The causal hierarchy of spacetimes* 

E. Minguzzi and M. Sánchez

Abstract. The full causal ladder of spacetimes is constructed, and their updated main properties are developed. Old concepts and alternative definitions of each level of the ladder are revisited, with emphasis in minimum hypotheses. The implications of the recently solved “folk questions on smoothability”, and alternative proposals (as recent isocausality), are also summarized.

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1. Introduction

Causality is an essential specific tool of Lorentzian Geometry, which appears as a fruitful interplay between relativistic motivations and geometric developments. Most of the goals of this theory are comprised in the so-called causal hierarchy of spacetimes: a ladder of spacetimes sharing increasingly better causal properties, each level with some specific results. This ladder and its main features were established at the end of the 70's, after the works of Carter, Geroch, Hawking, Kronheimer, Penrose, Sachs, Seifert, Wu and others (essentially, the last introduced level was in [28]) and were collected in the first version of Beem-Ehrlich book (1981) —later re-edited with Easley, [2]. Nevertheless, there are several reasons to write this revision. A first one is that the “folk questions on smoothability” of time functions and Cauchy hypersurfaces, which were left open in that epoch, have been solved only recently [4, 6, 7]. They affect to two levels of the ladder in an essential way —the equivalence between two classical definitions of stable causality and the structure of globally hyperbolic spacetimes. Even more, new results which fit typically on some of the levels, as well as some new viewpoints on the whole ladder, have been developed in the last years. So, we think that the full construction of the ladder from the lowest level to the highest one, may clarify the levels, avoid redundant hypotheses and simplify reasonings.

This paper is organized as follows. In Section 2 the typical ingredients of Causality are introduced: time-orientation, conformal properties, causal relations, maximizing properties of causal geodesics... Most of this introductory material is well-known and is collected in books such as [2, 27, 39, 40, 56]. Nevertheless, some aspects may be appreciated by specialists, as the introduction of globally hyperbolic neighborhoods (Theorem 2.14), the viewpoint of causal relation $I^+$, $J^+$, $E^+$ in $M \times M$ (Subsect. 2.4), or the conformal properties of lightlike pregeodesics (Theorem 2.36). The conformal invariance of some elements is stressed, even notationally (Remark 2.9).

In Section 3 the causal ladder is constructed. The nine levels are developed in subsections, from the lowest (non-totally vicious) to the highest one (globally hyperbolic). Essentially, our aims for each level are: (a) To give natural alternative definitions of the level (see, for example, Definitions 3.11 or 3.59 and further characterizations), with minimum hypotheses (see Definitions 3.63 or 3.70, with Prop. 3.64, Remark 3.72). (b) To check its strictly higher degree of specialization, in a standard way. (c) To explain geometric techniques or specific results of the level (for example, see Theorems 3.3, 3.89, 3.91 or Subsections 3.5, 3.7). In particular, we emphasize that only after the solution of the folk questions on smoothability, the classical characterization of causal stability in terms of the existence of a time function can be regarded as truly equivalent to the natural definition (see Theorem 3.56 and its proof). Even more, we detail the consequences of these folk questions for the structure of a globally hyperbolic spacetime (Theorem 3.78). Although the description of the smoothing procedure lies out of the scope of the present review (see [44], in addition to the original articles [4, 6, 7]), the main difficulties are stressed, Remark 3.77.

Finally, in Section 4 we explain briefly the recent proposal of isocausality by
García-Parrado and Senovilla [20]. This yields a partial ordering of spacetimes which was expected to refine the total order provided by the standard hierarchy. Even though, as proven later in [19], this ordering does not refine exactly the standard one, this is an alternative viewpoint, worth to be born in mind.

2. Elements of causality theory

Basic references for this section are [2, 27, 37, 39, 40, 56], other useful references will be [9, 10, 19, 21, 30, 31, 51].

2.1. First definitions and conventions.

Definition 2.1. A Lorentzian manifold is a smooth manifold $M$, of dimension $n_0 \geq 2$, endowed with a non-degenerate metric $g : M \rightarrow T^*M \otimes T^*M$ of signature $(-,+,\ldots,+)$. By smooth $M$ we mean $C^{r_0}$, $r_0 \in \{3,\ldots,\infty\}$. Except if otherwise explicitly said, the elements in $M$ will be also assumed smooth, i.e., as differentiable as permitted by $M$ ($C^{r_0-1}$ in the case of $g$, and $C^{r_0-3}$ for curvature tensor $R$). Manifolds are assumed Hausdorff and paracompact, even though the latter can be deduced from the existence of a non-degenerate metric (recall that the bundle of orthonormal references is always parallelizable; thus, it admits a positive definite-Riemannian metric, which implies paracompactness [52, vol II, Addendum 1, 34]).

The following convention includes many of the ones in the bibliography (the main discrepancies come from the causal character of vector 0, which somewhere else is regarded as spacelike [39]), and can be extended for any indefinite scalar product:

Definition 2.2. A tangent vector $v \in TM$ is classified as:

- **timelike**, if $g(v, v) < 0$.
- **lightlike**, if $g(v, v) = 0$ and $v \neq 0$.
- **causal**, if either timelike or lightlike, i.e., $g(v, v) \leq 0$ and $v \neq 0$.
- **null**, if $g(v, v) = 0$.
- **spacelike**, if $g(v, v) > 0$.
- **nonspacelike**, if $g(v, v) \leq 0$.

\footnote{We will not care about problems on differentiability (see the review [50 Sect. 6.1]). But notice that, essentially, $r_0 = 2$ suffices throughout the paper (the exponential map being only continuous), with the remarkable exception of Subsect. 2.6. Moreover, taking into account that \textit{globally hyperbolic neighborhoods} make sense for $r_0 = 1$, many elements are extendible to this case, see also [51].}
At each tangent space $T_pM$, $g_p$ is a (non-degenerate) scalar product, which admits an orthonormal basis $B_p = (e_0, e_1, \ldots, e_{n-1})$, $g_p(e_\mu, e_\nu) = \epsilon_\mu \delta_{\mu\nu}$, where $\delta_{\mu\nu}$ is Kronecker’s delta and $\epsilon_0 = -1, \epsilon_i = 1$ (Greek indexes $\mu, \nu$ run in $0, 1, \ldots, n-1$, while Latin indexes $i, j$ run in $1, \ldots, n-1$). Each $(T_pM, g_p), p \in M$ contains two causal cones. Definition 2.2 is naturally extended to vector fields $X \in \mathfrak{X}(M)$ and curves $\gamma : I \to M$ ($I \subset \mathbb{R}$ interval of extremes $-\infty \leq a < b \leq \infty$). Nevertheless, when $I = [a, b]$ we mean by timelike, lightlike or causal curve any piecewise smooth curve $\gamma : I \to M$, such that not only the tangent vectors are, respectively, timelike, lightlike or causal, but also the two lateral tangent vectors at each break lie in the same causal cone. The notion of causal curve will be extended below non-trivially to include less smooth ones, see Definition 3.15.

A time-orientation at $p$ is a choice of one of the two causal cones at $T_pM$, which will be called future cone, in opposition of the non-chosen one or past cone. In a similar way that for usual orientation in manifolds, a smooth choice of time-orientations at each $p \in M$ (i.e., a choice which coincides at some neighborhood $U_p$ with the causal cone selected by a -smooth- causal vector field on $U_p$) is called a time-orientation. The Lorentzian manifold is called time-orientable when one such time-orientation exists; no more generality is obtained either if smooth choices are weakened in $C^r$ ones, $r \in \{0, \ldots, r_0 - 1\}$, or if causal choices are strengthened in timelike ones. As the causal cones are convex, a standard partition of the unity argument yields easily:

**Proposition 2.3.** A Lorentzian manifold is time-orientable if and only if it admits a globally defined timelike vector field $X$ (which can be chosen complete).

Recall that this vector field $X$ can be defined to be future-directed at all the points and, then, any causal tangent vector $v_p \in T_pM$ is future directed if and only if $g(v_p, X_p) < 0$.

Easily one has: (a) any Lorentzian manifold admits a time-orientable double covering [40], [39, Lemma 7.17], and (b) let $g_R$ be any Riemannian metric on $M$ and $X \in \mathfrak{X}(M)$ non-vanishing, with $g_R$-associated 1-form $X^\flat$, then

$$g_L = g_R - \frac{2}{g_R(X, X)} X^\flat \otimes X^\flat$$

is a time-orientable Lorentzian metric. As a consequence, the possible existence of Lorentz metrics can be characterized [39, 5.37], [2, Sect. 3.1], [40, Sect. 1]:

**Theorem 2.4.** For any connected smooth manifold, the following properties are equivalent:

1. $M$ admits a Lorentz metric.
2. $M$ admits a time-orientable Lorentz metric.
3. $M$ admits a non-vanishing vector field $X$.
4. Either $M$ is non-compact or its Euler characteristic is 0.

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2 We use $\subset$ as a reflexive relation as in [40], that is, for any set $A$, $A \subset A$.

3 $X$ can be chosen complete because, given $X$ and an auxiliary complete Riemannian metric (which exists due to a theorem by Nomizu and Ozeki [38]) it can be replaced by the timelike vector field $X/|X|_R$, which is necessarily complete.
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The relevant new ingredient of a spacetime is a time-orientation:

**Definition 2.5.** A spacetime \((M, g)\) is a time-oriented connected Lorentz manifold.

The points of \(M\) are also called events. Notice that the time-orientation is implicitly assumed in the notation \((M, g)\) for a spacetime. In principle, \(M\) is not assumed to be an orientable manifold. Recall that orientability and time-orientability are logically independent. In fact, one can construct easily time-orientable and non-time-orientable Lorentz metrics on both a Möbius strip (or Klein bottle) and cylinder (or torus) by starting with the metric \(g\) on \(\mathbb{R}^2\)

\[
g(X_1, X_2) \equiv -1, \quad g(X_1, X_1) \equiv 0 \equiv g(X_2, X_2)
\]

\[
X_1 = \cos \pi x \partial_x + \sin \pi x \partial_y, \quad X_2 = -\sin \pi x \partial_x + \cos \pi x \partial_y,
\]

by making natural quotients (see figure 1).
2.2. Conformal/classical causal structure. The following algebraic result (Dajczer et al. criterion) has important consequences for the conformal structure of spacetimes, and has no analog in the positive definite case:

**Proposition 2.6.** Let \((V, g)\) be a real vector space with a non-degenerate indefinite scalar product, and let \(b\) be a bilinear symmetric form on \(V\). The following properties are equivalent:

1. \(b = c \cdot g\) for some \(c \in \mathbb{R}\),
2. \(b(v, v) = 0\) if \(g(v, v) = 0\),

The proof can be seen also in [2, Lemma 2.1], [56, App. D]. Obviously, 1 \(\Rightarrow\) 2, and the converse can be proved in dimension 2 easily; for higher dimensions, the problem is reduced to dimension 2, by grouping suitably the elements of a \(g\)-orthonormal basis. By the way, it is also known that any of the following conditions is equivalent to items 1, 2 (this yields bounds on the possible curvatures): (a) \(\exists a > 0 : \frac{b(v, v)}{g(v, v)} \leq a\) if \(g(v, v) \neq 0\), (b) \(\exists a > 0 : \frac{b(v, v)}{g(v, v)} \geq a\) if \(g(v, v) \neq 0\), (c) \(\exists a > 0 : |b(v, v)| \leq a|g(v, v)|\) if \(g(v, v) < 0\), (d) \(\exists a > 0 : |b(v, v)| \leq a|g(v, v)|\) if \(g(v, v) > 0\). In fact, any of these items implies item 2, by using that any lightlike vector can be approximated by both, timelike and spacelike ones. For some algebraic extensions to higher order tensors, see [3].

Two Lorentzian metrics \(g, g^*\) on the same manifold \(M\) are called pointwise conformal if \(g^* = e^{2u}g\) for some function \(u : M \to \mathbb{R}\). Proposition 2.6 yields directly:

**Lemma 2.7.** Two Lorentzian metrics \(g, g^*\) on a manifold \(M\) of dimension \(n_0 > 2\) are pointwise conformal if and only if both have the same lightlike vectors.

(The exceptional case \(n_0 = 2\) appears because a negative conformal factor keeps lightlike vectors unchanged, while exchanges timelike and spacelike vectors.)

Two spacetimes on the same manifold \(M\) are pointwise conformal if both, their metrics are pointwise conformal and their time-orientations agree at each event. The spacetime \((M, g)\) is called conformal to the spacetime \((M^*, g^*)\) if there exists a diffeomorphism \(\Phi : M \to M^*\) such that the pull-back spacetime on \(M\) obtained inducing the metric and the time-orientation through \(\Phi\) is pointwise conformal to \((M, g)\). Two spacetimes which only differ in the time-orientation are by definition not pointwise conformal and, moreover, they may be also non-conformal (see, for example, Figure 14 at the end). Clearly, the conformal relation is a relation of equivalence in the class of all the spacetimes. The following definition will be revisited in Section 4 in order to discuss what causality means.

**Definition 2.8.** The conformal or classical causal structure of the spacetime \((M, g)\) is the equivalence class \([[M, g]]\) for the conformal relation.

Several concepts in Lorentzian geometry do depend on the full metric structure of the spacetime \((M, g)\). Examples are the length of a curve, the time-separation between two events (see below), the non-lightlike geodesics or the geodesic completeness of a spacetime. Nevertheless, the conformal structure is particularly rich by itself, and its interplay with the metric becomes specially interesting.
Remark 2.9. The elements which come only from the conformal structure will be emphasized with the following conventions. For practical purposes, we will work with the relation of equivalence induced by the pointwise conformal relation in the spacetimes on the same $M$. For the spacetime $(M, g)$, its pointwise conformal class will be denoted as $(M, g)$ ($g$ denotes the set of all pointwise conformal metrics to $g$) where all the spacetimes in the class have the same time-orientation. When we refer to a spacetime as $(M, g)$, we emphasize that the considered properties hold for any $g$ in the class and, thus, depend only on the conformal structure. The boldface will be extended to equivalence classes of vectors and curves. So, $v$ denotes the equivalence class of vectors $v' = \alpha v$, $\alpha > 0$ and $g(v, w)$ is just the sign $(-1, 0, +1)$ of the scalar product $g(v, w)$.

Analogously, if $\gamma : I \to M$ is a curve then $\gamma$ is the equivalence class of curves coincident with $\gamma$ up to a strictly increasing reparametrization. Note that if $I$ is closed (or compact or open) for a representative of $\gamma$, the same holds for any representative $\gamma : I \to M$. Analogously, we say that $\gamma$ connects $p$ with $q$ if for a representative, $I = [a, b]$, $\gamma(a) = p$ and $\gamma(b) = q$, or write $p \in \gamma$ if $p$ belongs to the image of $\gamma$. If a future-directed causal curve $\gamma$ satisfies $\lim_{t \to b} \gamma(t) = q$ (resp. $\lim_{t \to a} \gamma(t) = p$), where $a, b$ $(-\infty \leq a < b \leq \infty)$ are the extremes of the interval $I$, the event $q$ (resp. $p$) is called the future (resp. past) endpoint of $\gamma$ (and the other way round if $\gamma$ is past-directed). These concepts are obviously extended to $\gamma$, so one can assume that $I$ is bounded when dealing with the endpoints of $\gamma$. A causal curve without future (resp. past) endpoint is said future (resp. past) inextendible.

2.3. Causal relations. Local properties. Given a spacetime $(M, g)$ the event $p$ is chronologically (resp. strictly causally; causally; horismotically) related to the event $q$, denoted $p \ll q$ (resp. $p < q$; $p \leq q$; $p \to q$) if there is a future-directed timelike (resp. causal; causal or constant; causal or constant, but not timelike) curve connecting $p$ with $q$. If $W \subset M$, given $p, q \in W$, the analogous relations for the spacetime $(W, g|_W)$ will be denoted $p \ll_W q$, $p <_W q$, $p \leq_W q$, $p \to_W q$.

From the viewpoint of set theory, relations $\ll, \leq, \to$ are written, regarded as subsets of $M \times M$, as:

$$I^+ = \{(p, q) : p \ll q\}, \quad J^+ = \{(p, q) : p \leq q\}, \quad E^+ = \{(p, q) : p \to q\}.$$  

Clearly, $E^\pm = J^\pm \setminus I^\pm$.

Note: all the definitions and properties extend naturally to the “minus” sign without further mention; for example, the sets (and binary relations) $I^{-}$, $J^{-}$, and $E^{-}$, are defined changing each $(p, q)$ above by $(q, p)$.

The chronological future of an event is defined as:

$$I^+(p) = \{q \in M : p \ll q\} = \pi_2(\pi_1^{-1}(p) \cap I^+) = \pi_1(\pi_2^{-1}(p) \cap I^-)$$

where $\pi_1$ and $\pi_2$ are the canonical projections to the factors of $M \times M$. Analogous expressions hold for the causal future $J^+(p)$ and horismos $E^+(p)$. By using
juxtapositions of curves, it is obvious that the relations \( \ll \) and \( \leq \) are transitive, but \( \rightarrow \) is not (see Proposition 2.31).

Every point of a spacetime admits an arbitrarily small (i.e. contained in any given neighborhood) convex neighborhood \( U \), that is, \( U \) is a (starshaped) normal neighborhood of any of its points \( p \in M \). This means that the the domain \( \tilde{U} \subset T_pM \) of the exponential map at \( p \), is chosen starshaped and yields a diffeomorphism onto \( U \), \( \exp_p : \tilde{U} \to U \). Thus, for any \( p, q \in U \), there exists a unique geodesic \( \gamma_{pq} : [0, 1] \to U \) which connects \( p \) with \( q \). Notice that one also has a diffeomorphism \[39] Lemma 5.9) between \( U \times U \) and and its image on \( TU \), which sends \( (p, q) \to \exp^{-1}_p q = \overrightarrow{pq} \in TM \). Such a convex \( U \) can be chosen simple, that is, with compact closure \( \overline{U} \) included in another open convex neighborhood \[40] p. 6).

Given an open subset \( U \) by \( I^+(p, U) \), \( J^+(p, U) \), \( E^+(p, U) \), will be denoted the corresponding future elements in \( U \) regarded as spacetime. If \( U \) is a convex neighborhood the causal relations in \( U \) are easily characterized\[39] Lemma 14.2]:

**Proposition 2.10.** Let \( (M, g) \) be a spacetime, \( \exp_p \) the exponential map at \( p \in M \), and \( U \) a convex neighborhood. Regarding \( U \) as a spacetime, given \( p \neq q, p, q \in U \):

1. \( q \in I^+(p, U) \) (resp. \( q \in J^+(p, U) \); \( q \in E^+(p, U) \)) \( \iff \overrightarrow{pq} = \exp^{-1}_p q \) is timelike (resp. causal; lightlike) and future-pointing.

2. \( I^+(p, U) \) is open in \( U \) (and \( M \)).

3. \( J^+(p, U) \) is the closure in \( U \) of \( I^+(p, U) \).

4. Causal relation \( J^+ \) is closed in \( U \times U \).

5. Any causal curve \( \gamma \) contained in a compact subset of \( U \) has two endpoints.

Notice from the first item (which can be regarded as a consequence of Theorem 2.20 below) that the study of the causal relations in \( U \) is reduced to the study of the causal character of tangent vectors type \( \overrightarrow{pq} \).

Nevertheless, convex neighborhoods depend on the metric structure. The following concept is purely conformal.

**Definition 2.11.** Let \( U, V \) be open subsets of a spacetime \( (M, g) \), with \( V \subset U \). \( V \) is called causally convex in \( U \) if any causal curve contained in \( U \) with endpoints in \( V \) is entirely contained in \( V \).

In particular, when this holds for \( U = M \), \( V \) is called causally convex.

**Remark 2.12.** Note that, in this case, if \( \leq_U, \leq_V \) denote, resp., the causal relations in \( U, V \) regarded as spacetimes, then the restriction of \( \leq_U \) to \( V \) agrees with \( \leq_V \) (this property does not characterize causal convexity, as can be checked from \( U = L^1, V = \{(t, x) \in \mathbb{R}^2 : |t|, |x| < 1\} \)). There are spacetimes such that the only open subset \( V \neq 0 \) of \( U = M \) which is causally convex is \( V = M \) (see the totally vicious ones in Sect. 3.1). Nevertheless, at least when \( U \) is also convex, the existence of arbitrarily small such \( V \) in \( U \), even with further properties, will be shown next.

Finally, note that if \( V \) is causally convex in \( U \) and \( W \) is an open set such that \( V \subset W \subset U \), then \( V \) is causally convex in \( W \).
It is also possible to prove that any point of \((M, g)\) admits a neighborhood with the best possible causal structure, i.e., which will belong to the top of the ladder, **global hyperbolicity** (see Section 3.11). Recall first the following result (by the best possible causal structure, i.e., which will belong to the top of the ladder).

**Remark 2.15.** (1) Of course, in Theorem 2.14 (which is formulated in a conformally invariant way) we can assume \(U' = V_1\). Nevertheless, it is clear from the proof that, for any representative \(g\) of the conformal class, \(U'\) can be chosen simple, which leads to the strongest local causal properties.

(2) The sequence \(\{V_n\}_n\) yields a topological basis at \(p\). Thus, an alternative formulation of Theorem 2.14 would ensure the existence of a (simple) \(U' \subset U\).
which admits arbitrarily small globally hyperbolic neighborhoods of \( p \), all of them causally convex at \( U' \).

(3) It also holds that each obtained neighborhood \( V = V_n \) of the sequence satisfies: each \( q \in V \) admits an arbitrarily small neighborhood which is causally convex in \( V \). In fact, this property is one of the alternative definitions of being strongly causal, see Sect. 3.4. Hence we have also proved that any spacetime \((M,g)\) is locally strongly causal, as any point \( p \) admits an arbitrarily small strongly causal neighborhood \( V \). Also, due to the last observation of Remark 2.12 any open set \( W \subset V \), \( p \in W \), is a strongly causal neighborhood of \( p \) as well. In particular, any spacetime \((M,g)\) admits arbitrarily small simple strongly causal neighborhoods.

### 2.4. Further properties of causal relations.

None of the properties in Proposition 2.16 but the second one, holds globally. In fact, given a timelike curve \( \gamma \) connecting the pair \((p,q)\) there are open neighborhoods \( U \ni p, V \ni q \) such that if \( \tilde{p} \in U \), \( \tilde{q} \in V \), then there exists a timelike curve \( \tilde{\gamma} \) connecting \( \tilde{p} \) and \( \tilde{q} \) (say, \( U, V \) can be chosen as \( I^-(p) \cap U_p, I^+(q) \cap U_q \), where \( U_p, U_q \) are convex neighborhoods of \( p, q \) which contains \( p_1, q_1 \), resp., and these points are chosen such that \( \gamma \) runs consecutively \( p, p_1, q_1, q \). Summing up,

**Proposition 2.16.** The set \( I^+ \) is open in \( M \times M \).

In what follows we claim that \( p \ll r \) and \( r \preceq q \) (or the other way round) implies \( p \ll q \) (see Proposition 2.11 for a more accurate result). In general, \( J^+(p) \subset \bar{I}^+(p) \) but the equality may not hold. Nevertheless, both closures as well as both boundaries (denoted with a dot in what follows) and interiors (denoted \( \text{Int} \)) coincide. Even more:

**Proposition 2.17.** It is \( \bar{J}^+ = \bar{I}^+, \text{Int}J^+ = I^+, \bar{J}^+ = \bar{I}^+ \).

**Proof.** Since \( I^+ \subset J^+ \), it is \( \bar{I}^+ \subset \bar{J}^+ \). Let \((p,q) \in \bar{J}^+ \) and let \( U \) and \( V \) be arbitrarily small neighborhoods of respectively \( p \) and \( q \). There are events \( p' \in U \), \( q' \in V \), such that \((p', q') \in J^+ \). Take events \( p'' \in U \cap I^-(p') \) and \( q'' \in V \cap I^+(q') \). Then \( p'' \) can be connected to \( q'' \) with the composition of a timelike, a causal, and finally a timelike curve, and, as claimed above, it follows \( p'' \ll q'' \). Since \( U \) and \( V \) are arbitrary \((p,q)\) is an accumulation point for points belonging to \( I^+ \). We conclude that \( \bar{I}^+ = \bar{J}^+ \).

Let us show that \( I^+ = \text{Int}J^+ \) from which it follows \( \bar{J}^+ = \bar{I}^+ \). Since \( I^+ \) is open and included in \( J^+ \), \( \bar{I}^+ \subset \text{Int} J^+ \). If \((p,q) \in \text{Int}J^+ \), then chosen normal convex neighborhoods \( U \ni p, V \ni q \), such that \( U \times V \) is included in \( \text{Int}J^+ \), and taken \( p' \in U \cap I^+(p), q' \in V \cap I^-(q) \), then \( q' \in J^+(p') \) and thus \( q \in I^+(p) \), i.e., \((p,q) \in I^+ \).

**Definition 2.18.** An open subset \( F \) (resp. \( P \)) is a future (resp. past) set if \( I^+(F) = F \) (resp. \( I^-(P) = P \)).

An example of future set is \( I^+(p) \) for any \( p \in M \). We have the following characterization:
Proposition 2.19. If $F$ is a future set then $\bar{F} = \{ p : I^+(p) \subset F \}$, and analogously in the past case.

Proof. (⊃). If $I^+(p) \subset F$ then $p \in \bar{I}^+(p) \subset \bar{F}$.

(⊂). Let $p \in \bar{F}$ and take any $q \in I^+(p)$. As $I^-(q) \ni p$ is open, $I^-(q) \cap F \neq \emptyset$. Thus, $q \in I^+(F) = F$, i.e. $I^+(p) \subset F$.

Remark 2.20. Even though the closure of $J^+$ in $M \times M$ induces a binary relation, this is not always transitive. As closedness becomes relevant for different purposes (for example, when one deals with limit of curves) Sorkin and Woolgar [51] defined the $K$-relation as the smallest one which contains $\ll$ and is: (i) transitive, and (ii) topologically closed. (That is, the corresponding set $K^+ \subset M \times M$ which defines the $K$-relation, is the intersection of all the closed subsets $C$ which contain $I^+$ such that $(p,q), (q,r) \in C \Rightarrow (p,r) \in C$). Among the applications, some results on positive mass and globally hyperbolic spacetimes with lower order of differentiability ($r_0 = 1$) have been obtained.

Notice that, in particular, $\bar{I}^+ \subset K^+$, but perhaps there exists $(p,q) \in K^+ \setminus \bar{J}^+$. In this case, $q \notin I^+(p)$ and hence there is a point $r \in I^+(q)$ not contained in $I^+(p)$. As a consequence $(p,q) \in K^+$ and $r \in I^+(q)$ do not imply $r \in I^+(p)$ (as happens when the causal relation $K^+$ is replaced with $J^+$, see Prop. 2.31). In particular, the relation $K^+$ does not define a causal space in the sense of Kronheimer and Penrose [30]. Nevertheless, this cannot happen if $(M,g)$ is causally simple, because then $\bar{I}^+ = \bar{J}^+ = J^+ = K^+$ (see Sect. 3.10), moreover, $(I^+, K^+)$ defines such a causal space if and only if $(M,g)$ is causally continuous [15].

Remark 2.21. In general $(p,q) \in \bar{I}^+$ does not imply $q \in \bar{I}^+(p)$ or $p \in \bar{I}^-(q)$ (see figure 2). For this reason it may be more useful to regard the causal relations as defined in $M \times M$, although it is customary to introduce them in $M$, that is, through $I^±(p)$, $J^±(p)$, $E^±(p)$.

![Figure 2](image-url)  

Figure 2. Minkowski spacetime without a spacelike half-line is an example of stably causal non-causally continuous spacetime (see Sects. 3.8–3.9). Here $(p,q) \in \bar{I}^+$, but neither $q \in \bar{I}^+(p)$, nor $p \in \bar{I}^-(q)$.

Recall that we have defined three binary relations $\ll, \leq, \rightarrow$ (and trivially a fourth one $<$) on any spacetime and, obviously, two of them determine the third. But starting with only one of them, one can define naturally a second (and then, a third) binary relation, which will coincide with the other causal-type relation in sufficiently well-behaved spacetimes:
Definition 2.22. Let $\ll,\leq,\rightarrow$ be the canonical binary relations of a spacetime $(M,g)$. We define the associated relations

1. starting at chronology $\ll$:
   - (a) $x \leq_{(\ll)} y \iff I^+(y) \subset I^+(x)$ and $I^-(x) \subset I^-(y)$.
   - (b) $x \rightarrow_{(\ll)} y \iff x \leq_{(\ll)} y$ and not $x \ll y$.

2. starting at horismos $\rightarrow$:
   - (a) $x \leq_{(\rightarrow)} y \iff x = x_1 \rightarrow x_2 \cdots \rightarrow x_n = y$ for some finite sequence $x_1, \ldots, x_n \in M$.
   - (b) $x \ll_{(\rightarrow)} y \iff x \leq_{(\rightarrow)} y$ and not $x \rightarrow_{(\rightarrow)} y$.

3. starting at causality $\leq$ (and, thus, $<$):
   - (a) $x \rightarrow_{(\leq)} y \iff x \leq y$ and $\leq$ is a total linear order in $J^+(p) \cap J^-(q)$ for any $p,q$ such that $x < p < q < y$ (i.e., the topological space $J^+(p) \cap J^-(q)$, ordered by $\leq$, is isomorphic to $[0,1]$ with its natural order; in particular, each two distinct $p',q' \in J^+(p) \cap J^-(q)$ satisfy either $p' \rightarrow q'$ or $q' \rightarrow p'$).
   - (b) $x \ll_{(\leq)} y \iff x \leq y$ and not $x \rightarrow_{(\leq)} y$.

As we will see, $\leq_{(\ll)} = \leq$ in causally simple spacetimes (Theorem 3.69), $\leq_{(\rightarrow)} = \leq$ in distinguishing spacetimes (Theorem 3.24), and $\rightarrow_{(\leq)} = \rightarrow$ in causal spacetimes (Theorem 3.9).

Some authors have studied the abstract properties of $\ll,\leq,\rightarrow$ and defined spaces which generalize (well-behaved) spacetimes with their canonical causal relations. Among them causal spaces by Kronheimer and Penrose [30], etiological spaces by Carter [10] and chronological spaces by Harris [25]. Among the applications to spacetimes, a better insight on the meaning of causal boundaries (whose classical construction by Geroch, Kronheimer and Penrose [23] relies on some types of future sets) is obtained, see [16, 21] and references therein.

2.5. Time-separation and maximizing geodesics. Let $(M,g)$ be a spacetime, fix $p,q \in M$ and let $\hat{C}(p,q)$ be the set of the future-directed causal curves which connect $p$ to $q$. The following concept is metric (non-conformally invariant) as it depends on the Lorentzian length $L(\gamma) = \int_{t_p}^{t_q} |\gamma'| dt$, $p = \gamma(t_p)$, $q = \gamma(t_q)$, $\gamma \in \hat{C}(p,q)$. Nevertheless, some of its properties will depend only on the conformal structure.

Definition 2.23. The time-separation (or Lorentzian distance) is the map $d : M \times M \rightarrow [0, +\infty]$ defined as:

$$d(p,q) = \begin{cases} 0, & \text{if } \hat{C}(p,q) = \emptyset \\ \sup \left\{ L(\alpha), \alpha \in \hat{C}(p,q) \right\}, & \text{if } \hat{C}(p,q) \neq \emptyset \end{cases}$$

Some simple properties are:
Proposition 2.24. Let $p, q, r \in M$:

1. $d(p, q) > 0 \iff p \in I^-(q)$

2. If there exists a closed timelike curve through $p$, $d(p, p) = +\infty$; otherwise: $d(p, p) = 0$.

3. $0 < d(p, q) < +\infty \implies d(q, p) = 0$ ($d$ is not symmetric)

4. $p \leq q \leq r \implies d(p, q) + d(q, r) \leq d(p, r)$.

Most of the proof of this proposition is straightforward; take into account Th. 2.30 below.

Of course, $d$ is not a true distance, but the last property suggests possible similitudes with the distance associated to a Riemannian metric. A first one is:

Proposition 2.25. In any spacetime, $d$ is lower semi-continuous, that is, given $p, q, p_m, q_m \in M, \{p_m\} \to p, \{q_m\} \to q$, the lower limit satisfies:

$$\liminf_m d(p_m, q_m) \geq d(p, q)$$

Nevertheless, $d$ may be no upper semi-continuous (see Figure 3).

![Figure 3](image_url)  
A classical example of spacetime for which $d$ is not upper semi-continuous. Here $\lim_n d(p_n, q) = 1 > 0 = d(p, q)$.

The main Riemannian similarities come from the maximizing properties of causal geodesics, which are consequences of an infinitesimal application of reversed triangle inequality. Concretely, the maximizing properties can be summarized in the following two results (see, for ex., [39, Lem. 5.34, 5.9], or around [50, Prop. 2.1]), the first one local (see also Proposition 2.10) and the second global:

**Theorem 2.26.** Let $U$ be a convex neighborhood of $(M, g)$, and $p, q \in U$. Assume there exists a causal curve $\alpha : [0, b] \to U$ from $p$ to $q$. Then, the radial segment $\gamma_{pq} : [0, 1] \to U$ from $p$ to $q$, (which has initial velocity $\dot{\gamma} = \exp_p^{-1}(q)$ and length $|\dot{\gamma}| = \sqrt{|g(\dot{\gamma}, \dot{\gamma})|}$), is causal and (up to reparametrization) maximizes strictly the length among all the causal curves in $U$ which connect $p$ to $q$.

In particular, if $\gamma_{pq}$ is lightlike then it is the unique causal curve contained entirely in $U$ which connects $p$ to $q$. 
Theorem 2.27. Assume that there exists a causal curve $\alpha : [0, b] \to M$ which connects $p$ to $q$, $p, q \in M$, with maximum length among all the causal curves which connect $p$ to $q$ in the spacetime $(M, g)$. Then, $\alpha$ is, up to a reparametrization, a causal geodesic without conjugate points (Def. 2.32) except, at most, the endpoints.

That is: (i) the length of a causal geodesic contained in a convex neighborhood is equal to the time-separation (computed in the neighborhood as a spacetime), of its endpoints, and (ii) if a causal curve in the spacetime has a length equal to the time-separation of its endpoints, then it is, up to a parametrization, a causal geodesic without conjugate points, except at most its endpoints.

Recall that if $p, q \in M$ satisfy $p < q$ and $d(p, q) = 0$, then these two properties are conformally invariant. So, Theorem 2.27 implies that any lightlike geodesic (and its first conjugate point) must be conformally invariant, up to a reparametrization. Next, we will see that this can be made much more precise.

2.6. Lightlike geodesics and conjugate events. It is known (see in these proceedings [9, Sect. 2.3]) that a curve $\gamma : I \to M$ with non-vanishing speed $\gamma'$ is a pregeodesic (i.e., it can be reparametrized as a geodesic for the Levi-Civita connection $\nabla$ of the spacetime) if and only if it satisfies

$$\nabla_{\gamma'} \gamma' = f \gamma'$$

for some function $f : I \to \mathbb{R}$. Explicitly, the reparametrization is $\tilde{\gamma}(\tilde{s}) = \gamma(s(\tilde{s}))$ where, for constants $\tilde{s}_0 \in \mathbb{R}, s_0 \in I, \tilde{s}_0' \neq 0$,

$$\tilde{s}(s) = \tilde{s}_0 + \tilde{s}_0' \int_{s_0}^{s} e^{\int_{s_0}^{r} f(r) \, dr} \, dt.$$  

(2)

If $\gamma$ is a lightlike geodesic for $g$ then it satisfies (1) for the Levi-Civita connection $\nabla^*$ of any conformal metric $g^* = e^{2u} g$, being $f = 2 \frac{d(du)}{dt}$ (see [9] or the proof of Theorem 2.36 below) and, thus, with the natural choice of $\tilde{s}_0$ in (2):

$$\tilde{\gamma}(\tilde{s}) = \gamma(s(\tilde{s})) \text{ lightlike geodesic with } \tilde{\gamma}' = e^{-2u} \gamma'.$$

(3)

That is, lightlike pregeodesics are (pointwise) conformally invariant and the following definition for the conformal class makes sense.

Definition 2.28. Given $(M, g)$, a lightlike curve $\gamma$ is a lightlike geodesic if for a choice of representatives (and hence for any choice), $g$ and $\gamma$, equation (1) holds.

Note that although the concept of lightlike geodesic makes sense given only the conformal structure, the definitions of timelike and spacelike geodesics do not. The fact that two events have a zero time-separation is also conformally invariant and, thus, the following definition makes also sense.

Definition 2.29. A lightlike curve $\gamma$ connecting two events $p$ and $q$ is maximizing if there is no timelike curve connecting $p$ and $q$. 
Recall that this concept is a pure conformal one, but the notion of *maximizing* for timelike curves depends on the metric.

The following result is standard, and relies on the possibility to deform any causal curve which is not a lightlike geodesic without conjugate points in a timelike one (see, for example, [2, Cor. 4.14], [27, Prop. 4.5.10] or [40, Prop. 2.20]). As discussed below Theorem 2.27, all these elements are conformally invariant, and the result is stated consequently.

**Theorem 2.30.** Let \((M, g)\) be a spacetime.

(i) Each two events \(p, q \in M\) connected by a causal curve \(\gamma\) which is not a maximizing lightlike curve are also connected by a timelike curve.

(ii) Any maximizing lightlike curve is a lightlike geodesic of \(g\) without conjugate points (i.e., when reparametrized as a lightlike geodesic for any \(g \in g\), does not have conjugate points) except at most the endpoints.

In fact, the timelike curve in (i) can be chosen arbitrarily close (in the \(C^0\) topology) to \(\gamma\). As a straightforward consequence, one has:

**Proposition 2.31.** Two events \(p, q\) are horismotically related if and only if they can be joined by a maximizing lightlike geodesic. Thus:

(i) If \(p \ll r\) and \(r \leq q\) then \(p \ll q\) (analogously, if \(p \leq r\) and \(r \ll q\) then \(p \ll q\).

(ii) If \(r \in E^+(p)\) and \(q \in E^+(r)\) then either \(q \in E^+(p)\) or \(p \ll q\).

The conformal invariance of conjugate points along lightlike geodesics is not only a consequence of maximizing properties, (which would be applicable only in a restricted way, for example, it would apply only for the first conjugate point) but a deeper one. Next, our aim is to show that the definition of Jacobi field in the lightlike case can be made independent of the metric and indeed depends only on the conformal structure. As a consequence the concept of conjugate point and its multiplicity, depends also only on the conformal structure for lightlike geodesics, while in the timelike case it requires the metric. We begin with the metric-dependent definition of Jacobi field, and show later that it can be made independent of the conformal factor in the lightlike geodesic case.

**Definition 2.32.** Let \(\gamma : I \rightarrow M\) be a geodesic of a spacetime (or any semi-Riemannian manifold), \((M, g)\). A vector field \(J\) on \(\gamma\) is a *Jacobi field* if it satisfies the *Jacobi equation*

\[ J'' + R(J, \gamma')\gamma' = 0 \]

where \(R\) is the (Riemann) curvature tensor, \(R(X, Y) = [\nabla_X, \nabla_Y] - \nabla_{[X,Y]}\). The events \(p = \gamma(s_p)\) and \(q = \gamma(s_q), s_p < s_q\) are said to be *conjugate* (of multiplicity \(m\)) if there exist \(m > 0\) independent Jacobi fields such that \(J(s_p) = 0 = J(s_q)\).

As in the (positive-definite) Riemannian case, one has:

**Lemma 2.33.** For any geodesic \(\gamma : I \rightarrow M\) of \((M, g)\):
(i) The variation vector field \( V \) of \( \gamma \) by means of a variation \((s,v) \mapsto \gamma^v(s)\) with geodesic longitudinal curves (at constant \( v \)), is a Jacobi field.

(ii) If \( J \) is a Jacobi field for \( \gamma \) then \( g(J, \gamma')(s) = as + b \) for suitable constants \( a \) and \( b \) and all \( s \in I \). Thus:

(a) If \( J \) vanishes at the endpoints, then \( g(J, \gamma') = 0 \).

(b) The only Jacobi fields proportional to \( \gamma' \), \( J(s) = f(s)\gamma'(s) \) satisfy \( f = cs + d \) for suitable constants \( c \) and \( d \), hence if they vanish at the endpoints they vanish everywhere.

(c) If \( J_1 \) and \( J_2 \) are two Jacobi fields vanishing at the endpoints and \( J_2 = J_1 + f\gamma' \) for some function \( f \), then they coincide.

As two causal vectors cannot be orthogonal, a straightforward consequence of the (a) part is:

Proposition 2.34. Let \( \gamma \) be lightlike and let \( J \) be a Jacobi field which vanishes at the endpoints but not everywhere, then \( J \) is spacelike and orthogonal to \( \gamma' \). In particular, no lightlike geodesic \( \gamma : I \to M \) in a 2-dimensional spacetime admits a pair of conjugate events.

Indeed, the last assertion follows because no spacelike vector field \( J \) exists which is orthogonal to \( \gamma' \).

In what follows \( \gamma \) will be always lightlike. We are interested in the case of conjugate points. It is convenient to introduce the space \( N(\gamma') \) of vector fields over \( \gamma \) orthogonal to \( \gamma' \) and the quotient space \( Q \) of vector fields of \( N(\gamma') \) defined up to additive terms type \( f\gamma' \). If \( X \in N(\gamma') \) is a vector field orthogonal to \( \gamma' \) then \([X] \in Q \) will denote its equivalence class. Let \( \pi : N(\gamma') \to Q \), \( \pi(X) = [X] \) be the natural projection. The covariant derivative, also denoted \( \gamma' = \nabla_{\gamma'} \), can be induced on \( Q \) by making it to commute with \( \pi \), i.e. \([X]' = [X']\). This definition is independent of the representative because:

\[
(i) \quad X' \in N(\gamma'), \text{ since } g(X', \gamma') = g(X, \gamma')' = 0 \\
(ii) \quad [X + f\gamma']' = [X' + (f\gamma')] = [X' + f\gamma'] = [X'],
\]

Even more, the curvature term in Jacobi equation can be projected to the map \( \mathcal{R} : Q \to Q \) defined as:

\[
\mathcal{R}[X] = \pi(\mathcal{R}(X, \gamma')\gamma'),
\]

which, again, is independent of the chosen representative \( X \).

Lemma 2.35. If \( J \in N(\gamma') \) is a Jacobi field then \([J] \in Q \) is a Jacobi class, that is, it solves the quotient Jacobi equation

\[
[J]'' + \mathcal{R}[J] = 0
\]

(where the zero must be understood in \( Q \), that is, as the class of any vector field pointwise proportional to \( \gamma' \)).
Conversely, if $[\tilde{J}] \in Q$ is a Jacobi class in the sense of Eq. (4) and $J_p, J_q \in TM$ are orthogonal to $\gamma$ at $\gamma(s_p), \gamma(s_q)$, $s_p < s_q$, with $[\tilde{J}]_p = [J_p], [\tilde{J}]_q = [J_q]$, then there exist a representative $J \in N(\gamma'), [J] = [\tilde{J}]$, which is a Jacobi field and fulfills the boundary conditions $J(s_p) = J_p, J(s_q) = J_q$. In particular if $[\tilde{J}]$ vanishes at the endpoints then there exists a representative $J$ which vanishes at the endpoints.

Proof. The first statement is obvious. For the converse, $\tilde{J}'' + R(\tilde{J}, \gamma')\gamma' = h\gamma'$ for some suitable function $h$. Let $J$ be another representative, $J = J + f\gamma'$, with $f'' = h$. Then $J$ is a Jacobi field, and given the initial conditions, $f_p = f(s_p)$ and $f_q = f(s_q)$, function

$$f(s) = \int_{s_p}^{s} \left( \int_{s_p}^{s'} h(s'') ds' \right) ds' + f_p + \frac{s - s_p}{s_q - s_p} \left[ f_q - f_p - \int_{s_p}^{s_q} \left( \int_{s_p}^{s'} h(s'') ds' \right) ds' \right],$$

solves the problem. \hfill \Box

These lemmas imply that in order to establish whether two events $p$ and $q$ are conjugate along a lightlike geodesic (and its multiplicity, i.e., the dimension of the space of Jacobi fields vanishing at the endpoints) it is easier to look for Jacobi conjugate along a lightlike geodesic (given by (3)), if and only if $[\tilde{J}]$ satisfies Eq. (5) on $\gamma$ (taking $\mathcal{R}[\tilde{J}]$ from (4)) if and only if $[\tilde{J}]$ satisfies Eq. (5) on $\tilde{\gamma}$ (where $\mathcal{R}^*\tilde{J}$ is defined as $\mathcal{R}^*\tilde{J} = \pi(\mathcal{R}^*(\tilde{X}, \gamma')\tilde{\gamma}')$, and $\mathcal{R}^*$ denotes the curvature tensor of $\gamma^*$).

Thus, the concept of conjugate events $p$ and $q$ along a lightlike geodesic $\gamma$, and its multiplicity, is well-defined for the conformal structure $(M, g)$.

Proof. We will put $[X]^\gamma = [\nabla^\gamma_X X]$, $X' = \nabla\gamma X$, and use index notation as in [50] App. D], $a, b, c, d = 0, \ldots, n - 1$, (see [49] for more intrinsic related computations). It is proved in that reference:

$$\nabla^a X^c = \nabla_a X^c + C^{ab}_{\alpha \beta} X^b,$$

where $C^{ab}_{\alpha \beta} = 2\delta^e_{(a} \partial_{b)} u - g_{ab} g^{cd} \partial_d u$, which implies that if $X \in N(\gamma')(= N(\tilde{\gamma}')$ up to reparametrizations),

$$[X]^\gamma = e^{-2u}[X' + C^{ab}_{\alpha \beta} X^b(\gamma')^a] = e^{-2u}[X' + u'X] = e^{-2u}([X'] + u'[X]),$$

and in particular

$$[X]^\gamma = e^{-4u}([X]' + u''[X] - (u')^2[X]).$$

We use the transformation of the Riemann tensor under conformal transformations (see, for example, [50])

$$(\mathcal{R}^*)^d_{\text{cab}} = R^d_{\text{cab}} - 2\delta^d_{\text{(a}} \nabla_{\text{b)}}(\partial_d(u) - 2(\partial_{\text{a}} u) \delta_{\text{b)}}(\partial_d u)
+ 2(\partial_{\text{a}} u) \delta_{\text{b)}} g_{\text{df}}(\partial_d u)(\partial_f u) + 2 g_{\text{a}} \delta_{\text{b)}} g_{\text{df}}(\partial_d u)(\partial_f u).$$
Using $\gamma_a' J^a = \gamma_a' \gamma^a = 0$, 

\[
(R^*)^d_{\ c ab}(\gamma)^c J^a (\gamma)^b = e^{-4u} \{ R^d_{\ c ab} - 2\delta^d_{[a} \nabla_{b]} \partial_c u + 2 g_{c[a} \nabla_{b]} \partial_c u - 2(\partial_{[a} u)\delta^d_{b]} \partial_c u \\
+ 2(\partial_{[a} u) g_{b]} g^{df} \partial_f u + 2 g_{c[a} \delta^d_{b]} g^{ef} (\partial_e u)(\partial_f u)\}\langle \gamma'\rangle^c J^a (\gamma')^b \\
= e^{-4u} \{ R^d_{\ c ab} - \delta^d_{a} \nabla_b \partial_c u + (\partial_b u)\delta^d_{a} \partial_c u\}\langle \gamma'\rangle^c J^a (\gamma')^b + f(\gamma')^d,
\]

for a suitable function $f$. This equation reads (up to reparametrizations) 

\[
R^*[J] = e^{-4u} (R[J] - u''[J] + (u')^2[J]),
\]

which together with Eq. (6) for $X = J$, gives the thesis.

\[\square\]

3. The causal hierarchy

As explained in the Introduction, the aim of this section is to construct the causal ladder, a hierarchy of spacetimes according to strictly increasing requirements on its conformal structure. Essentially, some alternative characterizations of each level will be studied, as well as some of its main properties, checking also that each level is strictly more restrictive than the previous one. At the top of this ladder globally hyperbolic spacetimes appear. Even though somewhat restrictive, this last hypothesis is, in some senses, as natural as completeness for Riemannian manifolds. Even more, according to the Strong Cosmic Censorship Hypothesis, the natural (generic) models for physically meaningful spacetimes are globally hyperbolic ones. So, these spacetimes are the main target of Causality Theory, and it is important to know exactly the generality and role of their hypotheses.

Most of the levels are related to the non-existence of travels to the past either for observers travelling through timelike curves (“grandfather’s paradox”), or for light beams, or for certain related curves. It is convenient to distinguish between the following notions, especially in the case of causal geodesics:

**Definition 3.1.** Let $\gamma : [a, b] \rightarrow M$ be a piecewise-smooth curve with non-vanishing velocity at any point:

(a) $\gamma$ is a loop (at $p$) if $\gamma(a) = \gamma(b) = p$;

(b) $\gamma$ is closed if it is smooth and $\gamma'(a) = \gamma'(b)$ (following our convention in Remark 2.9 for vectors).

(c) $\gamma$ is periodic if it is closed with $\gamma'(a) = \gamma'(b)$

Recall that if $\gamma$ is a lightlike geodesic, the properties of being closed or periodic are conformal invariant; moreover, such a closed $\gamma$ can be extended to a complete geodesic if and only if it is periodic (see in these proceedings [9]). For non-lightlike geodesics, the notions of closed and periodic become equivalent.

**3.1. Non-totally vicious spacetimes.** Recall that if $p \ll p$ then there exist a timelike loop at $p$ and, giving more and more rounds to it, one finds $d(p, p) = \infty$. Even more, if this property holds for all $p \in M$, then $I^+(p), I^-(p)$ are both, open and closed. So, one can check easily the following alternative definitions.
### Definition 3.2.

A spacetime \((M, g)\) is called **totally vicious** if it satisfies one of the following equivalent properties:

(i) \(d(p, q) = \infty, \forall p, q \in M\).

(ii) \(I^+(p) = I^-(p) = M, \forall p \in M\).

(iii) Chronological relation is reflexive: \(p \ll p, \forall p \in M\).

Accordingly, a spacetime is **non-totally vicious** if \(p \not\ll p\) for some \(p \in M\).

Of course, it is easy to construct non-totally vicious spacetimes. Nevertheless, totally vicious ones are interesting at least from the geometric viewpoint, and sometimes even in physical relativistic examples (Gödel spacetime is the most classical example). Let us consider an example. A spacetime \((M, g)\) is called **stationary** if it admits a timelike Killing vector field \(K\); classical Schwarzschild, Reissner Nordström or Kerr spacetimes (outside the event horizons) are examples of stationary spacetimes. This definition depends on the metric \(g\), but the fact that \(K\) is **conformal Killing** depends only on the conformal class \(g\). Moreover, if \(K\) is timelike and conformal Killing then it selects a unique representative \(g^K\) of \(g\) such that \(g^K(K, K) \equiv -1\) (and then \(K\) will be Killing for \(g^K\), as the conformal factor through the integral lines of \(K\) must be equal to 1; see also, for example, [43 Lemma 2.1]).

### Theorem 3.3. [47]

Any compact spacetime \((M, g)\) which admits a timelike conformal Killing vector field \(K\) is totally vicious.
Proof. (Sketch). In order to prove that each \( p \in M \) is crossed by a timelike loop, it is enough to prove that there exists a timelike vector field \( X \) with periodic integral curves. Recall that \( K \) is Killing not only for the selected metric \( g^K = -g/g(K, K) \) in the conformal class \( g \), but also for the associated Riemannian metric \( g_R \):

\[
g_R(u, v) = g^K(u, v) + 2g^K(u, K)g^K(v, K), \forall u, v \in T_pM, p \in M.
\]

Now, let \( G \) be the subgroup generated by \( K \) of the isometry group \( \text{Iso}(M, g_R) \). Then its closure \( \bar{G} \) satisfies:

- \( \bar{G} \) is compact, because so is \( \text{Iso}(M, g_R) \) (recall that \( g_R \) is Riemannian and \( M \) is compact).
- \( \bar{G} \) is abelian, because so is \( G \).
- As a consequence, \( \bar{G} \) is a \( k \)-torus, \( k \geq 1 \), and there exists a sequence of subgroups \( S_m \) diffeomorphic to \( S^1 \) which converges to \( G \).

Finally, notice that the corresponding infinitesimal generator \( K_m \) of \( S_m \) (which is Killing for \( g_R \)) have periodic integral curves and, for big \( m \), are timelike for \( g \). Thus, one can choose \( X = K_m \) for large \( m \).

**Remark 3.4.** The result is sharp: if \( K \) is allowed to be lightlike in some points there are counterexamples, see Figure 5.

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![Figure 5](image_url)

Figure 5. Non-totally vicious and non-chronological torus with a Killing vector field \( K = \partial_t \). The vector field \( K \) is timelike everywhere except when \( x = 1/3, 2/3 \), where it is lightlike.
3.2. Chronological spacetimes.

Definition 3.5. A spacetime \((M, g)\) is called \textit{chronological} if it satisfies one of the following equivalent properties:

(i) No timelike loop exists.

(ii) Chronological relation is \textit{irreflexive}, i.e., \(p \ll q \Rightarrow p \neq q\).

(iii) \(d(p, p) < \infty\) (and then equal to 0) for all \(p \in M\).

A chronological spacetime is clearly non-totally vicious (see Definition 3.2(iii)) but the converse does not hold, as Figure 5 shows. Notice that this example is compact and, in fact, as a general fact:

Theorem 3.6. No compact spacetime \((M, g)\) is chronological.

Proof. Recall the open covering of \(M\): \(\{I^+(p), p \in M\}\). Take a finite subcovering \(\{I^+(p_1), I^+(p_2), \ldots, I^+(p_m)\}\) and, without loss of generality, assume that, if \(i \neq j\) then \(p_i \notin I^+(p_j)\) (otherwise, \(I^+(p_i) \subset I^+(p_j)\), and \(I^+(p_i)\) can be removed). Then, \(p_1 \in I^+(p_1)\), as required.

3.3. Causal spacetimes.

Definition 3.7. A spacetime \((M, g)\) is called \textit{causal} if it satisfies one of the following equivalent properties:

(i) No causal loop exists.

(ii) Strict causal relation is \textit{irreflexive}, i.e., \(p < q \Rightarrow p \neq q\).

The following possibility is depicted in Figure 6.

Theorem 3.8. A chronological but non-causal spacetime \((M, g)\) admits a closed lightlike geodesic.

Proof. Take a causal loop \(\gamma\) at some \(p \in M\). If \(\gamma\) were not a lightlike geodesic loop then \(p \ll p\) (Theorem 2.30), in contradiction with chronology condition. And if \(\gamma\) were not closed, run it twice to obtain the same contradiction. 

Now, recall relation \(\rightarrow^{(\leq)}\) in Definition 2.22.

Theorem 3.9. In any causal spacetime \((M, g)\): \(x \rightarrow^{(\leq)} y \iff x \rightarrow y\).

Proof. \((\Rightarrow)\). If \(x \rightarrow y, x \neq y\), then \(x\) and \(y\) are connected by a (non-necessarily unique) maximizing lightlike geodesic contained in \(J^+(x) \cap J^-(y)\). Taken \(x < p < q < y\), the points \(p\) and \(q\) must lie on a unique maximizing lightlike geodesic \(\gamma\), which will also cross \(x\) and \(y\) (otherwise, there would be a broken causal curve joining \(x\) with \(y\), and hence \(y \in I^+(x)\)). Thus, \(J^+(p) \cap J^-(q)\) is nothing but the image of a portion of \(\gamma\), which can be either homeomorphic to a segment joining \(p\) to \(q\), or to a circumference (the latter excluded by the causality of \((M, g)\)).
If \( x \rightarrow^{(\leq)} y, x \neq y \), there are \((p_n, q_n), x < p_n < q_n < y, p_n \rightarrow x, q_n \rightarrow y\), such that \( J^+(p_n) \cap J^-(q_n) \) is linearly ordered; in particular, \( x < y \). But clearly \( x \not< y \) because, otherwise, as \( I^+ \) is open, \( q_n \in I^+(p_n) \) for large \( n \). That is, the open set \( I^+(p_n) \cap I^-(q_n) \) would be non empty, which clearly makes \( J^+(p_n) \cap J^-(q_n) \) non-isomorphic to \([0, 1]\).

![Figure 6. Chronological non-causal cylinder, and causal but non-distinguishing spacetime obtained by removing \{p\}. If one also removed the vertical half line below \( p \), a causal past-distinguishing but non-future distinguishing spacetime would be obtained.](image)

3.4. Distinguishing spacetimes. The set of parts of \( M \), i.e., the set of all the subsets of \( M \), will be denoted \( \mathcal{P}(M) \). Here it is regarded as a point set, but it will be topologized later (see Proposition 3.38).

The equivalence between some alternative definitions of distinguishing is somewhat subtler than in previous cases [30, 50]. So, we need the following previous result, which is proved below.

**Lemma 3.10.** The following properties are equivalent for \((M, g)\):

(i) \( I^+(p) = I^+(q) \) (resp. \( I^-(p) = I^-(q) \)) \( \Rightarrow p = q \),

(ii) The set-valued function \( I^+ \) (resp. \( I^- \)) : \( M \rightarrow \mathcal{P}(M), p \rightarrow I^+(p) \) (resp. \( p \rightarrow I^-(p) \)), is one to one,

(iii) Given any \( p \in M \) and any neighborhood \( U \ni p \) there exists a neighborhood \( V \subset U, p \in V \), which distinguishes \( p \) in \( U \) to the future (resp. past) i.e. such that any future-directed (resp. past-directed) causal curve \( \gamma : I = [a, b] \rightarrow M \) starting at \( p \) meets \( V \) at a connected subset of \( I \) (or, equivalently, if \( p = \gamma(a) \) and \( \gamma(b) \in V \) then \( \gamma \) is entirely contained in \( V \)).
(iv) Given any $p \in M$ and any neighborhood $U \ni p$ there exists a neighborhood $V \subset U$, $p \in V$, such that $J^+(p, V) = J^+(p) \cap V$ (resp. $J^-(p, V) = J^-(p) \cap V$).

**Definition 3.11.** A spacetime $(M, g)$ is called future (resp. past) distinguishing if it satisfies one of the equivalent properties in Lemma 3.10. A spacetime is distinguishing if it is both, future and past distinguishing.

**Proof.** (Lemma 3.10 for the future case.) (i) $\iff$ (ii) and (iii) $\implies$ (iv) Trivial.

No (i) $\implies$ no (iii). Let $p \neq q$ but $I^+(p) = I^+(q)$, take $U \ni p$ such that $q \notin U$ and any $V \ni p, V \subset U$. Then, choose $p' \in V, p \ll V p'$ and any $q' \notin U, q' \neq q$, on a future-directed timelike curve $\gamma_1$ which joins $q$ with $p'$. The required $\gamma$ is obtained by joining $p, q'$ with a future-directed timelike curve $\gamma_0$, and then $q'$ and $p'$ through $\gamma_1$.

No (iii) $\implies$ no (i). Let $U \ni p$ be a neighborhood where (iii) does not hold, that is, every $V \subset U$ intersects a suitable ($V$-dependent) future inextendible causal curve starting at $p$ in a disconnected set of its domain $I$. Take the sequence $\{V_n\}_n$ of nested globally hyperbolic neighborhoods in Theorem 2.14. They will be causally convex in some $U' \subset U$ and we can assume $U = U'$ (if (iii) does not hold for the pair $(p, U)$ then it does not hold for the pair $(p, U')$, $U' \subset U$), being $U$ also with closure contained in a simple neighborhood $W$. For each $V_n$, the causal curve $\gamma_n$ which escapes $V_n$ and then returns $V_n$ also escapes $U$ (because of causal convexity) and then returns to some point in the boundary $q_n \in \bar{U}$ which is the last one outside $U$, and to another point $p_n \in V_n$. As $W$ was simple, $\{q_n\} \to q \in \bar{U}$, up to a subsequence. Even more, $q \in J^-(p, W)$, because $q_n \in J^-(p_n, W)$, $(q_n, p_n) \to (q, p)$, and $J^-$ is closed on any convex neighborhood (see Prop. 2.10). Thus, $I^+(p) \subset I^+(q)$. Moreover, let $q' \in I^+(q)$ then, for large $n$, $(p \leq) q_n \ll q'$, that is $q' \in I^+(p)$, $I^+(q) \subset I^+(p)$.

No (iii) $\implies$ No (iv). Follow the reasoning in the last implication, with the same assumptions on $U$, and assuming that such a $V$ as in (iv) exists. Notice that connecting the obtained $q \in J^-(p, W)$ (satisfying $I^+(p) = I^+(q)$) with $p$ by means of the unique geodesic $\rho$ in $W$, one point $q_\rho \in (V \cap \rho) \setminus \{p\}$ will also satisfy $I^+(p) = I^+(q_\rho)$. But, as $U$ is convex, $J^+(p, U) = I^+(q_\rho, U)$ (use Prop. 2.10) and, even more, this holds arbitrarily close to $q_\rho$. Concretely, $(J^+(p) \cap V =) J^+(p, V) \not\supset I^+(q_\rho, V) \subset J^+(p, V) \cap V$, a contradiction.

**Remark 3.12.** (1) One can give easily another two alternative characterizations of being distinguishing, say (iii'), (iv'), just by replacing causal curves and futures in (iii), (iv) by timelike curves and chronological futures.

(2) Notice that, Lemma 3.10 also allows to define in a natural way what means to be distinguishing at $p$. In this case, for any neighborhood $U$ of $p$, a neighborhood $V$ which distinguishes $p$ in $U$ satisfies (iii) (and, thus, (iv)) for future and past causal curves. Notice also that, given $U$, one can find another neighborhood $U'$ and a sequence of nested neighborhoods $V_n \subset U'$ such that $\cap_n V_n = \{p\}$ and each $V_n$ is causally convex in $U'$ (see also Theorem 2.11).

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4These statements can be strengthened, as any convex subset $U$ is, in fact, causally simple and, thus, any open neighborhood of $U$ is not only distinguishing, but stably (and strongly) causal.
(3) Note that if $V$ future-distinguishes $p$ in $U$ then it also future-distinguishes in $U$ any other point $q$ on a future directed causal curve $\rho$ starting at $p$ contained in $V$.

(4) Obviously, any past or future distinguishing spacetime is causal (if $p,q$ lie on the same closed causal curve then $I^\pm(p) = I^\pm(q)$), but the converse does not hold (Fig. II).

A remarkable property of distinguishing spacetimes (complementary to Prop. 3.10 below) is the following [33].

**Proposition 3.13.** Let $(M_1, g_1), (M_2, g_2)$ be two spacetimes, $(M_1, g_1)$ distinguishing, and $f : M_1 \to M_2$ a diffeomorphism which preserves $\leq$, that is, such that:

$$p \leq q \Leftrightarrow f(p) \leq f(q).$$

Then $(M_2, g_2)$ is distinguishing and $g_1 = f^*g_2$.

**Proof.** Let us show that $(M_2, g_2)$ is distinguishing. First note that since $f$ is bijective it preserves also $<$. Take $p_2 \in M_2$, $U_2 \ni p_2$ and let $p_1 = f^{-1}(p_2)$, $U_1 = f^{-1}(U_2)$. Let $U_1 \subset U_1$ be a neighborhood which distinguishes $p_1$ in $U_1$, and let us check that $V_2 = f(U_1) \subset U_2$ distinguishes $p_2$ in $U_2$. Otherwise, there would be a causal curve $\gamma_2$ intersecting $V_2$ in a disconnected set of its domain. In particular one could choose points on the curve $p_2^1 < p_2^2 < p_2^3$ such that $p_2^1, p_2^2 \in V_2$, $p_2^3 \notin V_2$, hence $p_1^i = f^{-1}(p_2^i)$, $i = 1, 2, 3$, would satisfy the same property with respect to $V_1$ a contradiction.

Let $p \in M$, $g_1 \in g_1$, $g_2 \in g_2$, two metric representatives, $U_1 \ni p$, $U_2 \ni f(p)$ two simple neighborhoods with respect to the metric structures $(M, g_1)$ and $(M, g_2)$, and $V_1 \ni p$, $V_1 \subset U_1$ a neighborhood such that $V_2 = f(V_1) \subset U_2$ and both, $V_1$ and $V_2$ distinguish $p_1$, $p_2$ in $U_1$, $U_2$, respectively. Then:

$$f(J^+_1(p, V_1)) = f(J^+_1(p) \cap V_1) = f(J^+_1(p)) \cap f(V_1)$$

$$= J^+_2(f(p)) \cap V_2 = J^+_2(f(p), V_2) \subset U_2.$$  

The causal cones on $T_pM_1$ for the conformal structure $g_1$ are determined through the exponential diffeomorphism from the knowledge of $J^+_1(p, V)$ and since it coincides up to a pullback with $J^+_2(p, V)$ we conclude by Lemma 2.7 that $g_1 = f^*g_2$. □

**Remark 3.14.** (1) Again, an obvious timelike version of this result holds.

(2) Particularly interesting is the case in which $f$ is the identity map, as it states that, in a distinguishing spacetime, $J^+$ (as well as $I^+$) determines the metric, up to a conformal factor.

### 3.5. Continuous causal curves

When questions on convergences of curves are involved, the space of piecewise smooth causal curves is not big enough. So, the following extension of these curves (which becomes especially interesting in strongly causal spacetimes) is used.

**Definition 3.15.** A continuous curve $\gamma : I \to M$ is future-directed causal at $t_0 \in I$ if for any convex neighborhood $U \ni \gamma(t_0)$ there exist an interval $G \subset I$, $\gamma(G) \subset U$, such that $G$ is an open neighbourhood of $t_0$ in $I$, and satisfies: if $t' \in G$ and $t' < t_0$
The causal hierarchy of spacetimes

(resp. \( t_0 < t' \)) then \( \gamma(t') \prec_U \gamma(t_0) \prec_U \gamma(t) \). The continuous curve \( \gamma \) is said to be future-directed causal if it is so at any \( t \in I \).

The definition for past-directed is done dually.

Recall that it is enough to check this definition for one convex neighbourhood \( U \) of \( \gamma(t_0) \) and, if \( t_0 \) is not an extreme of \( I \), then \( G \) is just some open neighbourhood of \( t_0 \) included \( I \). In fact:

**Proposition 3.16.** A continuous curve \( \gamma : I \to M \) is a future-directed causal if and only if for each convex neighbourhood \( U \), given \( t, t' \in I, t < t' \) with \( \gamma([t, t']) \subset U \), it is \( \gamma(t) \prec_U \gamma(t') \).

**Proof.** For the left is trivial. For the converse, let \( \gamma([t_0, t_1]) \in U \), and assume by contradiction that \( \gamma(t_0) \nprec_U \gamma(t_1) \). Then there is a maximal \( \bar{t} \in (t_0, t_1) \), such that \( \gamma(t) \prec_U \gamma(t) \) for any \( t \in (t_0, \bar{t}) \). But since \( \gamma \) is also future-directed causal at \( \bar{t} \) (and the causal relation is transitive) a contradiction with maximality is obtained.

Note that several causal properties of piecewise smooth curves hold naturally in the continuous case. For instance, if \( \gamma \) is a continuous causal curve which connects \( p \) to \( q \in E^+(p) \) then \( \gamma \) is a maximal lightlike pregeodesic.

**Remark 3.17.** This definition could be extended naturally to continuous timelike curves. Nevertheless, recall that a possibility of confusion appears here. The curve in \( L^2 \), \( \gamma(t) = (\tan t, t) \), regarded as a continuous curve, would be future-directed timelike. Nevertheless, regarded as a (piecewise) smooth curve, it is not timelike at \( t = 0 \), as \( \gamma' \) is lightlike there.

**Remark 3.18.** A (future-directed, continuous) causal curve \( \gamma \) must be locally Lipschitzian (when suitably reparametrized, for the distance associated to any auxiliary Riemannian metric) and, thus, almost everywhere differentiable \([10, 2.26]\). If the interval \( I \) is compact, then \( \gamma \) becomes Lipschitzian with finite integral of its length. Therefore, it is absolutely continuous, and contained in a Sobolev space \( H^1 \), see \([2] \) Appendix\) for a detailed study.

Notice that a continuous curve \( \gamma \), a.e. differentiable, with timelike gradient (in the same time-orientation at each differentiable point) and finite integral of its length, is not necessarily a continuous causal curve. A counterexample in Lorentz-Minkowski spacetime \( L^2 \) can be constructed as follows. Consider a Cantor curve \( t \to x(t), t \in [0, 1] \) which is continuous, with 0 derivative a.e., and connects \( x(0) = 0, x(1) = 2 \). Now, the curve in natural coordinates of \( L^2 \), \( \gamma(t) = (x(t), t) \) satisfies all the required properties, but connects the non-causally related points \((0, 0), (2, 1)\). In order to be a causal curve, \( \gamma \) must be additionally (locally) Lipschitzian. In fact, it is possible to prove \([2]\): let \((M, g)\) be a spacetime, and \( \gamma : [a, b] \to M \) a continuous curve. Then \( \gamma \) is future-directed causal if and only if \( \gamma \) is \( H^1 \) (up to a reparametrization) and \( \gamma'(s) \) is a future-directed causal vector for \( s \in I \) a.e.

Continuous causal (and timelike) curves can be characterized in distinguishing spacetimes as follows:
Proposition 3.19. Let \((M, g)\) be a distinguishing spacetime. A continuous curve 
\(\gamma : I \to M\) is causal (either past or future directed) if and only if it is totally 
ordered by \(<\) (i.e. for any pair \(t_1, t_2 \in I\), \(t_1 < t_2\), \(p = \gamma(t_1)\), \(q = \gamma(t_2)\), either 
\(p < q\) or \(q < p\)).

A timelike version of the previous theorem (in fact easier to prove), where \(<\) 
and causal are replaced with \(\ll\) and timelike, also holds (for smooth versions see [20]).

Proof. To the right is trivial. For the converse, we claim first that the causal 
relation \(<\) is consistent along \(I\), that is, either \(t < t' \Rightarrow \gamma(t) < \gamma(t')\), or \(t < t' \Rightarrow \gamma(t') < \gamma(t)\) on all \(I\) (the first possibility will be assumed below). In fact, let us 
check that, in the case \(t_1 < t_2\) and \(\gamma(t_1) < \gamma(t_2)\), then \(t_1 < t_3 \Rightarrow \gamma(t_1) < \gamma(t_3)\). 
Otherwise, since the spacetime is distinguishing, defined \(L^\pm(p) = J^\pm(p) \setminus \{p\}\), it is, 
\(L^+(p) \cap L^-(p) = L^+(p) \cap L^-(p) = \emptyset\). Thus, putting \(p = \gamma(t_1)\), there is \(t\) included 
in either \([t_2, t_3]\) or \([t_3, t_2]\) such that \(r = \gamma(t) \in L^+(p) \cap L^-(p)\), which is impossible 
because either \(r \in L^+(p)\) or \(r \in L^-(p)\). Using a similar reasoning, \(t_3 < t_1\) implies 
\(\gamma(t_3) < \gamma(t_1)\), and the claim follows.

Now, let \(t_0 \in I\), \(p_0 = \gamma(t_0)\), \(U\) a convex neighborhood of \(p_0\) and \(V \supseteq p_0\), \(V \subset U\) 
a neighborhood which distinguishes \(p_0\) in \(U\). If \(t_0 < t\) we have \(p_0 < \gamma(t)\) and, thus, 
\(p_0 <_V \gamma(t)\), \(p_0 <_U \gamma(t)\), and the result follows. \(\square\)

Remark 3.20. This property does not characterize exactly distinguishing spacetime. 
Indeed, the reader may convince him/herself that in the causal past-distinguishing but non-future 
distinguishing spacetime obtained from Figure 3 by removing a vertical half-line, any continuous curve \(\gamma\) 
totally ordered by \(<\) is either a future or a past directed causal continuous curve.

3.6. Strongly causal spacetimes. The following equivalence is straightforward by using Theorem 3.19

Lemma 3.21. For any event \(p\) of a spacetime \((M, g)\), the following sentences are 
equivalent:

(i) Given any neighborhood \(U\) of \(p\) there exists a neighborhood \(V \subset U\), \(p \in V\) 
(which can be chosen globally hyperbolic), such that \(V\) is causally convex in \(M\) and thus in \(U\).

(ii) Given any neighborhood \(U\) of \(p\) there exists a neighborhood \(V \subset U\), \(p \in V\), 
such that any future-directed (and hence also any past-directed) causal curve 
\(\gamma : I \to M\) with endpoints at \(V\) is entirely contained in \(U\).

Definition 3.22. A spacetime \((M, g)\) is called strongly causal at \(p\) if it satisfies 
one of the equivalent properties in Lemma 3.21. A spacetime is strongly causal if 
it is strongly causal at \(p\), for any \(p \in M\).
Remark 3.23. (1) Item (i) in Lemma 3.21 collects the intuitive idea that, in a strongly causal spacetime, no causal “almost closed curve or loop” exist. Moreover, it also shows that strongly causal spacetimes are distinguishing (but the converse does not hold, see Fig. 7). Item (ii) is more frequently used as definition.

(2) Notice that $V$ in item (i) can be assumed included in a normal neighborhood, and a nested sequence $\{V_n\}_n$ of globally hyperbolic neighborhoods as in Theorem 2.14 can be taken. Then, chosen any representative $g \in g$, the causal relations and time-separation function on each $V_n$ regarded as a spacetime, agrees with the time-separation on $M$ restricted to $V_n$.

Finally, recalling the binary relations in Definition 2.22.

Theorem 3.24. In a strongly causal spacetime, $x \leq^{(\rightarrow)} y \iff x \leq y$.

Proof. $(\Rightarrow)$. It holds trivially in any spacetime.

$(\Leftarrow)$ Recall first the following claim: in any spacetime, if $p < q$ then $p, q$ can be connected by means of a piecewise smooth future-directed lightlike curve $\gamma$, such that each unbroken piece is a geodesic without conjugate points. In particular, the required implication holds trivially in any convex neighborhood $U$ regarded as a spacetime.

Thus, let $\gamma : [0, 1] \to M$ be one such one such unbroken future-directed geodesic piece of a curve connecting $x$ and $y$. Choose for each $p \in \gamma$ a convex neighborhood $U$ and a causally convex neighborhood $V \subset U$. Taking $\epsilon = 1/m$ for some large integer $m$ (a Lebesgue number of the covering), each consecutive $\gamma(ke), \gamma((k + 1)e)$, $(k = 0, \ldots, m - 1)$ lie in one such $V$ and, thus, satisfy $J^+(q, V) = J^+(q) \cap V$ for any $q \in \gamma([ke, (k + 1)e])$ (see Remark 3.12(3)). Therefore, $\gamma(ke) \leq^{(\rightarrow)} \gamma((k + 1)e)$, as required.

Finally, in order to prove the claim is sufficient to check that, given a timelike curve $\rho$, for any point $p = \rho(t_0)$ and a sufficiently small $\delta > 0$ the events $p = \rho(t_0)$ and $p_\delta = \rho(t_0 + \delta)$ can be connected by means of one such $\gamma$ with one break. This can be checked by taking a convex neighborhood $W$ of $p$ and noticing that, for small $\delta$, any past directed lightlike geodesic starting at $p_\delta$ will cross $E^+(p, W)$.

The properties below justify that strong causality is one of the most important assumptions on causality.

3.6.1. Characterization with Alexandrov’s topology. The following topology can be defined in any set with a binary relation type $\ll$.

Definition 3.25. Let $(M, g)$ be a spacetime. Alexandrov’s topology $\mathcal{A}$ on $M$ is the one which admits as a base the subsets:

$$B_{\mathcal{A}} = \{ I^+(p) \cap I^-(q) : p, q \in M \}$$

Remark 3.26. Its easy to check that $B_{\mathcal{A}}$ is always a base for some topology. Notice also that, for any $p, q \in M$, $I^+(p) \cap I^-(q)$ is open, thus the manifold topology is finer than Alexandrov’s.
Theorem 3.27. For a spacetime \((M, g)\), the following properties are equivalent:

(i) \((M, g)\) is strongly causal.

(ii) Alexandrov’s topology \(\mathcal{A}\) is equal to the original topology on \(M\).

(iii) Alexandrov’s topology is Hausdorff.

Proof. (i) \(\Rightarrow\) (ii). From Remark 3.26, we have just to show that for any open set \(U\) and \(x \in U\), there are \(p, q \in M\), such that \(x \in I^+(p) \cap I^-(q) \subset U\). To this end let \(V \subset U\), \(x \in V\) such that \(\ll_V\) agrees \(\ll\) on \(V\) (Remark 3.23(2)), and any pair \(p \ll_V x, q \gg_V x\) suffices.

(ii) \(\Rightarrow\) (iii). Trivial.

\[\neg (i) \Rightarrow \neg (iii).\] Assume that strong causality fails at \(p \in M\). Reasoning as Lemma 3.10 (implication \(\neg (iii) \Rightarrow \neg (i)\)), take a simple neighborhood \(U \ni p\), \(W \supset U\) convex, a sequence of nested globally hyperbolic neighborhoods \(\{V_n\}_n\) causally convex in \(U\), and a sequence of future-directed causal curves \(\{\gamma_n\}_n\), each one with endpoints \(p_n, p'_n \in V_n\), and such that \(\gamma_n\) escapes \(W\) and comes back at some last point \(q_n \in U\), \(\{q_n\} \to q \in \bar{U}\) up to a subsequence. So, \(q_n \leq_W p'_n, q \leq_W p\) (since \(J^+\) is closed in \(W\)), and hence \(q \leq p\).

Now, recall that, if \(q_1 \ll q \ll q_2\) then \(q_1 \ll p\) and \(p_n \ll q_2\) for large \(n\). Thus, \(p\) is an accumulation point of the Alexandrov open set \(I^+(q_1) \cap I^-(q_2)\) and, so, this open set is intersected by any (Alexandrov) open set which contains \(p\), as required.

3.6.2. Non-imprisoning spacetime. Strongly causal spacetimes will be non-imprisoning, that is, they will not contain any type of (partially) imprisoned causal curves, according to the following definitions.

Definition 3.28. Let \((M, g)\) be a spacetime, and let \(\gamma : [a, b) \to M\), be a causal curve with no endpoint at \(b\). Then:
(i) $\gamma$ is imprisoned (towards $b$) if, for some $\delta(\in (0, b-a))$, then $\gamma([b-\delta, b)) \subset K$, for some compact subset $K$.

(ii) $\gamma$ is partially imprisoned (towards $b$) if, for some sequence $\{t_m\} \nearrow b$, then $\gamma(t_m) \in K, \forall m \in \mathbb{N}$, for some compact subset $K$.

(Analogous definitions hold for $\gamma$ when defined on $(a, b]$.)

The following result is easy to prove.

**Proposition 3.29.** In a strongly causal spacetime, any causal curve $\gamma : [a, b) \to M$, with no endpoint at $b$ is a proper function (i.e., if $K \subset M$ is compact then $\gamma^{-1}(K)$ is compact).

**Remark 3.30.** Classical alternative statements of Proposition 3.29 are:

(a) If $\gamma$ crosses the compact subset $K$, it leaves $K$ at some point and never returns.

(b) Curve $\gamma$ is not partially imprisoned (nor imprisoned) in any compact $K$.

### 3.6.3. Limits of causal curves.

In the remaining of this subsection we will consider continuous causal curves, according to Definition 3.15.

**Definition 3.31.** Let $\{\gamma_k\}_k$ be a sequence of causal curves in a spacetime $(M, g)$.

- A curve $\gamma$ is a limit curve of $\{\gamma_k\}_k$ if there exists a subsequence $\{\gamma_{k_m}\}_m$ which distinguishes $\gamma$, i.e., such that:
  
  for all $p \in \gamma$, any neighborhood of $p$ intersects all $\{\gamma_{k_m}\}_m$ but a finite number of indexes.

- Assume that all the $\gamma_k$’s can be reparametrized in a compact interval $I = [a, b]$. A curve $\gamma : I \to M$ is a limit in the $C^0$ topology of $\{\gamma_k\}_k$ if:
  
  (i) $\{\gamma_k(a)\} \to \gamma(a)$, $\{\gamma_k(b)\} \to \gamma(b)$
  
  (ii) any neighborhood $U$ of $\gamma$ contains all $\gamma_k$’s, but a finite number of $\gamma_k$.

**Remark 3.32.** In general, these limits may be very bad behaved. For example, consider the quotient torus $T^2 = \mathbb{L}^2 / \mathbb{Z}^2$ and the projection $\gamma$ of the timelike curve $t \to (t, rt) \in \mathbb{L}^2$, where $r$ is an irrational number, $|r| < 1$. Then, any other curve $\rho$ in $T^2$ is a limit curve of the sequence $\{\gamma_n\}_n$ constantly equal to $\gamma$.

The properties of these limits are well-known (see for example [2, Ch. 3], [40, Ch. 6.7]), and remarkable ones appear in the strongly causal case. Summing up:

1. Any sequence of causal curves $\{\gamma_k\}_k$ without endpoints which admits a point of accumulation $p$, admits an inextendible causal limit curve $\gamma$ which crosses $p$. This result can be obtained by applying Arzela’s theorem [2, p. 76].

2. In strongly causal spacetimes:

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5An alternative approach to the present study of limits of curves is developed by O’Neill [49] by using the notion of quasi-limit. Here, we follow essentially [2] and [49], where we refer for detailed proofs.
• All limit curves are causal \cite[Lemma 2.39]{2} (and no inextendible limit curve $\gamma$ can be contained in a compact subset).

• Given a sequence $\{\gamma_k : I \to M\}, I = [a,b]$ which satisfies $\{\gamma_k(a)\} \to \gamma(a)$, $\{\gamma_k(b)\} \to \gamma(b)$, one has: $\gamma$ is a limit curve of $\{\gamma_k\}$ $\iff$ $\gamma$ is the limit of a subsequence $\{\gamma_{k_m}\}_m$ in the $C^0$ topology \cite[Prop. 3.34]{2}.

In this case the length $L$ for any metric $g$ in $g$ satisfies: $L(\gamma) \geq \lim m L(\gamma_k_m)$ \cite[Remark 3.35]{2} \cite[p. 54]{40}.

These properties have many applications for the geometry of the spacetime, for example \cite[Th. 8.10]{2}:

**Proposition 3.33.** If $(M,g)$ is a strongly causal spacetime then, for any $p \in M$, a future-directed geodesic ray $\gamma : [0,b) \to M$ starts at $p$ (that is, $\gamma$ is a f.-d. maximizing causal geodesic, $L(\gamma|_{[0,t]}) = d(p,\gamma(t))$, for all $t \in [0,b)$, with $p = \gamma(0)$ and no future endpoint).

**3.6.4. Isometries.** Finally, it is worth pointing out that, in the class of strongly causal spacetimes, the time-separation $d$ determines the metric (as the distance function of a Riemannian manifold determines the metric). Concretely (see \cite[Th. 4.17]{2}, which is extended to characterize homothetic maps and totally geodesic submanifolds):

**Theorem 3.34.** Let $(M,g)$, $(M',g')$ be two spacetimes with the same dimension, and let $(M,g)$ be strongly causal. If $f : M \to M'$ is an onto map (non-necessarily continuous) and $f$ preserves de time-separations $d,d'$, i.e.,

$$d(p,q) = d'(f(p),f(q)), \forall p,q \in M$$

then $f$ is a diffeomorphism and a metric isometry.

In particular, when $M = M'$ ($f = \text{identity}$), it holds: the time-separations of $g$ and $g'$ coincide if and only if $g = g'$.

**3.7. A break: volume functions, continuous $I^\pm$, reflectivity.**

**3.7.1. Admissible measures.** A pair of functions constructed from the volumes of $I^\pm(p), p \in M$ becomes very useful to study Causality. Nevertheless, for such a purpose the volumes must be finite and, thus, the natural measure of a spacetime associated to the metric may not be useful (even more, the representative $g$ in the conformal class $g$ of the spacetime must be irrelevant for the definition of the functions). An appropriate choice of the Borel measure (i.e. a measure on the $\sigma-$algebra generated by the open subsets of $M$) is:

The measure $m$ associated to any auxiliary (semi-)Riemannian metric $g_R$ with finite total volume $m(M)$.

Without loss of generality, it can be completed in the standard way, by adding to the Borel sigma algebra all the subsets of any subset of measure 0 (which are regarded as new subsets of measure 0); by Sard’s theorem, the subsets of measure 0 are intrinsic to the differentiable structure of $M$. It is worth pointing out:
• **Construction of \( m \).** Without loss of generality, we can assume that \( M \) is orientable (otherwise, reason with the orientable Lorentzian double-covering \( \Pi : \tilde{M} \to M \), and define the measure of any Borelian \( A \subset M \) as \(-\)one half of the measure of \( \Pi^{-1}(A) \)). Choose an orientation, and let \( \omega_g \) be the oriented volume element associated to the metric of the spacetime \( g \) (or any other semi-Riemannian metric). Fix any covering of \( M \) by open subsets with \( \omega_g \)-measure smaller than 1, and take a partition of the unity \( \{\rho_n\}_{n \in \mathbb{N}} \) subordinated to the covering. Define the measure \( m \) as the one associated to the volume element

\[
\omega_R = \sum_{n=1}^{\infty} 2^{-n} \rho_n \omega.
\]

(7)

It is easy to check that \( \omega_R \) is the measure associated to some pointwise conformal metric for any auxiliary (semi-)Riemannian metric (see [44] for more details).

• **Relevant properties of the measures.** The so-defined measure \( m \) satisfies:

1. Finiteness: \( m(M) < \infty \).
   
   This is straightforward from (7) and one can normalize \( m(M) = 1 \).
2. For any non-empty open subset \( U \), \( m(U) > 0 \).
3. The boundaries \( \dot{I}^{-}(p), \dot{I}^{+}(p) \) have measure 0, for any \( p \in M \).
   
   This holds for \( m \) because \( \dot{I}^{-}(p), \dot{I}^{+}(p) \) are closed, embedded, achronal topological hypersurfaces [27, Proposition 6.3.1]; thus, for any (differentiable) chart, they can be written as Lipschizian graphs, which have 0 measure.

Abstract measures satisfying these three properties were called *admissible* by Dieckmann [14], [2, Definition 3.19]; these properties are the only relevant ones for the applications below. Measure \( m \) constructed in (7) satisfies other interesting properties, as *regularity* (see [44] for a technical discussion).

Obviously, the third property cannot be deduced from the first and the second ones (choose a point \( q \in M \), and construct a new measure \( m' \) regarding \( q \) as an atom, say: \( m'(A) = m(A) + 1 \) if \( q \in A \), \( m'(A) = m(A) \) if \( q \notin A \), for all measurable subset \( A \)). Note that this third property implies \( m(I^{+}(p)) = m(\dot{I}^{-}(p)) = m(I^{+}(p)) = m(J^{+}(p)) \) for all \( p \), and analogously for \( I^{-} \).

### 3.7.2. Volume functions and their continuity.

In what follows, an admissible measure \( m \) on \( M \) is fixed. We have already regarded \( I^{+} \) (and analogously \( I^{-} \)) as a set-valued map in the set of parts of \( M \), \( I^{+} : M \to \mathcal{P}(M) \). As each \( I^{+}(p) \) is an open set, it lies in the \( \sigma \)-algebra of \( m \). So, one essentially takes the composition of \( m \) and \( I^{+} \) in the following definition.

**Definition 3.35.** Let \((M, g)\) be a spacetime with an admissible measure \( m \). The future \( t^{-} \) and past \( t^{+} \) volume functions associated to \( m \) are defined as:

\[
t^{-}(p) = m(I^{-}(p)), \quad t^{+}(p) = -m(I^{+}(p)), \quad \forall p \in M.
\]
Remark 3.36. Clearly, \( t^\pm \) satisfy:

1. They are both non-decreasing on any future-directed causal curve (in fact, the sign - is introduced for \( t^+ \) because of this reason). But perhaps they are not strictly increasing; in fact, they are constant on any causal loop.

2. They are not necessarily continuous (even though they are semi-continuous, see below). Fig. 8 shows a distinguishing counterexample.

The continuity of \( t^\pm \) is closely related to the continuity of \( I^\pm \). Nevertheless, we have to give an appropriate notion of what this means for a set-valued function. (In what follows, when there is no possibility of confusion we will make definitions and proofs for \( I^- \), \( t^- \), and the reasonings for \( I^+ \), \( t^+ \) will be analogous.)

Definition 3.37. Function \( I^- \), is inner (resp. outer) continuous at some \( p \in M \) if, for any compact subset \( K \subset I^-(p) \) (resp. \( K \subset M \setminus I^- (p) \)), there exists an open neighborhood \( U \ni p \) such that \( K \subset I^- (q) \) (resp. \( K \subset M \setminus I^- (q) \)) for all \( q \in U \).

As usual, \( I^\pm \) is (inner, outer) continuous when so is at each event \( p \in M \), and the spacetime is accordingly (future, past) inner or outer continuous. In order to understand better Definition 3.37 consider the following topology in \( P(M) \). For any compact \( K \subset M \), the subsets of \( M \) not intersecting \( K \) form a subset of \( P(M) \) which we define as open. These open sets are a base for the topology on \( P(M) \) considered in what follows.

Proposition 3.38. The set valued maps \( I^\pm : M \to P(M) \) satisfy:

(i) \( I^\pm \) are always inner continuous

(ii) \( I^\pm \) are outer continuous if and only if they are continuous as maps between topological spaces.

Proof. (i) Let \( K \subset I^-(q) \) be any compact subset. It is covered by the open sets \( \{ I^- (p) : p \in I^- (q) \} \), and admits a finite subcovering \( \{ I^- (p_1), \ldots, I^- (p_n) \} \). So, the neighborhood of \( q \), \( U = \bigcap_{i=1}^n I^+(p_i) \), has the required property.

(ii) Just check the definitions. \( \square \)

Nevertheless, it is easy to construct non-outer continuous examples (Fig. 8). And, in fact, this is related to the continuity of \( t^\pm \).

Lemma 3.39. The inner continuity of \( I^- \) (resp. \( I^+ \)) is equivalent to the lower (resp. upper) semi-continuity of \( t^- \) (resp. \( t^+ \)). Thus, it holds always.

Proof. As \( I^- \) is always inner continuous, only the implication to the right must be proved. Thus, let \( \{ p_n \} \to p \), fix \( \epsilon > 0 \) and let us prove \( t^- (p_n) > t(p) - \epsilon \) for large \( n \). There exists a compact subset \( K \subset I^- (p) \) such that \( m(K) > m(I^- (p)) - \epsilon = t^- (p) - \epsilon \) and, by inner continuity, \( K \subset I^- (p_n) \) for large \( n \). Thus, \( t^- (p_n) > m(K) > t^- (p) - \epsilon \), as required. \( \square \)

One can check this for any admissible measure (and it is obvious for any regular measure, as the explicitly constructed \( m \)), see [44, Lemma 3.7] for details.
Figure 8. \( M \subset \mathbb{L}^2 \), (in coordinates \( u, v, g = -2dudv \)), \( M = \{(u, v) \in \mathbb{L}^2 : |u|, |v| < 2\} \setminus \{(u, v) \in \mathbb{L}^2 : u = 0, -2 < v \leq 0\} \) is stably causal, \( I^+ \) is outer continuous but \( I^- \) is not. Correspondingly (for the canonical measure \( m \) of \( g \), \( t^+ \) is continuous but \( t^- \) is not as the sequence \( p_n = \left\{(1, 1/n)\right\} \) shows. As a consequence, \( M \) is non-causally simple (see Sect. 3.10), indeed, for instance, \( J^+(q) \) is not closed for \( q = (-1, -1) \).

Lemma 3.40. The following properties are equivalent: (i) \( I^- \) (resp. \( I^+ \)) is outer continuous at \( p \), and (ii) volume function \( t^- \) (resp. \( t^+ \)) is upper (resp. lower) semi-continuous at \( p \).

Proof. (i) \( \Rightarrow \) (ii) Completely analogous to the previous case, taking now \( K \) as a compact subset of \( M \setminus \bar{I}^-(p) \) with \( m(K) > m(M) - m(K) < t^-(p) + \epsilon \).

(ii) \( \Leftarrow \) (i) If \( I^- \) is not outer continuous, there exists a compact \( K \subset M \setminus \bar{I}^-(p) \) and a sequence \( \{p_n\} \rightarrow p \) such that each \( \bar{I}^- (p_n) \cap K \) contains at least one point \( r_n \).

Thus, \( r_n \rightarrow r \in K \), up to a subsequence, and choose \( s \ll r \in M \setminus \bar{I}^-(p) \) (\( s \) exists otherwise \( I^-(r) \subset I^-(p) \) then \( r \in \bar{I}^-(p) \) a contradiction). As the chronological relation is open, there exist neighborhoods \( U, V \subset M \setminus \bar{I}^-(p) \) of \( s, r \), resp., such that \( U \cap r \in V \setminus I^- (r) \), and, thus, \( U \subset I^-(p_n) \) for large \( n \). Now, choose a sequence \( \{q_j\} \rightarrow p \) satisfying

\[
p \ll q_j \ll q_{j-1}, \quad \text{for all } j.
\]

Then, \( U \subset I^-(q_j) \) for all \( j \) and, putting \( \epsilon = m(U) > 0 \):

\[
t^-(q_j) = m(I^-(q_j)) \geq m(I^-(p)) + m(U) = t^-(p) + \epsilon.
\]

Thus, the previous two lemmas yields directly:

Proposition 3.41. The following properties are equivalent for a spacetime:

(i) The set valued map \( I^- \) (resp. \( I^+ \)) is (outer) continuous.

(ii) Volume function \( t^- \) (resp. \( t^+ \)) is continuous.
3.7.3. Reflectivity. Continuity of $I^\pm$ (and, thus, $t^\pm$) can be also characterized in terms of reflectivity.

**Lemma 3.42.** Given any pair of events $(p, q) \in M \times M$ the following logical statements are equivalent:

(i) $I^+(p) \supset I^+(q) \Rightarrow I^-(p) \subset I^-(q)$,  
(resp. $I^-(p) \supset I^-(q) \Rightarrow I^+(p) \subset I^+(q)$)

(ii) $q \in \bar{I}^+(p) \Rightarrow p \in \bar{I}^-(q)$,  
(resp. $q \in \bar{I}^-(p) \Rightarrow p \in \bar{I}^+(q)$)

(iii) $q \in \dot{I}^+(p) \Rightarrow p \in \dot{I}^-(q)$,  
(resp. $q \in \dot{I}^-(p) \Rightarrow p \in \dot{I}^+(q)$).

**Proof.** (Equivalence in the past case). (i) $\Leftrightarrow$ (ii). Trivial from the equivalences: (a) $I^+(q) \subset I^+(p) \Leftrightarrow q \in \bar{I}^+(p)$, and (b) $I^-(p) \subset I^-(q) \Leftrightarrow p \in \bar{I}^-(q)$.

(ii) $\Leftrightarrow$ (iii). To the right, recall: $q \in \bar{I}^+(p) \Rightarrow q \in \bar{I}^+(p)$ but $(p, q) \notin I^+ \Rightarrow p \in I^-(q)$ but $(p, q) \notin I^- \Rightarrow p \in I^+(q)$. For the converse: $q \in \bar{I}^+(p) \Rightarrow q \in \bar{I}^+(p)$ or $(p, q) \in I^+ \Rightarrow p \in \bar{I}^-(q)$ or $(p, q) \in I^- \Rightarrow p \in \bar{I}^+(q)$. \hfill \Box

**Definition 3.43.** A spacetime $(M, g)$ is past (resp. future) reflecting at $q \in M$ if any of the corresponding equivalent items (i), (ii), (iii) in Lemma 3.42 holds for the pair $(p, q)$ for every $p \in M$. A spacetime is past (resp. future) reflecting if it is so at any $q \in M$, and reflecting if it is both, future and past reflecting.

**Remark 3.44.** Notice that if the items of Lemma 3.42 are required for $(p, q)$, for every $q \in M$, a different property, say (past) pseudo-reflectivity at $p$, would be obtained. Even though pseudo-reflectivity and reflectivity would be equivalent as spacetime properties (i.e. with no reference to a single point), they are different as properties for a single event (as can be checked in Figure 8), the former not to be considered in what follows.

Another characterization of reflectivity is the following.

**Proposition 3.45.** A spacetime $(M, g)$ is past reflecting at $q$ (resp. future reflecting at $p$) if and only if

$(p', q) \in \bar{I}^+ \Rightarrow p' \in \bar{I}^-(q)$,  
(resp. $(p, q') \in \bar{I}^+ \Rightarrow q' \in \bar{I}^+(p)$).

An analogous result holds with $I$ replaced with $\dot{I}$.

**Proof.** (Past case). Assume the spacetime is past reflecting at $q$ and let $(p, q) \in \bar{I}^+$, then there are sequences $p_n \to p$, $q_n \to q$, $q_n \in I^+(p_n)$. Take any $s \in I^-(p)$, so that $p \in I^+(s)$ and for large $n$, $q_n \in I^+(s)$ which implies $q \in \bar{I}^+(s)$. By using past reflectivity at $q$, $s \in \bar{I}^-(q)$ and taking the limit $s \to p$, $p \in \bar{I}^-(q)$.

Conversely, assume that $(p', q) \in \bar{I}^+ \Rightarrow p' \in \bar{I}^-(q)$ and consider any $p$ such that $q \in I^+(p)$. Then, $(p, q) \in I^+$ which implies $p \in \bar{I}^-(q)$, that is, the spacetime is past reflecting at $q$. \hfill \Box

**Lemma 3.46.** The following properties are equivalent: (i) $I^-$ (resp. $I^+$) is outer continuous at $p$, and (ii) the spacetime is past (resp. future) reflecting at $p$. 

Proof. (i) \(\Rightarrow\) (ii). Let \(I^-\) be outer continuous at \(q\), and assume there is a \(p\) such that \(q \in I^+(p)\) but \(p \notin I^-(q)\). By outer continuity there is a neighborhood \(V \ni q\) such that for every \(q' \in V\), \(p \notin I^-(q')\), but since \(q \in I^+(p)\) there is \(q' \in V\) such that \((p, q') \in I^+, a contradiction.

(ii) \(\Rightarrow\) (i). Let the spacetime be past reflecting at \(p\) and assume by contradiction that \(I^-\) is not outer continuous. Then there is a compact \(K\), \(K \cap \overline{I^-}(p) = \emptyset\), and a sequence \(p_n \to p\) such that \(K \cap \overline{I^-}(p_n) \neq \emptyset\). Taken \(r_n \in K \cap \overline{I^-}(p_n)\), up to a subsequence \(r_n \to r \in K\), and for any \(s \in I^-(r)\) we have for large \(n\), \(p_n \in I^+(s)\), which implies \(p \in I^+(s)\). By using reflexivity at \(p\), \(s \in I^-(p)\), and making \(s \to r\), \(r \in I^-(p)\), a contradiction because \(r \in K\).

The set \(R\) of points that do not comply these conditions has been studied in detail. The set \(R\) is a suitable union of null geodesics without past or future endpoint [14, Prop. 1.7]. Moreover, no point of \(R\) is isolated [55], and optimal bounds for its dimension are known [28, 12]. From Lemma 3.46, obviously:

**Proposition 3.47.** The following properties are equivalent for \((M, g)\):

(i) The set valued map \(I^-\) (resp. \(I^+\)) is (outer) continuous.

(ii) The spacetime is past (resp. future) reflecting.

### 3.8. Stably causal spacetimes.

Volume and time functions are essential in this and following levels. We start discussing their relations with previous ones.

#### 3.8.1. Time-type functions and characterization of some levels.

**Definition 3.48.** Let \((M, g)\) be a spacetime. A (non-necessarily continuous) function \(t : M \to \mathbb{R}\) is:

- A **generalized time function** if \(t\) is strictly increasing on any future-directed causal curve \(\gamma\).
- A **time function** if \(t\) is a continuous generalized time function.
- A **temporal function** if \(t\) is a smooth function with past-directed timelike gradient \(\nabla t\).

Notice that a temporal function is always a time function \((d(t \circ \gamma(s)/ds) = g(\dot{\gamma}(s), \nabla t) > 0)\), but even a smooth time function may be non-temporal. From Remark 3.36 volume functions are not far from being generalized time ones. In fact, the next two theorems characterize this property.

**Theorem 3.49.** A spacetime \((M, g)\) is chronological if and only if \(t^-\) (resp. if and only if \(t^+\)) is strictly increasing on any future-directed timelike curve.

**Proof.** (\(\Leftarrow\)). Obvious. (\(\Rightarrow\)). If \(p \ll q\) but \(t^-(p) = t^-(q)\), necessarily almost all the points in the open subset \(I^+(p) \cap I^-(q)\) lie in \(I^-(p)\). Thus, any point \(r\) in \(I^+(p) \cap I^-(q) \cap I^-(p)\) satisfies \(p \ll r \ll p\).
Remark 3.50. Notice that, as \( t^- \) is also constant on any causal loop, causal spacetimes cannot be characterized in this way. Figure 3.50 gives an example of causal non-distinguishing spacetime for which \( t^- \) is constant along a causal curve (the central almost closed circle).

Theorem 3.51. A spacetime \((M, g)\) is past (resp. future) distinguishing if and only if \( t^- \) (resp. \( t^+ \)) is a generalized time function.

Proof. \((\Rightarrow)\). To prove that \( t^- \) is strictly increasing on any future-directed causal curve, assume that \( p < q, p \neq q \), but \( t^-(p) = t^-(q) \). Then, almost all the points of \( I^-(q) \) are included in \( I^-(p) \). Choose a sequence \( \{q_n\}_n \subset I^-(p) \cap I^-(q) \) converging to \( q \). Recall that, necessarily then \( I^-(q_n) \subset I^-(p) \) for all \( n \), and \( I^-(q) = \cup_n I^-(q_n) \), but this implies \( I^-(q) \subset I^-(p) \) and, as the reversed inclusion is obvious, the spacetime is non-past distinguishing.

\((\Leftarrow)\). If \( I^-(p) = I^-(q) \) with \( p \neq q \), choose a sequence \( \{p_n\}_n \subset I^-(p) \) which converges to \( p \), and a sequence of timelike curves \( \gamma_n \) from \( q \) to \( p_n \). By construction, the limit curve \( \gamma \) of the sequence starting at \( q \) is a (non-constant) causal curve and \( I^-(p) \subset I^-\gamma(t) \subset I^-(q) \) for all \( t \). Thus, the equalities in the inclusions hold, and \( t^- \) is constant on \( \gamma \).

3.8.2. Stability of causality and chronology. Stable causality is related with the simple intuitive ideas that the spacetime must remain causal after opening slightly its lightcones, or equivalently, under small \((C^0)\) perturbations of the metric. Surprisingly, this is equivalent to the existence of time and temporal functions.

More precisely, let \( \text{Lor}(M) \) be the set of all the Lorentzian metrics on \( M \) (which will be assumed time-orientable in what follows, without loss of generality). A partial (strict) ordering \(<\) is defined in \( \text{Lor}(M) \):

\[ g < g' \text{ if and only if all the causal vectors for } g \text{ are timelike for } g'. \]

Notice that this ordering is naturally induced in the set \( \text{Con}(M) \) of all the classes of pointwise conformal metrics on \( M \). Even more, it induces naturally a topology in \( \text{Con}(M) \), the interval topology, which admits as a subbasis the subsets type

\[ U_{g_1, g_2} = \{ g : g_1 < g < g_2 \} \]

where \( g_1, g_2 \in \text{Con}(M) \), \( g_1 < g_2 \).

Remarkably, the interval topology coincides with the topology induced in \( \text{Con}(M) \) from the \( C^0 \) fine topology on \( \text{Lor}(M) \). Roughly, the \( C^0 \) topology on \( \text{Lor}(M) \) can be described by fixing a locally finite covering of \( M \) by open subsets of coordinate charts with closures also included in the chart. Now, for any positive continuous function \( \delta : M \to \mathbb{R} \) and \( g \in \text{Lor}(M) \) one defines \( U_\delta(g) \subset \text{Lor}(M) \) as the set containing metrics \( \hat{g} \) such that, in the fixed coordinates at each \( p \), \( |g_{ij}(p) - \hat{g}_{ij}(p)| < \delta(p) \) (in order to define the \( C^r \) topology on \( \text{Lor}(M) \), this inequality is also required for the partial derivatives of \( g_{ij} \) up to order \( r \)). A basis for the \( C^0 \)-fine topology is defined as the set of all such \( U_\delta(g) \) constructed for any \( \delta \) and \( g \) (see [41, 2, 41] for more detailed descriptions of this topology). Then, the
(quotient) $C^0$-topology in $\text{Con}(M)$ is defined as the finer one such that the natural projection $\text{Lor}(M) \to \text{Con}(M), g \to g$ is continuous.

A way to define directly the $C^0$-topology on $\text{Con}(M)$ which shows the relation with the interval one is as follows [2, 32]. Fix an auxiliary Riemannian metric $g_R$, and, for each $g \in \text{Con}(M)$, define the $g_R$-unit lightcone at $p \in M$ as:

$$C_p^{g_R}(p) = \{v \in T_p M : g(v, v) = 0, g_R(v, v) = 1\}.$$ 

Now, if $|\cdot|_R$ is the natural $g_R$-norm, one define naturally the distance of any vector $w \in T_p M$ to $C_p^{g_R}(p)$ as usual:

$$d_R(w, C_p^{g_R}(p)) = \text{Min}\{|w - v|_R : v \in C_p^{g_R}(p)\}.$$ 

Given a second $\tilde{g} \in \text{Con}(M)$ with associated $g_R$-unit lightcone $\tilde{C}_p^{g_R}(p)$, the maximum and minimum distances between the lightcones are, respectively:

$$|g - \tilde{g}|_R^M(p) = \text{Max}\{d_R(v, \tilde{C}_p^{g_R}) : v \in C_p^{g_R}(p)\}, \quad |g - \tilde{g}|_R^m(p) = \text{Min}\{d_R(w, \tilde{C}_p^{g_R}) : w \in C_p^{g_R}(p)\}.$$ 

Notice that

$$0 < |g - \tilde{g}|_R^m \iff \text{either } g < \tilde{g} \text{ or } \tilde{g} < g.$$ 

Now, for any positive continuous function $\delta : M \to \mathbb{R}$, let $U_\delta(g) = \{\tilde{g} \in \text{Con}(M) : |g - \tilde{g}|_R^M < \delta\}$. The sets $U_\delta(g)$ yields a basis for the $C^0$ topology.

**Definition 3.52.** A spacetime $(M, g)$ is stably causal if it satisfies, equivalently:

(i) There exists $\tilde{g} \in \text{Con}(M)$ such that $g < \tilde{g}$ and $\tilde{g}$ is causal.

(ii) There exists a neighborhood $U$ of $g$ in the quotient $C^0$ topology such that all the metrics in $U$ are causal.

**Remark 3.53.** (1) The equivalence of both definitions is clear because, if $\tilde{g}$ is causal, then so are all the spacetimes with smaller lightcones, and these spacetimes constitute a $C^0$ neighbourhood.

(2) A property of a metric $g$ is called $C^r$ stable ($r = 0, 1, \ldots, \infty$) if it holds for a $C^r$ neighborhood of $g$. As the $C^r$ topologies for $r > 0$ are finer than the $C^0$ one, stable causality means that the metric of the spacetime is not only causal, but also that this property is stable in all the $C^r$ topologies.

**Proposition 3.54.** ($C^0$) stable chronology and stable causality are equivalent properties for any spacetime $(M, g)$.

**Proof.** Obviously, the latter implies the former. Let us show than non-stably causal implies non-stably chronological. Indeed, if the spacetime is non-stably causal, any $g_1 > g$ admits a closed causal curve $\gamma_1$. But since this is also true for any $g_2$ such that $g < g_2 < g_1$, then the corresponding $\gamma_2$ is a closed timelike curve with respect to $g_1$. Thus, any $g_1 > g$ admits a closed timelike curve.

A nice property of bidimensional spacetimes is the following.
Theorem 3.55. Any simply connected 2-dimensional spacetime \((M, g)\) is stably causal.

Proof. As \(M\) has 0 Euler characteristic (Th. 2.4), necessarily \(M\) must be homeomorphic to \(\mathbb{R}^2\). Obviously, it is enough to prove that any spacetime constructed on \(\mathbb{R}^2\) is causal. Otherwise, by closing if necessary the lightlike cones in a tubular neighborhood of \(\gamma\), we can assume that there exists a lightlike closed curve \(\gamma\) (regarded as Jordan’s curve) bounds a domain \(D\). Thus, taking any timelike vector field \(X\), we have \(g(X, \gamma')\) never vanishes, i.e., \(X\) must point out either outwards or inwards \(\gamma \equiv D\). Thus, a standard topological argument says that \(X\) must vanish on some point of \(D\), a contradiction.

3.8.3. Time and temporal functions. The following characterization of stable causality in terms of time-type functions (see Definition 3.48) becomes specially useful. Nevertheless, it has been proved with rigor only recently [6, 44].

Theorem 3.56. For a spacetime \((M, g)\) the following properties are equivalent:

(i) To be stably causal.

(ii) To admit a time function \(t\)

(iii) To admit a temporal function \(T\)

Proof. (Sketch with comments; see [44, Sect. 4] for detailed proofs and discussions). (iii) \(\Rightarrow\) (i) As causality is a conformally invariant, choose the representative \(g\) with \(g(\nabla T, \nabla T) = -1\). Now, the metric can be written as

\[ g = -dT^2 + h \]

where \(h\) is the restriction of \(g\) to the bundle orthogonal to \(\nabla T\) (up to natural identifications). Then, consider the one parameter family of metrics

\[ g_\lambda = -\lambda dT^2 + h, \quad \lambda > 0. \]

Clearly, \(T\) is still a temporal function for each \(g_\lambda\). Thus, \(g_\lambda\) is always causal, and \(g = g_1 < g_2\), as required.

(i) \(\Rightarrow\) (ii) (Hawking [20], see also [27, Prop. 6.4.9] or [44, Theorem 4.13]). The fundamental idea is that, even though the past volume function \(t^-\) may be non-continuous (it is only a generalized time-function), an “average” of such functions for a 1-parameter family of metrics \(g_\lambda\) will work if \(g_\lambda\) satisfies: (i) \(g_0 = g\), (ii) \(g_\lambda\) is causal, for all \(\lambda \in [0, 2]\), and (iii) \(\lambda < \lambda' \Rightarrow g_\lambda < g_{\lambda'}\). Concretely, one checks that the following function is a time function:

\[ t(p) = \int_0^1 t^-_\lambda(p) d\lambda, \]

where \(t^-_\lambda\) is the past volume function for, say, \(g_\lambda = g + (\lambda/2)(\bar{g} - g)\), \(\lambda \in [0, 2]\) (\(\bar{g}\) is chosen causal with \(g < \bar{g}\)).
(ii) ⇒ (iii) This has been one of the “folk questions” on smoothability of the theory of Causality until its recent solution \[6\]. It becomes crucial because, otherwise, the implication (ii) ⇒ (i) was also open. We refer to the detailed exposition in \[44\] Sect. 4.6] (see also the comments on smoothability for globally hyperbolic spacetimes below, especially Remark 3.77).

**Proposition 3.57.** Stable causality implies strong causality.

**Proof.** Let \( t \) be a time function and let us see that condition (ii) in Lemma \[3.21\] holds at any \( p \in M \). Let \( U \ni p \) a neighborhood and assume, without loss of generality, that \( U \) is simple, its closure is included in another simple neighborhood \( \tilde{U} \), and \( t(p) = 0 \). For any \( q \in U \) put \( \epsilon^+_q = \text{Min}\{t(r) : r \in J^+(q, \tilde{U}) \cap \tilde{U}\} \), \( \epsilon^-_q = \text{Min}\{-t(r) : r \in J^-(q, \tilde{U}) \cap \tilde{U}\} \); the variation of \( \epsilon^+_q \) with \( q \) is continuous because \( \tilde{U} \) is convex. As \( \tilde{U} \) is compact, \(-\epsilon^-_q < t(q) < \epsilon^+_q\); in particular, \( \epsilon^-_q, \epsilon^+_q > 0 \). Thus, for a small neighborhood \( W \ni p, W \subset U \), one has \( \epsilon^-_q, \epsilon^+_q > 0 \) for all \( q \in W \). From the compactness of \( W \), necessarily \( \epsilon_W := \text{Min}\{\epsilon^+_q : q \in W\} > 0 \). The required neighborhood is \( V = W \cap t^{-1}(\epsilon_W/2) \). In fact, if a future-directed timelike curve starts at some \( q \in V \) and leaves \( U \) at some point \( q_U \), then \( t(q_U) \geq \epsilon_W \); thus, \( \gamma \) cannot return to \( V \).

**Remark 3.58.** (1) Stable causality implies strong causality but the converse does not hold (see figure 9).

(2) Between strong and stable causality, an infinite set of levels can be defined by using Carter’s “virtuosity” \[10\].

![Figure 9](image-url)  
Figure 9. An example of strongly causal non-stably causal spacetime. By opening slightly the causal cones there appear closed causal curves.

### 3.9. Causally continuous spacetimes.

Taking into account the characterizations of the continuity of \( I^\pm \) (Prop. 3.41, 3.47) as well as the behavior of \( t^\pm \) in distinguishing spacetimes (Th. 3.51), the following definitions of causal continuity (which can be also combined with the characterizations of reflectivity, Lemma 3.42, Prop. 3.45) hold.
**Definition 3.59.** A spacetime \((M, g)\) is causally continuous if (equivalently, and for any admissible measure):

(i) Maps \(I^\pm : M \to \mathcal{P}\) are: (a) one to one, and (b) continuous (i.e., \((M, g)\) is reflecting, Lemma 3.46).

(ii) \((M, g)\) is: (a) distinguishing, and (b) with continuous volume functions \(t^\pm\).

(iii) The volume functions \(t^\pm\) are time functions.

**Remark 3.60.** Trivially, totally vicious spacetimes have continuous \(I^\pm\). Even more, they are also continuous in the causal non-distinguishing spacetime of Fig. 6 (notice that the removed point in the circle does not affect to function \(t^\pm\)). Thus, the injectivity of these maps (i.e., the hypotheses “distinguishing”) is truly necessary for this level of the ladder.

Recall that a causally continuous spacetime not only admits a time function, but also the past and future volume functions are time functions. In particular:

**Proposition 3.61.** Any causally continuous spacetime is stably causal.

**Remark 3.62.** (1) The converse does not hold, as the example in Fig. 8 shows.

(2) Until stable causality, all the levels in the hierarchy of causality, except non–totally vicious, were inherited by open subsets. This is not the case neither for causal continuity (as the counterexample in figures 8 shows, being obtained from an open subset of \(\mathbb{R}^2\)) nor for the remaining levels of the ladder.

### 3.10. Causally simple spacetimes

There are different characterizations of causal simplicity (Prop. 3.68), we will start by the simplest one.

**Definition 3.63.** A spacetime \((M, g)\) is causally simple if it is:

(a) causal, and

(b) \(J^+(p), J^-(p)\) are closed for every \(p \in M\).

Typically, the condition of being distinguishing is imposed directly in the definition of causal simplicity instead of causality, but the former can be deduced from the latter [8 Sect. 2], as proven next. Nevertheless, “causality” cannot be weakened in “chronology”, see Remark 3.72(1).

**Proposition 3.64.** Conditions (a) and (b) in Definition 3.63 imply that the spacetime is distinguishing.

*Proof. Otherwise, if \(p \neq q\) and, say \(I^+(p) = I^+(q)\), any sequence \(\{q_n\} \to q\), with \(q < q_n\), shows \(q \in I^+(q) = I^+(p) = J^+(p) = J^+(q)\). Thus, \(p < q\) and, analogously, \(q < p\), i.e., the spacetime is not causal.*

---

7A counterexample for total-viciousness can be obtained from Figure 5, taking the open region determined by \(1/3 < x < 2/3\).
Condition (b) has also the following consequence.

**Proposition 3.65.** If a spacetime satisfies that \( J^+(p) \) (resp. \( J^-(p) \)) is closed for every \( p \), then \( I^- \) (resp. \( I^+ \)) is outer continuous. Thus, condition (b) in Definition 3.63 implies the reflectivity of \((M,g)\).

**Proof.** Recall first the equivalence between outer continuity and reflectivity (Prop 3.47), and let us prove the characterization of Lemma 3.42(ii) (for the future case). As now \( \bar{I}^\pm = J^\pm \), we have:

\[ q \in \bar{I}^+(p) \Rightarrow p \in J^-(q) = \bar{I}^-(p). \]

**Remark 3.66.** (1) By Propositions 3.64 and 3.65, causally simple implies causally continuous, but the converse does not hold. A counterexample can be obtained just by removing a point to \( L^2 \). On the other hand, a spacetime may have closed \( J^-(p) \) for every \( p \) but non-closed \( J^+(q) \) for some \( q \) (Fig. 8).

(2) Even though these spacetimes are almost at the top of the causal hierarchy, a metric in the pointwise conformal class of a causally simple spacetime may have a time-separation \( d \) with undesirable properties (see Fig. 10). For example:

(a) For some \( p, q \), perhaps \( d(p,q) = \infty \).

(b) Even if \( 0 < d(p,q) < \infty \), perhaps no causal geodesic connects \( p \) and \( q \).

(c) \( d \) may be discontinuous.

This will be remedied in the last step of the hierarchy.

Property (b) of Definition 3.63 can be also characterized in different ways.

**Lemma 3.67.** Let \( J^+(p) \) and \( J^-(p) \) be closed for every \( p \in M \), then:

1. \( J^+(K) \) and \( J^-(K) \) are closed for every compact \( K \subset M \).
2. \( J^+ \) (and hence \( J^- \)), regarded as a subset of \( M \times M \), is closed.

**Proof.** (1) Otherwise if, say, \( q \in J^+(K) \setminus J^+(K) \) there exists sequences \( q_n \to q \), \( p_n < q_n \), \( p_n \in K \), where, up to a subsequence, \( p_n \to p \in K \). Thus, \( (p,q) \in \bar{I}^+ \) and, by using Proposition 3.65 (recall Prop. 3.63), \( q \in J^+(p) = J^+(p) \subset J(K) \).

(2) Obviously, \( J^+ \subset \bar{I}^+ \) and, for the converse, use again \( (p,q) \in \bar{I}^+ \Rightarrow q \in J^+(p) \).

Thus, on the basis of these results, we have the following characterization.

**Proposition 3.68.** A spacetime \((M,g)\) is causally simple if it is causal and satisfies one of the following equivalent properties:

(i) \( J^+(p) \) and \( J^-(p) \) are closed for every \( p \in M \).

(ii) \( J^+(K) \) and \( J^-(K) \) are closed for every compact set \( K \).

(iii) \( J^+ \) is a closed subset of \( M \times M \).

Finally, notice that causal relations can be obtained now “starting at chronology” (Def. 2.22).
Figure 10. An example of causally simple non-globally hyperbolic spacetime, with a general metric conformal to the usual one, \( g = \Omega^2(t, x)(-dt^2 + dx^2) \), \( p = (0, -1) \), \( q = (0, 1) \), \( \Omega > 0 \). If \( \Omega = 1 \) then \( d(p, q) = 2 \) but no geodesic connects them (Remark 3.66 case 2b) while if \( \Omega^2 = 1/(t^2 + x^2) \), \( d(p, q) = +\infty \) (case 2a). If \( \Omega^2 = 1/(x + 1)^2 \) then \( d \) is discontinuous (case 2c) as \( d(p, q) < +\infty \) but \( d(p, q') = +\infty \) for \( q' \gg q \) (because the connecting causal curves can approach a finite segment on the left-hand side border). The causal diamond \( J^+(r) \cap J^-(s) \) is not compact and there are inextendible causal curves which, being “created by the naked singularity”, pass through \( s \).

**Theorem 3.69.** In a causally simple spacetime\( ^8 \), \( x \leq (\ll) y \Leftrightarrow x \leq y \).

**Proof.** To the left, it is trivial in any spacetime. So, let \( x \leq (\ll) y \). Since \( I^+(y) \subset I^+(x) \), \( y \in I^+(x) = J^+(x) = J^+(x) \), where \( J^+ \) is the usual causal relation. \( \square \)

### 3.11. Globally hyperbolic spacetimes.

There are at least four ways to consider global hyperbolicity: (1) by strengthening the notion of causal simplicity, (2) by using Cauchy hypersurfaces, (3) by splitting orthogonally the spacetime and (4) by using the space of causal curves connecting each two points. We will regard (1) as the basic definition and will study subsequently the other approaches, as well as some natural results under them.

#### 3.11.1. Strengthening causal simplicity.

**Definition 3.70.** A spacetime \( (M, g) \) is **globally hyperbolic** if:

(a) it is causal, and

(b) the intersections \( J^+(p) \cap J^-(q) \) are compact for all \( p, q \in M \).

Following \[3\] Sect. 3], the next result yields directly that a globally hyperbolic spacetime (according to our Definition 3.70) is causally simple.

---

\( ^8 \)Notice that, for this result, one can define \( x \leq (\ll) y \Leftrightarrow either \ I^+(y) \subset I^+(x) \ or \ I^-(x) \subset I^-(y) \).
Proposition 3.71. Condition (b) implies both, $J^+(p)$ and $J^-(p)$ are closed for all $p \in M$.

Proof. Assume that $J^+(p)$ is not closed and choose $r \in \bar{J}^+(p) \setminus J^+(p)$ and $q \in I^+(r)$. Take a sequence $\{r_n\} \to r$ with $r_n \in I^+(p)$ for all $n$ (Prop. 2.17), and notice that $r_n \ll q$ up to a finite number of $n$ (Prop. 2.16). Thus, $\{r_n\}_n \subset J^+(p) \cap J^-(q)$, but converges to a point out of this compact subset, a contradiction.

Remark 3.72. (1) As stressed in [8], the full consistency of the causal ladder yields that any globally hyperbolic spacetime is not only causally simple but also strongly causal. This last hypothesis is usually imposed in the definition of global hyperbolicity, instead of causality, but becomes somewhat redundant. Notice that causality does not follow from property (b) and cannot be weakened. Indeed, there are chronological non-causal spacetimes which satisfy it (see Fig. 6).

(2) The open subset $M = \{(t, x) \in \mathbb{L}^2 : 0 < x\}$ shows that a causally simple spacetime may be non-globally hyperbolic.

Notice that the two conditions in Definition 3.70 are natural from the physical (even philosophical) viewpoint:

1. Causality avoids paradoxes derived from trips to the past (grandfather’s paradox). For example, one cannot “send a laser beam which describes a causal loop in the spacetime and kills him/herself”.

2. The compactness of the diamonds $J^+(p) \cap J^-(q)$ can be interpreted as “there are no losses of information/energy in the spacetime”. In fact, otherwise one can find a sequence $\{r_n\}_n \subset J^+(p) \cap J^-(q)$ with no converging subsequence. Taking a sequence of causal curves $\{\gamma_n\}_n$, each one joining $p, r_n, q$, the limit curve $\gamma_p$ starting at $p$ cannot reach $q$. This can be interpreted as something which is suddenly lost or created in the boundary of the spacetime (see Fig. 10). That is, a singularity (this sudden loss/creation) is visible from $q$—there are “naked singularities”.

3.11.2. Cauchy hypersurfaces and Geroch’s theorem. Recall that a subset $A \subset M$ is called achronal (resp. acausal) if it is not crossed twice by any timelike (resp. causal) curve. The following notions are useful in relation to Cauchy hypersurfaces.

Definition 3.73. Let $A$ be an achronal subset of a spacetime $(M, g)$.

- The domain of dependence of $A$ is defined as $D(A) = D^+(A) \cup D^-(A)$, where $D^+(A)$ (resp. $D^-(A)$) is defined as the set of points $p \in M$ such that every past (resp. future) inextendible causal curve through $p$ intersects $A$.

- The Cauchy horizon of $A$ is defined as $H(A) = H^+(A) \cup H^-(A)$, where $H^+(A) = D^+(A) \setminus I^-(D^+(A)) = \{p \in D^+(A) : I^+(p) \text{ does not meet } D^+(A)\}$, and $H^-(A)$ is defined dually.
One can check that, if \( A \) is a closed subset, then \( \dot{D}(A) = A \cup H^+(A) \). Recall that \( D(A) \) can be interpreted as the part of the spacetime predictable from \( A \). A Cauchy hypersurface is defined as an achronal subset from where the full spacetime is predictable:

**Definition 3.74.** A Cauchy hypersurface of a spacetime \((M, g)\) is, alternatively:

(i) A subset \( S \subset M \) which is intersected exactly once by any inextendible timelike curve.

(ii) An achronal subset \( S \), with \( D(S) = M \).

(iii) An achronal subset \( S \), with \( H(S) = \emptyset \).

Some properties of any such Cauchy hypersurface \( S \) are the following:

1. Necessarily, \( S \) is a closed subset and an embedded topological hypersurface.
2. The spacetime \( M \) is the disjoint union \( M = I_-(S) \cup S \cup I_+(S) \).
3. Any inextendible causal curve \( \gamma \) crosses \( S \) and, if \( S \) is spacelike (at least \( C^1 \)) then \( \gamma \) crosses \( S \) exactly once (in general \( S \) may be non-achronal because \( \gamma \) may intersect \( S \) in a segment, i.e., in the image of an interval \([c, d], c < d \)).
4. If \( K \) is compact then \( J^\pm(K) \cap S \) is compact.

In what follows, a function \( t : M \to \mathbb{R} \) (in particular, a time or temporal one, according to Definition 3.48) will be called Cauchy if its levels \( S_c = t^{-1}(c) \) are Cauchy hypersurfaces; without loss of generality, we can assume that Cauchy functions are onto. Notice that the levels of a Cauchy time function are necessarily acausal Cauchy hypersurfaces.

The characterization of global hyperbolicity in terms of Cauchy hypersurfaces comes from the following celebrated Geroch’s theorem [22].

**Theorem 3.75.** \((M, g)\) is globally hyperbolic if and only if it admits a Cauchy hypersurface \( S \).

Even more, in this case: (i) the spacetime admits a Cauchy time function. (ii) all Cauchy hypersurfaces are homeomorphic to \( S \), and \( M \) is homeomorphic to \( \mathbb{R} \times S \).

The implication to the left is a (non-trivial) standard computation written in many references (for example, [39, 56]). For the implication to the right and the last assertion, recall first the following result:

**Lemma 3.76.** In a globally hyperbolic spacetime, the continuous function

\[
t(p) = \log \left( \frac{-t^-(p)}{t^+(p)} \right) = \log \left( \frac{m(I^-(p))}{m(I^+(p))} \right)
\]  

satisfies:

\[
\lim_{s \to a} t(\gamma(s)) = -\infty, \quad \lim_{s \to b} t(\gamma(s)) = \infty
\]

for any inextendible future-directed causal curve \( \gamma : (a, b) \to M \).
Proof. It is sufficient to check:
\[ \lim_{s \to a} t^- (\gamma(s)) = 0, \quad \lim_{s \to b} t^+ (\gamma(s)) = 0. \]

Reasoning for the former, it is enough to show that, fixed any compact subset \( K \), then \( K \cap I^- (\gamma(s_0)) = \emptyset \) for some \( s_0 \in (a, b) \) (and, thus, for any \( s < s_0 \)), see [44] for details. Choose any point on the curve, \( q = \gamma(c) \) for some \( c \in (a, b) \), and assume by contradiction the existence of a sequence \( p_j = \gamma(s_j) \), \( s_j \to a \), \( s_j \in (a, c) \), with an associate sequence \( r_j \in K \cap I^- (p_j) \). Up to a subsequence, \( \{r_j\} \to r \), and choosing \( p \ll r \), one has \( p \ll p_j \leq q \), and \( \gamma|_{[a, c]} \) lies in the compact subset \( J^+(p) \cap J^-(q) \). That is, \( \gamma \) is totally imprisoned to the past, in contradiction with strong causality, (see Proposition 3.29).

Proof. (Only Th. 3.75 \( \Rightarrow \).) As \( t \) in Lemma 3.76 is a time function, each level \( S_c \) is an acausal hypersurface. In order to check that any inextendible timelike curve \( \gamma \) crosses \( S_c \) (thus proving (i)), recall that \( \gamma \) can be reparametrized on all \( \mathbb{R} \) with \( t \), and [19] will also hold under any increasing continuous reparametrization of \( \gamma \). Thus, assuming that this reparametrization has been carried out, \( \gamma(c) \in S_c \).

For assertion (ii), it is enough to choose a complete timelike vector field \( X \), (Prop. 2.3) and project the full spacetime onto \( S \) by using its flow.

3.11.3. The folk questions on smoothability and the global orthogonal splitting. The statements of the results in Geroch’s theorem and its proof, suggest obvious problems on the smoothability of \( S \) and \( t \). In fact, these questions were regarded as “folk problems” because, on one hand, some proofs were announced and rapidly cited (see [5, Section 2] for a brief account) and, on the other, smoothability results yield useful simplifications and applications commonly employed. Nevertheless, they have remained fully open until very recently.

Remark 3.77. The solution to the problems on smoothability in [4, 6, 7] involves technical procedures very different to the expected approaches in previous attempts. These approaches can be summarized as:

(a) To smooth the Cauchy hypersurface \( S \) or the (Cauchy) time function \( t \) by using covolution [49]. The difficulty comes from the fact that, even when \( S, t \) are smooth, the tangent to \( S \) or the gradient of \( t \) may be degenerate, that is, close hypersurfaces or functions to \( S, t \) may be non-Cauchy or non-time functions. Therefore, \( S, t \) must be smoothed by taking into account that a \( C^\infty \) approximation may be insufficient.

(b) To choose an admissible measure \( m \) such that the volume functions \( t^+, t^- \) are directly not only continuous but also smooth [13]. Nevertheless, notice that those stably causal spacetimes which are not causally continuous, cannot admit continuous \( t^+, t^- \), but they do admit temporal time functions (Th. 3.56).

As a summary on these questions, assume that \((M, g)\) be globally hyperbolic:

1.- Must a (smooth) spacelike Cauchy hypersurface \( S \) exist? This is the simplest smoothability question, posed explicitly by Sachs and Wu in their review [42, p. 1155]. One difficulty of this problem (which makes useless naive approaches based
on covolution) is the following. Even if a Cauchy hypersurface $S$ is smooth at some point $p$, the tangent space $T_pS$ may be degenerate; so, the smoothing procedure of $S$ must “push” $T_pS$ in the right spacelike direction.

The existence of one such $S$ implies that the spacetime is not only homeomorphic but also diffeomorphic to $\mathbb{R} \times S$. Physical applications appear because spacelike Cauchy hypersurfaces are essential for almost any global problem in General Relativity (initial value problem for Einstein equation, singularity theorems, mass...), see [46]. For example, from the foundational viewpoint, they are necessary for the well-posedness of the initial value problem, as there is no a general reasonable way to pose well these conditions if the Cauchy hypersurface is not spacelike (or, at least, smooth).

This smoothability problem was solved in [4]. The idea starts recalling the following result, interesting in its own right (see also [18]):

Let $S$ be a Cauchy hypersurface. If a closed subset $N \subset M$ is a embedded spacelike (at least $C^1$) hypersurface which lies either in $I^+(S)$ or in $I^-(S)$ then it is achronal. If $N$ lies between two disjoint Cauchy hypersurfaces $S_1, S_2$ ($N \subset I^+(S_1) \cap I^-(S_2)$) then it is a Cauchy hypersurface (see Fig. 11).

Thus, as Geroch’s theorem ensures the existence of such $S_1, S_2$, the crux is to find a smooth function $t$ with a regular value $c$ such that $S_c = t^{-1}(c)$ lies between $S_1$ and $S_2$, and $\nabla t$ is timelike on $S_c$.

![Figure 11](image_url)

(A) The embedded spacelike hypersurface $N$ is achronal, because it lies in $I^+(S)$. But it is not (extendible to) a spacelike Cauchy hypersurface. (B) Now, as $N \subset M$ lies between two disjoint Cauchy hypersurfaces $S_1, S_2$, it would be a Cauchy hypersurface if it were closed.

2.- Must a Cauchy temporal function $T$ exist? This question is relevant not only as a natural extension of Geroch’s, but in much more depth, because in the affirmative case the smooth splitting $\mathbb{R} \times S$ of the spacetime can be strengthened in such a way
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that the metric has no cross terms between $\mathbb{R}$ and $\mathcal{S}$ (see (10) below for the explicit expression). This splitting is useful from practical purposes and also to introduce different techniques (Morse theory [54], variational methods [35], quantization...).

Notice that the constructive proof of Geroch’s Cauchy time function may yield a non-smooth one (Fig. 12). The freedom to choose an admissible measure $m$ may suggest that, perhaps, a wise choice of $m$ will yield directly a smooth Geroch’s function. Nevertheless, the related problem of smoothability in stably causal spacetimes (Th. 3.56) suggest that this cannot be the right approach (in this case, even $t^\pm$ may be non-continuous). The problem was solved affirmatively in [6] by different means, based on the construction of “time step functions”. We also refer to [44] for a sketch of these ideas.

Figure 12. $M \subset \mathbb{L}^2$, (coord. $u, v$). $M = \{(u, v) \in \mathbb{L}^2 : |u|, |v| < 2\} \setminus \{(u, v) \in \mathbb{L}^2 : u, v \geq 1\}$; $p = (0, 1), p_\epsilon = (0, 1 - \epsilon)$. Diagonal $S$ is a Cauchy hypersurface. For the natural $g$-measure, $t^+(p_\epsilon) = 2\epsilon + t^+(p)$ when $\epsilon > 0$, and $t^+$ is not smooth.

3.- If a spacelike Cauchy hypersurface $\mathcal{S}$ is prescribed, does a Cauchy temporal function $\mathcal{T}$ exist such that one of its levels is $\mathcal{S}$? This question has natural implications in classical General Relativity (even though was proposed explicitly by Bär, Ginoux and Pfaffle in the framework of quantization). For example, for the initial value problem, one poses initial data on a prescribed hypersurface which will be, a posteriori, a Cauchy hypersurface $\mathcal{S}$ of the solution spacetime. Now, in order to solve Einstein equation, one may assume that the spacetime will admit an orthogonal splitting as (10) below, with $\mathcal{S}$ one of the slices and being $\beta$, and the evolved metric $g_T$, the unknowns.

This problem was solved affirmatively in [7]. Notice that even a non-smooth Cauchy (resp. acausal Cauchy) hypersurface $\mathcal{S}$ can be regarded as a level of a time (resp. Cauchy time) function $t$ as follows. $I^+(\mathcal{S})$ and $I^-(\mathcal{S})$, regarded as spacetimes, are globally hyperbolic and, thus, we can take Cauchy temporal functions $T_{\mathcal{S}^\pm}$ on $I^\pm(\mathcal{S})$. Now, the required function is:
Function \( t \) is also smooth (and a Cauchy temporal function) everywhere except at most in \( S = t^{-1}(0) \). Nevertheless (replacing, if necessary, \( t \) by a function obtained technically by modifying \( t \) around \( S \)), one can assume that \( t \) is smooth even if \( S \) is not. Nevertheless, in this case the gradient of \( t \) on \( S \) will be 0 and, thus, \( t \) will not be a true Cauchy temporal function. Now, the crux is to show that, if \( S \) is spacelike, then it is possible to modify \( t \) in a neighborhood of \( S \), making its gradient everywhere timelike, and maintaining its other properties.

4.- Under which circumstances a spacelike submanifold \( A \) (with boundary) can be extended to a spacelike (or, at least, smooth) Cauchy hypersurface? As the previous question, this one is solved in [7] and has a natural classical meaning (but it was posed by Brunetti and Ruzzi motivated by quantization). Notice that an obvious requirement for \( A \) is achronality; moreover, compactness becomes also natural (the hyperbola \( t = \sqrt{x^2 + 1} \) would yield a counterexample, see Fig. 11). Even more: 

*any compact achronal \( K \subset M \), can be extended to a Cauchy hypersurface.* In fact, \( M' = M \setminus (I^+(K) \cup I^-(K)) \) would be a (possibly non-connected) globally hyperbolic spacetime and, then, would admit a Cauchy hypersurface \( S' \); the required Cauchy hypersurface of \( M \) would be \( S_K = S' \cup K \). Nevertheless, the corresponding Cauchy hypersurface \( S_A \) for the (smooth, compact, achronal) submanifold \( A \), may be non-smooth and even non-smoothable, see Figure 13. But it is possible to prove that, if \( A \) is not only achronal but also acausal, then \( S_A \) can be modified in a neighborhood of \( A \) to make it not only smooth but also spacelike.

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\[
t(p) = \begin{cases} 
\exp(T_S(p)), & \forall p \in I^+(S) \\
0, & \forall p \in S \\
-\exp(-T_S(p)), & \forall p \in I^-(S)
\end{cases}
\]

Figure 13. The canonical Lorentzian cylinder \((\mathbb{R} \times S^1, g = -dt^2 + d\theta^2)\) with the spacelike hypersurface \( A = \{ (\theta/2, \theta) : \theta \in [0, 4\pi/3] \} \). The spacelike achronal (but non-acausal) hypersurface \( A \) can not be extended to a smooth Cauchy hypersurface, although by adding the null geodesic segment between \( p \) and \( q \) one obtains a continuous Cauchy hypersurface \( S_A \).
As a summary of all these problems, it is possible to prove:

**Theorem 3.78.** A spacetime \((M, g)\) is globally hyperbolic if and only if it admits a (smooth) spacelike Cauchy hypersurface \(S\).

In this case it admits a Cauchy temporal function \(T\) and, thus, it is isometric to the smooth product manifold

\[ \mathbb{R} \times S, \quad \langle \cdot, \cdot \rangle = -\beta dT^2 + g_T \]  

where \(\beta : \mathbb{R} \times S \to \mathbb{R}\) is a positive smooth function, \(T : \mathbb{R} \times S \to \mathbb{R}\) the natural projection, each level at constant \(T\), \(S_T\), is a spacelike Cauchy hypersurface, and \(g_T\) is a Riemannian metric on each \(S_T\), which varies smoothly with \(T\).

Even more, if \(S\) a prescribed (topological) Cauchy hypersurface then there exists a smooth Cauchy function \(\tau : M \to \mathbb{R}\) such that \(S\) is one of its levels \((S = S_0)\).

If, additionally:

- \(S\) is also acausal then function \(\tau\) becomes a smooth Cauchy time function.
- If \(S\) is spacelike (and thus smooth and acausal), then \(\tau\) can be modified to obtain a Cauchy temporal function \(T : M \to \mathbb{R}\) such that \(S = T^{-1}(0)\).

Finally, if \(A \subset M\) is a compact achronal subset then it can be extended to a Cauchy hypersurface. If, additionally, \(A\) is acausal and a smooth spacelike submanifold with boundary, then it can be extended to a spacelike Cauchy hypersurface \(S \supset A\).

### 3.11.4. The space of causal curves.

The first definition of global hyperbolicity was given by Leray [31], and involves the compactness of the space of causal curves which connects any two points. More precisely, consider two events \(p, q\) of the spacetime \((M, g)\), and let \(C(p, q)\) be the set of all the continuous curves which are future-directed and causal (according to Definition 3.15) and connect \(p\) with \(q\), under the convention in Remark 2.39 i.e., two such curves are regarded as equal if they differ in a strictly monotonic reparametrization. For simplicity, \((M, g)\) will be assumed to be causal, and we will consider the \(C^0\) topology\(^9\) on \(C(p, q)\), that is, a basis of open neighborhood of \(\gamma \in C(p, q)\) is constructed in an open neighborhood \(U\) of the image of \(\gamma\).

**Theorem 3.79.** A spacetime \((M, g)\) is globally hyperbolic if and only if:

(i) it is causal, and

(ii) \(C(p, q)\) is compact for all \(p, q \in M\).

**Proof.** (\(\Rightarrow\)) Let \(\{r_n\}_n\) be a sequence in \(J^+(p) \cap J^-(q)\) and \(\gamma_n\) be a causal curve from \(p\) to \(q\) through \(r_n\) for each \(n\). Up to a subsequence \(\{\gamma_n\}_n\) converges to a curve

---

\(^9\)This sense of \(C^0\) topology agrees with the \(C^0\)-limit of curves, described in Definition 3.31. Even though this notion of limit had specially good properties for strongly causal spacetimes, we will not need a priori this hypothesis but only causality (recall also that the two extremes of the curves are fixed). Nevertheless, a posteriori, we will work with globally hyperbolic spacetimes, where strong causality holds.
$\gamma \in C(p, q)$. So, chosen any neighborhood $U \subset M$ of $\gamma$ with compact closure $\bar{U}$, all $\gamma_n(\ni r_n)$ lie in $U$ for large $n$ and, up to a subsequence, $\{r_n\} \to r \in \bar{U}$. But necessarily $r \in \gamma(\subset J^+(p) \cap J^-(q))$, as required.

(\Rightarrow) See for example [27, p. 208-9].

**Remark 3.80.** In fact, hypothesis (i) is somewhat redundant, because it is possible to define a natural topology on $C(p, q)$ even if the spacetime is not causal. But in this case, if there were a closed causal curve $\gamma$, parametrizing $\gamma$ by giving more and more rounds, a sequence of (non-equivalent) causal curves would be obtained, and the compactness assumption of $C(p, q)$ would be violated for this natural topology.

With this notion of global hyperbolicity at hand, it is not difficult to prove the main properties of the time-separation $d$ of a globally hyperbolic spacetime. Recall that $d$ is not conformally invariant, but the properties below will be so.

**Lemma 3.81.** Let $(M, g)$ be globally hyperbolic and $p < q$. Consider sequences:

\[ \{p_k\} \to p, \quad \{q_k\} \to q, \quad p_k \leq q_k \]

Then, for any sequence $\gamma_k$ of causal curves, each one from $p_k$ to $q_k$, there exists a limit in the $C^0$ topology $\gamma$ which joins $p$ to $q$.

**Proof.** Choose $p_1 \ll p$, and $q \ll q_1$, and, for large $n$, construct a sequence of causal curves $\{r_n\}_n$ starting at $p_1$, going to $p_n$, running $q_n$ and arriving at $q_1$. Then, use the compactness of $C(p_1, q_1)$.

**Remark 3.82.** From the properties in subsection 3.6.3, $\gamma$ is also a limit curve of the sequence, and $L(\gamma) \geq \lim_{m} L(\gamma_k)$.

**Theorem 3.83.** In any globally hyperbolic spacetime $(M, g)$:

1. $d$ is finite-valued
2. (Avez-Seifert [1, 28]). Each two causally related points can be joined by a causal geodesic which maximizes time-separation.
3. $d$ is continuous.

**Proof.** (1) Cover $J^+(p) \cap J^-(q)$ with a finite number $m$ of convex neighbourhoods $U_j$ such that each causal curve which leaves $U_j$ satisfies: (i) it never returns to $U_j$, (ii) its length is $\leq 1$. Then $d(p, q) \leq m$.

(2) Take a sequence of causal curves $\gamma_k$ with lengths converging to $d(p, q)$ and use Lemma 3.81 (this also yields an alternative proof of (1)).

(3) Otherwise (taking into account that $d$ is always lower semi-continuous) there are sequences $\{p_k\} \to p, \{q_k\} \to q, p_k \leq q_k$ with

\[ d(p_k, q_k) \geq d(p, q) + 2\delta \]

for some $\delta > 0$. Choose causal curves $\gamma_k$ from $p_k$ to $q_k$ satisfying

\[ L(\gamma_k) \geq d(p_k, q_k) - \delta. \]
Then the limit $\gamma$ yields the contradiction:

$$L(\gamma) \geq \limsup L(\gamma_k) \geq d(p, q) + \delta > d(p, q).$$

Remark 3.84. (1) The finiteness of $d$ holds for all the time-separations of metrics in $g$. In fact, the following characterization is classical: a strongly causal spacetime $(M, g)$ is globally hyperbolic if and only if the time-separation $d^\ast$ of any metric $g^\ast$ conformal to $g$ is finite. To check it, notice that when $(M, g)$ is not globally hyperbolic, there is a sequence $\{\gamma_k\}_{k} \subset C(p, q)$ which has a limit curve $\gamma$ starting at $p$ with no final endpoint. The conformal factor must be taken diverging fast along a neighborhood of $\gamma$ (see [2, Th. 4.30] for details).

(2) The existence of connecting causal geodesics in Avez-Seifert result can be made more precise: there exists a $d^\ast$–maximizing geodesic in each causal homotopy class and, if $p \ll q$, there is also a maximizing timelike geodesic in all the timelike homotopy classes included in each causal homotopy class, see the detailed study in [36, Sect. 2].

3.11.5. An application to closed geodesics and static spacetimes. Next, we will see some simple applications of the properties of globally hyperbolic spacetimes for the geodesics of some spacetimes. We refer to [45] for more results and extended proofs, especially regarding static spacetimes.

Proposition 3.85. If the universal covering $(\tilde{M}, \tilde{g})$ of a totally vicious spacetime $(M, g)$ is globally hyperbolic, then $(M, g)$ is geodesically connected through timelike geodesics (i.e., each $p, q \in M$ can be connected through a timelike geodesic).

Proof. By lifting to $\tilde{M}$ any timelike curve $\rho$ which connects $p, q$, one obtains two chronologically related points $\tilde{p}, \tilde{q} \in \tilde{M}$. So, they are connectable by means of a (maximizing) timelike geodesic $\tilde{\gamma}$, which projects in the required one.

Now, recall that a static spacetime is a stationary one such that the orthogonal distribution to its timelike Killing vector field $K$ is integrable. Locally, any static spacetime looks like a standard static spacetime i.e., the product $\mathbb{R} \times S$ endowed with the warped metric $g = -\beta dt^2 + g_S$, where $g_S$ is a Riemannian metric on $S$ and $\beta$ is a function which depends only on $S$. If $K$ is complete, any simply connected static spacetime is standard static, in particular:

Lemma 3.86. The universal covering $(\tilde{M}, \tilde{g})$ of a compact static spacetime is standard static.

These spacetimes have a good causal behaviour:

Proposition 3.87. Any standard static spacetime $(M, g)$ is causally continuous, and the following properties are equivalent:

(i) $(M, g)$ is globally hyperbolic.
(ii) The conformal metric $g^* = g_S/\beta$ is complete.

(iii) Each slice $t = \text{constant}$ is a Cauchy hypersurface.

In particular, the universal covering of a compact static spacetime is globally hyperbolic.

Proof. For the first assertion, it is enough to prove past (and analogously future) reflectivity $I^+(q) \subset I^+(p) \Rightarrow I^-(p) \subset I^-(q)$. Put $p = (t_p, x_p), q = (t_q, x_q)$. Assuming the first inclusion, it is enough to prove $p - \epsilon = (t_p - \epsilon, x_p) \in I^-(q)$, for all $\epsilon > 0$. As $q_\epsilon := (t_q + \epsilon, x_q) \in I^+(p)$, there exists a future-directed timelike curve $\gamma(s) = (s, x(s)), s \in [t_p, t_q + \epsilon]$ joining $p$ and $q$. Then, the future-directed timelike curve $\gamma_\epsilon(s) = (s - \epsilon, x(s))$ connects $p$ and $q$, as required.

The equivalences (i)—(iii) follows from standard computations valid for warped product spacetimes [2, Theorems 3.67, 3.69].

In particular, a standard static spacetime will be globally hyperbolic if $g_S$ is complete and $\beta$ is bounded (or at most quadratic). These conditions hold in the universal covering of a compact static spacetime, proving the last sentence. □

Thus, Proposition 3.85, 3.87, and Theorem 3.3 yields [47]:

Theorem 3.88. Any compact static spacetime is geodesically connected through timelike geodesics.

For closed geodesics, let us start with the following well-known result by Tipler [53] (in Beem’s formulation [2], later extended by Galloway [17].

Theorem 3.89. Any compact spacetime $(M, g)$, regularly covered by a spacetime $(\tilde M, \tilde g)$ which admits a compact Cauchy hypersurface $S$, contains a periodic timelike geodesic.

Proof. Take a timelike loop $\gamma$ in $M$ and a lift $\tilde \gamma : [0, 1] \rightarrow \tilde M$. Let $\psi : \tilde M \rightarrow M$ be a deck transformation which maps $\tilde \gamma(0)$ in $\tilde \gamma(1)$. The function $f : S \rightarrow \mathbb{R}$ $p \rightarrow d(p, \psi(p))$ admits a maximum $p_0$ (necessarily, $f(p_0) > 0$). The maximizing timelike geodesic from $p_0$ to $\psi(p_0)$ projects not only onto a geodesic loop, but also to a closed one (otherwise, a closed curve with bigger length could be obtained by means of a small deformation). □

Remark 3.90. The compactness of $S$ cannot be removed (Guediri’s counterexample, see [24] and references therein). Nevertheless, it can be replaced by the existence of a class of conjugacy $C$ of the fundamental group which contains a timelike curve and satisfies one of the following two conditions (see [17]):

(a) $C$ is finite, or

(b) The deck transformations satisfy a technical property of compatibility with an orthogonal globally hyperbolic splitting (roughly, $\phi(t, x) = (t + T_\phi, \phi^S(x))$ for some $T_\phi \in \mathbb{R}$ and some automorphism $\phi^S$ of $S$), which is always satisfied in the case of compact static spacetimes.

Thus, this possibility (b) yields [47]:

Theorem 3.91. Any compact static spacetime admits a closed timelike geodesic.
4. The “isocausal” ladder

4.1. Overview. Up to now, the causal structure of a spacetime is related to two notions: (a) its conformal structure, and (b) its position in the causal hierarchy. Nevertheless, in order to understand “when two spacetimes share the same causal structure” one can argue that the first one is too restrictive, and the latter too weak. For example: (a) most modifications of a Lorentzian metric around a point (say, any non-conformally flat perturbation of Minkowski spacetime in a small neighbourhood) imply a different conformal structure; but, one may have a very similar structure of future and past sets for all points, and (b) all globally hyperbolic spacetimes belong to the same level of the hierarchy, but clearly the causality of, say, Lorentz-Minkowski and Kruskal spacetimes behave in a very different way. It is not easy to find an intermediate notion, because “same causal structure” suggests “same causal relations” and, in any distinguishing spacetime, the conformal structure is determined by these relations (Prop. 3.13, Th. 3.9).

A fresh viewpoint was introduced by García-Parrado and Senovilla [20, 21] by taking into account the following two ideas: (i) the definition of most of the levels of the standard causal hierarchy prevents a bad behavior of some types of causal curves; thus, if the timecones of a metric \( g \) on \( M \) are included in the timecones of another one \( g' \) \( (g \prec g') \), then the causality of \( g \) will be at least as good as the causality of \( g' \), and (ii) perhaps for some diffeomorphisms \( \Phi, \Psi \) of \( M \) the pull-back metrics satisfy \( \Psi^* g \prec g' \prec \Phi^* g \); in this case (as the causality of \( g, \Psi^* g, \Phi^* g \) must be regarded equivalent), one says that \( g \) and \( g' \) are “isocausal”.

In this way, one introduces a partial (pre)order in the set of all the spacetimes, which was expected to refine the standard causal ladder. Nevertheless, this new order was carefully studied by García-Parrado and Sánchez [19], who observed that two of the levels of the standard ladder (causal continuity and causal simplicity) were not preserved by it. Thus, one obtains an alternative hierarchy of spacetimes, with common elements but also with relevant differences and complementary viewpoints. Next, we sketch this approach.

4.2. The ladder of isocausality.

**Definition 4.1.** Let \( V_i = (M_i, g_i), i = 1, 2, \) be two spacetimes. A diffeomorphism \( \Phi : M_1 \to M_2 \) is a causal mapping if the timecones of the pull-back metric \( \Phi^* g_2 \) include the cones of \( g_1 \), and the time-orientations are preserved by \( \Phi \). In this case, we write \( V_1 \prec_\Phi V_2 \), and \( V_1 \prec V_2 \) will mean that \( V_1 \prec_\Phi V_2 \) for some \( \Phi \).

The two spacetimes are isocausal, denoted \( V_1 \sim V_2 \), if \( V_1 \prec V_2 \) and \( V_2 \prec V_1 \).

**Remark 4.2.** (1) Recall that if \( V_1 \sim V_2 \) then \( V_1 \prec_\Phi V_2 \) and \( V_2 \prec_\Psi V_1 \) for some diffeomorphisms \( \Phi, \Psi \), but perhaps \( \Psi \neq \Phi^{-1} \).

(2) As in the case of conformal relations, one can consider, for practical purposes, a single differentiable manifold \( M \) in which two time-oriented Lorentzian metrics \( g_1, g_2 \) are defined, and study when the timecones of \( g_2 (\equiv \Phi^* g_2) \) are wider than the cones of \( g_1 \) (and with agreeing time-orientations), i.e. if the identity in \( M \) is a causal mapping. Nevertheless, the notation \( g_1 \prec g_2 \) means also the possibility
that the timecones of $\Phi^*g_2$ are (non-necessarily strictly) wider than the cones of $g_1$ for some $\Phi$.

(3) Even though the time-orientations can be usually handled in a simple way, their role cannot be overlooked. In fact, it is not difficult to find a Lorentzian manifold such that the two spacetimes obtained by choosing different time-orientations are not isocausal (see Fig. 14).

(4) One can check that, \textit{locally all the spacetimes are isocausal} [21, Theorem 4.4] (but, obviously, not necessarily conformal). This supports that the notion of “Causality” (which is appealing as a global concept) deals with properties invariant by isocausality, not only by conformal diffeomorphisms.

![Figure 14](image_url)

Figure 14. This spacetime, which admits a “black hole region” $-1 \leq x \leq 1$, is not isocausal (nor, thus, conformal) with the one obtained by reversing its time-orientation (which does not admit such a region).

Now, it is easy to check the following result:

\textbf{Theorem 4.3.} If $V_1 \prec V_2$ and $V_2$ is globally hyperbolic, causally stable, strongly causal, distinguishing, causal, chronological, or not totally vicious, then so is $V_1$.

\textit{Proof.} This is an exercise recalling that: (i) if $v$ is causal (or $\gamma$ is a closed timelike or causal curve) then $d\Phi(v)$ is causal (or $\Phi \circ \gamma$ is a closed timelike or causal curve), and (ii) if $d\Phi^{-1}(v')$ is non-causal (or $t' \circ \phi$ is a time function; $\phi^{-1}(S')$ is a Cauchy hypersurface, etc.) then $v'$ is non-causal (or $t'$ is a time function; $S'$ is Cauchy, etc.), see [20] for details.

The conditions appearing in this result comprise all the levels in the standard hierarchy of causality, except causally continuous and causally simple. Nevertheless, these levels are not necessarily preserved. In fact, there is an explicit counterexample [19, Section 3.2] which shows that $V_2$ may be causally simple and $V_1$ non-causally continuous, with $V_1 \prec V_2$.

Now, fix a manifold $M$, and define the \textit{isocausal structure} of the spacetime $(M,g)$ as its equivalence class $\coset(g)$ in the quotient set $\Con(M) \sim (\equiv \Lor(M)/\sim)$. 
A partial order $\preceq$ in $\text{Con}(M)/\sim$ can be defined by

$$\text{coset}(g_1) \preceq \text{coset}(g_2) \iff (M, g_1) \prec (M, g_2).$$

Isocausal structures can be naturally grouped in sets totally ordered by "$\preceq$", in the form

$$\cdots \preceq \text{coset}(g_1) \cdots \preceq \text{coset}(g_2) \cdots \preceq \cdots \preceq \text{coset}(g_m) \preceq \cdots.$$

Of course, some of the groups in a totally ordered chain may be empty; for example, if $M$ were compact no chain would contain chronological spacetimes. Furthermore the relation "$\preceq$" is not a total order and so a globally hyperbolic spacetime need not be related to, say, a causally stable spacetime (see [19] for exhaustive examples). Nevertheless, except for the two excluded levels (causal continuity and simplicity) relation $\preceq$ yields a refinement of the standard causal hierarchy, introducing further relations between elements of each level.

4.3. Some examples. In order to study the possible isocausality of two spacetimes, there are two basic naive ideas (see [19]):

1. In order to prove $V_1 \prec V_2$. Try to find an explicit causal mapping. For example, consider two Generalized Robertson-Walker (GRW) spacetimes on the same manifold $M$, that is $M$ is a warped product $I \times_{f_i} S$, where $I \subset \mathbb{R}$ is an interval, $S$ is a manifold endowed with a positive definite Riemannian metric $g_S$ and, with natural identifications:

$$g_i = -dt^2 + f_i^2(t)g_S.$$

Now, assume that $S$ is compact (i.e., the GRW spacetime is closed) and $I$ is unbounded. Then, it is not difficult to check that they are isocausal if both warping functions $f_i$ are bounded away from 0 and $\infty$, that is $0 < \inf(f_i) \leq \sup(f_i) < \infty$ for $i = 1, 2$. In fact, a causal mapping type $(t, x) \rightarrow (\varphi(t), x)$ can be found easily.

2. In order to prove $V_1 \not\prec V_2$. Try to find a causal invariant which would be transferred by the causal mapping (or its inverse), but not shared by both spacetimes. In fact, this is the reason why $V_1 \not\prec V_2$ if $V_2$ lies higher than $V_1$ in the standard ladder of causality (with causal continuity and simplicity removed). In this sense, criteria as the following are useful:

**Criterion.** Assume that $V_1 \prec V_2$ and that $V_1$ admits $j$ inextendible future-directed causal curves (or, in general, $j$ submanifolds at no point spacelike and closed as subsets of $V_1$) $\gamma_i, i = 1, \ldots, j$ satisfying either of the following conditions ($\downarrow$ will denote the common chronological past of all the points of the corresponding subset):

(a) $V_1 = I^+(\gamma_i) \cup \gamma_i \cup I^-(\gamma_i).$
(b) \( \gamma_i \subset \gamma_{i+1}, \forall i = 1, \ldots, j - 1, j > 1. \)

Then so does \( V_2. \)

In fact, if \( \Phi : V_1 \to V_2 \) is the causal mapping, the sets \( \Phi(\gamma_i), i = 1, \ldots, j \) satisfy condition 1 in \( V_2 \) whenever \( \gamma_i, i = 1, \ldots, j \) do in \( V_1. \) To prove the second point use the straightforward property:

\[
\Phi(\downarrow A) \subset \downarrow \Phi(A), \quad A \subset V_1,
\]

as required.

As a simple application, it is easy to show that there are infinitely many rectangles of \( \mathbb{L}^2 \), in standard Cartesian coordinates \((t, x)\), which are not isocausal (see [19, Figure 5])\(^{10}\).

By using these type of arguments one can study the isocausal structure of GRW spacetimes, obtaining as a typical result:

**Theorem 4.4.** Consider any GRW spacetime \( V = I \times f S, I \subset \mathbb{R} \) with \( S \) diffeomorphic to a \((n - 1)\)-sphere. Then \( V \) is isocausal to one and only one of the following types of product spacetimes:

1. \( \mathbb{R} \times S^{n-1} \), i.e., Einstein static universe, with metric

   \[
g = -dt^2 + g_0,
\]

   where \( g_0 \) represents the metric of the unit \((n - 1)\)-dimensional sphere.

2. \([-0, \infty[ \times S^{n-1} \) with metric as in the case 1.

3. \([-\infty, 0[ \times S^{n-1} \). The metric is as in the case 1.

4. \([0, L[ \times S^{n-1} \), for some \( L > 0. \)

Moreover, causal structures belonging to the above cases can be sorted as follows

\[
\text{coset}(g_4(L)) \leq \left\{ \begin{array}{c}
\text{coset}(g_2) \\
\text{coset}(g_3)
\end{array} \right\} \leq \text{coset}(g_1),
\]

where the roman subscripts mean that the representing metric belongs to the corresponding point of the above description.

Finally, it is worth pointing out the following question regarding stability (recall Section 3.8), also stressed in [19]. In the three first cases of Theorem 4.4 all the spacetimes are isocausal to a fixed one and, thus, the isocausal structure is \( C^0 \)-stable in the set of all the metrics on \( I \times S \). Nevertheless, in the last case there are different isocausal structures. And, in fact, classical de Sitter spacetime \( S^1_1 \) lies in

---

\(^{10}\) Notice also that, as these rectangles are neither conformal, this also re-proves the existence of infinitely many different simply-connected conformal Lorentz surfaces (in contrast with the Riemannian case), stressed by Weinstein [57].
this case and has a $C^r$-unstable isocausal structure for any $r \geq 0$ (very roughly, a criterion as the one explained above is applicable to $S^n_1$, but the number of curves $\gamma_i$ in this criterion varies under appropriate arbitrarily small $C^r$ perturbations). In contrast with this case, the isocausal structure of Lorentz-Minkowski $\mathbb{L}^n$ is again stable. In fact, a simple computation shows that any $g$ on $\mathbb{R}^n$ becomes isocausal to $\mathbb{L}^n$ if it satisfies: (i) $\partial_t$ is a $g$-timelike vector field, and (ii) there exists $0 < \theta_- \leq \theta_+ < \pi/2$ such that the Euclidean angle $\theta$ (for the usual Euclidean metric in $\mathbb{R}^n$) of any $g$-lightlike tangent vector and $\partial_t$ satisfies: $\theta_- \leq \theta \leq \theta_+$. Summing up:

**Theorem 4.5.**

1. The isocausal structure of Lorentz Minkowski $\mathbb{L}^n$ is stable in the $C^0$ (and, thus, in any $C^r$) topology.

2. The isocausal structure of $S^n_1$ is unstable in any $C^r$ topology.

The first result goes in the same direction that Christodoulou and Klainerman’s landmark result [11], who proved the nonlinear stability of four-dimensional Minkowski spacetime (a small amount of gravitational radiation added in the initial data of $\mathbb{L}^n$ will disperse to infinity without any singularities or black holes being formed). The second one suggests that the isocausal structure of de Sitter spacetime cannot be regarded as a physically reasonable one.

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