Circular Polarization in Compact Radio Sources: Constraints on Particle Acceleration and Electron–Positron Pairs

C.-I. Björnsson
Department of Astronomy, AlbaNova University Center, Stockholm University, SE-106 91 Stockholm, Sweden; bjornsson@astro.su.se
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Abstract

It is shown that the frequency distribution of the degree of circular polarization for a homogeneous source is sensitive to the properties of the synchrotron emitting plasma. Most of the circular polarization comes from the region around the turnover frequency, where the synchrotron radiation becomes optically thick. However, nearly circular characteristic waves result in circular polarization dominated by frequencies above the turnover frequency, while in the case of nearly linear characteristic waves, it is dominated by frequencies below. Observations argue in favor of nearly circular characteristic waves. This implies a low-energy cutoff in the electron distribution that is substantially below that corresponding to the turnover frequency and simultaneously provides an upper limit to the fraction of electron–positron pairs.

Key words: polarization – radiation mechanisms: non-thermal – radio continuum: galaxies

1. Introduction

The standard model for compact radio sources is well-established; energy generated close to the central black hole streams out in a jet-like structure (Blandford & Znajek 1977; Blandford & Payne 1982). However, several of its tenets lack a firm physical underpinning. This includes the launching of the jet and the means of energy transport. A central question here pertains to the material constituents of the plasma; i.e., whether it consists of electrons and protons, or if there is a significant fraction of electron–positron pairs. Another issue is the process by which the synchrotron-emitting particles are accelerated to relativistic energies. Diffusive shock acceleration, second-order Fermi acceleration in a turbulent plasma, or direct acceleration by the electric field generated in a reconnection process of the magnetic field have all been suggested to be the agent transferring energy to the radiating particles. Given that the injection of particles into the acceleration process is likely to be different for these mechanisms, the low-energy end of the particle distribution may be one way to distinguish between them.

Unfortunately, optical depth effects hide the low-energy electrons from direct view. Likewise, the presence of positrons can not be addressed by flux measurements alone, because their emitted flux is identical to that of the electrons. In contrast, both of these aspects of the plasma have a direct bearing on the observed circular polarization. Although this was realized early on (e.g., Pacholczyk 1973), the observed low level of circular polarization made it hard to draw any strong conclusions regarding the plasma properties. However, it was noted that, although the circular polarization varied more rapidly and with larger relative amplitude as compared to either the flux or linear polarization, it only rarely changed sign (Weiler & de Pater 1983; Komesaroff et al. 1984). This suggests the presence of a large-scale magnetic field. On the other hand, several theoretical arguments lead one to expect an important role for turbulence; e.g., in the acceleration process (Blandford et al. 2019; Zhdankin et al. 2018). The connection between the large and small scale properties of the magnetic field is another issue where observations of circular polarization have the potential to contribute significantly.

The low level of observed circular polarization narrowed down the type of questions that could be addressed. The increased accuracy with which circular polarization can now be measured has opened up new possibilities (Macquart et al. 2000; Rayner et al. 2000). Although VLBI-observations are still challenging (Homan et al. 2009), spatially resolved studies of circular polarization along the jet can be made (Wardle et al. 1998; Homan & Wardle 2004). Furthermore, polarization can be measured over a wide frequency range (O’Sullivan et al. 2013) as well as at high frequencies (Agudo et al. 2018b). In spite of this increase, both qualitatively and quantitatively, of the observations of circular polarization, no clear understanding of its origin has emerged (Vitrichchak et al. 2008; O’Sullivan et al. 2013). Hence, its use as a plasma diagnostic is still limited. However, because the circular polarization from the synchrotron emission process itself is quite simple, the observed rather complex behavior suggests that transport effects may play a crucial role.

The transfer equation for polarized light in a homogeneous medium has an analytical solution (e.g., Jones & O’Dell 1977). However, it is quite involved, and various approximations have been put forth to facilitate comparison to observations. The aim of the present paper is threefold: (1) To present an alternative form of the homogeneous solution, which uses the concept of characteristic waves. This is an extension of the discussion in Björnsson (1988). Such a formulation makes possible a more transparent and physical description of the polarization of the emergent radiation. Furthermore, it is argued that, even when the characteristic waves couple, this form of the solution can account—at least qualitatively—for some of the effects of inhomogeneities. (2) The high-frequency observations in the POLAMI survey (Agudo et al. 2018a; Thum et al. 2018) and the detailed, wideband observations of O’Sullivan et al. (2013) are discussed. It is shown how they can be given a relatively straightforward explanation; in particular, that both indicate the presence of nearly circular characteristic waves. (3) It is pointed out that some of the approximations in common use have limited validity, and hence, should be applied with care.

The outline of the paper is as follows. A short introduction to the transfer equation and the main properties of characteristic...
waves are given in Section 2. The formulation of the transfer equation for a light ray in terms of characteristic waves and its solution are presented in Section 3. The results for a homogeneous source are discussed in Section 4, where special attention is given to the two limits of nearly circular and nearly linear characteristic waves. Observations are discussed in Section 5, and the main points of the paper are summarized in Section 6.

2. Polarization Transfer in a Homogeneous Medium

Plane waves are the solution to Maxwell’s equations in a homogeneous medium. Hence, instead of considering the propagation of a general electromagnetic field, it is sufficient to restrict attention to its Fourier components, \( \exp\{i(k \cdot r - \omega t)\} \). Here, \( k = 2\pi/\lambda \) and \( \omega = 2\pi \nu \), where \( \lambda \) and \( \nu \) are the wavelength and frequency, respectively. However, the plane waves do not correspond to the physically measurable electric and magnetic fields \( E \) and \( B \), but rather to \( D \) and \( H \), whose relations to the physical fields are determined by the properties of the medium. In a plasma relevant for synchrotron sources, it is usually assumed that the permeability plays a negligible role, such that \( H = B \) and the influence of the plasma can be written in component form as \( D_l = (\delta_{l,m} + (4\pi i/\omega) \sigma_{l,m}) E_m \), where \( \sigma_{l,m} \) is the dielectric tensor. The indices \( l, m \) run over all three spatial coordinate \( (x, y, z) \), i.e., \( (l, m = x, y, z) \) and a repeated index implies summation. Because \( D \cdot k = 0 \), one finds

\[
E_x + \frac{4\pi i}{\omega} \sigma_{x,m} E_m = 0,
\]

where \( k \) has been chosen to lie along the z-axis (see Figure 1). Because \( |\sigma_{l,m}| \sim c \kappa \), where \( \kappa \) is the absorptivity of the plasma, the magnitude ratio between the longitudinal and transverse components of the electric field is \( |E_z/E_{(x,y)}| \sim \kappa/k \). This is usually a very small number, i.e., the distance over which the radiation is absorbed is much larger than its wavelength. The weak anisotropy limit then corresponds to neglecting the longitudinal component of the electric field, in which case the transfer equation can be written

\[
\left( \frac{\partial}{\partial t} - c \frac{k}{k} \cdot \frac{\partial}{\partial \mathbf{r}} \right) \left( \frac{\partial}{\partial t} + c \frac{k}{k} \cdot \frac{\partial}{\partial \mathbf{r}} \right) E = -4\pi \frac{\partial}{\partial t} J_l,
\]

where \( J_l = \sigma_{l,m} E_m \) \( (l, m = x, y) \) is the current induced by the electric field.

Because the magnitude of the right-hand side of Equation (2) is \( \sim \omega c k/\kappa |E| \), it can be seen that \( k c/\omega \sim 1 + \kappa/k \). The first operator on the LHS can then be evaluated to give \(-2i\nu \), while the second operator corresponds to the comoving derivative in a frame moving with velocity \( c \); i.e., \( c d/ds \). Hence, without loss of accuracy, Equation (2) can be written as a first-order differential equation

\[
\frac{d}{ds} E_l = -\frac{2\pi}{c} \sigma_{l,m} E_m.
\]

where \( s \) is the distance along a ray path.

The transfer equation in Equation (3) can be rewritten as

\[
\frac{d}{ds} E_l E_j^* = -\frac{2\pi}{c} (\sigma_{l,m} E_m E_j^* + \sigma_{j,m} E_m^* E_l),
\]

where \( (\cdot)^* \) denotes complex conjugate and \( (l, j, m = x, y) \). The transfer equation is usually written in terms of the Stokes parameters defined as \( I = |E_x|^2 + |E_y|^2 \), \( Q = |E_x|^2 - |E_y|^2 \), and \( U + iV = 2E_x E_y^* \). It is straightforward to show that Equation (4) is equivalent to the standard formulation.

In the homogeneous case, Equation (4) has an analytical solution (e.g., Jones & O’Dell 1977); however, it is rather complex. An alternative to the standard formulation is to start from Equation (3). The Stokes parameters are then calculated only after the radiation has been transported through the medium, rather than at the point of emission. Although the two methods are equivalent, as discussed briefly in Björnsson (1988), the latter solution is helpful when trying to understand how the physical properties of the plasma affect the polarization of the emerging radiation. The reason is that the standard solution is expressed in terms of the various plasma parameters (i.e., \( \sigma_{l,m} \)), while the alternative solution is expressed in terms of the polarization properties of the two characteristic waves \( (K_1^1, K_2^1) \) and their phase difference \( (\Delta k) \).

2.1. Characteristic Waves

The dielectric tensor in Equation (3) can be written

\[
\sigma_{l,m} = \frac{c \kappa}{4\pi} \left( \begin{array}{c} 1 \\ -\xi_V -i\xi_L \\ -\xi_L + i\xi_V \end{array} \right),
\]

where \( \xi_V = \xi_V + i\xi_L \) and \( \xi_L = \xi_L + i\xi_V \). The notation in this paper rather closely follows the one in Jones & O’Dell (1977), except that, in order to avoid confusion with the complex conjugate, \( (\cdot)^\star \) is used instead of \( (\cdot)^* \) to denote parameters accounting for the circular and linear birefringence of the plasma. All the \( \xi \)-parameters are normalized to the absorptivity; e.g., \( \xi_V = \kappa_V/k \) and \( \xi_L = \kappa_L/k \), where \( \kappa_V \) and \( \kappa_L \) are the absorption coefficients for the Stokes \( V \) and \( U \) parameters (see Appendix C). Furthermore, it proves convenient to use \( \phi = -\pi/4 \) (see Figure 1) instead of \( \phi = 0 \), as done in Jones & O’Dell (1977), because this renders \( K^1 = -K^2 \) (see below). As a result, the roles played by the Stokes parameters \( Q \) and \( U \) interchange; e.g., synchrotron emission has no \( Q \)-component. Because \( U + iV = 2E_x E_y^* \), this choice also brings forth the formal similarity between the linear and circular polarization.

The eigenvalues obtained by diagonalizing \( \sigma_{l,m} \) are given by

\[
\eta^{\pm 2} = \frac{c \kappa}{4\pi} (1 \mp i \sqrt{\xi_V^2 + \xi_L^2} )
\]
and Equation (3) can be solved directly for the two characteristic waves

\[ E^{i,2} = E^{i,2}_0 \exp \left( -\frac{2\pi}{c} y^{i,2} s \right) \]  

Their phase difference can be defined as

\[ \Delta k = -\frac{2\pi}{c} (\eta^i - \eta^2) = i\kappa \sqrt{\gamma_Y^2 + \gamma_L^2}. \]  

Likewise, the polarization of the two characteristic waves, \( K^{i,2} = E^{i,2}/E^{i,2}_x \), are obtained as

\[ K^{i,2} = \pm \frac{\delta k}{\gamma_Y - i\gamma_L} = \pm \frac{1 - \rho}{\sqrt{1 + \rho}}, \]  

where \( \delta k = \Delta k/\kappa \) is the normalized phase difference and

\[ \rho = \frac{\gamma_Y}{\gamma_L} = \frac{\hat{\xi}_U \hat{\xi}_U - \hat{\xi}_U \hat{\xi}_U + i(\hat{\xi}_V \hat{\xi}_U + \hat{\xi}_V \hat{\xi}_U)}{\hat{\xi}_U^2 + \hat{\xi}_U^2}. \]  

It should be noted that there is a sign ambiguity in Equations (7) and (8) when evaluating the square root. Below, it is shown that this sign always enters in the product of \( K^{i,2} \) and \( \Delta k \). Hence, the choice is physically unimportant as long as the same sign convention is used for both.

It is sometimes claimed that the characteristic waves are orthogonal (e.g., Kennett & Melrose 1998), which implies that their polarization vectors would point in opposite directions on the Poincaré sphere. The radiative transfer is then approximated as a rotation of the polarization vector of the emitted radiation around this axis (Kennett & Melrose 1998; Ruszkowski & Begelman 2002). The polarization of the characteristic waves are orthogonal when \( E^1 \cdot E^2 = 0 \) or \( 1 + K^1 K^2 = 0 \). Hence, it can be seen from Equation (8) that a necessary condition for the characteristic waves to be orthogonal is \( |K^{i,2}| = 1 \). Likewise, from Equation (8):

\[ |K^{i,2}|^2 = 1 - \frac{4\rho_i}{1 + [\rho]^2 - 2\rho_i} \]  

where the subscript “r” denotes the real part of \( \rho \). Therefore, in general, the characteristic waves are orthogonal—and hence, such a simplification should be used with care. However, one may note that, when absorption is neglected, the characteristic waves will be orthogonal, because then \( \rho_i = 0 \) (see Equation (9)).

3. Properties of the Transfer Equation

Although the transfer equation is trivial to solve when using characteristic waves (i.e., Equation (6)), there are a few aspects of the solution that need to be emphasized. The polarization properties of radiation are normally given in terms of the Stokes parameters and the emissivity \( (e) \) is specified for each one of them. Hence, the initial condition in Equation (6), i.e., \( E^{i,2}_0 \), need to be related to the emissivities of the individual Stokes parameters. This involves two steps: (1) Equation (6) presupposes 100% polarized radiation. The emissivities should therefore be divided into two 100% polarized waves. (2) Each of these waves is then written as a sum of the two characteristic waves.

3.1. Division into Two Characteristic Waves

Consider a 100% polarized wave, which initially has an electric field \( E_0 \) with polarization \( K_0 = E_y/E_x \). Its division into the two characteristic waves \( E^{i,2}_0 \) yields

\[ E_{x,0} = E^{i,1}_x + E^{i,2}_x \]

\[ K_{x,0} = K^{i,1}_x K^{i,2}_x + K^{i,2}_x K^{i,1}_x, \]

which can be solved to give

\[ E^{i,1}_{x,0} = -E^{i,2}_{x,0} \]

\[ K^{i,1}_{x,0} = K^{i,2}_{x,0} = K^{i,1}_0 - K^{i,2}_0 K^{i,1}_0 K^{i,2}_0. \]

The connection to the Stokes parameters is obtained from \( |E_{x,0}|^2 = (I_0 + Q_0)/2 \) and \( K^{i}_{x,0} = (U_0 + iV_0)/2|E_{x,0}|^2 \). Without loss of generality, \( E_{x,0} \) can be chosen to be real and one finds

\[ E^{i,1}_{x,0} = \frac{I_0}{8(1 + q_0)} \left( 1 + q_0 - \frac{u_0 - iV_0}{K^2} \right) \]

\[ E^{i,2}_{x,0} = \frac{I_0}{8(1 + q_0)} \left( 1 + q_0 - \frac{u_0 + iV_0}{K^2} \right), \]

where \( q_0 = Q_0/I_0, u_0 = U_0/I_0, v_0 = V_0/I_0, \) and \( K^1 = -K^2 \) has been used.

As the wave propagates through the plasma, its components vary according to \( E_x = E^{i,1}_x + E^{i,2}_x \) and \( E_y = K^{i,1}_x E^{i,1}_x + K^{i,2}_x E^{i,2}_x = K^2(-E^{i,1}_x + E^{i,2}_x) \), where now \( E^{i,1}_x = E^{i,2}_x \exp(-\kappa s/2 \pm \Delta ks/2) \). With \( U + iV = 2E_x E^*_y \), it is shown in Appendix A that, after traveling a distance \( s \), its circular polarization is

\[ V = I_0 \exp(-\kappa s) \left[ v_0 \left( \frac{K_i}{|K|} \right)^2 \cos(\Delta k s) + \left( \frac{K_r}{|K|} \right)^2 \sin(\Delta k s) \right] \]

\[ -u_0 \frac{K_i K_r}{|K|^2} \left[ \cos(\Delta k s) - \cos(\Delta k s) \right] \]

\[ + \frac{K_i}{2} \left[ 1 - \frac{V_0}{|K|^2} \right] q_0 \left[ 1 - \frac{1}{|K|^2} \right] \sin(\Delta k s), \]

\[ + \frac{K_r}{2} \left[ 1 - \frac{1}{|K|^2} \right] q_0 \left[ 1 - \frac{1}{|K|^2} \right] \sin(\Delta k s), \]

(14)

where the subscripts “r” and “i” denote the real and imaginary parts, respectively, of a quantity. Furthermore, \( K \equiv K^2 \) has been introduced to simplify the notation.

A number of general features of the circular polarization are apparent from Equation (14); they will also be relevant for a homogeneous source, i.e., when emission occurs along the ray path. The things to note for a synchrotron source are:
4. Circular Polarization from a Homogeneous Source

The circular polarization from a homogeneous source is obtained by integrating Equation (14) from \( s = 0 \) to \( s = s_{\text{max}} \). This is done in Appendix B. Here, \( s_{\text{max}} \) is the thickness of the source, such that its optical depth is \( \tau = n s_{\text{max}} \). The observed circular polarization in compact radio sources is usually on the order of one percent or smaller. Although inhomogeneities along a given sightline can severely affect the polarization (this will be discussed in a forthcoming paper), the simplest explanation is that the physical conditions are such that either \( |\rho| < 1 \) or \( |\rho| \gg 1 \) (see the discussion in Section 3.1). The plasma parameter with the least constrained value in compact radio sources is \( \xi_V \). This is due to its sensitivity to two virtually unknown quantities, i.e., the number of low-energy electrons (e.g., the low-energy cutoff of the relativistic electrons) and the fraction of electron-positron pairs in the plasma (see Appendix C). Hence, the two limits of \( |\rho| \) likely correspond to the two extremes \( |\xi_V| < 1 \) and \( |\xi_V| \gg 1 \).

4.1. Circular Polarization from Nearly Circular Characteristic Waves

It is convenient to write \( \Delta \kappa \), \( s_{\text{max}} = \delta k_1 \tau \) and \( \Delta \kappa \), \( s_{\text{max}} = \delta k_1 \tau \). When \( |\xi_V| \gg |\xi_U|, |\rho| \gg 1 \) and the characteristic waves are nearly circularly polarized (see Equation (8)). In most cases, this corresponds to \( |\xi_V| \gg 1 \). Expanding Equations (7) and (8) to the lowest order in \( |\rho|^{-1} \), one finds that \( |K|, |\delta k_1|, |\delta k_2|^{-1}, \) and \( |K|^{-2} - 1 \) are all \( \sim |\rho|^{-1} \). Furthermore, let \( v \) and \( u \) denote the normalized \( V \) and \( U \) emissivities, respectively. Then \( |v| \) and \( |\xi_V| \) are both small (see Appendix C). Assuming them to be on the same order of magnitude as \( |\rho|^{-1} \), the solution in Appendix B can be expanded to the lowest order in \( \rho^{-1} \). This yields

\[
V = S \left[ v \{1 - \exp(-\tau)\} - u K_t \{1 - \exp(-\tau)\} \right. \\
+ \left. \delta k_t \{1 - \exp(-\tau)(1 + \tau)\} \right]
+ u K_t \frac{\sin(\delta k_1 \tau)}{\delta k_1} \exp(-\tau),
\]

where \( S = \epsilon / \kappa \) is the source function.

The relevant plasma parameters are \( K_t = -\xi_U / \xi_V, \delta k_t = -\xi_V - \xi_U \xi_V, \delta k_1 = \xi_V, \) and \( \Lambda_1 = \kappa - 2 \xi_U / \xi_V \). With these expressions, it is straightforward to show that Equation (15) is identical to the solution given in Björnsson (1988). However, to illuminate the various physical mechanisms influencing the observed circular polarization, a better representation of the solution is

\[
V = S \left[ v \{1 - \exp(-\tau)(1 + \tau)\} + v \tau \exp(-\tau) \right]
- K_t \left\{ u \{1 - \exp(-\tau)(1 + \tau)\} \right.
+ \left. u \tau \exp(-\tau) \right\}.
\]

This shows explicitly the similarities between the \( V \) and \( U \) emissivities/absorptivities. For a thermal distribution of electrons, e.g., a relativistic Maxwellian, \( v = \xi_V \) and \( u = \xi_U \) (Jones & Hardee 1979). Hence, it is the nonthermal aspect of the electron distribution that causes the change of sign in the circular polarization at large optical depths. For a power-law distribution of relativistic electrons, both \( |v - \xi_V| \) and \( |u - \xi_U| \) are quite a bit smaller than \( |v| \) and \( |u| \), respectively (Jones & O’Dell 1977). Furthermore, for small optical depths, the nonthermal terms both vary as \( \tau^2 \), while the \( v \)- and \( u \)-terms vary as \( \tau \) (for the \( u \)-term, this is valid for \( \tau > \xi_V^{-1} \)). Therefore, it is expected that, for most electron distributions, the major contributions to the integrated circular polarization come from the \( v \)- and \( u \)-terms.

Although \( q = 0 \) for synchrotron radiation, the \( q \)-term has been kept in the general solution given in Appendix B. The reason for this is to illustrate the nature of the conversion of linear to circular polarization. It is sometimes said (Jones 1988; MacDonald & Marscher 2018) that the conversion acts only on the Stokes parameter \( Q \), and hence, that the conversion in a synchrotron source occurs in two steps; first, \( U \) is converted to
$Q$ through Faraday rotation, and then $Q$ is converted to $V$. However, no $q$-term appears in Equation (16). This implies that, even if there were a $Q$-term, its contribution to the circular polarization would be of order $\xi_V^{-2}$ and therefore negligible. Another way of seeing the same thing is to consider the magnitude of the transfer-induced circular polarization, i.e., $|K_r|$. Faraday rotation is $\alpha \xi_V$, but $K_r \propto \xi_V^{-1}$; i.e., larger Faraday rotation (larger $Q$) results in smaller circular polarization. Hence, the commonly used name “Faraday conversion” may be somewhat of a misnomer for this process, because the conversion of $U$ to $V$ occurs directly without any intermediate steps.

It is often assumed that absorption does not affect the conversion of $U$ to $V$ in the optically thin regime (e.g., Wardle et al. 1998; Enßlin 2003; O’Sullivan et al. 2013). The solution to the transfer equation is then given by the Faraday conversion term, $V/I = \mu \tau c / 6$, where $\tau_I = \xi_V \tau$ and $\tau_C = \xi_U \tau$. However, expanding Equation (16) for small optical depths and using $I = S\tau$, one finds $V/I = (\xi_V / \xi_U)\tau(u - \xi_U) + u(1 - \sin(\xi_U \tau) / \xi_U)$ for the conversion of linear to circular polarization. A rapid rise in circular polarization occurs at $\tau \sim \xi_V^{-1}$ such that, for $\xi_V \tau > 1$, the leading term is $u \xi_U / \xi_V$. For $\xi_V \tau < 1$, the circular polarization is substantially smaller because it is determined by higher-order terms. Among these is the Faraday conversion term, which is smaller by a factor $(\xi_V / \xi_U)^2$ as compared to $u \xi_U / \xi_V$. Furthermore, the contribution from the nonthermal term $(\chi_T (u - \xi_U))$ may become significant because it decreases with decreasing optical depth slower than the Faraday conversion term ($\tau$ versus $\tau^2$). It should also be noted that Equations (15) and (16) are correct only to first order in $|\rho|^{-1} \sim |\xi_U / \xi_V|$. Hence, second-order terms may dominate the Faraday conversion term $|\xi_U / \xi_V|$ versus $(\xi_V \tau)^2$. In this frequency range, Jones & O’Dell (1977) assumed that the observed circular polarization is unaffected by transfer effects; instead, it is given directly by the emission process. Therefore, it is unlikely that neglect of absorption is a viable approximation, even in the optically thin regime (see the discussion of the effects of absorption on the non-orthogonality of the characteristic waves at the end of Section 2.1). In general, then, the Faraday conversion term does not provide a good approximation to the transfer-induced circular polarization. This aspect of the circular polarization is discussed further in Section 4.2.

### 4.2. Circular Polarization from Nearly Linear Characteristic Waves

When $|\xi_U| \ll |\xi_V|$, the characteristic waves are nearly linearly polarized because $|\rho| \ll 1$. In this limit, $|K|$ and $|K|^2 - 1$ are both $\sim |\rho|$. Because $|\xi_U| \sim 1$ and $|\xi_V|$ cannot be assumed to be large in general, this corresponds in most cases to $|\xi_U| \ll 1$. The rather simpler forms of Equations (15) and (16) are due mainly to the properties of the phase difference $\delta k$ (i.e., $|\delta k| \ll 1$ and $|\delta k| \gg 1$). Here, on the other hand, one finds, to lowest order in $\rho$, $\delta k = -\xi_U$ and $\delta k = \xi_U$, the magnitudes of which are both expected to be on the order of unity. This leads to a somewhat more complex expression for $V$. Expanding the solution in Appendix B to lowest order in $\rho$ yields

$$V = S \left\{ \frac{(\nu - \xi_U - q_{\xi_U})}{1 + \xi_U^2} \right\} \left\{ 1 - \exp(-\tau) \left( \cos(\xi_U \tau) \right) + \frac{\sin(\xi_U \tau) \exp(-\tau)}{\xi_U} \right\}$$

$$+ K_r \left( \frac{(\nu - \xi_U)}{1 - \xi_U^2} \left( 1 - \exp(-\tau) \left( \cos(\xi_U \tau) + \frac{\sin(\xi_U \tau)}{\xi_U} \right) \right) \right)$$

$$+ \frac{\sin(\xi_U \tau) \exp(-\tau)}{\xi_U}$$

$$- K_r \left( \frac{(\nu - \xi_U)}{1 + \xi_U^2} \left( 1 - \exp(-\tau) \left( \cos(\xi_U \tau) + \frac{\sin(\xi_U \tau)}{\xi_U} \right) \right) \right)$$

$$+ \frac{\sin(\xi_U \tau) \exp(-\tau)}{\xi_U} \right\}$$

$$(17)$$

where $K_r = (\xi_V / \xi_U - \xi_V / \xi_U) / (\xi_U^2 + \xi_U^2)$ and $|K|^2 - 1/2 = (\xi_U / \xi_V - \xi_U / \xi_V) / (\xi_U^2 + \xi_U^2)$, which leads to $\xi_U / |K|^2 - 1/2 = \xi_U - \xi_V K_r$, have been used.

The structures of Equations (16) and (17) are rather similar. Although $|\delta k| \sim 1$ makes the variation of $V$ with $\tau$ more involved, their basic properties remain the same: e.g., the nonthermal terms (i.e., $\nu - \xi_U$ and $\nu - \xi_U$) are small compared to the $u$- and $\mu$-terms. They only become important at large optical depths, where they cause a change of sign. Furthermore, the amplitude of the conversion of linear to circular polarization is determined by $K_r$ rather than $K_r$.

This formal similarity between Equations (16) and (17) is due to the symmetric expressions of $\delta k$ and $K$ in the two limits (see Equations (7) and (8)). It is shown in Appendix B that the two limits can be related by just interchange of $\xi_V$ and $\xi_U$. Hence, several of the main properties, which derive from the birefringence of the plasma, can be obtained by interchanging $\xi_V$ and $\xi_U$. One example is the term giving the major contribution to the circular polarization from conversion of linear polarization. In Equation (16), it is $\propto \sin(\xi_U \tau) / \xi_U \tau$, while in Equation (17), the corresponding expression is $\propto \sinh(\xi_U \tau) / \xi_U \tau - \sin(\xi_U \tau) / \xi_U \tau$. For nearly circular characteristic waves, this term causes a rapid rise in circular polarization at $\tau \sim |\xi_U|^{-1}$ (see the discussion in Section 4.1). Likewise, for nearly linear characteristic waves, this increase occurs at $\tau \sim |\xi_U|^{-1}$. The major difference is the values of $|\xi_U|$ and $|\xi_V|$ in the two limits. As already mentioned, they are expected to be quite different; while $|\xi_U| \gg 1$ for circular characteristic waves, $|\xi_U|$ may not be much larger than unity for linear characteristic waves. Therefore, the observed circular polarization is expected to come from, on average, larger optical depths for nearly linear as compared to nearly circular characteristic waves.

The transition between these two limits was discussed in Björnsson (1990). There, it was shown that the optical depth where the circular polarization peaks decreases smoothly from $\tau > 1$ to $\tau < 1$ as the characteristic waves change from linear to circular (see also Jones & O’Dell (1977), for the latter limit).
Physically, this can be understood as follows. When there are few low-energy electrons, e.g., a power-law distribution of electrons with a low-energy cutoff close to the synchrotron self-absorption frequency, $|\xi_\ell| < |\xi_0|$, and the characteristic waves are linearly polarized. As the low-energy cutoff decreases, the magnitudes of both $\hat{\xi}_\ell$ and $\hat{\xi}_U$ increase. The value of $\hat{\xi}_\ell$ increases much faster than that of $\hat{\xi}_U$, which causes the characteristic waves to change from linear to circular. At the same time, as the value of $\hat{\xi}_U$ increases, so does the relative contribution to $V$ from the optically thin part of the spectrum (i.e., corresponding to $\tau \gtrsim |\xi_0^U|^{-1}$).

This shows that the frequency distribution of the circular polarization is expected to be quite different for nearly linear and nearly circular characteristic waves. Basically, this is due to the very different values of $\delta k_i$ in the two cases; i.e., it is a consequence of the increase in circular polarization at $\tau \sim |\delta k_i|^{-1}$, along with an increasing value of $|\delta k_i|$ as the characteristic waves change from nearly linear to nearly circular (see Equations (7) and (8)). The use of this property to distinguish observationally between linear and circular characteristic waves is discussed further in Section 5.

For nearly circular characteristic waves, the frequency range where $1 > \tau > |\xi_0^U|^{-1}$ should be rather large. On the other hand, the corresponding frequency range for nearly linear characteristic waves is expected to be much smaller. Hence, $\tau < |\xi_0^U|^{-1}$ may dominate the optically thin region.

Expanding Equation (17) to lowest order in $\hat{\xi}_U$ gives

$$\frac{V}{I} = \nu(1 - \tau) + (\nu - \xi_\ell)\frac{\tau}{2} + K_\nu u(\xi_\ell^2 + \hat{\xi}_U^2)\frac{\tau^2}{6} - q\hat{\xi}_\ell^2 \frac{\tau^2}{2}$$

$$= \nu(1 - \tau) + (\nu - \xi_\ell)\frac{\tau}{2} + u\xi_\ell\xi_U\frac{\tau^2}{6} + u\hat{\xi}_\ell\hat{\xi}_U\frac{\tau^2}{6} - q\hat{\xi}_\ell^2 \frac{\tau^2}{2}.$$  

(18)

It can be seen that the Faraday conversion term appears in Equation (18): $u\hat{\xi}_\ell\hat{\xi}_U\tau^2/6 = ur\tau/6$. Because $\tau r_\tau c = \hat{\xi}_U/(\hat{\xi}_U/\hat{\xi}_0)(\hat{\xi}_\ell/\hat{\xi}_0)^2 = (\hat{\xi}_\ell/\hat{\xi}_0)^2(\hat{\xi}_\ell/\hat{\xi}_0)^2$, this appearance is another example of the symmetry between nearly circular and nearly linear characteristic waves. If this term were the dominant one, the name “Faraday conversion” would be appropriate in this limit. However, as discussed already in Section 4.1, this is unlikely because: (1) it is really a third-order term, in the sense that both $\hat{\xi}_\ell/\hat{\xi}_0$ and $\hat{\xi}_U/\hat{\xi}_0$ are much smaller than unity; and (2) also keeping second-order terms in the expansion parameter (i.e., $\hat{\xi}_\ell/\hat{\xi}_0$) could give contributions to $V$ larger than the Faraday conversion term.

In contrast to circular characteristic waves, Equation (17) shows that a non-synchrotron $q$-term can affect the circular polarization. Formally, the $q$-term is similar to the Faraday conversion term because, roughly, $U\hat{\xi}_\ell/2$ is the $Q$-value produced by Faraday rotation of the synchrotron $U$-emission. Such an additional source of linearly polarized emission can give a significant contribution to $V$, because there are no restrictions on the value of $q$; for example, as shown by Hodge (1982), this term can dominate the observed circular polarization in inhomogeneous sources.

5. Discussion

The degree of circular polarization observed in compact radio sources varies, but it is rarely larger than $\sim 1\%$. This is roughly consistent with the level expected directly from the synchrotron emission process (see Appendix C). However, as discussed in the introduction, there are reasons to believe that the observed circular polarization is also affected by transport effects. This opens up a way to gain more detailed information about the source properties than is possible from the flux alone. As mentioned in Section 4, the rather low level of circular polarization makes it likely that the characteristic waves are either nearly linearly or nearly circularly polarized. It is important to be able to distinguish between the two, because this has implications for some of the most central issues regarding the properties of compact radio sources; e.g., the presence of electron–positron pairs and the acceleration process of the radiating particles.

The flat spectrum of compact radio sources has been called a “cosmic conspiracy” by Cotton et al. (1980). Blandford & Königl (1979) showed that a class of models in which relativistic electrons stream out in a jet with constant opening angle could account for the observations under two conditions: (1) the adiabatic losses of the electrons are compensated by a continuous re-acceleration, such that their low-energy cutoff stays constant; and (2) the strength of the magnetic field varies inversely with radius. This leads to a constant brightness temperature along the jet. Such an inhomogeneous jet has a self-similar structure which implies no change of polarization with frequency, because the parameters in the transfer equation stay constant along the jet.

Inhomogeneous sources can, roughly, be divided into two classes. In the first, the source is homogeneous along each sightline, but the source properties (e.g., the optical depth) vary between different sightlines. In its original form, the Blandford/Königl-jets belong to this class. In the second class, the source properties vary along a given sightline, e.g., due to turbulence.

The first class of sources can be seen as a superposition of many homogeneous sources. In practice, the polarization properties are obtained by integrating the results in Section 4 over the appropriate range of parameter values. On the other hand, the polarization properties of the second class of sources are more complicated to calculate here, because the characteristic waves do not propagate individually, i.e., they couple. In the present paper, it is assumed that the first kind of source model is sufficient to discuss the polarization properties of compact radio sources. The effects of coupling of the characteristic waves will be treated in a forthcoming paper.

Although neglecting coupling could be seen as a serious limitation to the validity of the conclusions, this may not be so—at least, not qualitatively. To understand why, recall that it was mentioned in Section 3.1 that the amplitude of the circular polarization is determined mainly by the polarization properties of the characteristic waves ($K$), while its variation with frequency/optical depth is determined in large part by their phase difference ($\Delta \phi$). Equation (3) shows that the accumulated phase difference along a ray path does not depend on whether the medium is homogeneous or not. The coupling between the characteristic waves is due to variation of the local value of $K$ along the ray path. As a result, the coupling is expected to affect mainly the amplitude of the circular polarization and less so its frequency/optical depth.
dependence. This can be seen explicitly in Björnsson (1990), where the circular polarization from a homogeneous medium is compared to that emerging from a medium in which coupling is important. Therefore, the discussion below focuses on the frequency/optical depth dependence of the circular polarization as a way to distinguish between nearly circular and nearly linear characteristic waves.

For flat spectrum radio sources, a substantial frequency dependence of the polarization is expected only in the region around the spectral turnover, where the emission becomes optically thin. This occurs normally at rather large frequencies (∼100 GHz) and it has only recently become possible to obtain high-quality observations of the circular polarization in this range for a fair number of sources (Thum et al. 2018). However, not all compact radio sources conform to the standard Blandford/König-jet model. Gigahertz-Peaked Spectrum sources are a class of objects that have lower turnover frequencies (∼few GHz) as well as a spectrum declining toward lower frequencies. It is clear that these sources are inhomogeneous, because their spectra normally are quite a bit flatter than the characteristic $\nu^{3/2}$ spectrum of homogeneous sources. Hence, they are expected to also show frequency-dependent polarization in the optically thick part of the spectrum. A good example of such a source is PKS B2126-158 (O’Sullivan et al. 2013).

### 5.1. The POLAMI Survey

In the POLAMI survey, a large number of compact radio sources have been observed multiple times at 3 and 1.3 mm (Agudo et al. 2018b). The spectral index ($\alpha$) indicates that the flux is mostly optically thin radiation. However, there is a tendency for the spectrum to flatten when the flux increases (Agudo et al. 2018a). This suggests that the turnover frequency (i.e., $\tau \sim 1$) is, on average, close to 3 mm. This sample is therefore a good starting point for a discussion of the origin of the circular polarization.

An important finding here is that the maximum amplitude of circular polarization is higher at 1 mm (2.6%) as compared to 3 mm (2.0%) (Thum et al. 2018). Furthermore, both of these values are, in turn, substantially larger than those found by others at longer wavelengths (i.e., optically thick frequencies). There are two implications from these observations, which both suggest the presence of nearly circular characteristic waves. As shown in Section 4, the observed peak of the degree of circular polarization in the optically thin regime is consistent with nearly circular characteristic waves, but hard to reconcile with nearly linear characteristic waves. Also, in an inhomogeneous source, the polarization at optically thick frequencies corresponds to an average over a range of optical depths. The sign change of the circular polarization at large optical depths (due to the $u - \xi_1$ term) is similar for both circular and linear characteristic waves. However, the relative contribution to the circular polarization from this nonthermal term is larger for nearly circular characteristic waves as compared to the nearly linear ones, because $K_u \propto \nu^{-1}$ and $K_1 \propto \nu$ (see Equations (16) and (17)). Hence, the relative increase of the circular polarization between the optically thick and thin parts of the spectrum should be larger for nearly circular characteristic waves as compared to nearly linear ones.

In the standard jet model, the spread in optical depth in the azimuthal direction is rather small and results in an averaging of possible rapid variations on small scales; see the integration over a thin shell done in Jones & O’Dell (1977). Hence, observations of an unresolved source are determined mainly by the radial variations of the jet properties.

For nearly circular characteristic waves, the circular birefringence is large (i.e., $|\xi_1| \gg 1$) and the main contribution to the linearly polarized flux comes from small optical depths, $\tau \sim |\xi_1|^{-1}$. Let $R_o$ be the radius where $\tau = 1$ for some frequency $\nu$ and $\xi_{1,o}$ the corresponding value of $\xi_1$. With $B \propto R^{-1}$, the radial variation of the optical depth is $\tau = (R/R_o)^{(5+2\alpha)/2}$ and $\xi_1 = \xi_{1,o}(R/R_o)^{(1+2\alpha)/2}$. The radius where most of the linearly polarized flux is emitted ($R_i$) is then obtained from $\tau(\xi_1) \sim 1$ as $R_i \sim R_o \xi_{1,o}$. Furthermore, the corresponding Stokes parameters are $U_L \sim |Q_L| \sim u S_o \xi_{1-o}^{\alpha/2}$, where $S_o$ is the source function at $R_o$. Because $U_L \sim |Q_L|$ in this limit, small variations of the radial jet properties could give rise to rather large variations in the polarization angle; in particular, this may account for the observed lack of a preferred polarization angle in many sources (Agudo et al. 2018a).

If conversion from linear polarization contributes significantly to the observed circular polarization, one can deduce $|\xi_{1-o}| \sim 10^2$ (see Equation (16)). Assuming $|\xi_{1-o}| \sim 1$, $R_i \sim 10 R_o$, and the linearly polarized flux comes from a radius much larger than that of either the total or the circularly polarized flux. In line with observations, this implies that variations in linear polarization should have a longer timescale than—and correlate weakly with—those in circular polarization or total flux. Furthermore, with $\alpha \approx 1$, the depolarization would also be $\sim 10$, which shows that Faraday rotation could be responsible for a larger fraction of the observed depolarization of the linear flux.

The total and circularly polarized fluxes come from roughly the same region (i.e., $\tau \sim 1$). However, their sensitivity to changes in the various plasma parameters are very different. The circular polarization varies more rapidly with optical depth than the total flux, but most importantly, the circular polarization is also sensitive to variations in plasma parameters that leave the total flux unaffected. As an example, for nearly circular characteristic waves, the magnitude of the circular polarization due to conversion from linear polarization is $\sim |\xi_1|/\xi_{1-o} \propto \gamma_i^{1/2} \ln \gamma_i$ (Appendix C), where $\gamma_i$ is the Lorentz factor at the lower cutoff in the energy distribution of the relativistic electrons. The more rapid and uncorrelated variations of the circular flux as compared to the total flux observed by POLAMI could then come from small changes in $\gamma_i$.

### 5.2. PKS B2126-158

O’Sullivan et al. (2013) have presented high-quality, multi-frequency polarization measurements of PKS B2126-158, which has a turnover frequency at 5.7 GHz. The source is inhomogeneous, because it has an inverted spectrum below this frequency ($\propto \nu^3$). This makes it an ideal object for frequency dependent polarization studies; in particular, in contrast to the flat spectrum sources, frequency-dependent polarization is expected in the optically thick part of the spectrum. The circular polarization peaks at a frequency above the turnover frequency, indicating nearly circular characteristic waves. Several of its properties are as expected for a homogeneous source with $|\xi_1| \gg 1$ (see Jones & O’Dell 1977); for example, a broad minimum in the degree of linear polarization coincides with the maximum in circular polarization, and there is a clear
indication of a $\sim 90^\circ$ swing in the polarization angle in the optically thick part of the spectrum (i.e., $Q$ changes sign).

However, there are two aspects of the observations that do not fit with a homogeneous source; namely, the lack of a sign change of the circular polarization in the optically thick part of the spectrum and the apparently smooth $\sim 90^\circ$ swing in the polarization angle (rather than an abrupt flip). In order to see how these can be accounted for by an inhomogeneous source structure, a few of its properties needs to be considered.

The range of optical depths in an inhomogeneous source, which contributes to the flux at a given frequency, depends on the slope in the optically thick part of the spectrum. For a flat spectrum, the polarization is independent of frequency and no change of sign is observed in either $Q$ or $V$. As the spectrum becomes more inverted, the relative importance of the large optical depths increases. Hence, for some value of the slope, sign changes will be observed for $Q$ and/or $V$. For $\tau[\xi_V] > 1$, it can be shown that $Q = S[\xi_V(1 - \exp(-\tau)) - u]/\xi_V$ in a homogeneous source. As compared to the circular polarization in Equation (16), there are two differences: (1) the sign change in $Q$ occurs at smaller optical depth than the corresponding change for $V$; and (2) because $Q \propto \xi_V^{-1}$ and $V \propto \xi_V/\xi_V$, the amplitude of $Q$ decreases with frequency somewhat faster than does the one for $V$ ($\nu^{-1.2}$ versus $\nu^{-1}$, where $\alpha = 0.7$ has been used). Hence, the relative importance of large optical depths is larger for $Q$ than for $V$. Both of these effects cause the sign change in $Q$ to occur at a higher frequency than for $V$. In a forthcoming paper, it will be discussed how the observed change of sign in $Q$ but not in $V$ can be made consistent with the observed spectrum.

In general then, sign changes in $V$ and $Q$ in inhomogeneous sources depend on the slope in the optically thick part of the spectrum. Observations with high spatial resolution may resolve some of the inhomogeneities, and hence could make it more likely to find such sign changes. This could be the case for the VLBA-observations of NGC 1275 (3C 84), where the sign of the circular polarization changed between the optically thick and thin parts of the source (Homan & Wardle 2004). Unfortunately, no linear polarization was detected, so the expected concurrent sign change in $Q$ could not be established.

In contrast to the circular polarization, the value of $Q$ is determined by contributions from two very different regions in the jet. The optically thin emission comes from radii much larger than that at $\tau \sim 1$, which emits $Q$-flux with opposite sign. As the spectrum becomes increasingly inverted, the relative contribution to $Q$ from the optically thin emission goes down. As mentioned above, $U \sim |Q|$ for this component, even when the total $Q$ changes sign—and hence, the value of $U$ will be non-negligible. This causes the $90^\circ$ flip in position angle observed in a homogeneous source to be replaced by a smooth $90^\circ$ swing in an inhomogeneous source.

### 5.3. Observational Implications

Both the POLAMI sample and the detailed observations of PKS B2126-158 are most straightforwardly understood for characteristic waves, which are nearly circular polarized. This conclusion rests on the observed frequency distribution of the circular polarization and implies $|\xi_V/\xi_V| \gg 1$. Its actual value is harder to estimate because, as discussed above, the magnitude of the circular polarization may be seriously affected by inhomogeneities along various lines of sight. However, the properties of the linear polarization in the POLAMI sample can be accounted for by a value of $|\xi_V|$ consistent with only minor contributions from inhomogeneities. Assuming this to be the case, the value of $|\xi_V/\xi_V| \sim 10^2$ can be used to constrain the properties of the synchrotron plasma.

In addition to the magnetic field direction, when the frequency dependence of the transfer coefficients are normalized to the turnover frequency, there are two free parameters (see Appendix C); namely, $\gamma_{\min}$ and the number of electron–positron pairs ($n_p$) relative to the excess number of electrons ($n_{\text{exc}}$). With $|\xi_V/\xi_V| \sim 10^2$, one finds $(\gamma_{\min}^3/\ln \gamma_{\min})(1 + 2n_p/n_{\text{exc}}) \sim 10^2$ (Equation (45)). Although the presence of nearly circular characteristic waves by itself is enough to show that $\gamma_{\min}$ is much below that corresponding to the turnover frequency (i.e., $\gamma_{\min} \ll \gamma_{\text{lab}} \approx 10^3$; see Appendix C), observations allow a fair fraction of electron–positron pairs. An upper limit from the relativistic particles is obtained for $\gamma_{\min} \sim 1$, i.e., the particles are injected into the acceleration process with transrelativistic energies. This gives $n_p/n_{\text{exc}} \lesssim 10^2$. The emission coefficient for the circular polarization depends on $n_{\text{exc}}/n_p$ but not $\gamma_{\min}$. Hence, the degeneracy between the two can be broken by direct observation of the circular polarization intrinsic to the synchrotron process. However, this may require observations in the frequency range corresponding to $|\xi_V| \tau \lesssim 1$ (see also below).

The conversion of linear to circular polarization is often described by the Faraday conversion term $u_{\tau_{C6}}/6$, which has a very steep frequency dependence ($\propto \nu^{-5}$). Because observations indicate a more modest frequency dependence of the circular polarization, this has limited more detailed modeling of the sources properties (e.g., O’Sullivan et al. 2013; Thum et al. 2018). However, it was shown in Section 4 that this term is unlikely to significantly affect the observed circular polarization. Instead, as argued above, the use of the full solution to the transfer equation allows a rather direct interpretation of the observations.

Nearby circular characteristic waves imply large Faraday depths over a wide range of frequencies. The apparent lack of observed Faraday rotation has been used to argue, instead, that the characteristic waves are linearly polarized (Wardle 1977). Although, in the standard jet model, polarization in the flat, optically thick part of the spectrum should be constant, Faraday rotation is expected in the optically thin part. However, even here, the polarization angle should remain constant until the transition to the Faraday thin regime occurs (i.e., $\xi_V/\tau \sim 1$). With the source parameters deduced above from the observed circular polarization (e.g., $|\xi_V| \sim 10^2$), this transition takes place at a frequency $|\xi_V|^{1/2} \sim 10$ larger than the turnover frequency. Accurate polarization measurements may be hard to obtain at such frequencies. Furthermore, the change in position angle should be smaller than for a homogeneous source. Because $U \sim |Q|$, the change in position angle is expected to be $\sim \pi/8$ rather than $\sim \pi/4$ for a homogeneous source.

### 6. Conclusions

The transfer equation of polarized light in a homogeneous medium can be solved analytically. However, the standard solution is complex and observations are usually discussed in terms of various approximations. The main conclusions in the present paper are:
(1) The use of characteristic waves allows an alternative way of expressing the transfer equation. The solution is more compact and transparent regarding the physical mechanisms determining the emerging polarization than in the standard formulation.

(2) The frequency dependence of the circular polarization is a direct way of establishing the properties of the characteristic waves.

(3) High-quality observations of circular polarization in compact radio sources indicate that the characteristic waves are nearly circularly polarized. This provides, for example, an upper limit to the fraction of electron–positron pairs.

(4) Several of the approximations in common use have limited applicability; for example, it has been shown that the Faraday conversion term is unlikely to have a significant impact on the observed circular polarization.

Appendix A

Propagation of a Polarized Light Ray

With the initial conditions given by Equation (13)

\[
U + iV = \frac{I_0 K^* \exp(-\kappa s)}{4(1 + q_o)} \left[ (1 + q_o - \sigma) \exp\left(\frac{\Delta k}{2} s\right) \right. \\
+ \left. (1 + q_o + \sigma) \exp\left(-\frac{\Delta k}{2} s\right) \right] \\
+ \left\{ (1 + q_o - \sigma^*) \exp\left(\frac{\Delta k^*}{2} s\right) \right. \\
+ \left. (1 + q_o + \sigma^*) \exp\left(-\frac{\Delta k^*}{2} s\right) \right\}, \tag{19}
\]

where, again, \( K^2 \equiv K \) has been used together with \( \sigma \equiv (u_o - i v_o)/K \). The terms in Equation (19) can be rearranged to give

\[
U + iV = \frac{I_0 K^* \exp(-\kappa s)}{4(1 + q_o)} \left[ (1 + q_o - \sigma) \exp\left(\frac{\Delta k}{2} s\right) \right. \\
+ \left. (1 + q_o + \sigma) \exp\left(-\frac{\Delta k}{2} s\right) \right] \\
+ \left\{ (1 + q_o - \sigma^*) \exp\left(\frac{\Delta k^*}{2} s\right) \right. \\
+ \left. (1 + q_o + \sigma^*) \exp\left(-\frac{\Delta k^*}{2} s\right) \right\]. \tag{20}
\]

Because the wave is 100% polarized, \( |\sigma|^2 = (1 - q_o^2)/|K|^2 \) and Equation (20) can be rewritten as

\[
U + iV = \frac{I_0 K^* \exp(-\kappa s)}{8(1 + q_o)} \left[ (1 + q_o - \sigma) \cosh(\Delta k_s s) \right. \\
+ \left. (1 + q_o + \sigma) \sinh(\Delta k_s s) \right] \\
+ \left\{ (1 + q_o - \sigma^*) \cosh(\Delta k^*_s s) \right. \\
+ \left. (1 + q_o + \sigma^*) \sinh(\Delta k^*_s s) \right\}. \tag{21}
\]

With the use of

\[
\sigma = \frac{K^* (u_o - i v_o)}{|K|^2} = \frac{(K_r u_o - K_i v_o) - i(K_i u_o + K_r v_o)}{|K|^2}, \tag{22}
\]

one finds

\[
V = I_0 \exp(-\kappa s) \left[ v_o \left\{ \left( \frac{K_i}{|K|} \right)^2 \cosh(\Delta k_s s) \right. \right. \\
+ \left. \left( \frac{K_r}{|K|} \right)^2 \cos(\Delta k^*_s s) \right\} \\
+ \left. \frac{u_o K_r K_i}{|K|^2} \left\{ -\cosh(\Delta k_s s) + \cos(\Delta k^*_s s) \right\} \right] \\
+ \frac{K_r}{2} \left\{ \left| K_r^2 + 1 \right| \left| K_i^2 - 1 \right| + q_o \left| K_i^2 + 1 \right| \left| K_r^2 - 1 \right| \right\} \sinh(\Delta k_s s) \\
+ \frac{K_i}{2} \left\{ \left| K_r^2 - 1 \right| \left| K_i^2 + 1 \right| q_o \left| K_i^2 + 1 \right| \sin(\Delta k_s s) \right\}, \tag{23}
\]

and

\[
U = I_0 \exp(-\kappa s) \left[ u_o \left\{ \left( \frac{K_i}{|K|} \right)^2 \cosh(\Delta k_s s) \right. \right. \\
+ \left. \left( \frac{K_r}{|K|} \right)^2 \cos(\Delta k^*_s s) \right\} \\
+ \left. \frac{v_o K_r K_i}{|K|^2} \left\{ -\cosh(\Delta k_s s) + \cos(\Delta k^*_s s) \right\} \right] \\
+ \frac{K_r}{2} \left\{ \left| K_r^2 + 1 \right| \left| K_i^2 - 1 \right| + q_o \left| K_i^2 + 1 \right| \left| K_r^2 - 1 \right| \right\} \sinh(\Delta k_s s) \\
+ \frac{K_i}{2} \left\{ \left| K_r^2 - 1 \right| \left| K_i^2 + 1 \right| q_o \left| K_i^2 + 1 \right| \sin(\Delta k_s s) \right\}. \tag{24}
\]

Likewise,

\[
|E_x|^2 = \frac{I_0 \exp(-\kappa s)}{8(1 + q_o)} \left[ (1 + q_o - \sigma) \cosh(\Delta k_s s) \right. \\
+ \left. (1 + q_o + \sigma) \sinh(\Delta k_s s) \right] \\
+ \left\{ (1 + q_o - \sigma^*) \cosh(\Delta k^*_s s) \right. \\
+ \left. (1 + q_o + \sigma^*) \sinh(\Delta k^*_s s) \right\}. \tag{25}
\]
\[ |E_y|^2 = \frac{I_0|K|^2 \exp(-\kappa s)}{8(1 + q_0)} \times \left\{ \left[ (1 + q_0 - \sigma) \exp\left(\frac{\Delta k_s}{2}\right) \right] \right. \\
\left. + \left[ 1 + q_0 + \sigma \right] \exp\left(-\frac{\Delta k_s}{2}\right) \right\} \\
\times \left\{ (1 + q_0 - \sigma^*) \exp\left(\frac{\Delta k^*_s}{2}\right) \right. \\
\left. + \left[ 1 + q_0 + \sigma^* \right] \exp\left(-\frac{\Delta k^*_s}{2}\right) \right\} \right\} \\
= \left[ \left( \frac{|K|^2 + 1}{2|K|^2} + q_0 \frac{|K|^2 - 1}{2|K|^2} \right) \cosh(\Delta k_s) \\
- \sigma_0 \sinh(\Delta k_s) \\
+ \left( \frac{|K|^2 - 1}{2|K|^2} + q_0 \frac{|K|^2 + 1}{2|K|^2} \right) \cos(\Delta k_s) + \sigma_0 \sin(\Delta k_s) \right] \\
(26)\]

which leads to

\[ |E_x|^2 = \frac{I_0 \exp(-\kappa s)}{2} \left[ \left( \frac{|K|^2 + 1}{2|K|^2} + q_0 \frac{|K|^2 - 1}{2|K|^2} \right) \cosh(\Delta k_s) \\
- \sigma_0 \sinh(\Delta k_s) \\
+ \left( \frac{|K|^2 - 1}{2|K|^2} + q_0 \frac{|K|^2 + 1}{2|K|^2} \right) \cos(\Delta k_s) + \sigma_0 \sin(\Delta k_s) \right] \\
(27)\]

and

\[ |E_y|^2 = \frac{I_0 |K|^2 \exp(-\kappa s)}{2} \left[ \left( \frac{|K|^2 + 1}{2|K|^2} + q_0 \frac{|K|^2 - 1}{2|K|^2} \right) \cosh(\Delta k_s) \\
- \sigma_0 \sinh(\Delta k_s) \\
+ \left( \frac{|K|^2 - 1}{2|K|^2} + q_0 \frac{|K|^2 + 1}{2|K|^2} \right) \cos(\Delta k_s) - \sigma_0 \sin(\Delta k_s) \right] \\
(28)\]

With the use of Equation (22), Equations (27) and (28) can be combined to give

\[ Q \equiv |E_x|^2 - |E_y|^2 = I_0 \exp(-\kappa s) \times \left\{ u_s \left\{ K_s(|K|^2 - 1) \sinh(\Delta k_s) - K_s(|K|^2 + 1) \sin(\Delta k_s) \right\} \\
- v_s \left\{ K_s(|K|^2 - 1) \sinh(\Delta k_s) + K_s(|K|^2 + 1) \sin(\Delta k_s) \right\} \\
- \left\{ \frac{|K|^4 - 1}{4|K|^2} + q_0 \frac{|K|^2 - 1}{2|K|^2} \right\} \cosh(\Delta k_s) \\
+ \left\{ \frac{|K|^4 - 1}{4|K|^2} + q_0 \frac{|K|^2 + 1}{2|K|^2} \right\} \cos(\Delta k_s) \right\} \\
= \left[ \left( \frac{|K|^2 + 1}{2|K|^2} + q_0 \frac{|K|^2 - 1}{2|K|^2} \right) \cosh(\Delta k_s) \\
+ \left( \frac{|K|^2 - 1}{2|K|^2} + q_0 \frac{|K|^2 + 1}{2|K|^2} \right) \cos(\Delta k_s) \right] \exp(-\kappa s) \\
\times \left[ \left( \frac{|K|^2 + 1}{2|K|^2} + q_0 \frac{|K|^2 - 1}{2|K|^2} \right) \sinh(\Delta k_s) \\
+ \left( \frac{|K|^2 - 1}{2|K|^2} + q_0 \frac{|K|^2 + 1}{2|K|^2} \right) \sin(\Delta k_s) \right] \\
(29)\]

and

\[ I \equiv |E_x|^2 + |E_y|^2 = I_0 \exp(-\kappa s) \times \left\{ u_s \left\{ \frac{K_s(|K|^2 + 1)}{2} \sinh(\Delta k_s) + \frac{K_s(|K|^2 - 1)}{2} \sin(\Delta k_s) \right\} \\
+ v_s \left\{ \frac{K_s(|K|^2 + 1)}{2} \sinh(\Delta k_s) + \frac{K_s(|K|^2 - 1)}{2} \sin(\Delta k_s) \right\} \\
+ \left\{ \frac{(|K|^2 + 1)^2}{4|K|^2} + q_0 \frac{|K|^4 - 1}{4|K|^2} \right\} \cosh(\Delta k_s) \\
- \left\{ \frac{(|K|^2 - 1)^2}{4|K|^2} + q_0 \frac{|K|^4 - 1}{4|K|^2} \right\} \cos(\Delta k_s) \right\} \\
(30)\]

where the terms have been grouped so as to emphasize the various physical mechanisms at play.

**Appendix B**

**General Solution to the Transfer Equation**

For a homogeneous source, Equations (23), (24), (29), and (30) need to be integrated through the emission region from \( s = 0 \) to \( s = s_{\text{max}} \), where \( s_{\text{max}} \) is the thickness of the source. Furthermore, the intensity is replaced by the emissivity (\( \epsilon \)) so that \( I_0 \rightarrow \epsilon \, ds = S \, d\tau \), where \( S \equiv \epsilon / \kappa \) is the source function. The polarization for a homogeneous source is then obtained directly from Equations (23), (24), (29), and (30) by substituting

\[ \exp(-\kappa s) \sinh(\Delta k_s) \rightarrow \frac{1 - \exp(-\tau) \{ \delta k_s \sinh(\delta k_s, \tau) + \cosh(\delta k_s, \tau) \}}{1 - \delta k_s^2} \]

\[ \exp(-\kappa s) \sin(\Delta k_s) \rightarrow \frac{\delta k_s - \exp(-\tau) \{ \delta k_s \cosh(\delta k_s, \tau) + \sin(\delta k_s, \tau) \}}{1 - \delta k_s^2} \]

\[ \exp(-\kappa s) \cos(\Delta k_s) \rightarrow \frac{1 - \exp(-\tau) \{ -\delta k_s \sin(\delta k_s, \tau) + \cos(\delta k_s, \tau) \}}{1 + \delta k_s^2} \]

\[ \exp(-\kappa s) \sin(\Delta k_s) \rightarrow \frac{\delta k_s - \exp(-\tau) \{ \delta k_s \cos(\delta k_s, \tau) + \sin(\delta k_s, \tau) \}}{1 + \delta k_s^2} \] (31)

where \( \tau = \kappa s_{\text{max}} \) is the optical depth of the source, \( \delta k_s = \Delta k_s / \kappa \), and \( \delta k_s = \Delta k_s / \kappa \).
The result for the circular polarization is
\[ V = \frac{S}{|K|^2} \left[ -uK_i + vK_i + \delta k \left( \frac{|K|^2 + 1}{2} + q \frac{|K|^2 - 1}{2} \right) \right] \]
\[ + \delta k \left( \frac{|K|^2 + 1}{2} + q \frac{|K|^2 - 1}{2} \right) \cosh(\delta k \tau) \]
\[ + \frac{K_i}{1 - \delta k_i^2} \left( |K|^2 - 1 \right) \sinh(\delta k \tau) \]
\[ + \frac{K_i}{1 + \delta k_i^2} \left( uK_i + vK_i + \delta k \left( |K|^2 - 1 \right) \right) \cos(\delta k \tau) \]
\[ - \frac{K_i}{1 + \delta k_i^2} \delta k_i (uK_i + vK_i) - |K|^2 - 1 \]
\[ - q \frac{|K|^2 + 1}{2} \sin(\delta k \tau) \] \]
\[ \] \.

(32)

where the various terms have now been grouped in order to emphasize the variation of the circular polarization with optical depth.

**B.1. Limiting Solution for \(|\rho| \gg 1\)**

From Equation (7), \( \delta k = i \sqrt{\gamma_Y \gamma_U} + \gamma_Y^2 = i \sqrt{\gamma_Y} \gamma_L + 1 + \gamma_L^2 / \gamma_Y \).

With \(|\rho| \gg 1\) and regarding \( |\xi_L| \) to be on the same order of smallness as \( |\hat{\xi}_U| \), one finds first to first order in \( \gamma_L / \gamma_Y \):

\[ \delta k = i \left( \gamma_Y + \frac{\gamma_Y^2}{2 \gamma_Y} \right) \]
\[ = -i \left( \gamma_U + \frac{\xi_U \hat{\xi}_U}{\xi_Y} + i \hat{\xi}_V \right). \]

Note that the real and imaginary parts are of different orders and that only the leading-order term has been retained for each of them. The same practice is followed below. Likewise, from Equation (8), the polarization of the characteristic wave \( K^2 = K = \delta k / (\gamma_Y - \gamma_L) \) is

\[ K = i \left( 1 + i \frac{\gamma_L}{\gamma_Y} \right) \]
\[ = -i \frac{\xi_U}{\xi_Y} + i \left( 1 - \frac{\xi_U}{\xi_Y} \right). \]

(34)

The degree of non-orthogonality between the characteristic waves is obtained from \( (|K|^2 - 1)/2 = -\xi_U / \hat{\xi}_U \).

Equations (33) and (34) show that \( |\delta k|, |\delta k|^2, |K|, \) and \( (|K|^2 - 1) \) are all of order \(|\rho|^{-1}\). The circular polarization to first order in \(|\rho|^{-1}\) is then obtained directly from Equation (32) as

\[ V = S \left[ -uK_i + v + \delta k \right] \left( \frac{|K|^2 + 1}{2} + q \frac{|K|^2 - 1}{2} \right) \]
\[ + \delta k \left( \frac{|K|^2 + 1}{2} + q \frac{|K|^2 - 1}{2} \right) \cosh(\delta k \tau) \]
\[ + \frac{K_i}{1 - \delta k_i^2} \left( |K|^2 - 1 \right) \sinh(\delta k \tau) \]
\[ + \frac{K_i}{1 + \delta k_i^2} \left( uK_i + vK_i + \delta k \left( |K|^2 - 1 \right) \right) \cos(\delta k \tau) \]
\[ - \frac{K_i}{1 + \delta k_i^2} \delta k_i (uK_i + vK_i) - |K|^2 - 1 \]
\[ - q \frac{|K|^2 + 1}{2} \sin(\delta k \tau) \] \]

(35)

**B.2. Limiting Solution for \(|\rho| \ll 1\)**

When \(|\rho| \ll 1\), one finds to first order in \( \gamma_Y / \gamma_L \),

\[ \delta k = i \left( \gamma_Y \gamma_L \gamma_U \gamma_Y^2 \right) \]
\[ = -\xi_U + i \hat{\xi}_U. \]

(36)

No higher-order terms are retained for \( \delta k \), because \( |\xi_U| \sim 1 \).

The polarization of the characteristic wave is given by

\[ K = -1 + i \gamma_Y \gamma_L \gamma_U \gamma_Y^2 \]
\[ = -1 + i \xi_U \xi_Y - \xi_Y \hat{\xi}_U + i \hat{\xi}_Y \xi_U + \hat{\xi}_Y \xi_Y. \]

(37)

Again, the degree of non-orthogonality of the characteristic waves is obtained from \( (|K|^2 - 1)/2 = (\xi_U \hat{\xi}_Y - \hat{\xi}_U \xi_Y) / \hat{\xi}_U^2 + \xi_U^2 \).

In this limit, the small quantities are \( |K| \) and \( |K|^2 - 1 \), which are both \( \sim |\rho| \). Expanding Equation (32) to first order in \(|\rho| \) yields

\[ V = S \left[ \frac{K_i}{1 - \delta k_i^2} (u + \delta k_i^2) + \frac{1}{1 + \delta k_i^2} \left( -uK_i + v \right) \right] \]
\[ - \delta k_i \left( \frac{|K|^2 - 1}{2} + q \right) \]
\[ - \exp(-\tau) \left\{ \frac{K_i}{1 - \delta k_i^2} (u + \delta k_i^2) \cosh(\delta k \tau) \right\} \]
\[ + (u \delta k_i + 1) \sinh(\delta k \tau) \]
\[ + \left[ \delta k_i (uK_i - v - \delta k_i \left( \frac{|K|^2 - 1}{2} + q \right) \cos(\delta k \tau) \right] \right\} \] \times \left[ \left( \delta k_i (uK_i - v) - \frac{|K|^2 - 1}{2} - q \sin(\delta k \tau) \right) \right] \] \]

(38)

It is useful to separate out the small contribution to the circular polarization due to the nonthermal terms (i.e., \( u - \xi_U \) and \( v - \xi_Y \)) from that resulting from \( u \) and \( v \). Because \( \delta k_i = -\xi_U \),
it is convenient to write

\[(u\delta k_i + 1)\sinh(\delta k_i\tau) - (u - \xi_U)\frac{\sinh(\xi_U\tau)}{\xi_U} - \frac{u(1 - \xi_U^2)}{\xi_U}\sin(\xi_U\tau).\]  

(39)

Furthermore, Equations (36) and (37) show that \(\delta k_i(|K|^2 - 1/2 - \delta k_iK_i = \xi_U\). With \(\delta k_i = \xi_U\), one can then write

\[
\left(\delta k_i(uK_i - v) - \frac{|K|^2 - 1/2}{2} - q\right)\sin(\delta k_i\tau)
= (K_i(\xi_U - u) + v - \xi_U - q\xi_U)\frac{\sin(\xi_U\tau)}{\xi_U}
+ \left(1 + \xi_U^2\right)(uK_i - v)\frac{\sin(\xi_U\tau)}{\xi_U}.
\]

(40)

With the use of Equations (39) and (40), the circular polarization can be expressed as

\[
V = S \left[ K_i \left\{ \frac{u - \xi_U}{1 - \xi_U^2} \left( 1 - \exp(-\tau) \cosh(\xi_U\tau) \right) \right. 
+ \frac{\sinh(\xi_U\tau)}{\xi_U} \exp(-\tau) \frac{u\sinh(\xi_U\tau)}{\xi_U} \right] 
+ K_i(\xi_U - u) + v - \xi_U - q\xi_U 
\times \left[ 1 - \exp(-\tau) \left( \cos(\xi_U\tau) + \frac{\sin(\xi_U\tau)}{\xi_U} \right) \right] 
\left. - \exp(-\tau) \frac{(uK_i - v)\sin(\xi_U\tau)}{\xi_U} \right].
\]

(41)

**Appendix C**

**Transfer Coefficients for a Synchrotron Source**

The transfer coefficients for a synchrotron source depend on the energy distribution of the radiating particles. Jones & O’Dell (1977) have given the expressions for a power-law distribution \(DN/d\gamma \propto \gamma^{-p}\) for \(\gamma > \gamma_{\text{min}}\) and \(p > 2\). The resulting optically thin spectral flux is then \(F(\nu) \propto \nu^{-\alpha}\), where \(\alpha = (p - 1)/2\). The magnitude of the transfer coefficients depends on \(p\). However, for realistic values of \(p\), this dependence is quite weak (for details, see Jones & O’Dell (1977)). Hence, neglecting factors of order unity, one can write

\[
u \lesssim \xi_U \lesssim 1
\]

\[|v| \lesssim |\xi_U| \lesssim \left(\frac{\nu}{\nu_{\text{abs}}}\right)^{1/2} \frac{n_{\text{exc}}}{n_{\text{exc}} + 2n_p} \left| \cot \theta \right|,
\]

where \(\text{sign}(v, \xi_U) = -\text{sign} \cot \theta\).

\[
\hat{\xi}_U \approx -\left(\frac{\nu}{\gamma_{\text{min}}^2 \nu_{\text{abs}}^{1+2}} \left( \frac{\nu}{\nu_{\text{abs}}} \right)^{(p-2)/2} - 1 \right)/(p - 2)
\]

\[
\hat{\xi}_V \approx \left(\frac{\nu}{\nu_{\text{abs}}}\right)^{p/2} \left( \frac{\ln \gamma_{\text{min}}}{\gamma_{\text{min}}} \right)^{1/2} \frac{n_{\text{exc}}}{n_{\text{exc}} + 2n_p} \cot \theta.
\]

(42)

It may be noted from Equation (44) that the limit of nearly circular characteristic waves (i.e., \(|\xi_U| \gg |\xi_V|\)) implies \(\gamma_{\text{min}} \ll \gamma_{\text{abs}} \approx 10^2\), such that

\[
\hat{\xi}_U \approx -\frac{10^{-4} \nu_{\text{abs}} \gamma_{\text{min}}^{-1}}{\nu} \frac{2n_p}{n_{\text{exc}}} \cot \theta.
\]

(45)

**ORCID iDs**

C.-I. Björnsson © https://orcid.org/0000-0002-3284-4481
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