Conformal Field Theories
Near a Boundary in General Dimensions

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The implications of restricted conformal invariance under conformal transformations preserving a plane boundary are discussed for general dimensions $d$. Calculations of the universal function of a conformal invariant $\xi$ which appears in the two point function of scalar operators in conformally invariant theories with a plane boundary are undertaken to first order in the $\varepsilon = 4 - d$ expansion for the the operator $\phi^2$ in $\phi^4$ theory. The form for the associated functions of $\xi$ for the two point functions for the basic field $\phi^2$ and the auxiliary field $\lambda$ in the the $N \to \infty$ limit of the $O(N)$ non linear sigma model for any $d$ in the range $2 < d < 4$ are also rederived. These results are obtained by integrating the two point functions over planes parallel to the boundary, defining a restricted two point function which may be obtained more simply. Assuming conformal invariance this transformation can be inverted to recover the full two point function. Consistency of the results is checked by considering the limit $d \to 4$ and also by analysis of the operator product expansions for $\phi^2 \phi^2$ and $\lambda \lambda$. Using this method the form of the two point function for the energy momentum tensor in the conformal $O(N)$ model with a plane boundary is also found. General results for the sum of the contributions of all derivative operators appearing in the operator product expansion, and also in a corresponding boundary operator expansion, to the two point functions are also derived making essential use of conformal invariance.
1 Introduction

In more than two dimensions it is not generally possible to construct explicitly non trivial conformal field theories, with a detailed knowledge of the spectrum of spins and scale dimensions of all operators in the theory and further the coefficients appearing in operator product expansions for each pair of operators, at least to the same degree as in two dimensions. Nevertheless for a very large class of quantum field theories scale invariance at possible critical points may also be extended to invariance under the full conformal group [1] which implies significant restrictions on the form of multi-point correlation functions [2]. In particular the functional form of two and three point functions is effectively unique assuming conformal covariance since to construct conformal invariants, and hence for arbitrary functions to be present, four or more points are required (for operators with spin there may be more than one linearly independent conformally covariant form for the three point function [3]). From an experimental viewpoint in a statistical physics context only two point functions are mostly relevant and in this case conformal invariance gives little more than just scale invariance. On the other hand for calculating critical exponents in the $1/N$ expansion using conformal invariance has proved essential in obtaining results to $O(1/N^2)$ and $O(1/N^3)$ [4].

For statistical mechanical problems involving a boundary then at a critical point there are new critical exponents, expressing the behaviour of physical quantities near or on the boundary, which are unrelated (at least in any simple general fashion) to bulk critical exponents [5]. Also for any particular bulk critical point there are a variety of possible boundary conditions with differing surface exponents. As Cardy first showed [6,1] there is still a residual conformal group consisting of conformal transformations leaving the boundary invariant. For a plane boundary in $d$ Euclidean dimensions the restricted conformal group is then $O(d-1,1)$. In this case the two point function, involving operators at $x, x'$, depends on functions of a single conformal invariant $\xi(x,x')$. These functions depend on the particular theory and associated boundary conditions, or rather on their corresponding universality class. However conformal invariance is a significantly stronger requirement than scale invariance as far as potential experimental implications are concerned [7]. If we define coordinates $x_\mu = (y,x)$, with $y$ measuring the perpendicular distance from the boundary, then scale invariance by itself, with conventional translational and rotational symmetries, only restricts the two point function to depend on functions of the two scale invariant variables $s^2/y^2, s^2/y'^2$ where $s = x - x'$.

For scalar fields there is a single function of $\xi$ in the associated two point function but for fields with spin there may be several. As a particular illustration we consider the energy momentum tensor $T_{\mu\nu}$, which is traceless in the conformal limit, and the two point function then contains three possible invariant functions. However the conservation equation $\partial_\mu T_{\mu\nu} = 0$ provides two first order linear differential equations linking these functions. In two dimensions the number of invariant functions is reduced to two and in this case the differential equations have a unique solution, satisfying appropriate boundary conditions, in accord with known results [8]. Under the restricted conformal group it is also possible to form non zero two point functions for fields of differing spin and scale dimensions, such as for $T_{\mu\nu}$ and a scalar field although in this case the functional dependence is entirely
In this paper we investigate conditions on the form of two point functions for operators in conformally invariant theories with a plane boundary. Previously we discussed such two point functions involving the energy momentum tensor and derived the necessary conditions arising from the conservation equations for $T_{\mu\nu}$ by considering a perpendicular configuration where the two points were restricted to be on a perpendicular to the boundary [9] (referred to subsequently as I). Here we construct a conformally covariant form for the general two point function $\langle T_{\mu\nu}(x)T_{\rho\sigma}(x') \rangle$ for arbitrary $x, x'$ by constructing in terms of the invariant scalar $\xi(x, x')$ vectors $X_\mu \propto \partial_\mu \xi$, $X'_\sigma \propto \partial'_\sigma \xi$ at $x, x'$ respectively.

In general a two point function in the presence of a boundary may be written, expressing invariance under translations parallel to the boundary, as $\langle O(x)O'(x') \rangle = G(y, y', x - x')$. In our discussions it is useful to consider a transformation of $G$ obtained by integrating over planes parallel to the boundary, $\int d^{d-1}x G(y, y', x) = G_{||}(y, y')$, which defines what is sometimes referred to as the parallel susceptibility. For a conformally invariant theory, when $\langle O(x)O'(x') \rangle$ is expressible in terms of a function $g(\xi)$, this procedure, which we refer to as parallel integration, defines a new function $\hat{g}(\rho)$, $\rho = (y - y')^2/4yy'$. Crucially we are able to show that the transformation $g \to \hat{g}$ is invertible with $g$ expressible in terms of an integral over $\hat{g}$, for $d = 3$ the result is simply $g(\xi) = -\hat{g}'(\xi)/\pi$. This then allows for significant simplifications in calculations since finding $G_{||}(y, y')$ is sufficient to determine the full two point function $\langle O(x)O'(x') \rangle$.

These methods are illustrated by application to the $O(N) \sigma$-model in the large $N$ limit. This is in the same universality class as an $N$ component scalar field theory with a renormalisable $\phi^4$ interaction at the non Gaussian Wilson fixed point. We are able to recover some results obtained some time ago for the arbitrary function of $\xi$ associated with the two point functions of the basic $N$ component fields $\phi$ and also the auxiliary field $\lambda$ which is a scalar under $O(N)$ [10,11].

We also consider constraints on the invariant function $g(\xi)$ arising from the operator product expansion in which the operators with lowest dimension and non vanishing one point functions are relevant for the limit $\xi \to 0$. With conformal invariance only scalar operators, or their derivatives, in the operator product expansion of $\mathcal{O}$ and $\mathcal{O}'$ can contribute to their two point function in the presence of a boundary. We derive an explicit form for $g(\xi)$, in terms of hypergeometric functions, resulting from all derivative operators formed from any particular scalar operator occurring in the operator product expansion. The derivative terms in the operator product expansion are determined by the requirement that they should reproduce exactly the appropriate full three point function for the conformal field theory without boundary. These results are applied to two point amplitudes involving $T_{\mu\nu}$ and the expressions obtained automatically satisfy the required conservation equations.

A similar boundary operator expansion, where a bulk operator $\mathcal{O}(x)$ is expanded in terms of boundary operators $\tilde{\mathcal{O}}(x)$ [12], is also investigated. This constrains the function $g(\xi)$ in the limit $\xi \to \infty$. Again, using conformal invariance, we are able to sum up explicitly the results for all derivative operators formed from a given boundary operator.
which in general also involves hypergeometric functions. In this case the essential input determining the derivative terms for a particular boundary operator $\hat{O}$ appearing in the expansion of $\mathcal{O}$ is the form of the two point function for $\hat{O}$ and also $\mathcal{O}$ at arbitrary points. The boundary operator expansion of the energy momentum tensor $T_{\mu\nu}$ defines a boundary scalar operator $\hat{T}$ which is given by the non singular limit of $T_{\bot\bot}(x)$ as $x \to (0, x)$ and therefore has dimension $d$ [8]. However for $d > 2$ it is necessary to also consider boundary operators $\hat{T}_{ij}$, symmetric traceless tensors whose dimension is not constrained by general principles (this again reflects the non uniqueness of $\langle T_{\mu\nu}(x)T_{\alpha\rho}(x') \rangle$ for $d > 2$).

In detail in section 2 we consider the general conditions stemming from conformal invariance for two point functions with a boundary. In particular we describe the notion of quasi-primary operators which have simple properties under conformal transformations invariance for two point functions with a boundary. In particular we describe the notion of the two point function for the operator $\phi^2$ to first order in $\varepsilon$. In section 4 we analyse the $O(N)$ model in the large $N$ limit and describe in detail the transformation $g \leftrightarrow \hat{g}$ mentioned above. Section 5 contains more calculations for the two point functions of $V_{\mu}$ or $T_{\mu\nu}$ in the $O(N)$ model to leading order in $1/N$. The result for the two point function of the energy momentum tensor is non trivial, involving generalised hypergeometric functions. Sections 6 and 7 respectively contain the details of our discussions of the consequences of the operator product and boundary operator expansions with applications to the results obtained in the $O(N)$ model. Various calculational details are relegated to five appendices.

2 Conformal Invariance with Plane Boundaries

In flat $d$-dimensional Euclidean space with coordinates $x_\mu \in \mathbb{R}^d$ a conformal transformation $g$ is defined by preservation of the line element up to a local scale factor

$$x_\mu \to x'^{g}_{\mu}(x), \quad dx'^{g}_{\mu} dx'^{g}_{\nu} = \Omega^g(x)^{-2} dx_\mu dx_\nu \Rightarrow d^d x'^g = \Omega^g(x)^{-d} d^d x. \quad (2.1)$$

Such transformations define a group, $(x^{g_2})^{g_1} = x^{g_2g_1}$, which is isomorphic to $O(d + 1, 1)$. For any conformal transformation we may define a local orthogonal matrix belonging to $O(d)$ by

$$\mathcal{R}^g_{\mu\alpha}(x) = \Omega^g(x) \frac{\partial x'^g_\mu}{\partial x_\alpha}, \quad (2.2)$$

satisfying $\mathcal{R}^{g_2}(x^{g_1}) \mathcal{R}^{g_1}(x) = \mathcal{R}^{g_2g_1}(x)$. For $d > 2$ arbitrary conformal transformations can be generated by combining translations and rotations, for which $\Omega^g = 1$, with inversions through the origin, $i$, taking $x_\mu \to x_\mu / x^2$, for which

$$\mathcal{R}^i_{\mu\nu}(x) = I_{\mu\nu}(x) \equiv \delta_{\mu\nu} - 2 \frac{x_\mu x_\nu}{x^2}, \quad \Omega^i(x) = x^2. \quad (2.3)$$

Under conformal transformations $I_{\mu\nu}(x - x') \to \mathcal{R}_{\mu\alpha}(x) \mathcal{R}_{\nu\beta}(x') I_{\alpha\beta}(x - x')$ so that it transforms as a vector at $x$ and also at $x'$. In consequence $I_{\mu\nu}(x - x')$ may be regarded
as representing a form of parallel transport for conformal transformations and it plays a crucial role in the construction of two point functions for vector and other tensor fields.

In a conformal field theory fields $O^i(x)$ which form the vector space on which some finite dimensional irreducible representation of $O(d)$ acts are defined as quasi-primary fields if they transform under the full conformal group according to \[13\]

$$O^i(x) \rightarrow O^g_i(x^g) = \Omega(x)\eta^i D^g_j(R^g(x)) O^j(x),$$

(2.4)

where $\eta$ is the scale dimension of the fields and, for any $R \in O(d)$, $D(R)$ is the corresponding matrix in this representation. Since the transformation rule in (2.4) depends on $x$, derivatives of quasi-primary fields are not in general quasi-primary but transform with extra inhomogeneous terms. However for $V_\mu$ a vector field of dimension $d - 1$ or $T_{\mu\nu}$ a symmetric traceless tensor field of dimension $d$ it is straightforward to show [3] that $\partial_\mu V_\mu$ and $\partial_\mu T_{\mu\nu}$ are quasi-primary fields. Hence it follows that in a conformal field theory a conserved vector current and the energy momentum tensor necessarily have dimensions $d - 1$ and $d$ respectively. For the full conformal group there are no invariants which can be constructed from two or three points so that the two or three point functions of quasi-primary fields are essentially uniquely determined, with no arbitrary functions present.

For a flat Euclidean space with a plane boundary, as was shown by Cardy [6], a non trivial subgroup of conformal transformations still remains. If we let $x_\mu = (y, \mathbf{x})$ and define a plane boundary by $y = 0$, so that $y$ is the perpendicular distance from $x$ to the boundary, then it is necessary to restrict the conformal group to those transformations leaving $y = 0$ invariant. This subgroup is then generated by $d - 1$ dimensional translations and $O(d - 1)$ rotations acting on $\mathbf{x}$ together with the inversion $x_\mu \rightarrow x_\mu / x^2$ again and forms the group $O(d, 1)$. Under such transformations for two points $x_\mu, x'_\mu$ it is easy to see that

$$(x - x')^2 \rightarrow \frac{(x - x')^2}{\Omega(x)\Omega(x')} , \quad y \rightarrow \frac{y}{\Omega(x)} , \quad y' \rightarrow \frac{y'}{\Omega(x')} .$$

(2.5)

Hence we may construct invariants

$$\xi = \frac{(x - x')^2}{4yy'} , \quad v^2 = \frac{(x - x')^2}{(x - x')^2 + 4yy'} = \frac{\xi}{\xi + 1} .$$

(2.6)

where $0 \leq \xi < \infty$ and $0 \leq v < 1$.

For a one point function of an operator $O$ in the neighbourhood of a plane boundary conformal invariance under transformations as in (2.4) implies that this can only be non zero for quasi-primary scalar fields, belonging to the singlet representation of $O(d)$, when we can write

$$\langle O(x) \rangle = \frac{A_O}{(2y)^\eta} .$$

(2.7)

For the two point function of quasi-primary operators the existence of the conformal invariants in (2.5) implies that there may be an arbitrary function present. In addition the
two point function may be non zero for operators of differing spins and scale dimension, unlike the case for conformal invariance without a boundary. For two scalar fields we may write in general

\[ \langle O_1(x)O_2(x') \rangle = \frac{1}{(2y)^{\eta_1}(2y')^{\eta_2}} f_{12}(\xi) = \frac{(2y')^{\eta_1-\eta_2}}{(x-x')^{2n}} F_{12}(v), \quad \xi^{\eta_1} f_{12}(\xi) = F_{12}(v). \]  

(2.8)

The functions \( f_{12}(\xi) \) or \( F_{12}(v) \) are constrained by the operator product expansion (OPE). If for \( s = x - x' \to 0 \)

\[ O_1(x)O_2(x') \sim \frac{C_{12}^3}{(s^2)^{\frac{1}{2}(\eta_1+\eta_2-\eta_3)}} O_3(x'), \quad \langle O_3(x') \rangle = \frac{A_3}{(2y')^{\eta_3}}, \]  

(2.9)

then for \( \xi \sim v^2 \to 0 \) there is a contribution to the functions \( f_{12}(\xi) \) or \( F_{12}(v) \) in (2.8) of the form

\[ f_{12}(\xi) \sim C_{12}^3 A_3 \xi^{-\frac{1}{2}(\eta_1+\eta_2-\eta_3)}, \quad F_{12}(v) \sim C_{12}^3 A_3 v^{\eta_1-\eta_2+\eta_3}. \]  

(2.10)

Later, in section 6, we determine the entire contribution of non leading derivative terms in the OPE.

There are additional constraints on the functions \( f_{12}(\xi) \) or \( F_{12}(v) \) arising from a boundary operator expansion (BOE). For conformal field theories with boundary it is natural to define a basis of boundary operators \( \hat{O}_n(x) \), which transform irreducibly under \( O(d-1) \) and have a well defined scale dimension \( \hat{\eta}_n \). The precise set of such operators depends of course on the particular conformal theory and also the precise boundary conditions, compatible with conformal invariance imposed. The BOE has the form

\[ O(x) = \sum_n \frac{B_{O\hat{O}_n}}{(2y)^{\eta-\hat{\eta}_n}} \hat{O}_n(x), \]  

(2.11)

where in (2.7) \( A_O = B_{O^1} \). The two point function of a bulk operator and a boundary operator is determined up to an overall constant

\[ \langle O(x)\hat{O}_n(x') \rangle = \frac{B_{O\hat{O}_n}}{(2y)^{\eta-\hat{\eta}_n}(s^2)^{\hat{\eta}_n}}, \quad \hat{s}_\mu = (y,x-x'). \]  

(2.12)

Combining (2.11) and (2.12) it is easy to find that in the limit \( \xi \to \infty \) or \( v \to 1 \)

\[ f_{12}(\xi) \sim B_{O_1\hat{O}} B_{O_2\hat{O}} \xi^{-\hat{\eta}}, \quad F_{12}(v) \sim B_{O_1\hat{O}} B_{O_2\hat{O}} (1-v^2)^{\hat{\eta}-\eta_1}, \]  

(2.13)

where \( \hat{\eta} \) is the scale dimension of the leading boundary operator \( \hat{O} \) which gives a non zero contribution when using (2.11) and (2.12) to take the limit \( y' \to 0 \). Clearly there are consistency conditions arising from considering the alternative limit \( y \to 0 \). For a statistical mechanical system at a critical point the invariant functions \( f(\xi) \) or \( F(v) \) appearing in two
point functions are universal, depending only on the universality class and the boundary conditions.*

For non scalar quasi-primary fields the expressions for the two point function in the neighbourhood of a plane boundary are more complicated to analyse. In I a conformal transformation was used to restrict \( x_\mu = (y, 0) \) and \( x'_\mu = (y', 0) \), on a perpendicular to the boundary, where the remaining invariance or little group is reduced to \( O(d - 1) \) rotations. However we now describe an alternative approach which gives equivalent results. This is based on defining vectors \( X_\mu, X'_\mu \), with zero scale dimension, under restricted conformal transformations preserving the boundary at \( x, x' \) respectively, so that \( X_\mu \rightarrow \mathcal{R}_{\mu\alpha}(x)X_\alpha, X'_\mu \rightarrow \mathcal{R}_{\mu\alpha}(x')X'_\beta \). Since \( \xi \) in (2.6) is a scalar these are explicitly given by

\[
X_\mu = y^\nu \xi \partial_\nu \xi = v \left( \frac{2y}{s^2} s_\mu - n_\mu \right), \quad X'_\mu = y'^\nu \xi \partial'_\nu \xi = v \left( -\frac{2y'}{s^2} s_\mu - n_\mu \right), \quad n_\mu = (1, 0),
\]

which satisfy

\[
X_\mu X_\mu = X'_\mu X'_\mu = 1, \quad X'_\mu = I_{\mu\nu}(s)X_{\nu}.
\]

As \( y \rightarrow 0 \) \( X_\mu \rightarrow -n_\mu \) and for \( y' \rightarrow 0 \) \( X'_\mu \rightarrow -I_{\mu\nu}(s)n_\nu \).

We may now easily construct invariant forms for two point amplitudes for tensor fields. For \( V_\mu \) a vector field of dimension \( d - 1 \) then

\[
\langle V_\mu(x)V_\nu(x') \rangle = \frac{1}{(s^2)^{d-1}} (I_{\mu\nu}(s)C(v) + X_\mu X'_\nu D(v)).
\]

To impose current conservation, \( \partial_\mu V_\mu = 0 \), we use

\[
\partial_\mu \left( \frac{1}{(s^2)^{d-1}} I_{\mu\nu}(s) \right) = 0, \quad \partial_\mu \left( \frac{1}{(s^2)^{d-1}} X_\mu \right) = -(d - 1) \frac{2y'}{s^2} v,
\]

\[
\partial_\mu X'_\nu = \frac{2y'}{s^2} v (-I_{\mu\nu}(s) + X_\mu X'_\nu), \quad \partial_\mu F(v) = \frac{2y'}{s^2} X_\mu v^2 F'(v),
\]

to obtain

\[
v \frac{d}{dv} (C + D) = (d - 1)D.
\]

Away from the boundary, for \( v \rightarrow 0 \), \( D \rightarrow 0 \) while \( C \) is a constant determined by the bulk conformal theory. Similarly for \( T_{\mu\nu} \) the traceless energy momentum tensor of dimension \( d \) and \( \mathcal{O} \) a scalar field of dimension \( \eta \) we may find expressions for the mixed two point functions

\[
\langle V_\mu(x)\mathcal{O}(x') \rangle = \frac{(2y')^{d-1-\eta}}{(s^2)^{d-1}} X_\mu C_{V\mathcal{O}}(v),
\]

\[
\langle T_{\mu\nu}(x)\mathcal{O}(x') \rangle = \frac{(2y')^{d-\eta}}{(s^2)^d} \left( X_\mu X_\nu - \frac{1}{d} \delta_{\mu\nu} \right) C_{T\mathcal{O}}(v).
\]

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* Note that the results (2.8) give for the limits of the two point function perpendicular and parallel to the boundary \( \langle \mathcal{O}_1(y, 0)\mathcal{O}_2(y', 0) \rangle \propto 1/[s^{\eta_\perp}] \) as \( y \rightarrow \infty \) and \( \langle \mathcal{O}_1(y, x)\mathcal{O}_2(y, x') \rangle \propto 1/[s^{\eta_{||}}] \) as \( |s| \rightarrow \infty \) where the surface critical exponents \( \eta_\perp = \eta_1 + \eta, \eta_{||} = 2\eta \) [10].
identities (2.23) are satisfied, using (2.7), if
\[ C_{\mu \nu} \delta_{\mu \nu} (2.24) \] is then also
\[ c \mu \nu (\delta_{\mu \nu} - \delta_{\mu \nu}) \]
Current conservation and also \( \partial_\mu T_{\mu \nu} = 0 \), using (2.17) and
\[
\partial_\mu \left( \frac{1}{(s^2)^d} (X_\mu X_\nu - \frac{1}{d} \delta_{\mu \nu}) \right) = -(d - 1) \frac{2y'}{(s^2)^{d+1}} v X_\nu ,
\] requires
\[
v \frac{d}{dv} C_{V O} = (d - 1) C_{V O} , \quad v \frac{d}{dv} C_{T O} = d C_{T O} ,
\]
which gives a unique functional form
\[
C_{V O}(v) = c_{V O} v^{d-1} , \quad C_{T O}(v) = c_{T O} v^d .
\]
The coefficients \( c_{V O} \) and \( c_{T O} \) are determined by Ward identities. The energy momentum tensor \( T_{\mu \nu} \) may be defined by the response to arbitrary infinitesimal reparameterisations \( x_\mu \to x_\mu + a_\mu(x) \) [6,14] and from conformal invariance it is possible to derive the relations (see eqs. (E.7,8) in ref. [3])
\[
\langle T_{\mu \nu}(x) O(x') \rangle = d C \langle O(x') \rangle \delta^d(s) , \quad \partial_\mu \langle T_{\mu \nu}(x) O(x') \rangle = \left( \frac{\eta}{d} + C \right) \langle O(x') \rangle \partial_\mu \delta^d(s) - \langle \partial_\mu O(x') \rangle \delta^d(s) ,
\]
where \( C \) is undetermined (reflecting the arbitrariness of \( \langle T_{\mu \nu}(x) O(x') \rangle \) when regularised up to terms \( \propto \delta_{\mu \nu} \delta^d(x - x')(2y')^{-\eta} \). The expression obtained in (2.19,22) can be shown to be compatible with (2.23) by considering the expansion for \( s \to 0 \) which has the form
\[
\frac{v^d}{(s^2)^d} (X_\mu X_\nu - \frac{1}{d} \delta_{\mu \nu}) \sim \frac{1}{(2y')^d(s^2)^\frac{d}{2}} \left\{ \frac{s_\mu s_\nu}{s^2} - \frac{1}{d} \delta_{\mu \nu} - \frac{1}{2y'} \left( n_\mu s_\nu + n_\nu s_\mu - \delta_{\mu \nu} n \cdot s + (d - 2)n \cdot s \frac{s_\mu s_\nu}{s^2} \right) \right\} .
\]
With a suitable regularisation of the terms \( O(s^{-d}) \) we may find *
\[
\partial_\mu \left( \frac{v^d}{(s^2)^d} (X_\mu X_\nu - \frac{1}{d} \delta_{\mu \nu}) \right) = - \frac{d - 1}{d} S_d \frac{1}{(2y)^d} \partial_\nu \delta^d(x - x') + c \frac{1}{(2y)^d} \partial_\nu \delta^d(x - x') ,
\]
for \( c \) arbitrary and \( S_d = 2\pi \frac{1}{d} / \Gamma(\frac{1}{2} d) \). Assuming the result (2.25) the regularised trace of (2.24) is then also \( d c \delta^d(x - x')(2y)^d \). With these formulae it is easy to check that the identities (2.23) are satisfied, using (2.7), if \( C A_O = c_{T O} c \) and [8,7]
\[
\frac{c_{T O}}{d - 1} \frac{A_O}{S_d} .
\]
* In the spirit of differential regularisation [15] we may obtain a well defined distribution for the \( O(s^{-d}) \) terms \( \frac{1}{s^d} \left( \frac{s_\mu s_\nu}{s^2} - \frac{1}{d} \delta_{\mu \nu} \right) = \frac{1}{d(d - 2)} \partial_\mu \partial_\nu - \frac{1}{d} \delta_{\mu \nu} \partial^2 \right) \frac{1}{s^d} + c \delta_{\mu \nu} \delta^d(s) \), where \( c \) is an arbitrary constant. The divergence and trace of this expression may be found using \( -\partial^2 s^{-d+2} = (d - 2) S_d \delta^d(s) \). The trace of the \( O(s^{-d+1}) \) terms in (2.24) is zero while the divergence is unambiguous being \( \propto \delta^d(s) \).
The two point function of the energy momentum tensor itself can also be written in a similar fashion in terms of a basis of conformally covariant symmetric traceless tensors at \( x, x' \) in the general form

\[
\langle T_{\mu\nu}(x)T_{\sigma\rho}(x') \rangle = \frac{1}{(s^2)^d} \left\{ \mathcal{I}_{\mu\nu,\sigma\rho}(s)C(v) + \left( X_\mu X_\nu - \frac{1}{d} \delta_{\mu\nu} \right) \left( X'_{\sigma} X'_{\rho} - \frac{1}{d} \delta_{\sigma\rho} \right) A(v) \right. \\
+ \left( X_\mu X'_{\sigma} I_{\nu\rho}(s) + \mu \leftrightarrow \nu, \sigma \leftrightarrow \rho \right. \\
- \frac{4}{d} \delta_{\mu\nu} X'_\sigma X'_\rho - \frac{4}{d} \delta_{\sigma\rho} X_\mu X_\nu + \frac{4}{d^2} \delta_{\mu\nu} \delta_{\sigma\rho} \left. \right) B(v) \right\},
\]

where \( \mathcal{I}_{\mu\nu,\sigma\rho} \) represents inversion on symmetric traceless tensors

\[
\mathcal{I}_{\mu\nu,\sigma\rho}(s) = \frac{1}{2} \left( I_{\mu\sigma}(s) I_{\nu\rho}(s) + I_{\mu\rho}(s) I_{\nu\sigma}(s) \right) - \frac{1}{d} \delta_{\mu\nu} \delta_{\sigma\rho}.
\]

If the two point function is defined on all \( \mathbb{R}^d \), with no boundary, then \( A, B \) are absent while \( C \) is a constant \( C_T \) and the conservation equation for \( T_{\mu\nu} \), giving \( \partial_\mu \langle T_{\mu\nu}(x)T_{\sigma\rho}(x') \rangle = 0 \), is satisfied by virtue of

\[
\partial_\mu \left( \frac{1}{(s^2)^d} \mathcal{I}_{\mu\nu,\sigma\rho}(s) \right) = 0.
\]

In the general case the conservation equation, using (2.29) with (2.17), (2.20) as well as

\[
\partial_\mu \left( \frac{1}{(s^2)^d} \left( X_\mu I_{\nu\sigma}(s) + X_\nu I_{\mu\sigma}(s) - \frac{2}{d} \delta_{\mu\nu} X'_{\sigma} \right) \right) \\
= \frac{2y^d}{(s^2)^{d+1}} \frac{v}{d} \left( (2 - d^2) I_{\nu\sigma}(s) + (d - 2) X_\nu X'_{\sigma} \right),
\]

leads to the conditions

\[
\left( v \frac{d}{dv} - d \right) C + 2B = - \frac{2}{d} (A + 4B) - dC, \tag{2.31a}
\]

\[
\left( v \frac{d}{dv} - d \right) (d - 1) A + 2(d - 2) B = 2A - 2(d^2 - 4)B. \tag{2.31b}
\]

These equations still leave the \( A, B, C \) undetermined up to an arbitrary function of \( v \). Away from the boundary, or as \( v, \xi \to 0 \), the terms involving \( X_\mu, X'_{\sigma} \) should disappear, hence in this limit \( A, B \to 0 \), while \( C \to C_T \) which is the constant determining the scale of energy momentum tensor two point function which has a simple form in the associated conformal field theory without boundary,

\[
\langle T_{\mu\nu}(x)T_{\sigma\rho}(x') \rangle_{\text{no boundary}} = \frac{C_T}{(s^2)^d} \mathcal{I}_{\mu\nu,\sigma\rho}(s).
\]

On the boundary if we assume \( T_{1i}(0, x) = 0 \), with \( i \) denoting components tangential to the boundary, then we must require \( C(1) + 2B(1) = 0 \). Since \( \langle T_{\mu\nu} \rangle = 0 \) with a plane boundary there are no Ward identity relations for this two point function.
When $d = 2$ there are only two linearly independent tensors instead of the three in (2.27) so that only $A, C + 2B$ are relevant. The differential equations then have a unique solution obeying the boundary conditions

$$A(v) = 4C_T v^4, \quad C(v) + 2B(v) = C_T (1 - v^4). \quad \text{(2.33)}$$

In two dimensions with complex coordinates $z = x + iy, \bar{z} = x - iy, \delta_{z\bar{z}} = \frac{1}{2}$ then

$$X_zX'_z = \frac{1}{2}i_{zz}(s) = -\frac{1}{4}\bar{z} - z', \quad X_{z\bar{z}} = \frac{1}{4}\bar{z} - z', \quad I_{z\bar{z}}(s) = 0, \quad \text{(2.34)}$$

and since $v^2 = |z - z'|^2/|z - \bar{z}'|^2$ we thereby obtain the universal form [1]

$$\langle T_{zz}(x)T_{zz}(z') \rangle = \frac{1}{4}C_{T} \frac{1}{(z - z')^4}, \quad \langle T_{zz}(z)T_{zz}(z') \rangle = \frac{1}{4}C_{T} \frac{1}{(z - z')^4}. \quad \text{(2.35)}$$

Up to a suitable normalisation factor $C_T$ is of course the Virasoro central charge.

For general $d$ the expression (2.27) for the energy momentum tensor two point function may be related to the results in I, where free field expressions were given for arbitrary $d$ and also the functional dependence on $v$ calculated to first order in the $\varepsilon$ expansion from $d = 4$ for the non Gaussian fixed point in $\phi^4$ field theory, by restricting (2.27) to the perpendicular configuration, when for $y > y', X_\mu = -X'_\mu = n_\mu, \ v = (y - y')/(y + y'),$ the general form can be written as

$$\langle T_{ij}(y, 0)T_{k\ell}(y', 0) \rangle = \frac{1}{(y - y')^{2d}} \gamma(v)\delta_{j,\ell}, \quad \text{(2.36)}$$

$$\langle T_{ij}(y, 0)T_{k\ell}(y', 0) \rangle = \frac{1}{(y - y')^{2d}} (\delta(v)\delta_{ij}\delta_{k\ell} + \epsilon(v)(\delta_{ik}\delta_{j\ell} + \delta_{i\ell}\delta_{jk})).$$

All other non zero components may be obtained trivially from (2.36) using $T_{\mu\nu} = T_{\nu\mu}$ and $T_{\mu\mu} = 0,$ for instance

$$\langle T_{11}(y, 0)T_{11}(y', 0) \rangle = \frac{1}{(y - y')^{2d}} \alpha(v), \quad \alpha = (d - 1)((d - 1)\delta + 2\varepsilon). \quad \text{(2.37)}$$

It is straightforward to show that

$$\alpha = \frac{d - 1}{d^2}((d - 1)(A + 4B) + dC), \quad \gamma = -B - \frac{1}{2}C, \quad \epsilon = \frac{1}{2}C, \quad \delta = \frac{1}{d^2}(A + 4B - dC). \quad \text{(2.38)}$$

Eqs. (2.31a,b) then translates into the results given in I, for instance (2.31b) becomes more simply

$$\left(v \frac{d}{dv} - d\right)\alpha(v) = 2(d - 1)\gamma(v). \quad \text{(2.39)}$$

On the boundary itself it is easy to see that

$$\langle T_{11}(0, x)T_{11}(0, x') \rangle = \frac{1}{(s^2)^d} \alpha(1). \quad \text{(2.40)}$$
3 Scalar Field Theory

The simplest conformally invariant field theory is that given by free massless scalar fields $\phi^\alpha(x)$, $\alpha = 1 \ldots N$ (for later convenience we assume $N$ components). To ensure compatibility with conformal invariance we may impose either Neumann or Dirichlet boundary conditions which for a plane boundary at $y = 0$ requires $\partial_y \phi^\alpha(0, x) = 0$ or $\phi^\alpha(0, x) = 0$ respectively. If

$$
\langle \phi^\alpha(x) \phi^\beta(x') \rangle = \delta^{\alpha\beta} G_\phi(x, x'), \quad G_\phi(x, x') = \frac{1}{s^{2\eta_\phi}} F_\phi(v),
$$

then in the free field case it is easy to see, using the method of images, that

$$
\eta_\phi = \frac{1}{2} d - 1, \quad F_\phi(v) = A(1 \pm v^{d-2}),
$$

where the upper/lower signs correspond to Neumann/Dirichlet boundary conditions and

$$
A = \frac{1}{(d-2) S_d}, \quad S_d = \frac{2 \pi^{\frac{d}{2}} d}{\Gamma\left(\frac{d}{2}\right)}.
$$

For the composite operator $\phi^2$ with dimension $\eta_{\phi^2}$ then in general

$$
\langle \phi^2(x) \phi^2(x') \rangle = \langle \phi^2(x) \rangle \langle \phi^2(x') \rangle + G_{\phi^2}(x, x'),
$$

$$
G_{\phi^2}(x, x') = \frac{1}{s^{2\eta_{\phi^2}}} F_{\phi^2}(v), \quad \langle \phi^2(x) \rangle = \frac{A_{\phi^2}}{(2y)^{\eta_{\phi^2}}},
$$

and in the free field case

$$
\eta_{\phi^2} = 2\eta_\phi = d - 2, \quad F_{\phi^2}(v) = 2NF_{\phi}(v)^2, \quad A_{\phi^2} = \pm NA.
$$

A more interesting case is the conformal field theory realised at the non Gaussian fixed point in the theory with interaction $\frac{1}{24} g (\phi^2)^2$ for which critical exponents and other universal quantities can be calculated in the $\varepsilon = 4 - d$ expansion. With minimal subtraction at the fixed point $g_*/16\pi^2 = 3\varepsilon/(N+8) + O(\varepsilon^2)$. For the $\langle \phi \phi \rangle$ two point function the $O(\varepsilon)$ corrections to the free field results for the universal function $F_\phi(v)$ have been calculated by Gompper and Wagner [16] and also in I*. Since $g_* = O(\varepsilon)$ it is sufficient to first order to evaluate the Feynman integrals just for $d = 4$. The results obtained at one loop give $\eta_\phi = \frac{1}{2} d - 1 + O(\varepsilon^2)$ and

$$
F(v)^{(1)} = \mp \frac{1}{2} \frac{N + 2}{N + 8} A \varepsilon \left( v^2 \ln \frac{1 - v^2}{v^2} \pm \ln(1 - v^2) \right).
$$

---

* In I the results in (C.1.2), and also (C.3), for $G(y, y')$ should have an additional $\pm$ sign. Other misprints in I are that the factor multiplying $F''$ in (C.5) should be $v^2(1 - v^2)^2$, in (C.8a) the tensor structure should be $\delta_{ik}\delta_{j\ell} + \delta_{il}\delta_{jk}$ and in the last line of (C.8b) the second factor should be $x^2 + z^2 - y'^2$, in (2.6) in the transformation equation for $T_{\mu\nu}$ replace $\Omega(\bar{x})$ by $\Omega(\bar{x})^d$, in (2.26) the correct equation is $A_{ij} = \eta \delta_{ij}((d - 1) S_d$ and in (2.29) in the formula for $a(1) 2d \to 2d$. Further in (A.2), and also subsequently on the same page, $e^{\mu i} e_{\mu j} \to e^{\mu i} e_{\nu j}$. 

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In the Dirichlet case it is easy to see that adding the result (3.6) to the free result (3.2) is compatible to the order calculated with the form

\[ F_{\phi}(v)_{\text{ord}} = C_{\phi} \left( 1 - v \eta_{\phi^2} \right) \hat{n} - \eta_{\phi}, \]  

(3.7)

where \( C_{\phi} = A(1 + O(\varepsilon)) \) and \( \hat{n} \) is the dimension of the surface operator \( \hat{\phi}_n(x) = \partial_1 \phi(0, x) \). The expression (3.6) implies the one loop corrections

\[ \eta_{\phi^2} = d - 2 + \frac{N + 2}{N + 8} \varepsilon + \ldots, \quad \hat{n} = \frac{1}{2} d - \frac{1}{2} \frac{N + 2}{N + 8} \varepsilon + \ldots, \]  

(3.8)
in accord with long established results. The subscript ord in (3.7) denotes that this function is appropriate to the ordinary transition in a statistical mechanical context. It is straightforward then to verify that the functional form in (3.7) is consistent with the limiting behaviours given by (2.10), since \( \phi^2 \) is here the lowest dimension operator appearing in the operator product \( \phi \phi \) beyond the identity, and also with (2.13) since in the Dirichlet case \( \hat{\phi}_n \) is the leading boundary operator appearing in the BOE of \( \phi \).

For the Neumann case, which is appropriate for the so called special transition, we may write

\[ F_{\phi}(v)_{\text{sp}} = C_{\phi} \left( 1 + v \eta_{\phi^2} \right) \left( 1 - v \eta_{\phi^2} \right) \hat{n} - \eta_{\phi}, \]  

(3.9)

where

\[ \hat{n} = \frac{1}{2} d - 1 - \frac{1}{2} \frac{N + 2}{N + 8} \varepsilon + \ldots, \]  

(3.10)
is the scale dimension of the the boundary operator \( \hat{\phi}(x) = \phi(0, x) \). (3.9) is again in accord with the usual OPE for \( \phi \phi \) and as \( y' \to 0 \) or \( v \to 1 \) it is easy to see, with (3.1),

\[ \langle \phi^\alpha(x) \phi^\beta(x') \rangle_{\text{sp}} \sim \delta^{\alpha\beta} \frac{2 C_{\phi}}{\hat{s}^2 \eta_{\phi}} \left( 4yy' \hat{s}^2 \right)^{\hat{n} - \eta_{\phi}} \left( 1 + (2\eta_{\phi} - \frac{1}{2} \eta_{\phi^2} - \hat{n}) \frac{2yy'}{\hat{s}^2} \right), \]  

(3.11)

where \( \hat{s}^2 \) is given in (2.12). It is crucial that, with the above results for scale dimensions, the next to leading corrections vanish, corresponding to the boundary operator \( \hat{\phi}_n = 0 \).

As \( v \to 0 \) the particular boundary conditions are irrelevant so we must require \( F_{\phi}(0)_{\text{sp}} = F_{\phi}(0)_{\text{ord}} \), as exemplified in (3.7.9). By considering the terms \( \propto v \eta_{\phi^2} \) we easily see, following (2.10), that

\[ \frac{A_{\phi^2,\text{sp}}}{A_{\phi^2,\text{ord}}} = -1 - \frac{N + 2}{N + 8} \varepsilon + O(\varepsilon^2), \]  

(3.12)

which is independent of the normalisation of the operator \( \phi^2 \).

We have also calculated the leading \( \varepsilon \) corrections for the two point function of the operator \( \phi^2 \), which represents the energy density in a statistical physics context. There are two Feynman graphs shown in figs. (1a,b), fig. (1a) corresponding to the one loop correction to \( G_{\phi}(x, x') \).
Details of the calculation are given in appendix A. After removing divergences the result is
\[
F_{\phi^2}(v)^{(1)} = 2A^2 N \frac{N + 2}{N + 8} \varepsilon \left(1 + v^4 - (1 - v^2)^2 \ln(1 - v^2) + 2v^4 \frac{2 - v^2}{1 - v^2} \ln v^2 \pm v^2 \ln v^2\right). \quad (3.13)
\]
In the Dirichlet case we therefore have to this order, combining with (3.2,5),
\[
F_{\phi^2}(v)_{\text{ord}} = C_{\phi^2} \left((1 - v^2)^{d - \eta_{\phi^2}} + \frac{N + 2}{N + 8} \varepsilon \left(2v^2 + v^4 \frac{3 - v^2}{1 - v^2} \ln v^2\right)\right), \quad (3.14)
\]
for \(C_{\phi^2} = 2A^2 N(1 + O(\varepsilon))\). In the limits \(v \rightarrow 0,1\) this behaves as
\[
F_{\phi^2}(v)_{\text{ord}} \sim C_{\phi^2} - 2C_{\phi^2} \left(1 - \frac{3}{2} \frac{N + 2}{N + 8} \varepsilon\right) v^{\eta_{\phi^2}}, \quad (3.15a)
\]
\[
F_{\phi^2}(v)_{\text{ord}} \sim C_{\phi^2} \left(1 + \frac{11}{6} \frac{N + 2}{N + 8} \varepsilon - \varepsilon\right) (1 - v^2)^{d - \eta_{\phi^2}}. \quad (3.15b)
\]
The first case corresponds to the leading contributions of the identity and \(\phi^2\) in the OPE as expected. For \(v \rightarrow 1\) the result corresponds to the boundary operator \(\hat{T}(x) = T_{11}(0, x)\) whose scale dimension is \(d\) at any conformal fixed point. In the Neumann case the results to this order from (3.13) can be simply expressed by
\[
F_{\phi^2}(v)_{\text{sp}} = 4C_{\phi^2} \left(1 - \frac{N + 2}{N + 8} \varepsilon\right) v^{\eta_{\phi^2}} + F_{\phi^2}(v)_{\text{ord}}. \quad (3.16)
\]
For \(v \rightarrow 1\) the first term is a constant and so corresponds to the contribution of a boundary operator \(\hat{\phi}^2\) with dimension \(\hat{\eta}_{\phi^2} = \eta_{\phi^2} + O(\varepsilon^2) = 2 + O(\varepsilon)\) which is now present in addition.
to the operator $\hat{T}$. For $v \to 0$ using $F_{\phi^2}(v) \sim C_{\phi^2} + C_{\phi^2} \phi^2 A_{\phi^2} v^{n_{\phi^2}}$ it is easy to see that (3.15a,16) are in accord with (3.12).

4 O(N) Model for Large N

Besides the $\varepsilon$ expansion it is also possible to obtain non trivial fixed points in quantum field theories by considering expansions in $1/N$ for theories with $N$-component fields [17]. Such large $N$ methods have the virtue of allowing calculations of critical exponents at conformally invariant fixed points for any dimension, at least in the range $2 < d < 4$. The simplest example is the non linear $O(N)$ $\sigma$ model for fields $\phi^\alpha \in S^{N-1}$. In a lagrangian formalism it is classically equivalent to remove the non linear constraint on $\phi$ but introduce an auxiliary field $\lambda$, without any kinetic term and with an interaction $L_I = \frac{1}{2} \lambda \phi^2$, whose field equation then enforces the condition $\phi^2 = \text{const.}$. The $1/N$ expansion for the corresponding quantum field theory may naturally be written in terms of propagators for $\phi, \lambda$ with a single vertex given by $L_I$. At the scale invariant fixed point Vasil’ev, Pis’mak and Khonkonen [4] formulated the $\sigma$ model in terms of a skeleton graph expansion for the two point functions for $\phi, \lambda$ which can be solved self consistently for the scaling dimensions $\eta_\phi, \eta_\lambda$ by imposing conformal invariance in a tractable $1/N$ expansion. Using these methods $\eta_\phi, \eta_\lambda$, which determine all other bulk critical indices, have been determined to $O(N^{-3}), O(N^{-2})$ respectively. To leading order $\eta_\phi = \frac{1}{2} d - 1, \eta_\lambda = 2$ and results are consistent with those from the $\varepsilon$ expansion for the renormalisable $\frac{1}{24} g (\phi^2)^2$, as considered in the previous section, with the identification $\eta_\lambda = \eta_\phi^2$. In recent years the $O(N)$ $\sigma$ model has been extensively investigated at its conformal fixed point so that the spectrum of operators present and their scaling dimensions are well understood [17,14].

Writing the two point functions for the basic fields $\phi^\alpha, \lambda$ as

$$\langle \phi^\alpha(x) \phi^\beta(x') \rangle = \delta^{\alpha\beta} G_\phi(x,x'), \quad \langle \lambda(x) \lambda(x') \rangle = \langle \lambda(x) \rangle \langle \lambda(x') \rangle + G_\lambda(x,x'), \quad (4.1)$$

since we assume manifest $O(N)$ invariance and $\langle \phi^\alpha \rangle = 0$. To zeroth order in $1/N$ [4] the basic equations determining these are equivalent to

$$(-\nabla^2 + \langle \lambda(x) \rangle) G_\phi(x,x') = \delta^d(x - x'), \quad (4.2a)$$

$$\int d^d x'' G_\phi(x,x'')^2 G_\lambda(x'',x') = -\frac{2}{N} \delta^d(x - x'). \quad (4.2b)$$

Alternatively in the renormalisable $(\phi^2)^2$ theory, with the identification $\lambda = \frac{1}{4} g \phi^2$, $G_\lambda$ represents the leading contribution at large $N$ due to summing all bubble diagrams. With no boundary present (4.1a,b) are trivial to solve at a fixed point. If, as required by conformal invariance,

$$G_\phi(x,x') = \frac{A}{s^2 \eta_\phi}, \quad G_\lambda(x,x') = \frac{B}{s^2 \eta_\lambda}, \quad (4.3)$$

then (4.2b) requires in general that

$$\eta_\lambda = d - 2 \eta_\phi, \quad A^2 B = -\frac{2}{N} f(2 \eta_\phi), \quad f(\alpha) = \frac{1}{\pi^d} \frac{\Gamma(d-\alpha) \Gamma(\alpha)}{\Gamma(\alpha-\frac{1}{2}d) \Gamma(\frac{1}{2}d - \alpha)}, \quad (4.4)$$

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while (4.2a) gives \( \eta_\phi = \frac{1}{2}d - 1 \), \( \langle \lambda \rangle = 0 \), with \( A \) as in (3.3), and hence \( \eta_\lambda = 2 \) for \( N \to \infty \). The relation between \( \eta_\lambda \) and \( \eta_\phi \) in (4.4) is necessary for the integral in (4.2b) to be conformally invariant, using the transformation of the measure in (2.1). It is also useful to note that for \( \eta_\phi = \frac{1}{2}d - 1 \) \( B > 0 \) for \( 2 < d < 4 \) and as \( d \to 4 \) \( B \sim 8\varepsilon^2/N \). In general the normalisations of the fields \( \phi, \lambda \) are arbitrary but with the above choices the scale of the vertex function is determined. Without a boundary, when \( \langle \lambda \rangle = 0 \), conformal invariance dictates the general form

\[
\langle \phi^\alpha(x_1)\phi^\beta(x_2)\lambda(x_3) \rangle = \delta^{\alpha\beta} \frac{C_{\phi\phi\lambda}}{(x_{12}^2)^{\frac{d'}{2}}\delta^2(x_{12}^2)^{\frac{d'}{2}}\eta_\lambda}, \quad x_{ij} = x_i - x_j. \tag{4.5}
\]

To leading order in \( 1/N \),

\[
\frac{C_{\phi\phi\lambda}}{AB} = \frac{1}{2\varepsilon}. \tag{4.6}
\]

The normalisation independent vertex coupling is then \( C_{\phi\phi\lambda}/(A^2B)^{\frac{1}{2}} = O(N^{-\frac{1}{2}}) \) which implies formally that the \( 1/N \) expansion is consistent.

The aim here is to extend these results to the situation for a plane boundary. The essential results in the large \( N \) limit have been obtained by Bray and Moore [10] and also Ohno and Okabe [11] some time ago but we differ in making essential use of conformal invariance which leads to a a more direct and perhaps simpler derivation of the \( \lambda \) two point function by solving (4.2b) (in both treatments Dirichlet boundary conditions, or the ordinary transition, is more straightforward than Neumann boundary conditions, as appropriate for the special transition). Following (2.4) if we write

\[
G_\phi(x,x') = \frac{1}{(4\pi)^{\frac{d'}{2}-1}}f_\phi(\xi) = \frac{1}{(s^2)^{\frac{d'}{2}-1}}F_\phi(v), \quad F_\phi(v) = \xi^{\frac{d'}{2}-1}f_\phi(\xi), \tag{4.7}
\]

then imposing (4.2a) requires \( F_\phi(0) = A \) and also, since \( \eta_\lambda = 2 \), if we take

\[
\langle \lambda(x) \rangle = \frac{A_\lambda}{4y^2}, \tag{4.8}
\]

\( F_\phi \) satisfies the differential equation

\[
\xi(1 + \xi) \frac{d^2}{d\xi^2}F_\phi + (\frac{1}{2}\varepsilon + 2\xi) \frac{d}{d\xi}F_\phi - \frac{1}{4}A_\lambda F_\phi = 0. \tag{4.9}
\]

This is easily seen to be soluble in general in terms of hypergeometric functions, the appropriate solution with the relevant behaviour as \( \xi \to 0 \) being

\[
AF(a,b;\tfrac{1}{2}\varepsilon; -\xi) = A(1 - v^2)^aF(a,\tfrac{1}{2}\varepsilon - b;\tfrac{1}{2}\varepsilon; v^2), \quad a + b = 1, \quad ab = -\frac{1}{4}A_\lambda. \tag{4.10}
\]

\* This may be obtained by calculating the lowest order contribution, using the conformal star triangle relation, \( \langle \phi^\alpha(x_1)\phi^\beta(x_2)\lambda(x_3) \rangle = -\delta^{\alpha\beta} \int d^d x \frac{A^2B}{((x_1-x_2)^2(x_2-x_3)^2)^{d/2-1}(x_3-x)^4} \) but the result is also in accord with a more systematic analysis based on requiring cancellation of shadow fields in the OPE [14].
For the particular applications of interest here we assume that the leading term for $G_\phi$ in the large $N$ limit is given by the $N \to \infty$ results of the previous section. It is important to note that the first order in $\varepsilon$ results for the scaling dimensions are exact in this limit\(^1\). Hence in the Dirichlet case we obtain

$$f_\phi(\xi)_{\text{ord}} = A(\xi(1 + \xi))^{-\frac{1}{2}d+1}, \quad F_\phi(v)_{\text{ord}} = A(1 - v^2)^{-\frac{1}{2}d-1}. \quad (4.11)$$

This solution is in exact agreement with the OPE for $\phi\phi$ where from (4.5) and (4.1,3) in the limit $s = x - x' \to 0$

$$\phi^\alpha(x)\phi^\beta(x') \sim \frac{A}{(s^2)^{\frac{1}{2}d-1}} \delta^{\alpha\beta} + \frac{C_{\phi\phi}^{\lambda}}{(s^2)^{-\frac{1}{2}d-1}} \delta^{\alpha\beta} \lambda(x'), \quad C_{\phi\phi}^{\lambda} = \frac{C_{\phi\phi}^{\lambda}}{B}. \quad (4.13)$$

Applying this to the two point function (4.1) with the general form (4.7) then we must require as $v \to 0$ $F_\phi(v) \sim A+C_{\phi\phi}^{\lambda} A_\lambda v^2$. Using the explicit expression (4.11) in conjunction with (4.6) is easily seen to recover (4.12).

Given an explicit formula for $G_\phi$ then $G_\lambda$ is determined by solving (4.2b). Given that $\eta_\lambda = 2$ to leading order in $1/N$ then conformal invariance dictates the general form

$$G_\lambda(x, x') = \frac{1}{(4yy')^2} f_\lambda(\xi) = \frac{1}{(s^2)^2} F_\lambda(v), \quad F_\lambda(v) = \xi^2 f_\lambda(\xi). \quad (4.14)$$

We first discuss the essential mathematical problem of inverting an operator represented by a kernel $G(x, x')$ which transforms as a scalar of dimension $\alpha$ at $x, x'$ on $\mathbb{R}_+^d = \{(y, x); y > 0\}$. Representing the inverse by a kernel $H(x, x')$ we require

$$\int_{\mathbb{R}_+^d} d^d x G(x_1, x) H(x, x_2) = \delta^d(x_1 - x_2), \quad x_1, x_2 \in \mathbb{R}_+^d. \quad (4.15)$$

Under a conformal transformation as in (2.1) $\delta^d(x_1 - x_2) \to \Omega^q(x_1)^d\delta^d(x_1 - x_2)$. Hence the integral (4.15) is compatible with restricted conformal transformations preserving the boundary if we take

$$G(x, x') = \frac{1}{(4yy')^\alpha} g(\xi), \quad H(x, x') = \frac{1}{(4yy')^{d-\alpha}} h(\xi). \quad (4.16)$$

Hence we may expect to reduce (4.15) to an equation for the single variable functions $g, h$. To achieve this it is convenient to define a transform, $g \to \hat{g}$, by integrating $G$ over planes parallel to the boundary*

$$\int d^{d-1} x G(x, x') = \frac{1}{(4yy')^{\alpha-\lambda}} \hat{g}(\rho), \quad \rho = \frac{(y - y')^2}{4yy'}, \quad \lambda = \frac{1}{2}(d - 1), \quad (4.17)$$

\(^1\) The two loop results [12,19,20,21] for the surface exponents $\hat{\eta}, \hat{\eta}_n$ vanish as $N \to \infty$.

* This is analogous to the radon transform [22].
where
\[ \hat{g}(\rho) = \frac{\pi^\lambda}{\Gamma(\lambda)} \int_0^\infty du \, u^{\lambda-1} g(u + \rho). \] (4.18)
This transform may be inverted, \( \hat{g} \to g \), by
\[ g(\xi) = \frac{1}{\pi^\lambda \Gamma(-\lambda)} \int_0^\infty d\rho \, \rho^{-\lambda-1} \hat{g}(\rho + \xi), \] (4.19)
where the integral of \( \rho^{-\lambda-1} \) is singular for \( d \) of interest here but may be defined by analytic continuation in \( \lambda \) from \( \text{Re}(\lambda) < 0 \). The inversion formula may be verified by using
\[ \int du \, (\rho - u)^{\mu-1} u^{\lambda-1} \sim B(\mu, \lambda) \rho_+^{\mu+\lambda-1} \sim \Gamma(-\lambda) \Gamma(\lambda) \delta(\rho) \text{ as } \mu \to -\lambda. \] (4.20)
For \( d = 3 \Rightarrow \lambda = 1 \) we use
\[ \frac{\rho_+^{-\lambda-1}}{\Gamma(-\lambda)} \sim \delta'(\rho) \text{ for } \lambda \to 1, \] (4.21)
to reduce (4.19) to the simple form
\[ g(\xi) = -\frac{1}{\pi} \hat{g}'(\xi), \] (4.22)
which is easy to check directly from (4.18).

With the definition (4.17) for \( \hat{g} \), and correspondingly for \( \hat{h} \), it is easy to see, by integrating over \( x_1 \), that (4.15) reduces to
\[ \int_0^\infty dy \frac{1}{y} \hat{g}(\rho_1) \hat{h}(\rho_2) = 4y_1 \delta(y_1 - y_2), \quad \rho_i = \frac{(y - y_i)^2}{4yy_i}. \] (4.23)
If \( y = e^{2\theta}, y_i = e^{2\theta_i} \), this may be simplified to
\[ \int_{-\infty}^\infty d\theta \hat{g}(\sinh^2(\theta - \theta_1)) \hat{h}(\sinh^2(\theta - \theta_2)) = \delta(\theta_1 - \theta_2). \] (4.24)
This is then straightforward to solve by Fourier transforms
\[ \tilde{g}(k) = \int d\theta \, e^{ik\theta} \hat{g}(\sinh^2 \theta), \] (4.25)
which finally gives
\[ \tilde{g}(k) \tilde{h}(k) = 1. \] (4.26)
Hence \( \tilde{h} \) may be determined and by taking the inverse Fourier transform and applying (4.19) it is possible to find \( h \).
For the problem of interest here then, given the result (4.11) for \( G_\phi \), we should take \( \alpha = d - 2 \) and, with \( f_\phi(\xi)^2_{\text{ord}} = A^2 g(\xi) \),

\[
g(\xi) = [\xi(\xi + 1)]^{-d+2}.
\]

(4.27)

For simplicity we describe here the application of the above method for the physically interesting case of \( d = 3 \) when the integrals are also relatively standard. General \( d \) is discussed in appendix B. For \( d = 3 \) we easily obtain

\[
\hat{g}(\rho) = \pi \ln \frac{\rho + 1}{\rho}, \quad \hat{g}(\sinh^2 \theta) = -2\pi \ln |\tanh \theta|,
\]

(4.28)

and therefore

\[
\tilde{g}(k) = \frac{2\pi^2}{k} \tanh \frac{1}{4} \pi k.
\]

(4.29)

Using (4.26) to give \( \tilde{h}(k) \) we then get

\[
\hat{h}(\rho) = -\frac{1}{2\pi^3} \frac{1}{\rho(\rho + 1)}, \quad h(\xi) = -\frac{1}{2\pi^4} \left( \frac{1}{\xi^2} - \frac{1}{(\xi + 1)^2} \right),
\]

(4.30)

with \( \hat{h} \to h \) given by (4.22). Hence for \( d = 3 \), when \( A = 1/4\pi \), the solution of (4.2b) gives in (4.14)

\[
f_\lambda(\xi)_{\text{ord}} = \frac{16}{\pi^2 N} \frac{1 + 2\xi}{\xi^2(1 + \xi)^2}, \quad F_\lambda(\nu)_{\text{ord}} = \frac{16}{\pi^2 N} (1 - \nu^4).
\]

(4.31)

For general \( d \) the results of appendix B lead to

\[
f_\lambda(\xi)_{\text{ord}} = B \frac{\Gamma(d)\Gamma(d-2)}{\Gamma(2d-4)} \xi^{-d} F(d-2, d; 2d-4; -\frac{1}{\xi}) = 2B \frac{Q^2_{d-3}(1 + 2\xi)}{\xi(1 + \xi)},
\]

(4.32)

\[
B = \frac{16}{\pi^2 N} \frac{\varepsilon \Gamma(d-2)}{\Gamma(2-\frac{1}{2}d) \Gamma(\frac{1}{2}d-1)^3},
\]

where \( Q^2_{d-3} \) is an associated Legendre function and, given \( A \) in (3.2), \( B \) is in accord with (4.4) for \( \eta_\phi = \frac{1}{2} d - 1 \). For \( d = 3 \), using \( F(1, 3; 2; z) = \frac{1}{2} ((1 - z)^{-2} + (1 - z)^{-1}) \), this is easily seen to be in agreement with (4.31). It is also of interest to consider the limit as \( d \to 4 \) when

\[
\xi^{-d+2} F(d-2, d; 2d-4; -\frac{1}{\xi}) = \frac{\xi^\varepsilon}{(1 + \xi)^2} F(d-2, -\varepsilon; 2d-4; -\frac{1}{\xi})
\]

\[
\sim \frac{1}{(1 + \xi)^2} \left( 1 + \varepsilon \ln \xi - 6\varepsilon \sum_{n=1}^{\infty} \frac{(-1)^n}{n(n + 2)(n + 3)} \frac{1}{\xi^n} \right)
\]

(4.33)

\[
= (1 - \nu^2)^2 (1 - \varepsilon \ln(1 - \nu^2)) + \varepsilon \left\{ 2\nu^2 + \nu^4 \frac{3 - \nu^2}{1 - \nu^2} \ln \nu^2 \right\}.
\]
With $B \sim 8\varepsilon^2/N$ this agrees exactly with the results in (3.4) and (3.14) in the $N \to \infty$ limit for $\lambda \to \frac{1}{6}g_\star \phi^2 \sim 8\pi^2\varepsilon\phi^2/N$.

Manifestly from (4.32) $f_\lambda(\xi)_{\text{ord}}$ may be expanded in powers $\xi^{-d-n}$ or $(2\xi+1)^{-d-2n}$ for $n = 0, 1, \ldots$ which, from later results, represents the contributions of boundary operators with dimensions $d+2n$ in the BOE. For $\xi \to \infty f_\lambda(\xi)_{\text{ord}} \propto \xi^{-d}$ as expected for the leading operator appearing in the BOE for $\lambda(x)$ being $T_{11}(0, x) \equiv \hat{T}(x)$. An alternative expansion of $f_\lambda(\xi)_{\text{ord}}$ valid for small $\xi$ also verifies the consistency of the leading $1/N$ expression for the $\lambda$ two point function with the OPE for $\lambda(x)\lambda(x')$. From (B.8) in the limit $\xi \to 0$ we get

$$f_\lambda(\xi)_{\text{ord}} = B \left( \frac{1}{\xi^2} (1 - (d-2)(d-3)\xi) + \frac{1}{2} (d-1)(d-2)(d-3)(d-4) \ln \frac{1}{\xi} + \ldots \right). \quad (4.34)$$

The leading operator, apart from the identity, appearing in the OPE for $\lambda\lambda$ is $\lambda$ itself and the relevant coefficient is determined from the three point function

$$\langle \lambda(x_1)\lambda(x_2)\lambda(x_3) \rangle = \frac{C_{\lambda\lambda\lambda}}{(x_{12}^2 x_{13}^2 x_{23}^2)^{\frac{1}{2} \eta_\lambda}}. \quad (4.35)$$

To leading order in $1/N$ [14], with our normalisations,

$$C_{\lambda\lambda\lambda} = B^2 \frac{d-3}{4-d} \quad (4.36)$$

and, with $\eta_\lambda = 2$, we can write

$$\lambda(x)\lambda(x') \sim \frac{B}{(x-x')^d} + \frac{C_{\lambda\lambda\lambda}^\lambda}{(x-x')^2} \lambda(x'), \quad C_{\lambda\lambda}^\lambda = \frac{C_{\lambda\lambda\lambda}}{B}, \quad (4.37)$$

which implies $F_\lambda(v) \sim B + C_{\lambda\lambda\lambda}^\lambda A_\lambda v^2$, using (4.8) for $\langle \lambda \rangle$. It is easy to see from this that (4.34) is compatible with the result (4.12) in this case. Note that $C_{\lambda\lambda\lambda} = 0$ for $d = 3$ which is reflected by the absence of a $v^2$ term in (4.31).

In the Neumann case if we take the $N \to \infty$ limit in (3.9) we get

$$f_\varphi(\xi)_{\text{sp}} = A \frac{1 + 2\xi}{(\xi(1 + \xi))^{\frac{1}{2}d-1}}, \quad F_\varphi(v)_{\text{sp}} = A(1 + v^2)(1 - v^2)^{-\frac{1}{2} \varepsilon}, \quad (4.38)$$

where the leading surface operator has dimension $d-3$ which agrees with the $N \to \infty$ limit of (3.10). The expression given in (4.38) corresponds to the general solution (4.10) if $a = -\frac{1}{2} \varepsilon$, $b = 1 + \frac{1}{2} \varepsilon$ so that in this case

$$A_{\lambda, \text{sp}} = (4-d)(6-d) = \varepsilon(2 + \varepsilon). \quad (4.39)$$
From (4.12,39) $A_{\lambda,sp}/A_{\lambda,ord} = -(2 + \varepsilon)/(2 - \varepsilon)$ which is in accord with the large $N$ limit of (3.12). The solution of (4.2b) to find $G_\lambda$ is more involved and is undertaken in appendix B. The conformal invariant function $f_\lambda$ in (4.14) is given by

$$f_\lambda(\xi)_{sp} = B \left\{ \frac{1}{3} \frac{6 - d}{d - 2} \frac{\Gamma(d)\Gamma(d - 2)}{\Gamma(2d - 5)} \right.$$ \hspace{2cm} (4.40) \\
\hspace{2cm} \times \frac{\xi + \frac{1}{2}}{[\xi(1 + \xi)]^{\frac{1}{2}(d+1)} \frac{3}{2} F_2 \left( \frac{1}{2} d + \frac{1}{2}, \frac{1}{2} d - \frac{3}{2}, \frac{3}{2}; d - \frac{5}{2}, \frac{5}{2}; -\frac{1}{4\xi(1 + \xi)} \right)} \\
+ \frac{\pi \Gamma \left( \frac{1}{2} d - 1 \right)^2}{\Gamma(d - 3) \Gamma \left( \frac{1}{2} d - \frac{3}{2} \right) \Gamma \left( \frac{7}{2} - \frac{1}{2} d \right)} \frac{8}{(1 + 2\xi)^2} \frac{1}{2} \frac{1}{2} d \right\} . \\
$$

The two terms correspond in the BOE to two classes of boundary operators with dimensions $d + 2n$ and $2 + 2n$, $n = 0, 1, \ldots$ respectively. Corresponding to (4.34) we also find for the singular behaviour as $\xi \to 0$

$$f_\lambda(\xi)_{sp} = B \left( \frac{1}{\xi^2} (1 + (d - 3)(6 - d)\xi) + \frac{(d - 1)(d - 3)(d - 4)(d - 6)^2}{2(d - 2)} \ln \frac{1}{\xi} + \ldots \right), \hspace{2cm} (4.41)$$

in which the first two terms may also be easily seen to be compatible with the OPE (4.37) together with (4.39).

In both (4.34) and (4.41) there appear $\ln \xi$ terms which are naively unexpected on the basis of the OPE. However these may be understood by considering the role of an operator, denoted as $\lambda^2$, which has dimension $\eta_{\lambda^2} = 4 + O(N^{-1})$. This may be defined in terms of the OPE for $\lambda \lambda$ where we take $C_{\lambda \lambda \lambda^2} = 1$ and we assume that, with this normalisation, $A_{\lambda^2} = A_{\lambda^2}(1 + O(N^{-1}))$. According to (2.10) we then expect from the OPE a contribution to the $\lambda$ two point function given by

$$f_{\lambda \lambda}(\xi)_{\lambda^2} \sim A_{\lambda^2}^2 \left( 1 + \frac{1}{2} (2\eta_{\lambda^1} - \eta_{\lambda^2,1}) \frac{1}{N} \ln \frac{1}{\xi} + \ldots \right), \hspace{2cm} (4.42)$$

where $\eta_{\lambda^1}, \eta_{\lambda^2,1}$ are the coefficients of the $1/N$ terms in the large $N$ expansion of the scale dimensions of $\lambda, \lambda^2$. The first term on the r.h.s. of (4.42) clearly represents the disconnected pieces $\langle \lambda \rangle \langle \lambda \rangle$ in (4.1), which are $O(1)$ as $N \to \infty$, while the second $O(N^{-1})$ term corresponds exactly to the $\ln \xi$ terms in (4.34,41), with (4.12,39), if

$$\left( 2\eta_{\lambda^1} - \eta_{\lambda^2,1} \right) \frac{1}{N} = \frac{(d - 1)(d - 3)}{(d - 2)(d - 4)} B, \hspace{2cm} (4.43)$$

which agrees with direct calculations.*

---

* $\eta_{\lambda^1}$ and $\eta_{\lambda^2,1}$ are given by (5.29) and (5.30) in the fifth paper listed in ref. [18] where $\eta_1(S)$ $\equiv$ $\eta_{\phi,1}$ with $B = 4d(d - 2)\eta_{\phi,1}/N$.
5 Two Point Functions for the Conserved Current and Energy Momentum Tensor in the Large N Limit

In the $O(N)$ model described in the previous section other operators may be defined in terms of the basic fields $\phi^{\alpha}$ through the OPE. Here we focus on the conserved current $J_\mu^\alpha = -J_\mu^{\beta\alpha}$, whose charges generate the $O(N)$ symmetry, and the energy momentum tensor $T_{\mu\nu}$ in this model. These have scale dimension $d - 1$ and $d$ respectively. The former may be defined through the OPE

$$
\phi^{[\alpha}(x)\phi^{\beta]}(x') \sim \frac{A}{S_dC_J}\frac{1}{(s^2)^{\eta_\phi-\frac{d}{2}+1}}s_\mu J^{\alpha\beta}_\mu(x'),
$$

(5.1)

where the coefficient is determined by $O(N)$ Ward identities [14] with $C_J$ setting the scale of the two point function for $J_\mu^{\alpha\beta}$. This may be written as

$$
\langle J^{\alpha\beta}_\mu(x)J^{\nu\delta}_{\nu'}(x') \rangle = (\delta^{\alpha\gamma}\delta^{\beta\delta} - \delta^{\alpha\delta}\delta^{\beta\gamma})G_{J\mu\nu}(x,x'),
$$

(5.2)

where, as in (2.16), conformal invariance dictates

$$
G_{J\mu\nu}(x,x') = \frac{1}{(s^2)^{d-1}}(I_{\mu\nu}(s)C_J(v) + X_\mu X_\nu D_J(v)), \quad C_J = C_J(0).
$$

(5.3)

In the $N \to \infty$ limit $\eta_\phi = \frac{1}{2}d - 1$ and $A$ is given by (3.3). The leading contributions to the two point function are then simply

$$
G_{J\mu\nu}(x,x') = \left(\frac{S_dC_J}{A}\right)^2 \frac{1}{2}(\partial_\mu G_\phi(x,x')\nabla_\nu G_\phi(x,x') - \partial_\mu G_\phi(x,x')\nabla_\nu G_\phi(x,x')).
$$

(5.4)

Substituting the conformally invariant form for $G_\phi$ in (3.1) then gives

$$
C_J(v) = \left(\frac{S_dC_J}{A}\right)^2 \frac{1}{2}((d-2)F_\phi(v)^2 - (1-v^2)vF_\phi(v)F'_\phi(v)),
$$

$$
D_J(v) = \left(\frac{S_dC_J}{A}\right)^2 \frac{1}{2}((1-v^2)vF_\phi(v) \frac{d}{dv}((1-v^2)vF_\phi(v)) - v^2(1-v^2)F'_\phi(v))^2)
$$

(5.5)

which leads to $C_J = 2/(d-2)S^2_d$. Using (4.9) we may verify that the general results (5.5) satisfy the conservation condition (2.18). For the Dirichlet case* using (4.11) we have

$$
C_J(v)_{\text{ord}} = C_J(1 + v^2)(1 - v^2)^{d-2}, \quad C_J(v)_{\text{ord}} + D_J(v)_{\text{ord}} = C_J(1 - v^2)^{d-1}.
$$

(5.6)

The calculation of two point functions involving the energy momentum tensor $T_{\mu\nu}$ is more involved. In a non trivial conformal field theory the natural definition of $T_{\mu\nu}$

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* For a free complex scalar field which has a conserved current $V_\mu = i\phi^*\delta^{\alpha}_{\mu}\phi$ then in (2.16) using (3.2) gives the corresponding results, if $C_V = \Gamma(\frac{d}{2})^2/(2\pi^d(d-2))$, $C(v) = C_V(\pm v^{d-2}(1 + v^2) + v^{2d-2})$ and $C(v) + D(v) = C_V(\pm (d-1)v^{d-2}(1 - v^2) - v^{2d-2})$. 

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is through its appearance in OPEs. In the simplest case we may extend (4.13) to give generally

\[ \phi^\alpha(x)\phi^\alpha(x') \sim \frac{NC_\phi}{s^{2\eta_\phi}} + \frac{NC_\phi^\lambda}{(s^2)^{\eta_\phi - \frac{1}{2}\eta_\lambda}} C^{\eta_\lambda,0}(s, \partial x') \lambda(x') \]

(5.7)

where \( C_\phi \) is the coefficient of the bulk two point function for \( \phi \), in (3.1) \( C_\phi = F_\phi(0) \) while \( C_T \) is given by the scale of the two point function for the energy momentum tensor, as in (2.32). The coefficient of the energy momentum tensor term in OPEs is determined through Ward identities [3,14]. In the term containing \( \lambda \) in (5.7) \( C^{\eta_\lambda,0}(s, \partial) = 1 + O(s) \) is introduced to include all derivatives of \( \lambda \) in the above OPE. It is determined by the requirement of reproducing the three point function (4.5) where in general \( C_\phi^\lambda = C_\phi^\lambda C_\lambda \). It is necessary to consider such derivative operators, unlike in (5.1), since \( T_{\mu\nu} \), which has dimension \( d \), is not the lowest dimension operator appearing in this OPE. Explicitly

\[
C^{a,b}(s, \partial) = \frac{1}{B(a_+, a_-)} \int_0^1 \alpha \alpha_{a+}^{-1}(1 - \alpha)^{a_- - 1} \times \sum_{m=0}^\infty \frac{1}{m!} \frac{1}{(a + 1 - \frac{1}{2}d)_m} \left[ -\frac{1}{4}s^2\alpha(1 - \alpha)\partial^2 \right] \frac{e^{\alpha s, \partial}}{m},
\]

(5.8)

for \( a_\pm = \frac{1}{2}(a \pm b) \) with the Pochhammer symbol \( (a)_m = \Gamma(a + m)/\Gamma(a) \) and we assume \( [s, \partial] = 0 \) to move all derivatives to the right. For our purposes we need to expand this to \( O(s^2) \)

\[
C^{a,0}(s, \partial) \sim 1 + \frac{1}{2} s \cdot \partial + \frac{1}{8(a + 1)} \left( (a + 2)s_\mu s_\nu \partial_\mu \partial_\nu - \frac{1}{4} a \right) \frac{s \cdot \partial^2}{a + 1 - \frac{1}{2}d} s^2 \partial^2).
\]

(5.9)

Initially we focus on determining the two point function \( \langle T_{\mu\nu}(x)\lambda(x') \rangle \) which as a consequence of (2.19,22) has a unique functional form in the conformally invariant limit. In order to use the OPE (5.7) to find this we need to determine an expression for the \( \langle \phi^\alpha \phi^\beta \rangle \) three point function. To leading order in \( 1/N \) the connected three point function in the \( O(N) \) \( \sigma \)-model with a plane boundary discussed previously is simply given by

\[
\langle \phi^\alpha(x_1)\phi^\beta(x_2)\lambda(x_3) \rangle_{\text{conn}} = -\delta^{\alpha\beta} \int_{\mathbb{R}^d_+} d^d r G_\phi(x_1, r)G_\phi(x_2, r)G_\lambda(r, x_3),
\]

(5.10)

where \( G_\phi, G_\lambda \) are given by the results of the section 4 for the ordinary and special critical points. The application of the OPE to this is based on the equation (verified in appendix C)

\[
\frac{1}{x_{\eta_1}, x_{\eta_2}} = C^{\eta_1, -\eta_1}(x_{12}, \partial_{x_2}) \frac{1}{x_{\eta_2}} + \rho(x_{12}) \frac{1}{x_{\eta_1}} C^{-\eta, -\eta_1}(x_{12}, \partial_{x_2}) \delta^d(x_2),
\]

\[ \eta = \eta_1 + \eta_2, \quad \rho = \pi^{\frac{1}{2}d} \frac{\Gamma(\frac{1}{2}d - \eta_1)\Gamma(\frac{1}{2}d - \eta_2)\Gamma(\eta - \frac{1}{2}d)}{\Gamma(\eta_1)\Gamma(\eta_2)\Gamma(d - \eta)} .
\]

(5.11)
Neglecting the $\delta$ function term, this result is the essential condition which determines the derivative operator $C^{a,b}(s, \partial)$ in the OPE for scalar fields in conformal theories. For our purpose we take $\eta_1 = \eta_2 = \frac{1}{2}d - 1$ and generalise this to write

$$G_{\phi}(x_1, r)G_{\phi}(x_2, r) \sim C^{d-2.0}(x_{12}, \partial_{x_2})G_{\phi}(x_2, r)^2 + \rho A^2(x_{12})^{\frac{d}{2}} C^{2.0}(x_{12}, \partial_{x_2})\delta^d(x_2 - r) - \frac{1}{2}(x_{12})_{\mu}(x_{12})_{\nu}t_{\mu\nu}(x_2, r) + \ldots,$$

(5.12)

where other terms are less singular and irrelevant here (for free fields without boundary (5.12) vanish). Using (5.12) in (5.10) shows that it is compatible with the OPE (5.7),

$$C\text{derivative operator at } d\text{ from (5.16) and using (4.9), we may also find}$$

$$(\text{Neglecting the } \delta\text{ function term, this result is the essential condition which determines the derivative operator } C^{a,b}(s, \partial) \text{ in the OPE for scalar fields in conformal theories. For our purpose we take } \eta_1 = \eta_2 = \frac{1}{2}d - 1 \text{ and generalise this to write})$$

$$G_{\phi}(x_1, r)G_{\phi}(x_2, r) \sim C^{d-2.0}(x_{12}, \partial_{x_2})G_{\phi}(x_2, r)^2 + \rho A^2(x_{12})^{\frac{d}{2}} C^{2.0}(x_{12}, \partial_{x_2})\delta^d(x_2 - r) - \frac{1}{2}(x_{12})_{\mu}(x_{12})_{\nu}t_{\mu\nu}(x_2, r) + \ldots,$$

(5.12)

where other terms are less singular and irrelevant here (for free fields without boundary (5.12) vanish). Using (5.12) in (5.10) shows that it is compatible with the OPE (5.7), taking $\eta_\phi = \frac{1}{2}d - 1, \eta_\lambda = 2$ and $C_\phi = A$, since the $G_{\phi}(x_2, r)^2$ term is absent due to (4.2b). In this case it is necessary that $\rho A^2 = -A/(2\varepsilon) = -C_{\phi\phi,\lambda}$ to leading order in $1/N$, which is in accord with (4.6). If we use the result for $C_T$ when $N \to \infty$, which is the same as for free scalar fields [3,14],

$$C_T = \frac{Nd}{(d - 1)S_d^2},$$

(5.13)

we may also obtain

$$\langle T_{\mu\nu}(x)\lambda(x') \rangle = -N \int_{\mathbb{R}_+^d} d^d r t_{\mu\nu}(x, r) G_\lambda(r, x').$$

(5.14)

Using (5.9,11) and the basic Green function equation (4.2a), with (4.8), gives

$$t_{\mu\nu}(x, x') = \hat{t}_{\mu\nu}(x, x') - \frac{1}{d} \delta_{\mu\nu} \frac{A_\lambda}{(2y)^2} G_{\phi}(x, x')^2,$$

$$\hat{t}_{\mu\nu} = - G_\phi D_{\mu\nu} G_\phi + \frac{d}{4(d - 1)} D_{\mu\nu}(G_\phi^2),$${

$$D_{\mu\nu} = \partial_\mu \partial_\nu - \frac{1}{d} \delta_{\mu\nu} \partial^2,$$

(5.15)

Using the form (4.7) for $G_\phi \hat{t}_{\mu\nu}$ becomes

$$\hat{t}_{\mu\nu}(x, x') = \frac{(2y)^2}{s^{2d}} \left( X_\mu X_\nu - \frac{1}{d} \delta_{\mu\nu} \right) f(v),$$

$$f(v) = - \frac{2}{d - 1} \xi(\xi + 1) \left( (d - 2) F_\phi \frac{d}{d\xi} \left( \xi^2 \frac{d}{d\xi} F_\phi \right) - d \xi^2 \left( \frac{d}{d\xi} F_\phi \right)^2 \right),$$

(5.16)

where $X_\mu$ is defined as in (2.14). Under conformal transformations $t_{\mu\nu}(x, x')$ transforms as a scalar with scale dimension $d - 2$ at $x'$, and also as a symmetric tensor of dimension $d$ at $x$, so that the integration in (5.14) preserves conformal invariance. From (5.15), or from (5.16) and using (4.9), we may also find

$$\left( (d - 2)n_\nu + y \partial_\nu \right)G_{\phi}(x, x')^2,$$

(5.17)

which in turn gives

$$\partial_\mu t_{\mu\nu}(x, x') = n_\nu \frac{2A_\lambda}{(2y)^3} G_{\phi}(x, x')^2,$$

(5.18)
where it is important to note that since \( f(v) \propto v^4 \) as \( v \to 0 \) and hence \( t_{\mu \nu}(x, x') = O(s^{4-2d}) \) there are no \( \delta \) function contributions. Since \( t_{\mu \mu} \) is easily found from (5.15) or (5.16) and using (5.18), and applying the basic equation (4.2b), we therefore can derive

\[
\langle T_{\mu \nu}(x) \lambda(x') \rangle = -\frac{2A_\lambda}{(2y)^2} \delta^d(s), \quad \partial_{\mu} \langle T_{\mu \nu}(x) \lambda(x') \rangle = n_\nu \frac{4A_\lambda}{(2y)^3} \delta^d(s).
\]  

(5.19)

This is in exact agreement with (2.23) for \( \mathcal{O} \rightarrow \lambda, \eta \rightarrow 2 \) if we take \( C = -2/d \). As a consequence of the general conformal invariance results given in (2.19,22,26) the integral (5.14) is completely determined therefore by

\[
\langle T_{\mu \nu}(x) \lambda(x') \rangle = -\frac{2dA_\lambda}{(d-1)S_d} \frac{(2y')^{d-2}}{s^{2d}} (X_\mu X_\nu - \frac{1}{d} \delta_{\mu \nu}) v^d.
\]  

(5.20)

The same procedures may also be applied to find the energy momentum tensor two point function if we start from \( \langle \phi^\alpha(x_1) \phi^\alpha(x_2) \phi^\alpha(x_3) \phi^\alpha(x_4) \rangle \) and consider the OPEs for \( x_{12}, x_{34} \sim 0 \). The relevant contributions for large \( N \) are

\[
N \left( G_\phi(x_1, x_3) G_\phi(x_2, x_4) + G_\phi(x_1, x_4) G_\phi(x_2, x_3) \right) + N^2 \int_{\mathbb{R}^d_+} d^d r \int_{\mathbb{R}^d_+} d^d r' G_\phi(x_1, r) G_\phi(x_2, r) G_\phi(x_3, r) G_\phi(x_4, r) G_\lambda(r, r').
\]  

(5.21)

For \( x_{12} = x_{34} = 0 \) this vanishes due to (4.2b). This is crucial in cancelling the \( O(1) \) contribution as \( x_{12} \sim x_{34} \rightarrow 0 \), which would correspond to a ‘shadow operator’ of dimension \( d - 2 \) in the OPE (for free fields this is represented by the operator \( \phi^2 \)) so that the leading behaviour is \( \propto (x_{12}^2)^{\frac{3}{2}}, (x_{34}^2)^{\frac{3}{2}} \) which is appropriate for \( \lambda \), with dimension 2, being the lowest dimension operator apart from the identity. Following a similar discussion to the above we may now obtain the large \( N \) expression for \( \langle T_{\mu \nu}(x) T_{\sigma \rho}(x') \rangle \) as

\[
G^f_{\mu \nu \sigma \rho}(x, x') = N \frac{2}{d} \delta_{\mu \nu} \frac{A_\lambda}{(2y')^2} t_{\sigma \rho}(x', x) + \delta_{\sigma \rho} \frac{A_\lambda}{(2y')^2} t_{\mu \nu}(x, x')
\]  

\[- N \frac{2}{2} \delta_{\mu \nu} \delta_{\sigma \rho} \frac{A_\lambda^2}{4yy'} G_\phi(x, x')^2 + N^2 \int_{\mathbb{R}^d_+} d^d r \int_{\mathbb{R}^d_+} d^d r' t_{\mu \nu}(x, r) t_{\sigma \rho}(x', r') G_\lambda(r, r'),
\]  

(5.22)

where, with the operator \( \mathcal{D}_{\mu \nu} \) defined in (5.15), the ‘free field’ part \( G^f \) is given by

\[
G^f_{\mu \nu \sigma \rho} = N \left\{ \frac{1}{d} G_\phi \mathcal{D}_{\mu \nu} \mathcal{D}''_{\sigma \rho} G_\phi + \mathcal{D}_{\mu \nu} G_\phi \mathcal{D}''_{\sigma \rho} G_\phi - \frac{2}{2(d-1)} \left( \mathcal{D}_{\mu \nu} (G_\phi \mathcal{D}''_{\sigma \rho} G_\phi) + \mathcal{D}''_{\sigma \rho} (G_\phi \mathcal{D}_{\mu \nu} G_\phi) \right) \right\}.
\]  

(5.23)

It is easy to verify that the trace on \( \mu \nu \) and \( \sigma \rho \) in (5.22) vanishes and hence the terms involving \( A_\lambda \) explicitly in (5.22) and also contained in \( t_{\mu \nu}, t_{\sigma \rho} \) can be dropped. Hence we can write

\[
\langle T_{\mu \nu}(x) T_{\sigma \rho}(x') \rangle = G^f_{\mu \nu \sigma \rho}(x, x') + G^\lambda_{\mu \nu \sigma \rho}(x, x'),
\]  

(5.24)
where
\[ G^\lambda_{\mu\nu\sigma}(x, x') = N^2 \int_{R^d_+} \int_{R^d_+} d^d r \int d^d r' \hat{\iota}_{\mu\nu}(x, r) \hat{\iota}_{\sigma\rho}(x', r') G_\lambda(r, r'). \] (5.25)

Writing \( r_\mu = (z, r) \) and using the result (5.14,20),
\[ G^\lambda_{\mu\nu\sigma}(x, x') = N \frac{2dA_\lambda}{(d-1)S_d} \int_0^\infty dz \int d^{d-1} r \left( \frac{2z\tilde{v}}{\tilde{s}^2} \right)^d f(\tilde{v}) \left( \tilde{X}_\mu \tilde{X}_\nu - \frac{1}{d} \delta_{\mu\nu} \right) \left( \tilde{X}'_\sigma \tilde{X}'_\rho - \frac{1}{d} \delta_{\sigma\rho} \right), \] (5.26)
for \( \tilde{s}^2 = (x - r)^2, \tilde{v}^2 = \tilde{s}^2/(\tilde{s}^2 + 4yz), \tilde{s}'^2, \tilde{v}'^2 \) similarly defined with \( x \to x' \), and \( \tilde{X}_\mu(x, r), \tilde{X}'_\sigma(x', r) \) vectors formed as in (2.14).

To verify the conservation equations we may use the definition (5.23) along with (4.2a,8) to find
\[ \partial_\mu G^f_{\mu\nu\sigma}(x, x') = \frac{N}{d} \frac{4A_\lambda}{(2y)^3} \left( (d - 2)n_\nu + y \partial_\nu \right) \hat{\iota}_\sigma(x', x), \] (5.27)
and using (5.17) in (5.25) it is easy to verify that this is cancelled by \( \partial_\mu G^\lambda_{\mu\nu\sigma} \). By virtue of conformal invariance \( G^f_{\mu\nu\sigma}(x, x') \) may be expressed in the form (2.27) but (5.27) now gives
\[
\begin{align*}
\left( v \frac{d}{dv} - d \right) \gamma^f(v) &= \frac{d}{(d-1)^2} \alpha^f(v) + \frac{(d+1)(d-2)}{d-1} \epsilon^f(v) + 2NA_\lambda \frac{1}{d} f(v), \\
\left( v \frac{d}{dv} - d \right) \alpha^f(v) &= 2(d-1) \gamma^f(v) + 2NA_\lambda \frac{d-1}{d^2} \left( vf'(v) - \frac{2d}{1-v^2} f(v) \right).
\end{align*}
\] (5.28)

By explicit calculation we find using the results in (4.11) or (4.38)
\[
\begin{align*}
\text{ord} \quad f(v)_{\text{ord}} &= 2A^2 \frac{(d-2)^2}{d-1} v^4 (1 - v^2)^{d-4}, \\
\text{sp} \quad f(v)_{\text{sp}} &= 2A^2 \frac{d}{d-1} v^4 (1 - v^2)^{d-6} \left\{ (d-3)(d-4)(1 + v^2)^2 - d(1 - v^2)^2 \right\}.
\end{align*}
\] (5.29a, b)

For the ordinary transition then (5.23), in conjunction with (2.27,38), gives
\[
\begin{align*}
\alpha^f(v)_{\text{ord}} &= \frac{N}{S_d^2} \left\{ (1 - v^2)^2 + \frac{2}{d-1} (1 - v^2)^{d-2} v^2 + 8 \frac{(d-2)^2}{d^2} (1 - v^2)^{d-4} v^4 \right\}, \\
\gamma^f(v)_{\text{ord}} &= -\frac{N}{S_d^2} \left( 1 + v^2 \right) \left\{ \frac{d}{2(d-1)} (1 - v^2)^{d-1} + \frac{d-2}{(d-1)^2} (1 - v^2)^{d-3} v^2 \right\}, \\
\epsilon^f(v)_{\text{ord}} &= \frac{N}{S_d^2} \left( 1 - v^2 \right)^{d-2} \left\{ \frac{d}{2(d-1)} (1 + v^2)^2 - \frac{d-2}{(d-1)^2} v^2 \right\}.
\end{align*}
\] (5.30)

It is easy to check that (5.29a) and (5.30) satisfy (5.28). The evaluation of the integral in (5.26) is discussed in appendix D. The results can be similarly expressed in the basis given by (2.27).
From (D.33.34) we obtain

\[
\alpha^\lambda(v)_{\text{ord}} = -\frac{N}{S_d^2} \frac{8}{d^2} (d-2)^2 v^4 (1-v^2)^{d-4} - \frac{N}{S_d^2} \frac{2}{d-1} v^2 (1-v^2)^{d-2} \frac{3}{d} F_2(1, d-1, \frac{1}{2} d - 2; d - \frac{3}{2}, \frac{1}{2} d; \frac{(1-v^2)^2}{4v^2}) \\
+ \alpha(1)_{\text{ord}} v^d \left( 1 + \frac{d}{d-1} \frac{(1-v^2)^2}{4v^2} \right),
\]

\[
\gamma^\lambda(v)_{\text{ord}} = \frac{N}{S_d^2} \frac{d-2}{(d-1)^2} v^2 (1+v^2) (1-v^2)^{d-3} \\
\times \frac{3}{d} F_2(1, d-1, \frac{1}{2} d - 2; d - \frac{3}{2}, \frac{1}{2} d; \frac{(1-v^2)^2}{4v^2}) \\
- \frac{d}{4(d-1)^2} \alpha(1)_{\text{ord}} v^{d-2} (1-v^4),
\]

\[
\epsilon^\lambda(v)_{\text{ord}} = -\frac{N}{S_d^2} \frac{d-2}{(d-1)^2(2d-3)} v^2 (1-v^2)^{d-2} \\
\times \frac{3}{d} F_2(1, d-1, \frac{1}{2} d - 2; d - \frac{1}{2}, \frac{1}{2} d; \frac{(1-v^2)^2}{4v^2}) \\
+ \frac{d}{4(d-1)^2(d+1)} \alpha(1)_{\text{ord}} v^{d-2} (1-v^2)^2,
\]

where

\[
\alpha(1)_{\text{ord}} = \frac{N}{S_d^2} 2^{d-2} \frac{\Gamma(\frac{1}{2}d)^2 \Gamma(3 - \frac{1}{2}d) \Gamma(d - \frac{3}{2})}{\Gamma(d-1) \Gamma(\frac{d}{2} + \frac{1}{2})}.
\]

These contributions vanish if \( d = 4 \) when \( \alpha(1)_{\text{ord}} = 2N/S_d^2 \) as expected since the integral (5.26) contains a factor \( 4 - d \) in \( A_\lambda \) and in agreement with the conformal invariant theory becoming then of course just a free scalar field theory. For the non trivial case when \( 2 < d < 4 \) it is crucial that adding (5.31) to (5.30) cancels the \( (1-v^2)^{d-4}, (1-v^2)^{d-2} \) terms in \( \alpha^\lambda(v) \) and also the \( (1-v^2)^{d-3} \) term in \( \gamma^\lambda(v) \) so that it vanishes as \( v \to 1 \). The final result for \( \alpha \) can be expressed more compactly as

\[
\alpha(v)_{\text{ord}} = \frac{N}{S_d^2} (1-v^2)^d \left\{ 1 + \frac{d-4}{d(2d-3)} \frac{3}{d} F_2(1, d, \frac{1}{2} d - 1; d - \frac{1}{2}, \frac{1}{2} d + 1; \frac{(1-v^2)^2}{4v^2}) \right\} \\
+ \alpha(1)_{\text{ord}} v^d \left( 1 + \frac{d}{d-1} \frac{(1-v^2)^2}{4v^2} \right).
\]

We have verified that expanding this to \( O(\varepsilon) \) gives the same expressions as the \( \varepsilon \) expansion results in \( I \) in the large \( N \) limit. When \( d = 2 \) the first term in (5.33) vanishes leaving only the contribution proportional to \( \alpha(1)_{\text{ord}} = 2N/S_d^2 \).

In order to consider how this result for \( \alpha(v)_{\text{ord}} \) behaves in the limit \( v \to 0 \), which is appropriate for the OPE, we may use the inversion formula for generalised hypergeometric
functions [28] to write this alternately as
\[
\alpha(v)_{\text{ord}} = \frac{N}{S_d^d} (1 - v^2)^d \\
\times \left\{ 1 + \frac{1}{2(d - 1)} \frac{4v^2}{(1 - v^2)^2} {}_{3}F_{2}(1, \frac{5}{2} - d, 1 - \frac{1}{2}d; 2 - d, 3 - \frac{1}{2}d; - \frac{4v^2}{(1 - v^2)^2}) \right\} \\
- \frac{N}{S_d^d} \frac{\Gamma(2d - 3)\Gamma(3 - d)}{\Gamma(d + 1)} \frac{4v^d}{d + 2} \frac{(1 - v^2)^d}{(1 - v^2)^2} F\left(\frac{3}{2}, \frac{1}{2}d; 2 + \frac{1}{2}d; - \frac{4v^2}{(1 - v^2)^2}\right).
\]

6 Operator Product Expansions

As described in section 2 there are constraints on the two point functions in the presence of a boundary, which in the conformal limit depend on a single variable function of \(\xi\) or \(v\), arising from the OPE. The coefficients appearing in this are a property of the conformal theory on \(\mathbb{R}^d\) without any boundary and dependence on boundary conditions arises solely through the one point functions of those operators appearing in the OPE. Since, as remarked earlier, only scalar operators have non zero one point functions, which have the simple form (2.7), we need only consider for our purposes the contributions of such operators to the OPE. In this section we determine the functional forms for the two point function in the neighbourhood of a boundary arising from all derivative operators formed from a given quasi-primary operator.

We initially consider the simplest case of just scalar operators when the general conformally invariant three point function without any boundary can be written as
\[
\langle O_1(x_1)O_2(x_2)O_3(x_3) \rangle_{\text{no boundary}} = \frac{C_{123}}{(x_{12})^{\frac{1}{2}>(\eta_1 + \eta_2 - \eta_3)}(x_{23})^{\frac{1}{2}>(\eta_2 + \eta_3 - \eta_1)}(x_{31})^{\frac{1}{2}>(\eta_3 + \eta_1 - \eta_2)}.
\]

If \(C_3\) denotes the normalisation factor for the two point function of the operator \(O_3\), so that in general \(\langle O_1(x)O_j(x') \rangle = \delta_{ij}C_i/s^{2n_i}\), and \(C_{123}^{-} = C_{123}/C_3\) we may rewrite the OPE (2.9) to include all derivatives of \(O_3\) as
\[
O_1(x)O_2(x') = \frac{C_{123}^{-}}{(s^2)^{\frac{1}{2}>(\eta_1 + \eta_2 - \eta_3)}C^{\eta_3, \eta_-}(s, \partial x')O_3(x'), \quad s = x - x', \quad \eta_- = \eta_1 - \eta_2,
\]
where \(C^{a,b}(s, \partial)\) is the derivative operator defined in (5.8). By virtue of (5.11) the OPE (6.2) exactly reproduces the three point function (6.1) for non coincident points. Of course the OPE of \(O_1\) and \(O_2\) in general contains contributions from infinitely many quasi-primary operators but such a summation is left implicit here.

For subsequent use it is convenient to generalise the expression (5.8) to
\[
C^{a,b}_n(s, \partial) = \frac{1}{B(a_+, a_-)} \int_0^1 d\alpha \alpha^{a_- - 1}(1 - \alpha)^{a_+ - 1} \\
\times \sum_{m=0} \frac{1}{m!} \frac{1}{(a_+ + 1 - n - \frac{1}{2}d)_m} \left[-\frac{1}{4}s^2(1 - \alpha)^2\right]^m e^{a\cdot s - \partial},
\]
which reduces to (5.8) when \( n = 0 \). The essential result for application of the OPE in the presence of a boundary is

\[
C_{n}^{a,b}(s, \partial_{x'}) \frac{1}{(2y')^{a}} = \frac{1}{(2y)^{a} + (2y')^{a}} F(a_{+}, a_{-}; a + 1 - n - \frac{1}{2}d; -\xi),
\]  

(6.4)

which is easily obtained by application of the explicit form (6.3).

Applying (6.2) to \( \langle O_{1}(x)O_{2}(x') \rangle \) then gives in (2.8) the equivalent results for the contribution of all derivative operators generated from \( O_{3} \)

\[
f_{12}(\xi)_{O_{3}} = C_{12}^{3} A_{3} \xi^{-\frac{1}{2}(\eta_{1} + \eta_{2} - \eta_{3})} F(\frac{1}{2}(\eta_{3} + \eta_{-}), \frac{1}{2}(\eta_{3} - \eta_{-}); \eta_{3} + 1 - \frac{1}{2}d; -\xi),
\]

\[
F_{12}(v)_{O_{3}} = C_{12}^{3} A_{3} v^{\eta_{3} + \eta_{-}} F(\frac{1}{2}(\eta_{3} + \eta_{-}), \frac{1}{2}(\eta_{3} + \eta_{-}) + 1 - \frac{1}{2}d; \eta_{3} + 1 - \frac{1}{2}d; v^{2}).
\]

(6.5)

The leading singular piece as \( \xi, v \to 0 \) of course coincides with (2.10). When \( \eta_{-} = 0 \) and \( \eta_{3} = d - 2 \) then \( F_{12}(v)_{O_{3}} = 1 \). This is appropriate for free scalar fields with \( O_{3} \to \phi^{2} \). In this case the above results show that the free field expression for \( \langle \phi^{a}(x)\phi^{b}(x') \rangle \), as given by (3.1,2,3) for either Dirichlet or Neumann boundary conditions, is reproduced solely by the contribution in the OPE of the identity and the operator \( \phi^{2} \) and its derivatives, taking \( C_{\phi\phi}\phi^{2} = 1/N \) and with \( A_{\phi^{2}} \) as in (3.5).

We may also derive the corresponding formulae for two point functions involving the energy momentum tensor \( T_{\mu\nu} \). The OPE is determined by knowledge of the three point function and detailed expressions in the relevant cases for conformal field theories in any dimension \( d \) were obtained by one of us recently. The construction given depends essentially on the result that for three points \( x_{1}, x_{2}, x_{3} \) it is possible to construct vectors \( X_{i}, i = 1, 2, 3 \) which transform homogeneously under conformal transformations, \( X_{i \mu} \to \Omega(x_{i})\mathcal{R}_{\mu \alpha}(x_{i})X_{i \alpha} \), at each \( x_{i} \). For \( X_{1} \) we have

\[
X_{1 \mu} = \frac{x_{1 \alpha}}{x_{1}^{2}} - \frac{x_{13 \mu}}{x_{13}^{2}}, \quad X_{1}^{2} = \frac{x_{23}^{2}}{x_{12}x_{13}}.
\]

(6.6)

while \( X_{2}, X_{3} \) are obtained by cyclic permutation.* A crucial result is that under ‘parallel transport’ by the inversion transformation, as defined in (2.3),

\[
I_{\mu \alpha}(x_{ij})X_{j \alpha} = -\frac{x_{ik}}{x_{jk}}X_{i \mu}, \quad k \neq i, j.
\]

(6.7)

For the three point function of the energy momentum tensor with two scalar fields \( O \) of dimension \( \eta \) we may then write [3]

\[
\langle T_{\mu \nu}(x_{1})O(x_{2})O(x_{3}) \rangle_{\text{no boundary}} = -\frac{\eta d}{d - 1} \frac{C_{O}}{S_{d}} \left( \frac{X_{1 \mu}X_{1 \nu}}{X_{1}^{2}} - \frac{1}{d} \delta_{\mu \nu} \right) \frac{(X_{2}^{2})^{\frac{1}{2}d}}{x_{23}^{2\eta}},
\]

(6.8)

* It is perhaps worth noting that if \( x_{1} \to x, x_{2} \to x', x_{3} \to \bar{x} \), for \( \bar{x} = (-y, x) \), then \( X_{1} \to X/(2yv) \) and \( X_{2} \to -(1 - v^{2})X'/v(2yv) \), with \( X, X' \) as given in (2.14).
where the normalisation is determined by Ward identities which relate this three point function to the two point function for $\mathcal{O}$. The result (6.8) determines the form of the OPE of $T_{\mu\nu}$ and $\mathcal{O}$ which can be written as

$$T_{\mu\nu}(x)\mathcal{O}(x') = -\frac{\eta d}{d-1} \frac{1}{S_d} \frac{1}{(s^2)^{\frac{1}{2}d}} A_{\mu\nu}(s, \partial x') \mathcal{O}(x'),$$  \hspace{1cm} (6.9)

where it is easy from (6.8) in the limit $x_{12} \to 0$ to see that

$$A_{\mu\nu}(s, \partial) = \frac{s_{\mu}s_{\nu}}{s^2} - \frac{1}{d} \delta_{\mu\nu} + \mathcal{O}(s).$$  \hspace{1cm} (6.10)

Using the full result (6.8), with the definition (6.6) for $X_1$, we may find after some work, in terms of $C_{n, b}^a(s, \partial)$ given by (6.3), that this may be extended to include derivative terms in the form

$$A_{\mu\nu}(s, \partial) = \frac{1}{\eta(\eta + 1)} \left( \frac{1}{4} s^2 \partial_{\mu} \partial_{\nu} C_{n+2, d-\eta}^a(s, \partial) - \frac{1}{d} \delta_{\mu\nu} C_{n, d-\eta}^a(s, \partial) + \text{terms } \propto s_\mu \text{ or } s_\nu. \right)$$  \hspace{1cm} (6.11)

In the presence of a boundary the OPE (6.9) gives

$$\langle T_{\mu\nu}(x)\mathcal{O}(x') \rangle = -\frac{\eta d}{d-1} \frac{A_{\mathcal{O}}}{S_d} \frac{1}{(s^2)^{\frac{1}{2}d}} A_{\mu\nu}(s, \partial x') \frac{1}{(2y')^\eta},$$  \hspace{1cm} (6.12)

and applying (6.4) with the explicit expression (6.11) leads to

$$A_{\mu\nu}(s, \partial x') \frac{1}{(2y')^\eta}$$

$$= \frac{1}{(2y)^{\frac{1}{2}d}(2y')^{\frac{1}{2}d}} \left( n_{\mu} n_{\nu} \xi \mathcal{F} \left( \frac{1}{2} d + 1, \eta + 1 - \frac{1}{2} d; \eta + 1 - \frac{1}{2} d; -\xi \right) - \frac{1}{d} \delta_{\mu\nu} \mathcal{F} \left( \frac{1}{2} d, \eta - \frac{1}{2} d; \eta - \frac{1}{2} d; -\xi \right) + \text{terms } \propto s_\mu \text{ or } s_\nu \right)$$  \hspace{1cm} (6.13)

$$= \frac{(2y')^{d-\eta}}{(s^2)^{\frac{1}{2}d}} \left( X_\mu X_\nu - \frac{1}{d} \delta_{\mu\nu} \right) v^d,$$

where in the last line we have completed the $s_\mu, s_\nu$ terms to achieve the desired general expression dictated by conformal invariance in accord with (2.19). Clearly this result agrees exactly with with the general expression obtained in (2.19,22) with also the result (2.26) for the coefficient $c_{T\mathcal{O}}$. Of course getting the correct form is a necessary consistency check of the above treatment.

In order to determine the OPE for two energy momentum tensors involving a scalar operator $\mathcal{O}$ we need the three point function [3]

$$\langle T_{\mu\nu}(x_1)T_{\sigma\rho}(x_2)\mathcal{O}(x_3) \rangle_{\text{no boundary}}$$

$$= \frac{1}{(x_{12}^2)^d} \left( -\frac{1}{x_{12}^2} \right)^{\frac{1}{2}d} \left\{ I_{\mu\nu, \sigma\rho}(x_{12}) C_\mathcal{O} + \left( X_{1\mu} X_{1\nu} - \frac{1}{d} \delta_{\mu\nu} \right) \left( X_{2\sigma} X_{2\rho} - \frac{1}{d} \delta_{\sigma\rho} \right) A_\mathcal{O} \right.$$  \hspace{1cm} (6.14)

$$+ \left( x_{12}^2 X_{1\mu} X_{2\sigma} I_{\nu\rho}(x_{12}) + \mu \leftrightarrow \nu, \sigma \leftrightarrow \rho \right) + \frac{4}{d} \delta_{\mu\nu} \frac{X_{2\sigma} X_{2\rho}}{X_2^2} + \frac{4}{d} \delta_{\sigma\rho} \frac{X_{1\mu} X_{1\nu}}{X_1^2} - \frac{4}{d^2} \delta_{\mu\nu} \delta_{\sigma\rho} \right\} B_\mathcal{O},$$

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where \( A_\mathcal{O}, B_\mathcal{O}, C_\mathcal{O} \) are constants and \( \mathcal{I}_{\mu\nu,\sigma\rho} \) is defined previously in (2.28). The three terms in (6.14) are separately conformally invariant. However imposing the conservation equation for \( T_{\mu\nu} \), using (2.29) and also

\[
\partial_\mu X_{2\sigma} = \frac{1}{x_{12}^2} I_{\mu\sigma}(x_{12}) , \quad \partial_\mu \left( \frac{1}{(x_{12}^2)^d} \left( \frac{X_{1\mu}X_{1\nu}}{X_1^2} - \frac{1}{d} \delta_{\mu\nu} \right) \right) = -(d-1) \frac{X_{1\nu}}{(x_{12}^2)^d},
\]

\[
\partial_\mu \left( \frac{1}{(x_{12}^2)^d} \left( x_{12}^2 \frac{X_{1\mu}I_{\nu\sigma}(x_{12}) + X_{1\nu}I_{\mu\sigma}(x_{12})}{x_{12}^2} + \frac{2}{d} \delta_{\mu\nu} \frac{X_{2\sigma}}{X_2^2} \right) \right)
= - \frac{x_{23}^2}{(x_{12}^2)^d x_{13}^2} \frac{1}{d} (d-2) \left( (d+1)I_{\nu\sigma}(x_{12}) + 2x_{12}^2 X_{1\nu} X_{2\sigma} \right),
\]

(6.15)

gives rise to the conditions

\[
\begin{align*}
\frac{1}{2} \eta d C_\mathcal{O} + A_\mathcal{O} + (d^2 - 4 - \eta d) B_\mathcal{O} &= 0, \\
(1 + \frac{1}{2}(d-\eta)(d-1)) A_\mathcal{O} + (\eta + 2)(d-2) B_\mathcal{O} &= 0.
\end{align*}
\]

(6.16)

In consequence there remains a single constant parameterising the three point function (6.14) but there are no Ward identities which relate this to any two point amplitude in this case.

From the three point function (6.14) we may derive the contribution of the operator \( \mathcal{O} \) to the OPE for two energy momentum tensors which can be written as

\[
T_{\mu\nu}(x)T_{\sigma\rho}(x') = \frac{1}{(s^2)^{-d+\frac{1}{2} \eta}} A_{\mu\nu\sigma\rho}(s, \partial_{x'}) \mathcal{O}(x'),
\]

(6.17)

where \( A_{\mu\nu\sigma\rho}(s, \partial) \) is determined by requiring it to reproduce the full expression on the r.h.s. of (6.14). In the presence of a boundary the operator \( \mathcal{O} \) then gives rise to an expression for the two point function \( \langle T_{\mu\nu}(x)T_{\sigma\rho}(x') \rangle \) using the result (2.7) for \( \langle \mathcal{O}(x') \rangle \). Following a similar, albeit more tedious, procedure to that outlined above for \( \langle T_{\mu\nu}(x)\mathcal{O}(x') \rangle \) the result is compatible with the form (2.26) where the corresponding form for the invariant functions \( A, B, C \) due to the operator \( \mathcal{O} \) in the OPE is given by

\[
\begin{align*}
A(v)_\mathcal{O} &= \frac{A_\mathcal{O}}{C_\mathcal{O}} A_\mathcal{O} \xi^{\frac{1}{2} \eta} \left( 1 + 4 \frac{d+2}{\eta^2} \xi \frac{d}{d\xi} + 4 \frac{(d+2)(d+4)}{\eta^2(\eta+2)^2} \xi^2 \frac{d^2}{d\xi^2} \right) F(\xi),
B(v)_\mathcal{O} &= - \frac{A_\mathcal{O}}{C_\mathcal{O}} \xi^{\frac{1}{2} \eta} \left( B_\mathcal{O} \left( 1 + \frac{2d}{\eta^2} \xi \frac{d}{d\xi} \right) + A_\mathcal{O} \frac{2}{\eta^2} \left( \xi \frac{d}{d\xi} + 2 \frac{d+2}{(\eta+2)^2} \xi^2 \frac{d^2}{d\xi^2} \right) \right) F(\xi),
C(v)_\mathcal{O} &= \frac{A_\mathcal{O}}{C_\mathcal{O}} \xi^{\frac{1}{2} \eta} \left( C_\mathcal{O} + B_\mathcal{O} \frac{8 \xi}{\eta^2} \xi \frac{d}{d\xi} + A_\mathcal{O} \frac{8}{\eta^2(\eta+2)^2} \xi^2 \frac{d^2}{d\xi^2} \right) F(\xi),
\end{align*}
\]

(6.18)

\[
f(\xi) = F\left( \frac{1}{2} \eta, \frac{1}{2} \eta; \eta + 1 - \frac{1}{2} d; -\xi \right).
\]
We may solve (6.16) by writing

\[ A_o = C_{TT} \left( \frac{d-2}{d-1} \eta(\eta + 2) \right), \]
\[ B_o = C_{TT} \left( \frac{d-2}{d-1} - \frac{1}{2}(d + 2 - \eta) \right) \eta, \]
\[ C_o = C_{TT} \left( - \frac{2d}{d-1} + (d - \eta)^2 \right), \]

and then, using various properties of hypergeometric functions and with the results in (2.36,37,38), we may obtain, with \( C_{TT}^{-1} = C_{TT}/C_o \),

\[ \alpha(v_o) = C_{TT} A_o (d - 2)(d + 1)v^\eta(1 - v^2) \]
\[ \times \left\{ \left( 1 - \frac{dv^2}{\eta + 2} \right) F(v^2) + \frac{2v^2}{\eta(\eta + 2)}(2 + d(1 - v^2))F'(v^2) \right\}, \]
\[ \gamma(v_o) = C_{TT} A_o \left( \frac{d-2}{d-1} \right)(d + 1)v^\eta(1 - v^2) \]
\[ \times \left\{ \left( \frac{1}{2}(\eta - d) - \frac{dv^2}{\eta + 2} \right) F(v^2) + \frac{v^2}{\eta + 2} \left( \eta - d - \frac{2dv^2}{\eta} \right) F'(v^2) \right\}, \]
\[ F(v^2) = F(\frac{1}{2} \eta, 1 - \frac{1}{2}d; \eta + 1 - \frac{1}{2}d; v^2). \]

Although these expressions are somewhat lengthy they satisfy differential equations equivalent to the conservation conditions (2.31a,b).

As a check on (6.20) we may consider the trivial case when \( O \to 1 \), the identity operator, with \( \eta = 0 \). In this case in (6.14) \( C_1 = C_T \), as defined in (2.32), and \( A_1 = 1 \) and from (6.19) \( C_T = C_{TT} 1 d(d - 2)(d + 1)/(d - 1) \). Using \( F(v^2) \sim 1 - \frac{1}{2} \eta \ln(1 - v^2) \) for \( \eta \sim 0 \) then taking the limit \( \eta \to 0 \) in (6.20) and expressing the coefficient in terms of \( C_T \) gives the expected results for \( \alpha_1, \gamma_1 \) independent of \( v \).

If \( \eta = d - 2 \) then \( F(v^2) = 1 \). As discussed earlier this case is relevant for the operator \( \phi^2 \) in free field theory when now \( C_{TT} \phi^2 = \frac{1}{4} d(d - 2)^2 A/(d - 1) \), \( A_{\phi^2} = \pm AN \), with \( A \) given in (3.3). In this case (6.20) becomes

\[ \alpha(v_{\phi^2}) = \pm \frac{N d}{4(d - 1)S^2 d} (d - 2)(d + 1)(1 - v^2)2v^{d-2}, \]
\[ \gamma(v_{\phi^2}) = \pm \frac{N d}{4(d - 1)^2 S^2 d} (d - 2)(d + 1)(1 - v^4)2v^{d-2}, \]

which is identical to the results calculated in I for free scalar field theory for the terms in \( \langle T_{\mu\nu}(x)T_{\sigma\rho}(x') \rangle \) dependent on the boundary conditions. In general however the expression obtained from a single operator \( O \) in the OPE, as given by (6.20), fails to satisfy the required behaviour as \( y, y' \to 0 \), in particular boundary conditions such as \( \gamma(1) = 0 \).*

* In two dimensions the contribution of general scalar operators from (6.20) is zero. Nevertheless the two point function (2.34) is reproduced by the OPE \( T_{zz}(z)T_{\bar{z}\bar{z}}(\bar{z}) = e^{(z - \bar{z})^2} \partial_z O(x') \) in terms of the quasi-primary operator \( O(x) = T_{zz}(z)T_{\bar{z}\bar{z}}(\bar{z}) \), which has dimension 4 and \( A_O = \frac{1}{4} C_T \).
In the $O(N)$ model for large $N$ the result for $\alpha(v)$ in (5.34) shows that the scalar operators appearing in the OPE of two energy momentum tensors have dimensions $2 + 2n$ and also $2d + 2n$ for $n = 0, 1, \ldots$.

7 Boundary Operator Expansion

In the previous section we described how the contribution of all derivative operators formed from a quasi-primary operator and appearing in the OPE for operators $O_1, O_2$ to the two point function $\langle O_1(x)O_2(x') \rangle$ in the presence of a boundary may be explicitly calculated. In this section we show how this can also be achieved for all derivative operators formed from a particular boundary operator $\hat{O}$ appearing in the BOE of $O_1$ or $O_2$. We assume a basis of boundary operators which have a well defined spin under $O(d-1)$ rotations and scale dimension under scale transformations. For $\hat{O}$ a scalar operator of scale dimension $\hat{\eta}$, so that the corresponding derivative operators have dimension $\hat{\eta} + n$, $n = 1, 2, \ldots$, the two point function on the boundary has the form

$$\langle \hat{O}(x)\hat{O}(x') \rangle = \frac{\hat{C}_{\hat{O}}}{s^{2\hat{\eta}}} , \quad s = x - x'. \quad (7.1)$$

The contribution to the BOE in (2.11) for a scalar operator $O$ of dimension $\eta$ involving derivatives of the operator $\hat{O}$ may then written as

$$O(x) = \frac{B_{O\hat{O}}}{(2y)^{\eta-\hat{\eta}}} D^{\hat{\eta}}(y^2\hat{\nabla}^2)\hat{O}(x), \quad (7.2)$$

where the differential operator $D^{\hat{\eta}}(y^2\hat{\nabla}^2)$, $\hat{\nabla}_i = \partial_i$, is determined by consistency with (2.12). Defining $B_{O\hat{O}} = B_{O\hat{O}}/\hat{C}_{\hat{O}}$ this requires

$$D^{\hat{\eta}}(y^2\hat{\nabla}^2) \frac{1}{s^{2\hat{\eta}}} = \frac{1}{(s^2 + y^2)^{\eta}} = \sum_{m=0}^{\infty} \frac{1}{m!}(\hat{\eta}^m(-y^2)^m) \frac{1}{(s^2)^{\eta+m}}. \quad (7.3)$$

It is easy to see that this is satisfied if we take

$$D^{\hat{\eta}}(y^2\hat{\nabla}^2) = \sum_{m=0}^{\infty} \frac{1}{m!} \frac{1}{(\hat{\eta} + \frac{3}{2} - \frac{1}{2}d)_m} (-\frac{1}{4}y^2\hat{\nabla}^2)^m. \quad (7.4)$$

For application to the BOE of two operators at points away from the boundary we extend the result (7.3) to

$$D^{\hat{\eta}}(y^2\hat{\nabla}^2) D^{\hat{\eta}'}(y'^2\hat{\nabla}'^2) \frac{1}{s^{2\hat{\eta}}} = \sum_{m,n} \frac{1}{m!n!} \frac{(\hat{\eta})_m(\hat{\eta} + \frac{3}{2} - \frac{1}{2}d)_m + (\hat{\eta}')_n(\hat{\eta} + \frac{3}{2} - \frac{1}{2}d)_n}{(s^2)^{\eta+m+n}} \left(\frac{-y^2}{s^2 + y^2 + y'^2}\right)^n \left(\frac{-y'^2}{s^2 + y^2 + y'^2}\right)^m.$$

$$= \frac{1}{(s^2 + y^2 + y'^2)^{\hat{\eta}}} F\left(\frac{1}{2}\hat{\eta}, \frac{1}{2}\hat{\eta} + \frac{1}{2}, \hat{\eta} + \frac{3}{2} - \frac{1}{2}d; \frac{1}{(2\xi + 1)^2}\right). \quad (7.5)$$
Applying this to \( \langle O_1(x)O_2(x') \rangle \) we can extend the leading order result (2.13) to give

\[
f_{12}(\xi) = B_{O_1}B_{O_2} \frac{1}{\xi} F(\eta, \eta + 1 - \frac{1}{2}d; 2\eta + 2 - d; -\frac{1}{\xi}), \tag{7.6}
\]

which therefore includes all derivatives of \( \hat{O} \) in the BOE of \( O_1 \) and \( O_2 \).

As a special case we may consider \( \hat{\eta} = \frac{1}{2}d - 1 \) when \( F(\hat{\eta}, \lambda; 2\lambda; z) \rightarrow \frac{1}{2}((1 - z)^{1-\hat{\eta}} + 1) \) as \( \lambda \rightarrow 0 \). This is relevant for free scalar field theory with Neumann boundary conditions for the surface operator \( \hat{\phi}(x) = \phi(0, x) \) when \( B_\phi = 1 \) and \( \hat{C}_\phi = 2A \) so that (7.6) gives

\[
f_\phi(\xi)_{sp, \phi} = A\left(\frac{1}{\xi^{\frac{1}{2}d-1}} + \frac{1}{(1 + \xi)^{\frac{1}{2}d-1}}\right), \tag{7.7}
\]

which is the exact result in this case. Alternatively if \( \hat{\eta} = \frac{1}{2}d \) then the hypergeometric function simplifies to \( F(\hat{\eta}, 1; 2; z) = ((1 - z)^{1-\hat{\eta}} - 1) / (\hat{\eta} - 1)z \). This is appropriate for free scalar field theory with Dirichlet boundary conditions for the surface operator \( \hat{\phi}_n(x) = \partial_1 \phi(0, x) \) when now \( B_\phi = 1/2 \) and \( \hat{C}_\phi = 2(d - 2)A \) so that (7.6) becomes

\[
f_\phi(\xi)_{ord, \phi} = A\left(\frac{1}{\xi^{\frac{1}{2}d-1}} - \frac{1}{(1 + \xi)^{\frac{1}{2}d-1}}\right), \tag{7.8}
\]

which is again of course the exact result.

Another case when simple formulae are obtained is when \( \hat{\eta} = d - 2 \). This is relevant for the boundary operator \( \phi^2(x) = \phi(0, x)^2 \) in free field theory with Neumann boundary conditions. Applying this to the two point function of \( \phi^2 \), with \( B_{\phi^2} = 1 \) and \( \hat{C}_{\phi^2} = 8NA^2 \), we then obtain

\[
F_{\phi^2}(v)_{sp, \phi^2} = 8NA^2 v^{d-2}, \quad F_{\phi^2}(v)_{sp} = F_{\phi^2}(v)_{sp, \phi^2} + 2NA^2(1 - v^{d-2})^2, \tag{7.9}
\]

where \( F_{\phi^2}(v)_{sp} \) is given by (3.2,5). The remaining part, after subtraction of the contribution arising from \( \phi^2 \), is identical with the result in the Dirichlet case for which the leading behaviour as \( v \rightarrow 1 \) is produced by an operator of dimension \( d \).

We may also extend this treatment to the BOE for operators with spin although we confine our attention here to the energy momentum tensor \( T_{\mu\nu} \). Following (2.14) for \( x_\mu = (y, x) \) and \( x' \) defining a point on the boundary we may define a vector under conformal transformations by

\[
\hat{X}_\mu = \frac{2y}{s^2} \hat{s}_\mu - n_\mu = -I_{\mu\nu}(s)n_\nu, \quad \hat{X}^2 = 1 \quad \hat{s}_\mu = (y, x - x'). \tag{7.10}
\]

Using

\[
\partial_\mu \left( \frac{(2y)^{\eta-d}}{(s^2)^{\eta}} \left( \hat{X}_\mu \hat{X}_\nu - \frac{1}{d}\delta_{\mu\nu} \right) \right) = (d - 1) \left( 1 - \frac{\hat{\eta}}{d} \right) \frac{1}{y} \frac{(2y)^{\eta-d}}{(s^2)^{\eta}} \hat{X}_\mu, \tag{7.11}
\]

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It is easy that $T_{\mu\nu}$ can only have a non-zero two-point function, invariant under conformal transformations and satisfying the conservation equation, with a scalar surface operator provided that this operator has dimension $d$. In any conformal theory there is such a surface operator $\hat{T}(x) = T_{11}(0, x)$, since $T_{11}$ is non-singular as the boundary is approached, and it is natural to assume that this is unique. The surface two-point function for $\hat{T}$ has the form (7.1) with a coefficient $\hat{C}_{\hat{T}}$. From the general result (2.27) or specifically from (2.40) we may then show that

$$\hat{C}_{\hat{T}} = \frac{d-1}{d} C(1) + \frac{(d-1)^2}{d^2} (A(1) + 4B(1)) = \alpha(1), \quad (7.12)$$

with $\alpha(v)$ defined in (2.37,38). The two-point function of $T_{\mu\nu}$ and $\hat{T}$ can then be written

$$\langle T_{\mu\nu}(x) \hat{T}(x') \rangle = \frac{d}{d-1} \hat{C}_{\hat{T}} \frac{1}{s^{2d}} \left( \hat{X}_\mu \hat{X}_\nu - \frac{1}{d} \delta_{\mu\nu} \right), \quad (7.13)$$

where we have taken $B_{T\hat{T}} = 1$.

From (7.13) we may deduce the form of the BOE for $T_{\mu\nu}$ involving $\hat{T}$,

$$T_{\mu\nu}(x) = D_{\mu\nu}(y, \hat{\nabla}) \hat{T}(x), \quad (7.14)$$

where we must require

$$D_{\mu\nu}(y, \hat{\nabla}) \frac{1}{s^{2d}} = \frac{d}{d-1} \frac{1}{s^{2d}} \left( \hat{X}_\mu \hat{X}_\nu - \frac{1}{d} \delta_{\mu\nu} \right). \quad (7.15)$$

Writing

$$D_{\mu\nu}(y, \hat{\nabla}) = \hat{D}_{\mu\nu}(y, \partial) D^d(y^2\hat{\nabla}^2), \quad (7.16)$$

with $D^d(y^2\hat{\nabla}^2)$ defined by (7.4), then we may take* 

$$\hat{D}_{ij}(y, \partial) = \frac{1}{(d-1)(d+1)} \left( y^2 \partial_i \partial_j - \delta_{ij} y \frac{\partial}{\partial y} - (d+1) \delta_{ij} \right), \quad (7.17)$$

$$\hat{D}_{i1}(y, \hat{\nabla}) = \frac{1}{(d-1)(d+1)} y \left( d + 1 + y \frac{\partial}{\partial y} \right) \partial_i, \quad \hat{D}_{11}(y, \partial) = -\hat{D}_{ii}(y, \partial).$$

With these results we may calculate the two-point function for the energy momentum tensor $T_{\mu\nu}$ and a scalar operator $\mathcal{O}$ of dimension $\eta$ by using the BOE involving $\hat{T}$ which

---

* Since $(y^2 \partial^2 + (d+2)y\partial y - \partial^2)/s^{2d} = 0$ there is some ambiguity in the definition of $\hat{D}_{\mu\nu}(y, \partial)$ but this does not affect the final results in the BOE.
gives

\[ \langle T_{\mu\nu}(x)O(x') \rangle = B_O \hat{T} \hat{C}_T (2y')^{d-\eta} D_{\mu\nu}(y, \hat{\nabla}) D^d(y'2\hat{\nabla}'2) \frac{1}{s^{2d}} \]

\[ = B_O \hat{T} \hat{C}_T (2y')^{d-\eta} \hat{D}_{\mu\nu}(y, \partial) \frac{1}{(4yy')^d} F(\xi) \]

\[ = B_O \hat{T} \hat{C}_T (2y')^{-\eta} (2y)^d \left\{ \left( X_{\mu}X_{\nu} - \frac{1}{d} \delta_{\mu\nu} \right) \xi(\xi + 1) \frac{d^2}{d\xi^2} F(\xi) \right\} \]

\[ - \left( n_\mu n_\nu - \frac{1}{d} \delta_{\mu\nu} \right) \left( \xi(\xi + 1) \frac{d^2}{d\xi^2} + d(\xi + \frac{1}{2}) \frac{d}{d\xi} - d \right) F(\xi) \] \hspace{1cm} (7.18)

\[ = \frac{d}{d-1} B_O \hat{T} \hat{C}_T (2y')^{-\eta} \left( \frac{v}{\xi} \right)^d \left( X_{\mu}X_{\nu} - \frac{1}{d} \delta_{\mu\nu} \right) , \]

for \( F(\xi) = \frac{1}{\xi^d} F(d, \frac{1}{2}d + 1; d + 2; 1 - \xi) . \)

It is easy to see that this is in agreement with the general form given by (2.19,22) and comparing with the result (2.26) shows that we must have

\[ B_O \hat{T} = B_O \hat{T} \hat{C}_T = -\eta \frac{A_O}{S_d} , \] \hspace{1cm} (7.19)

which was first obtained by Cardy [7] for consistency of the OPE and BOE.

We also consider the contribution of the boundary operator \( \hat{T} \) to the two point function of the energy momentum tensor where using (7.14) we obtain

\[ \langle T_{\mu\nu}(x)T_{\sigma\rho}(x') \rangle = \hat{C}_T D_{\mu\nu}(y, \hat{\nabla}) D_{\sigma\rho}(y', \hat{\nabla}') \frac{1}{s^{2d}} \]

\[ = \frac{d}{d-1} \hat{C}_T \hat{D}_{\sigma\rho}(y', \partial') \left( \frac{1}{(4yy')^d} \left( X_{\mu}X_{\nu} - \frac{1}{d} \delta_{\mu\nu} \right) \left[ \xi(\xi + 1) \right]^{-\frac{d}{2}} \right) , \] \hspace{1cm} (7.20)

using the result already obtained in (7.19). The evaluation of (7.20) is straightforward albeit tedious. Some details are given in appendix E. The final expression is of the necessary form shown in (2.27) where the coefficients are given by

\[ C(v)_{\hat{T}} = \frac{d}{(d-1)^2(d+1)^2} \hat{C}_T \frac{1}{4} v^{d-2} (1 - v^2)^2 , \]

\[ 2B(v)_{\hat{T}} + C(v)_{\hat{T}} = \frac{d}{(d-1)^2} \hat{C}_T \frac{1}{2} v^{d-2} (1 - v^4) , \]

\[ A(v)_{\hat{T}} + 4B(v)_{\hat{T}} = \frac{d^2}{(d-1)^2(d+1)^2} \hat{C}_T \frac{1}{4} v^{d-2} ((d + 2)(1 + v^4) + 2dv^2) . \] \hspace{1cm} (7.21)

These results satisfy the conservation equations (2.31a,b). Further for \( d = 2 \) they are exact since clearly only scalar operators on the boundary need be considered and in this case (7.21) coincides with (2.33) if \( C_T = \hat{C}_T \). From (2.37,38) we may also write

\[ \alpha(v)_{\hat{T}} = \frac{1}{d-1} \hat{C}_T \frac{1}{4} v^{d-2} (d(1 + v^4) + 2(d - 2)v^2) , \alpha(1) = \hat{C}_T . \] \hspace{1cm} (7.22)

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For application to a non trivial conformal field theory we may consider first results to first order in \( \varepsilon \) at the non Gaussian fixed point in scalar field theory. In ref. [23] and in I \( \hat{C} T \) was calculated to this order for Neumann/Dirichlet boundary conditions giving

\[
\hat{C} T = \frac{N}{S_d^2} \left( 2 + \frac{5}{3} \frac{N + 2}{N + 8} \varepsilon \right).
\]

(7.23)

The coefficient appearing in (3.15b) determines \((B_{\phi^2} T_{\text{ord}})^2 \hat{C} T\) and hence using (7.19) with \( \mathcal{O} \to \phi^2 \), together with (3.8) for \( \eta_{\phi^2} \), we find

\[
\frac{(A_{\phi^2, \text{ord}})^2}{C_{\phi^2}} = \frac{N}{2} (1 + O(\varepsilon^2)).
\]

(7.24)

It is also of interest to compare the expressions obtained in (7.21,22) with the results of the O(\( \varepsilon \)) calculations for \( \langle T_{\mu\nu}(x) T_{\sigma\rho}(x') \rangle \) in I. In the Neumann case for \( v \to 1 \)

\[
\alpha(v)_{\text{sp}} - \alpha(v)_{\hat{T}} \sim \frac{N}{S_d^2} \left( 20 \right) \frac{1}{3} (1 - v^2)^{2 - \frac{N+2}{N+8} \varepsilon},
\]

\[
\gamma(v)_{\text{sp}} - \gamma(v)_{\hat{T}} \sim \frac{N}{S_d^2} \left( 40 \right) \frac{1}{3} (1 - v^2)^{1 - \frac{N+2}{N+8} \varepsilon},
\]

\[
\epsilon(v)_{\text{sp}} - \epsilon(v)_{\hat{T}} \sim \frac{N}{S_d^2} \left( 8 \right) \frac{1}{3} (1 - v^2)^{-\frac{N+2}{N+8} \varepsilon},
\]

(7.25)

in terms of the functions defined in (2.36,37,38). These results correspond to the contributions expected to arise from a boundary operator which is a symmetric traceless tensor \( \hat{T}_{ij} \) of dimension \( \hat{\eta}_{\hat{T}} = d - \frac{N+2}{N+8} \varepsilon + O(\varepsilon^2) \). For such an operator we may define a two point function with the energy momentum tensor

\[
\langle T_{\mu\nu}(x) \hat{T}_{ij}(x') \rangle = B_{\hat{T}} \frac{(2y)^{\hat{\eta}_{\hat{T}}-d}}{s^2} \frac{\mathcal{I}_{\mu\nu,ij}(\hat{s})}{\mathcal{I} + \frac{1}{d-1} \left( \dot{X}_\mu \dot{X}_\nu - 1/d \delta_{\mu\nu} \right) \delta_{ij}},
\]

(7.26)

using \( \mathcal{I} \) which is defined in (2.28). This satisfies the required conformal transformation properties for \( T_{\mu\nu} \) and obeys the necessary conservation equation for arbitrary \( \hat{\eta}_{\hat{T}} \). In the Neumann case the boundary operator \( \hat{T}_{ij}(x) \) relevant for the limiting behaviour in (7.24) appears as the leading term in the BOE of the operator formed from the traceless part of \( T_{ij}(x) \). The above analysis shows that the coefficient, \( \propto (2y)^{-d+\hat{\eta}_{\hat{T}}} \), is singular for \( y \to 0 \) unlike the situation for \( T_{11}(x) = -T_{ii}(x) \) which tends smoothly to \( \hat{T}(x) \). \( \hat{T}_{ij} \) cannot contribute to the BOE of scalar operators so its role is not apparent in previous discussions. In the Dirichlet case, corresponding to (7.24), we have

\[
\alpha(v)_{\text{ord}} - \alpha(v)_{\hat{T}} \sim \frac{N}{S_d^2} \left( 1 - v^2 \right)^{4 - \frac{N+2}{N+8} \varepsilon},
\]

\[
\gamma(v)_{\text{ord}} - \gamma(v)_{\hat{T}} \sim \frac{N}{S_d^2} \left( 4 \right) \frac{1}{3} (1 - v^2)^{3 - \frac{N+2}{N+8} \varepsilon},
\]

\[
\epsilon(v)_{\text{ord}} - \epsilon(v)_{\hat{T}} \sim \frac{N}{S_d^2} \left( 12 \right) \frac{1}{5} (1 - v^2)^{2 - \frac{N+2}{N+8} \varepsilon},
\]

(7.27)
which represents the contribution for an operator \( \hat{T}_{ij} \) with dimension \( d + 2 - \frac{N+2}{N+3} \varepsilon \). This is as expected since such an operator should be constructed from the field \( \phi \) in terms of expressions of the form \( \partial_i \partial_1 \phi \partial_j \partial_1 \phi \).

In the critical \( O(N) \) model in the \( N \to \infty \) limit we may also use the results obtained in sections 4 and 5 to verify the consequences of the BOE. By considering the limit \( \xi \to \infty \) or \( v \to 1 \) of the \( \lambda \) two point function we may identify the contribution of the boundary operator \( \hat{T} \) of dimension \( d \).

Using now (7.19), with \( \eta = 2 \) and \( A_{\lambda, \text{ord}} \) given by (4.12), we then find

\[
\hat{C}_{\hat{T}, \text{ord}} = \frac{2N}{S_d^2} \frac{\Gamma(2d-3)\Gamma(3 - \frac{1}{2}d)\Gamma(\frac{1}{2}d)^3}{\Gamma(d)\Gamma(d-1)^2}.
\]  

(7.29)

Since \( \alpha(1) = \hat{C}_T \), this is identical with the result obtained by direct calculation in (5.32). The results for the energy momentum tensor two point function given in (5.30,31) and (5.33) exhibit explicitly the full contributions of the boundary operator \( \hat{T} \), as given in (7.21,22), while the remaining parts involving \( 3F_2 \) functions arise from non scalar boundary operators with dimension \( 2d - 2 + 2n \). For the special case (4.40) gives

\[
(B_{\lambda} \hat{T}_{\text{sp}})^2 \hat{C}_{\hat{T}, \text{sp}} = B \frac{1}{3} \frac{6 - d}{d - 2} \frac{\Gamma(d)\Gamma(d-2)}{\Gamma(2d-5)},
\]  

(7.30)

which then implies

\[
\hat{C}_{\hat{T}, \text{sp}} = \frac{2N}{S_d^2} \frac{6(6 - d)(\Gamma(2d-5)\Gamma(3 - \frac{1}{2}d)\Gamma(\frac{1}{2}d)^3)}{\Gamma(d)\Gamma(d-1)^2}.
\]  

(7.31)

It is easy to check that (7.29,31) are in accord with the \( \varepsilon \) expansion results in (7.23).

**8 Conclusion**

Although the critical behaviour of statistical systems with a boundary is relatively unexplored experimental investigation is feasible. In this paper we have discussed theoretically the form of the functional behaviour of two point functions at a conformal invariant critical point in dimensions \( d > 2 \). In particular the \( O(N) \) model, for \( N \to \infty \), provides a tractable non trivial example which may be analysed for \( 2 < d \leq 4 \). The results of calculations in this model, which have been here extended to include correlation functions involving the energy momentum tensor, are complementary to the \( \varepsilon \) expansion and provide a limiting case for other approximations.

We may perhaps note that the functions of the invariants \( v \) or \( \xi \) are initially defined on the physical region \( 0 \leq v \leq 1 \) or \( 0 \leq \xi < \infty \) but may be analytically continued to the whole complex plane. The singularities as \( v \to 0 \) and \( v \to 1 \) are well understood in terms of the OPE and BOE. It would be interesting to understand more directly the form of the singular behaviour as \( v \to \infty \) or \( \xi \to -1 \). In this context we may note that our results have simple transformation properties under \( v \to v^{-1} \) or \( \xi \to -1 - \xi \) (which is equivalent to taking \( y \to -y \) or \( y' \to -y' \)) which may merit further investigation.
Appendix A

Here we present some details of the perturbative calculation of the two point function of $\phi^2$ to first order in $\varepsilon = 4 - d$ assuming the renormalisable interaction $\mathcal{L}_{\text{int}} = \frac{1}{24}g(\phi^2)^2$ with $\phi^\alpha$ an $N$-component scalar field. The coupling is restricted to the non-gaussian fixed point $g = g_* = 48\pi^2\varepsilon/(N + 8)$ to lowest order. To simplify the calculation we choose a perpendicular configuration where $x_\mu = (y, 0)$ and $x'_\mu = (y', 0)$, with $y > y'$, so that $v = (y - y')/(y + y')$. For the free field case the basic propagator from (3.1,2) in this case is just

$$G(y, y') = \frac{A}{(y - y')^{d-2}}(1 \pm v^{d-2}), \quad A = \frac{1}{(d - 2)\Delta_d}.$$  \hspace{1cm} (A.1)

To first order in $g$ we may write

$$\langle \phi^2(y, 0)\phi^2(y', 0) \rangle = 2NG(y, y')^2 \mp 2\pi N(N + 2)g_*G(y, y')I_a - \frac{1}{3}N(N + 2)g_*I_b,$$  \hspace{1cm} (A.2)

where $I_a, I_b$ are integrals corresponding to fig. (1a,b). To this order we may restrict these to $d = 4$ since $g_* \propto \varepsilon$ and $I_a$ then has the form

$$I_a = A^4 \int_0^\infty dz \int d^3x \left( \frac{1}{X^2} \pm \frac{1}{X'^2} \right) \left( \frac{1}{X'^2} \pm \frac{1}{X'^2} \right) \frac{1}{4z^2}.$$  \hspace{1cm} (A.3)

with $X^2 = x^2 + (y - z)^2$, $X'^2 = x^2 + (y' - z)^2$, $\tilde{X}^2 = x^2 + (y + z)^2$, and $\tilde{X}'^2 = x^2 + (y' + z)^2$. For finiteness we exclude from the $z$-integral an $\varepsilon$ neighbourhood of the boundary $z = 0$. In the Dirichlet case the result is finite for $\varepsilon \to 0$, but in the Neumann case the divergence must be removed by the addition of a surface counter term $\propto \phi^2$. This calculation is equivalent to finding the one loop correction to $G_\phi(x, x')$ which was calculated previously [9,14]. Consequently we easily obtain the result

$$F_{\phi^2}(v)_a = \frac{2N(N + 2)A^2\varepsilon}{(N + 8)} \left( v^4 \ln \left( 1 - v^2 \right) + \ln \left( 1 - v^2 \right) \pm v^2 \ln \left( 1 - v^2 \right)^2 \right).$$  \hspace{1cm} (A.4)

For $I_b$ when $d = 4$ we have

$$I_b = A^4 \int_0^\infty dz \int d^3x \left( \frac{1}{X^2} \pm \frac{1}{X'^2} \right)^2 \left( \frac{1}{X'^2} \pm \frac{1}{X'^2} \right)^2.$$  \hspace{1cm} (A.5)

The integrals involved here are more difficult than the previous case. They contain the usual short distance divergence arising from the singular behaviour of $G_\phi(x, x')^2$ when $d = 4$ but this is removed by hand by restricting the integral over the vertex point ($z, x$) to exclude $z$ from $\varepsilon$ neighbourhoods of $y, y'$. Evaluating the integral then gives

$$F_{\phi^2}(v)_b = \frac{2N(N + 2)A^2\varepsilon}{(N + 8)} \left( 1 + v^4 + v^2 \ln \left( 1 - v^2 \right)^2 + v^4 \ln v^2 + \frac{v^4}{1 - v^2} \ln v^4 \right.$$

$$\left. \pm v^2 \ln \left( 1 - v^2 \right)^2 + \left( 1 \pm v^2 \right)^2 \ln \frac{4\varepsilon^2}{(y - y')^2} \right).$$  \hspace{1cm} (A.6)
The divergent term as $\epsilon \to 0$ is proportional to $G(y, y')^2 \ln ((y - y')^2/\epsilon^2)$ which of course reflects the modification of the singularity as $y \to y'$ from its free value arising from the one loop correction to the scaling dimension of $\phi^2$. The complete result to this order is now

$$F_{\phi^2}(v) = 2NA^2(1 \pm v^{d-2})^2 + F_{\phi^2}(v)_{a} + F_{\phi^2}(v)_{b}.$$  \hfill (A.7)

Dropping the divergent piece then gives the result quoted in (3.13).

**Appendix B**

In order discuss the evaluation of the integrals in section 4 for general $d$ when $g(\xi)$ is given by (4.27) it is convenient to work backwards and consider an expression for $\tilde{g}(k)$ of the form

$$\tilde{g}_{a,b}(k) = \frac{\Gamma(a + \frac{i}{4}k)\Gamma(a + \frac{i}{4}k)}{\Gamma(b - \frac{i}{4}k)\Gamma(b + \frac{i}{4}k)}.$$  \hfill (B.1)

Having found the corresponding $g_{a,b}(\xi)$ it is then trivial to find the associated $h(\xi)$, which determines the inverse kernel $H(x, x')$, since it is obviously equal to $g_{b,a}(\xi)$. The inverse Fourier transform can be found as a sum of residues of poles, at $\frac{i}{4}k = a + n$ for $\theta > 0$, giving a hypergeometric function

$$\hat{g}_{a,b}(\sinh^2 \theta) = \frac{1}{2\pi} \int dk \ e^{-ik\theta} \tilde{g}(k)$$

$$= \frac{4\Gamma(2a)}{\Gamma(b - a)\Gamma(b + a)} e^{-4a|\theta|} F(2a, a - b + 1; a + b; e^{-4|\theta|}).$$  \hfill (B.2)

$$= \frac{4\Gamma(2a)}{\Gamma(b - a)\Gamma(b + a)} \frac{1}{(4 \cosh^2 \theta)^{2a}} F(2a, a + b - \frac{1}{2}; 2a + 2b - 1; \frac{1}{\cosh^2 \theta}).$$

The transform $\hat{g} \to g$ is then straightforward since we can use

$$\frac{1}{\Gamma(\lambda)} \int_0^\infty du \ u^{\lambda-1} \ F(p + \lambda, (1 + \rho + u)^p + \lambda) = \frac{\Gamma(p)}{(1 + \rho)^p}.$$  \hfill (B.3)

Applying this term by term in (B.2), with $p = 2a + n$, we get

$$g_{a,b}(\xi) = \frac{\Gamma(2a + \lambda)}{4^{2a-1} \pi^2 \Gamma(b - a)\Gamma(b + a)} \frac{1}{(1 + \xi)^{2a + \lambda}} F(2a + \lambda, a + b - \frac{1}{2}; 2a + 2b - 1; \frac{1}{1 + \xi})$$

$$= \frac{\Gamma(2a + \lambda)}{4^{2a-1} \pi^2 \Gamma(b - a)\Gamma(b + a)} \frac{1}{\xi^{2a + \lambda}} F(2a + \lambda, a + b - \frac{1}{2}; 2a + 2b - 1; \frac{1}{\xi}).$$  \hfill (B.4)

For application in the Dirichlet case we take

$$g(\xi) = \frac{1}{\xi^{a}(1 + \xi)^{\alpha}} = \frac{\pi^{\lambda+1}}{4^\lambda} \frac{\Gamma(2\alpha)\Gamma(\frac{d}{2} - \alpha)}{\Gamma(\alpha + \frac{d}{2})\Gamma(\alpha)^2} g_{\alpha - \frac{1}{2}, \lambda, \frac{1}{2}(\lambda+1)}(\xi).$$  \hfill (B.5)
and hence in this case
\[ h(\xi) = f(\alpha) \frac{\Gamma(d)\Gamma(\alpha)}{\Gamma(2\alpha)\Gamma(d - \alpha)} \frac{1}{\xi^d} F(d, \alpha; 2\alpha; -\frac{1}{\xi}), \tag{B.6} \]

where \( f(\alpha) \) is as in (4.4). Alternatively (B.6) can be written as
\[ \xi^d h(\xi) = f(\alpha) \left( \nu^{2\alpha} F(\alpha, 2\alpha - d; \alpha - d + 1; \nu^2) + \frac{\Gamma(d)\Gamma(\alpha - d)}{\Gamma(2\alpha - d)\Gamma(d - \alpha)} \nu^{2d} F(d, \alpha; d - \alpha + 1; \nu^2) \right), \tag{B.7} \]

so that \( h(\xi) \sim f(\alpha) \xi^{\alpha-d} \) as \( \xi \to 0 \), assuming \( \alpha < d \). When \( \alpha = d - 2 \), as required to leading order in \( 1/N \), then
\[
\begin{align*}
  h(\xi) &= f(d - 2) \left\{ \frac{1}{\xi^2} (1 - (d - 2)(3 - d)\xi) + (d - 4) \left( \ln \frac{1}{\xi^2} F(d, 5 - d; 3; \xi) + \tilde{h}(\xi) \right) \right\}, \\
  \tilde{h}(\xi) &= \sum_{n=0} \xi^n \frac{(d)_n(5 - d)_n}{n!(n + 2)!} \left( \psi(n + 3) + \psi(n + 1) - \psi(d + n) - \psi(d - 4 - n) \right).
\end{align*}
\tag{B.8}

For the Neumann case we consider functions of the form
\[ g(\xi) = \frac{(1 + 2\xi)^2}{\xi^\alpha(1 + \xi)^\alpha}. \tag{B.9} \]

Writing \( (1 + 2\xi)^2 = 1 + 4\xi(1 + \xi) \) it is easy to see from the above discussion that the corresponding transform becomes
\[ \tilde{g}(k) = \frac{\pi^{\lambda+1}}{4^{\lambda+1}} \frac{\Gamma(2\alpha)\Gamma(\frac{1}{2}d - \alpha)}{\Gamma(\alpha + \frac{1}{2})\Gamma(\alpha)^2} \left( \mu^2 + \frac{1}{4}k^2 \right) \tilde{g}_{\alpha-1-\frac{1}{2}\lambda, \frac{1}{2}\lambda+1}(k), \quad \mu^2 = \lambda^2 - 2(\alpha - 1). \tag{B.10} \]

Defining, as in (4.26), \( \tilde{h}(k) = 1/\tilde{g}(k) \) then two approaches are possible in order to obtain \( h(\xi) \). Firstly finding the inverse Fourier transform may be simplified to the previous discussion by removing the pole at \( k^2 = -4\mu^2 \) which is equivalent to expressing \( \tilde{h} \) in terms of a differential equation,
\[ \left( \mu^2 - \frac{1}{4} \frac{d^2}{d\theta^2} \right) \tilde{h}(\sinh^2 \theta) = \frac{4^{\lambda+1}}{\pi^{\lambda+1}} \frac{\Gamma(\alpha + \frac{1}{2})\Gamma(\alpha)^2}{\Gamma(2\alpha)\Gamma(\frac{1}{2}d - \alpha)} \tilde{g}_{\frac{1}{2}((\lambda+1), \alpha-1-\frac{1}{2}\lambda}(\sinh^2 \theta), \tag{B.11} \]

which can further be rewritten as
\[ \left( \rho(1 + \rho) \frac{d^2}{d\rho^2} + (\rho + \frac{1}{2}) \frac{d}{d\rho} - \mu^2 \right) \tilde{h}(\rho) = -\pi^\lambda f(\alpha)(\alpha - \frac{1}{2}d - 1) \frac{\Gamma(\alpha - 1)\Gamma(\lambda + 1)}{\Gamma(2\alpha - 2)\Gamma(d - \alpha)} \]
\[ \times \frac{1}{(1 + \rho)^{\lambda+1}} F\left( \lambda + 1, \alpha - 1; 2\alpha - 2; \frac{1}{1 + \rho} \right). \tag{B.12} \]
By considering the definition of the transform (4.18), or its inverse (4.19), we can then derive an equivalent differential equation for \( h \)

\[
\left( \xi(1 + \xi) \frac{d^2}{d\xi^2} + d(\xi + \frac{1}{2}) \frac{d}{d\xi} + \lambda^2 - \mu^2 \right) h(\xi) = -f(\alpha)(\alpha - \frac{1}{2}d - 1) \frac{\Gamma(\alpha - 1)\Gamma(d)}{\Gamma(2\alpha - 2)\Gamma(d - \alpha)} \frac{1}{\xi^d} F(d, \alpha - 1; 2\alpha - 2; -\frac{1}{\xi}),
\]

(B.13)

which is essentially identical with an equation obtained, by very different means, by Ohno and Okabe [11] when \( \alpha = d - 2 \) which is appropriate for discussion of the large \( N \) limit in section 4. The homogeneous equation is easily solved in terms of standard hypergeometric functions. The relevant boundary conditions, which ensure a unique solution, are that as \( \xi \to \infty \), taking \( \mu > 0 \), \( h(\xi) \sim \xi^{-\mu-\lambda}, \xi^{-d} \) (with no \( \xi^{\mu-\lambda} \) behaviour) and as \( \xi \to 0 \) then \( h(\xi) \sim 1, \xi^{\alpha-d} \) (with no terms \( \propto \xi^{1-\frac{d}{2}} \)).

However a more direct approach is to take the inverse Fourier transform \( \hat{h} \to h \) when we obtain by contour integration

\[
\hat{h}(\sinh^2 \theta) = \frac{4^{\lambda+1}}{\pi^{\lambda+1}} \frac{\Gamma(\alpha + \frac{1}{2})\Gamma(\alpha)^2}{\Gamma(2\alpha)\Gamma(\frac{1}{2}d - \alpha)} \left( \frac{4\Gamma(\lambda + 1)}{\Gamma(\alpha - \frac{1}{2})\Gamma(\alpha - \lambda - \frac{3}{2})(\mu^2 - (\lambda + 1)^2)} G(\sinh^2 \theta) + \frac{\Gamma(\frac{1}{2}(\lambda + \mu + 1))\Gamma(\frac{1}{2}(\lambda - \mu + 1))}{\Gamma(\alpha - 1 - \frac{1}{2}(\lambda + \mu))\Gamma(\alpha - 1 - \frac{1}{2}(\lambda - \mu))} \frac{1}{\mu} e^{-2\mu|\theta|} \right),
\]

(B.14)

with

\[
G(\sinh^2 \theta) = e^{-2(\lambda+1)|\theta|} {}_4F_3(\lambda + 1, B, C_+, C_-; \alpha - \frac{1}{2}, C_+ + 1, C_- + 1; e^{-4|\theta|}),
\]

(B.15)

where

\[
B = \lambda + \frac{3}{2} - \alpha, \quad C_\pm = \frac{1}{2}(\lambda + 1 \pm \mu).
\]

(B.16)

The first term on the r.h.s of equation (B.14) represents a particular inhomogeneous solution to (B.11) while the second term is a solution to the homogeneous equation. It is possible to take the inverse transform \( \hat{h}(\rho) \to h(\xi) \) by noting that with \( \rho = \sinh^2 \theta \) then

\[
e^{-2a|\theta|} = (\sqrt{\rho} + \sqrt{1 + \rho})^{-2a} = \frac{1}{4^a(1 + \rho)^a} {}_2F(a, a + \frac{1}{2}; 2a + 1; \frac{1}{1 + \rho}),
\]

(B.17)

so that using (B.3) gives

\[
\frac{\Gamma(a)}{\Gamma(a + \lambda)} \frac{1}{\Gamma(-\lambda)} \int_\xi^\infty d\rho (\rho - \xi)^{-\lambda-1} \frac{1}{(1 + \rho)^a} F(a, a + \frac{1}{2}; 2a + 1; \frac{1}{1 + \rho})
\]

\[
= \frac{1}{(1 + \xi)^{a+\lambda}} F(a + \lambda, a + \frac{1}{2}; 2a + 1; \frac{1}{1 + \xi})
\]

(B.18)

\[
= \frac{\xi + \frac{1}{2}}{[\xi(\xi + 1)]^{\frac{1}{2}(a+\lambda+1)}} F(\frac{1}{2}(a + \lambda + 1), \frac{1}{2}(a - \lambda) + 1; a + 1; -\frac{1}{4\xi(\xi + 1)}).
\]
This formula can be used for both terms present in (B.14) if $G(\sinh^2 \theta)$, as given by (B.15), is expanded in a series of exponentials in $\theta$. The part of the final result for $h(\xi)$ corresponding to the $e^{-2\mu |\theta|}$ term in (B.16) is then quite simple,

\[
\frac{4^{2C-}}{\pi^d} \frac{\Gamma(\alpha + \frac{1}{2})\Gamma(\alpha)}{\Gamma(2\alpha)\Gamma(\frac{1}{2}d - \alpha)} \frac{\Gamma(C_+)\Gamma(C_-)}{\Gamma(\alpha - \frac{1}{2} - C_+}\Gamma(\alpha - \frac{1}{2} - C_-)\Gamma(\mu + 1)} \times \frac{\xi + \frac{1}{2}}{[\xi(\xi + 1)]^C_+} F(C_+, \mu + \frac{3}{2} - C_+; \mu + 1; -\frac{1}{4\xi(\xi + 1)}) ,
\]

but finding a nice form for the transform of the piece involving the $4F_3$ generalised hypergeometric function poses some challenges, a resolution of which is illustrated below.

After expanding $4F_3$ in (B.15) and using (B.18) for $a \rightarrow \lambda + 1 + 2n$ we may then define

\[
F(\xi) = \frac{\Gamma(\lambda + 1)}{\Gamma(d)} \frac{1}{\Gamma(-\lambda)} \int_0^\infty d\rho \rho^{\lambda - 1} G(\rho + \xi) = \frac{\xi + \frac{1}{2}}{(4\xi(1 + \xi))^{\lambda + 1}} \sum_{n=0}^{\infty} \frac{1}{n!} \frac{(\lambda + 1)n(B)_n(C_+)_n(C_-)_n}{(\alpha - \frac{1}{2})n(C_+ + 1)_n(C_- + 1)_n(\lambda + 1)_n} \frac{(d)_n}{(2n)!} (\lambda + 1 + n, \frac{3}{2} + n; \lambda + 2 + 2n; -\frac{1}{4\xi(1 + \xi)}) ,
\]

so that, in addition to (B.19), the remaining part of $h(\xi)$ becomes

\[
-\frac{4^{\lambda + 1}}{\pi^d} \frac{\Gamma(\alpha + \frac{1}{2})\Gamma(\alpha)}{\Gamma(2\alpha)\Gamma(\frac{1}{2}d - \alpha)} \frac{\Gamma(d)}{\Gamma(\alpha - \frac{1}{2})\Gamma(\alpha - \lambda - \frac{d}{2})} \frac{1}{C_+C_-} F(\xi) .
\]

The result (B.20) for $F(\xi)$ may be rewritten by expanding the hypergeometric function as

\[
F(\xi) = \frac{\xi + \frac{1}{2}}{(4\xi(1 + \xi))^{\lambda + 1}} \sum_{N=0}^{\infty} \frac{1}{N!} h_N \left( -\frac{1}{4\xi(1 + \xi)} \right)^N ,
\]

where $h_N$ is given by the finite sum

\[
h_N = \sum_{n=0}^{N} \frac{(-1)^n}{4^n} \binom{N}{n} \frac{(\lambda + 1)_n(B)_n(C_+)_n(C_-)_n(d)_{2n}(\lambda + 1 + n)_{N-n}(\frac{3}{2} + n)_{N-n}}{\Gamma(\alpha - \frac{1}{2})n(C_+ + 1)_n(C_- + 1)_n(\lambda + 1)_{2n}(\lambda + 2 + 2n)_{N-n}} .
\]

By using identities for the $\Gamma$ function, such as $(d)_n = 4^n (\frac{1}{2}d)_n (\lambda + 1)_n$, and (B.16) we can write this in the compact form

\[
h_N = \frac{(\frac{3}{2})_N(\lambda + 1)_N}{(\lambda + 2)_N} 7F_6(\lambda + 1, B, C_+, C_-, \lambda + \frac{1}{2}, \frac{1}{2} \lambda + \frac{3}{2}, -N; \alpha - \frac{1}{2}, C_+ + 1, C_- + 1, \frac{1}{2} \lambda + \frac{1}{2}, \frac{3}{2}, \lambda + 2 + N; 1) .
\]
All that remains is to find a closed expression for the sum of the terminating series represented by the \( _7F_6 \) generalized hypergeometric function. For the present this remains elusive for general \( \alpha \). However, if we take \( \alpha = d - 2 \), as required for section 4, then \( \mu = \frac{1}{2}(5 - d) \) so that \( C_+ = \frac{3}{2} \) and \( C_- = \lambda - \frac{1}{2} \). Consequently the \( _7F_6 \) series reduces to a finite \( _5F_4 \) series which is summable by a special case of Dougall’s theorem \([27]\)

\[
_5F_4(\lambda + 1, \frac{7}{2} - \lambda, \lambda - \frac{1}{2}, \frac{3}{2}\lambda + \frac{3}{2}, -N; d - \frac{3}{2}, \frac{5}{2}, \frac{1}{2}\lambda + \frac{1}{2}, \lambda + 2 + N; 1) = \frac{(\lambda + 2)\, _N(\lambda - \frac{1}{2})_N}{(d - \frac{5}{2})\, _N(\frac{5}{2})_N}, \tag{B.25}
\]

so that in this case

\[
F(\xi) = \frac{\xi + \frac{1}{2}}{[4\xi(1 + \xi)]^{\lambda + 1}} \, _3F_2(\lambda + 1, \lambda - 1, \frac{3}{2}; d - \frac{5}{2}, \frac{5}{2}; -\frac{1}{4\xi(1 + \xi)}). \tag{B.26}
\]

Hence, with \( f \) defined in (4.4), the final result becomes

\[
\begin{split}
    h(\xi) &= f(d - 2)\Gamma(\frac{1}{2}d - 1) \left\{ \frac{32}{3}(6 - d) \frac{\Gamma(\lambda + 1)}{\Gamma(d - \frac{5}{2})} F(\xi) \\
    &\quad + \frac{\sin \pi \lambda}{\lambda - 2} \frac{\Gamma(\lambda - \frac{1}{2})}{\Gamma(d - 3)} \frac{8}{(1 + 2\xi)^2} F\left(\frac{3}{2}, 1; 3 - \lambda; \frac{1}{(1 + 2\xi)^2}\right) \right\}.
\end{split} \tag{B.27}
\]

The form (B.27), with (B.26), for \( h(\xi) \) is not appropriate for considering the limit \( \xi \to 0 \). However by using the relation

\[
_3F_2(a, b; c, \rho + 1; -z) = \Gamma(\rho + 1) \frac{\Gamma(a - \rho)\Gamma(b - \rho)\Gamma(c)}{\Gamma(a)\Gamma(b)\Gamma(c - \rho)} \, z^\rho \\
- \frac{\Gamma(c)\Gamma(b - a)\rho}{\Gamma(b)\Gamma(c - a)} a - \rho \, z^{-a} _3F_2\left( a, a + 1 - c, a - \rho; a + 1 - b, a + 1 - \rho; -\frac{1}{z} \right) \\
- \frac{\Gamma(c)\Gamma(a - b)\rho}{\Gamma(a)\Gamma(c - b)} b - \rho \, z^{-b} _3F_2\left( b, b + 1 - c, b - \rho; b + 1 - a, b + 1 - \rho; -\frac{1}{z} \right), \tag{B.28}
\]

we may find an alternative expression for \( F(\xi) \),

\[
\begin{split}
F(\xi) &= \Gamma(\frac{\lambda + 1}{2}) \frac{\Gamma(d - \frac{5}{2}) \Gamma(\lambda - \frac{1}{2}) \Gamma(\lambda - \frac{3}{2})}{\Gamma(d - 4) \Gamma(\lambda - 1) \Gamma(\lambda + 1)} \frac{\xi + \frac{1}{2}}{[4\xi(1 + \xi)]^{\lambda - \frac{1}{2}}} \\
&\quad + \frac{\Gamma(d - \frac{5}{2})}{\Gamma(\lambda + 1) \Gamma(\frac{1}{2}d - 1)} \frac{3}{6 - d} \frac{\xi + \frac{1}{2}}{[4\xi(1 + \xi)]^{\lambda - \frac{1}{2}}} \left( 1 + (d - 3)(6 - d)\xi(\xi + 1) \right) \\
&\quad + \frac{\Gamma(d - \frac{5}{2})}{\Gamma(\lambda - 1) \Gamma(\frac{1}{2}d - 3)} \frac{3}{d - 2} \frac{\xi + \frac{1}{2}}{(\xi + \frac{1}{2})} \\
&\quad \times \left( \ln 4\xi(\xi + 1) _3F_2\left( \lambda + 1, \frac{1}{2}d, \frac{1}{2}d - 1; 3, \frac{1}{2}d; -4\xi(\xi + 1) \right) + \tilde{F}(\xi) \right),
\end{split} \tag{B.29}
\]

* This does not seem to appear in standard references but related results were found long ago \([28]\). The particular result may be derived by multiplying by \((-z)^\rho\) and differentiating when it reduces to a well known result for standard hypergeometric functions, the constant of integration is determined by taking \( z = 1 \).
where $\tilde{F}(\xi)$ is given by a power series in $\xi$, analogous to $\tilde{h}(\xi)$ in (B.8). With this form the behaviour for $\xi \sim 0$ is manifest. It can also be readily shown that

$$
\frac{1}{(2\xi + 1)^2} F\left(\frac{3}{2}, 1; 3 - \lambda; \frac{1}{(2\xi + 1)^2}\right) = \frac{2\sqrt{\pi}}{\sin \pi \lambda} \frac{\Gamma(\lambda - \frac{1}{2})}{\Gamma(\lambda - 2)} \left[4\xi(1 + \xi)\right]^{\lambda - \frac{1}{2}} \left(\frac{2\xi + 1}{\lambda - \frac{1}{2}} F(\lambda - 1, 1; \lambda + \frac{1}{2}; -4\xi(\xi + 1))\right).
$$

(B.30)

Hence in (B.27) the first terms on the r.h.s.’s of (B.29,30) cancel and

$$
h(\xi) \sim f(d - 2) \frac{2\xi + 1}{[\xi(\xi + 1)]^2} \left(1 + (d - 3)(6 - d)\xi(\xi + 1)\right) + f(d - 2) \frac{(d - 1)(d - 3)(d - 4)(d - 6)}{d - 2} \frac{\xi + \frac{1}{2}}{\ln \left[4\xi(\xi + 1)\right]^{-1}}.
$$

(B.31)

**Appendix C**

In order to justify (5.11), and also determine the appropriate form for the derivative operator which is exhibited in (5.8) and plays a crucial role in the OPE, we consider first the Fourier transform

$$
\int d^d r \frac{1}{(r^2 + \mu^2)^{\eta}} e^{ip \cdot r} = \frac{2\pi^{\frac{d}{2}}}{\Gamma(\eta)} \left(\frac{|p|}{2\mu}\right)^{\eta - \frac{d}{2}} K_{\eta - \frac{d}{2}}(\mu|p|)
$$

$$
= \frac{\pi^{\frac{d}{2}}}{\Gamma(\eta)} \left(\Gamma\left(\frac{1}{2}d - \eta\right)\left(\frac{1}{4}\mu^2\right)^{\eta - \frac{d}{2}} F_1(\eta - \frac{d}{2} - 1; \frac{1}{4}\mu^2\eta^2) + \Gamma(\eta - \frac{d}{2})(\mu^2)^{\frac{d}{2} - \eta} F_1\left(\frac{1}{2}d - \eta + 1; \frac{1}{4}\mu^2\eta^2\right)\right),
$$

(C.1)

where $F_1(\alpha; z) = \sum_n z^n/(n!(\alpha)_n)$ is defined by series expansion. Hence we may write

$$
\int d^d r \frac{1}{(x - r)^{2m}(x' - r)^{2n}} e^{ip \cdot r}
$$

$$
= \frac{1}{B(\eta_1, \eta_2)} \int_0^1 d\alpha \alpha^{n_1 - 1}(1 - \alpha)^{n_2 - 1} e^{ip \cdot (\alpha x + (1 - \alpha)x')} \int d^d r \frac{1}{(r^2 + \alpha(1 - \alpha)s^2)^{\eta}} e^{ip \cdot r}
$$

$$
= \frac{1}{B(\eta_1, \eta_2)} \frac{\pi^{\frac{d}{2}}}{\Gamma(\eta)} \int_0^1 d\alpha \alpha^{n_1 - 1}(1 - \alpha)^{n_2 - 1} e^{ip \cdot s + ip \cdot x'}
$$

$$
\times \left\{\Gamma\left(\frac{1}{2}d - \eta\right)\left(\frac{1}{4}\mu^2\eta^2\right)^{\eta - \frac{d}{2}} F_1(\eta - \frac{d}{2} - 1; \frac{1}{4}\alpha(1 - \alpha)s^2\eta^2) + \Gamma(\eta - \frac{d}{2})(\alpha(1 - \alpha)s^2)^{\frac{d}{2} - \eta} F_1\left(\frac{1}{2}d - \eta + 1; \frac{1}{4}\alpha(1 - \alpha)s^2\eta^2\right)\right\}
$$

(C.2)

$$
= C^{n_1 - n_2}(s, ip) \int d^d r \frac{1}{(x' - r)^{2n}} e^{ip \cdot r}
$$

$$
+ \pi^{\frac{d}{2}} B\left(\frac{1}{2}d - \eta_1, \frac{1}{2}d - \eta_2\right) \frac{\Gamma(\eta - \frac{1}{2}d)}{B(\eta_1, \eta_2)} \left(\frac{s^2}{\lambda}\right)^{\frac{d}{2} - \eta} C^{d - n_1, n_2 - n_1}(s, ip) e^{ip \cdot x'},
$$

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where \( s = x - x' \) and \( \eta = \eta_1 + \eta_2 \) and \( C^{a,b}(s,ip) \) is defined by (5.8). It is easy to see that this is then equivalent to (5.11).

**Appendix D**

For the calculations in section 5 it is necessary to evaluate conformally invariant integrals over \( \mathbb{R}^d_+ \). Techniques for dealing with conformal integrals on all \( \mathbb{R}^d \) are well known [29]. Here we show how the methods of section 4 can be used to calculate the form of the integral in (5.26) although a general discussion will be given elsewhere [30]. For \( x = (y, x) \) and \( x' = (y', x') \) the integrals to be considered are of the basic form

\[
F(\xi) = \int_0^\infty dz \int d^{d-1}r \frac{1}{(2z)^d} F_1(\xi) F_2(\xi'), \quad \xi = \frac{(x - r)^2}{4yz}, \quad \xi' = \frac{(x' - r)^2}{4y'z}, \quad r = (z, r),
\]

(D.1)

where restricted conformal invariance guarantees that the integral is a function of just the invariant \( \xi = (x - x')^2/4yy' \). This integral can be simplified by integrating over \( x \). Hence, considering the sequence of transformations \( F(\xi) \rightarrow \tilde{F}(\rho) \rightarrow \tilde{\tilde{F}}(k) \) defined in (4.18) and (4.25), we obtain

\[
\tilde{F}(\rho) = \int_0^\infty dz \frac{1}{2z} \tilde{F}_1(\rho) \tilde{F}_2(\rho'), \quad \rho = \frac{(y - z)^2}{4yz}, \quad \rho' = \frac{(y' - z)^2}{4y'z},
\]

(D.2)

\[
\tilde{F}(\sinh^2 \theta) = \frac{1}{2\pi} \int dk e^{-ik\theta} \tilde{\tilde{F}}_1(k) \tilde{\tilde{F}}_2(k).
\]

If \( \tilde{F}(\rho) \) can be determined then \( F(\xi) \) can be recovered by using (4.19). For our purposes it is necessary to extend this method to deal with integrals which transform as conformal tensors. For illustration we first consider

\[
F(\xi)\left(X_\mu X_\nu - \frac{1}{d} \delta_{\mu\nu}\right) = \int_0^\infty dz \int d^{d-1}r \frac{1}{(2z)^d} \left(\tilde{X}_\mu \tilde{X}_\nu - \frac{1}{d} \delta_{\mu\nu}\right) F_1(\tilde{\xi}) F_2(\tilde{\xi}'), \quad (D.3)
\]

for \( \tilde{X}_\mu = y[\tilde{\xi}(1 + \tilde{\xi})]^{-\frac{1}{2}} \partial_\mu \tilde{\xi} \) a conformal vector of scale 0 at \( x \). Again the form of the integral is dictated by conformal invariance to be given in terms of the single function \( F(\xi) \). To reduce (D.3) to the previous case we introduce the differential operator

\[
\tilde{D}_{\mu\nu} = \partial_\mu \partial_\nu + \frac{1}{y} (n_\mu \partial_\nu + n_\nu \partial_\mu) - \frac{1}{d} \delta_{\mu\nu} \left(\partial^2 + \frac{2}{y} n \cdot \partial\right),
\]

(D.4)

which is constructed to give

\[
\tilde{D}_{\mu\nu} F(\xi) = \frac{1}{y^2} \left(X_\mu X_\nu - \frac{1}{d} \delta_{\mu\nu}\right) \xi(1 + \xi) \mathcal{F}''(\xi).
\]

(D.5)

If we now set

\[
F(\xi) = 4\xi(1 + \xi) \mathcal{F}''(\xi)
\]

(D.6)
then to evaluate (D.3) it is sufficient to calculate

\[ F(\xi) = \int_0^{\infty} dz \int d^{d-1}r \frac{1}{(2z)^d} \mathcal{F}_1(\tilde{\xi}) \mathcal{F}_2(\tilde{\xi'}) , \]  

where \( \mathcal{F}_1 \) is given in terms of \( F_1 \) similarly to (D.6). From (D.6) and (4.18) we can compute the transform \( \hat{F} \) without explicitly finding \( F \) by solving (D.6) since

\[ \hat{F}(\rho) = \frac{\pi^\lambda}{\Gamma(\lambda + 2)} \int_0^{\infty} du u^{\lambda+1} F''(u + \rho) . \]  

For integrals involving the energy momentum tensor we are interested in integrals for which \( F_1 \to F_T \) where

\[ F_T(\xi) = [\xi(1 + \xi)]^{-\frac{1}{2}} . \]  

In this case the required transforms are particularly simple

\[ \hat{F}_T(\sinh^2 \theta) = \frac{1}{2} S_d \frac{1}{d(d+1)} e^{-(d+1)[\theta]} , \quad \hat{F}_T(k) = \frac{1}{d} S_d \frac{1}{k^2 + (d+1)^2} . \]  

It remains to treat integrals of the form given by (5.26) for which we consider the following expression

\[ G_{\mu\nu\sigma\rho} = \int d^d z \int d^{d-1}r \frac{1}{(2z)^d} \left( \tilde{X}_\mu \tilde{X}_\nu - \frac{1}{d} \delta_{\mu\nu} \right) \left( \tilde{X}'_\sigma \tilde{X}'_\rho - \frac{1}{d} \delta_{\sigma\rho} \right) F_T(\tilde{\xi}) H(\tilde{\xi}') . \]  

We now write using (D.5)

\[ G_{\mu\nu\sigma\rho} = (4yy')^2 \tilde{D}_{\mu\nu} \tilde{D}'_{\sigma\rho} G(\xi) \]  

where

\[ G(\xi) = \int d^d z \int d^{d-1}r \frac{1}{(2z)^d} F_T(\tilde{\xi}) H(\tilde{\xi}') , \quad 4\xi(1 + \xi) H''(\xi) = H(\xi) . \]  

Applying the transformations as described earlier we easily find

\[ \hat{G}(\sinh^2 \theta) = \frac{1}{2\pi} \int dk \ e^{-ik\theta} \hat{F}_T(k) \hat{H}(k) . \]  

The result in (D.10) for \( \hat{F}_T(k) \) then allows us to write

\[ \left( (d + 1)^2 - \frac{d^2}{d\theta^2} \right) \hat{G}(\sinh^2 \theta) = \frac{1}{d} S_d \hat{H}(\sinh^2 \theta) , \]  

which in turn translates into

\[ \left( \xi(1 + \xi) \frac{d^2}{d\xi^2} + d(\xi + \frac{1}{2}) \frac{d}{d\xi} - d \right) G(\xi) = -\frac{1}{4d} S_d H(\xi) . \]
This equation is more conveniently written as
\[
(\xi(1 + \xi) \frac{d^2}{d\xi^2} + (d + 4)(\xi + \frac{1}{2}) \frac{d}{d\xi} + d + 2)G''(\xi) = \frac{1}{d}S_d \frac{1}{16\xi(1 + \xi)} H(\xi). \tag{D.17}
\]

To evaluate (D.13) we use (2.17) and
\[
\tilde{D}_{\mu\nu}X'_{\sigma} = \frac{1}{4y^2} 2\xi + 1 \left( X_{\mu}I_{\nu\sigma}(s) + X_{\nu}I_{\mu\sigma}(s) - \frac{2}{d} \delta_{\mu\nu}X'_{\sigma} \right)
- \frac{1}{4y^2} 4\xi + 1 \left( X_{\mu}X_{\nu} - \frac{1}{d} \delta_{\mu\nu} \right)X'_{\sigma} \tag{D.18}
\]
to obtain
\[
G_{\mu\nu\sigma\rho} = I_{\mu\nu,\sigma\rho}(s) c(\xi) + \left( X_{\mu}X_{\nu} - \frac{1}{d} \delta_{\mu\nu} \right) \left( X'_{\sigma}X'_{\rho} - \frac{1}{d} \delta_{\sigma\rho} \right) a(\xi)
+ \left( X_{\mu}X'_{\sigma}I_{\nu\rho}(s) + \mu \leftrightarrow \nu, \sigma \leftrightarrow \rho \right)
- \frac{4}{d} \delta_{\mu\nu}X'_{\sigma}X'_{\rho} - \frac{4}{d} \delta_{\sigma\rho}X_{\mu}X_{\nu} + \frac{4}{d^2} \delta_{\mu\nu}\delta_{\sigma\rho} b(\xi), \tag{D.19}
\]
where the three invariant functions are given by
\[
c(\xi) = 8G''(\xi), \quad b(\xi) = -8(1 + \xi) \left( 1 + \xi \frac{d}{d\xi} \right)G''(\xi), \quad a(\xi) = 16(1 + \xi)^2 \left( 2 + 4\xi \frac{d}{d\xi} + \xi^2 \frac{d^2}{d\xi^2} \right)G''(\xi). \tag{D.20}
\]

An important test on the results (D.19,20) is provided by the conservation equation which may be obtained by using (2.25) (with the arbitrary constant c = 0) giving
\[
\frac{1}{(2y)^d}G_{\mu\nu\sigma\rho} = \frac{d - 1}{d^2}S_d \left\{ - \frac{1}{(2y)^{d+1}} v \left( - \left( I_{\nu\sigma}(s)X'_{\rho} + I_{\nu\rho}(s)X'_{\sigma} - \frac{2}{d} X_{\nu}\delta_{\sigma\rho} \right)H(\xi)
+ 2X_{\nu} \left( X'_{\sigma}X'_{\rho} - \frac{1}{d} \delta_{\sigma\rho} \right)(H(\xi) + \xi(1 + \xi)H'(\xi)) \right\}. \tag{D.21}
\]
This gives two differential equations relating \(a, b, c\) in (D.19) to \(H\) which, with the results (D.20), are equivalent to (D.17).

For presenting the results it is more convenient to define from (2.37,38)
\[
C(\xi) = -\frac{1}{2}c(\xi) - b(\xi) = 4(1 + 2\xi)G''(\xi) + 8(1 + \xi)\xi \frac{d}{d\xi}G''(\xi),
\]
\[
A(\xi) = \frac{1}{d^2}(d - 1)((d - 1)(a(\xi) + 4b(\xi)) + dc(\xi))
= -\frac{8}{d}(d - 1)^2(1 + \xi)\xi \left( (2\xi + 1) \frac{d}{d\xi} + 2 \right)G''(\xi)
+ \frac{8}{d}(d - 1)G''(\xi) - \frac{1}{d^3}(d - 1)^2 S_d H(\xi). \tag{D.22}
\]

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For our applications we consider for $H(\xi)$ functions of the form

$$H(\xi) = \frac{1}{[\xi(1 + \xi)]^\alpha}.$$  \hspace{1cm} (D.23)

From the definition of $\mathcal{H}(\xi)$ in terms of $H(\xi)$ in (D.13) and using (D.8) we find

$$\tilde{\mathcal{H}}(k) = \pi^{\frac{d}{2}} \frac{\Gamma(\lambda + \frac{3}{2} - \alpha)}{\Gamma(1 + \alpha)} 4^{\alpha - \lambda - \frac{5}{2}} g_{\alpha - \frac{1}{2}, \frac{1}{2}(\lambda + 3)}(k).$$  \hspace{1cm} (D.24)

With this result and (D.10) it is sufficient to find the inverse transform of functions of the form

$$\tilde{\mathcal{I}}(k) = \frac{\tilde{g}_{a,b}(k)}{4\pi k^2 + \mu^2}, \quad \tilde{g}_{a,b}(k) = \frac{\Gamma(a - \frac{i}{2}k)\Gamma(a + \frac{i}{2}k)}{\Gamma(b - \frac{i}{2}k)\Gamma(b + \frac{i}{2}k)},$$  \hspace{1cm} (D.25)

where we are ultimately interested in taking $\mu = \frac{1}{4}(d + 1)$, $a = \alpha - \frac{1}{2}\lambda$ and $b = \frac{1}{2}(\lambda + 3)$. By contour integration it is straightforward to obtain

$$\hat{\mathcal{I}}(\sinh^2 \theta) = \frac{1}{2\pi} \int \frac{dk}{k} e^{-ik\theta} \tilde{\mathcal{I}}(k) = \frac{4\Gamma(2a)}{\Gamma(b - a)\Gamma(b + a)(\mu^2 - a^2)} \times \sum_{n=0}^{\infty} \frac{1}{n!} \frac{(2a)_n(1 + a - b)_n(a - \mu)_n(a + \mu)_n}{(1 + a - \mu)_n(1 + a + \mu)_n} e^{-4(a + n)|\theta|}$$  \hspace{1cm} (D.26)

$$+ \frac{\Gamma(a - \mu)\Gamma(a + \mu)}{\Gamma(b - \mu)\Gamma(b + \mu)} \frac{2}{\mu} e^{-4\mu|\theta|}.$$

For the final inverse transform, we observe that it is only necessary to find $I''(\xi)$ which is given by inverting (D.8)

$$I''(\xi) = \frac{1}{\pi^\lambda \Gamma(-\lambda - 2)} \int_0^\infty d\rho \rho^{-\lambda - 3} \hat{\mathcal{I}}(\rho + \xi).$$  \hspace{1cm} (D.27)

This simplifies the calculation significantly. From (B.17) and (B.18) it follows that

$$I''(\xi) = \frac{1}{\pi^\lambda 4^{2a - 1}} \frac{\Gamma(2a + \lambda + 2)}{\Gamma(b - a)\Gamma(b + a)(\mu^2 - a^2)} \frac{1}{[\xi(1 + \xi)]^{a + \frac{1}{2}\lambda + 1}} \sum_{N=0}^{\infty} \frac{h_N}{N! [4\xi(1 + \xi)]^N}$$

$$+ \frac{1}{\pi^\lambda 4^{2\mu - 1}} \frac{\Gamma(a - \mu)\Gamma(a + \mu)}{\Gamma(2\mu + \lambda + 2)} \frac{1}{\Gamma(2\mu + 1)} \times \frac{F(\mu + \frac{1}{2}\lambda + 1, \mu - \frac{1}{2}\lambda - \frac{1}{2}; 2\mu + 1; - \frac{1}{4\xi(1 + \xi)})}{\frac{1}{[\xi(1 + \xi)]^{a + \frac{1}{2}\lambda + 1}}},$$  \hspace{1cm} (D.28)

$$h_N = \frac{(a - \frac{1}{2}\lambda - \frac{1}{2})_N}{(1 + 2a)_N} \frac{(2a)_n(1 + a - b)_n(a - \mu)_n(a + \mu)_n(a + \frac{1}{2}\lambda + \frac{3}{2})_n(-N)_n}{(a + b)_n(1 + a - \mu)_n(1 + a + \mu)_n(a - \frac{1}{2}\lambda - \frac{1}{2})_n(1 + 2a + N)_n}.$$
Now if we use the fact that $b = \frac{1}{2}(\lambda + 3)$ then the series in $h_N$ simplifies to a terminating \(_5F_4(;.;.;.;1)\) series and may be summed exactly once again by Dougall’s theorem \cite{27} giving

$$h_N = \frac{(1)_N(a - \frac{1}{2}\lambda - \frac{1}{2})N(a + \frac{1}{2}\lambda + 1)_N}{(1 + a - \mu)_N(1 + a + \mu)_N}.$$ \hspace{1cm} (D.29)

With this result the series in (D.28) can be written exactly as a \(_3F_2\) hypergeometric function. Putting it all together we obtain for $H$ given by (D.23) with general $\alpha$

$$G''(\xi) = \frac{S_d}{32d}\frac{1}{(d - 2\alpha)(\frac{1}{2} + \alpha)} \frac{1}{[\xi(1 + \xi)]^{\alpha + \alpha}}$$

$$\times \left\{ \begin{aligned}
&\frac{3F_2(1, 1 + \alpha, \alpha - \frac{1}{2}d; \frac{3}{2} + \alpha, 1 + \alpha - \frac{1}{2}d; -\frac{1}{\xi(1 + \xi)} - \frac{1}{\xi(1 + \xi)} - \frac{1}{\xi(1 + \xi)})}{\Gamma(1 + \alpha - \frac{1}{2}d)\Gamma(1 + \alpha - \frac{1}{2}d)\Gamma(\frac{3}{2} + \alpha)\Gamma(1 + \alpha - \frac{1}{2}d)} \\
&\frac{\Gamma(\frac{1}{2}d - 1)\Gamma(3 - \frac{1}{2}d)\Gamma(\frac{1}{2}d + \frac{3}{2})\Gamma(d - \frac{1}{2})}{\Gamma(d - 1)\Gamma(\frac{1}{2}d + \frac{3}{2})} [4\xi(1 + \xi)]^{\frac{1}{2}d - 2} \right\}. \hspace{1cm} (D.30)
\right.$$  

Note that when $\alpha = \frac{1}{2}d$ the two terms cancel so that there is no pole. Also from (B.28) $G''(\xi)$ has a leading behaviour for $\xi \to 0$ with terms $\propto \xi^{1+\alpha}$. We have checked that (D.30) satisfies (D.17) with the last term representing a solution of the homogeneous equation.

In order to evaluate the integral in (5.26) we need to take, for the ordinary case corresponding to (5.29a),

$$H(\xi)_{ord} = \frac{N}{4dA\lambda_{ord}} \frac{d - 2}{S_d} \frac{1}{(d - 1)^2} \frac{\alpha}{[\xi(1 + \xi)]^{d - 2}}. \hspace{1cm} (D.31)$$

Applying (D.30) gives then

$$G''(\xi)_{ord} = -\frac{N}{S_d} \frac{d - 2}{8(d - 1)^2} \frac{1}{[\xi(1 + \xi)]^{d - 1}}$$

$$\times \left\{ \begin{aligned}
&\frac{3F_2(1, d - 1, \frac{1}{2}d - 2; d - \frac{1}{2}, \frac{1}{2}d - 1; -\frac{1}{\xi(1 + \xi)})}{\Gamma(\frac{1}{2}d - 1)\Gamma(3 - \frac{1}{2}d)\Gamma(\frac{1}{2}d + 1)\Gamma(d - \frac{1}{2})} \\
&\frac{\Gamma(\frac{1}{2}d - 1)\Gamma(3 - \frac{1}{2}d)\Gamma(\frac{1}{2}d + \frac{3}{2})\Gamma(d - \frac{1}{2})}{\Gamma(d - 1)\Gamma(\frac{1}{2}d + \frac{3}{2})} [4\xi(1 + \xi)]^{\frac{1}{2}d - 2} \right\}. \hspace{1cm} (D.32)
\right.$$  

For $C, A$ defined as in (D.22) we have

$$C(\xi)_{ord} = \frac{N}{S_d} \frac{d - 2}{[\xi(1 + \xi)]^{d - 1}} \frac{2\xi + 1}{(d - 1)^2}$$

$$\times \left\{ \begin{aligned}
&\frac{3F_2(1, d - 1, \frac{1}{2}d - 2; d - \frac{3}{2}, \frac{1}{2}d - 1; -\frac{1}{\xi(1 + \xi)})}{\Gamma(\frac{1}{2}d - 1)\Gamma(3 - \frac{1}{2}d)\Gamma(\frac{1}{2}d + \frac{3}{2})\Gamma(d - \frac{1}{2})} \\
&\frac{\Gamma(\frac{1}{2}d - 1)\Gamma(3 - \frac{1}{2}d)\Gamma(\frac{1}{2}d + \frac{3}{2})\Gamma(d - \frac{1}{2})}{\Gamma(d - 1)\Gamma(\frac{1}{2}d + \frac{3}{2})} [4\xi(1 + \xi)]^{\frac{1}{2}d - 2} \right\}. \hspace{1cm} (D.33)$$

48
and

\[ A(\xi)_{\text{ord}} = -\frac{N}{S_d^2} \frac{8}{d^2} (d-2)^2 \frac{1}{[\xi(1+\xi)]^{d-2}} \]
\[ -\frac{N}{S_d^2} \frac{2}{d-1} \frac{1}{[\xi(1+\xi)]^{d-1}} \]
\[ \times 3F_2\left(1, d-1, \frac{1}{2}d-2; d-\frac{3}{2}, \frac{1}{2}d; -\frac{1}{4\xi(1+\xi)}\right) \]
\[ -\frac{\Gamma\left(\frac{1}{2}d\right)\Gamma\left(3-\frac{1}{2}d\right)\Gamma\left(d-\frac{3}{2}\right)}{\Gamma(d-1)\Gamma\left(\frac{1}{2}d-\frac{1}{2}\right)} \left[4\xi(1+\xi)\right]^{\frac{1}{2}d-1} \left(1 + \frac{d}{d-1} \frac{1}{4\xi(1+\xi)}\right) \right\}. \]

**Appendix E**

The crucial step in deriving (7.20) is to obtain, with \( \hat{D}_{\sigma\rho}(y, \partial) \) defined as in (7.16), the result for arbitrary \( H(\xi) \)

\[ \hat{D}_{\sigma\rho}(y', \partial') \left( \frac{1}{(4yy')^d} X_\mu X_\nu H(\xi) \right) \]
\[ = \frac{1}{(d^2-1)} \frac{1}{(4yy')^d} \left\{ T_{\mu\nu,\sigma\rho}(s) \frac{1}{2\xi(\xi+1)} H(\xi) \right. \]
\[ + \frac{1}{d} \left( X_\mu X_\nu I_{\rho\sigma}(s) + \mu \leftrightarrow \nu, \sigma \leftrightarrow \rho - \frac{4}{d} \delta_{\sigma\rho} X_\mu X_\nu \right) \left( \frac{1}{\xi+1} H(\xi) - H'(\xi) \right) \]
\[ + X_\mu X_\nu \left( X'_{\rho} X'_{\sigma} - \frac{1}{d} \delta_{\sigma\rho} \right) \left( \xi(\xi+1) H''(\xi) + 2H'(\xi) - \frac{2}{\xi+1} H(\xi) \right) \]
\[ - X_\mu X_\nu \left( n_\sigma n_\rho - \frac{1}{d} \delta_{\sigma\rho} \right) \left( \xi(\xi+1) H''(\xi) + d(\xi + \frac{1}{2}) H'(\xi) \right. \]
\[ \left. - dH(\xi) - \frac{1}{2} \frac{1}{\xi(\xi+1)} H(\xi) \right) \]
\[ - \delta_{\mu\nu} \left( n_\sigma n_\rho - \frac{1}{d} \delta_{\sigma\rho} \right) \frac{1}{2\xi(\xi+1)} H(\xi) \right\}. \]  

(E.1)

The terms dependent on \( n_\sigma n_\rho \) violate conformal invariance but they disappear on removing the trace in \( \mu\nu \) and \( H(\xi) \rightarrow [\xi(\xi+1)]^{-\frac{1}{2}d} \) as appropriate for application in (7.19).
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