I. INTRODUCTION.

Everyone who has ever studied perturbative quantum field theory (QFT) has come face to face with Feynman integrals and knows all too well that evaluating these loop integrals is often a hard task. Books on QFT generally teach us how to deal with them either in terms of the so-called $\alpha$-parametrization or by using Feynman parameters for the propagators. This approach, contrary to what we may think, is perhaps one of the most laborious techniques to solve them because the number of parametric integrals increases with the number of propagators. Moreover, the added parametric integrals are often very difficult to solve exactly even if there are no massive particles in the intermediate states. There are several techniques that have been developed over the years in order to solve Feynman integrals, and we can just mention a few ones: the Mellin-Barnes’ representation of massive propagators, the Gegenbauer polynomial approach in configuration space and integration by parts and some others. Yet in this arena, the simpler the method the better. A novel approach that was suggested some years ago has to do with the use of negative dimensions and seems to be a promising technique.

In mathematical physics, one of the most powerful and useful principles is that of analytic continuation. It is the underlying principle that allowed ’t Hooft et al to develop the elegant technique of dimensional regularization (DREG). Besides a great number of other uses in Physics, there is one that particularly is of interest and that most concerns us here: It is that of negative dimensional integration method (NDIM) which combines the two amazing features in its heart: The principle of analytic continuation and the technique of DREG. What is the advantage of such a combination over the plain DREG? The latter one allows us to calculate Feynman diagrams in the analytically continued $D$-dimensional space, while the former allows us to greatly simplify the technical difficulties associated with performing parametric integrals, just by analytically extending $D$ over negative values. This simplification comes about in virtue of the polynomial nature of the integrands. In a few words, this character emerges from the equivalence between negative dimensional bosonic integration and positive dimensional fermionic integration.

What is the price one pays for working them out in negative dimensions? Basically, the technical difficulties that arise are solving many systems of linear algebraic equations, performing gaussian/gaussian-type integrals and dealing with multi-indexed power series. For the former two, one can just ask whether could it be simpler than these? Yet the difficult part is not absent: the laborious piece comes in the form of multi-indexed power series. However, looking at it from the encouraging side, we can say that it allows us a standardized representation for Feynman integrals in terms of power series.

In a previous paper we carried out what we, as theoreticians, call a “lab-test”, that is, with a new approach one studies a well-known system. Employing NDIM we calculated a massless two-loop three point vertex, keeping two of the external legs on-shell. The NDIM technique led us to discover twelve different ways in which the result could be written down, i.e., a twelve-fold degeneracy for that particular integral. We also considered some two-loop self-energy diagrams.

Of course NDIM is not the only technique to calculate Feynman integrals. Recently, Fleischer et al studied asymptotic expansions of some two-loop vertex and Frink et al gave results for a general massive two-loop three point vertex. Ussyukina and Davydychev calculated the Feynman diagram we will study in this paper but they
did give the result in terms of only two dimensionless variables. On the other hand, we will write down twenty-one results in terms of not only two combinations of external momenta, but rather in many different ratio combinations for the external momenta. Our aim here is not to make any numerical calculations, of course, but analytical ones. The interested reader in the two-loop calculation “technology” in QFT can consult a good review on this subject by Davydychev [12].

The outline for our paper is as follows: in section 2 we calculate the vertex in question in euclidean space, in section 3 we consider five special cases where only one external leg is off-shell and in the last one, section 4, we conclude the work.

II. OFF-SHELL TWO-LOOP VERTEX.

This computation is performed following the few simple steps outlined in [8,9]. First of all, let us consider the integral,

\[ I = \int \int d^D r \, d^D q \exp \left[ -\alpha q^2 - \beta (q - p)^2 - \gamma r^2 - \omega (q - r - k)^2 \right], \]

which corresponds to the diagram of figure 1.

The general solution for the integral in negative D, defined by

\[ J_{NDIM} = \int \int d^D q \, d^D r \, (q^2)^i \left[ (q - p)^2 \right]^{j} \left[ (r^2) \right]^{l} \left[ (r - q + k)^2 \right]^{m}, \]

is given by the multiple series,

\[ S_{NDIM} = G(i, j, l, m; D) \sum_{n_1, \ldots, n_9 = 0} \frac{(p^2)^{n_1 + n_2 + n_3} (k^2)^{n_4} (t^2)^{n_5}}{n_1! n_2! n_3! n_4! n_5! n_6! n_7! n_8! n_9!}, \]

where

\[ G(i, j, l, m; D) = (-\pi)^D \Gamma(1 + i) \Gamma(1 + j) \Gamma(1 + l) \Gamma(1 + m) \Gamma(1 - \sigma - \frac{1}{2} D), \]

and for convenience we use the definition \( \sigma = i + j + l + m + D \). The system one must solve is,

\[
\begin{align*}
    n_1 + n_2 + n_3 + n_5 + n_6 &= i \\
    n_1 + n_2 + n_4 + n_7 + n_8 &= j \\
    n_1 + n_3 + n_4 + n_5 + n_7 + n_9 &= l \\
    n_2 + n_3 + n_4 + n_6 + n_8 + n_9 &= m \\
    n_1 + n_2 + n_3 + n_4 &= \sigma.
\end{align*}
\]

It is an easy matter to see that this system is composed of five equations with nine “unknowns” (the sum indices), so that it cannot be solved unless it is done in terms of four arbitrary “unknowns”. These, of course will label the four remnant summations, which means that the answer will be in terms of a fourfold summation series. From the combinatorics, it is a straightforward matter to see that there are many different ways we can choose those four indices; indeed, we can choose \( C_9^4 = 126 \) different ways. In other words, what we need to do is to solve 126 different systems. Of these, 45 are unsolvable systems, i.e., they are systems whose set solution is empty. There remains therefore 81 which have non-trivial solutions. Of course, the trivial solutions are of no interest at all. However, the non-trivial solutions generate a space of functions with different basis, characterized by their functional variable, according to the different possibilities allowed for ratios of external momenta. Each basis is a solution for the pertinent Feynman integral, which is connected by analytic continuation to all other basis defined by the other sets of solutions. We remind ourselves that a basis that generates a given space can be composed of one or several linearly independent functions combined in what is called linear combination.

Each representation of the Feynman integral will be given by a basis of functions generated by the solutions of the systems [13]. Of course, only linearly independent and non-degenerate solutions are relevant to define a basis.

It can be easily seen that the diagram we are dealing with here is symmetric under the exchange of external momenta \( k^2 \leftrightarrow l^2 \). This symmetry is reflected by the systems we have to solve, and the solutions display this fact. Therefore, solutions within this category will be given only once, that is, the ones which can be obtained by symmetry will not be written down explicitly.
A. Two Variables.

With the solutions in hands — it is an easy matter to write down a computer program to solve all the systems — and the general form of the results [8], we can start to build the power series representations of the Feynman graph.

We begin our analysis of the solutions for the systems by looking at the simpler ones having two variables, defined by ratios of external momenta. Of course, there are in fact four sums but two of them have unity argument, making it possible for us to actually sum the pertinent series as we shall shortly see. The variables are,

\[ (x, y), \quad (z, y^{-1}), \quad (x^{-1}, z), \quad (x^{-1}, z^{-1}), \]  \hspace{1cm} (5)

where we define the dimensionless ratios

\[ x = \frac{p^2}{k^2}, \]

\[ y = \frac{t^2}{k^2}, \]  \hspace{1cm} (6)

\[ z = \frac{p^2}{l^2}. \]

Note that the second pair of variables above is exactly symmetric to the first one by the interchange \( k^2 \leftrightarrow l^2 \), which means that we will not write this second solution explicitly. Also, each of the three pairs above appears twelve times among the total of 81 systems with non-trivial solutions. Each of these is therefore twelve-fold degenerate just like in the on-shell case calculated in [8]. A way of expressing the first solution in positive \( D \) is given by

\[ S_{AC} = \pi^D P_{AC} (p^2)^{i+j} + \frac{1}{2} D (k^2)^{i+m} + \frac{1}{2} D \sum_{n_4, n_6, n_7, n_9=0}^{\infty} \frac{(x)^{n_9} (y)^{n_4}}{n_4! n_9!} \]

\[ \times \frac{(-l - m - \frac{1}{2} D)|n_4 + n_9| (\frac{1}{2} D + i|n_4 + n_7 + n_9) \times (1 + j + \frac{1}{2} D)|n_4 + n_6 - n_7|}{(1 + i - l|n_4 - n_6 + n_7 + n_9)(1 - j + l - m - D)|n_4 - n_6|} \times (-1)^{n_7} \]

where

\[ P_{AC} = (-i - j - \frac{1}{2} D)(-j + l + \frac{1}{2} D)(-l + m + \frac{1}{2} D) \]

\[ \times (-m - l - \frac{1}{2} D)(l + m + D)(\sigma + \frac{1}{2} D)|i - \sigma). \]  \hspace{1cm} (8)

Using the property \( (a|b + c) = (a + b|c)(a|b) \) and the well-known summation formula [14, 15] of Gauss’ hypergeometric function \( {}_2 F_1 \) with unity argument,

\[ {}_2 F_1(a, b; c|1) = \frac{\Gamma(c)\Gamma(c - a - b)}{\Gamma(c - a)\Gamma(c - b)}, \]

we can sum the series in \( n_6 \) and in \( n_7 \) above. We then get,

\[ S_{AC} = \pi^D (p^2)^{i+j} + \frac{1}{2} D (k^2)^{i+m} + \frac{1}{2} D P_{AC} \sum_{n_4, n_9=0}^{\infty} \frac{(x)^{n_9} (y)^{n_4}}{n_4! n_9!} \]

\[ \times \frac{(-l - m - \frac{1}{2} D)|n_4 + n_9| (\frac{1}{2} D + i|n_4 + n_7 + n_9) \times (1 + j + \frac{1}{2} D)|n_4 + n_6 - n_7|}{(1 + i - l|n_4 - n_6 + n_7 + n_9)(1 - j + l - m - D)|n_4 - n_6|} \times (-1)^{n_7} \]

\[ \equiv \pi^D (p^2)^{i+j} + \frac{1}{2} D (k^2)^{i+m} + \frac{1}{2} D \times \]

\[ P_{AC} \times {}_2 F_1 \left( \begin{array}{c}
-i - j - \frac{1}{2} D, \frac{1}{2} D + i \\
1 + i + j + \frac{1}{2} D, 1 - j - l - m - D
\end{array} \right) \]

\[ x, y \}. \]  \hspace{1cm} (10)
The remnant double summation in $n_4$ and $n_9$ is by definition the Appel’s $F_4$ hypergeometric function of two variables. In the particular case where $i = j = l = m = −1$ we can simplify even more this result by using a reduction formula which relates the $F_4$ function to the gaussian hypergeometric function $2F_1$.

$$F_4\left(\alpha, \beta; 1 + \alpha - \beta, \frac{-u}{1-u}(1-w), \frac{-w}{1-w}\right) = (1-w)^\alpha \ 2F_1\left(\alpha, \beta; 1 + \alpha - \beta, \frac{-u(1-w)}{1-u}\right).$$ (11)

For this special case,

$$S_{AC}^1(-1, \cdots, -1) = \pi^D \frac{1}{(k^2)^{2-D/2}} \left(\frac{1}{p^2} - \frac{1}{k^2}\right)^{2-D/2} \times \frac{\Gamma(D-3)!\Gamma^2(2-\frac{1}{2}D)\Gamma^3(\frac{1}{2}D - 1)}{\Gamma(D-2)!\Gamma^2(\frac{1}{2}D - 4)} \times 2F_1\left(2 - \frac{1}{2}D, 1 - D; 1 + \frac{1}{2}D, \frac{kp p^2 - kp}{p^2 k^2 - kp}\right).$$ (12)

We note that in order to regularize the divergences we can follow the standard procedure of dimensional regularization.

The next solution (third one) also gives double series. Following the same steps we can sum two series of unity argument and the remaining two are by definition the Appel’s hypergeometric function $F_4$,

$$F_4\left(\alpha, \beta; 1 + \alpha - \beta, \frac{-u}{1-u}(1-w), \frac{-w}{1-w}\right) = (1-w)^\alpha \ 2F_1\left(\alpha, 1 + \alpha - \beta; \frac{1}{2}D, \frac{1}{2}D - 1 + \frac{1}{2}D; \frac{kp p^2 - kp}{p^2 k^2 - kp}\right).$$ (15)

Again, in the particular case where all the exponents of propagators are minus one, this $F_4$ function reduces to a gaussian hypergeometric function, too. Using another reduction formula, namely,

$$F_4\left(\alpha, \beta; \frac{-u}{1-u}(1-w), \frac{-w}{1-w}\right) = (1-w)^\alpha (1-w)^\alpha \ 2F_1\left(\alpha, 1 + \alpha - \beta; \frac{1}{2}D, 1 + \alpha - \beta; \frac{1}{2}D\right).$$ (16)
B. Three Variables.

In a manner similar to the previous results, there are solutions which have three remaining variables, meaning that one of the series with unity argument is summed out. There are six sets of these, determined by their variables, each appearing four times, i.e., a fourfold degeneracy. Just to keep our accounting straight, \(4 \times 6 = 24\) systems yielding solutions with three variables. These, added to the 36 of the previous subsection, gives us 60 from the total of 81 non-trivial systems.

The solutions within this category have functional dependencies given by:

\[
(x, x, y), \ (z, z, y^{-1})
\]

\[
(x, y, y), \ (z, y^{-1}, y^{-1})
\]

\[
(x^{-1}, x^{-1}, z), \ (z^{-1}, z^{-1}, x^{-1})
\]

Note that the above triplets are conveniently arranged into pairs connected by the symmetry \(k^2 \leftrightarrow t^2\), so that in the following, only three of them will be dealt with specifically.

We list below the triple power series representations provided by NDIM,

\[
S^{AC}_3 = f_3 \sum_{n_1, n_2, n_3=0}^{\infty} \frac{(z)^{n_1+n_2}(y^{-1})^{n_3}}{n_1!n_2!n_3!} \frac{(m + \frac{1}{2}D)n_1}{(1-i-j-\frac{1}{2}D)n_1+n_2)} \times \frac{(l+\frac{1}{2}D)n_2}{(l+m+D)n_1+n_2}(1-i-l-m-D)n_3),
\]

where

\[
f_3 = \pi^D((i^2)^\sigma (-j)|\sigma|(-l|l+m+\frac{1}{2}D)(-m|l+m+\frac{1}{2}D)
\]

\[
\times (l+m+D|i+j-l-m-\frac{1}{2}D)(\sigma + \frac{1}{2}D-2\sigma - \frac{1}{2}D),
\]

\[
S^{AC}_4 = f_4 \sum_{n_7, n_8, n_9=0}^{\infty} \frac{(y^{-1})^{n_7+n_8}(z)^{n_9}}{n_7!n_8!n_9!} \frac{(m + \frac{1}{2}D)n_7}{(1+i+j+\frac{1}{2}D)n_9)} \times \frac{(l+\frac{1}{2}D)n_8}{(l+m+D)n_7+n_8}(\sigma + \frac{1}{2}D)n_7+n_8)(1+i+l+m+D)n_7+n_8),
\]

where

\[
f_4 = \pi^D((k^2)^\sigma x^j z^{i+\frac{1}{2}D}(-j|\sigma)- \frac{1}{2}D)(-j|j+l+\frac{1}{2}D)
\]

\[
\times (-l|l-i-m-D)(-m|i+m+\frac{1}{2}D)(l+m+D|l-m-\frac{1}{2}D),
\]

\[
S^{AC}_5 = f_5 \sum_{n_4, n_7, n_8=0}^{\infty} \frac{(x^{-1})^{n_4+n_7}(z^{-1})^{n_8}}{n_4!n_7!n_8!} \frac{(l+\frac{1}{2}D)n_8}{(1-j+\sigma)n_7+n_8)} \times \frac{(m + \frac{1}{2}D)n_7}{(\sigma - j|n_7+n_8)(1+i-\sigma)n_4)}),
\]

where

\[
f_5 = \pi^D((k^2)^\sigma x^j(-l|\sigma)(-l|l+m+\frac{1}{2}D)(-m|l+m+\frac{1}{2}D)
\]

\[
\times (\sigma + \frac{1}{2}D|i-\sigma)(l+m+D|i+2j-2\sigma).
\]
It is not difficult to note that \( f_5 \) has only one pole in the particular case when \( i = j = l = m = -1 \) and \( D = 4 \). What is going on here? Looking at the series we observe that there is a factor \((1 + i - \sigma |n_4)\) in the denominator. For the special case in question it gives,

\[
\frac{1}{(1 + i - \sigma |n_4)} = \frac{\Gamma(1 + i - \sigma)}{\Gamma(1 + i - \sigma + n_4)} = \frac{\Gamma(4 - D)}{\Gamma(4 - D + n_4)},
\]

so that the only term which is not singular is the first one, \( n_4 = 0 \), while the others become divergent for \( D = 4 \). Then, in fact, we have a double pole as it should be.

C. Four Variables.

Lastly, we consider in this subsection solutions of systems of linear algebraic equations leading to four variables, or better, fourfold summation with four variables. There are 21 of these solutions, which completes the total of 81 nontrivial solutions for the systems. Again, the functional variables are given paired with their corresponding symmetries, as follows:

\[
(z^{-1}, z^{-1}, x^{-1}, x^{-1}),
\]

\[
(z, z, y^{-1}, y^{-1}), (x, x, y, y)
\]

\[
(y, y, z, y^{-1}), (y^{-1}, y^{-1}, x, y)
\]

\[
(x, y, z, y^{-1}),
\]

\[
(x, y, x^{-1}, z^{-1}), (z, y^{-1}, z^{-1}, x^{-1})
\]

\[
(x, z, x^{-1}, z^{-1}), (x, x, z^{-1}, x^{-1})
\]

\[
(z, z, x^{-1}, z^{-1}), (x, y, x^{-1}, x^{-1})
\]

To get the accounting straight, let us again mention that the first three appear just once and the remaining nine appear twice, totalling the needed 21 of this category. The next two solutions appear just one time; the first one being given by

\[
S^{AC}_6 = f_6 \sum_{\{n_i\}=0}^{\infty} \frac{(z^{-1})^{n_5+n_6}(x^{-1})^{n_7+n_8}}{n_5!n_6!n_7!n_8!} \frac{(l + \frac{1}{2}D|n_6 + n_8)}{(1 - i + \sigma |n_5 + n_6)}
\]

\[
\times \frac{(m + \frac{1}{2}D|n_5 + n_7)(\sigma + \frac{1}{2}D|n_5 + n_6 + n_7 + n_8)}{(1 - j + \sigma |n_7 + n_8)}, \tag{20}
\]

where

\[
f_6 = \pi^D \left( \frac{k^2 l^2}{p^2} \right)^\sigma z^i x^j (-i - l - m - D)
\]

\[
\times (-j - l - m - D)(-l[l + m + \frac{1}{2}D],(-m[l + m + \frac{1}{2}D]),
\]

and

\[
S^{AC}_7 = f_7 \sum_{\{n_i\}=0}^{\infty} \frac{(z)^{n_1+n_2}(y^{-1})^{n_3+n_4}}{n_1!n_2!n_3!n_4!} \frac{(-j|n_1 + n_2 + n_7 + n_8)}{(1 - j + \sigma |n_7 + n_8)}
\]

\[
\times \frac{(l + \frac{1}{2}D|n_2 + n_8)(m + \frac{1}{2}D|n_1 + n_7)}{(1 - i - j - \frac{1}{2}D|n_1 + n_2)}, \tag{21}
\]

where

6
\[ f_7 = \pi^D(k^2)^\sigma y^j (-i - l - m - D)(-l|l + m + \frac{1}{2}D) \times (-m|l + m + \frac{1}{2}D)(\sigma + \frac{1}{2}D)i + j - \sigma), \]

The next set of five solutions are such that each one of them appears twice but with exponents of propagators interchanged, i.e., \(l \leftrightarrow m\). This means that the space of functions here is generated by two linearly independent basis functions, and the series representation for the Feynman integral will be given by

\[ S_r^{AC} = S_r^{AC(1)} + S_r^{AC(2)}, \quad r=8, 9, 10, 11, 12 \]

where \(S_r^{AC(2)}\) is obtained from \(S_r^{AC(1)}\) by interchanging of the exponents \(l \leftrightarrow m\). Then, from

\[ S_8^{AC(1)} = f_8^{(1)} \sum_{\{n_i\}=0}^\infty \frac{(y)^{n_2+n_6}(y^{-1})^{n_7}(z)^{n_9}(-1)^{n_1+n_2+n_6}}{n_2!n_6!n_7!n_9!} \frac{(l+\frac{1}{2}D)n_2+n_6}{(1+i+j+\frac{1}{2}D)n_7+n_9-n_2} \times (\sigma - l|n_7 + n_9 - n_2) \]

\[ \times (1 - i + l|n_2 + n_6 - n_7 - n_9)(1 + i + m + \frac{1}{2}D)n_7 - n_6 - n_2), \]

where

\[ f_8^{(1)} = (-\pi)^D(i^2)^\sigma (y^{-1})^{i+m+\frac{1}{2}D}(z)^{i+j+\frac{1}{2}D}(-i - j - \frac{1}{2}D)(-j|j + l + \frac{1}{2}D) \times (-l|\sigma)(-m|l - i - \frac{1}{2}D)(\sigma + \frac{1}{2}D)i - l - \sigma - \frac{1}{2}D), \]

we have the general solution

\[ S_8^{AC} = S_8^{AC(1)} + S_8^{AC(2)}, \]

where the second term is obtained from the first by interchanging \(l \leftrightarrow m\).

In this solution, it is important to note that there is one pole (in the special case when the exponents of the propagators are minus one) of the form \(\Gamma(i - l)\). To regularize it one must introduce a small correction to one of the exponents (a suitable one, of course), that is, to take \(i = -1 - \delta\) and then expand all the factors and the power series around \(\delta = 0\). Then, this “apparent singularity” cancels out [11,17].

Another result is given by,

\[ S_9^{AC(1)} = f_9^{(1)} \sum_{\{n_i\}=0}^\infty \frac{(z)^{n_1+n_2+n_6}(y^{-1})^{n_7}(-1)^{n_1+n_2+n_6+n_7}}{n_1!n_2!n_6!n_7!} \frac{(l+\frac{1}{2}D)n_2+n_6}{(1+i+m+\frac{1}{2}D)n_7-n_2-n_6} \times (l+\frac{1}{2}D)n_2+n_6(m+\frac{1}{2}D)n_1+n_7) \]

\[ \times (1 - i - j - \frac{1}{2}D)n_1 + n_2)(1 + j + l + \frac{1}{2}D)n_6 - n_1 - n_7), \]

where

\[ f_9^{(1)} = \pi^D(k^2)^{i+m+D/2}(i^2)^{j+l+D/2}(-i - m - \frac{1}{2}D)(-j - l - \frac{1}{2}D) \times (-l|l + m + \frac{1}{2}D)(-m|l + m + \frac{1}{2}D)(\sigma + \frac{1}{2}D) - l - m - D), \]

so that we get,

\[ S_9^{AC} = S_9^{AC(1)} + S_9^{AC(2)}. \]

We note that in this result appears three gamma functions that diverge in four dimensions. What is the nature of this extra singularity? It can be a pinch singularity [13].

Next we have
\[ S_{10}^{AC(1)} = f_{10}^{(1)} \sum_{\{n_1\} = 0}^{\infty} \frac{(z)^{n_2}(z^{-1})^{n_5}(y^{-1})^{n_7}(l + \frac{1}{2}D|n_2 + n_8)}{n_2!n_5!n_7!n_8!} \frac{(-1)^{n_8}(l + \frac{1}{2}D|n_2 + n_8)}{(1 - j + \sigma|n_7 + n_8)} \]
\[ \times \frac{(m + \frac{1}{2}D|n_5 + n_7)(i + j + m + D|n_5 + n_7 - n_2)}{(1 + j + m + \frac{1}{2}D|n_5 - n_2 - n_8)}, \]  
\[ \text{(26)} \]
where
\[ f_{10}^{(1)} = \pi^D(x^2)^{j}y^{j}(p^2)^{-m-\frac{1}{2}D}(l^2)^{j+\frac{1}{2}D}(-i|l + m + \frac{1}{2}D)(-j|l - m - \frac{1}{2}D) \]
\[ \times (-l|2l + \frac{1}{2}D)(-m| - i - l - D)(\sigma + \frac{1}{2}D - l - \frac{1}{2}D), \]
yielding the solution
\[ S_{10}^{AC} = S_{10}^{AC(1)} + S_{10}^{AC(2)}. \]  
\[ \text{(27)} \]
Next we have
\[ S_{11}^{AC(1)} = f_{11}^{(1)} \sum_{\{n_1\} = 0}^{\infty} \frac{(z)^{n_2+n_8}(x^{-1})^{n_3}(z^{-1})^{n_5}(-1)^{n_2+n_3}}{n_2!n_3!n_5!n_8!} \]
\[ \times \frac{(l + \frac{1}{2}D|n_2 + n_8)(-j + \sigma|n_8 - n_3)}{(1 - j + l|n_2 + n_8 - n_3 - n_5)(1 + j + m + \frac{1}{2}D|n_5 - n_2 - n_8)}, \]  
\[ \text{(28)} \]
where
\[ f_{11}^{(1)} = \pi^D(p^2)^{j+l+\frac{1}{2}D}(l^2)^{j+m+\frac{1}{2}D}(-i|j - l - \frac{1}{2}D)(-j|\sigma) \]
\[ \times (-l|j)(-m| - j - \frac{1}{2}D)(\sigma + \frac{1}{2}D - l - \frac{1}{2}D), \]
yielding the general solution
\[ S_{11}^{AC} = S_{11}^{AC(1)} + S_{11}^{AC(2)}. \]  
\[ \text{(29)} \]
Finally, we have
\[ S_{12}^{AC(1)} = f_{12}^{(1)} \sum_{\{n_1\} = 0}^{\infty} \frac{(z)^{n_2}(y^{-1})^{n_3}(z^{-1})^{n_5+n_7}}{n_2!n_3!n_5!n_7!} \]
\[ \times \frac{(-1)^{n_5+n_7}(m + \frac{1}{2}D|n_5 + n_7)}{(1 + i + m + \frac{1}{2}D|n_5 - n_2 - n_3)} \]
\[ \times \frac{(-l + \sigma|n_5 + n_7 - n_2)(-j + \sigma|n_7 - n_3)}{(1 + m + \sigma + \frac{1}{2}D|n_5 + n_7 - n_2 - n_3)}, \]  
\[ \text{(30)} \]
where
\[ f_{12}^{(1)} = (-\pi^D(l^2)^{j}z^{m-\frac{1}{2}D}(-i|l + m + \frac{1}{2}D)(-j|\sigma) \]
\[ \times (-l|\sigma)(-m| - i - \frac{1}{2}D)(\sigma + \frac{1}{2}D - m - 2\sigma - D), \]
yielding the general solution
\[ S_{12}^{AC} = S_{12}^{AC(1)} + S_{12}^{AC(2)}. \]  
\[ \text{(31)} \]
The various power series we obtained with negative dimensional integration approach to solve the Feynman integral relative to the chosen two-loop vertex diagram are very similar to hypergeometric series. Of course hypergeometric functions of more than two variables are known and they are called Lauricella’s functions. But in Appel et al studied four Lauricella’s functions named $F_A$, $F_B$, $F_C$, and $F_D$ — even though they mention that there are other fourteen. So, we do not know what are the regions of convergence of them nor even how they are called. However, knowing the region of convergence is not an essential thing here, because our external legs are off-shell anyway. This question becomes meaningful in the case where one of the legs are put on-shell, since only in this regime we can attribute a value to the external momenta.

III. ON-SHELL LIMIT.

Of course, particular cases of on-shell external legs must be contained in the set of off-shell solutions. To check on this, let us take two legs on-shell, namely, let $k^2 = t^2 = 0$. Not all off-shell solutions $S^{AC}$ are suitable for taking this particular limit, because some of them either vanish or become divergent. It is easy to see that such a suitable solution is given by eq.(13), because in this limit only the first term in the solution is given by eq.(13), for arbitrary (negative) exponents of propagators and (positive) dimension can be read from the solution $S^{AC}$

$$S^{AC}_2(k^2, t^2 = 0) = \pi^D(p^2)^\sigma(-i|\sigma)(-j|\sigma)(-l| - m - \frac{1}{2}D)(-m|l + m + \frac{1}{2}D)$$

$$\times (l + m + D| - l - \frac{1}{2}D)(\sigma + \frac{1}{2}D| - 2\sigma - \frac{1}{2}D). \quad (32)$$

This result is valid for arbitrary $D$ and (negative) exponents of propagators. In order to confront this result with known one we still need to go further in specializing to the case where $i = j = l = m = -1$ to get

$$S^{AC}_2 = \pi^D(p^2)^{D-4}\frac{\Gamma^2(D - 3)\Gamma^2(\frac{1}{2}D - 1)\Gamma(2 - \frac{1}{2}D)\Gamma(4 - D)}{\Gamma(D - 2)\Gamma(\frac{7}{2}D - 4)}. \quad (33)$$

This is the very result obtained by Hathrell using standard procedures for calculating Feynman diagrams in positive $D$. Of course, a more straightforward way of getting this result using NDIM is to put the corresponding legs on-shell from the very beginning, and what we get then is twelve systems to solve with non-trivial solutions, exactly the number we have in question: a twelfold degeneracy giving the same correct result.

We can consider also other special case, namely, the one where $p^2 = k^2 = 0$. This one is interesting because it contributes to other two-loop three-point diagram if we apply the integration by parts technique. The general result, for arbitrary (negative) exponents of propagators and (positive) dimension can be read from the solution $S_3$, eq.(17),

$$S_3^{AC}(p^2, k^2 = 0) = \pi^D(t^2)^\sigma(-j|\sigma)(-l| + m + \frac{1}{2}D)(-m|l + m + \frac{1}{2}D)$$

$$\times (l + m + D|i + j - l - m - \frac{1}{2}D)(\sigma + \frac{1}{2}D| - 2\sigma - \frac{1}{2}D), \quad (34)$$

taking the same particular case of Kramer et al, i.e., $l = m = -2$ and $i = j = -1$, we obtain,

$$S_3^{AC} = \pi^D(t^2)^{D-6}\frac{\Gamma^3(\frac{7}{2}D - 2)\Gamma(D - 5)\Gamma(6 - D)}{\Gamma(D - 4)}, \quad (35)$$

which is the well-known result in euclidean space.

Two other simpler special cases can be read from this graph, namely, when $p = 0$, $k = t$ (see fig.2) and $k = 0$, $p = t$ (see fig. 3). The solution $S_3$ gives us the first one,

$$S_3^{AC}(p = 0, k = t) = f_3(-i - j|i)(i + l + m + D| - i), \quad (36)$$

note that in the $n_1$ and $n_2$ series only the first term contributes and the $n_3$ one reduces to a gaussian hypergeometric function with unit argument.
The second case (see fig.3), \( k = 0, p = t \), can be read from \( S_2 \),

\[
S_{2AC}(k = 0, p = -t) = \pi^D(p^2)\sigma P_{2AC}(-i + \sigma | -\sigma)(\frac{1}{2} D + j | \sigma),
\]

(37)

this result can be used to study two self-energy two-loop graphs and agrees with our previous results \(^1\).

Finally, let us check up on a solution that has four series. Let \( j = 0 \), we get the graph of figure 4. The solution that allows us to consider this limit is \( S_7 \),

\[
S_{7AC}(j = 0) = \pi^D(k^2)\sigma f_7(j = 0),
\]

(38)

observe that there is no sum in the result. This is due to the factor \((-j | n_1 + n_2 + n_7 + n_8)\) which leads to only one non-vanishing term, i.e., when \( n_1 = n_2 = n_7 = n_8 = 0 \) in the series.

IV. CONCLUSION.

We have shown in this paper how we can work out a two-loop vertex diagram with all external legs off-shell using the NDIM technique to solve the pertinent Feynman integral. Altogether, twenty-one distinct results are obtained via NDIM technique for the considered two-loop three-point vertex diagram. These are expressed in terms of power series which can be identified as hypergeometric functions. The simpler ones are Appel’s \( F_4 \) hypergeometric function with two variables, which for the particular cases where all the exponents of the propagators are set to minus one, can be transformed into even simpler ones of the gaussian hypergeometric function type. The technique provides simultaneously with several analytic continuation formulas between different results, because they arise from the same Feynman integral \(^4\).

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Figure 1: Off-shell two-loop three point vertex calculated with NDIM.
Figure 2: A special case ($p = 0$, $k = t$) of the diagram of figure 1.
Figure 3: Taking $k = 0$ and $p = t$ the graph of figure 1 reduces to this one[9].
The former we get setting $i = j = l = m = -1$ in eq. (37) and the latter making $i = -2$ and the others equal to minus one in the same equation.
Figure 4: A special case \(( j = 0 \) ) of the diagram of figure 1.