ALMOST-TORIC HYPERSURFACES

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Abstract. An almost-toric hypersurface is parameterized by monomials multiplied by polynomials in one extra variable. We determine the Newton polytope of such a hypersurface, and apply this to give an algorithm for computing the implicit equation.

1. Introduction and Formula

Toric varieties are parameterized by monomials. They form an important and rich class of examples in algebraic geometry and often provide a testing ground for theorems [CLS11]. Here we work with toric varieties that need not be normal. We fix an algebraically closed field $K$, and we set $K^* = K \setminus \{0\}$. Our first ingredient is an arbitrary projective toric variety of codimension 2 in $\mathbb{P}^{n+1}$. This is defined as follows.

Fix an $n \times (n+2)$ integer matrix $A$ whose columns span the lattice $\mathbb{Z}^n$:

$$A = \begin{bmatrix} a_0 & a_1 & \ldots & a_{n+1} \end{bmatrix}.$$  

The column vectors $a_i$ correspond to Laurent monomials $t^{a_i} = t_1^{a_{i1}} t_2^{a_{i2}} \ldots t_n^{a_{in}}$ in the variables $t = (t_1, t_2, \ldots, t_n)$. These $n+2$ monomials specify a monomial map

$$\Phi_A : (K^*)^n \to (K^*)^{n+2}, \quad t \to (t^{a_0}, t^{a_1}, \ldots, t^{a_{n+1}}).$$

Throughout this paper we assume that all columns of $A$ sum to the same positive integer $d$. Under this hypothesis, we obtain an induced map $\Phi_A : (K^*)^n \to \mathbb{P}^{n+1}$. The toric variety $X_A$ is the closure in $\mathbb{P}^{n+1}$ of the image of $\Phi_A$. The degree of $X_A$ is the normalized volume of the polytope conv$(A)$, and the equations defining $X_A$ form the toric ideal, a well-studied object in combinatorial commutative algebra and its applications [Stu96].

A natural extension of toric theory is the study of complex one $T$-varieties [IS11]. There are varieties with an action of a torus whose general orbits have codimension one. They can be viewed as a family of (possibly reducible) toric varieties over a curve. Our aim here is to explore such $T$-varieties from the point of view of symbolic computation. For simplicity we assume that $T$-varieties are rational, i.e. the underlying curve is rational, and we focus on projective hypersurfaces.

Our second ingredient is a vector of univariate polynomials in a new variable $x$:

$$f = (f_0(x), f_1(x), \ldots, f_{n+1}(x)) \in K[x]^{n+2}. \quad (1)$$

This vector specifies a parametric curve $Y_f \subset \mathbb{P}^{n+1}$, namely the closure of the set of points $(f_0(x) : f_1(x) : \ldots : f_{n+1}(x))$. Let $Z_{A,f}$ denote the Hadamard product in $\mathbb{P}^{n+1}$ of the toric variety $X_A$ and the curve $Y_f$. By definition, this is the Zariski closure of the set

$$\{ (t^{a_0} f_0(x) : t^{a_1} f_1(x) : \ldots : t^{a_{n+1}} f_{n+1}(x)) \in \mathbb{P}^{n+1} \mid t \in (K^*)^n, x \in K \} \subset \mathbb{P}^{n+1}.$$  

Under some mild hypotheses (see Theorem 1.1(a)), the variety $Z_{A,f}$ has codimension 1, and we call it the almost-toric hypersurface associated with $(A, f)$. We shall present a fast method for implicitizing $Z_{A,f}$. The output of our algorithm is the irreducible
polynomial in $K[u_0, u_1, \ldots, u_{n+1}]$ that vanishes on $Z_{A\cdot f}$. The torus action given by $A$ ensures that its Newton polytope $\text{Newt}(Z_{A\cdot f})$ lies in a plane in $\mathbb{R}^{n+2}$, so it is a polygon. Our first main result is the following combinatorial formula for this Newton polygon.

The ingredients in our formula are two matrices, which we now define. The Plücker matrix associated with $A$ is the $(n+2) \times (n+2)$-matrix $P_A = (p_{ij})$ with entries

$$p_{ij} = \begin{cases} \frac{1}{\delta}(-1)^{i+j}\det(A_{i,j}), & i < j; \\ -p_{ji}, & i > j; \\ 0, & i = j, \end{cases}$$

where $\delta$ is the greatest common divisor of all $\det(A_{i,j})$. Here $A_{i,j}$ is the $n \times n$ submatrix of $A$ obtained by deleting the columns $a_i$ and $a_j$. The Plücker matrix $P_A$ is skew-symmetric and has rank 2, and its row space is the kernel of $A$. The latter property implies that all rows and all columns of $P_A$ sum to zero.

The valuation matrix associated with $f$ is an integer matrix $V_f$ with $n+2$ rows that is defined as follows. Since $K$ is algebraically closed, each of our polynomials $f_i(x)$ factors into linear factors in $K[x]$. Let $g_1(x), g_2(x), \ldots, g_m(x)$ be the list of all distinct linear factors of $f_0(x)f_1(x)\cdots f_{n+1}(x)$. We write $\text{ord}_g f_i$ for the order of vanishing of $f_i(x)$ at the unique root of $g_j(x)$. We organize these numbers into the vectors

$$u_j = (\text{ord}_g f_0, \text{ord}_g f_1, \ldots, \text{ord}_g f_{n+1}) \in \mathbb{N}^{n+2} \quad \text{for } 1 \leq j \leq m.$$ 

We now aggregate these vectors according to the lines they span. Let $S = \{u_1, \ldots, u_m\}$. If two vectors in $S$ are linearly dependent, then we delete them and add their sum to the set. We repeat this procedure. After finitely many steps, we end up with a new set $S' = \{v_1, v_2, \ldots, v_{\ell}\}$ whose vectors span distinct lines. The valuation matrix is

$$V_f = \begin{bmatrix} v_1^T & v_2^T & \cdots & v_{\ell}^T \end{bmatrix} \begin{bmatrix} (-\sum_{j=1}^{\ell} v_j)^T \end{bmatrix}.$$ 

The last vector represents the valuation at $\infty$. It ensures that the rows of $V_f$ sum to zero. The following theorem allows us to derive the Newton polygon from $A$ and $f$.

**Theorem 1.1.** The edges of the Newton polygon of $Z_{A\cdot f}$ are the columns of the product of the Plücker matrix $P_A$ and the valuation matrix $V_f$. More precisely, given $A$ and $f$,

(a) if $\text{rank}(P_A \cdot V_f) = 0$ then $Z_{A\cdot f}$ is not a hypersurface;

(b) if $\text{rank}(P_A \cdot V_f) = 1$ then $Z_{A\cdot f}$ is a toric hypersurface;

(c) if $\text{rank}(P_A \cdot V_f) = 2$ then $Z_{A\cdot f}$ is a hypersurface but not toric. The directed edges of the Newton polygon of $Z_{A\cdot f}$ are the nonzero column vectors of $P_A \cdot V_f$.

The rest of this paper is organized as follows. In Section 2 we present the proof of Theorem 1.1, and we illustrate this result with several small examples. Earlier work of Philippon and Sombra [PS08, Proposition 4.1] yields an expression for the degree of the almost-toric hypersurface $Z_{A\cdot f}$ as a certain sum of integrals over $\text{conv}(A)$.

Section 3 is concerned with computational issues. Our primary aim is to give a fast algorithm for computing the implicit equation of the almost-toric hypersurface $Z_{A\cdot f}$. We develop such an algorithm and implement it in Maple 17. A case study of hard implicitization problems demonstrate that our method performs very well.

The mathematics under the hood of Theorem 1.1 is tropical algebraic geometry [MS15]. The proof relies on a technique known as tropical implicitization [STY07, ST08, SY08]. Thus this article offers a concrete demonstration that tropical implicitization can serve as an efficient and easy-to-use tool in computer algebra. It lays the foundation for future work that will extend toric algebra [Stu96] and its numerous applications to the almost-toric setting of complexity one T-varieties [IS11].
2. Proof, Examples, and the Philippon-Sombra Formula

In this section we prove Theorem 1.1. In our proof we use the technique of tropicalization. We then present examples to illustrate our main theorem. We also consider an easier task: finding the degree of the almost-toric hypersurface. We compare our result to existing results, including the Philippon-Sombra Formula in [PS08, Proposition 1.2] and another formula in [SY08].

First we prove some of the claims made in Section 1.

Lemma 2.1. Let $Z_{Af}$ be an almost-toric hypersurface. Then $\text{Newt}(Z_{Af})$ is at most 2-dimensional in $\mathbb{R}^{n+2}$.

Proof. Substituting variables $u_0, \ldots, u_{n+1}$ by the parameterization $u_i = t^{a_i} f_i(x)$ in the implicit equation $p(u_0, \ldots, u_{n+1})$ of $Z_{Af}$, we get another polynomial in variables $t_1, t_2, \ldots, t_n, x$. The latter polynomial is the zero polynomial. After this substitution, each term in $p$ becomes the product of a monomial in variables $t_1, t_2, \ldots, t_n$ and a polynomial in $x$. Since $p$ is the generator of the principal ideal corresponding to the almost-toric hypersurface, all such monomials in variables $t_1, t_2, \ldots, t_n$ are the same; otherwise the almost-toric hypersurface would vanish on a polynomial that contains some terms of $p$, a contradiction. Suppose after substitution the monomial is $\prod_{i=1}^{n} t_i^{a_i}$. If $v = (v_0, \ldots, v_{n+1})$ is a vertex of $\text{Newt}(Z_{Af})$, then $p$ contains a term $\prod_{i=0}^{n+1} u_i^{v_i}$. Therefore $v \cdot A^T = (a_1, \ldots, a_n)$. So vertices of $\text{Newt}(Z_{Af})$ satisfy $n$ independent linear equations and we conclude that $\text{Newt}(Z_{Af})$ is at most 2-dimensional. □

Remark 2.2. Even if $Z_{Af}$ is a hypersurface, its Newton polygon could be a degenerate polygon that has only two vertices.

Lemma 2.3. Let $P_A$ be a Plücker matrix. Then $P_A$ is skew-symmetric. The rank of $P_A$ is 2 and the entries in each row and column of $P_A$ sum to 0.

Proof. The first claim follows directly from the definition of $P_A$. For the second claim, note that the inner product of the $i$-th row of $P_A$ and the $j$-th row of $A$ is the determinant of the following $(n+1) \times (n+1)$ matrix up to sign: append the $j$-th row of $A$ to matrix $A$ and then delete the $i$-th column of the new matrix. Therefore $\text{row}(P_A) \subseteq \text{row}(A)^\perp$, which means the rank of $P_A$ is at most 2. Since $A$ has full rank, by the definition of $P_A$ there is a nonzero non-diagonal entry $p_{ij}$ of $P_A$. By the first claim, $p_{ij}$ is nonzero too. So we get a nonzero $2 \times 2$ minor in $P_A$. For the last claim, since $P_A$ is skew-symmetric it is enough to prove the claim for rows. It turns out that up to sign, the sum of entries in the $i$-th row is the determinant of the matrix formed by $A_{[i]}$ and the vector $1 = (1, 1, \ldots, 1) \in K^{n+1}$, where $A_{[i]}$ is the matrix obtained from $A$ by deleting the $i$-th column. Since each column of $A$ has sum $d$, the vector $1$ lies in the row space of $A_{[i]}$, so the matrix is singular and its determinant is zero. □

Next, we explore $Z_{Af}$. We consider its tropicalization (we follow the definition from [MS15, Definition 3.2.1]). Let $X_A, Y_f, Z_{Af}$ be defined as in Section 1. Note that $Z_{Af}$ is the Hadamard product $\overline{X_A \ast Y_f}$ of the varieties $X_A$ and $Y_f$. We have the following result relating the tropicalizations of $X_A, Y_f, Z_{Af}$:

Proposition 2.4. [Cue10, Corollary 3.3.6] \[ \text{trop}(Z_{Af}) = \text{trop}(X_A) + \text{trop}(Y_f), \]
where the sum is Minkowski sum.

So in order to find $\text{trop}(Z_{Af})$, it is enough to find $\text{trop}(X_A)$ and $\text{trop}(Y_f)$.
Lemma 2.5. If row($A$) is the row space of $A$ with real coefficients, then
trop($X_A$) = row($A$).

Proof. By Proposition 2.4, it is enough to prove the case when $n = 1$ and then use induction. For $n = 1$ we need to show that if $a_0, a_1, \ldots, a_{n+1}$ are integers and

$$X_A = \text{cl}(\{(t^{a_0} : t^{a_1} : \ldots : t^{a_{n+1}}) | t \in K^*\}),$$

then
trop($X_A$) = \{ $r \cdot a | r \in \mathbb{R}$ \},

where $a = (a_0, a_1, \ldots, a_{n+1})$. In this case the ideal $I(X_A)$ is generated by binomials as follows (cf. [Stu96, Corollary 4.3]):

$$I(X_A) = (x^u - x^v | a \cdot u = a \cdot v).$$

Then all points in trop($X_A$) are scalar multiples of $a$. □

Lemma 2.6. Let $Y_t$ be defined as in (1), and $S = \{g_1, \ldots, g_m, \infty\}$. Then
trop($Y_t$) = $\bigcup_{z \in S} \{ \lambda(\text{ord}_z f_0, \text{ord}_z f_1, \ldots, \text{ord}_z f_{n+1}) | \lambda \in \mathbb{R}^{n+2} \lambda \geq 0 \}.$

In addition, trop($Y_t$) is an 1-dimensional balanced polyhedral fan in $\mathbb{R}^{n+2}$ and the rays are spanned by the vectors $v_z$, where $v_z = (\text{ord}_z f_0, \text{ord}_z f_1, \ldots, \text{ord}_z f_{n+1})$.

Proof. Let $K' = K\{(t)\}$ be the field of Puiseux series[MS15, Example 2.1.3] in variable $t$ with coefficients in $K$. Then $K'$ has a nontrivial valuation and is algebraically closed. Let $Y'_t$ be a variety parameterized as in (1) but $x$ varies in $K'$ instead. Note that $Y_t$ and $Y'_t$ have the same ideal, so trop($Y_t$) = trop($Y'_t$).

By [MS15, Theorem 3.2.3], trop($Y'_t$) = val($Y'_t$), where

$$\text{val}(Y'_t) = \{ (\text{val}(f_0(x)), \ldots, \text{val}(f_{n+1}(x))) | x \in K' \}.$$

Since $Q$ is dense in $\mathbb{R}$, the right hand side of (2) is the closure of

$$B = \bigcup_{z \in S} \{ \lambda(\text{ord}_z f_0, \text{ord}_z f_1, \ldots, \text{ord}_z f_{n+1}) | \lambda \in \mathbb{Q}^+ \}.$$

It then suffices to show $B = \text{val}(Y'_t)$. We first show that $B \subseteq \text{val}(Y'_t)$. Fixing $z \in S$ and $\lambda > 0$, we get a vector

$$u = \lambda(\text{ord}_z f_0, \text{ord}_z f_1, \ldots, \text{ord}_z f_{n+1}) \in B.$$

- If $z \neq \infty$, then $z = g_j$ for some $1 \leq j \leq m$. Since each $g_j$ is linear, we may assume $g_j(x) = x - r_j$ where $r_j \in K$. For each $i$, we have $f_i(x) = (x - r_j)^{\text{ord}_z f_i} h_i(x)$, where $h_i(r_j) \neq 0$. Then $\text{val}(h_i(r_j)) = 0$. Now if we take $x = r_j + t^\lambda \in K'$, then $f_i(x) = t^{\text{ord}_z f_i} h_i(r_j + t^\lambda)$. Notice that $h_i(x)$ is a polynomial in $K[x]$. Then $t^\lambda$ divides $h_i(r_j + t^\lambda) - h_i(r_j)$ in $K'$, so $\text{val}(h_i(r_j + t^\lambda) - h_i(r_j)) > 0$. Then $\text{val}(h_i(r_j + t^\lambda)) = 0$. Thus $\text{val}(f_i(x)) = \text{ord}_z f_i$, which means $u \in \text{val}(Y'_t)$.

- If $z = \infty$, then $\text{ord}_z f_i = -\text{deg}(f_i)$. We take $x = t^{-\lambda} \in K'$. Then among all terms in $f_i(x)$, the term with smallest valuations is the leading term, because $\lambda > 0$. Then

$$\text{val}(f_i(x)) = (-\lambda) \text{deg}(f_i) = \lambda \text{ord}_\infty f_i.$$

So $u \in \text{val}(Y'_t)$, too.

We next show that $\text{val}(Y'_t) \subseteq B$. Suppose $u \in \text{val}(Y'_t)$. Then there exists $x_0 \in K'$ such that $u = (\text{val}(f_0(x_0)), \ldots, \text{val}(f_{n+1}(x_0)))$. We may assume that $u \neq 0$. We must deal with two cases.
Corollary 2.7. The tropicalization of Proposition 2.8. The edges of $P$ in $\text{Newt}(Z_{A,t})$ are parallel to the nonzero column vectors in $P_A \cdot V_f$.

Proof. By [MS15, Proposition 3.1.10], trop($Z_{A,t}$) is the support of an $(n+1)$-dimensional polyhedral fan, which is the $(n+1)$-skeleton of the normal fan of Newt($Z_{A,t}$). Since Newt($Z_{A,t}$) is a polygon, every cone in the $(n+1)$-skeleton of its normal fan is a cone spanned by row($A$) (which is exactly the orthogonal complement of the plane that contains the polygon) and a ray inside this plane that is orthogonal to the corresponding edge of the polygon. By Corollary 2.7, every one of these directed edges belongs to ker($A$) and is orthogonal to the corresponding column vectors of $V_f$. Let $v$ be a column vector of $V_f$. Since $P_A$ is symmetric, we have

$$v^T \cdot P_A \cdot v = 0.$$ 

Hence $P_A \cdot v$ is orthogonal to $v^T$, which means that there exists a scalar $c$ such that $c(P_A \cdot v)$ represents an edge of Newt($Z_{A,t}$).

It remains to show that the length of column vectors in $P_A \cdot V_f$ coincide with the edges of Newt($Z_{A,t}$). To analyze these lengths we recall the notion of multiplicity of a polyhedron which is maximal in a weighted polyhedral complex. We adopt the definition in [MS15, Definition 3.4.3]. We have the following result:

Lemma 2.9. [MS15, Lemma 3.4.6] The lattice length of any edge (defined as the number of lattice points on the edge minus 1) of Newt($Z_{A,t}$) is the multiplicity of the corresponding $(n+1)$-dimensional polyhedron in the normal fan of Newt($Z_{A,t}$).

Note that if an edge of Newt($Z_{A,t}$) is expressed by a vector, then its lattice length is the content of that vector. Therefore it is enough to find out mult($\sigma$) for each $\sigma$ in the $(n+1)$-skeleton of the normal fan of Newt($Z_{A,t}$). We cannot achieve this directly from the definition of multiplicity, because it involves the implicit polynomial of $Z_{A,t}$, which is unknown to us. However, we can find out those multiplicities with the help of another result in [MS15]. The following proposition will lead to our main theorem.
Proposition 2.10. For every maximal cell \( \sigma \) in the \((n+1)\)-skeleton of the normal fan of Newt\((Z_{A,f})\), \( \text{mult}(\sigma) \) is the content of the corresponding column vector of \( P_A \cdot V_f \).

Proof. Let \( \Sigma' \) be \( \text{trop}(Z_{A,f}) \), which is a pure weighted polyhedral complex in \( \mathbb{R}^{n+2} \). Suppose there are \( m \) roots \( z_1, \ldots, z_m \in K \) of \( \prod_{i=0}^{n+1} f_i(z) \). We define another pure weighted polyhedral complex \( \Sigma \subseteq \mathbb{R}^{n+m+2} \) as follows:

\[
\Sigma = \{ (u, v) \in \mathbb{R}^{n+m+2} \mid u \in \text{row}(A), v = \lambda \cdot e_i, 1 \leq i \leq m+1, \lambda \geq 0 \},
\]

where \( e_i \) is the vector with \( i \)-th component 1 and others 0 for \( 1 \leq i \leq m \), and \( e_{m+1} = -1 \). Note that

\[
\Sigma = \text{trop}(Z),
\]

where \( Z \) is the variety parameterized by

\[
\{(t^{a_0}_0 : \cdots : t^{a_{n+1}}_0, (x - z_1, x - z_2, \ldots, x - z_m)) \in \mathbb{P}^{n+1} \times \mathbb{C}^m \mid t \in (K^*)^n, x \in K \}.
\]

Note that we can get \( \Sigma' \) from \( \Sigma \) via a projection \( \phi \), where \( \phi \) keeps \( u \in \text{row}(A) \) fixed and sends each \( e_i \) to the transpose of the \( i \)-th column vector \( v_i \) in \( V_f \). So \( \phi \) maps a maximal cell \( \sigma \in \Sigma \) to a maximal cell \( \sigma' \in \Sigma' \). This correspondence of maximal cells is a bijection. Suppose \( \sigma' \) in the \((n+1)\)-skeleton of the normal fan of Newt\((Z_{A,f})\) corresponds to the vector \( v_i \). Then by [MS15, (3.6.2)]

\[
\text{mult}(\sigma') = \text{mult}(\sigma) \cdot [N_{\sigma'} : \phi(N_\sigma)],
\]

where \( N_{\sigma'} \subseteq \mathbb{R}^{n+2} \) is the lattice generated by all integer points in the span of row vectors of \( A \) and \( v_i^T \), and \( N_\sigma \subseteq \mathbb{R}^{n+m+2} \) is the lattice generated by all integer points in the space row\((A) \oplus \mathbb{R}e_i \). Let \( N \subseteq \mathbb{R}^{n+2} \) be the lattice generated by row vectors of \( A \) and \( v_i^T \). Then \( N \subseteq N_{\sigma'}, N_\sigma \) and \([N_{\sigma'} : \phi(N_\sigma)] = \frac{[N_{\sigma'} : N]}{[N_\sigma : N]} \). The lattice index \([N_{\sigma'} : N]\) is the greatest common divisor of all maximal minors of the matrix formed by all generating vectors of \( N_{\sigma'} \). Here this matrix is obtained from \( A \) by adding one row vector \( v_i^T \) and its maximal minors are the product of \( \delta \) and one entry in the \( i \)-th column of \( P_A \cdot V_f \), so \([N_{\sigma'} : N]\) is the product of \( \delta \) and the content of the \( i \)-th column vector of \( P_A \cdot V_f \). Similarly, \([N_\sigma : N]\) is the greatest common divisor of all maximal minors of \( A \), which is \( \delta \).

Finally the initial ideal \( \text{in}_\sigma(I(Z)) \) with respect to \( \sigma \) of the ideal of \( Z \) is the direct sum of two prime ideals: the first one is toric, the second one is generated by linear polynomials. Then the quotient ring of \( K[u_0, \ldots, u_{n+1}] \) modulo the toric ideal is a domain. Note that the quotient ring of \( K[u_0, \ldots, u_{n+1},v_1, \ldots, v_m] \) modulo \( \text{in}_\sigma(I(Z)) \) is the quotient ring of \( K[u_0, \ldots, u_{n+1}] \) modulo the toric ideal adjoining some variable(s) in \( \{v_1, \ldots, v_m\} \), which is also a domain. Hence \( \text{in}_\sigma(I(Z)) \) is also prime. Then by the definition of multiplicity, \( \text{mult}(\sigma) = 1 \). This finishes the proof.

Proof of Theorem (1.1). If \( \text{rank}(P_A \cdot V_f) = 0 \) then all column vectors of \( V_f \) belong to \( \text{row}(A) \), which means \( Z_{A,f} \) contains the toric variety \( X_A \) and has codimension 2, so it is not a hypersurface. If \( \text{rank}(P_A \cdot V_f) = 1 \), then essentially \( f \) provides one parameter not appearing in \( X_A \), which means that \( Z_{A,f} \) has codimension 1 and is a toric hypersurface. If \( \text{rank}(P_A \cdot V_f) = 2 \), then Newt\((Z_{A,f})\) is a nondegenerate polygon and with Proposition 2.8, Lemma 2.9 and Proposition 2.10 we have proved Theorem 1.1.

We illustrate Theorem 1.1 with the following example.

Example 2.11. Let \( Z_{A,f} \) admit the following parameterization over \( \mathbb{C} \):

\[
(t_1^2(x^2 + 1) : t_1t_2x^3(x - 1) : t_1t_3x(x + 1) : t_2^2(x - 2)(x^2 + 1) : t_3^2(x - 1)^2(x + 1)).
\]
Then the valuation matrix of Newt($Z_{A,f}$) is

$$V_f = \begin{bmatrix} 0 & 2 & 0 & 0 & -2 \\ 3 & 1 & 0 & 0 & -4 \\ 1 & 0 & 0 & 1 & -3 \\ 0 & 0 & 2 & 0 & 1 \\ 0 & 2 & 0 & 1 & 0 \end{bmatrix}.$$

Using ideal elimination in Macaulay 2 we can compute the implicit polynomial of Newt($Z_{A,f}$) in variables $u_0, u_1, u_2, u_3, u_4$: 

$$16u_1^4u_2^2u_3^2 - 40u_0u_1^4u_2^2u_3^2u_4 + 8u_0^3u_1^2u_2^2u_3^3u_4 - 16u_0^2u_1^6u_2^2u_3^2 + 20u_0^2u_1^4u_2^2u_3^2u_4$$

$$+ 15u_0^2u_1^2u_2^2u_3^2u_4^2 + u_0^2u_1^2u_2^2u_3^3u_4^2 + 54u_0^2u_1^2u_2^2u_3^3u_4^2 - 77u_0u_1^4u_2^10u_3^2u_4^3 + 379u_0u_1^2u_2^10u_3^4u_4^3$$

$$+ 5u_0^2u_1^6u_2^2u_3^4u_4 - 27u_0^2u_1^6u_2^2u_3^4u_4 - 29u_0^2u_1^2u_2^2u_3^2u_4^2 - 163u_0^2u_1^2u_2^2u_3^2u_4^2 - 12u_0^2u_3^2u_4^4$$

$$- 35u_0u_1^4u_2^6u_3^5u_4^2 - 425u_0^2u_1^2u_2^6u_3^2u_4^5 + 4u_0^2u_1^2u_2^6u_3^2u_4^5 + 87u_0^2u_1^2u_2^6u_3^2u_4^5 + 71u_0^2u_1^2u_2^6u_3^2u_4^5$$

$$+ 103u_0u_1^4u_2^6u_3^4u_4^7 + 115u_0u_1^4u_2^6u_3^4u_4^7 + 12u_0^2u_1^4u_2^6u_3^4u_4^7 + 4u_0^2u_1^4u_2^6u_3^4u_4^7.$$

The vertices of Newton polygon of this implicit polynomial are

$$(0, 4, 16, 2, 0), (2, 8, 8, 0, 4), (3, 8, 0, 6, 5), (7, 6, 0, 1, 8), (12, 4, 0, 2, 8), (4, 0, 12, 4, 2).$$

The directed edges are

$$(2, 4, -8, -2, 4), (-4, 4, 4, -2, -2), (-4, -4, 12, 2, -6)$$

$$(1, -2, 0, 1, 0), (4, -2, -6, 1, 3), (1, 0, -2, 0, 1),$$

and the product $P_A \cdot V_f$ is

$$\begin{bmatrix} -4 & -4 & 2 & 1 & 1 & 4 \\ -4 & 4 & 4 & -2 & 0 & -2 \\ 12 & 4 & -8 & 0 & -2 & -6 \\ 2 & -2 & -2 & 1 & 0 & 1 \\ -6 & -2 & 4 & 0 & 1 & 3 \end{bmatrix}.$$

The column vectors of this matrix are the directed edges of Newt($Z_{A,f}$).

Next we compare our result with existing work. The Philippon-Sombra formula [PS08, Proposition 4.1] computes the degree of $Z_{A,f}$ from $A$ and $f$. Let $B$ be a $2 \times (n + 2)$ matrix such that its entries are integers and its row vectors span the kernel of $A$, and $b_0, \ldots, b_{n+1}$ be the column vectors of $B$. Given any column vector $v = (v_0, \ldots, v_{n+1})^T \in \mathbb{R}^{n+2}$, we define a polytope as the convex hull of the following $n + 2$ vertices: $c_i$ is the vector formed by the $i$-th column vector of $A$ and $v_{i-1}$ for $1 \leq i \leq n + 2$. Then the vector $B \cdot v$ admits a triangulation $T$ of this polytope: for all $0 \leq i < j \leq n + 1$, the $n$-dimensional simplex formed by vertices excluding $c_i, c_j$.
belongs to this triangulation if and only if the vector $B \cdot v$ belongs to the nonnegative span of $b_i$ and $b_j$. Now we define a sum

$$\partial_v(A) = \frac{1}{\delta} \sum_{\sigma \in T} |A_{\sigma}| \sum_{i \in \sigma} v_i.$$ 

Here $A_{\sigma}$ is the $n \times n$ minor of $A$ that contains all columns corresponding to $\sigma$.

**Proposition 2.12.** The degree of an almost-toric hypersurface $Z_{A,f}$ is

$$\sum_v \partial_v(A).$$ 

$v$ is a column vector of $V_f$

This proposition is a direct corollary of [PS08, Proposition 4.1].

**Remark 2.13.** For some $v$, $\partial_v(A)$ is negative, so the computation of degree using the Philippon-Sombra formula is not very simple. The author attempted to obtain an alternative interpretation of the formula such that all summands are positive, but failed. However in Section 3 we present an algorithm to compute the implicit polynomial of Newt($Z_{A,f}$), and in step (3) we can compute the degree of this polynomial efficiently.

[STY07, Theorem 5.2] also provides an alternative way to compute the degree of $Z_{A,f}$, using tropical geometry. Applying this theorem to our almost-toric hypersurface $Z_{A,f}$ we have the following corollary.

**Corollary 2.14.** For a generic column vector $w \in \mathbb{R}^{n+2}$, the $i$-th coordinate of the vertex face$_w$(Newt($Z_{A,f}$)) is the number of intersection points, each counted with its intersection multiplicity, of the tropical hypersurface trop($Z_{A,f}$) with the half line $w + \mathbb{R}_{\geq 0} e_i$.

Here intersection multiplicity is defined in the remark after [STY07, Theorem 5.2] and $e_i \in \mathbb{R}^{n+2}$ is the column vector with $i$-th component 1 and others 0 for $1 \leq i \leq n + 2$. From Corollary 2.14 we get the following proposition.

**Proposition 2.15.** Let $p(u_0, \ldots, u_{n+1})$ be the implicit polynomial of $Z_{A,f}$. Then for a generic vector $w \in \mathbb{R}^{n+2}$, the initial monomial in$_wp$ is

$$\prod_{(i,j) \in S} u_i^{[e_i, P_A v_j]},$$

where

$$S = \{(i, j)|1 \leq i \leq n + 2, v_j \text{ is a column vector of } V_f, e_i^T P_A w, e_i^T P_A v_j, v_j^T P_A w \text{ have the same sign.}\}.$$ 

Hence the degree of $Z_{A,f}$ is

$$\sum_{(i,j) \in S} |e_i P_A v_j|.$$ 

**Proof.** If $w$ is generic, then trop($Z_{A,f}$) and $w + \mathbb{R}_{\geq 0} e_i$ have at most one intersection point. Let $v_j$ be the $j$-th column vector of matrix $V_f$ and $\text{col}(A^T) = \{u^T | u \in \text{row}(A)\}$. Suppose there is an intersection point of the maximal cone spanned by row($A$) and the half line $\mathbb{R}_{\geq 0} v_j$ and $w + \mathbb{R}_{\geq 0} e_i$. Since $w$ is generic, the 2-dimensional subspace spanned by $w, e_i$ has a unique common point with a translation of a codimension 2 subspace: $w - \text{col}(A^T)$. Then there exists a unique pair of nonzero (because $w$ is generic) real numbers $\lambda_1, \lambda_2$ and $u \in \text{row}(A)$ such that $u^T + \lambda_1 v_j = w + \lambda_2 e_i$. Then the intersection point exists if and only if $\lambda_1, \lambda_2 > 0$. Note that $\lambda_1 > 0$ if and only if $v_j$ and $w$ are on the same side of the hyperplane spanned by $\text{col}(A^T) = \{u^T | u \in \text{row}(A)\}$.
and $e_i$ (if they don’t span a hyperplane, then $e_i \in \text{col}(A^T)$ and no intersection point exists since $w$ is generic). So $\det([v_j \ e_i \ A^T])$ and $\det([w \ e_i \ A^T])$ have the same sign.

Note that for any column vectors $a, b \in \mathbb{R}^{n+2}$, $\det\left(\begin{bmatrix} b^T \\ a^T \\ A \end{bmatrix} \right) = \det([b \ a \ A^T]) = \delta a^T P_A b$. Hence $\lambda_1 > 0$ if and only if $e_i^T P_A w$ and $e_i^T P_A v_j$ have the same sign. Similarly, $\lambda_2 > 0$ if and only if $w$ and $e_i$ are on the opposite sides of the hyperplane spanned by $\text{col}(A^T)$ and $v_j$, which is equivalent to $e_i^T P_A v_j$ and $v_j^T P_A w$ having the same sign. So $(i, j) \in S$ if and only if there is an intersection point of the maximal cone spanned by row$(A)$ and the half line $\mathbb{R}_{\geq 0} v_j$ with the half line $w + \mathbb{R}_{\geq 0} e_i$. Next it suffices to show that the intersection multiplicity of this point is $|e_i P_A v_j|$. By the definition of intersection multiplicity, for this point it is the lattice index of the lattice spanned by $e_i, v_j$ and the transpose of row vectors of $A$, so the intersection multiplicity is $|e_i P_A v_j|$.

\begin{proof}
\end{proof}

\textbf{Remark 2.16.} Proposition 2.15 enables us to compute the degree of $Z_{A, f}$ without knowing $\text{Newt}(Z_{A, f})$.

3. Algorithm, Implementation and Case Study

\textbf{Algorithm to compute the implicit polynomial.} For an almost-toric hypersurface, we would like to compute its implicit polynomial in $n+2$ variables $u_0, u_1, \ldots, u_{n+1}$ from $A$ and $f$. An existing approach uses ideal elimination with Gröbner bases, which is inefficient when $n$ is large. Based on Theorem 1.1 we have the following alternative approach:

(1) Compute $P_A$ from $A$, factorize $f_0, f_1, \ldots, f_{n+1}$ over $K$ into irreducible factors to get $V_f$.

(2) Compute $P_A \cdot V_f$ and verify it has rank 2.

(3) Find $\text{Newt}(Z_{A, f})$ using Theorem 1.1.

(4) Determine all possible monomials in variables $u_0, u_1, \ldots, u_{n+1}$ that could appear in the implicit polynomial.

(5) Use linear algebra to compute the coefficients of these monomials.

We now explain our implementation of this method using the software Maple 17. Among the five steps, the first and second are trivial to implement (Maple 17 has the command \texttt{factor} which factors a polynomial into irreducible factors over a given field).

\textit{Step (3).} Theorem 1.1 tells us that the set of directed edges are the column vectors of $P_A \cdot V_f$. Then we need to arrange them in the correct order. We could project these vectors to a 2-dimensional space, by choosing two of the coordinates $1 \leq c_1 < c_2 \leq n+2$. There is still the problem of orientation: these directed edges admit two different arrangements. The correct orientation is determined by the sign of the $(P_A)_{c_1, c_2}$.

Now suppose all directed edges are arranged in correct order and are the column vectors of a matrix

$$[c_{i,j}]_{1 \leq i \leq n+2, 1 \leq j \leq m}.$$

If the vertex that corresponds to the first and $m$-th edges has coordinates $r_1, \ldots, r_{n+2}$, then the other vertices have coordinates

$$r_1 + \sum_{j=1}^{k} c_{1,j}, r_2 + \sum_{j=1}^{k} c_{2,j}, \ldots, r_{n+2} + \sum_{j=1}^{k} c_{n+2,j} , k = 1, \ldots, m - 1.$$
We notice that for $1 \leq i \leq n + 2$, the $i$-th coordinate of the vector of vertices corresponds to the exponents of $t_i$. Since the implicit polynomial is irreducible, the minimum of these exponents must be 0:

$$\min_{1 \leq k \leq m} \{r_i + \sum_{j=1}^{k} c_{i,j}\} = 0, \quad 1 \leq i \leq n + 2.$$  

Hence

$$r_i = -\min_{1 \leq k \leq m} \{\sum_{j=1}^{k} c_{i,j}\}, \quad 1 \leq i \leq n + 2.$$  

So Newt$(Z_{A,f})$ is uniquely determined by $P_A \cdot V_f$ and we can compute its vertices using the formula above.

Step (4). Given the vertices of Newt$(Z_{A,f})$, we need to find all lattice points of this polygon. Since all of them lie in the translation of a 2-dimensional subspace, projection onto 2 coordinates would work. We find all lattice points within a convex polygon (which may be degenerate), then recover the corresponding lattice points in Newt$(Z_{A,f})$. These lattice points correspond to all possible monomials in the implicit polynomial of $Z$: components of each vector are the exponents of variables $u_0, u_1, \ldots, u_{n+1}$.

Step (5). Given all monomials of the implicit polynomial $p(u_0, \ldots, u_{n+1})$, it is enough to find the coefficients of them. Consider the coefficients as undetermined unknowns. After the substitution $u_i = t^a_i f_i(x)$ we get another polynomial $q(t_1, \ldots, t_n, x)$. Each term in $q$ is the product of a coefficient, a monomial in the variables $t_1, \ldots, t_n$ and a polynomial in the variable $x$. We claim that all these monomials in variables $t_1, \ldots, t_n$ are the same. Assume the opposite situation. Then we can pick all terms in $q$ with a particular monomial in variables $t_1, \ldots, t_n$ and get the corresponding monomials in $p$, which form another polynomial $p'(u_0, \ldots, u_{n+1})$ with less terms than $p$. Since $p$ is the implicit polynomial, polynomial $q$ must be identically zero, then $p'$ vanishes everywhere too, so $p'$ belongs to the principal ideal of our hypersurface, a contradiction! So after the substitution we can cancel the unique monomial in variables $t_1, \ldots, t_n$ and get a univariate polynomial in $x$ with undetermined coefficients. Since this polynomial is identically zero, the undetermined coefficients satisfy a system of homogeneous linear equations.

Next we use interpolation. Suppose there are $k$ possible monomials in the implicit polynomial, then we replace $x$ by integers ranging from $-r$ to $r$, where $r = \lfloor \frac{k}{2} \rfloor$. Each interpolation gives a linear equation with $k$ coefficients. Then we use the `solve` command in Maple 17 to solve these coefficients. Since this is a homogeneous linear system, the solution space should be 1-dimensional. This leads us to add another equation, for example $a_1 = 1$ where $a_1$ is one of the coefficients, to guarantee the uniqueness of solution. After getting the solution, if all coefficients are rational, we normalize them so that their content is 1.

Example 3.1. Let $Z_{A,f}$ be the almost-toric surface in $\mathbb{P}^3$ parameterized by

$$(s^3x^2(x-1) : s^2t(x^2+1) : st^2x(x+1)^2 : t^3(x-1)(x-2)).$$
shows the time needed to find the implicit polynomial of $0.078s$.  and we choose $n + 2$ degree $d$ monomials randomly from all possible $\binom{n + d - 1}{d}$ choices. For the univariate polynomials, we choose $n + 2$ polynomials of the form $(x - 2)^*(x - 1)^*(x + 1)^*(x + 2)^*$, where each * is a random integer between 0 and $k$. Table 2 shows the time needed to find the implicit polynomial of almost-toric hypersurfaces given by randomly generated inputs. It turns out that our implementation improves the efficiency of finding the implicit polynomial of almost-toric hypersurfaces.  

| sample | degree | # of terms | our time cost | ideal elimination’s time cost |
|--------|--------|------------|---------------|-------------------------------|
| Example 2.11 | 22 | 24 | 0.094s | 1.8758s |
| Example 3.1 | 10 | 16 | 0.047s | 0.078s |

Table 1: Simple Examples

We then try some examples that both Macaulay 2 and Sagemath cannot solve in a reasonable time. We generate some samples as the input using the following method: let $n$ be the dimension of the torus, $d$ the degree of the homogeneous monomials and $k$ a positive integer. Then we choose $n + 2$ degree $d$ monomials randomly from all possible $\binom{n + d - 1}{d}$ choices. For the univariate polynomials, we choose $n + 2$ polynomials of the form $(x - 2)^*(x - 1)^*(x + 1)^*(x + 2)^*$, where each * is a random integer between 0 and $k$. Table 2 shows the time needed to find the implicit polynomial of almost-toric hypersurfaces given by randomly generated inputs. It turns out that our implementation improves the efficiency of finding the implicit polynomial of almost-toric hypersurfaces.

| sample | degree | # of terms | time cost | sample | degree | # of terms | time cost |
|--------|--------|------------|-----------|--------|--------|------------|-----------|
| 1 | 213 | 109 | 12.484s | 6 | 179 | 97 | 8.110s |
| 2 | 109 | 80 | 1.594s | 7 | 40 | 32 | 0.156s |
| 3 | 172 | 129 | 10.421s | 8 | 27 | 14 | 0.140s |
| 4 | 474 | 275 | 156.969s | 9 | 79 | 71 | 1.766s |
| 5 | 291 | 137 | 20.375s | 10 | 281 | 148 | 20.719s |

Table 2. $n = 4, d = 4, k = 5$
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