On $\alpha'$ corrections in $\mathcal{N} = 1$ F-theory compactifications

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We consider $\mathcal{N} = 1$ F-theory and Type IIB orientifold compactifications and derive new $\alpha'$ corrections to the four-dimensional effective action. They originate from higher derivative corrections to eleven-dimensional supergravity and survive the M-theory to F-theory limit. We find a correction to the Kähler moduli depending on a non-trivial intersection curve of seven-branes. We also analyze a four-dimensional higher curvature correction.

I. INTRODUCTION

F-theory is a formulation of Type IIB string theory with seven-branes at varying string coupling [3]. It captures string coupling dependent corrections in the geometry of an elliptically fibered higher-dimensional manifold. The general effective actions of F-theory compactifications have been studied using the duality with M-theory [2, 3]. M-theory is accessed through its long wave-length limitations [4] and the duality with M-theory [5-7]. The general effective actions of F-theory compactifications [2] with seven-branes at varying string coupling [1] have been studied using the duality with M-theory [2]. Starting with the two-derivative supergravity action one derives the classical F-theory effective action. Studying $\alpha'$ corrections to this action is of crucial importance for many questions both at the conceptual and phenomenological level. In particular, a central task is the analysis of moduli stabilization in four-dimensional (4d), $\mathcal{N} = 1$ F-theory compactifications [2].

In this work we study a set of $\alpha'$ corrections to 4d, $\mathcal{N} = 1$ F-theory effective actions arising from known higher-derivative terms in the 11d supergravity action. More precisely, we find all corrections induced by a classical Kaluza-Klein reduction of the purely gravitational M-theory $R^4$-terms investigated in [4-9] on elliptically fibered Calabi-Yau fourfolds. We implement the F-theory limit decompactifying the 3d F-theory reductions to four space-time dimensions and interpret the resulting corrections in F-theory. Two $\alpha'^2$ corrections are shown to survive the limit. We find a correction to the volume of the Calabi-Yau fourfold base and an $R^2$-term in the 4d effective action. Both only depend on the Kähler moduli of the $\mathcal{N} = 1$ reduction. The presence of a volume correction in the M-theory reduction on Calabi-Yau fourfolds has already been stressed in [10, 11]. We find here that, due to this correction, the right 4d, $\mathcal{N} = 1$ coordinates are shifted from their classical value. Moreover, the F-theory limit itself, which connects the 3d effective theory to the 4d one, appears to receive corrections as well. However, when written in terms of the corrected Kähler moduli, the functional dependence of the Kähler metric remains the classical one [12].

It was found in [13] that a general M-theory reduction on a Calabi-Yau fourfold also includes a warp factor. In this work we will neglect warping effects. There is no warp factor in six dimensions and we comment on the $\alpha'^2$ corrections in Calabi-Yau threefold reductions of F-theory.

To give an independent interpretation of these two $\alpha'^2$ corrections we take the Type IIB weak string coupling limit [15]. The F-theory volume correction is proportional to the volume of the intersection curve of the D7-branes with the O7-plane. A simple counting of powers of the string coupling suggests that this correction arises from tree-level string amplitudes involving oriented open strings with the topology of a disk, and non-orientable closed strings with the topology of a projective plane. This interpretation is at odds with the expectation that such a correction arises at open-string one-loop level [16].

This problem appears to be independent of the fact that the correction can be absorbed by redefining the Kähler coordinates on the moduli space. While we will clarify this point further in [13], it would also be crucial, on the one hand, to obtain an independent string derivation of this correction. Within our approach, on the other hand, one needs to check if there are further corrections in a fully backreacted M-theory reduction lifted to F-theory that have the same structure as the volume correction found here. The 4d higher curvature correction is matched with a higher curvature modification of the Dirac-Born-Infeld actions of D7-branes and O7-planes derived in [17]. Different $\alpha'$ corrections to F-theory effective actions and their weak coupling interpretations have been found in [18, 19]. A class of $\alpha'^2$ corrections in the heterotic string has been discussed recently in [20].

II. M-THEORY REDUCTION

Our starting point is the long wave-length limit of M-theory given by 11d supergravity. In particular, we focus on a well-known higher derivative correction to the Einstein-Hilbert term of the form [4, 9]

$$S_{(11)} \supset \frac{1}{(2\pi)^8 l_M^4} \int_{\mathcal{M}} *_{11} 1 \left\{ R_{(11)}^{(1)} + \frac{\pi^2 l_M^6}{3^2 2^{11}} J_0 \right\}, \quad (1)$$

where $*_{D} 1 = d^P X \sqrt{-G^{(D)}}$ is the $D$-dimensional volume element, $R_{(1)}^{(D)}$ is the $D$-dimensional Ricci scalar, and $l_M$ is the 11d Planck length. The correction is given by a Lorentz invariant combination of four powers of the Riemann tensor $R^{(11)}$ of the schematic form

$$J_0 = t_8 t_8 (R^{(11)})^4 - \frac{1}{4^4} c_{11}(R^{(11)})^4, \quad (2)$$
where the precise form of the individual terms is given in \[A5\] and \[A6\]. In this work we follow the conventions of \[B3\]. In these conventions the metric is dimensionless and only the space-time coordinates have dimensions.

If we now compactify this theory on a Calabi-Yau fourfold \(Y_4\), the resulting 3d effective action will include the curvature terms (before Weyl rescaling) of the form

\[
S_{(3)} \supset \frac{1}{(2\pi)^2 l_M^2} \int \ast_3 \left\{ \tilde{\nabla}_4 R_{a c}^{(3)} + l_M^2 \tilde{\nabla}_4 |R^{(3)}|^2 \right\},
\]

where \(R^{(D)} = \frac{1}{2} R^{(D)}_{\mu \nu} dx^\mu \wedge dx^\nu\) is the curvature two-form in \(D\) dimensions, and one has

\[
|R^{(D)}|^2 \ast_D 1 = \text{Tr} (R^{(D)} \wedge \ast_D R^{(D)})
\]

The volumes appearing in \[B5\] take the form

\[
\tilde{V}_4 = \frac{1}{4!} \int_{Y_4} J^4 + \frac{\pi^2}{24} \int_{Y_4} c_3(Y_4) \wedge J,
\]

\[
\tilde{V}_2 = \frac{\pi^2}{24} \int_{Y_4} c_2(Y_4) \wedge J \wedge J.
\]

as shown in appendix \[A2\]. All M-theory volumes are expressed in units of \(l_M\). The two-form \(J\) is the Kähler form of \(Y_4\), and \(c_2(Y_4), c_3(Y_4)\) denote the second and third Chern class of the tangent bundle of \(Y_4\), respectively. The quantum volume \(\tilde{V}_4\) contains an M-theoretic correction of order \(l_M^6\) to the classical volume \(V_4\) of the internal fourfold \(Y_4\) given by the first term in \[B5\]. Both corrections only depend on the Kähler structure of \(Y_4\) and do not introduce mixing with its complex structure. Note that the correction to the 3d Einstein-Hilbert term was already anticipated in \[B10, B11\].

We close this section by noting that the corrections in \[B5\] and \[B6\] are already present in 5d compactifications of M-theory on a Calabi-Yau threefold \(Y_3\). The classical volume of the threefold receives a correction proportional to the Euler number of the threefold. The four-derivative term in \[B5\] is induced due to a non-trivial integral \(\int c_2(Y_3) \wedge J\). This term is the supersymmetric completion of the mixed gauge-gravitational Chern-Simons term \(A^\Lambda \wedge \text{Tr} (R^{(5)} \wedge R^{(5)})\), where the vectors \(A^\Lambda\) are the supersymmetric partners of the Kähler structure deformations.

### III. LIFT TO F-THEORY

We next use the duality between M-theory and F-theory to lift the \(l_M\)-corrections in \[B5\] to \(\alpha'\)-corrections of the 4d effective theory arising from F-theory compactified on \(Y_3\). In order to do that, we first require that \(Y_4\) admits an elliptic fibration over a three-dimensional Kähler base \(B_3\). In this case we can use adjunction formulae to express Chern classes of \(Y_4\) in terms of Chern classes of \(B_3\). For simplicity, let us restrict to a smooth Weierstrass model, i.e. a geometry without non-Abelian singularities, that can be embedded in an ambient fibration with typical fibers being the weighted projective space \(\mathbb{P}^2_{4,3,1}\). This implies having just two types of divisors \(D_0, \Lambda = 1, \ldots, h^{1,1}(Y_4)\). There is the horizontal divisor corresponding to the 0-section \(D_0\), and the vertical divisors \(D_\alpha, \alpha = 1, \ldots, h^{1,1}(B_3)\), corresponding to elliptic fibrations over base divisors. Denoting the Poincaré-dual two-forms to the divisors by \(\omega_\Lambda = (\omega_0, \omega_\alpha)\), we expand

\[
J = v^0 \omega_0 + v^\alpha \omega_\alpha,
\]

where \(v^0\) is the volume of the elliptic fiber. Using adjunction formulae one derives

\[
c_3(Y_4) = c_3 - c_1 c_2 - 60 c_1^2 - 60 \omega_0 c_2^2, \tag{8}
\]

\[
c_2(Y_4) = c_2 + 11 c_1^2 + 12 \omega_0 c_1, \tag{9}
\]

where the \(c_i\) on the r.h.s. of these expressions denote the Chern classes of \(B_3\) pulled-back to \(Y_4\).

In order to take the F-theory limit of the expression \[B9\], we need the relation between the 11d Planck length \(l_3\) and the string length \(l_s\). Using M/F-theory duality one obtains

\[
2\pi l_s = \tilde{V}_4^{1/2} l_M. \tag{10}
\]

In the F-theory limit one sends \(v^0 \to 0\). Such operation decompactifies the fourth dimension by sending to infinity the radius of the 4d/3d circle in string units: \(r \sim \tilde{V}_4^{1/4} \to \infty\). Henceforth, all volumes of the base \(B_3\) will be expressed in units of \(l_s\).

We now have to retain the leading order terms in \[B9\] in the limit of vanishing fiber volume \(v^0 \to 0\). We introduce a small parameter \(\epsilon\) and express the scaling of the dimensionless fields by writing \(v^0 \sim \epsilon\). As explained in \[B20\] one finds \(v^0 \sim \epsilon^{-1/2}\) and infers the scaling behavior of the classical and quantum volume of \(Y_4\) to be \(V_4 \sim \tilde{V}_4 \sim \epsilon^{-1/2}\). In the following we use the subscript \(s\) to denote quantities of the base that are finite in the limit \(\epsilon \to 0\). In particular, one has

\[
2\pi l_s^0 = \sqrt{v^0} v^0, \tag{11}
\]

which holds in the strict \(\epsilon \to 0\) limit. Note that \(v^0\) are the volumes of two-cycles of the base in the Einstein frame. Inserting \[B11\] and \[B12\] into \[B9\], and neglecting all terms that vanish for \(\epsilon\) going to zero, we obtain

\[
S_{(4)} \supset \frac{1}{(2\pi)^2 l_s^2} \int \ast_4 \left\{ \tilde{V}_3^b R_{a c}^{(4)} + l_s^2 \tilde{V}_2^b |R^{(4)}|^2 \right\}, \tag{12}
\]

where

\[
\tilde{V}_3^b = \frac{1}{3!} \int_{B_3} J_3^b - \frac{5}{8} \int_{B_3} c_3(B_3) \wedge J_3, \tag{13}
\]

\[
\tilde{V}_2^b = \frac{1}{8} \int_{B_3} c_1(B_3) \wedge J_3^2. \tag{14}
\]
Here $\tilde{V}_k^4$ is now the quantum volume of the base $B_3$. While (14) enters the 4d effective action (12) as a higher derivative correction, (13) contains a correction to the classical volume of the base. Therefore, the $\alpha'$ correction in (13) would in principle induce a modified F-theory Kähler potential. Indeed, starting from the M-theory Kähler potential, the F-theory limit gives

$$K^M = -3 \log \tilde{V}_4 \rightarrow \log R + K^F. \quad (15)$$

Equation (15) contains the divergent term $\log R$ that is needed in the decompactification of the fourth dimension and one identifies $R = 1/\epsilon^2$. However, to determine $K^F$ explicitly one needs to identify the appropriate coordinates to perform the limit [13].

A first naive guess of coordinates is motivated by the reduction of the two-derivative action as in [3] and defines $L^A$ variables as two-cycle volumes normalized by the total quantum volume of the internal space, i.e. one sets

$$L^A = \frac{v^A}{V_4}, \quad L^0 = R, \quad L^b = \frac{v^b}{V_3}. \quad (16)$$

Splitting a factor of $R$ as demanded in (15) and dropping all terms in $V_4$ that vanish for $\epsilon \rightarrow 0$ one finds

$$K^F = \log \left[ \left( \frac{1}{3!} L^a_b L^b_c L^c_d - \frac{5}{8V_5^2} L^a_b k^b_c k^c_v \right) K_{\alpha \beta \gamma} \right]. \quad (17)$$

$K_{\alpha \beta \gamma}$ is the triple intersection matrix of the base and $k^a$ are the expansion coefficients in $c_1(B_3) = k^a \omega_a$. One thus would find that a correction quadratic in $k^a$ remains in $K^F$.

However, one realizes that one can modify the ansatz (10) in the presence of the higher curvature correction to

$$L^A = \frac{v^A}{V_4} \left( 1 + \lambda_1 \frac{\pi^2 \chi(J)}{V_4} + \lambda_2 \frac{\pi^2 \chi \Sigma}{V_4} \right), \quad (18)$$

where $\chi = \int_{Y_4} c_2(Y_4) \wedge \omega_4$, $\chi(J) = \int_{Y_4} c_2(Y_4) \wedge J$, and $\chi \Sigma$ is the inverse of $K_{\lambda \Sigma} = \int_{Y_4} \omega_4 \wedge \omega_4 \wedge J^2$. Remarkably, the two corrections proportional to the constants $\lambda_1, \lambda_2$ contain terms that scale with $\epsilon$ precisely as the original $v^A/V_4$. Furthermore, one finds that if

$$96 \lambda_1 + 4 \lambda_2 = 1 \quad (19)$$

is satisfied one can write

$$K^M = \log \left[ \frac{1}{4!} K_{\Sigma \Lambda \Gamma} L^\Sigma L^\Lambda L^F + O(\chi_\Sigma^2) \right], \quad (20)$$

where we suppressed corrections that are at least quadratic in the $\chi_\Sigma$. Performing the limit in these $L^\Sigma = (R, L^b)$ one finds

$$K^F = \log \left[ \frac{1}{3!} L^a_b L^b_c L^c_v K_{\alpha \beta \gamma} + O(k^a) \right] \quad (21)$$

where the $L^a_b$ are analogously modified by higher curvature corrections [15].

In order to evaluate the Kähler metric, the precise form of the $\mathcal{N} = 1$ complex Kähler coordinates is crucial. They can be obtained by dimensional reduction of other higher-curvature terms of the 11-dimensional action recently obtained in [12] as done in [13]. We will comment further on the Kähler coordinates and on their relation to the $L$-variables given in [15] in section V.

Before giving the Type IIB string interpretation of the $\alpha'$ corrections in (13), let us comment on some special cases. First of all, when the elliptic fibration is trivial, i.e. $Y_4 = X_3 \times T^2$ with $X_3$ being a Calabi-Yau threefold, then $c_2(Y_4) = c_2(X_3)$ and $c_3(Y_4) = c_3(X_3)$. Since these have no components along the fiber, all corrections in (13) go to zero and the $\alpha'$ corrections in (12) are absent in the resulting $\mathcal{N} = 2$ theory. Another $\mathcal{N} = 2$ corner of F-theory vacua is reached by taking $Y_4 = K3 \times K3$, a configuration studied in [19] with a focus on $\alpha'$ corrections. In this case $c_2(Y_4) = 0$ and the volume correction [15] vanishes identically. In contrast, both corrections are non-vanishing for 6d, $\mathcal{N} = 1$ vacua arising from F-theory on elliptically fibered Calabi-Yau threefolds with classical action derived in [22, 23]. The terms are generated by taking the F-theory limit of the 5d theory briefly discussed at the end of section II. Since the threefold volume is part of the 6d universal hypermultiplet, the volume correction descends to a modification of the hypermultiplet metric of the 4d, $\mathcal{N} = 2$ theory obtained upon further compactification on $T^2$. The impact of this correction will, however, crucially depend on the definition of the $\mathcal{N} = 2$ hypermultiplet coordinates. In summary, comparing all these setups one suspects that the volume correction [13] relies on the presence of intersecting seven-branes but its significance changes on backgrounds with different number of supercharges. This will indeed be confirmed by the analysis of section IV.

We close this section with two important remarks. Firstly, we stress that there are several additional $l_M$-corrections to the fourfold volume surviving the F-theory limit. To see this consider the case without seven-branes having a product geometry $Y_4 = X_3 \times T^2$. In this situation corrections involving the Type IIB axio-dilaton $\tau$ have been computed by integrating out the whole tower of $T^2$ Kaluza-Klein modes of the 11d supergravity multiplet [3]. This gives the following corrections to the fourfold volume

$$\Delta V_4^{\mathcal{N} = 2} \sim \frac{\chi(X_3)}{(4 \pi)^{1/2}} E_3/2 (\tau, \bar{\tau}), \quad (22)$$

that depends on $\tau$ through the non-holomorphic Eisenstein series $E_3/2$ (see also [24]) and has the correct scaling behavior to survive the F-theory limit.

Our second remark concerns the compactification of the 11d action (11) on $Y_4 \times S^1$, giving rise to Type IIA string theory in two dimensions. The resulting 2d string frame Einstein-Hilbert term takes the form

$$S_{(2)} \supset \frac{1}{(2\pi)^2} \int \ast_2 \theta_{1A} \tilde{V}_4^2 \mathcal{R}, \quad (23)$$
where we have used that the length of $S^1$ is $2\pi g_{11A} l_M = 2\pi g_{11A} l_s$ in terms of the Type IIA string coupling. $\tilde{V}_4$ denotes the quantum volume of the fourfold in units of $l_s$ and it takes the form

$$\tilde{V}_4 = V_4^r - \frac{1}{8} \left( \zeta(3) - g_{11A} \frac{1}{3} \right) \int_{Y_4} c_3(Y_4) \wedge J.$$  \hfill (24)

The above correction contains two pieces already present in the 10d Type IIA action as $R^4$ couplings \cite{5}. The first term in the brackets in (24) is tree-level in string perturbation theory and arises from integrating out $S^1$ Kaluza-Klein modes analogous to the derivation mentioned for (23). The second arises at one-loop of closed strings and is derived using mirror symmetry or localization techniques as done in \cite{25, 26}. However, it vanishes in the M-theory limit $g_{11A} \to \infty$, and hence is of no relevance for the present purposes.

\section{IV. STRING THEORY INTERPRETATION}

In this section we interpret the corrections in (12) in the weak string-coupling limit considered by Sen \cite{13}. This limit is performed in the complex structure moduli space of $Y_4$ and gives a weakly coupled description of F-theory in terms of Type IIB string theory on a Calabi-Yau threefold $X_3$ with an O7-plane and D7-branes. The Calabi-Yau threefold is a double cover of the base $B_3$ branched along the O7-plane. The class of this branching locus is the pull-back of $c_1(B_3)$ to $X_3$. When non-Abelian singularities are absent in F-theory, as in the case we consider, the corresponding Sen limit contains a single recombined D7-brane wrapping a divisor of class $8c_1(B_3)$, as required by seven-brane tadpole cancellation. This D7-brane has the characteristic Whitney-umbrella shape \cite{27, 28}.

We first discuss the volume correction in (13). For this correction the intersection curve of the D7-brane with the O7-plane plays a crucial role. It is a double curve with additional pinch point singularities. However, all we need in the following is its volume in $X_3$ given by

$$V_{D7\cap O7} = 8 \int_{X_3} c_1^2(B_3) \wedge J_b,$$ \hfill (25)

where we omitted the pullback map from $B_3$ to its double cover $X_3$ in the integrand. Since the intersection numbers of $X_3$ are twice the ones of $B_3$, we can immediately read off from (13) the induced correction to the classical volume of the Calabi-Yau threefold in units of $l_s$. Hence we find in the ten-dimensional Einstein frame the corrected threefold volume

$$\tilde{V}_3 = V_3 - \frac{5}{64} V_{D7\cap O7},$$ \hfill (26)

where $V_3$ is the classical volume of $X_3$ that is twice the classical volume of $B_3$. Note that the quantum correction in (26) can alternatively be expressed in terms of the volume of the self-intersection curve of the O7-plane by using tadpole cancellation. We stress that the correction is of order $\alpha'^2$ since two of the original six derivatives in M-theory have been absorbed by the integration on the elliptic fiber.

It is worth noting that the non-triviality of the elliptic fibration causes the appearance of an $\alpha'$ correction already at order two, a phenomenon also observed in \cite{19} for a different correction in F-theory compactifications on $K3 \times K3$.

In order to give the string theory interpretation of the correction in (26) we have to identify the string amplitude capturing it. We first look at the 4d effective action in the string frame with Einstein-Hilbert term

$$S^{(4)} \supset \frac{1}{(2\pi)^3 l_s^2} \int *^4 \{ \frac{V_3^2}{g_{11B}} - \frac{5V_{D7\cap O7}}{64 g_{11B}} \} R_{\text{sc}},$$ \hfill (27)

where the superscript $^*$ denotes quantities computed using the string frame metric. Let us indicate which string amplitude might generate the correction in (27). Recall that the power of the string coupling constant in a given amplitude coincides with $-\chi(\Sigma)$ modified by the number of insertions of vertex operators on the string world-sheet $\Sigma$. Both contributions in (27) are expected to arise from amplitudes with two graviton insertions and we study the relative $g_s$-power of the two terms. The general formula for the Euler number of Riemann surfaces, possibly non-orientable and with boundaries, is

$$\chi(\Sigma) = 2 - 2g - b - c,$$ \hfill (28)

where $g, b, c$ denote the genus, the number of boundaries, and the number of cross caps, respectively. Therefore, we immediately see that the volume correction in (26) should arise from a string amplitude that involves the sum over two topologies: The disk $(g = c = 0, b = 1)$ and the projective plane $(g = b = 0, c = 1)$. They correspond to the tree-level of orientable open strings and non-orientable closed strings, respectively. This interpretation seemingly contradicts the expectation that such a correction arises at open-string one-loop level \cite{16}. We hope to clarify this point further in future work. Let us stress that we also cannot exclude the possibility that there are further corrections in a fully backreacted M-theory reduction lifted to F-theory that have the same $\alpha'$-order and field dependence as the volume correction found here.

Let us next give a string theory interpretation of the 4d higher derivative correction in (12). In fact, at weak string coupling, the coefficient (13) can be written as

$$\tilde{V}_2^{\text{IIA}} = \frac{1}{96} (V_7 + 4V_{O7}),$$ \hfill (29)

where $V_7$ and $V_{O7}$ are the volumes of the D7-brane and the O7-plane in $X_3$, respectively. Both volumes are in
Remarkably, it turns out in \cite{13} that the correct choice in (1) including fluctuations of the Calabi-Yau metric. The relative factor in the volume split is in agreement with the relative factor in the higher curvature terms of the Chern-Simons actions of D7-branes and O7-planes. These have been studied to derive the 4d higher curvature term proportional to $\text{Tr} (R^{(4)} \wedge R^{(4)})$ in \cite{29}, which is the supersymmetric partner of the $\text{Tr} (R^{(4)} \wedge \ast_4 R^{(4)})$ term in \cite{12}. Translated to the string frame the higher derivative correction in \cite{12} becomes

$$S_4 \supset \frac{1}{96 (2 \pi)^4} \int \ast_4^* \left( \frac{\mathcal{V}_{D7} + 4 \mathcal{V}_{O7}}{g_{11B}} \right) |R^{(4)}|^2 ,$$

where we see that this correction has the same string loop order as the one in \cite{29}. This term is expected to directly arise from a higher curvature correction of the string-tree-level Dirac-Born-Infeld action on the D7-brane and O7-plane as discussed in \cite{17} (see, in particular, equation (3.5)).

V. REMARKS ON THE KÄHLER POTENTIAL AND TYPE IIB VACUA

In this final section we comment on the structure of the 4d, $\mathcal{N} = 1$ Kähler potential $K^F$ given in \cite{21} and analyze its properties. In order to derive the kinetic terms of the moduli and the scalar potential, one first needs to express $K^F$ in terms of the $h^{1,1}(B_3)$ correct $\mathcal{N} = 1$ complex coordinates $T_\alpha$. Starting in M-theory one has $h^{1,1}(Y_4)$ complex coordinates $T_\Lambda$. Classically the real parts of $T_\Lambda$ are the volumes of the divisors of $Y_4$, while the imaginary part is the integral of the M-theory six-form over the same divisors. Including higher curvature corrections also the $T_\Lambda$ might be shifted as \cite{31}

$$\text{Re} T_\Lambda = \frac{1}{3!} K_\Lambda \left( 1 + \frac{\pi^2}{4 Y_4} \chi(J) \right) + \kappa_2 \pi^2 \chi_\Lambda ,$$

where we abbreviated $K_\Lambda = \int Y_4 J^3 \wedge \omega_4$, and used $\chi_\Lambda$, $\chi(J)$ defined after \cite{18}. All variables are expressed in units of $l_M$. The shifts proportional to the constants $\kappa_1, \kappa_2$ could be expected in analogy to the proposal of \cite{3} that the $\text{Re} T_\Lambda$ are related to the periods of the mirror Calabi-Yau fourfold. To derive corrections to the $T_\Lambda$ the modification of their kinetic terms has to be computed \cite{13}. For the correction considered here one would have to dimensionally reduce the 11d higher curvature term in \cite{4} including fluctuations of the Calabi-Yau metric. Remarkably, it turns out in \cite{13} that the correct choice of $T_\Sigma$ is such that the Legendre dual variables to $\text{Re} T_\Lambda$ are

$$L^\Sigma = - \frac{\partial K^M}{\partial \text{Re} T_\Sigma} = -2 K^M_{\Sigma} ,$$

with $K^M$ as in \cite{20} and $L^\Sigma$ of the form \cite{18} with

$$\lambda_1 = \frac{1}{3 \cdot 24}, \quad \lambda_2 = \frac{1}{12}, \quad \kappa_1 = \frac{1}{24}, \quad \kappa_2 = -\frac{1}{24} ,$$

Hence \cite{19} is satisfied and $\text{Re} T_\Lambda L^\Lambda = 4$. This in turn implies that at linear order in $\chi_\Sigma$ the no-scale-like property of the corrected M-theory Kähler potential is still satisfied

$$K^M_{\Sigma} K^M T_\Lambda \tilde{T}_\Sigma K^M_{\Sigma} = 4 ,$$

where $K^T \tilde{T}$ is the inverse Kähler metric. The result \cite{31} can be attributed to the fact that $\text{Re} T_\Sigma$ and $L^\Sigma$ are Legendre dual variables via \cite{32} and $K^M$ takes the simple form \cite{20}. This is in contrast to the claim made in the previous version of this work and can be traced back to having not considered a sufficiently general ansatz for $L^\Sigma , T_\Sigma$. Clearly, the correct choice for $L^\Sigma , T_\Sigma$ can only be evaluated by a more complete reduction as done in \cite{13} in which we discovered the incompleteness in the original ansatz for \cite{31}.

In order to express the M-theory Kähler potential in terms of the $T_\Lambda$-moduli, one would have to solve \cite{31} for $L^\Lambda$, and insert the solution back into \cite{13}. Strictly speaking one would next have to perform a Legendre transformation replacing $\text{Re} T_\Lambda$ with $R$ and work with the modified kinetic potential $\tilde{K}(R, T^b_\alpha)$, where $T^b_\alpha$ are the 4d $\mathcal{N} = 1$ coordinates. However, in the F-theory limit one finds the expression \cite{15}, with $K^F$ given in \cite{21}, which at order $k^{a^2}$ still satisfies the no-scale property as

$$K^F_{\alpha} K^F T^{a+b} \alpha K^F_{c} = 3 .$$

Therefore, as was the case in 3d, also in 4d the quantum correction only affects the Kähler coordinates, whereas the Kähler metric remains classical. It turns out in \cite{13} that in order to derive the 4d quantum Kähler coordinates from the 3d ones given in \cite{31} one can also include $\alpha'$ corrections to the F-theory limit itself. This freedom allows to bring $\text{Re} T^b_\alpha$ into the form

$$\text{Re} T^b_\alpha / (2 \pi^2)^2 = \frac{1}{2} \kappa^b - \frac{5}{16} v^2 c K^b c \chi^b(J_b) + \frac{5}{8} (3c - 2) \chi^b \chi^b ,$$

where $c$ is a constant that parametrizes the freedom to modify the F-theory limit. Here we have defined $K^b = K_{\alpha \beta} \gamma^b \gamma^b \chi^b = \int B_3 \gamma^2(X_3) \wedge \omega_4$, and $\chi^b(J_b) = \chi^b \gamma^b$. The $\text{Re} T^b_\alpha$ as in \cite{31} are related to the $L^\alpha_\alpha$ variables via Legendre duality with $K^F$ as in \cite{21}. Note that when $c = 0$ the result \cite{30} for the corrected $\text{Re} T^b_\alpha$ contains just a constant shift from the classical value.

Before concluding, it is worth remarking that the fourfold volume \cite{3} will in principle get further corrections beyond linear order in $\chi_\Sigma$. Likewise, the system of Legendre dual coordinates will be modified and $L^\Sigma , \text{Re} T_\Sigma$ will have more general expressions reducing to \cite{18}, \cite{31} at linear order in $\chi_\Sigma$. However, since we have no control over them from a direct derivation of the moduli kinetic
term, we are not able to determine whether the Kähler metric will keep its classical form and the no-scale property will still be satisfied beyond linear order in $\chi_N$.

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Appendix A: Computation

We use the conventions of [30] for the definition of the Riemann tensor and related quantities. The background geometry is the product manifold $M_{11} = \mathbb{R}^{1,2} \times Y_4$, where the flat space has signature $\{ -, +, + \}$ and $Y_4$ is the Calabi-Yau fourfold. External indices are denoted by $\mu, \mu'$. For the coordinates on $Y_4$ we use real and complex indices denoted by $a, a'$ and $\alpha, \beta, \gamma, \delta$, respectively. Indices of the coordinates of the total space $M_{11}$ will be written in capital Latin letters $N, N'$. Furthermore, the convention for the totally anti-symmetric tensor in Lorentzian space in an orthonormal frame is $\epsilon_{012...10} = \epsilon_{012} = +1$.

The curvature two-form for Hermitian manifolds is defined as

$$R^\alpha_\beta = R^\alpha_{\beta \gamma \delta} d z^\gamma \wedge d z^\delta,$$

and one has

$$\text{Tr } R = R^\alpha_{\alpha \gamma \delta} dz^\gamma \wedge d z^\delta,$$

$$\text{Tr } R^2 = R^\alpha_{\beta \gamma \delta} R^\beta_{\alpha \gamma \delta} dz^\gamma \wedge d z^\delta + d z^\gamma_1 \wedge d z^\gamma_2 \wedge d z^\delta_1,$$

$$\text{Tr } R^3 = R^\alpha_{\beta \gamma \delta} R^\beta_{\gamma \delta \eta} R^\gamma_{\delta \eta \alpha} d z^\gamma \wedge d z^\delta \wedge d z^\eta.$$

The correction to the 11d Einstein-Hilbert term in (1) is given by $I_0$ schematically defined in (2) to be

$$I_0 = t_8 t_8 R^4 - \frac{1}{4!} \epsilon_1 \epsilon_1 R^4.$$

Following [31] we define in real coordinates

$$\epsilon_{S^8}^{a_1 a_2 \cdots a_8} = -2 \left( \delta^{[a_1 a_2} \delta^{a_3 a_4] \delta^{a_5 a_6} \delta^{a_7 a_8]} + \delta^{[a_1 a_3} \delta^{a_2 a_4] \delta^{a_5 a_6} \delta^{a_7 a_8]} + \delta^{[a_1 a_5} \delta^{a_2 a_6] \delta^{a_3 a_7} \delta^{a_8]} + \delta^{[a_7 a_8} \delta^{a_1 a_2] \delta^{a_3 a_4} \delta^{a_5 a_6]} \right) + 8 \left( \delta^{a_1} \delta^{a_2} \delta^{a_3} \delta^{a_4} \delta^{a_5} \delta^{a_6} \delta^{a_7} \delta^{a_8} \right).$$

The symbols $[ ]$, $[ ]$, $\lfloor \rfloor$, $\langle \rangle$ denote anti-symmetrization and indices in between two vertical lines $|$ are omitted in the respective bracket. The expression is anti-symmetrized in the following pairs of indices $(a_1 a_2), (a_3 a_4), (a_5 a_6), (a_7 a_8)$ respectively. Thus we have for the term $t_8 t_8 R^4$ in real coordinates

$$t_8 t_8 R^4 = t_8 a_1 \cdots a_8 R^{a_1 a_2} a_3 a_4 \cdots R^{a_7 a_8} a_9 a_10.$$

The second term in (A3) can be written as

$$\frac{1}{4!} \epsilon_1 \epsilon_1 R^4 = \frac{1}{4} E_8 (M_{11}),$$

where one uses the general definition in real coordinates

$$E_n (M_D) = \frac{1}{(D-n)!} \epsilon_{N_1 \cdots N_D} \epsilon_{N_{D-n} \cdots N_{D-n+1} \cdots N_D}$$

$$R^{N_{D-n+1} N_{D-n+2} N_{D-n+3} \cdots R_{N_{D-n} \cdots N_{D-n+1} \cdots N_{D-n}}} D,$$

where $n > 0$ and $D$ being the real dimension of the manifold $M_D$.

The Chern classes can be expressed in terms of the curvature two-form $R$ as

$$c_1 = i \text{Tr } R, \quad c_2 = \frac{1}{2!} (\text{Tr } R^2 - (\text{Tr } R)^2),$$

$$c_3 = \frac{1}{3} c_1 c_2 + \frac{1}{3} c_1 \text{Tr } R^2 - \frac{i}{3} \text{Tr } R^3,$$

$$c_4 = \frac{1}{24} \left( c_1^3 - 6 c_1^2 \text{Tr } R^2 - 8 i c_1 \text{Tr } R^3 \right)$$

$$+ \frac{1}{8} ((\text{Tr } R^2)^2 - 2 \text{Tr } R^4).$$

The Chern classes of a Calabi-Yau fourfold reduce to $c_3 (Y_4) = -\frac{i}{4} \text{Tr } R^3$ and $c_4 = \frac{1}{8} ((\text{Tr } R^2)^2 - 2 \text{Tr } R^4)$.

Let us first compute $E_8 (M_3 \times M_8)$ for a generic product space. By using the definition (A7), splitting indices and applying Schouten identities it is straightforward to show that

$$E_8 (M_3 \times M_8) = -E_8 (M_8) + 4 E_2 (M_3) E_6 (M_8),$$

where $E_2 (M_3) = -2 R_{a b}^{(3)}$ and

$$E_6 (M_8) = 6 R^{a_1 a_2} \cdots R^{a_5 a_6}.$$
Additionally we have
\[ \frac{1}{4} E_8(Y_4) * 8 \, 1 = 1536 \, c_4. \] (A14)

Finally we find for the reduction of \( t_s t_b R^4 \) the terms
\[ t_s t_b R^4 * 8 \, 1 = -96 (R^a_{\mu\nu}) (R^a_{\mu'} \nu') (R^a_{\beta\gamma}) \delta_i \delta_j \]
\[ (R^3)_{\alpha \beta \gamma} * 8 \, 1 + 1536 \, c_4 + \ldots, \] (A15)
where the dots indicate purely external terms. To rewrite this result let us consider the following terms
\[ c_2 \wedge J^2 = -2 \delta [\gamma_0 \delta]^{\alpha \beta} \delta_i (R^a_{\alpha \beta}) \delta_j (R^3)_{\alpha \beta \gamma} * 8 \, 1 \]
\[ = - (R^a_{\alpha \beta}) \delta_j (R^3)_{\alpha \beta \gamma} * 8 \, 1. \] (A16)

Furthermore, we find
\[ \text{Tr} \left[ (R^{(3)} \wedge * 3 R^{(3)}) \right] = \frac{1}{8} (R^a_{\mu \nu}) (R^a_{\mu'} \nu') * 8 \, 1, \] (A17)
with \( R^{(3)} \) being the real curvature two-form as in [4]. Hence we conclude
\[ t_s t_b R^4 * 8 \, 1 = 3 \cdot 2^8 \text{Tr} \left[ R^{(3)} \wedge * 3 R^{(3)} \right] \wedge (c_2 \wedge J^2) \]
\[ + 1536 \, c_4 + \ldots, \] (A18)
where we have omitted the same purely external terms as in [A15].

To conclude we use [A3], [A9], [A14] and [A18] to find for the internal terms of \( J_0 \)
\[ (J_0)_{\text{int}} * 8 \, 1 = \left[ (t_s t_b R^4)_{\text{int}} + \frac{1}{4} E_8(Y_4) \right] * 8 \, 1 = 3072 \, c_4, \] (A19)
which integrates to 3072 \( \chi \) on \( Y_4 \). The linear combination with a different relative sign in equation [A19] obviously vanishes on \( Y_4 \). This is of physical importance as discussed e.g. in [32].
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The conclusion that only the Kähler coordinates are corrected differs from the statements made in the previous version of this work. It resulted from an incomplete Ansatz for the Kähler coordinates. This incompleteness was discovered when performing the reduction of the terms recently obtained in [12]. The required complete reduction will be presented in detail in [13].

The crucial second term was not considered in the previous version of this work. Together, both shifts allow to revise the interpretation of the correction.