The problem of the gauge-invariant complete decomposition of
the nucleon spin demystified

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Abstract

The question whether the total gluon angular momentum in the nucleon can be decomposed into
its spin and orbital angular momentum parts without causing conflict with the gauge-invariance
principle has been an object of long-lasting debate. We give here a simple and transparent answer
to this question, including the uniqueness or nonuniqueness problem of the nucleon spin decom-
position, based only upon the most fundamental principles of physics, i.e. the Lorentz symmetry
and the gauge symmetry. We point out that the key factor is the existence of a particular spatial
direction in the parton physics, which is just the direction of parent nucleon momentum. This axis
is shown to play an analogous role as the direction of paraxial laser beam, which is used to measure
the spin and orbital angular momentum of a photon separately.
I. INTRODUCTION

To get a complete decomposition of nucleon spin is a fundamentally important homework of QCD. In fact, if our research ends up without accomplishing this task, tremendous efforts since the first discovery of the nucleon spin crisis would go up in smoke [1], [2]. Unfortunately, this is an extremely delicate and difficult problem, which has been rejecting a clear answer for more than 20 years since the first elaborate theoretical consideration in the seminal paper by Jaffe and Manohar [3]. (To overview controversial status of the problem, see two recent reviews [4], [5].) The central issue here is whether the total gluon angular momentum can be gauge-invariantly decomposed into its spin and orbital parts. It has long been believed that the answer is no [6], [7], just because many textbooks of electrodynamics clearly state that the total photon angular momentum cannot be gauge-invariantly decomposed into its spin and orbital parts [8] - [11]. On the other hand, we also know the fact that several experiments with use of the paraxial laser beam confirmed that the spin and the orbital angular momentum of a photon can separately be measured [12], [13]. Some years ago, motivated by the idea of transverse-longitudinal decomposition of the photon field, Chen et al. proposed the idea to decompose the gluon field into the physical and pure-gauge component [14], [15]. This enables them to get a gauge-invariant complete decomposition of the nucleon spin into four pieces, i.e. the spin and orbital angular momentum (OAM) parts of quarks and the spin and OAM parts of gluons. Although this is certainly a gauge-invariant decomposition of the nucleon spin, it is important to recognize that their decomposition of the gluon field is essentially the familiar transverse-longitudinal decomposition, which is given only after fixing the Lorentz frame of reference. Another point is that, even within the framework of this transverse-longitudinal decomposition, the way of gauge-invariant decomposition of the nucleon spin is not unique. In fact, we pointed out that another gauge-invariant complete decomposition of the nucleon spin exists [16]. The difference with the original decomposition by Chen et al. is characterized by the OAM parts of quarks and gluons. The quark and gluon OAMs in the Chen decomposition is essentially the “canonical” OAMs, although gauge-invariantized. On the other hand, the quark and gluon OAMs in our decomposition is the manifestly gauge-invariant “mechanical” OAMs. We thus propose to call these two decompositions the “canonical” and “mechanical” decompositions of the nucleon spin, respectively [4], [5].

In either case, these two decompositions are given in a fixed Lorentz frame, or given
in noncovariant forms, so that it is not very useful when investigating the relations with high-energy deep-inelastic-scattering (DIS) observables, which is the field of physics where the problem of the nucleon spin decomposition came into existence. In the paper [17], we therefore proposed more general form of nucleon spin decomposition, which has “seemingly” covariant appearance. It was shown that, based only upon a few general conditions, we can get two independent “seemingly” covariant forms of gauge-invariant complete decompositions of the nucleon spin, which was called in [17] the decomposition (I) and (II). The decomposition (II) was verified to contain the gauge-invariant Bashinsky-Jaffe decomposition [19] motivated by the light-cone gauge. It was also shown to contain the original Chen decomposition after an appropriate choice of the Lorentz frame. They both fall into the category of “canonical” decomposition. On the other hand, another decomposition (I) falls into the category of “mechanical” decomposition. It contains the noncovariant decomposition given in [16] after fixing the Lorentz frame, but it is more general [18].

Although most inclusive, the problem of this general framework is that the decomposition of the gluon field into the physical and pure-gauge component has large freedom or arbitrariness. In fact, it was criticized by several researchers that our formal decomposition of the gauge field into its physical and pure-gauge components is not unique at all and there can be infinitely many such decompositions, thereby being led them to the conclusion that there are in principle infinitely many decomposition of the nucleon spin [20]-[24]. This uniqueness or non-uniqueness problem of the nucleon spin decomposition is one of the two remaining issues of the gauge-invariant decomposition problem of the nucleon spin summarized below:

1. Are there infinitely many decompositions of the nucleon spin [25] ? If not, what physical principle favors one particular decomposition among many candidates ?

2. Among the two decompositions, i.e. the “canonical” type decomposition and the “mechanical” type decomposition, which can we say is more physical ? The more “physical” here means that it is closer to direct measurements.

Since the second issue was discussed in the recent review [5] in some detail, here we focus on the first issue, which was not solved satisfactorily at that time. We shall argue that all the delicacies of the problem stem from intricate interplays between gauge and Lorentz
symmetry. Then, we show that the key factor, which uniquely select a particular one from many possible gauge-invariant decompositions of the nucleon spin, is the Lorentz-frame independence, or more precisely, the Lorentz-boost-invariance in the momentum direction of the parent nucleon.

II. THE ROLE OF LORENTZ-INVARIENCE IN THE GAUGE-INARIANT NUCLEON SPIN DECOMPOSITION PROBLEM

As already emphasized, the transverse-longitudinal decomposition is given only noncovariantly, i.e. only after choosing a Lorentz-frame of reference. GSeeminglyh covariant extension of the gauge-invariant decomposition is then proposed in the paper [17]. As expected from already known decompositions given in noncovariant forms [16], we naturally find two “seemingly” covariant decompositions of the QCD angular momentum tensor, which are physically inequivalent. (The word “seemingly” is important here, because, as we shall see later, the decomposition of the gauge field into the physical and pure-gauge component is intrinsically noncovariant even though the appearance looks covariant.) The one is the “canonical” type decomposition given as

\[ M^{\lambda\mu\nu}_{QCD} = M^{\lambda\mu\nu}_{q-spin} + M^{\lambda\mu\nu}_{q-OAM} + M^{\lambda\mu\nu}_{G-spin} + M^{\lambda\mu\nu}_{G-OAM} + \text{boost} + \text{total divergence}, \]

where

\[ M^{\lambda\mu\nu}_{q-spin} = \frac{1}{2} \epsilon^{\lambda\mu\nu\sigma} \bar{\psi} \gamma_{\sigma} \gamma_{5} \psi, \]

\[ M^{\lambda\mu\nu}_{q-OAM} = \bar{\psi} \gamma^{\lambda} (x^{\mu} iD_{\nu pure}^{\nu} - x^{\nu} iD_{\mu pure}^{\mu}) \psi, \]

\[ M^{\lambda\mu\nu}_{G-spin} = 2 \text{Tr} \{ F^{\lambda\mu} A_{\mu pure}^{\nu phys} - F^{\lambda\mu} A_{\nu pure}^{\nu phys} \}, \]

\[ M^{\lambda\mu\nu}_{G-OAM} = 2 \text{Tr} \{ F^{\lambda\mu} (x^{\mu} D_{\nu pure}^{\nu} - x^{\nu} D_{\mu pure}^{\mu}) A_{\alpha phys}^{\alpha} \}. \]

Here, \( D_{\mu pure}^{\mu} = \partial^{\mu} - i g A_{\mu pure}^{\mu} \) is the pure-gauge covariant derivative for the fundamental representation of color SU(3), while \( D_{\mu pure}^{\mu} = \partial - i g [A_{\mu pure}^{\mu}, \cdot] \) is the pure-gauge covariant derivative for the adjoint representation. The other is the “mechanical” type decomposition given as

\[ M^{\lambda\mu\nu}_{QCD} = M^{\lambda\mu\nu}_{q-spin} + M^{\lambda\mu\nu}_{q-OAM} + M^{\lambda\mu\nu}_{G-spin} + M^{\lambda\mu\nu}_{G-OAM} + \text{boost} + \text{total divergence}, \]
where

\[ M_{q - \text{spin}}^{\lambda \mu \nu} = M_{q - \text{spin}}^{\lambda \mu \nu}, \]
\[ M_{q - \text{OAM}}^{\lambda \mu \nu} = \bar{\psi} \gamma^\lambda (x^\mu i D^\nu - x^\nu i D^\mu) \psi, \]
\[ M_{G - \text{spin}}^{\lambda \mu \nu} = M_{G - \text{spin}}^{\lambda \mu \nu}, \]
\[ M_{g - \text{OAM}}^{\lambda \mu \nu} = M_{G - \text{OAM}}^{\lambda \mu \nu} + 2 \text{Tr} \left[ (D_\alpha F^{\alpha \lambda}) (x^\mu A^{\nu}_{\text{phys}} - x^\nu A^{\mu}_{\text{phys}}) \right], \]

where \( D^\mu \) and \( D^\mu \) are the standard covariant derivative respectively acting on the fundamental and adjoint representations of color SU(3). To obtain these “seemingly” covariant decompositions, we need to impose a few general conditions only on the physical and pure-gauge components of the gluon field. They are the pure-gauge condition for the pure-gauge component of the gluon field

\[ F_{\text{pure}}^{\mu \nu} \equiv \partial^\mu A^{\nu}_{\text{pure}} - \partial^\nu A^{\mu}_{\text{pure}} - i g [A^{\mu}_{\text{pure}}, A^{\nu}_{\text{pure}}] = 0, \]

and the homogeneous (covariant) and inhomogeneous gauge transformation properties for the physical and pure-gauge components:

\[ A^{\mu}_{\text{phys}}(x) \rightarrow U(x) A^{\mu}_{\text{phys}}(x) U^{-1}(x), \]
\[ A^{\mu}_{\text{pure}}(x) \rightarrow U(x) \left( A^{\mu}_{\text{pure}}(x) + \frac{i}{g} \partial^\mu \right) U^{-1}(x). \]

Actually, these conditions are too general and they are not enough to fix the decomposition uniquely. Nevertheless, it was shown there that one of our decomposition, i.e. the “canonical” type decomposition (1), contains the LC-gauge motivated Bashinsky-Jaffe decomposition [19] as well as the Chen decomposition [14], [15] as special cases. The nonuniqueness of our formal decomposition is criticized by several researches [20] - [24]. This leads them to the claim that there are in principle infinitely many decomposition of the nucleon spin.

According to Ji et al. [20], [21], the arbitrariness of the decomposition comes from the path-dependence of the Wilson line, which is necessary for the decomposition of the gauge field into the physical and pure-gauge components. Another argument in favor of the existence of infinitely many decomposition of the nucleon spin was advocated by Lorce [22] - [24], based on what-he-call the Stueckelberg symmetry, which changes both of \( A^{\mu}_{\text{phys}} \) and \( A^{\mu}_{\text{pure}} \), while leaving their sum intact.

In a recent paper [26], Ji, Xu, and Zhao stepped further and showed that the total gluon helicity in a polarized proton is shown to be large momentum limit of a gauge-invariant
operator $\mathbf{E} \times \mathbf{A}_\perp$ with $\mathbf{A}_\perp$ being the transverse component of the gauge potential. (According to Ji et al., this operator $\mathbf{E} \times \mathbf{A}_\perp$ is just the gluon spin operator appearing in the Chen decomposition. This statement is not precise, however. The reason is that the definition of the color electric field is different in the two formulations, i.e. $E^i = F^{i0}$ in the definition of Chen et al., while $E^i = F^{i+}$ in that of Ji et al. This difference disappears only in the infinite-momentum-frame (IMF) limit.) Their argument goes as follows. First, they pointed out that, for the abelian gluon, the gluon spin operator $S_G$, which is deduced from the definition of the longitudinally polarized (abelian) gluon distribution as its first moment, can be expressed in the following form:

$$S_G = (\mathbf{E}(0) \times \mathbf{A}_{\text{phys}}(0))^3,$$

with

$$\mathbf{A}_{\text{phys}}(0) = \mathbf{A}(0) - \frac{1}{\sqrt{z}} \nabla A^+(\xi^-) \bigg|_{\xi^-=0}.$$

Next, they showed that the above operator is just the IMF limit of the operator $\mathbf{E} \times \mathbf{A}_\perp$. From this fact, they concluded that, to identify $\mathbf{E} \times \mathbf{A}_\perp$ as the (abelian) gluon helicity, one must have the following conditions, that is, the IMF and physical gauge, i.e. the light-cone gauge. The statement would be nothing wrong, but it has a danger of causing a little misunderstanding. In fact, the gluon spin, or more generally, the longitudinally polarized gluon distribution, must be a Lorentz-frame independent quantity. (This is clear from the fact that the measurement of these quantities can be carried out in the laboratory frame not in the IMF.) This especially means that the gluon spin or the longitudinally polarized gluon distribution should not depend on the magnitude of nucleon momentum. Since this boost-invariance of general collinear parton distributions (PDFs) along the direction of the nucleon momentum plays a decisive role in our subsequent argument, we think it useful to convince this property following Collins’ textbook \cite{27}. Let us start with the well-known definition of the simplest unpolarized PDF of the nucleon given as

$$\int \frac{d\lambda}{2\pi} e^{i\lambda k \cdot n} \langle P | \bar{\psi}(0) \gamma \not \! k \psi(\lambda n) | P \rangle,$$  \hspace{1cm} (14)

with $n$ being the standard light-like vector, while $P$ is the momentum of the nucleon. Since the r.h.s. is a scalar, it must be a function of $k \cdot n$ and $P \cdot n$ as

$$\bar{q}(k \cdot n, P \cdot n).$$  \hspace{1cm} (15)
The expression (14) is obviously invariant under the scaling of the 4-vector \( n \) by an arbitrary positive factor, which means that only the combination \( x \equiv k \cdot n / P \cdot n \) is allowed. This gives the standard definition of the unpolarized PDF given as

\[
q(x) = \int \frac{d\lambda}{2\pi} e^{i\lambda x (P \cdot n)} \langle P \mid \bar{\psi}(0) \not{\partial} \psi(\lambda n) \mid P \rangle, \tag{16}
\]

Now let us consider the Lorentz boost with a velocity \( v \) along the direction of the nucleon momentum \( P \), which we can take the 3-direction without loss of generality. It is given by

\[
x^0 \to \gamma (x^0 - v x^3), \quad x^1 \to x^1, \quad x^2 \to x^2, \quad x^3 \to \gamma (x^3 - v x^0), \tag{17}
\]

with \( \gamma = 1 / \sqrt{1 - v^2} \). Under this boost, \( k \cdot n \) and \( P \cdot n \) transform as

\[
k \cdot n \to \gamma (1 - v) k \cdot n, \quad P \cdot n \to \gamma (1 - v) P \cdot n, \tag{18}
\]

so that the ratio \( x = (k \cdot n) / (P \cdot n) \) is obviously invariant under it. This is only natural, because \( x \) is nothing but the Lorentz-invariant Bjorken variable. Nevertheless, what is important here is to clearly recognize the boost-invariance property of the general collinear PDFs along the direction of the nucleon momentum. Naturally, this also applies to the longitudinally polarized gluon distributions and also the gluon spin, which is defined as the first moment of the former.

To see the importance of the constraint of Lorentz-frame independence, it would be instructive to compare a noticeable difference between various definitions of the gphysicalh component of the gauge field \[28\]. Namely, we compare the LC gauge motivated definition, the temporal-gauge motivated one, the spatial axial-gauge motivated one, and the Coulomb-gauge motivated one given as

\[
A_{phys}^k = \frac{1}{D^+} F^{+k}, \tag{19}
\]

\[
A_{phys}^k = \frac{1}{D^0} F^{0k}, \tag{20}
\]

\[
A_{phys}^k = \frac{1}{D^3} F^{3k}, \tag{21}
\]

with \( k = 1, \) or \( 2, \) and \( D^+ \equiv \partial^+ - i g A^+, \) \( D^0 \equiv \partial^0 - i g A^0, \) \( D^3 \equiv \partial^3 - i g A^3, \) while

\[
A_{phys} = A - \nabla \frac{1}{\nabla^2} \nabla \cdot A. \tag{22}
\]

In principle, all these definitions of the physical component of the gluon give the corresponding definitions of the gluon spin by the relation,

\[
\Delta G = \frac{1}{2P_+} \langle PS \mid 2 \text{Tr} \left[ \tilde{F}^{+2} A_{phys}^1 - \tilde{F}^{\times+1} A_{phys}^2 \right] \mid PS \rangle, \tag{23}
\]
with \( \tilde{F}^{\mu
u} = \frac{1}{2} \varepsilon^{\mu\nu\alpha\beta} F_{\alpha\beta} \). Note however that a prominent feature of the LC gauge motivated choice of the physical component is that it is invariant under the Lorentz-boost along the 3-direction, i.e. the direction of nucleon momentum, which we have seen is a necessary condition for the definition of the gluon spin corresponding to the DIS measurements. In fact, under the Lorentz boost along the 3-direction given by (17), we can easily verify that the physical component in the LC-gauge motivated definition is invariant:

\[
A_k^{\text{phys}} \equiv \frac{1}{D^+} F^{+k} = \frac{1}{\partial^+ - i g A^+} F^{+k} \\
\rightarrow \frac{1}{\gamma (1-v) (\partial^+ - i g A^+)} \gamma (1-v) F^{+k} = \frac{1}{D^+} F^{+k} = A_k^{\text{phys}}. \tag{24}
\]

On the contrary, any other definitions of \( A_k^{\text{phys}} \) is not invariant under the boost. For example, the physical component with the temporal-gauge-motivated choice transforms as

\[
A_k^{\text{phys}} \equiv \frac{1}{D^0} F^{0k} = \frac{1}{\partial^0 - i g A^0} F^{0k} \\
\rightarrow \frac{1}{\gamma [ (\partial^0 - v \partial^3) - i g (A^0 - v A^3) ]} \gamma (F^{0k} - v F^{3k}) \\
= \frac{1}{D^0 - v D^3} (F^{0k} - v F^{3k}), \tag{25}
\]

so that it is not clearly boost-invariant. We therefore conclude that what plays a key role in the uniqueness problem of the gauge-invariant decomposition of the nucleon spin is the Lorentz-frame independence, or more precisely the Lorentz-boost invariance along the direction of parent nucleon. We should have noticed earlier that not only the gauge-invariance but also the Lorentz-frame independence is an important criterion of observability of the gluon spin accessed by the high-energy DIS measurements.

Still interesting observation is as follows. In the free field limit, we can always set both of the scalar and longitudinal components to zero:

\[
A^0 = A^3 = 0. \tag{26}
\]

In this case, all of the general axial-gauge motivated definitions of the physical component of the gluon reduce to the same expression:

\[
\text{LC} : \quad A_k^{\text{phys}} = \frac{1}{D^+} F^{+k} \rightarrow \frac{1}{\partial^+} \partial^+ A^k = A^k, \tag{27}
\]

\[
\text{temporal} : \quad A_k^{\text{phys}} = \frac{1}{D^0} F^{0k} \rightarrow \frac{1}{\partial^0} \partial^0 A^k = A^k, \tag{28}
\]

\[
\text{spatial axial} : \quad A_k^{\text{phys}} = \frac{1}{D^3} F^{3k} \rightarrow \frac{1}{\partial^3} \partial^3 A^k = A^k. \tag{29}
\]
This indicates perturbative equivalence of these three. In fact, the 1-loop anomalous dimension of the gluon spin operators was calculated in \[25\] and it turned out that they are all equal:

\[
\langle PS| \tilde{F}^{+k} A^k_{phys} | PS \rangle_G |_{A^+ = 0} = \left[ 1 + \frac{\alpha_S}{4\pi} \cdot \frac{\beta_0}{\varepsilon} \right] \langle PS| \tilde{F}^{+k} A^k_{phys} | PS \rangle_{tree}^G,
\]

\[
\langle PS| \tilde{F}^{0k} A^k_{phys} | PS \rangle_G |_{A^0 = 0} = \left[ 1 + \frac{\alpha_S}{4\pi} \cdot \frac{\beta_0}{\varepsilon} \right] \langle PS| \tilde{F}^{0k} A^k_{phys} | PS \rangle_{tree}^G,
\]

\[
\langle PS| \tilde{F}^{3k} A^k_{phys} | PS \rangle_G |_{A^3 = 0} = \left[ 1 + \frac{\alpha_S}{4\pi} \cdot \frac{\beta_0}{\varepsilon} \right] \langle PS| \tilde{F}^{3k} A^k_{phys} | PS \rangle_{tree}^G,
\]

where \(\beta_0 = 11 - 2n_f/3\). (We recall that, for the LC-gauge and temporal-gauge motivated choices, not only the divergent part but also the finite term in the 1-loop corrections to the gluon spin operator were calculated in \[28\] and shown to coincide including the finite part. However, the spatial axial-gauge motivated choice was not investigated by them, because they thought that this choice does not have any helicity interpretation. We shall later show that this case also has helicity interpretation.)

The analysis above also reveals the following fact. In the free field case, since we can eventually set \(A^0 = A^3 = 0\), the physical components of the gluon field is obviously the two transverse components \(A^1\) and \(A^2\). Under the presence of the quark-gluon interaction or the color-charged sources for the gluons, the situation is more complicated. It is true that, even in this case, the independent dynamical degrees of freedom, which should be quantized, are the two transverse degrees of freedom, and the other two components, i.e. the scalar component \(A^0\) and the longitudinal component \(A^3\) are the dependent fields, which can be expressed in terms of other dynamical degrees of freedom, i.e. the two transverse components of the gluon field and the quark field. (In the covariant treatment of gauge theories, the scalar and longitudinal components are also quantized. In this case, however, we must work in the Hilbert space with indefinite metric.) Nonetheless, because of the constraint of the Gauss law, we cannot set both of \(A^0\) and \(A^3\) to be zero at the same time. (One might say that they still contain physics at least in our problem of a strongly-coupled bound system of quarks and gluons.) This is thought to be the reason why the above mentioned various definitions of the physical component of the gluon lead to physically different definitions of the gluon spin in a nonperturbative sense.

Now, the definition of the gluon spin operator corresponding to DIS observations seems to be fixed, which also means that there is only one (or two) nucleon spin decomposition. The
explicit form of the gluon spin operator can readily be deduced from the standard expression of the longitudinally polarized gluon distribution $\Delta g(x)$ given by Manohar [29]:

$$\Delta g(x) = \frac{i}{4\pi x P^+} \int d\xi^- e^{ix P^+ \xi^-} \langle PS | \tilde{F}^{+\lambda}(0) L_{ab}[0, \xi^-] F^{+\lambda}_{0,\xi^-} | PS \rangle$$

$$= \frac{i}{4\pi x P^+} \int d\xi^- e^{ix P^+ \xi^-} \langle PS | 2 \text{Tr} \left[ \tilde{F}^{+\lambda}(0) L[0, \xi^-] F^{+\lambda}(\xi^-) L[\xi^-, 0] \right] | PS \rangle,$$  

where $a, b$ are color indices. Assuming the usual principle value prescription to handle the singularity in the distribution functions with use of the relation [30],

$$\int dx P \frac{1}{x} e^{i\lambda x} = i \pi \epsilon(\lambda),$$

the first moment of $\Delta g(x)$ becomes

$$\Delta G = \int dx \Delta g(x)$$

$$= \frac{1}{2P^+} \left( -\frac{1}{2} \right) \int d\xi^- \epsilon(\xi^-) \langle PS | \tilde{F}^{+\lambda}(0) L[0, \xi^-] F^{+\lambda}(\xi^-) L[\xi^-, 0] | PS \rangle.$$

This can also be expressed as

$$\Delta G = \frac{1}{2P^+} \langle PS | 2 \text{Tr} \left[ \tilde{F}^{+2}(0) A^1_{phys}(0) - \tilde{F}^{+1}(0) A^2_{phys}(0) \right] | PS \rangle,$$

with the definition of the physical component of the gluon field [31]:

$$A^k_{phys}(0) = -\frac{1}{2} \int d\xi^- \epsilon(\xi^-) L[0, \xi^-] F^{+k}(\xi^-) L[\xi^-, 0],$$

where $\epsilon(x)$ is a step function defined as $\epsilon(x) = +1$ for $x > 0$ and $\epsilon(x) = -1$ for $x < 0$. Hatta showed [31] that the above-defined physical component and the pure-gauge component given by $A^k_{pure} \equiv A^k - A^k_{phys}$ legitimately satisfy the general conditions (11), (12) and (13). We also point out that the above definition of the physical component is formally equivalent to (19). Anyhow, the above expression is gauge-invariant as well as Lorentz-boost invariant along the 3-direction. This especially means that we can work in arbitrary gauges, although $A^k_{phys}$ can be reduced to a local form only in the light-cone gauge.

What remains to be explained is the question about apparent contradiction with the standard textbook knowledge, which tells us that the total gluon angular momentum cannot be gauge-invariantly decomposed into its spin and orbital parts. A key to understand this dilemma is the existence of particular direction in the DIS physics, which is nothing but the direction of nucleon momentum. To understand it, we just recall that we have already
encountered a similar situation in the decomposition problem of the total photon angular momentum. In the clearly written papers [32], [33], Van Enk and Nienhuis argued that the total angular momentum of free electromagnetic field can certainly be decomposed into gspin + gobittal parts without causing conflict with gauge-invariance. This separation is based on the familiar transverse-longitudinal decomposition of the photon field. They clearly recognized that this separation is not Lorentz invariant. Nevertheless, the Lorentz-invariance is not essential in this problem. The reason is that the measurement of the spin and OAM of the photon is carried out by making use of the interaction between atoms and the paraxial laser beam of a photon and that this measurement is performed in a fixed laboratory frame. We now realize that a common feature of the gluon spin measurement and the photon spin measurement is the existence of a particular spatial direction. This particular direction in the photon spin and OAM measurement is nothing but the direction of paraxial laser beam, whereas it is the direction of nucleon momentum in the case of gluon spin measurements. We think that the final remark of the paper by Van Enk and Nienhuis is extremely enlightening to understand the physics behind and we quote it here for pedagogical reason. “The conclusion is that both ‘spin hand ‘orbital angular momentum of a photon are well defined and separately measurable. This concerns all three components. However, only the components along the propagation direction can be measured by detecting the change in internal and external angular momentum of an atom, respectively.” Also interesting to point out here is the following fact. In more general nonparaxial case, it was argued that there is no clear separation of the total photon angular momentum into its spin and OAM parts [34]. Although the separation is not still impossible, a peculiarity is that both the spin and OAM parts is generally dependent of the photon helicity [35].

Now we are ready to make a clear summary statement on the apparent contradiction between the two observations pointed out in Introduction. On the one hand, we know the classical statement in the standard textbooks of electrodynamics that the total angular momentum of a massless particle cannot be gauge-invariantly decomposed into its spin and orbital parts. On the other hand, we know that the spin and OAM of a photon can be separately measured. We conclude that what rescues this conflict is the existence of a particular spatial direction. Although the idea of transversality is certainly Lorentz-frame-dependent in general, the (gauge-invariant) transverse component of the photon or (gauge-covariant) physical components of the gluon can consistently be defined with respect to this
particular spatial direction.

Although we think that the argument above essentially demystifies the controversial points in the gauge-invariant decomposition problem of the nucleon spin, we think it instructive to inspect the physical contents of the resultant gluon spin operator in some more detail. In the LC gauge, the theoretical expression of the gluon spin reduces to the following form:

\[ \Delta G = \frac{1}{2} P^+ \langle PS | (E_\perp \times A_\perp)^3 + B_\perp \cdot A_\perp | PS \rangle, \]  

where we have omitted the color indices, for brevity. In the above equation, \( A_\perp \) should be understood to represent the physical component in the light-cone gauge. We emphasize that the presence of the 2nd term is crucial, because the 1st term alone is not invariant under the boost along the 3-direction. This can easily be verified from the following transformation properties of the relevant quantities under the boost along the 3-direction.

\[ E^1 \rightarrow \gamma (E^1 - vB^2), \quad E^2 \rightarrow \gamma (B^2 + vB^1), \]  

\[ B^1 \rightarrow \gamma (B^1 + vE^2), \quad B^2 \rightarrow \gamma (B^2 - vE^1), \]  

and

\[ A^{1,2} \rightarrow A^{1,2}. \]  

We emphasize again that the boost-invariance of the physical or transverse component \( A_\perp \) is guaranteed only for the light-cone gauge or light-cone gauge motivated choice. Jaffe once tried to estimate the contributions of both terms of (35) in the bag model as well as in the quark model [36]. Jaffe already recognized the fact that, since the sum is boost-invariant, the above \( \Delta G \) can be calculated in any Lorentz-frame including the rest frame of the nucleon, provided that the above \( A_\perp \) is the gauge potential in the LC gauge.

What is curious here is the physical meaning of the 2nd term of (35). Interestingly, it resembles the following quantity:

\[ S = \int B \cdot A \, d^3x, \]  

except the absence of the 3-component in \( B_\perp \cdot A_\perp \). In the field of space and laboratory plasma physics [37], the above \( S \) is called the magnetic helicity, and it is known to be a topological invariant of magnetic field configuration. This might indicates that, if a topological configuration of the gluon field play some role in the gluon spin, it is through this 2nd term.
Aside from such a speculation, a perturbative consideration manifests transparent physical meaning of the term $B_\perp \cdot A_\perp$. Using the familiar free field expansion of the gauge potential,

$$A_\perp(x,t) = \int d^3\tilde{k} \sum_{\lambda=\pm 1} \left[ a(k,\lambda) \varepsilon(k,\lambda) e^{-ik \cdot x} + a^\dagger(k,\lambda) \varepsilon^*(k,\lambda) e^{ik \cdot x} \right],$$  \hspace{1cm} (40)

with $d^3\tilde{k} = d^3k / (2\pi)^3 |k|$, and with $\lambda$ representing the two helicity states of the photon, one can easily verify the following two relations:

$$\int E_\perp \times A_\perp d^3x = \int d^3\tilde{k} \sum_{\lambda=\pm 1} \hat{k} \lambda a^\dagger(k,\lambda) a(k,\lambda),$$  \hspace{1cm} (41)

and

$$\int B_\perp \cdot A_\perp d^3x = \int d^3\tilde{k} \sum_{\lambda=\pm 1} \lambda a^\dagger(k,\lambda) a(k,\lambda).$$  \hspace{1cm} (42)

Then, despite its unfamiliar appearance, the term $B_\perp \cdot A_\perp$ also has the meaning of gluon helicity at least in a perturbative sense. Adding up the two terms, we thus find the following equation

$$\frac{1}{2} \int \left[ (E_\perp \times A_\perp)^3 + B_\perp \cdot A_\perp \right] d^3x = \sum_{\lambda=\pm 1} \lambda a^\dagger(k,\lambda) a(k,\lambda).$$  \hspace{1cm} (43)

As a result, we find that the sum of the two pieces in (35) reduces to the ordinary helicity operator of the gluon.

From the theoretical analysis so far, we are lead to very important understanding as follows. Namely, we must clearly distinguish the difference between $\lambda = +$ component and $\lambda = 0$ component of the QCD angular momentum tensor $M^{\lambda\mu\nu}$ in the nucleon spin decomposition problem. In the usual circumstances, we do not pay much attention to the difference between these two components. This is because we are too much accustomed to dealing with the covariant quantities. To be more specific, each term of the Ji decomposition

$$M^{\lambda\mu\nu}_{QCD} = M^{\lambda\mu\nu}_{q\text{-spin}} + M^{\lambda\mu\nu}_{q\text{-OAM}} + M^{\lambda\mu\nu}_G,$$

with

$$M^{\lambda\mu\nu}_G = 2 \text{Tr} \left[ x^\mu F^{\lambda\alpha} F_{\alpha}^{\nu} - x^\nu F^{\lambda\alpha} F_{\alpha}^{\mu} \right],$$  \hspace{1cm} (44)

is manifestly covariant as well as gauge-invariant. In this decomposition, it is not necessary to pay attention to the difference between the $\lambda = +$ component and the $\lambda = 0$ component,
because the nucleon matrix element of each term can be defined covariantly as well as gauge-invariantly. This is not the case, however, if we want to divide the gluon part into its spin and OAM parts as

\[ M_G^{\lambda \mu \nu} = M_{G-\text{spin}}^{\lambda \mu \nu} + M_{G-\text{OAM}}^{\lambda \mu \nu}. \] (45)

The reason is because the decomposition of the gluon field into the physical and pure-gauge components, which is necessary for obtaining the gauge-invariant decomposition of the gluon total angular momentum into its spin and OAM parts, is intrinsically noncovariant, since it inevitably calls for a particular spatial direction. (It may be helpful to remember the fact that the light-cone gauge fixing condition \( n^\mu A_\mu = 0 \) looks “seemingly” covariant, but of course the light-cone gauge is one of the representatives of the gauges called the noncovariant gauges.) After all, only \( M^{+12} \) component (not \( M^{012} \) component) with the light-cone gauge motivated choice of \( A_{\text{phys}}^k \) (with \( k = 1, 2 \)) seems to give physically accessible decomposition of the total gluon angular momentum, which is gauge-invariant as well as boost-invariant along the direction of the nucleon momentum. The difficulty of recognizing this simple fact is the reason why it has taken long time to reach above-explained transparent understanding of the gauge-invariant complete decomposition of the nucleon spin.

The understanding above that we have reached also clears away one widespread misconception \[20 - 24\], which is the idea of gauge-invariant-extension (GIE). It was often claimed that the Chen decomposition is a GIE of the Jaffe-Manohar decomposition based on the Coulomb gauge, while the Bashinsky-Jaffe (or Hatta) decomposition is another GIE of the Jaffe-Manohar decomposition based on the light-cone gauge. One should notice a strange or self-contradictory nature of this statement. In fact, if the two decompositions are the GIEs of a single object, they should give exactly the same physical answers after removing the unphysical gauge degrees of freedom by the way of gauge fixing. Now, the origin of this conceptual self-contradiction is clear. If one still prefers to using the words “GIE”, one may say that the Chen decomposition is a GIE of the Jaffe-Manohar decomposition based on the \( M^{012} \) component, while the Bashinsky-Jaffe decomposition is a GIE of the Jaffe-Manohar decomposition based on the \( M^{+12} \) component. The truth is that they are different even before GIE. Because of this reason, we think that word “GIE” is simply misleading and using it is not recommended. Without using the word “GIE”, one can simply say that there are various definitions of the relativistic spin decomposition of the nucleon, which are all
gauge-invariant. Which of them is physically relevant is a different question.

III. SUMMARY AND CONCLUSION

We have investigated the problem of the gauge-invariant complete decomposition of the nucleon spin. The central question here is whether the total gluon angular momentum can be decomposed into its spin and orbital parts without conflicting the gauge-invariance principle. Although many previous analyses show that the answer is certainly affirmative, no clear answer has ever been given as to the question whether it does not contradict the standard textbook statement that the total angular momentum of a massless particle like the photon or the gluon cannot be gauge-invariantly decomposed into its spin and orbital parts. We have shown that what enables to circumvent the conflict is the existence of a particular spatial direction in the physics that we are dealing with. It is the direction of the nucleon momentum in the parton physics, which plays the same role as the direction of paraxial laser beam in the physics of photon spin and OAM measurements. The physical or transverse components of the gluon or the photon, which are respectively gauge-covariant and gauge-invariant, can be consistently defined with respect to these axises, even though the concept of transversality is in general Lorentz-frame dependent.

The uniqueness or nonuniqueness problem of the gluon spin or the gauge-invariant complete decomposition of the nucleon spin can also be easily resolved once we notice the existence of this particular direction in the parton physics. We have shown that the physical requirement of the Lorentz-boost invariance along the direction of the parent nucleon momentum selects one favorable decomposition of the gluon spin from many possible candidates. It is the definition based on the light-cone gauge motivated choice of the physical component of the gluon. We then realize that, in some sense, the Lorentz symmetry plays more vital role than the gauge symmetry in the unambiguous definition of the gluon spin to be probed by the DIS measurements.

To sum up, we believe that the present analysis has succeeded to give us a simple and transparent solution to the long-standing problem of the gauge-invariant complete decomposition of the nucleon spin. However, one should not forget about the fact that there still exist two physically inequivalent decompositions of the nucleon spin, i.e. the “canonical” one and the “mechanical” one. How to relate each term of these decompositions to direct DIS
observable is one of the most important remaining tasks in the nucleon spin decomposition problem.

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