Decomposition of third-order constitutive tensors

Yakov Itin and Shulamit Reches
Mathematics Department, Jerusalem College of Technology, Israel

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Abstract

Third-order tensors are widely used as a mathematical tool for modeling physical properties of media in solid state physics. In most cases, they arise as constitutive tensors of proportionality between basic physics quantities. The constitutive tensor can be considered as a complete set of physical parameters of a medium. The algebraic features of the constitutive tensor can be seen as a tool for proper identification of natural material, as crystals, and for design the artificial nano-materials with prescribed properties.

In this paper, we study the algebraic properties of a generic 3-rd order tensor relative to its invariant decomposition. In a correspondence to different groups acted on the basic vector space, we present the hierarchy of types of tensor decomposition into invariant subtensors. In particular, we discuss the problem of non-uniqueness and reducibility of high-order tensor decomposition. For a generic 3-rd order tensor, these features are described explicitly. In the case of special tensors of a prescribed symmetry, the decomposition turns out to be irreducible and unique. We present the explicit results for two physically interesting models: the piezoelectric tensor as an example of a pair symmetry and the Hall tensor as an example of a pair skew-symmetry.

1 Introduction

Three-dimensional third-order tensors have a wide range of applications especially in solid state physics, see e.g. [4], [6], [7], [18], [23]. The well-known examples are the piezoelectric and piezomagnetic tensors that connect the electric and magnetic fields respectively to the elasticity stress tensor. The corresponded third-order tensors are symmetric in a pair of their indices and thus have at most 18 independent components. The electro-optical tensor and the second harmonic generation tensor in optics relate dielectric impermeability and dielectric polarization respectively.
to the exterior electric field. These quantities are third-order tensors of the same pair symmetry type. The Hall tensor is a third-order tensor of an alternative algebraic symmetry. It is skew-symmetric in a pair of its indices and thus has at most 9 independent components.

As it is well-known, the distinct components of a tensor themselves do not have an invariant meaning. Their magnitude depend on the coordinate system used. A covariant meaning, however, can be given to different subsets of the components that form sub-tensors of a given tensor. The separation of the whole set of tensor components into subsets that are tensors themselves is a useful procedure. It is usually termed as *irreducible decomposition*. In applications literature, this procedure is used sometimes rather ambiguously, without given the precise definitions and the necessary conditions. At another hand, the solid mathematical treatment that can be found in pure mathematical literature does not usually deal with the specific tensors encountered in applications. The goal of the current paper is to fill the indicated gap. Although we are studying here only the simplest case of the third-order tensor in three dimensional space, the basic algebraic problems of reducibility of the unique decomposition and of non-uniqueness of the irreducible decomposition is exhibited implicitly. In the case of physically meaningful third-order tensors with specific prescribed symmetries, these problems do completely disappear: The irreducible decomposition of the partially symmetric (or skew-symmetric) tensors is unique. It seems to be a good basis to guess that also the physically meaningful higher-order tensors must have additional symmetries and accept unique irreducible decomposition. This requirement may be a useful guiding principle in much more complicated situations in general relativity where fourth- and sixth-order tensors are naturally emerge, see [3], [10], [11].

Recently, the third-order tensors and their decomposition were studied intensively. Such procedure is usually based on the *harmonic decomposition* from representation theory, see e.g. [15], [1], [2], [19], [20], [22] and the references given therein. In this approach, a tensor is decomposed into rotational invariant sub-tensors, i.e., relative to the group $SO(3, \mathbb{R})$. In the current paper, we apply an alternative strategy, that can be traced to Weyl [25] and Schouten [21]. In particular, different types of decomposition are constructed in a correspondence with the geometric structure on the basic vector space $V$. As a result different types of decomposition are encountered relative to different groups of transformations. This way we are able to identify the sets of sub-tensors of similar algebraic and physical properties. This procedure is known to be relevant for fourth-order constitutive tensors in electromagnetism [8], linear elasticity [12], [13], [14], and gravity [9].

The organization of the current paper is as follows:

In Sect. 2, we present the basic facts about the definition of tensors and relevant groups of their transformations. Third-order tensor is defined as a multi-linear map of a Cartesian product of three copies of a vector spaces over a number field into the same field. In this paper, we restrict to the real numbers field $\mathbb{R}$ and to a 3–dimensional vector space $V$. The corresponded tensor is denoted by $T^{ijk}$ with indices changing in the range $i, j, k = 1, 2, 3$. In Sect. 3, we formulate a list of requirements for *irreducible decomposition of a tensor*. These conditions are presented in two equivalent forms – in term of sub-tensors and in term of subspaces of the tensor space. Our main results are derived in Sect. 4. Step-by-step, we provide the invariant decomposition of the tensor with respect to the group $GL(3, \mathbb{R})$ and its subgroups. The first level of the decomposition is derived by the tools of the symmetry group $S_3$, i.e., Young tableau. Already on this stage, we meet
the main problem: The irreducible decomposition is not unique in general. We construct different isotopic types of irreducible decomposition. Additional levels of decomposition are derived by use of traces (contractions) taken relatively to the metric tensor or permutation pseudo-tensor. In Sect. 5, we apply the general scheme to two most known classes of restricted 3-rd order tensors: the piezoelectric tensor and the Hall-effect tensor. In Conclusion section, we briefly discuss our results and propose some possible applications.

Notations: We denote third- and second-order tensors by capital Latin letters and strictly distinguish between the upper and lower indices. The indices take the values from the range \( i, j, k, \ldots = 1, 2, 3 \). Einstein’s summation rule for a pair of indices is assumed everywhere. The symmetrization of a tensor is denoted by \( T(ij) = (1/2)(T_{ij} + T_{ji}) \), while the antisymmetrization is presented as \( T[ij] = (1/2)(T_{ij} - T_{ji}) \). The vector spaces corresponding to specific tensors (tensor spaces) are denoted by the same letter in the bold font.

2 Third-order tensors and their transformation groups

In this section, we briefly recall some definitions and notations relevant to tensor algebra. We also discuss transformation groups related to various geometric structures defined on the basic vector space and their relation to 3 dimension tensors.

2.1 Third-order tensors

Let \( V \) denotes a 3-dimensional vector space over the real number field \( \mathbb{R} \) with the dual vector space \( V^* \). Most applications in solid-state physics deal with the spaces \( V \) and \( V^* \) that are isomorphic to the ordinary 3-dimensional space, \( V \cong V^* \cong \mathbb{R}^3 \). Denote a basis of \( V \) as \( e_i \) and the dual basis of \( V^* \) as \( \vartheta^i \). The duality of these two bases is defined by operator relations, as follows

\[
e_i(\vartheta^j) = \delta_i^j \quad \text{and} \quad \vartheta^i(e_j) = \delta^i_j,
\]

(1)

where \( \delta_i^j \) denotes Kronecker’s tensor (the set of components of the unit matrix). In these bases, the representations of a vector \( v \in V \) and a functional (1-form) \( \omega \in V^* \) are given by

\[
v = v^i e_i, \quad \text{and} \quad \omega = \omega_i \vartheta^i.
\]

(2)

Here and in the sequel, we use Einstein’s summation rule for two same-named indices in the upper and down positions. In 3-dimensional space, the indices change in the range \( i, j, \ldots = 1, 2, 3 \). The numerical values of the components \( v^i \) and \( \omega_i \) are defined module the transformations of the basis. Let the basis \( e_i \) of \( V \) be transformed into an arbitrary new basis \( e'_i \), then the basis \( \vartheta^i \) of \( V^* \) is transformed into the unique new basis \( \vartheta'^i \) due to the standard tensor law:

\[
e_i \rightarrow e'_i = R^i_j e_i \quad \text{and} \quad \vartheta^i \rightarrow \vartheta'^i = R^i_j \vartheta^j.
\]

(3)

The transformation matrices \( R^i_j \) and \( R^j_i \) are assumed to be inverse one to another. In tensor form, it is expressed as

\[
R^i_j R^j_l = \delta^i_l, \quad \text{and} \quad R^i_l R^l_j = \delta^i_j.
\]

(4)
Under the basis transformations (3), the components of the vector $v$ and the 1–form $\omega$ are transformed respectively as

$$v^i \rightarrow v'^i = R^i_j v^j, \quad \text{and} \quad \omega_i \rightarrow \omega'_i = R^i_j \omega_j.$$  

(5)

So in (2), the vector $v$ and the form $\omega$ themselves are invariant under arbitrary linear transformations of the basis.

Tensors are defined as multilinear map from the Cartesian product of vector spaces into the field $\mathbb{R}$. In particular, the covariant 3-rd order tensor is defined as

$$T : V^* \times V^* \times V^* \rightarrow \mathbb{R}. \quad (6)$$

With respect to the basis $e_i$, the tensor $T$ is represented by a set of 27 real components $T^{ijk}$ such that

$$T = T^{ijk} e_i \otimes e_j \otimes e_k, \quad (7)$$

where the tensor product notation $\otimes$ is used. Under the basis transformations (3), the components of the tensor are transformed as

$$T^{ijk} \rightarrow T'^{i'j'k'} = R^i_{i'} R^j_{j'} R^k_{k'} T^{ijk}. \quad (8)$$

Similarly, the contravariant 3-rd order tensor is defined as a multilinear map

$$T : V \times V \times V \rightarrow \mathbb{R}. \quad (9)$$

Relatively to the basis $\vartheta^i$ of the dual space $V^*$, this tensor is represented by a set of 27 real components $T_{ijk}$, as well,

$$T = T_{ijk} \vartheta^i \otimes \vartheta^j \otimes \vartheta^k. \quad (10)$$

Under the linear transformation (3) of the basis, the components of the contravariant tensor are transformed as

$$T_{ijk} \rightarrow T'_{i'j'k'} = R^i_{i'} R^j_{j'} R^k_{k'} T_{ijk}. \quad (11)$$

Another class of so-called mixed-type tensors is defined on the Cartesian product of the vector spaces $V$ and $V^*$. For instance, the linear map

$$T : V^* \times V \times V \rightarrow \mathbb{R} \quad (12)$$

introduces a tensor

$$T = T_{ijk} e_i \otimes \vartheta^j \otimes \vartheta^k. \quad (13)$$

with a “mixed” transformation law of the components

$$T_{ijk} \rightarrow T'_{i'j'k'} = R^i_{i'} R^j_{j'} R^k_{k'} T_{ijk}. \quad (14)$$

Analogously, one can define additional mixed-type tensors, such as $T^i_{j'k}$ and so on.
2.2 Groups relevant for decomposition of tensors

Since the Cartesian product of vector spaces is not abelian, the group of permutations (the symmetry group) acted on the set of indices is relevant for the algebra of tensors. In index notations, this group, $S_n$, acts on the components of an $n$-th order tensor by permutation of its indices. For the 3-order tensors, we are dealing with the symmetry (permutations) group $S_3$. It is a finite group of 6 independent elements:

$$S_3 = \{I, (12), (23), (13), (123), (132)\} \tag{15}$$

Here the cycle notations are used. The element $I$ denotes the identity permutation. The element $(12)$ means a permutation that exchanges the first and the second indices. The elements $(23)$ and $(13)$ are defined similarly. The element $(123)$ means that the first index goes into the second, the second into the third, and the third into the first. The element $(132)$ is defined in the same way.

A linear transformation of a basis of $V$ is naturally translated into the transformation of the tensor space. For a 3-dimensional space $V$, transformations form the group of invertible $3 \times 3$ matrices

$$GL(3, \mathbb{R}) = \{ (3 \times 3) - \text{matrices } G \text{ with } \det G \neq 0 \} \tag{16}$$

The relation between the action on tensors of these two groups, the permutation group $S_p$ and the general linear group $GL(n, \mathbb{R})$, is managed by the Schur-Weyl duality theorem. It states that due to commutativity of the simultaneous action of the groups $GL(n, \mathbb{R})$ and $S_p$, the decomposition of the tensor space relative to the symmetry group $S_p$ is invariant under the action of the general linear group $GL(n, \mathbb{R})$. An effective way to derive the $S_p$ decomposition is to apply the Young diagram technique. In our case, we are dealing with the groups $GL(3, \mathbb{R})$ and $S_3$.

For the vector space $V$ endowed with an additional geometric structure, such as a metric, the group of transformations is restricted to a subgroup of $GL(3, \mathbb{R})$ that preserves the geometric structure. Consequently, an additional decomposition of the tensor space is admissible. In this paper, we study the decomposition under the group $GL(3, \mathbb{R})$ and its subgroups:

- Orthogonal group $O(3, \mathbb{R})$ that preserves the scalar product structure on $V$;
- Special linear group $SL(3, \mathbb{R})$ that preserves the volume element, i.e., the orientation structure on $V$;
- Special orthogonal group $SO(3, \mathbb{R})$ that preserves the scalar product together with the orientation.

3 Irreducible decomposition of tensors

In a chosen basis, a tensor is presented by a large set of independent components. In order to characterize the algebraic properties of a tensor, it is useful to divide this set of components into some smaller subsets, with specific algebraic symmetry for each one of them, as

$$T^{i \cdots j} = \sum_{p=1}^{n} (p) T^{i \cdots j} = (1) T^{i \cdots j} + (2) T^{i \cdots j} + \cdots + (p) T^{i \cdots j} + \cdots + (n) T^{i \cdots j} \tag{17}$$
Such a procedure is called *decomposition of a tensor*. In order to get an algebraically meaningful decomposition, we would like the following conditions to be satisfied:

1. **Covariance:** All sub-tensors \((p)T^{i\cdots j}\) must be of the same order and of the same shape like the initial tensor \(T^{i\cdots j}\) is. In other words, all sub-tensors must have the same number of indices in the same positions.

2. **Independence:** The sub-tensors \((p)T^{i\cdots j}\) must be linearly independent, i.e., any equation of the form \(\sum_{p=1}^{n} \alpha_p (p)T^{i\cdots j} = 0\) yields \(\alpha_p = 0\) for all \(p\).

3. **Irreducibility:** The set of sub-tensors \((p)T^{i\cdots j}\) must be minimal, i.e., it can not be decomposed successively into some smaller set of sub-tensors.

4. **Uniqueness:** There is no alternative decomposition, i.e., if the tensor is written as \(T^{i\cdots j} = \sum_{p=1}^{n} (p)\tilde{T}^{i\cdots j}\), in an addition to (17), then \((p)\tilde{T}^{i\cdots j} = (p)T^{i\cdots j}\) for all \(p\).

Tensors of a specific order form a vector space of themselves, which can be called a tensor space. We denote the tensor spaces of the original tensor \(T^{i\cdots j}\) and of its sub-tensors \((p)T^{i\cdots j}\) by \(T\) and \((p)T\), correspondingly. The conditions above can be reformulated in term of tensor subspaces as follows:

1. **Covariance:** The subspaces \((p)T\) are invariant: It means that they are preserved under all prescribed transformations.

2. **Independence:** The intersections of the subspaces are trivial: For \(p \neq q\),
\[
(p)T \cap (q)T = \{0\}.
\] (18)

3. **Irreducibility:** The subspaces \((p)T\) are minimal, i.e., they do not contain any smaller non-zero invariant subspace.

4. **Uniqueness:** The subspaces \((p)T\) are unique (up to isomorphism).

Relaying on the conditions above, the decomposition of the tensor (17) is presented now as a resolution into the direct sum of the tensor space \(T\)
\[
T = \bigoplus_{p=1}^{n} (p)T = (1)T \oplus \cdots \oplus (p)T \oplus \cdots \oplus (n)T.
\] (19)

In particular, the dimensions of the subspaces satisfy
\[
\dim(T) = \sum_{p=1}^{n} \dim((p)T).
\] (20)

In order to clarify these issues, we present some simple examples. For a general (asymmetric) 2-nd order covariant tensor \(T^{ij}\), the decomposition into symmetric and skew-symmetric parts
\[
T^{ij} = S^{ij} + A^{ij}
\] (21)
where
\[ S^{ij} = T^{(ij)} := \frac{1}{2} (T^{ij} + T^{ji}), \quad A^{ij} = T^{[ij]} := \frac{1}{2} (T^{ij} - T^{ji}) \] (22)
is unique and irreducible under the action of the group \( GL(3, \mathbb{R}) \). The dimension of the total tensor space is distributed among the subspaces as \( 9 = 6 + 3 \).

Another example is a mixed-type second order tensor \( T^{ij} \). It is decomposed uniquely and \( GL(3, \mathbb{R}) \)-irreducibly into the scalar and traceless parts, respectively,

\[ T^{ij} = (1) T^{ij} + (2) T^{ij}, \] (23)

where
\[ (1) T^{ij} = \frac{1}{3} T^{m} m \delta^{j}, \quad (2) T^{i} = 0. \] (24)
The dimension of the total space is distributed now as \( 9 = 1 + 8 \).

When both tensors \( T^{ij} \) and \( T^{i} \) are considered relative to the sub-group \( O(3, \mathbb{R}) \), the decompositions given above turn our to be reducible. A finer decomposition can be derived by the use of the metric tensor \( g_{ij} = \text{diag}(1,1,1) \). For \( T^{ij} \), it takes the form

\[ T^{ij} = (1) T^{ij} + (2) T^{ij} + (3) T^{ij}, \] (25)

where
\[ (1) T^{ij} = \frac{1}{3} g_{mn} T^{m} g^{ij}, \quad (2) T^{ij} = T^{(ij)} - (1) T^{ij}, \quad (3) T^{ij} = T^{[ij]}. \] (26)

Here the dimension of the total space is distributed, as \( 9 = 1 + 5 + 3 \), respectively. This decomposition is unique and \( O(3, \mathbb{R}) \)-irreducible.

For higher order tensors the situation turns out to be much more complicated. Recall the well-known fact, see e.g. [16]:

*For a general tensor of order greater than 2, there is no unique irreducible decomposition. In other words, the unique decomposition is reducible, while the irreducible decomposition is not unique.*

We demonstrate these properties explicitly in the next section.

### 4 Decomposition of a covariant 3-rd order tensor

#### 4.1 \( GL(3, \mathbb{R}) \)-decomposition

In this section, we consider a bare vector space \( V \) without any additional structure. In this case, the decomposition of the tensor \( T^{ijk} \) has to be invariant under arbitrary invertible transformation of the basis in \( V \). Due to the Schur-Weyl duality theorem, such \( GL(3, \mathbb{R}) \)-decomposition is equivalent to the decomposition of \( T^{ijk} \) under the permutation group \( S_{3} \).
4.1.1 Straightforward decomposition

In three-dimensional space $V$, a general (non-restricted) third-order tensor $T^{ijk}$ has $3^3 = 27$ independent components. We are looking for an irreducible $GL(3, \mathbb{R})$-invariant decomposition of this tensor. Moreover, it is plausible to have a unique decomposition. First, we observe that $T^{ijk}$ can be readily decomposed into the sum of three independent $GL(3, \mathbb{R})$-invariant parts. Indeed, it is enough to define the totally symmetric part

$$S^{ijk} = T^{(ijk)} := \frac{1}{3!} \left( T^{ijk} + T^{jki} + T^{kij} + T^{kji} + T^{ikj} + T^{jk i} \right),$$

and the totally skew-symmetric part

$$A^{ijk} = T^{[ijk]} := \frac{1}{3!} \left( T^{ijk} + T^{jki} + T^{kij} - T^{jik} - T^{kji} - T^{ikj} \right).$$

Now we extract these two parts from $T^{ijk}$ to obtain the residue part $N^{ijk}$ given by

$$N^{ijk} := T^{ijk} - S^{ijk} - A^{ijk} = \frac{1}{3} \left( 2T^{ijk} - T^{jki} - T^{kji} \right).$$

Consequently, we obtain an invariant decomposition of $T^{ijk}$ into three invariant parts

$$T^{ijk} = S^{ijk} + A^{ijk} + N^{ijk}. \quad (30)$$

The linear independence of these three sub-tensors follows from the symmetry relations

$$N^{(ijk)} = A^{(ijk)} = 0,$$

and

$$N^{[ijk]} = S^{[ijk]} = 0. \quad (31)$$

These symmetry relations also show that the subspaces corresponding to the $S$, $A$, and $N$ tensors are mutually disjoint (up to the zero tensor).

The decomposition (30) is invariant under the action of the $GL(3, \mathbb{R})$ group. It means that the symmetries of the sub-tensors in (30) are preserved under any linear transformation of the basis. In particular, the transformed tensor $S^{i'j'k'} = R_i^{i'} R_j^{j'} R_k^{k'} S^{ijk}$ is totally symmetric while the transformed tensor $A^{i'j'k'} = R_i^{i'} R_j^{j'} R_k^{k'} A^{ijk}$ is totally skew-symmetric. Also the residue tensor $N^{ijk}$ preserves its symmetries (31) under these transformations. Moreover, we observe that the tensors $S^{ijk}$, $A^{ijk}$ and $N^{ijk}$ are defined uniquely.

Consequently, the tensor space of the third-order tensors is decomposed now into the direct sum of three subspaces

$$T = S \oplus A \oplus N. \quad (32)$$

Hence, the dimension of the total space is distributed among these subspaces as follows

$$27 = 10 + 1 + 16. \quad (33)$$

It is clear, that the sub-tensors $S^{ijk}$, and $A^{ijk}$ are irreducible. Indeed, any symmetrization (or antisymmetrization) of the indices in $S^{ijk}$ and $A^{ijk}$ preserves them (up to the total sign) or gives zero. The sub-tensor $N^{ijk}$, however, is reducible. For instance, one can decompose it into the sum of two non-zero parts $N^{(ijk)} + N^{[ijk]}$. Whether this decomposition is invariant? Is it minimal? How the invariant and minimal decomposition of $N^{ijk}$ can be derived? We discuss these issues in the sequel.
4.1.2 Young’s diagrams

To have an invariant irreducible decomposition of $N_{ijk}$, we apply the machinery of group theory. The technical tool used in the symmetry group decomposition is based on Young’s diagrams. The symmetry group that is relevant to the decomposition of third-order tensors in a space of an arbitrary dimension is $S_3$. For a generic tensor $T_{ijk}$ of 27 independent components, there are three different Young’s diagrams depicted as:

\[ \lambda_1 = \begin{array}{ccc} & & \\ \end{array}, \quad \lambda_2 = \begin{array}{c} \end{array}, \quad \lambda_3 = \begin{array}{cc} \end{array}. \quad (34) \]

Each diagram $\lambda$ has a corresponding dimension (the number of standard Young tableaux whose shape is a given Young diagram). This number can be calculated due to the so-called hook formula

\[ \dim \lambda = \frac{3!}{\prod_{(\alpha, \beta) \in \lambda} \text{hook}(\alpha, \beta)}. \quad (35) \]

Here, the pair of numbers $(\alpha, \beta)$ denotes the position of a cell in the diagram: $\alpha$ for the row and $\beta$ for the column. For a cell $(\alpha, \beta)$ in the diagram of a shape $\lambda$, the natural number “$\text{hook}(\alpha, \beta)$” is defined as the number of boxes that are in the same row to the right of it plus those boxes in the same column below it, plus one (for the box itself). It is called hook length. Using Eq.(35), we obtain:

\[ \dim \lambda_1 = 1, \quad \dim \lambda_2 = 1, \quad \dim \lambda_3 = 2. \quad (36) \]

The corresponding Young’s tableaux (the diagrams filling with the numbers 1,2,3) are given as

\[ \lambda_1 : \begin{array}{ccc} 1 & 2 & 3 \\ 2 & 3 \end{array}, \quad \lambda_2 : \begin{array}{c} 1 \\ 2 \\ 3 \end{array}, \quad \lambda_3 : \begin{array}{ll} 1 \ 2 \\ 1 \ 3 \\ 2 \end{array}. \quad (37) \]

The rows in these tableaux describe the symmetrization operated on the corresponding indices, while the columns mean the antisymmetrization. Consequently, the identity operator $I$ is decomposed into the sum of three Young operators:

\[ I = P_1 + P_2 + P_3, \quad (38) \]

where the symmetrization operator $P_1$ and antisymmetrization operator $P_2$ are defined, respectively, as

\[ P_1 = I + (12) + (13) + (23) + (123) + (321), \quad P_2 = I - (12) - (13) - (23) + (123) + (321). \quad (39) \]

The operator $P_3$ is represented as the sum of two operators described by the two tableaux $\lambda_3$ given in (37):

\[ P_3 = P_{3,1} + P_{3,2}. \quad (40) \]
Due to the rules mentioned above, the explicit expressions of these operators are given by

\[
P_{3,1} = (I + (12))(I - (13)) = I + (12) - (13) - (123),
\]
\[
P_{3,2} = (I + (13))(I - (12)) = I + (13) - (12) - (123).
\]  

Here we assumed the rule: the symmetrization operation is applied \textit{after} the antisymmetrization one. An alternative order yields two different operators that however equivalent (isotopic) to \(P_{3,1}\).

Hence we obtain the decomposition of the tensor \(T^{ijk}\) into four pieces:

\[
T^{ijk} = S^{ijk} + A^{ijk} + N^{ijk}
\]
\[
= S^{ijk} + A^{ijk} + \left( N^{ijk}_1 + N^{ijk}_2 \right)
\]  

Here \(S^{ijk}\) and \(A^{ijk}\) are the familiar totally symmetric and totally skew-symmetric parts, \((27)\) and \((28)\) respectively. The residue part \(N^{ijk}\) given in \((29)\) is presented now as a sum of two parts. The explicit expressions of these two sub-tensors are calculated due to the operators \((41)\) correspondingly to the two latter tableaux in \((37)\)

\[
N^{ijk}_1 = P_{3,1} \left( T^{ijk} \right) = \frac{1}{3} \left( T^{ijk} + T^{jik} - T^{kji} - T^{kij} \right)
\]  

and

\[
N^{ijk}_2 = P_{3,2} \left( T^{ijk} \right) = \frac{1}{3} \left( T^{ijk} - T^{jik} + T^{kji} - T^{jki} \right)
\]

It is clear that these two tensors are expressed via \(N^{ijk}\) only, i.e.,

\[
N^{ijk}_1 = \frac{1}{3} \left( N^{ijk} + N^{jik} - N^{kji} - N^{kij} \right)
\]

and

\[
N^{ijk}_2 = \frac{1}{3} \left( N^{ijk} - N^{jik} + N^{kji} - N^{jki} \right)
\]

Notice the symmetries of these two tensors following immediately from Eqs.\((41)\)

\[
N^{ijk}_1 = N^{jik}_1 \quad \text{and} \quad N^{ijk}_2 = N^{kji}_2.
\]  

4.1.3 Alternative decomposition

We can easily see that the decomposition \((42)\) is not unique. For example, if we assume an opposite order rule: the antisymmetrization applied after the symmetrization, we would obtain

\[
N^{ijk} = \tilde{N}^{ijk}_1 + \tilde{N}^{ijk}_2.
\]

The corresponding operators are

\[
\tilde{P}^1_3 = (I + (23))(I - (12)) = I + (23) - (12) - (132).
\]
\[
\tilde{P}^2_3 = (I + (12))(I - (23)) = I + (12) - (23) - (123).
\]
Consequently, these two alternative sub-tensors are expressed as

\[
\tilde{N}^{ijk}_{1} = \frac{1}{3} \left( T^{ijk} - T^{jik} + T^{ikj} - T^{kij} \right) = \frac{1}{3} \left( N^{ijk} - N^{jik} + N^{ikj} - N^{kij} \right),
\]

and

\[
\tilde{N}^{ijk}_{2} = \frac{1}{3} \left( T^{ijk} - T^{ikj} + T^{jik} - T^{jki} \right) = \frac{1}{3} \left( N^{ijk} - N^{jik} + N^{ikj} - N^{jki} \right). \tag{51}
\]

The \( \tilde{N} \)-decomposition is not the only possible alternative to the \( N \)-decomposition. Indeed, we can use various sequences of the permutation operators to produce different decompositions. Particularly, we use in the sequel a decomposition based on the operators

\[
\hat{P}^{1} = (I - (23))(I + (12)) = I - (23) + (12) - (132),
\]

\[
\hat{P}^{2} = (I - (12))(I + (23)) = I - (12) + (23) - (123). \tag{52}
\]

The corresponding subtensors are

\[
\hat{N}^{ijk}_{1} = \frac{1}{3} \left( T^{ijk} - T^{ikj} + T^{jik} - T^{kij} \right) = \frac{1}{3} \left( N^{ijk} - N^{jik} + N^{ikj} - N^{kij} \right), \tag{53}
\]

and

\[
\hat{N}^{ijk}_{2} = \frac{1}{3} \left( T^{ijk} - T^{jik} + T^{ikj} - T^{kji} \right) = \frac{1}{3} \left( N^{ijk} - N^{jik} + N^{ikj} - N^{jki} \right). \tag{54}
\]

All these isotopic decompositions of the tensor \( N^{ijk} \) are completely equivalent. All of them are irreducible and invariant under the \( GL(3, \mathbb{R}) \) transformations of the basis. For higher-order tensors, the non-uniqueness of the irreducible decomposition is a well-established fact, see e.g. [16].

### 4.1.4 Tensor space decomposition

With respect to the decomposition (42), the tensor space \( T \) is decomposed into the direct sum of the subspaces \( (p)T \) that correspond to the tableaux \( \lambda_p \) given in (37). The dimensions of these subspaces can be calculated by the following combinatorics formula, see, e.g. [5],

\[
\dim (p)T = \prod_{(\alpha, \beta) \in \lambda_p} \frac{3 + \beta - \alpha}{\operatorname{hook}(\alpha, \beta)}. \tag{55}
\]

Recall that the pair of integers \( (\alpha, \beta) \) numerates a cell of the diagram standing in the \( \alpha \)'s raw and \( \beta \)'s column. The summation is provided for all cells of the corresponding diagram. Accordingly, we calculate

\[
\dim S = 10, \quad \dim A = 1, \quad \dim N_1 = \dim N_2 = 8. \tag{56}
\]

Using the consideration above we derive the geometrical meaning of the decomposition (37):

**Proposition 1.** Let a third-order covariant tensor \( T^{ijk} \) and the corresponding tensor space \( T \) be given.
• The tensor space $T$ is decomposed into the direct sum of three subspaces

$$T = S \oplus A \oplus N.$$  \hfill(57)

This decomposition is uniquely but reducible.

• The subspace $N$ is decomposed additionally into the direct sum of two smaller subspaces

$$N = N_1 \oplus N_2.$$ \hfill (58)

This decomposition is irreducible but not unique. There are isotopic subspaces $N_1, \tilde{N}_1, \check{N}_1$ and so on.

• The dimension of the total tensor space is distributed among the subspaces accordingly to:

$$3 \times 3 \times 3 = 10 + 1 + (8 + 8).$$ \hfill (59)

• The subspaces $S$, $A$, and $N_1$, $N_2$ are invariant under $GL(3, \mathbb{R})$ transformations of the basis.

4.2 $O(3, \mathbb{R})$-decomposition

4.2.1 Metric tensor

In this section, we consider the vector space $V$ endowed with the scalar product structure. For arbitrary pair of vectors $x, y \in V$, the scalar product is defined by the use of an invertible matrix $g_{ij}$. In a chosen basis $e_i$, the vectors are represented as $x = x^i e_i$ and $y = y^i e_i$ while the scalar product is given by

$$(x, y) = g_{ij} x^i y^j.$$ \hfill (60)

In a transformed basis $e'_i = R^i_j e_j$, it is expressed as

$$(x, y) = g'_{ij} x'^i y'^j = \left( g'_{ij} R^i_j \right) x^i y^j.$$ \hfill (61)

Thus the matrix $g_{ij}$ is a tensor with the transformation law

$$g_{ij} \rightarrow g'_{ij} = R^i_p R^j_q g_{pq}.$$ \hfill (62)

In Euclidean space, the metric tensor has a fixed diagonal form $g_{ij} = \text{diag}(1, 1, 1)$. Such a metric is preserved under the transformations of basis restricted to a subgroup of $GL(3, \mathbb{R})$ which elements satisfy the requirement $R R^T = I$. This is the orthogonal group $O(3, \mathbb{R})$. 13
4.2.2 Trace vectors in symmetric and skew-symmetric parts

We are looking now for a decomposition of the tensor $T^{ijk}$ under the orthogonal group. Using the metric tensor $g_{ij}$, we can construct from $T^{ijk}$ various tensors of smaller orders. Here we refer to the sum of tensor components in some two fixed indices, one upper and one lower, as a *trace of a tensor*. The traces are tensors by themselves of the rank $p - 2$, where $p$ is the rank of the original tensor. From the tensor $T^{ijk}$, we can extract three traces as follows

$$u^k = g_{ij}T^{ijk}, \quad v^k = g_{ij}T^{ikj}, \quad w^k = g_{ij}T^{kij}.$$  

(63)

They are transformed as vectors under a transformation from $O(3, \mathbb{R})$. For a generic tensor $T^{ijk}$, these three vectors are linearly independent. Otherwise, we would have a linear relation between the components of $T^{ijk}$, then the tensor would not be general. We discuss such special tensors in the sequel.

In addition to the decomposition (42), we are able now to derive a successive decomposition of the tensor $T^{ijk}$ by using the trace vectors $u^k, v^k, w^k$. Let us start with the symmetric part (27). Due to the total symmetry of $S^{ijk}$, we have $g_{ij}S^{ijk} = g_{ij}S^{ikj} = g_{ij}S^{kij}$. Thus, there is only one independent trace vector that can be constructed from the symmetric part

$$\alpha^k = g_{ij}s^{ijk} = \frac{1}{3}(u^k + v^k + w^k).$$  

(64)

Now the tensor $S^{ijk}$ can be decomposed into the sum of two independent symmetric sub-tensors:

$$S^{ijk} = K^{ijk} + R^{ijk},$$  

(65)

where the *trace part* is defined as

$$K^{ijk} = \frac{1}{5}\left(\alpha^ig^{ijk} + \alpha^jg^{ik} + \alpha^kg^{ij}\right),$$  

(66)

while $R^{ijk} = S^{ijk} - K^{ijk}$ is the *totally traceless part*. The factor $(1/5)$ in (66) follows from the traceless relations

$$g_{ij}R^{ijk} = 0, \quad g_{ik}R^{ijk} = 0, \quad g_{jk}R^{ijk} = 0.$$  

(67)

In term of the corresponding tensor spaces, Eq. (65) means the decomposition of the space $\mathbf{S}$ into a direct sum of two invariant subspaces

$$\mathbf{S} = \mathbf{K} \oplus \mathbf{R}$$  

(68)

with the dimension reduction

$$10 = 3 + 7.$$  

(69)

The second part $A^{ijk}$ of the tensor $T^{ijk}$ is totally skew-symmetric. All its traces are zero so it does not have any invariant subspaces. It is trivial since the corresponding space $\mathbf{A}$ is 1-dimensional.
4.2.3 Trace vectors and decomposition of the mixed-symmetry part

We consider now the third part \( N^{ijk} \). It has two invariant subspaces \( N^{ijk}_1 \) and \( N^{ijk}_2 \), that are useful to consider separately. Due to the symmetry \( N^{ijk}_1 = N^{jik}_1 \), there are two different trace vectors

\[
g_{ij}N^{ijk}_1 = \frac{2}{3}(u^k - w^k), \quad \text{and} \quad g_{ij}N^{ijk}_1 = \frac{1}{3}(w^k - u^k).
\]

(70)

These vectors are linearly dependent, so we are left with only one independent vector

\[
\beta^k = g_{ij}N^{ijk}_1 = \frac{2}{3}(u^k - w^k)
\]

(71)

only. Now the tensor \( N^{ijk}_1 \) can be decomposed into the sum of two independent parts – the trace and the traceless one

\[
N^{ijk}_1 = M^{ijk}_1 + P^{ijk}_1.
\]

(72)

Due to the symmetry of \( N^{ijk}_1 \), the general form of the trace part can be considered as

\[
M^{ijk}_1 = x(\beta^i g^{jk} + \beta^j g^{ik}) + y\beta^k g^{ij}.
\]

(73)

We require the residue part \( P^{ijk}_1 \) to be totally traceless, i.e.,

\[
g_{ij}P^{ijk}_1 = 0, \quad g_{ik}P^{ijk}_1 = 0, \quad g_{jk}P^{ijk}_1 = 0.
\]

(74)

From these linear equations, it follows that \( x = -\frac{1}{4} \) and \( y = 1/2 \). Hence, we derived two parts of the tensor \( N^{ijk}_1 \). Explicitly,

\[
M^{ijk}_1 = \frac{1}{4}\left(2\beta^k g^{ij} - \beta^i g^{jk} - \beta^j g^{ik}\right),
\]

(75)

and

\[
P^{ijk}_1 = N^{ijk}_1 - M^{ijk}_1.
\]

(76)

Accordingly, we have a direct sum decomposition of the subspace \( N_1 \) into the sum of two subspaces with the corresponding dimensions

\[
N_1 = M_1 \oplus P_1, \quad 8 = 3 \oplus 5.
\]

(77)

For the tensor \( N^{ijk}_2 \) with the symmetry \( N^{ijk}_2 = N^{kji}_2 \), we can define two trace vectors

\[
g_{ij}N^{ijk}_2 = \frac{2}{3}(v^k - w^k), \quad \text{and} \quad g_{ij}N^{ijk}_2 = \frac{1}{3}(w^k - v^k),
\]

(78)

which turn out to be linearly dependent. We choose an independent vector

\[
\gamma^k = g_{ij}N^{ijk}_2 = \frac{2}{3}(v^k - w^k).
\]

(79)
Now, $N_{ijk}^2$ can be decomposed into the sum of two independent parts:

$$N_{ijk}^2 = M_{ijk}^2 + P_{ijk}^2,$$

(80)

where, similarly to $M_{ijk}^1$, we define the trace part

$$M_{ijk}^2 = \frac{1}{4} \left( 2\gamma^i g^{jk} - \gamma^j g^{ik} - \gamma^k g^{ij} \right),$$

(81)

while $P_{ijk}^2 = N_{ijk}^2 - M_{ijk}^2$ is a traceless tensor, i.e.,

$$g_{ij}P_{ijk}^2 = 0, \quad g_{ik}P_{ijk}^2 = 0, \quad g_{jk}P_{ijk}^2 = 0.$$

(82)

Consequently, we have the direct sum decomposition of the subspace $N_2$ into the sum of two subspaces with the corresponding dimensions

$$N_2 = M_2 \oplus P_2, \quad 8 = 3 \oplus 5$$

(83)

Recall that the decomposition of the tensor $N_{ijk}$ into two parts $N_{ijk}^1$ and $N_{ijk}^2$ is not unique but irreducible. Its trace decomposition, however, is unique but reducible. Indeed, it is enough to define

$$N_{ijk} = M_{ijk} + P_{ijk},$$

(84)

where the tensor $M_{ijk} = M_{ijk}^1 + M_{ijk}^2$ reads

$$M_{ijk} = \frac{1}{4} \left( (2\beta^k - \gamma^k) g^{ij} - (\beta^i + \gamma^i) g^{jk} + (2\gamma^j - \beta^j) g^{ik} \right),$$

(85)

or, equivalently,

$$M_{ijk} = \frac{1}{6} \left( (2u^k - v^k - w^k) g^{ij} + (2w^j - u^j - v^j) g^{ik} + (2v^j - u^j - w^j) g^{jk} \right).$$

(86)

This tensor has 6 independent components (the number of components of two independent vectors). Since the tensors $M_{ijk}$ and $P_{ijk}$ are independent, the second traceless tensor

$$P_{ijk} = P_{ijk}^1 + P_{ijk}^2$$

(87)

is left with 10 independent components.

Accordingly, the metric structure on the vector space $V$ provides a finer decomposition of the tensor $T_{ijk}$. In geometrical description, our results can be formulated as follows:

**Proposition 2.** Let a general tensor $T_{ijk}$ with the corresponding tensor space $T$ be given.

- The vector space of the tensor $T_{ijk}$ is decomposed uniquely into the sum of 5 independent invariant subspaces

$$T = S \oplus A \oplus N$$

$$= (K \oplus R) \oplus A \oplus (M \oplus P).$$

(88)
• The dimension of the total space is distributed between these subspaces as follows

\[ 27 = (3 + 7) + 1 + (6 + 10). \]  
\[ (89) \]

• The subspaces \( M \) and \( P \) are decomposed irreducible but not uniquely into the sum of two subspaces with the corresponding dimensions

\[ M = M_1 \oplus M_2 \quad 6 = 3 + 3 \]  
\[ (90) \]

and

\[ P = P_1 \oplus P_2 \quad 10 = 5 + 5. \]  
\[ (91) \]

• The decomposition is invariant under the action of the group \( O(3, \mathbb{R}) \).

4.2.4 Orthogonality of irreducible parts

With a metric tensor at hand we can define a scalar product of two tensors of the same index structure. For two third-order tensors \( A^{ijk} \) and \( B^{ijk} \), it is defined as

\[ (A, B) := A^{ijk} B^{mnp} g_{im} g_{jn} g_{kp} := A^{ijk} B_{ijk}. \]  
\[ (92) \]

In particular, the scalar square of the tensor \( T^{ijk} \) is defined as

\[ (T, T) := T^{ijk} T^{mnp} g_{im} g_{jn} g_{kp} = T^{ijk} T_{ijk}. \]  
\[ (93) \]

Since it is non-negative, the Euclidean norm of a tensor is given in the standard form \( ||T|| = (T, T)^{1/2} \). In Eqs. (92,93), we use the contravariant tensor

\[ T_{ijk} := g_{im} g_{jn} g_{kp} T^{mnp}. \]  
\[ (94) \]

Notice that for the Euclidean metric \( g_{ij} = \text{diag}(1,1,1) \), the tensors \( T_{ijk} \) and \( T^{ijk} \) have the same numerical components. So they can be referred to as different versions of the same tensor. This assumption is widely used in literature, even without notice that it is true only for Cartesian tensors with the Euclidean metric.

Let us observe the following properties of the scalar product of tensors that are useful in manipulation with the indices:

• For an arbitrary permutation of indices \( \sigma \),

\[ (\sigma A, \sigma B) = (A, B). \]  
\[ (95) \]

• For an arbitrary permutation of indices \( \sigma \) and its inverse \( \sigma^{-1} \),

\[ (\sigma A, B) = (A, \sigma^{-1} B). \]  
\[ (96) \]
For a linear combination $P = \sum \alpha_i \sigma_i$ of permutations $\sigma_i$ with arbitrary real coefficients $\alpha_i$,

$$(PA, B) = \left( \sum \alpha_i \sigma_i A, B \right) = \left( A, \sum \alpha_i \sigma_i^{-1} B \right).$$  \hspace{1cm} (97)

In particular, using (39) for the fully symmetric and skew-symmetric tensors, we have respectively

$A^{(ijk)}B_{ijk} = A^{ijk}B_{(ijk)} = A^{(ijk)}B_{(ijk)}$ \hspace{1cm} (98)

and

$A^{[ijk]}B_{ijk} = A^{[ijk]}B_{[ijk]} = A^{[ijk]}B_{[ijk]}$. \hspace{1cm} (99)

Now we are able to formulate the following statement:

**Proposition 3.** Let the tensor $T^{ijk}$ be uniquely decomposed as

$$T^{ijk} = S^{ijk} + A^{ijk} + N^{ijk}$$  \hspace{1cm} (100)

where

$S^{ijk} = K^{ijk} + R^{ijk}, \quad N^{ijk} = M^{ijk} + P^{ijk}$. \hspace{1cm} (101)

Then all five sub-tensors given above (and their corresponding subspaces) are mutually orthogonal one to another.

**Proof.** We provide a proof for two sequel levels of the decomposition. First, we prove that the tensors in (100) are mutually orthogonal. The relations

$$S^{ijk}A_{ijk} = S^{ijk}N_{ijk} = A^{ijk}N_{ijk} = 0$$ \hspace{1cm} (102)

follow immediately from Eqs. (98,99). For instance,

$$S^{ijk}N_{ijk} = S^{(ijk)}N_{ijk} = S^{(ijk)}N_{(ijk)} = 0.$$ \hspace{1cm} (103)

Thus the subspaces $S, A, N$ are mutually orthogonal one to another. On the second level, let us prove that the tensors $K^{ijk}$ and $R^{ijk}$ are orthogonal. Since the tensor $R_{ijk}$ is traceless,

$$K^{ijk}R_{ijk} = \frac{1}{5} \left( \alpha^i g^{jk} + \alpha^j g^{ik} + \alpha^k g^{ij} \right) R_{ijk} = 0.$$ \hspace{1cm} (104)

Similarly the tensors $M^{ijk}$ and $P^{ijk}$ are orthogonal one to another. Thus we conclude that all 5 independent parts of the tensor $T^{ijk}$ are mutually orthogonal.

The behaviour of the (non-unique) irreducible parts of $T^{ijk}$ is more complicated. Let us check the scalar product of the tensors $N^{ijk}_1$ and $N^{ijk}_2$. We use the operator representation of these tensors (41) and apply the fact that the simple cycle permutations (12), (13) and (23) are self inverse. Due to (96),

$$N^{ijk}_1 N^{ijk}_2 = (I + (12))(I - (13)) T^{ijk} N^{ijk}_2$$

$$= (I - (13)) T^{ijk}(I + (12)) N^{ijk}_2$$

$$= T^{ijk}(I - (13))(I + (12)) N^{ijk}_2$$

$$= T^{ijk}(I - (13))(I + (12))(I + (13))(I - (12)) T^{ijk} \neq 0.$$ \hspace{1cm} (105)
The product of the operators in the middle is not zero, thus the tensors $N_1^{ijk}$ and $N_2^{ijk}$ are not orthogonal.

We can, however, identify two orthogonal proper subspaces of the space $\mathbf{N}$. Since $(I + (12))(I - (12)) = 0$, we have

$$
N_1^{ijk} \hat{N}_2^{ijk} = (I + (12))(I - (13))T^{ijk}N_2^{ijk} \\
= (I - (13))T^{ijk}(I + (12))N_2^{ijk} \\
= T^{ijk}(I - (13))(I + (12))(I - (12))(I + (23))T^{ijk} = 0. 
$$

(106)

Notice, however, that we do not have here the direct sum decomposition,

$$
\mathbf{N} \neq \mathbf{N}_1 \oplus \hat{\mathbf{N}}_2. 
$$

(107)

### 4.3 $SL(3, \mathbb{R})$-decomposition

#### 4.3.1 Permutation tensor

Consider now a 3-dimensional vector space $V$ endowed with the volume element instead of the scalar product. The volume element structure is uniquely determined by the permutation pseudo-tensor $\varepsilon_{ijk}$ of Levi-Civita. It is defined as

$$
\varepsilon_{ijk} = \begin{cases} 
1 & \text{if } (ijk) \text{ is an even permutation of } (123); \\
-1 & \text{if } (ijk) \text{ is an odd permutation of } (123); \\
0 & \text{otherwise.}
\end{cases} 
$$

(108)

Notice the contraction relations

$$
\varepsilon_{ijk}\varepsilon^{mnk} = \delta_l^m \delta_j^n - \delta_j^m \delta_l^n, 
$$

(109)

and

$$
\varepsilon_{ijk}\varepsilon^{mjk} = 2\delta_i^m, \quad \varepsilon_{ijk}\varepsilon^{ijk} = 6
$$

(110)

that are useful for manipulations with the tensor $\varepsilon_{ijk}$. The upper indiced tensor $\varepsilon^{ijk}$ is defined with the same numerical values as in (108).

Under a linear transformation $e^i \rightarrow e'^i = R_i^j e^j$ of a basis in $V$, Levi-Civita’s pseudo-tensor is transformed as

$$
\varepsilon_{ijk} = (\det R) R_i^\ell R_j^\ell R_k^\ell \varepsilon_{\ell' j' k'}, 
$$

(111)

where $\det R$ denotes the determinant of the matrix $R_i^j$. Consequently, the numerical values of $\varepsilon_{ijk}$ are preserved only under transformations with matrices of unit determinant, i.e., $\det R = 1$. This condition defines a subgroup of $GL(3, \mathbb{R})$ – the *special linear group* $SL(3, \mathbb{R})$.

---

¹This definition is in a correspondence with the Euclidean signature. For Lorentzian signature, different sign assumptions for $\varepsilon^{ijk}$ and $\varepsilon_{ijk}$ must be involved.
4.3.2 Permutation tensor and symmetric and skew-symmetric parts

Now we try to apply the tensor $\varepsilon_{ijk}$ to a decomposition of a generic third-order tensor $T^{ijk}$. The full contraction of these two tensors produces a scalar

$$A = \frac{1}{6} \varepsilon_{ijk} T^{ijk}. \quad (112)$$

Notice that $A$ is a pseudo-tensor under transformations from $GL(3, \mathbb{R})$. Under transformations from $SL(3, \mathbb{R})$, it is a proper tensor.

Due to the symmetries of the parts $S^{ijk}$ and $N^{ijk}$, their full contraction with $\varepsilon_{ijk}$ vanish

$$\varepsilon_{ijk} S^{ijk} = 0, \quad \varepsilon_{ijk} N^{ijk} = 0. \quad (113)$$

Consequently, only the totally antisymmetric part of $T^{ijk}$ contributes into the lhs of Eq.(112), i.e.,

$$A = \frac{1}{6} \varepsilon_{ijk} A^{ijk}. \quad (114)$$

Accordingly to (110), the inverse relation is given by

$$A^{ijk} = A \varepsilon^{ijk}. \quad (115)$$

Thus the totally antisymmetric irreducible part $A^{ijk}$ of the tensor $T^{ijk}$ is completely expressed by the scalar $A$.

4.3.3 Pseudo-tensors

We define now partial contractions of the tensors $T^{ijk}$ and $\varepsilon_{ijk}$ with two indices summed. For a generic tensor $T^{ijk}$, there are three different possible contractions of this type

$$A^{m} = \varepsilon_{ijk} T^{mjk}, \quad B^{m} = \varepsilon_{ijk} T^{kmj}, \quad C^{m} = \varepsilon_{ijk} T^{jkm}. \quad (116)$$

Observe that these three tensors have the same trace

$$A^{i} = B^{i} = C^{i} = 6A. \quad (117)$$

Let us substitute the $GL(3, \mathbb{R})$-decomposition of the tensor, $T^{ij} = S^{ij} + A^{ij} + N^{ij}$ into Eqs.(116). Since a contraction of a symmetric and antisymmetric sets of the same indices is zero, the totally symmetric part $S^{ijk}$ does not contribute to the 2-indicied tensors (116). In order to derive the contributions of two additional 3-rd order tensors $A^{ijk}$ and $N^{ijk}$, it is convenient to calculate their sum

$$A^{m} + B^{m} + C^{m} = \varepsilon_{ijk} \left( T^{mjk} + T^{kmj} + T^{jkm} \right)$$

$$= \frac{1}{2} \varepsilon_{ijk} \left( T^{mjk} - T^{mkj} + T^{kmj} - T^{jmk} + T^{jkm} - T^{kmj} \right)$$

$$= 3 \varepsilon_{ijk} T^{[mjk]} = 3 \varepsilon_{ijk} A^{mjk}. \quad (118)$$
Due to (115) and (110), it means
\[ A_l^m + B_l^m + C_l^m = 3 A \varepsilon_{ijk} \varepsilon^{mjk} = 6 A \delta_l^m. \] (119)

Define the traceless combinations
\[ \tilde{A}_i^m := A_i^m - 2 A \delta_i^m, \quad \tilde{B}_i^m := B_i^m - 2 A \delta_i^m, \quad \tilde{C}_i^m := C_i^m - 2 A \delta_i^m. \] (120)

These tensors are traceless, \( \tilde{A}_i^i = \tilde{B}_i^i = \tilde{C}_i^i = 0 \), so every one of these tensors has 8 independent components. Eq. (119) yields
\[ \tilde{A}_i^m + \tilde{B}_i^m + \tilde{C}_i^m = 0. \] (121)

It means that only 16 components are independent. In fact, only the mixed symmetry part \( N_{ijk}^i \) of \( T_{ijk} \) contributes to the tensors (120). Indeed, the contribution of the symmetric tensor \( S_{ijk} \) multiplied with the antisymmetric Levi-Civita’s tensor is identically zero, while the tensor \( \Lambda_{ijk} \) compensates the scalar parts in (120). Consequently, (120) can be rewritten via the tensor \( N_{ijk}^i \) only
\[ \tilde{A}_i^m = \varepsilon_{ijk} N_{kmj}^i, \quad \tilde{B}_i^m = \varepsilon_{ijk} N_{kmj}^i, \quad \tilde{C}_i^m = \varepsilon_{ijk} N_{kmj}^i. \] (122)

Due to (121) the tensors \( \tilde{A}_i^m, \tilde{B}_i^m \) and \( \tilde{C}_i^m \) are linearly dependent, so we have two linearly independent traceless 2nd-order tensors of 8 independent components each. Every two of the tensors given in (120), or every two independent linear combinations of them, can be chosen as a representation of the irreducible part \( N_{ijk}^i \) of 16 independent components.

4.3.4 Pseudotensor representation of the mixed-symmetry part

In order to demonstrate the full equivalence between the 3-rd order tensor \( N_{ijk}^i \) and the 2-nd order tensors \( A_i^j, B_i^j, \) and \( C_i^j \), we have to solve the linear system (122). It is a system of 24 linear equations in 16 independent variables \( N_{ijk}^i \). Due to 8 constraints given in (121), this system is well-posed. Substituting the decomposition \( N_{ijk}^i = N_{1ijk}^i + N_{2ijk}^i \) and using the symmetries
\[ N_{1ijk}^i = N_{1ijk}, \quad N_{2ijk}^i = N_{2ji}^i \] (123)
we derive from (122)
\[ \tilde{B}_i^m = \varepsilon_{ijk} N_{1kmj}^i, \quad \tilde{C}_i^m = \varepsilon_{ijk} N_{2jkm}^i. \] (124)
of 16 linear equations in 16 independent variables. Moreover, our system is decomposed into a pair of 8 independent systems in 8 independent variables. A general form of a solution of the system
\[ \tilde{B}_i^m = \varepsilon_{ijk} N_{kmj}^i \] can be written as
\[ N_{1kmj}^i = x \tilde{B}_p^k \varepsilon^{pmj} + y \tilde{B}_p^m \varepsilon^{pkj} + z \tilde{B}_p^j \varepsilon^{pmk} \] (125)
with unknown numerical coefficients \( x, y, z \). Applying the symmetry of \( N_{1kmj}^i \) from (123), we obtain \( z = 0, x = y \). Thus the most general solution of the first equation in (124) reads
\[ N_{1kmj}^i = x \left( \tilde{B}_p^k \varepsilon^{pmj} + \tilde{B}_p^m \varepsilon^{pkj} \right). \] (126)
Contracting both sides of this equation with $\varepsilon_{ijk}$, we derive

$$\tilde{B}^m_i = x \left( \tilde{B}^k_p \varepsilon_{ijk} \varepsilon^{pmj} + \tilde{B}^m_p \varepsilon_{ijk} \varepsilon^{pkj} \right). \quad (127)$$

Due to the rules (109, 110) we have here

$$\tilde{B}^m_i = x \left( -\delta^m_i \delta^m_p + \delta^m_i \delta^m_p - \tilde{B}^m_p \delta^m_i \right) = -2x\tilde{B}^m_i. \quad (128)$$

Consequently, $x = -(1/2)$ and

$$N^{kmj}_1 = -\frac{1}{2} \left( \tilde{B}^k_p \varepsilon^{pmj} + \tilde{B}^m_p \varepsilon^{pkj} \right). \quad (129)$$

In the same fashion, we derive the solution of the second system in (124)

$$N^{kmj}_2 = -\frac{1}{2} \left( \tilde{C}^k_p \varepsilon^{pmj} + \tilde{C}^p_j \varepsilon^{pmk} \right). \quad (130)$$

Finally,

$$N^{kmj} = -\frac{1}{2} \left( (\tilde{B}^k_p \varepsilon^{pmj} + \tilde{B}^m_p \varepsilon^{pkj}) + (\tilde{C}^k_p \varepsilon^{pmj} + \tilde{C}^p_j \varepsilon^{pmk}) \right). \quad (131)$$

As a result, the $SL(3, \mathbb{R})$-structure on the basis vector space $V$ provides an additional specification of the 3-rd order tensor subspaces. In particular:

- The totally symmetric subspace $S$ is not sensitive to the additional $SL(3, \mathbb{R})$-structure.
- The totally skew-symmetric subspace $A$ is characterized by a unique parameter $A$. It is a pseudo-scalar under general linear transformations and a proper scalar under transformations from the group $SL(3, \mathbb{R})$.
- The mixed-symmetry subspace $N$ of dimension 16 is characterized by two independent traceless matrices of 8 components each. These matrices transform as pseudo-tensors under general linear transformations and as proper tensors under transformations from the group $SL(3, \mathbb{R})$. The choice of these matrices is not unique. This fact is in correspondence to the non-uniqueness of the decomposition $N^{ijk} = N_1^{ijk} + N_2^{ijk}$ that was derived at the level of $GL(3, \mathbb{R})$-structure.

### 4.4 $SO(3, \mathbb{R})$-decomposition

The $SO(3, \mathbb{R})$-structure is defined by proper orthogonal transformations that belong to the intersection of the special linear group and the orthogonal group, i.e.,

$$SO(3, \mathbb{R}) = SL(3, \mathbb{R}) \cap O(3, \mathbb{R}). \quad (132)$$

In this case, the metric tensor $g_{ij}$ and the permutation tensor $\varepsilon_{ijk}$ can be used simultaneously for decomposition of a 3-rd order tensor $T^{ijk}$. Recall that the $SL(3, \mathbb{R})$ group itself does not define
an additional decomposition of the tensor space. Consequently, the $SO(3, \mathbb{R})$ structure can provide only an alternative description of the subspaces that already derived by the $O(3, \mathbb{R})$-decomposition.

The tensor $\varepsilon_{ijk}$ is not relevant to the fully symmetric subspace $S$. Consequently, it’s decomposition remains the same as it is given in Eq. (65). Briefly this representation can be written as a pair of a vector and a fully traceless tensor

$$S^{ijk} = \{\alpha^i, R^{ijk}\},$$ (133)

with the dimension reduction $10 = 3 + 7$.

The skew-symmetric subspace $A$ of dimension 1 is represented in $SO(3, \mathbb{R})$-framework by a pseudo-scalar $A$, namely $A^{ijk} = Ae^{ijk}$.

The mixed-symmetric tensor $N^{ijk}$ is represented in the $SL(3, \mathbb{R})$-framework by two mixed-type tensors, say $\tilde{B}_{i^j}$ and $\tilde{C}_{i^j}$. These tensors are traceless, $\tilde{B}_{i^j} = \tilde{C}_{i^j} = 0$, thus their additional decomposition is impossible.

When the $O(3, \mathbb{R})$ group acts in addition to $SL(3, \mathbb{R})$, the pseudo-tensors $\tilde{B}_{i^j}$ and $\tilde{C}_{i^j}$ can be inverted into covariant (or contravariant) 2-nd order pseudo-tensors. It means that one can define

$$\tilde{B}_{ij} := g_{jm} \tilde{B}_{i^m} \quad \text{and} \quad \tilde{C}_{ij} := g_{jm} \tilde{C}_{i^m}$$ (134)

that are completely equivalent to $\tilde{B}_{i^j}$ and $\tilde{C}_{i^j}$. Now, the pseudo-tensors $\tilde{B}_{ij}$ and $\tilde{C}_{ij}$ can be irreducible decomposed into the sum of their symmetric and skew-symmetric parts

$$\tilde{B}_{ij} = \tilde{B}_{(ij)} + \tilde{B}_{[ij]} \quad \text{and} \quad \tilde{C}_{ij} = \tilde{C}_{(ij)} + \tilde{C}_{[ij]}.$$ (135)

Since the tensors are traceless, the dimensions of the corresponded parts are $8 = 5 + 3$, respectively. We denote the symmetric parts as

$$E_{ij} = \tilde{B}_{(ij)} \quad \text{and} \quad F_{ij} = \tilde{C}_{(ij)}.$$ (136)

As about the skew-symmetric parts of pseudo-tensors, they can be expressed as vectors. Indeed, we can define

$$\varepsilon^{ijk} \tilde{B}_{[ij]} \quad \text{and} \quad \varepsilon^{ijk} \tilde{C}_{[ij]}.$$ (137)

Notice, that being defined as contractions of two pseudo-tensors, these two vectors are proper. They can be expressed as linear combinations of the vectors $\beta^k$ and $\gamma^k$ defined above. Consequently, the mixed-type parts are decomposed irreducible as

$$N^{ijk}_1 = \{E_{ij}, \beta^i\} \quad \text{and} \quad N^{ijk}_2 = \{F_{ij}, \gamma^i\}.$$ (138)

We will provide the explicit expressions for these relations in the sequel.

### 4.5 Results:

In this section we derived different types of decomposition of a generic 3-nd order tensor, see Fig. 1 for a schematic representation. It is due to different structures on the vector space $V$ and correspondingly to different groups of transformations on $V$.

Our results are as follows:
Figure 1: Schematic representation of different types of decomposition of a generic 3-rd order tensor.

- On the \( \text{GL}(3, \mathbb{R}) \)-level, the decomposition is presented uniquely by three 3-rd order tensors. A successive decomposition into four 3-rd order tensors is irreducible but non-unique. Some explicit examples of equivalent irreducible decompositions are presented above.

- On the \( \text{O}(3, \mathbb{R}) \)-level with the metric tensor at hand, we can extract three vectors and 3 traceless 3-rd order tensors. These quantities provide the direct sum decomposition of the tensor space into 7 subspaces.

- The \( \text{SL}(3, \mathbb{R}) \)-structure with a permutation tensor \( \varepsilon_{ijk} \) defined allows us to express the totally skew-symmetric tensor by a pseudo-scalar. Moreover, we can define mixed 2-nd order pseudo-tensors as an alternative representation of the tensor \( N_{ijk}^i \).

- The \( \text{SO}(3, \mathbb{R}) \)-structure allows us to join the previous cases and yields the most finer decomposition. In particular, it adds a representation of the tensor \( N_{ijk}^j \) by three symmetric traceless 2-nth order tensors. Notably, the latter tensors have pseudo-tensorial nature.

5 Tensors with partial symmetries and their decomposition

In solid-state physics, the 3-rd order tensors emerge as constitutive tensors. They establish linear phenomenological relations between the primary physical variables, \([23]\). Due to the fundamental symmetries of these variables, constitutive tensors turn out to be partially symmetric.
5.1 Tensors with a symmetry in a pair of indices: Piezoelectric tensor

5.1.1 Definition

Most phenomenological models used in solid-state physics are dealing with the constitutive tensor that is symmetric in a pair of its indices. The well-known example is the piezoelectric tensor, see e.g., [7]. This tensor, $D^{ijk}$, is defined as a set of coefficients that relate the induced electric displacement vector $P^i$ to the second-order elasticity stress tensor $\sigma_{jk}$,

$$P^i = D^{ijk} \sigma_{jk}. \quad (139)$$

Due to the symmetry of the stress tensor $\sigma_{ij} = \sigma_{ji}$, the piezoelectric tensor satisfies the symmetry relation

$$D^{ijk} = D^{ikj}. \quad (140)$$

In general, such a pair-symmetric tensor has 18 independent components. The invariant decomposition of the piezoelectric tensor can provide a useful information about the piezoelectric phenomena as well as about the proper classification of the piezoelectric crystals. Moreover, it can serve as a useful theoretical tool for design novel piezoelectric materials with specific properties. Recently the algebraic properties of the pair-symmetric third-order tensor, especially in the case of the piezoelectric tensor, was studied intensively, see [20], [21]. Let us look at how the decomposition of the piezoelectric tensor $D^{ijk}$ can follow from the decomposition of the general tensor $T^{ijk}$ described above.

5.1.2 $GL(3,\mathbb{R})$-decomposition

We start with the permutation decomposition of the piezoelectric tensor. Due to the symmetry $D^{ijk}$ does not contain a totally skew-symmetric part $A^{ijk}$. Moreover, as the identity $T^{(ijk)} = T^{(ij,ik)}$ holds for an arbitrary tensor, the subspace corresponding to the totally symmetric part of $D^{ijk}$ lies into the tensor space defined by the tensor $D^{ijk}$. Consequently, the decomposition

$$D^{ijk} = S^{ijk} + N^{ijk}, \quad \text{where} \quad S^{ijk} = D^{(ijk)} \quad (141)$$

provides a unique decomposition of the space $D$ into the direct sum of two subspaces with the indicated dimensions

$$D = S \oplus N, \quad 18 = 10 + 8. \quad (142)$$

Here the fully symmetric part is reduced to

$$S^{ijk} = D^{(ijk)} = \frac{1}{3} \left( D^{ijk} + D^{jki} + D^{kij} \right), \quad (143)$$

while the residue part reads

$$N^{ijk} = \frac{1}{3} \left( 2D^{ijk} - D^{kij} - D^{jki} \right). \quad (144)$$

Notice that now $N^{ijk} = N^{ikj}$, i.e., this sub-tensor inhabits the symmetry of the piezoelectric tensor. In terms of tensor spaces, it means that space $N$ corresponding to $N^{ijk}$ is a proper subspace of the tensor space $D$, as it is indicated in Eq. (141).
In contrast to the general case, the successive decomposition of the tensor $N^{ijk}$ is impossible. Indeed, for a general 3-rd order tensor $T^{ijk}$, the space $N$ is of dimension 16 and it is decomposed (non-uniquely) into the direct sum of two subspaces $N_1, N_2$ both of dimension 8. However, in our current restricted case, the whole space $N$ is only of dimension 8. So it cannot contain two disjoint proper subspaces of dimension 8. Let us check what is going with the subspaces of $N^{ijk}$ in this restricted case. First we observe that the Young tableau generated subspaces $N^{ijk}_1$ and $N^{ijk}_2$ remain with the dimensions 8 also in our pair-symmetric case. But they do not have the fundamental symmetries (140), so their tensor spaces are not subspaces of $N$. In other words, even the equation $N^{ijk} = N^{ijk}_1 + N^{ijk}_2$ holds, $N \neq N_1 \oplus N_2$. The description is simplified when we project the equation above onto the subspace $D$. It means that we rearrange this equation into the form $N^{ijk} = N^{ijk}_1 + N^{ijk}_2$. For a pair-symmetric tensor, the equality $N^{ijk}_1 = N^{ijk}_2$ holds, so we indeed do not have here any additional decomposition.

An alternative decomposition $N^{ijk} = \tilde{N}^{ijk}_1 + \tilde{N}^{ijk}_2$ is even more suitable in our case. Indeed, now both sub-tensors $\tilde{N}^{ijk}_1$ and $\tilde{N}^{ijk}_2$ do have the fundamental symmetries (140) and produce corresponded subspaces. However in the pair-symmetric case, $\tilde{N}^{ijk}_2 = 0$ and $N^{ijk} = \tilde{N}^{ijk}_1$. Once more we do not have any invariant decomposition of the tensor $N^{ijk}$.

These facts demonstrate that the decomposition (141) is the unique irreducible $GL(3, \mathbb{R})$-invariant decomposition of the piezoelectric tensor.

### 5.1.3 $O(3, \mathbb{R})$-decomposition

Let us turn now to the decomposition of $D^{ijk}$ under the orthogonal group $O(3, \mathbb{R})$. With a metric tensor $g_{ij}$ at hand we can construct from a third-order tensor $D^{ijk} = D^{i(jk)}$ only two trace-type vectors $v^k, w^k$ such that:

$$v^k = g_{ij}D^{ijk} = g_{ij}D^{jki} \quad \text{and} \quad w^k = g_{ij}D^{kij}. \quad (145)$$

How these vectors contribute to the successive decomposition of the piezoelectric tensor?

Due to the fully symmetry of $S^{ijk}$, we have

$$g_{ij}S^{ijk} = g_{jk}S^{ijk} = g_{kj}S^{ijk}. \quad (146)$$

Thus only one trace vector can be constructed from $S^{ijk}$. We denote it as

$$\alpha^k := g_{ij}S^{ijk} = \frac{1}{3}g_{ij} \left( D^{ijk} + D^{jki} + D^{kij} \right) = \frac{1}{3}(2v^k + w^k) \quad (147)$$

Consequently, the tensor $S^{ijk}$ is decomposed into the sum of two independent symmetric parts:

$$S^{ijk} = K^{ijk} + R^{ijk} \quad (148)$$

where the vector part is given in (66) by

$$K^{ijk} = \frac{1}{5} \left( \alpha^i g^{jk} + \alpha^j g^{ik} + \alpha^k g^{ij} \right), \quad (149)$$
while the residue part $R^{ijk}$ satisfies the traceless relations
\[ g_{ij}R^{ijk} = 0, \quad g_{ik}R^{ijk} = 0, \quad g_{jk}R^{ijk} = 0. \] (150)

Hence, the tensor space $S$ is decomposed into the direct sum of two subspaces with the respective dimensions
\[ S = K \oplus R, \quad 10 = 3 \oplus 7. \] (151)

Now we consider the trace decomposition of the tensor $N^{ijk}$. Two possible vectors
\[ g_{ij}N^{ijk} = \frac{1}{3}(v^k - w^k), \quad g_{jk}N^{ijk} = \frac{2}{3}(w^i - v^i) \] (152)
turn out to be linearly dependent. Consequently we have here only one independent vector
\[ \beta^j = \frac{2}{3}(v^j - w^j) \] (153)

Thus the tensor $N^{ijk}$ can be decomposed into the sum:
\[ N^{ijk} = M^{ijk} + P^{ijk}, \] (154)
where the vector part
\[ M^{ijk} = \frac{1}{4} \left( 2g^{jk}\beta^i - g^{ik}\beta^j - g^{ij}\beta^k \right), \] (155)
while the residue part $P^{ijk} = N^{ijk} - M^{ijk}$ satisfies the traceless relation
\[ g_{ij}P^{ijk} = 0, \quad g_{ik}P^{ijk} = 0, \quad g_{jk}P^{ijk} = 0. \] (156)

Accordingly, we have a decomposition of the subspace $N$ into the direct sum of two invariant subspaces with the corresponding dimensions
\[ N = M \oplus P, \quad 8 = 3 \oplus 5. \] (157)

5.1.4 $SL(3, \mathbb{R})$-decomposition

In the $SL(3, \mathbb{R})$ background, the permutation pseudotensor $\varepsilon_{ijk}$ is available. Since the fully antisymmetric part of the piezoelectric tensor vanishes, the pseudo-scalar $A = 0$. Moreover, the contraction in two indices of the symmetric part $S^{ijk} = D^{ijk}$ with $\varepsilon_{ijk}$ is trivial as in the general case. Consequently, $\varepsilon_{ijk}$ acts only on the mixed-type tensor $N^{ijk}$. For the 2-nd order tensors defined in Eq.(116), we have
\[ A_i^m = \varepsilon_{ijk}D^{mjk} = 0, \quad B_i^m = \varepsilon_{ijk}D^{kmj} = \varepsilon_{ijk}D^{ijk} = -C_i^m. \] (158)

Since these tensors are traceless and satisfy the relations $A_i^m + B_i^m + C_i^m = 0$, there is only one nontrivial contraction, say $B_i^m$. In order to derive the contribution of this tensor to $N^{ijk}$, we start with the linear relations between these two tensors of 8 independent components
\[ B_i^m = \varepsilon_{ijk}D^{kmj} = \varepsilon_{ijk}N^{kmj}. \] (159)
To inverse this relation, we substitute $C_{im} = -B_{im}$ into Eq. (131). Consequently, we have

$$N^{kmj} = \frac{1}{2} \left( B^m_i \epsilon^{kpj} + B^j_i \epsilon^{kpm} \right).$$ (160)

Observe that this expression has a desired symmetry $N^{kmj} = N^{kjm}$. As a result, the representations of the mixed-symmetry part of the piezoelectric tensor by the 3-rd order tensor $N_{ijk}$ and by the 2-nd order pseudo-tensor $B_{ij}$ are completely equivalent. We are left with the decomposition of the piezoelectric tensor space into two subspaces with the dimensions $18 = 10 + 8$.

5.1.5 $SO(3, \mathbb{R})$-decomposition

On the $SO(3, \mathbb{R})$-level, the mixed pseudo-tensor $B_{ij}$ can be reverted into the contravariant tensor $B_{ij} = g_{jm}B^m_i$. This new tensor can be decomposed into the skew-symmetric and symmetric parts with the dimensional reduction $8 = 3 + 5$. Let us show that this is exactly the same decomposition that we already have on the $O(3, \mathbb{R})$-level. In particular, we can show that the skew-symmetric part $B_{[ij]} \sim \beta^i$, while the symmetric part $B_{(ij)} \sim P_{ijk}$.

Substituting into Eq. (160) the expression $B^m_i = g^{mr}B_{pr}$ we obtain

$$N^{kmj} = \frac{1}{2} \left( g^{mr} \epsilon^{kpj} + g^{jr} \epsilon^{kpm} \right) B_{pr} = \frac{1}{2} \left( g^{mr} \epsilon^{kpj} + g^{jr} \epsilon^{kpm} \right) (B_{[pr]} + B_{(pr)}).$$ (161)

In order to show that the first term gives the tensor $M^{ijk}$, we chose the representation,

$$B_{[pr]} = \frac{1}{2} \epsilon_{prs} \beta^s.$$ (162)

Then the first term in (161) is expressed as

$$\frac{1}{2} \left( g^{mr} \epsilon^{kpj} + g^{jr} \epsilon^{kpm} \right) B_{[pr]} = \frac{1}{4} \left( g^{mr} \epsilon^{kpj} + g^{jr} \epsilon^{kpm} \right) \epsilon_{prs} \beta^s =$$

$$-\frac{1}{4} \left( g^{mr} (\delta^k_s \delta^j_l - \delta^j_s \delta^k_l) + g^{jr} (\delta^k_s \delta^m_l - \delta^m_s \delta^k_l) \right) \beta^s =$$

$$\frac{1}{4} \left( 2g^{jm} \beta^k - 2g^{mk} \beta^j - g^{jk} \beta^m \right) = M^{ijk}.$$ (163)

Hence we identified the vector part of 3 independent components (155) with the first term of (161). The residue part is traceless and expressed by 5 independent components $B_{(ij)}$. So it is equal to $P^{ijk} = N^{ijk} - M^{ijk}$.

In the decomposition $N^{ijk} = M^{ijk} + P^{ijk}$, we derived the representations of the irreducible parts $M^{ijk}$ and $P^{ijk}$ by two independent matrices: The antisymmetric matrix $B_{[ij]}$ and the symmetric traceless matrix $B_{(ij)}$. Explicitly,

$$M^{ijk} = \frac{1}{2} \left( g^{mr} \epsilon^{kpj} + g^{jr} \epsilon^{kpm} \right) B_{[pr]},$$ (164)

$$P^{ijk} = \frac{1}{2} \left( g^{mr} \epsilon^{kpj} + g^{jr} \epsilon^{kpm} \right) B_{(pr)}.$$ (165)
5.1.6 Results

Let us summarize our results on the irreducible decomposition of the piezoelectric tensor (and tensors with similar pair symmetry). For a schematic representation, see Fig. 2. Notice that our approach allows to distinguish the vectors $\alpha_i$ and $\beta_i$. These two vectors are merely coming from two different irreducible parts $S$ and $N$.

**Proposition 4.** Let a piezoelectric-type tensor $D_{ijk}$ with the symmetry $D_{ijk} = D_{ikj}$ be given.

- **On the $GL(3, \mathbb{R})$-level**, the tensor is decomposed uniquely and irreducibly into the sum of two subtensors (143) and (144). The tensor space is decomposed into the direct sum of two subspaces:

$$D_{ijk} = S_{ijk} \oplus N_{ijk}, \quad 18 = 10 + 8.$$  \hspace{1cm} (166)

- **On the $O(3, \mathbb{R})$-level**, the fully symmetric part $S_{ijk}$ and the mixed part $N_{ijk}$ are decomposed both into the vector (trace) and the traceless parts. In term of subspaces,

$$S_{ijk} = K_{ijk} \oplus R_{ijk}, \quad 10 = 3 + 7. \hspace{1cm} (167)$$

$$N_{ijk} = M_{ijk} \oplus P_{ijk}, \quad 8 = 3 + 5. \hspace{1cm} (168)$$

Consequently,

$$D_{ijk} = (K_{ijk} \oplus R_{ijk}) \oplus (M_{ijk} \oplus P_{ijk}), \quad 18 = (3 + 7) + (3 + 5). \hspace{1cm} (169)$$

- The subspaces $K, R, M, P$ are mutually orthogonal one to another.
On the SL(3, \mathbb{R})-level, the tensor \( N^{ijk} \) is expressed as a pseudo-tensor \( B^i_{\ j} \).

On the SO(3, \mathbb{R})-level, the fully symmetric tensor \( S^{ijk} \) returns to the \( O(3, \mathbb{R}) \)-decomposition that is given by the vector and traceless parts. The pseudo-tensor \( B^i_{\ j} \) is decomposed irreducibly and uniquely into the sum of two pseudo-tensors \( B_{[ij]} \) and \( B_{(ij)} \) that span the spaces \( M \) and \( P \), respectively.

5.2 Tensors with skew-symmetry in a pair of indices: Hall tensor

5.2.1 Definition

An alternative constraint of the general third-order constitutive tensor emerges in the description of the Hall effect and the Faraday effect. In this case, the constitutive tensor is skew-symmetric in a pair of its indices. In particular, we consider here the Hall effect that is widely observed in conductors and semi-conductors. In anisotropic media, it is described as follows, see, e.g., [7], [24]

\[ E_i = \kappa_{ijk} I^j H^k, \]  

(170)

where \( I^j \) is an electric current density, \( H^k \) is a magnetic field, while \( E_i \) is an induced electric field. For physical details, see e.g. [7]. The set of the coefficients \( \kappa_{ijk} \) describes physical properties of the conductor. These coefficients form a contravariant third-order tensor referred to as the Hall tensor. Due to statistical arguments, the Onsager relations yield the basic symmetry of the Hall tensor [17]

\[ \kappa_{ijk} = -\kappa_{jik}. \]  

(171)

In general, the skew-symmetric tensor \( \kappa_{ijk} \) has 9 independent components. The decomposition of this tensor was studied recently in various publications, see, e.g., [24] and the references given therein. Let us look how this decomposition follows from the decomposition of the general third-order tensor (in the contravariant version).

5.2.2 \( GL(3, \mathbb{R}) \)-decomposition

We start with the permutation decomposition of the Hall tensor. Due to the skew-symmetry (171), this tensor does not contain the fully symmetric part \( S_{ijk} \). Thus it is decomposed uniquely into a sum of two independent parts: the fully skew-symmetric part \( A_{ijk} \), and the mixed-symmetry part \( N_{ijk} \). This decomposition with the indicated dimensions reads

\[ \kappa_{ijk} = A_{ijk} + N_{ijk}, \quad 9 = 1 + 8. \]  

(172)

Due to (171), the skew-symmetric part is reduced to

\[ A_{ijk} = \kappa_{[ijk]} = \frac{1}{3} \left( \kappa_{ijk} + \kappa_{jki} + \kappa_{kij} \right), \]  

(173)

while the residue mixed-symmetry part reads

\[ N_{ijk} = \kappa_{ijk} - \kappa_{[ijk]} = \frac{1}{3} \left( 2\kappa_{ijk} - \kappa_{kij} - \kappa_{jki} \right). \]  

(174)
Notice that $A_{ijk} = -A_{jik}$ and $N_{ijk} = -N_{jik}$, i.e., these partial tensors inhabit the skew-symmetry of the Hall tensor.

Observe that, similarly to the piezoelectric tensor, and in a contrast to the general case, the successive decomposition of the tensor $N^{ijk}$ is impossible. Indeed on the $GL(3, \mathbb{R})$-level, a subspace of the dimension 8 is the smallest one. Then it cannot contain $GL(3, \mathbb{R})$-invariant proper subspaces.

The Young tableau generated subspaces $N_{1^{ijk}}$ and $N_{2^{ijk}}$ do not vanish. However these tensors do not satisfy the skew-symmetry relation (171). Consequently, they do not form subspaces of the $\kappa$-space. They projection however are properly embedded into this space. This projection can be provided by antisymmetrization of the tensors in the first pair of their indices. In fact, we have $N_{1[ijk]} = 0$ and $N_{2[ijk]} = N_{ijk}$.

The alternative decomposition $N^{ijk} = \hat{N}^{1^{ijk}} + \hat{N}^{2^{ijk}}$ is even more suitable in our case. Indeed, these sub-tensors do have the skew-symmetries (171). Thus they produce subspaces of the $\kappa$-space. However, in the case of the Hall tensor, $\hat{N}^{2^{ijk}} = 0$ and $\hat{N}^{1^{ijk}} = N_{ijk}$.

These facts demonstrate that the decomposition (172) is a unique irreducible $GL(3, \mathbb{R})$-invariant decomposition of the Hall tensor.

### 5.2.3 $O(3,R)$-decomposition

Let us consider now the decomposition of $\kappa_{ijk}$ under the orthogonal group $O(3,\mathbb{R})$. With a covariant version of the metric tensor $g^{ij}$ at hand we can construct from $\kappa_{ijk}$ only one trace-type covector. Indeed for three vectors $u_k, v_k, w_k$ defined in Eq.(63), we have

$$u_k = g^{ij} \kappa_{ijk} = 0, \quad v_k = g^{ij} \kappa_{ikj} = -g^{ij} \kappa_{kij} = -w_k. \quad (175)$$

Consequently we have only one independent covector $v_k$ that contributes to the mixed-symmetry part $N_{ijk}$. Due to Eq.(86), we use in the decomposition

$$N_{ijk} = M_{ijk} + P_{ijk} \quad (176)$$

the vector part

$$M_{ijk} = \frac{1}{2} (v_j g_{ik} - v_i g_{jk}). \quad (177)$$

We can check straightforwardly that the residue part $P_{ijk} = N_{ijk} - M_{ijk}$ satisfies the traceless relation

$$g^{ij} P_{ijk} = g^{ij} p_{ikj} = g^{ij} p_{kij} = 0. \quad (178)$$

Accordingly, we have a direct sum decomposition of the subspace $N$ into the direct sum of two invariant subspaces with the corresponding dimensions

$$N = M \oplus P, \quad 8 = 3 \oplus 5. \quad (179)$$

### 5.2.4 $SL(3,R)$-decomposition

With the volume element structure defined on the basic space $V$, we are able to define the pseudoscalar

$$A = \frac{1}{6} \epsilon^{ijk} \kappa_{ijk} = \frac{1}{6} \epsilon^{ijk} A_{ijk}. \quad (180)$$
The inverse relation reads

\[ A_{ijk} = A \epsilon_{ijk} \]  

(181)

Similarly to Eq. (116) we define three pseudo-tensors

\[ A^i_m = \epsilon^{ijk} \kappa_{mjk}, \quad B^i_m = \epsilon^{ijk} \kappa_{mkj}, \quad C^i_m = \epsilon^{ijk} \kappa_{jmk}. \]  

(182)

Observe that

\[ B^i_m = \epsilon^{ijk} \kappa_{mkj} = -\epsilon^{ijk} \kappa_{mkj} = -\epsilon^{ikj} \kappa_{mkj} = \epsilon^{ijk} \kappa_{mjk} = A^i_m. \]  

(183)

Then also for the modified pseudo-tensors we have \( \tilde{B}^i_m = \tilde{A}^i_m \). We use the relation (121) to derive the second relation between the pseudotensors, \( \tilde{C}^i_m = -2 \tilde{A}^i_m \). Now we are able to express the mixed-symmetry part of the Hall tensor in term of the pseudo-tensor \( \tilde{A}^i_m \). Changing the position of the indices in Eq. (131) and substituting the relations above, we derive

\[ N_{kmj} = \frac{1}{2} \left( \tilde{A}^{p}_{k} \epsilon_{pmj} - \tilde{A}^{p}_{m} \epsilon_{pkj} - 2 \tilde{A}_{j}^{p} \epsilon_{kmp} \right). \]  

(184)

Notice that the lhs of this equation is skew-symmetric in the indices \( k \) and \( m \). Eq. (184) proves that the representations of the mixed-symmetry part by the third-order tensor \( N^{kmj} \) and the pseudo-tensor \( \tilde{A}^{i}_{p} \) are completely identical.

5.2.5 \( SO(3,R) \)-decomposition

On a space endowed with the metric and the volume element structures, the mixed-type pseudo-tensor \( \tilde{A}^{i}_{ij} \) can be reverted into the covariant tensor. Let us define

\[ \tilde{A}^{ij} := g^{mj} \tilde{A}^{i}_{m}. \]  

(185)

This traceless tensor of 8 independent components can be irreducibly decomposed into the sum of its symmetric and skew-symmetric parts of \( 8 = 3 + 5 \) independent components, respectively

\[ \tilde{A}^{ij} = \tilde{A}^{[ij]} + \tilde{A}^{(ij)}. \]  

(186)

This decomposition has to be in a correspondence with the \( O(3,\mathbb{R}) \)-decomposition (176). In particular, we can guess the relations of the form \( \tilde{A}^{[ij]} \sim v_{ij} \) and \( \tilde{A}^{(ij)} \sim p_{ijkl} \). Let us derive these expressions explicitly. Substituting into Eq. (184) the expression \( \tilde{A}^{p}_{m} = \tilde{A}^{pr}_{m} g_{mr} \) we obtain

\[ N_{kmj} = \frac{1}{2} \tilde{A}^{pr}_{j} \left( g_{kr} \epsilon_{pmj} - g_{mr} \epsilon_{pkj} - 2 g_{jr} \epsilon_{kmp} \right). \]  

(187)

Substituting here (186) we have

\[ N_{kmj} = \frac{1}{2} \left( \tilde{A}^{[pr]} + \tilde{A}^{(pr)} \right) \left( g_{kr} \epsilon_{pmj} - g_{mr} \epsilon_{pkj} - 2 g_{jr} \epsilon_{kmp} \right). \]  

(188)
We assume the skew-symmetric part of $\tilde{A}^{pr}$ to be parametrized as

$$\tilde{A}^{[pr]} = -\frac{1}{3} \varepsilon^{prs} v_s.$$  \hspace{1cm} (189)

This expression is contributed to the right-hand-side of Eq. (188) as

$$-\frac{1}{6} \varepsilon^{prs} (g_{kr} \varepsilon_{pmj} - g_{mr} \varepsilon_{pkj} - 2g_{jr} \varepsilon_{pkm}) v_s.$$  \hspace{1cm} (190)

Using (109) we calculate this expression to obtain that it is equal to the vector part of $N_{kmj}$

$$\frac{1}{2} (g_{jk} v_m - g_{mj} v_k) = M_{kmj}.$$  \hspace{1cm} (191)

Thus we have proved that the skew-symmetric part of the pseudo-tensor, $\tilde{A}^{[pr]}$, indeed equivalent to the vector part $v_k$. Since $\tilde{A}^{pr}$ contributes linearly to the right-hand-side of Eq. (188), the symmetric part of the pseudo-tensor, $\tilde{A}^{(pr)}$ is equivalent to the traceless part $P_{ijk}$. In particular, we have the identities

$$M_{kmj} = \frac{1}{2} \tilde{A}^{[pr]} (g_{kr} \varepsilon_{pmj} - g_{mr} \varepsilon_{pkj} - 2g_{jr} \varepsilon_{pkm}),$$  \hspace{1cm} (192)

and

$$P_{kmj} = \frac{1}{2} \tilde{A}^{(pr)} (g_{kr} \varepsilon_{pmj} - g_{mr} \varepsilon_{pkj} - 2g_{jr} \varepsilon_{pkm}).$$  \hspace{1cm} (193)

### 5.2.6 Results

In this section, we derived the invariant decomposition of the Hall tensor for different categories of the basic vector space. We derived explicitly the relation between the tensor and pseudo-tensor representations of the mixed-symmetry parts. The results are presented in Fig.3.

**Proposition 5.** Let a tensor $\kappa_{ijk}$ with the symmetry $\kappa_{ijk} = -\kappa_{jik}$ be given.

- **On the GL(3, $\mathbb{R}$)-level**, the tensor is decomposed uniquely and irreducibly into the sum of two subtensors. The tensor space is decomposed into the direct sum of two subspaces:

  $$\kappa = N \oplus A, \quad 18 = 10 + 8.$$  \hspace{1cm} (194)

- **On the O(3, $\mathbb{R}$)-level**, the mixed-symmetry part $S^{ijk}$ is decomposed into the vector (trace) and the traceless parts. In term of subspaces,

  $$S = M \oplus P, \quad 8 = 3 + 5.$$  \hspace{1cm} (195)

Consequently,

$$\kappa = (M \oplus P) \oplus A, \quad 9 = (3 + 5) + 1.$$  \hspace{1cm} (196)

- **The subspaces $M, P,$ and $A$ are mutually orthogonal one to another.**

- **On the SL(3, $\mathbb{R}$)-level**, the tensor $N^{ijk}$ is expressed as a pseudo-tensor $B^i_{jk}$.

- **On the SO(3, $\mathbb{R}$)-level**, the pseudo-tensor $B^i_{ij}$ is decomposed irreducibly and uniquely into the sum of two pseudo-tensors $B_{[ij]}$ and $B_{(ij)}$. These tensors span the spaces $M$ and $P$, respectively.
6 Conclusion

In this paper, we study an invariant decomposition of the 3-rd order tensor into smaller sub-tensors. Even this relatively simple case, demonstrates the principle problems of non-uniqueness of irreducible decomposition. We constructed explicitly different types of decomposition based on various geometrical structures defined on a basic vector space.

In the cases of a physical interest, the 3-rd order tensor emerges as a constitutive tensor with additional symmetries coming from the symmetries of the basic physical variables. We considered the pair symmetric tensor of a piezoelectric type and a skew-symmetric tensor of a Hall type. We show that for such tensors the irreducible decomposition is unique without ambiguity.

The problem of irreducible decomposition is not of the mathematical interest only. In fact, different irreducible invariant parts of the constitutive tensor have to represent different physical features of media. The examples of such type of presentation is known in gravity, solid state electromagnetism, and elasticity theory.

The irreducible decomposition presented here can be useful for investigation of the natural materials (as piezoelectric crystals) as well as for design the artificial nano-materials.

Data Availability Statement

Data sharing is not applicable to this article as no new data were created or analyzed in this study.
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