String Winding Modes From Charge Non-Conservation in Compact Chern-Simons Theory

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Abstract

In this letter we show how string winding modes can be constructed using topological membranes. We use the fact that monopole-instantons in compact topologically massive gauge theory lead to charge non-conservation inside the membrane which, in turn, enables us to construct vertex operators with different left and right momenta. The amount of charge non-conservation inside the membrane is interpreted as giving the momentum associated with the string winding mode and is shown to match precisely the full mass spectrum of compactified string theory.

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Space-time compactification is of central importance to string theory: it provides a mechanism for reducing higher-dimensional string theory down to four dimensions and is an alternative to the Chan-Paton method for introducing isospin. Compactification, therefore, is a key to performing meaningful four-dimensional string phenomenology. Space-time compactification also leads to the appearance of certain dualities between string theories at strong and weak coupling and therefore any new insight into the compactification mechanism may be helpful in finding a non-perturbative description of string theory.

In this letter we examine space-time compactification in topological membrane (TM) theory. In particular, we consider the simple example of string theory compactified on a circle, which can be described by a compact $U(1)$ topologically massive gauge theory (TMGT). The main difference between string theories with target-space containing the (non-compact) line $\mathbb{R}^1$ or (compact) circle $S^1$ is the existence of winding modes in the latter case, corresponding to the string wrapping around the circle (see, for example, [4, 5]). The major problem with introducing winding modes in TM theory is the fact that the corresponding worldsheet vertex operators must have different left and right momenta. As we shall discuss in detail later, worldsheet vertex operators are represented in TM theory by Wilson lines of charged particles propagating between the left and right boundaries (which represent the left and right string worldsheets) of the topological membrane. The charge along one of these Wilson lines is interpreted as giving the momentum of the corresponding vertex operator. If the vertex operator has different left and right momenta, then the charge along the Wilson line connecting left and right membrane boundaries must change accordingly. So, in order to construct string winding modes, we need some process which leads to charge non-conservation in compact TMGT. Fortunately, precisely this type of process was discussed by Lee [6], who considered the effect of monopole-instantons in compact Chern-Simons gauge theory. The presence of monopole-instantons in compact $U(1)$ TMGT was also discussed in [7] using a Hamiltonian approach, where it was also found that monopole-instantons induce a phase transition in the bulk matching precisely the BKT phase transition on the string worldsheet [8]. The purpose of this letter is to explicitly construct the string winding modes in TM theory using the fact that monopole-instantons lead to charge non-conservation in compact TMGT. We interpret the amount of charge non-conservation as giving the momentum of the string winding mode and show that the resultant spectrum matches precisely the mass spectrum for string theory compactified on a circle. We then generalize our argument to include the case of compactification on a $D$-dimensional torus.
We first recall how string scattering amplitudes can be expressed in terms of correlation functions of some 2d conformal field theory. For simplicity, we consider closed bosonic strings in the critical dimension which means that we may neglect 2d gravity. The scattering amplitude has the general form:

\[ A(p_1, \ldots, p_N) \sim \left\langle \prod_{i=1}^{N} V_i(p_i^\mu) \right\rangle, \]

where \( V_i(p_i^\mu) \) is the vertex operator corresponding to a particle of type \( i \) with momentum \( p_i^\mu \). Averaging is done using the path integral:

\[ \int D\mathcal{X}^\mu(\sigma) \exp \left\{ -\frac{1}{4\pi\alpha'} \int d^2\sigma \, \eta^{\alpha\beta} \partial_\alpha X^\mu \partial_\beta X^\mu \right\}, \]

where \( \alpha, \beta = 1,2 \) and \( \mu = 1, \ldots, 26 \). Compactification on the circle \( S^1 \) proceeds by identifying \( X_26 = X_26 + 2\pi R n \) where \( R \) is the radius of \( S^1 \) and \( n \) is the number of times the string winds around the circle. The compact part of the string action becomes

\[ S_{XY} = -\frac{R^2}{4\pi\alpha'} \int d^2\sigma (\partial_\alpha \theta)^2, \]

where \( \theta \in [0, 2\pi) \). The resulting mass spectrum, obtained from the on-shell condition \( 2(L_0 + \tilde{L}_0 - 2) = 0 \) (see [4] for details), is given by

\[ \alpha' M^2 = -4 + 2(N_R + N_L) + m^2 \frac{\alpha'}{R^2} + n^2 \frac{R^2}{\alpha'} \quad \text{with} \quad N_L - N_R = nm. \]

\( N_L \) and \( N_R \) are the numbers of left- and right-moving excitations and \( m \) is an integer describing the allowed momentum eigenvalues in the compact direction. The last two terms in the spectrum give the contributions of the compact momentum and winding energy to the 25-dimensional mass. It easy to see that there is a symmetry:

\[ R \leftrightarrow \alpha'/R \quad \text{and} \quad m \leftrightarrow n, \]

which leaves the spectrum (4) invariant. This is the famous \( T \)-duality of compactified string theory (for a review, see [5]).

Before proceeding to show how the compactified string spectrum arises in TM theory, we first recall how to obtain string scattering amplitudes (1) from topological membranes [2]. Using the normalizations of [10], it is known that the abelian TMGT

\[ S_{TMGT} = -\frac{1}{4e^2} \int_M F_{\mu\nu} F^{\mu\nu} + \frac{k}{8\pi} \int_M \epsilon^{\mu\nu\lambda} A_\mu \partial_\nu A_\lambda, \]
defined on a three-manifold \( \mathcal{M} \), induces the chiral string action

\[
S = \frac{k}{8\pi} \int_{\partial \mathcal{M}} \partial_z \theta \partial_{\bar{z}} \theta
\]  

(7)

where \( \theta \) is the pure gauge part of \( A_\mu \) on the boundary \( \partial \mathcal{M} \). When \( \mathcal{M} \) is the filled cylinder depicted in Figure 1, its two boundaries represent the left- and right-moving sectors of the string worldsheet. Worldsheet vertex operators can be constructed if we allow charged matter (either bosonic or fermionic) to propagate inside the topological membrane. The vertex operator

\[
V_q(z, \bar{z}) = V_q(z) V_q(\bar{z}) = e^{iq\theta(z)} e^{iq\theta(\bar{z})}
\]  

(8)

can be obtained from the bulk TMGT as the open Wilson line

\[
W_q[C] = \exp \left( iq \int_C A_\mu dx^\mu \right)
\]  

(9)

where the contour \( C \) connects left and right boundaries. Since the vector potential becomes pure gauge on the boundary, it is easy to see that the Wilson line (9) coincides on the left and right boundaries with the holomorphic and anti-holomorphic parts of the vertex operator (8), respectively. Moreover, the charge along the Wilson trajectory is to be interpreted as giving the momentum of the corresponding vertex operator. This suggests that the \( N \)-point function (1) should be related to the correlator of \( N \) Wilson lines in TMGT. For the simplest case, \( N = 2 \), the short-distance operator product expansion gives

\[
\langle V_q(z_1)V_{-q}(z_2) \rangle \sim (z_1 - z_2)^{-2\Delta},
\]  

(10)

where \( \Delta \) is the scaling dimension of the chiral vertex operator \( V_q(z) \). The corresponding three-dimensional picture is that of a charged particle-antiparticle pair propagating inside the membrane. The two-point function (10) should be related to the correlator of two Wilson lines in TMGT which, in the infrared (Chern-Simons) limit, is simply

\[
\langle W[C_1]W[C_2] \rangle = \exp \left\{ -4\pi i \left( q^2 / k \right) \gamma[C_1, C_2] \right\}
\]  

(11)

where \( \gamma[C_1, C_2] \) is the linking number of the curves \( C_1 \) and \( C_2 \). The relationship between the two- and three-dimensional correlation functions becomes clear if we adiabatically rotate \( V(z_1) \) around \( V(z_2) \). This induces a phase factor \( e^{-4\pi i \Delta} \) in (11). A similar phase factor appears in (11) due to the linking of the two Wilson lines (see Figure). Since

\[
\Delta = q^2 / k
\]  

(12)
Figure 1: String scattering amplitude as the correlator of Wilson lines in the bulk TMGT. Adiabatic rotation of the chiral vertex operators $V(z_1)$ and $V(z_2)$ corresponds to the linking of the two Wilson lines.

is the transmuted spin of the charged particle due to its interaction with the Chern-Simons gauge field, this establishes the equivalence between the anomalous spins in the two- and three-dimensional theories. Note that the spectrum of anomalous dimensions can also be obtained perturbatively from the TMGT $^{11}$. Vertex operators representing higher-spin states have the general form $^{4}$:

$$V_q(z, \bar{z}) \sim \prod_j \left(\partial_z^j \theta\right)^{m_j} e^{iq\theta(z)} \prod_k \left(\partial_{\bar{z}}^k \bar{\theta}\right)^{m_k} e^{iq\theta(\bar{z})},$$

(13)

where the number of left- and right-moving excitations are given by $N_L = \sum_j j m_j$ and $N_R = \sum_k k m_k$, respectively. Since the vector potential $A_\mu$ is pure gauge on the boundary, we can construct the pre-exponential (spin) factors in TM theory using the boundary value of $A_\mu$ and its derivatives. For details, we refer the reader to $^{2}$.

We now turn our attention to the topological membrane description of string compactification on $S^1$. Comparing the boundary action $^{7}$ with the compact part of the string action $^{3}$ shows that the compactification radius $R$ is related to the Chern-Simons coefficient $k$ by $k = 4R^2/\alpha'$. Moreover, if $\theta \in [0, 2\pi)$ then we must take the gauge group in $^{6}$ to be compact $U(1)$. As we shall see, there is a crucial difference between compact and non-compact TMGT which enables us to construct string winding modes in the for-
mer theory. The difference becomes apparent when we try to quantize the bulk theory. Quantizing in the $A_0 = 0$ gauge gives the equal-time commutation relations:

$$[E_i(x), E_j(y)] = -i\frac{k}{4\pi}\epsilon_{ij}\delta(x - y)$$
$$[E_i(x), B(y)] = -i\epsilon_{ij}\partial_j\delta(x - y)$$

(14)

where $E_i = \Pi_i - k/8\pi\epsilon_{ij}A^j$ and $B = \epsilon^{ij}\partial_iA_j$. The classical Gauss law is

$$\partial_iE^i + \frac{k}{4\pi}B = \rho,$$

(15)

where $\rho$ is the charge density of the charged matter. As usual, Gauss’ law generates time independent gauge transformations. The elements of the gauge group are the operators:

$$U = \exp\left\{-i\int d^2z \lambda(z)\left(\partial_iE^i + \frac{k}{4\pi}B - \rho\right)\right\},$$

(16)

with $UA_iU^{-1} = A_i + \partial_i\lambda$. The physical Hilbert space of the theory contains only those states which are invariant under the action of $U$, i.e. $U|\Psi\rangle = |\Psi\rangle$. If the gauge group is compact, however, we must be more careful. As described in [7] (see also [12]), we must also include in the gauge group operators of the form (16) where $\lambda(z)$ is now the angle in the complex plane. When $z$ is restricted to a simply connected region not containing the origin, $\lambda(z)$ is differentiable and the Cauchy-Riemann equations give

$$\partial_i\lambda(z) = -\epsilon_{ij}\partial_j\ln|z|.$$

(17)

So in the compact theory we must further restrict the Hilbert space to eigenstates of the operator

$$V(y) = \exp\left\{-i\int d^2z \left(E^i + \frac{k}{4\pi}\epsilon^{ij}A_j\right)\epsilon_{ik}\partial_k\ln|y - z| - \lambda(y - z)\rho\right\}.$$

(18)

Using the identity $\partial_k\partial_k\ln|z| = 2\pi\delta(z)$, it is straightforward to show that

$$[B(x), V^n(y)] = 2\pi n\delta(x - y) V^n(y),$$

(19)

and so $V^n(y)$ creates a pointlike magnetic vortex with flux $2\pi n$. Integrating the Gauss law (15) and noting that the electric field falls off exponentially (the photon in (6) is massive), shows that the operator $V^n(y)$ also creates electric charge

$$\Delta Q = nk/2.$$

(20)
This change of charge/flux is, however, unobservable far from the vortex because local observables such as the electric field fall off exponentially and the Aharonov-Bohm phase is unity. Therefore $V(y)$ is the operator for a monopole-instanton which interpolates between topologically inequivalent vacua.

Given that the compact theory contains monopole-instantons which carry quantized charge/flux, we ask what is spectrum of allowed charge in the compact TMGT? Since the charge along a Wilson trajectory specifies the momentum of the corresponding worldsheet vertex operator, finding the spectrum of charge is equivalent to finding the spectrum of allowed string momenta. To find the spectrum, we first write the gauge group generators in terms of the canonical momenta $\Pi^i = E^i + k/8\pi \epsilon^{ij} A_j$. Using the functional Schrödinger representation ($\Pi^i = i \delta/\delta A_i$) it is easy to see that physical states acquire a non-trivial phase under gauge transformations:

$$U \Psi[A_i] = \exp \left\{-i \int \lambda \left( \frac{k}{8\pi} B - \rho \right) \right\} \Psi[A_i + \partial_i \lambda]. \quad (21)$$

In the compact theory, $\lambda$ is defined modulo $2\pi$ and the phase factor in (21) must be invariant under $\lambda \to \lambda + 2\pi$. This implies that the charge $Q = \int d^2 z \rho$ must be quantized according to

$$Q = m + \frac{nk}{4}, \quad (22)$$

where $m$ is an integer. If $k = 0$ then we recover the more familiar charge quantization condition of compact electrodynamics. The extra term in (22) is due to the monopole-instanton background which, according to (19), carries magnetic flux $2\pi n$. Hence the spectrum of allowed charge (momenta) in the compact theory is

$$Q = m + nk/4. \quad (23)$$

To see how the presence of monopole-instantons in the compact TMGT generates the mass spectrum, consider the process illustrated in Figure 2. A particle of charge $Q = m + nk/4$ is inserted on the left boundary which then propagates through the bulk. The charged particle interacts with the monopole-instanton background which, according to (20), itself carries charge $nk/2$. The effect of the monopole-instanton is to change the charge of the particle, creating a new state of charge $Q = m - nk/4$, which propagates to the right boundary. Using (12), the scaling dimension of the corresponding worldsheet vertex operator is

$$\Delta + \bar{\Delta} = \frac{(m + nk/4)^2}{k} + \frac{(m - nk/4)^2}{k}. \quad (24)$$
Substituting $k = 4R^2/\alpha'$ we obtain

$$2(\Delta + \bar{\Delta}) = m^2 \frac{\alpha'}{R^2} + n^2 \frac{R^2}{\alpha'} ,$$

which matches precisely the last two terms of (14), i.e. the contribution of the compact momentum and winding energy to the 25-dimensional mass spectrum. The difference between the number of left- and right-moving excitations is given by

$$N_L - N_R = \Delta - \bar{\Delta} = nm ,$$

which also agrees with (14). Since $\Delta$ is the induced spin of the charged particle due to its interaction with the Chern-Simons gauge field, (26) implies that the angular momentum of the particle is not conserved as it propagates from left to right boundary. However, the total angular momentum in the membrane is conserved due to photon emission in the bulk, which can be seen as follows. Since $N_L \neq N_R$, then the vertex operator (13) corresponding to the string winding mode must have a different number of left and right pre-exponential (spin) factors. As discussed earlier, these spin factors are constructed in TM theory using the boundary value of $A_\mu$ and its derivatives. Hence, the three-dimensional interpretation of (26) is that there must be a different number of photon
operators on the left and right boundaries or, equivalently, that the charged particle must emit photons as it propagates between boundaries, thereby conserving the total angular momentum in the bulk. We hope to give the precise details of this process in a future paper, including the possibility that the emitted photons may themselves interact with the monopole-instanton background.

We now show how the above analysis generalizes to spacetime with higher compact dimension. Suppose the compactified space is a $D$ dimensional torus by identifying $X^I = X^I + 2\pi L^I$, where $I = 1 \ldots D$. The compact part of the string action now takes the form:

$$S = -\frac{1}{4\pi \alpha'} \int d^2\sigma \left( G_{IJ} \partial_\alpha \theta^I \partial_\alpha \theta^J + \epsilon^{\alpha\beta} B_{IJ} \partial_\alpha \theta^I \partial_\beta \theta^J \right),$$

(27)

where $G_{IJ}$ and $B_{IJ}$ are the graviton and antisymmetric tensor condensates. The allowed left- and right-moving momenta are (see, for example, Eq. (10) of [13])

$$P_I = m_I - G_{IJ} n^J/\alpha' - B_{IJ} n^J/\alpha',$$

$$\bar{P}_I = m_I + G_{IJ} n^J/\alpha' - B_{IJ} n^J/\alpha'.$$

(28)

The momentum associated with the winding mode is just the difference between the left and and right momenta, namely

$$\Delta P_I = 2G_{IJ} n^J/\alpha'.$$

(29)

The resultant mass spectrum is

$$\alpha'M^2 = -4 + 2(N_R + N_L) + \alpha' m_I G^{IJ} m_J + \frac{1}{\alpha'} n^I (G - BG^{-1} B)_{IJ} n^J + 2n^I B_{IK} G^{KJ} m_J,$$

(30)

which is invariant under

$$\frac{1}{\alpha'} (G \pm B)_{IJ} \rightarrow \alpha' (G \pm B)^{-1}_{IJ} = \alpha' (G \pm B)^{IJ}$$

(31)

with the simultaneous exchange of $m_I \leftrightarrow n^J$.

To obtain the topological membrane description of the above toroidal compactification, we consider the abelian TMGT with $D$ copies of $U(1)$:

$$S = -\frac{1}{4e^2} \int_M F_{\mu\nu}^I F^{\mu\nu}_I + \frac{K_{IJ}}{8\pi} \int_M \epsilon^{\mu\nu\lambda} A^I_\mu \partial_\nu A^J_\lambda,$$

(32)

where $K_{IJ}$ is some non-degenerate matrix (not necessarily symmetric). The induced chiral action on the boundary is

$$S = \frac{K_{IJ}}{8\pi} \int_{\partial M} \partial_+ \theta^I \partial_- \theta^J,$$

(33)
which matches the string action (27) if \( K_{IJ} = 4(G_{IJ} + B_{IJ})/\alpha' \). The scaling dimension (12) generalizes to

\[ \Delta = K^{IJ} Q_I Q_J, \tag{34} \]

where \( K^{IJ} = K^{-1}_{IJ} \). Quantization in the \( A^I_0 = 0 \) gauge proceeds analogously to the \( U(1) \) case except for the following modifications. The equal-time commutation relations are:

\[ [\Pi^i_I(x), A^j_J(y)] = -i \delta^{ij} \delta_{IJ} \delta(x - y) \tag{35} \]

where \( \Pi^i_I = E^i_I + (K_{JI}/8\pi) \epsilon^{ij} A^j_J \). The Gauss law (15) generalizes to

\[ \partial_i E^i_I + \tilde{G}^{IJ}_4 B^J = \rho_I, \tag{36} \]

where \( \tilde{G}^{IJ}_4 = 4G^{IJ}/\alpha' \) is the symmetric part of \( K^{IJ} \). The elements of the gauge group are

\[ U = \exp \left\{ -i \int \lambda^I \left( \partial_i E^i_I + \tilde{G}^{IJ}_4 B^J - \rho_I \right) \right\}, \tag{37} \]

where \( \lambda^I \) parameterizes each compact \( U(1) \) factor. Associated with each of these \( U(1) \) factors is the monopole-instanton operator

\[ V_I(y) = \exp \left\{ -i \int \delta^2 z \left( E^i_I + \tilde{G}^{IJ}_4/4\pi B^J - \rho_I \right) \epsilon_{ik} \partial_k \ln |y - z| - \lambda_I(y - z) \rho \right\}, \tag{38} \]

with

\[ [B_I(x), V^n_I(y)] = 2\pi n \delta_{IJ} \delta(x - y) V^n_I(y). \tag{39} \]

Hence the flux carried by each monopole-instanton is \( 2\pi n^J \). Integrating the Gauss law (36) shows that the monopole-instanton also carries charge

\[ \Delta Q_I = \tilde{G}^{IJ} n^J/2 = 2G^{IJ} n^J/\alpha'. \tag{40} \]

Just like the \( U(1) \) case, the charge non-conservation induced by the monopole-instanton matches the string winding momenta (29). However, now we encounter an apparent paradox: how does the bulk TMGT account for the extra terms in (28) involving the antisymmetric tensor \( B_{IJ} \)? The resolution of this paradox emerges if we write the gauge group generators in (37) in terms of the canonical momenta \( \Pi^i_I = E^i_I + (K_{JI}/8\pi) \epsilon^{ij} A^j_J \). Using the functional Schrödinger representation \( (\Pi_I^i = i \delta/\delta A^i_I) \), it is clear that the phase acquired by physical states under a gauge transformation

\[ U\Psi[A^i_I] = \exp \left\{ -i \int \lambda^I \left( K_{IJ}/8\pi B^J - \rho_I \right) \right\} \Psi[A^i_I + \partial_i \lambda^I]. \tag{41} \]
involves the full matrix $K_{IJ}$. Requiring that the phase factor in (41) be invariant under $\lambda^I \to \lambda^I + 2\pi$ gives the quantization condition

$$Q_I = m_I - \frac{K_{IJ}}{4} n^J, \quad (42)$$

where we have used the fact that the monopole-instanton associated with each compact $U(1)$ factor carries magnetic flux $2\pi n^J$. We are now in a position to show how the spectrum of left and right momenta (28) arises from the following process in the bulk. On the left boundary we insert the charge

$$Q_I = m_I - \frac{\tilde{G}_{IJ}}{4} n^J - \frac{\tilde{B}_{IJ}}{4} n^J, \quad (43)$$

where $\tilde{B}_{IJ} = 4B_{IJ}/\alpha'$ is the antisymmetric part of $K_{IJ}$. The charged matter then propagates through the bulk where, according to (40), interaction with the monopole-instanton background changes the charge by $\tilde{G}_{IJ} n^J/2$. This creates a new state with charge

$$\tilde{Q}_I = m_I + \frac{\tilde{G}_{IJ}}{4} n^J - \frac{\tilde{B}_{IJ}}{4} n^J \quad (44)$$

which propagates to the right boundary. Substituting $K_{IJ} = 4(G_{IJ} + B_{IJ})/\alpha'$ gives

$$Q_I = m_I - G_{IJ} n^J/\alpha' - B_{IJ} n^J/\alpha', \quad \tilde{Q}_I = m_I + G_{IJ} n^J/\alpha' - B_{IJ} n^J/\alpha', \quad (45)$$

which matches precisely the spectrum of left and right momenta (28). As a final check, the scaling dimension of the corresponding worldsheet vertex operator is

$$2(\Delta + \bar{\Delta}) = 2K^{IJ}(Q_I Q_J + \bar{Q}_I \bar{Q}_J)$$

$$= \alpha' m_I G^{IJ} m_J + \frac{1}{\alpha'} n^I(G - BG^{-1}B)_{IJ} n^J + 2n^I B_{IK} G^{KJ} m_J, \quad (46)$$

which indeed matches the last three terms of (30), i.e. the contribution of the compact momenta and winding energy to the $(26 - D)$-dimensional mass.

It is straightforward to show that the compactified spectrum (46) exhibits the $T$-duality (31). The question then arises: what is the meaning of this $T$-duality from the three-dimensional point of view? This question was addressed in [14] where it was shown that the three-dimensional analogue of $R \to \alpha'/R$ duality (and more generally $K_{IJ} \to K_{IJ}^{-1}$ duality) is the duality between large and small scales in the corresponding TMGT with the spontaneous breaking of gauge symmetry. This duality was shown to be a consequence
of the equivalence between TMGT and Chern-Simons gauge theory with a Proca mass term. In [14] however, it was not at all clear how to describe the string winding modes in three-dimensional terms. Now that we have shown how string winding modes correspond to monopole-instantons in the bulk TMGT, and motivated by the fact that T-duality relates winding modes in one string theory to momentum modes in the dual string theory, we suggest that the corresponding bulk duality should relate monopole-instantons (i.e. a non-perturbative bulk effect) in the original TMGT to a charge conserving (perturbative) process in the dual gauge theory. We hope to study the precise nature of this duality in future work.

We should point out that the three-dimensional analogue of $R \rightarrow \alpha'/R$ duality has also been discussed [15] in the context of the quantum Hall effect which, itself, can be described in terms of a Chern-Simons gauge theory. Moreover, it is interesting to note that the picture of charge transport between membrane boundaries presented in this letter also has a close counterpart in the quantum Hall effect. Indeed, tunneling between edge states has proved useful in classifying the internal topological orders in the quantum Hall effect [16] and has been supported by many experiments. Given that it remains an open question whether the quantum Hall effect is described by compact or non-compact Chern-Simons theory, and motivated by the connection with string winding modes described in this letter, it would be extremely interesting to search for some signal of charge non-conservation in a quantum Hall tunneling experiment.

In closing, we would like to mention a surprising (and possibly important) prediction that TM theory makes in relation to heterotic string theory. As described in [10], the different left and right sectors of the heterotic string can be obtained in TM theory by imposing certain boundary conditions on the bulk gauge fields. In particular, on one of the membrane boundaries we must fix both $A_z = A_{\bar{z}} = 0$ (N boundary conditions in the notation of [10]) which, in turn forces the magnetic field $B = 0$ on that boundary. The Gauss law (15) then implies that there can be no charged particles on that particular boundary — and this is a problem if we want to construct vertex operators from Wilson lines of charged particles propagating between left and right membrane boundaries. Clearly the problem is solved if the theory is compact, whereby a monopole-instanton transition can change the charge to zero along a Wilson trajectory. However, in the non-compact TMGT there is no known physical mechanism for changing the charge in the bulk and so this suggests that each $U(1)$ factor must be compact. Since each $U(1)$ factor corresponds to a particular space-time co-ordinate of the induced string theory, we must therefore conclude that if heterotic strings are described by topological membranes then
each and every space-time dimension must be compact! We emphasize that the radii of the four Minkowski space-time dimensions may still be large (this is a question of dynamics) but, nevertheless, a consistent TM theory of heterotic strings requires that they must be compact.

To conclude, we have shown how to construct string winding modes from a process of charge non-conservation in TM theory. Winding modes are the simplest example of solitonic states in string theory and we have shown that they arise in TM theory as a result of a non-perturbative effect (monopole-instantons) in the membrane bulk. We speculate that other (if not all) soliton states in string theory can be described in TM theory as some non-perturbative bulk process.

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