Renewal Monte Carlo: 
Renewal theory based reinforcement learning 

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Abstract—In this paper, we present an online reinforcement learning algorithm, called Renewal Monte Carlo (RMC), for infinite horizon Markov decision processes with a designated start state. RMC is a Monte Carlo algorithm and retains the advantages of Monte Carlo methods including low bias, simplicity, and ease of implementation while, at the same time, circumvents their key drawbacks of high variance and delayed (end of episode) updates. The key ideas behind RMC are as follows. First, under any reasonable policy, the reward process is ergodic. So, by renewal theory, the performance of a policy is equal to the ratio of expected discounted reward to the expected discounted time over a regenerative cycle. Second, by carefully examining the expression for performance gradient, we propose a stochastic approximation algorithm that only requires estimates of the expected discounted reward and discounted time over a regenerative cycle and their gradients. We propose two unbiased estimators for evaluating performance gradients—a likelihood ratio based estimator and a simultaneous perturbation based estimator—and show that for both estimators, RMC converges to a locally optimal policy. We generalize the RMC algorithm to post-decision state models and also present a variant that converges faster to an approximately optimal policy. We conclude by presenting numerical experiments on a randomly generated MDP, event-triggered communication, and inventory management.

Index Terms—Reinforcement learning, Markov decision processes, renewal theory, Monte Carlo methods, policy gradient, stochastic approximation

I. INTRODUCTION

In recent years, reinforcement learning [1]–[4] has emerged as a leading framework to learn how to act optimally in unknown environments. Policy gradient methods [5]–[10] have played a prominent role in the success of reinforcement learning. Such methods have two critical components: policy evaluation and policy improvement. In policy evaluation step, the performance of a parameterized policy is evaluated while in the policy improvement step, the policy parameters are updated using stochastic gradient ascent.

Policy gradient methods may be broadly classified as Monte Carlo methods and temporal difference methods. In Monte Carlo methods, performance of a policy is estimated using the discounted return of a single sample path; in temporal difference methods, the value(-action) function is guessed and this guess is iteratively improved using temporal differences. Monte Carlo methods are attractive because they have zero bias, are simple and easy to implement, and work for both discounted and average reward setups as well as for models with continuous state and action spaces. However, they suffer from various drawbacks. First, they have a high variance because a single sample path is used to estimate performance. Second, they are not asymptotically optimal for infinite horizon models because it is effectively assumed that the model is episodic; in infinite horizon models, the trajectory is arbitrarily truncated to treat the model as an episodic model. Third, the policy improvement step cannot be carried out in tandem with policy evaluation. One must wait until the end of the episode to estimate the performance and only then can the policy parameters be updated. It is for these reasons that Monte Carlo methods are largely ignored in the literature on policy gradient methods, which almost exclusively focuses on temporal difference methods such as actor-critic with eligibility traces [3].

In this paper, we propose a Monte Carlo method—which we call Renewal Monte Carlo (RMC)—for infinite horizon Markov decision processes with designated start state. Like Monte Carlo, RMC has low bias, is simple and easy to implement, and works for models with continuous state and action spaces. At the same time, it does not suffer from the drawbacks of typical Monte Carlo methods. RMC is a low-variance online algorithm that works for infinite horizon discounted and average reward setups. One doesn’t have to wait until the end of the episode to carry out the policy improvement step; it can be carried out whenever the system visits the start state (or a neighborhood of it).

Although renewal theory is commonly used to estimate performance of stochastic systems in the simulation optimization community [11], [12], those methods assume that the probability law of the primitive random variables and its weak derivate are known, which is not the case in reinforcement learning. Renewal theory is also commonly used in the engineering literature on queuing theory and systems and control for Markov decision processes (MDPs) with average reward criteria and a known system model. There is some prior work on using renewal theory for reinforcement learning [13], [14], where renewal theory based estimators for the average return and differential value function for average reward MDPs is developed. In RMC, renewal theory is used in a different manner for discounted reward MDPs (and the results generalize to average cost MDPs).

II. RMC ALGORITHM

Consider a Markov decision process (MDP) with state $S_t \in S$ and action $A_t \in A$. The system starts in an initial state $s_0 \in S$ and at time $t$:

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1) there is a controlled transition from $S_t$ to $S_{t+1}$ according to a transition kernel $P(A_t)$;
2) a per-step reward $R_t = r(S_t, A_t, S_{t+1})$ is received.

Future is discounted at a rate $\gamma \in (0, 1).

A (time-homogeneous and Markov) policy $\pi$ maps the current state to a distribution on actions, i.e., $A_t \sim \pi(S_t)$. We use $\pi(a|s)$ to denote $P(A_t = a|S_t = s)$. The performance of a policy $\pi$ is given by

$$J_\pi = E_{A_t \sim \pi(S_t)} \left[ \sum_{t=0}^{\infty} \gamma^t R_t \middle| S_0 = s_0 \right]. \quad (1)$$

We are interested in identifying an optimal policy, i.e., a policy that maximizes the performance. When $S$ and $A$ are Borel spaces, we assume that the model satisfies the standard conditions under which time-homogeneous Markov policies are optimal [15]. In the sequel, we present a sample path based online learning algorithm, which we call Renewal Monte Carlo (RMC), which identifies a locally optimal policy within the class of parameterized policies.

Suppose policies are parameterized by a closed and convex subset $\Theta$ of the Euclidean space. For example, $\Theta$ could be the weight vector in a Gibbs soft-max policy, or the weights of a deep neural network, or the thresholds in a control limit policy, and so on. Given $\theta \in \Theta$, we use $\pi_\theta$ to denote the policy parameterized by $\theta$ and $J_\theta$ to denote $J_{\pi_\theta}$. We assume that for all policies $\pi_\theta$, $\theta \in \Theta$, the designated start state $s_0$ is positive recurrent.

The typical approach for policy gradient based reinforcement learning is to start with an initial guess $\theta_0 \in \Theta$ and iteratively update it using stochastic gradient ascent. In particular, let $\nabla J_\theta$ be an unbiased estimator of $\nabla_\theta J_\theta|_{\theta = \theta_0}$, then update

$$\theta_{m+1} = \left[ \theta_m + \alpha_m \nabla J_{\theta_m} \right]_{\Theta} \quad (2)$$

where $[\theta]_{\Theta}$ denotes the projection of $\theta$ onto $\Theta$ and $\{\alpha_m\}_{m \geq 1}$ is a sequence of learning rates that satisfies the standard assumptions of

$$\sum_{m=1}^{\infty} \alpha_m = \infty \quad \text{and} \quad \sum_{m=1}^{\infty} \alpha_m^2 < \infty. \quad (3)$$

Under mild technical conditions [16], the above iteration converges to a $\theta^*$ that is locally optimal, i.e., $\nabla \theta J_\theta|_{\theta=\theta^*} = 0$. In RMC, we approximate $\nabla \theta J_\theta$ by a Renewal theory based estimator as explained below.

Let $\tau^{(n)}$ denote the stopping time when the system returns to the start state $s_0$ for the $n$-th time. In particular, let $\tau^{(0)} = 0$ and for $n \geq 1$ define

$$\tau^{(n)} = \inf \{ t > \tau^{(n-1)} : s_t = s_0 \}. \quad (4)$$

We call the sequence of $(S_t, A_t, R_t)$ from $\tau^{(n-1)}$ to $\tau^{(n)} - 1$ as the $n$-th regenerative cycle. Let $R^{(n)}$ and $T^{(n)}$ denote the total discounted reward and total discounted time of the $n$-th regenerative cycle, i.e.,

$$R^{(n)} = \Gamma^{(n)} \sum_{t=\tau^{(n-1)}}^{\tau^{(n)}-1} \gamma^t R_t \quad \text{and} \quad T^{(n)} = \Gamma^{(n)} \sum_{t=\tau^{(n-1)}}^{\tau^{(n)}-1} \gamma^t, \quad (5)$$

where $\Gamma^{(n)} = \gamma^{-(\tau^{(n-1)} - 1)}$. By the strong Markov property, $(R^{(n)})_{n \geq 1}$ and $T^{(n)}_{n \geq 1}$ are i.i.d. sequences. Let $\hat{R}_\theta$ and $\hat{T}_\theta$ denote $E[R^{(n)}]$ and $E[T^{(n)}]$, respectively. Define

$$\hat{R} = \frac{1}{N} \sum_{n=1}^{N} R^{(n)} \quad \text{and} \quad \hat{T} = \frac{1}{N} \sum_{n=1}^{N} T^{(n)}, \quad (6)$$

where $N$ is a large number. Then, $\hat{R}$ and $\hat{T}$ are unbiased and asymptotically consistent estimators of $R_\theta$ and $T_\theta$.

From ideas similar to standard Renewal theory [17], we have the following.

**Proposition 1 (Renewal Relationship)** The performance of policy $\pi_\theta$ is given by

$$J_\theta = \frac{R_\theta}{(1 - \gamma)T_\theta}. \quad (7)$$

**Proof** For ease of notation, define

$$\tilde{T}_\theta = E_{A_t \sim \pi_\theta(S_t)} \left[ \gamma^{\tau^{(n)} - \tau^{(n-1)}} \right]$$

Using the formula for geometric series, we get that $\tilde{T}_\theta = (1 - \tilde{T}_\theta)/(1 - \gamma)$. Hence,

$$T_\theta = 1 - (1 - \gamma)\tilde{T}_\theta. \quad (8)$$

Now, consider the performance:

$$J_\theta = E_{A_t \sim \pi_\theta(S_t)} \left[ \sum_{t=0}^{\tau^{(1)}-1} \gamma^t R_t + \gamma^{\tau^{(1)}} \sum_{t=\tau^{(1)}}^{\infty} \gamma^{t-\tau^{(1)}} R_t \middle| S_0 = s_0 \right]$$

$$= \hat{R}_\theta + E_{A_t \sim \pi_\theta(S_t)} \left[ \gamma^{\tau^{(1)}} \right] J_\theta$$

$$= \hat{R}_\theta + \tilde{T}_\theta J_\theta,$$

where the second expression in $(a)$ uses the independence of random variables from $(0, \tau^{(1)} - 1)$ to those from $\tau^{(1)}$ onwards due to the strong Markov property. Substituting $(a)$ in $(8)$ and rearranging terms, we get the result of the proposition. ■

Differentiating both sides of Equation $(a)$ with respect to $\theta$, we get that

$$\nabla_\theta J_\theta = \frac{H_\theta}{\Gamma^{(1)}_\theta (1 - \gamma)}, \quad \text{where} \quad H_\theta = T_\theta \nabla_\theta R_\theta - R_\theta \nabla_\theta T_\theta. \quad (9)$$

Therefore, instead of using stochastic gradient ascent to find the maximum of $J_\theta$, we can use stochastic approximation to find the root of $H_\theta$. In particular, let $H_m$ be an unbiased estimator of $H_{\theta_m}$. We then use the update

$$\theta_{m+1} = \left[ \theta_m + \alpha_m \tilde{H}_m \right]_{\Theta} \quad (10)$$

where $\{\alpha_m\}_{m \geq 1}$ satisfies the standard conditions on learning rates [3]. The above iteration converges to a locally optimal policy. Specifically, we have the following.
Theorem 1 Let $\hat{R}_m, \hat{T}_m, \nabla R_m$ and $\nabla T_m$ be unbiased estimators of $R_{\theta,m}, T_{\theta,m}, \nabla\theta R_{\theta,m}$, and $\nabla \theta T_{\theta,m}$, respectively such that $\hat{T}_m \perp \nabla R_m$ and $\hat{R}_m \perp \nabla T_m$. Then,

$$\hat{H}_m = \hat{T}_m \nabla R_m - \hat{R}_m \nabla T_m$$

(11)
is an unbiased estimator of $H_{\theta}$ and the sequence $\{\theta_m\}_{m \geq 1}$ generated by (10) converges almost surely and

$$\lim_{m \to \infty} \nabla \theta J_{\theta}|_{\theta = 0} = 0.$$

PROOF The unbiasedness of $\hat{H}_m$ follows immediately from the independence assumption. The convergence of the $\{\theta_m\}_{m \geq 1}$ follows from [16] Theorem 2.2 and the fact that the model satisfies conditions (A1)–(A4) of [16] pg 10–11.

In the remainder of this section, we present two methods for estimating the gradients of $R_{\theta}$ and $T_{\theta}$. The first is a likelihood ratio based gradient estimator which works when the policy is differentiable with respect to the policy parameters. The second is a simultaneous perturbation based gradient estimator that uses finite differences, which is useful when the policy is not differentiable with respect to the policy parameters.

A. Likelihood ratio based gradient estimator

One approach to estimate the performance gradient is to use likelihood ratio based estimates [12], [18], [19]. Suppose the policy $\pi_{\theta}(a|s)$ is differentiable with respect to $\theta$. For any time $t$, define the likelihood function

$$A_{t} = \nabla_{\theta} \log [\pi_{\theta}(A_{t} | S_{t})],$$

(12)
and for $\sigma \in \{\tau^{(n-1)}, \ldots, \tau^{(n)} - 1\}$, define

$$R_{\sigma}^{(n)} = \Gamma(n) \sum_{t=\sigma}^{\tau^{(n)}-1} \gamma^{t} \beta_{t}^{(n)} , \quad T_{\sigma}^{(n)} = \Gamma(n) \sum_{t=\sigma}^{\tau^{(n)}-1} \gamma^{t} .$$

(13)

In this notation $R_{\sigma}^{(n)} = R_{\tau^{(n-1)}}^{(n)}$ and $T_{\sigma}^{(n)} = T_{\tau^{(n-1)}}^{(n)}$. Then, define the following estimators for $\nabla_{\theta}R_{\theta}$ and $\nabla_{\theta}T_{\theta}$:

$$\hat{\nabla}R = \frac{1}{N} \sum_{n=1}^{N} \sum_{\sigma = \tau^{(n-1)}}^{\tau^{(n)}-1} R_{\sigma}^{(n)} A_{\sigma} ,$$

(14)

$$\hat{\nabla}T = \frac{1}{N} \sum_{n=1}^{N} \sum_{\sigma = \tau^{(n-1)}}^{\tau^{(n)}-1} T_{\sigma}^{(n)} A_{\sigma} ,$$

(15)

where $N$ is a large number.

Proposition 2 $\hat{\nabla}R$ and $\hat{\nabla}T$ defined above are unbiased and asymptotically consistent estimators of $\nabla_{\theta}R_{\theta}$ and $\nabla_{\theta}T_{\theta}$.

PROOF Let $P_{\theta}$ denote the probability induced on the sample paths when the system is following policy $\pi_{\theta}$. For $t \in \{\tau^{(n-1)}, \ldots, \tau^{(n)} - 1\}$, let $D_{t}^{(n)}$ denote the sample path $(S_{n}, A_{n}, S_{n+1})_{n=\tau^{(n-1)}}^{\tau^{(n)}}$ for the $n$-th regenerative cycle until time $t$. Then,

$$P_{\theta}(D_{t}^{(n)}) = \prod_{s=\tau^{(n-1)}}^{t} \pi_{\theta}(A_{s}|S_{s}) \pi(S_{s+1}|S_{s}, A_{s})$$

$^1$The notation $X \perp Y$ means that the random variables $X$ and $Y$ are independent.

Algorithm 1: RMC Algorithm with likelihood ratio based gradient estimates.

| input | Initial policy $\theta_0$, discount factor $\gamma$, initial state $s_0$, number of regenerative cycles $N$ |
|-------|--------------------------------------------------|
| for iteration $m = 0, 1, \ldots$ do |
| for regenerative cycle $n_1 = 1$ to $N$ do |
| Generate $n_1$-th regenerative cycle using policy $\pi_{\theta,m}$. |
| Compute $R^{(n_1)}$ and $T^{(n_1)}$ using (4). |
| Set $\hat{R}_m = \text{average}(R^{(n_1)} : n_1 \in \{1, \ldots, N\})$. |
| Set $\hat{T}_m = \text{average}(T^{(n_1)} : n_1 \in \{1, \ldots, N\})$. |
| for regenerative cycle $n_2 = 1$ to $N$ do |
| Generate $n_2$-th regenerative cycle using policy $\pi_{\theta,m}$. |
| Compute $R_{\sigma}^{(n_2)}$, $T_{\sigma}^{(n_2)}$ and $\Lambda_{\sigma}$ for all $\sigma$. |
| Compute $\hat{\nabla}R_{m}$ and $\hat{\nabla}T_{m}$ using (14) and (15). |
| Set $\hat{H}_{m} = \hat{T}_{m} \hat{\nabla}R_{m} - \hat{R}_{m} \hat{\nabla}T_{m}$. |
| Update $\theta_{m+1} = \theta_{m} + \alpha_{m} \hat{H}_{m}$. |

Therefore,

$$\nabla_{\theta} \log P_{\theta}(D_{t}^{(n)}) = \frac{1}{\gamma^{t}} \sum_{s=\tau^{(n-1)}}^{t} \nabla_{\theta} \log \pi_{\theta}(A_{s}|S_{s}) = \sum_{s=\tau^{(n-1)}}^{t} \Lambda_{s} .$$

(16)

Note that $R_{\theta}$ can be written as:

$$R_{\theta} = \Gamma(n) \sum_{t=\tau^{(n-1)}}^{\tau^{(n)-1}} \gamma^{t} E_{A_{t} \sim \pi_{\theta}}[R_{t}] .$$

Using the log derivative trick$^2$ we get

$$\nabla_{\theta} R_{\theta} = \Gamma(n) \sum_{t=\tau^{(n-1)}}^{\tau^{(n)-1}} \gamma^{t} E_{A_{t} \sim \pi_{\theta}}[R_{t}]$$

$$\iff (a) \Gamma(n) E_{A_{t} \sim \pi_{\theta}}[\Gamma(n) \sum_{t=\tau^{(n-1)}}^{\tau^{(n)-1}} \gamma^{t} R_{t}]$$

$= \sum_{t=\tau^{(n-1)}}^{\tau^{(n)-1}} \gamma^{t} R_{t} \sum_{t=\tau^{(n-1)}}^{\tau^{(n)-1}} \gamma^{t} A_{\sigma}$$

$= \gamma^{t} \sum_{t=\tau^{(n-1)}}^{\tau^{(n)-1}} \gamma^{t} R_{t} \sum_{t=\tau^{(n-1)}}^{\tau^{(n)-1}} \gamma^{t} A_{\sigma}$$

$= \sum_{t=\tau^{(n-1)}}^{\tau^{(n)-1}} \gamma^{t} R_{t} \sum_{t=\tau^{(n-1)}}^{\tau^{(n)-1}} \gamma^{t} A_{\sigma}$$

(17)

where (a) follows from (16), (b) follows from changing the order of summations, and (c) follows from the definition of $R_{\sigma}^{(n)}$ in (13). $\hat{\nabla}R$ is an unbiased and asymptotically consistent estimator of the right hand side of the first equation in (17). The result for $\hat{\nabla}T$ follows from a similar argument.

To satisfy the independence condition of Theorem 1 we use two independent sample paths: one to estimate $\hat{\nabla}R$ and the other to estimate $\hat{\nabla}T$. The complete algorithm in shown in Algorithm 1. An immediate consequence of Theorem 1 is the following.

$^2$Log-derivative trick: For any distribution $p(x|\theta)$ and any function $f$, $\nabla_{\theta} \log(p(X|\theta)) = \mathbb{E}_{X \sim p(X|\theta)}[f(X)] \nabla_{\theta} \log (p(X|\theta))$. 

Corollary 1 The sequence \( \{\theta_m\}_{m \geq 1} \) generated by Algorithm \ref{alg:RMC} converges to a local optimal.

Remark 1 Algorithm \ref{alg:RMC} is presented in its simplest form. It is possible to use standard variance reduction techniques such as subtracting a baseline \cite{19,20,21} to reduce variance.

Remark 2 In Algorithm \ref{alg:RMC} we use two separate runs to compute \((\bar{R}_m, \bar{T}_m)\) and \((\nabla \bar{R}_m, \nabla \bar{T}_m)\) to ensure that the independence conditions of Proposition \ref{prop:independence} are satisfied. In practice, we found that using a single run to compute both \((\bar{R}_m, \bar{T}_m)\) and \((\nabla \bar{R}_m, \nabla \bar{T}_m)\) has negligible effect on the accuracy of convergence (but speeds up convergence by a factor of two).

Remark 3 It has been reported in the literature \cite{22} that using a biased estimate of the gradient given by:

\[
R^{(n)}_\sigma = \Gamma^{(n)} \sum_{t=\sigma}^{\tau^{(n)}-1} \gamma^{t-\sigma} R_t,
\]

(and a similar expression for \(T^{(n)}_\sigma\)) leads to faster convergence. We call this variant RMC with biased gradients and, in our experiments, found that it does converge faster than RMC.

B. Simultaneous perturbation based gradient estimator

Another approach to estimate performance gradient is to use simultaneous perturbation based estimates \cite{23,24,25,26}. The general one-sided form of such estimates is

\[
\nabla R_\theta = \delta(\bar{R}_{\theta+c\delta} - \bar{R}_\theta)/c,
\]

where \(\delta\) is a random variable with the same dimension as \(\theta\) and \(c\) is a small constant. The expression for \(\nabla T_\theta\) is similar. When \(\delta_1 \sim \text{Rademacher}(\pm 1)\), the above method corresponds to simultaneous perturbation stochastic approximation (SPSA) \cite{23,24}; when \(\delta \sim \text{Normal}(0, I)\), the above method corresponds to smoothed function stochastic approximation (SFSA) \cite{25,26}.

Substituting the above estimates in \ref{alg:RMC} and simplifying, we get

\[
\tilde{H}_\theta = \delta(\tilde{T}_\theta \bar{R}_{\theta+c\delta} - \tilde{R}_\theta \tilde{T}_{\theta+c\delta})/c.
\]

The complete algorithm is shown in Algorithm \ref{alg:RMC SP}. Since \((\bar{R}_\theta, \bar{T}_\theta)\) and \((\bar{R}_{\theta+c\delta}, \bar{T}_{\theta+c\delta})\) are estimated from separate sample paths, \(\tilde{H}_\theta\) defined above is an unbiased estimator of \(H_\theta\). Then, an immediate consequence of Theorem \ref{thm:RMC} is the following.

Corollary 2 The sequence \( \{\theta_m\}_{m \geq 1} \) generated by Algorithm \ref{alg:RMC SP} converges to a local optimal.

III. RMC for Post-Decision State Model

In many models, the state dynamics can be split into two parts: a controlled evolution followed by an uncontrolled evolution. For example, many continuous state models have dynamics of the form

\[
S_{t+1} = f(S_t, A_t) + N_t,
\]

where \(\{N_t\}_{t \geq 0}\) is an independent noise process. For other examples, see the inventory control and event-triggered communication models in Sec \[V\]. Such models can be written in terms of a post-decision state model described below.

Consider a post-decision state MDP with pre-decision state \(S^-\in S^-\), post-decision state \(S^+\in S^+\), action \(A_t\in A\). The system starts at an initial state \(s^+_0\in S^+\) and at time \(t\):

1) there is a controlled transition from \(S^-_t\) to \(S^+_t\) according to a transition kernel \(P^-(A_t)\);
2) there is an uncontrolled transition from \(S^+_t\) to \(S^+_{t+1}\) according to a transition kernel \(P^+\);
3) a per-step reward \(R_t = r(S^-_t, A_t, S^+_t)\) is received.

Future is discounted at a rate \(\gamma \in (0, 1)\).

Remark 4 When \(S^+ = S^-\) and \(P^- = \text{identity}\), then the above model reduces to the standard MDP model, considered in Sec \[II\]. When \(P^+\) is a deterministic transition, the model reduces to a standard MDP model with post decision states \cite{27,28}.

As in Sec \[II\] we choose a (time-homogeneous and Markov) policy \(\pi\) that maps the current pre-decision state \(S^-\) to a distribution on actions, i.e., \(A_t \sim \pi(S^-_t)\). We use \(\pi(a|s^-)\) to denote \(\mathbb{P}(A_t = a|S^-_t = s^-)\).

The performance when the system starts in post-decision state \(s^+_0\in S^+\) and follows policy \(\pi\) is given by

\[
J_\pi = E_{A_t \sim \pi(S^-)} \left[ \sum_{t=0}^{\infty} \gamma^t R_t \bigg| S^+_0 = s^+_0 \right].
\]

As before, we are interested in identifying an optimal policy, i.e., a policy that maximizes the performance. When \(S\) and \(A\) are Borel spaces, we assume that the model satisfies the standard conditions under which time-homogeneous Markov

\begin{algorithm}
\caption{RMC Algorithm with simultaneous perturbation based gradient estimates.}
\begin{algorithmic}[1]
\State \textbf{input} : Initial policy \(\theta_0\), discount factor \(\gamma\), initial state \(s_0\), number of regenerative cycles \(N\), constant \(c\), perturbation distribution \(\Delta\).
\For {iteration \(m = 0, 1, \ldots\)}
\For {regenerative cycle \(n_1 = 1\) to \(N\)}
\State Generate \(n_1\)-th regenerative cycle using policy \(\pi_{\theta_m}\).
\State Compute \(R^{(n_1)}\) and \(T^{(n_1)}\) using \ref{alg:RMC SP}.
\State Set \(\bar{R}_m = \text{average}(R^{(n_1)} : n_1 \in \{1, \ldots, N\})\).
\State Set \(\bar{T}_m = \text{average}(T^{(n_1)} : n_1 \in \{1, \ldots, N\})\).
\State Sample \(\delta \sim \Delta\).
\State Set \(\theta'_m = \theta_m + c\delta\).
\EndFor
\For {regenerative cycle \(n_2 = 1\) to \(N\)}
\State Generate \(n_2\)-th regenerative cycle using policy \(\pi_{\theta_m}\).
\State Compute \(R^{(n_2)}\) and \(T^{(n_2)}\) using \ref{alg:RMC SP}.
\State Set \(\bar{R}'_m = \text{average}(R^{(n_2)} : n_2 \in \{1, \ldots, N\})\).
\State Set \(\bar{T}'_m = \text{average}(T^{(n_2)} : n_2 \in \{1, \ldots, N\})\).
\State Set \(\bar{H}_m = \delta(\bar{T}_m \bar{R}'_m - \bar{R}_m \bar{T}'_m)/c\).
\State Update \(\theta_{m+1} = [\theta_m + \alpha_m \bar{H}_m]_{\Theta}\).
\EndFor
\EndFor
\end{algorithmic}
\end{algorithm}
policies are optimal \([15]\). Let \(\tau^{(n)}\) denote the stopping times such that \(\tau^{(0)} = 0\) and for \(n \geq 1\),

\[
\tau^{(n)} = \inf\{t > \tau^{(n-1)} : s_{t-1}^n = s_0^n\}.
\]

The slightly unusual definition (using \(s_{t-1}^n = s_0^n\) rather than the more natural \(s_{t-1}^n = s_{t-1}^n\)) is to ensure that the formulas for \(R^{(n)}\) and \(T^{(n)}\) used in Sec. \[II\] remain valid for the post-decision state model as well. Thus, using arguments similar to Sec. \[II\], we can show that both variants of RMC presented in Sec. \[II\] converge to a locally optimal parameter \(\theta\) for the post-decision state model as well.

**IV. APPROXIMATE RMC**

In this section, we present an approximate version of RMC (for the basic model of Sec. \[II\]). Suppose that the state and action spaces \(S\) and \(A\) are separable metric spaces (with metrics \(d_S\) and \(d_A\)).

Given an approximation constant \(\rho \in \mathbb{R}_{>0}\), let \(B^\rho = \{s \in S : d_S(s, s_0) \leq \rho\}\) denote the ball of radius \(\rho\) centered around \(s_0\). Given a policy \(\pi\), let \(\tau^{(n)}\) denote the stopping times for successive visits to \(B^n\), i.e., \(\tau^{(0)} = 0\) and for \(n \geq 1\),

\[
\tau^{(n)} = \inf\{t > \tau^{(n-1)} : s_t \in B^n\}.
\]

Define \(R^{(n)}\) and \(T^{(n)}\) as in \([4]\) and let \(R^\rho_\theta\) and \(T^\rho_\theta\) denote the expected values of \(R^{(n)}\) and \(T^{(n)}\), respectively. Define

\[
J^\rho_\theta = \frac{R^\rho_\theta}{(1 - \gamma)T^\rho_\theta}.
\]

**Theorem 2** Given a policy \(\pi_0\), let \(V^\rho\) denote the value function and \(T^\rho_\theta = \mathbb{E}_{A_t \sim \pi_0(S_t)}[\gamma^{t-1} | S_0 = s_0]\) (which is always less than \(\gamma\)). Suppose the following condition is satisfied:

(C) The value function \(V^\rho\) is locally Lipschitz in \(B^\rho\), i.e., there exists a \(L^\rho\) such that for any \(s, s' \in B^\rho\),

\[
|V^\rho(s) - V^\rho(s')| \leq L^\rho d_S(s, s').
\]

Then

\[
|J_\theta - J^\rho_\theta| \leq \frac{L^\rho T^\rho_\theta}{(1 - \gamma)T^\rho_\theta} \leq \frac{\gamma}{(1 - \gamma)} L^\rho \rho.
\]  

**Proof** We follow an argument similar to Proposition \[1\]

\[
J_\theta = V^\rho(s_0) = \mathbb{E}_{A_t \sim \pi_0(S_t)} \left[ \sum_{t=0}^{\tau^{(1)} - 1} \gamma^t R_t + \gamma^{\tau^{(1)}} \sum_{t=\tau^{(1)}}^{\infty} \gamma^{t-\tau^{(1)}} R_t S_0 = s_{\tau^{(1)}} \right]^{(a)}
\]

\[
R^\rho_\theta + \mathbb{E}_{A_t \sim \pi_0(S_t)}[\gamma^{\tau^{(1)}} | S_0 = s_0] V^\rho(s_{\tau^{(1)}})
\]

where \((a)\) uses the strong Markov property. Since \(V^\rho\) is locally Lipschitz with constant \(L^\rho\) and \(s_{\tau^{(1)}} \in B^\rho\), we have that

\[
|J_\theta - V^\rho(s_{\tau^{(1)})})| = |V^\rho(s_0) - V^\rho(s_{\tau^{(1)})})| \leq L^\rho \rho.
\]

Substituting the above in \[(21)\] gives

\[
J_\theta \leq R^\rho_\theta + T^\rho_\theta(J_\theta + L^\rho \rho).
\]

**Proposition 3** Let \(V^\rho_\theta\) denote the value function for any policy \(\pi_\theta\). Suppose the model satisfies the following conditions:

1. The transition kernel \(P\) is Lipschitz, i.e., there exists a constant \(L_P\) such that for all \(s, s' \in S\) and \(a, a' \in A\),

\[
K(P(s, a), P(s', a')) \leq L_P[d_S(s, s') + d_A(a, a')]
\]

where \(K\) is the Kantorovich metric (also called Kantorovich-Monge-Rubinstein metric or Wasserstein distance) between probability measures.

2. The per-step reward \(r\) is Lipschitz, i.e., there exists a constant \(L_r\) such that for all \(s, s', s_+ \in S\) and \(a, a' \in A\),

\[
|r(s, a, +) - r(s', a', +)| \leq L_r[d_S(s, s') + d_A(a, a')]
\]

In addition, suppose the policy satisfies the following:

3. The policy \(\pi_\theta\) is Lipschitz, i.e., there exists a constant \(L_{\pi_\theta}\) such that for any \(s, s' \in S\),

\[
K(\pi_\theta(s), \pi_\theta(s')) \leq L_{\pi_\theta} d_S(s, s')
\]

4. \(\gamma L_P(1 + L_{\pi_\theta}) < 1\).

5. The value function \(V^\rho\) exists and is finite.

Then, \(V^\rho_\theta\) is Lipschitz. In particular, for any \(s, s' \in S\),

\[
|V^\rho_\theta(s) - V^\rho_\theta(s')| \leq L^\rho d_S(s, s'),
\]

where

\[
L^\rho = \frac{L_r(1 + L_{\pi_\theta})}{1 - \gamma L_p(1 + L_{\pi_\theta})}.
\]

**V. NUMERICAL EXPERIMENTS**

We conduct three experiments to evaluate the performance of RMC: a randomly generated MDP, event-triggered communication, and inventory management.
A. Randomized MDP (GARNET)

In this experiment, we study a randomly generated GARNET\((100, 10, 50)\) model\,[32], which is an MDP with 100 states, 10 actions, and a branching factor of 50 (which means that each row of all transition matrices has 50 non-zero elements, chosen Unif\([0, 1]\) and normalized to add to 1).

For each state-action pair, with probability \(p = 0.05\), the reward is chosen Unif\([10, 100]\), and with probability \(1 - p\), the reward is 0. Future is discounted by a factor of \(\gamma = 0.9\). The first state is chosen as start state. The policy is a Gibbs soft-max distribution parameterized by \(100 \times 10\) (states \(\times\) actions) parameters, where each parameter belongs to the interval \([-30, 30]\). The temperature of the Gibbs distribution is kept constant and equal to 1.

We compare the performance of RMC, RMC with biased gradient (denoted by RMC-B, see Remark 2), and actor critic with eligibility traces for the critic \,[3] (which we refer to as SARSA-\(\lambda\) and abbreviate as S-\(\lambda\) in the plots), with \(\lambda \in \{0, 0.25, 0.5, 0.75, 1\}\). For both the RMC algorithms, we use the same runs to estimate the gradients (see Remark 2 in Sec. II). Each algorithm\,[3] is run 100 times and the mean and standard deviation of the performance (as estimated by the algorithms themselves) is shown in Fig. 1a. The performance of the corresponding policy evaluated by Monte-Carlo evaluation over a horizon of 250 steps and averaged over 100 runs is shown in Fig. 1b. The optimal performance computed using value iteration is also shown.

The results show that SARSA-\(\lambda\) learns faster (this is expected because the critic is keeping track of the entire value function) but has higher variance and gets stuck in a local minima. On the other hand, RMC and RMC-B learn slower but have a low bias and do not get stuck in a local minima. The same qualitative behavior was observed for other randomly generated models. Policy gradient algorithms only guarantee convergence to a local optimum. We are not sure why RMC and SARSA differ in which local minima they converge to. Also, it was observed that RMC-B (which is RMC with biased evaluation of the gradient) learns faster than RMC.

B. Event-Triggered Communication

In this experiment, we study an event-triggered communication problem that arises in networked control systems\,[34], [35]. A transmitter observes a first-order autoregressive process \(\{X_t\}_{t \geq 1}\), i.e., \(X_{t+1} = \alpha X_t + W_t\), where \(\alpha, X_t, W_t \in \mathbb{R}\), and \(\{W_t\}_{t \geq 1}\) is an i.i.d. process. At each time, the transmitter uses an event-triggered policy (explained below) to determine whether to transmit or not (denoted by \(A_t = 1\) and \(A_t = 0\), respectively). Transmission takes place over an i.i.d. erasure channel with erasure probability \(p_d\). Let \(S_t^-\) and \(S_t^+\) denote the “error” between the source realization and it’s reconstruction at a receiver. It can be shown that \(S_t^-\) and \(S_t^+\) evolve as follows [34], [35]: when \(A_t = 0\), \(S_t^+ = S_t^-\); when \(A_t = 1\), \(S_t^+ = 0\) if the transmission is successful (w.p. \((1 - p_d)\)) and \(S_t^+ = S_t^-\) if the transmission is not successful (w.p. \(p_d\)); and \(S_{t+1} = \alpha S_t^+ + W_t\). Note that this is a post-decision state model, where the post-decision state resets to zero after every

---

3For all algorithms, the learning rate is chosen using ADAM [33] with default hyper-parameters and the \(\alpha\) parameter of ADAM equal to 0.05 for RMC, RMC-B, and the actor in SARSA-\(\lambda\) and the learning rate is equal to 0.1 for the critic in SARSA-\(\lambda\). For RMC and RMC-B, the policy parameters are updated after \(N = 5\) renewals.
successful transmission\textsuperscript{4}.

The per-step cost has two components: a communication cost of $\lambda A_t$, where $\lambda \in \mathbb{R}_{>0}$ and an estimation error $(S_t^+)^2$. The objective is to minimize the expected discounted cost.

An event-triggered policy is a threshold policy that chooses $A_t = 1$ whenever $|S_t| \geq \theta$, where $\theta$ is a design choice. Under certain conditions, such an event-triggered policy is known to be optimal \textsuperscript{34}, \textsuperscript{35}. When the system model is known, algorithms to compute the optimal $\theta$ are presented in \textsuperscript{36}, \textsuperscript{37}. In this section, we use RMC to identify the optimal policy when the model parameters are not known.

In our experiment we consider an event-triggered model with $\alpha = 1$, $\lambda = 500$, $p_d \in \{0, 0.1, 0.2\}$, $W_t \sim \mathcal{N}(0, 1)$, $\gamma = 0.9$, and use simultaneous perturbation variant of RMC\textsuperscript{2} to identify $\theta$. We run the algorithm 100 times and the result for different choices of $p_d$ are shown in Fig. \textsuperscript{2}\textsuperscript{4}. For $p_d = 0$, the optimal threshold computed using \textsuperscript{37} is also shown. The results show that RMC converges relatively quickly and has low bias across multiple runs.

C. Inventory Control

In this experiment, we study an inventory management problem that arises in operations research \textsuperscript{38}, \textsuperscript{39}. Let $S_t \in \mathbb{R}$ denote the volume of goods stored in a warehouse, $A_t \in \mathbb{R}_{\geq 0}$ denote the amount of goods ordered, and $D_t$ denotes the demand. The state evolves according to $S_{t+1} = S_t + A_t - D_t + 1$.

We work with the normalized cost function:

$$C(s) = a_p s (1 - \gamma) / \gamma + a_h s 1_{s \geq 0} - a_b s 1_{s < 0},$$

where $a_p$ is the procurement cost, $a_h$ is the holding cost, and $a_b$ is the backlog cost (see \textsuperscript{40}, Chapter 13 for details).

It is known that there exists a threshold $\theta$ such that the optimal policy is a base stock policy with threshold $\theta$ (i.e., whenever the current stock level falls below $\theta$, one orders up to $\theta$). Furthermore, for $s \leq \theta$, we have that \textsuperscript{40} Sec 13.2

$$V_\theta(s) = C(s) + \gamma (1 - \gamma) E[C(\theta - D)].$$ \hfill (22)

So, for $B^\theta \subset (0, \theta)$, the value function is locally Lipschitz, with

$$L_\theta = \left( a_h + \frac{1 - \gamma}{\gamma} a_p \right).$$

So, we can use approximate RMC to learn the optimal policy.

In our experiments, we consider an inventory management model with $a_h = 1$, $a_b = 1$, $a_p = 1.5$, $D_t \sim \text{Exp}(\lambda)$ with $\lambda = 0.025$, start state $s_0 = 1$, discount factor $\gamma = 0.9$, and use simultaneous perturbation variant of approximate RMC to identify $\theta$. We run the algorithm 100 times and the result is shown in Fig. \textsuperscript{3}\textsuperscript{4}. The optimal threshold and performance computed using \textsuperscript{40} Sec 13.2 is also shown. The result shows that RMC converges to an approximately optimal parameter value with total cost within the bound predicted in Theorem \textsuperscript{2}.

VI. CONCLUSIONS

We present a renewal theory based reinforcement learning algorithm called Renewal Monte Carlo. RMC retains the key advantages of Monte Carlo methods and has low bias, is simple and easy to implement, and works for models with continuous state and action spaces. In addition, due to the averaging over multiple renewals, RMC has low variance. We generalized the RMC algorithm to post-decision state models and also presented a variant that converges faster to an approximately optimal policy, where the renewal state is replaced by a renewal

\textsuperscript{4} Had we used the standard MDP model instead of the post-decision state model, this restart would not have always resulted in a renewal.

\textsuperscript{5} An event-triggered policy is a parametric policy but $p_\theta(a|s^-)$ is not differentiable in $\theta$. Therefore, the likelihood ratio method cannot be used to estimate performance gradient.

\textsuperscript{6} We choose the learning rate using ADAM with default hyper-parameters and the $\alpha$ parameter of ADAM equal to 0.01. We choose $\epsilon = 0.3$, $\rho = 100$, and $\Delta = \mathcal{N}(0, 1)$ in Algorithm \textsuperscript{2}.

\textsuperscript{7} We choose the learning rate using ADAM with default hyper-parameters and the $\alpha$ parameter of ADAM equal to 0.25. We choose $\epsilon = 3.0$, $\rho = 100$, and $\Delta = \mathcal{N}(0, 1)$ in Algorithm \textsuperscript{2} and choose $\rho = 0.5$ for approximate RMC. We bound the states within $[-100.0, 100.0]$. For $\text{Exp}(\lambda)$ demand, the optimal threshold is (see \textsuperscript{40} Sec 13.2)

$$\theta^* = \frac{1}{\lambda} \log \left( \frac{a_h + a_b}{a_h + a_p (1 - \gamma) / \gamma} \right).$$
set. The error in using such an approximation is bounded by the size of the renewal set.

In certain models, one is interested in the performance at a reference state that is not the start state. In such models, we can start with an arbitrary policy and ignore the trajectory until the reference state is visited for the first time and use RMC from that time onwards (assuming that the reference state is the new start state).

The results presented in this paper also apply to average reward models where the objective is to maximize

\[
J_\pi = \lim_{t_n \to \infty} \frac{1}{t_n} \mathbb{E}_{A_t \sim \pi(S_t)} \left[ \sum_{t=0}^{t_n-1} R_t \right] \bigg| S_0 = s_0 . \tag{23}
\]

Let the stopping times \( \tau^{(n)} \) be defined as before. Define the total reward \( R^{(n)} \) and duration \( T^{(n)} \) of the \( n \)-th regenerative cycle as

\[
R^{(n)} = \sum_{t=\tau^{(n-1)}}^{\tau^{(n)-1}} R_t \quad \text{and} \quad T^{(n)} = \tau^{(n)} - \tau^{(n-1)}
\]

Let \( R_\theta \) and \( T_\theta \) denote the expected values of \( R^{(n)} \) and \( T^{(n)} \) under policy \( \pi_\theta \). Then from standard renewal theory we have that the performance \( J_\theta \) is equal to \( R_\theta / T_\theta \) and, therefore

\[
\nabla_\theta J_\theta = H_\theta / T_\theta^2,
\]

where \( H_\theta \) is defined as in \( \langle 9 \rangle \). We can use both variants of RMC presented in Sec. \( \bowtie \) to obtain estimates of \( H_\theta \) and use these to update the policy parameters using \( \langle 10 \rangle \).

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