Semiclassical Asymptotics on Manifolds with Boundary

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Abstract. We discuss semiclassical asymptotics for the eigenvalues of the Witten Laplacian for compact manifolds with boundary in the presence of a general Riemannian metric. To this end, we modify and use the variational method suggested by Kordyukov, Mathai and Shubin (2005), with a more extended use of quadratic forms instead of the operators. We also utilize some important ideas and technical elements from Helffer and Nier (2006), who were the first to supply a complete proof of the full semi-classical asymptotic expansions for the eigenvalues with fixed numbers.

1. Introduction

A. In his famous paper [37], E. Witten introduced a deformation of the de Rham complex of differential forms on a compact closed manifold $M$. It is a new (“deformed”) complex which depends upon a given Morse function $f$ on $M$ and contains a small parameter $h > 0$ (“Planck’s constant”). The deformed differential is given by the formula

$$d_{h,f}\omega = he^{-f/h}d(e^{f/h}\omega) = hd\omega + df \wedge \omega,$$

where $\omega$ is an exterior differential form on $M$, $d_{h,f}^2 = 0$. Choosing a Riemannian metric $g$ on $M$, we can take the corresponding normalized deformed Laplacian

$$\Delta_{h,f,g}\omega = h^{-1}(d_{h,f}^*d_{h,f} + d_{h,f}d_{h,f}^*) = h\Delta\omega + (L_{\nabla f} + L_{\nabla f}^\ast)\omega + h^{-1}|\nabla f|^2\omega,$$

where $L_{\nabla f}$ is the Lie derivative along $\nabla f$, the adjoint operators $d_{h,f}^*$, $L_{\nabla f}^\ast$ (to $d_{h,f}$, $L_{\nabla f}$ respectively) are taken with respect to the scalar products defined by the metric $g$ (and by the corresponding smooth measure on $M$) on the exterior forms on $M$; $\Delta = d^*d + dd^*$ is the usual Laplacian on forms. The deformed Laplacian $\Delta_{h,f,g}$ is often called the Witten Laplacian.

Multiplication by $\sqrt{h}e^{f/h}$ defines an isomorphism between the deformed complex with the differential $d_{h,f}$ and the standard de Rham complex (with the differential $d$). In particular, the cohomology spaces of these complexes are isomorphic.

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By the Hodge theory, these cohomology spaces are naturally isomorphic to the corresponding spaces of harmonic forms, i.e. the kernels (null-spaces) of the Laplacians. It follows that

$$\dim \text{Ker} \Delta_{h,f,g}^{(p)} = \dim \text{Ker} \Delta^{(p)} = b_p(M),$$

where $\Delta_{h,f,g}^{(p)}$, $\Delta^{(p)}$ denote the restrictions of the corresponding Laplacians to $p$-forms, $b_p(M)$ is the $p$th real Betti number of $M$.

An important feature of the Laplacian $\Delta_{h,f,g}$ is as follows: for small $h$ the eigenforms corresponding to the bounded eigenvalues, are small outside a small neighborhood of the critical points of $f$ because the potential $V = |\nabla f|^2$ does not vanish there, and if eigenforms do not localize around the critical points, then the term $h^{-1}|\nabla f|^2$ will be larger than the sum of all other terms in the equation for the eigenfunction, provided $h$ is sufficiently small. Therefore, we can expect that only small neighborhoods of the critical points play a role in semiclassical asymptotics of the eigenvalues; in particular, we can expect that only “principal parts” of $f$ and $g$ are relevant. For example, we can hope that only quadratic parts of $f$ and constant (flat) metrics at every critical point contribute to the principal term in the semiclassical asymptotics of the eigenvalues (i.e. asymptotics as $h \to 0$).

Based on this idea, Witten gave an analytic proof of the Morse inequalities on compact smooth manifolds without boundary. In their simplest form (see [24]), these inequalities state that the number $m_p$ of the critical points with index $p$ of a Morse function $f$ can not be less than the Betti number $b_p$ of the underlying manifold: $m_p \geq b_p$, for all $p$. Semiclassical asymptotics of the eigenvalues relate these two numbers by including both of them into one mathematical object: the Witten deformation of the de Rham complex, where $m_p$ becomes the number of small eigenvalues (multiplicity counted) of $\Delta_{h,f,g}^{(p)}$ and $b_p$ is the multiplicity of the 0 as the eigenvalue of the same deformed Laplacian on $p$-forms. This immediately implies the Morse inequalities above.

Rigorous versions of Witten’s proof, with additional attention to details related to the quantum tunneling, appeared in papers by B. Simon [34] (see also the book [14]), B. Helffer and J. Sjöstrand [19] and others.

B. The definition of Morse function $f$ extends to manifolds with boundary if in addition we assume that $f$ has no critical points on the boundary and the restriction of $f$ to the boundary is also Morse. In its simplest form the Morse inequalities state that the number of critical points of index $p$ of $f$ plus the number of critical points of index $p - 1$ (resp. $p$) of $f|_{\partial M}$ with positive (resp. negative) outward normal derivative is not smaller than the $p$-th relative (resp. absolute) Betti number of the underlying manifold. A topological proof of this fact was obtained by E. Baiada and M. Morse in 1953 in [2]. For a modern topological treatment and generalizations to manifolds with corners see [16].

On manifolds with boundary the Witten Laplacian is defined by the same formula as in the case without boundary, but now we need to specify its domain. To obtain a differential complex, it is natural to choose the domain of the Witten differential $d_{h,f}$ as consisting of the forms with vanishing tangential (or normal)
components on the boundary. This defines the quadratic form of the corresponding Witten Laplacian, and we will mainly consider Witten Laplacian as the operator, defined by the closed quadratic form. The domain of the Laplacian requires additional vanishing conditions on the adjoint of the Witten differential (18).

K. C. Chang and J. Liu [15] were the first to use the method of the Witten Laplacian to give an analytic proof of Morse inequalities for compact manifolds $M$ with boundary by considering semiclassical asymptotics of small eigenvalues for the Witten Laplacian. Following the ideas in [14], Chang and Liu only had to study the case when the metric $g$ and the Morse function $f$ have canonical flat forms near the critical points. (This is sufficient to prove the Morse inequalities as a statement in differential topology.)

In 2006, B. Helffer and F. Nier [18] found semiclassical asymptotics of the Witten Laplacian on compact manifolds with boundary with the general Riemannian metric. They were mainly interested in obtaining very accurate asymptotics for the first (exponentially small) eigenvalue on functions. A new feature which appears here is an influence on the asymptotics of the behavior of the Morse function $f$ near some critical points of its restriction to the boundary. In particular, B. Helffer and F. Nier had to study the “rough” localization of the spectrum of the Witten Laplacian on forms.

In the present paper, in contrast to [15] and [18], we give a new proof of the semiclassical asymptotics for every eigenvalue of the Witten Laplacian with a fixed number (in increasing order) for compact manifolds with boundary in the presence of a general Riemannian metric. To this end, we modify a method suggested in [22] (where a similar result with some applications, including a vanishing result for the Quantum Hall conductivity, was obtained on regular coverings of compact manifolds without boundary). We will use some important technical elements from Helffer and Nier, as well as the technique of model operators, as formulated in [35].

The purpose of the present paper is to provide a new method of establishing semi-classical asymptotics of any eigenvalue of Witten’s Laplacian in the case of a smooth compact manifold with smooth boundary. We consider the boundary conditions obtained by choosing a domain for $d_{h,f}$. Namely, we take the domain of the corresponding quadratic form to consist of all forms of appropriate smoothness which have vanishing tangential (resp. normal) parts on the boundary. In this case eigenforms with bounded eigenvalues localize around the interior critical points and only those boundary critical points, i.e. critical points of $f|_{\partial M}$, which have a positive (resp. negative) outward normal derivative of $f$. In the spirit of [35] and [36], we construct the model operator which is the direct sum of two parts: one corresponding to the interior critical points and the other to the boundary critical points, as specified above. The part of the model operator corresponding to the interior critical points is the same as for manifolds without boundary. Namely, we choose coordinates such that the critical point $x_i$ is the origin. In these coordinates $\Delta_{h,f,g}$ can be written as

$$\Delta_{h,f,g} = -hA + B + h^{-1}V(x).$$
For each $x_i$, we obtain the model operator on $H^2(\mathbb{R}^n; AT^\ast \mathbb{R}^n)$ by replacing $A$ with its highest order terms with the coefficients frozen at the critical point, $B$ with $B(x_i)$ and $V$ with its quadratic part near the critical point. Then we take the direct sum of these operators over all interior critical points. At the relevant boundary critical points we construct the model operator in the same way, but this time for the Witten Laplacian on $\partial M$ with the function $f|_{\partial M}$ and the constant metric $g'$ which is obtained by restricting $g$ to the vectors tangent to $\partial M$ and then freezing it at the critical point. We prove that the spectrum of Witten Laplacian that is below an arbitrarily chosen constant $R$ concentrates around the part of the spectrum of the model operator that is below the same constant $R$ as $h \to 0$. Here we use techniques from [22]. In the proof, the part corresponding to the interior critical points is the same as the one in case when there is no boundary. The part corresponding to the boundary critical points is harder to treat, and we use appropriately modified ideas of Helffer and Nier [18].

C. In the last 25 years the method of Witten deformation was successfully applied to prove a number of significant results in topology and analysis. We provide a very brief review of literature. We note that our choices are highly subjective and are influenced by our own interests.

In 1982 J.-M. Bismut [4] modified the Witten deformation technique and combined it with intricate and deep probabilistic methods to produce a new proof of the degenerate Morse-Bott inequalities (see [6] for topological proof). A more accessible proof based on the adiabatic technique of Mazzeo-Melrose and Forman ([23], [17]) was given by I. Prokhorenkov in [29] (see also [20] for a different approaches to the proof).

A. V. Pazhitnov [28] used the method of Witten deformation to prove some of the Morse-Novikov inequalities when the gradient of Morse function is replaced by a closed 1-form. Novikov inequalities for vector fields were established by M. A. Shubin in [36]. Shubin’s results were extended by M. Braverman and M. Farber [8] to the case when 1-form (or corresponding vector field) has non-isolated zeros, and to the equivariant case in [9].

J. Alvarez López [1] used the method of Witten to prove Morse inequalities for the invariant cohomology of the space of orbits with applications to basic cohomology of Riemann foliations. V. Belfi, E. Park, and K. Richardson [3] used the Witten deformation of the basic Laplacian to prove an analog of Hopf index theorem for Riemannian foliations. Further applications of the method of Witten deformation to index theory were developed in [30] and [31].

The method of Witten Laplacian was also used to study the analytic torsion of the Witten complex. The analytic torsion was introduced by D. B. Ray and I. M. Singer [32]. For odd dimensional manifolds, D. B. Ray and I. M. Singer conjectured that the analytic torsion and the Reidemeister torsion coincide. Independently, J. Cheeger [13] and W. Müller [25] have proved this conjecture. The methods of J. Cheeger and W. Müller are both based on a combination of topological and analytical methods. Then J.M. Bismut and W. Zhang [5] suggested a purely analytical
proof of the Cheeger-Müller theorem and generalized it to the case where the metric is not flat. Later another analytic proof was suggested by D. Burghelea, L. Friedlander and T. Kappeler [11] which was shorter but based on application of the highly non-trivial Mayer-Vietoris type formula for the determinant of an elliptic operator. In this paper they generalize the theorem to the case of manifolds of any dimension (not necessarily odd). M. Braverman [7] found a short analytic proof by a direct way of analyzing the behaviour of the determinant of the Witten deformation of the Laplacian.

Finally, M. Braverman and V. Silantiev used the method of Witten deformation to extend Novikov Morse-type inequalities for closed 1-forms $\omega$ to manifolds with boundary in [10]. In the paper they require that the form $\omega$ is exact near the boundary of the manifold and that its critical set satisfies the condition of F. C. Kirwan (see [21]). The Witten deformation technique then is used to obtain discrete spectrum and to localize topological computations to the neighborhood of the critical set of $\omega$.

2. Preliminaries and the main theorem

Suppose that $M$ is a $C^\infty$ compact manifold, $\dim\mathbb{R}M = n$, with $C^\infty$ boundary $\partial M$, and $g$ is a Riemannian metric on $M$. Let $f$ be a Morse function on $M$, that is, $f$ is a Morse function on $M$ with no critical points on the boundary and $f|_{\partial M}$ is also a Morse function on $\partial M$.

We will use the following notations. The cotangent bundle on $M$ is denoted by $T^*M$, and the bundle of exterior forms is $\Lambda T^*M = \bigoplus_{k=0}^n \Lambda^kT^*M$. The spaces of smooth sections of the bundles $\Lambda T^*M$, $\Lambda^kT^*M$ are denoted by $\Lambda(M)$, $\Lambda^k(M)$ respectively. The elements of the sets $\Lambda(M)$ and $\Lambda^k(M)$ are called smooth differential forms on $M$ and smooth $k$-forms respectively. Given a manifold $M$ and a metric $g$ on $M$, we will use the notations $(\cdot, \cdot)_g$ and $\| \cdot \|_g$ for the $L^2$-inner product and the $L^2$-norm on $\Lambda(M)$ defined by the metric $g$.

It is convenient to use orientations of $M$ and $\partial M$ (if the orientations exist) to make global integrals well defined. If there is no orientation, we should define the integrals as sums of them over small disjoint pieces, each one located over a coordinate neighborhood. It is easy to see that every term (hence, every sum) does not depend upon the choice of orientations. For the sake of simplicity of notations, we will always assume both $M$ and $\partial M$ to be oriented.

Let $n$ be the outward unit normal vector field defined on the boundary of $M$ and $n^*$ be its dual 1-form with respect to the metric $g$. For any $\omega \in \Lambda^k(M)$, define the tangential part of $\omega$ as $t(\omega)(y) = i_{\bar{n}(y)}(\bar{n}^*(y) \wedge \omega(y))$ and the normal part of $\omega$ as $n(\omega)(y) = \omega(y) - t(\omega)(y)$ for any $y \in \partial M$. So $t\omega$ and $n\omega$ are sections of $\Lambda T^*M|_{\partial M}$. If $j : \partial M \rightarrow M$ is the inclusion map then $j^*$ defines isomorphism between $\Lambda(\partial M)$ and $\{t(\omega) : \omega \in \Lambda(M)\}$. By the Sobolev trace theorem, $t$ extends by continuity to a linear map of Sobolev spaces

$$t : H^1(M; \Lambda T^*M) \rightarrow H^{1/2}(\partial M; \Lambda T^*M) \subset L^2(\partial M; \Lambda T^*M),$$
and so does $n$.

The Witten deformation of the exterior derivative is defined by
\[ dh,f = e^{-f/h}hde^{f/h} = hd + df \wedge, \]
where $h > 0$ is a parameter The adjoint of $d_{h,f}$ with respect to the $L^2$-inner product $(\cdot, \cdot)_g$ is
\[ d^*_{h,f,g} = e^{f/h}hd^*e^{-f/h} = hd^* + i\nabla f. \]

Now define a quadratic form which is the closure of
\[ (h,f,g) = (\Delta, \omega)_g + ((\nabla f + \nabla^* \nabla f)\omega, \omega)_g + h^{-1}(\|\nabla f\|^2, \omega)_g \]
where $h > 0$ denotes the Sobolev space of forms. The closure is well defined and its domain is
\[ H^1(M; \Lambda^* M) : t(\omega) = 0 \]
where $H^1$ denotes the Sobolev space of forms.

The quadratic form $Q_{h,f,g}$ can be written as
\[ Q_{h,f,g}(\omega) = h(\Delta, \omega)_g + ((\nabla f + \nabla^* \nabla f)\omega, \omega)_g + h^{-1}(\|\nabla f\|^2, \omega)_g \]
\[ + \int_{\partial M} (t\omega) \wedge (\nabla f, \omega) - \int_{\partial M} (t\omega) \wedge (n\omega). \]

The Witten Laplacian is the elliptic self-adjoint operator associated with the closed quadratic form $Q_{h,f,g}$ by the Friedrichs construction [33, vol.1, section 8.6]. It is the operator defined by
\[ \Delta_{h,f,g} = h\Delta + ((\nabla f + \nabla^* \nabla f)\omega, \omega)_g + h^{-1}(\|\nabla f\|^2, \omega)_g \]
with the domain $D(\Delta_{h,f,g}) = \{ \omega \in H^2(M; \Lambda^* M) : t(\omega) = 0, \ t(d_{h,f}^* \omega) = 0 \}$ where $H^2$ is the Sobolev space of the corresponding sections.

On its domain, $Q_{h,f,g}$ can be also written as
\[ Q_{h,f,g}(\omega) = h(\|d\omega\|^2 + ||d^* \omega\|^2) + ((\nabla f + \nabla^* \nabla f)\omega, \omega)_g + h^{-1}(\|\nabla f\|^2, \omega)_g \]
\[ - \int_{\partial M} (\partial f)(x) < \omega, \omega >_g (x) d\mu_{\partial M}(x). \]

(see (2.15) in [18]) where $< \omega, \omega >_g (x)$ is the inner product on $\Lambda^* T_x M$ induced by the metric $g$ (see p. 226-227 [14]) and $\mu_{\partial M}$ is the measure on $\partial M$ defined by the metric $g$.

**Definition:** The set of critical points is a disjoint union of the sets $S_0$, $S_+$ and $S_-$ where $S_0$ is the set of interior points which are critical points of the Morse function $f$, $S_+$ and $S_-$ are the sets of boundary points which are critical points of $f|_{\partial M}$ with $\frac{\partial f}{\partial n} > 0$ and $\frac{\partial f}{\partial n} < 0$ respectively. Let $S = S_0 \cup S_+$. 
If \( y \in \partial M \) then \( y \in S_\pm \) if and only if \( \frac{\partial f}{\partial M}(y) = \pm |\nabla f(y)| \) respectively.

Assume that there exist \( N_0 \) interior critical points denoted by \( \{\bar{x}_1, \ldots, \bar{x}_{N_0}\} \) and \( N \) points in \( S_\pm \) denoted by \( \{\bar{y}_1, \ldots, \bar{y}_N\} \). In Section 4 we will prove that these points are the critical points we need to consider.

Now we want to form the model operator for the Witten Laplacian \( \Delta_{h,f,g} \) which provides the best approximation of \( \Delta_{h,f,g} \) by a direct sum of harmonic oscillators near the critical set \( S_0 \).

For each critical point \( \bar{x}_i \in S_0 \), we define the operator \( \Delta_i \) on \( H^2(\mathbb{R}^n; \Lambda T^*\mathbb{R}^n) \) to be
\[
\Delta_i = -hA_i + B_i + h^{-1}V_i(x),
\]
where the operator \( -A_i \) is the principal part of the Laplacian \( \Delta \) at \( \bar{x}_i \). It is an elliptic second order differential operator with constant coefficients. Operator \( B_i \) is the value at \( \bar{x}_i \) of the bounded self-adjoint zero order operator \( \mathcal{L}_{\nabla f} + \mathcal{L}_{\nabla f} \). Finally, \( V_i(x) \) is the quadratic part of the potential \( |\nabla f|^2 \) near \( \bar{x}_i \). In local coordinates \( x_1, \ldots, x_n \) near \( \bar{x}_i \), let \( g_i = g(\bar{x}_i) \), then
\[
A_i = \sum_{l,k=1}^n g_i^{lk} \frac{\partial^2}{\partial x_l \partial x_k},
\]
\[
B_i = \sum_{k,r=1}^n \frac{\partial^2 f}{\partial x_r \partial x_k}(\bar{x}_i) (a^*a a^* - a^*a^*a)
\]
and
\[
V_i(x) = \sum_{l,k,r,s=1}^n g_i^{rs} \frac{\partial^2 f}{\partial x_r \partial x_l}(\bar{x}_i) \frac{\partial^2 f}{\partial x_s \partial x_k}(\bar{x}_i)x_l x_k
\]
where \( a^k = (dx_k \wedge)^* \) and \( a^a k = (a^k)^* = dx_k \wedge \) are the fermionic annihilation and creation operators. Note that at the interior critical points, the model operator is the same as in the model operator for manifolds without boundary (see [35]).

For each boundary critical point \( \bar{y}_j \in S_\pm \) in a small neighborhood of \( \bar{y}_j \), let \( f'_j = f \mid \partial M \) and \( g'_j \) be the metric obtained by restricting \( g \) on the tangential vectors and freezing it at the critical point. Now define the operator \( \Delta_j \) on \( H^2(\mathbb{R}^{n-1}; \Lambda T^*\mathbb{R}^{n-1}) \) as
\[
\Delta_j = -hA'_j + B'_j + h^{-1}V'_j(x)
\]
where we define operators \( A'_j, B'_j, \) and \( V'_j \) as before by considering \( \partial M \) as a manifold without boundary. In local coordinates \( x_1, \ldots, x_{n-1} \) for \( \partial M \) near \( \bar{y}_j \) we have
\[
A'_j = \sum_{l,k=1}^{n-1} (g'_j)^{lk} \frac{\partial^2}{\partial x_l \partial x_k},
\]
\[
B'_j = \sum_{k,r=1}^{n-1} \frac{\partial^2 f'_j}{\partial x_r \partial x_k}(\bar{y}_j) (a^*a a^* - a^*a^*a)
\]
and

\[ V_j'(x) = \sum_{l,k,r,s=1}^{n-1} (g_j')_{rs} \frac{\partial^2 f_j'}{\partial x_r \partial x_l}(\bar{y}_j) \frac{\partial^2 f_j'}{\partial x_s \partial x_k}(\bar{y}_j) x_l x_k. \]

The **model operator** is defined by

\[ \Delta_{\text{mod}} = \left( \bigoplus_{i=1}^N \Delta_i \right) \oplus \left( \bigoplus_{j=1}^N \Delta_j \right). \]

The model operator does not depend on the choice of local coordinates on \( M \).

It follows from the general theory of elliptic operators on manifolds with boundary that the operator \( \Delta_{h,f,g} \) has discrete spectrum with eigenvalues

\[ \lambda_1(h) \leq \lambda_2(h) \leq \lambda_3(h) \leq \ldots \]

such that \( \lambda_l(h) \to \infty \) as \( l \to \infty \) for each \( h > 0 \).

The spectrum of the model operator \( \Delta_{\text{mod}} \) is also discrete and the eigenvalues are independent of \( h \) \([35]\). We list all elements of the spectrum of \( \Delta_{\text{mod}} \) in the increasing order as

\[ \mu_1 \leq \mu_2 \leq \mu_3 \leq \ldots \]

such that \( \mu_l \to \infty \) as \( l \to \infty \).

We will prove that up to any fixed real number \( R \notin \text{spec} \left( \Delta_{\text{mod}} \right) \), the spectrum of the operator \( \Delta_{h,f,g} \) concentrates near the spectrum of the model operator \( \Delta_{\text{mod}} \) as \( h \to 0 \). More precisely, our main result is

**Theorem 2.1.** For every positive number \( R \notin \text{spec} \left( \Delta_{\text{mod}} \right) \) there exist \( M > 0 \), \( h_0 > 0 \) and \( C > 0 \) such that both \( \Delta_{\text{mod}} \) and \( \Delta_{h,f,g} \) have exactly \( M \) eigenvalues less than \( R \) and

\[ |\lambda_l(h) - \mu_l| \leq Ch^{1/2}, \quad l = 1, 2, \ldots, M, \quad h \in (0, h_0). \]

**Remark 2.2.** One can replace the tangential boundary conditions (2.2) for the quadratic form \( Q_{h,f,g} \) with the normal boundary conditions (2.6)

\[ \text{D}(Q_{h,f,g}) = \{ \omega \in H^1(M; \Lambda T^* M) : n(\omega) = 0 \}. \]

The corresponding domain for the Witten Laplacian \( \Delta_{h,f,g} \) is \( \text{D}(\Delta_{h,f,g}) = \{ \omega \in H^2(M; \Lambda T^* M) : n(\omega) = 0, \ast n(d_{h,f} \omega) = 0 \} \). The case of the normal boundary conditions can be reduced to one of tangential boundary conditions by observing that the Hodge * operator maps the space \( \{ \omega \in H^2(M; \Lambda T^* M) : n(\omega) = 0, \ast n(d_{h,f} \omega) = 0 \} \) to the space \( \{ \omega \in H^2(M; \Lambda T^* M) : t(\omega) = 0, t(d_{h,f}(-f,g) \omega) = 0 \} \), and that \( \ast \Delta_{h,f,g} = \Delta_{h,(-f),g} \ast \).

We will prove our main result by comparing the Witten Laplacian to the model operator in a way suggested in Theorem 2.1 in [22]. We will give the abstract setting, which emphasizes the use of quadratic forms instead of operators, in the next section and proofs can be found in the appendix.
3. General results on equivalence of projections

Consider Hilbert spaces \( \mathcal{H}_1 \) and \( \mathcal{H}_2 \) equipped with inner products \((\cdot,\cdot)_1\) and \((\cdot,\cdot)_2\). Let \( Q_1 \) and \( Q_2 \) be closed bounded below quadratic forms with dense domains \( D(Q_1) \subset \mathcal{H}_1 \) and \( D(Q_2) \subset \mathcal{H}_2 \) respectively. Let \( A_1 \) and \( A_2 \) be the self-adjoint operators corresponding to the quadratic forms.

Let us take \( \lambda_{01}, \lambda_{02} \leq 0 \) such that
\[
Q_l(\omega) \geq \lambda_{0l} \|\omega\|_l^2, \quad \omega \in D(Q_l), \quad l = 1, 2.
\]

Let \( \mathcal{H}_0 \) be a Hilbert space, equipped with injective bounded linear maps \( i_1: \mathcal{H}_0 \rightarrow \mathcal{H}_1 \) and \( i_2: \mathcal{H}_0 \rightarrow \mathcal{H}_2 \). Assume that there are given bounded linear maps \( p_1: \mathcal{H}_1 \rightarrow \mathcal{H}_0 \) and \( p_2: \mathcal{H}_2 \rightarrow \mathcal{H}_0 \) such that \( p_1 \circ i_1 = \text{id}_{\mathcal{H}_0} \) and \( p_2 \circ i_2 = \text{id}_{\mathcal{H}_0} \), as in the following diagram:

Let \( J \) be a self-adjoint bounded operator in \( \mathcal{H}_0 \). Assume that \((i_2 J p_1)^* = i_1 J p_2\).

Since the operators \( i_l: \mathcal{H}_0 \rightarrow \mathcal{H}_l, \ l = 1, 2, \) are bounded and have bounded left-inverse operators \( p_l \), they are topological monomorphisms, i.e. they have closed images and the maps \( i_l: \mathcal{H}_0 \rightarrow \text{Im} i_l \) are topological isomorphisms. Therefore, we can assume that the estimate
\[
\rho^{-1} \|i_2 J \omega\|_2 \leq \|i_1 J \omega\|_1 \leq \rho \|i_2 J \omega\|_2, \quad \omega \in \mathcal{H}_0,
\]
holds with some \( \rho > 1 \). (Although we can choose the constant \( \rho \) in the estimate \((3.2)\) to be independent of \( J \), it may be possible to choose \( \rho \) closer to 1, due to the presence of \( J \).)

Define the bounded operators \( J_l \) in \( \mathcal{H}_l, \ l = 1, 2, \) by the formula \( J_l = i_l J p_l \). We assume that
\begin{itemize}
  \item the operator \( J_l, \ l = 1, 2, \) maps the domain of \( Q_l \) to itself;
  \item \( J_l \) is self-adjoint, and \( 0 \leq J_l \leq \text{id}_{\mathcal{H}_l}, \ l = 1, 2; \)
  \item for \( \omega \in \mathcal{H}_0, \ i_1 J \omega \in D(Q_1) \) if and only if \( i_2 J \omega \in D(Q_2) \).
\end{itemize}
Denote \( D = \{ \omega \in \mathcal{H}_0 : i_1 J \omega \in D(Q_1) \} = \{ \omega \in \mathcal{H}_0 : i_2 J \omega \in D(Q_2) \} \).

Introduce a self-adjoint positive bounded linear operator \( J'_l \) in \( \mathcal{H}_l \) by the formula \( J'_l^2 + J'_l^2 = \text{id}_{\mathcal{H}_l} \). We assume that
\begin{itemize}
  \item the operator \( J'_l, \ l = 1, 2, \) maps the domain of \( Q_l \) to itself;
\end{itemize}
• the quadratic forms $Q_l(\omega) - (Q_l(J_1\omega) + Q_l(J_1'\omega))$ are bounded i.e.

\begin{equation}
Q_l(J_1\omega) + Q_l(J_1'\omega) - Q_l(\omega) \leq \gamma_l\|\omega\|^2, \quad \omega \in D(Q_l), \quad l = 1, 2.
\end{equation}

Finally, we assume that

\begin{equation}
Q_l(J_1'\omega) \geq \alpha_l\|J_1'\omega\|^2, \quad \omega \in D(Q_l), \quad l = 1, 2,
\end{equation}

for some $\alpha_l > 0$, and

\begin{equation}
Q_2(i_2J_1\omega) \leq \beta_1Q_1(i_1J_1\omega) + \varepsilon_1\|i_1J_1\omega\|^2, \quad \omega \in D,
\end{equation}

\begin{equation}
Q_1(i_1J_1\omega) \leq \beta_2Q_2(i_2J_1\omega) + \varepsilon_2\|i_2J_1\omega\|^2, \quad \omega \in D,
\end{equation}

for some $\beta_1, \beta_2 \geq 1$ and $\varepsilon_1, \varepsilon_2 > 0$.

Denote by $E_l(\lambda), \ l = 1, 2$, the spectral projection of the operators $A_l$, corresponding to the semi-axis $(-\infty, \lambda]$.

**Theorem 3.1.** Under the assumptions in this section, let $b_1 > a_1$ and

\begin{equation}
a_2 = \rho \left[ \beta_1 \left( a_1 + \gamma_1 + \frac{(a_1 + \gamma_1 - \lambda_{01})^2}{a_1 - a_1 - \gamma_1} \right) + \varepsilon_1 \right],
\end{equation}

\begin{equation}
b_2 = \frac{\beta_2^{-1}(b_1 \rho^{-1} - \varepsilon_2)(\alpha_2 - \gamma_2) - \alpha_2\gamma_2 + 2\lambda_{02}\gamma_2 - \lambda_{02}^2}{\alpha_2 - 2\lambda_{02} + \beta_2^{-1}(b_1 \rho^{-1} - \varepsilon_2)}.
\end{equation}

Suppose that $\alpha_1 > a_1 + \gamma_1$, $\alpha_2 > b_2 + \gamma_2$ and $b_2 > a_2$. If the interval $(a_1, b_1)$ does not intersect with the spectrum of $A_1$, then:

1. the interval $(a_2, b_2)$ does not intersect with the spectrum of $A_2$;
2. the operator $E_2(\lambda_2)i_2J_1p_1E_1(\lambda_1) : \text{Im}E_1(\lambda_1) \to \text{Im}E_2(\lambda_2)$ is an isomorphism for any $\lambda_1 \in (a_1, b_1)$ and $\lambda_2 \in (a_2, b_2)$.

For the proof of this theorem see Appendix.

**Remark 3.2.** If the spectral projections $E_l(\lambda), \ l = 1, 2$ have finite rank for all $\lambda$, then the condition $(i_2J_1p_1)^* = i_1J_2p_2$ is not necessary and the condition that the operator $J_2$ is self-adjoint can be replaced by a weaker condition that $J_2$ is merely symmetric on $\text{Im}(J_2) \subset H_2$. The projections $E_l(\lambda), \ l = 1, 2$ have finite rank in the case of the Witten Laplacian $\Delta_{h,f,g}$.

**Remark 3.3.** Since $\rho > 1, \beta_1 \geq 1, \gamma_1 > 0$ and $\varepsilon_1 > 0$, we, clearly, have $a_2 > a_1$. The formula (3.8) is equivalent to the formula

\begin{equation}
b_1 = \rho \left[ \beta_2 \left( b_2 + \gamma_2 + \frac{(b_2 + \gamma_2 - \lambda_{02})^2}{\alpha_2 - b_2 - \gamma_2} \right) + \varepsilon_2 \right],
\end{equation}

which is obtained from (3.7), if we replace $\alpha_1, \beta_1, \gamma_1, \varepsilon_1, \lambda_{01}$ by $\alpha_2, \beta_2, \gamma_2, \varepsilon_2, \lambda_{02}$ accordingly and $a_1$ and $a_2$ by $b_2$ and $b_1$ accordingly. In particular, this implies that $b_1 > b_2$. 


4. Proof of the main theorem

We start by describing the setting for the application of Theorem 3.1.

Let $A_2$ be the Witten Laplacian $\Delta_{h,f,g}$ with the domain
$$D(\Delta_{h,f,g}) = \{ \omega \in H^2(M; \Lambda T^*M) : t(\omega) = 0, \ t(d_{h,f,g}^*\omega) = 0 \}$$
(see (2.4)). The operator $A_2$ corresponds to the quadratic form
$$Q_2(\omega) = Q_{h,f,g}(\omega) = \frac{1}{h} \left( \| d_{h,f,g} \omega \|_g^2 + \| d_{h,f,g}^*\omega \|_g^2 \right)$$
with the domain $D(Q_2) = \{ \omega \in H^1(M; \Lambda T^*M) : t(\omega) = 0 \}$ (see (2.4)). Let
$A_1 = \Delta_{\text{mod}}$ be the model operator (see (2.5)), and $Q_1$ be the quadratic form corresponding to the operator $A_1$ with the domain
$$D(Q_1) = H^1(\mathbb{R}^n, \Lambda T^*\mathbb{R}^n)^N_0 \oplus H^1(\mathbb{R}^{n-1}, \Lambda T^*\mathbb{R}^{n-1})^N.$$
We have
$$D(Q_2) \subset H_2 = L^2(M, \Lambda T^*M),$$
and
$$D(Q_1) \subset H_1 = L^2(\mathbb{R}^n, \Lambda T^*\mathbb{R}^n)^N_0 \oplus L^2(\mathbb{R}^{n-1}, \Lambda T^*\mathbb{R}^{n-1})^N
\cong (L^2(\mathbb{R}^n, \Lambda T^*\mathbb{R}^n) \otimes \mathbb{C}^N_0) \oplus (L^2(\mathbb{R}^{n-1}, \Lambda T^*\mathbb{R}^{n-1}) \otimes \mathbb{C}^N).$$

For each interior critical point $\bar{x}_i \in S_0$, we choose local coordinates $x_1, ..., x_n$. Let $B(\bar{x}_i, r) \subset M$ be the open ball around $\bar{x}_i$ with radius $r$ and $B_i(0, r)$ be the corresponding ball in $\mathbb{R}^n$ in these coordinates.

Recall that at each boundary critical point $\bar{y}_j \in S_+$, we have $\frac{\partial}{\partial n}(\bar{y}_j) = |\nabla f(\bar{y}_j)|$. Then it is possible to find local coordinates $x_1, x_2, ..., x_n$ near $\bar{y}_j$ such that in these coordinates $\bar{y}_j$ is the origin, $\partial M = \{ x_n = 0 \}$, $M = \{ x_n \leq 0 \}$,

$$f(x) = x_n + f'(x'),$$

and

$$g = g_{nn}(x)dx_n^2 + g'(x),$$

where $x = (x', x_n)$ (see (3.27) in [18] and Appendix B in [16]). Here $f' = f |_{\partial M}$, $g'$ is the restriction of the metric $g$ to the tangent space spanned by $\{ \partial/\partial x_1, \partial/\partial x_2, ..., \partial/\partial x_{n-1} \}$ and $x'$ is any coordinates on $\partial M$ such that $\bar{y}_j$ is the origin. Let $g_{ij} = g(0)$ be the constant metric in these coordinates. Furthermore, since $f |_{\partial M}$ is a Morse function on the boundary, the tangential coordinates $x_1, ..., x_{n-1}$ can be chosen so that

$$f'(x') = f(0) + \sum_{r=0}^{n-1} d_r x_r^2,$$

where the coefficients $d_r$ for $r = 1, ..., n-1$ in the expression of $f'$ are non vanishing real constants.

Let $C(\bar{y}_j, r) = \{ x \in M : |x'| < r, \ -r < x_n \leq 0 \}$ for some $r > 0$ and let $C_j(0, r)$ be the corresponding set in $\mathbb{R}^n$. Choose $r$ small enough so that around each boundary critical points $\bar{y}_j$ we can choose the special coordinates, all the sets
$B(\bar{x}_i, r)$ and $C(\bar{y}_j, r)$ are disjoint, and each $B(\bar{x}_i, r)$ is in the interior of $M$. Let $B(\bar{y}_j, r) \subset \partial M$ be the open ball around $\bar{y}_j$ with radius $r$ on the boundary $\partial M$, and let $B_j(0, r)$ be the corresponding set in $\partial \mathbb{R}^n = \mathbb{R}^{n-1}$. Let

$$\mathcal{H}_0 = (\oplus_{i=1}^N L^2(B_i(0, r), \Lambda T^* \mathbb{R}^n |_{B_i(0, r)}) \oplus (\oplus_{j=1}^N L^2(B_j(0, r), \Lambda T^* \mathbb{R}^{n-1} |_{B_j(0, r)})).$$

For some technical reasons that will become clear later, we choose $\kappa$ such that $\frac{1}{3} < \kappa < \frac{1}{2}$. Let $\psi \in C^\infty(\mathbb{R}^n)$ such that $0 \leq \psi \leq 1$, $\psi(x) = 1$ if $|x| \leq 1$, $\psi(x) = 0$ if $|x| \geq 2$. For small enough $h$, $\psi_i^{(h)}(x) = \psi(h^{-\kappa} x) \in C^\infty(B_i(0, r))$. Let $\phi \in C^\infty(\mathbb{R}^{n-1})$ such that $0 \leq \phi \leq 1$, $\phi(x) = 1$ if $|x| \leq 1$, $\phi(x) = 0$ if $|x| \geq 2$. For small enough $h$, $\phi_j^{(h)}(x) = \phi(h^{-\kappa} x) \in C^\infty(B_j(0, r))$. Let $J$ be the multiplication operator by $(\oplus_{i=1}^N \psi_i^{(h)}(x)) \oplus (\oplus_{j=1}^N \phi_j^{(h)}(x))$ in $\mathcal{H}_0$.

Let $i_1 : \mathcal{H}_0 \to \mathcal{H}_1$ be the natural inclusion and let $p_1 : \mathcal{H}_1 \to \mathcal{H}_0$ be the restriction map, then $p_1 \circ i_1 = \text{id}_{\mathcal{H}_0}$. Furthermore, the operators $J_1 = i_1 J p_1$ and $J_1'$ clearly satisfy the five properties listed after the definition of $J_1$ in Section 3. Indeed, the last property follows from the calculation

$$Q_1(J_1 \omega) + Q_1(J_1' \omega) - Q_1(\omega) = (A_1 J_1 \omega, J_1 \omega)_1 + (A_1 J_1' \omega, J_1' \omega)_1 - (A_1 \omega, \omega)_1$$

$$= (J_1 A_1 J_1 + J_1' A_1 J_1' - A_1) \omega, \omega)_1$$

$$= h \sum_{i=1}^N ((|d \psi_i^{(h)}|^2 + |d \bar{\psi}_i^{(h)}|^2) \omega, \omega)_1$$

$$+ h \sum_{j=1}^N ((|d \phi_j^{(h)}|^2 + |d \bar{\phi}_j^{(h)}|^2) \omega, \omega)_1,$$

where $\bar{\psi}_i^{(h)} = \sqrt{1 - (\psi_i^{(h)})^2}$. The last equality follows from IMS localization formula

$$J_1 A_1 J_1 + J_1' A_1 J_1' - A_1 = h \sum_{i=1}^N (|d \psi_i^{(h)}|^2 + |d \bar{\psi}_i^{(h)}|^2) + h \sum_{j=1}^N (|d \phi_j^{(h)}|^2 + |d \bar{\phi}_j^{(h)}|^2)$$

(see (11.37) in [14]). All of $|d \psi_i^{(h)}|$, $|d \bar{\psi}_i^{(h)}|$, $|d \phi_j^{(h)}|$ and $|d \bar{\phi}_j^{(h)}|$ are $O(h^{-\kappa})$; therefore, the inequality (3.3) is satisfied for $Q_1$ and $\gamma_1 = O(h^{1-2\kappa})$.

Now we will define the operators $i_2 : \mathcal{H}_0 \to \mathcal{H}_2$ and $p_2 : \mathcal{H}_2 \to \mathcal{H}_0$. Let $\phi_n \in C^\infty(\mathbb{R}_-) \subset C^\infty(\mathbb{R})$ such that $0 \leq \phi_n \leq 1$, $\phi_n(x) = 1$ if $-1 \leq x \leq 0$, $\phi_n(x) = 0$ if $x \leq -2$. For small enough $h$, $\phi_n^{(h)}(x) = \phi(h^{-\kappa} x) \in C^\infty((-r, 0])$. Let

$$\alpha = C(h) \phi_n^{(h)}(x_n) e^\frac{\pi}{r} dx_n$$

be a 1-form on $\mathbb{R}_-$. We choose the constant $C(h)$ so that the form $\alpha$ has $L^2$-norm one with respect to the metric $g_{nn}(0) dx_n^2$.

On $L^2(B_i(0, r), \Lambda T^* \mathbb{R}^n |_{B_i(0, r)})$ we define $i_2$ to be the inclusion given by the choice of the special coordinates near $\bar{x}_i$ in $B_i(0, r)$, and on $L^2(B_j(0, r), \Lambda T^* \mathbb{R}^{n-1} |_{B_j(0, r)})$ it is defined by

$$i_2 \omega = i(\alpha \wedge \omega)$$
where \( i : L^2(C_j(0, r), \Lambda T^*\mathbb{R}^n|_{C_j(0, r)}) \to L^2(M, \Lambda T^*M) \) is the inclusion given by the choice of the special coordinates near each \( \bar{y}_j \) in \( C_j(0, r) \).

Let \( L \) be the subspace of \( L^2(M, \Lambda T^*M) \) which contains only the forms which are of the form \( \alpha \wedge \omega'(x') \) in the special coordinates around the boundary critical points where \( \omega' \) is a form which depends only tangential components. The map \( p_2 \) is the composition of the restriction map
\[
p: L^2(M, \Lambda T^*M) \to \left( \bigoplus_{i=1}^N L^2(B_i(0, r), \Lambda T^*\mathbb{R}^n|_{B_i(0, r)}) \right),
\]
and the map \( r : L^2(C_j(0, r), \Lambda T^*\mathbb{R}^n|_{C_j(0, r)}) \to L^2(B_j(0, r), \Lambda T^*\mathbb{R}^n-1|_{B_j(0, r)}) \) defined by the natural extension of the map \( r(\omega(x)) = \frac{1}{c_i(x)}(i\partial_i/\partial x_n(\omega))(x', 0) \) on the subspace of the smooth forms \( \omega = \alpha \wedge \omega'(x') \) to the closure of this subspace in \( L^2(C_j(0, r), \Lambda T^*\mathbb{R}^n|_{C_j(0, r)}) \), and zero on the orthogonal complement. In other words, \( p_2 \) is defined by the formula
\[
p_2(\omega(x)) = \frac{1}{c_i(x)}[i\partial_i/\partial x_n(p(\omega))](x', 0)
\]
for any smooth \( \omega \in L \subset L^2(M, \Lambda T^*M) \) such that \( p(\omega) \in L^2(C_j(0, r), \Lambda T^*\mathbb{R}^n|_{C_j(0, r)}) \).

It is easy to check that \( p_2 \circ i_2 = \text{id}_{H_0} \) and the operator \( J_2 = i_2Jp_2 \) maps the domain of the quadratic form \( Q_2 \) to itself.

To show that the inequality (3.3) is satisfied for \( Q_2 \), first by (2.3) we have,
\[
Q_2(J_2\omega) + Q_2(J'_2\omega) - Q_2(\omega) = (A_2J_2\omega, J_2\omega)_{y} + (A_2J'_2\omega, J'_2\omega)_{y} - (A_2\omega, \omega)_{y}
- \int_{\partial M}(td^*_h,f,g J_2\omega) \wedge *nJ'_{2\omega} - \int_{\partial M}(td^*_h,f,g J'_2\omega) \wedge *n\bar{\omega}.
\]
On \( L \), we have
\[
(td^*_h,f,g J_2\omega) = d^*_h,f,g(\alpha) \wedge \phi_j^{(h)} \omega',
\]
and
\[
(td^*_h,f,g J'_2\omega) = d^*_h,f,g(\alpha) \wedge \bar{\phi}_j^{(h)} \omega'.
\]
Therefore, the multiplication with \( J_2 \) and \( J'_2 \) commutes with \( d^*_h,f,g \) and we have
\[
Q_2(J_2\omega) + Q_2(J'_2\omega) - Q_2(\omega) = (A_2J_2\omega, J_2\omega)_{y} + (A_2J'_2\omega, J'_2\omega)_{y} - (A_2\omega, \omega)_{y}
= h \sum_{i=1}^{N_0}((|d\psi_i^{(h)}|^2 + |d\bar{\psi}_i^{(h)}|^2)\omega, \omega)_{y} + \sum_{j=1}^{N_0}(|d\phi_j^{(h)}|^2 + |d\bar{\phi}_j^{(h)}|^2)\omega, \omega)_{y}
\]
by IMS localization formula (see(11.37) in [14]). Since all \(|d\psi_i^{(h)}|, |d\bar{\psi}_i^{(h)}|, |d\phi_j^{(h)}|\)
and \(|d\bar{\phi}_j^{(h)}|\) are \( O(h^{-c}) \), the inequality (3.3) is satisfied for \( Q_2 \) and \( \gamma_2 = O(h^{1-2c}) \).

We note that the operator \( J_2 \) is not self-adjoint, however it is symmetric on \( \text{Im}(J_2) \).
The following lemma provides the localization of eigenforms near the points in $S_0 \cup S_+$. The proof of this lemma is similar to the proof of Theorem 3.2.3 (p. 34 in [18]).

**Lemma 4.1.** Let $E$ be the complement of the union of balls $B(x_i, h^\kappa)$, $i = 1, \ldots, N_0$, and $C(y_j, h^\kappa)$, $j = 1, \ldots, N$. If $\omega \in D(Q_{h,f,g})$ such that $\text{supp}(\omega) \subseteq E$, then there exists $h_0 > 0$ such that for all $h \in (0, h_0)$

$$Q_{h,f,g}(\omega) \geq C h^{2\kappa-1} \| \omega \|^2$$

for some constant $C > 0$.

Since $\kappa < 1/2$, $h^{2\kappa-1} \to \infty$ as $h \to 0$, the lemma implies that for the eigenforms with bounded eigenvalues it is enough to consider only the forms supported in a small neighborhood of the critical points in $S_0 \cup S_+$.

**Lemma 4.2.** Let $\omega \in D(Q_1)$ such that

$$\text{supp}(\omega) \subseteq \{ x \in \mathbb{R}^n : \text{dist}(x, 0) \geq h^\kappa \}^{N_0} \cup \{ x \in \mathbb{R}^{n-1} : \text{dist}(x, 0) \geq h^\kappa \}^N.$$

Then there exists $h_0 > 0$ such that for all $h \in (0, h_0)$

$$Q_1(\omega) \geq C h^{2\kappa-1} \| \omega \|^2.$$

**Proof.** We can write the quadratic form $Q_1$ as

$$Q_1(\omega) = (\Delta_{mod}\omega, \omega)_1 = \sum_{i=1}^{N_0} (\Delta_i \omega, \omega)_{g_i} + \sum_{j=1}^{N} (\Delta_j \omega, \omega)_{g'_j}.$$

Suppose $\text{supp}(\omega)$ is not empty in the domain of $\Delta_i$ for some $i \in \{1, \ldots, N_0\}$. Since all the operators in the definition of the quadratic form $Q_1$ are positive operators, we have

$$Q_1(\omega) \geq (\Delta_i \omega, \omega)_{g_i} = h \left( \| d\omega \|^2_{g_i} + \| d^* \omega \|^2_{g'_i} \right) + \left( (\mathcal{L}_{\varphi_f} + \mathcal{L}_{\varphi_f}^\ast) \omega, \omega \right)_{g_i} + h^{-1} (V_i \omega, \omega)_{g_i} \geq h^{-1} (V_i \omega, \omega)_{g_i}.$$

We also have that $V_i \geq C_1 h^{2\kappa}$ for some $C_1 > 0$. Therefore, we conclude that

$$Q_1(\omega) \geq C h^{2\kappa-1} \| \omega \|^2.$$

If $\text{supp}(\omega)$ is empty in the domain of $\Delta_i$ for all $i$, then it is not empty in the domain of $\Delta_j$ for some $j \in \{1, \ldots, N\}$. Since $V' \geq C_2 h^{2\kappa}$ for some $C_2 > 0$, similar argument will lead us the inequality (4.9). \( \square \)

Our next goal is to obtain the estimates (3.6) and (3.7). In the setting of this section these estimates are equivalent to the inequalities

$$Q_2(i_2 \phi_j^{(h)}(x) \omega) \geq (1 - C_1 h^\kappa) Q_1(i_1 \phi_j^{(h)}(x) \omega) - C_2 h^{3\kappa-1} \| i_1 \phi_j^{(h)}(x) \omega \|^2_{g'_j}$$

and

$$Q_1(i_1 \phi_j^{(h)}(x) \omega) \geq (1 - C_1 h^\kappa) Q_2(i_2 \phi_j^{(h)}(x) \omega) - C_2 h^{3\kappa-1} \| i_2 \phi_j^{(h)}(x) \omega \|^2_{g'_j},$$

where $Q_1$ is the quadratic form associated with the operator $\Delta_{mod}$.
where \( \omega \in D \cap \bigoplus_{i=1}^{N} L^2(B_j(0, r), \Lambda T^*\mathbb{R}^{n-1}|_{B_j(0, r)}) \) and \( C_1, C_2 > 0 \) are some constants. Remember that

\[
D = \{ \omega \in \mathcal{H}_0 : i_1 J \omega \in D(Q_1) \} = \{ \omega \in \mathcal{H}_0 : i_2 J \omega \in D(Q_2) \}.
\]

In the proof of the inequalities we will use some intermediate quadratic forms in order to compare \( Q_1 \) and \( Q_2 \).

At each critical point \( \tilde{y}_j \in S^+ \), let \( x_1, \ldots, x_n \) be the special coordinates. In these coordinates \( f \) and \( g \) can be written as \((4.11)\) and \((4.12)\) respectively. We use the formula \((4.3)\) to extend \( f \) to \( \mathbb{R}^n \). Let \( \tilde{g}_j \) be an extension of the metric \( g \) to \( \mathbb{R}^n \) such that \( \tilde{g}_j(x) = g(x) \) if \( |x'| \leq h^\kappa \) and \( |x_n| \leq C \) for some positive constant \( C \), and \( \tilde{g}_j(x) = g(0) \) if \( |x'| \geq 2h^\kappa \) or \( |x_n| \geq 2C \).

Note that in the set \( |x'| \leq h^\kappa, |x_n| \leq h^\kappa, \) \( f = f_j \) and \( g = \tilde{g}_j \), therefore for each \( \tilde{y}_j \in S^+ \) and for any smooth \( \omega \in \mathcal{H}_0 \) we have

\[
\begin{align*}
Q_2(i_2(\phi_j(h)(x)\omega)) &= Q_{h,f_j,\tilde{g}_j}(\alpha \wedge i_1(\phi_j(h)(x)\omega)),
\end{align*}
\]
where \( Q_{h,f_j,\tilde{g}_j} \) is a quadratic form on \( H^1(\mathbb{R}^n; \Lambda T^*\mathbb{R}^n) \).

The proof of the following lemma is similar to the proof of Lemma 3.3.7 in [18].

**Lemma 4.3.** Let \( f = f_j, g_1 = \tilde{y}_j, \) and \( g_2(x', x_n) = \tilde{g}_j(x', 0) \). Then for some constant \( C \geq 0 \),

\[
(4.13) \quad Q_{h,f,g_1}(\omega) \geq (1 - Ch^\kappa)Q_{h,f,g_2}(\omega) - Ch^\kappa \| \omega \|^2_{g_2}
\]
for \( \omega \in H^1(\mathbb{R}^n; \Lambda T^*\mathbb{R}^n) \) such that \( t(\omega) = 0 \) and \( \text{supp} \omega \subset \{ x_n \geq -C_0 h^\kappa \} \) for some constant \( C_0 > 0 \).

Note that we also have

\[
(4.14) \quad Q_{h,f,j,\tilde{g}_2}(\omega) \geq (1 - Ch^\kappa)Q_{h,f,j,\tilde{g}_1}(\omega) - Ch^\kappa \| \omega \|^2_{g_1}
\]
since the argument in the proof of Lemma 3.3.7 is symmetric.

Now we want to freeze \( (g_2)_{nn} \). To do this we will need the following lemma.

**Lemma 4.4.** Let \( g_3(x) = \frac{(g_2)_{nn}(0)}{(g_2)_{nn}(x')} g_2(x) \). Then there exist \( C_1, C_2 > 0 \) and \( h_0 \) such that for any \( h \in (0, h_0) \)

\[
(4.15) \quad Q_{h,f,g_3}(\omega) \geq (1 - C_1 h^\kappa)Q_{h,f,g_2}(\omega) - C_2 h^{1-\kappa} \| \omega \|^2_{g_2}
\]
for \( \omega \) such that \( \text{supp} \omega \subset \{ x \in \mathbb{R}^n : |x'| \leq C_0 h^\kappa, x_n \geq -C_0 h^\kappa \} \) for some constant \( C_0 > 0 \).

**Proof.** Let \( e^{\varphi(x)} = \frac{(g_2)_{nn}(0)}{(g_2)_{nn}(x')}, \) then \( g_3(x) = e^{\varphi(x)} g_2(x) \). Note that \( \varphi(0) = 0 \).

Since \( g_2 = g_3 \) when \( |x'| \geq 2h^\kappa \), and both \( g_2 \) and \( g_3 \) are \( x_n \)-independent, \( \varphi(x) = O(h^\kappa) \) and \( e^{\varphi(x)} = 1 + O(h^\kappa) \) everywhere. Therefore,

\[
(4.16) \quad \min \{ e^{\varphi(x)} \} = 1 + O(h^\kappa).
\]
Given a metric $g$, the volume form is $V_g(x) = (\det g(x))^{1/n} dx_1 \wedge \ldots \wedge dx_n$. Then,

$$V_{g_2} (x) = e^{\varphi(x)} V_g (x).$$

For any $p$-forms $\omega$ and $\eta$

$$\omega \wedge \ast_{g_2} \eta = e^{(1/2-p)} \varphi(x) \omega \wedge \ast_{g_2} \eta,$$

see (pp. 26-27 [18]). Therefore,

$$\| \omega \|_{g_2} \geq e^{(1/2-p) \min \varphi(x)} \| \omega \|_{g_2}.$$

Since $d_{h,f}$ does not depend on $g$, we have

$$\| d_{h,f} \omega \|_{g_3} \geq e^{(1/2-p-1) \min \varphi(x)} \| d_{h,f} \omega \|_{g_2}.$$ 

Thus by (4.16),

$$\| d_{h,f} \omega \|_{g_3} \geq (1 - C_1 h^\kappa) \| d_{h,f} \omega \|_{g_2}. \quad (4.17)$$

We also have

$$\| d_{h,f,g_3} \omega \|_{g_3} \geq e^{(3p-1+n) \min \varphi(x)} \| d_{h,f,g_2} \omega + \omega \wedge (\bar{\nabla} \varphi(x)) \omega \|_{g_2}, \quad (4.18)$$

see the proof of Lemma 3.3.8 in [18].

Now we will use the following well known inequality,

$$\| f + g \|^2 \leq (1 + \varepsilon) \| f \|^2 + (1 + \frac{1}{\varepsilon}) \| g \|^2,$$

which implies that

$$\| f \|^2 \geq \frac{1}{1 + \varepsilon} \| f + g \|^2 - \frac{1}{\varepsilon} \| g \|^2.$$

Replacing $\varepsilon$ by $h^\kappa$, $f$ by $f + g$ and $g$ by $-g$, we get

$$\| f + g \|^2 \geq (1 - C_1 h^\kappa) \| f \|^2 - C_2 h^{-\kappa} \| g \|^2,$$

which implies that

$$\| d_{h,f,g_2} \omega + \omega \wedge (\bar{\nabla} \varphi(x)) \omega \|_{g_2}$$

$$\geq (1 - C_1 h^\kappa) \| d_{h,f,g_2} \omega \|_{g_2} - C_2 h^{-\kappa} \| \omega \|_{g_2} \geq (1 - C_1 h^\kappa) \| d_{h,f,g_2} \omega \|_{g_2} - C_2 h^{2-\kappa} \| \omega \|_{g_2}.$$

This together with the equation (4.16) and the inequality (4.18) imply

$$\| d_{h,f,g_3} \omega \|_{g_3} \geq (1 - C_1 h^\kappa) \| d_{h,f,g_2} \omega \|_{g_2} - C_2 h^{2-\kappa} \| \omega \|_{g_2}.$$

Together with (4.17) we have

$$Q_{h,f,g_3}(\omega) = \frac{1}{h}(\| d_{h,f} \omega \|_{g_3}^2 + \| d_{h,f,g_3} \omega \|_{g_2}^2)$$

$$\geq (1 - C_1 h^\kappa) Q_{h,f,g_2}(\omega) - C_2 h^{1-\kappa} \| \omega \|_{g_2}^2.$$ 

$\square$
Note that for the inequality
\[(4.19) \quad Q_{h,f,g_3}(\omega) \geq (1 - C_1 h^\kappa) Q_{h,f,g_3}(\omega) - C_2 h^{1-\kappa} \parallel \omega \parallel_{g_3}^2 \]
it is enough to see that
\[
\min \{e^{-\varphi(x)}\} = (\max \{e^{\varphi(x)}\})^{-1} = (1 + O(h^\kappa))^{-1} = 1 + O(h^\kappa).
\]

Now we will continue with the proof of the main result.

We combine formula (4.12) with inequalities (4.13), (4.14), (4.15), and (4.19), to conclude that there exist positive constants $C_1$ and $C_2$ such that
\[(4.20) \quad Q_2(i_2(\phi_j^{(h)}(x)\omega)) \geq (1 - C_1 h^\kappa) Q_{h,f,j,g_3}(i_1(\phi_j^{(h)}(x)\omega) \land \alpha) - C_2 h^\kappa \parallel i_1(\phi_j^{(h)}(x)\omega) \land \alpha \parallel_{g_3}^2 \]
and
\[(4.21) \quad Q_{h,f,j,g_3}(i_1(\phi_j^{(h)}(x)\omega) \land \alpha) \geq (1 - C_1 h^\kappa) Q_2(i_2(\phi_j^{(h)}(x)\omega)) - C_2 h^\kappa \parallel i_2(\phi_j^{(h)}(x)\omega) \parallel_{g_3}^2 \]
for all $\omega \in \mathcal{H}_0$.

Note that $(g_3)_{nn} = g_{nn}(0)$ and $(g_3)_{lk}$ depends only on $x'$ for $l, k = 1, \ldots, n - 1$. Therefore, in the metric $g_3$ the variables separate on the forms $\psi = i_1(\phi_j^{(h)}(x)\omega) \land \alpha$. In particular, let $f'(x) = f \mid_{\partial M}$, $g_3'$ be the restriction of the metric $g_3$ on the tangent space spanned by $\{\partial/\partial x_1, \partial/\partial x_2, \ldots, \partial/\partial x_{n-1}\}$, $f_j^{(n)} = x_n, g_3^{(n)} = g_{nn}(0)dx_n^2$. Then for any $\omega$ from the component of $\mathcal{H}_0$ corresponding to the boundary critical points we have
\[(4.22) \quad Q_{h,f,j,g_3}(i_1(\phi_j^{(h)}(x')\omega) \land \alpha) = Q_{h,f,j,g_3'}(i_1(\phi_j^{(h)}(x')\omega) \parallel \alpha \parallel_{g_3'}^2) + \parallel i_1(\phi_j^{(h)}(x')\omega) \parallel_{g_3'}^2 Q_{h,f,j,g_3}\]

We recall that from normalization we have $\parallel \alpha \parallel_{g_3}^2 = 1$. Moreover, $Q_{h,f,j,g_3}(\alpha) \leq C h^{3\kappa-1}$ for some $C > 0$. Indeed,
\[
Q_{h,f,j,g_3}(\alpha) = \frac{1}{h} \parallel d_{h,f,j,g_3}(\alpha) \parallel_{g_3}^2,
\]
where the norm and the adjoint are taken with respect to the metric $g_3$ on $\mathbb{R}^n$. Since $d_{h,f,j,g_3}(e^{\frac{\partial}{\partial x_n}}dx_n) = 0$, we have
\[
Q_{h,f,j,g_3}(\alpha) \leq C_1 h(c(h))^2 \parallel e^{\frac{\partial}{\partial x_n}}\phi_n^{(h)} \parallel_{g_3}^2,
\]
where $C_1$ is a constant independent of $h$, and $c(h)$ is normalization constant for $\phi_n^{(h)}(x_n)e^{\frac{\partial}{\partial x_n}}dx_n$ with respect to the metric $g_3$. Since the support of $\frac{\partial \phi_n^{(h)}}{\partial x_n}$ is in
Finally, $Q_{h,f_j^*,g_j^*}(\alpha) \leq P(h)e^{-k^{-1}}$

where $P(h)$ is a polynomial. Thus there exist $h_0 > 0$ small enough so that for any $h \in (0, h_0)$, $Q_{h,f_j^*,g_j^*}(\alpha) \leq Ch^{3k-1}$ for some $C > 0$. Therefore,

$$(4.23) \quad Q_{h,f_j^*,g_j^*}((i_1\phi_j^{(h)}(x) \land \alpha) \leq Q_{h,f_j^*,g_j^*}(i_1\phi_j^{(h)}(x) \land \alpha) + C\, h^{3k-1} \|i_1\phi_j^{(h)}(x)\omega\|_{g_j^*}^2,$$

and

$$(4.24) \quad Q_{h,f_j^*,g_j^*}(i_1\phi_j^{(h)}(x) \land \alpha) \leq Q_{h,f_j^*,g_j^*}((i_1\phi_j^{(h)}(x) \land \alpha) + C\, h^{3k-1} \|i_1\phi_j^{(h)}(x)\omega \land \alpha\|_{g_j^*}^2,$$

Now we will compare the quadratic forms $Q_{h,f_j^*,g_j^*}$ and $Q_j$. The quadratic forms $Q_{h,f_j^*,g_j^*}$ can be written as

$$Q_{h,f_j^*,g_j^*}(\omega) = h \sum_{l,k=1}^{n-1} \left( A_{2,1} \frac{\partial}{\partial x_l} \omega, \frac{\partial}{\partial x_k} \omega \right)_{g_j^*} + h \sum_{l=1}^{n-1} \left( A_{2,l} \frac{\partial}{\partial x_l} \omega, \omega \right)_{g_j^*} + (B_{2}\omega, \omega)_{g_j^*} + h^{-1} (V_2\omega, \omega)_{g_j^*},$$

where $A_{2,1} = (g_j^*)^{lk}(x')$, $A_{2,l}^{(1)}$ is a first order operator. Operator $B_2$ is bounded, and it can be written as

$$B_2 = \sum_{l,k=1}^{n-1} \frac{\partial^2 f_j^*}{\partial x_l \partial x_k}(x') (a^{*k}a^l - a^l a^{*k}) - \sum_{l=1}^{n-1} (\nabla f_j^*)_l(x') B_j^l(x').$$

Finally,

$$V_2 = \sum_{l,k=1}^{n-1} (g_j^*)^{kl}(x') \frac{\partial f_j^*}{\partial x_l}(x') \frac{\partial f_j^*}{\partial x_k}(x').$$

In coordinates the quadratic form $Q_1$ for the model operator $\Delta_{mod}$ (see (2.5)) on $L^2(\mathbb{R}^{n-1}, AT^*\mathbb{R}^{n-1})$ corresponding to $\tilde{y}_j$ can be written as

$$Q_1(\omega) = h(A_{2,1} \frac{\partial}{\partial x_l} \omega, \frac{\partial}{\partial x_k} \omega)_{g_j^*} + (B_1\omega, \omega)_{g_j^*} + h^{-1} (V_1\omega, \omega)_{g_j^*},$$

where $A_{2,1} = (g_j^*)^{lk}$,

$$B_1 = \sum_{l,k=1}^{n-1} \frac{\partial^2 f_j^*}{\partial x_l \partial x_k}(\tilde{y}_j) (a^{*k}a^l - a^l a^{*k}),$$

and

$$V_1 = \sum_{l,k,r,s=1}^{n-1} (g_j^*)^{rs} \frac{\partial^2 f_j^*}{\partial x_r \partial x_l}(\tilde{y}_j) \frac{\partial^2 f_j^*}{\partial x_s \partial x_k}(\tilde{y}_j) \frac{\partial^2 f_j^*}{\partial x_l \partial x_k}(\tilde{y}_j) x_l x_k.$$
Since \( g_j'(0) = g_j' \), \( f_j' \) is a Morse functions on \( \partial M \), and \( y_j \) corresponds to the origin, we have that

\[
(4.25) \quad A_{2,lk}(0) = A_{1,lk}(0), \quad B_2(0) = B_1(0), \quad V_2(0) = V_1(0) = 0, \\
\frac{\partial V_2}{\partial x^l}(0) = \frac{\partial V_1}{\partial x^l}(0) = 0, \quad l = 1, 2, \ldots, n, \\
\frac{\partial^2 V_2}{\partial x^l \partial x^k}(0) = \frac{\partial^2 V_1}{\partial x^l \partial x^k}(0), \quad l, k = 1, 2, \ldots, n.
\]

These equalities together with Lemma 2.11 \([22]\) imply the following lemma. (Lemma 2.11 is proved in \([22]\) in the case of operators, but its proof can be easily extended to include the case of quadratic forms).

Let \( \phi \in C^\infty_c(\mathbb{R}^{n-1}) \) be a function satisfying \( 0 \leq \phi \leq 1 \), \( \phi(x) = 1 \) if \( |x| \leq 1 \), \( \phi(x) = 0 \) if \( |x| \geq 2 \). Define \( \phi^{(h)}(x) = \phi(h^{-\kappa}x) \).

**Lemma 4.5.** Let \( 1/3 < \kappa < 1/2 \). There exist \( C > 0 \) and \( h_0 \) such that for any \( h \in (0, h_0) \),

\[
Q_{h,j',g_j'}(\phi^{(h)}(x) \omega) \leq (1 + Ch^\kappa)Q_j(\phi^{(h)}(x) \omega) + Ch^{3\kappa-1}(\phi^{(h)}(x) \omega, \phi^{(h)}(x) \omega)_{g_j'}
\]

for \( \omega \in C^\infty_c(\mathbb{R}^{n-1}; \Lambda T^*\mathbb{R}^{n-1}) \) such that \( \supp \omega \subset \{ x \in \mathbb{R}^{n-1} : |x| \leq Ch^\kappa \} \) for some constant \( C_0 > 0 \).

The proof of Lemma 2.11 \([22]\) is symmetric with respect to the quadratic forms \( Q_{h,j',g_j'} \) and \( Q_j \), so after dividing by \( (1 + Ch^\kappa) \), we have that

\[
Q_{h,j',g_j'}(i_1 \phi_j^{(h)}(x) \omega) \geq (1 - C_1 h^\kappa)Q_j(i_1 \phi_j^{(h)}(x) \omega) - C_2 h^{3\kappa-1} \| i_1 \phi_j^{(h)}(x) \omega \|_{g_j'}^2
\]

and

\[
Q_j(i_1 \phi_j^{(h)}(x) \omega) \geq (1 - C_1 h^\kappa)Q_{h,j',g_j'}(i_1 \phi_j^{(h)}(x) \omega) - C_2 h^{3\kappa-1} \| i_1 \phi_j^{(h)}(x) \omega \|_{g_j'}^2
\]

for any \( \omega \in D \) and for some \( C_1, C_2 > 0 \).

All the metrics we used differ from each other by multiplication by \( (1 + O(h^\kappa)) \) in \( h^\kappa \)-neighborhood of the points in \( S \). Therefore these two inequalities together with \((4.20), (4.21), (4.23), (4.24)\) imply \((4.10)\) and \((4.11)\).

The comparison of quadratic forms around the interior critical points is completely similar to the comparison of the forms \( Q_{h,j',g_j'} \) and \( Q_j \). Thus for any interior point \( x_i \in S_0 \), Lemma 2.11 \([22]\) implies that

\[
Q_{h,j,g}(i_2 \psi_i^{(h)}(x) \omega) \geq (1 - C_1 h^\kappa)Q_i(i_2 \psi_i^{(h)}(x) \omega) - C_2 h^{3\kappa-1} \| i_1 \psi_i^{(h)}(x) \omega \|_{g_i}^2
\]

and

\[
Q_i(i_1 \psi_i^{(h)}(x) \omega) \geq (1 - C_1 h^\kappa)Q_{h,j,g}(i_2 \psi_i^{(h)}(x) \omega) - C_2 h^{3\kappa-1} \| i_2 \psi_i^{(h)}(x) \omega \|_{g_i}^2
\]

for \( \omega \in D \cap \bigoplus_{i=1}^N L^2(B_i(0, r), \Lambda T^*\mathbb{R}^n|_{B_i(0, r)}) \) and for some \( C_1, C_2 > 0 \). Indeed, for interior critical point these estimates are the same as in the case of manifolds without boundary (see \([35]\)).
Therefore, we have
\[ Q_2(i_2 J \omega) \geq (1 - C_1 h^\kappa) Q_1(i_1 J \omega) - C_2 h^{3 \kappa - 1} \| i_1 J \omega \|^2_{g_j} \]
and
\[ Q_1(i_1 J \omega) \geq (1 - C_1 h^\kappa) Q_2(i_2 J \omega) - C_2 h^{3 \kappa - 1} \| i_2 J \omega \|^2_g \]
for \( \omega \in D \) and for some \( C_1, C_2 > 0 \). These inequalities imply (4.35) and (4.36).

The following lemma verifies inequality (4.4).

**Lemma 4.6.** Let \( \omega \in D(Q_2) \) then there exists \( h_0 \) such that for all \( h \in (0, h_0) \)
\[ Q_2(J_2^* \omega) \geq C h^{2 \kappa - 1} \| J_2^* \omega \|^2_g \]
for some \( C > 0 \).

**Proof.** Let \( \omega \) be a form supported on \( C(\bar{y}_j, h^\kappa) \). First we assume that \( \omega \)
restricts to a tangential form on the boundary, that is \( \omega = t \omega \). Then the boundary
integral term in (2.3) vanishes because of the boundary condition \( t \omega = 0 \). Therefore,
\[ Q_{h, f, g}(\omega) = h \left( \| d \omega \|^2_g + \| d^* \omega \|^2_g \right) + \left( (L \nabla f + L^\kappa f) \omega, \omega \right) + h^{-1} \left( |\nabla f|^2 \omega, \omega \right)_g. \]
Since \( |\nabla f| > C \) around \( \bar{y}_j \) for some positive constant \( C \), we have
\[ h^{-1} \left( |\nabla f|^2 \omega, \omega \right)_g > C h^{-1} \| \omega \|^2_g. \]
Therefore, for tangential forms we have that
\[ Q_2(J_2^* \omega) \geq C h^{2 \kappa - 1} \| J_2^* \omega \|^2_g \]
for some \( C > 0 \).

Now, let \( \omega \) be a form that restricts to a normal form on the boundary, that is
\( \omega = n \omega \). In the special local coordinates on \( C(\bar{y}_j, h^\kappa) \), consider the forms that can
be written as \( \omega' (\alpha) \wedge \alpha^\kappa(x_n) \), where \( \alpha \) is a 1-form that belongs to the \( L^2 \)-orthogonal
complement (with respect to the metric \( g_{n0}(0) dx_n^2 \)) of the one dimensional space generated by \( \alpha \)
in the space of all 1-forms supported in the interval \( (-2 h^\kappa, 0) \) in \( H^1(\mathbb{R}_-, \Lambda^1 \mathbb{R}_-) \). Since the inequalities (4.13) and (4.19) are valid for any differential
form with the support in a small neighborhood of critical points on the boundary, we have that
\[ Q_2(\omega) = Q_{h, f, g}(\omega) \geq (1 - C h^\kappa) Q_{h, f, g}(\omega) - C h^\kappa \| \omega \|^2_{g_3}. \]
Therefore,
\[ Q_2(\omega' \wedge \alpha) \geq (1 - C_1 h^\kappa) Q_{h, f, g}(\omega' \wedge \alpha) - C_2 h^\kappa \| \omega' \wedge \alpha \|^2_{g_3}. \]
After separating variables as in (4.22) and observing that \( Q_{h, f, g} \) is a positive quadratic form, we obtain
\[ Q_2(\omega' \wedge \alpha) \geq (1 - C_1 h^\kappa) \| \omega' \|^2_{g_3} Q_{h, f, g}(\omega' \wedge \alpha) - C_2 h^\kappa \| \omega' \wedge \alpha \|^2_{g_3}. \]
A simple calculation shows that the spectrum of the quadratic form \( Q_{h, f, g}(\omega) \) on \( H^1(\mathbb{R}_-, \Lambda^1 \mathbb{R}_-) \) is \( \{0\} \cup [C h^{-1}, \infty) \). The eigenspace corresponding to the 0
eigenvalue is the one dimensional space generated by the eigenform \( \exp(x_n/h) dx_n \). Since
the forms \( \alpha = C(h) \phi_n^{(h)}(x_n) \exp(x_n/h) dx_n \) (see (4.8) ) and \( C(h) \exp(x_n/h) dx_n \) are
equal for \(-h^{-\kappa} < x_n \leq 0\) and the function \(\exp(x_n/h)\) decreases exponentially fast when \(h \to 0\) and \(x_n \leq -h^{-\kappa}\), we conclude that

\[ Q_{h,f,g}(n)(\tilde{\alpha}) \geq h^{-1} \| \tilde{\alpha} \|_{g_3}^2 \]

which implies that

\[ Q_2(\omega' \land \tilde{\alpha}) \geq C h^{-1} \| \omega' \land \tilde{\alpha} \|_{g_3}^2. \]

Since any normal form supported in \(C(y_j, h^\kappa)\) which belongs to the image of \(J'_2\) can be approximated by the forms \(\omega'(x') \land \tilde{\alpha}(x_n)\) in special local coordinates and the metrics \(g\) and \(g_3\) differ from each other by \(O(h^\kappa)\), for any tangential form \(\omega\) we have

\[ Q_2(J'_2 \omega) \geq C h^{-1} \| J'_2 \omega \|_g^2 \]

for some \(C > 0\).

The inequalities (4.30) and (4.31) together with the lemma 4.4 imply the desired inequality. \(\Box\)

Now we can apply Theorem 3.1. From (4.28) and (4.29), \(\beta_l = 1 + O(h^\kappa)\) and \(\epsilon_l = O(h^{3\kappa-1})\) for \(l = 1, 2\). By the Lemma 4.6 \(\alpha_2 = O(h^{2\kappa-1})\) and by the Lemma 4.7 \(\alpha_1 = O(h^{2\kappa-1})\). The operators \(A_1 = \Delta_{mod}\) and \(A_2 = \Delta_{h,f,g}\) are elliptic operators with positive definite principal symbols, thus \(A_1\) and \(A_2\) are bounded from below so \(\lambda_{0l} = O(1)\). Moreover, we have \(\gamma_l = O(h^{1-2\kappa})\). Now assume that \((a_1, b_1)\) does not intersect with the spectrum of \(A_1\). For \(a_2, b_2\) given by formulas (3.7), (3.8) respectively, we have

\[ a_2 = a_1 + O(h^s), \quad b_2 = b_1 + O(h^s) \]

where \(s = \min\{3\kappa - 1, 1 - 2\kappa\}\). The best possible value of \(s\) is

\[ s = \max_{\kappa} \min\{3\kappa - 1, 1 - 2\kappa\} = \frac{1}{5} \]

which is attained when \(\kappa = 2/5\). By Theorem 3.1 the interval \((a_2, b_2)\) does not intersect with the spectrum of \(A_2\). Moreover, for any \(\lambda_1 \in (a_1, b_1)\) and \(\lambda_2 \in (a_2, b_2)\), \(\text{dim}(\text{Im}E_1(\lambda_1)) = \text{dim}(\text{Im}E_2(\lambda_2))\). Assume that there are \(M\) eigenvalues of the model operator \(A_1\) lower than \(R\) and let \(a_1\) be the highest eigenvalue of \(A_1\) lower than \(R\). Since \(R \notin \text{spec}(A_1)\), there exists \(h_0 > 0\) such that for all \(h \in (0, h_0)\), \(a_2 = a_1 + O(h^s)\) is less than \(R\). Then \(\text{dim}(\text{Im}E_1(R)) = \text{dim}(\text{Im}E_2(R))\) which implies \(A_2\) also has exactly \(M\) eigenvalues lower than \(R\). Since we can do this for any \(R \notin \text{spec}(A_1)\), we can conclude that the eigenvalues of \(A_2\) concentrates in the \(h^s\)-neighborhood of the eigenvalues of \(A_1\) and for any \(\lambda \in \text{spec}(A_1)\), \(A_2\) has exactly as many eigenvalues in the \(h^s\)-neighborhood of \(\lambda\) as the multiplicity of \(\lambda\). This implies Theorem 2.1.

**Remark 4.7.** There exists an isomorphism between the spectral spaces \(\text{Im}E_1(\lambda_1)\) and \(\text{Im}E_2(\lambda_1)\) which is given in the proof of Theorem 3.1 in the Appendix.
5. Eigenvalues of the Model Operator

Recall that the model operator does not depend on the choice of local coordinates, see Section 2. Thus we will choose local coordinates in which the model operator has an especially simple form.

At each critical point \( \bar{x}_i \in S_0 \), let us choose Morse coordinates \( x_1, \ldots, x_n \) for \( f \) near \( \bar{x}_i \). In these coordinates \( \bar{x}_i = 0 \), the metric at \( \bar{x}_i \) is Euclidean i.e. \( g_i = \sum_{r=1}^{n} dx^2_r \) and for some non vanishing real constants \( c_r, r = 1, \ldots, n-1 \),

\[
f(x) = f(0) + \frac{1}{2} \sum_{r=1}^{n} c_r x^2_r.
\]

Let \( f_i \) be the extension of \( f \) to \( \mathbb{R}^n \) with the same formula.

At each boundary critical point \( \bar{y}_j \in S_+ \), let us choose Morse coordinates \( x_1, \ldots, x_{n-1} \) for \( f |_{\partial M} \) near \( \bar{y}_j \). In these coordinates \( \bar{y}_j = 0 \), the metric at \( \bar{y}_j \) restricted to the tangential vectors is Euclidean i.e. \( g_j = \sum_{r=1}^{n-1} dx^2_r \) and for some non vanishing real constants \( d_r, r = 1, \ldots, n-1 \),

\[
f |_{\partial M} (x) = f(0) + \frac{1}{2} \sum_{r=1}^{n-1} d_r x^2_r.
\]

Let \( f_j \) be the extension of \( f |_{\partial M} \) to \( \mathbb{R}^{n-1} \) with the same formula.

In these coordinates the operators \( \Delta_i \) and \( \Delta_j \) can be written as

\[
\Delta_i = -h \sum_{k=1}^{n} \frac{\partial^2}{\partial x_k^2} + \sum_{k=1}^{n} c_k (a^* a^k - a^k a^* k) + h^{-1} \sum_{k=1}^{n} c_k^2 x^2_k
\]

and

\[
\Delta_j = -h \sum_{k=1}^{n-1} \frac{\partial^2}{\partial x_k^2} + \sum_{k=1}^{n-1} d_k (a^* a^k - a^k a^* k) + h^{-1} \sum_{k=1}^{n-1} d_k^2 x^2_k,
\]

where \( a^k = (dx^k)^* \) and \( a^* k = (a^k)^* \) are the fermionic creation and annihilation operators. The spectrum of the model operator \( \Delta_{mod} \) is the union of the spectra of the operators \( \oplus_{i=1}^{n_0} \Delta_i \) and \( \oplus_{j=1}^{n_1} \Delta_j \).

The spectra of the operators \( \Delta_i \) and \( \Delta_j \) are the same as in the case of manifolds without boundary (see [36]).

The spectrum of \( \Delta_i \) is

\[
(5.1) \quad \{ \sum_{l=1}^{n} (2k_l + 1)c_l + (c_{l_1} + \ldots + c_{l_k}) - (c_{l_{k+1}} + \ldots + c_{l_n}) \}
\]

where \( k_l \in \{0, 1, 2, \ldots\}, l_1 < \ldots < l_k, l_{k+1} < \ldots < l_n, \{l_1, \ldots, l_k\} \cup \{l_{k+1}, \ldots, l_n\} = \{1, \ldots, n\} \) (Corollary 2.22 in [36]).
The spectrum of $\Delta_j$ is
\begin{equation}
(\sum_{l=1}^{n-1}(2k_l+1)d_l + (d_{l_1} + \ldots + d_{l_k}) - (d_{l_{k+1}} + \ldots + d_{l_{n-1}}))
\end{equation}
where $k_l \in \{0, 1, 2, \ldots\}$, $l_{k+1} < \ldots < l_{n-1}$, $\{l_1, \ldots, l_k\} \cup \{l_{k+1}, \ldots, l_{n-1}\} = \{1, \ldots, n-1\}$
(Corollary 2.22 in [36]).

The spectrum of the model operator is the union of (5.1) and (5.2) over $i = 1, \ldots, N_0$ and $j = 1, \ldots, N$ respectively.

6. Appendix

6.1. Localization theorem for spectral projections. The goal of this Section is to prove Proposition 6.1 below, which we need for the proof of Theorem 3.1.

Let $Q$ be a closed bounded below quadratic form on a Hilbert space $\mathcal{H}$ with the domain $D(Q)$ which is assumed to be dense in $\mathcal{H}$. Let $A$ be the self-adjoint operator corresponding to the quadratic form.

Let us take $\lambda_0 \leq 0$ such that
\begin{equation}
Q(\omega) \geq \lambda_0 \|\omega\|^2, \quad \omega \in D(Q).
\end{equation}

Let $J$ be a self-adjoint bounded operator in $\mathcal{H}$ that maps the domain of $Q$ into itself, $J : D(Q) \rightarrow D(Q)$. We assume that $0 \leq J \leq \text{id}_\mathcal{H}$. Introduce a self-adjoint positive bounded operator $J'$ in $\mathcal{H}$ by the formula $J^2 + (J')^2 = \text{id}_\mathcal{H}$. We assume that $J'$ maps the domain of $Q$ into itself, the quadratic forms $Q(\omega) - (Q(J\omega) + Q(J'\omega))$ are bounded i.e. there exists $\gamma \geq 0$ such that
\begin{equation}
Q(J\omega) + Q(J'\omega) - Q(\omega) \leq \gamma \|\omega\|^2, \quad \omega \in D(Q).
\end{equation}

Finally, we assume that
\begin{equation}
Q(J'\omega) \geq \alpha \|J'\omega\|^2, \quad \omega \in D(Q),
\end{equation}
for some $\alpha > 0$.

Denote by $E(\lambda)$ the spectral projection of the operator $A$, corresponding to the semi-axis $(-\infty, \lambda]$. We have
\begin{equation}
Q(E(\lambda)\omega) \leq \lambda \|E(\lambda)\omega\|^2, \quad \omega \in \mathcal{H}.
\end{equation}

**Proposition 6.1.** If $\alpha > \lambda + \gamma$, then we have the following estimate
\begin{equation}
\|JE(\lambda)\omega\|^2 \geq \frac{\alpha - \lambda - \gamma}{\alpha - \lambda_0} \|E(\lambda)\omega\|^2, \quad \omega \in \mathcal{H}.
\end{equation}
Remark 6.2. Note that in the case \( \lambda < \lambda_0 \) the statement is trivial. In the opposite case \( \lambda \geq \lambda_0 \), since \( \alpha > \lambda + \gamma \) and \( \gamma \geq 0 \), the coefficient in the right-hand side of the formula (6.5) satisfies the estimate
\[
0 < \frac{\alpha - \lambda - \gamma}{\alpha - \lambda_0} \leq 1.
\]

Proof. (of Proposition 6.1) Combining (6.1), (6.2), (6.3) and (6.4) we get
\[
\| J' E(\lambda) \omega \|^2 \leq \frac{1}{\alpha} Q(J' E(\lambda) \omega)
\]
\[
\leq \frac{1}{\alpha} \left( Q(E(\lambda) \omega) - Q(JE(\lambda) \omega) + \gamma \| E(\lambda) \omega \|^2 \right)
\leq \frac{1}{\alpha} \left( (\lambda + \gamma) \| E(\lambda) \omega \|^2 - \lambda_0 \| JE(\lambda) \omega \|^2 \right).
\]
Hence, we have
\[
\| J E(\lambda) \omega \|^2 = \| E(\lambda) \omega \|^2 - \| J' E(\lambda) \omega \|^2 \geq \left( 1 - \frac{\lambda + \gamma}{\alpha} \right) \| E(\lambda) \omega \|^2 + \frac{\lambda_0}{\alpha} \| JE(\lambda) \omega \|^2,
\]
that immediately implies the required estimate (6.5). \( \square \)

Corollary 6.3. If \( \alpha > \lambda + \gamma \), then we have the following estimate:
\[
(6.6) \quad \| J' E(\lambda) \omega \|^2 \leq \frac{\lambda + \gamma - \lambda_0}{\alpha - \lambda_0} \| E(\lambda) \omega \|^2, \quad \omega \in \mathcal{H}.
\]

Proof. This follows immediately from the equality \( \| J \omega \|^2 + \| J' \omega \|^2 = \| \omega \|^2 \) for any \( \omega \in \mathcal{H} \). \( \square \)

Corollary 6.4. If \( \alpha > \lambda + \gamma \), then we have the following estimate
\[
(6.7) \quad Q(JE(\lambda) \omega) \leq \left( \lambda + \gamma - \lambda_0 \frac{\lambda + \gamma - \lambda_0}{\alpha - \lambda_0} \right) \| E(\lambda) \omega \|^2, \quad \omega \in \mathcal{H}.
\]

Proof. From (6.2), (6.3), (6.4) and (6.6) we get
\[
Q(JE(\lambda) \omega) \leq Q(E(\lambda) \omega) - Q(J' E(\lambda) \omega) + \gamma \| E(\lambda) \omega \|^2
\]
\[
\leq \left( (\lambda + \gamma) \| E(\lambda) \omega \|^2 - \lambda_0 \| J' E(\lambda) \omega \|^2 \right)
\leq \left( \lambda + \gamma - \lambda_0 \frac{\lambda + \gamma - \lambda_0}{\alpha - \lambda_0} \right) \| E(\lambda) \omega \|^2
\]
as desired. \( \square \)
6.2. Proof of Theorem 3.1

In this Section, we will use the notation of Section 3. We start with the following

**Proposition 6.5.** If \( \alpha_1 > \lambda_1 + \gamma_1 \) and
\[
\lambda_2 > \rho \left[ \beta_1 \left( \lambda_1 + \gamma_1 + \frac{(\lambda_1 + \gamma_1 - \lambda_01)^2}{\alpha_1 - \lambda_1 - \gamma_1} \right) + \varepsilon \right],
\]
then there exists \( \varepsilon_0 > 0 \) such that
\[
\|E_2(\lambda_2)i_2 Jp_1 E_1(\lambda_1)\omega\|_2^2 \geq \varepsilon_0 \|E_1(\lambda_1)\omega\|_{H_1}^2, \quad \omega \in H_1.
\]

**Proof.** Applying (3.5) to a function \( p_1 E_1(\lambda_1)\omega, \ \omega \in H_1 \) and taking into account that \( J_1 = i_1 Jp_1 \), we get
\[
Q_2(i_2 Jp_1 E_1(\lambda_1)\omega) \leq \beta_1 Q_1(J_1 E_1(\lambda_1)\omega) + \varepsilon_1 \|J_1 E_1(\lambda_1)\omega\|_{H_1}^2.
\]
Clearly, for any \( \lambda \) and \( l = 1, 2 \) we have the estimate
\[
Q_1((\text{id}_{H_l} - E_l(\lambda))\omega) \geq \lambda \|(\text{id}_{H_l} - E_l(\lambda))\omega\|_{H_1}^2, \quad \omega \in D(Q_l).
\]

By (6.9), (3.1) and (3.2), it follows that
\[
Q_2(i_2 Jp_1 E_1(\lambda_1)\omega)
= Q_2(E_2(\lambda_2)i_2 Jp_1 E_1(\lambda_1)\omega) + Q_2((\text{id}_{H_2} - E_2(\lambda_2))i_2 Jp_1 E_1(\lambda_1)\omega)
\geq \lambda_2 \|E_2(\lambda_2)i_2 Jp_1 E_1(\lambda_1)\omega\|_{H_2}^2 + \lambda_2 \|(\text{id}_{H_2} - E_2(\lambda_2))i_2 Jp_1 E_1(\lambda_1)\omega\|_{H_2}^2
= \lambda_2 \|i_2 Jp_1 E_1(\lambda_1)\omega\|_{H_2}^2 - (\lambda_2 - \lambda_02) \|E_2(\lambda_2)i_2 Jp_1 E_1(\lambda_1)\omega\|_{H_2}^2
\geq \lambda_2 \rho^{-1} \|J_1 E_1(\lambda_1)\omega\|_{H_1}^2 - (\lambda_2 - \lambda_02) \|E_2(\lambda_2)i_2 Jp_1 E_1(\lambda_1)\omega\|_{H_2}^2.
\]

On the other side, by (6.8), (6.7), we have
\[
Q_2(i_2 Jp_1 E_1(\lambda_1)\omega)
\leq \beta_1 \left( \lambda_1 + \gamma_1 - \lambda_01 \frac{\lambda_1 + \gamma_1 - \lambda_01}{\alpha_1 - \lambda_01} \right) \|E_1(\lambda_1)\omega\|_{H_1}^2 + \varepsilon_1 \|J_1 E_1(\lambda_1)\omega\|_{H_1}^2.
\]
Combining (6.10) and (6.11), we get
\[
\lambda_2 \rho^{-1} \|J_1 E_1(\lambda_1)\omega\|_{H_1}^2 - (\lambda_2 - \lambda_02) \|E_2(\lambda_2)i_2 Jp_1 E_1(\lambda_1)\omega\|_{H_2}^2
\leq \beta_1 \left( \lambda_1 + \gamma_1 - \lambda_01 \frac{\lambda_1 + \gamma_1 - \lambda_01}{\alpha_1 - \lambda_01} \right) \|E_1(\lambda_1)\omega\|_{H_1}^2 + \varepsilon_1 \|J_1 E_1(\lambda_1)\omega\|_{H_1}^2,
\]
that implies, due to (6.5),
\[
\|E_2(\lambda_2)i_2 Jp_1 E_1(\lambda_1)\omega\|_{H_2}^2 \geq \frac{1}{\lambda_2 - \lambda_02} \left[ (\lambda_2 \rho^{-1} - \varepsilon_1) \|J_1 E_1(\lambda_1)\omega\|_{H_1}^2 - \beta_1 \left( \lambda_1 + \gamma_1 - \lambda_01 \frac{\lambda_1 + \gamma_1 - \lambda_01}{\alpha_1 - \lambda_01} \right) \|E_1(\lambda_1)\omega\|_{H_1}^2 \right]
\geq \frac{1}{\lambda_2 - \lambda_02} \left[ (\lambda_2 \rho^{-1} - \varepsilon_1) \frac{\alpha_1 - \lambda_1 - \gamma_1}{\alpha_1 - \lambda_01} - \beta_1 \left( \lambda_1 + \gamma_1 - \lambda_01 \frac{\lambda_1 + \gamma_1 - \lambda_01}{\alpha_1 - \lambda_01} \right) \right] \|E_1(\lambda_1)\omega\|_{H_1}^2
\]
as desired. \( \square \)
Remark 6.6. Note that we only used estimate (3.5) (but not (3.6)) in the proof of Proposition 6.5. By using (3.5) we can get
\[ \| E_1(\lambda_1) i_1 J p_2 E_2(\lambda_2) \omega \|_2^2 \geq \varepsilon_0 \| E_2(\lambda_2) \omega \|_2^2, \quad \omega \in \mathcal{H}_2. \]

Proof. (of Theorem 6.1) As above, we will use the notation of Section 3. Take arbitrary \( \lambda_1 \in (a_1, b_1) \) and \( \lambda_2 \in (a_2, b_2) \).

Since
\[ \lambda_2 > a_2 = \rho \left[ \beta_1 \left( a_1 + \gamma_1 + \frac{(a_1 + \gamma_1 - \lambda_0)^2}{a_1 - a_1 - \gamma_1} \right) + \varepsilon_1 \right] \]
and (see Remark 6.3)
\[ b_1 = \rho \left[ \beta_2 \left( b_2 + \gamma_2 + \frac{(b_2 + \gamma_2 - \lambda_0^2)^2}{\alpha_2 - b_2 - \gamma_2} \right) + \varepsilon_2 \right] \]
\[ > \rho \left[ \beta_2 \left( \lambda_2 + \gamma_2 + \frac{(\lambda_2 + \gamma_2 - \lambda_0^2)^2}{\alpha_2 - \lambda_2 - \gamma_2} \right) + \varepsilon_2 \right], \]
it follows from Proposition 6.7 that the map
\[ E_2(\lambda_2) i_2 J p_1 E_1(\lambda_1) = E_2(\lambda_2) i_2 J p_1 E_1(a_1 + 0) : \text{Im} E_1(\lambda_1) \to \text{Im} E_2(\lambda_2) \]
is injective and from Remark 6.6 the map
\[ (E_2(\lambda_2) i_2 J p_1 E_1(\lambda_1))^* = E_1(\lambda_1) i_1 J p_2 E_2(\lambda_2) \]
\[ = E_1(b_1 - 0) i_1 J p_2 E_2(\lambda_2) : \text{Im} E_2(\lambda_2) \to \text{Im} E_1(\lambda_1) \]
is injective. Hence, the map \( E_2(\lambda_2) i_2 J p_1 E_1(\lambda_1) : \text{Im} E_1(\lambda_1) \to \text{Im} E_2(\lambda_2) \) is bijective.

Therefore \( \dim(\text{Im} E_2(\lambda_2)) = \dim(\text{Im} E_1(\lambda_1)) \). Since the spectrum of \( A_1 \) does not intersect with \((a_1, b_1) \), \( \dim(\text{Im} E_1(\lambda_1)) \) is constant for any \( \lambda_1 \in (a_1, b_1) \). Therefore \( \dim(\text{Im} E_2(\lambda_2)) \) is constant for any \( \lambda_2 \in (a_2, b_2) \). This implies that the interval \((a_2, b_2) \) does not intersect with the spectrum of \( A_2 \).

Remark 6.7. If the spectral projections \( E_l(\lambda) \), \( l = 1, 2 \) have finite rank for all \( \lambda \), then we do not need the condition \( i_2 J p_1 i_1 J p_2 = i_1 J p_2 \). In this case the part \( (E_2(\lambda_2) i_2 J p_1 E_1(\lambda_1))^* = E_1(\lambda_1) i_1 J p_2 E_2(\lambda_2) \) in the proof is not necessary to conclude \( \dim(\text{Im} E_2(\lambda_2)) = \dim(\text{Im} E_1(\lambda_1)) \). We can consider the maps
\[ E_2(\lambda_2) i_2 J p_1 E_1(\lambda_1) = E_2(\lambda_2) i_2 J p_1 E_1(a_1 + 0) : \text{Im} E_1(\lambda_1) \to \text{Im} E_2(\lambda_2) \]
and
\[ E_1(\lambda_1) i_1 J p_2 E_2(\lambda_2) = E_1(b_1 - 0) i_1 J p_2 E_2(\lambda_2) : \text{Im} E_2(\lambda_2) \to \text{Im} E_1(\lambda_1). \]
These maps are injective by Proposition 6.8 and Remark 6.6. Therefore, \( \dim(\text{Im} E_2(\lambda_2)) = \dim(\text{Im} E_1(\lambda_1)) \) because \( \text{Im} E_1(\lambda_1) \) and \( \text{Im} E_2(\lambda_2) \) are finite dimensional.

Denote the spectral projection of the operator \( A_l \) on the interval \((a, b) \) as \( E_l(a, b) \) for \( l = 1, 2 \).
Corollary 6.8. For any \( \lambda \in \text{spec}(A_1) \), there is a positive number \( \epsilon \) which is of order \( h^\kappa \) such that the spaces \( E_1((\lambda - \epsilon, \lambda + \epsilon)) \) and \( E_2((\lambda - \epsilon, \lambda + \epsilon)) \) are isomorphic.

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