Trace estimates for relativistic stable processes

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Abstract

In this paper, we study the asymptotic behavior, as the time \( t \) goes to zero, of the trace of the semigroup of a killed relativistic \( \alpha \)-stable process in bounded \( C^{1,1} \) open sets and bounded Lipschitz open sets. More precisely, we establish the asymptotic expansion in terms of \( t \) of the trace with an error bound of order \( t^{2/\alpha}t^{-d/\alpha} \) for \( C^{1,1} \) open sets and of order \( t^{1/\alpha}t^{-d/\alpha} \) for Lipschitz open sets. Compared with the corresponding expansions for stable processes, there are more terms between the orders \( t^{-d/\alpha} \) and \( t^{(2-d)/\alpha} \) for \( C^{1,1} \) open sets, and, when \( \alpha \in (0,1] \), between the orders \( t^{-d/\alpha} \) and \( t^{(1-d)/\alpha} \) for Lipschitz open sets.

1 Introduction and statement of the main results

For any \( m > 0 \) and \( \alpha \in (0,2) \), a relativistic \( \alpha \)-stable process \( X^m \) on \( \mathbb{R}^d \) with mass \( m \) is a Lévy process with characteristic function given by

\[
\mathbb{E} [\exp(i\xi \cdot (X^m_t - X^m_0))] = \exp(-t((|\xi|^2 + m^2/\alpha)^{\alpha/2} - m)), \quad \xi \in \mathbb{R}^d. \tag{1.1}
\]

The limiting case \( X^0 \), corresponding to \( m = 0 \), is a (rotationally) symmetric \( \alpha \)-stable process on \( \mathbb{R}^d \) which we will simply denote as \( X \). The infinitesimal generator of \( X^m \) is \( m - (m^2/\alpha - \Delta)^{\alpha/2} \). Note that when \( m = 1 \), this infinitesimal generator reduces to \( 1 - (1 - \Delta)^{\alpha/2} \). Thus the 1-resolvent kernel of the relativistic \( \alpha \)-stable process \( X^1 \) on \( \mathbb{R}^d \) is just the Bessel potential kernel. When \( \alpha = 1 \), the infinitesimal generator reduces to the so-called free relativistic Hamiltonian \( m - \sqrt{-\Delta + m^2} \). The operator \( m - \sqrt{-\Delta + m^2} \) is very important in mathematical physics due to its application to relativistic quantum mechanics.

In this paper, we will be interested in the asymptotic behavior of the trace of the semigroup associated with killed relativistic \( \alpha \)-stable processes in open sets of \( \mathbb{R}^d \). The process \( X^m \) has a transition density \( p^m(t,x,y) = p^m(t,y-x) \) given by the inverse Fourier transform

\[
p^m(t,x) = (2\pi)^{-d} \int_{\mathbb{R}^d} e^{-i\xi x} e^{-t(|\xi|^2 + m^2/\alpha)^{\alpha/2} + mt} d\xi.
\]

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For any open set $D$ in $\mathbb{R}^d$, the killed relativistic $\alpha$-stable process $X_t^{m,D}$ is defined by

$$X_t^{m,D} = \begin{cases} X_t^m & \text{if } t < \tau^m_D, \\ \partial & \text{if } t \geq \tau^m_D, \end{cases}$$

where $\tau^m_D = \inf \{ t > 0 : X_t^m \notin D \}$ is the first exit time of $X_t^m$ from $D$. The process $X_t^{m,D}$ is a strong Markov process with a transition density $p^m_D(t, x, y)$ given by

$$p^m_D(t, x, y) = p^m(t, x, y) - r^m_D(t, x, y),$$

with

$$r^m_D(t, x, y) = \mathbb{E}_x \left[ t > \tau^m_D; p^m(t - \tau^m_D, X^m_{\tau^m_D}, y) \right].$$

We denote by $(P_t^{m,D} : t \geq 0)$ the semigroup of $X_t^m$ on $L^2(D)$: for any $f \in L^2(D)$,

$$P_t^{m,D} f(x) := \mathbb{E}_x \left[ f(X_{t}^{m,D}) \right] = \int_D f(y) p^m_D(t, x, y) dy.$$  

Whenever $D$ is of finite volume, $P_t^{m,D}$ is a Hilbert-Schmidt operator mapping $L^2(D)$ into $L^\infty(D)$ for every $t > 0$. By general operator theory, there exist an orthonormal basis of eigenfunctions $\{\phi_n^{(m)}\}_{n=1}^\infty$ for $L^2(D)$ and corresponding eigenvalues $\{\lambda_n^{(m)}\}_{n=1}^\infty$ of the generator of the semigroup $P_t^{m,D}$ satisfying

$$0 < \lambda_1^{(m)} < \lambda_2^{(m)} \leq \lambda_3^{(m)} \leq \cdots$$

with $\lambda_n^{(m)} \to \infty$. By definition, we have

$$P_t^{m,D} \phi_n^{(m)}(x) = e^{-\lambda_n^{(m)} t} \phi_n^{(m)}(x), \quad x \in D, t > 0.$$  

We also have

$$p^m_D(t, x, y) = \sum_{n=1}^\infty e^{-\lambda_n^{(m)} t} \phi_n^{(m)}(x) \phi_n^{(m)}(y).$$

$\lambda_n^{(0)}$ will be simply denoted by $\lambda_n$.

In the remainder of this paper, we assume $d \geq 2$. We are interested in finding the asymptotic behavior, as $t \to 0$, of the trace defined by

$$Z^m_D(t) = \int_D p^m_D(t, x, x) dx = \sum_{n=1}^\infty e^{-\lambda_n^{(m)} t} \int_D (\phi_n^{(m)})^2(x) dx = \sum_{n=1}^\infty e^{-\lambda_n^{(m)} t}.$$  

It is shown in [2] that for any open set $D$ of finite volume, it holds that

$$\lim_{t \to 0} t^{d/\alpha} Z^0_D = C_1 |D|, \quad C_1 = \frac{\omega_D \Gamma(d/\alpha)}{(2\pi)^d \alpha}. \quad (1.2)$$
This is closely related to the growth of the eigenvalues of \( P_t^{0,D} \): if \( N^0(\lambda) \) is the number of eigenvalues \( \lambda_j \) such that \( \lambda_j \leq \lambda \), then it follows from the classical Karamata Tauberian theorem (see for example [10]) that

\[
N^0(\lambda) \sim \frac{C_1|D|}{\Gamma(d/\alpha + 1)} \lambda^{d/\alpha}, \quad \text{as } \lambda \to \infty.
\] (1.3)

This is the analogue for killed stable processes of the celebrated Weyl’s asymptotic formula for the eigenvalues of the Dirichlet Laplacian. We will see later in this paper that exactly the same formula is true for relativistic stable processes. That is, the first term in the expansion of \( Z^m_D(t) \) is the same as that of \( Z^0_D(t) \) and (1.3) is also true for relativistic stable processes.

Our main goal in this paper is to get the asymptotic expansion of \( Z^m_D(t) \) as \( t \to 0 \) under some additional assumptions on the smoothness of the boundary of \( D \). Our work is inspired by the paper [7] for Brownian motion and the papers [2, 3] for stable processes. The first theorem is an asymptotic expansion of \( Z^m_D(t) \) with error bound of order \( t^{2/\alpha}t^{-d/\alpha} \) in \( C^{1,1} \) open sets. To state the result precisely, we need some definitions. Recall that an open set \( D \) in \( \mathbb{R}^d \) is said to be a (uniform) \( C^{1,1} \) open set if there are (localization radius) \( R > 0 \) and \( \Lambda_0 \) such that for every \( z \in \partial D \), there exist a \( C^{1,1} \) function \( \phi = \phi_z : \mathbb{R}^d \to \mathbb{R} \) satisfying \( \phi(0, \cdots, 0) = 0 \), \( \nabla \phi(0) = (0, \cdots, 0) \), \( |\nabla \phi(x) - \nabla \phi(y)| \leq \Lambda_0 |x - z| \) and an orthonormal coordinate system \( CS_z : y = (y_1, \cdots, y_{d-1}, y_d) := (\tilde{y}, y_d) \) with origin at \( z \) such that \( B(z, R) \cap D = \{ y = (\tilde{y}, y_d) \in B(0, R) \} \). For \( x \in \mathbb{R}^d \), let \( \delta_D(x) \) denote the Euclidean distance between \( x \) and \( D^c \) and \( \delta_D(x) \) the Euclidean distance between \( x \) and \( \partial D \). It is well known that a \( C^{1,1} \) open set \( D \) satisfies both the uniform interior ball condition and the uniform exterior ball condition: there exists \( r_0 < R \) such that for every \( x \in D \) with \( \delta_D(x) < r_0 \) and \( y \in \mathbb{R}^d \setminus D \) with \( \delta_D(y) < r_0 \), there are \( z_x, z_y \in \partial D \) so that \( |x - z_x| = \delta_D(x) \), \( |y - z_y| = \delta_D(y) \) and that \( B(x_0, r_0) \subset D \) and \( B(y_0, r_0) \subset \mathbb{R}^d \setminus D \), where \( x_0 = z_x + r_0(x - z_x)/|x - z_x| \) and \( y_0 = z_y + r_0(y - z_y)/|y - z_y| \). In fact, \( D \) is a \( C^{1,1} \) open set if and only if \( D \) satisfies the uniform interior ball condition and the uniform exterior ball condition (see [1, Lemma 2.2]). In this paper we call the pair \( (r_0, \Lambda_0) \) the characteristics of the \( C^{1,1} \) open set \( D \). For any open set \( D \) in \( \mathbb{R}^d \), we use \( |D| \) to denote the \( d \)-dimensional Lebesgue measure of \( D \) and \( \mathcal{H}^{d-1}(\partial D) \) to denote the \((d-1)\)-dimensional Hausdorff measure of \( \partial D \). When \( D \) is a \( C^{1,1} \) open set, \( \mathcal{H}^{d-1}(\partial D) \) is equal to the surface measure \( |\partial D| \) of \( \partial D \). We will use \( H \) to denote the half space \( \{ x = (x_1, x_2, \cdots, x_d) : x_1 > 0 \} \).

The following is the first main result of this paper.

**Theorem 1.1** Suppose that \( D \) is a bounded \( C^{1,1} \) open set in \( \mathbb{R}^d \). Let \( k \) be the largest integer such that \( k < \frac{d}{\alpha} \). Then the trace \( Z^m_D(t) \) admits the following expansion

\[
Z^m_D(t) = C_1|D|t^{-\frac{d}{2}} - C_2|\partial D|t^{1-d/\alpha} + \frac{\omega_d \Gamma(d/\alpha)|D|}{(2\pi)^{d\alpha}}t^{-\frac{d}{2}} \sum_{n=1}^k \frac{m^n}{n!} t^n + O(t^{2/\alpha}t^{-d/\alpha}).
\]
where $C_1$ is given in (1.2) and

$$C_2 = \int_0^\infty r_H^0(1,(r,\tilde{0}),(r,\tilde{0}))dr.$$

The second main result of the paper is an asymptotic expansion of $Z_D^m(t)$ with error bound of order $t^{1/\alpha} - d/\alpha$ in Lipschitz open sets. Before we state the second main result, we recall the definition of Lipschitz open sets. An open set $D$ in $\mathbb{R}^d$ is called a Lipschitz open set if there exist constants $R_0$ (localization radius) and $\lambda > 0$ (Lipschitz constant) such that for every $z \in \partial D$ there exist a Lipschitz function $F : \mathbb{R}^{d-1} \to \mathbb{R}$ with Lipschitz constant $\lambda$ and an orthonormal coordinate system $y = (y_1, \cdots, y_d)$ such that $D \cap B(z, R_0) = \{y : y_d > F(y_1, \cdots, y_{d-1})\} \cap B(z, R_0)$. Here is the second main result.

**Theorem 1.2** Suppose that $D$ is a bounded Lipschitz open set in $\mathbb{R}^d$. Let $j$ be the largest integer such that $j \leq \frac{1}{\alpha}$. Then the trace $Z_D^m(t)$ admits the following expansion

$$t^{d/\alpha}Z_D^m(t) = C_1|D| - C_2\mathcal{H}^{d-1}(\partial D)t^{1/\alpha} + \frac{\omega_d \Gamma(d/\alpha)|D|}{(2\pi)^{d/2}}\sum_{n=1}^j \frac{m^n}{n!}t^n + o(t^{1/\alpha}),$$

where $C_1$ and $C_2$ are the same as in Theorem 1.1.

The asymptotic behaviors of the trace $Z_D(t)$ of the killed Brownian motion (i.e., killed symmetric $\alpha$-stable process with $\alpha = 2$) in bounded domains $D$ of $\mathbb{R}^d$ have been extensively studied by many authors. It is shown in [5] that, when $D$ is a bounded $C^{1,1}$ domain,

$$\left|Z_D(t) - (4\pi t)^{-d/2}\left(|D| - \frac{\sqrt{\pi t}}{2}|\partial D|\right)\right| \leq \frac{c|D|t^{1-d/2}}{R^2}, \quad t > 0.$$  

The following asymptotic result

$$Z_D(t) = (4\pi t)^{-d/2}\left(|D| - \frac{\sqrt{\pi t}}{2}|\partial D| + o(t^{1/2})\right), \quad t \to 0,$$  

was proved in [6] when $D$ is a bounded $C^1$ domain. (1.4) was subsequently extended to Lipschitz domains in [7].

The asymptotic behaviors of the trace $Z_D^0(t)$ of killed symmetric $\alpha$-stable processes, $0 < \alpha < 2$, in open sets of $\mathbb{R}^d$ have been studied in [2, 3]. It was shown in [2] that, for any bounded $C^{1,1}$ open set $D$,

$$\left|Z_D^0(t) - \frac{C_1|D|}{t^{d/\alpha}} + \frac{C_2|\partial D|t^{1/\alpha}}{t^{d/\alpha}}\right| \leq \frac{c|D|t^{2/\alpha}}{r_0^2d^{d/\alpha}},$$

where $C_1$ and $C_2$ are the same as in Theorem 1.1 and $c$ is a positive constant depending on $d$ and $\alpha$ only. It was shown in [3] that, when $D$ is a bounded Lipschitz domain, $Z_D^0(t)$ satisfies

$$t^{d/\alpha}Z_D^0(t) = C_1|D| - C_2\mathcal{H}^{d-1}(\partial D)t^{1/\alpha} + o(t^{1/\alpha}).$$
Remark 1.3 Note that the first term in the expansion of $Z^m_D(t)$ is exactly the same as in the case of $Z^0_D(t)$. However the rest of the terms are quite different. We note here that the coefficient of the term of order $t^{1/\alpha}t^{-d/\alpha}$ is the same in the stable process case, but in the case of relativistic stable processes for $C^{1,1}$ open sets, there are $k$ intermediate terms of the form $t^kt^{-d/\alpha}$, where $k$ is a positive integer less than $2/\alpha$. Since $0 < \alpha < 2$, there is at least one more term involved in the asymptotic expansion of $Z^m_D(t)$ than that of $Z^0_D(t)$ up to order of $t^{2/\alpha}t^{-d/\alpha}$. For Lipschitz open sets, when $\alpha \leq 1$ there are $j$ intermediate terms of the form $t^jt^{-d/\alpha}$, where $j$ is an integer that is less than or equal to $1/\alpha$.

Remark 1.4 In [4], an asymptotic expansion for the trace of relativistic $\alpha$-stable processes in bounded $C^{1,1}$ open sets was established. Compared with Theorem 1.1, the expansion of [4] does not contain the intermediate terms.

The rest of the paper is organized as follows. In Section 2, we recall some basic facts about relativistic stable processes and present several preliminary results which will be used in Sections 3 and 4. Theorem 1.1 is proved in Section 3, while Theorem 1.2 is proved in Section 4.

Throughout this paper, we will use $c$ to denote a positive constant depending (unless otherwise explicitly stated) only on $d$ and $\alpha$ but whose value may change from line to line, even within a single line. In this paper, the big O notation $f(t) = O(g(t))$ always means that there exist constants $C$ and $t_0 > 0$ such that $f(t) \leq Cg(t)$ for all $0 < t < t_0$.

2 Preliminaries

In this section, we recall some basic facts about relativistic $\alpha$-stable processes. From (1.1), one can easily see that $X^m$ has the following approximate scaling property:

$$\{m^{-1/\alpha}(X^m_{mt} - X^m_0), t \geq 0\}$$

has the same law as $$\{X^m_t - X^m_0, t \geq 0\}.$$ In terms of transition densities, this approximate scaling property can be written as

$$p^m(t, x, y) = m^{d/\alpha}p^1(mt, m^{1/\alpha}x, m^{1/\alpha}y).$$

(2.1)

It is well known that the transition density $p^m_D(t, x, y)$ of $X^m_D$ is continuous on $(0, \infty) \times D \times D$. Since both $p^m(t, x, y)$ and $p^m_D(t, x, y)$ are continuous on $(0, \infty) \times D \times D$, $r^m_D(t, x, y) = p^m(t, x, y) - p^m_D(t, x, y)$ is also continuous there. $p^m_D(t, x, y)$ and $r^m_D(t, x, y)$ also enjoy the following approximate scaling property:

$$p^m_D(t, x, y) = m^{-d/\alpha}p^m_D(t/m, x/m^{1/\alpha}, y/m^{1/\alpha}),$$

(2.2)

$$r^m_D(t, x, y) = m^{-d/\alpha}r^m_D(t/m, x/m^{1/\alpha}, y/m^{1/\alpha}).$$

(2.3)
The Lévy measure of the relativistic $\alpha$-stable process $X^m$ has a density

$$J^m(x) = j^m(|x|) := \frac{\alpha}{2\Gamma(1-\alpha/2)} \int_0^\infty (4\pi u)^{-d/2} e^{-|x|^2/4u}e^{-m^{2/\alpha}u^{-(1+\alpha/2)}} du,$$

which is continuous and radially decreasing on $\mathbb{R}^d \setminus \{0\}$ (see [13, Lemma 2]). Put $J^m(x, y) := j^m(|x - y|)$. Let $\mathcal{A}(d, -\alpha) := \alpha2^{\alpha-1}\pi^{-d/2}\Gamma(\frac{d+\alpha}{2})\Gamma(1 - \frac{\alpha}{2})^{-1}$. Using change of variables twice, first with $u = |x|^2v$ then with $v = 1/s$, we get

$$J^m(x, y) = \mathcal{A}(d, -\alpha)|x - y|^{-d-\alpha}\psi(m^{1/\alpha}|x - y|),$$

(2.4)

where

$$\psi(r) := 2^{-(d+\alpha)}\Gamma\left(\frac{d + \alpha}{2}\right)^{-1} \int_0^\infty s^{(d+\alpha)/2-1}e^{-s/4-r^2/s} ds,$$

(2.5)

which satisfies $\psi(0) = 1$ and

$$c_1^{-1}e^{-rt(d+\alpha-1)/2} \leq \psi(r) \leq c_1 e^{-rt(d+\alpha-1)/2} \quad \text{on } [1, \infty)$$

for some $c_1 > 1$ (see [9, pp. 276-277] for details). We denote the Lévy density of $X$ by

$$J(x, y) := J^0(x, y) = \mathcal{A}(d, -\alpha)|x - y|^{-d-\alpha}.$$

Note that from (2.4) and (2.5) we see that for any $x \in \mathbb{R}^d \setminus \{0\}$

$$j^m(|x|) \leq J^0(|x|).$$

It follows from [8, Theorem 4.1.] that, for any positive constants $M$ and $T$ there exists a constant $c > 1$ such that for all $m \in (0, M]$, $t \in (0, T]$, and $x, y \in \mathbb{R}^d$ we have

$$c^{-1}(t^{-d/\alpha} \wedge tJ^m(x, y)) \leq p^m(t, x, y) \leq c\left(t^{-d/\alpha} \wedge tJ^0(x, y)\right).$$

(2.6)

We will need a simple lemma from [11] about the relationship between $r_D^m(t, x, y)$ and $r_D^0(t, x, y)$. The lemma is true in much more general situations but we just need it when one of the processes is a symmetric $\alpha$-stable process and the other is a relativistic $\alpha$-stable process.

**Lemma 2.1** Suppose that $X$ and $Y$ are two Lévy processes with Lévy densities $J^X$ and $J^Y$, respectively. Suppose that $\sigma = J^X - J^Y$ is nonnegative on $\mathbb{R}^d$ with $\int_{\mathbb{R}^d} \sigma(x) dx = \ell < \infty$ and $D$ is an open set. Then for any $x \in D$ and $t > 0$,

$$p^Y_D(t, x, \cdot) \leq e^{\ell t}p^X_D(t, x, \cdot) \quad \text{a.s.}$$

If, in addition, $p^X(t, \cdot)$ and $p^Y(t, \cdot)$ are continuous, then we have for $x, y \in D$,

$$r^Y_D(t, x, y) \leq e^{2\ell t}r^X_D(t, x, y).$$
The next proposition is the (generalized) Ikeda-Watanabe formula for the relativistic stable process, which describes the joint distribution of \( \tau^m_D \) and \( X^m_{\tau^m_D} \).

**Proposition 2.2** (Proposition 2.7 [12]) Assume that \( D \) is an open subset of \( \mathbb{R}^d \) and \( A \) is a Borel set such that \( A \subset D^c \setminus \partial D \). If \( 0 \leq t_1 < t_2 < \infty \), then

\[
\mathbb{P}_x \left( X^m_{\tau^m_D} \in A, \ t_1 < \tau^m_D < t_2 \right) = \int_D \int_{t_1}^{t_2} p^m_D(s,x,y) ds \int_A J^m(y,z) dz dy, \quad x \in D.
\]

Now we state a simple lemma about the upper bound of \( r^m_D(t,x,y) \), which is an analogue of [2, Lemma 2.1] for stable processes.

**Lemma 2.3** Let \( M,T \) be positive constants. Then there exists a constant \( c = c(d,\alpha,M,T) \) such that for all \( m \in (0,M] \) and \( t \in (0,T] \) we have

\[
r^m_D(t,x,y) \leq c \left( t^{-d/\alpha} \wedge \frac{t\psi(m^{1/\alpha}\delta_D(x))}{\delta_D(x)^{d+\alpha}} \right).
\]

**Proof.** Since \( \psi \) is eventually decreasing and \( \psi(0) = 1 > 0 \), there exists a constant \( c_1 > 0 \) such that \( \psi(x) \leq c_1 \psi(y) \) for all \( 0 \leq y \leq x \). Now from the definition of \( r^m_D(t,x,y) \) and (2.6) we have

\[
r^m_D(t,x,y) = \mathbb{E}_y \left[ t > \tau^m_D; p^m(t - \tau^m_D, X^m_{\tau^m_D}, x) \right] \\
\leq \mathbb{E}_y \left[ c \left( t^{-d/\alpha} \wedge \frac{t\psi(m^{1/\alpha}|x - X^m_{\tau^m_D}|)}{|x - X^m_{\tau^m_D}|^{d+\alpha}} \right) \right] \\
\leq cc_1 \left( t^{-d/\alpha} \wedge \frac{t\psi(m^{1/\alpha}\delta_D(x))}{\delta_D(x)^{d+\alpha}} \right).
\]

We will need two results from [2]. The first result is about the difference \( p^m_F(t,x,y) - p^m_D(t,x,y) \) when \( D \subset F \). The proof in [2], given for stable processes, mainly uses the strong Markov property and it works for all strong Markov processes with transition densities.

**Proposition 2.4** (Proposition 2.3 [2]) Let \( D \) and \( F \) be open sets in \( \mathbb{R}^d \) such that \( D \subset F \). Then for any \( x,y \in \mathbb{R}^d \) we have

\[
p^m_F(t,x,y) - p^m_D(t,x,y) = \mathbb{E}_x \left[ \tau^m_D < t, X^m_{\tau^m_D} \in F \setminus D : p^m_F(t - \tau^m_D, X^m_{\tau^m_D}, y) \right].
\]
Now we introduce some notation. Recall that if $D$ is a $C^{1,1}$ open set with characteristics $(r_0, \Lambda_0)$, then for every $x \in D$ with $\delta_{\partial D}(x) < r_0$ and $y \in \mathbb{R}^d \setminus \bar{D}$ with $\delta_{\partial D}(y) < r_0$, there are $z_x, z_y \in \partial D$ so that $|x - z_x| = \delta_{\partial D}(x)$, $|y - z_y| = \delta_{\partial D}(y)$ and that $B(x_0, r_0) \subset D$ and $B(y_0, r_0) \subset \mathbb{R}^d \setminus \bar{D}$, where $x_0 = z_x + r_0(x - z_x)/|x - z_x|$ and $y_0 = z_y + r_0(y - z_y)/|y - z_y|$. Let $H(x)$ be the half-space containing $B(x_0, r_0)$ such that $\partial H(x)$ contains $z_x$ and is perpendicular to the segment $z_x z_y$. The next proposition says that, in case of the symmetric $\alpha$-stable process, for small $t$, the quantity $r_0^D(t, x, x)$ can be replaced by $r_0^H(t, x, x)$, which was a very crucial step in proving the main result in [2].

**Proposition 2.5 (Proposition 3.1 of [2])** Let $D \subset \mathbb{R}^d$ be a $C^{1,1}$ open set with characteristics $(r_0, \Lambda_0)$. Then, for any $x$ with $\delta_{\partial D}(x) < r_0/2$ and $t > 0$ with $t^{1/\alpha} \leq r_0/2$, we have

$$| r_0^D(t, x, x) - r_0^H(t, x, x) | \leq ct^{1/\alpha} r_0^{d/\alpha} \left( \frac{t^{1/\alpha}}{\delta_{\partial D}(x)} \right)^{d+\frac{d-2}{2}} \wedge 1.$$ 

We will need some facts about the “stability” of the surface area of the boundary of $C^{1,1}$ open sets. The following lemma is [5, Lemma 5].

**Lemma 2.6** Let $D$ be a bounded $C^{1,1}$ open set in $\mathbb{R}^d$ with characteristic $(r_0, \Lambda_0)$ and define for $0 \leq q < r_0$, 

$$D_q = \{ x \in D : \delta_D(x) > q \}.$$

Then

$$\left( \frac{r_0 - q}{r_0} \right)^{d-1} |\partial D| \leq |\partial D_q| \leq \left( \frac{r_0}{r_0 - q} \right)^{d-1} |\partial D|, \quad 0 \leq q < r_0.$$

The following result is [2, Corollary 2.14].

**Lemma 2.7** Let $D$ be a bounded $C^{1,1}$ open set in $\mathbb{R}^d$ with characteristic $(r_0, \Lambda_0)$. For any $0 < q \leq r_0/2$, we have

1. $2^{d+1} |\partial D| \leq |\partial D_q| \leq 2^{d-1} |\partial D|,$
2. $|\partial D| \leq \frac{2^d |D|}{r_0},$
3. $|\partial D_q| - |\partial D| \leq \frac{2^d dq |\partial D|}{r_0} \leq \frac{2^d dq |D|}{r_0}.$
3 Proof of Theorem 1.1

We first prove that \( \lim_{t \to 0} t^{d/\alpha} Z_D^m(t) \) exists and identify the limit.

**Lemma 3.1** The limit \( \lim_{t \to 0} t^{d/\alpha} Z_D^m(t) \) exists and is equal to \( C_1|D| \), where \( C_1 \) is the constant in Theorem 1.1.

**Proof.** By definition,

\[
t^{d/\alpha} Z_D^m(t) = t^{d/\alpha} \int_D p_D^m(t,x,x) dx
\]

\[
= t^{d/\alpha} \left( \int_D p^m(t,x,x) dx - \int_D r_D^m(t,x,x) dx \right). \tag{3.1}
\]

For the first integral on the right hand side of (3.1), note that, by the approximate scaling property (2.2) and the dominated convergence theorem, we have, as \( t \to 0 \),

\[
t^{d/\alpha} \left( \int_D p^m(t,x,x) dx \right) = \int_D p^m(1,x,x) dx = |D| p^m(1,0)
\]

\[
\to |D| \cdot p^0(1,0) = |D| \cdot \frac{\Gamma(d/\alpha) \omega_d}{(2\pi)^{d/2}}.
\]

It remains to show that \( \lim_{t \to 0} t^{d/\alpha} \int_D r_D^m(t,x,x) dx = 0 \). By Lemma 2.3 we have that

\[
t^{d/\alpha} r_D^m(t,x,y) \leq c, \quad (t,x,y) \in (0,1] \times D \times D,
\]

for some \( c > 0 \). Hence we have by the monotone convergence theorem,

\[
t^{d/\alpha} \int_{D \setminus D_{1/2}} r_D^m(t,x,x) \to 0 \quad \text{as} \quad t \to 0.
\]

For \( x \in D_{1/2} \) we have by Lemma 2.3 again for \( t \in (0,1] \),

\[
r_D^m(t,x,x) \leq c t^{d/2 + \frac{d}{2}}, \quad x \in D_{1/2}.
\]

Hence \( \lim_{t \to 0} t^{d/\alpha} \int_{D_{1/2}} r_D^m(t,x,x) dx = 0 \).

It follows from Lemma 3.1 that if \( N^m(\lambda) \) denotes the number of eigenvalues \( \lambda_j^m \) such that \( \lambda_j^m \leq \lambda \), then it follows from the classical Karamata Tauberian theorem (see for example [10]) that

\[
N^m(\lambda) \sim \frac{C_1|D|}{\Gamma(d/\alpha + 1)} \lambda^{d/\alpha}, \quad \text{as} \quad \lambda \to \infty.
\]
This is the analogue for killed relativistic stable processes of the celebrated Weyl’s asymptotic formula for the eigenvalues of the Dirichlet Laplacian and it is already proved in [4] (see [4, (1.10)]). This result has been known at least since 2009, see [4, Remark 1.2].

Now we focus on identifying the next terms in \( Z^m_D(t) \). For this, we need to find the order of \( t \) in \( Z^m_D(t) - C_1 t^{-d/\alpha} \). Note that by Lemma 3.1,

\[
Z^m_B(t) - C_1 t^{-d/\alpha} = \int_D p^m_B(t, x, x) - p^0(t, x, x) \, dx
\]

\[
= \int_D (p^m(t, x, x) - p^0(t, x, x)) \, dx - \int_D r^m_B(t, x, x) \, dx.
\]

The next lemma gives the orders of \( t \) in \( p^m(t, x, x) - p^0(t, x, x) \) up to \( t^{2} t^{-d/\alpha} \).

**Lemma 3.2** Let \( k \) be the largest integer such that \( k < \frac{2}{\alpha} \). Then we have

\[
p^m(t, x, x) - p^0(t, x, x) = t^{-d/\alpha} \omega_d \frac{\Gamma(d/\alpha)}{(2\pi)^{d/\alpha}} \sum_{n=1}^{k} m^n \frac{n^n}{n!} t^n + O(t^{2/\alpha} t^{-d/\alpha}).
\]

**Proof.** By the scaling property (2.1) we have

\[
p^m(t, x, x) - p^0(t, x, x) = p^m(1, 0) - p^0(1, 0)
\]

\[
= t^{-d/\alpha} (p^m(1, 0) - p^0(1, 0))
\]

\[
= t^{-d/\alpha} (2\pi)^{-d} \int_{\mathbb{R}^d} e^{-\left(||\xi||^2 + (mt)^{2/\alpha}\right)^{\alpha/2} + mt} - e^{-|\xi|^{\alpha}} d\xi.
\]

Note that for any \( x \geq 0 \) we have \( (1 + x)^{\alpha/2} \leq 1 + \frac{\alpha}{2} x \). Thus

\[
(\xi^2 + (mt)^{2/\alpha})^{\alpha/2} = \xi^{\alpha} \left(1 + \frac{(mt)^{2/\alpha}}{\xi^2}\right)^{\alpha/2} \leq \xi^{\alpha} \left(1 + \frac{\alpha}{2} \frac{(mt)^{2/\alpha}}{\xi^2}\right).
\]

Consequently

\[
0 \leq e^{-|\xi|^{\alpha}} - e^{-\left(||\xi||^2 + (mt)^{2/\alpha}\right)^{\alpha/2}}
\]

\[
\leq e^{-|\xi|^{\alpha}} - e^{-|\xi|^{\alpha} \left(1 + \frac{\alpha}{2} \frac{(mt)^{2/\alpha}}{||\xi||^2}\right)}
\]

\[
\leq e^{-|\xi|^{\alpha}} \left(\frac{\alpha}{2} \frac{(mt)^{2/\alpha}}{\xi^{2-\alpha}}\right),
\]

where we used \( 1 - e^{-x} \leq x \) for all \( x \geq 0 \) in the last inequality above. Therefore

\[
0 \leq \int_{\mathbb{R}^d} e^{-\left(||\xi||^2 + (mt)^{2/\alpha}\right)^{\alpha/2} + mt} - e^{-|\xi|^{\alpha}} d\xi
\]
\[
\int_{\mathbb{R}^d} \left| e^{-(|\xi|^2 + (mt)^{2/\alpha})^{\alpha/2} + mt} - e^{-|\xi|^\alpha} e^{mt} + e^{-|\xi|^\alpha} e^{mt} - e^{-|\xi|^\alpha} \right| d\xi \\
\leq \int_{\mathbb{R}^d} \left| e^{-|\xi|^\alpha} \right|^{\alpha/2} + \int_{\mathbb{R}^d} \left| e^{-|\xi|^\alpha} e^{mt} - e^{-|\xi|^\alpha} \right| d\xi \\
\leq \int_{\mathbb{R}^d} \left| e^{-|\xi|^\alpha} \right|^{\alpha/2} + \int_{\mathbb{R}^d} \left| e^{-|\xi|^\alpha} (e^{mt} - 1) \right| d\xi \\
= \frac{e^{mt}}{2} (mt)^{2/\alpha} \int_{\mathbb{R}^d} \frac{e^{-|\xi|^\alpha}}{|\xi|^{2-\alpha}} d\xi + \sum_{n=1}^{\infty} \frac{(mt)^n}{n!} \int_{\mathbb{R}^d} e^{-|\xi|^\alpha} d\xi.
\]

Since \( k + j \geq 2/\alpha \) for any \( j \geq 1 \), we have \( \sum_{n=k+1}^{\infty} \frac{(mt)^n}{n!} = O(t^{2/\alpha}) \). Therefore

\[
\int_{\mathbb{R}^d} \left( e^{-|\xi|^\alpha} (e^{mt} - 1) \right) d\xi = O(t^{2/\alpha}) + \frac{\omega_d \Gamma(d/\alpha)}{\alpha} \sum_{n=1}^{k} \frac{(mt)^n}{n!}
\]
and

\[
p^m(t, x, x) - p^0(t, x, x) = t^{-d/\alpha} \frac{\omega_d \Gamma(d/\alpha)}{(2\pi)^d \alpha} \sum_{n=1}^{k} \frac{m^n}{n!} t^n + O(t^{2/\alpha} t^{-d/\alpha}).
\]

Now we try to find the orders of \( t \) in the expansion of \( \int_{D} r^m_D(t, x, x)dx \) up to the order of \( t^{\frac{\alpha}{2}} t^{-\frac{\alpha}{d}} \). For this, we need to assume some regularity condition on the boundary of \( D \). Hence in the remainder of this section we assume that \( D \) is a bounded \( C^{1,1} \) open set with characteristic \((r_0, \lambda_0)\). We also assume that \( t^{1/\alpha} \leq \frac{r_0}{2} \).

We first deal with the contribution in \( D_{r_0/2} \).

**Lemma 3.3** There exists \( c = c(d, \alpha) > 0 \) such that

\[
\int_{D_{r_0/2}} r^m_D(t, x, x)dx \leq c e^{2mt |D|^{t^{2/\alpha}}}. \]

**Proof.** It follows from Lemma 2.1 that \( r^m_D(t, x, y) \leq e^{2mt} r^0_D(t, x, y) \). By [2, (3.2)] we know that

\[
\int_{D_{r_0/2}} r^0_D(t, x, y)dx \leq c |D|^{t^{2/\alpha}} \frac{r_0^{2d/\alpha}}{r_0^{2d/\alpha}}.
\]

The desired assertion follows immediately. \( \square \)

**Lemma 3.4** There exists \( c = c(d, \alpha) > 0 \) such that

\[
r^m_H(t, x, x) - r^m_H(t, x, x) \leq c e^{2mt} \left| \frac{t^{1/\alpha}}{\delta_D(x)} \right|^{d+\frac{\alpha}{2} - 1} \wedge 1
\]

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and
\[ \int_{D \setminus D_{1/2}} \left( r^m_D(t, x, x) - r^m_H(t, x, x) \right) dx \leq c e^{2m t^{2/\alpha}}. \]

**Proof.** If the first assertion of the lemma is right, then it is easy to see that
\[ \int_{D \setminus D_{1/2}} \left( \frac{t^{1/\alpha}}{\delta_D(x)} \right)^{d+\frac{\alpha}{2}-1} dx \leq c t^{1/\alpha}. \]

Hence we focus on proving the first assertion. By \([2, (3.4)]\), we know that
\[ r^0_D(t, x, x) - r^0_H(t, x, x) \leq c \frac{t^{1/\alpha}}{t^{d/\alpha}} \left( \frac{t^{1/\alpha}}{\delta_D(x)} \right)^{d+\frac{\alpha}{2}-1} \land 1. \]

Recall that \( J^m(x) \leq J^0(x) \) for any \( x \in \mathbb{R}^d \setminus \{0\} \). Now it follows from the generalized Ikeda-Watanabe formula and Lemma 2.1 that
\[
 r^m_D(t, x, x) - r^m_H(t, x, x) = E_x \left[ t > \tau^m_{D}, X^m_{\tau^m_{D}} \in H(x) \setminus D; p^m_H(t - \tau^m_{D}, X^m_{\tau^m_{D}}, x) \right] \\
= \int_D \int_0^t p^m_D(s, x, y) dy \int_{H(x) \setminus D} J^m(y, z) p^m_H(t - s, z, x) dz dy \\
\leq e^{2mt} \int_D \int_0^t p^0_D(s, x, y) dy \int_{H(x) \setminus D} J^0(y, z) p^0_H(t - s, z, x) dz dy \\
= e^{2mt} \mathbb{E}_x \left[ t > \tau^0_{D}, X^0_{\tau^0_{D}} \in H(x) \setminus D; p^0_H(t - \tau^0_{D}, X^0_{\tau^0_{D}}, x) \right] \\
= e^{2mt} \left( r^0_D(t, x, x) - r^0_H(t, x, x) \right) \\
\leq c e^{2mt} \frac{t^{1/\alpha}}{t^{d/\alpha}} \left( \frac{t^{1/\alpha}}{\delta_D(x)} \right)^{d+\frac{\alpha}{2}-1} \land 1. \]

\[ \square \]

**Lemma 3.5** There exists \( c = c(d, \alpha) > 0 \) such that
\[ \int_{D \setminus D_{1/2}} r^m_H(t, x, x) dx - t^{1/\alpha} \int_0^t \frac{1}{d D} \partial D \int r^m_H(t, u) du du \leq c t^{2/\alpha} t^{-d/\alpha}. \]

**Proof.** Using the scaling relation (2.3) we get
\[ \int_{D \setminus D_{1/2}} r^m_H(t, x, x) dx \]
\[
\int_0^{r_0/2} |\partial D_u| f_H^m(t, u)du = \int_0^{r_0/2} |\partial D_u| t^{-d/\alpha} f_H^{tm}(1, u/t^{1/\alpha})du = \int_0^{r_0/2} t^{1/\alpha} t^{-d/\alpha} |\partial D_u| f_H^{tm}(1, u)du.
\]

It follows from Corollary 2.7 that \( ||\partial D_q| - |\partial D|| \leq \frac{2d|q| |\partial D|}{r_0} \leq \frac{2d|q| |\partial D|}{r_0^d} \) for any \( q \leq r_0/2 \). Hence

\[
\left| \int_{D \setminus D_{r_0/2}} r_{H(x)}(t, x, x) - t^{1/\alpha} t^{-d/\alpha} \int_0^{r_0/2} \partial D| f_H^{tm}(1, u)du \right|
\leq t^{1/\alpha} t^{-d/\alpha} \int_0^{r_0/2} ||\partial D_u| - |\partial D|| f_H^{tm}(1, u)du
\leq c_1 t^{2/\alpha} t^{-d/\alpha} \int_0^{r_0/2} u f_H^{tm}(1, u)du
\leq c_2 t^{2/\alpha} t^{-d/\alpha}.
\]

\[\square\]

**Lemma 3.6** There exists \( c = c(d, \alpha) > 0 \) such that

\[ t^{1/\alpha} t^{-d/\alpha} \int_0^{r_0/2} |\partial D| f_H^{tm}(1, u)du - t^{1/\alpha} t^{-d/\alpha} \int_0^{r_0/2} |\partial D| f_H^{tm}(1, u)du \leq c t^{2/\alpha} t^{-d/\alpha}. \]

**Proof.** It follows from Lemma 2.1 that

\[
\int_0^{r_0/2} |\partial D| f_H^{tm}(1, u)du - t^{1/\alpha} t^{-d/\alpha} \int_0^{r_0/2} |\partial D| f_H^{tm}(1, u)du
= t^{1/\alpha} t^{-d/\alpha} \int_0^{r_0/2} |\partial D| f_H^{tm}(1, u)du
= t^{1/\alpha} t^{-d/\alpha} |\partial D| \int_0^{r_0/2} f_H^{tm}(1, u)du
\leq e^{2mt \alpha} t^{1/\alpha} t^{-d/\alpha} |\partial D| \int_0^{r_0/2} f_H^{0\alpha}(1, u)du.
\]

For \( q \geq r_0/(2t^{1/\alpha}) \) we have \( f_H^{0\alpha}(1, q) \leq cq^{-\alpha} \leq c q^{-2} \). Hence

\[
\int_0^{r_0/2} f_H^{0\alpha}(1, u)du \leq c \int_0^{r_0/2} \frac{dq}{q^2} \leq c \frac{t^{1/\alpha}}{r_0}
\]

and the result now follows. \[\square\]
Lemma 3.7 \( \lim_{t \downarrow 0} \int_0^\infty f_H^m(t,u)du = \int_0^\infty f_H^0(t,u)du. \)

**Proof.** This follows immediately from the continuity of \( m \mapsto r_D^m(t,x,y) \) and the dominated convergence theorem. \( \square \)

**Proof of Theorem 1.1** Combining Lemmas 3.1, 3.2, 3.3, 3.4, 3.5, 3.6, and 3.7, we immediately arrive at Theorem 1.1. \( \square \)

4 Proof of Theorem 1.2

In this section we always assume that \( D \) is a bounded Lipschitz open set in \( \mathbb{R}^d \). The argument of this section is similar to previous section and [3]. We will follow the argument in [3] closely, making necessary modifications for relativistic stable processes. Note that even though the main theorem in [3] is stated for a Lipschitz domain, it remains true for a bounded Lipschitz open set.

First we need two technical facts which play crucial roles later. The first proposition is [3, Proposition 2.9] and we will state it here for reader’s convenience.

**Proposition 4.1** (Proposition 2.9. [3]) Suppose that \( f : (0, \infty) \to [0, \infty) \) is continuous and satisfies \( f(r) \leq c(1 \wedge r^{-\beta}) \) for some \( \beta > 1 \). Furthermore, suppose that for any \( 0 < R_1 < R_2 < \infty \), \( f \) is Lipschitz on \([R_1, R_2]\). Then we have

\[
\lim_{\eta \to 0^+} \frac{1}{\eta} \int_D f\left( \frac{\delta_D(x)}{\eta} \right) dx = \mathcal{H}^{d-1}(\partial D) \int_0^\infty f(r)dr.
\]

**Lemma 4.2** Suppose that \( f : (0, \infty) \to [0, \infty) \) is continuous and satisfies \( f(r) \leq c_1(1 \wedge r^{-\beta}) \) for some \( \beta > 1 \). Furthermore, suppose that for any \( 0 < R_1 < R_2 < \infty \), \( f \) is Lipschitz on \([R_1, R_2]\). Let \( \{f^n : \eta > 0\} \) be continuous functions from \((0, \infty)\) to \([0, \infty)\) such that, for any \( 0 < L < M < \infty \), \( \lim_{\eta \to 0} f^n(r) = f(r) \) uniformly for \( r \in [L,M] \). Suppose that there exists \( c_2 > 0 \) such that \( f^n(r) \leq c_2 f(r) \) for all \( \eta \leq 1 \). Then we have

\[
\lim_{\eta \to 0^+} \frac{1}{\eta} \int_D f^n\left( \frac{\delta_D(x)}{\eta} \right) dx = \mathcal{H}^{d-1}(\partial D) \int_0^\infty f(r)dr.
\]

**Proof.** Let \( \psi_\eta(r) = \eta^{-1} |\{x \in D : \delta_D(x) < \eta r\}|. \) Note (cf. proof of [7, Proposition 1.1]) that \( \psi_\eta(r) \leq c \) for all \( \eta, r > 0 \) and that

\[
\eta^{-1} \int_D f\left( \frac{\delta_D(x)}{\eta} \right) dx = \int_0^\infty f(r)d\psi_\eta(r),
\]

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and
\[ \eta^{-1} \int_D f^\eta \left( \frac{\delta_D(x)}{\eta} \right) \, dx = \int_0^\infty f^\eta(r) \, d\psi_\eta(r). \]

It was shown in [3, Proposition 2.9.] that, for any \( 0 < R_1 < R_2 < \infty \) and \( \eta > 0 \), \( f \) satisfies
\[ \int_0^{R_1} f(r) \, d\psi_\eta(r) \leq cR_1, \quad (4.1) \]
\[ \int_{R_2}^{\infty} f(r) \, d\psi_\eta(r) \leq cR_2^{1-\beta} + cR_2^{1-\beta}, \quad (4.2) \]
\[ \lim_{\eta \to 0^+} \int_{R_1}^{R_2} f(r) \, d\psi_\eta(r) = \mathcal{H}^{d-1}(\partial D) \int_{R_1}^{R_2} f(r) \, dr. \]

Since \( f^\eta \leq c_2 f \) for \( \eta \leq 1 \) we have the same inequalities as (4.1) and (4.2) for \( f^\eta, \eta \leq 1 \). Hence it is enough to show that
\[ \lim_{\eta \to 0^+} \int_{R_1}^{R_2} f^\eta(r) \, d\psi_\eta(r) = \mathcal{H}^{d-1}(\partial D) \int_{R_1}^{R_2} f(r) \, dr. \]

For any partition \( R_1 = x_0 < x_1 < \cdots < x_n = R_2 \) of \([R_1, R_2]\), we have
\[
\left| \sum_{i=1}^{n} f^\eta(x_i) \left( \psi_n(x_i) - \psi_n(x_{i-1}) \right) - \sum_{i=1}^{n} f(x_i) \left( \psi_n(x_i) - \psi_n(x_{i-1}) \right) \right|
\leq \| f^\eta - f \|_{L^\infty([R_1, R_2])} \psi_n(R_2).
\]

Note that for any \( \eta > 0 \) the function \( r \to \psi_\eta(r) \) is nondecreasing and for any \( \eta > 0, r > 0 \) we have \( \psi_\eta(r) \leq c r \) for some constant \( c \). Since \( f^\eta \to f \) uniformly on \( r \in [R_1, R_2] \), taking supremum for all possible partitions gives
\[
\lim_{\eta \to 0^+} \int_{R_1}^{R_2} f^\eta(r) \, d\psi_\eta(r) = \lim_{\eta \to 0^+} \int_{R_1}^{R_2} f(r) \, d\psi_\eta(r) = \mathcal{H}^{d-1}(\partial D) \int_{R_1}^{R_2} f(r) \, dr.
\]

\[ \square \]

**Lemma 4.3** For any \( 0 < L < M < \infty \), \( p^m(t, x, y) \) converges uniformly to \( p^0(t, x, y) \) as \( m \to 0 \) for \( (t, x, y) \in [L, M] \times \mathbb{R}^d \times \mathbb{R}^d \).

**Proof.** Note that
\[
\left| p^m(t, x, y) - p^0(t, x, y) \right| = \left| (2\pi)^{-d} \int_{\mathbb{R}^d} e^{-i\xi(y-x)} \left( e^{-t((|\xi|^2+m^2)^{\alpha/2}-m)} - e^{-t|\xi|^\alpha} \right) d\xi \right|
\]

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\[
(2\pi)^{-d} \int_{\mathbb{R}^d} \left| e^{-i\xi(y-x)} \left( e^{-t((|\xi|^2+m^2/\alpha)^{\alpha/2}-m)} - e^{-t|\xi|^\alpha} \right) \right| d\xi 
\leq (2\pi)^{-d} \int_{\mathbb{R}^d} e^{-t((|\xi|^2+m^2/\alpha)^{\alpha/2}-m)} - e^{-t|\xi|^\alpha} d\xi 
= (2\pi)^{-d}(p^m(t,0) - p^0(t,0)).
\]

Now it follows from Lemma 3.2 that for \( t \in [L, M] \) and \( x, y \in \mathbb{R}^d \),
\[
\left| p^0(x,y) - p^m(x,y) \right| 
\leq t^{-d/\alpha}(2\pi)^{-d} e^{mt\alpha/2} \int_{\mathbb{R}^d} \frac{e^{-|\xi|^\alpha}}{|\xi|^{2-\alpha}} d\xi + \sum_{n=1}^\infty \frac{(mt)^n}{n!} \int_{\mathbb{R}^d} e^{-|\xi|^\alpha} d\xi 
\leq L^{-d/\alpha}(2\pi)^{-d} \frac{mM^{\alpha/2}}{\alpha} \int_{\mathbb{R}^d} \frac{e^{-|\xi|^\alpha}}{|\xi|^{2-\alpha}} d\xi + \sum_{n=1}^\infty \frac{(mM)^n}{n!} \int_{\mathbb{R}^d} e^{-|\xi|^\alpha} d\xi.
\]

The last quantity above converges to 0 as \( m \to 0 \).

For convenience, we define the following notation.
\( f^m_H(t,r) := r^m_H(t,(r,\tilde{0}),(r,\tilde{0})), \quad r > 0. \)

**Lemma 4.4** For any \( 0 < L < M < \infty \) and \( m > 0 \),
\[
\lim_{t \to 0} f^m_H(1,r) = f^0_H(1,r), \quad \text{uniformly in } r \in [L,M],
\]
that is, given \( \varepsilon > 0 \) there exists \( t_0 > 0 \) such that for \( 0 \leq t \leq t_0 \) we have
\[
\sup_{r \in [L,M]} \left| f^m_H(1,(r,\tilde{0}),(r,\tilde{0})) - f^0_H(1,(r,\tilde{0}),(r,\tilde{0})) \right| < \varepsilon.
\]

**Proof.** Recall that \( p^0_H(t,x,y) = E_x[\tau^0_H < t, p^0(t-\tau^0_H, X^0_H, y)] \) and \( p^m_H(t,x,y) = E_x[\tau^m_H < t, p^m(t-\tau^m_H, X^m_H, y)] \). It is well known that
\[
p^0(1,\tilde{0},(r,\tilde{0})) = t^{-d/\alpha} \wedge \frac{t}{|x-y|^{d+\alpha}}.
\]
Since \( |X^0_{\tau^0_H} - (r,\tilde{0})| > L \), we have, together with Lemma 2.1,
\[
p^0(1-\tau^0_H,X^0_{\tau^0_H},(r,\tilde{0}))) \leq 1 - \tau^0_H \frac{L^{d+\alpha}}{L^{d+\alpha}},
\]
\[
p^m(1-\tau^m_H,X^m_{\tau^m_H},(r,\tilde{0}))) \leq e^{tm} \frac{1 - \tau^m_H}{L^{d+\alpha}}.
\]
Since $m_0$, we also have

$$E_{(r, \tilde{0})} \left[ 1 - \delta_1 \leq \tau_H^0 < 1, p^0(1 - \tau_H^0, X_{\tau_H^0}^0, (r, \tilde{0})) \right] < \varepsilon, \quad (4.3)$$

$$E_{(r, \tilde{0})} \left[ 1 - \delta_1 \leq \tau_{tm}^0 < 1, p^{tm}(1 - \tau_{tm}^0, X_{\tau_{tm}^0}^{tm}, (r, \tilde{0})) \right] < \varepsilon. \quad (4.4)$$

Now let $V^m$ be a Lévy process with Lévy density $\sigma = J - J^m$ and define $T^m := \inf\{t > 0 : V^m_t \neq 0\}$. Then $V^m$ is a compound Poisson process and $T^m$ is an exponential random variable with parameter $m$ and independent of $X$ (See [13]). Then we have

$$E_{(r, \tilde{0})} \left[ \tau_{tm}^0 < 1 - \delta_1, p^{tm}(1 - \tau_{tm}^0, X_{\tau_{tm}^0}^{tm}, (r, \tilde{0})) \right] \leq c_{\frac{m}{L+\alpha}} \left( 1 - e^{-mt} \right). \quad (4.5)$$

Similarly we also have

$$E_{(r, \tilde{0})} \left[ T^{tm} \leq 1, \tau_H^0 < 1 - \delta_1, p^0(1 - \tau_H^0, X_{\tau_H^0}^0, (r, \tilde{0})) \right] \leq c_{\frac{m}{L+\alpha}} \left( 1 - e^{-mt} \right). \quad (4.6)$$

Take $t_1 > 0$ such that (4.5) and (4.6) is less than $\varepsilon$ for all $t \leq t_1$. Next note that for $T^{tm} > 1$ and $\tau_{tm}^0 < 1$, we have $\tau_{tm}^0 = \tau_H^0$ and $X_{\tau_{tm}^0}^{tm} = X_{\tau_H^0}^0$. Hence it follows that

$$|E_{(r, \tilde{0})} \left[ T^{tm} > 1, \tau_H^0 < 1 - \delta_1, p^0(1 - \tau_H^0, X_{\tau_H^0}^0, (r, \tilde{0})) \right] - E_{(r, \tilde{0})} \left[ T^{tm} > 1, \tau_H^0 < 1 - \delta_1, p^0(1 - \tau_H^0, X_{\tau_H^0}^0, (r, \tilde{0})) \right]| \leq \sup_{s \in [t_1, t_2]} |p^m(s, x, y) - p^0(s, x, y)| \leq \varepsilon. \quad (4.7)$$

It follows from Lemma 4.3 that there exists $t_2 > 0$ such that $\sup_{s \in [t_1, t_2]} |p^m(s, x, y) - p^0(s, x, y)| < \varepsilon$ for $0 \leq t \leq t_2$. Now let $t_0 = t_1 \land t_2$. Then for any $0 \leq t \leq t_0$ we have from (4.3), (4.4), (4.5), (4.6), and (4.7)

$$|\tau_{tm}^0(1, (r, \tilde{0}, (r, \tilde{0})) - \tau_H^0(1, (r, \tilde{0}, (r, \tilde{0}))| \leq |E_{(r, \tilde{0})}[\tau_{tm}^0 < 1, p^{tm}(1 - \tau_{tm}^0, X_{\tau_{tm}^0}^{tm}, (r, \tilde{0}))] - E_{(r, \tilde{0})}[\tau_H^0 < 1, p^0(1 - \tau_H^0, X_{\tau_H^0}^0, (r, \tilde{0}))]| \leq \varepsilon.$$
\[ \begin{align*}
&\leq |E(r,\tilde{0})| \left[ 1 > \tau_{tm} > 1 - \delta, \tau_{tm} < 1, p^{tm}(1 - \tau_{tm}, \mathcal{X}_{tm}^{r,\tilde{0}}, (r,\tilde{0})) \right] + \\
&\left| E(r,\tilde{0}) \right| \left[ 1 > \tau_{H} > 1 - \delta, \tau_{H} < 1, p^{0}(1 - \tau_{0}, \mathcal{X}_{t_{0}}^{r,\tilde{0}}, (r,\tilde{0})) \right] + \\
&\left| E(r,\tilde{0}) \right| \left[ T_{tm} \leq 1, \tau_{tm} < 1 - \delta, p^{tm}(1 - \tau_{tm}, \mathcal{X}_{tm}^{r,\tilde{0}}, (r,\tilde{0})) \right] + \\
&\left| E(r,\tilde{0}) \right| \left[ T_{tm} \leq 1, \tau_{0} < 1 - \delta, p^{0}(1 - \tau_{0}, \mathcal{X}_{t_{0}}^{r,\tilde{0}}, (r,\tilde{0})) \right] + \\
&\left| E(r,\tilde{0}) \right| \left[ T_{tm} > 1, \tau_{0} < 1 - \delta, p^{0}(1 - \tau_{0}, \mathcal{X}_{t_{0}}^{r,\tilde{0}}, (r,\tilde{0})) \right] \\
&< 5\varepsilon.
\end{align*} \]

As in [3], we need to divide the Lipschitz open set \( D \) into a good set and a bad set. We recall several geometric facts about the Lipschitz open set.

**Definition 4.5** Let \( \varepsilon, r > 0 \). We say that \( G \subset \partial D \) is \((\varepsilon,r)\)-good if for each point \( p \in G \), the unit inner normal \( \nu(p) \) exists and

\[ B(p,r) \cap \partial D \subset \{ x : |(x - p) \cdot \nu(p)| < \varepsilon |x - p| \}. \]

If \( G \) is an \((\varepsilon,r)\)-good subset of \( \partial D \), then using this definition we can construct a good subset \( G \) of the points near the boundary:

\[ G = \bigcup_{p \in G} \Gamma_{r}(p, \varepsilon), \]

where \( \Gamma_{r}(p, \varepsilon) = \{ x : (x - p) \cdot \nu(p) > \sqrt{1 - \varepsilon^2} |x - p| \} \cap B(p,r) \).

The next lemma is [3, Lemma 2.7] and it says the measure of the set of the bad points near the boundary is small. Note that even though [3, Lemma 2.7] is stated for a bounded Lipschitz domain, the proof remains true for a bounded Lipschitz open set.

**Lemma 4.6 (Lemma 2.7 in [3])** Suppose \( \varepsilon \in (0,1/2) \), \( r > 0 \) and that \( G \) is a measurable \((\varepsilon,r)\)-good subset of \( \partial D \). There exists \( s_0(\partial D, G) > 0 \) such that for all \( s < s_0 \)

\[ |\{ x \in D : \delta_{D}(x) < s \} \setminus G | \leq s \left[ \mathcal{H}^{d-1}(\partial D \setminus G) + \varepsilon \left( 3 + \mathcal{H}^{d-1}(\partial D) \right) \right]. \]

The next lemma is about the existence of a good subset \( G \subset \partial D \). Again the lemma remains true for a bounded Lipschitz open set \( D \).

**Lemma 4.7 (Lemma 2.8 in [3])** For any \( \varepsilon > 0 \) there exists \( r > 0 \) such that an \((\varepsilon,r)\)-good set \( G \subset \partial D \) exists and

\[ \mathcal{H}^{d-1}(\partial D \setminus G) < \varepsilon. \]
The two lemmas above imply that

$$\{|x \in D : \delta_D(x) < s \} \setminus \mathcal{G} \leq s \varepsilon \left(4 + \mathcal{H}^{d-1}(\partial D)\right).$$

For any $$\varepsilon \in (0, 1/4)$$, we fix the $$(\varepsilon, r)$$-good set from Lemma 4.7 and construct $$\mathcal{G}$$ from $$G$$. We choose $$r$$ to be smaller than the minimal distances between (finitely many) components of $$D$$. For any $$x \in \mathcal{G}$$, there exists $$p(x) \in \partial D$$ such that $$x \in \Gamma_r(p(x), \varepsilon)$$. Next we define inner and outer cones as follows

$$I_r(p(x)) = \{y : (y - p(x)) \cdot \nu(p(x)) > \varepsilon |y - p(x)|| \cap B(p(x), r),$$

(4.8)

$$U_r(p(x)) = \{y : (y - p(x)) \cdot \nu(p(x)) < -\varepsilon |y - p(x)|| \cap B(p(x), r).$$

(4.9)

It follows from [3, (2.20)] that there exists a half-space $$H^*(x)$$ such that

$$x \in H^*(x), \quad \delta_{H^*(x)}(x) = \delta_D(x), \quad I_r(p(x)) \subset H^*(x) \subset U_r(p(x))^c. \quad (4.10)$$

Now we are ready to prove Theorem 1.2.

**Proof of Theorem 1.2.** Fix $$\varepsilon \in (0, 1/4)$$, the $$(\varepsilon, r)$$-good set from Lemma 4.7 and the $$G$$ constructed from $$G$$. From the definition of the trace we have

$$-t^{d/\alpha} \int_D r^m_D(t, x, x) dx = t^{d/\alpha} \int_D (p^m_D(t, x, x) - p^m(t, x, x)) dx$$

$$= t^{d/\alpha} Z^m_D(t) - t^{d/\alpha} \int_D p^m(t, x, x) dx$$

$$= t^{d/\alpha} Z^m_D(t) - t^{d/\alpha} \int_D \left(p^0(t, x, x) - \left(p^0(t, x, x) - p^m(t, x, x)\right)\right) dx$$

$$= t^{d/\alpha} Z^m_D(t) - C_1 |D| + t^{d/\alpha} \int_D \left(p^0(t, x, x) - p^m(t, x, x)\right) dx.$$

Hence it follows from Lemma 3.2 that in order to prove Theorem 1.2 we must show that for given $$\varepsilon \in (0, 1/4)$$ there exists a $$t_0 > 0$$ such that for any $$0 < t < t_0$$,

$$\left| t^{d/\alpha} \int_D r^m_D(t, x, x) dx - C_2 \mathcal{H}^{d-1}(\partial D) t^{1/\alpha} \right| \leq c(\varepsilon) t^{1/\alpha},$$

where $$c(\varepsilon) \to 0$$ as $$\varepsilon \to 0$$. As in the proof of [3, Theorem 1.1.] we split the region of integration into three sets

$$D_1 = \{x \in D \setminus G : \delta_D(x) < s\},$$

$$D_2 = \{x \in D \cap G : \delta_D(x) < s\},$$

$$D_3 = \{x \in D : \delta_D(x) \geq s\},$$

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where $s$ must be smaller than the $s_0$ given by Lemma 4.6. For small enough $t$ we can take

$$s = t^{1/\alpha}/\sqrt{\epsilon}.$$ 

It is shown in [3, (3.2) and (3.4)] that

$$t^{d/\alpha} \int_{D_1 \cup D_3} r_D^0(t, x, x) dx \leq c(\epsilon) t^{1/\alpha}$$

(4.11)

where $c(\epsilon) \to 0$ as $\epsilon \to 0$. Hence it follows from Lemma 2.1 and (4.11) that

$$t^{d/\alpha} \int_{D_1 \cup D_3} r_D^m(t, x, x) dx \leq c(\epsilon)e^{2mt^{1/\alpha}}.$$ 

(4.12)

Now we deal with the integral on $D_2$. Let $H^*(x)$, $I_r(p(x))$, $U_r(p(x))$ be defined by (4.8), (4.9) and (4.10). We have

$$I_r(p(x)) \subset H^*(x) \subset U_r(p(x))^c.$$ 

Since $r$ is less than the minimal distances between components of $D$, we also have

$$I_r(p(x)) \subset D \subset U_r(p(x))^c.$$ 

Since $I_r(p(x)) \subset U_r(p(x))^c$, By an argument similar to that used in Lemma 3.4 we have

$$|r_D^m(t, x, x) - r_{H^*}^m(t, x, x)|$$

$$\leq r_{I_r(p(x))}^m(t, x, x) - r_{U_r(p(x))}^m(t, x, x)$$

$$\leq e^{2mt} \left( r_{I_r(p(x))}^0(t, x, x) - r_{U_r(p(x))}^0(t, x, x) \right).$$

(4.13)

Now it follows from [3, Proposition 3.1.] and (4.13) that

$$t^{d/\alpha} \int_{D_2} |r_D^m(t, x, x) - r_{H^*}^m(t, x, x)| dx$$

$$\leq ce^{2mt} (\epsilon^{1-\alpha/2} \vee \sqrt{\epsilon}) \mathcal{H}^{d-1}(\partial D)t^{1/\alpha} \int_0^\infty (r^{-d-\alpha+1} \wedge 1) dr.$$ 

Finally we will show that the integral

$$t^{d/\alpha} \int_{D_2} r_{H^*}^m(t, x, x) dx$$

gives the second term $C_2\mathcal{H}^{d-1}(\partial D)t^{1/\alpha}$ plus an error term of order $c(\epsilon)t^{1/\alpha}$. Recall that

$$r_{H^*}^m(t, x, x) = f^m_{H^*}(t, \delta_{H^*}(x)) = f^m_H(t, \delta_D(x)).$$
Hence we have
\[
t^{d/\alpha} \int_{D_2}^{m} r_{H^r}(t, x, x) dx = t^{d/\alpha} \int_{D_2} f_{H}^{m}(t, \delta_{D}(x)) dx = t^{d/\alpha} \int_{D_1 \cup D_3} f_{H}^{m}(t, \delta_{D}(x)) dx.
\]

By an argument similar to that used to get (4.12) we have that
\[
t^{d/\alpha} \int_{D_1 \cup D_3} f_{H}^{m}(t, \delta_{D}(x)) dx \leq c(\varepsilon) t^{1/\alpha},
\]
where \(c(\varepsilon) \to 0\) as \(\varepsilon \to 0\). From the (approximate) scaling property of the relativistic stable process, we have
\[
t^{d/\alpha} \int_{D_1 \cup D_3} f_{H}^{m}(t, \delta_{D}(x)) dx = \int_{D} f_{H}^{m}(1, \delta_{D}(x)) dx.
\]
Now apply Lemmas 4.2 and 4.4 to the function \(r \to f_{H}^{m}(1, r)\) and we get for small enough \(t\)
\[
\left| \int_{D} f_{H}^{m}(1, \delta_{D}(x)) dx - C_{2} H^{d-1}(\partial D) t^{1/\alpha} \right| \leq \varepsilon t^{1/\alpha}.
\]

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