An extended fuzzy supersphere and twisted chiral superfields

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Abstract

A noncommutative associative algebra of $N = 2$ fuzzy supersphere is introduced. It turns out to possess a nontrivial automorphism which relates twisted chiral to twisted anti-chiral superfields and hence makes possible to construct noncommutative nonlinear $\sigma$-models with extended supersymmetry.
1 Introduction

Supersymmetric nonlinear $\sigma$-models with $N = 2$ supersymmetry in two dimensions are important objects in modern mathematical physics. They possess a very rich structure interesting by itself and find also applications, for instance, in superstring theory. It is a well-known fact that the models with $N = 1$ supersymmetry can be constructed for an arbitrary geometry of the target space. However, the $N = 2$ case requires the target space to be Kähler [1] if we consider the case without torsion.

There exists a very convenient description of the $N = 2$ $\sigma$-models based on the $N = 2$ superspace. In this paper, we shall show that the $N = 2$ superspace can be constructed also on the noncommutative sphere. More precisely, we shall construct a noncommutative $N = 2$ supersphere. Note that the notation $N = 1$ or $N = 2$ refers usually to the Poincare-like superalgebras in which the anticommutators of the supercharges are the generators of translations of the underlying bosonic space. We shall see soon, however, that due to the fact that the two-sphere is conformally flat we can keep this terminology also for spherical worldsheets.

Noncommutative geometry [2] is the generalization of the ordinary geometry in which an algebra of functions which encodes the geometry of an ordinary space is replaced by certain noncommutative algebra. As an example we take a noncommutative (or fuzzy) sphere which is an object introduced by several researchers in the past [3, 4, 5, 6] with various motivations. Berezin himself has quantized the standard round symplectic structure on the two-sphere and he found that this can be done only for integer values of the inverse Planck constant. For example if $h = 1/n$, the quantized algebra of observables (=the fuzzy sphere) coincides simply with the algebra of $(n + 1) \times (n + 1)$ matrices. When $n \rightarrow \infty$ the size of the algebra approaches infinity; in fact, one recovers the standard algebra of functions on the commutative sphere. Effectively, the quantization cuts off the large angular momenta. This fact lead independently several authors [7, 8, 9] to use the fuzzy sphere as a regularization of fields theories formulated on the ordinary sphere.

It turns out that this regularization has an important advantage of preserving the standard $SO(3)$ invariance of the ordinary sphere. This is a quite remarkable fact because the regulated theory contains only a finite number of degrees of freedom and, even more importantly, the regulated sphere
continues to be a geometric object so it makes sense to formulate theories non-perturbatively directly on it.

The list of the virtues of the fuzzy regularization is not exhausted by the $SO(3)$ invariance and the finite number of degrees of freedom. In fact, one can introduce fuzzy monopole configurations and, perhaps even more remarkable, to regulate supersymmetric and supersymmetric gauge theories while manifestly preserving supersymmetry, supergauge symmetry and the finite number of degrees of freedom. It is indeed the purpose of this paper to show that models with extended supersymmetry are also regularisable by the method.

In section 2, we introduce the extended $N = 2$ supersphere and its non-commutative deformation. Moreover, we shall identify a nontrivial automorphism of the structure which will prove very useful in constructing $N = 2$ theories. Section 3 presents the construction of the commutative and non-commutative $N = 2$ supersymmetric nonlinear $\sigma$-models on the sphere.

## 2 $N = 2$ fuzzy supersphere

An $N = 1$ fuzzy supersphere has been constructed in with a goal to regularize $N = 1$ supersymmetric nonlinear $\sigma$-models. The reader may find an alternative more concise description of the structure in . In the $N = 2$ case, the construction begins in a similar way than in the $N = 1$ one but there is a point of depart in which new (and welcome) structural ingredients enter. Here are the details.

Consider the algebra of polynomial functions on the complex $C^{2|2}$ super-plane, i.e. algebra generated by finite sums of monomials in bosonic variables $\chi^\alpha, \bar{\chi}^\alpha, \alpha = 1, 2$ and in fermionic ones $\bar{a}^\alpha, a^\alpha, \alpha = 1, 2$. The algebra is equipped with the super-Poisson bracket

$$\{ f, g \} = \partial_{\bar{\chi}^\alpha} f \partial_{\chi^\alpha} g - \partial_{\chi^\alpha} f \partial_{\bar{\chi}^\alpha} g + (-1)^{f+1}[\partial_{a^\alpha} f \partial_{\bar{a}^\alpha} g + \partial_{\bar{a}^\alpha} f \partial_{a^\alpha} g].$$

(1)

and with the graded involution

$$\begin{align*}
(\chi^\alpha)^\dagger &= \bar{\chi}^\alpha, \\
(\bar{\chi}^\alpha)^\dagger &= \chi^\alpha, \\
(a^1)^\dagger &= \bar{a}^1, \\
(a^2)^\dagger &= -\bar{a}^2, \\
(\bar{a}^1)^\dagger &= -a^1, \\
(\bar{a}^2)^\dagger &= a^2,
\end{align*}$$

(2)

satisfying the following properties

$$\begin{align*}
(AB)^\dagger &= (-1)^{AB} B^\dagger A^\dagger, \\
(A^\dagger)^\dagger &= (-1)^A A.
\end{align*}$$

(3)
We can now apply the (super)symplectic reduction with respect to a moment map \( \bar{\chi}^\alpha \chi^\alpha + \bar{a}^\alpha a^\alpha - 1 \). The result is a smaller algebra \( A_\infty \), that by definition consists of all functions \( f \) with the property

\[
\{ f, \bar{\chi}^i \chi^i + \bar{a}^\alpha a^\alpha - 1 \} = 0. \tag{4}
\]

Moreover, two functions obeying (4) are considered to be equivalent if they differ just by a product of \( (\bar{\chi}^\alpha \chi^\alpha + \bar{a}^\alpha a^\alpha - 1) \) with some other such function.

The smaller algebra \( A_\infty \) (the reason for using of the subscript \( \infty \) will become clear soon) will be referred to as the algebra of superfunctions on an \( N = 2 \) supersphere. It will be sometimes more convenient to work with a different parametrization of \( A_\infty \), using the following coordinates

\[
z = \frac{\chi^1}{\chi^2}, \quad \bar{z} = \frac{\bar{\chi}^1}{\bar{\chi}^2}, \quad b^\alpha = \frac{a^\alpha}{\chi^2}, \quad \bar{b}^\alpha = \frac{\bar{a}^\alpha}{\bar{\chi}^2}. \tag{5}
\]

The Poisson bracket (1) then becomes

\[
\{ f, g \} = (1 + \bar{z} z + \bar{b}^\alpha b^\alpha) [(1 + \bar{z} z)(\partial_z f \partial_{\bar{z}} g - \partial_{\bar{z}} f \partial_z g) + \bar{b}^\beta \bar{z} (\partial_{\bar{b}}^\beta \partial_z f - \partial_z \partial_{\bar{b}}^\beta f) + (-1)^{(f+1)} (\partial_{\bar{b}}^\beta \partial_{\bar{b}}^\gamma f + \partial_{\bar{b}}^\gamma \partial_{\bar{b}}^\beta f)] \tag{6}
\]

A natural Berezin integral on \( A_\infty \) can be written as

\[
I(f) = \frac{1}{(2\pi i)^2} \int d\bar{\chi}^1 \wedge d\chi^1 \wedge d\bar{\chi}^2 \wedge d\chi^2 \wedge d\bar{a}^1 \wedge da^1 \wedge d\bar{a}^2 \wedge da^2 \delta(\bar{\chi}^\alpha \chi^\alpha + \bar{a}^\alpha a^\alpha - 1) f. \tag{7}
\]

It can be rewritten as

\[
I(f) \equiv \frac{1}{2\pi i} \int dz \wedge d\bar{z} \wedge d\bar{b}^1 \wedge db^1 \wedge d\bar{b}^2 \wedge db^2 f. \tag{8}
\]

(Note \( I(1) = 0 \).)

\(^1\)Note that in case we did not consider the fermionic variables \( \bar{a}^\alpha, a^\alpha \), we would obtain, as the result of the symplectic reduction, the algebra of functions on the standard bosonic sphere. In case of considering only one pair of fermionic variables \( \bar{a}, a \) we would obtain the \( N = 1 \) supersphere.
Strictly speaking, the generators $\bar{z}, z, \bar{b}^\alpha, b^\alpha$ are not elements of the algebra $A_\infty$. What is true is that $A_\infty$ is (finitely) linearly generated by the functions of the following form

$$\frac{z^k \bar{z}^l (\bar{b}_1^1)^m (b_1^1)^l (\bar{b}_2^2)^m (b_2^2)^l}{(1 + \bar{z}z + b^\alpha \bar{b}^\alpha)^m}, \quad k + \bar{l} + l^1 + l^2 \leq m,$$

where $k, \bar{k}, l^1, \bar{l}^1, m$ are non-negative integers. It is not difficult to understand the form (9) of the elements of $A_\infty$. Indeed, we first note that $\bar{z}, z, \bar{b}^\alpha, b^\alpha$ can be also interpreted as a local chart coordinates of the $N = 2$-supersphere obtained by the stereographic projection from the north pole. If we do the projection from the south pole, we obtain a complementary chart with local coordinates $\bar{w}, w, \bar{b}^\alpha_w, b^\alpha_w$. A transition rule on the overlap of the two charts reads

$$w = 1/z, \quad \bar{w} = 1/\bar{z}, \quad b^\alpha_w = b^\alpha / z, \quad \bar{b}^\alpha_w = \bar{b}^\alpha / \bar{z}. \quad (10)$$

It is now a simple matter to check that the functions of the form (9) will transform into

$$\frac{\bar{w}^{m-k-\bar{l}-l^2} w^{m-k-l^1-l^2} (\bar{b}_w^1)^m (b_w^1)^l (\bar{b}_w^2)^m (b_w^2)^l}{(1 + \bar{w}w + b^\alpha \bar{b}^\alpha)^m}. \quad (11)$$

Since $0 \leq m - k - \bar{l} - l^2 \leq m$ and $0 \leq m - k - l^1 - l^2 \leq m$ for $k + \bar{l} + l^2, k + l^1 + l^2 \leq m$; $k, \bar{k}, l^1, \bar{l}^1, m \geq 0$ we see that the elements of $A_\infty$ are form-invariant with respect to the coordinate transformation (10).

The reason to use the coordinates $\bar{z}, z, \bar{b}^\alpha, b^\alpha$ is simple: they will enable us to establish a connection between standard $N = 2$ supersymmetric nonlinear $\sigma$-models defined on the flat Euclidean space and their counterparts on the $N = 2$ supersphere. In fact, we shall see that the flat models in the coordinates $\bar{z}, z, \bar{b}^\alpha, b^\alpha$ and the spherical models in the same coordinates have the same field theoretical action! They differ, however, in the sense that the algebras of the superfields in both cases are different. In the flat case the superfield is an element of the algebra of superfunctions on the Euclidean $N = 2$ superspace while in the spherical case the superfield is an element of $A_\infty$.

Let us now introduce a Lie superalgebra $T$ which will turn out to contain all relevant structure of the $N = 2$ nonlinear $\sigma$-models on the sphere. It has seven even generators $R_\pm, R_3, Z_\pm, Z_3, C$ and eight odd ones $C_\pm, C_\pm^\dagger, C_\pm^\circ, C_\pm^{\dagger \circ}$. 4
We denote the corresponding Hamiltonians by small characters; they are given by

\[ r_3 = \frac{1}{2}(\bar{\chi}_1 \chi_1 - \bar{\chi}_2 \chi_2), \quad r_+ = \bar{\chi}_1 \chi_2, \quad r_- = \bar{\chi}_2 \chi_1; \]  
\[ z_3 = \frac{1}{2}(\bar{a}_1 a_1 - \bar{a}_2 a_2), \quad z_+ = \bar{a}_1 a_2, \quad z_- = \bar{a}_2 a_1; \]  
\[ c = \bar{\chi}_1 \chi_1 + \bar{\chi}_2 \chi_2 + \bar{a}_1 a_1 + \bar{a}_2 a_2; \]  
\[ c_+ = \bar{a}^2 \chi_1 + \bar{\chi}^2 a_1, \quad c_- = -\bar{a}^2 \chi_2 + \bar{\chi}^2 a_1; \]  
\[ c^\dagger_+ = \bar{a}^1 \chi_2 + \bar{\chi}^1 a_2, \quad c^\dagger_- = \bar{a}^1 \chi_1 - \bar{\chi}^1 a_2; \]  
\[ c_+^\circ = \bar{a}^2 \chi_1 + \bar{\chi}^2 a_2, \quad c_-^\circ = -\bar{a}^2 \chi_2 + \bar{\chi}^2 a_2; \]  
\[ c^\dagger_+^\circ = \bar{a}^1 \chi_2 + \bar{\chi}^1 a_2, \quad c^\dagger_-^\circ = \bar{a}^1 \chi_1 - \bar{\chi}^1 a_2. \]  

We should remember that these Hamiltonians are actually preimages of the true Hamiltonians in the process of the symplectic reduction. Since they anyway commute with the moment map \((c - 1)\) it is possible and technically preferable to work with them. A reader who wishes to work directly with expressions in terms of \(\bar{z}, z, \bar{b}^\alpha, b^\alpha\) coordinates can simply use the equations (12)-(18) and the following relation

\[ \frac{1}{\bar{\chi}^2 \chi^2} = 1 + \bar{z}z + \bar{b}^1 b^1 + \bar{b}^2 b^2. \]  

One obtains, for example,

\[ c_+ = \frac{z\bar{b}^2 + b^1}{1 + \bar{z}z + b^1\bar{b}^1 + b^2\bar{b}^2}, \quad c_- = \frac{\bar{z}b^1 - \bar{b}^2}{1 + \bar{z}z + b^1\bar{b}^1 + b^2\bar{b}^2} \]  

and so on for all Hamiltonians (12)-(18). In particular, the Hamiltonian \(c\) becomes simply

\[ c = 1. \]  

The last equation does not mean, however, that \(C\) gets detached from the superalgebra \(\mathcal{T}\). It rather means that \(\mathcal{T}\) is the central extension of \(\mathcal{T}/C\) by \(C\).

The (graded) commutation relations of the superalgebra \(\mathcal{T}\) are given by the Poisson brackets of the Hamiltonians (12)-(18). Though this is a correct
definition, we prefer to give an explicit list of nonvanishing commutators because many results of this paper depend directly on them. Here they are

\[
[R_3, R_{\pm}] = \pm R_{\pm}, \quad [Z_3, Z_{\pm}] = \pm Z_{\pm}, \quad [R_+, R_-] = 2R_3, \quad [Z_+, Z_-] = 2Z_3; \tag{22}
\]
\[
[R_3, C_{\pm}] = \mp \frac{1}{2} C_{\pm}, \quad [R_3, C^0_{\pm}] = \mp \frac{1}{2} C^0_{\pm}, \quad [R_3, C^1_{\pm}] = \pm \frac{1}{2} C^1_{\pm}, \quad [R_3, C^{\dagger 0}_{\pm}] = \pm \frac{1}{2} C^{\dagger 0}_{\pm}; \tag{23}
\]
\[
[R_{\pm}, C_{\pm}] = C_{\mp}, \quad [R_{\pm}, C^0_{\pm}] = C^0_{\mp}, \quad [R_{\pm}, C^1_{\pm}] = -C^1_{\mp}, \quad [R_{\pm}, C^{\dagger 0}_{\pm}] = -C^{\dagger 0}_{\mp}; \tag{24}
\]
\[
[Z_3, C_{\pm}] = \mp \frac{1}{2} C_{\pm}, \quad [Z_3, C^0_{\pm}] = \pm \frac{1}{2} C^0_{\mp}, \quad [Z_3, C^1_{\pm}] = \pm \frac{1}{2} C^1_{\mp}, \quad [Z_3, C^{\dagger 0}_{\pm}] = \mp \frac{1}{2} C^{\dagger 0}_{\mp}; \tag{25}
\]
\[
[Z_{\pm}, C_{\pm}] = \pm C^1_{\mp}, \quad [Z_{\pm}, C^0_{\mp}] = \mp C^0_{\mp}, \quad [Z_{\pm}, C^{\dagger 0}_{\mp}] = \pm C^{\dagger 0}_{\mp}, \quad [Z_{\pm}, C^{\dagger 1}_{\mp}] = \mp C^{\dagger 1}_{\pm}; \tag{26}
\]
\[
[C_{\pm}, C^0_{\pm}]_+ = \pm 2R_{\mp}, \quad [C^1_{\pm}, C^{\dagger 0}_{\pm}]_+ = \pm 2R_{\mp}, \quad [C_{\pm}, C^{\dagger 0}_{\pm}]_+ = 2Z_{\pm}, \quad [C^0_{\pm}, C^{\dagger 1}_{\pm}]_+ = 2Z_{\pm}; \tag{27}
\]
\[
[C_{\pm}, C^{\dagger 0}_{\mp}]_+ = [C^1_{\pm}, C^{\dagger 0}_{\mp}]_+ = 2(R_3 \mp Z_3), \quad [C_{\pm}, C^{\dagger 1}_{\mp}]_+ = [C^0_{\pm}, C^{\dagger 1}_{\pm}]_+ = C. \tag{28}
\]

We note that the commutation relations of \( T/C \) coincide with those of the anomaly free subalgebra of the \( N = 4 \) super-Virasoro algebra \([12]\).

Let us define an automorphism \( \phi \) of the algebra \( T \) which plays a crucial role in our construction. It is easy to verify that the commutation relations of \( T \) are invariant if

\[
R^0_{\pm} = R_{\pm}, \quad R^0_3 = R_3, \quad Z^0_+ = Z_-, \quad Z^0_3 = -Z_3, \quad C^0 = C. \tag{29}
\]

The action of the automorphism on the odd generators is given by the notation itself and by the claim that the automorphism is involutive i.e. it squares to the identity map.

It is a matter of a simple inspection to see that the associative algebra \( A_\infty \), which defines the \( N = 2 \) supersphere, is linearly and multiplicatively generated by four odd variables \( c_{\pm}, c^\dagger_{\pm} \) and three even ones \( l_{\pm}, l_3 \), defined by\(^2\)

\[
l_3 = r_3 - \frac{1}{2c} c^\dagger_+ c_+ + \frac{1}{2c} c^\dagger_- c_-; \tag{30}
\]
\[
l_{\pm} = r_{\pm} + \frac{1}{c} c^\dagger_{\pm} c_{\mp}. \tag{31}
\]

\(^2\)Although \( c = 1 \) in \( A_\infty \), it is useful to indicate \( c \) in (30) and (31) because these formulae hold also in the noncommutative case where \( c \neq 1 \).
Note that
\[ l_+^4 = l_-, \quad l_3^4 = l_3 \]  \hspace{1cm} (32)
and that \( l_\pm, l_3 \) are not independent variables as they are subject to the following relation
\[ l_3^2 + l_+ l_- = 1/4. \]  \hspace{1cm} (33)
But the relations (32) and (33) characterize the ordinary bosonic sphere! Moreover, the only nonvanishing Poisson brackets among the generators \( l_\pm, l_3, c_\pm, c^\pm_\pm \) turn out to be
\[ \{ l_3, l_\pm \} = \pm l_\pm, \quad \{ l_+, l_- \} = 2l_3; \]  \hspace{1cm} (34)
\[ \{ c_\pm, c^\pm_\pm \} = c. \]  \hspace{1cm} (35)

In other words, \( l \)'s and \( c \)'s completely decouple and we see that the algebra \( A_\infty \) is a direct product of the algebra \( B_\infty \) of the functions on the ordinary sphere and of the Grassmann algebra \( Gr_4 \) with four generators \( c_\pm, c^\pm_\pm \). This direct product concerns not only the associative multiplication but also the Poisson structure. The immediate conclusion of those facts is that it is very easy to quantize the \( N = 2 \) supersphere. The corresponding noncommutative algebra \( A_n \) is simply the ordinary bosonic fuzzy sphere \([4, 5, 6]\) tensored with a Clifford algebra \( Cf_4 \) with four generators \( c_\pm, c^\pm_\pm \). Here and in what follows we shall often use the same symbol for non-deformed generators and for their deformed counterparts. It should be clear from the context which usage we have in mind.

We remind that the bosonic fuzzy sphere is a \((n + 1) \times (n + 1)\) dimensional matrix algebra where the integer parameter \( n \) plays role of the inverse Planck constant \([4, 5, 6]\). Since the Poisson brackets (34) and (35) are to be replaced by commutators scaled by the inverse Planck constant, we get the following commutation relations for the noncommutative generators \( l_\pm, l_3, c_\pm, c^\pm_\pm \) and \( c \) of the \( N = 2 \) fuzzy supersphere
\[ [l_3, l_\pm] = \pm \frac{1}{n} l_\pm, \quad [l_+, l_-] = \frac{2}{n} l_3; \]  \hspace{1cm} (36)
\[ [c_\pm, c^\pm_\pm]_+ = \frac{1}{n} c. \]  \hspace{1cm} (37)
Moreover, we define the graded involution \( \dagger \) in the noncommutative case by (32) on \( l_\pm, l_3 \) and by the notation and the second property (3) on \( c_\pm, c^\pm_\pm \). It
is easy to find the explicit forms of the matrices $l_{\pm}, l_3, c_{\pm}, c^\pm_{\pm}$ and $c$:

$$
l_{\pm} = \frac{1}{n}(L_{\pm} \otimes 1), \quad l_3 = \frac{1}{n}(L_3 \otimes 1), \quad c = (1 + \frac{1}{n})(1 \otimes 1); \quad (38)
$$

$$
c_{\pm} = \frac{1}{\sqrt{n}}(1 \otimes \gamma_{\pm}), \quad c^\pm_{\pm} = \frac{1}{\sqrt{n}}(1 \otimes \gamma^\pm_{\pm}), \quad (39)
$$

where the first entry of the tensor product corresponds to the bosonic fuzzy sphere and the second entry to the Clifford algebra. $L_{\pm}, L_3$ are generators of $su(2)$ Lie algebra in the representation with spin $n/2$ and $\gamma$’s and $\gamma^\dagger$’s are standard Dirac matrices with respect to the Euclidean metric in four dimensions normalized according to

$$
[\gamma_{\pm}, \gamma^\pm_{\pm}]_+ = c = 1 + 1/n. \quad (40)
$$

Note that due to the tensor product structure of the $N = 2$ supersphere, the normalization of the central term $c$ must be 1 in the limit $n \to \infty$ but it is otherwise a free parameter of the construction. It is the choice $c = 1 + 1/n$ which makes possible to construct $N = 2$ supersymmetric $\sigma$-models on the $N = 2$ fuzzy supersphere.

Let us study the properties of the algebra $A_n$. It turns out that we shall need a non-commutative analogue of the automorphism $\circ$. Actually, we defined $\circ$ as the automorphism of the Lie superalgebra $T$ and not yet of $A_\infty$. On the other hand, $\circ$ can be directly defined to act on the whole algebra $A_\infty$ as the morphism of the (graded) multiplication. For instance,

$$
(l_3 c_+)^\circ = l_3^\circ c_+^\circ. \quad (41)
$$

Since we know that the Poisson bracket (6) is compatible with the associative multiplication in $A_\infty$, we conclude that $\circ$ is the automorphism of the both associative and Poisson structure of $A_\infty$.

We can equally well use another set of generators for describing the algebra $A_\infty$, namely, the set $c_+^{\circ}, c_\pm^{\circ}$ and $l_\pm^{\circ}, l_3^{\circ}$. Of course, $l_\pm^{\circ}, l_3^{\circ}$ are given by

$$
l_3^{\circ} = r_3 - \frac{1}{2c} c_+^{\circ} c_+^{\circ} + \frac{1}{2c} c_-^{\circ} c_-^{\circ}; \quad (42)
$$

$$
l_\pm^{\circ} = r_\pm + \frac{1}{c} c_\pm^{\circ} c_\pm^{\circ}. \quad (43)
$$
and they also turn out to fulfil the relations

\[ l_+^{\dagger} = l_-, \quad l_3^{\dagger} = l_3^{\dagger} \]  \hspace{1cm} (44)

and

\[ (l_3^{\dagger})^2 + l_+^{\dagger} l_- = 1/4. \]  \hspace{1cm} (45)

Moreover, the only nonvanishing Poisson brackets among the generators \( l_\pm, l_3, c_\pm, c_\mp \) are as follows

\[ \{ l_3^{\dagger}, l_\pm \} = \pm l_\pm, \quad \{ l_+^{\dagger}, l_- \} = 2l_3^{\dagger}, \]  \hspace{1cm} (46)

\[ \{ c_\pm^{\dagger}, c_\mp \} = c. \]  \hspace{1cm} (47)

The relations (46) and (47) are actually the direct consequences of the fact that \( \circ \) is the automorphism of the Poisson algebra \( A_\infty \). Nevertheless, we prefer to state them explicitly in order to stress the equal footing of the two sets of generators.

Of course, we can now construct the noncommutative deformation of the \( N = 2 \) supersphere, by quantizing the set of the new circled generators. Thus, the \( N = 2 \) fuzzy supersphere will be again nothing but the tensor product of the bosonic fuzzy sphere with the Clifford algebra \( C_{f4} \). A question arises: How the uncircled and the circled fuzzy superspheres fit together? Let us look for a key for answering this question in the commutative case \( A_\infty \), where the circled variables can be written in terms of the non-circled ones as follows

\[ c_\pm = \frac{1}{c}(2l_3 c_\mp + 2l_\pm c_\mp \mp \frac{1}{c} c_\dagger \dagger [c_\pm, c_\mp]); \]  \hspace{1cm} (48)

\[ c_\mp^{\dagger} = \frac{1}{c}(2l_3 c_\mp + 2l_\pm c_\pm \mp \frac{1}{c} c_\dagger \dagger [c_\pm, c_\mp]); \]  \hspace{1cm} (49)

\[ l_3^{\dagger} = l_3 + \frac{1}{2c} c_\dagger c_+ - \frac{1}{2c} c_\dagger c_- - \frac{1}{2c} c_\dagger c_\dagger c_+ + \frac{1}{2c} c_\dagger c_\dagger c_- \]  \hspace{1cm} (50)

\[ l_\pm^{\dagger} = l_\pm - \frac{1}{c} c_- c_+ + \frac{1}{c} c_- c_\dagger. \]  \hspace{1cm} (51)

Now we take the formulae (48)-(51) as the definition of the circled variables in the non-commutative case where \( l_\pm, l_3, c_\pm, c_\dagger \), \( c \) are given by (38) and (39).

The operator formulae (48) and (49) are remarkable since they involve cubic terms in the old uncircled generators. This causes that the usual ordering problem leads in this case to an operator rather than a \( c \)-number.
ambiguity. Indeed, writing the cubic terms in (48) and (49) requires the fixing of a certain ordering; in fact, the commutators (not anticommutators!) $[c^\pm_\pm, c_\pm]$ in (48) and (49) do the job. A slight change of the ordering in any of the definitions (48),(49) would completely destroy a crucial property of this maps, namely, the circled variables fulfil exactly the same properties as the noncircled ones. Explicitly,

$$[l^0_3, l^0_\pm] = \pm \frac{1}{n} l^0_\pm, \quad [l^0_\pm, l^0_-] = \frac{2}{n} l^0_3; \quad (52)$$

$$[c^0_\pm, c^{00}_\pm]_+ = \frac{1}{n} c \quad (53)$$

and all remaining graded commutators vanish. Remind that the relations (52) and (53) are not postulated but they are derived from the relations (36) and (37) and the definitions (48) and (49). The normalization and reality of $l^0_3, l^0_\pm$ is also correct since one can verify that

$$(l^0_3)^2 + l^0_+ l^0_- = l^2 + l_+ l_- = 1/4 + 1/2n \quad (54)$$

and

$$l^{0\dagger}_+ = l^-_3, \quad l^{0\dagger}_3 = l^0_3. \quad (55)$$

Thus we conclude that the uncircled $N = 2$ fuzzy supersphere is the same thing as the circled one. The mapping $\circ$ preserve the commutation relations among the generators therefore it can be extended to the whole supersphere as the automorphism of its associative product. Moreover, $\circ$ is an involutive automorphism since (51) and (52) are manifestly involutive and a tedious computation shows that the definitions (48) - (51) imply

$$c_\pm = \frac{1}{c} (2l^0_3 c^{0\dagger}_\pm \pm 2l^0_\pm c^{0\dagger}_\pm \pm \frac{1}{c} c^{00}_\pm [c^{00}_\pm, c^{0}_\pm]); \quad (56)$$

$$c^{0\dagger}_\pm = \frac{1}{c} (2l^0_3 c^0_- \pm 2l^0_\pm c^0_- \pm \frac{1}{c} c^0_\pm [c^0_\pm, c^0_-]). \quad (57)$$

For a completeness, we give explicit formulae for the even Hamiltonians $r_\pm, r_3, z_\pm, z_3$ in terms of the generators $l_3, l_\pm, c_\pm, c^{0\dagger}_\pm$ and $l^0_3, l^0_\pm, c^0_\pm, c^{00}_\pm$. They are valid in both commutative ($n \to \infty$) and noncommutative (finite $n$) cases:

$$r_3 = l_3 + \frac{1}{2c} c^{0\dagger}_+ c_+ - \frac{1}{2c} c^{\dagger}_- c_- = l^0_3 + \frac{1}{2c} c^{0\dagger}_+ c^0_+ - \frac{1}{2c} c^{\dagger}_- c^0_-; \quad (58)$$
The construction of the involutive automorphism $\circ$ is the main result of this section. In what follows, we shall always enjoy a freedom of choosing to work in one of the two $\circ$ related equivalent parametrization of the $N = 2$ fuzzy supersphere.

3 $N = 2$ nonlinear $\sigma$-models

3.1 The commutative case

The basic fact of life in the $N = 2$ flat Euclidean superspace is that a Lagrangian density of a field theoretic model does not involve derivatives. All dynamics is encoded in constraints imposed on $N = 2$ superfields in a way compatible with the $N = 2$ supersymmetry. For example, the Lagrangian of an $N = 2$ supersymmetric $\sigma$-model on the Euclidean plane is given by

$$S = \int d\bar{z} dz db^1 db^2 db^2 K(\bar{\Phi}\Phi).$$

(62)

Here $\Phi(\bar{z}, z, \bar{b}^1, b^1, \bar{b}^2, b^2)$ and $\bar{\Phi}(\bar{z}, z, \bar{b}^1, b^1, \bar{b}^2, b^2)$ are superfields on the plane. They are subject to the following constraints

$$D_+ \Phi = \bar{D}_- \Phi = 0; \quad D_- \bar{\Phi} = \bar{D}_+ \bar{\Phi} = 0$$

(63)

and $K(\bar{\Phi}\Phi)$ is the Kähler potential of a target Kähler manifold with complex coordinates $\bar{\Phi}, \Phi$. The supersymmetric covariant derivatives are defined as

$$D_+ = \partial_{\bar{z}} + b^1 \partial_z, \quad D_- = \partial_{\bar{z}} + b^2 \partial_z, \quad \bar{D}_+ = \partial_{\bar{z}} + \bar{b}^1 \partial_{\bar{z}}, \quad \bar{D}_- = \partial_{\bar{z}} + \bar{b}^2 \partial_{\bar{z}}.$$

(64)

Note that the flat measure in the integral (62) coincides with the $N = 2$ "round" measure (8). Because of this fact, the model (62) can be reinterpreted as a model on the $N = 2$ supersphere. For this interpretation, it is sufficient to declare that both $\bar{\Phi}$ and $\Phi$ are not the superfields on the plane.
but but they are rather elements of the algebra $\mathcal{A}_\infty$ i.e. of the algebra of the superfunctions on the $N = 2$ supersphere. More precisely, the superfields are linear combinations of the elements of $\mathcal{A}_\infty$ of the form (9) with coefficients being ordinary numbers, when $\bar{l}_1 + \bar{l}_2 + l_1 + l_2$ is even and Grassmann numbers when $\bar{l}_1 + \bar{l}_2 + l_1 + l_2$ is odd. These Grassmann numbers anticommute with the odd generators of $\mathcal{A}_\infty$. As a result, the superfields $\bar{\Phi}, \Phi$ are even. This remark is important when we calculate the Poisson brackets involving the superfields or when we use the graded involution.

The constraints (63) turn out to be equivalent to

$$\{c_\pm, \Phi\} = 0, \quad \{c_\pm^0, \bar{\Phi}\} = 0,$$

(65)

where $\{., .\}$ is the ”round” Poisson bracket (6) and the Hamiltonians $c_\pm, c_\pm^0$ are given in (15) and (17). The constraints (63) or (65) define so-called twisted chiral and twisted anti-chiral fields, respectively. In order to have another viable set of Poisson bracket constraints, giving so-called chiral and anti-chiral superfields, we would have to change the symplectic structure (6). This is easy but we shall not discuss it in this paper because the resulting picture is completely analogous to the twisted one. We just remark, that from the point of view of the Poisson bracket (6) on the $N = 2$ supersphere, the twisted fields are more ”natural” than the untwisted ones.

There is an inconspicuous but, in fact, an important detail that concerns the (graded) involution in (62) denoted by a bar. It acts on the generators $\bar{z}, z, \bar{b}_1, b_1, \bar{b}_2, b_2$ following the notation itself. This involution is not the same as the one denoted by $\dagger$ in section 2 (cf. (2)), although they coincide on the bosonic variables $\bar{z}, z$. In fact, $\dagger$ is rather a world-sheet involution. It is with respect to $\dagger$ that the generators $l_\pm, l$ or $l_\pm^0, l^0$ fulfil the correct reality conditions (32) of the bosonic generators of the (fuzzy) sphere. On the other hand, the bar involution sets the reality properties of fermionic fields if the supersymmetric action is written in components. These reality properties propagate to the quantization of field theoretical model and define an involution on the Hilbert space of the quantum field theory. We remark that all this is also a standard flat space supersymmetric story although many authors do not provide a detailed discussion of various involutions in game. Their approach is simple and pragmatic, once a Lagrangian is worked out in components, an involution on fermions is set which makes the action real. In our case, we have to be more careful since an experience [1, 6, 9] teaches us that
only superfields as whole are deformable; in other words, the notion of the component fields may lose sense after the noncommutative deformation.

Once we have defined the $\sigma$-model (62) on the commutative $N = 2$ supersphere, it is natural to ask what is its algebra of supersymmetry. There is a huge formal supersymmetry algebra of the theory (62) known as the $N = 2$ super-de-Witt algebra (whose central extension is $N = 2$ Virasoro algebra [12]). It is actually defined as the Lie superalgebra of vector fields that preserve the constraints (63) [13] and, explicitly, it is generated by even chiral (anti-chiral) vector fields $L_k, J_k, k \in \mathbb{Z}$ ($\bar{L}_k, \bar{J}_k, k \in \mathbb{Z}$) and odd chiral (anti-chiral) ones $G^\pm_{k+\frac{1}{2}}, k \in \mathbb{Z}$ ($\bar{G}^\pm_{k+\frac{1}{2}}, k \in \mathbb{Z}$). They are given by

\begin{align*}
L_k &= z^{-k+1} \partial_z + \frac{1}{2}(-k+1)z^{-k}(b^1 \partial_{b^1} + b^2 \partial_{b^2}); \\
J_k &= z^{-k}(b^1 \partial_{b^1} - b^2 \partial_{b^2}); \\
G^+_{k+\frac{1}{2}} &= (z^{-k} \partial_{b^2} + k z^{-k-1} b^1 b^2 \partial_{b^2} - z^{-k} b^1 \partial_z); \\
G^-_{k+\frac{1}{2}} &= (z^{-k} \partial_{b^1} + k z^{-k-1} b^2 b^1 \partial_{b^1} - z^{-k} b^2 \partial_z).
\end{align*}

The barred generators are given by the same formulas with $\bar{z}, \bar{b}^1, \bar{b}^2$ replacing $z, b^1, b^2$. It turns out that only eight chiral generators $L_{\pm 1}, L_0, J_0, G^\pm_{\pm \frac{1}{2}}$ and eight anti-chiral ones $\bar{L}_{\pm 1} \bar{L}_0, \bar{G}^\pm_{\pm \frac{1}{2}}$ preserve the algebra $A_\infty$ of the superfields on $N = 2$ supersphere. Obviously, they form a Lie subalgebra of the full de Witt algebra. It therefore seems that the algebra of supersymmetry of the model on the sphere has sixteen complex dimensions. However, it is not so because we have to impose two further condition which the supersymmetry algebra has to fulfil:

1) Since we are interested in the noncommutative deformation of the $N = 2$ $\sigma$-model (62) we have to consider only those generators which act by means of the Poisson bracket (6). This means that they are the Hamiltonian vector fields and, in the noncommutative case, they will act via the commutators. This reduces the supersymmetry algebra to an eight dimensional Lie superalgebra $spl(2, 1)$. It is generated by four even generators $R_\pm, R_3, Z_3$ and four odd ones $C^4_\pm, C^{\delta}_\pm$. Explicitly,

\begin{align*}
C^4_+ &= G^-_+ + \bar{G}^-_{-\frac{1}{2}}, \\
C^4_- &= G^-_- - \bar{G}^+_{\frac{1}{2}}, \\
C^{\delta}_+ &= G^+_{\frac{1}{2}} + \bar{G}^-_{-\frac{1}{2}}, \\
C^{\delta}_- &= G^-_{-\frac{1}{2}} - \bar{G}^+_{\frac{1}{2}}.
\end{align*}
Needless to say, the Hamiltonians of these generators of \( spl(2,1) \) are \( r_\pm, r_3, z_3 \) and \( c_\pm^1, c_\pm^0 \) of section 2, Eqs (12),(13),(16) and (18). Clearly, \( spl(2,1) \) is the Lie subalgebra of \( \mathcal{T} \) hence its commutation relations are contained in (22) - (28).

2) We require that a supersymmetric transformation \( \delta \) realized on both superfields \( \Phi \) and \( \Phi^- \) respect the conjugacy of the fields, in other words,

\[
\overline{\delta \Phi} = \delta \Phi. \tag{72}
\]

This reduces the supersymmetry algebra of the commutative model (62) to a certain real form of the \( spl(2,1) \) algebra. Explicitly, the \( spl(2,1) \) supersymmetry transformation is given by

\[
\delta \Phi = (\epsilon^+ C_+^1 + \epsilon^- C_+^{t_0} + \rho^+ C_+^{t_0} + \rho^- C_+^{\bar{t}_0} + \beta Z_3 + \alpha^3 R_3 + \alpha^+ R_+ + \alpha^- R_-) \Phi \tag{73}
\]

and in the same way for \( \Phi^- \). The Grassmann parameters \( \epsilon^\pm, \rho^\pm \) have to fulfil

\[
\overline{\epsilon^-} = \epsilon^+, \quad \overline{\rho^-} = \rho^+; \tag{74}
\]

and the bosonic ones \( \beta, \alpha^3, \alpha^\pm \)

\[
\overline{\beta} = -\beta, \quad \overline{\alpha^3} = -\alpha^3, \quad \overline{\alpha^\pm} = -\alpha^\mp. \tag{75}
\]

These conditions correspond precisely to the choice of the real form of the \( spl(2,1) \) superalgebra.

We have to check that the constraints (65) are compatible with the \( spl(2,1) \) supersymmetry. The most simple way to see it is to note that

1) the quadruples \( c_\pm, z_-, c \) and \( c_\pm^0, z_+, c \) form \( spl(2,1) \) multiplets under the adjoint action in \( \mathcal{T} \) (xy);

2) \( \{c, \Phi \} \) and \( \{c, \Phi^- \} \) trivially vanish;

3) The constraints \( \{c_\pm, \Phi \} = 0 \) and \( \{c_\pm^0, \Phi \} = 0 \) imply \( \{Z_-, \Phi \} = 0 \) and \( \{Z_+, \Phi \} = 0 \), respectively. This is true because of the explicit formulae (61).

We conclude that the model (62) on the commutative sphere is \( spl(2,1) \) supersymmetric, because also the measure of the integral (8) is invariant with respect to the Hamiltonian vector fields. Indeed, one can straightforwardly check that

\[
I(\{t, f\}) = 0 \tag{76}
\]

for whatever \( t, f \in \mathcal{A}_\infty. \)
3.2 The noncommutative case

Here are the ingredients needed for defining the noncommutative deformation of the $N = 2$ supersymmetric $\sigma$-model (62):

1) The bar involution in the noncommutative case for it coïdentifies the supersymmetry algebra and ensures the reality of the Lagrangian $K(\bar{\Phi}\Phi)$. It turns out that in the *commutative* case the bar involution can be expressed in terms of the automorphism $\circ$. Explicitely,

$$\bar{c}_\pm = \mp c_\mp^\circ, \quad c_\mp^\circ = \pm c_\mp^\circ, \quad \bar{l}_\pm = l_\mp^\circ, \quad \bar{l}_3 = l_3^\circ, \quad \bar{c} = c.$$  \hspace{1cm} (77)

Since the automorphism $\circ$ continues to make sense on the $N = 2$ fuzzy supersphere, we can use the relations (77) as the definition of the barred quantities in the noncommutative case. Note, however, that the barred involution *is not* an automorphism of the algebra $A_n$ although its action on the generators $l_\pm, l_3, c_\pm, c_\mp^\circ$ is expressible in terms of the automorphism $\circ$. The point is that the bar involution acting on the product of two generators is not a morphism of the associative product in $A_n$ since it is defined by the first rule of (3), e.g.

$$c_+ c_+^\circ = -c_\mp^\circ \bar{c}_+ = -c_\mp^\circ c_\mp^\circ.$$  \hspace{1cm} (78)

The automorphism $\circ$, in turn, does respect the multiplication, e.g.

$$(c_+ c_-^\circ)^\circ = c_+^\circ c_-^\circ.$$  \hspace{1cm} (79)

We actually use first rule of (3) to define the barred involution of all elements of $A_n$ hence the second rule of (3) has to be verified. For example, we can calculate $(\bar{c}_+)$ by using the formula (48) and the first rule of (3). But one can use also (77) and the second rule of (3), i.e.

$$\bar{c}_+ = \bar{c}_- = -c_-.$$  \hspace{1cm} (80)

The consistency of the definition requires that both ways of calculating must be equivalent. Fortunately, this is the case and we have the bar involution also in the noncommutative case.

2) We need also the noncommutative analogue of the integral (8). It is a simple exercise to show that the commutative measure in the variables
\( \bar{z}, z, \bar{b}^\alpha, b^\alpha \) can be rewritten in the variables \( l_\pm, l_3, c_\pm, c_\pm^\dagger \) as follows

\[
d\bar{z}dzd\bar{b}^1db^1d\bar{b}^2db^2 = dl_+dl_-dl_3\delta(l_+l_- + l_3^2 - \frac{1}{4})dc_+dc_-dc_+^\dagger dc_-^\dagger.
\] (81)

Thus we see that the measure is simply the direct product of the round measure on the bosonic sphere and of the flat measure in the remaining fermionic variables.

Upon the Berezin quantization, the integral over the bosonic measure becomes \( \frac{1}{n+1} \text{Tr} \). The fermionic measure, in turn, becomes the supertrace \( \text{STr} \) (not the trace!) over the Clifford algebra \( Cf_4 \). Indeed, the generators \( c_\pm, c_\pm^\dagger \) of the Clifford algebra satisfy the canonical anticommutation relation of a quantum mechanical system with two fermionic degrees of freedom. The Clifford algebra can be identified with the algebra of linear operators acting on the corresponding Fock space. The latter is naturally graded so we obtain the supertrace as

\[
\text{STr}(.) \equiv \text{Tr}(\Gamma.),
\] (82)

where \( \Gamma \) is the grading operator. It is a textbook fact that, upon the quantization of the fermionic system, the Berezin integral becomes the supertrace. It is easy to see it directly in the case of one fermionic oscillator only. Then the only nonzero Berezin integral is the one over \( c_\dagger^\dagger c_\dagger c \); this is also true for the supertrace in the quantum case.

The integral (denote it \( I_n \)) in the noncommutative case has a crucial property

\[
I_n(AB - (-1)^{AB}BA) = 0
\] (83)

for any \( A, B \in A_n \). This property plays the same role in the noncommutative case as (76) in the commutative one. Namely it will ensure the supersymmetry of the following action

\[
S_n = I_n(K(\bar{\Phi}\Phi)).
\] (84)

This is the action of the \( N = 2 \) supersymmetric nonlinear \( \sigma \)-model on the fuzzy sphere. The superfields \( \bar{\Phi} \) and \( \Phi \) are now elements of the fuzzy algebra \( A_n \). More precisely, if we take any element of \( A_n \), it can be written as a polynomial in the generators \( l_\pm, l_3, c_\pm, c_\pm^\dagger \). Now the coefficients in front of odd polynomials of the superfields have to anticommute with those polynomials, for example one has

\[
\eta c_+l_+l_- = -c_+l_+l_-\eta.
\] (85)
The coefficients in front of the even polynomials commute with them:

\[ rl_-c_+c_- = l_-c_+c_-r. \]  

(86)

These rules are, of course, standard in the superworld. In the commutative case, \( \eta \)'s in (85) were the Grassmann numbers belonging to some Grassmann algebra \( P \). In the noncommutative case, however, they are the Grassman numbers tensored with the grading \( \Gamma \) of the linear space \( H_n \) where \( \mathcal{A}_n \) acts. The tensoring with \( \Gamma \) is the representation of \( P \) (due to \( \Gamma^2 = 1 \)) and it ensures the correct commutative limit of the superfields \( \bar{\Phi}, \Phi \). On the other hand, \( r \) in (86) can be interpreted as a complex multiple of the unit element of \( \mathcal{A}_n \).

An important thing is that \( \eta \)'s are odd and \( r \)'s even. Thus the superfields \( \bar{\Phi}, \Phi \) are even.

The constraints in the noncommutative case are defined by the formulae

\[ n[c_\pm, \Phi] = 0, \quad n[c_\pm^\circ, \bar{\Phi}] = 0. \]

(87)

Note that the only change with respect to the commutative case is the replacement of the Poisson brackets by the commutators scaled by the inverse Planck constant \( n \). Of course, the Hamiltonians \( c_\pm, c_\pm^\circ \) are elements of \( \mathcal{A}_n \).

The \( spl(2,1) \) supersymmetry transformation \( \delta \) is again generated by the noncommutative Hamiltonians \( r_\pm, r_3, z_3, c_\pm^\dagger, c_\pm^{\dagger\circ} \) given by (12),(13),(16) and (18). Explicitly

\[ \delta \Phi = n(\epsilon^+[c_\dagger^\circ, \Phi] + \epsilon^-[c_\dagger^\circ, \Phi] + \rho^+[c_\dagger^{\dagger\circ}, \Phi] + \rho^-[c_\dagger, \Phi] \\
+ \beta[z_3, \Phi] + \alpha^3[r_3, \Phi] + \alpha^+[r_+, \Phi] + \alpha^-[r_-, \Phi]). \]

(88)

Here \( \alpha^\pm, \alpha^3, \beta \) are numbers and the parameters \( \epsilon^\pm \) and \( \rho^\pm \) are the quantities of the type \( \eta \) in (85). This is important for ensuring that a commutator of two supertransformations with different coefficients is again a supertransformation. The parameters of \( \delta \) are again to fulfil the same relations (74) and (75) as their counterparts in the commutative case. With this assignment we can easily check that we have also in the noncommutative case

\[ \overline{\delta} \Phi = \delta \bar{\Phi}. \]

(89)

One can prove that the constraints (87) are compatible with the supersymmetry transformation in the same way as in the commutative setting. Summing up, we have constructed the \( N = 2 \) \( spl(2,1) \) supersymmetric non-linear \( \sigma \)-model on the noncommutative sphere. Two \( N = 1 \) \( osp(2,1) \) subsupersymmetries can be obtained by setting respectively \( \epsilon^\alpha = \rho^\alpha \) and \( \epsilon^\alpha = -\rho^\alpha \).
3.3 Solving the constraints

In the flat space commutative case, one can do more than just define the $\sigma$-model by the action (62) and the constraints (63). Indeed, one can effectively solve (63) and cast the action (62) in terms of the solutions of the constraints. Here we are going to show that all this can be performed also on the commutative $N = 2$ supersphere and even on the noncommutative one. Indeed, due to the Poisson brackets/commutation relations (34),(35)/(36),(37) we immediately conclude, that any element of $A_{\infty}$ or $A_n$ of the form

$$\Phi(l_{\pm}, l_3, c_{\pm})$$  \hspace{1cm} (90)

solves the first set of the constraints in (87) and any element of the form

$$\tilde{\Phi}(l_{\pm}^0, l_3^0, c_{\pm}^0)$$  \hspace{1cm} (91)

solves the second set. Let us moreover show that every solution of (87) is of the form (90). Indeed, it is a simple matter to check that every element $\Psi$ of $A_{\infty}$ or $A_n$ can be unambiguously written as

$$\Psi = \Phi(l_{\pm}, l_3, c_{\pm}) + \Phi_-(l_{\pm}, l_3, c_{\pm})c_{\pm}^1 + \Phi_+(l_{\pm}, l_3, c_{\pm})c_{\pm}^1 + \Phi_{+-}(l_{\pm}, l_3, c_{\pm})c_{\pm}^1 c_{\pm}^1.$$  \hspace{1cm} (92)

Now the fact that $\Phi(l_{\pm}, l_3, c_{\pm})$ is the most general solution of (87) is the direct consequence of the Poisson brackets/commutation relations (34)-(37). The same argument holds also for the circled variables.

Finally, we remark how we can cast in components the action on the commutative $N = 2$ supersphere. We use the fact that $l_{\pm}, l_3 \in A_{\infty}$ are the generators of the ordinary sphere (cf. (32) and (33)). We can therefore introduce variables $u, u^\dagger$ such that

$$l_+ = \frac{u^\dagger}{1 + u^\dagger u}, \quad l_- = \frac{u}{1 + u^\dagger u}, \quad l = \frac{u^\dagger u - 1}{u^\dagger u + 1}.$$  \hspace{1cm} (93)

Then

$$c_+ = \frac{u b^2 + b^1}{1 + u^\dagger u} \quad c_- = \frac{-b^2 + u^\dagger b^1}{1 + u^\dagger u}.$$  \hspace{1cm} (94)

Comparing with (20), we have

$$u = z + b^1 b^2, \quad u^\dagger = \bar{z} + \bar{b}^2 \bar{b}^1$$  \hspace{1cm} (95)
and we arrive at
\[ \Phi = \Phi(z + b^1b^2, \bar{z} + \bar{b}^2\bar{b}^1, b^1, \bar{b}^2). \] (96)

Much in the same way, we obtain
\[ \Phi = \Phi(z + b^2b^1, \bar{z} + \bar{b}^1\bar{b}^2, \bar{b}^1, b^2). \] (97)

Expanding (96) and (97) in \( \bar{b}^{\alpha}, b^{\alpha} \), inserting in the action (62) and integrating over \( b^{\alpha}, \bar{b}^{\alpha} \), we obtain the standard action of the \( N = 2 \) supersymmetric \( \sigma \)-model in components.

4 Outlook

For the purpose of the quantization of the model, say by a path integral, it is sufficient to work directly in the superfield formalism. Nevertheless, it is perhaps of interest to know whether one can introduce the component fields also in the noncommutative case. In the \( N = 1 \) case it turned out \[ ] that one could not do that for a simple reason that the \( N = 1 \) supersphere is not the \( N = 0 \) supersphere tensored with some algebra. In the \( N = 2 \) case the question is more subtle. One has to find variables "between" the circled and the uncircled ones, like \( z, \bar{z}, b^{\alpha}, \bar{b}^{\alpha} \) in the commutative case, such that the \( N = 2 \) supersphere would be the product of the bosonic fuzzy sphere and the Clifford algebra in these intermediate variables. This is needed for being able to take the supertrace over the Clifford algebra separately and cast the action as the trace over the bosonic fuzzy sphere only. It is an open problem; personally we feel that it is not possible.

Another natural question concerns coordinate transformations on the Kähler target. Although for some simple manifolds (like complex projective spaces) one can completely define the Kähler potential working in one chart, one should anyway look for a more invariant definition of the theory. Of course, this problem does not concern only the theories on the noncommutative worldsheets but arises in general in the studies of quantum theory of the nonlinear \( \sigma \)-models.

References
[1] A. Sevrin and J. Troost, *Nucl.Phys.* **B492** (1997) 623 and references therein

[2] A. Connes, *Noncommutative geometry*, Academic Press, London (1994)

[3] F. Berezin, *Commun. Math. Phys.* **40** (1975) 153

[4] J. Hoppe, MIT PhD thesis, 1982 and *Elem. Part. Res. J. (Kyoto)* **80** (1989) 145

[5] J. Madore, *J. Math. Phys.* **32** (1991) 332 and *Class. Quant. Grav.* **9** (1992) 69

[6] H. Grosse, C. Klimčík and P. Prešnajder, *Commun. Math. Phys.* **185** (1997) 155

[7] H. Grosse, C. Klimčík et P. Prešnajder, *Commun. Math. Phys.* **178** (1996) 507

[8] S. Baez, A.P. Balachandran, B. Idri and S. Vaidya, *Monopoles and solitons in fuzzy physics*, hep-th/9811169

[9] C. Klimčík, *Commun. Math. Phys.* **199** (1998) 257

[10] C. Klimčík, *A nonperturbative regularization of the supersymmetric Schwinger model*, hep-th/9903112

[11] M. Scheunert, W. Nahm and V. Rittenberg, *J. Math. Phys* **18** (1977) 155

[12] M. Green, J. Schwarz and E. Witten, *Superstring theory*, Cambridge University Press (1997)

[13] A.S. Schwarz, *Symplectic formalism in conformal field theory*, in *Quantum symmetries*, Les Houches, Session LXIV, Eds. A. Connes, K. Gawędzki and J. Zinn-Justin, Elsevier Science B.V. (1998) 957