Optical-force-mediated coupling between levitated nanospheres can go ultrastrong

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We theoretically investigate the effect of optical-force-mediated interactions onto the quantum dynamics of a pair of dielectric nanospheres optically trapped in two neighboring optical tweezers. Thanks to the interference between the tweezer beams and the elastically scattered light by the other nanosphere, the effective inter-nanosphere coupling can reach the ultra-strong coupling regime. Experimentally accessible signatures of the entangled nature of the quantum ground state are highlighted, including entanglement witnesses based on position/momentum correlation functions and a sizable dynamical Casimir excitation upon a fast modulation of the tweezer’s properties.

Introduction.— Since the first proposal in [1], quantum optical systems in the so-called ultra-strong coupling (USC) regime have attracted a great interest from a wide community [2, 3]. As soon the strength of the coupling between two subsystems becomes comparable or larger that the characteristic frequency of each sub-system, the quantum ground state of the coupled system is deformed from the product of the individual sub-systems’ ones and strong quantum correlations emerge between them. Over the years, the USC regime has been investigated in many different systems, such as coupled harmonic oscillators of either mechanical or electromagnetic nature [4–6] or, in a more extensive way, matter excitations (e.g. intersubband electronic transitions in semiconductor heterostructures [7–10] or superconductor-based circuit-QED devices [11, 12]) coupled to electromagnetic resonators. In most cases, the signature was the Rabi splitting between the polariton branches and, in spite of strong experimental efforts [13–15], no evidence is yet available of the entangled nature of the quantum ground state.

 Levitated micro- and nanospheres have recently emerged as very promising candidates for the realization of extremely isolated mechanical systems [16, 17]. The extremely weak friction losses experienced by the nanosphere motion and the tremendous developments in opto-mechanical techniques [18] have allowed to cool the mechanical degrees of freedom very close to the quantum ground state and to measure mechanical quantities with a single excitation-quantum sensitivity [19–22]. These achievements suggest the potential of these systems as a platform where to investigate the quantum mechanics of macroscopic objects, with a number of applications to fundamental science and quantum technologies [23, 24].

In this work, we theoretically propose a system of two levitated nanospheres trapped in a pair of neighboring optical tweezers as a novel platform where to investigate quantum effects stemming from the USC between mechanical degrees of freedom arising from the so-called optical-binding mechanism [25]. Here the mechanical degrees of freedom of the two nanospheres are coupled by the optical force experienced by each nanosphere in the spatially oscillating intensity profile generated by the interference between the tweezer beams and the light that is elastically scattered by the other sphere. As compared to all previous set-ups, our proposed system allows for an almost instantaneous modulation of the mechanical parameters by acting on the intensity and polarization of the optical tweezers which leads to a sizable dynamical Casimir emission of mechanical excitations. This suggests that a full experimental characterization of the quantum entanglement features of USC is accessible in state-of-the-art systems.

Model.— We consider a system composed of two identical dielectric nanospheres with mass \(m\) and radius \(a\), trapped in two spatially separated optical tweezers focused along the \(z\)-axis. Their focus centers are kept at distance \(D\) along the \(x\)-axis, as depicted in Fig. 1. Each \(i = 1, 2\) optical tweezer is formed by a monochromatic Gaussian beam with electric field \(\vec{E}_{\text{tw},i}(t, \vec{r}) = \text{Re} \left[ \vec{E}_{\text{tw},i}(\vec{r}) e^{-i \omega_{\text{tw}} t} \right]\), with frequency \(\omega_{\text{tw}} = c k\), where \(c\) is the speed of light, \(k = 2\pi/\lambda\) the beam’s wave number.

Figure 1. Sketch of the proposed set-up. Two nanospheres are trapped in a pair of optical tweezers of waist \(w_0\) separated by a distance \(D\). The interaction between the two nanospheres located at \(\vec{r}_{1,2}\) is mediated by the radiation generated by the nanospheres’ oscillating polarization dipoles \(d_{1,2}\).
and $\vec{E}_{\text{tw},i}(\vec{r}) = \vec{e}_{\text{tw},i}E_i(\vec{r})e^{i[kz+\phi_i(\vec{r})]}$. Here, $\vec{e}_{\text{tw},i}$ is the polarization in the $(x,y)$-plane, $E_i(\vec{r})$ is the real amplitude and $\phi_i(\vec{r})$ is the phase (see Sup. Mat. [26]).

The dielectric nanospheres are described as point-like objects with positions $\vec{r}_i$, and dipole moments $d_i$, with $i = 1, 2$. Their motion is described by the Hamiltonian

$$H = \sum_{i=1}^{2} \left[ \frac{\vec{p}_i^2}{2m} - \frac{1}{2} \vec{d}_i(t, \vec{r}_i) \cdot \vec{E}(t, \vec{r}_i) \right],$$

where $\vec{p}_i$ is the momentum of the $i$-th sphere and $\vec{E}(t, \vec{r}_i)$ is the total electric field at its position. The nanospheres are not permanently polarized and their dipole moment is induced by the applied electric field $\vec{E}(t, \vec{r}_i)$. In the Rayleigh regime $a \ll \lambda$, we have $\vec{d}_i(t, \vec{r}_i) = \alpha \vec{E}(t, \vec{r}_i)$ with the Clausius-Mossotti dielectric polarizability $\alpha = 4\pi a^3\varepsilon_0(\varepsilon_r - 1)/(\varepsilon_r + 2)$ written in terms of the vacuum permittivity $\varepsilon_0$ and relative dielectric permittivities [27].

The total electric field acting on the $i$-th sphere reads

$$\vec{E}(t, \vec{r}_i) = \text{Re}[\vec{E}_{\text{tot}}(t, \vec{r}_i)e^{-i\omega_{\text{tw}}t}]$$

The first term in (2) describes the tweezers’ field, while the second one takes into account the elastic radiation emitted by the nanospheres’ dipoles, $\vec{d}_i(t, \vec{r}_i) = \text{Re}[\vec{d}_i(t, \vec{r}_i)e^{-i\omega_{\text{tw}}t}]$ and expressed here by the Green’s dyadic tensor $G$ [26]. In (2), the other tweezers’ field was neglected under the assumption that their distance is larger $D \gg w_0$ or at least comparable to the tweezer waist $w_0$, and the nanospheres are well localised in the center of their respective beam. As a second consequence of this condition, we assume that the elastically scattered field is in the $D \gg \lambda$ far-field regime, so that its relative amplitude compared to the tweezer fields is quantified by the (typically small) dimensionless parameter $\eta_f = \alpha k^2/(4\pi\varepsilon_0D)$. As highlighted in the Sup. Mat. [26], the far-field regime still holds even if the distance is comparable to the wavelength, $D \sim \lambda$, due to the extreme weakness of the near-field component of the interaction. For instance, we consider the following parameters inspired from recent experiments [19, 21] $\varepsilon_r = 2.1$, $D = 3.2\mu m$, $a = 110nm$ and $\lambda = 1064nm$ we obtain $\eta_f \sim 0.004$. Under the Rotating Wave and far-field approximations, we can derive [26] an explicit expression for the Hamiltonian at first order in $\eta_f$:

$$H \approx \sum_{i=1}^{2} \left( \frac{\vec{p}_i^2}{2m} + U_{\text{trap},i}(\vec{r}_i) \right) + U_{\text{int}}(\vec{r}_1, \vec{r}_2),$$

where $U_{\text{trap},i}(\vec{r}_i) = -\alpha |E_{\text{tw},i}(\vec{r}_i)|^2/4$ is the usual optical trapping potential in the tweezer [17, 26, 28]. The interaction potential is

$$U_{\text{int}}(\vec{r}_1, \vec{r}_2) = -\alpha \nu \eta_f \frac{D}{\Delta r} \cos(k\Delta r) \times$$

$$\times \cos \left( \left( k - \frac{1}{2a} \right) \Delta z \right) E_1(\vec{r}_1)E_2(\vec{r}_2),$$

where $\Delta r_{12} = |\vec{r}_1 - \vec{r}_2|$, $\Delta z = z_1 - z_2$, and the $1/2a$ term in the cosine comes from the phase difference $\phi_{12} = \phi_1(\vec{r}_1) - \phi_2(\vec{r}_2)$. The dipole polarization pattern of the scattered light is included by the factor $\nu = e_{\text{tw}1}^T\vec{M}_f(\Delta \vec{r})e_{\text{tw}2}$, where $\vec{M}_f(\Delta \vec{r})$ is the far-field tensor, which is the projector on the plane orthogonal to the distance between the nanospheres, approximately oriented along $(x, z)$. The interaction between the nanospheres is strongest $\nu \approx 1$ when both polarizations are directed along $y$, while it vanishes $\nu \approx 0$ when the two polarizations are orthogonal or both directed along $x$. In all cases, the motion along the $y$-axis remains effectively decoupled.

Due to the interference between the tweezer and the elastically scattered radiation, the potential in Eq. (4) has an oscillatory behaviour with wavelength $\sim \lambda$. Given its radiative origin, it scales as the inverse distance between the nanospheres $\sim 1/\Delta r_{12}$, so in the far-field it dominates over the electrostatic dipole-dipole interaction. Since for realistic parameters one has $\eta_f \ll 1$, the overall amplitude of the interaction potential remains small compared to that of the trap potential. However, its fast oscillations in space make that it has a dominant effect in the low energy dynamics of the nanospheres, as we are going to see in the following.

In Fig. 2 a) we show the exact total potential along the $z$ axis without any approximations, as derived in [26], while in Fig. 2 b) we compare a cut along the $z_1 = -z_2$ direction of the exact total potential with the approximated form of Eqs.(3-4). As in the range of parameters relevant for our discussion the accuracy of the approximated form is almost perfect, we can base our analysis on the analytical potential Eq.(3).

**Ultra-strong coupling regime**— Under suitable conditions that we are going to verify a posteriori, the potential can be linearised around the equilibrium position of the nanospheres. With no loss of generality, we assume that the nanospheres distance is approximately an integer multiple of the tweezer’s wavelength, $kD \approx 2\pi\ell$, so that [26] the equilibrium positions approximately corre-
spond to the centers of each beam $r_{0,1}^2 = (\pm D/2, 0, 0)$.

Since the motions in the different directions are approximately decoupled [26], the linearised dynamics can be approximated by $H_{\text{lin}} = H_{\text{lin},x} + H_{\text{lin},y} + H_{\text{lin},z}$ with

$$H_{\text{lin},u} = \frac{P_{u,1}^2}{2m} + \frac{P_{u,2}^2}{2m} + \frac{m\omega_{M,u}^2}{2} \left( u_1^2 + u_2^2 + \eta_{au} (u_1 - u_2)^2 \right),$$

(5)

where $u_1$ is the small displacement from equilibrium position of the $i$-th nanosphere along the direction $u = x, y, z$, and $P_{u,i}$ its respective canonical momentum. The exact expressions of the dimensionless couplings $\eta_{au}$, determined by the second derivatives of $U_{\text{int}}(r_1, r_2)$ along $u_1$ and $u_2$, are given in [26]. The coupling along the $z$ direction is the most favorable: $\eta_{zz} \approx \nu \eta_f (k w_0)^4/2$.

The coupling along $x$ also depends on $k w_0$, but with the less favorable scaling $\eta_{xx} \approx \nu \eta_f (k w_0)^2$. As noticed before, the the coupling along $y$ remains negligible, indeed $\eta_{yy} \approx \eta_f (w_0/D)^2$.

We focus then our attention on the $z$ direction, setting $\eta_{zz} = \eta$. For a Gaussian tweezer beam, the mechanical frequency along $z$ is $\omega_{M,z} = \sqrt{\alpha E(0)^2/(2m z_0^2)}$, where $z_0 = \pi w_0^2/\lambda$ is the Rayleigh length. Using the values listed above and $w_0 \sim 1 \mu m$ and the realistic value given in the Sup. Mat. [26] for $E_0$ we find a mechanical frequency $\omega_M \approx 35$ KHz. The frequency scale of the interaction is instead given by $g = \eta \omega_M$ and, using the same parameters, we find the value

$$g = \frac{\eta \omega_M}{\omega} \approx \frac{(2\pi)^6}{2} \frac{\varepsilon_r - 1}{\varepsilon_r + 2} \left( \frac{\lambda}{D} \right)^2 \left( \frac{w_0}{\lambda} \right)^4 \approx 2. \quad (6)$$

This suggests that the system can reach the ultra-strong coupling (USC) regime where the interaction strength is comparable or even larger than the eigenfrequencies of the individual resonators [1–3]. For $(k w_0)^4 > 1/\eta f$ the oscillations of short period $\lambda$ compared to the Rayleigh length $z_0$ can compensate for the weaker intensity of the scattered light, so that the dynamics is dominated by the interaction between the nanospheres.

To wrap up, we validate a posteriori the quality of the linearization procedure by comparing the mechanical zero point motion of the nanospheres along $z$ to the Rayleigh length: $z_M/z_0 = \sqrt{\hbar/(2m \omega_M z_0^2)} \sim 10^{-6}$; and comparing the zero point motion of the relative coordinate to the characteristic length $\lambda$ of the interaction potential: $z_M/\lambda = z_M/z_0 \sqrt{z_0^2/(\eta \lambda^2)} \sim 10^{-5}$. As a last remark, one can redo this whole analysis for the $x$ direction: USC can also be achieved, though with more demanding values of the parameters [26].

Quantum ground state.— One of the main reasons of interest of the USC regime is the non-trivial structure of its ground state, populated by a sizable amount of virtual excitations [1] and displaying a robust ground-state entanglement between each individual oscillator [29] without the need of any additional coupling element [30]. As first proposed in [1], a direct way to measure such excitations is to suddenly switch off the interaction such that the coupled ground state becomes an excited state with respect to the new uncoupled system. By relaxing into the new uncoupled ground state, the virtual excitations are turned to real and they can be observed as energy emission from one part of the system. This phenomenon belongs to the family of the dynamical Casimir effects, whose first experimental evidence was reported only recently in a circuit-QED setup [31, 32]. In this context, the nanopshere setup allows modulating the USC at an extremely fast rate. If one of the two tweezers is removed – which can be performed in an optical time-scale – the interaction is completely switched off: one nanosphere would both lose its dipole and fall due to gravity, whereas the motion of the remaining trapped nanosphere would be excited by the dynamical Casimir effect. Even better, one can suddenly change the polarizations of the two tweezers beams, making them orthogonal. According to (4), this results in an effective turn-off of the interaction with the advantage of both trapping potentials remaining active. Given the ability to measure mechanical quantities at a single quantum level [19–22], measurement of the ensuing dynamical Casimir excitation appears a realistic perspective within reach of forthcoming experiments.

To make our analysis more quantitative, we start from analyzing in detail the consequences of the USC interaction on the quantum ground state wavefunction,

$$|\psi_{\text{GS}}(z_1, z_2)|^2 = \frac{m \omega_M \sqrt{1 + 2\eta}}{\pi \hbar} e^{-\frac{1}{2} (z_1^2 + z_2^2) + B z_1 z_2}, \quad (7)$$

where $z_{\parallel} = \sqrt{m \omega_M / \hbar z_0}$, and $A, B = \sqrt{1 + 2\eta} \pm 1$. In Fig. 3(a) we see how the coupling changes the shape of the ground state wave function in the USC regime, giving rise to a correlated squeezed state [33]. This correlation can be further explored by calculating explicitly the nor-
Figure 4. a) Number $\Delta n$ of absorbed or emitted excitations after a sudden switch-off of the interaction as a function of the coupling constant $\eta$ and the bare occupation number $N_T(\omega_M)$ (a convenient proxy for temperature). b) Value of the entanglement indicator. Entanglement occurs below the white line.

Finite temperature effects. — In any realistic situations the system is affected by various sources of noise and decoherence, which may hinder isolating the dynamical Casimir excitation from other spurious excitations. We assume that the two interacting nanospheres can be cooled down to a low temperature $T$ determined by the cooling procedure and the small residual interaction with the environment [37]. As already mentioned, the number of excitations is not conserved in the process of modulating the interaction strength. In order to thermalize again, the system must then absorb or emit the missing/exceeding excitations. This is quantified by the quantity $\Delta n = n_1(\eta, T) - n_1(\eta = 0, T)$ plotted in Fig. 4(a), where $N_T(\omega)$ is the standard Bose law $N_T(\omega) = 1/(\exp[\hbar \omega/(k_B T)] - 1)$. In the classical, high-temperature regime, one finds [26] that $\Delta n \leq 0$: since switch-off of the interaction lowers the oscillation frequencies, the nanosphere must re-absorb excitations while re-thermalizing. The dynamical Casimir emission with $\Delta n > 0$ is recovered for small enough temperatures and high enough couplings, yet within the reach of state-of-the-art experiments.

The presence of entanglement between the two nanospheres can be assessed in terms of the Simon-Duan inequality [38, 39] based on the indicator $V$ involving [26] the normalized position $\tilde{z}$, and momentum $\hat{P}_i = \hbar \sqrt{1/n_M}$ correlators,

$$V = \left( \langle \tilde{z}_1 - \tilde{z}_2 \rangle^2 + \langle \hat{P}_1 + \hat{P}_2 \rangle^2 \right) = 1 + 2N_T(\omega_M) + \frac{1 + 2N_T(\omega_M\sqrt{1 + 2\eta})}{\sqrt{1 + 2\eta}} :$$

it was shown that for any separable state one has $V \geq 2$, while $V < 2$ indicates entanglement between the two objects. This criterion is particularly convenient in our system as it only involves the measurement of two quadratures, instead of the full tomography of the system, and is already widely used in other optomechanical systems [40, 41]. In Fig. 4(b) we report the value of $V$ as a function of the coupling constant $\eta$ and of the temperature. Exactly at $\eta = 0$ there is no entanglement for any given occupancy (temperature), and $V \geq 2$. At any given non-zero value of $\eta$, a clear signature of entanglement is found at low enough temperatures. For instance, for $\eta = 2$, entanglement is found for a bare occupancy $N_T(\omega_M) \leq 0.26$ and for a realistic value $N_T(\omega_M) = 0.1$ the indicator has already dropped to a significantly low value $V < 1.7$. We emphasize that ground state cooling for the low frequency $\omega$-mode of a free-space levitated particle (no cavity) has been already achieved [42] prospecting the possibility of cooling modes with frequency below 100 KHz.

Conclusions.— In this work we have proposed a platform to study the basic quantum physics of the ultra-strong coupling regime in a novel mechanical context. Our proposed system is based on a pair of dielectric nanospheres trapped in optical tweezers. The mechanical coupling between the nanospheres occurs via the optical forces and is dramatically reinforced by the fast-oscillating interference fringes between the tweezer beams and the elastically scattered light. Observable signatures of ground-state entanglement and of the dynamical Casimir emission are pointed out: estimates based on parameters from on-going experiments are promising in view of a prompt experimental implementation. In addition to providing a textbook realization of a most celebrated model of quantum physics, our proposal introduces new tools for the quantum manipulation of the mechanical degrees of freedom of macroscopic objects, a system of high interest for quantum technologies.

Note added.— A preprint reporting experimental evidences and a theoretical description of strong optical-binding between levitating nanospheres in the classical regime [43] has recently appeared.

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In this Supplemental Material, we report the detailed calculation of the approximations used in the main text and a detailed comparison between different possible experimental parameters. It includes: (i) derivation of the single particle linearised Hamiltonian from the general electrodynamical model, (ii) definition of the radiation emitted by the nanoparticles in terms of dyadic Green’s function, (iii) derivation of the general electrodynamical Hamiltonian for two particles and (iv) derivation of the coupled linearised model, (v) comparison between possible experimental parameters and (vi) complete description of the equilibrium properties of the linearised two particles model.
I. ELECTRODYNAMICS OF A SINGLE TRAPPED NANOSPHERE

We describe here in details the dynamics of a single trapped nanoparticle.

The tweezer trapping beam can be described as an external classical electromagnetic field \( \hat{\mathbf{E}}_{\text{tw}} \) which is unperturbed by the nanoparticle motion. Specifically we consider a monochromatic Gaussian beam with frequency \( \omega_{\text{tw}} \) (and wavelength \( \lambda = 2\pi/k = 2\pi c/\omega_{\text{tw}} \), where \( c \) is the speed of light), polarised orthogonal to the \( z \) direction. The beam has a circular Gaussian profile at the focus plane \( z = 0 \) with waist \( w_0 \). Using the complex notation \( \hat{\mathbf{E}}_{\text{tw}}(\mathbf{r}) = \text{Re}[\hat{\mathbf{E}}_{\text{tw}}(\mathbf{r})e^{-i\omega_{\text{tw}}t}] \), we have

\[
\hat{\mathbf{E}}_{\text{tw}}(\mathbf{r}) = \bar{e}_{\text{tw}} E_0 \int_{-\infty}^{+\infty} dk_x dk_y f(k_x, k_y)e^{i(k_x x + k_y y)}e^{i k_z z} \approx \bar{e}_{\text{tw}} E_0 e^{-\frac{w_0^2}{\pi} \left[ k_x^2 + k_y^2 \right] / \left[ 1 + \left( z/z_0 \right)^2 \right]^2} e^{i k_x x + i k_y y},
\]

where \( \bar{e}_{\text{tw}} \) is the tweezer polarization in the \((x, y)\) plane, and in the last approximated equality we used that

\[
f(k_x, k_y) = \int_{-\infty}^{+\infty} \frac{dx dy}{4\pi^2} e^{-\frac{x^2+y^2}{w_0^2}} e^{i(k_x x + k_y y)} = \frac{w_0^2}{\pi} e^{-\frac{w_0^2 k_z^2}{4}},
\]

together with the paraxial approximation \( k_z = \sqrt{k^2 - k_1^2} \approx k[1 - k_1^2/(2k^2)] \), where \( k_1^2 = k_x^2 + k_y^2 \). The length scale \( z_0 = k w_0^2/2 \) is the Rayleigh length whereas the phase shift scaling with \( k \) in the exponential function is called Gouy phase shift [S1]. From the Gouy phase we define

\[
\varphi(\mathbf{r}) = -\arctan(z/z_0) + \frac{k}{2} \left[ \frac{x^2 + y^2}{z \left( 1 + (z_0/z)^2 \right)} \right].
\]

We consider as nanoparticle an homogeneous dielectric sphere of radius \( a \) much smaller than the wavelength of the Gaussian beam \( a \ll \lambda \) (Rayleigh’s regime). When the nanoparticle is embedded in an electric field has an induced electric dipole moment \( \mathbf{d} \). For a single nanosphere the only electric field present in the system is due to the tweezer, so we have that

\[
\bar{d} \approx \alpha \bar{\mathbf{E}}_{\text{tw}}(\mathbf{r}),
\]

where we used the complex notation \( \bar{d} = \text{Re}[d e^{-i\omega_{\text{tw}}t}] \). The polarizability \( \alpha \) of the nanosphere can be expressed in terms of its relative dielectric permittivity \( \varepsilon_r \) using the well-known Clausius-Mossotti relation [S2]

\[
\alpha = 4\pi \varepsilon_0 \left( \frac{\varepsilon_r - 1}{\varepsilon_r + 2} \right) a^3,
\]

with \( \varepsilon_0 \) is the vacuum permittivity. The potential energy of the nanosphere is then given by the electrodynamical energy of a dipole in an electric field

\[
U_{\text{tw}}(\mathbf{r}) = -\frac{1}{2} \bar{d} \cdot \hat{\mathbf{E}}_{\text{tw}} \approx -\frac{1}{8} \bar{d} \cdot \hat{\mathbf{E}}_{\text{tw}}^{*} + \text{c.c.} = -\frac{|\bar{d}(\mathbf{r})|^2}{4\alpha} = -\frac{\alpha}{4} |\mathbf{E}_{\text{tw}}(\mathbf{r})|^2,
\]

where we have neglected the fast rotating terms \( \sim \exp(\pm i2\omega_{\text{tw}}t) \) in the second approximated equality using the rotating wave approximation (RWA). The Hamiltonian of the dielectric nanosphere is then given by

\[
H = \frac{\hat{p}_m^2}{2m} + U_{\text{tw}}(\mathbf{r}) = \frac{\bar{p}_m^2}{2m} - \frac{\alpha}{4} |\mathbf{E}_{\text{tw}}(\mathbf{r})|^2
\]

where \( \bar{r} \) is the nanosphere position and \( \bar{p} \) is momentum. Assuming that the displacement of the motion of the nanosphere is smaller than the characteristic lengths \( w_0 \) and \( z_0 \), we linearise the potential \( U_{\text{tw}}(\mathbf{r}) \) around the origin \((x = y = z = 0)\)

\[
\frac{\partial^2 U_{\text{tw}}}{\partial x^2} \bigg|_0 = \frac{\partial^2 U_{\text{tw}}}{\partial y^2} \bigg|_0 = \frac{\alpha E_0^2}{w_0^2}, \quad \frac{\partial^2 U_{\text{tw}}}{\partial z^2} \bigg|_0 = \frac{\alpha E_0^2}{2z_0^2},
\]

(S8)
whereas the mixed partial derivatives vanish. Finally the linearised trapping potential reads

\[ U_{\text{har}} = \frac{m}{2} \left( \omega_{\perp}^2 x^2 + \omega_z^2 z^2 \right). \]  

(S9)

whose frequencies are given by

\[ \omega_{\perp} = \sqrt{\frac{\alpha E_0^2}{2m\omega_0^2}}, \]

\[ \omega_z = \sqrt{\frac{\alpha E_0^2}{2m\omega_z^2}}. \]  

(S10)

II. THE ELECTRIC FIELD GENERATED BY AN OSCILLATING DIPOLE

Under the action of the oscillating electric field of the trapping beam laser, the dipole oscillates and emits electromagnetic radiation. The electric field generated at position \( \vec{r} \) by the oscillating dipoles, assuming the dipole at the origin, is given by [S2]

\[ \vec{E}_d(\vec{r}) = \frac{e^{ikr}}{4\pi\varepsilon_0 r} \left[ \left( 1 - \frac{ikr}{r^2} \right) \left( 3(\vec{n} \cdot \vec{d})\vec{n} - \vec{d} \right) + k^2 \left( \vec{n} \times \vec{d} \right) \times \vec{n} \right], \]  

(S11)

where \( \vec{n} = \vec{r}/r \). The previous expression can be written using the dyadic tensor \( \vec{G}(\vec{r} - \vec{r}') = G(\vec{r} - \vec{r}') \vec{d}(\vec{r}') \) with \( \vec{G}(\vec{r}) \) given by

\[ \vec{G}(\vec{r}) = \frac{e^{ikr}}{4\pi\varepsilon_0 r} \left[ \left( 1 - \frac{ikr}{r^2} \right) \hat{M}_n(\vec{r}) + k^2 \hat{M}_f(\vec{r}) \right], \]  

(S12)

where the near-field tensor is

\[ \hat{M}_n(\vec{r}) = \frac{1}{r^2} \begin{bmatrix} 3x^2 - r^2 & 3xy & 3xz \\ 3xy & 3y^2 - r^2 & 3yz \\ 3xz & 3yz & 3z^2 - r^2 \end{bmatrix}, \]  

(S13)

and the far-field tensor is

\[ \hat{M}_f(\vec{r}) = \frac{1}{r^2} \begin{bmatrix} r^2 - x^2 & -xy & -xz \\ -xy & r^2 - y^2 & -yz \\ -xz & -yz & r^2 - z^2 \end{bmatrix}. \]  

(S14)

In the next section we discuss the interaction between two dipoles trapped in two optical tweezers whose centers of the focus plane is at distance \( D \). Then, for the following analysis, it is useful to introduce two dimensionless constants

\[ \eta_n = \frac{\alpha}{4\pi\varepsilon_0 D^3}, \quad \eta_f = \frac{\alpha k^2}{4\pi\varepsilon_0 D}; \]  

(S15)

giving the amplitude of the near-field and far-field components of the full dyadic tensor. The dimensionless propagator reads

\[ \alpha \vec{G}(\vec{r}) = e^{ikr} \left[ \eta_n \left( \frac{D}{r} \right)^3 \left( 1 - \frac{ikr}{r} \right) \hat{M}_n(\vec{r}) + \eta_f \left( \frac{D}{r} \right) \hat{M}_f(\vec{r}) \right]. \]  

(S16)

In general, the two constants are small, and the near-field one is much smaller than the far-field one: \( \eta_n \ll \eta_f \ll 1 \) when \( \lambda \ll D \) since \( \eta_n = \eta_f \frac{\lambda^2}{4\pi\varepsilon_0 D^2} \). It is worth to notice that also when \( D \sim \lambda \) the near-field amplitude remains much smaller than the far-field amplitude by a factor \( \sim 1/(4\pi^2) \sim 1/40 \).
III. ELECTRODYNAMICS OF TWO TRAPPED NANOSPHERES

Following the case of a single nanosphere described above, we derive here the Hamiltonian for two nanospheres. The major difference with respect to the single particle case is that here we need to consider the total electric field acting on the individual dipole, which is given by sum of the two tweezer beams plus the electric field generated by the two oscillating dipoles discussed in the previous section. This field is simply proportional the induced dipole with constant $1/\alpha$. Then the total potential energy given by the sum of the two polarizable particle potentials can be cast in the following form

$$U = -\frac{1}{4\alpha} \left( |\vec{d}_1|^2 + |\vec{d}_2|^2 \right).$$  \hspace{1cm} \text{(S17)}

As in the single particle case, we consider that the two particles are trapped in two separated Gaussian beams with distance $D$ between their centers in the focus plane on the $x$-axis. The beams are assumed parallel and they have electric field with polarization $\vec{e}_{tw,1}, \vec{e}_{tw,2}$ in the $(x,y)$-plane. We introduce here the simplified notation

$$\vec{E}_{tw,1}(x,y,z) = \vec{E}_{tw}(x-D/2, y, z),$$  \hspace{1cm} \text{(S18)}

$$\vec{E}_{tw,2}(x,y,z) = \vec{E}_{tw}(x+D/2, y, z)$$  \hspace{1cm} \text{(S19)}

with $\vec{E}_{tw}(x,y,z)$ given by Eq. (S1). The two dipoles $\vec{d}_1$ and $\vec{d}_2$ are at the positions $\vec{r}_1$ and $\vec{r}_2$. Hereafter we set notation

$$\vec{r}_1 = (x_1 - D/2, y_1, z_1), \hspace{1cm} \vec{r}_2 = (x_2 + D/2, y_2, z_2),$$  \hspace{1cm} \text{(S20)}

where the coordinates $(x_1, y_1, z_1)$ and $(x_2, y_2, z_2)$ represent the displacements of the nanoparticles from the center of the corresponding optical trap. The dipoles are given by the equations

$$\begin{pmatrix} \vec{d}_1 \\ \vec{d}_2 \end{pmatrix} = \alpha \begin{pmatrix} \vec{E}_{tw,1}(\vec{r}_1) + \vec{E}_{tw,2}(\vec{r}_1) \\ \vec{E}_{tw,1}(\vec{r}_2) + \vec{E}_{tw,2}(\vec{r}_2) \end{pmatrix} + \alpha \begin{pmatrix} 0 & \vec{G}_{12} \\ \vec{G}_{21} & 0 \end{pmatrix} \begin{pmatrix} \vec{d}_1 \\ \vec{d}_2 \end{pmatrix}$$  \hspace{1cm} \text{(S21)}

with the notation for the dyadic tensors $\vec{G}_{12} = \vec{G}(\vec{r}_1 - \vec{r}_2)$ and $\vec{G}_{21} = \vec{G}(\vec{r}_2 - \vec{r}_1)$ with $\vec{G}$ given in the previous section. By inverting matrices of size $6 \times 6$, one can easily obtain the formal solution for the Eq. (S21). Alternatively, one can write a closed-form expression, for example, for the first dipole and then invert matrices $3 \times 3$. Proceeding in the latter way, one obtains the solution for $\vec{d}_1$

$$\vec{d}_1 = \left[ I - \alpha^2 \vec{G}_{12} \vec{G}_{21} \right]^{-1} \alpha \left[ \vec{E}_{tw,1}(\vec{r}_1) + \vec{E}_{tw,2}(\vec{r}_1) + \alpha \vec{G}_{12} \left( \vec{E}_{tw,1}(\vec{r}_2) + \vec{E}_{tw,2}(\vec{r}_2) \right) \right]$$  \hspace{1cm} \text{(S22)}

with $I$ is the $3 \times 3$ identity matrix. A similar expression holds for the second dipole. Inserting the solution for the dipoles in Eq. (S17) we have the full potential.

In order to further simplify this expression we can use is the fact that the two Gaussian beam are exponentially confined around their center in the focus plane. Then we can neglect the electric field of the first beam $\vec{E}_{tw,1}$ on the distant dipole $\vec{d}_2$, and viceversa.

The second observation is that the dominant contribution of the electric field generated by a dipole at $D \gtrsim \lambda$ is given by the radiative far-field component proportional to $\eta_f \gg \eta_n$. Even when $D \simeq \lambda$ the near-field component is $\sim 40$ times smaller than $\eta_f$.

$$\alpha \vec{G}(\vec{r}) \simeq e^{ikr} \eta_f \left( \frac{D}{r} \right) \vec{M}_f(\vec{r}), \text{ for } r \gtrsim \lambda.$$  \hspace{1cm} \text{(S23)}

Moreover the leading component of the dipole vectors is in the direction of the tweezer polarization. This can be seen rewriting the tweezer electric field as

$$\vec{E}_{tw,i}(\vec{r}) = \vec{e}_{tw,i}(\vec{r}) e^{i\theta_i(\vec{r})}.$$  \hspace{1cm} \text{(S24)}

Here $E_i$ is a real amplitude and the phase is $\theta_i(\vec{r}) = k z + \varphi_i(\vec{r})$, where $\varphi_i(\vec{r})$ is the $i$-th tweezer phase, given by Eq. (S3). It is worth noticing that this phase can have different shapes for the two tweezer, but in this work we consider
it identical for both tweezers for simplicity. So we have \( \vec{d}_t \approx \bar{e}_{tw,i} \vec{d}_i \). Thus, the leading components of the radiative fields acting on the \( i \)-th dipole is also along the \( \bar{e}_{tw} \) beam’s polarization. Projecting each dipole equations on its own beam polarization we have two scalar equations

\[
\begin{pmatrix}
    d_1 \\
    d_2
\end{pmatrix} = \alpha \begin{pmatrix}
    E_1(r_1) & e^{i \theta_1(r_1)} \\
    E_2(r_2) & e^{i \theta_2(r_2)}
\end{pmatrix} + \alpha \begin{pmatrix}
    0 & \nu \eta_f e^{i k \Delta r} \left( \frac{D}{\Delta r} \right) \\
    0 & 0
\end{pmatrix} \begin{pmatrix}
    d_1 \\
    d_2
\end{pmatrix},
\]

with

\[
\Delta r = |\vec{r}_1 - \vec{r}_2| = \sqrt{(x_1 - x_2 - D)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2},
\]

and where we introduced the polarizations projector

\[
\nu = \bar{e}_{tw}^T \hat{M}_f (\vec{r}_1 - \vec{r}_2) \bar{e}_{tw}. \]

The solution of the last equation is

\[
\begin{pmatrix}
    d_1 \\
    d_2
\end{pmatrix} = \begin{pmatrix}
    1 \\
    -\alpha \nu \eta_f e^{i k \Delta r} \left( \frac{D}{\Delta r} \right) \\
    -1 \\
    1
\end{pmatrix}^{-1} \begin{pmatrix}
    E_1(r_1) e^{i \theta_1(r_1)} \\
    E_2(r_2) e^{i \theta_2(r_2)}
\end{pmatrix} \begin{pmatrix}
    d_1 \\
    d_2
\end{pmatrix},
\]

Inserting the solution for the dipoles in Eq. (S17) and using the fact that \( \eta_f \ll 1 \), we have the approximated potential energy to the lowest order in \( \eta_f \), which can be written as

\[
U \approx U_{\text{trap},1}(\vec{r}_1) + U_{\text{trap},2}(\vec{r}_2) + U_{\text{int}}(\vec{r}_1, \vec{r}_2),
\]

where

\[
U_{\text{trap},i} = -\frac{\alpha}{4} |E_1(\vec{r}_i)|^2,
\]

and

\[
U_{\text{int}}(\vec{r}_1, \vec{r}_2) = -\alpha \nu \eta_f \left( \frac{D}{\Delta r} \right) \cos (k \Delta r) \cos (k [z_1 - z_2] + \varphi_{12}) E_1(\vec{r}_1) E_2(\vec{r}_2)
\]

\[
\approx -\alpha \nu \eta_f \left( \frac{D}{\Delta r} \right) \cos (k \Delta r) \cos \left( \frac{k - \frac{1}{z_0}}{\Delta z} \right) E_1(\vec{r}_1) E_2(\vec{r}_2)
\]

which is the energy potential reported in the main text. Here we have introduced \( \varphi_{12} = \varphi_1(\vec{r}) - \varphi_2(\vec{r}) \), assuming for simplicity that both tweezers have the same phase. Its linearization leads to

\[
\varphi_{12} = -\arctan(z_1/z_0) + \arctan(z_2/z_0) + \frac{k}{2} \left[ \frac{x_1^2 + y_1^2}{z_1 (1 + (z_0/z_1)^2)} - \frac{x_2^2 + y_2^2}{z_2 (1 + (z_0/z_2)^2)} \right] \approx -\frac{\Delta z}{z_0},
\]

as it remains small in the limit \( x_1, x_2 \ll w_0, (y_1, y_2) \ll w_0 \) and \( (z_1, z_2) \ll z_0 \).

IV. TWO PARTICLES LINEARISED DYNAMICS IN THE FAR-FIELD REGIME

We discuss here the linearisation of Eq. (S31). The linearisation of the single particle dynamics is already discussed in Sec. I, so here we focus on the interaction part.

A. First derivatives

In general the dipole-dipole interaction leads to a force acting on the two dipoles which shifts the position equilibrium from the origin of the optical beam. Assuming that the particles are well confined in the optical trap, we can estimate this force by taking the \( x \) or \( z \) derivative of Eq. (S31) at the center of the beam. The force acting on the \( x \) direction, for instance, is given by

\[
F_1 = -\left. \frac{\partial U_{\text{int}}}{\partial x_1} \right|_0 = \nu \eta_f \alpha E_0^2 k \left[ \sin (kD) + \frac{\cos (kD)}{kD} \right],
\]
with \( F_2 = -F_1 \). This force leads to a shift from the equilibrium of the two nanoparticles which reads
\[
\frac{\Delta x_{sh}}{w_0} = \pm \nu \eta f k w_0 \left( \sin(kD) + \frac{\cos(kD)}{kD} \right) \approx \nu \eta f \frac{w_0}{D} \quad \text{for} \quad kD \approx 2\pi \ell .
\] (S34)

Due to the typical small values of \( \eta f \) (see the table in Sec. V) this shift is always small with respect to the waist \( w_0 \) and the wavelength \( \lambda \). In this way it does not affect the validity of the linear approximation and it can be simply absorbed redefining the nanospheres displacements as \( (x_i - \Delta x_{sh}) \rightarrow x_i \). Moreover, within our approximations, the forces in the directions \( y \) and \( z \) vanish. This can immediately seen by considering the first derivative of the interaction potential with respect to \( z \)
\[
\frac{\partial}{\partial z} U_{int} = -\alpha \nu \eta f E_0^2 D \cos \left( \left( k - \frac{1}{z_0} \right) \Delta z \right) \frac{\partial}{\partial \Delta r} \left( \frac{\cos(k \Delta r)}{\Delta r} \right) \frac{\Delta z}{\Delta r} + \alpha \nu \eta f E_0^2 D \frac{\cos(k \Delta r)}{\Delta r} \left( k - \frac{1}{z_0} \right) \sin \left( \left( k - \frac{1}{z_0} \right) \Delta z \right) .
\] (S35)

When we set \( \Delta z = 0 \) is then clear that \( \partial U_{int} / \partial \Delta z \bigg|_0 = 0 \).

**B. Second derivatives in \( x \)**

For the second derivative in \( x_1 \) at the center, it is possible to show that
\[
\frac{\partial^2 U_{int}}{\partial x_1^2} \bigg|_0 = -\alpha \nu \eta f \left[ D \left. \frac{\partial^2 \left( \frac{\cos(k \Delta r)}{\Delta r} \right)}{\partial \Delta x^2} \right|_0 E_0^2 + \cos(kD)E_0 \left. \frac{\partial^2 E_1(\vec{r}_1)}{\partial x_1^2} \right|_0 \right] .
\] (S36)

The second term in Eq. (S36) yields a small frequency renormalization
\[
\tilde{\omega}_z^2 = \omega_{\perp}^2 \left[ 1 + 2 \nu \eta f \cos(kD) \right] ,
\] (S37)
which remains small as long as \( \eta f \ll 1 \).

By calculating the second mix derivatives in \( x_1 \) and \( x_2 \), it is possible to show that the first term in Eq. (S36) corresponds to an interaction proportional to \( (\Delta x)^2 \) as described in the main text with dimensionless coupling constant
\[
\eta_{xx} = \nu \eta f k^2 w_0^2 \left[ \cos(kD) \left( 1 - \frac{2}{(kD)^2} \right) - \frac{2 \sin(kD)}{kD} \right] \approx \nu \eta f k^2 w_0^2 \left( 1 - \frac{2}{(kD)^2} \right) \quad \text{for} \quad kD \approx 2\pi \ell ,
\] (S38)

which can reach the USC regime.

**C. Second derivatives in \( y \)**

For the second derivative in \( y_1 \) at the center, it is also possible to show that
\[
\frac{\partial^2 U_{int}}{\partial y_1^2} \bigg|_0 = -\alpha \nu \eta f \left[ D \left. \frac{\partial^2 \left( \frac{\cos(k \Delta r)}{\Delta r} \right)}{\partial \Delta y^2} \right|_0 E_0^2 + \cos(kD)E_0 \left. \frac{\partial^2 E_1(\vec{r}_1)}{\partial y_1^2} \right|_0 \right] .
\] (S39)

Again, the second term in Eq. (S39) yields a small frequency renormalization
\[
\tilde{\omega}_y^2 = \omega_{\perp}^2 \left[ 1 + 2 \nu \eta f \cos(kD) \right] ,
\] (S40)

which remains small as long as \( \eta f \ll 1 \). As in the \( x \) direction, by calculating the second mix derivatives in \( y_1 \) and \( y_2 \), it is possible to show that the first term in Eq. (S39) corresponds to an interaction proportional to \( (\Delta y)^2 \)
\[
\eta_{yy} = \frac{\nu \eta f k w_0^2}{2} \left[ \sin(kD) + \frac{\cos(kD)}{kD} \right] \approx \eta f \frac{w_0^2}{D^2} \quad \text{for} \quad kD \approx 2\pi \ell ,
\] (S41)

with remains small as long as the distance is close to the antinode.
D. Second derivatives in $z$

Finally, for the second derivative in $z_1$ at the center, we have that

$$\frac{\partial^2 U_{\text{int}}}{\partial z_1^2} \bigg|_0 = -\alpha \nu \eta_j E_0^2 \left[ D \frac{\partial^2 \left( \frac{\cos(k\Delta r)}{\Delta r} \right)}{\partial \Delta z^2} \bigg|_0 - \cos(kD) \left( k - \frac{1}{z_0} \right)^2 - \frac{\cos(kD)}{z_0^2} \right].$$

(S42)

In this case, the third term in Eq. (S42) yields a small frequency renormalization

$$\tilde{\omega}_z^2 = \omega_z^2 \left[ 1 + \frac{1}{2} \nu \eta_j \cos(kD) \right],$$

which remains small as long as $\eta_j \ll 1$. By calculating the second mix derivatives in $z_1$ and $z_2$, it is possible to show that the first term and the second term in Eq. (S42) correspond to an interaction proportional to $(\Delta z)^2$

$$\eta_{zz} = 2 \nu \eta_j k^2 z_0^2 \left[ \cos(kD) \left( \left( \frac{1}{kz_0} \right)^2 + \frac{1}{(kD)^2} \right) + \frac{\sin(kD)}{kD} \right] \approx \frac{\nu \eta_j k^4 w_0^4}{2} \left[ \frac{1 - \frac{2}{k^2 w_0^2}}{(kD)^4} + \frac{1}{(kD)^2} \right].$$

(S44)

In the last approximation we used the fact that $kD \approx 2\pi \ell$. As $k w_0 > 1$, we have $\eta_{zz} > \eta_{zz}$ and the system in the $z$ direction reaches more easily the USC regime respect to the $x$ direction. For this reason we focus mainly on this direction, taking $\eta = \eta_{zz}$ and $\omega_M = \omega_z$ as the adimensional coupling and the mechanical frequency to describe the linearised dynamics in the following section VI.

E. Mixed derivatives $xz$

The mixed derivatives can be easily computed by considering Eq. (S35). It is clear that deriving it a second time with respect to $x$ does not change the linear dependence from $\Delta z$ in the first term and from $\sin((k - 1/z_0)\Delta z)$ in the second term. We can thus immediately say that this mixed derivative is zero when calculated on the equilibrium position, where $\Delta z = 0$,

$$\frac{\partial^2}{\partial x_i \partial z_j} U_{\text{int}} \bigg|_0 = 0.$$

(S45)

When then conclude that all the different directions $x, y, z$ are decoupled between each others.

V. IMPLEMENTATION AND EXPERIMENTAL PARAMETERS

In this section we collect the some typical parameters inspired from the most recent experiments with levitated nanospheres [S3–S5] and we propose new parameters for future experiments.

We fix the values of relative permittivity $\varepsilon_r = 2.1$ and density $\rho = 2 \cdot 10^3$ Kg/m$^3$ of the silica nanospheres. For the laser beam, we use the standard wavelength $\lambda = 1064$ nm. For the Gaussian beam, we have the relation $P_{tw} = (\pi/4) w_0^2 c \varepsilon_0 E_0^2$ one obtains $\alpha E_0^2 / m = 12/\pi (\varepsilon_r - 1)/(\varepsilon_r + 2) P_{tw} / (cw_0^2 \rho)$, where $c$ is the speed of light.

Inspired by [S5] we fix the distance between the tweezer to $D = 3\lambda \approx 3.2 \mu m$. Using such parameters and relations, and plugging in into Eqs. (S10)-(S15)-(S38)-(S44) one obtains the following table for the trapping frequencies and the adimensional near/far-field couplings and $x/z$ linearised couplings.

| Set | $\alpha$ [jJum] | $P_{tw}$ [W] | $w_0$ [jJum] | $\omega_{\perp}/(2\pi)$ [KHz] | $\omega_{z}/(2\pi)$ [KHz] | $\eta_n$ | $\eta_j$ | $\eta_{xx}$ | $\eta_{zz}$ |
|-----|----------------|-------------|--------------|-----------------|-----------------|--------|--------|---------|---------|
| 1   | 0.071          | 0.4         | 0.73         | 246.9           | 81              | $0.3 \times 10^{-5}$ | $1.0 \times 10^{-3}$ | 0.019    | 0.14    |
| 2   | 0.085          | 0.3         | 1            | 114             | 27.3            | $0.5 \times 10^{-5}$ | $1.8 \times 10^{-5}$ | 0.06     | 0.97     |
| 3   | 0.11           | 0.5         | 1            | 147.1           | 35.2            | $1.1 \times 10^{-5}$ | $3.9 \times 10^{-5}$ | 0.14     | 2.14     |

Table I. Collection of system’s parameters with the respective linearised trapping frequencies and adimensional couplings. The set 1 corresponds to Ref. [S3] Delić et al., whereas the set 2 corresponds to Ref. [S4] Ranfagni et al. The set 3 corresponds to our proposal.
VI. NORMAL MODES OF THE TWO COUPLED HARMONIC OSCILLATORS HAMILTONIAN

As in the main text, it is particularly useful to rewrite the main-text Eq. (5) only using adimensional variables by introducing the following rescaling $\tilde{P}_i = P_i / P_M$ and $\tilde{z}_i = z_i / z_M$, where $i = 1, 2$ and

$$
P_M = \sqrt{\hbar m \omega_M},$$

$$
z_M = \sqrt{\frac{\hbar}{m \omega_M}}.$$

(S46)

The linearised Hamiltonian reads

$$
H_{\text{lin}} = \frac{\omega_M}{2} \left[ \tilde{P}_1^2 + \tilde{P}_2^2 + (1 + \eta) \tilde{z}_1^2 + (1 + \eta) \tilde{z}_2^2 - 2\eta \tilde{z}_1 \tilde{z}_2 \right] = \frac{\omega_M}{2} \tilde{q}^T M \tilde{q},
$$

(S47)

where $\tilde{q}^T = (\tilde{P}_1, \tilde{P}_2, \tilde{z}_1, \tilde{z}_2)$ and

$$
M = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 + \eta & 2\eta \\
0 & 0 & 2\eta & 1 + \eta
\end{pmatrix}
$$

(S48)

Diagonalising $M$ means to find the normal modes and then to diagonalise the whole linearised Hamiltonian. In order to do so we need the following canonical transformation

$$
U = \begin{pmatrix}
R & 0_{2 \times 2} \\
0_{2 \times 2} & R
\end{pmatrix},
$$

(S49)

where

$$
R = \begin{pmatrix}
1/\sqrt{2} & 1/\sqrt{2} \\
-1/\sqrt{2} & 1/\sqrt{2}
\end{pmatrix}.
$$

(S50)

In such a way $U M U^\dagger = \text{diag } (1, 1, 1, 1 + 2\eta)$. We notice that in the more general case of two oscillators with different bare frequencies $R$ is a two-dimensional rotation parameterised by an angle $\theta$. In our specific case, of two identical oscillators $\theta = \pi/4$. We introduce new coordinates $\tilde{v} = U \tilde{q}$, where $\tilde{v}^T = (\tilde{\pi}_r, \tilde{\pi}_c, \tilde{\xi}_r, \tilde{\xi}_c)$. In our specific case they correspond respectively to the relative and center of mass momentum and position of the two nanospheres. The Hamiltonian is then expressed as the sum of two independent harmonic oscillators, describing the relative motion and the center of mass motion of the two nanospheres

$$
U H U^\dagger = \frac{\omega_M}{2} \left( \tilde{\pi}_r^2 + \Lambda_r^2 \tilde{\xi}_r^2 + \tilde{\pi}_c^2 + \Lambda_{c\text{m}}^2 \tilde{\xi}_{c\text{m}}^2 \right),
$$

(S51)

where

$$
\Lambda_r^2 = 1 + 2\eta,
$$

(S52)

$$
\Lambda_{c\text{m}}^2 = 1.
$$

(S53)

The Hamiltonian is then fully diagonalised by considering the normal modes annihilation/creation operators

$$
b_{r/c\text{m}} = \sqrt{\frac{\Lambda_{r/c\text{m}}}{2}} \tilde{\xi}_{r/c\text{m}} + i \sqrt{\frac{1}{2\Lambda_{r/c\text{m}}}} \tilde{\pi}_{r/c\text{m}},
$$

(S54)

giving back $H = \omega_r b_r^\dagger b_r + \omega_{c\text{m}} b_{c\text{m}}^\dagger b_{c\text{m}}$, where the normal modes eigenfrequencies are simply given by

$$
\omega_{r/c\text{m}} = \omega_M \Lambda_{r/c\text{m}}.
$$

(S55)

We can immediately see that the mode related to the center of mass motion keeps the same mechanical frequency $\omega_{c\text{m}} = \omega_M$ as the uncoupled nanospheres, while the relative motion gets blue (red) shifted by the repulsive (attractive) interaction.
VII. THE FINITE TEMPERATURE COVARIANCE MATRIX OF TWO COUPLED HARMONIC OSCILLATORS

Since the linearised dynamics is described by a quadratic Bosonic Hamiltonian everything we need to know about the equilibrium properties of the system is encoded in the covariance matrix [S6, S7]. Here we consider only equilibrium states which are symmetric, and all the expectation values of operators which are odd in each nanosphere parity vanishes, in particular \( \langle z_i \rangle = \langle \hat{P}_1 \rangle = \langle z_i \hat{P}_2 \rangle = 0 \). The symmetrized momentum-position mixed products also vanishes \( \langle z_i \hat{P}_1 \rangle + \langle \hat{P}_1 z_i \rangle = 0 \) and the covariance matrix takes the form of

\[
\mathcal{C} = \begin{pmatrix}
\langle \hat{z}_1^2 \rangle & \langle \hat{z}_1 \hat{z}_2 \rangle & 0 & 0 \\
\langle \hat{z}_1 \hat{z}_2 \rangle & \langle \hat{z}_2^2 \rangle & 0 & 0 \\
0 & 0 & \langle \hat{P}_1^2 \rangle & \langle \hat{P}_1 \hat{P}_2 \rangle \\
0 & 0 & \langle \hat{P}_1 \hat{P}_2 \rangle & \langle \hat{P}_2^2 \rangle \\
\end{pmatrix} = U \begin{pmatrix}
\langle \hat{z}_1^2 \rangle & 0 & 0 & 0 \\
0 & \langle \hat{z}_2^2 \rangle & 0 & 0 \\
0 & 0 & \langle \pi_{1cm}^2 \rangle & 0 \\
0 & 0 & 0 & \langle \pi_{2cm}^2 \rangle \\
\end{pmatrix} U^\dagger. \tag{S56}
\]

Since we consider the average only on equilibrium states \( \langle \cdot \rangle = \text{Tr} \left[ \cdot e^{-\beta H} / Z \right] \), with \( Z = \text{Tr} \left[ e^{-\beta H} \right] \) and \( \beta = 1/(k_B T) \) where \( k_B \) is the Boltzman constant and \( T \) the temperature. The covariance matrix in the normal modes basis can be solved analytically, considering that

\[
\begin{align*}
\langle \hat{z}_1^2 \rangle &= \frac{1}{2} \left( 1 + 2 N_T^{r/cm} \right), \\
\langle \hat{z}_2^2 \rangle &= \left( 1 + 2 N_T^{r/cm} \right), \\
\langle \pi_{1cm}^2 \rangle &= \frac{1}{2} \left( 1 + 2 N_T^{r/cm} \right), \\
\langle \pi_{2cm}^2 \rangle &= \frac{1}{2} \left( 1 + 2 N_T^{r/cm} \right), \tag{S57}
\end{align*}
\]

where we introduced the usual Bose thermal population

\[
N_T^{r/cm} = \frac{1}{e^{\beta \omega_{r/cm}} - 1}. \tag{S58}
\]

VIII. THE COUPLED GROUND STATE, DYNAMICAL CASIMIR EXCITATIONS AND ENTANGLEMENT

In this section we characterise the quantities that can be measured in order to explore the ultrastrongly coupled ground state. First we focus on the excitations arising in the spheres in thermal equilibrium after a sudden switch off of the interaction, then we describe the measure of the entanglement between the two nanospheres.

When the temperature is low enough a sizable contribution in the excitation of the system comes from the virtual excitations present in the ground state that are turned real by switching off the coupling. These play no role while the system is in equilibrium, but they are fundamental when the system is perturbed non-adiabatically. For this reason this kind of experiment realises a toy model for dynamical Casimir emission, typically studied in the context of QED [S8].

A central quantity to characterise the Casimir excitation of one sphere, when the coupling is switched off, is the excitation number

\[
n_1(\eta, T) = \langle \hat{a}_1^\dagger \hat{a}_1 \rangle = \frac{1 + 2 N_T^{r/cm}}{4} + \frac{1 + \eta}{4 \sqrt{1 + 2 \eta}} \left( 1 + 2 N_T^{r/cm} \right) - \frac{1}{2}, \tag{S59}
\]

which scales as

\[
n_1 \sim \frac{\eta^2}{8}, \tag{S60}
\]

as the temperature goes to zero \( T \sim 0 \) and at vanishingly small coupling \( \eta \ll 1 \). As in the main text, we define the excitation emitted/absorbed after quenching the interaction as

\[
\Delta n = n_1(\eta, T) - n_1(\eta = 0, T) = n_1(\eta, T) - N_T(\omega_M). \tag{S61}
\]

In the high-temperature limit \( T \gg \hbar \omega_M \sqrt{1 + 2 \eta} / k_B \) we find that

\[
\Delta n \approx \frac{k_B T}{2 \hbar \omega_M} \frac{1 - \sqrt{1 + \eta}}{\sqrt{1 + \eta}} \leq 0, \tag{S62}
\]
which can be also easily derived by just applying the equipartition theorem valid in the classical regime. This inequality implies that $\Delta n > 0$ is a clear sign of the dynamical Casimir emission, which can happen only if the system is completely in the low temperature regime where quantum effects are dominant.

Another fundamental quantity is the Von Neumann entropy of the reduced density matrix of a single sphere. It describes the amount of heat released by the sphere, due to breaking the correlations of the coupled ground-state. The Von Neumann entropy of a single sphere can be calculated from the eigenvalues of the symplectic reduced covariance matrix $\mathbf{C}_1$

$$C_1 = \begin{pmatrix} 0 & i \langle \hat{P}_1^2 \rangle \\ -i \langle \hat{z}_1^2 \rangle & 0 \end{pmatrix}. \tag{S63}$$

The eigenvalues of $C_1$ are

$$\lambda_{\pm} = \pm \sqrt{\langle \hat{z}_1^2 \rangle \langle \hat{P}_1^2 \rangle}, \tag{S64}$$

and the Von Neumann entropy is given by [S6, S7]

$$S_1 = \left( \lambda_+ + \frac{1}{2} \right) \log \left( \lambda_+ + \frac{1}{2} \right) - \left( \lambda_- + \frac{1}{2} \right) \log \left( \lambda_- + \frac{1}{2} \right) \tag{S65}$$

At temperature $T = 0$, so when the system is in the ground state, we have

$$\lambda_{\pm} = \pm \frac{1 + \sqrt{1 + 2\eta}}{4\sqrt{1 + 2\eta}}. \tag{S66}$$

Expanding around small coupling $\eta \ll 1$ at $T \sim 0$ we find

$$S_1 \sim \left(1 + \frac{\eta^2}{16}\right) \log \left[1 + \frac{\eta^2}{16}\right] - \frac{\eta^2}{16} \log \left[\frac{\eta^2}{16}\right]. \tag{S67}$$

The presence of a large amount of virtual excitations in the ground state is related to the presence of strong correlations between the two nanospheres. This is a clear hint that the system at low enough temperature is in an entangled state. In order to characterise the degree of entanglement at a given temperature $T$, we consider here the separability estimator from [S10, S11], which quantifies how much the density matrix of a system can be written as a separable state. In our case, considering only equilibrium states, it reads

$$V = 2 \langle \hat{\tilde{z}}_r^2 \rangle + 2 \langle \hat{\pi}_{cm}^2 \rangle = \frac{1 + 2N_T(\omega_r)}{\sqrt{1 + 2\eta}} + 1 + 2N_T(\omega_{cm}) \tag{S68}$$

For any separable states it holds

$$V \geq 2. \tag{S69}$$

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