Synchronous Lagrangian variational principles in General Relativity

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The problem of formulating synchronous variational principles in the context of General Relativity is discussed. Based on the analogy with classical relativistic particle dynamics, the existence of variational principles is pointed out in relativistic classical field theory which are either asynchronous or synchronous. The historical Einstein-Hilbert and Palatini variational formulations are found to belong to the first category. Nevertheless, it is shown that an alternative route exists which permits one to cast these principles in terms of equivalent synchronous Lagrangian variational formulations. The advantage is twofold. First, synchronous approaches allow one to overcome the lack of gauge symmetry of the asynchronous principles. Second, the property of manifest covariance of the theory is also restored at all levels, including the symbolic Euler-Lagrange equations, with the variational Lagrangian density being now identified with a 4-scalar. As an application, a joint synchronous variational principle holding both for the non-vacuum Einstein and Maxwell equations is displayed, with the matter source being described by means of a Vlasov kinetic treatment.

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I. INTRODUCTION

Variational approaches to the standard formulation of General Relativity (SF-GR) have been a popular subject of research since 1915 following the original mathematical formulation of this type first established independently by Einstein and Hilbert [1–3]. Nevertheless, the topic still represents an open fundamental theoretical challenge because of a number of critical issues which, to date, remain unsolved. Here we refer, in particular, to the following key mathematical requirements: 1) the fulfillment of the property of manifest covariance with respect to the group of local point transformations; 2) the property of gauge invariance of the related continuum Lagrangian formulations; 3) the proper prescription of the functional setting to be adopted in these variational approaches in order to fulfill properties 1) and 2).

The fact that these properties are actually mandatory in this context stems from elementary physical/mathematical arguments. Indeed, regarding the first requirement, the manifest covariance property is a necessary condition following from the general covariance principle (GCP) which requires the identical fulfillment of the property of covariance with respect to arbitrary local point transformations (i.e., smooth and invertible local coordinate transformations associated with the 4–position). This implies, in turn, that it must always be possible to cast an arbitrary relativistic continuum field theory satisfying GCP, such as SF-GR, in a form which fulfills the property of manifest covariance at all levels. In other words, this means representing all involved quantities exclusively by means of 4–tensor continuum fields, starting from the variational action functional, its variational Lagrangian density, and including as well the corresponding Lagrangian generalized coordinates, generalized velocities and momenta. As a consequence, the related equations should all be manifestly covariant too, including in particular the symbolic Euler-Lagrange equations determined in terms of the variational Lagrangian. A GR theory which fulfills such a property will be referred to as manifestly covariant.

The second requirement, by no means less physically-relevant, requires that a gauge-representation is given to the relevant variational principles holding in SF-GR. In turn, this demands the precise definition of the notion of gauge invariance to be adopted in this context. This should follow by analogy with the corresponding well-known flat space-time gauge theories available for continuum fields. Nevertheless, such a property is generally not met in variational GR approaches to be found in the literature.

The third requirement concerns the precise specification of the functional class of variations which is involved in the prescription of the continuum variational functionals. In principle, this should be determined in such a way to meet the previous requirements 1) and 2). More precisely, this refers to the prescription of: 1) the boundary conditions for the continuum fields; 2) the variations for the same continuum fields; 3) the variational action functional. In previous literature this issue has not always being treated unambiguously. In fact, implicitly, most of the literature adopts the
same functional setting originally proposed by Einstein (see for example Refs. [1, 4]), and recalled below.

Given the fundamental importance of variational principles for the axiomatic foundations of classical relativistic field theory, in this paper we intend to address these topics in order to propose their possible consistent solution. As indicated below, this is realized via the introduction of a new representation for the variational action functional and of the corresponding Lagrangian density associated with the continuum classical fields. The resulting variational principle for the Einstein equations is referred to as synchronous Lagrangian variational principle. As a consequence, it is found that by construction the same Lagrangian density exhibits the correct (i.e., manifest) covariance and gauge invariance properties. The modified setting is shown to be of general validity for classical fields. In particular, it applies also to the non-vacuum Einstein equations, when classical sources represented by the electromagnetic (EM) field and matter distributions are taken into account. For this purpose a Vlasov kinetic description of the field sources is introduced, which permits one to set the corresponding fluid descriptions on a rigorous basis (see for example the related discussion in Refs. [5–7]). This feature is remarkable in itself. In fact, as shown below, it affords the establishment of a joint synchronous variational principle in which all classical sources for the Einstein and Maxwell equations, including the same Vlasov equation, are dealt with by means of a single variational functional.

In the framework of SF-GR, the starting point is provided by the Einstein equations for the physical observable associated with the structure of the space-time, namely its symmetric 2-rank metric tensor, which arises in the presence of external sources. Its solution coincides with the extremal field indicated below, and therefore is denoted as \( g_{\mu\nu}(r) \). For the same reason, in the paper all barred continuum fields will be intended either as functions of \( \bar{g}_{\mu\nu} \) or independent extremal functions. Notice also that here and in the following, \( r \) stands for the local functional dependence in terms of the 4–position \( r^\mu \). Neglecting for the moment the contribution associated with the cosmological constant \( \Lambda \), such an equation is given by the manifestly covariant 2nd-order PDE

\[
\mathcal{T}_{\mu\nu} - \frac{1}{2} \mathcal{R} g_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu},
\]

(1)

to be supplemented by suitable boundary conditions \([3, 9]\). This problem determines uniquely the metric tensor, which can be equivalently expressed either in terms of its covariant (\( \bar{g}_{\mu\nu} \)) or its contravariant (\( \bar{g}^{\mu\nu} \)) components. The latter ones are by construction the inverse of each other, so that they are related by the condition \( \bar{g}_{\mu\alpha} \bar{g}^{\alpha\beta} = \delta^\beta_\alpha \). The notation is standard. Thus, first \( G \) is the universal gravitational constant and \( c \) is the speed of light in vacuum, while \( \mathcal{R} g_{\mu\nu} \equiv \bar{R}_{\mu\nu} - \frac{1}{2} \bar{R} \bar{g}_{\mu\nu} \) denotes the symmetric Einstein tensor. Furthermore, \( \bar{R} = \bar{g}_{\mu\nu} \bar{T}^{\mu\nu} \) is the Ricci 4–scalar, while \( \bar{R}_{ik} \) is the related Ricci curvature 4–tensor

\[
\bar{R}_{ik} = \frac{\partial \bar{\Gamma}_{ik}^l}{\partial x^l} - \frac{\partial \bar{\Gamma}_{il}^k}{\partial x^k} + \bar{\Gamma}_{ik}^l \bar{\Gamma}^m_l - \bar{\Gamma}_{il}^m \bar{\Gamma}^m_k.
\]

(2)

Here \( \bar{\Gamma}_{ik} \) denote a suitable coordinate representation of the Levi-Civita connection functions, and hence are inherently non-tensorial in character. More precisely, \( \bar{\Gamma}_{ik} \) identify the extremal Christoffel symbols, which are evaluated with respect to the extremal field \( \bar{g}_{\mu\nu} \) and defined as usual as

\[
\bar{\Gamma}_{ik}^l = \frac{1}{2} \bar{g}^{lm} \left( \frac{\partial \bar{g}_{il}}{\partial x^m} + \frac{\partial \bar{g}_{mk}}{\partial x^i} - \frac{\partial \bar{g}_{ik}}{\partial x^m} \right).
\]

(3)

Eq. (1) is completed by the metric compatibility condition in terms of the covariant derivative of \( \bar{g}^{\mu\nu} \):

\[
\nabla_\alpha \bar{g}^{\mu\nu} = 0,
\]

(4)

where by construction the covariant derivative operator \( \nabla_\alpha \) is defined with respect to the connections provided by the extremal Christoffel symbols \([2, 3]\). Finally the symmetric stress-energy tensor \( \bar{T}_{\mu\nu} \) takes into account the contribution of external sources. Its precise form is determined by the prescription of the external mass distribution and the EM field.

II. GOALS

The target of the paper is the construction of a new variational principle for the non-vacuum Einstein equations, referred to here as synchronous Lagrangian variational principle, which allows one to meet the physical requirements indicated above. For this purpose, we find it useful to present a brief introduction dealing with single-particle relativistic dynamics, in which case the adoption of synchronous variational principles is well-known. In fact, the distinction between synchronous and asynchronous variational principles occurring in such a context permits one to
identify the route to follow also in the case of continuum field theory. Indeed, the physical analogy between the two problems provides a strong argument in support of the validity of synchronous Lagrangian variational principles also in the context of GR for relativistic field theory. For greater generality, in the present work matter and charge current source terms will be treated by means of a kinetic Vlasov description. A related issue is therefore that of looking for a variational principle providing at the same time also a synchronous variational principle of similar type for the Vlasov equation. In particular, for the Vlasov equation the case of a conservative mean-field vector field acting on single-particle dynamics will be considered. The result allows one to determine a joint variational principle for the non-vacuum Einstein and Maxwell equations, together with the species Vlasov equations.

The structure of the presentation is as follows. In Section III and IV the main physical motivations for the present investigation are discussed, which arise from the analysis of standard literature approaches to the variational formulation of GR. In Section V synchronous and asynchronous Lagrangian variational principles of relativistic classical dynamics, together with their basic features are recalled. Section VI deals with the formulation of synchronous Lagrangian variational principles for the vacuum Einstein equations (THM.2), while the extension to the non-vacuum case (i.e., in the presence of sources) is considered in Section VII (THM.3). As an application of the theory, in Section VIII first a joint synchronous variational principle for the non-vacuum Einstein-Maxwell equations is established, with the Vlasov source being included (THM.4). Then, a synchronous variational principle is constructed for the Vlasov kinetic equation. Concluding remarks are drawn in Section IX. Finally, in the Appendix a constrained asynchronous variational principle is established (THM.1), which summarizes the Einstein-Hilbert and Palatini variational approaches.

III. PHYSICAL MOTIVATIONS

As originally pointed out by Einstein and Hilbert [1], the non-vacuum equation (1) is intrinsically variational in character so that, depending on the functional setting, it could be cast in terms of a multiplicity of equivalent variational forms. In particular, it follows that the same equation can always be represented by means of a suitable Lagrangian variational principle (see below). In the Einstein and Hilbert original approach as well as in the subsequent literature its construction actually implicitly relies on the validity of a number of common underlying hypotheses. These will be referred to in a short way as Einstein-Hilbert (EH) axioms, being realized as follows:

**EH axiom 1** - There exists a suitable functional class \( \{Z\} \equiv \{Z = (Z_1\ldots Z_n) : f(Z) = 0\} \) of smooth real fields \( Z_i(r) \), for \( i = 1, n \), to be denoted as continuum Lagrangian generalized coordinates, which are defined in the physical spacetime \( \mathbb{D}^4 \equiv (\mathbb{R}^4, g_{\mu\nu}) \), where \( \mathbb{Q}^4 \) is a \( C^k \)-differentiable Lorentzian manifold, with \( k \geq 3 \), endowed with the metric tensor \( g_{\mu\nu}(r) \) yet to be determined. Furthermore, \( Z_i(r) \) are required to satisfy appropriate boundary conditions on a prescribed boundary \( \partial\mathbb{D}^4 \), namely \( Z_i|_{\partial\mathbb{D}^4} = Z_i|_{\mathbb{D}} \), where the fields \( Z_i|_{\mathbb{D}} \) are considered prescribed, namely for all \( i = 1, n \), to be the same for all fields \( Z_i(r) \). Finally, the same fields are also possibly subject to further suitable constraints, symbolically represented here by equations of the form \( f(Z) = 0 \). An arbitrary Lagrangian coordinate \( Z_i(r) \in \{Z\} \) will also be referred to as variational field.

**EH axiom 2** - There exists a 4–scalar functional \( S(Z) \) defined in \( \{Z\} \) and taking values in \( \mathbb{R} \) of the form

\[
S(Z) = \int_{\mathbb{D}^4} d^4x \sqrt{-g} L(Z, \mathcal{D}Z).
\]

This will be denoted as continuum Lagrangian action. Here the notation is as follows. First, \( \mathbb{D}^4 \subseteq \mathbb{D}^4 \) is the open interior subset of \( \mathbb{D}^4 \). Second, \( d\Omega \equiv d^4x \sqrt{-g} \) identifies the 4–scalar 4–volume element, with \( d^4x \equiv \prod_{i=0}^{3} dx^i \) being the canonical measure and \( g \) the determinant of \( g_{\mu\nu} \), both to be considered as variational quantities, namely functions of the variational fields. Third, \( L(Z, \mathcal{D}Z) \) is referred to as *variational field Lagrangian* and is identified with a real 4–scalar smooth function of the variational fields \( Z \) and their “generalized velocities” \( \mathcal{D}Z \) determined via an appropriate differential operator \( \mathcal{D} \) to be identified below.

**EH axiom 3** - The notion of variation \( \delta Z_i \) must be introduced for all the continuum Lagrangian fields \( Z_i(r) \) belonging to \( \{Z\} \). Thus, if \( Z_i \) and \( Z'_i \) are two arbitrary realizations of the \( i \)-th field, both belonging to \( \{Z\} \), then for all \( Z_i \) the variation \( \delta Z_i(r) \) is prescribed by means of the scalar variation operator \( \delta \) so that

\[
\delta Z_i(r) = Z_i(r) - Z'_i(r),
\]

where \( Z_i(r) - Z'_i(r) \) is considered infinitesimal and such that on the boundary it satisfies for all \( i = 1, n \) the constraint equation \( \delta Z_i(r)|_{\mathbb{D}^4} = 0 \). This shall be referred to as variation of the function \( Z_i(r) \). It follows that a generic variational field \( Z_{i\ell}(r) \in \{Z\} \) can always be expressed in terms of the parametric representation

\[
Z_{i\ell}(r) = Z_i(r) - \alpha \delta Z_i(r),
\]
being $\alpha \in [-1,1]$ an arbitrary real number which is left invariant by the $\delta-$operator. Similarly, the variation of the generalized velocities $\delta D Z_i$ must also generally be taken into account. This is identified with

$$\delta D Z_i (r) = D Z_i (r) - D Z_i' (r).$$

(8)

Requiring that the differential operators $D$ and $\delta$ commute, this implies that

$$\delta D Z_i (r) = D \delta Z_i (r).$$

(9)

**EH axiom 4** - The continuum Lagrangian action functional $S(Z)$ is assumed to satisfy the variational principle

$$\delta S (Z) = 0,$$

(10)

to hold for arbitrary variations $\delta Z \equiv \{ \delta Z_1, ... \delta Z_n \}$. Here $\delta Z (r)$ denotes the variation of the functional $S (Z)$, to be identified with its Frechet derivative (see below). Then, provided the differential operator $D$ is suitably defined, the related notion of functional derivative $\frac{\delta S}{\delta Z_i}$ follows, for all $i = 1, n$. Hence, the Euler-Lagrange equations corresponding to Eq.(10) are given, for all $i = 1, n$, by

$$\frac{\delta S}{\delta Z_i} = 0,$$

(11)

where the variations are performed with respect to the *variational Lagrangian density*, namely

$$\mathcal{L}(Z, DZ) \equiv \sqrt{-g} L(Z, DZ).$$

(12)

The latter is not a 4-scalar. Nevertheless, by construction it is a smoothly-differentiable ordinary real function, which depends on the variables $(Z, DZ)$. Notice, in fact, that the variation of the action functional $\delta S (Z)$ acts here, by assumption, also on the coefficient $\sqrt{-g}$ of the 4-scalar volume element which appears in the functional $S (Z)$ defined according to Eq.(5). For this reason Eq.(10) will be referred to here as an *asynchronous variational principle*.

We stress that in the previous axioms the actual prescription of the functional class $\{Z\}$ and of the variational field Lagrangian $L(Z, DZ)$ remains arbitrary. We summarize in the next paragraph the choices adopted for them in previous literature for the Einstein equations, starting from the Einstein-Hilbert original approach.

### A. Previous variational approaches

In this section we briefly outline two examples of literature treatments which actually permit the identification of a variational Lagrangian density, namely the Einstein-Hilbert and the Palatini variational approaches. In both cases the variational field Lagrangian is identified with

$$L = L_{EH} + L_F,$$

(13)

where $L_{EH}$ denotes the Einstein-Hilbert vacuum field Lagrangian

$$L_{EH} (Z, DZ) \equiv -\frac{c^2 G}{16\pi} R,$$

(14)

with $R$ being the Ricci 4-scalar, which is assumed to be a function of the variational fields defined in the functional class, while $L_F$ is a prescribed external source field Lagrangian. In the case the latter is identified with the EM field, $L_F$ is denoted as $L_{EM}$ and considered as a local smooth scalar function of the EM 4-potential $A_\mu$. The two approaches are characterized by different choices of the functional class $\{Z\}$. For definiteness, let us consider the case of the vacuum Einstein equations. In the original Einstein-Hilbert variational approach, also found in Ref.[2], $\{Z\} \equiv \{Z\}_E$ is identified with the ensemble of generalized coordinates $g_{\mu \nu} (r)$ which are symmetric 4-tensors in the indices $\mu, \nu$, and defined as

$$\{Z\}_E = \left\{ \begin{array}{l} Z_1 \equiv g_{\mu \nu} : g_{\mu \nu} (r) = g_{\nu \mu} (r) \in C^k (\mathbb{R}^4) \\ f_k (Z_1) = g^{ak} g_{b\delta} - \delta_{a}^{\delta} = 0 \\ \Gamma^\nu_{\alpha \beta} = \Gamma^\nu_{(C)\alpha \beta} (g) \\ g_{\mu \nu} (r) |_{\partial \Omega^4} = g_{\mu \nu} (r) \\ \omega^\nu (Z_1, DZ_1) |_{\partial \Omega^4} = 0 \\ \delta (d\Omega) = d^4x \delta (\sqrt{-g}) \neq 0 \end{array} \right\}.$$

(15)
Notice that hereon $k \geq 3$ (to warrant the existence of $C^4$ solutions for $g_{\mu\nu}$ in $\mathbb{D}^4$ which are continuous on $\partial\mathbb{D}^4$), $D = \partial_u$ and $w^\mu$ is the 4-vector $w^\mu = g^{\alpha\beta} \delta \Gamma^\mu_{(C)\alpha\beta} - g^\alpha{}_{\mu} \delta \Gamma^\beta_{(C)\alpha\beta}$, which depends both on $g_{\mu\nu}$ and its partial derivatives. Here the constraint $f_1(Z_1) = 0$ warrants that the variational coordinates $g^{\mu\nu}(r)$ and $g_{\mu\nu}(r)$ raise and lower indexes (in particular in the Lagrangian density $\mathcal{L}(Z, DZ)$). Finally, $\Gamma^l_{\alpha\beta}(g)$ are the Christoffel symbols evaluated in terms of the variational field $g_{\mu\nu}$, namely

$$\Gamma^l_{\alpha\beta}(g) = \frac{1}{2} g^{lm} \left( \frac{\partial g_{lm}}{\partial x^\beta} + \frac{\partial g_{mk}}{\partial x^\beta} - \frac{\partial g_{ik}}{\partial x^\beta} \right).$$

(16)

These will be referred to as prescribed Christoffel symbols. We remark here that the choice of the boundary condition for $w^\mu$ involves the prescription of the partial derivative of $g_{\mu\nu}$ on the boundary. An alternative possible definition of $\{Z\}_E$ which avoids such a type of boundary condition can be found in Ref.\cite{9}. In this case however the variational functional $S(Z)$ needs to be modified by means of the introduction of a surface contribution. This feature actually prevents the introduction of a Lagrangian density as indicated above.

The set of Euler-Lagrange equations associated with the Einstein-Hilbert variational principle and based on the choices indicated above is well-known. Written in the so-called symbolic representation, namely expressed in terms of the variational Lagrangian density, these are given by the PDE

$$\frac{1}{\sqrt{-g}} \frac{\partial \mathcal{L}}{\partial g^{\mu\nu}} = 0,$$

(17)

where $\mathcal{L}$ is now $\mathcal{L} = \sqrt{-g} \mathcal{L}_{EH}(Z, DZ)$ and the partial derivative with respect to the continuum Lagrangian coordinate $g^{\mu\nu}$ must be performed keeping constant the connections. We notice that $\sqrt{-g}$ and $\mathcal{L}$ are separately not 4—tensors, so that both $\frac{\partial \mathcal{L}}{\partial g^{\mu\nu}}$ and $\frac{1}{\sqrt{-g}} \frac{\partial \mathcal{L}}{\partial g^{\mu\nu}}$ are not 4—tensors too. Hence, the symbolic representation of the Euler-Lagrange equation given by Eq.\textcolor{red}{(17)} is not manifestly covariant. Nevertheless, once the explicit calculation is performed, the previous equation delivers

$$\frac{\partial \mathcal{L}}{\partial g^{\mu\nu}} - \frac{1}{2} L g_{\mu\nu} = 0,$$

(18)

which coincides with the Einstein vacuum equation, and hence recovers the property of manifest covariance.

The second approach to be mentioned is the one referred to in the literature as the Palatini variational principle \cite{8,9}. This is realized by considering both the metric tensor $g_{\mu\nu}$ and the connections $\Gamma^\mu_{\alpha\nu}$ as independent continuum Lagrangian coordinates. Lagrangian coordinates of this type will be referred to as superabundant ones. In contrast, the ten independent components of the variational metric tensor $g_{\mu\nu}(r)$ will be denoted as essential Lagrangian coordinates. As a consequence, the functional class can now be identified with $\{Z\} \equiv \{Z\}_\text{Pal}$, represented by the ensemble of variational coordinates $g_{\mu\nu}(r)$ and $\Gamma^\mu_{\alpha\nu}(r)$ which are both symmetric in the lower indices $\mu, \nu$, and is defined as

$$\{Z\}_\text{Pal} \equiv \left\{ [Z_1, Z_2] \equiv \{g_{\mu\nu}(r), \Gamma^\mu_{\alpha\nu}(r)\} \in C^k(\mathbb{D}^4) \right\}$$

\begin{align*}
   f_1(Z_1) &= g^{\alpha k} g_{\beta k} - \delta^\alpha_{\beta} = 0 \\
   g_{\mu\nu}(r) \big|_{\partial\mathbb{D}^4} &= g_{\mu\nu}(r) \\
   \Gamma^\mu_{\alpha\nu}(r) \big|_{\partial\mathbb{D}^4} &= \Gamma^\mu_{\alpha\nu}(r) \\
   \delta(\partial\Omega) &= d^4x \delta(\sqrt{-g}) \not= 0
\end{align*}

(19)

with $k \geq 3$. The Euler-Lagrange equation corresponding to the functional setting $\{Z\}_\text{Pal}$ and the same choice of the variational Lagrangian given above follow in a similar way. In particular the variation with respect to $\delta g^{\mu\nu}$ recovers again the symbolic Euler-Lagrange equation given by Eq.\textcolor{red}{(17)}. Instead, the remaining extremal equation is obtained by considering the variation with respect to $\delta \Gamma^\beta_{\alpha\gamma}$. In symbolic form this is expressed by the independent PDE

$$\nabla_\alpha \left[ \frac{1}{\sqrt{-g}} \frac{\partial \mathcal{L}}{\partial R_{\mu\nu}} \right] = 0.$$

(20)

Therefore, once again the symbolic Euler-Lagrange equations \textcolor{red}{(17)} and \textcolor{red}{(20)} violate the property of manifest covariance. Nevertheless, once the calculation of the partial derivatives is explicitly carried out, the resulting PDEs recover the required covariance property. In particular, as a result, it is immediate to show that Eq.\textcolor{red}{(20)} simply reduces to Eq.\textcolor{red}{(17)}.

In conclusion, it follows that both the Einstein-Hilbert and Palatini variational approaches do not fulfill the property of manifest covariance of the theory. Nevertheless, this feature is not-withstanding, as both theories uniquely prescribe the vacuum Einstein equations in the framework of mathematically-consistent approaches, i.e., being obtained in terms of well-defined variational principles.
B. An alternative constrained variational principle

In this section we point out an alternative variational formulation for the Einstein vacuum equations, which is still based on the adoption of an asynchronous variational principle (see below) and is useful to display the relationship with the Einstein and Palatini approaches recalled above. This is obtained by suitably prescribing the functional class \( \{ Z \} \), while leaving unchanged the Einstein-Hilbert action. In fact, one notices that formally Eq. (17) recovers identically the Einstein vacuum equation provided the Ricci tensor \( R_{\mu \nu} \) is kept constant during variations, namely is identified with its extremal value \( \bar{R}_{\mu \nu} \):

\[
R_{\mu \nu} = \bar{R}_{\mu \nu}.
\]

Such a constraint can be satisfied adopting the method of the Lagrange multipliers and introducing the functional class:

\[
\{ Z \}_c = \left\{ \begin{array}{l}
\{ Z_1 (r), Z_2 (r), Z_3 (r) \equiv \{ g_{\mu \nu}, R_{\mu \nu}, \lambda^{\mu \nu} \} \\
\bar{Z}_2 (r) \equiv \bar{R}_{\mu \nu} \\
Z (r), \bar{Z} (r) \in C^k (D^4) \\
Z (r) \vert_{\partial D^4} = \bar{Z} \vert_{\partial D^4} \\
\delta \bar{Z}_2 (r) = 0 \\
\delta (d\Omega) = d^4x (\sqrt{-g}) \neq 0
\end{array} \right\},
\]

where again \( k \geq 3 \) and the variational fields \( \{ g_{\mu \nu}, R_{\mu \nu}, \lambda^{\mu \nu} \} \) are all assumed symmetric in the indices \( \mu, \nu \). This means, in other words, that the Einstein-Hilbert variational principle can be replaced by an equivalent one in which the boundary condition \( w^\mu (Z_1, DZ_1) \vert_{\partial D^4} = 0 \) needs not be imposed anymore, as this instead is replaced by the constraint equation (21).

On the basis of these premises, one can prove that THM.1 reported in Appendix holds. The notable consequence is to reproduce exactly the Euler-Lagrange equation (17) by means of a constrained Lagrangian variational principle. In this regard, the following comments are made:

1) In analogy to the customary Einstein-Hilbert principle, for both functionals \( S_c (Z, \bar{Z}) \) and \( S_{E-c} (Z, \bar{Z}) \) introduced in THM.1 the continuum Lagrangian coordinates are tensorial in character.

2) The asynchronous constrained principle established in THM.1 can be extended to the case of non-vacuum Einstein equations.

3) The proposition T11 of the theorem can be generalized also to the case in which the Ricci tensor \( R_{\mu \nu} \) is not considered as an independent continuum Lagrangian coordinate, but a function of the same variational metric tensor. In fact, thanks to the presence of the constraint and the Lagrange multiplier, any contribution arising from the Ricci tensor vanishes in the extremal equation. This feature permits one to reach an equivalent representation of the action in terms of \( S_{E-c} (Z, \bar{Z}) \) (see proposition T12).

4) THM.1 can in principle be extended also to the Palatini approach. The fundamental reason for this conclusion is that the second Euler-Lagrange equation holding in such a case [i.e., Eq. (20)] is identically fulfilled by the extremal continuum Lagrangian coordinates. Hence, the variational principle must hold also when the Palatini functional class (19), subject to the constraint (21), is considered.

These conclusions show that the constrained principle given by THM.1 encompasses both the Einstein-Hilbert and Palatini approaches and provides a convenient framework for the variational treatment of the contributions associated with the Christoffel symbols.

IV. GAUGE INvariance PROPERTIES

A further critical issue inherent in certain literature formulations lies in the lack of gauge invariance properties. This feature is related to the adoption of a non-tensorial variational Lagrangian density \( L \). This happens in particular in the case of the two variational approaches indicated above. Indeed, this feature gives rise to continuum field theories which are intrinsically non-gauge invariant. It is important to stress, however, that the property of gauge invariance should be regarded as a mandatory feature of variational field theories in general (see also related discussion in Section VI ). This demands that gauge invariance should be fulfilled both by variational \((Z)\) and extremal \((\bar{Z})\) continuum fields, the latter being identified with the solutions of the Euler-Lagrange equations determined by the variational principle. As a consequence, also the variational functional \( S(Z) \) and the corresponding variational Lagrangian \( L(Z, D\bar{Z}) \), together with the corresponding extremal quantities \( S(\bar{Z}) \) and \( L(\bar{Z}, D\bar{Z}) \), should be necessarily determined up to a suitable gauge contribution. However, this property is violated both in the Einstein-Hilbert and Palatini approaches as well as in THM.1.
To illustrate the issue, consider for example the trivial gauge transformation acting on the variational field Lagrangians considered in these approaches:

$$L(Z, \mathcal{D}Z) \rightarrow L(Z, \mathcal{D}Z) + C,$$

with $C$ being an arbitrary constant 4-scalar. It follows that the Lagrangian density $\mathcal{L}$ transforms as

$$\mathcal{L}(Z, \mathcal{D}Z) \rightarrow \mathcal{L}(Z, \mathcal{D}Z) + \sqrt{-g}C.$$

It is immediate to show that, in all approaches discussed above, the introduction of the additive constant $C$ changes in a non-trivial way the form of the Einstein equations, which becomes

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} - \frac{1}{2}Cg_{\mu\nu} = \frac{8\pi G}{c^4}T_{\mu\nu},$$

where $C$ plays a role analogous to the so-called cosmological constant $\Lambda$.

Additional difficulties may arise when a general gauge transformation of the type

$$L(Z, \mathcal{D}Z) \rightarrow L(Z, \mathcal{D}Z) + \nabla_\alpha C^\alpha(Z)$$

is introduced. First we notice that in the Einstein-Hilbert approach the term $\nabla_\alpha C^\alpha(Z)$ remains truly a gauge function, where $\nabla_\alpha$ must be intended as a function of the prescribed Christoffel symbols $\Gamma^\alpha_{\beta\gamma}(g)$. This happens because in the functional setting $\{Z\}_E$ the following identity

$$\nabla_\alpha C^\alpha = \frac{1}{\sqrt{-g}} \frac{\partial}{\partial x^\alpha} (\sqrt{-g}C^\alpha)$$

holds. Hence, thanks to Gauss theorem, the same gauge term does not contribute to the Einstein-Hilbert action functional. Instead, when the Palatini approach is considered, the gauge function $\nabla_\alpha C^\alpha(Z)$ must be excluded “a priori”, since it depends intrinsically on the variational connections entering through the definition of $\nabla_\alpha$, which by construction are considered independent of the prescribed Christoffel symbols $\Gamma^\alpha_{\beta\gamma}(g)$. Therefore, Eq. (27) does not hold anymore, so that Gauss theorem cannot apply because the volume element of integration $d^4x\sqrt{-g}$ and the differential operator $\nabla_\alpha$ remain in this case independent. The implication is that both the Einstein-Hilbert and Palatini approaches are unsatisfactory because they violate at least in part the gauge invariance symmetries (23) and (26). The same conclusion can be reached for the constrained variational principles considered in THM.1.

However, the violation of gauge invariance displayed here is a serious inconsistency. Indeed, it is in conflict, for example, with the gauge-invariance property of the Maxwell equations when they are considered in the flat space-time. In such a case in fact, one finds that the corresponding variational and field Lagrangian densities coincide, namely

$$\mathcal{L}_{EM}(Z, \mathcal{D}Z) \equiv \mathcal{L}_{EM}(Z, \mathcal{D}Z).$$

Such a property is also satisfied in curved space-time when the metric tensor is considered extremal. Remarkably, this definition warrants that the variational Lagrangian for the EM field has the same tensorial character both in curved and flat space-times, so that in both cases it is actually a 4-scalar. Similar considerations apply in principle also to other possible field Lagrangians, such as those describing classical scalar fields. This feature assures that, when the metric tensor and the connections are extremal, the variational Lagrangian density $\mathcal{L}_{EM}(Z, \mathcal{D}Z)$ recovers the customary flat-space-time gauge symmetry, so that both the transformations

$$\mathcal{L}_{EM} \rightarrow \mathcal{L}_{EM} + C,$$

$$\mathcal{L}_{EM} \rightarrow \mathcal{L}_{EM} + \nabla_\alpha C^\alpha(Z),$$

leave invariant the Maxwell equations. Notice that here the gauge $\nabla_\alpha C^\alpha(Z)$ must be regarded as extremal both with respect to the metric tensor and the connections.

In the following, when $\mathcal{L}(Z, \mathcal{D}Z)$ satisfies the property $\mathcal{L}(Z, \mathcal{D}Z) = L(Z, \mathcal{D}Z)$, the corresponding variational principle will be referred to as standard Lagrangian variational approach, with $\mathcal{L}(Z, \mathcal{D}Z)$ being denoted as standard variational Lagrangian density. In such a case the following characteristic properties are expected to be fulfilled:

1) The continuum field theory is manifestly covariant at all levels. In particular, this means that the continuum Lagrangian coordinates have all a well-defined tensorial character, so that the corresponding symbolic Euler-Lagrange equations are manifestly covariant too.

2) The property of gauge invariance, in the sense of Eqs. (29)-(30), both for the functionals and the related variational Lagrangian densities, is warranted.
From the previous analysis a number of serious discrepancies emerges for the variational formulations of GR considered so far and the field theories for other classical fields, which concern the fundamental properties of manifest covariance at all levels and gauge symmetries. The issue is whether full consistency with these basic principles can be ultimately reached also for the variational formulation of the Einstein equations. The achievement would be of basic relevance for several reasons, i.e., at least:

**Requirement #1** - To assure a unified variational treatment valid for all classical fields in the context of GR.

**Requirement #2** - To warrant the gauge invariance property of the variational functional and of the related variational Lagrangian.

**Requirement #3** - To permit the manifest covariance of the theory at all levels and in particular to assure that both the variational Lagrangian and the associated symbolic Euler-Lagrange equations, corresponding to the Einstein equations, exhibit a tensorial structure and hence are manifestly covariant too.

V. LAGRANGIAN VARIATIONAL PRINCIPLES IN CLASSICAL MECHANICS

In this section we briefly summarize the basic features of variational formulations available in relativistic classical mechanics for single-particle Lagrangian dynamics. This analysis is useful to introduce the concept of synchronous and asynchronous Lagrangian variational principles, to be extended also to classical field theory. In this regard, we shall denote by \( r^\mu (s) \) the Lagrangian world-line trajectory of a charged point particle with rest mass \( m_o \), charge \( q_o \) and proper time \( s \), so that the corresponding 4-velocity is \( u^\mu (s) = \frac{dr^\mu (s)}{ds} \), while

\[
ds^2 = g_{\mu \nu} (r) dr^\nu (s) dr^\mu (s).
\]

(31)

Here the metric tensor \( g_{\mu \nu} (r) \) and the Faraday tensor \( F_{\mu \nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \) of the external EM fields are considered prescribed functions of \( r \), namely extremal in the sense indicated in Section I, while omitting for brevity in this section the barred notation.

For convenience, we start recalling the standard notions of synchronous and asynchronous variations in the context of single-particle dynamics in GR. For this purpose, first we consider the customary asynchronous principle, which can be found for example in Ref.[2]. The action functional in this case is identified with

\[
S_{pA}(r) = - \int_{s_1}^{s_2} ds \left( g_{\mu \nu} (r(s)) \frac{dr^\nu (s)}{ds} + q A_\mu (r(s)) \right) \frac{dr^\mu (s)}{ds},
\]

(32)

where \( q = \frac{q_o}{m_o c} \) is the normalized charge and \( s_1 \) and \( s_2 \) are fixed boundary values. In the functional \( S_{pA}(r) \), the function \( r^\mu (s) \) is assumed to belong to the asynchronous functional class:

\[
\{ r^\mu \}_A = \left\{ \begin{array}{l}
\r^\mu (s) \in C^2 (\mathbb{R}) \\
\delta (ds) \neq 0 \\
r^\mu (s_k) = r^\mu_k, \, k = 1, 2 \end{array} \right\},
\]

(33)

where in particular we require

\[
\delta (ds) = \delta \left( g_{\mu \nu} (r(s)) dr^\nu (s) dr^\mu (s) \right).
\]

(34)

Here \( \delta \) denotes the Frechet derivative which, when acting on the vector function \( r^\mu (s) \), is defined simply as the virtual displacement

\[
\delta r^\mu (s) = r^\mu (s) - r^\mu_0 (s),
\]

(35)

where \( r^\mu (s) \) and \( r^\mu_0 (s) \) identify two arbitrary functions belonging to the functional class \( \{ r^\mu \}_A \). In particular, we notice that \( r^\mu_0 (s) \) can be always identified with the extremal curve \( r^\mu_{extr}(s) \), solution of the initial-value problem associated with the Euler-Lagrange equations given below. Hence, the well-known asynchronous Hamilton variational principle is recovered. This is given by the variational equation

\[
\delta S_{pA} (r) \equiv \left. \frac{d}{d\alpha} \Psi (\alpha) \right|_{\alpha = 0} = 0,
\]

(36)

to hold for arbitrary displacements \( \delta r^\mu (s) \). Here \( \Psi (\alpha) \) is the smooth real function \( \Psi (\alpha) = S_{pA}(r + \alpha \delta r) \), being \( \alpha \in [-1, 1] \) to be considered independent of \( r(s) \) and \( s \). The corresponding Euler-Lagrange equation becomes

\[
\frac{\delta S_{pA}(r)}{\delta r^\mu (s)} = g_{\mu \nu} \frac{D}{Ds} \frac{dr^\nu (s)}{ds} - q F_{\mu \nu} \frac{dr^\nu (s)}{ds} = 0.
\]

(37)
Let us now consider the corresponding synchronous variational principle, which can be found for example in Refs. [8, 10] and has been adopted also in Refs. [11–16] for the treatment of the non-local EM interaction characterizing extended particle dynamics in the presence of EM radiation-reaction phenomena (see also discussion below). However, for the illustration of the theory and consistent with the purpose of this section, in the following we restrict to the treatment of local interactions occurring for point-like classical particles. In this case, the functional is expressed in terms of superabundant variables \( r^\mu (s) \) and \( u^\mu (s) \) and is identified with

\[
S_{ps}(r, u) = - \int_{s_1}^{s_2} ds L_{ps} \left( r(s), \frac{dr(s)}{ds}, u(s) \right),
\]

where \( L_{ps} \) is the 4–scalar Lagrangian

\[
L_{ps} \left( r(s), \frac{dr(s)}{ds}, u(s) \right) \equiv (u_\mu (s) + qA_\mu (r(s))) \frac{dr^\mu (s)}{ds} - \frac{1}{2} u^\mu (s) u_\mu (s),
\]

which is linear in \( \frac{dr(s)}{ds} \). In addition, the functions \( r^\mu (s) \) and \( u^\mu (s) \) are required to belong to the synchronous functional class defined as

\[
\{r^\mu, u^\mu\}_s = \left\{ \begin{array}{l} r^\mu(s), u^\mu(s) \in C^2 (\mathbb{R}) \\
\delta(ds) = 0 \\
r^\mu(s_k) = r^\mu_k, \ k = 1, 2 \\
u^\mu(s_k) = u^\mu_k, \ k = 1, 2 \end{array} \right\}. \tag{39}
\]

Notice that here the generic functions \( u^\mu (s) \) in \( \{r^\mu, u^\mu\}_s \) are not required to satisfy the kinematic constraint \( u^\mu (s) u_\mu (s) = 1 \), while the line element \( ds \) is by construction required to be determined by Eq. (31) in which \( r^\mu (s) \) is an extremal curve (see definition below). Furthermore, here \( \delta \) denotes again the variation operator which, when acting on the functions \( r^\mu (s) \) and \( u^\mu (s) \), determines the position and velocity virtual displacements

\[
\delta r^\mu(s) = r^\mu(s) - r^\mu_{extr}(s), \tag{40}
\delta u^\mu(s) = u^\mu(s) - u^\mu_{extr}(s), \tag{41}
\]

where \( u^\mu_{extr}(s) \equiv \frac{d}{ds} r^\mu_{extr}(s) \), and \( r^\mu_{extr}(s) \) denotes the extremal curve. Here, \( r^\mu(s) \) and \( u^\mu(s) \) are considered independent, so that \( \delta r^\mu(s) \) and \( \delta u^\mu(s) \) are independent too. In this case it is immediate to show that the corresponding synchronous Hamilton variational principle takes the form

\[
\delta S_{ps}(r, u) \equiv \left. \frac{d}{d\alpha} \Psi(\alpha) \right|_{\alpha=0} = 0, \tag{42}
\]

to hold for arbitrary independent displacements \( \delta r^\mu(s) \) and \( \delta u^\mu(s) \). Here \( \Psi(\alpha) \) is the smooth real function \( \Psi(\alpha) = S_{ps}(r+\alpha \delta r, u+\alpha \delta u) \), being \( \alpha \in [-1,1] \) to be considered independent of \( r(s), u(s) \) and \( s \). In this case the corresponding Euler-Lagrange equations deliver

\[
\frac{\delta S_{ps}(r, u)}{\delta r^\mu(s)} \equiv \frac{D}{Ds} u_\mu - qF_{\mu\nu} u^\nu = 0, \tag{43}
\frac{\delta S_{ps}(r, u)}{\delta u^\mu(s)} \equiv u_\mu - g_{\mu\nu} \frac{dr^\nu(s)}{ds} = 0, \tag{44}
\]

which can be combined to recover Eq. (37) and imply also the kinematic constraint (mass-shell constraint) \( u^\mu (s) u_\mu (s) = 1 \). Notice that Eqs. (43) and (44) determine the extremal curves \( r^\mu(s) \) and \( u^\mu(s) \) which belong to the functional class \( \{r^\mu, u^\mu\}_s \) and are solutions of the same equations.

Introducing for the particle state the symbolic representation \( \mathbf{x} \equiv \{r^\mu, u^\mu\} \), it is possible to cast Eqs. (43) and (44) in the equivalent Lagrangian form as

\[
\frac{d}{ds} \mathbf{x}(s) = \mathbf{X}(\mathbf{x}(s)), \tag{45}
\]

where \( \mathbf{X}(\mathbf{x}(s)) \) is the vector field

\[
\mathbf{X}(\mathbf{x}(s)) \equiv \{u^\mu, G^\mu\}. \tag{46}
\]
In the case of EM interactions considered above, the 4-vector $G^{\mu}$ is given by $G^{\mu} \equiv qF^{\mu \nu}u^\nu$, so that $X(x(s))$ is conservative, namely

$$\frac{\partial}{\partial x} \cdot X(x) \equiv \frac{\partial}{\partial u^\mu}G^{\mu} = 0. \quad (47)$$

The following remarks are in order regarding the comparison between the two variational approaches given in this section:

1) A basic feature of the synchronous approach lies in the adoption of superabundant variables from the start.

2) The two approaches differ for the treatment of the line element $ds$, which is assumed to be held fixed in the synchronous principle, in the sense that $\delta ds = 0$.

3) An alternative possible definition for the synchronous functional $S_{pS}$ can be achieved based on a constrained variational principle, in terms of a Lagrange-multiplier approach which warrants that the kinematic constraint for $u^\mu(s)$ is satisfied by the extremal curves only.

4) The mass-shell constraint acting on the set of superabundant variables is satisfied identically only by the extremal curves and not by generic curves of $\{r^\mu, u^\mu\}$ $S$. Therefore, Eq. (12) should be regarded in a proper sense as an unconstrained variational principle.

5) Both for the synchronous and asynchronous functionals considered above, the Lagrangians are 4-scalars which are defined up to an arbitrary but suitable gauge function, namely an exact differential. Furthermore, the corresponding Euler-Lagrange equations are manifestly covariant. Therefore, in both cases a standard variational Lagrangian formulation exists, which satisfies both the principle of manifest covariance at all levels and gauge symmetry.

6) A further notable property of the synchronous variational principle (12) to be mentioned concerns the possibility of introducing arbitrary extended phase-space transformations of the particle state $x \equiv \{r^\mu, u^\mu\}$. This is realized by a diffeomorphism of the form

$$x(s) \rightarrow z(s) \equiv z(x(s)), \quad (48)$$

$$z(s) \rightarrow x(s) \equiv x(z(s)). \quad (49)$$

Notice that, when a synchronous variation is performed on a generic curve $x(s)$, a synchronous variation is generated on $z(s)$, and vice versa. Then it follows that, denoting $S_{pS}(r, u) \equiv S_{pS}(x)$ and introducing the transformed functional

$$S_{pS}(x(z)) = \hat{S}_{pS}(z), \quad (50)$$

the two synchronous variational principles $\delta S_{pS}(x) = 0$ and $\delta \hat{S}_{pS}(z) = 0$, to hold respectively for arbitrary variations $\delta x(s)$ and $\delta z(s)$, are manifestly equivalent. The variables $z(s)$ are denoted as hybrid variables, because they can differ from customary Lagrangian coordinates and related conjugate momenta. As a consequence, it is always possible to cast the synchronous Hamilton variational principle in hybrid form. This feature is of basic importance, for example, for the construction of gyrokinetic theory [17–21].

A point worth to be further discussed here concerns the case of non-local interactions investigated in Refs. [11–16]. These include EM interactions arising in N-body systems of point-like relativistic particles as well as the so-called EM radiation-reaction self-force in extended particles. In all such cases, the adoption of the synchronous Hamilton variational principle is mandatory, since it is the only one that permits to recover standard Lagrangian and Hamiltonian formulation for relativistic particle dynamics (regarding the precise definition of these notions we refer in particular to Ref. [12]).

Finally, it is worth pointing out that a strong physical motivation behind the adoption of synchronous rather than asynchronous variational principles actually exists in this context. In fact, as pointed out for example in Ref. [10], only in the first case a Hamiltonian variational formulation is achievable.

**VI. SYNCHRONOUS LAGRANGIAN VARIATIONAL PRINCIPLES FOR THE EINSTEIN EQUATIONS IN VACUUM**

Let us now pose the problem of the construction of a Lagrangian variational principle, which holds for the Einstein equations in vacuum and satisfies the Requirements #1–#3 indicated above. We intend to show that the goal is realized by imposing, in analogy with the discussion given on relativistic particle dynamics, that the invariant 4-volume element $d\Omega \equiv d^4r\sqrt{-g}$ is kept fixed (i.e., prescribed) during arbitrary variations performed on the action functional, with $d^4r$ identifying the corresponding configuration-space volume element. As a consequence, in contrast to the variational approaches considered in Section III, in this case $d\Omega$ must remain constant when variations of the fields (i.e., generalized coordinates) are performed. This feature departs from standard approaches used in previous
literature for continuum field theories, where instead the factor $\sqrt{-g}$ is considered variational and hence the invariant volume element is not preserved by the functional variations. In continuum field theory, the two features are formally analogous to particle dynamics (see discussion in Section V) as far as the treatment of the line element $ds$ is concerned. For this reason, the two possible routes in which $d\Omega$ is preserved or not during variations (in the sense indicated above) are referred to here respectively as *synchronous and asynchronous Lagrangian variational principles*.

In order to determine a possible realization of the new synchronous approach, as a starting point we consider invoking proposition T1 of the same theorem, we now identify the functional class $\{Z\}$ for the continuum Lagrangian coordinates with the synchronous functional class

$$\{Z\}_{E-S} \equiv \begin{cases} Z_1 (r) \equiv g_{\mu\nu} (r) \\ \hat{Z}_1 (r) \equiv \hat{g}_{\mu\nu} (r) \\ \hat{Z}_2 (r) \equiv \hat{R}_{\mu\nu} (r) \\ Z (r) , \hat{Z} (r) \in C^k (\mathbb{D}^4) \\ g_{\mu\nu} (r) |_{\partial \mathbb{D}^4} = g_{\mu\nu\hat{0}} (r) \\ \delta \hat{Z} (r) = 0 \\ \delta (d\Omega) = 0 \\ g^\mu_\nu = \hat{g}^\alpha_\beta \hat{\gamma}^\beta_\nu \\ g^\mu_\nu = \hat{g}^\alpha_\beta \hat{\gamma}^\beta_\nu \\ \nabla_\alpha \equiv \nabla_\alpha \end{cases}.$$  \hspace{1cm} (51)

Here the notation is as follows. First, $Z_1 (r) \equiv g_{\mu\nu} (r)$ is the Lagrangian variational field. Second, $\hat{Z}_1 (r) \equiv \hat{g}_{\mu\nu} (r)$ and $\hat{Z}_2 (r) \equiv \hat{R}_{\mu\nu} (r)$ are respectively a prescribed metric tensor (to be defined below) and the corresponding prescribed Ricci tensor, namely the Ricci tensor determined in terms of the same $\hat{g}_{\mu\nu} (r)$. Similarly, $\nabla_\alpha$ denotes the covariant derivative in which the connections are identified with the extremal Christoffel symbols evaluated in terms of $\hat{g}_{\mu\nu} (r)$. This implies in particular that the variations $\delta \hat{Z}$ must vanish identically, namely

$$\delta \hat{Z} \equiv 0.$$  \hspace{1cm} (52)

In detail, the following additional assumptions are made:

1. First, $d\Omega \equiv d^4r \sqrt{-g}$, where $\hat{g}$ is the determinant of $\hat{g}_{\mu\nu} (r)$, so that $\delta (d\Omega) = 0$. For this reason $\delta$ is referred to here as the *synchronous variation operator*.

2. Second, $\hat{g}_{\mu\nu}$ and $\hat{g}^{\mu\nu}$ are the covariant and contravariant components of the same 2-rank tensor. By construction we assume that: a) $\hat{g}_{\mu\nu}$ and $\hat{g}^{\mu\nu}$ lower and raise tensorial indices, so that necessarily $\hat{g}_{\mu\nu} \hat{g}^{\nu\beta} = \delta^\beta_\alpha$; b) they satisfy the differential constraints $\nabla_\alpha \hat{g}_{\mu\nu} = 0$ and $\nabla_\alpha \hat{g}^{\mu\nu} = 0$; c) their synchronous variations vanish identically, namely $\delta \hat{g}_{\mu\nu} = 0$ and $\delta \hat{g}^{\mu\nu} = 0$. In other words, $\hat{g}_{\mu\nu}$ and $\hat{g}^{\mu\nu}$ are considered prescribed, namely held fixed during arbitrary synchronous variations.

3. The operator $\delta$ acts on the variational function $g^{\mu\nu}$ of $\{Z\}_{E-S}$ in such a way to preserve its tensorial character, namely so that

$$\delta \hat{g}^{\mu\nu} (r) = g^{\mu\nu} (r) - g^{\mu\nu}_1 (r)$$  \hspace{1cm} (53)

is a 2-rank tensor. This means that the difference $g^{\mu\nu} (r) - g^{\mu\nu}_1 (r)$ must be considered as infinitesimal, with $g^{\mu\nu} (r)$ and $g^{\mu\nu}_1 (r)$ being two arbitrary functions belonging to $\{Z\}_{E-S}$.

4. Introducing a functional of the form

$$S_1 (Z, \hat{Z}) = \int_{\mathbb{D}^4} d\Omega L_1 (Z, \hat{Z}),$$  \hspace{1cm} (54)

we denote as *synchronous variation* $\delta S_1 (Z, \hat{Z})$ the corresponding *synchronous Frechet derivative*, namely

$$\delta S_1 (Z, \hat{Z}) \equiv \frac{d}{d\alpha} \Psi (\alpha) \bigg|_{\alpha = 0},$$  \hspace{1cm} (55)

where $\Psi (\alpha)$ is the smooth real function defined as $\Psi (\alpha) = S_1 (Z + \alpha \delta Z, \hat{Z})$, with $\alpha \in [-1, 1]$ to be considered independent of both $\hat{Z}$ and $r^\nu$. 


Then, the following result applies.

**THM.2 - Synchronous variational principle (vacuum Einstein equations)**

Let us introduce the modified Einstein-Hilbert Lagrangian density, to be identified with 4-scalar variational Lagrangian density

\[ L_1 (Z, \hat{Z}) = L_{EH-S} (Z, \hat{Z}) h (Z, \hat{Z}), \]  

with \( Z, \hat{Z} \) belonging to synchronous functional class \( \{ Z \}_{E-S} \) and the corresponding action functional to be referred to as modified Einstein action. Here the notation is as follows:

A) \( L_{EH-S} (Z, \hat{Z}) \) is the vacuum Einstein-Hilbert Lagrangian density expressed in the functional setting \( \{ Z \}_{E-S} \) and thus prescribed as

\[ L_{EH-S} (Z, \hat{Z}) = -\frac{\alpha^3}{16\pi G} g^\mu\nu \hat{R}_\mu\nu. \]  

B) Furthermore, \( h (Z, \hat{Z}) \) is the 4-scalar correction factor

\[ h (Z, \hat{Z}) \equiv \left( 2 - \frac{1}{4} g_{\alpha\beta} g_{\alpha\beta} \right), \]

where in the functional setting \( \{ Z \}_{E-S} \) by definition \( g_{\alpha\beta} = \hat{g}_{\alpha\mu} \hat{g}_{\beta\nu} g^{\mu\nu} \).

Then, the following propositions hold:

T2₁) Let us introduce the modified Einstein action \( S_1 (Z, \hat{Z}) \) defined in terms of Eqs.(54) and (56), and the corresponding synchronous variation \( \delta S_1 (Z, \hat{Z}) \). Then, the synchronous variational principle

\[ \delta S_1 (Z, \hat{Z}) = 0, \]

holding for arbitrary synchronous variations \( \delta g^{\mu\nu}(r) \), determines the manifestly covariant symbolic Euler-Lagrange equation

\[ \frac{\partial L_1 (Z, \hat{Z})}{\partial g^{\mu\nu}} = 0, \]

whose solution identifies the extremal field \( g^{\mu\nu}(r) = \overline{g}^{\mu\nu}(r) \). The previous equation then coincides with the vacuum Einstein equations upon requiring that \( \hat{g}^{\mu\nu}(r) \) coincides identically with \( \overline{g}^{\mu\nu}(r) \), namely \( \hat{g}^{\mu\nu}(r) = \overline{g}^{\mu\nu}(r) \).

T2₂) The variational Lagrangian density \( L_1 (Z, \hat{Z}) \) is gauge symmetric, namely it is defined up to arbitrary gauge transformations of the type

\[ L_1 \to L_1 + C^\alpha (Z), \]

\[ L_1 \to L_1 + \nabla_\alpha C^\alpha (Z). \]

**Proof -** The proof of T2₁ follows by explicit evaluation of the synchronous Frechet derivative (see definition above) in the functional class \( \{ Z \}_{E-S} \). In detail, one has that by construction the synchronous variation of the modified Einstein action is just:

\[ \delta S_1 (Z, \hat{Z}) = \int_{\mathbb{S}^4} d\Omega \delta L_1 (Z, \hat{Z}), \]

while

\[ \delta L_1 (Z, \hat{Z}) = h (Z, \hat{Z}) \delta L_{EH-S} (Z, \hat{Z}) + L_{EH-S} (Z, \hat{Z}) \delta h (Z, \hat{Z}). \]

Since in the functional class \( \{ Z \}_{E-S} \) the Ricci tensor is held fixed, it follows that the only terms which contribute explicitly to the variations of the functional \( \delta S_1 (Z, \hat{Z}) \) can be expressed in the form:

\[ \delta S_1 (Z, \hat{Z}) = \int_{\mathbb{S}^4} d\Omega \delta g^{\mu\nu}(r) [A + B] = 0, \]
where respectively

\[ A \equiv h\left(Z, \hat{Z}\right) \frac{\partial L_{EH-S}\left(Z, \hat{Z}\right)}{\partial \delta g^{\mu\nu}}, \]

(66)

\[ B \equiv L_{EH-S}\left(Z, \hat{Z}\right) \frac{\partial h\left(Z, \hat{Z}\right)}{\partial \delta g^{\mu\nu}}. \]

(67)

Explicit calculation gives

\[ \frac{\partial L_{EH-S}\left(Z, \hat{Z}\right)}{\partial \delta g^{\mu\nu}} = \hat{R}_{\mu\nu}, \]

(68)

\[ \frac{\partial h\left(Z, \hat{Z}\right)}{\partial \delta g^{\mu\nu}} = -\frac{1}{2} \delta g^{\mu\nu}. \]

(69)

Due to the arbitrariness of \( \delta g^{\mu\nu}(r) \) then it follows

\[ h\left(Z, \hat{Z}\right) \hat{R}_{\mu\nu} - \frac{1}{2} \left(g^{\alpha\beta}\hat{R}_{\alpha\beta}\right) g_{\mu\nu} = 0. \]

(70)

This equation coincides with the vacuum Einstein equations provided \( g_{\mu\nu}(r) = \hat{g}_{\mu\nu}(r) = \hat{g}_{\mu\nu}(r) \), since by construction \( h\left(Z_1 \equiv \hat{Z}_1 = \hat{g}^{\alpha k}(r)\right) = 1 \) in such a case.

The proof of T2 is straightforward in the case the gauge function is identified with the constant \( C \), because synchronous variations of a constant always vanish identically. Second, in Eq.(62) the covariant derivative must be considered as prescribed in terms of the fixed metric tensor. This condition is mandatory because in the synchronous principle the invariant volume element depends on the same prescribed metric tensor, which is held fixed too. This represents a consistency condition for the synchronous approach and warrants the validity of the Gauss theorem, so that one can write

\[ \hat{\nabla}_\alpha C^\alpha(Z) = \frac{1}{\sqrt{-g}} \frac{\partial}{\partial x^\alpha} \left( \sqrt{-g} C^\alpha(Z) \right). \]

(71)

Therefore, the gauge invariance property follows as a direct consequence of the boundary conditions imposed on the generalized coordinates \( \{Z\} \).

Q.E.D.

As an immediate consequence of THM.2, we can conclude that the synchronous variational principle consistently realizes the Requirements #1-#3 posed in Section III. This recovers the correct gauge invariance properties of the theory and overcomes at once the lack of gauge invariance characteristic of all asynchronous principles displayed above. In particular, in such a framework, one has that \( L_1\left(Z, \hat{Z}\right) = L_1\left(Z, \hat{Z}\right) \) also for the gravitational field. According to the notation introduced here, this implies that the corresponding variational principle defines a standard Lagrangian variational approach.

It is important to remark that an equivalent synchronous variational principle can in principle be obtained also adopting different choices both for the functional class \( \{Z\}_{E-S} \) as well as the variational action functional. A possible example is obtained by extending the functional class \( \{Z\}_{E-S} \) in such a way that \( g^{\alpha\beta}(r) \) and \( g_{\alpha\beta}(r) \) are treated as independent superabundant Lagrangian coordinates, so that it is not necessary to raise and lower indices in the Lagrangian density. This is obtained by identifying the functional class with

\[ \{Z\}_{E-S} = \left\{ \left( Z_1, Z_2 \right) \equiv (g_{\mu\nu}(r), g^{\mu\nu}(r)), \right. \]

\[ \left. \left( \hat{Z}_1, \hat{Z}_2 \right) \equiv (\hat{g}_{\mu\nu}(r), \hat{g}^{\mu\nu}(r)), \right. \]

\[ \left( \hat{Z}_3, \hat{Z}_4 \right) \equiv \left( \hat{R}_{\mu\nu}(r), \hat{R}^{\mu\nu}(r) \right) \}

\[ Z(r), \hat{Z}(r) \in C^k(D^4) \]

\[ g_{\mu\nu}(r)|_{\partial D^4} = g_{\mu\nu}(r) \]

\[ g^{\mu\nu}(r)|_{\partial D^4} = g^{\mu\nu}(r) \]

\[ \delta \hat{Z}(r) = 0 \]

\[ \delta (d\Omega) = 0 \]

\[ \nabla_\alpha \equiv \hat{\nabla}_\alpha \]

(72)
and requiring again that the fields \((Z (r), \tilde{Z} (r))\) are symmetric in the indices \(\mu, \nu\). Similarly, also the variational Lagrangian density must be modified by replacing it with the symmetrized form

\[
L_{1-sym} (Z, \tilde{Z}) \equiv \left( - \frac{c^3}{16\pi G} \right) \frac{1}{2} \left( \tilde{R}_{\mu\nu} g^{\mu\nu} + \tilde{R}^{\mu\nu} g_{\mu\nu} \right) h (Z),
\]  

(73)

where now \(h (Z)\) depends only on variational quantities. Under the same assumptions holding for THM.2, the resulting Euler-Lagrange equations coincide with the mutually-equivalent covariant and contravariant representations of the vacuum Einstein equations.

In view of THM.2, it is worth pointing out how the synchronous variational principles determined here can be extended to take into account the presence of a non-vanishing cosmological constant \(\Lambda\). In such a case the extremal vacuum Einstein equations become

\[
\tilde{R}_{\mu\nu} - \frac{1}{2} \tilde{g}_{\mu\nu} + \Lambda \tilde{g}_{\mu\nu} = 0.
\]

(74)

The synchronous variational Lagrangian density to be adopted becomes correspondingly

\[
L_{1\Lambda} \equiv L_1 (Z, \tilde{Z}) - 2\Lambda h (Z, \tilde{Z}).
\]

(75)

In a similar way it is possible in principle to adopt a synchronous form of the variational principle to treat also so-called metric \(f (R)\)-models considered in modified theories of GR [22].

To conclude this section, a discussion concerning the physical meaning of the synchronous variational principle is proposed. First, it must be remarked that the latter is actually based on the introduction of superabundant variables. More precisely, this is realized by allowing the prescribed metric tensor \(\tilde{g}_{\beta\gamma}\) to be independent both of the variational and extremal metric tensors \(g^{\beta\gamma}\) and \(\tilde{g}^{\beta\gamma}\), the latter being determined as a solution of the Einstein equations. The physical interpretation of such a representation is as follows: 1) \(g_{\mu\nu}\) is a physical continuum field, whose dynamical equations are determined by the Euler-Lagrange equations following from the synchronous variational principle. In this sense, \(g_{\mu\nu}\) has no geometrical interpretation, namely it does not raise or lower indices nor it appears in the prescribed 4-volume element (for the same consistency of the synchronous principle) or in the prescribed Ricci tensor \(\tilde{R}_{\mu\nu}\). 2) \(\tilde{g}_{\mu\nu}\) plays the role of a geometrical continuum field in the variational functional. It determines a number of geometric properties: the invariant 4-volume element, the covariant derivatives, the Ricci tensor \(\tilde{R}_{\mu\nu}\), and finally it raises/lowers tensor indices. On the other hand, the physical and geometrical characters which distinguish the two fields are reconciled when the extremal condition \(g_{\mu\nu} (r) = \tilde{g}_{\mu\nu} (r) = \tilde{g}_{\mu\nu} (r)\) is set in the Euler-Lagrange equations.

\section*{VII. SYNCHRONOUS VARIATIONAL PRINCIPLES FOR THE NON-VACUUM EINSTEIN EQUATIONS}

THM.2 provides a unique recipe for constructing also the appropriate form of the field Lagrangian \(L_{E}\) which carries the non-vacuum source in the Einstein equations. The issue here is to determine however the correct representation for \(L_{E}\) which is consistent with the synchronous variational principle, preserves the correct definition of the corresponding stress-energy tensor and at the same time does not modify the form of the external source field equations. Therefore, the procedure should provide in principle, besides the non-vacuum Einstein equations, also a joint synchronous variational principle for classical matter and the EM field.

In the literature, the Lagrangian density of the EM field is identified with

\[
L_{EM} (A_{\mu}, g_{\mu\nu}) = - \frac{1}{16\pi c} F_{\mu\nu} F^{\mu\nu},
\]

(67)

where \(F_{\mu\nu} = \partial_{\mu} A_{\nu} - \partial_{\nu} A_{\mu}\). This representation is consistent with the assumptions underlying the asynchronous principle, and in particular the requirement that the variational \(g_{\mu\nu}\) lowers and raises indices. In the context of the synchronous variational principle instead, the variational \(g_{\mu\nu}\) cannot be used to lower and raise indices. Therefore, the corresponding synchronous Lagrangian density becomes

\[
L_{EM-S} (A_{\mu}, g_{\mu\nu}) = - \frac{1}{16\pi c} g^{\mu\alpha} g^{\nu\beta} F_{\mu\nu} F_{\alpha\beta}.
\]

(77)

In addition we identify the field Lagrangian density \(L_{E}\) with

\[
L_{E} = L_{EM-S} + L_{V},
\]

(78)
where $L_V$ is a generic matter Lagrangian density, which provides the sources for both the gravitational and EM fields. Its representation will be specified below in Section VIII in terms of a kinetic Vlasov description of source matter.

Here we first pose the problem of the consistent treatment of both the variational ($L_{1F}$) and field ($L_F$) Lagrangians in the context of the synchronous variational formulation developed in THM.2. The following proposition holds.

**THM.3 - Stress-energy tensor in the synchronous variational principle**

Given validity of THM.2, the variational Lagrangian density $L_{1F}$, which carries the contribution of the external sources, in the context of the synchronous variational principle is given by the 4–scalar

$$L_{1F} (Z, \hat{Z}) = L_F (Z, \hat{Z}) h (Z, \hat{Z}),$$

(79)

with $L_F (Z, \hat{Z})$ being of the form given by Eq. (78). The corresponding action integral is therefore

$$S_{1F} (Z, \hat{Z}) = \int_{\Sigma^4} d\Omega L_{1F} (Z, \hat{Z}).$$

(80)

As a consequence, the variational derivative $\frac{\delta S_{1F}(Z, \hat{Z})}{\delta g_{\mu\nu}}$ must give the correct extremal stress-energy tensor entering the non-vacuum Einstein equations, which are defined as

$$T_{\mu\nu} (r) = -2 \frac{\partial L_{1F} (Z, \hat{Z})}{\partial g_{\mu\nu}} + g_{\mu\nu} L_{1F} (Z, \hat{Z}).$$

(81)

Therefore, for the consistency of the synchronous variational principle with the same Einstein equations, it must be

$$T_{\mu\nu} (r) = -2 \frac{\delta S_{1F} (Z, \hat{Z})}{\delta g_{\mu\nu}}.$$

(82)

**Proof** - Invoking the definition and evaluating the Frechet derivative, one obtains

$$\delta S_{1F} (Z, \hat{Z}) = \int_{\Sigma^4} d\Omega \delta L_{1F} (Z, \hat{Z})$$

$$= \int_{\Sigma^4} d\Omega \left[ L_F (Z, \hat{Z}) \delta h (Z, \hat{Z}) + h (Z, \hat{Z}) \delta L_F (Z, \hat{Z}) \right],$$

(83)

where respectively

$$\delta h (Z, \hat{Z}) = - \frac{1}{2} g_{\mu\nu} \delta g^{\mu\nu},$$

(84)

$$\delta L_F (Z, \hat{Z}) = \frac{\partial L_F (Z, \hat{Z})}{\partial g^{\mu\nu}} \delta g_{\mu\nu}. $$

(85)

The proof then follows by elementary algebra by recalling that, after performing the partial derivatives, the extremal value of $h (\mathbf{g}_{\mu\nu}) = 1$ must be adopted.

Q.E.D.

As a consequence of THM.3, the synchronous variational principle with the definition preserves the correct form of the stress-energy tensor.

**VIII. KINETIC DESCRIPTION OF MATTER SOURCE**

In this section we present an application of the theory of synchronous variational principles developed in the previous sections. The study-case considered concerns the treatment of matter source by means of a kinetic description and its application to the non-vacuum variational principle for the Einstein equations. For definiteness, we consider here a general configuration corresponding to a physical system represented by a multi-species plasma or a neutral matter. In the framework of a Vlasov description of collisionless systems, possible realizations are provided by kinetic
equilibria arising in relativistic plasmas [20, 21, 22], dark matter halos [24], accretion-disc plasmas [5, 6, 19, 25–30] and relativistic plasmas subject to EM radiation-reaction effects [12, 31, 32].

To start with, we notice that the (1-particle) kinetic distribution function (KDF) $f(x, \mathbf{\gamma}^\alpha \beta(r), \mathbf{x}_r(r)) \equiv f(x)$ must be considered as held fixed (i.e., prescribed) in the context of a variational formulation for the non-vacuum Einstein and Maxwell equations, since a variational formulation of the Vlasov equation is not generally achievable when both $g_{\mu \nu}$ and $A_\mu$ are non-extremal fields in the corresponding functional dependence. Nevertheless:

1) As shown here, both the 4-current $J^\mu(r)$ and the stress-energy tensor produced by the Vlasov source $\Pi_{\mu \nu}(r)$ can still be considered variational in a proper sense, by a suitable identification of the corresponding Vlasov field Lagrangian $L_V = L_V\left(Z, \tilde{Z}, f\right)$.

2) We intend to point out that in such a context also the KDF can be determined via a prescribed variational principle.

In particular, let us recall first the definitions of the extremal 4-current $J^\mu(r)$ and stress-energy tensor $\Pi_{\mu \nu}(r)$ in terms of the kinetic distribution function $f(x)$ [33]. The covariant components are provided by

$$J_\mu(r) = \sum_{\text{species}} q_\alpha c \int_{\mathbb{U}_4^4} d\eta \delta(u^\alpha u_\alpha - 1) \Theta(u^0) u_\mu f(x), \quad (86)$$

$$\Pi_{\mu \nu}(r) = \sum_{\text{species}} m_\alpha c^2 \int_{\mathbb{U}_4^4} d\eta \delta(u^\alpha u_\alpha - 1) \Theta(u^0) u_\mu u_\nu f(x), \quad (87)$$

where $d\eta \equiv d^4u_\alpha/\sqrt{-g}$ is the invariant 4-velocity volume element, with $d^4u_\alpha \equiv \prod_{i=0,3} du_i$ and $\mathbb{U}_4$ is the 4-velocity space.

We now introduce the following definition of the Vlasov functional as

$$S_{1V}\left(Z, \tilde{Z}, f\right) = \int_{\mathbb{U}_4^4} d\Omega L_{1V}\left(Z, \tilde{Z}, f\right), \quad (88)$$

where $L_{1V}\left(Z, \tilde{Z}, f\right)$ is the variational Vlasov Lagrangian density, which in accordance to the prescription given above in THM.3, is taken of the form

$$L_{1V}\left(Z, \tilde{Z}, f\right) = L_V\left(Z, \tilde{Z}, f\right) h\left(Z, \tilde{Z}\right), \quad (89)$$

in which the dependence in terms of the KDF $f$ characteristic of the kinetic Vlasov description is explicitly displayed. In particular, here the Vlasov-source field Lagrangian density $L_V\left(Z, \tilde{Z}, f\right)$ is defined as follows:

$$L_V\left(Z, \tilde{Z}, f\right) = -\sum_{\text{species}} \int_{\mathbb{U}_4^4} d\eta \left[G_{\Pi} + G_J\right] f(x) \delta(u^i u^k \tilde{g}_{jk}(r) - 1), \quad (90)$$

where the two 4-scalars are

$$G_{\Pi} = \frac{m_\alpha c^2}{4} \left[2u_\mu u_\nu g^{\mu \nu}(r) - h\left(Z, \tilde{Z}\right)\right], \quad (91)$$

$$G_J = \frac{1}{c^2} q_\alpha c g^{\mu \nu}(r) u_\nu (r) \left(A_\mu - \tilde{A}_\mu\right), \quad (92)$$

and the last one is defined in terms of the variational and extremal fields $A_\mu$ and $\tilde{A}_\mu$ as indicated, to warrant that $G_J$ is observable. We intend to show that these two variational contributions provide the correct source terms for the Einstein and Maxwell equations.

In view of THMs 2-3 and the definitions for the source Lagrangian $L_V$, it is possible to formulate a single synchronous variational principle holding for classical fields, which determines both the non-vacuum Einstein equations as well as the dynamical equation for the EM 4-potential $A_\mu$ and corresponding to the non-homogeneous Maxwell equation. This is given by the following proposition, which extends the conclusions of THM.2.

**THM.4 - Non-vacuum Einstein-Maxwell synchronous variational principle**

*Given validity of THMs 2 and 3, the total variational Lagrangian density $L_{tot}$, which carries the contributions of the gravitational and EM fields as well as the additional matter and current sources is given by the 4-scalar

$$L_{tot}\left(Z, \tilde{Z}, f\right) = L_1\left(Z, \tilde{Z}\right) + L_{1F}\left(Z, \tilde{Z}, f\right), \quad (93)$$

where*
where \( L_1 \left( Z, \tilde{Z} \right) \) is given by Eq. (56), \( L_{1F} \left( Z, \tilde{Z}, f \right) \) is defined by Eq. (77), with \( L_F \left( Z, \tilde{Z} \right) \) being prescribed by Eq. (78) in which \( L_{EM-S} \) is defined by Eq. (77) and \( L_V \left( Z, \tilde{Z}, f \right) \) by Eq. (30). The Lagrangian \( L_{tot} \left( Z, \tilde{Z}, f \right) \) is prescribed in the following synchronous functional class:

\[
\{ Z \}_{tot} = \begin{cases}
Z_1 (r) = g_{\mu \nu} (r) \\
\dot{Z}_1 (r), \dot{\tilde{Z}}_2 (r) = \left( \dot{g}_{\mu \nu} (r), \dot{\tilde{R}}_{\mu \nu} (r) \right) \\
Z_3 \equiv A_\mu (r) \\
Z (r), \tilde{Z} (r) \in \mathbb{C} \ (\mathbb{D}^4) \\
Z (r) \big|_{g_{\alpha \beta}} = Z_\mathbb{D} (r) \\
\delta \tilde{Z} (r) = 0 \\
\delta (d\Omega) = 0 \\
g_{\mu \nu} = g^{\alpha \beta} g_{\mu \alpha} g_{\nu \beta} \\
g^{\alpha \beta} = g^{\alpha \mu} g^{\beta \nu} g_{\mu \nu} \\
\nabla_\alpha &\equiv \nabla_\alpha \end{cases}
\]  

(94)

where \( A_\mu (r) \) is subject to the Coulomb gauge \( \nabla_\mu A_\mu = 0 \). The corresponding action integral defined in \( \{ Z \}_{tot} \) is therefore

\[
S_{tot} \left( Z, \tilde{Z}, f \right) = \int_{\mathbb{D}^4} d\Omega L_{tot} \left( Z, \tilde{Z}, f \right). 
\]  

(95)

Then, the following propositions hold:

T41) The synchronous variational principle is provided by the Frechet derivative of \( S_{tot} \left( Z, \tilde{Z}, f \right) \), namely \( \delta S_{tot} \left( Z, \tilde{Z}, f \right) = 0 \), to hold for arbitrary independent synchronous variations of the fields \( \delta Z \). In particular, these include:

1) The variations \( \delta g^{\mu \nu} \), which provide the non-vacuum Einstein equations (7) (see THMs 2 and 3).

2) The variations \( \delta A_\mu \), which provide the non-homogeneous Maxwell equations.

T42) The total Lagrangian is defined up to an arbitrary gauge, to be identified either with a constant real \( C \) or an exact differential of the form \( \tilde{\nabla}_\mu C^\nu (Z) \), with \( C^\nu (Z) \) being a real \( 4 \)--vector field.

T43) The conservation law \( \nabla_\mu T^{\mu \nu} (r) = 0 \) and \( \nabla_\mu J^{\mu} (r) = 0 \) are identically satisfied for the extremal fields.

Proof - The proof of T41 is a consequence of THMs 2 and 3. Let us evaluate in particular first the Einstein equations. The vacuum contribution (i.e., the rhs of Eq. (11)) is provided by THM 2. Instead, the non-vacuum contribution coming from \( L_{1F} \) determines the stress-energy tensor of the general form

\[
T^{\mu \nu} (r) = T^{\mu \nu}_{EM} (r) + T^{\mu \nu}_{V} (r). 
\]  

(96)

Here the first term \( T^{\mu \nu}_{EM} (r) \) represents the customary stress-energy tensor associated with the EM field. A straightforward calculation gives in fact

\[
T^{\mu \nu}_{EM} (r) = -2 \left. \frac{\partial \left( L_{EM-S} h \right)}{\partial g^{\mu \nu}} \right|_{ext} = \frac{1}{4\pi} \left( -F_{\mu \nu} F^{\mu \nu} + \frac{1}{4} F^{\alpha \beta} F_{\alpha \beta} g_{\mu \nu} \right), 
\]  

(97)

where on the rhs all quantities must be intended as extremal ones. Instead, \( T^{\mu \nu}_{V} (r) \) arises from the variational Vlasov Lagrangian density \( L_{1V} \left( Z, \tilde{Z}, f \right) \). This gives, after explicit calculation

\[
T^{\mu \nu}_{V} (r) = -2 \left. \frac{\partial \left( L_V h \right)}{\partial g^{\mu \nu}} \right|_{ext} = -2\Pi_{\mu \nu} (r), 
\]  

(98)

where \( \Pi_{\mu \nu} (r) \) is defined by Eq. (57). In particular, we notice that the moment associated with \( G_{\dot{\mu}} \) does not contribute at this stage. This completes the proof for the Einstein equations. The treatment of the Maxwell equation is analogous.

In particular, denoting \( S_{EM-S} \left( Z, \tilde{Z} \right) \equiv \int_{\mathbb{D}^4} d\Omega L_{EM-S} h \left( Z, \tilde{Z} \right) \), the vacuum contribution originated by \( L_{EM-S} \) gives for the extremal \( g^{\mu \nu} (r) \)

\[
\left. \frac{\delta S_{EM-S} \left( Z, \tilde{Z} \right)}{\delta A_\mu (r)} \right|_{\extr} = -\frac{1}{4\pi c} \nabla_\mu F^{\mu \nu} (r). 
\]  

(99)
Instead, the Vlasov source functional gives
\[
\frac{\delta S_{1V}}{\delta A_\mu (r)} = - \frac{1}{c^2} J^\mu (r),
\]
(100)
where \( J^\mu (r) \) is defined by Eq.(86). As a result, the customary form of the non-homogeneous Maxwell equation is recovered, namely
\[
\nabla_\nu F^{\mu \nu} (r) = - \frac{4\pi}{c} J^\mu (r).
\]
(101)
This completes the proof of T4\(_1\). Next, the proof of proposition T4\(_2\) is an immediate consequence of THM.2, which extends manifestly its validity when sources are present. Finally, concerning proposition T4\(_3\), the conservation laws follow from the very structure of the Einstein and Maxwell equations.

Q.E.D.

We point out that in the present approach the form of the KDF has remained arbitrary. This permits in principle the investigation of a variety of physical problems ranging from kinetic equilibria to dynamically-evolving matter distributions.

As a final target, we pose the problem of setting up a variational principle for the Vlasov equation holding in the case of conservative vector fields \( \mathbf{X}(\mathbf{x}) \) (see Eq.(47)). In accordance with the assumption indicated above, we shall assume that the Vlasov kinetic equation holds for each species belonging to the source matter. The treatment of this issue is intrinsically related to the variational description of single-particle dynamics presented in Section V and permits the extension of such a theory to collisionless multi-species systems in the framework of statistical physics.

To start with, we introduce the evolution operator (i.e., the flow \( T_{s-o} \)) associated with the classical dynamical system corresponding to the 1-particle system, which is generated by the dynamical equations (43) and (44). The initial and generic states at proper times \( s_o \) and \( s \) are then related by the following bijection
\[
x(s_o) \equiv x_o \leftrightarrow x(s),
\]
(102)
where in terms of the evolution operator and its inverse
\[
x(s) = T_{s-s_o} x(s_o),
\]
(103)
\[
x(s_o) = T_{s_o-s} x(s).
\]
(104)
Then, for an arbitrary species, the Vlasov equation must hold. In the integral (Lagrangian) form this is expressed as
\[
f(x(s)) = f_o(x(s_o)),
\]
(105)
where \( f_o(x(s_o)) \) is an initial KDF at \( s = s_o \). The implications of Eq.(105) are straightforward. First we notice that such an equation is defined only on the mass-shell. As a consequence, for all \( x \) belonging to the mass-shell, there exists a phase-space trajectory such that \( x(s) = x \). On the other hand, \( x(s_o) \) can be represented in terms of \( x \) via Eq.(104). Hence, Eq.(105) delivers:
\[
f(x) = f_o(T_{s_o-s} x),
\]
(106)
which uniquely determines \( f(x) \) in terms of the initial KDF \( f_o \). Now, assuming without loss of generality that \( f_o \) is a smooth differentiable function, by differentiating Eq.(105) with respect to the proper time \( s \), one determines the Lagrangian differential Vlasov equation
\[
\frac{d}{ds} f(x(s)) = 0.
\]
(107)
Let us now go back to the functional \( S_{1V}(Z, \tilde{Z}, f) \). Noting that again the KDF is evaluated on the mass-shell, the previous integral representation (105) can be adopted. Hence, for an arbitrary state \( x \), let us consider the Lagrangian phase-space trajectory \( x(s) \) having the initial condition \( x \equiv x(s) \). It follows that the KDF can be represented as
\[
f(x(s)) = \int_{s_o}^{s} ds' \left[ \frac{d}{ds'} f(x(s')) \right] + f_o(x(s_o)).
\]
(108)
where $\Psi(\delta s)$ necessarily be identified with the Frechet derivative and where have been discussed in THM.1. In particular, it has been shown that the Einstein-Hilbert and Palatini variational approaches can be summarized in a variational principles for the Einstein equations, the origins of the aforementioned difficulties have been analyzed. In the variational theory, in which both the Lagrangian variables and the symbolic Euler-Lagrange equations are tensorial in character. The second issue is related to the violation of gauge symmetry arising in the same formulations.

Relativity has been addressed. The motivations of the investigation are related to two different critical aspects of previous literature approaches to the variational formulations of GR. The first one is the lack of a manifestly-covariant dynamical systems, the present variational principle represents an alternative to the approach described in Ref.[34].

To conclude the analysis, we notice that the variational principle (111) defined above is synchronous with respect to $s$.

Hence, the variational principle for the Vlasov equation becomes simply $dS_{1V}(z, f) = 0$. This implies that the variational derivative $\delta S_{1V}(z, f) / \delta s(x)$ recovers the correct Vlasov equation written in Lagrangian form, namely Eq.(107).

To conclude the analysis, we notice that the variational principle (111) defined above is synchronous with respect to the phase-space volume element $d\Omega d\eta$, because the latter remains unchanged during the variation. For conservative dynamical systems, the present variational principle represents an alternative to the approach described in Ref.[34].

IX. CONCLUSIONS

In this paper the problem of formulating synchronous Lagrangian variational approaches in the context of General Relativity has been addressed. The motivations of the investigations are related to two different critical aspects of previous literature approaches to the variational formulations of GR. The first one is the lack of a manifestly-covariant variational theory, in which both the Lagrangian variables and the symbolic Euler-Lagrange equations are tensorial in character. The second issue is related to the violation of gauge symmetry arising in the same formulations.

In this regard, a number of results has been achieved. Starting from the analysis of historical literature on the variational principles for the Einstein equations, the origins of the aforementioned difficulties have been analyzed. In particular, it has been shown that the Einstein-Hilbert and Palatini variational approaches can be summarized in a single theorem realized by means of as constrained variational principle. Its various possible equivalent realizations have been discussed in THM.1.

In order to overcome these limitations, a preliminary discussion has concerned the variational principles in relativistic classical mechanics. In such a context two different variational approaches are available, respectively based...
on asynchronous and synchronous Hamilton variational principles. The basic difference between the two approaches lies on the constraint to be placed on the line element $ds$, which is considered variational in the case of asynchronous principles, and extremal (i.e., prescribed during variations) for synchronous ones. This feature suggests in a natural way its possible extension in the context of continuum classical fields. As shown in this paper, this is achieved by means of the introduction of constrained synchronous variational principles, in which the configuration-space 4–scalar volume element $d\Omega$ is left invariant by this kind of variations. This approach differs from those typically adopted in the literature, in which instead the volume element is not preserved during variations, and are therefore referred to here as asynchronous principles.

A number of related theorems have been proved. The first one (THM.2) deals with the vacuum Einstein equations, for which a synchronous Lagrangian variational principle has been established. As discussed in the paper, this involves necessarily the adoption of a constrained formulation, which warrants the conservation of the 4–scalar volume element. This demands however that also the Christoffel symbols must be considered prescribed, i.e., fixed, during synchronous variations. In turn this implies that both the covariant derivative and the Ricci tensor itself must be considered as prescribed in the same way.

The theorem has several remarkable implications. The first one concerns the physical interpretation of the variational constraints. Indeed, the synchronous principle requires the existence of a prescribed field $\tilde{g}_{\mu\nu}(r)$ which in a sense effectively determines the geometric properties associated with the functional setting, namely which prescribes the volume element as well as the Ricci curvature tensor and the covariant derivative operator, and at the same time raises and lowers tensor indices. Then, as shown here, the Einstein equations are recovered once the $\tilde{g}_{\mu\nu}(r)$ is identified with the extremal metric tensor. A further physically-meaningful aspect of the theory concerns the fulfillment of both gauge symmetry and the property of manifest covariance of the theory. The analysis of the gauge invariance property of this approach is revealing. This concerns general gauge functions which can be either arbitrary constants or exact differentials. Its validity in fact is a unique consequence of the adoption of the synchronous variational principle, because in such a case the variational Lagrangian and the Lagrangian density coincide and are realized in terms of a 4–scalar field. On the contrary, asynchronous principles to be found in the previous literature are never fully gauge invariant, since the variational Lagrangian density is not a 4–scalar. As a side consequence, this feature prevents the possibility of satisfying the manifest covariance also in the case of symbolic Euler-Lagrange equations stemming from asynchronous variational principles.

As a further interesting development the case of the non-vacuum Einstein and Maxwell equations have been treated in terms of a single synchronous variational principle. For this purpose, first it has been proved that such a principle preserves the correct prescription of the stress-energy tensor expected for the source fields (THM.3). Then, a joint synchronous variational principle has been established both for the Einstein and Maxwell equations. Its basic feature has been shown again to realize the properties of manifest covariance and gauge symmetry even in the presence of classical external sources (THM.4). In this reference, the case of a matter source described in the framework of a Vlasov kinetic theory has been considered and the corresponding Vlasov variational Lagrangian density uniquely determined. As a notable consequence, it has been shown that in terms of the action functional, a variational principle can be achieved also for the Vlasov equation. The latter applies in the case of particle dynamics subject to a conservative 4–vector force.

The theory presented here exhibits intriguing features, providing a novel route for the variational treatment of classical continuum field theory in the context of General Relativity. In authors’ view the new variational approach might be susceptible of developments concerning the Hamiltonian treatment of General Relativity and quantum gravity.

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XI. APPENDIX - CONSTRAINED VARIATIONAL PRINCIPLE FOR THE EINSTEIN EQUATIONS

In this Appendix we report the proof of a theorem concerning an alternative constrained variational principle of asynchronous type for the Einstein equations. This is given by adopting the functional class defined by Eq.(22).

THM.1 - Constrained Einstein-Hilbert variational principle
Let us consider for simplicity and without loss of generality the case of the vacuum Einstein equations. We define the constrained action integral \( S_c \) as

\[
S_c (Z, \overline{Z}) = \int d^4 x \sqrt{-g} \left[ L_c (Z) + \lambda^{\mu \nu} \left( R_{\mu \nu} - \overline{R}_{\mu \nu} \right) \right],
\]

(114)

where \( L_c (Z) \) is defined as

\[
L_c (Z) = - \frac{c^3}{16 \pi G} g^{\mu \nu} R_{\mu \nu},
\]

(115)

with \( R_{\mu \nu} \) and \( g^{\mu \nu} \) to be considered as independent variational functions. In addition, \( \lambda^{\mu \nu} \) is a Lagrange multiplier to be included in \( \{ \lambda \} \) and \( R_{\mu \nu}, \overline{R}_{\mu \nu} \) are the variational and extremal Ricci tensors, the latter being defined in particular by Eq.(2) and evaluated in terms of the extremal metric tensor \( \overline{g}_{\mu \nu} \). Here the functional class is defined as in Eq.(122). Then, the following propositions hold:

T1) The variational principle

\[
\delta S_c (Z, \overline{Z}) = 0,
\]

(116)

to hold for arbitrary variations of the generalized coordinates \( \delta Z \) is such that: a) it yields as extremal equations the vacuum Einstein equations, once \( g_{\mu \nu} = \overline{g}_{\mu \nu} \) is identified with the extremal metric tensor; b) it determines the extremal value \( R_{\mu \nu} \), which must coincide with \( \overline{R}_{\mu \nu} \); c) it yields the extremal value of \( \lambda^{\mu \nu} \) as \( \overline{\lambda}^{\mu \nu} = \frac{c^3}{16 \pi G} \overline{g}^{\mu \nu} \).

T12) The action functional can be equivalently replaced with

\[
S_{E-c} (Z, \overline{Z}) = \int d^4 x \sqrt{-g} L_{E-c} (Z, \overline{Z}),
\]

(117)

to be denoted as constrained Einstein action, where now

\[
L_{E-c} (Z, \overline{Z}) = - \frac{c^3}{16 \pi G} g^{\mu \nu} \overline{R}_{\mu \nu},
\]

(118)

and the functional class becomes

\[
\{ Z \}_E \equiv \left\{ \begin{array}{l}
Z (r) \equiv g_{\mu \nu} (r) \\
\overline{Z} (r) \equiv \overline{R}_{\mu \nu} \\
Z (r), \overline{Z} (r) \in C^k (D^4) \\
Z (r)_{|_{\partial D^4}} = Z (r) \\
\delta \overline{Z} (r) = 0 \\
\delta (d\Omega) = d^4 x \delta (\sqrt{-g}) \neq 0
\end{array} \right\},
\]

(119)

with \( k \geq 3 \), while both \( g_{\mu \nu} (r) \) and \( \overline{R}_{\mu \nu} \) are assumed symmetric in the indices \( \mu, \nu \).

Proof – Let us start from proposition T11. The variation of the Lagrange multiplier gives

\[
\frac{\delta S_c (Z, \overline{Z})}{\delta \lambda^{\mu \nu}} = R_{\mu \nu} - \overline{R}_{\mu \nu} = 0,
\]

(120)

which provides the representation for the extremal Ricci tensor in terms of the Christoffel symbols carried by \( \overline{R}_{\mu \nu} \). Then, the independent variation with respect to \( R_{\mu \nu} \) yields

\[
\frac{\delta S_c (Z, \overline{Z})}{\delta R_{\mu \nu}} = - \frac{c^3}{16 \pi G} g^{\mu \nu} + \lambda^{\mu \nu} = 0,
\]

(121)

which determines the extremal value of the Lagrange multiplier. Finally, the independent variation with respect to \( g^{\mu \nu} \) gives

\[
\frac{\delta S_c (Z, \overline{Z})}{\delta g^{\mu \nu}} = \partial \left[ \sqrt{-g} L_c (Z) \right] = 0,
\]

(122)

where we have already taken into account the validity of Eq.(120). This equation can be written explicitly as

\[
\frac{\partial \left[ \sqrt{-g} L_c (Z) \right]}{\partial g^{\mu \nu}} = \overline{R}_{\mu \nu} - \frac{1}{2} (g^{\alpha \beta} \overline{R}_{\alpha \beta}) g_{\mu \nu} = 0,
\]

(123)
which recovers the vacuum Einstein equations upon identifying $g_{\mu\nu}$ with the extremal metric tensor. This completes the proof of $T1_1$.

As for proposition $T1_2$, one obviously has

$$
\frac{\delta S_{E-E}}{\delta g^{\mu\nu}}(Z, \overline{Z}) = \frac{\delta S_c(Z, \overline{Z})}{\delta g^{\mu\nu}} = 0,
$$

(124)

which recovers the Einstein equations in vacuum, in which $\overline{R}_{\mu\nu}$ is already represented in terms of the extremal metric tensor.

Q.E.D.

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