Bilinearization of a Generalized Derivative Nonlinear Schrödinger Equation

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Abstract

A generalized derivative nonlinear Schrödinger equation,

\[ iqt + q_{xx} + 2i\gamma|q|^2q_x + 2i(\gamma - 1)q^2q_x^* + (\gamma - 1)(\gamma - 2)|q|^4q = 0, \]

is studied by means of Hirota’s bilinear formalism. Soliton solutions are constructed as quotients of Wronski-type determinants. A relationship between the bilinear structure and gauge transformation is also discussed.

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§1 Introduction

The nonlinear Schrödinger (NLS) equation is one of the most generic soliton equations. Various modifications of the equation have been proposed. One of these attempts is to study the effects of higher order perturbations. Extended version of the NLS equation with higher order nonlinearity has been proposed by various authors. It has been revealed that there exist “integrable” cases even for extended equations.

Among them, there are two celebrated equations with derivative-type nonlinearities, which are called the derivative nonlinear Schrödinger (DNLS) equations. One is the Chen-Lee-Liu (CLL) equation \[1, 2\],

\[
i Q_T + Q_{XX} + 2i|Q|^2Q_X = 0, \tag{1}
\]

and the other is the Kaup-Newell (KN) equation \[3\],

\[
i Q_T + Q_{XX} + 2i(|Q|^2Q)_X = 0. \tag{2}
\]

It is known that the CLL and the KN equations may be transformed into each other by a “gauge transformation” \[4\]. The method of gauge transformation can be applied to some generalized case. Kundu \[5\] proposed the following equation:

\[
i Q_T + Q_{XX} + \beta|Q|^2Q + i(4\delta + 2\alpha)|Q|^2Q_X + \delta(4\delta + \alpha)Q^4Q = 0, \tag{3}
\]

where \(Q^*\) denotes the complex conjugate of \(Q\). If we set

\[
Q = \hat{Q} \exp \left( -2i\delta \int^X |\hat{Q}|^2 dX \right), \tag{4}
\]

then eq. (3) is gauge-equivalent to

\[
i \hat{Q}_T + \hat{Q}_{XX} + \beta|\hat{Q}|^2\hat{Q} + i\alpha (|\hat{Q}|^2\hat{Q})_X = 0, \tag{5}
\]

which is a hybrid of the NLS equation and the DNLS equation of KN-type. In case of \(\alpha > 0\), eq. (3) is transformed into a normalized form;

\[
i q_t + q_{xx} + 2\gamma |q|^2q_x + 2i(\gamma - 1)q^2q_x^* + (\gamma - 1)(\gamma - 2)|q|^4q = 0, \tag{6}
\]

by means of the change of variables,

\[
Q(X, T) = \sqrt{\frac{2}{\alpha}} q(x, t) \exp \left( \frac{i\beta x}{\alpha} + \frac{i\beta^2 t}{\alpha^2} \right),
\]

\[
X = x + \frac{2\beta}{\alpha} t, \quad T = t, \quad \gamma = \frac{4\delta}{\alpha} + 2.
\]

Clarkson and Cosgrove applied the Painlevé test to this type of equations and check the integrability \[1\].

The aim of this paper is to construct multi-soliton solutions for eq. (6). Soliton solutions for the KN equation \((\gamma = 2)\) are already known \[3\], and solutions for eq. (6) are constructed by the gauge transformation \(4\) in principle. However, to obtain the explicit form of the solutions, one must integrate eq. (4) in practice. The integration becomes very complicated in multi-soliton case. In this paper, we study eq. (6) by means of Hirota’s bilinear formalism, and construct soliton solutions through the Wronskian technique \[7\]. One can get clear insights for the algebraic structure through the bilinear method.
§2  Bilinearization

In ref.2, Nakamura and Chen bilinearized the CLL equation ($\gamma = 1$) and constructed soliton solutions. Hirota found that the CLL equation and the KN equation ($\gamma = 2$) can be transformed into the same bilinear equations[3]. Here we generalize their results and show that eq.(3) shares the same bilinear structure for all $\gamma$.

For the moment, we will forget the complex structure of eq.(6), and start with the coupled equations;

\[
\begin{align*}
    u_{x_2} - u_{x_1} + 2\gamma uv u_{x_1} + 2(\gamma - 1)u^2 v_{x_1} + (\gamma - 1)(\gamma - 2)u^3 v^2 &= 0, \quad (7a) \\
    -v_{x_2} - v_{x_1} - 2\gamma uv v_{x_1} - 2(\gamma - 1)v^2 u_{x_1} + (\gamma - 1)(\gamma - 2)v^2 u^3 &= 0. \quad (7b)
\end{align*}
\]

Note that in case of $v = \pm u^\ast$, eqs.(7) are reduced to eq.(3) by setting $q = u$, $x_1 = \pm ix$ and $x_2 = -it$. Using the dependent variable transformations,

\[
u = \frac{f^{\gamma-1}g}{f^\gamma}, \quad v = \frac{f^{\gamma-1}\tilde{g}}{f^\gamma}, \quad (8)
\]

eqs.(7) may be decoupled into the following bilinear equations;

\[
\begin{align*}
    (D^2_{x_1} + D_{x_2}) f \cdot \tilde{f} &= 0, \quad (9a) \\
    (D^2_{x_1} - D_{x_2}) f \cdot \tilde{g} &= 0, \quad (9b) \\
    (D^2_{x_1} + D_{x_2}) \tilde{f} \cdot g &= 0, \quad (9c) \\
    D^2_{x_1} f \cdot f + 2g\tilde{h} &= 0, \quad (9d) \\
    D^2_{x_1} \tilde{f} \cdot \tilde{f} + 2\tilde{g}h &= 0, \quad (9e) \\
    D_{x_1} f \cdot \tilde{f} - g\tilde{g} &= 0, \quad (9f) \\
    D_{x_1} \tilde{f} \cdot g - \tilde{f}h &= 0, \quad (9g) \\
    D_{x_1} \tilde{f} \cdot g + f\tilde{h} &= 0, \quad (9h)
\end{align*}
\]

where we have introduced auxiliary independent variables $h$ and $\tilde{h}$. In eqs.(9) we use the $D$-operator of Hirota[4], defined by

\[
D^m_x D^n_t F(x, y) \cdot G(x, t) = (\partial_x - \partial_{x'})^m (\partial_t - \partial_{t'})^n F(x, y) F(x', t')|_{x'=x, t'=t}.
\]

On decoupling eqs.(7) into eqs.(9), the following properties of the $D$-operator, valid for any functions $F(x)$, $G(x)$ and $H(x)$, are useful;

\[
\begin{align*}
    \frac{\partial}{\partial x} \left( \frac{f^{\gamma-1}G}{H^\gamma} \right) &= \frac{f^{\gamma-1}}{H^{\gamma+1}}D_x G \cdot H + (\gamma - 1) \frac{f^{\gamma-2}G}{H^{\gamma+1}} D_x F \cdot H, \quad (10a) \\
    \frac{\partial^2}{\partial x^2} \left( \frac{f^{\gamma-1}G}{H^\gamma} \right) &= \frac{f^{\gamma-1}}{H^{\gamma+1}}D^2_x G \cdot H + (\gamma - 1) \frac{f^{\gamma-2}G}{H^{\gamma+1}} D^2_x F \cdot H \\
    &\quad - \frac{f^{\gamma-1}G}{H^{\gamma+1}} D^2_x H \cdot H + 2(\gamma - 1) \frac{f^{\gamma-2}G}{H^{\gamma+1}} (D_x F \cdot H)(D_x G \cdot H) \\
    &\quad + (\gamma - 1)(\gamma - 2) \frac{f^{\gamma-3}G}{H^{\gamma+1}} (D_x F \cdot H)^2, \quad (10b) \\
    D^2_x F \cdot G &= \frac{G}{2F} D^2_x F \cdot F + \frac{F}{2G} D^2_x G \cdot G + \frac{1}{FG} (D_x F \cdot G)^2. \quad (10c)
\end{align*}
\]
Relations (10a, b) are obtained from an expansion of the following identity;

$$
\exp \left( \epsilon \frac{\partial}{\partial x} \right) F^\gamma G^{\gamma-1} = \frac{(\exp (\epsilon D_x) F \cdot H)^{\gamma-1} (\exp (\epsilon D_x) G \cdot H)}{(\cosh (\epsilon D_x) G \cdot H)^\gamma},
$$

and (10c) is from

$$(\exp(\epsilon D_x) F \cdot F) (\exp(\epsilon D_x) G \cdot G) = (\exp(\epsilon D_x) F \cdot G) (\exp(\epsilon D_x) G \cdot F).$$

The latter identity is a special case of eq.(VII) of ref.9 ($\delta = 0, a = b, c = d$). Additional formulas listed in the appendix of ref.9 are also useful.

We note that eqs.(9) are overdetermined. However, once we have a set of solutions for eqs.(9), we can construct a solution of eqs.(7) through the transformation (8)

§3 Construction of Solutions

It is known that solutions of bilinear equations are expressed as quotients of Wronski-type determinants[7]. We will show that solutions of the bilinear equations (9) can also be constructed by the Wronskian technique.

In the following, we use abbreviation,

$$
\bar{n} = \left( \frac{\partial}{\partial x_1^{(1)}} \right)^n \left( \begin{array}{c} \varphi_1(x^{(1)}) \\ \varphi_2(x^{(1)}) \\ \vdots \\ \varphi_{2N}(x^{(1)}) \end{array} \right), \quad \bar{n} = \left( \frac{\partial}{\partial x_1^{(2)}} \right)^n \left( \begin{array}{c} \psi_1(x^{(2)}) \\ \psi_2(x^{(2)}) \\ \vdots \\ \psi_{2N}(x^{(2)}) \end{array} \right). \tag{11}
$$

Define the $\tau$-function,

$$
\tau_{m,n;i} = \frac{m}{m}, \frac{m+1}{m+1}, \ldots, \frac{m+N-1}{m+N-1}; \frac{n}{n}, \frac{n+1}{n+1}, \ldots, \frac{n+N-1+i}{n+N-1+i}, \tag{12}
$$

where the elements of the determinant satisfy the linear differential equations (dispersion relations);

$$
\frac{\partial \varphi_j}{\partial x_n^{(1)}} = \delta_{ji} \frac{\partial^n \varphi_j}{\partial x_1^{(1)n}}, \quad \frac{\partial \psi_j}{\partial x_n^{(1)}} = \delta_{ji} \frac{\partial^n \psi_j}{\partial x_1^{(2)n}}. \tag{13}
$$

This type of determinant is called “double-Wronskian”[4].

Hereafter we assume this $\tau$-function has the $A_1^{(1)}$-symmetry[10];

$$
\tau_{m,n;i} = \tau_{m+1,n+1;i}. \tag{14}
$$

Solutions for the overdetermined equations (9) can be constructed owing to this symmetry. Under this assumption, the $\tau$-function depends only on the difference of $x_j^{(1)}$ and $x_j^{(2)}$ for all $j$. Set

$$
\begin{align*}
\tau &= \tau_{0,0;0}, & \tilde{\tau} &= \tau_{1,0;0}, \\
\tilde{\tau} &= \tau_{0,0;-1}, & \tilde{\tilde{\tau}} &= \tau_{1,0;-1}, \\
\tilde{\tau} &= \tau_{0,0;1}, & \tilde{\tilde{\tilde{\tau}}} &= \tau_{1,0;1},
\end{align*}
$$

3
and
\[ x_1 = x^{(1)}_1 - x^{(2)}_1, \quad x_2 = x^{(1)}_2 - x^{(2)}_2. \]  \hfill (16)

Then one can show that \( f, g, h, \tilde{f}, \tilde{g}, \) and \( \bar{h} \) above satisfy eqs.(9). Here we only prove the first of eqs.(9) by using the Laplace expansion. Let us consider the following identity for \( 4N \times 4N \) determinant,
\[
\begin{vmatrix}
\bar{0} & 1, \ldots, N-2 & 0 & N-1, N, N+1 & 0, \ldots, N-1 & 0 \\
\emptyset & 1, \ldots, N-2 & 0 & N-1, N, N+1 & 0 & 0 \\
\emptyset & 1, \ldots, N-2 & 0 & N-1, N, N+1 & 0 & 0 \\
\emptyset & 1, \ldots, N-2 & 0 & N-1, N, N+1 & 0 & 0 \\
0 & 1, \ldots, N-2 & 0 & N-1, N, N+1 & 0 & 0 \\
0 & 1, \ldots, N-2 & 0 & N-1, N, N+1 & 0 & 0 \\
\end{vmatrix}
= 0. \hfill (17)
\]

Applying the Laplace expansion into \( 2N \times 2N \) minors to the left-hand side of eq.(17), one finds
\[
\begin{align*}
&\begin{vmatrix}
\bar{0}, \ldots, N-2, N-1; 0, \ldots, N-1 \\
1, \ldots, N-2, N, N+1; 0, \ldots, N-1 \\
- \begin{vmatrix}
\bar{0}, \ldots, N-2, N; 0, \ldots, N-1 \\
1, \ldots, N-2, N-1, N+1; 0, \ldots, N-1 \\
+ \begin{vmatrix}
\bar{0}, \ldots, N-2, N+1; 0, \ldots, N-1 \\
1, \ldots, N-2, N-1, N; 0, \ldots, N-1
\end{vmatrix}
&= 0.
\end{align*}
\hfill (18)
\]

From eq.(18), we have
\[
(D^2_{x_1^{(1)}} + D^2_{x_2^{(2)}})\tau_{0,0;0,0} = 0. \hfill (19)
\]

On the other hand, if we apply the Laplace expansion to
\[
\begin{vmatrix}
1, \ldots, N \\
\emptyset & 1, \ldots, N-2 & 0 & N-1, N, N+1 & 0, \ldots, N-1 & 0 \\
\emptyset & 1, \ldots, N-2 & 0 & N-1, N, N+1 & 0 & 0 \\
\emptyset & 1, \ldots, N-2 & 0 & N-1, N, N+1 & 0 & 0 \\
\emptyset & 1, \ldots, N-2 & 0 & N-1, N, N+1 & 0 & 0 \\
0 & 1, \ldots, N-2 & 0 & N-1, N, N+1 & 0 & 0 \\
0 & 1, \ldots, N-2 & 0 & N-1, N, N+1 & 0 & 0 \\
\end{vmatrix}
= 0, \hfill (20)
\]

we get
\[
(D^2_{x_1^{(2)}} + D^2_{x_2^{(1)}})\tau_{1,0;0,0} = 0. \hfill (21)
\]

Under the conditions (14)–(16), both eqs.(19) and (21) reduce to eq.(14). The remaining equations in eqs.(9) can be proved in the same way.

In order to construct soliton solutions of eq.(6), one may choose \( \varphi_j(x^{(1)}) \) and \( \psi_j(x^{(2)}) \) in eqs.(13) as
\[
\begin{align*}
\varphi_j(x^{(1)}) &= a_j \exp(p_jx_1^{(1)} + p_j^2x_2^{(1)}), \\
\psi_j(x^{(2)}) &= b_j \exp(p_jx_1^{(2)} + p_j^2x_2^{(2)}).
\end{align*} \hfill (22a)
\]

If we impose the conditions,
\[
p_{N+j} = p_j^*, \quad |p_j|^2 = 1, \quad \frac{b_{N+j}}{a_{N+j}} = \frac{a_j}{b_j} p_j^*, \quad j = 1, 2, \ldots, N, \hfill (23)
\]
on \( \varphi_j(x^{(1)}) \) and \( \psi_j(x^{(2)}) \) above, and assume \( x_1^{(1)}, x_1^{(2)}, x_2^{(1)} \) and \( x_2^{(2)} \) are pure imaginary, then the functions \( f, g, \tilde{f} \) and \( \tilde{g} \) defined by eqs.(15) satisfy the relations;
\[
\left( \frac{g}{f} \right)^* = -\frac{\tilde{g}}{\tilde{f}}, \quad \left( \frac{f}{\tilde{f}} \right)^* = \frac{\tilde{f}}{f}. \hfill (24)
\]
The reduction condition (14) is satisfied under the assumptions (22), (23). Hence we have \( v = -u^* \).

To prove the reality condition (24), we start with the linear equation:

\[
\begin{pmatrix}
0, 1, \ldots, N-1; 0, 1, \ldots, N-1
\end{pmatrix}
\begin{pmatrix}
w_1
\end{pmatrix}
= \begin{pmatrix}
N
\end{pmatrix},
\]  

for

\[
\begin{pmatrix}
w^1_1 \\
w^2_1 \\
\vdots \\
w^{2N}_1
\end{pmatrix}.
\]

We find that \( w^{(2N)}_1 = (-)^{N-1}g/f \) by solving eq.(25). Next, by solving

\[
\begin{pmatrix}
1, 2, \ldots, N; 0, 1, \ldots, N-1
\end{pmatrix}
\begin{pmatrix}
w_2
\end{pmatrix}
= \begin{pmatrix}
N
\end{pmatrix},
\]

for

\[
\begin{pmatrix}
w^1_2 \\
w^2_2 \\
\vdots \\
w^{2N}_2
\end{pmatrix},
\]

we get \( w^{(N)}_2 = (-)^N \tilde{g}/\tilde{f} \). Under the conditions (23), eq.(24) is transformed into the complex conjugate of eq.(23) by suitable rearrangement of rows and columns. By using this fact, we obtain the first one of eqs.(24). The second is similarly proved, but in this case we use the symmetry (14).

As an example of the solutions, we give the explicit form of the 1-soliton solution,

\[
q(x, t) = \frac{p - p^*}{ce^\phi + \frac{1}{c}pe^{-\phi^*}} \left( \frac{ce^{\phi} + \frac{1}{c}pe^{-\phi^*}}{cp^*e^{\phi} + \frac{1}{c^*}e^{-\phi^*}} \right)^\gamma,
\]

where \( c \) and \( p \) are complex constants. The parameter \( p \) is chosen to satisfy \( |p|^2 = 1 \). The phase \( \phi(x, t) \) is defined by

\[
\phi(x, t) = ipx + ip^2t + \phi^{(0)},
\]

with \( \phi^{(0)} \) an arbitrary constant. If \( \gamma \) is an integer, eq.(27) becomes rational w.r.t. exponential functions, as is always the case of soliton solutions. But in general case, eq.(27) is not rational. This type of soliton solution has not been obtained so far as the authors know.

§4 Relation with the Gauge Transformation

In this section, we discuss the relationship between the gauge transformation method[4, 5] and our theory.
We set \( \hat{u} \) and \( \hat{v} \) as a solution of eqs. (7) with \( \gamma = 2 \),
\[
\hat{u}_{x_2} - \hat{u}_{x_1 x_1} + 4 \hat{u} \hat{v} \hat{u}_{x_1} + 2 \hat{u}^2 \hat{v}_{x_1} = 0, \tag{28a}
\]
\[
-\hat{v}_{x_2} - \hat{v}_{x_1 x_1} - 4 \hat{u} \hat{v} \hat{v}_{x_1} - 2 \hat{v}^2 \hat{u}_{x_1} = 0. \tag{28b}
\]
As is shown in the preceding section, the solution of the equations above is represented as eqs. (8) with \( \gamma = 2 \);
\[
\hat{u} = \frac{fg}{f^2}, \quad \hat{v} = \frac{\tilde{f} \tilde{g}}{f^2}. \tag{29}
\]
In this case, the gauge transformation between eqs. (28) (eqs. (7) with \( \gamma = 2 \)) and eqs. (7) (arbitrary \( \gamma \)) is written as
\[
u = \hat{u} \exp \left( -\left( \gamma - 2 \right) \int_{x_1}^{x_2} \hat{u} \hat{v} dx_1 \right), \tag{30a}
\]
\[
v = \hat{v} \exp \left( \left( \gamma - 2 \right) \int_{x_1}^{x_2} \hat{u} \hat{v} dx_1 \right). \tag{30b}
\]
If we substitute \( u, v \) of eqs. (8) and \( \hat{u}, \hat{v} \) of eqs. (29), eqs. (30) are transformed into
\[
\int_{x_1}^{x_2} \frac{g \tilde{g}}{f \tilde{f}} dx_1 = \log \frac{\tilde{f}}{f}. \tag{31}
\]
Differentiating both side of eq. (31), one finds
\[
\frac{g \tilde{g}}{f \tilde{f}} = \frac{D x_1 f \cdot \tilde{f}}{f \tilde{f}}. \tag{32}
\]
Equation (32) is nothing but eq. (14). Hence we conclude that the gauge transformation (30) is a direct consequence of the bilinear structure of the equations. We remark that the bilinear variables, \( f, g, \tilde{f} \) and \( \tilde{g} \), clarify the structure of the gauge transformation.

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