Balanced Tanner Units And Their Properties
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Abstract— We introduce a new notation called Balanced Tanner Units for regular bipartite graphs. Several enumeration problems for labeled and unlabeled Balanced Tanner Units have been introduced. A general algorithm for enumerating all non-isomorphic \((m,2)\) Balanced Tanner Units has been described and a mathematical proof has been provided for its completeness. An abstraction of \(m\) Symmetric Permutation Tree in order to visualize a labeled \((m,r)\) BTU and enumerate its automorphism group has been introduced. An algorithm to generate the partition associated with two compatible permutations has been derived and we raise open questions for girth.

Keywords Balanced Tanner Unit, Permutation Groups, Cycle Index, \((m,r)\) BTU, Girth Maximum \((m,r)\) BTU

I. Introduction

We introduce new notation for regular bipartite graphs called Balanced Tanner Units (BTUs). The focus of this paper is to study \((m,r)\) BTUs, and prove the necessary results that allows us to create families of graphs, which in turn will allow us to create \((m,r)\) BTUs of maximum girth. The scope of this paper includes the preliminary enumeration results for \((m,2)\) BTUs, abstraction of the \(m\) Symmetric Permutation Tree, interpretation of a labeled \((m,r)\) Balanced Tanner Unit (BTU), permutations, partitions between permutations and finally the family of BTUs \(\Phi(\beta_1, \beta_2, \ldots, \beta_{r-1})\) has been introduced. An upper bound on the number of non-isomorphic graphs in the family of BTUs \(\Phi(\beta_1, \beta_2, \ldots, \beta_{r-1})\) has been derived using mathematical results from combinatorics.

II. Balanced Tanner Unit

A \((m,r)\) Balanced Tanner Unit (BTU) is a regular bipartite graph that can be represented by a \(m \times m\) square matrix with \(r\) non-zero elements in each of its rows and columns. Every \((m,r)\) BTU has a regular bipartite graph representation and a matrix representation. There exists a one-one correspondence between a BTU and its equivalent regular bipartite graph representation. Hence, for the rest of this paper, by \((m,r)\) BTU, we simultaneously refer to its matrix representation as well as its equivalent regular bi-partite graph representation. We divide the two sets of nodes in the regular bi-partite graph as CN nodes and VN nodes each of which are \(m\) in number. In its matrix representation element \((i,j)\) takes the value 1 if CN node \(i\) \((1 \leq i \leq m)\) is connected to VN node \(j\) \((1 \leq j \leq m)\) and is 0 otherwise.

II.1 Labeled \((m,r)\) BTUs

A \((m,r)\) BTU is labeled when each CN node and VN node is labeled with unique labels from the set \(\{1,2,\ldots,m\}\). Any \((m,r)\) BTU that is written in matrix form is by definition a labeled \((m,r)\) BTU.

II.2 Isomorphic \((m,r)\) BTUs

Two labeled \((m,r)\) BTUs \(A\) and \(B\) are isomorphic to each other if there exists a set of row or column exchanges that can transform \(A\) into \(B\).

$$
\begin{pmatrix}
0 & 0 & 0 & 0 & 1 & 1 \\
1 & 0 & 0 & 0 & 0 & 1 \\
1 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 1 & 0 & 0 \\
0 & 0 & 1 & 1 & 0 & 0 \\
0 & 1 & 1 & 0 & 0 & 0 \\
\end{pmatrix}
\cong
\begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 1 \\
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 1 & 0 & 0 \\
0 & 0 & 1 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 1 \\
\end{pmatrix}
$$

II.3 Non-isomorphic \((m,r)\) BTUs

Two labeled \((m,r)\) BTUs \(A\) and \(B\) are non-isomorphic if there does not exist any row or column exchanges which
BTUs.

Balanced Tanner Units. BTU is defined as the length of the

BTU Enumeration is Enumeration

BTUs.

A subset of this problem would be to enumerate all labeled \((m, r)\) BTUs that are isomorphic to a given labeled \((m, r)\) BTU.

II.4 Labeled \((m, r)\) BTU Enumeration Problem

Labeled \((m, r)\) BTU Enumeration refers to enumeration of all distinct labeled \((m, r)\) BTUs. A subset of this problem would be to enumerate all labeled \((m, r)\) BTUs that are isomorphic to a given labeled \((m, r)\) BTU.

II.5 Non-Isomorphic \((m, r)\) BTU Enumeration Problem

Non-Isomorphic \((m, r)\) BTU Enumeration is Enumeration of all distinct non-isomorphic \((m, r)\) BTUs.

II.6 Definition of \(E(m, r)\)

Let \(E(m, r)\) where \(r \leq m\) be a \(E : \mathbb{N}^2 \to \mathbb{N}\) function that represents enumerations of number of distinct non-isomorphic \((m, r)\) Balanced Tanner Units.

II.7 Girth of a \((m, r)\) BTU

Girth of a \((m, r)\) BTU is defined as the length of the smallest cycle in its equivalent regular bipartite graph representation.

III Partitions

III.1 List of partitions

Let \(P_2(m)\) be a set of partitions of \(m\) using natural numbers that are greater or equal to 2. Let \(sP_2(m) \subset P_2(m)\) where \(\{a_1, a_2, \ldots, a_k\} \in sP_2(m)\) if and only if \(a_1 \geq a_2 \geq \ldots \geq a_k\).

III.2 Partition Component

If \(\beta \in P_2(m)\) refers to \(\sum_{j=1}^{y} q_j = m\), then each \(q_j\) for \(1 \leq j \leq y\) is referred to as a partition component of \(\beta\).

III.3 Examples of \(P_2(m)\) for various values of \(m\)

Let us consider the following partitions of \(m\) that consist of numbers greater than or equal to 2.

$$
\begin{array}{|c|c|}
\hline
m & \text{Examples} \\
\hline
4 & \{(2, 2); (4)\} \\
5 & \{(3, 2); (5)\} \\
6 & \{(2, 2, 2); (3, 3); (4, 2); (6)\} \\
7 & \{(3, 2, 2); (4, 3); (5, 2); (7)\} \\
8 & \{(2, 2, 2, 2); (4, 2, 2); (6, 2); (4, 4); (5, 3); (3, 3, 2); (8)\} \\
9 & \{(3, 2, 2, 2); (4, 3, 2); (6, 3); (5, 4); (6, 3); (7, 2); (3, 3, 3); (9)\} \\
\hline
\end{array}
$$

IV General Algorithm for enumeration of all non-isomorphic \((m, 2)\) BTUs

A general algorithm for enumerating all non-isomorphic \((m, 2)\) Balanced Tanner Units is described in this section. It is clear that we have only one non-isomorphic graph for \(r = 1\), the canonical form of which could be represented as \(I_m\), which is a \(m \times m\) identity matrix. It is clear that we have \(m!\) labeled \((m, 1)\) BTUs that are isomorphic to each other by considering all the \(m!\) permutations on labels.

IV.1 Theorem 1

Each element \(\beta \in P_2(m)\) corresponds to a non-isomorphic \((m, 2)\) Balanced Tanner Unit. The number of non-isomorphic \((m, 2)\) Balanced Tanner Units is precisely the number of elements in \(P_2(m)\), i.e., \(E(m, 2) = p(m, 2)\) where \(m \in \mathbb{N}\).

**Proof** Let us consider a labeled \((m, r)\) Balanced Tanner Unit with \(CN_1, CN_2, \ldots, CN_m\) and \(VN\) nodes \(VN_1, VN_2, \ldots, VN_m\). We have this labeled Balanced Tanner Unit by in steps by initially creating a labeled \((m, 1)\) Balanced Tanner Unit, and then a labeled \((m, 2)\) Balanced Tanner Unit.

For \(r = 1\), without loss of generality, we connect node \(CN_i\) to node \(VN_i\) \(\forall 1 \leq i \leq m\). We have only one non-isomorphic graph for \(r = 1\) since can permute the VN nodes among themselves before any connection is made.
and if \( CN_i \) is connected to \( VN_j \), we permute \( VN_i \) and \( VN_j ; j > i \), and hence the first edge connection is made between each \( CN_i \) and \( VN_j \forall 1 \leq i \leq m \). Hence, without loss of generality, for \( r = 1 \), the first edge connection is made between each \( CN_i \) and \( VN_j \forall 1 \leq i \leq m \).

This could be expressed as:

For \( (i = 1; i \leq m; i++) \)

\[
\text{Connect } CN_i \text{ with } VN_i ;
\]

For \( r = 2 \), Starting from \( CN_1 \), we connect the second edge from \( CN_1 \) to \( VN_{i(1)} \) where \( 1 \leq i(1) \leq m \); and then \( CN_{i(1)} \) to an arbitrary VN node \( VN_{i(2)} ; i(2) \neq i(1) \neq 1 \).

Connect, \( CN_{i(2)} \) to \( VN_{i(3)} \), and so on until \( CN_{i(k)} \) to \( VN_1 \), until \( VN_1 \) is reached for some positive integer \( k ; 1 \leq k \leq m \).

If all the CN and VN nodes do not have two edges, let \( CN_{i(1,0)} \) be the CN node which does not have two edges such that \( i(1,0) \) is the minimum value of indexes for all nodes that do not yet have two edges.

Starting from \( CN_{i(1,0)} \), we connect the second edge to an arbitrary VN node which has only one edge, \( VN_{i(1,0)} ; i(1,0) \neq (1,0) \).

Similarly, we connect \( CN_{i(1,1)} \) to \( VN_{i(2,2)} ; i(2,2) \neq i(1,1) \neq i(1,0) \), and so on until \( VN_{i(1,0)} \) is reached.

We continue the above process until all the CN and VN nodes have two edges each and we have a \( (m, 2) \) Balanced Tanner Unit.

We can establish a one-one onto map corresponding with this structure and a partition of \( m \) that consist of numbers that are greater than or equal to 2.

Thus, we have established a mapping between an arbitrary unlabeled \( (m, 2) \) Balanced Tanner Unit and partitions of \( m \) that consist of numbers that are greater than or equal to 2.

### IV.2 Mapping between partitions and enumerations for \( r = 2 \)

If we consider a partition \( (p_1, p_2, \ldots, p_y) \in P_2(m) \) where \( \sum_{i=1}^{y} p_i = m \quad p_i \geq 2, 1 \leq i \leq y \in \mathbb{N} \), now consider a \( (m, 2) \) Balanced Tanner Unit.

Without loss of generality, we assume that the first edge are connected from \( CN_j \) to \( VN_j \) for \( 1 \leq j \leq m \).

For the second edge for each VN and CN node,

for each \( 1 \leq j < p_1 \) we connect \( CN_j \) to \( VN_{j+1} \) and then connect \( VN_j \) to \( CN_{p_1} \).

For each \( p_1 + p_2 + \ldots + p_i \leq j < p_1 + p_2 + \ldots + p_i + p_{i+1} \)

we connect \( CN_j \) to \( VN_{j+1} \) and then

Connect \( VN_{p_1 + p_2 + \ldots + p_i} \) to \( CN_{p_1 + p_2 + \ldots + p_i + p_{i+1}} \)

For each \( p_1 + p_2 + \ldots + p_y = m \)

we connect \( CN_j \) to \( VN_{j+1} \) and then connect \( VN_{p_1 + p_2 + \ldots + p_y} \) to \( CN_m \).

Thus, any partition of \( m \), that consists of natural numbers that are greater or equal to 2 could be mapped to a possible structure of the \((m, 2)\) Balanced Tanner Unit. Thus, any possible structure of the \((m, 2)\) Balanced Tanner Unit could be mapped to a partition of \( m \), that consists of natural numbers that are greater or equal to 2.

### IV.3 Values of \( E(m, 2) \forall m > 2, m \in \mathbb{N} \)

Thus \( E(m, 2) = p(m, 2), \forall m > 2, m \in \mathbb{N} \) where \( p(m, r) \) represents the number of partitions of \( m \) using natural numbers that are greater or equal to \( r \).

### IV.4 Canonical forms for \( r = 2 \)

Non-isomorphic forms of \((m, 2)\) Balanced Tanner Units correspond to each of the partitions of \( m \) that consists of natural numbers that are greater or equal to 2. The canonical forms for \( r = 2 \) correspond to the matrices generated by the following algorithm. If \( \beta \in P_2(m), \#(\beta) \) is the number of terms in \( \beta \).

\[
\forall \beta \in P_2(m)
\]

\[
\begin{array}{l}
\text{for} \ (i = 0; i < m; i++)
\end{array}
\]

\[
\begin{array}{l}
\text{Connect } CN_i \text{ with } VN_i ;
\end{array}
\]

\[
\begin{array}{l}
s=0;
k=0;
t=0;
\text{for} \ (j = 0; j < \#(\beta); j++)
\end{array}
\]

\[
\begin{array}{l}
u = \beta \cdot j ;
\end{array}
\]

\[
\begin{array}{l}
f(z = 0; z < u; z++)
\end{array}
\]

\[
\begin{array}{l}
l = (z + 1) \% u + s ;
\end{array}
\]

Connect \( CN_t \) with \( VN_t ;
\]

\[
\begin{array}{l}
t++;;
\end{array}
\]

\[
\begin{array}{l}
s=s+u;
\end{array}
\]

### IV.5 Theorem

The number of elements in the set \( P_2(m) \) is given by \( p(m, 2) = p(m) - p(m - 1) \) where \( p(m) \) is the number of unrestricted partitions of \( m \in \mathbb{N} \).

**Proof** If \( P(m) \) is the set of all unrestricted partitions of \( m \in \mathbb{N} \), we can establish a one-one onto map between
the set \( P(m-1) \) and subset \( A(m) \subset P(m) \) that contains partitions of \( m \) with at least one 1 in it since all partitions of \( m-1 \) with 1 appended are partitions of \( m \) with at least one 1. Hence, the set \( P_2(m) \) is generated when \( A(m) \) is removed from \( P(m) \) and therefore we obtain the equation \( E(m,2) = p(m,2) = p(m)-p(m-1) \). The earliest equation for \( p(m) \) was obtained by Ramanujan and Hardy in 1918 and has been described in [10].

IV.6 Sample enumeration all non-isomorphic \((6,2)\) BTUs

For example, \( m = 6 \), we obtain \( P_2(6) = \{(6),(4,2),(3,3),(2,2,2)\} \) and therefore all non-isomorphic \((6,2)\) BTUs can be enumerated

\[
\Psi((6)) = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 1 \\
1 & 1 & 0 & 0 & 0 & 0 \\
0 & 1 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 1 & 0 \\
0 & 0 & 0 & 0 & 1 & 1 \\
\end{bmatrix}
\]

\[
\Psi((2,2,2)) = \begin{bmatrix}
1 & 1 & 0 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 1 & 0 \\
0 & 0 & 0 & 0 & 1 & 1 \\
\end{bmatrix}
\]

\[
\Psi((4,2)) = \begin{bmatrix}
1 & 0 & 0 & 1 & 0 & 0 \\
1 & 1 & 0 & 0 & 0 & 0 \\
0 & 1 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 1 \\
0 & 0 & 0 & 0 & 1 & 1 \\
\end{bmatrix}
\]

\[
\Psi((3,3)) = \begin{bmatrix}
1 & 0 & 1 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 & 0 & 0 \\
0 & 1 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 1 \\
0 & 0 & 0 & 1 & 1 & 0 \\
0 & 0 & 0 & 0 & 1 & 1 \\
\end{bmatrix}
\]

IV.7 Algorithm to enumerate all non-isomorphic \((m,2)\) BTUs

1. Enumerate all elements of the set \( P_2(m) \).
2. For each instance of \( \beta \in P_2(m) \), construct a \((m,2)\) BTU with \( \Psi(\beta) \).

V Rank of a \((m,r)\) BTU

The well known set \( GF(2) \), is the Galois field of two elements that consists of 0 and 1 with operations of modular addition and modular multiplication.

V.1 Theorem If a \((m,2)\) BTU is constructed with partition \( \beta \in P_2(m) \) that refers to \( \sum_{i=1}^{k} q_i = m \), rank of the \((m,2)\) BTU is \( m - k \) in \( GF(2) \).

**Proof** A partition \( \sum_{i=1}^{k} q_i = m \) will correspond to \( k \) components in the \((m,2)\) BTU, and for each component, we have the first row as sum of all other rows in \( GF(2) \) as per definition of \( \Psi \). Hence rank of each component is \( q_i - 1 \). Hence, rank of the \((m,2)\) BTU is \( \sum_{i=1}^{k} (q_i - 1) = m - k \).

V.2 Theorem

A \((m,r)\) Balanced Tanner Unit is not full rank in \( GF(2) \) if \( r \) is an even positive integer.

**Proof** One can verify that the sum of all rows of the \((m,r)\) Balanced Tanner Unit is 0 in \( GF(2) \) if \( r \) is even. Hence, a \((m,r)\) Balanced Tanner Unit is not full rank if \( r \) is even.

VI Properties of \( E(m,r) \)

VI.1 Theorem

The Enumerations of \((m,r)\) Balanced Tanner Units correspond to enumerations of partitions in the following manner.

\[
E(m,r) = E(m,m-r)\forall r; 0 \leq r \leq m.
\]

**Proof** We can establish a one-one onto map between the set of all non-isomorphic \((m,r)\) BTUs and the set of all non-isomorphic \((m,m-r)\) BTUs for \( 0 \leq r \leq m \) in the following manner. Let us assume that \( E(m,r) = z \).

If \( \{A_1, A_2, \ldots, A_z\} \) are the canonical forms for a non-isomorphic \((m,r)\) BTUs, then we obtain matrices for \((m,m-r)\) BTUs \( \{B_1, B_2, \ldots, B_z\} \) by mapping all 0 s in each \( A_i \) to 1 s and all 1 in each \( A_i \) to 0 s. In the set \( \{B_1, B_2, \ldots, B_z\} \), we observe that \( B_i \neq B_j \) for...
VI.2 Number of Non-Isomorphic Graphs Checked Manually

The following have been checked empirically by manually working out all possible non-isomorphic graphs for Balanced Tanner Units.

| Number of non-isomorphic \((m, r)\) Balanced Tanner Units |
|-------------------------------------------------------------|
| \(E(4, 1) = 1\)                                            |
| \(E(4, 2) = 2\)                                            |
| \(E(5, 1) = 1\)                                            |
| \(E(5, 2) = 2\)                                            |
| \(E(6, 1) = 1\)                                            |
| \(E(6, 2) = 4\)                                            |
| \(E(6, 3) = 7\)                                            |
| \(E(6, 4) = 4\)                                            |

VI.3 Summarizing Properties of Enumeration Function \(E(m, r)\)

\[
E(m, r) = E(m, m-r) \\
E(m, 1) = E(m, m-1) = 1. \\
E(m, 2) = p(m, 2) = p(m) - p(m-1) \\
E(m, 2) = E(m, m-2) = p(m, 2) \\
E(m, r + 1) > E(m, r) \text{ for } 2 < r < m/2 \\
E(m, r + 1) < E(m, r) \text{ for } m/2 < r < m
\]

VII Groups and Permutation Groups

VII.1 Permutation

If \(X\) is a nonempty set, a bijection \(\pi : X \to X\) is called a permutation of \(X\). An example of a permutation on a set: \(\pi : X \to X\)

\[
\pi = \left(\begin{array}{c}
\alpha_1 \alpha_2 \ldots \alpha_n \\
\beta_1 \beta_2 \ldots \beta_n
\end{array} \right) : \alpha_i, \beta_i \in X \forall 1 \leq i \leq n \\
\beta_i \text{ is the image of } \alpha_i \text{ under } \pi.
\]

VII.2 Permutation Group and Object Set

A set of permutations \(P\) of \(X\) with a binary operation \(* : P \times P \to P\) form a permutation group if the following conditions are satisfied.

- \(\exists I \in P \forall \pi \in P; \pi * I = I * \pi = \pi\)
- \(\forall \pi \in P, \exists \pi^{-1} \in P; \pi * \pi^{-1} = I\)
- \(\forall A, B, C \in P, (A * B) * C = A * (B * C)\)

\(X\) is referred to as the object set of this permutation group.

VII.3 Order and Degree of a Permutation Group

Order refers to the number of permutations in the permutation group. Degree refers to the number of elements in the object set \(X\).

VII.4 Symmetric Group

The Symmetric Group on an object set \(X\) consists of all possible permutations on the elements of \(X\). A symmetric group with degree \(n\) has an order of \(n!\). For example, \(S_3 = \{e, (23), (12), (123), (132), (13)\}\).

Part – II Symmetric Permutation Tree

VIII Symmetric Permutation Tree and its properties

A \(m\) Symmetric permutation tree \(S_{PT}\{m\}\) is defined as a labeled tree with the following properties:

1. \(S_{PT}\{m\}\) has a single root node labeled \(0\).
2. \(S_{PT}\{m\}\) has \(m\) nodes at depth \(1\) from the root node.
3. \(S_{PT}\{m\}\) has nodes at depths ranging from \(1\) to \(m\), with each node having labels chosen from \(\{1, 2, \ldots , m\}\). The root node \(0\) has \(m\) successor nodes. Each node at depth \(i\) has \(m-i+1\) successor nodes at depth \(i+1\). Each node at depth \(m-1\) has \(1\) successor node at depth \(m\).
4. No successor node in \(S_{PT}\{m\}\) has the same node label as any of its ancestor nodes.
5. No two successor nodes that share a common parent node have the same label.
6. The sequence of nodes in the path traversal from the node at depth 1 to the leaf node at depth \( m \) in \( S_{PT}\{m\} \) represents the permutation represented by the leaf node.

7. \( S_{PT}\{m\} \) has \( m! \) leaf nodes each of which represent an element of the symmetric group of degree \( m \) denoted by \( S_m \).

VIII.1 Some properties of symmetric permutation tree \( S_{PT}\{m\} \)

1. Number of nodes

| Node | Number of nodes at specified depth | Number of successor nodes per node at specified depth |
|------|-----------------------------------|-----------------------------------------------------|
| 1    | \( m \)                           | \( m - 1 \)                                         |
| 2    | \( m \ast (m - 1) \)              | \( m - 2 \)                                         |
| \( i \) | \( \prod_{i=0}^{i-1} (m - l) \) | \( m - i + 1 \)                                    |
| \( m - 1 \) | \( m! \)               | 1                                                   |
| \( m \) | \( m! \)                | 0                                                   |

VIII.2 Theorem

There exists an one-one onto map between the set of permutations represented by the leaf nodes of a symmetric permutation tree \( S_{PT}\{m\} \) and elements of \( S_m \), the Symmetric group of degree \( m \).

Proof \( S_{PT}\{m\} \) and \( S_m \) have \( m! \) elements each, and and since The sequence of nodes in the path traversal from the node at depth 1 to the leaf node at depth \( m \) in \( S_{PT}\{m\} \) represents the permutation represented by the leaf node, we can establish a one-one onto map between \( S_{PT}\{m\} \) and \( S_m \).

VIII.3 Theorem

Given any node other than the root node of a \( m \) symmetric permutation tree \( S_{PT}\{m\} \), the set of all of its descendant nodes and the set of all sibling nodes for each node from depths 1 to \( m \) have the same number nodes and both sets have the same set of distinct labels.

Proof The set of all sibling nodes at any depth in a complete \( m \) symmetric permutation tree contains all nodes from the set \( \{1, 2, \ldots, m\} \) except node labels of ancestor nodes. The descendant nodes at any depth of a \( m \) symmetric permutation tree contains all nodes from the set \( \{1, 2, \ldots, m\} \) except node labels of ancestor nodes.

Hence, the sibling nodes and the descendant nodes for each node in a complete \( m \) symmetric permutation tree contains the same number of elements with precisely the same labels.

VIII.4 Properties of a \( m \) Symmetric Permutation Tree

1. One node with label \( i; 1 \leq i \leq m \) at depth 1.
2. \( m - 1 \) nodes with label \( i; 1 \leq i \leq m \) at depth 2.
3. \( (m - 1) \ast (m - 2) \) nodes with label \( i; 1 \leq i \leq m \) at depth 3.
4. \( (m - 1) \ast (m - 2) \ast \ldots \ast (m - j + 1) \) nodes with label \( i; 1 \leq i \leq m \) at depth \( j; 1 \leq j \leq m \).
5. \( (m - 1)! \) nodes with label \( i; 1 \leq i \leq m \) at depth \( m \).

VIII.5 Permutation Interpretation of a labeled \( (m, r) \) BTU

Permutation Interpretation of a labeled \( (m, r) \) BTU is obtained in the following manner.

1. We split all the non-zero elements labeled \( (m, r) \) BTU into \( r \) sets such that each set contains exactly one non-zero element or 1 in exactly one row and column. This decomposition into \( r \) sets is clearly not unique.
2. We associate permutations \( p_1, p_2, \ldots, p_r \in S_m \) with each of the \( r \) sets.
3. For each of the \( p_l; 1 \leq l \leq r \) sets, if a column \( j \) contains a 1 at row \( i \), then the value of the label in the \( j^{th} \) location of the permutation is \( i \).

VIII.6 Compatible Permutations

1. Two permutations on a set of \( s \) elements represented by \( (x_1, x_2, \ldots, x_s); x_p \neq x_q \forall p \neq q; 1 \leq p \leq s; 1 \leq q \leq s; p, q \in \mathbb{N} \) where \( 1 \leq x_i \leq s; i \in \mathbb{N}; 1 \leq i \leq s \) and \( (y_1, y_2, \ldots, y_s); y_p \neq y_q \forall p \neq q; 1 \leq p \leq s; 1 \leq q \leq s; p, q \in \mathbb{N} \) where \( 1 \leq y_i \leq m; i \in \mathbb{N}; 1 \leq i \leq s \) are compatible if and only if \( x_i \neq y_i \forall i \in \mathbb{N}; 1 \leq i \leq s \).
2. A set of \( r \) permutations on a set of \( s \) elements represented by \((x_{1,1}, x_{1,2}, \ldots, x_{1,s}); x_{i,p} \neq x_{i,q} \forall p \neq q; 1 \leq p \leq s; 1 \leq q \leq s; p, q \in \mathbb{N}\) where \( 1 \leq x_{i,\alpha} \leq s \forall i \leq r; 1 \leq \alpha \leq s; i, \alpha \in \mathbb{N} \) are compatible if and only if \( x_{i,\alpha} \neq x_{j,\alpha} \forall i \neq j; 1 \leq \alpha \leq s; 1 \leq i \leq r; 1 \leq j \leq r; i, j, \alpha \in \mathbb{N} \).

VIII.7 Notation

\( p_i \notin C(I_{m}, p_2, \ldots, p_{i-1}) : p_i \) is compatible with permutations \( I_{m}, p_2, \ldots, p_{i-1} \).

VIII.8 Theorem

Any set of \( r \) compatible permutations of a set of \( s \) elements yields a labeled \((s, r)\) Balanced Tanner Unit.

**Proof** Let us consider a set of \( r \) compatible permutations of a set of \( s \) elements represented by \((x_{1,1}, x_{1,2}, \ldots, x_{1,s}); x_{i,p} \neq x_{i,q} \forall p \neq q; 1 \leq i \leq r; 1 \leq p \leq s; 1 \leq q \leq s \) where \( 1 \leq x_{i,j} \leq s \forall j \in \mathbb{N} \) . By definition, we have \( x_{i,p} \neq x_{i,q} \forall p \neq q; p, q \in \mathbb{N}; 1 \leq p \leq s; 1 \leq q \leq s; \forall i \in \mathbb{N} \); \( 1 \leq i \leq r \). This guarantees that all connections made by the algorithm below are distinct, without any repeated connections between any two set of CN and VN nodes, and we hence obtain a labeled \((s, r)\) Balanced Tanner Unit.

Let us start with \( s \) CN nodes each having distinct labels from \( 1, 2, \ldots, s \), and \( s \) VN nodes each having distinct labels from \( 1, 2, \ldots, s \).

1. for \((j = 1; j \leq r; j++)\) {
   2. for \((i = 1; i \leq s; i++)\) {
   3. Connect CN node \( i \) to VN node \( x_{i,j} \);
   }
1. }

VIII.9 General Conjecture on Tree structure after first permutation

If the first chosen permutation is \((x_1, x_2, \ldots, x_s)\) and if the second permutation chosen is \((y_1, y_2, \ldots, y_s)\) where \( x_i \neq y_i; 1 \leq i \leq s \)

Then the number of successors on the path traversed \((y_1, y_2, \ldots, y_s)\) at depths \( 1, 2, \ldots, s \) are \( s-1 + \delta_{x_1,y_1}, s-2 + \delta_{x_2,y_2}, s-3 + \delta_{x_3,y_3}, \ldots, 1 \)

In general for depth \( i \) the number of successors on the path traversed are \( f(i) = m-i + \delta_{x_i,y_i} \) where \( \delta_{x_i,y_i} = 1 \) if \( x_i = y_j \) for some \( j \) satisfying \( s \geq i \geq j \geq 1 \)

\( \delta_{x_i,y_i} = 0 \) if \( x_i \neq y_j \forall j \) satisfying \( s \geq i \geq j \geq 1 \)

**Definition**

\( \delta_{x_i,y_i} = 1 \) if \( x_i = y_j \) for some \( j \) satisfying \( s \geq i \geq j \geq 1 \)

\( \delta_{x_i,y_i} = 0 \) if \( x_i \neq y_j \forall j \) satisfying \( s \geq i \geq j \geq 1 \).

VIII.10 Conjecture

Given any two compatible partitions \( a, b \in S_m \), we can compute the corresponding partition \( \beta(a, b) \in P_2(m) \).

VIII.11 Partitions

\( \beta : S_m \times U_m(a) \rightarrow P_2(m) \) where \( \alpha \) is the first element chosen from \( S_m \), and \( U_m(a) \cup C(a) = S_m \), where \( C(a) \) is the set of permutations that are not compatible with \( \alpha \).

VIII.12 Permutations to Partitions Mapping Problem

Given two compatible permutations \( q_1, q_2 \in S_t \) where \( S_t \) is the Symmetric group of degree \( t \), to determine the partition \( \beta \in P_2(t) \), represented by \( \beta\{q_1, q_2\} \), where \( P_2(t) \) is the set of all partitions of the number \( t \) that consist of numbers greater than or equal to \( 2 \).

VIII.13 Partitions to Permutations Satisfiability Problem

Given a partition \( \beta \in P_2(t) \), where \( P_2(t) \) is the set of all partitions of the number \( t \) that consist of numbers greater than or equal to \( 2 \), and a permutation \( q_1 \in S_t \), where \( S_t \) is the Symmetric group of degree \( t \), to determine a compatible permutation \( q_2 \in S_t \), such that \( \beta \in P_2(t) \) is the partition represented by \( \beta\{q_1, q_2\} \).

VIII.14 Partitions to Permutations Enumeration Problem

Given a partition \( \beta \in P_2(t) \), where \( P_2(t) \) is the set of all partitions of the number \( t \) that consist of numbers greater than or equal to \( 2 \), and a permutation \( q_1 \in S_t \), where \( S_t \) is the Symmetric group of degree \( t \), to determine all possible compatible permutation \( q_2 \in S_t \), such that \( \beta \in P_2(t) \) is the partition represented by \( \beta\{q_1, q_2\} \).

VIII.15 Partitions to Permutations Ordered Enumeration Problem

Given a partition \( \beta \in P_2(t) \), where \( P_2(t) \) is the set of all partitions of the number \( t \) that consist of numbers greater than or equal to \( 2 \), and a permutation \( q_1 \in S_t \), where \( S_t \) is the Symmetric group of degree \( t \), to determine all possible compatible permutation \( q_2 \in S_t \), such that \( \beta \in P_2(t) \) is the partition represented by \( \beta\{q_1, q_2\} \) satisfying the condition that \( q_2 \) is after \( q_1 \) on a \( t \) symmetric permutation tree.
IX.2 Algorithm to calculate $\beta_k < \beta_i$ at depth $b$ mutation for some $\beta \in P_2(m)$.

IX Cycles

IX.1 Conjecture on Criterion for a Cycle for two permutations

Two compatible permutations $a, b \in S_m$ have a cycle with labeled nodes $u$ and $v$ if Label $u$ at depth $i$ for permutation $a$ at depth $j$ is Label at depth $j$ for permutation $b$ where $j < i$ and if Label $v$ at depth $i$ for permutation $b$ at depth $j$ is Label at depth $k$ for permutation $a$ where $k < i$.

IX.2 Algorithm to calculate $\beta(a, b) \in P_2(m)$ given compatible permutations $a, b \in S_m$

1. Given compatible permutations $a, b \in S_m$, we construct the corresponding $(m, 2)$ BTU and exhaustively enumerate all cycles by graph traversal which is not very computationally intensive for a $(m, 2)$ BTU and obtain $\beta(a, b) \in P_2(m)$.

   - Label_List(a) = All labels in permutation $a$ in the increasing order of depth;
   - Label_List(b) = All labels in permutation $b$ in the increasing order of depth;
   - Cycle_List(a, b) = NULL;
   - for (i = 0; i <= m; i++) {
     if (Label at depth $i$ for permutation $a$ at depth $j$ is Label at depth $j$ for permutation $b$ where $j < i$) {
       if (Label at depth $i$ for permutation $b$ at depth $j$ is Label at depth $k$ for permutation $a$ where $k < i$) {
         $u =$ Label at depth $i$ for permutation $a$ is at depth $j$;
         $v =$ Label at depth $i$ for permutation $b$ is at depth $k$;
         CycleInit();
         Cycle_Length = 0;
         Add_label_2_Cycle( $u$ );
         $d = u$;
         while { $d ! = v$ } {
           $c =$ Depth of Label $d$ in permutation $a$;
           $d =$ Label at depth $c$ in permutation $b$;
           Add_label_2_Cycle( $c$ );
           Add_label_2_Cycle( $d$ );
           Cycle_Length++;
       }
     }
   }
   Cycle LENGTH++;
   Add_Partition_component( Cycle LENGTH);
}\n
We can now construct $\Psi(\beta(a, b))$ which is isomorphic to the labeled $(m, 2)$ represented by the compatible permutations $a, b \in S_m$.

IX.3 Algorithm to analyze cycles arising from partitions associated with constituent permutations in a labeled $(m, r)$ BTU

1. Decompose the labeled $(m, r)$ BTU into permutations $p_1, p_2, \ldots, p_r \in S_m$. This decomposition of a BTU into compatible permutations is not unique.

2. Calculate partitions $\alpha_{i, j}$ corresponding to $p_i$ and $p_j$ where $i \neq j; 1 \leq i \leq r; 1 \leq j \leq r$. Clearly, $\alpha_{i, j} = \alpha_{j, i}$; $\forall i \neq j; 1 \leq i \leq r; 1 \leq j \leq r$.

IX.4 Conjecture

Given a labeled $(m, r)$ BTU with compatible permutations $p_1, p_2, \ldots, p_r$ where

1. $p_i$ is between $p_{i-1}$ and $p_{i+1}$ on a complete $m$ symmetric permutation tree for all integer values of $i$ given by $2 \leq i \leq r - 1$.

2. Partitions between between permutations $p_{i-1}$ and $p_i$ represented by $\alpha_{i-1, i} \in P_2(m)$ for all integer values of $i$ given by $2 \leq i \leq r - 1$.

Partitions $\alpha_{i-1, i} \in P_2(m)$ are invariant with any of the following operations on permutations $p_1, p_2, \ldots, p_r$.

1. Position $i$ exchanged with position $j$ where $i \neq j; 1 \leq i \leq m; 1 \leq j \leq m$ in each of $p_1, p_2, \ldots, p_r$.

2. Position with value $i$ exchanged with position with value $j$ where $i \neq j; 1 \leq i \leq m; 1 \leq j \leq m$ in each of $p_1, p_2, \ldots, p_r$. 
IX.5 Criterion for 4 cycles in a labeled \((m, r)\) BTU

If the permutation representation for a labeled \((m, r)\) BTU is given by \(\{x_{i,j}: 1 \leq i \leq r; 1 \leq j \leq m\}\), the labeled \((m, r)\) BTU has a cycle of length 4 if

\[\exists l_1, l_2 \text{ such that } l_1 \neq l_2; 1 \leq l_1 \leq m; 1 \leq l_2 \leq m\]

satisfying \(x_{l_1, l_1} = x_{l_2, l_2} = x_{l_1, l_2} = x_{l_2, l_1}\) for some values of \(j_1, j_2, j_1 \neq j_2; 1 \leq j_1 \leq r; 1 \leq j_2 \leq r\) and \(j_3, j_4; j_3 \neq j_4; 1 \leq j_3 \leq r; 1 \leq j_4 \leq r\).

IX.6 Definition Cycle termination positions

Given two compatible permutations \(p_1, p_2 \in S_m; p_2 \notin C(p_1)\), the cycle termination positions are the depth positions \(1 < j \leq m\) where the cycles terminate. Each of these cycles correspond to the cycles in the partition between \(p_1\) and \(p_2\).

IX.7 Generalized Cycle Traversal

If labels \(l_1\) and \(l_2\) occur at the same depth, labels \(l_2\) and \(l_3\) occur at the same depth, \ldots, and finally labels \(l_r\) and \(l_1\) occur at the same depth, in the permutation representation of a labeled \((m, r)\) BTU, then there exists a cycle connecting the labels \(l_1, l_2, \ldots, l_r\).

IX.8 Partitions for a labeled \((m, r)\) BTU

The number of partitions for a labeled \((m, r)\) BTU specified by a set of permutations \(p_1, p_2, \ldots, p_r\) is \(r^*(r-1)/2\) by considering all combinations of two partitions \(p_i, p_j\) such that \(i \neq j, 1 \leq i \leq r, 1 \leq j \leq r\).

IX.9 Definition \(\Phi(\beta_1, \beta_2, \ldots, \beta_{r-1})\)

\(\Phi(\beta_1, \beta_2, \ldots, \beta_{r-1})\) is the set of all labeled \((m, r)\) BTUs created with compatible permutations \(\{p_1, p_2, \ldots, p_r\}\) such that \(p_1 = I_m\) and the partition between \(p_i\) and \(p_{i+1}\) is \(\beta_i\) for \(1 \leq i \leq r-1\).

IX.10 Important facts about \(\Phi(\beta_1, \beta_2, \ldots, \beta_{r-1})\)

1. Every labeled \((m, r)\) BTU lies in a unique family of \(\Phi(\beta_1, \beta_2, \ldots, \beta_{r-1})\).

2. Of the \(r^*(r-1)/2\) partitions associated with a labeled \((m, r)\) BTU, the family of \(\Phi(\beta_1, \beta_2, \ldots, \beta_{r-1})\) specifies only \(r\) partitions and hence \(r^*(r-1)/2 - r^2/2 - 3*r/r/2\) partitions remain unspecified.

3. However, many labeled \((m, r)\) BTUs can correspond to the same non-isomorphic \((m, r)\) BTU. Labeled \((m, r)\) BTUs in multiple families \(\Phi(\beta_1, \beta_2, \ldots, \beta_{r-1})\) can be isomorphic to each other.

\(\Phi(\beta_1, \beta_2, \ldots, \beta_{r-1})\) is a very useful construct for graph construction in a practical context.

IX.11 Cycles Conjecture

For a \((m, r)\) BTU which is a member of \(\Phi(\beta_1, \beta_2, \ldots, \beta_{r-1})\) where \(\beta_1, \beta_2, \ldots, \beta_{r-1} \in P_2(m)\),

1. The known cycles are due to \(\beta_1, \beta_2, \ldots, \beta_{r-1} \in P_2(m)\) where \(\beta_i\) is the partition between \(p_{i+1}\) and \(p_i\) for \(1 \leq i \leq r-1\).

2. The additional cycles due to other two combinations of permutations \(p_j\) and \(p_k\) where \(j \neq k, j \neq k + 1, 1 \leq j \leq r; 1 \leq k \leq r\).

3. Additional Cycles caused due to combinations of three or more permutations.

Part – IV Automorphism Group and Permutation Enumeration

X Labeled Enumeration - Enumeration Of Labeled \((m, r)\) BTUs

X.1 Isomorphism

Two labeled \((m, r)\) BTUs \(A\) and \(B\) are isomorphic to each other if there exists an ordered set of row and column exchange operations that transform \(A\) into \(B\).

X.2 Automorphism Group

The set of all labeled \((m, r)\) BTUs that are isomorphic to a given labeled \((m, r)\) BTU is called its Automorphism Group with the group operation defined as the isomorphism between two labeled \((m, r)\) BTUs.

X.3 Permutation Representation of a labeled \((m, r)\) BTU

Permutation Representation of a labeled \((m, r)\) BTU is specified by specifying the following

\[\{x_{i,j}: 1 \leq i \leq r; 1 \leq j \leq m\}\]

such that \(x_{i,j} \neq x_{t,j}\) for \(t \neq i; 1 \leq t \leq r\) and \(x_{i,j} \neq x_{i,s}\) for \(s \neq j; 1 \leq s \leq m\). Hence, \(p_i = (x_{i,1}, x_{i,m}) \in S_m; 1 \leq i \leq r\) represent compatible \(p_j \notin C(p_1, p_2, \ldots, p_{j-1}); 1 < j \leq r\).
X.4 Two distinct kinds of permutations operations

An element \(a \in S_m\) could be interpreted in the following two distinct ways

1. **Permutations on Depth** This is a permutation of depths in the permutation representation of a labeled \((m, r)\) BTU.

2. **Permutations on Labels** This is a permutation of labels in the permutation representation of a labeled \((m, r)\) BTU, located irrespective of the depths that they are located in.

X.5 Theorem

Given a labeled \((m, r)\) BTU with compatible permutations \(p_1, p_2, \ldots, p_r\). Isomorphism is also preserved with the following operations on permutations \(p_1, p_2, \ldots, p_r\):

1. Position \(i\) exchanged with Position \(j\) where \(i \neq j; 1 \leq i \leq m; 1 \leq j \leq m\) in each of \(p_1, p_2, \ldots, p_r\).

2. Position with value \(i\) exchanged with Position with value \(j\) where \(i \neq j; 1 \leq i \leq m; 1 \leq j \leq m\) in each of \(p_1, p_2, \ldots, p_r\).

**Proof** Operation \#1 is equivalent to column exchange of a matrix representation of the \((m, r)\) BTU and Operation \#2 is equivalent to row exchange of a matrix representation of the \((m, r)\) BTU.

X.6 Theorem

Given the permutation representation of a labeled \((m, r)\) BTU, isomorphism is preserved with permutations on Depth and Permutations on Labels.

**Proof** Permutations on Labels is equivalent to column exchange of a matrix representation of the \((m, r)\) BTU and Permutations on Depth is equivalent to row exchange of a matrix representation of the \((m, r)\) BTU.

X.7 Theorem

The Automorphism Group of a labeled \((m, r)\) BTU can be enumerated by enumerating all distinct labeled \((m, r)\) BTUs generated by combinations of permutations on depth and permutations on labels on the equivalent permutation representation of the labeled \((m, r)\) BTU. This follows directly from the definition of the Automorphism Group of a labeled \((m, r)\) BTU, and the fact that all isomorphisms could be generated by row and column operations on a labeled \((m, r)\) BTU that results in a distinct labeled \((m, r)\) BTU which in turn are equivalent to permutations on depth and permutations on labels.

X.8 Two important enumeration problems

1. Enumeration of the Automorphism Group of a labeled \((m, r)\) BTU.

2. Enumeration of all non-isomorphic \((m, r)\) BTUs in \(\Phi(\beta_1, \beta_2, \ldots, \beta_{r-1})\).

X.9 Enumeration Of Labeled \((m, r)\) BTUs

Enumeration Of Labeled \((m, r)\) BTUs consists of

1. Enumerating unique non-isomorphic instances of \((m, r)\) BTUs which we refer to as canonical forms.

2. Enumerating the Automorphism Group for each canonical form of \((m, r)\) BTU.

X.10 Partition Set of of a labeled \((m, r)\) BTU with first permutation \(p_1 = I_m\), the identity permutation

Partition Set of a labeled \((m, r)\) BTU with \(p_1 = I_m\) \(\{p_1, p_2, \ldots, p_r\}; p_{i+1} \notin C(p_1, p_2, \ldots, p_i)\) for \(1 < i \leq r-1\) is generated by combinations of permutations on depths and permutations on labels that result in \(p_1 = I_m\) and a different value for one or more of \(p_2, \ldots, p_r\).

X.11 Automorphism Group of a labeled \((m, r)\) BTU

For each element in the Partition Set of a labeled \((m, r)\) BTU, we apply all \(m!\) Permutation on depths in order to obtain the Automorphism Group of a labeled \((m, r)\) BTU.

X.12 Theorem

Given \(\beta_1 \in P_2(m)\) given by \(\sum_{i=1}^{y_1} p_{1,i} = m\), consider all possible distinct labeled partitions on the set \(\{1, 2, \ldots, m\}\) such that we have a subset with \(p_{1,1}\) elements, a subset with \(p_{1,2}\) elements, \ldots, a subset with \(p_{1,y_1}\) elements. For each of the above subsets, we consider distinct labeled graphs that arise from variations of permutations within each subset corresponding to \(p_{1,i}; 1 \leq i < y_1\) is \(\left\{p_{1,i}\right\}\).

**Proof** If a set has \(x\) elements, and if we consider all permutations on this set, with all circular permutations removed, this has \(x!\) distinct elements. We notice that all circular permutations in each subset results in the same labeled graph, and hence if we remove all
XI Permutation Enumeration Formulae

XI.1 General Permutation Enumeration formula

Permutation Enumeration Formulae for compatible permutations \( \{p_r; \beta_{r-1}(p_r, p_{r-1}), p_r \notin C(p_1 = I_m, p_2, \ldots, p_{r-1})\} \) is given by \( f(\beta_{r-1}) = (m - r + 1) \sum_{j \text{distinct}p_{r-1,j}} (p_{r-1,j-1})! \prod_{i=1}^{p_{r-1,j}} (p_{r-1,i}) \beta_{r-1} \in P_k(m) \).

Proof For each of the possible \( m - r + 1 \) choices for \( p_r \) at depth 1 of the \( m \) symmetric permutation tree, the number of distinct permutations that satisfy the constraint that the partition with \( p_{r-1} \) is \( \beta_{r-1} \) are \( \sum_{j \text{distinct}p_{r-1,j}} (p_{r-1,j-1})! \prod_{i=1}^{p_{r-1,j}} (p_{r-1,i}) \beta_{r-1} \in P_k(m) \) since we get different permutations only for each distinct \( p_{r-1,j} \) and hence the result follows.

XI.2 Corollary

Permutation Enumeration Formulae for compatible permutations \( \{p_2; \beta_1(p_2, p_1), p_2 \notin C(p_1 = I_m)\} \) is given by \( f(\beta_1) = (m - 1) \sum_{j \text{distinct}p_1,j} (p_{1,j-1})! \prod_{i=1}^{p_{1,i}} (p_{1,i}) \beta_1 \in P_k(m) \).

XI.3 Corollary

Permutation Enumeration Formulae for compatible permutations \( \{p_2; \beta_1(p_2, p_1) = (m), p_2 \notin C(p_1)\} \) is given by \( f(\beta_1 = (m)) = (m - 1)! \beta_1 \in P_k(m) \).

XII Towards Exhaustive Enumeration

XII.1 Thought Experiment to visualize all labeled \((m, r)\) BTUs that can be created by choosing permutations on a \( m \) symmetric permutation tree

1. Start with a set of \( r \) compatible partitions \( q_1, q_2, \ldots, q_r \), where without loss of generality, \( q_2, \ldots, q_r \) occur in the order of traversal of all leaf nodes of a complete \( m \) symmetric permutation tree from first element \( I_m \) until the last element.

2. Vary \( q_r \) for all possible compatible leaf nodes on the complete \( m \) symmetric permutation tree.

3. For each of the above \( q_r \) above, we vary \( q_{r-1} \) for all possible compatible leaf nodes on the complete \( m \) symmetric permutation tree.

4. This process continues until for each of the possible \( q_3 \), we vary \( q_2 \) for all possible compatible leaf nodes on the complete \( m \) symmetric permutation tree.

XII.2 Exhaustive Enumeration of a \( m \) symmetric permutation tree

\( f \) is a simple concatenation function that converts a permutation on \( n-1 \) elements to a permutation on \( n \) elements.

\[
P((x_1, x_2, \ldots, x_n), n) \quad \text{for} \quad j = 1; j < n; j++) \quad \{ \\
\text{f}(x_j, P((x_1, x_2, \ldots, x_{j-1}, x_{j+1}, \ldots, x_n), n - 1)); \\
\}
\]

XII.3 Sample Construction of a labeled \((m, r)\) BTU

Construct \((m, r)\) \{
InitForbiddenList(\( m \)); 
p_1 = I_m; 
Add_ForbiddenList(p_1); 
EnumerateCP(p_1, 1); 
Randomly choose \( p_2 \) from enumerated compatible permutations; 
\ldots
EnumerateCP(p_1, p_2, \ldots, p_{r-1}, r - 1); 
Randomly choose \( p_r \) from enumerated compatible permutations; 
\} 
Check(label \( x \), depth \( y \)) \{
if(LabelList[y] contains label \( x \)) \{ 
return TRUE; 
else 
return FALSE; 
\}
Add_ForbiddenLabel(label \( x \), depth \( y \)) \{
Add label \( x \) to LabelList[y]; 
\}
InitLabelList(int \( k \)) \{ // \( k \) refers to depth 
LabelList[k] = NULL; 
\}
InitForbiddenList(int \( m \)) \{
for \( i = 1; i < m; i++ \) \{
    InitLabelList \( (i) \);
\}
\}
// Enumerate the list of remaining compatible partitions after removing \( p_1, p_2, \ldots, p_t \)
EnumerateCP \( (p_1, p_2, \ldots, p_t, t) \) \{
for \( i = 0; i < t; i++ \) \{
    Add_ForbiddenList \( (p_i) \);
\}
P((1, 2, \ldots, m), m, m) ;
\}
f is a simple concatenation function that creates a permutation.
P((x_1, x_2, \ldots, x_n), n, l) ;
for \( j = 1; j < n; j++ \) \{
if (check \( (x_j, j + l - n) == \) TRUE ) \{
    f(x_j, P(x_1, x_2, \ldots, x_{j-1}, x_{j+1}, \ldots, x_n), n - 1, l);
\}
\}
\}
\}
Part – V Cycle Index

The definitions, notation and examples in this Part have been motivated by standard textbooks such as [9] and [10].

XIII Cycle Index of A Permutation Group

Let \( A \) be a permutation group with object set \( X = \{1, 2, \ldots, n\} \). Each permutation \( \alpha \) in \( A \) can be written as a unique product of disjoint cycles. For each integer \( k \) such that \( 1 \leq k \leq n \), let \( j_k(\alpha) \) be the number of cycles of length \( k \) in the disjoint cycle decomposition of \( \alpha \). Then the cycle index of \( A \), denoted by \( Z(A) \) is by the following the polynomial in the variables \( s_1, s_2, \ldots, s_n \) defined by
\[
Z(A) = |A|^{-1} \sum_{\alpha \in A} \prod_{k=1}^{n} s_k^{j_k(\alpha)}
\]

For example, we consider the symmetric group \( S_n \) on \( n \) objects.
\[
S_3 = \{e, (23), (12), (123), (132), (13)\} \quad \text{for } n = 3
\]
The identity permutation \( (1)(2)(3) \) has three cycles of length 1, that we represent as \( s_1^3 \). Since the three permutations \( (1)(2), (2)(13), (3)(12) \) each have one cycle of length 1 and one of length 2, we obtain the term \( 3s_1s_2 \) is obtained. Finally, the permutations \( (123) \) and \( (132) \) contribute to the term \( 2s_3 \) and hence
\[
Z(S_3) = \left(\frac{1}{3!}\right)(s_1^3 + 3s_1s_2 + 2s_3)
\]
The cyclic permutation group \( C_3 \) consists of the identity permutation and two permutations with 3-cycles \( C_3 = \{e, (123), (132)\} \), and hence \( Z(C_3) = \left(\frac{1}{3!}\right)(s_1^3 + 2s_3) \).

We shall denote a partition of \( n \) by the vector \( \mathbf{j} = (j_1, j_2, \ldots, j_n) \) where \( j_k \) is the number of parts equal to \( k \). Hence \( n = \sum_{k=1}^{n} k \mathbf{j}_k \). Let \( h(\mathbf{j}) \) be the number of permutations in \( S_n \) whose cycle decomposition determines partition \( \mathbf{j} \), so that for each \( k \) we have \( j_k = j_k(\alpha) \). Then we have \( h(\mathbf{j}) = \frac{n!}{\prod_{k=1}^{n} k^{j_k} j_k!} \).

XIV Superposition of Cycle Index

The ring of rational polynomials in the following variables \( s_1, s_2, \ldots, s_d \) is denoted by \( R \). The operation \( \mathcal{G} \) is defined for a sequence of variables \( s_1^i, s_2^j, \ldots, s_d^k = s_1^{i_1}s_2^{j_1}\ldots s_d^{k_1}, \ldots, s_1^{i_m}s_2^{j_2}\ldots s_d^{k_m} \) of \( m \geq 2 \) monomials in \( R \) by
\[
(s_1^{i_1}s_2^{j_1}\ldots s_d^{k_1}) \cap (s_1^{i_2}s_2^{j_2}\ldots s_d^{k_2}) \cap \ldots = (\prod_{k=1}^{m} i_k^{k_k})(m-1)
\]
which reduces to \( (s_1^{i_1}s_2^{j_1}\ldots s_d^{k_1}) \cap (s_1^{i_2}s_2^{j_2}\ldots s_d^{k_2}) \cap \ldots \).

XIV.1 Examples of superposition from [9]

1. In order to calculate the number of superpositions of two cycles of order \( n \), we calculate \( Z(D_n) \cap Z(D_n) \) with \( n = 5 \) since \( Z(D_5) = \frac{1}{4!}(s_1^5 + 4s_1s_2^3 + 5s_1s_2s_3^2) \). Therefore, \( Z(D_5) \cap Z(D_5) \) becomes \( \frac{1}{25}(s_1^5 + 16s_1s_2^3 + 25s_1s_2s_3^2) \) which after simplification is \( \frac{1}{100}(120 + 80 + 200) = 4 \).

2. Consider \( Z(S_3) \cap Z(S_3) \). Since \( Z(S_3) = \left(\frac{1}{3!}\right)(s_1^3 + 3s_1s_2 + 2s_3) \).

Therefore, \( Z(S_3) \cap Z(S_3) \) becomes
\[
\frac{1}{3!}(s_1^3 + 3s_1s_2^3 + 9s_1s_2^2 + 4s_3 + 3s_4 + 3s_5 + 3s_6)
\]
which can be simplified as
\[
\frac{1}{6}(36) = 6.
\]

3. \( Z(D_3) = \frac{1}{6}s_1^3 + \frac{1}{2}s_1s_2 + \frac{1}{3}s_3 \).

Therefore, \( Z(D_3) \cap Z(D_3) \) becomes \( \frac{1}{36}(s_1^3 + s_2^3) + \frac{1}{6}(s_1s_2 + s_1^2s_2) + \frac{1}{3}(s_1s_3 + s_1^2s_3) + \frac{1}{6}(s_3 + s_1s_3) + \frac{1}{3}(s_3 + s_1s_3) \) which can be simplified as \( \frac{1}{6} + \frac{1}{2} + \frac{1}{3} = 1 \).
XV Cycle Index Form Of Redfield Enumeration Theorem

The following theorem is the well known cycle index form of the Redfield Enumeration Theorem which can be found in standard textbooks such as [8] and [9]. The permutations of the object set considered for cycle index should not be confused with permutations on a m symmetric permutation. Though the two concepts are related, they are really not the same.

Theorem The number of different superpositions of m graphs $G_i$ with the same set of unlabeled vertexes is $Z(G_1) \cap \ldots \cap Z(G_m)$ where $Z(G_i)$ is the cycle index of the automorphism group of $G_i$.

XVI Upper Bound Theorem

Theorem The number of non-isomorphic $(m, r)$ BTUs in $\Phi(\beta_1, \beta_2, \ldots, \beta_{r-1})$ is less than or equal to $Z(\beta_1) \cap Z(\beta_2) \cap \ldots \cap Z(\beta_{r-1})$ where $Z(\beta_i)$ is the cycle index of the automorphism group of the $(m, 2)$ BTU generated by $\Psi(\beta_i)$ for $1 \leq i \leq r - 1$.

Proof Directly follows by applying Redfield Enumeration Theorem, and the fact that all superpositions of $\Psi(\beta_i)$ for $1 \leq i \leq r - 1$ are $(m, r)$ BTUs.

XVII Girth maximum BTU of degree 2 and The Cage Problem

XVII.1 Known Cycle Theorem for a $(m, 2)$ BTU

The cycle lengths of a $(m, 2)$ BTU that is isomorphic to $\Psi(\beta)$ for some $\beta \in P_2(m)$ given by $\sum_{i=1}^{y} q_i = m$ are $\{2 * q_i; 1 \leq i \leq y \}$. 

Proof A $(m, 2)$ BTU that is isomorphic to $\Psi(\beta)$ has no other cycles other than that of $\beta \in P_2(m)$ given by $\sum_{i=1}^{y} q_i = m$. The cycle length for a partition component $q_i$ is $2 * q_i$. Hence, it follows that the cycle lengths are $\{2 * q_i; 1 \leq i \leq y \}$.

XVII.2 Maximum possible girth of a $(m, 2)$ BTU

The maximum possible girth of a $(m, 2)$ BTU is $2 * m$.

Proof This directly follows when we consider that every $(m, 2)$ BTU can be mapped to $\Psi(\beta)$ where $\beta \in P_2(m)$. It is clear that girth of a $(m, 2)$ BTU is $2 * \min(q_i); 1 \leq i \leq y$ where $\sum_{i=1}^{y} q_i = m$ represents $\beta \in P_2(m)$. Hence, it follows that the maximum possible girth of a $(m, 2)$ BTU is $2 * m$.

XVIII Cage Problem for regular bipartite graphs of degree 2

The Cage Problem has been described in detail in [11].

XVIII.1 Theorem

Given an even natural number $g$, the regular bipartite graphs of degree 2 with minimum order is a $(m, 2)$ BTU constructed with the partition $(m) \in P_2(m)$ where $m = g/2$.

Proof We know that a $(m, 2)$ BTU has a maximum girth of $2 * m$ for partition $(m) \in P_2(m)$. Hence, it follows that the regular bipartite graphs of degree 2 with minimum order is a $(m, 2)$ BTU constructed with the partition $(m) \in P_2(m)$ where $m = g/2$.

XVIII.2 Corollary

For a specified even natural number $g \geq 4$, every $(g/2, 2)$ BTU constructed with the partition $(g/2) \in P_2(g/2)$ is a $(2, g)$-cage for regular bipartite graphs, the regular bipartite graph of minimum order for specified girth $g \geq 4$.

XIX Open Questions on girth maximum $(m, r)$ BTU

1. How do we construct a girth maximum $(m, r)$ BTU?

2. How do we choose $\beta_1, \beta_2, \ldots, \beta_{r-1} \in P_2(m)$ such that the girth maximum $(m, r)$ BTU lies in the family of graphs $\Phi(\beta_1, \beta_2, \ldots, \beta_{r-1})$?

Answers to these Questions have been explored in [1].

XX CONCLUSION

This paper describes a general algorithm for all non-isomorphic $(m, 2)$ Balanced Tanner Units and a mathematical proof has been provided for its completeness. Several results for $E(m, r)$ have been proved. An abstraction of Symmetric Permutation Tree in order to visualize a labeled $(m, r)$ BTU and enumerate its automorphism group has been introduced. An algorithm to generate the partition associated with two compatible permutations has been introduced. The relationship between Automorphism Group and permutation enumeration problem has used to derive formulae for compatible permutations. A family of labeled $(m, r)$ BTUs referred to as $\Phi(\beta_1, \beta_2, \ldots, \beta_{r-1})$ have been introduced and an upper bound on the number of non-isomorphic graphs in this family has been derived. We raise open questions about the construction of a girth maximum $(m, r)$ BTU.
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