A RIEMANN MAPPING THEOREM FOR TWO-CONNECTED DOMAINS IN THE PLANE

STEVEN R. BELL, ERSIN DEGER, & THOMAS TEGTMeyer

Abstract. We show how to express a conformal map Φ of a general two connected domain in the plane such that neither boundary component is a point to a representative domain of the form \( \mathcal{A}_r = \{z : |z + 1/z| < 2r\} \), where \( r > 1 \) is a constant. The domain \( \mathcal{A}_r \) has the virtue of having an explicit algebraic Bergman kernel function, and we shall explain why it is the best analogue of the unit disc in the two connected setting. The map Φ will be given as a simple and explicit algebraic function of an Ahlfors map of the domain associated to a specially chosen point. It will follow that the conformal map Φ can be found by solving the same extremal problem that determines a Riemann map in the simply connected case.

1. Main results

The Riemann map \( f_a \) associated to a point \( a \) in a simply connected domain \( \Omega \neq \mathbb{C} \) in the complex plane maps \( \Omega \) one-to-one onto the unit disc \( \{z : |z| < 1\} \) and is the solution to an extremal problem: among all holomorphic mappings of \( \Omega \) into the unit disc, \( f_a \) is the map such that \( f_a'(a) \) is real and as large as possible. The objects of potential theory associated to the unit disc are particularly simple. The Bergman and Szegő kernel functions are explicit and simple rational functions on the disc, for example. The Poisson kernel is also very simple. Consequently, the Bergman, Poisson, and Szegő kernels associated to \( \Omega \) can be expressed very simply in terms of a Riemann map.

It was noted in [B4] that any two-connected domain \( \Omega \) in the plane such that neither boundary component is a point can be mapped conformally via a mapping \( \Phi \) to a representative domain of the form

\[ \mathcal{A}_r := \{z : |z + 1/z| < 2r\}, \]

where \( r \) is a constant greater than one. See [J-T] for a nice proof of this fact which also proves a much more general theorem. See Figure 1 for a picture of \( \mathcal{A}_r \). The dotted circle in the figure is the unit circle. Notice that \( 1/z \) is an automorphism of \( \mathcal{A}_r \) that fixes the unit circle.

In this paper, we show that the conformal map \( \Phi : \Omega \to \mathcal{A}_r \) can be expressed simply in terms of an Ahlfors map of \( \Omega \). The Bergman kernel associated to a representative domain \( \mathcal{A}_r \) is an algebraic function (see [B4, B5]). In fact, the Bergman kernel has recently been written down explicitly in [D]. The Szegő

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kernel is also algebraic and the Poisson kernel can be expressed in terms of rather simple functions (see [B6]). It is proved in [B3] that neither the Bergman nor Szegő kernel can be rational in a two-connected domain. Thus, the representative domain \( \mathcal{A}_r \) can be thought of as perhaps the best analogue of the unit disc in the two-connected setting. We shall show that the conformal map \( \Phi \) can be determined by solving the same extremal problem as the one that determines a Riemann map in the simply connected case, but we must be careful to choose the base point properly. In fact, we show how to find a point \( a \) in \( \Omega \) so that \( \Phi \) is given by \( cf_a + \sqrt{c^2f_a^2 - 1} \) where \( f_a \) is the Ahlfors mapping associated to \( a \), and \( c \) is a complex constant that we will determine. We will explain along the way how \( cf_a + \sqrt{c^2f_a^2 - 1} \) can be understood to be a single valued holomorphic function on \( \Omega \). The Ahlfors map \( f_a \) is the solution to the extremal problem: among all holomorphic mappings of \( \Omega \) into the unit disc, \( f_a \) is the map such that \( f_a'(a) \) is real and as large as possible. See [B2, p. 49] for the basic properties of the Ahlfors map.

In order to choose \( a \), we will inadvertently discover a new way to compute the modulus of the domain \( \Omega \).

![Figure 1. The domain \( \mathcal{A}_r \) with \( r = 1.05 \).](image)

Let \( a_\rho \) denote the annulus \( \{ z : \frac{1}{\rho} < |z| < \rho \} \) when \( \rho > 1 \), and let

\[
J(z) := \frac{1}{2} \left( z + \frac{1}{z} \right)
\]

denote the Joukowsky map. Note that \( \frac{1}{2}J(z) \) is a proper holomorphic map of \( \mathcal{A}_r \) onto the unit disc which is a 2-to-one (counting multiplicities) branched covering map. Several authors have found explicit biholomorphic maps between \( a_\rho \) and \( \mathcal{A}_r \) and have determined the relationship between \( \rho \) and \( r \) (see [J-T, JOT, C-M, D]). Since we shall need such a map, and since we can construct one in a minimum of space here using the method from [D], we shall include the construction for
completeness. The biholomorphic mapping \( \Psi : a_\rho \to A_r \) will be an explicit map involving the Jacobi \( \text{sn} \) function. There is a standard conformal map of the annulus \( \{ z : \frac{1}{\rho} < |z| < 1 \} \) onto the unit disc minus a slit from \(-L\) to \(+L\) along the real axis. See Nehari [N, p. 293]) for a construction where the relationship between \( L \) and \( \rho \) is given explicitly. The mapping is given by a constant times \[
\text{sn} \left( K + \frac{2iK}{\pi} \log \rho z ; \rho^{-4} \right),
\]
where \( K \) is a constant that can be determined explicitly. The map takes the unit circle to the unit circle and the circle of radius \( 1/\rho \) to the slit from \(-L\) to \(+L\). By composing with an inversion of the annulus, we get a conformal map that maps the annulus \( \{ z : \frac{1}{\rho} < |z| < 1 \} \) one-to-one onto the unit disc minus a slit from \(-L\) to \(+L\) along the real axis. The Joukowsky map \( J(z) \) is a biholomorphic mapping of \( \{ z \in A_r : |z| > 1 \} \) onto the disc of radius \( r \) minus a slit from \(-1\) to \(+1\). It maps the outer boundary of \( A_r \) to the unit circle and the unit circle two-to-one onto the slit. Thus, by scaling properly and then composing, we can write down a biholomorphic mapping from the annulus \( \{ z : \frac{1}{\rho} < |z| < 1 \} \) onto \( \{ z \in A_r : |z| > 1 \} \) which maps the unit circle to the unit circle. We may now extend this map by reflection to a biholomorphic map from \( a_\rho \) onto \( A_r \). Finally, we may compose with the automorphism \( 1/z \) and a rotation, if necessary, to obtain a conformal mapping \( \Psi : a_\rho \to A_r \) such that the outer boundary of \( a_\rho \) maps to the outer boundary of \( A_r \), and such that \( \Psi(i) = i \). Note that \( \Psi \) is uniquely determined by these conditions. We shall need to know that \( \Psi(-i) = -i \). Since \(-1/\Psi(-1/z)\) satisfies the same properties, namely that it is a conformal map that takes the outer boundary to the outer boundary and \( \Psi(i) = i \), it must be that \( \Psi(z) \equiv -1/\Psi(-1/z) \). From this, we deduce that \( \Psi(-i) \equiv -1/\Psi(-i) \), and hence that \( \Psi(-i) \) is a square root of \(-1\). Because \( \Psi \) is one-to-one, it cannot be that \( \Psi(-i) \) is \( i \), and so \( \Psi(-i) = -i \).

It is well known that any two-connected domain \( \Omega \) in the plane such that neither boundary component is a point can be mapped biholomorphically via a map \( F \) to a unique annulus of the form \( a_\rho \). Let \( C_1 \) denote the unit circle. We shall call \( \rho^2 \) the modulus of \( \Omega \) and we shall call the set \( F^{-1}(C_1) \) the median of the two-connected domain. The median is the unique connected one-dimensional set that is left invariant by every automorphism of the two-connected domain. The biholomorphic map \( \Psi \) of \( a_\rho \) onto \( A_r \) preserves the unit circle. Hence, the median of \( A_r \) is also the unit circle.

Our main results will follow from two key lemmas, which we now list.

**Lemma 1.1.** Suppose that \( \Omega \) is a two-connected domain in the plane such that neither boundary component is a point. The Ahlfors map \( f_a \) associated to any point \( a \) in \( \Omega \) is such that it has two distinct and simple branch points on the median of \( \Omega \).

Lemma 1.1 was proved by McCullough and Mair in [M-M] and independently and by other means in [Te].
Lemma 1.2. The Ahlfors map associated to the point \( i \) in \( \mathcal{A}_r \) is given by \( \frac{1}{r}J(z) \).

Lemma 1.2 was proved in [D]. For completeness, and because it will not take much space, we will give alternate proofs of the lemmas in the next section.

Suppose that \( \Omega \) is a two-connected domain in the plane such that neither boundary component is a point. We now describe the procedure to determine the mapping \( \Phi \), assuming the truth of the lemmas. Pick any point \( P \) in \( \Omega \). Lemma 1.1 guarantees the existence of a point \( a \) on the median of \( \Omega \) such that \( f'_P(a) = 0 \). (Note that the Ahlfors map is eminently numerically computable if the boundary of \( \Omega \) is sufficiently smooth (see [K-S, K-T, Tr, B1]), and the zeroes of a function with only two simple zeroes can easily be located numerically as well.) Now the Ahlfors map \( f_a \) associated to the point \( a \) is such that there are two distinct points \( p_1 \) and \( p_2 \) on the median of \( \Omega \) such that \( f'_a(p_i) = 0 \) for \( i = 1, 2 \). The points \( p_i \) are also different from \( a \) because \( f'_a(a) \neq 0 \). We now claim that there exists an \( r > 1 \) and a biholomorphic mapping from \( \Omega \) to \( \mathcal{A}_r \) such that \( \Phi(a) = i \). We may also arrange for the outer boundary of \( \Omega \) to get mapped to the outer boundary of \( \mathcal{A}_r \). To see this, we use the biholomorphic mapping \( F \) of \( \Omega \) onto an annulus \( a_\rho \). By composing with \( 1/z \), if necessary, we may map the outer boundary of \( \Omega \) to the outer boundary of \( a_\rho \). Since \( a \) is on the median, \( F(a) \) is on the unit circle, and we may compose with a rotation so that \( F(a) = i \). Now we may compose with the biholomorphic map \( \Psi \) constructed above of \( a_\rho \) onto \( \mathcal{A}_r \), noting that \( i \) is fixed under \( \Psi \) and that outer boundaries are preserved, to obtain \( \Phi \). (At the moment, we can only say that the number \( r \) exists. We will explain how to find it momentarily.) Lemma 1.2 states that \( \frac{1}{r}J(z) \) is the Ahlfors map of \( \mathcal{A}_r \) associated to the point \( i \). The extremal property of Ahlfors maps makes it easy to see that there is a unimodular complex number \( \lambda \) such that

\[
\frac{1}{r}J(\Phi(z)) = \lambda f_a(z).
\]

Since the branch points of \( J(z) \) occur at \( \pm 1 \), this last identity reveals that \( \Phi(p_i) = \pm 1 \) for \( i = 1, 2 \). We may assume that \( \Phi(p_1) = -1 \) and \( \Phi(p_2) = 1 \). Since \( J(-1) = -1 \) and \( J(1) = 1 \), equation (1.1) now gives \( \frac{-1}{r} = \lambda f_a(p_1) \) and \( \frac{1}{r} = \lambda f_a(p_2) \). Either one of these two equations determines both \( \lambda \) and \( r \). We can now solve equation (1.1) locally for \( \Phi \) to obtain

\[
\Phi(z) = J^{-1}(c f_a(z)),
\]

where \( c = r \lambda \) and, of course, \( J^{-1}(w) = w + \sqrt{w^2 - 1} \). This formula extends by analytic continuation from any point in \( \Omega \) that is not a branch point of \( f_a \). The apparent singularities at the branch points are removable because the construction aligned the two algebraic singularities of \( J^{-1} \) to occur at the two images of the branch points under \( c f_a \). We shall explain how to compute \( \Phi \) from its boundary values in §3. The modulus \( \rho^2 \) of \( \Omega \) can now be gotten from \( r \) via any of the formulas relating \( \rho \) and \( r \) proved in [J-T, JOT, C-M, D].

In the last section of this paper, we explore how the mapping \( \Phi \) can be used to study the Bergman kernel of two-connected domains.
2. Proofs of the Lemmas

To prove Lemmas 1.1 and 1.2, we will make use of the simple fact that if $G : \Omega_1 \to \Omega_2$ is a biholomorphism between bounded domains, the Ahlfors maps transform according to the following formula. Let $f_j(z; w)$ denote the Ahlfors map $f_a$ associated to the point $w$ in $\Omega_j$. The extremal property of the Ahlfors maps can be exploited to show that

$$f_1(z; w) = \lambda f_2(G(z); G(w)),$$

where $\lambda$ is the unimodular constant that makes the derivative of $f_1(z; w)$ in the $z$ variable at $z = w$ real and positive.

**Proof of Lemma 1.1.** The biholomorphic map $F$ from $\Omega$ to $a_\rho$ together with equation (2.1) reveals that it is sufficient to prove Lemma 1.1 for an annulus $a_\rho$ with base point $a > 0$. The function $h(z) = e^{iz}$ maps the horizontal strip $H = \{ z : -\log \rho < \Im z < \log \rho \}$ onto $a_\rho$, takes the real line onto $C_1$, and is $2\pi$-periodic. Consider $g(z) = f_a(h(z))$. Notice that $g$ is $2\pi$-periodic, maps $H$ onto the unit disk, and maps the boundary of $H$ onto the unit circle. It can be extended via a sequence of reflections to a function on $\mathbb{C}$. Because of these reflections, $g$ has a second period of $(4 \log \rho)i$, making $g$ an elliptic function. Because $f_a$ is two-to-one, $g$ is a second order elliptic function.

Any second order elliptic function $\phi$ satisfies the equation $\phi(w - z) = \phi(z)$, where $w$ is the sum of two noncongruent zeroes of $\phi$ (see [H]). From [T-T], we know that if $a$ is real, $f_a$ has zeroes at $a$ and $-1/a$. (We will prove this fact below to make this paper self contained.) It now follows that $g$ will have zeroes at $-i \log a$ and $i \log a + \pi$. Thus, we can choose $w = \pi$. The mapping $z \mapsto \pi - z$ has $\pi/2$ as a fixed point, while the point $3\pi/2$ gets mapped to the congruent point $-\pi/2$. These two points will be branch points of $g$, and $i$ and $-i$ will be the branch points of $f_a$, lying on the median of $a_\rho$.

**Proof of Lemma 1.2.** The biholomorphism $\Psi$ (see §1) from $a_\rho$ onto $A_r$ mapping $i$ to $i$ maps $-i$ to $-i$. Let $f_i$ denote the Ahlfors map of $a_\rho$ with base point $i$. From [T-T], we know that $f_i$ will have $i$ and $-i$ as its zeroes. (We include a proof of this fact in the next paragraph.) Equation (2.1) tells us that the Ahlfors map of $A_r$ with base point $i$ will have $i$ and $-i$ as its zeroes. Because this Ahlfors map has the same zeroes as $\frac{-1}{2} J(z)$, and both functions map the boundary of $A_r$ to the unit circle, they are the same up to multiplication by a unimodular constant. Since $J'(i) = 1 > 0$, this constant must be 1. Lemma 1.2 is proved.

We shall now flesh out the proofs of the lemmas by giving a shorter and simpler proof than the one given in [T-T] that the Ahlfors map $f_a$ associated to a point $a$ in $a_\rho$ has simple zeroes at $a$ and $-1/\bar{a}$. Thus $f_a$ has simple zeroes at $a$ and $-1/a$ if $a$ is real, and $f_i$ has simple zeroes at $i$ and $-i$. The Ahlfors map $f_a$ always has a simple zero at $a$. Since the Ahlfors map is two-to-one (counting multiplicities), $f_a$ must have exactly one other simple zero distinct from $a$. The Ahlfors map is given via

$$f_a(z) = S(z, a)/L(z, a)$$
where \( S(z,a) \) is the Szegő kernel and \( L(z,a) \) is the Garabedian kernel. For the basic properties of the Szegő and Garabedian kernels, see [B2, p. 49]. We note here that \( S(z,a) = \overline{S(a,z)} \) and that \( L(z,a) \neq 0 \) if \( a \in \Omega \) and \( z \in \overline{\Omega} \) with \( z \neq a \). Also, \( L(z,a) \) has a single simple pole at \( a \) in the \( z \) variable, and \( L(z,a) = -L(a,z) \). It follows from equation (2.2) that the other zero of \( f_a \) is the one and only zero of \( S(z,a) \) in the \( z \) variable. Let \( Z(a) \) denote this zero. It is proved in [B4] that \( Z(a) \) is a proper antiholomorphic self-correspondence of the domain. In the two-connected case that we are in now, it is rather easy to see that \( Z(a) \) is an antiholomorphic function of \( a \) that maps \( a \rho \), one-to-one onto itself. To see this, let \( S'(z,a) \) denote the derivative of \( S(z,a) \) in the \( z \) variable and notice that the residue theorem yields

\[
Z(a) = \frac{1}{2\pi i} \int_{z \in \partial \Omega} \frac{zS'(z,a)}{S(z,a)} \, dz.
\]

(See [B2] for proofs that \( S(z,a) \) is smooth up to the boundary and non-vanishing on the boundary in the \( z \) variable when \( a \) is held fixed in \( a \rho \).) It can be read off from this last formula that \( Z(a) \) is antiholomorphic in \( a \). Now hold \( z \) fixed in \( a \rho \) and let \( a \) tend to a boundary point \( a_b \). Equation (2.2) shows that \( f_a(z) \) tends to \( S(z,a_b)/L(z,a_b) \). But \( S(z,a_b) = -\frac{1}{i}L(z,a_b)T(a_b) \) where \( T(a_b) \) is the complex unit tangent vector at \( a_b \) (see [B2, p. 107]), and so \( f_a(z) \) tends to \( iT(a_b) \), a unimodular constant. This shows that the zeroes of \( f_a(z) \) must tend to the boundary as \( a \) tends to the boundary. Consequently, \( Z(a) \) is a proper antiholomorphic self mapping of the annulus. The only such maps are one-to-one and onto (see [M-R]). We shall say that \( Z(a) \) is an antiholomorphic automorphism of \( a \rho \). Note that \( Z(a) \) must satisfy \( Z(Z(a)) = a \). The only such antiholomorphic automorphisms are \( Z(a) = \lambda \bar{a} \), where \( \lambda \) is a unimodular constant, and \( Z(a) = 1/\bar{a} \), and \( Z(a) = -1/\bar{a} \). Since \( f_a \) does not have a double zero at \( a \), \( Z(a) \) cannot have a fixed point. Only \( Z(a) = -1/\bar{a} \) is without fixed points in \( a \rho \), so \( f_a \) has zeroes at \( a \) and \( -1/\bar{a} \). The proof is complete.

3. How to Compute \( \Phi \) and the Median of \( \Omega \).

Suppose that \( \Omega \) is a bounded two connected domain bounded by two non-intersecting \( C^\infty \) Jordan curves. The Kerzman-Stein-Trummer method [K-S, K-T] can be used to compute the boundary values of Ahlfors mappings associated to \( \Omega \). (See [B1] and [B2], Chapter 26, for a description of how to do this in the multiply connected setting.) Thus, we may pick a point \( P \) in \( \Omega \) and compute \( f_P \). We shall explain how to compute the two branch points of \( f_P \) on the median of \( \Omega \) below. Hence, we may find a point \( a \) on the median of \( \Omega \). Let \( f \) denote the Ahlfors mapping associated to the point \( a \) and let \( p_1 \) and \( p_2 \) denote the two simple zeroes of \( f' \) on the median of \( \Omega \). We shall now show how to compute the points \( p_1 \) and \( p_2 \) and the function \( \Phi \) using only the boundary values of the Ahlfors map \( f \). The residue theorem yields that

\[
p_1^n + p_2^n = \frac{1}{2\pi i} \int_{\partial \Omega} \frac{f''(z)z^n}{f'(z)} \, dz.
\]
The two points can be determined from the values of $p_1^n + p_2^n$ for $n$ equal to one and two. Indeed the coefficients $A$ and $B$ of the polynomial $z^2 - Az + B = (z - p_1)(z - p_2)$ are given as $A = p_1 + p_2$ and $B = \frac{1}{2}[(p_1 + p_2)^2 - (p_1^2 + p_2^2)]$, and so $p_1$ and $p_2$ can be found via the quadratic formula. Therefore, the point $a$ itself and the branch points of $f_a$ can readily be found numerically.

We may calculate $c = r\lambda = -1/f(p1)$ by the procedure described in §1. There are two holomorphic branches of $J^{-1}(w) = w + \sqrt{w^2 - 1}$ near the boundary of the disc of radius $r$ corresponding to the two choices of the square root. One branch $j_o$ maps the circle of radius $r$ one-to-one onto the outer boundary of $\mathcal{A}_r$, and the other branch $j_i$ maps the circle of radius $r$ one-to-one onto the inner boundary of $\mathcal{A}_r$. The mapping $\Phi$ that maps $\Omega$ one-to-one onto $\mathcal{A}_r$ can be determined from its boundary values on $\partial \Omega$. The boundary values of $\Phi$ on the outer boundary of $\Omega$ can be taken to be $j_o(cf(z))$ and the boundary values on the inner boundary are then $j_i(cf(z))$. The extension of $\Phi$ to $\Omega$ can be gotten from the boundary values of $\Phi$ via the Cauchy integral formula.

We now turn to a method to determine the median of $\Omega$. Equation (1.1) shows that $J(\Phi(z)) = cf(z)$ where $c = r\lambda$ and $f$ is shorthand for $f_a$. This shows that $f$ maps the median of $\Omega$ to the line segment $\mathcal{L}$ from $-1/c$ to $1/c$. Hence, the median of $\Omega$ is given by $f^{-1}(\mathcal{L})$. We already know the points $p_1,$ $p_2,$ and $a$ on the median. To compute other points $w$ in this set such that $f(w) = \tau$ for some $\tau$ in $\mathcal{L}$ where $\tau$ is not one of theEndpoints of $\mathcal{L}$, we may again use the residue theorem to obtain

$$w^n_1 + w^n_2 = \frac{1}{2\pi i} \int_{\partial \Omega} \frac{f'(z)z^n}{f(z) - \tau} \, dz,$$

where $w_1$ and $w_2$ are the two points in $f^{-1}(\tau)$. Finally, we may use the Newton identity for $n = 2$ and the quadratic formula as we did above to determine $p_1$ and $p_2$ to get $w_1$ and $w_2$. Note that $p_1$ and $p_2$ correspond to the inverse images of the two endpoints of $\mathcal{L}$, and so we may generate the rest of the median by letting $\tau$ move between the endpoints.

4. OTHER CONSEQUENCES.

We describe in this section how the Bergman kernel of a two-connected domain can be expressed rather concretely in terms of an Ahlfors map. It was proved in [D] that the Bergman kernel associated to $\mathcal{A}_r$ is given by

$$K_r(z, w) = C_1 \frac{2k^2S(z, \bar{w}) + kC(z, \bar{w})D(z, \bar{w}) + C_2}{z\bar{w}\sqrt{1 - k^2J(z)^2\sqrt{1 - k^2J(\bar{w})^2}}},$$

where $k = 1/r^2$, and $C_1$ and $C_2$ are constants that only depend on $r$, and

$$S(z, w) = -\left( \frac{J(z)w_2 - 1}{2w_2} \sqrt{1 - k^2J(z)^2} + J(w)\frac{w_2 - 1}{2w_2} \sqrt{1 - k^2J(w)^2} \right)^2,$$

and

$$C(z, w) = \frac{\frac{-w_2 - 1}{2w} - J(z)J(w)\sqrt{1 - k^2J(z)^2}}{1 - k^2J(z)^2J(w)^2}.$$
and
\[ D(z, w) = \frac{\sqrt{1 - k^2 J(z)^2} \sqrt{1 - k^2 J(w)^2} + k^2 J(z)J(w) \frac{z^2 - 1}{2z} \frac{w^2 - 1}{2w}}{1 - k^2 J(z)^2 J(w)^2}. \]

We showed that the biholomorphic map \( \Phi \) that we constructed above from a two-connected domain to its representative domain \( \mathcal{A}_r \) is such that \( \frac{1}{r} J(\Phi(z)) = \lambda f_a(z) \), where \( f_a \) is an Ahlfors map associated to \( \Omega \) for a point \( a \) on the median of \( \Omega \) and \( \lambda \) is a unimodular constant. Differentiate this identity to see that
\[
\Phi'(z) = 2 cf_a'(z)/(1 - \Phi(z)^{-2}) = 2 cf_a'(z)/(1 - J^{-1}(cf_a(z))^{-2}),
\]
where \( J^{-1}(w) = w + \sqrt{w^2 - 1} \). The Bergman kernel associated to \( \Omega \) is given by
\[
\Phi'(z)K_r(\Phi(z), \Phi(w))\overline{\Phi'(w)}.
\]

We may read off from this and the formulas above that the Bergman kernel associated to \( \Omega \) is \( f_a'(z)\overline{f_a'(w)} \) times a rational function of \( f_a(z) \), and \( f_b(w) \), and simple algebraic functions of \( f_a \) of the form \( \sqrt{Af_a(z)^2 - 1} \) and the conjugate of \( \sqrt{Af_a(w)^2 - 1} \), where \( A \) is equal to \( c, \lambda, \) or one. The degree of the rational function does not depend on the modulus of the domain, and the coefficients of the rational function depend in a straightforward way on the modulus.

It was proved in \([B5]\) that the Bergman kernel associated to a two-connected domain such that neither boundary component is a point can be expressed in terms of two Ahlfors maps \( f_a \) and \( f_b \) as
\[
f_a'(z)R(z, w)\overline{f_a'(w)}
\]
where \( R(z, w) \) is a rational function of \( f_a(z), f_b(z), \overline{f_a(w)}, \) and \( \overline{f_b(w)} \). The rational function has been proved to exist, but it has never been given explicitly. We are now in a position to write down this rational function for the domain \( \mathcal{A}_r \) using the points \( a = i \) and \( b = 1 \). It was proved in \([D]\) that the Ahlfors map associated to the point 1 of \( \mathcal{A}_r \) is
\[
g(z) = \frac{1}{2r} \frac{z - \frac{1}{r}}{\sqrt{1 - k^2 J(z)^2}}
\]
where \( k = 1/r^2 \). The square root in the formula can be taken to be the principal branch because the modulus of \( kJ(z) \) is less than one in \( \mathcal{A}_r \). We shall prove this result here because all the ingredients are on the table. It is a pleasant exercise to check that \( g \) is a two-to-one branched covering map of \( \mathcal{A}_r \) onto the unit disc. Notice that \( g \) has zeroes at 1 and \( -1 \). We proved above that the Ahlfors map \( f_1 \) has zeroes at 1 and \( -1 \). Hence \( f_1/g \) is a unimodular constant. Checking derivatives at the point 1 yields that the unimodular constant is 1. So \( g \) is equal to the Ahlfors map \( f_1 \). Recall that the Ahlfors map \( f_i \) associated to \( i \) is just \( \frac{1}{r} J(z) \).
To determine the rational function, we make note that the terms \( \frac{z^2 - 1}{2z} \) can be multiplied by unity in the form of

\[
\frac{\sqrt{1 - k^2 J(z)^2}}{\sqrt{1 - k^2 J(z)^2}}
\]

to see that

\[
\frac{z^2 - 1}{2z} = r \frac{1}{2r} \left( z - \frac{1}{z} \right) = rf_1(z) \sqrt{1 - k^2 J(z)^2} = rf_1(z) \sqrt{1 - r^2 f_i(z)^2}.
\]

The terms in the denominator in the expression for \( K(z, w) \) can be manipulated by multiplying by unity in the form of \( f_i^p / f_i^p \), noting that \( f_i^p(z) = \frac{1}{r}(1 - \frac{1}{z^2}) \), to see that

\[
z \sqrt{1 - k^2 J(z)} = \frac{2}{f_i^p(z)} f_1(z)(1 - r^2 f_i(z)^2).
\]

When these procedures are carried out in each of the expressions that comprise the Bergman kernel, and when all the terms are combined, the square roots all become squared and we obtain

\[
K(z, w) = f_i^p(z) Q(f_i(z), f_1(z), f_i(w), f_1(w)) f_i^p(w),
\]

where \( Q(z_1, z_2, w_1, w_2) \) is a rational function of four variables given by

\[
C_1 \left( \frac{\sigma(z_1, z_2, w_1, w_2) + \delta(z_1, z_2, w_1, w_2) + C_2}{q(z_1, z_2, w_1, w_2)} \right),
\]

where

\[
\sigma(z_1, z_2, w_1, w_2) = -2 \left( \frac{z_1 w_2 (1 - r^2 w_1^2) + w_1 z_2 (1 - r^2 z_1^2)}{1 - z_1^2 w_1^2} \right)^2
\]

and

\[
\delta(z_1, z_2, w_1, w_2) = - \left( \frac{(z_2 w_2 + z_1 w_1)(1 - r^2 z_1^2)(1 - r^2 w_1^2)(1 + z_1 w_1 z_2 w_2) + C_2}{(1 - z_1^2 w_1^2)^2} \right),
\]

and

\[
q(z_1, z_2, w_1, w_2) = z_2 w_2 (1 - r^2 z_1^2)(1 - r^2 w_1^2).
\]

That is more than twice as complicated as any formula on the one-connected unit disc, but it could be worse. A fascinating feature that can be read off is that the coefficients depend very simply on \( r \) and the degree of the rational function does not depend on \( r \).

For the general two connected domain \( \Omega \), we use the mapping \( \Phi \) and the transformation formula for the Bergman kernels to read off that

\[
K(z, w) = f_i^p(z) R(z, w) f_i^p(w),
\]

where \( R(z, w) \) is a rational function of \( f_a(z), f_b(z), f_a(w), \) and \( f_b(w) \). We take \( a \) to be the point we chose in the construction of \( \Phi \) and \( b \) to be equal to \( p_2 \). We obtain a formula for the rational function and we note that the degree does not depend on \( r \).

It will be interesting to investigate if similar properties hold for domains of higher connectivity using techniques of Crowdy and Marshall [C-M].
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Mathematics Department, Purdue University, West Lafayette, IN 47907
E-mail address: bell@math.purdue.edu

Mathematics Department, University of Cincinnati, Cincinnati, OH 45221
E-mail address: ersin.deger@uc.edu

Mathematics Department, Truman State University, Kirksville, MO 63501
E-mail address: tegt@truman.edu