ESTIMATES FOR SOLUTIONS OF KDV ON THE PHASE SPACE OF
PERIODIC DISTRIBUTIONS IN TERMS OF ACTION VARIABLES

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ABSTRACT. We consider the KdV equation on the Sobolev space of periodic distributions. We obtain estimates of the solution of the KdV in terms of action variables.

1. Introduction and main results

Consider the KdV equation
\[ \partial_t \psi = -\psi'''_{xxx} + 6\psi\psi'_{x} \] (1.1)
on the Sobolev space of zero-meanvalue 1-periodic distributions \( H_{-1} = \{ \psi = q' : q \in H \} \), where the real Hilbert space \( H = H_0 \) consists of zero-meanvalue functions \( q \in L^2(T), T = \mathbb{R}/\mathbb{Z} \).
The space \( H_{-1} \) is equipped with the norm \( \| \psi \|_{-1}^2 = \| q \|_2^2 = \int_0^1 q^2(x)dx \) for \( \psi = q' \in H_{-1} \). The initial value problem for KdV in the phase space of periodic distributions was solved by Kappeler and Topalov [KT], see also [B], [CT]. That problem in various Sobolev spaces was studied by many authors, see references in [B], [CT], [KT]. The action-angle variables for the periodic KdV are studied by Veselov-Novikov [VN], Kuksin [Ku], Kappeler-Pöschel [KP]. The action-angle variables for the case \( \psi \in H_{-1} \) were constructed by Kappeler-Möhr-Topalov [KMT] and were essentially used in [KT]. In Sobolev spaces estimates for the potential \( \psi \) and for the KdV–Hamiltonian in terms of the action variables were obtained by Korotyaev [K2].

We describe the motivation of the present paper. Introduce the real Hilbert spaces \( \ell^2_m, m \in \mathbb{R}, \) of sequences \( (f_n)_{n}^{\infty}, \) equipped with the norm \( \| f \|_m^2 = \sum_{n \geq 1} (2\pi n)^{2m} f_n^2. \) Recall that the KDV equation on \( H_{-1} \) admits action-angle variables \( A_n \geq 0, \phi_n \in [0, 2\pi), n \geq 1 \) such that (see [KT]):

1. for each \( q \in H_{-1} \) there exist actions \( A_n \geq 0 \) such that \( \sum_{n \geq 1} \frac{A_n}{n} < \infty \) and angles \( \phi_n \in [0, 2\pi), n \geq 1. \)
2. The mapping \( \Psi : H_{-1} \to \ell^2_{\frac{1}{2}} \oplus \ell^2_{\frac{1}{2}} \) given by \( q \to \Psi(q) = (|A_n|^{\frac{1}{2}} \cos \phi_n, |A_n|^{\frac{1}{2}} \sin \phi_n)_{n=1}^{\infty} \) is a real analytic isomorphism between \( H_{-1} \) and \( \ell^2_{\frac{1}{2}} \oplus \ell^2_{\frac{1}{2}}. \)
3. This mapping is symplectic.

For the case of periodic distributions no estimates for the potential \( q \) in terms of action variables \( A_n \) were known. Our main goal in this paper is to obtain them.

We recall some results on the action variables for the KdV. If \( \psi \in H_0, \) then the following identity hold true (see [MT], [K4]):
\[ \| \psi \|_2^2 = 4 \sum_{n \geq 1} (\pi n) A_n. \] (1.2)
Moreover, if $\psi' \in H$, then the Hamiltonian $\mathcal{H}(\psi) = \frac{1}{2} \int_0^1 (\psi'^2 + 2\psi^3) dx$ obey the following estimates:

$$8P_3 - 8P_1 P_{-1} \leq \mathcal{H}(\psi) \leq 8P_3,$$

(1.3)

where $P_j = \sum_{n \geq 1} (\pi n)^j A_n$, $j \in \mathbb{R}$, see [K6] and see [2,8] for definition of the actions $A_n$.

We formulate our main result.

**Theorem 1.1.** Let $\psi \in H_{-1}$ and let $P_{-1} = \sum_{n \geq 1} \frac{4}{\pi n}$. Then the following estimates hold true:

$$\|\psi\|_{-1}^2 \leq 3P_{-1}(1 + P_{-1}),$$

(1.4)

$$P_{-1} \leq \|\psi\|_{-1}^2 (1 + \|\psi\|_{-1})^\beta.$$

(1.5)

**Conjecture.** The estimate (1.3) is sharp. It means that an estimate $\|\psi\|_{-1} \leq CP_{-1}(1 + P_{-1})^\beta$ with some $C > 0$ and $\beta < 1$ is not correct.

We formulate a simple corollary, which follows directly from estimate (1.4), (1.5).

**Corollary 1.2.** Let $\psi(x, t)$ be a solution of (1.1) such that $\psi(\cdot, 0) \in H_{-1}$. Then for all time $t$ the following estimates hold true:

$$\|\psi(\cdot, t)\|_{-1} \leq 3\|\psi(\cdot, 0)\|_{-1}(1 + \|\psi(\cdot, 0)\|_{-1})^\beta,$$

(1.6)

$$\|\psi(\cdot, 0)\|_{-1} \leq 14\max\{\|\psi(\cdot, t)\|_{-1}, \|\psi(\cdot, t)\|_{-1}^\beta\}.$$

(1.7)

**Example.** We now discuss relation of estimates (1.6) with the inverse cascade of energy in the KdV equation. Let an initial condition $\psi(\cdot, 0) \in H_0$ satisfies

$$\|\psi(\cdot, 0)\|_{-1} = \varepsilon \in [0, 1/4], \quad \|\psi(\cdot, 0)\| = C = \text{const}.$$

Then for any $N \geq 1$ and every $t$ estimate (1.6) yields

$$\|\psi(\cdot, t)\|_{-1} \leq 6\varepsilon, \quad \|P_N \psi(\cdot, t)\| \leq 6(2\pi N)\varepsilon,$$

(1.9)

where $P_N f, f \in H_0$ is given by $P_N f = \sum_{|n| \leq N} e^{2\pi i n x} \int_0^1 f(s) e^{-2\pi i n s} ds$. Let in addition, $\delta = 6(2\pi N)\varepsilon$ be small enough. Then (1.9) gives

$$\|(I - P_N) \psi(\cdot, t)\|^2 \geq C^2 - \delta^2, \quad \|P_N \psi(\cdot, t)\| \leq \delta, \text{ any } t \geq 0.$$

(1.10)

Thus we deduce that in our case the inverse cascade of energy is impossible. It means that if the initial condition is such that $\|\psi(\cdot, 0)\| = 1$ and $\|P_N \psi(\cdot, 0)\| = 0$ for some $N_0 \gg N$, then $\|P_N \psi(\cdot, t)\| \leq \frac{6N}{N_0}$ will be small for all time $t$, since $\|\psi(\cdot, 0)\|_{-1} \leq \frac{1}{2\pi N}$. That is, if the energy of a solution was initially concentrated in high modes, then a substantial part of the energy cannot flow to low modes.

Note that the function $\psi(x, 0)$ with the property (1.8) may be a finite trigonometric polynomial.

2. PROOF OF THE MAIN THEOREM

Our main ingredients to study the KdV equation (similar to [KT]) are the spectral properties of the Schrödinger operator $T = -\frac{d^2}{dx^2} + \psi + q_0$, where $\psi \in H_{-1}$ is a 1-periodic distribution with zero mean-value and $q_0 \in \mathbb{R}$ is a constant. It is well known [K3] that the spectrum of $T$ is absolutely continuous and consists of intervals $\mathcal{S}_n = [\lambda_n^+, \lambda_n^-]$, where $\lambda_n^+ < \lambda_n^-$, $n \geq 1$. We take a constant $q_0$ such that $\lambda_0^+ = 0$. The intervals $\mathcal{S}_n$ and $\mathcal{S}_{n+1}$ are separated by the gap $\gamma_n = (\lambda_n^-, \lambda_n^+)$. If a gap degenerates, that is $\gamma_n = \emptyset$, then the corresponding segments $\mathcal{S}_n$ and $\mathcal{S}_{n+1}$ merge. The sequence $\lambda_0^+ < \lambda_1^- \leq \lambda_2^+ < \ldots$ is the spectrum of the equation.
\[-y'' + (\psi + q_0) y = \lambda y\] with the 2-periodic boundary conditions, i.e., \(y(x+2) = y(x), x \in \mathbb{R}\). If \(\lambda_n^- = \lambda_n^+\) for some \(n\), then this number \(\lambda_n^\pm\) is the double eigenvalue of this equation with the 2-periodic boundary conditions. The lowest eigenvalue \(\lambda_1^+\) is always simple and the corresponding eigenfunction is 1-periodic. The eigenfunctions, corresponding to the eigenvalue \(\lambda_n^\pm\), are 1-periodic, when \(n\) is even and are antiperiodic, i.e., \(y(x+1) = -y(x), x \in \mathbb{R}\), when \(n\) is odd.

We cannot introduce the standard fundamental solutions for the operator \(T\) since the perturbation \(\psi \in H_{-1}\) is very strong. But we can do this using another representation of \(T\) given by \(T = \mathcal{U} T_w \mathcal{U}^{-1}\). Here \(T_w\) is the self-adjoint periodic operator acting in \(L^2(\mathbb{R}, w^2(x)dx)\) and given by

\[
T_w f = -\frac{1}{w^2}(w^2 f')' = -f'' - 2pf', \quad w(x) = e^{\int_0^x p(s)ds}, \quad p \in H.
\]  

\(\mathcal{U}\) is the unitary transformation \(\mathcal{U} : L^2(\mathbb{R}, w^2dx) \to L^2(\mathbb{R}, dx)\), given by the multiplication by \(w\). Note that

\[
T = -\frac{d^2}{dx^2} + q' + q_0 \geq 0, \quad q' = p'(x) + p^2(x) - ||p||^2, \quad q_0 = ||p||^2 = \int_0^1 p^2(x)dx,
\]

where \(\psi = q'\) is a 1-periodic potential (distribution). Thus, if \(p' \in H\), then \(T_w\) corresponds to the Hill operator \(T\) with \(L^2\)-potential. The operator \(T_w\) is well studied, see [K5] and references therein. In fact the direct spectral problem for \(T_w\) is equivalent to that for \(T\) [K3].

The operator \(T_w\) has the standard fundamental solutions \(\varphi(x, \lambda), \vartheta(x, \lambda)\), which satisfy the equation \(-y'' - 2py' = \lambda y, \lambda \in \mathbb{C}\) and the conditions \(\varphi'(0, \lambda) = \vartheta(0, \lambda) = 1, \varphi(0, \lambda) = \vartheta'(0, \lambda) = 0\). Here and below we use the notation \(f' = \frac{d}{dx}f\). Introduce the Lyapunov function \(\Delta(\lambda) = \frac{1}{2}(\varphi'(1, \lambda) + \vartheta(1, \lambda))\). Note that \(\Delta(\lambda_n^\pm) = (-1)^n\), \(n \geq 1\), and that for each \(n \geq 1\) there exists a unique point \(\lambda_n \in [\lambda_n^-, \lambda_n^+]\) such that \(\Delta'(\lambda_n) = 0\).

Now we recall results, crucial for the present paper. For each \(\psi \in H_{-1}\) there exists a unique conformal mapping (the quasimomentum) \(k : \mathbb{Z} \to \mathcal{K}(h)\) with asymptotics \(k(z) = z + o(1)\) as \(|z| \to \infty\) (see Fig. 1 and 2) and such that (see [K3])

\[
\cos k(z) = \Delta(z), \quad z \in \mathbb{Z} = \mathbb{C} \setminus \cup \Sigma_n, \quad g_n = (z_n^-, z_n^+) = -g_n, \quad z_n^\pm = \sqrt{\lambda_n^\pm} \geq 0, \quad n \geq 1,
\]

\[
\mathcal{K}(h) = \mathbb{C} \setminus \cup \Gamma_n, \quad \Gamma_n = (\pi n - ih_n, \pi n + ih_n), \quad h_0 = 0, \quad h_n = h_{-n} \geq 0, \quad h_n \geq 0 \quad \text{is defined by the equation} \quad \cosh h_n = (-1)^n \Delta'(\lambda_n) \geq 1.
\]

Here \(g_0 = 0\) and \(\Gamma_n\) is the vertical cut, \(z_n = \sqrt{\lambda_n} \in [z_n^-, z_n^+]\), \(n \geq 1\), \(\Delta'(z_n^2) = 0\). Moreover, we have \((h_n)_{n=1}^{\infty} \in \ell^2\) iff \(\psi \in H_{-1}\) (and \((nh_n)_{n=1}^{\infty} \in \ell^2\) iff \(\psi \in H\)), see [K3], [K1].

Due to [MO1], the quantities \(v = \text{Im} k(z)\) and \(u = \text{Re} k(z), z \in \mathbb{Z}\), possess the following properties:

1) \(v(z) \geq \text{Im} z > 0\) and \(v(z) = -v(\overline{z})\) for all \(z \in \mathbb{C}_+ = \{\text{Im} z > 0\}\).
2) \(v(z) = 0\) for all \(z \in \sigma_n = [z_n^-, z_n^+] = -\sigma_n, n \geq 1\).
3) If some \(g_n \neq 0, n \in \mathbb{Z}\), then the function \(v(z+i0) > 0\) for all \(z \in g_n\) and \(v(z+i0)\) has a maximum at \(z_n \in g_n\) such that \(\Delta'(z_n^2) = 0\) and \(v'(z_n+i0) = h_n > 0, v'(z_n) = 0\), and

\[
v(z+i0) = -v(z-i0) > v_n(z) = \frac{1}{2} \frac{1}{|z-z_n^-|} (z-z_n^+)^2 > 0, \quad v''(z+i0) < 0,
\]

for all \(z \in g_n \neq 0\), see Fig. 3.
4) \(u' > 0\) on \(\mathbb{R} \setminus \cup \Sigma_n\) and \(u(z) = \pi n\) for all \(z \in g_n \neq 0, n \in \mathbb{Z}\).
5) The function \( k(z) \) maps a horizontal cut (a "gap") \( \Gamma_n \) onto a vertical cut \( \Gamma_n \) and a spectral band \( \sigma_n \) onto the segment \( [\pi(n-1), \pi n] \) for all \( \pm n \in \mathbb{N} \).

The heights \( h_n, \ n \geq 1 \) are so-called Marchenko-Ostrovski parameters \([MO1]\). In spirit, such result goes back to the classical Hilbert Theorem (for a finite number of cuts, see e.g. \([J]\)) in the conformal mapping theory. A similar theorem for the Hill operator is technically more complicated (there is an infinite number of cuts) and was proved by Marchenko-Ostrovski \([MO1]\) for the case \( \psi \in H \). For additional properties of the conformal mapping we also refer to our previous papers \([K1]-[K6]\). Note that the inverse problems for the operator \( H \) with \( \psi \in H_{-1} \) in terms of the Marchenko-Ostrovski parameters \( h_n, n \geq 1 \) and gap-lengths were solved by Korotyev in \([K3]\).

For the sake of the reader, we briefly recall the results existing in the literature about estimates. In the case \( h = (h_n)_1^\infty \) and \( \psi \in H \) Marchenko and Ostrovski \([MO1-2]\) obtained the estimates:

\[
\|\psi\|_1^2 \leq C(1 + \sup_{n \geq 1} h_n)\|h\|_1^1 \text{ and } \|h\|_1 \leq C\|\psi\|_1 \exp( C_1\|\psi\|_1) \text{ for some absolute constants } C, C_1.
\]

These estimates are not sharp since they used the Bernstein inequality. Using the harmonic measure argument Garnett and Trubowitz \([GT]\) obtained \( \|\gamma\| \leq (4 + \|h\|_1)\|h\|_1 \) for the case \( \psi \in H \) and \( \gamma = (|\gamma_n|)_{n \geq 1} \), where \( |\gamma_n| \) is a gap length. Using the conformal mapping theory, Korotyev \([K1,K6]\) obtained estimates of potentials (and the Hamiltonian of the KDV) in terms of gap lengths, actions variables, effective masses, the heights \( h = (h_n)_1^\infty \) for large class of potentials. In fact in order to get new estimates new results from the conformal mapping theory were obtained. Note that estimates simplify the proof for the inverse spectral problem, see \([KK],[K3]\). We recall only few results from these estimates:

\( I \). Let \( \psi \in H_{-1} \). Then the following estimates hold true (see \([K3]\)):

\[
\|\gamma\|_{-1} \leq \sqrt{2}\|\psi\|_{-1}(1 + \|\psi\|_{-1}), \quad \|\psi\|_{-1} \leq 8\pi\|\gamma\|_{-1}(1 + \|\gamma\|_{-1}),
\]

\[
\frac{\sqrt{\pi}}{\sqrt{8}}\|\psi\|_{-1} \leq \|h\|_0 \leq \pi\|\psi\|_{-1}(1 + \|\psi\|_{-1})^{1/2}.
\]

\( II \) If \( \psi \in H \), then the following estimates hold true (see \([K1]\)):

\[
\|\psi\| \leq 2\|\gamma\|_0(1 + \|\gamma\|_0^{1/5}), \quad \|\gamma\|_0 \leq 2\|\psi\|(1 + \|\psi\|^{1/3}).
\]

If \( \psi \in H_{-1} \), then the quasimomentum \( k(\cdot) \) has asymptotics

\[
k(z) = z - \frac{Q_0 + o(1)}{z} \quad \text{as } z \to +i\infty,
\]

where \( Q_0 = \frac{1}{\pi} \int \Re v(z + i0)dz \geq 0 \) and \( p \) (defined in \((2.2)) satisfy the identities from \([K5]\):

\[
Q_0 = \frac{1}{\pi} \int \Re v(z + i0)dz = \frac{1}{2\pi} \int \Re \int \left| z'(k) - 1 \right|^2dudv = \frac{\|p\|^2}{2}, \quad k = u + iv.
\]

Due to \([FM]\) we define the action \( A_n, n \geq 1 \) by

\[
A_n = \frac{(-1)^n+12}{\pi} \int_{\gamma_n} \frac{\lambda\Delta'(\lambda)d\lambda}{(\Delta^2(\lambda) - 1)^{1/2}} \geq 0.
\]

We rewrite \( A_n \) in the more convenient form. The differentiation of \( \Delta(z^2) = \cos k(z) \) gives \( k'(z) = -\frac{\Delta'(z^2)2z}{\sin k(z)} \), which together with \( \sin k(z) = \sqrt{1 - \Delta^2(z^2)} \) yield

\[
A_n = -\frac{1}{i\pi} \int_{\gamma_n} z^2 \frac{\Delta'(z^2)2z}{\sin k(z)}dz = \frac{1}{i\pi} \int_{\gamma_n} z^2 k'(z)dz = -\frac{2}{i\pi} \int_{\gamma_n} z k(z)dz = \frac{4}{\pi} \int \Re v(z + i0)dz \geq 0,
\]
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Figure 1. Domain $\mathcal{Z} = \mathbb{C} \setminus \cup g_n$, where $z = \sqrt{\lambda}$ and momentum gaps $g_n = (z_n^-, z_n^+)$

Figure 2. $k$-plane and cuts $\Gamma_n = (\pi n - ih_n, \pi n + ih_n), n \in \mathbb{Z}$

Figure 3. The graph of $v(z + i0), z \in g_n \cup \sigma_n \cup \sigma_{n+1}$ and $|h_n| = v(z_n + i0) > 0$

which gives

$$A_n = \frac{4}{\pi} \int_{g_n} zv(z + i0)dz \geq 0. \quad (2.9)$$

Below we need results about the Riccati mapping (see Theorem 1.2 in [K3]).

Theorem 2.1. The Riccati map $R : H \to H$ given by $p \to q = R(p), q' = p'(x) + p^2(x) - \|p\|^2$ is a real analytic isomorphism of $H$ onto itself. Moreover, the following estimates hold true:

$$\|q\| \leq \|p\|(1 + 2\|p\|), \quad (2.10)$$

$$\|p\| \leq \sqrt{2}\|q\|(1 + 2\|q\|). \quad (2.11)$$
In order to show (1.4), we need

**Lemma 2.2.** The following estimate holds true:

$$\|p\|^2 \leq \sum_{n \geq 1} \frac{A_n}{\pi n} = P_{-1}. \quad (2.12)$$

**Proof.** Using the following identity for $Q_0 = \frac{1}{\pi} \int_{\mathbb{R}} v(z+i0)dz$ (see Theorem 2.3 from [K4])

$$Q_0 = \frac{2}{\pi} \int_{g_+} zv \frac{dz}{u}, \quad (2.13)$$

we obtain

$$Q_0^2 \leq \frac{2}{\pi} \int_{g_+} vudz \frac{2}{z} \int_{g_+} \frac{zvdz}{u} = Q_0 \frac{2}{\pi} \int_{g_+} \frac{zvdz}{u}, \quad g_+ = \cup_{n \geq 1} g_n,$$

which together with the identity for $A_n$ (2.9) yields

$$Q_0 \leq \frac{2}{\pi} \int_{g_+} \frac{zvdz}{u} = \frac{1}{2} \sum_{n \geq 1} \frac{A_n}{\pi n},$$

since $u|_{g_n} = \pi n$ and the identity (2.7) gives (2.12).

We show the estimate (1.4). Using (2.10), (2.12) we obtain

$$\|\psi\|^2_{L^1} = \|q\|^2 \leq \|p\|^2 (1 + 2\|p\|)^2 \leq \|p\|^2 (1 + \|p\|^2) \leq 5P_{-1}(1 + P_{-1}),$$

which gives (1.4).

We show the estimate (1.5). The estimate $v|_{g_n} \leq h_n$ and the identity for $A_n$ (2.9) gives

$$A_n = \frac{4}{\pi} \int_{g_n} zv(z)dz \leq \frac{4h_n}{\pi} \int_{g_n} zdz = \frac{4h_n}{\pi} |\gamma_n|,$$

and then

$$P_{-1} = \sum_{n \geq 1} \frac{A_n}{\pi n} \leq \sum_{n \geq 1} \frac{4h_n}{\pi} \frac{|\gamma_n|}{\pi n} \leq \frac{4}{\pi} \|h\|_0 \|\gamma\|_{-1}. \quad (2.14)$$

Substituting estimates (2.3), (2.6) into (2.14) we obtain (1.5).

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