An Agent-Based Distributed Control Policy for Networked SIR Epidemics

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Abstract—This paper revisits a longstanding problem of interest concerning the distributed control of an epidemic process on human contact networks. Due to the stochastic nature and combinatorial complexity of the problem, finding optimal policies are intractable even for small networks. A potentially even larger problem is such policies are notoriously brittle when confronted with small disturbances or uncooperative agents in the network. Unlike the vast majority of related works in this area, we employ a bottom-up agent-based modeling and control approach where we are inspired by principles of self-organization and emergence to better understand how effectively simple control strategies can address this problem. More specifically, rather than attempting to design top-down (either centralized or decentralized) optimal control strategies, we aim to understand how simple strategies work in different conditions. More specifically, based on the locally available information to a particular person, how should that person make use of this information to better protect their self? How can that person socialize as much as possible while ensuring some desired level of safety? We show the optimal solution of this problem to be a form of threshold on the chance of infection of the neighbors of that person. Simulations illustrate our results.

Index Terms—Optimal Control, agent based model.

1 INTRODUCTION

Modeling epidemic processes has been a longstanding research area with the earliest models proposed by Bernoulli in 1760 [1], [2]. Most current works in the literature focus on deterministic (generally mean-field) models that are only good for tracking aggregate numbers of infectious, while exact stochastic models (Markov Chain models) are much more suitable for understanding the spread of a disease at the person-to-person level. However, in general, deterministic mean field model is simpler to analyze but its predictions is inaccurate in small networks. On the other hand, exact Markov chain model gives accurate predictions but its intractable to analyze even for small networks. The majority of existing works only focus on the deterministic models [3]–[6]. While few works in the literature that instead study exact Markov chain compartmental models rather than deterministic approximations. More specifically, the work [7] investigates the connections between the exact Markov chain models (2^N dimensional Markov Chain) and their mean-field approximations for the SIS compartmental model. In [8]–[10] the authors extend this type of analysis to the slightly more complicated SIRS model (3^N dimensional Markov chain). In [11], [12] the exact SIR model is analyzed for various specific small graph structures or graphs with some special properties (e.g., no loops). In our previous work [13], we studied the SEIR exact stochastic epidemic spreading model and its deterministic MFA epidemic spreading model. Another work that show the relation between stochastic and deterministic SIS, SIR in discrete time [14]. Many of these established results are discussed in the book [15, Chapter 2].

Beside modeling, controlling a disease spreading has also been a longstanding research area, which has a great attention from researchers. The vast majority of works are considering a centralized controller, i.e., top-down, strategies, e.g., controlling the network structure, or controlling the decay rate of the pandemic on a network. While these control methods can help eradicating the disease, we are instead more interested in designing a bottom-up controller that can help eradicating the virus at local level, i.e., at person-person level.

Instead of works studying the connections between exact Markov chain models and their deterministic approximations, there are even fewer works in which controlling the exact Markov chain model of interest. One of the earliest works that consider a very similar setup is [16] where a similar exact model is considered but the available control actions came in the form of curative resources (e.g., individuals can pay some cost to recover from an infection faster such as by going to a doctor). Instead, in this paper we only consider Non-Pharmaceutical Interventions (NPIs), or more specifically the act of avoiding social interactions with chosen people.

Some of, the top-down, works have focused on minimizing the spectral radius of the adjacency matrix to suppress the epidemic [17], [18]. In [19], proposes an optimal control problem for a centralized network controller that regulates the infection levels in the network via adapting the curing rates of the nodes, where they used the heterogeneous SIS linearized MFA networked model. Optimal resource allocation problem in [20], [21]. In [22] an optimal control problem was solved for rumor spreading on node level. In [23] the authors studied an optimal link removal to minimize the
spread of infection via quarantining with limited resources. Other work proposed algorithm with approximation, to minimize the number of infected people, based on the idea of bounding the number of infection by supermodular function. An optimal control problem with data-driven model for the spread of COVID-19 and minimize the economic costs associated with implementing NPIs. In the authors proposed an optimal control problem formulation to minimize the total number of infectious during the spread of SIR epidemics by controlling the contact rate, they ended up solving the problem for centralized well-mixed homogeneous SIR model. In a centralized optimal control problem to control the contact rate with between agents were presented and solved numerically using SQP. More related to our problem, the authors in studied an optimal control problem for an individual who is trying to avoid getting infected by controlling their contact rate, where they consider well mixed homogeneous SIR model, and solved the problem numerically. Another work proposed in where an agent controls its rate of contact with others such that partial observability of viral status is taken into consideration.

Statement of Contributions: First, we formulate a local stochastic optimal control problem from the point of view of a single person in a social network. We then consider the mean-field approximation of the problem for which a control strategy is proposed where interactions with certain people are stopped when their chance of infection exceeds some threshold. We show that the optimal solution to the relaxed problem is a controller of this form. We also show that the optimal solution to the relaxed problem is a suboptimal solution on the exact stochastic problem, and we verify the effectiveness of our solutions on the original stochastic problem, rather than the relaxed problem. Finally, we take the problem a step further, by analyzing the solution of a single person interacting with a lumped population, we found that the optimal control strategy also depends on the chance of infection of the lumped node, such that, the single individual will fully quarantine if the chance of infection of the lumped node exceed some threshold. We show numerically how such a strong assumption can increase the cost of disconnection on the individual from considering a heterogeneous population.

Preliminaries and notations: We denote by \( \mathbb{R}, \mathbb{R}_{>0}, \mathbb{R}_{\geq 0}, \) and \( \mathbb{Z}_{\geq 0} \) the set of real, positive real, non-negative real, and non-negative integer numbers, respectively.

We denote an n-dimensional column vector with each entry equal to 1 by \( 1_n \). We say that the matrix \( A \in \mathbb{R}^{n \times n} \) is symmetric if \( A = A^T \). If a vector \( x \in \mathbb{R}^N \), we denote the diagonal matrix of \( x \) by diag(\( x \)) \( \in \mathbb{R}^{N \times N} \), where all the off diagonal entries are zeros while the main diagonal contains all the elements of \( x \). The Cartesian product of two sets \( A \) and \( B \) is denoted by \( A \times B = \{(a,b) | a \in A, b \in B \} \) which represents the set of all points \( (a,b) \) where \( a \in A \) and \( b \in B \).

Graph Theory: an unweighted undirected graph \( G = (V, E, A) \), where the set of vertices \( V = \{1, \ldots, N \} \) captures all the nodes in a network, the edge set \( E \subset V \times V \) denoting the interactions between the different nodes, and the adjacency matrix \( A = [a_{ij}] \in \mathbb{R}^{N \times N} \), where \( a_{ij} = 1 \) if \((i, j) \in E\), and \( a_{ij} = 0 \), otherwise. Unirected graphs implies a symmetric adjacency matrix \( A \).

## 2 Problem Formulation

While the majority of works on this topic generally consider top-down intervention methods for containing a disease outbreak at a macroscopic level (e.g., involving a centralized decision maker), here we are interested in formalizing the problem in a distributed manner from an individual person’s point of view. In particular, with limited information about the evolving state of a disease spreading through a population, an individual must balance the following constraints against each other: the desire to stay healthy versus the desire to maintain regular social interactions. Let us now formalize this problem of interest.

Consider a group of \( N \) people (i.e., nodes), interacting according to an unweighted undirected graph \( G = (V, E, A) \). An interaction, in this problem context, is any activity that regularly brings individuals close enough to spread a disease. If \((i, j) \in E\), then person \( i \) and person \( j \) have contact with each other and the disease can spread between them. Imagining the problem from the point of view of node 1 (without loss of generality), we are interested in analyzing and formulating a stochastic optimal control problem trading off the person’s desire to socialize against the desire to stay healthy. More specifically, the person should socialize as much as possible while ensuring that their probability of remaining healthy after some period of time \( T \) is greater than some personal threshold \( T_{i}^{S} \in [0, 1] \). A more cautious person would select a larger \( T_{i}^{S} \) for instance.

### 2.1 Compartmental Modeling

We use a Markov Process to keep track of the exact Markov states and the state of each node \( i \in V \) in the whole network. We start with Figure 1, which shows the Susceptible-Infected-Removed (SIR) compartmental model for a single person. A person in the Infected state will naturally move to the Removed state over time. However, a person that is in the Susceptible compartment can only move to the Infected compartment through interactions with infected individuals.

The rate at which an individual might transition from one compartment to the next are defined by \( \beta(t), \delta > 0 \), where \( \beta(t) \) depends on the states of the individual’s neighbors. The term \( \beta(t) \) will be explained soon, whereas the recovery rate \( \delta \) is a fixed constant that doesn’t depend on interactions with other people. Note that our model does not distinguish between people who have recovered or have died and we lump these individuals in the ‘Removed’ state. The Markov process for the entire system is denoted by \( X(t) \), where \( X(t) = [X_1(t), \ldots, X_N(t)] \in C^N \) is the entire Markov state and \( X_i(t) \in C \) denotes the state of individual \( i \). The total number of possible states of the network is then \( n = 3^N \).

Now, for convenience, we define \( N_{i} \) as the set of all neighbors of node \( i \). We also define \( N_{i}^{I} \) as the set of infected neighbors
of node \( i \). The total infection rate felt by node \( i \) is then given by \( \beta_{\text{eff}}(t) = \beta \sum_{j \in \mathcal{N}_i} (1 - \max(u_{ij}(t), u_{ji}(t))) \), where \( u_{ij}(t) \in [0, 1] \) is the control input which will be formalized in the next section. In other words, each infected neighbor contributes \( \beta > 0 \) to the total rate of infection.

The Markov process is defined by the following Poisson rates, as presented graphically in Figure 1:

\[
X_i : S \rightarrow I \text{ with rate } \beta_{\text{eff}}(t),
\]

\[
X_i : I \rightarrow R \text{ with rate } \delta.
\]

### 2.2 Control Mechanism

Before we introduce the problem from a point of view of node “1”, we formulate the real world problem for the entire network as a distributed optimal control. In order for all nodes \( i \) to be able to protect themselves, we allow the ability to disconnect its own links from neighbors (e.g., avoiding contacts with specific people); or even disconnecting entirely (e.g., quarantine) when needed. To model this, we define the weighted undirected subgraph \( \mathcal{G}(U(t)) = (\mathcal{V}, E(t), \hat{A}(t)) \), which has the same set of vertices of the static graph \( \mathcal{G} \), but a different set of edges \( E(t) \subset \mathcal{E} \), and weighted adjacency matrix \( \hat{A}(t) = (A - U(t)) \in [0, 1]^{N \times N} \), where \( U(t) = [u_{ij}(t)] \in [0, 1]^{N \times N} \) is defined as the action matrix, where \( u_{ij}(t) = 0 \) if \( a_{ij} = 0 \). If \( a_{ij} = 1 \), then the value \( u_{ij} \in [0, 1] \) represents the level of caution between persons \( i \) and \( j \), where the link \( (i, j) \) will depend on the more cautious person, with \( u_{ij} = 1 \) meaning that they are not interacting at all and thus the infection cannot spread between the two. Further, we define the set \( \mathcal{U}_i = \{u_{ij} \in [0, 1]| j \in \mathcal{N}_i \} \), which captures all the control inputs of node “1”, and the set \( \mathcal{U} = \mathcal{U}_1 \times \mathcal{U}_2 \times \cdots \times \mathcal{U}_N \), which captures all the control inputs for the whole network.

The global cost, (e.g., global social cost) of using this control is given as a linear function of this input so the total cost of interest to minimize is

\[
J_G = \int_0^T \sum_{i \in \mathcal{V}} \sum_{j \in \mathcal{N}_i} c_{ij} \max(u_{ij}(t), u_{ji}(t)) dt
\]  

(1)

where \( T > 0 \) is the time horizon, and \( c_{ij} = c_{ji} > 0 \) is the cost associated with disconnecting edge \((i, j)\) from the graph.

The question then is how should person \( i \) actively control their own links \( u_{ij}(t) \) for \( j \in \mathcal{N}_i \) to minimize the global cost \( J_G \) while satisfying the constraint that its probability of being healthy at any time \( t \in [0, T] \), (i.e., over some time horizon), is greater than or equal to \( \mathcal{T}_i^S \),

\[
\Pr[X_i(t) = S] \geq \mathcal{T}_i^S.
\]  

(2)

This is formalized in Problem 1.

### Problem 1 (Maintaining Global Safety)

\[
\begin{align*}
\text{minimize} & \quad \int_0^T \sum_{i \in \mathcal{V}} \sum_{j \in \mathcal{N}_i} c_{ij} \max(u_{ij}(t), u_{ji}(t)) dt, \\
\text{subject to} & \quad \Pr[X_i(t) = S] \geq \mathcal{T}_i^S, \forall i \in \mathcal{V}.
\end{align*}
\]

It’s worth noting here, that problem 1 is a centralized problem due to the global cost. The reason behind that, is that all individuals in the network have a global objective which is minimizing some global cost (e.g. global social cost). Such that all individuals have to follow some strategy where their ultimate objective is the global cost and not their own cost. However, our goal in this paper, instead is to design local optimal control problem for each individual in the population, based on their own personal needs. As a result, these combined local strategy will impact a global quantity somehow (such as economy, number of infectious, country health cost,...). Indeed, local, from individual perspective, means that this individual cares more about their selves and their objective is more of something that can be beneficial for them, (e.g., social interaction).

To this end, we go back to our problem from the point of view of node 1 (without loss of generality). Since node 1 is the one trying to satisfy its safety, thus, we consider the control input related to node 1, (i.e., we set \( u_{ij} = 0 \) for all \( i \in \{2, \ldots, N\} \)). The social cost of using this control is given as a linear function of this input so the total cost of interest to minimize is

\[
J = \int_0^T \sum_{j \in \mathcal{N}_1} c_{1j} \max(u_{1j}(t), u_{j1}(t)) dt
\]

where \( T > 0 \) is the time horizon, and \( c_{1j} > 0 \) is the cost associated with disconnecting edge \((1, j)\) from the graph. Note that since we consider a point of view of node “1”, then we have \( u_{1j} = u_{j1} \). This can simplify the cost function by dropping \( \max(\cdot) \) operator. The social cost becomes,

\[
J = \int_0^T \sum_{j \in \mathcal{N}_1} c_{1j} u_{1j} dt
\]  

(3)

The question then is how should person 1 actively control their own links \( u_{1j}(t) \) for \( j \in \mathcal{N}_1 \) to minimize the cost \( J \) while satisfying the constraint that its probability of being healthy at any time \( t \in [0, T] \) (i.e., over some time horizon), is greater than or equal to \( \mathcal{T}_1^S \),

\[
\Pr[X_1(t) = S] \geq \mathcal{T}_1^S.
\]  

(4)

This is formalized in our main Problem 2.

### Problem 2 (Maintaining Individual Safety)

\[
\begin{align*}
\text{minimize} & \quad \int_0^T \sum_{j \in \mathcal{N}_1} c_{1j} u_{1j}(t) dt, \\
\text{subject to} & \quad \Pr[X_1(t) = S] \geq \mathcal{T}_1^S.
\end{align*}
\]
3 Optimal Control Formulation

Here we show how Problem 2 can be formulated as a stochastic optimal control problem as presented in Problem 2.

3.1 Continuous Time Exact Markov Chain Model (CT-Exact)

Since we are considering the spread at individual level we formalize the networked Continuous-Time Exact Markov Chain/Process (i.e., stochastic) SIR model of interest. Indeed, this model [30], is not new and it’s known in the literature, but we will introduce it and link it to our optimal control problem. Now, in order to analyze the Markov process exactly, we need to consider the probability distribution across the total number of possible states in the entire network \( n = 3^N \). We refer to the collection of all distinct states as state space denoted \( \mathcal{Y} = \{ Y_1, \ldots, Y_n \} \), where each Markov state \( Y \in \mathcal{Y} \) corresponds to exactly one element in \( C^N \). Define \( g : C^N \rightarrow \mathcal{Y} \) as the bijective map from each state \( X(t) \in C^N \) to a unique Markov state \( Y(t) = g(X(t)) \in \mathcal{Y} \). Then given a Markov state \( Y \in \mathcal{Y} \), we can extract the compartmental state of node \( i \) by \( g^{-1}_{Y_i}(Y) \in \mathcal{C} \).

Since this is a probabilistic model, let \( \pi(t) \in [0,1]^n \) be the belief state (i.e., probability distribution) of the Markov process, capturing the probability of the Markov process being in each of the states, i.e., \( \pi_m(t) = \text{Pr}[g(X(t)) = Y_m] \). Since this is a probability distribution, we have the property \( \mathcal{T}^S \pi(t) = 1 \) at all times.

Now, given some initial belief state \( \pi(0) = \pi_0 \), we wish to propagate the belief state forward in time, so that we know what the probability of being in each state is. This can be accomplished using the transition matrix of the time varying graph \( P(U(t)) \in \mathbb{R}^{n \times n} \) of the continuous time Markov process, where only one node \( i \in V \) can change state at a time. Note we write \( P \), as a function of \( U(t) \), to emphasize that it explicitly depends on the control input. Each entry \( p_{rc} \) in the transition matrix, represents the rate of change from Markov state \( Y_r \) to Markov state \( Y_c \) defined by

\[
p_{rc} = \begin{cases} 
\delta_i, & \text{if } g_i^{-1}(Y_r) = I, \ g_i^{-1}(Y_c) = R, \\
\beta_{ei}(t), & \text{if } g_i^{-1}(Y_r) = j, \ g_i^{-1}(Y_c) = I, \\
\gamma_i^{-1}(Y_r) = j, \ g_i^{-1}(Y_c) = I, \\
- \sum_{b=1}^{m} g_{bc}(t), & \text{if } r = c, \\
0, & \text{otherwise},
\end{cases}
\]

where \( g_i^{-1}(Y_r), g_i^{-1}(Y_c) \) are the compartmental state of node \( i \) in Markov state \( Y_r \) and \( Y_c \), respectively, for any node \( i, j \in \mathcal{V} \), and \( c, r = 1, \ldots, n \).

Now, we start writing the forward propagation equation of continuous time exact Markov chain model by applying the law of total probability,

\[
\dot{\pi}_c(t) = \sum_{r=1}^{n} \pi_r(t)p_{rc}, \quad (5)
\]

Now, we can define the forward propagation of the belief state \( \pi(t) \) in matrix form as

\[
\dot{\pi}^T(t) = \pi^T(t)P(U(t)). \quad (6)
\]

In order to extract the probability of a given node \( i \) being in a compartmental state \( S, I, R \in \mathcal{C} \), we simply sum over the distribution of Markov states \( Y \in \mathcal{Y} \) that represent \( X_i = S, X_i = I \), or \( X_i = R \),

\[
x_i^S(t) \triangleq \text{Pr}[X_i(t) = S] = \sum_{m \in \{1, \ldots, n\}|g_i^{-1}(Y_m) = S} \pi_m, \\
x_i^I(t) \triangleq \text{Pr}[X_i(t) = I] = \sum_{m \in \{1, \ldots, n\}|g_i^{-1}(Y_m) = I} \pi_m, \\
x_i^R(t) \triangleq \text{Pr}[X_i(t) = R] = \sum_{m \in \{1, \ldots, n\}|g_i^{-1}(Y_m) = R} \pi_m. \quad (7, 8, 9)
\]

Where \( x_i^S(t) + x_i^I(t) + x_i^R(t) = 1 \) holds at all times, due to the probabilistic nature of the model.

We can now formalize the constraint of Problem 2 and present this in Problem 3.

Problem 3 (Agent-Based Stochastic Optimal Control)

\[
\text{minimize} \int_0^T \sum_{j \in N_i} c_{ij} u_{ij}(t) dt, \\
\text{subject to} \ x_i^S(t) \geq \mathcal{T}_i^S, \\
\dot{x}_i^S(t) = \pi^T(t)P(U(t)), \\
\pi(0) = \pi_0, \\
u_{ij}(t) \in [0, 1], \forall j \in N_i, t \in [0, T].
\]

It is worth noting that the problem is feasible as long as the probability of node 1 being Susceptible is initially greater than or equal to the threshold \( \mathcal{T}_1^S \). This is easy to see as simply disconnecting entirely from all neighbors \( u_{ij}(t) = 1 \) will satisfy the constraint (but with a worst-case cost).

Remark 1 (POMDP) Problem 3 can also be looked at as a Partially Observable Markov Decision Process (POMDP), with the tuple of \( (\mathcal{Y}, \mathcal{U}, P(U(t)), J, O, B) \) where \( \mathcal{Y} \) is the
state space, \( \mathcal{U} \) is the action (control) set, \( P(U(t)) \) is the transition matrix, \( J \) is the decision running cost, which were predefined. While \( \mathcal{O} \) represents a finite observation set, and \( B \) observation function, which are not considered in this work scoo. General talking, Covid-19 Antigens testing results can be considered as the observation set for all people on network. This formulation and its solution techniques is beyond this work scoo. But it’s worth to be mentioned since it can be similar to the problem formulation [3]. For more details about POMDP we refer reader to [31] Part II. 

Solving Problem [3] exactly is generally intractable due to the \( O(3^N) \) size complexity of the problem, but is indeed of great interest. To the best of our knowledge, no one have ever solved the original problem before or even show the effectiveness of the solution to the relaxed problem on the original problem. Instead for now, we resort to relaxing the problem to provide a sub-optimal solution to Problems [2] and [3] instead.

### 4 Sub-optimal Solution Approach

In order to obtain a sub-optimal feasible solution to Problem [3], we first relax the problem and then solve the relaxed problem optimally by proposing a simplifying assumption. While this is not as good of a solution we hope to eventually obtain, we believe it is the best solution that currently exists in the literature for Problem 3 as the majority of works consider relaxed problems to begin with (e.g., mean-field or lumped degree models). To leverage existing results, we similarly perform a Mean Field Approximation to relax the problem [10], [32], but unlike other works we evaluate the performance of its solution on the original Problem [3].

#### 4.1 Continuous Time Mean Field Approximation (CT-MFA)

The MFA approximation aims to reduce the number of states from \( 3^N \) to \( 2N \), by closing the system using a moment closure. In other words, it aims to reduce the size complexity of the problem form being exponential \( O(3^N) \) to polynomial \( O(3N) \). This can be done by assuming an independent random variables among the nodes in a network. In order to derive the MFA for the exact model, we need to close the exact model [7], where we consider the first moment closure. This results the first order MFA of the continuous time exact model, which is known as the “N-Intertwined Mean field approximation (NIMFA)” [32]. Note that we use (”) notation to differentiate between the relaxed and the exact states.

\[
\dot{\tilde{x}}^S_i = -\tilde{x}^S_i(t) \sum_{j \in N_i} \beta (1 - \max(u_{ij}(t), u_{ji}(t))) \tilde{x}^I_j(t),
\]

\[
\dot{\tilde{x}}^I_i = -\delta \tilde{x}^I_i(t) + \tilde{x}^S_i(t) \sum_{j \in N_i} \beta (1 - \max(u_{ij}(t), u_{ji}(t))) \tilde{x}^I_j(t),
\]

\[
\dot{\tilde{x}}^R_i = \delta \tilde{x}^I_i(t).
\]

Due to the probabilistic nature of the model, \( \tilde{x}^S_i(t) + \tilde{x}^I_i(t) + \tilde{x}^R_i(t) = 1 \) will always be valid for all time \( t \). Thus, one can reduce the number of the states to \( 2N \), such that \( \tilde{x}^R_i(t) = 1 - \tilde{x}^S_i(t) - \tilde{x}^I_i(t) \). This drops the need of keep tracking the recovered states.

One can also write the dynamics (10) in matrix form

\[
\begin{align*}
\dot{\tilde{x}}^S &= -\beta \text{diag}(\tilde{x}^S) A(t) \tilde{x}^I(t), \\
\dot{\tilde{x}}^I &= -\delta \tilde{x}^I(t) + \beta \text{diag}(\tilde{x}^S) A(t) \tilde{x}^I(t), \\
\dot{\tilde{x}}^R &= \delta \tilde{x}^I(t).
\end{align*}
\]

where \( \tilde{x}^S \in \mathbb{R}^N, \tilde{x}^I \in \mathbb{R}^N, \tilde{x}^R \in \mathbb{R}^N, \) and \( \text{diag}(\tilde{x}^S(t)) \in \mathbb{R}^{N \times N} \).

#### Theorem 1 (Relation between exact [7] and relaxed (MFA) models [10], [33], [34])

At any time \( t \in [0, T] \), the solution to the relaxed model (10), \( \tilde{x}_i^S(t) \), will always satisfy the personal threshold in Problem 3. Such that

\[
x_1^S(t) \geq \tilde{x}_1^S(t) \geq T_1^S.
\]

**Proof:** It’s known that for any \( i \in \mathcal{V} \) and for all time \( t \in [0, T] \), the probability of being Susceptible, \( x_i^S(t) \), in the relaxed MFA model (10) lower bounds the exact one, \( \tilde{x}_i^S(t) \), in [7], [33], [34]. Which can be written mathematically as

\[
x_i^S(t) \geq \tilde{x}_i^S(t),
\]

This leads to the validity of the inequality (12). □

Another note here, as the number of agents \( N \) on a network increases, the solution to MFA relaxed model (10) should become closer to the exact solution of (7), such that as \( N \to \infty \) the exact solution should match the relaxed MFA solution. This gives a good reason of the popularity of the MFA models over the exact model, which is intractable and computationally costly for a network that contains large number of agents. For more details about the relation between the exact and MFA models we refer the reader to our previous work in [13].

Now, we write the relaxed problem, from point of view of node “1”, as follows,

#### Problem 4 (Relaxed Agent-Based Stochastic Optimal Control)

\[
\begin{align*}
\text{minimize} & \quad \int_0^T \sum_{j \in N_i} c_{ij} u_{ij}(t) dt, \\
\text{subject to} & \quad \tilde{x}_i^S(t) \geq T_1^S, \\
& \quad \tilde{x}_i^S(t) = -\tilde{x}_i^S(t) \sum_{j \in N_i} \beta (1 - \max(u_{ij}(t), u_{ji}(t))) \tilde{x}_j^I(t), \\
& \quad \tilde{x}_i^I(t) = \tilde{x}_i^S(t) \sum_{j \in N_i} \beta (1 - \max(u_{ij}(t), u_{ji}(t))) \tilde{x}_j^I(t) \\
& \quad - \delta \tilde{x}_i^I(t), \\
& \quad \tilde{x}(0) = x_0, \quad i \in \mathcal{V}, \\
& \quad u_{ij}(t) \in [0, 1], \quad \forall j \in N_i, t \in [0, T].
\end{align*}
\]
constraints as we elaborated the probabilistic nature of the model. Note that we can drop max(·) operator, in dynamics constraints, and replace it with $u_{i j}$, since node 1 is the one trying to satisfy its safety, thus, we consider the control input related to node 1, (i.e., we set $u_{i j} = 0$ for all $i \in \{2, \ldots, N\}$), and $u_{1 j} = u_{j 1}$ in this case. Therefore, the dynamics constraints in Problem [4] can be written as,

\[
\dot{x}^S_j = -\dot{x}^S_j(t) \sum_{j \in N_i} \beta(1 - u_{i j}(t)) \dot{x}^I_j(t),
\]

\[
\dot{x}^I_i = -\delta \dot{x}^I_i(t) + \dot{x}^S_i(t) \sum_{j \in N_i} \beta(1 - u_{i j}(t)) \dot{x}^I_j(t).
\]

An important note here, based on the results in Theorem 5.2 of [35], and the terminal constraint is

\[
\psi(x(T)) = \dot{x}^S_i(T) - \dot{T}^S_i = 0.
\]

This means the Hamiltonian is

\[
H = \sum_{j \in N_i} c_{1 j} u_{i j} + \sum_{i \in V} \lambda_{x^S_i} \dot{x}^S_i + \sum_{i \in V} \lambda_{x^I_i} \dot{x}^I_i,
\]

where $\lambda \in \mathbb{R}^{2 N}$ is the costate, $\lambda_{x^S_i} \in \{\lambda_1, \ldots, \lambda_N\}$ and $\lambda_{x^I_i} \in \{\lambda_{N + 1}, \ldots, \lambda_{2 N}\}$. This means that the costate dynamics is

\[
\dot{\lambda}_{x^S_i} = - \frac{\partial H}{\partial x^S_i},
\]

\[
\dot{\lambda}_{x^I_i} = - \frac{\partial H}{\partial x^I_i}.
\]

The other boundary condition is

\[
v \frac{\partial x^I_l}{\partial x}(T) = \lambda(T) = 0,
\]

which implies $\lambda_i(T) = v$, for some $v \in \mathbb{R}$, and $\lambda_i(T) = 0$, for $i = 2, \ldots, 2 N$.

We need to use Pontryagin’s minimum principle (Chapter 5.2 of [35]) to get an expression for the optimal control $u^*$

\[
H(\dot{x}^S_i, \dot{x}^I_i, u^*, \lambda^*) \leq H(\dot{x}^S_i, \dot{x}^I_i, u, \lambda^*)
\]

\[
\sum_{j \in N_i} c_{1 j} u_{i j} + \sum_{i \in V} \lambda_{x^S_i} \dot{x}^S_i + \lambda_{x^I_i} \dot{x}^I_i \leq \sum_{j \in N_i} c_{1 j} u_{i j} + \sum_{i \in V} \lambda_{x^S_i} \dot{x}^S_i + \lambda_{x^I_i} \dot{x}^I_i,
\]

where $\dot{x}^S_i = [\dot{x}^S_1, \ldots, \dot{x}^S_N]^T$, $\dot{x}^I_i = [\dot{x}^I_1, \ldots, \dot{x}^I_N]^T$. After some analysis and simplifications, the inequality becomes as follows,

\[
\sum_{j \in N_i} (c_{1 j} + \sum_{i \in V} (\lambda_{x^S_i} - \lambda_{x^I_i}) \dot{x}^S_i \dot{x}^I_j) u_{i j} \leq \sum_{j \in N_i} (c_{1 j} + \sum_{i \in V} (\lambda_{x^S_i} - \lambda_{x^I_i}) \dot{x}^S_i \dot{x}^I_j) u_{i j},
\]

for all $j \in N_i$, and using the fact that the $u_{i j}$’s are independent. Therefore, if $c_{1 j} + \sum_{i \in V} (\lambda_{x^S_i} - \lambda_{x^I_i}) \dot{x}^S_i \dot{x}^I_j < 0$, then $u_{i j} = 1$, and if $c_{1 j} + \sum_{i \in V} (\lambda_{x^S_i} - \lambda_{x^I_i}) \dot{x}^S_i \dot{x}^I_j > 0$, then $u_{i j} = 0$. Therefore, we can write the input as

\[
u_{i j} = \frac{1}{2} \left[ 1 - \frac{1}{2} \text{sgn} \left( c_{1 j} + \sum_{i \in V} (\lambda_{x^S_i} - \lambda_{x^I_i}) \dot{x}^S_i \dot{x}^I_j \right) \right].
\]

Now, note that in order to find closed form solution of the optimal input $u^*_{i j}$, we need to get a closed form solution of the states $x^S_i, x^I_i$, and the co-states $\lambda_i$ by methods of solving odes. Getting such a closed form solutions are generally intractable [19]. However, one can turn to numerical optimization techniques to get an approximated numerical solution to problem [4]. Therefore, we use the Open Control Library (OpenOCL) [36], and CasADI-software tools [37] to solve the problem numerically. Note that, OpenOCL provides a modeling language that helps to implement constrained optimal control problems. It implements direct collocations methods, and interfaces CasADI to solve a non-linear program. However, solving problem [4] approximately using this numerical optimization techniques can be computationally costly and intractable to the optimizer. For example, the time complexity to solve problem [4] using the above mentioned tools can be worse than $O(N \log(N))$. Note that, we verify this time complexity by running the optimization for two types of graphs with different number of nodes.

Further, to avoid the burden of the numerical optimization approach, we present next our main result in the paper, where we show how to get an analytical solution to the problem. Although it’s not a closed form solution, but is indeed of great interest, where we are able to show the analytical form of the optimal solution to Problem [4] Now, we start by proposing some simplifying assumptions before we start the analysis.

### 4.2 One-way Infection to Node 1

**Remark 2 (Simplifying Assumption)** This assumption is motivated by the fact that, for the problem of keeping node 1 safe, we do not wish to consider cases where infection spreads from node 1, because then the safety of that node would already have been compromised.

Using the simplifying assumption that node 1 cannot infect its neighbors, but can be infected by them, we have the dynamics

\[
\dot{x}^S_1 = -\dot{x}^S_1(t) \sum_{j \in N_i} \beta(1 - u_{i j}(t)) \dot{x}^I_j(t),
\]

\[
\dot{x}^I_i = -\delta \dot{x}^I_i(t) + \dot{x}^S_1(t) \sum_{j \in N_i} \beta \dot{x}^I_j(t),
\]

\[
\dot{x}^S_i = -\dot{x}^S_i(t) \sum_{j \in N_i \setminus \{i\}} \beta \dot{x}^I_j(t),
\]

where $i \neq 1$, such that $i \in \{2, \ldots, N\}$.

This greatly simplifies the problem, because (19) and (20) do not depend on $\dot{x}^S_1$ or $u_1$. Note that one would need to consider a digraph to model this one way infection. However, for the sake of clarity in presenting the results
to the reader, we stick to the notation of excluding node 1 from the set of neighbors as in \( \{18\} \), and \( \{19\} \). Also note that the dynamics size will be at most \( 2N - 1 \) for the complete graph, since we don’t need to keep track of the recovered state of any node nor the infected state of node 1.

**Theorem 2 (Problem 4 solution)** We can solve Problem 4 with the dynamics constraints replaced by \( \{18\}, \{19\}, \{20\} \), using the input form

\[
u_{ij}(t) = \begin{cases} 1, & \text{if } \tilde{x}_j^1(t) > T_j^f, \\ 0, & \text{if } \tilde{x}_j^1(t) < T_j^f. \end{cases} \tag{21}
\]

where \( T_j^f = \frac{-\alpha_j}{\beta N_j} > 0 \) is a threshold on the probability of node 1’s neighbor \( j \) being infected, for \( j \in N_1 \).

*Proof:* See appendix.

**Remark 3 (Optimal Infection Threshold)** Note that in this work we show that such a threshold exist, but we do not yet have a closed form solution for the optimal threshold \( T_j^f \) for which \( \{21\} \) is optimal, thus we rely on numerical optimization techniques to solve the problem.

We notify here that the solution \( \{21\} \) to problem 4 solves the original problem \( \{3\} \) suboptimally, i.e., the solution \( \{21\} \) guarantees the safety threshold in problem \( \{3\} \). We will also verify this results in the simulation section.

### 4.3 Node“1” vs Lumped population

As we presented in the literature review, the majority of the researches focus the work on the homogeneity assumption. That is for a good reason, it can greatly simplifies the problem for large network. Therefore, we would like here to propose an assumption that the network is well mixed, to see how bad such an assumption can be compared to the heterogeneous networked relaxed MFA model.

**Remark 4 (Stronger Simplifying Assumption)** We propose here an assumption that node“1” interactions can be controlled by controlling one input rather than the multi-inputs. This assumption can be achieved by assuming a well mixed homogeneous network, such that node“1” is interacting with a lumped population. Therefore, this assumption can greatly simplify the problem.

Note that, it’s convenient here to consider a well mixed homogeneous network. The assumption of a well mixed homogeneous network, considers that every individual in the network is in contact with all other individuals, i.e., complete graph. This means, that node “1” is considered to connect to all the individuals on the network homogeneously. The well-mixed homogeneous network model is popular in the literature due to its simplicity, some of the optimal control problems that considers this model can be found in \( \{26\}, \{27\} \). Such assumption can greatly reduce the computation cost of finding multi-input’s optimal solution, specifically for large networks. Note that we show by simulation how bad such an assumption can affect the cost.

Now, we denote the lumped node’s (population) average probabilities of being susceptible, infected, and recovered by \( S_{LP}(t), I_{LP}(t), \) and \( R_{LP}(t) \), respectively. Due to the complete graph assumption in the lumped node, we can write the equations that describes the states of the lumped node as follows

\[
S_{LP}(t) = \sum_{i \in V \setminus 1} \frac{\tilde{x}_i^S(t)}{N - 1},
\]

\[
I_{LP}(t) = \sum_{i \in V \setminus 1} \frac{\tilde{x}_i^I(t)}{N - 1},
\]

\[
R_{LP}(t) = \sum_{i \in V \setminus 1} \frac{\tilde{x}_i^R(t)}{N - 1}.
\]

Since we are dealing with average probabilities, thus, \( S_{LP}(t) + I_{LP}(t) + R_{LP}(t) = 1 \) holds at any time \( t \), due to the probabilistic nature of the states in the lumped node. This can justify dropping the need of keep tracking of the average probability of recovered individuals in the lumped node. Using this simplifying assumption, i.e., homogeneously mixed network, we can write the dynamics of the system we wish to control,

\[
\begin{align*}
\dot{\tilde{x}}_1^S &= -\beta \tilde{x}_1^S (1 - u) I_{LP}, \\
\dot{I}_{LP} &= -\delta I_{LP} + \beta S_{LP}(I_{LP} + (1 - u) \tilde{x}_1^I), \\
\dot{S}_{LP} &= -\beta S_{LP}(I_{LP} + (1 - u) \tilde{x}_1^I), \\
\dot{\tilde{x}}_1^I &= -\delta \tilde{x}_1^I + \beta (1 - u) \tilde{x}_1^S I_{LP}.
\end{align*}
\]

Note that due to homogeneously mixed network assumption, the index of the control \( u \) is dropped, since only one input is being controlled.

We notify here, the problem \( \{4\} \) using the constraint dynamics \( \{22\} \) can also be solved numerically using CasADi and Open Optimal Control Library (OpenOCL) software package.

Note that we can get the form of the optimal control solution if we assume here that node“1”, is safe, thus, the chance node “1” spreading the infection to the lumped node can be neglected since it won’t have a significant effect in spreading the infection to the lumped node. We also can drop the index on the control input since we only have one input in this problem. Considering this assumption we can simplify the dynamics in \( \{22\} \) as follows,

\[
\begin{align*}
\dot{\tilde{x}}_1^S &= -\beta (1 - u) \tilde{x}_1^S I_{LP}, \\
\dot{I}_{LP} &= -\delta I_{LP} + \beta S_{LP} I_{LP}, \\
\dot{S}_{LP} &= -\beta S_{LP} I_{LP}.
\end{align*}
\]

Also the cost function becomes

\[ J = \int_0^T c_{av} u(t) dt, \]

where \( c_{av} = \frac{N - 1}{|N|} \sum_{j \in N_1} c_{1j} \), is the average of all edge’s associated cost. Note that in \( c_{av} \), we multiply by \( N - 1 \), because of the complete graph assumption. Means that node “1” will be considered to have a link that connects with all neighbors.
Theorem 3 (Problem solution with strong assumption)

We can solve Problem with the dynamics constraints replaced by using the input form

\[ u^*(t) = \begin{cases} 1, & \text{if } I_{LP}(t) > I^*, \\ 0, & \text{if } I_{LP}(t) < I^*. \end{cases} \]

(24)

where \( I^* = \frac{\ln(1 - \Theta)}{\beta T_{\text{av}}} > 0 \) is a threshold on the probability of node 1’s lumped node neighbor being infected.

Proof: See appendix.

Remark 5 (Lumped Node Optimal Infection Threshold)

Note that in this work we show that such a threshold exist, and node”1” will decide to connect or disconnect with the lumped node based on the chance of infection of the lumped node being below or above not this infection. However, we do not have a closed form solution for the optimal threshold \( I^* \) for which (24) is optimal, thus we rely sweep method to find the threshold or numerical optimization techniques to solve the problem.

5 Simulation

Consider a network of 5 nodes \( G \), such that we are given the adjacency matrix \( A \), the initial conditions are \( x^S(0) = [1, 1, 0.62, 0.99, 0.01] \), \( x^I(0) = [0, 0.38, 0.01, 0.99] \), the rates of the model \( \beta = 0.2, \delta = 1/10 \), safety threshold of \( T_{1}^S = 0.7 \), the parameter \( c_{12} = 10 \), and the terminal time \( T = 11 \) days.

In the first numerical simulation, the graph of \( A \) can be visually represented as in Figure 2(a). Note that the nodes colors in the graph represent the initial state of the nodes. Green color represents a Susceptible state, and grey color represent nodes that are’t 100% susceptible or infected, i.e., refer the initial condition (distribution) of the nodes. Further, the enlarged node (i.e., node”1”) is the node that we keep its safety above a given safety threshold \( (T_{1}^S) \).

At the beginning, we find the optimal \( T_{1}^{I*} \), by sweeping the \( \tilde{x}_{1}^{S}(T) \) and the cost \( J(T) \) over different values of \( T_{2}^I \in [0, 1] \) as can be seen in Figure 2(b). Such that the red dot is the optimal infection threshold of node “2”, i.e., \( T_{2}^{I*} = 0.4672 \).

Since now we know \( T_{2}^{I*} \), therefore, we implement the “Threshold Strategy”, which is the strategy of the solution in (21). Further, we compare “Threshold Strategy” (blue line) against “OCL Strategy” (red line). The “OCL Strategy” is the numerical optimization technique to solve Problem where we use OpenOCL (and CasADi) software packages, which uses collocation method.

The comparison can be seen in Figure 2 (d) Probability of node”1” being susceptible (\( \tilde{x}_{1}^{S}(t) \)) against time(days), (e) the cost \( J(t) \) against time(days), and (f) the control, \( u_{12}(t) \) (left) and the probability of node”2” being infected (\( \tilde{x}_{2}^{I}(t) \)) (right) against time(days).

We would like also to show the results mentioned in Theorem which elaborate that solving problem based on the “Threshold Strategy” is a sub-optimal solution to the original problem. Figure verifies the results in Theorem 4.

We clearly can see the bound on the \( x_{1}^{S}(t) \) by \( \tilde{x}_{1}^{S}(t) \), and further we can see \( x_{1}^{S}(T) > \tilde{x}_{1}^{S}(T) \), which is quantified by the red line in the figure.

Secondly, as a second numerical simulation, we would like to show a case when node “1” has more than one neighbor. We use the same given initial conditions, and the new \( A \) matrix is given and the graph is shown in Figure 2(a). The \( c_{12} = 10 \), and \( c_{14} = 1 \) are given, which also can be seen in the figure. This means that disconnecting from node “2” will cost more than disconnecting from node “4”. We sweep to find the optimal infection thresholds, where \( T_{2}^{I*} = 0.466 \) and \( T_{4}^{I*} = 0.07 \). Now, we show the results of implementing “Threshold Strategy” vs “OCL Strategy” in Figure 2 (b) \( \tilde{x}_{1}^{S}(t) \) against time(days), (c) \( J(t) \) against time(days), (d) \( u_{12}(t) \) and \( \tilde{x}_{2}^{I}(t) \) against time, (f) \( u_{14}(t) \) and \( \tilde{x}_{4}^{I}(t) \) against time.

Thirdly, as a third numerical example, we show a case when all agents in the network is trying to keep themselves above some personal threshold, such that \( \tilde{T}^S = [0.7, 0.4, 0.3, 0.3, 0.15] \), and \( c_{12} = c_{21} = 10, c_{14} = c_{41} = 1, c_{23} = c_{32} = 1, c_{34} = c_{43} = 1, \) and \( c_{25} = c_{52} = 1 \). Such that in this example every individual is running his local strategy based on the given information. The question would someone ask is what would the global cost for the entire network in this case be. Figure shows the probability of all nodes being susceptible, and the global cost over
time. As we can see from Figure 5 (b), each need has its own personal threshold, that is different from others in the network.

Lastly, we compare the “Threshold Strategy” and “OCL Strategy” with “Node vs LP Strategy” (orange line). This last strategy suggest the well mixed homogeneous network. Such that node”1” interactions can be lumped into one link,i.e., one control input, and other nodes on the graph can be lumped into one node by assuming a complete graph. The lumped graph can be seen in Figure 6 (a), where $S_{LP} = \frac{1}{3} \sum_{i=2}^{5} \tilde{x}_i^S(t) = 0.6550$, $I_{LP} = \frac{1}{3} \sum_{i=2}^{5} \tilde{x}_i^I(t) = 0.345$, and $c_{av} = \frac{4c_{12}+c_{14}}{2}$. The results of the comparison between the three strategies can be found in Figure 6 (b) and (c).

The main take away message of this last comparison is, it show us how bad homogeneity assumption can be, and why it’s important to consider the heterogeneity into the problem.

6 Conclusions

In this paper, we formulate a real world problem, of how to optimally control individual interactions with their neighbors. We formulated the problem in two different scenarios. The problem in its exact Markov chain model is very intractable even for small network due to the number of states to be tracked. We relaxed the problem using the relaxed MFA model. Finding closed form analytic solution for the relaxed problem is not possible without some assumptions. Interestingly, under some valid assumptions, we found the optimal solution to the problem to be a form of optimal infection threshold of each neighbor. To compare our solution against some other solutions, we solved the relaxed problem numerically using Open OCL software package. Where the solution was very close to our threshold solution. Due to the time complexity of using the numerical optimization methods, we designed a homogeneous case, and we showed how well that homogeneous case compared to the heterogeneous case.

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Fig. 5. (a) Graph of 5 Node (b) Probability of all nodes being susceptible over time ($x_i^S(t)$), where $i \in V$, using 'OCL Strategy'. (c) the global cost of the controller over time $J_G(t)$.

Fig. 6. (a) graph of Node "1" interacting with Lump Node, inside the lumped node a complete graph is considered (b) Probability of node "1" being susceptible over time ($x_1^S(t)$), for different strategies. (c) the total cost of the controller over time $J(t)$.

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7 APPENDIX

7.1 Proof of Theorem \[3\]

For convenience, we define \( \chi^I = [\tilde{x}_1^I, \ldots, \tilde{x}_N^I]^T \in \mathbb{R}^{N-1} \), \( \chi^S = [\tilde{x}_2^S, \ldots, \tilde{x}_N^S]^T \in \mathbb{R}^{N-1} \), and \( \chi = [\chi^I, \chi^S]^T \). Setting up an optimal control problem as in Table 3.2-1 of [35], we have the cost function

\[ J = \int_0^T \sum_{j \in N_1} c_{1j} u_{1j}(t) dt, \]

and the terminal constraint is

\[ \psi(x(T)) = \tilde{x}^S_1(T) - T^S_1 = 0. \]

This means the Hamiltonian is

\[ H = \sum_{j \in N_1} c_{1j} u_{1j} + \lambda_1 \dot{x}_1^S + \lambda_1^T \dot{x}_1 \]

where \( \lambda \in \mathbb{R}^{2N-1} \) is the costate, and \( \lambda_1 \triangleq [\lambda_2, \ldots, \lambda_{2N-1}]^T \).

This means that the costate dynamics is

\[ \dot{\lambda}_1 = -\frac{\partial H}{\partial \dot{x}_1^S}, \]

\[ \dot{\lambda}_1 = -\frac{\partial H}{\partial \dot{x}_1}. \]

The other boundary condition is

\[ v \frac{\partial u}{\partial x} (T) - \lambda(T) = 0, \]

which implies \( \lambda_1(T) = v \), for some \( v \in \mathbb{R} \), and \( \lambda_i(T) = 0 \), for \( i = 2, \ldots, 2N-1 \).

We need to use Pontryagin’s minimum principle (Chapter 5.2 of [35]) to get an expression for the optimal control \( u^* \)

\[ H([\dot{x}^S_1, \chi^T], u^*, \lambda^*) \leq H([\dot{x}^S_1, \chi^T], u_1, \lambda^*) \]

\[ \sum_{j \in N_1} c_{1j} u_{1j}^* + \lambda_1^* \dot{x}_1^S + \lambda_1^T \dot{x}_1 \leq \sum_{j \in N_1} c_{1j} u_{1j} + \lambda_1 \dot{x}_1^S + \lambda_1^T \dot{x}_1, \]

since the only term of the states that depends on the input is \( \dot{x}_1^S \), the inequality becomes as follows,

\[ \sum_{j \in N_1} (c_{1j} + \lambda_1^* \beta \dot{x}_1^S \tilde{x}_j^S) u_{1j} \leq \sum_{j \in N_1} (c_{1j} + \lambda_1 \beta \dot{x}_1^S \tilde{x}_j^S) u_{1j} \]

for all \( j \in N_1 \), and using the fact that the \( u_{1j}^* \)'s are independent. Therefore, if \( c_{1j} + \lambda_1^* \beta \dot{x}_1^S \tilde{x}_j^S < 0 \), then \( u_{1j} = 1 \), and if \( c_{1j} + \lambda_1^* \beta \dot{x}_1^S \tilde{x}_j^S > 0 \), then \( u_{1j} = 0 \). Therefore, we can write the input as

\[ u_{1j} = \frac{1}{2} - \frac{1}{2} \text{sgn} \left( c_{1j} + \lambda_1 \beta \dot{x}_1^S \tilde{x}_j^S \right). \]

(25)

Now, note that

\[ \dot{\lambda}_1 = -\frac{\partial H}{\partial \dot{x}_1^S} = -\lambda_1 \frac{\partial}{\partial \dot{x}_1^S} \dot{x}_1^S, \]

because only \( \dot{x}_1^S \) depends on \( \dot{x}_1^S \). Now, from [18], we have

\[ \dot{\lambda}_1 = -\frac{\lambda_1 \dot{x}_1^S}{\dot{x}_1^S}. \]

This implies that \( 0 = \dot{\lambda}_1 \dot{x}_1^S + \lambda_1 \dot{x}_1^S = \frac{\partial}{\partial \dot{x}_1^S} (\lambda_1 \dot{x}_1^S (t)) \), indicating that the quantity \( \lambda_1 \dot{x}_1^S (t) \) is constant with respect to time. At the terminal time we know, \( \lambda_1(T) \dot{x}_1^S (T) = \dot{T}^S_1 \), and so \( \lambda_1(T) \dot{x}_1^S (t) = \dot{T}^S_1, \forall t \in [0, T] \). Therefore, the optimal input \( u_{1j} \) can be written as

\[ u_{1j} = \begin{cases} 1, & \text{if } v \tilde{x}_j^S < -\frac{c_{1j}}{\beta \tilde{x}_j^S}, \\ 0, & \text{if } v \tilde{x}_j^S > \frac{c_{1j}}{\beta \tilde{x}_j^S}. \end{cases} \]

Note that \( c_{1j}, \beta, H^S, v \) are all constant in time and \( c_{1j}, \beta, H^S, \tilde{x}_j^S, \tilde{x}_j^T > 0, \) so we have the trivial solution \( u_{1j}^* (t) = 0, \forall t \in [0, T] \) if \( v > 0 \). In the nontrivial case where \( v < 0 \), then, we can write the optimal solution as

\[ u_{1j} = \begin{cases} 1, & \text{if } \tilde{x}_j^S > T^j_1, \\ 0, & \text{if } \tilde{x}_j^S < T^j_1, \end{cases} \]

where \( T^j_1 \triangleq -\frac{c_{1j}}{\beta \tilde{x}_j^S} \). Noting that, since \( \tilde{x}_j^S \in [0, 1] \), we can recover the trivial case with a proper selection of \( T^j_1 \), we have proven that the optimal input has the threshold form.

7.2 Proof of Theorem \[4\]

Setting up an optimal control problem as in Table 3.2-1 of [35], the cost function with lumped population becomes

\[ J = \int_0^T c_{av} u(t) dt, \]

where \( c_{av} = \frac{N-1}{|\chi|} \sum_{j \in N_1} c_{1j} \), is the average of all edge’s associated cost.

The terminal constraint is

\[ \psi(x(T)) = \tilde{x}^S_1(T) - T^S_1 = 0. \]

This means the Hamiltonian is

\[ H = c_{av} u + \lambda_1 \dot{x}_1^S + \lambda_2 I^L, \lambda_3 S^L, \]

where \( \lambda \in \mathbb{R}^3 \) is the costate, and \( \lambda \triangleq [\lambda_1, \lambda_2, \lambda_3]^T \). This means that the costate dynamics is

\[ \dot{\lambda}_1 = -\frac{\partial H}{\partial \dot{x}_1^S}, \]

\[ \dot{\lambda}_2 = -\frac{\partial H}{\partial I^L}, \]

\[ \dot{\lambda}_3 = -\frac{\partial H}{\partial S^L}. \]

The other boundary condition is

\[ v \frac{\partial \psi}{\partial x} (T) - \lambda(T) = 0, \]

which implies \( \lambda_1(T) = v \), for some \( v \in \mathbb{R} \), and \( \lambda_i(T) = 0 \), for \( i = 2, 3 \).
We need to use Pontryagin’s minimum principle (Chapter 5.2 of [35]) to get an expression for the optimal control $u^*$

$$H(x_1^{S*}, I_{LN}, S_{LN}, u^*, \lambda^*) \leq H(x_1^{S*}, I_{LN}, S_{LN}, u, \lambda^*)$$

$$c_{av} u^* + \lambda_1^* \dot{x}_1^{S*} + \lambda_2^* \dot{I}_{LN} + \lambda_3^* S_{LN} \leq c_{av} u + \lambda_1^* \dot{x}_1^{S*} + \lambda_2^* \dot{I}_{LN} + \lambda_3^* S_{LN},$$

since the only term of the states that depends on the input $c$ is $\dot{x}_1^{S*}$, because only $0 = \dot{\lambda}^*$ since the only term of the states that depends on the input $c$ is $\dot{x}_1^{S*}$, the inequality becomes as follows,

$$(c_{av} + \lambda_1^* \beta \dot{x}_1^{S*} I_{LN}) u^* \leq c_{av} + \lambda_1^* \beta \dot{x}_1^{S*} I_{LN} u.$$ \hspace{1cm} (27)

Now, if $c_{av} + \lambda_1^* \beta \dot{x}_1^{S*} I_{LN} < 0$, then $u^* = 1$, and if $c_{av} + \lambda_1^* \beta \dot{x}_1^{S*} I_{LN} > 0$, then $u^* = 0$. Therefore, we can write the input as

$$u^* = \frac{1}{2} - \frac{1}{2} \text{sng} \left( c_{av} + \lambda_1^* \beta \dot{x}_1^{S*} I_{LN} \right).$$ \hspace{1cm} (28)

Now, note that

$$\dot{\lambda}_1 = -\frac{\partial H}{\partial x_1^{S*}} = -\lambda_1 \frac{\partial}{\partial x_1^{S*}} \dot{x}_1^{S*},$$

because only $\dot{x}_1^{S*}$ depends on $\dot{x}_1^{S*}$. Now, from [18], we have

$$\frac{\partial}{\partial x_1^{S*}} \dot{x}_1^{S*} = -\beta(1 - u(t)) I_{LN},$$

which means

$$\dot{\lambda}_1 = -\lambda_1 \frac{\dot{x}_1^{S*}}{x_1^{S*}}.$$  

This implies that $0 = \dot{\lambda}_1 \dot{x}_1^{S*} + \lambda_1 \dot{x}_1^{S*} = \frac{d}{dt}(\lambda_1(t) \dot{x}_1^{S*}(t))$, indicating that the quantity $\lambda_1(t) \dot{x}_1^{S*}(t)$ is constant with respect to time. At the terminal time we know, $\lambda_1(T) \dot{x}_1^{S*}(T) = v T_1^S$, and so $\lambda_1(t) \dot{x}_1^{S*}(t) = v T_1^S, \forall t \in [0, T]$. Therefore, the optimal input (28) can be written as

$$u^* = \begin{cases} 1, & \text{if } v I_{LN} < \frac{-c_{av}}{\beta T_1^S}, \\ 0, & \text{if } v I_{LN} > \frac{-c_{av}}{\beta T_1^S}. \end{cases}$$

Note that $c_{av}, \beta, T_1^S, v$ are all constant in time and $c_{av}, \beta, T_1^S, I_{LN} > 0$, so we have the trivial solution $u^*(t) = 0, \forall t \in [0, T]$ if $v \geq 0$. In the nontrivial case where $v < 0$, then, we can write the optimal solution as

$$u^* = \begin{cases} 1, & \text{if } I_{LN} > I^*, \\ 0, & \text{if } I_{LN} < I^*, \end{cases}$$

where $I^* = \frac{-c_{av}}{\beta T_1^S}$. Noting that, since $I^* \in [0, 1]$, we can recover the trivial case with a proper selection of $I^*$, we have proven that the optimal input has the threshold form.