INTRINSIC CHIRALITY OF GRAPHS IN 3-MANIFOLDS

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Abstract. The main result of this paper is that for every closed, connected, orientable, irreducible 3-manifold $M$, there is an integer $n_M$ such that any abstract graph with no automorphism of order 2 which has a 3-connected minor whose genus is more than $n_M$ has no achiral embedding in $M$. By contrast, the paper also proves that for every graph $\gamma$, there are infinitely many closed, connected, orientable, irreducible 3-manifolds $M$ such that some embedding of $\gamma$ in $M$ is pointwise fixed by an orientation reversing involution of $M$.

1. Introduction

The study of chirality originally developed as a tool to help predict and explain molecular behavior. In particular, a molecule is said to be chiral if it can chemically interconvert with its mirror image at room temperature, and otherwise it is said to be achiral. Since small molecules are normally rigid, whether or not a small molecule is chiral can be determined from a geometric model. However, a molecule which is flexible or can rotate around specific bonds can be achiral even if a rigid model of it is geometrically distinct from its mirror image. The existence of such non-rigid molecules was the original motivation for studying chirality of spatial graphs from a topological perspective. However, the topological chirality of spatial graphs is interesting to consider whether or not they represent molecular structures.

In particular, we say that a graph embedded in $S^3$ is achiral, if there is an orientation reversing homeomorphism of $S^3$ taking the graph to itself. Otherwise, we say the embedded graph is chiral. We can think of knots with vertices as examples of graphs where some embeddings are chiral and others are not. By contrast, there are abstract graphs which have the property that no matter how they are embedded in $S^3$, they are topologically chiral. In this case, the graph is said to be intrinsically chiral in $S^3$. In chemical terms, a molecule would be intrinsically chiral if it and all of its topological stereoisomers are chiral. Molecular Möbius ladder with an odd number of rungs (at least three) were the first molecules that were shown to be intrinsically chiral [2]. More generally, the following theorem provides a method...
for showing that many graphs (molecular and otherwise) are intrinsically chiral in $S^3$.

**Theorem 1.** Every non-planar abstract graph $\gamma$ with no automorphism of order 2 is intrinsically chiral in $S^3$.

It makes sense to call such graphs *intrinsically* chiral because the chirality of such a graph depends only on the abstract graph and not on the embedding of the graph in $S^3$. However, we can define chirality for graphs embedded in any 3-manifold, and ask whether a graph which is intrinsically chiral in $S^3$ would be intrinsically chiral in a different 3-manifold. We prove the following theorem which shows that no graph can be intrinsically chiral in every 3-manifold.

**Theorem 2.** For every graph $\gamma$, there are infinitely many closed, connected, orientable, irreducible 3-manifolds $M$ such that some embedding of $\gamma$ in $M$ is pointwise fixed by an orientation reversing involution of $M$.

The proof of this result can be thought of as a generalization of the fact that every planar graph has an embedding in $S^3$ which is pointwise fixed by a reflection of $S^3$. On the other hand, our main result is the following generalization of Theorem 1 which shows that for any “nice” 3-manifold $M$, any 3-connected abstract graph with large enough genus and no involution is intrinsically chiral in $M$.

**Theorem 3.** For every closed, connected, orientable, irreducible 3-manifold $M$, there is an integer $n_M$ such that any abstract graph with no automorphism of order 2 which has a 3-connected minor $\lambda$ with genus$(\lambda) > n_M$ is intrinsically chiral in $M$.

Note that by contrast with our result about embeddings of graphs in 3-manifolds without boundary, Ikeda [9] has shown in the theorem below that for “nice” 3-manifolds with aspherical boundary, any abstract graph with large enough genus which has a certain type of involution has an achiral hyperbolic embedding in the double of the manifold.

**Ikeda’s Theorem.** [9] Let $M$ be a compact, connected, orientable, 3-manifold with non-empty aspherical boundary. Then there is an integer $n_M$ such that for any abstract graph $\lambda$ with genus$(\lambda) > n_M$ and no vertices of valence 1, any automorphism of order 2 of $\lambda$ that does not restrict to an orientation preserving automorphism of a cycle in $\lambda$ can be induced by an orientation reversing involution of some hyperbolic embedding of $\lambda$ in the double of $M$.

In Section 2, we prove Theorem 1. In Section 3, we determine the value of $n_M$ for for a given manifold $M$. In Section 4, we prove Theorem 3 making use of a proposition, which we then prove in Section 5.
2. Achiral embeddings

The goal of this section is to prove Theorem 2. To that end, we prove the following proposition. Note that we use \( \dim\mathbb{Z}(H_1(M,\mathbb{Z})) \) to denote the dimension of the first \( \mathbb{Z} \)-homology group of \( M \) and \( \dim\mathbb{Z}_2(H_1(M,\mathbb{Z}_2)) \) to denote the dimension of the first \( \mathbb{Z}_2 \)-homology group of \( M \).

**Proposition 1.** Let \( S \) be a closed, orientable surface. Then for infinitely many closed, connected, orientable, irreducible 3-manifolds \( Q \) such that \( \dim\mathbb{Z}_2(H_1(Q,\mathbb{Z}_2)) = \text{genus}(S) \), there is an embedding of \( S \) in \( Q \) which is pointwise fixed by an orientation reversing involution of \( Q \).

The proof of Proposition 1 will make use of the idea of a disk-busting curve, which is a simple closed curve in a handlebody that intersects every essential, properly embedded disk in the handlebody. For example, a core of a solid torus is disk busting in the solid torus. For a genus 2 handlebody with fundamental group generated by \( a \) and \( b \) an example of a disk-busting curve is one that includes into the fundamental group of the handlebody as \( abab^{-1} \) (see Figure 1). All handlebodies have disk-busting curves and Richard Strong [16] gives an algorithm to recognize them. For more on disk-busting curves see [16] or [8].

![Figure 1. A disk-busting curve in a genus 2 handlebody.](image)

**Proof.** Let \( g \) be the genus of a closed orientable surface \( S \). Let \( M \) be the manifold obtained by gluing genus \( g \) handlebodies \( V_1 \) and \( V_2 \) together along \( S \) in such a way that there is an orientation reversing involution \( h \) interchanging \( V_1 \) and \( V_2 \) which pointwise fixes the surface \( S \). Now \( M \) is a closed, connected, orientable 3-manifold \( M \) such that \( \dim\mathbb{Z}_2(H_1(M,\mathbb{Z}_2)) = \text{genus}(S) \). However, \( M \) is reducible.

In order to create an irreducible 3-manifold, we first remove neighborhoods \( N_1 \) and \( N_2 \) of identical disk busting curves in the interiors of the handlebodies \( V_1 \) and \( V_2 \) such that \( N_1 \) and \( N_2 \) are interchanged by the involution \( h \). Note that since \( \text{cl}(M - (S \cup N_1 \cup N_2)) \) consists of two identical handlebodies from which neighborhoods of disk busting curves have been removed, the inclusion of \( S \) in each component of \( \text{cl}(M - (S \cup N_1 \cup N_2)) \) is incompressible. Now we sew in identical knot complements \( Q_1 \) and \( Q_2 \) along \( \partial N_1 \) and \( \partial N_2 \) respectively so that the Seifert surfaces of the \( Q_i \) are glued in where the meridians of the \( N_i \) were.
Let \( Q \) denote the 3-manifold obtained in this way. Then the restriction \( h|_{\text{cl}(M - (S \cup N_1 \cup N_2))} \) can be extended to an orientation reversing involution of \( Q \) that pointwise fixes \( S \). Note that the components of \( \text{cl}(Q - (S \cup \partial Q_1 \cup \partial Q_2)) \) that contain \( S \) are homeomorphic to the corresponding components of \( \text{cl}(M - (S \cup \partial N_1 \cup \partial N_2)) \).

**Claim 1:** The surfaces \( S, \partial Q_1, \) and \( \partial Q_2 \) are each incompressible in \( Q \)

**Proof of Claim 1:** Assume one of \( S, \partial Q_1, \) or \( \partial Q_2 \) is compressible in \( Q \). Then there is a compressing disk \( D \) for one of these surfaces that meets \( S \cup \partial Q_1 \cup \partial Q_2 \) transversally in a minimal number of components. If the interior of \( D \) intersects one of the surfaces \( S, \partial Q_1, \) or \( \partial Q_2 \), then an innermost loop on \( D \) bounds a compressing disk for that surface whose interior is disjoint from the other surfaces. This implies that one of \( S, \partial Q_1, \) or \( \partial Q_2 \) is compressible in a component of \( \text{cl}(Q - (S \cup \partial Q_1 \cup \partial Q_2)) \). However, this is a contradiction because both \( Q_i \) and \( \text{cl}(V_i - N_i) \) have incompressible boundary.

**Claim 2:** \( Q \) is irreducible.

**Proof of Claim:** Let \( F \) be a sphere in \( Q \) and assume without loss of generality that \( F \) intersects \( S \cup \partial Q_1 \cup \partial Q_2 \) transversally in a minimal number of components. If \( F \) intersects any of \( \partial Q_1, \partial Q_2, \) or \( S \), then there is an innermost loop on \( F \) bounding a disk \( D \) that is a compressing disk for \( \partial Q_1, \partial Q_2, \) or \( S \) in \( \text{cl}(Q - (S \cup \partial Q_1 \cup \partial Q_2)) \). But this violates Claim 1. Thus \( F \) is contained in some \( Q_i \) or \( V_i - Q_i \). However, both the knot complements \( Q_i \) and the \( V_i - Q_i \) are irreducible. Therefore \( Q \) is irreducible.

Now, recall that \( \dim_{\mathbb{Z}_2}(H_1(M, \mathbb{Z}_2)) = \text{genus}(S) \). Also, it can be seen using a Meyer-Vietoris sequence that replacing the handlebodies \( N_1 \) and \( N_2 \) by the knot complements \( Q_1 \) and \( Q_2 \) does not change \( H_1 \), since a meridional disk of \( N_i \) is replaced by a Seifert surface in \( Q_i \). Thus \( \dim_{\mathbb{Z}_2}(H_1(Q, \mathbb{Z}_2)) = \text{genus}(S) \), and hence \( Q \) has the properties required by the proposition.

In order to prove that we can find infinitely many such 3-manifolds \( Q \), we first show as follows that if \( Q_1 \) is the complement of a connected sum of distinct knots \( K_1 \# K_2 \# \ldots \# K_n \), then \( Q \) contains at least \( 2n \) disjoint non-parallel incompressible tori. The first such torus is a boundary parallel torus in \( Q_1 \). The second such torus is a follow-swallow torus that swallows \( K_1 \) and follows the other \( n - 1 \) tori. The third swallows \( K_1 \# K_2 \) and follows the other \( n - 2 \) and so on, each time swallowing one more knot and following one less until the \( n^{th} \) torus swallows all of the \( K_j \) with \( j \neq n \). Since we have \( n \) such disjoint tori in each \( Q_i \), this gives us \( 2n \) such disjoint tori in \( Q \). These tori cannot be parallel to each other since they bound distinct knot complements in \( Q_1 \).

To see that these \( 2n \) tori are incompressible in \( Q \), suppose there is a compressing disk for one of the tori that intersects \( \partial Q_1 \cup \partial Q_2 \cup S \) transversally in a minimal number of components. An innermost loop on the disk would be a compressing disk for \( \partial Q_1, \partial Q_2 \) or \( S \), again contradicting Claim 1.
Thus the torus would have to be compressible in one of the $Q_i$. But it is well known that follow-swallow tori are incompressible in knot complements, so the $2n$ tori must be incompressible in $Q$ as well.

Thus $Q$ contains $2n$ disjoint, non-parallel, incompressible tori. On the other hand, it follows from Kneser-Haken finiteness, that for a given compact, orientable manifold, such as $Q$, there is some finite constant $t_1$ such that $Q$ cannot contain more than $t_1$ disjoint closed, non-parallel, incompressible surfaces (see, for example, Proposition 1.7 in [5]). Thus $t_1 > 2n$.

To get a manifold $Q'$ which is not homeomorphic to $Q$, we replace each $Q_i$ by the complement of a knot that is the connected sum of more than $\frac{1}{2}t_1$ knots. Now $Q'$ will contain $t_2$ disjoint, non-parallel, incompressible tori with $t_2 > t_1$, and thus $Q'$ is distinct from $Q$. By repeating this process, we can create an infinite sequence of such manifolds each containing more disjoint, non-parallel, incompressible tori than the previous manifold did.

Since every graph embeds in a closed orientable surface, the following theorem is an immediate consequence of Proposition 1.

**Theorem 2.** For every graph $\gamma$, there are infinitely many closed, connected, orientable, irreducible 3-manifolds $M$ such that some embedding of $\gamma$ in $M$ is pointwise fixed by an orientation reversing involution of $M$.

### 3. Determining the Value of $n_M$ for a Given $M$

Let $M$ be a closed, connected, orientable, irreducible 3-manifold. We will associate several constants with $M$ that will help us determine the value of $n_M$. First, observe that we can apply the characteristic decomposition theorem of Jaco-Shalen [10] and Johannson [12] to $M$ to find a unique minimal family $\Omega$ of disjoint incompressible tori such that the closures of the components obtained by splitting $M$ along $\Omega$ are atoroidal or Seifert fibered. If $\Omega \neq \emptyset$ let $t = |\Omega|$, and otherwise let $t = 1$.

For each Seifert fibered component $M_i$, let $g_i$ denote the genus of the base surface $F_i$, let $b_i$ denote the number of boundary components of $F_i$, and let $w_i = \max\{b_i + 3g_i - 3, 1\}$. By using a standard pants decomposition argument (see, for example, [6]), we see that the surface $F_i$ can contain at most $w_i$ disjoint, non-parallel, non-boundary parallel, essential circles. This implies that $M_i$ contains at most $w_i$ disjoint, non-parallel, non-boundary parallel, incompressible vertical tori (see Figure 2). Let $w = \sum w_i$ taken over all Seifert fibered components $M_i$. Then $M$ contains at most $w$ disjoint, non-parallel, non-boundary parallel, incompressible tori which are vertical in some Seifert fibered component.

Now let $T$ be an incompressible torus in $M$. Then $T$ can be isotoped to be disjoint from the tori in the characteristic family $\Omega$ (see for example [11]). Thus without loss of generality, we assume that $T$ is contained in either an atoroidal or a Seifert fibered component of $M - \Omega$. If $T$ is in an atoroidal
component, then \( T \) is parallel to a torus in \( \Omega \). Hence there are at most \( t = |\Omega| \) disjoint, non-parallel, such tori in \( M \). If \( T \) is in a Seifert fibered component, then by Waldhausen \([18]\), \( T \) must be parallel to either a vertical or horizontal torus. As we saw above there are at most \( w \) disjoint, non-parallel, such tori which are vertical in some Seifert fibered component. If \( T \) is parallel to a horizontal torus in some Seifert fibered component, then \( M \) is Seifert fibered with base surface a torus. In this case, there is only one such torus, and \( w = w_i = \max\{b_i + 3g_i - 3, 1\} = 1 \). Thus there are at most \( w \) disjoint, non-parallel tori in \( M \) which are either parallel to either a vertical or horizontal torus in some Seifert fibered component. Hence altogether, \( M \) has at most \( N_M = t + w \) disjoint, non-parallel incompressible tori. Note that since \( w \geq 1 \) and \( t \geq 1 \), we have \( N_M \geq 2 \).

Now for any closed, orientable, connected, irreducible 3-manifold \( M \), we define \( n_M = \dim\mathbb{Z}_2(H_1(M, \mathbb{Z}_2)) + N_M \). We will refer to the constants \( n_M \) and \( N_M \) in the statements and proofs of Theorem \([8]\) and Proposition \([2]\).

4. Intrinsic Chirality

The goal of this section is to prove our main result. We begin with the following definition.

**Definition 3.** Let the *genus* of a graph \( \gamma \) be defined as the minimum value of

\[
\frac{2 - \chi(S)}{2} = \frac{\dim\mathbb{Z}_2(H_1(S, \mathbb{Z}_2))}{2}
\]

over any surface \( S \) in which \( \gamma \) embeds.

It is worth pointing out that the genus of a non-orientable surface is not consistently defined in the literature. Some papers use our preferred definition, but others define the projective plane as having genus 1 instead of \( \frac{1}{2} \).

In the following proposition and subsequent theorem, we refer to the values of \( N_M \) and \( n_M \) which were defined in Section 3.
Proposition 2. Let \(\gamma\) be a 3-connected graph with genus at least 2, and let \(\Gamma\) be an embedding of \(\gamma\) in a closed, connected, orientable, irreducible 3-manifold \(M\) such that \((M, \Gamma)\) has an orientation reversing homeomorphism fixing every vertex of \(\Gamma\).

Then there is an embedding \(\Gamma'\) of \(\gamma\) in a closed, connected, orientable 3-manifold \(M'\) such that \((M', \Gamma')\) has an orientation reversing involution pointwise fixing \(\Gamma'\) and \(\dim\mathbb{Z}_2(H_1(M', \mathbb{Z}_2)) \leq n_M\).

The point of this proposition is that if we have an embedding \(\Gamma\) of a graph \(\gamma\) in a manifold \(M\) such that \((M, \Gamma)\) has an orientation reversing homeomorphism, then we can find another manifold \(M'\) and an embedding \(\Gamma'\) of \(\gamma\) in \(M'\) such that \((M', \Gamma')\) has an orientation reversing involution.

Furthermore, even though \(M'\) might be homologically more complicated than \(M\), there is a bound on \(\dim\mathbb{Z}_2(H_1(M', \mathbb{Z}_2))\) which depends only on \(M\) and not on the graph \(\gamma\) or a particular embedding of \(\gamma\) in \(M\).

Proposition 2 will be proved in the next section. We now prove Theorem 3 (restated below) by making use of Proposition 2 together with the following result of Kobayashi.

Kobayashi’s Theorem. [13] Let \(X\) be a closed, orientable, 3-manifold admitting an orientation reversing involution \(h\). Then
\[
\dim\mathbb{Z}_2(H_1(\text{fix}(h), \mathbb{Z}_2)) \leq \dim\mathbb{Z}_2(H_1(X, \mathbb{Z}_2)) + \dim\mathbb{Z}(H_1(X, \mathbb{Z})).
\]

Theorem 3. For every closed, connected, orientable, irreducible 3-manifold \(M\), there is an integer \(n_M\) such that any abstract graph with no automorphism of order 2 which has a 3-connected minor \(\lambda\) with genus(\(\lambda\)) > \(n_M\) is intrinsically chiral in \(M\).

Proof. Let \(\gamma\) be a graph with no automorphism of order 2. Suppose for the sake of contradiction that \(\gamma\) has an achiral embedding \(\Gamma\) in the manifold \(M\). Let \(n_M\) and \(N_M\) be the constants associated with \(M\) that were defined in Section 3. Let \(\lambda\) be a 3-connected minor of \(\gamma\). We will now show that \(\lambda\) satisfies the inequality
\[
\text{genus}(\lambda) \leq \dim\mathbb{Z}_2(H_1(M, \mathbb{Z}_2)) + N_M = n_M.
\]

First observe that if \(\text{genus}(\lambda) \leq 1\), then the above inequality is immediate since we saw in Section 3 that \(N_M \geq 2\). Thus we assume that \(\text{genus}(\lambda) \geq 2\).

Since \(\Gamma\) is an achiral embedding of \(\gamma\) in \(M\), there is an orientation reversing homeomorphism \(f\) of the pair \((M, \Gamma)\). Let \(\varphi\) denote the automorphism that \(f\) induces on the graph \(\Gamma\). Now even though \(f\) does not necessarily have finite order, \(\varphi\) has finite order because \(\Gamma\) has a finite number of vertices. Hence we can express the order of \(\varphi\) as \(2^r q\), where \(r \geq 0\) and \(q\) is odd. Since \(f\) is orientation reversing and \(q\) is odd, \(g = f^q\) is also an orientation reversing homeomorphism of \((M, \Gamma)\). Now it follows that \(g\) induces the automorphism \(\varphi^q\) on \(\Gamma\) and \(\text{order}(\varphi^q) = 2^r\). In particular, \(g^{2^r}\) fixes every vertex of \(\Gamma\).
If \( r \geq 1 \), then \( g^{2r-1} \) would induce an order two automorphism on \( \Gamma \). As we assumed that no such automorphism exists, we must have \( r = 0 \). Thus \( g = g^2 \) is an orientation reversing homeomorphism of \((M, \Gamma)\) which fixes every vertex of \( \Gamma \).

Now let \( \lambda \) be a 3-connected minor of the abstract graph \( \gamma \). Then by deleting and/or contracting some edges of the embedding \( \Gamma \) of \( \gamma \) in \( M \), we obtain an embedding \( \Lambda \) of \( \lambda \) in \( M \). Furthermore, by composing the homeomorphism \( g \) with an isotopy in a neighborhood of each edge that was contracted, we obtain an orientation reversing homeomorphism of \((M, \Lambda)\) which fixes every vertex of \( \Lambda \).

Since \( \lambda \) is a 3-connected graph with \( \text{genus}(\lambda) \geq 2 \), we can now apply Proposition 2 to get an embedding \( \Lambda' \) of \( \lambda \) in a 3-manifold \( M' \) such that \((M', \Lambda')\) has an orientation reversing involution \( h \) pointwise fixing \( \Lambda' \)

\[
\dim_{\mathbb{Z}_2}(H_1(M', \mathbb{Z}_2)) \leq n_M = \dim_{\mathbb{Z}_2}(H_1(M, \mathbb{Z}_2)) + N_M.
\]

Let \( F \) be the component of the fixed point set \( \text{fix}(h) \) containing \( \Lambda' \), and let \( x \in \Lambda' \). We put a metric \( d \) on \( M' \), and define a new metric \( d' \) by \( d'(x, y) = d(x, y) + d(h(x), h(y)) \). Then any neighborhood of \( x \) with respect to the metric \( d' \) will be setwise invariant under \( h \). Now we can pick a neighborhood \( N(x) \) with respect to \( d' \) such that \( N(x) \) is homeomorphic to a ball. Then by Smith theory \[15\], since \( h|N(x) \) is an orientation reversing involution of the ball \( N(x) \), the fix point set of \( h|N(x) \) is either a single point or a properly embedded disk. Since \( N(x) \cap \Lambda \) contains more than one point, \( \text{fix}(h|N(x)) \) is a properly embedded disk, and hence \( F \) is a closed surface. Thus

\[
\text{genus}(\lambda) \leq \text{genus}(F) = \frac{2 - \chi(F)}{2} = \frac{\dim_{\mathbb{Z}_2}(H_1(F, \mathbb{Z}_2))}{2} \leq \frac{\dim_{\mathbb{Z}_2}(H_1(\text{fix}(h), \mathbb{Z}_2))}{2}.
\]

Hence we have the inequality

\[2\text{genus}(\lambda) \leq \dim_{\mathbb{Z}_2}(H_1(\text{fix}(h), \mathbb{Z}_2)).\]

Also, since \( h \) is an orientation reversing involution and \( M' \) is a closed orientable manifold, we can apply Kobayashi’s Theorem \[13\] to obtain the inequality

\[\dim_{\mathbb{Z}_2}(H_1(\text{fix}(h), \mathbb{Z}_2)) \leq \dim_{\mathbb{Z}_2}(H_1(M', \mathbb{Z}_2)) + \dim_{\mathbb{Z}}(H_1(M', \mathbb{Z})).\]

It follows that

\[\dim_{\mathbb{Z}_2}(H_1(\text{fix}(h), \mathbb{Z}_2)) \leq 2\dim_{\mathbb{Z}_2}(H_1(M', \mathbb{Z}_2)).\]

Combining the above inequalities, we now have

\[\text{genus}(\lambda) \leq \dim_{\mathbb{Z}_2}(H_1(M', \mathbb{Z}_2)).\]
But $M'$ was given by Proposition 2 such that
\[ \dim_{\mathbb{Z}_2}(H_1(M',\mathbb{Z}_2)) \leq \dim_{\mathbb{Z}_2}(H_1(M,\mathbb{Z}_2)) + N_M. \]

Hence we obtain the required inequality
\[ \text{genus}(\lambda) \leq \dim_{\mathbb{Z}_2}(H_1(M,\mathbb{Z}_2)) + N_M = n_M. \]

It now follows that if $\gamma$ has a 3-connected minor whose genus is greater than $n_M$, then $\gamma$ must be intrinsically chiral in $M$. □

5. Proof of Proposition 2

For the sake of completeness, we begin by proving the following elementary lemma that we will use in the proof of Proposition 2.

Lemma 1. Let $S$ be a punctured sphere with $n \geq 3$ boundary components $\{c_1, c_2, \ldots, c_n\}$. Let $\{s_1, s_2, \ldots, s_m\}$ be disjoint embedded loops on $S$ each parallel to some $c_i$, and let $\{\alpha_1, \ldots, \alpha_r\}$ be disjoint annuli on $S - \bigcup_{i=1}^{m} s_i$. Then the closure of some component of $S - (\bigcup_{i=1}^{m} s_i \cup \bigcup_{i=1}^{r} \alpha_i)$ is a sphere with at least three holes.

Proof. First note that $\text{cl}(S - \bigcup_{i=1}^{m} s_i)$ consists of $n$ annuli and a sphere with $n \geq 3$ holes. Call the closure of this sphere with holes $F_1$. Any annuli $\alpha_i$ that are not in $F_1$ are thrown out of the collection $\{\alpha_1, \ldots, \alpha_r\}$ to get a possibly smaller collection $\{\alpha_1, \ldots, \alpha_t\}$. Now it suffices to prove that the closure of some component of $F_1 - (\bigcup_{i=1}^{m} s_i \cup \bigcup_{i=1}^{r} \alpha_i)$ is a sphere with at least three holes.

Observe that the total number of boundary components of $\text{cl}(F_1 - \alpha_1)$ is $n + 2$. There are three possible cases for these components. First, one component of $\text{cl}(F_1 - \alpha_1)$ could be a disk $D_1$. In this case, the total number of boundary components of $\text{cl}(F_1 - (\alpha_1 \cup D_1))$ is $n + 1 \geq 3$. Second, one component of $\text{cl}(F_1 - \alpha_1)$ could be an annulus $A_1$, in which case the total number of boundary components of $\text{cl}(F_1 - (\alpha_1 \cup A_1))$ is still $n \geq 3$. Finally, if neither component of $\text{cl}(F_1 - \alpha_1)$ is a disk or an annulus, then $\alpha_1$ splits $F$ into two punctured spheres each with at least three boundary components. Let $F_2$ denote the closure of one of these spheres with at least three holes.

We repeat the above paragraph inductively to conclude that the closure of some component of $S - (\bigcup_{i=1}^{m} s_i \cup \bigcup_{i=1}^{r} \alpha_i)$ is a sphere with at least three holes. □

We will also use the well known “Half Lives, Half Dies” Theorem, which we state below. See [7] or [5] for a proof of this theorem.

Theorem 4. (Half Lives, Half Dies) Let $M$ be a compact orientable 3-manifold. Then the following equation holds with any field coefficients
\[ \dim(\text{Kernel}(H_1(\partial M) \to H_1(M))) = \frac{1}{2} \dim H_1(\partial M). \]
Corollary 1. Let \( M \) be a manifold which has a torus boundary component \( T \). Then for any pair of generators \( a \) and \( b \) of \( H_1(T, \mathbb{Z}_2) \), at least one of \( a \) and \( b \) is non-trivial in \( H_1(M, \mathbb{Z}_2) \).

Proof. Suppose for the sake of contradiction that the generators \( a \) and \( b \) are both trivial in \( H_1(M, \mathbb{Z}_2) \). Attach handlebodies to all boundary components of \( M \) except \( T \) to form a new manifold \( J \) with a single boundary component. Then \( a \) and \( b \) are both trivial in \( H_1(J, \mathbb{Z}_2) \). Since \( a \) and \( b \) generate the homology of the only boundary component of \( J \), we see that \( \dim (\text{Kernel}(H_1(\partial J, \mathbb{Z}_2))) \to H_1(J, \mathbb{Z}_2)) = \dim_{\mathbb{Z}_2}(H_1(\partial J, \mathbb{Z}_2)) = 2 \). But this contradicts the Half Lives, Half Dies Theorem. Thus at least one of the generators of \( H_1(T, \mathbb{Z}_2) \) must have been non-trivial in \( M \). \( \square \)

Note that it is tempting to assume that the Half Lives, Half Dies Theorem implies that one of \( a \) or \( b \) must be trivial in \( J \). But this is not always true. In particular, let \( M \) denote the product of a torus and an interval. Then no single non-trivial curve in the boundary of \( M \) is in the kernel.

Now we are ready to prove Proposition 2. Recall that the definition of the constant \( n_M \) is given in Section 3.

Proposition 2. Let \( \gamma \) be a 3-connected graph with genus at least 2, and let \( \Gamma \) be an embedding of \( \gamma \) in a closed, connected, orientable, irreducible 3-manifold \( M \) such that \((M, \Gamma)\) has an orientation reversing homeomorphism \( g \) fixing every vertex of \( \Gamma \).

Then there is an embedding \( \Gamma' \) of \( \gamma \) in a closed, connected, orientable 3-manifold \( M' \) such that \((M', \Gamma')\) has an orientation reversing involution pointwise fixing \( \Gamma' \) and \( \dim_{\mathbb{Z}_2}(H_1(M', \mathbb{Z}_2)) \leq n_M \).

Proof. Let \( \Lambda \) denote either \( \Gamma \) or \( \Gamma \) with one edge deleted. Suppose that \( \Lambda \) is contained in a ball \( B \) in \( M \). Since \( g \) fixes every vertex of \( \Lambda \), without loss of generality, we can assume that \( \Lambda \) is pointwise fixed by \( g \). Furthermore, since \( g(B) \) is isotopic to \( B \) in \( M \), we can assume that \( g \) leaves \( B \) setwise invariant. Let \( f \) be an embedding of \((B, \Lambda)\) in \( S^3 \). Then \( f \circ g \circ f^{-1} \) is an orientation reversing homeomorphism of \( f(B) \) pointwise fixing \( f(\Lambda) \). Now, \( f \circ g \circ f^{-1} \) can be extended to an orientation reversing homeomorphism of \( S^3 \) pointwise fixing \( f(\Lambda) \).

However, Jiang and Wang [11] showed that no graph containing one of the graphs \( K_{3,3} \) or \( K_5 \) has an embedding in \( S^3 \) which is pointwise fixed by an orientation reversing homeomorphism of \( S^3 \). Thus \( \Lambda \) cannot contain \( K_{3,3} \) or \( K_5 \), and hence is abstractly planar. But this implies that genus(\( \Gamma \)) \leq 1, which is contrary to hypothesis. Thus neither \( \Gamma \) nor \( \Gamma \) with an edge deleted can be contained in a ball in \( M \). We will use this result later in the proof.

Since the remainder of the proof is quite lengthy, we break it into steps.

Step 1: We define a neighborhood \( N(\Gamma) \).
Let $V$ and $E$ be the sets of vertices and edges of $\Gamma$ respectively. For each vertex $v \in V$, define $N(v)$ to be a ball around $v$ in $M$ (i.e., a 0-handle containing $v$), and let $N(V)$ denote the union of the balls around the vertices. Also, for each edge $e \in E$, let $N(e) = D \times I$ be a solid tube around $cl(e - N(V))$ in $M$ (i.e., a 1-handle containing the portion of $e$ outside of the 0-handles). Let $N(E)$ denote the union of the tubes around the edges. Then for each $e$ in $N(E)$ the intersection $N(V) \cap N(e)$ follows the standard convention for attaching 1-handles to 0-handles. In other words, in $N(e)$ this intersection consists of the disks $D$ in the boundaries of the balls. Then $N(\Gamma) = N(E) \cup N(V)$ is a neighborhood of $\Gamma$.

For convenience we introduce the following terminology. For each vertex $v$, we let $\partial' N(v)$ denote the sphere with holes $\partial N(v) \cap \partial N(\Gamma)$, and for each edge $e$ we let $\partial' N(e)$ denote the annulus $\partial N(e) \cap \partial N(\Gamma)$. Thus $\partial N(\Gamma) = \partial' N(E) \cup \partial' N(V)$.

Now since $g(\Gamma) = \Gamma$ fixing each vertex of $\Gamma$, we know that $g(N(\Gamma))$ is isotopic to $N(\Gamma)$ setwise fixing $\Gamma$ and fixing each vertex. Thus we can modify $g$ by an isotopy (and by an abuse of notation, still refer to the map as $g$) so that for each vertex $v$ and edge $e$ we have $g(N(v)) = N(v)$ and $g(N(e)) = N(e)$. Because this modification was by an isotopy, our new $g$ is still orientation reversing.

**Step 2:** We split $cl(M - N(\Gamma))$ along a family $\tau$ of JSJ tori and choose an invariant component $X$.

Since $M$ is irreducible and we have assumed that $\Gamma$ is not contained in a ball, $cl(M - N(\Gamma))$ is irreducible. Thus we can apply the Characteristic Decomposition Theorem of Jaco-Shalen \cite{10} and Johannson \cite{12} to get a minimal family of incompressible tori $\tau$ for $cl(M - N(\Gamma))$ such that each closed up component of $M - (N(\Gamma) \cup \tau)$ is either Seifert fibered or atoriodal. Since the characteristic family $\tau$ is unique up to isotopy, we can again modify $g$ by an isotopy (and again by an abuse of notation still refer to the map as $g$) so that $g(\tau) = \tau$ and still have $g(N(v)) = N(v)$ and $g(N(e)) = N(e)$ for each vertex $v$ and edge $e$. Let $X$ be the closed up component of $M - (N(\Gamma) \cup \tau)$ containing $\partial N(\Gamma)$ (see for example Figure \ref{fig:jsj}). Then $g(X) = X$.

Also, since $\Gamma$ is 3-connected, $\text{genus}(\partial N(\Gamma)) > 1$. Thus the component $X$ is not Seifert fibered, and hence is atoroidal. Let $P$ denote the set of torus boundary components of $X$ together with the annuli that make up the components of $\partial' N(E)$. Since $\Gamma$ is 3-connected, $\partial X - P = \partial N(\Gamma) - \partial' N(E) = \partial' N(V)$ is incompressible in $cl(M - N(\Gamma))$. It follows that $\partial X - P$ is incompressible in $X$. Furthermore, $X$ is irreducible since $cl(M - N(\Gamma))$ is irreducible and $X$ is a component of the JSJ decomposition of $cl(M - N(\Gamma))$.

**Step 3:** We show that any sphere obtained by capping off an annulus in the JSJ decomposition of $(X, P)$ bounds a ball in $M$ intersecting at most one edge of $\Gamma - N(V)$. 
We now apply the Characteristic Decomposition Theorem for Pared Manifolds [10, 12] to the pared manifold \((X, P)\). Since \(X\) is atoroidal, this gives us a characteristic family \(\sigma\) of incompressible annuli in \(X\) with boundaries in \(\partial X - P\) such that if \(W\) is the closure of any component of \(X - \sigma\), then the pared manifold \((W, W \cap (P \cup \sigma))\) is either simple, Seifert fibered, or I-fibered (see [1] for the necessary definitions). Once again, since the characteristic family \(\sigma\) is unique up to isotopy, we can modify \(g\) by an isotopy (and again by an abuse of notation, still refer to the map as \(g\)) so that \(g(\sigma) = \sigma\).

Let \(A\) be an annulus component of \(P \cup \sigma\), and let \(S\) denote the sphere obtained by capping off \(A\) by a pair of disjoint disks \(D_1\) and \(D_2\) in \(\partial N(v_1)\) and \(\partial N(v_2)\), where \(v_1\) and \(v_2\) may or may not be distinct vertices. Suppose that each component of \(M - S\) intersects more than one edge of \(\Gamma - N(V)\). Then by removing the vertices \(v_1\) and \(v_2\) and the edges that contain them we would obtain two non-empty subgraphs (see Figure 4). But this contradicts our hypothesis that \(\Gamma\) is 3-connected. Thus one of the components of \(M - S\) meets \(\Gamma - N(V)\) in at most one edge of \(\Gamma\).

Figure 3. \(X\) is the closed up component of \(M - (N(\Gamma) \cup \tau)\) between the grey incompressible torus and the black \(\partial N(\Gamma)\).

Figure 4. There is more than one edge on each side of this capped off annulus.
Now, since $M$ is irreducible, one of the closed up components of $M - S$ is a ball $B$. However, we assumed at the beginning of our proof that neither $\Gamma$ nor $\Gamma$ with an edge removed can be contained in a ball in $M$. Thus $B$ must be the closed up component of $M - S$ intersecting at most one edge of $\Gamma - N(V)$. Furthermore, since the annulus $A$ is incompressible in $X$, if $v_1 \neq v_2$ then there is some edge $e$ with vertices $v_1$ and $v_2$ such that $B$ contains $\text{cl}(e - (N(v_1) \cup N(v_2)))$. On the other hand, if $v_1 = v_2$, then since $\Gamma$ is a graph $B$ must be disjoint from $\Gamma - N(V)$.

Note that since a ball cannot contain an incompressible torus, no torus boundary component of $X$ can occur in one of these balls. It follows that every torus component of $\partial X$ must also be a component of $\partial W$.

**Step 4:** We define a collection of balls $U_{e_1}, \ldots, U_{e_n}, V_{F_1}, \ldots, V_{F_m}$ in $M$ such that every annulus in $P \cup \sigma$ is contained in some $U_{e_i}$ if its boundaries are in distinct components of $\partial N(V)$, and in some $V_{F_i}$ if its boundaries are in a single component of $\partial N(V)$.

Let $A$ be an annulus in $P \cup \sigma$ with one boundary in $\partial N(v_1)$ and the other boundary in $\partial N(v_2)$ with $v_1 \neq v_2$. By capping off $A$ with disks in $N(v_1)$ and $N(v_2)$ we obtain a sphere, which as we saw in Step 3, bounds a ball $B$ that contains $\text{cl}(e - (N(v_1) \cup N(v_2)))$ for some edge $e$ in $\Gamma$. Now let $C_e$ denote the collection of all annuli in $P \cup \sigma$ with one boundary in $\partial N(v_1)$ and the other boundary in $\partial N(v_2)$. By capping off the annuli in $C_e$ with pairwise disjoint disks in $N(v_1)$ and $N(v_2)$, we obtain a collection of disjoint spheres which bound nested balls containing $\text{cl}(e - (N(v_1) \cup N(v_2)))$. Let $A_e$ denote the annulus in $C_e$ which when capped off in this way is outermost with respect to this nesting. Observe that the boundaries of $A_e$ also bound disks $D_1 \subseteq \partial N(v_1)$ and $D_2 \subseteq \partial N(v_2)$ which each meet $\Gamma$ in a single point of $e$. Now the sphere $A_e \cup D_1 \cup D_2$ bounds a ball $U_e$ in $M$ which contains both $\text{cl}(e - (N(v_1) \cup N(v_2)))$ and every annulus in $C_e$.

We repeat the above paragraph for each annulus in $P \cup \sigma$ with boundaries in distinct components of $\partial N(V)$ to get a collection of pairwise disjoint balls $U_{e_1}, \ldots, U_{e_n}$ such that $U_{e_1} \cup \cdots \cup U_{e_n}$ contains both $\text{cl}(\Gamma - N(V))$ and every annulus in $P \cup \sigma$ with boundaries in distinct components of $\partial N(V)$. Observe that for every edge $e$, the annulus $\partial N(e)$ is in $P$. Thus every edge $e$ is contained in some $U_e$.

Next we consider an annulus $F$ in $\sigma$ which has both boundaries in a single component of $\partial N(V)$. We saw in Step 3 that if we cap off $F$ by any pair of disjoint disks in $N(V)$ we obtain a sphere which bounds a ball that is disjoint from $\Gamma - N(V)$. We can now cap off every such annulus by pairwise disjoint disks properly embedded in $N(V)$ such that the balls we get are disjoint from the set of vertices $V$. Since the annuli in $\sigma$ are disjoint and the pairs of disks are disjoint, pairs of balls we obtain in this way will either be nested or disjoint. Thus for each annulus in $\sigma$ with boundaries in a single component of $\partial N(V)$, we can choose an annulus $F \in \sigma$ which when capped
off bounds an outermost ball $V_F$ with respect to the nesting of the collection (see Figure 5).

In this way we get a collection of disjoint balls $V_{F_1}, \ldots, V_{F_m}$ such that $V_{F_1} \cup \cdots \cup V_{F_m}$ contains every annulus in $\sigma$ with boundaries in a single component of $\partial N(V)$. Furthermore, each such ball $V_{F_i}$ is disjoint from $\Gamma - N(V)$, $V$, and from $U_{e_1} \cup \cdots \cup U_{e_n}$. Note that each $U_{e_j} \subseteq X$ and has only one boundary component; and each $V_{F_i} \cap X$ has only one boundary component. Thus both $X - U_{e_j}$ and $X - V_{F_i}$ have a single component. It follows that the manifold

$$W = \text{cl}(X - (U_{e_1} \cup \cdots \cup U_{e_n} \cup V_{F_1} \cup \cdots \cup V_{F_m}))$$

is the closure of a single component of $X - \sigma$ (see Figure 6).

**Step 5:** We show that $g(W) = W$ and $(W, W \cap (P \cup \sigma))$ is simple as a pared manifold.
Recall that $g$ fixes each vertex and leaves each edge setwise invariant. Also, $g(N(\Gamma)) = N(\Gamma)$, $g(P) = P$, and $g(\sigma) = \sigma$. Now, since the sets of balls $\{U_{e_1}, \ldots, U_{e_n}\}$ and $\{V_{F_1}, \ldots, V_{F_m}\}$ were chosen to be outermost, each of these sets is invariant under $g$. Furthermore, we know from Step 4 that each $U_{e_j}$ intersects $\Gamma - N(V)$ only in $e_j$. Now since each edge is fixed by $g$, both $A_{e_j}$ and $U_{e_j}$ are setwise invariant under $g$, and since each vertex is fixed by $g$, each boundary component of $A_{e_j}$ is also setwise invariant under $g$. Now let $v$ be a vertex such that some $F_i$ has both its boundary components in $\partial N(v)$. Since $F_i$ is incompressible in $X$, each component of $\partial F_i$ bounds a disk in $\partial N(v)$ which contains a boundary component of some $U_{e_j}$. Since the boundary components of each $U_{e_j}$ are setwise invariant under $g$, it follows that $F_i$ and its boundary components are also setwise invariant under $g$. Thus $V_{F_i}$ is setwise invariant under $g$. Finally, since all of the $U_{e_j}$ and $V_{F_i}$ are invariant under $g$, we know that $W$ must be setwise invariant under $g$ as well.

To show that the pared manifold $(W, W \cap (P \cup \sigma))$ is simple, first recall that $W$ is the closure of a single component of $X - \sigma$. Hence by JSJ for pared manifolds [10, 12], $(W, W \cap (P \cup \sigma))$ is either $I$-fibered, Seifert fibered, or simple as a pared manifold. We see that $(W, W \cap (P \cup \sigma))$ cannot be Seifert fibered or $I$-fibered as follows.

First observe that for every vertex $v$, there is some edge $e$ such that the ball $U_e$ meets $\partial N(v)$. It follows that $\partial W$ meets every component of $\partial N(V)$. Furthermore, since every vertex $v$ has valence at least three, $\partial' N(v)$ is a sphere with at least three holes. Also each $U_{e_j}$ contains at most one boundary component of $\partial' N(v)$, and each $V_{F_i}$ contains no boundary components of $\partial' N(v)$. Hence by Lemma 1 some component of $W \cap \partial N(v)$ is a sphere with at least three holes. It follows that the component of $\partial W$ meeting $\partial N(\Gamma)$ has genus more than one, and thus the pared manifold $(W, W \cap (P \cup \sigma))$ cannot be Seifert fibered.

Next suppose for the sake of contradiction that the pared manifold $(W, W \cap (P \cup \sigma))$ is $I$-fibered. By definition of $I$-fibered for pared manifolds, this means that there is an $I$-bundle map of $W$ over a base surface $Y$ such that $W \cap (P \cup \sigma)$ is in the pre-image of $\partial Y$. It follows that $Y$ must be homeomorphic to a component of $\partial' N(V)$. This means that the base surface $Y$ must be a sphere with holes. Now since $M$ is orientable, in fact $W$ is a product $Y \times I$. Thus $W \cap (P \cup \sigma) = \partial Y \times I$, and $Y_0 = Y \times \{0\}$ and $Y_1 = Y \times \{1\}$ are components of $\partial' N(V) \cap W$. However, since $\partial W$ meets every component of $\partial N(V)$, this means that $\Gamma$ contains at most two vertices. But this contradicts our hypothesis that $\Gamma$ is 3-connected. Therefore, the pared $(W, W \cap (P \cup \sigma))$ is not $I$-fibered, and since it is also not Seifert fibered, it must be simple.

**Step 6:** We prove that $g|_W$ is isotopic to an orientation reversing involution $h$ of $(W, W \cap (P \cup \sigma))$. 

Now it follows from Thurston’s Hyperbolization Theorem for Pared Manifolds [17] applied to the simple pared manifold \((W, W \cap (P \cup \sigma))\) that \(W - (W \cap (P \cup \sigma))\) admits a finite volume complete hyperbolic metric with totally geodesic boundary. Let \(D\) denote the double of \(W - (W \cap (P \cup \sigma))\) along its boundary. Then \(D\) is a finite volume hyperbolic manifold, and \(g|W\) can be doubled to obtain an orientation reversing homeomorphism of \(D\) (which we still call \(g\)) taking each copy of \(W - (W \cap (P \cup \sigma))\) to itself.

Now by Mostow’s Rigidity Theorem [14] applied to \(D\), the homeomorphism \(g : D \to D\) is homotopic to an orientation reversing finite order isometry \(h : D \to D\) that restricts to an isometry of \(W - (W \cap (P \cup \sigma))\). By removing horocyclic neighborhoods of the cusps of \(W - (W \cap (P \cup \sigma))\), we obtain a copy of the pair \((W, W \cap (P \cup \sigma))\) which is contained in \(W - (W \cap (P \cup \sigma))\) and is setwise invariant under \(h\). We abuse notation and now consider \(h\) to be an orientation reversing finite order isometry of \((W, W \cap (P \cup \sigma))\) instead of this copy. Furthermore, \(h\) induces isometries on the collection of tori and annuli in \(W \cap (P \cup \sigma)\) with respect to a flat metric. Furthermore, the sets \(\partial N(V) \cap W\), \(\partial N(E) \cap W\), and \(\tau \cap W\) are each setwise invariant under \(h\). Finally, it follows from Waldhausen’s Isotopy Theorem [19] that \(h\) is isotopic to \(g|W\) by an isotopy leaving \(W \cap (P \cup \sigma)\) setwise invariant.

Now, recall that the boundary components of \(W\) consist of tori in \(\tau\), and the union of spheres with holes in \(\partial N(V)\) together with annuli in \(P \cup \sigma\). Recall from Step 5 that \(g\) leaves setwise invariant each annulus \(A_{e_j} \subseteq \partial U_{e_j}\) with boundaries in distinct components of \(\partial N(V)\), each annulus \(F_i \subseteq \partial V_{F_i}\) with both boundaries in a single component of \(\partial N(V)\), each component of \(\partial A_{e_j}\), and each component of \(\partial F_i\). Since \(h\) is isotopic to \(g|W\) by an isotopy leaving \(W \cap (P \cup \sigma)\) setwise invariant, \(h\) leaves invariant the same sets as \(g\). It follows that for each vertex \(v\), we have \(h(\partial N(v) \cap W) = \partial N(v) \cap W\), and \(h\) takes each component of \(W \cap \partial N(v)\) to itself, leaving each boundary component setwise invariant.

Since \(h\) has finite order, \(h\) restricts to a finite order homeomorphism of every component of \(W \cap \partial N(V)\). We saw in Step 5 that for every vertex \(v\), at least one component \(C_v\) of \(W \cap \partial N(v)\) is a sphere with at least three holes. Since \(h\) restricts to a finite order homeomorphism of \(C_v\) taking each boundary component of \(C_v\) to itself, \(h\) must be a reflection of \(C_v\) which also reflects each component of \(\partial C_v\). Now \(h^2\) is a finite order, orientation preserving isometry of \(W\) that pointwise fixes the surface \(C_v\). It follows that \(h^2\) is the identity, and hence \(h\) is an involution of \(W\).

**Step 7:** We extend \(h\) to an orientation reversing involution of \(X \cup N(\Gamma)\) which pointwise fixes a new embedding \(\Gamma'\) of \(\gamma\).

Observe that since every annulus in \(P \cup \sigma\) is incompressible in \(W\), no component of \(W \cap \partial N(V)\) can be a disk. Thus every component of \(W \cap \partial N(V)\) is either an annulus or a sphere with at least three holes. As we saw in Step 6, for each vertex \(v\), \(h\) reflects some component \(C_v\) of \(W \cap \partial N(v)\) which
is a sphere with at least three holes, and \( h \) reflects every component of \( \partial C_v \). Let \( b_0 \) denote some boundary component of \( C_v \). Then \( b_0 \) is also a boundary component of either an annulus \( A_{e_j} \) or an annulus \( F_i \). Since \( h \) reflects \( b_0 \), we know that \( h \) must also reflect the annulus \( A_{e_j} \) or \( F_i \), whichever contains \( b_0 \) in its boundary. Since the boundaries of the annulus are not interchanged, \( h \) must also reflect each boundary component of \( A_{e_j} \) or \( F_i \). Below we extend \( h \) to \( U_{e_j} \) or \( V_{F_i} \).

First we consider the case where \( b_0 \) is in the boundary of an annulus \( A_{e_j} \subseteq \partial U_{e_j} \). Let \( D_j \) and \( D_j' \) denote the disks in \( \text{cl}(\partial U_{e_j} - A_{e_j}) \). Then \( D_j \) and \( D_j' \) each meet \( \Gamma \) in a single point of \( e_j \). Since \( h \) reflects the annulus \( A_{e_j} \) together with each boundary component of \( A_{e_j} \), we can extend \( h \) radially to the disks \( D_j \) and \( D_j' \) to get a reflection of the sphere \( A_{e_j} \cup D_j \cup D_j' \) pointwise fixing a circle containing the points \( D_j \cap e_j \) and \( D_j' \cap e_j \). Recall that the sphere \( A_{e_j} \cup D_i \cup D_i' \) bounds the ball \( U_{e_j} \) in \( M \). Now, we can express \( U_{e_j} \) as a product \( D_j \times I \) whose core \( \overline{e_j} \) has endpoints \( D_j \cap e_j \) and \( D_j' \cap e_j \) (see Figure 7). Now we extend \( h \) from a reflection of the sphere \( A_{e_j} \cup D_j \cup D_j' \) to a reflection of the product \( D_j \times I \) which pointwise fixes the core \( \overline{e_j} \).

**Figure 7.** We can think of \( U_{e_j} \) as a product \( D_j \times I \) with core \( \overline{e_j} \).

Next we consider the case where \( b_0 \) is a boundary component of an annulus \( F_i \subseteq \partial V_{F_i} \), which has both boundaries in a single \( \partial N(v) \). Recall that \( \text{cl}(\partial V_{F_i} - F_i) \) consists of disks \( D_i \) and \( D_i' \) properly embedded in \( N(v) \). Since \( h \) reflects the annulus \( F_i \) together with each of its boundary components, we can extend \( h \) radially to the disks \( D_i \) and \( D_i' \) to get a reflection of the sphere \( F_i \cup D_i \cup D_i' \) pointwise fixing a circle containing all of the points of \( \Gamma \cap D_i \) and \( \Gamma \cap D_i' \). Recall that \( V_{F_i} \cap \Gamma \) is a collection of one or more arcs. Thus we can extend \( h \) to a reflection of the ball \( V_{F_i} \) which pointwise fixing a disk containing \( V_{F_i} \cap \Gamma \) (see Figure 9).

Now the extension of \( h \) reflects the sphere with holes \( C_v \), and one of the balls \( U_{e_j} \) or \( V_{F_i} \) depending on whether \( b_0 \) is a boundary component of \( A_{e_j} \) or \( F_i \), respectively. Next we let \( S_1 \) denote the union of \( C_v \) together with the annulus \( A_{e_j} \) or \( F_i \) glued along \( b_0 \). Now \( h \) reflects \( S_1 \) taking every boundary component of \( S_1 \) to itself, and hence reflecting every boundary component of \( S_1 \). Let \( b_1 \) be a boundary component of \( S_1 \). If \( b_1 \) is not the other boundary of the annulus \( A_{e_j} \) or \( F_i \), then we repeat the above argument with \( b_1 \) in place of \( b_0 \). If \( b_1 \) is the other boundary of the annulus \( A_{e_j} \) or \( F_i \), then \( b_1 \) is also a boundary of some other component \( S_1' \) of \( W \cap \partial N(V) \), as illustrated in Figure 9. In this case, since \( b_1 \) is reflected by \( h \) and every component of
Figure 8. We extend $h$ to a reflection of the ball $V_{F_i}$ which pointwise fixes $\Gamma \cap V_{F_i}$.

$S'_1$ is invariant under $h$, we know that $h$ must reflect $S_2 = S_1 \cup S'_1$. Now let $b_2$ denote a boundary component of $S_2$, and repeat the above argument with $b_2$ in place of $b_0$.

Figure 9. In this illustration, we have three choices for the boundary component $b_2$ of $S_2 = S_0 \cup A_{e_j} \cup S'_1$.

In general, for a given surface $S_n$ obtained in this way, the surface $S_{n+1}$ is the union of $S_n$ together with either an annulus of the form $A_{e_j}$ or $F_i$ or a sphere with at least two holes contained in $W \cap \partial N(V)$. Furthermore, $S_{n+1}$ is reflected by $h$. This process will only stop when the surface that we obtain has no boundary components. Since $\partial W$ has only one component which intersects $\partial N(V)$, the closed surface that we obtain in this way must be $\partial W$. Thus we have extended $h$ to an orientation reversing involution of each of the balls $U_{e_1}, \ldots, U_{e_n}, V_{F_1}, \ldots, V_{F_m}$.

Now let $N = N(V) - (V_{F_1} \cup \cdots \cup V_{F_m})$. Then $N$ is a collection of disjoint balls (for example in Figure 8 $N(v) - V_{F_i}$ is two balls one of which contains $v$). Also, $h$ is a reflection of each component of $\partial N$ that fixes each point in $\partial N \cap \Gamma$. Now we extend $h$ radially to a reflection of each ball of $N$ in such a way that $h$ pointwise fixes each component of $N \cap \Gamma$. Thus $h$ is defined as a reflection of each component of $N(V)$ which pointwise fixes $N(V) \cap \Gamma$.

We have now extended $h$ to an orientation reversing involution of the manifold.
Recall from the end of Step 3 that \( \partial X \) and \( \partial W \) have the same collection of tori in their boundary components. Furthermore, we have filled in the boundary component of \( W \) meeting \( \partial N(\Gamma) \) with a collection of balls in \( X \) and \( N(\Gamma) \). Thus in fact \( Y = X \cup N(\Gamma) \).

Finally, we define a new embedding \( \Gamma' \) of \( \gamma \) in \( X \cup N(\Gamma) \) as follows. Let \( \Gamma' \cap N(V) = \Gamma \cap N(V) \). Then for each edge \( e_j \) define an embedding of \( e_j \) in \( \Gamma' \) as the core \( \overline{\epsilon_j} \) of \( U_{e_j} = D_j \times I \), which we know is pointwise fixed by \( h \) according to the way we extended \( h \) to \( U_{e_j} \) (recall Figure 7).

**Step 8:** We prove that if an essential curve in a component of \( \partial(X \cup N(\Gamma)) \) compresses in \( M \), then it compresses in \( X \cup N(\Gamma) \).

Let \( \{T_1, \ldots, T_r\} \) denote the set of boundary components of \( X \cup N(\Gamma) \). These tori are contained in the characteristic family \( \tau \), and hence are incompressible in \( \text{cl}(M - N(\Gamma)) \).

Suppose that an essential curve \( \lambda_i \) on some \( T_i \) compresses in \( M \). Let \( D_i \) be a compressing disk for \( \lambda_i \) whose intersection with the set of tori \( \{T_1, \ldots, T_r\} \) is minimal. Let \( D = D_i \) if the interior of \( D_i \) is disjoint from \( T_i \). Otherwise, there exists some \( D \) in the interior of \( D_i \) such that \( D_i \) is a compressing disk for \( T_i \) whose interior is disjoint from \( T_i \). In either case, the intersection of \( D \) with \( \{T_1, \ldots, T_r\} \) is minimal.

Suppose that \( D \) contains at least one curve of intersection in its interior. Hence there is an innermost disk \( \Delta \) on \( D \) which is a compressing disk for some \( T_j \) with \( j \neq i \). Since \( T_j \) compresses in \( M \) but is incompressible in \( \text{cl}(M - N(\Gamma)) \), we know that \( \Delta \) intersects \( \Gamma \).

Since \( M \) is irreducible, any compressible torus is separating in \( M \). Thus we can let \( X_j \) denote the closed up component of \( M - T_j \) containing \( X \) and let \( V_j \) denote the closed up component of \( M - T_j \) whose interior is disjoint from \( X \). Now let \( S \) denote the region of \( D \) which is adjacent to the innermost disk \( \Delta \). Then \( S \subseteq V_j \), since \( \Delta \subseteq X_j \). Also, \( \partial D \subseteq T_i \subseteq X \subseteq X_j \) implies that \( S \) is adjacent to another region of \( D \) which is contained in \( X_j \). In particular, there must be another circle of intersection \( \alpha \) of \( D \cap T_j \) which bounds a disk \( \overline{D} \subseteq D \) containing \( \Delta \cup S \). We illustrate the abstract disk \( D \) and its intersections with \( T_j \) in Figure 11. The white regions in the figure are contained in \( V_j \), and the grey regions are contained in \( X_j \). Note we do not include any circles of intersection of \( D \) with any \( T_k \) with \( k \neq j \).

Now, since the intersection of \( D \) with the tori \( T_1, \ldots, T_r \) is minimal, all of the curves of intersection of \( D \cap T_j \) must be essential on \( T_j \). In particular, \( \partial \Delta \) and \( \alpha \) must both be essential on \( T_j \). Since there cannot be two essential, disjoint, non-parallel curves on a torus, this means that \( \alpha \) is parallel to \( \partial \Delta \) on \( T_j \). It follows that \( \alpha \) must bound a disk \( \overline{\Delta} \) which is parallel to \( \Delta \) in \( M \). In particular, since the interior of \( \Delta \) is disjoint from \( T_1, \ldots, T_r \), the interior of \( \overline{\Delta} \) is as well. But now by replacing the disk \( \overline{D} \) with the disk \( \overline{\Delta} \) in the
compressing disk $D$ we obtain a new compressing disk $D'$ which has fewer curves of intersection with $T_1, \ldots, T_r$ than $D$ has. From this contradiction we conclude that the interior of $D$ must be disjoint from $T_1 \cup \cdots \cup T_r$, and hence $D \subseteq X \cup N(\Gamma)$.

If $\partial D = \lambda_i$, then $\lambda_i$ compresses in $X \cup N(\Gamma)$ as required. Otherwise, the compression disk $D$ was contained in the interior of the original disk $D_1$ and $\partial D \subseteq T_i$. In this case, since the intersection of $D_1$ with the set of tori $\{T_1, \ldots, T_r\}$ was minimal, $\partial D$ is essential in $T_i$. But now $\partial D$ and $\lambda_i$ are disjoint essential curves on $T_i$. Hence as we saw above, the disks $D$ and $D_1$ must be parallel in $M$. Now, since $D \subseteq X \cup N(\Gamma)$, it follows that $\lambda_i$ must compress in $X \cup N(\Gamma)$ as well.

**Step 9:** We fill each component $T_i$ of $\partial(X \cup N(\Gamma))$ with a solid torus such that the manifold $M'$ that we get satisfies the condition below.

**Condition:** If $T_i$ is compressible in $M$, then both generators of $H_1(T_i, \mathbb{Z}_2)$ are trivial in $H_1(M', \mathbb{Z}_2)$, and if $T_i$ is incompressible in $M$ then at least one generator of $H_1(T_i, \mathbb{Z}_2)$ is trivial in $H_1(M', \mathbb{Z}_2)$.

Let $T_i$ be a component of $\partial(X \cup N(\Gamma))$. By Corollary 1 there is a curve $\mu_i$ on $T_i$ which is non-trivial in $H_1(X \cup N(\Gamma), \mathbb{Z}_2)$. Also, we know from Step 8 that if some essential curve $\lambda_i$ on $T_i$ compresses in $M$, then $\lambda_i$ also compresses in $X \cup N(\Gamma)$. In particular, $\lambda_i$ is not homologous in $T_i$ to $\mu_i$.

Now suppose that for some $j \neq i$, the involution $h$ interchanges $T_i$ and $T_j$. Since $h : X \cup N(\Gamma) \to X \cup N(\Gamma)$ is a homeomorphism and $\mu_i$ is a curve on $T_i$ which is non-trivial in $H_1(X \cup N(\Gamma), \mathbb{Z}_2)$, we know that $h(\mu_i)$ is a curve on $T_j$ which is also non-trivial in $H_1(X \cup N(\Gamma), \mathbb{Z}_2)$. Now we fill $X \cup N(\Gamma)$ along $T_i$ by adding a solid torus $V_i$ with its meridian attached to the non-trivial curve $\mu_i$, and we fill along $T_j$ by adding a solid torus $V_j$ with its meridian attached to $h(\mu_i)$. Then we extend the involution $h$ radially on $V_i \cup V_j$ (abusing notation and still calling the involution $h$). We repeat
this process for every component of $X \cup N(\Gamma)$ which is not setwise invariant under $h$. As a result, for every $T_i$ along which we have glued a solid torus $V_i$, the curve $\mu_i$ on $T_i$ is trivial in $H_1(X \cup N(\Gamma) \cup V_i, \mathbb{Z}_2)$. Furthermore, if $T_i$ is compressible in $M$, then there is an essential curve $\lambda_i$ on $T_i$ which compresses in $X \cup N(\Gamma)$. Now, $\lambda_i$ is not homologous to $\mu_i$ in $H_1(T_i, \mathbb{Z}_2)$, and together they generate $H_1(T_i, \mathbb{Z}_2)$. Furthermore, both $\lambda_i$ and $\mu_i$ are trivial in $H_1(X \cup N(\Gamma) \cup V_i, \mathbb{Z}_2)$.

Let $Z$ be the manifold that we have obtained by filling all of the boundary components of $X \cup N(\Gamma)$ which are not setwise fixed by $h$, and let $T_i$ be a component of $\partial Z$. Recall from Step 7 that $\Gamma'$ is an embedding of $\gamma$ in $X \cup N(\Gamma)$ which is pointwise fixed by $h$. Now $h : Z \rightarrow Z$ is an orientation reversing involution pointwise fixing $\Gamma'$, and $T_i$ is setwise invariant under $h$. Since $h|T_i$ is an order 2 isometry of a torus, $h|T_i$ is either a reflection pointwise fixing two parallel circles on $T_i$, a rotation pointwise fixing four points of $T_i$, or a rotation fixing no points of $T_i$. In each of these cases, there is a pair of generators of $H_1(T_i, \mathbb{Z}_2)$ each of which is homologous to its image under $h$. It follows that for any given generator $a_i$ of $H_1(T_i, \mathbb{Z}_2)$ (which may or may not be homologous to $h(a_i)$ in $H_1(T_i, \mathbb{Z}_2)$), there is a curve $b_i$ on $T_i$ such that $(a_i, b_i) = H_1(T_i, \mathbb{Z}_2)$ and $h(b_i)$ is homologous to $b_i$ in $H_1(T_i, \mathbb{Z}_2)$.

Now suppose that some essential curve $\lambda_i$ on $T_i$ compresses in $M$. Then by Step 8, $\lambda_i$ also compresses in $X \cup N(\Gamma)$, and hence in $Z$. We can now pick a curve $b_i$ on $T_i$ such that $(\lambda_i, b_i) = H_1(T_i, \mathbb{Z}_2)$ and $h(b_i)$ is homologous to $b_i$ in $H_1(T_i, \mathbb{Z}_2)$. Since $\lambda_i$ is null homologous in $Z$, by Corollary 1 $b_i$ is non-trivial in $H_1(Z, \mathbb{Z}_2)$. Now we fill $Z$ along $T_i$ by adding a solid torus $V_i$ with its meridian attached to the non-trivial curve $b_i$. Since $h(b_i)$ is homologous to $b_i$ in $H_1(T_i, \mathbb{Z}_2)$, we can extend $h$ radially to the solid torus $V_i$. Then $h : Z \cup V_i \rightarrow Z \cup V_i$ is an orientation reversing involution, and both $\lambda_i$ and $b_i$ are trivial in $H_1(Z \cup V_i, \mathbb{Z}_2)$.

Now suppose that some $T_i$ is incompressible in $M$. As we saw above, there is a pair of generators of $H_1(T_i, \mathbb{Z}_2)$ each of which is homologous in $T_i$ to its image under $h$. By Corollary 1 at most one of these generators is null homologous in $Z$. So there is some curve $b_i$ on $T_i$ which is non-trivial in $H_1(Z, \mathbb{Z}_2)$ and homologous to $h(b_i)$ on $T_i$. Now fill $T_i$ by adding a solid torus $V_i$ with its meridian attached to the curve $b_i$ and extend $h$ to $V_i$. Then $h : Z \cup V_i \rightarrow Z \cup V_i$ is again an orientation reversing involution, and $b_i$ is trivial in $H_1(Z \cup V_i, \mathbb{Z}_2)$.

In this way, we glue a solid torus to each of the $T_i$ in $\partial(X \cup N(\Gamma))$ to obtain a closed manifold $M'$ satisfying the required condition. Since $\Gamma' \subseteq X \cup N(\Gamma)$, this gives us an embedding $\Gamma'$ of $\gamma$ in $M'$. Furthermore, we have extended $h$ to an orientation reversing involution of $(M', \Gamma')$ which pointwise fixes $\Gamma'$.

**Step 10:** We prove that there are at most $N_M$ tori in $\partial(X \cup N(\Gamma))$ which are incompressible in $M$. 
First suppose that some pair of distinct components \( T_i \) and \( T_j \) of \( \partial(X \cup N(\Gamma)) \) are parallel in \( M \). Then \( T_i \) and \( T_j \) co-bound a region \( R \) in \( M \) which is homeomorphic to a product of a torus and an interval. However, since \( T_i \) and \( T_j \) are tori in the characteristic family for \( \text{cl}(M - N(\Gamma)) \), they cannot be parallel in \( \text{cl}(M - N(\Gamma)) \). Thus \( R \) intersects \( \Gamma \). But since \( \partial R = T_i \cup T_j \) and \( \Gamma \) is disjoint from \( T_i \cup T_j \), this implies that \( \Gamma \subseteq R \).

Suppose that \( X \cup N(\Gamma) \) has a boundary component \( T_k \) which is distinct from \( T_i \) and \( T_j \). Since \( X \cup N(\Gamma) \) is a connected set which contains \( \Gamma \) and has \( T_i \), \( T_j \), and \( T_k \) among its boundary components, \( T_k \) must be contained in \( R \). But since \( R \cong T \times I \), either \( T_k \) is parallel in \( M \) to both \( T_i \) and \( T_j \) or \( T_k \) bounds a solid torus \( V \subseteq R \). The former would imply that \( T_k \) is parallel to one of \( T_i \) or \( T_j \) in \( \text{cl}(M - N(\Gamma)) \), which is impossible since all three are in the characteristic family for \( \text{cl}(M - N(\Gamma)) \). However, the latter would imply that \( N(\Gamma) \subseteq V \) because otherwise \( T_k \) would be compressible in \( \text{cl}(M - N(\Gamma)) \). But, this is impossible since \( \partial N(\Gamma) \), \( T_k \), \( T_i \), and \( T_j \) are all boundary components of \( X \). Hence we must have \( \partial(X \cup N(\Gamma)) = T_i \cup T_j \).

Since we saw in Section 3 that \( N_M \geq 2 \), it now follows that \( \partial(X \cup N(\Gamma)) \) has at most \( N_M \) components as required. Thus we can assume that no pair of distinct components of \( \partial(X \cup N(\Gamma)) \) are parallel in \( M \).

Let \( T_i \) be a component of \( \partial(X \cup N(\Gamma)) \) which is incompressible in \( M \). Since \( \Omega \) is the characteristic family of tori for \( M \), we know that \( T_i \) can be isotoped to be disjoint from \( \Omega \) (see for example [4]). Thus, without loss of generality, we can assume that \( T_i \) is contained in a closed up component of \( M - \Omega \) which is either atoroidal or Seifert fibered. If \( T_i \) is in an atoroidal component, then \( T_i \) is parallel to a torus in \( \Omega \). If \( T_i \) is in a Seifert fibered component, then it follows from Waldhausen [18] that \( T_i \) is parallel to either a torus in \( \Omega \) or a vertical or horizontal torus of the fibration.

Since no pair of distinct components of \( \partial(X \cup N(\Gamma)) \) are parallel in \( M \), there are at most \( t = |\Omega| \) incompressible tori in \( \partial(X \cup N(\Gamma)) \) that are parallel to a torus in \( \Omega \), and at most \( w \) (see Section 3 for the definition of \( w \)) incompressible tori in \( \partial(X \cup N(\Gamma)) \) that are parallel to a vertical or horizontal torus in some Seifert fibered closed up component of \( M - \Omega \). Hence there are at most \( N_M = t + w \) tori in \( \partial(X \cup N(\Gamma)) \) that are incompressible in \( M \).

Step 11: We prove the inequality \( \dim_{\mathbb{Z}_2}(H_1(M', \mathbb{Z}_2)) \leq n_M \).

Recall that \( M' \) is obtained from \( X \cup N(\Gamma) \) by adding a collection of solid tori \( V_1, \ldots, V_r \) along the tori \( T_1, \ldots, T_r \).

Let \([\beta]_{M'}\) be a non-trivial element of \( H_1(M', \mathbb{Z}_2) \). Now for each solid torus \( V_i \), let \( C_i \) denote its core. Then by general position we can choose a representative curve for \([\beta]_{M'}\) that is disjoint from \( C_1 \cup \cdots \cup C_r \). Hence we can assume that \( \beta \) is disjoint from \( V_1, \ldots, V_r \). Now since \( \beta \subseteq X \cup N(\Gamma) \subseteq M \), we can also consider the element \([\beta]_{M} \in H_1(M, \mathbb{Z}_2)\).

Suppose that \([\beta]_{M} \) is trivial in \( H_1(M, \mathbb{Z}_2) \). It follows that \( \beta \) is homologous in \( X \cup N(\Gamma) \) to a collection of curves on \( T_1 \cup \cdots \cup T_r \) which are trivial
in $M$ but non-trivial in $M'$. Recall from the condition in Step 9 that if $T_i$ is compressible in $M$, then both generators of $H_1(T_i; \mathbb{Z}_2)$ are trivial in $H_1(M'; \mathbb{Z}_2)$, and if $T_i$ is incompressible in $M$ then at least one generator of $H_1(T_i; \mathbb{Z}_2)$ is trivial in $H_1(M'; \mathbb{Z}_2)$. For each $T_i$ which is incompressible in $M$, if there is a generator of $H_1(T_i; \mathbb{Z}_2)$ that is not trivial in $H_1(M'; \mathbb{Z}_2)$ we denote it by $\beta_i$. Then every curve on $T_i$ is either trivial in $H_1(M'; \mathbb{Z}_2)$ or homologous to $\beta_i$ in $H_1(M'; \mathbb{Z}_2)$.

Returning now to the curve $\beta$ which is homologous in $M'$ to a sum of curves on $T_1 \cup \ldots \cup T_r$. Since for any $T_i$ which is compressible in $M$ both generators of $H_1(T_i; \mathbb{Z}_2)$ are trivial in $H_1(M'; \mathbb{Z}_2)$, it now follows that $\beta$ is homologous in $H_1(M'; \mathbb{Z}_2)$ to a sum of $\beta_i$’s on $T_i$’s that are incompressible in $M$. But by Step 10, there are at most $N_M$ such tori. Hence there are at most $N_M$ such $\beta_i$ that are not trivial in $H_1(M'; \mathbb{Z}_2)$. It follows that these $N_M$ curves generate every non-trivial $[\beta]_{M'}$ which is trivial in $M$. This gives us the required inequality:

$$\dim_{\mathbb{Z}_2}(H_1(M'; \mathbb{Z}_2)) \leq \dim_{\mathbb{Z}_2}(H_1(M; \mathbb{Z}_2)) + N_M = n_M.$$ 

Hence the proposition follows. \qed

References cited

[1] F. Bonahon, *Geometric structures on 3-manifolds*, Handbook of Geometric Topology, 93–164, North-Holland, Amsterdam, 2002.
[2] E. Flapan, *Symmetries of Möbius ladders*, Mathematische Annalen 283 (1989), 271–283.
[3] E. Flapan, *Rigidity of graph symmetries in the 3-sphere*, Journal of Knot Theory and its Ramifications 4 (1995), 373–388.
[4] A. Hatcher, *The classification of 3-manifolds- a brief overview* http://www.math.cornell.edu/~hatcher/3M/3Mdownloads.html
[5] A. Hatcher, *Notes on basic 3-manifold topology*, http://www.math.cornell.edu/~hatcher/3M/3Mdownloads.html
[6] A. Hatcher, P. Lochak and L. Schneps, *On the Teichmüller tower of mapping class groups*, J. Reine Angew. Math. 521 (2000), 1-24.
[7] H. Howards, *Generating disjoint incompressible surfaces*, Topology and its Applications 58 (2011), 325-343.
[8] H. Howards, *Surfaces and 3-manifolds*, Dissertation University of California San Diego, mathematics (1997).
[9] T. Ikeda, Rigidly achiral hyperbolic spatial graphs in 3-manifolds Journal of Knot Theory and its Ramifications 22 (2013), 12 pages.
[10] W. Jaco and P. Shalen, *Seifert fibred spaces in 3-manifolds*, Memoirs Amer. Math. Soc. 220, Amer. Math. Soc., Providence (1979).
[11] B. Jiang and S. Wang, *Achirality and planarity*, Communications in Contemporary Mathematics 2 (2000), 299-305.

[12] K. Johannson, *Homotopy equivalences of 3–manifolds with boundaries*, Lecture Notes in Mathematics 761, Springer-Verlag, New York, Berlin, Heidelberg (1979).

[13] M. Kobayashi, *Fixed point sets of orientation reversing involutions on 3-manifolds* Osaka J. Math. 25 (1988), 877–879.

[14] G. Mostow, *Strong rigidity of locally symmetric spaces*, Annals of Mathematics Studies 78, *Princeton University Press*, Princeton, NJ, 1973.

[15] P. A. Smith, *Transformations of finite period. II*, Ann. of Math. 40 (1939), 690-711. 1

[16] Richard Strong, *Diskbusting elements of the free group*, Mathematical Research Letters 4 (1997), 2010.

[17] W. Thurston, *Three-dimensional manifolds, Kleinian groups and hyperbolic geometry*, Bull. Amer. Math. Soc. 6 (1982), 357-381.

[18] F. Waldhausen, *Eine Klasse von 3-dimensionaler Mannigfaltigkeiten I*, Inventiones 3 (1967), 308-333.

[19] F. Waldhausen, *On irreducible 3-manifolds which are sufficiently large*, Ann. Math. 87 (1968), 56–88.

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