THE CLASSIFICATION OF THICK REPRESENTATIONS OF SIMPLE LIE GROUPS

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Abstract. We characterize finite-dimensional thick representations over $\mathbb{C}$ of connected complex semi-simple Lie groups by irreducible representations which are weight multiplicity-free and whose weight posets are totally ordered sets. Moreover, using this characterization, we give the classification of thick representations over $\mathbb{C}$ of connected complex simple Lie groups.

1. Introduction

In our previous paper [8], we have introduced $m$-thickness and thickness of group representations. Let $\rho : G \to \text{GL}(V)$ be a finite-dimensional representation of a group $G$. If for any subspaces $V_1$ and $V_2$ of $V$ with $\dim V_1 = m$ and $\dim V_2 = \dim V - m$ there exists $g \in G$ such that $(\rho(g)V_1) \oplus V_2 = V$, we say that a representation $\rho : G \to \text{GL}(V)$ is $m$-thick. We also say that a representation $\rho : G \to \text{GL}(V)$ is thick if $\rho$ is $m$-thick for each $0 < m < \dim V$ (Definition 2.1). In [8, Proposition 2.7], we proved that 1-thickness is equivalent to irreducibility (Proposition 2.8). Hence $m$-thickness is a natural generalization of irreducibility of group representations.

Let $G$ be a connected semi-simple Lie group over $\mathbb{C}$, $B$ a Borel subgroup of $G$, $T$ a maximal torus which is contained in $B$. Denote their Lie algebras by $\mathfrak{g}$, $\mathfrak{b}$ and $\mathfrak{t}$, respectively. Let $\rho : G \to \text{GL}(V)$ be a finite-dimensional irreducible representation of $G$ over $\mathbb{C}$. We denote the set of $\mathfrak{t}$-weights in $V$ by $W(V)$. Choosing a set of simple roots for $(\mathfrak{g}, \mathfrak{t})$, we can regard $W(V)$ as a partially ordered set (poset) with respect to the usual root order. This poset $W(V)$ is called the weight poset. A representation $\rho : G \to \text{GL}(V)$ is said to be weight multiplicity-free if the weight spaces in $V$ are all one-dimensional. We give the following characterization of thickness.

Theorem 1.1 (Theorem 3.5). An irreducible representation $\rho : G \to \text{GL}(V)$ of a connected semi-simple Lie group $G$ is thick if and only if it is weight multiplicity-free and its weight poset is a totally ordered set.

Using this characterization, we can classify the complex thick representations of connected semi-simple Lie groups.
Theorem 1.2 (Theorems 3.11 and 3.12). If a representation of a connected semisimple Lie group is thick, then it is geometrically equivalent to one of the following list:

\[ e, \ SL_n(n \geq 2), \ S^mSL_2(m \geq 2), \ SO_{2n}(n \geq 2), \ Sp_{2n}(n \geq 2), \ G_2. \]

Here the irreducible representation of a connected simple Lie group \( G \) of the highest weight \( \omega_1 \), where \( \omega_1 \) is the first fundamental weight, is denoted by \( G^{\omega_1} \). Similarly, \( S^mG \) stands for the \( m \)-th symmetric power of \( G \). Let \( e \) denote the trivial 1-dimensional representation for any group \( G \). For the definition of geometric equivalence, see Definition 3.8.

We denote by \( \omega_i \) the \( i \)-th fundamental weight for a connected simple Lie group \( G \). In §3, all Lie groups are assumed to be over \( \mathbb{C} \) and all representations are finite-dimensional over \( \mathbb{C} \).

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2. PRELIMINARIES

A representation of a group \( G \) on a vector space \( V \) is a homomorphism \( \rho : G \to GL(V) \). Then such a map \( \rho \) gives \( V \) the structure of a \( G \)-module. We sometimes call \( V \) itself a representation of \( G \) and write \( gv \) for \( \rho(g)(v) \). We recall several definitions and results in our previous paper [8].

Definition 2.1 ([8, Definition 2.1]). Let \( G \) be a group. Let \( V \) be a finite-dimensional vector space over a field \( k \). We say that a representation \( \rho : G \to GL(V) \) is \( m \)-thick if for any subspaces \( V_1 \) and \( V_2 \) of \( V \) with \( \dim V_1 = m \) and \( \dim V_2 = \dim V - m \), there exists \( g \in G \) such that \( (\rho(g)V_1) \oplus V_2 = V \). We also say that a representation \( \rho : G \to GL(V) \) is thick if \( \rho \) is \( m \)-thick for each \( 0 < m < \dim V \).

Definition 2.2 ([8, Definition 2.3]). Let \( G \) be a group. Let \( V \) be a finite-dimensional vector space over a field \( k \). We say that a representation \( \rho : G \to GL(V) \) is \( m \)-dense if the induced representation \( \wedge^m \rho : G \to GL(\wedge^m V) \) is irreducible. We also say that a representation \( \rho : G \to GL(V) \) is dense if \( \rho \) is \( m \)-dense for each \( 0 < m < \dim V \).

We show several examples. See [8] for details.

Example 2.3 (cf. [8, Proposition 6.5]). Let \( V \) be the standard representation of \( SL_n \) and \( V^* \) the dual representation of \( V \). Then \( V \) and \( V^* \) are dense.

Example 2.4 ([8, Proposition 6.10]). The standard representation of \( SO_{2n} \) is \( m \)-dense for each \( 0 < m < 2n \) with \( m \neq n \), but not \( n \)-thick.

Example 2.5 ([8, Proposition 6.11]). The standard representation of \( SO_{2n+1} \) is dense.
Example 2.6 ([8, Proposition 6.18]). The standard representation of $\text{Sp}_{2n}$ is thick, but not $m$-dense for each $1 < m < 2n - 1$.

Let $V$ be a finite-dimensional representation of a group $G$. For positive integers $i$ and $j$ with $i + j = \dim V$, let us consider the $G$-equivariant perfect pairing $\Lambda^i V \otimes \Lambda^j V \rightarrow \Lambda^{\dim V} V \cong k$. For a $G$-invariant subspace $W$ of $\Lambda^i V$, put $W^\perp := \{ y \in \Lambda^j V \mid x \wedge y = 0 \text{ for any } x \in W \}$. Then $W^\perp$ is also $G$-invariant. In particular, $\Lambda^i V$ is irreducible if and only if so is $\Lambda^j V$.

Proposition 2.7 ([8, Proposition 2.6]). Let $V$ be an $n$-dimensional representation of a group $G$. For each $0 < m < n$, $V$ is $m$-thick (resp. $m$-dense) if and only if $V$ is $(n - m)$-thick (resp. $(n - m)$-dense).

Proposition 2.8 ([8, Proposition 2.7]). For any finite-dimensional representation $V$ of a group $G$, the following implications hold for $0 < m < \dim V$:

\[
\begin{align*}
\text{m-dense} & \implies \text{m-thick} \\
\downarrow & \\
\text{1-dense} & \iff \text{1-thick} \iff \text{irreducible}.
\end{align*}
\]

Corollary 2.9 ([8, Corollary 2.8]). For any finite-dimensional representation of a group $G$, the following implications hold:

\[
\text{dense} \Rightarrow \text{thick} \Rightarrow \text{irreducible}.
\]

Corollary 2.10 ([8, Corollary 2.9]). For any representation $V$ of a group $G$ with $\dim V \leq 3$, the following implications hold:

\[
\text{dense} \iff \text{thick} \iff \text{irreducible}.
\]

Definition 2.11 ([8, Definition 2.10]). Let $V$ be an $n$-dimensional vector space over a field $k$. For a $d$-dimensional subspace $V'$ of $V$ with $0 < d < n$, we can consider a point $[\Lambda^d V']$ in the projective space $\mathbb{P}(\Lambda^d V)$. In the sequel, we identify $[\Lambda^d V']$ with a non-zero vector $\Lambda^d V' \in \Lambda^d V$ (which is determined by $[\Lambda^d V']$ up to scalar) for simplicity. For a vector subspace $W \subset \Lambda^d V$, we say that $W$ is realizable if $W$ contains a non-zero vector $\Lambda^d V'$ obtained by a $d$-dimensional subspace $V'$ of $V$.

We have the following criterion of thickness.

Proposition 2.12 ([8, Proposition 2.11]). Let $V$ be an $n$-dimensional representation of a group $G$. For $0 < m < n$, $V$ is not $m$-thick if and only if there exist $G$-invariant realizable subspaces $W_1 \subseteq \Lambda^m V$ and $W_2 \subseteq \Lambda^{n-m} V$ such that $W_1^\perp = W_2$.

3. The classification of thick representations of simple Lie groups

Let $G$ be a connected semi-simple Lie group over the complex number field $\mathbb{C}$, $B$ a Borel subgroup of $G$, $T$ a maximal torus which is contained in $B$, $B^-$ a Borel subgroup of $G$ opposite to $B$ relative to $T = B \cap B^-$. Denote their Lie algebras by
Proposition 3.2. If a representation $V$ is not WMF, then there exists a weight $\varphi \in W(V)$ such that the dimension of $V_{\varphi}$ is larger than one. Let $W^+(\varphi)$ be the set of all weights strictly larger than $\varphi$, and $Y^+(\varphi)$ the subspace of $V$ which is spanned by all weight spaces for weights in $W^+(\varphi)$. Because the dimension of $V_{\varphi}$ is larger than one, we can choose two linear independent $\varphi$-weight vectors $v$ and $w$. Let $W_{\varphi}^{a,b}(+) = \mathbb{C}(av + bw) \oplus Y^+(\varphi)$ for $a, b \in \mathbb{C}$. The subspace $W_{\varphi}^{a,b}(+)$ is $B$-invariant. Let $n$ be the dimension of $V$, and $d$ the dimension of $W_{\varphi}^{a,b}(+)$ for $(a, b) \in \mathbb{C}^2 \setminus \{(0, 0)\}$. The elements $\bigwedge^d W_{\varphi}^{a,b}(+)$ for $(a, b) \in \mathbb{C}^2 \setminus \{(0, 0)\}$ are distinct $B$-eigenvectors in $\bigwedge^d V$ with the same weight. Let $U_{\varphi}^{a,b}(+)$ be the irreducible $G$-submodule in $\bigwedge^d V$ with the highest weight vector $\bigwedge^d W_{\varphi}^{a,b}(+)$. Let $U_{\varphi}(+)$ be the direct sum $U_{\varphi}^{1,0}(+) \oplus U_{\varphi}^{0,1}(+) \subset \bigwedge^d V$. Any irreducible $G$-submodule of $U_{\varphi}(+)$ is equal to $U_{\varphi}^{a,b}(+)$ for some $(a, b) \in \mathbb{C}^2 \setminus \{(0, 0)\}$. Hence any irreducible $G$-submodule of $U_{\varphi}(+)$ is realizable.

Let $Y^-(\varphi)$ be the subspace of $V$ which is spanned by all weight spaces for weights in $W(V) \setminus \{W^+(\varphi), \varphi\}$. We take a basis $\{v, w, u_1, \ldots, u_s\}$ for $V_{\varphi}$ which contains $v, w$. Let $W_{\varphi}(-)$ be the subspace of $V$ which is spanned by $\{w, u_1, \ldots, u_s\}$ and $Y^-(\varphi)$. The subspace $W_{\varphi}(-)$ is invariant under the action of the opposite Borel subgroup $B^-$. The equalities $\dim W_{\varphi}(-) = \dim V - \dim W_{\varphi}^{a,b}(+) = n - d$ hold for $(a, b) \in \mathbb{C}^2 \setminus \{(0, 0)\}$. Then $\bigwedge^{n-d} W_{\varphi}(-)$ is a $B^-$-eigenvector in $\bigwedge^{n-d} V$. Let $U_{\varphi}(-)$ be the irreducible $G$-submodule with the lowest weight vector $\bigwedge^{n-d} W_{\varphi}(-)$ in $\bigwedge^{n-d} V$. Obviously, $U_{\varphi}(-)$ is realizable. The irreducibility of $U_{\varphi}(-)$ shows the irreducibility of $\bigwedge^d V/(U_{\varphi}(-))$. Then $(U_{\varphi}(-))^+ \cap U_{\varphi}(+) \neq \{0\}$ because $U_{\varphi}(+)$ is not irreducible. Hence $(U_{\varphi}(-))^+ \cap U_{\varphi}(+) \subset U_{\varphi}^{a,b}(+)$. Therefore $(U_{\varphi}(-))^+$ is realizable. Putting $W_1 = (U_{\varphi}(-))^+$ and $W_2 = U_{\varphi}(-)$, we see that $V$ is not thick by Proposition 2.12. Hence if $V$ is thick, then it is WMF.

Proposition 3.3. If a representation $V$ of $G$ is thick, its weight poset $W(V)$ is a totally ordered set.
Proof. Let $V$ be a thick representation of $G$. By Proposition 3.1, $V$ is WMF. For any weight $\phi \in W(V)$, let $W^+(\phi)$ be the set of all weights strictly larger than $\phi$, and $Y^+(\phi)$ the subspace of $V$ which is spanned by all weight spaces for weights in $W^+(\phi)$. Note that the irreducible representation $V$ has a highest weight $\omega$ and that each weight of $V$ has the form $\omega - \sum_{i=1}^{l} m_i \alpha_i \ (m_i \in \mathbb{N})$, where $\Pi = \{\alpha_1, \ldots, \alpha_l\}$.

Suppose that the weight poset $W(V)$ is not a totally ordered set. There exists an integer $d > 1$ such that $W(V)$ has the $(d - 1)$-st highest weight, but not the $d$-th highest weight. Let $\mathcal{O}$ be the $(d - 1)$-st highest weight, and $\psi_1, \psi_2$ maximal weights in $W(V) \setminus (W^+(\mathcal{O}) \cup \{\mathcal{O}\})$. Then the subset $W^+(\mathcal{O}) \cup \{\mathcal{O}\}$ is a totally ordered set, $\varphi$ covers $\psi_1, \psi_2$, and $W^+(\psi_1) = W^+(\psi_2) = W^+(\mathcal{O}) \cup \{\mathcal{O}\}$. Because $V$ is WMF, there exists a unique $\psi_1$-weight vector $v_i$ up to scalar for each $i = 1, 2$. Let $W_{\psi_1}(+) = \mathcal{O} \oplus Y^+(\psi_1)$. The subspaces $W_{\psi_1}(+)$ are $B$-invariant for each $i = 1, 2$. Let $n$ be the dimension of $V$. Note that $\dim W_{\psi_1}(+) = d$ for $i = 1, 2$. The elements $\Lambda^d W_{\psi_1}(+)$ and $\Lambda^d W_{\psi_2}(+)$ are distinct $B$-eigenvectors with distinct weights in $\Lambda^d V$. Let $U_{\psi_1}(+)$ be the irreducible $G$-submodule of $\Lambda^d V$ with the highest weight vector $\Lambda^d W_{\psi_1}(+)$ for each $i = 1, 2$. Then $U_{\psi_1}(+) \oplus U_{\psi_2}(+)$ are realizable and not isomorphic to each other as $G$-modules. Let $Y^-(\psi_1)$ be the subspace of $V$ which is spanned by all weight spaces for weights in $W(V) \setminus (W^+(\psi_1) \cup \{\psi_1\})$. The subspace $Y^-(\psi_1)$ is invariant under the action of the opposite Borel subgroup $B^-$. The equalities $\dim Y^-(\psi_1) = \dim V - \dim W_{\psi_1}(+) = n - d$ hold. Then $\Lambda^{n-d} Y^-(\psi_1)$ is a $B^-$-eigenvector in $\Lambda^{n-d} V$. Let $U_{\psi_1}(\psi_1)$ be the irreducible $G$-submodule of $\Lambda^{n-d} V$ with the lowest weight vector $\Lambda^{n-d} Y^-(\psi_1)$. Then $U_{\psi_1}(\psi_1)$ is realizable. The irreducibility of $U_{\psi_1}(\psi_1)$ shows the irreducibility of $\Lambda^d V/ (U_{\psi_1}(\psi_1))^\perp$. Then $(U_{\psi_1}(\psi_1))^\perp \cap (U_{\psi_1}(+) \oplus U_{\psi_2}(+)) \neq \{0\}$. Because $U_{\psi_1}(\psi_1)$ is not isomorphic to $U_{\psi_2}(+)$, $U_{\psi_1}(+) \subset (U_{\psi_1}(\psi_1))^\perp$ or $U_{\psi_2}(+) \subset (U_{\psi_1}(\psi_1))^\perp$. In particular, $(U_{\psi_1}(\psi_1))^\perp$ is realizable. Putting $W_1 = (U_{\psi_1}(\psi_1))^\perp$ and $W_2 = U_{\psi_1}(\psi_1)$, we see that $V$ is not thick by Proposition 2.12. This is a contradiction. Hence $W(V)$ is a totally ordered set. \qed

Let us denote the Grassmann variety which is the set of all $k$-dimensional subspaces of a vector space $V$ by $\mathbb{G}(k, V) \subset \mathbb{P}(\Lambda^k V)$.

Lemma 3.3. Let $V$ be a representation of $G$, and $W$ a $G$-invariant realizable subspace of $\Lambda^k V$. Then there exists $[v] \in \mathbb{P}(W) \cap \mathbb{G}(k, V)$ such that $[v]$ is $B^-$-invariant.

Proof. Let $X$ be $\mathbb{P}(W) \cap \mathbb{G}(k, V)$. Because $W$ is realizable, $X$ is not empty. Note that $X$ is $G$-invariant and compact. We take a $G$-orbit $O$ in $X$ whose dimension is minimal. The orbit $O$ is closed and then compact. There is a parabolic subgroup $P$ of $G$ such that the orbit $O$ is isomorphic to $G/P$. Then there is a point $[v] \in O \subset \mathbb{P}(W) \cap \mathbb{G}(k, V)$ such that $[v]$ is $B^-$-invariant. \qed

Lemma 3.4. Assume that an irreducible representation $V$ of $G$ is weight multiplicity-free, its weight poset $W(V)$ is a totally ordered set $\{\varphi_1 > \varphi_2 > \cdots > \varphi_n\}$, and
$W$ is a $G$-invariant realizable subspace of $\bigwedge^k V$. Let $v_i$ be a nonzero vector in the $\varphi_i$-weight space $V_{\varphi_i}$ ($i = 1, 2, \ldots, n$). Then $W$ contains $v_1 \wedge v_2 \wedge \cdots \wedge v_k$ and $v_{n-(k-1)} \wedge v_{n-(k-2)} \wedge \cdots \wedge v_n$.

**Proof.** Because $V$ is weight multiplicity-free, $\{v_1, \ldots, v_n\}$ is a basis of $V$. By Lemma 3.3, there exists $[v] \in \mathbb{P}(W) \cap \text{Grass}(k, V)$ such that $v$ is a highest weight vector of an irreducible subrepresentation of $W$ with respect to $B$. We can put

$$v = \left( p_{1,1}v_1 + p_{1,2}v_2 + \cdots + p_{1,n}v_n \right)$$

$$\wedge \left( p_{2,1}v_1 + p_{2,2}v_2 + \cdots + p_{2,n}v_n \right)$$

$$\vdots$$

$$\wedge \left( p_{k,1}v_1 + p_{k,2}v_2 + \cdots + p_{k,n}v_n \right)$$

up to scalar multiplication, where $P = (p_{i,j})$ is in reduced row echelon form. Remark that $P$ is uniquely determined. Let $X_\alpha$ be a root vector for a positive root $\alpha \in \Delta^+$. Then $X_\alpha v = 0$ holds for any $\alpha \in \Delta^+$. If $p_{1,1} = p_{1,2} = \cdots = p_{1,i} = 0$ and $p_{1,i+1} = 1$ for $i \geq 1$, there is a positive root $\alpha \in \Delta^+$ such that $X_\alpha v_{i+1} = cv_i$ for a nonzero constant $c$. Then $X_\alpha v$ is not 0. This is a contradiction. So $p_{1,1} = 1$. Similarly, we can show that $p_{22} = \cdots = p_{kk} = 1$. Because $v$ is a highest weight vector, for any $t \in t$ there is a constant $c$ such that $tv = cv$. Then by the uniqueness of $P$ we can show that $p_{ij} = 0$ for $i = 1, \ldots, k$ and $j = k+1, \ldots, n$. Then $v = v_1 \wedge v_2 \wedge \cdots \wedge v_k \in W$. A similar argument with respect to $B^-$ shows that $v_{n-(k-1)} \wedge v_{n-(k-2)} \wedge \cdots \wedge v_n \in W$. □

**Theorem 3.5.** An irreducible representation $V$ of a connected semi-simple Lie group $G$ is thick if and only if it is weight multiplicity-free and its weight poset is a totally ordered set.

**Proof.** The “only if” part can be proved by Propositions 3.1 and 3.2. Let us prove the “if” part. Let us use the notations in Lemma 3.4. Assume that $W_1 \subseteq \bigwedge^k V$ and $W_2 \subseteq \bigwedge^{n-k} V$ are $G$-invariant realizable subspaces. By Lemma 3.3, $v_1 \wedge v_2 \wedge \cdots \wedge v_k \in W_1$ and $v_{k+1} \wedge v_{k+2} \wedge \cdots \wedge v_n \in W_2$. Since $(v_1 \wedge v_2 \wedge \cdots \wedge v_k) \wedge (v_{k+1} \wedge v_{k+2} \wedge \cdots \wedge v_n) \neq 0$, $W_1^\perp \neq W_2$. By Proposition 2.12 $V$ is thick. □

By [4] Theorem 4.6.3, we have Howe’s classification of irreducible representations of connected simple Lie groups which are weight multiplicity-free. We also refer to Panyushev’s paper [9] Table 1] for the weight posets of weight multiplicity-free representations. Thus, we have

**Theorem 3.6.** The thick representations of connected simple Lie groups are those on the following list:

1. the trivial 1-dimensional representation for any groups
2. $A_n$ ($n \geq 1$)
   - the standard representation $V$ for $A_n$ ($n \geq 1$) with highest weight $\omega_1$
   - the dual representation $V^*$ of $V$ for $A_n$ ($n \geq 1$) with highest weight $\omega_n$
The dense representations of connected simple Lie groups are those

- the symmetric tensor $S^m(V)$ $(m \geq 2)$ of $V$ for $A_1$ with highest weight $m\omega_1$

(3) $B_n$ $(n \geq 2)$
- the standard representation $V$ for $B_n$ $(n \geq 2)$ with highest weight $\omega_1$
- the spin representation for $B_2$ with highest weight $\omega_2$

(4) $C_n$ $(n \geq 3)$
- the standard representation $V$ for $C_n$ $(n \geq 3)$ with highest weight $\omega_1$

(5) $G_2$
- the 7-dimensional representation $V$ for $G_2$ with highest weight $\omega_1$.

Proof. By Theorem 3.5, it suffices to list up all irreducible representations which are weight multiplicity-free and whose weight posets are totally ordered sets. Using [4, Theorem 4.6.3] and [9, Table 1], we can obtain the list of thick representations of connected simple Lie groups.

We also have the list of dense representations:

Theorem 3.7. The dense representations of connected simple Lie groups are those on the following list:

(1) the trivial 1-dimensional representation for any groups
(2) $A_n$ $(n \geq 1)$
- the standard representation $V$ for $A_n$ $(n \geq 1)$ with highest weight $\omega_1$
- the dual representation $V^*$ of $V$ for $A_n$ $(n \geq 1)$ with highest weight $\omega_n$
- the symmetric tensor $S^2(V)$ of $V$ for $A_1$ with highest weight $2\omega_1$

(3) $B_n$ $(n \geq 2)$
- the standard representation $V$ for $B_n$ $(n \geq 2)$ with highest weight $\omega_1$.

Proof. It suffices to verify whether thick representations in the list of Theorems 3.5 are dense or not. It is well-known that the standard representations $V$ of $A_n$ and $B_n$ are dense. We also see that the dual representation $V^*$ of $V$ for $A_n$ is dense. (For $A_n$, see Example 2.5 or [3, §15.2]. For $B_n$, see Example 2.5 or [3, Theorem 19.14].)

By Corollary 2.10, $S^2(V)$ for $A_1$ is dense since $\dim S^2(V) = 3$.

Conversely, let us show that $S^m(V)$ for $A_1$ is not dense if $m \geq 3$. Let $\{\varphi_1 > \varphi_2\}$ be the weight poset of the standard representation $V$ of $A_1$. The weight poset of $S^m(V)$ is $\{(m - k)\varphi_1 + k\varphi_2 \mid k = 0, 1, 2, \ldots, m\}$. Thereby, the weight poset of $\bigwedge^2 S^m(V)$ is $\{(2m - k_1 - k_2)\varphi_1 + (k_1 + k_2)\varphi_2 \mid 0 \leq k_1 < k_2 \leq m\}$. If $m \geq 3$, then $\dim \bigwedge^2 S^m(V)_{(2m-3)\varphi_1+3\varphi_2} = 2$ for the cases $(k_1, k_2) = (0, 3), (1, 2)$. Since $\bigwedge^2 S^m(V)$ is not weight multiplicity-free and any irreducible representations $S^m(V)$ of $A_1$ are weight multiplicity-free, $\bigwedge^2 S^m(V)$ is not irreducible. Hence $S^m(V) (m \geq 3)$ is not dense. It is well-known that the first fundamental representations of $C_n$ and $G_2$ are not dense. (For $C_n$, see Example 2.6 or [3, §17.2]. For $G_2$, see [3, §22.3].) The spin representation for $B_2$ with highest weight $\omega_2$ is not dense since it is equivalent to
the first fundamental representation for \( C_2 \) (for \( C_2 \), see Example 2.6 or [3] §16.2). Therefore, we obtain the list of dense representations.

According to [11] §6, [7] §5, and so on, we introduce the notion of geometric equivalence for simplifying the classification of thick representations.

**Definition 3.8** (cf. [11] §6, [7] §5). For two representations \( \rho : G \to GL(V) \) and \( \rho' : G' \to GL(V') \), we say that they are geometrically equivalent if there exists a \( \mathbb{C} \)-linear isomorphism \( f : V \to V' \) such that \( \rho'(G') = f \rho(G) f^{-1} \).

We prove the following proposition which was known in [7] §5.

**Proposition 3.9** ([7] §5). Let \( G \) be a connected semi-simple Lie group over \( \mathbb{C} \). Let \( \rho^* : G \to GL(V^*) \) be the dual representation of a finite-dimensional irreducible representation \( \rho : G \to GL(V) \) over \( \mathbb{C} \). Then \( \rho \) and \( \rho^* \) are geometrically equivalent.

**Proof.** Let \( \mathfrak{h} \) be a Cartan subalgebra of the Lie algebra \( \mathfrak{g} \) of \( G \). Fix a set of simple roots \( \Pi \) of the root system \( \Delta \). Let \( \lambda \) be the highest weight of \( V \) with respect to \( \Pi \) and \( w_0 \) the longest element of the Weyl group \( W \). Then \( -w_0(\lambda) \) is the highest weight of \( V^* \) (see [5] Exercises 10.9 and 21.6). Let \( \phi' : \mathfrak{h} \to \mathfrak{h} \) be the isomorphism whose dual \( \phi'^* : \mathfrak{h}^* \to \mathfrak{h}^* \) is given by \( \mu \mapsto -w_0(\mu) \). There exists a Lie algebra isomorphism \( \phi : \mathfrak{g} \to \mathfrak{g} \) extending \( \phi' \) (see [5] Theorem 18.4 (b)].) Take a universal cover \( \pi : \tilde{G} \to G \). The dual representation \( \tilde{\rho}^* \) of \( \tilde{\rho} = \rho \circ \pi : \tilde{G} \to GL(V) \) can be identified with \( \rho^* \circ \pi \).

By [2] Chapter III, §6, Theorem 1], there exists an automorphism \( \psi : \tilde{G} \to \tilde{G} \) such that \( d\psi = \phi \). Since \( \tilde{\rho} \circ \psi \) and \( \tilde{\rho}^* \) have the same highest weight \( -w_0(\lambda) \), there exists an isomorphism \( f : V \to V^* \) such that \( (\rho^* \circ \pi)(\tilde{g}) = f(\tilde{\rho}(\tilde{g})) = f(\tilde{\rho}(\psi)(\tilde{g})) = f^{-1} \) for any \( \tilde{g} \in \tilde{G} \). Hence \( \rho^*(G) = f((\rho \circ \psi)(\tilde{G})) = f(\rho(G) f^{-1} = f \rho(G) f^{-1} \). Therefore \( \rho \) and \( \rho^* \) are geometrically equivalent.

**Remark 3.10.** Assume that two representations \( \rho : G \to GL(V) \) and \( \rho' : G' \to GL(V') \) are geometrically equivalent. Then \( \rho \) is thick (resp. dense) if and only if so is \( \rho' \).

According to [6] §3.1], we denote the irreducible representation of a connected simple Lie group \( G \) with highest weight \( \omega_1 \) by \( G \). Similarly, \( S^mG \) stands for the \( m \)-th symmetric power of \( G \). In addition, let \( e \) denote the trivial 1-dimensional representation for any groups \( G \). Then we have:

**Theorem 3.11.** If a representation of a connected simple Lie group is thick, then it is geometrically equivalent to one of the following list:

\[ e, \ SL_n(n \geq 2), \ S^mSL_2(m \geq 2), \ SO_{2n+1}(n \geq 2), \ Sp_{2n}(n \geq 2), \ G_2. \]

If a representation of a connected simple Lie group is dense, then it is geometrically equivalent to one of the following list:

\[ e, \ SL_n(n \geq 2), \ S^2SL_2, \ SO_{2n+1}(n \geq 2). \]
Proof. The last fundamental representation of $B_2$ with highest weight $\omega_2$ is geometrically equivalent to the first fundamental representation of $C_3$ with highest weight $\omega_1$, that is, $\text{Sp}_4$. By Theorems 3.6 and 3.7 we have the classification above.  

Theorem 3.11 also shows the list of geometric equivalence classes of thick (or dense) representations of connected semi-simple Lie groups.

**Theorem 3.12.** Any thick representation $V$ of a connected semi-simple Lie group $G$ is geometrically equivalent to one of the list in Theorem 3.11. In particular, the list of geometric equivalence classes of thick representations (resp. dense representations) of connected semi-simple Lie groups is the same as that of thick representations (resp. dense representations) of connected simple Lie groups in Theorem 3.11.

**Proof.** Let $\rho : G \to \text{GL}(V)$ be a thick representation of a connected semi-simple Lie group $G$. Take a universal cover $\pi : \tilde{G} = G_1 \times G_2 \times \cdots \times G_r \to G$, where $G_i$ is a simply-connected simple Lie group for each $i = 1, 2, \ldots, r$. We have a thick representation $\tilde{\rho} = \rho \circ \pi : \tilde{G} \to \text{GL}(V)$. Since $V$ is an irreducible representation of $\tilde{G}$, there exist irreducible representations $V_i$ of $G_i$ $(1 \leq i \leq r)$ such that $V \cong V_1 \otimes V_2 \otimes \cdots \otimes V_r$ as representations of $\tilde{G}$. By Theorem 3.5 $V$ is WNF as a representation of $\tilde{G}$ and the weight poset $W_{\tilde{G}}(V)$ is a totally ordered set. Here, weights in $W_{\tilde{G}}(V)$ are with respect to a maximal torus $T = T_1 \times T_2 \times \cdots \times T_r$ of $\tilde{G}$, where $T_i$ is a maximal torus of $G_i$. The order in $W_{\tilde{G}}(V)$ is defined with respect to a set $\Pi = \Pi_1 \sqcup \Pi_2 \sqcup \cdots \sqcup \Pi_r$ of simple roots of $\tilde{G}$, where $\Pi_i$ is a set of simple roots of $G_i$. Let $W_{G_i}(V_i)$ be the weight poset (with respect to $T_i$ and $\Pi_i$) of the $G_i$-module $V_i$. We can write $W_{\tilde{G}}(V) = \{ \sum_{i=1}^r \psi_i | \psi_i \in W_{G_i}(V_i) \}$.

Suppose that there exists $1 \leq i < j \leq r$ such that $\tilde{\rho}(G_i) \neq \{e\}$ and $\tilde{\rho}(G_j) \neq \{e\}$. Then $\xi W_{G_i}(V_i) \geq 2$ and $\xi W_{G_j}(V_j) \geq 2$. Choose $\varphi_1, \varphi_2 \in W_{G_i}(V_i)$ and $\varphi_1, \varphi_2 \in W_{G_j}(V_j)$ such that $\varphi_1 > \varphi_2$ and $\varphi_1 > \varphi_2$. Let $\xi = \sum_{k \neq i, j} \psi_k$ be the sum of the highest weights $\psi_k \in W_{G_k}(V_k)$ for $k \neq i, j$. For $\eta_1 = \xi + \varphi_1 + \varphi_2$ and $\eta_2 = \xi + \varphi_2 + \varphi_1$ we have $\eta_1 > \eta_2$ nor $\eta_1 < \eta_2$ holds. This implies that $W_{\tilde{G}}(V)$ is not totally ordered, which is a contradiction. Hence, any $G_k$ satisfy $\tilde{\rho}(G_k) = \{e\}$ except some $G_i$. Since $V_k = \mathbb{C}$ except for $k = i$, the representation $V$ of $\tilde{G}$ is geometrically equivalent to the representation $V_i$ of $G_i$. In particular, the representation $V$ of $G$ is geometrically equivalent to a thick representation $V_i$ of a connected simple Lie group $G_i$. Therefore, Theorem 3.11 also shows the lists of geometric equivalence classes of thick and dense representations of connected semi-simple Lie groups. 

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