbar} \Delta \tau_1 \hat{H}_0(\tau_1) \right) \cdots \left( 1 - \frac{i}{\hbar} \Delta \tau_{N-1} \hat{H}_0(\tau_{N-1}) \right). \tag{B.8b}
\]
where, in the discrete expression in (B.8b), the parametrization sequence is ordered:

\[ t' = t_0 < t_1 < t_2 < \cdots < t_{N-1} < t_N = t. \]  

(B.9)

Thus, by (B.8b), we have

\[ \mathcal{T} e^{-\frac{i}{\hbar} \int \mathcal{L}(\tau) d\tau} = e^{-\frac{i}{\hbar} \Delta t_n \mathcal{H}_t(\tau_{n-1}) - \frac{i}{\hbar} \Delta t_{n-1} \mathcal{H}_t(\tau_{n-2}) - \cdots - \frac{i}{\hbar} \Delta t_1 \mathcal{H}_t(\tau_0)} + O(\epsilon^2), \]

(B.10)

which is identical to the operator in the middle in (B.3a) (up to \( O(\epsilon^2) \)) except that the parametrization sequence \( \{\tau_n\} \) is no longer generic but has to be ordered as in (B.9). Consequently, everything in the first case can be exactly repeated and thus we also obtain (B.6) for the second case, but this time \( \{\tau_n\} \) is ordered and the paths \( \tilde{\gamma}_0(\tau) \) are not allowed to move backward in time.

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