Non-canonical quantum optics (II): Poincaré covariant formalism and thermodynamic limit

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The paper contains further development of the idea of field quantization introduced in M. Czachor, J. Phys. A: Math. Gen. 33 (2000) 8081-8103. The formalism is extended to the relativistic domain. The link to the standard theory is obtained via a thermodynamic limit. Unitary representations of the Poincaré group at the level of fields and states are explicitly given. Non-canonical multi-photon and coherent states are introduced. In the thermodynamic limit the statistics of photons in a coherent state is Poissonian. The $S$ matrix of radiation fields produced by a classical current is given by a non-canonical coherent-state displacement operator, a fact automatically eliminating the infrared catastrophe. Field operators are shown to be operators and not operator-valued distributions, and can be multiplied at the same point in configuration space. An exactly solvable example is used to compare predictions of the standard theory with those of non-canonical quantum optics, and explicitly shows the mechanism of automatic ultraviolet regularization occurring in the non-canonical theory. Similar conclusions are obtained in perturbation theory, where one finds the standard Feynman diagrams, but the Feynman rules are modified. A comparison with the Dicke-Hepp-Lieb model allows to identify the physical structure behind the non-canonical algebra as corresponding to an ensemble of indefinite-frequency oscillators with constant density $N/V$.

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I. INTRODUCTION

The idea of “non-canonical quantization” of electromagnetic fields was introduced in [1]. One begins with the observation that even in nonrelativistic quantum mechanics it is natural to treat the frequency $\omega$ characterizing a harmonic oscillator as an eigenvalue and not a parameter. A replacement of $\omega$ by an operator $\hat{\omega}$ leads to an indefinite-frequency oscillator with altered (non-)canonical commutation relations, and forms a natural departure point for a new version of the ‘old fashioned’ field quantization. An analysis of physical structures associated with such indefinite-frequency operators was discussed in detail in [2]. In the present paper we show that the non-canonically quantized electromagnetic field may be regarded as an ensemble of indefinite-frequency oscillators of constant density.

We extend the formalism to the relativistic domain. Representations of both the non-canonical commutation relations and the Poincaré group are explicitly constructed. Multi-photon and coherent states are defined. Their properties make them similar to those from the canonical formalism if one performs an appropriately defined thermodynamic limit $N \to \infty$. Vacuum states transform as a massless scalar field. Radiation fields produced by a classical current lead to the correct form of the $S$ matrix. The $S$ matrix is proportional to a non-canonical coherent-state displacement operator, a fact eliminating the infrared catastrophe.

Commutation relations for vector potentials are found showing certain deviations from locality due to nontrivial structure of vacua but simultaneously allow for multiplication of field operators at the same point in configuration space, a first hint suggesting the ultraviolet finiteness of the theory. Two other arguments in favor of the ultraviolet finiteness are based on analysis of perturbation theory and the spontaneous emission survival amplitude.

An example of the spontaneous-emission survival amplitude is exactly solvable in both canonical and non-canonical theories and can be employed to illustrate the links between the two approaches. Of particular interest is the analogy between our method of quantization and the Hepp-Lieb treatment of the Dicke model. We show that that the Dicke-Hepp-Lieb Hamiltonian has the structure of the same type as our multi-oscillator non-canonical Hamiltonian. This leads to the conclusion that the formal requirement that the RHS of non-CCR commutators satisfies the resolution of unity is equivalent to the physical requirement that the electromagnetic field is an ensemble of indefinite-frequency harmonic oscillators of constant density $N/V$.

The fact that the RHS of commutation relations is modified by the presence of an element from the center of the algebra makes our “non-canonical” fields analogous to the so-called generalized free fields [3]. The essential difference between the two approaches is that, first of all, our fields do interact with charges and we do not demand the Poincaré invariance of field commutators but employ the algebra which is only covariant. As a result we loose Poincaré invariance of vacuum and locality. This leads to analogies with nonlocal quantum field theories discussed
in [3]. The loss of locality implies also interesting analogies between our approach and quantum field theory in non-commutative space [3,8]. Still, there are also differences between what we propose and these approaches. For example, the formalism of [3] is based on nonlocal distributions and regularizations of Hamiltonians. Similarly to our approach vacua in such theories are Poincare non-invariant, a fact used to circumvent limitations imposed by the Haag theorem [3]. However, in our approach the nonlocal distributions occur at the level of amplitudes or averages, and not at the algebra of field operators. Analogously, perturbative expansions of amplitudes are automatically regularized in spite of the fact that there is no regularization at the level of operators.

II. NOTATION

In order to control covariance properties of fields in generalized frameworks it is best to work in a manifestly covariant formalism. The most convenient is the one based on spinors and passive unitary transformations.

A. Spinor convention and fields

We take \( c = 1 \) and \( \hbar = 1 \). The index notation we use in the paper is consistent with the Penrose-Rindler spinor and world-tensor convention [11]. The electromagnetic field-tensor and its dual are

\[
F_{ab} = \begin{pmatrix}
0 & E_1 & E_2 & E_3 \\
-E_1 & 0 & -B_3 & B_2 \\
-E_2 & B_3 & 0 & -B_1 \\
-E_3 & -B_2 & B_1 & 0
\end{pmatrix}
\]

\[
{}^*F_{ab} = \begin{pmatrix}
0 & -B_1 & -B_2 & -B_3 \\
-B_1 & 0 & -E_3 & E_2 \\
-B_2 & E_3 & 0 & -E_1 \\
-B_3 & -E_2 & E_1 & 0
\end{pmatrix}
\]

Self-dual and anti-self-dual parts of \( F_{ab} \) are related to the electromagnetic spinor by

\[
{}^+F_{ab} = \frac{1}{2} (F_{ab} - {}^*F_{ab}) = \varepsilon_{AB} \bar{\varphi}_{A'B'} = \begin{pmatrix}
0 & F_1 & F_2 & F_3 \\
-F_1 & 0 & iF_3 & -iF_2 \\
-F_2 & -iF_3 & 0 & iF_1 \\
-F_3 & iF_2 & -iF_1 & 0
\end{pmatrix}
\]

\[
{}^-F_{ab} = \frac{1}{2} (F_{ab} + {}^*F_{ab}) = \varepsilon_{A'B'} \varphi_{AB}
\]

where \( \mathbf{F} = (\mathbf{E} + i\mathbf{B})/2 \) is the Riemann-Silberstein vector [11]. Denote \( k \cdot x = k_a x^a \). The electromagnetic spinor has the following Fourier representation [11,12]

\[
\varphi_{AB}(x) = \int \frac{d\Gamma(k)\pi_A(k)\pi_B(k)}{4\pi^2} \left( f(k,+)e^{-ik \cdot x} + \overline{f(k,+)}e^{ik \cdot x} \right)
\]

\[
\bar{\varphi}_{A'B'}(x) = \int \frac{d\Gamma(k)\bar{\pi}_{A'}(k)\bar{\pi}_{B'}(k)}{4\pi^2} \left( f(k,+)e^{-ik \cdot x} + \overline{f(k,+)}e^{ik \cdot x} \right)
\]

where the spinor field \( \pi_A(k) \) is related to the future-pointing 4-momentum by

\[
k^a = \bar{\pi}^A(k)\bar{\pi}^{A'}(k) = (\not{k}, k) = (\not{|k|}, k)
\]

and the invariant measure on the light-cone is \( d\Gamma(k) = [(2\pi)^2\hbar^2]^{-1}d^3k \). Anti-self-dual and self-dual parts of the field tensor are

\[
{}^-F_{ab}(x) = \int \frac{d\Gamma(k)e_{A'B'}\pi_A(k)\pi_B(k)}{4\pi^2} \left( f(k,+)e^{-ik \cdot x} + \overline{f(k,+)}e^{ik \cdot x} \right)
\]

\[
{}^+F_{ab}(x) = \int \frac{d\Gamma(k)e_{AB}\bar{\pi}_{A'}(k)\bar{\pi}_{B'}(k)}{4\pi^2} \left( f(k,)e^{ik \cdot x} + \overline{f(k,)}e^{-ik \cdot x} \right)
\]
The latter form is used by the Bialynicki-Birulas in [12].

The sign in the amplitude \( f(k, \pm) \) corresponds to the value of helicity of positive-frequency fields. The four-vector potential \( A_a(x) \) is related to the electromagnetic spinor by

\[ \varphi_{XY}(x) = \nabla(X Y'A_{Y'})Y'(x) \]  

In the Lorenz gauge \( \nabla^a A_a = 0 \) we do not have to symmetrize the unprimed indices and

\[ \varphi_{XY}(x) = \nabla X Y' A_{Y'}Y'(x). \]

One of the possible Lorenz gauges is

\[ A_a(x) = i \int d\Gamma(k) \left( m_a(k) \left( f(k, +)e^{-ik \cdot x} - \bar{f}(k, -)e^{ik \cdot x} \right) - \bar{m}_a(k) \left( \bar{f}(k, +)e^{ik \cdot x} - f(k, -)e^{-ik \cdot x} \right) \right) \]  

where \( \omega_A^1 = 1 \), i.e. \( \omega_A = \omega_A(k) \) is a spin-frame partner of \( \pi_A(k) \). In (14) we have introduced the null vectors

\[ m_a(k) = \omega_A(k) \bar{\pi}_A(k) \]  
\[ \bar{m}_a(k) = \pi_A(k) \omega_A(k) \]  

which, together with

\[ k_a = \pi_A(k) \bar{\pi}_A(k) \]  
\[ \omega_a(k) = \omega_A(k) \bar{\omega}_A(k) \]

form a null tetrad [9].

A change of gauge is in the Fourier domain represented by a shift by a multiple of \( k^a \). The form (14) shows that gauge freedom is related to the nonuniqueness of \( \omega_A(k) \) which can be shifted by a multiple of \( \pi_A(k) \).

**B. Momentum representation**

Consider the momentum-space basis normalized by

\[ \langle p | p' \rangle = (2\pi)^32p_0\delta^{(3)}(p - p') = \delta_T(p, p'), \quad p_0 > 0. \]  

The identity operator in momentum space is \( \int d\Gamma(p) |p\rangle \langle p| \). We can use the following explicit realization of \( |p\rangle = |f_p\rangle \) in terms of distributions

\[ f_p(k) = (2\pi)^32p_0\delta^{(3)}(p - k) = \delta_T(p, k) \]

Since

\[ \int d\Gamma(k) F(k) \delta_T(p, k) = \int d^3k \delta^{(3)}(p - k) F(k) = F(p) \]

the Fourier transform of \( f_p(k) \) is

\[ \hat{f}_p(x) = \int d\Gamma(k) f_p(k) e^{-ik \cdot x} = e^{-ip \cdot x} \]

If 1 is the identity operator occurring at the right-hand-side of CCR \( [a_s, a_{s'}^\dagger] = \delta_{ss'}1 \), we denote

\[ I_k = |k\rangle \langle k| \otimes 1, \]  
\[ I = \int d\Gamma(k) |k\rangle \langle k| \otimes 1. \]
C. Multi-particle conventions

Let \( A \) be an operator \( A: \mathcal{H} \to \mathcal{H} \) where \( \mathcal{H} \) is a one-particle Hilbert space. The multi-particle Hilbert space

\[
\mathcal{H} = \bigotimes_{n=1}^{\infty} \mathcal{H}^n
\]  

(25)

is the Hilbert space of states corresponding to an indefinite number of bosonic particles; \( \otimes_{n}^{\infty} \mathcal{H} \) stands for a space of symmetric states in \( \mathcal{H} \otimes \ldots \otimes \mathcal{H} \). We introduce the following notation for operators defined at the multi-particle level:

\[
\oplus_{\alpha_n} A = \alpha_1 A + \alpha_2 (A \otimes I + I \otimes A) + \alpha_3 (A \otimes I \otimes I + I \otimes A \otimes I + I \otimes I \otimes A) + \ldots
\]  

(26)

Here \( \oplus_{\alpha_n} A: \mathcal{H} \to \mathcal{H} \) \( \alpha_n \) are real or complex parameters, and \( I \) is the identity operator in \( \mathcal{H} \).

The following properties follow directly from the definition

\[
\left[ \oplus_{\alpha_n} A, \oplus_{\beta_n} B \right] = \oplus_{\alpha_n \beta_n} [A, B]
\]  

(27)

\[
e^{\oplus_{\alpha_n} A} = \oplus_{n=1}^{\infty} e^{\alpha_n A} \otimes \ldots \otimes e^{\alpha_n A}
\]  

(28)

\[
e^{\oplus_{\alpha_n} A} \oplus_{\beta_n} B e^{-\oplus_{\alpha_n} A} = \oplus_{\beta_n} e^{A} B e^{-A}
\]  

(29)

Identity operators in \( \mathcal{H} \) and \( \mathcal{H} \) are related by

\[
\hat{I} = \oplus_{n=1}^{\infty} I
\]  

(30)

We will often use the operator

\[
\hat{L}_k = \oplus_{n=1}^{\infty} I_k
\]  

(31)

III. POINCARÉ TRANSFORMATIONS OF CLASSICAL ELECTROMAGNETIC FIELDS

Denote, respectively, by \( \Lambda \) and \( y \) the \( SL(2, C) \) and 4-translation parts of a Poincaré transformation \( \Lambda \) \( (\Lambda, y) \). The spinor representation of the Poincaré group acts in the space of anti-self-dual electromagnetic fields in 4-position representation as follows:

\[
\hat{F}_{ab}(x) \mapsto (T_{\Lambda, y} \hat{F})_{ab}(x) = \Lambda_a e^{\Lambda_b \hat{F}_{cd}(\Lambda^{-1}(x-y))}
\]  

(32)

\[
= \int d\Gamma(k) \varepsilon_{A'B'} \Lambda_A C_{\pi C}(\mathbf{k}) \Lambda_B D_{\pi D}(\mathbf{k}) \left( f(\mathbf{k}, -) e^{-i k \cdot \Lambda^{-1}(x-y)} + \overline{f(\mathbf{k}, +)} e^{i k \cdot \Lambda^{-1}(x-y)} \right)
\]  

(33)

where \( \Lambda^{-1} \mathbf{k} \) is the spacelike part of \( \Lambda^{-1 a} b k_b \). The transformed field

\[
(\Lambda \pi)_A(\mathbf{k}) = \Lambda_A C_{\pi C}(\Lambda^{-1} \mathbf{k})
\]  

(36)

satisfies

\[
k^a = \pi^A(\mathbf{k}) \bar{\pi}^{A'}(\mathbf{k}) = (\Lambda \pi)^A(\mathbf{k}) \overline{\Lambda \pi}^{A'}(\mathbf{k})
\]  

(37)

Now, if \( \omega_A(\mathbf{k}) \) is a spin-frame partner of \( \pi_A(\mathbf{k}) \), i.e. \( \omega_A(\mathbf{k}) \pi^A(\mathbf{k}) = 1 \), one can write

\[
\pi^A(\mathbf{k}) = (\Lambda \pi)^A(\mathbf{k}) \bar{\omega}_A(\mathbf{k}) \overline{\Lambda \pi}^{A'}(\mathbf{k})
\]  

(38)

which shows that \( \pi^A(\mathbf{k}) \) and \( (\Lambda \pi)_A(\mathbf{k}) = \Lambda_A C_{\pi C}(\Lambda^{-1} \mathbf{k}) \) are proportional to each other, the proportionality factor being
\( \lambda(\Lambda, \mathbf{k}) = \tilde{\omega}_A(\mathbf{k}) A^{-1}(\Lambda) \). (39)

The form (34) showed that the gauge freedom is related to shifts

\( \omega_A(\mathbf{k}) \mapsto \omega_A(\mathbf{k}) + \text{scalar} \times \pi_A(\mathbf{k}) \) (40)

which do not affect \( \lambda(\Lambda, \mathbf{k}) \) making it independent of gauge. Using again

\( k^a = |\lambda(\Lambda, \mathbf{k})|^2 \pi^A(\mathbf{k}) \pi^A(\mathbf{k}) = |\lambda(\Lambda, \mathbf{k})|^2 k^a \) (41)

one concludes that \( \lambda(\Lambda, \mathbf{k}) \) is a phase factor

\[ \lambda(\Lambda, \mathbf{k}) = e^{i \Theta(\Lambda, \mathbf{k})} \] (42)

and we find

\[ (T_{\Lambda, y} \tilde{F})_{ab}(x) = \int d\Gamma(\mathbf{k}) e_{AB} \pi_A(\mathbf{k}) \pi_B(\mathbf{k}) e^{-2i \Theta(\Lambda, \mathbf{k})} \left( f(\Lambda^{-1} \mathbf{k}, -) e^{-ik \cdot (x-y)} + f(\Lambda^{-1} \mathbf{k}, +) e^{ik \cdot (x-y)} \right) \] (43)

\[ = \int d\Gamma(\mathbf{k}) e_{ab}(\mathbf{k}) \left( e^{-2i \Theta(\Lambda, \mathbf{k})} e^{ik \cdot y} f(\Lambda^{-1} \mathbf{k}, -) e^{-ik \cdot x} + e^{-2i \Theta(\Lambda, \mathbf{k})} e^{-ik \cdot y} f(\Lambda^{-1} \mathbf{k}, +) e^{ik \cdot x} \right) \] (44)

\[ = \int d\Gamma(\mathbf{k}) e_{ab}(\mathbf{k}) \left( (T_{\Lambda, y} f)(\mathbf{k}, -) e^{-ik \cdot x} + (T_{\Lambda, y} f)(\mathbf{k}, +) e^{ik \cdot x} \right) \] (45)

We have obtained therefore the passive transformation of the classical wave function

\[ f(\mathbf{k}, \pm) \mapsto (T_{\Lambda, y} f)(\mathbf{k}, \pm) = e^{\pm 2i \Theta(\Lambda, \mathbf{k})} e^{ik \cdot y} f(\Lambda^{-1} \mathbf{k}, \pm). \] (46)

which is simply the unitary zero-mass spin-1 representation the Poincaré group. The above derivation clearly shows that the rule (46) is obtained without any particular assumption about the choice of \( f(\mathbf{k}, \pm) \). In particular, the derivation remains valid even if one replaces functions \( f \) by operators, independently of their algebraic properties. The above ‘passive’ viewpoint on the structure of unitary representations is particularly useful if one aims at generalizations of CCR. The passive derivation of all the non-tachyonic unitary representations of the Poincaré group can be found in [13].

**IV. NON-CANONICAL QUANTIZATION**

We follow the strategy described in [1] and [2]. Let \( a_s \) be canonical annihilation operators satisfying CCR \([a_s, a_{s'}^\dagger] = \delta_{ss'} 1\). Define the 1-oscillator non-canonical creation and annihilation operators [15]

\[ a(f)^\dagger = \sum_s \int d\Gamma(k) f(k, s)|k\rangle \langle k| \odot a_s^\dagger \] (47)

\[ = \sum_s \int d\Gamma(k) f(k, s)a(k, s)^\dagger \] (48)

\[ a(f) = \sum_s \int d\Gamma(k) \overline{f(k, s)}|k\rangle \langle k| \odot a_s \] (49)

\[ = \sum_s \int d\Gamma(k) \overline{f(k, s)}a(k, s) \] (50)

satisfying the non-CCR algebra

\[ [a(k, s), a(k', s')^\dagger] = \delta_{ss'} \delta_{\Gamma} (k, k') |k\rangle \langle k| \otimes 1 \] (51)

\[ = \delta_{ss'} \delta_{\Gamma} (k, k') I_k. \] (52)

Taking, in particular, \( f_{p,r}(k, s) = \delta_{rs} \delta_\Gamma (p, k) \) one finds

\[ a(f_{p,r}) = |p\rangle \langle p| \otimes a_r = a(p, r). \] (53)
The one-oscillator quantization is
\[
\hat{F}_{ab}(x) = \varepsilon_{A'B'} \hat{\phi}_{AB}(x)
\]
\[
= \int d\Gamma(k) \varepsilon_{A'B'} \pi_A(k) \pi_B(k)(a(k, -)e^{-ikx} + a(k, +)e^{ikx})
\]
\[
= \int d\Gamma(k) e_{ab}(k)(a(k, -)e^{-ikx} + a(k, +)e^{ikx})
\]
(54)
(55)
(56)

Spinor transformations of \(\hat{F}_{ab}(x)\) lead to the passive transformation
\[
a(k, \pm) \mapsto (T_{\Lambda,y}a)(k, \pm) = e^{\pm i\Theta(\Lambda, k)} e^{ikx} a(\Lambda^{-1}k, \pm).
\]
(57)

The quantization procedure is gauge independent since we work at the gauge-independent level of \(\hat{F}_{ab}(x)\).

Multi-oscillator fields are defined in terms of
\[
a(p, s) = \oplus_{\sqrt{a}} a(p, s)
\]
and
\[
a(f) = \sum_{s} \int d\Gamma(k) f(k, s) a(k, s)\]
\[
a(f)^\dagger = \sum_{s} \int d\Gamma(k) f(k, s)^\dagger a(k, s)^\dagger.
\]
(58)
(59)
(60)

The fact that the coefficients \(\frac{1}{\sqrt{a}}\) are found in the multi-oscillator definition may appear awkward. At least three different formal arguments for such a choice of the multi-particle extension of field operators were given in [1]. Below, in Sec. X, we will show that this very special choice of the non-CCR representation corresponds physically to an ensemble of oscillators uniformly distributed in space.

The non-CCR algebra is
\[
[a(f), a(g)] = \sum_{s} \int d\Gamma(k) f(k, s) a(k, s) \hat{L}_k
\]
(61)

The right-hand-side of the above formula is in the center of the non-CCR algebra, i.e.
\[
[[a(f), a(g)^\dagger], a(h)] = 0
\]
\[
[a(f), a(g)^\dagger], a(h)^\dagger] = 0
\]
(62)
(63)

Useful is also the formula
\[
[a(f, r), a(f', r')^\dagger] = \delta_{r, r'} \delta_{r, r'} \hat{L}_p
\]
(64)
(65)

The presence of \(L_p\) at the right-hand-sides of non-CCR will influence orthogonality properties of multi-photon states, as we shall see later.

At the multi-oscillator level the electromagnetic field tensor operator is
\[
F_{ab}(x) = \int d\Gamma(k) e_{ab}(k)(a(k, -)e^{-ikx} + a(k, +)e^{ikx})
\]
\[
+ \int d\Gamma(k) \pi_{ab}(k)(a(k, -)^\dagger e^{ikx} + a(k, +)^\dagger e^{-ikx}).
\]
(66)
(67)

The four-potential operator is (in our choice of gauge)
\[
A_a(x) = i \int d\Gamma(k) \left( m_a(k)(a(k, +)e^{-ikx} - a(k, -)^\dagger e^{ikx}) + \bar{m}_a(k)(a(k, -)e^{-ikx} - a(k, +)^\dagger e^{ikx}) \right).
\]
(68)

It is well known that field “operators” of the standard theory (let us denote them by \(\hat{A}_a(x)\)) are in fact operator-valued distributions. As a consequence the operator products of the form \(\hat{A}_a(x) \hat{A}_b(y)\) are ill defined and lead to ultraviolet-divergent expressions if \(x = y\). The techniques of dealing with ultraviolet divergences are based on appropriate regularizations of products of distributions taken at “diagonals” in configuration space \(\hat{A}_a(x) \hat{A}_a(x)\). In Sec. VI w shall see that the non-canonically quantized \(\hat{A}_a(x)\) is an operator and there is no difficulty with \(\hat{A}_a(x) \hat{A}_a(x)\).
V. ACTION OF THE POINCARÉ GROUP ON FIELD OPERATORS

We are interested in finding the representation of the group in terms of unitary similarity transformations, i.e.

\[ a(k, \pm) \mapsto e^{\pm 2i\Theta(\Lambda, k)} e^{ik \cdot y} a(\Lambda^{-1} k, \pm) = U^\dagger_{\Lambda, y} a(k, \pm) U_{\Lambda, y} \]

(69)

It is sufficient to find an appropriate representation at the one-oscillator level. Indeed, assume we have found \( U_{\Lambda, y} \) satisfying

\[ e^{\pm 2i\Theta(\Lambda, k)} e^{ik \cdot y} a(\Lambda^{-1} k, \pm) = U^\dagger_{\Lambda, y} a(k, \pm) U_{\Lambda, y}. \]

(70)

Then

\[ U_{\Lambda, y} = \bigoplus_{N=1}^\infty U_{\Lambda, y} \otimes \ldots \otimes U_{\Lambda, y}. \]

(71)

A. Four-translations

The definition of four momentum for a single harmonic oscillator is

\[ P_a = \int d\Gamma(k) k_a |k\rangle \langle k| \otimes h \]

(72)

where

\[ h = \frac{1}{2} \sum_s (a_s^\dagger a_s + a_s a_s^\dagger) = \sum_s h_s. \]

(73)

One immediately verifies that

\[ e^{ip \cdot x} a(k, s) e^{-ip \cdot x} = a(k, s) e^{-ix \cdot k} \]

(74)

\[ e^{ip \cdot x} a(k, s) \dagger e^{-ip \cdot x} = a(k, s) \dagger e^{ix \cdot k} \]

(75)

implying

\[ U_{1, y} = e^{iy \cdot P}. \]

(76)

Consequently, the generator of four-translations corresponding to \( U_{1, y} = e^{iy \cdot P} \) is \( P_a = \oplus_1 P_a \) and

\[ e^{ip \cdot x} a(k, s) e^{-ip \cdot x} = a(k, s) e^{ix \cdot k} \]

(77)

\[ e^{ip \cdot x} a(k, s) \dagger e^{-ip \cdot x} = a(k, s) \dagger e^{-ix \cdot k}. \]

(78)

The \( x \)-dependence of field operators can be introduced via \( P \):

\[ F_{ab}(x) = e^{ix \cdot P} F_{ab}(x) e^{-ix \cdot P}. \]

(79)

B. Rotations and boosts

To find an analogous representation of

\[ a(k, \pm) \mapsto e^{\pm 2i\Theta(\Lambda, k)} a(\Lambda^{-1} k, \pm) = U^\dagger_{\Lambda, 0} a(k, \pm) U_{\Lambda, 0} \]

(80)

we define

\[ U_{\Lambda, 0} = \exp \left( \sum_s 2is \int d\Gamma(k) \Theta(\Lambda, k) |k\rangle \langle k| \otimes \sum_r \int d\Gamma(p) |p, r\rangle \langle \Lambda^{-1} p, r| \otimes 1 \right). \]

(81)

Finally the transformations of the field tensor are

\[ U^\dagger_{\Lambda, 0} F_{ab}(x) U_{\Lambda, 0} = \Lambda_a^c \Lambda_b^d F_{cd}(\Lambda^{-1} x) \]

(82)

\[ U^\dagger_{1, y} F_{ab}(x) U_{1, y} = F_{ab}(x - y) \]

(83)

The zero-energy part of \( P \) can be removed by a unitary transformation leading to a vacuum picture dynamics (cf. [2]). We will describe this in more detail after having discussed the properties of non-canonical states.
VI. STATES AND THEIR POINCARE TRANSFORMATIONS

It is clear that in order to control transformation properties of states it is sufficient to discuss single-oscillator representations. We shall start with single-oscillator states and then extend them to many oscillators.

A. Representation in the one-oscillator sector

The one-oscillator Hilbert space consists of functions $f$ satisfying

$$\sum_{n_+, n_-=0}^{\infty} \int d\Gamma(k) |f(k, n_+, n_-)|^2 < \infty. \quad (84)$$

We will write them in the Dirac notation as

$$|f\rangle = \sum_{n_\pm} \int d\Gamma(k) f(k, n_+, n_-) |k, n_+, n_-\rangle. \quad (85)$$

The representation of the Poincaré group is

$$|f\rangle \mapsto U_{\Lambda, y}|f\rangle = U_{1, y} U_{\Lambda, 0} |f\rangle = \sum_{n_\pm} \int d\Gamma(k) f(k, n_+, n_-) e^{2i(n_+ - n_-) \Theta(\Lambda, k)} e^{ik \cdot y (n_+ + n_- + 1/2)} |k, n_+, n_-\rangle. \quad (86)$$

The latter formula can be written as

$$f(k, n_+, n_-) \mapsto U_{\Lambda, y} f(k, n_+, n_-) = e^{ik \cdot y (n_+ + n_- + 1/2)} e^{2i(n_+ - n_-) \Theta(\Lambda, k)} f(\Lambda^{-1} k, n_+, n_-) \quad (87)$$

or

$$U_{\Lambda, y} f = |U_{\Lambda, y} f\rangle. \quad (88)$$

The form (87) is very similar to the zero-mass spin-1 representation (84), the difference being in the multiplier $n_+ + n_- + 1/2$. One can check by a straightforward calculation that (87) defines a representation of the group.

B. Generators and vacuum picture

Denote by $K_a$ and $L_{ab} + S_{ab}$ the generators of 4-translations and $SL(2, \mathbb{C})$ of the standard zero-mass spin-1 unitary representation of the Poincaré group. $L_{ab}$ denotes the orbital part of the generator. The generators of (87) are then

$$P_a = K_a \otimes h \quad (89)$$

$$J_{ab} = L_{ab} \otimes I + S_{ab} \otimes h_s. \quad (90)$$

$S_{ab} \otimes s$ are matrix elements of $S_{ab}$ (which is a diagonal $p$-dependent matrix). Denote by $\tilde{S}_{ab}$ the generators of the ($1/2, 1/2$) spinor representation of $SL(2, \mathbb{C})$, i.e. $\Lambda = \exp (i\xi^{ab} \tilde{S}_{ab}/2)$. The generators of the unitary representation are defined by

$$P_a f(k, n_+, n_-) = -i \frac{\partial}{\partial y^a} U_{\Lambda, y} f(k, n_+, n_-)|_{\xi, y=0} \quad (91)$$

$$J_{ab} f(k, n_+, n_-) = -i \frac{\partial}{\partial \xi_{ab}} U_{\Lambda, y} f(k, n_+, n_-)|_{\xi, y=0} \quad (92)$$

In what follows we will work in a “vacuum picture”, i.e with unitary transformations

$$f(k, n_+, n_-) \mapsto V_{\Lambda, y} f(k, n_+, n_-) = e^{i(n_+ + n_-) k \cdot y} e^{2i(n_+ - n_-) \Theta(\Lambda, k)} f(\Lambda^{-1} k, n_+, n_-). \quad (93)$$

The transition

$$U_{\Lambda, y} \mapsto V_{\Lambda, y} = W^*_y U_{\Lambda, y} \quad (94)$$

is performed by means of the unitary transformation which commutes with non-CCR creation and annihilation operators.

Let us stress that the fact that we “remove” the zero-energy parts from generators does not mean that energy of vacuum is zero. The vacuum picture is in a sense a choice of representation co-moving with vacuum.
C. Vacuum states

Vacuum states are all the states which are annihilated by all annihilation operators. At the one-oscillator level these are states of the form

\[ |O\rangle = \int d\Gamma(k)O(k)|k,0,0\rangle. \tag{95} \]

Even in the vacuum picture the vacuum states are not Poincaré invariant since

\[ V_{\Lambda,y}O(k) = O(\Lambda^{-1}k) \tag{96} \]

which means they transform as a 4-translation-invariant scalar field. We will often meet the expression \[ Z(k) = |O(k)|^2 \]
describing the probability density of the “zero modes”.

D. Coherent states

An analogue of the standard coherent (or “semiclassical”) state is at the 1-oscillator level

\[ |O_\alpha\rangle = \int d\Gamma(k)O(k)|\alpha(k,+),\alpha(k,-)\rangle \tag{97} \]

where

\[ a_s|\alpha(k,+),\alpha(k,-)\rangle = \alpha(k,s)|\alpha(k,+),\alpha(k,-)\rangle \tag{98} \]

Its explicit form in the basis of eigenstates of the oscillator is

\[ |O_\alpha\rangle = \int d\Gamma(k)O(k) \sum_{n_+,n_-=0}^\infty \alpha(k,+)^{n_+}/\sqrt{n_+!} \alpha(k,-)^{n_-}/\sqrt{n_-!} e^{-|\alpha(k,\pm)|^2/2} |n_+,n_-\rangle \tag{99} \]

\[ = \sum_{n_+,n_-} \int d\Gamma(k)O_\alpha(k,n_+,n_-)|k,n_+,n_-\rangle \tag{100} \]

where

\[ O_\alpha(k,n_+,n_-) = \frac{1}{\sqrt{n_+!n_-!}} O(k)\alpha(k,+)^{n_+}\alpha(k,-)^{n_-} e^{-\sum_\pm |\alpha(k,\pm)|^2/2} \tag{101} \]

The average of the 1-oscillator field operator evaluated in such a coherent state is

\[ \langle O_\alpha|\hat{F}_{ab}(x)|O_\alpha\rangle = \int d\Gamma(k)e_{ab}(k)Z(k)\left(\alpha(k,-)e^{-ik\cdot x} + \alpha(k,+)^*e^{ik\cdot x}\right). \tag{102} \]

The Poincaré transformation of the state implies

\[ \langle O_\alpha|V_{\Lambda,y}^d - \hat{F}_{ab}(x)V_{\Lambda,y}|O_\alpha\rangle = \Lambda_a c^{cd} \Lambda_b ^d \langle O_\alpha|\hat{F}_{cd}(\Lambda^{-1}(x-y))|O_\alpha\rangle \tag{103} \]

The coherent-state wave function transforms by

\[ V_{\Lambda,y}O_\alpha(k,n_+,n_-) = e^{i(n_+n_-)k\cdot y} e^{i(n_+n_-)\Theta(\Lambda,k)}O_\alpha(\Lambda^{-1}k,n_+,n_-) \tag{104} \]

\[ = O(\Lambda^{-1}k) \prod_{s=\pm} e^{in_s k\cdot y} e^{i(n_s-n_-)\Theta(\Lambda,k)} \frac{1}{\sqrt{n_s!}} \alpha(\Lambda^{-1}k,s)^{n_s} e^{-|\alpha(\Lambda^{-1}k,s)|^2/2} \tag{105} \]

\[ = O(\Lambda^{-1}k) \prod_{s=\pm} \frac{1}{\sqrt{n_s!}} \left(T_{\Lambda,y}\alpha(k,s)\right)^{n_s} e^{-|T_{\Lambda,y}\alpha(k,s)|^2/2} \tag{106} \]

where

\[ \alpha(k,s) \rightarrow T_{\Lambda,y}\alpha(k,s) \tag{107} \]
is the spin-1 massless unitary representation \([40]\). Using this result we get again

\[
\langle O_\alpha | V_{\Lambda,y}^\dagger \hat{T}_{\alpha\beta}(x)V_{\Lambda,y} | O_\alpha \rangle = \int d\Gamma(k) e_{ab}(k) Z(\Lambda^{-1}k) \left( T_{\Lambda,y} \alpha(k, -) e^{-ikx} + T_{\Lambda,y} \alpha(k, +) e^{ikx} \right)
\]

(108)

(109)

showing that those somewhat counter-intuitive forms of \(U_{\Lambda,y}\) and \(V_{\Lambda,y}\) are consistent with passive \(T_{\Lambda,y}\) transformations of classical wave functions.

With \(a(\beta)\) and \(a(\beta)^\dagger\) given by \([43]-[50]\) we define the displacement operator

\[
\mathcal{D}(\beta) = e^{a(\beta)^\dagger - a(\beta)}
\]

(110)

\[
= \exp \left( \sum_s \int d\Gamma(k) (\beta(k, s) a(k, s)^\dagger - \overline{\beta(k, s)} a(k, s)) \right)
\]

(111)

\[
= \int d\Gamma(k) |k, s\rangle \langle k, s| \otimes e^{\sum_s (\beta(k, s)a_s^\dagger - \overline{\beta(k, s)} a_s)}
\]

(112)

which performs a shift of the classical wave function

\[
\mathcal{D}(\beta) | O_\alpha \rangle = | O_{\alpha + \beta} \rangle
\]

(113)

and commutes with \(I_k\):

\[
\mathcal{D}(\beta)^\dagger I_k \mathcal{D}(\beta) = I_k.
\]

(114)

Vacuum states are also coherent states corresponding to \(\alpha = 0\).

### E. Multi-oscillator coherent states

Consider a family \(\alpha_N(k, s), N = 1, 2, \ldots\) of functions and the state

\[
| O_N \rangle = \bigoplus_{N=1}^{\infty} \sqrt{p_N} | O_{\alpha_N} \rangle \otimes \ldots \otimes | O_{\alpha_N} \rangle
\]

(115)

where

\[
| O_{\alpha_N} \rangle = \int d\Gamma(k) O(k) | k\rangle | \alpha_N(k, +), \alpha_N(k, -) \rangle
\]

(116)

and \(\sum_{N=1}^{\infty} p_N = 1\). Taking, for example, \(\alpha_N(k, s) = \alpha(k, s)/\sqrt{N}\) we find

\[
\langle O_\alpha | - \hat{F}_{ab}(x) | O_N \rangle = \int d\Gamma(k) e_{ab}(k) Z(k) \left( e^{ikx} + \overline{\alpha(k, +)} e^{ikx} \right) = \langle O_\alpha | - \hat{F}_{ab}(x) | O_N \rangle
\]

(117)

i.e. the same result as in the 1-oscillator case.

A multi-oscillator displacement operator is

\[
\mathcal{D}(\beta) = \bigoplus_{N=1}^{\infty} \mathcal{D}(\beta_N) \otimes \ldots \otimes \mathcal{D}(\beta_N) = e^{\mathcal{A}(\beta)^\dagger - \mathcal{A}(\beta)},
\]

(118)

\(\beta_N(k, s) = \beta(k, s)/\sqrt{N}\), implying

\[
\mathcal{D}(\beta) | O_N \rangle = | O_{\alpha + \beta} \rangle
\]

(119)

\[
\mathcal{D}(\beta)^\dagger \mathcal{A}(p, s) \mathcal{D}(\beta) = \mathcal{A}(p, s) I_p
\]

(120)

\[
\mathcal{D}(\beta)^\dagger I_p \mathcal{D}(\beta) = I_p.
\]

(121)

The fact that \(\alpha_N(k, s) = \alpha(k, s)/\sqrt{N}\) will be shown to be of crucial importance for the question of statistics of excitations of multi-oscillator coherent states. Let us note that a similar property of coherent states was found in \([4]\) when we employed the definition in terms of eigenstates of annihilation operators.
F. Multi-oscillator vacua

Vacuum consists of states with \( n_{\pm} = 0 \), i.e. with all the oscillators in their ground states. Of particular interest, due to its simplicity, is the following vacuum state

\[
|O\rangle = \bigoplus_{N=1}^{\infty} \sqrt{p_N} |O\rangle \otimes \ldots \otimes |O\rangle
\]  

(122)

where

\[
|O\rangle = \int d\Gamma(k) O(k)|k,0\rangle
\]  

(123)

Such a vacuum is simultaneously a particular case of a coherent state with \( \alpha(k, s) = 0 \). Coherent states are related to the vacuum state via the displacement operator

\[
\mathcal{D}(\alpha)|O\rangle = \bigoplus_{N=1}^{\infty} \sqrt{p_N} |O_{\alpha N}\rangle \otimes \ldots \otimes |O_{\alpha N}\rangle
\]  

(124)

\[
= |O_{\alpha}\rangle.
\]  

(125)

G. Normalized 1-photon states

Consider the vector

\[
\mathfrak{a}(f)^\dagger|O\rangle.
\]  

(126)

Choosing the particular form (122) we find

\[
\langle O|\mathfrak{a}(f)\mathfrak{a}(g)^\dagger|O\rangle = \sum_s \int d\Gamma(k) Z(k) f(k, s) g(k, s)
\]  

(127)

\[
= \langle fO|gO\rangle = :\langle f|g\rangle Z.
\]  

(128)

\( fO \) denotes the pointlike product \( fO(k, s) = O(k)f(k, s) \). Since anyway only the modulus \( |O(k)| = Z(k)^{\frac{1}{2}} \) occurs in the above scalar products one can also work with \( f_B(k, s) = Z(k)^{\frac{1}{2}} f(k, s) \). The relation between \( f_B \) and \( f \) resembles the one between the bare and renormalized fields [18]. We believe this is more than just an analogy.

Thinking of bases in the Hilbert space one can take functions \( f_i \) satisfying

\[
\langle f_i|f_j\rangle_Z = \delta_{ij} = \langle f_B i | f_B j \rangle.
\]  

(129)

H. Normalization of multi-photon states

Normalization of multi-photon states is more complicated. In this section we will discuss this point in detail since the argument we give is very characteristic for the non-canonical framework. It will be used in Sec. X to show that in the thermodynamic limit of a large number of oscillators the non-CCR perturbation theory tends to the CCR one but in a version which is automatically regularized. We will also use a similar trick to show that the multi-oscillator coherent states have, again in the thermodynamic limit, Poissonian statistics of excitations.

Denote by \( \sum_\sigma \) the sum over all the permutations of the set \( \{1, \ldots, m\} \).

**Theorem 1.** Consider the vacuum state (122) with \( p_N = 1 \) for some \( N \). Then

\[
\lim_{N \to \infty} \langle O|\mathfrak{a}(f_1) \ldots \mathfrak{a}(f_m)\mathfrak{a}(g_1)^\dagger \ldots \mathfrak{a}(g_m)^\dagger|O\rangle = \sum_\sigma \langle f_1|g_{\sigma(1)}\rangle Z \ldots \langle f_m|g_{\sigma(m)}\rangle Z
\]  

\[
= \sum_\sigma \sum_{s_1 \ldots s_m} \int d\Gamma(k_1) Z(k_1) \ldots d\Gamma(k_m) Z(k_m) f_{\sigma(1)}(k_1, s_1) \ldots f_{\sigma(m)}(k_m, s_m) g_{\sigma(1)}(k_1, s_1) \ldots g_{\sigma(m)}(k_m, s_m)
\]  

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Proof: The scalar product of two general unnormalized multi-photon states is
\[
\langle O | \mathbf{a}(f_1) \cdots \mathbf{a}(f_m) | \mathbf{g}(g_1) \cdots \mathbf{g}(g_m) \rangle | O \rangle
\]
\[
= \sum_{\sigma s_1 \cdots s_m} \int d\Gamma(k_1) \cdots d\Gamma(k_m) f_1(k_1, s_1) \cdots f_m(k_1, s_m) g_1(k_1, s_1) \cdots g_m(k_1, s_m) \langle O | \mathbf{L}_{k_1} \cdots \mathbf{L}_{k_m} | O \rangle
\]
\[
= \sum_{\sigma s_1 \cdots s_m} \int d\Gamma(k_1) \cdots d\Gamma(k_m) f_1(k_1, s_1) \cdots f_m(k_1, s_m) g_1(k_1, s_1) \cdots g_m(k_1, s_m)
\]
\[
\times \frac{1}{N^m} \langle O | \cdots | O \rangle \left( I_{k_1} \otimes \cdots \otimes I + \cdots + I \otimes \cdots \otimes I_{k_1} \right) \cdots \left( I_{k_m} \otimes \cdots \otimes I + \cdots + I \otimes \cdots \otimes I_{k_m} \right) | O \rangle \cdots | O \rangle \quad (130)
\]
Further analysis of (130) can be simplified by the following notation:

\[
1_{k_j} = I_{k_j} \otimes \cdots \otimes I
\]
\[
2_{k_j} = I \otimes I_{k_j} \otimes \cdots \otimes I
\]
\[
\vdots
\]
\[
N_{k_j} = I \otimes \cdots \otimes I_{k_j}
\]
with \( j = 1, \ldots, m; \) the sums-integrals \( \sum_{s_j} \int d\Gamma(k_j) \) are denoted by \( \sum_{k_j} \). Then (130) can be written as

\[
\sum_{\sigma} \sum_{k_1 \cdots k_m} \tilde{f}_1(k_1) \cdots \tilde{f}_m(k_m) g_{\sigma(1)}(k_1) \cdots g_{\sigma(m)}(k_m) \frac{1}{N^m} \sum_{A \cdots Z = 1}^{N} \langle O | \cdots | O \rangle A_{k_1} \cdots Z_{k_m} | O \rangle \cdots | O \rangle (131)
\]
Since \( m \) is fixed and we are interested in the limit \( N \to \infty \) we can assume that \( N > m \). Each element of the sum over \( A_{k_1} \cdots Z_{k_m} \) in (131) can be associated with a unique point \( (A, \ldots, Z) \) in an \( m \)-dimensional lattice embedded in a cube with edges of length \( N \).

Of particular interest are those points of the cube, the coordinates of which are all different. Let us denote the subset of such points by \( C_0 \). For \( (A, \ldots, Z) \in C_0 \)

\[
\langle O | \cdots | O \rangle A_{k_1} \cdots Z_{k_m} | O \rangle \cdots | O \rangle = Z(k_1) \cdots Z(k_m) (132)
\]
no matter what \( N \) one considers and what are the numerical components in \( (A, \ldots, Z) \). (This makes sense only for \( N \geq m \); otherwise \( C_0 \) would be empty). Therefore each element of \( C_0 \) produces an identical contribution (132) to (131). Let us denote the number of points in \( C_0 \) by \( N_0 \).

The sum (131) can be now written as

\[
\sum_{\sigma} \sum_{k_1 \cdots k_m} \tilde{f}_1(k_1) \cdots \tilde{f}_m(k_m) g_{\sigma(1)}(k_1) \cdots g_{\sigma(m)}(k_m) P_0 Z(k_1) \cdots Z(k_m)
\]
\[
+ \sum_{\sigma} \sum_{k_1 \cdots k_m} \tilde{f}_1(k_1) \cdots \tilde{f}_m(k_m) g_{\sigma(1)}(k_1) \cdots g_{\sigma(m)}(k_m) \frac{1}{N^m} \sum_{(A \cdots Z) \notin C_0}^{N} \langle O | \cdots | O \rangle A_{k_1} \cdots Z_{k_m} | O \rangle \cdots | O \rangle (133)
\]
The coefficient \( P_0 = \frac{N_0}{N^m} \) represents a probability of \( C_0 \) in the cube. The elements of the remaining sum over \( (A, \ldots, Z) \notin C_0 \) can be also grouped into classes according to the values of \( \langle O | \cdots | O \rangle A_{k_1} \cdots Z_{k_m} | O \rangle \cdots | O \rangle \). There are \( m-1 \) such different classes, each class has its associated probability \( P_j \), \( 0 < j \leq m-1 \), which will appear in the sum in an analogous role as \( P_0 \).

The proof is completed by the observation that

\[
\lim_{N \to \infty} P_0 = 1, \quad (134)
\]
\[
\lim_{N \to \infty} P_j = 0, \quad 0 < j. \quad (135)
\]
Indeed, the probabilities are unchanged if one rescales the cube to \([0,1]^m\). The probabilities are computed by means of an \( m \)-dimensional uniformly distributed measure. \( N \to \infty \) corresponds to the continuum limit, and in this limit the sets of points of which at least two coordinates are equal are of \( m \)-dimensional measure zero. ■
Comments: (a) The thermodynamic limit is naturally equipped with the scalar product yielding orthogonality relation of the form \( \langle \| \| \rangle \). However, for small \( N \) there will be differences if \( m \) is large. On the other hand if \( N \) is sufficiently large then the values of \( m \) for which the corrections are non-negligible must be also large. But then a classical limit will be justified and the use of non-canonical coherent states should again give the correct description. (b) Concrete values of \( P_j \) for some small \( m \) were given in \[1\]. For \( m = 2: P_0 = 1−1/N, P_1 = 1/N; \) for \( m = 3: P_0 = 1−3/N+2/N^2, P_1 = 3/N−3/N^2, P_2 = 1/N^2 \). In general we do not have to assume that \( p_N = 1 \). If \( P_N \) are general probabilities then the coefficients involve averages. For \( m = 2: P_0 = 1−(1/N), P_1 = (1/N); \) for \( m = 3: P_0 = 1−(3/N)+(2/N^2), P_1 = (3/N)−(3/N^2), P_2 = (1/N^2) \), where \( (1/N) = \sum_N P_N/N \) etc. The normalization in terms of \( \langle | \rangle \) is then obtained under the assumption that all those averages vanish, which can hold only approximately, meaning that the probability \( P_N \) is peaked in a region of large \( N \).

I. States generated by field operators and a first hint indicating absence of ultraviolet divergences

Consider the single-oscillator vector potential operator \( A_a(x) \) which is related to \( \frac{\hat{A}_a(x)}{\sqrt{x}} \) by \( \hat{A}_a(x) = \frac{1}{\sqrt{\pi}} A_a(x) \) and acting with \( A_a(x) \) on a single-oscillator vacuum \( |O\rangle \) define the vector

\[
|A_a(x)\rangle = A_a(x)|O\rangle
\]

(136)

\[
= -i \int d\Gamma(k)e^{ik \cdot x} (m_a(k)a(k,−)^i + \bar{m}_a(k)a(k,+)^i)|O\rangle
\]

(137)

\[
= -i \int d\Gamma(k)e^{ik \cdot x} O_M(k)|k\rangle \left( m_a(k)^i a_−^\dagger|0,0\rangle + \bar{m}_a(k)^i a_+^\dagger|0,0\rangle \right)
\]

(138)

Its multi-particle analogue is

\[
|A_a(x)\rangle = \bigoplus_{N=1}^{\infty} \sqrt{\frac{\sum_{N=1}^{\infty} |A_a(x)\rangle \langle O| \ldots \langle O| \langle O| \ldots \langle O| \langle A_a(x)\rangle}.
\]

(140)

The positive definite scalar product

\[
\langle A_a(y)|(-g^{ab})|A_b(x)\rangle = \langle A_a(y)|(-g^{ab})|A_b(x)\rangle = 2 \int d\Gamma(k)e^{-ik \cdot (x−y)} Z(k)
\]

(141)

shows that there is no ultraviolet divergence at \( x = y \) since \( \int d\Gamma(k)Z(k) = 1 \).

It is easy to understand that the same property will hold also for general states. To see this let us write the single-oscillator field operator as a function of the operator \( \hat{k}_a = \int d\Gamma(k)k_a|k\rangle\langle k| \), i.e.

\[
A_a(x) = im_a(k)\left( e^{-ik \cdot x} \otimes a_− - e^{ik \cdot x} \otimes a_+^\dagger \right) + i\bar{m}_a(k)\left( e^{-ik \cdot x} \otimes a_− - e^{ik \cdot x} \otimes a_+^\dagger \right).
\]

(142)

The operators \( m_a(k) \) and \( \bar{m}_a(k) \) are functions of the operator \( \hat{k} \) and are defined in the standard way via the spectral theorem. Moreover, they are complex combinations of two bounded operators representing directions of transverse polarizations. The remaining operators \( (a_j, a_+^\dagger, \) and \( e^{\pm ik \cdot x} \) are also well behaved. Particularly striking is the fact that the distribution \( \int d\Gamma(k)e^{ik \cdot x} \) is replaced by the unitary operator

\[
e^{ik \cdot x} = \int d\Gamma(k)e^{ik \cdot x}|k\rangle\langle k|.
\]

(143)

The latter property is at the very heart of the regularities encountered in the non-canonical formalism.

Now, it is widely known that configuration-space renormalization of ultraviolet divergences can be reduced to an appropriate treatment of products of field operators on the diagonals \( x = y \). The formula (141) is a strong indication that such divergences may be absent in the non-canonical framework. Actually, the analysis of perturbation theory in nonrelativistic quantum optics given in \[1\] and further elaborated in Sec. X of the present paper, shows that \( Z(k) \) occurs in exactly those places where ultraviolet form-factors are expected to appear. The same property will hold for non-canonical quantized fermionic fields.
VII. STATISTICS OF EXCITATIONS

It is an experimental fact that laser beams produce Poissonian statistics of photocounts. At the theoretical level of canonical quantum optics the Poisson distribution follows trivially from the form of canonical coherent states. In the non-canonical case the exact Poisson statistics is characteristic of the single-oscillator \((N = 1)\) sector. For \(1 < N < \infty\) the statistics of excitations is non-Poissonian. At the other extreme is the thermodynamic limit for multi-oscillator states. In what follows we will show that in the limit \(N \to \infty\) one recovers the same Poisson distribution as for \(N = 1\). This, at a first glance unexpected, result justifying our definitions in terms of displacement operators is a consequence of certain classical universality properties of the Poisson distribution.

We will also return to the question of thermal states and the Planck formula. In [1] it was argued that non-CCR quantization implies deviations from the black-body law. However, a consistent interpretation of the field in terms of the thermodynamic limit shows that no deviations should be expected.

A. Multi-oscillator coherent states

To study the thermodynamic limit of multi-oscillator coherent states we simplify the discussion by taking an exactly \(N\)-oscillator coherent state \(\ket{\Omega_N} = |O_{\alpha_N} \rangle \otimes \cdots \otimes |O_{\alpha_N} \rangle\) \((N \gg 1\) is fixed and \(p_N = 1\)), i.e.

\[
\ket{\Omega_N} = |O_{\alpha_N} \rangle \otimes \cdots \otimes |O_{\alpha_N} \rangle
\]

where

\[
|O_{\alpha_N} \rangle = \int d\Gamma(k)\mathcal{O}(k)|k, s\rangle|\alpha(k, +)\rangle/\sqrt{N}|\alpha(k, -)\rangle/\sqrt{N}.
\]

The average number of excitations in this state is

\[
\langle n \rangle = \sum_{s} \int d\Gamma(k)Z(k)|\alpha(k, s)|^2.
\]

The simplest case is the one where \(\alpha(k, s) = \alpha = \text{const}\). Then \(\langle n \rangle = |\alpha|^2\) and the statistics of excitations of single-oscillator coherent states \(\ket{O_{\alpha_N}}\) is Poissonian with the distribution \(p_n = e^{-|\alpha N|^2}/|\alpha N|^{2n}/n!\).

\(m\) excitations distributed in the ensemble of \(N\) oscillators can be represented by the ordered \(m\)-tuple \((j_1, \ldots, j_m)\), \(1 \leq j_1 \leq \ldots \leq j_m \leq N\). For example, for \(m = 10\), \(N = 12\), the point \((2, 2, 2, 5, 5, 7, 7, 7, 11, 11)\) represents 10 excitations distributed in the ensemble of 12 oscillators as follows: 3 excitations in 2nd oscillator, 2 in the 5th one, 3 in the 7th, and 2 in the 11th. Such points form a subset of the cube \([0, N]^m\), the interior of the set corresponding to points whose all the indices are different. The latter means that the interior represents situations where there are \(m\) oscillators excited, and each of them is in the first excited state. The boundary of this set consists of points representing at least one oscillator in a higher excited state. Probabilities of events represented by points with the same numbers of repeated indices must be identical due to symmetries. Intuitively, the Poissonian statistics of the thermodynamic limit follows from the fact that the probability of finding a point belonging to the boundary tends to zero as \(N\) increases. The statistics is dominated by Bernoulli-type processes with probabilities related to the two lowest energy levels of a single oscillator in a coherent state.

To make the argument more formal we introduce the following notation:

\[
X^{(N)}_{N_1 \ldots N_k} = \left\{ x \in \mathbb{N}^m : m \geq 1, x = (j_1, \ldots, j_m), 1 \leq j_1 \leq \ldots \leq j_m \leq N \right\}
\]

\[
X^{(N)}_{N_1 \ldots N_k} = \left\{ x \in X^{(N)}_{N_1 \ldots N_k} : x = \underbrace{(i_1, \ldots, i_1)}_{n_1}, \ldots, \underbrace{(i_k, \ldots, i_k)}_{n_k}, i_1 < \ldots < i_k \right\}
\]

\[
Y_m = \bigcup_{(n_1 \ldots n_k) \neq (1 \ldots 1)} X^{(N)}_{n_1 \ldots n_k}
\]

If we add a single-element set \(X^{(N)}_{0 \ldots 0} \) containing the event representing \(N\) oscillators in their ground states we can represent the set of all the events by the disjoint sum

\[
X^{(N)} = \bigcup_{m = 0}^{\infty} X^{(N)}_{m}
\]
The probability of finding the partition \( m = n_1 + \ldots + n_k \) is
\[
P(X_{n_1 \ldots n_k}^{(N)}) = N_{n_1 \ldots n_k} p_1 \cdots p_{n_k} p_0^{N-k}
\]
\[
= N_{n_1 \ldots n_k} \frac{e^{-N|\alpha|^2} |\alpha_N|^{2m}}{n_1! \ldots n_k!}
\]
where \( N_{n_1 \ldots n_k} \) is the number of elements of \( X_{n_1 \ldots n_k}^{(N)} \subset X_m^{(N)} \). The probability that \( m \) excitations are found is
\[
P(X_m^{(N)}) = \sum_{n_1 \ldots n_k} P(X_{n_1 \ldots n_k}^{(N)}),
\]
the sum being over all the partitions of \( m \). Denote by \( P(Y_m^{(N)}|X_m^{(N)}) \) the conditional probability of finding at least one oscillator in the 2nd or higher excited state under the condition that the sum of excitations is \( m > 1 \). We first prove the following

**Lemma 1.**
\[
\lim_{N \to \infty} P(Y_m^{(N)}|X_m^{(N)}) = 0.
\]

**Proof:** Since \( Y_m^{(N)} \cap X_m^{(N)} = Y_m^{(N)} \) one finds
\[
P(Y_m^{(N)}|X_m^{(N)}) = \frac{\sum_{(n_1 \ldots n_k) \neq (1 \ldots 1)} P(X_{n_1 \ldots n_k}^{(N)})}{\sum_{n_1 \ldots n_k} P(X_{n_1 \ldots n_k}^{(N)})}
\]
\[
= \left[ 1 + N_{1 \ldots 1} \left( \sum_{(n_1 \ldots n_k) \neq (1 \ldots 1)} \frac{N_{n_1 \ldots n_k}}{n_1! \ldots n_k!} \right)^{-1} \right]^{-1}
\]
\[
< \left[ 1 + N_{1 \ldots 1} \left( \sum_{(n_1 \ldots n_k) \neq (1 \ldots 1)} N_{n_1 \ldots n_k} \right)^{-1} \right]^{-1}
\]
\[
\lim_{N \to \infty} \frac{\sum_{(n_1 \ldots n_k) \neq (1 \ldots 1)} N_{n_1 \ldots n_k}}{N_{1 \ldots 1}} = 0
\]
on the basis of the geometric argument we gave in the previous section. This ends the proof.

The main result of this section is the following version of the well known Poisson theorem:

**Theorem 2.** Assume that \( \alpha(k, s) = \alpha = \text{const.} \) Then
\[
\lim_{N \to \infty} P(X_m^{(N)}) = e^{-|\alpha|^2 |\alpha|^{2m}} \frac{m!}{N!}.
\]

**Proof:** As an immediate consequence of the lemma we find
\[
\lim_{N \to \infty} P(X_m^{(N)}) = \lim_{N \to \infty} P(X_m^{(N)} - Y_m^{(N)})
\]
which means that in the thermodynamic limit we can treat excitations of the oscillators to 2nd and higher excited levels as events whose probability is zero. The probabilities of ground and first excited states follow from the single-oscillator Poisson distributions but conditioned by the fact that only the lowest two levels are taken into account.

We thus arrive at the standard Poisson process with
\[
P_N = \frac{p_1}{p_0 + p_1} = \frac{|\alpha_N|^2}{1 + |\alpha_N|^2} = \frac{|\alpha|^2 / N}{1 + |\alpha|^2 / N}
\]
and \( \lim_{N \to \infty} NP_N = |\alpha|^2 \).
B. Thermal states

A single-oscillator free-field Hamiltonian $H$ has the usual eigenvalues

$$E(\omega, n) = \hbar \omega \left(n + \frac{1}{2}\right).$$

The eigenvalues of the free-field Hamiltonian $H$ at the multi-oscillator level are sums of the single-oscillator ones. In [1] it was assumed that the Boltzmann-Gibbs distribution of thermal radiation should be constructed in terms of $H$. Let us note, however, that such a construction makes use of $H$ as if it was a Hamiltonian of a single element of a statistical ensemble. The discussion of the thermodynamic limit we have given above, as well as the results of [2], suggest that $H$ is the Hamiltonian of the entire ensemble of systems described by $H$, and it is $H$ and not $H$ which should be used in the Boltzmann-Gibbs distribution. Then, of course, the result will be the standard one and no deviations from the Planck formula will occur.

VIII. COMMUTATORS OF 4-POTENTIALS AND LOCALITY

A straightforward calculation shows that the multi-oscillator vector potential operator satisfies, for any space-time points $x, y, z$, the commutators

$$[\mathbf{A}_a(x), \mathbf{A}_b(y)] = \int d\Gamma(k) \mathcal{L}_k \left(m_a(k)\bar{m}_b(k)e^{ik(y-x)} - \bar{m}_a(k)m_b(k)e^{ik(x-y)}\right)$$

$$+ \int d\Gamma(k) \mathcal{L}_k \left(\bar{m}_a(k)m_b(k)e^{ik(y-x)} - m_a(k)\bar{m}_b(k)e^{ik(x-y)}\right)$$

$$[[\mathbf{A}_a(x), \mathbf{A}_b(y)], \mathbf{A}_c(z)] = 0.$$ (158)

To obtain more insight as to the meaning of the commutator (158) consider its coherent-state average evaluated in a state of the form (157):$\langle Q_a, [A_a(x), A_b(y)] | Q_a \rangle = \int d\Gamma(k) Z(k) \left(m_a(k)\bar{m}_b(k)e^{ik(y-x)} - \bar{m}_a(k)m_b(k)e^{ik(x-y)}\right)$

$$+ \int d\Gamma(k) Z(k) \left(\bar{m}_a(k)m_b(k)e^{ik(y-x)} - m_a(k)\bar{m}_b(k)e^{ik(x-y)}\right).$$ (160)

The Minkowski-space metric tensor can be decomposed [9] in terms of null tetrads as follows

$$g_{ab} = k_a\omega_b + \omega_a k_b - m_a\bar{m}_b - \bar{m}_a m_b.$$ (162)

With the help of this identity we can write

$$\langle Q_a, [A_a(x), A_b(y)] | Q_a \rangle = \int d\Gamma(k) Z(k) \left(k_a\omega_b(k) + k_b\omega_a(k)\right) \left(e^{ik(y-x)} - e^{ik(x-y)}\right)$$

$$+ g_{ab} \int d\Gamma(k) Z(k) \left(e^{ik(y-x)} - e^{ik(x-y)}\right)$$

It is known that terms such as the first integral vanish if the potential couples to a conserved current. The same property guarantees gauge independence of the formalism. Therefore we can concentrate only on the explicitly gauge independent term proportional to $g_{ab}$. Denote

$$D_Z(x) = i \int d\Gamma(k) Z(k) \left(e^{-ikx} - e^{ikx}\right)$$

For $Z(k) = \text{const} = Z$ we get $D_Z(x)$ proportional to the Jordan-Pauli function,

$$D_Z(x) = Z D(x)$$ (165)

which vanishes for spacelike $x$. However, the choice of constant $Z(k)$ is excluded by the requirement of square-integrability of $O$. Therefore the requirement that the vacuum state be square-integrable seems to introduce some kind of nonlocality into the formalism.
There are two possibilities one can contemplate. First of all, one can perform the calculations with arbitrary $O$ and then perform a renormalization (we have seen that such a step is necessary even in the nonrelativistic case). After the renormalization we can go to the “flat” pointwise limit $Z(k) \to 0, \| O \| = 1$, corresponding to the uniform distribution of all the frequencies. Second, performing the calculations in a preferred frame we can consider equal-time commutation relations

$$\langle O_a, [A_a(t, x), A_b(t, y)]_O \rangle = \int d\Gamma(k)Z(k)(k_a \omega_b(k) + k_b \omega_a(k)) \left( e^{i\vec{k} \cdot (\vec{y}-\vec{x})} - e^{i\vec{k} \cdot (\vec{x}-\vec{y})} \right) + g_{ab} \int d\Gamma(k)Z(k) \left( e^{i\vec{k} \cdot (\vec{x}-\vec{y})} - e^{i\vec{k} \cdot (\vec{y}-\vec{x})} \right)$$

(166)

The last term will vanish if

$$Z(k) = Z(-k)$$

(167)
i.e. if the vacuum is 3-inversion invariant. This can hold, however only in one reference frame unless $O$ is constant, which we exclude.

One can conclude that non-canonically quantized electrodynamics is not a local quantum field theory, at least in the strict standard sense. This is not very surprising since the presence of $Z(k)$ in the integrals introduces some kinds of effective extended structures, a consequence of nontrivial structures of non-canonical vacua. The issue requires further studies. In particular, it is important to understand an influence of the thermodynamic limit of effective extended structures, a consequence of non-canonical vacua. The issue requires further studies. In particular, it is important to understand an influence of the thermodynamic limit $N \to \infty$ on locality problems in the context of relativistic perturbation theory.

There is an intriguing analogy between the kind of non-locality we have obtained and the one encountered in field theory in non-commutative space-time [6-8].

IX. RADIATION FIELDS ASSOCIATED WITH CLASSICAL CURRENTS

The problem of radiation fields is interesting for several reasons. First of all, the radiation fields satisfy homogeneous Maxwell equations so that the theory we have developed can be directly applied. Second, this is one of the simplest ways of addressing the question of infrared divergences within the non-canonical framework.

It is widely known [223] that in the canonical theory the scattering matrix corresponding to radiation fields produced by a classical transversal current is given, up to a phase, by a coherent-state displacement operator $e^{-i\int d^4y J(y) \cdot A_{\alpha}(y)}$. One of the consequences of such an approach is the Poissonian statistics of photons emitted by classical currents. An unwanted by-product of the construction is the infrared catastrophe.

Starting with the same $S$ matrix but expressed in terms of non-canonical “in” fields we obtain the non-canonical displacement operator. Photon statistics is Poissonian in the thermodynamic limit but the infrared catastrophe is automatically eliminated.

A. Classical radiation field

Let us assume that we deal with a classical transversal current $J_a(x)$ whose Fourier transform is $\tilde{J}_a(k) = \int d^4xe^{i\vec{k} \cdot \vec{x}} J_a(x)$. Transversality means here that

$$\tilde{J}_a(|k|, \vec{k}) = m_a(k)\tilde{J}_{1\alpha}(|k|, \vec{k}) + m_a(k)\tilde{J}_{0\alpha}(|k|, \vec{k}).$$

(168)

Formally, a solution of Maxwell equations

$$\Box A_a(x) = J_a(x)I$$

(169)
can be written as

$$A_a(x) = A_{a\text{in}}(x) + \int d^4y D_{\text{ret}}(x-y)J_a(y)I$$

(170)

$$= A_{a\text{out}}(x) + \int d^4y D_{\text{adv}}(x-y)J_a(y)I$$

(171)

Here $A_{a\text{in}}$ and $A_{a\text{out}}$ are solutions of homogeneous equations. $D_{\text{ret}}$ and $D_{\text{adv}}$ are the retarded and advanced Green functions whose difference is the Jordan-Pauli function.
\[ D(x) = i \int d\Gamma(k)(e^{-ik\cdot x} - e^{ik\cdot x}). \] (172)

The 4-potential of the radiation field is

\[ A_{\text{rad}}(x) = A_{\text{out}}(x) - A_{\text{in}}(x) = \int d^4y D(x - y)J_a(y) \] (173)

and leads to the field spinors

\[ \zeta_{XY}^{\text{rad}}(x) = \int d\Gamma(k) \left( \pi(X \pi^Y \bar{J}Y)^{(k)}(k)e^{-ik\cdot x} + \pi(X \pi^Y \bar{J}Y)^{(k)}(k)e^{ik\cdot x} \right) \] (174)

\[ \zeta_{X'Y'}^{\text{rad}}(x) = \int d\Gamma(k) \left( \pi(X' \pi^{Y'} \bar{J}'Y')^{(k)}(k)e^{ik\cdot x} + \pi(X' \pi^{Y'} \bar{J}'Y')^{(k)}(k)e^{-ik\cdot x} \right) \] (175)

Comparing these formulas with expressions (13) and (14) valid for all solutions of free Maxwell equations one finds

\[ f(k, +) = -\bar{m}^a(k)\bar{J}_a(|k|, k) = \bar{J}_0^+(|k|, k) = j(k, +) \] (176)

\[ f(k, -) = -m^a(k)\bar{J}_a(|k|, k) = \bar{J}_0^-(|k|, k) = j(k, -) \] (177)

\[ \bar{a}(k, s)_{\text{out}} = \bar{a}(k, s)_{\text{in}} + j(k, s) \] (178)

**B. Non-canonical radiation field**

Formula (178) is analogous to the one from the canonical theory. It is clear that although \( \bar{a}(k, s)_{\text{in}} \) and \( \bar{a}(k, s)_{\text{out}} \) cannot be simultaneously of the form given by (53) and (58), they do satisfy the non-CCR algebra (55). In spite of this the result (178) is not very satisfactory. Indeed, one expects that the scattering matrix describing a non-canonical quantum field interacting with a classical current is

\[ S = e^{i\phi}e^{-i \int d^4y J(y) A_a(y)} \] (179)

with some phase \( \phi \). Then

\[ A_{\text{out}}(x) = S^\dagger A_{\text{in}}(x)S \] (180)

\[ = A_{\text{in}}(x) - i \int d^4y J_b(y)[A_{\text{in}}(x), A_{\text{in}}(y)]. \] (181)

Employing (158), (176), (177) one can write

\[ A_{\text{rad}}(x) = i \int d^4y J_b(y) \int d\Gamma(k) \mathcal{L}_k \left( (e^{ik\cdot (x+y)}m_{a\bar{m}}b - e^{-ik\cdot (x-y)}m_{a\bar{m}}b) + (e^{ik\cdot (x+y)}m_{\bar{a}m}b - e^{-ik\cdot (x-y)}m_{\bar{a}m}b) \right) \] (182)

\[ = i \int d\Gamma(k) \mathcal{L}_k \left( m_a(e^{-ik\cdot x}j(k, +) - e^{ik\cdot x}j(k, -)) + \bar{m}_a(e^{-ik\cdot x}j(k, -) - e^{ik\cdot x}j(k, +)) \right), \] (183)

where \( m_a = m_a(k) \), and

\[ \bar{a}(k, s)_{\text{out}} = \bar{a}(k, s)_{\text{in}} + j(k, s) \mathcal{L}_k \] (184)

\[ = \mathcal{D}(j)^\dagger \bar{a}(k, s)_{\text{in}} \mathcal{D}(j). \] (185)

Consequently, the \( S \) matrix is in the non-canonical theory proportional to the *non-canonical* displacement operator

\[ S = e^{i\phi} \mathcal{D}(j). \] (186)

This fact will be shown to eliminate the infrared catastrophe.
C. Propagators

Evaluating the average of (181) in a coherent state $|\alpha\rangle$ one finds

$$\langle O_\alpha | A_{\text{rad}}(x) | O_\alpha \rangle = \int d^4y D_Z(x - y) J_a(y) + \text{gauge term.}$$  \hspace{1cm} (187)

The irrelevant gauge term is a remainder of the first part of (163). As expected the radiation field does not depend on what $\alpha$ one takes in the coherent state, but does depend on the vacuum structure. The presence of the regularized function $D_Z(x - y)$ instead of $D(x - y)$ implies that the radiation signal propagates in a neighborhood of the light cone. Any deviation from $c$ in velocity of signal propagation can be regarded as an indication of a non-constant vacuum wave function $O(k)$. To have a feel of the scale of the nonlocality assume that $O(k)$ is constant up to, roughly, the Planck scale. Corrections of the order of the classical electron radius can be seen in the distance travelled by light if the photon travels 100 light years. Moreover, even in the orthodox canonical quantum electrodynamics a detailed analysis of signal propagation leads to small deviations from velocity of light, especially at small distances \[20\]. It may be difficult to experimentally distinguish between the two effects. A similar effect was predicted for Maxwell fields in non-commutative space-time \[6,7\].

Using (162) one can rewrite (158) as

$$[A_a(x), A_b(y)] = g_{ab} \int d\Gamma(k) I_k(e^{ik \cdot (x - y)} - e^{ik \cdot (y - x)}) + \ldots$$  \hspace{1cm} (188)

where the dots stand for all the terms which are gauge dependent and do not contribute to physically meaningful quantities. We can therefore identify

$$D(x) = i \int d\Gamma(k) D_z \left(e^{-ik \cdot x} - e^{ik \cdot x}\right)$$  \hspace{1cm} (189)

as the operator responsible for the appearance of the smeared out Jordan-Pauli function $D_z$ in the coherent-state average (187). $D(x)$ is a translation-invariant scalar-field operator solution of the d’Alembert equation, i.e.

$$\Box D(x) = 0,$$  \hspace{1cm} (190)

$$V_{\Lambda, y} D(x) V_{\Lambda, y} = D(x).$$  \hspace{1cm} (191)

The operator analogues of retarded and advanced Green functions are

$$D_{\text{ret}}(x) = \Theta(x_0) D(x),$$  \hspace{1cm} (192)

$$D_{\text{adv}}(x) = -\Theta(-x_0) D(x),$$  \hspace{1cm} (193)

$$D(x) = D_{\text{ret}}(x) - D_{\text{adv}}(x).$$  \hspace{1cm} (194)

Eq. (190) implies that the operators

$$\int d^4 y D_{\text{ret}}(x - y) J_a(y)$$  \hspace{1cm} (195)

and

$$\int d^4 y D_{\text{adv}}(x - y) J_a(y)$$  \hspace{1cm} (196)

differ by at most a solution of the homogeneous equation

$$\Box A_a(x) = 0.$$  \hspace{1cm} (197)

(190) implies also that one can define

$$\delta(x) := \Box D_{\text{adv}}(x) = \Box D_{\text{ret}}(x).$$  \hspace{1cm} (198)

It follows that having a solution $A_{\text{in}}(x)$ of (197) one can define another solution $A_{\text{out}}(x)$ of (197) by means of
\[ A_a(x) = A_{\text{in}}(x) + \int d^4 y D_{\text{ret}}(x - y) J_a(y) \]
\[ = A_{\text{out}}(x) + \int d^4 y D_{\text{adv}}(x - y) J_a(y), \]

simultaneously guaranteeing that the correct \( S \)-matrix conditions \([179], [180]\) are fulfilled up to, perhaps, a gauge transformation. \( A_a(x) \) is a solution of

\[ \Box A_a(x) = J_a(x) \] \hspace{1cm} (201)

where

\[ J_a(x) = \int d^4 y \delta(x - y) J_a(y). \] \hspace{1cm} (202)

**D. The problem of infrared catastrophe**

To close this part of the discussion let us consider the issue of infrared catastrophe. We have to compute an average number of photons in the state \( D(j)|O \rangle \). The number-of-photons operator is

\[ \underline{n} = \oplus_1 (1 \otimes \sum_s a_s^\dagger a_s). \] \hspace{1cm} (203)

The 1 in the above formula is the identity in the \( k \) space and \( a_s \) satisfy CCR. The average reads

\[ \langle n \rangle = \langle Q_j | \underline{n} | Q_j \rangle = \sum_s \int d\Gamma(k) Z(k)|j(k, s)|^2. \] \hspace{1cm} (204)

The four-momentum of the radiation field is

\[ \langle P_a \rangle = \langle Q_j | \underline{P}_a | Q_j \rangle = \sum_s \int d\Gamma(k) k_a Z(k)|j(k, s)|^2. \] \hspace{1cm} (205)

\( O(k) \) belongs to a carrier space of an appropriate unitary representation of the Poincaré group. As such this is a differentiable function vanishing at the origin \( k = 0 \) of the light cone. This is a consequence of the fact that the cases \( k = 0 \) and \( k \neq 0, k^2 = 0 \) correspond to representations of the Poincaré group induced from \( SL(2, C) \) and \( E(2) \), respectively (for another justification see [11]).

Hence, the regularization of the infrared divergence is implied by relativistic properties of the field. It is quite remarkable that all the divergences are regularized automatically by the same property of the formalism: The nontrivial structure of the vacuum state. In the case of ultraviolet and vacuum divergences the regularization is a consequence of square integrability of \( O \).

**X. SELECTED QUESTIONS OF PHYSICAL INTERPRETATION**

In this section we want to address three problems which allow to better understand on physical grounds the formal structures characteristic of our choice of field operators. First we will discuss in a simplified model (scalar field, discrete spectrum of frequencies) the relationship between the non-CCR approach and the canonical one. We will perform this in two versions: One using an exact result, and the second one based on perturbation theory. Making use of the same model we will interpret the factor \( 1/\sqrt{N} \) occurring in the definition of multi-oscillator field operators, and finally we will look more closely at the structure of the state space.
A. Canonical limit and the second hint indicating absence of ultraviolet divergences: Exact example

We now want to make a link between the non-CCR theory with nonunique vacuum and the one based on CCR and the unique vacuum. One may regard it as a test of experimental consequences of the non-canonical approach. The analysis of Poisson statistics of coherent states indicates that the link may be provided by the thermodynamic limit $N \to \infty$. A similar conclusion follows from the analysis of orthogonality properties of multi-photon states.

In what follows we shall discuss this in more detail on a simplified model of a non-CCR scalar field in dipole approximation interacting with a two-level atom located at $x = 0$. It is known that without a regularization of the coupling parameters by $g_k \to g_k Z_k^2$, with $Z_k^2 \to 0$ for $|k| \to \infty$, the model is highly ultraviolet divergent (even more than its relativistic counterpart). On the other hand the vast majority of experimental tests of quantum radiation fields is based on this type of calculation.

We will compare predictions of two models: The CCR one with regularized coupling constants, and the non-CCR quantized field without regularization of the Hamiltonian.

Consider the interaction-picture Hamiltonian

$$H(t) = \frac{i \hbar \alpha}{2} \sum_k \sqrt{\frac{\hbar}{2 \omega_k}} (R_+ a_k e^{-i\Delta_k t} - R_- a_k^\dagger e^{i\Delta_k t}).$$

(206)

The coupling constant $\alpha$ depends linearly on the ratio $e_0/\sqrt{V}$ (charge $e_0$ and quantization volume $V$). We assume the CCR algebra $[a_k, a_k^\dagger] = \delta_{kk'}$ with the unique vacuum $|0_{\text{CCR}}\rangle$, and $R_\pm = |\pm\rangle\langle\mp|$.

Of particular interest is the vacuum-to-vacuum amplitude

$$F(t) = \langle O | U(t) | O \rangle$$

(207)

which can be exactly computed in both models. In the CCR case, taking $|O\rangle = |0_{\text{CCR}}\rangle + \rangle$, in order to compute the amplitude the Hamiltonian $H(t)$ has to be replaced by its regularized version

$$H_{\text{reg}}(t) = \frac{i \hbar \alpha}{2} \sum_k Z_k^\dagger \sqrt{\frac{\hbar}{2 \omega_k}} (R_+ a_k e^{-i\Delta_k t} - R_- a_k^\dagger e^{i\Delta_k t}).$$

(208)

Evaluating the vacuum average of the Dyson expansion one finds the Volterra-type integral equation

$$F(t) = 1 - C \int_0^t dt_1 \int_0^{t_1} dt_2 f_Z(t_1 - t_2) F(t_2)$$

(209)

where $C = \alpha^2 \hbar/8$ and

$$f_Z(\tau) = \sum_k Z_k e^{-i\Delta_k \tau}/\omega_k.$$

(210)

The amplitude expressed in terms of the Laplace transform, with integration along a contour located to the right of all the poles of the integrand, is

$$F(t) = \frac{1}{2\pi i} \int_\Gamma dz \frac{\exp(z t)}{z + C \sum_k Z_k z \frac{1}{\omega_k} \frac{1}{i\Delta_k + z}}.$$

(211)

Let us now take the interaction-picture Hamiltonian

$$H(t) = \frac{i \hbar \alpha}{2} \sum_k \sqrt{\frac{\hbar}{2 \omega_k}} (R_+ a_k e^{-i\Delta_k t} - R_- a_k^\dagger e^{i\Delta_k t}),$$

(212)

with the non-CCR operators $[a_k, a_k^\dagger] = \delta_{kk'} \mathbf{L}_k$, $\sum_k \mathbf{L}_k = \mathbf{L}$, and the non-canonical vacuum

$$|O\rangle = \bigotimes_{N=1}^{\infty} \sqrt{p_N} \sum_{k_1, k_2} \ldots O_{k_N} |k_1 \ldots k_N\rangle,$$

(213)

which is of the form $|\text{122}\rangle$. Let us note that now we do not introduce any regularization of the coupling parameters.
Setting $Z_k = |O_k|^2$ and denoting the non-canonical amplitude by $F'$ one can write

$$F'(t) = \sum_{N=1}^{\infty} p_N \sum_{k_1, \ldots, k_N} Z_{k_1} \cdots Z_{k_N} \langle k_1 \ldots k_N | U(t) | k_1 \ldots k_N \rangle$$

$$= \sum_{N=1}^{\infty} p_N \sum_{k_1, \ldots, k_N} Z_{k_1} \cdots Z_{k_N} F(t)_{k_1 \ldots k_N}. \tag{215}$$

The Dyson expansion leads to a similar Volterra-type equation as before

$$F(t)_{k_1 \ldots k_N} = 1 - \frac{C}{N} \int_0^t \int_0^{t_1} dt_1 dt_2 f(t_1 - t_2)_{k_1 \ldots k_N} F(t_2)_{k_1 \ldots k_N} \tag{216}$$

with

$$f(\tau)_{k_1 \ldots k_N} = \sum_{j=1}^{N} \frac{e^{-i\Delta\omega_j \tau}}{\omega_j}. \tag{217}$$

$\omega_j = \omega_j = |k_j|$, and the same constant $C$. The solution is

$$F(t)_{k_1 \ldots k_N} = \frac{1}{2\pi i} \int_{\Gamma} dz \frac{e^{zt}}{z + \frac{C}{N} \sum_{j=1}^{N} \frac{1}{\omega_j i\Delta\omega_j + z}} \tag{218}$$

where $\Gamma$ is any contour parallel to the imaginary axis and to the right of all the poles of the integrand.

The solution (218) looks very similar to (211) but there are also evident differences. For example the sum in the denominator of the integrand in (218) is finite, involves no regularizations, and $C$ is replaced by $C/N$.

To see the link between $F(t)$ and $F'(t)$ let us assume that there are exactly $N$ oscillators (i.e. $p_N = 1$ for some $N$ and zero otherwise) and consider the thermodynamic limit $N \rightarrow \infty$. This is the same type of reasoning we have employed in discussion of the Poisson statistics of coherent states and normalization of multi-photon states.

**Theorem 3.** Under the above assumptions

$$\lim_{N \rightarrow \infty} F'(t) = F(t). \tag{219}$$

*Proof:* The poles of the integrand in (218) are equal to the eigenvalues of

$$\begin{pmatrix}
0 & -\sqrt{\frac{C}{\omega_1 N}} & \cdots & -\sqrt{\frac{C}{\omega_N N}} \\
\sqrt{\frac{C}{\omega_1 N}} & i\Delta\omega_1 & 0 & \cdots & 0 \\
\vdots & & & & \vdots \\
\sqrt{\frac{C}{\omega_N N}} & 0 & \cdots & i\Delta\omega_N
\end{pmatrix} \tag{220}$$

and, hence, are purely imaginary. The parameters $C$, $N$ and $\omega_1, \ldots, \omega_N$ in (218) are fixed, integration is over any contour localized to the right of all the poles, and the poles are imaginary. We assume that the spectrum of $\omega$s contains a minimal $\omega > 0$ (as in cavity). It follows that the contour can be shifted sufficiently far to the right so that the inequality

$$\left| \frac{C}{N} \sum_{j=1}^{N} \frac{1}{\omega_j z i\Delta\omega_j + z} \right| < 1 \tag{221}$$

is satisfied for any choice of $\omega_1, \ldots, \omega_N$, and

$$\frac{1}{1 + \frac{C}{N} \sum_{j=1}^{N} \frac{1}{\omega_j z i\Delta\omega_j + z}} = \sum_{n=0}^{\infty} \left( \frac{C}{N} \sum_{j=1}^{N} \frac{1}{\omega_j z i\Delta\omega_j + z} \right)^n \tag{222}$$

The amplitude of interest can be thus written as
\[ F'(t) = \frac{1}{2\pi i} \sum_{k_1 \ldots k_N} Z_{k_1} \ldots Z_{k_N} \sum_{n=0}^{\infty} (-C)^n \int_{\Gamma} dz \frac{e^{zt}}{z^{n+1}} \frac{1}{N^n} \left( \sum_{j=1}^{N} \frac{1}{\omega_j i\Delta \omega_j + z} \right)^n \] (223)

\[ = \frac{1}{2\pi i} \sum_{k_1 \ldots k_N} Z_{k_1} \ldots Z_{k_N} \sum_{n=0}^{N} (-C)^n \int_{\Gamma} dz \frac{e^{zt}}{z^{n+1}} \frac{1}{N^n} \left( \sum_{j=1}^{N} \frac{1}{\omega_j i\Delta \omega_j + z} \right)^n \] (224)

The convergence of the geometric series guarantees that for any \( \varepsilon \) there exists \( N_\varepsilon \) such that

\[ |F'_{n>N_{\varepsilon}}(t)_{k_1 \ldots k_N}| < \varepsilon, \quad \sum_{k_1 \ldots k_{N_\varepsilon}} Z_{k_1} \ldots Z_{k_{N_\varepsilon}} F'_{n>N_{\varepsilon}}(t)_{k_1 \ldots k_{N_\varepsilon}} < \varepsilon. \] (225)

Therefore

\[ \lim_{N \to \infty} F'(t) = \frac{1}{2\pi i} \lim_{N \to \infty} \sum_{n=0}^{N} (-C)^n \int_{\Gamma} dz \frac{e^{zt}}{z^{n+1}} \sum_{k_1 \ldots k_N} Z_{k_1} \ldots Z_{k_N} \frac{1}{N^n} \left( \sum_{j=1}^{N} \frac{1}{\omega_j i\Delta \omega_j + z} \right)^n \] (226)

\[ \quad = \frac{1}{2\pi i} \lim_{N \to \infty} \sum_{n=0}^{N} (-C)^n \int_{\Gamma} dz \frac{e^{zt}}{z^{n+1}} \left[ \left( \sum_{k} Z_k \frac{1}{\omega_k i\Delta \omega_k + z} \right)^n P_0 + \ldots + \sum_{k} Z_k \left( \frac{1}{\omega_k i\Delta \omega_k + z} \right)^n P_N \right] \] (227)

\[ = \frac{1}{2\pi i} \lim_{N \to \infty} \int_{\Gamma} dz \frac{e^{zt}}{z} \left[ f_{n}^0 P_0 + \ldots + f_{n}^N P_N \right], \] (228)

where \( P_0 \ldots P_N \) are the probabilities employed in the proof of Theorem 1 and \( \lim_{N \to \infty} f_{n}^N < \infty \) for all \( j \). Using \( \lim_{N \to \infty} P_0 = 1 \) and \( \lim_{N \to \infty} P_j = 0 \) for \( j > 0 \) we find

\[ \lim_{N \to \infty} F'(t) = \frac{1}{2\pi i} \int_{\Gamma} dz \frac{e^{zt}}{z + C \sum_k Z_k \frac{1}{\omega_k i\Delta \omega_k + z}}. \] (229)

It is important to note that \( F'(t) \) obtained as the limiting value of \( F'(t) \) contains \( Z_k = |O_k|^2 \) satisfying the normalization condition \( \sum_k Z_k = 1 \). Therefore although the thermodynamic limit regularizes the non-canonical amplitude as if it originated from the regularized canonical Hamiltonian \( H_{\text{reg}} \), the regularization involves functions \( Z_k \) which are somewhat unusual from the viewpoint of standard quantum optics.

Indeed, the typical choice one finds in the literature is of the form [2]

\[ Z_k(\Lambda) = \frac{\Lambda}{\sqrt{k^2 + \Lambda^2}} \approx \begin{cases} 1 & \text{for } |k| \ll \Lambda \\ 0 & \text{for } |k| \gg \Lambda \end{cases} \] (230)

and it is essential for the standard interpretation that \( Z_k \to 1 \) with \( \Lambda \to \infty \). The regularization we arrive at must be square summable (or square integrable) and normalized. So assume

\[ Z_k(\Lambda_1, \Lambda_2) = \begin{cases} Z = \text{const} & \text{for } |k| \in [\Lambda_1, \Lambda_2] \\ \text{tends to} 0 & \text{for } |k| \notin [\Lambda_1, \Lambda_2] \end{cases} \] (231)

But since \( \sum_k Z_k = 1 \), the limit \( \Lambda_2 \to \infty \) is accompanied by \( Z \to 0 \). An agreement with the standard formalism is then obtained if one performs the latter limit under the constraint \( CZ = C_{\text{exp}} \), i.e. renormalizes the coupling parameter \( \alpha \) by \( \alpha_{\text{exp}} = \alpha Z^\beta \), which is essentially the charge renormalization \( e_{\text{exp}} = e_0 Z^\beta \).

All the experiments so far suggest the following

**Conjecture:** The thermodynamic limit \( N \to \infty \) maps predictions of the non-canonical theory into those of the canonical theory regularized by square-integrable formfactors. Elimination of regularization must be accompanied by charge renormalization.
B. Canonical limit and the third hint suggesting absence of ultraviolet divergences: Perturbation theory

Let us now discuss the thermodynamic limit of a perturbative expansion to order $n$ of an arbitrary amplitude

$$ F_{fi}(t) = \langle \Psi_f | U(t) | \Psi_i \rangle. \quad (232) $$

Examples of low order perturbation theory in concrete examples were explicitly treated in [1]. Here we want to discuss an arbitrary order of a general amplitude corresponding to the Hamiltonian (212) which has led to the Conjecture from the previous section. It is sufficient to discuss the amplitude for the general multiphoton states discussed in Theorem 1.

The interaction Hamiltonian (212) is applicable to fields with discrete frequency (i.e. cavity) spectrum. This is technically a simple case since momentum eigenvectors are normalized in volume $V$ by the Kroenecker deltas. Performing the expansion to order $n$ of the amplitude we arrive, for each Feynman diagram in momentum space, at the term proportional to

$$ \langle Q | a(f_1) \ldots a(f_m) \text{Mono}(a(g_1), \ldots, a(g_n)) a(g_1)\ldots a(g_{m\prime})\dagger | Q \rangle \quad (233) $$

where $\text{Mono}(\ldots)$ is a mononomial of order $n$ in creation and annihilation operators. Due to the Kroenecker-delta normalization the term (233) vanishes if and only if an analogous term does in the canonical theory (cf. the discussion in [1]). For this reason the non-CCR amplitude produces the same Feynman diagrams, but the Feynman rules are modified by the non-CCR algebra.

The latter modifications are easy to identify. Simply, each time a commutator of creation and annihilation operators produces a delta in the standard theory, say $\delta_{k\k'}$, here it is replaced by $\delta_{k\k} L_k$. The Wick-theorem-type manipulations one has to perform under the vacuum average are of the same type as those we have met in the proof of Theorem 1 while normalizing multi-photon states.

The essential and final step follows from the main property of the thermodynamic limit: $\lim_{N \to \infty} P_0 = 1$ and $\lim_{N \to \infty} P_j = 0$ for $j > 0$. Therefore only those terms count in the limit which have each vacuum average of the product $L_{k_1} \ldots L_{k_n}$ replaced by $Z_{k_1} \ldots Z_{k_n}$. Choosing the initial and final wavepackets $f_i, g_j$ normalized with respect to the scalar product $\langle \cdot | \cdot \rangle_Z$ from Theorem 1 we find that the modification of the amplitude is of precisely the type we have formulated in the Conjecture: Coupling constants in momentum space are regularized according to $g_k \to g_k Z_k^2$ and the ultraviolet infinities are regularized by square summability of $Z_k$.

C. The physical meaning of the parameters occuring in the multi-oscillator non-CCR representation

The Hamiltonian (212) can be also written as

$$ H(t) = \frac{\hbar \alpha}{2} \sum_k \sqrt{\frac{\hbar}{2\omega_k}} \left( R_+ \otimes \frac{1}{\sqrt{N}} a_k e^{-i\Delta_k t} - R_- \otimes \frac{1}{\sqrt{N}} a_k^\dagger e^{i\Delta_k t} \right). \quad (234) $$

The choice of $\alpha_N = \frac{1}{\sqrt{N}}$ was motivated by purely formal reasons. With this choice: (1) the RHS of field commutators satisfied the resolution of unity, (2) the statistics of coherent-state excitations was Poissonian in the thermodynamic limit, (3) the formfactors $Z(k)$ were regularizing the interaction term in the thermodynamic limit, (4) non-canonical coherent-state averages at the multi-oscillator and single-oscillator levels were identical, (5) normalization of single-photon and multi-photon states could be given in terms of the same scalar product $\langle \cdot | \cdot \rangle_Z$ in the thermodynamic limit.

The restriction $H(t)_N$ of $H(t)$ to an $N$-oscillator subspace reads

$$ H(t)_N = \sum_{j=1}^N \frac{1}{\sqrt{N}} \frac{\hbar \alpha}{2} \sum_k \sqrt{\frac{\hbar}{2\omega_k}} \left( R_+ a_k^{(j)} e^{-i\Delta_k t} - R_- a_k^{(j)\dagger} e^{i\Delta_k t} \right) \quad (235) $$

where $a_k^{(j)} = I \otimes \ldots a_k \ldots \otimes I$ with the single-oscillator annihilation operator at the $j$th place.

It is clear that $H(t)_N$ represents the interaction Hamiltonian of a single 2-level system interacting with $N$ indefinite-frequency harmonic oscillators. The factor $\frac{1}{\sqrt{N}}$ means that the strength of the coupling decreases with growing $N$, a property whose physical interpretation is well known and occurs, for example, in the Hepp-Lieb treatment of the Dicke model [22,23].
The Dicke model is in a sense dual to the one we consider. It represents a single harmonic oscillator interacting with $N$ 2-level systems via the interaction-picture Hamiltonian

$$H(t)_{\text{Dicke}} = \sum_{j=1}^{N} i \frac{\hbar \alpha}{\sqrt{N}} \sqrt{\frac{\hbar}{2 \Omega}} \left( R_{+}^{(j)} a_{j} e^{-i \Delta \Omega t} - R_{-}^{(j)} a_{j}^{\dagger} e^{i \Delta \Omega t} \right).$$

(236)

Here $R_{(j)}^{(j)} = I \otimes \ldots R_{+} \ldots \otimes I$, and $[a_{j}, a_{j}^{\dagger}] = I$.

Now, assuming the particular representation of $[a_{j}, a_{j}^{\dagger}] = I$ given by $\Omega = \sum_{k} \omega_{k} |k\rangle \langle k| \otimes I$, $I = 1 \otimes 1$, $[\hat{a}_{k}, \hat{a}_{k}^{\dagger}] = \delta_{kl} 1$, $a_{k} = |k\rangle \langle k| \otimes \hat{a}_{k}$, $a_{1} = \sum_{k} a_{k}$, we can rewrite (236) as

$$H(t)_{\text{Dicke}} = \sum_{j=1}^{N} i \frac{\hbar \alpha}{\sqrt{N}} \sum_{k} \sqrt{\frac{\hbar}{2 \omega_{k}}} \left( R_{+}^{(j)} a_{k} e^{-i \Delta \Omega t} - R_{-}^{(j)} a_{k}^{\dagger} e^{i \Delta \Omega t} \right).$$

(237)

Although the Hamiltonian (236) is formally a single-mode one, the form (237) resembles the non-canonical interaction term discussed in the previous subsection. What is important, the $N$-dependent coupling constant can be derived on physical grounds if one assumes a constant density $N/V$ of (bosonic) atoms. To see this, one writes the parameters of the Dicke-Hepp-Lieb Hamiltonian as

$$\alpha(\hbar/2)^{3/2} = \omega_{0} d \sqrt{2 \pi \hbar \rho}$$

(238)

where $\rho = N/V$ is the density of atoms, $d$ the dipole moment, and $\omega_{0}$ the atomic frequency. The thermodynamic limit of the Hepp-Lieb approach is performed under the assumption $\alpha = \text{const}$ which is equivalent to the constant density $\rho$.

This is precisely what happens in our approach to the thermodynamic limit. It follows that the choice of the non-CCR algebra whose RHS satisfies the resolution of unity is physically equivalent to the requirement that the electromagnetic field consists of indefinite-frequency oscillators of constant density $N/V$. The thermodynamic limit is then equivalent to the infinite volume limit $V \to \infty$.

Finally, let us note that taking the direct sum over $N$ we find

$$H(t)_{\text{Dicke}} = \bigoplus_{N=1}^{\infty} H(t)_{\text{Dicke}} = i \frac{\hbar \alpha}{2} \sum_{k} \sqrt{\frac{\hbar}{2 \omega_{k}}} \left( R_{+} a_{k} e^{-i \Delta \Omega t} - R_{-} a_{k}^{\dagger} e^{i \Delta \Omega t} \right)$$

(239)

where $R_{\pm} = \bigoplus_{N} R_{\pm}$, i.e. a structure analogous to the interaction Hamiltonian in non-canonical quantum optics.

D. Structure of the space of states

The Conjecture we have formulated above implies that probabilities have to be computed as if the vacuum was unique. Otherwise it is essential for the non-canonical construction to have an infinite dimensional space of different vacua. Otherwise it would be impossible to associate the spectrum of frequencies with a single harmonic oscillator. There is no contradiction if we treat the Fock space generated from a vacuum state $|\Omega\rangle$ as a fiber over $\mathcal{O}$ in a vector bundle with Fock fibers and the base space consisting of vacua. Vacuum fluctuations represented by the zero-energy part of the Hamiltonian generate a motion (a flow) in the base space and thus play a role of bundle connection. The “vacuum picture” corresponds to a choice of connection.

XI. CONCLUSIONS AND FURTHER PERSPECTIVES

Fields with the property of having at the RHS of the commutator an element from a nontrivial center of the algebra are usually termed the generalized free fields. In the context of our formalism we find this term misleading for two main reasons. First of all the generalized free fields are “free” whereas the examples we discussed include scattering of radiation, spontaneous emission and interaction picture perturbation theory. All these processes involve interactions with charges. Secondly, we arrived at the particular non-CCR representation on the basis of a concrete physical model of the indefinite frequency oscillator. The mathematical structure of the model does not follow the typical limitations imposed on generalized free fields. For example, the vacuum states are Poincaré non-invariant. The same concerns
the RHS of the non-CCR algebra which is only Poincaré covariant and satisfies the resolution of unity. The issues of locality have to be formulated in terms of certain operator generalizations of Green and Jordan-Pauli functions which lead to very special locality properties and eliminate problems with multiplication of field operators at the same point in configuration space. The notion of the thermodynamic limit is typical of the representation we use and does not seem to have been investigated in the context of axiomatic quantum fields.

The construction we are developing has a slightly different logic than the usual axiomatic quantum field theory. It seems that what we are doing is somehow in-between the generalized free fields, nonlocal quantum field theory, and noncommutative geometry.

An issue which has not been addressed so far is how to quantize fermions. Some results on the Dirac equation are known already and will be presented in a separate paper.

Another problem is to embed the concrete non-canonical quantization procedure we have proposed into a more abstract scheme of quantizations in a $C^*$-algebraic setting. The fact that the right-hand-side of commutation relations is not an identity but rather an operator belonging to the center of the algebra suggests directions for generalizations. It seems there is a link with the work of Streater on non-abelian cocycles [24]. An appropriate version of a coherent-state quantization based on the formalism of Naudts and Kuna [25] is in preparation.

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[14] In this paper we speak of the Poincaré group we mean the semidirect product of 4-translations and $SL(2, C)$, i.e. the universal covering space of the Poincaré group.
[15] This representation is in-between those introduced in [1] and [2]. The difference is that here two CCR operators are used to describe the polarization degree of freedom, as opposed to the one operator used in [1]. The reason for such a modification becomes clear when one introduces non-canonical (anti)commutation relations for massive particles (M. Czachor, in preparation). In the massless case there is no essential difference between the two approaches. That two CCR operators should be employed in the context of polarizations was stressed by J. Naudts (private communication).
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