EQUIVARIANT K-THEORY AND RESOLUTION
I: ABELIAN ACTIONS

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Abstract. The smooth action of a compact Lie group on a compact manifold can be resolved to an iterated space, as made explicit by Pierre Albin and the second author. On the resolution the lifted action has fixed isotropy type, in an iterated sense, with connecting fibrations and this structure descends to a resolution of the quotient. For an abelian group action the equivariant K-theory can then be described in terms of bundles over the base with morphisms covering the connecting maps. A similar model is given, in terms of appropriately twisted deRham forms over the base as an iterated space, for delocalized equivariant cohomology in the sense of Baum, Brylinski and MacPherson. This approach allows a direct proof of their equivariant version of the Atiyah-Hirzebruch isomorphism.

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Introduction

One intention of this note is to demonstrate that real blow-up can be an effective tool in the analysis of smooth group actions, particularly in the compact case. To do so, we describe equivariant K-theory in terms of resolved spaces and in consequence introduce (here only in the abelian case) a geometric model for the delocalized equivariant cohomology of Baum, Brylinski and MacPherson [2], designed to realize an equivariant form of the Atiyah-Hirzebruch isomorphism

\[ \text{Ch} : K^*_G(M) \otimes \mathbb{C} \xrightarrow{\sim} H^*_{\text{di,G}}(M). \]

The more general case of the action by a non-abelian compact Lie group will be treated subsequently. That the non-abelian case is more intricate can be seen from the computation of the equivariant K-theory in case of an action with single isotropy type by Wassermann [7]. See also the paper of Rosu [5].
Resolution of a group action, as described by Pierre Albin and the second author in [1], replaces it by a tree of actions each with unique isotropy type and with connecting equivariant fibrations. This results in a similar resolution of the quotient, which we call an ‘iterated space’ corresponding to its smooth stratification. The description given here of the various cohomology theories is directly in terms of smooth ‘iterated’ objects, bundles or forms, over these iterated spaces with augmented ‘pull-back’ morphisms covering the connecting fibrations. Resolution may be thought of as replacing the ‘analytic complexity’ of strata by the ‘combinatorial complexity’ of iterated fibrations. The perceived advantage of this is that many standard arguments can be transferred directly to this iterated setting, since the spaces are smooth. The objects which appear here have local product structures.

The case of a compact abelian group, $G$, acting, with single isotropy group, on a compact manifold (with corners), $M$, is relatively simple and forms the core of our iterative approach. If the action is free then each equivariant bundle is equivariantly isomorphic to the pull-back of a bundle over the base; equivariant bundles descend to bundles. Equivariant K-theory is then identified, as a ring, with the ordinary K-theory of the base. However the structure of $K_G(M)$ as a module over the representation ring of $G$ is lost in this identification. Tensor product and descent defines an action of irreducible representations of $G$ on smooth bundles over the base

\[ \sigma : \hat{G} \times \text{Bun}(Y) \rightarrow \text{Bun}(Y) \]

which projects to give the action of $\hat{G}$ on $K_G(M)$. In realizing equivariant K-theory and delocalized equivariant cohomology over the resolved space we need to retain aspects of $\sigma$.

For an abelian action with fixed isotropy group $B \subset G$ there is a similar reduction to objects on the base. Equivariant bundles may be decomposed over the dual group, $\hat{B}$, giving a finite number of coefficient bundles. Lifting an element of $\hat{B}$ into $\hat{G}$ and taking the tensor product with the inverse gives the coefficient bundle an action of $\hat{G}/\hat{B}$. The case of a principal action then applies and results in a collection of bundles $W_{\hat{g}}$ over the base, $Y$, indexed by $\hat{g} \in \hat{G}$. We assemble these into a bundle over $\hat{G} \times Y$ – allowed to have different dimensions over different components – with two additional properties. First its support projects to a finite subset of $\hat{B}$ and more significantly it is ‘twisted’ under the action of $\hat{G}/\hat{B}$ on $\hat{G}$ in the sense that

\[ \sigma(\hat{h}) \otimes W_{\hat{g}} = W_{\hat{g}}. \]

In this setting of a single isotropy group, the delocalized equivariant cohomology is given in terms of a twisted deRham complex. These forms are given by finite sums of formal products

\[ \sum_i \hat{g}_i \otimes u_i, \quad u_i \in C^\infty(Y; \Lambda^*) \]

where the twisting law (3) is replaced by its cohomological image

\[ (\hat{h} \hat{g}) \otimes \text{Ch}(\hat{h}) \wedge v \simeq \hat{g} \otimes v, \quad \hat{h} \in \hat{G}/\hat{B}, \quad v \in C^\infty(Y; \Lambda^*). \]

Here $\text{Ch}(\hat{h})$ is the Chern character of the bundle, with connection, given by descent from the representation $\hat{h}$ interpreted as a trivial bundle with equivariant action and with product connection. The reduced bundles may be given connections, consistent
with the connection on $\hat{h}$ and (3) for which the Chern character is a delocalized form in the sense of (1). For discussions of the equivariant Chern character see the book [3] of Berline, Getzler and Vergne and the paper of Getzler [4].

These definitions of reduced bundles and delocalized deRham forms are extended to iterated objects over the resolution of the quotient, $Y\ast$, by adding morphisms covering the connecting fibrations. This leads directly to the Atiyah-Hirzebruch-Baum-Brylinski-MacPherson isomorphism (1), proved here using the six-term exact sequences which result from successive pruning of the isotropy tree.

In outline the paper proceeds as follows. In §1 we recall from [1] the resolution $X\ast$ of any compact Lie group action on a compact manifold, with the quotient an iterated space $Y\ast$. The lifting of equivariant bundles to iterated equivariant bundles on $X\ast$ is described in §2 and the reduction to twisted iterated bundles over $Y\ast$ is discussed, for abelian actions, in §3 – the non-abelian case is much more intricate because of the appearance of ‘Mackey twisting’. The realization of equivariant K-theory in terms of reduced bundles is contained in §4 and this leads to the geometric model for delocalized (abelian) equivariant cohomology in §5. The relative sequences obtained by successive pruning of the isotropy tree are introduced in §6 and used to establish (1) in §7. Examples of circle actions are considered in §8.

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1. Resolution

In [1], the resolution of the smooth action of a compact Lie group action on a compact manifold, $M$, was described. The action stratifies $M$, into smooth submanifolds, with the isotropy group lying in a fixed conjugacy class of closed subgroups of $G$ on each stratum. For convenience we shall assume, without loss of generality, that the quotient, $M/G$, is connected. If $M$ is not connected then $G$ acts on the set of components and we may consider each orbit separately and so assume that $G$ acts transitively on the set of components. Similarly, we declare the strata, $M_\alpha \subset M$, to consist of the images under the action of $G$ of the individual components of the manifolds where the isotropy class is fixed. Thus the labelling index, $\alpha \in A$, records a little more than the isotropy type.

We recall both the resolution of such a group action and the consequent resolution of the quotient in terms of ‘iterated spaces’. This is essentially the notion of a ‘resolved stratified space’.

For present purposes the category $\mathfrak{Man}$ has as objects the compact manifolds with corners, not necessarily connected. Each such manifold has a finite collection, $\mathcal{M}_1(M)$, of boundary hypersurfaces $H \subset M$. By definition of a ‘manifold with corners’ we require that these boundary hypersurfaces are embedded – they are themselves manifolds with corners having no boundary faces identified in $M$. As a result each boundary hypersurface has a global defining function $0 \leq \rho \in C^\infty(M)$, vanishing simply and precisely on $H$. As morphisms we will take ‘smooth interior b-maps’ which is to say smooth maps in the usual sense $M_1 \to M_2$ such that the pull-back of a boundary defining function for a boundary hypersurface of $M_2$ is the product of powers of boundary defining functions for hypersurfaces of $M_1$ (including the case that the pull-back is strictly positive). Certainly all smooth diffeomorphisms are interior b-maps. A smooth $G$ action on $X$ is required to be
boundary free in the sense that

\[(1.1) \quad g \in G, \; H \in \mathcal{M}_1(M) \Rightarrow \text{either } gH = H \text{ or } gH \cap H = \emptyset.\]

In fact the morphisms we are most concerned with here are fibre bundles, which we call ‘fibrations’. In this compact context, these are simply the surjective interior b-maps with surjective differentials. The implicit function theorem applies to show that for such a map each point in the base has an open neighbourhood \(U\) with inverse image diffeomorphic to the product \(U \times Z\) with \(Z\) a fixed (over components of the base) compact manifold with corners with the map reducing to projection. Note that the b-map condition is used here; without such an assumption the fibres can be cut off by boundaries.

**Definition 1.** The category, \(\mathfrak{Man}\), has as objects, \(X_*\), iterated spaces in the following sense. There is a ‘principal’ manifold with corners \(X_0\) which is the root of a tree \(X_\alpha\) of manifolds corresponding to a partial order (‘depth’) \(\alpha \leq \beta \in A\). The boundary hypersurfaces of \(X_0\) are partitioned into subsets, with elements which do not intersect, forming ‘collective boundary hypersurfaces’ \(H_\alpha(X_0) \subset \mathcal{M}_1(X_0)\). These carry fibrations

\[(1.2) \quad \psi_\alpha : H_\alpha(X_0) \rightarrow X_\alpha.\]

Under the partial order on the \(H_\alpha\) two (always collective) hypersurfaces are related if and only if they intersect and any collection with non-trivial total intersection forms a chain. For each \(\alpha\) the set of boundary hypersurfaces of \(X_\alpha\) is also partitioned into collective boundary hypersurfaces

\[(1.3) \quad H_\beta(X_\alpha) = \psi_\alpha(H_\beta), \; \beta > \alpha\]

and \(\psi_\beta\) restricted to \(H_\alpha\) factors through a fibration

\[(1.4) \quad \psi_{\beta,\alpha} : H_\beta(X_\alpha) \rightarrow X_\beta, \; \beta > \alpha;\]

with the base index denoted 0, \(\psi_0 = \psi_{\alpha,0}\).

A smooth \(G\)-action on an iterated space is a boundary free \(G\) action on each \(X_\alpha\) with respect to which all the fibrations \(\psi_{\alpha,\beta}\) are \(G\)-equivariant.

It follows that in an iterated space, for any chain

\[(1.5) \quad \alpha_1 < \alpha_2 < \cdots < \alpha_k\]

there is a sequence of fibrations

\[(1.6) \quad \bigcap_{1 \leq j \leq k} H_{\alpha_j}(X_0) \xrightarrow{\psi_{\alpha_1}} \bigcap_{2 \leq k \leq k} H_{\alpha_j}(X_{\alpha_1}) \xrightarrow{\psi_{\alpha_2,\alpha_1}} \cdots \xrightarrow{\psi_{\alpha_k,\alpha_{k-1}}} H_{\alpha_k}(X_{\alpha_{k-1}}) \xrightarrow{\psi_{\alpha_{k,\alpha_{k-1}}}} X_{\alpha_0}\]

with composite the restriction of \(\psi_{\alpha_k}\). It is also follows that the fibres of the restricted fibrations have strictly increasing codimension as submanifolds of the fibres in the hypersurfaces.

Resolution is accomplished in [1] by radial blow up (which corresponds to a sequence of interior b-maps) of successive smooth centres corresponding to the tree of isotropy types, in (any) order of decreasing codimension. This results in a well-defined iterated space, \(X_*\), with \(G\)-action in the sense described above with principal space \(X = X_0\) and iterated blow-down map

\[(1.7) \quad \beta : X \rightarrow M\]
giving the resolution of $M$. The $X_\alpha$ are the resolutions of the isotropy types $\overline{M}_\alpha$ in the same sense. The important property of the resolution is that the $G$-action on each (smooth, compact) $X_\alpha$ now has fixed isotropy type and the 'change of isotropy type' occurs within the fibrations $\psi_{\alpha,\beta}$.

Since the action on each $X_\alpha$ has fixed isotropy type the quotients
\begin{equation}
Y_\alpha = X_\alpha / G
\end{equation}
are all smooth manifolds with corners having boundary hypersurfaces $H_\beta(Y_\alpha)$, $\beta > \alpha$, labelled by the index set
\begin{equation}
A_\alpha = \{ \beta \in A; \beta > \alpha \}
\end{equation}
and forming a tree with the corresponding intersection relations and base $\alpha$. The $G$-equivariant fibrations (1.4) descend to give $Y_\ast$ the structure of an iterated space
\begin{equation}
H_\beta(X_\alpha) \xrightarrow{f_G} H_\beta(Y_\alpha), \; \beta > \alpha, \; \phi_\alpha = \phi_{\alpha,0}.
\end{equation}

2. Lifting

Let $\mathfrak{Bun}(M)$ denote the category of finite-dimensional, smooth, complex, vector bundles over a compact manifold $M$, with bundle maps as morphisms. Similarly if $M$ is a smooth $G$-space let $\mathfrak{Bun}_G(M)$ denote the category of bundles with equivariant $G$-action covering the action on $M$ and with morphisms the bundle maps intertwining the actions. Thus the equivariant K-theory of $M$ can be realized (see Segal [6]) as the Grothendieck group
\begin{equation}
K_G(M) = \mathfrak{Bun}_G(M) \ominus \mathfrak{Bun}_G(M) / \simeq
\end{equation}
with the relation of stable $G$-equivariant bundle isomorphism.

In general if $F : M \rightarrow N$ is a smooth $G$-equivariant map of $G$-spaces then pull-back defines a functor
\begin{equation}
F^* : \mathfrak{Bun}_G(N) \rightarrow \mathfrak{Bun}_G(M).
\end{equation}
In particular this applies to the blow-down map in the resolution of the action.

Definition 2. If $X_\ast$ is an iterated space we denote by $\mathfrak{Bun}(X_\ast)$ the category with objects ‘iterated bundles’ consisting of a bundle $B_\alpha \in \mathfrak{Bun}(X_\alpha)$ for each $\alpha \in A$ and with pull-back isomorphisms specified over each $H_\alpha(X_0)$,
\begin{equation}
\mu_\alpha : \phi_\alpha^* B_\alpha \simeq B_0 \big|_{H_\alpha(X_0)}
\end{equation}
which factor through intermediate bundle isomorphisms $\mu_{\alpha,\beta}$, $\alpha < \beta$, covering the sequence (1.6) over each boundary face of $X_0$. The morphisms are bundle maps between the corresponding bundles which commute with the connecting morphisms (2.3).

If $X_\ast$ is an iterated space with $G$-action, $\mathfrak{Bun}_G(X_\ast)$ denotes the category in which the bundles carry $G$-actions covering the actions on the $X_\alpha$ and the connecting isomorphisms, (2.3), are $G$-equivariant; morphisms are then required to be $G$-equivariant.
Lemma 1. If the iterated G-space $X_\ast$ is the resolution of $M$, with compact G-action, then pull-back under the iterated blow-down map defines a functor
\begin{equation}
\beta^* : \text{Bun}_G(M) \to \text{Bun}_G(X_\ast)
\end{equation}
and every iterated bundle in $\text{Bun}_G(X_\ast)$ is isomorphic to the image of a bundle in $\text{Bun}_G(M)$.

Proof. The lifting of the objects, G-equivariant bundles, and corresponding morphisms under $\beta$ is simply iterated pull-back. It only remains to show that every G-equivariant iterated bundle in $\text{Bun}_c(X_\ast)$ is isomorphic to such a pull-back. As shown in [1] the resolution $X_\ast$ can be ‘rigidified’ by choosing product decompositions near all boundary hypersurfaces with G-invariant smooth defining functions consistent near all corners, i.e. so that the various retractions commute.

In the simple setting of a compact manifold with boundary, $M$, suppose $V$ is a smooth vector bundle over $M$, $U$ is a vector bundle over the boundary $H$ and $T : V|_H \to U$ is a bundle isomorphism. Then $V$ can be modified near $H$ to an isomorphic bundle $\tilde{V}$ which has fibres over $H$ identified with those of $U$ and outside a small collar neighbourhood of $H$ has fibres identified with $V$. This can be accomplished by a rotation in the isomorphism bundle of $V \oplus U$ and in particular carries over to the equivariant case. Indeed the standard construction has the virtue of leaving the original bundle unchanged over any set in the collar over an open set on which $T$ is already an identification. This allows the bundle isomorphisms to be ‘removed’ inductively over the isotropy tree.

Once the isomorphisms are reduced to the identity the bundles themselves can be similarly modified in equivariant collars around the boundary hypersurfaces of $X_0$ to be constant along the normal fibrations and hence to be the pull-backs of smooth bundles on the base. Alternatively the topological bundles obtained by direct projection can be smoothed over $M$. \hfill \Box

Pulling back a G-connection from a bundle on $M$ we find:

Corollary 2. For a G-bundle $W_\ast \in \text{Bun}_G(X_\ast)$ there are a G-equivariant connection on each $W_{\ast \alpha}$ which are intertwined by the $\mu_{\alpha,\beta}$

Now, we can therefore identify
\begin{equation}
K_G(M) = \text{Bun}_G(X_\ast) \ominus \text{Bun}_G(X_\ast) \simeq \text{Bun}_G(M)/\simeq
\end{equation}
as the Grothendieck group of iterated G-bundles on the resolution up to stable isomorphism.

Finite dimensional representations of a compact Lie group, $G$, can be decomposed into direct sums of tensor products with respect to a fixed set $\hat{G}$ of irreducibles, which can be identified with the set of characters. This allows the representation category to be identified with the $\text{Bun}_c(\hat{G})$ with objects the finitely supported ‘bundles’ over $\hat{G}$ and morphisms being bundle maps. Here, for a non-connected space, the objects in $\text{Bun}_c$ are permitted to have different dimensions over different components but in this case, where there may be infinitely many components, the bundles must have dimension 0 outside a compact set. So the objects consists of a (complex) vector spaces associated to a finite number of characters. Each object in $\text{Bun}_c(\hat{G})$ defines an equivariant bundle over any G-space and tensor product with these bundles induces an action of the representation ring, $R(G) = \hat{G}(\mathbb{Z})$ on $K_G(M)$. Aspects of this action are particularly important in the sequel.
Proposition 3. For the action of a compact Lie group on an iterated space $X_*$, taking the tensor product with a (finite-dimensional) representation gives a functor

$$\sigma : \mathfrak{Bun}(\hat{G}) \times \mathfrak{Bun}_G(X_*) \to \mathfrak{Bun}_G(X_*).$$

Proof. Given an element $(V, E) \in \mathfrak{Bun}(\hat{G}) \times \mathfrak{Bun}_G(X_*)$, the corresponding object in $\mathfrak{Bun}_G(X_*)$ is the tensor product of $E$ and $V$, with $V$ thought as the trivial iterated bundle over $X_*$ with the implied $G$-action. Given an element $V$ of $\mathfrak{Bun}(\hat{G})$ and an equivariant iterated bundle map $E \to F$, we obtain an equivariant bundle map $V \otimes E \to V \otimes F$ and similarly morphism of representations $V_1 \to V_2$ and an equivariant iterated bundle $E$, we obtain an induced equivariant bundle map $V_1 \otimes E \to V_2 \otimes E$. 

3. Reduction

The abelian case is considerably simpler than the general one and has been more widely studied. From this point on, in this paper, we shall assume that $G$ is compact and abelian. One fundamental simplification is that all (complex) irreducible representations in the abelian case are one-dimensional (and of course all 1-dimensional representations are irreducible). In this case $\hat{G}$ is a discrete abelian group.

As recalled in the Introduction, if a compact Lie group acts freely on a compact manifold, $X$, then the quotient, $Y$, is a compact manifold and $X$ is a principal bundle over it. For an equivariant bundle over $X$, the action over each orbit gives descent data for the bundle, defining a vector bundle over the base. This gives an equivalence of categories

$$\mathfrak{Bun}_G(X) \cong \mathfrak{Bun}(Y) \text{ if } G \text{ acts freely.}$$

For such a free action, tensor product with representations gives a ‘quantization’ of the dual group

$$\sigma : \hat{G} \to \mathfrak{Bun}(Y)$$

corresponding to (2.6).

We need to understand this operation in the more general case of an action with a fixed isotropy group $B \subset G$, necessarily a closed subgroup. There is then a short exact sequence

$$B \to G \to G/B$$

that is split since the groups are abelian. The dual sequence

$$\hat{G}/\hat{B} \to \hat{G} \to \hat{B}$$

is also exact and split so there exists a group homomorphism $\tau$ as indicated, giving a right inverse. Two such maps $\tau$, $\tau'$ are related by a group homomorphism

$$\mu : \hat{B} \to G/\hat{B}.$$
For an action with isotropy group $B$, the quotient $G/B$ acts freely on $X$ and the discussion above gives the equivalence of categories and shift functor

\[(3.6) \quad \text{Bun}_{G/B} (X) \cong \text{Bun} (Y), \quad \sigma : \widehat{G/B} \to \text{Bun} (Y), \]

\[Y = X/G = X/(G/B).\]

It is still the case that $G$-equivariant bundles descend to the quotient but only after decomposition under the action of $B$. Consider the space $\widehat{G} \times Y$, which has a natural action by $\widehat{G/B}$, with $\hat{g} \times Y$ mapped to $(\hat{h} \otimes \hat{g}) \times Y$.

**Definition 3.** For a compact abelian $G$-action on a compact manifold $X$ with fixed isotropy group $B \subset G$ and base $Y = X/G$, let $\text{Bun}_G^B (\widehat{G} \times Y)$ denote the category of bundles over $\widehat{G} \times Y$ with support which is finite when projected to $\widehat{B}$ and which satisfy the transformation law

\[(3.7) \quad \sigma (\hat{h}) \otimes W_{\hat{h}\otimes \hat{g}} = W_{\hat{g}} \forall \hat{h} \in \widehat{G/B}; \; \hat{g} \in \widehat{G}.\]

Morphisms are bundle maps over each $\hat{g} \times Y$ which are natural with respect to (3.7).

Note that we could eliminate the action (3.7) at the expense of choosing a splitting group homomorphism $\tau : \widehat{B} \to \widehat{G}$ as in (3.4), reducing elements of $\text{Bun}_G^B (\widehat{G} \times Y)$ to arbitrary elements of $\text{Bun}_G (\widehat{B} \times Y)$.

**Proposition 4.** For an action of a compact abelian group with fixed isotropy group $B \subset G$ there is an equivalence of categories

\[(3.8) \quad R : \text{Bun}_G (X) \cong \text{Bun}_G^B (\widehat{G} \times Y), \quad Y = X/G\]

where $W \in \text{Bun}_G^B (\widehat{G} \times Y)$ corresponds to the $G$-equivariant bundle

\[(3.9) \quad \bigoplus_{\hat{b} \in \hat{B}} \tau (\hat{b}) \otimes \pi^* (W_{\tau (\hat{b})})\]

for a splitting homomorphism $\tau$ as in (3.4).

Elements of $\text{Bun}_G^B (\widehat{G} \times Y)$ are our ‘reduced bundles’ in this simple case.

In order to define (3.9) we to pass to the restriction of an element of $\text{Bun}_G^B (\widehat{G} \times Y)$ to the image $\tau (\hat{B}) \times U$ given by a splitting homomorphism $\tau$. As a consequence of (3.7) the result is independent of the choice of $\tau$.

**Proof.** The isotropy group acts on the fibres of an equivariant bundle $U \in \text{Bun}_G (X)$ which therefore decomposes into a direct sum of $B$-equivariant bundles

\[(3.10) \quad U = \bigoplus_{\hat{b} \in \hat{B}, \text{finite}} U_{\hat{b}}\]

where the action of $B$ on each term factors through the irreducible representation $\hat{b}$. If $\hat{g} \in \widehat{G}$ is a representation which restricts to $\hat{b}$ then the action of $B$ on $\hat{g}^{-1} \otimes U_{\hat{b}}$ is trivial. This bundle therefore has an equivariant $G/B$-action and so descends to a bundle $W_{\hat{g}}$ over $Y$. Doing this for every $\hat{g}$, we define a bundle over $\widehat{G} \times Y$, supported over a finite subset of $\widehat{B}$. Clearly these bundles satisfy (3.7). Conversely each element of $\text{Bun}_G^B (\widehat{G} \times Y)$ defines an element of $\text{Bun}_G (X)$. \qed
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Note that the category $\mathcal{B}\text{un}^B(\hat{G} \times Y)$ is not determined by the groups and base $Y$ alone since it depends on the ‘shift’ isomorphism $\sigma$ which retains some information about the principal bundle, namely the images under descent to $Y$ of the trivial $G$-bundles given by elements of $\hat{G}/B$.

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Consider next a principal $G$-bundle, for $G$ compact abelian, and a $G$-equivariant fibration giving a commutative diagram

\[
\begin{array}{ccc}
X & \xrightarrow{\pi} & X_1 \\
\downarrow /G & & \downarrow /G \\
Y & \xrightarrow{\pi_1} & Y_1
\end{array}
\]

where the $G$-action on $X_1$ has fixed isotropy group $B$; thus $\pi_1$ is a fibration of smooth compact manifolds. In view of the identification of equivariant bundles in Proposition 4, the pull-back map descends to an ‘augmented pull back map’

\[
\begin{array}{ccc}
\mathcal{B}\text{un}_G(X) & \xleftarrow{\pi^*} & \mathcal{B}\text{un}_G(X_1) \\
\downarrow & & \downarrow \\
\mathcal{B}\text{un}(Y) & \xleftarrow{\pi_1^*} & \mathcal{B}\text{un}_B^c(\hat{G} \times Y_1)
\end{array}
\]

given by pull-back followed by summation over a splitting $\tau : \hat{B} \rightarrow \hat{G}$:

\[
\begin{array}{cc}
\pi_1^\#: & \mathcal{B}\text{un}_c^B(\hat{G} \times Y_1) \xrightarrow{\pi_1} \mathcal{B}\text{un}_c^B(\hat{G} \times Y) \xrightarrow{\sigma^\#} \mathcal{B}\text{un}(Y) = \mathcal{B}\text{un}_c^c(\hat{G} \times Y), \\
& \sigma^\#(V) = \bigoplus_{b \in \hat{B}} \sigma(\tau(b))V_{\tau(b)}.
\end{array}
\]

As implicitly indicated by the notation, $\sigma^\#(V)$ is independent of the section $\tau$.

We need this in the more general case of an equivariant fibration between two actions with fixed isotropy groups. For nested closed subgroups, $K \subset B \subset G$, we choose iterated splittings

\[
\begin{align*}
\tau' : & \hat{K} \rightarrow \hat{B}, \tau_1 : \hat{B} \rightarrow \hat{G} \implies \tau = \tau_1 \tau' : \hat{K} \rightarrow \hat{G}.
\end{align*}
\]

Then

\[
\begin{align*}
\hat{b}' : \hat{b} \in \hat{B}, \hat{b}'|_K & = \hat{b} \iff \exists \hat{h} \in \hat{G}/K \text{ s.t. } \tau_1(\hat{b}') = \hat{h}\tau(\hat{b}).
\end{align*}
\]

**Proposition 5.** If (4.1) is an equivariant fibration between actions of a compact abelian Lie group $G$ with fixed isotropy groups $B \supset K$ then pull back of equivariant bundles descends to the augmented pull-back map

\[
\begin{array}{ccc}
\pi_1^\#: & \mathcal{B}\text{un}_c^B(\hat{G} \times Y_1) \rightarrow \mathcal{B}\text{un}_c^K(\hat{G} \times Y) \\
\end{array}
\]

given by pull back on the fibres

\[
\begin{array}{ccc}
\pi_1^* : & \mathcal{B}\text{un}_c^B(\hat{G} \times Y_1) \rightarrow \mathcal{B}\text{un}_c^B(\hat{G} \times Y)
\end{array}
\]
followed by summation to give the value at the image $\tau(\hat{k})$ using (4.3)

$$\left(\sigma^\#(V)\right)_{\tau(\hat{k})} = \bigoplus_{\{\hat{b}\in\hat{B}\mid_k = \hat{k}\}} \sigma(\hat{h})V_{\hat{h},\tau(\hat{k})},$$

(4.7)

$$\forall \ V \in \mathfrak{Bun}_c^B(\hat{G} \times Y), \ \hat{k} \in \hat{K}.$$  

\textbf{Proof.} An equivariant fibration can be factored through the fibre product

$$\hat{X}_1 = X_1 \times_{Y_1} Y \longrightarrow Y, \ X \longrightarrow \hat{X}_1 \longrightarrow X_1$$

where the $G$-action on $\hat{X}_1$ has isotropy group $B$. Thus it suffices to consider the two cases of the pull-back of an action under a fibration and the quotient of an action with isotropy group $K$ by a larger subgroup $B$. In the first case the augmented pull-back is simply the pull-back as in (5) with (4.7) being the identity. In the second case the base is unchanged, so (5) is the identity and the summation is over those elements of $\hat{B}$ with fixed restriction to $K$. \qed

Now we pass to the general case of the action of a compact abelian Lie group $G$ on a compact manifold $M$ with resolution $X_*$ and resolved quotient $Y_*$ as discussed above. The isotropy groups $B_\alpha \subset G$ form a tree with root $B_0$ the principal isotropy group. Generalizing the choice (4.4) we can choose iterative splittings by proceeding stepwise along chains

$$(\tau_{\beta,\alpha}: \hat{B}_\alpha \longrightarrow \hat{B}_\beta \ \forall \ \beta > \alpha, \ \tau_{\gamma,\beta} \circ \tau_{\beta,\alpha} = \tau_{\gamma,\alpha}, \ \gamma > \beta > \alpha).$$

Using notation as for the fibration maps we set $\tau_\alpha = \tau_{\alpha,0}$. Then the formulæ (4.6) and (4.7) are valid for any pair and are consistent along chains.

\textbf{Definition 4.} Reduced bundles $W_*$ in the case of an abelian action, consist of the following data

1. A bundle $W_\alpha \in \mathfrak{Bun}_c^{B_\alpha}(\hat{G} \times Y_*)$ for each element of the tree.
2. For each non-principal isotropy type $\alpha > 0$ (so $B_\alpha \supset B_0$) a bundle isomorphism

$$(\tau_{\alpha}: \pi^\#_\alpha W_\alpha \simeq W_0|_{H_\alpha(Y_0)})$$

(4.9)

3. The consistency conditions that for any chain $\alpha_*, \ \alpha_k > \cdots > \alpha_1 > 0$ the isomorphisms (4.9) restricted to the boundary face, of codimension $k$,

$$H_{\alpha_*}(Y_0) = \bigcap_{j} H_{\alpha_j}(Y_0)$$

form a chain, corresponding to isomorphisms for each $\alpha < \beta$

$$T_{\alpha,\beta}: \pi^\#_{\alpha\beta} W_\beta \simeq W_\alpha|_{H_\beta(Y_\alpha)}.$$  

(4.10)

Morphisms between such data consist of bundle maps at each level of the tree intertwining the isomorphisms $T_\alpha$ in (4.9).

We denote by $\mathfrak{Bun}_c^{B_*}(\hat{G} \times Y_*)$ the category of such reduced bundles and the corresponding Grothendieck group of pairs of reduced bundles up to stable isomorphism by

$$(4.11) \quad K_{\text{red}}(Y_*) = \mathfrak{Bun}_c^{B_*}(\hat{G} \times Y_*) \oplus \mathfrak{Bun}_c^{B_*}(\hat{G} \times Y_*)/ \simeq.$$
Theorem 6. The equivariant K-theory for the action of a compact abelian group on a compact manifold $M$ is naturally identified with the reduced K-theory of the resolved quotient.

Proof. This follows from the equivalence of the categories of $G$-equivariant iterated bundles over $X_\ast$ and reduced bundles over $Y_\ast$ which in turn follows from Propositions 4 and 5. □

Definition 5. An iterated connection $\nabla_\ast$ on a reduced bundle $W_\ast \in \text{Bun}^B_c(\hat{G} \times Y_\ast)$ is a connection $\nabla_{\hat{g},\alpha}$ on each bundle $W_{\hat{g},\alpha} \in \text{Bun}(Y_\alpha)$ satisfying
\[
\nabla_{\hat{g}} \otimes \nabla_{\hat{g},\alpha} = \nabla_{\hat{g},\alpha}, \quad \forall \alpha \in A, \quad \hat{g} \in \hat{G}
\]
under the transformation law and compatible under augmented pull-back isomorphisms.

Lemma 7. Any reduced bundle can be equipped with an iterated connection in the sense of Definition 5.

Proof. Such a connection can be obtained following the reduction procedure from a $G$-connection on the corresponding iterated $G$-bundle over $X_\ast$. It is also straightforward to construct such a connection directly. □

The odd version of reduced bundles may be defined by ‘suspension’ – simply taking the product with an interval and demanding that all bundles be trivialized over the end points leading to a category
\[
\text{Bun}^B_c(\hat{G} \times ([0,1] \times Y_\ast); \{0\} \cup \{1\} \times Y_\ast).
\]
This leads to the odd version of equivariant K-theory
\[
K^1_G(M) = K^1_{\text{red}}(Y_\ast) = \text{Bun}^B_c(\hat{G} \times ([0,1] \times Y_\ast); \{0\} \cup \{1\} \times Y_\ast) \otimes \text{Bun}^B_c(\hat{G} \times Y_\ast; \{0\} \cup \{1\} \times Y_\ast)/ \simeq .
\]
The isotropy tree can also be ‘pruned’ by choosing any subtree
\[
A' \subset A, \quad \alpha \in A', \quad \beta \in A, \quad \beta < \alpha \implies \beta \in A'.
\]
If $P = A \setminus A'$ is the complement of a tree then reduced bundles which are trivialized on the elements of $P$ form a subcategory
\[
\text{Bun}^B_c(\hat{G} \times Y_\ast; P).
\]
These correspond to $G$ bundles over $M$ which are trivialized over the corresponding isotropy types. We denote by $K^*_{\text{red}}(Y_\ast; P)$ the Grothendieck groups of these relative spaces of bundles and their suspended versions.

5. Delocalized equivariant cohomology

If $\rho \in \hat{G}$ is an irreducible representation of a compact abelian group on a complex line, $E$, then, the corresponding trivial line bundle over a $G$-space, $X$, is $G$-equivariant,
\[
E \in \text{Bun}_G(X).
\]
The deRham differential defines a $G$-equivariant connection on $E$. If the action of $G$ is free, so $X \to Y$ is a principal $G$-bundle, then $E$ descends to a bundle, $\tilde{E}$, with connection. The Chern character therefore defines a multiplicative map

$$\text{Ch} : \hat{G} \to \mathcal{C}^\infty(Y; \Lambda^{2*}).$$

Let $R(G)$ be the representation algebra with complex coefficients, so the vector space of formal finite linear combinations of elements of $\hat{G}$. Then the map (5.2) extends to a map of algebras

$$\text{Ch} : R(G) \to \mathcal{C}^\infty(Y; \Lambda^{2*}) \otimes \text{Ch} R(G).$$

This and (5.2), for $G/B$ lead to:

**Definition 6.** For a compact abelian group $G$ acting with fixed isotropy group $B$ on a compact manifold $X$, the space of twisted forms over the base $Y$ is defined as

$$\mathcal{C}^\infty(Y; \Lambda^*_{\text{ad}}) = \mathcal{C}^\infty(Y; \Lambda^*) \otimes \text{Ch} R(G).$$

Thus an element of this space is a finite linear combination of formal products $u_i \otimes \hat{g}_i$, $u_i \in \mathcal{C}^\infty(Y; \Lambda^*)$, $\hat{g}_i \in \hat{G}$ under the equivalence relation

$$u \otimes \hat{g} \simeq \text{Ch}(\hat{h}) \wedge u \otimes \hat{h} \hat{g}, \forall \hat{h} \in G/B, u \in \mathcal{C}^\infty(Y; \Lambda^*).$$

Since the Chern character is closed, the deRham differential descends

$$d : \mathcal{C}^\infty(Y; \Lambda^*_{\text{ad}}) \to \mathcal{C}^\infty(Y; \Lambda^*_{\text{ad}}), \ d^2 = 0.$$

**Lemma 8.** Suppose that $\pi_1 : X \to X_1$ is a $G$-equivariant fibration for actions with fixed isotropy groups $K \subset B$ and $\tilde{\pi}_1 : Y \to Y_1$ is the induced fibration, then there is a natural augmented pull-back

$$\pi_1^\# : \mathcal{C}^\infty(Y_1; \Lambda^*_{\text{ad}}) \to \mathcal{C}^\infty(Y; \Lambda^*_{\text{ad}})$$

which intertwines the action of $d$.

**Proof.** To define (5.8) it suffices to consider three elementary cases.

First suppose that $\pi$ is simply an isomorphism of principle bundles covering the identity map $\tilde{\pi}_1$. The only appearance of the bundle in (5.7), (5.6) is through the Chern character and this is invariant under such a transformation.

Secondly suppose that $K = B$ but that $\pi_1$ is a $G$-equivariant fibration. Then, after a bundle isomorphism, this corresponds to $X_1$ being the pull-back of the principal $G/B$ bundle over $Y_1$ under a fibration $\tilde{\pi}_1$. The bundle $E$ corresponding to representations of $G/B$ and their connections pull back naturally and in this case (5.8) corresponds to the pull-back of the coefficient forms.

Finally then consider the case that $X$ is a principal $G/K$ bundle and that $K \subset B \subset G$ is a second closed subgroup with

$$\pi_1 : X \to X_1 = X/B,$$

so $Y = Y_1$. The equivalence relation (5.4), now for $\hat{i} \in \hat{G}/B$ means that any element of $\mathcal{C}^\infty(X_1; \Lambda^*_{\text{ad}})$ can be represented by a finite sum

$$u_i \otimes \hat{g}_i.$$
where the $\hat{g}_i \in \hat{G}$ exhaust $\hat{B}$ under restriction. These can be chosen, and relabelled, to be $\hat{f}_{kj}\hat{g}_j$ where the $\hat{g}_j \in \hat{G}$ restrict to exhaust $\hat{K}$ and $\hat{f}_{kj} \in \hat{G}/B$. Then

\begin{equation}
\pi_1^\# : \sum_{\text{finite}} u_{jk} \otimes \hat{f}_{kj}\hat{g}_j = \sum_{\text{finite}} (\sum_k \text{Ch}(f_{kj})^{-1} \wedge u_{jk}) \otimes \hat{g}_j.
\end{equation}

For elements of $\hat{G}/B$ the construction of the Chern character factors through the projection to $X_1$.

The general case corresponds to a composite of these three cases. \hfill \square

Our model for the delocalized equivariant cohomology of Baum, Brylinski and MacPherson in the case of a smooth action of a compact abelian Lie group $G$ on a compact manifold $M$ is the following data on the resolved quotient.

**Definition 7.** An element of the delocalized deRham complex $C^\infty(Y_\ast; \Lambda^*_dl)$ consists of:

1. For each $\alpha \in A$ a twisted smooth form $u_\alpha \in C^\infty(Y_\alpha; \Lambda^*_dl)$.
2. Compatibility conditions at all boundary faces

\begin{equation}
\left.u_\alpha\right|_{H_{\beta}(Y_\alpha)} = \pi_1^\# u_\beta, \; \beta > \alpha.
\end{equation}

including the boundary hypersurfaces of the principal quotient corresponding to $\alpha = 0$.

Again the relative versions corresponding to a subtree $A' \subset A$, $A = A' \sqcup P$ are similarly defined by demanding that the forms vanish over the boundary hypersurfaces indexed by $P$.

If $\nabla_\ast$ is an iterated connection on an iterated bundle $W_\ast \in \text{Bun}^B_\ast (\hat{G} \times Y_\ast)$, as in Definition 5 and Lemma 7 then the Chern character of each bundle $W_\alpha$ is a form on $\hat{G} \times Y_\alpha$:

\begin{equation}
\text{Ch}(W_\alpha, \nabla_\ast)_{\hat{g}} = \text{Ch}((W_\alpha)_{\hat{g}}, \nabla_\ast) \text{ on } \{\hat{g}\} \times Y_\alpha.
\end{equation}

**Proposition 9.** The Chern character of a reduced bundle with compatible connection is an element of $C^\infty(Y_\ast; \Lambda^*_dl)$.

**Proof.** The forms (5.13) shift correctly under the action of $\hat{G}/B$ in view of the corresponding property for the connections and the iterative relations over the boundary fibrations similarly follow from the standard properties of the Chern character under pull-back. \hfill \square

The verification of the ‘Atiyah-Hirzebruch-Baum-Brylinski-MacPherson’ isomorphism (7.1) is given in §7 below. The iterative proof uses the six-term exact sequences arising from pruning the isotropy tree at successive levels. As with the whole approach here, this is based on reduction to the case of a fixed isotropy group where the result reduces in essence to the Atiyah-Hirzebruch isomorphism.

**Proposition 10.** If $G$ is a compact abelian group acting on a compact manifold with fixed isotropy group $B$ then the Chern character gives an isomorphism of $K_G(M) \otimes \mathbb{C}$ and $H^\text{even}_{dl}(Y)$.

**Proof.** The Atiyah-Hirzebruch isomorphism is valid rationally. This amounts to the two statements that for a compact manifold (with corners) the range of the Chern character

\begin{equation}
\text{Ch} : K(Y) \longrightarrow H^\text{even}(Y)
\end{equation}
spans the cohomology (with complex coefficients) and that the null space consists of torsion elements. At the bundle level this means that if the Chern character for a pair of bundles $V_+ \oplus V_-$ is exact then there for some integers $p$ and $N$

$$I : V^p_+ \oplus \mathbb{C}^N \rightarrow V^p_+ \oplus \mathbb{C}^N.$$  

A given connection on the $V_\pm$ lifts to a connection which can then be deformed to commute with $I$ and so have zero Chern character.

Now, in the equivariant case we can consider a splitting homomorphism $\tau : \hat{B}^{-} \rightarrow \hat{G}$ and then pull a pair of bundles $V_\pm \in \mathfrak{Bun}^B_{\text{red}}(\hat{G} \times Y)$ back to $\hat{B} \times Y$ where the Chern character is given by

$$\sum_{b \in \hat{B}} \tau(\hat{g}) \otimes (\text{Ch}(V_{+,\tau(\hat{b})}) - \text{Ch}(V_{-,\tau(\hat{b})}).$$

The vanishing of the class $H_{\text{even}}(Y; \Lambda_{\text{dl}})$ is equivalent to the exactness of each of the deRham classes $\text{Ch}(V_{+,\tau(\hat{b})}) - \text{Ch}(V_{-,\tau(\hat{b})})$. Thus the vanishing of the Chern character in delocalized cohomology implies that each of the pairs $V_{\pm,\tau(\hat{b})}$ is stably trivial in the sense of (5.15).

Since $\hat{B}$ is finite and we may always further stabilize (5.15) by taking powers and adding trivial bundles, we may take $p$ to be the product of the integers for each $\hat{b}$ and similarly increase $N$. This however amounts to a stable trivialization of the whole bundle $V^p_+ \oplus V^p_-$ as an element $\mathfrak{Bun}^B_{\text{red}}(\hat{G} \times Y)$ and proves the injectivity of (7.1) in this case.

The surjectivity is a direct consequence of Atiyah-Hirzebruch isomorphism and the definition of delocalized forms. □

6. The relative sequences

Our proof of Theorem 13 is based on induction over pruning and the six-term exact sequences which results from passing from one subtree to another with one more element

$$A'' = A' \cup \{\alpha\}, \quad \alpha \notin A', \quad \beta < \alpha \implies \beta \in A'.$$

Reduced bundles ‘supported’ on subtrees and the corresponding K-groups are discussed above.

**Proposition 11.** For any subtrees $A' = A \setminus P$ and $A' \setminus \{\alpha\}$ there is a six term exact sequence

$$K^0_{\text{red}}(Y_\alpha; P \cup \{\alpha\}) \xrightarrow{\quad} K^0_{\text{red}}(Y_*; P) \xrightarrow{\quad} K^0_{\text{red}}(Y_\alpha; P) \xleftarrow{\quad} K^1_{\text{red}}(Y_\alpha; P) \xrightarrow{\quad} K^1_{\text{red}}(Y_*; P) \xrightarrow{\quad} K^1_{\text{red}}(Y_\alpha; P \cup \{\alpha\}).$$

Proof. The upper left arrow is given by inclusion and the upper right arrow given by restriction of the reduced bundle data to be non-trivial only on $Y_\alpha$. The arrows in the bottom row are defined accordingly. Exactness in the middle of the top and the bottom row are immediate from the definitions.

To define the connecting homomorphisms on the left, consider an element in $K^1_{\text{red}}(Y_\alpha; P)$. Choosing splittings as in (4.8) this can be represented by a pair of bundles over $\hat{B}_\alpha \times Y_\alpha \times [0,1]$ with identifications at all boundary hypersurfaces.
of $Y_\alpha$, since these correspond to deeper strata, and at the ends of the interval. Using the augmented pull-back map, this element lifts to a pair of bundles over $\tilde{B} \times H_\alpha \times [0,1]$. Now, we can identify $\tilde{B} \times H_\alpha \times [0,1]$ with a collar neighborhood of $\tilde{B} \times H_\alpha$ in $\tilde{B} \times Y$. Since the bundles are trivial over the ends of the interval, this defines an element of $K^0_{\text{red}}(Y; P \cup \{\alpha\})$ which is independent of choices so defines a homomorphism. For the connecting homomorphism on the right the construction is the same after tensoring with the Bott bundle on $[0,1]^2$ and using one variable as the normal to the boundary and the other as the suspension variable.

To check exactness at the top left corner, suppose an element of $K^0_{\text{red}}(Y; P)$ maps to the trivial element of $K^0_{\text{red}}(Y; P \cup \{\alpha\})$ under inclusion of reduced bundle data. Then, for a stabilized representative $V_\alpha$, inside $Y_\alpha$ there is a homotopy of the trivial bundle to itself (respecting triviality of the bundle over deeper strata) that lifts to a homotopy from $V_\alpha$ to the reduced bundle data corresponding to the trivial element. Such a homotopy induces a pair of bundles over $\tilde{B}_\alpha \times Y_\alpha \times [0,1]$ trivial at the endpoints and trivial at all strata deeper than $Y_\alpha$ and hence an element of $K^1_{\text{red}}(Y_\alpha; P)$. A similar argument shows that any element in the kernel of the upper left arrow is of the form discussed in the construction of the connecting homomorphism.

Finally we prove exactness at the bottom left corner. Suppose that an element in $K^1_{\text{red}}(Y_\alpha; P)$ inserted into the neck near $\tilde{B} \times H_\alpha$ is homotopic to the trivial element, the homotopy preserving the appropriate trivializations corresponding to greater depth. This is equivalent to the existence of a bundle $V_{[0,1]}$ over $\tilde{B} \times Y \times [0,1]$ trivial at $\tilde{B} \times Y \times \{1\}$ and equal to the given element at $\tilde{B} \times Y \times \{0\}$. Denote by $V_t$ the bundle above $\tilde{B} \times Y \times \{t\}$; we need to show is that this data allows one to extend the lift of the given bundle from $\tilde{B} \times H_\alpha \times [0,1]$ to $b \times Y \times [0,1]$ such that the bundle data is trivial everywhere (after a homotopy) except at $\tilde{B}_\alpha \times Y_\alpha \times [0,1]$. Fixing a collar neighborhood $\tilde{B} \times H_\alpha \times [0,1] \times [0,1] \subset \tilde{B} \times Y \times [0,1]$ and extending the lifted bundle over $\tilde{B} \times Y \times \{t\}$ by embedding it into the bundle $V_0 |_{\tilde{B} \times H_\alpha \times [t,1]}$. This can be seen to be a bundle over $\tilde{B} \times Y \times [0,1]$ extending the lift and be nullhomotopic due to the existence of the nullhomotopy in the beginning. The only issue is that the bundle is not trivial over $\tilde{B} \times Y \times \{0\}$. To fix this, we use the above nullhomotopy to shift the bundle into the required one. Namely, over $\tilde{B} \times Y \times \{t\}$ shift the bundle to $V_1 - t |_{\tilde{B} \times H_\alpha \times [t,1]}$. This is still a bundle that now has the appropriate triviality conditions and it is still a lift because the above homotopy does not affect $\tilde{B} \times H_\alpha$.

\begin{proposition}
For any subtrees $A' = A \setminus P$ and $A' \setminus \{\alpha\}$ there is a six term exact sequence
\begin{equation}
\begin{align*}
H^0_{\text{dl}}(Y_*; P \cup \{\alpha\}) &\longrightarrow H^0_{\text{dl}}(Y_*; P) \longrightarrow H^0_{\text{dl}}(Y_\alpha; P) \\
&\quad \downarrow \quad \quad \downarrow \quad \quad \downarrow \quad \quad \downarrow \quad \quad \downarrow \quad \quad \downarrow \\
H^1_{\text{dl}}(Y_\alpha; P) &\longrightarrow H^1_{\text{dl}}(Y_*; P) \longrightarrow H^1_{\text{dl}}(Y_*; P \cup \{\alpha\}).
\end{align*}
\end{equation}
\end{proposition}

\begin{proof}
This can be proved by combining standard arguments for the long exact sequence for the cohomology of a manifold relative to its boundary with the arguments as in the case of K-theory above.
\end{proof}
7. The isomorphism

**Theorem 13** (See [2]). For the action of a compact abelian Lie group on a compact manifold the Chern character defines an isomorphism

\[ K^*_G(X) \otimes \mathbb{C} \to H^*_{G,\text{dil}}(X). \]

**Proof.** For any subtrees \( A' = A \setminus P \) and \( A' \setminus P' \), \( P' = P \cup \{ \alpha \} \) the exact sequences \((6.2)\) and \((6.3)\) combine to form a commutative diagramme

\[
\begin{array}{cccc}
K^0_{\text{red}}(Y_\alpha; P') & \xrightarrow{\text{Ch}_0} & K^0_{\text{red}}(Y_\alpha; P) & \xrightarrow{\text{Ch}_0} & K^0_{\text{red}}(Y_\alpha; P') \\
H^0_{\text{dil}}(Y_\alpha; P') & \xrightarrow{\text{Ch}_1} & H^0_{\text{dil}}(Y_\alpha; P) & \xrightarrow{\text{Ch}_1} & H^0_{\text{dil}}(Y_\alpha; P') \\
H^1_{\text{dil}}(Y_\alpha; P) & \xrightarrow{\text{Ch}_2} & H^1_{\text{dil}}(Y_\alpha; P) & \xrightarrow{\text{Ch}_2} & H^1_{\text{dil}}(Y_\alpha; P') \\
K^1_{\text{red}}(Y_\alpha; P) & \xrightarrow{\text{Ch}_3} & K^1_{\text{red}}(Y_\alpha; P) & \xrightarrow{\text{Ch}_3} & K^1_{\text{red}}(Y_\alpha; P').
\end{array}
\]

Tensoring the K-theory part with \( \mathbb{C} \) therefore also gives a commutative diagramme with both six-term sequences exact. Now we may proceed by induction using the fives lemma repeatedly

\[ A_0 = \{ 0 \} \subset A_1 \cdots \subset A_N = A \]

where the initial inductive step is given by Proposition 10 for \( A_0 \) and the central column is always exact.

\[ \square \]

8. Examples

The examples considered here are covered by Proposition 3.19 in Segal’s paper [6]. We illustrate here how the same conclusions can be reached by resolution.

Consider the standard circle action on the 2-sphere given by rotation around an axis. The two poles are fixed points and on the complement the action is free. Radial blow up of the two poles replaces the 2-sphere by a compact cylinder \( I \times S \) with free circle action and quotient, an interval, \( I \). Thus, from Theorem 13, equivariant bundles up to isomorphism are in 1-1 correspondence with ‘reduced bundles’ consisting of a bundle over \( I \) with isomorphisms with reduced bundles over \( Y_\alpha = \{ I, \{ N \}, \{ S \} \} \). Over the end-points these are simply (finitely-supported) bundles over \( \mathbb{Z} = U(1) \), i.e. a finite collection of vector spaces. Over the principal space \( I \) we simply have a vector bundle. The pull-back map is trivial and the augmented pull-back map reduces to summation. Thus reduced bundles in this case amount to a bundle over \( I \) with decompositions into subspaces over the end-points.
These decompositions are unrelated up to isomorphism, except that dimensions must sum to the dimension of the bundle over $I$. Thus the equivariant K-theory in this case is

\[(8.1) \quad K^{1}_{U(1)}(\mathbb{S}^{2}) = \mathcal{R}(U(1)) \oplus \mathcal{R}(U(1))/ \simeq \]

where the relation is given by equality of the images of the dimension maps. The action of $U(1)$ is the diagonal action on the representation rings.

The odd equivariant K-theory is given by the even equivariant K-theory of $\mathbb{S}^{2} \times (0,1)$. The triviality of these bundles over the corners $\{0\} \times \partial I$ and $\{1\} \times \partial I$ implies there triviality over the end-points so the odd equivariant K-theory corresponds to pairs of bundles supported on $(0,1) \times (I \setminus \partial I)$. Thus

\[K^{1}_{U(1)}(\mathbb{S}^{2}) = K^{0}((-1,1) \times (0,1)) = K^{0}((\mathbb{R}^{2}) = \mathbb{Z}.\]

If $J$ is a generator of the odd K-theory and $H$ is the Hopf bundle then, as an algebra, the total K-theory is generated by $H \oplus I = J$. Since $H^{2} = 2H - J$ this recovers the result of Segal alluded to above.

**Lemma 14.** For a product group action by compact abelian groups $A \subset G$ where $A$ acts trivially the equivariant K-theory is

\[(8.2) \quad K_{A \times G}(X) = \mathcal{R}(A) \otimes K_{G}(X).\]

**Proof.** This follows immediately from the decomposition of bundles under the action of $A$ and the naturality of the lift of representations for a product. \hfill \square

An immediate corollary of this Lemma in combination and the calculation above shows it follows that for the rotation around on the sphere around an axis with $n$ times the usual speed the equivariant K-theory is

\[(8.3) \quad \mathcal{R}(\mathbb{Z}^{n}) \otimes (\mathcal{I}(U(1)) \oplus \mathcal{I}(U(1))/ \simeq ).\]

Consider next 2-dimensional complex projective space $\mathbb{P}^{2}$ with the circle action

\[(8.4) \quad U(1) \ni e^{i\theta} : \mathbb{P}^{2} \ni [z_{1} : z_{2} : z_{3}] \longrightarrow [e^{i\theta}z_{1} : z_{2} : e^{-i\theta}z_{3}] \in \mathbb{P}^{2}.\]

This is principally free and has three fixed points; at $[0 : 0 : 1]$, $[0 : 1 : 0]$ and $[1 : 0 : 0]$. On complement of the fixed points the isotropy group is $\mathbb{Z}_{2} = \{1,-1\} \subset U(1)$ on the the sphere $\mathbb{P} = \{[z_{1} : 0 : z_{3}]\}.$

This sphere is precisely the complement of the affine space given by $z_{2} \neq 0$, i.e. $\{[z_{1} : 1 : z_{3}]\}$ around the isolated fixed point at $[0 : 1 : 0]$. If we blow this point up in the complex sense we replace $\mathbb{C}^{2}$ by the canonical bundle over projective space

\[(8.5) \quad [\mathbb{C}^{2} ; \{0\}]_{\mathbb{C}} = KP.\]

The complement of the zero section is covered by the two coordinate patches

\[(8.6) \quad U'_{1} = \{(z_{1}, z_{3}) ; |z_{1}| > \frac{1}{2}|z_{3}| > 0\}, \quad U'_{3} = \{(z_{1}, z_{3}) ; |z_{3}| > \frac{1}{2}|z_{1}| > 0\};\]

These project to dense subsets of the two regions of $\mathbb{P}^{2}$:

\[(8.7) \quad U_{1} = \{[1 : \zeta_{2} : \zeta_{3}] , \zeta_{2} = \frac{1}{z_{1}}, \zeta_{3} = \frac{z_{3}}{z_{1}}, |\zeta_{3}| < \frac{1}{2}\}, \quad U_{3} = \{[\eta_{1} : \eta_{2} : 1]), \eta_{2} = \frac{1}{z_{3}}, \eta_{1} = \frac{z_{1}}{z_{3}}, |\eta_{1}| < \frac{1}{2}\};\]
which together cover \( \mathbb{P}^2 \setminus \{[0 : 1 : 0]\} \). The real blow-up of \( \mathbb{P} \), respectively \( \{\zeta_2 = 0\} \subset U_1 \) and \( \{\eta_2 = 0\} \subset U_3 \) replaces them by

\[
\begin{align*}
\widetilde{U}_1 &= \{0, \infty\}_R \times \mathbb{S} \times \{\|\zeta_3\| < 2\} \supseteq \{r, e^{i\delta}, \zeta_3\} \mapsto [1 : re^{i\delta}; \zeta_3] \in U_1 \\
\widetilde{U}_3 &= [0, \infty)_R \times \mathbb{S} \times \{\|\zeta_3\| < 2\} \ni (R, e^{i\gamma}, \zeta_1) \mapsto \{[\zeta_1 : R e^{i\gamma}; 1]\} \in U_3.
\end{align*}
\]

The intersection of \( U_1 \) and \( U_3 \) corresponds to the two regions and transition map

\[
\begin{align*}
\widetilde{U}_1 &\ni \{(r, e^{i\delta}, \zeta_3) \in [0, \infty)_R \times \mathbb{S} \times \mathbb{C}; 1/2 < \|\zeta_3\| < 2\} \ni (r, e^{i\delta}, \zeta_3) \mapsto \\
\left(\frac{r}{|\zeta_3|}, e^{i\frac{\delta}{|\zeta_3|}}, 1/\zeta_1\right) \in \widetilde{U}_1 &\ni \{(R, e^{i\gamma}, \eta_1) \in [0, \infty)_R \times \mathbb{S} \times \mathbb{C}; 1/2 < |\eta_1| < 2\}.
\end{align*}
\]

each \( \widetilde{U}_i \) is the product of a boundary variable, a circle and an open disk with the transition map patching this to the product of a boundary variable and a circle bundle over the sphere.

The circle actions are therefore

\[
\begin{align*}
\widetilde{U}_1 &\ni \{r, e^{i\delta}, \zeta_3\} \mapsto (r, e^{i(\delta - \theta)}, e^{-2i\theta} \zeta_3) \\
\widetilde{U}_3 &\ni (R, e^{i\gamma}, \eta_1) \mapsto (R, e^{i(\gamma + \theta)}, e^{2i\theta} \eta_1).
\end{align*}
\]

The remaining fixed points at \([1; 0; 0]\) \( \in U_1 \) and \([0; 0; 1]\) \( \in U_2 \) lift under the blow-up of \( \mathbb{P} \) to the circles

\[
\{(0, e^{i\delta}, 0)\} \subset \widetilde{U}_1, \{(0, e^{i\gamma}, 0)\} \subset \widetilde{U}_3
\]
on which the circle actions are now free. Nevertheless, our prescription calls for these to be blown up.

Thus identifies

\[
[\mathbb{P}^2; \{[0 : 1 : 0]\}]_C = (\mathbb{P}K)\mathbb{P}
\]
as projective compactification of the canonical bundle of \( \mathbb{P} \) where the exceptional divisor corresponding to the blow-up of \([0 : 1 : 0]\) is the zero section and the section at infinity is the projective line at \( \{\zeta_2 = 0\} \).

The real blow-up of these two projective lines in (8.5) shows that

\[
[\mathbb{P}^2; \{[0 : 1 : 0]\}], \mathbb{P}|_{\mathbb{R}} = [0, 1] \times S^1 \longrightarrow \mathbb{P}
\]
is the corresponding bundle of cylinders over \( \mathbb{P} \), so the product of a radial interval and the Hopf bundle \( S^1 \longrightarrow \mathbb{P} \).

The circle action on \( \mathbb{P}^2 \) restricted to the affine \( \{\zeta_2 \neq 0\} \) in (8.13) is the action on the 3-sphere as

\[
(Z_1, Z_2) \longrightarrow (e^{i\theta} Z_1, e^{-i\theta} Z_3)
\]
which is the Hopf fibration after conjugating the second variable.

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