Reducing spatially correlated noise and decoherence with quantum error correcting codes

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It is shown that the noise process in quantum computation can be described by spatially correlated decoherence and dissipation. We demonstrate that the conventional quantum error correcting codes correcting for single-qubit errors are applicable for reducing spatially correlated noise.

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Quantum computers hold the promise for solving many hard problems much more effectively than their classical counterparts [1-2]. However, in practice the inevitable noise and decoherence process will diminish the advantages of quantum computation [4]. To overcome the fragility of quantum information, many kinds of techniques have been discovered to combat noise and decoherence in quantum computers [5-18]. Among these techniques, quantum error correction is the most important one [5-12].

In quantum error correction schemes, the input state is encoded into a state in a larger Hilbert space so that it can be recovered from a certain class of errors. The error operators are identified from physical consideration. Conventionally, it is assumed that qubits (quantum bits) are decohered independently. In this circumstance, after a short time interval, the most important errors are those described by single-qubit operators [19]. Hence, quantum error correcting codes (QECCs) need only to correct single-qubit or few-qubit errors [5-12].

The assumption of independent decoherence plays an important role in quantum error correction schemes [5,10]. Except independent decoherence, there are possibly other kinds of decoherence, for example, collective decoherence [15,16]. Alternate schemes, called quantum error avoiding codes, have been proposed for reducing collective decoherence [16-18]. Are quantum error correction techniques applicable for reducing collective decoherence? The answer is shown to be positive for a special collective dephasing model [20]. In this paper, we consider the most general type of noise of many qubits. It is shown that to a good approximation, the general noise process can be described by spatially correlated decoherence and dissipation, which includes independent decoherence and collective decoherence as its special cases. The master equation is derived, and we identify the error operators for general spatially correlated noise, using the quantum trajectory approach [21,22]. The error operators for spatially correlated noise are no longer single-qubit operators. They are either a sum of single-qubit operators (the quantum jump errors) or a tensor product of them (the effective evolution error). However, we demonstrate that the conventional QECCs correcting for single-qubit errors remain valid for reducing this kind of spatially correlated noise.

We consider noise and decoherence of \( L \) qubits. The \( l \) qubit is described by the Pauli operator \( \vec{\sigma}_l \). The most general noise process of \( L \) qubits can be described by the following interaction Hamiltonian (setting \( \hbar = 1 \)):

\[
H_I(t) = g_1 \sum_{l,\alpha} \sigma^\alpha_l \Gamma^\alpha_l(t) + g_2 \sum_{l,l',\alpha_1,\alpha_1'} \sigma^{\alpha_1}_l \otimes \sigma^{\alpha_1'}_{l'} \Gamma^{\alpha_1\alpha_1'}_{ll'}(t)
\]  

(1)
\[
+ \cdots + g_L \sum_{\alpha_1, \alpha_2, \ldots, \alpha_L} \sigma_1^{\alpha_1} \otimes \sigma_2^{\alpha_2} \otimes \cdots \otimes \sigma_L^{\alpha_L} \Gamma_{12 \cdots L}^{\alpha_1, \alpha_2, \ldots, \alpha_L} (t),
\]

where \( \alpha_l = 1, 2, 3, \) and \( l = 1, 2, \ldots, L \) (\( \sigma_i^{1,2,3} \) correspond to \( \sigma_i^{x,y,z} \), respectively). All the \( \Gamma_{\alpha}^{\alpha_i} (t) \), \( \Gamma_{ll'}^{\alpha_i, \alpha_i'} (t) \), and \( \Gamma_{12 \cdots N}^{\alpha_1, \alpha_2, \ldots, \alpha_N} (t) \), generally dependent of time, are noise terms, which may be classical stochastic variables or stochastic quantum operators, corresponding to classical noise or quantum noise, respectively. The Hamiltonian (1) includes all possible interaction terms between the qubits and the noisy environment, and it is very complicate. Fortunately, in practice this complicate description is not necessary. The coupling coefficients \( g_1, g_2, \ldots \) and \( g_n \) normally satisfy the condition \( g_1 \gg g_2 \gg \cdots \gg g_L \). Hence, the most important noise always comes from the first term of the right hand side of Eq. (1), if this term does not reduce to zero due to some special symmetry. As a good approximation, we can safely drop off all the high order nonlinearities in Eq. (1). The interaction Hamiltonian is then simplified to

\[
H_I (t) = g_1 \sum_{\alpha, \alpha} \sigma_1^{\alpha} \Gamma_{\alpha}^{\alpha} (t).
\]

The explicit expressions for the noise terms \( \Gamma_{\alpha}^{\alpha} (t) \) depend on the concrete physical model of quantum computation. However, it is reasonable to assume that \( \Gamma_{\alpha}^{\alpha} (t) \) satisfy the following conditions:

\[
\langle (\Gamma_{\alpha}^{\alpha} (t)) \rangle_{\text{env}} = 0,
\]

\[
\langle \Gamma_{\alpha}^{\alpha} (t) \Gamma_{\beta}^{\alpha} (t') \rangle_{\text{env}} = f_{\alpha \beta}^{\alpha \beta} (t - t'),
\]

where \( \langle \rangle_{\text{env}} \) denotes average over the environment. In Eq. (4), all the \( f_{\alpha \beta}^{\alpha \beta} (t - t') \) are correlation functions. From the Hermiticity of \( \Gamma_{\alpha}^{\alpha} (t) \), we have

\[
f_{\alpha \beta}^{\alpha \beta} (t) = \left[ f_{\alpha \beta}^{\alpha \beta} (-t) \right]^*.
\]

The general form of the master equation with the interaction Hamiltonian \( H_I (t) \) is expressed as [23]

\[
\frac{d}{dt} \rho (t) = -\int_0^\infty d\tau tr_{\text{env}} \{ [H_I (t), [H_I (t - \tau), \rho (t) \otimes \rho_{\text{env}}]] \},
\]

where \( \rho_{\text{env}} \) is the density operator of the environment, and \( \rho (t) \) denotes the reduced density operator of the qubits in the interaction picture. Substituting the Hamiltonian (2) into Eq. (6), and using the conditions (3) and (4), we derive the following master equation for noise and decoherence of \( L \) qubits:
\[
\frac{d}{dt} \rho(t) = -\frac{i}{2} \sum_{l,l',\alpha,\beta} B_{l'l}^{\beta\alpha} \left[ \sigma_l^\beta \sigma_l'^\alpha, \rho(t) \right] + \sum_{l,l',\alpha,\beta} A_{l'l}^{\beta\alpha} \left[ \sigma_l^\alpha \rho(t) \sigma_l'^\beta - \frac{1}{2} \sigma_l^\beta \sigma_l'^\alpha \rho(t) - \frac{1}{2} \rho(t) \sigma_l^\beta \sigma_l'^\alpha \right],
\]

where the coefficients
\[
A_{l'l}^{\beta\alpha} = g_1^2 \int_{-\infty}^{+\infty} f_{l'l}^{\beta\alpha} (\tau) d\tau,
\]
\[
B_{l'l}^{\beta\alpha} = -ig_1^2 \int_{0}^{\infty} \left[ f_{l'l}^{\beta\alpha} (\tau) - f_{l'l}^{\beta\alpha} (-\tau) \right] d\tau.
\]

From Eq. (5), it follows that the matrixes \( A = \left[ A_{l'l}^{\beta\alpha} \right] \) and \( B = \left[ B_{l'l}^{\beta\alpha} \right] \) are Hermitian. The first term of the right hand side of Eq. (7) represents the environment-induced Lamb phase shift, and the second term represents decoherence and dissipation of the qubits. If the coefficients \( A_{l'l}^{\beta\alpha} \) and \( B_{l'l}^{\beta\alpha} \) are directly proportional to \( \delta_{ll'} \), Eq. (7) describes independent decoherence (in the terminology of Refs. [15-19]). In contrast, if the correlation terms \( A_{l'l}^{\beta\alpha} \) and \( B_{l'l}^{\beta\alpha} \) with \( l \neq l' \) attain the maximum, Eq. (7) represents collective decoherence [15,16]. In general circumstances, the qubits are subject to spatially correlated decoherence.

We are interested in the problem that to what extent the noise described by Eq. (7) can be reduced by the conventional quantum error correction techniques. To examine the problem, we need first to identify all the first-order error operators for spatially correlated noise. It is convenient to use the quantum trajectory approach to attain this goal. The quantum trajectory approach is a recently-developed numerical simulation method for solving complicate open quantum systems [21,22]. In this approach, the evolution of the dissipative system is represented by an ensemble of wave functions that propagate according to the effective Hamiltonian interrupted by random quantum jumps [24]. To use the language of quantum trajectories, we need to re-express the master equation (7) in a diagonal form. The correlation matrix \( A \) is Hermitian, hence it can diagonalized by a unitary matrix \( U = \left[ U_{kl}^{\gamma \alpha} \right] \), i.e., we have
\[
A_{l'l}^{\beta\alpha} = \sum_{k,\gamma} U_{l'k}^{\gamma \beta} \xi_k^{\gamma} U_{kl}^{\gamma \alpha},
\]
where \( \xi_k^\gamma \), with \( k = 1, 2, \ldots, L \) and \( \gamma = 1, 2, 3 \), are eigenvalues of the positive-definite Hermitian matrix \( A \), which should be positive real numbers. Define the operators \( s_k^{\gamma} \) by the equation
\[
s_k^{\gamma} = \sum_{l,\alpha} U_{kl}^{\gamma \alpha} \sigma_l^\alpha.
\]
In general, \( s_k^\gamma \) are no longer Hermitian operators. With the transformations (10) and (11), the master equation (7) is rewritten as

\[
\frac{d}{dt} \rho(t) = -iH_{eff} \rho(t) + i\rho(t) H_{eff}^\dagger + \sum_{k,\gamma} \xi_k^\gamma s_k^\gamma \rho(t) (s_k^\gamma)^\dagger,
\]

where the non-Hermitian effective Hamiltonian is

\[
H_{eff} = \frac{1}{2} \sum_{l,l',\alpha,\beta} B_{\alpha l}^{\beta l'} \sigma_{l'}^{\beta} \sigma_{l}^{\alpha} - i \sum_{k,\gamma} \xi_k^\gamma (s_k^\gamma)^\dagger s_k^\gamma.
\]

(13)

The first term of the effective Hamiltonian is the Hermitian Lamb phase shift, which in general cannot be simplified by introducing the operators \( s_k^\gamma \), since the matrixes \( A \) and \( B \) do not necessarily commute with each other. The second term of the right hand side of Eq. (13) is the non-Hermitian damping Hamiltonian.

Suppose that in a finite time \( T_0 \), we perform \( N \) times error corrections. In a short time interval \( \Delta t = \frac{T_0}{N} \), we need to find all the error operators up to the first order of \( \Delta t \). In the language of quantum trajectories, the system evolution described by Eq. (12) is represented by an ensemble of pure states that evolve according to the effective Hamiltonian (13), interrupted at random times by quantum jumps \( s_k^\gamma \). Up to the first order of \( \Delta t \), the normalized state after \( \Delta t \) will be either

\[
|\Psi_k^\gamma (\Delta t)\rangle = \sqrt{\frac{\xi_k^\gamma \Delta t}{p_k}} (s_k^\gamma)^\dagger |\Psi(0)\rangle
\]

(14)

with probability \( p_k^\gamma = \xi_k^\gamma \Delta t \langle \Psi(0)| (s_k^\gamma)^\dagger s_k^\gamma |\Psi(0)\rangle \) in case of a jump in decay channel \((k,\gamma)\) at a random time in the interval \( \Delta t \), or

\[
|\Psi_0 (\Delta t)\rangle = \frac{1}{\sqrt{p_0}} \exp (-iH_{eff} \Delta t) |\Psi(0)\rangle
\]

(15)

with probability \( p_0 = 1 - \sum_{k,\gamma} \xi_k^\gamma \Delta t \langle \Psi(0)| (s_k^\gamma)^\dagger s_k^\gamma |\Psi(0)\rangle = o\left(\Delta t^2\right) \) if no jump occurred. Let \( Q_0 = \sqrt{1 - \frac{\Delta t}{2} \sum_{k,\gamma} \xi_k^\gamma s_k^\gamma (s_k^\gamma)^\dagger} \), \( Q_{3(k-1)+\gamma} = p_k^\gamma \), and \( Q_{3(k-1)+\gamma} = p_k^\gamma \), where \( k = 1,2,\ldots,L \) and \( \gamma = 1,2,3 \). With this notation, the system state after a short time \( \Delta t \) is then represented by the following density operator

\[
\rho (\Delta t) = \sum_{n=0}^{3L} p_n Q_n |\Psi(0)\rangle \langle \Psi(0)| + o\left(\Delta t^2\right),
\]

(16)
where $\rho(0) = |\Psi(0)\rangle\langle\Psi(0)|$. In the above equation, $Q_0$ represents the effective evolution error, and $Q_k$ ($k = 1, 2, \cdots, 3L$) represent the quantum jump errors. All the $Q_n$ ($n = 0, 1, \cdots, 3L$) make a complete set of the first-order error operators.

For independent decoherence, the correlation coefficients $A_{\beta\alpha}^l$ and $B_{\beta\alpha}^l$ are directly proportional to $\delta_{ll'}$. All the first-order errors $Q_n$ then reduce to single-qubit operators. For general spatially correlated decoherence, the first-order errors are no longer single-qubit operators. The quantum jump errors $Q_k$ ($k = 1, 2, \cdots, 3L$) are expressed as sums of single-qubit operators, and more seriously, the effective evolution error $Q_0$ includes the terms that are tensor products of single-qubit operators. Hence, it is not clear that this kind of decoherence can be reduced by the conventional QECCs. In fact, it has been suggested that to combat the effective evolution error, more involved and less efficient QECCs need be devised [25].

However, here we show that the conventional QECCs correcting for single-qubit errors remain applicable for reducing general spatially correlated decoherence, if the error correction procedure is repeated frequently enough. This can be demonstrated by the following explicit calculation of the state fidelity after error correction.

In QECCs that correct single-qubit errors, the error operators are represented by $\sigma_{l\alpha}^\alpha$ with $l = 1, 2, \cdots, L$ and $\alpha = 1, 2, 3$ [11]. Let $R_0 = I$, denoting the identity operator, and $R_{3(l-1)+\alpha}^l = \sigma_{l\alpha}^\alpha$. We only consider orthogonal QECCs. Most of the discovered QECCs belong to this class [5-12]. For orthogonal QECCs, the encoded input state $|\Psi(0)\rangle$ should satisfy the condition [10]

$$
\langle\Psi(0)| R_{n}^{l} R_{m'}^{n'} |\Psi(0)\rangle = \delta_{nn'} \quad (n = 0, 1, \cdots, 3L). \quad (17)
$$

During the error correction procedure, we first detect the error syndrome. If there is a $R_n$ error, i.e., the state becomes $R_n |\Psi(0)\rangle$, we apply the recovery operator $R_n^{-1} = R_n$ to the state and thus get the correct initial state $|\Psi(0)\rangle$. In the case of spatially correlated decoherence, the error operators are represented by $Q_n$ ($n = 0, 1, \cdots, 3L$), but we still adopt the above error correction procedure. If there is a $Q_n$ error, which occurs with probability $p_n$, we detect the error syndrome and with probability $|\langle\Psi(0)| R_n^l Q_n |\Psi(0)\rangle|^2$ find that the error is $R_m$. After this detection, the state is collapsed into $R_m |\Psi(0)\rangle$. We apply the recovery operator and thus get the initial state. The whole error correction procedure described above yields the following average state fidelity after error correction

$$
F_a(\Delta t) = \sum_{n=0}^{3L} \sum_{m=0}^{3L} p_n |\langle\Psi(0)| R_m^l Q_n |\Psi(0)\rangle|^2 \quad (18)
$$
\[ = 1 - o (\Delta t^2) . \]

In deriving Eq. (18), we have used Eqs. (11) and (17), together with the identity \( \sum_{l,\alpha} |U_{kl}^\gamma\alpha|^2 = 1 \) (from the unitarity of the matrix \( U \)). After the whole time \( T_0 \), the final average state fidelity \( F_a (T_0) \) is then approximated by

\[ F_a (T_0) \simeq [F_a (\Delta t)]^N \simeq 1 - o (N \Delta t^2) . \tag{19} \]

Since \( N \Delta t^2 \propto \frac{1}{N} \), it can be made arbitrarily small by a frequent repetition of the error correction procedure. This demonstrates that the QECCs devised to correct single-qubit errors are applicable for reducing spatially correlated decoherence.

Before ending the paper, we should emphasize that we have omitted all the other terms except the first one in the Hamiltonian (1). If these omitted terms become important due to some special reason, the QECCs that correct for single-qubit errors may not work well any more. For example, if the second term in the Hamiltonian (1) is not negligible, the quantum jump errors will include not only the terms that can be expressed as sums of single-qubit operators, but also the terms that are tensor products of them. To combat this kind of decoherence, the QECCs need at least having the ability to correct two-qubit errors.

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