Singular Hochschild Cohomology of Radical Square Zero Algebras
Zhengfang WANG *

Abstract

Following the approach developed by Cibils and Sánchez for ordinary Hochschild cohomology, we compute the singular Hochschild cohomology of radical square zero algebras. We show that the singular Hochschild cohomology of radical square zero algebras has a very combinatorial structure. A PROP interpretation for the Gerstenhaber bracket is given in the case of radical square zero algebras. This interpretation allows us to completely determine the Gerstenhaber algebra structures for several examples.

1 Introduction

Following Buchweitz [Buch] and Orlov [Orl] the singular category $\mathcal{D}_{sg}(A)$ of an associative algebra $A$ is the localisation of the bounded derived category of $A$ at the full subcategory of compact objects. For more details see e.g. [Zim]. In the recent paper [Wang], we defined the singular Hochschild cohomology groups $\text{HH}^i_{sg}(A, A)$ of an associative $k$-algebra $A$ over a commutative base ring $k$, as morphisms from $A$ to $A[i]$ in the singular category $\mathcal{D}_{sg}(A \otimes_k A^{op})$. That is, for any $i \in \mathbb{Z}$,

$$\text{HH}^i_{sg}(A, A) := \text{Hom}_{\mathcal{D}_{sg}(A \otimes_k A^{op})}(A, A[i]),$$

where we denote as usual by $[i]$ the $i$-th iterate of the suspension functor. Moreover, we proved that the singular Hochschild cohomology ring $\text{HH}^*_{sg}(A, A)$ is a Gerstenhaber algebra if $A$ is a $k$-algebra such that $A$ is projective as a $k$-module (cf. [Wang, Theorem 4.2]). The singular Hochschild cohomology group $\text{HH}^i_{sg}(A, A)$ can be computed by the bar resolution of $A$, roughly speaking, it is realized as the colimit of an inductive system associated to the bar resolution (cf. [Wang, Proposition 3.1]).

A radical square zero algebra is a finite dimensional algebra over a field $k$ such that the square of its Jacobson radical is zero. From Gabriel’s theorem it follows that every radical square zero algebra over an algebraically closed field $k$ is Morita equivalent to a path algebra modulo relations of the form $kQ/\langle Q_2 \rangle$, where $Q$ is a finite quiver and $Q_2$ is the set of all paths in $Q$ of length 2.

We follow [Cib1, Cib2, Cib3] and [San1, San2] to compute the singular Hochschild cohomology of radical square zero algebras. We will study the dimension of $\text{HH}^*_{sg}(A, A)$

*zhengfang.wang@imj-prg.fr, Université Paris Diderot-Paris 7, Institut de Mathématiques de Jussieu-Paris Rive Gauche CNRS UMR 7586, Bâtiment Sophie Germain, Case 7012, 75205 Paris Cedex 13, France
and the Gerstenhaber algebra structure on $\text{HH}^*(A, A)$. Recall that in [Cib1], Cibils constructs a reduced bar resolution (cf. Lemma 2.1 below) of $A$ as an $A$-$A$-bimodule for any finite dimensional $k$-algebra $A$, where $k$ is a perfect field. By this projective resolution, Cibils gave a very combinatorial construction of the Hochschild cohomology groups $\text{HH}^*(A, A)$ in the case of a radical square zero algebra $A$. Following Cibils’s construction, in [San1, San2] Sánchez studied the Lie module structure on the Hochschild cohomology groups $\text{HH}^1(A, A)$ over the Lie algebra $\text{HH}^1(A, A)$. In this paper, we will generalize these results to the singular Hochschild cohomology $\text{HH}^*_\text{sg}(A, A)$ of a radical square zero algebra $A$. We also give examples for algebras $A$ whose Hochschild cohomology (or singular Hochschild cohomology) do not admit a BV algebra structure (cf. Remarks 3.19 and 5.8).

This paper is organized as follows. Section 2 is devoted to recall some notions and results from [Cib1, Cib2, Cib3]. In Section 3, we develop the general theory on the singular Hochschild cohomology groups $\text{HH}^*_\text{sg}(A, A)$ for any radical square zero algebra $A$. In Section 4, we consider examples of the radical square zero algebra $A$ associated to the c-crown quiver. The dimension of $\text{HH}^*_\text{sg}(A, A)$ and the Gerstenhaber algebra structure on $\text{HH}^*_\text{sg}(A, A)$ will be computed explicitly. Section 5 considers the radical square zero algebra $A$ associated to the two loops quiver. We give a description on the Gerstenhaber algebra structure on $\text{HH}^*_\text{sg}(A, A)$. In Section 6, we give a PROP interpretation for the Gerstenhaber bracket in the case of radical square zero algebras. This generalizes the construction in Section 5. In Appendix A, we use PROP theory to construct a new Lie algebra structure on $\mathfrak{g}(\mathfrak{u}_n(k))$.

In this paper, we frequently use some notions on quivers without recalling. We refer to [Cib1, Cib3, Boe, Mich, Zim] for details.

**Acknowledgement**

This work is a part of author’s PhD thesis, I would like to thank my PhD supervisor Alexander Zimmermann for introducing this interesting topic and for his many valuable suggestions for improvement. I also would like to thank Huafeng Zhang for many useful discussions during this project.

## 2 Background and Notation

In this section, we assume that $k$ is a field and $A$ is a finite dimensional $k$-algebra. Recall that the Jacobson radical $\text{rad}(A)$ is defined as the intersection of all maximal left ideals of $A$. Wedderburn-Malcev theorem says that there exists a semi-simple subalgebra $E$ of $A$ such that $A \cong E \oplus \text{rad}(A)$.

Let $Q = (Q_0, Q_1, s, t)$ be a finite (i.e. $|Q_0 \cup Q_1| < +\infty$) and connected quiver. For $n \in \mathbb{Z}_{\geq 0}$, we define $Q_n$ to be the set of all paths in $Q$ of length $n$. The trivial path is denoted by $e$, for a vertex $i$ in $Q_0$. We consider the path algebra $kQ$ of $Q$ over the field $k$. As a vector space,

$$kQ = \bigoplus_{n \in \mathbb{Z}_{\geq 0}} kQ_n.$$  

Let $p$ and $q$ be two paths in $Q$, then their product $pq$ is the concatenation of the paths $p$ and $q$ if $t(q) = s(p)$ and 0 otherwise. Then the radical square zero algebra of $Q$ is defined as

$$A_Q := kQ/\langle Q_2 \rangle,$$
where \( \langle Q_2 \rangle \) is the two-sided ideal in \( kQ \) generated by \( Q_2 \), the set of paths of length 2. Hence \( A_Q \) is a finite-dimensional \( k \)-algebra with the Jacobson radical \( \text{rad}(A_Q) = kQ_1 \) and the Wedderburn-Malcev decomposition is \( A_Q = kQ_0 \oplus kQ_1 \), where \( E = kQ_0 \) is a commutative semi-simple algebra, which is isomorphic to

\[
\bigoplus_{e_n \in Q_0} ke_n.
\]

We remark that \( \text{rad}^2(A_Q) = 0 \).

For a finite-dimensional \( k \)-algebra \( A \) (not necessarily, radical square zero algebra) with a Wedderburn-Malcev decomposition \( A = E \oplus \text{rad}(A) \), we have a projective resolution of the \( A \)-\( A \)-bimodule \( A \).

We abbreviate in the sequel

\[
a_{i,j} := a_i \otimes a_{i+1} \otimes \cdots \otimes a_j
\]

for \( i < j \).

**Lemma 2.1** (Lemma 2.1 \[Cib1\]). Let \( A \) be a finite dimensional \( k \)-algebra and let

\[
A = E \oplus \text{rad}(A)
\]

be a Wedderburn-Malcev decomposition of \( A \). Put \( r := \text{rad}(A) \). Then the following is a projective resolution of the \( A \)-\( A \)-bimodule \( A \):

\[
\mathcal{R}(A) : \cdots \xrightarrow{d_{i+1}} A \otimes_E r \otimes_E r \otimes_E A \xrightarrow{d_i} A \otimes_E r \otimes_E A \xrightarrow{d_2} A \otimes_E A \xrightarrow{\epsilon} A \xrightarrow{0}
\]

(1)

where \( \epsilon(a \otimes b) = ab \) and

\[
d_i(a \otimes x_{1,i} \otimes b) = ax_1 \otimes x_{2,i} \otimes b + \sum_{j=1}^{i} (-1)^j a \otimes x_{1,j-1} \otimes x_j x_{j+1} \otimes x_{j+2,i} \otimes b + (-1)^i a \otimes x_{1,i-1} \otimes x_i b
\]

where all tensor products \( \otimes \) represent \( \otimes_E \).

**Proposition 2.2** (Proposition 2.2. \[Cib1\]). Let \( A \) be a finite-dimensional \( k \)-algebra and \( E \) be a subalgebra of \( A \) such that \( A = E \oplus \text{rad}(A) \) and let \( M \) be a \( A \)-\( A \)-bimodule. Then the Hochschild cohomology vector spaces \( \text{HH}^i(A, M) \) are the cohomology groups of the complex:

\[
\begin{array}{ccc}
0 & \xrightarrow{\delta} & M^E \\
\delta & & \text{Hom}_{E-E}(r, M) \xrightarrow{\delta} \text{Hom}_{E-E}(r \otimes_E r, M) \xrightarrow{\delta} \\
\delta & & \cdots & \text{Hom}_{E-E}(r \otimes_E r, M) \xrightarrow{\delta} \cdots
\end{array}
\]

(2)

where we denote

\[
r := \text{rad}(A),
\]

\[
M^E := \{ m \in M \mid sm = ms \text{ for all } s \in E \}.
\]
for $m \in M^E$ and $x \in \text{rad}(A)$,

$$\delta(m)(x) = mx - xm$$

and for $f \in \text{Hom}_{E-E}(r \otimes_{E^1} M)$,

$$\delta(f)(x_{1,i+1}) = x_1f(x_{2,i+1}) + \sum_{j} (-1)^j f(x_{1,j-1} \otimes x_j x_{j+1} \otimes x_{j+2,i+1}) + (-1)^{i+1} f(x_{1,i})x_{i+1},$$

where all tensor products $\otimes$ represent $\otimes_{E}$.

**Remark 2.3.** If $A$ is a radical square zero algebra (i.e. $\text{rad}^2(A) = 0$), then the differential $\delta(f)$ just has two terms, namely, we have

$$\delta(f)(x_{1,i+1}) = x_1f(x_{2,i+1}) + (-1)^{i+1} f(x_{1,i})x_{i+1}.$$

Next we will apply the projective resolution $R(A)$ in Lemma 2.1 to compute the singular Hochschild cohomology of radical square zero algebras.

### 3 General theory for radical square zero algebras

Now we fix a finite and connected quiver $Q = (Q_0, Q_1, s, t)$ and let $A := kQ/\langle Q_2 \rangle$ be the radical square zero algebra of $Q$ over a field $k$.

**Lemma 3.1** (Lemma 2.1 [Cib3]). Let $k$ be a field. Let $r = kQ_1$ be the Jacobson radical of $A$ and $E = kQ_0$. Then $r \otimes_{E^1} E^n$ has a basis given by $Q_n$, the set of paths of length $n$.

**Remark 3.2.** Let $p_i \in r$, then

$$p_1 \otimes_E \cdots \otimes_E p_n \neq 0 \in r \otimes_{E^n}$$

if and only if for $1 \leq i \leq n - 1$, $s(p_i) = t(p_{i+1})$, that is, $p_1p_2 \cdots p_n$ is a path of length $n$ in $kQ$.

We follow Cibils to introduce the notation $X//Y$ for two sets $X$ and $Y$ of paths in $kQ$. Namely,

$$X//Y = \{(\gamma, \gamma') \in X \times Y \mid \text{such that } s(\gamma) = s(\gamma') \text{ and } t(\gamma) = t(\gamma')\}.$$

For instance, $Q_n//Q_0$ is the set of oriented cycles of length $n$. In the following, we will denote by $k(B)$, the vector space spanned by a given set $B$.

**Lemma 3.3** (Lemma 2.2 [Cib3]). For the algebra $A := kQ/\langle Q_2 \rangle$, the vector space $\text{Hom}_{E-E}(r \otimes_{E^n} A, A)$ is isomorphic to $k(Q_n//Q_0) \oplus k(Q_n//Q_1)$.

Recall the projective resolution $R(A)$ defined in Lemma 2.1. We denote the image of the $i$-th differential

$$A \otimes_E r \otimes_{E^1} \cdots \otimes_{E^i} A \xrightarrow{d_i} A \otimes_E r \otimes_{E^{i-1}} \cdots \otimes_{E^1} A$$

by $\Omega^i(A)$. In particular, $\Omega^0(A) = A$.

Then we have the following lemma analogous to Lemma 3.1 and 3.3 above.
Lemma 3.4. Let $p \in \mathbb{Z}_{\geq 0}$ and let $\Omega^p(A)$ be the image of the $p$-th differential in the projective resolution $R(A)$. Then $\Omega^p(A)$ has a basis given by $Q_p \cup Q_{p+1}$ and the vector space

$$\text{Hom}_{E-E}(r^{\otimes E^m}, \Omega^p(A))$$

is isomorphic to $k(Q_m/ / Q_p) \oplus k(Q_m/ / Q_{p+1})$.

Proof. The proof is analogous to the ones of Lemma 3.1 and 3.3. For the reader’s convenience, we give the proof here. Since we have $A \cong r \oplus E$,

$$A \otimes_E r^{\otimes E^p} \otimes_E A \cong r \otimes_E r^{\otimes E^p} \otimes_E r \bigoplus E \otimes_E r^{\otimes E^p} \otimes_E r \bigoplus E \otimes_E r^{\otimes E^p} \otimes_E E.$$ 

So we have the following isomorphism of $k$-linear vector spaces

$$A \otimes_E r^{\otimes E^p} \otimes_E A \cong k(Q_{p+2} \cup eQ_{p+1} \cup Q_{p+1}e \cup eQ_pe),$$

where we use the word $e$ to distinguish the differences between $eQ_{p+1}$ and $Q_{p+1}e$, more precisely, $eQ_{p+1}$ is the basis corresponding to $E \otimes_E r^{\otimes E^p} \otimes_E r$, $Q_{p+1}e$ is the one corresponding to $r \otimes_E r^{\otimes E^p} \otimes_E E$ and $eQ_pe$ corresponds to the basis of $E \otimes_E r^{\otimes E^p} \otimes_E E$. Moreover, we have the following commutative diagram,

$$A \otimes_E r^{\otimes E^p-1} \otimes_E A \xrightarrow{d_{p-1}} A \otimes_E r^{\otimes E^p-2} \otimes_E A \cong k(Q_{p+3} \cup eQ_{p+2} \cup Q_{p+2}e \cup eQ_{p+2}e),$$

where

$$d_{p-1} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ \text{id} & 0 & 0 & 0 \\ (-1)^{p-1} \text{id} & 0 & 0 & 0 \\ 0 & (-1)^{p-1} \text{id} & 0 & 0 \end{pmatrix}.$$ 

Since $R(A)$ is exact, the image of the $p$-th differential equals to the kernel of the $(p-1)$-th differential. Let us consider the kernel of $d_{p-1}$. Note that

$$\overline{d}_{p-1}(\gamma) = 0$$

for any $\gamma \in Q_{p+1}$ and

$$\overline{d}_{p-1}(e\gamma_p + (-1)^p \gamma_p e) = 0,$$

for any $\gamma_p \in Q_p$. We also note that

$$\overline{d}_{p-1}(x) = 0$$

if and only if

$$x = \sum_{\gamma \in Q_{p+1}} a_{\gamma} \gamma + \sum_{\gamma' \in Q_p} b_{\gamma'}(e\gamma' + (-1)^p \gamma' e)$$

where $a_{\gamma}, b_{\gamma'} \in k$. So the kernel of the $(p-1)$-th differential has a basis given by $(Q_{p+1} \cup Q_p)$. Thus we have

$$\text{Hom}_{E-E}(r^{\otimes E^m}, \Omega^p(A)) \xrightarrow{\cong} \text{Hom}_{E-E}(k(Q_m), k(Q_{p+1})) \xrightarrow{\cong} k(Q_m/ / Q_p) \oplus k(Q_m/ / Q_{p+1}).$$

5
Now let us translate the differential $\delta$ in Proposition 2.2 through these linear isomorphisms. Let
\[
D_{m,p} : k(Q_m//Q_p) \to k(Q_{m+1}//Q_{p+1})
\]
defined by the following formula
\[
D_{m,p}(\gamma_m, \gamma'_p) := \sum_{a \in Q_1} (a \gamma_m, a \gamma'_p) + (-1)^{p+m+1} \sum_{a \in Q_1} (\gamma_m a, \gamma'_p a).
\] (3)

**Proposition 3.5.** We have the following commutative diagram, where the vertical maps are given by the linear isomorphisms of Lemma 3.4
\[
\begin{array}{ccc}
\Hom_{E-E}(r \otimes E^m, \Omega^p(A)) & \xrightarrow{\delta} & \Hom_{E-E}(r \otimes E^{m+1}, \Omega^p(A)) \\
\downarrow \cong & & \downarrow \cong \\
k(Q_m//Q_p) \oplus k(Q_m//Q_{p+1}) & \xrightarrow{(0, D_{m,p})} & k(Q_{m+1}//Q_p) \oplus k(Q_{m+1}//Q_{p+1}).
\end{array}
\] (4)

**Proof.** Let $(\gamma_m, \gamma'_p) \in Q_m//Q_p$. Then by the vertical isomorphism, $(\gamma_m, \gamma'_p)$ corresponds the element $\eta(\gamma_m, \gamma'_p) \in \Hom_{E-E}(r \otimes E^m, \Omega^p(A))$ defined as follows. For any $\alpha_m \in Q_m$,
\[
\eta(\gamma_m, \gamma'_p)(\alpha_m) := \begin{cases} 
\gamma'_p + (-1)^p \gamma'_p \alpha_m & \text{if } \alpha_m = \gamma_m; \\
0 & \text{otherwise}.
\end{cases}
\]
Then from Remark 2.3 it follows that
\[
\delta(\eta(\gamma_m, \gamma'_p)) = 0.
\]
Similarly, let $(\gamma_m, \gamma'_p) \in Q_m//Q_p$. The vertical isomorphism sends $(\gamma_m, \gamma'_p)$ to the element $\eta(\gamma_m, \gamma'_p) \in \Hom_{E-E}(r \otimes E^m, \Omega^p(A))$, which is defined as follows,
\[
\eta(\gamma_m, \gamma'_p)(\alpha_m) := \begin{cases} 
e Gamma' + (-1)^p \gamma'_p e & \text{if } a \alpha_m = \gamma_m; \\
0 & \text{otherwise}.
\end{cases}
\]
Hence from Remark 2.3 we have, for any $\alpha_{m+1} \in Q_{m+1}$,
\[
\delta(\eta(\gamma_m, \gamma'_p))(\alpha_{m+1}) = \begin{cases} 
a \gamma'_p & \text{if } \alpha_{m+1} = a \gamma_m \text{ for some } a \in Q_1; \\
(-1)^{p+m+1} \gamma'_p a & \text{if } \alpha_{m+1} = \gamma_m a \text{ for some } a \in Q_1; \\
0 & \text{otherwise}.
\end{cases}
\]
Therefore, we have verified the commutativity of Diagram (4).

**Remark 3.6.** From Proposition 3.5 it follows that
\[
HH^m(A, \Omega^p(A)) \cong \frac{k(Q_m//Q_{p+1})}{\text{Im}(D_{m-1,p})} \oplus \text{Ker}(D_{m,p}).
\] (5)

Recall that the connecting homomorphism (cf. [Wang, Proposition 4.7]),
\[
\theta_{m,p} : HH^m(A, \Omega^p(A)) \to HH^{m+1}(A, \Omega^{p+1}(A))
\]
is defined as follows,
\[
\theta_{m,p}(f)(a_{1,m+1}) = -f(a_{1,m}) \otimes E a_{m+1}.
\] (6)
Lemma 3.7. Let $A$ be the radical square zero $k$-algebra of a quiver $Q$ over a field $k$. Then under the isomorphism (5) above, we have the following commutative diagram.

$$HH^m(A, \Omega^p(A)) \xrightarrow{\theta_{m,p}} HH^{m+1}(A, \Omega^{p+1}(A))$$

where

$$E_{m,p+1} : \frac{k(Q_m/Q_{p+1})}{Im(D_{m-1,p})} \xrightarrow{k(Q_{m+1}/Q_{p+2})} \frac{Im(D_{m,p+1})}{Im(D_{m-1,p})}$$

is defined as,

$$E_{m,p+1}(\gamma_m, \gamma'_p) := - \sum_{a \in Q_1} (\gamma_m a, \gamma'_p a).$$

Proof. First, let us prove that $E_{m,p+1}$ is well-defined on $\frac{k(Q_m/Q_{p+1})}{Im(D_{m-1,p})}$ and $E_{m,p}$ is well-defined on $Ker(D_{m,p})$. Let $(\gamma_{m-1}, \gamma'_p) \in Q_{m-1}/Q_p$. We need to prove that

$$E_{m,p+1}(D_{m-1,p}((\gamma_{m-1}, \gamma'_p))) \in Im(D_{m,p+1}).$$

Indeed, we have

$$E_{m,p+1}(D_{m-1,p}((\gamma_{m-1}, \gamma'_p))) = \sum_{a \in Q_1} E_{m,p+1}(a\gamma_{m-1}, a\gamma'_p) + (-1)^{p+m} \sum_{a \in Q_1} E_{m,p+1}(\gamma_{m-1} a, \gamma'_p a)$$

$$= \sum_{a,b \in Q_1} (-a\gamma_{m-1} b, a\gamma'_p b) + (-1)^{p+m} \sum_{a,b \in Q_1} (\gamma_{m-1} ab, \gamma'_p ab)$$

$$= \sum_{a \in Q_1} D_{m,p+1}(\gamma_{m-1} a, \gamma'_p a).$$

Similarly, we can also prove that $E_{m,p}$ is well-defined on $Ker(D_{m,p})$.

Let $(\gamma_m, \gamma'_p) \in Q_m/Q_{p+1}$, then by the vertical isomorphism, $(\gamma_m, \gamma'_p)$ is sent to an element in $HH^m(A, \Omega^p(A))$ represented by the element

$$\eta(\gamma_m, \gamma'_p) \in Hom_E(E \otimes \mathcal{E}^m, \Omega^p(A)),$$

where $\eta(\gamma_m, \gamma'_p)$ is defined in the proof of Proposition 3.5. From Formula (6), it follows that

$$\theta_{m,p}(\eta(\gamma_m, \gamma'_p)) (a_{m+1}) = \begin{cases} -\gamma_{m+1} a & \text{if } a_{m+1} = \gamma_m a \text{ for some } a \in Q_1; \\ 0 & \text{otherwise.} \end{cases}$$

Similarly, let

$$z := \sum_{(\gamma_m, \gamma'_p) \in Q_m/Q_p} x(\gamma_m, \gamma'_p)(\gamma'_m, \gamma'_p)$$

be an element in $Ker(D_{m,p})$. Then $z$ is sent to an element in $HH^m(A, \Omega^p(A))$ represented by

$$\eta_z := \sum_{(\gamma_m, \gamma'_p) \in Q_m/Q_p} x(\gamma_m, \gamma'_p)(\gamma_m, \gamma'_p) \in Hom_E(E \otimes \mathcal{E}^m, \Omega^p(A)).$$
Formula (6) implies
\[ \theta_{m,p}(\eta)(a_{m+1}) = \begin{cases} -\gamma_p a & \text{if } a_{m+1} = \gamma a \text{ for some } a \in Q_1; \\ 0 & \text{otherwise.} \end{cases} \]

Therefore, we have verified the commutativity of Diagram (7).

\[ \text{Proposition 3.8.} \quad \text{Let } k \text{ be a field. Let } A \text{ be the radical square zero } k\text{-algebra of a quiver } Q. \text{ Then for } m \in \mathbb{Z}, \text{ we have} \]
\[ \text{HH}_{sg}^m(A, A) \cong \lim_{\substack{p \in \mathbb{Z}_{>0} \\ m+p>0}} k(Q_{m+p}/Q_{p+1}) \oplus \lim_{\substack{p \in \mathbb{Z}_{>0} \\ m+p>0}} \text{Ker}(D_{m+p,p}). \quad (8) \]

\[ \text{Proof.} \quad \text{From [Wang, Proposition 3.1], it follows that} \]
\[ \text{HH}_{sg}^m(A, A) \cong \lim_{\substack{p \in \mathbb{Z}_{>0} \\ m+p>0}} \text{HH}_{sg}^{m+p}(A, \Omega^p(A)). \]

So from Lemma 3.7 we have
\[ \text{HH}_{sg}^m(A, A) \cong \lim_{\substack{p \in \mathbb{Z}_{>0} \\ m+p>0}} k(Q_{m+p}/Q_{p+1}) \oplus \lim_{\substack{p \in \mathbb{Z}_{>0} \\ m+p>0}} \text{Ker}(D_{m+p,p}). \]

\[ \text{Corollary 3.9.} \quad \text{Let } k \text{ be a field. Let } Q \text{ be a finite and connected quiver and } A \text{ be its radical square zero } k\text{-algebra. If } Q \text{ has no oriented cycles, then for any } m \in \mathbb{Z}, \text{ we have} \]
\[ \text{HH}_{sg}^m(A, A) = 0. \]

\[ \text{Proof.} \quad \text{If } Q \text{ has no oriented cycles, then for } p \gg 0, \]
\[ k(Q_{m+p}/Q_{p+1}) = 0. \]

Hence the right hand side of the isomorphism (8) vanishes. So \( \text{HH}_{sg}^m(A, A) = 0. \)

\[ \text{Remark 3.10.} \quad \text{In general, for } m, p \in \mathbb{Z}_{>0}, \text{ the map } D_{m,p} \text{ is not injective. We are interested in the case when } D_{m,p} \text{ is injective.} \]

\[ \text{Definition 3.11.} \quad \text{Let } c \in \mathbb{Z}_{\geq 0}. \text{ A } c\text{-crown is a quiver with } c\text{-vertices cyclically labelled by the cyclic group of order } c, \text{ and } c \text{ arrows } a_0, \ldots, a_{c-1} \text{ such that } s(a_i) = i \text{ and } t(a_i) = i+1. \]

For instance, a 1-crown is a loop, and a 2-crown is the two-way quiver.

Now let us consider a finite and connected quiver \( Q = (Q_0, Q_1, s, t) \) without sources or sinks (cf. e.g. [Mich], that is, for any vertex \( e \in Q_0 \), there exist arrows \( p, q \in Q_1 \) such that \( s(p) = t(q) = e \). For instance, \( c \)-crowns and the following quivers have no sources or sinks.
Proposition 3.12. Let $k$ be a field. Let $Q = (Q_0, Q_1, s, t)$ be a finite and connected quiver without sources or sinks. Suppose that $Q$ is not a crown. Then we have the following two cases for $m, p \in \mathbb{Z}_{>0}$:

1. If $m = p$, then we have $\text{Ker}(D_{m,m})$ is a one-dimensional $k$-vector space with a basis $\sum_{\gamma_m \in Q_m} (\gamma_m, \gamma_m)$.

2. If $m \neq p$, then the map $D_{m,p} : k(Q_m/Q_p) \to k(Q_{m+1}/Q_{p+1})$

is injective.

Proof. This proof is analogous to the one of [Cib3, Theorem 3.1]. Suppose that $x = \sum_{(\gamma_m, \beta_p) \in Q_m/Q_p} x_{(\gamma_m, \beta_p)}(\gamma_m, \beta_p)$ is an element in $\text{Ker}(D_{m,p})$, where $x_{(\gamma_m, \beta_p)} \in k$. Let us fix an element $(\gamma_m, \beta_p)$ such that $x_{(\gamma_m, \beta_p)} \neq 0$. We consider the contribution of $x_{(\gamma_m, \beta_p)}(\gamma_m, \beta_p)$ in $D_{m,p}(x)$, which is

$$x_{(\gamma_m, \beta_p)} \left( \sum_{a \in Q_1 s(\gamma_m)} (a \gamma_m, a \beta_p) + (-1)^{p+m+1} \sum_{a \in t(\gamma_m) Q_1} (\gamma_ma, \beta_pa) \right). \tag{9}$$

Then from (9), we have the following observation.

Observation 3.13. If $x_{(\gamma_m, \beta_p)} \neq 0$, then for $1 \leq i \leq \min\{m, p\}$,

$$f_i(\gamma_m) = f_i(\beta_p)$$

and

$$l_i(\gamma_m) = l_i(\beta_p),$$

where $f_i(\gamma)$ and $l_i(\gamma)$ denote the first $i$ arrows and last $i$ arrows of a path $\gamma$ respectively.

Suppose $m = p$. From Observation 3.13 above, it follows that $x_{(\gamma_m, \beta_m)} \neq 0$ if and only if $\gamma_m = \beta_m$. Hence we can write $x \in \text{Ker}(D_{m,m})$ as follows,

$$x = \sum_{(\gamma_m, \gamma_m) \in Q_m/Q_p} x_{(\gamma_m, \gamma_m)}(\gamma_m, \gamma_m).$$

We claim that, for any $(\gamma_m, \gamma_m), (\gamma'_m, \gamma'_m) \in Q_m/Q_m$,

$$x_{(\gamma_m, \gamma_m)} = x_{(\gamma'_m, \gamma'_m)}.$$

Indeed, we define an equivalence relation on $Q_m$ as follows. For $\gamma_m := a_1 \cdots a_m, \gamma'_m := b_1 \cdots b_m \in Q_m$,

we say that $\gamma_m \sim \gamma'_m$. 

9
if we have
\[ a_i = b_{i+1} \]
for \( 1 \geq i \geq m - 1 \), or
\[ b_i = a_{i+1} \]
for \( 1 \geq i \geq m - 1 \). Then we can extend \( \sim \) to an equivalence relation \( \sim_c \) on \( Q_m \). Observe that if \( \gamma_m \sim_c \gamma'_m \), then
\[ x(\gamma_m, \gamma_m) = x(\gamma'_m, \gamma'_m). \]
So it is sufficient to show that for any \( \gamma_m, \gamma'_m \in Q_m \),
\[ \gamma_m \sim_c \gamma'_m. \]

Since \( Q \) is a finite and connected quiver without sources or sinks, we have the following two observations.

**Observation 3.14.** If \( \gamma_m \) intersects with \( \gamma'_m \) (that is, there exist a vertex \( e \in Q_0 \) such that \( e \in \gamma_m \) and \( e \in \gamma'_m \)), then
\[ \gamma_m \sim_c \gamma'_m. \]

**Observation 3.15.** Given any path \( \beta \) of length smaller than \( m \), we can extend \( \beta \) to a path of length \( m \).

Now let us prove \( \gamma_m \sim_c \gamma'_m \) for any \( \gamma_m, \gamma'_m \in Q_m \) by the two observations above. Since \( Q \) is finite and connected, there exists a non-oriented path connecting \( \gamma_m \) and \( \gamma'_m \).

Let us use induction on the length \( l(\beta) \) of the (non-oriented) path \( \beta \) connecting \( \gamma_m \) and \( \gamma'_m \). It is clear for \( l(\beta) = 0 \) by Observation 3.14. Assume that
\[ \gamma_m \sim_c \gamma'_m \]
for any \( \gamma_m, \gamma'_m \in Q_m \) such that there exists a path of length smaller than \( (l - 1) \) between them. Now by Observation 3.15, the arrow \( a \) in Diagram (10) can be extended to a path \( \gamma'_m \) of length \( m \). From Observation 3.14 again it follows that
\[ \gamma_m \sim_c \gamma'_m. \]

Note that the length of the (non-oriented) path connecting \( \gamma'_m \) and \( \gamma'_m \) is \( (l - 1) \), thus by induction hypothesis, we have
\[ \gamma'_m \sim_c \gamma'_m. \]

Since \( \sim_c \) is transitive, we have
\[ \gamma_m \sim_c \gamma'_m. \]
for any $\gamma_m, \gamma'_m \in Q_m$. Therefore we have

$$x \in k \left( \sum_{\gamma_m \in Q_m} (\gamma_m, \gamma_m) \right)$$

if $x \in \text{Ker}(D_{m,m})$. On the other hand, for any $\lambda \in k$, we observe that

$$D_{m,m}(\lambda \sum_{\gamma_m \in Q_m} (\gamma_m, \gamma_m)) = 0.$$

Indeed,

$$D_{m,m}(\lambda \sum_{\gamma_m \in Q_m} (\gamma_m, \gamma_m)) = \lambda \sum_{\gamma_{m+1} \in Q_{m+1}} (\gamma_{m+1}, \gamma_{m+1}) - \lambda \sum_{\gamma_{m+1} \in Q_{m+1}} (\gamma_{m+1}, \gamma_{m+1})$$

$$= 0.$$

Therefore

$$\text{Ker}(D_{m,m}) = k \left( \sum_{\gamma_m \in Q_m} (\gamma_m, \gamma_m) \right).$$

Suppose $m \neq p$. Without loss of generality, we may assume that $m < p$. Then there will be two cases.

1. If $p \geq 2m$, let us write $\gamma_m = a_1a_2 \cdots a_m$ (possibly $a_i = a_j$ for some $i, j \in \{1, 2, \cdots, m\}$). If $x_{(\gamma_m, \beta_p)} \neq 0$, then from Observation 3.13 it follows that

$$\beta_p = a_1 \cdots a_mb_1b_2 \cdots b_{p-2}a_1 \cdots a_m$$

is an oriented cycle (possibly with self-intersection).

Since $Q$ is not a crown, there exists a vertex $e$ such that there is an another arrow $c_1$ starting at $e$ and $c_1 \neq b_2$ (or there is an another arrow $c_1$ ending at $e$ and $c_1 \neq b_1$, respectively) (cf. Diagram (11)). Let us just consider the first case, namely where there is an arrow $c_1$ starting at $e$ and $c_1 \neq b_2$, the other case being analogous. From (9), we obtain that

$$(-1)^{p+m+1} x_{(\gamma_m, \beta_p)} (a_1 \cdots a_mb_1, a_1 \cdots a_mb_1 \cdots b_{p-2}a_1 \cdots a_mb_1)$$

is a nonzero summand of $D_{m,p}(x)$. Since $D_{m,p}(x) = 0$, we get that $x_{(\gamma'_m, \beta'_p)} \neq 0$, where

$$(\gamma'_m, \beta'_p) := (a_2 \cdots a_mb_1, a_2 \cdots a_mb_1 \cdots b_{p-2}a_1 \cdots a_mb_1).$$
Thus it follows that
\[
(-1)^{p+m+1} x_{(\gamma_m, \beta_p)}(a_2 \cdots a_m b_1 c_1, a_2 \cdots a_m b_1 \cdots b_{p-2m} a_1 \cdots a_m b_1 c_1) \]
is a nonzero summand of \( D_{m,p}(x) \). By this induction process, we have that
\[
x(c_1 c_2 \cdots c_{m-1} c_m b_2 \cdots b_{p-2m} a_1 \cdots a_m b_1 c_1 \cdots c_{m-1} c_m) \neq 0.
\]
Since \( c_1 \neq b_2 \), from Observation 3.13 it follows that
\[
x(c_1 c_2 \cdots c_{m-1} c_m b_2 \cdots b_{p-2m} a_1 \cdots a_m b_1 c_1 \cdots c_{m-1} c_m) = 0.
\]
Contradiction! Therefore, we have \( x = 0 \). For the remaining cases, we can apply an analogous induction process to argue \( x = 0 \).

2. If \( m < p < 2m \), the proof is similar to the one of Case 1. From Observation 3.13 it follows that \( \gamma_m \) is an oriented cycle (possibly with self-intersection).

Suppose \( s(\gamma_m) = t(\gamma_m) = e \) in Diagram (12). Since \( Q \) is not a crown, there exists an arrow \( c_1 \) such that \( c_1 \neq a_4 \), as we have shown in Diagram (12). Then by the induction process similar to the one in Case 1, we can show that \( x = 0 \).

Therefore, we have proved that
\[
\text{Ker}(D_{m,p}) = 0
\]
for \( m \neq p \). So the proof is completed.

**Proposition 3.16.** Let \( k \) be a field. Let \( Q \) be a finite and connected quiver without sources or sinks and \( A \) be its radical square zero \( k \)-algebra. Suppose that \( Q \) is not a crown. Then we have the following:

1. For \( m, p \in \mathbb{Z}_{>0} \),
\[
\text{HH}^m(A, \Omega^p(A)) = \begin{cases} 
\frac{k(Q_m/Q_{m+1})}{\text{Im}(D_{m-1,p})} & \text{if } m \neq p, \\
\frac{k(Q_m/Q_{m+1})}{\text{Im}(D_{m-1,m})} \oplus \text{Ker}(D_{m,m}) & \text{if } m = p.
\end{cases}
\]

2. The connecting homomorphism
\[
\theta_{m,p} : \text{HH}^m(A, \Omega^p(A)) \rightarrow \text{HH}^{m+1}(A, \Omega^{p+1}(A))
\]
is injective for any \( m, p \in \mathbb{Z}_{>0} \).
3. For any \( m \in \mathbb{Z} \), the singular Hochschild cohomology \( \text{HH}^m_{sg}(A, A) \) has a filtration
\[
\cdots \subset \text{HH}^{m+p}(A, \Omega^p(A)) \subset \text{HH}^{m+p+1}(A, \Omega^{p+1}(A)) \subset \cdots
\]
Moreover, this filtration respects the Gerstenhaber algebra structure, that is, for \( m, n \in \mathbb{Z} \),
\[
\text{HH}^{m+p}(A, \Omega^p(A)) \cup \text{HH}^{n+q}(A, \Omega^q(A)) \subset \text{HH}^{m+p+n+q}(A, \Omega^{p+q}(A)),
\]
\[
[\text{HH}^{m+p}(A, \Omega^p(A)), \text{HH}^{n+q}(A, \Omega^q(A))] \subset \text{HH}^{m+p+n+q-1}(A, \Omega^{p+q}(A)).
\]

Proof. It is sufficient to verify that \( \theta_{m,p} \) is injective for \( m, p \in \mathbb{Z}_{>0} \). Recall that we have the long exact sequence
\[
\cdots \rightarrow \text{HH}^m(A, A \otimes r^{\otimes p} \otimes A) \rightarrow \text{HH}^m(A, \Omega^p(A)) \xrightarrow{\theta_{m,p}} \text{HH}^{m+1}(A, \Omega^{p+1}(A)) \rightarrow \cdots
\]
where \( d \) is induced by the differential in the resolution \( R(A) \) (cf. Lemma 2.41). Hence \( \theta_{m,p} \) is injective if and only if \( d = 0 \). Now let us show that \( d = 0 \) for any \( m, p \in \mathbb{Z}_{>0} \). Note that \( \text{Hom}_{E-E}(r^{\otimes m}, A \otimes E r^{\otimes p} \otimes E A) \) has a decomposition with respect to the decomposition \( A \cong E \oplus r \). Namely,
\[
\text{Hom}_{E-E}(r^{\otimes m}, A \otimes E r^{\otimes p} \otimes E A) \cong \bigoplus \text{Hom}_{E-E}(r^{\otimes m}, r \otimes E r^{\otimes p} \otimes E A)
\]
\[
\bigoplus \text{Hom}_{E-E}(r^{\otimes m}, r \otimes E r^{\otimes p} \otimes E E)
\]
\[
\bigoplus \text{Hom}_{E-E}(r^{\otimes m}, E \otimes E r^{\otimes p} \otimes E E),
\]
hence \( \text{Hom}_{E-E}(r^{\otimes m}, A \otimes E r^{\otimes p} \otimes E A) \) has a basis
\[
S_{m,p} := (Q_m/\langle Q_{p+2} \rangle) \cup (Q_m/\langle eQ_{p+1} \rangle) \cup (Q_m/\langle Q_{p+1} e \rangle) \cup (Q_m/\langle Q_p \rangle),
\]
where we use the word \( e \) to distinguish the differences between \( (Q_m/\langle eQ_{p+1} \rangle) \) and \( (Q_m/\langle Q_{p+1} e \rangle) \), more precisely, \( (Q_m/\langle eQ_{p+1} \rangle) \) is the basis corresponding to \( \text{Hom}_{E-E}(r^{\otimes m}, E \otimes E r^{\otimes p} \otimes E r) \) and \( (Q_m/\langle Q_{p+1} e \rangle) \) is the one corresponding to \( \text{Hom}_{E-E}(r^{\otimes m}, r \otimes E r^{\otimes p} \otimes E E) \). Moreover, we can write down the differential.

\[
\begin{pmatrix}
0 & D_{m,p+1} & 0 \\
0 & 0 & E_{m,p} \\
0 & 0 & 0
\end{pmatrix}
\]

where
\[
D_{m,p+1}(\gamma_m, e\beta_{p+1}) := \sum_{a \in Q_1} (a\gamma_m, a\beta_{p+1});
\]
\[
D'_{m,p+1}(\gamma_m, \beta_{p+1} e) := \sum_{a \in Q_1} (-1)^{m+1}(\gamma_m a, \beta_{p+1} a);
\]
\[
E_{m,p}(\gamma_m, \beta_p) := \sum_{a \in Q_1} (a\gamma_m, a\beta_{p+1});
\]
\[
E'_{m,p}(\gamma_m, \beta_p) := \sum_{a \in Q_1} (-1)^{m+1}(\gamma_m a, \beta_{p+1} a).
\]
Now suppose $x \in \text{HH}^m(A, A \otimes r^p \otimes A)$ is a nonzero element. Then from Diagram (13), we can write $x$ as follows,

$$x = \sum_{(\gamma_m, \beta_{p+1}) \in Q_m/Q_{p+1}} x_{(\gamma_m, \beta_{p+1})} + \sum_{(\gamma_m, \beta_{p+1}) \in Q_m/Q_{p+1}} x_{(\gamma_m, \epsilon \beta_{p+1})} + \sum_{(\gamma_m, \beta_{p+1}) \in Q_m/Q_{p+1}} x_{(\gamma_m, \beta_{p+1} \epsilon)}.$$

Now $\delta(x) = 0$ is equivalent to

$$\sum_{a \in Q_1} (x_{(\gamma_m, \epsilon \beta_{p+1})} (a \gamma_m, a \beta_{p+1}) + (-1)^{p+m} x_{(\gamma_m, \beta_{p+1} \epsilon)} (\gamma_m a, \beta_{p+1} a)) = 0. \quad (14)$$

We have the following observation.

**Observation 3.17.** Suppose that $x_{(\gamma_m, \epsilon \beta_{p+1})} \neq 0$.

Then $t(\gamma_m) = t(\beta_{p+1})$. Moreover, for those $a \in Q_1$ such that $a \gamma_m \neq 0$,

$$x(a f_{m-1}(\gamma_m), a f_p(\beta_{p+1} \epsilon)) = (-1)^{p+m} x_{(\gamma_m, \epsilon \beta_{p+1})},$$

where we denote by $f_{m-1}(\gamma_m)$ the path formed by the first $m-1$ arrows in $\gamma_m$. Similarly, suppose that $x_{(\gamma'_m, \beta'_p \epsilon)} \neq 0$.

Then $s(\gamma'_m) = s(\beta'_p \epsilon)$. Moreover, for those $a \in Q_1$ such that $\gamma'_m a \neq 0$,

$$x(l_{m-1}(\gamma'_m), a, e_p(\beta'_p \epsilon)) = (-1)^{m+p} x_{(\gamma'_m, \beta'_p \epsilon)},$$

where we denote by $l_{m-1}(\gamma'_m)$ the path formed by the last $m-1$ arrows in $\gamma'_m$.

Now let us compute

$$d(x) = \sum x_{(\gamma_m, \epsilon \beta_{p+1})} + \sum (-1)^{p} x_{(\gamma_m, \beta_{p+1} \epsilon)} \in \text{HH}^m(A, \Omega^p(A)).$$

We claim that

$$d(x) = (-1)^m \sum x_{(\gamma_m, \epsilon \beta_{p+1})} D_{m-1,p}(f_{m-1}(\gamma_m), f_p(\beta_{p+1})).$$
Indeed, we have that
\[
\sum_{(\gamma_m, e\beta_{p+1})} x_{(\gamma_m, e\beta_{p+1})} D_{m-1, p}(f_{m-1}(\gamma_m), f_p(\beta_{p+1}))
\]
\[
= \sum_{(\gamma_m, e\beta_{p+1}), \ a \in Q_1} x_{(\gamma_m, e\beta_{p+1})}(af_{m-1}(\gamma_m), f_p(\beta_{p+1})a) + (-1)^{p+m} \sum_{(\gamma_m, e\beta_{p+1}), \ a \in Q_1} x_{(\gamma_m, e\beta_{p+1})}(af_{m-1}(\gamma_m), f_p(\beta_{p+1})a)
\]
\[
= (-1)^m \sum_{(\gamma_m, e\beta_{p+1}), \ a \in Q_1} x_{(\gamma_m, e\beta_{p+1})}(af_{m-1}(\gamma_m), f_p(\beta_{p+1})a) + (-1)^m \sum_{(\gamma_m, e\beta_{p+1}), \ a \in Q_1} x_{(\gamma_m, e\beta_{p+1})}(af_{m-1}(\gamma_m), f_p(\beta_{p+1})a)
\]
\[
= (-1)^m d(x),
\]
where the second identity comes from Observation 3.13. So it follows that \(d(x) = 0\) in \(\text{HH}^m(A, \Omega^p(A))\). Therefore we have showed that \(\theta_{m,p}\) is injective.

As a corollary, we have the following result.

**Corollary 3.18.** Let \(Q\) be a finite and connected quiver without sources or sinks and \(A\) be its radical square zero \(k\)-algebra over a field \(k\). Suppose that \(Q\) is not a crown. Then for any \(m, n, p, q \in \mathbb{Z}_{>0}\) such that \(m \neq p\) and \(n \neq q\),
\[
\text{HH}^m(A, \Omega^p(A)) \cup \text{HH}^n(A, \Omega^q(A)) = 0.
\]
In particular, for \(m, n \in \mathbb{Z}\) and \(mn \neq 0\),
\[
\text{HH}^m_{sg}(A, A) \cup \text{HH}^n_{sg}(A, A) = 0.
\]

**Proof.** From Proposition 3.16 it follows that
\[
\text{HH}^m(A, \Omega^p(A)) = \frac{k(Q_m/\mathbb{Q}_{p+1})}{\text{Im}(D_{m-1, p})},
\]
for \(m \neq p\). Note that we have the following identity on the level of chains, for \(m \neq p\) and \(n \neq q\),
\[
k(Q_m/\mathbb{Q}_{p+1}) \cup k(Q_n/\mathbb{Q}_{q+1}) = 0.
\]
Hence on the level of cohomology groups, we have for \(m \neq p\) and \(n \neq q\),
\[
\text{HH}^m(A, \Omega^p(A)) \cup \text{HH}^n(A, \Omega^q(A)) = 0.
\]
In particular,
\[
\text{HH}^m_{sg}(A, A) \cup \text{HH}^n_{sg}(A, A) = 0,
\]
for \(mn \neq 0\).
Remark 3.19. From Corollary 3.18 above, it follows that in general,

$$(\text{HH}^*(A, A), \cup, [\cdot, \cdot])$$

and

$$(\text{HH}^*_{sg}(A, A), \cup, [\cdot, \cdot])$$

have no BV algebra structures since the cup products vanish, however, the Lie brackets does not vanish in general (cf. Section 5).

In the rest of this section, we will consider the Gerstenhaber bracket $[\cdot, \cdot]$ (cf. [Wang]) on the total space

$$\bigoplus_{m,p \in \mathbb{Z}_{\geq 0}} \text{Hom}_{E-E}(r \otimes E^m, \Omega^p(A)).$$

First let us recall the construction of the Gerstenhaber bracket $[\cdot, \cdot]$ on $\text{HH}^*_{sg}(A, A)$. Let $k$ be a field and $A$ be a finite dimensional $k$-algebra (not necessarily radical square zero) with a Wedderburn-Malcev decomposition $A := E \oplus r$. For $m \in \mathbb{Z}_{>0}$ and $p \in \mathbb{Z}_{\geq 0}$, denote

$$C^m(A, \Omega^p(A)) := \text{Hom}_{E-E}(r \otimes E^m, \Omega^p(A)).$$

Then we have a double complex $C^*(A, \Omega^*(A))$. Take two elements $f \in C^m(A, \Omega^p(A))$ and $g \in C^n(A, \Omega^q(A))$. Denote

$$\bullet, g := \begin{cases} d((f \otimes \text{id} \otimes \text{id}) \otimes \text{id} - g \otimes \text{id} \otimes \text{id}) \otimes 1) & \text{if } 1 \leq i \leq m, \\ d((\text{id} \otimes f \otimes \text{id} \otimes \text{id}) \otimes 1) & \text{if } -q \leq i \leq -1, \end{cases}$$

where $\otimes$ represents $\otimes_E$. We also denote

$$f \bullet g := \sum_{i=1}^{m} (-1)^{p+q+i-1}(q-n-1)f \bullet, g + \sum_{i=1}^{q} (-1)^{p+q+i-1}(p-m-1)f \bullet, g.$$

Then we define

$$[f, g] := f \bullet g - (-1)^{(m-p-1)(n-q-1)}g \bullet f. \quad (15)$$

Note that

$$[f, g] \in C^{m+n-1}(A, \Omega^{p+q}(A)).$$

Then from Proposition 4.6 in [Wang], it follows that $[\cdot, \cdot]$ defines a differential graded Lie algebra structure on the total complex

$$\bigoplus_{m \in \mathbb{Z}_{>0}, p \in \mathbb{Z}_{\geq 0}} C^m(A, \Omega^p(A))$$

and thus $[\cdot, \cdot]$ defines a graded Lie algebra structure on the cohomology groups

$$\bigoplus_{m \in \mathbb{Z}_{>0}, p \in \mathbb{Z}_{\geq 0}} \text{HH}^m(A, \Omega^p(A)).$$

Let $A$ be a radical square zero algebra with a Wedderburn-Malcev decomposition $A = E \oplus r$. Then given

$$f \in \text{Hom}_{E-E}(r \otimes E^m, \Omega^p(A))$$

16
and

\[ g \in \text{Hom}_{E^{-E}}(r \otimes^p n, \Omega^p(A)) \]

the Gerstenhaber bracket \([\cdot, \cdot]\) can be written as

\[
[f, g](a_{1,m+n-1}) = \sum_{i=1}^{m} (-1)^{(i-1)(q-n-1)}(f \otimes \text{id})(a_{1,i-1} \otimes g(a_{i,i+n-1} \otimes a_{i+n,m+n-1}) +
\sum_{i=1}^{q} (-1)^{(p-m-1)}(\text{id} \otimes f)(g(a_{1,n}) \otimes a_{n+1,m+n-1}) - (-1)^{(m-p-1)(n-q-1)}
\sum_{i=1}^{n} (-1)^{(i-1)(p-m-1)}(g \otimes \text{id})(a_{i,i-1} \otimes f(a_{i,i+m-1}) \otimes a_{i+m,m+n-1}) -
(-1)^{(m-p-1)(n-q-1)} \sum_{i=1}^{p} (-1)^{(q-n-1)}(\text{id} \otimes g)(f(a_{1,m}) \otimes a_{m+1,m+n-1}),
\]  

(16)

for any \(a_{1,m+n-1} \in r \otimes^m n \otimes n - 1\). In particular, for \(m = n = 1\), we have

\[
[f, g](a) = \sum_{i=0}^{q} (-1)^{ip}(\text{id} \otimes f)(g(a)) - (-1)^{pq} \sum_{i=0}^{p} (-1)^{iq}(\text{id} \otimes \text{id})(f(a)).
\]  

(17)

Next we consider the radical square zero algebra for a quiver \(Q\). First, let us follow \cite{San1} and \cite{San2} to introduce the notation \(\diamond\): Given two paths \(\alpha \in Q_m\) and \(\beta \in Q_n\). Suppose that

\[\alpha = a_1a_2 \cdots a_m,\]
\[\beta = b_1b_2 \cdots b_n,\]

where \(a_i, b_j \in Q_1\). Let \(i = 1, \cdots, m\). If \(a_i/\beta\), we denote by

\[\alpha \diamond_i \beta\]

the path in \(Q_{m+n-1}\) obtained by replacing the arrow \(a_i\) by the path \(\beta\). Namely, we define

\[\alpha \diamond_i \beta := \begin{cases} a_1 \cdots a_{i-1}b_1 \cdots b_n a_{i+1} \cdots a_m, & \text{if } a_i/\beta, \\ 0 & \text{otherwise}. \end{cases}\]

**Lemma 3.20.** Let \(k\) be a field. Let \(Q\) be any finite and connected quiver. Then via the linear isomorphism in Lemma \ref{lem:linear_isomorphism}, we have the following

1. Let

\[(x, y) \in k(Q_1/\beta) \subset \text{Hom}_{E^{-E}}(r, A)\]

and

\[(\gamma_m, \beta_{p+1}) \in k(Q_m/\beta_{p+1}) \subset \text{Hom}_{E^{-E}}(r \otimes^m, \Omega^p(A))\]

Then

\[[(x, y), (\gamma_m, \beta_{p+1})] = \sum_{i=1}^{p+1} \delta_{b_i,x}(\gamma_m, \beta_{p+1} \diamond_i y) - \sum_{i=1}^{m} \delta_{a_i,y}(\gamma_m \diamond_i x, \beta_{p+1}),\]

where \(a_i, b_i\) are the \(i\)-th arrow in \(\gamma_m\) and \(\beta_{p+1}\), respectively.
2. Let
\[(x, \gamma_{p+1}) \in k(Q_1/Q_{p+1}) \subset \text{Hom}_{E-(r, \Omega_p(A))}\]
and
\[(y, \beta_{q+1}) \in k(Q_1/Q_{q+1}) \subset \text{Hom}_{E-(r, \Omega_q(A))}.\]
Then
\[[x, \gamma_{p+1}], (y, \beta_{q+1}) = p \sum_{i=1}^{q+1} (-1)^i \delta_{x,b_i}(y, \beta_{q+1} \circ_i \gamma_{p+1}) -
(-1)^{pq} \sum_{i=1}^{p+1} (-1)^{i-q} \delta_{y,a_i}(x, \gamma_{p+1} \circ_i \beta_{q+1}),\]
where \(a_i, b_i\) are the \(i\)-th arrow in \(\gamma_{p+1}\) and \(\beta_{q+1}\), respectively.

Proof. This is a direct consequence of Formula (16).

Remark 3.21. For general cases, the formula for the Gerstenhaber bracket \([\cdot, \cdot]\) is quite complicate. But however, we can use PROP theory to describe it in Section 6.

4 \(c\)-crown

We shall use the notion of a \(c\)-crown from Definition 3.11. We will study the case of particular \(c\)-crowns in the following subsections. In this section, we assume that the base field \(k\) is not of characteristic two.

4.1 The case: \(c = 1\)

Let us first consider the 1-crown (i.e. the one loop quiver).

\[Q := \bullet \xrightarrow{a} \bullet\]

Its radical square zero algebra is \(A = k[a]/(a^2)\), the algebra of dual numbers. Since \(A\) is a commutative symmetric algebra, from [Buch Corollary 6.4.1], we have that

\[\text{HH}^m_{\text{sg}}(A, A) = \begin{cases} 
\text{HH}^m(A, A) & \text{for } m > 0, \\
\text{Tor}^{A^e}_{-m-1}(A, A) & \text{for } m < -1.
\end{cases}\]

Proposition 4.1 (Proposition 3.4 [Cib3]). Let \(Q\) be the one loop quiver and \(A\) be its radical square zero algebra over a field \(k\). Assume that \(k\) is not of characteristic two. Then for every \(n > 0\), we have

\[\dim \text{HH}^n(A, A) = 1.\]

Remark 4.2. Since \(\text{HH}^n(A, A) \cong \text{Tor}^{A^e}_m(A, A)^e\), we also have

\[\dim \text{Tor}^{A^e}_m(A, A) = 1.\]
Lemma 4.3. For any \( n \in \mathbb{Z} \), we have
\[
\dim \text{HH}^n_{sg}(A, A) = 1.
\]

Proof. From Proposition 4.1 and Remark 4.2, it is sufficient to show that
\[
\dim \text{HH}^0_{sg}(A, A) = \dim \text{HH}^{-1}_{sg}(A, A) = 1.
\]
Recall that from [Buch, Corollary 6.4.1], we have an exact sequence,
\[
0 \longrightarrow \text{HH}^{-1}_{sg}(A, A) \longrightarrow \text{HH}_0(A, A) \longrightarrow \text{HH}^0(A, A) \longrightarrow \text{HH}^0_{sg}(A, A) \longrightarrow 0
\]
where \( \mu_a \) is the map multiplied by \( a \). Hence we have
\[
\dim \text{Ker}(\mu_a) = \dim \text{HH}^{-1}_{sg}(A, A) = 1
\]
and
\[
\dim \text{coker}(\mu_a) = \dim \text{HH}^0_{sg}(A, A) = 1.
\]

Remark 4.4. The graded Lie algebra structure on the positive side (i.e. the Hochschild cohomology HH\(^{\ast}\)(A, A)) of HH\(^{\ast}_{sg}\)(A, A) has been investigated in [San1]. Next we will describe the graded Lie algebra structure on the negative side.

Recall that we have for \( m \in \mathbb{Z}_{\geq 0} \),
\[
\text{HH}^{-m}_{sg}(A, A) \cong \text{HH}^1(A, \Omega^{m+1}(A))
\]
since \( A \) is a symmetric algebra. From Remark 3.6 it follows that
\[
\text{HH}^1(A, \Omega^{m+1}(A)) \cong \frac{k(Q_1//Q_{m+2})}{\text{Im}(D_{0,m+1})} \oplus \text{Ker}(D_{1,m+1}).
\]
Recall that
\[
D_{i,m+1} : k(Q_i//Q_{m+1}) \rightarrow k(Q_{i+1}//Q_{m+2})
\]
is defined as follows,
\[
D_{i,m+1}(\gamma_i, \beta_{m+1}) := \sum_{a \in Q_1} (a_{\gamma_i}, a_{\beta_{m+1}}) + (-1)^{i+m} \sum_{a \in Q_1} (\gamma_i a, \beta_{m+1} a).
\]
Note that if \( m \) is odd, then
\[
D_{0,m+1} = 0
\]
and \( D_{1,m+1} \) is a bijection. Similarly if \( m \) is even, then \( D_{0,m+1} \) is a bijection and
\[
D_{1,m+1} = 0.
\]
Hence we have
\[
\text{HH}^1(A, \Omega^{m+1}(A)) = \begin{cases} 
  k(Q_i//Q_{m+2}) & \text{if } m \text{ is odd}, \\
  k(Q_i//Q_{m+1}) & \text{if } m \text{ is even}.
\end{cases}
\]
(18)
Proposition 4.5. Let $Q$ be the one loop quiver

$$Q := \bullet \xrightarrow{a} \bullet$$

and $A$ be its radical square zero algebra over a field $k$. Assume that $k$ is not of characteristic two. Then for $m, n \in \mathbb{Z}_{\geq 1}$, we have the following cases:

1. If both $m$ and $n$ are odd, then we have the following commutative diagram,

$$\text{HH}_{1}^{1}(A, \Omega^{m+1}(A)) \times \text{HH}_{1}^{1}(A, \Omega^{n+1}(A)) \xrightarrow{[-]} \text{HH}_{1}^{1}(A, \Omega^{m+n+2}(A))$$

$$\cong$$

$$k(Q_1//Q_{m+2}) \times k(Q_1//Q_{n+2}) \xrightarrow{\{\cdot, \cdot\}} k(Q_1//Q_{m+n+3})$$

where the bracket $\{\cdot, \cdot\}$ is defined as follows,

$$\{(a, a^{m+2}), (a, a^{n+2})\} = (n - m)(a, a^{m+n+3}).$$

2. If both $m$ and $n$ are even, then we have the following commutative diagram,

$$\text{HH}_{sg}^{1}(A, \Omega^{m+1}(A)) \times \text{HH}_{sg}^{1}(A, \Omega^{n+1}(A)) \xrightarrow{[-]} \text{HH}_{1}^{1}(A, \Omega^{m+n+2}(A))$$

$$\cong$$

$$k(Q_1//Q_{m+1}) \times k(Q_1//Q_{n+1}) \xrightarrow{\{\cdot, \cdot\}} k(Q_1//Q_{m+n+3})$$

where

$$\{(a, a^{m+1}), (a, a^{n+1})\} = 0.$$

3. If $m$ is even and $n$ is odd, then the following diagram commutes,

$$\text{HH}_{sg}^{1}(A, \Omega^{m+1}(A)) \times \text{HH}_{sg}^{1}(A, \Omega^{n+1}(A)) \xrightarrow{[-]} \text{HH}_{1}^{1}(A, \Omega^{m+n+2}(A))$$

$$\cong$$

$$k(Q_1//Q_{m+1}) \times k(Q_1//Q_{n+2}) \xrightarrow{\{\cdot, \cdot\}} k(Q_1//Q_{m+n+2})$$

where

$$\{(a, a^{m+1}), (a, a^{n+2})\} := -m(a, a^{m+n+2}).$$

Proof. The assertion (1) comes from Lemma 3.20. Now let us prove the assertion (2). Recall that $(a, a^{m+1})$ represents the element in $\text{Hom}_{E-E}(r, \Omega^{m+1}(A))$, which sends $a$ to

$$ea^{m+1} + (-1)^{m+1}a^{m+1}e \in \Omega^{m+1}(A).$$

From Formula (17) it follows that

$$[(a, a^{m+1}), (a, a^{n+1})] \in k(Q_1//Q_{m+n+2}) \subset \text{Hom}_{E-E}(r, \Omega^{m+n+2}(A)).$$

So from Formula (18), we have

$$[(a, a^{m+1}), (a, a^{n+1})] = 0$$

20
in $\text{HH}^1(A, \Omega^{m+n+2}(A))$. It remains to verify the assertion (3). From Formula (17) again, we have

$$[(a, a^{m+1}), (a, a^{n+2})](a) = \sum_{i=0}^{n+1} (-1)^{i(m+1)}(\text{id} \otimes (a, e^{a^{m+1}} + (-1)^{m+1}a^{m+1}e))(a^{n+2}) -$$

$$\sum_{i=0}^{m+1} (-1)^{(i+m+1)(n+1)}(\text{id} \otimes (a, a^{n+2}))(e^{a^{m+1}} + (-1)^{m+1}a^{m+1}e)$$

$$= -m(e^{a^{m+n+2}} - a^{m+n+2}e),$$

hence we have

$$[(a, a^{m+1}), (a, a^{n+2})] = -m(a, a^{m+n+2}) \in \text{HH}^1(A, \Omega^{m+n+2}(A)).$$

Therefore, we have completed the proof. ■

From [Wang, Corollary 5.20] it follows that $(\text{HH}^*_{sg}(A, A), \cup, [\cdot, \cdot], \Delta)$ is a BV algebra.

Now let us describe the BV algebra structure in this concrete example. First, we can write the formula for the Connes $B$-operator.

**Lemma 4.6.** Let $Q$ be the one loop quiver and $A$ be its radical square zero algebra. Then for $m \in \mathbb{Z}_{\geq 0}$, we have

1. If $m$ is even, then we have the following commutative diagram,

$$\begin{array}{c}
\text{HH}^1(A, \Omega^{m+1}(A)) \\ \downarrow \cong \end{array} \xrightarrow{B} \begin{array}{c}
\text{HH}^1(A, \Omega^{m+2}(A)) \\ \downarrow \cong \\
\text{k}(Q_1//Q_{m+1}) \\ \Delta = 0 \\
\text{k}(Q_1//Q_{m+3})
\end{array}$$

2. If $m$ is odd, then the following diagram commutes,

$$\begin{array}{c}
\text{HH}^1(A, \Omega^{m+1}(A)) \\ \downarrow \cong \end{array} \xrightarrow{B} \begin{array}{c}
\text{HH}^1(A, \Omega^{m+2}(A)) \\ \downarrow \cong \\
\text{k}(Q_1//Q_{m+2}) \\ \Delta \\
\text{k}(Q_1//Q_{m+2})
\end{array}$$

where $\Delta$ is defined as follows,

$$\Delta((a, a^{m+2})) = m(a, a^{m+2}).$$

**Proof.** The proof is completely analogous to the one of Proposition [4.5]. ■

Similarly, we can also write down the formula for the cup product $\cup$.

**Lemma 4.7.** Let $Q$ be the one loop quiver and $A$ be its radical square zero $k$-algebra, where $k$ is not of characteristic two. Then we have the following cases for $m, n \in \mathbb{Z}_{>0}$,

1. If both $m$ and $n$ are odd, then we have the following commutative diagram,

$$\begin{array}{c}
\text{HH}^1_{sg}(A, \Omega^{m+1}(A)) \times \text{HH}^1_{sg}(A, \Omega^{n+1}(A)) \\ \downarrow \cong \end{array} \xrightarrow{\cup} \begin{array}{c}
\text{HH}^1(A, \Omega^{m+n+1}(A)) \\ \downarrow \cong \\
k(Q_1//Q_{m+2}) \times k(Q_1//Q_{n+2}) \\ 0 \\
k(Q_1//Q_{m+n+1})
\end{array}$$

where we used the connecting isomorphism

$$\theta_{1, m+n+1} : \text{HH}^1(A, \Omega^{m+n+1}(A)) \to \text{HH}^2(A, \Omega^{m+n+2}(A)).$$
2. If both $m$ and $n$ are even, then
\[
\HH_{sg}(A, \Omega^{m+1}(A)) \times \HH_{sg}(A, \Omega^{n+1}(A)) \cong \HH^1(A, \Omega^{m+n+1})
\]
where
\[
k(Q_1//Q_{m+1}) \times k(Q_1//Q_{n+1}) \cong k(Q_1//Q_{m+n+1})
\]
\[
(a, a^{m+1}) \cup' (a, a^{n+1}) = (a, a^{m+n+1}).
\]

3. If $m$ is even and $n$ is odd, then
\[
\HH_{sg}(A, \Omega^{m+1}(A)) \times \HH_{sg}(A, \Omega^{n+1}(A)) \cong \HH^1(A, \Omega^{m+n+1})
\]
where
\[
k(Q_1//Q_{m+1}) \times k(Q_1//Q_{n+2}) \cong k(Q_1//Q_{m+n+2})
\]
\[
(a, a^{m+1}) \cup' (a, a^{n+2}) = -(a, a^{m+n+2}).
\]

Proof. The proof is completely analogous to the one of Proposition 4.5.

Remark 4.8. Let us recall the Witt algebra $\mathcal{W}$ (cf. e.g. [Zim]). As a vector space,
\[
\mathcal{W} = \bigoplus_{n \in \mathbb{Z}} k\langle L_n \rangle,
\]
where $k\langle L_n \rangle$ is a one-dimensional $k$-vector space with a basis $L_n$. The Lie bracket is defined
\[
[L_m, L_n] = (m - n) L_{m+n}.
\]
Clearly, the even part of $\mathcal{W}$
\[
\mathcal{W}^{\text{even}} := \bigoplus_{m \in \mathbb{Z}} k\langle L_{2m} \rangle
\]
is a Lie subalgebra of $\mathcal{W}$. Let us construct a natural representation $\mathcal{M}$ of $\mathcal{W}$. As a vector space,
\[
\mathcal{M} = \bigoplus_{n \in \mathbb{Z}} k\langle M_n \rangle,
\]
where $k\langle M_n \rangle$ is a one-dimensional $k$-vector space with a basis $M_n$. Define the action of $L_m$ on $M_n$ as follows:
\[
[L_m, M_n] := -n M_{m+n}.
\]
Then one can check that it induces a representation on $\mathcal{M}$ of $\mathcal{W}$. Denote
\[
\mathcal{M}^{\text{even}} := \bigoplus_{n \in \mathbb{Z}} k\langle M_{2n+1} \rangle.
\]
Trivially $\mathcal{M}^{\text{even}}$ is a representation of $\mathcal{W}^{\text{even}}$.

We can give a geometric interpretation of the Witt algebra $\mathcal{W}$ and its representation $\mathcal{M}$. Recall that $\mathcal{W}$ can be considered as the Lie algebra of vector fields with Laurent polynomial coefficients, i.e. those of the form
\[
f(t) \frac{d}{dt},
\]
with \( f(t) \in k[t, t^{-1}] \). Then
\[
L_n := -t^{n+1} \frac{d}{dt}
\]
for \( n \in \mathbb{Z} \) is a basis of \( \mathcal{W} \). It is straightforward to verify that
\[
[L_m, L_n] = (m - n)L_{m+n}.
\]
Note that \( \mathcal{M} \) can be considered as the Laurent polynomial ring \( k[t, t^{-1}] \). Clearly it has a basis
\[
M_n := t^n
\]
for \( n \in \mathbb{Z} \) and \( \mathcal{W} \) acts on \( \mathcal{M} \) by derivations, that is, for any \( g(t) \in k[t, t^{-1}] \), we define
\[
[L_m, g(t)] := -t^{m+1} \frac{dg(t)}{dt}.
\]
Hence we have,
\[
[L_m, M_n] = -nt^{n+m} = -nM_{m+n}.
\]
It is straightforward to verify that this action defines a Lie module structure on \( \mathcal{M} \) over the Witt algebra \( \mathcal{W} \). On the other hand, note that \( \mathcal{M} \) is a commutative algebra and we can also define an action of \( \mathcal{M} \) on \( \mathcal{W} \) as follows. For \( f(t) \in \mathcal{M} \) and \( g(t) \frac{d}{dt} \in \mathcal{W} \),
\[
f(t) \cdot g(t) \frac{d}{dt} := f(t)g(t) \frac{d}{dt} \in \mathcal{W}.
\]
In particular, we have
\[
M_n \cdot L_m = L_{m+n}.
\]
Clearly, by this action \( \mathcal{W} \) is a module over the commutative algebra \( \mathcal{M} \).

Moreover, we can construct a BV algebra \( (\mathcal{M} \times \mathcal{W}, \cup, [, ,], \Delta) \) as follows. The grading is
\[
(\mathcal{M} \times \mathcal{W})_n := \begin{cases} 
M_m & \text{if } n = 2m; \\
L_m & \text{if } n = 2m + 1.
\end{cases}
\]
As a graded commutative algebra,
\[
(\mathcal{M} \times \mathcal{W}, \cup) \cong (\mathcal{M} \times \mathcal{W}, \cdot)
\]
and as a graded Lie algebra,
\[
(\mathcal{M} \times \mathcal{W}, [, ,]) \cong (\mathcal{M} \times \mathcal{W}, [, ,]).
\]
The BV operator is defined as follows:
\[
\Delta_{2m}(M_m) := 0, \\
\Delta_{2m+1}(L_m) := -mM_m.
\]
Then one can check that \( (\mathcal{M} \times \mathcal{W}, \cup, [, ,], \Delta) \) is a BV algebra. Similarly, we can also construct a BV algebra \( (\mathcal{M}^{\text{even}} \times \mathcal{W}^{\text{even}}, \cup, [, ,], \Delta) \). The grading is
\[
(\mathcal{M}^{\text{even}} \times \mathcal{W}^{\text{even}})_n = \begin{cases} 
k(L_{n-1}) & \text{if } n \text{ is odd}, \\
k(M_n) & \text{if } n \text{ is even}.
\end{cases}
\]
As a graded commutative algebra,
\[(\mathcal{M}^{\text{even}} \times \mathcal{W}^{\text{even}}, \cup) \cong (\mathcal{M}^{\text{even}} \times \mathcal{W}^{\text{even}}, \cdot)\]
and as a graded Lie algebra,
\[(\mathcal{M}^{\text{even}} \times \mathcal{W}^{\text{even}}, [, ,]) \cong \mathcal{M}^{\text{even}} \times (\mathcal{W}^{\text{even}}, [, ,]).\]

The BV operator is defined by
\[\Delta_2 m(M_{2m}) := 0,\]
\[\Delta_{2m+1}(L_{2m}) := -2mM_{2m}.\]

Now let us describe the Gerstenhaber algebra structure on $\text{HH}^*_{sg}(A, A)$. Denote, for $m \in \mathbb{Z}$,
\[L_{2m} := \text{HH}^{2m+1}_{sg}(A, A)\]
and
\[M_{2m} := \text{HH}^{2m}_{sg}(A, A).\]

Then we have the following proposition.

**Proposition 4.9.** Let $Q$ be the one loop quiver

\[Q := \bullet \quad \circlearrowright\]

and $A$ be its radical square zero algebra over a field $k$, where $k$ is not of characteristic two. Then we have a Lie algebra isomorphism
\[(\text{HH}^{\text{odd}}_{sg}(A, A)[1], [, ,]) \cong (\mathcal{W}^{\text{even}}, [, ,]).\]

and an isomorphism of commutative algebras
\[(\text{HH}^{\text{even}}_{sg}(A, A), \cup) \cong (\mathcal{M}^{\text{even}}, \cdot).\]

Moreover, $(\text{HH}^*_{sg}(A, A), \cup, [, ,], \Delta)$ is isomorphic to the BV algebra $(\mathcal{M}^{\text{even}} \times \mathcal{W}^{\text{even}}, [, ,], \Delta)$.

**Proof.** The statements are a direct consequence of Proposition 4.5, Lemma 4.6 and 4.7.

4.2 The case: $c \geq 2$

In this section, we fix $Q$ to be a $c$-crown with $c \geq 2$. Denote by $A$ its radical square zero $k$-algebra, where $k$ is not of characteristic two.

**Proposition 4.10** (Proposition 3.3 [CILb3]). Let $Q$ be a $c$-crown with $c \geq 2$. Let $n$ be an even multiple of $c$. Then
\[\dim_k \text{HH}^n(A, A) = \dim_k \text{HH}^{n+1}(A, A) = 1.\]

The cohomology vanishes in all other degrees.
Next let us consider the singular Hochschild cohomology $\text{HH}^*_{sg}(A, A)$. Note that $A$ is a self-injective algebra (but not a symmetric algebra). From [Buch, Corollary 6.4.1], it follows that for $m \geq 1$, 
$$\text{HH}^m_{sg}(A, A) \cong \text{HH}^m(A, A).$$
For the negative side of $\text{HH}^*_{sg}(A, A)$, we have for $m > 1$, 
$$\text{HH}^{-m}_{sg}(A, A) \cong \text{Tor}^A_{m-1}(A, \text{Hom}_A^e(A, A^e)) \cong \text{HH}^1(A, \Omega^{m+1}(A)).$$
From Remark 3.6, we have that 
$$\text{HH}^1(A, \Omega^{m+1}(A)) \cong \frac{k(Q_1//Q_{m+2})}{\text{Im}(D_{0,m+1})} \oplus \text{Ker}(D_{1,m+1}).$$
Recall that 
$$D_{0,m+1}(e, \gamma_{m+1}) = \sum_{a \in Q_1} (a, a\gamma_{m+1}) + (-1)^m \sum_{a \in Q_1} (a, \gamma_{m+1}a),$$
$$D_{1,m+1}(x, \gamma_{m+1}) = \sum_{a \in Q_1} (xa, a\gamma_{m+1}) + (-1)^{m+1} \sum_{a \in Q_1} (xa, \gamma_{m+1}a).$$
Anaglous to Proposition 4.10 above, we have the following proposition.

**Proposition 4.11.** Let $Q$ be a $c$-crown with $c \geq 2$. If $m$ is an even multiple of $c$, then 
$$\dim_k \text{HH}^m_{sg}(A, A) = \dim_k \text{HH}^{m+1}_{sg}(A, A) = 1.$$ 
The singular Hochschild cohomology vanishes in all other degrees.

The Lie algebra structure on the positive side of $\text{HH}^*_{sg}(A, A)$ has been obtained in [San1]. Similarly, we obtain the following proposition for the negative side of $\text{HH}^*_{sg}(A, A)$.

**Proposition 4.12.** Let $Q$ be a $c$-crown with $c \in 2\mathbb{Z}_{>0}$ and $A$ be its radical square zero algebra over a field $k$, where $k$ is not of characteristic two. Denote by $\gamma$ the oriented cycle (i.e. $\gamma = a_1a_2\cdots a_c$). Then we have the following cases:

1. Let $(a, a\gamma^p) \in \text{HH}^1(A, \Omega^p(A))$ and $(a, a\gamma^q) \in \text{HH}^1(A, \Omega^q(A))$ be nontrivial elements respectively, where $a \in Q_1$. Then 
$$[(a, a\gamma^p), (a, a\gamma^q)] = (q-p)(a, a\gamma^{p+q}),$$

2. Let 
$$x := \sum_{a \in Q_1} (a, a\gamma^p) \in \text{HH}^1(A, \Omega^{p+1}(A))$$
and 
$$y := \sum_{a \in Q_1} (a, a\gamma^q) \in \text{HH}^1(A, \Omega^{q+1})$$
be nontrivial elements, respectively, then 
$$[x, y] = 0.$$
3. Let
\[ x := \sum_{a \in Q_1} (a, a^{\gamma^p}) \in \text{HH}^1(A, \Omega^{\gamma^p+1}(A)) \]
and \((b, b^{\gamma^q}) \in \text{HH}^1(A, \Omega^{\gamma^q}(A))\) be nontrivial elements, respectively, then
\[ [x, (b, b^{\gamma^q})] = -p \sum_{a \in Q_1} (a, a^{\gamma^{p+q}}). \]

Let us denote, for \(p \in \mathbb{Z}\),
\[ L_p := \text{HH}^{cp+1}_{\text{sg}}(A, A) \]
and
\[ M_p := \text{HH}^p_{\text{sg}}(A, A). \]

Then we have the following proposition.

**Proposition 4.13.** Let \(Q\) be a \(c\)-crown with \(c \in 2\mathbb{Z}_{>0}\) and \(A\) be its radical square zero \(k\)-algebra. Assume that \(k\) is not of characteristic two. Then
\[ (L := \bigoplus_{i \in \mathbb{Z}} L_i, [\cdot, \cdot]) \]
is isomorphic to the Witt algebra \(W\) and
\[ (M := \bigoplus_{i \in \mathbb{Z}} M_i, \cup) \]
is isomorphic to the graded commutative algebra \(M\). Moreover, \((\text{HH}^*_{\text{sg}}(A, A), \cup, [\cdot, \cdot])\) is isomorphic to the Gerstenhaber algebra \((M \times W, \cup, [\cdot, \cdot])\).

**Proof.** The proof is completely analogous to the one of Proposition 4.9. □

**Remark 4.14.** From Remark 4.8, it follows that \((M \times W, \cup, [\cdot, \cdot])\) is a BV algebra. Hence the singular Hochschild cohomology \(\text{HH}^*_{\text{sg}}(A, A)\) has also a BV algebra structure. In fact, this is a corollary of [LZZ, Theorem 0.1] and [Vol, Corollary 3] since in this case, the Nakayama automorphism of \(A\) has finite order.

**5 Two loops quiver**

In this section, we consider the two loops quiver \(Q\), namely,

\[ Q := \begin{tikzpicture}[baseline=(current bounding box.center)]
\node (a) at (0,0) {$a$};
\node (b) at (1,0) {$b$};
\draw[->] (a) to [out=0,in=90] (b);
\draw[->] (b) to [out=270,in=180] (a);
\end{tikzpicture} \]

and its radical square zero algebra is
\[ A \cong k[x, y]/(x^2, y^2, xy), \]
where \(k\) is a field of characteristic zero. Recall that
Proposition 5.1 (Proposition 4.4. [San2]). Assume that $Q$ is the two loops quiver with the loops $a$ and $b$. Let $A$ be its radical square zero algebra over a field $k$ of characteristic zero. Then $\text{HH}^1(A, A) \cong k(Q_1//Q_1)$ and the elements in $k(Q_1//Q_1)$

$$H := (a, a) - (b, b),$$
$$E := (b, a),$$
$$F := (a, b)$$

generate a copy of the Lie algebra $\mathfrak{sl}_2(k)$ in $\text{HH}^1(A, A)$. Moreover, the Lie algebra $\text{HH}^1(A, A)$ is isomorphic to $\mathfrak{sl}_2 \times k$, where

$$I := (a, a) + (b, b)$$

is a non-zero element such that $[I, \text{HH}^1(A, A)] = 0$.

Remark 5.2. If we denote the path $a$ by 1 and the path $b$ by 2. We also denote

$$(x, y) := E_{y,x},$$

where $\{x, y\} \subset \{1, 2\}$. Then we have

$$H = E_{1,1} - E_{2,2}$$
$$E = E_{1,2}$$
$$F = E_{2,1},$$

where $E_{i,j}$ is the basis of $\mathfrak{gl}_2(k)$, which is defined as follows,

$$E_{i,j}(e_k) = \delta_{i,k}e_j.$$

In [San2], the author also gave a description of $\text{HH}^m(A, A)$ as a Lie module over $\text{HH}^1(A, A)$, for any $m \in \mathbb{Z}_{>0}$. Next we will completely describe the Gerstenhaber algebra structures on $\text{HH}^*(A, A)$ and $\text{HH}^*_{\text{sg}}(A, A)$.

To describe the Gerstenhaber algebra structure on $\text{HH}^*_{\text{sg}}(A, A)$, let us first work on a more general setting. Let $V$ be a finite-dimensional vector space over a field $k$. Define, for $m, p \in \mathbb{Z}_{>0}$,

$$T^{m,p}(V) := T_0^{m,p}(V) \oplus T_1^{m,p}(V),$$

where

$$T_0^{m,p}(V) := \text{Hom}_k(V^\otimes m, V^\otimes p)$$

and

$$T_1^{m,p}(V) := \text{Hom}_k(V^\otimes m, V^\otimes p+1),$$

and denote

$$T^{*,*}(V) := \bigoplus_{m,p \in \mathbb{Z}_{>0}} T^{m,p}(V),$$

$$T^{*,*}(V) := \bigoplus_{m,p \in \mathbb{Z}_{>0}} T^{m,p}(V),$$

27
We will define a bracket $\{ \cdot, \cdot \}$ on $T^{*,*}(V)$ as follows. Let $f \in T_1^{m,p}(V)$ and $g \in T_1^{n,q}(V)$, we define $\{ f, g \} \in T_1^{m+n-1,p+q}(V)$ as follows,

$$\{ f, g \} := \sum_{i=1}^{m}(1)(i-1)(q-n-1)(f \otimes \text{id}^{\otimes q})(\text{id}^{\otimes i-1} \otimes g \otimes \text{id}^{\otimes m-i}) +$$

$$\sum_{i=1}^{q-1}(1)(i-1)(p-m-1)(\text{id}^{\otimes i} \otimes f \otimes \text{id}^{\otimes q-i})(g \otimes \text{id}^{\otimes m-i}) -$$

$$\sum_{i=1}^{n}(1)(n-q+i)(p-m-1)(g \otimes \text{id}^{\otimes p})(\text{id}^{\otimes i-1} \otimes f \otimes \text{id}^{\otimes m-i}) -$$

$$\sum_{i=1}^{p-1}(1)(m-p+i-1)(q-n-1)(\text{id}^{\otimes i} \otimes g \otimes \text{id}^{\otimes p-i})(f \otimes \text{id}^{\otimes m-i}).$$

(19)

Let $f \in T_0^{m,p}(V)$ and $g \in T_0^{n,q}(V)$, we define $\{ f, g \} \in T_0^{m+n-1,p+q}(V)$ as follows,

$$\{ f, g \} := \sum_{i=1}^{m-1}(1)(i-1)(q-n-1)(f \otimes \text{id}^{\otimes q})(\text{id}^{\otimes i-1} \otimes g \otimes \text{id}^{m-i}) -$$

$$\sum_{i=1}^{p-2}(1)(q-n-1)(p-m+i-1)(\text{id}^{\otimes i} \otimes g \otimes \text{id}^{\otimes p-i-1})(f \otimes \text{id}^{\otimes m-i}).$$

and define $\{ g, f \} \in T_0^{m+n-1,p+q}(V)$ as follows,

$$\{ g, f \} := -\{ f, g \}.$$

Define $\{ f, g \} = 0$ if $f \in T_0^{m,p}(V)$ and $g \in T_0^{n,q}(V)$. Clearly, $\{ \cdot, \cdot \}$ is skew-symmetry, that is, for $f \in T^{m,p}(V)$ and $g \in T^{n,q}(V)$,

$$\{ f, g \} = -\{ g, f \}.$$

For $m, p \in \mathbb{Z}_{>0}$, we have the following canonical embedding,

$$\theta_{m,p}: T_1^{m,p}(V) \rightarrow T_1^{m+1,p+1}(V)$$

$$\phi_{m-1,p}: T_0^{m-1,p}(V) \rightarrow T_0^{m,p}(V)$$

$$f \mapsto -f \otimes \text{id}_V.$$

Similarly, we also have the following morphism for $m, p \in \mathbb{Z}_{>0}$,

$$f \mapsto \text{id}_V \otimes f + (-1)^{p+m}f \otimes \text{id}_V.$$

Then for any $p \in \mathbb{Z}_{\geq 0}$, we have a complex

$$\cdots \longrightarrow T^{m,p}(V) \xrightarrow{\phi_{m,p}} T^{m+1,p}(V) \xrightarrow{\phi_{m+1,p}} T^{m+2,p}(V) \longrightarrow \cdots.$$  

(20)

Remark 5.3. If $m \neq p$, then $\phi_{m,p}$ is injective and if $m = p$, $\text{Ker}(\phi_{m,m})$ is a one-dimensional $k$-vector space with a basis $\{ \text{id}_V \otimes \}$. Let us denote the homology group of the complex (20) by

$$K^{m,p}(V) := \text{Ker}(\phi_{m,p}) \oplus \frac{T_1^{m,p}(V)}{\text{Im}(\phi_{m-1,p})}.$$  

28
Then we have
\[ K_{m,p}(V) = \begin{cases} \frac{T_{m-p}(V)}{\text{Im}(\phi_{m-1,p})} & \text{if } m \neq p, \\
\text{id}_{V^\otimes m} \oplus \frac{T_{m,m}(V)}{\text{Im}(\phi_{m-1,m})} & \text{if } m = p. \end{cases} \]

**Lemma 5.4.** For \( m, p \in \mathbb{Z}_{\geq 0} \), \( \theta_{m,p} \) induces a morphism (still denoted by \( \theta_{m,p} \)),
\[ \theta_{m,p} : K_{m,p}(V) \to K_{m+1,p+1}(V). \]

**Remark 5.5.** For \( m = p \), we define
\[ \theta_{m,m}|_{\text{Ker}(\phi_{m,m})} := \text{id}. \]

We have an inductive system
\[ \cdots \to K_{m,p}(V) \overset{\theta_{m,p}}{\to} K_{m+1,p+1}(V) \overset{\theta_{m+1,p+1}}{\to} \cdots \]

Let us denote the colimit of this inductive system by \( K^*_{\text{sg}}(V) \) and we denote
\[ K^*_{\text{sg}}(V) := \bigoplus_{n \in \mathbb{Z}} K^n_{\text{sg}}(V). \]

Now let us go back to the two loops quiver \( Q \). Recall that
\[ A = r \oplus E \]
is the Wedderburn-Malcev decomposition of \( A \), where \( r = kQ_1 \) and \( E = kQ_0 \).

Let \( V := k \oplus k \). Let \( e_1, e_2 \in k^2 \) are the canonical basis of \( V \). Then
\[ \text{End}(V) \cong \text{gl}_2(k). \]

We can identify \( k(Q_m/Q_p) \) with \( \text{Hom}_k(V^\otimes m, V^\otimes p) \) as follows,
\[ k(Q_m/Q_p) \to \text{Hom}_k(V^\otimes m, V^\otimes p) \]
\[ (x_1 \cdots x_m, y_1 \cdots y_p) \mapsto \delta_{e_{x_1} \cdots e_{x_m}, e_{y_1} \cdots e_{y_p}} \]

where
\[ x_i, y_j \in \{1, 2\} \]
for \( 1 \leq i \leq m \) and \( 1 \leq j \leq p \), and \( \delta_{e_{x_1} \cdots e_{x_m}, e_{y_1} \cdots e_{y_p}} \) is defined as follows,
\[ \delta_{e_{x_1} \cdots e_{x_m}, e_{y_1} \cdots e_{y_p}}(e_{x'_1} \cdots e_{x'_m}) = \begin{cases} e_{y_1} \cdots e_{y_p} & \text{if } x_i = x'_i \text{ for } 1 \leq i \leq m; \\
0 & \text{otherwise.} \end{cases} \]

Hence we have the following isomorphism for \( m, p \in \mathbb{Z}_{\geq 0} \).
\[ F_{m,p} : k(Q_m/Q_p) \oplus k(Q_m/Q_{p+1}) \to T^{m,p}(V). \]
Proposition 5.6. Let $Q$ be the two loops quiver and $A$ be its radical square zero algebra over a field $k$ of characteristic zero. Then

1. for any $p \in \mathbb{Z}_{\geq 0}$, we have a chain homomorphism,

$$
\cdots \rightarrow k(Q_m/\mathcal{Q}_p) \oplus k(Q_m/\mathcal{Q}_{p+1}) \rightarrow k(Q_m+1/\mathcal{Q}_p) \oplus k(Q_m+1/\mathcal{Q}_{p+1}) \rightarrow \cdots
$$

$$
\cdots \rightarrow T^{m,p}(V) \rightarrow T^{m+1,p}(V) \rightarrow \cdots
$$

where we recall that $D_{m,p}$ is defined in Proposition 3.5. As a consequence, $F_{m,p}$ induces an isomorphism for $m, p \in \mathbb{Z}_{\geq 0}$,

$$
F_{m,p} : \text{HH}^m(A, \Omega^p(A)) \rightarrow K^{m,p}(V).
$$

2. The following diagram commutes, for $m, p \in \mathbb{Z}_{>0}$.

$$
\begin{array}{ccc}
\text{HH}^m(A, \Omega^p(A)) & \stackrel{\theta_{m,p}}{\rightarrow} & \text{HH}^{m+1}(A, \Omega^{p+1}(A)) \\
F_{m,p} & \downarrow & F_{m+1,p+1} \\
K^{m,p}(V) & \stackrel{\phi_{m,p}}{\rightarrow} & K^{m+1,p+1}(V)
\end{array}
$$

As a consequence, $\phi_{m,p}$ is injective for $m, p \in \mathbb{Z}_{\geq 0}$.

3. The following diagram commutes for $m, n \in \mathbb{Z}_{>0}$ and $p, q \in \mathbb{Z}_{\geq 0}$

$$
\begin{array}{ccc}
\text{Hom}_{E-E}(r^{\otimes E^m}, \Omega^p(A)) \times \text{Hom}_{E-E}(r^{\otimes E^n}, \Omega^q(A)) & \stackrel{[\cdot, \cdot]}{\rightarrow} & \text{Hom}_{E-E}(r^{\otimes E^{m+n}}, \Omega^{p+q}(A)) \\
\cong & \downarrow & \cong \\
T^{m,p}(V) \times T^{n,q}(V) & \stackrel{\{\cdot, \cdot\}}{\rightarrow} & T^{m+n-1,p+q}(V)
\end{array}
$$

where $[\cdot, \cdot]$ is the Gerstenhaber bracket defined by Formula (15) in Section 3, $\{\cdot, \cdot\}$ is the Lie bracket defined in (19), and we identify

$$
\text{Hom}_{E-E}(r^{\otimes E^m}, \Omega^p(A))
$$

with

$$
k(Q_m/\mathcal{Q}_p) \oplus k(Q_m/\mathcal{Q}_{p+1})
$$

for $m, p \in \mathbb{Z}_{\geq 0}$ by Lemma 3.4

Proof. The proofs of Assertions (1) and (2) are straightforward from the definitions. Let us verify the Assertion (3). Let

$$(x_1 \cdots x_m, y_1 \cdots y_{p+1}) \in \text{Hom}_{E-E}(r^{\otimes E^m}, \Omega^p(A))$$

and

$$(a_1 \cdots a_n, b_1 \cdots b_{q+1}) \in \text{Hom}_{E-E}(r^{\otimes E^n}, \Omega^q(A))$$

By Formula (21), they correspond, respectively, the elements

$$
\delta_{e_{x_1} \otimes \cdots \otimes e_{x_m}, e_{y_1} \otimes \cdots \otimes e_{y_{p+1}}} \in T^{m,p}
$$
and
\[ \delta_{e_{a_1} \otimes \ldots \otimes e_{a_n} \otimes e_{b_1} \otimes \ldots \otimes e_{b_{q+1}}} \in T^{m,q}. \]

Then under this correspondence, we observe that Formula (21) and (16) have the same type.

Therefore, as a consequence, we obtain the following corollary.

**Corollary 5.7.** Let \( V \) be a finite dimensional vector space of dimensional 2 over a field \( k \) of characteristic zero. Then

1. \( T^{*,*}(V)[1] \) is a differential graded Lie algebra, where the grading is defined as follows, for \( n \in \mathbb{Z} \),

\[ T^{*,*}(V)_{n} := \bigoplus_{m,p \in \mathbb{Z} \geq 0, m-p = n} T^{m,p}(V) \]

and the differential is induced by

\[ T^{m,p}(V) \left( \phi_{m,p}^0 0 \right) \rightarrow T^{m+1,p}(V), \]

In particular, \( K^{*,*}(V) \) is a \( \mathbb{Z} \)-graded Lie algebra with the induced grading and induced Lie bracket,

2. The morphism in Lemma 5.4

\[ \theta_{m,p} : K^{m,p}(V) \rightarrow K^{m+1,p+1}(V) \]

induces a module homomorphism of graded Lie algebra,

\[ \theta : K^{*,*}(V) \rightarrow K^{*,*}(V), \]

that is,

\[ \theta([f, g]) = [\theta(f), g] = -(-1)^{(m-p-1)(n-q)} [f, \theta(g)]. \]

Hence \( K^{*,*}_{sg}(V) \) is a graded Lie algebra which is isomorphic to \( HH^{*,*}_{sg}(A, A) \), where \( A \) is the radical square zero algebra of the two loops quiver \( Q \).

**Proof.** This is an immediate consequence of Proposition 5.6.

**Remark 5.8.** Since the cup product of \( HH^{*,*}(A, A) \) vanishes except in degree zero and the Lie bracket does not, in general, it is impossible to endow \( (HH^{*,*}_{sg}(A, A), \cup, [\cdot, \cdot]) \) or \( (HH^{*}(A, A), \cup, [\cdot, \cdot]) \) with a BV algebra structure.

### 6 A PROP interpretation of the Gerstenhaber bracket

In this section, we will give an interpretation of the Gerstenhaber bracket from the point of view of PROP theory, in the case of radical square zero algebras. In fact, this interpretation generalizes the construction in Section 5.
6.1 Definition of PROP

Let us start with the definition of PROP. For more details on PROP and operad theory, we refer the reader to [Fio, KaMa, Lod, LoVa, Mar, Val1]. Let $k$ be a commutative base ring, we denote by $k\text{-Mod}$ the category of $k$-modules. Note that $k\text{-Mod}$ is a tensor category with the tensor product $\otimes_k$. For simplicity, we write $\otimes_k$ as $\otimes$ in this section.

**Definition 6.1.** A strict $k$-linear tensor category (strict $k$-linear monoidal category) is a triple $(\mathcal{C}, \otimes, 1)$, where $\mathcal{C}$ is a $k$-linear category, $1$ a distinguished object and

$$\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$$

is a $k$-linear functor, satisfying

$$(X \otimes Y) \otimes Z = X \otimes (Y \otimes Z)$$

and

$$X \otimes 1 = X = 1 \otimes X$$

for any $X, Y, Z \in \mathcal{C}$.

**Definition 6.2.** A $k$-linear non-symmetric PROP is a strict monoidal category $\mathcal{P} = (P, \odot, 1)$ enriched over $k\text{-Mod}$ such that

1. the objects are indexed by the set $\mathbb{Z}_{\geq 0}$ and
2. the tensor product satisfies

$$m \odot n = m + n,$$

for any $m, n \in \mathbb{Z}_{\geq 0}$ (hence the unit $1$ equals $0$).

**Remark 6.3.** For a (non-symmetric) PROP $\mathcal{P}$, denote

$$\mathcal{P}(m, n) := \text{Hom}_\mathcal{P}(m, n).$$

Each $\mathcal{P}(m, n)$ is a $k$-module. Therefore a PROP is a collection

$$\mathcal{P} = \{\mathcal{P}(m, n)\}_{m, n \in \mathbb{Z}_{\geq 0}}$$

of $k$-modules, together with two types of compositions:

1. (horizontal composition)

$$\odot : \mathcal{P}(m_1, n_1) \otimes_k \cdots \otimes_k \mathcal{P}(m_s, n_s) \to \mathcal{P}(m_1 + \cdots + m_s, n_1 + \cdots + n_s)$$

induced by the tensor product $\otimes$ of $P$, for all $m_1, \cdots, m_s, n_1, \cdots, n_s \geq 0$ and

2. (vertical composition)

$$\circ : \mathcal{P}(m, n) \otimes_k \mathcal{P}(n, p) \to \mathcal{P}(m, p),$$

given by the categorical composition, for all $m, n, p \geq 0$. 
and the following compatibility: For any $f_i \in \mathcal{P}(m_i, p_i)$ and $g_i \in \mathcal{P}(p_i, q_i)$, where $i = 1, 2$,

$$(g_1 \circ f_1) \odot (g_2 \circ f_2) = (g_1 \circ g_2) \circ (f_1 \circ f_2).$$

We remark that in Definition 6.2, the category $k\text{-Mod}$ can be replaced by any arbitrary symmetric monoidal category. Given a (non-symmetric) PROP $\mathcal{P}$, we can construct an operad $\mathcal{O}_\mathcal{P}$ as follows (cf. Page 8, [Fio]): For $n \in \mathbb{Z}_{\geq 0}$, we define

$$\mathcal{O}_\mathcal{P}(n) := \mathcal{P}(n, 1),$$

and the structural product map is the following composition,

$$\mathcal{P}(n, 1) \otimes \mathcal{P}(j_1, 1) \otimes \cdots \otimes \mathcal{P}(j_n, 1) \to \mathcal{P}(j_1 + \cdots + j_n, n) \to \mathcal{P}(j_1 + \cdots + j_n, 1)$$

for $n, j_1, \cdots, j_n \in \mathbb{Z}_{\geq 0}$.

**Example 6.4.**

1. The endomorphism PROP $\text{End}_V$ of a $k$-module $V$ is the following collection

$$\text{End}_V = \{\text{End}_V(m, n)\}_{m,n \geq 0},$$

where $\text{End}_V(m, n)$ is the space of $k$-linear maps $\text{Hom}_k(V \otimes^m, V \otimes^n)$. The horizontal composition is induced by the tensor product $\otimes_k$ and the vertical composition is induced by the ordinary composition of $k$-linear maps. The associated operad $\mathcal{O}_{\text{End}_V}$ is the linear operad of $V$, that is, for $n \in \mathbb{Z}_{\geq 0}$,

$$\mathcal{O}_{\text{End}_V}(n) := \text{Hom}_k(V \otimes^k, V),$$

with the natural structural product map.

2. The direct sum PROP of a $k$-module $V$ is defined as a PROP by the following collection

$$\mathcal{P}_\oplus := \{\text{Hom}_k(V \otimes^m, V \otimes^n)\}_{m,n \geq 0},$$

where $V \otimes^0 := 0$. The horizontal composition is induced by the direct sum in $k\text{-Mod}$,

$$\oplus : \text{Hom}_k(V \otimes^{m_1}, V \otimes^{n_1}) \otimes \cdots \otimes \text{Hom}_k(V \otimes^{m_s}, V \otimes^{n_s}) \to \text{Hom}_k(V \otimes^{m_1 + \cdots + m_s}, V \otimes^{n_1 + \cdots + n_s})$$

for any $m_1, \cdots, m_s, n_1, \cdots, n_s \in \mathbb{Z}_{\geq 0}$. The vertical composition is induced by the ordinary composition of $k$-linear maps. In Appendix A, we apply this PROP in the case of $V := k$ to give a new Lie algebra structure on $\mathfrak{gl}_\infty(k)$.

3. The $r$-multiple loops quiver PROP is defined by the following collection

$$\mathcal{P}_r(m, p)_{m,p \in \mathbb{Z}_{\geq 0}} := \text{Hom}_k(V \otimes^m, V \otimes^{p-1}) \oplus \text{Hom}_k(V \otimes^m, V \otimes^p),$$

where $V$ is a $k$-vector space of dimension $r$ and we use the notation

$$V \otimes^{-1} := 0.$$ 

The horizontal composition is

$$\mu : \mathcal{P}_r(m, p) \otimes_k \mathcal{P}_r(n, q) \to \mathcal{P}_r(m + n, p + q),$$

33
where
\[
\mu(f \otimes g) := \begin{cases} 
0 & \text{if } f \in \text{Hom}_k(V^\otimes m, V^\otimes p-1) \text{ and } g \in \text{Hom}_k(V^\otimes n, V^\otimes q-1), \\
g \otimes_k g & \text{otherwise}.
\end{cases}
\]

and the vertical composition is defined as follows,
\[
\nu : P_r(m, p) \otimes_k P_r(p, n) \rightarrow P_r(m, n),
\]
where
\[
\nu(f \otimes g) := \begin{cases} 
0 & \text{if } f \in \text{Hom}_k(V^\otimes m, V^\otimes p-1), \\
g \circ f & \text{otherwise}.
\end{cases}
\]

It is straightforward to verify that \(P_r\) is a well-defined PROP. Note that when \(r = 2\), \(P_2\) corresponds to the construction of Section 5.

### 6.2 A Lie bracket on PROP

Recall that in [KaMa], the authors proved that the positive total space
\[
\bigoplus_{n \in \mathbb{Z}_{>0}} O(n)
\]
of a (non-symmetric) operad \(O\) is endowed with a \(\mathbb{Z}\)-graded Lie algebra structure. Next we will extend this result to a PROP \(P\). Namely, we will show that there is a \(\mathbb{Z}\)-graded Lie algebra structure on the positive total space
\[
\bigoplus_{m,p \in \mathbb{Z}_{>0}} P(m, p)
\]
of a (non-symmetric) PROP \(P\). Let us denote the identity element (with respect to the categorical composition) in \(P(1, 1)\) by
\[
id \in P(1, 1).
\]

Let \(m, n, p, q \in \mathbb{Z}_{>0}\). Let \(f \in P(m, p)\) and \(g \in P(n, q)\). Define
\[
\begin{align*}
\forint = & \begin{cases} 
(f \circ \text{id}^\otimes q^{-1}) \circ (\text{id}^\otimes i^{-1} \circ g \circ \text{id}^\otimes m^{-i}) & \text{for } 1 \leq i \leq m, \\
(\text{id}^\otimes i \circ f \circ \text{id}^\otimes q^{-1+i}) \circ (g \circ \text{id}^\otimes m^{-1}) & \text{for } -q+1 \leq i \leq -1.
\end{cases}
\end{align*}
\]

Then we denote
\[
\begin{align*}
f \oplus g := & \sum_{i=1}^{m} (-1)^{(i-1)(q-n)} \forint + \sum_{i=1}^{q-1} (-1)^{(p-m)} \forint \oplus g \\
\end{align*}
\]
and the Lie bracket is defined as follows,
\[
[f, g] := f \oplus g - (-1)^{(m-p)(n-q)} g \oplus f.
\]

Note that
\[
[f, g] \in P(m + n - 1, p + q - 1).
\]
In the following, we will follow \([\text{MeVa, Val1, Val2}]\) to give the planar graph presentations of elements in \(\mathcal{P}\) and the operation \(\star\).

Let \(f \in \mathcal{P}(m, p)\). We associate to \(f\) the following graph:

\[
\begin{array}{c}
1 \\
\downarrow \\
\vdots
\end{array} \quad \begin{array}{c}
f \\
\uparrow \\
1 \quad 2
\end{array} \quad \begin{array}{c}
m \\
\downarrow \\
\vdots
\end{array}
\]

where the inputs of the vertex \(f\) are labelled by integers \(\{1, \ldots, m\}\) from left to right and the outputs are labelled by integers \(\{1, \ldots, p\}\) from left to right. Let \(g \in \mathcal{P}(n, q)\). For \(1 \leq i \leq m\), we define the graph presentation of the operation \(f \star_i g\) as follows:

\[
\begin{array}{c}
1 \quad 2 \\
\downarrow \downarrow \\
i-1 \quad i
\end{array} \quad \begin{array}{c}
i+1 \\
\downarrow \downarrow \\
\quad i+n-1
\end{array} \quad \begin{array}{c}
m \\
\downarrow \downarrow \\
i+p-1
\end{array}
\]

Similarly, for \(1 \leq i \leq q\), we define the graph presentation of \(f \star_{-i} g\) as follows:

\[
\begin{array}{c}
1 \quad 2 \\
\downarrow \downarrow \\
i+1 \quad i
\end{array} \quad \begin{array}{c}
i+1 \\
\downarrow \downarrow \\
\quad i+p
\end{array} \quad \begin{array}{c}
q+1 \\
\downarrow \downarrow \\
\vdots \vdots
\end{array}
\]

**Remark 6.5.** Analogous to \([\text{MeVa}]\), we use the following graph to present \(f \star g\):

\[
\begin{array}{c}
g \\
\downarrow
\end{array} \quad \begin{array}{c}
f
\end{array}
\]

Namely, it is the sum of all graphs presented in Diagram (25) and (26).
Proposition 6.6. Let $k$ be a commutative ring. Let $\mathcal{P}$ be a $k$-linear (non-symmetric) PROP. Then the Lie bracket $[\cdot,\cdot]$ defined above gives a $\mathbb{Z}$-graded Lie algebra structure on the positive total space

$$\mathcal{P} := \bigoplus_{m,p \in \mathbb{Z}_{>0}} \mathcal{P}(m,p)$$

where the $\mathbb{Z}$-grading is defined as follows: for $n \in \mathbb{Z}$,

$$\mathcal{P}_n := \bigoplus_{m,p \in \mathbb{Z}_{>0}} \mathcal{P}(m,p).$$

In particular, the following natural embedding of the total space is a homomorphism of graded Lie algebras

$$\mathcal{O}_\mathcal{P} \to \mathcal{P},$$

where the Lie bracket of $\mathcal{O}_\mathcal{P}$ is defined in [KaMa].

Proof. The proof is completely analogous to the one of [McVe, Proposition 4]. Let $f \in \mathcal{P}(m,p), g \in \mathcal{P}(n,q), h \in \mathcal{P}(l,r)$. Then $(f \star g) \star h$ is spanned by graphs of the form.

Similarly, $f \star (g \star h)$ is spanned by graphs of the form.

Note that we have the same coefficient for the same type in $(f \star g) \star h$ and $f \star (g \star h)$, so we obtain that $(f \star g) \star h - f \star (g \star h)$ is spanned by

Let us compare the coefficients of those graphs. Note that the coefficient of $(f \star g) \star h - f \star (g \star h)$ in the Jacobi identity is $(-1)^{(m-p)(l-r)}$ and the one of $(g \star f) \star h - g \star (f \star h)$ is $-(-1)^{(m-p)(n-q+l-r)}$. So the coefficient of the following graph from $(f \star g) \star h - f \star (g \star h)$

is $(-1)^{(m-p)(l-r)}$ and the coefficient of the following graph from $(g \star f) \star h - g \star (f \star h)$

is $-(-1)^{(m-p)(n-q+l-r)}$. 

36
is $-(1)^{(m-p)(q+l-r)}$. On the other hand, we also note that these two graphs have the same type up to coefficient $-(1)^{(m-p)(n-q)}$. Note that

$$(1)^{(m-p)(l-r)} - (1)^{(m-p)(n-q)}(1)^{(m-p)(n-q+l-r)} = 0,$$

hence the sum of these two graphs is zero in the Jacobi identity. Similarly, we can check that the other graphs will be cancelled by comparing the coefficients. Therefore we obtain the Jacobi identity. 

**Remark 6.7.** The graphs above have a differential meaning from the ones in [MeVa] although they have the same shapes. In general, the $\mathbb{Z}$-graded Lie algebra structure on the positive total space of $P$ is also differential from the one constructed in [MeVa, Proposition 4], since the graphs presented in (25) and (26) are not connected. But these two differential $\mathbb{Z}$-graded Lie algebra structures induce exactly the same $\mathbb{Z}$-graded Lie algebra structure on the positive total space of $\mathcal{O}_P$ constructed in [KaMa].

### 6.3 An interpretation of Gerstenhaber bracket

Let us go back to radical square zero algebras. Let $k$ be a field. Let $Q$ be a finite and connected quiver, $A$ be its radical square zero algebra over $k$ and $A = E \oplus r$ be the Wedderburn-Malcev decomposition. Recall that we have the following space for $m, p \in \mathbb{Z}_{\geq 0}$

$$\mathcal{P}(m, p + 1) := \text{Hom}_{E-E}(r^\otimes E^m, \Omega^p(A))$$

where $\Omega^p(A)$ is the $p$-th kernel in the projective resolution $\mathcal{R}(A)$ (cf. Lemma 2.1). Let us denote

$$\mathcal{P}(m, 0) := \text{Hom}_{E-E}(r^\otimes E^m, E)$$

for $m \geq 0$. Recall that from Lemma 3.4, we have the following isomorphism for $m, p \in \mathbb{Z}_{\geq 0}$,

$$\text{Hom}_{E-E}(r^\otimes E^m, \Omega^p(A)) \cong k(Q_m//Q_p \cup Q_{p+1}).$$

Next we will give a $k$-linear PROP structure on the collection

$$\{\mathcal{P}(m, p)\}_{m, p \in \mathbb{Z}_{\geq 0}}.$$

First, let us define the horizontal composition as follows, for $m, p, n, q \in \mathbb{Z}_{\geq 0}$,

$$\mathcal{P}(m, p + 1) \otimes_k \mathcal{P}(n, q + 1) \cong \mathcal{P}(m + n, p + q + 2)$$

$$k(Q_m//Q_p \cup Q_{p+1}) \otimes_k k(Q_n//Q_q \cup Q_{q+1}) \overset{\mu}{\longrightarrow} k(Q_{m+n}//Q_{p+q+1} \cup Q_{p+q+2})$$

where $\mu$ is defined in the following way,

$$\mu((\alpha, \beta) \otimes_k (\alpha', \beta')) := \begin{cases} 0 & \text{if } (\alpha, \beta) \in k(Q_m//Q_p) \text{ and } (\alpha', \beta') \in k(Q_n//Q_q), \\ (\alpha \otimes_E \alpha', \beta \otimes_E \beta') & \text{otherwise}. \end{cases}$$
The vertical composition is defined as follows, for \( m, p, n \in \mathbb{Z}_{\geq 0} \),

\[
\mathcal{P}_A(m, p + 1) \otimes_k \mathcal{P}_A(p + 1, n + 1) \xrightarrow{\cong} \mathcal{P}_A(m, n + 1)
\]

\[
k(Q_m/\mathbb{Q}_p \cup Q_{p+1}) \otimes_k k(Q_{p+1}/\mathbb{Q}_n \cup Q_{n+1}) \xrightarrow{\nu} k(Q_m/\mathbb{Q}_n \cup Q_{n+1})
\]

where \( \nu \) is defined in the following way,

\[
\nu((\alpha, \beta) \otimes_k (\beta', \gamma')) := \delta_{\beta, \beta'}(\alpha, \gamma').
\]

From Proposition 6.6 it follows that the positive total space

\[
\bigoplus_{m, p \in \mathbb{Z}_{>0}, m - p = n} \mathcal{P}_A(m, p)
\]

is a \( \mathbb{Z} \)-graded Lie algebra.

**Proposition 6.8.** Let \( Q = (Q_0, Q_1, s, t) \) be a finite connected quiver and \( A \) be its radical square zero algebra over a field \( k \). Then the Lie algebra structure on the positive total space

\[
\bigoplus_{m, p \in \mathbb{Z}_{>0}, m - p = n} \mathcal{P}_A(m, p)
\]

coincides with the Gerstenhaber Lie algebra structure defined in (15) of Section 3.

**Proof.** It is clear that these two Lie brackets coincide in the positive total spaces of \( \mathcal{P}_A \). \( \blacksquare \)

**Remark 6.9.** Section 4 provides a concrete example in the case of two loops quiver. Similarly, for a \( r \)-multiple loops quiver \( Q \ (r \geq 2) \),

\[
Q := \begin{array}{c}
\vdots \\
1 \\
2 \\
\end{array}
\]

let \( A \) be its radical square zero algebra, then the PROP \( \mathcal{P}_A \) constructed above is isomorphic to the PROP \( \mathcal{P}_r \) defined in Example 6.4.

**A Another Lie algebra structure on \( \mathfrak{gl}_\infty(k) \)**

Let \( k \) be a field. For \( n \in \mathbb{Z}_{>0} \), we have a natural embedding,

\[
\theta_n : \mathfrak{gl}_n(k) \rightarrow \mathfrak{gl}_{n+1}(k) \quad A \rightarrow \begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix}
\]

\[ (27) \]
then we have
\[ \mathfrak{gl}_\infty(k) := \lim_{n \to \infty} \mathfrak{gl}_n(k). \]

Next we will give a Lie bracket on the graded space
\[ \bigoplus_{m \in \mathbb{Z}_{>0}} \mathfrak{gl}_m(k). \]

Let \( A \in \mathfrak{gl}_m(k) \) and \( B \in \mathfrak{gl}_n(k) \), define \( \{A, B\} \in \mathfrak{gl}_{m+n-1}(k) \) as follows,
\[
\{A, B\} := \sum_{i=0}^{n-1} \left[ \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & A & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \right] \left[ \begin{pmatrix} B & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \right] + \sum_{i=1}^{m-1} \left[ \begin{pmatrix} 0 & 0 & 0 & 0 \\ A & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \right] \left[ \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ B & 0 & 0 & 0 \end{pmatrix} \right]
\]

where \( 0_i \) indicates the zero matrix of size \( i \times i \) and \([\cdot, \cdot]\) is the Lie bracket in \( \mathfrak{gl}_{m+n-1}(k) \).

Clearly, \( \{\cdot, \cdot\} \) is skew-symmetric, namely,
\[
\{A, B\} = -\{B, A\}
\]
for any \( A \in \mathfrak{gl}_m(k) \) and \( B \in \mathfrak{gl}_n(k) \). Note that \( \{\mathfrak{gl}_1(k), \mathfrak{gl}_n(k)\} = 0 \).

**Lemma A.1.** Let \( A \in \mathfrak{gl}_m(k) \) and \( B \in \mathfrak{gl}_n(k) \), then we have
\[
\{\theta_m(A), B\} = \theta_{m+n-1}(\{A, B\}),
\]
or equivalently,
\[
\{A, \theta_n(B)\} = \theta_{m+n-1}(\{A, B\}).
\]

So the bracket \( \{\cdot, \cdot\} \) induces a well-defined bracket (still denoted by \( \{\cdot, \cdot\} \)) on \( \mathfrak{gl}_\infty(k) \).

**Proposition A.2.** The bracket \( \{\cdot, \cdot\} \) defines a Lie algebra structure on the graded space
\[ \bigoplus_{m \in \mathbb{Z}_{>0}} \mathfrak{gl}_m(k). \]

In particular, it induces a Lie algebra structure on \( \mathfrak{gl}_\infty(k) \).

**Proof.** It is sufficient to verify the Jacobi identity. Let \( A \in \mathfrak{gl}_m(k), B \in \mathfrak{gl}_n(k) \) and
Recall that for $C \in \mathfrak{gl}(p)$. Then we have

$$\{\{A, B\}, C\} = \sum_{i=1}^{n-1} \sum_{j=0}^{p-1} \left\{ \begin{pmatrix} 0_j & 0 \end{pmatrix} \right\} \left( \begin{pmatrix} 0 & 0 \ 0 & 0 \end{pmatrix} \right) \left( \begin{pmatrix} B & 0 \ 0 & 0 \end{pmatrix} \right) \left( \begin{pmatrix} 0 \ 0 \end{pmatrix} \right) \right), \left( \begin{pmatrix} 0 & 0 \ 0 & 0 \end{pmatrix} \right) +$$

$$\sum_{i=1}^{n-1} \sum_{j=1}^{m+n-2} \left\{ \begin{pmatrix} 0 & 0 \ 0 & 0 \end{pmatrix} \right\} \left( \begin{pmatrix} B & 0 \ 0 & 0 \end{pmatrix} \right) \left( \begin{pmatrix} 0 \ 0 \end{pmatrix} \right) \right), \left( \begin{pmatrix} 0 & 0 \ 0 & 0 \end{pmatrix} \right) +$$

$$\sum_{i=0}^{n-1} \sum_{j=1}^{m+n-2} \left\{ \begin{pmatrix} 0 & 0 \ 0 & 0 \end{pmatrix} \right\} \left( \begin{pmatrix} B & 0 \ 0 & 0 \end{pmatrix} \right) \left( \begin{pmatrix} 0 \ 0 \end{pmatrix} \right) \right), \left( \begin{pmatrix} 0 & 0 \ 0 & 0 \end{pmatrix} \right) +$$

$$\sum_{i=0}^{n-1} \sum_{j=0}^{p-1} \left\{ \begin{pmatrix} 0 & 0 \ 0 & 0 \end{pmatrix} \right\} \left( \begin{pmatrix} B & 0 \ 0 & 0 \end{pmatrix} \right) \left( \begin{pmatrix} 0 \ 0 \end{pmatrix} \right) \right), \left( \begin{pmatrix} 0 & 0 \ 0 & 0 \end{pmatrix} \right) \right)$$

all the matrices on the right hand side of the second identity above are of size $(m + n + k - 2) \times (m + n + k - 2)$, hence, from the Jacobi identity of $\mathfrak{gl}(m+n+k-2)$, it follows that

$$\{\{A, B\}, C\} + \{\{B, C\}, A\} + \{\{C, A\}, B\} = 0.$$

Therefore, the graded space

$$\bigoplus_{m \in \mathbb{Z}_{>0}} \mathfrak{gl}(m)$$

is a Lie algebra. From Lemma A.1, it follows that $\mathfrak{gl}(k)$ is a Lie algebra.

**Remark A.3.** Recall that for $n \in \mathbb{Z}_{>0}$, the natural embedding $\theta_n$ defined in (28) induces the following embedding on the level of Lie groups.

$$\theta_n: \quad GL_n(k) \rightarrow GL_{n+1}(k) \quad A \mapsto \begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix}$$

(28)

Denote

$$GL_\infty(k) := \lim_{n \in \mathbb{Z}_{>0}} GL_n(k).$$

Then $\mathfrak{gl}_\infty(k)$ is the Lie algebra of $GL_\infty(k)$. So it is interesting to ask whether $GL_\infty(k)$ has also a new Lie group structure which is compatible with the new Lie algebra structure on $\mathfrak{gl}_\infty(k)$.  

40
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