FIRST ORDER DIFFERENTIAL EQUATION
SUDORDINATION ASSOCIATED WITH CASSINI CURVE

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Abstract: We denote \( p(z) \) as analytic functions defined on the open unit disk with \( p(0) = 1 \). In this paper, we determined the condition for \( \beta \) so that the results hold for the expressions \( 1 + \beta z p'(z) \), \( 1 + \beta z p'(z)/p(z) \) and \( 1 + \beta z p'(z)/p^2(z) \) are subordinate to \( \sqrt{1 + cz} \).

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1. Introduction

Let \( A \) be the class of all the analytic functions of the form
\[
f(z) = z + \sum_{n=2}^{\infty} a_n z^n \quad (z \in D),
\]
in a unit disk \( D = \{z \in \mathbb{C} : |z| < 1\} \) and normalized by the condition \( f(0) = 0 = f'(0) - 1 \). We denote \( S \) as the subset of \( A \) of univalent functions. Also, we denoted \( C \) as the class of convex functions and \( S^* \) as the class of starlike functions. An analytic function \( f \) is subordinate to an analytic function \( g \), we
write \( f(z) \prec g(z) \) for \( z \in D \), if there exists an analytic function \( w \) in \( D \) such that \( w(0) = 0 \) and \( |w(z)| < 1 \) for \( |z| < 1 \) and \( f(z) = g(w(z)) \). In particular, if \( g \) is univalent in \( D \), we say that \( f(z) \prec g(z) \) is equivalent to \( f(0) = g(0) \) and \( f(D) \subset g(D) \).

Goluzin [1] found that if the first order differential subordination \( zp'(z) \prec zq'(z) \) holds and \( zq'(z) \) is convex, then \( p(z) \prec q(z) \) holds where \( q \) is the best dominant. Eventually, researchers continued to study about this and the general theory is discussed detailed by Miller and Mocanu in [2]. Nunokawa et al. [3] proved that if \( 1 + zp'(z) \prec 1 + z \) hold, the subordination \( p(z) \prec 1 + z \) also hold.

There are more results obtained by many other researchers, see [4], [5], [6], [7], [8], [9] and [10].

Sokół and Stankiewicz [11] introduced a class called \( S^*_L \) which consists the function of \( f \in A \) such that \( w(z) := zf'(z)/f(z) \) lies in the region bounded by the right half of the lemniscate of Bernoulli given by \( |w^2 - 1| < 1 \). This class is associated with the function \( \sqrt{1+z} \).

Besides, Aouf et al. [12] defined the class \( S^*(q_c) \) for \( c \in (0,1] \) as:

\[
S^*(q_c) = \left\{ f \in A : \left| \left[ \frac{zf'(z)}{f(z)} \right]^2 - 1 \right| < c, z \in D \right\}.
\]

It can be established that

\[
f \in S^*(q_c) \iff \frac{zf'(z)}{f(z)} < \sqrt{1+cz} \quad (z \in D).
\]

We also denoted \( \theta_c \) as the set of all points in the right half-plane such that the product of the distances from each point to the focuses -1 and 1 is less than \( c \):

\[
\theta_c := \{ w \in \mathbb{C} : Re \ w > 0, |w^2 - 1| < c \},
\]

thus the boundary \( \partial \theta_c \) is the right loop of the Cassinian ovals \((x^2 + y^2)^2 - 2(x^2 - y^2) = c^2 - 1 \) and for \( c = 1 \), \( S^*(q_1) \equiv S^*_L \).

For an analytic function \( p(z) = 1 + c_1 z + c_2 z^2 + \cdots \), we determine the condition of \( \beta \) so that \( p(z) \prec P(z) \) where \( P(z) \) is a function with positive real part like \( \sqrt{1+z} \) and \( \varphi_0(z) := 1 + \frac{k}{k} ((k + z)/(k - z)) \) \( (k = \sqrt{2} + 1) \), whenever \( 1 + \beta z p'(z)/p'(z) \prec \sqrt{1+cz} \), where \( j = 0, 1, 2 \) (please see [13] for more about \( \varphi_0(z) \)).
2. Preliminary results and definitions

Our results deal with classes of $S^*(q_c)$ associated with $S_L^*$ and $\varphi_0(z)$ respectively. Some sufficient conditions with for functions belong to the above defined classes can be obtained by applying the application on starlike functions with positive real part. The first result gives a bound of $\beta$ so that $1 + \beta z p'(z) \prec \sqrt{1 + cz}$ implies that the function $p$ is subordinate to the $\sqrt{1+z}$ function.

Before to get our result, we need the following lemma to prove the theorems.

**Lemma 1.** ([14]) Let $q$ be analytic in $D$ and let $\psi$ and $v$ be analytic in domain $U$ containing $q(D)$ with $\psi(w) \neq 0$ when $w \in q(D)$. Set $Q(z) := zq'(z)\psi(q(z))$ and $h(z) := v(q(z)) + Q(z)$. Suppose that:

i. either $h$ is convex or $Q$ is starlike univalent in $D$, and

ii. $\text{Re} \left( zh'(z)/Q(z) \right) > 0$ for $z \in D$.

If $p$ is analytic in $D$, with $p(0) = q(0)$, $p(D) \subseteq U$ and

$$v(p(z)) + zp'(z)\psi(p(z)) \prec v(q(z)) + zq'(z)\psi(q(z)),$$

then $p(z) \prec q(z)$ and $q$ is best dominant.

3. Main results

**Theorem 1.** Let the function $p$ be analytic in $D$, $p(0) = 1$ and $1 + \beta z p'(z) \prec \sqrt{1 + cz}$, $c \in (0, 1]$. Then the following subordination results hold:

(a) If $\beta \geq \frac{2 \left( \sqrt{1 + c} - \ln(1 + \sqrt{1 + c}) + \ln 2 - 1 \right)}{\sqrt{2} - 1}$, then $p(z) \prec \sqrt{1 + z}$.

(b) If $\beta \geq \frac{2 \left( \sqrt{1 - c} - \ln(1 - \sqrt{1 - c}) + \ln 2 - 1 \right)}{\sqrt{2} - 3}$, then $p(z) \prec \varphi_0(z)$.

**Proof.** The function $q_B : \overline{D} \to \mathbb{C}$ defined by

$$q_B(z) = 1 + \frac{2}{\beta} \left[ \sqrt{1 + cz} - \ln(1 + \sqrt{1 + cz}) + \ln 2 - 1 \right]$$
is analytic and it is the solution of $1 + \beta z p'(z) = \sqrt{1 + c z}$. Let $v(w) = 1$ and $\psi(w) = \beta$. So the function $Q : \mathbb{D} \to \mathbb{C}$ is defined by $Q(z) = z q_B'(z) \psi(q_B(z)) = \beta z q_B'(z)$. Since $\sqrt{1 + c z} - 1$ is starlike function in $D$, it follows that function $Q$ is starlike. Besides, the function $h(z) = v(q_B(z)) + Q(z)$ satisfies $\text{Re}(zh'(z)/Q(z)) > 0$ for $z \in D$. Thus, by using Lemma 1, it shows $1 + \beta z p'(z) < 1 + \beta z q_B'(z)$ implies $p(z) < q_B(z)$. We can say that $p(z) < P(z)$ for appropriate $P$ and this holds if the subordination $q_B(z) < P(z)$ holds. If $q_B(z) < P(z)$, then $P(-1) < q_B(-1) < q_B(1) < P(1)$. This gives a necessary condition for $p < P$ hold. This necessary condition is sufficient.

(a). By taking $P(z) = \sqrt{1 + z}$, the inequalities $q_B(-1) \geq P(-1)$ and $q_B(1) \leq P(1)$ reduce to $\beta \geq \beta_1$ and $\beta \geq \beta_2$, where

$$
\beta_1 = 2 \left[ \ln(1 + \sqrt{1 - c}) + 1 - \ln 2 - \sqrt{1 - c} \right]
$$

and

$$
\beta_2 = 2 \left[ \frac{\sqrt{1 + c} - \ln(1 + \sqrt{1 + c}) + \ln 2 - 1}{\sqrt{2} - 1} \right],
$$

respectively. The subordination $q_B(z) < \sqrt{1 + z}$ holds if $\beta \geq \max\{\beta_1, \beta_2\} = \beta_2$.

(b). Consider $P(z) = \varphi_0(z)$, then the inequalities $q_B(-1) \geq \varphi_0(-1)$ and $q_B(-1) \leq \varphi_0(1)$ reduce to $\beta \geq \beta_1$ and $\beta \geq \beta_2$, where

$$
\beta_1 = 2 \left[ \frac{\sqrt{1 - c} - \ln(1 + \sqrt{1 - c}) + \ln 2 - 1}{2\sqrt{2} - 3} \right]
$$

and

$$
\beta_2 = 2 \left[ \frac{\sqrt{1 + c} - \ln(1 + \sqrt{1 + c}) + \ln 2 - 1}{\sqrt{2} - 1} \right],
$$

respectively. Thus, the subordination $q_B(z) < \varphi_0(z)$ holds if $\beta \geq \max\{\beta_1, \beta_2\} = \beta_1$.

When $c = 1$, we may get Corollary 5. The next result gives bound on $\beta$ so that $1 + \beta z p'(z)/p(z) < \sqrt{1 + c z}$ implies $p$ is subordinate to $\varphi_0(z)$ function.

**Theorem 2.** Let the function $p$ be analytic in $D$, $p(0) = 1$ and $1 + \beta z p'(z)/p(z) < \sqrt{1 + c z}$, $c \in (0, 1]$. Then the following subordination result holds:

If $\beta \geq \frac{2 \left[ \sqrt{1 - c} - \ln(\sqrt{1 - c} + 1) + \ln 2 - 1 \right]}{\ln(2\sqrt{2} - 2)}$, then $p(z) < \varphi_0(z)$.
**Proof.** The function $q_B : \overline{D} \rightarrow \mathbb{C}$ defined by

$$q_B(z) = \exp \left\{ 1 + \frac{2}{\beta} \left[ \sqrt{1 + cz} - \ln(1 + \sqrt{1 + cz}) + \ln 2 - 1 \right] \right\}$$

is analytic and is the solution of $1 + \beta z p'(z)/p(z) = \sqrt{1 + cz}$. Define $v(w) = 1$ and $\psi(w) = \beta/w$. The function $Q : \overline{D} \rightarrow \mathbb{C}$ defined by $Q(z) := zq_B'(z)\psi(q_B(z)) = \beta zq_B'(z)/q_B(z) = \sqrt{1 + cz} - 1$ is starlike in $D$. The function $h(z) := v(q_B(z)) + Q(z) = 1 + Q(z)$ satisfies $\text{Re}(zh'(z)/Q(z)) > 0$ for $z \in D$. Therefore, by using Lemma 1, we get that $1 + \beta z p'(z)/p(z) \prec \sqrt{1 + cz}$ implies $p$ is subordinate to $\phi_0(z)$.

Next, we determine a bound on $\beta$ so that $1 + \beta z p'(z)/p^2(z) \prec \sqrt{1 + cz}$ implies $p$ is subordinate to $\phi_0(z)$.

**Theorem 3.** Let the function $p$ be analytic in $D$, $p(0) = 1$ and $1 + \beta z p'(z)/p^2(z) \prec \sqrt{1 + cz}, c \in (0, 1]$. Then the following subordination results hold:

If $\beta \geq \frac{4 \left( \sqrt{2} - 1 \right) \left( \sqrt{1 - c} - \ln(\sqrt{1 - c} + 1) + \ln 2 - 1 \right)}{2\sqrt{2} - 3}$,

then $p(z) \prec \phi_0(z)$.

**Proof.** The function $q_B : \overline{D} \rightarrow \mathbb{C}$ defined by

$$q_B(z) = \left( 1 + \frac{2}{\beta} \left[ \sqrt{1 + cz} - \ln(1 + \sqrt{1 + cz}) + \ln 2 - 1 \right] \right)^{-1}$$

is analytic. It is the solution of $1 + \beta z p'(z)/p^2(z) = \sqrt{1 + cz}$. Define $v(w) = 1$ and $\psi(w) = \beta/w^2$. The function $Q : \overline{D} \rightarrow \mathbb{C}$ defined by

$$Q(z) := zq_B'(z)\psi(q_B(z)) = \beta zq_B'(z)/q_B^2(z) = \sqrt{1 + cz} - 1$$

is starlike in $D$, so $Q$ is starlike function. The function $h(z) := v(q_B(z)) + Q(z) = 1 + Q(z)$ satisfies $\text{Re}(zh'(z)/Q(z)) > 0$ for $z \in D$. Therefore, by using Lemma 1, we get that

$$1 + \beta z p'(z)/p^2(z) \prec 1 + \beta \frac{zq_B'(z)}{q_B^2(z)}$$
implies \( p(z) \preceq q_B(z) \). As the similar lines of the proof of Theorem 2 the proof of the result is completed.

Also, let \( c = 1 \), we have the result in Corollary 7.

4. Corollaries

**Corollary 4.** ([10]) Let the function \( p \) be analytic in \( D \), \( p(0) = 1 \) and \( 1 + \beta z p'(z) \prec \sqrt{1 + cz} \). Then the following subordination results hold:

(a) If \( \beta \geq \frac{2[\sqrt{2} - 1 + \ln 2 - \ln (1 + \sqrt{2})]}{\sqrt{2} - 1} \approx 1.09116 \), then \( p(z) \prec \sqrt{1 + z} \).

(b) If \( \beta \geq \frac{2(1 - \ln 2)}{3 - 2\sqrt{2}} \approx 3.57694 \), then \( p(z) \prec \varphi_0(z) \).

**Corollary 5.** ([10]) Let the function \( p \) be analytic in \( D \), \( p(0) = 1 \) and \( 1 + \beta z p'(z)/p(z) \prec \sqrt{1 + cz} \). Then the following subordination results hold:

If \( \beta \geq \frac{2(\ln 2 - 1)}{\ln(2\sqrt{2} - 2)} \approx 3.26047 \), then \( p(z) \prec \varphi_0(z) \).

**Corollary 6.** ([10]) Let the function \( p \) be analytic in \( D \), \( p(0) = 1 \) and \( 1 + \beta z p'(z)/p^2(z) \prec \sqrt{1 + cz} \). Then the following subordination results hold:

If \( \beta \geq 4(1 + \sqrt{2})(1 - \ln 2) \approx 2.96323 \), then \( p(z) \prec \varphi_0(z) \).

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