On Some New Generalized Difference Sequence Spaces of Non-Absolute Type

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Abstract
In this study, we define a new triangle matrix \( \widehat{W} = \{ w_{nk}(r, s, t) \} \) which derived by using multiplication of \( \lambda = (\lambda_{nk}) \) triangle matrix with \( B(r, s, t) \) triple band matrix. Also, we introduce the sequence spaces \( c_0^\lambda(\widehat{B}), c^\lambda(\widehat{B}), \ell_\infty^\lambda(\widehat{B}) \) and \( \ell_p^\lambda(\widehat{B}) \) by using matrix domain of this matrix on the sequence spaces \( c_0, c, \ell_\infty \) and \( \ell_p \) of the matrix \( \widehat{W} \), respectively. Moreover, we show that norm isomorphic to the spaces \( c_0, c, \ell_\infty \) and \( \ell_p \), respectively. Furthermore, we es-tablish some inclusion relations concerning with those spaces and determine \( \alpha-, \beta-, \gamma- \) duals of those spaces and construct their Schauder basis. Finally, we characterize the classes \( (\mu_1^\lambda(\widehat{B}) : \mu_2) \) of infinite matrices , where \( \mu_1 \in \{ c, c_0, \ell_p \} \) and \( \mu_2 \in \{ \ell_\infty, c, c_0, \ell_p \} \).

Key words: Matrix domain of a sequence space, Matrix transformations , Schauder basis, \( \alpha-, \beta- \) and \( \gamma- \) duals
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1. Preliminaries, background and notation

By a sequence spaces , we understand a linear subspace of the space \( \omega \) of all complex sequences which contains \( \phi \), the set of all finitely non-zero sequences. We write \( \ell_\infty, c, c_0 \) and \( \ell_p \) for the classical sequence spaces of

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all bounded, convergent, null and absolutely $p$-summable sequences, respectively, where $1 \leq p < \infty$. Also by $bs$ and $cs$, we denote the spaces of all bounded and convergent series, respectively. We assume throughout unless stated otherwise that $p, q > 1$ with $p^{-1} + q^{-1} = 1$ and use the convention that any term with negative subscript is equal to zero. We denote throughout that the collection of all finite subsets of $\mathbb{N}$ by $\mathcal{F}$.

Let $A = (a_{nk})$ be an infinite matrix of complex numbers $a_{nk}$, where $n, k \in \mathbb{N}$. Then, $A$ defines a matrix mapping from $X$ to $Y$ and is denote by $A : X \to Y$ if for every sequence $x = (x_k) \in X$ the sequence $Ax = \{(Ax)_n\}$, the $A$-transform of $x$, is in $Y$ where

$$\left(\begin{array}{c}
(Ax)_n = \sum_{k} a_{nk}x_k, \quad (n \in \mathbb{N})
\end{array}\right) \quad (1)$$

By $(X : Y)$, denote the class of all matrices $A$ such that $A : X \to Y$. Thus, $A \in (X : Y)$ if and only if the series on the right side (1) converges each $n \in \mathbb{N}$ and $x \in X$, and we have $Ax = \{(Ax)_n\}_{n \in \mathbb{N}} \in Y$ for all $x \in X$. A sequence $x \in \omega$ is said to be $A$-summable to $l$ if $Ax$ converges to $l$, which is called the $A$-limit of $x$.

A matrix $A = (a_{nk})$ is called a triangle if $a_{nk} = 0$ for $k > n$ and $a_{nn} \neq 0$ for all $n, k \in \mathbb{N}$. It is trivial that $A(Bx) = (AB)x$ holds for the triangle matrices $A, B$ and a sequence $x$. Further, a triangle matrix $U$ uniquely has an inverse $U^{-1} = V$ which is also triangle matrix. Then, $x = U(Vx) = V(Ux)$ holds for all $x \in \omega$.

Let us give the definition of some triangle limitation matrices which are needed in the text. Let $q = (q_k)$ be a sequence of positive reals and write

$$Q_n = \sum_{k=0}^{n} q_k, \quad (n \in \mathbb{N}).$$

Then the Cesàro mean of order one, Riesz mean with respect to the sequence $q = (q_k)$ and Euler mean of order $r$ with $0 < r < 1$ are respectively defined by the matrices $C = (c_{nk}), R^q = (r^q_{nk})$ and $E^r = (e^r_{nk})$; where

$$c_{nk} = \begin{cases} 
\frac{1}{n+1}, & (0 \leq k \leq n), \\
0, & (k > n),
\end{cases} \quad r^q_{nk} = \begin{cases} 
\frac{q_k}{Q_n}, & (0 \leq k \leq n), \\
0, & (k > n),
\end{cases}$$

and

$$e^r_{nk} = \begin{cases} 
\binom{n}{k} (1-r)^{n-k} r^k, & (0 \leq k \leq n) \\
0, & (k > n)
\end{cases}$$
for all \( k, n \in \mathbb{N} \). We write \( U \) for the set of all sequences \( u = (u_k) \) such that \( u_k \neq 0 \) for all \( k \in \mathbb{N} \). For \( u \in U \), let \( 1/u = (1/u_k) \). Let \( z, u, v \in U \), and define the summation matrix \( S = (s_{nk}) \), the difference matrix \( \Delta = (\Delta_{nk}^{(1)}) \), the generalized weighted mean or factorable matrix \( G(u, v) = (g_{nk}) \), \( A_u^r = \{a_{nk}^r(u)\} \), \( \Delta^{(m)} = (\Delta_{nk}^{(m)}) \) by

\[
\begin{align*}
    s_{nk} &= \begin{cases} 
        1, & (0 \leq k \leq n), \\
        0, & (k > n),
    \end{cases} \quad
    \Delta_{nk}^{(1)} = \begin{cases} 
        (-1)^{n-k}, & (n - 1 \leq k \leq n), \\
        0, & (0 \leq k < n - 1 \text{ or } k > n),
    \end{cases} \\
    g_{nk} &= \begin{cases} 
        u_nv_k, & (0 \leq k \leq n), \\
        0, & (k > n),
    \end{cases} \quad
    a_{nk}^r(u) = \begin{cases} 
        1 + r^k/n + 1 u_k, & (0 \leq k \leq n), \\
        0, & (k > n),
    \end{cases}
\end{align*}
\]

\( \Delta_{nk}^{(m)} = \begin{cases} 
    (-1)^{n-k} \left( \frac{m}{n - k} \right), & (\max\{0, n - m\} \leq k \leq n) \\
    0, & (0 \leq k < n - 1 \text{ or } k > n)
\end{cases} \) for all \( k, n \in \mathbb{N} \), where \( u_n \) depends only on \( n \) and \( v_k \) only on \( k \). Let \( r \) and \( s \) be non-zero real numbers, and define the generalized difference matrix \( B(r, s) = \{b_{nk}(r, s)\} \) by

\[
    b_{nk}(r, s) = \begin{cases} 
        r, & (k = n), \\
        s, & (k = n - 1), \\
        0, & (0 \leq k < n - 1 \text{ or } k > n),
    \end{cases}
\]

for all \( k, n \in \mathbb{N} \). The \( B(r, s) \)-transform of a sequence \( x = (x_k) \) is

\[
    B(r, s)_k(x) = rx_k + sx_{k-1} \quad \text{for all } k \in \mathbb{N}.
\]

We note that the matrix \( B(r, s) \) can be reduced to the difference matrices \( \Delta \) in case \( r = 1 \) and \( s = -1 \).}

For a sequence space \( X \), the matrix domain \( X_A \) of an infinite matrix \( A \) is defined by

\[
    X_A = \{x = (x_k) \in \omega : Ax \in X\}, \tag{2}
\]

which is a sequence space. If \( A \) is triangle, then one can easily observe that the sequence space \( X_A \) and \( X \) are linearly isomorphic, i.e., \( X_A \cong X \).

Although in the most cases the new sequence space \( X_A \) generated in the limitation matrix \( A \) from a sequence space \( X \) is the expansion or the
contraction of the original space $X$, it may be observed in some cases that those space overlap. Indeed, one can easily see that the inclusion $X_S \subset X$ strictly holds for $X \in \{ \ell_\infty, c, c_0 \}$. As this, one can deduce that the inclusion $X \subset X_{\Delta^1}$ also strictly holds for $X \in \{ \ell_\infty, c, c_0, \ell_p \}$. However, if we define $X = c_0 \oplus \text{span}\{z\}$ with $z = ((-1))^k$, i.e., $x \in X$ if and only if $x := \beta + \alpha z$ for some $\beta \in c_0$ and some $\alpha \in \mathbb{C}$, and consider the matrix $A$ with the rows $A_n$ defined by $A_n = ((-1))^n e^{(n)}$ for all $n \in \mathbb{N}$, we have $Ae = z \in X$ but $Az = e \notin X$ which lead us to the consequences that $z \in X \setminus X_A$ and $e \in X_A \setminus X$, where $e = (1, 1, 1, \ldots)$ and $e^{(n)}$ is a sequence whose only non-zero term is a 1 in $n$th place for each $n \in \mathbb{N}$. That is to say that the sequence spaces $X_A$ and $X$ overlap but neither contains to other.

The approach constructing a new sequence space by means of the matrix domain of a particular limitation method has recently been employed by Wang [1], Ng and Lee [2], Malkowsky [3], Altay and Başar [4], Malkowsky and Savaş [5], Başarır [6], Aydın and Başar [7], Başar et al. [8], Şengönül and Başar [9], Altay [10], Polat and Başar [11], and Malkowsky et al. [12]. $\Delta, \Delta^2$ and $\Delta^m$ are the transposes of the matrices $\Delta^{(1)}, \Delta^{(2)}$ and $\Delta^{(m)}$, respectively, and $c_0(u, p)$ are the spaces consisting of the sequences $x = (x_k)$ such that $ux = (u_k x_k)$ in the spaces $c_0(p)$ and $c(p)$ for $u \in \mathcal{U}$, respectively, studied by Başarır [6]. More recently, the generalized difference matrix $B(r, s) = \{b_{nk}(r, s)\}$ has been used by Kirişçi and Başar [13] for generalizing the difference spaces $\ell_\infty(\Delta), c(\Delta), c_0(\Delta)$, and $bv_p$. Finally, the new technique for deducing certain topological properties, for example $AB-$, $KB-$, $AD-$ properties, etc., and determining the $\beta-$ and $\gamma-$ duals of the domain of a triangle matrix in a sequence space has been given by Altay and Başar [14].

Let $r, s$ and $t$ be non-zero real numbers, and define the generalized difference matrix $\hat{B} = B(r, s, t) = \{b_{nk}(r, s, t)\}$ by

$$b_{nk}(r, s, t) = \begin{cases} r, & (k = n) \\ s, & (k = n - 1) \\ t, & (k = n - 2) \\ 0, & (0 \leq k < n - 1 \text{ or } k > n) \end{cases}$$

for all $n, k \in \mathbb{N}$. The inverse of $B(r, s, t) = \{b_{nk}(r, s, t)\}$, which is denote
\(B^{-1}(r, s, t) = \{d_{nk}(r, s, t)\}\) is given by

\[
d_{nk}(r, s, t) = \begin{cases} 
\frac{1}{r} \sum_{v=0}^{n-k} \left( \frac{-s + \sqrt{s^2 - 4tr}}{2r} \right)^{n-k-v} \left( \frac{-s - \sqrt{s^2 - 4tr}}{2r} \right)^v, & (0 \leq k \leq n), \\
0, & (k > n),
\end{cases}
\]

\(k \leq n\),

(4)

We should record here that \(B(r, s, 0) = B(r, s)\), \(B(1, -2, 1) = \Delta^{(2)}\) and \(B(1, -1, 0) = \Delta^{(1)}\). So, the results related to the matrix domain of the triple band matrix \(B(r, s, t)\) are more general and more comprehensive than the consequences on the matrix domain of \(B(r, s), \Delta^{(2)}\) and \(\Delta^{(1)}\), and include them. We assume throughout that \(\lambda = (\lambda_k)_{k=0}^{\infty}\) is a strictly increasing sequence of positive reals tending to \(\infty\), that is

\[0 < \lambda_0 < \lambda_1 < ... \quad \text{and} \quad \lim_{k \to \infty} \lambda_k = \infty.\]

(5)

The main purpose of the present paper is to introduce the sequence space \(\mu^\lambda(\hat{B})\) and to determine the \(\alpha-, \beta-\) and \(\gamma-\) duals of the space, where \(\mu\) denotes the any of the classical spaces \(\ell_\infty, c, c_0\) or \(\ell_p\), and \(\hat{B}\) is the triple band matrix \(B(r, s, t)\) and the sequence \(\lambda = (\lambda_k)\) is defined in (5). Furthermore, the Schauder bases for the spaces \(c_0^\lambda(\hat{B}), c^\lambda(\hat{B})\) and \(\ell_p^\lambda(\hat{B})\) are given, and some topological properties of the spaces \(c_0^\lambda(\hat{B}), c^\lambda(\hat{B})\) and \(\ell_p^\lambda(\hat{B})\) are examined. Finally, some classes of matrix mappings on the space \(\mu^\lambda(\hat{B})\) are characterized.

The paper is organized as follows: In Section 2, the \(BK-\) spaces \(c_0^\lambda(\hat{B}), c^\lambda(\hat{B}), \ell_\infty^\lambda(\hat{B})\) and \(\ell_p^\lambda(\hat{B})\) of generalized difference sequences are introduced and the Schauder bases of the spaces \(c_0^\lambda(\hat{B}), c^\lambda(\hat{B})\) and \(\ell_p^\lambda(\hat{B})\) are given. In Section 3, are examined some inclusion relations concerning with those space. In Section 4, the \(\alpha-, \beta-\) and \(\gamma-\) duals of the generalized difference sequence space \(\mu^\lambda(\hat{B})\) of non-absolute type is determined, respectively. In Section 5, the classes \((\mu_1^\lambda(\hat{B}) : \mu_2)\) of infinite matrices are characterized, where \(\mu_1 \in \{c, c_0, \ell_p\}\) and \(\mu_2 \in \{\ell_\infty, c, c_0, \ell_p\}\).

2. The Difference Sequence Spaces \(c_0^\lambda(\hat{B}), c^\lambda(\hat{B}), \ell_\infty^\lambda(\hat{B})\) and \(\ell_p^\lambda(\hat{B})\) of Non-Absolute Type

The difference sequence spaces have been studied by several authors in different ways [15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26]. In the present
section, we introduce the spaces $c_0^\lambda(\hat{B}), c^\lambda(\hat{B}), \ell_\infty^\lambda(\hat{B})$ and $\ell_p^\lambda(\hat{B})$ and show that these spaces are $BK-$ spaces of non-absolute type which are norm isomorphic to the spaces $c_0, c, \ell_\infty$ and $\ell_p,$ respectively. Furthermore, we give the basis for the spaces $c_0^\lambda(\hat{B}), c^\lambda(\hat{B})$ and $\ell_p^\lambda(\hat{B})$.

Let $\lambda = (\lambda_k)_{k=0}^\infty$ be a strictly increasing sequence of positive reals tending to infinity, that is

$$0 < \lambda_0 < \lambda_1 < \lambda_2 < \ldots \text{ and } \lambda_k \to \infty \text{ as } k \to \infty.$$ 

We say that a sequence $x = (x_k) \in \omega$ is $\lambda-$ convergent to the number $l \in \mathbb{C},$ called the $\lambda-$ limit of $x$, if $\Lambda_n(x) \to l$ as $n \to \infty$ where

$$\Lambda_n(x) = \frac{1}{\lambda_n} \sum_{k=0}^{n} (\lambda_k - \lambda_{k-1})x_k; \quad (n \in \mathbb{N}). \quad (6)$$

In particular, we say that $x$ is a $\lambda-$ null sequence if $\Lambda_n(x) \to 0$ as $n \to \infty.$

Further, we say that $x$ is $\lambda-$ bounded if $\sup_{n \in \mathbb{N}} |\Lambda_n(x)| < \infty.$ Recently, Mursaleen and Noman [27, 28] studied the sequence spaces $c_0^\lambda, c^\lambda, \ell_\infty^\lambda$ and $\ell_p^\lambda$ of non-absolute type as follows:

$$c_0^\lambda = \{ x = (x_k) \in \omega : \lim_{n \to \infty} \frac{1}{\lambda_n} \sum_{k=0}^{n} (\lambda_k - \lambda_{k-1})x_k = 0 \}$$

$$c^\lambda = \{ x = (x_k) \in \omega : \lim_{n \to \infty} \frac{1}{\lambda_n} \sum_{k=0}^{n} (\lambda_k - \lambda_{k-1})x_k \text{ exists} \}$$

$$\ell_\infty^\lambda = \{ x = (x_k) \in \omega : \sup_{n \in \mathbb{N}} \left| \frac{1}{\lambda_n} \sum_{k=0}^{n} (\lambda_k - \lambda_{k-1})x_k \right| < \infty \}$$

and

$$\ell_p^\lambda = \{ x = (x_k) \in \omega : \sum_{n=0}^{\infty} \left| \frac{1}{\lambda_n} \sum_{k=0}^{n} (\lambda_k - \lambda_{k-1})x_k \right|^p < \infty \}. \quad (7)$$

On the other hand, we define the matrix $\hat{\Lambda} = (\hat{\lambda}_{nk})$ for all $n, k \in \mathbb{N}$ by

$$\hat{\lambda}_{nk} = \begin{cases} \frac{\lambda_k - \lambda_{k-1}}{\lambda_n}, & (0 \leq k \leq n) \\ 0, & (k > n). \end{cases}$$
Then, it can be easily seen that the equality

$$\widehat{\Lambda}_n(x) = \frac{1}{\lambda_n} \sum_{k=0}^{n} (\lambda_k - \lambda_{k-1}) x_k$$  \hspace{1cm} (8)$$

holds for all \( n \in \mathbb{N} \) and every \( x = (x_k) \in \omega \), which leads us together with to the fact that

$$c_0^\lambda = \{c_0\}_{\widehat{\Lambda}}, \quad c^\lambda = c_{\widehat{\Lambda}}, \quad \ell^\lambda = \{\ell_n\}_{\widehat{\Lambda}}, \quad \ell^\lambda_p = \{\ell_p\}_{\widehat{\Lambda}}.$$  

More recently, Sönmez \[29\] has defined the sequence spaces \( \ell^\lambda(\widehat{B}), c(\widehat{B}), c_0(\widehat{B}) \) and \( \ell^\lambda_p(\widehat{B}) \) as follows:

$$\ell^\lambda(\widehat{B}) = \{ x = (x_k) \in \omega : \sup_{k \in \mathbb{N}} |r x_k + s x_{k-1} + t x_{k-2}| < \infty \},$$

$$c(\widehat{B}) = \{ x = (x_k) \in \omega : \exists l \in \mathbb{C} \ni \lim_{k \to \infty} |r x_k + s x_{k-1} + t x_{k-2} - l| = 0 \},$$

$$c_0(\widehat{B}) = \{ x = (x_k) \in \omega : \lim_{k \to \infty} |r x_k + s x_{k-1} + t x_{k-2}| = 0 \},$$

$$\ell^\lambda_p(\widehat{B}) = \{ x = (x_k) \in \omega : \sum_k |r x_k + s x_{k-1} + t x_{k-2}|^p < \infty \}.$$

In fact, the sequence spaces \( \ell^\lambda(\widehat{B}), c(\widehat{B}), c_0(\widehat{B}) \) and \( \ell^\lambda_p(\widehat{B}) \) can be consider as the set of all sequences whose \( B(r, s, t) \) - transforms are in the spaces \( \ell^\lambda, c, c_0 \) and \( \ell^\lambda_p \), respectively. That is,

$$\ell^\lambda(\widehat{B}) = \{ \ell^\lambda \}_{B(r,s,t)}, \quad c(\widehat{B}) = c_{B(r,s,t)}, \quad c_0(\widehat{B}) = \{ c_0 \}_{B(r,s,t)}, \quad \ell^\lambda_p(\widehat{B}) = \{ \ell^\lambda_p \}_{B(r,s,t)}.$$

Now, we introduce the difference sequence spaces \( \ell^\lambda(\widehat{B}), c^\lambda(\widehat{B}), c_0^\lambda(\widehat{B}) \) and \( \ell^\lambda_p(\widehat{B}) \) as follows:

$$\ell^\lambda(\widehat{B}) = \{ x = (x_n) \in \omega : \sup_{n \in \mathbb{N}} \left| \frac{1}{\lambda_n} \sum_{k=0}^{n} (\lambda_k - \lambda_{k-1})(r x_k + s x_{k-1} + t x_{k-2}) \right| < \infty \},$$

$$c^\lambda(\widehat{B}) = \{ x = (x_n) \in \omega : \lim_{n \to \infty} \frac{1}{\lambda_n} \sum_{k=0}^{n} (\lambda_k - \lambda_{k-1})(r x_k + s x_{k-1} + t x_{k-2}) \text{ exists} \},$$

$$c_0^\lambda(\widehat{B}) = \{ x = (x_n) \in \omega : \lim_{n \to \infty} \frac{1}{\lambda_n} \sum_{k=0}^{n} (\lambda_k - \lambda_{k-1})(r x_k + s x_{k-1} + t x_{k-2}) = 0 \},$$

$$\ell^\lambda_p(\widehat{B}) = \{ x = (x_n) \in \omega : \sum_n \left| \frac{1}{\lambda_n} \sum_{k=0}^{n} (\lambda_k - \lambda_{k-1})(r x_k + s x_{k-1} + t x_{k-2}) \right|^p < \infty \}.$$

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On the other hand, we define the triangle matrix \( \hat{W} = \{ w_{nk}^\lambda \} = \hat{\Lambda} \hat{B} \) by

\[
w_{nk}^\lambda = \begin{cases} 
\frac{r(\lambda_k - \lambda_{k-1}) + s(\lambda_{k+1} - \lambda_k) + t(\lambda_{k+2} - \lambda_{k+1})}{\lambda_n}, & (k < n - 1) \\
\frac{r(\lambda_{n-1} - \lambda_{n-2}) + s(\lambda_n - \lambda_{n-1})}{\lambda_n}, & (k = n - 1) \\
\frac{r(\lambda_n - \lambda_{n-1})}{\lambda_n}, & (k = n) \\
0, & \text{(otherwise)}
\end{cases}
\]

for all \( k, n \in \mathbb{N} \). Then, it can be easily seen that the equality

\[ \hat{W}_n(x) = \frac{1}{\lambda_n} \sum_{k=0}^{n} (\lambda_k - \lambda_{k-1})(rx_k + sx_{k-1} + tx_{k-2}) \]

holds for all \( n \in \mathbb{N} \) and every \( x = (x_k) \in \omega \). In fact, the sequence spaces \( c_0^\lambda(\hat{B}), c^\lambda(\hat{B}), \ell_\infty^\lambda(\hat{B}) \) and \( \ell_p^\lambda(\hat{B}) \) can be considered as the set of all sequences whose \( \hat{W} - \) transforms are in the spaces \( c_0, c, \ell_\infty \) and \( \ell_p, \) respectively. That is,

\[ c_0^\lambda(\hat{B}) = \{ c_0 \hat{W} \}, \quad c^\lambda(\hat{B}) = c \hat{W}, \quad \ell_\infty^\lambda(\hat{B}) = \{ \ell_\infty \hat{W} \}, \quad \ell_p^\lambda(\hat{B}) = \{ \ell_p \hat{W} \}. \]

Further, for any sequence \( x = (x_k) \) we define the sequence \( y_k(\lambda) = \{ y_k(\lambda) \} \) which will be used, as the \( \hat{W} \)-transform of \( x \) and so we have

\[ y_k(\lambda) = \sum_{j=0}^{k-2} \frac{r(\lambda_j - \lambda_{j-1}) + s(\lambda_{j+1} - \lambda_j) + t(\lambda_{j+2} - \lambda_{j+1})}{\lambda_k} x_j \]

\[ + \frac{r(\lambda_{k-1} - \lambda_{k-2}) + s(\lambda_k - \lambda_{k-1})}{\lambda_k} x_{k-1} + \frac{r(\lambda_k - \lambda_{k-1})}{\lambda_k} x_k; \quad (k \in \mathbb{N}). \]

Since the proof may also be obtained in the similar way as for the other spaces, to avoid the repetition of the similar statements, we give the proof only for one of those spaces. Now, we may begin with the following theorem which is essential in the study.

**Theorem 1.** (i) The difference sequence spaces \( c_0^\lambda(\hat{B}), c^\lambda(\hat{B}) \) and \( \ell_\infty^\lambda(\hat{B}) \) are \( BK \)-space with the norm \( \| x \|_{c_0^\lambda(\hat{B})} = \| x \|_{c^\lambda(\hat{B})} = \| x \|_{\ell_\infty^\lambda(\hat{B})} = \| \hat{W}(x) \|_\infty \), that is,

\[ \| x \|_{c_0^\lambda(\hat{B})} = \| x \|_{c^\lambda(\hat{B})} = \| x \|_{\ell_\infty^\lambda(\hat{B})} = \sup_{n \in \mathbb{N}} | \hat{W}_n(x) |. \]
\( \text{(ii) Let } 1 \leq p < \infty. \text{ Then } \ell_p^\lambda(\hat{B}) \text{ is a } BK-\text{ space with the norm} \)
\[
\|x\|_{\ell_p^\lambda(\hat{B})} = \|\hat{W}x\|_p, \text{ that is,}
\]
\[
\|x\|_{\ell_p^\lambda(\hat{B})} = \left( \sum_n |\hat{W}_n(x)|^p \right)^{1/p}.
\]

\textbf{Proof.} Since \((11)\) holds and \(c_0, c \text{ and } \ell_\infty \text{ are } BK-\text{ spaces with respect to their natural norms (see } [30, \text{ pp. 16-17}] \text{)} \text{ and the matrix } \hat{W} \text{ is a triangle, Theorem 4.3.12 Wilansky [31, pp. 63] gives the fact that} c_0^\lambda(\hat{B}), c^\lambda(\hat{B}) \text{ and} \ell_\infty^\lambda(\hat{B}) \text{ are } BK-\text{ spaces with the given norms. This completes the proof.}

\textbf{Remark 1.} \text{Let } \mu \in \{\ell_\infty, c, c_0, \ell_p\} \text{ and } |x| = |x_k|. \text{ Then the absolute property does not hold on the space } \mu^\lambda(\hat{B}), \text{ that is, } \|x\|_{\mu^\lambda(\hat{B})} \neq \|x\|_{\mu^\lambda(\hat{B})}. \text{ This can be shown that for at least one sequence in those space. Hence } \mu^\lambda(\hat{B}) \text{ is the sequence space of non-absolutely type.}

\textbf{Theorem 2.} \text{The sequence spaces } c_0^\lambda(\hat{B}), c^\lambda(\hat{B}), \ell_\infty^\lambda(\hat{B}) \text{ and } \ell_p^\lambda(\hat{B}) \text{ of non-absolutely type are norm isomorphic to the spaces } c_0, c, \ell_\infty \text{ and } \ell_p, \text{ respectively, that is, } c_0^\lambda(\hat{B}) \cong c_0, c^\lambda(\hat{B}) \cong c, \ell_\infty^\lambda(\hat{B}) \cong \ell_\infty \text{ and } \ell_p^\lambda(\hat{B}) \cong \ell_p.

\textbf{Proof.} \text{We prove the theorem for the space } c_0^\lambda(\hat{B}). \text{ To prove our assertion we should show the existence of a linear bijection between the spaces } c_0^\lambda(\hat{B}) \text{ and } c_0. \text{ Let } T : c_0^\lambda(\hat{B}) \to c_0 \text{ be defined by } (10). \text{ Then, } T(x) = y_k(\lambda) = \hat{W}(x) \in c_0 \text{ for every } x \in c_0^\lambda(\hat{B}) \text{ and the linearity of } T \text{ is clear. Further, it is trivial that } x = 0 \text{ whenever } Tx = \theta \text{ and hence } T \text{ is injective.}

\text{Moreover, let } y = (y_k) \in c_0 \text{ and define the sequence } x = x_k(\lambda) \text{ by}
\]
\[
x_k(\lambda) = \sum_{j=0}^{k} \frac{1}{r} \sum_{v=0}^{k-j} \left( \frac{-s + \sqrt{s^2 - 4tr}}{2r} \right)^{k-j-v} \left( \frac{-s - \sqrt{s^2 - 4tr}}{2r} \right)^v \sum_{i=j-1}^{j} (-1)^{j-i} \frac{\lambda_i}{\lambda_j - \lambda_{j-1}} y_i.
\]
\]

Then we obtain
\[
rx_k(\lambda) + sx_{k-1}(\lambda) + tx_{k-2}(\lambda) = \sum_{j=k-1}^{k} (-1)^{k-j} \frac{\lambda_j}{\lambda_k - \lambda_{k-1}} y_j \quad \text{for all } k \in \mathbb{N}.
\]

Hence, for every \( n \in \mathbb{N} \) we get by \((10)\)
\[
\hat{W}_n(x) = \frac{1}{\lambda_n} \sum_{k=0}^{n} (\lambda_k - \lambda_{k-1})(rx_k + sx_{k-1} + tx_{k-2}) = \frac{1}{\lambda_n} \sum_{k=0}^{n} \sum_{j=k-1}^{k} (-1)^{k-j} \lambda_j y_j = y_n.
\]

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Thus, we deduce that $x^T$ shows that $c_0$, we conclude that $\hat{W}(x) \in c_0$. Thus, we deduce that $x \in c_0^\lambda(\hat{B})$ and $Tx = y$. Hence $T$ is surjective.

Moreover one can easily see for every $x \in c_0^\lambda(\hat{B})$ that

$$\|Tx\|_\infty = \|y_k(\lambda)\|_\infty = \|\hat{W}x\|_\infty = \|x\|_{c_0^\lambda(\hat{B})}$$

which means that $T$ is norm preserving. Consequently $T$ is a linear bijection which show that the spaces $c_0^\lambda(\hat{B})$ and $c_0$ are linearly isomorphic, as desired.

Let $(X, \|\cdot\|)$ be a normed space. A sequence $(b_k)$ of elements of $X$ is called a Schauder basis for $X$ if and only if, for each $x \in X$ there exists a unique sequence $(\alpha_k)$ of scalars such that $x = \sum_k \alpha_k b_k$, i.e. such that

$$\lim_{n \to \infty} \|x - \sum_{k=0}^n \alpha_k b_k\| = 0.$$

Because of the isomorphism $T$, defined in the proof of Theorem 2, is onto the inverse image of the basis of those space $c_0, c$ and $\ell_p$ are the basis of new spaces $c_0^\lambda(\hat{B}), c^\lambda(\hat{B})$ and $\ell_p^\lambda(\hat{B})$ respectively. Therefore, we have the following:

**Theorem 3.** Let $\alpha_k(\lambda) = \hat{W}_k(x)$ for all $k \in \mathbb{N}$ and $\lim_{k \to \infty} \hat{W}_k(x) = 1$. Define the sequence $b^{(k)}(\lambda) = \{b^{(k)}_n(\lambda)\}_{n \in \mathbb{N}}$ for every fixed $k \in \mathbb{N}$ by

$$b^{(k)}_n(\lambda) = \begin{cases} \frac{d_{nk}(r, s, t)\lambda_k}{\lambda_k - \lambda_{k-1}} - \frac{d_{nk+1}(r, s, t)\lambda_k}{\lambda_{k+1} - \lambda_k}, & (n > k) \\ \frac{1}{r} \frac{\lambda_k - \lambda_{k-1}}{\lambda_k - \lambda_{k+1}}, & (n = k) \\ 0, & (n < k). \end{cases} \quad (14)$$

Then, the following statements hold:

(i) The sequence $\{b^{(k)}_n(\lambda)\}_{n \in \mathbb{N}}$ is a basis for the spaces $c_0^\lambda(\hat{B})$ and any $x \in c_0^\lambda(\hat{B})$ has a unique representation of the norm $x = \sum_k \alpha_k(\lambda)b^{(k)}(\lambda)$.

(ii) The sequence $\{b^{(k)}_n(\lambda)\}_{n \in \mathbb{N}}$ is a basis for the spaces $\ell_p^\lambda(\hat{B})$ and any $x \in \ell_p^\lambda(\hat{B})$ has a unique representation of the norm $x = \sum_k \alpha_k(\lambda)b^{(k)}(\lambda)$.

(iii) The sequence $\{b, b^{(0)}(\lambda), b^{(1)}(\lambda), \ldots\}$ is a basis for the space $c^\lambda(\hat{B})$ and any $x \in c^\lambda(\hat{B})$ has a unique representation of the norm $x = lb + \sum_k [\alpha_k(\lambda) - l]b^{(k)}(\lambda)$, where $b = (b_k) = \left(\sum_{j=0}^k d_{kj}\right)$.
3. The Inclusion Relations

In the present section, we prove some inclusion relations concerning with the spaces $c_0^\lambda(\hat{B}), c^\lambda(\hat{B}), \ell_\infty^\lambda(\hat{B})$ and $\ell_p^\lambda(\hat{B})$.

**Theorem 4.** The inclusion $c_0^\lambda(\hat{B}) \subset c^\lambda(\hat{B})$ strictly holds.

**Proof.** It is obvious that the inclusion $c_0^\lambda(\hat{B}) \subset c^\lambda(\hat{B})$ holds. Further to show that this inclusion is strict, consider the sequence $x = (x_k)$ defined by

$$x_k = \sum_{j=0}^{k} d_{kj}$$

for all $k \in \mathbb{N}$. Then

$$rx_k + sx_{k-1} + tx_{k-2} = \sum_{j=k-1}^{k} (-1)^{k-j} \frac{\lambda_j}{\lambda_k - \lambda_{k-1}}$$

Then we obtain by (10) that

$$\hat{W}(x) = \frac{1}{\lambda_n} \sum_{k=0}^{n} (\lambda_k - \lambda_{k-1})(rx_k + sx_{k-1} + tx_{k-2}) = \frac{1}{\lambda_n} \sum_{k=0}^{n} (\lambda_k - \lambda_{k-1}) = 1$$

for all $n \in \mathbb{N}$, which shows that $\hat{W}(x) = e$ and hence $\hat{W}(x) \in c^\lambda \setminus c_0$, where $e = (1, 1, 1, ...)$). Thus, this sequence $x$ is in $c^\lambda(\hat{B})$ but not in $c_0^\lambda(\hat{B})$. Hence, the inclusion $c_0^\lambda(\hat{B}) \subset c^\lambda(\hat{B})$ is strict and this completes the proof of the theorem.

**Theorem 5.** If $r + s + t = 0$ then the inclusion $c \subset c_0^\lambda(\hat{B})$ strictly hold.

**Proof.** Suppose that $r + s + t = 0$ and $x \in c$. Then $\hat{B}x = B(r, s, t)(x) = (rx_k + sx_{k-1} + tx_{k-2}) \in c_0$ and hence $\hat{B}(x) \in c_0^\lambda$ since the inclusion $c_0 \subset c_0^\lambda$ [27]. This show that $x \in c_0^\lambda(\hat{B})$, i.e., $c \subset c_0^\lambda(\hat{B})$ holds. Further consider the sequence $y = (y_k)$ defined by $y_k = \ln(k+3)$ for all $k \in \mathbb{N}$. Then it is trivial that $y \notin c$. On the other hand, it can easily seen that $\hat{B}y \in c_0$. Hence $\hat{B}y \in c_0^\lambda$ which means that $y \in c_0^\lambda(\hat{B})$. Thus the sequence $y \in c_0^\lambda(\hat{B}) \setminus c$. Hence, the inclusion $c \subset c_0^\lambda(\hat{B})$ is strict. This completes the proof.
Lemma 1. $A \in (\ell_{\infty} : c_0)$ if and only if $\lim_{n \to \infty} \sum_k |a_{nk}| = 0$.

Theorem 6. Let $z = (z_k)$ be defined by

$$z_k = \left| \frac{r(\lambda_k - \lambda_{k-1}) + s(\lambda_{k+1} - \lambda_k) + t(\lambda_{k+2} - \lambda_{k+1})}{\lambda_k - \lambda_{k-1}} \right|$$

for all $k \in \mathbb{N}$. Then the inclusion $\ell_{\infty} \subset c_0^\lambda(\hat{B})$ strictly holds if and only if $z \in c_0^\lambda$.

**Proof.** Let $\ell_{\infty}$ be a subset of $c_0^\lambda(\hat{B})$. Then we obtain that $\hat{W}(x) \in c_0$ for every $x \in \ell_{\infty}$ and the matrix $\hat{W} = \{w_{nk}^\lambda\}$ is in the class $(\ell_{\infty} : c_0)$. By using Lemma 1, it follows that

$$\lim_{n \to \infty} \sum_k |w_{nk}^\lambda| = 0.$$  \hspace{1cm} (15)

Now, by taking into account the definition of matrix $\hat{W} = \{w_{nk}^\lambda\}$ given by (14), we have for every $n \in \mathbb{N}$ that

$$\sum_k |w_{nk}^\lambda| = \frac{1}{\lambda_n} \sum_{k=0}^{n-2} \left| r(\lambda_k - \lambda_{k-1}) + s(\lambda_{k+1} - \lambda_k) + t(\lambda_{k+2} - \lambda_{k+1}) \right| \lambda_k - \lambda_{k-1} + \left| r(\lambda_{n-1} - \lambda_{n-2}) + s(\lambda_n - \lambda_{n-1}) \right| \lambda_n$$

\hspace{2cm} (16)

Thus, the condition (16) implies that

$$\lim_{n \to \infty} \left| r \left( \frac{\lambda_n - \lambda_{n-1}}{\lambda_n} \right) \right| = 0$$ \hspace{1cm} (17)

$$\lim_{n \to \infty} \left| \frac{r(\lambda_{n-1} - \lambda_{n-2}) + s(\lambda_n - \lambda_{n-1})}{\lambda_n} \right| = 0$$ \hspace{1cm} (18)

$$\lim_{n \to \infty} \frac{1}{\lambda_n} \sum_{k=0}^{n-2} \left| r(\lambda_k - \lambda_{k-1}) + s(\lambda_{k+1} - \lambda_k) + t(\lambda_{k+2} - \lambda_{k+1}) \right| = 0.$$ \hspace{1cm} (19)

Now, we have for every $n \geq 1$ that

$$\frac{1}{\lambda_n} \sum_{k=0}^{n-2} \left| r(\lambda_k - \lambda_{k-1}) + s(\lambda_{k+1} - \lambda_k) + t(\lambda_{k+2} - \lambda_{k+1}) \right| = \frac{\lambda_n - 2}{\lambda_n} \sum_{k=0}^{n-2} (\lambda_k - \lambda_{k-1}) z_k$$
and since \( \lim_{n \to \infty} \frac{\lambda_n}{\lambda_n} = 1 \) by (17) and (18); we obtain that by (19)

\[
\lim_{n \to \infty} \frac{1}{\lambda_{n-2}} \sum_{k=0}^{n-2} (\lambda_k - \lambda_{k-1})z_k = 0
\]  \hspace{1cm} (20)

which shows that \( z = (z_k) \in c_0^\lambda \) where the sequence \( z = (z_k) \) is defined by

\[
z_k = \left| \frac{r(\lambda_k - \lambda_{k-1}) + s(\lambda_{k+1} - \lambda_k) + t(\lambda_{k+2} - \lambda_{k+1})}{\lambda_k - \lambda_{k-1}} \right|
\]

for all \( k \in \mathbb{N} \).

Conversely, we suppose that \( z = (z_k) \in c_0^\lambda \). Then we have (20). Further, for every \( n \geq 2 \), we derive that,

\[
\frac{1}{\lambda_n} \sum_{k=0}^{n-2} |r(\lambda_k - \lambda_{k-1}) + s(\lambda_{k+1} - \lambda_k) + t(\lambda_{k+2} - \lambda_{k+1})| \leq \frac{1}{\lambda_{n-2}} \sum_{k=0}^{n-2} (\lambda_k - \lambda_{k-1})z_k.
\]  \hspace{1cm} (21)

Then, (20) and (21) together imply that (19) holds. On the other hand, we have for every \( n \geq 2 \) that

\[
\left| \frac{r\lambda_{n-2} + s(\lambda_{n-1} - \lambda_0) + t(\lambda_n - \lambda_1)}{\lambda_n} \right|
\]

\[
= \left| \frac{1}{\lambda_n} \sum_{k=0}^{n-2} r(\lambda_k - \lambda_{k-1}) + s(\lambda_{k+1} - \lambda_k) + t(\lambda_{k+2} - \lambda_{k+1}) \right|
\]

\[
\leq \frac{1}{\lambda_n} \sum_{k=0}^{n-2} |r(\lambda_k - \lambda_{k-1}) + s(\lambda_{k+1} - \lambda_k) + t(\lambda_{k+2} - \lambda_{k+1})|
\]

Therefore, it follows by (19)

\[
\lim_{n \to \infty} \frac{r\lambda_{n-2} + s(\lambda_{n-1} - \lambda_0) + t(\lambda_n - \lambda_1)}{\lambda_n} = 0.
\]
Particularly if (i) $r = 0$, $s = -1$, $t = 1$, then we obtain $\lim_{n \to \infty}[(\lambda_n - \lambda_{n-1})/\lambda_n] = 0$, which shows that (17) holds. (ii) $r = -1$, $s = 1$, $t = 0$, then we obtain $\lim_{n \to \infty}[(\lambda_{n-1} - \lambda_{n-2})/\lambda_n] = 0$, which shows that (18) holds. Thus, we deduce by the relation (16) that (15) holds. This leads us with Lemma 11 to the consequence that, $\hat{W}(x) \in (\ell_\infty : c_0)$. Hence, the inclusion $\ell_\infty \subset c_0^\lambda(\hat{B})$ holds. Lastly, it is obvious that the sequence $y$, defined in the proof of Theorem 5, is in $c_0^\lambda(\hat{B})$ but not in $\ell_\infty$, so the inclusion $\ell_\infty \subset c_0^\lambda(\hat{B})$ is strict.

**Theorem 7.** Let $1 \leq p < \infty$. Then the inclusions $\ell_\infty \subset \ell_p^\lambda(\hat{B})$ and $\ell_p^\lambda(\hat{B}) \subset \ell_\infty^\lambda(\hat{B})$ are strictly holds.

**Proof.** To prove the validity of the inclusion $\ell_\infty \subset \ell_p^\lambda(\hat{B})$, it suffices to show that, for every $x \in \ell_p^\lambda(\hat{B})$, there exists a positive real number $K$ such that $\|x\|_{\ell_p^\lambda(\hat{B})} \leq K\|x\|_{\ell_\infty}$. Let $x \in \ell_\infty$. Then, we have

$$\|x\|_{\ell_p^\lambda(\hat{B})} = \sup_{n \in \mathbb{N}} \left|\frac{1}{\lambda_n} \sum_{k=0}^{n} (\lambda_k - \lambda_{k-1})(r x_k + s x_{k-1} + t x_{k-2})\right|$$

$$\leq \sup_{0 \leq k \leq n} |r x_k + s x_{k-1} + t x_{k-2}| \sup_{n \in \mathbb{N}} \left|\frac{1}{\lambda_n} \sum_{k=0}^{n} (\lambda_k - \lambda_{k-1})\right|$$

$$\leq \max\{|r| + |s| + |t|\}\|x\|_{\ell_\infty}$$

so that $x \in \ell_p^\lambda(\hat{B})$ and hence, $\ell_\infty \subset \ell_p^\lambda(\hat{B})$. Furthermore, we consider $x = (x_k)$ defined by $x_k = \sum_{j=0}^{k} d_{kj}$ for all $k \in \mathbb{N}$. Then we have $\hat{W}(x) = 1$. Thus, we deduce that $\|x\|_{\ell_p^\lambda(\hat{B})} = \|\hat{W} x\|_{\ell_\infty} = 1$. Hence $x = (x_k) \in \ell_p^\lambda(\hat{B})$.

On the other hand, we know that the inclusion $\ell_\infty \subset c_0^\lambda(\hat{B})$ is strict from Theorem 5. Since $c_0^\lambda(\hat{B}) \subset \ell_\infty(\hat{B})$, the inclusion $\ell_\infty \subset \ell_\infty^\lambda(\hat{B})$ strictly holds.

Now, we suppose that $x \in \ell_p^\lambda(\hat{B})$. Since $\hat{W}(x) \in \ell_p \subset \ell_\infty$, we have $x \in \ell_\infty^\lambda(\hat{B})$. Hence, $\ell_p^\lambda(\hat{B}) \subset \ell_\infty^\lambda(\hat{B})$. Further, to show that this inclusion is strict, we consider the sequence $x = (x_k)$ defined by $x = (1, 1, 1, ...)$ and assume that $r + s + t = 1$. Then, we have

$$\|\hat{W} x\|_{\ell_\infty} = \sup_{n \in \mathbb{N}} \left|\frac{1}{\lambda_n} \sum_{k=0}^{n} (\lambda_k - \lambda_{k-1})(r x_k + s x_{k-1} + t x_{k-2})\right| = 1.$$
But, for $1 \leq p < \infty$

$$\sum_{n=0}^{\infty} \left| \frac{1}{\lambda_n} \sum_{k=0}^{n} (\lambda_k - \lambda_{k-1})(rx_k + sx_{k-1} + tx_{k-2}) \right|^p = \sum_{n=0}^{\infty} 1 = \infty.$$  

Hence, we have $x \notin \ell_p^\lambda(\hat{B})$, which we want to show.

4. The $\alpha$-, $\beta$- and $\gamma$-Duals of The Spaces $c_0^\lambda(\hat{B})$, $c^\lambda(\hat{B})$, $\ell_\infty^\lambda(\hat{B})$ and $\ell_p^\lambda(\hat{B})$

In this section, we determine the $\alpha$-, $\beta$- and $\gamma$-duals of the generalized difference sequence spaces $c_0^\lambda(\hat{B})$, $c^\lambda(\hat{B})$, $\ell_\infty^\lambda(\hat{B})$ and $\ell_p^\lambda(\hat{B})$ of non-absolute type.

We shall firstly give the definition of $\alpha$-, $\beta$- and $\gamma$-duals of a sequences space and after quote the lemmas due to Stieglitz and Tietz [32] which are needed in proving the theorems given in Section 4 and 5.

For the sequence spaces $\lambda$ and $\mu$, define the set $S(\lambda, \mu)$ by

$$S(\lambda, \mu) = \{ z = (z_k) \in w : xz = (x_kz_k) \in \mu \text{ for all } x \in \lambda \}. \quad (22)$$

With the notation of (22), the $\alpha$-, $\beta$- and $\gamma$-duals of a sequences space $\lambda$, which are respectively denoted by

$$\lambda^\alpha = S(\lambda, \ell_1), \quad \lambda^\beta = S(\lambda, cs) \quad \text{and} \quad \lambda^\gamma = S(\lambda, bs).$$

**Lemma 2.** (i) $A \in (c_0 : \ell_1) = (c : \ell_1) = (\ell_\infty : \ell_1)$ if and only if

$$\sup_{K \in F} \sum_n \left| \sum_{k \in K} a_{nk} \right| < \infty.$$

(ii) Let $1 < p < \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$. Then $A \in (\ell_p : \ell_1)$ if and only if

$$\sup_{N \in F} \sum_k \left| \sum_{n \in N} a_{nk} \right|^q < \infty;$$

**Lemma 3.** $A \in (c_0 : c)$ if and only if

$$\lim_{n \to \infty} a_{nk} = \alpha_k \text{ for each fixed } k \in \mathbb{N}, \quad (23)$$

$$\sup_{n \in \mathbb{N}} \sum_k |a_{nk}| < \infty. \quad (24)$$

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Lemma 4. \( A \in (c : c) \) if and only if (23) and (24) hold, and
\[
\lim_{n \to \infty} \sum_{k} a_{nk} \text{ exists.} \tag{25}
\]

Lemma 5. \( A \in (\ell_\infty : c) \) if an only if (23) holds and
\[
\lim_{n \to \infty} \sum_{k} |a_{nk}| = \sum_{k} |\alpha_k|.
\]

Lemma 6. Let \( 1 < p < \infty \). Then, \( A \in (\ell_p : c) \) if and only if (23) hold and
\[
\sup_{n \in \mathbb{N}} \sum_{k} |a_{nk}|^q < \infty \quad \left( \frac{1}{p} + \frac{1}{q} = 1 \right). \tag{26}
\]

Lemma 7. \( A \in (c : \ell_\infty) = (c_0 : \ell_\infty) = (\ell_\infty : \ell_\infty) \) if and only if (24) holds.

Lemma 8. Let \( 1 < p < \infty \). Then, \( A \in (\ell_p : \ell_\infty) \) if and only if (26) holds.

Now we consider the following sets:

\[
\begin{align*}
 f^\lambda_1 &= \left\{ a = (a_n) \in w : \sup_{K \in \mathcal{F}} \sum_n \left| \sum_{k \in K} f^\lambda_{nk} \right| < \infty \right\}, \\
 f^\lambda_2 &= \left\{ a = (a_n) \in w : \sum_{j=k}^\infty d_{jk} a_j \text{ exists for each } k \in \mathbb{N} \right\}, \\
 f^\lambda_3 &= \left\{ a = (a_n) \in w : \sup_{n \in \mathbb{N}} \sum_{k=0}^{n-1} |\tilde{g}_k(n)|^q < \infty \right\}, \\
 f^\lambda_4 &= \left\{ a = (a_n) \in w : \sup_{n \in \mathbb{N}} \left| \frac{1}{r(\lambda_n - \lambda_{n-1})} a_n \right| < \infty \right\}, \\
 f^\lambda_5 &= \left\{ a = (a_n) \in w : \lim_{n \to \infty} \sum_{k=0}^n \left( \sum_{j=0}^k d_{kj} \right) a_k \text{ exists} \right\}, \\
 f^\lambda_6 &= \left\{ a = (a_n) \in w : \sup_{N \in \mathcal{F}} \sum_{k=0}^\infty \left( \sum_{n \in N} f^\lambda_{nk} \right)^q < \infty \right\}, \\
 f^\lambda_7 &= \left\{ a = (a_n) \in w : \lim_{n \to \infty} \sum_k |v^\lambda_{nk}| = \sum_k \left| \lim_{n \to \infty} v^\lambda_{nk} \right| \right\},
\end{align*}
\]
where the matrices $F^\lambda = (f^\lambda_{nk})$ and $V^\lambda = (v^\lambda_{nk})$ are defined as follows,

$$
f^\lambda_{nk} = \begin{cases} 
\frac{d_{nk}\lambda_k}{\lambda_k - \lambda_{k-1}} - \frac{d_{n,k+1}\lambda_k}{\lambda_{k+1} - \lambda_k} a_n, & (k < n) \\
\frac{1}{\lambda_n} a_n, & (k = n) \\
\frac{1}{r (\lambda_n - \lambda_{n-1})} a_n, & (k > n)
\end{cases}
$$

and

$$
v^\lambda_{nk} = \begin{cases} 
\hat{g}_k(n), & (k < n) \\
\frac{1}{\lambda_n} a_n, & (k = n) \\
\frac{1}{r (\lambda_n - \lambda_{n-1})} a_n, & (k > n)
\end{cases}
$$

for all $k, n \in \mathbb{N}$ and the $\hat{g}_k(n)$ is defined as follows

$$
\hat{g}_k(n) = \lambda_k \left( \frac{1}{\lambda_k - \lambda_{k-1}} \sum_{j=k}^{n} d_{jk}a_j - \frac{1}{\lambda_{k+1} - \lambda_k} \sum_{j=k+1}^{n} d_{j,k+1}a_j \right) \quad \text{for } k < n.
$$

**Theorem 8.**
(i) $\{c^\lambda_0(\hat{B})\}^\alpha = \{c^\lambda(\hat{B})\}^\alpha = \{\ell^\lambda_\infty(\hat{B})\}^\alpha = f^\lambda_1$.

(ii) Let $1 < p < \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$. Then, $\{\ell^\lambda_p(\hat{B})\}^\alpha = f^\lambda_0$.

**Proof.** We prove the theorem for the space $c^\lambda_0(\hat{B})$. Let $a = (a_n) \in w$. Then, we obtain the equality

$$
a_nx_n = \sum_{k=0}^{n} d_{nk} \sum_{j=k-1}^{k} (-1)^{k-j} \lambda_j \frac{a_ny_i}{\lambda_k - \lambda_{k-1}} = F^\lambda_n(y) \quad \text{for all } n \in \mathbb{N}, \quad (27)
$$

by relation (13). Thus we observe by (27) that $ax = (a_nx_n) \in \ell_1$ whenever $x = (x_k) \in c^\lambda_0(\hat{B})$ if and only if $F^\lambda y \in \ell_1$ whenever $y = (y_k) \in c_0$. This means that the sequence $a = (a_n) \in \{c^\lambda_0(\hat{B})\}^\alpha$ if and only if $F^\lambda \in (c_0 : \ell_1)$. Therefore we obtain by Lemma 2 with $F^\lambda$ instead of $A$ that $a = (a_n) \in \{c^\lambda_0(\hat{B})\}^\alpha$ if and only if

$$
\sup_{K \in \mathcal{F}} \sum_{n} \left| \sum_{k \in K} f^\lambda_{nk} \right| < \infty
$$

which leads us to the consequence that $\{c^\lambda_0(\hat{B})\}^\alpha = f^\lambda_1$. This completes the proof.
Theorem 9. (i) \( \{c^0_0(\hat{B})\}^\beta = f_2^\lambda \cap f_3^\lambda \cap f_4^\lambda \) (with \( q = 1 \)).
(ii) \( \{c^\lambda(\hat{B})\}^\beta = f_2^\lambda \cap f_3^\lambda \cap f_4^\lambda \cap f_5^\lambda \) (with \( q = 1 \)).
(iii) \( \{\ell_{\infty}^\lambda(\hat{B})\}^\beta = f_2^\lambda \cap f_3^\lambda \cap f_4^\lambda \)
(iv) Let \( 1 < p < \infty \) and \( \frac{1}{p} + \frac{1}{q} = 1 \). Then, \( \{\ell_p^\lambda(\hat{B})\}^\beta = f_2^\lambda \cap f_3^\lambda \cap f_4^\lambda \).

Proof. Consider the equality
\[
\sum_{k=0}^{n} a_k x_k = \sum_{k=0}^{n} \left[ \sum_{j=0}^{k} d_{kj} \sum_{i=j-1}^{i} (-1)^{i-j} \frac{\lambda_i}{\lambda_j - \lambda_{j-1}} y_i \right] a_k
\]
\[
= \sum_{k=0}^{n} (\lambda_k y_k - \lambda_{k-1} y_{k-1}) \left[ \frac{1}{\lambda_k - \lambda_{k-1}} \sum_{j=k}^{n} d_{jk} a_j \right]
\]
\[
= \sum_{k=0}^{n-1} \lambda_k \left[ \sum_{j=0}^{n} d_{jk} a_j \lambda_k - \lambda_{k-1} - \frac{\sum_{j=k+1}^{n} d_{j,k+1} a_j}{\lambda_{k+1} - \lambda_k} \right] y_k + \frac{1}{r} \frac{a_n \lambda_n}{\lambda_n - \lambda_{n-1}} y_n
\]
\[
= \sum_{k=0}^{n} \widehat{g}_k(n) y_k + \frac{1}{r} \frac{a_n \lambda_n}{\lambda_n - \lambda_{n-1}} y_n = V_\lambda^n(y); \quad (n \in \mathbb{N}). \tag{28}
\]

Then we deduce by (28) that \( ax = (a_k x_k) \in cs \) whenever \( x = (x_k) \in c^0_0(\hat{B}) \)
if and only if \( V^\lambda y \in c \) whenever \( y = (y_k) \in c_0 \). This means that \( a = (a_k) \in \{c^0_0(\hat{B})\}^\beta \) if and only if \( V^\lambda \in (c_0 : c) \). Therefore, by using Lemma 3, we obtain :
\[
\sum_{j=k}^{\infty} d_{jk} a_j \quad \text{exists for each} \quad k \in \mathbb{N}, \tag{29}
\]
\[
\sup_{n \in \mathbb{N}} \sum_{k=0}^{n-1} \left| \frac{\lambda_k}{r} \frac{a_k \lambda_k}{\lambda_k - \lambda_{k-1}} \right| < \infty, \tag{30}
\]
\[
\sup_{k \in \mathbb{N}} \left| \frac{\lambda_k}{r} \frac{a_k \lambda_k}{\lambda_k - \lambda_{k-1}} \right| < \infty. \tag{31}
\]

Hence, we conclude that \( \{c^0_0(\hat{B})\}^\beta = f_2^\lambda \cap f_3^\lambda \cap f_4^\lambda \).

Finally, we ended up this section with the following theorem which determines the \( \gamma \)-duals of sequence spaces \( c^0_0(\hat{B}), c^\lambda(\hat{B}), \ell_{\infty}^\lambda(\hat{B}) \) and \( \ell_p^\lambda(\hat{B}) \).
Theorem 10. (i) \( \{c_0^\lambda(\hat{B})\}^\gamma = \{c^\lambda(\hat{B})\}^\gamma = \{\ell_\infty^\lambda(\hat{B})\}^\gamma = f_3^\lambda \cap f_4^\lambda \) (with \( q = 1 \)).

(ii) Let \( 1 < p < \infty \) and \( \frac{1}{p} + \frac{1}{q} = 1 \). Then, \( \{\ell_p^\lambda(\hat{B})\}^\gamma = f_3^\lambda \cap f_4^\lambda \).

5. Some Matrix Transformations Related To The Spaces \( c_0^\lambda(\hat{B}), c^\lambda(\hat{B}), \ell_\infty^\lambda(\hat{B}) \) and \( \ell_p^\lambda(\hat{B}) \)

In this final section, we state some results which characterize various matrix mappings on the spaces \( c_0^\lambda(\hat{B}), c^\lambda(\hat{B}), \ell_\infty^\lambda(\hat{B}) \) and \( \ell_p^\lambda(\hat{B}) \). We shall write throughout for brevity that

\[
\hat{g}_{nk}(m) = \lambda_k \left( \frac{1}{\lambda_k - \lambda_{k-1}} \sum_{j=k}^{m} d_{jk} a_{nj} - \frac{1}{\lambda_{k+1} - \lambda_k} \sum_{j=k+1}^{m} d_{j,k+1} a_{nj} \right) \quad \text{for } k < m.
\]

and

\[
\hat{g}_{nk} = \lambda_k \left( \frac{1}{\lambda_k - \lambda_{k-1}} \sum_{j=k}^{\infty} d_{jk} a_{nj} - \frac{1}{\lambda_{k+1} - \lambda_k} \sum_{j=k+1}^{\infty} d_{j,k+1} a_{nj} \right)
\]

for all \( k, m, n \in \mathbb{N} \) provided the convergence of the series.

We shall begin with lemmas which are needed in the proof of our theorems.

Lemma 9. The matrix mappings between the BK-spaces are continuous.

Lemma 10. \( A = (a_{nk}) \in (c : \ell_p) \) if and only if

\[
\sup_{F \in F} \sum_n \left| \sum_{k \in F} a_{nk} \right|^p < \infty; \quad (1 < p < \infty)
\] (32)

Lemma 11. \( A = (a_{nk}) \in (c : c_0) \) if and only if (24) holds and

\[
\lim_{n \to \infty} a_{nk} = 0 \quad \text{for each fixed } k \in \mathbb{N}
\] (33)

\[
\lim_{n \to \infty} \sum_k a_{nk} = 0
\]
Lemma 12. $A \in (c_0 : c_0)$ if and only if (24) and (33) hold.

Now, we may give our matrix transformations.

Theorem 11. Let $A = (a_{nk})$ be an infinite matrix and $1 < p < \infty$. Then, $A \in (c^\lambda(\hat{B}) : \ell_p)$ if and only if

$$\sup_{F \in F} \sum_n \left| \sum_{k \in F} \hat{g}_{nk} \right|^p < \infty; \quad (1 < p < \infty) \quad (34)$$

$$\sum_{j=k}^{\infty} d_{jk} a_{nj} \quad \text{exists for all fixed } k, n \in \mathbb{N} \quad (35)$$

$$\sup_{m \in \mathbb{N}} \sum_{k=0}^{m-1} |\hat{g}_{nk}(m)| < \infty \quad (36)$$

$$\lim_{k \to \infty} \frac{1}{r \left( \lambda_k - \lambda_{k-1} \right)} a_{nk} = a_n \quad \text{exists for each fixed } n \in \mathbb{N} \quad (37)$$

$$\left\{ \sum_{j=k}^{\infty} d_{jk} a_{nj} \right\} \in cs; \quad (n \in \mathbb{N}) \quad (38)$$

$$(a_n) \in \ell_p \quad (39)$$

Proof. If the conditions (34)-(39) hold and $x = (x_k)$ be any sequence in the space $c^\lambda(\hat{B})$ then by using Theorem 9 we have that $\{a_{nk}\}_{k \in \mathbb{N}} \in \{c^\lambda(\hat{B})\}^\beta$ for all $n \in \mathbb{N}$. Hence, $A-$transform of $x$ exists,i.e., $Ax$ exists. Furthermore, since the associated sequence $y = (y_k)$ is in the space $c$, we may write $\lim_k y_k = l$ for some suitable $l$. Also, the matrix $\hat{A} = (\hat{g}_{nk})$ is in the class $(c : \ell_p)$ by Lemma 10 and condition (34) where $1 < p < \infty$.

Now, we may consider the $m$th partial sum of the series $\sum_k a_{nk} x_k$ which is derived by using the relation (13):

$$\sum_{k=0}^{m} a_{nk} x_k = \sum_{k=0}^{m-1} \hat{g}_{nk}(m) y_k + \frac{1}{r \left( \lambda_m - \lambda_{m-1} \right)} a_{nm} y_m; \quad \text{for all } n, m \in \mathbb{N} \quad (40)$$

Then, since $y \in c$ and $\hat{A} \in (c : \ell_p)$; $\hat{A}y$ exists and so the series $\sum_k \hat{g}_{nk} y_k$ converges for every $n \in \mathbb{N}$. Also, it follows by (35) we have that $\lim_{m \to \infty} \hat{g}_{nk}(m) = \hat{g}_{nk}$. Therefore, if we pass to limit in (40) as $m \to \infty$, then we obtain by (37) that

$$\sum_k a_{nk} x_k = \sum_k \hat{g}_{nk} y_k + l a_n \quad \text{for all } n \in \mathbb{N} \quad (41)$$

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which can be written as follows:

\[ A_n(x) = \hat{A}_n(y) + la_n \quad \text{for all} \quad n \in \mathbb{N}. \]  

(42)

This yields by taking \( \ell_p \)-norm that

\[ \|Ax\|_{\ell_p} \leq \|\hat{A}y\|_{\ell_p} + |l|\|a_n\|_{\ell_p} < \infty. \]

Consequently, we have that \( Ax \in \ell_p \), i.e., \( A \in (c^\lambda(\hat{B}) : \ell_p) \).

Conversely, if \( A \in (c^\lambda(\hat{B}) : \ell_p) \), we have \( \{a_{nk}\}_{k \in \mathbb{N}} \in \{c^\lambda(\hat{B})\}^\beta \) for all \( n \in \mathbb{N} \) which implies with Theorem 9 that the conditions (35), (36) and (38) are necessary.

On the other hand, since \( c^\lambda(\hat{B}) \) and \( \ell_p \) are \( BK \)-spaces, we have by Lemma 9 that there is a constant \( M > 0 \) such that

\[ \|Ax\|_p \leq M\|x\|_{c^\lambda(\hat{B})} \]  

holds for all \( x \in c^\lambda(\hat{B}) \). Now \( F \in \mathcal{F} \). Then, the sequence \( z = \sum_{k \in F} b^{(k)}(\lambda) \) is in \( c^\lambda(\hat{B}) \), where the sequences \( b^{(k)}(\lambda) = \{b^{(k)}_n(\lambda)\}_{n \in \mathbb{N}} \) is defined by (14) for every fixed \( k \in \mathbb{N} \).

Since \( \hat{W}(b^{(k)}(\lambda)) = e^{(k)} \) for each fixed \( k \in \mathbb{N} \), we have

\[ \|z\|_{c^\lambda(\hat{B})} = \|\hat{W}(z)\|_{\ell_\infty} = \left\| \sum_{k \in F} \hat{W}(b^{(k)}(\lambda)) \right\|_{\ell_\infty} = \left\| \sum_{k \in F} e^{(k)} \right\|_{\ell_\infty} = 1. \]

Furthermore, for every \( n \in \mathbb{N} \), we obtain by (14) that

\[ A_n(z) = \sum_{k \in F} A_n(b^{(k)}(\lambda)) = \sum_{k \in F} \sum_{j} a_{nj} b^{(k)}_j(\lambda) = \sum_{k \in F} \hat{g}_{nk}. \]

Hence, since the inequality [43] is satisfied for the sequence \( z \in c^\lambda(\hat{B}) \), we have for any \( F \in \mathcal{F} \) that

\[ \left( \sum_n \left| \sum_{k \in F} \hat{g}_{nk} \right|^p \right)^{1/p} \leq M \]

which shows the necessity of (34). Thus, it follows by Lemma 10 that \( \hat{A} = (\hat{g}_{nk}) \in (c : \ell_p) \).

Moreover, we consider the sequence \( x = x_k \) defined by (13) for every \( k \in \mathbb{N} \) and suppose that \( y = (y_k) \in c^\lambda c_0 \). Then, since \( x \in c^\lambda(\hat{B}) \) such
that \( y = \hat{W}(x) \) by (12), the transforms \( Ax \) and \( \hat{A}y \) exists. Hence, the series 
\[
\sum_{k} a_{nk} x_k \]
and 
\[
\sum \hat{g}_{nk} y_k \]
converges for every \( n \in \mathbb{N} \). So we infer that 
\[
\lim_{m \to \infty} \sum_{k=0}^{m-1} \hat{g}_{nk}(m) y_k = \sum \hat{g}_{nk} y_k; \quad (n \in \mathbb{N}).
\]
Consequently, we obtain from (40) that 
\[
\lim_{m \to \infty} \frac{1}{r} \frac{\lambda_m}{\lambda_m - \lambda_{m-1}} a_{nm} y_m \quad \text{exists for each fixed} \quad n \in \mathbb{N}.
\]
Hence, we deduce that 
\[
\lim_{m \to \infty} \frac{1}{r} \frac{\lambda_m}{\lambda_m - \lambda_{m-1}} a_{nm} \quad \text{exists for each fixed} \quad n \in \mathbb{N}.
\]
which leads us to the necessity of (37) and so the relation (40) holds, where 
\( l = \lim_{k} y_k \).

Finally, since \( Ax \in \ell_p \) and \( \hat{A}y \in \ell_p \); the necessity of (39) is immediate by (40). This completes the proof.

**Theorem 12.** In order that \( A = (a_{nk}) \in (c^\lambda(\hat{B}) : \ell_\infty) \) where \( A = (a_{nk}) \) be an infinite matrix, it is necessary and sufficient that (37) and (38) hold, and

\[
\sup_{n \in \mathbb{N}} \sum_{k} |\hat{g}_{nk}| < \infty, \quad (44)
\]

\[
(a_n) \in \ell_\infty. \quad (45)
\]

**PROOF.** This is an immediate consequence of Lemma 7 and Theorem 11.

**Theorem 13.** (i) In order that \( A = (a_{nk}) \in (c^\lambda(\hat{B}) : \ell_p) \), it is necessary and sufficient that (37), (35) and (36) hold and

\[
\sup_{k} \left| \frac{1}{r} \frac{\lambda_k}{\lambda_k - \lambda_{k-1}} a_{nk} \right| < \infty \quad (n \in \mathbb{N}). \quad (46)
\]

(ii) In order that \( A \in (c_0^\lambda(\hat{B}) : \ell_\infty) \), it is necessary and sufficient that (36), (44) and (46) hold.

**PROOF.** This may be obtained by proceedings as in Theorem 11, above. So, we omit the detail.
Theorem 14. In order that $A = (a_{nk}) \in (c^\lambda(\hat{B}) : c)$, it is necessary and sufficient that (37), (38), (44) hold and

\begin{align}
\lim_{n} a_n &= a \quad (47) \\
\lim_{n} \hat{g}_{nk} &= \alpha_k \quad \text{for each fixed } k \in \mathbb{N} \quad (48) \\
\lim_{n} \sum_{k} \hat{g}_{nk} &= \alpha \quad (49)
\end{align}

Proof. Suppose that $A$ satisfies conditions (37), (38), (44), (47), (48) and (49), and let $x = (x_k)$ be a sequence in the space $c^\lambda(\hat{B})$. Since (44) implies (36), we have by Theorem 9 that \{a_{nk}\}_{k \in \mathbb{N}} \in \{c^\lambda(\hat{B})\}^3$ for all $n \in \mathbb{N}$. Hence, $Ax$ exists. We also observe from (41) and (48) that

$$\sum_{j=0}^{k} |\alpha_j| \leq \sup_{n \in \mathbb{N}} \sum_{j} |\hat{g}_{nk}| < \infty$$

holds for every $k \in \mathbb{N}$. So $(\alpha_k) \in \ell_1$ and hence the series $\sum_k \alpha_k (y_k - l)$ converges, where $y = (y_k) \in c$ is the sequence connected with $x = (x_k)$ by the relation (12) such that $\lim_k y_k = l$. Also, it is obvious by combining Lemma 4 with the conditions (44), (48) and (49) that the matrix $\hat{A} = (\hat{g}_{nk})$ is in the class $(c : c)$.

Now, by following the similar way used in the proof of Theorem 11 we obtain that the relation (41) holds, which can be written as follows:

$$\sum_{k} a_{nk} x_k = \sum_{k} \hat{g}_{nk} (y_k - l) + l \sum_{k} \hat{g}_{nk} + la_n \quad \text{for all } n \in \mathbb{N} \quad (50)$$

If we pass the limit in (50) as $n \to \infty$ we have that

$$\lim_{n} A_n (x) = \sum_{k} \alpha_k (y_k - l) + l(\alpha + a),$$

which shows that $Ax \in c$, i.e., $A \in (c^\lambda(\hat{B}) : c)$.

Conversely, suppose that $A \in (c^\lambda(\hat{B}) : c)$. Since the inclusion $c \subset \ell_\infty$ holds; $A \in (c^\lambda(\hat{B}) : \ell_\infty)$. Therefore, the necessity of conditions (37), (38) and (44) are obvious from Theorem 12. Furthermore, consider the sequence $b^{(k)}(\lambda) = \{b_n^{(k)}(\lambda)\}_{n \in \mathbb{N}} \in c^\lambda(\hat{B})$ defined by (14) for every fixed $k \in \mathbb{N}$. Then,
one can see that $A b^{(k)}(\lambda) = \{g_{nk}\}_{n \in \mathbb{N}}$ and hence $\{g_{nk}\}_{n \in \mathbb{N}} \in c$ for every $k \in \mathbb{N}$ which shows that the necessity of (48). Let $z = \sum_k b^{(k)}(\lambda)$. Then, since the linear transformation $T : c^{\lambda}(\hat{B}) \to c$, defined as in the proof of Theorem 2 by analogy, is continuous and $\hat{W}(b^{(k)}(\lambda)) = e^{(k)}$ for each fixed $k \in \mathbb{N}$, we obtain that
\[ \hat{W}_n(z) = \sum_k \hat{W}_n(b^{(k)}(\lambda)) = \sum_k \delta_{nk} = 1 \quad \text{for each } n \in \mathbb{N} \]
which shows that $\hat{W}_n(z) = e \in c$ and hence $z \in c^{\lambda}(\hat{B})$. On the other hand, since $c^{\lambda}(\hat{B})$ and $c$ are the $BK$—spaces , Lemma 9 implies the continuity of the matrix mapping $A : c^{\lambda}(\hat{B}) \to c$. Thus, we have for every $n \in \mathbb{N}$ that
\[ A_n(z) = \sum_k A_n(b^{(k)}(\lambda)) = \sum_k \hat{g}_{nk}. \]
This shows the necessity of (49).

Now, it follows by (44), (48) and (49) with Lemma 4 that $\hat{A} = (\hat{g}_{nk}) \in (c : c)$. So by (37), (38) and relation (42) holds for all $x \in c^{\lambda}(\hat{B})$ and $y \in c$.

Finally, the necessity of (47) is immediately by (42) since $Ax \in c$ and $\hat{A}x \in c$. This completes the proof.

**Theorem 15.** In order that $A = (a_{nk}) \in (c^{\lambda}(\hat{B}) : c_0)$, it is necessary and sufficient that (37), (38), (44) hold and
\[ \lim_n a_n = 0 \]
\[ \lim_n \hat{g}_{nk} = 0 \quad \text{for each fixed } k \in \mathbb{N} \]
\[ \lim_n \sum_k \hat{g}_{nk} = 0 \]

**Proof.** This is obtained in the similar way used in the proof of Theorem 14 with Lemma 11 instead of Lemma 4 and so we omit the detail.

**Theorem 16.** In order that $A = (a_{nk}) \in (c^{\lambda}(\hat{B}) : c)$, it is necessary and sufficient that (35), (44), (46) and (48) hold.

**Proof.** This is an immediate consequence of Lemma 3, Theorem 9 and Theorem 13(iii).
Theorem 17. In order that $A = (a_{nk}) \in (c_0^\lambda(\hat{B}) : c_0)$, it is necessary and sufficient that (35), (44) and (48) hold.

**Proof.** This is an immediate consequence of Lemma 12, Theorem 9 and Theorem 16.

Theorem 18. Let $1 < p < \infty$. $A = (a_{nk}) \in (\ell^\lambda_p(\hat{B}) : \ell_\infty)$ if and only if (36) and (46) hold and

$$\sup_{n \in \mathbb{N}} \sum_{k=0}^\infty |\hat{g}_{nk}|^q < \infty \quad (n \in \mathbb{N}). \quad (51)$$

**Proof.** Suppose that $A = (a_{nk})$ satisfies the conditions (36), (46) and (51) and take any $x = (x_k) \in \ell^\lambda_p(\hat{B})$. Then, it is obvious that $\{a_{nk}\}_{k \in \mathbb{N}} \in \{\ell^\lambda_p(\hat{B})\}^\beta$ for each $n \in \mathbb{N}$. Hence, $A_n(x) = \sum_{k=0}^\infty a_{nk}x_k$ is convergent for all $x \in \ell^\lambda_p(\hat{B})$, i.e., $Ax$ exists. Also, it is obvious by combining Lemma 8 with the condition (51) that $\hat{A} = (\hat{g}_{nk}) \in (\ell_p : \ell_\infty)$.

Now, we shall show that $Ax \in \ell_\infty$ for all $x \in \ell^\lambda_p(\hat{B})$. By using the relation (40) we have that

$$\sum_{k=0}^\infty a_{nk}x_k = \sum_{k=0}^\infty \hat{g}_{nk}y_k. \quad (52)$$

By applying Hölder’s inequality to (52) we obtain that

$$|(Ax)_n| = \left| \sum_{k=0}^\infty a_{nk}x_k \right| \leq \sum_{k=0}^\infty |\hat{g}_{nk}| |y_k|$$

$$\leq \|y\|_{\ell_p} \left( \sum_{k=0}^\infty |\hat{g}_{nk}|^q \right)^{1/q}$$

which gives us by taking supremum over $n \in \mathbb{N}$ in this last inequality that

$$\sup_{n \in \mathbb{N}} |(Ax)_n| \leq \|y\|_{\ell_p} \sup_{n \in \mathbb{N}} \left( \sum_{k=0}^\infty |\hat{g}_{nk}|^q \right)^{1/q} < \infty$$

which implies that $Ax \in \ell_\infty$, i.e., $A \in (\ell^\lambda_p(\hat{B}) : \ell_\infty)$. 25
Conversely, suppose that \( A = (a_{nk}) \in (\ell^\lambda_p(\hat{B}) : \ell_\infty) \). Thus, \( Ax \) exists and bounded for all \( x \in \ell^\lambda_p(\hat{B}) \). Also, \( \{a_{nk}\}_{k \in \mathbb{N}} \in \{\ell^\lambda_p(\hat{B})\}^\beta \) for all \( n \in \mathbb{N} \) which is implies the necessity of (36) and (46). If we define the sequences \( \hat{g}_n \) such that \( \hat{g}_n = \{\hat{g}_{nk}\}_{k \in \mathbb{N}} \) then we have that
\[
\|\hat{g}_n\|_{\ell_q} < \infty
\]
So, bearing in mind (46) one can easily obtain the relation (52) by using the relation (40). On the other hand, the sequences \( a_n = \{a_{nk}\}_{k \in \mathbb{N}} \) define the continuous linear functionals on the space \( \ell^\lambda_p(\hat{B}) \) as follows
\[
f_n(x) = \sum_{k=0}^{\infty} a_{nk}x_k; \quad (n \in \mathbb{N}).
\]
Since the spaces \( \ell^\lambda_p(\hat{B}) \) and \( \ell^\lambda_p \) are linear isomorphic; we have that
\[
\|f_n\| = \|\hat{g}_n\|_{\ell_q}.
\]
Also, since \( \{f_n(x)\}_{n \in \mathbb{N}} \in \ell_\infty \); it is obvious that
\[
\sup_{n \in \mathbb{N}} \|f_n(x)\| < \infty.
\]
Therefore, by using the Banach-Steinhaus Theorem [31, see, pp:1-2] we obtain that
\[
\sup_{n \in \mathbb{N}} \|f_n\| = \|\hat{g}_n\|_{\ell_q} = \sup_{n \in \mathbb{N}} \left( \sum_{k=0}^{\infty} |\hat{g}_{nk}|^q \right)^{1/q} < \infty
\]
which shows that the necessity of (51). This step completes the proof.

**Theorem 19.** Let \( 1 < p < \infty \). \( A = (a_{nk}) \in (\ell^\lambda_p(\hat{B}) : c) \) if and only if (36), (46), (48) and (51) hold.

**Proof.** Suppose that the conditions (36), (46), (48) and (51) hold. Then, \( \{a_{nk}\}_{k \in \mathbb{N}} \in \{\ell^\lambda_p(\hat{B})\}^\beta \) for each \( n \in \mathbb{N} \). Hence, the series \( A_nx = \sum_{k=0}^{\infty} a_{nk}x_k \) is convergent for each \( n \in \mathbb{N} \) and for all \( x \in \ell^\lambda_p(\hat{B}) \) which is implies that \( Ax \) exists. By using (48) we obtain that
\[
|\hat{g}_{nk}|^q \to |\alpha_k|^q; \quad (n \to \infty).
\]
Since
\[
\left( \sum_{k=0}^{m} |\hat{g}_{nk}|^q \right)^{1/q} \leq \sup_{n \in \mathbb{N}} \left( \sum_{k=0}^{\infty} |\hat{g}_{nk}|^q \right)^{1/q} = M
\]
for all \( m > 0 \). In this situation we see by passing to the limit in last inequality as \( n \to \infty \) that
\[
\left( \sum_{k=0}^{m} |\alpha_k|^q \right)^{1/q} \leq \sup_{n \in \mathbb{N}} \left( \sum_{k=0}^{\infty} |\hat{g}_{nk}|^q \right)^{1/q}.
\]  \( (53) \)

Because of \( (53) \) holds for all integer \( m > 0 \); we have that
\[
\left( \sum_{k=0}^{\infty} |\alpha_k|^q \right)^{1/q} < \infty.
\]

We remember that \( x = (x_k) \) and \( y = (y_k) \) are associated sequences by the relation \( (13) \) where \( y = (y_k) \in \ell_p \) for \( x = (x_k) \in \ell_p^\lambda(\hat{B}) \).

Let \( \varepsilon \) be any positive number. Then, there exists a number \( N \) such that
\[
\left( \sum_{k=N}^{\infty} |y_k|^q \right)^{1/q} < \frac{\varepsilon}{4M}.
\]

On the other hand, there is an integer \( N_1 \) such that
\[
\left| \sum_{k=0}^{N} \{\hat{g}_{nk} - \alpha_k\} y_k \right| \leq \frac{\varepsilon}{2}
\]
whenever \( n \geq N_1 \). Therefore, we obtain
\[
\left| \sum_{k=0}^{\infty} \{\hat{g}_{nk} - \alpha_k\} y_k \right| \leq \left| \sum_{k=0}^{N} \{\hat{g}_{nk} - \alpha_k\} y_k \right| + \left| \sum_{k=N+1}^{\infty} \{\hat{g}_{nk} - \alpha_k\} y_k \right|
\leq \frac{\varepsilon}{2} + \left( \sum_{k=N+1}^{\infty} |\hat{g}_{nk}| + |\alpha_k| \right)^{q/1} \left( \sum_{k=N+1}^{\infty} |y_k|^p \right)^{1/p}
\leq \frac{\varepsilon}{2} + 2M \frac{\varepsilon}{4M} = \varepsilon,
\]

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which shows that
\[
\lim_{n \to \infty} \sum_{k=0}^{\infty} \hat{g}_{nk}y_k = \sum_{k=0}^{\infty} \alpha_k y_k.
\]

Since
\[
\lim_{n \to \infty} (Ax)_n = \lim_{n \to \infty} \sum_{k=0}^{\infty} \hat{g}_{nk}y_k = \sum_{k=0}^{\infty} \alpha_k y_k
\]

we have that \( Ax \in c \) which implies that \( A \in (\ell_p^A(\hat{B}) : c) \).

The necessity part can be proved by using the similar way in the proof of Theorem 14, so we omit the detail.

**Theorem 20.** Let \( 1 < p < \infty \). In order that \( A = (a_{nk}) \in (\ell_p^A(\hat{B}) : c_0) \), it is necessary and sufficient that (30), (40), (47) hold and
\[
\lim_{n} \hat{g}_{nk} = 0 \quad \text{for each fixed } k \in \mathbb{N}
\]

**Proof.** This result, can be proved similarly as the proof of Theorem 19.

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