A HAUSDORFF-YOUNG INEQUALITY FOR MEASURED GROUPOIDS

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Abstract. The classical Hausdorff-Young inequality for locally compact abelian groups states that, for 1 \( \leq p \leq 2 \), the \( L^p \)-norm of a function dominates the \( L^q \)-norm of its Fourier transform, where \( 1/p + 1/q = 1 \). By using the theory of non-commutative \( L^p \)-spaces and by reinterpreting the Fourier transform, R. Kunze (1958) [resp. M. Terp (1980)] extended this inequality to unimodular [resp. non-unimodular] groups. The analysis of the \( L^p \)-spaces of the von Neumann algebra of a measured groupoid provides a further extension of the Hausdorff-Young inequality to measured groupoids.

1. Introduction.

The classical Hausdorff-Young inequality for a locally compact abelian group \( G \) says that, for 1 \( \leq p \leq 2 \) and for a compactly supported continuous function \( f \) on \( G \), the \( L^p \)-norm of \( f \) dominates the \( L^q \)-norm of its Fourier transform \( \hat{f} \), where \( q \) is the conjugate exponent (1/p + 1/q = 1):

\[ \|\hat{f}\|_q \leq \|f\|_p. \]

The Fourier transform \( \hat{f} \) is the function on the dual group \( \hat{G} \) defined by

\[ \hat{f}(\chi) = \int f(x)\chi(x)dx. \]

The Haar measure of \( \hat{G} \) is normalized so that the Fourier transform extends to an isometry from \( L^2(G) \) onto \( L^2(\hat{G}) \). Thus, the inequality holds for \( p = 2 \). The proof is usually given as one of the first applications of the complex interpolation method, since the inequality also holds for \( p = 1 \).

This inequality has been generalized to non-abelian locally compact groups. The main difficulty is to give a meaning to the space \( L^q(\hat{G}) \) and to the Fourier transform \( F_p : L^p(G) \to L^q(\hat{G}) \) when \( G \) is no longer abelian. The theory of operator algebras provides a solution. In the abelian case, the function \( \hat{f} \) can be seen as the multiplication operator \( M(\hat{f}) \) on \( L^2(\hat{G}) \), i.e. \( M(\hat{f})\xi = \hat{f}\xi \). Up to unitary equivalence, this is the convolution operator \( L(f) \) on \( L^2(G) \), i.e. \( L(f)\xi = f*\xi \). This still makes sense when \( G \) is not abelian, provided one is willing to deal with unbounded operators. When \( f \) is continuous and compactly supported, \( L(f) \) is a bounded operator on \( L^2(G) \). The von Neumann algebra generated by these operators is the group von Neumann algebra \( VN(G) \), which will play the rôle of the von Neumann algebra \( L^\infty(G) \).

The case of unimodular groups was solved by R. Kunze in [10]. In that case, the von Neumann algebra \( M = VN(G) \) has a trace \( \tau \) and the non-commutative integration theory of I. Segal ([14]) is available. In fact, Kunze defines and studies
non-commutative $L^p$-spaces $L^p(M, \tau)$ for a general von Neumann algebra $M$ endowed with a trace $\tau$. Its elements are unbounded operators on the Hilbert space $L^2(M, \tau)$ of the GNS construction affiliated with $M$. He then defines $L^q(G)$ as $L^q(VN(G), \tau)$. For $1 \leq p \leq 2$, the $p$-Fourier transform $F_p : L^p(G) \to L^q(G)$ is given by $F_p(f) = L(f)$ just as above.

One had to wait for the Tomita-Takesaki theory and its developments to tackle the non-unimodular case. Relying on the work of U. Haagerup ([2]) on one hand and of A. Connes ([4]) and M. Hilsum ([8]) on the other on non-commutative $L^p$-spaces, M. Terp solved the general case of a locally compact group in a paper [17] unfortunately never published.

She also developed in [18], in the case of a weight instead of a state, the interpolation approach of H. Kosaki to non-commutative $L^p$-spaces, a framework particularly well-fitted to the generalization of the Hausdorff-Young inequality.

In the meantime, other Hausdorff-Young type inequalities were found. In particular, B. Russo gave in [13] the following Hausdorff-Young inequality for integral operators. Let $X$ be a Borel space endowed with a $\sigma$-finite measure $dx$. Given $f \in L^2(X \times X)$, let $L(f)$ be the integral operator on $L^2(X)$ having $f$ for kernel. Then, for $1 \leq p \leq 2$ and $q$ conjugate exponent of $p$, the following inequality holds:

$$\|L(f)\|_q \leq \max (\|f\|_{p,q}, \|f\|_{p,q}),$$

where $\|L(f)\|_q$ is the $q$-Schatten norm of $L(f)$, $\|f\|_{p,q} = (\int (\int |f(x, y)|^p dx)^{q/p} dy)^{1/q}$ is the mixed norm introduced in [2] and $f^*(x, y) = \bar{f}(y, x)$.

When $p = 1$, $q = \infty$; the corresponding norm is the usual operator norm and the inequality becomes

$$\|L(f)\| \leq \max (\sup_y (\int |f(x, y)| dx), \sup_x (\int |f(x, y)| dy))$$

which is a well-known consequence of the Cauchy-Schwarz inequality. When $p = 2$, $q = 2$; the corresponding norm is the Hilbert-Schmidt norm; both terms on the right hand side are equal by Fubini’s theorem; as well known, we have the equality

$$\|L(f)\|_2 = \left(\int \int |f(x, y)|^2 dxdy\right)^{1/2}.$$

As in the classical case, the case $1 < p < 2$ is obtained by complex interpolation. Although the mixed norms $\|f\|_{p,q}$ are interpolation norms just as the usual $L^p$-norms are, one cannot conclude directly, due to the presence of a max of these norms.

The similarity between the group case and the case of integral operators is striking. It can still be made stronger by introducing the measured groupoid $G = X \times X$. Then the formulation is almost identical to the group case: the relevant von Neumann algebra is $M = VN(G)$ (it is the algebra of all bounded operators on $L^2(X)$ but acting by left multiplication on $L^2(G)$), $L(f)$ is the left convolution operator on $L^2(G)$ and the Schatten space $S_q$ is the non-commutative $L^q$-space of $M$ endowed with its natural trace. The main difference is that the usual $L^p$-norm of $f$ has been replaced by the maximum of the mixed norms $\|f\|_{p,q}$ and $\|f\|_{p,q}$.

Let us fix our notation for groupoids. The unit space of the groupoid $G$ will usually be denoted by $X = G^{(0)}$. We shall use greek letters like $\gamma, \ldots$ for elements of $G$ and roman letters $x, y, \ldots$ for elements of $G^{(0)}$. The range and source maps will be denoted by $r, s : G \to G^{(0)}$. For $x \in G^{(0)}$, we define $G^x = r^{-1}(x)$ and $G_x = s^{-1}(x)$. The inverse of $\gamma \in G$ is denoted by $\gamma^{-1}$. A measured groupoid (see [5], [6]) consists of a Borel groupoid $G$, a measurable Haar system $\lambda$ made of measures $\lambda^x$ on $G^x$ satisfying the left invariance and the measurability property, and a quasi-invariant measure $\mu$ on $G^{(0)}$. The quasi-invariance of $\mu$ is expressed
by the equivalence of the measures \( \nu = \mu \circ \lambda \) and \( \nu^{-1} = \mu \circ \tilde{\lambda} \), where \( \tilde{\lambda} \) is the right invariant Haar system consisting of the measures \( \lambda_x = (\lambda^x)^{-1} \) on \( G_x \). By definition, the modular function is the Radon-Nikodym derivative \( \delta = d\nu/d\nu^{-1} \).

We say that the measured groupoid is unimodular if \( \delta = 1 \). To avoid technicalities, we assume that \( G \) is a second countable locally compact Hausdorff topological groupoid, that the measures are Radon measures, that \( \lambda \) is a continuous Haar system and that the modular function is continuous. Let us give some examples of measured groupoids: locally compact groups (with a chosen left Haar measure), measured spaces and more generally transformation groups \((X, G)\) equipped with a quasi-invariant measure (with the appropriate assumptions to fit our hypotheses).

As further examples: the above measured groupoid \( X \times X \) and the holonomy groupoid of a foliation equipped with a Lebesgue measure. The mixed norms still make sense in this context. For \( f \) non-negative and measurable on \( G \) or in \( C_c(G) \), one can define

\[
\|f\|_{p,q} = \left( \int \left( \int |f|^p d\lambda^x \right)^{q/p} d\mu(x) \right)^{1/q}.
\]

In the case of a group, \( \|f\|_{p,q} = \|f\|_p \) while in the case of a space, \( \|f\|_{p,q} = \|f\|_q \).

On the other hand, we have the regular representation of \( G \) on \( L^2(G) \) which defines the von Neumann algebra \( M = VN(G) \) and provides the operator \( L(f) \). Thus, we have all the ingredients to express a Hausdorff inequality for measured groupoids.

As we shall see, it will have the following form

\[
\|\mathcal{F}_p(f)\|_{L^q(M)} \leq \max(\|f\|_{p,q}, \|f^*\|_{p,q}),
\]

where \( 1 \leq p \leq 2 \), \( \mathcal{F}_p(f) = \Delta^{1/2q}L(f)\Delta^{1/2q} \), \( \Delta \) is the operator of multiplication by \( \delta \) on \( L^2(G) \) and \( f^*(\gamma) = \overline{f(\gamma^{-1})} \).

2. Non-commutative \( L^p \)-spaces for measured groupoids.

Let \((G, \lambda, \mu)\) be a measured groupoid. As said above, we assume that \( G \) is a second countable locally compact Hausdorff topological groupoid, that the measures are Radon measures, that \( \lambda \) is a continuous Haar system and that the modular function \( \delta \) is continuous. We denote by \( \nu \) the measure \( \nu = \mu \circ \lambda \) on \( G \), by \( \nu^{-1} \) its image under the inverse map and by \( \nu_0 \) the symmetric measure \( \nu_0 = \delta^{-1/2} \nu \).

It is convenient for our purpose to present the von Neumann algebra \( M = VN(G) \) of the measured groupoid as the left algebra of a modular Hilbert algebra (see [15]). We endow the space \( C_c(G) \) of compactly supported continuous functions on \( G \) with the convolution product

\[
f \ast g(\gamma) = \int f(\gamma \eta)g(\eta^{-1}) d\lambda^x(\gamma)(\eta),
\]

the involution

\[
f^*(\gamma) = \overline{f(\gamma^{-1})}
\]

the inner product

\[
(f \langle g) = \int f(\gamma)\overline{g(\gamma)} d\nu^{-1}(\gamma)
\]

and the complex one-parameter group \( \Delta(z) \) of automorphisms of \( C_c(G) \)

\[
\Delta(z)f = \delta^zf.
\]

The axioms (I) to (VIII) of [15] Definition 2.1] are readily verified. The left and right convolution operators \( L(f) \) and \( R(g) \) are respectively defined by

\[
L(f)\xi = f \ast \xi \quad R(g)\eta = \eta \ast g.
\]

They extend to bounded linear operators, still denoted by \( L(f) \) and \( R(g) \), on the completed Hilbert space \( H = L^2(G, \nu^{-1}) \). The map \( L \) is a representation of the
$*$-algebra $C_c(G)$. We define $M = VN(G)$ as the von Neumann algebra generated by $\{L(f), f \in C_c(G)\}$. According to the general theory, the von Neumann algebra generated by $\{R(g), g \in C_c(G)\}$ is the commutant $M'$ of $M$. The polar decomposition $S = J\Delta^{1/2}$ of the closure $S$ of the operator $\xi \mapsto \xi^*$, viewed as an unbounded antilinear operator on $H$ provides the ingredients of the Tomita-Takesaki theory. The modular operator $\Delta$ is the operator of multiplication by $\delta$; it is self-adjoint positive; its domain $\text{Dom}(\Delta)$ is the subspace of the $\xi$'s in $H$ such that $\delta \xi \in H$; it contains $C_c(G)$ as an essential domain. The modular involution $J$ is given by $J\xi = \delta^{1/2}\xi^*$. The canonical weight $\varphi_0$ on $M$ is defined by

$$\varphi_0(L(f)^*L(g)) = (g|f) \quad \text{for } f, g \in C_c(G).$$

Another useful formula is

$$\varphi_0(L(f)) = \int_{G(0)} f d\mu \quad \text{for } f \in C_c(G).$$

The modular group of the weight $\varphi_0$ is given by $\sigma_t(T) = \Delta^t T \Delta^{-t}$ for $T \in M$.

The spatial theory (see [4, 8]) uses the dual weight $\psi_0$ on $M'$, defined by $\psi_0(T) = \varphi_0(JTJ)$. We have

$$\psi_0(R(Jf)^*R(Jg)) = (g|f) \quad \text{for } f, g \in C_c(G).$$

In our context, it can be expressed as follows. A vector $\xi \in H$ is called left bounded if the operator $L(\xi)$ defined by $L(\xi) f = \xi^* f$ for $f \in C_c(G)$ is bounded. The space of left bounded operators is denoted by $\mathcal{A}_l$. One associates to each normal semi-finite weight $\varphi$ on $M$ a self-adjoint positive operator $T$ on $H$ such that

$$||T^{1/2}\xi||^2 = \varphi(L(\xi)L(\xi)^*)$$

for all $\xi \in \mathcal{A}_l$. The operator $T$ is called the spatial Radon-Nikodym derivative of the weight $\varphi$ and is denoted by $\frac{d\varphi}{d\nu(0)}$. It is called integrable if $\varphi$ is a state.

One then defines its integral as $\int T d\psi_0 = \varphi(1)$. The correspondence $\varphi \mapsto \frac{d\varphi}{d\psi(0)}$ extends to $\varphi \in M_\lambda$. The space $L^1(M, \psi_0)$ is defined as its image. On the other hand, if $\varphi \in M_\lambda$, $f \mapsto \varphi(L(f))$ is a Radon measure on $C_c(G)$ which is absolutely continuous with respect to $\nu = \mu \circ \lambda$. We denote by $\frac{d\varphi}{d\psi(0)}$ its usual Radon-Nikodym derivative with respect to the symmetric measure $\nu_0$. Explicitly, if $\varphi(T) = (T \xi|\eta)$ with $\xi, \eta \in H$, one finds that $\frac{d\varphi}{d\nu(0)}$ is the coefficient

$$(\xi', \eta')(\gamma) = (L(\gamma)\xi' \circ s(\gamma)|\eta' \circ r(\gamma))$$

of the left regular representation of $G$ on the field of Hilbert spaces $x \mapsto L^2(G, \lambda_x)$, where $\xi' = \delta^{-1/2}\xi$ and $\eta' = \delta^{-1/2}\eta$. We define the space $L^1(G)$ as the image of the map $\varphi \in M_\lambda \mapsto \frac{d\varphi}{d\psi(0)}$. The three spaces $M_\lambda$, $L^1(M, \psi_0)$ and $L^1(\hat{G})$ are isomorphic as linear spaces, and also as Banach spaces with the appropriate norms. When $\varphi(T) = (T \xi|\xi)$ with $\xi \in C_c(G)$, the spatial Radon-Nikodym derivative $T = \frac{d\varphi}{d\psi(0)}$ is the closure of the operator $\Delta^{1/2}L(\hat{F})\Delta^{1/2}$ defined on $C_c(G)$, where $F = \frac{d\varphi}{d\nu(0)}$ and $\hat{F}(\gamma) = F(\gamma^{-1})$. When $G$ is a locally compact group, $L^1(\hat{G})$ is the usual Fourier algebra $A(G)$.

The Hilbert space $H = L^2(G, \nu^{-1})$ admits the direct integral decomposition

$$H = \int^H H_x d\mu_x,$$

where $H_x = L^2(G_x, \lambda_x)$. A decomposable operator $T = \int^H T_x d\mu_x$
is called $\alpha$-homogeneous if for a.e. $\gamma \in G$,

$$R(\gamma)T_\gamma(\Delta) \subset \delta^{-\alpha}(\gamma)T_\gamma(\Delta)R(\gamma),$$

where $R(\gamma)$ is the isometry $H_{\gamma}(\cdot) \rightarrow H_{\gamma}(\cdot)$ given by $R(\gamma)\xi(\gamma') = \xi(\gamma' \gamma)$. The elements of $M$ (and more generally the operators affiliated with $M$) are 0-homogeneous. The operator $\Delta$ as well as the elements of $L^1(M, \psi_0)$ are $(-1)$-homogeneous. For $p \in [0, \infty]$, $L^p(M, \psi_0)$ is the set of closed densely defined $(-1/p)$-homogeneous operators $H$ such that $T^p[H]$ belongs to $L^1(M, \psi_0)$. The $p$-norm of $T \in L^p(M, \psi_0)$ is defined as $\|T\|_p = \left(\int |T|^p dv_0\right)^{1/p}$. For $p = \infty$, we define $L^\infty(M, \psi_0) = M$.

The well-known isomorphism of $L^2(X \times X)$ onto the space of Hilbert-Schmidt operators on $L^2(X)$ which associates to $f$ the integral operator $L(f)$ with kernel $f$ and the Fourier-Plancherel transform of $L^2(G)$ onto $L^2(\hat{G})$, where $G$ is a locally compact abelian group can be viewed as particular cases of the following general result for measured groupoids, which is proved in [3].

**Theorem 2.1.** Let $(G, \lambda, \mu)$ be a measured groupoid. There exists a unique isometry $\mathcal{F}_2 : L^2(G, \nu_0) \rightarrow L^2(M, \psi_0)$ such that, for $f \in C_c(G)$, $\mathcal{F}_2(f)$ is the closure of the operator $\Delta^{1/4}L(f)\Delta^{1/4}$ defined on $C_c(G)$.

In [13], M. Terp defines the non-commutative $L^p$-spaces by the complex interpolation method in a framework perfectly well-fitted to ours. The data consists of a von Neumann algebra $M$ and a (normal, faithful and semi-finite) weight $\varphi$. She defines the “intersection” $L$ of $M$ and its predual $M_*$ and then embeds $M$ and $M_*$ into $L^*$, turning $(M, M_*)$ into a compatible pair of Banach spaces. She also defines (Section 2.3) for all $p \in [1, \infty]$ a linear, norm-decreasing, injective map $\mu_p : L \rightarrow L^p(M, \psi)$ which has dense range for $p < \infty$ and where $\psi$ is the dual weight of $M^\prime$ in the GNS representation of $(M, \varphi)$. We apply this to $(M = VN(G), \varphi = \varphi_0)$. We observe that $C_c(G) \ast C_c(G)$ is contained in $L$ and that for $f \in C_c(G) \ast C_c(G)$, $\mu_p(f)$ is the closure of the operator $\Delta^{1/2p}L(f)\Delta^{1/2p}$ defined on $C_c(G)$. For $p = 1$, $f \mapsto \mu_1(f)$ extends to the isomorphism $L^1(\hat{G}) \rightarrow L^1(M, \psi_0)$ described earlier. For $p = 2$, $f \mapsto \mu_2(f)$ extends to the above Fourier-Plancherel isometry $\mathcal{F}_2 : L^2(G, \nu_0) \rightarrow L^2(M, \psi_0)$. For $p = \infty$, $\mu_\infty = L$ is the left regular representation.

3. A HAUSSFORD-Young INEQUALITY.

We use the same notation as in the previous section. Because of our assumptions, $\Delta$ as well as $\Delta^\gamma$, where $z \in C$, admit $C_c(G)$ as an essential domain.

**Lemma 3.1.** Let $z \in \mathbb{C}$ and $f \in C_c(G)$. We have the commutation rule

$$L(f)\Delta^{-z} \subset \Delta^{-z}L(\delta^z f).$$

**Proof.** We have seen that $\Delta^z$ implements the automorphism $\Delta(z)$ of the modular Hilbert algebra $C_c(G)$. Therefore, $\Delta^z L(f) \Delta^{-z} \subset L(\delta^z f)$. \hfill $\square$

Since $\Delta^z$ as well as $L(f)$ leave $C_c(G)$ invariant, $C_c(G)$ is contained in the domain of $\Delta^z L(f)\Delta^z$. Therefore $\Delta^z L(f)\Delta^z$ admits an adjoint; moreover the equality

$$(\Delta^z L(f)\Delta^z \xi|\eta) = (\xi|\Delta^z L(f^*)\Delta^z \eta)$$

for $\xi, \eta \in C_c(G)$ shows that this adjoint is an extension of $\Delta^z L(f^*)\Delta^z$. In particular, $\Delta^z L(f)\Delta^z$ is closable.

**Definition 3.1.** Given $\alpha \in \mathbb{C}$ and $\beta = (1 - 2\alpha)^{-1}$, we define the $\beta$-Fourier transform $\mathcal{F}_\beta(f)$ of $f \in C_c(G)$ as the closure of the operator $\Delta^\alpha L(f)\Delta^\alpha$ defined on $C_c(G)$. 

We have met particular cases of this definition: when \( f \) belongs to \( C_c(G) \ast C_c(G) \) and \( \alpha = 1/2q \), where \( q \in [1, \infty) \), then \( \beta = p \) is the conjugate exponent of \( q \) and \( \mathcal{F}_p(f) = \mu_q(f) \), where \( \mu_q \) is Terqs symmetric embedding recalled earlier. In that case, \( \mathcal{F}_p(f) \) belongs to \( L^q(M, \psi_0) \). On the other hand, \( \mathcal{F}_2(f) \) belongs to \( L^2(M, \psi_0) \) and \( \mathcal{F}_1(f) = L(f) \) belongs to \( L^\infty(M, \psi_0) = M \) for all \( f \in C_c(G) \).

Our goal is to show that, for \( 1 \leq p \leq 2 \), \( f \in C_c(G) \), \( \mathcal{F}_p(f) \) belongs to \( L^q(M, \psi_0) \) where \( q \) is the conjugate exponent of \( p \) and satisfies the following Hausdorff-Young inequality:

\[
\|\mathcal{F}_p(f)\|_q \leq \max(\|f\|_{p,q}, \|f^*\|_{p,q})
\]

where \( \|f\|_{p,q} \) denotes the following mixed norm, which generalizes the mixed norms of [2].

**Definition 3.2.** Given \( p, q \in [1, \infty] \) and \( f \in C_c(G) \), the mixed norm \( \|f\|_{p,q} \) of \( f \) is defined as

\[
\|f\|_{p,q} = (\int (\int |f|^p d\lambda^x)^{q/p} d\mu(x))^{1/q}
\]

with the obvious modifications when \( p \) or \( q \) is infinite.

We first verify that this inequality holds when \( p = 1 \) or \( p = 2 \). Indeed, for \( p = 1 \), this is the well-known inequality

\[
\|L(f)\| \leq \max(\|f\|_{1,\infty}, \|f^*\|_{1,\infty})
\]

which is obtained by the Cauchy-Schwarz inequality. For \( p = 2 \), we have

\[
\|\mathcal{F}_2(f)\|_2^2 = \int |f|^2 d\nu_0 \leq (\int |f|^2 d\nu)^{1/2}(\int |f|^2 d\nu^{-1})^{1/2} = \|f\|_{2,2}\|f^*\|_{2,2}
\]

again by Cauchy-Schwarz. We want to deduce the general case \( 1 \leq p \leq 2 \) from these two extremal cases by interpolation. Although the norm \( \|f\|_{p,q} \) interpolates between 1 and 2 (in fact between 1 and \( \infty \)), we cannot interpolate directly because our formula involves the maximum of two mixed norms. We proceed exactly as in [17]. We shall only give the necessary modifications. It relies on the following characterization of \( L^p(M, \psi_0) \).

**Proposition 3.2.** [17, Proposition 2.3] Let \( p \in [1, \infty] \) and define \( q \) by \( 1/p + 1/q = 1 \). Let \( T \) be a closed densely defined \((-1/p)\)-homogeneous operator on \( H \). Then, the following conditions are equivalent:

(i) \( T \in L^p(M, \psi_0) \),

(ii) there exists a constant \( C \geq 0 \) such that

\[
\forall S \in L^p(M, \psi_0), \forall \xi \in A_I \cap \text{Dom}(T), \forall \eta \in A_I \cap \text{Dom}(S) : \\
\|(T\xi, S\eta)\| \leq C\|S\|_q\|L(\xi)\|\|L(\eta)\|.
\]

If \( T \in L^p(M, \psi_0) \), then \( \|T\|_p \) is the smallest \( C \) satisfying (ii).

In the sequel, we assume that \( 1 < p < 2 \), we define \( q \) by \( 1/p + 1/q = 1 \) and we let \( f \in C_c(G) \) such that \( \max(\|f\|_{p,q}, \|f^*\|_{p,q}) \leq 1 \). We let \( M(x,y) = \max(\|f\|^p, \|f^*\|^p) \). We choose \( \epsilon > 0 \) such that \( \epsilon^{(q-p)} \|f\|^p \leq 1 \) and we set \( M_\epsilon(x,y) = \max(M(x,y), \epsilon) \). We define for \( z \in C \) and \( \gamma \in G \)

\[
f_z(\gamma) = \begin{cases} 
\text{sgn}(\gamma)|f(\gamma)|^p M_\epsilon(v(\gamma), s(\gamma))^{q-z(p+q)} & \text{ if } f(\gamma) \neq 0 \\
0 & \text{ if } f(\gamma) = 0.
\end{cases}
\]

Then \( f_z \) belongs to \( C_c(G) \). We denote by \( \Omega \) the open strip \( \{z \in C : 1/2 \leq \Re z \leq 1\} \).

We have the following lemma which is proved exactly as [17, Lemma 4.1].

**Lemma 3.3.** Let \( p \in [1, 2], \Omega \subset C, f, f_z \in C_c(G) \) be as above. Let \( \xi \in A_I \). Then
(i) for each \( z \in \Omega \), the function
\[
\xi_z = F_{1/z}(f_z)\xi = \Delta^{\frac{1}{2z}} L(f_z) \Delta^{\frac{1}{2z}} \xi
\]
is defined and belongs to \( H \);
(ii) the function \( z \mapsto \xi_z \) with values in \( H \) is bounded on \( \Omega \);
(iii) for each \( \eta \in H \), the scalar function \( z \mapsto (\xi_z, \eta) \) is continuous on \( \Omega \) and analytic on \( \Omega \).

**Proof.** The proof given in [17, Lemma 4.1] applies with little change. To prove (i) and (ii), one introduces \( \eta \in C_c(G) \) and considers the integral
\[
H_\eta(z) = \int \xi(\gamma) F_{1/z}(f_z)\eta(\gamma) d\nu^{-1}(\gamma).
\]
By the use of Fubini's theorem, one checks that this is well defined and that
\[
H_\eta(z) = \int \xi_z(\gamma) \eta(\gamma) d\nu^{-1}(\gamma).
\]
Then, one proves that there exists \( C \geq 0 \) such that \( |H_\eta(z)| \leq \|\eta\|_2 \). To do that, one applies the Phragmen-Lindelöf principle. The function \( z \mapsto H_\eta(z) \) is bounded continuous on \( \Omega \) and analytic on \( \Omega \). It suffices to estimate it on the lines \( \Re z = 1 \) and \( \Re z = 1/2 \).

Suppose that \( z = 1 + it \), where \( t \) is real. Then,
\[
\xi_z = \Delta^{-it/2} L(f_z) \Delta^{-it/2} \xi
\]
belongs to \( H \) and we have
\[
|H_\eta(z)| \leq \|L(f_z)\| \|\xi\|_2 \|\eta\|_2.
\]
In order to estimate the operator norm of \( L(f_z) \), we use the inequality
\[
\|L(f_z)\| \leq \max(\|f_z\|_{1,\infty}, \|f_z^*\|_{1,\infty}).
\]
Let us fix \( x \in G^{(0)} \) and estimate
\[
\int |f_z|d\lambda^x = \int |f(\gamma)|^p M_e(x, s(\gamma))^{-p} d\lambda^x(\gamma).
\]
We write \( G^x \) as the disjoint union \( A \cup B \), where \( A = \{ \gamma \in G^x : M(x, s(\gamma)) \leq \epsilon \} \) and \( B = G^x \setminus A \). If \( \gamma \in A \), \( M_e(x, s(\gamma)) = M(x, s(\gamma)) \geq \|f_z\|_p \). Therefore
\[
\int_A |f_z|d\lambda^x \leq \|f_z\|^p \int_A |f(\gamma)|^p d\lambda^x(\gamma) \leq 1.
\]
On the other hand, if \( B \) is non-empty, then \( |f_z|_p < \epsilon \) and for all \( \gamma \in B \), \( M_e(x, s(\gamma)) = \epsilon \). Therefore,
\[
\int_B |f_z|d\lambda^x \leq \epsilon^{-p} \int_B |f(\gamma)|^p d\lambda^x(\gamma) \leq \epsilon^{-p} \epsilon^p \leq 1.
\]
This shows that \( \|f_z\|_{1,\infty} \leq 2 \). One shows similarly that \( \|f_z^*\|_{1,\infty} \leq 2 \). This gives \( \|L(f_z)\| \leq 2 \) and
\[
|H_\eta(z)| \leq 2\|\xi\|_2 \|\eta\|_2.
\]
Suppose next that \( z = \frac{1}{2} + it \), where \( t \) is real. Then,
\[
\xi_z = \Delta^{-it} F_2(\delta^{it/2} f_z)\xi.
\]
This also belongs to \( H \) because the domain of operators in \( L^2(M, \psi_0) \) contains \( \mathcal{A}_1 \).

We deduce the inequality
\[
|H_\eta(z)| \leq \|F_2(\delta^{it/2} f_z)\| \|\eta\|_2.
\]
As a corollary of Proposition 3.2, we have the inequality
\[
\|F_2(\delta^{it/2} f_z)\| \|\eta\|_2 \leq \|F_2(\delta^{it/2} f_z)\| \|L(\xi)\|.
\]
We use the Plancherel Theorem \([21]\) to evaluate the norm
\[
\|F_2(\hat{\delta}t^2 f_z)\|_2 = \left( \int |f_z|^2 \, d\nu \right)^{1/2} = \left( \int |f_z|^2 \, d\nu \int |f_z|^2 \, d\nu^{-1} \right)^{1/4}.
\]
Let us estimate the integral
\[
\int |f_z|^2 \, d\nu = \int |f(\gamma)|^p M_s(r(\gamma), s(\gamma))^{(q-p)} d(\mu \circ \lambda)(\gamma).
\]
We write the domain of summation as the union of three parts
\[
A = \{ \gamma : \|f^r(\gamma)\|_p \geq \max (\|f_s(\gamma)\|_p, \epsilon) \},
\]
\[
B = \{ \gamma : \|f_s(\gamma)\|_p \geq \max (\|f^r(\gamma)\|_p, \epsilon) \}
\]
and
\[
C = \{ \gamma : M(r(\gamma), s(\gamma)) \leq \epsilon \}.
\]
If \(\gamma \in A\), \(M_s(r(\gamma), s(\gamma)) = \|f^r(\gamma)\|_p\), therefore
\[
\int_A |f_z|^2 d(\mu \circ \lambda) \leq \int (\int |f(\gamma)|^p d\lambda^x(\gamma)) \|f^r\|_p^{-p} \|f^r\|_p \|d\mu(x) = \|f\|_{p,q}^2 \leq 1.
\]
Similarly,
\[
\int_B |f_z|^2 d(\mu \circ \lambda) \leq \|f^r\|_{p,q}^2 \leq 1.
\]
Finally, if \(\gamma \in C\), \(M_s(r(\gamma), s(\gamma)) = \epsilon\) and
\[
\int_C |f_z|^2 d(\mu \circ \lambda) = \epsilon^{(q-p)} \int_R |f|^p d(\mu \circ \lambda) \leq \epsilon^{(q-p)} \|f\|_{p,p}.
\]
Because of our choice of \(\epsilon > 0\), this is majorized by 1. The whole integral is majorized by 3. One shows in the same fashion that the integral \(\int |f_z|^2 \, d\nu^{-1}\) is also majorized by 3. Therefore, \(\|F_2(\hat{\delta}t^2 f_z)\|_2 \leq \sqrt{3} \leq 2\). This gives
\[
|H_\eta(z)| \leq 2\|L(\xi)\|\|\eta\|_2
\]
on the line \(\Re z = 1/2\).

This proves the desired inequality with \(C = \max(\|\xi\|_2, \|L(\xi)\|)\) and (i) and (ii). We have already proved (iii) for \(\eta \in C_c(G)\). The general case follows by approximation.

The proof of the following lemma, which is exactly [17, Lemma 4.2], does not require any modification.

**Lemma 3.4.** Let \(p \in [1, 2]\) and let \(q\) be defined by \(1/p + 1/q = 1\). Let \(f \in C_c(G)\) be as above and let \(S \in L^p(M, \psi_0)\). Then, for all \(\xi \in A_f\) and \(\eta \in A_f \cap \text{Dom}(S)\), we have:
\[
|\langle F_p(f)\xi, S\eta \rangle| \leq 2\|S\|_p \|L(\xi)\|\|L(\eta)\|.
\]
Note that \(\xi \in \text{Dom}(F_p(f))\) by Lemma 3.3

**Proof.** We just recall how the proof goes. We assume \(\|S\|_p \leq 1\). By [17, Lemma 2.5], it suffices to consider the case \(\eta \in A_f \cap \text{Dom}(|S|^p)\). Let \(S = U|S|\) be the polar decomposition of \(S\). For \(z \in \overline{G}\), one defines
\[
\eta_z = U|S|^{p_2} \eta.
\]
Then the function \(z \mapsto \eta_z \in H\) is bounded continuous on \(\overline{G}\) and analytic on \(\Omega\). We then define \(H(z) = \langle \xi_z, \eta_z \rangle\). The scalar function \(z \mapsto H(z)\) satisfies the same properties. One proves the estimate
\[
|H(z)| \leq 2\|L(\xi)\|\|L(\eta)\|
\]
for \(z \in \Omega\) by checking it on the lines \(\Re z = 1\) and \(\Re z = 1/2\) as in the previous lemma.
For $z = 1 + it$, where $t$ is real, one has
\[
|H(z)| = \langle L(\delta^{it/2} f_z) \xi, \Delta^{it} U[S^{it}] \eta \rangle \leq \|L(\delta^{it/2} f_z)\| \|L(\xi)\| \|L(\eta)\|.
\]
by applying Proposition 3.2 with $p = \infty$. Then one uses the estimate
\[
\|L(\delta^{it/2} f_z)\| \leq 2
\]
established in the previous lemma.

For $z = \frac{1}{2} + it$, where $t$ is real, one simply uses Cauchy-Schwarz' inequality
\[
|H(z)| \leq \|\xi\|_2 \|\eta\|_2.
\]
We have established earlier the estimate $\|\xi\|_2 \leq 2\|L(\xi)\|$. On the other hand,
\[
\|\eta\|_2 = \|U[S^{it}]^{-p\eta}\|_2
\]
can be also estimated by Proposition 3.2 with $p = 2$:
\[
\|U[S^{it}]^{-p\eta}\|_2 \leq \|U[S^{it}]^{-p\eta}\|_2 \leq \|L([S^{it}\eta]\|_2) \leq \|L(\eta)\|.
\]
In particular, we have
\[
|H(\frac{1}{p})| = (\mathcal{F}_p(f) \xi, \mathcal{F}_p(\eta)) \leq 2\|L(\xi)\| \|L(\eta)\|.
\]

One deduces from Proposition 3.2 that for $f \in C_c(G)$, $\mathcal{F}_p(f)$ belongs to $L^q(M, \psi_0)$ for all $p \in [1, 2]$ where $1/p + 1/q = 1$ and
\[
\|\mathcal{F}_p(f)\|_{L^q(M)} \leq 2 \max(\|f\|_{p,q}, \|f^*\|_{p,q}).
\]
This inequality can be improved.

**Theorem 3.5. (Hausdorff-Young) Let $p \in [1, 2]$ and $f \in C_c(G)$. Then $\mathcal{F}_p(f)$ belongs to $L^q(M, \psi_0)$ where $1/p + 1/q = 1$ and
\[
\|\mathcal{F}_p(f)\|_{L^q(M)} \leq \max(\|f\|_{p,q}, \|f^*\|_{p,q})
\]
where $1/p + 1/q = 1$.

**Proof.** We first observe that the mixed norms as well as the $L^p$ norms behave well with respect to products of groupoids. Explicitly, let $(G_i, \lambda_i, \mu_i), i = 1, 2$ be two measured groupoids and let $(G = G_1 \times G_2, \lambda = \lambda_1 \otimes \lambda_2, \mu = \mu_1 \otimes \mu_2)$ be their product. Given $1 \leq p, q \leq \infty$ and $f_i \in C_c(G_i)$, we have
- $\|f_1 \otimes f_2\|_{p,q} = \|f_1\|_{p,q} \|f_2\|_{p,q}$;
- $\mathcal{F}_p(f_1 \otimes f_2) = \mathcal{F}_p(1)(f_1) \otimes \mathcal{F}_p(2)(f_2)$;
- $\|T_i \in L^q(M_i, \psi_0^{(i)})\|_{L^q(M_i, \psi_0^{(i)})}, T = T_1 \otimes T_2$ belongs to $L^q(M, \psi_0)$ and $\|T\|_{L^q(M)} = \|T_1\|_{L^q(M^{(1)})} \|T_2\|_{L^q(M^{(2)})}$ where the indices $1, 2$ refer to the corresponding groupoids. The Hilbert space, the von Neumann algebra and the canonical weights of the product groupoid are also tensor products: $H = H^{(1)} \otimes H^{(2)}$, $M = M^{(1)} \otimes M^{(2)}$, $\varphi = \varphi^{(1)} \otimes \varphi^{(2)}$ and $\psi = \psi^{(1)} \otimes \psi^{(2)}$.

Given $f \in C_c(G)$, we form
\[
F = (f^* \otimes f) \otimes (f^* \otimes f) \otimes \ldots (f^* \otimes f)
\]
where $f^* \otimes f$ is repeated $n$ times. It belongs to $C_c(G^{2n})$. According to the above $\mathcal{F}_p(F)$ belongs to $L^q(M^{(2)\otimes 2n}, \psi_0^{(2\otimes 2n)})$ and
\[
\|\mathcal{F}_p(F)\|_{L^q(M^{(2)\otimes 2n})} \leq \max(\|F\|_{p,q}, \|F^*\|_{p,q})
\]
Since $\|F\|_{p,q} = \|F^*\|_{p,q} = \|f\|_{p,q} \|f^*\|_{p,q}$ and $\|\mathcal{F}_p(F)\|_{L^q(M^{2n})} = \|\mathcal{F}_p(f)\|_{L^q(M)}$, this gives
\[
\|\mathcal{F}_p(f)\|_{L^q(M)} \leq 2^{1/2n} \|f\|_{p,q} \|f^*\|_{p,q}^{1/2n}.
\]
Passing to the limit, one gets:

$$\|F_p(f)\|_{L^q(M)} \leq \|f\|^{1/2}_{p,q} \|f^*\|^{1/2}_{p,q} \leq \max(\|f\|_{p,q}, \|f^*\|_{p,q}).$$

□

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