Equatorial light bending around Ker-Newman black holes

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Abstract

We study the deflection angle of a light ray as it traverses on the equatorial plane of a charged spinning black hole. We provide the detailed analysis on the light ray’s trajectory, and derive the closed-form expression of the deflection angle due to the black hole in terms of elliptic integrals. In particular, the geodesic equation of the light ray along the radial direction can be used to define an appropriate “effective potential”. The non-zero charge of the black hole shows stronger repulsive effects to prevent light rays falling into the black hole as compared with the Kerr case. As a result, the radius of the innermost circular motion of light rays with the critical impact parameter decreases as charge $Q$ of the black hole increases for both direct and retrograde motions. Additionally, the deflection angle decreases when $Q$ increases with the fixed impact parameter. These results will have a direct consequence on constructing the apparent shape of a rotating charged black hole.

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I. INTRODUCTION

General relativity provides a unified description of gravity as a geometric property of spacetime [1, 2]. In particular, the presence of matter and radiation with energy and momentum can curve spacetime, and the light ray’s trajectory will be deflected as a chief effect [3]. One of the very important consequences of general relativity is the bending of a light ray in the presence of a gravitational field. In the light of the first image of the black hole captured by the Event Horizon Telescope [4–6], these rays will yield the apparent shadow of the black hole for an observer in the asymptotic region, and the understanding of the shadows becomes very important for measuring the properties of astrophysical black holes.

Light deflection in weak gravitational field of Schwarzschild black holes was known in 1919, and served as the starting point to develop gravitational lensing theory. Nevertheless, light deflection in strong gravitational field of Schwarzschild black holes was not studied until several decades ago by Darwin [7]. It was then re-examined in [8–11], and extended to the Reissner-Nordstrom spacetime [12], and to any spherically symmetric black holes [13]. Black hole lenses were also explored numerically by [14–16]. Kerr black hole lenses were analyzed in [17–23], which, in particular, found rotating black hole apparent shapes or shadows with an optical deformation rather than being the circles as in the case of non-rotating ones [24–30]. The bending angle of light rays due to Kerr black holes on the equatorial plane was studied analytically in [32–35] using the null geodesic equations. The deflections produced in the presence of a rotating black hole explicitly depend on the direction of motion of the light relative to the spin direction of the black hole. Especially, the authors of [32] derived the closed-form expression of the equatorial light deflection angle in terms of elliptic integrals. However, the strong gravitation field gives rise to the large bending of light near a black hole. The bending angle can be larger than $2\pi$, showing the possibility that light rays might go around the center of the black hole several times before reaching the observer. Apart from a primary image, a theoretically infinite sequence of images, which we term relativistic images, might be formed, and are usually greatly demagnified. The closed-form expression of the light bending angle in an exact result [32] or in some sort of asymptotic approximations might be of great help to study these images [34, 35], although the observation of relativistic images is a very difficult task.

Another known asymptotically flat, stationary solution of the Einstein-Maxwell field
equations in general relativity is the Kerr-Newman metric, a generalization of the Kerr metric, which describes spacetime in the exterior of a rotating charged black hole. Apart from gravitation fields, both electric and magnetic fields exist intrinsically from the black hole. Although one might not expect that astrophysical black holes have a large residue electric charge, some accretion scenarios were proposed to investigate the possibility of the spinning charged back holes [31]. It is then still of great interest to extend the previous studies to a Kerr-Newman black hole [36–39]. The central thread of this paper is to try to achieve the exact expression of the light deflection angle by a Kerr-Newman black hole in the equatorial plane, an extension of the work in [32] for the Kerr black hole case. In Sec.II, we focus on circular trajectories of light rays arriving from and returning to the spatial infinity. Their null geodestic equations along the radial direction on the equatorial plane of the black hole can be analogously realized as particle motion in the effective potential. Sec. III explores the effects of the black hole charge on the circular trajectories via this effective potential. In particular, we solve the geodesic equations to find the radius of innermost circular trajectories and its corresponding impact parameter in terms of the black hole’s parameters. Sec. IV derives a closed-form expression for the equatorial light deflection angle. We have verified that, by taking the limit of $Q = 0$, our result reduces to the case of Kerr black hole, obtained by work [32]. All results will be summarized in the closing section.

II. GEODESTIC EQUATIONS AND INMOST CIRCULAR TRAJECTORIES OF LIGHT RAYS IN KERR-NEWMAN SPACETIME

In this paper, we thoroughly study the light bending due to Kerr-Newmann black hole, in which spacetime outside a black hole with the gravitational mass $M$, charge $Q$, and angular momentum per unit mass $a = J/M$ is described by the Kerr-Newman metric as

\[
\begin{align*}
    ds^2 &= g_{\mu\nu}dx^\mu dx^\nu \\
    &= -\frac{(\Delta - a^2 \sin^2 \theta)}{\Sigma} dt^2 + \frac{a \sin^2 \theta (Q^2 - 2Mr)}{\Sigma} (dt d\phi + d\phi dt) \\
    &\quad + \frac{\Sigma}{\Delta} dr^2 + \Sigma d\theta^2 + \frac{\sin^2 \theta}{\Sigma} \left((r^2 + a^2)^2 - a^2 \Delta \sin^2 \theta\right) d\phi^2,
\end{align*}
\]

where

\[
\begin{align*}
    \Sigma &= r^2 + a^2 \cos^2 \theta, \quad \Delta = r^2 + a^2 + Q^2 - 2Mr.
\end{align*}
\]
FIG. 1: Sign convention for orbits as viewed from above. The spin axis of the black hole points out of the page in this figure. The red (blue) solid line shows the direct (retrograde) orbit with the radius of closest approach $r_0$. The suffix "c" is added in the case of the intermost circular motion.

The event horizon $R_H$ can be found by solving $\Delta(r) = 0$, and is given by

$$R_H = M + \sqrt{M^2 - (Q^2 + a^2)}$$  \hspace{1cm} (3)

with the condition $M^2 > Q^2 + a^2$. The Lagrangian of a particle is then

$$\mathcal{L} = \frac{1}{2} g_{\mu\nu} u^\mu u^\nu,$$  \hspace{1cm} (4)

with the 4-velocity $u^\mu = dx^\mu/d\lambda$ defined in terms of an affine parameter $\lambda$.

Due to the fact that the metric of Kerr-Newman black hole is independent of $t$ and $\phi$, the associated Killing vectors are $\xi^\mu_{(t)}$ and $\xi^\mu_{(\phi)}$ given, respectively, by

$$\xi^\mu_{(t)} = \delta^\mu_t, \quad \xi^\mu_{(\phi)} = \delta^\mu_\phi.$$  \hspace{1cm} (5)

Then, together with 4-velocity of light rays, the conserved quantities, namely energy and azimuthal angular momentum, along a geodesic, can be constructed by the above Killing vectors

$$\varepsilon \equiv -\xi^\mu_{(t)} u_\mu = \frac{1}{\Sigma} \left[ a \left( \ell - \varepsilon a \sin^2 \theta \right) + \frac{(r^2 + a^2) \left[ \varepsilon (r^2 + a^2) - a\ell \right]}{\Delta} \right],$$  \hspace{1cm} (6)

$$\ell \equiv \xi^\mu_{(\phi)} u_\mu = \frac{1}{\Sigma} \left[ \frac{\ell - \varepsilon a \sin^2 \theta}{\sin^2 \theta} + \frac{a [\varepsilon (r^2 + a^2) - a\ell]}{\Delta} \right],$$  \hspace{1cm} (7)

where $\varepsilon$ and $\ell$ are the light ray’s energy and azimuthal angular momentum evaluated at spatial infinity. Light rays travel along null world lines obeying the condition $u^\mu u_\mu = 0$. 

Additionally, there exists a Carter constant

$$\kappa = u_\mu u_\nu K^{\mu\nu} - (\ell - a\varepsilon)^2,$$  \hspace{1cm} (8)

where

$$K^{\mu\nu} = \Delta k^\mu q^\nu + r^2 g^{\mu\nu},$$

$$q^\mu = \frac{1}{\Delta} [ (r^2 + a^2) \delta_t^\mu + \Delta \delta_t^\mu + a a_t^\mu ],$$

$$k^\mu = \frac{1}{\Delta} [ (r^2 + a^2) \delta_t^\mu - \Delta \delta_t^\mu + a a_t^\mu ].$$ \hspace{1cm} (9)

Using these three constants of motion together with \( u_\mu u^\mu = 0 \), we are able to write down the general geodesic equations for light rays in terms of \( \varepsilon, \ell, \) and \( \kappa \) as

$$\Sigma \dot{t} = -a \left( a \varepsilon \sin^2 \theta - \ell \right) + \left( r^2 + a^2 \right) \frac{[\varepsilon (r^2 + a^2) - a \ell]}{\Delta},$$ \hspace{1cm} (10)

$$\Sigma \dot{\phi} = - \left( a \varepsilon - \frac{\ell}{\sin^2 \theta} \right) + \frac{a [\varepsilon (r^2 + a^2) - a \ell]}{\Delta},$$ \hspace{1cm} (11)

$$\Sigma^2 \dot{r}^2 = (\varepsilon (r^2 + a^2) - a \ell)^2 - \Delta \left[ (\ell - a \varepsilon)^2 + \kappa \right],$$ \hspace{1cm} (12)

$$\Sigma^2 \dot{\theta}^2 = \kappa + \cos^2 \theta \left( a^2 \varepsilon^2 - \frac{\ell^2}{\sin^2 \theta} \right).$$ \hspace{1cm} (13)

The over dot means the derivative with respect to the affine parameter \( \lambda \). To indicate whether the light ray is traversing along the direction of frame dragging or opposite to it, we define the following impact parameter :

$$b_s = s \left| \frac{\ell}{\varepsilon} \right| \equiv s b,$$ \hspace{1cm} (14)

where \( s = \text{Sign}(\ell/\varepsilon) \) and \( b \) is the positive magnitude. The parameter \( s = +1 \) for \( b_s > 0 \) will be referred to as direct orbits; and those with \( s = -1 \) for \( b_s < 0 \) as retrograde orbits (see Fig.(1) for the sign convention).

Here we restrict the light rays traveling on the equatorial plane of the black hole by choosing \( \theta = \pi/2 \), and \( \dot{\theta} = 0 \), so that \( \kappa = 0 \) in Eq. (13). Rewriting the equation of motion along the radial direction, (12) allows us to define the function \( W_{\text{eff}} \) from

$$\frac{1}{b^2} = \frac{\dot{r}^2}{r^2} + W_{\text{eff}}(r),$$ \hspace{1cm} (15)

where

$$W_{\text{eff}}(r) = \frac{1}{r^2} \left[ 1 - \frac{a^2}{b^2} + \left( -2M + \frac{Q^2}{r^2} \right) \left( 1 - \frac{a}{b_s} \right)^2 \right].$$ \hspace{1cm} (16)
FIG. 2: The “effective potential” $W_{\text{eff}}$ as a function of $r/M$ for four sets of parameters. (a) $a/M = 0.5$, $Q/M = 0.3$, and $b = 8M$; (b) $a/M = 0.5$, $Q/M = 0.6$, and $b = 8M$; (c) $a/M = 0.5$, $Q/M = 0.3$, and $b = 6M$; (d) $a/M = 0.5$, $Q/M = 0.6$, and $b = 6M$. The plot convention used henceforth: Kerr-Newman direct (red), Kerr-Newman retrograde (blue), Kerr direct (dashed with $Q = 0$), Kerr retrograde (dotted with $Q = 0$), Reissner-Nordström (purple with $a = 0$), and Schwartzschild (black with $Q = 0, a = 0$).

The above equation is analogous to that of particle motion in the effective potential $W_{\text{eff}}(r)$ shown in Fig.(2) with the kinetic energy $r^2/\ell^2$ and constant total energy $1/b^2$ [2, 37].

In the limits of $Q = 0$ and $Q = 0, a = 0$, the effective potential $W_{\text{eff}}$ reduces to the ones for the Kerr and Schwarzchild black holes respectively [2]. In the case of the Kerr-Newman
black hole, the non-zero charge of the black hole seems to give repulsive effects to the light rays as seen from its contributions to the centrifugal potential of the $1/r^2$ term. These repulsive forces in turn affect light rays so as to prevent them from collapsing into the event horizon, and thus as will be seen later, shift the innermost circular trajectories of the light rays toward the black hole. Also, the presence of black hole’s charge is found to decrease the deflection angle due to this additional repulsive effect on the light rays, as compared with the Kerr’s case for the the same impact parameter $b$.

To see it, let us consider a light ray that starts in the asymptotic region to approach the black hole, and then turn back to the asymptotic region to reach the observer. Such light rays have a turning point, the radius of closest approach to a black hole $r_0$, which crucially depends on the impact parameter $b$, determined by

$$\left. \frac{r^2}{\ell^2} \right|_{r=r_0} = \frac{1}{b^2} - W_{\text{eff}}(r_0) = 0.$$  \hspace{1cm} (17)

Eq. (17) leads to a quartic equation in $r_0$

$$r_0^4 - b^2 \left(1 - \frac{a^2}{b^2}\right) r_0^2 + 2Mb^2 \left(1 - \frac{a}{b_s}\right)^2 r_0 - Q^2b^2 \left(1 - \frac{a}{b_s}\right)^2 = 0.$$  \hspace{1cm} (18)

We express Ferrari’s solutions in terms of trigonometric functions, where one of the three roots of (18) gives the analytical result of $r_0$ as

$$r_0(b_s) = \frac{b}{\sqrt{6}} \sqrt{1 - \omega_s^2} \left\{ \sqrt{1 + \sqrt{1 - \Omega_s \cos \left(\frac{2\Theta}{3}\right)}} + \sqrt{2 - \sqrt{1 - \Omega_s \cos \left(\frac{2\Theta}{3}\right)} - \frac{3\sqrt{6}M(1 - \omega_s)^2}{b(1 - \omega_s^3)^2 \sqrt{1 + \sqrt{1 - \Omega_s \cos \left(\frac{2\Theta}{3}\right)}}} \right\},$$  \hspace{1cm} (19)

with $\omega_s$, $\Omega_s$ and $\Theta$, depending explicitly on the black hole’s parameters, as well as the impact parameter $b_s$, defined as

$$\omega_s = \frac{a}{b_s},$$

$$\Omega_s = \frac{12Q^2}{b^2 (1 + \omega_s)^2},$$

$$\Theta = \arccos \left( \frac{3\sqrt{3}M (1 - \omega_s)^2}{b (1 - \omega_s^3)^2 (1 - \Omega_s)^{\frac{3}{2}}} \sqrt{1 - \frac{b^2 (1 + \omega_s)^3}{54M^2 (1 - \omega_s)} \left[ 1 + 3 \Omega_s - (1 - \Omega_s)^{\frac{3}{2}} \right]} \right).$$  \hspace{1cm} (20)
In the case of Kerr black hole with $Q = 0$, we have

$$
\Omega_s = 0, \\
\Theta = \arccos \left( \frac{3\sqrt{3}M(1 - \omega_s)^2}{b(1 - \omega_s^2)^{\frac{3}{2}}} \right).
$$

Equation (19) reduces to $r_0(b)$ in [32] as

$$
r_0(b_s) = \frac{b}{\sqrt{6}} \sqrt{1 - \omega_s^2} \left\{ \sqrt{1 + \cos \left( \frac{2\Theta}{3} \right)} + \sqrt{2 - \cos \left( \frac{2\Theta}{3} \right)} - \frac{3\sqrt{6}M(1 - \omega_s)^2}{b(1 - \omega_s^2)^{\frac{3}{2}} \sqrt{1 + \cos \left( \frac{2\Theta}{3} \right)}} \right\}
$$

$$
= \frac{2b}{\sqrt{3}} \sqrt{1 - \frac{a^2}{b^2}} \cos \left[ \frac{1}{3} \arccos \left( -\frac{3\sqrt{3}M \left( 1 - \frac{a}{b} \right)^2}{b \left( 1 - \frac{a^2}{b^2} \right)^{\frac{3}{2}}} \right) \right].
$$

The distance of closest approach for the light ray to travel around Kerr-Newman black holes, given in Eq.(19), certainly generalize that of the Kerr or the Schwarzschild black holes, depicted in Fig.(3).

![FIG. 3: The distance of closest approach $r_0/M$ as a function of the impact parameter $b/M$ for (a) $a/M = 0.5$ and $Q/M = 0.3$; (b) $a/M = 0.5$ and $Q/M = 0.6$. The plots show the results for the Schwarzschild, Reissner-Nordström, Kerr and Kerr-Newman black holes for comparison.](image)
It is anticipated that for both direct and retrograde motions, the repulsive effects from the charge of the black hole pushes the distance of closest approach $r_0$ being away from the black hole for a fixed impact parameter $b$, which can be compared with the Kerr case with the same impact parameter $b$.

III. CRITICAL IMPACT PARAMETERS AND INNERMOST CIRCULAR ORBITS

Again, consider the light rays coming in from spatial infinity with the impact parameter $b$. The plots in Fig. (2) show the shape of the effective potential $W_{\text{eff}}(r)$ that vanishes at large $r$ and has one maximum. The behavior of the light ray trajectories depends on whether $1/b^2$ is greater or less than the maximum height of $W_{\text{eff}}(r)$. The innermost trajectories of light rays have a direct consequence on the apparent shape of the black hole with the smallest radius $r_c$ when the turning point $r_0$ is located at the maximum of $W_{\text{eff}}(r)$ for a particular choice of $b_c$ obeying

$$\frac{dW_{\text{eff}}(r)}{dr} \bigg|_{r=r_{sc}} = 0,$$

with the value

$$r_{sc} = \frac{3M}{2} \left( 1 - \frac{a}{b_{sc}} \right) \left[ 1 + \sqrt{1 - \frac{8Q^2 (1 + a/b_{sc})}{9M^2 (1 - a/b_{sc})}} \right],$$

which is a circular motion forming a photon sphere. However, these circular trajectories are unstable because any small change in $b$ results in the trajectory moving away from the maximum. The radius of the circular photon orbit $r_{sc}$ above is consistent with the finding in [3], and the known result of $r_{sc}$ in the limit $Q = 0$ giving in [32]. We then substitute (22) into (18) for obtaining the corresponding critical impact parameter $b_{sc}$.

At this step, it finds more convenient to express the equations in terms of $y_+$ and $y_-$, respectively [32],

$$y_+ = b_+ + a,$$

$$y_- = -(b_- + a),$$

$$\frac{dW_{\text{eff}}(r)}{dr} \bigg|_{r=r_{sc}} = 0,$$
which obey the quartic equations

\[
\begin{align*}
(Q^2 - M^2)y_+^4 + 2M^2ay_+^3 + (27M^4 - 36M^2Q^2 + 8Q^4)y_+^2 \\
- (108M^4a - 72M^2Q^2a)y_+ + (108M^4a^2 + 16Q^6) = 0, \\
(Q^2 - M^2)y_-^4 - 2M^2ay_-^3 + (27M^4 - 36M^2Q^2 + 8Q^4)y_-^2 \\
+ (108M^4a - 72M^2Q^2a)y_- + (108M^4a^2 + 16Q^6) = 0,
\end{align*}
\]

where we have used Eq.(18). The solutions of the critical value \( b_{sc} \) in a Kerr-Newman black hole are found analytically to be

\[
b_{sc} = -a + \frac{M^2a}{2(M^2 - Q^2)} + \frac{s}{2\sqrt{3}(M^2 - Q^2)} \left[ \sqrt{V + (M^2 - Q^2)} \left( U + \frac{P}{U} \right) \\
+ \sqrt{2V - (M^2 - Q^2)} \left( U + \frac{P}{U} \right) - \frac{6\sqrt{3}M^2a [(M^2 - Q^2)(9M^2 - 8Q^2)^2 - M^4a^2]}{\sqrt{V + (M^2 - Q^2)} (U + \frac{P}{U})} \right]
\]

\begin{equation}
\tag{26}
\end{equation}

where

\[
\begin{align*}
P &= (3M^2 - 4Q^2) \left[ 9(3M^2 - 4Q^2)^3 + 8Q^2(9M^2 - 8Q^2)^2 - 216M^4a^2 \right], \\
U &= \left\{ - \left[ 3(3M^2 - 2Q^2)^2 - 4Q^4 \right] \left[ 9M^2(9M^2 - 8Q^2)^3 - 8 \left[ 3(3M^2 - 2Q^2)^2 - 4Q^4 \right]^2 \right] \\
&\quad + 108M^4a^2 \left[ 9(3M^2 - 4Q^2)^3 + 4Q^2(9M^2 - 8Q^2)^2 - 54M^4a^2 \right] \\
&\quad + 24\sqrt{3}M^2 \sqrt{(M^2 - a^2 - Q^2)[Q^2(9M^2 - 8Q^2)^2 - 27M^4a^2]^3} \right\}^{\frac{1}{3}}, \\
V &= 3M^4a^2 + (M^2 - Q^2) \left[ 6(3M^2 - 2Q^2)^2 - 8Q^4 \right].
\end{align*}
\]

\begin{equation}
\tag{27}
\end{equation}

Although one can numerically check the consistency between the expression of (26) in the limit of \( Q = 0 \) and that of the Kerr case given by [32]

\[
b_{sc} \to -a + s6M \cos \left[ \frac{1}{3} \cos^{-1} \left( \frac{-sa}{M} \right) \right], \quad \tag{28}
\]

direct simplification to recover (28), by setting \( Q = 0 \) in (26), is not so trivial to achieve. An alternative consistency check is to consider that the above two quartic equations (24) and (25) in the limit of Kerr case, \( Q = 0 \) lead to

\[
\begin{align*}
- M^2 (y_+ - 2a) \left( y_+^3 - 27M^2y_+ + 54M^2a \right) &= 0, \\
- M^2 (y_- + 2a) \left( y_-^3 - 27M^2y_- - 54M^2a \right) &= 0.
\end{align*}
\]
FIG. 4: The critical impact parameter $b_{sc}/M$ as a function of the spin parameter $a/M$ for (a) $Q/M = 0.3$, (b) $Q/M = 0.6$. The plots show the Schwarzschild, Reissner-Nordström, Kerr and Kerr-Newman black holes for comparison.

Their solutions certainly give (28) using (23). Then, the radius of innermost circular trajectories $r_{sc}$ with the impact parameter $b_{sc}$ can be obtained through (22), which will be a tedious function of the black hole’s parameters.

According to [39], in fact, the equation to determine the $r_c$ in terms of the black hole’s parameters can be derived directly. Let us consider a particle with mass $m$ moving around the Kerr-Newman black hole. There exists the circular motion of the particle with the radius $r$ when the energy $E$ and azimuthal angular momentum $L$ satisfying [36, 40]

$$\frac{E}{m} = \frac{a\sqrt{Mr - Q^2} + (Q^2 + r^2 - 2Mr)}{r\sqrt{2Q^2 + r^2 - 3Mr + 2a(Mr - Q^2)^{1/2}}}, \quad (29)$$

$$\frac{L}{m} = \frac{a(Q^2 - 2Mr) + (a^2 + r^2)\sqrt{Mr - Q^2}}{r\sqrt{2Q^2 + r^2 - 3Mr + 2a(Mr - Q^2)^{1/2}}}. \quad (30)$$

Apparently, the radius of the circular motion can not be arbitrarily small. In particular, the case of massless limit ($m \to 0$), and the conditions of finite values of $E$ and $L$ require the
radius of the circular photon orbit to obey

\[ 2Q^2 + r_c^2 - 3Mr_c + 2a(Mr_c - Q^2)^{1/2} = 0. \]  

(31)

In the limit of \( Q = 0 \), we have one of the roots given by

\[ r_{sc} = 2M \left\{ 1 + \cos \left[ \frac{2}{3} \cos^{-1} \left( -\frac{sa}{M} \right) \right] \right\}, \]  

(32)

in agreement with the Kerr case in [3, 32]. In the general situation of finite \( Q \), the solution of \( r_c \) becomes

\[
\begin{align*}
r_{sc} &= \frac{3M}{2} + \frac{1}{2\sqrt{3}} \sqrt{9M^2 - 8Q^2 + U_c + \frac{P_c}{U_c}} \\
&\quad - \frac{s}{2} \sqrt{6M^2 - \frac{16Q^2}{3} - \frac{1}{3} \left( U_c + \frac{P_c}{U_c} \right)} + \frac{8\sqrt{3}Ma^2}{\sqrt{9M^2 - 8Q^2 + U_c + \frac{P_c}{U_c}}},
\end{align*}
\]

(33)
FIG. 6: The critical impact parameter $b_{sc}/M$ as a function of charge $Q/M$ for (a) $a/M = 0.3$, (b) $a/M = 0.6$. The plots show the results for the Schwarzschild, Reissner-Nordström, Kerr and Kerr-Newman black holes for comparison.

where

$$P_c = (9M^2 - 8Q^2)^2 - 24a^2(3M^2 - 2Q^2),$$

$$U_c = \left\{ (9M^2 - 8Q^2)^3 - 36a^2(9M^2 - 8Q^2)(3M^2 - 2Q^2) + 216M^2a^4 
+ 24\sqrt{3a^2}\sqrt{(M^2 - a^2 - Q^2)} [Q^2(9M^2 - 8Q^2)^2 - 27M^4a^2] \right\}^{\frac{1}{3}}. \quad (34)$$

For the Reissner-Nordstrom black-holes, $a \to 0$, we find

$$P_c = U_c^2 = (9M^2 - 8Q^2)^2$$

with

$$r_{sc} = \frac{3M}{2} \left( 1 + \sqrt{ \frac{8Q^2}{9M^2} } \right)$$

as anticipated. Combining (22) with (18) can derive the following useful relation

$$b_{sc} = a + \frac{r_{sc}^2}{\sqrt{Mr_{sc} - Q^2}}. \quad (35)$$
Thus, substituting the solution of $r_{sc}$ in (33) into the above relation can obtain the result of $b_{sc}$ in terms of the black hole’s parameters in stead.

Plugging in the value for all parameters, we reproduce the critical impact parameter $b_{sc}$, and the corresponding radius of the innermost circular motion $r_{sc}$ in Figs.(4) and (5), plotted as a function of $a$ for the Kerr black hole [32], and additionally, for the the Kerr-Newman black holes. Due to the fact that the charge of black hole gives the repulsive effects to the light rays that prevent them from collapsing into the black hole, it is found that the circular orbits exist for a smaller value of the radius $r_{sc}$ with the smaller impact parameter $b_{sc}$ as compared with the Kerr case for the same $a$. Also, the radius of the innermost circular motion of light rays with the critical impact parameter decreases as charge $Q$ of the black hole increases for both direct and retrograde motions seen in Figs.(6) and (7).
IV. THE EXACT EXPRESSION OF EQUATORIAL LIGHT DEFLECTION ANGLE

To obtain the closed-form deflection angle of a light ray due to the Kerr-Newman black hole, we introduce

\[ u = \frac{1}{r} \]

and rewrite (11) and (12), with further combinations, as

\[ \left( \frac{du}{d\phi} \right)^2 = \frac{1 - 2Mu + (a^2 + Q^2)u^2}{1 - (2Mu - Q^2u^2) \left( 1 - \frac{a}{b_s} \right)} B(u) , \]

where the quantity \( B(u) \) is a quartic polynomial

\[ B(u) = -Q^2 \left( 1 - \frac{a}{b_s} \right)^2 u^4 + 2M \left( 1 - \frac{a}{b_s} \right)^2 u^3 - \left( 1 - \frac{a^2}{b^2} \right) u^2 + \frac{1}{b^2} . \]

The deflection angle \( \hat{\alpha} \) as the light ray proceeds in from spatial infinity and back out again is just twice the deflection angle from the turning point \( r = r_0 \) to infinity. Integrating the equation of motion for \( \phi \) in (37) and subtracting \( \pi \) from it lead to

\[ \hat{\alpha} = -\pi + 2 \int_0^{1/r_0} \frac{1 - (2Mu - Q^2u^2) \left( 1 - \frac{a}{b_s} \right)}{1 - 2Mu + (a^2 + Q^2)u^2} \frac{du}{\sqrt{B(u)}} . \]

With \( B(u) \) that generally has three real positive roots \( u_2, u_3 \) and \( u_4 \) and one real negative root \( u_1 \), we can rewrite (38) as

\[ B(u) = -Q^2 (1 - \omega) (u - u_1)(u - u_2)(u - u_3)(u - u_4) . \]

By extending the approach of [32], we parametrize the four roots of \( B(u) = 0 \) as follows:

\[ u_1 = \frac{X - 2M - Y}{4Mr_0} , \]

\[ u_2 = \frac{1}{r_0} , \]

\[ u_3 = \frac{X - 2M + Y}{4Mr_0} , \]

\[ u_4 = \frac{2M}{Q^2} - \frac{X}{2Mr_0} , \]

where we have introduced two functions \( X \) and \( Y \), to be determined later. We assume that the functions \( X \) and \( Y \) in the limit of \( Q = 0 \) smoothly reduce to the corresponding functions
in a Kerr black hole in [32]. If so, by taking the $Q = 0$ limit, the function $B(u)$ in a Kerr black hole in [32] can be recovered from (40), where the root $u_4$ is removed. Comparing the coefficients of different powers of $u$ in $B(u)$ to those in the original polynomial in (38), we obtain the equations to determine the functions $X$ and $Y$ as

\begin{align}
Q^2 \left[ Y^2 - (X - 2M)(X + 6M) + 4X^2 \right] &= 16M^2 r_0 \left( X - r_0 \frac{1 + \omega}{1 - \omega} \right), \\
Y^2 - (X - 2M)^2 &= \frac{8M(X - 2M)(Q^2 X - 4M^2 r_0)}{Q^2(X - 2M) - 4M^2 r_0}, \\
\left[ Y^2 - (X - 2M)^2 \right] \left( \frac{1}{8Mr_0^3} - \frac{Q^2 X}{32M^3 r_0^4} \right) &= \frac{1}{b^2(1 - \omega)^2}.
\end{align}

Notice that these equations are given in terms of the black hole’s parameters as well as the distance of closest approach $r_0$ of a light ray with the impact parameter $b$. The substitution (45) into (47) reproduces (18), being $1/r_0$ one of the roots. The combination of (45) and (46), with further rearrangement, gives a cubic equation of $X$

\begin{align}
\frac{Q^2}{2M} X^3 - (Q^2 + 4Mr_0)X^2 + \left( 4M^2 r_0 + 2MQ^2 + \frac{8M^3 r_0^2}{Q^2} + \frac{2Mr_0^2(1 + \omega)}{(1 - \omega)^3} \right) X
&= 4M^2 Q^2 + \frac{4M^2 r_0^2(1 + \omega)}{(1 - \omega)} + \frac{8M^3 r_0^4(1 + \omega)}{Q^2(1 - \omega)}.\tag{48}
\end{align}

We can solve directly the cubic equation (48), and one of three roots is

\begin{align}
X(M, Q, r_0, \omega) &= \frac{2M(Q^2 + 4Mr_0)}{3Q^2} \\
&\quad + \frac{8M^2 r_0}{3Q^2} \sqrt{1 + \frac{Q^2}{2M^2 r_0} \left( M - \frac{3r_0(1 + \omega)}{2(1 - \omega)} - \frac{Q^2}{r_0} \right) \cos \left( \frac{\Theta_X}{3} + \frac{2\pi}{3} \right)}, \tag{49}
\end{align}

where

\begin{align}
\Theta_X &= \arccos \left[ \frac{-8M^3 r_0^3 - 3MQ^2 r_0^2 \left( 2M - \frac{3r_0(1 + \omega)}{(1 - \omega)^3} \right) - 3Q^4 r_0 \left( 5M - \frac{3r_0(1 + \omega)}{(1 - \omega)^3} \right) + 10Q^6}{4M^2 r_0^2 + Q^2 r_0 \left( 2M - \frac{3r_0(1 + \omega)}{(1 - \omega)^3} \right) - 2Q^4} \right]^{\frac{3}{2}}.\tag{50}
\end{align}

The limit of $Q = 0$ leads to $\Theta_X = \pi$. In this case the equation (49) can be simplified enormously as

\begin{align}
X(M, Q = 0, r_0, \omega) &= r_0 \frac{1 + \frac{a}{b}}{(1 - \frac{a}{b})}, \tag{51}
\end{align}

which agrees with the result in [32]. Next, using the solutions of $r_0$ in (19) and $X$ in (48), and (46) one can find the function $Y$ in terms of $(a, Q, b, s, r_0)$. Thus, we have successfully
generalize the expressions of the Kerr black hole case in [32] to those of a Kerr-Newman black hole. They become very crucial, as seen later, to obtain a representation the deflection angle of light in terms of the elliptic functions.

To proceed, we first write the function in the square bracket of (37) as

\[
\frac{1 - 2M u (1 - \omega_s)}{1 - 2M u + (a^2 + Q^2)u^2} + \frac{Q^2 u^2 (1 - \omega_s)}{1 - 2M u + (a^2 + Q^2)u^2} = \frac{C_+}{u_+ - u} + \frac{C_-}{u_- - u} + \frac{C_{Q+} u}{u_+ - u} + \frac{C_{Q-} u}{u_- - u},
\]

(52)

where

\[
u_\pm = \frac{M \pm \sqrt{M^2 - (a^2 + Q^2)}}{a^2 + Q^2}.
\]

Solving for \(C_+, C_-, C_{Q+}\) and \(C_{Q-}\), we obtain

\[
C_+ = \frac{2M(1 - \omega_s)\left( M + \sqrt{M^2 - (a^2 + Q^2)} \right) - (a^2 + Q^2)}{2(a^2 + Q^2)\sqrt{M^2 - (a^2 + Q^2)}},
\]

\[
C_- = \frac{(a^2 + Q^2) - 2M(1 - \omega_s)\left( M - \sqrt{M^2 - (a^2 + Q^2)} \right)}{2(a^2 + Q^2)\sqrt{M^2 - (a^2 + Q^2)}},
\]

\[
C_{Q+} = \frac{-Q^2(1 - \omega_s)\left( M + \sqrt{M^2 - (a^2 + Q^2)} \right)}{2(a^2 + Q^2)\sqrt{M^2 - (a^2 + Q^2)}},
\]

\[
C_{Q-} = \frac{Q^2(1 - \omega_s)\left( M - \sqrt{M^2 - (a^2 + Q^2)} \right)}{2(a^2 + Q^2)\sqrt{M^2 - (a^2 + Q^2)}}.
\]

(54)

The integral of deflection angle in (39) is then

\[
\hat{\alpha} = -\pi + 2 \int_0^{1/r_0} \left( \frac{C_+}{u_+ - u} + \frac{C_-}{u_- - u} + \frac{C_{Q+} u}{u_+ - u} + \frac{C_{Q-} u}{u_- - u} \right) \frac{1}{\sqrt{B(u)}} \, du
\]

(55)

where \(B(u)\) has been written as a product of the roots in (40). The exact expression of bending angle consequently is given in terms of elliptical integrals as [42]

\[
\hat{\alpha} = -\pi + \frac{4}{(1 - \omega_s)\sqrt{Q^2(u_4 - u_2)(u_3 - u_1)}} \left\{ \frac{C_+ + C_{Q+} u_3}{u_+ - u_1} \left[ \Pi(n_+, k) - \Pi(n_+, \psi_0, k) \right] + \frac{C_- + C_{Q-} u_1}{u_- - u_1} \left[ \Pi(n_-, k) - \Pi(n_-, \psi_0, k) \right] \\
- \frac{C_+ + C_{Q+} u_4}{u_+ - u_4} \left[ \Pi(n_+, k) - \Pi(n_+, \psi_0, k) - K(k) + F(\psi_0, k) \right] \\
- \frac{C_- + C_{Q-} u_4}{u_- - u_4} \left[ \Pi(n_-, k) - \Pi(n_-, \psi_0, k) - K(k) + F(\psi_0, k) \right] \right\}.
\]

(56)
FIG. 8: Deflection angle as a function of impact parameter $b/M$ for four sets of parameters. (a) $a/M = 0.5$, $Q/M = 0.6$; (b) $a/M = 0.5$, $Q/M = 0.3$; (c) $a/M = 0.9$, $Q/M = 0.3$; (d) $a/M = 0.5$, $Q/M = 0.8$. The plots show the results for the Schwarzschild, Reissner-Nordström, Kerr and Kerr-Newman black holes for comparison.

In (56),

\[ n_{\pm} = \frac{u_2 - u_1}{u_{\pm} - u_1} \left[ 1 + \frac{2MQ^2(1 - r_0u_{\pm})}{4M^2r_0 - Q^2(X + 2M)} \right], \]
\[ k^2 = \frac{(Y + 6M - X) [8M^2r_0 - Q^2(Y - 2M + 3X)]}{4Y [4M^2r_0 - Q^2(X + 2M)]}, \]
\[ \psi_0 = \arcsin \sqrt{\frac{(Y + 2M - X) [4M^2r_0 - Q^2(X + 2M)]}{(Y + 6M - X)(4M^2r_0 - Q^2X)}}, \]
and $\Pi(n_+, k)$ and $\Pi(n_+, \psi_0, k)$ are, respectively, the complete and the incomplete elliptic integrals of the third kind. Additionally, $K(k)$, the complete elliptic integral of the first kind, and $F(\psi_0, k)$, the incomplete elliptic integral of the first kind, are also involved. In particular, the condition $0 \leq k^2 \leq 1$ has been checked numerically. This is one of main results of the work.

It is quite straightforwardly to find that in the $Q = 0$ limit, the whole expression of (55) reduces to that of the Kerr black hole in [32]. Numerical studies in Fig (8) reproduce the deflection angle of light rays due to the Schwarzschild ($Q \rightarrow 0, a \rightarrow 0$) and Kerr black holes ($Q \rightarrow 0$) as a function of the impact parameter $b$ in [32]. However, in the Kerr-Newmann case, one of the remarkable results is the suppression of the bending angle compared with the Kerr case due to the repulsive effects from the black hole’s charge to the light rays for a fixed impact parameter in both direct and retrograde motions. These effects can be realized in Fig.(9), and also in particular in Fig.(10) where the deflection angle $\hat{\alpha}$ decreases as the charge $Q$ of the black hole increases. This will result in the modification of the apparent
 FIG. 10: Deflection angle as a function of charge $Q/M$ for (a) $a/M = 0.2$, and $b = 7M$; (b) $a/M = 0.5$, and $b = 7M$. The plots show the results for the Schwarzschild, Reissner-Nordström, Kerr and Kerr-Newman black holes for comparison.

shadows from the Kerr case as it has been seen in [37]. Our approach gives an closed-form expression of the deflection angle of the light rays due to the Kerr-Newman black hole that certainly generalizes the Kerr case in [32], and also explore the effect from the charge of the black hole to the light bending with an appropriate “effective potential” defined from the radial motion of a light ray.

V. SUMMARY AND OUTLOOK

In summary, the dynamics of light rays traveling around a Kerr-Newman black hole is studied with emphasis on how the charge of black holes influences their trajectories. We first examine the innermost circular orbits on the equatorial plane. It is found that the presence of the charge of the black results in the “effective potential”, which is defined from the equation of motion of light rays along the radial direction, with stronger repulsive effects as compared with the Kerr back hole. As a result, the radius of the innermost circular motion of light rays decreases as charge $Q$ of the black hole increases for both direct and retrograde
motions, and certainly give a direct consequence on constructing the apparent shape of a rotating charged black hole. We then derive the closed-form expression of the deflection angle in terms of elliptic integrals by generalizing the work of [32]. The non-zero charge of black hole decreases the deflection angle also due to the additional repulsive effects on the light rays given by the “effective potential” of the photon radial motion. In the next step, we will study this analytical solution in the form of series expansion in the strong and weak deflection regimes, and compare with the works of [34, 41] for the Kerr case. The accurate and efficient approximation schemes can be an attractive alternative to computationally involved elliptical integrals used in black hole simulations.

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