Thermodynamics of large N gauge theory from top down holography

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Abstract: By considering fluxes on $D7$ branes and explicitly computing their backreaction on the geometry with and without a black hole, we show how the UV divergence of Klebanov-Strassler model can be regulated. Using the form of the metric and fluxes in the extremal and non-extremal limit, we compute the on-shell gravity action including the localized sources up to linear order in perturbation parameter $\frac{g_s M^2}{N}, g_s N_f$ where $N, M$ and $N_f$ are units of $D3, D5$ and $D7$ charges in the dual gauge theory. Using the gravitational description, we show how the gauge theory undergoes a first-order Hawking-Page like phase transition and compute the critical temperature $T_c$. Finally, we obtain the equation of state for the gauge theory by computing thermodynamic state functions of the black hole and exhibit how black holes in deformed cone geometry can lead to results that are qualitatively similar to lattice QCD simulations.
1. Introduction

Asymptotic freedom guarantees that at high temperatures, nuclear matter is best described as a weakly interacting gas of quarks and gluons. The weak nature of the coupling allows a perturbative description of QCD where observables in principal can be computed with increasing accuracy as temperature is increased. However at low temperatures, nuclear matter is color neutral, indicating color degrees of freedom are strongly coupled and confined inside hadrons. The strong coupling regime of QCD is analyzed by either studying the theory on the lattice or resorting to effective field theories- both of which have their success and limitations.

On the other hand, gauge theories naturally arise from excitations of open strings ending on branes [1] while gravitons can be described by excitations of closed strings. By studying the interaction between open and closed strings, one can relate the Hilbert space of the gauge theory with that of gravity. The best studied example is the AdS/CFT correspondence [2]: Here the gauge theory is four dimensional maximally SUSY $\mathcal{N} = 4$ conformal field theory and the gravitons describe a ten dimensional space $AdS_5 \times S^5$ i.e. five dimensional anti-deSitter space times a compact five sphere. When the t’Hooft coupling for the gauge theory is large, Hilbert space of the gauge theory is conjectured to be contained in the Hilbert space of gravitons described by classical action of $AdS_5 \times S^5$ geometry. Thus a strongly coupled quantum
gauge theory gets a classical description in terms of weakly coupled gravity. Expectation values in gauge theory which are otherwise extremely difficult to compute due to strong coupling, can easily be computed using the dual holographic description [3].

Since QCD is a gauge theory, the obvious question becomes is there a holographic description for a QCD like theory? Any model that attempts to mimic QCD should feature its two key attributes: deconfinement at high temperatures and confinement at low temperatures. Thus a holographic description of QCD, if it exists, should incorporate both the deconfined and confined phase while allowing us to study the thermodynamics near the critical region where phase transition occurs. Since QCD coupling is large near the critical temperature $T_c$, the corresponding 'tHooft coupling is also large and we expect the holographic description to be most accurate near $T_c$. While for $T \gg T_c$, QCD coupling is small rendering pQCD techniques to be most reliable and we do not require a holographic description. In addition, lattice QCD simulations suggest that the conformal anomaly is largest near $T_c$. Thus we should look for a holographic description near $T_c$ where the gauge coupling is large and the theory is highly non-conformal.

The AdS/CFT correspondence only considers conformal field theories, where there is no phase transition and no critical temperature $T_c$. Black holes in $AdS_5 \times S^5$, describing a thermal CFT can mimic large $T \gg T_c$ regime of QCD. But as already mentioned, at large $T$ we can simply use perturbative QCD to study the thermodynamics quite accurately. The regime near $T \sim T_c$ where pQCD breaks down is where holographic techniques can be most valuable. But since black holes in AdS space cannot describe a non-conformal theory, we must find a generalization of AdS/CFT correspondence to describe thermal gauge theories that undergo phase transitions.

Obtaining a geometric description of a gauge theory that resembles thermal QCD near phase transition is a formidable task. Before attempting to find a holographic map between QCD and gravity, we must first explore the general scope of gauge/gravity correspondence and understand how gauge theories with non-trivial Renormalization Group (RG) flows arise in string theory. In principal excitations of D branes placed in various geometries give rise to gauge theories. The fluxes and scalar fields sourced by the branes back-react and warp the geometry. This warped geometry is referred as the ‘dual geometry’. The fluxes and dilaton field in the dual geometry are used to obtain the RG flow of the gauge theory. While all the thermodynamic state functions of the gauge theory can be obtained by identifying the partition function of the dual geometry with that of the gauge theory.

Since QCD is non supersymmetric and non-conformal, the first objective is to find gauge theories with RG flows arising from D brane configurations with minimal

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1 Analysis of high temperature phase and phase transitions can be found in [4]-[7]. For recent developments in lattice QCD, please consult [8]-[12] and [13]-[18] for the large N limit.
SUSY. A great deal of progress has been made in that direction: In [20, 19] RG flows that connected conformal fixed points at IR and UV was incorporated, and [21] connected the UV $\mathcal{N} = 4$ conformal fixed point to a $\mathcal{N} = 1$ confining theory. But the model with QCD like RG flow and minimal supersymmetry is the Klebanov-Strassler (KS) model [22] which was obtained by considering IR modifications to Klebanov-Tseytlin (KT) model [23] (with an extension [24] to incorporate fundamental matter by considering D7 branes). However, at the highest energies the gauge theory is nothing like QCD: the effective degrees of freedom diverges and the gauge theory is best described in terms of bifundamental fields. Only at the lowest energies the gauge theory resembles $\mathcal{N} = 1$ SUSY QCD which confines. In the limit when effective $D3$ brane charge is large, the gauge theory with large 'tHooft coupling has an equivalent description in terms of warped deformed cone. We can learn about this gauge theory which is very different from QCD in the UV, using the dual description.

In a series of papers [30]-[37], we proposed the general procedure to modify the UV dynamics of KS theory. In this paper we demonstrate how UV modifications are realized with an exact calculation of fluxes and metric as follows: We first consider world volume fluxes $\tilde{F}_2$ on D7 branes embedded in KS geometry with or without a black hole. These fluxes $\tilde{F}_2$ induce anti-D3 and anti-D5 charges such that the total effective $D5$ charge vanishes, while the effective $D3$ charge no longer diverges in the far UV. This way, the UV divergence of KS theory is removed. The resulting dual geometry takes the form of a warped deformed cone at small radial distances and $AdS_5 \times T^{1,1}$ far away from the tip of the cone. Thus, we have confinement at IR, dual to deformed cone at small radial distance and conformal gauge theory at UV, dual to AdS space.

It is worth mentioning the great deal of effort given in computing the black hole geometry in the presence of flux and scalar fields. For example in [25, 26, 27, 28] the cascading picture of the original KS model was extended to incorporate black-hole without any fundamental matter, while fundamental matter was accounted for in [29]. However, since these black holes are obtained in KS geometry, the dual gauge theories are UV divergent and quite distinct from QCD.

Additionally, most of the attempts are based on obtaining an effective lower dimensional action from KK reducing ten dimensional supergravity action. Dimensional reduction of a generic ten dimensional action can be quite challenging specially when there are non-trivial fluxes and scalar fields. It is also highly non-trivial to obtain a consistent truncation. Furthermore, it is not clear how RG flow of the dual gauge theory can be obtained since the fields in the effective action are not the dilaton or the flux in the original ten dimensional action.

In our approach, we directly work with the ten dimensional geometry, avoiding the difficulty of KK reduction. With the UV divergences of KS theory removed, we study the thermodynamics of the gauge theory by directly identifying the gauge theory partition function with that of the ten dimensional geometry. The thermal
gauge theory arising from the brane excitations has a rich phase structure and as temperature is altered, we expect phase transitions. In this work, we make progress in that direction and obtain ten dimensional geometry (with or without a black hole) that arises from low energy limit of type IIB superstring theory including localized sources.

The role of localized sources is crucial in our analysis, since they allow us to modify the geometry at large radial distances. They also give rise to a radial scale $r_0$ and consequently an energy scale $\Lambda_0$: Warped geometry in small $r$ region i.e. $r < r_0$ correspond to IR modes ($\Lambda < \Lambda_0$) of the gauge theory while inclusion of large $r$ region i.e. $r > r_0$ correspond to including UV modes ($\Lambda > \Lambda_0$) of the gauge theory. For any given temperature of the dual gauge theory, there are two geometries—extremal (without black hole) and non-extremal (with black hole) but the geometry with lower on-shell action is preferred. At a critical temperature $T_c$, both geometries are equally likely and we have a phase transition. We evaluate critical temperature using a perturbative analysis and $T_c$ depends on the boundary conditions as well as the scale $r_0$. Thus the localized sources directly influence the thermodynamics of the gauge theory.

An alternative approach to construct gravitational description of gauge theories is to start with non-critical string theory and consider the resultant dual geometry \cite{38}. For a QCD like gauge theory that confines in the IR and becomes free in the UV, one can obtain a five dimensional dual geometry \cite{39}. The gravity action includes dilaton field and an effective potential for the dilaton. However, the geometry has large curvature and higher order terms in $\alpha'$ need to be included, which will modify the classical gravity action. By considering part of the higher order terms, one can find an effective dilaton potential which in turn can reproduce the QCD beta function. In a bottom up scenario, this effective potential is tuned to fit lattice QCD results for the conformal anomaly and the Polyakov loop. Since the potential is not derived directly from an underlying brane configuration, there is no guarantee that the geometry is in fact a holographic image of a gauge theory.

On the other hand, in our top down approach we proceed by first analyzing brane excitations in conifold geometries where the field theory has global and local symmetries common to that of QCD. At low energies, the excitations give rise to a four dimensional gauge theory which decouples from gravity and can be described holographically by the low energy limit of critical superstring theory i.e supergravity in ten dimensions. As we study a gauge theory arising from strings ending on branes, we know the field content of the theory and in some cases, the exact superpotential at zero temperature. Hence our top down approach is distinct from the bottom up models where a precise knowledge of the gauge theory is lacking or the gravity action is incomplete. To make meaningful quantitative comparisons with QCD, one must identify the gauge theory for which the dual gravity is being constructed. While in bottom up models this identification is not clear, in our top down approach, it is
automatic. Thus the phenomenology that results from this gravity description can be directly compared to that of QCD as the gauge theory resembles large N QCD.

The paper is organized as follows: In section 2.1, we obtain exact values for type IIB fluxes in warped ten dimensional geometry in the presence of D7 branes both in extremal (no black hole) and non-extremal (black hole) limit. The metric and fluxes are evaluated as a Taylor series in perturbative parameter $\mathcal{O}\left(\frac{g_s M^2}{N}\right)$, $\mathcal{O}(g_s N_f)$ where $N, M$ and $N_f$ are number of D3, D5 and D7 branes in the dual gauge theory. For the metric, terms up to linear order are evaluated while the fluxes are obtained at zeroth order. In section 2.2 we propose a brane configuration that can source such fluxes and demonstrate how UV divergence of KS theory can be removed. Using the metric and flux, in section 3.1 the on-shell gravity action is exactly evaluated up to linear order and Hawking-Page like transition is analyzed. In section 3.2, the effect of localized source is incorporated and thermodynamic state functions are obtained. Finally in section 3.3 connections to QCD are established by considering small black holes in deformed cone geometry.

2. Gauge/String duality: From branes to geometry

As already mentioned in the introduction, holographic map between gauge theory and gravity can be constructed by studying excitations of branes placed in certain geometries. The gauge theory arises from open strings ending on the branes while the interactions between open and closed strings leave a holographic imprint of the gauge theory on the geometry. This imprint is captured by the warped dual geometry. At the lowest energies, open and closed string sector decouples and we are left with a gauge theory living in flat four dimensional space which can be described by the dual geometry.

The dual geometry has a classical action with fluxes and localized sources. The classical action is enough to describe the geometry, since the curvature will be small everywhere which in turn can be guaranteed by considering large $M$. In the following section, we analyze this classical action and then in section 2.2, we will describe the brane configuration that can give rise to such geometry.

2.1 Geometry: Fluxes and localized sources in type IIB theory

Consider the type IIB action including $N_f$ number of coincident Dp branes in string frame:\footnote{For $N_f \neq 1$, the abelian action we wrote down gets modified and we need to consider the non-abelian action [40]. Approximating the non-abelian action by taking $N_f$ copies of the abelian action means that we are not distinguishing between different flavors and ignoring their interactions. For our purpose, we can simply set $N_f = 1$ and consider the abelian case.}

$$S_{\text{total}} = S_{\text{SUGRA}} + N_f S_{Dp}$$
\[ S_{\text{SUGRA}} = \frac{1}{2\kappa_{10}^2} \int d^{10}x \sqrt{G} \left( e^{-2\phi} \left( R_{\phi} + 4(\nabla \phi)^2 \right) - \frac{F_1^2}{2} - \frac{|F_3|^2}{4 \cdot 5!} - \frac{G_3 \cdot \tilde{G}_3}{12} \right) \]

\[ + \frac{1}{8i\kappa_{10}^2} \int C_4 \wedge G_3 \wedge \tilde{G}_3 \frac{\tau}{\text{Im} \tau} \]  

(2.1)

where \( \tau = C_0 + i e^{-\phi} \), \( F_1 = dC_0 \) and \( G^s = \det g_{MN}^s \), \( M, N = 0, \ldots, 9 \), \( g_{MN}^s \) is the metric in string frame and \( G_3 = F_3 - \tau H_3 \). Here the action for a \( p \) brane, upto quadratic order in flux \( \tilde{F}_{ab} \equiv B_{ab} + F_{ab} \) is given by

\[ S_{Dp} = - \int d^{p+1}x T_p \sqrt{-f} \left( 1 + \frac{1}{4} \tilde{F}_{ab} \tilde{F}_{ab} \right) + \mu_p \int \left( C e^{\tilde{F}} \right)_{p+1} \]  

(2.2)

Here \( f^s = \det f_{ab}^s \), \( f_{ab}^s = g_{MN}^s \partial_a X^M \partial_b X^N \) is the pull back metric, \( B_{ab} = \hat{B}_{MN} \partial_a X^M \partial_b X^N \). By taking the real part of the action and minimizing it, we can obtain the real valued fluxes \( F_3, H_3 \), the metric \( g_{MN}^s \) and the scalar fields \( \phi, C_0 \). We will consider the following action

\[ S_{\text{SUGRA}} = \frac{1}{2\kappa_{10}^2} \int d^{10}x \sqrt{G} \left( e^{-2\phi} \left( R_{\phi} + 4(\nabla \phi)^2 \right) - \frac{F_1^2}{2} - \frac{|F_3|^2}{4 \cdot 5!} - \frac{G_3 \cdot \tilde{G}_3}{12} \right) \]

\[ - \frac{1}{4\kappa_{10}^2} \int C_4 \wedge H_3 \wedge F_3 \]  

(2.3)

We can simplify equations resulting from variation of the above action by absorbing the scalar field in the definition of the metric. This is done by going to the Einstein frame defined through \( g_{MN} = g_{MN}^s e^{-\phi/2} \), where the action (2.3) takes the following form

\[ S_{\text{SUGRA}} = \frac{1}{2\kappa_{10}^2} \int d^{10}x \sqrt{G} \left( R + \frac{\partial_M \tau \partial^M \tau}{2 \text{Im} \tau} - \frac{|F_3|^2}{4 \cdot 5!} - \frac{G_3 \cdot \tilde{G}_3}{12 \text{Im} \tau} \right) \]

\[ - \frac{1}{4\kappa_{10}^2} \int C_4 \wedge H_3 \wedge F_3 \]  

(2.4)

\[ S_{Dp} = - \int d^{p+1}x T_p e^{\frac{d(p+1)}{4}} \sqrt{-f} \left( 1 + \frac{1}{4} \tilde{F}_{ab} \tilde{F}_{ab} \right) + \mu_p \int \left( C e^{\tilde{F}} \right)_{p+1} \]  

(2.5)

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\(^3\text{We will be considering } D7 \text{ branes for which the Chern-Simons action contains } \sim \mu \gamma^2 \int [C_4 \wedge \text{tr}(R \wedge R) - e^{-\phi} \text{tr}(R \wedge *R)] \text{ where } R \text{ is the pullback of the curvature two form. However we consider geometries such that } R \wedge R \text{ and } R \wedge *R \text{ is zero on the world volume of the brane.} \)
where \( f = \det f_{ab} \), \( f_{ab} = g_{MN} \partial_a X^M \partial_b X^N \) and \( \widetilde{F}_{ab} \) is raised or lowered with the pullback metric \( f_{ab} \) in Einstein frame. The background warped metric takes the following familiar form

\[
\begin{aligned}
    ds^2 &= g_{MN} \, dx^M dx^N \equiv g_{\mu\nu} \, dx^\mu dx^\nu + g_{mn} \, dx^m dx^n \\
    &= -e^{2A+2B} \, dt^2 + e^{2A} (dx^2 + dy^2 + dz^2) + e^{-2A-2B} \tilde{g}_{mn} \, dx^m dx^n
\end{aligned}
\]

(2.6)

where the internal unwarped metric is given by

\[
\begin{aligned}
    \tilde{g}_{mn} dx^m dx^n &= \frac{1}{2} A^{4/3} K(\rho) \left[ \frac{1}{3K^3(\rho)} \left( d\rho^2 + e^{2B} (g^5)^2 \right) + \cosh^2 \left( \frac{\rho}{2} \right) e^{2B} \left[ (g^3)^2 + (g^4)^2 \right] \\
    &\quad + \sinh^2 \left( \frac{\rho}{2} \right) e^{2B} \left[ (g^1)^2 + (g^2)^2 \right] \right] \\
    K(\rho) &= \frac{(\sinh(2\rho) - 2\rho)^{1/3}}{2^{1/3} \sinh \rho}
\end{aligned}
\]

(2.7)

Here \( \tilde{g}_{mn} \) is the metric of the base of deformed cone while \( \tilde{g}_{mn}^1 \) is the perturbation due to the presence of fluxes and localized sources. Also \( A \) is a constant, \( g^i, i = 1, \ldots, 5 \) are one forms given by

\[
\begin{aligned}
    g^1 &= \frac{e^1 - e^3}{\sqrt{2}}, \quad g^2 = \frac{e^2 - e^4}{\sqrt{2}} \\
    g^3 &= \frac{e^1 + e^3}{\sqrt{2}}, \quad g^4 = \frac{e^2 + e^4}{\sqrt{2}}, \quad g^5 = e^5 \\
    e^1 &\equiv -\sin \theta_1 \, d\phi_1, \quad e^2 \equiv d\theta_1 \\
    e^3 &\equiv \cos \psi \sin \theta_2 \, d\phi_2 - \sin \psi \, d\theta_2, \\
    e^4 &\equiv \sin \psi \sin \theta_2 \, d\phi_2 + \cos \psi \, d\theta_2, \\
    e^5 &\equiv d\psi + \cos \theta_1 \, d\phi_1 + \cos \theta_2 \, d\phi_2
\end{aligned}
\]

(2.8)

and \( m.n = 4, \ldots, 9 \) denote the internal ‘cone’ direction while \( \mu, \nu = 0, \ldots, 3 \) run over Minkowski directions. The warp factor \( A(x^m), B(x^m) \) are functions of the cone coordinate \( x^m = \rho, \psi, \phi_1, \phi_2, \theta_1, \theta_2 \). Observe that with a change of coordinates

\[
r^3 = A^2 e^\rho
\]

(2.9)

for large \( \rho \), the metric becomes

\[
\tilde{g}_{mn} dx^m dx^n \sim dr^2 + r^2 e^{2B} \left( \frac{1}{9} (g^5)^2 + \frac{1}{6} \sum_{i=1}^4 (g^i)^2 \right)
\]

(2.10)

which is the metric of regular cone with base \( T^{1,1} \). Thus only for small radial coordinate \( \rho \), the internal metric is a deformed cone while at large \( \rho \), we really have a regular cone with topology of \( R \times T^{1,1} \).
We will now consider \( p = 7 \), that is we embed D7 branes in the large \( \rho \) region
where \( r \) is the more convenient radial coordinate. Adding these sources in the large
\( \rho \) region means that we are only modifying the UV of the dual gauge theory and we
expect that the IR of gauge theory remain mostly unaltered. The effect of this brane
embedding for the gauge theory will be discussed in detail in section 2.2.

The D7 branes fill up Minkowski space \((t, x, y, z)\), stretching along \( r \) direction
and filling up \( S^3 \) inside the \( T^{1,1} = S^3 \times S^2 \). We consider two branches:

- Branch I with parametrization \((\sigma^0, \sigma^1, ..., \sigma^7) = (t, x, y, z, r, \psi, \phi_2, \theta_2)\). The brane
  fills up 4D Minkowski space and stretches along the \( r \) direction, filling up an \( S^3 \)
  inside \( T^{1,1} = S^3 \times S^2 \). It is a point \((\phi_1(\sigma^a), \theta_1(\sigma^a))\) inside \( S^2 \) and we pick a profile
  such that \( \theta_1(\sigma^a) = \pi/2 \) and \( \phi_1(\sigma^a) \equiv \tilde{\phi}_1(r) \) is only a function of the \( r \).
The DBI part of the world volume action for this branch takes the form

\[
S_I = -|\mu_I| \left( V_4 \int d\Omega^I \, dr \, \frac{r^3 e^{\phi}}{18} \sqrt{1 + \frac{e^{2B_I r^2}}{6} \phi_1^2} + \int \frac{1}{2} F_2^I \wedge \ast_f F_2^I \right) 
\]

(2.11)

where \( V_4 \equiv \int d^4x, d\Omega^I \equiv d\psi \, d\phi_2 \, d\theta_2 \sin(\theta_2) \), we have denoted world volume flux on
branch I with \( \tilde{F}_2^I \) and \( \ast_f \) is the Hodge star with respect to the pullback metric \( f \)
of the branch. In obtaining the above action from (2.5), we have used the definition of tension of Dp brane, \( T_p = |\mu_p| e^{-\phi} \).

Now observe that Branch I of D7 brane is a point
on an \( S^2 \) with volume form

\[
\Omega_1 \equiv \sin(\theta_1) d\phi_1 \wedge d\theta_1 
\]

(2.12)

Thus the Chern-Simons action for Branch I can be written as

\[
S_{CS}^I = \mu_I \int \Gamma_1 \left( C_8 + C_6 \wedge \tilde{F}_2^I + \frac{1}{4} C_4 \wedge \tilde{F}_2^I \wedge \tilde{F}_2^I + \frac{1}{6} C_2 \wedge \tilde{F}_2^I \wedge \tilde{F}_2^I \wedge \tilde{F}_2 \right) \wedge \Omega_1 
\]

(2.13)

where \( C_4 \) is the pullback of the RR form to the world volume \(^4\)

\[
\Gamma_1 = \delta(\theta_1 - \pi/2) \delta \left( \phi_1 - \tilde{\phi}_1(r) \right) 
\]

(2.14)

- Branch II with parametrization \((\sigma^0, \sigma^1, ..., \sigma^7) = (t, x, y, z, r, \psi, \phi_1, \theta_1)\). Again the brane
  fills up 4D Minkowski space and stretches along the \( r \) direction, fills up an \( S^3 \)

\(^4\)In deriving (2.13), we put an additional factor of \( 1/2 \) in front of \( C_4 \), starting with the Chern-
Simons action (2.5). This is because five form \( F_5 = dC_4 \) is self dual, that is \( \ast_5 dC_4 = dC_4 \).
Then \( C_4 \) includes both electric and magnetic flux since typically \( \ast E = B \), where \( E \) and \( B \) are electric
and magnetic fluxes. Defining \( C_4 = C_4^{RR} + C_4^{NS} \), we can get \( \ast dC_4^{NS} = \ast dC_4^{RR} = dC_4^{NS} \)
and \( dC_4^{RR} = 1/2 dC_4, \, dC_4^{NS} = 1/2 \ast dC_4 \). The Chern-Simons term is \( 1/2 C_4^{RR} \wedge \tilde{F}_2 \wedge \tilde{F}_2 \), and since
\( C_4^{RR} = 1/2 C_4 \), we get the expansion (2.13)
inside $T^{1,1} = S^3 \times S^2$ but now is a point $(\phi_2(\sigma^\alpha), \theta_2(\sigma^\alpha))$ inside $S^2$. Again we pick a profile such that $\theta_2(\sigma^\alpha) = \pi/2$ and $\phi_2(\sigma^\alpha) \equiv \tilde{\phi}_2(r)$ is only a function of the $r$. The DBI part of the world volume action for this branch takes the form

$$S_{II} = -|\mu_7| \left( V_4 \int d\Omega^{II} dr \frac{r^3 e^\phi}{18 \sqrt{1 + \frac{e^{2B} r^2}{6} \phi_2^2 + \int \frac{1}{2} \tilde{F}_2^{II} \wedge * \tilde{F}_2^{II}} \right)$$

(2.15)

where $d\Omega^{II} \equiv d\psi d\phi_1 d\theta_1 \sin(\theta_1)$. Observing that Branch II of D7 brane is a point on an $S^2$ with volume form

$$\Omega_2 \equiv \sin(\theta_2) d\phi_2 \wedge d\theta_2$$

(2.16)

the Chern-Simons action for Branch II can be written as

$$S_{CS}^{II} = \mu_7 \int \Gamma_2 \left( C_8 + C_6 \wedge \tilde{F}_2^{II} + \frac{1}{4} C_4 \wedge \tilde{F}_2^{II} \wedge \tilde{F}_2^{II} + \frac{1}{6} C_2 \wedge \tilde{F}_2^{II} \wedge \tilde{F}_2^{II} \wedge \tilde{F}_2^{II} \right) \wedge \Omega_2$$

(2.17)

where

$$\Gamma_2 = \delta(\theta_2 - \pi/2) \delta \left( \phi_2 - \tilde{\phi}_2(r) \right)$$

(2.18)

Note that $C_2$ has no legs in the $t, x, y, z$ direction and thus the last term in both (2.13) and (2.17) do not contribute. Now varying the action (2.5) with respect to $\tilde{F}_2$, we get the following equation

$$d \left( * \tilde{F}_2 \right) = d \left[ \frac{2 \mu_7}{|\mu_7|} \left( C_6 + \frac{1}{2} C_4 \wedge \tilde{F}_2 \right) \right]$$

(2.19)

where $\tilde{F}_2 = \tilde{F}_2^{I}$ or $\tilde{F}_2^{II}$.

Note that $G_3 = F_3 - \tau H_3$, and $*_{10} F_3 = dC_6 + I_7$ where $I_7$ is some seven form. Then the action $S_{\text{total}}$ (where $S_{\text{SUGRA}}$ is given by (2.4)) to be stationary under variation $C_6$ and $C_2$ gives

$$d \left( \frac{\tilde{F}_3}{10 \mu_7} \right) = 4 \kappa_{10}^2 \mu_7 N_f \left( \Gamma_1 \tilde{F}_2^{II} \wedge \Omega_1 + \Gamma_2 \tilde{F}_2^{II} \wedge \Omega_2 \right)$$

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where $\tilde{F}_3 = F_3 - C_0 H_3$. In deriving the above equation from (2.4) and (2.5), we used the relation $G_3 \cdot \tilde{G}_3 = |\tilde{F}_3|^2 + |\text{Im}\tau H_3|^2$ and $\int \sqrt{G} |\tilde{F}_3|^2 = \int 6\tilde{F}_3 \wedge \ast_{10} \tilde{F}_3$.

Now observe that $\bar{F}_5 = dC_4 + I_5$, where $I_5$ is some five form. Then for the action to be stationary under variation of $C_4$ gives the Bianchi identity

$$d \bar{F}_5 = H_3 \wedge F_3 - \mu_7 \kappa_{10} N_f \left( \Gamma_1 \bar{F}^I \wedge \bar{F}^I \wedge \Omega_1 + \Gamma_2 \bar{F}^{II} \wedge \bar{F}^{II} \wedge \Omega_2 \right)$$

(2.21)

Now Lorentz invariance and self-duality fixes the five form to be

$$\bar{F}_5 = (1 + \ast_{10}) d\alpha \wedge dt \wedge dx \wedge dy \wedge dz$$

(2.22)

where $\alpha(x^m)$ is a scalar function. Variation of the action with respect to the space time metric $g_{MN}$ gives the Einstein equations

$$R_{\mu\nu} = -g_{\mu\nu} \left[ \frac{G_3 \cdot \tilde{G}_3}{48 \text{Im}\tau} + \frac{\tilde{F}_3^2}{8 \cdot 5!} \right] + \frac{\tilde{F}_{\mu a b c d} \tilde{F}_{\nu a b c d}}{4 \cdot 4!} + \kappa_{10}^2 N_f \left( T_{\mu\nu}^{\text{loc}} - \frac{1}{8} g_{\mu\nu} T^{\text{loc}} \right)$$

$$R_{mn} = -g_{mn} \left[ \frac{G_3 \cdot \tilde{G}_3}{48 \text{Im}\tau} + \frac{\tilde{F}_3^2}{8 \cdot 5!} \right] + \frac{\tilde{F}_{ma b c d} \tilde{F}_{n a b c d}}{4 \cdot 4!} + \frac{G_{m c} G_{n b c}}{4 \text{Im}\tau} + \frac{\partial_m \tau \partial_n \tau}{2 |\text{Im}\tau|^2} + \kappa_{10}^2 N_f \left( T_{mn}^{\text{loc}} - \frac{1}{8} g_{mn} T^{\text{loc}} \right)$$

(2.23)

where $\tilde{R}_{mn}$ is the Ricci tensor for the metric $\tilde{g}_{mn}$ and $T^{\text{loc}}_{MN}$ is defined through

$$T^{\text{loc}}_{MN} = -\frac{2}{\sqrt{G}} \frac{\delta S_{D7}}{\delta g^{MN}}$$

(2.24)

We want to solve the flux equations (2.19),(2.20), (2.21), the Einstein equations (2.23) and simultaneously find the embedding $\tilde{\phi}(r)$ that minimizes the action. But before we do so, observe that $\Omega_i$ is closed form, i.e. $d\Omega_i = 0$. Then taking derivative of the first equation in (2.20) gives

$$\Gamma_1 d \left( \text{Im}\tau \tilde{F}_2^I \right) \wedge \Omega_1 + \Gamma_2 d \left( \text{Im}\tau \tilde{F}_2^{II} \right) \wedge \Omega_2 = 0$$

(2.25)

which has the solution

$$d \left( \text{Im}\tau \tilde{F}_2^I \right) = 0, \quad d \left( \text{Im}\tau \tilde{F}_2^{II} \right) = 0$$

(2.26)

On the other hand, the four form is given by

$$C_4 = \alpha(x^m) dt \wedge dx \wedge dy \wedge dz$$

(2.27)
The scalar function \( \alpha(x^m) \) can be obtained from (2.21), which is the Bianchi identity while the warp factor \( A, B \) can be obtained from the Einstein equations. We now proceed as follows: The Ricci tensor in the Minkowski direction takes the following simple form

\[
R_{\mu\nu} = -\frac{1}{2} \left[ \partial_m (g^{mn} \partial_n g_{\mu\nu}) + g^{mn} \Gamma^M_{nM} \partial_m g_{\mu\nu} - g^{nm} g^{\nu\prime \mu\prime} \partial_m g_{\nu\prime \mu\prime} \partial_n g_{\nu\prime \mu\prime} \right]
\]

(2.28)

where \( \nu', \mu' = 0, \ldots, 3 \) and \( \Gamma^M_{nM} \) is the Christoffel symbol. Now using the ansatz (2.6) for the metric, (2.28) can be written as

\[
R_{tt} = e^{2(A+B)} \left[ \tilde{\nabla}^2 (4A + B) - 3g^{mn} \partial_m B \partial_n (4A + B) \right]
\]

\[
R_{ij} = -\eta_{ij} e^{2(A+B)} \left[ \tilde{\nabla}^2 A - 3g^{mn} \partial_n B \partial_m A \right]
\]

(2.29)

where the Laplacian is defined as

\[
\tilde{\nabla}^2 = g^{mn} \partial_m \partial_n + \partial_m \tilde{g}^{mn} \partial_n + \frac{1}{2} \tilde{g}^{mn} \tilde{g}^{pq} \partial_n \tilde{g}_{pq} \partial_m
\]

(2.30)

The set of equations can be simplified by taking the trace of the first equation in (2.23) and using (2.29). Doing this we get

\[
\tilde{\nabla}^2 (4A + B) - 3g^{mn} \partial_m B \partial_n (4A + B) = e^{-2A-2B} G_{mnp} \tilde{G}^{mnp} \frac{12i \text{Im} \tau}{2} + e^{-10A-4B} \partial_m \alpha \partial^m \alpha
\]

\[
+ \frac{k^2_{10} N_f}{2} e^{-2A-2B} (T_{m} - T_{\nu})_{\text{loc}}
\]

(2.31)

On the other hand using (2.23) in (2.29), one gets

\[
R^t_t - R^x_x = 0
\]

(2.32)

which in turn would immediately imply

\[
\tilde{\nabla}^2 B - 3g^{mn} \partial_m B \partial_n B = 0
\]

(2.33)

Now let’s look at the Bianchi identity. Using (2.22) in (2.21) gives

\[
\tilde{\nabla}^2 \alpha - 3e^{-2A-2B} \partial_m B \partial^m \alpha = e^{2A-B} *_6 \tilde{G}_3 \cdot \tilde{G}_3 \frac{12i \text{Im} \tau}{2} + 2e^{-6A-3B} \partial_m e^{4A+B} \partial^m \alpha + \mathcal{L}_{\text{loc}}
\]

(2.34)

where \(*_6\) is the Hodge star for the metric \( g_{mn} \) and

\[
\mathcal{L}_{\text{loc}} \sim \mu \tau \kappa_{10}^2 N_f \tilde{F}_{ab} \tilde{F}_{ab} e^{4A} \Gamma_i
\]

(2.35)

Now taking the trace of the first equation in (2.23), subtracting it from (2.34), we get

\[
\tilde{\nabla}^2 [e^{4A} (e^B - \gamma)] = \frac{e^{2A-B}}{24 \text{Im} \tau} |i \tilde{G}_3 - *_6 \tilde{G}_3|^2 + e^{-6A-3B} |\partial e^{4A} (e^B - \gamma)|^2
\]
\[ +3 e^{-2A-2B} \partial_m B \partial^m \left[ e^{4A} (e^B - \gamma) \right] + \frac{k_{10}^2 N_f}{2} e^{2A-B} (T^m_{\mu} - T^\mu_{\mu})_{\text{loc}} - \mathcal{L}_{\text{loc}} \]  

(2.36)

where \( \gamma \equiv \alpha e^{-4A} \). Now, we can ignore the localized term

\[ \mathcal{I}_{\text{loc}} = \frac{k_{10}^2 N_f}{2} e^{2A-B} (T^m_{\mu} - T^\mu_{\mu})_{\text{loc}} - \mathcal{L}_{\text{loc}} \]  

(2.37)

which is in fact second or higher order in our perturbation, as we shall see in what follows. Solving (2.31), (2.33) and (2.36) together will give the scalar functions \( \alpha, A \) and \( B \). We can solve the system perturbatively, order by order in our perturbative parameter

\[ \epsilon \equiv \mathcal{O} \left( \frac{g_s \tilde{M}^2}{N} \right), \mathcal{O} (g_s N_f) \]  

(2.38)

where \( \tilde{M} \) is a unit less constant defined through \( G_3 \sim \tilde{M} G_3 \), \( G_3 \) being a 3-form and \( N_f \) is the number of D7 branes.

Now writing \( *_{10} dC_0 = 2 |\text{Im} \tau|^2 dC_8 \), we get the following equation from \( S_{\text{total}} \), by varying with respect to \( C_8 \)

\[ dF_1 \sim 4 \kappa_{10}^2 \mu_7 N_f (\Gamma_1 \Omega_1 + \Gamma_1 \Omega_2) \]  

(2.39)

which leads to

\[ C_0 \sim \mathcal{O} (N_f) \]  

(2.40)

Using this normalization of the axion field, we get following scaling of the dilaton field, using F-theory

\[ \frac{1}{\text{Im} \tau} = e^\phi = g_s (1 + \mathcal{O} (g_s N_f) \mathcal{J}(x^m) ) \]  

(2.41)

where \( \mathcal{J}(x^m) \) is some function describing the running of dilaton. We solve (2.31), (2.33) and (2.36) by only considering terms up to \( \mathcal{O} (\epsilon) \) i.e. linear order in our perturbation. In the large \( \rho \) region, by the switching to \( r \) coordinate, we find following scaling of the solution with our perturbative parameter

\[ \gamma = 1 + \sum_{l=1}^{\infty} \mathcal{O} \left( \frac{g_s \tilde{M}^2}{N} \right) \mathcal{O} \left( \frac{\tilde{r}_h}{r} \right)^{4l} \]  

\[ e^{-4A} = \frac{27 \pi N \alpha'}{4 r^4} \left[ 1 + \sum_{j=0}^{\infty} \mathcal{O} \left( \frac{\tilde{r}_h}{r} \right)^{4j} \mathcal{O} \left( \frac{g_s \tilde{M}^2}{N} \right) \right] \]  

\[ e^{2B} = 1 - \frac{\tilde{r}_h}{r^4} + \sum_{l=1}^{\infty} \mathcal{O} \left( \frac{\tilde{r}_h}{r} \right)^{4l} \mathcal{O} \left( \frac{g_s \tilde{M}^2}{N} \right) \]  

(2.42)
It is instructive to note that at zeroth order in our perturbation, the above solution gives an AdS warp factor and a Schwarzschild black hole with horizon radius $\tilde{r}_h$. When perturbation is included, the true horizon surface $x^m = x^m_h$ defined through the relation $e^{B(x^m = x^m_h)} = 0$, would have radial location $r_h$ given by

$$r_h = \tilde{r}_h \left( 1 + O \left( \frac{g_s \tilde{M}^2}{N} \right) \right)$$

(2.43)

Using (2.42) and the form of $C_4$ as given in (2.27), equation (2.19) can be solved with

$$\star_4 \tilde{F}_2 = \frac{\mu_7}{\mu_7} \tilde{F}_2 \left( 1 + \sum_{l=1}^{\infty} O \left( \frac{g_s \tilde{M}^2}{N} \right) O \left( \frac{\tilde{r}_h}{r} \right)^{4l} \right)$$

$$C_6 = \frac{|\mu_7|^4 A_4}{2 \mu_7} \left( e^B - 1 \right) dt \wedge dx \wedge dy \wedge dz \wedge \tilde{F}_2$$

(2.44)

where $\star_4$ is the Hodge star for the metric $f_{\alpha\beta}$, $\alpha, \beta \neq 0, 1, 2, 3$.

Combining (2.26) and (2.44), we see that $\tilde{F}_2$ is self dual (or anti-self dual) while $\text{Im} \tau \tilde{F}_2$ is closed at zeroth order in our perturbation. Thus it takes the following form

$$\tilde{F}_2^I = \left( 1 + \frac{\mu_7}{|\mu_7|} \star_4 \right) \frac{\mathcal{M} \alpha' \sqrt{1 + \frac{r^2 e^{2B} \hat{\phi}_I^2}{6}}}{re^B \text{Im} \tau} \left[ dr \wedge d\psi + a \ dr \wedge d\phi_2 \right]$$

$$\tilde{F}_2^{II} = \left( 1 + \frac{\mu_7}{|\mu_7|} \star_4 \right) \frac{-\mathcal{M} \alpha' \sqrt{1 + \frac{r^2 e^{2B} \hat{\phi}_I^2}{6}}}{re^B \text{Im} \tau} \left[ dr \wedge d\psi + a \ dr \wedge d\phi_1 \right]$$

(2.45)

where $\mathcal{M}, a$ are constants. The factor $(1 + \star_4)$ makes the flux self dual while the function $f_i \equiv \sqrt{\frac{1 + \frac{r^2 e^{2B} \hat{\phi}_I^2}{6}}{re^B \text{Im} \tau}}$ is exactly chosen for closure. One can readily check that using $\tilde{g}_{00}$ as the internal metric, indeed the above flux satisfies closure and self duality.

The form in (2.45) also makes it clear that localized term that we ignored to obtain (2.36), are

$$\mathcal{L}_{\text{loc}} \sim O \left( \frac{g_s^2 N_f M^2}{N^2} \right)$$

$$\sim O \left( \frac{g_s M^2}{N} \right) O \left( g_s N_f \right) O \left( \frac{1}{N} \right)$$

$$\kappa_1^{2N_f} e^{2A} (T^m_m - T^{\mu}_{\mu}) = \kappa_1^{2N_f} |\mu_7| e^{4A} \sqrt{\tilde{F}_2^I} \tilde{F}_2^{II}$$

$$\sim O \left( \frac{g_s^2 N_f M^2}{N^2} \right) \sim O \left( \frac{g_s M^2}{N} \right) O \left( g_s N_f \right) O \left( \frac{1}{N} \right)$$

(2.46)

Then viewing (2.36) as an equation for $\gamma$ gives that the localized terms are of second order in our perturbation- which justifies ignoring them. On the other hand, solving
the second equation in \((2.23)\), one obtains that

\[
\tilde{g}_{mn}^1 \sim \sum_{l=1} O \left( \frac{r_h}{r} \right)^{4l} O \left( \frac{g_s M^2}{N} \right) + O (g_s N_f) + \text{higher order} \quad (2.47)
\]

Thus at zeroth order in our perturbation, \((2.45)\) is an exact solution of \((2.26)\) and \((2.44)\). Although the world volume flux is evaluated at zeroth order in our perturbative parameter, the flux \(\tilde{F}_2\) gives rise to linear order terms \(O \left( \frac{g_s M^2}{N} \right)\) in the supergravity action and eventually regularizes it. This is not surprising since the background magnetic field \(B_2 \sim O (g_s M)\) in the KT solution also gives rise to linear order terms \(O \left( \frac{g_s M^2}{N} \right)\) in the supergravity action.

On the other hand, using the form of the flux, one readily observes that the action \(S_I, S_{II}\) are now only a functional of \(\tilde{\phi}_1(r)\) and \(\tilde{\phi}_2(r)\) respectively. Variation of the action with respect to \(\tilde{\phi}_i(r)\) gives the following equations

\[
\frac{e^\phi r^5 \tilde{\Phi}_i'}{\sqrt{1 + \frac{r^2}{6} \tilde{\Phi}_i^2}} = C_i
\]

\[
\tilde{\phi}_i = \int dr \ e^{-B} \frac{\partial \tilde{\Phi}_i}{\partial r}
\]

where \(i = 1, 2\) and \(C_i\) are constants. The above equation has a simple solution

\[
\cos \left( \frac{4}{\sqrt{6} \tilde{\Phi}_i} \right) = \frac{r_0^4}{r^4} \quad (2.49)
\]

where \(r_0\) is a constant. Note that \(\tilde{\Phi}_i(r_0) = 0\) and \(\tilde{\Phi}_i(\infty) = \pm \frac{\sqrt{6} \pi}{8}\). However, we pick \(\tilde{\Phi}_i(\infty) = + \frac{\sqrt{6} \pi}{8}\) for both branches I and II of the D7 branes. We can invert \(\tilde{\Phi}_i(r)\) to obtain \(r(\tilde{\Phi}_i)\) and observe that \(r\) has a minimum given by \(r_0\). Thus both branches of D7 brane extend from \(r = \infty\) to \(r = r_0\). For \(r_0 = 0\), we have \(\tilde{\phi}_i = \text{constant}\), and the D7 brane stretches from the tip \(r = 0\) to \(r = \infty\). Observe that each branch is similar to the embedding considered in \([41]\). The stability of such embeddings were studied in \([42]\) and thus we expect our embedding to be stable under perturbations. A heuristic argument for stability could be the observation that the D7 world volume \((2.11)\) and \((2.15)\) are independent of warp factors at linear order in \(O(\epsilon)\) and thus the brane sees a flat geometry. Hence there is no gravitational pull and the embedding is stable.

\(5\)We are essentially ignoring the running of the dilaton on the world volume of the brane. Note that the dilaton field sourced by Branch I on Branch I is zero but the field sourced by Branch II is finite on Branch I. Here we are neglecting the running of the field on Branch I sourced by Branch II and vice versa. If we account for the running, the brane profile will get \(O(g_s N_f)\) corrections which can be ignored since we only consider the profile at zeroth order.

\[–14–\]
Note that the $D7$ brane embedding (2.48) is well defined for $r_h < r_0$ with $r_h$ being the horizon i.e. $e^B(r_h) = 0$. When $r_0 = r_h$, (2.48) implies that $\tilde{\phi}_i(r_h)$ is divergent and thus the embedding is not well defined. Thus we will first consider the localized sources to be outside of the horizon. Then whether we have a black hole or not, the boundary values are $\tilde{\phi}(r_0) = 0$ and $\tilde{\phi}(\infty) = +\sqrt{\frac{3}{8}}$ for both branches I and II of the D7 branes. However, the shape of the embedding are quite different in the presence of the black hole. Again, by inverting, we can write $r(\tilde{\phi})$ and conclude that the D7 brane ends at $r = r_0$.

As we consider larger horizons, we will have $r_h = r_0$ and then the D7 brane will simply fall into the black hole and there will be no action for the brane. Note that the supergravity background without any localized sources already has non-zero three form flux $G^0_3$, sourced by the $D5$ branes of the dual gauge theory. So even at the absence of any localized sources in (2.4), we consider a charged black hole. The addition of localized source for small black holes (that is $r_h < r_0$) then alters $G^0_3 \to G_3$ by inducing additional charges and modifies the total charge of the black hole. When $r_h \geq r_0$, the localized sources fall into the black hole and we do not have an action for them. However, we must account for their induced charges and consider $G_3 \neq G^0_3$ even without any localized sources outside the horizon. We can choose boundary conditions such that the induced charge is negative and exactly cancels the background charge. Thus when $r_h \geq r_0$, the induced charge fall into the black hole and we can simply consider $G_3 = 0$ outside the horizon. We will elaborate these points in the following sections.

Now for the case $r_h < r_0$, that is the brane is outside the horizon and acts as a localized source for the world volume fluxes $\tilde{F}_2$, we can solve the two equations in

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{D7_Brane_Embedding.png}
\caption{D7 brane embedding; $i = 1$ or 2 corresponds to Branch I or II.}
\end{figure}
(2.20). The solution is

\[ F_3 = \frac{M \alpha'}{2} \omega_3 + 4 \kappa_{10}^2 M \mathcal{N} \alpha' \mu \tau \left( F(r) \tilde{\omega}_3^{-1} + H(r) \tilde{\omega}_3^{-2} \right) \]

\[ H_3 = \frac{\ast_6 (e^B F_3)}{\text{Im} \tau} \]

(2.50)

where \( \ast_6 \) is the Hodge star for the metric \( g_{mn} \) and we have neglected second and higher order in \( \mathcal{O}(\alpha_s) \). Also we have defined the three forms \( \omega_3, \tilde{\omega}_3^{-1}, \tilde{\omega}_3^{-2} \) and the scalar functions \( F(r), H(r) \) as follows

\[
\omega_3 = g^5 \wedge (g^1 \wedge g^2 + g^3 \wedge g^4)
\]

\[
\tilde{\omega}_3^{-1} = \Gamma_1 (d\psi + a \, d\phi_2) \wedge \Omega_1 - \Gamma_2 (d\psi + a \, d\phi_1) \wedge \Omega_2
\]

\[
\tilde{\omega}_3^{-2} = \frac{\text{Im} \tau}{\mathcal{M}} \left[ \Gamma_1 \cos(\theta_1) F^{I}_{\phi_2} d\phi_1 \wedge d\phi_2 \wedge d\theta_2 - \Gamma_2 \cos(\theta_2) F^{II}_{\phi_1} d\phi_1 \wedge d\phi_2 \wedge d\theta_1 + d\psi \wedge \cos(\theta_1) F^{I}_{\psi_2} d\phi_2 \wedge d\phi_1 \right]
\]

\[
H(r) = 1 \quad \text{for} \ r \geq r_0
\]

\[
= 0 \quad \text{for} \ r < r_0
\]

\[
F(r) = \int^r du \sqrt{1 + \frac{u^2 e^{2B(u)}}{6} \left( \frac{d\phi_1}{du} \right)^2 H(u)}
\]

(2.51)

The fluxes in (2.50) are the key results of our analysis. They represent the total RR and NS-NS three form flux in the presence of world volume fluxes on D7 branes. For small black holes that is \( r_h < r_0, \) (2.50) gives the fluxes in the presence of a black hole while by setting \( \tilde{r}_h = 0, \) one obtains the fluxes in vacuum.

Now using the internal metric as \( \tilde{g}_{mn} \) with \( g_{mn} = e^{-2A - 2B + \ast_6 g_{mn}} \) and the world volume flux \( \tilde{F}_2 \) as in (2.45), one can readily check that indeed the fluxes in (2.50) solves the equations (2.20) up to linear order in our perturbation. In the extremal limit, \( \tilde{r}_h = 0, B = 0 \) and one readily gets that \( \ast_6 G_3 = iG_3 \), that is \( G_3 \) is ISD in the absence of a black hole.

Note that the \( G_3 \) above is obtained using \( \tilde{F}_2 \) that is zeroth order in our perturbative parameter \( \epsilon \). Thus the solution we presented here for \( G_3, \tilde{F}_2 \) are exact up to zeroth order in \( \epsilon \). For higher order corrections to the flux, we must exactly solve the Einstein equations (2.23) to all order in \( \epsilon \) to obtain \( \tilde{g}_{mn}^I \) and use the metric to find closed, self dual \( \tilde{F}_2 \). Then we can solve (2.20) using \( \tilde{F}_2 \) at higher order and obtain \( G_3 \) that will include higher order terms in \( \mathcal{O}(\epsilon) \).

Now observe that the integrand appearing in the definition of \( F(r) \) is only non-vanishing for \( r_0 \leq r \) and thus we can choose boundary conditions such that \( F(r) = 0 \) for \( r \leq r_0 \). Then we can choose \( \mathcal{M} \) such that \( \lim_{r \to \infty} \int F_3 \to 0 \) and eliminate the UV divergences of Klebanov-Strassler theory as shown in the following section.
2.2 Brane engineering: From non-conformal confining IR to conformal UV

The supergravity solution along with fluxes presented in the previous section may arise from a specific configuration of branes placed in conifold geometries. Before going into the exact set up of branes, we briefly review two related configurations of branes that give rise to Klebanov-Witten and Klebanov-Tseytlin/Klebanov-Strassler model: Place N D3 branes at the tip of a regular cone [See Fig 2(a)]. At zero temperature, the gauge group is $SU(N) \times SU(N)$ with bi-fundamental fields $A_i, B_j, i, j = 1, 2$. This gauge theory has a conformal fixed plane and the number of D3 branes remains the same at all energy scales. This is the Klebanov-Witten model [43].

Now if we put another stack of D5 branes that wraps the vanishing two cycle at the tip of the cone [See Fig 2(b)] the gauge theory becomes $SU(N + M) \times SU(N)$ with the bi-fundamental fields, and it is no longer conformal. The $SU(M + N)$ sector has $2N$ effective flavors while the $SU(N)$ sector has $2(N + M)$ effective flavors thus it is dual to the $SU(N - M) \times SU(N)$ gauge theory under an Seiberg duality. Under a series of such dualities which is called cascading, at the far IR region the gauge theory can be described by $SU(M) \times SU(K)$ group, where $N = lM + K$, $l, 0 \leq K < M$ are positive integers. Now the number of ‘actual’ D3 branes $N$ is no longer the relevant quantity, rather $N \pm pM$ where $p$ is an integer describes the D3 brane charge. We take $K = 0$ in all our analysis, so at the bottom of the cascade, we are left with $\mathcal{N} = 1$ SUSY $SU(M)$ strongly coupled gauge theory which looks very much like strongly coupled SUSY QCD.

Due to the strong coupling at the IR, the gauge theory develops non-perturbative superpotential [44] and breaks the $Z_{2M}$ R symmetry down to $Z_2$ group. Since the complex fields $A_i, B_j, i, j = 1, 2$ also describe the complex coordinates of the cone, the breaking of the R symmetry modifies the geometry from a regular to a deformed cone [22]. Thus to capture the IR modification of the gauge theory, we must consider the warped deformed cone. The warped deformed cone dual to the confining gauge theory was proposed by Klebanov-Strassler, while the large ‘r’ region of the geometry was studied by Klebanov-Tseytlin. Now observe that the gauge group becomes $SU(M)$ only at the far IR while at high energies, it can be described by $SU(k(\Lambda)M) \times SU((k(\Lambda) - 1)M)$ group, with $k(\Lambda)$ increasing with energy. Thus the UV of the gauge theory has divergent effective degrees of freedom and looks nothing like QCD. Although the confined phase of the gauge theory may resembles $\mathcal{N} = 1$ SUSY QCD, the deconfined phase of the gauge theory is quite different.

Note that this UV divergence resulting from the Seiberg duality cascade is solely due to the presence of $D5$ brane charge. On the other hand, confinement is also a result of the $D5$ branes. To obtain a gauge theory that confines in the IR but does not have diverging degrees of freedom at the UV, we need to annihilate the effect $D5$ branes at high energies while keeping the theory unchanged at the IR.
This can be done by adding $M$ anti-five branes separated from each other and from the $D3/D5$ branes at the tip of the cone. To obtain this separation, we must blow up one of the $S^2$’s at the tip and give it a finite size - which essentially means putting a resolution parameter. The brane setup is sketched in the left figure of Fig. 3. The separation gives masses $\Lambda_0$ to the $D5/\bar{D}5$ strings and at scales less than the mass, the gauge group is $SU(N+M) \times SU(N) \times U(1)^M$ where the additional $U(1)$ groups arise due to the massless strings ending on the same $\bar{D}5$ brane. At scales much larger than $\Lambda_0$, $D5/\bar{D}5$ strings are excited and we have $SU(N+M) \times SU(N+M)$ gauge theory. Essentially $M$ pairs of $D5/\bar{D}5$ branes with fluxes are equivalent to $M$ number of $D3$ branes and hence they contribute an additional $M$ units of $D3$ charge to Klebanov-Witten theory, resulting in $SU(N+M) \times SU(N+M)$. For $\Lambda < \Lambda_0$, i.e. at low energy, gauge theory is best understood as arising from the set up of Fig. 2(b) (since the modes from $\bar{D}5$ branes are not excited), while at high energy $\Lambda \gg \Lambda_0$, the gauge theory is best described as arising from left figure of Fig. 3.

Now of course the presence of anti branes will create tachyonic modes and system will be unstable. To stabilize the system, we need to add world volume fluxes on the $D5/\bar{D}5$ branes. Alternatively, we can introduce $D7$ branes and absorb the anti D5 branes as gauge fluxes on the $D7$ branes. Then a stable configuration of $D7$ branes with gauge fluxes in the presence of coincident $D3/D5$ branes will be equivalent to
stable configuration of coincident $D3/D5$ branes and anti-D5 along with $D7$ branes.

In the dual geometry, these $D7$ branes appear as stable embeddings in warped deformed conifold geometry with world volume fluxes. The embedding $\tilde{\phi}_\alpha(r)$ presented in the previous section along with the world volume flux $\tilde{F}_2$ are precisely the values that give a stable configuration. Thus the gravity solution presented in the previous section describe a gauge theory that arises from anti-D5 branes absorbed in $D7$ branes with coincident $D3/D5$ branes at the tip of a regular cone. In the next section we will see how different regions of the dual geometry represent different regimes of the gauge theory, from the UV modes to the IR modes. In Fig 3 we sketched the brane configuration and the dual geometry.

In fact, with the solution for the flux presented in the previous section, the total effective $D5$ charge in the gauge theory can be obtained for $r_h < r_0$:

$$M_{\text{eff}}^{\text{total}}(r) = \frac{1}{4\kappa_{10}^2 \mu_5} \int F_3 = \frac{1}{4\kappa_{10}^2 \mu_5} \int \left[ \frac{M\alpha'}{2} \omega_3 + 4\kappa_{10}^2 \mathcal{M}_f \alpha' \mu_7 F(r) \omega_3^{-1} \right]$$

(2.52)

This includes the background charge $M$ arising from $D5$ branes at the tip of the regular cone and the induced $D5$ charges due to the presence of the world volume flux $\tilde{F}_2$ on the $D7$ branes. The latter is given by

$$M_{\text{eff}}^{\text{induced}}(r) = \frac{\mathcal{M}_f \alpha' \mu_7}{\mu_5} \int F(r) \omega_3^{-1}$$

(2.53)
From the form of $F(r)$ given in (2.51), one obtains that $|M_{\text{induced}}(r)|$ increases with $r$ and $M_{\text{induced}}(r) = 0$ for $r \leq r_0$. When we consider the Gaussian surface integral of (2.53) for $r > r_0$, we enclose the induced charges and larger $r$ encloses more induced charge. We can choose the sign of the induced charge to be negative and then decrease total $D5$ charge for increasing $r$. In fact we can choose $\mathcal{M}$, $a$ such that

$$M_{\text{induced}}(r \to \infty) = -\frac{M\alpha'}{8\kappa_{10}^2\mu_5}\int \omega_3 \quad (2.54)$$

This automatically leads to

$$M_{\text{total}}(r \to \infty) = 0 \quad (2.55)$$

On the other hand, we also demand

$$\lim_{r \to \infty} \int B_2 \to 0 \quad (2.56)$$

The above condition guarantees that at far UV, the gauge couplings $(g_1, g_2)$ of $SU(N + M) \times SU(N + M)$ theory become identical and we essentially have two copies of the same gauge group $SU(N + M)^6$.

The conditions (2.54), (2.56) gives the value of the constants $\mathcal{M}$ and $a$. Observe that (2.54) can be satisfied with any value of $a$ by considering orientation of the Gaussian surface. In particular, the surface $d\phi_1 \wedge d\theta_1$ has opposite orientation to the surface $d\phi_2 \wedge d\theta_2$. Then using the symmetry between $\theta_1$ and $\theta_2$, one readily obtains that the constant $a$ is not determined by (2.54) and we get

$$\mathcal{M} \sim -\frac{M}{\lim_{r \to \infty} F(r)} \quad (2.57)$$

since $\int \omega_3^{-1}$ is finite. This also means that $\tilde{M}$ in our perturbative parameter (2.38) is of $O(M)$. With the explicit form of $B_2$ given in the appendix, we can use (2.56) to determine $a$.

For the dual gauge theory, (2.53) along with (2.57) implies that for energy scale $\Lambda > \Lambda_0 \sim r_0$, we have induced anti-$D5$ branes with charge $M_{\text{induced}} < 0$ and the total effective $D5$ charge at scale $\Lambda \to \infty$ has vanished. Thus at the far UV, we are only left with $D3$ charges and Seiberg cascade has terminated. Since we also have $D7$ brane, the axio-dilaton field $\tau$ will be running according to (2.41) and the gauge coupling will also run. But the running of the gauge coupling $g_{YM}^2$ is of $O(g_2^2)$ and can be neglected. Alternatively, by considering additional assembly of $D7/\bar{D7}$ branes the as discussed in detail in [32][34] [35] [36], we can effectively make the axio-dilaton field constant. Then at the far UV ($\Lambda \gg \Lambda_0$), we end up with a gauge theory with color symmetry $SU(N + M) \times SU(N + M)$, where gauge couplings $(g_1, g_2)$ are

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$^6$Since $\frac{1}{g_1} - \frac{1}{g_2} \sim \int B_2$, when $\int B_2 \to 0$, $g_1 = g_2$
identical and do not run. At the far IR ($\Lambda \ll \Lambda_0$), the gauge theory cascades down to a single group $SU(M)$ and confines. This way we obtain a gauge theory that is UV conformal and IR confining- which are common features with QCD. In the next section, we will explore the thermodynamics of the gauge theory and make direct connections to QCD.

Coming back to the flux analysis, since $\int_{r \to \infty} F_3 = 0$, $\int_{r \to \infty} B_2 = 0$, we readily get

$$\int G_3 \wedge *_6 \tilde{G}_3$$

is negligible in the large $r$ region. Thus the bulk action (2.4) gets negligible contribution from $G_3$ for $r \gg r_0$. By integrating (2.4) over the angular coordinates $\psi, \phi_i, \theta_i$, we can obtain an action that describes $AdS_5$ geometry for $r \gg r_0$. Hence, addition of the localized sources has allowed us to patch together a deformed warped cone at small $r$ to an $AdS_5 \times T^{1,1}$ like geometry at large $r$.

Finally, observe that we consider the induced charge (2.53) when the source is outside the horizon, that is we still have small black holes $r_h < r_0$. When $r_h \geq r_0$, the D7 brane falls into the black hole, along with the effective anti-D5 charge. These anti-D5 charges will neutralize the D5 charge and thus for large black holes $r_h \geq r_0$, we can simply consider $G_3 = 0$ outside the horizon. Then the only non-trivial flux in (2.4) would be $\tilde{F}_5$, as will be discussed in detail in section 3.2.

3. Thermodynamics of the gauge theory

Using the gravity solution of the previous section, we can obtain the partition function of the gauge theory using the identification

$$Z_{\text{gauge}} = e^{-F/T} = Z_{\text{gravity}} \simeq e^{-S_{\text{ren}}^{\text{gravity}}}$$

$$S_{\text{ren}}^{\text{gravity}} = S_{\text{bulk}} + S_{\text{GH}} + S_{\text{counter}}$$

where $S_{\text{bulk}} = S_{\text{SUGRA}} + S_{\text{loc}}$, $S_{\text{loc}}$ is the action for the localized sources (branes, anti-branes, or other localized manifolds), $S_{\text{GH}}$ is the Gibbons-Hawking boundary term and $S_{\text{counter}}$ is counter terms required to regularize the action. Then using thermodynamic identities we can directly obtain free energy, pressure and entropy from the partition function.

However, the action $S_{\text{total}}$ gives rise to geometries with or without a black hole and we denote them by $X^2$ and $X^1$. The on shell values of the action including the Gibbons-Hawking terms for the two geometries $X^1$ and $X^2$ are distinct and we denote them by $S^1$ and $S^2$. The temperature of the dual gauge theory corresponding to the manifold $X^1, X^2$ can be obtained by identifying the periodicity of Euclidean time. However, for a manifold with singularity, this period is fixed by the nature
of the singularity, while for a regular manifold, the period is arbitrary. For a given temperature of the dual gauge theory, the geometry with less on-shell value for the action will be preferred. Using self-duality of five form flux, and ignoring second order terms in $O(\epsilon)$ that arise from the axio-dilaton field, we readily obtain the bulk Euclidean on-shell action:

$$S_{\text{bulk}} = \frac{1}{2\kappa^2_{10}} \left[ \int d^8 x \int_0^\beta d\tilde{\tau} \int_{r_h}^\infty dr \sqrt{G_{2}} \left( -\frac{G_3 \cdot \tilde{G}_3}{24\text{Im}\tau} \right) - i \int \frac{C_4 \wedge G_3 \wedge \tilde{G}_3}{4\text{Im}\tau} \right] + S_{\text{loc}}$$

$$S_{\text{loc}} = N_f S_{Dp} \quad (3.2)$$

The above action is obtained from the on-shell value of (2.4,2.5) after wick rotation $t = i\tilde{\tau}$ and imposing periodicity $\beta$. In particular we used $S^E = -i S^M$, where $S^M$ is the action using Minkowski metric and then wick rotating the on-shell value and $S^E$ is the Euclidean action.

In order to study the gauge theory at different energy scales, it is useful to identify two regions in the dual geometry according to the radial distance $r$. This is particularly instructive since gravitons coming from the small $r$ region are red shifted compared to large $r$ region as measured by an observer at a fixed $r = r_c$. Thus, small $r$ region is dual to low energy modes while large $r$ region accounts for the high energy modes of the gauge theory.

Before discussing these two regions in detail, we would like to point out the connection between our gravitational description and Wilsonian Renormalization Group (RG) flow. The RG flow of the gauge couplings $(g_1, g_2)$ can be obtained from the dual flux $B_2$ and the dilaton field $e^\phi$ from the following relation:

$$\frac{1}{g_1^2} + \frac{1}{g_2^2} = e^{-\phi}$$

$$\frac{1}{g_1^2} - \frac{1}{g_2^2} = e^{-\phi} \int_{S^2} B_2$$

(3.3)

By replacing the radial coordinate with energy scale i.e. $r \rightarrow \Lambda$, and using the expression for $B_2, e^\phi$ as given in (A.2),(2.41) in extremal limit ($r_h = 0$), one readily obtains the running of the gauge couplings $(g_1(\Lambda), g_2(\Lambda))$ with scale $\Lambda$. The flux $B_2$ and dilaton field $e^\phi$ was obtained using the bulk action (2.4), (2.5) which describes the entire geometry, from $\rho = 0$ to $r = \infty$. However, if we consider the action by restricting the radial integral up to $r = r_0$ and we neglect the localized sources, then the action (denoted by $S_{R_1}$) describes the fluxes in small $r$ region, i.e. $r < r_0$. We denote this small $r$ region $r < r_0$ as region 1. In particular, the action $S_{R_1}$ and $S_{\text{SUGRA}}, S_{D7}$ in (2.4), (2.5) both give identical result for flux and the dilaton field in region 1, up to linear order in $O(\epsilon)$. Thus whether we use $S_{R_1}$ or $S_{\text{SUGRA}} + S_{D7}$ as
the action, both will describe identical RG flow according to (3.3). Hence \( S_{R_1} \) can be thought of as the Wilsonian effective action that describes the IR of the gauge theory.

However, we need to be careful in carrying out this analogy. The on-shell values of \( S_{R_1} \) and \( S_{SUGRA} + S_{D7} \) are distinct and so are the corresponding partition functions. On the other hand, in Wilsonian RG flow, the partition function is independent of the flow. Thus the ‘flow’ in going from \( S_{SUGRA} + S_{D7} \) to \( S_{R_1} \) is not identical to the Wilsonian flow, even though the beta functions follow the analogy.

In the following we first discuss region 1 and then the inclusion of large \( r \) region, denoted by region 2.

### 3.1 Region 1, \( r < r_0 \)

**Figure 4:** Region 1 with boundary \( r_b = r_0 \), with or without a black hole.

First consider this region for extremal manifold \( X^1 \). The manifold is regular and the metric is given by (2.6) with the internal metric (2.7) in the limit \( B = 0 \). The warp factor \( h(\rho) \equiv e^{-4A} \) has no singularities in this region and thus we can impose any periodicity \( \beta \) of Euclidean time \( \tilde{\tau} = -it \), after wick rotation. Thus, region 1 can describe any temperature \( T_1 \equiv \beta_1^{-1} \). Furthermore, this region does not include the localized sources and in the extremal limit \( B = 0 \), three form flux is ISD: \( *_6 G_3 = iG_3 \) with \( C_4 = e^{4A} \tilde{r} \wedge dx \wedge dy \wedge dz \). Then for geometry \( X^1 \) in region 1, we have \( G_3 \wedge *_10 \tilde{G}_3 = -iC_4 \wedge G_3 \wedge \tilde{G}_3 \). Thus if we restrict the integral in (3.2) to region 1, we get

\[
S_{R_1}^{(1)} = \frac{1}{2\kappa_1^2} \left[ \int d^8 x \int_0^\beta d\tilde{\tau} \int_{\rho=0}^{r=r_0} dr \sqrt{G_2} \left( -\frac{G_3 \cdot \tilde{G}_3}{24\text{Im}\tau} \right) 
- i \int C_4 \wedge G_3 \wedge \tilde{G}_3 \frac{4\text{Im}\tau}{4\text{Im}\tau} \right]
\]
Next we can consider the non-extremal manifold $X^2$ in region 1 which has a horizon $r_h \leq r_0$. Again, using our explicit solutions for the fluxes given in (2.50), we can evaluate the on-shell action up to linear order in $O(\epsilon)$ for region 1 to obtain [37]

$$S^2_{R_1} = \frac{\beta_2 g_s M^2 V_8}{2 \kappa_{10}^2 N_{\text{eff}}(\rho)} \int_{r_h}^{r_0} dr \frac{3 \hat{F}_5^4}{16 r}$$

$$= \frac{3 \beta_2 g_s M^2 V_8 \hat{r}_h^4}{32 \kappa_{10}^2 N_{\text{eff}}(\rho)} \log \left( \frac{r_0}{\hat{r}_h} \right)$$

(3.5)

Before analyzing the action, it is crucial to note that the above result is valid only for a large black holes, that is $r_h$ is large. For a small black hole, we need to go back to radial coordinate $\rho$ and consider the warp factor at small $\rho$

$$e^{-4A(\rho)} = h(\rho) = c_i \rho^i$$

$$e^{2B(\rho)} = 1 + \sum_{i=1} b_i \rho^i$$

(3.6)

where $c_0 \neq 0$, $c_i, b_i \sim O(g_s M^2)$ are constants and we have the boundary conditions $h(\rho_h) \neq 0, e^{2B(\rho_h)} = 0$. In the extremal limit when horizon $\rho_h = 0$, we have $e^{2B(\rho)} = 1$ and $h(\rho)$ is the Klebanov-Strassler warp factor. In the non-extremal limit, we must solve (2.31), (2.33) and (2.36) using the internal metric of the deformed cone $\tilde{g}_{mn}$ (with $\tilde{g}_{mn}^0$ given by (2.7) and $\tilde{g}_{mn}^1$ obtained from solving the second Einstein equation in (2.23)). In solving these system of coupled equations in small $\rho$ region, the relevant parameter is $g_s M^2 N_{\text{eff}}(\rho)$ with

$$N_{\text{eff}}(\rho) = \frac{1}{2 \kappa_{10}^2 T_3} \int_{T^{1,1}, \rho} \tilde{F}_5$$

$$\int_{T^{1,1}, \rho} \tilde{F}_5 - \int_{T^{1,1}, \rho_{\mu}} \tilde{F}_5 = \int_{\rho_{\mu}}^{\rho_{\nu}} d\tilde{F}_5$$

(3.7)

In the above, the surface integral is taken over $\rho = \text{constant}$ surface which is a warped $T^{1,1}$. When $\rho_{\mu} \to 0$, warped $T^{1,1}$ shrinks to a three sphere $S^3$ and the surface integral vanishes, $\int_{T^{1,1}, \rho_{\mu} \to 0} \tilde{F}_5 \to 0$. Then we get

$$N_{\text{eff}}(\rho) = \int_0^\rho d\tilde{F}_5$$

(3.8)

Then using (2.21), one readily gets that $N_{\text{eff}}(\rho)$ decreases with $\rho$ and for small enough $\rho$,

$$\frac{g_s M^2}{N_{\text{eff}}(\rho)} > 1$$

(3.9)
becomes large. Thus for small black holes, we cannot solve the system of equations (2.31), (2.33), (2.36) and the Einstein equations perturbatively, since there is no small perturbative parameter like (2.38). Hence, we do not have exact expressions for the on-shell action (3.2) when the black hole horizon $\rho_h$ is small.

However, the region of the geometry near small horizons is crucial in determining the low temperature dynamics of the gauge theory. Especially small black holes will determine the thermodynamics of the gauge theory near critical phase transition temperature, which we will see shortly. In fact the small $\rho$ region is dual to QCD like $SU(M)$ pure glue theory while large $\rho$ region is dual to a bi-fundamental gauge theory, very different from QCD. Thus, the black hole geometry dual to QCD like gauge theory near phase transition cannot be described by our perturbative approach. Although we do not have an exact solution for the metric near horizon for small black holes i.e. $\rho_h$ is small, we can find the form of the warp factor $A(\rho)$ and black hole factor $B(\rho)$. Using these forms, we can qualitatively understand the behaviors of thermodynamic state functions. We will elaborate the issue in the following sections, which in fact makes it clear the connection between lattice QCD and our holographic model.

Only at large $\rho$, (3.6) takes the form (2.42), with $N \equiv N(r_l) \gg 1$ for some large $r_l$. Then the parameter $\epsilon$ is very small and our perturbative analysis is valid with the on-shell action given by (3.5). Since $X^2$ has black hole singularity, the periodicity $\beta_2$ is not arbitrary and the temperature of the field theory living at surface $r_b$ is given by

$$T_2(r_b) = \frac{\beta_2^{-1}}{\sqrt{g(r_b)}}$$

$$\beta_2 = \frac{4\pi \sqrt{h(r_b)}}{|g'(r_b)|}$$

(3.10)

where $g = e^{2B}$ and prime denotes derivative with respect to $r$. The two geometries describe the same field theory at the same temperature on the hyper surface $r = r_b$ if

$$T = T_1(r_b) = T_2(r_b) \Rightarrow \beta_1 = \beta_2 e^{B(r_b)}$$

(3.11)

Now the Gibbons-Hawking term at boundary $r_b = r_0$ for $X^1$, $X^2$ is given by [37]

$$S_{R_1, GH}^1 = \frac{1}{108 \kappa_{10}^2} \left[ 4r_0^4 + \frac{729 g_s M^2}{16 N} r_h^4 \right] \beta_1 V_8$$

$$S_{R_1, GH}^2 = \frac{1}{108 \kappa_{10}^2} \left[ 4r_0^4 - 2r_h^4 + \frac{729 g_s M^2}{16 N} (r_0^4 - (1 + d)r_h^4) \right] \beta_2 V_8 + \tilde{T}$$

$$\tilde{T} = \mathcal{O} \left( \frac{g_s M^2}{N} \right) \sum_{l=1}^{\infty} \mathcal{O} \left( \frac{r_h^{4(l+1)}}{r_0^{4l}} \right)$$

(3.12)
Then we get the action difference,

$$
\Delta S = S_{R_1}^2 - S_{R_1}^1 + S_{R_1,\text{GH}}^2 - S_{R_1,\text{GH}}^1
= \frac{3g_sM^2\beta_2V_{8r_h}}{32\kappa_{10}^2N} \left( \log \left( \frac{r_0}{\tilde{r}_h} \right) - \frac{9}{4} - \frac{9}{2} \left[ d - \alpha^1 \right] \right) + \mathcal{I}

\mathcal{I} \equiv \tilde{\mathcal{I}} + O \left( \frac{g_sM^2}{N} \right) \sum_{l=1}^\infty O \left( \frac{\tilde{r}_h^{4(l+1)}}{r_0^{2l}} \right)
$$

(3.13)

where $\alpha^1$ arises from the following expansion

$$
e^{2B} = g(r) \equiv 1 - \frac{\tilde{r}_h^4}{r^4} \left( 1 + \frac{729\alpha^1 g_sM^2}{8N} \right) + \frac{\tilde{r}_h^8}{r^8} O \left( \frac{g_sM^2}{N} \right) + \ldots
$$

(3.14)

Note that $d$ is determined by the Einstein equations and the flux equations once boundary conditions are imposed. Both $d$ and $\mathcal{I}$ arise from the expansion of $g(r)$ and are sensitive to the near horizon geometry and in particular can be obtained from the horizon values of the metric. Furthermore, $d, \mathcal{I}$ are related to $h(r_h)$, which in turn determines the number of effective degrees of freedom at a temperature $T \approx r_h/L^2$. Thus from the gauge theory side, $d, \mathcal{I}$ are both related to the effective colors at the thermal scale.

Without explicitly solving the Einstein equations near the black hole horizon, we can speculate two boundary conditions by ignoring $\mathcal{I}$ (which is small since $r_0 > \tilde{r}_h$ always):

• $d \leq \alpha^1 - 1/2$: In this case $\Delta S > 0$, which means extremal geometry is favored over black hole. In that case all black holes in region 1 have higher free energy compared to vacuum and the dual gauge theory is described by extremal geometry $X^1$. However note that, since there is no black hole horizon in the vacuum geometry $X^1$, the Euclidean renormalized on-shell action $S_{\text{gravity}}^{\text{ren}} = \beta F$ is independent of horizon with $F$ independent of $T$. Then using thermodynamic identity, one readily gets

$$
s = -\frac{\partial F}{\partial T} = 0
$$

(3.15)

Thus for $d \leq \alpha^1 - 1/2$ and we ignore $\mathcal{I}$, region 1 corresponds to confined phase.

It is possible that for $d \leq \alpha^1 - 1/2$ we can still have $\Delta S(\tilde{r}_h = r_h^c) = 0$ if $\mathcal{I}(\tilde{r}_h = r_h^c)$ is not small. Then $r_h^c$ would give the critical horizon. $\mathcal{I}$ becomes more and more significant for larger $\tilde{r}_h$, but since we do not have an exact expression for $\mathcal{I}$, we cannot directly analyze this case. In fact for a gauge theory that behaves similar to QCD near the critical phase transition temperature, it is likely that we cannot ignore $\mathcal{I}$. In particular, we will find out later in this section that ignoring $\mathcal{I}$ results in a conformal anomaly that only matches the lattice QCD behavior for temperature much larger than $T_c$ and not near $T_c$. 

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• $d > \alpha^1 - 1/2$: In this case, using (3.13) and ignoring $I$, it is possible to obtain $\Delta S = 0$ in region 1, with the following value for critical horizon

$$r_h^c = \frac{r_0}{\exp \left( \frac{9}{4} [1 + 2(d - \alpha^1)] \right)}$$

(3.16)

Then the corresponding critical temperature is

$$T_c = \frac{r_h^c \left( 1 + O \left( \frac{g_s M^2}{N} \right) \right)}{\pi L^2(r_h^c)} \sim \frac{2r_0}{\exp \left( \frac{9}{4} [1 + 2(d - \alpha^1)] \right) \sqrt{27\pi N\alpha'}}$$

(3.17)

$L^4(r) \equiv r^4 h(r)$

We need to keep in mind that the derivation of critical horizon (3.16) uses the form of the action (3.5), which is only valid for large black holes. But the black hole cannot be too large, since we are also ignoring $I$, which is only justified for $\tilde{r}_h \ll r_0$. Note that the notion of ‘largeness’ and ‘smallness’ in region 1 is relative to the scale $r_0$: If $r_h \ll r_0$, we have a small black hole while if $r_h \lesssim r_0$, we have a ‘large’ black hole in region 1.

If $d$ is very large, $r_h^c$ will be small and (3.5) will no longer give the exact on-shell action, since our perturbative analysis will break down. Only for small $d > \alpha^1 - 1/2$, we will have large enough $r_h^c$ such that our perturbative analysis holds (but in that case $I$ can become large and not negligible) and $T_c$ is given by (3.17). Each set of value for $(d, I)$ fixes the geometry and describes a particular gauge theory, which may or may not resemble QCD near critical temperature. Small (but $d > \alpha^1 - 1/2$) values of $d$ which makes our perturbation in $O \left( \frac{g_s M^2}{N} \right)$ exact, also makes $I$ large and thus computation of $T_c$ less and less exact. Furthermore, with small $d$, the width of the conformal anomaly is small and distinct from lattice QCD results. This mismatch with lattice QCD is in fact not unexpected and can be attributed to the fact that we ignored $I$. We will elaborate the issue in detail in the following section.

Coming back to our perturbative gravity analysis, for $T > T_c$, the black hole is preferred and describes the deconfined phase which has non-zero entropy. On the other hand for $T < T_c$, we have a confined phase with zero entropy described by the extremal geometry. Thus at $T = T_c$, there is a first order phase transition in the gauge theory and we have obtained a gravitational description of it in terms of Hawking-Page phase transition between two geometries.

Now to obtain the thermodynamic state function of the gauge theory, we have to obtain the partition function using (3.1) and thus we need the counter terms for $r_0 \to \infty$. The divergent part $\lim_{r_0 \to \infty} \beta_2 r_0^4$ is dependent on temperature and thus the counter term will also be dependent on it. Also observe that the regularization scheme adopted in [45, 46] identifies $\Delta S$ as the regularized gravity action. Hence regularization is not just subtraction of the infinite part $\lim_{r_0 \to \infty} \beta_2 r_0^4$, but also a finite part arising from the vacuum action. Thus, even if $r_0$ is finite, which is the case we are in fact considering, we need counter terms that depend only on temperature.
There is an alternative approach to obtain thermodynamic state function. We can use Wald’s formula for the gravity Lagrangian

$$\mathcal{L}_{\text{bulk}} = R + \frac{\partial M_\tau \partial \bar{M}_\tau}{2|\text{Im}\tau|^2} - \frac{|\bar{F}_3|^2}{4 \cdot 5!} - \frac{G_3 \cdot \bar{G}_3}{12 \text{Im}\tau}$$

(3.18)

to compute the entropy associated with the geometry $X^2$ in region 1. Using the expansion for the warp factors (2.42), the form of the internal metric (2.7,2.47) and considering only up to linear terms in $O(\epsilon)$, we get the Wald entropy [47]-[50]

$$s_{R_1} = \frac{\pi r_h^3 \sqrt{27\pi N V_S \alpha'}}{108 \kappa_{10}^2} \left( 1 + \frac{a_0 g_s M^2}{N} + \frac{a_1 g_s M^2}{N} \log \frac{r_h}{r^*_s} \right)$$

(3.19)

where $a_0, a_1$ are constants independent of $M, N$ and is determined by the value of the warp factor $h$ and internal metric $\bar{g}_{mn}$ at the horizon. Using the definition of temperature (3.10) in terms of the horizon, we can obtain entropy as a function of temperature,

$$s_{R_1} = \frac{27 \pi^6 T^3 N^2 V_S}{32 \kappa_{10}^2} \left( 1 + \frac{b_0 g_s M^2}{N} + \frac{b_1 g_s M^2}{N} \log \left( T \sqrt{N \alpha'} \right) \right)$$

(3.20)

where $b_0, b_1$ are a constants independent of $M, N$. They are determined by using the definition of temperature (3.10) and the horizon value of the metric.

Once entropy is known as a function of temperature, we can readily obtain the free energy of the gauge theory dual to region 1,

$$F_{R_1} = -\int dT \ s_{R_1}$$

$$= -\frac{27 \pi^6 T^4 N^2 V_S \alpha'^4}{128 \kappa_{10}^2} \left( 1 + \frac{b_0 g_s M^2}{N} - \frac{b_1 g_s M^2}{4N} \right) + \frac{b_1 g_s M^2}{N} \log \left( T \sqrt{N \alpha'} \right)$$

(3.21)

### 3.2 Region 1 and region 2, $r > r_0$

We will now consider the on-shell action for both the extremal $X^1$ and non-extremal black hole geometry $X^2$, by including the large $r$ region of the geometry. Inclusion of localized terms makes $X^1$ a singular manifold, since we are taking the back reaction of the local source. Removing the singularity imposes that the period $\beta_1 \to \infty$ and thus $X^1$ now only describes zero temperature. In particular $X^1$ with D7 branes will describe a gauge theory with mesons and gluons at zero temperature.

On the other hand, $X^2$ can have any horizon $r_h$. For $r_h < r_0$, the inclusion of localized sources gives rise to two singularities to the manifold $X^2$: one at the black
Figure 5: D7 brane in Region 2, with boundary $r_b \to \infty$, with or without a black hole.

hole horizon and the other at the location of the brane. We can only remove one of the singularities by imposing a single periodicity of Euclidean time throughout the manifold - but then the metric is not smooth since the other singularity remains. Identifying temperature this way does not result in a unique temperature and thus $X^2$ does not describe a gauge theory at thermal equilibrium. $X^2$ without localized sources describe gluonic medium at thermal equilibrium and the temperature can be identified from the black hole singularity of the manifold. Insertion of D7 brane into the geometry is equivalent to immersing mesons into a gluonic thermal bath. When the system is not at thermal equilibrium, mesons and gluons have different temperatures. Thus $X^2$ with D7 branes outside the horizon represents a system that is not at thermal equilibrium.

Analytic continuation $t \to i\tilde{\tau}$ to Euclidean metric and then identifying temperature with period of $\tilde{\tau}$ is applicable for a system at thermal equilibrium. Since $X^2$ including D7 branes do not describe a system at thermal equilibrium, Euclidean action of singular geometry $X^2$ does not represent the free energy of the system. Thus for $r_h < r_0$ and we include region 2, we do not have multiple description of gauge theory in terms of $X^1, X^2$ and there is no Hawking-Page like transition.

For $r_h \geq r_0$, the localized sources will fall into the horizon and now we have a unique definition of temperature as the only singularity is the black hole singularity. Thus the black hole geometry with horizon $r_h \geq r_0$ can describe non-zero temperature gauge theory with the temperature given by the Hawking temperature (3.10).

Since $M_{\text{eff}}^{\text{induced}} \sim -M$ for $r \gg r_0$, when $r_h \to r_0$ from below, the black hole absorbs $-M$ units of $D5$ charge and thus the total charge outside the horizon $\lim_{r_h \to r_0} M_{\text{eff}}^{\text{total}}(r_h) \to 0$. This means when we consider the case $r_h \geq r_0$, we can neglect three form flux and consider $\tilde{F}_5$ to be the only non-zero flux. Thus for

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7We thank Martin Kruczenski for elaborating this point.
For $r_h \geq r_0$, we have an exact black hole solution in $\text{AdS}_5 \times T^{1,1}$ background, with the following warp factors and flux strength

\[
e^{-4A_{\text{AdS}}} = h_{\text{AdS}} = \frac{27\pi \tilde{N}_{\text{eff}} \alpha^2}{4r^4}
\]
\[
e^{2B_{\text{AdS}}} = 1 - \frac{r^4}{\tilde{r}_h^4}
\]
\[
\tilde{F}^5_{\text{AdS}} = (1 + \ast_{10}) \frac{\partial h_{\text{AdS}}^{-1}}{\partial r} \wedge dt \wedge dx \wedge dy \wedge dz
\]

where $\tilde{N}_{\text{eff}}$ is a free parameter related to the five form flux strength,

\[
\tilde{N}_{\text{eff}} = \frac{1}{2\kappa_4^2 \mu_3} \int \tilde{F}^5_{\text{AdS}}
\]

The Wald entropy and the corresponding free energy of $\text{AdS}_5 \times T^{1,1}$ is easily computed

\[
s_{R_1+R_2} = \frac{\pi r_h^3 \sqrt{27\pi \tilde{N}_{\text{eff}} V_5 \alpha'}}{108 \kappa_{10}^2} = \frac{27}{32} \pi^2 \tilde{N}_{\text{eff}}^2 V_3 T_{\text{AdS}}^2
\]
\[
F_{R_1+R_2} = -\frac{27}{128} \pi^2 \tilde{N}_{\text{eff}}^2 V_3 T_{\text{AdS}}^4
\]

In the above, the temperature is defined through (3.10) using the AdS metric (3.22):

\[
T_{\text{AdS}}(r_h) = \frac{2r_h}{\pi \sqrt{27\pi \tilde{N}_{\text{eff}} \alpha'}}
\]

(3.24) describes entropy and free energy of the gauge theory for black hole horizons $r_h \geq r_0$ in terms of temperature $T_{\text{AdS}}$ and effective degrees of freedom $\tilde{N}_{\text{eff}}$. While (3.19,3.21) describes entropy and free energy of the gauge theory for black hole horizons $r_h < r_0$. To compare temperatures described by these different size black holes, we must first obtain the exact expression for temperature described by region 1. The explicit forms for the metric in region 1 is given by

\[
h(r) = \frac{27\pi N \alpha^2}{4r^4} \left[ 1 + \frac{g_s M^2}{N} \log \left( \frac{r}{r_s} \right) + \frac{g_s M^2}{N} c_l \left( \frac{\tilde{r}_h}{r} \right)^4 l \right]
\]
\[
g(r) = 1 - \frac{\tilde{r}_h^4}{r^4} + \frac{g_s M^2}{N} d_l \left( \frac{\tilde{r}_h}{r} \right)^4 l
\]

where $c_l, d_l$ are constants independent of $M, N$. The horizon radius $r_h$ is such that $g(r_h) = 0$, which gives that $r_h$ is the solution of the following equation

\[
\frac{\tilde{r}_h^4}{r_h^4} = 1 + \frac{g_s M^2}{N} d_l \left( \frac{\tilde{r}_h}{r_h} \right)^4 l
\]
Putting everything together, we get that the temperature described exclusively by region 1 without the inclusion of region 2 is

\[
T(r_h) = \frac{2r_h \left( 1 + \frac{g_s M^2}{N} d_l (1 + l) \left( \frac{\tilde{r}_h}{r_h} \right)^{4l} \right)}{\pi \sqrt{27 \pi N \alpha'}} \left[ 1 + \frac{g_s M^2}{N} \log \left( \frac{r_h}{r_*} \right) + \frac{g_s M^2}{N} c_l \left( \frac{\tilde{r}_h}{r_{00}} \right)^{4l} \right]^{1/2} \tag{3.28}
\]

As \( r_h \to r_0^+ \), black holes in \( AdS_5 \times T^{1,1} \) geometry should describe the same temperature as the black holes in region 1, with \( r_h \to r_0^- \). This means we must have

\[
T_{AdS}(r_h \to r_0^+) = T(r_h \to r_0^-) \equiv T_0 \tag{3.29}
\]

The above condition can be used to relate \( \bar{N}_{\text{eff}} \) with \( N \) to give

\[
\bar{N}_{\text{eff}} = N \left[ 1 + \frac{g_s M^2}{N} \log \left( \frac{r_h}{r_*} \right) + \frac{g_s M^2}{N} c_l \left( \frac{\tilde{r}_h}{r_{00}} \right)^{4l} \right] \frac{1}{1 + \frac{g_s M^2}{N} d_l (1 + l) \left( \frac{\tilde{r}_h}{r_{00}} \right)^{4l}} \equiv N \left[ 1 + \frac{2e_0 g_s M^2}{N} + \frac{2e_1 g_s M^2}{N} \log \left( \frac{r_0}{r_*} \right) \right] \tag{3.30}
\]

where we have introduced constants \( e_0, e_1 \) independent of \( M, N \) and determined by above equation. Expanding only up to linear order in \( O(\epsilon) \) and using (3.30) in (3.19,3.24), we readily get the entropy difference

\[
\Delta s = s_{R_1 + R_2}(r_h \to r_0^+) - s_{R_1}(r_h \to r_0^-)
= \frac{\pi r_0^3 \sqrt{27 \pi N V_{5\alpha'}}}{108 \kappa_{10}^2} \left[ \frac{g_s M^2}{N} \left[ e_0 - a_0 \right] + \frac{g_s M^2}{N} \left[ e_1 - a_1 \right] \log \left( \frac{r_0}{r_*} \right) \right] \tag{3.31}
\]

As entropy always increases with temperature, we must have \( \Delta s \geq 0 \), which will automatically lead to

\[
\Delta F = F_{R_1 + R_2}(r_h \to r_0^+) - F_{R_1}(r_h \to r_0^-) < 0 \tag{3.32}
\]

Finally we can compute the internal energy and pressure and evaluate the conformal anomaly. At low energies for the case \( d > \alpha^1 - 1/2 \) and ignoring \( I \), we can have a regime \( T_c \leq T \leq T_0 \). Then the internal energy and pressure is given by black hole in region 1 and we get

\[
e_{R_1} = \frac{1}{V_3} (F_{R_1} + T s_{R_1})
= \frac{27 \pi^6 T^4 N^2 V_5 \alpha'}{128 \kappa_{10}^2} \left( 3 + \frac{3b_0 g_s M^2}{N} + \frac{b_1 g_s M^2}{4N} + \frac{3b_1 g_s M^2}{N} \log \left( T \sqrt{N \alpha'} \right) \right)
\]
\[ p_{R_1} = -\frac{\partial F_{R_1}}{\partial V_3} \] (3.33)

This readily gives the conformal anomaly

\[ \Delta_{R_1} \equiv \frac{e_{R_1} - 3p_{R_1}}{T^4} = \frac{27\pi^6 N^2 \alpha'^4 V_5 b_1 g_s M^2}{512\kappa_{10}^2 N} \] (3.34)

On the other hand, with \( T > T_0 \), the gauge theory is described by black hole in \( AdS_5 \times T^{1,1} \), where

\[ \Delta_{R_1 + R_2} = \Delta_{AdS} = 0 \] (3.35)

While for \( T < T_c \), for the confined phase, we can take both internal energy and pressure to be zero to obtain \( \Delta_{\text{confine}} = 0 \).

![Figure 6: \( \frac{s}{N^2 T^3} \) as a function of \( \frac{T}{T_c} \) with boundary condition \( d - \alpha^1 = \frac{4 \log(3)}{18} - \frac{1}{2} \).](image)

We can summarize the scenario as follows:

- \( d > \alpha^1 - 1/2 \): The low energy regime of the gauge theory is given by region 1 and we get a first order Hawking-Page like transition with critical temperature \( T_c \) given by (3.17). In deriving \( T_c \), we have ignored \( I \), which is only valid if \( r_h^c \ll r_0 \). On the other hand, if \( r_h^c \) is too small, our perturbative analysis is invalid since there is no small perturbative parameter. Thus (3.17) is only an approximation which gets more and more accurate if \( r_h^c \) is in narrow regime were \( I \) is negligible and the perturbative parameter (2.38) exists.

For \( T < T_c \) we have confinement while \( T > T_c \) describes a deconfined phase. Region 1 describes all temperatures up to \( T_0 \) and the entropy of the gauge theory is given by (3.19). To incorporate the high temperature modes \( T > T_0 \) of the gauge theory, we add a UV cap to the geometry by considering back reactions of localized world volume fluxes sourced by on D7 branes. Then the entropy is given by (3.24).
By using relation between horizon and temperature, we can obtain entropy as a function of temperatures for all $T$. The result is plotted in Fig 6. Whatever the boundary condition, we must satisfy the thermodynamic condition (3.31) and Fig 6 is the scenario where $\Delta s > 0$, both across $T_c$ and $T_0$. The jump at $T_c$ corresponds to the first order Hawking-Page transition while the jump at $3T_c = T_0$ is consistent with the scenario that $D7$ brane has fallen into the black hole, resulting in increased entropy. This discontinuity at $T_0$ can also be understood in the following way: For $T < T_0$, we are using the gravity action $S_{R_1}$ of region 1, neglecting the geometry of region 2 with localized sources. Region 1 with a black hole is holographic dual to a gauge theory at thermal equilibrium. Only considering the action $S_{R_1}$ is equivalent to the scenario that this gauge theory does not interact with localized sources of region 2. When $T > T_0$, gauge theory dual to region 1 comes in thermal equilibrium with the localized sources of region 2 and the to total system is described by an $AdS_5 \times T^{1,1}$ black hole. We expect the entropy of the combined system to be larger than the entropy gauge theory dual to region 1 and thus the discontinuity at $T_0$ is reasonable. In Fig 7, we have plotted the anomaly, which is just a box of width $3T_c$.

- $d \leq \alpha^1 - 1/2$: Again, low energy regime of the gauge theory is given by region 1. However now, for all $T < T_0$ only the geometry $X^1$ describes non-zero temperature and we have the confined phase. For $T > T_0$, we add region 2 and only the black hole geometry describes non-zero temperature with entropy given by (3.24). That is we have thermal CFT for $T > T_0$ and a confined theory for $T < T_0$. Thus for the condition $d \leq \alpha^1 - 1/2$, we can treat $T_0$ as the critical temperature of deconfinement. Note that we are ignoring $T$ and using our perturbation in $O(\epsilon)$, both of which is only valid if $r_h$ is a narrow regime.

The conformal anomaly of Fig 7 looks very different from what is expected in QCD. The discrepancy can be attributed to the break down of our perturbative
analysis as follows:

- The width of $3T_c$ is obtained by choosing $d = \alpha + \frac{4\log(3)}{18} - \frac{1}{2}$. This value of $d$ could be too large such that $r_h^6$ is too small and our perturbative analysis in $O(\epsilon)$ breaks down. Thus it is possible that for an anomaly of width $3T_c$, the computation of anomaly need to be modified.

- It is possible that $r_h^6$ is large such that we cannot ignore $I$. The derivation of $T_c$ ignores $I$ entirely and for $r_h \lesssim r_0$, $I$ can be large enough to invalidate the computation of $T_c$. Although large horizon makes our perturbation in $O(\epsilon)$ exact, it also makes $I$ large and thus computation of $T_c$ less and less exact. In other words, if we accounted for $I$, then the computation of $T_c$ along with the free energy of the black hole in region 1 could get significant modifications. These modifications could drastically alter the anomaly and bring it closer to what is seen in lattice QCD.

In light of the above discussions, it would be ideal to consider small black holes, for which $I$ can be neglected. However the fluxes in section 2 were computed using a perturbative series in $O(\epsilon)$, where we only kept up to linear order terms in the action. For small black holes, fluxes cannot be written as such a series since there is no small parameter like $(2.38)$. In other words, if we were to write the fluxes as a series, the relevant parameter is then $\tilde{\epsilon} \equiv \frac{g_s M^2}{N(\rho_c)}$. For small black holes, $\rho_c$ will be small and $\tilde{\epsilon}$ will be large. Then higher order terms in $\tilde{\epsilon}$ are equally or more important than the linear order term and thus we cannot exactly compute the on-shell gravity action. Consequently we cannot obtain the thermodynamics near critical temperature. On the other hand, small black holes are dual to QCD like gauge theories near critical temperature. Thus in the following section we consider small black holes in deformed warped cone which in fact give rise to QCD like features.

### 3.3 Connection to QCD

As already discussed in the previous sections, the perturbative analysis used to derive the black hole solutions and the resulting thermodynamic state functions of the gauge theory breaks down when the horizon $r_h$ is small. However, small horizon could give rise to temperatures that are above or near the deconfinement temperature of QCD. This is because large black holes describe ultra violate modes of dual gauge theories with gauge group $SU(N + M) \times SU(N)$, which is nothing like QCD. Only at low energy the gauge group cascades down to $SU(M)$, which can resemble QCD. Thus to determine the thermodynamics of the strongly coupled gauge theory that behaves like QCD, we must obtain black hole solutions with $r_h$ small.

On the other hand, when $r > r_h$ is small, the internal metric (2.7) does not take the simple form (2.10) of a regular cone and we need to obtain black holes in warped deformed cone where the radial coordinate is $\rho$ and horizon is $\rho_h$. When there is no horizon and we restrict to region 1, where there is no localized source, the internal metric is given by $\tilde{g}^0_{mn}$. When horizon $\rho_h \neq 0$, the internal metric is $\tilde{g}_{mn} = \tilde{g}^0_{mn} + \tilde{g}^1_{mn}$ with $\tilde{g}^1_{mn} \neq 0$. For very large horizons, $\tilde{g}^1_{mn}$ is small and given by (2.47), but for small
horizons the higher order terms are equally or more important than the leading term in the expansion and we do not have exact expressions. Without knowing the form of \( \tilde{g}_{mn}^1 \), we cannot find the horizon which is obtained by solving (2.33).

However, we can still obtain the form of the function \( B(x^m) \) by first noting that key quantity that enters (2.33) and crucially depending on \( \tilde{g}_{mn}^1 \) is

\[
H_n \equiv \tilde{g}^{pq} \partial_n \tilde{g}_{pq} \tag{3.36}
\]

where \( p, q = 4, \ldots, 9 \) runs over the cone directions. If \( n = \rho \) and we use \( \tilde{g}_{pq} = \tilde{g}_{pq}^0 \), then one readily gets that \( H_\rho = H_\rho(\rho) \) is only a function of \( \rho \). Then, we can solve (2.33) to obtain that \( B(\rho) \) is only a function of \( \rho \). This leads to a horizon that is an eight dimensional surface described by \( \rho = \rho_h \) where \( e^{B(\rho_h)} = 0 \). In region 1, there are no localized sources and no source for asymmetry. Thus we expect the horizon to be a surface given by \( \rho = \rho_h \) even when \( \tilde{g}_{pq}^1 \) corrections are considered and \( \tilde{g}_{pq} = \tilde{g}_{pq}^0 + \tilde{g}_{pq}^1 \).

Thus including the metric corrections \( \tilde{g}_{pq}^1 \), we expect

\[
\tilde{g}^{mn} \tilde{g}^{pq} \partial_n \tilde{g}_{pq} \partial_mB = \tilde{g}^{pp} \tilde{g}^{pq} \partial_p \tilde{g}_{pq} \partial_B = \left( \tilde{g}^{pp,0} \tilde{g}^{pq,0} \partial_p \tilde{g}_{pq}^0 + Q(\rho) \right) \partial_B \tag{3.37}
\]

where \( Q(\rho) \) arises due to corrections \( \tilde{g}_{pq}^1 \). If we further assume \( \tilde{g}_{\rho \rho} = \tilde{g}_{\rho \rho}^0 \), i.e the perturbations \( \tilde{g}_{pq}^1 \) are only in the compact direction, (2.33) drastically simplifies to give

\[
\left[ 32 \rho \cos(\rho) + 4 \rho \cosh(3\rho) - 5 \sinh(\rho) - 12 \sinh(3\rho) + \sinh(5\rho) + \tilde{Q} \right] g'(\rho) + 2 \left[ -2 \rho + \sinh(2\rho) \right] \sinh(3\rho) g''(\rho) = 0 \tag{3.38}
\]

where prime denotes a derivative with respect to \( \rho \) and \( \tilde{Q} \neq 0 \) only when \( \tilde{g}_{pq}^1 \neq 0 \). We need to solve the above equation along with (2.31), and (2.36) to obtain the scalar functions \( g(\rho), h(\rho) \) and \( \gamma(\rho) \). When we are in large \( \rho \) region such that \( \tilde{g}_{mn}^0 \) is given by (2.10), the above equation has a simple solution in \( r \) coordinate as written in (2.42). For small \( \rho \), there are corrections to \( g(r) \) and we can write

\[
g(\rho) = 1 - \exp\left( \frac{4\rho^0_h}{3} - \frac{4\rho}{3} \right) + G(\rho) \tag{3.39}
\]

where \( \rho^0_h \) is a constant related to \( \tilde{r}_h \). Plugging in the above form (3.39) in (3.38), we can obtain a second order linear differential equation for \( G(\rho) \), which can be exactly solved if \( \tilde{Q}(\rho) \) was known. We can numerically solve (3.38) by plugging in a Taylor series for \( \tilde{Q} \)

\[
\tilde{Q} = q_i \rho^i \tag{3.40}
\]

\[
\left[ 32 \rho \cos(\rho) + 4 \rho \cosh(3\rho) - 5 \sinh(\rho) - 12 \sinh(3\rho) + \sinh(5\rho) + \tilde{Q} \right] g'(\rho) + 2 \left[ -2 \rho + \sinh(2\rho) \right] \sinh(3\rho) g''(\rho) = 0
\]
where $q_i$ are constants. In fig 8 we plotted $g(\rho)$ obtained by solving (3.38) with $\tilde{Q} = 0$ and the boundary condition $G(100) = 0, G'(10) = \frac{5}{10000}$ with the choice $\rho_h^0 = 1$. The solution is dependent on these boundary conditions and the choice for $\tilde{Q}$ and is only presented to demonstrate the qualitative feature of $g(\rho)$. Of course the Einstein equations in (2.23) will determine $\tilde{g}_{mn}$ and consequently $\tilde{Q}$. But since we do not have a solution to the set of flux equations and Einstein equations for small $\rho$, we only present a numerical solution to (3.38) ignoring $\tilde{Q}$ to understand the form of the function $g(\rho)$. Solution to (2.23) will give $q_i$ which are not necessarily zero, and thus the numerical solution presented in the plot will be altered. From our numerical solution, we find that the horizon is at

$$\rho = \rho_h \simeq 2.5\rho_h^0$$

(3.41)

Now, to obtain temperature, we need to find the warp factor for small $\rho$ in the presence of a black hole. We need to solve (2.31) and (2.36) with $g(\rho)$ given by (3.39). When $\rho_h = 0$, $h(\rho)$ is given by the Klebanov-Strassler solution

$$h_{KS}(\rho) = \alpha_0 \frac{2^{2/3}}{4} \int_{\rho}^{\infty} d\zeta \frac{\zeta \coth(\zeta) - 1}{\sinh^2(\zeta)} (\sinh(2\zeta) - 2\zeta)^{1/3}$$

(3.42)

In the presence of the black hole, this solution is altered and we expect a regular solution

$$h(\rho) = h_{KS}(\rho) + \tilde{c}_i \rho^i$$

(3.43)

where $\tilde{c}_i$ are constants such that $h(\rho_h) \neq 0$. 

![Figure 8: $g(\rho)$ as a function of $\rho$. The zero of the function gives the horizon $\rho_h$.](image)
With the form of $g(\rho) = e^{2B}$, $h(\rho) = e^{-2A}$ known, we can find the temperature

$$T(\rho_h) = \frac{g'(\rho_h)}{4\pi \sqrt{\hat{g}_{\rho\rho}(\rho_h)h(\rho_h)}}$$

(3.44)

In Fig 9, we have plotted the points $T(\rho_h)$ for various values of horizon $\rho_h$. In deriving the points shown in red dots, we numerically solved (3.38) for various choices of $\rho^0_h$, i.e $\rho^0_h = 1, 2, 2.2, ..., 6$, every time solving the differential equation with the condition $G(100) = 0, G'(10) = 0$. Each solution give a value for the horizon $\rho_h$ and then for each $\rho_h$, we evaluated (3.44) using

$$h(\rho) \sim h_{KS}(\rho) \sim \frac{2^{1/3}(g_sM)^2}{\rho e^{-4\psi/3}}$$

which is again an approximation. This approximation gets better for larger $\rho$ and since $\rho > \rho_h > 2$, we expect the approximation to be a reasonable one. Also note that we are ignoring $\tilde{c}_i$ since we do not know their exact values. Of course solving the Einstein equations will determine the constants $\tilde{c}_i$ and the solution for $T(\rho_h)$ will depend on $\tilde{c}_i$. However, we expect the form of $T(\rho_h)$ to remain unchanged, that is it should behave as a Taylor series

$$T(\rho_h) = t_i \rho^i_h$$

(3.45)

where $t_i$ are constants. For our plot in Fig 9, we were able to fit all the points with $T = 0.234 - 0.05 \rho_h + 0.126 \rho^2_h$ and the fit is shown in the blue curve. Of course, when $\tilde{c}_i$ are included, the coefficients will change. In generating this plot, we have set the constants such that

$$g^2_sM^2 = \frac{16}{A^{4/3}2^{1/3}}$$

(3.46)

which can lead to large $g_sM$ for a relatively small value of $A$. Also, to simplify and avoid keeping track of $\alpha'$, we set $\alpha' = 1$ in obtaining the plots and in what follows.

Using Wald’s formula, we can find the entropy of the black hole,

$$s \sim \sqrt{h(\rho_h) \sinh(\rho_h)(\sinh(2\rho_h) - 2\rho_h)^{1/3}}$$

(3.47)

Both $s$ and $T$ are functions of horizon $\rho_h$ and thus we can plot $s$ as a function of $T$. This is done in Fig 10 and 11 where temperature is obtained from the fit in Fig 9 and entropy is obtained using the scaling in (3.47). For small $T$, we see that $s \sim T$ while for larger $T$, $s \sim T^2$. These scalings arise by considering black holes in deformed cone, that is for small black holes. On the other hand for large black holes i.e. $r_h$ large, the entropy and free energy scales as (3.20), (3.21) and (3.24). Combining all these results, we find that the black hole has the following scaling of free energy with temperature

$$F \sim -T^2$$

for small $T_0 > T > T_c$.
Figure 9: $T(\rho_h)$ as a function of $\rho_h$. Solid line is the polynomial fit while the points are generated using the numerical solution.

Figure 10: $s(T)$ as a function of $T$, for small $T$.

\[\sim -T^3 \quad \text{for intermediate } T_0 > T > T_c\]
\[\sim -T^4 \left( 1 + \frac{b_0 g_s M^2}{N} - \frac{b_1 g_s M^2}{4N} + \frac{b_3 g_s M^2}{N} \log \left(T \sqrt{N \alpha'}\right) \right) \quad \text{for large } T \lesssim T_0\]
\[\sim -T^4 \quad \text{for large } T > T_0\]

Finally using these scalings, we get the conformal anomaly for the dual gauge theory,

\[\Delta \sim \frac{1}{T^2} \quad \text{for small } T_0 > T > T_c\]
\[ s(T) \]

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure11.png}
\caption{$s(T)$ as a function of $T$, for large $T$.}
\end{figure}

\[
\begin{align*}
\sim & \frac{1}{T} \quad \text{for intermediate } T_0 > T > T_c \\
\sim & \frac{27 \pi^6 N^2 \alpha' V b_1 g_s M^2}{512 \kappa^3 N} \quad \text{for large } T \lesssim T_0 \\
\sim & 0 \quad \text{for large } T > T_0
\end{align*}
\] (3.49)

Since we do not have exact numerical solutions for small black holes in region 1 (we only have exact actions for large black holes in region 1), we cannot compute exact on-shell action. Thus we cannot obtain the critical temperature $T_c$ above which black holes in deformed cone are preferred over vacuum. However, we have obtained the scalings of free energy, entropy and conformal anomaly for the black holes which can be done without computing $T_c$. If $T_c \ll T_0$ is small, the deconfined phase of the gauge theory has the scalings given by (3.48, 3.49), which is qualitatively similar to the lattice QCD simulations. On the other hand, the value of $T_c$ depends on values of the metric near the horizon and boundary $r = r_b$ and these boundary conditions can be altered. Thus by picking a certain class of boundary conditions, we expect to obtain $T_c \ll T_0$ and thus the scalings can describe the deconfined phase of a gauge theory that arises from the brane configuration of Fig 3. This way, we can obtain a black hole description for a thermal gauge theory that confines at IR, becomes conformal at the UV and behaves similar to QCD near $T_c$.

4. Conclusion

In this paper we demonstrated how the UV divergence of Klebanov-Strassler (KS) model can be eliminated by considering world volume fluxes on $D7$ branes. The
back reaction of the world volume fluxes modifies the KS fluxes $F_3, H_3$ for large $r$ and the resultant fluxes are explicitly given by (2.50). These fluxes are evaluated in the presence of a black hole and by taking the limit $\tilde{r}_h \to 0$, we can obtain the fluxes in vacuum.

Using these fluxes, we then showed that the effective $D5$ charge in the dual gauge theory vanishes in the far UV. The metric and fluxes are similar to the KS model in small ‘r’ region and thus the gauge theory confines in far IR. Hence we end up with a gauge theory that is UV conformal and IR confining—similar to QCD. The RG flow of the gauge theory can be directly obtained by using (3.3) with fluxes and dilation field given by (A.2), (2.41). A more detailed analysis of RG flow will be presented in our upcoming work [51].

The presence of the localized $D7$ branes along with world volume fluxes introduced an additional scale $r_0$ in the theory. The fluxes and dilaton field we obtained are identical to the KS model (up to linear order $O(\epsilon)$) in region 1 i.e. $r < r_0$ while they are modified in region 2 i.e. $r > r_0$. Thus $r_0$ can be thought of as the radial scale up to which KS solution can be used to study the gauge theory. This was done in section 3.1 and 3.3 by considering the gravity action $S_{R_1}$ for region 1 and neglecting the localized sources. Although in region 1 we neglected the localized sources, which was crucial in analyzing Hawking-Page transition, the scale $r_0$ arising from the sources explicitly entered our analysis. In particular the critical temperature $T_c$ given in (3.17) explicitly depends on $r_0$ and thus the localized sources implicitly affect the thermodynamics of the gauge theory. By considering the black hole entropy in region 1, we were able to compute thermodynamic state function of the gauge theory. Since we obtain identical RG flow irrespective of which gravitational action we use to describe region 1 ($S_{R_1}$ or $S_{SUGRA} + S_{D7}$), we expect that Wald entropy obtained from $S_{R_1}$ will be similar to the one obtained from $S_{SUGRA} + S_{D7}$. This is reasonable since the entropy depends on the near horizon behavior of the metric and both actions $S_{R_1}$ and $S_{SUGRA} + S_{D7}$ can result in identical horizon values.

In section 3.2 we studied the thermodynamics of the gauge theory when region 2 and localized sources are included. When $D7$ branes is outside the horizon and we consider back reaction, we argued that the dual gauge theory is not at thermal equilibrium and there is no unique temperature. In this scenario, there are no Hawking-Page transitions between geometries. On the other hand, when we consider larger horizons, the $D7$ brane along with world volume fluxes fall into the black hole. This black hole which absorbed the $D7$ branes corresponds to a gauge theory at thermal equilibrium and the temperature is given by the Hawking temperature of the black hole. We argue for large horizons, we will end up with Schwarzschild black holes since the total $D5$ charge will be neutralized. Thus at large temperatures, we will obtain a thermal CFT with a dual description in terms of Schwarzschild black holes in $AdS_5 \times T^{1,1}$.

Finally in section 3.3, we try to make connections to QCD by considering small
black holes in deformed cone geometry. Using the form of the warp factors near the tip of the deformed cone, we worked out a numerical solution for the black hole. Then computing the entropy and temperature of such a black hole, we were able to obtain the scaling of conformal anomaly. This scaling is qualitatively similar to QCD near critical temperature. For an exact analysis, we need to numerically obtain black hole solution in warped deformed conifold, which is rather challenging and beyond the scope of our current analysis. However, the qualitative agreement between scalings of conformal anomaly indicates we indeed have a gravitational description of a QCD like theory.

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**A. Appendix : Three forms and $B_2$**

Here we explicitly write down the expressions for three forms and their Hodge dual, using the unperturbed metric $\tilde{g}_{\mu
u}$

\[ *_6 \tilde{\omega}_3^1 = \frac{3\Gamma_1 \sin \theta_2 e^{-B(r)}}{r} dr \wedge \left[ \left( 1 + \frac{6}{9} \left( \cot^2 \theta_2 - \frac{a \cot \theta_2}{\sin \theta_2} \right) \right) d\phi_2 \wedge d\theta_2 + \frac{6}{9} \left( \frac{\cos \theta_1 \cot \theta_2}{\sin \theta_2} \right) d\psi \wedge d\theta_2 \right] - \frac{3\Gamma_2 \sin \theta_1 e^{-B(r)}}{r} dr \wedge \left[ \left( 1 + \frac{6}{9} \left( \cot^2 \theta_1 - \frac{a \cot \theta_1}{\sin \theta_1} \right) \right) d\phi_1 \wedge d\theta_1 + \frac{6}{9} \left( \frac{\cos \theta_2 \cot \theta_1}{\sin \theta_1} - \frac{a \cos \theta_2}{\sin^2 \theta_1} \right) d\psi \wedge d\theta_1 \right] \]

Using the form (A.1) and the definition $H_3 = dB_2 + J_3$, for some $J_3$, we readily find

\[
B_2 = \frac{3g_s M \alpha'}{2} \log \left( \frac{r}{r_s} \right) \left( g^1 \wedge g^2 + g^3 \wedge g^4 \right) + 12\mathcal{K}(r)\kappa_{10}^2 g_s \mathcal{M} N_f \alpha' \mu_7 \left( \Gamma_1 \sin \theta_2 \right) \left[ \left( 1 + \frac{6}{9} \left( \cot^2 \theta_2 - \frac{a \cot \theta_2}{\sin \theta_2} \right) \right) d\phi_2 \wedge d\theta_2 + \frac{6}{9} \left( \frac{\cot \theta_2}{\sin \theta_2} - \frac{a \cot \theta_2}{\sin^2 \theta_2} \right) d\psi \wedge d\theta_2 \right] - \Gamma_2 \sin \theta_1 \left[ \left( 1 + \frac{6}{9} \left( \cot^2 \theta_1 - \frac{a \cot \theta_1}{\sin \theta_1} \right) \right) d\phi_1 \wedge d\theta_1 + \frac{6}{9} \left( \frac{\cot \theta_1}{\sin \theta_1} - \frac{a \cot \theta_1}{\sin^2 \theta_1} \right) d\psi \wedge d\theta_1 \right] \]

\[ - 41 - \]
$$\mathcal{K}(r) = \int^r du \frac{F(u)}{u} \left(1 + \frac{\hat{r}_h^4}{w^4 e^{2B}}\right) \quad (A.2)$$

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