MULTIFRACTALITY OF JUMP DIFFUSION PROCESSES

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Abstract. We study the local regularity and multifractal nature of the sample paths of a class of jump diffusion processes, which are solutions to the stochastic differential equation

\[ \mathcal{M}_t = \int_0^t \sigma(\mathcal{M}_s) \, dB_s + \int_0^t b(\mathcal{M}_s) \, ds + \int_{[0,1] \setminus \{0\}} G(\mathcal{M}_s, z) \, \tilde{N}(dsdz). \]

This class includes the symmetric stable-like processes. We prove that the Markov processes \( \mathcal{M} \) have remarkable multifractal properties, and that their multifractal spectrum is not only random, but also depends on time. We explain, especially by computing their tangent processes at all points, why these jump diffusion processes locally look like a Lévy process. New techniques are developed to study these processes.

1. Introduction

Multifractal properties are now identified as important features of sample paths of stochastic processes. The variation of the regularity of random measures and processes has been observed considerably since mid-70's, e.g. fast and slow points of Brownian motion [28, 29], fast points of fractional Brownian motion [25], multiplicative cascades [24], Lévy processes [22, 13, 2] and Lévy processes in multifractal time [5], among many other examples. Multifractal analysis turns out to be a relevant approach to provide organized information about the distribution of singularities and to describe the roughness of the object under consideration.

The regularity exponent we consider is the pointwise Hölder exponent. Let us recall some relevant definitions in this context.

**Definition 1.** Let \( f \in L^\infty_{loc}(\mathbb{R}) \), \( x_0 \in \mathbb{R} \), \( h \in \mathbb{R}^+ \). The function \( f \) belongs to \( C^h(x_0) \) if there exist two positive constants \( C, M > 0 \), a polynomial \( P \) with degree less than \( h \), such that when \( |x - x_0| < M \),

\[ |f(x) - P(x - x_0)| \leq C|x - x_0|^h. \]

The pointwise Hölder exponent of \( f \) at \( x_0 \) is

\[ H_f(x_0) = \sup \{ h \geq 0 : f \in C^h(x_0) \}. \]

We aim at describing the distribution of the singularities of a function, via the computation of its multifractal spectrum. \( \dim_H \) stands for the Hausdorff dimension and by convention \( \dim_H \emptyset = -\infty \), see [16] for more on dimensions.

**Definition 2.** Let \( f \in L^\infty_{loc}(\mathbb{R}) \). For \( h \geq 0 \), the iso-Hölder set is

\[ E_f(h) = \{ x \in \mathbb{R} : H_f(x) = h \} \]

and the multifractal spectrum of \( f \) is the mapping \( D_f : \mathbb{R}^+ \to [0,1] \cup \{-\infty\} \) defined by

\[ h \mapsto D_f(h) = \dim_H E_f(h). \]

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We also define, for any open set $A \subset \mathbb{R}^+$, the local spectrum of $f$ on $A$ as
\begin{equation}
D_f(A, h) = \dim_H(A \cap E_f(h)).
\end{equation}

The aforementioned stochastic processes have homogeneous multifractal spectra, meaning that there is no dependency on the region where the spectra are computed: $D_f(\mathbb{R}^+, h) = D_f(A, h)$, for all open sets $A \in \mathbb{R}^+$. However, many types of real signals have multifractal characteristics that change as time passes: financial, geographical and meteorologic data provide such examples. Recently, Barral, Fournier, Jaffard and Seuret [4] showed that natural theoretical objects also share this feature. This is also the case for the class of jump diffusions that we will consider in this paper.

We also focus in the pointwise multifractal spectrum at a given point.

**Definition 3.** Let $f \in L^\infty_{loc}(\mathbb{R})$, $t_0 \in \mathbb{R}^+$, and let $I_n(t_0) = [t_0 - 1/n, t_0 + 1/n]$ for every $n \geq 1$. The pointwise multifractal spectrum of $f$ at $t_0$ is the mapping defined by
\begin{equation}
\forall h \geq 0, \quad D_f(t_0, h) = \lim_{n \to +\infty} D_f(I_n, h).
\end{equation}

The local spectrum $D_f(A, h)$ on any open set $A$ can be completely recovered from the pointwise spectrum $D_f(t_0, h)$, for $t_0 \in A$, as stated by next Proposition. Hence the pointwise multifractal spectrum results are finer than the multifractal spectrum results on an interval.

**Proposition 4** (Proposition 2, [3]). Let $f \in L^\infty_{loc}(\mathbb{R})$. Then for any open interval $I = (a, b) \subset \mathbb{R}_+$, for any $h \geq 0$, we have $D_f(I, h) = \sup_{t \in I} D_f(t, h)$. Consequently, the mapping $t \mapsto D_f(t, h)$ is upper semi-continuous.

The multifractal analysis of a Lévy process was performed by Jaffard [22].

**Theorem 5** ([22]). Let $(\mathcal{L}_t)_{t \geq 0}$ be a Lévy process of index $\beta \in (0, 2)$, with a non-zero Brownian component. Almost surely, at every $t > 0$, the sample path of $\mathcal{L}$ has the (deterministic) pointwise spectrum
\begin{equation}
D_{\mathcal{L}}(t_0, h) = D_{\mathcal{L}}(h) = \begin{cases}
\beta h & \text{if } h \in [0, 1/2), \\
1 & \text{if } h = 1/2, \\
-\infty & \text{if } h > 1/2.
\end{cases}
\end{equation}

In particular, Lévy processes are homogeneously multifractal.

Examples of stochastic process with varying pointwise spectrum were given in [4 14 20]. Here we deal with a general class of Markov processes, $\mathcal{M}$, defined below by Equation (2), called the jump diffusions. This large class of processes, which are both semimartingales and Markov processes, was studied by Çinlar and Jacod [11], and includes the symmetric stable-like processes (constructed by Bass [7], see also Negoro [27]). With well-chosen coefficients, they can be viewed as solutions to SDE driven by stable processes which attracted much interest in the last decade [8 19 26 18]. Furthermore, they are useful in the modeling of option pricing [12], population evolution [19] and other physical phenomena. We investigate their multifractal nature and determine their pointwise multifractal spectra, by using a localized Diophantine approximation theorem [9].

We work in the probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})$ satisfying the usual conditions. The process $\mathcal{M}$ is defined as the solution to the one-dimensional stochastic differential equation with jump:
\begin{equation}
\mathcal{M}_t = \int_0^t \sigma(\mathcal{M}_s-)dB_s + \int_0^t b(\mathcal{M}_s)ds + \int_0^t \int_{C(0,1)} G(\mathcal{M}_s-, z)\tilde{N}(dsdz),
\end{equation}
where:
Definition 6. The set \( \mathcal{G} \) is the set of those functions \( G : \mathbb{R} \times C(0,1) \to [-1,1] \) satisfying:

1. Symmetry : \( \forall (x,y) \in \mathbb{R}^2, \forall z \in C(0,1) \),
   \[ G(x,z) = \text{sign}(z)|G(x,|z|)| \quad \text{and} \quad G(x,z)G(y,z) > 0. \]

2. Asymptotically stable-like : \( \forall x \in \mathbb{R} \),
   \[ \lim_{z \to 0} \frac{\log |G(x,z)|}{\log |z|} \quad \text{exists and denoted by} \quad \frac{1}{\beta(x)}. \]

3. Lipschitz condition : there exists \( C > 0 \) such that \( \forall (x,y) \in \mathbb{R}^2, \forall z \in C(0,1) \),
   \[ \left| \frac{\log |G(y,z)| - \log |G(x,z)|}{\log |z|} \right| \leq C|x - y|. \]

4. Boundedness : \( \text{Range } \beta \subset (0,2) \) and for all \( \varepsilon > 0 \), there exists \( r_\varepsilon > 0 \) such that for every \( z \in C(0,r_\varepsilon) \) and every \( x \),
   \[ |G(x,z)| \leq |z|^{1/\beta(x)}. \]

The reader should keep in mind that when \( G \in \mathcal{G} \), one has intuitively

\[ G(x,z) \sim |z|^{1/\beta(x)} \]

for some function \( \tilde{\beta} \) which ranges in \( [\varepsilon,2 - \varepsilon] \) for some \( \varepsilon > 0 \). We comment these assumptions in Section 2.1.

We introduce the following notation. For the rest of the paper, we set

\[ t \in \mathbb{R}^+ \mapsto \beta(t) = \tilde{\beta}(\mathcal{M}(t)). \]

The quantity \( \beta(t) \) is key: it shall be understood as the local Blumenthal-Getoor index of \( \mathcal{M} \) at time \( t \), and governs the local behavior of \( \mathcal{M} \) at \( t \).

We state now the multifractal properties of \( \mathcal{M} \). When the diffusion part does not vanish, the pointwise multifractal spectrum of \( \mathcal{M} \) takes a simple form, which is the main result of this paper.

Theorem 7. Assume that \( G \in \mathcal{G} \) and \( \sigma \not\equiv 0 \). Then, with probability one, for every \( t \in \mathbb{R}^+ \), the pointwise multifractal spectrum of \( \mathcal{M} \) at \( t \) is

\[
D_{\mathcal{M}}(t,h) = \begin{cases} 
  h \cdot (\beta(t) \vee \beta(t-)) & \text{if } h < 1/2, \\
  1 & \text{if } h = 1/2, \\
  -\infty & \text{if } h > 1/2.
\end{cases}
\]

In particular, if \( t \) is a continuous time for \( \mathcal{M} \), the formula reduces to \( D_{\mathcal{M}}(t,h) = h \cdot \beta(t) \) when \( h < 1/2 \).
We prove this result in Section 7. The case when the diffusion part vanishes will also be entirely treated in Theorems 26 and 27 of Section 7. It is more complicated to state, since many cases must be distinguished according to various correlations between \( t, M_t \) and \( \beta \). The presence of the Brownian component eliminates these difficulties.

Observe that the pointwise spectrum is linear up to the exponent \( h = 1/2 \). Recalling Theorem 5, Corollary 8 implies that the multifractal spectrum of \( M \) looks like that of a Lévy process, except that the slope of the linear part of the spectrum is random and depends on the interval on which we compute the spectrum. This remarkable property reflects the fact that the “local Blumenthal-Getoor” index of a jump diffusion \( M \) depends on time.

From the pointwise spectrum of \( M \) we deduce its local spectrum.

**Corollary 8.** Assume that \( G \in G, \sigma \neq 0 \). For \( I = (a, b) \subset \mathbb{R}^+ \), let

\[
\gamma_I := \sup \{ \beta(s) : s \in I \}.
\]

With probability one, the local multifractal spectrum of \( M \) on \( I \) is

\[
D_M(I, h) = \begin{cases} 
  h \cdot \gamma_I & \text{if } h < 1/2, \\
  1 & \text{if } h = 1/2, \\
  -\infty & \text{if } h > 1/2.
\end{cases}
\]

We can also give the statement for the local multifractal spectrum when the diffusion vanishes (this is a corollary of Theorems 26 and 27).

**Corollary 9.** Assume that \( G \in G, \sigma \equiv 0 \) and \( (H) \) holds (see Theorem 11 below for the definition of \( (H) \)). Let \( I = (a, b) \subset \mathbb{R}^+ \) and

\[
\gamma_I(h) := \sup \{ \beta(s) : s \in I, \beta(s) \leq 1/h \},
\]

\[
\bar{\gamma}_I := \inf \{ \beta(s) : s \in I \}.
\]

With probability one, the local multifractal spectrum of \( M \) on \( I \) is

\[
D_M(I, h) = \begin{cases} 
  h \cdot \gamma_I(h) & \text{if } h < 1/\bar{\gamma}_I \text{ and } h \notin (\beta(J))^{-1}, \\
  -\infty & \text{if } h > 1/\bar{\gamma}_I.
\end{cases}
\]

Both corollaries are consequences of Proposition 4. The difference between the corollaries follows from the fact that the diffusion has regularity 1/2 at every point, so the complicated part of the multifractal spectrum \( (h > 1/2) \) in Corollary 9 disappears.

Observe that we do not give the value of the spectrum on the countable set \((\beta(J))^{-1}\). This is due to the occurrence of various delicate situations depending on the trajectory of \( M \), which are described in Section 7.

In order to compute \( D_M \), we have to investigate the pointwise exponent of the process \( M \) at every point. To state our main result in this direction, let us introduce the Poisson point process \( \mathcal{P} \) associated with the process \( M \) and the notion of the approximation rate by \( \mathcal{P} \).

The Poisson random measure \( N \) can be derived from a Lévy process \( \mathcal{L} \) with characteristic triplet \((0, 0, \pi(dz))\), meaning that there are no Brownian component and no drift. For all \( s, t \in \mathbb{R}^+ \) and every Borel set \( A \in \mathcal{B}(C(0, 1)) \), denote \( N([s, t], A) = \sharp \{ u \in [s, t] : \Delta \mathcal{L}_u \in A \} \). Then, almost surely, \( N \) is a Poisson random measure with intensity \( dt \otimes \pi(dz) \). Let

\[
(3) \quad \mathcal{P} = (T_n, Z_n)_{n \geq 0}.
\]
denote the sequence of jump times and jump sizes of $\mathcal{L}$. Then almost surely,

$$N = \sum_{n \geq 1} 1_{(T_n, Z_n)}.$$  

We can assume that $(|Z_n|)_{n \in \mathbb{N}}$ forms a decreasing sequence by rearrangement. Let $J := \{t \in \mathbb{R}^+ : \Delta M_t \neq 0\}$ be the set of locations of the jumps, where $\Delta M_t := M_t - M_{t-}$. Using the property of Poisson integral,

$$J = \{T_n : n \in \mathbb{N}\}$$

and for every $n \in \mathbb{N}$,

$$\Delta M_{T_n} = G(M_{T_n-}, Z_n).$$

See for instance Section 2.3 of [1] for details.

The approximation rate $\delta_t$ by $\mathcal{P}$ describes how close to the jump points $T_n$ a point $t$ is. Intuitively, the larger $\delta_t$ is, the closer to large jumps $t$ is.

**Definition 10.** *The approximation rate of $t \in \mathbb{R}^+$ by $\mathcal{P}$ is defined by*

$$\delta_t := \sup\{\delta \geq 0 : |T_n - t| \leq (Z_n)^\delta \text{ for infinitely many } n\}.$$  

This random approximation rate plays a key role when investigating the pointwise regularity of $\mathcal{M}$, as stated by the following result.

**Theorem 11.** *Let $G \in \mathcal{G}$ be an admissible function.*

1. *Assume that $\sigma \neq 0$. Then almost surely*

$$\forall t \notin J, \quad H_\mathcal{M}(t) = \frac{1}{\delta_t \beta(t)} \wedge \frac{1}{2}.$$ 

2. *Assume that $\sigma \equiv 0$ and that the following assumption (H) holds:

   (a) *either $\bar{b} \in C^\infty(\mathbb{R})$ and $\text{Range } \beta \subset [1, 2)$.*

   (b) *or $\bar{b} \in C^\infty(\mathbb{R})$ and $x \mapsto \tilde{b}(x) := \int_0^1 G(x, z) \frac{dz}{z^2} \in C^\infty(\mathbb{R})$.*
Then, almost surely, \( \forall t \notin J, H_M(t) = \frac{1}{\delta(t)b(t)} \).

This theorem is proved in Section 5 and 6.

Observe that the terms in the right hand side of (2) yield a semimartingale decomposition of the process \( M \), see [9]. To simplify notations, we write

\[
M_t = \mathcal{X}_t + \mathcal{Y}_t + \mathcal{Z}_t,
\]

where:

- \( \mathcal{X}_t = \int_0^t \sigma(M_s-) dB_s \) is the diffusion term.
- \( \mathcal{Y}_t = \int_0^t b(M_s) ds \) is the drift term.
- \( \mathcal{Z}_t = \int_0^t \int_{C(0,1)} G(M_s-, z) \tilde{N}(dsdz) \) is the jump term.

Let us make additional remarks on our main results:

- The drift part of a jump diffusion is not as simple as that of a Lévy process, which is linear hence belongs to \( C^\infty(\mathbb{R}) \). The regularity of the drift depends on that of \( M \) which varies along time. However, this has no consequence on the statement of Theorem 11 because we are able to prove that the drift is always more regular than \( M \) (see Section 5).

- The sum of the diffusion \( \mathcal{X}_t \) and the jump term \( \mathcal{Z}_t \) has, almost surely, everywhere a pointwise exponent less than \( 1/2 \). Indeed, \( \mathcal{X} \) has an exponent everywhere equal to \( 1/2 \) (see Proposition 15). When the pointwise exponent of \( \mathcal{Z}_t \) is not equal to \( 1/2 \), we use the fact \( H_{\mathcal{X}+\mathcal{Z}}(t) \geq H_\mathcal{X}(t) \wedge H_\mathcal{Z}(t) \) with equality if \( H_\mathcal{X}(t) \neq H_\mathcal{Z}(t) \). When \( H_\mathcal{X}(t) = H_\mathcal{Z}(t) \), equality is not true in general. But it does hold in our context, because the irregularity generated by jump discontinuities cannot be compensated by a continuous term (see Section 6 for details).

- One key argument in our proof consists in constructing simultaneously with \( M \) a family of martingales \( (\mathcal{P}_j)_{j \geq 1} \), whose local index does not vary much, such that \( \mathcal{Z} = \sum_j \mathcal{P}_j \) plus a process with only large jumps whose regularity controlled. The increments of the \( \mathcal{P}_j \) are easier to control, and using this decomposition we are able to estimate the values of the increments of \( M \) on all dyadic intervals, see Proposition 21.

This paper is organized as follows. In next Section, first properties of the process \( M \) are given. In Section 3, we determine the pointwise exponent of the diffusion term. In Section 4, we deal with the pointwise regularity of the jump term. We prove Theorem 11 in Sections 6-7 and we compute the local spectrum of \( M \) in different situations (Theorems 7, 26 and 27 in Section 7. We show the existence of tangent processes of the process \( M \) in some special cases in Section 8. Appendices 1 and 2 contain auxiliary results.

We mention some possible extensions. It would be important to relax the assumptions on \( G \) (for instance, the symmetry assumption in \( G \) can certainly be dropped). It would also be interesting to determine the regularity of the drift term associated to \( M \), even in the case where \( b \) is \( C^\infty \). More generally, the present article opens the way to a systematic multifractal study of other classes of Markov processes. Other dimensional properties of stochastic processes, such as dimensions of the image, of the graph of \( M \), are important mathematical properties with application in physics for modeling purposes, and will be investigated in a future paper.
2. Properties of $M$

We comment the assumptions on $M$, then show the existence of a unique pathwise solution to (2), and that the jump part of the process is a $L^2$-martingale. We also compute its generator.

2.1. Comments. Let us comment the assumptions made on the parameters of the problem, especially on the set $G$ given by Definition 6.

The symmetry (part 1.) facilitates some computations and statement of the results. Looked at from a generator point of view (see Proposition 14 and the discussion below it), Part 2 is a natural extension of the assumption that $M$ is a stable-like process. Parts 3 and 4 in Definition 6 ensure the existence of $M$, and enable us to perform multifractal analysis.

It is worth mentioning that the intensity measure can be any stable Lévy measure, i.e. $|z|^{-1-\alpha}dz$ with $\alpha \in (0,2)$ (in this case, the definition of the set $G$ must be adapted). With a suitable change of measure for the Poisson integral, it suffices to consider our specific Lévy measure $\pi(dz)$.

2.2. Basic properties.

**Proposition 12.** The SDE (2) has a unique pathwise solution.

By the classical Picard iteration procedure (see e.g. [1] Theorem 6.2.3), it is enough to check that there are $K_0, K_1 < +\infty$ such that:

(i) Growth condition : $\forall x \in \mathbb{R}$,

$$|\sigma(x)|^2 + |b(x)|^2 + \int_{C(0,1)} |G(x, z)|^2 \pi(dz) \leq K_0 (1 + x^2).$$

(ii) Lipschitz condition : $\forall (x, y) \in \mathbb{R}^2$,

$$|\sigma(x) - \sigma(y)|^2 + |b(x) - b(y)|^2 + \int_{C(0,1)} |G(x, z) - G(y, z)|^2 \pi(dz) \leq K_1 |x - y|^2.$$

We check both conditions for every $G \in G$ in Appendix 1.

**Proposition 13.** The jump term $Z$ is a martingale and there is a strictly positive sequence $(\alpha_n)_{n \geq 1}$ with $\alpha_n \to_{n \to +\infty} 0$ such that a.s.,

$$Z_t = \lim_{n \to +\infty} \int_0^t \int_{C(0,1)} G(M_{s-}, z) \tilde{N}(dsdz),$$

where the convergence is uniform on $[0,1]$.

**Proof.** By a classical argument using uniform convergence on every compact set in probability of a sequence of $L^2$-martingale, it is enough to show the following (see e.g. Theorem 4.2.3 of Applebaum [1])

$$\forall t > 0, \int_0^t \int_{C(0,1)} \mathbb{E}[G(M_{s-}, z)^2] ds \pi(dz) < +\infty.$$ 

Observe that the fourth property of the functions in $G$ gives that, for any small $\varepsilon_0 > 0$, there exists $z_0 \in (0,1]$ such that

$$\forall x, \forall z \in C(0, z_0), \quad |G(x, z)| \leq |z|^\frac{1}{2} + \varepsilon_0.$$
Using (4) and the uniform boundedness of \( G \), one obtains that for all \( t > 0 \),
\[
\int_0^t \int_{C(0,1)} \mathbb{E}[G(M_{s-}, z)^2] \, ds \pi(dz) = \int_0^t \int_{C(0,20)} \mathbb{E}[G(M_{s-}, z)^2] \, ds \pi(dz) + \int_0^t \int_{C(20,1)} \mathbb{E}[G(M_{s-}, z)^2] \, ds \pi(dz) \\
\leq \int_0^t \int_{C(0,20)} (|z|^{1+\varepsilon_0})^2 \, ds \, \frac{dz}{z^2} + t \int_{C(20,1)} \frac{dz}{z^2} \leq t \left( \frac{z_0^{2\varepsilon_0}}{\varepsilon_0} + \frac{2}{z_0} - 2 \right) < +\infty.
\]

\[ \square \]

**Proposition 14.** The generator of the process \( \mathcal{M} \) is
\[
\mathcal{L} f(x) := b(x) f'(x) + \frac{1}{2} \sigma^2(x) f''(x) + \int_{C(0,1)} [f(x + G(x, z)) - f(x) - G(x, z) f'(x)] \, \frac{dz}{z^2}.
\]

**Proof.** Let us denote by \( t \mapsto \langle X \rangle_t \) the quadratic variation process of \( X \). We compute the generator through Itô’s formula (for semimartingales). For every twice continuously differentiable function \( f \) with compact support,
\[
\begin{align*}
 f(M_t) - f(0) &= \int_0^t f'(M_s) \sigma(M_s) \, dB_s + \int_0^t f'(M_s) b(M_s) \, ds + \frac{1}{2} \int_0^t f''(M_s) \sigma^2(M_s) \, d\langle B \rangle_s \\
&+ \int_0^t \int_{C(0,1)} [f(M_s + G(M_s, z)) - f(M_s)] \, \tilde{N}(dsdz) \\
&+ \int_0^t \int_{C(0,1)} [f(M_s + G(M_s, z)) - f(M_s) - G(M_s, z) f'(M_s)] \, \frac{dz}{z^2} \, ds.
\end{align*}
\]

Taking expectation yields \( \mathbb{E}[f(M_t)] = f(0) + \int_0^t \mathbb{E}[\mathcal{L} f(M_s)] \, ds \), so that the generator of \( \mathcal{M} \) is indeed \( \mathcal{L} \).

\[ \square \]

**2.3. Relation with stable-like processes.** We clarify the relation between our processes in the case \( \sigma = b \equiv 0 \) and the symmetric stable-like processes.

The infinitesimal generator governs the short time behavior of the semi-group of a Markov process, hence characterizes the Markov process under consideration. A famous class of Markov processes consists of symmetric \( \alpha \)-stable processes, whose generator is
\[
\mathcal{L}^\alpha f(x) = \int_{[-1,1]} [f(x + u) - f(x) - uf'(x)] \, |u|^{-1-\alpha} \, du
\]
with \( \alpha \in (0, 2) \) and this measure controls the density of the jumps. Note that we integrate on \([-1,1]\) instead of on \( \mathbb{R} \), because the number of large jumps, which is a.s. finite, does not influence the sample path properties outside these points and we do not want to worry about the integrability of the process. One natural way to enrich the family of the \( \alpha \)-stable processes consists in adding a dependency of the jump measure on the location. Let
\[
\mathcal{L}^{\tilde{\beta}(\cdot)} f(x) = \int_{[-1,1]} [f(x + u) - f(x) - uf'(x)] \, \tilde{\beta}(x) |u|^{-1-\tilde{\beta}(x)} \, du.
\]
Assume that \( \text{Range } \beta \subset (0, 2) \). The corresponding processes were first constructed by Bass \([7]\) by solving a martingale problem, and was called stable-like processes. There exist other constructions by using pseudo-differential operator or Hill-Yosida-Ray theorem, see \([9]\) and references therein.

Let \( u = \text{sign}(z)|z|^{1/\beta(x)} \). This change of variable yields
\[
L^{\tilde{\beta}}(x)f(x) = \int_{[-1,1]} \left[ f(x + \text{sign}(z)|z|^{1/\beta(x)}) - f(x) - \text{sign}(z)|z|^{1/\beta(x)}f'(x) \right] \frac{dz}{z^2}.
\]

The class of jump-diffusions \( \mathcal{M} \) has a jump generator behaving asymptotically like \( L^{\tilde{\beta}}(\cdot) \). In particular, take a Lipschitz continuous function \( \tilde{\beta} \) and
\[
G(x,z) = G_0(x,z) = \text{sign}(z)|z|^{1/\tilde{\beta}(x)},
\]
satisfying \( \text{Range } \beta \subset (0, 2) \). Then \( G \in \mathcal{G} \) and we recover \( L^{\tilde{\beta}}(\cdot) \).

3. POINTWISE REGULARITY OF THE DIFFUSION TERM

**Proposition 15.** With probability one, for every \( t \geq 0 \), \( H_X(t) = \frac{1}{2} \).

By Dambis-Dubins-Swartz theorem, \( X \) (which is a local martingale) can be written as a Brownian motion subordinated in time where the subordinated process is a bi-Lipschitz continuous function. The Hölder regularity of Brownian motion can thus be inherited by the martingale in question. This is somewhat classical, we will include a complete proof in Appendix 2.

**Remark 16.** The condition that \( \sigma \) stays away from 0 cannot be dropped. Indeed, when \( \sigma(M_t) = 0 \), the process \( X \) may gain more regularity at \( t \) and the computation of \( H_M(t) \) involves the regularity of \( \sigma(M_\cdot) \) at time \( t \).

4. POINTWISE REGULARITY OF THE JUMP TERM

In the rest of the paper, we restrict our study to the time interval \([0, 1] \), the extension to \( \mathbb{R}^+ \) is straightforward.

Following Jaffard \([22]\), we introduce a family of limsup sets on which we can control the growth of the jump part \( Z \). For every \( \delta \geq 1 \), let
\[
A_\delta = \limsup_{n \to +\infty} B(T_n, |Z_n|^{1/\delta}),
\]
where \( \mathcal{P} = (T_n, Z_n)_{n \geq 1} \) is the point process \([3]\) generating the Poisson measure \( N \) in \([2]\). We first prove a covering property for the system \( \mathcal{P} \).

**Proposition 17.** With probability one, \([0,1] \subset A_1 \).

**Proof.** The Poisson random measure \( N \) has intensity \( ds \pi(dz) \) where \( \pi(dz) = dz/z^2 \). Using Shepp’s theorem \([32]\) (and a reformulated version by Bertoin \([10]\)), it suffices to prove that
\[
S = \int_0^1 \exp \left( 2 \int_t^1 \pi((u,1)) \, du \right) \, dt = +\infty.
\]
But \( \pi((u,1)) = u^{-1} - 1 \), so that \( S = \int_0^1 e^{2(t-1-\log t)} \, dt = +\infty. \) \( \square \)

It is clear from Proposition \([17]\) that a.s., the approximation rate \( \delta_t \) by the system of points \( \mathcal{P} \) (see Definition \([10]\)) is well-defined, always greater than 1, and random because it depends on \((T_n, Z_n)_{n \geq 1}\). Another consequence of this proposition is that the set of jumps is dense in \([0, 1]\). These considerations lead to the the upper bound for the pointwise exponent of the jump term.
Proposition 18. With probability one, for all \( t \in [0, 1] \), \( H_Z(t) \leq \frac{1}{\beta(t) \delta t} \).

This proposition is based on two lemmas. The first is observed by Jaffard [21] which sheds light on the importance of the jump times.

Lemma 19. Let \( f : \mathbb{R} \rightarrow \mathbb{R} \) be a càdlàg function discontinuous on a dense set of points, and let \( t \in \mathbb{R} \). Let \( (t_n)_{n \geq 1} \) be a real sequence converging to \( t \) such that, at each \( t_n \), \( |f(t_n) - f(t_n^-)| = \delta_n > 0 \). Then

\[
H_f(t) \leq \liminf_{n \to +\infty} \frac{\log \delta_n}{\log |t_n - t|}.
\]

The second lemma establishes a first link between the pointwise regularity and the approximation rate.

Lemma 20. For all \( \delta \geq 1 \), almost surely

\[
\forall t \in A_\delta, \quad H_Z(t) \leq \frac{1}{\beta(t) \delta}.
\]

Proof. Recall that almost surely the set of jumps \( J \) is

\[
J = \{ t \in [0, 1] : \Delta M_t \neq 0 \} = \{ T_n : n \in \mathbb{N} \}
\]

and that for all \( n \in \mathbb{N} \), \( \Delta Z_{T_n} = G(M_{T_n^-}, Z_n) \). Consider \( t \in A_\delta \setminus J \). Necessarily, \( t \) is a continuous time of \( M \) and there is an infinite number of \( n \) such that

\[
t \in B(T_n, |Z_n|) \delta.
\]

Applying Lemma 19 to the process \( Z \) and the jumps satisfying (6), one has

\[
H_Z(t) \leq \liminf_{n \to +\infty} \frac{\log |G(M_{T_n^-}, Z_n)|}{\log |T_n - t|} \leq \liminf_{n \to +\infty} \frac{\log |G(M_{T_n^-}, Z_n)|}{\delta \log |Z_n|} \leq \liminf_{n \to +\infty} \frac{-\log |G(M_{T_n^-}, Z_n)| - \log |G(M_t, Z_n)|}{\delta \log |Z_n|} + \liminf_{n \to +\infty} \frac{\log |G(M_t, Z_n)|}{\delta \log |Z_n|} \leq \frac{1}{\beta(t) \delta} + \frac{C}{\delta} \liminf_{n \to +\infty} |M_{T_n^-} - M_t| \leq \frac{1}{\beta(t) \delta},
\]

where we used (6) for the second inequality, the Lipschitz condition of \( G \) for the last inequality and the continuity of \( M \) at time \( t \) for the last equality.

For all \( t \in J \), \( H_Z(t) = 0 \), which completes the proof.

Proof of Proposition 18.: It follows from Lemma 20 that a.s., for all rational number \( \delta \geq 1 \), (5) holds. Using the monotonicity of \( \delta \to A_\delta \) and the density of rational numbers in \([0,1]\), we deduce that (5) holds for all \( \delta \geq 1 \), a.s.

If \( \delta_t < +\infty \), then \( t \in A_{\delta_t - \varepsilon} \), for every \( \varepsilon > 0 \). Hence, \( H_Z(t) \leq \frac{1}{\beta(t) (\delta_t - \varepsilon)} \) as a consequence of Lemma 20. Letting \( \varepsilon \) tend to 0, we obtain the result.

If \( \delta_t = +\infty \), then \( t \in \bigcap_{\delta \geq 1} A_\delta \), meaning that \( t \in B(T_n, |Z_n|) \) for infinitely many integers \( n \), for all \( \delta \geq 1 \). We deduce by Lemma 20 that \( H_Z(t) \leq \frac{1}{\beta(t) \delta} \), for all \( \delta \geq 1 \), thus \( H_Z(t) = 0 \), which completes the proof.
Recall that we want to prove that the upper bound that we obtain in Proposition [18] is in fact optimal. Let us first make a useful remark which determines the configuration of the jumps around a point \( t \). Let \( t \notin A_δ \cup J \). Then there exists a random integer \( n_0 \), such that
\[
\forall n \geq n_0, \quad |T_n - t| \geq |Z_n|^δ.
\]
Let \( s > t \) sufficiently close to \( t \) such that \([t, s]\) does not contain those \( T_n \) which violate (7). It is possible because the cardinality of such \( T_n \) is finite. For each \( s \), there exists a unique integer \( m \) (which depends on \( s \)) such that
\[
2^{-m-1} \leq |s - t| < 2^{-m}.
\]
Assume that \( T_n \in [t, s] \), then \( 2^{-m} > |t - s| \geq |T_n - s| \geq |Z_n|^δ \), so that \( |Z_n| \leq 2^{-m/δ} \). This means that in an interval of length \( 2^{-m} \) with one extreme point in the complementary of \( A_δ \cup J \), there is no jump whose corresponding Poisson jump size is larger than \( 2^{-m/δ} \). Therefore, to consider the increment of our process near such time \( t \), we can split the increment of the compensated Poisson integral \( Z_t - Z_s \) into two parts:
\[
\int_s^t \int_{C(0,2^{-m/δ})} G(\mathcal{M}_{u-}, z) \tilde{N}(dudz) + \int_s^t \int_{C(2^{-m/δ}, 1)} G(\mathcal{M}_{u-}, z) \tilde{N}(dudz).
\]

The last remark motivates Proposition [21] Before stating it, we define the random quantities
\[
\beta_{s,t}^m = \left( \sup_{u\in[s,t]} \beta(u) + \frac{2}{m} \right) \quad \text{and} \quad \beta_{s,t}^m = \left( \sup_{u\in[s,t]\pm2^{-m}} \beta(u) + \frac{2}{m} \right),
\]
where
\[
[s,t] = \begin{cases} [s,t] & \text{if } s < t, \\ [t,s] & \text{otherwise}, \\ s,t & \text{otherwise,} \\ \end{cases} \quad [s,t] \pm 2^{-m} = \begin{cases} [s-2^{-m}, t+2^{-m}] & \text{if } s < t, \\ [t-2^{-m}, s+2^{-m}] & \text{otherwise}. \end{cases}
\]

**Proposition 21.** For every \( \delta > 1 \), \( \varepsilon > 0 \), and every large integer \( m \),
\[
\mathbb{P} \left( \sup_{[s-t] \leq 2^{-m}} \int_s^t \int_{C(0,2^{-m/δ})} G(\mathcal{M}_{u-}, z) \tilde{N}(dudz) \geq 6m^2 \right) \leq C_δ e^{-m}.
\]

Balança [2] proved a similar result for Lévy processes. Here, the idea is to exploit the martingale nature of our process and some “freezing” procedure for the local upper index process \( t \mapsto \beta(t) \). Intuitively, in the neighborhood of a continuity point \( t \), since \( \beta(t) \) is also continuous at \( t \), one may say that our process behaves locally like a Lévy process with Blumenthal-Getoor’s index \( \beta(t) \). A good way to make explicit this intuition is to cut the index process in the spirit of Lebesgue integral. Roughly speaking, we decompose the first term of (8) as a sum of \( m \) processes \( \mathcal{P}_j \), whose local index takes value in \([2j/m, 2(j+1)/m]\). When \( m \) becomes large, the local behavior of these processes is comparable with that of some Lévy process, in probability.

Proposition [21] brings information about the uniform increment estimate of \( Z \). In contrast with the Lévy case, it is remarkable that the exponent depends on two parameters: the approximation rate \( \delta \), and \( \beta(t) \), both random and correlated with \( \mathcal{M} \). This observation complicates in the proof.

The proof is decomposed into several lemmas. The following lemma gives an increment estimate in the first dyadic interval with “frozen” index.
Lemma 22. Let $\varepsilon > 0$ be small. For every large integer $m$ and every $j = \{0, \cdots, m - 1\}$,
\[
\mathbb{P} \left( \sup_{t \leq 2^{-m}} \left| 2^{m(2j + 2 + 2m\varepsilon)/m} \int_0^t \int_{C(0, 2^{-m/\delta})} G(M_s, z) 1_{\beta(s) \in \left[ \frac{2j + 2}{m}, \frac{2j + 2 + 2}{m} \right]} \tilde{N}(dsdz) \right| \geq 2m \right) 
\leq C_\delta e^{-2m}.
\]

Proof. Let
\[
H_j(M_s, z) := 2^{m(2j + 2 + 2m\varepsilon)/m} G(M_s, z) 1_{\beta(s) \in \left[ \frac{2j + 2}{m}, \frac{2j + 2 + 2}{m} \right]} C(0, 2^{-m/\delta})(z),
\]
\[
P_i^j := \int_0^t \int_{C(0, 1)} H_j(M_s, z) \tilde{N}(dsdz).
\]
For every $t \leq 2^{-m}$, one has
\[
\int_0^t \int_{C(0, 1)} \mathbb{E}[H_j(M_s, z)^2] \frac{dz}{z^2} ds
= \int_0^t \mathbb{E} \left[ 2^{m(2j + 2 + 2m\varepsilon)/m} 1_{\beta(s) \in \left[ \frac{2j + 2}{m}, \frac{2j + 2 + 2}{m} \right]} \int_{C(0, 2^{-m/\delta})} G(M_s, z)^2 \frac{dz}{z^2} \right] ds
\leq 2 \int_0^t \mathbb{E} \left[ 2^{m(2j + 2 + 2m\varepsilon)/m} 1_{\beta(s) \in \left[ \frac{2j + 2}{m}, \frac{2j + 2 + 2}{m} \right]} \int_{C(0, 2^{-m/\delta})} z^{\beta(s) + \varepsilon/2} 2^{-m/\delta} ds \right]
\leq C \int_0^t \mathbb{E} \left[ 2^{m(2j + 2 + 2m\varepsilon)/m} 1_{\beta(s) \in \left[ \frac{2j + 2}{m}, \frac{2j + 2 + 2}{m} \right]} \right] ds,
\]
where we used the property of class $G$ for the first inequality. Hence,
\[
\int_0^t \int_{C(0, 1)} \mathbb{E}[H_j(M_s, z)^2] \frac{dz}{z^2} ds \leq C \int_0^t \mathbb{E} \left[ 2^m \right] du = C t 2^{m/\delta} \leq C.
\]
Hence, $(t \mapsto P_i^j)_{t \leq 2^{-m}}$ is a martingale by Proposition 13. One deduces by convexity and Jensen’s inequality that $t \mapsto e^{P_i^j}$ and $t \mapsto e^{-P_i^j}$ are positive submartingales. By Doob’s maximal inequality for positive martingales,
\[
\mathbb{P} \left( \sup_{t \leq 2^{-m}} |P_i^j| \geq 2m \right) \leq \mathbb{P} \left( \sup_{t \leq 2^{-m}} e^{P_i^j} \leq e^{2m} \right) + \mathbb{P} \left( \sup_{t \leq 2^{-m}} e^{-P_i^j} \leq e^{2m} \right)
\leq e^{-2m} \left( \mathbb{E} \left[ e^{P_i^j} \right] + \mathbb{E} \left[ e^{-P_i^j} \right] \right).
\]

We now compute $\mathbb{E}[e^{P_i^j}]$ and $\mathbb{E}[e^{-P_i^j}]$. If these expectations are finite and independent of the value of $m$, then the proof is done. We only study the positive submartingales $e^{P_i^j}$ ($e^{-P_i^j}$ can be studied similarly). By Itô’s Formula for compensated Poisson integral,
\[
e^{P_i^j} = 1 + \int_0^t \int_{C(0, 1)} e^{P_i^j} (e^{H_j(M_s, z)} - 1) \tilde{N}(dsdz)
+ \int_0^t \int_{C(0, 1)} e^{P_i^j} (e^{H_j(M_s, z)} - 1 - H_j(M_s, z)) \frac{dz}{z^2} ds.
\]

(9)
Note that for all $s \in [0, 1]$, using the property of $G \in \mathcal{G}$,
\[
|H_j(M_{s-}, z)| \leq 2^{\frac{m}{2(2j+2m+e)/m}} |z|^{\frac{m}{2(j+e/2)}} 1_{|z| \leq 2^{-m/\delta}} 1_{|s-e| \leq \frac{2j+2}{m}} \leq 1.
\]
Then, since $|e^u - 1 - u| \leq |u|^2$ for $|u| \leq 1$, taking expectation in (13) yields
\[
\mathbb{E}[e^{P^j_t}]
= 1 + \mathbb{E} \left[ \int_0^t \int_{C(0,1)} e^{P^j_{s-}} \left( e^{H_j(M_{s-}, z)} - 1 - H_j(M_{s-}, z) \right) \frac{dz}{z^2} ds \right]
\leq 1 + \mathbb{E} \left[ \int_0^t \int_{C(0,1)} e^{P^j_{s-}} H_j(M_{s-}, z)^2 \frac{dz}{z^2} ds \right]
= 1 + \mathbb{E} \left[ \int_0^t \int_{C(0,2^{-m/\delta})} e^{P^j_{s-}} 2^{\frac{2m}{2(2j+2m+e)/m}} 1_{|s-e| \leq \frac{2j+2}{m}} G(M_{s-}, z)^2 \frac{dz}{z^2} ds \right]
\leq 1 + 2\mathbb{E} \left[ \int_0^t e^{P^j_{s-}} 2^{\frac{2m}{2(2j+2m+e)/m}} 1_{|s-e| \leq \frac{2j+2}{m}} \int_0^{2^{-m/\delta}} z^{\frac{2}{2(j+e/2)+2}} \frac{dz}{z^2} ds \right],
\]
where we used the definition of the class $\mathcal{G}$. We deduce that
\[
\mathbb{E}[e^{P^j_t}] \leq 1 + C \mathbb{E} \left[ \int_0^t e^{P^j_{s-}} 2^{\frac{2m}{2(2j+2m+e)/m}} 1_{|s-e| \leq \frac{2j+2}{m}} 2^{- \frac{m}{2(j+e/2)+2}} 1_{|s-e| \leq \frac{2j+2}{m}} \frac{dz}{z^2} ds \right]
\leq 1 + C \mathbb{E} \left[ \int_0^t e^{P^j_{s-}} 2^{\frac{2m}{2(2j+2m+e)/m}} 1_{|s-e| \leq \frac{2j+2}{m}} \frac{dz}{z^2} ds \right]
\leq 1 + C \mathbb{E} [e^{P^j_0}] 2^{m/\delta} ds = 1 + C \int_0^t \mathbb{E}[e^{P^j_t}] 2^{m/\delta} ds.
\]
By Gronwall’s inequality applied to $s \mapsto \mathbb{E}[e^{P^j_t}]$, one concludes that
\[
\mathbb{E}[e^{P^j_{2^{-m}}}] \leq e_0^{2^{-m}} C 2^{m/\delta} \leq e^{C 2^{m/\delta} 2^{-m}} \leq e^C.
\]

Now we can consider the whole jump process.

**Lemma 23.** Let $\varepsilon > 0$ be small. For $m$ large, for every integer $i \in \{0, \ldots, 2^m - 1\}$, one has
\[
\mathbb{P} \left( \sup_{t \leq 2^{-m}} \left| \int_{2^{-m}}^{2^{-m}+t} \int_{C(0,2^{-m/\delta})} G(M_{s-}, z) \tilde{N}(dsdz) \right| \geq 2m2^{- \frac{m}{2(j+e/2)+2}} \right) \leq C \delta e^{-m}.
\]

**Proof.** It suffices to show this inequality for the first dyadic interval, namely $i = 0$. For every $j$, let us introduce the event
\[
A_j = \left\{ \sup_{t \leq 2^{-m}} \left| \int_0^t \int_{C(0,2^{-m/\delta})} G(M_{s-}, z) 1_{|s-e| \leq \frac{2j+2}{m}} \tilde{N}(dsdz) \right| \geq 2m2^{- \frac{m}{2(j+e/2)+2}} \right\}.
\]
We have
\[
A_j = \left( A_j \cap \left\{ \sup_{s \leq 2^{-m}} \beta(s-) < \frac{2j}{m} \right\} \right) \cup \left( A_j \cap \left\{ \sup_{s \leq 2^{-m}} \beta(s-) \geq \frac{2j}{m} \right\} \right).
\]
Under the event \( \left\{ \sup_{s \leq 2^{-m}} \beta(s-) < \frac{2j}{m} \right\} \), the compensated Poisson integral in \( A_j \) is zero, thus the first event is a null set. One has

\[
P(A_j) = P\left( A_j \cap \left\{ \sup_{s \leq 2^{-m}} \beta(s-) \geq \frac{2j}{m} \right\} \right)
= P\left( A_j \cap \left\{ \sup_{s \leq 2^{-m}} \beta(s-) + \frac{2}{m} + \varepsilon \geq \frac{2j + 2}{m} + \varepsilon \right\} \right)
\leq P\left( \sup_{t \leq 2^{-m}} \left| \int_0^t \int_{C(0, 2^{-m}/\delta)} G(\mathcal{M}_{s-}, z) 1_{[\beta(s-) \in [\frac{2j}{m}, \frac{2j+2}{m}]} N(dsdz) \right| \geq 2m2^{-\delta(2j+2+m\varepsilon)/m} \right).
\]

Using Lemma 22, one concludes that \( P(A_j) \leq C \delta e^{-2m} \).

In order to get the increment estimate with “moving” index, we note that

\[
\left\{ \sup_{t \leq 2^{-m}} \left| \int_0^t \int_{C(0, 2^{-m}/\delta)} G(\mathcal{M}_{s-}, z) N(dsdz) \right| \geq 2m2^{-\delta(jm_0, 2^{-m} + \varepsilon)} \right\} \subset \bigcup_{j=1}^m A_j.
\]

One deduces that

\[
P\left( \sup_{t \leq 2^{-m}} \left| \int_0^t \int_{C(0, 2^{-m}/\delta)} G(\mathcal{M}_{s-}, z) N(dsdz) \right| \geq 2m2^{-\delta(jm_0, 2^{-m} + \varepsilon)} \right) \leq C \delta e^{-2m},
\]

which completes the proof of Lemma 23. \( \square \)

We prove Proposition 21 using a classical discretization procedure.

**Proof of Proposition 21**: We discretize the first term of (8) in the time domain. Let \( s, t \in [0, 1] \) such that \( |s-t| \leq 2^{-m} \). There exists \( i \in \{0, \ldots, 2^m - 1\} \) such that \( |s, t| \subset [(i-1)/2^m, (i+1)/2^m] \), thus

\[
2^{\delta(\beta_{i, t+\varepsilon})} \int_s^t \int_{C(0, 2^{-m}/\delta)} G(\mathcal{M}_{s-}, z) N(dsdz)
\leq 2^{\delta(\beta_{i-1, (i+1)2^{-m}+\varepsilon})} \sup_{t \leq 2^{-m}} \left| \int_{i-1}^{i+1} \int_{C(0, 2^{-m}/\delta)} G(\mathcal{M}_{s-}, z) N(dsdz) \right|
+ 2 \cdot 2^{\delta(\beta_{i-1, (i+1)2^{-m}, i, (i+1)2^{-m}+\varepsilon})} \sup_{t \leq 2^{-m}} \left| \int_{i-1}^{i+1} \int_{C(0, 2^{-m}/\delta)} G(\mathcal{M}_{s-}, z) N(dsdz) \right|
\]

Therefore,

\[
\left\{ \sup_{|s-t| \leq 2^{-m}} \left| \int_s^t \int_{C(0, 2^{-m}/\delta)} G(\mathcal{M}_{s-}, z) N(dsdz) \right| \geq 6m^2 \right\} \subset \bigcup_{i=0}^{2^m-1} \left\{ 2^{\delta(\beta_{i, t+\varepsilon})} \sup_{t \leq 2^{-m}} \left| \int_{i-1}^{i+1} \int_{C(0, 2^{-m}/\delta)} G(\mathcal{M}_{s-}, z) N(dsdz) \right| \geq 2m^2 \right\}
\]
so that by Lemma 23 one sees that for every \( m \) large enough,
\[
\mathbb{P} \left( \sup_{|s-t| \leq 2^{-m}, s,t \in [0,1]} 2^m H_{s,t} \right) \left( \int_s^t \int_{C(0,2^{-m/4})} G(\mathcal{M}_{s,z}, \tilde{N}(dudz) \right) \geq 6m^2 \right) \\
\leq 2^m C_\delta e^{-2m} \leq C_\delta e^{-m}.
\]

\[Q.E.D.\]

5. Proof of Theorem 11 (ii) Jumps without diffusion

Recall that we assume that \((\mathcal{H})\) holds, and in this case, one has
\[
\mathcal{M}_t = \int_0^t b(\mathcal{M}_u)du + \int_0^t \int_{C(0,1)} G(\mathcal{M}_{u,z}, \tilde{N}(dudz) = \mathcal{Y}_t + \mathcal{Z}_t.
\]

**Case (a) of \((\mathcal{H})\):** Let \( b \in C^\infty(\mathbb{R}) \) and Range \( \beta \subset [1,2) \). Under the conditions on \( b \), the drift term \( \mathcal{Y} \) does not influence the pointwise regularity of \( \mathcal{M} \). This follows from the next Lemma.

**Lemma 24.** Let \( f, g : \mathbb{R} \to \mathbb{R} \) and \( F(\cdot) = \int_0^\cdot f(y)dy. \)

(i) Let \( g \in C^\infty(\mathbb{R}) \) and \( x \in \mathbb{R} \), then \( H_{gof}(x) \geq H_f(x). \)

(ii) \( \forall x \in \mathbb{R}, H_{F(x)} \geq H_f(x) + 1. \)

It is a simple exercise to check the lemma. Hence, for every \( t \), one has \( H_\beta(t) \geq H_{b(\mathcal{M}_t)}(t) + 1 \geq H_\mathcal{M}(t) + 1 \), which yields \( H_\mathcal{M}(t) = H_\mathcal{Z}(t) \). Therefore, it is enough to prove that a.s. for every \( t \in [0,1] \setminus \mathcal{J}, H_\mathcal{Z}(t) = \frac{1}{\delta(\beta(t))}. \)

The upper bound is obtained by Proposition 18, so it remains us to get the lower bound, which will be deduced from the following property:

\[
\forall \delta > 1, \forall \varepsilon > 0, \text{ almost surely, } \forall t \notin \mathcal{J} \cup A_\delta, H_\mathcal{Z}(t) \geq \frac{1}{\delta(\beta(t) + \varepsilon)}.
\]

Indeed, assume that \((10)\) holds true. This implies that, almost surely, for all rational pair \( \varepsilon > 0 \) and \( \delta > 1 \), one has \( H_\mathcal{Z}(t) \geq \frac{1}{\delta(\beta(t) + \varepsilon)} \) for all points \( t \notin \mathcal{J} \cup A_\delta. \) The monotonicity of the mapping \( \delta \mapsto A_\delta \) yields that if \( \delta' > \delta \), \( t \notin A_{\delta'} \Rightarrow (t \notin A_\delta). \) One deduces that almost surely, for every \( \delta > 1 \) real, for every rational \( \delta' > \delta \) and \( \varepsilon > 0 \), if \( t \notin \mathcal{J} \cup A_\delta \), the exponent \( H_\mathcal{Z}(t) \) satisfies \( H_\mathcal{Z}(t) \geq \frac{1}{\delta(\beta(t) + \varepsilon)} \). Using the density of the rational numbers in \( \mathbb{R} \) and taking \( \varepsilon \) arbitrarily small in \( \mathbb{Q} \) yields that almost surely,

\[\forall \delta > 1, \forall t \notin \mathcal{J} \cup A_\delta, H_\mathcal{Z}(t) \geq \frac{1}{\delta(\beta(t))}.\]

We deduce the lower bound for \( H_\mathcal{Z}(t) \). Let \( \varepsilon' > 0 \). If \( t \notin J \) and \( \delta_t < +\infty \), one has \( t \notin A_{\delta_t + \varepsilon'} \) by the definition of the approximation rate \( \delta_t \). hence \( H_\mathcal{Z}(t) \geq \frac{1}{(\delta_t + \varepsilon')\beta(t)}. \) If \( t \notin J \) but \( \delta_t = +\infty \), then \( \frac{1}{(\delta_t + \varepsilon')\beta(t)} = 0 \), the desired inequality is trivial. Letting \( \varepsilon' \to 0 \) yields that a.s., \( \forall t \notin J \),

\[H_\mathcal{Z}(t) \geq \frac{1}{\delta(\beta(t))}.\]

Now we prove \((10)\). Applying Proposition 21 and Borel-Cantelli lemma, on sees that \( \forall \varepsilon > 0, \forall \delta > 1 \), almost surely, \( \exists m(\omega), \forall m \geq m(\omega), \)

\[
\sup_{|s-t| \leq 2^{-m}, s,t \in [0,1]} 2^m H_{s,t} \left( \int_s^t \int_{C(0,2^{-m/4})} G(\mathcal{M}_{s,z}, \tilde{N}(dudz) \right) \leq 6m^2.
\]


For every $t \notin J \cup A_\delta$, pick a point $s$ close to $t$ such that $|s - t| < 2^{-m(\omega)}$. Hence

$$2^{-m-1} \leq |s - t| < 2^{-m}.$$  

for some unique $m \geq m(\omega)$, and

$$\left| 2^m \left[ \frac{\beta^m}{2^m \epsilon + \epsilon} \right] \int_s^t \int_{C(0, 2^{-m/\delta})} G(M_{u-, z}) \tilde{N}(dudz) \right| \leq 6m^2$$

where we used (11). Therefore, by (12), one has

$$\left| \int_s^t \int_{C(0, 2^{-m/\delta})} G(M_{u-, z}) \tilde{N}(dudz) \right| \leq 6m^2 2^{-m(\beta(\delta + \epsilon))}$$

where we used the continuity of $\beta := \tilde{\beta} \circ M$ at $t$ (because $\tilde{\beta}$ is Lipschitz and $M$ is continuous at $t$) and the fact that $\tilde{\beta}^m \leq \beta(t) + \epsilon$ for large $m$.

Recalling (8), in the interval $[s, t]$ with $t \notin J \cup A_\delta$ and $|s - t| \sim 2^{-m}$, there is no time whose corresponding jump size is larger than $2^{-m/\delta}$. Hence,

$$\int_s^t \int_{C(2^{-m/\delta}, 1)} G(M_{u-, z}) \tilde{N}(dudz) = \int_s^t \int_{C(2^{-m/\delta}, 1)} G(M_{u-, z}) \frac{dz}{z^2} du.$$

By the fact that $G \in \mathcal{G}$, for every $\epsilon > 0$, there exists $z(\epsilon) > 0$, such that for all $z \in C(0, z(\epsilon))$ and $x \in [0, 1]$, one has $|G(x, z)| \leq |z| \frac{1}{z^{\beta(x) + \epsilon}}$. Hence

$$\left| \int_s^t \int_{C(2^{-m/\delta}, z(\epsilon))} G(M_{u-, z}) \frac{dz}{z^2} du \right| \leq 2 \left| \int_s^t \int_{2^{-m/\delta}}^{z(\epsilon)} z^{\frac{1}{\beta(t) + \epsilon}} \frac{dz}{z^2} du \right|$$

$$\leq 2 \left| \int_s^t \int_{2^{-m/\delta}}^{z(\epsilon)} z^{\frac{1}{\beta(t) + \epsilon}} \frac{dz}{z^2} du \right|$$

$$\leq C |s - t|^{\frac{1}{\delta} + \frac{1}{\beta(t) + \epsilon} - 1}$$

$$= C |s - t|^{\frac{1}{\delta} + \frac{1}{\beta(t) + \epsilon}}$$

and

$$\left| \int_s^t \int_{C(z(\epsilon), 1)} G(M_{u-, z}) \frac{dz}{z^2} du \right| \leq \int_s^t \int_{C(z(\epsilon), 1)} \frac{dz}{z^2} du \leq Cz(\epsilon)^{-1}|s - t|$$
by the uniform boundedness of $G$. Combining the estimates (13)-(16),

\[
|Z_t - Z_s| \leq \left| \int_s^t \int_{C(0,2^{-m/\delta})} G(M_{u^-}, z) \tilde{N}(dudz) \right| + \left| \int_s^t \int_{C(2^{-m/\delta}, 1)} G(M_{u^-}, z) \tilde{N}(dudz) \right| \\
\leq \left| \int_s^t \int_{C(0,2^{-m/\delta})} G(M_{u^-}, z) \tilde{N}(dudz) \right| + \left| \int_s^t \int_{C(2^{-m/\delta}, z(\varepsilon))} G(M_{u^-}, z) \frac{dz}{z^2}du \right| \\
\quad + \left| \int_s^t \int_{C(\varepsilon, 1)} G(M_{u^-}, z) \frac{dz}{z^2}du \right| \\
\leq 6|s - t|^{\frac{1}{2\log(1+t)}} \left( \log \frac{1}{|s - t|} \right)^2 + C|s - t|^{1 - \frac{1}{2\log(1+t)}} + C\varepsilon^{-1}|s - t| \\
\leq 8|s - t|^{\frac{1}{2\log(1+t)}} \left( \log \frac{1}{|s - t|} \right)^2
\]

for $s$ sufficiently close to $t$, where we used the fact that $1 > \frac{1}{\delta(\beta(t) + \varepsilon)}$ and $1 - 1/\delta > 0$. The desired lower bound is obtained.

**Case (b) of (H):** $b \in C^\infty$ and $x \mapsto \tilde{b}(x) := \int_0^1 G(x, z)dz/z^2 \in C^\infty$, whenever the integral is well defined.

If $t$ is such that $\beta(t) \geq 1$, we follow the exact same lines of Case (a) to obtain the exponent.

If $t$ is such that $\beta(t) < 1$, by continuity, for every $u \in [t - \varepsilon, t + \varepsilon]$ with $\varepsilon$ sufficiently small, $\beta(u) < 1$. Then we write

\[
M_u - M_{u-\varepsilon} = Y_u - Y_{u-\varepsilon} + Z_u - Z_{u-\varepsilon} \\
= \int_{u-\varepsilon}^u b(M_s) ds + \int_{u-\varepsilon}^u \int_{C(0,1)} G(M_{s-}, z) \tilde{N}(dudz) \\
= \int_{u-\varepsilon}^u \tilde{b}(M_s) ds + \int_{u-\varepsilon}^u \int_{C(0,1)} G(M_{s-}, z) N(dudz) \\
:= \tilde{Y}_u - \tilde{Z}_u
\]

where $\tilde{b}(x) = b(x) - \tilde{b}(x)$. Using that $\beta(t) < 1$, one easily checks that both integrals in the last line are well defined for every $u \in [t - \varepsilon, t + \varepsilon]$. Under the conditions we impose on $b$ and $\tilde{b}$, it follows that $h_{\tilde{Z}}(t) \geq H_M(t) + 1$, always due to Lemma 24. Therefore, $H_M(t) = H_{\tilde{Z}}(t)$.

Notice that the upper bound for $H_{\tilde{Z}}(t)$ is also an upper bound for $H_{\tilde{Z}}(t)$ since Lemma 19 depends only on the system of Poisson points $\mathcal{P}$. We only need to obtain the lower bound for $H_{\tilde{Z}}(t)$. By the same arguments as those developed in case (a), it suffices to prove (10) for $\tilde{Z}$.
But for $s$ close to $t$, if $m$ is the integer given by (12) (since $s \in [t - \varepsilon, t + \varepsilon]$ for $m$ large), one has

$$| \mathcal{Z}_t - \mathcal{Z}_s | = \left| \int_s^t \int_{C(0,1)} G(\mathcal{M}_u, z) N(dudz) \right|$$

$$= \left| \int_s^t \int_{C(0,2^{-m/\delta})} G(\mathcal{M}_u, z) N(dudz) \right|$$

$$\leq \left| \int_s^t \int_{C(0,2^{-m/\delta})} G(\mathcal{M}_u, z) \bar{N}(dudz) \right|$$

$$+ \left| \int_s^t \int_{C(0,2^{-m/\delta})} G(\mathcal{M}_u, z) \frac{dz}{z^2} du \right|$$

$$: = I_1 + I_2.$$

We estimate $I_1$ by (13). For $I_2$, we apply (12) to get

$$I_2 \leq 2 \int_s^t \int_0^{2^{-m/\delta}} z^{\frac{1}{\beta(t) + 1} - 1} \frac{dz}{z^2} du \leq C \int_s^t 2^{-\frac{m}{\delta}(\frac{1}{\beta(t) + \varepsilon} - 1)} du$$

$$\leq C|s - t|^{1 + \frac{1}{\beta(t) + \varepsilon}} \leq C|s - t|^{1 + \frac{1}{\beta(t) + \varepsilon}},$$

so that

$$| \mathcal{Z}_t - \mathcal{Z}_s | \leq 6|s - t|^{\frac{1}{\beta(t) + \varepsilon}} \left( \log \frac{1}{|s - t|} \right)^2 + C|s - t|^{1 + \frac{1}{\beta(t) + \varepsilon}}$$

or

$$\leq 7|s - t|^{\frac{1}{\beta(t) + \varepsilon}} \left( \log \frac{1}{|s - t|} \right)^2,$$

which yields (10). From this we deduce the desired lower bound.

### 6. Proof of Theorem 11 (1) Jump with a Non-Zero Diffusion

Along the proof, whenever $\beta(t) < 1$, we focus on a small interval $[t - \varepsilon, t + \varepsilon]$ and study $\bar{Y}$ and $\mathcal{Z}$, as we did in the previous section.

Recall that we want to show that almost surely, $\forall t \notin J, H_M(t) = \frac{1}{\delta(t) + \varepsilon}$. It is enough to show that $\forall \delta > 1, \forall \varepsilon > 0$, almost surely,

$$\forall t \notin J \cup A_\delta, \quad \beta(t) \geq 1 \Rightarrow H_Z(t) \geq \frac{1}{\delta(\beta(t) + \varepsilon)},$$

$$\beta(t) < 1 \Rightarrow H_{\mathcal{Z}}(t) \geq \frac{1}{\delta(\beta(t) + \varepsilon)}.$$

Indeed, assume that (18) holds true.

Then, for every $\varepsilon > 0$, almost surely, $\forall \delta > 1$ rational, $\forall t \notin J \cup A_\delta$, one has

$$\beta(t) \geq 1 \quad \Rightarrow \quad H_Z(t) \geq \frac{1}{\delta(\beta(t) + \varepsilon)},$$

and

$$\beta(t) < 1 \quad \Rightarrow \quad H_{\mathcal{Z}}(t) \geq \frac{1}{\delta(\beta(t) + \varepsilon)}.$$
By the same argument as the one used in the proof of Theorem 11 (ii), one can remove the (rational) restriction on \( \delta \) and remove \( \varepsilon \). Hence, almost surely, \( \forall \delta > 1, \forall t \notin J \cup A_\delta \), one has

\[
\beta(t) \geq 1 \implies H_Z(t) \geq \frac{1}{\delta \beta(t)} \\
\text{and} \quad \beta(t) < 1 \implies H_Z(t) \geq \frac{1}{\delta \beta(t)}.
\]

Then by the definition of \( \delta t \) and Proposition 18 (upper bound for the exponent of the jump part), one deduces that \( \forall \varepsilon' > 0 \), almost surely one has

\[
\forall t \notin J, \quad \beta(t) \geq 1 \implies \frac{1}{(\alpha + \varepsilon')\beta(t)} \leq H_Z(t) \leq \frac{1}{\alpha \beta(t)}, \\
\beta(t) < 1 \implies \frac{1}{(\alpha + \varepsilon')\beta(t)} \leq H_Z(t) \leq \frac{1}{\alpha \beta(t)}.
\]

Letting \( \varepsilon' \to 0 \) yields almost surely

\[
\forall t \notin J, \quad \beta(t) \geq 1 \implies H_Z(t) = \frac{1}{\alpha \beta(t)}, \\
\beta(t) < 1 \implies H_Z(t) = \frac{1}{\alpha \beta(t)}.
\]

But almost surely, for every \( t \), \( H_{X+Y}(t) = H_{X+Y} = \frac{1}{2} \) (use Proposition 15 and the facts that \( H_Y(t) \geq 1 \), \( H_Y(t) \geq 1 \)).

Consider now a time \( t \notin J \) where the process is continuous. Four cases must be distinguished.

- If \( \beta(t) \geq 1 \) and \( \frac{1}{2} \neq \frac{1}{\alpha \beta(t)} \), then \( H_M(t) = H_{X+Y+Z}(t) = \frac{1}{2} \wedge \frac{1}{\delta \beta(t)} \).
- If \( \beta(t) \geq 1 \) and \( \frac{1}{2} = \frac{1}{\alpha \beta(t)} \), then by by Lemma 19 for \( M \),
  \[
  \frac{1}{2} \wedge \frac{1}{\delta \beta(t)} \geq H_M(t) = H_{X+Y+Z}(t) \geq H_{X+Y}(t) \wedge H_Z(t) = \frac{1}{2} \wedge \frac{1}{\delta \beta(t)}.
  \]
- If \( \beta(t) < 1 \) and \( \frac{1}{2} \neq \frac{1}{\alpha \beta(t)} \), then \( H_M(t) = H_{X+Y+Z}(t) = \frac{1}{2} \wedge \frac{1}{\delta \beta(t)} \).
- If \( \beta(t) < 1 \) and \( \frac{1}{2} = \frac{1}{\alpha \beta(t)} \), then
  \[
  \frac{1}{2} \wedge \frac{1}{\delta \beta(t)} \geq H_M(t) = H_{X+Y+Z}(t) \geq H_{X+Y}(t) \wedge H_Z(t) = \frac{1}{2} \wedge \frac{1}{\delta \beta(t)}.
  \]

In all cases, if \( t \) is not a jump time, \( H_M(t) = \frac{1}{2} \wedge \frac{1}{\delta \beta(t)} \).

Finally, we prove 18. Let \( \delta > 1 \) and \( \varepsilon > 0 \), and consider \( t \notin J \cup A_\delta \).

- If \( \beta(t) \geq 1 \), we use 13-16 to get, for \( s \) sufficiently close to \( t \),
  \[
  |Z_t - Z_s| \leq C|s - t|^\frac{1}{2(\beta(t)+2\varepsilon)} \left( \log \frac{1}{|s - t|} \right)^2.
  \]

- If \( \beta(t) < 1 \), using 17 for \( s \) close to \( t \), one gets
  \[
  |Z_t - Z_s| \leq 7|s - t|^\frac{1}{2(\beta(t)+2\varepsilon)} \left( \log \frac{1}{|s - t|} \right)^2.
  \]

7. Computation of the pointwise multifractal spectrum

In this section, we compute the pointwise spectrum of \( M \) in all possible settings, i.e. Theorem 7 for jumps with diffusion and Theorems 20-21 for jumps without diffusion. The main tool comes from geometric measure theory, the so-called ubiquity theorem, which consists in determining the Hausdorff dimension of some limsup sets. This theory finds its origin in Diophantine approximation and the localized version developed by Barral and Seuret [3] is very useful in studying random objects with varying spectra.
Theorem 25 [5]. Consider a Poisson point process $S$ with intensity $dt \otimes 1_{z \in C(0,1)}dz/z^2$. Let $I = (a, b) \subset [0, 1]$ and $f : I \to [1, +\infty)$ be continuous at every $t \in I \setminus C$, for some countable set $C \subset [0, 1]$. Consider
\[
S(I, f) = \{ t \in I : \delta_t \geq f(t) \} \quad \text{and} \quad \tilde{S}(I, f) = \{ t \in I : \delta_t = f(t) \}.
\]
Then almost surely for every $I = (a, b) \subset [0, 1]$
\[
\dim_H S(I, f) = \dim_H \tilde{S}(I, f) = \sup \{ 1/f(t) : t \in I \setminus C \}.
\]

7.1. Proof of Theorem 7: Pointwise spectrum of $\mathcal{M}$ when $\sigma \neq 0$.

When the diffusion term does exist, the computation of the pointwise spectrum is easier to state. Fix one point $t \in \mathbb{R}^+$. Let
\[
I_t^n := \left[ t - \frac{1}{n}, t + \frac{1}{n} \right] \cap \mathbb{R}^+.
\]

- If $h > 1/2$, then $D_M(t, h) = -\infty$ by item 1. of Theorem 11.
- If $h < 1/2$, then
\[
E_M(h) \cap I_t^n = \left\{ s \in I_t^n : h = \frac{1}{\delta_s \beta(s)} \wedge \frac{1}{2} \right\}
\]
\[
= \left\{ s \in I_t^n : h = \frac{1}{\delta_s \beta(s)} \right\} = \left\{ s \in I_t^n : \delta_s = \frac{1}{h \beta(s)} \right\}.
\]

But $\text{Range } \beta \subset (0, 2)$ so that $\frac{1}{h \beta(s)} > 1$, $\forall s \in I_t^n$. This yields that $\dim_H E_M(h) \cap I_t^n = \sup \{ h \beta(s) : s \in I_t^n \}$ by Theorem 25. Hence
\[
D_M(t, h) = \lim_{n \to +\infty} \dim_H E_M(h) \cap I_t^n = h \cdot (\beta(t) \vee \beta(t-)).
\]

- If $h = 1/2$. For every $h' < 1/2$, let $\tilde{E}_{h'} = \left\{ s \in \mathbb{R}^+ : \delta_s \geq \frac{1}{h' \beta(s)} \right\}$. Clearly, for every $h' < 1/2$, $E_M(h') \subset \tilde{E}_{h'}$ and Theorem 25 gives
\[
(19) \quad \dim_H E_M(h') = \dim_H \tilde{E}_{h'}, \text{ for all } h' < 1/2.
\]

Now decompose
\[
I_t^n = \left( \bigcup_{h' < 1/2} (E_M(h') \cap I_t^n) \right) \cup (E_M(1/2) \cap I_t^n)
\]
\[
\subset \left( \bigcup_{h' < 1/2} (\tilde{E}_{h'} \cap I_t^n) \right) \cup (E_M(1/2) \cap I_t^n).
\]

Hence
\[
1 = \dim_H(I_t^n) \leq \left( \dim_H \bigcup_{h' < 1/2} (\tilde{E}_{h'} \cap I_t^n) \right) \vee (\dim_H E_M(1/2) \cap I_t^n)
\]
\[
= \left( \lim_{h' \uparrow 1/2} \dim_H (\tilde{E}_{h'} \cap I_t^n) \right) \vee (\dim_H E_M(1/2) \cap I_t^n)
\]
\[
= \left( \frac{1}{2} \sup \{ \beta(s) : s \in I_t^n \} \right) \vee (\dim_H E_M(1/2) \cap I_t^n),
\]
where we used the monotonicity of the sets $(E_{t'})_{h < 1/2}$ and (19). Since $\text{Range } \beta \subset (0, 2)$, $\dim H E_M(1/2) \cap I_t^n = 1$, which yields $D_M(t, 1/2) = 1$.

7.2. Definitions and statement of the results when $\sigma \equiv 0$. When the diffusion does not vanish, it eliminates all the problems we encounter at some specific points. To deal with all possible situations, we need to introduce some notations. For $t \in J$, we define

$$I^n_{t+} := \begin{cases} (t, t + \frac{1}{n}], & \text{if } \beta(t) > \beta(t^-), \\ (t - \frac{1}{n}, t), & \text{otherwise.} \end{cases}$$

For all $t \in J$, we need the mapping $\beta^{t+}(s) : I^n_{t+} \cup \{t\} \to \mathbb{R}$ defined by

$$\beta^{t+}(s) = \begin{cases} \beta(s) & \text{if } s \in I^n_{t+}, \\ \lim_{I^n_{t+} \ni u \to t} \beta(u) & \text{if } s = t. \end{cases}$$

The map $\beta^{t+}$ coincides with $\beta$ except at $t$. Similarly, we set

$$I^n_{t-} := \left[t - \frac{1}{n}, t + \frac{1}{n}\right] \setminus I^n_{t+} \setminus \{t\}, \quad \beta^{t-}(s) = \begin{cases} \beta(s) & \text{if } s \in I^n_{t-}, \\ \lim_{I^n_{t+} \ni u \to t} \beta(u) & \text{if } s = t. \end{cases}$$

Throughout this section, we write $t \in LM(F)$ to mean that $t$ is a strict local minimum for a mapping $F$.

Finally, we introduce two functions $F_{\text{cont}}$ and $F_{\text{jump}}$ (see Figure 2) which correspond to different cases of the pointwise spectra.

- For a time $t$ where the process is continuous, we will use

$$F_{\text{cont}}(c, \gamma, h) = \begin{cases} \gamma h & \text{if } h \in [0, 1/\gamma), \\ c & \text{if } h = 1/\gamma, \\ -\infty & \text{otherwise.} \end{cases}$$

There will be only three possible values for $c$ (1, 0 and $-\infty$).
Assume that 

\[ F_{jump}(c_1, c_2, \gamma_1, \gamma_2, h) = \begin{cases} 
\gamma_1 \cdot h & \text{if } h \in [0, 1/\gamma_1), \\
c_1 & \text{if } h = 1/\gamma_1, \\
\gamma_2 \cdot h & \text{if } h \in [1/\gamma_1, 1/\gamma_2), \\
c_2 & \text{if } h = 1/\gamma_2, \\
-\infty & \text{otherwise,}
\end{cases} \]

when \( \gamma_1 > \gamma_2 \). There will be three possible values for \( c_2 \) (1, 0 and \(-\infty\)) and two for \( c_1 \) (1 and \( h \cdot \gamma_2 \)).

The several cases in the theorems below correspond to assigning a precise value to the discontinuous points of the pointwise spectrum, and various scenarios may occur, depending on the fact that \( t \) is or not a local minimum for the functions \( \beta^{t+} \) and \( \beta^{t-} \). The reader shall keep in mind the following heuristics:

- If \( t \) is a jump time for the process, we will use the function \( F_{jump} \)

\[ F_{jump}(c_1, c_2, \gamma_1, \gamma_2, h) = \begin{cases} 
\gamma_1 \cdot h & \text{if } h \in [0, 1/\gamma_1), \\
c_1 & \text{if } h = 1/\gamma_1, \\
\gamma_2 \cdot h & \text{if } h \in [1/\gamma_1, 1/\gamma_2), \\
c_2 & \text{if } h = 1/\gamma_2, \\
-\infty & \text{otherwise,}
\end{cases} \]

when \( \gamma_1 > \gamma_2 \). There will be three possible values for \( c_2 \) (1, 0 and \(-\infty\)) and two for \( c_1 \) (1 and \( h \cdot \gamma_2 \)).

Theorem 26. Assume that \( \sigma \equiv 0 \) and (H) holds (see Theorem [17]). Then, with probability one,

1. for every \( t \notin J \), the pointwise spectrum of \( \mathcal{M} \) at time \( t \) is given by

\[ D_{\mathcal{M}}(t, h) = \begin{cases} 
F_{cont}(1, \beta(t), h) & \text{if } t \notin \text{LM}(\beta), \\
F_{cont}(0, \beta(t), h) & \text{if } t \in \text{LM}(\beta) \text{ and } \delta_t = 1, \\
F_{cont}(\infty, \beta(t), h) & \text{if } t \in \text{LM}(\beta) \text{ and } \delta_t \neq 1.
\end{cases} \]

2. Assume that \( t \in J \) and that \( t \notin \text{LM}(\beta^{t-}) \cup \text{LM}(\beta^{t+}) \). Define for \( t \in J \)

\[ \beta_m(t) = \beta(t) \land \beta(t-) \quad \text{and} \quad \beta_M(t) = \beta(t) \lor \beta(t-). \]

We have

\[ D_{\mathcal{M}}(t, h) = F_{jump}(1, 1, \beta_M(t), \beta_m(t)). \]

This theorem covers the most frequent cases, i.e. when \( t \) is a continuous time or \( t \) is a jump time and not a strict local minimum for \( \beta^{t+} \) and \( \beta^{t-} \).

Next theorem covers all the "annoying" cases, i.e. when \( t \) is a jump time and is a minimum for at least one of the two functions \( \beta^{t+} \) and \( \beta^{t-} \). Observe that this concerns at most a countable number of times.

Theorem 27. Assume that \( \sigma \equiv 0 \) and (H) holds. Almost surely:

1. If \( t \notin \text{LM}(\beta^{t+}) \), then

\[ D_{\mathcal{M}}(t, h) = \begin{cases} 
F_{jump}(1, 0, \beta_M(t), \beta_m(t), h) & \text{if } t \in \text{LM}(\beta^{t-}), \Delta \beta(t) > 0 \text{ and } \delta_t = 1, \\
F_{jump}(1, -\infty, \beta_M(t), \beta_m(t), h) & \text{if } t \in \text{LM}(\beta^{t-}), \Delta \beta(t) < 0 \text{ or } \delta_t \neq 1.
\end{cases} \]

2. If \( t \in \text{LM}(\beta^{t+}) \), then

\[ D_{\mathcal{M}}(t, h) = \begin{cases} 
F_{jump}(h \cdot \beta_m(t), 1, \beta_M(t), \beta_m(t), h) & \text{if } t \notin \text{LM}(\beta^{t-}),
\end{cases} \]

\[ F_{jump}(h \cdot \beta_m(t), 0, \beta_M(t), \beta_m(t), h) & \text{if } t \in \text{LM}(\beta^{t-}), \Delta \beta(t) > 0, \delta_t = 1. \\
F_{jump}(h \cdot \beta_m(t), -\infty, \beta_M(t), \beta_m(t), h) & \text{if } t \in \text{LM}(\beta^{t-}), \Delta \beta(t) < 0 \text{ or } \delta_t \neq 1. \]
When \( t \) is a jump time, the behaviors of \( \mathcal{M} \) on the right hand-side and on the left hand-side of \( t \) may differ a lot. So the pointwise spectrum reflects the superposition of two local behaviors, which explains the formulas above. Though not easy to read, these formulas are simple consequences of these complications that may arise as very special cases.

7.3. First part of the proof of Theorems 26 and 27: the linear parts.

We start with an easy lemma.

**Lemma 28.** Assume that \( \sigma \equiv 0 \) and \((\mathcal{H})\) holds. Almost surely, for Lebesgue-almost every \( t \in [0,1] \), \( h_\mathcal{M}(t) = 1/\beta(t) \).

**Proof.** Using Theorem 11 one sees that for \( I = (a,b) \subset [0,1] \),

\[
\dim_{\mathcal{H}} \{ t \in I : h_\mathcal{M}(t) = \kappa/\beta(t) \} = \dim_{\mathcal{H}} \{ t \in I : \delta_t = 1/\kappa \}.
\]

Let \( \kappa \in (0,1) \). We apply Theorem 25 to the Poisson system \( \mathcal{P} \) and the mapping \( f(t) \equiv 1/\kappa \); this yields directly \( \dim_{\mathcal{H}} \{ t \in I : H_\mathcal{M}(t) = \kappa/\beta(t) \} = \kappa \). Still by Theorem 25 one has

\[
(21) \quad \dim_{\mathcal{H}} \{ t \in I : h_\mathcal{M}(t) \leq \kappa/\beta(t) \} = \kappa.
\]

Next, let us decompose the interval \( I \) as

\[
(22) \quad I = \left\{ t \in I : H_\mathcal{M}(t) = 1/\beta(t) \right\} \cup \left( \bigcup_{n \geq 1} S_n \right)
\]

where \( S_n := \{ t \in I : H_\mathcal{M}(t) \leq (1 - 1/n)/\beta(t) \} \). For every \( n \geq 1 \), the Lebesgue measure of \( S_n \) is zero since it has Hausdorff dimension strictly less than 1, by (21). We deduce by (22) that for Lebesgue-a.e. \( t \in I \), \( H_\mathcal{M}(t) = 1/\beta(t) \). Since this holds for any interval \((a,b) \subset [0,1]\), the conclusion follows. \( \square \)

We only prove the result for \( t \in J \). If \( t \) is a continuous time for \( \mathcal{M} \), the pointwise spectrum at \( t \) is obtained directly since in this case \( \beta(t) = \beta_M(t) = \beta_m(t) \). We treat separately three linear parts of \( F_{\text{jump}} \).

- If \( h < \frac{1}{\beta_M(t)} \), there exists \( \varepsilon > 0 \) such that \( h < \frac{1}{\beta_M(t)+\varepsilon} \). But \( \forall s \in I^n_t, \beta(s) < \beta_M(t)+\varepsilon/2 \) by the càdlàg property of the sample paths, which implies

\[
\frac{1}{h\beta(s)} > \frac{\beta_M(t) + \varepsilon}{\beta_M(t) + \varepsilon/2} > 1
\]

for all \( s \in I^n_t \) with \( n \) large enough. Theorem 25 implies that

\[
\dim_{\mathcal{H}} E_\mathcal{M}(h) \cap I^n_t = \dim_{\mathcal{H}} \left\{ s \in I^n_t : \delta_s = \frac{1}{h\beta(s)} \right\} = \sup \{ h\beta(s) : s \in I^n_t \} = h \cdot \sup \{ \beta(s) : s \in I^n_t \}.
\]

for large \( n \), which yields \( D_\mathcal{M}(t,h) = \lim_{n \to +\infty} \sup \{ h\beta(s) : s \in I^n_t \} = h \cdot (\beta_M(t)) \).

- If \( h \in \left( \frac{1}{\beta_M(t)}, \frac{1}{\beta_M(t)} \right) \), there exists \( \varepsilon > 0 \) small enough so that \( h \) is in the interval \( \left( \frac{1}{\beta_M(t)-\varepsilon}, \frac{1}{\beta_M(t)+\varepsilon} \right) \).

Let us consider separately \( I^n_{t+} \) and \( I^n_{t-} \). For all \( s \in I^n_{t+} \),

\[
\frac{1}{h\beta(s)} \leq \frac{\beta_M(t) - \varepsilon}{\beta_M(t) - \varepsilon/2} < 1
\]
by the càdlàg property of the sample paths, for \( n \) large enough. Hence \( E_{\mathcal{M}}(h) \cap I_{t+}^n = \left\{ s \in I_{t+}^n : \delta_s = \frac{1}{h\beta(s)} \right\} = \emptyset \), because \( \delta_s \geq 1 \) almost surely. For all \( s \in I_{t-}^n \),

\[
\frac{1}{h\beta(s)} > \frac{\beta_m(t) + \varepsilon}{\beta_m(t) + \varepsilon/2} > 1
\]

which implies by Theorem 25 that

\[
\dim_{\mathcal{H}} E_{\mathcal{M}}(h) \cap I_{t-}^n = \dim_{\mathcal{H}} E_{\mathcal{M}}(h) \cap I_{t-}^n = h \cdot \sup \left\{ \beta(s) : s \in I_{t-}^n \right\}
\]

for large \( n \). Using the càdlàg property of the sample paths, one concludes that \( D_{\mathcal{M}}(t, h) = \lim_{n \to +\infty} \sup \left\{ h\beta(s) : s \in I_{t-}^n \right\} = h \cdot (\beta_m(t)) \).

- If \( h > \frac{1}{\beta_m(t)} \), one can choose \( \varepsilon > 0 \) small enough so that \( h > \frac{1}{\beta_m(t) - \varepsilon} \). But \( \forall s \in I_{t-}^n \), \( \beta(s) > \beta_m(t) - \varepsilon \) for large \( n \). So,

\[
h > \frac{1}{\beta(s)} \geq \frac{1}{\delta_s\beta(s)} = H_{\mathcal{M}}(s)
\]

yields \( E_{\mathcal{M}}(h) \cap I_{t-}^n = \emptyset \). Hence, \( D_{\mathcal{M}}(t, h) = \lim_{n \to +\infty} \dim_{\mathcal{H}} E_{\mathcal{M}}(h) \cap I_{t-}^n = -\infty \).

### 7.4. Second part of the proof of Theorems 26 and 27: the points of discontinuities of \( F_{\text{cont}} \) and \( F_{\text{jump}} \)

There are two possible discontinuities for \( F_{\text{jump}} \), which are \( \frac{1}{\beta_m(t)} \) and \( \frac{1}{\beta_m(t)} \).

- If \( h = \frac{1}{\beta_m(t)} \), we distinguish between two cases.

**Case 1:** \( t \in LM(\beta^+) \). Then \( \forall s \in I_{t+}^n \),

\[
\frac{1}{h\beta(s)} = \frac{\beta_M(t)}{\beta_M(s)} < 1,
\]

which implies \( E_{\mathcal{M}}(h) \cap I_{t+}^n = \emptyset \). Notice that

\[
E_{\mathcal{M}}(h) \cap I_{t-}^n = \left\{ s \in I_{t-}^n : \delta_s = \frac{\beta_M(t)}{\beta(s)} \right\}.
\]

For every \( s \in I_{t-}^n \), one has

\[
\frac{\beta_M(t)}{\beta(s)} \geq \frac{\beta_M(t)}{\beta_m(t) + |\Delta \beta_t|/2} > 1.
\]

This ensures that \( \dim_{\mathcal{H}} E_{\mathcal{M}}(h) \cap I_{t-}^n = \sup \left\{ \frac{\beta(s)}{\beta_M(t)} : s \in I_{t-}^n \right\} \), still by Theorem 25. Therefore,

\[
\dim_{\mathcal{H}} E_{\mathcal{M}}(h) \cap I_{t-}^n = \dim_{\mathcal{H}} E_{\mathcal{M}}(h) \cap (I_{t-}^n \cup \{t\}) = \dim_{\mathcal{H}} E_{\mathcal{M}}(h) \cap I_{t-}^n,
\]

which yields \( D_{\mathcal{M}}(t, h) = \lim_{n \to +\infty} \dim_{\mathcal{H}} E_{\mathcal{M}}(h) \cap I_{t-}^n = \frac{\beta_m(t)}{\beta_M(t)} = h \cdot \beta_m(t) \).

**Case 2:** \( t \notin LM(\beta^+) \). In this case, either \( t \) is not a local minimum for \( \beta^+ \), or \( \beta^+ \) is locally constant near \( t \). If \( t \) is not a local minimum for \( \beta^+ \), one can extract a monotone sequence \( \{s_k\} \subset I_{t+}^n \) tending to \( t \) such that

\[
(23) \quad \beta(s_k) < \beta_M(t).
\]

Since \( \beta \) is càdlàg and the cardinality of \( J \) is at most countable, we can choose \( s_k \) as continuous points for \( \beta \). Let us first compute the pointwise spectrum of \( \mathcal{M} \) on points \( s_k \) and deduce the result by a regularity restriction of the pointwise spectrum.

Fix \( k \geq 1 \) and let \( p \) be large enough. For every \( s \in I_{s_k}^p \), one has \( \frac{\beta_M(t)}{\beta(s)} > 1 \) by \( (23) \). Further, Theorem 25 ensures that

\[
\dim_{\mathcal{H}} E_{\mathcal{M}}(h) \cap I_{s_k}^p = \sup \left\{ h\beta(s) : s \in I_{s_k}^p \right\}.
\]
which yields that $D_M(s_k, h) = h\beta(s_k)$, for every integer $k \geq 1$. Hence
\[ 1 \geq D_M(t_h) = \limsup_{s \to t} D_M(s_h) \geq \limsup_{k \to +\infty} D_M(s_k, h) = h\beta_M(t) = 1 \]
where we used Proposition 4. If $\beta^+$ is locally constant near $t$, then $E_M(h) \cap I_{t+}^n = \{ s \in I_{t+}^n : \delta_s = \frac{1}{\beta(m)} = 1 \}$ for $n$ large. Applying Lemma 28, one deduces that $\text{Leb}(E_M(h) \cap I_{t+}^n) = \text{Leb}(I_{t+}^n)$ where Leb denotes the Lebesgue measure, and so $\dim_{\dot{H}}(E_M(h) \cap I_{t+}^n) = 1$. One concludes that
\[ 1 \geq D_M(t_h) = \lim_{n \to +\infty} \dim_{\dot{H}}(E_M(h) \cap I_{t}^n) \geq \lim_{n \to +\infty} \dim_{\dot{H}}(E_M(h) \cap I_{t+}^n) = 1. \]

- **If** $h = \frac{1}{\beta(m)}$: As before, we distinguish two cases.

**Case 1**: $t \in LM(\beta^-)$. Then $\forall s \in I_{t-}^n$, one has $\frac{\beta_m(t)}{\beta(s)} < 1$, which implies that $E_M(h) \cap I_{t-}^n = \emptyset$. Notice that $E_M(h) \cap I_{t+}^n = \{ s \in I_{t+}^n : \delta_s = \frac{\beta_m(t)}{\beta(s)} \}$ and that $\forall s \in I_{t+}^n$,
\[ \frac{\beta_m(t)}{\beta(s)} < \frac{\beta_m(t)}{\beta_M(t) - |\Delta \beta_t|/2} < 1. \]
Hence $E_M(h) \cap I_{t+}^n = \emptyset$, for large $n$. But
\[ E_M(h) \cap \{ t \} = \begin{cases} \{ t \} & \text{if } \beta(t-) > \beta(t) \text{ and } \delta_t = 1, \\ \emptyset & \text{otherwise.} \end{cases} \]
Hence,
\[ \dim_{\dot{H}} E_M(h) \cap I_{t}^n = \dim_{\dot{H}} E_M(h) \cap \{ t \} = \begin{cases} 0 & \text{if } \beta(t-) > \beta(t) \text{ and } \delta_t = 1, \\ -\infty & \text{otherwise,} \end{cases} \]
for large $n$, which yields
\[ D_M(t, h) = \begin{cases} 0 & \text{if } \beta(t-) > \beta(t) \text{ and } \delta_t = 1, \\ -\infty & \text{otherwise.} \end{cases} \]

**Case 2**: $t \notin LM(\beta^-)$. Then, either $t$ is not a local minimum for $\beta^-$, or $\beta^-$ is locally constant near $t$. If $t$ is not a local minimum for $\beta^-$. By a similar argument as in the second case of the last situation, we can prove that $d(s_k, h) = h\beta(s_k)$ where $\{ s_k \} \subset I_{t-} \setminus J$ is a strictly monotone sequence tending to $t$ satisfying $\beta(s_k) < \beta_m(t)$. Therefore,
\[ 1 \geq D_M(t, h) = \limsup_{s \to t} D_M(s_h) \geq \limsup_{k \to +\infty} D_M(s_k, h) = h(\beta_m(t)) = 1. \]
If $\beta^-$ is locally constant near $t$, then $E_M(h) \cap I_{t-}^n = \{ s \in I_{t-}^n : \delta_s = \frac{\beta_n(t)}{\beta(s)} = 1 \}$ for $n$ large. Still by Lemma 28, one has $\text{Leb}(E_M(h) \cap I_{t-}^n) = \text{Leb}(I_{t-}^n)$, which yields $\dim_{\dot{H}}(E_M(h) \cap I_{t-}^n) = 1$. Hence,
\[ 1 \geq D_M(t, h) \geq \lim_{n \to +\infty} \dim_{\dot{H}}(E_M(h) \cap I_{t-}^n) = 1. \]

8. Existence of tangent processes

In order to describe the local structure of stochastic processes which are often rough (not differentiable), several authors consider the tangent processes associated with them, see for instance [17]. Precisely, given a stochastic process $X$ and $t_0$ a fixed time, one wonders if there exist two sequences $(\alpha_n)_{n \geq 1}$, $(r_n)_{n \geq 1}$ decreasing to zero such that the sequence of process $(r_n X_{t_0 + \alpha_n t})_{t \geq 0}$ converges in law to some limit process $(Y_t)_{t \geq 0}$, and call it, if exists, a tangent process. One observes in Theorem 7 and Theorem 28 that the pointwise spectrum of the process $M$ looks like (but not exactly) the spectrum of some Lévy process. Then natural questions concern the
connections between the pointwise spectrum of the process at \( t_0 \) and its tangent process at this point. In the stable-like case, we show the existence of tangent processes of \( \mathcal{M} \), which are some stable Lévy processes. Their spectra coincide with the pointwise spectra of \( \mathcal{M} \) at time \( t \) except for one value of \( h \). Here, the scaling \((r_n, \alpha_n)\) must be carefully chosen and plays an important role.

Throughout this section, the Skorohod space of càdlàg functions on \([0, 1]\) is endowed with the uniform convergence topology. We consider the function \( G_0(x, z) = \text{sign}(z)|z|^{\gamma}/\tilde{\beta}(x) \) with \( \tilde{\beta} \) Lipschitz continuous and \( \text{Range} \tilde{\beta} \subset (0, 2) \), and the pure jump diffusion

\[
\mathcal{M}_t = \int_0^t \int_{C(0,1)} G_0(\mathcal{M}_{s-}, z) \tilde{N}(dz). 
\]

**Proposition 29.** Let \( t_0 \geq 0 \) be fixed, conditionally on \( \mathcal{F}_{t_0} \), the family of processes \( \left( \frac{\mathcal{M}_{t_0+\alpha}-\mathcal{M}_{t_0}}{\alpha^{1/\beta(t_0)}} \right)_{t \in [0, 1]} \) converges in law to a stable Lévy process with Lévy measure \( \beta(t_0)u^{-1-\beta(t_0)} \) du, when \( \alpha \to 0 \).

The next lemma gives some moment estimate for \( \mathcal{M} \) near 0. The second point was proved in [1], we still prove it for the sake of completeness. Let us introduce the stopping times for every \( \eta > 0 \)

\[
\tau_\eta := \inf\{t > 0 : \beta(t) > \tilde{\beta}(0) + \eta\}. 
\]

**Lemma 30.** Let \( \eta > 0 \) be small.

(i) If \( \tilde{\beta}(0) \geq 1 \), for every \( \gamma \in (\tilde{\beta}(0) + \eta, 2) \), there exists a constant \( c_\gamma \) such that \( \forall \alpha > 0 \),

\[
\mathbb{E}[|\mathcal{M}_{\alpha \wedge \tau_\eta}|^\gamma] \leq c_\gamma \alpha.
\]

(ii) If \( \tilde{\beta}(0) < 1 \), for every \( \gamma \in (\tilde{\beta}(0) + \eta, 1 \wedge 2\tilde{\beta}(0)) \), the same moment inequality holds.

**Proof.** (i) Since \( \mathcal{M} \) is a martingale, by Burkholder-Davis-Gundy inequality and the symmetry of \( G_0 \) in \( z \), we have

\[
\mathbb{E}[|\mathcal{M}_{\alpha \wedge \tau_\eta}|^\gamma] \leq \mathbb{E}\left[ \sup_{0 \leq \xi \leq \alpha \wedge \tau_\eta} |\mathcal{M}_t|^\gamma \right] 
\]

\[
\leq c_\gamma \mathbb{E}\left[ \int_0^{\alpha \wedge \tau_\eta} \int_{C(0,1)} |G_0(\mathcal{M}_{s-}, z)|^2 N(dsdz) \right]^{\gamma/2} 
\]

\[
\leq c_\gamma \mathbb{E}\left[ \int_0^{\alpha \wedge \tau_\eta} \int_0^1 |G_0(\mathcal{M}_{s-}, z)|^\gamma N(dsdz) \right] 
\]

\[
= c_\gamma \mathbb{E}\left[ \int_0^{\alpha \wedge \tau_\eta} \int_0^1 |G_0(\mathcal{M}_{s-}, z)|^\gamma dz/z^2 \right]. 
\]

For every \( s \in [0, \tau_\eta) \), one has

\[
\int_0^1 |G_0(\mathcal{M}_{s-}, z)|^\gamma dz/z^2 = \int_0^1 |z|^{\gamma/\beta(s-)}dz/z^2 \leq \int_0^1 |z|^{\gamma/(\tilde{\beta}(0)+\eta)}dz/z^2 < +\infty,
\]

where we used that \( \gamma > \tilde{\beta}(0) + \eta \). Hence,

\[
\mathbb{E}[|\mathcal{M}_{\alpha \wedge \tau_\eta}|^\gamma] \leq c_\gamma \mathbb{E}\left[ \int_0^{\alpha \wedge \tau_\eta} ds \right] \leq c_\gamma \alpha.
\]
(ii) For every \( s \in [0, \tau_\eta) \) with \( \eta \) small enough, it makes sense to separate the compensated Poisson measure, i.e. the difference of the Poisson measure and its intensity. Using \((a + b)^\gamma \leq c_\gamma (a^\gamma + b^\gamma)\) for all \((a, b) \in \mathbb{R}^2_+\), subadditivity and symmetry, we have

\[
\mathbb{E}(|\mathcal{M}_{\alpha^\wedge \tau_\eta}|^\gamma) \leq c_\gamma \mathbb{E} \left[ \int_0^{\alpha^\wedge \tau_\eta} \int_{C(0,1)} |G_0(\mathcal{M}_{s-}, z)| N(dsdz) \right]^{\gamma} \\
+ c_\gamma \mathbb{E} \left[ \int_0^{\alpha^\wedge \tau_\eta} \int_{C(0,1)} |G_0(\mathcal{M}_{s-}, z)| dz/z^2 ds \right]^{\gamma} \\
\leq c_\gamma \mathbb{E} \left[ \int_0^{\alpha^\wedge \tau_\eta} \int_0^1 |G_0(\mathcal{M}_{s-}, z)|^\gamma dz/z^2 ds \right].
\]

Repeating the arguments of the first point yields the result. \( \square \)

**Lemma 31.** Let \( x_0 \) be fixed. For all \( \gamma > \tilde{\beta}(x_0) \), there exist strictly positive constants \( C_\gamma \) and \( \delta \) such that for all \( x \in B(x_0, \delta) \)

\[
\int_{C(0,1)} |G_0(x, z) - G_0(x_0, z)|^\gamma \pi(dz) \leq C_\gamma |x - x_0|^\gamma.
\]

It is easy to check Lemma 31. Now we prove Proposition 29 using the self-similarity of the limit process and last two lemmas.

**Proof.** By the Markov property, it is enough to prove the proposition for \( t_0 = 0 \). Let us introduce

\[
\mathcal{L}_t = \int_0^t \int_{C(0,1)} G_0(0, z) \tilde{N}(dsdz), \quad \mathcal{S}_t = \int_0^t \int_{C(0, +\infty)} G_0(0, z) \tilde{N}(dsdz).
\]

Note that \( \mathcal{L} \) and \( \mathcal{S} \) are pure jump Lévy processes whose Lévy measure are \( \tilde{\beta}(0)|z|^{\tilde{\beta}(0)-1}1_{C(0,1)}dz \)
and \( \tilde{\beta}(0)|z|^{-\tilde{\beta}(0)-1}dz \), respectively. As is well known, \( \mathcal{S} \) is \( 1/\tilde{\beta}(0) \) self-similar, meaning that for every \( \alpha > 0 \), one has

\[
(\alpha^{-1/\tilde{\beta}(0)} \mathcal{S}_t)_{t \in [0,1]} = (\mathcal{S}_t)_{t \in [0,1]}
\]

in law, see for instance Chapter 3 of Sato [31]. Observe that \( \forall \delta > 0 \),

\[
\mathbb{P} \left( \sup_{0 \leq t \leq 1} |\alpha^{-1/\tilde{\beta}(0)}(\mathcal{L}_t - \mathcal{S}_t)| \leq \delta \right) \geq \mathbb{P} (N([0, \alpha), D(1, +\infty)) = 0) = e^{-\alpha} \\
\rightarrow_{\alpha \rightarrow 0} 1.
\]

This computation yields that

\[
\alpha^{-1/\tilde{\beta}(0)} \sup_{t \in [0,1]} |\mathcal{L}_t - \mathcal{S}_t| \rightarrow 0
\]

in probability, when \( \alpha \rightarrow 0 \). Recall that the self-similarity of \( \mathcal{S} \) ensures that \( (\alpha^{-1/\tilde{\beta}(0)} \mathcal{S}_t)_{t \in [0,1]} \)
converges (equals) in law to \( (\mathcal{S}_t)_{t \in [0,1]} \), thus the process \( (\alpha^{-1/\tilde{\beta}(0)} \mathcal{L}_t)_{t \in [0,1]} \)
converges in law to \( (\mathcal{S}_t)_{t \in [0,1]} \). To conclude, it remains to prove the following

\[
\alpha^{-1/\tilde{\beta}(0)} \Delta_\alpha \rightarrow 0 \text{ in probability},
\]

where \( \Delta_\alpha := \sup_{0 \leq t \leq \alpha} |\mathcal{M}_t - \mathcal{L}_t| \). There are two cases.
Proof of Proposition 12: By the Lipschitz assumption on \( \sigma \) and \( b \), it suffices to show that \( G \in \mathcal{G} \) satisfies those two conditions.

(i) We check the growth condition. We divide the integral into two parts, use (4) and the uniform boundedness of \( G \) to get

\[
\int_{C(0,1)} G(x, z)^2 \frac{dz}{z^2} = \int_{C(0, z_0)} G(x, z)^2 \frac{dz}{z^2} + \int_{C(z_0, 1)} G(x, z)^2 \frac{dz}{z^2}.
\]

where we used Lemma \( \text{31} \) and Lemma \( \text{30} \) Hence, for every \( \delta > 0 \), one has

\[
\mathbb{P} \left( \alpha^{-1/\beta(0)} \Delta_\alpha \geq \delta \right) \leq \mathbb{P}(\tau_\eta \leq \alpha) + \mathbb{P} \left( \alpha^{-1/\beta(0)} \Delta_\alpha \wedge \tau_\eta \geq \delta \right),
\]

where \( \lim_{\alpha \downarrow 0^+} \mathbb{P}(\tau_\eta \leq \alpha) = \mathbb{P}(\tau_\eta = 0) = 0 \) and

\[
\mathbb{P} \left( \alpha^{-1/\beta(0)} \Delta_\alpha \wedge \tau_\eta \geq \delta \right) \leq \delta^{-\gamma} \alpha^{-\gamma/\beta(0)} \mathbb{E}[|\Delta_\alpha \wedge \tau_\eta|^\gamma] \leq c_\delta \gamma \alpha^{2-\gamma/\beta(0)} \rightarrow 0,
\]

since \( 2\beta(0) \geq 2 > \gamma > \beta(0) + \eta \).

Case 2: \( \beta(0) < 1 \). As in Lemma \( \text{31} \) for every \( s \in [0, \tau_\eta) \) with \( \eta \) small enough, it makes sense to separate the compensated Poisson measure. By subadditivity, for every \( \gamma \in (\beta(0) + \eta, 1 \wedge 2\beta(0)) \),

\[
\mathbb{E}[|\Delta_\alpha \wedge \tau_\eta|^\gamma] \leq c_\gamma \mathbb{E} \left[ \int_0^{\alpha \wedge \tau_\eta} \int_{C(0,1)} |G_0(\mathcal{M}_{s-}, z) - G_0(0, z)| N(dsdz) \right]^{\gamma/2} \]

\[
+ c_\gamma \mathbb{E} \left[ \int_0^{\alpha \wedge \tau_\eta} \int_{C(0,1)} |G_0(\mathcal{M}_{s-}, z) - G_0(0, z)| dz/z^2 ds \right]^{\gamma/2}
\]

\[
\leq c_\gamma \int_0^{\alpha \wedge \tau_\eta} \int_{C(0,1)} |G_0(\mathcal{M}_{s-}, z) - G_0(0, z)|^\gamma dz/z^2 ds
\]

\[
\leq c_\gamma \int_0^{\alpha \wedge \tau_\eta} |\mathcal{M}_s|^\gamma ds \leq c_\gamma \int_0^{\alpha \wedge \tau_\eta} \mathbb{E}[|\mathcal{M}_{s \wedge \tau_\eta}|^\gamma] ds \leq c_\gamma \alpha^2,
\]

where we used again Lemma \( \text{31} \) and Lemma \( \text{30} \). Repeating the computations (24), (25) and using \( \gamma \in (\beta(0) + \eta, 1 \wedge 2\beta(0)) \) yield the result. \( \square \)

Appendix 1

Proof of Proposition 12: By the Lipschitz assumption on \( \sigma \) and \( b \), it suffices to show that \( G \in \mathcal{G} \) satisfies those two conditions.

(i) We check the growth condition. We divide the integral into two parts, use (4) and the uniform boundedness of \( G \) to get

\[
\int_{C(0,1)} G(x, z)^2 \frac{dz}{z^2} = \int_{C(0, z_0)} G(x, z)^2 \frac{dz}{z^2} + \int_{C(z_0, 1)} G(x, z)^2 \frac{dz}{z^2}.
\]

\[
\leq \int_{C(0, z_0)} |z|^{1+2\varepsilon_0} \frac{dz}{z^2} + \int_{C(z_0, 1)} \frac{dz}{z^2}
\]

\[
= \frac{2\varepsilon_0}{2\varepsilon_0} + 2 \left( \frac{1}{z_0} - 1 \right) := K_0 \leq K_0(1 + x^2).
\]
Indeed, still by (4) and the uniform boundedness of $G$. To conclude, it remains to show that this integral is finite and independent of the value of $x$. Theorem (Dambis-Dubins-Swartz, Th. 5.1.6 [30])

Indeed, still by (4) and the uniform boundedness of $G$,

$$
\int_{C(0,1)} |G(x, z) - G(y, z)|^2 \frac{dz}{z^2} = 2 \int_0^1 |G(x, z)|^2 \left(1 - e^{\log |G(y, z)| - \log |G(x, z)|}\right)^2 \frac{dz}{z^2}
$$

where we used the inequality $1 - e^{-u} \leq u$ for all $u > 0$ and the Lipschitz condition on $G \in \mathcal{G}$.

To conclude, it remains to show that this integral is finite and independent of the value of $x$. Indeed, still by (4) and the uniform boundedness of $G$,

$$
\int_0^1 G(x, z)^2 (\log z)^2 \frac{dz}{z^2} = \int_0^{z_0} G(y, z)^2 (\log z)^2 \frac{dz}{z^2} + \int_{z_0}^1 G(y, z)^2 (\log z)^2 \frac{dz}{z^2}
$$

\[ \leq \int_0^{z_0} |z|^{1+2\varepsilon_0} (\log z)^2 dz + \int_{z_0}^1 (\log z)^2 \frac{dz}{z^2} \]

\[ \leq c_1^2 \left( \int_0^{z_0} z^{\varepsilon_0-1} dz + \int_{z_0}^1 z^{-2-\varepsilon_0} dz \right) \]

\[ \leq c_1^2 \frac{z_0^{\varepsilon_0}}{\varepsilon_0} + c_1^2 \frac{1}{1+\varepsilon_0} \left( \frac{1}{(z_0)^{1+\varepsilon_0}} - 1 \right) : = K_1/C < +\infty, \]

\[ \square \]

**APPENDIX 2**

Recall the martingale representation theorem.

**Theorem** (Dambis-Dubins-Swartz, Th. 5.1.6 [30]). Let $M$ be a $(\mathcal{F}_t, \mathbb{P})$-continuous local martingale such that $M_0 = 0$ and $\langle M \rangle_{+\infty} = +\infty$. Let

\[ T_t = \inf\{s \geq 0 : \langle M \rangle_s > t\}, \]

then $B_t = M_{T_t}$ is a $(\mathcal{F}_{T_t})$-Brownian motion and a.s. $\forall t \in \mathbb{R}^+$, $M_t = B_{\langle M \rangle_t}$.

**Proof of Proposition**. Recall that $X_t = \int_0^t \sigma(M_s) dB_s$ is a local martingale starting from 0. The quadratic variation process of $X$

$$
\langle X \rangle_t = \int_0^t \sigma(M_s)^2 ds,
$$

satisfies $\langle X \rangle_{+\infty} = +\infty$ almost surely, since $\sigma$ stays away from 0 by assumption. Applying Theorem of Dambis-Dubins-Swartz to $X$, one can find a standard Brownian motion $\tilde{B}$ on $(\mathcal{F}, \mathbb{P})$ such that a.s. $\forall t$, $X_t = \tilde{B}_{\langle X \rangle_t}$.

First computation yields a.s. for every $t \in \mathbb{R}^+$, $\forall r > 0$, for all $u \in B(t, r)$,

$$
(26) \quad c|u - t| \leq |\langle X \rangle_u - \langle X \rangle_t| \leq \int_t^u C(1 + |M_s|)^2 ds \leq C|u - t|,
$$

where we used that $\sigma$ stays away from 0 to find the constants $c, C \in \mathbb{R}^+$.
By the properties of Lévy’s modulus (Theorem 1.2.7 of [30]), for every \( \varepsilon > 0 \), a.s. for every \( t \), for \( u \) sufficiently close to \( t \), one has by (26)

\[
|X_u - X_t| = |\widetilde{B}(\langle X \rangle)_u - \widetilde{B}(\langle X \rangle)_t| \leq C' |\langle X \rangle_u - \langle X \rangle_t|^{\frac{1}{2} - \varepsilon} \leq C' |u - t|^{\frac{1}{2} - \varepsilon}.
\]

Hence, almost surely, \( \forall t, H_X(t) \geq \frac{1}{2} - \varepsilon \).

On the other hand, Dvoretzky [15] proved that, for a standard Brownian motion \( B \), there exists a constant \( K > 0 \), such that almost surely

\[
\forall t, \limsup_{h \to 0^+} \frac{|B_{t+h} - B_t|}{h^{1/2}} \geq K.
\]

Applying Dvoretzky’s Theorem to our Brownian motion \( \widetilde{B} \), we get that almost surely for every \( t \geq 0 \), there exists a positive sequence \( (h_n)_{n \geq 1} \) converging to zero such that

\[
|\widetilde{B}(\langle X \rangle)_t + h_n - \widetilde{B}(\langle X \rangle)_t| \geq K h_n^{1/2}.
\]

(27)

As \( t \mapsto \langle X \rangle_t \) is a strictly increasing (always by the assumption that \( \sigma \) stays away from 0) continuous function, there exists a sequence \( (u_n)_{n \geq 1} \) such that \( \langle X \rangle_t + h_n = \langle X \rangle_{u_n} \). By the first inequality of (26),

\[
|h_n| \geq c |u_n - t|.
\]

(28)

It follows form (27) and (28) that

\[
|X_{u_n} - X_t| = |\widetilde{B}(\langle X \rangle)_{u_n} - \widetilde{B}(\langle X \rangle)_t| = |\widetilde{B}(\langle X \rangle)_{t+h_n} - \widetilde{B}(\langle X \rangle)_t| \geq K c |u - t|^{1/2},
\]

This yields a.s. \( \forall t, H_X(t) \leq 1/2 \), and letting \( \varepsilon \) tend to 0 gives the result. \( \square \)

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References

[1] D. Applebaum, Lévy Processes and Stochastic Calculus, Second Edition. Cambridge Studies in Advanced Mathematics, 116. Cambridge Univ. Press, Cambridge, 2009.
[2] P. Balança, Fine regularity of Lévy processes and linear (multi)fractional stable motion. Electron. J. Probab. 19 (2014), no. 101, 1–37.
[3] J. Barral, A. Durand, S. Jaffard and S. Seuret, Local multifractal analysis. Fractal geometry and dynamical systems in pure and applied mathematics. II. Fractals in applied mathematics, 31–64, Contemp. Math., 601, AMS, Providence, RI, 2013.
[4] J. Barral, N. Fournier, S. Jaffard and S. Seuret. A pure jump Markov process with a random singularity spectrum, Ann. Probab. 38 (2010), no. 5, 1924-1946.
[5] J. Barral and S. Seuret, The singularity spectrum of Lévy processes in multifractal time, Adv. Math., 14 (1), 437-468, 2007.
[6] J. Barral and S. Seuret, A localized Jarnik-Besicovitch theorem, Adv. Math. 226 (2011), no. 4, 3191-3215.
[7] R. Bass, Uniqueness in law for pure jump Markov processes. Probab. Theory Related Fields 79 (1988), no. 2, 271–287.
[8] R. Bass, Stochastic differential equations driven by symmetric stable processes. Sémin. de Prob., XXXVI, 302–313, Lec. Notes in Math., 1801, Springer, Berlin, 2003.
[9] B. Böttcher, R. Schilling and J. Wang, Lévy matters. III. Lévy-type processes: construction, approximation and sample path properties. With a short biography of Paul Lévy by Jean Jacod. Lecture Notes in Mathematics, 2099. Lévy Matters. Springer, Cham, 2013. xviii+199 pp.
[10] J. Bertoin, On nowhere differentiability for Lévy processes, Stochastics and stochastics reports, 50, 205–210, 1994.
[11] E. Çinlar and J. Jacod, Representation of semimartingale Markov processes in terms of Wiener processes and Poisson random measures. Seminar on Stochastic Processes, 1981 (Evanston, Ill., 1981), pp. 159–242, Progr. Prob. Statist., 1, Birkhäuser, Boston, Mass., 1981.

[12] R. Cont and P. Tankov, Financial modelling with jump processes. Chapman Hall/CRC Financial Mathematics Series. Boca Raton, FL, 2004. xvi+535 pp.

[13] A. Durand, Singularity sets of Lévy processes. Prob. Th. Rel. Fields 143 (2009), no. 3-4, 517–544.

[14] A. Durand, Random wavelet series based on a tree-indexed Markov chain. Comm. Math. Phys. 283 (2008), no. 2, 451–477.

[15] A. Dworetkzky, On the oscillation of the Brownian motion process. Israel J. Math. 1 1963 212–214.

[16] K. Falconer, Fractal geometry. Mathematical foundations and applications., Second edition. John Wiley and Sons, Inc., Hoboken, NJ, 2003.

[17] K. Falconer, The local structure of random processes. J. London Math. Soc. (2) 67 (2003), no. 3, 657–672.

[18] N. Fournier, On pathwise uniqueness for stochastic differential equations driven by stable Lévy processes. Ann. IHP Probab. Stat. 49 (2013), no. 1, 138–159.

[19] Z. Fu and Z. Li, Stochastic equations of non-negative processes with jumps. Stochastic Process. Appl. 120 (2010), no. 3, 306–330.

[20] R. Garra, E. Orsingher and F. Polito, Fractional diffusions with time-varying coefficients, http://arxiv.org/abs/1501.04806

[21] S. Jaffard, Old Friends Revisited : the Multifractal Nature of Some Classical Functions. J. Fourier Anal. Appl. 3 (1), 1–22, 1997.

[22] S. Jaffard, The multifractal nature of Lévy processes, Prob. Theory Rel. Fields, 114 (2), 207–227, 1999.

[23] S. Jaffard, On lacunary wavelet series, Ann. Appl. Probab., 10 (1), 313–329, 2000.

[24] J.-P. Kahane and J. Peyrière, Sur certaines martingales de Benoit Mandelbrot. Advances in Math. 22 (1976), no. 2, 131–145.

[25] D. Khoshnevisan and Z. Shi, Fast sets and points for fractional Brownian motion. Sémi. de Prob., XXXIV, 393–416, Lec. Notes in Math., 1729, Springer, Berlin, 2000.

[26] Z. Li and L. Mytnik, Strong solutions for stochastic differential equations with jumps. Ann. Inst. Henri Poincaré Probab. Stat. 47 (2011), no. 4, 1055–1067.

[27] A. Negoro, Stable-like processes: construction of the transition density and the behavior of sample paths near t = 0. Osaka J. Math. 31 (1994), no. 1, 189–214.

[28] S. Orey, S.J. Taylor, How often on a Brownian path does the law of the iterated logarithm fail? Proc. London Math. Soc. 28 (3), 174-192, 1974.

[29] E. Perkins, On the Hausdorff dimension of the Brownian slow points, Z. Wahrsch. Verw Geb. 64, 369–389, 1983.

[30] D. Revuz and M. Yor, Continuous martingales and Brownian motion. Third edition. Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], 293. Springer-Verlag, Berlin, 1999. xiv+602 pp.

[31] K. Sato, Lévy processes and infinitely divisible distributions., Cambridge Studies in Advanced Mathematics, 68. Cambridge University Press, Cambridge, 1999.

[32] L.A. Shepp, Covering the line with random intervals, Z. Wahrsch. Verw. Gebiete 23, 163–170, 1972.

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