Construction of Permutation Snarks

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Abstract

A permutation snark is a snark which has a 2-factor \( F_2 \) consisting of two chordless circuits; \( F_2 \) is called the permutation 2-factor of \( G \). We construct an infinite family \( \mathcal{H} \) of cyclically 5-edge connected permutation snarks. Moreover, we prove for every member \( G \in \mathcal{H} \) that the permutation 2-factor given by the construction of \( G \) is not contained in any circuit double cover of \( G \).

Keywords: circuit double cover, cycle permutation graph, snark

1 Introduction and main result

A circuit is defined to be a 2-regular 2-connected graph. A circuit double cover (CDC) of a cubic graph \( G \) is a set \( S \) of circuits of \( G \) such that every edge of \( G \) is covered by exactly two circuits of \( S \). A 2-regular subgraph \( D \) of \( G \) is said to be contained in \( S \) if every circuit of \( D \) is an element of \( S \).

A cubic graph \( G \) with a 2-factor \( F_2 \) which consists of two chordless circuits is called a cycle permutation graph and \( F_2 \) is called the permutation 2-factor of \( G \). If \( G \) is also a snark, then we say \( G \) is a permutation snark. The Petersen graph has been for a long time the only known cyclically 5-edge connected permutation snark. In [2] twelve new cyclically 5-edge connected permutation snarks have been discovered by computer search. Here, we present the first infinite family of cyclically 5-edge connected permutation snarks.

We state the main theorem, see Theorem 2.16 and Corollary 2.25.

Theorem 1.1 For every \( n \in \mathbb{N} \), there is a cyclically 5-edge connected permutation snarks \( G \) of order \( 10 + 24n \). Moreover, \( G \) has a permutation 2-factor which is not contained in any CDC of \( G \).

Applying the above theorem we obtain infinitely many counterexamples to the following conjectures.

*supported by the FWF project P20543.
Conjecture 1.2 [6] Let $G$ be a cyclically 5-edge-connected cycle permutation graph. If $G$ is a snark, then $G$ must be the Petersen graph.

Conjecture 1.3 [3] If $G$ is an essentially 6-edge-connected 4-regular graph with a transition system $T$, then $(G,T)$ has no compatible cycle decomposition if and only if $(G,T)$ is the bad loop or the bad $K_5$.

Conjecture 1.4 [5, 6] Let $G$ be a cyclically 5-edge-connected cubic graph and $D$ be a set of pairwise disjoint circuits of $G$. Then $D$ is a subset of a CDC, unless $G$ is the Petersen graph.

Conjecture 1.5 [6] Let $G$ be a cycle permutation graph with the cordless circuits $C_1$ and $C_2$ where $C_1 \cup C_2$ is a 2-factor. If $G$ is cyclically 5-edge-connected and there is no CDC which includes both $C_1$ and $C_2$, then $G$ must be the Petersen graph.

Note that finitely many counterexamples to the above conjectures were found in [2] by computer search.

2 Definitions and proofs

We refer to [6] for the definition of a multitpole and an half-edge. Moreover, for terminology not defined here we refer to [1]. We use a more general definition of a CDC than the one stated in the introduction.

Definition 2.1 We say a set $S = \{A_1, A_2, ..., A_m\}$ is a path circuit double cover (PCDC) of a graph $G$ if the following is true

1. $A_i$ is a subgraph of $G$ where every component of $A_i$ is either a circuit or a path with both endvertices being vertices of degree 1 in $G$, $\forall i \in \{1, 2, ..., m\}$.
2. $\sum_{i=1}^{m} |e \cap E(A_i)| = 2 \ \forall e \in E(G)$.

If no $A_i$ contains a path as a component, then we call $S$ a CDC of $G$ and if $|S| = k$, then we call $S$ a $k$-CDC of $G$. Obviously, a PCDC is a CDC if $G$ contains no vertex of degree 1. For a survey on CDC’s, see [6, 7].

Later we need the following known lemma [6].

Lemma 2.2 Let $G$ be a 3-edge colorable cubic graph and $D$ be a 2-regular subgraph of $G$. Then $G$ has a 4-CDC $S$ with $D \in S$. 
Definition 2.3 Let $A \in S$ be given where $S$ is a PCDC of a graph $G$. Let $e$ be an half-edge or edge of $A$, then $[e]$ denotes the unique element of $S$ which contains $e$ and which is not $A$. We say $[e]$ refers to $A$.

Definition 2.4 Let $Q^i$ with $i \in \{1, 2, ..., 4\}$ be a cyclically 5-edge connected permutation snark with a permutation 2-factor $F^i$ such that $F^i$ is not contained in any CDC of $Q^i$. The two circuits of $F^i$ are denoted by $C_1^i$ and $C_2^i$. We may assume w.l.o.g. that $C_1^i$ (or $C_2^i$) contains a subpath which has the following vertices in the following consecutive order: $x_1^i, x_2^i, z_2^i, x_6^i (x_4^i, z_1^i, x_5^i)$, such that $z_1^i z_2^i \in E(Q^i)$.

Definition 2.5 Let $\tilde{Q}^i$ be the graph which is obtained from $Q^i$ by removing the edge $x_1^i x_2^i$, the vertices $z_1^i, z_2^i$ and by adding the vertex $y_j^i, j = 1, 2, ..., 6$ and the edges $e_3^i := x_2^i y_3^i, e_s^i := x_s^i y_s^i, s = 1, 2, 4, 5, 6$, see Figure 1.

By an end-edge of a graph $G$, we mean an edge which is incident with a vertex of $G$ with degree 1 in $G$.

Definition 2.6 The six end-edges of $\tilde{Q}^i$ together with the remaining edges of $F^i$ in $Q^i$ induce the following three paths in $\tilde{Q}^i$: the path $A_1^i$ with end-edges $e_1^i$ and $e_6^i$, the path $A_2^i$ with end-edges $e_4^i$ and $e_5^i$ and the path $A_3^i$ with end-edges $e_2^i$ and $e_3^i$. Moreover, set $A^i := A_1^i \cup A_2^i \cup A_3^i$ and $A^i := \{A_1^i, A_2^i, A_3^i\}$.

![Figure 1: The graph $\tilde{Q}^i$, $i \in \{1, 2, 3, 4\}$.

We recall that $[\cdot]$ refers to the given element of a PCDC or CDC. We need the following propositions and lemmas for proving Theorem 2.16.
Proposition 2.7 Let $\tilde{Q}^i$ and $A^i$ with $i \in \{1, 2, ..., 4\}$ be defined as above. Then every PCDC $S$ of $\tilde{Q}^i$ with $A^i \in S$ satisfies the following.

1. If $[e^1_i] \neq [e^5_i]$, then $[e^1_i] \notin \{[e^2_i], [e^3_i]\}$. If $[e^1_i] \in \{[e^2_i], [e^3_i]\}$, then $[e^1_i] = [e^5_i]$.
2. $[e^2_i] \neq [e^5_i]$.

Proof. Suppose by contradiction that one of the two conclusions of (1) is not fulfilled. Then $S$ implies a CDC of $Q^i$ containing $P^i$ which contradicts the definition of $Q^i$. If (2) is not fulfilled, then the edge $e \notin A^i$ which is incident with $x^i_2$ cannot be covered by $S$ which is impossible. Hence, the proof is finished.

Corollary 2.8 Let $S$ with $A^i \in S$ be a PCDC of $\tilde{Q}^i$ with $i \in \{1, 2, ..., 4\}$. If $[e^1_i] \neq [e^5_i]$, then

1. $|\{[e^1_i], [e^2_i], [e^3_i]\}| = |\{[e^4_i], [e^5_i], [e^6_i]\}| = 3$.
2. $|\{[e^1_i], [e^2_i], ..., [e^6_i]\}| = 3$.

Proof. Since by Proposition 2.7, $[e^2_i] \neq [e^5_i]$ and $[e^1_i] \notin \{[e^2_i], [e^3_i]\}$, the corollary follows.

Definition 2.9 Denote by $P^i$ with $i \in \{1, 2, 3, 4\}$ the connected multipole which is obtained from $\tilde{Q}^i$ by transforming every end-edge of $\tilde{Q}^i$ into an half-edge except for $i = 1$, $e^1_2$; for $i = 2$, $e^3_2$; for $i = 3$, $e^3_2$; and for $i = 4$, $e^3_4$.

Definition 2.10 Denote by $H(Q^1, Q^2, Q^3, Q^4)$ or in short by $H$ the cubic graph which is constructed from $P^1, P^2, P^3$ and $P^4$, as illustrated in Figure 2, by gluing together half-edges, identifying vertices of degree 1 and by adding the edge $\alpha$.

Note that we keep in $H$ the edge labels of $P^i$, respectively, of $Q^i$, see Figure 2.

Definition 2.11 Let $A \in A^i$ (Def. 2.6), i.e. $A \subseteq \tilde{Q}^i$, $i \in \{1, 2, 3, 4\}$. Then $A \subseteq H$ is defined to be the path in $H$ containing all edges of $H$ which have the same edge-labels as $A \subseteq \tilde{Q}^i$; if an edge $e$ of $A \subseteq \tilde{Q}^i$ corresponds to a half-edge of $H$ then the edge of $H$ which contains $e$ is defined to be part of $A \subseteq H$.

Definition 2.12 Denote by $F$ the permutation 2-factor of $H$ with $F := A_1 \cup A_2 \cup A_3 \cup A_4$ and $A_i \subseteq H$, $i = 1, 2, 3, 4$, see Figure 1, 2 and 3.
Figure 2: The graph $H$.

Figure 3: The permutation 2-factor $F$ of $H$ and $\alpha \notin E(F)$.
Lemma 2.13 Let $S$ be a CDC of $H$ with $F \in S$. Then $[e_i^4] \neq [e_j^5]$ for some $i \in \{1, 2, 3, 4\}$.

Proof by contradiction. Consider $P^3$ in Figure 2. Since $[e_i^3] = [e_j^3]$ we obtain $[e_i^3] = [e_i^4]$. Therefore and since $[e_i^4] = [e_j^3]$ it follows that $[e_i^3] = [e_j^3]$. Consider $P^1$. By analogous arguments, $[e_i^2] = [e_j^2]$. Since $[e_i^2] = [e_j^2]$ we obtain $[e_i^2] = [e_j^2]$ which is impossible.

Lemma 2.14 Let $S$ be a CDC of $H$ with $F \in S$ and let $[e_i^4] \neq [e_j^5]$ for some $i \in \{1, 2, 3, 4\}$. Then $[e_i^4] \neq [e_j^5]$ for all $i \in \{1, 2, 3, 4\}$.

Proof. Let $[e_i^4] \neq [e_j^5]$. Then by Proposition 2.7 (1), $[e_i^4] \neq [e_j^5]$. Hence $[e_i^4] \neq [e_j^5]$ and by Proposition 2.7 (1), $[e_i^4] \neq [e_j^5]$. Hence $[e_i^4] \neq [e_j^5]$ and thus by Proposition 2.7 (1), $[e_i^4] \neq [e_j^5]$ implying $[e_i^4] \neq [e_j^5]$. Each of the three remaining cases to consider, i.e. $[e_i^4] \neq [e_j^5]$, $i = 2, 3, 4$, can be proven analogously.

Corollary 2.8 Lemma 2.13 and Lemma 2.14 imply the following proposition.

Proposition 2.15 Let $S$ be a CDC of $H$ with $F \in S$. Then the following is true for $i = 1, 2, 3, 4$.

1. $[e_i^4] \neq [e_j^5]$.
2. $|\{[e_i^4], [e_j^5], [e_k^6]\}| = |\{[e_i^4], [e_j^5], [e_k^6]\}| = 3$.
3. $|\{[e_i^4], [e_j^5], ..., [e_k^6]\}| = 3$.

For the proof of the next theorem we form a new cubic graph $H'$ from $H$. Consider for this purpose $P^i$, $i = 1, 2, 3, 4$ in Figure 2 as a vertex of degree 6 and split every $P^i$ into two vertices $v^i$ and $w^i$ such that $v^i$ ($w^i$) is incident with $e_j^i$, $j = 1, 2, 3$ ($e_j^i$, $j = 4, 5, 6$) to obtain $H'$. For reasons of convenience we do not use the edge-labels of $H$ for $H'$, see Figure 4.

Theorem 2.16 Let $S$ be CDC of $H$, then $F \not\in S$.

Proof. Note that for every CDC $S$ of $H$ with $F \in S$, $\{[e_i^4], [e_j^5]\} = \{[e_k^6], [e_l^7]\}$.

(2) and (3) in Proposition 2.15 imply that it suffices to show that there is no proper edge-coloring $f : E(H') \rightarrow \mathbb{N}$ such that $v^i$ and $w^i$ in $H'$ are incident with the same colors, $\forall i \in \{0, 1, 2, 3, 4\}$. Let $v \in V(H')$, then $E_v$ denotes the edge-set containing all edges of $H'$ incident with $v$. We proceed by contradiction. There are two cases to consider.

Case 1. $f(a_1) = f(a_2) = 1$ and $f(a_3) = f(a_4) = 2$.
Since $f(a_1) = 1$ and $a_1 \in E_v$, there is $x \in E_v$ with $f(x) = 1$. Since
f(a_3) = 1, f(a_{11}) = 1. Since f(a_2) = 2 and since a_2 \in E_{v^3} \text{ there is } y \in E_{w^3} \text{ with } f(y) = 2 \text{ which is impossible since } f(a_{11}) = 1 \text{ and } f(a_4) = 2.

Case 2. f(a_1) = f(a_4) = 1 \text{ and } f(a_2) = f(a_3) = 2.

Since f(a_3) = 2 \text{ and } a_3 \in E_{v^2} \text{ there is } x \in E_{w^2} \text{ with } f(x) = 2. \text{ Since } f(a_2) = 2, f(a_{12}) = 2. \text{ Since } f(a_4) = 1 \text{ and } a_4 \in E_{v^4} \text{ there is } y \in E_{w^4} \text{ with } f(y) = 1 \text{ which is impossible since } f(a_1) = 1 \text{ and } f(a_{12}) = 2.

The cyclic edge-connectivity of a graph G which contains two vertex-disjoint circuits is denoted by \lambda_c(G); it is the minimum number of edges one needs to delete from G in order to obtain two components such that each of them contains a circuit. In order to show that \lambda_c(H) > 4, we need several results.

Definition 2.17 Let G be a graph with a given 2-factor F_2 consisting of two chordless circuits. We call e \in E(G) a spoke if e \notin E(F_2).

Proposition 2.18 Let G be a cubic graph with a 2-factor F_2 consisting of two chordless circuits C_1, C_2.

(1) Let \mid V(G) \mid \geq 8, \text{ then } \lambda_c(G) \geq 4.

(2) Let \mid V(G) \mid \geq 10. \text{ Then every cyclic 4-edge cut } E_0 \text{ of } G \text{ is a matching and } |E_0 \cap C_i| = 2, i = 1, 2.

Proof. Suppose by contradiction that M is a cyclic 3-edge cut of G. Obviously, M is matching. First we show that M contains no spoke and consider...
two cases.

**Case 1.** $M$ contains two or three spokes.

It is straightforward to see that $G - M$ is connected and thus this is impossible.

**Case 2.** $M$ contains exactly one spoke.

If $|M \cap E(C_i)| = 1$ for $i = 1, 2$ then $C_1 - M \subseteq G$ is a path which is connected by more than one spoke to $C_2 - M \subseteq G$. Thus $G - M$ is connected. Hence w.l.o.g. $|M \cap E(C_1)| = 2$. Since both paths of $C_1 - M$ contain more than one vertex, both paths in $G$ are connected by more than one spoke to $C_2$ and thus $G - M$ is connected. Hence $M$ contains no spoke.

Suppose $|M \cap E(C_1)| = 2$ and thus $|M \cap E(C_2)| = 1$. Then both paths of $C_1 - M \subseteq G$ are connected by more than one spoke to the path $C_2 - M \subseteq G$. Hence $G - M$ is connected which contradicts the assumption and thus finishes the first part of the proof.

Suppose $E_0$ contains a spoke $s$. For every subdivision $Z'$ of a cubic graph $Z$, $\lambda_c(Z') = \lambda_c(Z)$. Thus and by the first statement of the Proposition, $G - s$ is cyclically 4-edge connected. Hence $s \not\in E_0$.

Suppose $E_0$ is not a matching and let $a_1$ be adjacent with $a_2$ where $\{a_1, a_2\} \subseteq E_0 \cap E(C_1)$. Let $s$ be the unique spoke which is adjacent with $a_1$ and $a_2$. Then $E'_0 := E_0 - a_1 \cup s$ is a cyclic 4-edge cut of $G$. This contradicts the previous observation that a spoke is not contained in any cyclic 4-edge cut. Hence $E_0$ is a matching. Suppose $E_0$ contains no (one) edge of $C_1$ and thus four (three) edges of $C_2$. $C_2 - E_0$ consists of four (three) paths where each of them is connected by a spoke to $C_1 - E_0$ which is connected in both case. Hence $G - E_0$ is connected which contradicts the assumption and thus finishes the proof.

Let $V'$ be a subset of vertices of a graph $G$, then we denote by $\langle V' \rangle$ the vertex induced subgraph of $G$.

**Lemma 2.19** Let $G$ with $|V(G)| \geq 10$ be a cubic graph which contains a 2-factor $F_2$ consisting of two chordless circuits $C_1, C_2$. Then the following is true and the analogous holds for $C_2$.

(1) $\lambda_c(G) = 4$ if and only if $C_1$ contains a path $L_1$ with $1 < |V(L_1)| < |V(C_1)| - 1$ such that $L_2 := \langle N(V(L_1)) \cap V(C_2) \rangle$ is a path of $C_2$. In particular, the four distinct end-edges of $F_2 - E(L_1) - E(L_2)$ form a cyclic 4-edge cut of $G$.

(2) Let $E_0 := \{a_1, a_2, b_1, b_2\}$ be a cyclic 4-edge cut of $G$ where by Prop. 2.18.

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w.l.o.g. \( \{a_1, a_2\} \subseteq E(C_1) \) and \( \{b_1, b_2\} \subseteq E(C_2) \). Denote the two paths of \( C_1 - a_1 - a_2 \) \( (C_2 - b_1 - b_2) \) by \( L'_1 \) and \( L''_1 \) \( (L'_2 \) and \( L''_2 \)). Then, \( \{N(V(L'_1)) \cap V(C_2)\}, \{N(V(L''_1)) \cap V(C_2)\} \) = \( \{L'_2, L''_2\} \).

Proof. We first prove \( (2) \). The four paths \( L'_1, L''_1, L'_2, L''_2 \) decompose the graph \( C_1 \cup C_2 - E_0 \). Every neighbor of a vertex of \( L'_1 \) or \( L''_1 \) in \( C_2 \) is thus contained in \( L'_2 \) or \( L''_2 \). Hence the equality in \( (2) \) does not hold if and only if \( L'_1 \) or \( L''_1 \) contains two distinct vertices \( x, y \) such that \( N(x) \cap V(C_2) \subseteq V(L'_2) \) and \( N(y) \cap V(C_2) \subseteq V(L''_2) \). Let w.l.o.g. \( \{x, y\} \subseteq V(L'_1) \) and suppose by contradiction that \( x \) and \( y \) have the before described properties. Then \( L'_1 \subseteq G - E_0 \) is connected to \( L'_2 \) and \( L''_2 \). Since \( L''_1 \subseteq G - E_0 \) is connected to \( L'_2 \) or \( L''_2 \), \( G - E_0 \) is connected which contradicts the assumption and thus finishes this part of the proof.

We prove \( (1) \). Let \( \lambda_c(G) = 4 \) and let \( E_0 \) be a cyclic 4-edge cut of \( G \) as defined in \( (2) \). By Proposition 2.18 \( \{a_1, a_2\} \) is a matching of \( G \). Hence the inequality in \( (1) \) is satisfied by setting \( L_1 := L'_1 \). It is straightforward to check that the four end-edges of \( F_2 - E(L_1) - E(L_2) \) form a cyclic 4-edge cut of \( G \). Hence the proof is finished.

**Lemma 2.20** Let \( A \subseteq H \) and \( A \in A^i, i \in \{1, 2, 3, 4\} \). Then there is no cyclic 4-edge cut \( E_0 \) of \( H \) such that \( |E_0 \cap E(A)| = 2 \).

Proof by contradiction. Set \( E_0 \cap E(A) = \{a_1, a_2\} \). By Proposition 2.18 \( E_0 \) is a matching. Hence \( A \neq A^i \). There are two cases to consider.

**Case 1.** \( A = A^i_1 \). Set \( B := A^i_2 \). \( A \) and \( B \) belong to different components of \( F \subseteq H \), see Figure 1 and Figure 3. Denote by \( A^* \) the unique path of \( A - a_1 - a_2 \) which connects one endvertex of \( a_1 \) with one endvertex of \( a_2 \). Since \( E_0 \) is a cyclic 4-edge cut and by Lemma 2.19 (2) and by the structure of \( P^i, \) \( B^* := (N(V(A^*)) \cap V(B)) \) is a subpath of \( B \), see Figure 1. Since \( \{a_1, a_2\} \) is a matching of \( H \), \( |V(A^*)| \geq 1 \). Since \( |V(A^*)| = |V(B^*)| \), \( |V(B^*)| \geq 1 \).

Denote by \( w \) the neighbor of \( x^i_2 \) in \( B \). Consider the graph \( Q^i \) which was defined for constructing \( P^i \), see Figure 1. Then \( B^* \) is a path of \( C^i_2 \subseteq Q^i \) (Def. 2.14) with \( 1 < |V(B^*)| < |V(C^i_2)| - 1 \) since \( \{x^i_1, w\} \cap V(B^*) = \emptyset \). Moreover, \( A^* \subseteq Q^i \) is a path of \( C^i_1 \) and every vertex of \( A^* \) is adjacent to a vertex of \( B^* \subseteq Q^i \) and vice versa. Therefore and by Lemma 2.19 (1), \( \lambda_c(Q^i) = 4 \) which is a contradiction to Def. 2.4

**Case 2.** \( A = A^i_2 \). Set \( B := A^i_1 \). Let \( A^* \) be the unique path of \( A - a_1 - a_2 \) which connects one endvertex of \( a_1 \) with one endvertex of \( a_2 \). Let \( C \) denote the component of \( F \subseteq H \) such that \( A^* \not\subseteq C \). Since \( E_0 \) is a cyclic 4-edge cut

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and by Lemma 2.19 (2), $B^* := \langle N(V(A^*)) \cap V(C) \rangle$ is a subpath of $C$. Denote by $w$ the neighbor of $x^i_2$ in $A$. Suppose $w \in V(A^*)$. Then $x^i_2 \in V(B^*)$. Since $|V(B^*)| > 1$, $B^*$ contains a vertex $y \neq x^i_2$. Since $N(V(A^*) - w) \cap V(C) \subseteq V(B)$ and since $x^i_2$ is not adjacent to a vertex of $B$, $B^*$ is not a path which is a contradiction. Hence, $w \notin V(A^*)$. Similar to Case 1, $A^* \subseteq Q^i$ is a path of $C^i_2$ with $\{z^i_1, w\} \cap V(A^*) = \emptyset$ where every vertex of $A^*$ is adjacent to a vertex of the path $B^* \subseteq Q^i$ and vice versa. Hence, by applying the same arguments as in Case 1, the proof is finished.

We need the following observations and notations for the proof of Theorem 2.23.

**Definition 2.21** Let $X$ be a path with $|V(X)| > 2$. Then $oX$ is the subpath of $X$ which contains all vertices of $X$ except the two endvertices of $X$.

Let $R$ be a set of subgraphs of $H$; then $E(R)$ denotes the union of the edge-sets of the subgraphs of $R$. Note that the vertices of the graph $J$ defined below are not the vertices of $H'$.

![Figure 5: The graph $J$.](image)

**Definition 2.22** Let $C_1, C_2$ denote the two circuits of $F \subseteq H$ where $e^i_j \in E(C_1)$, see Figure 3. Set $R := \{oA^i_j \mid i \in \{1, 2, 3, 4\} \text{ and } j \in \{1, 2, 3\}\}$. Let
Let \( J := H/R \) be the graph which is obtained from \( H \) by contracting every element of \( R \) in \( H \) to a distinct vertex and by then replacing every multiple-edge by a single edge, see Figure 5. The edges of \( E(F) - E(R) \) induce a 2-factor of \( J \) which consists of two circuits \( D_1, D_2 \), see Figure 5.

Note that only subpaths of \( F \subseteq H \) are contracted in the transformation from \( H \) into \( J \) and that \( C_k \subseteq H \) is transformed into \( D_k \subseteq J \), \( k = 1, 2 \). We keep the labels of the edges respectively half-edges of \( E(F) - E(R) \subseteq E(H) \) for \( E(D_1) \cup E(D_2) \subseteq E(J) \), see Figure 5. Every \( oA_j^1 \subseteq H \) corresponds to a vertex of \( J \) and \( v_4w_6 \in E(J) \) corresponds to \( \alpha \in E(H) \). Set \( \mathbb{H} := \{ V(oA_i^j) \mid i = 1, 2, 3, 4 \text{ and } j = 1, 2, 3 \} \cup \{ v_4 \} \cup \{ w_6 \} \). Then \( \mathbb{H} \) is a vertex partition of \( V(H) \).

Every \( v \in V(J) \) corresponds to an element \( \hat{v} \) of \( \mathbb{H} \) and vice versa.

Let \( h : V(H) \rightarrow V(J) \) be the mapping where \( h(x) \) is defined to be the unique \( v \in V(J) \) such that \( x \in \hat{v} \).

Let \( e \in E(C_k) - E(R) \), then \( h'(e) \) denotes the corresponding edge of \( D_k \subseteq J \), \( k = 1, 2 \).

Let \( X \) be a subpath of \( C_k \subseteq H \), then \( X/R \) denotes the subpath of \( D_k \subseteq J \) with \( V(X/R) := \{ h(v) \mid v \in V(X) \} \) and \( E(X/R) := \{ h'(x) \mid x \in E(X) \cap (E(F) - E(R)) \} \).

**Theorem 2.23** \( H \) is a cyclically 5-edge connected permutation snark.

Proof. By Theorem 2.16 \( F \) is not contained in a CDC. Hence Lemma 2.20 implies that \( H \) is not 3-edge colorable. It remains to show that \( \lambda_e(H) > 4 \).

Suppose by contradiction that \( E_0 := \{ a_1, a_2, a_3, a_4 \} \) is a cyclic 4-edge cut of \( H \) and let by Proposition 2.18 (2), w.l.o.g. \( \{ a_1, a_2 \} \) be a matching of \( E(C_1) \).

Let \( X, X' \) denote the two components of \( C_1 - a_1 - a_2 \). \( X/R \) and \( X'/R' \) are edge disjoint paths of \( D_1 \subseteq J \) with \( |E(X/R)| + |E(X'/R')| \leq 7 \). Let w.l.o.g. \( |E(X/R)| \leq 3 \). Then, \( 2 \leq |V(X/R)| \leq 4 \) since by Lemma 2.20 \( \{ a_1, a_2 \} \not\subseteq E(A_j^i) \) for \( i = 1, 2, 3, 4 \) and \( j = 1, 2, 3 \). Set \( Y := X/R \).

By Lemma 2.19 (2), \( X^N := \langle N(X) \cap V(C_2) \rangle \) is a path in \( C_2 \). Hence \( Y^* := X^N/R \) is a path of \( D_2 \subseteq J \). Since only subpaths of \( F \subseteq H \) are contracted in the construction of \( J \) and since every vertex of \( X \subseteq C_1 \) is adjacent to a vertex of \( X^N \subseteq C_2 \) and vice versa, every vertex of \( Y \) is adjacent to a vertex of \( Y^* \) and vice versa. We make the following observation.

If \( v \) is an endvertex of \( Y \), then \( |(N(v) \cap V(D_2)) \cap V(Y^*)| \geq 1 \).

If \( |V(Y)| > 2 \) and \( v \) is an inner vertex of \( Y \), then \( N(v) \cap V(D_2) \subseteq V(Y^*) \).
Set $U := N(V(oY)) \cap V(D_2)$. Denote by $v_s$ and $v_t$ the two distinct endvertices of $Y$. Then, $Y^* = (S \cup T \cup U)$ for some $S \subseteq N(v_s) \cap V(D_2)$ and for some $T \subseteq N(v_t) \cap V(D_2)$.

If $V(Y) = \{v_1, v_2\}$ then $V(Y^*) \in \{\{w_1, w_3\}, \{w_1, w_3\}, \{w_1, w_3\}\}$. We abbreviate this conclusion by only writing the indices: $12 \sim 15, 13, 135$. We recall that $2 \leq |V(Y)| \leq 4$. Analogously, we consider for $Y$ the following cases where $Y \neq \{\{v_6, v_7, v_1, v_2\}\}$.

$23 \sim 13, 15, 135; 34 \sim 16; 45 \sim 46; 56 \sim 42, 47, 247; 67 \sim 24, 47, 247; 71 \sim 14.$

$123 \sim 135; 234 \sim 136, 156, 356; 345 \sim 146; 456 \sim 2467, 246, 467; 567 \sim 247; 671 \sim 124, 147, 1247; 712 \sim 134, 145, 1345.$

$1234 \sim 1356; 2345 \sim 1346, 1456, 13456; 3456 \sim 1246, 1467, 12467; 4567 \sim 2467; 5671 \sim 1247; 7123 \sim 1345.$

In none of the above cases, $V(Y^*)$ can be the vertex set of a path in $D_2$ contradicting the assumption that $Y^*$ is a path.

Thus it remains to consider $Y := \{v_6, v_7, v_1, v_2\}$. Since $v_1 \in V(Y) \subseteq V(D_1)$ (see the half-edges incident with $v_1$ in Figure 5), the path $X \subseteq C_1$ contains all vertices of $oA_1^1$ (Figure 1). The vertex $x_1^1 \in V(H)$ (Figure 1) corresponds to $v_3 \in V(D_1)$ (Figure 5) with $v_3 \notin V(Y)$. Hence, $x_1^1 \in V(C_1)$ and $x_2^1 \notin V(X)$. Moreover, $x_2^1$ is neither adjacent to $x_4^1$ nor to $x_5^1$; otherwise $Q^1$ contains a circuit of length 4 which contradicts Def. 2.4. Hence, $\{x_4^1, x_5^1\} \subseteq V(X^N)$.

By the structure of $Q_1$ (Figure 1) and since $oA_1^1 \subseteq X$, $X^N$ contains every vertex of $oA_2^1 - w$ where $w$ denotes the neighbor of $x_2^1$ in $oA_2^1$. Thus, and since $X^N$ is a subpath of $C_2$, and since $\{x_4^1, x_5^1\} \subseteq V(X^N)$, $|V(X^N)| = |V(C_2)| - 1$. Since $|V(X)| = |V(X^N)|$ and since $\{a_1, a_2\}$ in the definition of $X$ is a matching, this is impossible which finishes the proof.

Denote by $P_{10}$ the Petersen graph.

**Definition 2.24** Set $\mathcal{H} := \bigcup_{n=0}^{\infty} \{H_n\}$ where $H_n := H(H_{n-1}, P_{10}, P_{10}, P_{10})$ and $H_0 := P_{10}$, see Definition 2.10.

Note that in the above definition we use the graph $H_n$, respectively, $P_{10}$ as $Q^1$ and thus suppose that two subpaths in the known permutation 2-factors of $H_n$ and $P_{10}$ are chosen as the paths specified in Definition 2.4.

**Corollary 2.25** $\mathcal{H}$ is an infinite set of cyclically 5-edge connected permutation snarks where $H_n \in \mathcal{H}$ has $10 + 24n$ vertices.
Hence and by Theorem 2.16 we obtain the following corollary.

**Corollary 2.26** For every \( n \in \mathbb{N} \), there is a counterexample of order \( 10 + 24n \) to Conjecture 1.2, Conjecture 1.4 and Conjecture 2.4.

**Corollary 2.27** For every \( n \in \mathbb{N} \), there is a counterexample \( G \) of order \( 5 + 12n \) to Conjecture 1.3.

**Proof.** Set \( H := H_n \), see Definition 2.24. Contract every spoke of \( H \) with respect to \( F \) to obtain a 4-regular graph \( G \) of order \( 5 + 12n \). Then \( G = C'_1 \cup C'_2 \) where \( C'_1 \) and \( C'_2 \) are two edge-disjoint hamiltonian circuits of \( G \) which correspond to \( C_1 \) and \( C_2 \) in \( H \). Hence, every edge-cut \( E_0 \) of \( G \) has even size. Suppose that \( E_0 \) is an essential 4-edge cut of \( G \). Since \( G \) is 4-regular, every component of \( G - E_0 \) has more than 2 vertices. It is straightforward to see that then \( E_0 \) corresponds to a cyclic 4-edge cut of \( H \) which contradicts \( \lambda_e(H) = 5 \). Thus and since \( E_0 \) is of even size, \( G \) is essentially 6-edge connected. By defining that every pair of two edges which are adjacent and part of \( C'_i \) for some \( i \in \{1, 2\} \) from a transition, we obtain a transition system \( T(G) \) of \( G \). Since every compatible cycle decomposition of \( T(G) \) would imply a CDC \( S \) of \( H \) which contains \( F \) and thus would contradict Theorem 2.16, the proof is finished.

**Acknowledgement.** A. Hoffmann-Ostenhof thanks R. Häggkvist for the invitation to the University of Umea where part of the work has been done.

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