Richardson extrapolation for the iterated Galerkin solution of Urysohn integral equations with Green’s kernels

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1. Introduction

Let $\mathcal{X} = L^\infty[0, 1]$ and consider the following Urysohn integral operator

$$\mathcal{K}(x)(s) = \int_0^1 \kappa(s, t, x(t)) \, dt, \quad s \in [0, 1], \quad x \in \mathcal{X},$$

where $\kappa(s, t, u)$ is a real-valued continuous function defined on $\Omega = [0, 1] \times [0, 1] \times \mathbb{R}$. Then $\mathcal{K}$ is a compact operator from $L^\infty[0, 1]$ to $C[0, 1]$. Consider the Urysohn integral equation

$$x(s) - \int_0^1 \kappa(s, t, x(t)) \, dt = f(s), \quad s \in [0, 1],$$

where $f \in \mathcal{X}$ is given and $x$ is the unknown to be determined. We assume that $\varphi$ is an isolated solution of the above equation and consider its numerical approximations.

We are interested in approximate solutions which converge to $\varphi$ uniformly. We consider some projection methods associated with a sequence of orthogonal projections converging to the identity operator point-wise.

In this paper, we consider the case when the kernel $\kappa$ of the integral operator $\mathcal{K}$ is of the type of Green’s function in its domain. We allow the partial derivatives of the kernel $\kappa$ to have jump discontinuities along the diagonal $s = t$. For $r \geq 1$, let $\mathcal{X}_n^r$ be a space of piecewise polynomials of degree $\leq r - 1$ with respect to a uniform partition of $[0, 1]$ with $n$ subintervals each of length $h = \frac{1}{n}$. Let $\pi_n$ be the restriction to $L^\infty[0, 1]$ of the orthogonal projection from $L^2[0, 1]$ onto $\mathcal{X}_n^r$. The Galerkin
method is a classical projection method for the approximate solution of an integral equation. In this method, (2) is approximated by

\[ x_G^n - \pi_n K(x_G^n) = \pi_n f. \]  

The above projection method has been studied extensively in the research literature (see [6–8] for details).

The iterated Galerkin solution is defined by

\[ x_S^n = K(x_G^n) + f. \]

Note that

\[ x_G^n = \pi_n x_S^n \]

and then the iterated Galerkin solution satisfies the following equation:

\[ x_S^n - K(\pi_n x_S^n) = f. \]  

From Atkinson–Potra [3], we quote the following orders of convergence:

If \( r = 1 \), then

\[ \|x_G^n - \varphi\|_\infty = O(h), \quad \|x_S^n - \varphi\|_\infty = O(h^2), \]  

whereas if \( r \geq 2 \), then

\[ \|x_G^n - \varphi\|_\infty = O(h^r), \quad \|x_S^n - \varphi\|_\infty = O(h^{r+2}). \]

Asymptotic error analysis and extrapolation methods are classical topics in numerical analysis. Richardson extrapolation is the popular one. In [5], the Hammerstein integral equation with Green’s function type of kernel is considered. The composite trapezoidal quadrature method is used to approximate the integral operator, and then an asymptotic error expansion is obtained for the approximate solution at the node points. Richardson extrapolation is applied to improve the orders of convergence. In [12], the authors have defined the Nyström operator based on the composite midpoint and the composite modified Simpson rules to approximate the integral operator of a Hammerstein integral equation with Green’s function-type kernel. Asymptotic expansions for the approximate solution at the node points as well as at the partition points are obtained and Richardson extrapolation is used to obtain approximate solutions with higher orders of convergence. The Hammerstein integral equation is a special case of the Urysohn integral equation. The case when the kernel of the Urysohn integral equation is sufficiently smooth, asymptotic error analyses are investigated for various projection methods in [10]. In the case of a linear integral equation of the second kind with a smooth kernel, an asymptotic series expansion for the iterated Galerkin solution is proved by McLean [14]. In the case of integral equation with Green’s kernel, i) asymptotic error analysis for the Nyström solution (associated with the midpoint rule and the modified Simpson’s rule) at the partition points, and ii) asymptotic expansion for the iterated collocation solution at the partition points, are treated in [11].

In [15], we considered a Fredholm integral equation with a kernel of the type of Green’s function and then performed asymptotic error analysis for the iterated Galerkin solution at the partition points. Richardson extrapolation is applied to obtain an approximate solution with a higher rate of convergence.

In this paper, we shall analyse asymptotic expansion for the iterated Galerkin solution of the Urysohn integral equation with Green’s function-type kernel. We will use Richardson extrapolation to improve the order of convergence.

The paper is organized as follows. In Section 2, notation is set and some preliminary results are proved for later use. In Section 3, asymptotic error analysis for the iterated Galerkin solution at the partition points is investigated. A numerical illustration is given in Section 4.
2. Preliminaries

In this section, we describe the Urysohn integral operator with Green’s function-type kernel, its Fréchet derivatives and related preliminary results. We introduce the following notations.

For an integer \( \alpha \geq 0 \), let \( C^\alpha[0,1] \) denotes the space of all real-valued \( \alpha \)-times continuously differentiable functions on \([0,1]\) with the following norm:

\[
\|x\|_{\alpha,\infty} = \max_{0 \leq j \leq \alpha} \|x^{(j)}\|_{\infty},
\]

where \( x^{(j)} \) is the \( j \)th derivative of the function \( x \), and

\[
\|x^{(j)}\|_{\infty} = \sup_{0 \leq t \leq 1} |x^{(j)}(t)|.
\]

Define

\[
\|\kappa\|_{\alpha,\infty} = \max_{0 \leq i+j+k \leq \alpha} \|D^{(i,j,k)}\kappa(s,t,u)\|_{\infty},
\]

where

\[
D^{(i,j,k)}\kappa(s,t,u) = \frac{\partial^{i+j+k}\kappa}{\partial s^i \partial t^j \partial u^k}(s,t,u).
\]

2.1. Properties of the kernel (Green’s function type)

Let \( r \geq 1 \) be an integer and assume that the kernel \( \kappa \) has the following properties.

(1) For \( i = 1, 2, 3, 4 \), the functions \( \kappa, \frac{\partial^i\kappa}{\partial u^i} \in C(\Omega) \), where \( C(\Omega) \) denotes the space of all real-valued continuous function on \( \Omega = [0,1] \times [0,1] \times \mathbb{R} \).

(2) Let

\[
\Omega_1 = \{(s,t,u) : 0 \leq t \leq s \leq 1, \ u \in \mathbb{R}\}, \quad \Omega_2 = \{(s,t,u) : 0 \leq s \leq t \leq 1, \ u \in \mathbb{R}\}.
\]

There are two functions \( \kappa_j \in C^r(\Omega_j), j = 1, 2 \), such that

\[
\kappa(s,t,u) = \begin{cases} \kappa_1(s,t,u), & (s,t,u) \in \Omega_1, \\ \kappa_2(s,t,u), & (s,t,u) \in \Omega_2. \end{cases}
\]

(3) Denote \( \ell(s,t,u) = \frac{\partial\kappa}{\partial u}(s,t,u) \) and \( q(s,t,u) = \frac{\partial^2\kappa}{\partial u^2}(s,t,u) \), for all \( (s,t,u) \in \Omega \). The partial derivatives of \( \ell(s,t,u) \) and \( q(s,t,u) \) with respect to \( s \) and \( t \) have jump discontinuities on \( s = t \).

(4) There are functions \( \ell_j, q_j \in C^r(\Omega_j), j = 1, 2 \), with

\[
\ell(s,t,u) = \begin{cases} \ell_1(s,t,u), & (s,t,u) \in \Omega_1, \\ \ell_2(s,t,u), & (s,t,u) \in \Omega_2, \end{cases} \quad q(s,t,u) = \begin{cases} q_1(s,t,u), & (s,t,u) \in \Omega_1, \\ q_2(s,t,u), & (s,t,u) \in \Omega_2. \end{cases}
\]

Following Atkinson–Potra [3], if the kernel \( \kappa \) satisfies the above conditions, then we say that \( \kappa \) is of class \( \mathcal{D}_q(r,0) \).
Under the above assumptions, the operator $K$ is four times Fréchet differentiable, and its Fréchet derivatives at $x \in \mathcal{X}$ are given by

$$K'(x)v_1(s) = \int_0^1 \frac{\partial K}{\partial u}(s, t, x(t)) \ v_1(t) \ dt,$$

$$K^{(i)}(x)(v_1, \ldots, v_i)(s) = \int_0^1 \frac{\partial^i K}{\partial u^i}(s, t, x(t)) \ v_1(t) \cdots v_i(t) \ dt, \quad i = 2, 3, 4,$$

where

$$\frac{\partial^i K}{\partial u^i}(s, t, x(t)) = \frac{\partial^i K}{\partial u^i}(s, t, u) \big|_{u=x(t)}, \quad i = 1, 2, 3, 4,$$

and $v_1, v_2, v_3, v_4 \in \mathcal{X}$. Note that $K'(x) : \mathcal{X} \to \mathcal{X}$ is linear and $K^{(i)}(x) : \mathcal{X}^i \to \mathcal{X}$ are multi-linear operators, where $\mathcal{X}^i$ is the Cartesian product of $i$ copies of $\mathcal{X}$ (see [16]). The norms of these operators are defined by

$$\left\| K^{(i)}(x) \right\| = \sup_{\|r\|_\infty \leq 1} \left\| K^{(i)}(x)(v_1, \ldots, v_i) \right\|_\infty, \quad i = 1, 2, 3, 4.$$

It follows that

$$\left\| K^{(i)}(x) \right\| \leq \sup_{0 \leq s, t \leq 1} \left| \frac{\partial^i K}{\partial u^i}(s, t, x(t)) \right|, \quad i = 1, 2, 3, 4.$$

We rewrite Equation (2) as

$$x - K(x) = f, \quad x \in \mathcal{X}. \quad (8)$$

Let

$$T(x) = K(x) + f, \quad x \in \mathcal{X}. \quad (9)$$

Assume that $\varphi$ is a fixed point of $T$. Since $K$ is compact, $K'(\varphi)$ is a compact linear operator (see [6]). Assume that 1 is not an eigenvalue of $K'(\varphi)$. Then, $\varphi$ is an isolated solution of Equation (8). Let $f \in C^\alpha[0, 1]$, then by Corollary 3.2 of [3], it follows that $\varphi \in C^\alpha[0, 1]$.

### 2.2. Approximating space and projection operator

Let $n \in \mathbb{N}$ and consider the following uniform partition of $[0, 1]$:

$$0 = t_0 < t_1 < \cdots < t_{n-1} < t_n = 1, \quad (10)$$

where

$$t_j = \frac{j}{n}, \quad j = 0, \ldots, n. \quad (11)$$

Let

$$\Delta_j = [t_{j-1}, t_j] \quad \text{and} \quad h = t_j - t_{j-1} = \frac{1}{n}, \quad j = 1, \ldots, n.$$

Consider a finite-dimensional approximating space as

$$\mathcal{X}_n = \left\{ x \in L^\infty[0, 1] : x|_{\Delta_j} \text{ is a polynomial of degree } \leq r - 1, \quad j = 1, 2, \ldots, n \right\}.$$

As no continuity conditions are imposed at the partition points, the dimension of $\mathcal{X}_n$ is $nr$ and $\mathcal{X}_n \subset L^\infty[0, 1]$.
Let \( \pi_n \) be the restriction to \( L^\infty[0,1] \) of the orthogonal projection from \( L^2[0,1] \) onto \( \mathcal{P}_n \), which converges to the identity operator pointwise. Then
\[
\sup_{n \geq 1} \| \pi_n \|_{L^\infty[0,1] \to L^\infty[0,1]} < \infty. \tag{12}
\]
If \( x \in C^\alpha[0,1] \), it is well known that
\[
\|(I - \pi_n)x\|_\infty \leq C_1 \|x^{(\beta)}\|_\infty h^\beta, \tag{13}
\]
where \( \beta = \min\{\alpha, r\} \) and \( C_1 \) is a constant independent of \( h \) (see [1,4]). Denote
\[
\pi_{n,j}x = \pi_nx|_{\Delta_j}, \quad j = 1, 2, \ldots, n.
\]
For \( x \in C^\alpha(\Delta_j) \), we have
\[
\|(I - \pi_{n,j})x\|_{\Delta_j, \infty} \leq C_2 \|x^{(\beta)}\|_{\Delta_j, \infty} h^\beta, \tag{14}
\]
where \( \beta = \min\{\alpha, r\} \) and \( C_2 \) is a constant independent of \( h \) (see [3, Corollary 4.3]).

### 2.3. Asymptotic expansions and the higher order terms

Let \( \varphi \in C^{2r+2}[0,1] \). For \( \delta > 0 \), let
\[
B(\varphi, \delta) = \{ x \in \mathcal{X} : \| x - \varphi \|_\infty \leq \delta \}
\]
denote the closed \( \delta \)-neighbourhood, where \( \varphi \) is the unique solution of (8). Without loss of generality, we assume that the Galerkin solution \( x^G_n \) and the iterated Galerkin solution \( x^S_n \) belong to the above neighbourhood.

Denote
\[
\ell_*(s,t) = \frac{\partial K}{\partial u}(s,t,\varphi(t)), \quad s, t \in [0,1],
\]
\[
q_*(s,t) = \frac{\partial^2 K}{\partial u^2}(s,t,\varphi(t)), \quad s, t \in [0,1].
\]
It follows that
\[
K'(\varphi)v(s) = \int_0^1 \ell_*(s,t) v(t) \, dt, \quad v \in \mathcal{X}, \quad s \in [0,1],
\]
\[
K''(\varphi)(v_1, v_2)(s) = \int_0^1 q_*(s,t) v_1(t) v_2(t) \, dt, \quad v_1, v_2 \in \mathcal{X}, \quad s \in [0,1],
\]
where the kernels \( \ell_*(\cdot, \cdot), \; q_*(\cdot, \cdot) \in C[0,1] \times C[0,1] \) are of the type of Green’s function as mentioned in Section 2.1. Then
\[
\| K'(\varphi) \| \leq \sup_{0 \leq t \leq 1} \int_0^1 |\ell_*(s,t)| \, ds,
\]
\[
\| K''(\varphi) \| \leq \sup_{0 \leq t \leq 1} \int_0^1 |q_*(s,t)| \, ds.
\]
(see [1]).

By assumption, \( I - K'(\varphi) \) is invertible. Let
\[
\mathcal{M} = (I - K'(\varphi))^{-1} K'(\varphi),
\]

\[
\| \mathcal{M} \| \leq \sup_{0 \leq t \leq 1} \int_0^1 |\ell_*(s,t)| \, ds
\]
\[
\| \mathcal{M} \| \leq \sup_{0 \leq t \leq 1} \int_0^1 |q_*(s,t)| \, ds
\]
\[ M_2 = (I - K'(\varphi))^{-1} K''(\varphi), \]
\[ M_3 = (I - K'(\varphi))^{-1} K^{(3)}(\varphi). \]

Then \( M, M_2 \) and \( M_3 \) are, respectively, compact linear, bi-linear and tri-linear integral operators (see [17]). For \( \nu \in X \), let
\[ M\nu(s) = \int_0^1 m(s, t) \nu(t) \, dt, \quad s \in [0, 1], \]

Note that the kernels of \( M, M_2 \) and \( M_3 \) inherit the same smoothness properties as the kernels of \( K'(\varphi), K''(\varphi) \) and \( K^{(3)}(\varphi) \), respectively (see [3, Lemma 5.1]). Hence, the kernels of the above three operators are of the type of Green’s function as mentioned in Section 2.1. Now we will mention some results on asymptotic expansion which will be used to prove our main result.

We quote the following result from [15].
\[ M\varphi(t_i) = M\pi_n\varphi(t_i) + (A_{2r}\varphi)(t_i) h^{2r} + O(h^{2r+2}), \quad i = 0, 1, \ldots, n, \quad (17) \]

where
\[ (A_{2r}\varphi)(t_i) = \bar{b}_{2r,2} \int_0^1 m(t_i, t) \psi^{(2r)}(t) \, dt + \sum_{p=1}^{2r-1} \bar{b}_{2r,p} \left\{ \left[ \left( \frac{\partial}{\partial t} \right)^{2r-p-1} m(t_i, t) \psi^{(p)}(t) \right]_{t=t_i-1}^{t=t_i+1} \right\} \]

with
\[ \bar{b}_{2r,p} = \int_0^1 \int_0^1 \Lambda_r(\sigma, \tau) \frac{(\sigma - \tau)^p B_{2r-p}(\tau)}{(2r-p)!} \, d\sigma \, d\tau, \]

\[ \Lambda_r(\sigma, \tau) = \sum_{q=0}^{r-1} e_q(\sigma) e_q(\tau), \quad \{e_0, e_1, e_2, \ldots\} \] is the sequence of orthonormal polynomials in \( L^2[0, 1] \) and \( B_k \) is the Bernoulli polynomial of degree \( k \geq 0 \).

As in [10, Lemma 2.4], for all \( s \in [0, 1] \), it can be shown that
\[ M_2(\pi_n \varphi - \varphi)^2(s) = \mathcal{V}_1(\varphi)(s) h^{2r} + O(h^{2r+2}) \quad (18) \]

and
\[ M_3(\pi_n \varphi - \varphi)^3(s) = \mathcal{V}_2(\varphi)(s) h^{3r} + O(h^{3r+1}), \quad (19) \]

where
\[ \mathcal{V}_1(\varphi) = \left( \int_0^1 [\chi_r(t)]^2 \, dt \right) M_2 \left( \psi^{(r)} \right)^2, \]
\[ \mathcal{V}_2(\varphi) = \left( \int_0^1 [\chi_r(t)]^3 \, dt \right) M_3 \left( \psi^{(r)} \right)^3 \quad \text{and} \quad \mathcal{V}_2(\varphi) = 0 \quad \text{for} \quad r = 1 \]

with
\[ \chi_r(t) = \int_0^1 \Lambda_r(\sigma, t) \frac{(\sigma - t)^r}{r!} \, d\sigma, \]

are independent of \( h \). In the proof of Lemma 2.4 in [10], the authors used Euler–McLaurin expansion for a smooth kernel. Since, the kernels of \( M_2 \) and \( M_3 \) are of the type of Green’s function, we use the
extended Euler–McLaurin summation formula from [11]. Note that
\[ \| M_3 (\pi_n \varphi - \varphi)^3 \|_\infty = O (h^4) \quad \text{for } r = 1. \] (20)

Let
\[
C_3 = \max \left\{ \sup_{0 \leq s \leq 1} \left| D^{(1,0)} q_1 (s, t, u) \right|, \sup_{0 \leq s \leq 1} \left| D^{(1,0)} q_2 (s, t, u) \right| \right\},
\]
\[
C_4 = \max \left\{ \sup_{0 \leq s \leq 1} \left| D^{(1,0)} \ell_n, 1 (s, t) \right|, \sup_{0 \leq s \leq 1} \left| D^{(1,0)} \ell_n, 2 (s, t) \right| \right\},
\]
where \( D^{(i,j)} \ell_n (s, t) = \frac{\partial^{i+j} \ell_n}{\partial p^i \partial t^j} (s, t), i, j \in \mathbb{N} \).

We prove the following preliminary result which is needed later on.

Let \( s \in [0, 1] \). By definition
\[
K'' (x) (v_1, v_2) (s) = \int_0^1 q (s, t, x (t)) v_1 (t) v_2 (t) \, dt
\]
\[
= \int_0^s q (s, t, x (t)) v_1 (t) v_2 (t) \, dt + \int_s^1 q (s, t, x (t)) v_1 (t) v_2 (t) \, dt.
\]

It follows that
\[
\left( K'' (x) (v_1, v_2) \right)' (s)
\]
\[
= \int_0^s \frac{\partial q_1}{\partial s} (s, t, x (t)) v_1 (t) v_2 (t) \, dt + q_1 (s, s, x (s)) v_1 (s) v_2 (s)
\]
\[
+ \int_s^1 \frac{\partial q_2}{\partial s} (s, t, x (t)) v_1 (t) v_2 (t) \, dt - q_2 (s, s, x (s)) v_1 (s) v_2 (s).
\]

Since \( q \) is continuous on \( \Omega \),
\[
\left( K'' (x) (v_1, v_2) \right)' (s)
\]
\[
= \int_0^s \frac{\partial q_1}{\partial s} (s, t, x (t)) v_1 (t) v_2 (t) \, dt + \int_s^1 \frac{\partial q_2}{\partial s} (s, t, x (t)) v_1 (t) v_2 (t) \, dt.
\]

Taking supremum over the set \( \{ s \in [0, 1] \} \), we obtain
\[ \| (K'' (x) (v_1, v_2))' \|_\infty \leq C_3 \| v_1 \|_\infty \| v_2 \|_\infty. \] (21)

Let \( x \in B (\varphi, \delta) \). Then by (13) and the above inequality (21), we obtain
\[ \| (I - \pi_n) (K'' (x) (v_1, v_2)) \|_\infty \leq C_1 C_3 \| v_1 \|_\infty \| v_2 \|_\infty h. \] (22)

Similarly, for any \( v \in \mathcal{X} \), it can be shown that the function \( K' (\varphi) v \) satisfies the following inequality:
\[ \| (K' (\varphi) v) \|_\infty \leq C_4 \| v \|_\infty. \] (23)

It follows that
\[ \| (I - \pi_n) (K' (\varphi) v) \|_\infty \leq C_1 C_4 \| v \|_\infty h. \] (24)
The following crucial estimate
\[
\| K'(\varphi)(I - \pi_n)\varphi \|_\infty = \begin{cases} O(h^2), & r = 1, \\ O(h^{r+2}), & r \geq 2, \end{cases}
\] (25)
follows from Lemma 9 of [4]. From Theorem 3.1 of [9], we have
\[
\| (I - \pi_n)K'(\varphi)(I - \pi_n)\varphi \|_\infty = O(h^{r+2}), \quad r \geq 1.
\] (26)
In order to prove our main result, we need to establish the following lemmas and propositions. Since \( \pi_n x_n^S = x_n^G \),
\[
x_n^G - \varphi = \pi_n(x_n^S - \varphi) - (I - \pi_n)\varphi.
\] (27)
We use the above relation between Galerkin and iterated Galerkin solution several times in the following lemmas and propositions.

**Lemma 2.1:** Let \( x_n^G \) be the Galerkin solution defined by Equation (3). Then for \( r \geq 1 \),
\[
(I - K'(\varphi))^{-1} K''(\varphi)(x_n^G - \varphi)^2(s) = (V_1(\varphi))(s) h^{2r} + O(h^{2r+2}), \quad s \in [0, 1],
\] (28)
where \( V_1 \) is defined by (18).

**Proof:** Using (27), we write
\[
K''(\varphi)(x_n^G - \varphi)^2 = K''(\varphi) \left( \pi_n(x_n^S - \varphi) - (I - \pi_n)\varphi \right)^2
\]
\[
= K''(\varphi) \left( \pi_n(x_n^S - \varphi) \right)^2
\]
\[
- 2K''(\varphi) \left( \pi_n(x_n^S - \varphi), (I - \pi_n)\varphi \right)
+ K''(\varphi) ((I - \pi_n)\varphi)^2.
\]
It follows that
\[
(I - K'(\varphi))^{-1} K''(\varphi)(x_n^G - \varphi)^2 = M_2 \left( \pi_n(x_n^S - \varphi) \right)^2
\]
\[
- 2(I - K'(\varphi))^{-1} K''(\varphi) \left( \pi_n(x_n^S - \varphi), (I - \pi_n)\varphi \right)
+ M_2 ((I - \pi_n)\varphi)^2.
\] (29)
Since \( M_2 = (I - K'(\varphi))^{-1} K''(\varphi) \) is bounded, from (5), (6), (12), (13) and (16), it is easy to see that
\[
\left\| M_2 \left( \pi_n(x_n^S - \varphi) \right)^2 \right\|_\infty = \begin{cases} O(h^4), & r = 1, \\ O(h^{2r+4}), & r \geq 2, \end{cases}
\] (30)
and
\[
\left\| K''(\varphi) \left( \pi_n(x_n^S - \varphi), (I - \pi_n)\varphi \right) \right\|_\infty = \begin{cases} O(h^3), & r = 1, \\ O(h^{2r+2}), & r \geq 2. \end{cases}
\]
When \( r = 1 \), that is, when \( S_n \) is the space of piecewise constant functions, the order of the term \( \| K''(\varphi)(\pi_n(x_n^S - \varphi), (I - \pi_n)\varphi) \|_\infty \) can be improved to \( h^4 \) in the following way. Note that
\[
K''(\varphi) \left( \pi_n(x_n^S - \varphi), (I - \pi_n)\varphi \right)(s) = \int_0^1 q(s, t) (\pi_n(x_n^S - \varphi))(t) (I - \pi_n)\varphi(t) \, dt
\]
\[
= \sum_{j=1}^{n} \int_{t_{j-1}}^{t_{j}} q_s(s, t) (\pi_{n,j}(x_n^S - \varphi))(t) (I - \tau_{n,j})\varphi(t) \, dt.
\]

Recall that the range of \( \pi_n \) is \( \mathcal{X}_n \) and \( \pi_{n,j}x = \pi_n x|_{\Delta_j}, \) \( j = 1, 2, \ldots, n. \) It follows that \( \pi_{n,j}(x_n^S - \varphi) \) is a constant on \([t_{j-1}, t_{j}]\). Then

\[
\mathcal{K}''(\varphi) \left( \pi_n(x_n^S - \varphi), (I - \pi_n)\varphi \right) (s) = \sum_{j=1}^{n} (\pi_{n,j}(x_n^S - \varphi)) \left( \frac{t_{j-1} + t_j}{2} \right) \int_{t_{j-1}}^{t_{j}} q_s(s, t) (I - \pi_{n,j})\varphi(t) \, dt.
\]

As in the proof of Lemma 9 of [4], it can be shown that

\[
\left| \int_{t_{j-1}}^{t_{j}} q_s(s, t) (I - \pi_{n,j})\varphi(t) \, dt \right| = O(h^3).
\]

Since \( \{\pi_{n,j}\} \) is uniformly bounded and \( \|x_n^S - \varphi\|_{\infty} = O(h^2), \) from the above two estimates it is easy to see that

\[
\|\mathcal{K}''(\varphi) \left( \pi_n(x_n^S - \varphi), (I - \pi_n)\varphi \right)\|_{\infty} = O(h^4), \quad \text{when } r = 1.
\]

Hence,

\[
\|\mathcal{K}''(\varphi) \left( \pi_n(x_n^S - \varphi), (I - \pi_n)\varphi \right)\|_{\infty} = \begin{cases} O(h^4), & r = 1, \\ O(h^{2r+2}), & r \geq 2. \end{cases} \tag{31}
\]

Hence, (28) follows from (18), (29), (30) and (31), the proof of the proposition is complete. \( \blacksquare \)

**Lemma 2.2:** Let \( s \in [0, 1] \) and \( r \geq 1. \) Then

\[
(I - \mathcal{K}'(\varphi))^{-1} \mathcal{K}^{(3)}(\varphi)(x_n^G - \varphi)^3(s) = \begin{cases} O(h^4), & r = 1, \\ \mathcal{V}_2(\varphi)(s) h^{3r} + O(h^{3r+1}), & r \geq 2, \end{cases} \tag{32}
\]

where \( \mathcal{V}_2 \) is defined by (19).

**Proof:** We write

\[
(I - \mathcal{K}'(\varphi))^{-1} \mathcal{K}^{(3)}(\varphi)(x_n^G - \varphi)^3 = \mathcal{M}_3 \left( \pi_n(x_n^S - \varphi) - (I - \pi_n)\varphi \right)^3
\]

\[
= \mathcal{M}_3 \left( \pi_n(x_n^S - \varphi) \right)^3
\]

\[
- 3 \mathcal{M}_3 \left( \pi_n(x_n^S - \varphi), \pi_n(x_n^S - \varphi), (I - \pi_n)\varphi \right)
\]

\[
+ 3 \mathcal{M}_3 \left( \pi_n(x_n^S - \varphi), (I - \pi_n)\varphi, (I - \pi_n)\varphi \right)
\]

\[
- \mathcal{M}_3 ((I - \pi_n)\varphi)^3. \tag{33}
\]

Since \( \mathcal{M}_3 = (I - \mathcal{K}'(\varphi))^{-1} \mathcal{K}^{(3)}(\varphi) \) is bounded, from (6), (12), (13) we have

\[
\|\mathcal{M}_3 \left( \pi_n(x_n^S - \varphi) \right)^3\|_{\infty} = \begin{cases} O(h^6), & r = 1, \\ O(h^{3r+6}), & r \geq 2, \end{cases} \tag{34}
\]
\[ \| M_3 \left( \pi_n(x_n^S - \varphi), \pi_n(x_n^S - \varphi), (I - \pi_n)\varphi \right) \|_\infty = \begin{cases} O(\epsilon^5), & r = 1, \\ O(\epsilon^{3r+4}), & r \geq 2, \end{cases} \]  
(35)

\[ \| M_3 \left( \pi_n(x_n^S - \varphi), (I - \pi_n)\varphi, (I - \pi_n)\varphi \right) \|_\infty = \begin{cases} O(\epsilon^4), & r = 1, \\ O(\epsilon^{3r+2}), & r \geq 2. \end{cases} \]  
(36)

On the other hand, from (19) we have

\[ M_3 ((I - \pi_n)\varphi)^3(s) = \begin{cases} O(\epsilon^4), \\ (V_2(\varphi))(s) \epsilon^{3r} + O(\epsilon^{3r+1}), \end{cases} \]  
(37)

Hence, (32) follows from (33), (34), (35), (36) and the above equation. The proof of the lemma is complete. \[ \blacksquare \]

3. The main result

Recall that the iterated Galerkin solution is defined by

\[ x_n^S - K(\pi_n x_n^S) = f \]

and the exact solution as \( \varphi - K(\varphi) = f \).

In this section, we prove our main result about the asymptotic series expansion for the iterated Galerkin solution \( x_n^S \) at the partition points \( t_i, i = 0, 1, \ldots, n \). That is, we will prove the following:

\[ \varphi(t_i) - x_n^S(t_i) = A_2r(t_i) \epsilon^{2r} + O(\epsilon^{2r+2}), \]  
(38)

where \( A_2r \) is a function independent of \( n \). Then, we can apply Richardson extrapolation to obtain a higher order approximation of \( \varphi \) at the partition points. From [5, Section 5], it can be shown that a continuous function can be reconstructed from the extrapolated discrete values at the partition points, and it approximates the exact solution \( \varphi \) to higher order in the uniform norm. We will not discuss this thing here. Our main aim is to prove (37).

Recall that

\[ M = (I - K'(\varphi))^{-1} K'(\varphi) \]

with

\[ (Mx)(s) = \int_0^1 m(s, t)x(t) \, dt, \quad s \in [0, 1], \quad x \in \mathcal{X}, \]  
(39)

where the kernel \( m \) is of the type of Green’s function.

We quote the following expression for the error in the iterated Galerkin solution from [2, equation (2.28)]:

\[ x_n^S - \varphi = (I - K'(\varphi))^{-1} \left\{ \left[ K(x_n^G) - K(\varphi) - K'(\varphi)(x_n^G - \varphi) \right] \right\} \]

\[ - M(I - \pi_n) \left[ K(x_n^G) - K(\varphi) - K'(\varphi)(x_n^G - \varphi) \right] \]

\[ - M(I - \pi_n) K'(\varphi)(x_n^G - \varphi) \]

\[ - M(I - \pi_n) \varphi. \]  
(40)

By the following propositions, we will prove that the second and third terms on the right-hand side of the above equation are of the order \( 2r + 2 \) or higher, and the first and the last term has an asymptotic expansion at the partition points.
Let
\[ C_5 = \max_{0 \leq i \leq 4} \left( \sup_{s, t \in [0,1]} \left| \frac{\partial^i \kappa}{\partial u^i} (s, t, \varphi) \right| \right). \]

Let us investigate the first term on the right-hand side of Equation (39) for an asymptotic expansion.

**Proposition 3.1:** Let \( x_n^G \) be the Galerkin solution defined by Equation (3) and \( s \in [0,1] \). Then for \( r \geq 1 \),
\[
(I - K'(\varphi))^{-1} \left[ \kappa(x_n^G) - \kappa(\varphi) - \kappa'(\varphi)(x_n^G - \varphi) \right] (s) = \mathcal{V}_1(\varphi)(s) h^{2r} + O(h^{2r+2}),
\]
where \( \mathcal{V}_1 \) is defined by (18).

**Proof:** Using the generalized Taylor series expansion (see [13]) in the neighbourhood \( B(\varphi, \delta) \), we obtain
\[
\kappa(x_n^G) - \kappa(\varphi) - \kappa'(\varphi)(x_n^G - \varphi) = \frac{1}{2} \kappa''(\varphi) (x_n^G - \varphi)^2 + \frac{1}{6} \kappa^{(3)}(\varphi) (x_n^G - \varphi)^3 + \mathcal{R}_4(x_n^G, \varphi),
\]
where
\[
\mathcal{R}_4(x_n^G, \varphi) = \frac{1}{6} \int_0^1 \kappa^{(4)} \left( \varphi + \theta(x_n^G - \varphi) \right) (x_n^G - \varphi)^4 (1 - \theta)^3 \, d\theta.
\]
By (7), we have
\[
\kappa^{(4)} \left( \varphi + \theta(x_n^G - \varphi) \right) (x_n^G - \varphi)^4 (s) = \int_0^1 \frac{\partial^4 \kappa}{\partial u^4} \left( s, t, \varphi(t) + \theta(x_n^G - \varphi)(t) \right) (x_n^G - \varphi)^4 (t) \, dt.
\]
Since \( x_n^G \in B(\varphi, \delta) \) and \( \theta \in (0,1) \),
\[
\varphi + \theta(x_n^G - \varphi) \in B(\varphi, \delta).
\]
It follows that
\[
\left\| \kappa^{(4)} \left( \varphi + \theta(x_n^G - \varphi) \right) (x_n^G - \varphi)^4 \right\|_\infty \leq C_5 \left\| x_n^G - \varphi \right\|_\infty^4.
\]
Using (6) and the above estimate, we obtain
\[
\mathcal{R}_4(x_n^G, \varphi) = O(h^{4r}). \tag{40}
\]

Note that
\[
(I - K'(\varphi))^{-1} \left[ \kappa(x_n^G) - \kappa(\varphi) - \kappa'(\varphi)(x_n^G - \varphi) \right]
= \frac{1}{2} \left( I - K'(\varphi) \right)^{-1} \kappa''(\varphi)(x_n^G - \varphi)^2 + \frac{1}{6} \left( I - K'(\varphi) \right)^{-1} \kappa^{(3)}(\varphi)(x_n^G - \varphi)^3
+ \left( I - K'(\varphi) \right)^{-1} \mathcal{R}_4(x_n^G, \varphi).
\]
Thus, by Lemma 2.1, Lemma 2.2, Equation (40) and the above estimate, we obtain the following.
For \( r = 1 \),
\[
(I - \mathcal{K}'(\varphi))^{-1} \left[ \mathcal{K}(x_n^G) - \mathcal{K}(\varphi) - \mathcal{K}'(\varphi)(x_n^G - \varphi) \right](s) = \mathcal{V}_1(\varphi)(s) h^2 + O(h^4),
\]
for \( r \geq 2 \),
\[
(I - \mathcal{K}'(\varphi))^{-1} \left[ \mathcal{K}(x_n^G) - \mathcal{K}(\varphi) - \mathcal{K}'(\varphi)(x_n^G - \varphi) \right](s)
= \mathcal{V}_1(\varphi)(s) h^{2r} + \mathcal{V}_2(\varphi)(s) h^{3r} + O(h^{2r+2}).
\]
Since \( 3r \geq 2r + 2 \) for \( r \geq 2 \),
\[
(I - \mathcal{K}'(\varphi))^{-1} \left[ \mathcal{K}(x_n^G) - \mathcal{K}(\varphi) - \mathcal{K}'(\varphi)(x_n^G - \varphi) \right](s)
= \mathcal{V}_1(\varphi)(s) h^{2r} + O(h^{2r+2}), \quad r \geq 2.
\]
This completes the proof. ■

Now we investigate the second term on the R.H.S. of Equation (39).

**Proposition 3.2:** Let \( \{ t_i : i = 0, 1, \ldots, n \} \) be the set of all partition points defined by (11). Then for \( r \geq 1 \),
\[
\mathcal{M}(I - \pi_n) \left[ \mathcal{K}(x_n^G) - \mathcal{K}(\varphi) - \mathcal{K}'(\varphi)(x_n^G - \varphi) \right](t_i) = O(h^{2r+2}).
\]

**Proof:** By the generalized Taylor theorem, we obtain
\[
\mathcal{K}(x_n^G) - \mathcal{K}(\varphi) - \mathcal{K}'(\varphi)(x_n^G - \varphi) = \int_0^1 \mathcal{K}'' \left( \varphi + \theta(x_n^G - \varphi) \right) (x_n^G - \varphi)^2 (1 - \theta) \, d\theta.
\]
Let
\[
\mathcal{R}_2(x_n^G, \varphi) = \int_0^1 \mathcal{K}'' \left( \varphi + \theta(x_n^G - \varphi) \right) (x_n^G - \varphi)^2 (1 - \theta) \, d\theta. \tag{41}
\]
Note that
\[
\mathcal{K}'' \left( \varphi + \theta(x_n^G - \varphi) \right) (x_n^G - \varphi)^2 \leq C_5 \| x_n^G - \varphi \|_\infty^2.
\]
It follows that
\[
\left\| \mathcal{K}'' \left( \varphi + \theta(x_n^G - \varphi) \right) (x_n^G - \varphi)^2 \right\|_\infty \leq C_5 \| x_n^G - \varphi \|_\infty^2.
\]
Since \( \| x_n^G - \varphi \|_\infty = O(h^r) \),
\[
\left\| \mathcal{K}'' \left( \varphi + \theta(x_n^G - \varphi) \right) (x_n^G - \varphi)^2 \right\|_\infty = O(h^{2r}). \tag{42}
\]
Let \( s \in [0, 1] \) be fixed and \( m_s(t) = m(s, t), \quad t \in [0, 1] \), then
\[
\mathcal{M}(I - \pi_n) \left[ \mathcal{K}(x_n^G) - \mathcal{K}(\varphi) - \mathcal{K}'(\varphi)(x_n^G - \varphi) \right](s)
\]
\[
=m_s, (I - \pi_n) \left[ \mathcal{K}(x_n^G) - \mathcal{K}(\varphi) - \mathcal{K}'(\varphi)(x_n^G - \varphi) \right],
\]
where \(\langle \cdot, \cdot \rangle\) is the usual inner product in \(L^2[0, 1]\), i.e.
\[
\langle x, y \rangle = \int_0^1 x(t) y(t) \, dt, \quad x, y \in L^2[0, 1].
\]
Since \(I - \pi_n\) is self-adjoint,
\[
\mathcal{M}(I - \pi_n) \left[ \mathcal{K}(x_n^G) - \mathcal{K}(\varphi) - \mathcal{K}'(\varphi)(x_n^G - \varphi) \right] (t_i) = \left\langle (I - \pi_n) m_{t_i}, (I - \pi_n) \left[ \mathcal{K}(x_n^G) - \mathcal{K}(\varphi) - \mathcal{K}'(\varphi)(x_n^G - \varphi) \right] \right\rangle.
\]
Note that, \(m_{t_i}\) is continuous on \([t_{j-1}, t_j]\) and \(r\) times continuously differentiable on \((t_{j-1}, t_j)\) for all \(j = 1, 2, \ldots, n\). Therefore, by (14)
\[
\| (I - \pi_{n,j}) m_{t_i} \|_{\Delta_j, \infty} = O(h^{r}).
\]
Using (12), (42), (43) and the above estimate, we obtain
\[
\mathcal{M}(I - \pi_n) \left[ \mathcal{K}(x_n^G) - \mathcal{K}(\varphi) - \mathcal{K}'(\varphi)(x_n^G - \varphi) \right] (t_i) = O(h^{3r}), \quad r \geq 2.
\]
Consider the case when \(r = 1\).
From (41), it is easy to see that
\[
\left( \mathcal{R}_2(x_n^G, \varphi) \right)' = \int_0^1 \left( K'' \left( \varphi + \theta(x_n^G - \varphi) \right) (x_n^G - \varphi)^2 \right)' (1 - \theta) \, d\theta.
\]
This implies
\[
\left\| \left( \mathcal{R}_2(x_n^G, \varphi) \right)' \right\|_{\infty} \leq \frac{1}{2} \left\| \left( K'' \left( \varphi + \theta(x_n^G - \varphi) \right) (x_n^G - \varphi)^2 \right)' \right\|_{\infty}, \quad 0 < \theta < 1.
\]
Since \(\varphi + \theta(x_n^G - \varphi) \in B(\varphi, \delta)\), by (21) we have
\[
\left\| \left( \mathcal{R}_2(x_n^G, \varphi) \right)' \right\|_{\infty} \leq C_3 \| x_n^G - \varphi \|^2.
\]
From (13), it follows that
\[
\left\| (I - \pi_n) \left[ \mathcal{K}(x_n^G) - \mathcal{K}(\varphi) - \mathcal{K}'(\varphi)(x_n^G - \varphi) \right] \right\|_{\infty} = \left\| (I - \pi_n) \mathcal{R}_2(x_n^G, \varphi) \right\|_{\infty} \leq C_1 C_3 \| x_n^G - \varphi \|^2 h.
\]
By (5), (43), (44) and the above estimate, we obtain
\[
\mathcal{M}(I - \pi_n) \left[ \mathcal{K}(x_n^G) - \mathcal{K}(\varphi) - \mathcal{K}'(\varphi)(x_n^G - \varphi) \right] (t_i) = O(h^4), \quad r = 1.
\]
Hence, the required result follows from (45) and (46).

Next, we investigate the third term on the R.H.S. of Equation (39).
Proposition 3.3: Let \( \{t_i : i = 0, 1, \ldots, n\} \) be the set of all partition points defined by (11). Then

\[
\mathcal{M}(I - \pi_n)K'(\varphi)(x_n^G - \varphi)(t_i) = O(h^{2r+2}), \quad \text{for } r \geq 1.
\]

Proof: From (38), we have

\[
\mathcal{M}(I - \pi_n)K'(\varphi)(x_n^G - \varphi)(t_i) = \int_0^1 m(t_i, t)(I - \pi_n)K'(\varphi)(x_n^G - \varphi)(t) \, dt
\]

\[
= \left\{ m_{t_i}, (I - \pi_n)K'(\varphi)(x_n^G - \varphi) \right\}
\]

\[
= \left\{ (I - \pi_n)m_{t_i}, (I - \pi_n)K'(\varphi)(x_n^G - \varphi) \right\}.
\] (47)

It is easy to see that \( m_{t_i} \) is continuous on \([t_{j-1}, t_j]\) and \( r \) times continuously differentiable on \((t_{j-1}, t_j)\) for all \( j = 1, 2, \ldots, n \). Therefore, by (14)

\[
\| (I - \pi_{n,j})m_{t_i} \|_{\Delta_j, \infty} = O(h^r).
\] (48)

Note that

\[
K'(\varphi)(x_n^G - \varphi) = K'(\varphi) \left( \pi_n(x_n^S - \varphi) \right) - K'(\varphi)(I - \pi_n)\varphi.
\] (49)

Thus, by (6), (12) and (25), we obtain

\[
\left\| K'(\varphi)(x_n^G - \varphi) \right\|_{\infty} = O(h^{r+2}), \quad r \geq 2.
\]

Hence, from (47), (48) and the above estimate, we obtain

\[
\mathcal{M}(I - \pi_n)K'(\varphi)(x_n^G - \varphi)(t_i) = O(h^{2r+2}), \quad r \geq 2.
\] (50)

Note that when \( r = 1 \), that is, when \( \mathcal{H}_n \) is the space of piecewise constant functions. It is easy to see from (5), (25) and (49) that

\[
\left\| K'(\varphi)(x_n^G - \varphi) \right\|_{\infty} = O(h^2),
\]

which is not equal to \( O(h^{2r+2}) \) with \( r = 1 \). We consider this case separately to improve the order.

Note that

\[
(I - \pi_n)K'(\varphi)(x_n^G - \varphi) = (I - \pi_n)K'(\varphi) \left( \pi_n(x_n^S - \varphi) \right) - (I - \pi_n)K'(\varphi)(I - \pi_n)\varphi.
\] (51)

From (24), we have

\[
\left\| (I - \pi_n)K'(\varphi) \left( \pi_n(x_n^S - \varphi) \right) \right\|_{\infty} \leq C_1 C_4 \| \pi_n \| \left\| x_n^S - \varphi \right\|_{\infty} h.
\]

From (5) and (12), it follows that

\[
\left\| (I - \pi_n)K'(\varphi) \left( \pi_n(x_n^S - \varphi) \right) \right\|_{\infty} = O(h^3).
\]

By (26), (47), (48), (51) and the above estimate, we obtain

\[
\mathcal{M}(I - \pi_n)K'(\varphi)(x_n^G - \varphi)(t_i) = O(h^4), \quad r = 1.
\] (52)

Hence, the required result follows from (50) and (52).
Now, we prove our main theorem.

**Theorem 3.4:** Let $f \in C^{2r+2}[0,1]$, and the kernel of Urysohn integral operator (1) be of class $G_4(r,0)$. Let $\phi$ be a fixed point of the operator $T$ defined by (9), with 1 not an eigenvalue of $K'(\phi)$. For $r \geq 1$, let $X_n$ be the space of piecewise polynomials of degree $\leq r-1$ with respect to partition (10) and $\pi_n$ be the orthogonal projection defined by (12)–(13). Let $x_n^S$ be the iterated Galerkin solution defined by (4). Then, for $i = 0, 1, \ldots, n$,

$$x_n^S(t_i) - \phi(t_i) = -\zeta_{2r}(t_i) h^{2r} + O(h^{2r+2}),$$

where $\zeta_{2r}$ is a function bounded by a constant independent of $h$.

**Proof:** From Equation (39)

$$x_n^S(t_i) - \phi(t_i) = (I - K'(\phi))^{-1} \left[ K(x_n^G) - K(\phi) - K'(\phi)(x_n^G - \phi) \right](t_i)$$

$$- \mathcal{M}(I - \pi_n) \left[ K(x_n^G) - K(\phi) - K'(\phi)(x_n^G - \phi) \right](t_i)$$

$$- \mathcal{M}(I - \pi_n)K'(\phi)(x_n^G - \phi)(t_i)$$

$$- \mathcal{M}(I - \pi_n)\phi(t_i).$$

Let $\zeta_{2r} = A_{2r} - Y_1(\phi)$. Hence, the proof of this theorem follows from Equation (17), Proposition 3.1 – 3.3. ■

We can now apply one step of Richardson extrapolation and obtain an approximation of $\phi$ of the order $h^{2r+2}$ at the partition points.

Define

$$x_n^{EX} = \frac{2^{2r}x_n^S - x_n^S}{2^{2r} - 1}.$$  

Then under the assumptions of Theorem 3.4, we have the following result:

$$x_n^{EX}(t_i) - \phi(t_i) = O(h^{2r+2}), \quad i = 0, 1, 2, \ldots, n. \quad (53)$$

4. **Numerical illustration**

For the sake of numerical illustration, we consider the following example of a non-linear Hammerstein integral equation from [12]. Consider

$$\phi(s) - \int_0^1 \kappa(s,t) [\psi(t,\phi(t))] \, dt = f(s), \quad 0 \leq s \leq 1, \quad (54)$$

where

$$\kappa(s,t) = \frac{1}{\gamma \sinh \gamma} \begin{cases} \sinh \gamma s \sinh \gamma(1-t), & 0 \leq t \leq s \leq 1, \\ \gamma(1-s) \sinh \gamma t, & 0 \leq s \leq t \leq 1, \end{cases}$$

with $\gamma = \sqrt{12}$, and

$$\psi(t,\phi(t)) = \gamma^2 \phi(t) - 2 (\phi(t))^3, \quad t \in [0,1].$$

We have $f(s) = \frac{1}{\sinh \gamma} \{ 2 \sinh \gamma (1-s) + \frac{2}{3} \sinh \gamma s \}$. The exact solution of (54) is given by $\phi(s) = \frac{2}{2s+1}, s \in [0,1]$. 
Let $\mathcal{X}_n$ be the space of piecewise constant polynomials with respect to uniform partition (10) of the interval $[0, 1]$ considered before. Let $\pi_n : L^\infty[0, 1] \rightarrow \mathcal{X}_n$ be the orthogonal projection defined by (12)–(13).

In this case, it is given by
\[
(\pi_n \varphi)(s) = \frac{1}{h} \int_{(i-1)h}^{ih} \varphi(t) \, dt, \quad s \in [(i-1)h, ih],
\]
where $h = \frac{1}{n}$. In the definition of the projection operator defined above, if we replace the integral by the right-hand rule, then $(\pi_n \varphi)(ih-) = \varphi(ih-)$. Let $x_n^0$ be the Sloan solution defined by (4). Then $x_n^G(ih) = \frac{\pi_n x_n^0(ih-) + \pi_n x_n^0(ih+)}{2}$ at the partition points is obtained by solving the approximate system of non-linear equations which gives the values of $\pi_n x_n^0(ih-)$ and $\pi_n x_n^0(ih+)$. The system is as follows:
\[
\alpha_j = h \sum_{l=1}^{n} \kappa(s_l, s_l) \left[ \begin{array}{c}
\gamma^2 \alpha_l - 2 \frac{\alpha_l^3}{h} \end{array} \right] + \frac{f(s_j)}{\sqrt{h}}, \quad j = 1, 2, \ldots, n,
\]
where $\alpha_l = \pi_n x_n^G(s_l)$ and $s_l = (l - \frac{1}{2})h$ for $l = 1, 2, \ldots, n$. The above system is obtained by replacing all the integrals by numerical integration formula.

We have used Picard’s iteration to solve the above system of non-linear equations.

Let $t_i = (i - 1)/20, i = 1, 2, \ldots, 21$ be the partition points with step size $h = \frac{1}{20}$. It is easy to see that
\[
E_1^n(t_i) = |\varphi(t_i) - x_n^S(t_i)| = O(h^2).
\]
We define
\[
x_n^{EX}(t_i) = \frac{4x_n^S(t_i) - x_n^S(t_i)}{3}.
\]
Then $E_2^n(t_i) = |\varphi(t_i) - x_n^{EX}(t_i)| = O(h^4)$. The orders of convergence are calculated using the formula:
\[
\alpha_1 = \frac{\log(E_1^n(t_i)/E_2^n(t_i))}{\log(2)},
\]

| $t_i$ | $E_1^n(t_i)$ : $n = 20$ | $E_1^n(t_i)$ : $n = 40$ | $E_1^n(t_i)$ : $n = 80$ | $\alpha_1$ | $\alpha_2$ |
|-------|----------------|----------------|----------------|---------|---------|
| 0.05  | $8.6 \times 10^{-3}$ | $2.15 \times 10^{-3}$ | $5.37 \times 10^{-4}$ | 2.00 | 2.00 |
| 0.1   | $7.56 \times 10^{-3}$ | $1.89 \times 10^{-3}$ | $4.72 \times 10^{-4}$ | 2.00 | 2.00 |
| 0.15  | $6.79 \times 10^{-3}$ | $1.7 \times 10^{-3}$ | $4.24 \times 10^{-4}$ | 2.00 | 2.00 |
| 0.2   | $6.22 \times 10^{-3}$ | $1.55 \times 10^{-3}$ | $3.89 \times 10^{-4}$ | 2.00 | 2.00 |
| 0.25  | $5.78 \times 10^{-3}$ | $1.44 \times 10^{-3}$ | $3.61 \times 10^{-4}$ | 2.00 | 2.00 |
| 0.3   | $5.45 \times 10^{-3}$ | $1.36 \times 10^{-3}$ | $3.4 \times 10^{-4}$ | 2.00 | 2.00 |
| 0.35  | $5.19 \times 10^{-3}$ | $1.3 \times 10^{-3}$ | $3.24 \times 10^{-4}$ | 2.00 | 2.00 |
| 0.4   | $4.98 \times 10^{-3}$ | $1.25 \times 10^{-3}$ | $3.11 \times 10^{-4}$ | 2.00 | 2.00 |
| 0.45  | $4.82 \times 10^{-3}$ | $1.2 \times 10^{-3}$ | $3.01 \times 10^{-4}$ | 2.00 | 2.00 |
| 0.5   | $4.68 \times 10^{-3}$ | $1.17 \times 10^{-3}$ | $2.92 \times 10^{-4}$ | 2.00 | 2.00 |
| 0.55  | $4.55 \times 10^{-3}$ | $1.14 \times 10^{-3}$ | $2.84 \times 10^{-4}$ | 2.00 | 2.00 |
| 0.6   | $4.44 \times 10^{-3}$ | $1.11 \times 10^{-3}$ | $2.77 \times 10^{-4}$ | 2.00 | 2.00 |
| 0.65  | $4.33 \times 10^{-3}$ | $1.08 \times 10^{-3}$ | $2.7 \times 10^{-4}$ | 2.00 | 2.00 |
| 0.7   | $4.22 \times 10^{-3}$ | $1.05 \times 10^{-3}$ | $2.64 \times 10^{-4}$ | 2.00 | 2.00 |
| 0.75  | $4.10 \times 10^{-3}$ | $1.02 \times 10^{-3}$ | $2.56 \times 10^{-4}$ | 2.00 | 2.00 |
| 0.8   | $3.98 \times 10^{-3}$ | $9.94 \times 10^{-4}$ | $2.48 \times 10^{-4}$ | 2.00 | 2.00 |
| 0.85  | $3.84 \times 10^{-3}$ | $9.6 \times 10^{-4}$ | $2.4 \times 10^{-4}$ | 2.00 | 2.00 |
| 0.9   | $3.69 \times 10^{-3}$ | $9.22 \times 10^{-4}$ | $2.3 \times 10^{-4}$ | 2.00 | 2.00 |
| 0.95  | $3.52 \times 10^{-3}$ | $8.8 \times 10^{-4}$ | $2.2 \times 10^{-4}$ | 2.00 | 2.00 |
Table 2. Error in the extrapolated solution.

| $t_i$ | $\mathcal{E}_2(t_i) : n = 20$ | $\mathcal{E}_5(t_i) : n = 40$ | $\beta$ |
|-------|-------------------------------|-------------------------------|---------|
| 0.05  | $2.98 \times 10^{-6}$         | $1.87 \times 10^{-7}$         | 3.99    |
| 0.1   | $2.23 \times 10^{-6}$         | $1.41 \times 10^{-7}$         | 3.99    |
| 0.15  | $1.59 \times 10^{-6}$         | $1.01 \times 10^{-7}$         | 3.99    |
| 0.2   | $1.09 \times 10^{-6}$         | $6.94 \times 10^{-8}$         | 3.97    |
| 0.25  | $7.13 \times 10^{-7}$         | $4.58 \times 10^{-8}$         | 3.96    |
| 0.3   | $4.46 \times 10^{-7}$         | $2.91 \times 10^{-8}$         | 3.94    |
| 0.35  | $2.7 \times 10^{-7}$          | $4.58 \times 10^{-8}$         | 3.91    |
| 0.4   | $1.69 \times 10^{-7}$         | $4.58 \times 10^{-8}$         | 3.86    |
| 0.45  | $1.3 \times 10^{-7}$          | $4.58 \times 10^{-8}$         | 3.83    |
| 0.5   | $1.41 \times 10^{-7}$         | $4.58 \times 10^{-8}$         | 3.85    |
| 0.55  | $1.91 \times 10^{-7}$         | $4.58 \times 10^{-8}$         | 3.89    |
| 0.6   | $2.72 \times 10^{-7}$         | $4.58 \times 10^{-8}$         | 3.93    |
| 0.65  | $3.75 \times 10^{-7}$         | $4.58 \times 10^{-8}$         | 3.95    |
| 0.7   | $4.95 \times 10^{-7}$         | $4.58 \times 10^{-8}$         | 3.97    |
| 0.75  | $6.26 \times 10^{-7}$         | $4.58 \times 10^{-8}$         | 3.98    |
| 0.8   | $7.6 \times 10^{-7}$          | $4.58 \times 10^{-8}$         | 3.99    |
| 0.85  | $8.94 \times 10^{-7}$         | $4.58 \times 10^{-8}$         | 3.99    |
| 0.9   | $1.02 \times 10^{-6}$         | $4.58 \times 10^{-8}$         | 3.99    |
| 0.95  | $1.14 \times 10^{-6}$         | $4.58 \times 10^{-8}$         | 4       |

Figure 1. Iterated Galerkin solution at the partition points.
\[ \beta = \frac{\log(E_2^n(t_i)/E_2^{2n}(t_i))}{\log(2)}, \quad n = 40 \]
\[ \alpha_2 = \frac{\log(E_1^n(t_i)/E_1^{2n}(t_i))}{\log(2)}, \quad n = 20. \]

We expect \( \alpha_1 = \alpha_2 = 2 \) and \( \beta = 4 \) (Tables 1 and 2).

This illustrates result (53). The graphical representation of the iterated Galerkin solution \( x_n^S \) and the extrapolated solution \( x_n^{EX} \) at the partition points are as follows.

Figure 1 represents the iterated Galerkin solution at the partition points and Figure 2 represents the corresponding extrapolated solution at the partition points.

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