On the Impossibility of Decomposing Binary Matroids

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Abstract

We show that there exist \( k \)-colorable matroids that are not \((b, c)\)-decomposable when \( b \) and \( c \) are constants. A matroid is \((b, c)\)-decomposable, if its ground set of elements can be partitioned into sets \( X_1, X_2, \ldots, X_l \) with the following two properties. Each set \( X_i \) has size at most \( ck \). Moreover, for all sets \( Y \) such that \(|Y \cap X_i| \leq 1\) it is the case that \( Y \) is \( b \)-colorable. A \((b, c)\)-decomposition is a strict generalization of a partition decomposition and, thus, our result refutes a conjecture from [BSY19].

Keywords— Matroid, Matroid Coloring, Matroid Decomposition, Matroid Intersection

1 Introduction

Consider a matroid \( M = (S, \mathcal{I}) \) where \( S \) is the ground set of elements and \( \mathcal{I} \) is the collection of independent sets. \( M \) is said to be \( k \)-colorable if \( S \) can be partitioned in \( k \) sets \( C_1, C_2, \ldots, C_k \) such that \( C_i \in \mathcal{I} \) for all \( i \in [k] \). The smallest \( k \) for which \( M \) is \( k \)-colorable is known as the coloring number of the matroid \( M \). An optimal coloring of a matroid can be computed in polynomial time [Edm65]. This is not necessarily the case anymore if we consider, instead of a single matroid, the intersection of \( h \) matroids. Consider a collection of \( h \) matroids on the same ground set \( M_i = (S, \mathcal{I}_i) \) for \( i \in [h] \). The intersection of \( M_1, M_2, \ldots, M_h \) is said to be \( k \)-colorable if \( S \) can be partitioned in \( h \) sets \( X_1, X_2, \ldots, X_h \) such that \( X_j \in \bigcap_{i=1}^{h} \mathcal{I}_i \) for all \( j \). That is, each \( X_j \) is independent in all of the \( h \) matroids. The coloring number of the intersection of \( M_1, M_2, \ldots, M_h \) is the smallest \( k \) for which the given intersection is \( k \)-colorable. Matroid intersection coloring is known to be NP-hard for \( h \geq 3 \) [OBS17].

[IMP20] showed that if each of the \( k \)-colorable matroids \( M_1, \ldots, M_h \) is \((b, c)\)-decomposable, the intersection of these matroids can be colored with \( k \cdot b \cdot c \cdot b^h \) colors.

Definition 1 \(((b, c)\)-decomposable). A \( k \)-colorable matroid \( M = (S, \mathcal{I}) \) is \((b, c)\)-decomposable if there is a partition \( X = \{X_1, X_2, \ldots, X_l\} \) of \( S \) such that:

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• For all $i \in [\ell]$, it is the case that $|X_i| \leq c \cdot k$, and
• every set $Y = \{v_1, \ldots, v_\ell\}$, such that $v_i \in X_i$, is $b$-colorable.

We refer to $X$ as a $(b,c)$-decomposition.

If $b = 1$ then $X = \{X_1, X_2, \ldots, X_\ell\}$ represents a partition matroid, and thus \[\text{{BSY19}}\] called the $(1,c)$-decomposition a partition reduction. Furthermore, \[\text{{IMP20}}\] showed that if the $(b,c)$-partitions are given for a collection of matroids on the same ground set, or can be efficiently computed, then the coloring of their intersection can be efficiently computed. Note that if $h, b$ and $c$ are all $O(1)$ then the resulting coloring is an $O(1)$-approximation to an optimal coloring as the coloring number for each individual matroid lower bounds the coloring number for the intersection.

Furthermore, \[\text{{BSY19}}, \text{{IMP20}}, \text{{LMP21}}\] showed that many common types of matroids, including transversal matroids, laminar matroids, graphic matroids and gammoids, have $(1,2)$-decompositions. Moreover, they showed that these decompositions can be computed efficiently from the standard representations of these matroids. Thus \[\text{{BSY19}}\] reasonably conjectured that every matroid is $(1,2)$-decomposable. If this conjecture held, and such decompositions could be found efficiently, then the result from \[\text{{IMP20}}\] would yield an efficient $O(1)$-approximation algorithm for coloring the intersection of $O(1)$ arbitrary matroids.

This paper’s main result is that there are matroids that are not $(O(1), O(1))$-decomposable. This refutes the conjecture from \[\text{{BSY19}}\]. In particular, we show that the binary matroid, consisting of the $2^n - 1$ nonzero vectors of dimension $n$, is not $(O(1), O(1))$-decomposable.

Before proving our main result in Section 2 we review related work and basic definitions.

### 1.1 Other Related Work

\[\text{{AB06}}\] showed that for two matroids $M_1$ and $M_2$, with coloring numbers $k_1$ and $k_2$, the coloring number $k$ of $M_1 \cap M_2$ is at most $2 \max(k_1, k_2)$. The proof in \[\text{{AB06}}\] uses topological arguments that do not directly give an efficient algorithm for finding the coloring. \[\text{{BSY19}}\] also showed how to use the existence of $(1,c)$-decompositions to prove the existence of certain list colorings.

Motivated by applications to the matroid secretary problem, \[\text{{AKKG21}}\] independently showed that the same binary matroid that we consider is not $(1, O(1))$-decomposable.

### 1.2 Definitions

A hereditary set system is a pair $M = (S, \mathcal{I})$ where $S$ is a universe of $n$ elements and $\mathcal{I} \subseteq 2^S$ is a collection of subsets of $S$ with the property that if $A \subseteq B \subseteq S$ and $B \in \mathcal{I}$ then $A \in \mathcal{I}$. The sets in $\mathcal{I}$ are called independent. A subset $R$ of $S$ is $k$-colorable if $R$ can be partitioned into $k$ independent sets. The coloring number of $M$ is the smallest $k$ such that $S$ is $k$-colorable. The rank $r(X)$ of a subset $X$ of $S$ is the maximum cardinality of an independent subset of $X$. A matroid is an hereditary set system with the additional properties that $\emptyset \in \mathcal{I}$ and if $A \in \mathcal{I}$, $B \in \mathcal{I}$, and $|A| < |B|$ then there exists an $s \in B \setminus A$ such that $A \cup \{s\} \in \mathcal{I}$. The intersection of matroids $(S, \mathcal{I}_1), \ldots, (S, \mathcal{I}_h)$ is a hereditary set system with universe $S$ where a set $I \subseteq S$ is independent if and only if for all $i \in [1, h]$ it is the case that $I \in \mathcal{I}_i$. A flat $F$ of $M$ is subset of $S$ such that for all elements $y \in S \setminus F$ it is the case that adding $y$ to $F$ strictly increases the rank.

### 2 Main Result: Binary Matroids are Not Decomposable

This section focuses on showing that binary matroids are not $(b,c)$-decomposable for constants $b$ and $c$. 
Definition 2. Let $M = (S, \mathcal{I})$ be the binary matroid where $S$ consists of all $n$ dimensional vectors with entries that are either 0 or 1, with the exception of the all zero vector. A subset $R$ of $S$ is independent if and only if the elements of $R$ are linearly independent over the field with the elements 0 and 1 with addition and multiplication modulo 2.

Note that $S$ contains $2^n - 1$ elements and has rank $n$.

Lemma 3. The coloring number of any rank $d$ flat of $M$ is $\lceil (2^d - 1)/d \rceil$. Thus, by taking $d = n$, the coloring number $k$ of $M$ is precisely $\lceil 2^n/n \rceil$.

Proof. It is well known that a matroid can be colored with $k$ colors if and only if for every subset $R$ of elements, $k \cdot r(R) \geq |R|$, that is, $k$ times the rank of $R$ is at least the cardinality of $R$ [Edm65]. The maximum value of $|R|/r(R)$ over subsets $R$ of a rank $d$ flat $F$ occurs when $R = F$. Thus this maximum is $(2^d - 1)/d$. □

Lemma 4. If $d \leq n/2$ then the number of distinct rank $d$ flats of $M$ is at least $\frac{2^{dn}}{2^{d^2+d}}$.

Proof. Consider the process of picking one by one a collection of $d$ vectors to form a basis of a rank $d$ flat $F$. When considering the $i$th choice, there are $(2^n - 1) - (2^{i-1} - 1)$ choices of elements of $S$ that are linearly independent from the previous choices. As the order of the $d$ vectors chosen does not matter, the number possible collections of elements that form a basis of rank $d$ flat is the following.

$$\prod_{i=1}^{d} \frac{(2^n - 1) - (2^{i-1} - 1)}{d!}$$

Similarly for a particular rank $d$ flat $F$ there are

$$\prod_{i=1}^{d} \frac{(2^d - 1) - (2^{i-1} - 1)}{d!}$$

collections of elements from $F$ that form a basis for $F$. Thus there are

$$\prod_{i=1}^{d} \frac{(2^n - 1) - (2^{i-1} - 1)}{(2^d - 1) - (2^{i-1} - 1)} = \prod_{i=1}^{d} \frac{2^n - 2^{i-1}}{2^d - 2^{i-1}}$$

flats of rank $d$. Lower bounding each term in the product in the numerator by $2^n - 2^d$, and upper bounding each term in the product in the denominator by $2^d$, we can conclude that there are at least

$$\left(\frac{2^n - 2^d}{2^d}\right)^d$$

flats of rank $d$. Then if $d \leq n/2$, this is at least $\frac{2^{dn}}{2^{d^2+d}}$. □

Theorem 5. If $M$ admits a $(b,c)$-decomposition then it must be the case that $4c^22^{d^2+d} \geq n$, where $d$ is the minimum integer such that $(2^d - 1)/d > b$. In particular, for sufficiently large $n$, $M$ admits no $(O(1), O(1))$-decomposition.

Proof. Consider an arbitrary $(b,c)$-decomposition $X = \{X_1, X_2, \ldots, X_\ell\}$ of $M$. As $(2^d - 1)/d > b$, a flat of rank $d$ is not $b$-colorable by Lemma 3. Thus for each rank $d$ flat $F$, at least two elements of $F$ must be in the same part in $X$. Otherwise, we get a contradiction to the definition of $(b,c)$-decomposability. To see this, consider setting $Y$ to $F$ in the definition of the $(b,c)$-decomposition. That is, each element of $F$ is
selected to be in $Y$ as this includes at most one element in any part $X_i$. The resulting representatives would not be $b$-colorable by the above characterization of $F$. If two elements of a rank $d$ flat $F$ are in the same part $X_i \in X$ then we say that $F$ is covered by part $X_i$.

Since $X$ is a $(b,c)$-decomposition, the cardinality of each part of $X$ is at most $ck$. Each pair of elements $x, y$ in a part $X_i \in X$ can be contained in at most $\binom{2n}{d-2}$ rank $d$ flats. To see this note that each rank $d$ flat $F$ can be represented by $d$ independent basis vectors in $F$, and since $x$ and $y$ are already specified, there are at most $d - 2$ more choices for these basis vectors. There are at most $\binom{ck}{2}$ possible pairs of elements from a part $X_i \in X$, and $X_i$ can cover at most $\binom{ck}{2} \binom{2n}{d-2}$ different flats. Thus in aggregate, all the parts of $X$ can cover at most $\ell \binom{ck}{2} \binom{2n}{d-2}$ flats. Then using the fact that $\ell$ is at most $n$, $k$ is at most $2 \cdot 2^n / n$, and upper bounding $\binom{ck}{2}$ by $x^b$, we can conclude that in aggregate all the parts of $X$ can cover at most $\ell \binom{ck}{2} \binom{2n}{d-2} \leq n(ck)^2(2^n)^{d-2} \leq 4c^2 2^{nd} / n$ flats. Since each of the flats must be covered by some part of $X$, and since by Lemma 4 the number of rank $d$ flats is at least $\frac{2^{nd}}{2^d + d}$, it must be the case that

$$4c^2 2^{nd} / n \geq \frac{2^{nd}}{2^d + d}$$

or equivalently $4c^2 2^{nd} + d \geq n$.  

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