A technique for avoiding infinite integrals in the calculation of the one-loop diagram contribution to the vacuum polarization component of an atomic energy level is presented. This makes renormalization unnecessary. Infinite integrals do not occur because, as it is shown, no delta functions are required for the Green’s functions. Thus there are none to overlap. This procedure is shown to produce the same formula as the one obtained by dimensional renormalization.

I. INTRODUCTION

Divergent integrals appear in the calculations of the Lamb shift in a number of papers and text books. They are removed by one of the processes called “renormalization”. The present author once asked Richard Feynman if these infinite integrals represent a defect in the fundamental equations of QED or do they originate in the method used to solve the equations. He paused for a moment and then replied, “I don’t know”. The same question to Paul Dirac produced the answer, “The basic equations are at fault.” We shall see evidence here, however, that there may be no defect in the fundamental equations, but that the infinities may arise from the use of faulty Green’s functions as they do in many one-loop calculations.

Actually, this is a solution to a very old problem that held up the development of quantum electrodynamics (QED) in the years preceding 1947. Feynman\(^1\) noted that the infinite integrals that were blocking the calculations in those days were present because of the overlapping of the delta functions in the photon or fermion Green’s functions. He and a number of others have invented forms of renormalization to subtract the infinite terms. However, they were unable to avoid the appearance of these infinities in the first place. A means for avoiding the infinity in the electron self-energy has been published\(^2\). However, the renormalization procedure for the vacuum polarization contribution to the Lamb shift absorbs an infinite term into the electric charge. No way of guaranteeing a finite charge from this procedure has been suggested. A way of avoiding this problem will be shown here.

We note that Green’s functions are often used to solve problems in potential theory involving Poisson’s equation. These Green’s functions often have the form, \(|r_2 - r_1|^{-1}\). This is to be compared with Feynman’s\(^3\) propagator which is,

\[
K_+(2,1) = i(i\partial + m)I_+(2,1) \tag{1}
\]

where

\[
I_+(2,1) = -(4\pi)^{-1}\delta(s^2) + (m/8\pi s)H_1^{(2)}(ms) \tag{2}
\]

where \(s = (t^2 - x^2)^{1/2}\) for \(t^2 > x^2\) and \(s = -i(x^2 - t^2)^{1/2}\) for \(t^2 < x^2\). \(H_1^{(2)}\) is the Hankel function. The three dimensional Green’s function used in potential theory has no delta function while the four dimensional Green’s functions for the electron and photon do. When we consider the vacuum polarization electron-positron loops, it will be clear
that the delta functions from the two fermion propagators will have the same argument. This is an example of a case where an infinity occurs.

In Section II, a new fermion Green’s function will be derived in a manner similar to the derivations often used for potential theory. In Section III, they will be applied to obtain the formula for the contribution of the fundamental single-loop vacuum polarization to the Lamb shift. This formula is identical to that derived using dimensional renormalization. Some implications of this result will be discussed in Section IV.

II. THE FERMION GREEN’S FUNCTION

In order to ease the visualization of the steps required to define a covariant Green’s function, we make use of a Wick rotation to obtain a complex time coordinate defined by \( x^4 = i ct \) and \( \gamma^4 = i \gamma^0 \) so that \( \not\! \partial = \gamma^4 \partial_4 + \gamma^j \partial_j \). We define \( R^2 = (x^1)^2 + (x^2)^2 + (x^3)^2 + (x^4)^2 \) and \( \not\! \partial^2 = (\partial_4)^2 + \nabla^2 \).

We wish to solve the equation,

\[
(i \not\! \partial_1 - m) \Psi(1) = iS(1),
\]

where the underlines indicate matrices. We require the Green’s function \( K(2,1) \) to be a solution of

\[
K(2,1)(i \gamma \cdot \not\! \partial_1 + m) = 0
\]

where the arrow indicates differentiation to the left.

Our problem is to solve Eq. (3) for \( \Psi \) at a point 2 given its values at points numbered 1 on a surface enclosing it and the source \( S(1) \) lying inside the surface. See Figure 1. To avoid any singularity at point 2, we introduce a small sphere with radius \( \epsilon \), centered at the point 2, and shift point 2 to its surface. The radius, \( \epsilon \), will be made vanishingly small when all other calculations have been finished. To obtain an integral equation, we multiply Eq. (3) by \( K(2,1) \) from the left and integrate \( x_1 \) over the region exterior to the sphere containing point 2 and within the outer surface. Then we multiply Eq. (4) from the right by \( \Psi \) and again limit it to the region exterior to the sphere and within the outer surface. Finally we add the altered Eq. (4) to the altered Eq. (3) and divide by \( i \). The result is

\[
\int K(2,1) \gamma \cdot (\not\! \partial_1 + \not\! \partial_1) \Psi(1) d^4 x_1 = \int K(2,1) S(1) d^4 x_1
\]

The left hand side of this equation is equivalent to

\[
\int \partial_{1\mu} \cdot [K(2,1) \gamma^\mu \Psi(1)] d^4 x_1 = \int \Sigma [K(2,1) \gamma^\mu \Psi(1)] d\Sigma_{1\mu}
\]

\[
= \int_{\Sigma_0} [K(2,1) \gamma^\mu \Psi(1)] d\Sigma_{1\mu} - \int_{\sigma} [K(2,1) \gamma^\mu \Psi(1)] d\sigma_{1\mu}
\]

where we have made use of Gauss’s theorem, and where \( \Sigma \) is the surface in four dimensions that limits the integration. It is made up of the outer surface \( \Sigma_0 \) and the infinitesimal spherical surface \( \sigma \) containing the point 2. The derivation of the surface integral from the volume integral usually gives an outward pointing surface normal 4-vector for \( d\Sigma \) and this is the case for \( \Sigma_0 \), but we reverse the inward pointing surface normal 4-vector on \( \sigma \) so that the surface normal 4-vector \( d\sigma \) on this infinitesimal spherical surface points outward. This has reversed the sign in front of the integral over \( \sigma \) in the last equation.
When the radius $\epsilon$ gets very small, the point 1 in the integral over $\sigma$ approaches 2 so that $\Psi$ can be factored out of it. Replacing the left hand side of Eq. (5) by the right hand side of Eq. (6) then gives

$$\int_{\Sigma_0} [K(2, 1)\gamma^\mu \Psi(1)]d\Sigma_{1\mu} - \left[ \int_{\sigma} K(2, 1)d\sigma_{1\mu} \right] \gamma^\mu \Psi(2) = \int K(2, 1)S(1)d^4x_1. \quad (7)$$

In order to determine $K(2, 1)$, it is convenient to write it in terms of a scalar function so that

$$K(2, 1) = \phi(R_{21})(-i\gamma \cdot \nabla_2 + m). \quad (8)$$

Then if we multiply this equation from the right by $(i\gamma \cdot \nabla_2 + m)$, Eq. (4) shows that $\phi$ satisfies

$$(\Box^2 - m^2)\phi(R_{21}) = 0 \quad (9)$$

This is the equation satisfied by $\phi$ in the region between the surface $\sigma$ and the surface $\Sigma_0$. The integral over $\sigma$ in Eq. (7) is then, according to Eq. (8),

$$\int_{\sigma} K(2, 1)d\sigma_{1\mu} = -i\gamma^\eta \int_{\sigma} \partial_1\eta\phi(\epsilon)d\sigma_{1\mu} + m\phi(\epsilon) \int_{\sigma} d\sigma_{1\mu}. \quad (10)$$

The last integral vanishes because $d\sigma_{1\mu}$ is a component of a vector of constant length that will point in every direction with equal weight. Since

$$\partial_1\eta\phi(R_{21}) = x_1\eta\phi'(R_{21})/R_{21},$$

Eq. (10) reduces to

$$\int_{\sigma} K(2, 1)d\sigma_{1\mu} = -i\gamma^\eta \int_{\sigma} x_1\eta d\sigma_{1\mu} \phi'(\epsilon)/\epsilon. \quad (12)$$

Since $\sigma$ is a four-dimensional sphere, the value of the last integral is the same for any value of $\eta$. Let us choose four for $\eta$. The integral over the spatial dimensions will then vanish for values of $\mu$ unequal to four. We shall use $\theta$ and $\phi$ for the usual angles in three dimensions and add $\chi$ as the angle between an arbitrary vector and $x^4$. (See Ref. 4, Appendix B, for angles in four dimensions.) For a point on $\sigma$,

$$x_4 = g_44x^4 = x^4 = \epsilon \cos \chi$$

and

$$d\sigma^4_1 = \epsilon^3 \sin \theta \sin^2 \chi \cos \chi d\theta d\phi d\chi.$$ 

From these two equations, it is clear that

$$\int_{\sigma} x_1^4d\sigma^4_1 = \epsilon^4 \pi^2/2 \quad (13)$$

and

$$\int_{\sigma} x_1\eta d\sigma_{1\mu} = g_\eta \epsilon^4 \pi^2/2. \quad (14)$$

We can substitute this expression into Eq. (12) to obtain

$$\int_{\sigma} K(2, 1) d\sigma_{1\mu} = -i\gamma^\mu \epsilon^3 (\pi^2/2)\phi'(\epsilon).$$
If we reduce Eq. (9) to a radial equation, it will be Bessel’s equation. We need a solution \( \phi \) with the property that, when substituted into Eq. (12) along with the value of the integral in Eq. (14), it will cancel \( \epsilon^4 \) and leave a constant. Such a function is

\[
\phi(R) = aH_1^{(1)}(imR)/R = aH_1^{(2)}(-imR)/R
\]

where \( H_1^{(1)} \) is a Hankel function. The Bessel equation is linear so that \( a \) is an arbitrary constant. This function \( \phi(\epsilon) \) has the limiting form \( -2a/(\pi m \epsilon^2) \) as \( \epsilon \) vanishes. We now substitute this into Eq. (15) and the result, in turn, into Eq. (7). Then this equation can be rearranged to give

\[
\Psi(2) = -\int K(2,1)S(1)d^4x_1 + \int_{\Sigma_o} K(2,1)\gamma^\mu\psi(1)d\Sigma_1\mu
\]

where the constant \( a \) has been assigned the value \( im^2/8\pi \).

If we have a boundary \( \Sigma_o \) in the form of an initial flat constant time surface, a large sphere in space at intermediate times, and a final flat constant time surface, we can undo the Wick rotation and obtain the result,

\[
\Psi(2) = \int K(2,1)S(1)d^4x_1 - \int_f K(2,1)\gamma^0\psi(1)d^3x_1 + \int_i K(2,1)\gamma^0\psi(1)d^3x_1,
\]

where the subscript \( i \) indicates the initial and \( f \) indicates the final times. This equation agrees with Feynman’s \( ^3 \) Eq.(19) (on his page 754) if there is no source \( S \). However, Feynman’s Green’s function is given in Eqs. (1) and (2), and ours replaces Eq.(2) by

\[
I_+(2,1) = \phi(R_{21}) = (m/8\pi)H_1^{(2)}(-imR_{2,1})/R(2,1).
\]

This differs from Feynman’s by only a delta function. A similar derivation can be carried out for the photon, and, again, the Green’s function has no delta function although Feynman’s does. As Feynman points out, the overlapping of these delta functions produces many, perhaps all, of the infinities in QED. We will see in the next section that our Green’s function for fermions gives the single loop vacuum polarization for the hydrogen lamb shift without infinities.

### III. THE VACUUM POLARIZATION FOR THE LAMB SHIFT

A large number of text books present the calculation of the vacuum polarization contribution to the Lamb shift, and we need not repeat all of their analysis. The portion that interests us is the single-loop contribution to \( \Pi^{\mu\nu} \) that is represented by Fig. 2. It is a loop involving an electron and a positron propagator connecting a bare and a dressed vertex. In this case, the excluded part of the volume is enclosed by an infinitesimal sphere centered on one of the vertices in space-time. Its radius \( \epsilon \) is made vanishingly small at the end of the calculation. The setting up of the expressions to be evaluated would be most direct in configuration space-time since the inner excluded region is a sphere in that space. However these expressions are, as usual, most easy to evaluate in four-momentum space. For this purpose, we can use Feynman’s propagator and then subtract out the contributions that originate from the excluded sphere. This excludes the delta functions since they lie inside the spherical surface.
The expression (to lowest order) to be evaluated in this manner is:

\[ \Pi^{\mu\nu}(\eta) = i4\pi \alpha Tr \left[ \int S(p) \gamma^\mu S(p - \eta) \gamma^\nu d^4p/(2\pi)^4 \right] \] (20)

where an overline indicates a four vector, Tr indicates a trace, and \( S^{-1}(p) = \not{p} - m \). The four-vector \( \eta \) is the momentum of the photon connected to each of the vertices. Our first step is to perform the indicated trace. The resulting integral has an integrand with two nonidentical factors in its denominators. They can be made identical with the aid of Feynman’s transforms. The results can finally be reduced to

\[ \Pi^{\mu\nu}(\eta) = \frac{m^2\alpha}{2\pi} \int_{-1}^{1} \{ g^{\mu\nu}[(1 - \eta) M(\eta) + \frac{1}{2}L(\eta)] + (q^\mu q^\nu/q^2) 2\eta M(\eta) \} d\beta. \] (21)

where

\[ \eta = \frac{q^2(1 - \beta^2)}{4m^2}, \] (22)

\[ M(\eta) = M_0 + M_1(\eta), \] (23)

and

\[ L(\eta) = L_0 + \eta L_1 + L_2(\eta). \] (24)

In addition,

\[ M_0 = \int_0^\infty \frac{u \, du}{(1 + u)^2}, \] (25)

\[ L_0 = \int_0^\infty \frac{u^2 \, du}{(1 + u)^2}, \] (26)

\[ L_1 = -2 \int_0^\infty \frac{u^2 \, du}{(1 + u)^3}, \] (27)

\[ M_1(\eta) = \int_0^\infty \left[ \frac{1}{(1 + \eta + u)^2} - \frac{1}{(1 + u)^2} \right] \, du = -\ln(1 + \eta), \] (28)

\[ L_2(\eta) = \int_0^\infty \left[ \frac{1}{(1 + \eta + u)^2} - \frac{1}{(1 + u)^2} + \frac{2\eta}{(1 + u)^3} \right] u^2 \, du \]

\[ = 2[(1 + \eta) \ln(1 + \eta) - \eta]. \] (29)

The integrals \( M_0, L_0, \) and \( L_1 \) are independent of \( \eta \) and, therefore, of \( q^2 \). When they are substituted into \( \Pi^{\mu\nu} \) and multiply another term that is independent of \( \eta \), the Fourier transform to space-time produces a delta function located at the center of the spherical excluded region. For this reason, they should not have been included in the expression for \( \Pi^{\mu\nu} \). Similarly, when \( M_0, L_1, \) or \( M_0 \) multiply the \( q^{2n} \) in \( \eta^n \), this produces a \( \Delta^{2n} \) operating on a delta function. Thus when they are transformed back to space-time, they are also confined to the center of the sphere. They, too, should not have been included in \( \Pi^{\mu\nu} \). The same is true for the for the term \( 2\eta \) that appears in \( L_2 \). This leaves only \( M_1(\eta) \) in Eq(28) and the \( 2(1 + \eta) \ln(1 + \eta) \) portion of \( L_2 \) to be substituted into \( \Pi^{\mu\nu} \). Substituting these symbols into \( \Pi^{\mu\nu} \) then gives

\[ \Pi^{\mu\nu} = \frac{m^2\alpha}{2\pi} \int_{-1}^{1} \left\{ g^{\mu\nu} \left[ -(1 - \eta) \ln(1 + \eta) + [(1 + \eta) \ln(1 + \eta) - \eta] - \frac{q^\mu q^\nu}{q^2} 2\eta \ln(1 + \eta) \right] \right\} d\beta. \] (30)
The $\eta$ should not be shown because it produces the derivative of a delta function in the sphere, and the remainder of the integrand cancels down to leave

$$\Pi^{\mu\nu} = \frac{m^2\alpha}{\pi} \int_{-1}^{1} \left( g^{\mu\nu} - \frac{q^\mu q^\nu}{q^2} \right) \eta \ln(1 + \eta) d\beta. \quad (31)$$

Since $\eta$ depends upon $\beta^2$, the integral can be written as twice the integral from zero to one.

In order to make a comparison with the work of others, it is convenient to define $x$ so that, from Eq. (22),

$$\eta = \frac{q^2}{4m^2} (1 - \beta^2) = \frac{q^2}{m^2} x (1 - x). \quad (32)$$

from which $\beta = 1 - 2x$. Substituting these expressions into Eq. (31) then gives

$$\Pi^{\mu\nu} = \left( q^2 g^{\mu\nu} - q^\mu q^\nu \right) \pi(q^2) \quad (33)$$

where

$$\pi(q^2) = \frac{2\alpha}{\pi} \int_{0}^{1} x(1 - x) \ln \left( 1 + \frac{q^2}{m^2} x(1 - x) \right) dx. \quad (34)$$

To convert the fine structure constant $\alpha$ to $e^2$ where $e$ is the charge associated with each vertex, we divide by $4\pi$, with the result that our equation becomes

$$\pi(q^2) = \frac{e^2}{2\pi^2} \int_{0}^{1} x(1 - x) \ln \left( 1 + \frac{q^2}{m^2} x(1 - x) \right) dx. \quad (35)$$

This equation is identical to Weinberg’s\textsuperscript{5} Eq (11.2.22) obtained by using dimensional renormalization. He uses it to derive a contribution of -27.13MHz to the Lamb shift of hydrogen. This number is in agreement with experiment.

**IV. DISCUSSION**

The method presented here for the evaluation of the one loop vacuum polarization contribution to the photon self energy avoids any infinities and, therefore, has no need for renormalization. The same is true for the self energy diagram for the electron presented in Reference 2. It is, therefore possible to compute the Lamb shift for the hydrogen atom to the lowest order without renormalization. If such finite techniques can be shown to work to all orders, it will indicate that the equations of quantum electrodynamics are without defects and that the presence of infinities in calculations with them are due to faulty techniques used to solve them.
V. ACKNOWLEDGEMENTS

This is the third paper in a series that would not have been possible without the mathematical techniques introduced by the late H. S. Green.
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Figure 1  The outer surface $\Sigma_0$ includes $t_i$, $t_f$, and the space cylindrical surface. The inner sphere $\sigma$ has a radius $\epsilon$ that goes to zero in the end. The point $S$ lies in the source.
Figure 2  Vacuum Polarization Feynman diagram with surface $\sigma$ enclosing the excluded region.