The six-dimensional Delaunay polytopes

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Abstract

Given a lattice $L$, a full dimensional polytope $P$ is called a Delaunay polytope if the set of its vertices is $S \cap L$ with $S$ being an empty sphere of the lattice. Extending our previous work [DD01] on the hypermetric cone $HYP_7$, we classify the six-dimensional Delaunay polytopes according to their combinatorial type. The list of 6241 combinatorial types is obtained by a study of the set of faces of the polyhedral cone $HYP_7$.

1 Introduction

A distance vector $(d_{ij})_{0 \leq i < j \leq n} \in \mathbb{R}^N$ with $N = \binom{n+1}{2}$ is called an $(n+1)$-hypermetric if it satisfies the following hypermetric inequalities:

$$H(b)d = \sum_{0 \leq i < j \leq n} b_ib_jd_{ij} \leq 0 \text{ for any } b = (b_i)_{0 \leq i \leq n} \in \mathbb{Z}^{n+1} \text{ with } \sum_{i=0}^{n} b_i = 1 . \quad (1)$$

The set of distance vectors satisfying (1) is called the hypermetric cone and denoted by $HYP_{n+1}$.

In fact, $HYP_{n+1}$ is a polyhedral cone (see [DeLa97] p. 199). Lovasz (see [DeLa97] p. 201-205) gave another proof of it and bound $\max |b_i| \leq n!2^n \binom{2n}{n}^{-1}$ for any vector $b = (b_i)_{0 \leq i \leq n-1}$ defining a facet of $HYP_n$.

There is a one-to-one correspondence between non-degenerate elements of $HYP_{n+1}$ and semi-metrics on affine basis of $n$-dimensional Delaunay polytopes. So, the enumeration of combinatorial types of $n$-dimensional Delaunay polytopes is reduced to the enumeration of non-degenerate faces of $HYP_{n+1}$ under an equivalence relation called geometrical equivalence (see Remark [DGL92] and Section [DGL92]). Moreover, the dimension of faces of $HYP_{n+1}$ allows to define the notion of rank of a Delaunay polytope (see [DGL92]).

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We have the inclusion $\text{CUT}_n \subset \text{HYP}_n$, where $\text{CUT}_n$ (see Section 3 below and Chapter 4 of [DeLa97]) is the cone generated by all cut semi-metrics on $n$ points. One has $\text{HYP}_n = \text{CUT}_n$ for $n \leq 6$; so, the enumeration of combinatorial types of Delaunay polytopes of dimension less or equal to five correspond to the study of faces of $\text{CUT}_6$. This study was done by Fedorov ([Fe85]), Erdahl and Ryshkov ([RyEr87] and [RyEr88]), and Kononenko ([Ko99]) in dimension three, four and five, respectively.

In the case of dimensional six, the inclusion $\text{CUT}_7 \subset \text{HYP}_7$ is strict, but the description of facets (i.e. faces of rank 20, corresponding to repartitioning polytopes) and extreme rays (i.e. faces of rank 1, corresponding to extreme Delaunay polytopes) of the cone $\text{HYP}_7$ is still possible (see [DD01]). In Section 3 we present the number of combinatorial types of six-dimensional Delaunay polytope for every given rank. In Section 4 we present our method for the computation of those combinatorial types using the face lattice of $\text{HYP}_7$.

Voronoi [Vo08] defines a partition of the cone $\text{PSD}_n$ of positive semi-definite quadratic forms by $L$-type domains. Two forms of the same $L$-type domain have affinely equivalent Voronoi polytopes. Every vertex of a Voronoi polytope correspond to a center of a Delaunay polytope. All Delaunay polytopes form a partition of the space $\mathbb{R}^n$, which is dual to the partition by Voronoi polytopes. It is proved in [DGL93] that the hypermetric cone $\text{HYP}_n$ is the union of a finite number of $L$-type domains.

While Voronoi theory of $L$-type domains describes the combinatorial structure of lattices, the theory of hypermetrics ([DGL92] and [DeLa97]) describes combinatorial structure of one Delaunay polytope in a lattice. While the hypermetric cone $\text{HYP}_n$ has the symmetry group $\text{Sym}(n)$, a $L$-type domain, in general, has trivial group.

2 Delaunay polytopes and hypermetrics

Here we present the main notions for hypermetrics, which are needed to our study; the presentation is slightly simplified by the systematic use of affine bases. For the complete theory, with proofs, see [DGL92] and Chapters 13–16 of [DeLa97].

A family $v_0, \ldots, v_n$ of $n+1$ vertices of $\mathbb{R}^n$ is called independent if the family $(v_i - v_0)_{1 \leq i \leq n}$ has linear rank $n$. Let $L \subset \mathbb{R}^n$ be a $n$-dimensional lattice and let $S = S(c, r)$ be a sphere in $\mathbb{R}^n$ with center $c$ and radius $r$. Then $S$ is said to be an empty sphere in $L$ if the following two conditions hold:

(i) $\|v - c\| \geq r$ for all $v \in L$ and

(ii) the set $S \cap L$ contain an independent set of size $n+1$.

The center $c$ of $S$ is called a hole in [CS99]. The polytope $D$, which is defined as the convex hull of the set $S \cap L$ is called a Delaunay polytope, or (in original terms of Voronoi, who introduced them in [Vo08]) $L$-polytope.

**Definition 1** Let $D$ be a $n$-dimensional Delaunay polytope with vertex-set $V$. 
(i) A family \( v_0, \ldots, v_n \) of vertices of \( D \) is called an affine basis if for all \( v \in V \) there exist an unique family \((b_i)_{0 \leq i \leq n} \in \mathbb{Z}^{n+1}\), such that

\[
\sum_{i=0}^{n} b_i v_i = v \quad \text{and} \quad \sum_{i=0}^{n} b_i = 1.
\]

(ii) The Delaunay polytope \( D \) is called basic if it has at least one affine basis. The vertices of an affine basis are called basic vertices.

All known Delaunay polytopes are basic. We prove in Theorem 1 that all six-dimensional Delaunay polytopes are basic too. We will always assume that Delaunay polytopes are basic.

For every family \( A = \{v_0, \ldots, v_m\} \) of vertices of a Delaunay polytope \( P \) one can define a distance function \( d_A \) by \( d_A(i, j) = \|v_i - v_j\|^2 \). The function \( d_A \) turns out to be an hypermetric by the following formula (see [As82] and [DeLa97] p. 195):

\[
\sum_{0 \leq i, j \leq m} b_i b_j d_A(i, j) = 2(r^2 - \sum_{i=0}^{m} b_i v_i - c)^2 \leq 0.
\]

On the other hand, Assouad has shown in [As82] that every \( d \in HY P_{n+1} \) can be expressed as \( d_A \) with \( A \) being a family of vertices of a Delaunay polytope \( D \) of dimension less or equal to \( n \).

A ray \( d \in HY P_{n+1} \) is called non-degenerate if \( d = d_B \) with \( B \) an affine basis of a \( n \)-dimensional Delaunay polytope \( D \). For a given ray \( d \in HY P_{n+1} \) the annulator is defined by

\[
Ann(d) = \{ b \in \mathbb{Z}^{n+1} : \sum_{i=0}^{n} b_i = 1 \text{ and } H(b)d = 0 \}.
\]

We call basic vectors the vectors \( e_i = (0, \ldots, 0, 1, 0, \ldots, 0) \in \mathbb{Z}^{n+1} \) with \( 0 \leq i \leq n \).

Using Proposition below, those vectors \( e_i \) are identified with basic vertices \( v_i \). One has \( H(b) = 0 \) if and only if \( b = e_i \) for some \( 0 \leq i \leq n \).

**Proposition 1** Let \( D \) be an \( n \)-dimensional Delaunay polytope with vertex-set \( V \); let \( d = d_A \) with \( A = \{v_0, \ldots, v_n\} \) a family of \( V \). Then there is equivalence between following properties:

(i) the family \( A \) is independent and

(ii) \( \det [d(0, i) + d(0, j) - d(i, j)]_{1 \leq i, j \leq n} \neq 0 \).

Also one has equivalence between properties:

(iii) \( A \) is an affine basis and

(iv) the following mapping is one-to-one

\[
b \in Ann(d) \mapsto \sum_{i=0}^{n} b_i v_i \in V.
\]
Proof. The family \((v_i)_{0 \leq i \leq n}\) is independent if and only if the matrix \(((v_i - v_0), (v_j - v_0))_{1 \leq i,j \leq n}\) is positive definite. The formula \(2(v_i - v_0). (v_j - v_0) = \|v_i - v_0\|^2 + \|v_j - v_0\|^2 - \|v_i - v_j\|^2\), gives the first equivalence. The second equivalence is obvious.

By above Proposition, the set \(Ann(d)\) is finite if \(d\) is non-degenerate, since the hypermetric vectors \(b \in Ann(d)\) correspond to lattice points \(v\), which belong to a sphere. We will consider only non-degenerate hypermetrics (in the hypermetric cone \(HYP_{n+1}\)) associated with affine bases of \(n\)-dimensional Delaunay polytopes.

Using above mapping, we identify vertices \(v\) of \(D\) with hypermetric vectors \(b\). For example, for the affine basis \(B = \{v_0, \ldots, v_3\}\) with \(v_0 = (0, 0, 0), v_1 = (1, 0, 0), v_2 = (0, 1, 0)\) and \(v_3 = (0, 0, 1)\) of the \(3\)-cube, one has \((1, 1, 0) = v_1 + v_2 - v_0, (0, 1, 1) = v_2 + v_3 - v_0, (1, 0, 1) = v_1 + v_3 - v_0\) and \((1, 1, 1) = v_1 + v_2 + v_3 - 2v_0\). Note that, while \((-2, 1, 1, 1) \in Ann(d_B)\), the inequality \(H(-2, 1, 1, 1)d \geq 0\) does not define a facet of the cone \(HYP_4\), since \(H(-2, 1, 1, 1) = H(-1, 1, 1, 0) + H(-1, 0, 1, 1) + H(-1, 1, 0, 1)\).

The face \(F(d)\), associated to an hypermetric \(d \in HYP_{n+1}\), is the minimal face, containing the vector \(d\). It can also be defined as:

\[
F(d) = \{e \in HYP_{n+1} : H(b)e = 0 \text{ for all } b \in Ann(d)\}.
\]

The rank of a \(n\)-dimensional Delaunay polytope \(D\) is defined as dimension of \(F(d_B)\), where \(B\) is an affine basis of \(D\) (see [DeLa97] p. 217); its corank is, by definition, \(\binom{n+1}{2} - \text{rank } D\). The rank of a Delaunay polytope \(D\) is equal to the topological dimension of the set of affine bijections \(T\) of \(\mathbb{R}^n\) (up to translations and orthogonal transformations), for which \(T(D)\) is again a Delaunay polytope (see [DeLa97] p. 225).

If \(D\) is an \(n\)-simplex, then its unique affine basis defines an hypermetric \(d\), which has \(Ann(d) = \{e_0, \ldots, e_n\}\) and \(F(d) = HYP_{n+1}\). If \(D\) is a \(n\)-dimensional Delaunay polytope with \(n + 2\) vertices \(v_0, \ldots, v_{n+1}\) (such polytope is called repartitioning polytope) and affine basis \(B = \{v_0, \ldots, v_n\}\), then, by writing \(v_{n+1} = \sum_{i=0}^{n} b_i v_i\), one obtains \(F(d_B) = \{d \in HYP_{n+1}, H(b)d = 0\}\). So, \(F(d)\) is a facet of \(HYP_{n+1}\). On the other hand, if \(d \in HYP_{n+1}\) with \(F(d)\) being a facet, then \(Ann(d) = \{e_0, \ldots, e_n, b\}\) and \(d = d_B\) with \(B\) being an affine basis of a repartitioning polytope.

Define two Delaunay polytopes to be affinely equivalent, if there is an affine bijective mapping transforming one into another; the equivalence classes by this relation are called combinatorial type. By Theorem 15.2.1 in [DeLa97] p. 222, two vectors \(d, d'\) of \(HYP_{n+1}\), such that \(F(d) = F(d')\), correspond to affinely equivalent Delaunay polytopes. Moreover, if two Delaunay polytopes \(D, D'\) are affinely equivalent, then an affine basis \(B\) of \(D\) induces an affine basis \(B'\) of \(D'\), such that \(F(d_B) = F(d_{B'})\).

A face of \(HYP_{n+1}\) is called non-degenerate, if one of its interior vectors \(d\) is non-degenerate (by Theorem 15.2.1 of [DeLa97], this is equivalent to all interior vectors being non-degenerate). One can consider only non-degenerate faces, since we already assumed, that all Delaunay polytopes are basic. Given a face \(F\) of \(HYP_{n+1}\), we define

\[
Ann(F) = \{b \in \mathbb{Z}^{n+1} : \sum_{i} b_i = 1 \text{ and } H(b)d = 0 \text{ for all } d \in F\}.
\]

We will call \(\{b^0, \ldots, b^n\} \subset Ann(F)\) an affine basis of the combinatorial type if \(\det \{b^0, \ldots, b^n\} = \pm 1\). Any affine basis of \(Ann(F)\) corresponds to an affine basis of a Delaunay polytope \(D\), if \(D\) has this combinatorial type.
**Definition 2** Two faces \( F, F' \) of \( HYP_{n+1} \) are said to be geometrically equivalent if there exist an affine basis \( B = \{b^0, \ldots, b^n\} \) of \( Ann(F) \), such that the mapping

\[
\begin{align*}
\phi_{F',F} : Ann(F') & \rightarrow Ann(F) \\
b & \mapsto b^0 b + \ldots + b_n b^n
\end{align*}
\]

is bijective.

If \( \pi \) is a permutation of \( \{0, \ldots, n\} \) and \( F \) a face of \( HYP_{n+1} \), then \( \{e_{\pi(0)}, \ldots, e_{\pi(n)}\} \) is an affine basis of \( Ann(F) \); so, two faces \( F \) and \( F' \), which are equivalent, by a symmetry of \( HYP_{n+1} \), are also geometrically equivalent. The reverse is not true, in general.

**Remark 1** Let \( D \) be a \( n \)-dimensional repartitioning polytope with an affine basis \( B = \{v_0, \ldots, v_n\} \); we write \( v_{n+1} = \sum_{i=0}^{n} b_i v_i \). The family \( B' = (v_0, \ldots, v_{i-1}, v_{n+1}, v_{i+1}, \ldots, v_n) \) is an affine basis if and only if \( |b_i| = 1 \).

If \( b_i = -1 \), then \( F(d_{B'}) = F(d_B) = \{d \in HYP_{n+1}, H(b)d = 0\} \), while \( F(d_{B'}) \neq F(d_B) = \{d \in HYP_{n+1}, H(b')d = 0\} \) with \( b' = (-b_0, \ldots, -b_{i-1}, b_i, -b_{i+1}, \ldots, -b_n) \).

So, the study of combinatorial types of \( n \)-dimensional Delaunay polytopes is reduced to the study of non-degenerate faces of the hypermetric cone \( HYP_{n+1} \) under the geometrical equivalence.

Consider, as an example, the two-dimensional case. We have \( HYP_3 = CUT_3 \) and \( CUT_3 \) has three facets, which correspond to \( H(-1,1,1) \) and its permutations.

One can check that a triangle \( T = \{v_0, v_1, v_2\} \) satisfies \( d_T = (d_01, d_02, d_{12}) \in HYP_3 \) if and only if it is an obtuse triangle. The vector \( d_T \) is non-degenerate if and only if three vertices of \( T \) are not aligned. Moreover, \( d_T \) is incident to an hypermetric facet, say, \( H(-1,1,1) \), if and only if the vertex \( v_0 \) has angle \( \frac{\pi}{2} \), in which case the Delaunay polytope has four vertices: \( v_0, v_1, v_2 \) and \( v_1 + v_2 - v_0 \). So, there are two combinatorial types of Delaunay polytopes in dimension two: obtuse triangles and rectangles.

Take a face \( F \) of the hypermetric cone \( HYP_{n+1} \). To every \( b = (b_i)_{0 \leq i \leq n} \in Ann(F) \), we associate a vertex \( (b_1, \ldots, b_n) \in \mathbb{Z}^n \); let the set of such vertices being denoted by \( V \). Every distance vector \( d \in F \) correspond to a Gram matrix \( G \). The set \( V \) is the vertex-set of a Delaunay polytope, which is described by the scalar product defined by \( G \). So, we can encode, in our computations, the combinatorial type of Delaunay polytope by the set \( Ann(F) \).

**Proposition 2** If \( D \) is a Delaunay polytope and \( B, B' \) are two affine bases of \( D \), then the faces \( F(d_B), F(d_{B'}) \) are equivalent up to a linear mapping. This linear mapping preserve the non-degeneracy.

**Proof.** If \( (v_0, \ldots, v_n) \) and \( (v'_0, \ldots, v'_n) \) are two affine bases of \( D \), then one can express \( v'_i \) in terms of \( (v_j)_{0 \leq j \leq n} \) as follows

\[
v'_i = \sum_{j=0}^{n} \alpha_{ij} v_j \quad \text{with} \quad \sum_{j=0}^{n} \alpha_{ij} = 1.
\]
One can express \( ||v'_{i_1} - v'_{i_2}||^2 \) in terms of \( ||v_{j_1} - v_{j_2}||^2 \). This induces a linear mapping \( \phi \) from \( F(d_B) \) to \( F(d_B') \); expressing \( v_j \) in terms of \( (v'_{i})_{0 \leq i \leq n} \), one get the reverse mapping \( \phi^{-1} \) and so, the linear equivalence.

If \( d \in F(d_B) \), then \( d \) is non-degenerate if and only if \( \phi(d) \) is non-degenerate.

**Definition 3** Let \( \mathcal{B} \) be an affine basis of a Delaunay polytope \( D \); then

(i) If rank \( D = 1 \), then \( D \) is called extreme.

(ii) If all sub-faces of \( F(d_B) \) are degenerate, then \( D \) is called maximal.

Above definition of maximality is independent of the choice of affine basis \( \mathcal{B} \), since, by above Proposition, the linear equivalence between two faces preserves the non-degeneracy. Obviously, any extreme Delaunay polytope is maximal. We present in Corollary 3 the list of all maximal six-dimensional Delaunay polytopes.

Let \( \gamma_n = \{0,1\}^n \) be the vertex-set of the Delaunay polytope of the lattice \( \mathbb{Z}^n \) and let

\[
    h\gamma_n = \{x \in \gamma_n : \sum_{i=1}^{n} x_i \text{ is even}\}
\]

be the vertex-set of the Delaunay polytope with center \( c = (1/2,\ldots,1/2) \) (this polytope is called half \( n \)-cube) of the root lattice

\[
    D_n = \{x \in \mathbb{Z}^n : \sum_{i=1}^{n} x_i \text{ is even}\}.
\]

**Proposition 3** The \( n \)-cube \( \gamma_n \) and the half \( n \)-cube \( h\gamma_n \) have rank \( n \) and are maximal Delaunay polytopes.

**Proof.** One can define the following distance functions on \( \gamma_n \) and \( h\gamma_n \):

\[
    d_i : \gamma_n \times \gamma_n \rightarrow \mathbb{R} \quad (x, x') \mapsto (x_i - x'_i)^2.
\]

If \( d \) is the distance function of the \( n \)-dimensional Delaunay polytope \( \gamma_n \) or \( h\gamma_n \), then this distance function is expressed as

\[
    d = \sum_{i=1}^{n} \lambda_i d_i \quad \text{with} \quad \lambda_i > 0,
\]

which proves that the rank of \( \gamma_n \) and of \( h\gamma_n \) is \( n \). Now, if a face \( F \) is included in \( F(\gamma_n) \) or \( F(h\gamma_n) \), then one of the coefficients \( \lambda_i \) becomes zero and the face \( F \) corresponds to \( \gamma_{n-1} \) or \( h\gamma_{n-1} \), which is of lower dimension \( n - 1 \). So, the Delaunay polytopes \( \gamma_n \) and \( h\gamma_n \) are maximal.
3 The case of dimension six

Call cut cone and denote by $CUT_{n+1}$ the cone generated by all cuts $\delta_S \in \mathbb{R}^N$ (where $S$ is a subset of \{0, \ldots, n\}), defined by

$$(\delta_S)_{ij} = 1 \text{ if } |S \cap \{i, j\}| = 1 \text{ and } (\delta_S)_{ij} = 0, \text{ otherwise.}$$

Clearly, $H(b)\delta_S = b(S)(1-\delta(S))$ with $b(S) = \sum_{a \in S} b_a$; this proves that all hypermetric inequalities are valid on $CUT_{n+1}$. So, $CUT_{n+1} \subset HYP_{n+1}$. Moreover, a cut $\delta_S$ is incident to the face, defined by $H(b)$, if and only if $b(S) = 0$ or 1.

The list of 3773 facets of $HYP_7$ was found by Baranovskii [Ba99] using the method described in [Ba70], i.e. he found, by hand, that for all other hypermetric vectors $b$, one can express $H(b)$ as a sum of terms $H(b')$ with $b'$ belonging to his list of 3773 elements. While this result was announced in [Ba99], the detailed computations were not published. In [DD01], another method was proposed: if the Baranovskii’s list was not complete, then, in our computation [DD01] of the extreme rays of $HYP_7$, we should find some extreme rays, which are not hypermetric. But this was not the case; so, the list is complete.

The list of representatives of 14 orbits of facets is given below:

| $b^1$ = (1, 1, -1, 0, 0, 0, 0) | $b^2$ = (1, 1, 1, -1, 0, 0, 0) |
| $b^3$ = (1, 1, 1, 1, -1, 2, 0) | $b^4$ = (2, 1, 1, -1, -1, 0) |
| $b^5$ = (1, 1, 1, 1, -1, -1, -1) | $b^6$ = (2, 1, 1, 1, -1, -1, -2) |
| $b^7$ = (2, 2, 1, -1, -1, 1, -2) | $b^{8}$ = (1, 1, 1, 1, 1, -2, -2) |
| $b^{9}$ = (3, 1, 1, 1, -1, -1, -1) | $b^{10}$ = (1, 1, 1, 1, 1, -1, -3) |
| $b^{11}$ = (2, 2, 1, 1, -1, -1, -3) | $b^{12}$ = (3, 1, 1, 1, 1, -2, -2) |
| $b^{13}$ = (3, 2, 1, -1, -1, -1, -2) | $b^{14}$ = (2, 1, 1, 1, 1, -2, -3) |

It gives the total of 3773 inequalities. The first ten orbits are the orbits of hypermetric facets of the cut cone $CUT_7$; first four of them come as 0-extension of facets of the cone $HYP_6 = CUT_6$ (see [DeLa97], Chapter 7). Last four orbits consist of some 19-dimensional simplex-faces of $CUT_7$, becoming 20-dimensional, i.e. simplex-facets in $HYP_7$.

Using Remark [1], we obtain that the list of 14 orbits of facets fall into nine equivalence classes $b^1$, $b^2$, $\{b^3, b^4\}$, $b^5$, $b^6$, $\{b^7, b^8\}$, $\{b^9, b^{10}\}$, $\{b^{11}, b^{12}\}$, $\{b^{13}, b^{14}\}$. So, there are nine combinatorial types of six-dimensional Delaunay polytopes of rank 20 (i.e. repartitioning polytopes).

In [DeLa97] p. 229 another notion, called switching by root of $b$, is defined: if $A \subset \{0, \ldots, n\}$ and $b(A) = 0$, then define $b^A$ by $b^A_i = -b_i$ if $i \in A$ and $b^A_i = b_i$ if $i \notin A$. It is proved that the switching by root of a facet-defining vector of $HYP_{n+1}$ is again a facet-defining vector. Three following vectors define facets of $HYP_7$, which are switching by root equivalent:

$$b^1 = (2, 2, 1, -1, -1, -1, -1), b^2 = (2, -2, 1, 1, -1, -1, -1), b^3 = (-2, -2, 1, 1, 1, 1, 1).$$

On the other hand, $b^1$ and $b^3$ are geometrically equivalent by Remark [1] while $b^1$ and $b^2$ are not geometrically equivalent.
Remind that $E_6$, $E_7$, $E_8$ are root lattices defined by

$$E_6 = \{x \in E_7 : x_1 + x_2 = 0\}, \quad E_7 = \{x \in E_8 : x_1 + \cdots + x_8 = 0\}, \quad E_8 = \{x \in \mathbb{R}^8 : x \in \mathbb{Z}^8 \cup (\frac{1}{2} + \mathbb{Z})^8 \text{ and } \sum x_i \in 2\mathbb{Z}\}.$$

Unique type of Delaunay polytope of $E_6$ is called Schlafli polytope (see [Cox63]) and denoted by $Sch$. Its skeleton graph is 27-vertex (strongly regular) graph, called the Schlafli graph, whose symmetry group has size 51840 and is isomorphic to the group of isometry, preserving the Schlafli polytope. This group is denoted by $Aut(Sch)$.

In [DGL92] were found 26 orbits of non-cut extreme rays of $HYP_7$ by classifying the affine bases of the Schlafli polytope of the root lattice $E_6$. For every non-cut extreme ray $(\mathbb{R}^7, v)$ of $HYP_7$, there exist a facet-inducing inequality $f(x) \geq 0$ of $CUT_7$, which is non-hypermetric, so that $f(v) < 0$. This property establish a bijection between the 26 orbits of non-hypermetric facets of the cut cone $CUT_7$ (see [Gr90]) and the 26 orbits of non-cut extreme rays of $HYP_7$ and proves that $HYP_7$ has 29 orbits of extreme rays: three orbits of non-zero cuts and 26 orbits coming from $Sch$ (see [DD01]).

**Proposition 4** Let $B$ be an affine basis of Schlafli polytope; then

(i) The distance vector $d_B$ is incident to 20 hypermetric faces of $HYP_7$, which are all facets of $HYP_7$.

(ii) If $F$ is a face of $HYP_7$, containing the vector $d_B$, then it is non-degenerate and $|Ann(F)| = 7 + \text{corank}(F)$.

**Proof.** The Schlafli polytope is six-dimensional and has 27 vertices. So, for every affine basis $B$, the vector $d_B$ satisfies $H(b)d_B = 0$ for 27 different $b \in Ann(d_B)$; we write $Ann(d_B) = \{e_0, \ldots, e_6\} = \{b^1, \ldots, b^{20}\}$. On the other hand, it is known ([DGL92] and [DeLa97], p. 239), that the Schlafli polytope has rank 1. So, the rank of the matrix $[H(b^1), \ldots, H(b^{20})]$ must be $(6+1)/2 - 1 = 20$. So, the family $[H(b^i)]_{1 \leq i \leq 20}$ is linearly independent. If one of $H(b^i)$ is not a facet of $HYP_7$, then it can be expressed in terms of $[H(b^i)]_{j \neq i}$; this contradicts to linear independence and so, (i) holds.

The extreme ray $d_B$ is non-degenerate and is a sub-face of $F$; so, $F$ is also non-degenerate. Every hypermetric face, containing $F$, contains $d_B$; so, one has $Ann(F) - \{e_0, \ldots, e_6\} = \{b^1, \ldots, b^k\} \subset \{b^1, \ldots, b^{20}\}$. The linear independence of the family $[H(b^i)]_{1 \leq i \leq 20}$ implies that $k = \text{corank}(F)$ and so, (ii) holds.

Above Proposition is not true for the 56-vertex Gosset polytope ([Cox63]): the Gosset polytope has 374 orbits of affine bases. Each extreme ray, corresponding to an affine basis $\{v_0, \ldots, v_7\}$ of the Gosset polytope, is incident to 48 ($= 56 - 8$) hypermetric faces of $HYP_8$. But amongst these 48 face-defining inequalities, the number of facets varies from 27 ($= (7+1)/2 - 1$) to 41. See [DeLa97], p. 230 for general lower bounds (on the number of vertices of a Delaunay polytope) as a function of its rank.

**Theorem 1** All six-dimensional Delaunay polytopes are basic.
Proof. The simplex is a basic polytope, since there are 7 vertices and they form an affine basis.

Assume that $D$ is a non-simplicial Delaunay polytope of a lattice $L$ generated by the vectors $w_1, \ldots, w_6$. Denote by $V$ the volume of the simplex formed by the vectors $0, w_1, \ldots, w_6$. Take a family of 7 independent vertices $\mathcal{A} = \{v_0, \ldots, v_6\}$ in the vertex-set of $D$ and denote the volume of the corresponding simplex by $V'$.

In [RyBa98], it was proved that the relative volume $k = \frac{V'}{V}$ is 1, 2 or 3. If $k = 1$, then $\mathcal{A}$ is an affine basis and we are done. Assume now that $k > 1$, i.e. that $\mathcal{A}$ is not an affine basis. Then, there exists a vertex $v$ of $D$, which is written uniquely as $v = \sum_{i=0}^{6} b_i v_i$ with $b_i$ being fractional and $\sum_{i=0}^{6} b_i = 1$.

The distance vector $d_A$ satisfies $H(b_i) d_A = 0$ with $b_i = (b_{i0}, \ldots, b_{i6})$ being a fractional hypermetric vector. But one can express $H(b_i)$ as $\sum_{l=1}^{N} \lambda_l H(b'_l)$ with $\lambda_l > 0$ and $b'_l$ being a permutation of one of the following vectors (see [RyBa98]):

| Case $k = 2$ | Case $k = 3$ |
|--------------|--------------|
| $\frac{1}{3}(-1, -1, 1, 1, 1, 0)$ | $\frac{1}{3}(-1, -1, 1, 1, 2, 2)$ |
| $\frac{1}{3}(-1, -1, -1, 1, 1, 2)$ | $\frac{1}{3}(-1, -1, -1, 1, 1, 1)$ |
| $\frac{1}{3}(-2, -1, -1, 1, 1, 3)$ | $\frac{1}{3}(-2, -1, 1, 1, 1, 2)$ |
| $\frac{1}{3}(-2, -1, 1, 1, 1, 1)$ | |
| $\frac{1}{3}(-1, -1, -1, -1, 1, 2, 3)$ | |
| $\frac{1}{3}(-3, -1, 1, 1, 1, 1, 2)$ | |
| $(-1, 1, 1, 0, 0, 0)$ | |

Since $\lambda > 0$, one has $H(b'_l) d = 0$, i.e. the vectors $w'_l = \sum_{i=0}^{6} b'_l v_i$ with $1 \leq l \leq N$ are vertices of the Delaunay polytope $D$.

Since $b$ is fractional, at least one of $b'_l$ is fractional, say, $b'_{l0}$. But all fractional hypermetric vectors of above Table have one coordinate with absolute value equal to $\frac{1}{k}$, say, $|b'_{l0}| = \frac{1}{k}$. So, the family $\{v_0, \ldots, v_{i-1}, v'_i, v_{i+1}, \ldots, v_6\}$ defines a simplex of relative volume $k|b'_{l0}| = 1$, i.e. it is an affine basis.

Theorem 2 The 6241 combinatorial types of Delaunay polytopes are partitioned by rank in the following way:
| rank | Nr. in HYP7 | Nr. in CUT7 |
|------|-------------|-------------|
| 21   | 1 (simplex) | 0           |
| 20   | 9 (repart.) | 1           |
| 19   | 30          | 2           |
| 18   | 95          | 8           |
| 17   | 233         | 28          |
| 16   | 500         | 95          |
| 15   | 814         | 241         |
| 14   | 1092        | 434         |
| 13   | 1145        | 527         |
| 12   | 981         | 481         |
| 11   | 686         | 325         |
| 10   | 417         | 183         |
| 9    | 218         | 83          |
| 8    | 108         | 35          |
| 7    | 52          | 13          |
| 6    | 21          | 3           |
| 5    | 8           | 0           |
| 4    | 4           | 0           |
| 3    | 2           | 0           |
| 2    | 1           | 0           |
| 1    | 1 (Schlafli)| 0           |

**Proof.** The proof is purely computational and the method is described in next Section.

**Corollary 1.** All maximal six-dimensional Delaunay polytopes are: Schlafli polytope, 6-cube, half 6-cube and direct product of half 5-cube with 1-cube.

**Proof.** This result follows directly from the computation of above Theorem.

## 4 Computational methods

Our computation of combinatorial types of six-dimensional Delaunay polytopes used the face-lattice of $HYP_7$; combinatorial types of Delaunay polytopes of corank $i + 1$ were found from combinatorial types of Delaunay polytopes of corank $i$. We start from the list of combinatorial types of corank 1, i.e. the nine repartitioning polytopes. The plan of our computation was as follows:

(i) Take the list of combinatorial types of Delaunay polytopes of corank $i$.

(ii) For each of them, find all sub-faces, using our knowledge of facets and extreme rays of $HYP_7$; we obtain faces of corank $i + 1$.

(iii) For every face $F$ of corank $i + 1$, find extreme rays $(\mathbb{R}_+ f_i)_{1 \leq i \leq N}$, contained in it, and define $d = \sum_{i=1}^{N} f_i$. The distance vector $d$ is in the interior of $F$; so, using Proposition 1, one can test if this ray is non-degenerate or not and this tells us if the face is non-degenerate or not.

(iv) Find the classes of geometrical equivalence amongst the non-degenerate faces and so, the list of combinatorial types of corank $i + 1$. 

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Above procedure finds all combinatorial types of Delaunay polytopes from corank 1 (i.e. repartitioning polytopes) till corank 20 (i.e. the Schlafl i polytope). But in order to describe completely our method, we need to precise how we find the classes under geometrical equivalence.

The first algorithm for the problem of geometrical equivalence is the following: given two faces $F$ and $F'$, find all affine bases of $\text{Ann}(F)$ until one finds an equivalence $\phi_{F,F'}$ (see Definition 2). This method works for corank 1 or 2 and was used by Kononenko in the five-dimensional case. But the number of affine bases becomes too important to be workable in corank 3.

So, one needs another, more efficient method. The first idea is to split the set of faces of $HYP_7$ into two classes: faces, which contain (one or more) Schlafli extreme rays, and those, which are generated only by cuts.

**Definition 4** Let $F$ be a face of $HYP_7$, which contains a Schlafli extreme ray $d_B$ corresponding to an affine basis $B = \{v_0, \ldots, v_6\}$. Then, every $b \in \text{Ann}(F)$ defines a vertex $v = \sum_{i=0}^{6} b_i v_i$ of Sch. All such vertices are denoted by $S(F, d_B)$.

So, every combinatorial type of faces, which contains at least one Schlafli extreme ray, can be interpreted as a set of vertices of Sch.

**Proposition 5** Let $F$ and $F'$ be two faces, which contain at least one Schlafli extreme ray; then it holds:

(i) If $F$ and $F'$ are geometrically equivalent, then for every Schlafli extreme ray $d_B$ in $F$, there exist an affine basis $B'$ of Sch, such that $S(F, d_B) = S(F', d_{B'})$.

(ii) If $F$ and $F'$ contain a Schlafli extreme ray $d_B$ and $d_{B'}$, such that $S(F, d_B)$ is identical to $S(F', d_{B'})$ up to an element of $\text{Aut}(Sch)$, then $F$ and $F'$ are geometrically equivalent.

**Proof.** For every distance vector $d_B \in F$, one gets an identification of $\text{Ann}(F)$ with $S(F, d_B)$. If $F'$ is geometrically equivalent to $F$, then the vectors $e_0, \ldots, e_6$ in $\text{Ann}(F')$ are identified with vertices $v'_0, \ldots, v'_6$ in $S(F, d_B)$. These vertices form an affine basis $B'$ and one has $S(F', d_{B'}) = S(F, d_B)$; so, (i) holds.

Since $S(F, d_B)$ is isomorphic to $S(F', d_{B'})$ by an element of $\text{Aut}(Sch)$, one can, without loss of generality, assume that they are identical. Now, $B$ is identified with vertices $v_0, \ldots, v_6$ of $S(F, d_B)$ and $B'$ is identified with vertices $v'_0, \ldots, v'_6$ of $S(F, d_B)$. Then, the expression of $v'_j$ in terms of $v_i$ determine an affine basis of $\text{Ann}(F)$, for which the mapping $\phi_{F,F'}$ is well-defined and bijective.

Above Proposition express the geometrical equivalence in terms of the existence of an element of $\text{Aut}(Sch)$ mapping a set of vertices into another set of vertices. Those sets of vertices are identified with corresponding sets of vertices of the Schlafli graph (the Schlafli graph and the Schlafli polytope have the same symmetry group $\text{Aut}(Sch)$). So, the problem is expressed in graph-theoretic terms and can be solved, using, for example, the nauty program ([]MK[]). Therefore, one can build the geometrical equivalence classes.
Now, we extend above method to the case of faces generated by cuts. Let $F$ be a face, generated by cuts $\{\delta(S_i)\}_{1 \leq i \leq N}$; we first need to find $Ann(F)$, i.e. all vectors $b \in \mathbb{Z}^{n+1}$ with $\sum_{i=0}^{n} b_i = 1$, having $H(b)\delta(S_i) = 0$.

Those equations can be rewritten as $\sum_{x \in S_i} b_x = x_i$ with $x_i = 0$ or 1, i.e. a linear system in $b$. This linear system has rank $n + 1$, because of Proposition below; so, one can find the set $Ann(F)$ for every face, generated by cuts.

**Proposition 6** Let $\mathcal{F}$ be a face of $HY P_{n+1}$ generated by cuts $(\delta(S_i))_{1 \leq i \leq N}$. Then the following properties are equivalent:

(i) the face $F$ is non-degenerate and

(ii) the linear system, formed by the equations $\sum_{x \in S_i} b_x = 0$ and $\sum_{i=0}^{n} b_i = 0$, has solution set $\{0\}$.

**Proof.** If the face $F$ is degenerate, then there exist a vertex $v$, which can be expressed in two different forms $v = \sum_{i=0}^{n} b_i v_i = \sum_{i=0}^{n} b'_i v_i$. So, after denoting $\alpha_i = b_i - b'_i$, one gets $v = \sum_{i=0}^{n} (b_i + k\alpha_i) v_i$ with $k \in \mathbb{Z}$ and $b + k\alpha$ belongs to $Ann(F)$. Therefore, one gets $(b + k\alpha)(S_i) = 0$ or 1, and $\sum_{i=0}^{n} b_i + k\alpha_i = 1$ for all $k \in \mathbb{Z}$. This is possible only if $\sum_{x \in S_i} \alpha_x = 0$ and $\sum_{i=0}^{n} \alpha_i = 0$.

Let the solution set be non-zero, i.e. suppose that one can find an integer-valued non-zero solution $\alpha$. This implies that the vectors $e_0 + k\alpha$ belong to $Ann(F)$ for every $k \in \mathbb{Z}$. So, $Ann(F)$ is infinite and $F$ is degenerate.

If we write $Ann(F) = \{b^1, \ldots, b^v\}$, then every cut $\delta(S_i) \in F$ with $1 \leq i \leq N$ defines an Euclidean semi-metric on the set $\{e_0, \ldots, e_n\}$; this semi-metric can be uniquely extended to $Ann(F)$ by $\delta_i(b, b') = |b(S_i) - b'(S_i)|$.

So, to every face, generated by cuts, one can associate a set of semi-metrics on $Ann(F)$, which are, in fact, cut semi-metrics.

A combinatorial type of Delaunay polytope encodes all possible embeddings of $Ann(F)$ into the vertex-set $V$ of a Delaunay polytope of a lattice. These embeddings are completely described by the distance vector $d_V$ on their vertices. This distance vector is expressed as $\sum_{i=1}^{N} \lambda_i \delta_i$ with $\lambda_i > 0$.

Therefore, the combinatorial type of a face, generated by cuts, corresponds to the description of all semi-metrics on this set. This information can be expressed in graph-theoretic terms; so, we can test if two faces, generated by cuts, are isomorphic, using the nauty program ([MK]). So, again one can build the geometrical equivalence classes and our method is completely described.

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