A note on the incidence coloring of outerplanar graphs.

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Abstract

In this note we prove that every outerplanar graph is \( \Delta + 2 \) colorable. This is slightly stronger than an unpublished result of Wang Shudong, Ma Fangfang, Xu Jin, and Yan Lijun proving the same for 2-connected outerplanar graphs.

1 Definitions and notations.

A graph is outerplanar if it can be embedded in the plane without crossing edges, in such a way that all the vertices are on the boundary of the exterior region.

An incidence of a simple graph \( G \) is a pair \((v, vw)\) of an edge \( vw \) and one of its vertices. Two incidences \((v, vw)\) and \((\hat{v}, \hat{v}w)\) are adjacent if \( v = \hat{v} \), or \( w = \hat{v} \) or \( v = \hat{w} \).

Following Wang, Ma, Xu, and Yan, we define \((k, l)\)-incidence colorings to be a proper colorings of incidences of a given graph \( G \) with at most \( k \) colors such that for any vertex \( v \) of \( G \) the number of colors used in coloring all incidences \((u, uv)\) is at most \( l \). This notion also appears in \cite{2}.

The maximum degree of a vertex in \( G \) is denoted by \( \Delta \).

Finally, the neighbourhood \( N(v) \) of a vertex \( v \) is a set of all vertices adjacent to \( v \) in \( G \).

2 The proof.

Theorem 1. Any outerplanar graph \( G \) has a \((\Delta + 2, 2)\)-incidence coloring.

Proof. It suffices to prove the theorem for connected graphs. We will need the following lemma.
Lemma 2. For every connected simple outerplanar graph $G$ at least one of the following holds:

Case 1: $G$ has a vertex of degree 1.
Case 2: $G$ has two adjacent vertices of degree 2.
Case 3: $G$ has a vertex $u$ of degree 2 with $N(u) = (v, w)$ and $vw \in G$.
Case 4: $G$ contains a vertex $u$ of degree 2 with $G - u$ disconnected.

The proof is based on the proof of Proposition 7.1.15 in (4, p.254).

Proof. Suppose $G$ has no vertex of degree 1.

The following procedure exhibits $G$ as a subgraph of an outerplanar graph $H$ such that the boundary of the unbounded face of $H$ is a cycle, i.e., a 2-connected outerplanar graph:

If boundary of $G$ is not a cycle then it is a walk that visits some vertex $u$ twice. If $\ldots, v, u, w, \ldots$ is such a visit we add the edge $vw$. We continue in this way until we get to $H$.

Now the weak dual of $H$ is a tree and its leaves correspond to faces with exactly one internal edge. Take one such face $F$ with the internal edge $e = ab$.

Case A) There are at least 4 edges in the boundary of $F$. Then there are 2 adjacent vertices $u, v$ on the boundary of $F$ different from $a, b$. Both of these are of degree 2 in $H$, so of degree at most 2 in $G$. Since $G$ is connected and has no degree 1 vertices, they are both of degree 2. This is Case 2 of the lemma.

Case B) There are 3 edges in the boundary of $F$. Denote the vertex not on the edge $e$ by $u$. Again $u$ is of degree 2 in $H$, hence also in $G$. If $e$ is in $G$ we are in Case 3 of the lemma.

If $e$ is not in $G$ then it was added in passing from $G$ to $H$, which means that $v$ was traversed twice in the walk of the unbounded face of $G$. Then $G - u$ is disconnected, we are in Case 4.

We shall now prove the theorem by induction on order of $G$. If $\Delta = 2$ it is obvious, so we assume $\Delta \geq 3$. Note that the case $\Delta = 3$ follows from (3), but the resulting simplification in the proof is minor, and we prefer to keep the argument self-contained. We now have four cases, corresponding to the cases in the lemma:

Case 1: The graph $G$ has a vertex $u$ of degree 1. Let’s denote the vertex adjacent to $u$ by $v$. Then $G^* = G - u$ is an outerplanar graph of smaller order and maximum degree at most $\Delta$. Hence by induction hypothesis $G^*$ can be $(\Delta + 2, 2)$-incidence colored by a coloring $\sigma^*$. We extend it to a coloring $\sigma$ of $G$. The degree of $v$ in $G^*$ is at most $\Delta - 1$, so there are at most $\Delta - 1$ colors used by incidences $(v, vw)$ outgoing from $v$, and at most 2 used by the incidences $(w, uv)$ incoming into $v$. Hence there is at least one color left to color $(v, vu)$. The incidence $(u, uv)$ can be colored by one of the colors incoming into $v$.

Case 2: The graph $G$ has two adjacent vertices $u, v$ of degree 2. Denote the other vertex adjacent to $u$ by $w$, the one adjacent to $v$ by $x$. Consider $G^* = G - u$. Again, $G$ is outerplanar, has smaller order and maximum degree at most $\Delta$ and so can be $(\Delta + 2, 2)$-incidence colored by a coloring $\sigma^*$. 

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Degree of \( w \) in \( G^* \) is at most \( \Delta - 1 \), so there is at least one color \( \alpha \) available to color \((w, \, wu)\). One of the incoming colors of \( w \) can be used to color \((u, \, uw)\). Now we need to color \((u, \, uv)\) and \((v, \, vu)\). There are at most 4 prohibited colors and at least 5 available (as \( \Delta \geq 3 \)). If the color of \((w, \, wu)\) or \((u, \, uw)\) is the same as the color of \((x, \, xv)\) then there are at most 3 prohibited colors, and we can use 2 remaining ones to finish the coloring. If all \((w, \, wu)\), \((u, \, uw)\) and \((x, \, xv)\) have distinct colors, we can use the color of \((x, \, xv)\) to color \((u, \, uv)\), and have a color left to finish coloring \((v, \, vu)\). Resulting coloring is in fact a \((\Delta + 2, \, 2)\) coloring.

Case 3: The graph \( G \) has a vertex \( u \) of degree 2 with \( N(u) = (v, \, w) \) and \( vw \in G \). Consider \( G^* = G - u \). Again, \( G \) is outerplanar, has smaller order and maximum degree at most \( \Delta \) and so can be \((\Delta + 2, \, 2)\)-incidence colored by a coloring \( \sigma^* \). Suppose \((v, \, vw)\) is colored by color \( \alpha \) and \((w, \, wv)\) by color \( \beta \).

We now assign color \( \alpha \) to \((u, \, uw)\) and color \( \beta \) to \((u, \, uv)\). This does not produce any conflicts since \( \alpha \) already was an incoming color for \( w \) and \( \beta \) for \( v \), and \( \alpha \neq \beta \). Finally, the vertex \( v \) has degree at most \( \Delta - 1 \) in \( G^* \) so there is at least one color \( \gamma, \, \gamma \neq \alpha, \, \beta \), that can be used to color \((v, \, vu)\). Similarly, there is a color \( \delta, \, \delta \neq \alpha, \, \beta \), that can be used to color \((w, \, wu)\) (it is possible that \( \delta = \gamma \)). This produces a \((\Delta + 2, \, 2)\)-incidence coloring of \( G \).

Case 4: The graph \( G \) has a vertex \( u \) of degree 2 such that \( G - u \) is disconnected.

Again \( G^* \) is outerplanar, of smaller order and maximal degree at most \( \Delta \), hence \((\Delta + 2, \, 2)\) colorable.

Denote \( N(u) = (v, \, w) \). Let a \((\Delta + 2, \, 2)\) coloring of the component of \( G^* \) containing \( v \) be \( \sigma_1 \) and a \((\Delta + 2, \, 2)\) coloring of the component of \( G^* \) containing \( w \) be \( \sigma_2 \). Other components of \( G - u \) are components of \( G \), they can be \((\Delta + 2, \, 2)\) colored and left unmodified. Since degrees of \( v \) and \( w \) are at most \( \Delta - 1 \) there exists a way to color incidences \((v, \, vu)\) by \( \alpha \) and \((w, \, wu)\) by \( \beta \), and then to assign one of the colors \( \gamma \) incoming to \( v \) to the incidence \((u, \, uv)\) and one of the colors \( \delta \) incoming to \( w \) to the incidence \((u, \, uw)\). The problem is that while \( \alpha \neq \gamma \) and \( \beta \neq \delta \) there may be other equalities, so we get adjacent incidences at \( u \) colored in the same way. However, the set of colors has at least 4 elements. Hence for any colors \( \beta, \, \delta \) exists a permutation of colors sending \( \beta, \, \delta \) to colors different from \( \alpha, \, \gamma \). Composing \( \sigma_2 \) (together with the colorings of \((w, \, wu)\) and \((u, \, uw))\) with this permutation gives a \((\Delta + 2, \, 2)\) coloring of \( G \).

This completes the proof.

3 Questions on the incidence coloring of planar and higher-genus graphs.

Even though not every graph is \((\Delta + 2)\)-colorable (c.f. [1]), the counterexamples known to me are not planar. The question of whether planar graphs are \((\Delta + 2)\)-colorable is unsolved. The bound of \( \Delta + 7 \) was obtained in [2]. More generally, in the same paper it is shown that any \( k \)-degenerate graph has a \((\Delta + 2k - 1, \, k)\)
incidence coloring. Any graph of positive genus $g$ has a vertex of degree at most $d = \frac{1}{2}(7 + \sqrt{1 + 48g})$, and hence is $d$-degenerate, producing a bound of $\Delta + 6 + \sqrt{1 + 48g}$ on the incidence coloring number. Planar graphs are 5-degenerate, and outerplanar graphs are 2-degenerate, so the resulting bounds of $\Delta + 9$ and $\Delta + 3$, respectively, are not optimal. The higher-genus bounds are probably not tight either.

References

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