Dating a random walk: Statistics of the duration
time of a random walk given its present position

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Abstract

We consider the distribution of the duration time, the time elapsed
since it began, of a diffusion process given its present position, under
the assumption that the process began at the origin. For unbiased
diffusion, the distribution does not exist (it is identically zero) for one
and two dimensional systems. We find the explicit expression for the
distribution for three and higher dimensions and discuss the behavior
of the duration time statistics: we find that the expected duration
time exists only for dimensions five and higher, whereas the variance
becomes finite for seven dimensions and above. We then turn to the
case of biased diffusion. The drift velocity introduces a new time scale
and the resulting statistics arise from the interplay of the diffusive time
scale and the drift time scale. For these systems all the moments exist
and explicit expressions are presented and discussed for the expected
duration time and its variance for all dimensions.

1 Introduction

The dating of events and objects, that is, the determination of the time
elapsed since an event occurred or the time since an object was made (or
last used), is fundamental in areas as diverse a geology and archeology, to
epidemiology and forensic science. A classic example is Carbon dating, in
which the ratio of $^{14}C$ to $^{12}C$ is used to estimate the age of organic samples
in archeology and geology [1, 2]. On the other hand, in forensic investigations,
the problem of estimating the time elapsed since the occurrence of an
event arises, for example, in the determination of the post mortem interval
(PMI), that is, the time since a person has died. The estimation of the PMI
is achieved by a variety of methods, ranging from the temperature of the body, the study of the different stages the body passes through after death -algor, livor and rigor mortis- to the different stages of body decomposition. Other interesting forensic techniques include using the development of flies and other insects, as well as their larvae in the body to establish the PMI\[3\]. Indeed, the guilt or innocence of a suspect, may hinge on the accurate determination of the time of the events before, during and after the crime. In the context of engineering, the determination of the time at which faults and errors occur in a production process may help to pinpoint the causes behind each fault and correct or prevent them from happening again in the future\[4\]. From the perspective of epidemiology, outbreaks are detected when there are already several infected individuals. To make things harder, in some cases, these individuals may have already moved from the place where they caught the disease. In this situation, knowing the time at which the epidemic began may be useful in identifying the source of the disease \[5\]. Other situations related to the problem of dating the beginning of a diffusion process would be the determination of when a leak occurred given the dispersion of contaminant particles emanated from the leak or establishing when a batch of marked or counterfeit bills were placed in circulation.

The common feature of these examples is that given the present state of a system -the ratio of radioactive isotopes in an archeological sample, the number or age of fly larvae in a body, the number of faulty products in a production line, the number and location of infected individuals in an epidemic, the spatial distribution of pollutants or of fake bills- one must infer when the evolution that led to this state began. In all these cases, the amount of time elapsed is determined through a precise knowledge of the initial conditions and time evolution of the process in question. Of course, since many random factors intervene in these systems, the inferred time can, at best, be determined probabilistically. However, it appears that not much attention has been devoted to the determination of the statistics of the time elapsed given the present state of a stochastic process; not even for one of the simplest and most widely used of such processes: the random walk. Random walks are, of course, the paradigm of stochastic transport. They have been used to model the motion of living organisms, particles in suspensions, atoms and molecules in and on solids, light in turbid media, among many other things \[6\]. Indeed, random walks represent the microscopic process that gives rise to diffusive transport at macroscopic scales. In this work we will be interested in the question of, given the present position of one (or several) diffusing particles, when did the process begin? That is, we want to find the probability distribution \( P(t|\vec{x},\vec{x}_0) \) that the process has evolved for a time \( t \),
given that it began at position \( \bar{x}_0 \) and is presently at position \( \bar{x} \). We will refer to this as the distribution of “duration times”.

It should be noted that a process consisting of \( M \) independent \( d \)-dimensional diffusing particles is formally equivalent to the case of a single particle diffusing in a \( N = M \times d \) dimensional space. Thus, in what follows, we will always refer to single diffusing particles in \( N \)-dimensional spaces. Indeed, and perhaps not unexpectedly, the properties of the duration time distribution will be strongly dependent on the dimension of the space in which the walk takes place, becoming more sharply peaked the higher the dimension. On the other hand, we will also show that for unbiased diffusion, the duration time distribution is not even defined (it is identically zero) for 1 and 2-dimensional systems. This, we argue, is related to Polya’s recurrence theorem [6, 11, 12]; namely that unbiased random walks return to their initial position with probability 1 and thus return infinitely many times, in one and two dimensional spaces, whereas their is a finite probability that the walker will never return to its initial position in dimensions three and higher. We also show that the distribution of duration times for unbiased diffusion exists but does not have a mean in 3 and 4 dimensions, whereas the variance of the distribution exists for unbiased random walks in 7 and higher dimensions. Next, we turn to the distribution and the statistics of the duration time of biased random walks. Since these are always transient, the distribution and its moments exist in all dimensions, we obtain their explicit expressions and discuss the limiting behaviors.

2 Inferring the distribution of duration time for unbiased diffusion

A central quantity in the theory of random walks is \( P(\bar{x}|t, \bar{x}_0) \), the probability of finding a \( N \)-dimensional random walker at position \( \bar{x} \) given that it started at position \( \bar{x}_0 \) and a time \( t \) has elapsed [6]. This probability describes the transport properties of this process as a function of time and has been extensively studied for over a century [6, 7, 8, 9, 10, 11]. In what follows, the explicit expressions for the distributions of duration times will be obtained from Bayes’ theorem [13] using \( P(\bar{x}|t, \bar{x}_0) \) and an exponential prior duration time distribution \( p(t) = e^{-t/\lambda}/\lambda \). The choice of this prior distribution -essentially the marginal distribution of duration times- is arbitrary, but ensures that the duration time is not negative. Thus we take the limit \( \lambda \to \infty \) to have the most un-informative distribution with positive domain. The idea behind this approach is, first, to consider the duration
time \( t \) as a random variable. Then, since we have interpreted \( P(\bar{x}|t, x_0) \) as
the conditional position of the walker given that the duration time of the
walk is \( t \), we multiply by the exponential prior distribution to construct a
joint distribution of elapsed times \( t \) and positions. From this joint distribu-
tion we can finally calculate the distribution of elapsed times conditioned
on the present position of the random walk by dividing over the marginal
distribution of the walker’s positions. However, the exponential prior distri-
bution effectively introduces a time scale \( \lambda \) which is not intrinsic to the
diffusion process. While this choice may be realistic in certain applications
(for instance when the walkers are atoms which decay radioactively, it is an
exogenous time scale generated from additional knowledge of the system).
Thus by taking the limit \( \lambda \to \infty \) we get rid of this extrinsic time scale and
we are left with the situation in which any (unconditioned) positive duration
time is, in some sense, equally likely. That is, the marginal time distribu-
tion, the exponential, becomes vanishingly small, though still normalized,
ensuring only that the process began sometime in the past. Other choices,
for example taking \( t \) uniformly distributed in the interval \([0, T]\) and the tak-
ing the limit \( T \to \infty \), give the same results. Under these conditions then,
the statistics of the duration time of the random walk arise solely from the
diffusion process.

For ease of calculation, we first consider the case of a \( N \)-dimensional un-
baised diffusion process, which corresponds to the continuum limit of sym-
metric random walks with jump distributions with finite second moment [6, 
9, 12, 14, 15]. The probability density for finding the walker at position \( \bar{x} \)
assuming it began at the origin is given by:

\[
P(\bar{x}|t, 0) = \frac{1}{(4\pi Dt)^{N/2}} e^{-\frac{1}{4D} \sum_{i=1}^{N} x_i^2}
\]  

Thus, using an exponential prior, we have that the distribution of duration
times is [16]:

\[
P_\lambda(t|\bar{x}, 0) = \frac{e^{-\frac{1}{4D} \sum_{i=1}^{N} x_i^2} e^{-t/\lambda}}{t^{N/2} \int_0^\infty e^{-\frac{1}{4D} \sum_{i=1}^{N} x_i^2} e^{-\tau/\lambda} \frac{d\tau}{\tau^{N/2}}} 
= \frac{e^{-\frac{1}{4D} \sum_{i=1}^{N} x_i^2} e^{-t/\lambda}}{2t^{N/2} \left( \frac{\lambda \sum_{i=1}^{N} x_i^2}{4D} \right)^{(1-N/2)/2} K_{1-N/2} \left( \sqrt{\frac{\sum_{i=1}^{N} x_i^2}{4D\lambda}} \right)}
\]

where \( K_\nu(z) \) denotes the modified Bessel function of order \( \nu \) [17]. In the
limit $\lambda \to \infty$ we obtain:

$$P(t|\bar{x},0) = \frac{1}{t} \left( \frac{\sum_{i=1}^{N} x_i^2}{4Dt} \right)^{\frac{N}{2}-1} e^{-\frac{\sum_{i=1}^{N} x_i^2}{4Dt}} \frac{\sum_{n=1}^{N} x_i^2}{\Gamma\left(\frac{N}{2} - 1\right)}, \quad N > 2 \quad (4)$$

while $P(t|\bar{x},0) \equiv 0$ for $N = 1, 2$. As mentioned above, the reason the duration time distribution vanishes for one and two dimensional random walks is because they are recurrent [6]. That is, with probability one, the walks return to the vicinity of every point in the system, and thus, they return an infinite number of times (though the average time between successive returns diverges). Therefore, by knowing the walker’s position, there is no way of knowing whether it has visited that point a finite or infinite number of times in the past. Indeed, since it may have visited its present position arbitrarily many times in the past, essentially any arbitrarily large time may have elapsed since the walk began and the distribution of duration times vanishes accordingly, so no information on the duration of the process can be inferred from the position of the walker in these cases. On the other hand, for three and higher dimensional systems, there is a finite probability that the walker never returns to the vicinity of a point, and if it does, it returns in a finite mean time[6]. Thus, the number of previous visits to any given point is finite and the duration time for the process has a vanishing probability of being indefinitely long.

When the duration time distribution is defined, it is straightforward to see that the mean duration time does not exist for 3 and 4 dimensional systems; only becoming finite for $N \geq 5$:

$$\langle t|\bar{x} \rangle = \frac{2}{N-4} \left( \frac{\sum_{i=1}^{N} x_i^2}{4D} \right) \quad N \geq 5 \quad (5)$$

The variance becomes finite for 7 and higher dimensional systems:

$$\frac{\langle t^2|\bar{x} \rangle - \langle t|\bar{x} \rangle^2}{\langle t|\bar{x} \rangle^2} = \frac{2}{N-6} \quad N \geq 7 \quad (6)$$

so the higher the dimension of the system, the more precisely the time elapsed since the beginning of the process can be determined or, alternatively, the more accurately can the walker be used as a clock.

Another statistic that can be readily computed for systems of any dimension $N$ is $t_{mpv}$, the most probable duration time, which is given by

$$t_{mpv}(\bar{x}) = \frac{1}{N} \left( \frac{\sum_{i=1}^{N} x_i^2}{2D} \right) \quad (7)$$
3 The case of biased diffusion

A more interesting case is that in which we attempt to date the beginning of a biased diffusion process. In this case the process is always transient and the distribution of duration times is defined in every dimension. In contrast to the unbiased case, the biased case has two distinct time scales (and hence infinitely many): one given by the mean square displacement divided by the diffusion constant, another given by the net displacement divided by the drift velocity. The interplay of these scales will determine the statistics of the duration time. Repeating the procedure described above, the distribution of duration times given the position of the particle diffusing in a N-dimensional system is given by:

$$P_{\lambda}(t|\bar{x},0) = \frac{e^{-\frac{1}{4D}\frac{\sum_{i=1}^{N} x_i^2 - \bar{t} \sum_{i=1}^{N} v_i^2}{2\pi}}}{2\sqrt[4]{\frac{\sum_{i=1}^{N} x_i^2}{\sum_{i=1}^{N} v_i^2}}^{(1-N/2)/2} K_{1-N/2} \left( \frac{1}{\sqrt{D}\sqrt{\left(\sum_{i=1}^{N} x_i^2\right) \left(\sum_{i=1}^{N} v_i^2\right)}} \right)}$$

The exponential factors induce cutoffs which insure that all the moments of the distribution exist. In particular, the mean duration time can be shown to be

$$<t|\bar{x}> = \left(\frac{\sum_{i=1}^{N} x_i^2}{\sum_{i=1}^{N} v_i^2}\right)^{1/2} \frac{K_{2-N/2} \left( \frac{1}{\sqrt{D}\sqrt{\left(\sum_{i=1}^{N} x_i^2\right) \left(\sum_{i=1}^{N} v_i^2\right)}} \right)}{K_{1-N/2} \left( \frac{1}{\sqrt{D}\sqrt{\left(\sum_{i=1}^{N} x_i^2\right) \left(\sum_{i=1}^{N} v_i^2\right)}} \right)}$$

and the variance is

$$<t^2|\bar{x}> - <t|\bar{x}>^2 = \left(\frac{\sum_{i=1}^{N} x_i^2}{\sum_{i=1}^{N} v_i^2}\right) \left[ \frac{K_{3-N/2} \left( \frac{1}{\sqrt{D}\sqrt{\left(\sum_{i=1}^{N} x_i^2\right) \left(\sum_{i=1}^{N} v_i^2\right)}} \right)}{K_{1-N/2} \left( \frac{1}{\sqrt{D}\sqrt{\left(\sum_{i=1}^{N} x_i^2\right) \left(\sum_{i=1}^{N} v_i^2\right)}} \right)} \right]$$

$$- \left(\frac{K_{2-N/2} \left( \frac{1}{\sqrt{D}\sqrt{\left(\sum_{i=1}^{N} x_i^2\right) \left(\sum_{i=1}^{N} v_i^2\right)}} \right)}{K_{1-N/2} \left( \frac{1}{\sqrt{D}\sqrt{\left(\sum_{i=1}^{N} x_i^2\right) \left(\sum_{i=1}^{N} v_i^2\right)}} \right)} \right)^2$$

which are exact, but admittedly not the most informative of formulae. To have a better understanding of the behavior of these statistics, we focus on the limits of large and small values of the arguments of the Bessel functions.
For large values of the arguments of the Bessel functions, to leading order, we have [17]

\[ < t | \bar{x} > \sim \left( \frac{\sum_{i=1}^{N} x_i^2}{\sum_{i=1}^{N} v_i^2} \right)^{1/2} \text{ for } \left( \sum_{i=1}^{N} x_i^2 \right) \left( \sum_{i=1}^{N} v_i^2 \right) \gg 4D^2 \] (11)

whereas in the same limit, the variance behaves as

\[ < t^2 | \bar{x} > - < t | \bar{x} >^2 \sim 2D \frac{\left( \sum_{i=1}^{N} x_i^2 \right)^{1/2}}{\left( \sum_{i=1}^{N} v_i^2 \right)^{3/2}} \] (12)

so the relative fluctuations are small:

\[ \frac{< t^2 | \bar{x} > - < t | \bar{x} >^2}{< t | \bar{x} >^2} \sim 2D \frac{\left( \sum_{i=1}^{N} x_i^2 \right)^{1/2}}{\left( \sum_{i=1}^{N} v_i^2 \right)^{1/2}} < 1 \] (13)

Naively, this behavior can be understood by noting that for the systems under consideration, very large displacements are mostly due to the effect of the drift, the diffusive contribution playing an increasingly negligible role, so the duration time can be accurately estimated as the net displacement divided by the magnitude of the drift velocity.

A richer variety of phenomena occur in the regime in which the arguments of the Bessel functions are small. In this case both the diffusion and the drift can contribute to the displacement to different degrees depending on the dimension of the system. In this regime, the mean duration time is given by

\[< t | \bar{x} > \sim \begin{cases} \frac{1}{2} \frac{4D}{\sum_{i=1}^{N} v_i^2} & \text{N=1} \\ \frac{4D}{\left( \sum_{i=1}^{N} v_i^2 \right) \ln \left[ 2D / \left( \sum_{i=1}^{N} x_i^2 \right)^{1/2} \left( \sum_{i=1}^{N} v_i^2 \right)^{1/2} \right]} & \text{N=2} \\ \left( \frac{\sum_{i=1}^{N} x_i^2}{\sum_{i=1}^{N} v_i^2} \right)^{1/2} & \text{N=3} \\ \frac{\sum_{i=1}^{N} x_i^2}{4D} \ln \left[ 2D / \left( \sum_{i=1}^{N} x_i^2 \right)^{1/2} \left( \sum_{i=1}^{N} v_i^2 \right)^{1/2} \right] & \text{N=4} \\ \frac{2}{N-4} \frac{\sum_{i=1}^{N} x_i^2}{4D} & \text{N \geq 5} \end{cases} \]
Interestingly, for small enough displacements in one dimensional systems, the expected duration time is independent of the displacement, whereas for 5 or higher dimensional systems, the duration time is independent of the drift velocity as long as \(\sum_{i=1}^{N} x_i^2 \sum_{i=1}^{N} v_i^2 \ll 4D^2\). In this same limit, the behavior of the variance has to be evaluated case by case up to six dimensional systems; to leading order we have:

\[
< t^2 | \bar{x} > - < t | \bar{x} >^2 \sim \begin{cases} \\
\frac{8D^2}{(\sum_{i=1}^{N} v_i^2)^2} & N=1 \\
\frac{8D^2}{(\sum_{i=1}^{N} v_i^2)^4} \ln \left[ 2D / \left( \sum_{i=1}^{N} x_i^2 \right)^{1/2} (\sum_{i=1}^{N} v_i^2)^{1/2} \right] & N=2 \\
\frac{2D(\sum_{i=1}^{N} x_i^2)^{1/2}}{(\sum_{i=1}^{N} v_i^2)^{1/2}} & N=3 \\
\frac{\left( \sum_{i=1}^{N} x_i^2 \right)^{3/2}}{2D(\sum_{i=1}^{N} v_i^2)^{1/2}} & N=4 \\
\frac{\left( \sum_{i=1}^{N} x_i^2 \right)^2}{8D} \ln \left[ \frac{2D}{(\sum_{i=1}^{N} x_i^2)^{1/2} (\sum_{i=1}^{N} v_i^2)^{1/2}} \right] & N=5 \\
\frac{\left( \sum_{i=1}^{N} x_i^2 \right)^2}{2D^2(N-4)^2(N-6)} & N \geq 7 
\end{cases}
\]

On the other hand, the most probable duration time is given by the expression:

\[
t_{mpv}(\bar{x}) = \frac{1}{\sum_{i=1}^{N} v_i^2} \left[ (D^2N^2 + \sum_{i=1}^{N} x_i^2) \left( \sum_{i=1}^{N} v_i^2 \right)^{1/2} - DN \right] \quad (14)
\]

This quantity ranges from

\[
t_{mpv}(\bar{x}) \sim \frac{\sum_{i=1}^{N} x_i^2}{2DN} \quad \text{for } D^2N^2 >> \left( \sum_{i=1}^{N} x_i^2 \right) \left( \sum_{i=1}^{N} v_i^2 \right) \quad (15)
\]

to

\[
t_{mpv}(\bar{x}) \sim \frac{\left( \sum_{i=1}^{N} x_i^2 \right)^{1/2}}{\left( \sum_{i=1}^{N} v_i^2 \right)^{1/2}} \quad \text{for } D^2N^2 << \left( \sum_{i=1}^{N} x_i^2 \right) \left( \sum_{i=1}^{N} v_i^2 \right) \quad (16)
\]
as expected. Note that this large displacement behavior coincides with the mean value $< t | \bar{x} >$ in the same limit.

It is worth noting that the term in $\bar{x} \cdot \bar{v}$ cancels out in the distribution of duration times Eq. (8), so that if the walker is biased in only one component, that particular component plays no special role in determining the duration time. Indeed, somewhat surprisingly, even though the drift breaks the symmetry of the systems, the duration time distribution remains rotationally invariant around the origin.

4 Summary and Conclusions

In summary, we have considered the question of the duration of a diffusion process given its present state under the assumption that the process began at the origin. This is tantamount to establishing when the process began, given the present position of the diffusing particle. We find that in the case of unbiased diffusion, the distribution does not exist for one and two dimensional systems and argued that this is due to the recurrent nature of unbiased random walks in these dimensions. We find the explicit distribution for three and higher dimensions and discuss the behavior of the duration time statistics for these systems: since the distribution decays slowly, it turns out that the expected duration time exists only for dimensions five and higher, whereas the variance becomes finite for seven dimensions and above. Not surprisingly, the system becomes a better clock as the dimension increases.

Next we focused on the statistics of the duration time for a N-dimensional biased diffusion process given the position of the diffusing particle. The drift velocity in these systems introduces a new time scale and the resulting statistics arise from the interplay of the diffusive time scale and the drift time scale. For these systems all the moments exist and explicit expressions are presented for the expected duration time and its variance for all dimensions.

Future work will consider the statistics of duration time for process that evolve by anomalous diffusion, due, say, to a power law distribution of jump lengths in the case of superdiffusion, or to a long tailed distribution of waiting times for subdiffusive systems [6]. Other interesting and more complicated extensions could involve considering the statistics of the duration time for diffusive processes in other geometries; for example, by evaluating the effect that a reflecting wall has on the distribution of duration times, or by considering diffusion on a nontrivial substrate, like a fractal [18, 19]. Also, the dating of the beginning of other stochastic processes not related to transport might be of interest. For example, when did the “first” ancestor of a family
line live, given the present number of descendants.

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