ON COHEN-MACAULAYNESS AND DEPTH OF IDEALS IN INVARIANT RINGS

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Abstract. We investigate the presence of Cohen-Macaulay ideals in invariant rings and show that an ideal of an invariant ring corresponding to a modular representation of a $p$-group is not Cohen-Macaulay unless the invariant ring itself is. As an intermediate result, we obtain that non-Cohen-Macaulay factorial rings cannot contain Cohen-Macaulay ideals. For modular cyclic groups of prime order, we show that the quotient of the invariant ring modulo the transfer ideal is always Cohen-Macaulay, extending a result of Fleischmann.

1. Introduction

The depth and Cohen-Macaulay property of invariant rings have always been among the major interests of invariant theorists, see the references below. In this paper, we consider ideals of invariant rings (as modules over the latter), and investigate their depth and Cohen-Macaulayness. The original goal of this paper was to find filtrations of the invariant rings with Cohen-Macaulay quotients (a weakening of being “sequentially Cohen-Macaulay” as introduced in [15, section III.2]). However, the results of this paper show that in many cases, invariant rings fail to contain any Cohen-Macaulay ideal, so the goal is missed in the first step already.

Before we go into more details, we fix our setup. Let $V$ be a finite dimensional representation of a group $G$ over a field $K$. The representation is called modular if the characteristic of $K$ divides the order of $G$. Otherwise, it is called non-modular.

There is an induced action on the symmetric algebra $K[V] := S(V^*)$ given by $\sigma(f) = f \circ \sigma^{-1}$ for $\sigma \in G$ and $f \in K[V]$. We let $K[V]^G := \{ f \in K[V] \mid \sigma(f) = f \text{ for all } \sigma \in G \}$ denote the subalgebra of invariant polynomials in $K[V]$. For any non-modular representation, $K[V]^G$ is a Cohen-Macaulay ring [11]. In the modular case, on the other hand, $K[V]^G$ almost always fails to be Cohen-Macaulay, see [12]. The depth of $K[V]^G$ has attracted much attention and has been determined for various families of representations, see for example [3,8,10,13,17]. In this paper we consider ideals of $K[V]^G$ as modules over $K[V]^G$. We show that, if $V$ is a modular representation of a $p$-group, then $K[V]^G$ does not contain a Cohen-Macaulay ideal unless $K[V]^G$ is Cohen-Macaulay itself. In fact, our results hold in a broader generality. We first show that a Cohen-Macaulay ideal in an affine domain can not have height bigger than one. If, in addition, the affine domain is factorial, then only principal ideals can be Cohen-Macaulay. So we get the desired implication for the groups and their representations whose invariants are factorial. We also include an example that
shows that the condition that the affine domain is factorial can not be dropped. We then restrict to modular representations of a cyclic group of prime order. Our main result here is that the quotient $K[V]^G/I^G$ of the invariant ring modulo the transfer ideal $I^G$ is Cohen-Macaulay. Note that this extends results of Fleischmann [9] in this case, namely that $K[V]^G/\sqrt{I^G}$ is Cohen-Macaulay, and that $\sqrt{I^G} = I^G$ if $V$ is projective. This also allows us to compute the depth of the transfer ideal. We end with a reduction result that reduces computing the depth of a $K[V]^G$-module to computing a grade of the transfer ideal.

We refer the reader to [1,4,6] for more background in modular invariant theory.

2. Preliminaries

In this section we summarize our notation as well as some basic results that we use in our computations. Let $R = \bigoplus_{d=0}^{\infty} R_d$ be a graded affine $K$-algebra such that $R_0 = K$, and $M = \bigoplus_{d=0}^{\infty} M_d$ a finitely generated graded nonzero $R$-module. We call $R_+ := \bigoplus_{d=1}^{\infty} R_d$ the maximal homogeneous ideal of $R$. A sequence of homogeneous elements $a_1, \ldots, a_k \in R_+$ is called $M$-regular if each $a_i$ is a nonzero divisor on $M/(a_1, \ldots, a_{i-1})M$ for $i = 1, \ldots, k$. For a homogeneous ideal $I \subseteq R_+$, the maximal length of an $M$-regular sequence lying in $I$ is called the grade of $I$ on $M$, denoted by $\text{grade}(I, M)$. Furthermore, one calls $\text{depth}(M) := \text{grade}(R_+, M)$ the depth of $M$. Recall that we have $\text{depth}(M) \leq \dim(M)$ (where $\dim(M) := \dim(R/\text{Ann}_R(M))$ denotes the Krull dimension), and $M$ is called Cohen-Macaulay if equality holds.

By Noether-Normalization, $R$ contains a homogeneous system of parameters (hsop), i.e., algebraically independent homogeneous elements $a_1, \ldots, a_n \in R$ such that $R$ is finitely generated as a module over the (polynomial) subalgebra $A := K[a_1, \ldots, a_n]$. Note that $n = \dim(R)$ is uniquely determined. Any subset of an hsop is called a partial hsop (phsop). If $R$ is also a domain, then homogeneous elements $a_1, \ldots, a_k \in R_+$ form a phsop if and only if $\text{height}(a_1, \ldots, a_k) = k$, (see [12, Lemma 1.5], [2, Theorem A. 16]). Note that $M$ is also an $A$-module, and from the graded Auslander-Buchsbaum formula [6, Exercise 19.8] we get that $M$ is free as an $A$-module if and only if its depth as an $A$-module is equal to $\dim(A)$. But since the depths of $M$ as an $A$- and as an $R$-module are equal (see [5, Lemma 3.7.2] or [2, Exercise 1.2.26]), this is also equivalent to the condition that the depth of $M$ as an $R$-module is $\text{dim}(R) = \text{dim}(A)$. In other words: $M$ is free as an $A$-module if and only if $M$ is Cohen-Macaulay and $\text{dim}(M) = \text{dim}(R)$, i.e., $M$ is maximal Cohen-Macaulay.

We include the following standard facts about depth for the reader’s convenience.

Lemma 1. [2, Proposition 1.2.9] Assume that $I$ is a homogeneous nonzero proper ideal of the graded affine ring $R$. Then we have the following inequalities.

(a) $\text{depth}(R) \geq \min\{\text{depth}(I), \text{depth}(R/I)\}$.

(b) $\text{depth}(I) \geq \min\{\text{depth}(R), \text{depth}(R/I) + 1\}$.

(c) $\text{depth}(R/I) \geq \min\{\text{depth}(I) - 1, \text{depth}(R)\}$.

This lemma implies that $\text{depth}(I)$ and $\text{depth}(R/I)$ are often strongly related:

Lemma 2. Assume that $I$ is a homogeneous nonzero proper ideal of the graded affine ring $R$.

(a) If one of the following conditions

(i) $\text{depth}(R) > \text{depth}(I)$,
(ii) $\text{depth}(R) > \text{depth}(R/I)$,

(iii) $R$ is a Cohen-Macaulay domain

holds, then

$$\text{depth}(I) = \text{depth}(R/I) + 1.$$  

(b) If $\text{depth}(I) > \text{depth}(R)$, then $\text{depth}(R/I) = \text{depth}(R)$. 

(c) If $\text{depth}(R/I) > \text{depth}(R)$, then $\text{depth}(I) = \text{depth}(R)$. 

Proof. (a) Assume first $\text{depth}(R) > \text{depth}(I)$. From Lemma 1 (b) it follows that $\text{depth}(I) \geq \text{depth}(R/I) + 1$, and from Lemma 1 (c) we get $\text{depth}(R/I) \geq \text{depth}(I) - 1$, implying the desired equality. Secondly, assume $\text{depth}(R) > \text{depth}(R/I)$. From Lemma 1 (b) it follows that $\text{depth}(I) \geq \text{depth}(R/I) + 1$, and from Lemma 1 (c) we have $\text{depth}(R/I) \geq \text{depth}(I) - 1$ so we obtain the result again. Finally assume $R$ is a Cohen-Macaulay domain. As $R$ is a domain and $I \neq \{0\}$, it follows that $\dim(R/I) < \dim(R)$. Hence we have the inequality $\text{depth}(R/I) \leq \dim(R/I) < \dim(R) = \text{depth}(R)$, so the assertion follows from (ii).

Statement (b) follows similarly from Lemma 1 (a) and (c). Statement (c) follows from Lemma 1 (a) and (b). ∎

For example, if $R$ has positive depth, then the homogeneous maximal ideal always has depth one by the above lemma, as its quotient is zero-dimensional. Now we note that for any given number $1 \leq k \leq \text{depth}(R)$, there exists an ideal of depth $k$:

**Lemma 3.** Assume that the homogeneous elements $a_1, \ldots, a_k$ of positive degree form a regular sequence of $R$. Then

$$\text{depth}((a_1, \ldots, a_k)R) = \text{depth}(R) + 1 - k.$$  

Proof. We have $\text{depth}(R/(a_1, \ldots, a_k)R) = \text{depth}(R) - k < \text{depth}(R)$, and the result follows from the previous lemma. ∎

3. Cohen-Macaulay ideals in affine domains

The main result of this section is Theorem 5 where it is shown that only principal ideals can be Cohen-Macaulay in factorial affine domains. Nevertheless even when the affine domain is not factorial, the height of a Cohen-Macaulay ideal can be at most one.

**Lemma 4.** Assume that $R$ is a graded affine domain, and $I \neq R$ a homogeneous ideal of height at least 2. Then $I$ is not Cohen-Macaulay as an $R$-module.

Proof. As $I$ is homogeneous of height at least two, it contains a phsop $p, q$ of $R$. We extend this phsop to an hsp of $R$ and consider the $K$-subalgebra $A$ of $R$ generated by this hsp, i.e., $A = K[\text{hsp}] = K[p, q, \ldots]$. Then $A$ is isomorphic to a polynomial ring over $K$ in $\dim(R)$ variables. Assume by way of contradiction that $I$ is Cohen-Macaulay as an $R$-module. Then $I$ is free as an $A$-module, i.e., there exist elements $g_1, \ldots, g_m \in I$ such that $I = \bigoplus_{i=1}^{m} A g_i$. As $p, q$ are elements of $I$, we can find unique elements $a_i, b_i \in A$ for $i = 1, \ldots, m$ such that $p = \sum_{i=1}^{m} a_i g_i$ and $q = \sum_{i=1}^{m} b_i g_i$. Multiplying both equations with $p$ and $q$ respectively, we get $\sum_{i=1}^{m} (qa_i) g_i = pq = \sum_{i=1}^{m} (pb_i) g_i$. As $qa_i, pb_i \in A$ and $g_1, \ldots, g_m$ is a free $A$-basis of $I$, we get $qa_i = pb_i$ for all $i = 1, \ldots, m$. As $p, q$ are different variables in the polynomial ring $A$, it follows $p|a_i$ and $q|b_i$ for all $i$. Therefore there exist $b'_i \in A$ such that $b_i = q b'_i$ for $i = 1, \ldots, m$. Hence we get $q = \sum_{i=1}^{m} b'_i$, and
as we are in a domain dividing by \( q \) yields 1 = \( \sum_{i=1}^{m} b'_i g_i \in \bigoplus_{i=1}^{m} A g_i = I \). This implies \( I = R \), contradicting the hypothesis \( I \neq R \) of the lemma. \( \square \)

**Theorem 5.** Assume that \( R \) is a graded factorial affine domain. If \( I \neq R \) is a homogeneous ideal which is not principal, then \( I \) is not Cohen-Macaulay as an \( R \)-module. Therefore, if \( R \) is not Cohen-Macaulay, then \( R \) does not contain any nonzero homogeneous Cohen-Macaulay ideal (as an \( R \)-module).

**Proof.** Assume by way of contradiction that \( I \) is Cohen-Macaulay and not principal. Let \( a_1, \ldots , a_n \) denote a finite set of generators of \( I \). As we are in a factorial ring, we can consider the greatest common divisor \( d \) of those elements. Then we have \( I = (a_1, \ldots , a_n) \subsetneq (d) \), where the inclusion is strict as \( I \) is not a principal ideal by assumption. As \( R \) is a domain and \( d \mid a_i \) for all \( i \), the elements \( \frac{a_i}{d} \in R \) are well defined, and we can consider the ideal \( J := (\frac{a_1}{d}, \ldots , \frac{a_n}{d}) = \frac{1}{d} I \). Note that from \( I \subsetneq (d) \) it follows that \( J \subsetneq (1) = R \), so \( J \) is a proper ideal of \( R \). Multiplication by \( d \) yields an \( R \)-module isomorphism from \( J \) to \( I \), and therefore \( J \) is also Cohen-Macaulay as an \( R \)-module. From Lemma 4 it follows that the height of \( J \) is at most 1. But \( R \) is a domain and \( J \neq 0 \), so the height of \( J \) is 1. It follows that there exists a prime ideal \( \wp \) of \( R \) of height one such that \( I \subsetneq \wp \). As \( R \) is factorial, height one primes are principal, and so \( \wp \) is generated by a prime element \( p \), so we have \( J \subsetneq \wp = (p) \), which implies that \( p \) is a common divisor of \( \frac{a_1}{d}, \ldots , \frac{a_n}{d} \). This is a contradiction, as \( d \) is the greatest common divisor of \( a_1, \ldots , a_n \).

Now the second assertion of the theorem follows from the first and the fact that principal ideals of \( R \) are isomorphic to \( R \) as \( R \)-modules. \( \square \)

We demonstrate two examples of affine domains with non-principal Cohen-Macaulay ideals. First one is a Cohen-Macaulay ring, the second one is not. Therefore, a non-Cohen-Macaulay ring may contain a Cohen-Macaulay ideal, and the hypothesis of \( R \) being factorial can not be dropped out in the previous theorem.

**Example 6.** Consider the subalgebra \( R = K[x^2, y^2, xy] \) of the polynomial ring \( K[x, y] \) in two variables. Note that \( R \) is not factorial as the equality \( x^2 \cdot y^2 = (xy) \cdot (xy) \) shows. We claim that the ideal \( I = (x^2, xy) \) of \( R \) is Cohen-Macaulay and not principal. Clearly \( I \) is not principal, because \( R \) is a graded ring that starts in degree 2. We now consider the lhp \( x^2, y^2 \) of \( R \) and the subalgebra \( A = K[x^2, y^2] \) generated by the lhp. We claim that we have the direct sum decompositions \( R = A \oplus Ay \) and \( I = Ax^2 \oplus Ay \). In both cases, the directness of the sum follows as in the first summands all \( x \) degrees are even, while in the second summands all \( x \) degrees are odd. As both sums contain the respective ideal generators, it only remains to show that both sums are invariant under multiplication with \( xy \). For the sum for \( R \), this follows from \( xy \cdot xy = x^2y^2 \in A \). For the sum for \( I \), we have \( xy \cdot x^2 = x^3 \cdot xy \in Ay \) and \( xy \cdot xy = y^2 \cdot x^2 \in Ax^2 \). Therefore \( R \) and \( I \) are free \( A \)-modules, so \( R \) and \( I \) are Cohen-Macaulay as \( R \)-modules. Also note that \( I \subsetneq \sqrt{(x^2)} \), as \( (xy)^2 = y^2 \cdot x^2 \in (x^2) \). Thus, \( \text{height}(I) \leq \text{height}((x^2)) = 1 \), as predicted by Lemma 4.

We state the following example as a proposition.

**Proposition 7.** Consider the subalgebra \( R := K[x^4, x^3y, xy^3, y^4] \) of the polynomial ring \( K[x, y] \). Then the ideal \( I := (x^4, x^3y) \) of \( R \) is of height one and Cohen-Macaulay as an \( R \)-module. (While \( R \) is not Cohen-Macaulay and not factorial).
As $\epsilon$ and claim that consider the subalgebra $A$.
We set $I$ that
$2.5.4$, and as $x^4 \cdot y^4 = (x^2y) \cdot (xy^3)$, $R$ is also not factorial. Also since $(x^3y)^4 = y^4x^8 \cdot x^4 \in (x^4)$ we have $I \subseteq \sqrt{(x^4)}$, which shows that the height of $I$ is $1$. We consider the subalgebra $A := K[x^4, y^4]$ of $R$ generated by an hsop, and we will show that $I$ is a free $A$-module, which implies that $I$ is Cohen-Macaulay as an $R$-module. We set
\[ a := x^4, \quad p := x^3y, \quad q := xy^3, \quad b := y^4 \]
and claim that
\[ I = (a, p) = Aa \oplus Aaq \oplus Ap \oplus Ap^2. \]
The inclusion “$\supseteq$” is clear. We first show that the sum on the right hand side is indeed direct. Let $\epsilon$ denote the map from the set of monomials of $K[x, y]$ to $\mathbb{N}^6_0$ given by $\epsilon(x^i y^j) := (i, j)$. We compute the epsilon values of the $A$-module generators modulo $4$:
\[ \epsilon(a) = (4, 0), \quad \epsilon(aq) = (5, 3) \equiv (1, 3), \quad \epsilon(p) = (3, 1), \quad \epsilon(p^2) = (6, 2) \equiv (2, 2). \]
As $\epsilon(m) \equiv (0, 0)$ for any monomial $m$ in $A$, it follows that the $\epsilon$-values of monomials in $Aa, Aaq, Ap, Ap^2$ fall into different congruence classes modulo $4$, so the sum is indeed direct.
We now verify the inclusion “$\subseteq$”. Clearly, $a, p \in S := Aa \oplus Aaq \oplus Ap \oplus Ap^2$, and $S$ is closed under multiplication with $a$ and $b$. It remains to show that $S$ is closed under multiplication with $p$ and $q$, which follows from
\[
\begin{align*}
p(Aa) &= Aa \subseteq Ap, & q(Aa) &= Aaq, \\
p(Aaq) &= Aax^4y^4 \subseteq Aa, & q(Aaq) &= A^6x^6y^6 = Abp^2 \subseteq Ap^2, \\
p(Ap) &= Ap^2, & q(Ap) &= Apq = A^4x^4y^4 \subseteq Aa, \\
p(Ap^2) &= A^6x^5y^3 = A^2aq \subseteq Aaq, & q(Ap^2) &= A^2x^7y^5 = A^2x^4y^4p \subseteq Ap.
\end{align*}
\]
\[
\square
\]
Remark 8. We thank Roger Wiegand from whom we learned that there are theorems that say that, for some special classes of rings, non-free maximal Cohen-Macaulay modules have high ranks. Since the rank of an ideal in a domain is one, and non-principal ideals are non-free, Lemma [4] and Theorem [5] readily follows for such rings whose non-free maximal Cohen-Macaulay modules are known to have a high rank. But we can not expect that a non-free maximal Cohen-Macaulay module will always have rank $> 1$ as the previous two examples demonstrate.

4. DEPTH OF IDEALS AND QUOTIENT OF THE TRANSFER IN INVARIANT RINGS

We start with an application of Theorem [5] to modular invariant rings.

Theorem 9. Assume that $K$ is of positive characteristic $p$ and $G$ is a finite $p$-group. For any finite dimensional linear representation $V$ of $G$ over $K$ such that the invariant ring $K[V]^G$ is not Cohen-Macaulay, no nonzero homogeneous ideal of $K[V]^G$ is Cohen-Macaulay (as a $K[V]^G$-module).

Proof. As a $p$-group, $G$ does not admit any non-trivial group homomorphism $\varphi : G \to K^*$, as for every $\sigma \in G$ we have $(\varphi(\sigma))^{|G|} = 1 = 0$, which implies $\varphi(\sigma) = 1$. Therefore $K[V]^G$ is factorial by a theorem of Nakajima (see [13, Theorem 2.11], [11, Corollary 3.9.3]). The claim now follows from Theorem [5]. \[
\square
\]
For the rest of the paper we specialize to a cyclic group $G$ of prime order $p$ equal to the characteristic of the field $K$, which we assume to be algebraically closed. Fix a generator $\sigma$ of $G$. There are exactly $p$ indecomposable $G$-modules $V_1, \ldots, V_p$ over $K$ and each indecomposable module $V_i$ is afforded by a Jordan block of dimension $i$ with 1’s on the diagonal. Let $V$ be an arbitrary $G$-module over $K$. Assume that $V$ has $l$ summands and so we can write $V = \sum_{1 \leq j \leq l} V_{n_j}$. Notice that $l = \dim V^G$. We also assume that none of these summands is trivial, i.e., $n_j > 1$ for $1 \leq j \leq l$. We set $K[V] = K[x_{i,j} \mid 1 \leq i \leq n_j, 1 \leq j \leq l]$ and the action of $\sigma$ is given by $\sigma(x_{i,j}) = x_{i,j} + x_{i-1,j}$ for $1 < i \leq n_j$ and $\sigma(x_{1,j}) = x_{1,j}$. For $f \in K[V]$ let $N(f) = \prod_{x \in G} \sigma(f)$ denote the norm of $f$. Notice that for $1 \leq i \leq n_j$, $N(x_{i,j})$ is monic of degree $p$ as a polynomial in $x_{i,j}$. By a famous theorem of Ellingsrud and Skjelbred [7], \[ \text{depth}(K[V]^G) = \min\{ \dim_k (V^G) + 2, \dim_k (V) \} = \min\{ l + 2, \dim_k (V) \}. \] In [8], this result is extended to some other classes of groups, and the proof is also made more elementary and explicit. Restricting the results of [8] to our case, we get the following description of a maximal $K[V]^G$-regular sequence, which allows to explicitly construct an ideal of a given depth at most that of the invariant ring.

**Proposition 10.** A maximal $K[V]^G$-regular sequence is given by

\[
\begin{align*}
x_{1,1}, x_{1,2}, N(x_{n_1,1}), \ldots, N(x_{n_1,l}) & \quad \text{if } l > 1; \\
x_{1,1}, N(x_{2,1}), N(x_{n_1,1}) & \quad \text{if } l = 1, n_1 > 2; \\
x_{1,1}, N(x_{2,1}) & \quad \text{if } l = 1, n_1 = 2.
\end{align*}
\]

Let $I_k$ denote the ideal of $K[V]^G$ generated by the first $k$ elements of the sequence. Then we have $\text{depth} I_k = \text{depth}(K[V]^G) + 1 - k$ for $1 \leq k \leq \text{depth}(K[V]^G)$.

**Proof.** Let $b$ denote the second element of the sequence, i.e., $x_{1,2}$ or $N(x_{2,1}) = x_{2,1}^p - x_{2,1}x_{1,1}^{-1}$ if $x_{1,1}$ and $b$ are coprime in $K[V]$, and both are invariant, they form a regular sequence in $K[V]^G$. Proceeding by induction, we assume that the elements $x_{1,1}, b, N(x_{n_1,1}), \ldots, N(x_{n_{k-1},k-1})$ form a regular sequence for some $k < l$. Consider the standard basis vector $e_{n_k, k} \in V$ corresponding to the variable $x_{n_k,k}$. Then $e_{n_k, k}$ is a fixed point, and $U := K e_{n_k, k}$ is a 1-dimensional submodule of $V$. Since no element of the regular sequence $x_{1,1}, b, N(x_{n_1,1}), \ldots, N(x_{n_{k-1},k-1})$ contains the variable $x_{n_k,k}$, [8 Corollary 17] applies to $U$ and $x_{n_k,k}$, so the regular sequence can be extended by the element $N(x_{n_k,k})$. Since the length of the given sequence equals $\text{depth}(K[V]^G)$ in each case, we are done. The final statement now follows from Lemma [8].

The transfer ideal $I^G$ is defined as the image of the transfer map $\text{Tr} : K[V] \to K[V]^G$ given by $\text{Tr}(f) = \sum_{i=0}^{p-1} \sigma^i(f)$. The transfer ideal often plays an important role in computing the invariant ring and its various aspects have been subject to research. The vanishing set of $I^G$ equals the fixed point space $V^G$ (see [4, Theorem 9.0.10]), in particular we have $\dim(K[V]^G/I^G) = \dim(V^G) = l$. We will show that $K[V]^G/I^G$ is Cohen-Macaulay, which also allows us to compute the depth of the transfer ideal. To do this we prove that $N(x_{n_1,1}), \ldots, N(x_{n_1,l})$ is a $K[V]^G/I^G$-regular sequence. Let $f \in K[V]$ and $1 \leq j_1 < j_2 < \cdots < j_r \leq l$ be arbitrary. Since $N(x_{n_{j_1},j_1})$ is a monic polynomial of degree $p$ in $x_{n_{j_1},j_1}$, we can write $f = q_1 N(x_{n_{j_1},j_1}) + r_1$, where $\deg x_{n_{j_1}, j_1} > 1 < p$. Next we divide $r_1$ by $N(x_{n_{j_2},j_2})$ and we get a decomposition $f = q_1 N(x_{n_{j_1},j_1}) + q_2 N(x_{n_{j_2},j_2}) + r_2$, where...
deg_{x_{n_1,j_1}} r_2, \deg_{x_{n_2,j_2}} r_2 < p \text{ and } \deg_{x_{n_1,j_1}} q_2 < p. \text{ In this way we get a decomposition}
\[ f = q_1 N(x_{n_1,j_1}) + \cdots + q_t N(x_{n_t,j_t}) + r, \]
where \( \deg_{x_{n_1,j_1}} r < p \) for \( 1 \leq i \leq t \) and \( \deg_{x_{n_1,j_1}} q_i < p \) for \( i < i' \). This is called the norm decomposition and \( r \) is called the remainder of \( f \) with respect to \( N(x_{n_1,j_1}), \ldots, N(x_{n_t,j_t}) \). Notice that \( r \) is unique. If \( f \in K[V]^G \) is an invariant, then the quotients \( q_i \) for \( 1 \leq i \leq t \) and the remainder \( r \) are also invariant, see [10] Proposition 2.1.

**Theorem 11.** The algebra \( K[V]^G/I^G \) is Cohen-Macaulay, and an hsop is given by the set \( \{ N(x_{n,j}) + I^G \mid 1 \leq j \leq l \} \). In particular, we have \( \operatorname{depth}(I^G) = l + 1 \).

**Proof.** Let \( f := f + I^G \) for \( f \in K[V]^G \). We show that \( N(x_{n_1,1}), \ldots, N(x_{n_l,1}) \) forms a regular sequence for \( K[V]^G/I^G \). As its length \( l \) equals the dimension of \( K[V]^G/I^G \), it follows that this ring is Cohen-Macaulay. First we show that \( N(x_{n_i,i}) \) is a \( K[V]^G/I^G \)-regular element for \( 1 \leq i \leq l \). Assume \( f N(x_{n_i,i}) \in I^G \) for some invariant \( f \). Then \( f N(x_{n_i,i}) = \operatorname{Tr}(g) \) for some \( g \in K[V] \). Consider the norm decomposition \( g = q N(x_{n_i,i}) + r \) with respect to \( N(x_{n_i,i}) \). So \( f N(x_{n_i,i}) = \operatorname{Tr}(q N(x_{n_i,i}) + r) = \operatorname{Tr}(q) N(x_{n_i,i}) + \operatorname{Tr}(r) \), hence \( 0 = (f - \operatorname{Tr}(q)) N(x_{n_i,i}) + \operatorname{Tr}(r) \). Note that the group action preserves the \( x_{n_i,i} \)-degree, so we have \( \deg_{x_{n_i,i}}(\operatorname{Tr}(r)) \leq \deg_{x_{n_i,i}}(N(x_{n_i,i})) \). So we get that \( f - \operatorname{Tr}(q) = 0 \) and \( \operatorname{Tr}(r) = 0 \).

Therefore \( f \in I^G \), and \( \overline{N(x_{n_i,i})} \) is a \( K[V]^G/I^G \)-regular element. Assume now by induction that \( \overline{N(x_{n_1,1}), \ldots, N(x_{n_{j-1},1})} \) is a \( K[V]^G/I^G \)-regular sequence, and we have
\begin{equation}
(1) \quad f N(x_{n,j}) = f_1 N(x_{n_1,1}) + \cdots + f_{j-1} N(x_{n_{j-1},1}) + \operatorname{Tr}(t),
\end{equation}
where \( f, f_i \in K[V]^G \) for \( 1 \leq i \leq j - 1 \) and \( t \in K[V] \). Consider the norm decompositions of \( f \) and \( t \) with respect to \( N(x_{n_1,1}), \ldots, N(x_{n_{j-1},1}) \). Since the quotient and the remainder in the decomposition of \( f \) are invariants, we can replace \( f \) by its remainder. As for \( \operatorname{Tr}(t) \), notice that \( \operatorname{Tr}(t) \) and the transfer of the remainder of \( t \) differ by a combination of \( N(x_{n_1,1}), \ldots, N(x_{n_{j-1},1}) \) by invariants. So we can replace \( \operatorname{Tr}(t) \) with the transfer of the remainder of \( t \). Moreover, by considering the norm decomposition of \( f_i \) with respect to \( N(x_{n_1,1}), \ldots, N(x_{n_{j-1},1}) \) for \( 1 \leq i \leq j - 1 \), we replace \( f_i \) with its corresponding remainder. Therefore we may assume that \( \deg_{x_{n_{i',i}}'} f_i < p \) for \( 1 \leq i' < i \) and \( 1 \leq i \leq j - 1 \). Notice also that the degree of \( f \) and \( \operatorname{Tr}(t) \) with respect to any variable \( x_{n_i,i'} \) is \( < p \) for \( 1 \leq i' \leq j \). Now considering Equation (1) as a polynomial equation in the variable \( x_{n_{i,1}} \) gives that \( f_1 = 0 \). Then comparing the coefficients of \( x_{n_2,2} \) gives \( f_2 = 0 \).

Along the same way we get \( f_1 = f_2 = \cdots = f_{j-1} = 0 \). So Equation (1) becomes
\[ f N(x_{n,j}) = \operatorname{Tr}(t). \]
But since \( \overline{N(x_{n_{j,1}})} \) is a \( K[V]^G/I^G \)-regular element, we have \( f \in I^G \) as desired. This shows that \( \overline{N(x_{n_1,1}), \ldots, N(x_{n_l,1})} \) is a regular sequence. From \( \operatorname{depth}(K[V]^G/I^G) = l \leq \operatorname{depth}(K[V]^G) = \min\{l + 2, \dim_K(V)\} \) (we assume a non-trivial action) and Lemma 2 it now follows that \( \operatorname{depth}(I^G) = l + 1 \). \( \Box \)

We also prove a reduction result for the depth of a module over the invariant ring, which is based on the following lemma. The statement is probably folklore, but for the convenience of the reader and the lack of a reference, we provide a proof.
Lemma 12. Assume that $R$ is a graded affine ring and $M$ is a finitely generated graded nonzero $R$-module. If $h_1, \ldots, h_r \in R_+$ form a homogeneous $M$-regular sequence and $I$ is a homogeneous ideal of $R$ such that $\sqrt{I} + (h_1, \ldots, h_r)R = R_+$, then

$$\text{depth}(M) = \text{grade}(I, M/(h_1, \ldots, h_r)M) + r.$$ 

Proof. As the homogeneous elements $h_1, \ldots, h_r \in R_+$ form an $M$-regular sequence, we have that $\text{depth}(M) = \text{depth}(M/(h_1, \ldots, h_r)M) + r$. We show that

$$\text{grade}(I, M/(h_1, \ldots, h_r)M) \geq \text{grade}(R_+, M/(h_1, \ldots, h_r)M),$$

as the reverse inequality is obvious. Let $f_1, \ldots, f_d \in R_+$ be a maximal homogeneous $M/(h_1, \ldots, h_r)M$-regular sequence. Since taking powers does not hurt regularity, we assume that all elements in the sequence are contained in $I + (h_1, \ldots, h_r)R$. Therefore for $1 \leq i \leq d$ we can write $f_i = g_i + b_i$ with homogeneous elements $g_i \in I$ and $b_i \in (h_1, \ldots, h_r)R$. Since $b_i$ is in the annihilator of $M/(h_1, \ldots, h_r, f_1, \ldots, f_{i-1})M = M/(h_1, \ldots, h_r, g_1, \ldots, g_{i-1})M$ it follows that $g_i$ is regular on $M/(h_1, \ldots, h_r, g_1, \ldots, g_{i-1})M$ as well. Hence the elements $g_1, \ldots, g_d$ of $I$ form an $M/(h_1, \ldots, h_r)M$-regular sequence. \hfill \Box

We recall that for any ideal $I$ of the invariant ring $K[V]^G$, we have $\sqrt{I} = \sqrt{IK[V] \cap K[V]^G}$. This holds generally when $G$ is a reductive group [15, Lemma 3.4.2], and an elementary proof for finite groups can be found in [4, Lemma 12.1.1].

Proposition 13. Let $M$ be a finitely generated graded $K[V]^G$-module on which the norms $N(x_{n_1,1}), \ldots, N(x_{n_1,l})$ form an $M$-regular sequence. Then

$$\text{depth}(M) = \text{grade}(I^G, M/(N(x_{n_1,1}), \ldots, N(x_{n_1,l}))M) + l.$$ 

Proof. We have already mentioned that the zero set of $I^G$ is given by $V^G = \bigoplus_{i=1}^l Ke_{n_1,i}$. As for an element $v = \sum_{i=1}^l \lambda_i e_{n_1,i} \in V^G$, we have $N(x_{n_1,i})(v) = \lambda_i^p$, the common zero set of $I^G + (N(x_{n_1,1}), \ldots, N(x_{n_1,l}))$ is zero, hence by the Nullstellensatz $\sqrt{I^G + (N(x_{n_1,1}), \ldots, N(x_{n_1,l}))(K[V] = K[V]^G}$. From the paragraph before the proposition we obtain that the radical ideal of $I^G + (N(x_{n_1,1}), \ldots, N(x_{n_1,l}))$ equals $K[V]^G$, and the lemma above applies. \hfill \Box

Examples where the proposition applies include the case $l = 1$ and $M = I$ a nonzero homogenous ideal of $K[V]^G$. The corollary also applies for arbitrary $l$ and $M = K[V]^G$ by Proposition 10. In the “non-trivial” cases where $\text{depth}(K[V]^G) = l + 2$, it follows from $\text{depth}(K[V]^G) = \text{grade}(I^G, K[V]^G/(N(x_{n_1,1}), \ldots, N(x_{n_1,l}))) + l$, that

$$\text{grade}(I^G, K[V]^G/(N(x_{n_1,1}), \ldots, N(x_{n_1,l}))) = 2.$$ 

Therefore, there is a maximal $K[V]^G$-regular sequence consisting of the $l$ norms and two transfers. Also compare with the known fact that $\text{grade}(I^G, K[V]^G) = 2$ in these cases, see [3] Propositions 20 and 22. As $\text{depth}(K[V]^G) = l + 2$, this also shows that $\text{depth}(M) \neq \text{grade}(I^G, M)$ in general.

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