A Note on New Bernstein-type Inequalities for the Log-likelihood Function of Bernoulli Variables

Yunpeng Zhao *

September 4, 2019

Abstract

We prove a new Bernstein-type inequality for the log-likelihood function of Bernoulli variables. In contrast to classical Bernstein’s inequality and Hoeffding’s inequality when applied to the log-likelihood, the new bound is independent of the parameters of the Bernoulli variables and therefore does not blow up as the parameters approach 0 or 1. The new inequality strengthens certain theoretical results on likelihood-based methods for community detection in networks and can be applied to other likelihood-based methods for binary data.

Keywords: Concentration inequality; Bernstein-type inequality; Bernoulli distribution; moment generating function

1 Introduction

Let $X_1, X_2, ..., X_n$ be independent Bernoulli random variables, where $X_i$ takes the value 1 with probability $p_i$, denoted by $\text{Ber}(p_i)$. We are interested in deriving a concentration bound, which decays exponentially and is independent of parameters $p_i$, for the joint log-likelihood function of $X_1, X_2, ..., X_n$. That is,

$$\mathbb{P}\left(\left|\frac{1}{n}\sum_{i=1}^{n}(X_i \log p_i + (1 - X_i) \log (1 - p_i)) - \frac{1}{n}\sum_{i=1}^{n}(p_i \log p_i + (1 - p_i) \log (1 - p_i))\right| \geq \epsilon\right) \leq c_1 e^{-c_2 n},$$

where $c_1$ and $c_2$ are constants that only depend on $\epsilon$.

This research is motivated by theoretical studies of likelihood-based methods for binary data, in particular likelihood-based methods for community detection in networks. For example, Theorem

*School of Mathematical and Natural Sciences, Arizona State University, AZ, 85306. Email: yunpeng.zhao@asu.edu.
2 in Choi et al. (2012) uses on an inequality of this type and so does Theorem 2 in Paul and Chen (2016).

We begin with classical results. By symmetry, we only consider
\[ P \left( \left| \frac{1}{n} \sum_{i=1}^{n} X_i \log p_i - \frac{1}{n} \sum_{i=1}^{n} p_i \log p_i \right| \geq \epsilon \right). \]

Since \( X_i \log p_i \equiv p_i \log p_i \) almost surely when \( p_i = 1 \) or 0 (using the convention \( 0 \log 0 = 0 \)), the term can be dropped. Without loss of generality, assume \( p_i \in (0, 1) \) for \( i = 1, ..., n \). Noticing that \( X_i \log p_i \in [\log p_i, 0] \) for \( i = 1, ..., n \), we have Hoeffding’s inequality (Hoeffding, 1963): for all \( \epsilon > 0, \)
\[ P \left( \left| \frac{1}{n} \sum_{i=1}^{n} (X_i - p_i) \log p_i \right| \geq \epsilon \right) \leq 2 \exp \left\{ -\frac{2n^2 \epsilon^2}{\sum_{i=1}^{n} (\log p_i)^2} \right\}. \]

Let \( p_{(1)} \) be the smallest value among \( p_1, ..., p_n \). Then \( |(X_i - p_i) \log p_i| \leq |\log p_{(1)}| \) for \( i = 1, ..., n \). Bernstein’s inequality (see Dubhashi and Panconesi (2009), Theorem 1.2) gives: for all \( \epsilon > 0, \)
\[ P \left( \left| \frac{1}{n} \sum_{i=1}^{n} (X_i - p_i) \log p_i \right| \geq \epsilon \right) \leq 2 \exp \left\{ -\frac{n^2 \epsilon^2 / 2}{\sum_{i=1}^{n} \text{Var}(X_i \log p_i) + |\log p_{(1)}| n \epsilon / 3} \right\}. \]

Note that both inequalities depend on \( p_1, ..., p_n \). As a result, when \( p_{(1)} \) goes to 0 fast enough as \( n \) grows, the bounds can be trivial due to the divergence of \( |\log p_{(1)}| \). When applying these inequalities, technical assumptions are therefore needed to control the rate of the parameters going to the boundaries, for example, the condition on \( P_{ij} \) in Theorem 2 of Choi et al. (2012).

In this note, we prove a Bernstein-type inequality where the bound is independent of \( p_1, ..., p_n \). In other words, we show that \( \sum_{i=1}^{n} (X_i - p_i) \log p_i \) is in fact well-behaved when the parameters are near the boundary. The results such as in Choi et al. (2012) and Paul and Chen (2016) can therefore be strengthened by removing the technical assumptions. The new inequality is particularly useful in cases where those assumptions are not convenient to be made.

\section{Main Result}

\textbf{Theorem 1.} Let \( X_i \) be independent \( \text{Ber}(p_i) \) for \( i = 1, ..., n \) where \( p_i \in [0, 1] \). For all \( t > 0, \)
\[ P \left( \left| \sum_{i=1}^{n} (X_i - p_i) \log p_i \right| \geq t \right) \leq 2 \exp \left\{ -\frac{t^2}{2(n + t)} \right\}. \]

\textbf{Proof.} Let \( Y_i = (X_i - p_i) \log p_i \). Let \( G(p_i, \lambda) \) be the moment generating function of \( Y_i \), which is
\[ G(p_i, \lambda) = E[e^{\lambda Y_i}] = p_i e^{\lambda(1-p_i) \log p_i} + (1 - p_i) e^{-\lambda p_i \log p_i}. \]
The key step is to derive an exponential upper bound for $G(p_i, \lambda)$ which is independent of $p_i$. First consider the case where $p_i \in (0, 1)$.

We prove a Bernstein’s condition (see Wainwright (2019), p. 27 for an introduction) for the moments of $Y_i$. That is, find constants $\sigma^2$ and $b$, such that

$$|\mathbb{E}[Y_i^m]| \leq \frac{1}{2} m! \sigma^2 b^{m-2} \quad \text{for } m = 3, 4, .... \quad (3)$$

Different from Wainwright (2019), here we look for constants $\sigma^2$ and $b$ which are independent of $p_i$.

Consider

$$\mathbb{E}[Y_i^m] = p_i (1-p_i)^m \left( \log p_i \right)^m + (1-p_i)(-p_i \log p_i)^m.$$

By taking the first and the second derivatives of $p_i (\log p_i)^m$, one can easily check that its optimum is achieved at $p_i = e^{-m}$. Therefore,

$$|A_1| \leq |p_i (\log p_i)^m| \leq \left( \frac{m}{e} \right)^m \leq \frac{m!}{\sqrt{2\pi m}},$$

where the last inequality follows from Stirling’s formula (Robbins, 1955). Similarly,

$$|A_2| \leq (1-p_i)(-p_i \log p_i)^m \leq e^{-m}.$$

It follows that

$$|\mathbb{E}[Y_i^m]| \leq \frac{m!}{\sqrt{2\pi m}} + \frac{1}{e^m} \leq \frac{1}{2} m! \quad \text{for } m = 3, 4, ....$$

Therefore, the Bernstein’s condition (3) holds when $\sigma^2 = 1$ and $b = 1$.

We now use the Bernstein’s condition to derive an upper bound for $G(p_i, \lambda)$. The argument is similar to Wainwright (2019), pp. 27-28. We give the details for completeness.

By the power series expansion of the exponential function and Fubini’s theorem (for exchanging the expectation and summation),

$$G(p_i, \lambda) = \mathbb{E}[e^{\lambda Y_i}] = 1 + \frac{\lambda^2 \text{Var}(Y_i)}{2} + \sum_{m=3}^{\infty} \frac{\lambda^m \mathbb{E}[Y_i^m]}{m!}$$

$$\leq 1 + \frac{\lambda^2}{2} + \frac{\lambda^2}{2} \sum_{m=1}^{\infty} |\lambda|^m,$$

where the inequality follows from the Bernstein’s condition (3) and $\text{Var}(Y_i) = p_i (1-p_i)(\log p_i)^2 \leq 1$. For any $|\lambda| < 1$, the geometric series converges, and

$$G(p_i, \lambda) \leq 1 + \frac{\lambda^2}{2} \frac{1}{1-|\lambda|} \leq \exp \left\{ \frac{\lambda^2}{2(1-|\lambda|)} \right\}, \quad (4)$$

where the second inequality follows from $1 + s \leq e^s$. Notice that $G(p_i, \lambda) \equiv 1$ for $p_i = 0$ or 1 so the inequality holds for all $p_i \in [0, 1]$. 

3
The rest of the proof follows from a standard argument using the Chernoff bound, which can be found in a standard textbook on concentration inequalities, for example, Dubhashi and Panconesi (2009), Chapter 1. We give the details for readers who are unfamiliar with this technique. For $-1 < \lambda < 0$,

$$
P \left( \sum_{i=1}^{n} Y_i \leq -t \right) = P \left( e^{\lambda \sum_{i=1}^{n} Y_i} \geq e^{-\lambda t} \right) \leq \prod_{i=1}^{n} \frac{E \left[ e^{\lambda Y_i} \right]}{e^{-\lambda t}} \leq \exp \left\{ \frac{n\lambda^2}{2(1 - |\lambda|)} + \lambda t \right\},
$$

where the first inequality is Markov’s inequality and the second inequality follows from (4). By setting $\lambda = -\frac{1}{t+n} \in (-1, 0)$, we obtain

$$
P \left( \sum_{i=1}^{n} Y_i \leq -t \right) \leq \exp \left\{ -\frac{t^2}{2(n + t)} \right\}.
$$

The bound for the right tail can be obtained similarly by setting $\lambda = \frac{t}{t+n}$.

Remark 1. $E[Y_i^m]$ is dominated by the term $p_i(1-p_i)^m(\log p_i)^m$, which has a bump near the boundary, – that is, its value achieves the order of $(m/e)^m$ at $p_i = e^{-m}$. This value is, however, still bounded by $m!$, which implies the left-tail bound of $\sum_{i=1}^{n} Y_i$ is well-behaved when the parameters are near the boundary.

Remark 2. The constant $\sigma^2 = 1$ in the Bernstein’s condition (3) is not the optimal value. We simply choose this value for obtaining a nice form in (2). On the contrary, $b = 1$ is optimal because $1/\sqrt{2\pi m}$ dominates $b^{m-2}$ for any $0 < b < 1$. This fact can also be seen from the following proposition:

**Proposition 1.** For $\lambda < -1$, $\lim_{p \to 0^+} G(p, \lambda) = \infty$, which implies $G(p, \lambda)$ cannot be bounded by any function that takes finite values. For $\lambda \geq -1$, $\lim_{p \to 0^+} G(p, \lambda) < \infty$.

**Proof.** The result is obvious by noticing that $p e^{\lambda \log p} = p^{\lambda+1}$.

We now prove (1). We state a slightly more general result for multinoulli variables. Let $X_i = (X_{i1}, ..., X_{iK})$ be a multinoulli variable with $p_{ik} = P(X_{ik} = 1)$, and assume $X_1, ..., X_n$ are independent.

**Corollary 1.** For $p_{ik} \in [0, 1]$, $i = 1, ..., n$, $k = 1, ..., K$, and all $\epsilon > 0$,

$$
P \left( \left| \frac{1}{n} \sum_{i=1}^{n} \sum_{k=1}^{K} (X_{ik} - p_{ik}) \log p_{ik} \right| \geq \epsilon \right) \leq 2K \exp \left\{ -\frac{n\epsilon^2}{2K(K + \epsilon)} \right\}.
$$

**Proof.** The result is obvious by noticing that

$$
P \left( \left| \frac{1}{n} \sum_{i=1}^{n} \sum_{k=1}^{K} (X_{ik} - p_{ik}) \log p_{ik} \right| \geq \epsilon \right) \leq \sum_{k=1}^{K} P \left( \left| \sum_{i=1}^{n} (X_{ik} - p_{ik}) \log p_{ik} \right| \geq \frac{n\epsilon}{K} \right),
$$

and setting $t = n\epsilon/K$ in (2).
3 Extension to Grouped Observations

We now extend our result to a setup where the observations are grouped into different classes. In fact, this is the setup that can be directly applied to the community detection literature, for example, Theorem 2 in [1] and Theorem 2 in [2]. We will also apply the result in a working paper by the author and collaborators on the theory of hub models, a special latent class model for binary data proposed by [3].

Let $X^{(1)}_1, X^{(1)}_2, ..., X^{(n_1)}_1, X^{(2)}_1, X^{(2)}_2, ..., X^{(n_2)}_1, X^{(I)}_1, X^{(I)}_2, ..., X^{(n_I)}_1$ be independent Bernoulli variables, where $p^{(i)}_j$ is the parameter for $X^{(i)}_j$. Let $\sum_{i=1}^{n_i} = n$. And let $\bar{p}^{(i)} = \frac{1}{n_i} \sum_{j=1}^{n_i} p^{(i)}_j$ for $i = 1, ..., I$, where $\bar{p}^{(i)} \in [0, 1]$.

**Theorem 2.** For all $t > 0$,

$$\mathbb{P} \left( \left\| \sum_{i=1}^{I} \sum_{j=1}^{n_i} (X^{(i)}_j - p^{(i)}_j) \log \bar{p}^{(i)} \right\| \geq t \right) \leq 2 \exp \left\{ - \frac{t^2}{2(n + t)} \right\}. \quad (5)$$

Note that here the model assumption on $X^{(i)}_j$ is identical to the setup in Section 2 where each Bernoulli variable has its own parameter. The function we consider in the inequality is, however, defined differently. Moreover, this theorem reduces to Theorem 1 when $n_i \equiv 1$ for $i = 1, ..., I$.

**Proof.** Let $Z^{(i)} = \sum_{j=1}^{n_i} (X^{(i)}_j - p^{(i)}_j) \log \bar{p}^{(i)} = \sum_{j=1}^{n_i} (X^{(i)}_j - \bar{p}^{(i)}) \log \bar{p}^{(i)}$. Consider the moment generating function $\mathbb{E}[e^{\lambda Z^{(i)}}]$ for $\bar{p}^{(i)} \in (0, 1)$.

$$\mathbb{E}[e^{\lambda Z^{(i)}}] = \prod_{j=1}^{n_i} \left( p^{(i)}_j e^{\lambda (1-p^{(i)}_j) \log \bar{p}^{(i)}} + (1 - p^{(i)}_j) e^{-\lambda \bar{p}^{(i)} \log \bar{p}^{(i)}} \right) \leq \left( \bar{p}^{(i)} e^{\lambda (1-\bar{p}^{(i)}) \log \bar{p}^{(i)}} + (1 - \bar{p}^{(i)}) e^{-\lambda \bar{p}^{(i)} \log \bar{p}^{(i)}} \right)^{n_i} = (G(\bar{p}^{(i)}, \lambda))^{n_i},$$

where the inequality follows from the inequality of arithmetic and geometric means: $\sqrt[n]{\prod_{i=1}^{n_i} a_i} \leq \frac{\sum_{i=1}^{n_i} a_i}{n}$ for non-negative $a_1, ..., a_n$. From (4), $G(\bar{p}^{(i)}, \lambda) \leq \exp \left\{ - \frac{\lambda^2}{2(1-\lambda)} \right\}$. It follows that $\mathbb{E}[e^{\lambda Z^{(i)}}] \leq \exp \left\{ \frac{n_i \lambda^2}{2(1-\lambda)} \right\}$. The inequality also holds for $\bar{p}^{(i)} = 0$ or 1 as $\mathbb{E}[e^{\lambda Z^{(i)}}] \equiv 1$. The rest of the proof follows from the standard argument using the Chernoff bound as shown in the proof of Theorem 1. \( \square \)

We conclude this note with a corollary that is easily proved by the same argument for Corollary 1. Let $X^{(1)}_1, X^{(1)}_2, ..., X^{(n_1)}_1, X^{(2)}_1, X^{(2)}_2, ..., X^{(n_2)}_1, X^{(I)}_1, X^{(I)}_2, ..., X^{(n_I)}_1$ be independent multinoulli variables, where each $X^{(i)}_j = (X^{(i)}_{1j}, ..., X^{(i)}_{kj})$, and $p^{(i)}_{jk} = \mathbb{P}(X^{(i)}_{jk} = 1)$ for $k = 1, ..., I$. As before, let $n = \sum_{i=1}^{I} n_i$. And let $\bar{p}^{(i)}_{jk} = \frac{1}{n_i} \sum_{j=1}^{n_i} p^{(i)}_{jk}$ for $i = 1, ..., I$ and $k = 1, ..., K$, where $\bar{p}^{(i)}_{jk} \in [0, 1]$.

**Corollary 2.** For all $\epsilon > 0$,

$$\mathbb{P} \left( \frac{1}{n} \left\| \sum_{i=1}^{I} \sum_{j=1}^{n_i} \sum_{k=1}^{K} (X^{(i)}_{jk} - \bar{p}^{(i)}_{jk}) \log \bar{p}^{(i)}_{jk} \right\| \geq \epsilon \right) \leq 2K \exp \left\{ - \frac{ne^2}{2K(K + \epsilon)} \right\}.$$

5
Acknowledgements

This research was supported by the National Science Foundation grant DMS-1840203.

References

Choi, D. S., Wolfe, P. J., and Airoldi, E. M. (2012). Stochastic blockmodels with a growing number of classes. *Biometrika*, 99(2):273–284.

Dubhashi, D. P. and Panconesi, A. (2009). *Concentration of measure for the analysis of randomized algorithms*. Cambridge University Press.

Hoeffding, W. (1963). Probability inequalities for sums of bounded random variables. *Journal of the American Statistical Association*, 58(301):13–30.

Paul, S. and Chen, Y. (2016). Consistent community detection in multi-relational data through restricted multi-layer stochastic blockmodel. *Electronic Journal of Statistics*, 10(2):3807–3870.

Robbins, H. (1955). A remark on stirling’s formula. *The American mathematical monthly*, 62(1):26–29.

Wainwright, M. J. (2019). *High-Dimensional Statistics: A Non-Asymptotic Viewpoint*. Cambridge Series in Statistical and Probabilistic Mathematics. Cambridge University Press.

Zhao, Y. and Weko, C. (2019). Network inference from grouped observations using hub models. *Statistica Sinica*, 29(1):225–244.