Large $N$ Phase Transition In The Heat Kernel On The $U(N)$ Group

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Abstract

The large $N$ phase transition point is investigated in the heat kernel on the $U(N)$ group with respect to arbitrary boundary conditions. A simple functional relation is found relating the density of eigenvalues of the boundary field to the saddle point shape of the typical Young tableaux in the large $N$ limit of the character expansion of the heat kernel. Both strong coupling and weak coupling phases are investigated for some particular cases of the boundary holonomy.

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# 1 Introduction

Heat kernels on classical groups are widely used in mathematical physics, especially in quantum field theory [1] [2]. For example it serves as a building block (disc partition function with an arbitrary boundary holonomy) for calculations of the partition function of Yang-Mills theory on 2D manifolds with various topologies [3] [4] [5] [6]. The large $N$ limit of the heat kernel is of special interest since it is related to the so-called planar limit in some quantum field theories, QCD$_2$ being the most interesting among them [7] [8] [9] [10]. This limit not only can simplify the analytical investigation of a theory but can also give rise to new qualitative phenomena, like large $N$ phase transitions [11].

Our task in this paper is to calculate, as explicitly as possible, the heat kernel for the $U(N)$ group (as an example) in the large $N$ limit. The method will be closely related to the recent calculation of the partition function of Yang Mills theory on the two dimensional sphere [11].

To give a physical taste to the heat kernel let us derive the heat kernel as the partition function of a simple quantum mechanical model; the one dimensional principal chiral field with initial and final holonomies specified. The functional integral representation of the partition function is

$$Z(T, \theta_0, \theta_T) = \int \mathcal{D}g(t) e^{N \text{tr} \int_0^T dt \partial_t g^{-1} \partial_t g},$$  \hspace{1cm} (1)$$

where $[\mathcal{D}g(t)]_H$ is the Haar measure on the unitary group $U(N)$, and the holonomies at the end points of the time interval are $g(0) = e^{i\theta_0}$ and $g(T) = e^{i\theta_T}$ (with $e^{i\theta_0}$ and $e^{i\theta_T}$ the eigenvalues of $g(0)$ and $g(T)$). Due to the invariance of the Haar measure at each moment of time, we have, after the introduction of a new variable (connection) $A(t) = ig^{-1}\partial_t g(t)$ the following representation of $Z(T, \theta_0, \theta_T)$:

$$Z(T, \theta_0, \theta_T) = \int \mathcal{D}A(t) e^{-N \text{tr} \int_0^T dt A(t)^2} \delta\left(\left[Te^{i\int_0^T dt A(t)}\right]e^{i\theta_0}, e^{i\theta_T}\right).$$  \hspace{1cm} (2)$$

Using the character expansion of the group $\delta$-function: $\delta(U, U') = \sum_R \chi_R(U)\chi_R(U')^*$,
and the fact that the character is the trace of the matrix element in a given representation \( R \): 
\[
\chi_R(T \exp(i \int Adt)) = tr_R[T \exp(i \int A^a \tau^a_R dt)] \quad (\tau^a_R \text{ is the } a\text{th generator of } U(N) \text{ in the } R\text{th irreducible representation}),
\]
we arrive after a simple gaussian integration in \( A(t) \) (independant at each moment \( t \)) at the conclusion that

\[
Z(T, \theta_0, \theta_T) = \sum_R e^{-TC_2(R)} \chi_R(e^{i\theta_0}) \chi_R(e^{i\theta_T})
\]  

(3)

where \( C_2(R) \) is the second Casimir of the group \( U(N) \) in the representation \( R \). This is also equivalent to the partition function of two dimensional QCD on the cylinder with the boundary holonomies specified at either end of the cylinder by the two distributions \( e^{i\theta_0} \) and \( e^{i\theta_T} \). For the rest of this article we restrict our attention to the case where one of the holonomies is the unit matrix (two dimensional QCD on the disc), and the heat kernel takes the form

\[
Z(T, \theta) = \sum_R d_R e^{-TC_2(R)} \chi_R(e^{i\theta}).
\]  

(4)

In section(2) we derive an explicit formula for the partition function for the heat kernel with fixed boundary field. The solution is written directly in terms of the eigenvalues of the boundary field. The calculation is valid only in the weak coupling phase. In section(3) we find a simple inversion relation between the density of eigenvalues of the boundary field and the density of heighest weights labeling the saddle point Young tableau. In some important particular cases of boundary holonomy, such as the semicircle distribution, it is shown to be explicitly solvable. From this it is then trivial to calculate the strong coupling transition point. We then discuss in section(4) how to calculate the partition function in the strong coupling phase. An alternative approach is discussed in the appendix. The appendix also contains an explicit solution for the heat kernel action illustrating the procedure developed in section(3) for a nontrivial boundary field (the density of the \( \theta \)'s being given by an inverted semicircle distribution \( \sigma(\theta) = \frac{1}{\pi \sqrt{g^2 - \theta^2}} \)).
2 Weak coupling phase

To study the action in the large $N$ limit we proceed as in [11]. The sum over representations $R$ for the group $U(N)$ is the sum over all Young tableaux labeled by the components of highest weight $n_1, n_2, ..., n_N$. The $n_i$ are integers, and the sum over representations is given by summing over all \{n_i\} satisfying the inequality:

$$\infty \geq n_1 \geq n_2 \geq ... \geq n_N \geq -\infty.$$  \hfill (5)

With this labeling for the representation, the explicit expressions for the dimension, second Casimir and character are:

$$d_R = \prod_{i>j} (1 - \frac{n_i - n_j}{i-j}), \hfill (6)$$

$$C_2(R) = \sum_{i=1}^{N} n_i(n_i - 2i + N + 1), \hfill (7)$$

$$\chi_R(e^{i\theta}) = \det_{k,l}(e^{i(n_k-k+N)\theta_l}) \Delta(e^{i\theta}), \hfill (8)$$

where the determinant is of the matrix whose $(k,l)$ element is $e^{i(n_k-k+N)\theta_l}$ and $\Delta(e^{i\theta})$ is the Vandermondt determinant $\Delta(x) = \prod_{i>j}(x_i - x_j)$.

It is convenient to introduce a new set of labels, \{h_i\} defined by

$$h_i = -n_i + i - (N+1)/2,$$  \hfill (9)

which satisfy the inequality

$$h_{i+1} > h_i.$$  \hfill (10)

The heat kernel is then given by

$$Z = \sum_{\{h_i\} \text{constrained}} \Delta(h)e^{-\frac{\pi}{N} \sum_{i=1}^{N} h_i^2} \det_{k,l}(e^{-ih_k\theta_l}) f(\theta) \hfill (11)$$

where

$$f(\theta) = \frac{\text{const}}{\Delta(e^{i\theta})} e^{-\frac{\pi}{N}(N^2-1)+i\frac{N-1}{2} \sum_{i=1}^{N} \theta_i}, \hfill (12)$$

and the sum over the \{h_i\} is an ordered sum subject to the inequality (10). The determinant consists of $N!$ terms of alternating sign. By interchanging the \{h_i\},
each of these terms can be made equal to \( e^{-i \sum_{i=1}^{N} h_i \theta_i} \), the ordered sum becomes unconstrained, (each of the \( N! \) terms contributing a different segment of the sum over all \( \{h_i\} \)) and the partition function becomes:

\[
Z = \sum_{\{h_i\}} \Delta(h) \ e^{-\sum_{i=1}^{N} (\frac{T}{2} h_i^2 + i h_i \theta_i)} \ f(\theta).
\] (13)

In the large \( N \) limit we redefine the \( \{h_i\} \), dividing them by \( N \), and then treat them as continuum variables so that the multiple sum becomes a multiple integral:

\[
Z = \int_{-\infty}^{\infty} \left( \prod_i dh_i \right) \Delta(h) \ e^{-N \sum_i (\frac{T}{2} h_i^2 + i h_i \theta_i)} \ f(\theta).
\] (14)

It is important to realise that by changing the sum into an integral we may have lost the inequality (10). The expression for \( Z \) we now obtain is therefore only valid in the weak coupling phase where (10) is satisfied. By shifting the \( \{h_i\} \) we can complete the square and do the gaussian integral:

\[
Z = \int_{-\infty}^{\infty} \left( \prod_i dh_i \right) \Delta(h - \frac{i}{2T} \theta) \ e^{-N \sum_i (\frac{T}{2} h_i^2 + i h_i \theta_i)} \ f(\theta)
\] (15)

\[
= \Delta(i \theta) e^{-\frac{N}{2T} \sum_i \theta_i^2} \ T^{-N^2/2} f(\theta).
\] (16)

The last step is accomplished by noting that the answer is a completely antisymmetric polynomial of the \( \theta_i \) of the same degree as the Vandermondt determinant, ie., it is the Vandermondt determinant up to a constant. We have thus restored the partition function in the weak coupling phase [2]. We will see in the next section that the weak coupling solution will be valid only for small enough \( T \), due to the constraint (10).

### 3 Strong Coupling Transition Point

It is expected that in the large \( N \) limit the integral above is dominated by the contribution from a single set of \( \{h_i\} \) ie., a single representation. In terms of the continuum variables, this corresponds to a density \( \rho(h) = \frac{1}{N} \frac{\partial}{\partial h} \) for the distribution of \( h \)’s. As is discussed in [1], for this distribution to correspond to an allowed Young
tableau $\rho(h)$ must satisfy the constraint $\rho(h) \leq 1$, following directly from (13). The point at which $\rho(h) = 1$, for some $h$, corresponds to the strong coupling transition point beyond which the weak coupling solution is no longer valid, and the strong coupling methods discussed in the following section have to be used. The rest of this section is entirely devoted to finding the distribution $\rho(h)$ corresponding to the weak coupling solution, to allow us to test for this condition.

To find the density we calculate the resolvent, $H(h)$, which is defined to be:

$$H(h) = \left\langle \frac{1}{N} \sum_{i=1}^{N} \frac{1}{h - h_i} \right\rangle_{h'},$$

where $h$ lies in the complex plane and the average is taken with respect to $h'$. In the large $N$ continuum limit this is given as the integral along the support of $\rho(h)$

$$H(h) = \int dh \frac{\rho(h')}{h - h'},$$

The discontinuity across the cut of the resolvent gives $2\pi i$ times the density, $\rho(h)$.

To proceed we take the Fourier transform of each term in the sum (17):

$$H(h) = \frac{1}{N} \sum_{i=1}^{N} i \int \frac{dk}{2\pi} \text{sign}(k) e^{-ikh} \left\langle e^{ih'k} \right\rangle_{h'},$$

The expectation value, $\left\langle e^{ih'k} \right\rangle_{h'}$, can be calculated from the formulae (14) and (16) for $Z$ by realising that the term $e^{ih'k}$ can be absorbed into the definition of the $\theta$.

We therefore define a new distribution of $\tilde{\theta}$’s which is identical to the old distribution except for the $i$th $\theta$ which is shifted by $-k/N$;

$$\tilde{\theta}_j = \theta_j - \frac{k}{N} \delta_{ij}.$$

This allows us immediately to write down the result

$$\left\langle e^{ih'k} \right\rangle_{h'} = \frac{\Delta(\tilde{\theta}) e^{-\frac{N}{4} \sum_j \theta_j^2} \prod_{j \neq i} \left( 1 - \frac{k}{N(\theta_i - \theta_j)} \right) e^{-\frac{k}{N} \left( \theta_i^2 - 2\theta_i \theta_j \right)}}{\Delta(\theta) e^{-\frac{N}{4} \sum_j \theta_j^2} \prod_{j \neq i} \left( 1 - \frac{k}{N(\theta_i - \theta_j)} \right) e^{-\frac{k}{N} \left( \theta_i^2 - 2\theta_i \theta_j \right)}}$$

Substituting this into equation (19) we notice that the sum can be replaced by a contour integration in $\theta$ (encircling all the $\theta$ eigenvalues) if we also unrestrict the
product, so that the product is over all $j$:

$$H(h) = \int_{-\infty}^{\infty} dk \frac{\text{sign}(k) e^{-ikh}}{2k} \frac{1}{2\pi} \oint d\theta \prod_{j=1}^{N} \left(1 - \frac{k}{N(\theta - \theta_j)}\right) e^{-\frac{1}{4\pi} \left(\frac{k^2}{N} - 2\theta k\right)}. \quad (22)$$

We can then exponentiate the unrestricted product and keep only the lowest order in $1/N$ to arrive at

$$H(h) = \frac{1}{2\pi} \oint d\theta \int_{0}^{\infty} dk \left(e^{-\frac{k^2}{4NT}} - \cos k(h + \frac{\theta}{2T} - i\Theta(\theta))\right) \quad (23)$$

where

$$\Theta(\theta) = \int d\theta' \sigma(\theta') = \frac{1}{N} \sum_{j} \frac{1}{\theta - \theta_j} + O\left(\frac{1}{N}\right) \quad (24)$$

is the resolvent for the distribution of $\theta$. The apparent divergence of the $k$ integral about $k = 0$ is canceled by the contour integration in $\theta$ (we can add any non-$\theta$ dependant function of $k$ into the $k$ integrand, e.g. $(e^{-\frac{k^2}{4NT}} \cos(k))$, since it will be ignored by the contour integral). The gaussian exponential term suppresses the divergent behaviour of the cos (it’s argument is complex) and permits us to do the integral unambiguously in the large $N$ limit, the result being:

$$H(h) = \frac{1}{2\pi} \oint_C d\theta \log[h + i\frac{\theta}{2T} - i\Theta(\theta)] \quad (25)$$

where the contour $C$ goes anticlockwise around the cut of the resolvent $\Theta(\theta)$. We would like to thank Matthias Staudacher for simplifying this derivation.

In the appendix we show how to find an analagous formula (the roles of $H(h)$ and $\Theta(\theta)$ are interchanged) using either Migdal’s method [13] or some results developped by Matytsin [14].

It is more convenient to study the derivative of $H(h)$:

$$H'(h) = \frac{1}{2\pi} \oint_C d\theta \frac{1}{[h + i\frac{\theta}{2T} - i\Theta(\theta)]} \quad (26)$$

In this case the cut point(s) of the logarithm becomes a pole(s) and the contour can be inflated out from the cut to catch both the pole(s) and also a contribution from infinity:

$$H'(h) = 2T - \sum_{\text{poles}} \frac{i}{[\frac{1}{2T} - i\Theta'(\theta(h))]} \quad (27)$$
where the $\theta(h)$ satisfy
\[ h + \frac{i\theta(h)}{2T} - i\Theta(\theta(h)) = 0 \] (28)
and the sum over the zeros means the sum over all the $\theta(h)$ which satisfy equation (28).

Taking the derivative with respect to $h$ of equation (28) gives an identity which allows us to reduce equation (27) to
\[ H'(h) = 2T + i\sum_{\text{zeros}} \theta'(h) \] (29)

Integrating up with respect to $h$ we arrive at the very simple formula
\[ H(h) = 2Th + i\sum_{\text{zeros}} \theta(h) \] (30)

Some general comments are now in order. Firstly, we only need to solve the inversion equation (28) in some neighbourhood of $h$, as analytic continuation then defines $H(h)$ over the whole complex plane. The cutpoints are completely specified, although the actual path of the cuts between the cutpoints maybe ambiguous, in which case they are chosen to lie on the real axis. For densities $\sigma(\theta)$ which are non-singular, ie. finite on all of their support, we can choose the neighbourhood of $h$ to be far from the origin so that there is only one solution of equation (28). In this case we have
\[ H(h) = 2Th + i\theta(h) = 2Ti\Theta(\theta(h)) \] (31)
where the last equality follows directly from (28). Studying the large $h$ behaviour of the right hand side we find that $H(h)$ behaves as $\frac{1}{h}$ for large $h$ as is required. For this reason we have neglected the possible constant term in equation (30) coming from the integration.

We are now in a position to find the strong coupling transition point, ie. to test the inequality $\rho(h) \leq 1$. If the distribution of $h$’s reaches it’s maximum at $h = 0$ we can derive a very simple expression for the strong coupling transition point. Setting $h = 0$ in equation (31) we have at the strong coupling transition point $\Theta(\theta(0)) = \pm \pi/2T$, which implies (from equation (28)) that $\theta(0) = \pm \pi$ which leads in turn to the simple
expression for the strong coupling transition point:

\[ T_c = \frac{\pi}{2\Theta(\pi)} \]  

(32)

We would like to thank Andrei Matytsin for this comment [12].

3.1 An example: The semicircle distribution

To illustrate the method discussed above we present the simplest example, the case where the spectrum of \( \theta \)'s is given by a semicircle distribution:

\[ \sigma(\theta) = \frac{2}{\pi c} \sqrt{c - \theta^2} \quad \text{and} \quad \Theta(\theta) = \frac{2}{c} (\theta - \sqrt{\theta^2 - c}) \]  

(33)

Inverting equation (28) gives

\[ \theta(h) = 2ih(T - \frac{1}{c h}) + \frac{2i}{c h} \sqrt{h^2 - c h} \]  

(34)

and leads to a semicircle distribution for the \( h \)'s

\[ \rho(h) = \frac{2}{\pi c h} \sqrt{c h - h^2} \]  

(35)

where the constant \( c_h \) is related to \( c \) by

\[ c_h = \frac{2}{T} - \frac{c}{4T^2}. \]  

(36)

The strong coupling transition point occurs for the value of \( T \) at which \( \rho(h = 0) = 1 \), giving:

\[ c_h = \frac{4}{\pi^2} \implies T = \frac{\pi}{\frac{4}{(\pi - \sqrt{\pi^2 - c})}} \]  

(37)

This can also be retrieved directly from equation (32). As a check we set \( c = 0 \) so that \( M(\theta) \) is the unit matrix to give the partition function of \( QCD_2 \) on the sphere and obtain the critical point discovered in [11] which is \( 2T_{\text{crit}} = \pi^2 \), where \( 2T \) is the area.

Another exactly solvable distribution \( \sigma(\theta) = \frac{1}{\pi \sqrt{\theta^2 - c^2}} \) is studied in the appendix. This distribution is singular at the end points and provides an illustration of the case where there are several zeros to equation (28).
4 The Strong Coupling Phase

In this section we present the analogue of the method described in [11] for calculating the distribution of $h$’s and the free energy in the strong coupling phase. In [11] the procedure was to start with the saddle point equation and look for the density $\rho(h)$ which satisfies it subject to the constraint that for a finite interval $[-b,b]$ $\rho(h)$ takes on it’s maximal value, ie. $\rho(h) = 1$ for $h \epsilon [-b,b]$. The actual value of $b$ is then found from consistency arguments.

Although we derived our results from a completely different direction we still have a saddle point equation. It is the real part of equation (30) ie.

$$P \int dh' \rho(h) \frac{h}{h-h'} = \Re H(h)_{\text{weak}} = 2Th - \Im \sum_{\text{zeros}} \theta(h) \quad (38)$$

The procedure is thus to solve the problem in the weak coupling phase by the inversion described above, and obtain $H(h)_{\text{weak}}$ as an explicit expression of $h$, take the real part of this equation and solve (38) for $\rho(h)$ subject to the constraint that $\rho(h) \leq 1$.

To do this we follow exactly the method discussed in [11], which we repeat here almost verbatim. For simplicity’s sake, we assume here that the distribution of $h$’s is symmetric and that there is only one interval, $[-b,b]$, for which $\rho(h) = 1$.

$$\rho(h) = \begin{cases} 1 & \text{for } h \epsilon [-b,b]; \\ \tilde{\rho}(h) & \text{elsewhere}. \end{cases} \quad (39)$$

Substituting into the real part of (38) we get

$$\Re H(h)_{\text{weak}} - \log \frac{h-b}{h+b} = P \int dh \frac{\tilde{\rho}(h')}{h-h'} \quad (40)$$

We define the resolvent for the distribution $\tilde{\rho}(h)$

$$\tilde{H}(h) = \int \frac{dh' \tilde{\rho}(h')}{h-h'} \quad (41)$$

where the integral runs along the support of $\tilde{\rho}(h)$, $[-a,-b]$ and $[b,a]$ (with $a$ and $b$ to be found later from the large $h$ behaviour of $\tilde{H}(h)$), and generate the full analytic function $\tilde{H}(h)$ from $\Re H(h)_{\text{weak}}$ by performing the contour integral

$$\tilde{H}(h) = -\frac{1}{2\pi i} \sqrt{(a^2-h^2)(b^2-h^2)} \oint ds \frac{\Re H(s)_{\text{weak}} - \log \frac{s-b}{s+b}}{(s-h)\sqrt{(a^2-h^2)(b^2-h^2)}} \quad (42)$$
where the contour encircles the cut of the square root but does not enclose any other cuts or singularities of the integrand. By inflating the contour we catch the pole at \( h = s \), the cut of the log and any singularities of the analytic continuation of \( \Re H(h)_{\text{weak}} \). The full resolvent \( H(h)_{\text{strong}} \) for the strong coupling phase involves adding to \( \tilde{H}(h) \) the contribution from the constant part of the full density \( \rho(h) \). This precisely cancels out the log term generated by the pole at \( h = s \). We thus arrive at the expression for the resolvent for the strong coupling phase

\[
H(h)_{\text{strong}} = \Re H(h)_{\text{weak}} - \sqrt{(a^2 - h^2)(b^2 - h^2)} \int_{-b}^{b} ds \frac{1}{(s-h)\sqrt{(a^2 - h^2)(b^2 - h^2)}} + \text{( contribution from singularities of } \Re H(h)_{\text{weak}}) \quad (43)
\]

The last two terms give plus or minus \( i\pi \) times the full density \( \rho(h) \). As is discussed in [11] the second term can be written entirely in terms of the complete elliptic integral of the third kind \( \Pi[\phi, x] \):

\[
\frac{1}{\pi} \frac{b - a}{b + a} \sqrt{\frac{(a + h)(b + h)}{(a - h)(b - h)}} \Pi\left[\frac{2b}{a + b h + b}, \frac{2\sqrt{a b}}{a + b}\right] \quad (44)
\]

The coefficients \( a, b \) are found by ensuring that for large \( |h| \) \( H(h)_{\text{strong}} = \frac{1}{h} + O\left(\frac{1}{h^2}\right) \). The large \( h \) limit of the elliptic integral is discussed in [11].

For the semicircle distribution given as an example at the end of section (3) we have \( \Re H(h)_{\text{weak}} = \frac{2c}{h} h \). All the analysis conducted in [11] for the strong coupling phase is as before if we make the replacement in the notation of [11] of \( A/2 \to T/(1 - \frac{c}{2T}) \).

5 Appendix

5.1 An alternative approach

Here we present an alternative method to finding the results of section(2). The character for \( U(N) \) is, up to a factor of \( i \) and a couple of trivial Vandermondt factors, the Itzykson-Zuber determinant

\[
\chi_R(e^{-i\theta}) = \Delta(h)I(h, -i\theta) \frac{\Delta(i\theta)}{\Delta(e^{i\theta})} \prod_i e^{i\frac{N-1}{2}\theta_i} \quad (45)
\]
where
\[ I(\phi, \psi) = \int dU e^{N tr(U\phi U^\dagger \psi)} \to e^{N^2 F_0[\rho, \sigma]} \] as \( N \to \infty. \) (46)

Migdal [13], in the context of induced QCD, and more recently Matytsin [14] derived some useful results for the large \( N \) limit of the Itzykson Zuber integral: \( I[\rho(\phi), \sigma(\psi)] = e^{N^2 F_0[\rho, \sigma]} \). We start from a useful formula which comes from taking the large \( N \) limit of \( \frac{tr}{N} \frac{\partial}{\partial \sigma} I(\phi, \psi). \) From Matytsin’s paper [14] we quote the result (with the minor change of an extra factor of \( -i \))

\[ \int d\theta' \sigma(\theta')(-i\theta')^q = \frac{1}{2\pi i} \frac{1}{q + 1} \int dh' \left[ \left( \frac{\partial}{\partial h} \frac{\delta F_0[\rho, \sigma]}{\delta \rho(h')} + \mathcal{V}(h') + i\pi \rho(h') \right)^{q+1} \right. \]
\[ \left. - \left( \frac{\partial}{\partial h} \frac{\delta F_0[\rho, \sigma]}{\delta \rho(h')} + \mathcal{V}(h') - i\pi \rho(h') \right)^{q+1} \right]. \] (47)

The integral runs along the support of \( \rho(h) \). We recognise \( \mathcal{V}(h) \equiv i\rho(h) \) as the value of the resolvent \( H(h) \) above and below the cut and the equation simplifies to

\[ \int d\theta' \sigma(\theta')(\theta')^q = \frac{-1}{2\pi} \frac{1}{q + 1} \oint_C dh \left[ i\mathcal{F}(h) + iH(h) \right]^{q+1} \] (48)

where for convenience we have defined \( \mathcal{F} = \frac{\partial}{\partial h} \frac{\delta F_0[\rho, \sigma]}{\delta \rho(h')} \), and the contour integration circles the cut. Multiplying both sides of the above equation by \( \frac{1}{\theta^{q+1}} \) and summing from \( q = 0 \) to \( \infty \) we obtain

\[ \Theta(\theta) = \oint_C \frac{dh}{2\pi} \log[\theta - i\mathcal{F}(h) - iH(h)]. \] (49)

This is the analogue of equation(25). It is an equation derived entirely from the properties of the character (Itzykson Zuber integral) itself. To proceed further one has to replace \( \mathcal{F}(h) \) by its functional form derived from the saddle point equation evaluated for the particular action studied. It is important to remember that \( \mathcal{F} \) is defined in terms of the Itzykson Zuber integral which is related to the character by the equation(15). The action has first to be written in terms of the Itzykson Zuber integral and then the saddle point equation evaluated.

For the heat kernel we rewrite the action in terms of the Itzykson Zuber integral,

\[ Z = \int dh \rho(h) \Delta^2(h) e^{-\Phi} \sum_i h_i^2 I(\rho, \sigma) \times \Delta(i\theta) f(\theta) \] (50)
and then find the saddle point equation:

$$\mathcal{F}(h) = 2Th - 2\mathcal{V}(h)$$  \hspace{1cm} (51)$$

Substituting into equation (49) and using the identity $H^\pm(h) - 2\mathcal{V}(h) = -H^\mp(h)$, where $H^\pm$ is the value of $H(h)$ immediately above and below the cut, we obtain

$$\Theta(\theta) = -\oint_C \frac{dh'}{2\pi} \log[\theta - i2Th + iH(h)]$$ \hspace{1cm} (52)$$

As earlier we take the derivative, expand the contour out to infinity, picking up the poles on the way, and arrive at the equation

$$\Theta(\theta) = \frac{\theta}{2T} - i \sum_{\text{zeros}} h(\theta)$$ \hspace{1cm} (53)$$

where $h(\theta)$ satisfies

$$\theta - i2Th(\theta) + iH(h(\theta)) = 0,$$ \hspace{1cm} (54)$$

and the sum over zeros is the sum over all the solutions to equation (54). We have thus found the analogue of equations (30) and (28). In the case where there are single solutions to equations (28) and (54), we see that equations (53) and (54) are identical, respectively, to equations (28) and (31).

### 5.2 An example: The inverted semicircle distribution

We present here the exact solution for a nontrivial density. The spectrum of eigenvalues is

$$\theta_k = g \cos\left(\frac{\pi k}{N}\right)$$ \hspace{1cm} (55)$$

and the corresponding density and resolvent

$$\sigma(\theta) = \frac{1}{\pi \sqrt{g^2 - \theta^2}} \quad \text{and} \quad \Theta(\theta) = \frac{1}{\sqrt{\theta^2 - g^2}}.$$ \hspace{1cm} (56)$$

This density was used in the large $N$ limit of the 2D chiral model studied in [15], where it corresponded to an external magnetic field precisely tuned to excite the full spectrum of mass states of the theory. The heat kernel action investigated in this
article corresponds to the 1D chiral model in an external magnetic field. Following
the procedure of section(3), we solve equation(28). For convenience we introduce the
rescaled variables; $x = \frac{\theta}{\sqrt{2T}}, \eta = h\sqrt{2T}$ and $b^2 = \frac{g^2}{2T}$; in terms of which the equation is
\[
\eta + ix - \frac{i}{\sqrt{x^2 - b^2}} = 0 \quad (57)
\]
We square up the square root and arrive at the quartic
\[
x^4 - 2i\eta x^3 - (\eta^2 + b^2)x^2 + 2b\eta x + (b^2\eta^2 - 1) = 0, \quad (58)
\]
Although the quartic equation has 4 solutions only three satisfy the equation (57). This can be intuitively understood by looking at $\eta$ far from the origin. In this case there is one solution in which $x$ is far from the origin ($x \approx -i\eta$) and the only two other possible solutions are the ones caught in the singularities of the end points of the square root. The final expression for the resolvent $H(h)$ involves summing up all three of these solutions. Using a trivial identity $x_1 + x_2 + x_3 + x_4 = 2b^2i\eta$ for the roots of a quartic allows us to express $H(h)$ in terms of the single quartic solution (single sheet), $x_4$, that does not satisfy the inversion equation(57). Using standard formulae for the roots of a quartic, the solution for the resolvent can be expressed as
\[
H(h) = -\sqrt{\frac{T}{2}}(\eta + i(\sqrt{z_1} + \sqrt{z_2} - \sqrt{z_3})) \quad (59)
\]
with $z_1$, $z_2$ and $z_3$ defined by
\[
z_1 = \frac{2}{3}b^2 - \frac{1}{3}\eta^2 + \frac{1}{3}(F^+ + F^-) \quad (60)
\]
\[
z_2 = \frac{2}{3}b^2 - \frac{1}{3}\eta^2 + \frac{1}{3}(\omega F^+ + \omega^2 F^-) \quad (61)
\]
\[
z_3 = \frac{2}{3}b^2 - \frac{1}{3}\eta^2 + \frac{1}{3}(\omega^2 F^+ + \omega F^-) \quad (62)
\]
where $\omega = \frac{1}{2}(-1 + i\sqrt{3})$ is the cube root of 1, and $F^\pm$ are given by
\[
F^\pm = -\sqrt[3]{k^3 - 18k + 54b^2 \pm \sqrt{(k^3 - 18k + 54b^2)^2 - (k^2 - 12)^3}} \quad \text{with} \quad k = \eta^2 + b^2 \quad (63)
\]
All the cut structure resides in the square root term inside the definition of $F^\pm$. The $\sqrt[z]{z}$ terms do not contribute cuts since for $z \approx 0$, $z \propto \eta^2$, and the square root undoes the square. Similarly the cube roots do not contribute any cuts since, close to their zeros, the arguments are proportional to $k^3$ and the root undoes the cube. The zeros of the square root in equation (63) and thus the cutpoints for the quartic are

$$\eta_1 = \pm i \sqrt{b^2 - \frac{1}{3\beta^2} - (B^+ + B^-)}$$

$$\eta_2 = \pm \sqrt{-b^2 + \frac{1}{3\beta^2} + (\omega B^+ + \omega^2 B^-)}$$

$$\eta_3 = \pm \sqrt{-b^2 + \frac{1}{3\beta^2} + (\omega^2 B^+ + \omega B^-)}$$

where $\omega = \frac{1}{2}(-1 + i\sqrt{3})$ and $B^\pm$ are defined by

$$B^\pm = -\frac{1}{\beta^2} + \frac{8\beta}{\beta^2} \pm \frac{8}{\beta^2} (\beta^2 - 1)^\frac{3}{2} \text{ with } \beta = \sqrt{\frac{27}{4} b^2}. \quad (67)$$

On the physical sheet (the sheet of the quartic corresponding to the resolvent) there is the single pair of cut points $\pm \eta_3$. The other cut points connect together, in pairs, the three other non-physical sheets. The cut structure for $H(h)$ is therefore given by a single cut or single interval for the support of the density running from $-\eta_3$ to $\eta_3$.

For $b^4 \leq \frac{4}{27}$ the support of the density lies entirely along the real axis, but for $b^4 > \frac{4}{27}$ the end points move out into the complex plane. This is a signal that there is no longer a single Young tableau contributing to the sum. Probably it can be interpreted as the manifestation of the fact that a “density” of terms of alternating sign contributes in the large $N$ limit rather than a single term.

There are two cases where everything simplifies (the simplification reduces the inversion of equation (64) to the solving of a quadratic). The first is when $b = 0$, which corresponds to all the eigenvalues lying at the origin, ie. when $M(\theta)$ is the unit matrix. In this case both the cube root and the square root around the $z_i$ can be taken explicitly and a semicircle distribution recovered as before. The other more useful case is when $h=0$. Again the cube roots and square roots around the $z_i$ can be taken explicitly and we get a simple expression for the discontinuity across the cut
at the origin ie

\[
\rho(h = 0) = \frac{\sqrt{2\pi}}{\pi} \sqrt{\frac{b^4}{4} + 1 - \frac{b^2}{2}} 
\]

(68)

\[
= \frac{1}{\pi} \sqrt{\frac{a^4}{4} + 4T^2 - \frac{a^2}{2}}.
\]

(69)

Since the density for the \( h \)'s reaches it's maximum at the origin we have found the critical point for the distribution ie. \( \rho(h)_{\text{max}} = 1 \) at \( T = \frac{\pi}{2} \sqrt{\pi^2 - g^2} \). This result could also have been obtained directly from equation(62).

**Note**

Since the completion of this work it has come to our attention that D. Gross and A. Matytsin \[16\] have also been studying the large \( N \) limit of equation(3) and have found, amongst other things, a similar simple inversion relation for the density of heighest weights for the case where \( g(0) = g(T) \).†

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