Mathematical modeling and analysis of motion of a low-orbital space tether system

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Abstract. The paper analyzes the motion of a low-orbital space tether system, which consists of a main spacecraft and a sub-spacecraft connected by a tether. The total tether length is several ten kilometers. We considered deployment phase, the free motion phase and the stabilization motion phase on a low and nearly circular orbit. In the stabilization motion phase the orbital altitude of the system is kept in a given range by a corrective thruster. The corrective thruster is located on the main spacecraft. The simulation for all motion phases of the system is based on a mathematical model with distributed parameters, in which the tether is considered as a sequence of point masses with one-sided elastic mechanical coupling.

1. Introduction

In recent years, space tether systems (STSs) have received much attention due to their potential applications [1, 2]. The analysis of the motion of the STS is very complicated if the tether length is of the order of several ten kilometers, and it is necessary to take into account the aerodynamic force acting on the elements of the STS [3].

The STS in this work consists of a main spacecraft and a sub-spacecraft. These two spacecrafts are connected by a tether. A dynamic deployment control law is proposed, which generalizes the control laws used in [2, 4, 5]. The control law in this paper takes into account the effect of aerodynamic force on all elements of the STS. A discrete model of motion is used to estimate the applicability of the proposed nominal program. In this model the tether is considered as a sequence of point masses with one-sided elastic mechanical coupling. The proposed nominal deployment program makes it possible to dramatically reduce the swing amplitude of the system with respect to the local vertical. After the deployment phase, the free motion phase begins, and this phase finishes when the orbital altitude of the system center of mass descends to a predetermined altitude. The stabilization motion phase begins after the free motion phase. In the stabilization phase the orbital altitude of the STS is kept in a given range. The stabilization is accomplished by a corrective thruster located on the main spacecraft.
2. Mathematical model of motion of STS with distributed parameters

The coordinate systems $OXYZ$, $OX_a Y_a Z_a$, $CX_y Y_y Z_y$, $CX_z Y_z Z_z$ are used for deriving the motion equations of the STS with an extensible tether (Figure 1). The geocentric right-hand coordinate system $OXYZ$ is associated with the orbital plane of the system center of mass $C$, where axis $OX$ is in the direction along the nodal line, and axis $OZ$ is in the direction of the angular momentum vector of the orbital motion. The geocentric orbital moving coordinate system $OX_a Y_a Z_a$ rotates with respect to the coordinate system $OXYZ$ with the angular velocity $\Omega = d\gamma / dt$, where $\gamma$ is the argument of latitude. The axes of the coordinate systems $OX_a Y_a Z_a$ and $CX_y Y_y Z_y$, $CX_z Y_z Z_z$ are parallel, and the only difference between these two coordinate systems is in the coordinates of the origin. The coordinate system $CX_y Y_y Z_y$ is associated with the tether and axis $CX_z$ is directed along the tether. The relative positions of the coordinate systems $CX_y Y_y Z_y$ and $CX_z Y_z Z_z$ are described by angles $\theta$ and $\beta$ (Figure 2).

![Figure 1. Coordinate systems.](image1)

![Figure 2. Relative positions of coordinate systems $CX_y Y_y Z_y$ and $CX_z Y_z Z_z$.](image2)

The STS with distributed parameters is a mechanical system. In this mechanical system the STS is considered as $n$ point masses with one-sided elastic mechanical coupling (Figure 3). The motion equations of the STS are presented as [6]:

\begin{align*}
    &\text{Equation 1:} \\
    &\text{Equation 2:}
\end{align*}
\[ \frac{dr_k}{dt} = V_k, \quad m_k \frac{dV_k}{dt} = G_k + R_k + T_k - T_{k-1} \]  

where \( r_k \) \((k = 1, 2, ..., n)\) are the positions of the main spacecraft \((k = 1)\), point masses of the tether \((k = 2, 3, ..., n-1)\) and sub-spacecraft \((k = n)\), \( m_k = m(n-2)^{-1} \) \((k = 2, 3, ..., n-1)\) are the tether’s point mass, \( m_i \) is the tether’s mass, \( n \) is the number of the STS’s point masses, \( V_k \) is the absolute velocity, \( G_k \) is the gravitational force, \( R_k \) is the aerodynamic force.

The tensional force \( T_k \) is calculated according to the Hooke’s law taking into account that the mechanical coupling is one-sided:

\[ T_k = T_k \frac{r_{k+1} - r_k}{|r_{k+1} - r_k|}, \quad (k = 1, 2, ..., n) \]

\[ T_k = \begin{cases} 
  c |r_{k+1} - r_k| - \Delta L_k, & \text{if } |r_k - r_{k+1}| - \Delta L_k \geq 0 \\
  0, & \text{if } |r_k - r_{k+1}| - \Delta L_k < 0 
\end{cases} \]

where \( \Delta L_k \) is the unstrained length of the tether element, \( c \) is the stiffness of the tether. In addition, only one tensional force acts on the end-bodies, i.e. \( T_0 = T_n = 0 \).

The gravitational force is calculated as:

\[ G_k = -\mu \frac{m_k r_k}{r_k^3}, \quad (k = 1, 2, ..., n) \]

where \( \mu \) is the Earth’s gravitational parameter.

In contrast to the models used in [5, 6], mathematical model (1) takes into consideration the aerodynamic force. The aerodynamic force significantly affects the motion of the low-orbital STS. This force acting on the tether element is calculated as [2]:

\[ R_{c,k} = -\frac{1}{2} c_r \rho D \Delta L_k V_{c,k} \sin(\alpha_k), \quad (k = 1, 2, ..., n) \]

where \( V_{c,k} \) is the velocity of the tether element center with respect to the atmosphere, \( c_r \) is the drag coefficient of the tether, \( \alpha_k \) is the angle of attack of the tether element. The atmospheric density \( \rho \) is calculated based on standard atmosphere parameters of GOST 25645.166 – 2004 [7].

The velocity and angle of attack of the tether element are calculated as:

\[ V_{c,k} = \frac{V_{c,k} + V_{c,k+1}}{2}, \quad \cos \alpha_k = \frac{(r_{k+1} - r_k) \cdot V_{c,k}}{|r_{k+1} - r_k| V_{c,k}}, \quad (k = 1, 2, ..., n) \]

where \( V_{t,k} \) is the velocity of the tether’s point mass with respect to the atmosphere.

The aerodynamic forces acting on the tether’s point masses and end-bodies are calculated as:

\[ R_k = \frac{(R_{c,k+1} + R_{c,k})}{2}, \quad (k = 2, 3, ..., n) \]

\[ R_i = -\frac{1}{2} c_r \rho g_i |V_{t,i} V_{t,i} + R_{c,i}}{2} \]

Figure 3. Model with distributed parameters for STS.
The control law is presented as \[5\]:

\[ R_c = -\frac{1}{2} c_n \rho_a S |V_{t,a}| V_{t,a} + \frac{R_{\text{d,eq}}}{2}, \]

where \( c_t \) and \( c_n \) are the drag coefficients of the main spacecraft and sub-spacecraft, respectively.

The relationship between the absolute and relative velocities is presented as:

\[ V_{t,a} = V_k - \Omega_a \times r_k, \quad (k = 1, 2, ..., n) \]

where \( \Omega_a \) is the angular velocity of the Earth’s self-rotation.

The dimension of the motion model (1) increases during the deployment process, as it is necessary to add new point masses due to the continual increase in the tether length [4]. In the stabilization phase the dimension of the model (1) does not change, and the thrust of the corrective thruster is taken into account. The motion model (1) provides more detailed insight into the system dynamics as this model makes it possible to analyze the tether shape and other features of the STS, such as the tether extensibility and the possibility of tether becoming slack.

3. Deployment phase

The deployment of the STS into the vertical position is analyzed in this section. It is assumed that the tether is deployed from the main spacecraft, and the deployment mechanism works only for braking.

The nominal control law for the STS deployment into the vertical position is designed under the conditions of the system’s motion at the end of the deployment process \( L' = L'' = 0 \), \( L = L_{\text{end}} \), where \( L_{\text{end}} \) is the total tether length, \( L' = dL/dt \), \( L'' = d^2L/dt^2 \). The control law is presented as [5]:

\[ T_p = \nu_c \Omega^2 \cos^2 \theta_t[a(L - L_{\text{end}}) + bL' / \Omega] + Q_t, \]

where \( \nu_c = (m_0^2 - LP)(m_0 + LP/2)M^{-1} \), \( \nu \) and \( b \) are the parameters of the control law, \( \theta_t \) is the deflection angle of the tether from the local vertical \( \Omega \) is defined taking into account the aerodynamic force), \( m_0 \) is the initial mass of the main spacecraft, \( M \) is the mass of the STS, \( \rho_1 \) is the line density of the tether material, \( Q_t \) is the generalized aerodynamic force, which is defined taking into account the aerodynamic force acting on the main spacecraft, sub-spacecraft and tether [5]. The tether in this case is considered as an inextensible rigid rod. The calculation for the aerodynamic force acting on the tether is based on the integration over the tether length.

The motion equation (1) is associated with the equation that takes into account the dynamics of the control mechanism operation. The latter can be described as [4]:

\[ m_c \frac{dV_c}{dt} = T_p - F_c, \quad \frac{dl}{dt} = V_d, \]

where the coefficient \( m_c \) describes the inertia of the control mechanism (it is assumed that \( m_c = \text{const} \) during the deployment process), \( l \) is the unstrained length of the tether deployed from the control mechanism, \( V_d \) is the deployment rate.

\[ F_c = T_p + p_1 (l - L) + p_2 (V_d - L') \]

where \( p_1 \) and \( p_2 \) are the feedback coefficients, \( T_p \) is the nominal tensional force determined in (11), \( L \) and \( L' \) are the nominal values. The limitation \( F_c \geq F_{\text{min}} \) is considered for calculating (13).

The nominal motion of the STS is analyzed with a lumped-parameter system in the orbital moving coordinate system \( OX, Y, Z \) [5]. The parameters used for numerical simulations in this paper are presented in Table 1.

The trajectories of the sub-spacecraft with respect to the main spacecraft are shown in Figure 4. The control force of the deployment mechanism and the nominal control force are shown in Figure 5. Figure 4(a) and Figure 5(a) correspond to the case without consideration of the aerodynamic force for determining the nominal variables \( L(t) \) and \( L'(t) \). Figure 4(b) and Figure 5(b) correspond to the case
taking into account the contribution of the aerodynamic force. The results of numerical simulations show that using the nominal deployment program without taking into account the aerodynamic force leads to the large control error in bringing the sub-spacecraft into the vertical (see Figure 5(a)). This control error causes a large swing amplitude with respect to the local vertical (see Figure 4(a)). Taking into account the aerodynamic force for designing the nominal program dramatically reduces the swing amplitude (see Figure 4(b)) and control error (see Figure 5(b)). In this case, the swing amplitude with respect to the vertical is reduced approximately 3 times, as shown in Figure 4. The swing of the STS is caused by the joint effect of the gravitational and aerodynamic forces [2].

| Parameter                                      | Value | Unit  |
|------------------------------------------------|-------|-------|
| Initial altitude of circular orbit: $H_e$     | 270   | km    |
| Total tether length: $L_{end}$                 | 30    | km    |
| Line density of tether material: $\rho_t$      | 0.2   | kg/km |
| Tether diameter: $D_t$                         | 0.6   | mm    |
| Drag coefficients of the main spacecraft and sub-spacecraft: $c_k$ ($k = 1, n$) | 2.4   |       |
| Drag coefficient of the tether: $c_t$          | 2.2   |       |
| Relative velocity of separating sub-spacecraft from main spacecraft along the local vertical downward: $\Delta V$ | 2     | m/s   |
| Cross-sectional area of the main spacecraft: $S_1$ | 3.14  | m²    |
| Ballistic coefficient of the main spacecraft: $\sigma_1$ | $3 \cdot 10^3$ | m²/kg |
| Initial mass of the main spacecraft: $m_1^0$   | 2500  | kg    |
| Cross-sectional area of the sub-spacecraft: $S_n$ | 0.785 | m²    |
| Ballistic coefficient of the sub-spacecraft: $\sigma_n$ | 0.094 | m²/kg |
| Mass of sub-spacecraft: $m_n$                  | 20    | kg    |
| Parameter of the control law: $a$              | 4     |       |
| Parameter of the control law: $b$              | 5     |       |
| Inertia of control mechanism: $m_e$            | 0.2   | kg    |
| Stiffness of the tether: $c$                   | 7070  | N     |
| Minimum control force for deployment mechanism: $F_{min}$ | 0.01  | N     |
| Feedback coefficient: $p_1$                    | 0.243 | kg/s² |
| Feedback coefficient: $p_2$                    | 7.824 | kg/s  |

4. Free motion phase
The free motion phase begins just after the deployment phase. The mathematical model of the free motion phase is described by (1).

Figure 6 shows the change in the tether shape and the trajectory of the sub-spacecraft with respect to the main spacecraft (dashed line) in the free motion phase. The orbital altitude of the system center of mass descends from about 270 km to 110 km (the dense layers of the atmosphere). The STS is under the influence of aerodynamic force and gravitational force. The tether swings with respect to the local vertical. In this case the maximum slack of the tether is 0.8 km. Also, the deflection angle of the
tether from the local vertical increases during the process of orbital altitude descent. The deflection angle reaches almost $0.5\pi$ when the STS reaches the dense layers of the atmosphere.

Figure 4. Trajectories of the sub-spacecraft with respect to the main spacecraft (i – perturbed trajectory; ii – nominal trajectory): (a) without consideration of the aerodynamic force; (b) with consideration of the aerodynamic force.

Figure 5. Control force of the deployment mechanism ($F_c$) and nominal control force ($T_p$): (a) without consideration of the aerodynamic force; (b) with consideration of the aerodynamic force.

5. Stabilization motion phase

In the free motion phase the orbital altitude of the STS descends. In order to ensure that the low-orbital STS can fly and work for a sufficiently long time, it is necessary to stabilize the orbital parameters of the STS. In this paper, it is proposed to use a corrective thruster located on the main spacecraft for the stabilization motion. The low-thrust liquid thruster 11D428AF-16 is used for numerical simulation [8]. The thrust is 157 N, and the specific impulse is 306 s. During the stabilization motion phase the thruster works periodically. The thrust direction is opposite to the velocity direction of the main spacecraft with respect to the atmosphere.

Figure 7 shows that the orbital altitudes of the main spacecraft and the sub-spacecraft are maintained in a given range, where $\tau = t_p^{-1}$ is the non-dimensional time, $t_p$ is the period of motion of
the system on the initial circular orbit. In the free motion phase and stabilization motion phase the orbital eccentricity changes a little and is close to zero. The range of stabilization orbital altitude is selected based on the restriction on the deflection angle of the tether from the local vertical. In this paper the deflection angle is smaller than $30^\circ$.

![Figure 6. Tether shape in the free motion phase.](image)

![Figure 7. Orbital altitudes of the main spacecraft and sub-spacecraft.](image)

6. Conclusions
Taking the aerodynamic force into account for designing the nominal deployment program for the low-orbital STS can reduce the control error and swing amplitude of the STS with respect to the vertical.

The proposed method can stabilize the flight of the STS in a given range of orbital altitudes by means of using a corrective thruster located on the main spacecraft. The thrust direction is opposite to the direction of the relative velocity of the main spacecraft.

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