An alternative non-negative gravitational energy tensor to the Bel-Robinson tensor

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Abstract
The Bel-Robinson tensor $B_{\alpha\beta\mu\nu}$ gives a positive definite gravitational energy in the small sphere limit approximation. However, there is an alternative tensor $V_{\alpha\beta\mu\nu}$ which was proposed recently that offers the same positivity as $B_{\alpha\beta\mu\nu}$ does. These two tensors are a basis for expressions which have the desirable non-negative gravitational energy in the small sphere region limit.

1 Introduction

The Bel-Robinson tensor $B_{\alpha\beta\mu\nu}$ has many nice properties. It is completely symmetric, completely trace free and completely divergence free. It is usually regarded as being related to gravitational energy. In particular, the gravitational energy-momentum density in the small sphere vacuum limit is generally expected to be proportional to the Bel-Robinson tensor. This expectation is related to the requirement of energy positivity [1].

However, we recently found another tensor [2], $V_{\alpha\beta\mu\nu}$, which is also quadratic in the curvature, and which enjoys the same positivity properties as $B_{\alpha\beta\mu\nu}$. Moreover, we found that $B_{\alpha\beta\mu\nu}$ and $V_{\alpha\beta\mu\nu}$ are a basis for expressions which have the desirable non-negative gravitational energy in the small sphere vacuum limit. In this work, we examine some properties of $V_{\alpha\beta\mu\nu}$ and some other quadratic in curvature tensors, $S_{\alpha\beta\mu\nu}$, $K_{\alpha\beta\mu\nu}$ and $W_{\alpha\beta\mu\nu}$, which have shown up in the expansion of energy in the small sphere limit.

We found that $V_{\alpha\beta\mu\nu}$ fulfills the weak energy condition in the small sphere limit. We found another $V'_{\alpha\beta\mu\nu}$, which does not satisfy the pseudotensor conservation of energy-momentum restriction, but does satisfy the weak energy condition.

The gravitational energy expression in the small region limit can be investigated through the pseudotensors. In general, the expansion of the pseudotensor expression can be represented by the tensors $B_{\alpha\beta\mu\nu}$, $S_{\alpha\beta\mu\nu}$ and $K_{\alpha\beta\mu\nu}$ if we consider the second order terms [2] [3]. Even though a pseudotensor is not a tensorial object, this does not imply that it is useless. The second order expansion expression provides guidance whether the gravitational energy expression is positive or not [2].

For the zeroth order term, the pseudotensor gives the mass density as the equivalence principle demands. The non-vanishing second order terms contribute the gravitational energy-momentum in a small region limit; these terms are quadratic in the curvature tensor.
2 Quadratic curvature tensors

There are three basic tensors that commonly occur in the gravitational pseudotensor expression \[3, 4\]

\[ B_{\alpha\beta\mu\nu} := R_{\alpha\lambda\mu\sigma} R_{\beta\lambda\nu}^{\,\sigma} + R_{\alpha\lambda\nu\sigma} R_{\beta\lambda\mu}^{\,\sigma} - \frac{1}{8} g_{\alpha\beta} g_{\mu\nu} R^2, \]  
\[ S_{\alpha\beta\mu\nu} := R_{\alpha\mu\lambda\sigma} R_{\beta\nu}^{\lambda\sigma} + R_{\alpha\nu\lambda\sigma} R_{\beta\mu}^{\lambda\sigma} + \frac{1}{4} g_{\alpha\beta} g_{\mu\nu} R^2, \]  
\[ K_{\alpha\beta\mu\nu} := R_{\alpha\lambda\beta\sigma} R_{\mu\nu}^{\lambda\sigma} + R_{\alpha\lambda\beta\sigma} R_{\nu\mu}^{\lambda\sigma} - \frac{3}{8} g_{\alpha\beta} g_{\mu\nu} R^2, \]  

where \( R^2 = R_{\rho\tau\xi\kappa} R^{\rho\tau\xi\kappa} \). Some properties of \( S_{\alpha\beta\mu\nu} \) and \( K_{\alpha\beta\mu\nu} \) \[2\] are

\[ S_{\alpha\beta\mu\nu} \equiv S_{(\alpha\beta)(\mu\nu)} \equiv S_{\mu\nu\alpha\beta}, \quad S_{\alpha\beta\nu}^{\mu} \equiv \frac{3}{2} g_{\alpha\beta} R^2, \quad S_{\alpha\mu\beta}^{\mu} \equiv 0, \]  
\[ K_{\alpha\beta\mu\nu} \equiv K_{(\alpha\beta)(\mu\nu)} \equiv K_{\mu\nu\alpha\beta}, \quad K_{\alpha\beta\nu}^{\mu} \equiv -\frac{3}{2} g_{\alpha\beta} R^2, \quad K_{\alpha\mu\beta}^{\mu} \equiv 0. \]  

Note that unlike \( B_{\alpha\beta\mu\nu} \), both \( S_{\alpha\beta\mu\nu} \) and \( K_{\alpha\beta\mu\nu} \) are neither totally symmetric nor totally trace free.

For the quadratic curvature tensors, there are 4 independent basis \[5\] expressions, we may use

\[ \tilde{B}_{\alpha\beta\mu\nu} := R_{\alpha\lambda\mu\sigma} R_{\beta\lambda\nu}^{\,\sigma} + R_{\alpha\lambda\nu\sigma} R_{\beta\lambda\mu}^{\,\sigma} + \frac{1}{8} g_{\alpha\beta} g_{\mu\nu} R^2, \]  
\[ \tilde{S}_{\alpha\beta\mu\nu} := R_{\alpha\mu\lambda\sigma} R_{\beta\nu}^{\lambda\sigma} + R_{\alpha\nu\lambda\sigma} R_{\beta\mu}^{\lambda\sigma} - \frac{1}{4} g_{\alpha\beta} g_{\mu\nu} R^2, \]  
\[ \tilde{K}_{\alpha\beta\mu\nu} := R_{\alpha\lambda\beta\sigma} R_{\mu\nu}^{\lambda\sigma} + R_{\alpha\lambda\beta\sigma} R_{\nu\mu}^{\lambda\sigma} + \frac{3}{8} g_{\alpha\beta} g_{\mu\nu} R^2, \]  
\[ \tilde{T}_{\alpha\beta\mu\nu} := -\frac{1}{8} g_{\alpha\beta} g_{\mu\nu} R^2. \]  

These four tensors fulfill the symmetry \( \tilde{M}_{\alpha\beta\mu\nu} = \tilde{M}_{(\alpha\beta)(\mu\nu)} = \tilde{M}_{\mu\nu\alpha\beta} \). Although there exists some other tensors different from \( \tilde{B}_{\alpha\beta\mu\nu}, \tilde{S}_{\alpha\beta\mu\nu}, \tilde{K}_{\alpha\beta\mu\nu} \) and \( \tilde{T}_{\alpha\beta\mu\nu} \), they are just linear combinations of these four. For instance

\[ \tilde{T}_{\alpha\beta\mu\nu} + \tilde{T}_{\alpha\beta\mu} \equiv \tilde{B}_{\alpha\beta\mu\nu} + \frac{1}{2} \tilde{S}_{\alpha\beta\mu\nu} - \tilde{K}_{\alpha\beta\mu\nu} + 2 \tilde{T}_{\alpha\beta\mu\nu}. \]  

The above equality can be obtained by making use of the completely symmetric property of the Bel-Robinson tensor. Using \[10\], we can rewrite the Bel-Robinson tensor in a different representation \[5\]:

\[ B_{\alpha\beta\mu\nu} \equiv -\frac{1}{2} S_{\alpha\beta\mu\nu} + K_{\alpha\beta\mu\nu} + \frac{5}{8} g_{\alpha\beta} g_{\mu\nu} R^2 - \frac{1}{8} (g_{\alpha\mu} g_{\beta\nu} + g_{\alpha\nu} g_{\beta\mu}) R^2. \]  

This equation will be used in the next section.
3 An alternative non-negative gravitational energy tensor

Using the Riemann normal coordinate Taylor series expansion, consider all the possible combinations of the small region energy-momentum density in vacuum, the pseudotensor has the form

\[ 2 \kappa t_{\alpha}^{\beta} = 2G_{\alpha}^{\beta} + \left( a_1 \bar{B}_{\alpha \mu \nu} + a_2 \bar{S}_{\alpha \mu \nu} + a_3 \bar{K}_{\alpha \beta \mu \nu} + a_4 \bar{T}_{\alpha \beta \mu \nu} \right) x^\mu x^\nu + O(\text{Ricci}, x) + O(x^3), \]

(12)

where \( \kappa = 8\pi G/c^4 \) (with \( c = 1 \) for simplicity) and \( \bar{a}_1 \) to \( \bar{a}_4 \) are real numbers. From now on, the second order term will be kept but the others are dropped, because we are mainly interested in the gravitational energy. The essential purpose of the present paper is to prove that \( B_{\alpha \beta \mu \nu} \) and \( V_{\alpha \beta \mu \nu} \) are a basis for positive gravitational energy in the small sphere limit. There are two physical conditions which can constrain the unlimited combinations between \( \bar{B}_{\alpha \beta \mu \nu} \), \( \bar{S}_{\alpha \beta \mu \nu} \), \( \bar{K}_{\alpha \beta \mu \nu} \) and \( \bar{T}_{\alpha \beta \mu \nu} \). The first one is the conservation of the energy-momentum density and the second is the positive gravitational energy in the small sphere limit.

(i) First condition: energy-momentum conservation. Consider (12) as follows

\[ 0 = \partial_\beta t_\alpha^{\beta} = \frac{1}{4} (\bar{a}_1 - 2\bar{a}_2 + 3\bar{a}_3 - \bar{a}_4) g_{\alpha \beta} x^\beta R^2. \]

(13)

Therefore, the constraint for the conservation of the energy-momentum density is

\[ \bar{a}_1 - 2\bar{a}_2 + 3\bar{a}_3 - \bar{a}_4 = 0. \]

(14)

Although there are an infinite number of combinations which can fulfill the above constraint, it has removed one degree of freedom. As each single tensor of \( \bar{B}_{\alpha \beta \mu \nu} \), \( \bar{S}_{\alpha \beta \mu \nu} \), \( \bar{K}_{\alpha \beta \mu \nu} \) or \( \bar{T}_{\alpha \beta \mu \nu} \) cannot satisfy the conservation requirement, but the sum with the others do. One can simplify the situation to eliminate \( \bar{T}_{\alpha \beta \mu \nu} \) which is being absorbed by \( \bar{B}_{\alpha \beta \mu \nu} \), \( \bar{S}_{\alpha \beta \mu \nu} \) and \( \bar{K}_{\alpha \beta \mu \nu} \). Consequently there are only 3 basis tensors left. Rewrite (12) as

\[ 2 \kappa t_{\alpha}^{\beta} = (a_1 B_{\alpha \beta \mu \nu} + a_2 S_{\alpha \beta \mu \nu} + a_3 K_{\alpha \beta \mu \nu}) x^\mu x^\nu, \]

(15)

where \( a_1 \) to \( a_3 \) are constants. This is the reason why these three tensors in (15) always appear to this order when one investigates the gravitational energy using the pseudotensor. Now, every tensor of \( B_{\alpha \beta \mu \nu} \), \( S_{\alpha \beta \mu \nu} \) and \( K_{\alpha \beta \mu \nu} \) satisfies the condition of the energy-momentum density conservation.

(ii) Second condition: non-negative gravitational energy in the small sphere limit. The purpose of the pseudotensor is for determining the gravitational energy-momentum, the associated energy-momentum can be calculated as

\[ 2 \kappa P_{\mu} = \int_{t=0}^{t} t^\rho \kappa x^\xi x^\kappa d\Sigma_\rho = t^0 \mu lm \int_{t=0}^{t} x^l x^m d^3x = t^0 \mu lm \frac{4\pi r^5}{15}, \]

(16)
where \( l, m = 1, 2, 3 \). Using this calculation method, the energy-momentum in the small sphere limit for (15) becomes

\[
2\kappa P_\mu = (-E, \vec{P}) = -\frac{r^5}{60} \left( a_1 B_{\mu 0 l}^l + a_2 S_{\mu 0 l}^l + a_3 K_{\mu 0 l}^l \right).
\]

The “energy-momentum” values associated with \( B_{\alpha\beta\mu\nu}, S_{\alpha\beta\mu\nu} \) and \( K_{\alpha\beta\mu\nu} \) are proportional to

\[
B_{\mu 0 l}^l = (E_{ab}E_{ab}^b + H_{ab}H_{ab}^b, 2\epsilon_{c}^{ab}E_{ad}H_{db}^d),
\]

\[
S_{\mu 0 l}^l = -10(E_{ab}E_{ab}^b - H_{ab}H_{ab}^b, 0),
\]

\[
K_{\mu 0 l}^l = B_{\mu 0 l}^l - S_{\mu 0 l}^l.
\]

where the electric part \( E_{ab} \) and magnetic part \( H_{ab} \) are defined in terms of the Weyl tensor as follows:

\[
E_{ab} := C_{a0b0}, \quad H_{ab} := *C_{a0b0}.
\]

Referring to (15), we are interested in the positive gravitational energy within a small sphere limit, the Bel-Robinson tensor already satisfies this condition. Precisely

\[
B_{00 l}^l = E_{ab}E_{ab}^b + H_{ab}H_{ab}^b \geq 0.
\]

The rest of the job is to find the coefficients \( a_2 \) and \( a_3 \). Using (20), rewrite (17) as

\[
2\kappa P_\mu = 2\kappa (-E, \vec{P}) = -\frac{r^5}{60} \left( [a_1 + a_3]B_{\mu 0 l}^l + (a_2 - a_3)S_{\mu 0 l}^l \right).
\]

Equation (19) shows that \( S_{\mu 0 l}^l \) cannot ensure positivity, since we should allow for any magnitude of \(|E_{ab}|\) and \(|H_{ab}|\). In other words, for \( S_{\alpha\beta\mu\nu} \) the sign of the “energy” density is uncertain. Therefore the only possibility for (23) to guarantee positivity is when \( a_2 = a_3 \). Recall the new tensor \( V_{\alpha\beta\mu\nu} \) which is defined as

\[
V_{\alpha\beta\mu\nu} := S_{\alpha\beta\mu\nu} + K_{\alpha\beta\mu\nu}.
\]

Additionally, referring to (20),

\[
V_{\mu 0 l}^l = S_{\mu 0 l}^l + K_{\mu 0 l}^l = B_{\mu 0 l}^l.
\]

Consequently (17) becomes

\[
2\kappa P_\mu = -\frac{r^5}{60} \left( [a_1 + a_3]B_{\mu 0 l}^l + (a_2 - a_3)S_{\mu 0 l}^l \right) = -\frac{r^5}{60} (a_1 + a_2)B_{\mu 0 l}^l.
\]

Hence the proof is completed. Indeed \( B_{\alpha\beta\mu\nu} \) and \( V_{\alpha\beta\mu\nu} \) are a basis for expressions which have non-negative gravitational “energy” density in vacuum.

Note that \( B_{\alpha\beta\mu\nu} \) and \( V_{\alpha\beta\mu\nu} \) are different tensors since they are defined by different fundamental quadratic curvatures, explicitly

\[
B_{\alpha\beta\mu\nu} = \bar{B}_{\alpha\beta\mu\nu} + \bar{T}_{\alpha\mu\nu},
\]

\[
V_{\alpha\beta\mu\nu} = \bar{S}_{\alpha\beta\mu\nu} + \bar{K}_{\alpha\beta\mu\nu} + \bar{T}_{\alpha\mu\nu}.
\]
In particular $V_{\alpha\beta\mu\nu}$ is totally trace free but not totally symmetric. The following list shows some properties

\begin{align*}
V_{\alpha\beta\mu\nu} &\equiv V_{(\alpha\beta)(\mu\nu)} \equiv V_{\mu\nu\alpha\beta}, \quad V_{\alpha\beta\mu} \equiv 0 \equiv V_{\alpha\mu\beta}, \\
V_{0000} &\equiv V_{00l} \equiv V_{mml} \equiv V_{mlm} \equiv E_{ab}E^{ab} + H_{ab}H^{ab} \equiv B_{0000}, \\
V_{\mu00l} &\equiv V_{\mu0l0} \equiv (E_{ab}E^{ab} + H_{ab}H^{ab}, 2\epsilon_{c}^{ab}E_{ad}H_{db}) \equiv B_{\mu00l}.
\end{align*}

(29)

It is known that $B_{\alpha\beta\mu\nu}$ has the dominant energy property \[8\]

\[B_{\alpha\beta\mu\nu} w_{1}^{\alpha} w_{2}^{\beta} w_{3}^{\mu} w_{4}^{\nu} \geq 0,\] (32)

where $w_{1}, w_{2}, w_{3}, w_{4}$ are any future-pointing causal vectors. Using (11), rewrite $V_{\alpha\beta\mu\nu}$ as

\[V_{\alpha\beta\mu\nu} := B_{\alpha\beta\mu\nu} + W_{\alpha\beta\mu\nu},\] (33)

where

\[W_{\alpha\beta\mu\nu} := \frac{3}{2}S_{\alpha\beta\mu\nu} - \frac{5}{8}g_{\alpha\beta}g_{\mu\nu}R^{2} + \frac{1}{8}(g_{\alpha\mu}g_{\beta\nu} + g_{\alpha\nu}g_{\beta\mu})R^{2}.\] (34)

This tensor has some interesting properties

\[W_{\alpha\beta\mu\nu} u_{1}^{\alpha} u_{2}^{\beta} u_{3}^{\mu} u_{4}^{\nu} = 0, \quad W_{\alpha\beta\mu\nu} u_{1}^{\alpha} u_{2}^{\beta} u_{3}^{\mu} u_{4}^{\nu} = 0,\] (35)

where $t$ is a timelike unit normal vector and $u$ can be timelike or null. $V_{\alpha\beta\mu\nu}$ contains more information than $B_{\alpha\beta\mu\nu}$, however it seems that $B_{\alpha\beta\mu\nu}$ is the important part of $V_{\alpha\beta\mu\nu}$ and $W_{\alpha\beta\mu\nu}$ is a kind of gauge freedom (i.e., it has no important physical effect).

A physical reasonable energy-momentum tensor has to fulfill the energy condition. The local energy density measured by the observer with a 4-velocity should be non-negative. The energy condition must be true for all timelike unit normal vectors \[9\]. In fact, we found $V_{\alpha\beta\mu\nu}$ has the non-negative “energy” property

\[V_{\alpha\beta\mu\nu} t^{\alpha} t^{\beta} t^{\mu} t^{\nu} = B_{\alpha\beta\mu\nu} t^{\alpha} t^{\beta} t^{\mu} t^{\nu} = E_{ab}E^{ab} + H_{ab}H^{ab} \geq 0.\] (36)

This is called the weak energy condition. Because of the continuity \[9\], the above inequalities must still be true if the timelike vector $t$ is replaced by a null vector $v$. Indeed, we found

\[V_{\alpha\beta\mu\nu} v^{\alpha} v^{\beta} v^{\mu} v^{\nu} = B_{\alpha\beta\mu\nu} v^{\alpha} v^{\beta} v^{\mu} v^{\nu} \geq 0.\] (37)

Therefore the statement is correct according to \[9\] for $V_{\alpha\beta\mu\nu}$, which is based on the fact that $B_{\alpha\beta\mu\nu}$ has the dominant energy property.

Following from (16), the energy-momentum density for $V_{\alpha\beta\mu\nu}$ in the small sphere limit is

\[2\kappa P_{\mu} = \frac{4\pi r^{5}}{15} (V_{\mu\alpha}^{0} - V_{\mu0}^{0}) = -\frac{4\pi r^{5}}{15} V_{0\mu0}.\] (38)

Or, more covariantly,

\[2\kappa P_{\mu} u^{\mu} = -\frac{4\pi r^{5}}{15} V_{\mu\alpha\beta\gamma} u^{\alpha} t^{\beta} t^{\gamma},\] (39)

where

\[V_{\alpha\beta\mu\nu} t^{\alpha} t^{\beta} t^{\mu} t^{\nu} = B_{\alpha\beta\mu\nu} t^{\alpha} t^{\beta} t^{\mu} t^{\nu} = (E_{ab}E^{ab} + H_{ab}H^{ab}, 2\epsilon_{c}^{ab}E_{ad}H_{db}).\] (40)
The physical meaning (non-spacelike energy-momentum) is here simpler and clearer than that of the dominant energy condition \((32)\). Obviously, \(V_{\alpha\beta\mu\nu}\) can play the same role as \(B_{\alpha\beta\mu\nu}\), it ensures positivity in the small sphere limit. In other words, the “energy-momentum” density according to \(B_{\alpha\beta\mu\nu}\) and \(V_{\alpha\beta\mu\nu}\) are on equal footing at the small sphere region limit.

Moreover, from the technical point of view, if we are just interested in the positive energy and relax the restriction on the pseudotensor constraint, which means the conservation of the energy-momentum, there are an infinite number of combinations that have the weak energy condition, not including \(B_{\alpha\beta\mu\nu}\). We define

\[
V'_{\alpha\beta\mu\nu} = \widetilde{K}_{\alpha\beta\mu\nu} + s\widetilde{S}_{\alpha\beta\mu\nu} + t_1\tilde{T}_{\alpha\beta\mu\nu} + t_2\tilde{T}_{\alpha\mu\beta\nu} + t_3\tilde{T}_{\alpha\nu\beta\mu}. \tag{41}
\]

where \(s, t_1, t_2, t_3\) are real numbers and \(t_1 + t_2 + t_3 = 1\). Note that the energy-momentum contribution for \(\widetilde{S}_{\alpha\beta\mu\nu}\) can be ignored simply because

\[
\widetilde{S}_{\alpha\beta\mu\nu}u^\alpha u^\beta u^\mu u^\nu = 0 = \widetilde{S}_{\alpha\beta\mu\nu}u^\alpha u^\beta u^\mu u^\nu. \tag{42}
\]

On the other hand,

\[
\tilde{T}_{\alpha\beta\mu\nu}u^\alpha u^\beta u^\mu u^\nu \equiv \tilde{T}_{\alpha\mu\beta\nu}u^\alpha u^\beta u^\mu u^\nu \equiv \tilde{T}_{\alpha\nu\beta\mu}u^\alpha u^\beta u^\mu u^\nu. \tag{43}
\]

Once again, \(u\) can be timelike or null. Similarly, from the weak energy condition and using the continuity property, we found the results of \(V'_{\alpha\beta\mu\nu}\) as follows

\[
V'_{\alpha\beta\mu\nu}t^\alpha t^\beta t^\mu t^\nu \equiv B_{\alpha\beta\mu\nu}t^\alpha t^\beta t^\mu t^\nu \geq 0, \tag{44}
\]

\[
V'_{\alpha\beta\mu\nu}v^\alpha v^\beta v^\mu v^\nu \equiv B_{\alpha\beta\mu\nu}v^\alpha v^\beta v^\mu v^\nu \geq 0. \tag{45}
\]

This illustrates that there exists an infinite number of other tensors which have positivity if we exclude the conservation of the energy-momentum density requirement according to the pseudotensor restriction.

Furthermore, in order to obtain the dominant energy condition, \(B_{\alpha\beta\mu\nu}\) is the unique tensor that has the suitable combination according to the 4 fundamental quadratic curvature combinations, namely from \((6)\) to \((9)\). The Bel-Robinson tensor has more nice properties than the other quadratic curvature combinations generally. However, concerning gravitational energy at the small sphere limit, \(V_{\alpha\beta\mu\nu}\) becomes the only alternative choice to compare with \(B_{\alpha\beta\mu\nu}\).

\section{Conclusion}

According to the four fundamental quadratic curvature tensors \([1]\), we construct all the possible combinations in the pseudotensor expression. We recovered that \(B_{\alpha\beta\mu\nu}\) gives a definite positive gravitational energy in the small sphere limit approximation. However, we found a unique alternative, the new tensor \(V_{\alpha\beta\mu\nu}\), which also contributes the same non-negative gravitational energy density at the same region limit. These two tensors can be classified as a basis for expressions which have the desirable non-negative gravitational energy in the small sphere region limit. Moreover, we found that the tensor \(W_{\alpha\beta\mu\nu}\), associated with \(V_{\alpha\beta\mu\nu}\), behaves as a kind of gauge freedom.
Relaxing the restriction of the energy-momentum conservation requirement for the pseudotensor, $V'_{\alpha\beta\mu\nu}$ demonstrates that there are an infinite number of ways to obtain positivity, namely the weak energy condition. For the conserved expressions $B_{\alpha\beta\mu\nu}$ satisfies the dominant energy condition while $V_{\alpha\beta\mu\nu}$ satisfies the weak energy condition.

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