LOGARITHMIC EULER-MARUYAMA SCHEME FOR
MULTI-DIMENSIONAL STOCHASTIC DELAY EQUATIONS WITH JUMPS

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1. Introduction

In [2] we introduced a logarithmic Euler-Maruyama scheme for a single stochastic delay equation, which preserve positivity if the solution to the original equation is positive. The convergence rate was also obtained for such scheme. This scheme is important for simulation of the paths of the equation. It plays important role in option pricing for example since we often cannot obtain the explicit pricing value and we need to use Monte-Carlo to complete the evaluation. Naturally our next question would be what will be the analogous scheme for a system of stochastic delay equations and if such schemes converges. This type of problems is very important since there is always more than one stock in the real market. Now in more than one dimension, the problem of positive solution and the numerical schemes which preserve the positivity are much more complicated. In this paper we shall extend our work in [2] to a system of stochastic delay differential equations. The problems of existence and uniqueness of a positive are solved. The multi-dimensional logarithmic Euler-Maruyama scheme are constructed which preserve the positivity of the approximate solutions. The scheme is proved to be convergent with rate 0.5.

2. Positivity

We consider the following system of stochastic delayed differential equations:

\[
\begin{cases}
    dS_i(t) = \sum_{j=1}^{d} f_{ij}(S(t-b))S_j(t)dt \\
    \quad + S_i(t-b) \sum_{j=1}^{d} g_{ij}(S(t-b))dZ_j(t), & i = 1, \ldots, d, \\
    S_i(t) = \phi_i(t), & t \in [-b, 0], i = 1, \ldots, d,
\end{cases}
\]  

(2.1)

where \( S(t) = (S_1(t), \ldots, S_d(t))^T \) and

(i) \( f_{ij}, g_{ij} : \mathbb{R}^d \to \mathbb{R} \) are some given bounded measurable functions for all \( 0 \leq i, j \leq d \).

(ii) \( b > 0 \) is a given number representing the delay of the equation.

(iii) \( \phi_i : [-b, 0] \to \mathbb{R} \) is a (deterministic) measurable function for all \( 0 \leq i \leq d \).

(iv) \( Z_j(t) = \sum_{k=1}^{N_j(t)} Y_{j,k} \) are Lévy processes, where \( Y_{j,k} \) are i.i.d random variables, \( N_{\ell}(t) \) are independent Poisson random processes which are also independent of \( Y_{j,k} \) for \( j, \ell, = 1, 2, \ldots, d, k = 1, 2, \ldots \).

Let \( |\cdot| \) Euclidean norm in \( \mathbb{R}^d \). If \( A \) is a \( d \times m \) matrix, we denote

\[ |A| = \sup_{|x| \leq 1} |Ax|. \]

For example, if \( A = I + M \) is a \( d \times d \) matrix, where \( M = (m_{ij})_{1 \leq i, j \leq d} \) is a matrix, then we can bound the norm of \( A \) as follows. Let \( 0 \leq \lambda_1 \leq \cdots \leq \lambda_d \) be eigenvalues of \( M^T M \) (since \( M^T M \)
is a positive definite matrix, we can assume that its eigenvalues are all positive). Then
\[
|I + M| = \sup_{|x| \leq 1} \sqrt{|x|^2 + x^T M x} \leq \sqrt{1 + \max_{1 \leq i \leq d} \lambda_i |x|}.
\]

But \( \sum_{i=1}^d \lambda_i = \text{Tr}(M^T M) \). Thus we have
\[
|I + M| \leq \sqrt{1 + \text{Tr}(M^T M)} |x|.
\] (2.2)

To study the above stochastic differential equation, it is common to introduce the Poisson random measure associated with the Lévy process \( Z_j(t) \). We write the jumps of the process \( Z_j \) at time \( t \) by
\[
\Delta Z_j(t) := Z_j(t) - Z_j(t^-) \quad \text{if } \Delta Z_j(t) \neq 0 \quad j = 1, 2, \ldots, d.
\]

Denote \( \mathbb{R}_0 := \mathbb{R} \setminus \{ 0 \} \) and let \( \mathcal{B}(\mathbb{R}_0) \) be the Borel \( \sigma \)-algebra generated by the family of all Borel subsets \( U \subset \mathbb{R} \), such that \( \overline{U} \subset \mathbb{R}_0 \). For any \( t > 0 \) and for any \( U \in \mathcal{B}(\mathbb{R}_0) \) we define the Poisson random measure, \( N_j : [0, T] \times \mathcal{B}(\mathbb{R}_0) \times \Omega \rightarrow \mathbb{R} \) (without confusion we use the same notation \( N \)), associated with the Lévy process \( Z_j(t) \) by
\[
N_j(t, U) := \sum_{0 \leq s \leq t, \Delta Z_j(s) \neq 0} \chi_U(\Delta Z_j(s)), \quad j = 1, 2, \ldots, d,
\] (2.3)
where \( \chi_U \) is the indicator function of \( U \). The associated Lévy measure \( \nu \) of the Lévy process \( Z_j \) is given by
\[
\nu_j(U) := \mathbb{E}[N_j(1, U)] \quad j = 1, 2, \ldots, d.
\] (2.4)

We now define the compensated Poisson random measure \( \tilde{N}_j \) associated with the Lévy process \( Z_j(t) \) by
\[
\tilde{N}_j(dt, dz) := N_j(dt, dz) - \mathbb{E}[N_j(dt, dz)] = N_j(dt, dz) - \nu_j(dz) dt.
\] (2.5)

We assume that the process \( Z_j(t) \) has only bounded negative jumps to guarantee that the solution \( S(t) \) to (2.1) is positive. This means that there is an interval \( J = [-R, \infty) \) bounded from the left such that \( \Delta Z_j(t) \in J \) for all \( t > 0 \) and for all \( j = 1, 2, \ldots, d \).

With these notations, we can write
\[
Z_j(t) = \int_{[0,t] \times J} z N_j(ds, dz) \quad \text{or} \quad dZ_j(t) = \int_J z \tilde{N}_j(dt, dz)
\]
and write (2.1) as
\[
\begin{cases}
    dS_i(t) = \sum_{j=1}^d f_{ij}(S(t-b)) S_j(t) dt + S_i(t^-) \sum_{j=1}^d \int_J zg_{ij}(S(t-b)) \nu_j(dz) dt \\
    \quad \quad \quad \quad + S_i(t^-) \sum_{j=1}^d \int_J zg_{ij}(S(t-b)) \tilde{N}_j(dz, dt), \\
    S_i(t) = \phi_i(t), \quad t \in [-b,0], \ i = 1, \ldots, d.
\end{cases}
\] (2.6)

In fact we can consider a slightly more general version of system of equations than (2.6):
\[
\begin{cases}
    dS_i(t) = \sum_{j=1}^d f_{ij}(S(t-b)) S_j(t) dt \\
    \quad \quad \quad \quad + S_i(t^-) \sum_{j=1}^d \int_J g_{ij}(z, S(t-b)) \tilde{N}_j(dz, dt), \quad i = 1, \ldots, d, \\
    S_i(t) = \phi_i(t), \quad t \in [-b,0], \ i = 1, \ldots, d.
\end{cases}
\] (2.7)
First, we discuss the existence, uniqueness and positivity of (2.7).

**Theorem 2.1.** Suppose that \( f_{ij} : \mathbb{R}^d \rightarrow \mathbb{R} \) and \( g_{ij} : \mathcal{J} \times \mathbb{R}^d \rightarrow \mathbb{R} \), \( 1 \leq i, j \leq d \) are bounded measurable functions such that there is a constant \( \alpha_0 > 1 \) satisfying \( g_{ij}(z, x) \geq \alpha_0 > -1 \) for all \( 1 \leq i, j \leq d \), for all \( z \in \mathcal{J} \) and for all \( x \in \mathbb{R} \), where \( \mathcal{J} = [-R, \infty) \) is the common supporting set of the Poisson measures \( \tilde{N}_j(t, dz), j = 1, \ldots, d \). If for all \( i \neq j \), \( f_{ij}(x) \geq 0 \) for all \( x \in \mathbb{R} \), and \( \phi_i(0) \geq 0 \), \( i = 1, \ldots, d \), then, the stochastic differential delay equation (2.7) admits a unique pathwise solution such that \( S_i(t) \geq 0 \) almost surely for all \( i = 1, \ldots, d \) and for all \( t > 0 \).

**Proof** The theorem is stated and proved in [2, Theorem 1] following the method of [3] (where the case of Brownian motion was dealt with). In fact, the existence and uniqueness are routine and easy. The main point is to show the positivity of the solution. The idea in [2] was to decompose the solution to (2.7) as product of some nonnegative processes. Here we give a slightly different decomposition which will prove the positivity and will be very useful in our numerical scheme.

Denote \( \tilde{f}_{ij}(t) = f_{ij}(S(t - b)) \) and \( \tilde{g}_{ij}(t, z) = g_{ij}(z, S(t - b)) \). Let \( Y_i(t) \) be the solution to the stochastic differential equation

\[
\frac{dY_i(t)}{dt} = \tilde{f}_{ii}(t)Y_i(t)dt + Y_i(t- \sum_{j=1}^{d} \int_{\mathcal{J}} \tilde{g}_{ij}(t, z)\tilde{N}_j(dt, dz)
\]

with initial conditions \( Y_i(0) = \phi_i(0) \). Since this is a scalar equation for \( Y_i(t) \), its explicit solution can be represented

\[
Y_i(t) = \phi_i(0) \exp \left\{ \sum_{j=1}^{d} \log \left[ 1 + \tilde{g}_{ij}(s, z) \right] \tilde{N}_j(ds, dz) + \int_{0}^{t} \tilde{f}_{ii}(s)ds + \sum_{j=1}^{d} \int_{[0, t] \times \mathcal{J}} \left( \log \left[ 1 + \tilde{g}_{ij}(s, z) \right] - \tilde{g}_{ij}(s, z) \right) dsv_j(dz) \right\}
\]

(2.8)

where \( \nu_j \) is the associated Lévy measure for \( \tilde{N}_j(ds, dz) \). Let \( p_i(t) \) be the solution to the following system of equations

\[
\frac{dp_i(t)}{dt} = \sum_{j=1, j \neq i}^{d} \tilde{f}_{ij}(t)p_j(t)dt, \quad p_i(0) = 1, \quad i = 1, \ldots, d.
\]

Since by the assumption that \( \tilde{f}_{ij}(t) \geq 0 \) almost surely for all \( i \neq j \), Theorem [1, p.173] implies that \( p_i(t) \geq 0 \) for all \( t \geq 0 \) almost surely. Now it is easy to check by the Itô formula that \( \tilde{S}_i(t) = p_i(t)Y_i(t) \) satisfies (2.7) and \( \tilde{S}_i(t) \geq 0 \) almost surely. By the uniqueness of the solution we see that \( S_i(t) = \tilde{S}_i(t) \) for \( i = 1, \ldots, d \). The theorem is then proved. \( \blacksquare \)

3. CONVERGENCE RATE OF LOGARITHMIC EULER-MARUYAMA SCHEME

In this section we construct numerical scheme to approximate (2.1) by positive value processes.

Motivated by the proof of Theorem 2.1 we shall decompose equation (2.1) into the following system:

\[
\begin{cases}
\frac{dX_i(t)}{dt} = f_{ii}((S(t - b)))X_i(t)dt + X_i(t-) \sum_{j=1}^{d} g_{ij}(S(t - b))dZ_j(t) & \text{(3.1a)} \\
\frac{dp_i(t)}{dt} = \sum_{j=1, j \neq i}^{d} f_{ij}((S(t - b)))p_j(t)dt, & \text{(3.1b)} \\
S_i(t) = p_i(t) \cdot X_i(t), \quad i = 1, 2, \ldots, d. & \text{(3.1c)}
\end{cases}
\]

The reason is, as in the proof of Theorem 2.1, that \( X_i(t) \) and \( p_i(t) \) are all positive.
Consider a finite time interval \([0, T]\) for some fixed \(T > 0\) and let \(\pi\) be a partition of the time interval \([0, T]\):

\[
\pi : 0 = t_0 < t_1 < \cdots < t_n = T.
\]

Let \(\Delta_k = t_{k+1} - t_k\) and \(\Delta = \max_{0 \leq k \leq n-1} (t_{k+1} - t_k)\) and assume \(\Delta < b\).

We shall now construct explicit logarithmic Euler-Maruyama recursive scheme to numerically solve (3.1a)-(3.1c). By the expression (2.8) the solution \(X\) on \([t_k, t_{k+1}]\) to Equation (3.1a) is given by

\[
X_i(t) = X_i(t_k) \exp \left\{ \int_{t_k}^{t} f_{ii}(s) ds + \sum_{j=1}^{d} \int_{t_k}^{t} \log \left[ 1 + g_{ij}(s) ds \right] \right\},
\]

where \(Z_j(t) = \sum_{k=1}^{N_{(t)}} Y_{j,k}\). If we denote by \(F(x)\) the \(d \times d\) matrix whose diagonal elements are all zero and whose off diagonal entries are \(f_{ij}(x)\), namely,

\[
F_{ij}(x) = \begin{cases} 
0 & \text{when } i = j \\
 f_{ij}(x) & \text{when } i \neq j.
\end{cases}
\]

With this notation we can write (3.1b) as a matrix form:

\[
\frac{dp(t)}{dt} = F((S(t - b))p(t)), \quad p(t) = (p_1(t), \ldots, p_d(t))^T, \quad (3.2)
\]

and its solution on the sub-interval \([t_k, t_{k+1}]\) is given by

\[
p(t) = \exp \left( \tilde{F}(S(t - b)) \right) p(t_k), \quad t \in [t_k, t_{k+1}], \quad (3.3)
\]

where the exponential of a matrix is in the usual sense: \(e^A = \sum_{k=0}^{\infty} A^k/k!\), the integral of a matrix is entry-wise. Here due to the noncommutativity \(\tilde{F}(S(t - b))\) is complicated to determine and we give the following formula for the sake of completeness:

\[
\tilde{F}(S(t - b)) = \sum_{r=1}^{\infty} \sum_{\sigma \in P_r} \left( \frac{(-1)^{e(\sigma)}}{r^2} \right) \int_{T_r(t)} [\cdots [F(S(u_{\sigma(1)} - b)F(S(u_{\sigma(2)} - b)) \cdots]F(S(u_{\sigma(r)} - b)]du_1 \cdots du_r
\]

is given by the Campbell-Baker-Hausdorff-Dynkin Formula (see e.g. [4], [5]), where \(P_r\) is the set of all permutations of \(\{1, 2, \ldots, r\}\), \(e(\sigma)\) is the number of errors in ordering consecutive terms in \(\{\sigma(1), \ldots, \sigma(r)\}\), \([AB] = AB - BA\) denotes the commutator of the matrices, and \(T_r(t) = \{0 < u_1 < \cdots < u_r < t\}\).

Analogously to [2] we propose the following logarithmic scheme to approximate the solution:

\[
\begin{cases}
X_i^\pi(t) = X_i^\pi(t_k) \exp \left( f_{ii}(S^\pi(t_k - b))(t - t_k) + \sum_{j=1}^{d} \ln \left( 1 + g_{ij}(S^\pi(t_k - b))(Z_j(t) - Z_j(t_k)) \right) \right), \\
p^\pi(t) = \left[ F(S^\pi(t_k - b)))(t - t_k) + I \right]p^\pi(t_k), \\
S_i^\pi(t) = p_i^\pi(t)X_i^\pi(t), \\
X_i^\pi(0) = \phi_i(0), \quad p^\pi(0) = 1, \quad t_k \leq t \leq t_{k+1}, \quad k = 1, 2, \ldots, n - 1.
\end{cases} \quad (3.5a, 3.5b, 3.5c, 3.5d)
\]

We introduce step processes

\[
\begin{cases}
v_1(t) = \sum_{k=0}^{\infty} \mathbb{1}_{[t_k, t_{k+1})}(t)S_i^\pi(t_k) \\
v_2(t) = \sum_{k=0}^{\infty} \mathbb{1}_{[t_k, t_{k+1})}(t)S_i(t_k - b).
\end{cases}
\]
Using the above step process we can write the continuous interpolation for \(X_t\) as

\[
X_t^\pi(t) = \exp \left( \int_0^t f_i(v_2(u))du + \sum_{j=1}^d \sum_{0 \leq u \leq t, \Delta Z(u) \neq 0} \ln \left( 1 + g_{ij}(v_2(u))Y_{j,N_i(u)} \right) \right). \tag{3.6}
\]

Denote \([t] = \max\{k, t_k < t\}\). From (3.5b) we have

\[
p^\pi(t) = \left[ \int_{[t]} F(v_2(u))du + I \right] \prod_{k=1}^{[t]} \left[ \int_{t_{k-1}}^{t_k} F(v_2(u))du + I \right]. \tag{3.7}
\]

We first show that \(p^\pi(t_k) \geq 0\).

**Lemma 3.1.** If \(\phi(0) \geq 0\) a.s., then \(p^\pi(t_k) \geq 0\) a.s. with \(p^\pi(t) = \phi(t)\) for all \(t \in [-b, 0]\).

**Proof** This can be seen from (3.5b) and by induction. Assume \(p^\pi(t_k) \geq 0\) a.s. Since by our definition of \(F(S^\pi(t_k - b))\) we know all of its components are positive, we see from (3.5b) that \(p^\pi(t) \geq 0\) a.s. for all \(t_k \leq t \leq t_{k+1}\).

Similarly we will have

**Lemma 3.2.** If \(\phi(0) \geq 0\) a.s., then \(X^\pi(t) \geq 0\) a.s. , hence \(S^\pi(t) \geq 0\) a.s. for all \(0 \leq t \leq T\).

To obtain the convergence of the logarithmic Euler–Maruyama scheme (3.5a)-(3.5d), we make the following assumptions:

**A1** The initial data \(\phi_i(0) > 0\) and it is Hölder continuous, i.e. there exist constant \(\rho > 0\) and \(\gamma \in [1/2, 1]\) such that for \(t, s \in [-b, 0]\)

\[
|\phi_i(t) - \phi_i(s)| \leq \rho |t - s|^{\gamma}. \quad i = 1, 2, \cdots, d. \tag{3.8}
\]

**A2** \(f_{ij}\) is bounded. \(f_{ij}\) and \(g_{ij}\) are global Lipschitz for \(i, j = 1, 2, \cdots, d\). This means that there exists a constant \(\rho > 0\) such that

\[
\begin{align*}
|g_{ij}(x_1) - g_{ij}(x_2)| &\leq \rho |x_1 - x_2|, & \forall \ x_1, x_2 \in \mathbb{R}^d; \\
|f_{ij}(x_1) - f_{ij}(x_2)| &\leq \rho |x_1 - x_2|, & \forall \ x_1, x_2 \in \mathbb{R}^d; \\
f_{ij}(x) &\leq \rho, & \forall \ x \in \mathbb{R}^d.
\end{align*}
\]

**A3** The support \(\mathcal{J}\) of the Poisson random measure \(N_j\) (associated with \(Z\)) is contained in \([-R, \infty)\) for each \(j = 1, 2, \cdots, d\) for some \(R > 0\) and there are constants \(\alpha_0 > 1\) and \(\rho > 0\) satisfying \(-\rho \leq g_{ij}(x) \leq \frac{\alpha_0}{R}\) for all \(x \in \mathbb{R}^d\) and for all \(i, j = 1, 2, \cdots, d\).

**A4** For any \(q > 1\) there is a \(\rho_q > 0\)

\[
\int_{\mathcal{J}} (1 + |z|)^q \nu_i(dz) \leq \rho_q, \quad \forall \ x \in \mathbb{R}. \quad i = 1, 2, \cdots, d. \tag{3.9}
\]

**Lemma 3.3.** Let Assumptions (A1)–(A4) be satisfied. Then, for any \(q \geq 1\), there exists \(K_q\), independent of the partition \(\pi\), such that

\[
\mathbb{E} \left[ \sup_{1 \leq i \leq d} \sup_{0 \leq t \leq T} |X_i(t)|^q \right] \leq \mathbb{E} \left[ \sup_{1 \leq i \leq d} \sup_{0 \leq t \leq T} |X_i^\pi(t)|^q \right] \leq K_q.
\]
Proof. From our definition of $X_{i}^{\pi}$ and boundedness of $f_{ij}$ for all $i, j$ we have

$$E\left[\sup_{0 \leq t \leq T} |X_{i}^{\pi}(t)|^q \right] = E\left[\sup_{0 \leq t \leq T} \exp \left( q \int_0^t f_{ii}(v_{2}(u))du + q \sum_{j=1}^{d} \sum_{0 \leq n \leq t, \Delta z(u) \neq 0} \ln(1 + g_{ij}(v_{2}(u))Y_{j,N_{j}(u)}) \right) \right]$$

$$= E\left[\sup_{0 \leq t \leq T} \exp \left( q \int_0^t f_{ii}(v_{2}(u))du + q \sum_{j=1}^{d} \int_T \ln(1 + z_{j}g_{ij}(v_{2}(u)))N_{j}(du, dz) \right) \right]$$

$$\leq K E\left[\sup_{0 \leq t \leq T} \exp \left( q \sum_{j=1}^{d} \int_T \ln(1 + z_{j}g_{ij}(v_{2}(u)))N_{j}(du, dz) \right) \right]$$

$$=: K I,$$

where $\mathbb{T}_t = [0, t] \times J$. Denote $h_j = ((1 + z_jg_{i,j}(v_{2}(u))^{2q} - 1))/z_j$. Then,

$$I = E\left[\sup_{0 \leq t \leq T} \exp \left( \frac{1}{2} \sum_{j=1}^{d} \int_{\mathbb{T}_t} \ln(1 + z_j h_j)N_j(du, dz_j) \right) \right]$$

$$= E\left[\sup_{0 \leq t \leq T} \exp \left( \sum_{j=1}^{d} \left( \frac{1}{2} \int_{\mathbb{T}_t} \ln(1 + z_j h_j)N_j(du, dz_j) + \frac{1}{2} \int_{\mathbb{T}_t} \ln(1 + z_j h_j)\nu_j(dz_j)du \right) \right) \right]$$

$$= E\left[\sup_{0 \leq t \leq T} \exp \left( \sum_{j=1}^{d} \left( \frac{1}{2} \int_{\mathbb{T}_t} \ln(1 + z_j h_j)N_j(du, dz_j) + \frac{1}{2} \int_{\mathbb{T}_t} [\ln(1 + z_j h_j) - z_j h_j] \nu_j(dz_j)du \right) \right) \right]$$

$$\leq C_q E\left[\sup_{0 \leq t \leq T} \exp \left( \sum_{j=1}^{d} \left( \frac{1}{2} \int_{\mathbb{T}_t} \ln(1 + z_j h_j)N_j(du, dz_j) + \frac{1}{2} \int_{\mathbb{T}_t} [\ln(1 + z_j h_j) - z_j h_j] \nu_j(dz_j)du \right) \right) \right] ,$$

where we used Assumption (A4) and the boundedness of $g_{ij}$. Write for $k = 1, 2, \cdots, d$

$$M_{k,t} := \exp \left( \int_{\mathbb{T}_t} \ln(1 + z_k h_k) \tilde{N}_k(du, dz_k) + \int_{\mathbb{T}_t} [\ln(1 + z_k h_k) - z_k h_k] \nu_k(dz_k)du \right) .$$

Then $(M_{k,t}, 0 \leq t \leq T)$ is an exponential martingale. Now an application of the Cauchy–Schwartz inequality yield

$$I \leq C_q \left\{ E\left[\sup_{0 \leq t \leq T} M_{1,t} \right] \right\}^{d/2},$$

which proves

$$E\left[\sup_{0 \leq t \leq T} |X_{i}^{\pi}(t)|^q \right] \leq K_q < \infty.$$

In the same way, we can show $E\left[\sup_{0 \leq t \leq T} |X_{i}(t)|^q \right] \leq K_q < \infty$. This completes the proof of the lemma. □

Lemma 3.4. Assume Assumptions (A1)–(A4). Then for $\Delta < 1$, there is a constant $K > 0$, independent of $\pi$, such that

$$E \sup_{0 \leq t \leq T} \left| S^{\pi}(t) - v_{2}(t) \right|^p \leq K \Delta^{p/2}.$$
Proof. Let $|t| = t_k$ if $t \in [t_k, t_{k+1})$ for some $k$. We have $v_2 = (v_{21}, v_{22}, \ldots, v_{2d})$ for which we write in short $v_2 = (\bar{v}_1, \bar{v}_2, \ldots, \bar{v}_d)$. For any $i = 1, \ldots, d,$

\[
E \sup_{0 \leq t \leq T} \left| S^\pi_i(t) - \bar{v}_i(t) \right|^p = E \sup_{0 \leq t \leq T} \left| p^\pi_i(t)X^\pi(t) - p^\pi_i([t])X^\pi([t]) \right|^p.
\]

By Assumption 2 we can bound
\[
E \sup_{0 \leq t \leq T} \left| p^\pi_i(t)X^\pi(t) - p^\pi_i([t])X^\pi([t]) \right|^p \leq C \left( E \sup_{0 \leq t \leq T} \left| p^\pi_i(t) - p^\pi_i([t]) \right|^{2p} \right)^{1/2} \left( E \sup_{0 \leq t \leq T} \left| X^\pi(t) \right|^{2p} \right)^{1/2}.
\]

(3.11)

\[
+ C \left( E \sup_{0 \leq t \leq T} \left| X^\pi(t) - X^\pi([t]) \right|^{2p} \right)^{1/2} \left( E \sup_{0 \leq t \leq T} \left| p^\pi_i([t]) \right|^{2p} \right)^{1/2}.
\]

(3.12)

By Assumption 2 we can bound $E \sup_{0 \leq t \leq T} \left| p^\pi_i([t]) \right|^{2p}$ and by lemma (3.3) we can bound $E \sup_{0 \leq t \leq T} \left| X^\pi_i(t) \right|^{2p}$. We now bound the other two components.

\[
E \sup_{0 \leq t \leq T} \left| p^\pi_i(t) - p^\pi_i([t]) \right|^{2p} \leq \sum_{j,j \neq i} \sum_{0 \leq t \leq T} \left| \int_{[t]} f_{ij}(v_2(u))du \right|^{2p}.
\]

(3.13)

By Assumption 2 it is easy to see that for some constant $C_1$

\[
E \sup_{0 \leq t \leq T} \left| p^\pi_i(t) - p^\pi_i([t]) \right|^{2p} \leq C_1 \Delta^{2p}.
\]

(3.14)

For $E \sup_{0 \leq t \leq T} \left| X^\pi_i(t) - X^\pi([t]) \right|^{2p}$ we use the expression for $X^\pi_i(t)$, boundedness of $f_{ij}$ for all $i, j$ and use $|e^x - e^y| \leq |e^x + e^y||x - y|$ to obtain

\[
E \sup_{0 \leq t \leq T} \left| X^\pi_i(t) - X^\pi([t]) \right|^{2p} \leq \left\{ E \sup_{0 \leq t \leq T} \left| X^\pi_i(t) + X^\pi([t]) \right|^{2p} \right\}^{1/2} \cdot K \left\{ E \sup_{0 \leq t \leq T} \left[ \sum_{j=1}^d \sum_{[t] \leq s \leq t} \ln(1 + g_{ij}(v_2(s))Y_{j,N(s)}) \right] \right\}^{2p}.
\]

The first factor is bounded and now, we want to bound the second factor:

\[
I := E \sup_{0 \leq t \leq T} \left| \sum_{j=1}^d \sum_{[t] \leq s \leq t} \ln(1 + g_{ij}(v_2(s))Y_{j,N(s)}) \right|^{2p}.
\]

(We use the same notation $I$ to denote different quantities in different occasions and this does not cause ambiguity.) We write the above sum as an integral:

\[
I = E \sup_{0 \leq t \leq T} \left| \sum_{j=1}^d \int_{[t]} \int_{[t]} \ln(1 + z_j g_{ij}(v_2(s))N_j(ds, dz_j))^{2p} \right|
\]

\[
= E \sup_{0 \leq t \leq T} \left| \sum_{j=1}^d \int_{[t]} \int_{[t]} \ln(1 + z_j g_{ij}(v_2(s))\tilde{N}_j(ds, dz_j))^{2p} \right|
\]

\[
+ \left| \sum_{j=1}^d \int_{[t]} \int_{[t]} \ln(1 + z_j g_{ij}(v_2(s))v_j(dz_j))^{2p} \right|
\]

\[
\leq C_p \left( \Delta^{2p} + E \sup_{0 \leq t \leq T} \left| \int_{[t]} \int_{[t]} \ln(1 + z_j g_{ij}(v_2(s))\tilde{N}_j(ds, dz_j))^{2p} \right| \right).
\]
By the Burkholder–Davis–Gundy inequality, we have
\[
\mathbb{E} \sup_{0 \leq t \leq T} \left| \int_\mathcal{J} \int_0^t \ln(1 + z_j g_{ij}(v_2(s))) \tilde{N}_j(ds, dz_j) \right|^{2p} \leq \mathbb{E} \sup_{0 \leq t \leq T} \left( \int_\mathcal{J} \int_0^t \left| \ln(1 + z_j g_{ij}(v_2(s))) \right|^2 v_j(dz_j) ds \right)^p \leq K_p \Delta^p. \tag{3.15}
\]

Plugging above, (3.14), in (3.12) we get for some \( K, K_1, K_2 > 0 \)
\[
\mathbb{E} \sup_{0 \leq t \leq T} \left| S^r_t(t) - v_t(t) \right|^p \leq K_1 \Delta^p + K_2 \Delta^p/2 \leq K \Delta^p/2. \tag{3.16}
\]
This proves the lemma. \( \square \)

**Theorem 3.5.** Assume that Assumptions (A1)–(A4) are true. Then, there is a constant \( K_{pd,T} \), independent of \( \pi \) such that
\[
\mathbb{E} \left[ \sup_{0 \leq t \leq T} \left[ |S(t) - S^\pi(t)|^p \right] \right] \leq K_{pd,T} \Delta^{p/2}. \tag{3.17}
\]

**Proof** First, we want to bound
\[
I_1 := \mathbb{E} \left( \sup_{0 \leq t \leq r} |p(t) - p^\pi(t)|^p \right). \tag{3.18}
\]
From (3.4), we see that when \( t \in [t_k, t_{k+1}] \),
\[
\tilde{F}(S(t-b)) = \int_{t_k}^t F(S(u-b)) du + O(\Delta^2).
\]
Thus
\[
\exp(\tilde{F}(S(t-b))) = I + \int_{t_k}^t F(S(u-b)) du + O(\Delta^2).
\]
Thus we have a formula for \( p(t) \) which is analogous to the one for \( p^\pi(t) \) (Equation (3.7)):
\[
\begin{align*}
p(t) &= \left[ I + \int_{[t]}^t F(S(u-b)) du + O(\Delta^2) \right] \prod_{k=0}^{[t]} \left[ I + \int_{t_k}^{t_{k+1}} F(S(u-b)) du + O(\Delta^2) \right] \\
&= \rho([t], t) \prod_{k=0}^{[t]} \rho(t_k, t_{k+1}), \tag{3.19}
\end{align*}
\]
where
\[
\rho(r, s) = I + \int_r^s F(S(u-b)) du + O(\Delta^2).
\]
We can also write
\[
p^\pi(t) = \rho^\pi([t], t) \prod_{k=0}^{[t]} \rho^\pi(t_k, t_{k+1}), \tag{3.20}
\]
where
\[
\rho^\pi(r, s) = I + F(S^\pi([s] - b))(s - r).
\]
When \( r, s \in [t_k, t_{k+1}] \), \( r < s \), we have by the Lipschitz condition
\[
|\rho(r, s) - \rho^\pi(r, s)| \leq |F(S(t_k - b)) - F(S^\pi(t_k - b))|(s - r) + \int_r^s |F(S(u-b)) - F(S^\pi(t_k - b))| du + O(\Delta^2) \leq C|S(t_k - b) - S^\pi(t_k - b)| + O(\Delta^{3/2}). \tag{3.21}
\]
We also have

$$|\rho^{\pi}(r, s)| = |I + F(S^{\pi}([s] - b))(s - r) \leq |I + C(s - r)| \leq e^{C(s-r)}. \quad (3.22)$$

In the same way we have

$$|\rho(r, s)| \leq e^{C(s-r)}. \quad (3.23)$$

Thus

$$|p^{\pi}(t) - p(t)| \leq |\rho([t], t) - \rho^{\pi}([t], t)| \prod_{k=0}^{[t]} \rho^{\pi}(t_k, t_{k+1})$$

$$+ \sum_{\ell=0}^{[t]} |\rho(t_\ell, t_{\ell+1}) - \rho^{\pi}(t_\ell, t_{\ell+1})| \rho([t], t) \prod_{k=0, k\neq \ell}^{[t]} \rho^{\pi}(t_k, t_{k+1})$$

$$\leq \left[C |S(t_k - b) - S^{\pi}(t_k - b)| + O(\Delta^{3/2}) \right] \prod_{k=0}^{[t]} e^{C(t_{k+1} - t_k)}$$

$$+ \sum_{\ell=0}^{[t]} \left[C |S(t_\ell - b) - S^{\pi}(t_\ell - b)| + O(\Delta^{3/2}) \right] \rho([t], t) \prod_{k=0, k\neq \ell}^{[t]} e^{C(t_{\ell+1} - t_\ell)} \quad (3.24)$$

Thus we have for some $C > 0$

$$I_1 \leq C \mathbb{E} \sup_{0 \leq t \leq r} \left|S(t - b) - S^{\pi}(t - b)\right|^p + K_1 \mathbb{E} \sup_{0 \leq t \leq r} \left|v_2(u) - S^{\pi}(t - b)\right|^p.$$

Then by lemma 3.4 we have

$$I_1 \leq C \mathbb{E} \sup_{0 \leq t \leq r} \left|S(t - b) - S^{\pi}(t - b)\right|^p + C \Delta^{b/2}. \quad (3.25)$$

We now bound $\mathbb{E} \sup_{0 \leq t \leq r} |X(t) - X^{\pi}(t)|^p$. Denote

$$A_{i,t} = \sum_{j=1}^{d} \sum_{0 \leq u \leq t \atop \Delta Z(u) \neq 0} \ln \left(1 + g_{ij}(S(u - b))Y_{j,N_j(u)} \right)$$

$$A_{i,t}^{\pi} = \sum_{j=1}^{d} \sum_{0 \leq u \leq t \atop \Delta Z(u) \neq 0} \ln \left(1 + g_{ij}(v_2(u))Y_{j,N_j(u)} \right) \quad (3.26)$$
and denote $I_2 = \mathbb{E}\left( \sup_{0 \leq t \leq r} |X(t) - X^\pi(t)|^p \right)$. Then,

$$I_2 = \mathbb{E}\left( \sup_{0 \leq t \leq r} |X(t) - X^\pi(t)|^p \right)$$

$$\leq \left( \mathbb{E} \sup_{0 \leq t \leq r} \sum_{i=1}^{d} \sum_{0 \leq u \leq t \atop \Delta S(u) \neq 0} \sum_{j=1}^{d} \left[ \ln(1 + g_{ij} (S(u - b)) Y_{j,N_j(u)}) - \ln(1 + g_{ij} (v(u)) Y_{j,N(u)}) \right] \right)^{1/2} \left( \mathbb{E}\left( \left| \exp(A_{i,t}) + \exp(A_{i,t}^\pi) \right|^2 \right) \right)^{1/2}$$

$$+ \int_0^t \left( f_{ii}(S(u - b)) - f_{ii}(v(u)) \right) du \right|^p \right) \right)^{1/2} \left( \mathbb{E}\left( \left| \exp(A_{i,t}) + \exp(A_{i,t}^\pi) \right|^2 \right) \right)^{1/2}$$

$$= \left( \sum_{i=1}^{d} \mathbb{E} \sup_{0 \leq t \leq r} \left| \int_{\mathbb{J} \times [0,t]} \sum_{j=1}^{d} \left[ \ln(1 + z_j g_{ij} (S(u - b))) - \ln(1 + z_j g_{ij} (v(u))) \right] \nu_j(z) du \right|^p \right)^{1/2}.$$ 

Then for some $C_1 > 0$ we have

$$I_2 \leq \left( C_1 \mathbb{E} \sup_{0 \leq t \leq r} \left| \int_{\mathbb{J} \times [0,t]} \sum_{j=1}^{d} \left[ \ln(1 + z_j g_{ij} (S(u - b))) - \ln(1 + z_j g_{ij} (v(u))) \right] \nu_j(z) du \right|^p \right)^{1/2}$$

$$+ \left( C_1 \mathbb{E} \sup_{0 \leq t \leq r} \left| \int_{\mathbb{J} \times [0,t]} \sum_{j=1}^{d} \left[ \ln(1 + z_j g_{ij} (S(u - b))) - \ln(1 + z_j g_{ij} (v(u))) \right] \nu_j(z) du \right|^p \right)^{1/2}$$

$$+ \left( C_1 \mathbb{E} \sup_{0 \leq t \leq r} \left| \int_0^t \left( f_{ii}(S(u - b)) - f_{ii}(v(u)) \right) du \right|^p \right)^{1/2} \left( \mathbb{E}\left( \left| \exp(A_{i,t}) + \exp(A_{i,t}^\pi) \right|^2 \right) \right)^{1/2}$$

$$= C_1 \left( I_{21}^{1/2} + I_{22}^{1/2} + I_{23}^{1/2} \right) \cdot \left( \mathbb{E}\left( \left| \exp(A_{i,t}) + \exp(A_{i,t}^\pi) \right|^2 \right) \right)^{1/2}.$$ 

Using the Lipschitz condition on $g_{ij}$, $\int_{\mathbb{J}} z_j \nu_j(z) (dz) = K_\nu < \infty$ for $j = 1, 2, \ldots, d$, lemma (3.4) and Assumption 3 we have

$$I_{22} \leq \mathbb{E} \sup_{0 \leq t \leq r} \left| \int_{\mathbb{J} \times [0,t]} \sum_{j=1}^{d} \left[ \ln(1 + z_j g_{ij} (S(u - b))) - \ln(1 + z_j g_{ij} (v(u))) \right] \nu_j(z) du \right|^p \right|^p$$

$$\leq C \mathbb{E} \sup_{0 \leq t \leq r} \left| S(t - b) - S^\pi(t - b) \right|^p + C \mathbb{E} \sup_{0 \leq t \leq r} \left| v(u) - S^\pi(t - b) \right|^p$$

$$= C \mathbb{E} \sup_{0 \leq t \leq r} \left| S(t - b) - S^\pi(t - b) \right|^p + C \Delta^p.$$ 

Using the Burkholder-Davis-Gundy inequality we have

$$I_{21} \leq \sum_{i=1}^{d} \mathbb{E}\left( \int_{\mathbb{J}} \int_0^t \sum_{j=1}^{d} \left[ \ln(1 + z_j g_{ij} (S(u - b))) - \ln(1 + z_j g_{ij} (v(u))) \right] \nu_j(z) du \right)^p.$$ 

Similar to the bound for $I_{22}$ we have

$$I_{21} \leq C \mathbb{E} \sup_{0 \leq t \leq r} \left| S(t - b) - S^\pi(t - b) \right|^p + C \Delta^p.$$
Similar to the bound for $I_{22}$ using assumption (A2) we have
\[
I_{23} \leq C \mathbb{E} \sup_{0 \leq t \leq r} \left| S(t - b) - S^\pi(t - b) \right|^{2p} + C \Delta^p.
\]
Combining the bounds for $I_{21}, I_{22}, I_{23}$ with the help of lemma (3.3), we get for some $K_2 > 0$
\[
I_2 \leq K_2 \left( \mathbb{E} \sup_{0 \leq t \leq r} \left| S(t - b) - S^\pi(t - b) \right|^{2p} \right)^{1/2} + K_2 \Delta^{p/2}. \tag{3.27}
\]
We write $I_3 = \mathbb{E} \left( \sup_{0 \leq t \leq r} |S(t) - S^\pi(t)|^p \right)$. Then we have
\[
I_3 = \mathbb{E} \left( \sup_{0 \leq t \leq r} |S(t) - S^\pi(t)|^p \right) \leq \mathbb{E} \left( \sup_{0 \leq t \leq r} \left| (p(t) - p^\pi(t))X(t) - (X(t) - X^\pi(t))p^\pi(t) \right|^p \right)
\]
\[
\leq 2^{p-1}\mathbb{E} \left( \sup_{0 \leq t \leq r} |p(t) - p^\pi(t))X(t)|^p \right) + 2^{p-1}\mathbb{E} \left( \sup_{0 \leq t \leq r} |p^\pi(t)(X(t) - X^\pi(t))|^p \right).
\]
\[
=: C(I_{31} + I_{32}).
\]
We now bound $I_{31}, I_{32}$
\[
I_{31} \leq C \left( \mathbb{E} \left( \sup_{0 \leq t \leq r} \left| X(t) \right|^{2p} \right) \right)^{1/2} \left( \mathbb{E} \left( \sup_{0 \leq t \leq r} \left| (p(t) - p^\pi(t)) \right|^{2p} \right) \right)^{1/2}. \tag{3.28}
\]
Using the Lemmas 3.3 and 3.25 we will have f
\[
I_{31} \leq C \left( \mathbb{E} \sup_{0 \leq t \leq r} \left| S(t - b) - S^\pi(t - b) \right|^{2p} + \Delta^p \right)^{1/2}.
\]
Using assumption 2 we can show that $p^\pi$ is bounded, hence we can write using (3.27)
\[
I_{32} \leq C \left( \mathbb{E} \sup_{0 \leq t \leq r} \left| S(t - b) - S^\pi(t - b) \right|^{4p} + \Delta^p \right)^{1/2}. \tag{3.29}
\]
Hence we have for some $K_3 > 0$
\[
I_3 \leq K_3 \left( \mathbb{E} \sup_{0 \leq t \leq r} \left| S(t - b) - S^\pi(t - b) \right|^{2p} \right)^{1/2} + K_3 \Delta^{p/2}. \tag{3.30}
\]
Therefore we get
\[
\mathbb{E} \left[ \sup_{0 \leq t \leq r} \left| S(t) - S^\pi(t) \right|^p \right]
\]
\[
\leq C \mathbb{E} \left( \sup_{0 \leq t \leq r} \left| S(t) - S^\pi(t) \right|^{2p} \right)^{1/2} + K \Delta^{p/2}. \tag{3.31}
\]
Taking $r = b$, we have
\[
\mathbb{E} \left[ \sup_{0 \leq t \leq b} \left| S(t) - S^\pi(t) \right|^p \right] \leq C \Delta^{p/2} \tag{3.32}
\]
for any $p \geq 2$. Now, taking $r = 2b$ in (3.31), we have
\[
\mathbb{E} \left[ \sup_{0 \leq t \leq 2b} \left| S(t) - S^\pi(t) \right|^p \right] \leq C \left( \mathbb{E} \sup_{-b \leq t \leq b} \left| S(t) - S^\pi(t) \right|^{2p} \right)^{1/2} + K \Delta^{p/2}
\]
\[
\leq C [K \Delta^p]^{1/2} + K \Delta^{p/2} \leq C \Delta^{p/2}. \tag{3.33}
\]
Continuing this way, we obtain for any positive integer $k \in \mathbb{N}$,
\[
I_{0 \leq t \leq kb} \leq C_{p,k,d,T} \Delta^{p/2} \tag{3.34}
\]
Now, since $T$ is finite, we can choose a $k$ such that $(k - 1)b < T \leq kb$. This completes the proof of the theorem.
References

[1] Bellman, Richard, Introduction to matrix analysis, McGraw-Hill Book Co., Inc., New York-Toronto-London, 1960.
[2] Agrawal, Nishant; Hu, Yaozhong. 2020. Jump Models with Delay—Option Pricing and Logarithmic Euler–Maruyama Scheme Mathematics 8, no. 11: 1932.
[3] Hu, Yaozhong. Multi-dimensional geometric Brownian motions, Onsager-Machlup functions, and applications to mathematical finance, Acta Math. Sci. Ser. B (Engl. Ed.), 2000, no.3, 341–358.
[4] Hu, Y. Calculs formels sur les EDS de Stratonovitch. Séminaire de Probabilités, XXIV, 1988/89, 453-460, Lecture Notes in Math., 1426, Springer, Berlin, 1990.
[5] Strichartz, R. S. The Campbell-Baker-Hausdorff-Dynkin formula and solutions of differential equations. J. Funct. Anal. 72 (1987), 320-345.

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