\emph{c-NILPOTENT MULTIPLIER OF FINITE $p$-GROUPS}

FARANGIS JOHARI, MOHSEN PARVIZI, AND PEYMAN NIROOMAND

Abstract. The aim of this work is to find some exact sequences on the $c$-nilpotent multiplier of a group $G$. We also give an upper bound for the $c$-nilpotent multiplier of finite $p$-groups and give the explicit structure of groups whose take the upper bound. Finally, we will get the exact structure of the $c$-nilpotent multiplier and determine $c$-capable groups in the class of extra-special and generalized extra-special $p$-groups. It lets us to have a vast improvement over the last results on this topic.

1. INTRODUCTION AND MOTIVATION

Let $G$ be a group presented as the quotient $F/R$ of a free group $F$ by a normal subgroup $R$. The $c$-nilpotent multiplier of $G$ is defined as

$$\mathcal{M}^{(c)}(G) = \frac{\gamma_{c+1}(F) \cap R}{\gamma_{c+1}(R, F)}$$

is the Baer invariant of $G$ with respect to the variety of nilpotent groups of class at most $c$ in which $\gamma_{c+1}(F)$ is the $(c + 1)$-th term of the lower central series of $F$ and $\gamma_1(R, F) = R, \gamma_{c+1}(R, F) = [\gamma_c(R, F), F]$, inductively. The $c$-nilpotent multiplier of $G$ was born in the work of Baer [1] (see also [5, 6, 12, 23] for more details in this topic). It is readily verified that the $c$-nilpotent multiplier $\mathcal{M}^{(c)}(G)$ is abelian and independent of the choice of the free presentation of $G$ (see [22]). The 1-nilpotent multiplier of $G$ is denoted by $\mathcal{M}(G)$ and it is called the Schur multiplier of $G$. There are many stories involving this concept and it can be found for instance in [2, 9, 10, 19, 20, 21, 29, 30, 34]. The main reason to study the $c$-nilpotent multiplier comes from the isologism theory of P. Hall [14, 15]. It is an instrument for classification of groups into isologism classes. Some results about the Baer invariant can be found in [5, 6, 12, 23, 24, 26, 27, 28].

The aim of paper is to obtain some more inequalities on the $c$-nilpotent multiplier of a finite $p$-group $G$. For instance, among the results for non-abelian $p$-groups, we give a vast improvement over the result due to Moghaddam [26, 28]. The results also gives an extension of the results obtained recently in [32]. In the class of extra-special $p$-groups and generalized extra-special $p$-groups, we characterize the explicit structure of the $c$-nilpotent multiplier. In the other direction, it seems finding a suitable upper bound may be useful to know more about the $c$-nilpotent multiplier and $c$-capability of groups. It is shown in [27, 28] that for a $p$-group $G$ of order $p^n$,

$$|\mathcal{M}^{(c)}(G)||\gamma_{c+1}(G)| \leq |\mathcal{M}^{(c)}(\mathbb{Z}_{p^n})|,$$

\textbf{Date:} March 13, 2022.

\textbf{Key words and phrases.} $c$-Nilpotent multiplier, Capability, $p$-Groups.

\textbf{Mathematics Subject Classification 2010.} Primary 20C25; Secondary 20D15.

This research was supported by a grant from Ferdowsi University of Mashhad-Graduate Studys (No. 31659).
where $Z_p(n) = Z_p \oplus Z_p \oplus \cdots \oplus Z_p$.

Using this inequality and Corollary 2.10 we have $|\mathcal{M}^{(c)}(G)|$ is bounded by $p^{\chi_c+1(n-1)}$, where $c \geq 2$. We will show that the bound is attained exactly when $G$ is elementary abelian similar to the result of \cite[Corollary 2]{2} due to Berkovich. Although we will reduce the upper bound as much as possible in the case of non-abelian $p$-groups, and as a result we will generalize the work of Berkovich \cite{2} to the $c$-nilpotent multipliers of finite $p$-groups. Finally we are able to identify which of generalized extra-special $p$-groups are $c$-capable. It also generalizes the result about the capability and 2-capability of extra-special $p$-groups in \cite{4,32}.

2. Preliminaries

In this section, we state the concepts and results which will be used in the next sections.

Notation. We use techniques involving the concept of basic commutators. Here is the definition. Let $X$ be an arbitrary subset of a free group, and select an arbitrary total order for $X$. The basic commutators on $X$, their weight $\text{wt}$, and the order among them are defined as follows:

(i) The elements of $X$ are basic commutators of weight one, ordered according to the total order previously chosen.

(ii) Having defined the basic commutators of weight less than $n$, a basic commutator of weight $n$ is $c_k = [c_i, c_j]$, where:

(a) $c_i$ and $c_j$ are basic commutators and $\text{wt}(c_i) + \text{wt}(c_j) = n$, and

(b) $c_i > c_j$, and if $c_i = [c_s, c_t]$, then $c_j \geq c_t$.

(iii) The basic commutators of weight $n$ follow those of weight less than $n$. The basic commutators of weight $n$ are ordered among themselves in any total order, but the most common used total order is lexicographic order; that is, if $[b_1, a_1]$ and $[b_2, a_2]$ are basic commutators of weight $n$. Then $[b_1, a_1] < [b_2, a_2]$ if and only if $b_1 < b_2$ or $b_1 = b_2$ and $a_1 < a_2$.

The number of basic commutators is given in the following:

Theorem 2.1. (Witt Formula). The number of basic commutators of weight $n$ on $d$ generators is given by the following formula:

$$\chi_n(d) = \frac{1}{n} \sum_{m|n} \mu(m) d^{n/m},$$

where $\mu(m)$ is the Möbius function, which is defined to be

$$\mu(m) = \begin{cases} 1 & \text{if } m = 1, \\ 0 & \text{if } m = p_1^{\alpha_1} \cdots p_k^{\alpha_k} \exists \alpha_i > 1, \\ (-1)^s & \text{if } m = p_1 \cdots p_s, \end{cases}$$

where the $p_i$, $1 \leq i \leq k$, are the distinct primes dividing $m$.

Theorem 2.2. (\cite{13} M. Hall and \cite{17} Theorem 11.15(a)) Let $F$ be a free group on $\{x_1, x_2, \ldots, x_d\}$. Then for all $i$, $1 \leq i \leq n$,

$$\frac{\gamma_n(F)}{\gamma_{n+i}(F)}$$

is a free abelian group freely generated by the basic commutators of weights $n, n + 1, \ldots, n + i - 1$ on the letters $\{x_1, x_2, \ldots, x_d\}$. 
Recall that from [5] if \( N \) is a normal subgroup of \( G \), then we can form the nonabelian tensor product \( (N \otimes G) \otimes G \). Therefore \( N \otimes_c G \) is defined recursively by \( N \otimes_0 G = N \) and \( N \otimes_{c+1} G = (N \otimes_c G) \otimes G \). Moreover, the exterior product \( N \wedge_c G \) is defined recursively by \( N \wedge_0 G = N \) and \( N \wedge_{c+1} G = (N \wedge_c G) \wedge G \).

Here we give some results concerning \( M^{(c)}(G) \) which are used in the proof of the main results. This result are proved in [27] for any variety, but we need them only for the variety of nilpotent groups of class at most \( c \).

**Theorem 2.3.** [27] Lemma 2.2] Let \( G \) be a finite group, \( B \trianglelefteq G \) and \( A = G/B \). Then there exists a finite group \( L \) with a normal subgroup \( M \) such that

\[
(\text{i}) \quad \gamma_{c+1}(G) \cap B \cong L/M,
\]

\[
(\text{ii}) \quad M \cong M^{(c)}(G),
\]

\[
(\text{iii}) \quad M^{(c)}(A) \text{ is a homomorphic image of } L.
\]

**Corollary 2.4.** Let \( G \) be a finite group, \( B \trianglelefteq G \) and \( A = G/B \). Then

\[
\left| M^{(c)}(A) \right| \divides \frac{|M^{(c)}(G)|}{|\gamma_{c+1}(G) \cap B|}.
\]

**Proof.** Let \( 1 \to R \to F \to G \to 1 \) be a free presentation of \( G \) and suppose \( B = S/R \) so that \( A = G/B \cong F/S \). By Theorem 2.3, we have

\[
\left| M^{(c)}(A) \right| \left| \gamma_{c+1}(S,F) \right| = \left| M^{(c)}(G) \right| \left| \gamma_{c+1}(G) \cap B \right|.
\]

Since

\[
\gamma_{c+1}(B,G) \cong \left[ \frac{S}{R} , \frac{F}{R} \right] = \frac{\gamma_{c+1}(S,F)}{\gamma_{c+1}(R,F)} \cong \frac{\gamma_{c+1}(S,F)}{R \cap \gamma_{c+1}(S,F)},
\]

\[
\frac{\gamma_{c+1}(S,F)}{\gamma_{c+1}(R,F)} = \frac{\gamma_{c+1}(S,F)}{R \cap \gamma_{c+1}(S,F)} \frac{R \cap \gamma_{c+1}(S,F)}{\gamma_{c+1}(R,F)} = \frac{\gamma_{c+1}(B,G)}{\gamma_{c+1}(R,F)},
\]

It follows that

\[
\left| M^{(c)}(A) \right| \divides \frac{|M^{(c)}(G)|}{|\gamma_{c+1}(G) \cap B|}.
\]

\( \square \)

For a group \( G \), the quotient \( G/G' \) is denoted by \( G^{ab} \).

**Lemma 2.5.** Let \( G \) be a finite group and \( 1 \to R \to F \to G \to 1 \) be a free presentation of \( G \). Let \( B \) be a central subgroup of \( G \) with \( B = S/R \). Put \( A = G/B \cong F/S \). Then

\[
\frac{\gamma_{c+1}(S,F)}{\gamma_{c+1}(R,F)} \text{ is a homomorphic image of } B \otimes_c \left( \frac{G}{B} \right)^{ab}.
\]

**Proof.** Define

\[
\theta : B \times \left( \frac{G}{B} \right)^{ab} \times \ldots \times \left( \frac{G}{B} \right)^{ab} \twoheadrightarrow \frac{\gamma_{c+1}(S,F)}{\gamma_{c+1}(R,F)} \frac{\gamma_{c+1}(S,F)}{\gamma_{c+1}(R,F)}
\]

\[
(sR, f_1 F', \ldots , f_c F') \mapsto \left[ s, f_1, \ldots , f_c \right] \gamma_{c+1}(R,F) \gamma_{c+1}(S)
\]
where \( f_1, \ldots, f_c \in F, s \in S \). Put \( T = \gamma_{c+1}(R, F) \gamma_{c+1}(S) \). We claim that \( \theta \) is well-defined. Let \( f_i = f'_i x_i y_i \) where \( f'_i \in F, x_i \in F^*, y_i \in S \) and \( 1 \leq i \leq c \) and also \( s = s_1 r' \) with \( r' \in R, s_1 \in S \). It is easy to see that \([s, f_1, \ldots, f_c] \equiv [s_1, f'_1, \ldots, f'_c] \mod T\). This follows that \( \theta \) is well-defined. Therefore there exists a unique homomorphism

\[
\overline{\theta} : B \otimes \left( \frac{G}{B} \right)^{ab} \otimes \ldots \otimes \left( \frac{G}{B} \right)^{ab} \xrightarrow[c\text{-times}]{\gamma_{c+1}(S, F)} \frac{\gamma_{c+1}(S, F)}{\gamma_{c+1}(R, F) \gamma_{c+1}(S)},
\]

such that \( \text{Im} \overline{\theta} = \frac{\gamma_{c+1}(S, F)}{\gamma_{c+1}(R, F) \gamma_{c+1}(S)} \). \( \square \)

**Theorem 2.6** Let \( G \) be a finite group, \( B \subseteq Z(G) \) and \( A = G/B \). Then

\[
|\mathcal{M}^{(c)}(G)| \left| \gamma_{c+1}(G) \cap B \right| \text{ divides } |\mathcal{M}^{(c)}(A)| \left| \mathcal{M}^{(c)}(B) \right| \left| B \otimes c \left( \frac{G}{B} \right)^{ab} \right|.
\]

**Proof.** Let \( 1 \to R \to F \to G \to 1 \) be a free presentation of \( G \), and \( B \) be a central subgroup of \( G \). Put \( B = S/R \) so that \( A \cong F/S \). Clearly, \([F, S] \subseteq R\). By Theorem 2.3 and its proof,

\[
\left| \mathcal{M}^{(c)}(G) \right| \left| \gamma_{c+1}(G) \cap B \right| = \left| \mathcal{M}^{(c)}(A) \right| \left| \frac{\gamma_{c+1}(S, F)}{\gamma_{c+1}(R, F) \gamma_{c+1}(S)} \right|.
\]

Since

\[
\frac{\gamma_{c+1}(S, F)}{\gamma_{c+1}(R, F) \gamma_{c+1}(S)} \cong \frac{\gamma_{c+1}(S, F)}{\gamma_{c+1}(R, F) \gamma_{c+1}(S)} \cong \frac{\gamma_{c+1}(S)}{\gamma_{c+1}(S) \gamma_{c+1}(R, F)} \cong \frac{\gamma_{c+1}(S)}{\gamma_{c+1}(R, S)}
\]

we have

\[
\left| \mathcal{M}^{(c)}(G) \right| \left| \gamma_{c+1}(G) \cap B \right| = \left| \mathcal{M}^{(c)}(A) \right| \left| \frac{\gamma_{c+1}(S, F)}{\gamma_{c+1}(R, F) \gamma_{c+1}(S)} \right| \left| \frac{\gamma_{c+1}(R, F) \gamma_{c+1}(S)}{\gamma_{c+1}(R, F)} \right|.
\]

But

\[
\frac{\gamma_{c+1}(R, F) \gamma_{c+1}(S)}{\gamma_{c+1}(R, F)} \cong \frac{\gamma_{c+1}(S)}{\gamma_{c+1}(S) \gamma_{c+1}(R, F)} \cong \frac{\gamma_{c+1}(S)}{\gamma_{c+1}(R, S)}
\]

Now since \( \gamma_{c+1}(S) \subseteq \gamma_{c+1}(S, F) \subseteq [S, F] \subseteq R \), we have

\[
\gamma_{c+1}(S) = \gamma_{c+1}(S) \cap \gamma_{c+1}(R, S) = \mathcal{M}^{(c)}(B).
\]

Therefore

\[
\left| \mathcal{M}^{(c)}(G) \right| \left| \gamma_{c+1}(G) \cap B \right| = \left| \mathcal{M}^{(c)}(A) \right| \left| \frac{\gamma_{c+1}(S, F)}{\gamma_{c+1}(R, F) \gamma_{c+1}(S)} \right| \left| \frac{\gamma_{c+1}(S)}{\gamma_{c+1}(R, S)} \right|.
\]

Finally by Lemma 2.3

\[
\left| \mathcal{M}^{(c)}(G) \right| \left| \gamma_{c+1}(G) \cap B \right| \text{ divides } \left| \mathcal{M}^{(c)}(A) \right| \left| \frac{\gamma_{c+1}(S, F)}{\gamma_{c+1}(R, F)} \right| \left| \frac{\gamma_{c+1}(S)}{\gamma_{c+1}(R, S)} \right| \left| B \otimes c \left( \frac{G}{B} \right)^{ab} \right|.
\]
We just need the results of the next three propositions, so they come without proof.

**Proposition 2.7.** [5, Theorem 2.6] Let \( G \) be a group and \( B \) be a central subgroup of \( G \). Then the sequence
\[
B \cap_c G \rightarrow \mathcal{M}^c(G) \rightarrow \mathcal{M}^c(G/B) \rightarrow B \cap \gamma_{c+1}(G) \rightarrow 1
\]
is exact.

G. Ellis in [7] has generalized the formula of M. R. R. Moghaddam in [26], for the \( c \)-nilpotent multiplier of the direct product of two groups. In the next theorem \( \Gamma_{c+1}(A, B) \) for abelian groups \( A \) and \( B \) is defined as follows. Let \( A \) and \( B \) be \( d \)-generator and \( d' \)-generator abelian groups with generating sets \( \{a_1, \ldots, a_d\} \) and \( \{b_1, \ldots, b_{d'}\} \), respectively. For each basic commutator of weight \( c+1 \) on \( \{a_1, \ldots, a_d, b_1, \ldots, b_{d'}\} \) which is ordered as \( a_1 < a_2 < \cdots < a_d < b_1 < b_2 < \cdots < b_{d'} \) such as [\( x_{i_1}, \ldots, x_{i_{c+1}} \)] (with any bracketing), we correspond an iterating tensor product \( X_1 \otimes \cdots \otimes X_{c+1} \) in which \( X_{i_j} = A \) if \( x_{i_j} \in \{a_1, \ldots, a_d\} \) and \( X_{i_j} = B \) if \( x_{i_j} \in \{b_1, \ldots, b_{d'}\} \). Now \( \Gamma_{c+1}(A, B) \) is the direct sum of all such iterated tensor products involving at least one \( A \) and one \( B \). (See [7] for more information).

**Proposition 2.8.** [7, Theorem 5] Let \( G \) and \( H \) be finite groups. Then there is an isomorphism
\[
\mathcal{M}^c(G \times H) \cong \mathcal{M}^c(G) \oplus \mathcal{M}^c(H) \oplus \Gamma_{c+1}(G^{ab}, H^{ab}).
\]

Let \( \mathbb{Z}_n^{(m)} \) denote the direct product of \( m \) copies of \( \mathbb{Z}_n \).

**Proposition 2.9.** [21, Theorem 2.4] Let \( G \cong \mathbb{Z}_{n_1} \oplus \cdots \oplus \mathbb{Z}_{n_k} \), where \( n_{i+1} \mid n_i \), \( 1 \leq i \leq k - 1 \). Then
\[
\mathcal{M}^c(G) \cong \mathbb{Z}_{n_2}^{(\chi_{c+1}(2))} \oplus \mathbb{Z}_{n_3}^{(\chi_{c+1}(3) - \chi_{c+1}(2))} \oplus \cdots \oplus \mathbb{Z}_{n_k}^{(\chi_{c+1}(k) - \chi_{c+1}(k-1))},
\]
where \( \chi_r(s) \) is the number of all basic commutators of weight \( r \) on \( s \) letters.

**Corollary 2.10.** Let \( G \cong \mathbb{Z}_{p^{m_1}} \oplus \cdots \oplus \mathbb{Z}_{p^{m_k}} \), where \( m_1 \geq \ldots \geq m_k \). Then
\[
\mathcal{M}^c(G) \cong \bigoplus_{i=2}^k \mathbb{Z}_{p^{m_i}}^{(\chi_{c+1}(i) - \chi_{c+1}(i-1))}.
\]

3. \( c \)-nilpotent multipliers of finite \( p \)-groups

In this section, we intend to obtain an upper bound for the order of the \( c \)-nilpotent multiplier of a finite \( p \)-group. While we’re working on the \( p \)-groups, the extra-special \( p \)-groups can not be overlooked. At first, we compute the \( c \)-nilpotent multipliers of these groups. The following theorem of Beyl et al. to see the analogies between the Schur multiplier and the \( c \)-nilpotent multiplier of extra-special \( p \)-groups (see Theorem 3.3).

**Theorem 3.1.** (Beyl and Tappe 1982) [21, Theorem 3.3.6] Let \( G \) be an extra-special \( p \)-group of order \( p^{2n+1} \).

1. If \( n > 1 \), then \( G \) is unipotent and \( \mathcal{M}(G) \) is an elementary abelian \( p \)-group of order \( p^{2n^2-n-1} \).
(ii) Suppose that $G$ of order $p^3$ and $p$ is odd. Then $\mathcal{M}(G) \cong \mathbb{Z}_p \times \mathbb{Z}_p$ if $G$ is of exponent $p$ and $\mathcal{M}(G) = 0$ if $G$ is of exponent $p^2$.

(iii) The quaternion group of order 8, $Q_8$, has trivial multiplier, whereas the multiplier of the dihedral group of order 8, $D_8$, is of order 2.

The concept of epicenter $Z^*(G)$ is defined by Beyl et al. in [4]. It gives a criterion for detecting capable groups. In fact $G$ is capable if and only if $Z^*(G) = 1$. Ellis in [8] defined the exterior center $Z^e(G)$ of $G$ the set of all elements $g$ of $G$ for which $g \wedge h = 1_{G\wedge G}$ for all $h \in G$ and he showed $Z^*(G) = Z^e(G)$.

Using the next lemma, the $c$-nilpotent multiplier of non-capable extra-special $p$-groups can be easily computed.

Lemma 3.2. (See [4] Theorem 4.2) Let $N$ be a central subgroup of a group $G$. Then $N \subseteq Z^e(G)$ if and only if $\mathcal{M}(G) \to \mathcal{M}(G/N)$ is a monomorphism.

Theorem 3.3. Let $G$ be a non-capable extra-special $p$-group of order $p^n$ and $c \geq 2$. Then

$$\mathcal{M}^{(c)}(G) \cong \mathcal{M}^{(c)}(G/G').$$

Proof. Since $G$ is a non-capable extra-special $p$-group, $Z^*(G) = G'$. Now by virtue of Proposition 2.7, the sequence

$$G' \wedge_e G \xrightarrow{\tau} \mathcal{M}^{(c)}(G) \longrightarrow \mathcal{M}^{(c)}(G^{ab}) \longrightarrow G' \cap \gamma_{c+1}(G) \longrightarrow 1$$

is exact. The rest of proof is obtained by the fact that $G$ is nilpotent of class 2 and $\text{Im} \, \tau = 0$. Therefore

$$\mathcal{M}^{(c)}(G) \cong \mathcal{M}^{(c)}(G/G').$$

Corollary 3.4. Let $G$ be a non-capable extra-special $p$-group of order $p^{2n+1}$ and $c \geq 2$. Then

$$\mathcal{M}^{(c)}(G) \cong \mathbb{Z}_p^{\chi_{c+1}(2n)}.$$ 

Proof. By Corollary 2.10 and Theorem 3.3 we have $\mathcal{M}^{(c)}(G) \cong \mathbb{Z}_p^{\chi_{c+1}(2n)}$. □

There are only two capable extra-special $p$-groups, $D_8$ and $E_1$, in which $E_1$ is the extra-special $p$-group of order $p^3$ and exponent $p$ for odd $p$, in [4]. In the following we compute $\mathcal{M}^{(c)}(E_1)$ which has the following presentation.

$$E_1 = \langle x, y | x^p = y^p = [y, x]^p = [y, x, y] = [y, x, x] = 1 \rangle.$$

We have $\mathcal{M}^{(c)}(E_1) = \frac{R \cap \gamma_{c+1}(F)}{[R, x \ F]}$ in which $F$ is the free group on the set $\{x, y\}$ and $R = \langle x^p, y^p, [y, x]^p, [y, x, y], [y, x, x] \rangle^F$.

Since $E_1$ is nilpotent of class 2, we have $\gamma_{c+1}(F) \subseteq \gamma_3(F) \subseteq R$. Therefore for every $c \geq 2$,

$$\mathcal{M}^{(c)}(E_1) = \frac{\gamma_{c+1}(F)}{[R, x \ F]} \cong \frac{\gamma_{c+1}(F)}{[R, x \ F]}.$$

We know that $\frac{\gamma_{c+1}(F)}{[R, x \ F]}$ is the free abelian group with the basis of all basic commutators of weights $c + 1$ and $c + 2$ on $\{x, y\}$, by Theorem 2.2.
Theorem 3.5. With the above notations and assumptions we have
\[ [R_{c+1} F] \equiv (\gamma_{c+1}(F))^p \mod \gamma_{c+3}(F), \text{ for all } c \geq 2. \]

Proof. Using induction on c, the result is true for \( c = 2 \) (Theorem 3.6). Assume that the result holds for \( c \geq 2 \). By hypothesis, we have
\[ [R_{c+1} F] \equiv \left[ (\gamma_{c+1}(F))^p, F \right] \mod \gamma_{c+4}(F). \]

It follows that
\[ [R_{c+1} F] \equiv (\gamma_{c+2}(F))^p \mod \gamma_{c+4}(F). \]

\[ \square \]

Theorem 3.6. With that above notations and assumptions we have
\[ \mathcal{M}^c(E_1) \cong \mathbb{Z}_p^{\chi_{c+1}(2) + \chi_{c+2}(2)}, \text{ for } c \geq 2. \]

Proof. By Theorem 2.2, we know that \( \frac{\gamma_{c+1}(F)}{\gamma_{c+3}(F)} \) is the free abelian group with the basis of all basic commutators of weight \( c + 1 \) and \( c + 2 \) on \( \{x, y\} \). By Theorem 3.5 we have
\[ \frac{[R_{c+1} F]}{\gamma_{c+3}(F)} = \left( \frac{\gamma_{c+1}(F)}{\gamma_{c+3}(F)} \right)^p, \]
so the result holds.

\[ \square \]

Theorem 3.7. [11, Theorem C] Let \( G \) be a group and \( 1 \to R \to F \to G \to 1 \) be a free presentation of \( G \). Put \( \gamma_{c+1}(G) = \frac{\gamma_{c+1}(F)}{\gamma_{c+1}(R, F)} \), then

(i) \( 0 \to \mathcal{M}^c(G) \to \gamma_{c+1}(G) \to \gamma_{c+1}(G) \to 0 \) is exact.

(ii) For \( n \geq 2 \), we have
\[ \gamma_{c+1}(D_n) \cong \left\{ \begin{array}{ll} \mathbb{Z}_n & \text{n is odd} \\ \mathbb{Z}_n \times \mathbb{Z}_2^{\chi_{c+1}(2) - 1} & \text{n is even} \end{array} \right. \]

where \( D_n \) is dihedral group of order \( 2n \).

It is known that \( \mathcal{M}^c(D_8) \cong \mathbb{Z}_4 \times \mathbb{Z}_2^{\chi_{c+1}(2) - 1} \). Also we know that \( E_2 \), the extra-special \( p \)-group of order \( p^3 \) and exponent \( p^2 \) for odd \( p \), and \( Q_8 \) are not capable groups, so we can summarize the explicit structure of the \( c \)-nilpotent multipliers of all extra-special \( p \)-groups as follows, for \( c \geq 2 \).

Theorem 3.8. Let \( G \) be an extra-special \( p \)-group of order \( p^{2n+1} \) and \( c \geq 2 \).

(i) If \( n > 1 \), then \( \mathcal{M}^c(G) \) is an elementary abelian \( p \)-group of order \( p^{\chi_{c+1}(2n)} \).

(ii) Suppose that \( G \) of order \( p^3 \) and \( p \) is odd. Then \( \mathcal{M}^c(G) \cong \mathbb{Z}_p^{\chi_{c+1}(2) + \chi_{c+2}(2)} \) if \( G \) is of exponent \( p \) and \( \mathcal{M}^c(G) \cong \mathbb{Z}_p^{\chi_{c+1}(2)} \) if \( G \) is of exponent \( p^2 \).

(iii) The quaternion group of order 8, \( Q_8 \), has \( \mathbb{Z}_2^{\chi_{c+1}(2)} \) as \( c \)-nilpotent multiplier and the \( c \)-nilpotent multiplier of the dihedral group of order 8, \( D_8 \), is \( \mathbb{Z}_4 \times \mathbb{Z}_2^{\chi_{c+1}(2) - 1} \).

In the proof of some of next theorems, we need to work with the \( c \)-nilpotent multiplier of finite abelian \( p \)-groups. For abelian \( p \)-groups we have.
Lemma 3.13. the order of its derived subgroup. Then the following conditions are equivalent:
(1) \(|M^{(c)}(G)| = p^{|c+1(n)}| if and only if \(m_i = 1\) for all \(i\).
(2) \(|M^{(c)}(G)| \leq p^{|c+1(n-1)}| if and only if \(m_1 \geq 2\).

Proof. Using Corollary 2.4 we have \(|M^{(c)}(G)| = p^{\sum_{i=2}^{k}(c+1(i)-c+1(i-1))m_i}\). Define
\[S(k : m_1, \ldots, m_k) = \sum_{i=2}^{k}(c+1(i) - c+1(i - 1))m_i.\]
It is easy to see that \(S(n : 1, \ldots, 1) = c+1(n)\) and \(S(n : 2, 1, \ldots, 1) = c+1(n - 1)\). Assume that for some \(j\), we have \(m_j > 1\) an straightforward computation shows that
\[S(k : m_1, \ldots, m_k) - S(k + 1 : m_1, \ldots, m_j - 1, \ldots, m_k, 1) =
(c+1(j - 1) - c+1(j)) + (c+1(k) - c+1(k + 1))\]
which is negative. Hence the maximum value of \(S(k : m_1, \ldots, m_k)\) is \(S(n : 1, \ldots, 1)\) and the next largest value of \(S(k : m_1, \ldots, m_k)\) is \(S(n - 1 : 2, 1, \ldots, 1)\). □

Here we recall the notion of central product of two groups.

Definition 3.10. The group \(G\) is a central product of \(A\) and \(B\), if \(G = AB\), where \(A\) and \(B\) are normal subgroups of \(G\) and \(A \subseteq C_G(B)\).

We denote the central product of two groups \(A\) and \(B\) by \(A \cdot B\).
Recall that a group \(G\) is said to be minimal non-abelian if it is non-abelian but all its proper subgroups are abelian.
The number minimal generators of group \(G\) and the Frattini subgroup of \(G\) are denoted by \(d(G)\) and \(Φ(G)\), respectively.
This lemma gives information on minimal non-abelian \(p\)-groups.

Lemma 3.11. [Lemma 2.2] Assume that \(G\) is a finite nonabelian \(p\)-group. Then the following conditions are equivalent:
1. \(G\) is minimal non-abelian;
2. \(d(G) = 2\) and \(|G'| = p|\)
3. \(d(G) = 2\) and \(Φ(G) = Z(G)\).

For \(p\)-groups with small derived subgroup we have:

Lemma 3.12. [Lemma 4.2] Let \(G\) be a \(p\)-group with \(|G'| = p\). Then \(G = (A_1 \cdot A_2 \cdot \ldots \cdot A_s)Z(G)\), the central product, where \(A_1, \ldots, A_s\) are minimal non-abelian groups. Furthermore \(G/Z(G)\) is an elementary abelian group of even rank.

The order of the \(c\)-nilpotent multiplier of a finite \(p\)-group depends somehow on the order of its derived subgroup.

Lemma 3.13. Let \(G\) be a non-abelian finite \(p\)-group of order \(p^n\) whose derived subgroup is of order \(p\) and \(c \geq 2\). Then \(|M^{(c)}(G)| \leq p^{c+1(n-1)+c+2(2)},\) and the equality holds if and only if \(G \cong E_1 \times Z_p^{(n-3)}\).
Proof. By Lemma 3.12, $G = (A_1 \cdot A_2 \cdot \ldots \cdot A_s)Z(G)$ where $A_i$'s are minimal nonabelian groups. First suppose that $Z(G)$ is not elementary abelian. Then by Theorem 2.6, we have

$$|\mathcal{M}^{(c)}(G)| |\gamma_{c+1}(G) \cap Z(G)| \leq |\mathcal{M}^{(c)}(G/Z(G))| |\mathcal{M}^{(c)}(Z(G))| |Z(G) \otimes_c G/Z(G)|.$$ 

Using Proposition 2.8 and Theorem 3.9(ii), since $Z(G) \otimes_c G/Z(G)$ is the direct summand of $\Gamma_{c+1}(Z(G), G/Z(G))$, we have

$$|Z(G) \otimes_c G/Z(G)| \leq |\Gamma_{c+1}(Z(G), G/Z(G))|.$$ 

Therefore

$$|\mathcal{M}^{(c)}(G)| \leq |\mathcal{M}^{(c)}(Z(G) \times G/Z(G))| \leq p^{\chi_{c+1}(n-1)} \leq p^{\chi_{c+1}(n-1)+\chi_{c+2}(2)}.$$ 

Now assume that $Z(G)$ is elementary abelian. If $|Z(G)| = p$, then $G$ is extra-special due to Theorem 3.8. We may assume that $|Z(G)| \geq p^2$. If $G/G'$ is not elementary abelian, then by a similar way in the previous case and using Theorem 2.6 and Proposition 2.8, we have

$$|\mathcal{M}^{(c)}(G)| |\gamma_{c+1}(G) \cap G'| \leq |\mathcal{M}^{(c)}(G/G')| |\mathcal{M}^{(c)}(G')| |G' \otimes_c G/G'|.$$ 

Thus Theorem 3.9(ii), implies $|\mathcal{M}^{(c)}(G)| \leq p^{\chi_{c+1}(n-1)} \leq p^{\chi_{c+1}(n-1)+\chi_{c+2}(2)}$.

Now assume that $G/G'$ is elementary abelian. It is shown that [29, Lemma 2.2], $G$ is a central product of an extra-special $p$-group $H$ of order $p^{2k+1}$ and $Z(G)$ of order $p^{n-2k}$. Still in the case that $Z(G)$ is elementary abelian, suppose that $T$ be a complement of $G'$ in $Z(G)$, we have $Z(G) = G' \times T$ and so $G = H \times T$. By Proposition 2.8, we have

$$|\mathcal{M}^{(c)}(G)| = |\mathcal{M}^{(c)}(H \times T)| = |\mathcal{M}^{(c)}(H)| |\mathcal{M}^{(c)}(T)| |\Gamma_{c+1}(H^{ab}, T)|.$$ 

If $H$ is not capable, then Theorems 3.3 and 3.8 imply

$$|\mathcal{M}^{(c)}(H/H')| = |\mathcal{M}^{(c)}(H)| = p^{\chi_{c+1}(2k)}.$$ 

By Theorem 3.9(ii), we have

$$|\mathcal{M}^{(c)}(G)| = |\mathcal{M}^{(c)}(T \times H^{ab})| = p^{\chi_{c+1}(n-1)}.$$ 

Now assume that $H = E_1$ or $H = D_8$. By Proposition 2.8 and Theorem 3.9(ii), we have

$$|\mathcal{M}^{(c)}(T)| = p^{\chi_{c+1}(n-3)}, \quad |\Gamma_{c+1}(H^{ab}, T)| = p^{\chi_{c+1}(n-1)-\chi_{c+1}(n-3)-\chi_{c+1}(2)}.$$ 

Thus

$$|\mathcal{M}^{(c)}(G)| = \begin{cases} 2^{\chi_{c+1}(n-1)+1} & \text{if } p = 2, \\ p^{\chi_{c+1}(n-1)+\chi_{c+1}(2)} & \text{if } p > 2. \end{cases}$$ 

Now it is easy to see that $|\mathcal{M}^{(c)}(G)| = p^{\chi_{c+1}(n-1)+\chi_{c+1}(2)}$ if and only if $G \cong E_1 \times Z_p^{(n-3)}$. \qed

Now we are in a position to summarize the results in the following theorem.
Theorem 3.14. Let $G$ be a $p$-group of order $p^n$ with $|G'| = p^m (m \geq 1)$ and $c \geq 2$. Then

$$|\mathcal{M}^{(c)}(G)| \leq p^{c+1(n-m)+\chi_{c+2}(2)+(m-2)(n-m)^e},$$

In particular, if $|G'| = p$, then we have $|\mathcal{M}^{(c)}(G)| \leq p^{c+1(n-1)+\chi_{c+2}(2)}$ and the equality holds in last one if and only if $G \cong E_1 \times \mathbb{Z}_p^{(n-3)}$.

Proof. Let $G$ be an arbitrary non-abelian $p$-group of order $p^n$. We proceed by induction on $m$. The case $m = 1$ follows from Lemma 3.13. Therefore, we may assume that $m \geq 2$. Let $B$ a central subgroup of order $p$ in $G'$, by Theorem 2.6, we have

$$|\mathcal{M}^{(c)}(G)| \leq |\mathcal{M}^{(c)}(G)| \cap_{c+1}(G) \cap B \leq |\mathcal{M}^{(c)}(G/B)| \cap B \otimes_c G^{ab}.$$  

Using induction hypothesis

$$|\mathcal{M}^{(c)}(G/B)| \leq p^{c+1(n-m)+\chi_{c+2}(2)+(m-2)(n-m)^e},$$

and so

$$|\mathcal{M}^{(c)}(G)| \leq p^{c+1(n-m)+\chi_{c+2}(2)+(m-2)(n-m)^e} + (n-m)^e,$$

which completes the proof. \qed

4. $c$-Capability of Extra-Special $p$-Groups

The first study of the capability of extra-special $p$-groups was by Beyl et al. in [4]. They showed that in extra-special $p$-groups, only $E_1$ and $D_8$ are capable. Recently the last two authors, proved that for extra-special $p$-groups, the notions “capable” and “$2$-capable” are equivalent. Here we generalize it and show that for these groups, “capability” and “$c$-capability” are equivalent. Recall that if $G$ is a $c$-capable group, that is $G \cong E/Z_c(E)$ for some group $E$; then $G \cong E/\mathbb{Z}_{c-1}(E)/Z(E/\mathbb{Z}_{c-1}(E))$ which shows $G$ is capable too. So we just have to prove the $c$-capability of $D_8$ and $E_1$. To do this we need the following proposition which can be found in [3] Lemma 2.1(iv) and Proposition 1.2. For the statement of this fact we need terminology from [5] as below.

Let $F/R$ be a free presentation for $G$ and $\pi : F/[R, c F] \to G$ be the canonical surjection. The $c$-central subgroup $Z^*_c(G)$ of $G$ is the image in $G$ of the section term of the upper central series of $F/[R, c F]$. More precisely it is equal to $\pi(Z_c(F/[R, c F]))$.

Proposition 4.1. [11] Proposition 12]

(i) A group $G$ is $c$-capable if and only if $Z^*_c(G)$ is trivial;
(ii) If $N$ is normal subgroup of $G$ contained in $Z^*_c(G)$, then the canonical

$$\mathcal{M}^{(c)}(G) \to \mathcal{M}^{(c)}(G/N),$$

homomorphisms is injection.
(iii) $Z^*_{c+1}(G)$ contains $Z^*_c(G)$ for $c \geq 0$.

Proposition 4.1 yields the sequences

$$1 = Z^*_0(G) \subseteq Z^*_1(G) \subseteq Z^*_2(G) \subseteq Z^*_3(G) \subseteq \ldots,$$

we recall that $Z^*_c(G)$ the epicentre of $G$. Beyl’s results can be extended to $c$-capability as follows.
Theorem 4.2. An extra-special $p$-group is $c$-capable if and only if $G$ is isomorphic to one of the groups $D_8$ or $E_1$.

Proof. Let $G$ be an extra-special $p$-group of order $p^3$ and exponent $p$. We will show there is no nontrivial normal subgroup of $G$ for which the natural homomorphism $\mathcal{M}^c(G) \rightarrow \mathcal{M}^c(G/N)$ is injective. Let $N$ be a nontrivial normal subgroup of $G$, so $G/N$ is an abelian $p$-group of order at most $p^2$ and hence, Theorem 3.9 shows $|\mathcal{M}^c(G/N)| \leq p^{\chi+1}(2)$. Since $|\mathcal{M}(G)| = p^{\chi+1}(2)+\chi+2(2)$ by Theorem 3.14 $\mathcal{M}^c(G) \rightarrow \mathcal{M}^c(G/N)$ fails to be injective and the result holds. For the case $G = D_8$, we have $D_2^{2+1}/Z_c(D_2^{2+1}) \cong D_8$. The proof is complete. \hfill $\square$

5. $c$-Capability and $c$-nilpotent multiplier of generalized extra-special $p$-groups

Jafari et al. in [18] studied the Schur multiplier of generalized extra-special $p$-groups. We obtain the $c$-nilpotent multiplier of a generalized extra-special $p$-groups. Niroomand and Parvizi in [31] first studied the capability of generalized extra-special $p$-groups. We obtain the following.

Definition 5.1. [33] Definition 3.1 Let $G$ be a finite $p$-group. Then $G$ is called generalized extra-special $p$-group if $\Phi(G) = G' \cong Z_p$.

The structure of generalized extra-special $p$-groups is as follows. Here by $E_{p^{2m+1}}$ we mean the extra-special $p$-group of order $p^{2m+1}$.

Lemma 5.2. [33] Lemma 3.2 Let $G$ be a generalized extra-special $p$-group. Then $Z(G) \cong \Phi(G) \times A$ then $G \cong E_{p^{2m+1}} \times A$, and if $Z(G) \cong Z_p \times A$ then $G \cong (E_{p^{2m+1}} \times Z_p^2) \times A$, in which $A$ is an elementary abelian $p$-group.

The capability of the generalized extra-special $p$-groups is determined in the following.

Proposition 5.3. [31] Theorem 3.4 Let $G$ be a generalized extra-special $p$-group. $G$ is capable if and only if $G = E_1 \times Z_p^{(n-3)}$ or $G = D_8 \times Z_2^{(n-3)}$.

Proposition 5.4. [18] Lemma 2.9 Let $G$ be the generalized extra-special $p$-group and isomorphic to $E_{p^{2m+1}} \cdot B$ such that $B \cong Z_p^2$. Then $|\mathcal{M}(G)| = p^{2m^2+m-4}$.

The following proposition will be used in the next investigation.

Proposition 5.5. Let $G = E_{p^{2m+1}} \cdot B$ such that $B \cong Z_p^2$. Then $|\mathcal{M}^c(G)| = p^{\chi+1(2m)}$, for all $c \geq 2$.

Proof. By Proposition 5.3 since $G$ is a non-capable generalized extra-special $p$-group, $Z^c(G) = G'$. Now by Proposition 4.3 the homomorphism $\mathcal{M}(c)(G) \rightarrow \mathcal{M}(c)(G^ab)$ is an isomorphism. Therefore $|\mathcal{M}(c)(G)| = |\mathcal{M}(c)(G^ab)|$. Now the result holds by Theorem 3.14. \hfill $\square$

The following theorem determines the $c$-capability of the generalized extra-special $p$-groups.
Theorem 5.6. Let $G$ be a generalized extra-special $p$-group and $|G| = p^n$. $G$ is $c$-capable if and only if $G = E_1 \times \mathbb{Z}_p^{(n-3)}$ or $G = D_8 \times \mathbb{Z}_2^{(n-3)}$.

Proof. Let $G = E_1 \times \mathbb{Z}_p^{(n-3)}$ be a generalized extra-special $p$-group. We will show there is no nontrivial normal subgroup of $G$ for which the natural homomorphism $M^{(c)}(G) \rightarrow M^{(c)}(G/N)$ is injective. Let $N$ be a nontrivial normal subgroup of $G$, so $G/N$ is an abelian $p$-group of order at most $p^{n-1}$ and hence Theorem 3.9 shows $|M^{(c)}(G/N)| \leq p^{\chi_c(n-1)}$. Since $|M^{(c)}(G)| = p^{\chi_c(n-1) + \chi_c + 2}$ by Theorem 3.14, $M^{(c)}(G) \rightarrow M^{(c)}(G/N)$ fails to be injective and the result holds. Now Proposition 4.1 shows $G = E_1 \times \mathbb{Z}_p^{(n-3)}$ is $c$-capable. The proof for the case $G = D_8 \times \mathbb{Z}_2^{(n-3)}$ is completely similar expect that $M^{(c)}(D_8) \cong \mathbb{Z}_4 \times \mathbb{Z}_2^{(\chi_c+1)(2)-1}$. □

Corollary 5.7. Let $G$ be a generalized extra-special $p$-group. Then $G$ is $c$-capable if and only if $G$ is capable.

References

[1] R. Baer, Representations of groups as quotient groups, I, II, and III. Trans. Amer. Math. Soc. 54 (1945) 295-419.
[2] Ya. G. Berkovich, On the order of the commutator subgroups and the Schur multiplier of a finite $p$-group, J. Algebra 144 (1991) 269-272.
[3] Ya. G. Berkovich, Zvonimir, J. Groups of Prime Power Order. Vol. 2. de Gruyter Expositions in Mathematics, 47. Walter de Gruyter GmbH & Co. KG, Berlin, 2008.
[4] F. R. Beyl, U. Felgner, P. Schmid, On groups occurring as center factor groups, J. Algebra 61 (1970) 161-177.
[5] J. Burns, G. Ellis, On the nilpotent multipliers of a group, Math. Z. 226 (1997) 405-428.
[6] J. Burns, G. Ellis, Inequalities for Baer invariants of finite groups. Canad. Math. Bull. 41 No. 4, (1998) 385-391.
[7] G. Ellis, On groups with a finite nilpotent upper central quotient, Arch. Math. 70 (1998) 89-96.
[8] G. Ellis, Tensor products and $q$-crossed modules, J. London Math. Soc. 51(2) (1995) 243-258.
[9] G. Ellis, On the Schur multiplier of $p$-groups, Comm. Algebra 9 (1999) 4173-4177.
[10] G. Ellis, J. Wiegold, A bound on the Schur multiplier of a prime power group, Bull. Austral. Math. Soc. 60 (1999) 191-196.
[11] G. Ellis, On the relation between upper central quotient and lower central series of a group, A.M.S. Soc. 353 (2001) 4219-4234.
[12] A. Fröhlich, Baer invariants of algebras. Trans. Amer. Math. Soc. 109, (1962) 221-244.
[13] M. Hall, The Theory of Groups, MacMillan Company, NewYork, 1959.
[14] P. Hall, The classificatin of prime-power groups, J. Reine Angew. Math. 182 (1940) 130-141.
[15] P. Hall, Verbal and marginal subgroups, J. Reine Angew. Math. 182 (1940) 156-157.
[16] H. Heineken, D. Nikolova, Class two nilpotent capable groups, Bull. Austral. Math. Soc. 54 (1996) 347-352.
[17] B. Huppert, N. Blackburn, Finite Groups II (Springer, Berlin, 1982).
[18] S. H. Jafari, F. Saeedi, E. Khamseh, Characterization of finite $p$-groups by their non-abelian tensor square, Comm. Algebra, 41 (2013) 1954-1963.
[19] M. R. Jones, Multiplicutors of $p$-groups, Math. Z. 127 (1972) 165-166.
[20] M. R. Jones, Some inequalities for the multiplicator of a finite group, Proc. Amer. Math. Soc. 39 (1973) 450-456.
[21] G. Karpilovsky, The Schur Multiplier, N.S. 2, London Math. Soc. Monogr, 1987.
[22] C. R. Leedham-Green, S. MacKay, Baer invariants, isologism, varietal laws and homology, Acta Math. 137 (1976) 99-150.
[23] J. L. MacDonald, Group derived functors. J. Algebra 10 (1968) 448-477.
[24] B. Mashayekhy, M. A. Sanati, On the order of nilpotent multipliers of finite $p$-groups, Comm. Algebra 33(7) (2005) 2079-2087.
[25] B. Mashayekhy, F. Mohammadzadeh, Some inequalities for nilpotent multipliers of powerful $p$-groups, Bull. Iranian Math. Soc. Vol. 33, No. 2 (2007) 61-71.
[26] M. R. R. Moghaddam, The Baer invariant of a direct product, Arch. Math. 33 (1980) 504-511.
[27] M. R. R. Moghaddam, Some inequalities for Baer invariant of a finite group, Bull. Iranian Math. Soc. 9 (1981) 5-10.
[28] M. R. R. Moghaddam, On the Schur-Baer property. J. Austral. Math. Soc. Ser. A 31 (1981) 343-361.
[29] P. Niroomand, On the order of Schur multiplier of non-abelian $p$-groups, J. Algebra 322 (2009) 4479-4482.
[30] P. Niroomand, The Schur multiplier of $p$-groups with large derived subgroup, Arch. Math. 95 (2010) 101-103.
[31] P. Niroomand, M. Parvizi, A remark on the capability of finite $p$-groups, J. Adv. Res. Pure Math (2013) 91-95.
[32] P. Niroomand, M. Parvizi, On the 2-nilpotent multiplier of finite $p$-groups, Glasg. Math J. 57 (2015) 201-210.
[33] R. Stancu, Almost all generalized extra-special $p$-groups are resistant. J. Algebra 249 (2002) 120-126.
[34] X. Zhou, On the order of Schur multipliers of finite $p$-groups, Comm. Algebra 1 (1994) 1-8.
[35] Xu, M. Y., An, L. J. and Zhang, Q. H., Finite $p$-groups all of whose non-abelian proper subgroups are generated by two elements, J. Algebra, 319, 2008, 3603-3620

Department of Pure Mathematics, Ferdowsi University of Mashhad, Mashhad, Iran
E-mail address: farangisjohary@yahoo.com

Department of Pure Mathematics, Ferdowsi University of Mashhad, Mashhad, Iran
E-mail address: parvizi@um.ac.ir

School of Mathematics and Computer Science, Damghan University, Damghan, Iran
E-mail address: niroomand@du.ac.ir, pniroomand@yahoo.com