ANTIFLAG TRANSITIVE COLLINEATION GROUPS

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Abstract. We present known results concerning antiflag transitive collineation groups of finite projective spaces and finite polar spaces.

Preface

This is a revision of a paper by P. J. Cameron and W. M. Kantor, “2-Transitive and Antiflag Transitive Collineation Groups of Finite Projective Spaces”, J. Algebra 60 (1979) 384–422. All theorems in that paper are corrected by adding further imprimitive antiflag transitive subgroups to various statements:

- $SL(\frac{1}{2}n, 16)$ or $Sp(\frac{1}{2}n, 16) < G < \Gamma L(n, 4)$, or $G_2(16) < G < \Gamma L(12, 4)$;
- $Sp(\frac{1}{2}n, 16) < G < \Gamma Sp(n, 4) \cong \Gamma O(n + 1, 4)$, or $G_2(16) < G < \Gamma Sp(12, 4) \cong \Gamma O(13, 4)$;
- $SU(\frac{1}{2}n, 4) < G < \Gamma O^\epsilon(n, 4)$, where $\epsilon = (-)^{\frac{1}{2}n}$; and
- $G \cong A_9$ inside $\Omega^+(8, 2)$.

This revision uses the same methodology as the original paper. In particular, it does not involve more recent group theory.

The actual results can be deduced from more recent results that depend on the Classification of the Finite Simple Groups. Liebeck [29] completed results of Hering concerning transitive finite linear groups, so that paper implies Theorems I-III. The transitivity results in Theorems IV-V are implicitly very special cases of Guralnick-Penttila-Praeger-Saxl [30].

However, I strongly believe that the aforementioned Classification should not be invoked when it is not needed. Moreover, there are surprising and entertaining parts of the original proofs (especially the appearance of generalized hexagons). The notation, methodology and relatively minimal background of the original paper are retained here. As much as possible the original paper has been left unchanged; for example, the numbering of intermediate results is not altered.

1. Introduction

An unpublished result of Perin [20] states that a subgroup of $\Gamma L(n, q)$, $n \geq 3$, that induces a primitive rank 3 group of even order on the set of points of $PG(n - 1, q)$, necessarily preserves a symplectic polarity. (Such groups are essentially known, if $q > 3$, by another theorem of Perin [19].) The present paper extends both Perin’s result and his method, in order to deal with some familiar problems concerning collineation groups of finite projective spaces; among these, 2-transitive collineation groups [25], and both the case of semilinear groups and the case $q \leq 3$ of Perin’s theorem [19].

An antiflag is an ordered pair consisting of a hyperplane and a point not on it; if the underlying vector space is endowed with a symplectic, unitary or orthogonal
geometry, both the point and the pole of the hyperplane are assumed to be isotropic or singular. Our main results are the following four theorems.

**Theorem I.** If \( G \leq \Gamma L(n, q) \), \( n \geq 3 \), and \( G \) is 2-transitive on the set of points of \( PG(n-1, q) \), then either \( G \geq SL(n, q) \), or \( G \) is \( A_7 \) inside \( SL(4, 2) \).

**Theorem II.** If \( G \leq \Gamma L(n, q) \) and \( G \) is transitive on antiflags and primitive but not 2-transitive on points, then \( G \) preserves a symplectic polarity, and one of the following holds:

(i) \( G \geq Sp(n, q) \);
(ii) \( G \) is \( A_6 \) inside \( Sp(4, 2) \); or
(iii) \( G \geq G_2(q) \), \( q \) even, and \( G \) acts on the generalized hexagon associated with \( G_2(q) \), which is itself embedded naturally in \( PG(5, q) \).

**Theorem III.** If \( G \leq \Gamma L(n, q) \) and \( G \) is transitive on antiflags and imprimitive on points, then \( q = 2 \) or 4 and \( \Gamma L(\frac{1}{2}n, q^2) \geq G \geq SL(\frac{1}{2}n, q^2) \), \( Sp(\frac{1}{2}n, q^2) \), or \( G_2(q^2) \) (with \( n = 12 \)) \(^1\) In each case, \( G \) is embedded naturally in \( \Gamma L(n, q) \).

**Theorem IV.** If \( G \leq \Gamma Sp(n, q) \), \( \Gamma O^\pm(n, q) \) or \( \Gamma U(n, q) \), for a classical geometry of rank at least 3, and \( G \) is transitive on antiflags, then one of the following holds (and the embedding of \( G \) is the natural one):

(i) \( G \geq Sp(n, q) \), \( O^\pm(n, q) \) resp. \( SU(n, q) \);
(ii) \( G \geq G_2(q) \) inside \( \Gamma O(7, q) \) (or \( \Gamma Sp(6, q) \), \( q \) even);
(iii) \( \Omega(7, q) \leq G/Z(G) \leq \Gamma O^+(8, q) \), with \( G/Z(G) \) conjugate in \( Aut(\mathcal{P}O^+(8, q)) \) to a group fixing a nonsingular 1-space;
(iv) For \( q = 2 \) or 4, \( Sp(\frac{1}{2}n, q^2) \leq G \leq \Gamma Sp(n, q) \leq \Gamma O(n + 1, q) \); \(^2\)
(v) For \( q = 2 \) or 4, \( G_2(q^2) \leq G \leq \Gamma Sp(12, q) \leq \Gamma O(13, q) \); \(^2\)
(vi) For \( q = 2 \) or 4, \( SU(\frac{1}{2}n, q) \leq G \leq \Gamma O^+(n, q) \), where \( \epsilon = (-)^n \); \(^3\) or
(vii) \( G \leq A_9 \) inside \( \Omega^+(8, 2) \) \(^3\).

Theorem I solves a problem posed by Hall and Wagner \([25]\), which has been studied by Higman \([8, 10]\), Perin \([19]\), Kantor \([13]\) and Kornya \([15]\). An independent and alternative approach to this theorem is given by Orchel \([16]\); we are grateful to Orchel for sending us a copy of his thesis.

If \( G \) is 2-transitive, then \( G \) is antiflag transitive; and also \( G^H_H \) is antiflag transitive for each hyperplane \( H \). This elementary fact allows us to use induction. (Indeed, Theorems I-III are proved simultaneously by induction in Part I of this paper.) The groups in Theorem III and Theorem IV(iv-vi) must contain both the indicated quasisimple group and \( Aut(GF(q^2)) \). Another problem, solved in Theorems II and IV, is that of primitive rank 3 subgroups of classical groups. This was posed by Higman and McLaughlin \([11]\), and solved by Perin \([19]\) (for linear groups) and Kantor and Liebler \([14]\) except in the cases \( Sp(2m, 2) \equiv \Omega(2m + 1, 2) \) and \( Sp(2m, 3) \).

Here, induction is made possible by fact that the stabilizer of a point \( x \) is antiflag transitive on \( x^+ / x \).

The striking occurrence of \( G_2(q) \) in these theorems is related to a crucial element of our approach. This case is obtained from a general embedding theorem for

\(^1\) We are grateful to Nick Inglis and Jan Saxl for pointing out that the case \( \Gamma L(\frac{1}{2}n, 16) \) had been omitted. This led to the other groups over \( GF(16) \).
\(^2\) The case \( q = 4 \) had been omitted.
\(^3\) This case had been omitted.
metrically regular graphs (3.1), in which the Feit-Higman theorem [7] on generalized polygons arises unexpectedly but naturally. Other familiar geometric objects and theorems come into play later on: the characterizations of projective spaces due to Veblen and Young [24] and Ostrom and Wagner [18], as well as translation planes, arise in Theorem III, while Tits’ classification of polar spaces [23] and the triality automorphism of $P\Omega^+(8, q)$ are used for Theorem IV.

All of the proofs require familiarity with the geometry of the classical groups. On the other hand, group-theoretic classification theorems have been avoided. Moreover, knowledge of $G_2(q)$ is not assumed for Theorem I, and what is required for Theorems II-IV is contained in the Appendix, where we have given a new and elementary proof of the existence of the generalized hexagons of type $G_2(q)$.

This paper began as an attempt to extend Perin’s result [20] to rank 4 subgroups of classical groups. As in Perin [19], one case with $q = 2$ is left open:

**Theorem V.** Suppose $G \leq \Gamma Sp(n, q)$ ($n \geq 6$), $\Gamma O^\pm(n, q)$ ($n \geq 7$), or $\Gamma U(n, q)$ ($n \geq 6$). If $G$ induces a primitive rank 4 group on the set of isotropic or singular points, then one of the following holds:

(i) $G \geq G_2(q)$ is embedded naturally in $\Gamma O(7, q)$ (or $\Gamma Sp(6, q)$, $q$ even);
(ii) $G \geq \Omega(7, q)$, $q$ even, or $2.\Omega(7, q)$, $q$ odd, each embedded irreducibly in $\Gamma O^+(8, q)$; or
(iii) $G \leq O^+(2m, 2)$, and $G$ is transitive on the pairs $(x, L)$ with $L$ a totally singular line and $x$ a point of $L$.

The examples (ii) (and (iii) in Theorem IV) are obtained by applying the triality automorphism to the more natural $\Omega(7, q)$ inside $P\Omega^+(8, q)$. As for (iii), examples are $A_7$ and $S_7$ inside $O^+(6, 2)$.

Other results in a similar spirit are given in Section 8, as corollaries to Theorem I. Some further results are of interest independent of their application to the above theorems. A general result on embedding metrically regular graphs in projective spaces is proved in Section 3; this is used several times, and is crucial for all of the theorems. Theorem 10.3 characterizes nonsingular quadrics of dimension $2m - 1$ contained in an $O^+(2m, q)$ quadric for $m \geq 3$. In Section 12, parameter restrictions are obtained for rank 4 subgroups of rank 3 groups (and their combinatorial analogues). Finally, the Appendix gives an elementary construction and characterization of the $G_2(q)$ hexagon.

The paper falls into two parts. The first (Sections 2-8) deals with antiflag transitive collineation groups of projective spaces (Theorems I-III); we note that Sections 3 and 5, on the primitive, not 2-transitive case, are virtually self-contained. The second part (Sections 9-14) contains the proofs of Theorems IV and V, concerning polar spaces.

I. THEOREMS I-III

2. PRELIMINARIES

A point (hyperplane) of a vector space $V$ is a subspace of dimension 1 (codimension 1). If $V$ is $n$-dimensional over $GF(q)$, the set of points (equipped with the structure of projective geometry) is denoted by $PG(n - 1, q)$; but in this paper, its dimension will always be $n$. The notation $SL(V) = SL(n, q)$, $GL(n, q)$ and $\Gamma L(n, q)$ is standard.
If, in addition, $V$ is equipped with a symplectic, unitary or orthogonal geometry, then $ΓSp(n, q)$, $ΓU(n, q)$ and $ΓO^±(n, q)$ denote the groups of semilinear maps preserving the geometry projectively. For example, $ΓO^±(n, q)$ consists of all invertible semilinear maps $g$ such that $φ(vθ) = cφ(v)θ$ for all $v ∈ V$, where $φ$ is the quadratic form defining the geometry, $c$ is a scalar, and $θ$ is a field automorphism. The groups $Sp(n, q)$, $SU(n, q)$ and $Ω^±(n, q)$ are defined as usual. We use totally isotropic or totally singular (abbreviated t.i. or t.s.) subspaces of these geometries. There is some ambiguity in the terminology “t.i. or t.s. subspace” since orthogonal geometries have both types of subspaces in characteristic 2; but in this case we always refer to t.s. subspaces. We will occasionally require the fact that $Sp(2n, q) ∼= Ω(2n + 1, q)$ when $q$ is even. (Explicitly, if $V$ is the natural $Sp(2n, q)$-module, then there is a nondegenerate $2n + 1$-dimensional orthogonal space $V'$ such that $V'/\text{rad} V = V$, with the natural map $V \to V'$ inducing a bijection between singular and isotropic points.) The reader is referred to Dieudonné [6] for further information concerning all of these groups.

Points will be denoted $x, y, z$, lines $L, L'$ and hyperplanes $H, H'$. We will generally identify a subspace $Δ$ of $V$ with its set of points; $|Δ|$ denotes its number of points, and $x ∈ Δ$ will be used instead of $x ⊆ Δ$. Similarly, for subspaces $Δ$ and $Σ$, $Δ − Σ$ denotes the set of points in $Δ$ but not $Σ$. On the other hand, the dimension $\text{dim}_F Δ$ of a subspace denotes the vector space dimension. If $A ⊆ GL(V)$ and $W$ is a subspace of $V$ then $C_W(A) = \{ w ∈ W \mid w^a = w, a ∈ A \}$ and $[W, A] = \{ w^a - w \mid w ∈ W, a ∈ A \}$ are vector subspaces that will be studied as sets of points; we expect that the context will make it clear whether a subspace is being viewed as a set of points after being obtained as a set of vectors.

We generally consider semilinear groups; but when discussing transitivity we always consider the induced (projective) group on 1-spaces (points) rather than transitivity on vectors. If $Δ$ is any subset of $V$, then $G_Δ$ and $C_G(Δ)$ are respectively the setwise and vector-wise stabilizers of $Δ$ in the semilinear group $G$: $G_ΔΣ = G_Δ ∩ G_Σ$. Moreover, $G_Δ^*$ is the semilinear group induced on $Δ$ if $Δ$ is a subspace; this group will usually be viewed projectively. Similarly, if $x ∈ H$, then $G_H^{xH}$ is the group induced by $G_{xH}$ on the space $H/x$.

The rank of a transitive permutation group is the total number of orbits of the stabilizer of a point.

The remainder of this section lists further definitions and results required in the proofs of Theorems I-V.

**Theorem 2.1** (Ostrom-Wagner [18], Ostrom [17]). If a projective plane $P$ of prime power order $q$ admits a collineation group $G$ transitive on non-incident point-line pairs, then $P$ is desarguesian and $G ≥ PSL(3, q)$.

Of course, (2.1) is true without the prime power assumption, but we only need the stated case, which is much easier to prove. The next result is needed for (2.1), and is also used elsewhere in our argument.

**Theorem 2.2** ([4, pp. 122, 130-34]). Let $A$ be an affine translation plane of order $q$, $L$ a line, $x ∈ L$, and $E$ the group of elations with center $x$ and axis $L$. Then

(i) $E$ is semiregular on the set of lines different from $L$ on $x$; and
(ii) If $|E| = q$ for each $L$ and $x$, then $A$ is desarguesian.
Additional, more elementary results concerning translation planes will also be required; the reader is referred to Dembowski [4, Chap. 4] for further information concerning perspectivities and Baer involutions.

Consider next a geometry $\mathcal{G}$ of points, with certain subsets called “lines”, such that any two points are on at most one line, each line has at least three points, and each point is on at least three lines. Call $\mathcal{P}$ and $\mathcal{L}$ the sets of points and lines. If $a, b \in P \cup L$, the distance $\partial(a, b)$ between them is the smallest number $k$ for which there is a sequence $a = a_0, a_1, \ldots, a_k = b$, with each $a_i \in P \cup L$ and $a_i$ incident with $a_{i+1}$ for $i = 0, \ldots, k - 1$. Such a sequence is called a “path” from $a$ to $b$. Now $\mathcal{G}$ is a generalized $n$-gon $(n \geq 3)$ if

(i) whenever $\partial(a, b) < n$ there is a unique shortest path from $a$ to $b$;

(ii) for all $a$ and $b$, $\partial(a, b) \leq n$; and

(iii) there exist $a$ and $b$ with $\partial(a, b) = n$.

A generalized $n$-gon has parameters $s, t$ if each line has exactly $s + 1$ points and each point is on exactly $t + 1$ lines.

**Theorem 2.3** (Feit-Higman [7]). Generalized $n$-gons can exist only for $n = 3, 4, 6$ or $8$; those with $n = 8$ cannot have parameters $s, s$.

Generalized quadrangles enter our considerations as the geometries of points and lines in low-dimensional symplectic, unitary, and orthogonal geometries. Generalized hexagons are much less familiar; the ones we need are discussed in the Appendix (see also Sections 3, 5 below).

Generalized $n$-gons are special cases of metrically regular graphs. Let $\Gamma$ be a connected graph defined on a set $X$ of vertices. If $x, y \in X$, let $d(x, y)$ denote the distance between them. Let $d$ be the diameter, and $\Gamma_i(x)$ the set of points at distance $i$ from $x$, for $0 \leq i \leq d$. Then $\Gamma$ is metrically regular if

(i) $|\Gamma_i(x)|$ depends only on $i$, not on $x$; and

(ii) if $d(x, y) = i$, the numbers of points at distance $1$ from $x$ and distance $i - 1$ (resp. $i, i + 1$) from $y$ depend only on $i$, and not on $x$ and $y$.

(Condition (i) follows from (ii) here.)

If $\mathcal{G}$ is a geometry as previously defined, its point graph $\Gamma$ is obtained by joining two points of $\mathcal{G}$ by an edge precisely when they are distinct and collinear. This graph may be metrically regular; for example, it is so when $\mathcal{G}$ is a generalized $n$-gon. (Here the distances $d$ and $\partial$ in graph and geometry are related by $d(x, y) = \frac{1}{2} \partial(x, y)$ for $x, y \in \mathcal{P}$.)

If $n$ is an integer then $n_p$ denotes the largest power of $p$ dividing $n$ (where $p$, as always, is a prime).

If $q$ is a power $p^r$ of $p$, and $k \geq 2$, a primitive divisor of $q^k - 1$ is a prime $r \mid q^k - 1$ such that $r \not\mid p^i - 1$ for $1 < p^i < q^k$. Note that $r \equiv 1 \pmod{ek}$, by Fermat’s theorem.

**Theorem 2.4** (Zsigmondy [28]). If $q > 1$ is a power of $p$ and $k > 1$, then $q^k - 1$ has a primitive divisor unless either

(i) $k = 2$ and $q$ is a Mersenne prime, or

(ii) $q^k = 64$.

3. Embedding Metrically regular graphs in projective spaces

In this section we will prove a general result concerning certain embeddings in projective spaces. Let $\mathcal{G}$ be a geometry, with point set $\Omega$ and point graph $\Gamma$. For
$x \in \Omega$, let $W_i(x)$ be the set of points distant at most $i$ from $x$. We assume the following axioms (for all $x \in \Omega$):

(a) $\Omega$ is a set of points spanning $PG(n-1, q)$;
(b) each line $L$ of $\mathcal{G}$ (or $\mathcal{G}$-line) is a line of $PG(n-1, q)$;
(c) $\Omega$ is the union of the set of $\mathcal{G}$-lines;
(d) $\Gamma$ is metrically regular with diameter $d \geq 2$;
(e) $W_1(x)$ is a subspace of $PG(n-1, q)$;
(f) $W_i(x) = \Omega \cap U_i(x)$ for some subspace $U_i(x)$ for each $i$; and
(g) $|W_2(x)| = (q^h - 1)/(q - 1)$ for some $h$.

Note that (a)-(d) are among the embedding hypotheses in Buekenhout-Lef`evre [1].

In (3.1) and (3.2) we will determine all geometries satisfying (a)-(g). For Theorem I, a complete classification is not required; the weaker result (3.1) suffices.

**Theorem 3.1.** If $\mathcal{G}$ satisfies (a)-(g), then either

(i) $d = 2$ and $\mathcal{G}$ consists of the totally isotropic points and lines of a symplectic polarity $x \leftrightarrow W_1(x)$; or

(ii) $d = 3$, $\mathcal{G}$ is a generalized hexagon with parameters $q, q$, and each $W_1(x)$ has dimension $3$. (Moreover, if $W_2(x)$ and $W_3(x)$ are subspaces for all $x$, then $n = 6$ and $x \leftrightarrow W_2(x)$ is a symplectic polarity.)

**Proof.** Set $m = \dim W_1(x)$ (recalling from Section 2 that “dim” means vector space dimension). If $d(x, y) = i \geq 1$, let

$$e_i = \dim W_1(x) \cap W_{i-1}(y),$$

$$f_i = \dim W_1(x) \cap W_i(y).$$

(Note that both $W_1(x) \cap W_{i-1}(y)$ and $W_1(x) \cap W_i(y)$ are subspaces. For if $W_j(y) = \Omega \cap U_j(y)$, then $W_1(x) \cap W_j(y) = W_1(x) \cap \Omega \cap U_j(y) = W_1(x) \cap U_j(y)$.) These dimensions depend only on $i$, not $x$ or $y$. For, if $\Gamma_i(x) = W_i(x) - W_{i-1}(x)$ is the set of points at distance $i$ from $x$, then

$$|\Gamma_1(x) \cap \Gamma_{i-1}(y)| = (q^{e_i} - 1)/(q - 1),$$

$$|\Gamma_1(x) \cap \Gamma_i(y)| = (q^{f_i} - 1)/(q - 1) - (q^{e_i} - 1)/(q - 1) - 1,$$

and

$$|\Gamma_1(x) \cap \Gamma_{i+1}(y)| = (q^m - 1)/(q - 1) - (q^{f_i} - 1)/(q - 1)$$

(provided also that $i < d$). By (g), $|\Gamma_2(x)| = (q^h - q^m)/(q - 1)$. By (d) these imply the stated independence.

Counting pairs $(y, z)$ with $d(x, y) = 1 = d(y, z)$ and $d(x, z) = 2$ yields

$$|\Gamma_1(x)||\Gamma_2(x) \cap \Gamma_1(y)| = |\Gamma_2(x)||\Gamma_1(x) \cap \Gamma_1(z)|,$$

whence $(q^m - q)(q^m - q^{f_1}) = (q^h - q^m)(q^{e_2} - 1)$. Equating powers of $q$ yields $1 + f_1 = m$. There are then two possibilities:

(i) $m - 1 = e_2, 1 = m - f_1 = h - m$; or

(ii) $m - 1 = h - m, 1 = m - f_1 = e_2$.

Suppose (i) holds. Each point is on exactly $(q^{m-1} - 1)/(q - 1) = (q^{e_2} - 1)/(q - 1)$ $\mathcal{G}$-lines. Thus, if $d(x, z) = 2$, each of the $\mathcal{G}$-lines on $z$ contains a point of the $e_2$-space $W_1(x) \cap W_1(z)$. Consequently, the graph has diameter $d = 2$. Moreover, $\Omega$ is a subspace. (For if $x$ and $y$ are distinct points of $\Omega$ but $(x, y)$ is not a $\mathcal{G}$-line, then there is a point $z \in W_1(x) \cap W_1(y)$; then $x$ and $y$ are in the subspace $W_1(z)$, then $x \leftrightarrow W_2(z)$ is a symplectic polarity.)
all of whose points are in $\Omega$.) Now (a) yields $h = n$, so $m = n - 1$ and $W_1(x)$ is a hyperplane. Since $y \in W_1(x)$ implies that $x \in W_1(y)$, it follows that $x \leftrightarrow W_1(x)$ is a symplectic polarity, so (3.1i) holds.

From now on, assume that case (ii) occurs. Since $e_2 = 1$ there is a unique point joined to two given points at distance 2. The restriction of the relation “joined or equal” to $G_1(x)$ is thus an equivalence relation, so $G_1(x)$ is a disjoint union of complete graphs, each of size $(q^{f_1} - q^{e_1})/(q - 1) = q(q^{m-2} - 1)/(q - 1)$. Since $|G_1(x)| = q(q^{m-1} - 1)/(q - 1)$, this implies that $m - 2 \mid m - 1$, whence $m = 3$. Then $f_1 = m - 1 = 2$ (and of course $e_2 = 1$).

We next determine the sequences $\{e_i\}$, $\{f_i\}$. Both are nondecreasing: if $d(x, y) = i$, $d(y, z) = 1$ and $d(x, z) = i + 1 \leq d$, then $W_1(x) \cap W_{i-1}(y) \subseteq W_1(x) \cap W_1(z)$ and $W_1(x) \cap W_i(y) \subseteq W_1(x) \cap W_{i+1}(z)$. Also, $e_i < f_i$ since $W_1(x) \cap W_{i-1}(y) \subset W_1(x) \cap W_i(y)$. If $f_i = 3$ for some $i$, then $G_1(x) \subseteq W_3(y)$ when $d(x, y) = i$, and so $i = d$; and conversely $f_d = \dim(W_1(x) \cap W_d(y)) = \dim W_1(x) = 3$. Thus, $e_i = 1$ and $f_i = 2$ for $i < d$, while $f_d = 3$ and $e_d = 1$ or 2.

We will show that $\mathcal{G}$ is a generalized $(2d + 1)$-gon or $2d$-gon (with parameters $q, q$) according as $e_d = 1$ or $e_d = 2$. Thus, we must verify axioms (i)–(iii) given in Section 2, where $\partial$ was defined. For convenience, we separate the two cases.

Case $e_d = 1$. Since $e_1 = 1$ for all $i \geq 1$ there is a unique shortest path joining any two points. Also, a $\mathcal{G}$-line $L$ contains a unique point nearest $x$, unless $L \subseteq \Gamma_d(x)$. (For, if $y \in L$ with $d(x, y) = i < d$ minimal, and $u \in W_1(y) \cap W_{i-1}(x)$, then $L \neq \langle y, u \rangle = W_1(y) \cap W_i(x)$ since $f_i = 2$. If $L$ contains a second point in $W_i(x)$ then $L \subseteq U_i(x) \cap \Omega = W_i(x)$ by (f), whereas $L \neq W_1(y) \cap W_i(x)$. Thus, there is a unique shortest path between $x$ and $L$ if $\partial(x, L) < 2d + 1$ (since then $L \not\subseteq \Gamma_d(x)$).

Let $L$ and $L'$ be two $\mathcal{G}$-lines. If $L' \not\subseteq \Gamma_d(x)$ for all $x \in L$ and $L \not\subseteq \Gamma_d(x')$ for all $x' \in L'$, then there is a unique shortest path between $L$ and $L'$. (By the preceding paragraph, two shortest paths would go between points $x_j$ of $L$ and $x'_j$ of $L'$ for $j = 1, 2$, where $x_1 \neq x'_1$, $x'_2 \neq x_2'$, and hence produce two shortest paths from $x_1$ to $x'_2$.)

Suppose $L' \subseteq \Gamma_d(x)$ for some $x \in L$. Then there is a unique shortest path from $x$ to each of the $q + 1$ points of $L'$, no two such paths using the same $\mathcal{G}$-line through $x$ (since this would produce a point $y \in W_1(x)$ with $\partial(y, L') < 2d$ and two shortest paths from $y$ to $L'$). Then these paths use all $q + 1$ $\mathcal{G}$-lines through $x$, and hence $L$ must occur among them. Thus, $\partial(L, L') = 2d$ and a unique shortest path again exists from $L$ to $L'$. Consequently, axioms (i) and (ii) hold with $n = 2d + 1$. Since $f_d = 3$ and $e_d = 1$, so does axiom (iii) (using $y$ and any of $q$ $\mathcal{G}$-lines on $x$ if $d(x, y) = d$).

Case $e_d = 2$. This time, there is a unique shortest path from $x$ to $x'$ unless $x' \in \Gamma_d(x)$. As above, any $\mathcal{G}$-line $L$ contains a unique point closest to $x$, and there is a unique shortest path from $x$ to $L$. (For, it is not possible for a closest point $y \in L$ to have distance $d$ from $x$, as this would imply that $W_1(y) \cap W_{d-1}(x)$ has dimension $e_d = 2$ and hence would meet $L \subset W_1(y)$ at a point at distance $d - 1$ from $x$.) Finally, let $L$ and $L'$ be $\mathcal{G}$-lines with $\partial(L, L') < 2d$. Then only one shortest path can exist between $L$ and $L'$: as above, two would go between points $x_j$ of $L$ and $x'_j$ of $L'$ for $j = 1, 2$, where $x_1 \neq x'_2$, $x'_2 \neq x_2'$, and hence produce two shortest paths from $x_1$ to $x'_2$. Thus, as above axioms (i)-(iii) again hold.

Since $e_2 = 1$, we have $d \geq 3$. The Feit-Higman Theorem (2.3) now shows that $d = 3$ and $e_3 = 2$. 
It remains to prove the parenthetical remark in (3.1i). A generalized hexagon with parameters $q, q$ has $|\Omega| = (q^6 - 1)/(q - 1)$ points. Since $\Omega = W_3(x)$ is a subspace we have $n = 6$. Since $2 = m - 1 = h - m$ it follows that $W_2(x)$ is a hyperplane and $x \leftrightarrow W_2(x)$ is a symplectic polarity, as required.

**Theorem 3.2.** Suppose the hypotheses and conclusions of (3.1ii) hold (but not necessarily the hypothesis in the parenthetical portion). Then

(i) If $n = 6$, then $q$ is even; and

(ii) otherwise $n = 7$ and $\Omega$ is the set of singular points of a geometry of type $O(7, q)$.

In either case the embedding of $\mathcal{G}$ is unique.

We defer the proof to (A.1iii) in the Appendix.

4. A REFORMULATION OF ANTIFLAG TRANSITIVITY

Sometimes the following criterion for antiflag transitivity is convenient.

**Lemma 4.1.** A subgroup $G$ of $\Gamma L(n, q)$ is antiflag transitive if and only if $G^L_x$ is 2-transitive for every line $L$.

**Proof.** Suppose $G_x$ has $s$ orbits of hyperplanes on $x$, $t$ orbits of hyperplanes not on $x$, and $s' + 1$ point-orbits in all. Then $s + t = s' + 1$, and $G_x$ has $s$ orbits of lines through $x$. Each such line-orbit defines at least one point-orbit other than $\{x\}$. Thus $t - 1 = s' - s \geq 0$, with equality if and only if $G^L_x$ is transitive for every line $L$ through $x$, as required.

From Dickson’s list of subgroups of $SL(2, q)$ [5, Chap. 12], it is seen that only when $q = 4$ is there a 2-transitive subgroup $H$ of $\Gamma L(2, q)$ for which $H \cap GL(2, q)$ is not 2-transitive. We deduce the following.

**Corollary 4.2.** If $q \neq 4$ and $G \leq \Gamma L(n, q)$ is antiflag transitive then so is $G \cap GL(n, q)$, and $G^L_x \cap GL(2, q) \geq SL(2, q)$ for any line $L$.

5. THE HEART OF THEOREM II

Suppose $G \leq \Gamma L(n, q)$ is antiflag transitive but not 2-transitive on the points of $V$. The following lemma incorporates Perin’s main idea [20].

**Lemma 5.1.** If $x$ is a point, then there is a subspace $W(x)$ (different from $x$ and $V$) containing $x$, such that $G_x$ fixes $W(x)$ and is transitive on $V - W(x)$.

**Proof.** A Sylow $p$-subgroup of $G$ fixes a hyperplane $H$ and a point $x \in H$, and is transitive on $V - H$. Then

$$W(x) = \bigcap \{H^g \mid g \in G_x\}$$

is a $G_x$-invariant subspace; $G_x$ is transitive on the pairs $(H^g, y)$ for $g \in G_x, y \notin H^g$, and hence is transitive on $V - W(x)$. Finally, $W(x) \neq x$ since $G$ is not 2-transitive.

**Theorem 5.2.** Suppose $G \leq \Gamma L(n, q)$ is primitive but not 2-transitive on points, and is antiflag transitive. Then $G$ preserves a symplectic polarity, and either

(i) $G$ has rank 3 on points; or

(ii) $G$ has rank 4 on points, $G \leq \Gamma Sp(6, q)$, and $G$ acts on a generalized hexagon with parameters $q, q$ consisting of the points and some of the totally isotropic lines of $V$.  


The proof involves an iteration of (5.1), followed by (3.1). Let \( d + 1 \) denote the rank of \( G \) in its action on points.

**Lemma 5.3.** There are subspaces

\[
x = W_0(x) \subset W_1(x) \subset W_2(x) \subset \cdots \subset W_{d-1}(x) \subset W_d(x) = V
\]

with the properties

(i) \( G_x \) fixes \( W_i(x) \) and is transitive on \( W_i(x) - W_{i-1}(x) \) for \( 1 \leq i \leq d \);

(ii) if \( y \in W_i(x) \) and \( 0 \leq i \leq d-1 \), then \( W_i(y) \subseteq W_{i+1}(x) \);

(iii) \( W_i(x^g) = W_i(x)^g \) for all \( g \in G \); and

(iv) \( d > 1 \).

**Proof.** Set \( W_d(x) = V \) and \( W_{d-1}(x) = W(x) \) (cf. (5.1)), where \( d > 1 \) by hypothesis. Since \( W_d(x) - W_{d-1}(x) \) is the largest orbit of \( G_x \), certainly \( W_{d-1}(x^g) = W_{d-1}(x)^g \) for all \( g \in G \).

Now proceed by “backwards induction”. Suppose \( W_j(x) \) has been defined for \( j = i+1, \ldots, d \), and behaves as in (i), where \( i + 1 < d \); we need to define \( W_i(x) \). Set \( m_{i+1} = \dim W_{i+1}(x) \). A Sylow \( p \)-subgroup \( P \) of \( G_x \) fixes a line \( L \) on \( x \); since all \( P \)-orbits on \( V - W_{i+1}(x) \) have length at least \( q^{m_i+1} \), necessarily \( L \subseteq W_{i+1}(x) \). If \( y \in L - x \) then all \( P \)-orbits on \( W_{i+1}(y) - W_i(x) \) have length at least \( q^{m_{i+1}-1} \). (By primitivity, \( W_{i+1}(y) \neq W_i(x) \).) It follows that \( W_{i+1}(x) \cap W_{i+1}(y) \) is a hyperplane of \( W_{i+1}(x) \), and that \( G_{xy} \) is transitive on \( W_{i+1}(y) - W_{i+1}(x) \). Then

\[
W_i(y) = \bigcap \{ W_{i+1}(x)^g \mid g \in G_y \}
\]

is a subspace of \( W_{i+1}(y) \), and \( G_y \) is transitive on \( W_{i+1}(y) - W_i(y) \). Then (iii) holds, since \( G_y \) has only one orbit of size \( |W_{i+1}(x) - W_i(x)| \).

This process terminates when \( W_0(x) = x \). Then \( W_1(x) - x \) consists of all points \( y \) for which \( \langle x, y \rangle \) is fixed by some Sylow \( p \)-subgroup of \( G \). Now (ii) follows from the definition of \( W_i(y) \). Thus, all parts of (5.3) are proved.

Let \( \mathcal{G} \) be the geometry with line set \( \{ \langle x, y \rangle \mid x \neq y \in W_1(x) \} \), and \( \Gamma \) its point graph. By (5.3ii) and induction on \( i \), we see that \( W_i(x) \) is the set of points at distance at most \( i \) from \( x \) (relative to the metric \( d \) in \( \Gamma \)). Also, \( G \) is transitive on the pairs \( (x, y) \) with \( y \in W_{i+1}(x) - W_i(x) \) for each \( i \). Consequently, \( \Gamma \) is metrically regular, and (3.1) applies. Since all \( W_i(x) \) are subspaces, (5.2) follows.

By (3.2), the generalized hexagon in (5.2ii) must be the one associated with \( G_2(q) \). However, as stated in Section 1, we will make the proof of Theorem I, and most of Theorems II and III, independent of the known existence and uniqueness of the \( G_2(q) \) hexagon. The required information is easily proved (frequently in the spirit of other of our arguments), and is collected in the following lemma (where \( q \) may be even or odd).

**Lemma 5.4.** If \( G \) is as in (5.2ii), then the following statements hold:

(a) \( G \) has exactly two orbits of t.i. lines;

(b) \( G \) has exactly two orbits of t.i. planes;

(c) there is a t.i. plane \( E \) such that \( G_E^r \cong SL(3, q) \);

(d) there is an element \( t \in G \cap SL(V) \) with \( t^p = 1 \) and \( \dim C_V(t) \geq 4 \);

(e) \( |G| = q^6(q^6 - 1)(q^2 - 1)c \), where \( c \mid (q - 1)e \) if \( q = p^e \) and \( c \mid q - 1 \) if \( G \leq GL(V) \); and
Theorem 6.1. Let $G \leq \Gamma L(n,q), n \geq 2$, be antiflag transitive. Then one of the following holds:

(i) $G \geq SL(n,q)$;

(ii) $G$ is $A_7$ inside $SL(4,2)$;
(iii) $G \supseteq Sp(n, q);$
(iv) $G$ is $A_6$ inside $SL(4, 2);$
(v) $G < GL(2, 4)$ has order 20 modulo scalars;
(vi) $G \leq \Gamma Sp(6, q) < GL(6, q),$ and $G$ acts as a rank 4 group on the points of a generalized hexagon with parameters $q, q,$ whose points and lines consist of all points and certain totally isotropic lines for $Sp(6, q);$
(vii) For $q = 2$ or 4, $G \supseteq SL(\frac{q}{2}n, q^2),$ embedded naturally in $GL(n, q);$
(viii) For $q = 2$ or 4, $G \supseteq Sp(\frac{q}{2}n, q^2),$ embedded naturally in $GL(n, q); or$
(ix) For $q = 2$ or 4, $G$ is a subgroup of $\Gamma Sp(6, q^2),$ itself embedded naturally in $\Gamma L(12, q),$ such that $G$ acts on a generalized hexagon in $PG(5, q^2)$ as in (vi).

Note that 2-transitive subgroups of $\Gamma L(n, q)$ are automatically antiflag transitive (Wagner [25, p. 416], or (4.1)).

The theorem will be proved by induction on $n$ in Sections 6, 7. The case $n = 2$ is omitted, while (2.1) handles $n = 3.$ We therefore assume $n \geq 4.$ By (4.2), if $q \neq 4$ we may assume that $n \geq 6$ or $n = 4$ but $q$ is not a Mersenne prime.

In the remainder of this section we will consider only groups $G$ that are primitive on the points of the projective space. Then either (5.2) applies, or $G$ is 2-transitive. In either case, induction or known results almost always produce sufficiently large groups of transvections for $G$ to be identified; case (5.2ii) is exactly (6.1vi), and will be considered in the Appendix.

Proposition 6.2. If (5.2i) holds then either $G \supseteq Sp(n, q)$ or $G$ is $A_6$ inside $Sp(4, 2).$

Proof. We will follow Perin [19] when possible, but we include semilinear groups and the cases $Sp(n, 2)$ and $Sp(n, 3)$ not dealt with in [19]. His method works primarily when $q > 4$ and when either $n \geq 6$ or $n = 4$ but $q$ is not a Mersenne prime.

If $G$ contains the group of all transvections with a given center, then $G$ contains all transvections by transitivity and $G \supseteq Sp(n, q).$

Assume that $q > 4,$ and either $n \geq 6$ or $n = 4$ and $q$ is not a Mersenne prime. We have $|x^+ - x| = q(q^{n-2} - 1)/(q - 1).$ Let $r$ be a primitive divisor of $q^{n-2} - 1$ (see (2.4); use $r = 3$ if $q^{n-2} - 1 = 8^p - 1$ with $n = 4, q = 8$) and $R = Syl_r(G_x).$ Then $r > 2$ and $R < GL(V)$ is completely reducible, so $U = C_V(R)$ is a nonsingular 2-space. Moreover, $N_G(R)^U \geq SL(2, q)$ (by (4.2), since $G_U = C_G(U)N_G(R)$ by the Frattini argument), while $N_G(R)^U$ is solvable. Then $C_G(U^-)$ contains $SL(2, q) = SL(2, q')$ and hence contains a full transvection group, so $G \supseteq Sp(n, q).$ Note that the same argument handles the case $G_U \supseteq SL(2, q) = SL(2, 4).$

It remains to consider the possibility that either $q \leq 4$ or that $n = 4$ and $q$ is a Mersenne prime.

Let $x$ and $y$ be distinct points of the t.i. line $L.$ There is a Sylow $p$-subgroup $P$ of $G$ fixing $x$ and $L,$ and transitive on $V - x^+.$ Then all orbits of $P_y$ on $V - x^+$ have length at least $q^{n-1}/q,$ so $P_y$ is transitive on $y^+ - x^+.$ Since $G_y$ is already transitive on $y^+ / y$ by (5.2i), it is antiflag transitive there.

By our inductive hypothesis concerning (6.1), $K = G_U^{y^+ / y}$ satisfies one of the following conditions:

(a) $K \supseteq Sp(n - 2, q);$
(b) $K = A_6, n - 2 = 4, q = 2;$
(c) $K$ acts on a generalized hexagon as in (6.1vi), $n - 2 = 6;$
(d) For $q = 2$ or 4, $Sp(\frac{q}{2}(n - 2), q^2) \triangleleft K \leq \Gamma Sp(\frac{q}{2}(n - 2), q^2)$ with $\frac{q}{2}(n - 2)$ even;
(ε) For \( q = 2 \) or \( 4 \), \( K < \Gamma Sp(6, q^2) \) acts on a generalized hexagon over \( GF(q^2) \) as in (6.1vi), \( n - 2 = 12 \); or
(ζ) \( K < \Gamma L(2, 4) \) has order 20 modulo scalars, \( n - 2 = 2 \).

In particular, if \( q \) is odd then \( K \geq Sp(n - 2, q) \).

If \( Q = Q_p(Sp(V)_{y}) \) and \( T \) is the group of transvections in \( Q \), then \( Q/T \) and \( y^+ \cap T/y \) are naturally \( \Gamma Sp(V)_{y} \)-isomorphic projective modules (via \( uT \to [V, uT]/y \) for \( u \in Q \)). Moreover, \( T \) is the Frattini subgroup of \( Q \) if \( q \) is odd, while \( Q \) is naturally an \( O(n - 1, q) \)-space if \( q \) is even.

The case \( G \leq Sp(n, q) = Sp(4, 2) \cong S_{56} \) is easily handled and so will be excluded. Note that \( [G \cap Q] = 4 \) if \( G = Sp(4, 2)' \cong A_{6} \).

If \( G \cap Q \leq T \) we will show that \( G \geq T \) and hence \( G \) contains \( Sp(n, q) \). Let \( r \) be a primitive divisor of \( q^{n-2} - 1 \) and \( R \in \text{Syl}_{r}(G \cap Sp(n, q)) \) (using \( r = 3 \) if \( q^{n-2} - 1 = 2^8 - 1 \) when \( n = 4, q = 8 \), or \( r = 7 \) when \( n = 8, q = 2 \)). The \( R \)-invariant subgroup \( W = [G \cap Q, R] \) projects onto a subspace \( WT/T \) of the \( GF(q) \)-space \( Q/T \); in view of the action of \( R \) on \( y^+ \cap T/y \) and hence on \( Q/T \), we have \( WT/T = Q/T \). If \( q \) is odd it follows that \( W \) contains the Frattini subgroup \( T \) of \( Q \), so that \( G > T \). If \( q \) is even then \( W \) is a nonsingular hyperplane of the orthogonal \( GF(q) \)-space \( Q \). If \( G \) does not contain \( T \) then each element of \( G_{y} \) leaves the hyperplane \( W \) invariant, while acting antiflag transitively on \( y^+ \cap T/y \) and hence on the 1-spaces of the orthogonal space \( W \), so we are in case (ζ). Then \( G^\perp_{y}L \) is \( Z_{4} \) by (4.1). Since \( W \) is elementary abelian, \( |W^{L'}| \leq 2 \) and \( |C_W(L)| \geq 8 \). If \( L' \) is a second t.i. line containing \( y \) then we obtain the contradiction \( 4 \leq |C_W(L) \cap C_W(L')| = |C_W([L, L'])| = |C_W(y^+)| = 1 \).

If \( n = 4 \) then \( q^3 \leq |G_{y}|p = |G \cap Q||K|p \leq |G \cap Q|q^{e_p} \) where \( q = p^f \), so that \( |G \cap Q| > |T| \) and we have seen that \( G \geq Sp(4, q) \). This takes care of dimension \( n = 4 \), including (ζ). From now on \( n \geq 6 \) and \( q \leq 4 \).

If \( n = 6 \) then the same argument yields \( q^5 \leq |G \cap Q||Sp(4, q)|p^{e_p} = |G \cap Q|q^{e_p} \), so \( G \cap Q \neq 1 \). We have already handled the cases \( G \cap Q \leq T \) and \( G \cap Q > T \). It remains to eliminate the possibility \( 1 \neq G \cap Q < T \), where \( p|e \) and hence \( q = 4 \). Since the above inequality shows that \((α)\) holds, if \( E \subset y^+ \) is a nonsingular 2-space then some \( g \in G_{y} \cap Sp(n, 4) \) induces an element of order 3 on \( E/y \) and hence acts in that manner on a nonsingular 2-space \( D \subset (y, E) \), fixing a point \( z \in D \). Some \( h \in G \) satisfies \( z^h = y \in D^h \), and then \( q^{h}D^h \) acts nontrivially on \((G \cap T)D^h \), which contradicts the assumption \( 1 < |G \cap Q| < 4 \).

Now \( n > 6 \). If \( q = 3 \) then \( K \geq Sp(n - 2, 3) \). Let \( r \) be a primitive divisor of \( 3^{n-1} - 1 \) and \( R \in \text{Syl}_{r}(G_{y}) \). Then \( U = C_{V}(R) \) is a nonsingular 4-space. Since \( R \) is a Sylow subgroup of the stabilizer of two perpendicular points of \( U \) and of the stabilizer of two non-perpendicular points of \( U \), by the Frattini argument \( N_{G}(R)^{U} \) has rank 3 and hence contains \( Sp(4, 3) \) by induction. Also \( N_{G}(R)^{U^+} \) is solvable (lying in \( \Gamma L(1, 3^{n-4}) \)). Then \( C_{G}(U^+) \) contains transvection groups and \( G \geq Sp(n, 3) \).

Now \( q = 2 \) or \( 4 \). In (α) let \( r \) be a primitive divisor of \( q^{(n - 2) - 2} - 1 \) (use \( r = 7 \) if \( q^{(n - 2) - 2} = 2^{(10 - 2) - 2} - 1 \), in (γ) let \( r = q + 1 \), in (δ) let \( r \) be a primitive divisor of \( q^{2} + 1 \), and in (ε) let \( r = q^{2} + 1 \). Let \( R \in \text{Syl}_{r}(G_{y}L) \). Then \( C_{y^+}y(R) \) is a nonsingular 2-space over \( GF(q) \) in (α) and (γ) (cf. (5.4f)) or over \( GF(q^{2}) \) in (δ) and (ε), so \( U = C_{V}(R) \) is nonsingular of dimension 4 or 6. As above, by the Frattini argument \( N_{G}(R)^{U} \) has rank 3 and hence contains \( A_{4}, Sp(4, q) \) or \( Sp(6, q) \) by induction. Also \( N_{G}(R)^{U^+} \) is solvable or is a subgroup of \( \Gamma Sp(2, q^{2}) \) in (γ) or of \( \Gamma Sp(2, q^{4}) \) in (ε). Then \( N_{G}(R) \) has a subgroup \( N \) inducing the identity on \( U^+ \).
and $A_6$, $Sp(4, q)$ or $Sp(6, q)$ on $U$. In the last two cases we obtain $G \trianglerighteq Sp(n, q)$ as usual; in the $A_6$ case a $G$-conjugate of $N$ meets $Q$ in a subgroup of size 4 and hence $G \cap Q \not\leq T$, which was handled above.

The next primitive case is Theorem I.

**Proposition 6.3.** If $G \leq \Gamma L(n, q)$ $(n \geq 3)$ is 2-transitive on points, then either $G \geq SL(n, q)$ or $G$ is $A_7$ inside $SL(4, 2)$.

**Proof.** In view of Wagner [25, Theorem 4], we may assume that $n \geq 6$. We recall the following additional facts from Wagner [25, pp. 414, 416]: $G$ is 2-transitive on hyperplanes, and if $H$ is a hyperplane, then $G_H^H$ is antiflag transitive.

Once again, we will run through the possibilities provided by induction for $G_H^H$ and, dually, $G_x^V$. If either is 2-transitive, then by induction $G$ is transitive on complete flags (i.e., maximal increasing sequences of subspaces, one of each dimension), and the result follows from Wagner [25, Theorem 3] or Higman [8, Theorem 1]; so suppose not. Let $H$ be a hyperplane and $x \in H$.

Suppose $G_x^V$ is primitive and hence is contained in $\Gamma Sp(n-1, q)$ by (5.2). Then $G_xH$ fixes a line $\Delta$ on $H$ and $x$. By (4.1), $(G_H^H)^H_\Delta$ is 2-transitive, so $G_xH < G_{\Delta H}$ and $G_H^H$ is imprimitive.

Thus, we may assume that $K = G_x^V$ is imprimitive. By Theorem III, $q = 2$ or 4, $n - 1 \geq 6 - 1$ is even and $K \leq \Gamma L(\frac{1}{2}(n - 1), q^2)$ behaves as follows:

- $(\alpha) \quad K \trianglerighteq SL(\frac{1}{2}(n - 1), q^2)$;
- $(\beta) \quad K \trianglerighteq Sp(\frac{1}{2}(n - 1), q^2)$; or
- $(\gamma) \quad K < \Gamma Sp(6, q^2)$ acts on a generalized hexagon over $GF(q^2)$ as in (6.1vi), $n - 1 = 12$.

Let $r$ be a primitive divisor of $(q^2)^\frac{1}{2}(n-1)-1$ in $(\alpha)$ or of $(q^2)^\frac{1}{2}(n-1)-2$ in $(\beta)$, and let $r = q^2 + 1$ in $(\gamma)$. (Use $r = 7$ if $(q^2)^\frac{1}{2}(n-1)-1 = (2^2)^3 - 1$ in $(\alpha)$.) Let $R \in Syl_r(G_x)$. Then dim $C_V(R)$ is 1 + 2 in $(\alpha)$ and 1 + 4 in $(\beta)$ and $(\gamma)$ (using (5.4f) in $(\gamma)$). By the Frattini argument, $N_G(R)$ is 2-transitive on $U = C_V(R)$, inducing at least $SL(U)$ by induction. Moreover, $N_G(R)^{[V, R]}$ is solvable, except perhaps in $(\gamma)$ with $N_G(R)^{[V, R]} \leq \Gamma L(2, q^4)$. As usual, $C_G([V, R]) \geq SL(U)$, so $G$ contains a full transvection group and $G \geq SL(V)$, which contradicts the behavior of $K$.

Now (5.2), (3.2), (6.2) and (6.3) complete the inductive step in (6.1) when $G$ is primitive on points.

Having dealt with the primitive case, we record an elementary corollary for use in the next section.

**Lemma 6.4.** Suppose $G$ is as in (6.1) and is primitive on points. If $F \leq G$ with $F$ antiflag transitive and $|G:F|$ a power of $p$, then $F$ is also primitive on points.

**Proof.** Let $P \in Syl_p(G_x)$. Then $P$ fixes a unique line $L$ on $x$. (In case (6.1vi), by (5.3i) the $p$-parts of the nontrivial orbit lengths of $G_x$ are $q$, $q^3$ and $q^3$.)

Clearly $G = PF_F$ and $P \cap F \in Syl_p(F_x)$. If $F$ is imprimitive then, by (6.1vii-ix), there is a unique line containing $x$ fixed by $F$, and it is also the unique line fixed by $P \cap F$; this line must be $L$. Thus, $G_x = PF_x$ fixes $L$, contradicting (4.1) and the primitivity of $G$. 

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7. The imprimitive case; completion of the proof

Continuing our proof of (6.1), we now turn to the case of an antiflag transitive subgroup $G$ of $PL(n,q)$ that is *imprimitive on points*. The method here is entirely different from that of Sections 5, 6; we build a new projective space on which $G$ continues to act antiflag transitively.

If $\Delta$ is a nontrivial imprimitivity block for the action of $G$ on points, then $\Delta$ is the set of points of a subspace. (For, every hyperplane of $\Delta$ does not contain some point of $\Delta$. Then $G_\Delta$ is transitive on the hyperplanes of $\langle \Delta \rangle$, hence on its points, and thus $\Delta$ must contain all points of $\langle \Delta \rangle$.) We usually identify $\Delta$ with $\langle \Delta \rangle$.

By Remark 3 at the end of Section 5, $G$ is also imprimitive on hyperplanes, and a block of imprimitivity consists of all hyperplanes containing a subspace $\Sigma$. The next result (independent of the aforementioned Remark) shows that there is a close connection between blocks of points and hyperplanes. It is due to Orchel [16], and simplifies and improves a result in an earlier version of this paper.

**Lemma 7.1** (Orchel). Let $\Delta$ be a block of imprimitivity for $G$ acting on points, and $\delta = \dim \Delta$. Let $H$ be a hyperplane, and let $\Sigma$ be the union of the members of $\Delta^G$ contained in $H$. Then $\Sigma$ is a subspace of dimension $n - \delta$ partitioned by $\Delta^G \cap \Sigma$, and the set of hyperplanes containing $\Sigma$ is a block of imprimitivity for $G$ acting on hyperplanes.

**Proof.** We have $|\Delta^G| = (q^n - 1)/(q^\delta - 1)$. Then $(q^n - 1)/({q - 1}) = |H \cap \Delta^G|$. $(q^\delta - 1)/({q - 1}) + ((q^n - 1)/(q^\delta - 1) - |H \cap \Delta^G|)(q^\delta - 1)/({q - 1})$, so $|H \cap \Delta^G| = (q^n - \delta - 1)/(q^\delta - 1)$. The union $\Sigma$ of the members of $H \cap \Delta^G$ has cardinality $(q^n - \delta - 1)/({q - 1})$.

Let $P \in Syl_p(G_H)$. Then $P$ is transitive on $V - H$, and hence on $\Delta^G - (H \cap \Delta^G)$. Let $\Sigma'$ be a subspace of $H$ of dimension $n - \delta$ fixed by $P$. If $\Sigma' \cap \Delta' \neq 0$ for one (and hence all) $\Delta' \in \Delta^G - (H \cap \Delta^G)$, then $|\Sigma'| \geq |\Delta^G - (H \cap \Delta^G)| = q^n - \delta$, which is false; so $\Sigma \supseteq \Sigma'$, and comparing cardinalities shows that $\Sigma = \Sigma'$ is a subspace. Moreover, if $\Sigma \cap \Delta' \neq 0$ for $\Delta' \in \Delta^G$, then $\Delta' \cap H$ and hence $\Delta' \subseteq \Sigma$, so $\Sigma$ is partitioned by $\Delta^G \cap \Sigma$.

Now, if $H'$ is any hyperplane containing $\Sigma$, then $H'$ contains all $\Delta^q \subset H$, so $H' \cap \Delta^G$ contains $H \cap \Delta^G$ and hence has union containing $\Sigma$. As any element of $G$ sending $H$ to $H'$ sends $H \cap \Delta^G$ to $H' \cap \Delta^G$, $G_\Sigma$ is transitive on the set of such hyperplanes $H'$. This proves the lemma.

**Notation.** Let $\Delta$ be a minimal proper block of imprimitivity, and define $\Sigma$ as in (7.1). Let $\mathcal{L}$ be the set of all intersections of members of $\Sigma^G$.

**Lemma 7.2.** If $n > 2\delta$ then $\mathcal{L}$ is the lattice of subspaces of a projective space $PG(n/\delta - 1, q^\delta)$ on which $G$ acts as an antiflag transitive collineation group.

**Proof.** By (7.1), if $W \in \mathcal{L}$ then $W \cap \Delta^G$ partitions $W$. If $W = \langle \Delta_1, \ldots, \Delta_k \rangle$ with $\Delta_i \in \Delta^G$ and $k$ minimal, then $\dim W = k\delta$ and $|W \cap \Delta^G| = (q^{k\delta} - 1)/(q^\delta - 1)$. Call $W$ a Point, Line, or Plane if $k = 1, 2$ or 3, respectively. Then two Points are on a unique Line (containing $q^\delta + 1$ Points), and three Points not on a Line are in a unique Plane (containing $q^{2\delta} + q^\delta + 1$ Points). The Veblen and Young axioms [24] imply that $\mathcal{L}$ is a projective space.

By (7.1), $H \cap \Delta^G = \Sigma \cap \Delta^G$, and $G_H$ is transitive on the $q^{n-\delta}$ Points not in $\Sigma$. Thus, $G$ acts antiflag transitively on $\mathcal{L}$. 
Theorem. Let $\mathfrak{A}$ denote the set of all cosets of members of $\mathcal{L}$. (Since $0 \in \mathcal{L}$, all vectors of $V$ are in $\mathfrak{A}$.)

**Lemma 7.3.** If $n > 2\delta$ then $\mathfrak{A}$ is the lattice of subspaces of $AG(n/\delta, q^\delta)$.

**Proof.** Form $\mathfrak{A} \cup \mathcal{L}$ by attaching $\mathcal{L}$ “at infinity” as follows: adjoin $U \in \mathcal{L}$ to $W + v$ if $U \subseteq W \in \mathcal{L}$. Thus, $\mathfrak{A} \cup \mathcal{L}$ will have two types of “points” (vectors and members of $\Delta^G$), and two types of “lines” (cosets of members of $\Delta^G$, and Lines of $\mathcal{L}$). If $\langle \Delta, \Delta' \rangle$ is a Line of $\mathcal{L}$, then it and any vector determine a translation plane of order $q^\delta$ in a standard manner [4, p. 133]; $\langle \Delta, \Delta' \rangle$ plays the role of line at infinity. By (7.2), $\mathfrak{A} \cup \mathcal{L}$ satisfies the Veblen and Young axioms, and hence is $PG(n/\delta, q^\delta)$. This proves the lemma.

**Lemma 7.4.** If $n = 2\delta$ then $\mathfrak{A}$ is $AG(2, q^\delta)$.

**Proof.** As above, $\mathfrak{A}$ is an affine translation plane of order $q^\delta$. But here $\Delta^G$ is merely its line at infinity, so proving that $\mathfrak{A}$ is desarguesian will be more difficult. We will use standard properties of collineations of finite projective planes [4, Chap. 4]. Using dimensions, $V = \Delta \oplus \Delta'$ for distinct $\Delta, \Delta' \in \Delta^G$.

Let $x \in \Delta$ and $P \in \text{Syl}_{p}(G_x)$. The group $E = C_P(\Delta)$ consists of all elations of $\mathfrak{A}$ with axis $\Delta$: it is semiregular on the set $\Delta^G - \{\Delta\}$ of lines $\neq \mathfrak{A}$ through the point 0 of $\mathfrak{A}$, and $\mathfrak{A}$ is desarguesian if $|E| = q^\delta$, by (2.2). We may thus assume that $|E| < q^\delta$ and aim at a contradiction.

Let $H \supset \Delta$ be a hyperplane fixed by $P$, so $H \cap \Delta^G = \{\Delta\}$ by (7.1). Since $P$ is transitive on $V - H$ and hence on $\Delta^G - \{\Delta\}$, $P_{\Delta'}$ is transitive on $H' = \Delta'$ (since $\Delta'$ is a block). Now $G_\Delta$ is transitive on the pairs $(x, \Delta')$ with $x \in \Delta$ and $\Delta' \in \Delta^G - \{\Delta\}$, so $G^\Delta_{\Delta'}$ is transitive and hence $G^\Delta_{\Delta'}$ is antiflag transitive since $P_{\Delta'}^{-H}$ is transitive. Moreover, $G_\Delta = P \cdot G_{\Delta'}$ since $P$ is transitive on $\Delta^G - \{\Delta\}$.

Then $G^\Delta_{\Delta'} = P_{\Delta \Delta'} G^\Delta_{\Delta'}$; since $G^\Delta_{\Delta'}$ is primitive by the minimality of $\Delta$, $G^\Delta_{\Delta'}$ is primitive by (6.4).

We claim that $C_G(\Delta)_{\Delta'} = 1$. For, $C_G(\Delta) \leq G_\Delta$, where $G_\Delta$ is transitive on $\Delta^G - \{\Delta\}$ and $C_{G(\Delta)}_{\Delta'}$ consists of homologies of $\mathfrak{A}$ with axis $\Delta$. Thus, if $C_G(\Delta)_{\Delta'} \neq 1$, then this holds for every $\Delta' \in \Delta^G - \{\Delta\}$.

Then in the action of $C_G(\Delta)$ on $\Delta^G - \{\Delta\}$, the stabilizer of any two points is trivial, but the stabilizer of any point is nontrivial. This implies that $C_G(\Delta)$ acts as a transitive Frobenius group on $\Delta^G - \{\Delta\}$, with kernel $E$ of order $q^\delta$, contrary to assumption.

It follows that $C_G(\Delta) = E$, and $|G^\Delta_{\Delta'} : G^\Delta_{\Delta'}| = (|G_{\Delta}||E|)/|G_{\Delta \Delta'}| = q^\delta/|E|$ is a power of $p$.

Suppose $q$ is odd. By (4.2) we may assume that $G \leq GL(n, q)$. By induction, both $G^\Delta_{\Delta'}$ and $G^\Delta_{\Delta'}$ have normal subgroups $SL(\delta, q)$ or $Sp(\delta, q)$ or a group as in (5.4). The known orders (cf. (5.4e)) do not allow for distinct subgroups of one of these types to have index a power of $p$ in one another (since $G \leq GL(n, q)$). It follows that $G^\Delta_{\Delta'} = G^\Delta_{\Delta'}$, and $q^\delta/|E| = 1$, contrary to assumption.

Consequently, $p$ is even. Since $G_{\Delta \Delta'}$ has even order it has an involution $t$. Then $t$ is a Baer involution (since it fixes $\Delta$ and $\Delta'$), and $\dim C_{\Delta}(t) = \frac{1}{2}\delta$ and $|C_{E}(t)| \leq q^{\delta/2}$ (since $C_{E}(t)$ acts on the Baer subplane for $t$). Induction for $G^\Delta_{\Delta'}$, together with this restriction on involutions in $G_{\Delta \Delta'}$ (cf. (5.4d)), imply that either (a) $\delta = 2$, or (b) $\delta = 4$, $q = 2$, $G^\Delta_{\Delta'} = A_6$ or $A_7$.

(a) The argument used for $q$ odd applies, unless $q = 4$, $G^\Delta_{\Delta'} = SL(2, 4)$ and $G^\Delta_{\Delta'} = SL(2, 4)$ (modulo scalars). Here, $4^2/|E| = q^\delta/|E| = |G^\Delta_{\Delta'} : G^\Delta_{\Delta'}| = 2$, so $G_{\Delta \Delta'} \cong SL(2, 4)$ centralizes $E$. Choosing $t$ in this $SL(2, 4)$ contradicts $|C_{E}(t)| \leq 4$. 


Proof of (6.1). Each translation \( v \rightarrow v + c \) permutes the members of \( \mathfrak{A} \), sending each hyperplane of \( \mathfrak{A} \) to itself or a disjoint hyperplane. Thus, these form the group of translations of the affine space \( \mathfrak{A} \). The corresponding group of scalar transformations acts homogeneously on \( V \), and hence is uniquely determined up to \( GL(V) \)-conjugacy. Then the group \( G^+ \) of all collineations of \( \mathfrak{A} \) induced by elements of \( GL(n, q) \) is \( GL(n/\delta, q^\delta) \).

In particular, \( (G^+_A)^\Delta = \Gamma L(1, q^\delta) \) has order \( (q^\delta - 1)\delta e \), where \( q = p^e \). Since this group is antiflag transitive, \( q^\delta - 1 \) divides \( \delta e \), whence \( q = \delta = 2 \) or \( e = \delta = 2, \ q = 4 \).

Thus, \( G \) is an antiflag transitive subgroup of \( \Gamma L(1/2n, q^2) \), where \( \Gamma L(1/2n, q^2) \) is embedded naturally in \( \Gamma L(V) \). Moreover, \( G \) acts primitively on the set \( \Delta^G \) of points of \( PG(n/\delta - 1, q^\delta) \). For otherwise, there is an imprimitivity block \( \Lambda \subset \Delta \), and \( G \) lies in \( \Gamma L(1/2n, q^2) \). Then \( q = 2, |\Lambda| = 16 \) and \( G^+_A \) lies in \( \Gamma L(1, 16) \), which we have just seen is not antiflag transitive.

This primitivity and (6.1.iii,vi) produce (6.1vii-ix), finishing our proof of (6.1).

Remark. Examples of (6.1vii-ix) occur. For, let \( F = GF(q^\delta) \) if \( K = GF(q) \) with \( q = 2 \) or 4, and \( V = V(1/2n, q^2) = Fv \oplus W \) with \( n \geq 4 \) even and \( W \) an \( F \)-hyperplane.

\( \)Let \( b \in F - K \) and \( \sigma \in Aut(F) \) of order \( \log_2 q^2 = q \). Then the \( K \)-hyperplanes containing \( W \) are \( Kv \oplus W \) and \( Kb^i \oplus W, 0 \leq i < q \), where \( \langle \sigma \rangle \) fixes the first of these and is transitive on the remaining ones. Since \( SL(1/2n, q^2) \) contains a subgroup of order \( q + 1 \) transitive on the 1-dimensional \( K \)-subspaces of \( Fv \), \( SL(1/2n, q^2) \sigma \) is antiflag transitive. The symplectic and \( G_2 \) cases are similar. Moreover, any antiflag transitive instance of (6.1vii-ix) contains one of the groups generated by \( SL(1/2n, q^2) \), \( Sp(1/2n, q^2) \) or \( G_2(q^2) \) together with a group of \( q \) field automorphisms.

Now the proofs of (6.1) and Theorem I are complete. Moreover, for Theorems II and III, we only have to identify the groups occurring in (6.1vii) – the hexagon \( \mathcal{G} \) is already known to be both unique and correctly embedded, by (5.2) and (3.2). It is known that the group of automorphisms of \( \mathcal{G} \) induced by elements of \( Sp(6, q) \) is \( G_2(q) \); this is stated in Tits [22, (11.3)] and proved in Tits [23, (5.9)]. We observe, independent of this, that \( G \cap Sp(6, q) = G_2(q) \): in view of \( G_2(q) \leq Aut(\mathcal{G}) \) and (A.6iii), if \( S \) is the group of scalar transformations of \( V \) then \( |GS \cap GL(6, q)| = |G_2(q)S| \) and \( G_2(q) \cap S = 1 \).

8. Corollaries

In this section we give some consequences of Theorems I-III. The affine group \( AGL(n, q) \) is defined as the group

\[ \{ v \rightarrow v^g + c \mid g \in GL(n, q), c \in V \} = T \times GL(n, q) \]

of all collineations of the affine space \( AG(n, q) \) based on \( V \), an \( n \)-space over \( GF(q) \). (\( T \) denotes the translation group.)
Proposition 8.1. Let $G \leq AG(n, q), n \geq 3$, be transitive on ordered non-collinear triples of points of $AG(n, q)$. Then $G = T \times G_0$, where $T$ is the translation group, and $G_0 \cong SL(n, q)$ or $G_0$ is $A_7$ (with $n = 4$, $q = 2$).

Proof. The hypothesis implies that $G_0$ (the stabilizer of 0) is projectively one of the groups of Theorem I; it remains only to show that $G$ contains $T$. If not, then $G \cap T = 1$ (since $G_0$ is transitive on points), and so $|G| \leq |GL(n, q)|$ since $|AG(n, q)| = |GL(n, q)||T|$. But then $|G: G_0| = q^n$ contradicts $|GL(n, q): G_0| \leq (q - 1)e$ (resp. $|GL(n, q): G_0| = 8$) if $G_0 \geq SL(n, q), q = p^e$ (resp. $G_0 = A_7$).

Corollary 8.2. The only proper 3-transitive subgroup of $AG(n, 2)$ is $V_4 \times A_7$ when $n = 4$.

This corollary improves various results in the literature (for example Cameron [3, Theorem 1]); and also Jordan’s theorem (Wielandt [26, (9.9)]):

Corollary 8.3. A normal subgroup $N$ of a 3-transitive group $G$ is 2-transitive, unless it is elementary abelian of order $2^n$ and either $G = N \times GL(n, 2)$ or $n = 4$ and $G = N \times A_7$.

From results of Perin [19] and Kantor [12], we deduce the following

Proposition 8.4. Suppose $G \leq GL(n, q)$ is transitive on the $j$-subspaces of $PG(n - 1, q)$ for some $j$ with $2 \leq j \leq n - 2$. Then $G$ is transitive on the $i$-subspaces for all $i$ with $1 \leq i \leq n - 1$, and one of the following occurs:

(i) $G \cong SL(n, q)$;
(ii) $G$ is $A_7$ inside $GL(4, 2)$; or
(iii) $G$ is $GL(1, 2^5)$ inside $GL(5, 2)$.

Remark. A “$t$-$(v, k, \lambda)$ design in a finite vector space” is a collection of $k$-subspaces or “blocks” in a $v$-space, any $t$-space being contained in precisely $\lambda$ blocks. No nontrivial examples are known with $t \geq 2$; and (8.4) shows that none can be constructed by the analogue of the familiar construction of $t$-designs from $t$-homogeneous groups (Dembowski [4, (2.4.4)]).

To motivate the next result, we sketch the deduction of Perin’s Theorem [20] (mentioned in Section 1) from Theorem II. Suppose $G \leq GL(n, q), n \geq 4$, and suppose $G$ acts as a primitive rank 3 group of even order on the points of $PG(n - 1, q)$. For a point $x$, $G_x$ has three orbits on points, and hence three orbits on hyperplanes. If $G$ is antiflag transitive, then $G \leq GSp(n, q)$ by Theorem II (and indeed $G$ is known). Otherwise, $G_x$ is transitive on the hyperplanes through $x$, and so also on the lines through $x$, in contradiction to Kantor [12].

Proposition 8.5. Suppose $G \leq GL(n, q), n \geq 4$, and $G$ acts as a primitive rank 4 group on the points of $PG(n - 1, q)$. Then either $q = 2, 3, 4, 5$ or 9, or $G \cong G_2(q)$, $q$ even, embedded naturally in $GSp(6, q)$.

Proof. By Theorem II, we may assume that $G$ is not antiflag transitive; by the previous argument and Kantor [12], we may assume it is not transitive on incident point-hyperplane pairs. Thus, of the four $G_x$-orbits on hyperplanes, two consist of hyperplanes containing $x$. Then $G_x$ has two orbits on lines containing $x$. There are thus two $G$-orbits on lines, with $G_x$ transitive on the lines of each orbit which pass through $x$. Consequently, $G_L$ is transitive for each line $L$. 


Since $G_x$ has three orbits on points different from $x$, it follows that, for suitable $L$ and $M$ from different line-orbits, $G_{L}^{L - x}$ is transitive while $G_{M}^{M - x}$ has two orbits. Thus, $G_{L}^{L}$ is 2-transitive while $G_{M}^{M}$ has rank 3. But, using Dickson’s list of subgroups of $PSL(2, q)$ [5, Chap. 12], we see that $PGL(2, q)$ has a rank 3 subgroup only if $q = 2, 3, 4, 5$ or 9.

**Proposition 8.6.** Let $G$ be an irreducible subgroup of $PGL(n, q), n \geq 4$. Suppose $G_x$ is transitive on the lines through $x$, for some point $x$. Then $G$ is 2-transitive on points (and Theorem I applies).

**Proof.** By Kantor [12], it is enough to show that $G$ is transitive on points. So let $X = x^G$ and assume $X$ is not the set of all points. If $L$ is a line and $L \cap X \neq 0$, then $l = |L \cap X|$ is independent of $L$, and $1 < l < q - 1$. If dim $W = m$ and $W \cap X \neq 0$, then $|W \cap X| = 1 + (l - 1)(q^{m - 1} - 1)/(q - 1)$.

There is an $(n - 2)$-space $U$ disjoint from $X$ (for otherwise the hyperplane sections of $X$ would be the blocks of a design having the same $b, r, \lambda$ as $PG(n - 1, q)$ and hence $|X| = v = b$). The hyperplanes containing $U$ partition $X$ into sets of cardinality $k = 1 + (l - 1)(q^{n - 2} - 1)/(q - 1)$; so $k$ divides $|X| = 1 + (l - 1)(q^{n - 1} - 1)/(q - 1)$ and hence also $(l - 1)q^{n - 2}$. Then $(q - 1)(l - 1) = 0 \pmod k$. Since $k > (q^{n - 2} - 1)/(q - 1) > q$, we have $l = q$. But then the complement of $X$ contains one or all points of each line, and so is a hyperplane fixed by $G$, contradicting irreducibility.

II. THEOREMS IV AND V

9. THE GEOMETRY OF PRIMITIVE ANTIFLAG TRANSITIVE GROUPS

The proof of Theorem IV occupies Sections 9-11. The present section contains notation and the analogue of (5.3). The primitive case is concluded in Section 10; there the method is different from that of Section 6. Unlike Theorems I-III, the primitive case does not depend on the imprimitive one. Finally, Section 11 corresponds to Section 7.

The symplectic case is covered by Theorems II and III; so we will exclude the case $G \leq \Gamma Sp(2m, q)$ for the remainder of the proof. Also, in view of the isomorphism between the $Sp(2m, q)$ and $O(2m + 1, q)$ geometries when $q$ is even, we will also exclude the case $G \leq \Gamma O(2m + 1, q), q$ even. Thus, the geometry is associated with a nondegenerate sesquilinear form.

In the proof, $\Omega$ denotes the set of t.i. or t.s. points of the appropriate classical geometry, defined on a vector space $V$ over $GF(q)$. (This assumption involves a slight change of notation in the unitary case: $G$ will be a subgroup of $GU(n, q^{1/2})$. This may lead to the impression of minor discrepancies between the statement of Theorem IV and parts of Sections 9-11: the notation for the name of the group will remain the same as in Section 2, only the meaning of “$q$” will change.)

In general our convention is to refer only to 1-spaces in $\Omega$, though there will be situations where other 1-spaces will be mentioned. Thus, in general we identify a subspace with the set of members of $\Omega$ it contains; some care is needed when dealing with anisotropic subspaces. Similarly, in general if $S$ is a subset of $\Omega$, then $S^\perp$ is the set of points of $\Omega$ collinear with (i.e., perpendicular to) every point of $S$. The subspace 0 plays the role of 0, so $0^\perp = \Omega$. This convention has odd-looking consequences, such as: a t.i. or t.s. subspace $W$ is maximal if and only if $W^\perp = W$. (However, if $W$ is nonsingular and if no point is collinear with every point of $W$, then $W^\perp$ will denote an anisotropic vector subspace.) The notation $\langle X \rangle$ usually
refers to a vector subspace, not just a set of points; the meaning will be clear from the context. The dimension of a t.i. or t.s. subspace is its vector space dimension (cf. Section 2), and the rank \( r \) of the geometry is the maximal such dimension.

We begin with two preliminary lemmas.

**Lemma 9.1.** There do not exist subspaces \( T, W \) with \( T \cup T^\perp = W^\perp \) and \( T^\perp \neq W^\perp \).

**Proof.** If \( T \cup T^\perp = W^\perp \) then \( T \cap T^\perp = (W^\perp)^\perp = W \). Let \( t_1 \in T - W \) and \( t_2 \in T^\perp - W \), and observe that a point of \( \langle t_1, t_2 \rangle \) is not in \( T \cup T^\perp \).

**Lemma 9.2.** Suppose \( T, W \) are t.i. or t.s. subspaces with \( \dim T = i - 1 \), \( \dim W = i \), and \( T \subset W \). Then \( |T^\perp - W^\perp| = q^{2r-i+c} \), where \( c \geq -1 \) depends on the type of \( V \) but not on \( r = \dim(V) \) or \( i \), and is given in the following table.

| Type of \( V \) | \( \text{Sp}(2r,q) \) | \( O^+(2r,q) \) | \( O(2r+1,q) \) | \( O^-(2r+2,q) \) | \( U(2r,q^{1/2}) \) | \( U(2r+1,q^{1/2}) \) |
|---------------|----------------|----------------|----------------|----------------|----------------|----------------|
| \( c \)       | 0              | \(-1\)         | 0              | \(-\frac{1}{2}\) | \( \frac{1}{2} \) |

**Proof.** For \( i = 1 \), \( |T^\perp - W^\perp| = |\Omega - W^\perp| \) is the number of points not perpendicular to the point \( W \), and is easily computed. For \( i \geq 2 \), \( T^\perp/T \) has rank \( r - i + 1 \) and the same type as \( V \); each of its points outside \( W^\perp/T \) corresponds to a coset (containing \( q^i-1 \) points) of \( T \) outside \( W^\perp \).

Throughout the rest of this section and the next, \( G \) will be assumed to act antiflag transitively on the geometry and primitively on the set \( \Omega \) of points. Let \( d + 1 \) denote the rank of \( G \) on points.

**Lemma 9.3.** For each point \( x \) there is a chain of \( G_x \)-invariant subspaces \( 0 = W_{-1}(x) \subset x = W_0(x) \subset W_1(x) \subset \cdots \subset W_d(x) = V \) with the following properties:

(i) \( W_i(x)^\perp = W_{d-i-1}(x) \) (in particular, \( W_i(x) \) is t.i. or t.s. if and only if \( i \leq \frac{d}{2}(d - 1) \));

(ii) \( G_x \) is transitive on \( W_i(x) - W_{i-1}(x) \) for each \( i \);

(iii) if \( y \in W_i(x) \) and \( 0 \leq i \leq d - 1 \), then \( W_i(y) \subseteq W_{i+1}(x) \);

(iv) \( W_i(x^g) = W_i(x)^g \) for all \( i, x, g \); and

(v) \( W_i(x) \cap W_i(y) \) is a hyperplane of \( W_i(x) \) if \( y \in W_i(x) - x \) and \( d \geq 4 \).

**Proof.** Let \( L \) be a line on \( x \) fixed by some \( P \in \text{Syl}_p(G_x) \). For \( y \in L - x \), all \( P_y \)-orbits on \( V - x^\perp \) have length at least \( q^{(2r-1+c)-1} \) by (9.2), so \( P_y \) is transitive on \( y^\perp - L^\perp \) (again by (9.2)). Set \( W_i(y) = (L^g \mid g \in G_y) \). Then \( y \in L \subset y^\perp \), so \( y \in W_i(y) \subset y^\perp \). Moreover,

\[
W_i(y)^\perp = \bigcap \{ (L^g)^9 \mid g \in G_y \},
\]

and \( G_y \) is transitive on \( y^\perp - W_i(y)^\perp \). (In particular, \( W_i(y)^\perp \) does not depend on \( x \); it is the unique \( G_y \)-invariant subspace \( U \) of \( y^\perp \) such that \( G_y \) is transitive on \( y^\perp - U \).) Define \( W_i(y)^\perp = W_i(y)^9 \) for all \( g \in G \).

If \( W_i(x)^\perp = x \), we are finished (and \( d = 2 \)). So suppose \( W_i(x)^\perp \neq x \). Then \( W_i(x) \cup W_i(x)^\perp \neq x^\perp \), by (9.1). Since \( G_x \) is transitive on \( x^\perp - W_i(x)^\perp \), it follows that \( W_i(x) \subseteq W_i(x)^\perp \), that is, \( W_i(x) \) is t.i. or t.s. (in the characteristic 2 orthogonal case \( W_i(x) \) is t.s. since it is totally isotropic and spanned by t.s. subspaces). Also, \( G_x \) is transitive on \( W_i(x) - x \). (For, \( W_i(x) \) is naturally isomorphic to the dual space of \( V/W_1(x)^\perp \). Now \( G_x \) has two orbits on the points of \( V/W_1(x)^\perp \), namely those
in $x^\perp/W_1(x)^\perp$ and those not in $x^\perp/W_1(x)^\perp$; so it has two orbits on the points of $W_1(x)$, namely $x$ and $W_1(x) - x$.

Now proceed by induction, assuming that $1 \leq i \leq \frac{1}{4}(d - 1)$ and that subspaces $W_j(x)$ and $W_{d-j-1}(x)$ have been defined for $-1 \leq j \leq i$ satisfying (i)-(iv). Set $m = \dim W_i(x)$. By (ii) and (9.2), the $P$-orbits on $V - W_i(x)^\perp$ have length at least $q^{2r - m + c}$, and hence the $P_y$-orbits have length at least $q^{2r - m + c - 1}$. We may assume that $m \neq r$, since otherwise we are finished. Again by (9.2), $q^{2r - m + c - 1} \geq q^m > |W_i(x)|$. As above, $W_i(y) \subseteq W_i(x)^\perp$ and $⟨W_i(y), W_i(x)⟩$ is t.i. or t.s., where $W_i(x) \neq W_i(y)$ by primitivity. Since $P_y$ acts on $W_i(y)^\perp - ⟨W_i(y), W_i(x)⟩$ with orbit lengths at least $q^{2r - (m+1) + c}$, (9.2) implies that $W_i(y)$ is a hyperplane of $⟨W_i(y), W_i(x)⟩$ and $P_y$ is transitive on $W_i(y)^\perp - ⟨W_i(y), W_i(x)⟩$.

Set $W_{i+1}(y) = ⟨W_i(y), W_i(x)^g | g \in G_y⟩ \subseteq W_i(y)^\perp$. Then $G_y$ fixes $W_{i+1}(y)$ and is transitive on both $W_i(y)^\perp - W_{i+1}(y)^\perp$ and $W_{i+1}(y) - W_i(y)$. (For, as before, $W_{i+1}(y)$ is naturally isomorphic to the dual space of $V/W_{i+1}(y)^\perp$, and $G_y$ has exactly $i + 1$ point-orbits $(W_{j-1}(y)^\perp/W_{i+1}(y)^\perp) - (W_{j-1}(y)^\perp/W_{i+1}(y)^\perp)$, $0 \leq j \leq i$, on $V/W_{i+1}(y)^\perp$, while acting on $i + 1$ subsets $W_{j-1}(y) - W_{j-1}(y)$ of $W_{i+1}(y)$.)

Now (i,ii,iv) hold, while (iii) follows from the definition of $W_{i+1}(y)$ if $i < \frac{1}{4}(d - 1)$, and from $W_{d-i-1}(y) \supseteq W_{d-(i+1)}(y)$ if $i > \frac{1}{2}(d - 1)$. This completes the inductive step.

Finally, (v) was proved in our argument when $i = 1$, since $m \neq r$ in that case.

Definition. The geometry $\mathcal{G}$ consists of the points of $\Omega$, together with those lines ($\mathcal{G}$-lines) joining $x$ to points of $W_1(x)$ for all $x \in \Omega$. The point graph of $\mathcal{G}$ is $\Gamma$. By (4.1), if $y$ is a point of a $\mathcal{G}$-line $L$ then $L \subseteq W_1(y)$.

Lemma 9.4. (i) $\Gamma$ is metrically regular.

(ii) $d \leq 4$.

(iii) If $V$ has type $O(2r + 1, q)$, then the conclusions of Theorem IV hold.

(iv) If $d = 2$ then the conclusions of Theorem IV hold.

Proof. (i) This follows from (9.3ii-iv).

(ii) If $d \geq 5$ then $W_2(x)$ is t.i. or t.s., and hence satisfies axiom (g) in Section 3, so (3.1) yields a contradiction.

(iii) Recall that $W_2(x)$ is either $x^\perp$ or t.s., and hence $|W_2(x)| = (q^h - 1)/(q - 1)$ for some $h$. If $d = 2$ then $G$ has rank 3 on points, and Kantor-Liebler [14, (1.3)] applies, since $q$ is odd. If $d = 3$ then (3.1) and (3.2) show that $\mathcal{G}$ is the generalized hexagon associated with $G_2(q)$, embedded naturally in $V$ of type $O(7, q)$. Then $G \cong G_2(q)$ as at the end of Section 7.

(iv) Use Kantor-Liebler [14, (1.3), (6.1)] (since we have excluded the symplectic case).

Notation. $e_2 = e$ and $f_1 = f$ are defined as in Section 3; $W(x) = W_1(x)$, and $m = \dim W(x)$.

Lemma 9.5. $d = 4$ is impossible.

Proof. If $d = 4$ then the chain in (9.3) is

$$0 \subset x \subset W(x) \subset W(x)^\perp \subset x^\perp \subset V;$$

the differences being orbits of $G_2$. By (9.3v), $f = m - 1$. Let $N_{r-m}$ denote the number of points of $W(x)^\perp/W(x)$. Then $|W(x)^\perp - W(x)| = q^m N_{r-m}$, as in the
proof of (9.2). As in Section 3, a count of pairs \((y, z)\) with \(d(x, y) = d(y, z) = 1, d(x, z) = 2\), yields

\[(q^m - q)(q^m - q^{m-1}) = q^m N_{r-m}(q-1)(q^e - 1).\]

Thus, \(e\) divides \(m - 1\).

Since \(W(x) \neq W(y)\) for \(x \neq y\), \(W(x)\) is not a clique. Let \(y, z \in W(x)\) be nonadjacent points. Then

\[e = \dim W(y) \cap W(z) \geq \dim W(x) \cap W(y) \cap W(z) \geq m - 2,\]

since \(W(x) \cap W(y)\) and \(W(x) \cap W(z)\) are hyperplanes of \(W(x)\). Now \(q^{m-1} - 1 = N_{r-m}(q^e - 1) > q^e - 1, e \mid m - 1\) and \(e \geq m - 2\) force \(m \leq 3\). Clearly \(m > 2\), since \(\Gamma\) is connected. Thus, \(m = 3\) and \(N_{r-m} = (q^2 - 1)/(q^e - 1)\). Then \(e = 1\) and \(|W_2(x)| = |W_1(x)| + q^m N_{r-m} = (q^3 - 1)/(q - 1)\), in contradiction to (3.1).

10. The case \(d = 3\)

In this section we continue the proof of Theorem IV in the primitive case. By Section 9 we may assume that \(d = 3\) and \(V\) is not of type \(O(2r + 1, q)\). The chain of subspaces in (9.3) is now

\[0 \subset x \subset W(x) \subset x^\perp \subset V;\]

with \(W(x)\) maximal t.i. or t.s. Set \(k = |x^\perp - x|\) and \(v_i = (q^i - 1)/(q - 1)\).

**Lemma 10.1.** \(V\) has type \(O^+(2r, q)\) with \(r = 4, 5\) or \(6\), while \(f = r - 2\) and \(e = 2\).

**Proof.** As usual, count the pairs \((y, z)\) with \(d(x, y) = d(y, z) = 1, d(x, z) = 2\), this time obtaining

\[(v_r - 1)(v_r - v_f) = (k - (v_r - 1))v_e.\]

In particular, \(k \leq (v_r - 1)(v_r - v_f + 1) \leq v_r(v_r - 1)\). However, \(k\) is easily computed for each type, and the types \(O^-(2r + 2, q)\) and \(U(2r + 1, q^{1/2})\) fail to satisfy this inequality. Moreover, in the case \(U(2r, q^{1/2})\), we have \(k = q(q^{r-1} - 1)(q^{r-3/2} + 1)/(q - 1)\), whence

\[v_r - v_f = q^{r-3/2}v_e,\]

and \(f = r - 3/2\), which is absurd.

Thus, \(V\) has type \(O^+(2r, q)\). This time,

\[k - v_r + 1 = q^{r-1}(q^{r-1} - 1)/(q - 1) = q^{r-2}(v_r - 1),\]

so

\[q^{r-2}v_e = v_r - v_f,\]

whence \(f = r - 2, e = r - f = 2\). Since \(W(x) \cap W(y) \supseteq (x, y)\) for \(y \in W(x) - x, f \geq 2\), so \(r \geq 4\).

Let \(y, z\) be nonadjacent vertices in \(W(x)\). Then

\[2 = e \geq \dim W(x) \cap W(y) \cap W(z) \geq 2(r-2) - r \geq r - 4,\]

whence \(r \leq 6\), as required.
Lemma 10.2. \( r = 4 \).

Proof. Suppose \( r = 5 \) or \( 6 \). If \( \langle x, y \rangle \) is a \( \mathcal{G} \)-line then \( \dim W(x) \cap W(y) = f = r - 2 > 2 \), so there is a point \( z \in W(x) \cap W(y) - \langle x, y \rangle \). Call the span of three noncollinear but pairwise adjacent points a special plane; note that all lines of a special plane belong to \( \mathcal{G} \) (since \( \langle x, y \rangle \subset W(z) \)). Then

\[
| W(x) \cap W(y) - \langle x, y \rangle | = \frac{q^f - q^2}{q - 1},
\]

so \( \langle x, y \rangle \) lies in exactly \( \frac{q^f - 2}{q - 1} \) special planes. If \( f = r - 2 = 3 \), this number is 1, so the number of special planes is

\[
v_5(q^4 + 1) \cdot v_4 \cdot 1/(q^2 + q + 1)(q + 1),
\]

which is not an integer. So \( r = 6 \) and \( f = 4 \).

In this case, we will show that the \( \mathcal{G} \)-lines and special planes that pass through \( x \) form a generalized pentagon with parameters \( q, q \), contradicting the Feit-Higman Theorem (2.3).

Any special plane through \( x \) contains \( q + 1 \) \( \mathcal{G} \)-lines through \( x \), and any such \( \mathcal{G} \)-line lies in \( \frac{q^f - 2}{q - 1} = q + 1 \) special planes. If \( \langle x, y \rangle \) and \( \langle x, z \rangle \) are \( \mathcal{G} \)-lines through \( x \) not contained in a special plane, tightness in the inequalities \((*)\) shows that \( W(x) \cap W(y) \cap W(z) \) is a \( \mathcal{G} \)-line through \( x \), the unique such \( \mathcal{G} \)-line lying in special planes with both \( \langle x, y \rangle \) and \( \langle x, z \rangle \). Now elementary counting verifies axioms (i)-(iii) for a generalized pentagon in Section 2, which yields the desired contradiction.

There are several ways to handle the case \( r = 4 \). One is to show that \( \mathcal{G} \) is a "dual polar space" (of type \( O(7, q) \)) in the sense of Cameron [3]; another is to quote transitivity results in Kantor-Liebler [14, Sect. 5]. The method used here involves triality, a concept which we now briefly discuss; we will see that triality is involved in the embedding appearing in Theorem IV(iii). We refer to [22] for further discussion of triality.

Let \( \mathcal{P} \) be the set of points of the geometry of type \( O^+(8, q) \), \( \mathcal{L} \) the set of lines, and \( \mathcal{M}_1 \) and \( \mathcal{M}_2 \) the two families of solids (maximal t.s. subspaces); thus, any t.s. plane lies in a unique member of each family. More generally, two solids lie in the same family if and only if their intersection has even dimension. The geometry admits a "triality automorphism" \( \tau \) mapping \( \mathcal{L} \to \mathcal{L} \) and \( \mathcal{P} \to \mathcal{M}_1 \to \mathcal{M}_2 \to \mathcal{P} \) and preserving the natural incidence between \( \mathcal{P} \cup \mathcal{M}_1 \cup \mathcal{M}_2 \) and \( \mathcal{L} \) (defined by inclusion or reverse inclusion). Also, \( \tau \) preserves the "incidence" on \( \mathcal{P} \cup \mathcal{M}_1 \cup \mathcal{M}_2 \), in which a solid is incident with a point contained in it, and two solids are incident if they meet in a plane. This "automorphism" induces an automorphism of \( P\Omega^+(8, q) \).

Before continuing with the proof, we outline the way in which the examples of Theorem IV(iii) arise. Let \( v \) be a nonsingular vector, so that \( v^\perp \cap \mathcal{P} \) carries a geometry of type \( O(7, q) \). If \( M_i \in \mathcal{M}_i \) \((i = 1, 2)\), then \( v^\perp \cap M_i \) is a plane, contained in a unique member \( M^* \) of \( \mathcal{M}_{3-i} \); thus \( v \) induces bijections between \( \mathcal{M}_1 \), \( \mathcal{M}_2 \) and the set of planes (maximal t.s. subspaces) of \( v^\perp \cap \mathcal{P} \). These bijections are invariant under \( G = \Omega^+(8, q)_v \), which acts transitively on each set. Now apply triality: \( G^\tau \) is an irreducible subgroup of \( \Omega^+(8, q) \), transitive on \( \mathcal{M}_2 = \mathcal{P} \), and preserving a "geometry" on \( \mathcal{P} \) isomorphic to the dual polar space of t.s. planes of \( v^\perp \cap \mathcal{P} \). (Strictly, here and below, in place of \( G^\tau \) we use the inverse image in \( \Omega^+(8, q) \) of \( (G/Z)^\tau \), where \( Z = Z(\Omega^+(8, q)) \).) \( G \) is transitively on disjoint pairs of t.s. planes of \( v^\perp \cap \mathcal{P} \), and hence on disjoint pairs of elements of \( \mathcal{M}_2 \); hence \( G^\tau \) is transitive
on nonperpendicular members of $\mathcal{P}$, that is, antiflag transitive. Note that $G^r$ and $G^{r^{-1}}$ lie in different conjugacy classes in $\Omega^+(8, q)$. Note also that $G^r = \Omega(7, q)$ only if $q$ is even; for $q$ odd, $G^r$ contains the element $-1 \in \Omega^+(8, q)$.

The process can be continued one further time. If $w \in V$ is a nonsingular vector, then $(G^r)^w$ acts transitively (and even antiflag transitively) on $w^+ \cap \mathcal{P}$, preserving a geometry that is the $G_2(q)$ hexagon, naturally embedded.

We return to the proof. There are $(q^4 - 1)(q^3 + 1)/(q - 1) = (q + 1)(q^2 + 1)(q^3 + 1)$ points, and equally many subspaces $W(x)$. Since $f = e = 2$, $\dim W(x) \cap W(y) = 2$ or $0$ for $x \neq y$, and so all subspaces $W(x)$ belong to the same family; without loss of generality, $\{W(x) \mid x \in \mathcal{P}\} = \mathcal{M}_1$.

Call $M \in \mathcal{M}_2$ special if it contains a $\mathcal{G}$-line $L$. If $x$ is a point of $M - L$, then $\langle x, L \rangle$ is contained in a unique member $W(y)$ of $\mathcal{M}_1$, and $M \cap W(y) = \langle x, L \rangle$. Since $\mathcal{G}$ has no triangles (as $f = 2$), we have $y \in L$, and $\langle x, y \rangle$ is a $\mathcal{G}$-line. Thus, the points and $\mathcal{G}$-lines in $M$ form a generalized quadrangle using all points of $M$. Then $x \leftrightarrow M \cap W(x)$ is a symplectic polarity of $M$ whose absolute lines are the $\mathcal{G}$-lines in $M$; so the quadrangle is of type $Sp(4, q)$.

Let $A$ be the set of special solids, so $A^r$ is a set of points. We claim that, if $U$ is a solid, then $U \cap A^r$ is a t.s. plane. For, $U^r \cap A$ is the set of special solids incident with $U^{r^{-1}}$, which is either a point $x$ or a solid $W(x)$, so $U^{r^{-1}} \cap A$ is the set of special solids $M$ such that $M \cap W(x)$ is a plane of $W(x)$ containing $x$. That set is the set of special solids incident with the incident pair $x, W(x)$, and hence the set of solids incident with the pair $x, W(x)$. Its $\tau$-image is the set of points incident with an incident pair $x^\tau, W(x)^\tau$ of solids, and hence is a t.s. plane of $V$.

The following result now identifies $A^r$ (and hence $A$).

**Theorem 10.3.** Let $\Phi$ be a subset of $\Omega$, the point set of a geometry of type $O^+(2r, q)$, $r \geq 3$. Suppose that, for every t.s. $r$-space $U$ of $\Omega$, $\Phi \cap U$ is an $(r - 1)$-space. Then $\Phi = \Omega \cap v^\perp$ for some nonsingular vector $v$.

**Proof.** We treat first the case $r = 3$. Identify $\Omega$ (the Klein quadric) with the set of lines of $PG(3, q)$. Then a t.s. plane of $\Omega$ is either the set of lines on a point or the set of lines in a plane; and a line of $\Omega$ is the set of lines in a plane $E$ and on a point $x \in E$. Thus, under this identification, $\Phi$ is a set of lines of $PG(3, q)$ having the property that the members of $\Phi$ on a point $x$ all lie in a plane $E$, while those in a plane $E$ all contain a point $x$. Then $x \leftrightarrow E$ is a symplectic polarity, and $\Phi$ is its set of t.i. lines. A symplectic polarity of $PG(3, q)$ can be identified with a point $v$ outside the Klein quadric $\Omega$, its t.i. lines corresponding to points of $\Omega \cap v^\perp$.

For $r > 3$ we use induction. If $x, y$ are nonperpendicular points of $\Phi$, then $\Omega \cap \langle x, y \rangle^{\perp} = \Omega$ is of type $O^+(2r - 2, q)$. We claim that $\Phi \cap \langle x, y \rangle^{\perp} = \Phi'$ satisfies the conditions of the theorem in $\Omega'$ (with $r - 1$ replacing $r$). If $U$ is a t.s. $(r - 1)$-space in $\Omega'$, then $\langle x, U \rangle$ is a t.s. $r$-space, and $\Phi \cap \langle x, U \rangle$ is an $(r - 1)$-space containing $x$ by hypothesis; so $\Phi \cap U = \Phi' \cap U$ is an $(r - 2)$-space. By induction, $\Phi \cap \langle x, y \rangle^{\perp} = \Omega \cap \langle x, y, v \rangle^{\perp}$ for a nonsingular vector $v \in \langle x, y \rangle^{\perp}$.

For $a, b \in \Phi$ distinct and perpendicular, a t.s. $r$-space $U$ containing $a$ and $b$ produces an $(r - 1)$-space $\Phi \cap U$, so $(a, b) \subseteq \Phi \cap U \subseteq \Phi$.

In particular, $\Phi \cap x^\perp \supseteq \langle x, b \rangle \mid b \in \Phi \cap \langle x, y \rangle^{\perp} = \Omega \cap \langle x, v \rangle^{\perp}$ and $\Phi \cap y^\perp \supseteq \Omega \cap \langle y, v \rangle^{\perp}$. Every point of $\Omega \cap v^\perp$ lies on a line meeting $\Omega \cap \langle x, v \rangle^{\perp}$ and $\Omega \cap \langle y, v \rangle^{\perp}$ in different points, so $\Omega \cap v^\perp \subseteq \Phi$. Finally, $\Phi \subseteq \Omega \cap v^\perp$, since each point of $\Omega - v^\perp$.
is in a t.s. r-space properly containing a t.s. \((r-1)\)-space of \(v^\perp\), and hence cannot lie in \(\Phi\).

Remark. The theorem fails for \(r = 2, q > 3\): \(\Omega\) is a ruled quadric (a \((q+1)\times(q+1)\) square lattice), and there are \((q+1)!\) sets \(\Phi\) satisfying the hypothesis of (10.3), only \((q+1)q(q-1)\) of which are conics.

Completion of the proof of the primitive case of Theorem IV. It remains to identify \(G\). Let \(H\) be the group induced by \(G^r\) on the \(O(7, q)\) geometry \(\Lambda^r < v^\perp\). Then \(H\) is transitive and has rank 4 on the set of t.s. planes contained in \(\Lambda^r\). (For, \(H\) has rank 4 on \(\mathcal{P}^r = \mathcal{M}_1\) and hence on the set of planes \(W(x) \cap \Lambda^r\).) We will use the action on these planes to show that \(H\) contains \(\Omega(v^\perp)\).

If \(E\) is a plane, then \(H_E\) is transitive on the t.s. planes meeting \(E\) in a line, so \(H^r_E\) is line-transitive. Also, \(H_E\) is transitive on the \(q^6\) t.s. planes disjoint from \(E\). Since any point outside \(E\) lies on \(q^3\) such planes, every point-orbit outside \(E\) of a Sylow \(p\)-subgroup \(P\) of \(H_E\) has length divisible by \(q^3\). Let \(L\) be a line of \(E\) fixed by \(P\). Since \(L\) only lies in \(q\) t.s. planes \(E' \neq E\), each of which has \(q^2\) points outside \(E\), it follows that \(P_{E'}\) is transitive on \(E' - E\). By (2.1), \(H^r_E \geq SL(E)\). If \(x\) is any point of \(E\) then \(C_H(x)_E\) is transitive on \(E/x\). Since \(E\) can be any t.s. plane of \(v^\perp\) on \(x\), it follows that \(C_H(x)\) is transitive on \(x^+/x\).

Let \(Q\) denote the centralizer of both \(x\) and \(x^+/x\) in \(\Omega(7, q)\). We have \(q^0 \leq |H|_p = |H \cap Q||H_{x^+/x}|_p \leq |H \cap Q|q^4e\) since \(H_{x^+/x} \leq \Gamma O(5, q)\), so \(H \cap Q \neq 1\). But \(Q\) is elementary abelian of order \(q^5\), and is \(C_H(x)\)-isomorphic to \(x^+/x\). Then \(C_H(x)\) acts irreducibly on \(Q\), and hence \(H \cap Q = Q\). If \(h \in H\) and \(x^h \notin x^\perp\), then \(H \geq \langle Q, Q^h \rangle = \Omega(7, q)\).

This completes the primitive case of Theorem IV.

11. The imprimitive case

Throughout this section (which corresponds roughly to Section 7), \(G\) satisfies the hypotheses of Theorem IV and is imprimitive on points. We are assuming that \(V\) has rank \(r \geq 3\).

Let \(\Delta\) be a proper block of imprimitivity for \(G\) on \(\Omega\). Then \(G_{\Delta^\perp}\) is transitive, while \(G_{\Delta} = G_{x_\Delta}\) is transitive on \(V - x^\perp\) for \(x \in \Delta\). Thus, \(\Delta \subseteq \Delta^\perp\), and \(G_{\Delta}\) is transitive on \(V - \Delta^\perp\). Then \(\langle \Delta \rangle\) is t.i. or t.s., and (by the duality between \(V/\langle \Delta \rangle^\perp\) and \(\langle \Delta \rangle\)) \(G_{\Delta}\) is transitive on \(\langle \Delta \rangle\). Thus, \(\Delta = \langle \Delta \rangle\) is a t.i. or t.s. subspace and \(G_{\Delta}\) is antiflag transitive.

From now on, \(\Delta\) will be a minimal proper block of imprimitivity and \(x \in \Delta\). Set \(\delta = \dim \Delta\).

Lemma 11.1. \(x^\perp \cap \Delta^G = \Delta^\perp \cap \Delta^G\) partitions \(\Delta^\perp\).

Proof. This is clear if \(\Delta\) is a maximal t.i. or t.s. subspace, so assume that \(\delta < r\). Suppose \(\Delta' \in \Delta^G, \Delta' \cap \Delta^\perp \neq 0\) and \(\Delta' \not\subset \Delta^\perp\). Let \(y \in \Delta - \Delta^\perp\). Since \(G_y\) is transitive on \(\Omega - y^\perp\), it is transitive on \(\Delta^G - (y^\perp \cap \Delta^G)\), so every member of \(\Delta^G - (y^\perp \cap \Delta^G)\) meets \(\Delta^\perp - \Delta\). Since \(|\Delta' - y^\perp| = q^{r-1}\),

\[
|\Delta^\perp - \Delta| \geq |\Delta^G - (y^\perp \cap \Delta^G)| = |\Omega - y^\perp|/q^{\delta-1}.
\]

A check of each classical geometry (computing \(|\Delta^\perp - \Delta|\) as in (9.2)) shows that this inequality does not hold except in the case \(O^+(2r, 2)\).
Consider that case. We will use a different inequality that is stronger in that case. Since $G_\Delta$ is transitive on $\Delta$, every member of $\Delta^G - (\Delta^\perp \cap \Delta^G)$ arises for some $y$ as above, so $G_\Delta$ is transitive on this set. If $k = \dim(\Delta^\perp \cap \Delta')$ then

$$|\Delta^\perp - \Delta| \geq |\Delta^G - (\Delta^\perp \cap \Delta^G)|(2^k - 1) = \{\Omega - \Delta^\perp|/\Omega - \Delta^\perp|\} (2^k - 1)$$

(by counting in two ways the pairs $(z, \Delta''$) with $z \in \Delta'' = \Delta^\perp$ and $\Delta'' \in \Delta^G - (\Delta^\perp \cap \Delta^G)$). Also, $|\Delta|/\Omega$ implies that $r \neq \delta|r$. But this condition together with $|\Delta^\perp - \Delta| \geq \{\Omega - \Delta^\perp|/\Omega^\perp|\} (2^k - 1)$ never hold.

Thus, $\Delta' \in \Delta^G$ and $\Delta' \cap \Delta^\perp \neq 0$ imply that $\Delta' \subseteq \Delta^\perp$, so $\Delta^\perp$ is partitioned by $\Delta^\perp \cap \Delta^G$. Finally, if $\Delta' \in \Delta^G$ then either $\Delta' \subseteq \Delta^\perp \subseteq x^\perp$ or $\Delta' \cap \Delta^\perp = 0$, $\langle \Delta', \Delta^\perp \rangle = V$ and $\Delta^\perp \subseteq x^\perp$, so $\Delta' \nsubseteq x^\perp$.

**Corollary 11.2.** If $W$ is an intersection of subspaces $(\Delta^g)^\perp$, $g \in G$, then $W$ is partitioned by $W \cap \Delta^G$.

**Lemma 11.3.** Either

(i) there is a subspace $\Delta' \in \Delta^\perp \cap \Delta^G$, $\Delta' \neq \Delta$, or

(ii) $\dim \Delta = 4$, $V$ has type $O^+(8,2)$ and $G \equiv A_9$ is unique up to conjugacy in $O^+(8,2)$.

**Description of the example in (11.3ii).** Let $W = GF(2)^9$ be the permutation module for $H = A_9$ over $GF(2)$, and let $wt(v)$ be the number of nonzero coordinates of $v \in W$. Then $V = \{v \in W | wt(v) \equiv 0 (\mod 2)\}$ is an $O^+(8,2)$-space with quadratic form $\varphi(v) \equiv \frac{1}{2}wt(v) (\mod 2)$. Clearly, $H = H' \leq O^+(8,2)$, and $S = \{v | v \in V, wt(v) = 8\}$ is an $H$-orbit of 9 pairwise nonperpendicular points. Applying a triality automorphism $r$ (see Section 10 following the proof of (10.2)) produces the desired $H^r$-invariant set $S^r$ of t.s. 4-spaces in (11.3ii).

**Proof of Lemma.** Assume that (i) does not hold. By (11.1), $\Delta^\perp = \Delta$, so $\Delta$ is a maximal t.i. or t.s. subspace and $\delta = r$.

For distinct $\Delta, \Delta' \in \Delta^G$, $G^\Delta_{\Delta'}$ is antiflag transitive. (For, if $x \in \Delta$ then $G_{\Delta \Delta'}$ is transitive on the points $y \in \Delta' - x^\perp$ and hence on the hyperplanes $y^\perp \cap \Delta$ of $\Delta$ not on $x$, as asserted.) Since $\Delta$ is a minimal block, $G^\Delta_{\Delta'}$ is primitive. Moreover, $G_\Delta$ is transitive on $\Delta^G - (\Delta^\perp \cap \Delta^G) = \Delta^G - \{\Delta\}$, where $|\Delta^G| - 1 = q^{2r-1+c}/|\Delta' - (x^\perp \cap \Delta')| = q^{2r-1+c}/q^{c-1} = q^{2r-1+c}$ for $c$ in (9.2). Thus, $G^\Delta_{\Delta'}$ is primitive by (6.4), and hence is as in Theorem I or II.

For $g \in G_{\Delta \Delta'} \cap GL(V)$ of order $p$, let $k = \dim C_\Delta(g)$, in which case $g$ also centralizes a $k$-space of $\Delta$ (since $\Delta$ and $\Delta'$ are dual $(g)$-modules), as well as the anisotropic $(n-2r)$-space $\langle \Delta, \Delta' \rangle^\perp$ unless $V$ has type $O^-(2r+2,q)$ with $q$ even, in which case $g$ centralizes at least a 1-space of $\langle \Delta, \Delta' \rangle^\perp$. We claim that $k \leq (r+1)/2$. For otherwise, if $g$ centralizes $\langle \Delta, \Delta' \rangle^\perp$ then $\dim C_\Delta(g) \geq 2k + (n-2r) > n - r$; while if $g$ centralizes a 1-space of the anisotropic 2-space $\langle \Delta, \Delta' \rangle^\perp$ then once again $\dim C_\Delta(g) \geq 2k + 1 > n - r$. Thus, $C_\Delta(g)$ meets every member of $\Delta^G$ nontrivially and hence in a $k$-space. Now $C_\Delta(g)$ is a subspace having a non-zero t.i. or t.s. radical since $|g| = p$, and having exactly $(q^{r+c} + 1)(q^{r-1} - 1)$ t.i. or t.s. points with $k > (r + 1)/2$, which is impossible.

Since $\delta = r > 2$, it follows that $G^\Delta_{\Delta'}$ cannot contain nontrivial translations; and it cannot contain $G_2(q)$ by (5.4d). By Theorems I and II, the only remaining possibilities are $\delta = 4$, $q = 2$, and $G^\Delta_{\Delta'} = A_6$ or $A_7$. Now $G$ acts on $\Delta^G$ as a 2-transitive group of degree $q^{r+c} + 1 = 2^3 + 1$ or $2^5 + 1$ (for $V$ of type $O^+(8,2)$
and $V > 0$ that $\dim V > 8$. Thus, $\dim V > 8$.

From now on we will assume that (11.3i) holds. Then $\Delta$ is not a maximal t.i. or t.s. subspace.

Lemma 11.4.

(i) If $W \neq 0$, $V$ is a subspace such that $W = \langle W \cap G \rangle$ and $W^\perp = \langle W \cap \Omega \rangle$, then $W$ is an intersection of subspaces $(\Delta')^\perp$, $g \in G$, and is partitioned by $W \cap \Delta'$. 

(ii) If $W_i = (W_i \cap \Delta')$ for $i = 1, 2$, then $\langle W_1, W_2 \rangle \cap \Delta'$ partitions $\langle W_1, W_2 \rangle$.

Proof. (i) Since $W = \langle W \cap G \rangle$, $W^\perp$ is an intersection of subspaces $(\Delta')^\perp$, $g \in G$, and hence by (11.2) is partitioned by subspaces $\Delta'$. Since the bilinear form defining the geometry is nondegenerate (cf. the beginning of Part II), $W = W^\perp$ is an intersection of subspaces $(\Delta')^\perp$, and (11.2) applies again.

(ii) The set $S$ of points of $(W_1, W_2)$ lying in a member of $(W_1, W_2) \cap \Delta'$ clearly spans $(W_1, W_2)$. We claim that, if $s_1, s_2 \in S$ are perpendicular then $(s_1, s_2) \subseteq S$, so $\langle S \rangle = (W_1, W_2)$. Let $s_i \in \Delta_i \in (W_1, W_2) \cap \Delta'$ for $i = 1, 2$. Since $\delta < r$ by (11.3i), the subspace $\langle \Delta_1, \Delta_2 \rangle^\perp$ is spanned by its points; by (i), $\langle \Delta_1, \Delta_2 \rangle$ is partitioned by $\langle \Delta_1, \Delta_2 \rangle \cap \Delta'$, so $(s_1, s_2) \subseteq S$.

Lemma 11.5. $V$ is orthogonal and $\delta = 2$.

Proof. Choose $\Delta' \in \Delta'^\perp$, $\Delta' \not\subseteq \Delta$. Then $\Delta' \cap \Delta^\perp = 0$ by (11.1), so $W = \langle \Delta, \Delta' \rangle$ is nonsingular. By (11.3), $\delta < r$, so $W^\perp = \langle W^\perp \cap \Omega \rangle$ and $W^\perp \cap \Delta'$ partitions $W^\perp$ by (11.4i). If $y_1, y_2 \in W - x^\perp$, and $y_1 \in \Delta_i \in \Delta'$ ($i = 1, 2$), then $\Delta_i \subseteq W$ by (11.4i), so $W = \langle \Delta, \Delta, \Delta_i \rangle$ ($i = 1, 2$): an element of $G_x$ mapping $y_1$ to $y_2$ also maps $\Delta_1$ to $\Delta_2$ and so fixes $W$. Then $G^W$ is antiflag transitive and imprimitive. If we are in case (11.3i) for $G^W$, then induction implies that $\dim W = 2\delta = 4$. The possibility that $W$ is a 4-dimensional symplectic space was excluded at the start of Section 9, while the 4-dimensional unitary possibility is eliminated by Kantor-Liebler [14, (5.12)]. Thus, $V$ is orthogonal.

It remains to consider the possibility (11.3ii) for $G^W$, where we are assuming that $\dim V > 8$. Let $\Delta_1 \in W^\perp \cap \Delta'$, $\Delta_1' \in (W^\perp \cap \Delta') - \Delta_1^\perp$, $W_1 = \langle \Delta_1, \Delta_1' \rangle$ and $V' = \langle W_1 \rangle$, so $\dim V' \geq 16$. By (11.4ii), $V'$ is partitioned by $V' \cap \Delta'$. Let $\Delta_2 \in V' \cap \Delta'$, $\Delta_2 \not\subseteq W, W'$. Then $\Delta_2 \cap W'$ has dimension 12 and so meets both $W$ and $W'$ in subspaces of dimension $\geq 4$. By (11.4i), $\Delta_2 \cap W'$ contains some $\Delta_3 \in \Delta'$ and $\Delta_2^\perp \cap W'$ contains some $\Delta_4 \in \Delta'$. Then $\langle \Delta_2, \Delta_3, \Delta_4 \rangle$ is a t.s. subspace of $V'$ of dimension $> 8$, which is not possible. This rules out (11.3ii).

Definition. Let $\mathcal{L}$ be the set of all t.s. subspaces that are intersections of members of $(\Delta')^\perp$. By (11.4i), $\Delta' \subseteq \mathcal{L}$. Clearly $\mathcal{L}$ is closed under intersections.

Lemma 11.6. $\mathcal{L}$ is the set of all t.i. subspaces of a classical geometry of type $U(\frac{1}{2}n, q)$.
Proof. By (11.2), each member of $\mathcal{L}$ is partitioned by the members of $\Delta^G$ it contains. If $M$ is a maximal member of $\mathcal{L}$, then $M$ is a maximal t.s. subspace. (For if $x \in M^\perp - M$, then by (11.2) the member of $\Delta^G$ containing $x$ would be in $M^\perp$ and, together with $M$, would span a member of $\mathcal{L}$ by (11.4i).)

Assume that $r = \dim M > 4$. Then exactly as in the proof of (7.2), $M$ is a projective space with $M \cap \Delta^G$ its set of points and $q^d + 1 = q^d + 1$ points per line. If $\Delta \not\subseteq M = M^\perp$ then $\Delta^\perp \cap M = (\Delta, M^\perp)^\perp$ has dimension $n - (2 + n - r) = r - 2$, so $\Delta^\perp \cap M$ is a hyperplane of our new projective space $M$.

Note that any $N \in \mathcal{L}$ of dimension $r - \delta = r - 2$ lies in at least two maximal members of $\mathcal{L}$: by (11.4i), those maximal members induce a partition of $N^\perp - N$. If $M$ and $M'$ are r-spaces in $\mathcal{L}$ with nonzero intersection, let $N \subseteq M$ be an $(r - 2)$-space in $\mathcal{L}$ with $M' \cap N \subseteq M' \cap M$. Then $N^\perp \cap M = M' \cap M$ since $\langle N, N^\perp \cap M \rangle$ is t.s. and contains $\langle N, M' \cap M \rangle = M'$.

If $M'' \not\subseteq M'$ is an r-space in $\mathcal{L}$ containing $N$, then $M'' \cap M \subseteq N^\perp \cap M = M' \cap M$; and $M'' \cap M \subseteq M' \cap M$, since otherwise $M'' \not\subseteq \langle N, M'' \cap M \rangle = M'$. Continuing, we find that there exist disjoint r-spaces in $\mathcal{L}$.

It follows from Tits [23] that $\mathcal{L}$ is a classical polar space since $r/2 > 2$.

Now if $M$ and $M'$ are disjoint maximal subspaces of $\mathcal{L}$ and $\langle M, M' \rangle \not\subseteq V$, then there is a member of $\Delta^G$ disjoint from $\langle M, M' \rangle$. So $n = \dim V = 2r$ or $2r + 2$, where $r$ is even. If $n = 2r + 2$ then $V$ has type $O^-(2r + 2, q)$ and so has $(q^{r+1} + 1)(q^r - 1)/q - 1$ points; then $|\Delta^G| = (q^{r+1} + 1)(q^r - 1)/q^2 - 1$, and $\mathcal{L}$ is of type $U(r + 1, q)$. Similarly, if $n = 2r$, then $V$ has type $O^+(2r, q)$, and the same argument shows $\mathcal{L}$ has type $U(r, q)$. In all dimensions the results of Tits [23] show that the embedding of $\mathcal{L}$ in $V(\frac{n}{2}, q^2)$ is the natural one.

Next suppose that $r = 4$. Then $\mathcal{L}$ is the lattice of points and lines of a geometry $\mathcal{G}$. Arguing as above, we find that $\mathcal{G}$ is a generalized quadrangle with $s = q^2$, and $t = q$ or $q^3$ according as $V$ has type $O^+(8, q)$ or $O^-(10, q)$.

If $\Delta \not\subseteq \Delta^\perp$, then $\langle \Delta, \Delta' \rangle$ has type $O^+(4, q)$ and $|\langle \Delta, \Delta' \rangle \cap \Delta^G| = q + 1$ by (11.4i). For any $\Delta'' \in \langle \Delta, \Delta' \rangle \cap \Delta^G$, $(\Delta'')^\perp \supseteq \Delta^\perp \cap (\Delta')^\perp$. Thus, if $t = q^3$ then a theorem of Thas [21] and its proof identify the quadrangle as that of type $U(5, q)$, with uniqueness of the embedding.

If $t = q$, the points and lines of the quadrangle are certain lines and solids of the $O^+(8, q)$ geometry. Any two of the solids are disjoint or meet in a line, so they all belong to the same class. Applying the triality map (cf. Section 10), the dual quadrangle is embedded as a set of points and lines in an $O^+(8, q)$ geometry, satisfying the hypotheses of Buekenhout-Lefèvre [1, Theorem 1]. Thus the dual of $\mathcal{L}$ is of type $O^-(6, q)$ in its natural embedding, and $\mathcal{L}$ is of type $U(4, q)$ also embedded naturally. This proves (11.6).

We can now complete the proof of Theorem IV. By (11.6), $\mathcal{L}$ is embedded naturally in a projective space derived from a vector space $V(\frac{n}{2}, q^2)$. Proceeding as in Section 7, we obtain the original space $V$ by restricting the scalars; repeat the argument in that section (Proof of (6.1), second paragraph) to show that either $q = \delta = 2$ or $\epsilon = \delta = 2$, $q = 4$, and that $G$ is primitive and antiflag transitive on the $U(\frac{n}{2}, q)$ geometry. If $\frac{n}{2} \geq 6$, then this geometry has rank $\geq 3$; Section 10 does not provide any unitary examples so (9.4iv) implies that $SU(\frac{n}{2}, q) \leq G \leq GU(\frac{n}{2}, q) < GO(\frac{n}{2}, q)$, as required. If $\frac{n}{2} = 4$ or 5 then $G \leq GU(\frac{n}{2}, q)$ with $q = 2$ or 4, and $G \supseteq SU(\frac{n}{2}, q)$ by Kantor-Liebler [14, (5.12)]. (As in Section 7, $G$ must contain $q = 2$ or 4 field automorphisms in order to have $G^A_3$ antiflag transitive.)
12. Rank 4 subgroups of rank 3 groups

In this section, $G$ will denote a primitive rank 3 permutation group on a set $X$, and $H$ a subgroup of $G$ having rank 4 on $X$.

Let $k, l, \lambda, \mu$ be the usual parameters for $G$, as defined in Higman [9], and let $I, A, B$ be the adjacency matrices corresponding to the orbits $\{x\}, \Delta(x)$ and $\Gamma(x)$ of $G_x$, $x \in X$. If $k, r, s$ are the eigenvalues of $A$, then $\lambda = k + r + rs$, $\mu = k + rs$, $k(k - \lambda - 1) = l\mu$.

We assume that $H_x$ splits $\Gamma(x)$ into two orbits $\Gamma_1(x)$ and $\Gamma_2(x)$, of lengths $j$, $l - j$ and with adjacency matrices $C, B - C$ respectively. Set $jt = |\Gamma_1(x) \cap \Delta(y)|$ for $y \in \Gamma_2(x)$. Then, with respect to the $\Delta$-graph, the intersection numbers for $H$ are as in the following diagram.

Then $AC = (k - \lambda - 1)(j/l)A + (k - \mu - (l - j)t)C + jt(B - C)$. Applying this to an eigenvector of $A$ and $C$ with eigenvalues $r, \theta$, respectively, yields

$$r\theta = (k - \lambda - 1)(j/l)r + (k - \mu - (l - j)t)\theta + jt(-r - 1 - \theta).$$

(Since $A + B + I$ is the all $-1$ matrix, $-r - 1$ is an eigenvalue of $B$.) Simplifying, $$(r(s + 1) + lt)\theta = -(j/l)(r + 1)(r(s + 1) + lt).$$

Similarly, if $\varphi$ is an eigenvalue of $C$ corresponding to the eigenvalue $s$ of $A$, $$(s(r + 1) + lt)\varphi = -(j/l)(s + 1)(s(r + 1) + lt).$$

But the centralizer algebra of $H$ has dimension 4, so exactly one of the eigenspaces of $A$ must split into two eigenspaces for $C$. If this corresponds to $r$, then $\theta$ is not unique, so

$$r(s + 1) + lt = 0.$$  

Since $r \neq s$, it follows that

$$\varphi = -j(s + 1)/l.$$  

But $\varphi$ must be an integer, so

$$l/(l, s + 1) \text{ divides } j.$$
Also, for \( y \in \Gamma_1(x) \), \(|\Delta(x) \cap \Gamma_2(y)| = j(l - j)t/k \), so
\[
(12.4) \quad kl \text{ divides } j(l - j)r(s + 1).
\]

Remark. Of course, the same results hold in a more general situation (involving association schemes).

13. Theorem V

The proof of Theorem V follows (and was inspired by) the pattern of Perin’s Theorem [20] discussed in Section 8. Suppose that \( G \) satisfies the hypotheses of Theorem V. If \( G_x \) is transitive on the points outside \( x^\perp \), then \( G \) is antiflag transitive, and Theorem IV applies. So we may assume that \( G_x \) is transitive on \( x^\perp - x \) and splits \( V - x^\perp \) into two orbits. Then \( G \) is transitive on t.i. or t.s. lines. We use the notation of the last section.

Suppose first that \( G \leq \Gamma Sp(2m, q) \). One or both of \( q^{m-1} - 1 \) and \( q^{2(m-1)} - 1 \) have a primitive divisor \( r \) (see (2.4)); let \( R \in \text{Syl}_r(G_x) \). Then \( W = C_V(R) \) is a nonsingular 2-space and \( N_G(T)^R \) has rank at most 3. If \( G_x \) has two orbits on the nonsingular 2-spaces containing \( x \), then the stabilizer of any projective line (singular or not) acts 2-transitively on it. By (4.1), \( G \) is antiflag transitive, contrary to assumption. So \( G_x \) is transitive on the \( q^{2m-2} \) nonsingular 2-spaces containing \( x \), and \( G^W_W \) has rank 3; call the subdegrees 1, \( h, q - h \). As in (8.5), \( (q, h) = (2, 1), (3, 1), (4, 2), (5, 1) \) or (9, 3).

We have \( k = q(q^{2m-2} - 1)/(q-1), l = q^{2m-1}, j = q^{2m-2}h \). Also, \( r, s = \pm q^{m-1} \). By (12.4),
\[
q^{2m}(q^{2m-2} - 1)/(q - 1) \text{ divides } q^{4m-4}h(q - h)q^{m-1}(q^{m-1} + 1),
\]
whence
\[
q^{m-1} \mp 1 \text{ divides } (q - 1)h(q - h).
\]

This is impossible if \( m \geq 4 \); and none of the specific values of \( q \) and \( h \) satisfy it when \( m = 3 \). So this case cannot occur.

The case \( V \) unitary is ruled out by Kantor-Liebler [14, (6.1)].

Suppose \( G \leq \Gamma O(2m + 1, q), m \geq 3, q \) odd. Let \( r \) be a primitive divisor of \( q^{2m-2} - 1 \), and \( R \in \text{Syl}_r(G_x) \). Then \( W = C_V(R) \) is a nonsingular 3-space, and \( N_G(R)^W \) has rank 2 or 3. (Rank 4 does not occur since \( W \) does not contain any t.s. line and hence does not contain any point of \( x^\perp - x \).) If \( q > 5 \) then \( N_G(R)^W \) contains \( \Omega(3, q) \) or (if \( q = 9 \)) \( A_5 \), using [5, Chap. 12], and hence so does \( C_G(W^\perp)^W \) since \( N_G(R)^W \) is solvable. Then the argument used in (6.2) (i.e., using a group behaving like \( Q \)) shows that \( G \geq \Omega(2m + 1, q) \), which is a contradiction since that group has rank 3 on points. Thus, \( q \leq 5 \).

Suppose \( G \leq \Gamma O(2m + 1, 5), m \geq 3 \). In addition to \( r \) we will use a primitive divisor \( r^o \) of \( 5^{m-1} - 1 \); let \( R^o \in \text{Syl}_{r^o}(G_x) \) and \( W^o = C_V(R^o) \). As above, we may assume that \( N_G(R)^W \) and \( N_G(R^o)^W \) do not contain \( \Omega(3, 5) \); both are rank 3 groups that therefore contain \( S_4 \). Since \( N_G(R)^W \leq \Gamma O^+(2, 5^{m-1}) \) and \( N_G(R^o)^W \leq \Gamma O^+(2, 5^{m-1}) \) are metacyclic, \( C_G(W^\perp) \cap \Omega(V) \) and \( C_G(W^\perp^o) \cap \Omega(V) \) contain normal subgroups \( A \) and \( B \), respectively, isomorphic to \( Z_2^2 \). In view of the behavior of \( R^W \) and \( R^{W^o} \), \( W \) and \( W^o \) are not isometric.

Let \( b_1, b_2, b_3 \) be an orthogonal basis of \( W \) with respect to which \( A \) is diagonal; \( N_G(R) \) acts transitively on \( \{ \langle b_1 \rangle, \langle b_2 \rangle, \langle b_3 \rangle \} \). Then \( T = \langle b_1 + 2b_2, b_2 + 2b_3 \rangle \) is an \( O^+(2, 5) \)-space. Since \( N_G(R^o)^W \) contains representatives of both \( G \)-orbits of
\(O^+(2,5)\)-spaces, we may assume that \(T \subset W^\circ\). Then \(F = \langle W, W^\circ \rangle\) is a 4-space containing non-isometric nonsingular 3-spaces, and hence is nonsingular. We have a group \(H = \langle A, B \rangle^F \leq \Omega(F)\); \(A\) and \(B\) are not conjugate (since \([V, A] = W\), \([V, B] = W^\circ\)), and \(A \cap B = 1\) (every nontrivial element of \(A\) moves \(T\)). However, \(\Omega(F) = \Omega^\pm(4, 5)\) has no such subgroup \(H\).

The case \(G \leq \Gamma O(2m + 1, 3), m \geq 3\), is harder. This time choose a primitive divisor \(r | 3^{m-2} - 1\) or \(r | 3^m - 1\) according to whether \(m\) is odd or even. Then \(W = C_V(R)\) is a nonsingular 3-space; \(N_G(R)^W\) has rank 2 or 3 and so contains \(\Omega(3, 3)\) or \(D_8\), while \(N_G(R)^{W^\perp} \leq \Gamma O^\pm(2, 3^m-1)\) is metacyclic, with a normal cyclic subgroup of order dividing \(3^m - 1\) ± 1 \(\equiv 2 \pmod{4}\) in view of our choice of \(r\). If \(t\) is the square of an element of order 4 in \(N_G(R)\), or if \(N_G(R)^W \geq \Omega(3, 3)\) and \(t \in N_G(R)'\) has order 2, then \(t \in C_G(W^\perp)\) is an involution with \(t^W\) inducing \(-1\) on an anisotropic 2-space. Let \(b, b' \in W\) be linearly independent vectors with \(b' = -b, b'' = -b'\), where (since \(q = 3\) and in view of the action of \(N_G(R)^W\)) we may assume that they are perpendicular and \(\phi(b) = \phi(b') = 1\) for the quadratic form \(\phi\) on \(V\).

Any two \(G_{(b)}\)-conjugates of the reflection \(-t b^\perp\) commute. (For otherwise, the product of two such non-commuting conjugates of \(t\) has order 3 and centralizes \(y^\perp/y\) for some point \(y \in b^\perp\). Then the argument used in (6.2) yields the contradiction
\[G \geq \Omega(2m + 1, 3),\]
It follows that \(b^\perp = U_1 \perp U_2\), where \(U_1 = \langle b'^{G_{(b)}} \rangle\) is spanned by pairwise perpendicular members of \(\langle b'^{G_{(b)}} \rangle\).

Let \(\mathcal{N}_1 = \{v | v \in V, \phi(v) = 1\}\). Since \(G_x\) has two orbits of \(y \in \Omega - x^\perp\) and each \(\langle x, y \rangle\) contains a unique \(\langle b \rangle \in \mathcal{N}_1, G_x\) has two orbits on \(\mathcal{N}_1 - x^\perp\). This proves that \(G_{(b)}\) has at most two orbits on \(\Omega - b^\perp\); and there are two orbits if and only if \(G\) is transitive on \(\mathcal{N}_1\).

Suppose that \(G\) is intransitive on \(\mathcal{N}_1\). Then \(G_{(b)}\) is transitive on \(\Omega - b^\perp\), but leaves invariant \(U_1\) and \(U_2\). Then \(U_2 = 0\) and \(G_{(b)}\) is monomial on \(U_1 = b^\perp\) with respect to an orthonormal basis. Since \(G_{(b)}\) is transitive on \(\Omega - b^\perp\) and \(2m = n - 1 > 4\), this is impossible.

Thus, \(G\) is transitive on \(\mathcal{N}_1\). Let \(s\) be a primitive divisor of \(3^m - 1\) or \(3^{2m-1}\) such that \(s | U_1(b^\perp)\). If \(S \in \text{Syl}_s(G)\) then \(C_V(S) \in \mathcal{N}_1\). We may assume that \(S\) fixes \(b\) and hence has no proper nonsingular invariant subspace \(U_2\) in \(b^\perp\). Once again \(G_{(b)}\) is monomial on \(U_1 = b^\perp\) with respect to an orthonormal basis. Members of \(\Omega - b^\perp\) look like \(b + u\) with \(u \in b^\perp\) and \(\phi(u) = -1\), where \(u\) has \(k\) nonzero coordinates with \(k \equiv 2 \pmod{3}\). Since there are only two such orbits, \(k\) can only be 2 or 5, so \(\dim b^\perp = n - 1 < 8\) and we are in an \(O(7,3)\) geometry. Since \(G\) is transitive on \(\mathcal{N}_1, G_x\) has an orbit on \(\mathcal{N}_1 - x^\perp\) of length
\[1/2 \cdot 3^3(3^3 \pm 1) \cdot 2^2(7^{-1}) \cdot \{(3^6 - 1)/(3 - 1)\},\]
which is not an integer.

Finally, consider the case \(G \leq \Gamma O^\pm(2m, q), m \geq 3\), where \(q > 2\) (by hypothesis), in which \(x^\perp/x\) has \((q^m - 1)/(q - 1)\) points. If \(m = 3\), use Kantor-Liebler \([14, (5.12)\) and (5.14)]. Assume that \(m \geq 4\), and use \(r \mid q^{m-1} \pm 1\) and \(R \in \text{Syl}_r(G_x)\) as before, temporarily excluding the case \(O^- (8, q)\) with \(q\) a Mersenne prime. This time \(W = C_V(R)\) is a nonsingular 4-space of type \(O^-(4, q)\) since \(V\) has type \(O^\pm (2m, q)\) and \([V, R]\) has type \(O^\pm (2m - 4, q)\). Then \(N_G(R)^W\) has rank 2 or 3 and hence contains \(\Omega^- (4, q)\) or (if \(q = 3\)) \(A_5 [5, \text{Chap. 12}]\). As in (6.2) we obtain the contradiction \(G \geq \Omega^\pm (2m, q)\).

This leaves the excluded possibility \(G \leq \Gamma O^- (8, q)\) with \(q\) a Mersenne prime. We may assume that \(-1 \in G\). If \(L\) is a line then \(G^L_x\) is 2-transitive and hence
contains $SL(2,q)$. Then there is an involution $t \in G$ such that $-t = 1$ on $L$ and $W = C_V(-t)$ has type $O^+(4,q)$. Let $-t \in R \in \text{Syl}_2(C_G(L))$, so $W = C_V(R)$ and $R \in \text{Syl}_2(C_G(W))$. By the Frattini argument, if $N = N_G(R)$ then $N^W$ is transitive on lines while $N^W = 2$ is 2-transitive. Then $N^W \geq \Omega^+(4,q)$.2; clearly $N^W \geq \Omega^+(4,q)$. Thus, if $q \geq 3$ then $C_N(W^+)$ contains $\Omega^+(4,q)$, hence a long root group, and then all long root groups by line-transitivity; but this produces the usual contradiction $G \geq \Omega(V)$. If $q = 3$ then $C_N(W^+)$ contains an involution centralizing a 6-space, and a simpler version of the argument used above for $\Omega(2m + 1,3)$ produces a contradiction. This completes the proof of Theorem V.

Remark. If $G < O^+(2m,2)$, the argument breaks down when $r | 2^{m-2} \pm 1$, $\dim W = 4$, and $|N_G(R)| = 10$ or 20.

14. Concluding remarks

1. The method used in our proofs for employing $p$-groups also works for suitable permutation representations of the exceptional Chevalley groups.

2. After classifying antiflag transitive groups, it is natural to ask about transitivity on incident point-hyperplane pairs (where the hyperplane is not the polar of the point in the case of a classical geometry). If a group $G$ is transitive on all such pairs in $\text{PG}(n-1,q)$, then it is transitive on incident point-line pairs, and hence 2-transitive on points (Kantor [12]); so Theorem I applies. However, for classical geometries, results are known only in the unitary case (Kantor-Liebler [14, (6.1)]).

3. The proofs of Theorems I-III do not depend on “modern” group-theoretic classification theorems. Theorem IV requires results summarized in Kantor-Liebler [14] that only use older group theory; most of the required results used nothing more than elementary properties of classical groups, such as concrete sets of generators.

4. It should be noted that [14] produces a proof of the rank 2 analogue of Theorem IV, as follows. We assume that $V$ does not have type $O^+(4,q)$. The primitive case proceeds as in Sections 9, 10. In the imprimitive case, the block $\Delta$ of Section 11 is a t.i. or t.s. line. If $x \in \Delta$ then $G_x$ is transitive on $\Delta^G - \{\Delta\}$ and hence on $x^+ - \Delta$. It follows easily that $G$ has one orbit of points and two orbits each of lines and incident point-line pairs. Now [14, Sect. 5] applies.

Appendix A. The $G_2(q)$ generalized hexagon

This appendix contains new and elementary proofs of the existence and uniqueness statements in Section 3, as well as further properties of the generalized hexagons (including antiflag transitivity).

Assume that $\mathcal{G}$ is as in (3.2), and set $W(x) = W_1(x)$. We will prove several properties of $\mathcal{G}$, from which an explicit construction will easily follow.

Lemma A.1. (i) For any points $x, y$ of $\mathcal{G}$ such that $d(x, y) = 1 \text{ or } 2$, all 1-spaces of $\langle x, y \rangle$ are points of $\mathcal{G}$.

(ii) If $z \notin \langle x, y \rangle$ and $d(x, y) = 1 \text{ or } 2$, then $W_2(z) \cap \langle x, y \rangle$ is either $\langle x, y \rangle$ or a point.

(iii) Either $\dim V = 6$ and $V$ is symplectic, or $\dim V = 7$ and $V$ is orthogonal. In either case, the points and lines of $\mathcal{G}$ consist of all points and certain t.i. or t.s. lines of $V$. Moreover, $W_2(x)$ consists of all points of $x^+$ (i.e., $d(x, y) \leq 2 \iff y \in x^+$).
Proof. (i) \((x, y) \subseteq W(u)\) if \(u \in W(x) \cap W(y)\).

(ii) If \(d(x, y) = 1\), this follows from the axioms for a generalized hexagon. Suppose \(d(x, y) = 2\), and set \(u = W(x) \cap W(y)\). We must show that the subspace \(W_2(z) \cap (x, y)\) is nonzero (cf. (f) in Section 2). This is clear if \(d(u, z) \leq 2\), while if \(d(u, z) = 3\) it follows from the fact that \(W(u) \cap W_2(z)\) is a subspace meeting each line on \(u\).

(iii) As in Yanushka [27, Sect. 3], this follows from (ii): the points and lines are the points and lines of a polar space (Tits [23]). Moreover, \(\mathcal{G}\) has exactly \((q^6 - 1)/(q - 1)\) points (and \(|W_2(x)| = (q^6 - 1)/(q - 1)\)).

Two points are opposite if they are at distance 3.

Lemma A.2. Let \(a\) and \(b\) be opposite points, and set \(H = \langle W(a), W(b) \rangle\).

(i) \(H = E \oplus F\), where \(E\) and \(F\) are t.i. or t.s. planes such that, for \(e \in E\), \(f \in F\), \(\langle e, f \rangle\) is a \(\mathcal{G}\)-line if and only if it is a (t.i. or t.s.) line (call these \(E\mid F\)-lines).

(ii) If \(e \in E\), then \(W(e) = \langle e, e^\perp \cap F \rangle\).

(iii) If \(x\) is a point on no \(E\mid F\)-line, then \(W(x)\) meets exactly \(q + 1\) \(E\mid F\)-lines, and the points of intersection lie on a t.i. or t.s. line.

(iv) If \(V\) has type \(Sp(6, q)\), then \(q\) is even.

Proof. (i) Since \(W(a) \cap W(b) = 0\), \(\dim H = 6\). Let \(a = x_1, x_2, x_3, b = x_4, x_5, x_6\) be the vertices of an ordinary hexagon in \(\mathcal{G}\). Then \(x_2, x_6 \in W(a)\) and \(x_3, x_5 \in W(b)\). Set \(E = \langle x_2, x_4, x_6 \rangle\) and \(F = \langle x_1, x_3, x_5 \rangle\). Then \(E\) and \(F\) are t.i. or t.s. (by (A.1iii)) and \(H = E \oplus F\). Also \(W(x_{2i}) = \langle x_{2i}, x_{2i}^\perp \cap F \rangle\) for each \(i\). We can thus vary \(x_2, x_6 \in W(a) \cap E\), and also move around the ordinary hexagon, in order to show that each t.i. or t.s. line \(\langle e, f \rangle\) is a \(\mathcal{G}\)-line (for \(e \in E, f \in F\)).

(ii) This is clear from the above proof. (In fact, the points of \(E \cup F\) and the \(E\mid F\)-lines form a degenerate subhexagon with \(s = 1, t = q\).)

(iii) Let \(E_x = E \cap x^\perp, F_x = E \cap x^\perp, U = \langle E_x, F_x \rangle, e = F_x^\perp \cap E\) and \(f = E_x^\perp \cap F\).

The pair \(e, f\) corresponds to a flag of \(E\) (and of \(F\)) if and only if \(e\) and \(f\) are perpendicular; and then \(e \in E_x\), \(f \in F_x\) and (for \(V\) symplectic resp. orthogonal) \(U^\perp\) is \(\langle e, f \rangle\) or \(\langle e, f \rangle \perp H^\perp\), which cannot contain the point \(x\) lying in no \(\langle E\mid F \rangle\)-line.

Thus, \(e, f\) corresponds to an antiflag of \(E\). It follows easily that \(U\) is nonsingular. If \(z \in E_x\) and \(u = W(x) \cap W(z)\) (cf. (A.1iii)), then \(\langle z, u \rangle\) is a \(\mathcal{G}\)-line and hence (by (ii)) an \(\langle E\mid F \rangle\)-line, so \(u \in W(x) \cap U\). Thus, \(W(x) \cap U\) is the desired set of points, and is a t.i. or t.s. line.

(iv) If \(V\) has type \(Sp(6, q)\), then \(U\) has type \(Sp(4, q)\). But the \(Sp(4, q)\) quadrangle contains six lines forming a \(3 \times 3\) grid (such as \(E \cap x^\perp, F \cap x^\perp, W(x) \cap U\), and any three \(E\mid F\)-lines in \(U\)) if and only if \(q\) is even.

Remark. Because of (A.2iv), and the isomorphism between the \(Sp(6, q)\) and \(O(7, q)\) geometries when \(q\) is even, we will assume from now on that \(V\) has type \(O(7, q)\). Then \(H\) has type \(O^+(6, q)\), and the line mentioned in (iii) is \(W(x) \cap H\).

Also, \(O(7, q) = SO(7, q) \times \{\pm 1\}\), so we may where necessary assume that linear automorphisms of \(\mathcal{G}\) have determinant 1.

The next lemma is more technical, and concerns generating \(\mathcal{G}\).
Lemma A.3. Let $S$ be a set of points, containing at least one pair $a, b$ of opposite points, and such that $W(a) \cap b \subseteq S$ for any such pair. Then either $S = E \cup F$ for some $E, F$ as in (A.2i), or $S$ consists of all points of $\mathcal{G}$.

Proof. Certainly $S \supseteq E \cup F$ if $E$ and $F$ are obtained as in (A.2i). (Each line of $F$ on $b$ is $W(a') \cap F$ for some $a' \in W(b) \cap E = W(b) \cap a^\perp$.) Let $\mathcal{G}_0$ consist of $S$ together with the set of lines meeting it at least twice. We will show that $\mathcal{G}_0$ is a (possibly degenerate) subhexagon.

Let $L$ be a line of $\mathcal{G}_0$ and $x \in S - L$; we must show that the unique point $u$ of $L$ nearest $x$ lies in $S$. Let $y \in L - u$. Since $x$ is opposite some point of $E$ or $F$, our hypothesis implies that each line on $x$ meets $S - \{x\}$. If $d(x, u) = 1$, pick $z \in S \cap W(x)$ with $d(y, z) = 3$, so $u \in W(y) \cap z \subseteq S$. If $d(x, u) = 2$ then $d(x, y) = 3$, so $u \in W(y) \cap x \subseteq S$.

Thus, $\mathcal{G}_0$ is a subhexagon. Let $a \in S$. Then $S \cap W(a)$ has the following properties: it meets every line on $a$ at least twice; if $x, y \in S \cap W(a)$ and $W(a) = \langle a, x, y \rangle$, then $\langle x, y \rangle \subseteq S$. (For, since $\mathcal{G}_0$ is a subhexagon, there is a point $b \in x \cap y \cap S$ opposite $a$, and then $\langle x, y \rangle = W(a) \cap b \perp$.) Thus $S \cap W(a)$ is a subplane of $W(a)$ (possibly degenerate: just $\{a\} \cup \langle x, y \rangle$).

If each line of $\mathcal{G}_0$ on $a$ has size 2, then $S = E \cup F$. So suppose that some line of $\mathcal{G}_0$ on a has at least three points. Then $S \cap W(a)$ is nondegenerate, and hence is all of $W(a)$. Thus $\mathcal{G} = \mathcal{G}_0$.

Lemma A.4. Suppose $\mathcal{G}$ and $\mathcal{G}'$ are both embedded in $V$ as in Section 3. Let $x_1, \ldots, x_6$ and $y_1, \ldots, y_6$ be the vertices of ordinary hexagons in $\mathcal{G}$ resp. $\mathcal{G}'$. Then there is an element of $GL(V)$ mapping $x_i$ to $y_i$ ($i = 1, \ldots, 6$) and inducing an isomorphism of $\mathcal{G}$ onto $\mathcal{G}'$.

Proof. The orthogonal geometries determined by $\mathcal{G}$ and $\mathcal{G}'$ as in (A.1iii) are equivalent under $GL(V)$; we may suppose that they are equal. There is an orthogonal transformation taking $x_i$ to $y_i$ ($i = 1, \ldots, 6$), so we may assume that $x_i = y_i$ for each $i$. Set $E = \langle x_2, x_3, x_6 \rangle, F = \langle x_1, x_3, x_5 \rangle$. By (A.2ii), if $e \in E, f \in F$, then $W(e)$ and $W(f)$ are the same whether computed in $\mathcal{G}$ or $\mathcal{G}'$.

Pick a point $x$ on no $E|F$-line, so $W(x) \cap H$ is the t.s. line in (A.2iii), and hence is one of the $q - 1$ lines $\neq E \cap x \perp, F \cap x \perp$ in $U$ meeting each $E|F$-line of $U = \langle E \cap x \perp, F \cap x \perp \rangle$. But $O(7, q)_{EFU}$ is transitive on these $q - 1$ lines, so we may assume that $W(x) = \langle x, W(x) \cap H \rangle$ is the same in $\mathcal{G}$ and $\mathcal{G}'$ for the chosen $x$.

We will show that (A.3) applies to the set of points $u$ of $V$ such that $W(u)$ is the same in both $\mathcal{G}$ and $\mathcal{G}'$. Let $a, b \in S$ be opposite. Then $A = W(a) \cap b \perp$ and $B = W(b) \cap a \perp$ are t.s. lines. If $u \in A$ then $L = W(b) \cap u \perp$ is a line on $b$; let $v = L \cap B$. In $\mathcal{G}$ (and $\mathcal{G}'$) there is a unique shortest path $a, u, v, b$; since $w \in W(b) \cap a \perp$ we have $w = v$. Then $W(u) = \langle u, a, v \rangle$ in both $\mathcal{G}$ and $\mathcal{G}'$, so $u \in S$.

Now $\mathcal{G} = \mathcal{G}'$ by (A.3).

Corollary A.5. The group $\text{Aut}_V(\mathcal{G})$ of automorphisms of $\mathcal{G}$ induced by elements of $\text{SL}(V)$ is transitive on the set of ordered ordinary hexagons of $\mathcal{G}$. In particular, $\text{Aut}_V(\mathcal{G})$ is antiflag transitive.

Corollary A.6. (i) There is a subgroup $K \cong \text{SL}(3, q)$ of $\text{Aut}_V(\mathcal{G})$ fixing $E$ and $F$ (cf. (A.2)) and centralizing $H^\perp$.

(ii) The stabilizer of $E$ in $\text{Aut}_V(\mathcal{G})$ induces $\text{SL}(3, q)$ on it.
(iii) |Aut₁(⟨G⟩)| = (q⁶ - 1)q⁶(q² - 1) and Aut₁(⟨G⟩) contains no nontrivial scalar transformations.

Proof. (i) Use (A.4) and (2.1) (compare (5.4c)).

(ii) The plane E uniquely determines the plane F = ⟨W(a) ∩ W(b) | a, b ∈ E, a ≠ b⟩. Let J = Aut₁(⟨G⟩)EF and C = C(J(E ∩ U) for the O⁺(4, q)-space U = ⟨E ∩ U, F ∩ U⟩ in the proof of (A.2ii) and (A.4). Both J and K₁ fix the antiflag (E ∩ U ⊥, E ∩ U) of E and induce GL(2, q) on E ∩ U. Then J = CK₁. We will show that C = 1, so that Aut₁(⟨G⟩)EF = Aut₁(⟨G⟩)_EF = K and (ii) holds.

Since C is 1 on E ∩ U, fixes F ∩ U and acts inside O(U) = O⁺(4, q), it is 1 on U. Then C fixes each 2-space W(y) ∩ U for y ∈ U ⊥ = ⟨E, F⟩, and then fixes the unique point y joined by ⟨G⟩-lines to all points of W(y) ∩ U. Since C fixes U ⊥ ∩ E and U ⊥ ∩ F, C < SL(V) centralizes U and fixes all points of U ⊥, so C = 1.

(iii) There are [(q⁶ - 1)/(q - 1)] ⟨q + 1⟩q · qq · q ordered hexagons in ⟨G⟩. The stabilizer in Aut₁(⟨G⟩) of one of them is the stabilizer in K of a triangle in E and hence has order (q - 1)². The final assertion is clear since Aut₁(⟨G⟩) < O(V) ∩ SL(V).

Theorem A.7. Each O(7, q) space has one and only one isomorphism type of generalized hexagons embedded as in Section 3. An Sp(6, q) space has such a hexagon if and only if q is even.

Proof. Uniqueness follows from (A.4), and the assertion about Sp(6, q) from (A.2iv). The preceding results (especially (A.1), (A.2) and (A.6)) tell us exactly how ⟨G⟩ must look, and hence how to construct ⟨G⟩.

Construction. Let V be a vector space carrying a geometry of type O(7, q), and E and F t.s. planes such that H = ⟨E, F⟩ is nonsingular of dimension 6. Let K < O(7, q) fix E and F, centralize H ⊥, and induce SL(3, q) on both E and F. If {e₁, e₂, e₃} is a basis for E and {f₁, f₂, f₃} the dual basis for F, then the matrices of e and f with respect to these bases are inverse transposes of one another for all g ∈ K. We may assume that H ⊥ = ⟨d⟩ with φ(d) = -1.

We must use the E[F]-lines ⟨e, f⟩, with e ∈ E, f ∈ e ⊥ ∩ F, as ⟨G⟩-lines; set W(e) = ⟨e, e ⊥ ∩ F⟩, W(f) = ⟨f, f ⊥ ∩ E⟩ as in (A.2ii). Note that K is transitive on the (q⁶ + q + 1)(q + 1)(q - 1) points of H not in E ∪ F, on the (q⁶ + q + 1)(q² - q²) points of V - H, and on the (q⁶ + q + 1)(q² - q²) lines of H not meeting E ∪ F.

We will use the E[F]-line ⟨e₁, f₁⟩, the point u = ⟨e₁ + f₁⟩, and the t.s. plane W(u) = ⟨e₁, f₂, e₃ + f₃ + d⟩. Write W(u°) = W(u) for all g ∈ K. The new points must be the t.s. points of V - H, and the new G-lines must be the lines of W(u°) through u°, for all g ∈ K. We must show that this is well-defined and yields a generalized hexagon. This will be done in several steps.

(1) If u° = u then W(u°) = W(u); so W(u°) is well-defined. For, |K_u| = q³(q - 1), and K_u fixes W(⟨e₁⟩)/⟨e₁, f₂⟩ and W(⟨f₂⟩)/⟨e₁, f₂⟩. Thus, each p-element of K_u fixes every plane containing ⟨e₁, f₂⟩. Suppose |g|q - 1. Since gE and gF are diagonalizable, we may assume that our dual bases {e₁, e₂, e₃} and {f₁, f₂, f₃} have been chosen so that g fixes each ⟨e₁⟩, ⟨f₁⟩. If gE = diag(α, β, γ) then gF = diag(α⁻¹, β⁻¹, γ⁻¹) and αβγ = 1. Since u° = u = ⟨e₁ + f₂⟩ we have β⁻¹ = α, whence e² = e₁, f² = f₂. Then W(u°) = ⟨αe₁, β⁻¹f₂, e₃ + f₃ + d⟩ = W(u).

(2) If W(u°) = W(u) then u° = u. For, g fixes W(u) ∩ E = ⟨e₁⟩ and W(u) ∩ F = ⟨f₂⟩. Hence, K(⟨e₁⟩)(⟨f₂⟩) is the stabilizer of a flag of PG(2, q), of order q³(q - 1); each of its p-elements fixes u. If |g|q - 1 then g is diagonalizable and we may assume that
our dual bases \{e_1, e_2, e_3\} and \{f_1, f_2, f_3\} have been chosen so that \(g\) fixes each \((e_i)\), \((f_i)\). Since \(g\) fixes \(W(u) = \langle e_1, f_2, e_3 + f_3 + d\rangle\), if \(g^E = \text{diag}(\alpha, \beta, \gamma)\) with \(\alpha \beta \gamma = 1\) then \((e_3 + f_3 + d)^g = \gamma e_3 + \gamma^{-1} f_3 + d\), so \(\gamma = 1\), whence \(\beta^{-1} = \alpha\) and \(u^g = u\).

(3) If \(L\) is a \(G\)-line on \(u\) then \(L \subset W(u)\). (For, we may assume \(L \not\subset H\) and \(u^g \in L \subset W(u^g)\) for some \(g \in G\), so \(u = L \cap H = u^g\) and \(L \subset W(u)\).) The total number of \(G\)-lines is then

\[
(q^2 + q + 1)(q + 1) + (q^2 + q + 1)(q + 1) \cdot (q - 1)q = \frac{(q^6 - 1)}{(q - 1)}.
\]

Since \(K\) is transitive on \(V - H\), each point \(x \notin H\) lies on

\[
(q^2 + q + 1)(q + 1)q / (q^2 + q + 1)(q^3 - q^2) = q + 1
\]

\(G\)-lines.

(4) Let \(x \in V - H\). Then \(K_x \cong SL(2, q)\) acts on the \(O^+(4, q)\)-space \(U = \langle E \cap x^-, F \cap x^+ \rangle\); it fixes each of the \(q - 1\) lines \(M \neq E \cap x^-, F \cap x^+\) of the same type as \(E \cap x^+\) that partition the points of \(U\), and \(K_x^M \cong SL(2, q)\).

If \(L\) is a \(G\)-line on \(x\) then \(y = L \cap H\) is singular but not in \(E \cup F\), and \(W(y)\) contains \(x\) and points \(e \in E\) and \(f \in F\). Since \(W(y)\) is t.s. it follows that \(y \in \langle e, f \rangle\) lies in \(\langle E \cap x^-, F \cap x^+\rangle = U\) and hence on one of the lines \(M\).

Define \(W(x) = \langle x, M \rangle\); this is a t.s. plane. Since \(K_x^M\) is transitive, all lines of \(W(x)\) on \(x\) are \(G\)-lines. By the transitivity of \(K\) on the t.s. lines of \(H\) not meeting \(E \cup F\), each such line occurs as \(W(z) \cap H\) for some \(z \in V - H\). Since the numbers of such \(z\) and such t.s. lines are the same, distinct points \(z\) yield distinct \(W(z)\). It follows that \(W(a) \neq W(b)\) for any distinct points \(a, b\) of \(V\).

(5) Points \(a, b\) are perpendicular if and only if \(d(a, b) \leq 2\). For, if \(1 \leq d(a, b) \leq 2\) then \(a, b \in W(c)\) for some \(c\), and \(W(c)\) is t.s. But the number of such ordered pairs is \((q^6 - 1)/(q - 1))(q + 1)q + (q^6 - 1)/(q - 1))q^2\), which is the same as the number of ordered pairs of distinct perpendicular points.

(6) \(G\) has no \(k\)-gons for \(k \leq 5\). For, let \(a_1, \ldots, a_k\) be the vertices of a \(k\)-gon. Then \(d(a_i, a_j) \leq 2\) for all \(i, j \) so \(a_1, \ldots, a_k\) is a t.s. plane by (5), which must be both \(W(a_1)\) and \(W(a_2)\), contradicting (4).

(7) \(G\) is a generalized hexagon. Since each \(G\)-line is on \(q + 1\) points, and each point is on \(q + 1\) \(G\)-lines, this follows from the same type of elementary counting argument as in the proof of (10.2).

This completes the proof of (A.7).

Remarks. Further properties of the group \(G_2(q) = \text{Aut}_V(G)\) are found in (5.4). Additional information, such as simplicity when \(q \neq 2\) and identification with \(PSU(3, 3) \times \mathbb{Z}_2\) if \(q = 2\), are left to the reader, and can be found in Tits [22].
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