Novel Decision Making Framework Based on Complex $q$-Rung Orthopair Fuzzy Information

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Abstract

Assessing uncertainty in decision-making is a major challenge for decision makers (DMs) and the $q$-rung orthopair fuzzy set ($q$-ROFS), as the direct extension of intuitionistic fuzzy set (IFS) and pythagorean fuzzy set (PFS) play a crucial role in this aspect. The complex $q$-rung orthopair fuzzy set ($C_q$-ROFS) is a strong tool to deal with imprecision, vagueness and fuzziness by expanding the scope of membership degree (MD) and non-membership degree (NMD) of $q$-ROFS from real to complex unit disc. In this paper, we develop some new $C_q$-ROF Hamacher aggregation operators (AOs), i.e., the $C_q$-ROF weighted averaging ($C_q$-ROFWHA) operator, the $C_q$-ROFH weighted geometric ($C_q$-ROFHWG) operator, the $C_q$-ROFH ordered weighted averaging ($C_q$-ROFHOWA) operator and the $C_q$-ROFH ordered weighted geometric ($C_q$-ROFHOWG) operator. Subsequently, we establish a novel $C_q$-ROFH graph framework based on the Hamacher operator called $C_q$-ROFH graphs ($C_q$-ROFHGs) and evaluate its energy and Randić energy. In particular, we compute the energy of a splitting $C_q$-ROFHG and shadow $C_q$-ROFHG. Further, we describe the notions of $C_q$-ROFH digraphs ($C_q$-ROFHDGs). Moreover, an algorithm is given to solve multiple attribute group decision making (MAGDM) problems and the main steps are discussed clearly. Finally, a numerical instance related to the facade clothing systems (FCS) selection is presented to show the effectiveness of the developed concepts in decision-making circumstances. In order to verify the effectiveness of our proposed scheme, a comparative analysis with previous approaches is provided.

Keywords: Complex $q$-rung orthopair fuzzy set; Complex $q$-rung orthopair fuzzy graph; Hamacher operator; Energy; Randić energy, Facade clothing system.

1 Introduction

In the real world, as a complex cognitive computing method, multiple attribute decision making (MADM) intends to make scientific decisions from a finite number of alternatives by utilizing the multiple attributes perspective, and then rank a series of alternatives or pursue the appropriate one through effective information aggregation rules and decision analysis tools. A lot of soft computing methods have been used for MADM research over the last decades, and most of them are addressed in the form of generalized fuzzy sets (FSs) [1, 2, 3, 4, 5]. MAGDM, an integrative research field that combines group decision-making with MADM, usually provides structures for acquiring group preference information through individual preference information and specifically evaluating various alternatives through different theoretical decision-making templates. For some MAGDM problems, DMs experience problems in depicting attribute values of alternatives by utilizing crisp numbers. To describe the uncertainties, a novel notion of FS has been initiated by Zadeh [6] whose element has only MD in $[0, 1]$. Further IFSs [7], PFSs [8] and Fermatean FSs (FFSs) [9], whose elements are pairs of
fuzzy numbers, have been proposed. Each of them demonstrates the MD and NMD. The restriction of MD and NMD is that, the sum and square sum of both, would be belong to [0,1]. Yager revealed that the current frameworks of IFSs and PFSs are not capable enough to represent human opinion in a more realistic setting and has invented the concept of $q$-ROFS [10] which effectively broadens the scope of information by developing a new subjective constraints, where the $q$th sum of MD and NMD belongs to $[0,1]$. If $q = 3$, $q = 2$, and $q = 1$, then the $q$-ROFS is reduced to the FFS, PFS and IFS, respectively.

If DMs change the codomain of FSs from [0,1] to unit disc, then to tackle certain type of problems, Ramot et al. [11] design the idea of complex fuzzy sets (CFSs) expressed by complex valued mapping with codomain as unit circle in the complex plane. Moreover, to represent the complex valued NMD, Alkouri and Salleh [12, 13] generalized the idea of CFS to complex IFSs (CIFs) and proposed the concept of CIF relations and a distance measure in CIF circumstances. Further, Rani and Garg set forward the CIF robust averaging-geometric AOs [14], explored certain series of distance measures between the two CIFs [15], Archimedean t-norm and t-conorm based generalized CIF Bonferroni mean AOs [16], exponential, logarithmic and compensative generalized AOs with CIF information [17], CIF power AOs [18] and presented their applications in the field of decision-making. The idea of $q$-ROFS deals with only one aspect at a time, that sometimes causes data loss. In real life, however, we notice complex natural phenomena where in measuring the second dimension of the expression of the MD and NMD becomes essential. Complete facts are projected into a collection by creating the second dimension, which prevents any loss of information. Liu et al. [19] put forward an efficient and powerful tool to express unclear anomalies, called Cq-ROFSs and developed the weighted averaging operator and weighted geometric operator based on Cq-ROF circumstances. The amplitude term represents the degree to which an object belongs in a Cq-ROF and the phase terms are usually related to periodicity. The CS and traditional $q$-ROFs theories are differentiated by certain phase terms. Garg et al. [20] presented multiple forms of operators such as power averaging, power weighted averaging, hybrid averaging, power geometric, power weighted geometric, and power hybrid geometric operator in the context of Cq-ROFSs. Liu et al. [21] introduced the concept of Cq-RO linguistic (Cq-ROL) sets and developed operators like the Cq-ROL Heronian mean, Cq-ROL weighted geometric Heronian mean, and Cq-ROL weighted geometric Heronian mean operator. Cq-ROFs are incredibly versatile and efficient as compared to many current FSs theories.

Graphs can be used to design numerous types of relations and methodologies of physical, biological, social and information technology, and has a massive variety of valuable applications. Graph theory is eventually the study of relationships and offers a useful resource to quantify and simplify the several components of dynamic systems. Studying graphs in a system offers responses to a broad variety of configurations, networking, optimization, matching and operational issues. Graph properties in relation to the characteristic polynomial and matrix values associated with the graph are studied in spectral graph theory, such as its adjacency matrix, Harmonic matrix, Zagreb matrix and geometric-arithmetic matrix. The idea of the graph energy was set up by Gutman [22] and explore lower and upper limits. Vaidya and Popat [23] proposed the energy of splitting and shadow graphs. The Randić matrix $R(G) = (\eta_{ij})$ of a graph $G$ whose vertex $\eta_i$ has degree $d_i$ is defined by $\eta_{ij} = \frac{1}{\sqrt{d_i d_j}}$ if the vertices $\eta_i$ and $\eta_j$ are adjacent and $\eta_{ij} = 0$, otherwise. The sum of absolute values of the eigenvalues of $R(G)$ is the Randić energy. Graph vertices and edges uncertainties are typical in this context due to a number of factors, such as noise measurements and conflicting sources of information. In order to deal with such complexities in objects and connections, Rosenfeld [24] originated the concept of fuzzy graphs (FGs) and established its framework. Anjali and Mathew [25] set forward the energy of a FG. Akram et al. [26, 27, 28] developed several novel concepts of graphs in generalized fuzzy circumstances. Thirunavukarasu et al. [29] determine the energy of complex fuzzy graphs (CFSs). Luqman et al. developed the concepts of complex fuzzy hypergraphs [30] and complex neutrosophic hypergraphs [31]. Naz et al. put forward the concepts of Pythagorean fuzzy graphs (PFGs) [32] and complex PFs [33] as well as its pertinent applications in decision making. Akram et al. [34] designed a new definition of $q$-ROF graphs ($q$-ROFGs) and presented its use in the soil ecosystem. Yin et al. [35] elaborated some product operations on $q$-ROFGs. Further, Akram et al. [36] presented $q$-runge orthopair fuzzy graphs under Hamacher operators. Moreover, Guleria and Bajaj [37] designed the notion of T-spherical FGs along with the operations. Recently, Naz et al. [38, 39] extended $q$-ROFGs to the dual hesitant $q$-ROF scenario and proposed several types of energy like geometric arithmetic energy, atom bond connectivity energy, Zegrab energy and harmonic energy.

Information AO performs a significant role in the decision-making process, especially in MADM. In 1978, Hamacher [40] introduced the Hamacher operations like Hamacher product and Hamacher sum, which are more
and more general and flexible as compared to the algebraic and Einstein product and sum. Inspired by the theory of $q$-ROFGs, it is essential to expand $q$-ROFG to complex $q$-ROFG (C$q$-ROFG), since C$q$-ROFG is a powerful concept for dealing with uncertain and unpredictable information, and it is also a general form of FGs, whose restriction is just like $q$-ROFG, however the MD and NMD range are bounded to unit disc in a complex plane rather than $[0,1]$. Moreover, Hamacher operators are more flexible and parameterized. So, we define the C$q$-ROF relations, and put forward the innovative concept of C$q$-ROFGs utilizing Hamacher operator. The newly proposed C$q$-ROF Hamacher graphs (C$q$-ROFHGs) are extremely versatile and efficient and can coordinate the expert decision-making opinions in a complex state compared to many existing FSs theories. We also establish the energy and Randić energy of the developed C$q$-ROFHGs and C$q$-ROF Hamacher digraphs (C$q$-ROFHDGs) and provide their pertinent application in MAGDM.

The format of the paper is as follows. Section 2 reviews some fundamental concepts of C$q$-ROFSs. Section 3 puts forward some C$q$-ROF Hamacher AOs. Section 4 proposes the innovative idea of C$q$-ROFHGs and C$q$-ROFHDGs, and examines their energy. The concept of splitting C$q$-ROFHG and shadow C$q$-ROFHG with their energies are also discussed in this section. Section 5 refers to Randić energy of C$q$-ROFHGs and C$q$-ROFHDGs. Further in Section 6, a novel MAGDM approach is established based on energy and Randić energy of C$q$-ROFHGDs. In Section 7, a case study and, an appropriate comparative analysis are discussed to illustrate the usefulness and effectiveness of the established ideas of C$q$-ROFHGs in decision making. Finally, Section 8 concludes the entire paper and points out several future research topics. The graphical interpretation of the paper is given in Fig. 1.
2 Preliminaries

In this section, the basic notions like Cq-ROFSs along its operations and t-norms are reviewed to better understand the next sections.

Definition 2.1. [41] A Cq-ROFS \( \mathcal{L} \) is defined as:

\[
\mathcal{L} = \{(s, \tilde{\tau}_E(s), \tilde{\varrho}_E(s)) : s \in R\},
\]

where \( \tilde{\tau}_E, \tilde{\varrho}_E : R \rightarrow \{c : c \in C, |c| \leq 1 \} \) are the complex-valued membership and non-membership functions, respectively and defined as:

\[
\begin{align*}
\tilde{\tau}_E(s) &= \tilde{\varrho}_E(s)e^{i2\pi \omega_\varrho(s)}; \\
\tilde{\varrho}_E(s) &= \tilde{R}_E(s)e^{i2\pi \omega_{\tilde{R}_E}(s)},
\end{align*}
\]

where \( 0 \leq \tilde{\varrho}_E(s), \tilde{R}_E(s), \tilde{\varrho}_E(s) + \tilde{R}_E(s) \leq 1 \) and \( 0 \leq \omega_\varrho(s), \omega_{\tilde{R}_E}(s), \omega_{\tilde{\varrho}_E}(s) + \omega_{\tilde{R}_E}(s) \leq 1 \). Further, \( \tilde{\tau}_E(s) = (1 - (\tilde{\varrho}_E(s) + \tilde{R}_E(s)))^{\frac{1}{\omega_\varrho}} \) and \( \omega_{\tilde{\varrho}_E}(s) = (1 - (\omega_{\tilde{\varrho}_E}(s) + \omega_{\tilde{R}_E}(s)))^{\frac{1}{\omega_{\tilde{R}_E}}} \) are complex hesitancy degree of \( s \). For simplicity, the pair \( \tilde{\mathcal{I}} = (\tilde{\varrho}(s, \omega_\varrho), (\tilde{R}_E(s, \omega_{\tilde{R}_E})) \) is called the Cq-ROF number (Cq-ROFN), where \( 0 \leq \tilde{\varrho}, \tilde{R}, \tilde{\varrho} + \tilde{R} \leq 1 \), and \( 0 \leq \omega_\varrho, \omega_{\tilde{R}_E}, \omega_{\tilde{\varrho}_E} + \omega_{\tilde{R}_E} \leq 1 \).

Definition 2.2. [41] Let \( \mathcal{L} = \{(s, (\tilde{\varrho}_E(s), \omega_{\tilde{R}_E}(s)), (\tilde{R}_E(s), \omega_{\tilde{\varrho}_E}(s))) : s \in R\}, \mathcal{L}_1 = \{(s, (\tilde{\varrho}_E(s), \omega_{\tilde{R}_E}(s)), (\tilde{R}_E(s), \omega_{\tilde{\varrho}_E}(s))) : s \in R\} \) be the Cq-ROFSs in \( R \), then

(i) \( \mathcal{L}_1 \subseteq \mathcal{L}_2 \) if and only if \( \tilde{\varrho}_E(s) \leq \tilde{\varrho}_E(s), \tilde{R}_E(s) \geq \tilde{R}_E(s) \) for amplitude terms and \( \omega_{\tilde{R}_E}(s) \leq \omega_{\tilde{R}_E}(s), \omega_{\tilde{\varrho}_E}(s) \geq \omega_{\tilde{\varrho}_E}(s) \) for phase terms, for all \( s \in R \); 

(ii) \( \mathcal{L}_1 = \mathcal{L}_2 \) if and only if \( \tilde{\varrho}_E(s) = \tilde{\varrho}_E(s), \tilde{R}_E(s) = \tilde{R}_E(s) \) for amplitude terms and \( \omega_{\tilde{R}_E}(s) = \omega_{\tilde{R}_E}(s), \omega_{\tilde{\varrho}_E}(s) = \omega_{\tilde{\varrho}_E}(s) \) for phase terms, for all \( s \in R \); 

(iii) \( \overline{\mathcal{L}} = \{(s, (\tilde{R}_E(s), \omega_{\tilde{\varrho}_E}(s)), (\tilde{\varrho}_E(s), \omega_{\tilde{R}_E}(s))) : s \in R\} \).

Definition 2.3. [42] Let \( \mathcal{L} = \{(s, (\tilde{\varrho}_E(s), \omega_{\tilde{R}_E}(s)), (\tilde{R}_E(s), \omega_{\tilde{\varrho}_E}(s))) : s \in R\}, \mathcal{L}_1 = \{(s, (\tilde{\varrho}_E(s), \omega_{\tilde{R}_E}(s)), (\tilde{R}_E(s), \omega_{\tilde{\varrho}_E}(s))) : s \in R\}, \mathcal{L}_2 = \{(s, (\tilde{\varrho}_E(s), \omega_{\tilde{R}_E}(s)), (\tilde{R}_E(s), \omega_{\tilde{\varrho}_E}(s))) : s \in R\} \) be the Cq-ROFSs in \( R \), then

1. \( \mathcal{L}_1 \oplus \mathcal{L}_2 = \left( \sqrt[\lambda]{
\frac{\tilde{\varrho}_E(s) + \tilde{R}_E(s) - \tilde{\varrho}_E(s) \tilde{R}_E(s)}{\omega_{\varrho}\tilde{R}_E(s) + \omega_{\tilde{\varrho}_E}(s)}
, \sqrt[\lambda]{
\frac{\tilde{\varrho}_E(s) + \tilde{R}_E(s) - \tilde{\varrho}_E(s) \tilde{R}_E(s)}{\omega_{\varrho}\tilde{R}_E(s) + \omega_{\tilde{\varrho}_E}(s)}
} \right) ;
\)

2. \( \mathcal{L}_1 \otimes \mathcal{L}_2 = \left( \sqrt[\lambda]{
\frac{\tilde{\varrho}_E(s) \tilde{R}_E(s) - \tilde{\varrho}_E(s) \tilde{R}_E(s)}{\omega_{\varrho}\tilde{R}_E(s) + \omega_{\tilde{\varrho}_E}(s)}
, \sqrt[\lambda]{
\frac{\tilde{\varrho}_E(s) \tilde{R}_E(s) - \tilde{\varrho}_E(s) \tilde{R}_E(s)}{\omega_{\varrho}\tilde{R}_E(s) + \omega_{\tilde{\varrho}_E}(s)}
} \right) ;
\)

3. \( \mathcal{L} \lambda = \left( \sqrt[\lambda]{1 - (1 - \tilde{\varrho}_E(s))\lambda}, \sqrt[\lambda]{1 - (1 - \tilde{R}_E(s))\lambda} \right), \lambda > 0; \)

4. \( \mathcal{L} \lambda = \left( (\tilde{\varrho}_E(s))\lambda, (\omega_{\tilde{R}_E}(s))\lambda \right), \lambda > 0. \)

Example 2.1. Suppose a fixed set \( R \) has only one element \( s \), \( \tilde{\varrho}_E(s) = 0.5, \omega_{\tilde{\varrho}_E}(s) = 0.8, \tilde{R}_E(s) = 0.8, \omega_{\tilde{R}_E}(s) = 0.9 \). Then \( \overline{\mathcal{I}} = \{(s, (0.5, 0.8), (0.8, 0.9))\} \) is a C5-ROFN, represented as \( \overline{\mathcal{I}} = ((0.5, 0.8), (0.8, 0.9)) \) for simplicity.

Definition 2.4. The score function \( F \) and the accuracy function \( \mathcal{H} \) of a Cq-ROFN \( \overline{\mathcal{I}} = ((\tilde{\varrho}, \omega_\varrho), (\tilde{R}_E, \omega_{\tilde{R}_E})) \), are defined as \( F(\overline{\mathcal{I}}) = \frac{1}{2}((1 + \tilde{\varrho} - \tilde{R}) + (1 + \omega_\varrho - \omega_{\tilde{R}_E})) \), \( F(\overline{\mathcal{I}}) \in [0, 1] \) and \( \mathcal{H}(\overline{\mathcal{I}}) = (\tilde{\varrho} + \tilde{R}) + (\omega_\varrho + \omega_{\tilde{R}_E}) \), \( \mathcal{H}(\overline{\mathcal{I}}) \in [0, 1] \), respectively.

Definition 2.5. Let \( \overline{\mathcal{I}}_1 = ((\tilde{\varrho}_1, \omega_{\tilde{\varrho}_1}), (\tilde{R}_1, \omega_{\tilde{R}_1})) \) and \( \overline{\mathcal{I}}_2 = ((\tilde{\varrho}_2, \omega_{\tilde{\varrho}_2}), (\tilde{R}_2, \omega_{\tilde{R}_2})) \) be two Cq-ROFNs, then
1. if $F(\tilde{y}_1) > F(\tilde{y}_2)$, then $\tilde{y}_1 > \tilde{y}_2$;
2. if $F(\tilde{y}_1) = F(\tilde{y}_2)$, then
   - if $\tilde{y}_1 > \tilde{y}_2$, then $\tilde{y}_1 > \tilde{y}_2$;
   - if $\tilde{y}_1 = \tilde{y}_2$, then $\tilde{y}_1 = \tilde{y}_2$.

To extend the existing t-norm (TN) and t-conorm (TCN) operations, the Hamacher product and the
Hamacher sum defined by Hamacher [40] are as follows:

\[
T^H_N(s, t) = \begin{cases} \frac{st}{s+t-st} & \text{if } N > 0, \\ \frac{s+t-st}{s+t} & \text{if } N = 0, \end{cases}
\]

\[
(T^s)^H_N(s, t) = \begin{cases} \frac{s+t-st-(1-N)st}{s+t-st} & \text{if } N > 0, \\ \frac{s-t-st}{s+t-st} & \text{if } N = 0. \end{cases}
\]

Here $P(s, t) \leq \frac{st}{s+t-st} \leq M(s, t)$ and $M^*(s, t) \leq \frac{s+t-2st}{s+t-st} \leq P^*(s, t)$.

3 Complex q-rung orthopair fuzzy Hamacher aggregation operators

Some Hamacher operations, that is, the Hamacher product and the Hamacher sum of two Cq-ROFNs $\tilde{y}_1$ and $\tilde{y}_2$, $N > 0$, are defined as follows:

**Definition 3.1.** Let $\tilde{y} = ((\tilde{y}_1, \omega_{y_1}), \tilde{R}, \omega_{\tilde{R}})$, $\tilde{y}_1 = ((\tilde{y}_1, \omega_{y_1}), (\tilde{R}_1, \omega_{\tilde{R}_1}))$, and $\tilde{y}_2 = ((\tilde{y}_2, \omega_{y_2}), (\tilde{R}_2, \omega_{\tilde{R}_2}))$ be the Cq-ROFNs, then their basic Hamacher operations can be defined as:

1. $\tilde{y}_1 \oplus \tilde{y}_2 = \left(\frac{\sqrt[3]{\tilde{R}_1 \tilde{R}_2^{(1+\omega_{\tilde{R}_2}^{(1+\omega_{\tilde{R}_2}^{(1+\omega_{\tilde{R}_2}^{(1+\omega_{\tilde{R}_2}^{1)})})})}} \omega_{y_1} \omega_{y_2}}{\sqrt[3]{\tilde{R}_1 \tilde{R}_2^{(1+\omega_{\tilde{R}_2}^{(1+\omega_{\tilde{R}_2}^{(1+\omega_{\tilde{R}_2}^{1)})})}} \omega_{y_1} \omega_{y_2}}, \frac{\sqrt[3]{\tilde{R}_1 \tilde{R}_2^{(1+\omega_{\tilde{R}_2}^{(1+\omega_{\tilde{R}_2}^{(1+\omega_{\tilde{R}_2}^{1)})})}} \omega_{y_1} \omega_{y_2}}{\sqrt[3]{\tilde{R}_1 \tilde{R}_2^{(1+\omega_{\tilde{R}_2}^{(1+\omega_{\tilde{R}_2}^{(1+\omega_{\tilde{R}_2}^{1)})})}} \omega_{y_1} \omega_{y_2}}\right)$, $N > 0$;

2. $\tilde{y}_1 \otimes \tilde{y}_2 = \left(\frac{\tilde{R}_1 \tilde{R}_2^{(1+\omega_{\tilde{R}_2}^{(1+\omega_{\tilde{R}_2}^{(1+\omega_{\tilde{R}_2}^{1)})})}} \omega_{y_1} \omega_{y_2}}{\sqrt[3]{\tilde{R}_1 \tilde{R}_2^{(1+\omega_{\tilde{R}_2}^{(1+\omega_{\tilde{R}_2}^{(1+\omega_{\tilde{R}_2}^{1)})})}} \omega_{y_1} \omega_{y_2}}, \frac{\tilde{R}_1 \tilde{R}_2^{(1+\omega_{\tilde{R}_2}^{(1+\omega_{\tilde{R}_2}^{(1+\omega_{\tilde{R}_2}^{1)})})}} \omega_{y_1} \omega_{y_2}}{\sqrt[3]{\tilde{R}_1 \tilde{R}_2^{(1+\omega_{\tilde{R}_2}^{(1+\omega_{\tilde{R}_2}^{(1+\omega_{\tilde{R}_2}^{1)})})}} \omega_{y_1} \omega_{y_2}}\right)$, $N > 0$;

3. $\lambda \tilde{y}_1 = \left(\frac{\psi \tilde{R}_1^{(1+\omega_{\tilde{R}_2}^{(1+\omega_{\tilde{R}_2}^{(1+\omega_{\tilde{R}_2}^{1)})})}} \psi \tilde{R}_1^{(1+\omega_{\tilde{R}_2}^{(1+\omega_{\tilde{R}_2}^{(1+\omega_{\tilde{R}_2}^{1)})})}} \omega_{y_1} \omega_{y_2}}{\sqrt[3]{\tilde{R}_1 \tilde{R}_2^{(1+\omega_{\tilde{R}_2}^{(1+\omega_{\tilde{R}_2}^{(1+\omega_{\tilde{R}_2}^{1)})})}} \omega_{y_1} \omega_{y_2}}, \frac{\psi \tilde{R}_1^{(1+\omega_{\tilde{R}_2}^{(1+\omega_{\tilde{R}_2}^{(1+\omega_{\tilde{R}_2}^{1)})})}} \psi \tilde{R}_1^{(1+\omega_{\tilde{R}_2}^{(1+\omega_{\tilde{R}_2}^{(1+\omega_{\tilde{R}_2}^{1)})})}} \omega_{y_1} \omega_{y_2}}{\sqrt[3]{\tilde{R}_1 \tilde{R}_2^{(1+\omega_{\tilde{R}_2}^{(1+\omega_{\tilde{R}_2}^{(1+\omega_{\tilde{R}_2}^{1)})})}} \omega_{y_1} \omega_{y_2}}\right)$, $N > 0$;

4. $\tilde{y}_1^\lambda = \left(\frac{\psi \tilde{R}_1^{(1+\omega_{\tilde{R}_2}^{(1+\omega_{\tilde{R}_2}^{(1+\omega_{\tilde{R}_2}^{1)})})}} \psi \tilde{R}_1^{(1+\omega_{\tilde{R}_2}^{(1+\omega_{\tilde{R}_2}^{(1+\omega_{\tilde{R}_2}^{1)})})}} \omega_{y_1} \omega_{y_2}}{\sqrt[3]{\tilde{R}_1 \tilde{R}_2^{(1+\omega_{\tilde{R}_2}^{(1+\omega_{\tilde{R}_2}^{(1+\omega_{\tilde{R}_2}^{1)})})}} \omega_{y_1} \omega_{y_2}}, \frac{\psi \tilde{R}_1^{(1+\omega_{\tilde{R}_2}^{(1+\omega_{\tilde{R}_2}^{(1+\omega_{\tilde{R}_2}^{1)})})}} \psi \tilde{R}_1^{(1+\omega_{\tilde{R}_2}^{(1+\omega_{\tilde{R}_2}^{(1+\omega_{\tilde{R}_2}^{1)})})}} \omega_{y_1} \omega_{y_2}}{\sqrt[3]{\tilde{R}_1 \tilde{R}_2^{(1+\omega_{\tilde{R}_2}^{(1+\omega_{\tilde{R}_2}^{(1+\omega_{\tilde{R}_2}^{1)})})}} \omega_{y_1} \omega_{y_2}}\right)$, $N > 0$.

Utilizing the Hamacher operations among the Cq-ROFNs, in this section, we develop the weighted averaging and geometric Hamacher AOs with Cq-ROFS, such as Cq-ROFHWA operator, Cq-ROF Hamacher weighted ordered averaging (Cq-ROFHWA) operator, Cq-ROFHWG operator, and Cq-ROF Hamacher ordered weighted geometric (Cq-ROFHOWG) operator.

**Definition 3.2.** Consider $\tilde{y}_j = ((\tilde{y}_j, \omega_{y_j}), (\tilde{R}_j, \omega_{\tilde{R}_j})) (j = 1, 2, \ldots, n)$ is a collection of Cq-ROFNs, then the Cq-ROFHWA operator is described as:

$$Cq-ROFHWA_w(\tilde{y}_1, \tilde{y}_2, \ldots, \tilde{y}_n) = \oplus_{j=1}^{n} (w_j \tilde{y}_j)$$

where $w = (w_1, w_2, \ldots, w_n)^T$ be the $\tilde{y}_j (j = 1, 2, \ldots, n)$, weight vector and $\omega_j > 0, \sum_{j=1}^{n} \omega_j = 1$.  


Theorem 3.1. Let \( \tilde{J}_j = ((\tilde{\psi}_j, \omega_{\tilde{\psi}_j}), (\tilde{\mathcal{R}}_j, \omega_{\tilde{\mathcal{R}}_j})) \) \((j = 1, 2, \ldots, n)\) be a collection of Cq-ROFNs, where \( \aleph > 0 \). Then its aggregated value by utilizing Cq-ROFHWA operator is also a Cq-ROFN, and

\[
\text{Cq-ROFHWA}_w(\tilde{J}_1, \tilde{J}_2, \ldots, \tilde{J}_n) = \bigoplus_{j=1}^{n} (w_j \tilde{J}_j)
\]

\[
= \left( \sqrt{\prod_{j=1}^{n} (\tilde{\mathcal{R}}_j)^{\omega_j}}, \prod_{j=1}^{n} (1 + (\aleph - 1)(\tilde{\mathcal{R}}_j)^{\omega_j} + (\aleph - 1) \prod_{j=1}^{n} (1 - (\tilde{\mathcal{R}}_j)^{\omega_j}) \right)
\]

When \( \aleph = 1 \), Cq-ROFHWA operator reduces to the Cq-ROF weighted averaging (Cq-ROFWA) operator as follows:

\[
\text{Cq-ROFWA}_w(\tilde{J}_1, \tilde{J}_2, \ldots, \tilde{J}_n) = \left( \sqrt{\prod_{j=1}^{n} (\tilde{\mathcal{R}}_j)^{\omega_j}}, \prod_{j=1}^{n} (1 - (\tilde{\psi}_j)^{\omega_j}), \prod_{j=1}^{n} (1 - (\omega_{\tilde{\psi}_j})^{\omega_j}) \right)
\]

Definition 3.3. Let \( \tilde{J}_j = ((\tilde{\psi}_j, \omega_{\tilde{\psi}_j}), (\tilde{\mathcal{R}}_j, \omega_{\tilde{\mathcal{R}}_j})) \) \((j = 1, 2, \ldots, n)\) be a collection of Cq-ROFNs, then the Cq-ROFHOWA operator is

\[
\text{Cq-ROFHOWA}_w(\tilde{J}_1, \tilde{J}_2, \ldots, \tilde{J}_n) = \bigoplus_{j=1}^{n} (w_j \tilde{J}_j)
\]

\[
= \left( \sqrt{\prod_{j=1}^{n} (\tilde{\mathcal{R}}_j)^{\omega_j}}, \prod_{j=1}^{n} (1 + (\aleph - 1)(\tilde{\mathcal{R}}_j)^{\omega_j} + (\aleph - 1) \prod_{j=1}^{n} (1 - (\tilde{\mathcal{R}}_j)^{\omega_j}) \right)
\]

where \((\mathcal{V}(1), \mathcal{V}(2), \ldots, \mathcal{V}(n))\) is a permutation of \((1, 2, \ldots, n)\), such that \(\tilde{J}_{\mathcal{V}(j-1)} \geq \tilde{J}_{\mathcal{V}(j)}\) for every \(j = 2, \ldots, n\), and \(w = (w_1, w_2, \ldots, w_n)^T\) is the weight vector such that \(w_j > 0\), \(\sum_{j=1}^{n} w_j = 1\), \(\aleph > 0\).

When \( \aleph = 1 \), Cq-ROFHOWA operator reduces to the Cq-ROF ordered weighted averaging (Cq-ROFOWA) operator as follows:

\[
\text{Cq-ROFOWA}_w(\tilde{J}_1, \tilde{J}_2, \ldots, \tilde{J}_n) = \left( \sqrt{\prod_{j=1}^{n} (\tilde{\mathcal{R}}_j)^{\omega_j}}, \prod_{j=1}^{n} (1 - (\tilde{\psi}_j)^{\omega_j}), \prod_{j=1}^{n} (1 - (\omega_{\tilde{\psi}_j})^{\omega_j}) \right)
\]

Definition 3.4. Let \( \tilde{J}_j = ((\tilde{\psi}_j, \omega_{\tilde{\psi}_j}), (\tilde{\mathcal{R}}_j, \omega_{\tilde{\mathcal{R}}_j})) \) \((j = 1, 2, \ldots, n)\) be a collection of Cq-ROFNs, then the Cq-ROFHWG operator is formally defined as:

\[
\text{Cq-ROFHWG}_w(\tilde{J}_1, \tilde{J}_2, \ldots, \tilde{J}_n) = \bigoplus_{j=1}^{n} (w_j \tilde{J}_j)
\]
Theorem 3.2. Let \( \mathbf{\tilde{\mathbf{i}}}_j = ((\tilde{\varphi}_j, \omega_{\tilde{\varphi}_j}), (\tilde{R}_j, \omega_{\tilde{R}_j})) \) \( (j = 1, 2, \ldots, n) \) be a collection of \( C_q \)-ROFNs, where \( N > 0 \). Then its aggregated value by utilizing \( C_q \)-ROFHGW operator is also a \( C_q \)-ROFN, and
\[
\begin{align*}
C_q\text{-ROFHGW}(\mathbf{\tilde{\mathbf{i}}}_1, \mathbf{\tilde{\mathbf{i}}}_2, \ldots, \mathbf{\tilde{\mathbf{i}}}_n) & = \left( \sqrt[n]{\prod_{j=1}^{n} \left( 1 + (N-1)(1 - (\tilde{\varphi}_j)^q) \right)^{\omega_{\tilde{\varphi}_j}}}, \sqrt[n]{\prod_{j=1}^{n} \left( 1 + (N-1)(1 - (\tilde{R}_j)^q) \right)^{\omega_{\tilde{R}_j}}} \right), \\
\end{align*}
\] When \( N = 1 \), \( C_q \)-ROFHGW operator is converted to the \( C_q \)-ROF weighted geometric (\( C_q \)-ROFWG) operator as follows:
\[
\begin{align*}
C_q\text{-ROFWG}_w(\mathbf{\tilde{\mathbf{i}}}_1, \mathbf{\tilde{\mathbf{i}}}_2, \ldots, \mathbf{\tilde{\mathbf{i}}}_n) & = \left( \left( \prod_{j=1}^{n} \left( \tilde{R}_j \right)^{\omega_{\tilde{R}_j}}, \prod_{j=1}^{n} \left( \tilde{\varphi}_j \right)^{\omega_{\tilde{\varphi}_j}} \right), \\
\left( \sqrt{1 - \prod_{j=1}^{n} \left( 1 - (\tilde{\varphi}_j)^q \right)^{\omega_{\tilde{\varphi}_j}}}, \sqrt{1 - \prod_{j=1}^{n} \left( 1 - (\tilde{R}_j)^q \right)^{\omega_{\tilde{R}_j}}} \right). \\
\end{align*}
\]
Definition 3.5. Let \( \mathbf{\tilde{\mathbf{i}}}_j = ((\tilde{\varphi}_j, \omega_{\tilde{\varphi}_j}), (\tilde{R}_j, \omega_{\tilde{R}_j})) \) \( (j = 1, 2, \ldots, n) \) be a collection of \( C_q \)-ROFNs, then the \( C_q \)-ROFHOWG operator is described as:
\[
\begin{align*}
C_q\text{-ROFHOWG}(\mathbf{\tilde{\mathbf{i}}}_1, \mathbf{\tilde{\mathbf{i}}}_2, \ldots, \mathbf{\tilde{\mathbf{i}}}_n) & = \left( \left( \prod_{j=1}^{n} \left( \omega_{\tilde{R}_j} \right)^{\omega_{\tilde{R}_j}}, \prod_{j=1}^{n} \left( \tilde{R}_j \right)^{\omega_{\tilde{R}_j}} \right), \\
\left( \sqrt{1 - \prod_{j=1}^{n} \left( 1 - (\tilde{\varphi}_j)^q \right)^{\omega_{\tilde{\varphi}_j}}}, \sqrt{1 - \prod_{j=1}^{n} \left( 1 - (\tilde{R}_j)^q \right)^{\omega_{\tilde{R}_j}}} \right). \\
\end{align*}
\]
When \( N = 1 \), \( C_q \)-ROFHOWG operator transforms to the \( C_q \)-ROF ordered weighted geometric (\( C_q \)-ROFOWG) operator as:
\[
\begin{align*}
C_q\text{-ROFOWG}_w(\mathbf{\tilde{\mathbf{i}}}_1, \mathbf{\tilde{\mathbf{i}}}_2, \ldots, \mathbf{\tilde{\mathbf{i}}}_n) & = \left( \left( \prod_{j=1}^{n} \left( \tilde{R}_j \right)^{\omega_{\tilde{R}_j}}, \prod_{j=1}^{n} \left( \tilde{R}_j \right)^{\omega_{\tilde{R}_j}} \right), \\
\left( \sqrt{1 - \prod_{j=1}^{n} \left( 1 - (\tilde{\varphi}_j)^q \right)^{\omega_{\tilde{\varphi}_j}}}, \sqrt{1 - \prod_{j=1}^{n} \left( 1 - (\tilde{R}_j)^q \right)^{\omega_{\tilde{R}_j}}} \right). \\
\end{align*}
\]

4 Complex \( q \)-rung orthopair fuzzy Hamacher graphs

In this section, a new concept of \( C_q \)-ROFG based on Hamacher operator termed as \( C_q \)-ROF Hamacher graphs (\( C_q \)-ROFGH) is formed first, and then its energy along with the relevant properties is determined. Subsequently, inspired by the theory of splitting energy and shadow energy of a graph, we determine the energy of splitting \( C_q \)-ROFGH and the energy of shadow \( C_q \)-ROFGH.

Definition 4.1. A \( C_q \)-ROFS \( \Xi \) in \( \mathbb{R} \times \mathbb{R} \) is said to be a \( C_q \)-ROF relation (\( C_q \)-ROFR) in \( \mathbb{R} \), denoted by
\[
\Xi = \{ (st, (\tilde{\varphi}_\Xi(st), \omega_{\tilde{\varphi}_\Xi(st)), (\tilde{R}_\Xi(st), \omega_{\tilde{R}_\Xi(st))) \mid st \in \mathbb{R} \times \mathbb{R} \},
\]
where \( \tilde{\varphi}_\Xi, \tilde{\varphi}_\Xi, \omega_{\tilde{\varphi}_\Xi}, \omega_{\tilde{R}_\Xi} : \mathbb{R} \times \mathbb{R} \rightarrow [0, 1] \) indicate the membership and non-membership function of \( \Xi \), respectively, such that \( 0 \leq \tilde{\varphi}_\Xi^q(st) + \tilde{R}_\Xi^q(st) \leq 1 \) and \( 0 \leq \omega_{\tilde{\varphi}_\Xi}(st) + \omega_{\tilde{R}_\Xi}(st) \leq 1 \) for all \( st \in \mathbb{R} \times \mathbb{R} \).
We define the C$q$-ROF preference relations (C$q$-ROFPRs) as follows:

**Definition 4.2.** A C$q$-ROFPR on the set $R$ is given by a matrix $\mathcal{R} = (\tilde{\mathcal{R}}_{ij})_{n \times n}$, where $\tilde{\mathcal{R}}_{ij} = ((\tilde{\varphi}_{ij}, \omega_{\tilde{\varphi}_{ij}}), (\tilde{\mathcal{R}}_{ij}, \omega_{\tilde{\mathcal{R}}_{ij}}))$ $(i, j = 1, 2, \ldots, n)$, where $(\tilde{\varphi}_{ij}, \omega_{\tilde{\varphi}_{ij}})$ and $(\tilde{\mathcal{R}}_{ij}, \omega_{\tilde{\mathcal{R}}_{ij}})$ represent the complex MD and complex NMD, and $\pi_{ij} = \sqrt{(1 - \tilde{\varphi}_{ij}^q - \tilde{\mathcal{R}}_{ij}^q) + (1 - \omega_{\tilde{\varphi}_{ij}}^q - \omega_{\tilde{\mathcal{R}}_{ij}}^q)}$ indicates the hesitation degree, subject to the following conditions:

\[
\begin{align*}
0 \leq \tilde{\varphi}_{ij}, \tilde{\mathcal{R}}_{ij}, \tilde{\varphi}_{ij}^q + \tilde{\mathcal{R}}_{ij}^q & \leq 1, \quad \tilde{\varphi}_{ii} = \tilde{\mathcal{R}}_{ii} = 0.5, \\
0 \leq \omega_{\tilde{\varphi}_{ij}}, \omega_{\tilde{\mathcal{R}}_{ij}}, \omega_{\tilde{\varphi}_{ij}}^q + \omega_{\tilde{\mathcal{R}}_{ij}}^q & \leq 1, \quad \omega_{\tilde{\varphi}_{ii}} = \omega_{\tilde{\mathcal{R}}_{ii}} = 0.5, \quad \text{for all } i, j = 1, 2, \ldots, n.
\end{align*}
\]

**Definition 4.3.** A C$q$-ROFHG on a non-empty set $R$ is an ordered pair $\mathfrak{G} = (\mathcal{L}, \Xi)$, where $\mathcal{L}$ is a C$q$-ROFS on $R$ and $\Xi$ is a C$q$-ROFR on $R$ such that:

\[
\begin{align*}
\bar{\varphi}_\Xi(st) & \leq \frac{\bar{\varphi}_\mathcal{L}(s)\bar{\varphi}_\mathcal{L}(t)}{\bar{\varphi}_\mathcal{L}(s) + \bar{\varphi}_\mathcal{L}(t) - \bar{\varphi}_\mathcal{L}(s)\bar{\varphi}_\mathcal{L}(t)}, \\
\bar{\omega}_\Xi(st) & \leq \frac{\bar{\varphi}_\mathcal{L}(s)\bar{\omega}_\mathcal{L}(t)}{\bar{\varphi}_\mathcal{L}(s) + \bar{\varphi}_\mathcal{L}(t) - \bar{\varphi}_\mathcal{L}(s)\bar{\varphi}_\mathcal{L}(t)},
\end{align*}
\]

for amplitude terms
\[
\begin{align*}
\bar{\varphi}_\mathcal{R}(st) & \leq \frac{\bar{\varphi}_\mathcal{L}(s) + \bar{\mathcal{R}}_\mathcal{L}(t) - 2\bar{\varphi}_\mathcal{L}(s)\bar{\mathcal{R}}_\mathcal{L}(t)}{1 - \bar{\mathcal{R}}_\mathcal{L}(s)\bar{\mathcal{R}}_\mathcal{L}(t)}, \\
\bar{\omega}_\mathcal{R}(st) & \leq \frac{\bar{\varphi}_\mathcal{L}(s) + \bar{\mathcal{R}}_\mathcal{L}(t) - 2\bar{\varphi}_\mathcal{L}(s)\bar{\mathcal{R}}_\mathcal{L}(t)}{1 - \bar{\mathcal{R}}_\mathcal{L}(s)\bar{\mathcal{R}}_\mathcal{L}(t)},
\end{align*}
\]

for phase terms

where $0 \leq \bar{\varphi}_\mathcal{R}^2(st) + \bar{\mathcal{R}}_\mathcal{R}^2(st) \leq 1$ and $0 \leq \bar{\varphi}_\mathcal{R}^q(st) + \bar{\mathcal{R}}_\mathcal{R}^q(st) \leq 1$ for all $s, t \in R$. We call $\mathcal{L}$ as a C$q$-ROFS of vertices and $\Xi$ as a C$q$-ROFS of edges in $\mathfrak{G}$, respectively. Here, we consider $\Xi$ as a symmetric C$q$-ROFR on $\mathcal{L}$.

In case of not symmetric on $\mathcal{L}$, $\mathfrak{D} = (\mathcal{L}, \Xi)$ is called C$q$-ROF Hamacher digraph (C$q$-ROFHDG).

**Example 4.1.** Consider a graph $\mathcal{G} = (\bar{V}, \bar{E})$, where $\bar{V} = \{s_1, s_2, s_3, s_4, s_5, s_6\}$ and $\bar{E} = \{s_1s_2, s_1s_3, s_1s_4, s_1s_5, s_1s_6\}$ are the vertex and edge set of $\mathcal{G}$, respectively. Let $\mathfrak{G} = (\mathcal{L}, \Xi)$ be a C3-ROFHG on $\bar{V}$, as presented in Fig. 2, defined by:

\[
\mathcal{L} = \left(\begin{array}{cccccc}
(0.9, 0.7) & (0.6, 0.8) & (0.7, 0.9) & (0.7, 0.5) & (0.8, 0.7) & (0.5, 0.8) \\
(0.7, 0.9) & (0.6, 0.8) & (0.8, 0.5) & (0.5, 0.6) & (0.9, 0.6) & (0.6, 0.9) \\
(0.7, 0.9) & (0.5, 0.6) & (0.8, 0.5) & (0.5, 0.6) & (0.9, 0.6) & (0.8, 0.6) \\
(0.7, 0.9) & (0.7, 0.5) & (0.8, 0.5) & (0.5, 0.6) & (0.9, 0.6) & (0.8, 0.6) \\
(0.7, 0.9) & (0.7, 0.5) & (0.8, 0.5) & (0.5, 0.6) & (0.9, 0.6) & (0.8, 0.6) \\
(0.7, 0.9) & (0.7, 0.5) & (0.8, 0.5) & (0.5, 0.6) & (0.9, 0.6) & (0.8, 0.6)
\end{array}\right),
\]

\[
\Xi = \left(\begin{array}{cccccc}
(0.9, 0.62) & (0.75, 0.82) & (0.72, 0.52) & (0.66, 0.88) & (0.71, 0.40) & (0.70, 0.90) \\
(0.63, 0.62) & (0.75, 0.82) & (0.72, 0.52) & (0.66, 0.88) & (0.71, 0.40) & (0.70, 0.90) \\
(0.9, 0.62) & (0.75, 0.82) & (0.72, 0.52) & (0.66, 0.88) & (0.71, 0.40) & (0.70, 0.90) \\
(0.9, 0.62) & (0.75, 0.82) & (0.72, 0.52) & (0.66, 0.88) & (0.71, 0.40) & (0.70, 0.90) \\
(0.9, 0.62) & (0.75, 0.82) & (0.72, 0.52) & (0.66, 0.88) & (0.71, 0.40) & (0.70, 0.90) \\
(0.9, 0.62) & (0.75, 0.82) & (0.72, 0.52) & (0.66, 0.88) & (0.71, 0.40) & (0.70, 0.90)
\end{array}\right).
\]

**Figure 2:** C3-ROFHG.

**Definition 4.4.** The adjacency matrix (AM) $A(\mathfrak{G}) = \left( A(\bar{\varphi}_\Xi(s_is_j), \omega_{\bar{\varphi}_\Xi(s_is_j)), A(\bar{\mathcal{R}}_\Xi(s_is_j), \omega_{\bar{\mathcal{R}}_\Xi(s_is_j))} \right)$ of a C$q$-ROFHG $\mathfrak{G} = (\mathcal{L}, \Xi)$ is a square matrix $A(\mathfrak{G}) = [a_{ij}]$, $a_{ij} = \left( \bar{\varphi}_\Xi(s_is_j), \omega_{\bar{\varphi}_\Xi(s_is_j)), \bar{\mathcal{R}}_\Xi(s_is_j), \omega_{\bar{\mathcal{R}}_\Xi(s_is_j)) \right)$, where $(\bar{\varphi}_\Xi(s_is_j), \omega_{\bar{\varphi}_\Xi(s_is_j)))$ and $(\bar{\mathcal{R}}_\Xi(s_is_j), \omega_{\bar{\mathcal{R}}_\Xi(s_is_j)))$ indicate the strength of relationship and non-relationship between $s_i$ and $s_j$, respectively, in complex scenario.
Definition 4.5. The spectrum of AM of a Cq-ROFHG $A(\mathcal{G})$ is characterized as $(\mathcal{P}, \mathcal{Q})$, where $\mathcal{P}$ and $\mathcal{Q}$ represent the sets of eigenvalues of $A(\tilde{\psi}_{\mathcal{E}}(s_i\mathcal{S}), \omega_{\psi_{\mathcal{E}}}(s_i\mathcal{S}))$ and $A(\tilde{\mathcal{R}}_{\mathcal{E}}(s_i\mathcal{S}), \omega_{\mathcal{R}_E}(s_i\mathcal{S}))$, respectively.

Definition 4.6. The energy of a Cq-ROFHG $\mathcal{G} = (\mathcal{L}, \Xi)$ is defined as:

\[
E(\mathcal{G}) = \left( E(\tilde{\psi}_{\mathcal{E}}(s_i\mathcal{S}), \omega_{\psi_{\mathcal{E}}}(s_i\mathcal{S}))(\mathcal{G}), E(\tilde{\mathcal{R}}_{\mathcal{E}}(s_i\mathcal{S}), \omega_{\mathcal{R}_E}(s_i\mathcal{S}))(\mathcal{G}) \right)
\]

\[
= \left( \left( \sum_{j=1}^{n} |\tilde{\psi}_j|, \sum_{j=1}^{n} |\omega_{\tilde{\psi}_j}| \right), \left( \sum_{j=1}^{n} |\tilde{\chi}_j|, \sum_{j=1}^{n} |\omega_{\tilde{\chi}_j}| \right) \right).
\]

Theorem 4.1. Let $\mathcal{G} = (\mathcal{L}, \Xi)$ be a Cq-ROFHG and let $A(\mathcal{G})$ be its AM. If $\tilde{\psi}_1 \geq \tilde{\psi}_2 \geq \ldots \geq \tilde{\psi}_n$ and $\tilde{\chi}_1 \geq \tilde{\chi}_2 \geq \ldots \geq \tilde{\chi}_n$ are the eigenvalues of $A(\tilde{\psi}_{\mathcal{E}}(s_i\mathcal{S}), \omega_{\psi_{\mathcal{E}}}(s_i\mathcal{S}))$ and $A(\tilde{\mathcal{R}}_{\mathcal{E}}(s_i\mathcal{S}), \omega_{\mathcal{R}_E}(s_i\mathcal{S}))$, respectively, then:

(i) \[
\left( \sum_{j=1}^{n} \tilde{\psi}_j, \sum_{j=1}^{n} \omega_{\tilde{\psi}_j} \right) = (0, 0) \quad \text{and} \quad \left( \sum_{j=1}^{n} \tilde{\chi}_j, \sum_{j=1}^{n} \omega_{\tilde{\chi}_j} \right) = (0, 0).
\]

(ii) \[
\sum_{j=1}^{n} \tilde{\psi}_j^2 + \sum_{j=1}^{n} \omega_{\tilde{\psi}_j}^2 = 2 \left( \sum_{1 \leq i < j \leq n} \tilde{\psi}_{\mathcal{E}}^2(s_i\mathcal{S}), \omega_{\psi_{\mathcal{E}}}(s_i\mathcal{S}) \right) \quad \text{and}
\]

\[
\sum_{j=1}^{n} \tilde{\chi}_j^2 + \sum_{j=1}^{n} \omega_{\tilde{\chi}_j}^2 = 2 \left( \sum_{1 \leq i < j \leq n} \tilde{\mathcal{R}}_{\mathcal{E}}^2(s_i\mathcal{S}), \omega_{\mathcal{R}_E}(s_i\mathcal{S}) \right).
\]

Proof. (i) Obvious.

(ii) Since

\[
tr \left( A^2(\tilde{\psi}_{\mathcal{E}}(s_i\mathcal{S}), \omega_{\psi_{\mathcal{E}}}(s_i\mathcal{S}))(\mathcal{G}) \right) = \left( \sum_{j=1}^{n} \tilde{\psi}_j^2, \sum_{j=1}^{n} \omega_{\tilde{\psi}_j}^2 \right)
\]

where:

\[
tr \left( A^2(\tilde{\psi}_{\mathcal{E}}(s_i\mathcal{S}), \omega_{\psi_{\mathcal{E}}}(s_i\mathcal{S}))(\mathcal{G}) \right) = (0 + (\tilde{\psi}_{\mathcal{E}}^2(s_1\mathcal{S}), \omega_{\psi_{\mathcal{E}}}(s_1\mathcal{S}))) + \ldots + (\tilde{\psi}_{\mathcal{E}}^2(s_n\mathcal{S}), \omega_{\psi_{\mathcal{E}}}(s_n\mathcal{S}))
\]

\[
: + ((\tilde{\psi}_{\mathcal{E}}^2(s_n\mathcal{S}), \omega_{\psi_{\mathcal{E}}}(s_n\mathcal{S})))
\]

\[
= 2 \left( \sum_{1 \leq i < j \leq n} \tilde{\psi}_{\mathcal{E}}^2(s_i\mathcal{S}), \sum_{1 \leq i < j \leq n} \omega_{\psi_{\mathcal{E}}}(s_i\mathcal{S}) \right).
\]

Hence:

\[
\left( \sum_{j=1}^{n} \tilde{\psi}_j^2, \sum_{j=1}^{n} \omega_{\tilde{\psi}_j}^2 \right) = 2 \left( \sum_{1 \leq i < j \leq n} \tilde{\psi}_{\mathcal{E}}^2(s_i\mathcal{S}), \sum_{1 \leq i < j \leq n} \omega_{\psi_{\mathcal{E}}}(s_i\mathcal{S}) \right).
\]

Analogously,

\[
\left( \sum_{j=1}^{n} \tilde{\chi}_j^2, \sum_{j=1}^{n} \omega_{\tilde{\chi}_j}^2 \right) = 2 \left( \sum_{1 \leq i < j \leq n} \tilde{\mathcal{R}}_{\mathcal{E}}^2(s_i\mathcal{S}), \sum_{1 \leq i < j \leq n} \omega_{\mathcal{R}_E}(s_i\mathcal{S}) \right). \quad \square
\]
Example 4.2. Let $\mathcal{G} = (\mathcal{L}, \Xi)$ be a $C_4$-ROFHG on $\hat{V} = \{s_1, s_2, s_3, s_4, s_5, s_6, s_7\}$ and $\hat{E} = \{s_1s_2, s_2s_3, s_3s_4, s_3s_5, s_3s_6, s_3s_7, s_2s_6, s_1s_3\}$, as given in Fig. 3:

\[\begin{align*}
\begin{array}{c}
\text{Figure 3: C4-ROFHG.}
\end{array}
\end{align*}\]

The $A(\mathcal{G})$, $\text{Spec}(\mathcal{G})$, and $E(\mathcal{G})$ of a $C_4$-ROFHG given in Fig. 3 are:

\[
A(\mathcal{G}) = \begin{bmatrix}
(0,0) & (0.0) & (0.0) & (0.0) & (0.0) & (0.0) & (0.0) \\
(0.33,0.52),(0.92,0.90) & (0.33,0.52),(0.92,0.90) & (0.33,0.52),(0.92,0.90) & (0.33,0.52),(0.92,0.90) & (0.33,0.52),(0.92,0.90) & (0.33,0.52),(0.92,0.90) & (0.33,0.52),(0.92,0.90) \\
(0.46,0.31),(0.90,0.92) & (0.46,0.31),(0.90,0.92) & (0.46,0.31),(0.90,0.92) & (0.46,0.31),(0.90,0.92) & (0.46,0.31),(0.90,0.92) & (0.46,0.31),(0.90,0.92) & (0.46,0.31),(0.90,0.92) \\
(0.0) & (0.0) & (0.0) & (0.0) & (0.0) & (0.0) & (0.0) \\
(0.0) & (0.0) & (0.0) & (0.0) & (0.0) & (0.0) & (0.0) \\
(0.0) & (0.0) & (0.0) & (0.0) & (0.0) & (0.0) & (0.0) \\
(0.0) & (0.0) & (0.0) & (0.0) & (0.0) & (0.0) & (0.0) \\
(0.0) & (0.0) & (0.0) & (0.0) & (0.0) & (0.0) & (0.0) \\
(0.0) & (0.0) & (0.0) & (0.0) & (0.0) & (0.0) & (0.0) \\
\end{bmatrix}.
\]

\[
\text{Spec}(\check{\varphi}(s_is_j), \omega_{\check{\varphi}}(s_is_j)) = \{-1.6444, -0.8067, -0.5294, -0.6710, -0.0000, -0.0000, 0.0000, 0.0000, 0.3639, 0.3246, 1.8099, 1.1532\}.
\]

\[
\text{Spec}(\check{\varphi}(s_is_j), \omega_{\check{\varphi}}(s_is_j)) = \{-1.6131, -1.8712, -1.1651, -1.1599, -0.0000, -0.0000, 0.0000, -0.0000, 0.4480, 0.6122, 2.3302, 2.4189\}.
\]

Now, $E(\check{\varphi}(s_is_j), \omega_{\check{\varphi}}(s_is_j)) = (4.3475, 2.9555)$ and $E(\check{\varphi}(s_is_j), \omega_{\check{\varphi}}(s_is_j)) = (5.5563, 6.0622)$. Therefore, $E(\mathcal{G}) = ((4.3475, 2.9555),(5.5563, 6.0622))$.

Theorem 4.2. Let $\mathcal{G} = (\mathcal{L}, \Xi)$ be a $C_q$-ROFHG on $n$ vertices and $A(\mathcal{G}) = (\check{\varphi}(s_is_j), \omega_{\check{\varphi}}(s_is_j))$, $A(\check{\varphi}(s_is_j), \omega_{\check{\varphi}}(s_is_j))$ be the AM of $\mathcal{G}$. Then:

\[
\begin{align*}
&\text{if} \quad 2 \left( \sum_{1 \leq i < j \leq n} \check{\varphi}^2(s_is_j) + \sum_{1 \leq i < j \leq n} \omega^2_{\check{\varphi}}(s_is_j) \right) + n(n - 1) \left| \det(A(\check{\varphi}(s_is_j), \omega_{\check{\varphi}}(s_is_j))) \right|^2 \leq E(\check{\varphi}(s_is_j), \omega_{\check{\varphi}}(s_is_j)) \leq 2n \left( \sum_{1 \leq i < k \leq n} \check{\varphi}^2(s_is_j) + \sum_{1 \leq i < k \leq n} \omega^2_{\check{\varphi}}(s_is_j) \right).
\end{align*}
\]
(ii) \[
\sqrt{2 \left( \sum_{1 \leq i < j \leq n} \tilde{R}^2_{\Psi}(\psi_i, \psi_j) \right) + n(n-1) \text{det}(A(\tilde{R}_{\Psi}(\psi_i, \psi_j)))} \leq E(\tilde{R}_{\Psi}(\psi_i, \psi_j)) \leq \sqrt{2 \left( \sum_{1 \leq i < j \leq n} \tilde{R}^2_{\Psi}(\psi_i, \psi_j) \right) + n(n-1) \text{det}(A(\tilde{R}_{\Psi}(\psi_i, \psi_j)))}.
\]

Now we determine the energy of a splitting Cq-ROFHG and a shadow Cq-ROFHG.

**Definition 4.7.** The splitting Cq-ROFHG \( S(\Phi) \) of a Cq-ROFHG \( \Phi \) is attained by adding to each vertex \( \Phi \) another vertex \( \Phi' \), such that \( \Phi' \) is adjacent to each vertex that is adjacent to \( \Phi \) in \( \Phi \), and MD and NMD remain unchanged.

**Theorem 4.3.** Let \( S(\Phi) \) be a splitting Cq-ROFHG of a Cq-ROFHG \( \Phi \). Then \( E(S(\Phi)) = \sqrt{2} E(\Phi) \).

**Proof.** Consider a Cq-ROFHG with set of vertices \( \{s_1, s_2, \ldots, s_n\} \). Then its AM is \( A(\Phi) = (A(\tilde{\psi}(s_i, s_j), \tilde{\omega}(s_i, s_j)))(\Phi), A(\tilde{\psi}(s_i, s_j), \tilde{\omega}(s_i, s_j))(\Phi)) \), where:

\[
A(\tilde{\psi}(s_i, s_j), \tilde{\omega}(s_i, s_j))(\Phi) = \begin{bmatrix}
0 & (\tilde{\psi}(s_1, s_2), \tilde{\omega}(s_1, s_2)) & \cdots & (\tilde{\psi}(s_1, s_n), \tilde{\omega}(s_1, s_n)) \\
(\tilde{\psi}(s_2, s_1), \tilde{\omega}(s_2, s_1)) & 0 & \cdots & (\tilde{\psi}(s_2, s_n), \tilde{\omega}(s_2, s_n)) \\
\vdots & \vdots & \ddots & \vdots \\
(\tilde{\psi}(s_n, s_1), \tilde{\omega}(s_n, s_1)) & (\tilde{\psi}(s_n, s_2), \tilde{\omega}(s_n, s_2)) & \cdots & 0
\end{bmatrix}.
\]

To obtain \( S(\tilde{\psi}(s_i, s_j), \tilde{\omega}(s_i, s_j))(\Phi) \), let \( s_1', s_2', \ldots, s_n' \) be the vertices corresponding to \( s_1, s_2, \ldots, s_n \), which are included in \( \Phi \), such that \( N(s_i) = N(s'_i), i = 1, 2, \ldots, n \). At that point we can represent \( A(\tilde{\psi}(s_i, s_j), \tilde{\omega}(s_i, s_j))(\Phi) \) as a block matrix as follows:

\[
A(\tilde{\psi}(s_i, s_j), \tilde{\omega}(s_i, s_j))(\Phi) = \begin{bmatrix}
0 & (\tilde{\psi}(s_1, s_2), \tilde{\omega}(s_1, s_2)) & \cdots & (\tilde{\psi}(s_1, s_n), \tilde{\omega}(s_1, s_n)) \\
(\tilde{\psi}(s_2, s_1), \tilde{\omega}(s_2, s_1)) & 0 & \cdots & (\tilde{\psi}(s_2, s_n), \tilde{\omega}(s_2, s_n)) \\
\vdots & \vdots & \ddots & \vdots \\
(\tilde{\psi}(s_n, s_1), \tilde{\omega}(s_n, s_1)) & (\tilde{\psi}(s_n, s_2), \tilde{\omega}(s_n, s_2)) & \cdots & 0
\end{bmatrix}.
\]

i.e.,

\[
A(\tilde{\psi}(s_i, s_j), \tilde{\omega}(s_i, s_j))(\Phi) = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \otimes A(\tilde{\psi}(s_i, s_j), \tilde{\omega}(s_i, s_j))(\Phi).
\]

Since \( \frac{1 + \sqrt{5}}{2} \) and \( \tilde{\psi}_i, \tilde{\omega}_i, (i = 1, 2, \ldots, n) \), are the eigenvalues of \( \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \) and \( A(\tilde{\psi}(s_i, s_j), \tilde{\omega}(s_i, s_j))(\Phi) \), respectively. So,

\[
E(\tilde{\psi}(s_i, s_j), \tilde{\omega}(s_i, s_j))(\Phi) = \sum_{i=1}^{n} \left( \frac{1 + \sqrt{5}}{2} \right)^i = \sum_{i=1}^{n} \left( \frac{1 + \sqrt{5}}{2} + \sqrt{5} - 1 \right)^i = \sqrt{5} \sum_{i=1}^{n} (|\tilde{\psi}_i|, |\tilde{\omega}_i|)
\]

Similarly, we can show that \( E(\tilde{\psi}(s_i, s_j), \tilde{\omega}(s_i, s_j))(\Phi) = \sqrt{5} \sum_{i=1}^{n} (|\tilde{\psi}_i|, |\tilde{\omega}_i|) \).

Hence, \( E(\Phi) = \sqrt{5} E(\Phi) \).
Definition 4.8. The shadow \( C_q \)-ROFHG \( S\mathcal{H}(\mathcal{G}) \) of a connected \( C_q \)-ROFHG \( \mathcal{G} \) is designed by taking two duplicates of \( \mathcal{G} \), say \( \mathcal{G}' \) and \( \mathcal{G}'' \). Connect each vertex \( s' \) in \( \mathcal{G}' \) to the neighbors of the corresponding vertex \( s'' \) in \( \mathcal{G}'' \) with the same MD and NMD.

Theorem 4.4. Let \( S\mathcal{H}(\mathcal{G}) \) be a shadow \( C_q \)-ROFHG of a \( C_q \)-ROFHG \( \mathcal{G} \). Then \( E(S\mathcal{H}(\mathcal{G})) = 2E(\mathcal{G}) \).

Proof. Consider a \( C_q \)-ROFHG with set of vertices \( \{v_1, v_2, \ldots, v_n\} \). Then its AM is as follows:

\[
A(\tilde{\psi}(s_i, s_j), \omega_{\tilde{\psi}(s_i, s_j)})(\mathcal{G}) = \begin{pmatrix}
0 & (\tilde{\psi}(s_1s_2), \omega_{\tilde{\psi}(s_1s_2)}) & \cdots & (\tilde{\psi}(s_1s_n), \omega_{\tilde{\psi}(s_1s_n)}) \\
(\tilde{\psi}(s_2s_1), \omega_{\tilde{\psi}(s_2s_1)}) & 0 & \cdots & (\tilde{\psi}(s_2s_n), \omega_{\tilde{\psi}(s_2s_n)}) \\
\vdots & \vdots & \ddots & \vdots \\
(\tilde{\psi}(s_ns_1), \omega_{\tilde{\psi}(s_ns_1)}) & (\tilde{\psi}(s_ns_2), \omega_{\tilde{\psi}(s_ns_2)}) & \cdots & 0
\end{pmatrix}.
\]

To obtain \( S\mathcal{H}(\tilde{\psi}(s_i, s_j), \omega_{\tilde{\psi}(s_i, s_j)})(\mathcal{G}) \), let \( s'_1, s'_2, \ldots, s'_n \) be the vertices corresponding to \( s_1, s_2, \ldots, s_n \), which are added in \( \mathcal{G} \), such that, \( N(s_i) = N(s'_i), i = 1, 2, \ldots, n \). Then we can represent \( A(\tilde{\psi}(s_i, s_j), \omega_{\tilde{\psi}(s_i, s_j)})(S\mathcal{H}(\mathcal{G})) \) as a block matrix as follows:

\[
A(\tilde{\psi}(s_i, s_j), \omega_{\tilde{\psi}(s_i, s_j)})(S\mathcal{H}(\mathcal{G})) = \begin{pmatrix}
0 & (\tilde{\psi}(s_1s_2), \omega_{\tilde{\psi}(s_1s_2)}) & \cdots & (\tilde{\psi}(s_1s_n), \omega_{\tilde{\psi}(s_1s_n)}) \\
(\tilde{\psi}(s_2s_1), \omega_{\tilde{\psi}(s_2s_1)}) & 0 & \cdots & (\tilde{\psi}(s_2s_n), \omega_{\tilde{\psi}(s_2s_n)}) \\
\vdots & \vdots & \ddots & \vdots \\
(\tilde{\psi}(s_ns_1), \omega_{\tilde{\psi}(s_ns_1)}) & (\tilde{\psi}(s_ns_2), \omega_{\tilde{\psi}(s_ns_2)}) & \cdots & 0
\end{pmatrix}.
\]

i.e.,

\[
A(\tilde{\psi}(s_i, s_j), \omega_{\tilde{\psi}(s_i, s_j)})(S\mathcal{H}(\mathcal{G})) = \begin{pmatrix}
A(\tilde{\psi}(s_i, s_j), \omega_{\tilde{\psi}(s_i, s_j)})(\mathcal{G}) & A(\tilde{\psi}(s_i, s_j), \omega_{\tilde{\psi}(s_i, s_j)})(\mathcal{G}) \\
A(\tilde{\psi}(s_i, s_j), \omega_{\tilde{\psi}(s_i, s_j)})(\mathcal{G}) & A(\tilde{\psi}(s_i, s_j), \omega_{\tilde{\psi}(s_i, s_j)})(\mathcal{G})
\end{pmatrix} = \begin{pmatrix}
1 & 1 \\
1 & 1
\end{pmatrix} \otimes A(\tilde{\psi}(s_i, s_j), \omega_{\tilde{\psi}(s_i, s_j)})(\mathcal{G}).
\]

Since 0, 2 and \( \langle \tilde{\psi}_i, \omega_{\tilde{\psi}_i} \rangle, i = 1, 2, \ldots, n \), are the eigenvalues of \( \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \) and \( A(\tilde{\psi}(s_i, s_j), \omega_{\tilde{\psi}(s_i, s_j)})(\mathcal{G}) \), respectively. So,

\[
E(\tilde{\psi}(s_i, s_j), \omega_{\tilde{\psi}(s_i, s_j)})(S\mathcal{H}(\mathcal{G})) = \sum_{i=1}^{n} \left| \tilde{\psi}_i \right| \left| \omega_{\tilde{\psi}_i} \right| = 2 \left( \sum_{i=1}^{n} \left| \tilde{\psi}_i \right| \right) = 2 \left( \sum_{i=1}^{n} |\tilde{\psi}_i| \right).
\]

Analogously, \( E(\tilde{\mathcal{H}}(s_i, s_j), \omega_{\tilde{\mathcal{H}}(s_i, s_j)})(S\mathcal{H}(\mathcal{G})) = 2 \left( \sum_{i=1}^{n} |\tilde{x}_i| \right) \).

Hence, \( E(S\mathcal{H}(\mathcal{G})) = 2E(\mathcal{G}) \).

Example 4.3. Let \( \mathcal{G} = (E, \Xi) \) be a \( C2 \)-ROFHG on \( \tilde{V} = \{s_1, s_2, s_3, s_4\} \) and \( \tilde{E} = \{s_1s_2, s_1s_3, s_1s_4\} \), as in Fig. 4, defined by:
The $A(\mathfrak{G})$ and $E(\mathfrak{G})$ of a C2-ROFHG, given in Fig. 4, are:

$$A(\mathfrak{G}) = \begin{pmatrix}
(0, 0) & (0, 0) & (0, 0) & (0, 0) \\
(0, 0) & (0.54, 0.50, (0.71, 0.76)) & (0, 0) & (0.54, 0.50, (0.71, 0.76)) \\
(0, 0) & (0, 0) & (0.54, 0.50, (0.71, 0.76)) & (0, 0) \\
(0.54, 0.50, (0.71, 0.76)) & (0.54, 0.50, (0.71, 0.76)) & (0, 0) & (0, 0)
\end{pmatrix}$$

$$E(\mathfrak{G}) = ((1.5981, 1.7174), (2.5912, 2.7078)).$$

The splitting C2-ROFHG of the C2-ROFHG shown in Fig. 4, is presented in Fig. 5.
The $A(S(\mathfrak{G}))$ and the $E(S(\mathfrak{G}))$ of a splitting C2-ROFHG, given in Fig. 5, are:

$$A(S(\mathfrak{G})) = \begin{pmatrix}
(0, 0) & (0, 0) & (0, 0) & (0, 0) \\
(0, 0) & (0.54, 0.50, (0.71, 0.76)) & (0, 0) & (0.54, 0.50, (0.71, 0.76)) \\
(0, 0) & (0, 0) & (0.54, 0.50, (0.71, 0.76)) & (0, 0) \\
(0.54, 0.50, (0.71, 0.76)) & (0.54, 0.50, (0.71, 0.76)) & (0, 0) & (0, 0)
\end{pmatrix}$$

$$E(S(\mathfrak{G})) = ((3.5735, 3.8403), (5.7941, 6.0548))$$

$$= \sqrt{5}((1.5981, 1.7174), (2.5912, 2.7078))$$

$$= \sqrt{5}E(\mathfrak{G}).$$
Figure 5: Splitting C2-ROFHG.

Figure 6: Shadow C2-ROFHG.
The $A(SH(\Theta))$ and the $E(SH(\Theta))$ of a shadow C2-ROFHG, given in Fig. 6, are:

$$A(SH(\Theta)) = \begin{pmatrix}
(0,0) & (0.54,0.50),(0.71,0.76) & (0.45,0.43),(0.72,0.85) & (0.38,0.55),(0.81,0.73) \\
(0.54,0.50),(0.71,0.76) & (0.0) & (0.0) & (0.0) \\
(0.45,0.43),(0.72,0.85) & (0.0) & (0.0) & (0.0) \\
(0.38,0.55),(0.81,0.73) & (0.0) & (0.0) & (0.0) \\
\end{pmatrix}$$

$$E(SH(\Theta)) = ((3.1962, 3.4349), (5.1824, 5.4155)) = 2((1.5981,1.7174), (2.5912,2.7078)) = 2E(\Theta).$$

**Definition 4.9.** The energy of a C$q$-ROFHG $\mathcal{D} = (\mathcal{L}, \Xi)$ is defined as:

$$E(\mathcal{D}) = \left(E(\bar{\psi}(s_i,s_j),\omega_{\bar{\psi}(s_i,s_j)}), E(\bar{\psi}(s_i,s_j),\omega_{\bar{\psi}(s_i,s_j)})\right)$$

$$= \left(\sum_{i=1}^{n} |\text{Re}(\bar{\psi}_i)|, \sum_{i=1}^{n} |\text{Re}(\bar{\psi}_i)|, \sum_{i=1}^{n} |\text{Re}(\bar{\psi}_i)|, \sum_{i=1}^{n} |\text{Re}(\tilde{\chi}_i)|, \sum_{i=1}^{n} |\text{Re}(\tilde{\chi}_i)|\right)$$

where $\mathcal{Y}$ and $\mathcal{Z}$ represent the sets of eigenvalues of $A(\bar{\psi}(s_i,s_j),\omega_{\bar{\psi}(s_i,s_j)})$ and $A(\bar{\psi}(s_i,s_j),\omega_{\bar{\psi}(s_i,s_j)})$, and $\text{Re}(\bar{\psi}_i)$ and $\text{Re}(\tilde{\chi}_i)$ represent the real parts of the eigenvalues $\bar{\psi}_i$ and $\tilde{\chi}_i$, respectively.

## 5 Randić energy of C$q$-ROFHGs

In this section, the novel concept of the Randić energy (RE) of a C$q$-ROFHG is introduced and its relevant properties are discussed in detail.

**Definition 5.1.** Let $\mathcal{G} = (\mathcal{L}, \Xi)$ be a C$q$-ROFHG on $n$ vertices. The Randić matrix (RM), $R(\theta) = (R(\hat{\psi}(s_i,s_j),\omega_{\hat{\psi}(s_i,s_j)}), R(\bar{\psi}(s_i,s_j),\omega_{\bar{\psi}(s_i,s_j)})) = [a'_{ij}]$, of $\mathcal{G}$ is a square matrix defined as:

$$a'_{ij} = \begin{cases} 
0 & \text{if } i = j, \\
\frac{1}{d_{\mathcal{G}}(s_i)d_{\mathcal{G}}(s_j)} & \text{if the nodes } s_i \text{ and } s_j \text{ of the C$q$-ROFHG } \mathcal{G} \text{ are adjacent,} \\
0 & \text{if the nodes } s_i \text{ and } s_j \text{ of the C$q$-ROFHG } \mathcal{G} \text{ are non-adjacent.}
\end{cases}$$

**Definition 5.2.** The RE of a C$q$-ROFHG $\mathcal{D} = (\mathcal{L}, \Xi)$ is defined as:

$$RE(\mathcal{G}) = \left(RE(\hat{\psi}(s_i,s_j),\omega_{\hat{\psi}(s_i,s_j)}), RE(\bar{\psi}(s_i,s_j),\omega_{\bar{\psi}(s_i,s_j)})\right)$$

$$= \left(\sum_{j=1}^{n} |\tilde{\psi}_j|, \sum_{j=1}^{n} |\omega_{\tilde{\psi}_j}|, \sum_{j=1}^{n} |\bar{\psi}_j|, \sum_{j=1}^{n} |\omega_{\bar{\psi}_j}|\right).$$

where $\mathcal{Y}_R$ and $\mathcal{Z}_R$ are the sets of Randić eigenvalues of $R(\hat{\psi}(s_i,s_j),\omega_{\hat{\psi}(s_i,s_j)})(\mathcal{G})$ and $R(\bar{\psi}(s_i,s_j),\omega_{\bar{\psi}(s_i,s_j)})(\mathcal{G})$, respectively.
Lemma 5.1. Let $\mathcal{G} = (\mathcal{L}, \Xi)$ be a Cq-ROFHG on $n$ vertices and $R(\mathcal{G}) = (R(\wp_{\Xi}(s_i, s_j)), \wp_{\Xi}(s_i, s_j)))$, $R(\wp_{\Xi}(s_i, s_j), \wp_{\Xi}(s_i, s_j)))$ be the RM of $\mathcal{G}$. Then

1. $tr(R(\mathcal{G})) = 0$

2. $tr(R^2(\mathcal{G})) = 2 \sum_{i \neq j} \frac{1}{d_{\mathcal{G}}(s_i) d_{\mathcal{G}}(s_j)}$

3. $tr(R^3(\mathcal{G})) = 2 \sum_{i \neq j} \frac{1}{d_{\mathcal{G}}(s_i) d_{\mathcal{G}}(s_j)} \left( \sum_{k \neq j} \frac{1}{d_{\mathcal{G}}(s_k)} \right)$

4. $tr(R^4(\mathcal{G})) = \sum_{i=1}^{n} \left( \sum_{i \neq j} \frac{1}{d_{\mathcal{G}}(s_i) d_{\mathcal{G}}(s_j)} \right)^2 + \sum_{i \neq j} \frac{1}{d_{\mathcal{G}}(s_i) d_{\mathcal{G}}(s_j)} \left( \sum_{k \neq j} \frac{1}{d_{\mathcal{G}}(s_k)} \right)^2$

Proof:

1. Obvious.

2. For matrix $R^2(\wp_{\Xi}(s_i, s_j), \wp_{\Xi}(s_i, s_j))((\mathcal{G}))$. If $i = j$

$$R^2(\wp_{\Xi}(s_i, s_j), \wp_{\Xi}(s_i, s_j))((\mathcal{G})) = \sum_{j=1}^{n} R(\wp_{\Xi}(s_i, s_j), \wp_{\Xi}(s_i, s_j))((\mathcal{G})) R(\wp_{\Xi}(s_i, s_i), \wp_{\Xi}(s_i, s_i))((\mathcal{G})) = \sum_{j=1}^{n} (R(\wp_{\Xi}(s_i, s_j), \wp_{\Xi}(s_i, s_j)))^2((\mathcal{G})) = \sum_{i \neq j} R(\wp_{\Xi}(s_i, s_j), \wp_{\Xi}(s_i, s_j))^2((\mathcal{G}))$$

Whereas if $i \neq j$

$$R^2(\wp_{\Xi}(s_i, s_j), \wp_{\Xi}(s_i, s_j))((\mathcal{G})) = \sum_{k=1}^{n} R(\wp_{\Xi}(s_i, s_k), \wp_{\Xi}(s_i, s_k))((\mathcal{G})) R(\wp_{\Xi}(s_i, s_j), \wp_{\Xi}(s_i, s_j))((\mathcal{G})) = R(\wp_{\Xi}(s_i, s_i), \wp_{\Xi}(s_i, s_i))((\mathcal{G})) R(\wp_{\Xi}(s_i, s_j), \wp_{\Xi}(s_i, s_j))((\mathcal{G})) + \sum_{k \neq i, j} R(\wp_{\Xi}(s_i, s_k), \wp_{\Xi}(s_i, s_k))((\mathcal{G})) R(\wp_{\Xi}(s_i, s_j), \wp_{\Xi}(s_i, s_j))((\mathcal{G}))$$

Therefore

$$tr(R^2(\wp_{\Xi}(s_i, s_j), \wp_{\Xi}(s_i, s_j))((\mathcal{G}))) = \sum_{i=1}^{n} \sum_{i \neq j} \frac{1}{d(\wp_{\Xi}(s_i, s_j)) d(\wp_{\Xi}(s_j, s_j))} \sum_{k \neq j} \frac{1}{d(\wp_{\Xi}(s_k, s_k))}.$$
3. Now, we determine the matrix $R^3(\tilde{\varphi}(s_i, s_j), \omega_\tilde{\varphi}(s_i, s_j))(\mathfrak{G})$.

$$R^3(\tilde{\varphi}(s_i, s_j), \omega_\tilde{\varphi}(s_i, s_j))(\mathfrak{G}) = \sum_{j=1}^{n} \frac{1}{d(\tilde{\varphi}(s_i, s_j), d(\tilde{\varphi}(s_j, s_j)) | R^2(\tilde{\varphi}(s_j, s_k), \omega_\tilde{\varphi}(s_j, s_k))(\mathfrak{G})|}$$

$$= \sum_{i<j} \left( \sum_{k=1}^{n} \frac{1}{d(\tilde{\varphi}(s_i, s_j), d(\tilde{\varphi}(s_k, s_k)) \sum_{k=1}^{n} \frac{1}{d(\tilde{\varphi}(s_k, s_k))} \right.$$}

Therefore

$$tr(R^3(\tilde{\varphi}(s_i, s_j), \omega_\tilde{\varphi}(s_i, s_j))(\mathfrak{G})) = \sum_{i<j} \left( \sum_{k=1}^{n} \frac{1}{d(\tilde{\varphi}(s_i, s_j), d(\tilde{\varphi}(s_k, s_k)) \sum_{k=1}^{n} \frac{1}{d(\tilde{\varphi}(s_k, s_k))} \right.$$}

Similarly, we can show that $tr(R^3(\tilde{\varphi}(s_i, s_j), \omega_\tilde{\varphi}(s_i, s_j))(\mathfrak{G})) = \sum_{i<j} \left( \sum_{k=1}^{n} \frac{1}{d(\tilde{\varphi}(s_i, s_j), d(\tilde{\varphi}(s_k, s_k)) \sum_{k=1}^{n} \frac{1}{d(\tilde{\varphi}(s_k, s_k))} \right.$$}

Hence $tr(R^3(\mathfrak{G})) = \sum_{i<j} \left( \sum_{k=1}^{n} \frac{1}{d(\tilde{\varphi}(s_i, s_j), d(\tilde{\varphi}(s_k, s_k)) \sum_{k=1}^{n} \frac{1}{d(\tilde{\varphi}(s_k, s_k))} \right.$}

4. We now calculate $tr(R^4(\tilde{\varphi}(s_i, s_j), \omega_\tilde{\varphi}(s_i, s_j))(\mathfrak{G}))$. Because $tr(R^4(\tilde{\varphi}(s_i, s_j), \omega_\tilde{\varphi}(s_i, s_j))(\mathfrak{G})) = \|R^2(\tilde{\varphi}(s_i, s_j), \omega_\tilde{\varphi}(s_i, s_j))(\mathfrak{G})\|_F^2$, where $\|R^2(\tilde{\varphi}(s_i, s_j), \omega_\tilde{\varphi}(s_i, s_j))(\mathfrak{G})\|_F$ denotes the Frobenius norm of $R^2(\tilde{\varphi}(s_i, s_j), \omega_\tilde{\varphi}(s_i, s_j))(\mathfrak{G})$, we obtain

$$tr(R^4(\tilde{\varphi}(s_i, s_j), \omega_\tilde{\varphi}(s_i, s_j))(\mathfrak{G})) = \sum_{i=1}^{n} \frac{1}{d(\tilde{\varphi}(s_i, s_j), d(\tilde{\varphi}(s_i, s_j))) (\mathfrak{G})^2}$$

Similarly, we can show that

$$tr(R^4(\tilde{\varphi}(s_i, s_j), \omega_\tilde{\varphi}(s_i, s_j))(\mathfrak{G})) = \sum_{i=1}^{n} \frac{1}{d(\tilde{\varphi}(s_i, s_j), d(\tilde{\varphi}(s_i, s_j))) (\mathfrak{G})^2}$$

Hence $tr(R^4(\mathfrak{G})) = \sum_{i=1}^{n} \frac{1}{d(\tilde{\varphi}(s_i, s_j), d(\tilde{\varphi}(s_i, s_j))) (\mathfrak{G})^2}$. 

\[\square\]

**Theorem 5.1.** Let $\mathfrak{G} = (\mathfrak{L}, \Xi)$ be a Cq-ROFHG on $n$ vertices. Then

$$RE(\mathfrak{G}) \leq \sqrt{2n \sum_{i<j} \frac{1}{d(\tilde{\varphi}(s_i), d(\tilde{\varphi}(s_j)))}.}$$
Furthermore, $RE(\mathcal{G}) = \sqrt{2n \sum_{i \sim j} \frac{1}{d(\mathcal{G})} d_{\mathcal{G}}(s_i)d_{\mathcal{G}}(s_j)}$ if and only if $\mathcal{G}$ is a Cq-ROFHG with only end vertices, or isolated vertices.

Proof. The variance of the numbers $|\tilde{\delta}_i, \omega_{\delta_i}| = \frac{1}{n} \left( \sum_{i=1}^{n} |\tilde{\delta}_i|^2, \sum_{i=1}^{n} |\omega_{\delta_i}|^2 \right) - \left( \frac{1}{n} \left( \sum_{i=1}^{n} |\tilde{\delta}_i|, \sum_{i=1}^{n} |\omega_{\delta_i}| \right) \right)^2 \geq 0$, $i = 1, 2, \ldots, n$. Now,

$$\left( \sum_{i=1}^{n} |\tilde{\delta}_i|^2, \sum_{i=1}^{n} |\omega_{\delta_i}|^2 \right) = \left( \sum_{i=1}^{n} \tilde{\delta}_i^2, \sum_{i=1}^{n} \omega_{\delta_i}^2 \right) = tr(R^2(\tilde{\varphi}(s_i, s_j), \omega_{\varphi}(s_i, s_j)))(\mathcal{G})$$

therefore

$$\frac{1}{n} tr(R^2(\tilde{\varphi}(s_i, s_j), \omega_{\varphi}(s_i, s_j)))(\mathcal{G}) \geq \left( \frac{1}{n} RE(\tilde{\varphi}(s_i, s_j), \omega_{\varphi}(s_i, s_j)) \right)^2$$

$$\iff RE(\tilde{\varphi}(s_i, s_j), \omega_{\varphi}(s_i, s_j)) \leq \frac{2n \sum_{i,j \sim} d(\mathcal{G})}{d(\tilde{\varphi}, \omega_{\varphi})(s_i)d(\tilde{\varphi}, \omega_{\varphi})(s_j)}.$$ 

If $\mathcal{G}$ is a Cq-ROFHG with only isolated vertices, i.e., without edges, then $(\tilde{\delta}_i, \omega_{\delta_i}) = (0, 0)$ for all $i = 1, 2, \ldots, n$, and therefore $RE(\tilde{\varphi}(s_i, s_j), \omega_{\varphi}(s_i, s_j))(\mathcal{G}) = 0$. Since no vertices are adjacent, $\sum_{i,j \sim} d(\mathcal{G}) = 0$. If $\mathcal{G}$ is a Cq-ROFHG with only end vertices, i.e. incident with one edge, then $(\tilde{\delta}_i, \omega_{\delta_i}) = \pm d(\mathcal{G})$, so the variance of $(|\tilde{\delta}_i|, |\omega_{\delta_i}|) = 0$, $i = 1, 2, \ldots, n$. Thus $RE(\tilde{\varphi}(s_i, s_j), \omega_{\varphi}(s_i, s_j))(\mathcal{G}) = \sqrt{2n \sum_{i,j \sim} d(\mathcal{G})}$. Analogously, we can show that

$$\iff RE(\tilde{\varphi}(s_i, s_j), \omega_{\varphi}(s_i, s_j))(\mathcal{G}) \leq \frac{2n \sum_{i,j \sim} d(\mathcal{G})}{d(\tilde{\varphi}, \omega_{\varphi})(s_i)d(\tilde{\varphi}, \omega_{\varphi})(s_j)}.$$ 

Hence

$$RE(\mathcal{G}) \leq \frac{2n \sum_{i,j \sim} d(\mathcal{G})}{d(\tilde{\varphi}, \omega_{\varphi})(s_i)d(\tilde{\varphi}, \omega_{\varphi})(s_j)}.$$ 

Theorem 5.2. Let $\mathcal{G} = (\mathcal{L}, \Xi)$ be a Cq-ROFHG on $n$ vertices and at least one edge. Then

$$RE(\mathcal{G}) \geq \frac{2 \sum_{i \sim j} \frac{1}{d(\mathcal{G})} d_{\mathcal{G}}(s_i)d_{\mathcal{G}}(s_j)}{\sqrt{\left( \frac{2 \sum_{i \sim j} \frac{1}{d(\mathcal{G})} d_{\mathcal{G}}(s_i)d_{\mathcal{G}}(s_j)}{\sqrt{n} \sum_{i \sim j} \frac{1}{d(\mathcal{G})} d_{\mathcal{G}}(s_i)d_{\mathcal{G}}(s_j)} \right)^2 + \sum_{i \sim j} \frac{1}{d(\mathcal{G})} d_{\mathcal{G}}(s_i)d_{\mathcal{G}}(s_j) \left( \frac{2 \sum_{i \sim j} \frac{1}{d(\mathcal{G})} d_{\mathcal{G}}(s_i)d_{\mathcal{G}}(s_j)}{\sqrt{n} \sum_{i \sim j} \frac{1}{d(\mathcal{G})} d_{\mathcal{G}}(s_i)d_{\mathcal{G}}(s_j)} \right)^2}}.$$ 

Proof. According to the Hölder inequality

$$\sum_{i=1}^{n} l_i m_i \leq \left( \sum_{i=1}^{n} l_i^p \right)^{\frac{1}{p}} \left( \sum_{i=1}^{n} m_i^q \right)^{\frac{1}{q}}$$

which holds for any real numbers $l_i, m_i \geq 0$, $i = 1, 2, \ldots, n$. Setting $l_i = (|\tilde{\delta}_i|^4, |\omega_{\delta_i}|^4)$, $m_i = (|\tilde{\delta}_i|^4, |\omega_{\delta_i}|^4)$, $p = \frac{1}{2}$, and $q = 3$, we obtain

$$\left( \sum_{i=1}^{n} |\tilde{\delta}_i|^4, \sum_{i=1}^{n} |\omega_{\delta_i}|^4 \right) \leq \left( \sum_{i=1}^{n} |\tilde{\delta}_i|, \sum_{i=1}^{n} |\omega_{\delta_i}| \right)^{\frac{3}{4}} \left( \sum_{i=1}^{n} |\tilde{\delta}_i|^4, \sum_{i=1}^{n} |\omega_{\delta_i}|^4 \right)^{\frac{1}{4}}.$$
If the Cq-ROFHG $\mathfrak{G}$ has at least one edge, then all $(\delta_i, \omega_{s_i})$'s are not equal to zero. Then \( \left( \sum_{i=1}^{n} |\delta_i|^4, \sum_{i=1}^{n} |\omega_{s_i}|^4 \right) \neq 0 \) and

\[
RE(\tilde{\nu}(s, s_j), \omega_{\tilde{\nu}(s, s_j)})(\mathfrak{G}) = \left( \sum_{i=1}^{n} |\delta_i|^2, \sum_{i=1}^{n} |\omega_{s_i}|^2 \right)^{\frac{1}{2}} \geq \sqrt{\frac{\text{tr}(R^2(\tilde{\nu}(s, s_j), \omega_{\tilde{\nu}(s, s_j)})(\mathfrak{G}))^2}{\text{tr}(R^4(\tilde{\nu}(s, s_j), \omega_{\tilde{\nu}(s, s_j)})(\mathfrak{G}))}}
\]

Similarly,

\[
RE(\tilde{\nu}(s, s_j), \omega_{\tilde{\nu}(s, s_j)})(\mathfrak{G}) \geq 2 \sum_{i \sim j} \frac{1}{d_{\tilde{\nu}}(s_i)d_{\tilde{\nu}}(s_j)} \sqrt{\sum_{i \sim j} \frac{1}{d_{\tilde{\nu}}(s_i)d_{\tilde{\nu}}(s_j)}} \geq \sqrt{\frac{\text{tr}(R^2(\tilde{\nu}(s, s_j), \omega_{\tilde{\nu}(s, s_j)})(\mathfrak{G}))^2}{\text{tr}(R^4(\tilde{\nu}(s, s_j), \omega_{\tilde{\nu}(s, s_j)})(\mathfrak{G}))}}
\]

Hence

\[
RE(\mathfrak{G}) \geq 2 \sum_{i \sim j} \frac{1}{d_{\tilde{\nu}}(s_i)d_{\tilde{\nu}}(s_j)} \sqrt{\sum_{i \sim j} \frac{1}{d_{\tilde{\nu}}(s_i)d_{\tilde{\nu}}(s_j)}} \geq \sqrt{\frac{\text{tr}(R^2(\tilde{\nu}(s, s_j), \omega_{\tilde{\nu}(s, s_j)})(\mathfrak{G}))^2}{\text{tr}(R^4(\tilde{\nu}(s, s_j), \omega_{\tilde{\nu}(s, s_j)})(\mathfrak{G}))}}
\]

**Theorem 5.3.** Let $\mathfrak{G} = (\mathcal{L}, \Xi)$ be a Cq-ROFHG on $n$ vertices. If $\mathfrak{G}$ is regular of degree $((p, \omega_p), (r, \omega_r))$ where $p, r > 0$, then

\[
RE(\mathfrak{G}) = \frac{1}{((p, \omega_p), (r, \omega_r))} E(\mathfrak{G}).
\]

**Proof.** Assume that $\mathfrak{G}$ is a regular Cq-ROFHG of degree $((p, \omega_p), (r, \omega_r))$ and $p, r > 0$, i.e., $d_{\tilde{\nu}}(s_1) = d_{\tilde{\nu}}(s_2) = \ldots = d_{\tilde{\nu}}(s_n) = (p, \omega_p)$. Then all non zero entries of $R(\tilde{\nu}(s, s_i), \omega_{\tilde{\nu}(s, s_i)})(\mathfrak{G})$ are equal to $\frac{1}{(p, \omega_p)}$, implying that $R(\tilde{\nu}(s, s_i), \omega_{\tilde{\nu}(s, s_i)})(\mathfrak{G}) = \frac{1}{(p, \omega_p)} A(\tilde{\nu}(s, s_i), \omega_{\tilde{\nu}(s, s_i)})(\mathfrak{G})$. Therefore, for all $i = 1, 2, \ldots, n$

\[
(\delta_i, \omega_{s_i}) = \frac{1}{(p, \omega_p)} (\tilde{\nu}_i, \omega_{s_i})
\]
\[
\left( \sum_{i=1}^{n} \delta_i, \sum_{i=1}^{n} \omega_i \right) = \frac{1}{\langle p_i, \omega_{p_i} \rangle} \left( \sum_{i=1}^{n} \psi_i, \sum_{i=1}^{n} \omega_{\psi_i} \right).
\]

\[
RE(\tilde{\psi}(s_i s_j), \omega_{\tilde{\psi}}(s_i s_j))(\mathcal{G}) = \frac{1}{\langle p_i, \omega_{p_i} \rangle} E(\tilde{\psi}(s_i s_j), \omega_{\tilde{\psi}}(s_i s_j))(\mathcal{G}).
\]

Similarly, we can show that
\[
RE(\tilde{\mathcal{R}}(s_i s_j), \omega_{\tilde{\mathcal{R}}}(s_i s_j))(\mathcal{G}) = \frac{1}{\langle r_i, \omega_{r_i} \rangle} E(\tilde{\mathcal{R}}(s_i s_j), \omega_{\tilde{\mathcal{R}}}(s_i s_j))(\mathcal{G}).
\]

Hence
\[
RE(\mathcal{G}) = \frac{1}{\langle p_i, \omega_{p_i} \rangle, \langle r_i, \omega_{r_i} \rangle} E(\mathcal{G}).
\]

**Example 5.1.** Let \( \mathcal{G} = (\mathcal{L}, \mathcal{E}) \) be a C5-ROFHG on \( \mathcal{V} = \{s_1, s_2, s_3, s_4, s_5, s_6\} \) and \( \tilde{\mathcal{E}} = \{s_1 s_2, s_2 s_3, s_3 s_4, s_4 s_5, s_5 s_6, s_6 s_1, s_2 s_6, s_3 s_5\} \), as in Fig. 7, defined by:

\[
\mathcal{L} = \left( \begin{array}{cccccc}
\varnothing & s_1 & s_2 & s_3 & s_4 & s_5 \\
\varnothing & (0.0, 0.9, 0.8) & (0.8, 0.7) & (0.7, 0.9) & (0.6, 0.8) & (0.9, 0.7) \\
\varnothing & (0.8, 0.9) & (0.5, 0.8) & (0.7, 0.8) & (0.9, 0.6) & \end{array} \right),
\]

\[
\mathcal{M} = \left( \begin{array}{cccccc}
s_1 & s_2 & s_3 & s_4 & s_5 & s_6 \\
(0.72, 0.59) & (0.82, 0.92) & (0.50, 0.58) & (0.90, 0.91) & (0.54, 0.65) & (0.91, 0.75) \\
(0.57, 0.71) & (0.63, 0.65) & (0.92, 0.83) & (0.59, 0.60) & (0.91, 0.91) & (0.51, 0.72) & (0.91, 0.81) \\
\end{array} \right).
\]

**Figure 7:** C5-ROFHG.

The \( A(\mathcal{G}) \), \( R(\mathcal{G}) \), spectrum, and \( RE(\mathcal{G}) \) of the C5-ROFHG, shown in Fig. 7 are as follows:

\[
A(\mathcal{G}) = \left( \begin{array}{cccccc}
(0, 0) & (0.72, 0.59) & (0.82, 0.92) & (0, 0) \\
(0.72, 0.59) & (0, 0) & (0, 0) & (0.50, 0.58) & (0.90, 0.91) \\
(0, 0) & (0.50, 0.58) & (0, 0) & (0, 0) & (0.54, 0.65) & (0.91, 0.75) \\
(0, 0) & (0, 0) & (0, 0) & (0, 0) & (0.51, 0.72) & (0.91, 0.81) \\
(0.63, 0.65) & (0.92, 0.83) & (0.59, 0.60) & (0.91, 0.91) & (0, 0) \\
(0.54, 0.65) & (0.91, 0.75) & (0.51, 0.72) & (0.91, 0.81) & (0, 0) & (0, 0) \\
(0.72, 0.59) & (0.91, 0.75) & (0.51, 0.72) & (0.91, 0.81) & (0, 0) & (0, 0) \\
(0, 0) & (0, 0) & (0, 0) & (0, 0) & (0, 0) & (0, 0) \\
\end{array} \right).
\]
Now we will define the RM and RE of Cq-ROFHDG.

**Definition 5.3.** Let $\mathcal{D} = (\mathcal{L}, \Xi)$ be a Cq-ROFHDG on $n$ vertices. The RM, $R(\mathcal{D}) = (R(\bar{\varphi}_\Xi(s_i s_j), \omega_{\bar{\varphi}_\Xi(s_i s_j)}), R(\bar{\Theta}_\Xi(s_i s_j), \omega_{\bar{\Theta}_\Xi(s_i s_j)})) = [s_{ij}]$, of $\mathcal{D}$ is a $n \times n$ matrix defined as:

$$s_{ij} = \begin{cases} 0 & \text{if } i = j, \\ \frac{1}{d_{\bar{\varphi}_\Xi}(s_i)d_{\bar{\varphi}_\Xi}(s_j)} & \text{if the nodes } s_i \text{ and } s_j \text{ of the Cq-ROFHDG } \Xi \text{ are adjacent,} \\ 0 & \text{if the nodes } s_i \text{ and } s_j \text{ of the Cq-ROFHDG } \Xi \text{ are non adjacent.} \end{cases}$$

**Definition 5.4.** The RE of a Cq-ROFHDG $\mathcal{D} = (\mathcal{L}, \Xi)$ is defined as:

$$RE(\mathcal{D}) = \left( RE(\bar{\varphi}_\Xi(s_i s_j), \omega_{\bar{\varphi}_\Xi(s_i s_j)}), RE(\bar{\Theta}_\Xi(s_i s_j), \omega_{\bar{\Theta}_\Xi(s_i s_j)}) \right)$$

$$= \left( \sum_{i=1}^{n} |\text{Re}(\delta_i)|, \sum_{i=1}^{n} |\omega_{\text{Re}}(\delta_i)|, \sum_{i=1}^{n} |\text{Re}(\eta_i)|, \sum_{i=1}^{n} |\omega_{\text{Re}}(\eta_i)| \right)$$

where $\mathcal{Y}_R, \mathcal{Z}_R$ are the sets of Randić eigenvalues of $R(\bar{\varphi}_\Xi(s_i s_j), \omega_{\bar{\varphi}_\Xi(s_i s_j)})(\mathcal{D})$ and $R(\bar{\Theta}_\Xi(s_i s_j), \omega_{\bar{\Theta}_\Xi(s_i s_j)})(\mathcal{D})$, whereas $\text{Re}(\delta_i)$ and $\text{Re}(\eta_i)$ express the real parts of the eigenvalues $\delta_i$ and $\eta_i$, respectively.

6 Novel MAGDM method based on proposed concepts of energy and Randić energy of Cq-ROFGs

This section proposes a new MAGDM approach based on Cq-ROFDG to solve MAGDM problems where there are relationships between attributes. First of all, we present this kind of problem. We will then develop the procedure of the proposed method in depth.

The strategy is outlined in the given Algorithm 1.
Algorithm 1: The algorithm for the optimal facade clothing system (FCS) selection.

**INPUT:** A discrete set of FCSs (alternatives) $X = \{\mathbb{F}_1, \mathbb{F}_2, \ldots, \mathbb{F}_n\}$, a set of specialists $e = \{e_1, e_2, \ldots, e_s\}$ and construction of $Cq$-ROFPR $\mathbb{A}_{k} = (\mathbb{A}_{ij})_{n \times n}$ ($i, j = 1, 2, \ldots, n$) for each specialist.

**OUTPUT:** The most advantageous FCS selection.

**Step 1.** Determine the $E(\mathbb{D}_k)$ and $RE(\mathbb{D}_k)$ ($k = 1, 2, \ldots, s$) of each $Cq$-ROFDG.

**Step 2.** On the basis of $E(\mathbb{D}_k)$ and $RE(\mathbb{D}_k)$, calculate the weight vector of experts.

$$\mathbf{w}_k = \left( \frac{E((\mathbb{D}_\circ)_k)}{\sum_{l=1}^{s} E((\mathbb{D}_\circ)_l)}, \frac{E((\mathbb{D}_\ominus)_k)}{\sum_{l=1}^{s} E((\mathbb{D}_\ominus)_l)}, \frac{E((\mathbb{D}_\ominus)_k)}{\sum_{l=1}^{s} E((\mathbb{D}_\ominus)_l)}, \frac{E((\mathbb{D}_\ominus)_k)}{\sum_{l=1}^{s} E((\mathbb{D}_\ominus)_l)} \right),$$

(6.1)

and

$$\mathbf{w}_k = \left( \frac{RE((\mathbb{D}_\circ)_k)}{\sum_{l=1}^{s} RE((\mathbb{D}_\circ)_l)}, \frac{RE((\mathbb{D}_\ominus)_k)}{\sum_{l=1}^{s} RE((\mathbb{D}_\ominus)_l)}, \frac{RE((\mathbb{D}_\ominus)_k)}{\sum_{l=1}^{s} RE((\mathbb{D}_\ominus)_l)}, \frac{RE((\mathbb{D}_\ominus)_k)}{\sum_{l=1}^{s} RE((\mathbb{D}_\ominus)_l)} \right),$$

(6.2)

**Step 3.** Aggregate all $\mathbb{A}_{ij}^{(k)}$ ($i, j = 1, 2, \ldots, n$) corresponding to the FCS $\mathbb{F}_i$, and get the $Cq$-ROF element ($Cq$-ROFE) $\mathbb{A}_{ij}^{(k)}$ of the FCSs $\mathbb{F}_i$ over all the other FCSs for the specialist $e_k$ by using the $Cq$-ROFA operator.

**Step 4.** Aggregate all $\mathbb{A}_{ij}^{(k)}$ ($k = 1, 2, \ldots, s$) into a fused $Cq$-ROFN $\mathbb{A}_i$ for the FCS $\mathbb{F}_i$ using the $Cq$-ROFWA operator.

**Step 5.** Determine the score functions $f(\mathbb{A}_i)$ ($i = 1, 2, \ldots, n$), utilizing Def. 2.4.

**Step 6.** Rank all the FCSs $\mathbb{F}_i$ according to $f(\mathbb{A}_i)$ ($i = 1, 2, \ldots, n$).

**Step 7.** Output the best FCS.

Fig. 8 shows the flow chat of MAGDM based on $Cq$-ROFHWA operator ($K = 1$).
Construct Cq-ROFPRs for each decision maker

Compute energy and Randić energy of each Cq-ROFDG

Determine the score function

Aggregate all Cq-ROFNs into collective Cq-ROFNs

Calculate weight vector

Rank the given alternatives based on score values

Select the optimal alternative

End

Figure 8: Flow chart of MAGDM based on Cq-ROFHWA operator (n = 1).

7 Numerical Example

The prior section presents a new MAGDM process. In order to further explain the methodology of the suggested decision-making approach, we are applying it to a real decision-making problem.

Suppose a group of DMs compare alternatives FCSs for the surface clothing of building compliant with their practical properties. The gathering interacts with four experts; e₁ = architect, e₂ = structural designer, e₃ = constructor and e₄ = adviser, independently. The experts analyze four choices frameworks which are:

- ℳ₁: “natural stone clothing”;
- ℳ₂: “plastic painting”;
- ℳ₃: “compact laminate clothing”;
- ℳ₄: “wood clothing”.

The experts compare each pair of attributes ℳᵢ and ℳⱼ (i, j = 1, 2, 3, 4), and provide Cq-ROFNs $\mathcal{F}_{ij}^k = (\tilde{\varphi}_{ij}^k, \omega_{ij}^k, \tilde{\rho}_{ij}^k, \omega_{ij}^k)$ (k = 1, 2, 3, 4), composed of the complex MD $(\tilde{\varphi}_{ij}^k, \omega_{ij}^k)$ to which ℳᵢ is preferable to ℳⱼ and the complex NMD $(\tilde{\rho}_{ij}^k, \omega_{ij}^k)$ to which ℳᵢ is not preferable to ℳⱼ, and then develop the Cq-ROFPRs $\mathcal{A}_k = (\mathcal{F}_{ij}^k)_{4 \times 4}$ (k = 1, 2, 3, 4) as follows:

Step 1. The Cq-ROFDGs $\mathcal{D}_k$ according to Cq-ROFPRs in Tables 1-4, are shown in Fig. 9.
Table 1: Cq-ROFPs of the architect.

| $A_1$ | $A_2$ | $A_3$ | $A_4$ |
|-------|-------|-------|-------|
| $T_1$ | $T_2$ | $T_3$ | $T_4$ |
| $(0.5,0.5),(0.5,0.5)$ | $(0.6,0.8),(0.8,0.7)$ | $(0.8,0.6),(0.4,0.9)$ | $(0.7,0.9),(0.7,0.5)$ |
| $(0.8,0.7),(0.6,0.8)$ | $(0.5,0.5),(0.5,0.5)$ | $(0.7,0.7),(0.8,0.6)$ | $(0.6,0.7),(0.6,0.8)$ |
| $(0.4,0.9),(0.8,0.6)$ | $(0.8,0.6),(0.7,0.7)$ | $(0.5,0.5),(0.5,0.5)$ | $(0.9,0.6),(0.6,0.8)$ |
| $(0.7,0.5),(0.7,0.9)$ | $(0.6,0.8),(0.6,0.7)$ | $(0.6,0.8),(0.9,0.6)$ | $(0.5,0.5),(0.5,0.5)$ |

Table 2: Cq-ROFPs of the structural designer.

| $A_2$ | $A_1$ | $A_3$ | $A_4$ |
|-------|-------|-------|-------|
| $T_1$ | $T_2$ | $T_3$ | $T_4$ |
| $(0.5,0.5),(0.5,0.5)$ | $(0.6,0.6),(0.7,0.8)$ | $(0.7,0.9),(0.8,0.3)$ | $(0.5,0.9),(0.9,0.6)$ |
| $(0.7,0.8),(0.6,0.6)$ | $(0.5,0.5),(0.5,0.5)$ | $(0.8,0.6),(0.6,0.8)$ | $(0.7,0.8),(0.7,0.7)$ |
| $(0.8,0.3),(0.7,0.9)$ | $(0.6,0.8),(0.6,0.6)$ | $(0.5,0.5),(0.5,0.5)$ | $(0.5,0.6),(0.7,0.6)$ |
| $(0.9,0.6),(0.5,0.9)$ | $(0.7,0.7),(0.7,0.8)$ | $(0.7,0.6),(0.5,0.6)$ | $(0.5,0.5),(0.5,0.5)$ |

Table 3: Cq-ROFPs of the constructor.

| $A_3$ | $A_1$ | $A_2$ | $A_4$ |
|-------|-------|-------|-------|
| $T_1$ | $T_2$ | $T_3$ | $T_4$ |
| $(0.5,0.5),(0.5,0.5)$ | $(0.7,0.8),(0.5,0.4)$ | $(0.8,0.5),(0.6,0.9)$ | $(0.5,0.6),(0.8,0.7)$ |
| $(0.5,0.4),(0.7,0.8)$ | $(0.5,0.5),(0.5,0.5)$ | $(0.9,0.6),(0.4,0.8)$ | $(0.7,0.7),(0.8,0.7)$ |
| $(0.6,0.9),(0.8,0.5)$ | $(0.4,0.8),(0.9,0.6)$ | $(0.5,0.5),(0.5,0.5)$ | $(0.6,0.9),(0.8,0.6)$ |
| $(0.8,0.7),(0.5,0.6)$ | $(0.8,0.7),(0.7,0.7)$ | $(0.8,0.6),(0.6,0.9)$ | $(0.5,0.5),(0.5,0.5)$ |

Table 4: Cq-ROFPs of the adviser.

| $A_4$ | $A_1$ | $A_2$ | $A_3$ | $A_4$ |
|-------|-------|-------|-------|-------|
| $T_1$ | $T_2$ | $T_3$ | $T_4$ | $T_1$ |
| $(0.5,0.5),(0.5,0.5)$ | $(0.6,0.6),(0.9,0.8)$ | $(0.7,0.8),(0.8,0.6)$ | $(0.5,0.9),(0.8,0.5)$ |
| $(0.9,0.8),(0.6,0.6)$ | $(0.5,0.5),(0.5,0.5)$ | $(0.8,0.6),(0.7,0.8)$ | $(0.5,0.4),(0.6,0.7)$ |
| $(0.8,0.6),(0.7,0.8)$ | $(0.7,0.8),(0.8,0.6)$ | $(0.5,0.5),(0.5,0.5)$ | $(0.5,0.6),(0.9,0.9)$ |
| $(0.8,0.5),(0.5,0.9)$ | $(0.6,0.7),(0.5,0.4)$ | $(0.9,0.9),(0.5,0.6)$ | $(0.5,0.5),(0.5,0.5)$ |

Figure 9: C3-ROFDG.

Step 2. The energy of each C3-ROFDG is determined in Table 5:
**Step 3.** Utilizing Equation (6.1), each expert’s weight (see Table 6) can be calculated as:

\[
\omega_{ij} = \left(\prod_{j=1}^{n} (1 - (\omega_i)^q)\right)^{1/n}, \quad \left(\prod_{j=1}^{n} (1 - (\omega_{ij})^q)\right)^{1/n}, \quad \left(\prod_{j=1}^{n} (1 - (\omega_{ji})^q)\right)^{1/n}
\]

The final results are shown in Table 7.

**Table 7: The fused results of the experts \(\epsilon_k\) (\(k = 1, 2, 3, 4\)).**

| Experts | \(z_{i1}^{(k)}\) | \(z_{i2}^{(k)}\) | \(z_{i3}^{(k)}\) | \(z_{i4}^{(k)}\) |
|---------|-----------------|-----------------|-----------------|-----------------|
| \(\epsilon_1\) | (0.6805, 0.6760), (0.5785, 0.6300) | (0.6805, 0.6651), (0.6160, 0.6620) | (0.7526, 0.7257), (0.6402, 0.6402) | (0.6118, 0.7024), (0.6593, 0.6593) |
| \(\epsilon_2\) | (0.5921, 0.8073), (0.7085, 0.5180) | (0.7012, 0.7142), (0.5958, 0.6402) | (0.6418, 0.6253), (0.6654, 0.6344) | (0.7573, 0.6118), (0.5439, 0.6817) |
| \(\epsilon_3\) | (0.6665, 0.6418), (0.5886, 0.5958) | (0.7321, 0.5790), (0.5785, 0.6880) | (0.5402, 0.8336), (0.7326, 0.5477) | (0.7579, 0.6401), (0.5692, 0.6593) |
| \(\epsilon_4\) | (0.5921, 0.7670), (0.7326, 0.5886) | (0.7582, 0.6315), (0.5958, 0.6402) | (0.6665, 0.6575), (0.7085, 0.6817) | (0.7670, 0.7321), (0.5000, 0.5733) |

**Step 5.** Compute a collective Cq-ROFE \(z_i\) (\(i = 1, 2, 3, 4\)) of the FCS \(\xi_i\) over all the other FCSs based on the Cq-ROFHWA operator with \(n = 1\) (Table 8).

---

**Table 5: Energy of each C3-ROFDG.**

| Energy | \(\tilde{\phi}\) | \(\omega_\phi\) | \(\Re\) | \(\omega_\Re\) |
|--------|------------------|-----------------|---------|-----------------|
| \(E(\xi_1)\) | 4.0943 | 4.3907 | 4.0943 | 4.3907 |
| \(E(\xi_2)\) | 4.0586 | 4.1770 | 4.0586 | 4.1770 |
| \(E(\xi_3)\) | 3.9962 | 4.1055 | 3.9962 | 4.1055 |
| \(E(\xi_4)\) | 4.0969 | 4.2162 | 4.0969 | 4.2162 |

**Table 6: Weight of each C3-ROFDG.**

| Weights | \(\tilde{\phi}\) | \(\omega_\phi\) | \(\Re\) | \(\omega_\Re\) |
|---------|------------------|-----------------|---------|-----------------|
| \(\varpi_1\) | 0.2520 | 0.2600 | 0.2520 | 0.2600 |
| \(\varpi_2\) | 0.2498 | 0.2473 | 0.2498 | 0.2473 |
| \(\varpi_3\) | 0.2460 | 0.2431 | 0.2460 | 0.2431 |
| \(\varpi_4\) | 0.2522 | 0.2496 | 0.2522 | 0.2496 |

**Table 8: Energy of each C3-ROFDG.**
Step 6. Determine the score functions $f(\bar{x}_i)$ of $\bar{x}_i(i = 1, 2, 3, 4)$, utilizing Def. 2.4.

$$f(\bar{x}_1) = 0.5549, \ f(\bar{x}_2) = 0.5393, \ f(\bar{x}_3) = 0.5285, \ f(\bar{x}_4) = 0.5649.$$  

Step 7. Rank all the FCSs $\bar{\gamma}_i(i = 1, 2, 3, 4)$ according to the values of $f(\bar{x}_i)(i = 1, 2, 3, 4)$. Then, $\bar{\gamma}_4 \succ \bar{\gamma}_1 \succ \bar{\gamma}_2 \succ \bar{\gamma}_3$.

Step 8. Thus, the optimal FCS is $\bar{\gamma}_4$.

Now, the RMs of the Cq-ROFDGs $R(\mathcal{D}_k) = R^R_k (k = 1, 2, 3, 4)$, (Fig. 9) are shown in Tables 9-12:

Table 8: Aggregated C3-ROFN.

| Aggregated Values | $\varphi$ | $\omega_\varphi$ | $R$ | $\omega_R$ |
|-------------------|----------|-----------------|-----|----------|
| $\mathcal{D}_1$   | 0.6367   | 0.7558          | 0.6487 | 0.5822  |
| $\mathcal{D}_2$   | 0.7203   | 0.6537          | 0.5965 | 0.6572  |
| $\mathcal{D}_3$   | 0.6646   | 0.7284          | 0.6855 | 0.6247  |
| $\mathcal{D}_4$   | 0.7327   | 0.6779          | 0.5652 | 0.6420  |

Table 9: RM of the Cq-ROFDG $\mathcal{D}_1$.

| $R^R_1$ | $\mathcal{D}_1$ | $\mathcal{D}_2$ | $\mathcal{D}_3$ | $\mathcal{D}_4$ |
|---------|-----------------|-----------------|-----------------|-----------------|
| $\mathcal{D}_1$ | (0.50, 0.50), (0.50, 0.50) | (0.48, 0.46), (0.51, 0.47) | (0.48, 0.46), (0.50, 0.48) | (0.50, 0.46), (0.49, 0.47) |
| $\mathcal{D}_2$ | (0.51, 0.47), (0.48, 0.46) | (0.50, 0.50), (0.50, 0.50) | (0.48, 0.48), (0.49, 0.47) | (0.49, 0.48), (0.48, 0.45) |
| $\mathcal{D}_3$ | (0.50, 0.48), (0.48, 0.46) | (0.49, 0.47), (0.48, 0.48) | (0.50, 0.50), (0.50, 0.50) | (0.50, 0.50), (0.47, 0.47) |
| $\mathcal{D}_4$ | (0.49, 0.47), (0.50, 0.46) | (0.48, 0.45), (0.49, 0.48) | (0.47, 0.47), (0.50, 0.48) | (0.50, 0.50), (0.50, 0.50) |

Step 1. The Randić energy of each Cq-ROFDG is calculated in Table 13:

Table 13: Randić energy of each C3-ROFDG.

| $RE(\mathcal{D}_k)$ | $\varphi$ | $\omega_\varphi$ | $R$ | $\omega_R$ |
|---------------------|----------|-----------------|-----|----------|
| $RE(\mathcal{D}_1)$ | 2.9345   | 2.8147          | 2.9345 | 2.8147  |
| $RE(\mathcal{D}_2)$ | 2.9547   | 2.9713          | 2.9547 | 2.9713  |
| $RE(\mathcal{D}_3)$ | 3.0075   | 2.9699          | 3.0075 | 2.9699  |
| $RE(\mathcal{D}_4)$ | 2.9368   | 2.9489          | 2.9368 | 2.9489  |
Step 2. Utilizing Equation (6.2), each expert’s weight (see Table 14) can be calculated as:

| Weights | $\omega_1$ | $\omega_2$ | $\omega_3$ | $\omega_4$ |
|---------|-------------|-------------|-------------|-------------|
| $\gamma_1$ | 0.2507 | 0.2405 | 0.2507 | 0.2405 |
| $\gamma_2$ | 0.2497 | 0.2539 | 0.2497 | 0.2539 |
| $\gamma_3$ | 0.2542 | 0.2537 | 0.2542 | 0.2537 |
| $\gamma_4$ | 0.2482 | 0.2519 | 0.2482 | 0.2519 |

Step 3. Now, we will utilize above calculated weights and determine a collective $C_q$-ROFE $\tilde{\gamma}_i$ ($i = 1, 2, 3, 4$) of the FCS $\gamma_i$ over all the other FCSs based on the $C_q$-ROFHW operator with $N = 1$ (Table 15):

$$\tilde{\gamma}_i = C_q$-ROFHWA($\tilde{\gamma}_1^{(1)}, \tilde{\gamma}_2^{(2)}, \ldots, \tilde{\gamma}_4^{(n)}$)

| Aggregated values | $\tilde{\omega}_1$ | $\tilde{\omega}_2$ | $\tilde{\omega}_3$ | $\tilde{\omega}_4$ |
|-------------------|------------------|------------------|------------------|------------------|
| $\tilde{\gamma}_1$ | 0.6376 | 0.7551 | 0.6471 | 0.5810 |
| $\tilde{\gamma}_2$ | 0.7208 | 0.6532 | 0.5955 | 0.6573 |
| $\tilde{\gamma}_3$ | 0.6642 | 0.7293 | 0.6851 | 0.6237 |
| $\tilde{\gamma}_4$ | 0.7334 | 0.6768 | 0.5645 | 0.6419 |

Step 4. Determine the score functions $F(\tilde{\gamma}_i)$ of the $C_q$-ROFE $\tilde{\gamma}_i$ ($i = 1, 2, 3, 4$), utilizing Def. 2.4.

$$F(\tilde{\gamma}_1) = 0.5556, F(\tilde{\gamma}_2) = 0.5395, F(\tilde{\gamma}_3) = 0.5292, F(\tilde{\gamma}_4) = 0.5651.$$

Step 5. Rank all the FCSs $\gamma_i$ ($i = 1, 2, 3, 4$) according to the values of $F(\tilde{\gamma}_i)$ ($i = 1, 2, 3, 4$). Then, $\gamma_4 \succ \gamma_1 \succ \gamma_2 \succ \gamma_3$.

Step 6. Thus, the optimal FCS is $\gamma_4$ among the four given FCSs.

7.1 Comparison analysis

To examine the consequences of the technique utilized, we will compare our results with previous findings in the literature.

- We compare our approach with the $C_q$-ROF weighted averaging (Cq-ROFWA) operator suggested by Liu et al. [41], by taking parameters $N = 1$ and $q = 3$. The ranking results which obtained from Cq-ROFWA operator are listed as: $\gamma_4 \succ \gamma_1 \succ \gamma_2 \succ \gamma_3$. We get the same ranking outcomes as in literature. Nevertheless, the methodology of graph adopted based on the Hamacher TN and TCN, and by taking parameter $N = 1$, Hamacher TNs and TCNs are converted into the algebraic TNs and TCNs. As the algebraic TNs and TCNs are special instance of the Hamacher TNs and TCNs. So, the scheme described in this article is more comprehensive than the methodology proposed by Liu et al. [41].

- Comparing with the $C_q$-ROF Einstein weighted averaging (Cq-ROFEWA) operator by taking parameters $N = 2$, and $q = 3$. The ranking outcomes by utilizing Cq-ROFEWA operator are obtained as: $\gamma_4 \succ \gamma_1 \succ \gamma_2 \succ \gamma_3$. But, the approach proposed in this article based on the Hamacher TN and TCN, and the Einstein TN and TCN are just a special case of the Hamacher TN and TCN when we take the parameter $N = 2$. Hence, the proposed approach is more effective than the Cq-ROFEWA operator.

- We compare our proposed approach with the CIF Hamacher weighted averaging (CIFHWA) operator by taking $q = 1$. Utilizing CIFHWA operator, the ranking outcomes are acquired as: $\gamma_4 \succ \gamma_1 \succ \gamma_2 \succ \gamma_3$. The CIFHWA operator just aggregates the CIFNs, and the CIF must meet the conditions that $0 \leq \tilde{\omega}_\phi + \tilde{\omega}_\bar{\phi} \leq 1$ and $0 \leq \omega_\phi + \omega_\bar{\phi} \leq 1$. Clearly, most of the assessment values do not follow the limit of condition described as: $0 \leq \tilde{\omega}_\phi + \tilde{\omega}_\bar{\phi} \leq 1$ and $0 \leq \omega_\phi + \omega_\bar{\phi} \leq 1$, so, this example, shows that the CIFHWA operator is not reasonable in described instance.
Further, we compare our developed strategy with the CPF Hamacher weighted averaging (CPFHWA) operator (i.e., taking $q=2$). The CPFHWA operator gives the ranking results as: $\ell_4 > \ell_1 > \ell_2 > \ell_3$. The CPFHWA operator just aggregates the CPFNs and has wider range than CIFHWA operator, and the CPFN should satisfy the limit of condition described as: $0 \leq \phi^2 + \Re^2 \leq 1$ and $0 \leq \omega_\phi^2 + \omega_\Re^2 \leq 1$. Clearly, most of the assessment values do not follow the limit of condition described as: $0 \leq \phi^2 + \Re^2 \leq 1$ and $0 \leq \omega_\phi^2 + \omega_\Re^2 \leq 1$ in this example. Hence, the CPFHWA operator is not appropriate for that described example.

Now developed approach will be compared with the complex fermatean fuzzy Hamacher weighted averaging (CFFHWA) operator by taking $q=3$. According to CFFHWA operator, we get the ranking results as: $\ell_4 > \ell_1 > \ell_2 > \ell_3$. The CFFHWA operator satisfies the conditional limit of $0 \leq \phi^3 + \Re^3 \leq 1$ and $0 \leq \omega_\phi^3 + \omega_\Re^3 \leq 1$. However, the space of application of the CFFHWA operator is broader than the CPFHWA operator but limited than the Cq-ROF Hamacher weighted averaging operator discussed in our proposed scheme. Clearly, the assessment values in this decision-making problem meet the limit of conditions $0 \leq \phi^q + \Re^q \leq 1$ and $0 \leq \omega_\phi^q + \omega_\Re^q \leq 1$. So, in this example, the CFFHWA operator cannot completely deal the decision making problem.

Detailed evaluation results gained by using different MAGDM approaches are given in Tables 16, 17 and Figures 10,11.

**Table 16: Comparison of decision results by utilizing different approaches (Energy).**

| Approaches            | Parameter | $f(\ell_1)$ | $f(\ell_2)$ | $f(\ell_3)$ | $f(\ell_4)$ | Order relation |
|-----------------------|-----------|-------------|-------------|-------------|-------------|----------------|
| Cq-ROFWA operator     | $N=1, q=3$| 0.5549      | 0.5393      | 0.5285      | 0.5649      | $\ell_4 > \ell_1 > \ell_2 > \ell_3$ |
| Cq-ROFWE operator     | $N=2, q=3$| 0.5532      | 0.5386      | 0.5257      | 0.5633      | $\ell_4 > \ell_1 > \ell_2 > \ell_3$ |
| CIFHWA operator       | $q=1, N=3$| 0.5379      | 0.5288      | 0.5161      | 0.5482      | $\ell_4 > \ell_1 > \ell_2 > \ell_3$ |
| CPFHWA operator       | $q=2, N=3$| 0.5511      | 0.5381      | 0.5223      | 0.5630      | $\ell_4 > \ell_1 > \ell_2 > \ell_3$ |
| CFFHWA operator       | $q=3, \Re=3$| 0.5524     | 0.5382      | 0.5240      | 0.5625      | $\ell_4 > \ell_1 > \ell_2 > \ell_3$ |
Figure 10: Comparison with some existing approaches (energy).

Table 17: Comparison of decision results by utilizing different approaches (Randić energy).

| Approaches            | Parameter | $f(\mathcal{J}_1)$ | $f(\mathcal{J}_2)$ | $f(\mathcal{J}_3)$ | $f(\mathcal{J}_4)$ | Order relation          |
|-----------------------|-----------|--------------------|--------------------|--------------------|--------------------|-------------------------|
| Cq-ROFWA operator     | $N = 1$, $q = 3$ | 0.5558             | 0.5398             | 0.5299             | 0.5654             | $\gamma_4 > \gamma_1 > \gamma_2 > \gamma_3$ |
| Cq-ROFWE operator     | $N = 2$, $q = 3$ | 0.5543             | 0.5394             | 0.5271             | 0.5641             | $\gamma_4 > \gamma_1 > \gamma_2 > \gamma_3$ |
| CIFHWA operator       | $q = 1$, $N = 3$ | 0.5392             | 0.5299             | 0.5175             | 0.5492             | $\gamma_4 > \gamma_1 > \gamma_2 > \gamma_3$ |
| CPFHWA operator       | $q = 2$, $N = 3$ | 0.5524             | 0.5392             | 0.5239             | 0.5640             | $\gamma_4 > \gamma_1 > \gamma_2 > \gamma_3$ |
| CFFHWA operator       | $q = 3$, $N = 3$ | 0.5535             | 0.5391             | 0.5256             | 0.5633             | $\gamma_4 > \gamma_1 > \gamma_2 > \gamma_3$ |
The q-ROFG deals with one-dimensional information at a time, which often results in data loss. But the Cq-ROFG is a strong way of dealing with ambiguous information compared to the q-ROFG, since it incorporates two-dimensional information in a single element. Thus, the loss of data can be avoided by adding the second dimension of MD and NMD. If we consider the phase term of MD and NMD to be zero, then the $C_q$-ROFG is transformed to $q$-ROFG, and if we take $q=1$ and $q=2$ then the $q$-ROFG is converted to IFG and PFG, respectively. The comparison of $C_q$-ROFG with existing FG theories are shown in Table 18.

![Figure 11: Comparison with some existing approaches (Randić energy).](image)

| Model | $\phi$ | $\pi$ | Periodicity | Represents two-dimensional information |
|-------|--------|--------|-------------|--------------------------------------|
| FG    | ✓      | ×      | ×           | ×                                    |
| IFG   | ✓      | ✓      | ✓           | ×                                    |
| PFG   | ✓      | ✓      | ✓           | ×                                    |
| FFG   | ✓      | ✓      | ✓           | ×                                    |
| q-ROFG| ✓      | ✓      | ✓           | ×                                    |
| CIFG  | ✓      | ✓      | ✓           | ✓                                    |
| CFFG  | ✓      | ✓      | ✓           | ✓                                    |
| CPFG  | ✓      | ✓      | ✓           | ✓                                    |
| $C_q$-ROFG | ✓ | ✓ | ✓ | ✓ |

The merits of our approach are summarized in the following points:

1. Our proposed scheme estimate that the sum of $q$th power of MD and NMD closed in complex plane of unit disc. The CIFS and CPFS loses their ability, when they tend to deal with such kind of information $((0.8,0.7),(0.9,0.8))$, provided by decision-making experts. Here, the Cq-ROFS proves their ability, due to its flexibility of $q$th power of MD and NMD in complex plane of unit disc.

2. Our proposed scheme is more broad than CIFS and CPFS. The notions of CIFS and CPFS can be originated
from Cq-ROFS with the specific qth powers, such as q=1 and q=2. So, the proposed scheme is more preferable than CIFS and CPFS.

(3) The new framework is evidently apparent and in the MAGDM environment, the Cq-ROF approach can be utilized effectively with the key role of minor data loss.

(4) Usage of graph theory is one of the crucial aspect of proposed scheme, which shows its superiority on other existing methods.

(5) For depicting information in realistic decision-making problem, the Cq-ROFSs approach can be implemented effectively.

(6) Under Cq-ROF domain, to address the MAGDM problems, the Hamacher operator is a more powerful tool.

8 Conclusions

The Cq-ROFS is an effective way to portray ambiguous data and is better than the CIFSs and the CPFSs. Its prominent feature is that the total of the qth power of the amplitude term (similar to the phase term) of the complex-valued MD and the qth power of the amplitude term (similar to the phase term) of the complex-valued NMD is equal to or less than 1. In this article, some new Cq-ROF Hamacher operations and Cq-ROF Hamacher aggregation operators, such as Cq-ROFHW A operator, Cq-ROFHOWA operator, Cq-ROFHWG operator and Cq-ROFHOWG operator have been developed for aggregating Cq-ROFNs. Subsequently, the novel idea of Cq-ROFGs utilizing Hamacher operator called Cq-ROFHG is set forward and its energy and Randić energy is computed. In particular, the energy of a splitting Cq-ROFHG and shadow Cq-ROFHG has been developed. Finally, a quantitative example relating to the selection of FCSs has been provided to show the credibility of the concepts set out in the decision-making process. A Cq-ROFHG can well depict the network fuzziness. In future, our research work will be extended to: (1) Linguistic Cq-ROFGs; (2) 2-Tuple linguistic Cq-ROFGs; (3) Complex spherical fuzzy graphs; and (4) 2-Tuple linguistic complex spherical fuzzy graphs.

Conflicts of Interest: The authors declare no conflict of interest.

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