Entropy of Spin Factors

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Abstract. Recently it has been demonstrated that the Shannon entropy or the von Neuman entropy are the only entropy functions that generate a local Bregman divergences as long as the state space has rank 3 or higher. In this paper we will study the properties of Bregman divergences for convex bodies of rank 2. The two most important convex bodies of rank 2 can be identified with the bit and the qubit. We demonstrate that if a convex body of rank 2 has a Bregman divergence that satisfies sufficiency then the convex body is spectral and if the Bregman divergence is monotone then the convex body has the shape of a ball. A ball can be represented as the state space of a spin factor, which is the most simple type of Jordan algebra. We also study the existence of recovery maps for Bregman divergences on spin factors. In general the convex bodies of rank 2 appear as faces of state spaces of higher rank. Therefore our results give strong restrictions on which convex bodies could be the state space of a physical system with a well-behaved entropy function.

1 Introduction

Although quantum physics has been around for more than a century the foundation of the theory is still somewhat obscure. Quantum theory operates at distances and energy levels that are very far from everyday experience and much of our intuition does not carry over to the quantum world. Nevertheless, the mathematical models of quantum physics have an impressive predictive power. These years many scientists try to contribute to the development of quantum computers and it becomes more important to pinpoint the nature of the quantum resources that may speed up the processing of a quantum computer compared with a classic computer. There is also an interest in extending quantum physics to be able to describe gravity on the quantum level and maybe the foundation of quantum theory has to be modified in order to be able to describe gravity. Therefore the foundation of quantum theory is not only of philosophical interest but it is also important for application of the existing theory and for extending the theory.

A computer has to consist of some components and the smallest component must be a memory cell. In a classical computer each memory cell can store one bit. In a quantum computer the memory cells can store one qubit. In this paper
we will focus on such minimal memory cells and demonstrate that under certain assumptions any such memory cell can be represented as a so-called spin factor. We formalize the memory cell by requiring that the state space has rank 2. In some recent papers it was proved that a local Bregman divergence on a state space of rank at least 3 is proportional to information divergence and the state space must be spectral [8,10]. Further, on a state space of rank at least 3 locality of a Bregman divergence is equivalent to the conditions called sufficiency and monotonicity. If the rank of the state space is 2 the situation is quite different. First of all the condition called locality reduce almost to a triviality. Therefore it is of interest to study sufficiency and monotonicity on state spaces of rank 2.

The paper is organized as follows. In the first part we study convex bodies and use mathematical terminology without reference to physics. The convex bodies may or may not correspond to state spaces of physical systems. In Section 2 some basic terminology regarding convex sets is established and the rank of a set is defined. In Section 3 regret and Bregman divergences are defined, but for a detailed motivation we refer to [8]. In Section 4 spectral sets are defined and it is proved that a spectral set of rank 2 has central symmetry. In Section 5 sufficiency of a regret function is defined and it is proved that a convex body of rank 2 with a regret function that satisfies sufficiency is spectral.

Spin factors are introduced in Section 6. Spin factors appear as sections of state spaces of physical systems described by density matrices on complex Hilbert spaces. Therefore we will borrow some terminology from physics. In Section 7 monotonicity of a Bregman divergence is introduced. It is proved that a convex body with a sufficient Bregman divergence that is monotone under dilations can be represented as a spin factor. For general spin factors we have not obtained a simple characterization of the monotone Bregman, but some partial results are presented in Section 8. In Section 9 it is proved that equality in the inequality for a monotone Bregman divergence implies the existence of a recovery map.

In this paper we focus on finite dimensional convex bodies. Many of the results can easily be generalized to bounded convex set in separable Hilbert spaces, but that would require that topological considerations are taken into account.

2 Convex Bodies of Rank 2

In this paper we will work within a category where the objects are convex bodies, i.e. finite dimensional convex compact sets. The morphisms will be affinities, i.e. affine maps between convex bodies. The convex bodies are candidates for state spaces of physical systems, so a point in a convex bodies might be interpreted as a state that may represent our knowledge of the physical system. A convex combination \( \sum p_i \sigma_i \) is interpreted as a state where the system is prepared in state \( \sigma_i \) with probability \( p_i \). In classical physics the state space is a simplex and in the standard formalism of quantum physics the state space is isomorphic to the density matrices on a complex Hilbert space.
A bijective affinity will be called an **isomorphism**. Let $K$ and $L$ denote convex bodies. An affinity $S : K \rightarrow L$ is called a **section** if there exists an affinity $R : L \rightarrow K$ such that $R \circ S = id_K$, and such an affinity $R$ is called a **retraction**. Often we will identify a section $S : K \rightarrow L$ with the set $S(K)$ as a subset of $L$.

Note that the affinity $S \circ R : L \rightarrow L$ is idempotent and that any idempotent affinity determines a section/retraction pair. We say that $\sigma_0$ and $\sigma_1$ are **mutually singular** if there exists a section $S : [0,1] \rightarrow K$ such that $S(0) = \sigma_0$ and $S(1) = \sigma_1$. Such a section is illustrated on Figure 1. A retraction $R : K \rightarrow [0,1]$ is a special case of a test [13, p. 15] (or an effect as it is often called in generalized probabilistic theories [2]). We say that $\sigma_0, \sigma_1 \in K$ are **orthogonal** if $\sigma_0$ and $\sigma_1$ belong to a face $F$ of $K$ such that $\sigma_0$ and $\sigma_1$ are mutually singular in $F$.

![Fig. 1. A retraction with orthogonal points $\sigma_0$ and $\sigma_1$. The corresponding section is obtained by reversing the arrows.](image)

The following result was stated in [9] without a detailed proof.

**Theorem 1.** If $\sigma$ is a point in a convex body $K$ then $\sigma$ can be written as a convex combination $\sigma = (1 - t) \cdot \sigma_0 + t \cdot \sigma_1$ where $\sigma_0$ and $\sigma_1$ orthogonal.

**Proof.** Without loss of generality we may assume that $\sigma$ is an algebraically interior point of $K$. For any $\sigma_0$ on the boundary of $K$ there exists a $\sigma_1$ on the boundary of $K$ and $t_{\sigma_0} \in [0,1]$ such that $(1 - t_{\sigma_0}) \cdot \sigma_0 + t_{\sigma_0} \cdot \sigma_1 = \sigma$. Let $R$ denote a retraction $R : K \rightarrow [0,1]$ such that $R(\sigma_0) = 0$. Let $S$ denote a section corresponding to $R$ such that $S(0) = \sigma_0$. Let $\pi_1$ denote the point $S(1)$. There exists a point $\pi_0$ on the boundary such that $\sigma = (1 - t_{\pi_0}) \cdot \pi_0 + t_{\pi_0} \cdot \pi_1$. Then

\[ R(\sigma) = R((1 - t_{\pi_0}) \cdot \pi_0 + t_{\pi_0} \cdot \pi_1) \]
\[ = (1 - t_\pi) \cdot R(\pi_0) + t_\pi \cdot R(\pi_1) \]
\[ \geq t_\pi \]
and
\[
R(\sigma) = R\left((1 - t_{\sigma_0}) \cdot \sigma_0 + t_{\sigma_0} \cdot \sigma_1\right) \\
= (1 - t_{\sigma_0}) \cdot 0 + t_{\sigma_0} \cdot R(\sigma_1) \\
= t_{\sigma_0} \cdot R(\sigma_1) .
\]
(4)
(5)
(6)
(7)

Therefore
\[
t_{\sigma_0} \cdot R(\sigma_1) \geq t_{\pi_0}
\]
(8)

Since \(t_{\sigma_0}\) is a continuous function of \(\sigma_0\) the function we may choose \(\sigma_0\) such that \(t_{\sigma_0}\) is minimal, but if \(t_{\sigma_0}\) is minimal Inequality (8) implies that \(R(\sigma_1) = 1\) so that \(\sigma_0\) and \(\sigma_1\) are orthogonal.

Iterated use of Theorem 1 leads to an extended version of Caratheodory’s theorem [9, Thm. 2].

**Theorem 2** (Orthogonal Caratheodory Theorem). Let \(K\) denote a convex body of dimension \(d\). Then any point \(\sigma \in K\) has a decomposition \(\sigma = \sum_{i=1}^{n} t_i \cdot \sigma_i\) where \(t^n_i\) is a probability vector and \(\sigma_i\) are orthogonal extreme points in \(K\) and \(n \leq d + 1\).

The Caratheodory number of a convex body is the maximal number of extreme points needed to decompose a point into extreme points. We need a similar definition related to orthogonal decompositions.
Definition 3. The rank of a convex body $K$ is the maximal number of orthogonal extreme points needed in an orthogonal decomposition of a point in $K$.

If $K$ has rank 1 then it is a singleton. Some examples of convex bodies of rank 2 are illustrated in Figure 3. Clearly the Caratheodory number lower bounds the rank of a convex body. Figure 5 provides an example where the Carathodory number is different from the rank. The rest of this paper will focus on convex bodies of rank 2. Convex bodies of rank 2 satisfy weak spectrality as defined in [4].

Fig. 3. Convex bodies of rank 2. The convex body to the left has a smooth strictly convex boundary so that any point on the boundary has exactly one orthogonal point. The body in the middle has a set of three extreme points that are orthogonal, but any point can be written as a convex combination of just two points. The convex body to the right is centrally symmetric without 1-dimensional proper faces, i.e. it is a spectral set.

If $K$ is a convex body it is sometimes convenient to consider the cone $K_+$ generated by $K$. The cone $K_+$ consist of elements of the form $x \cdot \sigma$ where $x \geq 0$ and $\sigma \in K$. Elements of the cone are called positive elements and such elements can be multiplied by positive constants via $x \cdot (y \cdot \sigma) = (x \cdot y) \cdot \sigma$ and can be added as follows.

$$x \cdot \rho + y \cdot \sigma = (x + y) \cdot \left( \frac{x}{x+y} \cdot \rho + \frac{y}{x+y} \cdot \sigma \right). \quad (9)$$

For a point $\sigma \in K$ the trace of $x \cdot \sigma \in K_+$ is defined by $\text{tr} [x \cdot \sigma] = x$. The cone $K_+$ can be embedded in a real vector space by taking the affine hull of the cone and use the apex of the cone as origin of the vector space and the trace extends to a linear function on this vector space. In this way a convex body $K$ can be identified with the set of positive elements in a vector space with trace 1.

Lemma 4. Let $K$ be a convex body and let $\Phi : K \to K$ be an affinity. Let $\Phi_{\mu} = \sum_{n=0}^{\infty} \frac{\mu^n}{n!} e^{-\mu} \Phi^n$. Then $\Phi_{\infty} = \lim_{\mu \to \infty} \Phi_{\mu}$ is a retraction of $K$ onto the set of fix-points of $\Phi$.

Proof. Since $K$ is compact the affinity $\Phi$ has a fix-point that we will call $s_0$. The affinity can be extended to a positive trace preserving affinity of the real
vector space generated by $\mathcal{K}$ into itself. Since $\Phi$ maps a convex body into itself all the eigenvalues of $\Phi$ are numerically upper bounded by 1. The affinity can be extended to a complexification of the vector space. On this complexification of the vector space there exist a basis in which the affinity $\Phi$ has the Jordan normal form with blocks of the form
\[
\begin{pmatrix}
\lambda & 1 & 0 & \cdots & 0 & 0 \\
0 & \lambda & 1 & \ddots & \ddots & \\
0 & 0 & \lambda & \ddots & \ddots & \\
\vdots & \vdots & \ddots & \ddots & \ddots & \\
0 & 0 & 0 & \cdots & \lambda & 1 \\
0 & 0 & 0 & \cdots & 0 & \lambda
\end{pmatrix}
\]
(10)

and $\Phi^n$ has blocks of the form
\[
\begin{pmatrix}
\lambda^n \binom{n}{n-1} \lambda^{n-1} (n-2) & \cdots & (n-\ell) & \cdots \\
0 & \lambda^n & \ddots & \ddots & \ddots \\
0 & 0 & \lambda^n & \ddots & \ddots \\
\vdots & \vdots & \ddots & \ddots & \ddots \\
0 & 0 & 0 & \cdots & \lambda^n \\
0 & 0 & 0 & \cdots & 0
\end{pmatrix}
\]
(11)

Now
\[
\sum_{n=0}^{\infty} \frac{\mu^n}{n!} e^{-\mu} \binom{n}{n-j} \lambda^{n-j} = \sum_{n=j}^{\infty} \frac{\mu^n}{(n-j)!} e^{-\mu} \lambda^{n-j}
\]
(12)
\[
= \frac{\mu^j}{j!} \sum_{n=j}^{\infty} \frac{\mu^{n-j}}{(n-j)!} e^{-\mu} \lambda^{n-j}
\]
(13)
tends to zero for $\mu$ tending to infinity except if $\lambda = 1$. If $\lambda = 1$ then there is no uniform upper bound on $\Phi^n$ except if the Jordan block is diagonal. Therefore $\Phi_n = \sum_{n=0}^{\infty} \frac{\mu^n}{n!} e^{-\mu} \Phi^n$ converges to a map $\Phi_\infty$ that is diagonal with eigenvalues 0 and 1, i.e. a idempotent. Since $\Phi$ and $\Phi_\infty$ commute they have the same fix-points.

**Proposition 5.** Let $\Phi$ denote an affinity $\mathcal{K} \to \mathcal{L}$ and let $\Psi$ denote an affinity $\mathcal{L} \to \mathcal{K}$. Then the set of fix-points of $\Psi \circ \Phi$ is a section of $\mathcal{K}$ and the set of fix-points of $\Phi \circ \Psi$ is a section of $\mathcal{L}$. The affinities $\Phi$ and $\Psi$ restricted to the fix-point sets are isomorphisms between these sets.

### 3 Regret and Bregman Divergences

Consider a payoff function where the payoff may represent extracted energy or how much data can be compressed or something else. Our payoff depends both
of the state of the system and of some choice that we can make. Let \( F(\sigma) \) denote the maximal mean payoff when our knowledge is represented by \( \sigma \). Then \( F \) is a convex function on the convex body.

The two most important examples are squared the Euclidean norm squared \( F(\vec{v}) = \|\vec{v}\|^2 \) defined on a vector space and minus the von Neuman entropy \( F(\sigma) = \text{tr}[\sigma \ln(\sigma)] \). Note that Shannon entropy may be considered as a special case of von Neuman entropy when all operators commute. We may also consider \( F(\sigma) = -S_\alpha(\sigma) \) where the Tsallis entropy of order \( \alpha > 0 \) is defined by

\[
S_\alpha(\sigma) = -\text{tr}[\sigma \log_\alpha(\sigma)]
\]

and where the logarithm of order \( \alpha \neq 1 \) is given by

\[
\log_\alpha(x) = \frac{x^{\alpha-1} - 1}{\alpha - 1}
\]

and \( \log_1(x) = \ln(x) \). We will study such entropy functions via the corresponding regret functions that are defined by:

**Definition 6.** Let \( F \) denote a convex function defined on a convex body \( \mathcal{K} \). For \( \rho, \sigma \in \mathcal{K} \) we define the regret function \( D_F \) by

\[
D_F(\rho, \sigma) = F(\rho) - \left( F(\sigma) + \lim_{t \to 0^+} \frac{F((1-t)\cdot \sigma + t \cdot \rho) - F(\sigma)}{t} \right).
\]

The regret function \( D_F \) is strict if \( D_F(\rho, \sigma) = 0 \) implies that \( \rho = \sigma \). If \( F \) is differentiable the regret function is called a Bregman divergence.

The interpretation of the regret function is that \( D_F(\rho, \sigma) \) tells how much more payoff one could have obtained if the state is \( \rho \) but one act as if the state was \( \sigma \). This is illustrated in Figure 4.

![Fig. 4. The regret equals the vertical distance between the curve and the tangent.](image)

The two most important examples of Bregman divergences are squared Euclidean distance \( \|\vec{v} - \vec{w}\|^2 \) that is generated by the squared Euclidean norm and
information divergence

\[ D(\rho\|\sigma) = \text{tr}[\rho(\ln(\rho) - \ln(\sigma)) - \rho + \sigma] \]  

(17)

that is generated by minus the von Neuman entropy. The Bregman divergence generated by \(-S_{\alpha}\) is called the Bregman divergence of order \(\alpha\) and is denoted \(D_{\alpha}(\rho, \sigma)\). Various examples of payoff functions and corresponding regret functions are discussed in [8] where some basic properties of regret functions are also discussed. If \(F\) is differentiable the regret function is a Bregman divergence and the formula (16) reduces to

\[ D_{F}(\rho, \sigma) = F(\rho) - (F(\sigma) + \langle \nabla F(\sigma) \mid \rho - \sigma \rangle) \]  

(18)

Bregman divergences were introduced in [5], but they only gained popularity after their properties were investigated in great detail in [3]. A Bregman divergence satisfies the Bregman equation

\[ \sum t_i \cdot D_{F}(\rho_i, \sigma) = \sum t_i \cdot D_{F}(\rho_i, \bar{\rho}) + D_{F}(\bar{\rho}, \sigma) \]  

(19)

where \((t_1, t_2, \ldots)\) is a probability vector and \(\bar{\rho} = \sum t_i \cdot \rho_i\).

Assume that \(D_{F}\) is a Bregman divergence on the convex body \(K\). If the state is not know exactly but we know that \(s\) is one of the states \(s_1, s_2, \ldots, s_n\) then the minimax regret is defined as

\[ C_{F} = \inf_{\sigma \in K} \sup_{\rho \in K} D_{F}(\rho, \sigma). \]  

(20)

The point \(\sigma\) that achieves the minimax regret will be denoted by \(\sigma_{\text{opt}}\).

**Theorem 7.** If \(K\) is a convex body with a Bregman divergence \(D_{F}\) and with a probability vector \((t_1, t_2, \ldots, t_n)\) on the points \(\rho_1, \rho_2, \ldots, \rho_n\) with \(\bar{\rho} = \sum t_i \cdot \rho_i\) and \(\sigma_{\text{opt}}\) achieves the minimax regret then

\[ C_{F} \geq \sum t_i \cdot D_{F}(\rho_i, \bar{\rho}) + D_{F}(\bar{\rho}, \sigma_{\text{opt}}). \]  

(21)

**Proof.** If \(\sigma_{\text{opt}}\) is optimal then

\[ C_{F} = \sum t_i \cdot C_{F} \]  

\[ \geq \sum t_i \cdot D_{F}(\rho_i, \sigma_{\text{opt}}) \]  

(23)

\[ = \sum t_i \cdot D_{F}(\rho_i, \bar{\rho}) + D_{F}(\bar{\rho}, \sigma_{\text{opt}}) \]  

(24)

which proves Inequality (21). \(\square\)

One can formulate a minimax theorem for divergence, but we will prove a result that is stronger than a minimax theorem in the sense that it gives an upper bound on how close a specific strategy is to the optimal strategy. First we need the following lemma.
 Lemma 8. Let $\mathcal{K}$ be a convex body with a Bregman divergence $D_F$ that is lower semi-continuous. Let $\mathcal{L}$ denote a closed convex subset of $\mathcal{K}$. For any $\sigma \in \mathcal{K}$ there exists a point $\sigma^* \in \mathcal{L}$ such that

$$D_F(\rho, \sigma) \geq D_F(\rho, \sigma^*) + D_F(\sigma^*, \sigma)$$

for all $\rho \in \mathcal{L}$. In particular $\sigma^*$ minimizes $D_F(\rho, \sigma)$ under the constraint that $\rho \in \mathcal{L}$.

Proof. Using that $\mathcal{L}$ is closed and lower semicontinuity of $D_F$ we find a point $\sigma^* \in \mathcal{L}$ that minimizes $D_F(\rho, \sigma)$ under the constraint that $\rho \in \mathcal{L}$. Define

$$\rho_t = (1-t) \cdot \sigma^* + t \cdot \rho.$$  \hspace{1cm} (26)

Then according to the Bregman equation

$$(1-t) \cdot D_F(\sigma^*, \sigma) + t \cdot D_F(\rho, \sigma) = (1-t) \cdot D_F(\sigma^*, \rho_t) + t \cdot D_F(\rho, \rho_t) + D_F(\rho_t, \sigma) \geq t \cdot D_F(\rho, \rho_t) + D_F(\sigma^*, \sigma)$$ \hspace{1cm} (27)

After reorganizing the terms and dividing by $t$ we get

$$D_F(\rho, \sigma) \geq D_F(\rho, \rho_t) + D_F(\sigma^*, \sigma).$$  \hspace{1cm} (28)

Inequality (25) is obtained by letting $t$ tend to zero and using lower semicontinuity.

Theorem 9. If $\mathcal{K}$ is a convex body with a Bregman divergence $D_F$ that is lower semi-continuous in both variables and such that $F$ is continuously differentiable $C^1$. Then

$$C_F = \sup_{\bar{\sigma}} \sum_i t_i \cdot D_F(\rho_i, \bar{\rho})$$ \hspace{1cm} (29)

where the supremum is taken over all probability vectors $\bar{\sigma}$ supported on $\mathcal{K}$. Further the following inequality holds

$$\sup_{\rho \in \mathcal{K}} D_F(\rho, \sigma) \geq C_F + D_F(\sigma_{\text{opt}}, \sigma)$$ \hspace{1cm} (30)

for all $\sigma$.

Proof. First we prove the theorem for a convex polytope $\mathcal{L} \subseteq \mathcal{K}$. Assume that $\rho_1, \rho_2, \ldots, \rho_n$ are the extreme points of $\mathcal{L}$. Let $\sigma_{\text{opt}}(\mathcal{L})$ denote a point that minimizes that $\sup_{\rho \in \mathcal{K}} D_F(\rho, \sigma)$. Let $J$ denote the set of indices $i$ for which

$$D_F(\rho_i, \sigma) = \sup_{\rho \in \mathcal{L}} D_F(\rho, \sigma)$$ \hspace{1cm} (31)

Let $\mathcal{M}$ denote the convex hull of $\rho_i$, $i \in J$. Let $\pi$ denote the projection of $\sigma_{\text{opt}}$ on $\mathcal{M}$. The there exists a mixture such that $\sum_{i \in J} t_i \cdot \rho_i = \pi$. Then for any $\sigma$

$$\sup_{\rho \in \mathcal{L}} D_F(\rho, \sigma) \geq \sum_{i \in J} t_i \cdot D_F(\rho_i, \sigma) \geq \sum_{i \in J} t_i \cdot D_F(\rho_i, \bar{\rho}) + D_F(\bar{\rho}, \sigma).$$ \hspace{1cm} (33)
Since all divergences $D_F (\rho_i, \sigma)$ where $i \in J$ can be decreased by moving $\sigma$ from $\sigma_{opt}$ towards $\pi$ and the divergences $D_F (\rho_i, \sigma)$ where $i \notin J$ are below $C (\mathcal{L})$ as long as $\sigma$ is only moved a little towards $\pi$ we have that $\pi = \sigma_{opt}$ and that (29) holds. Inequality (30) follows from inequality (32) when $\bar{\rho} = \rho_{opt}$.

Let $\mathcal{L}_1 \subseteq \mathcal{L}_1 \subseteq \cdots \subseteq \mathcal{K}$ denote an increasing sequence of polytopes such that the union contain the interior of $\mathcal{K}$. We have

$$C_F (\mathcal{L}_1) \leq C_F (\mathcal{L}_1) \leq \cdots \leq C_F (\mathcal{K})$$  \hspace{1cm} (34)

Let $\sigma_{opt,i}$ denote a point that is optimal for $\mathcal{L}_i$. By compactness of $\mathcal{K}$ we may assume that $\sigma_i \to \sigma_\infty$ for $i \to \infty$ for some point $\sigma_\infty \in \mathcal{K}$. Otherwise we just replace the sequence by a subsequence. For any $\rho \in \mathcal{L}_i$ we have

$$D_F (\rho, \sigma_i) \leq \lim_{i \to \infty} \inf_{\rho \in L_i} D_F (\rho, \sigma_i) \leq \lim_{i \to \infty} C_F (\mathcal{L}_i) \hspace{1cm} (35)$$

By lower semi-continuity

$$D_F (\rho, \sigma_\infty) \leq \lim_{i \to \infty} C_F (\mathcal{L}_i) \hspace{1cm} (36)$$

By taking the supremum over all interior points $\rho \in \mathcal{K}$ we obtain

$$C_F (\mathcal{K}) \leq \sup_{\rho \in \mathcal{K}} D_F (\rho, \sigma_\infty) = \lim_{i \to \infty} C_F (\mathcal{L}_i) = \sup_{i} \sum_{i} t_i \cdot D_F (\rho_i, \bar{\rho}) \hspace{1cm} (37)$$

which in combination with (34) proves (29) and also proves that $\sigma_\infty$ is optimal. We also have

$$\sup_{\rho \in \mathcal{K}} D_F (\rho, \sigma) = \lim_{i \to \infty} \sup_{\rho \in \mathcal{L}_i} D_F (\rho, \sigma) \geq \lim_{i \to \infty} \inf_{\rho \in \mathcal{L}_i} (C_F (\mathcal{L}_i) + D_F (\sigma_i, \sigma)) \geq C_F + D_F (\sigma_\infty, \sigma)$$  \hspace{1cm} (38)

which proves Inequality (30).

\section*{4 Spectral Sets}

Let $\mathcal{K}$ denote a convex body of rank 2. Then $\sigma \in \mathcal{K}$ is said to have unique spectrality if all orthogonal decompositions $\sigma = (1 - t) \cdot \sigma_0 + t \cdot \sigma_1$ have the same coefficients $\{1 - t, t\}$ and the set $\{1 - t, t\}$ is called the spectrum of $\sigma$. If all elements of $\mathcal{K}$ have unique spectrality we say that $\mathcal{K}$ is spectral. A convex body $\mathcal{K}$ is said to be centrally symmetric with center $c$ if for any point $\sigma \in \mathcal{K}$ there exists a centrally inverted point $\tilde{\sigma}$ in $\mathcal{K}$, i.e. a point $\tilde{\sigma} \in \mathcal{K}$ such that $\frac{1}{2} \sigma + \frac{1}{2} \tilde{\sigma} = c$. 

Theorem 10. A spectral set $\mathcal{K}$ of rank 2 is centrally symmetric.

Proof. Let $S : [0, 1] \to \mathcal{K}$ denote a section. Let $\pi_0 \in \mathcal{K}$ denote an arbitrary extreme point and let $\pi_1$ denote a point on the boundary such that $(1 - s) \cdot \pi_0 + s \cdot \pi_1 = S(\frac{1}{2})$ where $0 \leq s \leq \frac{1}{2}$. Then $S(\frac{1}{2})$ can be written as a mixture $(1 - t) \cdot \sigma_0 + t \cdot \sigma_1$ of points on the boundary such that $t$ is minimal. As in the proof of Theorem 1 we see that $\sigma_0$ and $\sigma_1$ are orthogonal. Since $\mathcal{K}$ is spectral we have $t = \frac{1}{2}$. Since $t \leq s \leq \frac{1}{2}$ we have $s = \frac{1}{2}$ implying that $\mathcal{K}$ is symmetric around $S(\frac{1}{2})$.

Proposition 11. Let $S : \mathcal{L} \to \mathcal{K}$ denote a section with retraction $R : \mathcal{K} \to \mathcal{L}$. If $\mathcal{K}$ is a spectral set of rank 2 and $\mathcal{L}$ is not a singleton then $\mathcal{L}$ is also a spectral set of rank 2. If $c$ is the center of $\mathcal{K}$ then $R(c)$ is the center of $\mathcal{L}$ and $S(R(c)) = c$, i.e. the section goes through the center of $\mathcal{K}$.

Proof. Let $\sigma \to \tilde{\sigma}$ denote reflection in the point $c \in \mathcal{K}$. If $\rho \in \mathcal{L}$ then
\[
R(c) = R \left( \frac{1}{2} \cdot S(\rho) + \frac{1}{2} \cdot \tilde{S}(\rho) \right) \quad (44)
\]
\[
= \frac{1}{2} \cdot R(S(\rho)) + \frac{1}{2} \cdot R(\tilde{S}(\rho)) \quad (45)
\]
\[
= \frac{1}{2} \cdot \rho + \frac{1}{2} \cdot R(\tilde{S}(\rho)) \quad (46)
\]
so that $\mathcal{L}$ is centrally symmetric around $R(c)$. If $\mathcal{F}$ is a proper face of $\mathcal{L}$ then $S(\mathcal{F})$ is a proper face of $\mathcal{K}$ implying that $S(\mathcal{F})$ is a singleton. Therefore $\mathcal{F} = R(S(\mathcal{F}))$ is a singleton implying that $\mathcal{L}$ has rank 2. If $\rho \in \mathcal{L}$ is an extreme point then $\tilde{\rho} = R(\tilde{S}(\rho))$ is also an extreme point of $\mathcal{L}$. Now
\[
S(R(c)) = S \left( \frac{1}{2} \cdot \rho + \frac{1}{2} \cdot R(\tilde{S}(\rho)) \right) \quad (47)
\]
\[
= \frac{1}{2} \cdot S(\rho) + \frac{1}{2} \cdot S \left( R(\tilde{S}(\rho)) \right) \quad (48)
\]
Since $\tilde{\rho} \in \mathcal{L}$ is an extreme point and $\tilde{\rho} = R(\tilde{S}(\rho))$ we have that $R^{-1}(\tilde{\rho})$ is a proper face of $\mathcal{K}$ and thereby a singleton. Therefore $S \left( R(\tilde{S}(\rho)) \right) = \tilde{S}(\rho)$ and $S(R(c)) = \frac{1}{2} \cdot S(\rho) + \frac{1}{2} \cdot \tilde{S}(\rho) = c$.

Corollary 12. If $\sigma$ is an extreme point of a spectral set $\mathcal{K}$ of rank 2 then there exists a unique element in $\mathcal{K}$ that is orthogonal to $\sigma$.

If a centrally symmetric set has a proper face that is not an extreme point then the set is not spectral as illustrated in Figure 5.

Let $\rho = x \cdot \sigma_0 + y \cdot \sigma_1$ denote an orthogonal decomposition of an element of the vector space generated by a spectral set of rank 2. Then we may define
\[
f(\rho) = f(x) \cdot \sigma_0 + f(y) \cdot \sigma_1. \quad (49)
\]
Fig. 5. A centrally symmetric convex body with non-trivial faces AB and CD. The Caratheodory number is 2, but the rank is 3. The points A, B, C, and D are orthogonal extreme points and any point in the interior of the square □ABCD has several orthogonal decompositions with different mixing coefficients with weights on A, B, C, and D. Points in the convex body but outside the triangles can be decomposed as a mixture of two orthogonal extreme points on the semi circles.

If \( \rho = x \cdot \rho_0 + y \cdot \rho_1 \) is another orthogonal decomposition then \( x = y \) and

\[
 f(x) \cdot \sigma_0 + f(y) \cdot \sigma_1 = 2 f(x) \cdot \frac{\sigma_0 + \sigma_1}{2}
\]

(50)

and

\[
 f(x) \cdot \rho_0 + f(y) \cdot \rho_1 = 2 f(x) \cdot \frac{\rho_0 + \rho_1}{2}.
\]

(51)

Since

\[
 \frac{\sigma_0 + \sigma_1}{2} = \frac{\rho_0 + \rho_1}{2} = c
\]

(52)

different orthogonal decompositions will result in the same value of \( f(\rho) \). Note in particular that for the constant function \( f(x) = \frac{1}{2} \) we have \( f(\rho) = c \). In this sense \( c = \frac{1}{2} \) and from now on we will use \( \frac{1}{2} \) in bold face instead of \( c \) as notation for the center of a spectral set. If \( \frac{1}{2} \cdot \rho + \frac{1}{2} \cdot \sigma = c \) then \( \frac{1}{2} \cdot \rho + \frac{1}{2} \cdot \sigma = c \) so that \( \rho + \sigma = 1 \) so that the central inversion of \( \rho \) equals \( 1 - \rho \). We note that if \( f(x) \geq 0 \) for all \( x \) then \( f(\rho) \) is element in the positive cone. Therefore \( \sum_i \rho_i^2 = 0 \) implies that \( \rho_i = 0 \) for all \( i \), where \( \rho_i^2 \) is defined via Equation (49). Note also that if \( \Phi \) is an isomorphism then \( \Phi(f(\rho)) = f(\Phi(\rho)) \).

5 Sufficient Regret Functions

There are a number of equivalent ways of defining sufficiency, and the present definition of sufficiency is based on [18]. We refer to [14] where the notion of sufficiency is discussed in great detail.

Definition 13. Let \((\sigma_\theta)_{\theta}\) denote a family of points in a convex body \( \mathcal{K} \) and let \( \Phi \) denote an affinity \( \mathcal{K} \to \mathcal{L} \) where \( \mathcal{K} \) and \( \mathcal{L} \) denote convex bodies. Then \( \Phi \) is said to be sufficient for \((\sigma_\theta)_{\theta}\) if there exists an affinity \( \Psi : \mathcal{L} \to \mathcal{K} \) such that \( \Psi(\Phi(\sigma_\theta)) = \sigma_\theta \), i.e. the states \( \sigma_\theta \) are fix-points of \( \Psi \circ \Phi \).
The notion of sufficiency as a property of general divergences was introduced in [11]. It was shown in [15] that a Bregman divergence on the simplex of distributions on an alphabet that is not binary determines the divergence up to a multiplicative factor. In [8] this result was extended to $C^*$-algebras. Here we are interested in the binary case and its generalization that is convex bodies of rank 2.

**Definition 14.** We say that the regret function $D_F$ on the convex body $K$ satisfies sufficiency if

$$D_F(\Phi(\rho), \Phi(\sigma)) = D_F(\rho, \sigma)$$

for any affinity $K \to K$ that is sufficient for $(\rho, \sigma)$.

**Lemma 15.** If a strict regret function on a convex body of rank 2 satisfies sufficiency, then the convex body is spectral and the regret function is generated by a function of the form

$$F(\sigma) = \text{tr}[f(\sigma)]$$

for some convex function $f: [0, 1] \to \mathbb{R}$.

**Proof.** For $i = 1, 2$ assume that $S_i: [0, 1] \to K$ are sections with retractions $R_i: K \to [0, 1]$. Then $S_2 \circ R_1$ is sufficient for the pair $(S_1(t), S_1(1/2))$ with recovery map $S_1 \circ R_2$ implying that

$$D_F(S_1(t), S_1(1/2)) = D_F(S_2(t), S_2(1/2)).$$

Define $f(t) = D_F(S_1(t), S_1(1/2))$. Then $D_F(S_2(t), S_2(1/2)) = f(t)$ for any section $S_2$, so this divergence is completely determined by the spectrum $(t, 1 - t)$. In particular all orthogonal decompositions have the same spectrum so that the convex body is spectral.

Let $K$ denote a spectral convex set of rank 2 with center $\frac{1}{2}$. If the Bregman divergence $D_F$ satisfies sufficiency then $D_F(\rho, \sigma) = D_F(1 - \rho, 1 - \sigma)$ and

$$D_F(\rho, \sigma) = \frac{D_F(\rho, \sigma) + D_F(1 - \rho, 1 - \sigma)}{2}$$

$$= \frac{D_F(\rho, \sigma) + D_{\tilde{F}}(\rho, \sigma)}{2}$$

$$= D_{\frac{F + \tilde{F}}{2}}(\rho, \sigma)$$

where $\tilde{F}(\sigma)$ is defined as $F(1 - \sigma)$. Now $\frac{F + \tilde{F}}{2}$ is convex and invariant under central inversion. Therefore a regret function on a spectral set of rank 2 is generated by a function that is invariant under central inversion.

Let $F$ denote a convex function that is invariant under central inversion and assume that $D_F$ satisfies sufficiency. If $\sigma_0$ and $\sigma_1$ are orthogonal we may define
f(t) = \frac{1}{2} \cdot F((1 - t) \cdot \sigma_0 + t \cdot \sigma_1) \text{ for } t \in [0, 1]. \text{ Then}
\begin{align*}
\text{tr}[f(\sigma)] &= \text{tr}[f(1 - t) \cdot \sigma_0 + f(t) \cdot \sigma_1] \\
&= f(1 - t) \cdot 1 + f(t) \cdot 1 \\
&= 2 \cdot f(t) \\
&= 2 \cdot \frac{1}{2} \cdot F((1 - t) \cdot \sigma_0 + (t) \cdot \sigma_1) \\
&= F(\sigma),
\end{align*}
which proves Eq. (54).

**Proposition 16.** Let \( K \) denote a spectral convex set of rank 2. If \( f : [0, 1] \to \mathbb{R} \) is convex then \( F(\sigma) = \text{tr}[f(\sigma)] \) defines a convex function on \( K \) and the regret function \( D_F \) satisfies sufficiency.

**Proof.** Let \( \rho_0 \) and \( \rho_1 \) denote points in \( K \). Let \( \sigma \) denote a point that is co-linear with \( \frac{1}{2} \) and \( \rho_0 \) and such that \( F(\sigma) = F(\rho_1) \). Then
\begin{align*}
F((1 - t) \cdot \rho_0 + t \cdot \rho_1) &\leq F((1 - t) \cdot \rho_0 + t \cdot \sigma) \\
&= \text{tr}[f((1 - t) \cdot \rho_0 + t \cdot \sigma)] \\
&\leq \text{tr}[(1 - t) \cdot f(\rho_0) + t \cdot f(\sigma)] \\
&= (1 - t) \cdot F(\rho_0) + t \cdot F(\sigma) \\
&= (1 - t) \cdot F(\rho_0) + t \cdot F(\rho_1),
\end{align*}
which proves that \( F \) is convex.

Now we will prove that \( D_F \) satisfies sufficiency. Let \( \rho, \sigma \in K \) denote two points and let \( \Phi : K \to K \) denote an affinity that is sufficient for \( \rho, \sigma \) with recovery map \( \Psi \). Then \( \Phi \circ \Psi \) and \( \Psi \circ \Phi \) are retractions and the fixpoint set of \( \Psi \circ \Phi \) and \( \Phi \circ \Psi \) are isomorphic convex bodies. Accoring to Proposition 11 the center of \( K \) a fixpoint under retractions and we see that a decomposition into orthogonal extreme point in a fixpoint set is also an orthogonal decomposition in \( K \). Therefore \( \text{tr}[f(\sigma)] \) has the same value when the calculation is done within the fixpoint set of \( \Psi \circ \Phi \), which proves the proposition.

**Theorem 17.** Let \( K \) denote a convex body of rank 2 with a sufficient Bregman divergence \( D_F \) that is strict. Then the center of \( K \) the unique point that achieves the minimax regret.

**Proof.** Let \( S : [0, 1] \to K \) denote a section. Then \( S(1/2) = \frac{1}{2} \) and
\begin{align*}
C_F &\geq \frac{1}{2} \cdot D_F(S(0), S(1/2)) + \frac{1}{2} \cdot D_F(S(1), S(1/2)) + D_F(S(1/2), \sigma_{opt}) \\
&= D_F\left(S(1), \frac{1}{2}\right) + D_F\left(\frac{1}{2}, \sigma_{opt}\right)
\end{align*}
Further we have
\begin{align*}
\sup_{\rho \in K} D_F\left(\rho, \frac{1}{2}\right) &\geq C_F + D_F\left(\sigma_{opt}, \frac{1}{2}\right).
\end{align*}
Now \( \rho = S_\rho(t) \) for some section \( S_\rho \) and some \( t \in [0, 1] \). Therefore

\[
D_F \left( \rho, \frac{1}{2} \right) = D_F \left( S_\rho(t), S_\rho(1/2) \right)
\]

(72)
\[
= D_F \left( S_\rho(t), S_\rho(1/2) \right)
\]

(73)
\[
= D_F \left( S(t), S(1/2) \right)
\]

(74)
\[
\leq D_F \left( S(1), \frac{1}{2} \right).
\]

(75)

Therefore \( C_F = D_F \left( S(1), \frac{1}{2} \right) \) and \( D_F \left( \sigma_{opt}, \frac{1}{2} \right) = 0 \) implying \( \sigma_{opt} = \frac{1}{2} \). □

If the Bregman divergence is based on Shannon entropy then the minimax regret is called the capacity and the result is that a convex body of rank 2 has a capacity of 1 bit.

### 6 Spin Factors

We say that a convex body is a **Hilbert ball** if the convex body can be embedded as a unit ball in a \( d \) dimensional real Hilbert space \( \mathcal{H} \) with some inner product that will be denoted \( \langle \cdot | \cdot \rangle \). The positive elements are the elements \((\vec{v}, s)\) where \( \|\vec{v}\|_2 \leq s \). The trace of the spin factor is \( \text{tr} [(\vec{v}, s)] = 2s \).

The direct sum \( \mathcal{H} \oplus \mathbb{R} \) can be equipped a product \( \bullet \) by

\[
(\vec{v}, s) \bullet (\vec{w}, t) = (t \cdot \vec{v} + s \cdot \vec{w}, \langle \vec{v} | \vec{w} \rangle + s \cdot t)
\]

(76)

This product is distributive and \((\vec{v}, 1) \bullet (-\vec{v}, 1) = 0 \). Therefore \( x^2 \) defined via (49) will be equal to \( x \bullet x \) and \((\mathcal{H} \oplus \mathbb{R}, \bullet)\) becomes a formally real Jordan algebra of the type that is called a **spin factor** and is denoted \( J_{\text{Spin}_d} \). The unit of a spin factor is \((\vec{0}, 1)\) and will be denoted \( 1 \). See [16] for general results on Jordan algebras.

Let \( \mathcal{M}_n(\mathbb{F}) \) denote \( n \times n \) matrices over \( \mathbb{F} \) where \( \mathbb{F} \) may denote the real numbers \( \mathbb{R} \) or the complex numbers \( \mathbb{C} \) or the quaternions \( \mathbb{H} \) or the octonions \( \mathbb{O} \). Let \( (\mathcal{M}_n(\mathbb{F}))_h \) denote the set of self-adjoint matrices of \( \mathcal{M}_n(\mathbb{F}) \). Then \( (\mathcal{M}_n(\mathbb{F}))_h \) is a formally real Jordan algebra with a Jordan product \( \bullet \) is given by

\[
x \bullet y = \frac{1}{2} (xy + yx)
\]

(77)

except for \( \mathbb{F} = \mathbb{O} \) where one only get a Jordan algebra when \( n \leq 3 \). The self-adjoint \( 2 \times 2 \) matrices with real, complex, quaternionic or octonionic entries can be identified with spin factors with dimension \( d = 2, 3, 5, \) or \( d = 9 \). The most important examples of spin factors are the bit \( J_{\text{Spin}_1} \) and the qubit \( J_{\text{Spin}_3} \).

We introduce the Pauli matrices

\[
\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}
\]

(78)
and observe that $\sigma_1 \cdot \sigma_3 = 0$. Let $v_1, v_2, \ldots, v_d$ denote a basis of the Hilbert space $\mathcal{H}$. Let the function $S : JSpin_d \rightarrow (\mathcal{M}_2 (\mathbb{R}))^\otimes (d-1)$ be defined by

$$S (1) = 1 \otimes 1 \otimes \cdots \otimes 1,$$

$$S (v_1) = \sigma_1 \otimes 1 \otimes \cdots \otimes 1,$$

$$S (v_2) = \sigma_3 \otimes \sigma_1 \otimes 1 \otimes \cdots \otimes 1,$$

$$S (v_3) = \sigma_3 \otimes \sigma_3 \otimes \sigma_1 \otimes \cdots \otimes 1,$$

$$\vdots$$

$$S (v_{d-1}) = \sigma_3 \otimes \sigma_3 \otimes \cdots \otimes 1,$$

$$S (v_d) = \sigma_3 \otimes \sigma_3 \otimes \cdots \otimes \sigma_3.$$  

Then $S$ can be linearly extended and one easily checks that

$$S (x \cdot y) = S (x) \cdot S (y).$$  

Now $S (JSpin_d)$ is a linear subspace of the real Hilbert space $(\mathcal{M}_2 (\mathbb{R}))^\otimes (d-1)$ so there exists a projection of $(\mathcal{M}_2 (\mathbb{R}))^\otimes (d-1)$ onto $S (JSpin_d)$ and this projection maps symmetric matrices in $(\mathcal{M}_2 (\mathbb{R}))^\otimes (d-1)$ into symmetric matrices. Therefore $S$ is a section with a retraction generated by the projection. In this way $JSpin_d$ is a section of a Jordan algebra of symmetric matrices with real entries. The Jordan algebra $\mathcal{M}_n (\mathbb{R})_h$ is obviously a section of $\mathcal{M}_n (\mathbb{C})_h$ so $JSpin_d$ is a section of $\mathcal{M}_n (\mathbb{C})_h$. Note that the projection of $\mathcal{M}_n (\mathbb{C})_h$ on a spin factor is not necessarily completely positive.

Since the standard formalism of quantum theory represents states as density matrices in $\mathcal{M}_n (\mathbb{C})_h$ we see that spin factors appear as sections of state spaces of the usual formalism of quantum theory. Therefore the points in the Hilbert ball are called states and the Hilbert ball is called the state space of the spin factor. The extreme points in the state space are called pure states.

The positive cone of a spin factor is self-dual in the sense that any positive functional $\phi : JSpin_d \rightarrow \mathbb{R}$ is given by $\phi (x) = \text{tr} [x \cdot y]$ for some uniquely determined positive element $y$. We recall the definition of the polar set of a convex body $K \subseteq \mathbb{R}^d$

$$K^\circ = \{ y \in \mathbb{R}^d \mid \langle x, y \rangle \leq 1 \text{ for all } x \in K \}.$$  

**Proposition 18.** Assume that the cone generated by a spectral convex body $K$ of rank 2 is self-dual. Then it can be represented as a spin factor.

**Proof.** If $\phi$ is a test on $K$ then $2 \cdot \phi - 1$ maps $K$ into $[0, 1]$, which an element in the polar set of $K$ embedded in a Hilbert space with the center as the origin. Since the cone is assumed to be self-dual the set $K$ is self-polar and Hilbert balls are the the only self-polar sets. The result follows because a Hilbert ball can be represented as the state space of a spin factor. 

A convex body $K$ of rank 2 is said to have symmetric transission probabilities if for any extreme points $\sigma_1$ and $\sigma_2$ there exists retractions $R_1 : K \rightarrow [-1, 1]$ and $R_2 : K \rightarrow [-1, 1]$ such that $R_0 (\sigma_1) = 1$, and $R_1 (\sigma_2) = R_2 (\sigma_1)$. 

\[ \text{Proof.} \]
Theorem 19. A spectral convex body $K$ of rank 2 with symmetric transmission probabilities can be represented by a spin factor.

Proof. For almost all extreme points $\sigma$ of $K$ a retraction $R : K \rightarrow [-1, 1]$ with $R(\sigma) = 1$ is uniquely determined. Let $\sigma_1$ and $\sigma_2$ be two extreme points that are not antipodal and with unique retractions $R_1$ and $R_2$. Let $L$ denote the intersection of $K$ with the affine span of $\sigma_1$, $\sigma_2$ and the center. Embed $L$ in a 2-dimensional coordinate system with the center of $L$ as origin of the coordinate system. Let $\sigma$ denote an extreme point with a unique retraction $R$. Then $R_1(\sigma) \cdot \sigma - \sigma_1$ is parallel with $R_2(\sigma) \cdot \sigma - \sigma_2$ because

$$R(R_i(\sigma) \cdot \sigma - \sigma_i) = R_i(\sigma) \cdot R(\sigma) - R(\sigma_i)$$

$$= R_i(\sigma) \cdot 1 - R_i(\sigma)$$

$$= 0.$$  \hspace{1cm} (88)

Therefore the determinant of $R_1(\sigma) \cdot \sigma - \sigma_1$ and $R_2(\sigma) \cdot \sigma - \sigma_2$ is zero, but the determinant can be calculated as

$$\det (R_1(\sigma) \cdot \sigma - \sigma_1, R_2(\sigma) \cdot \sigma - \sigma_2)$$

$$= 0 - \det (R_1(\sigma) \cdot \sigma, \sigma_2) - \det (\sigma_1, R_2(\sigma) \cdot \sigma) + \det (\sigma_1, \sigma)$$

$$= \det (\sigma, R_2(\sigma) \cdot \sigma_1 - R_1(\sigma) \cdot \sigma_2) - \det (\sigma_2, \sigma_1). \hspace{1cm} (89)$$

This means that $\sigma$ satisfies the following equation

$$\det (\sigma, R_2(\sigma) \cdot \sigma_1 - R_1(\sigma) \cdot \sigma_2) = \det (\sigma_2, \sigma_1). \hspace{1cm} (90)$$

This is a quadratic equation in the coordinates of $\sigma$, which implies that $\sigma$ lies on a conic section. Since $L$ is bounded this conic section must be a circle or an ellipsoid. Almost all extreme points of $L$ have unique retractions. Therefore almost all extreme points lie on a circle or an ellipsoid which by convexity implies that all extreme points of $L$ lie on an ellipsoid or a circle. Since this holds for almost all pairs $\sigma_1$ and $\sigma_2$ the convex set $K$ must be an ellipsoid, which can be mapped into a ball. \qed

Definition 20. Let $A \subseteq \text{JSpin}_d$ denote a subalgebra of a spin factor. Then $E : \text{JSpin}_d \rightarrow A$ is called a conditional expectation if $E(1) = 1$ and $E(a \cdot x) = a \cdot E(x)$ for any $a \in A$.

Theorem 21. Let $K$ denote the state space of a spin factor and assume that $\Phi : K \rightarrow K$ is an idempotent that preserves the center. Then $\Phi$ is a conditional expectation of the spin factor into a sub-algebra of the spin factor.

Proof. Assume that the spin factor is based on the Hilbert space $H = H_1 \oplus H_2$ and that the idempotent $\Phi$ is the identity on $H_1$ and maps $H_2$ into the origin. Let $v, w_1 \in H_1$ and $w_2 \in H_2$ and $s, t \in R$. Then

$$\Phi((v, s) \bullet (w_1 + w_2, t)) = \Phi((v, s) \bullet (w_1, t)) + \Phi((v, s) \bullet (w_2, 0))$$

$$= (v, s) \bullet (w_1, t) + \Phi(s \cdot w_2, \langle v, w_2 \rangle)$$

$$= (v, s) \bullet \Phi(w_1 + w_2, t), \hspace{1cm} (93)$$

which proves the theorem. \qed
7 Monotonicity under dilations

Next we introduce the notion of monotonicity. In thermodynamics monotonicity is associated with decrease of free energy in a closed system and in information theory it is associated with the data processing inequality.

**Definition 22.** Let $D_F$ denote a regret function on the convex body $K$. Then $D_F$ is said to be monotone if

$$D_F(\Phi(\rho), \Phi(\sigma)) \leq D_F(\rho, \sigma)$$

for any affinity $\Phi : K \rightarrow K$.

A simple example of a monotone regret function is squared Euclidean distance in a Hilbert ball, but later we shall see that there are many other examples. All monotone regret functions are Bregman divergences [8, Prop. 6] that satisfy sufficiency [8, Prop. 8]. We shall demonstrate that a convex body of rank 2 with a monotone Bregman divergence can be represented by a spin factor.

We will need to express the Bregman divergence as an integral involving a different type of divergence. Define

$$D^F(x, y) = \left. \frac{d^2}{ds^2} F(x_s) \right|_{s=1}$$

where $x_s = (1 - s) \cdot x + s \cdot y$. If $F$ is $C^2$ and $H(y)$ is the Hesse matrix of $F$ calculated in the point $y$ then

$$D^F(x, y) = \langle x - y | H(y) | x - y \rangle .$$

Since

$$F(x_s) = F(y) + \langle \nabla F(y) | x_s - y \rangle + D_F(x_s, y)$$

we also have

$$D^F(x, y) = \left. \frac{d^2}{ds^2} D_F(x_s, y) \right|_{s=1} .$$

It is also easy to verify that

$$D^F(x, y) = \left. \frac{d}{ds} D_F(y, x_s) \right|_{s=1} .$$

**Proposition 23.** Let $F : [0, 1] \rightarrow \mathbb{R}$ denote a twice differentiable convex function. If $x_s = (1 - s) \cdot x + s \cdot y$. Then

$$D_F(x, y) = \int_0^1 \frac{D_F(x_s)}{s} \, ds .$$

where $D_F(x, x_s)$ is given by one of the equations (97), (98), (100), or (101).

A similar result appear in [12] as Eq. 2.118. In the context of complex matrices the result was proved as Proposition 23 in [19].

We will need the following lemma.
Lemma 24. Let $F$ denote a convex function defined on a convex body $K$. Then for almost all $y \in K$ we have

$$\lim_{x \to y} \frac{D_F(x, y) - \frac{1}{2}D_F^2(x, y)}{\|x - y\|^2} = 0.$$  \hspace{1cm} (103)

Proof. According to our definitions

$$D_F(x, y) - \frac{1}{2}D_F^2(x, y) = F(x) - \left(F(y) + \langle \nabla F(y) \mid x - y \rangle + \frac{1}{2} \langle x - y \mid H(y) \mid x - y \rangle \right)$$  \hspace{1cm} (104)

and we see that Lemma 24 states that a convex function is twice differentiable almost everywhere, which is exactly Alexandrov’s theorem [1]. \hfill \square

Lemma 25. If $F$ is twice differentiable then $D_F$ is a monotone Bregman divergence if and only if $D_F^2$ is monotone.

Proof. Assume that $D_F$ is monotone and that $\Phi$ is some affinity and that $x_s = (1 - s) \cdot x + s \cdot y$. Then

$$D_F(\Phi(x_s), \Phi(y)) \leq D_F(x_s, y).$$  \hspace{1cm} (105)

Since

$$D_F(x_s, y) = 0$$  \hspace{1cm} (106)

$$\frac{d}{ds}D_F(x_s, y)_{|s=1} = 0$$  \hspace{1cm} (107)

and

$$D_F(\Phi(x_s), \Phi(y)) = 0$$  \hspace{1cm} (108)

$$\frac{d}{ds}D_F(\Phi(x_s), \Phi(y))_{|s=1} = 0$$  \hspace{1cm} (109)

we must have

$$D_F^2(\Phi(x), \Phi(y)) = \frac{d^2}{ds^2}D_F(\Phi(x_s), \Phi(y))_{|s=1}$$  \hspace{1cm} (110)

$$\leq \frac{d^2}{ds^2}D_F(x_s, y)_{|s=1}$$  \hspace{1cm} (111)

$$\leq D_F^2(x, y).$$  \hspace{1cm} (112)

If $D_F$ is monotone then Proposition 23 implies that $D_F$ is monotone. \hfill \square

Theorem 26. Let $K$ denote a convex body with a sufficient regret function $D_F$ that is monotone under dilations. Then $F$ is $C^2$. In particular $D_F$ is a Bregman divergence.
Proof. Since $D_F$ is monotone under dilation we have that $D^F$ is monotone under dilations whenever $D^F$ is defined. Let $y$ be a point where $F$ is differentiable and let $0 < r < 1$ and $z$ be a point such that $F$ is differentiable in $(1-r) \cdot z + r \cdot y$. Then

\begin{align*}
  D^F ((1-r) \cdot z + r \cdot x, (1-r) \cdot z + r \cdot y) &\leq D^F (x, y) \tag{113} \\
  \langle r \cdot x - r \cdot y, H((1-r) \cdot z + r \cdot y) \rangle &\leq \langle x - y, H(y) \rangle |x - y| \tag{114} \\
  r^2 \cdot \langle x - y, H((1-r) \cdot z + r \cdot y) \rangle &\leq \langle x - y, H(y) \rangle |x - y| \tag{115} \\
  r^2 \cdot H((1-r) \cdot z + r \cdot y) &\leq H(y). \tag{116}
\end{align*}

Let $L_1 \subseteq K$ denote a ball around $y$ with radius $R_1$ and let $L_2$ denote a ball around $y$ with radius $R_2 < R_1$. Then for any $w \in L_2$ there exists a $z \in L_1$ such that $w = (1-r) \cdot z + r \cdot y$ where $r \geq \frac{R_2}{R_1}$ implying that

\[
  \left(1 - \frac{R_2}{R_1}\right)^2 \cdot H(w) \leq H(y). \tag{117}
\]

There also exists a $\tilde{z} \in L_1$ such that $y = (1-\tilde{r}) \cdot z + \tilde{r} \cdot w$ where $\tilde{r} \geq 1 - \frac{R_2}{R_1 + R_2}$ implying that

\[
  \left(1 - \frac{R_2}{R_1 + R_2}\right)^2 \cdot H(y) \leq H(w). \tag{118}
\]

We see that if $R_2$ is small Then $y \to H(y)$ is uniformly continuous on any compact subset of the interior of $K$ restricted to points where $F$ is twice differentiable. Therefore $H$ has a unique continuous extension to $K$ and we can use the extension of $H$ to get an extension of $D^F$. The last thing we need to prove is that the unique extended function $H$ actually gives the Hesse matrix in any interior point in $K$. Let $x, y \in K$. Introduce $x_r = (1-r) z + r \cdot x$ and $x_r = (1-r) z + r \cdot x$. Then

\begin{align*}
  D_F (x, y) - \frac{1}{2} D^F (x, y) &\geq D_F ((1-r) z + r \cdot x, (1-r) z + r \cdot y) - \frac{1}{2} D^F (x, y) \\
  &= D_F (x_r, y_r) - \frac{1}{2} D^F (x_r, y_r) + \frac{1}{2} D_F (x_r, y_r) - \frac{1}{2} D^F (x_r, y_r) - \frac{1}{2} D^F (x, y) \\
  &\geq D_F (x_r, y_r) - \frac{1}{2} D^F (x_r, y_r) + \frac{1}{2} \langle x - y, H(y_r) - H(y) \rangle |x - y| \\
  &\geq D_F (x_r, y_r) - \frac{1}{2} D^F (x_r, y_r) - \frac{1}{2} \|x - y\|^2 \cdot H((1-r) z + r \cdot y) - H(y) \|\|x - y\|^2 \\
\end{align*}

Therefore

\[
  \frac{D_F (x, y) - \frac{1}{2} D^F (x, y)}{\|x - y\|^2} \geq r^2 \frac{D_F (x_r, y_r) - \frac{1}{2} D^F (x_r, y_r)}{\|x - y\|^2} - \frac{1}{2} \|x - y\|^2 \cdot H((1-r) z + r \cdot y) - H(y) \| \tag{119}
\]
and
\[
\liminf_{x \to y} \frac{D_F(x, y) - \frac{1}{2} D_F^2(x, y)}{\|x - y\|^2} \geq -\frac{1}{2} \|r \cdot H((1 - r) z + r \cdot y - H(y))\|. \tag{121}
\]

Since this holds for all positive \( r < 1 \) we have
\[
\liminf_{x \to y} \frac{D_F(x, y) - \frac{1}{2} D_F^2(x, y)}{\|x - y\|^2} \geq 0. \tag{122}
\]

One can prove that \( \limsup \) is less that 0 in the same way.

**Theorem 27.** Assume that \( f : [0, 1] \to \mathbb{R} \) is a convex symmetric function and that the function \( F \) is defined as \( F(\sigma) = \text{tr}[f(\sigma)] \). If the Bregman divergence \( D_F \) is monotone under dilations then \( y \to y^2 \cdot f''(y) \) is an increasing function.

**Proof.** Assume that \( D_F \) is monotone under dilations. Let \( S : [0, 1] \to K \) denote a section. Then a dilation around \( S(0) \) commutes with the retraction corresponding to the section \( S \). Therefore \( D_F \) restricted to \( S([0, 1]) \) is monotone, so we may without loss of generality assume that the convex body is the interval \([0, 1]\).

Then \( F \) is \( C^2 \) and \( D_F \) is monotone.
\[
D_F(r \cdot x, r \cdot y) = F''(r \cdot y) \cdot (r \cdot x - r \cdot y)^2 \tag{123}
\]
\[
= F''(r \cdot y) \cdot (r \cdot y)^2 \cdot \left(\frac{x}{y} - 1\right)^2. \tag{124}
\]

Therefore \( y^2 \cdot F''(y) \) and \( y^2 \cdot f''(y) \) are increasing.

**Theorem 28.** Let \( K \) denote a convex body of rank 2 with a sufficient and strict regret function \( D_F \) that is monotone under dilations. Then \( K \) can be represented by a spin factor.

**Proof.** First we note that \( K \) is a spectral set with a center that we will denote \( c \). We will embed \( K \) in a vector space with \( c \) as the origin. If \( \sigma \) and \( \rho \) are points on the boundary and \( \lambda \in [0, \frac{1}{2}] \) then
\[
D_F((1 - \lambda) \sigma + \lambda \cdot c, c) = D_F((1 - \lambda) \rho + \lambda \cdot c, c). \tag{125}
\]

Therefore
\[
D_F(\sigma, c) = k \tag{126}
\]

for some constant \( k \). Equation (126) can be written in terms of the Hesse matrix as
\[
\langle \sigma - c | H(c) | \sigma - c \rangle = k, \tag{127}
\]

and this is the equation for an ellipsoid. The result follows because any ellipsoid is isomorphic to a ball.
One easily check that if \( f : C^2([0, 1]) \) then \( F(\sigma) = \text{tr}[f(\sigma)] \) defines a \( C^2 \)-function on any spin factor.

**Theorem 29.** Assume that \( f : C^3([0, 1]) \) is a convex symmetric function and that the function \( F \) is defined as \( F(\sigma) = \text{tr}[f(\sigma)] \) on a spin factor. If \( y \rightarrow y^2f(y) \) is an increasing function then the Bregman divergence \( D_F \) is monotone under dilations.

**Proof.** Assume that \( y \rightarrow y^2f(y) \) is an increasing function. It is sufficient to prove that \( D^F(x, y) \) is decreasing under dilations. Let \( x \rightarrow (1-r)z+rx \) denote a dilation around \( z \) by a factor of \( r \in [0, 1] \). Then

\[
D^F((1-r)z+rx, (1-r)z+ry) = r^2 \langle x-y | H((1-r)z+ry) | x-y \rangle .
\]

(128)

so it is sufficient to prove that \( r \rightarrow r^2H((1-r)z+ry) \) is an increasing matrix function. Since \( f \) is \( C^3 \) we may differentiate with respect to \( r \) and we have to prove the inequality

\[
2rH((1-r)z+ry) + r^2 \frac{d}{dr}H((1-r)z+ry) \geq 0.
\]

(129)

Without loss of generality we may assume \( r = 1 \) so that we have to prove that

\[
2H(y) + \frac{d}{dr}H((1-r)z+ry)_{r=1} \geq 0.
\]

(130)

If \( y = (y_1, y_2, \ldots, y_d) \) and \( H = (H_{i,j}) \) then

\[
\frac{d}{dr}H((1-r)z+ry)_{r=1} = \left( \frac{d}{dr}H_{i,j}((1-r)z+ry) \right)_{r=1}
\]

(131)

\[
= \langle \nabla H_{i,j}((1-r)z+ry) | y-z \rangle_{r=1}
\]

(132)

\[
= \langle \nabla H_{i,j}(y) | y-z \rangle .
\]

(133)

Since inequality (130) is invariant under rotations that leave the center and \( y \) invariant the same must be the case for the inequality

\[
2(H_{i,j}) + \langle \nabla H_{i,j}(y) | y-z \rangle \geq 0 .
\]

(134)

but this inequality is linear in \( z \) so we may take the mean under all rotated versions of this inequality. If \( \bar{z} \) denotes the mean of rotated versions of \( z \) we have to prove that

\[
2(H_{i,j}) + \langle \nabla H_{i,j}(y) | y-\bar{z} \rangle \geq 0 .
\]

(135)

Since \( \bar{z} \) is collinear with the \( y \) and the center we have reduced the problem to dilations of a one-dimensional spin factor which is covered in Theorem 30. \( \square \)

For the Tsallis entropy of order \( \alpha \) we have \( F(x) = \frac{x^\alpha+(1-x)^\alpha-1}{\alpha-1} \) so that

\[
F''(x) = \alpha \left( x^{\alpha-2} + (1-x)^{\alpha-2} \right)
\]

and

\[
x^2 F''(x) = x^2 \alpha \left( x^{\alpha-2} + (1-x)^{\alpha-2} \right)
\]

(136)

\[
= \alpha \left( x^{\alpha} + x^2 (1-x)^{\alpha-2} \right) .
\]

(137)
The derivative is
\[
\alpha \left( \alpha x^{\alpha - 1} + 2x (1 - x)^{\alpha - 2} - x^2 (\alpha - 2) (1 - x)^{\alpha - 3} \right)
\]
\[
= \alpha \left( \alpha x^{\alpha - 1} + (2x (1 - x) - x^2 (\alpha - 2)) (1 - x)^{\alpha - 3} \right)
\]
\[
= \alpha \left( \alpha x^{\alpha - 1} + x (2 - \alpha x) (1 - x)^{\alpha - 3} \right)
\]
\[
= \alpha x^{\alpha - 1} \left( \alpha + \left( \frac{2}{x} - \alpha \right) \left( \frac{1}{x} - 1 \right)^{\alpha - 3} \right).
\]

(138)

(139)

(140)

(141)

Set \( z = \frac{1}{x} - 1 \) so that \( x = \frac{1}{z + 1} \) which gives
\[
\alpha + \left( \frac{2}{x} - \alpha \right) \left( \frac{1}{x} - 1 \right)^{\alpha - 3} = \alpha + (2z + 2 - \alpha) z^{\alpha - 3}.
\]

(142)

For \( \alpha \leq 2 \) the derivative is always positive. For \( \alpha < 3 \) and \( z \) tending to zero the derivative tends to \(-\infty\) if \( 2 - \alpha \) is negative so we do not have monotonicity for \( 2 < \alpha < 3 \).

For \( \alpha \geq 3 \) we calculate the derivative in order to determine the minimum.
\[
2x^{\alpha - 3} + (2z + 2 - \alpha) (\alpha - 3) z^{\alpha - 4} = 0,
\]

(143)

which has the solution \( z = \frac{\alpha - 3}{2} \). Plugging this solution the expression in Equation (142) gives the value
\[
\alpha + \left( 2 \cdot \frac{\alpha - 3}{2} + 2 - \alpha \right) \left( \frac{\alpha - 3}{2} \right)^{\alpha - 3} = \alpha - \left( \frac{\alpha - 3}{2} \right)^{\alpha - 3}.
\]

(144)

Numerical calculations show that this function is positive for values of \( \alpha \) between 3 and 6.43779.

8 Monotonicity of Bregman divergences on Spin Factors

A binary system can be represented as the spin factor \( JSpin_1 \) or as the interval \([0,1]\).

**Theorem 30.** Let \( F : [0,1] \to \mathbb{R} \) denote a convex and symmetric function. Then \( D_F \) is monotone if and only if \( F \in C^2([0,1]) \) and \( y \to y^2 \cdot F''(y) \) is increasing.

**Proof.** The convex body \([0,1]\) has the identity and a reflection as the only isomorphisms. Any affinity can be decomposed into an isomorphism and two dilations where each dilation is a dilation around one of the extreme points \([0,1]\). Therefore \( D_F \) is monotone if and only if it is monotone under dilations.

Next we will study monotonicity of Bregman divergences in spin factors \( JSpin_d \) for \( d \geq 2 \).
Lemma 31. Let $D_F$ denote a Bregman divergence on $\text{JSpin}_d$ where $d \geq 2$. If $D_F$ satisfies sufficiency and the restriction to $\text{JSpin}_2$ is monotone, then $D_F$ is monotone on $\text{JSpin}_d$.

Proof. Assume that $D_F$ satisfies sufficiency and that the restriction of $D_F$ to $\text{JSpin}_2$ is monotone. Let $\rho_1, \sigma \in \text{JSpin}_d$ and let $\Phi : \text{JSpin}_d \to \text{JSpin}_d$ denote a positive trace preserving affinity. Let $\Delta$ denote the disc spanned of $\rho_1, \sigma$ and $\frac{1}{2}$. Then $\Phi(\rho_1), \Phi(\sigma)$ and $\Phi\left(\frac{1}{2}\right)$ spans a disc $\tilde{\Delta}$ in $\text{JSpin}_d$. The restriction of $\Phi$ to $\Delta$ can be written as $\Phi|_{\Delta} = \Phi_2 \circ \Phi_1$ where $\Phi_1$ is an affinity $\Delta \to \Delta$ and $\Phi_2$ is an isomorphism $\Delta \to \tilde{\Delta}$. Essentially $\Phi_2$ maps a great circle into a small circle where the great circle is the boundary of $\Delta$ and the small circle is the boundary of the $\tilde{\Delta}$. According to our assumptions $\Phi_1$ is monotone so it is sufficient to prove that $\Phi_2$ is monotone.

![Fig. 6. Illustration of $\Delta$ and the relative position of the states mentioned in the proof of Lemma 31.](image)

Let $\rho_2$ denote a state such that

\begin{align*}
D_F(\rho_2, \sigma) &= D_F(\rho_1, \sigma) \\
D_F\left(\rho_2, \frac{1}{2}\right) &= D_F\left(\rho_1, \frac{1}{2}\right).
\end{align*}

(145)
(146)

Then

\begin{align*}
D_F(\rho_1, \sigma) &= \frac{1}{2} \cdot D_F(\rho_1, \sigma) + \frac{1}{2} \cdot D_F(\rho_2, \sigma) \\
&= \frac{1}{2} \cdot D_F(\rho_1, \tilde{\sigma}) + \frac{1}{2} \cdot D_F(\rho_2, \tilde{\sigma}) + D_F(\tilde{\sigma}, \sigma)
\end{align*}

(147)
(148)

where $\tilde{\sigma} = \frac{1}{2} \cdot \rho_1 + \frac{1}{2} \cdot \rho_2$. Now $\tilde{\sigma}, \sigma$ and $\frac{1}{2}$ are co-linear and so are $\Phi_2(\tilde{\sigma}), \Phi_2(\sigma),$ and $\Phi_2\left(\frac{1}{2}\right)$ so the restriction of $\Phi$ to the span of $\tilde{\sigma}, \sigma$ and $\frac{1}{2}$ is an interval and the span of $\Phi_2(\tilde{\sigma}), \Phi_2(\sigma), \Phi_2\left(\frac{1}{2}\right),$ and $\frac{1}{2}$ is a disc so by assumption the restriction...
is monotone implying that
\[ D_F (\Phi_2 (\rho), \Phi_2 (\sigma)) \leq D_F (\rho, \sigma). \] (149)

Let \( \bar{\pi} \in \Delta \) denote a state that is collinear with \( \bar{\rho} \) and \( \frac{1}{2} \) and such that
\[ D_F (\bar{\pi}, \frac{1}{2}) = D_F (\Phi_2 (\bar{\rho}), \frac{1}{2}). \] Then there exists an affinity \( \Psi : \Delta \to \Delta \) such that \( \Psi (\bar{\rho}) = \bar{\pi} \) and for \( i = 1, 2 \)
\[ D_F (\Phi (\rho_i), \Phi (\bar{\rho})) = D_F (\Psi (\rho_i), \Psi (\bar{\rho})). \] (150)

Since \( \Psi \) is monotone
\[ D_F (\Phi (\rho_1), \Phi (\bar{\rho})) = D_F (\Phi (\rho_2), \Phi (\bar{\rho})) \leq D_F (\rho_1, \bar{\rho}). \] (151)

Therefore
\[
D_F (\Phi (\rho_1), \Phi (\sigma)) = \frac{1}{2} D_F (\Phi (\rho_1), \Phi (\sigma)) + \frac{1}{2} D_F (\Phi (\rho_2), \Phi (\sigma)) \\
= \frac{1}{2} D_F (\Phi (\rho_1), \Phi (\bar{\rho})) + \frac{1}{2} D_F (\Phi (\rho_2), \Phi (\bar{\rho})) + D_F (\Phi (\bar{\rho}), \Phi (\sigma)) \\
\leq \frac{1}{2} D_F (\rho_1, \bar{\rho}) + \frac{1}{2} D_F (\rho_2, \bar{\rho}) + D_F (\rho, \sigma) = D_F (\rho_1, \rho). \] (152)

**Theorem 32.** Information divergence is monotone on spin factors.

**Proof.** According to Lemma 31 we just have to check monotonicity on spin factors of dimension 2, but these are sections of qubits. Müller-Hermes and Reeb [17] proved that quantum relative entropy is monotone on density matrices on complex Hilbert spaces. In particular quantum relative entropy is monotone on qubits. Therefore information divergence is monotone on any spin factor. \( \square \)

We will need the following lemma.

**Lemma 33.** Let \( \Phi : \mathcal{K} \to \mathcal{K} \) denote an affinity of a centrally symmetric set into itself. Let \( \Psi_r \) denote a dilation around the center \( c \) with a factor \( r \in [0, 1] \). Then \( \Psi_r \circ \Phi \circ \Psi_r^{-1} \) maps \( \mathcal{K} \) into itself.

**Proof.** Embed \( \mathcal{K} \) in a vector space \( V \) with origin in the center of \( \mathcal{K} \). Then \( \Phi \) is given by \( \Phi (\bar{v}) = A\bar{v} + \bar{b} \) and \( \Psi_r (\bar{v}) = r \cdot \bar{v} \). Then
\[
\Psi_r \circ \Phi \circ \Psi_r^{-1} (\bar{v}) = r \cdot \left( A \left( \frac{1}{r} \cdot \bar{v} \right) + \bar{b} \right) \\
= A\bar{v} + r \cdot \bar{b}. \] (153)

Assume that \( \bar{v} \in \mathcal{K} \). Then \( \Phi (\bar{v}) \in \mathcal{K} \) and \( -\Phi (\bar{v}) \in \mathcal{K} \). Hence for \( (1 - t) \cdot \Phi (\bar{v}) + t \cdot (-\Phi (\bar{v})) \in \mathcal{K} \). Now
\[
(1 - t) \cdot \Phi (\bar{v}) + t \cdot (-\Phi (\bar{v})) = (1 - t) \cdot \left( A\bar{v} + \bar{b} \right) + t \cdot \left( -A (-\bar{v}) + \bar{b} \right) \\
= A\bar{v} + (1 - 2t) \cdot \bar{b}. \] (155)
For \( t = \frac{1-r}{2} \) we get
\[
(\Psi_r \circ \Phi \circ \Psi_r^{-1}) (\vec{v}) = (1-t) \cdot \Phi (\vec{v}) + t \cdot (-\Phi (-\vec{v})) \in K,
\] (157)
which completes the proof. \( \square \)

**Theorem 34.** If \( D_F \) is a monotone Bregman divergence on a spin factor and \( F_r (x) = F \left( (1-r) \cdot \frac{1}{2} + r \cdot x \right) \) then the Bregman divergence \( D_{F_t} \) is also monotone.

**Proof.** We have
\[
D_{F_t} (\rho, \sigma) = D_F \left( (1-r) \cdot \frac{1}{2} + r \cdot \rho, (1-r) \cdot \frac{1}{2} + r \cdot \sigma \right)
= D_F (\Psi_r (\rho), \Psi_r (\sigma))
\] (158)
(159)
where \( \Psi_r \) denotes a dilation around \( \frac{1}{2} \) by a factor \( r \in ]0,1[ \). Let \( \Phi \) denote an affinity of the state space into itself. Then according to Lemma 33
\[
D_{F_r} (\Phi (\rho), \Phi (\sigma)) = D_F (\Psi_r (\Phi (\rho)), \Psi_r (\Phi (\sigma)))
= D_F (\left( \Psi_r \circ \Phi \circ \Psi_r^{-1} \right) (\Psi_r (\rho)), \left( \Psi_r \circ \Phi \circ \Psi_r^{-1} \right) (\Psi_r (\sigma)))
\] (160)
(161)
(162)
(163)
\[
= D_F (\Psi_r (\rho), \Psi_r (\sigma))
\] (164)
which proves the theorem. \( \square \)

In [19] joint convexity of Bregman divergences on complex density matrices was studied (see also [20]).

**Theorem 35.** The Bregman divergence \( D_F \) given by \( F (x) = \text{tr}[f(x)] \) is jointly convex if and only if \( f \) has the form
\[
f (x) = a (x) + \frac{\gamma}{2} q (x) + \int_0^{\infty} e_\lambda \ d\mu (\lambda)
\] (165)
where \( a \) is affine and
\[
q (x) = x^2
\] (166)
and
\[
e_\lambda (x) = (\lambda + x) \ln (\lambda + x)
\] (167)

This result is related to the matrix entropy class introduced in [6] and further studied in [7]. The function \( q \) generates the Bregman divergence \( D_q (\rho, \sigma) = \text{tr} \left[ (\rho - \sigma)^2 \right] \) and the function \( e_\lambda \) generates the Bregman divergence
\[
D_{e_\lambda} (\rho, \sigma) = D (\rho + \lambda \| \sigma + \lambda).
\] (168)
We note that
\[
D(\rho + \lambda \| \sigma + \lambda) = (1 + 2\lambda) \cdot D \left( \frac{1}{1 + 2\lambda} \cdot \rho + \frac{2\lambda}{1 + 2\lambda} \cdot c \left\| \frac{1}{1 + 2\lambda} \cdot \sigma + \frac{2\lambda}{1 + 2\lambda} \cdot c \right\| \right),
\] (169)
which implies that 
\[2\lambda (1 + 2\lambda) \cdot D_{e\lambda}(\rho, \sigma) \to \text{tr} [\rho - \sigma]^2 \] so the Bregman divergence \(D_2\) may be considered as a limiting case. Now
\[
D_f(\rho, \sigma) = \frac{\gamma}{2} \text{tr} [\rho - \sigma]^2 + \int_0^\infty D(\rho + \lambda \| \sigma + \lambda) \, d\mu(\lambda).
\] (170)
Note that the Bregman divergence of order \(\alpha\) can be written in this way for \(\alpha \in [1, 2]\).

**Theorem 36.** Any Bregman divergence based on a function of the form (165) is monotone on spin factors.

**Proof.** The result follows from Equation (169) and Equation (170) in combination with Theorem 32.

### 9 Strict monotonicity

**Definition 37.** We say that a regret function is strictly monotone if
\[
D_F(\Phi(\rho), \Phi(\sigma)) = D_F(\rho, \sigma)
\] (171)
implies that \(\Phi\) is sufficient for \(\rho, \sigma\).

In [10] it was proved that strict monotonicity implies monotonicity. As we shall see in Theorem 39 on convex bodies of rank 2 strictness and monotonicity is equivalent to strict monotonicity as long as the Bregman divergence is based on an analytic function.

**Lemma 38.** Let \(\sigma\) denote a point in a convex body \(K\) with a monotone Bregman divergence \(D_F\). If \(\Phi: K \to K\) is an affinity then the set
\[
C = \{\rho \in C \mid D_F(\Phi(\rho), \Phi(\sigma)) = D_F(\rho, \sigma)\}
\] (172)
is a convex body that contains \(\sigma\).

**Proof.** Assume that \(\rho_0, \rho_1 \in C\) and \(t \in [0, 1]\) and \(\tilde{\rho} = (1 - t) \cdot \rho_0 + t \cdot \rho_1\). Then according to the Bregman identity
\[
(1 - t) \cdot D_F(\Phi(\rho_0), \Phi(\sigma)) + t \cdot D_F(\Phi(\rho_1), \Phi(\sigma))
\]
\[
= (1 - t) \cdot D_F(\Phi(\rho_0), \Phi(\tilde{\rho})) + t \cdot D_F(\Phi(\rho_1), \Phi(\tilde{\rho})) + D_F(\Phi(\tilde{\rho}), \Phi(\sigma))
\]
\[
\leq (1 - t) \cdot D_F(\rho_0, \tilde{\rho}) + t \cdot D_F(\rho_1, \tilde{\rho}) + D_F(\tilde{\rho}, \sigma)
\]
\[
= (1 - t) \cdot D_F(\rho_0, \sigma) + t \cdot D_F(\rho_1, \sigma).
\] (173)
Therefore the inequality must hold with equality and
\[ D_F(\Phi(\bar{\rho}), \Phi(\sigma)) = D_F(\bar{\rho}, \sigma), \]  
which proves the lemma. \(\square\)

**Theorem 39.** Let \(D_F\) denote a monotone Bregman divergence that is strict on a spin factor based on an analytic function \(f\). Then \(D_F\) is strictly monotone.

**Proof.** Assume that \(D_F\) is monotone and that
\[ D_F(\Phi(\rho), \Phi(\sigma)) = D_F(\rho, \sigma) =: (175) \]
Let \(\rho_0\) and \(\rho_1\) denote extreme points such that \(\rho\) and \(\sigma\) lie on the line segment between \(\rho_0\) and \(\rho_1\). Lemma 25 implies that
\[ D_F(\Phi(\rho_{i}), \Phi(\sigma_{t})) = D_F(\rho_{i}, \sigma_{t}) \] for all \(s \in [0,1]\) where \(\sigma_{t} = (1-s) \rho + s \sigma\). Since \(f\) is assumed to be analytic the identity (176) must hold for all \(t\) for which \((1-s) \rho + s \sigma \geq 0\). The identity (176) also holds if \(\rho\) is replaced by any point \(\rho'\) on the line segment between \(\rho_0\) and \(\rho_1\) because both sides of Equation (176) are quadratic functions in the first variable. Using Proposition 23 we see that Equation (175) can be extended to any pair of points on the line segment between \(\rho_0\) and \(\rho_1\). In particular
\[ D_F(\Phi(\rho_{i}), \Phi(\bar{\rho})) = D_F(\rho_{i}, \bar{\rho}) \] for \(i = 0, 1\) and \(\bar{\rho} = \frac{1}{2} \rho_0 + \frac{1}{2} \rho_1\). Since both \(\rho_{i}\) and \(\Phi(\rho_{i})\) are extreme points we have
\[ D_F\left(\Phi(\rho_{i}), \frac{1}{2}\right) = D_F\left(\rho_{i}, \frac{1}{2}\right) \] (178)
we have \(D_F(\bar{\rho}, \frac{1}{2}) = D_F(\Phi(\bar{\rho}), \frac{1}{2})\). Therefore the points \(\bar{\rho}\) and \(\Phi(\bar{\rho})\) have the same distance to the center \(\frac{1}{2}\). Therefore there exists a rotation \(\Psi\) that maps \(\Phi(\rho_{i})\) into \(\rho_{i}\). Since \(\Psi\) is a recovery map of the states \(\rho_{i}\) it is also a recovery map of \(\rho\) and \(\sigma\). \(\square\)

An affinity in a Hilbert ball has a unique extension to a positive trace preserving map in the corresponding spin factor. Here we shall study such maps with respect to existence of recovery maps and with respect to monotonicity of Bregman divergences. Let \(\Phi\) denote a positive trace preserving map of \(JSpin_{d}\) into itself. Then the adjoint map \(\Phi^{*}\) is defined by
\[ \langle \Phi^{*}(x), y \rangle = \langle x, \Phi(y) \rangle. \] (179)
If \(\Phi(\sigma)\) is not singular then we may define
\[ \Psi(\rho) = \sigma^{-1/2} \Phi^{*} \left( (\Phi(\sigma))^{-1/2} \rho (\Phi(\sigma))^{-1/2} \right)^{1/2} \sigma^{1/2} \] (180)
We observe that $\Psi(\Phi(\sigma)) = \sigma$. If $\Phi$ is an isomorphism then $\Phi(y) = O^*yO$ where $O$ is an orthogonal map on $JSpin_d$ as a Hilbert space. Then

$$\langle \Phi^*(x), y \rangle = \langle x, \Phi(y) \rangle$$

(181)

$$= \text{tr} [xO^*yO]$$

(182)

$$= \text{tr} [OxO^*y]$$

(183)

$$= \langle OxO^*, y \rangle$$

(184)

so that $\Phi^*(x) = OxO^*$. Then

$$\Psi(\Phi(\rho)) = \sigma^{1/2}\Phi^*\left(\Phi\left(\sigma^{-1/2}\right)\Phi(\rho)\Phi\left(\sigma^{-1/2}\right)\right)\sigma^{1/2}$$

(185)

$$= \sigma^{1/2}O\left(O^*\left(\sigma^{-1/2}\right)O\right)O^*\rho O\left(O^*\left(\sigma^{-1/2}\right)O\right)O^*\sigma^{1/2}$$

(186)

$$= \rho.$$  

(187)

Therefore $\Psi$ is a recovery map. This formula extends to any $\rho$ for which there exists a recovery map because $\Phi$ is an isomorphism between two sections of the state space that contain $\rho$ and $\sigma$.

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