Ward-Takahashi Identity for Yang-Mills Theory in the Exact Renormalization Group

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We give a functional derivation of the Ward-Takahashi identity for Yang-Mills theory in the framework of the exact renormalization group. The identity realizes non-abelian gauge symmetry nontrivially despite the presence of a momentum cutoff. The cutoff deforms the gauge transformation by introducing composite operators. In our functional method, which is an extension of the method used in our previous work on QED, these composite operators are expressed in terms of the Wilson action that depends on both a UV cutoff and an IR cutoff.

§1. Introduction

Exact renormalization group (ERG)1)–6) has found applications in a variety of fields and is regarded as a powerful tool to elucidate non-perturbative physics.6) Despite its many successes, there remains an important question in its application to a field theory with symmetry, in particular, gauge symmetry. A momentum cutoff introduced in the ERG approach is often in conflict with the gauge symmetry present in a field theory. There are mainly two different approaches to this problem. One is to construct a manifestly gauge invariant regularization scheme.** The other is to introduce identities for the Green functions that constrain symmetry breaking terms induced by the momentum cutoff. In the latter approach, the identity is often written for the Legendre effective action, \( \Gamma \), the generating functional of the 1PI part of the connected cutoff Green functions. The identity for \( \Gamma \), called the modified Slavnov-Taylor identity,16),17) has been extensively used to discuss gauge symmetries in ERG.*** However, symmetry breaking terms in this identity involve the inverse of the full propagator, and are consequently somewhat complicated. For the discussion of symmetry, it becomes more convenient to use the Ward-Takahashi (WT) identity for the Wilson action \( S \), the generating functional of the cutoff connected Green functions. This is because the nontrivial Jacobian of the generalized gauge transformation takes a simple algebraic form; cancellation between the Jacobian and symmetry breaking terms in \( S \) can be seen easily. The WT identity is suitable to demonstrate the presence of exact gauge symmetries in ERG.

Recently, one of the present authors (H.S.) derived the WT identity for QED.18) The identity was subsequently elevated to the quantum master equation (QME).19)
showing the presence of a nilpotent BRST symmetry, or gauge symmetry, in a cutoff theory. Furthermore, we have obtained the master action, a solution to the QME, in terms of the Wilson action satisfying the WT identity. A more direct derivation of the master action for QED has also been given in 20). The progress described above is a concrete realization of the generic idea proposed in Ref. 21) for symmetry realization in ERG. It is important to stress that the results obtained in 18)–20) can be readily generalized to global as well as local symmetries that are realized linearly in the continuum limit.1) Apparently, the presence of QME is not restricted to linear symmetries such as QED. It is natural to expect that we can extend the works 18)–20) to theories with nonlinear (gauge) symmetry such as Yang-Mills theory. The WT identity for the Wilson action \( S \) in Yang-Mills (YM) theory was given by Becchi in his pioneering work.25) This was actually the first step towards finding exact symmetries in cutoff theories within the framework of perturbation theory. He introduced the notion of composite operators to describe generic nonlinear terms appearing in generalized BRST transformations. One of the present authors (H.S.) has recently elaborated on Becchi’s program by showing how to solve QME perturbatively for YM theory.26),27) The purpose of this note is to give a functional derivation of the generic WT identity for nonlinear gauge symmetries, and to apply the results concretely to the pure \( SU(2) \) YM theory. For simplicity, we do not introduce source terms or anti-fields that generate the BRST transformation. In contrast to 25) and 26), the BRST transformations we derive are given by composite operators expressed in terms of functional derivatives of the Wilson action, and our result is more similar to the WT identity for QED given in 18).

In what follows, the Wilson action is obtained by a functional integral over fields of momemtum between the usual IR cutoff \( \Lambda \) and an UV cutoff \( \Lambda_0 \gg \Lambda \). The necessity of the UV cutoff becomes more apparent when we deal with nonlinear symmetries. A finite \( \Lambda_0 \) is necessary to write down the WT identity explicitly, even though \( \Lambda_0 \) can be eventually sent to infinity as the theory is renormalizable. Our approach should be compared with that given in 25) and 26), where the BRST invariance of the continuum limit (\( \Lambda_0 \to \infty \)) is used as an input to obtain the identity at a finite value of an IR cutoff \( \Lambda \). These two approaches differ in the choice of initial conditions for the ERG differential equation, and they are physically equivalent in the limit \( \Lambda_0 \to \infty \).

In \S 2, we consider a generic theory with BRST symmetry, and explain our functional derivation of the WT identity as well as the BRST transformation for the IR theory. Then, in \S 3, we apply the results to the pure \( SU(2) \) YM theory.

1) The general expression of the WT identity for linear symmetries on the lattice has recently been given in 22) as a generalization of the Ginsparg-Wilson relation23) for lattice chiral symmetry. Related problems are considered in Ref. 24).

2) See also 27) where QME is solved perturbatively for the Wess-Zumino model. Supersymmetry transformation is nonlinear in the absence of auxiliary fields.
§2. A general functional integral derivation of the WT identity

Consider an action $S[\phi]$, a functional of fields $\phi^A$. The theory is assumed to have some symmetry, that is written as a BRST symmetry. The Grassmann parity for $\phi^A$ is expressed as $\epsilon(\phi^A) = \epsilon_A$: $\epsilon_A = 0$ (1) if the field $\phi^A$ is Grassmann even (odd). When we consider a gauge theory, $S[\phi]$ is the gauge fixed action, and $\phi^A$ represent collectively gauge and matter fields as well as ghosts, antighosts, and B-fields. The index $A$ represents the Lorentz indices of vector fields, the spinor indices of fermions, and indices distinguishing different types of generic fields.

In order to regularize the theory, we introduce an IR momentum cutoff $\Lambda$ and a UV cutoff $\Lambda_0 > \Lambda$ through a positive function that behaves as

$$K\left(\frac{p^2}{\Lambda^2}\right) \simeq \begin{cases} 1, & (p^2 < \Lambda^2) \\ 0, & (p^2 > \Lambda^2) \end{cases}$$

(2.1)

In the following, we use two functions $K(p) \equiv K(p^2/\Lambda^2)$ and $K_0(p) \equiv K(p^2/\Lambda_0^2)$.

By introducing sources $J_A$, the generating functional is written as

$$Z_\phi[J] = \int \mathcal{D}\phi \exp \left(-S[\phi; \Lambda_0] + K_0^{-1}J \cdot \phi \right),$$

(2.2)

where the action $S$ defined at the scale $\Lambda_0$ is written as the sum of the kinetic and interaction terms

$$S[\phi; \Lambda_0] = \frac{1}{2} \phi \cdot K_0^{-1}D \cdot \phi + S_I[\phi; \Lambda_0].$$

(2.3)

In this paper we use the matrix notation in momentum space:

$$J \cdot \phi = \int \frac{d^4p}{(2\pi)^4} J_A(-p)\phi^A(p),$$

$$\phi \cdot D \cdot \phi = \int \frac{d^4p}{(2\pi)^4} \phi^A(-p)D_{AB}(p)\phi^B(p).$$

(2.4)

In performing the functional integral (2.2), we decompose the fields $\phi^A$ with the propagator $K_0(p) \left(D_{AB}(p)\right)^{-1}$ into the sum of the IR fields $\Phi^A$ with the propagator $K(p) \left(D_{AB}(p)\right)^{-1}$ and the UV fields $\chi^A$ with $(K_0(p) - K(p)) \left(D_{AB}(p)\right)^{-1}$. Note that $\Phi^A$ carry the momenta below $\Lambda$, and $\chi^A$ between $\Lambda_0$ and $\Lambda$. The integration over the UV fields $\chi^A$ gives us the generating functional for the IR fields $\Phi^A$:

$$Z_\Phi[J] = \int \mathcal{D}\Phi \exp \left(-S[\Phi; A] + J \cdot K^{-1}\Phi \right),$$

(2.5)

where

$$S[\Phi; A] \equiv \frac{1}{2} \Phi \cdot K^{-1}D \cdot \Phi + S_I[\Phi; A]$$

(2.6)

is the Wilson action, and its interaction part $S_I$ is defined by

$$\exp(-S_I[\Phi; A]) \equiv \int \mathcal{D}\chi \exp \left[-\frac{1}{2} \chi \cdot (K_0 - K)^{-1}D \cdot \chi - S_I[\Phi + \chi; \Lambda_0] \right].$$

(2.7)
Two generating functionals (2.2) and (2.6) are related as
\[ Z_\phi[J] = N_J Z_\Phi[J], \tag{2.8} \]
where the normalization factor \( N_J \) is given by
\[ \ln N_J = -\frac{(-)^{\epsilon_A}}{2} J_A K_0^{-1} K^{-1} \left( K_0 - K \right) \left( D^{-1} \right)^{AB} J_B. \tag{2.9} \]

We next consider how the symmetry realization is affected by the presence of cutoffs. We write the BRST transformation with an anticommuting constant \( \lambda \) as
\[ \phi^A \rightarrow \phi'^A = \phi^A + \delta \lambda \phi^A, \quad \delta \lambda \phi^A = \delta \phi^A \lambda = R^A[\phi; A_0] \lambda. \tag{2.10} \]

The generating functional (2.2) is invariant under the change of the integration variable by the BRST transformation (2.10). This trivial observation produces the relation
\[ \int D\phi \left( K_0^{-1} J \cdot \delta \phi - \Sigma[\phi; A_0] \right) \exp \left( -S[\phi; A_0] + K_0^{-1} J \cdot \phi \right) = 0, \tag{2.11} \]
where \( \Sigma[\phi; A_0] \) is the WT operator given as*)
\[ \Sigma[\phi; A_0] = \frac{\partial^r S}{\partial \phi^A} \delta \phi^A - \frac{\partial^r}{\partial \phi^A} \delta \phi^A. \tag{2.12} \]
\( \Sigma[\phi, A_0] \) is the sum of the change of the original gauge fixed action \( S[\phi; A_0] \)
\[ \delta \lambda S = \frac{\partial^r S}{\partial \phi^A} \delta \lambda \phi^A, \tag{2.13} \]
and that of the functional measure \( D\phi \)
\[ \delta \lambda \ln D\phi = (-)^{\epsilon_A} \frac{\partial^r}{\partial \phi^A} \delta \lambda \phi^A = \left( \frac{\partial^r}{\partial \phi^A} \delta \phi^A \right) \lambda. \tag{2.14} \]
The relation (2.11) may be rewritten as
\[ \langle \Sigma[\phi; A_0] \rangle_{\phi, K_0^{-1} J} = K_0^{-1} J \cdot \langle \delta \phi \rangle_{\phi, K_0^{-1} J} = K_0^{-1} J \cdot \langle R[\phi; A_0] \rangle_{\phi, K_0^{-1} J} = K_0^{-1} J \cdot R[K_0 \partial^j; A_0] Z_\phi[J], \tag{2.15} \]
where the field \( \phi^A \) is replaced by the functional derivative with respect to \( J^A \), and \( R^A[\partial^j] \) is called the Slavnov operator. Note that (2.15) is valid whether or not the theory is invariant under the BRST transformation (2.10).

For an anomaly-free renormalizable theory, we assume that the operator (2.12) behaves as
\[ \Sigma[\phi; A_0] = O(1/A_0^2). \tag*{\text{*) The left and right derivatives are indicated by the superscripts, } l \text{ and } r, \text{ respectively.}} \]
WT Identity for YM in ERG

for a large but finite value of $\Lambda_0$. To be more precise, we assume

$$\langle \Sigma[\phi; \Lambda_0] \rangle_{\phi, K_0^{-1} J} = O(1/\Lambda_0^2)$$  \hspace{1cm} (2.16)$$
or equivalently,

$$K_0^{-1} J \cdot \mathcal{R}[K_0 \delta J; \Lambda_0] \mathcal{Z}[J] = O(1/\Lambda_0^2)$$ \hspace{1cm} (2.17)$$
for an arbitrary source $J$. This is the statement of the WT identity for the original theory defined at the UV scale $\Lambda_0$.

We now wish to transform the above into an equivalent condition on the Wilson action $S$ with a finite IR cutoff $\Lambda$. As a preparation let us first summarize important properties of composite operators in the ERG framework. In general, given an operator $O[\phi; \Lambda_0]$ at the UV scale $\Lambda_0$, we can define the corresponding IR composite operator $O[\Phi; \Lambda]$ by

$$O[\Phi; \Lambda] \exp(-S_I[\Phi; \Lambda]) \equiv \int \mathcal{D} \chi O[\Phi + \chi; \Lambda_0] \cdot \exp\left[-\frac{1}{2} \chi \cdot (K_0 - K)^{-1} \mathcal{D} \cdot \chi - S_I[\Phi + \chi; \Lambda_0]\right].$$ \hspace{1cm} (2.18)$$

This operator has two important properties:

1. The $\Lambda$ dependence is given by the ERG flow equation:

$$\dot{\mathcal{O}} = \frac{\partial^r \mathcal{O}}{\partial \Phi^A} \left( \dot{K} D^{-1} \right)^{AB} \frac{\partial^l S_I}{\partial \Phi^B} - (-)^{c_A(c_O+1)} \left( \dot{K} D^{-1} \right)^{AB} \frac{\partial^l \partial^r \mathcal{O}}{\partial \Phi^B \partial \Phi^A},$$ \hspace{1cm} (2.19)$$

where the dot denotes the logarithmic derivative $\Lambda \frac{\partial}{\partial \Lambda}$.

2. The expectation value in the presence of an arbitrary source satisfies

$$\langle O[\Phi; \Lambda] \rangle_{\Phi, K^{-1} J} = N_J^{-1} \langle O[\phi; \Lambda_0] \rangle_{\phi, K_0^{-1} J}.$$ \hspace{1cm} (2.20)$$

Now, we apply (2.18) and (2.20). Using (2.18) we first define the WT operator $\Sigma[\Phi; \Lambda]$ for the IR theory by

$$\Sigma[\Phi; \Lambda] \exp(-S_I[\Phi; \Lambda]) \equiv \int \mathcal{D} \chi \Sigma[\Phi + \chi; \Lambda_0] \times \exp\left[-\frac{1}{2} \chi \cdot (K_0 - K)^{-1} \mathcal{D} \cdot \chi - S_I[\Phi + \chi; \Lambda_0]\right].$$ \hspace{1cm} (2.21)$$

Then, (2.20) implies

$$\langle \Sigma[\Phi; \Lambda] \rangle_{\Phi, K^{-1} J} = N_J^{-1} \langle \Sigma[\phi; \Lambda_0] \rangle_{\phi, K_0^{-1} J}.$$ \hspace{1cm} (2.22)$$

We also define the IR operator $\delta \Phi^A$ corresponding to $\delta \phi^A$ by

$$\delta \Phi^A[\Phi; \Lambda] \exp(-S_I[\Phi; \Lambda]) \equiv K K_0^{-1} \int \mathcal{D} \chi \delta \Phi^A[\Phi + \chi; \Lambda_0] \times \exp\left[-\frac{1}{2} \chi \cdot (K_0 - K)^{-1} \mathcal{D} \cdot \chi - S_I[\Phi + \chi; \Lambda_0]\right]$$ \hspace{1cm} (2.23)$$

\textsuperscript{1) Later in Eq. (3.1), we write the gauge fixed action $S[\phi; \Lambda_0]$ explicitly. Thanks to renormalizability, this action has only a finite number of parameters. Our assumption is that we can tune the parameters so that Eq. (2.16) holds.}
(2.20) implies
\[ \langle \delta \Phi^A \rangle_{\Phi, K^{-1} J} = KK_0^{-1} N_J^{-1} \langle \delta \Phi^A \rangle_{\phi, K_0^{-1} J}. \] (2.24)

Replacing the field by a functional derivative with respect to a source, we obtain
\[ R^A[K \partial^J; \Lambda] Z_{\phi}[J] = KK_0^{-1} N_J^{-1} R^A[K_0 \partial^J; \Lambda_0] Z_{\phi}[J], \] (2.25)

where we denote
\[ \delta \Phi^A[\Phi; \Lambda] = R^A[\Phi; \Lambda]. \] (2.26)

Now, using (2.15), we obtain
\[ N_J^{-1} \langle \Sigma[\phi; \Lambda_0] \rangle_{\phi, K_0^{-1} J} = N_J^{-1} K_0^{-1} J \cdot R[K \partial^J; \Lambda] Z_{\phi}[J]. \] (2.27)

Hence, using (2.22) and (2.25), we obtain
\[ \langle \Sigma[\Phi; \Lambda] \rangle_{\Phi, K^{-1} J} = K^{-1} J \cdot R[K \partial^J; \Lambda] Z_{\phi}[J]. \] (2.28)

This implies
\[ \Sigma[\Phi; \Lambda] = \partial^r S[\Phi; \Lambda] \delta \Phi^A - \partial^r \partial \Phi^A \delta \Phi^A. \] (2.29)

This relation is analogous to the original WT operator (2.12), but note here that \( S \) is
the Wilson action at an IR scale \( \Lambda \), and that \( \delta \Phi^A \) are composite operators expressed
in terms of \( S \). For \( \Lambda = \Lambda_0 \), \( \Sigma[\Phi; \Lambda] \) becomes the original \( \Sigma[\phi; \Lambda_0] \), as is obvious from
the definition.

Consequently, the WT identity for the Wilson action \( S \) is given either as
\[ \Sigma[\Phi; \Lambda] = O(1/\Lambda_0^2) \] (2.30)
or
\[ K^{-1} J \cdot R[K \partial^J; \Lambda] Z_{\phi}[J] = O(1/\Lambda_0^2). \] (2.31)

Before concluding this section, we consider a particular class of BRST transformation
\[ \delta \phi^A = R^A[\phi; \Lambda_0] = K_0 \left( R^{(1)A} B(A_0) \phi^B + \frac{1}{2} R^{(2)A} B C(A_0) \phi^B \phi^C \right), \] (2.32)

which are at most quadratic in fields. The BRST transformation for YM theories belongs to this class.

In rewriting the expectation value
\[ \langle \delta \phi^A \rangle_{\phi, K_0^{-1} J} = R^A[K_0 \partial^J; \Lambda_0] Z_{\phi}[J] \]

for the IR theory, we need to compute two things. First,
\[ K_0 \frac{\partial^J}{\partial J_A} Z_{\phi}[J] = K_0 \frac{\partial^J}{\partial J_A} N_J Z_{\phi}[J] \]
\[ = N_J \left[ (-)^{A+1} \left( \frac{K_0 - K}{K} \right)^{A B} J_B + K_0 \frac{\partial^J}{\partial J_A} \right] Z_{\phi}[J] \]
\[\begin{align*}
&= N_J \left\langle K_0 K^{-1} \Phi^A - (K_0 - K) (D^{-1})^{AB} \frac{\partial^l S}{\partial \Phi^B} \right\rangle_{\Phi, K^{-1} J} \\
&= N_J \left\langle \Phi^A - (K_0 - K) (D^{-1})^{AB} \frac{\partial^l S^I}{\partial \Phi^B} \right\rangle_{\Phi, K^{-1} J} \\
&\equiv N_J \left\langle [\Phi^A]_{\text{com}} \right\rangle_{\Phi, K^{-1} J} ,
\end{align*}\]

where we define
\[\begin{align*}
[\Phi^A]_{\text{com}} &\equiv K_0 K^{-1} \Phi^A - (K_0 - K) (D^{-1})^{AB} \frac{\partial^l S}{\partial \Phi^B} \\
&= \Phi^A - (K_0 - K) (D^{-1})^{AB} \frac{\partial^l S^I}{\partial \Phi^B} .
\end{align*}\]

Second,
\[\begin{align*}
K_0^2 \frac{\partial^l}{\partial J_A} \frac{\partial^l}{\partial J_B} Z_{\Phi}[J] &= K_0^2 \frac{\partial^l}{\partial J_A} \frac{\partial^l}{\partial J_B} N_J Z_{\Phi}[J] \\
&\equiv N_J \left\langle [\Phi^A, \Phi^B]_{\text{com}} \right\rangle_{\Phi, K^{-1} J} ,
\end{align*}\]

where we define
\[\begin{align*}
[\Phi^A, \Phi^B]_{\text{com}} &\equiv [\Phi^A]_{\text{com}} [\Phi^B]_{\text{com}} \\
&- (K_0 - K) (D^{-1})^{AC} (K_0 - K) (D^{-1})^{BD} \frac{\partial^l \partial^l S^I}{\partial \Phi^C \partial \Phi^D} .
\end{align*}\]

It is important to note that nontrivial contributions arise from derivatives \( \partial_J \) acting on the normalization factor \( N_J \).

Hence, using (2.25), we obtain
\[\begin{align*}
\left\langle \delta \Phi^A \right\rangle_{\Phi, K^{-1} J} &= K \left\langle \mathcal{R}^{(1)}_{B} (A_0) [\Phi^B]_{\text{com}} + \frac{1}{2} \mathcal{R}^{(2)}_{BC} (A_0) [\Phi^B \Phi^C]_{\text{com}} \right\rangle_{\Phi, K^{-1} J} .
\end{align*}\]

Since this is valid for arbitrary \( J \), we obtain the operator equality
\[\begin{align*}
\delta \Phi^A &= K \left( \mathcal{R}^{(1)}_{B} (A_0) [\Phi^B]_{\text{com}} + \frac{1}{2} \mathcal{R}^{(2)}_{BC} (A_0) [\Phi^B \Phi^C]_{\text{com}} \right) .
\end{align*}\]

It is important to stress the necessity of the cutoff function \( K_0 \) to make (2.38) UV finite.\(^\ast\)

\section*{§3. Pure Yang-Mills theory: WT identity and BRST transformation}

Let us find the explicit form of \( \Sigma[\Phi; A] \) for the \( SU(2) \) pure Yang-Mills theory. We use the following notations:

\begin{align*}
\text{UV fields} &\equiv \{ a^a_\mu, b^a, c^a, \bar{c}^a \} , \\
\text{Source terms} &\equiv \{ J^a_\mu, J^a_B, J^a_c, J^a_\bar{c} \} .
\end{align*}

\(^\ast\) The potential UV divergence is hard to see in the matrix notation. It is hidden in the loop momentum integral contained in \( \mathcal{R}^{(2)}_{BC} [\Phi^B \Phi^C]_{\text{com}} \). In the next section, where we apply the above results to a pure YM theory, we will elaborate on this point. See the remark right after (3.10).
As a UV action we take
\[ S[\phi; \Lambda_0] = \frac{1}{2} \phi \cdot K_0^{-1} D \cdot \phi + S_I[\phi; \Lambda_0], \tag{3.1} \]
where
\[ \frac{1}{2} \phi \cdot K_0^{-1} D \cdot \phi = \int_p K_0^{-1}(p) \left[ \frac{1}{2} a_\mu(-p) \cdot (p^2 \delta_{\mu\nu} - p_\mu p_\nu) a_\nu(p) \\
+ \bar{c}(-p)i p^2 \cdot c(p) - b(-p) \cdot (i p_\mu a_\mu(p) + \frac{\xi}{2} b(p)) \right] \]
and
\[ S_I[\phi; \Lambda_0] = \int_p \left[ \frac{a_q}{2} A_0^q a_\mu(-p) a_\mu(p) + \frac{z_1}{2} p^2 a_\mu(p) a_\mu(-p) + \frac{z_2}{2} p_\mu p_\nu a_\mu(-p) a_\nu(p) \right] \\
+ z_3 \int_{p,q} p_\nu a_\mu(-p) \cdot [a_\mu(q) \times a_\nu(p - q)] \\
+ \frac{z_4}{8} \int_{p_1, \ldots, p_4} \delta \left( \sum p_i \right) a_\mu(p_1) \cdot a_\mu(p_2) a_\nu(p_3) \cdot a_\nu(p_4) \\
+ \frac{z_5}{8} \int_{p_1, \ldots, p_4} \delta \left( \sum p_i \right) a_\mu(p_1) \cdot a_\nu(p_2) a_\mu(p_3) \cdot a_\nu(p_4) \\
- \int_p \left[ p_\mu \bar{c}(-p) \cdot (-i) z_6 p_\mu c(p) + z_7 \int_q a_\mu(p - q) \times c(q) \right]. \tag{3.2} \]
The BRST transformation is given by
\[ \delta a_\mu(p) = K_0(p) \left[ (1 + z_6 K_0(p))(-i)p_\mu c(p) + z_7 K_0(p) \int_q a_\mu(p - q) \times c(q) \right], \]
\[ \delta \bar{c}(p) = i K_0(p) b(p), \]
\[ \delta c(p) = K_0(p) \frac{z_8}{2} \int_q c(q) \times c(p - q), \tag{3.3} \]
where we have chosen \( \delta a_\mu(p) \) so that
\[ p_\mu \delta a_\mu(p) = -K_0(p)^2 \frac{\partial}{\partial \bar{c}(-p)} S[\phi; \Lambda_0] \tag{3.4} \]
is the operator for the ghost equation of motion. Note \( z_8 \) is an independent parameter that does not appear in the action. We tune the eight \( z \) coefficients to satisfy the WT identity.**

Before proceeding further, we would like to comment on the particular form of \( S[\phi; \Lambda_0] \) in Eq. (3.2). The functional integration to obtain the Wilson action \( S_I[\Phi; \Lambda] \)

*) For the SU(2) group adjoint indices we use the notation \( A \cdot B = A^a B^a \) and \( (A \times B)^a = \epsilon^{abc} A^b B^c \).

**) The logarithmic dependence of \( z \)'s is determined by renormalizability. It is the part independent of \( \Lambda_0 \) that must be tuned.
from the bare action $S[\phi; A_0]$ generates terms that depend on the UV cutoff $A_0$. We choose the $A_0$ dependence of $S_f[\phi; A_0]$ such that the Wilson action $S_f[\Phi; A]$ for any finite $A$ has a limit as we take $A_0$ to infinity. The possibility of making such a choice amounts to renormalizability of the theory, and we only need to keep a finite number of those terms whose mass dimensions are four or less. In particular the gauge mass term in the bare action (3.2) is necessary in order to get a finite gauge mass term in the Wilson action, and the gauge mass term of the Wilson action is necessary to cancel the contribution from the Jacobian (the second term of the right-hand of (2.29)) to the WT operator. For the role of the gauge mass term in the Wilson action of QED, we refer the reader to Ref. 18.

Now let us compute

$$\left(K_0^{-1} J \cdot R[K_0 \partial J, A_0]\right) Z_\phi[J] = N_J^{-1} \left(K_0^{-1} J \cdot R[K_0 \partial J, A_0]\right) N_J Z_\Phi[J], \quad (3.5)$$

where the Slavnov operator is given by

$$K_0^{-1} J \cdot R[K_0 \partial J, A_0] = \int_p J_\mu(-p) \cdot \left\{ (1 + z_0 K_0(p))(-i p_\mu)K_0(p) \frac{\partial^l}{\partial J_c(-p)} + z_7 K_0(p) \int_k K_0(p - k)K_0(k) \frac{\partial^l}{\partial J_\mu(-p + k)} \times \frac{\partial^l}{\partial J_c(-k)} \right\}$$

$$+ \int_p \left[J_c(-p) \cdot i K_0(p) \frac{\partial^l}{\partial J_B(-p)} + \frac{z_8}{2} J_c(-p) \int_k K_0(p - k)K_0(k) \frac{\partial^l}{\partial J_c(-p + k)} \times \frac{\partial^l}{\partial J_c(-k)}\right]. \quad (3.6)$$

For the pure YM case, the source dependent normalization factor $N_J$ takes the form

$$\ln N_J = -\frac{(-)^{\xi A}}{2} \int_p \left[K_0 - K \right] (D^{-1})^{AB} J_B$$

$$= \int_p \left[K_0 - K \right] (D^{-1}) \left(J_c(-p)\frac{-i}{p^2} J_c(p) - J_B(-p)\frac{-i p_\mu}{p^2} J_\mu(p)\right)$$

$$- \frac{1}{2} J_\mu(-p) \frac{1}{p^2} \left(\delta_{\mu\nu} - (1 - \xi) \frac{p_\mu p_\nu}{p^2}\right) J_\nu(p). \quad (3.7)$$

It is easy to see that the derivatives with respect to $J_A$ in (3.6) give rise to composite operators. The following composite operators appear in the WT identity:

$$[A_\mu(p)]_{\text{com}} \equiv A_\mu(p) - \frac{K_0(p) - K(p)}{p^2} \left(\delta_{\mu\nu} - (1 - \xi) \frac{p_\mu p_\nu}{p^2}\right) \frac{\partial S_I}{\partial A_\nu(-p)}$$

$$[B(p)]_{\text{com}} \equiv B(p) + i \frac{K_0(p) - K(p)}{p^2} \frac{\partial S_I}{\partial A_\mu(-p)}$$

$$[C(p)]_{\text{com}} \equiv C(p) + i \frac{K_0(p) - K(p)}{p^2} \frac{\partial S_I}{\partial C(-p)}$$

$$[A_\mu(q) \times C(p - q)]_{\text{com}} \equiv [A_\mu(q)]_{\text{com}} \times [C(p - q)]_{\text{com}}$$
We obtain the WT identity Eq. (2.29). To be concrete, we have
\[ \Lambda \]
with the BRST transformation given as
\[ q \]
Here, the integrals over \( \Lambda \) cutoff that of these integrals over \( q \). Note that the logarithmic dependence of \( z_0, z_7, z_8 \) on \( A_0 \) is chosen to cancel that of these integrals over \( q \). Hence, we can define further the composite operators
\[ [(A_\mu \times C)(p)]_{com} \equiv -iz_6p_\mu [C(p)]_{com} + z_7 \int_q [A_\mu(q) \times C(p-q)]_{com}, \]
\[ [(C \times C)(p)]_{com} \equiv z_8 \int_q [C(q) \times C(p-q)]_{com}, \]
which have a limit as \( A_0 \to \infty \). Then, in this limit we obtain \( \Sigma[\Phi, A] = 0 \) with
\[
\begin{cases}
\delta A_\mu(p) & = K(p)(-ip_\mu[C(p)]_{com} + [(A_\mu \times C)(p)]_{com}), \\
\delta C(p) & = K(p)i[B(p)]_{com}, \\
\delta C(p) & = K(p)\frac{1}{2} [(C \times C)(p)]_{com}.
\end{cases}
\]
nonlinear even classically, the deformation of the symmetry due to the presence of a momentum cutoff becomes highly nontrivial. In our functional method, the deformation of the BRST symmetry is described by some contributions generated by the Slavnov operator acting on the normalization factor which is quadratic in sources. The Slavnov operator is characterized by the first and second functional derivatives of sources, and generates particular combinations of factors with functional derivatives of the Wilson action. They satisfy the ERG flow equations for the composite operators. In our functional method, however, we need to introduce a UV cutoff $\Lambda_0$ in addition to an IR cutoff $\Lambda$ to make the WT identity well-defined. This is the price we pay for insisting on expressing the nonlinear BRST transformation explicitly in terms of the Wilson action. Alternatively, we can construct the composite operators by using their flow equations without introducing $\Lambda_0$, as discussed in 25) and 26). These two approaches will be equivalent at least within the framework of perturbation theory.

It should be remarked that even in the QED case, the BRST transformation for the Wilson action is not nilpotent. This observation motivated us to elevate the WT identity to QME in the Batalin-Vilkovisky formalism\(^{(28),29)}\) in our earlier paper.\(^{(19)}\) Including the antifield contributions, the nilpotency is recovered, and the BRST invariance of the system becomes easy to see. Naturally, we have a similar situation here for YM theory: the BRST transformation in Eq. (3.10) is not nilpotent. So our next immediate task is to reformulate the theory in the BV formalism.

The general ERG formalism guarantees the existence of QME for YM theory. In fact in the approach by Becchi\(^{(25)}\) in which the renormalized theory is constructed directly without introducing an UV theory at scale $\Lambda_0$, a quantum master action satisfying QME has been constructed.\(^{(26)}\) But Becchi’s approach applies only to perturbation theory. In a forthcoming paper,\(^{(30)}\) we plan to construct a quantum master action with both UV and IR cutoffs. The advantage of this construction is its applicability beyond perturbation theory.

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