Morawetz estimates as well as spacetime bounds based on pseudoconformal conservation law and interaction Morawetz estimates for a quasilinear Schrödinger equation

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Abstract

In this paper, we consider the Cauchy problem of the quasilinear Schrödinger equation

\[
\begin{align*}
\begin{cases}
iu_t &= \Delta u + 2uh'(|u|^2)\Delta h(|u|^2) + V(x)u + F(|u|^2)u + (W*|u|^2)u, \quad x \in \mathbb{R}^N, \quad t > 0 \\
u(x,0) &= u_0(x), \quad x \in \mathbb{R}^N
\end{cases}
\end{align*}
\]

Here \(h(s), F(s), V(x)\) and \(W(x)\) are some real functions. \(V(x) \in L^{p_1}(\mathbb{R}^N) + L^\infty(\mathbb{R}^N), \quad p_1 > \max(1, \frac{N}{2}), \quad \text{and} \quad W(x) \in L^{p_2}(\mathbb{R}^N) + L^\infty(\mathbb{R}^N), \quad p_2 > \max(1, \frac{N}{4}), \quad W(x) \text{ is even.}
\]

Keywords: Quasilinear Schrödinger equation; Pseudoconformal conservation law; Morawetz estimate; Spacetime bound.

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1 Introduction

In this paper, we consider the following Cauchy problem:

\[
\begin{align*}
\begin{cases}
iu_t &= \Delta u + 2uh'(|u|^2)\Delta h(|u|^2) + V(x)u + F(|u|^2)u + (W*|u|^2)u, \quad x \in \mathbb{R}^N, \quad t > 0 \\
u(x,0) &= u_0(x), \quad x \in \mathbb{R}^N
\end{cases}
\end{align*}
\]

Here \(h(s), F(s), V(x)\) and \(W(x)\) are some real functions. \(V(x) \in L^{p_1}(\mathbb{R}^N) + L^\infty(\mathbb{R}^N), \quad p_1 > \max(1, \frac{N}{2}) \) and \(W(x) \in L^{p_2}(\mathbb{R}^N) + L^\infty(\mathbb{R}^N), \quad p_2 > \max(1, \frac{N}{4}), \quad W(x) \text{ is even.}
\]

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Equation (1.1) can be used to model a lot of physical phenomena, such as the superfluid film equation in plasma physics if \( h(s) = s \), physics phenomenon in dissipative quantum mechanics if \( h(s) = \sqrt{s} \) and the self-channelling of a high-power ultra short laser in matter if \( h(s) = \sqrt{1+s} \). It also appears in condensed matter theory and nonlinear optical theory, see [1, 3, 4, 16, 19, 20, 21, 22, 27]. In convenience, we always assume that \( h(s) \geq 0 \) for all \( s \geq 0 \), \( V(x) \leq 0 \) and \( W(x) \leq 0 \) for all \( x \in \mathbb{R}^N \). In the case if \( F(s) \leq 0 \) for all \( s \geq 0 \) or changes sign, we say that (1.1) is in defocusing case, if \( F(s) \geq 0 \) for all \( s \geq 0 \), we say that (1.1) is in combined defocusing and focusing case.

There are many interesting topics on (1.1), such as local wellposedness, global wellposedness, asymptotic behaviour for the solution. First, we say something about the local wellposedness of the solution of (1.1). Besides the assumptions on \( V(x) \) and \( W(x) \), suppose that there exist \( r > 1 \) and \( C > 0 \) such that
\[
|G(\|u\|^2)|^r \leq C[h(\|u\|^2)]^{2^*},
\]
where \( G(s) = \int_0^s F(\eta)d\eta \). Then (1.1) is local wellposedness in the energy space
\[
\mathbf{X} = \{ w \in H^1(\mathbb{R}^N), \quad \int_{\mathbb{R}^N} |\nabla h(\|w\|^2)|^2dx < +\infty \} \tag{1.2}
\]
by the results of [6, 7, 18, 26, 29, 30], and the conditions on global wellposedness and blowup for the solution of (1.1) had been established. We are concerned with the asymptotic behaviour and spacetime bound estimates for the solution in this paper. It is well known that pseudoconformal conservation law is essential for the study of the asymptotic behaviour of the solution and Morawetz estimate is an important tool to construct scattering operator on the energy space, see [2, 5, 8, 9, 12, 13, 14, 23].

However, two more interesting questions are as follows: 1. What is the relationship between pseudoconformal conservation law and Morawetz estimate? 2. How to establish the link between pseudoconformal conservation law and spacetime bound estimate?

To obtain the answers, we will establish Morawetz estimates based on pseudoconformal conservation law in this paper, which reveals the relationship between pseudoconformal conservation law and Morawetz estimate. We also give the spacetime bounds based on pseudoconformal conservation law, which illustrates the link between pseudoconformal conservation law and spacetime bound estimates.

Before we state our main results, we define the mass and energy of (1.1) below.

\[
m(u) = \left( \int_{\mathbb{R}^N} |u(x,t)|^2dx \right)^{\frac{1}{2}}, \tag{1.3}
\]
\[
E(u) = \frac{1}{2} \int_{\mathbb{R}^N} [\|
abla u\|^2 + \|
abla h(\|u\|^2)\|^2 - G(\|u\|^2) - V(x)|u|^2 - \frac{1}{2}(W * |u|^2)|u|^2]dx \tag{1.4}
\]

Mass and energy conservation laws will be proved in Section 2.

Now we state pseudo-conformal conservation law as follow.

**Theorem 1.** (Pseudoconformal conservation law) Let \( u(x,t) \) be the solution of (1.1) in energy space \( \mathbf{X} \), \( u_0 \in \mathbf{X} \) and \( xu_0 \in L^2(\mathbb{R}^N) \). Then in the time interval \([0,t]\)
when it exists,
\[
P(t) = \int_{\mathbb{R}^N} |(x - 2it\nabla)u|^2 dx + 4t^2 \int_{\mathbb{R}^N} |\nabla h(|u|^2)|^2 dx - 4t^2 \int_{\mathbb{R}^N} G(|u|^2) dx
\]
\[
- 4t^2 \int_{\mathbb{R}^N} V(x)|u|^2 dx - 2t^2 \int_{\mathbb{R}^N} (W * |u|^2)|u|^2 dx
\]
\[
= \int_{\mathbb{R}^N} |xu_0|^2 dx + 4 \int_0^t \tau \theta(\tau) d\tau. \quad (1.5)
\]

Here
\[
\theta(t) = \int_{\mathbb{R}^N} -4N[2h''(|u|^2)h'(|u|^2)|u|^2 + (h'(|u|^2))]|u|^2|\nabla u|^2 dx
\]
\[
+ \int_{\mathbb{R}^N} [NF(|u|^2)|u|^2 - (N + 2)G(|u|^2)] dx
\]
\[
- \int_{\mathbb{R}^N} [2V + (x \cdot \nabla V)]|u|^2 dx - \int_{\mathbb{R}^N} \left( |W + \frac{(x \cdot \nabla W)}{2}| * |u|^2 \right) |u|^2 dx. \quad (1.6)
\]

**Remark 1.1.** Although we assume that \( V(x) \leq 0 \) and \( W(x) \leq 0 \) for all \( x \in \mathbb{R}^N \) in this paper, Theorem 1 hold for \( V(x), W(x) \geq 0 \) or change sign for \( x \in \mathbb{R}^N \).

Based on the pseudo-conformal conservation law, we can get Morawetz estimates for the solution of (1.1) in energy space \( X \) below.

**Theorem 2.** (Morawetz estimates based on pseudoconformal conservation law) Let \( u(x, t) \) be the solution of (1.1) in energy space \( X \), \( u_0 \in X \) and \( xu_0 \in L^2(\mathbb{R}^N) \), the space dimension \( N \geq 1 \) in defocusing case, \( N \geq 3 \) in combined defocusing and focusing case. In addition, suppose that in combined defocusing and focusing case, there exist \( c_1, c_1', c_2, c_2' > 0, 0 < \gamma_1, \tilde{\gamma}_1 < 1 \) and \( \gamma_2, \tilde{\gamma}_2 > 1 \) such that
\[
C_r(u_0) := \sum_{j=1}^2 (c_j\|u_0\|_{L^2}^2)^\frac{2}{C_j} (c_jC_s)^\frac{2}{C_j} < 1, \quad \frac{2^*(1 - \gamma_1)}{2(\gamma_2 - \gamma_1)} = 1, \quad \frac{2^*(1 - \tilde{\gamma}_1)}{2(\tilde{\gamma}_2 - \tilde{\gamma}_1)} = 1, \quad (1.7)
\]
\[
||G(s)||^{\gamma_1} \leq c_1 s, \quad ||G(s)||^{\tilde{\gamma}_1} \leq c_1' [h(s)]^{2^*} \text{ for } 0 \leq s \leq 1, \quad (1.8)
\]
\[
||G(s)||^{\gamma_2} \leq c_2 s, \quad ||G(s)||^{\tilde{\gamma}_2} \leq c_2' [h(s)]^{2^*} \text{ for } s > 1. \quad (1.9)
\]

Here \( C_s \) denotes the best constant in the Sobolev’s inequality
\[
\int_{\mathbb{R}^N} w^2 dx \leq C_s \left( \int_{\mathbb{R}^N} |\nabla w|^2 dx \right)^{\frac{2}{2^*}} \quad \text{ for any } w \in H^1(\mathbb{R}^N), \quad (1.10)
\]

1. Assume that \( 2h''(s)h'(s)s + (h'(s))^2 \geq 0 \) and \( |NF(s) - (N + 2)G(s)| \leq 0 \) for all \( s \geq 0, [2V + (x \cdot \nabla V)] \geq 0 \) and \( [2W + (x \cdot \nabla W)] \geq 0 \) for all \( x \in \mathbb{R}^N \). Then \( u(x, t) \) is global existence, and

**Estimate (A):**
\[
\int_0^\infty \int_{\mathbb{R}^N} \left[ |\nabla h(|u|^2)|^2 + |G(|u|^2)| + |V(x)||u|^2 + \frac{1}{2}(W * |u|^2)|u|^2 \right] dx dt
\leq M_1(u_0, \theta)
\] (1.11)

for any \(a(x) \geq 0, \frac{1}{a(x)} \in L^{\frac{1}{1-\theta}}(\mathbb{R}^N), \frac{1}{2} < \theta < 1;\)

**Estimate (B):**
\[
\int_0^\infty \int_{\mathbb{R}^N} t^2 \left[ |\nabla h(|u|^2)|^2 + |G(|u|^2)| + |V(x)||u|^2 + \frac{1}{2}(W * |u|^2)|u|^2 \right] dx dt
\leq M_2(u_0, k)
\] (1.12)

if \(b(x) \geq 0, 1 < k < 3\) or \(b(x) \geq b > 0, 1 < k.\)

Especially, let \(b(x) \equiv 0, k = 2, \) then

**Estimate (C):**
\[
\int_0^\infty \int_{\mathbb{R}^N} \left[ |\nabla h(|u|^2)|^2 + |G(|u|^2)| + |V(x)||u|^2 + (W * |u|^2)|u|^2 \right] dx dt
\leq M_3(u_0).
\] (1.13)

2. Assume that
(i) \(-k_1(h'(s))^2 \leq 2h''(s)h'(s)s + (h'(s))^2 \leq 0\) for some \(k_1 > 0;\)
(ii) \(-k_2|G(s)| \leq NF(s)s - (N + 2)G(s) \leq 0\) for some \(k_2 > 0;\)
(iii) \(-k_3|V| \leq 2V + (x \cdot \nabla V) \leq 0\) for some \(k_3 > 0;\)
(iv) \(-k_4|W| \leq 2W + (x \cdot \nabla W) \leq 0\) for some \(k_4 > 0.\)

Let
\[
l = \max(Nk_1, k_2, k_3, 2k_4).
\] (1.14)

Then \(u(x, t)\) is global existence, and

**Estimate (D):**
\[
\int_0^\infty \int_{\mathbb{R}^N} t^2 \left[ |\nabla h(|u|^2)|^2 + |G(|u|^2)| + |V(x)||u|^2 + \frac{1}{2}(W * |u|^2)|u|^2 \right] (c(x) + t)^k dx dt
\leq M_4(u_0, k, l)
\] (1.15)

Here \(l + 1 < k < 3\) if \(l < 2\) in defocusing case, \(k > 1 + \frac{C_r(u_0)}{c - C_r(u_0)}\) in combined defocusing and focusing case, if \(c(x) \geq 0.\) While \(l + 1 < k\) in defocusing case, \(k > 1 + \frac{C_r(u_0)}{1 - C_r(u_0)}\) in combined defocusing and focusing case, if \(c(x) \geq c > 0.\)

Especially, if \(c(x) \equiv 0, l < 1\) and \(k = 2, \) then

**Estimate (E):**
\[
\int_0^\infty \int_{\mathbb{R}^N} \left[ |\nabla h(|u|^2)|^2 + |G(|u|^2)| + |V(x)||u|^2 + \frac{1}{2}(W * |u|^2)|u|^2 \right] dx dt
\leq M_5(u_0, l).
\] (1.16)
Remark 1.2. The assumptions of Case 2 can be weaken as: Assume that at least one of (i)–(iv) holds. And the corresponding value of \( l \) can be take one of \( Nk_1, k_2, k_3 \) and \( 2k_4 \). For example, if (i) holds, while \( NF(s)s - (N+2)G(s) \geq 0, 2V + (x \cdot \nabla V) \geq 0, 2W + (x \cdot \nabla W) \geq 0 \), we can take \( l = Nk_1 \); If (i) and (ii) hold, while \( 2V + (x \cdot \nabla V) \geq 0, 2W + (x \cdot \nabla W) \geq 0 \), we can take \( l = \max(Nk_1, k_2) \), and so on.

Based on the pseudo-conformal conservation law, we also can obtain the spacetime bound estimates for the solution below.

**Theorem 3. (Space-time bounds based on pseudo-conformal conservation law)** Let \( u(x, t) \) be the solution of (1.1) in energy space \( X \), \( u_0 \in X \) and \( xu_0 \in L^2(\mathbb{R}^N) \), the space dimension \( N \geq 1 \) in defocusing case, \( N \geq 3 \) in combined defocusing and focusing case. Besides the assumptions of Theorem 2, suppose that there exist \( c_3, c_4, 0 < \vartheta < 1 \) and \( \gamma > 1 \) such that

\[
||G(s)||^\vartheta \leq c_3 s, \quad ||G(s)||^\gamma \leq c_4 [h(s)]^{2^*} \quad \text{if} \quad N \geq 3.
\]

In addition, suppose that

\[
C'_r(u_0) := (c_3 ||u_0||_{L^2})^\frac{2}{3} (c_4 C_s)^\frac{N - 2}{N} < 1, \quad \frac{2^*(1 - \vartheta)}{2(\gamma - \vartheta)} = 1
\] (1.18)

in combined defocusing and focusing case. Then \( u(x, t) \) is global existence, and

\[
\begin{align*}
\left( \int_0^\infty \left( \int_{\mathbb{R}^N} [\nabla h(|u|^2)]^2 + |G(|u|^2)| + |V(x)||u|^2 + \frac{1}{2}(|W| * |u|^2)|dx \right)^p dt \right)^{\frac{1}{p}} \\
\leq C(u_0, p).
\end{align*}
\] (1.19)

Here \( p > \frac{1}{2} \) under the assumptions of Theorem 2 in Case 1,

\[
p > \frac{1}{(2 - l)}, \quad 0 < l < 2
\]

under the assumptions of Theorem 2 in defocusing subcase of Case 2, and

\[
p > \max \left( \frac{1}{2}, \frac{[1 - C'_r(u_0)]}{2(1 - C'_r(u_0)) - l(1 + C'_r(u_0))} \right), \quad 0 < l < \frac{2[1 - C'_r(u_0)]}{[1 + C'_r(u_0)]}
\]

under the assumptions of Theorem 2 in combined defocusing and focusing subcase of Case 2.

Moreover, if \( N \geq 3 \), then

\[
\|G(|u|^2)\|_{L^r(\mathbb{R}^N)} = \left( \int_0^\infty \left( \int_{\mathbb{R}^N} [G(|u|^2)]^r dx \right)^{\frac{q}{r}} dt \right)^{\frac{1}{q}} \leq C(u_0, r, q, \vartheta, \gamma).
\] (1.20)

Here \( 1 \leq r < \gamma, q > \frac{r(\gamma - \vartheta)}{2^*(r - \vartheta)} \) under the assumptions of Theorem 2 in Case 1,

\[
q > \frac{2r(r - \vartheta)}{(2 - l)2^* q(\gamma - \vartheta) - 2r(r - \vartheta)}, \quad 0 < l < 2
\]
under the assumptions of Theorem 2 in defocusing subcase of Case 2,
\[ q > \frac{2r(\gamma - \vartheta)[1 - C'_r(u_0)]}{2^r(r - \vartheta)[2(1 - C'_r(u_0)) - l(1 + C'_r(u_0))]}, \quad 0 < l < \frac{2[1 - C'_r(u_0)]}{1 + C'_r(u_0)} \]
under the assumptions of Theorem 2 in combined defocusing and focusing subcase of Case 2.

**Remark 1.3.** To obtain Morawetz estimates and spacetime bounds, the key technique is to use pseudoconformal conservation law obtaining the bound for
\[ \int_{\mathbb{R}^N} [||\nabla h(||u||^2)||^2 + |G(||u||^2)||] + |V(x)||u||^2 + \frac{1}{2}(|W| * ||u||^2)||u||^2]dx \]
when $t \geq 1$. Therefore, we call Theorem 2 as Morawetz estimates based on pseudoconformal conservation law and Theorem 3 as spacetime bounds based on pseudoconformal conservation law. The values of $M_1(u_0, \theta)$, $M_2(u_0, k)$, $M_4(u_0, k, l)$, $M_5(u_0, l)$, $C(u_0, p)$ and $C(u_0, r, q, \theta, \gamma)$ will be calculated explicitly in Section 3 and Section 4.

**Remark 1.4.** By the basic inequalities $(\frac{A+B}{2})^K \leq \frac{A^K + B^K}{2} \leq (\frac{A+B}{2})^K$ for $A, B \geq 0$ and $K > 1$, Estimate (B) and Estimate (D) are essentially the same type. However, we prefer to write them in the forms at present because the assumptions on $h(s)$, $V(x)$, $G(s)$ and $W(x)$ are different and the discussions in the proofs are discrepant. Especially, we can take $b(x) = |x|^r$ and $c(x) = |x|$ in Estimate (B) and Estimate (D) respectively. We also can replace $b(x) + t^k$ and $(c(x) + t)^k$ by more general function $f(x, t)$ and obtain the corresponding estimates.

In convenience, we will use $C$, $C'$, and so on, to denote some constants in the sequel, the values of it may vary line to line.

**Remark 1.5.** If $h(s) = s^\alpha$, $F(||u||^2) = \mp ||u||^{2\beta}$, $V(x) = -\frac{1}{|x|^m}$ and $W(x) = -\frac{1}{|x|^n}$, then (1.1) becomes
\[
\begin{align*}
  \begin{cases}
    iu_t = \Delta u + 2a|x|^{2\alpha - 2}u\Delta(||u||^{2\alpha}) - \frac{u}{|x|^m} \mp ||u||^{2\beta}u - \left(\frac{1}{|x|^m} * ||u||^2\right)u, x \in \mathbb{R}^N \setminus \{0\}, t > 0 \\
    u(x, 0) = u_0(x), \quad x \in \mathbb{R}^N.
  \end{cases}
\end{align*}
\]
(1.21)

We especially have
\[
\begin{align*}
  &\int_0^\infty \int_{\mathbb{R}^N} \left[|\nabla(||u||^2\alpha)|^2 + \frac{|u|^{2\beta + 2}}{\beta + 1} + \frac{1}{|x|^m}|u|^2 + \frac{1}{2}\left(\frac{1}{|x|^m} * ||u||^2\right)||u||^2\right]dxdt \leq C \quad (1.22) \\
  &\int_0^\infty \left(\int_{\mathbb{R}^N} \left[|\nabla(||u||^2\alpha)|^2 + \frac{|u|^{2\beta + 2}}{\beta + 1} + \frac{1}{|x|^m}|u|^2 + \frac{1}{2}\left(\frac{1}{|x|^m} * ||u||^2\right)||u||^2\right]dx\right)^p dt \leq C \quad (1.23) \\
  &\|u\|_{L^p(\mathbb{R}^+)L^p(\mathbb{R}^N)} = \left(\int_0^\infty \left(\int_{\mathbb{R}^N} |u|^p dx\right)^\frac{q}{p} dt\right)^\frac{1}{q} \leq C \quad (1.24)
\end{align*}
\]
under certain assumptions.

The organization of this paper is as follows. In Section 2, we will prove mass and energy conservation laws and some equalities, and give the proof of Theorem 1.
In Section 3, we will prove Theorem 2 and establish Morawetz estimates based on pseudoconformal conservation law. In Section 4. We prove Theorem 3 and give space-time bound estimates for the solution. In Section 5, we establish interaction Morawetz estimates for the solution.

2 Preliminaries and the proof of Theorem 1

In this section, we first prove a lemma as follows.

**Lemma 2.1.** Assume that $u$ is the solution of (1.1). Then in the time interval $[0, t]$ when it exists, $u$ satisfies

(i) Mass conservation:

$$m(u) = \left( \int_{\mathbb{R}^N} |u(x, t)|^2 dx \right)^{\frac{1}{2}} = \left( \int_{\mathbb{R}^N} |u_0(x)|^2 dx \right)^{\frac{1}{2}} = m(u_0);$$

(ii) Energy conservation:

$$E(u) = \frac{1}{2} \int_{\mathbb{R}^N} [\nabla u]^2 + \nabla h(|u|^2)]^2 dx - \frac{1}{2} \int_{\mathbb{R}^N} V(x)|u|^2 dx$$

$$- \frac{1}{2} \int_{\mathbb{R}^N} G(|u|^2) dx - \frac{1}{4} \int_{\mathbb{R}^N} (W * |u|^2)|u|^2 dx$$

$$= E(u_0);$$

(2.1)

(iii) 

$$\frac{d}{dt} \int_{\mathbb{R}^N} |x|^2|u|^2 dx = -4 \Im \int_{\mathbb{R}^N} \bar{\tilde{u}}(x \cdot \nabla u) dx;$$

(iv)

$$\frac{d}{dt} \Im \int_{\mathbb{R}^N} \bar{\tilde{u}}(x \cdot \nabla u) dx = -2 \int_{\mathbb{R}^N} |\nabla u|^2 dx - (N + 2) \int_{\mathbb{R}^N} |\nabla h(|u|^2)|^2 dx$$

$$- 8N \int_{\mathbb{R}^N} h''(|u|^2)h'(|u|^2)|u|^4 |\nabla u|^2 dx - \int_{\mathbb{R}^N} (x \cdot \nabla V)|u|^2 dx$$

$$+ N \int_{\mathbb{R}^N} |u|^2 F(|u|^2) - G(|u|^2)] dx$$

$$- \int_{\mathbb{R}^N} [(x \cdot \nabla W) * |u|^2]|u|^2 dx.$$ 

(2.2)

**Proof:** (i) Multiplying (1.1) by $2 \bar{\tilde{u}}$, taking the imaginary part of the result, we get

$$\frac{\partial}{\partial t} |u|^2 = \Im(2\bar{\tilde{u}} \Delta u) = \nabla \cdot (2 \Im \nabla u).$$

(2.3)

Integrating (2.3) over $\mathbb{R}^N \times [0, t]$, we have

$$\int_{\mathbb{R}^N} |u|^2 dx = \int_{\mathbb{R}^N} |u_0|^2 dx,$$
which implies mass conservation law.

(ii) Multiplying (1.1) by $2\bar{u}_t$, taking the real part of the result, then integrating it over $\mathbb{R}^N \times [0, t]$, we obtain

$$
\int_{\mathbb{R}^N} [\nabla |u|^2 + |\nabla h(|u|^2)|^2]dx - \int_{\mathbb{R}^N} V(x)|u|^2dx
- \int_{\mathbb{R}^N} G(|u|^2)dx - \frac{1}{2} \int_{\mathbb{R}^N} (W * |u|^2)|u|^2dx
= \int_{\mathbb{R}^N} [\nabla |u_0|^2 + |\nabla h(|u_0|^2)|^2]dx - \int_{\mathbb{R}^N} V(x)|u_0|^2dx
- \int_{\mathbb{R}^N} G(|u_0|^2)dx - \frac{1}{2} \int_{\mathbb{R}^N} (W * |u_0|^2)|u|^2dx,
$$

which implies energy conservation law.

(iii) Multiplying (2.3) by $|x|^2$ and integrating it over $\mathbb{R}^N$, we get

$$
\frac{d}{dt}\int_{\mathbb{R}^N} |x|^2|u|^2dx = \int_{\mathbb{R}^N} |x|^2\nabla \cdot (2\Im(\bar{u}\nabla u))dx = -4\Im \int_{\mathbb{R}^N} \bar{u}(x \cdot \nabla u)dx.
$$

(iv) Let $a(x, t) = \Re u(x, t)$ and $b(x, t) = \Im u(x, t)$. Then

$$
\frac{d}{dt}\Im \bar{u}(x \cdot \nabla u) = \sum_{k=1}^N [x_k(b_t)_{x_k}a - x_k(a_t)_{x_k}b] + \sum_{k=1}^N (x_kb_{x_k}a_t - x_ba_{x_k}b_t).
$$

Since

$$
\int_{\mathbb{R}^N} \sum_{k=1}^N [x_k(b_t)_{x_k}a - x_k(a_t)_{x_k}b]dx + \int_{\mathbb{R}^N} \sum_{k=1}^N (x_kb_{x_k}a_t - x_ba_{x_k}b_t)dx
= \int_{\mathbb{R}^N} \sum_{k=1}^N [x_k(b_t)_{x_k}a - x_k(a_t)_{x_k}b]dx + \int_{\mathbb{R}^N} \sum_{k=1}^N (x_kb_{x_k}\Delta a + x_kb_{x_k}\Delta b)dx
+ \frac{1}{2} \int_{\mathbb{R}^N} \sum_{k=1}^N x_k(|u|^2)_{x_k}[2h'(|u|^2)\Delta h(|u|^2) + V(x) + F(|u|^2) + (W * |u|^2)]dx
= N \int_{\mathbb{R}^N} (a_t b - ab_t)dx + \int_{\mathbb{R}^N} \sum_{k=1}^N (x_kb_{x_k}a_t - x_ba_{x_k}b_t)dx
+ \frac{N - 2}{2} \int_{\mathbb{R}^N} |\nabla u|^2dx + \frac{N - 2}{2} \int_{\mathbb{R}^N} |\nabla h(|u|^2)|^2dx
- \frac{1}{2} \int_{\mathbb{R}^N} [NV + (x \cdot \nabla V)]|u|^2dx - \frac{N}{2} \int_{\mathbb{R}^N} G(|u|^2)dx
- \frac{1}{2} \int_{\mathbb{R}^N} ([NW + \frac{1}{2}(x \cdot \nabla W)] * |u|^2)|u|^2dx,
$$
we have
\[
\frac{d}{dt} \mathfrak{I} \int_{\mathbb{R}^N} \bar{u} (x \cdot \nabla u) dx = N \int_{\mathbb{R}^N} \left( [a \Delta a + b \Delta b] + 2 |u|^2 h'(|u|^2) \Delta h(|u|^2) \right) dx \\
+ N \int_{\mathbb{R}^N} \left( V(x)|u|^2 + F(|u|^2)|u|^2 + (W * |u|^2)|u|^2 \right) dx \\
+ (N - 2) \int_{\mathbb{R}^N} |\nabla u|^2 dx + (N - 2) \int_{\mathbb{R}^N} |\nabla h(|u|^2)|^2 dx \\
- \int_{\mathbb{R}^N} [NV + (x \cdot \nabla V)]|u|^2 dx - N \int_{\mathbb{R}^N} G(|u|^2) dx \\
- \int_{\mathbb{R}^N} \left( [NW + 1/2(x \cdot \nabla W)] * |u|^2 \right) |u|^2 dx \\
= -2 \int_{\mathbb{R}^N} |\nabla u|^2 dx - (N + 2) \int_{\mathbb{R}^N} |\nabla h(|u|^2)|^2 dx \\
- 8N \int_{\mathbb{R}^N} h'(|u|^2) h''(|u|^2)|u|^4 |\nabla u|^2 dx - \int_{\mathbb{R}^N} (x \cdot \nabla V)|u|^2 dx \\
+ N \int_{\mathbb{R}^N} |u|^2 F(|u|^2) - G(|u|^2)|u|^2 dx - \int_{\mathbb{R}^N} \left( \frac{(x \cdot \nabla W)}{2} \right)^2 * |u|^2 |u|^2 dx.
\]

Lemma 2.1 is proved. \(\square\)

Now we give the proof of Theorem 1.

**Proof of Theorem 1**: Assume that \(u\) is the solution of (1.1), \(u_0 \in X\) and \(xu_0 \in L^2(\mathbb{R}^N)\). Using energy conservation law, we get

\[
P(t) = \int_{\mathbb{R}^N} |xu|^2 dx + 4t \mathfrak{I} \int_{\mathbb{R}^N} \bar{u} (x \cdot \nabla u) dx + 4t^2 \int_{\mathbb{R}^N} |\nabla u|^2 dx \\
+ 4t^2 \int_{\mathbb{R}^N} |\nabla h(|u|^2)|^2 dx - 4t^2 \int_{\mathbb{R}^N} G(|u|^2) dx \\
- 4t^2 \int_{\mathbb{R}^N} V(x)|u|^2 dx - 2t^2 \int_{\mathbb{R}^N} (W * |u|^2)|u|^2 dx \\
= \int_{\mathbb{R}^N} |xu|^2 dx + 4t \mathfrak{I} \int_{\mathbb{R}^N} \bar{u} (x \cdot \nabla u) dx + 8t^2 E(u_0). \tag{2.4}
\]

Recalling that

\[
\frac{d}{dt} \int_{\mathbb{R}^N} |x|^2 |u|^2 dx = -4 \mathfrak{I} \int_{\mathbb{R}^N} \bar{u} (x \cdot \nabla u) dx,
\]

we get
P'(t) = \frac{d}{dt} \int_{\mathbb{R}^N} |xu|^2 dx + 4 \Theta \int_{\mathbb{R}^N} \bar{u} (x \cdot \nabla u) dx + 4t \frac{d}{dt} \Theta \int_{\mathbb{R}^N} \bar{u} (x \cdot \nabla u) dx + 16tE(u_0)
= 4t \left\{ -2 \int_{\mathbb{R}^N} |
abla u|^2 dx - (N + 2) \int_{\mathbb{R}^N} |\nabla h(|u|^2)|^2 dx \\
- 8N \int_{\mathbb{R}^N} h''(|u|^2) h'(|u|^2) |u|^4 |
abla u|^2 dx - \int_{\mathbb{R}^N} (x \cdot \nabla V) |u|^2 dx \\
+ N \int_{\mathbb{R}^N} |u|^2 F(|u|^2) - G(|u|^2) dx - \int_{\mathbb{R}^N} \frac{x \cdot \nabla W}{2} * |u|^2 |u|^2 dx \right\}
+ 8t \int_{\mathbb{R}^N} |\nabla u|^2 + |\nabla h(|u|^2)|^2 - V(x)|u|^2 - G(|u|^2) - \frac{1}{2} (W * |u|^2)|u|^2 dx
= 4t \int_{\mathbb{R}^N} -4N[2h''(|u|^2) h'(|u|^2) |u|^2 + (h'(|u|^2))^2] |u|^2 |\nabla u|^2 dx
+ 4t \int_{\mathbb{R}^N} [N |u|^2 F(|u|^2) - (N + 2) G(|u|^2)] dx - 4t \int_{\mathbb{R}^N} [2V + (x \cdot \nabla V)] |u|^2 dx
- 4t \int_{\mathbb{R}^N} \left( W + \frac{x \cdot \nabla W}{2} \right) * |u|^2 dx. \tag{2.5}

Integrating (2.5) from 0 to t, we obtain

\[ P(t) = P(0) + 4 \int_0^t \tau \Theta(\tau) d\tau = \int_{\mathbb{R}^N} |xu_0|^2 dx + 4 \int_0^t \tau \Theta(\tau) d\tau. \]

That is,

\[ P(t) = \int_{\mathbb{R}^N} [(x - 2it \nabla) u]|2 dx + 4t^2 \int_{\mathbb{R}^N} |\nabla h(|u|^2)|^2 dx - 4t^2 \int_{\mathbb{R}^N} G(|u|^2) dx \\
- 4t^2 \int_{\mathbb{R}^N} V(x) |u|^2 dx - 4t^2 \int_{\mathbb{R}^N} (W * |u|^2)|u|^2 dx \\
= \int_{\mathbb{R}^N} |xu_0|^2 dx + 4 \int_0^t \tau \Theta(\tau) d\tau, \]

where \( \Theta(\tau) \) is defined by (1.6).

\[ \square \]

3 Morawetz estimates based on pseudoconformal conservation law

We divide this section into two subsection and prove Theorem 2 in two cases, which will establish Morawetz estimates based on pseudoconformal conservation law.
3.1 The proof of Theorem 2 in Case 1

In this subsection, we prove Theorem 2 in Case 1.

The proof of Theorem 2 in Case 1: First, we give estimates for
\[
\int_{\mathbb{R}^N} \left[ |\nabla h(|u|^2)|^2 + |G(|u|^2)| + |V(x)||u|^2 + \frac{1}{2}(|W| * |u|^2)|u|^2 \right] dx := \int_{\mathbb{R}^N} \Phi(u) dx \tag{3.1}
\]
in two subcases.

Subcase (1). defocusing case, \( N \geq 1 \). By energy conservation law, we get
\[
\int_{\mathbb{R}^N} \Phi(u) dx \leq 2E(u_0) \text{ for any } t \geq 0 \text{ (especially for } 0 \leq t \leq 1 \text{).} \tag{3.2}
\]

Using (1.5) and (1.6), we have
\[
4t^2 \int_{\mathbb{R}^N} \Phi(u) dx \leq \int_{\mathbb{R}^N} |xu_0|^2 dx, \quad \int_{\mathbb{R}^N} \Phi(u) dx \leq \frac{C(u_0)}{4t^2} \text{ for any } t \geq 1. \tag{3.3}
\]

Here
\[
C(u_0) = \int_{\mathbb{R}^N} |xu_0|^2 dx. \tag{3.4}
\]

Subcase (2). combined defocusing and focusing case, \( N \geq 3 \). Under the assumptions (1.7)–(1.9), by the result of (6.27) in [29], we obtain
\[
\int_{\mathbb{R}^N} |G(|u|^2)| dx \leq C_r(u_0) \int_{\mathbb{R}^N} |\nabla h(|u|^2)|^2 dx. \tag{3.5}
\]

By energy conservation law, we get
\[
[1 - C_r(u_0)] \int_{\mathbb{R}^N} \left[ |\nabla h(|u|^2)|^2 + |V(x)||u|^2 + \frac{1}{2}(|W| * |u|^2)|u|^2 \right] dx \\
\leq \int_{\mathbb{R}^N} \left[ |\nabla h(|u|^2)|^2 + G(|u|^2) + |V(x)||u|^2 + \frac{1}{2}(|W| * |u|^2)|u|^2 \right] dx \\
= 2E(u_0),
\]
and
\[
\int_{\mathbb{R}^N} \left[ |\nabla h(|u|^2)|^2 + |V(x)||u|^2 + \frac{1}{2}(|W| * |u|^2)|u|^2 \right] dx \leq \frac{2E(u_0)}{1 - C_r(u_0)}. \tag{3.6}
\]
consequently,
\[
\int_{\mathbb{R}^N} \Phi(u) dx \leq \frac{2E(u_0)[1 + C_r(u_0)]}{1 - C_r(u_0)} \text{ for any } t \geq 0 \text{ (especially for } 0 \leq t \leq 1 \text{).} \tag{3.7}
\]
Using (1.5) and (1.6), we obtain
\[
[1 - C_r(u_0)]\left(4t^2 \int_{\mathbb{R}^N} |\nabla h(|u|^2)|^2 \, dx + 4t^2 \int_{\mathbb{R}^N} |V(x)||u|^2 \, dx + 2t^2 \int_{\mathbb{R}^N} (|W| * |u|^2)|u|^2 \, dx \right)
\leq [1 - C_r(u_0)]4t^2 \int_{\mathbb{R}^N} |\nabla h(|u|^2)|^2 \, dx + 4t^2 \int_{\mathbb{R}^N} |V(x)||u|^2 \, dx + 2t^2 \int_{\mathbb{R}^N} (|W| * |u|^2)|u|^2 \, dx
\leq \int_{\mathbb{R}^N} |xu_0|^2 \, dx,
\]
which implies that
\[
\int_{\mathbb{R}^N} ||\nabla h(|u|^2)||^2 \, dx + |V(x)||u|^2 + \frac{1}{2}(|W| * |u|^2)|u|^2 \, dx \leq \frac{C(u_0)}{4[1 - C_r(u_0)]t^2}, \tag{3.8}
\]
and consequently
\[
\int_{\mathbb{R}^N} \Phi(u) \, dx \leq \frac{C(u_0)[1 + C_r(u_0)]}{4[1 - C_r(u_0)]t^2} \text{ for any } t \geq 1. \tag{3.9}
\]
Based on the bounds for \(\int_{\mathbb{R}^N} \Phi(u) \, dx\) and by energy conservation law, we have
\[
\int_{\mathbb{R}^N} |\nabla u|^2 \, dx + \int_{\mathbb{R}^N} |\nabla h(|u|^2)|^2 \, dx < +\infty,
\]
i.e., the solution is global existence under the assumptions of Case 1 in Theorem 2.

Now Morawetz estimates can be proved below.

**Estimate (A):**
For any \(\frac{1}{2} < \theta < 1\) and \(\frac{1}{a(x)} \in L^{\frac{1}{\gamma'}}(\mathbb{R}^N)\), using (3.2)–(3.3), we have
\[
\int_0^\infty \int_{\mathbb{R}^N} \frac{[\Phi(u)]^\theta}{a(x)} \, dx \, dt = \int_0^1 \int_{\mathbb{R}^N} \frac{[\Phi(u)]^\theta}{a(x)} \, dx \, dt + \int_1^\infty \int_{\mathbb{R}^N} \frac{[\Phi(u)]^\theta}{a(x)} \, dx \, dt
\leq \int_0^1 \left(\int_{\mathbb{R}^N} \Phi(u) \, dx\right)^\theta \left(\int_{\mathbb{R}^N} \frac{1}{[a(x)]^{1-\theta}} \, dx\right)^{1-\theta} \, dt
+ \int_1^\infty \left(\int_{\mathbb{R}^N} \Phi(u) \, dx\right)^\theta \left(\int_{\mathbb{R}^N} \frac{1}{[a(x)]^{1-\theta}} \, dx\right)^{1-\theta} \, dt
\leq \left[\int_0^1 C \, dt + \int_1^\infty \frac{C'}{t^{2\theta}} \, dt\right] \left(\int_{\mathbb{R}^N} \frac{1}{[a(x)]^{1-\theta}} \, dx\right)^{1-\theta}
\leq M_1(u_0, \theta). \tag{3.10}
\]
Here
\[
M_1(u_0, \theta) = \begin{cases} 
\left(\frac{[2E(u_0)]^\theta}{2\theta-1} + \frac{1}{2\theta-1} \left[\frac{C(u_0)}{4}\right]^{\theta}\right) \left(\int_{\mathbb{R}^N} \frac{1}{[a(x)]^{1-\theta}} \, dx\right)^{1-\theta} & \text{in defocusing case} \\
\left(\frac{1 + C_r(u_0)}{1 - C_r(u_0)}\right)^\theta \left(\frac{[2E(u_0)]^\theta}{2\theta-1} + \frac{1}{2\theta-1} \left[\frac{C(u_0)}{4}\right]^{\theta}\right) \left(\int_{\mathbb{R}^N} \frac{1}{[a(x)]^{1-\theta}} \, dx\right)^{1-\theta} & \text{in combined defocusing and focusing case}
\end{cases}
\]
3.2 The proof of Theorem 2 in Case 2

In this subsection, we prove Theorem 2 in Case 2.

The proof of Theorem 2 in Case 2:

Estimate (D): We prove it in two subcases.
Subcase (i). defocusing case, $N \geq 1$. By energy conservation law, we also have

$$\int_{\mathbb{R}^N} \Phi(u) dx \leq 2E(u_0) \quad \text{for any} \ t \geq 0(\text{especially for} \ 0 \leq t \leq 1).$$
Letting

\[
A(t) = 4 \int_0^t \tau \int_{\mathbb{R}^N} \Phi(u) dx \, d\tau,
\]

using (1.5) and (1.6), we have

\[
tA'(t) \leq \int_{\mathbb{R}^N} |xu_0|^2 dx + tA(t) = C(u_0) + tA(t),
\]
i.e.,

\[
A'(t) \leq \frac{t}{2} A(t) + \frac{C(u_0)}{t}.
\]  

(3.17)

Applying Gronwall inequality to (3.17), we get

\[
A(t) \leq e^{\int_1^t \frac{1}{\eta} d\eta} [A(1) + \int_1^t \frac{C(u_0)}{\eta} e^{-\int_\eta^t \frac{1}{\xi} d\xi} d\eta]
\]

\[
= t[A(1) + \frac{C(u_0)}{l} - \frac{C(u_0)}{lt}] \leq [4E(u_0) + \frac{C(u_0)}{l}][t]
\]

(3.18)

for any \( t \geq 1 \). (3.17) and (3.18) mean that

\[
\int_{\mathbb{R}^N} \Phi(u) dx \leq \frac{C(u_0)}{4l^2} + \frac{4E(u_0) + C(u_0)}{4l^{2-t}}
\]

for any \( t \geq 1 \). (3.19)

In defocusing case, we obtain

\[
\int_0^\infty \int_{\mathbb{R}^N} \frac{t^2 \Phi(u)}{c(x) + t} dx dt = \int_0^1 \int_{\mathbb{R}^N} \frac{t^2 \Phi(u)}{c(x) + t} dx dt + \int_1^\infty \int_{\mathbb{R}^N} \frac{t^2 \Phi(u)}{c(x) + t} dx dt
\]

\[
\leq \int_0^1 t^{2-k} \int_{\mathbb{R}^N} \Phi(u) dx dt + \int_1^\infty \frac{1}{tk} \int_{\mathbb{R}^N} t^2 \Phi(u) dx dt
\]

\[
\leq 2E(u_0) \int_0^1 t^{2-k} dt + \frac{C(u_0)}{4} \int_1^\infty \frac{1}{tk} dt
\]

\[
+ [4E(u_0) + C(u_0)] \int_1^\infty \frac{1}{tk-l} dt
\]

\[
\leq \frac{2E(u_0)}{3-k} + \frac{C(u_0)}{4(k-1)} + \frac{4E(u_0) + C(u_0)}{4k - (l+1)}
\]

(3.20)

for \( c(x) \geq 0, \ l+1 < k < 3 \) if \( l < 2 \).

Similarly, we have

\[
\int_0^\infty \int_{\mathbb{R}^N} \frac{t^2 \Phi(u)}{c(x) + t} dx dt \leq 2E(u_0) \int_0^1 \frac{t^2}{(c+t)^k} dt + \frac{C(u_0)}{4} \int_1^\infty \frac{1}{tk} dt
\]

\[
+ [4E(u_0) + C(u_0)] \int_1^\infty \frac{1}{tk-l} dt
\]

\[
\leq \frac{2E(u_0)}{3c^k} + \frac{C(u_0)}{4(k-1)} + \frac{4E(u_0) + C(u_0)}{4k - (l+1)}
\]

(3.21)
for $c(x) \geq c > 0$, $l + 1 < k$.

**Subcase (ii).** combined defocusing and focusing case, $N \geq 3$. Recall that (3.7)

$$\int_{\mathbb{R}^N} \Phi(u) \, dx \leq \frac{2E(u_0)[1 + C_r(u_0)]}{[1 - C_r(u_0)]}$$

for any $t \geq 0$ (especially for $0 < t \leq 1$).

Using (1.5) and (1.6), we get

$$[1 - C_r(u_0)]4t^2 \int_{\mathbb{R}^N} \nabla h(|u|^2)^2 \, dx + 4t^2 \int_{\mathbb{R}^N} |V(x)||u|^2 \, dx + 2t^2 \int_{\mathbb{R}^N} (|W| \ast |u|^2)|u|^2 \, dx$$

$$\leq C(u_0) + 4l \int_{0}^{t} \tau \int_{\mathbb{R}^N} \Phi(u) \, dx \, d\tau$$

$$\leq C(u_0) + 4l[1 + C_r(u_0)] \int_{0}^{t} \tau \int_{\mathbb{R}^N} (\nabla h(|u|^2)^2 \, dx + |V(x)||u|^2 + \frac{1}{2}(|W| \ast |u|^2)|u|^2 \, dx d\tau.$$  (3.23)

Letting

$$B(t) = 4 \int_{0}^{t} \tau \int_{\mathbb{R}^N} (\nabla h(|u|^2)^2 \, dx + |V(x)||u|^2 + \frac{1}{2}(|W| \ast |u|^2)|u|^2 \, dx d\tau,$$

we have from (3.23)

$$B'(t) \leq \frac{C(u_0)}{[1 - C_r(u_0)]} t + \frac{l[1 + C_r(u_0)]}{[1 - C_r(u_0)]} B(t).$$  (3.24)

Applying Gronwall inequality to (3.24), and using (3.6), we obtain

$$B(t) \leq \frac{4lE(u_0)[1 + C_r(u_0)] + C(u_0)[1 - C_r(u_0)]}{l[1 - C_r(u_0)]} \left[ \frac{1 + C_r(u_0)}{1 - C_r(u_0)} \right],$$

and

$$\int_{\mathbb{R}^N} \nabla h(|u|^2)^2 \, dx + |V(x)||u|^2 + \frac{1}{2}(|W| \ast |u|^2)|u|^2 \, dx$$

$$\leq \frac{C(u_0)}{4[1 - C_r(u_0)]t^2} + \frac{4lE(u_0)[1 + C_r(u_0)] + C(u_0)[1 - C_r(u_0)]}{4[1 - C_r(u_0)]t^2 - \frac{[1 + C_r(u_0)]}{1 - C_r(u_0)}}.$$  (3.25)

for $t \geq 1$. Consequently,

$$\int_{\mathbb{R}^N} \Phi(u) \, dx \leq \frac{[1 + C_r(u_0)]}{4[1 - C_r(u_0)]} \left( \frac{C(u_0)}{t^2} + \frac{4lE(u_0)[1 + C_r(u_0)] + C(u_0)[1 - C_r(u_0)]}{[1 - C_r(u_0)]t^2 - \frac{[1 + C_r(u_0)]}{1 - C_r(u_0)}} \right).$$  (3.26)

for any $t \geq 1$. 

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Similar to (3.24), in combined defocusing and focusing case,

\[
\int_0^\infty \int_{\mathbb{R}^N} \frac{t^2 \Phi(u)}{(c(x) + t)^k} dx dt \leq \frac{2E(u_0)[1 + C_r(u_0)]}{[1 - C_r(u_0)]} \int_0^1 t^{2-k} dt + \int_1^\infty \frac{4E(u_0)[1 + C_r(u_0)]^2 + C(u_0)[1 - C_r^2(u_0)]}{4[1 - C_r(u_0)]^2} \int_0^1 t^{2-k} dt \\
= \frac{1 + C_r(u_0)}{1 - C_r(u_0)} \left( \frac{2E(u_0)}{(3-k)} + \frac{C(u_0)}{4(k-1)} + \frac{4E(u_0)[1 + C_r(u_0)] + C(u_0)[1 - C_r(u_0)]}{4((k-1)[1 - C_r(u_0)] - l[1 + C_r(u_0)])} \right)
\]

(3.27)

for \( c(x) \geq 0, k > 1 + \frac{l[1+C_r(u_0)]}{1-C_r(u_0)} \). Combining (3.21) and (3.27), we have

\[
\int_0^\infty \int_{\mathbb{R}^N} \frac{t^2 \Phi(u)}{(c(x) + t)^k} dx dt \leq M_4(u_0, k, l).
\]

(3.28)

Here

\[
M_4(u_0, k, l) = \begin{cases} \\
\frac{2E(u_0)}{3-k} + \frac{C(u_0)}{4(k-1)} + \frac{4E(u_0)+C(u_0)}{4[l-1]} & \text{in defocusing case} \\
\frac{1 + C_r(u_0)}{1 - C_r(u_0)} \left( \frac{2E(u_0)}{(3-k)} + \frac{C(u_0)}{4(k-1)} + \frac{4E(u_0)[1 + C_r(u_0)] + C(u_0)[1 - C_r(u_0)]}{4((k-1)[1 - C_r(u_0)] - l[1 + C_r(u_0)])} \right) & \text{in combined defocusing and focusing case} \\
\end{cases}
\]

(3.29)

if \( c(x) \geq 0 \).

Similarly to (3.22), (3.27) in combined defocusing and focusing case,

\[
\int_0^\infty \int_{\mathbb{R}^N} \frac{t^2 \Phi(u)}{(c(x) + t)^k} dx dt \leq \frac{[1 + C_r(u_0)]}{[1 - C_r(u_0)]} \left( \frac{2E(u_0)}{3k} + \frac{C(u_0)}{4(k-1)} \\
+ \frac{4E(u_0)[1 + C_r(u_0)] + C(u_0)[1 - C_r(u_0)]}{4((k-1)[1 - C_r(u_0)] - l[1 + C_r(u_0)])} \right)
\]

(3.30)

for \( c(x) \geq c > 0, k > 1 + \frac{l[1+C_r(u_0)]}{1-C_r(u_0)} \). Combining (3.22) and (3.30), we get

\[
\int_0^\infty \int_{\mathbb{R}^N} \frac{t^2 \Phi(u)}{(c(x) + t)^k} dx dt \leq M_4(u_0, k, l).
\]

(3.31)

Here

\[
M_4(u_0, k, l) = \begin{cases} \\
\frac{2E(u_0)}{3k} + \frac{4E(u_0)+C(u_0)}{4[l-1]} & \text{in defocusing case} \\
\frac{1 + C_r(u_0)}{1 - C_r(u_0)} \left( \frac{2E(u_0)}{3k} + \frac{C(u_0)}{4(k-1)} + \frac{4E(u_0)[1 + C_r(u_0)] + C(u_0)[1 - C_r(u_0)]}{4((k-1)[1 - C_r(u_0)] - l[1 + C_r(u_0)])} \right) & \text{in combined defocusing and focusing case} \\
\end{cases}
\]

(3.32)

if \( c(x) \geq c > 0 \).

**Estimate (E):**

Especially, if \( c(x) \equiv 0, k = 2, l < \frac{1-C_r(u_0)}{1+C_r(u_0)} \), by the discussions above, we have

\[
\int_0^\infty \int_{\mathbb{R}^N} \Phi(u)dx dt \leq M_5(u_0, l).
\]

(3.33)
Here

\[
M_5(u_0, l) = \begin{cases} 
2E(u_0) + \frac{C(u_0)}{4} + \frac{4E(u_0) + C(u_0)}{4l(1-l)} & \text{in defocusing case} \\
\frac{[1+C_r(u_0)]}{[1-C_r(u_0)]} \left(2E(u_0) + \frac{C(u_0)}{4} + \frac{4E(u_0)[1+C_r(u_0)] + C(u_0)[1-C_r(u_0)]}{4[1-C_r(u_0)] - l[1+C_r(u_0)]} \right) & \text{in combined defocusing and focusing case}
\end{cases}
\]

(3.34)

**Remark 3.1.** We use (1.21) as a typical example to illustrate the results of Theorem 2. Since

\[
2h''(s)h'(s)s + (h'(s))^2 = (2\alpha - 1)\alpha^2 s^{2\alpha - 2},
\]

\[
NF(s)s - (N + 2)G(s) = \mp[N - \frac{N + 2}{\beta + 1}]s^{\beta + 1},
\]

\[
2V + (x \cdot \nabla V) = \frac{m - 2}{|x|^m}, \quad 2W + (x \cdot \nabla W) = \frac{n - 2}{|x|^n},
\]

\[
V(x) \in L^{p_1}(\mathbb{R}^N) + L^{\infty}(\mathbb{R}^N), \quad p_1 > \frac{N}{2}, \quad p_1 > 1 \quad \text{and} \quad W(x) \in L^{p_2}(\mathbb{R}^N) + L^{\infty}(\mathbb{R}^N), \quad p_2 > \frac{N}{4}, \quad p_2 > 1,
\]

we have to require that \(0 < m < 2, 0 < n < 4\) if \(N \geq 3, 0 < m < 1, 0 < n < 4\) if \(N = 1, 2\). That is, the assumptions of Case 1 cannot be satisfied, yet at least one of the assumptions of Case 2 can be satisfied, and

\[
k_1 = 2\alpha - 1, \quad k_2 = \frac{|N\beta - 2|}{\beta + 1}, \quad k_3 = 2 - m, \quad k_4 = |n - 2|,
\]

(1.22) holds for suitable \(\alpha, \beta, m, n\).

By energy conservation law, by (3.3), (3.9), (3.20) and (3.26), we have

**Corollary 3.1.** Let \(u(x, t)\) be the solution of (1.1). Under the assumptions of Theorem 2, we have

\[
\lim_{t \to +\infty} \int_{\mathbb{R}^N} |\nabla h(|u|^2)|^2 + |V(x)|u|^2 + |G(|u|^2)| + \frac{1}{2}(W * |u|^2)|u|^2 dx = 0,
\]

(3.35)

\[
\lim_{t \to +\infty} \int_{\mathbb{R}^N} |\nabla u|^2 dx = 2E(u_0).
\]

(3.36)

4 Spacetime bound estimates based on pseudoconformal conservation law

In this section, we prove Theorem 3 and establish spacetime bound estimates based on pseudoconformal conservation law.
Proof of Theorem 3: Note that
\[
\int_{\mathbb{R}^N} |G(|u|^2)|^r dx \leq \left( \int_{\mathbb{R}^N} |G(|u|^2)|^\theta dx \right)^{\frac{1}{\theta}} \left( \int_{\mathbb{R}^N} |G(|u|^2)|^\gamma dx \right)^{\frac{1}{\gamma}}
\]
\[
\leq \left( c_3 \int_{\mathbb{R}^N} |u|^2 dx \right)^{\frac{1}{\tau_1}} \left( c_4 \int_{\mathbb{R}^N} |h(|u|^2)|^{2^*} dx \right)^{\frac{1}{\gamma}}
\]
\[
\leq \left( c_3 \|u_0\|^2 \right)^{\frac{1}{\tau_1}} \left( c_4 C_s \left( \int_{\mathbb{R}^N} |\nabla h(|u|^2)|^2 dx \right)^{\frac{2^*}{2}} \right)^{\frac{1}{\gamma}}
\] (4.1)
if \( N \geq 3 \). Here
\[
\frac{1}{\tau_1} = \frac{\gamma - r}{\gamma - \theta}, \quad \frac{1}{\tau_2} = \frac{r - \theta}{\gamma - \theta}, \quad r \geq 1.
\] (4.2)

By (1.17), (1.18), and taking \( r = 1 \) in (4.1), we have
\[
\int_{\mathbb{R}^N} |G(|u|^2)| dx \leq C'_r(u_0) \int_{\mathbb{R}^N} |\nabla h(|u|^2)|^2 dx.
\] (4.3)

We prove (1.19) in three subcase.

Subcase (1). Case 1 of Theorem 2. Similar to (3.2), (3.3), (3.7) and (3.9), we have
\[
\int_{\mathbb{R}^N} \Phi(u) dx \leq C \quad \text{for} \quad 0 \leq t \leq 1, \quad \int_{\mathbb{R}^N} \Phi(u) dx \leq \frac{C'}{t^2} \quad \text{for} \quad t \geq 1.
\]
Consequently,
\[
\left( \int_0^{+\infty} \left( \int_{\mathbb{R}^N} \Phi(u) dx \right)^p dt \right)^{\frac{1}{p}} \leq \left( \int_0^1 C^p dt + \int_{1}^{+\infty} \frac{C'^p}{t^{2p}} dt \right)^{\frac{1}{p}}
\]
\[
\leq \tilde{c}_1 \left( \int_0^1 C^p dt \right)^{\frac{1}{p}} + \tilde{c}_1 \left( \int_{1}^{+\infty} \frac{C'^p}{t^{2p}} dt \right)^{\frac{1}{p}} \leq \tilde{c}_1 C + \frac{\tilde{c}_1 C'}{(2p - 1)^{\frac{1}{p}}} := C(u_0, p).
\] (4.4)

Here \( \tilde{c}_1 = 1 \) if \( p > 1 \), \( \tilde{c}_1 = 2 \frac{1-p}{p} \) if \( \frac{1}{2} < p \leq 1 \), and
\[
C(u_0, p) = \begin{cases} 
(2E(u_0) + \frac{C(u_0)}{4}) \tilde{c}_1 & \text{in defocusing case} \\
\tilde{c}_1 [1 + C'_r(u_0)] \left( 2E(u_0) + \frac{C(u_0)}{4(2p-1)p} \right) & \text{in combined \( 2^\text{nd} \) case} 
\end{cases}
\] (4.5)

Subcase (2). defocusing subcase in Case 2 of Theorem 2. By (3.2) and (3.20),
we have
\[
\left( \int_0^{\infty} \left( \int_{\mathbb{R}^N} \Phi(u) dx \right)^p dt \right)^{\frac{1}{p}} \\
\leq \left( \int_0^1 \left[ 2E(u_0) \right]^p dt + \int_1^{\infty} \left( \frac{C(u_0)}{4t^2} + \frac{4lE(u_0) + C(u_0)}{4t(2-l)} \right)^p dt \right)^{\frac{1}{p}} \\
\leq \tilde{c}_1 \left( \int_0^1 \left[ 2E(u_0) \right]^p dt \right)^{\frac{1}{p}} + \tilde{c}_1 \left( \int_1^{\infty} \left( \frac{C(u_0)}{4t^2} + \frac{4lE(u_0) + C(u_0)}{4t(2-l)} \right)^p dt \right)^{\frac{1}{p}} \\
\leq 2E(u_0)\tilde{c}_1 + \frac{\tilde{c}_1^2\tilde{c}_2C(u_0)}{4} \left( \int_1^{\infty} \frac{1}{2t^2p} dt \right)^{\frac{1}{p}} + \frac{\tilde{c}_1^2\tilde{c}_2[4lE(u_0) + C(u_0)]}{4} \left( \int_1^{\infty} \frac{1}{t(2-l)p} dt \right)^{\frac{1}{p}} \\
\leq 2E(u_0)\tilde{c}_1 + \frac{\tilde{c}_1^2\tilde{c}_2C(u_0)}{4[2p - 1]^\frac{1}{p}} + \frac{\tilde{c}_1^2\tilde{c}_2[4lE(u_0) + C(u_0)]}{4[(2-l)p - 1]^\frac{1}{p}}. \\
\tag{4.6}
\]

Here \( \tilde{c}_2 = 1 \) if \( p < 1 \), \( \tilde{c}_2 = 2^\frac{1}{p} \) if \( p \geq 1 \).

**Subcase (3).** combined defocusing and focusing subcase in Case 2 of Theorem 2.

Similar to (3.9) and (3.26), we have
\[
\int_{\mathbb{R}^N} \Phi(u) dx \leq \frac{2E(u_0)[1 + C'_r(u_0)]}{1 - C'_r(u_0)}
\]
for \( 0 \leq t \leq 1 \) and
\[
\int_{\mathbb{R}^N} \Phi(u) dx \leq \frac{\left[ 1 + C'_r(u_0) \right]}{4[1 - C'_r(u_0)]} \left( \frac{C(u_0)}{t^2} + \frac{(1 + C_r(u_0)) + C(u_0)[1 - C'_r(u_0)]}{[1 - C'_r(u_0)]^2 - \frac{(1 + C'_r(u_0))}{1 - C'_r(u_0)}} \right)
\]
for \( t \geq 1 \). Consequently,
\[
\left( \int_0^{\infty} \left( \int_{\mathbb{R}^N} \Phi(u) dx \right)^p dt \right)^{\frac{1}{p}} \\
\leq \tilde{c}_1 \left( \int_0^1 \left( \frac{2E(u_0)[1 + C'_r(u_0)]}{1 - C'_r(u_0)} \right)^p dt \right)^{\frac{1}{p}} + \tilde{c}_1 \left( \int_1^{\infty} \left( \frac{\tilde{C}_1}{t^2} + \frac{\tilde{C}_2}{t^2 - \frac{(1 + C'_r(u_0))}{1 - C'_r(u_0)}} \right)^p dt \right)^{\frac{1}{p}} \\
\leq 2E(u_0)[1 + C'_r(u_0)]\frac{\tilde{c}_1}{1 - C'_r(u_0)} + \frac{\tilde{c}_1^2\tilde{c}_2\tilde{C}_1}{(2p - 1)^\frac{1}{p}} + \frac{\tilde{c}_1^2\tilde{c}_2\tilde{C}_2\tilde{C}_3}{(2p - 1)^\frac{1}{p}}, \\
\tag{4.8}
\]
where
\[
\tilde{C}_1 = \frac{C(u_0)[1 + C'_r(u_0)]}{4[1 - C'_r(u_0)]}; \\
\tilde{C}_2 = \frac{4lE(u_0)[1 + C'_r(u_0)]^2 + C(u_0)[1 - C'_r(u_0)]}{4[1 - C'_r(u_0)]^2}; \\
\tilde{C}_3 = \frac{[1 - C'_r(u_0)]}{(2[1 - C'_r(u_0)] - l[1 + C'_r(u_0)])p - [1 - C'_r(u_0)]}.
\]
Note that the facts
\[
\int_{\mathbb{R}^N} |\nabla h(|u|^2)|^2 \, dx \leq \int_{\mathbb{R}^N} \Phi(u) \, dx,
\]
\[
\int_{\mathbb{R}^N} |\nabla h(|u|^2)|^2 \, dx \leq \int_{\mathbb{R}^N} |\nabla h(|u|^2)|^2 + |V(x)||u|^2 + \frac{1}{2}(W * |u|^2)|u|^2 \, dx
\]
and
\[
\left( \int_0^\infty \left( \int_{\mathbb{R}^N} |G(|u|^2)|^r \, dx \right)^{\frac{2}{r'}} \, dt \right)^{\frac{r}{r'}} \leq C(u_0, r, \vartheta, \gamma) \left( \int_0^\infty \left( \int_{\mathbb{R}^N} |\nabla h(|u|^2)|^2 \, dx \right)^{\frac{2^*}{2^* - r_{\tau_2}}} \, dt \right)^{\frac{1}{q}},
\] (4.9)

where
\[
C(u_0, r, \vartheta, \gamma) = (c_3\|u_0\|_{L^2})^{\frac{1}{r_{\tau_2}}} (c_4C_s)^{\frac{1}{r_{\tau_2}}}.
\] (4.10)

We divide four subcases to prove (1.20).

Subcase (4). defocusing subcase in Case 1 of Theorem 2. By (3.2), (3.3) and (4.9), we get
\[
\left( \int_0^\infty \left( \int_{\mathbb{R}^N} |G(|u|^2)|^r \, dx \right)^{\frac{2}{r'}} \, dt \right)^{\frac{r}{r'}} \leq C(u_0, r, \vartheta, \gamma) \left( \int_0^\infty \left( \int_{\mathbb{R}^N} |\nabla h(|u|^2)|^2 \, dx \right)^{\frac{2^*}{2^* - r_{\tau_2}}} \, dt \right)^{\frac{1}{q}}
\]
\[
= C(u_0, r, \vartheta, \gamma) \bar{c}_3 \left( [2E(u_0)]^{\frac{2^*}{2^* - r_{\tau_2}}} + \left( \frac{C(u_0)}{4} \right)^{\frac{2^*}{2^* - r_{\tau_2}}} \left( \frac{r_{\tau_2}}{2^* q - r_{\tau_2}} \right)^{\frac{1}{q}} \right).
\] (4.11)

Here \( \bar{c}_3 = 1 \) if \( q > 1 \), \( \bar{c}_3 = 2^{1-q} \) if \( q \leq 1 \).

Subcase (5). combined defocusing and focusing subcase in Case 1 of Theorem 2.

Similar to (3.6) and (3.8), we have
\[
\int_{\mathbb{R}^N} |\nabla h(|u|^2)|^2 + |V(x)||u|^2 + \frac{1}{2}(W * |u|^2)|u|^2 \, dx \leq \frac{2E(u_0)}{1 - C'_r(u_0)}
\] (4.12)
for any \( t > 0 \)(especially for \( 0 < t \leq 1 \)) and
\[
\int_{\mathbb{R}^N} |\nabla h(|u|^2)|^2 + |V(x)||u|^2 + \frac{1}{2}(W * |u|^2)|u|^2 \, dx \leq \frac{C(u_0)}{4(1 - C'_r(u_0)) t^2}
\] (4.13)
for \( t \geq 1 \).
By (4.9), we get
\[
\left(\int_0^\infty \left(\int_{\mathbb{R}^N} |G(|u|^2)|^\gamma dx\right)^{\frac{2}{\gamma}} dt\right)^{\frac{1}{\gamma}}
\]
\[
\leq C(u_0, r, \vartheta, \gamma) \left( \int_0^1 \left\{ \frac{2E(u_0)}{1 - C_r''(u_0)} \right\}^{\frac{2q}{2r\tau_2}} dt + \int_1^{+\infty} \left( \frac{C(u_0)}{4(1 - C_r''(u_0))^{\frac{1}{2}}} \right)^{\frac{2q}{2r\tau_2}} dt \right)^{\frac{1}{\gamma}}
\]
\[
= C(u_0, r, \vartheta, \gamma) \tilde{c}_3 \left( \left\{ \frac{2E(u_0)}{1 - C_r''(u_0)} \right\}^{\frac{2q}{2r\tau_2}} + \left( \frac{C(u_0)}{4(1 - C_r''(u_0))} \right)^{\frac{2q}{2r\tau_2}} \left( \frac{r\tau_2}{2^*q - r\tau_2} \right)^{\frac{1}{\gamma}} \right)^{\frac{1}{\gamma}}. \quad (4.14)
\]

**Subcase (6).** Defocusing subcase in Case 2 of Theorem 2. By (3.2), (3.20) and (4.9), we get
\[
\left(\int_0^\infty \left(\int_{\mathbb{R}^N} |G(|u|^2)|^\gamma dx\right)^{\frac{2}{\gamma}} dt\right)^{\frac{1}{\gamma}}
\]
\[
\leq C(u_0, r, \vartheta, \gamma) \left( \int_0^1 \left\{ \frac{2E(u_0)}{1 - C_r''(u_0)} \right\}^{\frac{2q}{2r\tau_2}} dt + \int_1^{+\infty} \left( \frac{C(u_0)}{4t^2} + \frac{4E(u_0) + C(u_0)}{4t^{2-l}} \right)^{\frac{2q}{2r\tau_2}} dt \right)^{\frac{1}{\gamma}}
\]
\[
\leq C(u_0, r, \vartheta, \gamma) \tilde{c}_3 \left( \left\{ \frac{2E(u_0)}{1 - C_r''(u_0)} \right\}^{\frac{2q}{2r\tau_2}} + \tilde{c}_3 \left( \tilde{C} \int_1^{+\infty} \left( \frac{C(u_0)}{4t^2} \right)^{\frac{2q}{2r\tau_2}} dt \right)^{\frac{1}{\gamma}} \right) + C(u_0, r, \vartheta, \gamma) \tilde{c}_3^2 \left( \int_1^{+\infty} \tilde{C} \left( \frac{4E(u_0) + C(u_0)}{4t^{2-l}} \right)^{\frac{2q}{2r\tau_2}} dt \right)^{\frac{1}{\gamma}}
\]
\[
= C(u_0, r, \vartheta, \gamma) \tilde{c}_3 \left( \frac{2E(u_0)}{1 - C_r''(u_0)} + \tilde{c}_3 \tilde{C} \left( \frac{C(u_0)}{4} \right)^{\frac{2q}{2r\tau_2}} \left( \frac{r\tau_2}{2^*q - r\tau_2} \right)^{\frac{1}{\gamma}} \right) + C(u_0, r, \vartheta, \gamma) \tilde{c}_3^2 \tilde{C} \left( \frac{4E(u_0) + C(u_0)}{4} \right)^{\frac{2q}{2r\tau_2}} \left( \frac{2r\tau_2}{(2-l)2^*q - 2r\tau_2} \right)^{\frac{1}{\gamma}} \quad (4.15)
\]

Here \( \tilde{C} = 1 \) if \( \frac{2q}{2r\tau_2} \leq 1 \) and \( \tilde{C} = \frac{2^{2q}}{2r\tau_2} \) if \( \frac{2^q}{2r\tau_2} > 1 \).

**Subcase (7).** Combined defocusing and focusing subcase in Case 2 of Theorem 2. Similar to (3.25), we have
\[
\int_{\mathbb{R}^N} \left| \nabla h(|u|^2) \right|^2 + |V(x)||u|^2 + \frac{1}{2}(W * |u|^2)|u|^2 dx
\]
\[
\leq \frac{C(u_0)}{4(1 - C_r''(u_0))^{\frac{1}{2}}} + \left( \frac{4E(u_0)[1 + C_r'(u_0)] + C(u_0)[1 - C_r'(u_0)]}{4(1 - C_r''(u_0))^{\frac{1}{2}}} \right)^{\frac{1}{2}} \frac{1}{t^{2 - \frac{1}{4}(1 + C_r'(u_0))}}. \quad (4.16)
\]
for $t \geq 1$.

By (4.9), (4.12) and (4.16), we obtain

$$
\left( \int_0^\infty \left( \int_{\mathbb{R}^N} |G(|u|^2)|^\frac{q}{2} \, dx \right)^\frac{1}{q} \, dt \right) \leq C(u_0, r, \vartheta, \gamma) \tilde{c}_3 \left\{ \left( \int_0^1 \left[ \frac{2E(u_0)}{1 - C'_r(u_0)} \right] \frac{q}{2} \, dt \right)^\frac{1}{q} + 
\int_1^{+\infty} \left( \frac{C(u_0)}{4(1 - C'_r(u_0))^2} + \frac{4E(u_0)[1 + C'_r(u_0)] + C(u_0)[1 - C'_r(u_0)]}{4(1 - C'_r(u_0))^2} \right) \frac{q}{2} \, dt \right\} \frac{1}{q} \right\}
$$

$$
\leq C(u_0, r, \vartheta, \gamma) \tilde{c}_3 \left\{ \left[ \frac{2E(u_0)}{1 - C'_r(u_0)} \right] \frac{q}{2} + \tilde{C}_3 \tilde{c}_3 \left( \frac{C(u_0)}{4(1 - C'_r(u_0))^2} \right) \frac{q}{2} \left( \int_1^{+\infty} \frac{1}{t^{2 - \frac{q}{2} + \frac{|1 + C'_r(u_0)|}{1 - C'_r(u_0)}}} \, dt \right) \right\} \frac{1}{q} \right\} 
$$

$$
= C(u_0, r, \vartheta, \gamma) \tilde{c}_3 \left\{ \left[ \frac{2E(u_0)}{1 - C'_r(u_0)} \right] \frac{q}{2} + \tilde{C}_3 \tilde{c}_3 \left( \frac{C(u_0)}{4(1 - C'_r(u_0))^2} \right) \frac{q}{2} \left( \int_1^{+\infty} \frac{1}{t^{2 - \frac{q}{2} + \frac{|1 + C'_r(u_0)|}{1 - C'_r(u_0)}}} \, dt \right) \right\} \frac{1}{q} \right\} 
$$

$$
+ C(u_0, r, \vartheta, \gamma) \tilde{C}_3 \tilde{c}_3 \left( \frac{4E(u_0)[1 + C'_r(u_0)] + C(u_0)[1 - C'_r(u_0)]}{4(1 - C'_r(u_0))^2} \right) \frac{q}{2} \left( \int_1^{+\infty} \frac{1}{t^{2 - \frac{q}{2} + \frac{|1 + C'_r(u_0)|}{1 - C'_r(u_0)}}} \, dt \right) \right\} \frac{1}{q} \right\} 
$$

$$
\times \left( \frac{2r_2[1 - C'_r(u_0)]}{(22^q - 2r_2)[1 - C'_r(u_0)] - 2^q \gamma [1 + C'_r(u_0)]} \right)^\frac{1}{q} \right). \quad (4.17)
$$

As a corollary of Theorem 2 and Theorem 3, we give the time-space bounds for (1.21) below.

**Corollary 4.1.** Let $u(x, t)$ be the global solution of (1.21) in energy space $X$, $u_0 \in X$ and $u_0 \in L^2(\mathbb{R}^N)$. In addition, suppose that $0 < \beta = 2\alpha - 1 + \frac{2}{N}$ in combined defocusing and focusing case. Then

\[ (A). \quad \int_0^\infty \int_{\mathbb{R}^N} \left[ |\nabla |u|^{2\alpha}|^2 + \frac{|u|^{2\beta + 2}}{\beta + 1} + \frac{1}{|x|^m}|u|^2 + \frac{1}{2} \left( \frac{1}{|x|^n} \ast |u|^2 \right)|u|^2 \right] \, dx \, dt \leq M(u_0, \alpha, \beta, m, n). \quad (4.18) \]

\[ (B). \quad \int_0^\infty \left( \int_{\mathbb{R}^N} \left[ |\nabla |u|^{2\alpha}|^2 + \frac{|u|^{2\beta + 2}}{\beta + 1} + \frac{1}{|x|^m}|u|^2 + \frac{1}{2} \left( \frac{1}{|x|^n} \ast |u|^2 \right)|u|^2 \right] \, dx \right)^p \, dt \leq C(u_0, p, \alpha, \beta, m, n), \quad (4.19) \]

\[ (C). \quad \|u\|_{L^q(\mathbb{R}^+)} L^r(\mathbb{R}^N) = \left( \int_0^\infty \left( \int_{\mathbb{R}^N} |u|^r \, dx \right)^\frac{q}{r} \right)^\frac{1}{q} \leq C(u_0, q, r, \alpha, \beta, m, n). \quad (4.20) \]
Here $\tilde{r} = (2\beta + 2)r$, $q$ and $r$ are taken the same values in (1.19).

**Proof:** Note that $h(s) = s^\alpha$, $|G(s)| = s^{\beta + 1}/\beta + 1$, $V(x) = -\frac{1}{|x|^m}$ and $W(x) = -\frac{1}{|x|^N}$.

1. The assumptions on $h(s)$, $G(s)$, $V(x)$ and $W(x)$ in Case 1 of Theorem 2 imply that
   \[ \alpha \geq \frac{1}{2}, \quad m \geq 2, \quad n \geq 2, \]
   and $\beta \geq \frac{2}{N}$ in defocusing case, $\beta \leq \frac{2}{N}$ in combined defocusing and focusing case.

2. The assumptions on $h(s)$, $G(s)$, $V(x)$ and $W(x)$ in Case 2 of Theorem 2 imply that
   \[ \alpha \leq \frac{1}{2}, \quad m \leq 2, \quad n \leq 2, \]
   and $\beta \leq \frac{2}{N}$ in defocusing case, $\beta \geq \frac{2}{N}$ in combined defocusing and focusing case. And
   \[ k_1 = 1 - 2\alpha, \quad k_3 = 2 - m, \quad k_4 = 2 - n, \]
   and $k_2 = \frac{2-N\beta}{\beta+1}$ in defocusing case, $k_2 = \frac{N\beta-2}{\beta+1}$ in combined defocusing and focusing case. And
   \[ l = \max(1-2\alpha, \frac{|N\beta-2|}{\beta+1}, 2-m, 2-n). \]

3. The assumptions $h(s)$, $G(s)$, $V(x)$ and $W(x)$ in combined defocusing and focusing case of Theorem 3, (1.17) and (1.18) imply that
   \[ \beta = 2\alpha - 1 + \frac{2}{N}, \quad \theta = \frac{1}{\beta+1}, \quad \gamma = \frac{\alpha}{\beta+1}, \quad c_3 = \left(\frac{1}{\beta+1}\right)^{\frac{1}{\beta+1}}, \]
   \[ c_4 = \left(\frac{1}{\beta+1}\right)^{\frac{2\alpha}{\beta+1}}, \quad C'_r(u_0) = \left(\frac{1}{\beta+1}\right)^{\frac{2\alpha+2}{\beta+1}}\|u_0\|_{L^2_{\beta+1}} \frac{4}{N} \frac{1}{C_s^{\frac{N-2}{N}}}. \]

(1.18) and (1.19) are the direct results of Theorem 2 and Theorem 3.

Since $\tilde{r} = (2\beta + 2)r$, $q$ and $r$ are taken the same values in (1.19), we have $2 < \tilde{r} < 2\alpha \cdot 2^\ast$ and
\[ |u|^{\tilde{r}} = \left(|u|^{2\beta+2}\right)^{\frac{\tilde{r}}{2\beta+2}} = \left(|u|^{2\beta+2}\right)^{\beta+1} = (\beta + 1)^r [G(|u|^2)]^r. \]
Now (4.20) can be deduced by (4.22) directly. \hfill $\Box$

## 5 Interaction Morawetz inequality

Although we don’t consider the local wellposedness result in $H^{1/2}(\mathbb{R}^N)$ in this paper, we can give some priori estimates for the $H^{1/2}$ solution of (1.1). In fact, since $H^{1/2}(\mathbb{R}^N) \cap X \neq \emptyset$, we at least can establish interaction Morawetz estimates for the solution belonging to $H^{1/2} \cap X$. Inspired by [8, 9, 10, 28], we divide into three subsections to discuss it.

We assume that
\[ G(s) - F(s)s \geq 0, \quad 2|h'(s)|^2 + h''(s)h'(s)s \geq 0 \quad \text{for all } s \geq 0, \quad (5.1) \]
\[ x \cdot \nabla V(x) \geq 0, \quad x \cdot \nabla W(x) \geq 0 \quad \text{for all } x \in \mathbb{R}^N, \quad N \geq 2. \quad (5.2) \]
5.1 Interaction Morawetz inequality in Dimension $N \geq 3$

Let

$$M_a^\otimes(t) = 2 \int_{\mathbb{R}^N \otimes \mathbb{R}^N} \nabla a(x-y) \cdot \Re \left( (u(t,x)u(t,y))\nabla (\bar{u}(t,x)\bar{u}(t,y)) \right) dx dy,$$

where $\nabla = (\partial_x, \partial_y)$ and

$$\tilde{H}(|u|^2) = \int_0^{[u]^2} [h'(s)]^2 sds.$$  \hspace{1cm} (5.4)

Then

$$\frac{d}{dt} M_a^\otimes(t) = - \int_{\mathbb{R}^N \otimes \mathbb{R}^N} \left[ 4 \tilde{H}(|u(t,x)|^2) + |u(t,x)|^2 \Delta_x (\Delta_x a(x,y)) dx |u(t,y)|^2 dy ight]$$

$$+ 4 \int_{\mathbb{R}^N \otimes \mathbb{R}^N} \sum_{j,k=1}^N \Re(\partial_{x_j} u(t,x) \partial_{x_j} \bar{u}(t,x))\partial_{x_k} \partial_{x_k} a(x,y) dx |u(t,y)|^2 dy$$

$$+ 4 \int_{\mathbb{R}^N \otimes \mathbb{R}^N} \sum_{j,k=1}^N \partial_{x_k} h(|u(t,x)|^2)\partial_{x_j} h(|u(t,x)|^2)\partial_{x_k} \partial_{x_j} a(x,y) dx |u(t,y)|^2 dy$$

$$+ \int_{\mathbb{R}^N \otimes \mathbb{R}^N} h'(|u|^2)[2h'(|u|^2) + h''(|u|^2)|u|^2]\nabla_x (|u(t,x)|^2)\Delta_x a(x,y) dx |u(t,y)|^2 dy$$

$$+ 2 \int_{\mathbb{R}^N \otimes \mathbb{R}^N} [G(|u(t,x)|^2) - F(|u(t,x)|^2)|u(t,x)|^2] \Delta_x a(x,y) dy |u(t,y)|^2 dx$$

$$+ 2 \int_{\mathbb{R}^N \otimes \mathbb{R}^N} (\nabla_x V(x) \cdot \nabla_x a(x,y))|u(t,x)|^2 dx |u(t,y)|^2 dy$$

$$+ \int_{\mathbb{R}^N \otimes \mathbb{R}^N} [(\nabla_x W(x) \cdot \nabla_x a(x,y)) * |u(t,x)|^2]|u(t,x)|^2 dx |u(t,y)|^2 dy$$

$$- \int_{\mathbb{R}^N \otimes \mathbb{R}^N} \left[ 4 \tilde{H}(|u(t,y)|^2) + |u(t,y)|^2 \Delta_y (\Delta_y a(x,y)) dy |u(t,x)|^2 dx ight]$$

$$+ 4 \int_{\mathbb{R}^N \otimes \mathbb{R}^N} \sum_{j,k=1}^N \Re(\partial_{y_j} u(t,y) \partial_{y_j} \bar{u}(t,y))\partial_{y_k} \partial_{y_k} a(x,y) dy |u(t,x)|^2 dx$$

$$+ 4 \int_{\mathbb{R}^N \otimes \mathbb{R}^N} \sum_{j,k=1}^N \partial_{y_k} h(|u(t,y)|^2)\partial_{y_j} h(|u(t,y)|^2)\partial_{y_k} \partial_{y_j} a(x,y) dy |u(t,x)|^2 dx$$

$$+ \int_{\mathbb{R}^N \otimes \mathbb{R}^N} h'(|u|^2)[2h'(|u|^2) + h''(|u|^2)|u|^2]\nabla_y (|u(t,y)|^2)\Delta_y a(x,y) dy |u(t,x)|^2 dx$$

$$+ 2 \int_{\mathbb{R}^N \otimes \mathbb{R}^N} [G(|u(t,y)|^2) - F(|u(t,y)|^2)|u(t,y)|^2] \Delta_y a(x-y) dy |u(t,x)|^2 dx$$

$$+ 2 \int_{\mathbb{R}^N \otimes \mathbb{R}^N} (\nabla_y V(y) \cdot \nabla_y a(x,y))|u(t,y)|^2 dy |u(t,x)|^2 dx$$

$$+ \int_{\mathbb{R}^N \otimes \mathbb{R}^N} [(\nabla_y W(y) \cdot \nabla_y a(x,y)) * |u(t,y)|^2]|u(t,y)|^2 dy |u(t,x)|^2 dx.$$  \hspace{1cm} (5.5)
If $a(x, y) = |x - y|$, then $a(x, y)$ is convex with respect to both $x$ and $y$, and

$$-\Delta_x \Delta_x a(x, y) = -\Delta_y \Delta_y a(x, y) \begin{cases} 8\pi \delta(x - y) & \text{if } N = 3, \\ \frac{(N-1)(N-3)}{|x-y|^3} & \text{if } N \geq 4. \end{cases} \quad (5.6)$$

Under the assumptions on $h(s)$, $F(s)$, $V(x)$ and $W(x)$, we get

$$\frac{d}{dt} M_a^{\otimes 2}(t) \geq - \int_{\mathbb{R}^N \otimes \mathbb{R}^N} [4 \tilde{H}(|u(t, x)|^2) + |u(t, x)|^2] \Delta_x a(x, y) dx |u(t, y)|^2 dy$$

$$- \int_{\mathbb{R}^N \otimes \mathbb{R}^N} [4 \tilde{H}(|u(t, y)|^2) + |u(t, y)|^2] \Delta_y a(x, y) dy |u(t, x)|^2 dx. \quad (5.7)$$

Therefore, if $N = 3$, by the property of the function $\delta(z)$, we have

$$\int_0^T \int_{\mathbb{R}^3} |u(t, x)|^4 + \tilde{H}(|u(t, x)|^2) |u(t, x)|^2 dx dt \leq C \sup_{t \in [0, T]} |M_a^{\otimes 2}(t)|. \quad (5.8)$$

If $N \geq 4$, we obtain

$$\int_0^T \int_{\mathbb{R}^N \otimes \mathbb{R}^N} \frac{|u(t, x)|^2 |u(t, y)|^2}{|x-y|^3} + \frac{\sqrt{|u(t, x)|^2} \tilde{H}(|u(t, x)|^2)}{|x-y|^3} \frac{\sqrt{|u(t, y)|^2} \tilde{H}(|u(t, y)|^2)}{|x-y|^3} dx dy dt$$

$$\leq \int_0^T \int_{\mathbb{R}^N \otimes \mathbb{R}^N} \frac{|u(t, x)|^2 |u(t, y)|^2}{|x-y|^3} + \frac{|u(t, x)|^2 \tilde{H}(|u(t, x)|^2)}{|x-y|^3} \frac{|u(t, y)|^2 \tilde{H}(|u(t, y)|^2)}{|x-y|^3} dx dy dt$$

$$\leq C \sup_{t \in [0, T]} |M_a^{\otimes 2}(t)|. \quad (5.9)$$

Similar to the proof of Theorem 2.17 in [10], using Plancherel theorem, we get

$$\int_{\mathbb{R}^N \otimes \mathbb{R}^N} \frac{|u(t, x)|^2 |u(t, y)|^2}{|x-y|^3} dx dy \simeq \int_{\mathbb{R}^N} |D^{-\frac{N-3}{2}}(|u(x)|^2)|^2 dx \quad (5.10)$$

and

$$\int_{\mathbb{R}^N \otimes \mathbb{R}^N} \frac{\sqrt{|u(t, x)|^2} \tilde{H}(|u(t, x)|^2)}{|x-y|^3} \frac{\sqrt{|u(t, y)|^2} \tilde{H}(|u(t, y)|^2)}{|x-y|^3} dx dy$$

$$\simeq \int_{\mathbb{R}^N} \left| D^{-\frac{N-3}{2}} \left( \sqrt{|u(t, x)|^2} \tilde{H}(|u(t, x)|^2) \right) \right|^2 dx. \quad (5.11)$$

By the results of [8, 9, 10] and using mass conservation law, we have

$$\sup_{t \in [0, T]} |M_a^{\otimes 2}(t)| \leq C' \sup_{t \in [0, T]} \|u(t, x)\|_{H^{1/2}_2}^2 \|u\|_{L^2_2}^2 \leq C \sup_{t \in [0, T]} \|u(t, x)\|_{H^{1/2}_2}^2. \quad (5.12)$$

Combining (5.8), (5.12), we have

$$\|D^{-\frac{N-3}{2}} \left( |u(x)|^2 + \sqrt{|u(t, x)|^2} \tilde{H}(|u(t, x)|^2) \right) \|^2_{L^2_2 L^2_2} \leq C \sup_{t \in [0, T]} \|u(t, x)\|_{H^{1/2}_2}^2, \quad (5.13)$$

which is the interaction Morawetz estimates for (1.1) in the case of $N \geq 3$. 

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5.2 Interaction Morawetz inequality in dimension $N = 2$

Inspired by [8, 9, 10, 28], we will choose $a(x, y) = a(|x - y|)$ in the Morawetz action when $N = 2$, where $a(r)$ is a radial function satisfies

$$\Delta a(r) = \int_{r}^{\infty} s \log(s/r) w_{r_0}(s) ds,$$

where $r_0$ is a small positive number and

$$w_{r_0}(s) := \begin{cases} \frac{1}{s^3} & \text{if } s \geq r_0, \\ 0 & \text{otherwise.} \end{cases}$$

(5.15)

Let

$$M^\otimes_2(t) = 2 \int_{\mathbb{R}^2 \otimes \mathbb{R}^2} \nabla a(|x - y|) \cdot \Im \left( u(t, x)u(t, y)\nabla (u(t, x)u(t, y)) \right) dxdy.$$  

(5.16)

Similar to (5.5) and (5.7), and using

$$-\Delta \Delta a(|x|) = \frac{2\pi}{r_0} \delta(|x|) - w_{r_0}(|x|),$$

we can get

$$\int_{0}^{T} \int_{\mathbb{R}^2} \frac{2\pi}{r_0} \left[ |u(t, x)|^4 + 4\widetilde{H}(|u(t, x)|^2) |u(t, x)|^2 \right] dx dt$$

$$+ \int_{0}^{T} \int_{\mathbb{R}^2} \frac{2\pi}{r_0} \left[ |u(t, y)|^4 + 4\widetilde{H}(|u(t, y)|^2) |u(t, y)|^2 \right] dy dt$$

$$- 2 \int_{0}^{T} \int_{\mathbb{R}^2 \otimes \mathbb{R}^2} w_{r_0}(|x - y|) |u(t, x)|^2 |u(t, y)|^2 dx dy dt$$

$$- 4 \int_{0}^{T} \int_{\mathbb{R}^2 \otimes \mathbb{R}^2} w_{r_0}(|x - y|) \widetilde{H}(|u(t, x)|^2) |u(t, y)|^2 dx dy dt$$

$$- 4 \int_{0}^{T} \int_{\mathbb{R}^2 \otimes \mathbb{R}^2} w_{r_0}(|x - y|) \widetilde{H}(|u(t, y)|^2) |u(t, x)|^2 dx dy dt$$

$$\leq C \sup_{t \in [0, T]} |M^\otimes_2(t)|.$$  

(5.17)

Noticing that

$$\int_{\mathbb{R}^2} w_{r_0}(|x - y|) dx = \int_{\mathbb{R}^2} w_{r_0}(|x - y|) dy = \frac{2\pi}{r_0},$$

(5.18)

using (5.17), we have

$$\int_{0}^{T} \int_{\mathbb{R}^2 \otimes \mathbb{R}^2} w_{r_0}(|x - y|) \left[ |u(t, x)|^2 - |u(t, y)|^2 \right]^2 dx dy dt$$

$$+ 4 \int_{0}^{T} \int_{\mathbb{R}^2 \otimes \mathbb{R}^2} w_{r_0}(|x - y|) \left[ |u(t, x)|^2 - |u(t, y)|^2 \right] \left[ \widetilde{H}(|u(t, x)|^2) - \widetilde{H}(|u(t, y)|^2) \right] dx dy dt$$

$$\leq C' \sup_{t \in [0, T]} |M^\otimes_2(t)|.$$  

(5.19)
Let $r_0 \to 0$, we get
\begin{align*}
&\int_0^T \int_{\mathbb{R}^2 \otimes \mathbb{R}^2} \frac{|u(t,x)|^2 - |u(t,y)|^2|^2}{|x-y|^3} dxdydt \\
&+ \int_0^T \int_{\mathbb{R}^2 \otimes \mathbb{R}^2} \frac{|u(t,x)|^2 - |u(t,y)|^2|\tilde{H}(|u(t,x)|^2) - \tilde{H}(|u(t,y)|^2)|}{|x-y|^3} dxdydt \\
&\leq C' \sup_{t \in [0,T]} |M_{\alpha}^{\otimes 2}(t)| \leq C \sup_{t \in [0,T]} \|u(t,x)\|^2_{H^\frac{1}{2}}. \tag{5.20}
\end{align*}

Note that
\begin{align*}
\int_{\mathbb{R}^2 \otimes \mathbb{R}^2} \frac{|u(t,x)|^2 - |u(t,y)|^2|^2}{|x-y|^3} dxdy \sim \|u\|^2_{H^\frac{1}{2}}.
\end{align*}

We have
\begin{align*}
\|D^\frac{1}{2}(|u|^2)\|_{L^2_t L^2_x}^2 \leq C \sup_{t \in [0,T]} \|u(t,x)\|^2_{H^\frac{1}{2}}. \tag{5.21}
\end{align*}

Especially, if $h(|u|^2) = |u|^2$, we have
\begin{align*}
\|D^\frac{1}{2}(|u|^2)\|_{L^2_t L^2_x}^2 + \|D^\frac{1}{2}(|u|^3)\|_{L^2_t L^2_x}^2 \leq C \sup_{t \in [0,T]} \|u(t,x)\|^2_{H^\frac{1}{2}}. \tag{5.22}
\end{align*}

### 5.3 Interaction Morawetz inequality in dimension $N = 1$

Besides the assumptions on $h(s)$ and $F(s)$ in (5.1), we assume that
\begin{align*}
V'(x) \geq 0, \quad W'(x) \geq 0 \quad \text{for all} \quad x \in \mathbb{R}.
\end{align*}

Let
\begin{align*}
a(x) := erf\left(\frac{a}{\epsilon}\right) = \int_0^\frac{a}{\epsilon} e^{-t^2} dt \tag{5.23}
\end{align*}
and
\begin{align*}
M_{\alpha}(t) = \frac{1}{2} \int_\mathbb{R} \int_\mathbb{R} a(x-y)|u(t,x)|^2 \Im(u(t,x)\bar{u}_x(t,x))dxdy. \tag{5.24}
\end{align*}

In convenience, we denote
\begin{align*}
\rho(x) = \frac{1}{2} |u(t,x)|^2, \quad p(x) = \Im(u(t,x)\bar{u}_x(t,x)). \tag{5.25}
\end{align*}

After some elementary computations, we obtain
\[
\frac{d}{dt} M_a(t) = \int_{\mathbb{R}} \int_{\mathbb{R}} \frac{1}{\varepsilon} e^{-\frac{(x-y)^2}{\varepsilon^2}} \frac{\rho(y)}{\rho(x)} (\partial_x \rho(x))^2 \, dx \, dy \\
+ \frac{1}{2} \int_{\mathbb{R}} \int_{\mathbb{R}} \frac{1}{\varepsilon} e^{-\frac{(x-y)^2}{\varepsilon^2}} \left( \sqrt{\frac{\rho(y)}{\rho(x)}} p(x) - \sqrt{\frac{\rho(x)}{\rho(y)}} p(y) \right)^2 \, dx \, dy \\
+ \int_{\mathbb{R}} \int_{\mathbb{R}} \frac{1}{\varepsilon} e^{-\frac{(x-y)^2}{\varepsilon^2}} [-\partial_{xx} \rho(x)] \rho(y) \, dx \, dy \\
- \int_{\mathbb{R}} \int_{\mathbb{R}} \frac{1}{\varepsilon} e^{-\frac{(x-y)^2}{\varepsilon^2}} h'(\abs{u(t,x)}^2) \abs{u(t,x)}^2 (h(\abs{u(t,x)}^2))_{xx} \, dx \, dy \\
+ \frac{1}{2} \int_{\mathbb{R}} \int_{\mathbb{R}} \frac{1}{\varepsilon} e^{-\frac{(x-y)^2}{\varepsilon^2}} [(h(\abs{u(t,x)}^2))_x]^2 \, dx \, dy \\
+ \frac{1}{2} \int_{\mathbb{R}} \int_{\mathbb{R}} \frac{1}{\varepsilon} e^{-\frac{(x-y)^2}{\varepsilon^2}} [G(\abs{u(t,x)}^2) - F(\abs{u(t,x)}^2) \abs{u(t,x)}^2] \, dx \, dy \\
+ \frac{1}{2} \int_{\mathbb{R}} \int_{\mathbb{R}} a(x-y) [V_x + \frac{1}{2} (W_x * \abs{u}^2)] \abs{u(t,x)}^2 \, dx \, dy. \tag{5.26}
\]

Noticing that
\[
\int_{\mathbb{R}} \int_{\mathbb{R}} \frac{1}{\varepsilon} e^{-\frac{(x-y)^2}{\varepsilon^2}} [-\partial_{xx} \rho(x)] \rho(y) \, dx \, dy = \int_{\mathbb{R}} \xi^2 \rho^2(\xi) e^{-\xi^2} \, d\xi \geq 0,
\]
\(V'(x) \geq 0\) and \(W'(x) \geq 0\), from (5.26), we can get
\[
\int_{\mathbb{R}} \int_{\mathbb{R}} \frac{1}{\varepsilon} e^{-\frac{(x-y)^2}{\varepsilon^2}} \frac{\rho(y)}{\rho(x)} (\partial_x \rho(x))^2 \, dx \, dy \\
- \int_{\mathbb{R}} \int_{\mathbb{R}} \frac{1}{\varepsilon} e^{-\frac{(x-y)^2}{\varepsilon^2}} h'(\abs{u(t,x)}^2) \abs{u(t,x)}^2 (h(\abs{u(t,x)}^2))_{xx} \, dx \, dy \\
+ \frac{1}{2} \int_{\mathbb{R}} \int_{\mathbb{R}} \frac{1}{\varepsilon} e^{-\frac{(x-y)^2}{\varepsilon^2}} [(h(\abs{u(t,x)}^2))_x]^2 \, dx \, dy \\
+ \frac{1}{2} \int_{\mathbb{R}} \int_{\mathbb{R}} \frac{1}{\varepsilon} e^{-\frac{(x-y)^2}{\varepsilon^2}} [G(\abs{u(t,x)}^2) - F(\abs{u(t,x)}^2) \abs{u(t,x)}^2] \, dx \, dy \\
\leq \frac{d}{dt} M_a(t). \tag{5.27}
\]
Letting $\epsilon \to 0$, and integrating (5.27) from 0 to $T$, we have
\[
\int_0^T \int_{\mathbb{R}} (\partial_x \rho(x))^2 dx dt - \int_0^T \int_{\mathbb{R}} |u(t,x)|^4 h'(|u(t,x)|^2)(h(|u(t,x)|^2))_{xx} dx dt \\
+ \frac{1}{2} \int_0^T \int_{\mathbb{R}} |u(t,x)|^2 [(h(|u(t,x)|^2))]_x^2 dx dt \\
+ \frac{1}{2} \int_0^T \int_{\mathbb{R}} [G(|u(t,x)|^2) - F(|u(t,x)|^2)|u(t,x)|^2]u(t,x)^2 dx dt \\
= \int_0^T \int_{\mathbb{R}} (\partial_x \rho(x))^2 dx dt + \int_0^T \int_{\mathbb{R}} [|u(t,x)|^4 h'(|u(t,x)|^2)]x [h(|u(t,x)|^2)]_{xx} dx dt \\
+ \frac{1}{2} \int_0^T \int_{\mathbb{R}} |u(t,x)|^2 [(h(|u(t,x)|^2))]_x^2 dx dt \\
+ \frac{1}{2} \int_0^T \int_{\mathbb{R}} [G(|u(t,x)|^2) - F(|u(t,x)|^2)|u(t,x)|^2]u(t,x)^2 dx dt \\
\leq \lim_{\epsilon \to 0} \sup_{t \in [0,T]} |M(t)|.
\]

Therefore,
\[
\int_0^T \int_{\mathbb{R}} \left\{ 5h'(|u|^2) + 2h''(|u|^2)|u|^2 h'(|u|^2)|u|^2 + 1 \right\} [\partial_x |u(t,x)|^2]^2 dx dt \\
+ \int_0^T \int_{\mathbb{R}} [G(|u(t,x)|^2) - F(|u(t,x)|^2)|u(t,x)|^2]u(t,x)^2 dx dt \\
\leq C' \sup_{t \in [0,T]} |M(t)| \leq C' \sup_{t \in [0,T]} \|u(t,x)\|^2_{H_x^{\frac{1}{2}}} \|u(t,x)\|^2_{L_x^2} \leq C \sup_{t \in [0,T]} \|u(t,x)\|^2_{H_x^{\frac{1}{2}}}. \quad (5.28)
\]

**Theorem 4. (Interaction Morawetz estimates)** Let $u(x,t)$ be the $H_x^{\frac{1}{2}}$ solution of (1.1) on the space-time slab $\mathbb{R}^N \times [0,T]$. 

1. If $N \geq 3$, under the assumptions of (5.1) and (5.2), then
\[
\|D^{-\frac{N-3}{2}} \left( |u(x)|^2 + \sqrt{|u(x)|^2} \tilde{H}(|u(x)|^2) \right) \|^2_{L_t^2 L_x^2} \leq C \sup_{t \in [0,T]} \|u(t,x)\|^2_{H^{1/2}}. \quad (5.29)
\]

Here $\tilde{H}(|u|^2) = \int_0^{|u|^2} [h'(s)]^2 ds$.

2. If $N = 2$, under the assumptions of (5.1) and (5.2), then
\[
\int_0^T \int_{\mathbb{R}^2 \times \mathbb{R}^2} \frac{|u(t,x)|^2 - |u(t,y)|^2|^2}{|x-y|^3} dxdydt \\
+ \int_0^T \int_{\mathbb{R}^2 \times \mathbb{R}^2} \frac{|u(t,x)|^2 - |u(t,y)|^2|\tilde{H}(|u(t,x)|^2) - \tilde{H}(|u(t,y)|^2)|}{|x-y|^3} dxdydt \\
\leq C \sup_{t \in [0,T]} \|u(t,x)\|^2_{H_x^{\frac{1}{2}}}. \quad (5.30)
\]

Here $\tilde{H}(|u|^2) = \int_0^{|u|^2} [h'(s)]^2 ds$. 

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3. If \( N = 1 \), \( G(s) \geq F(s)s \) for \( s \geq 0 \) and \( V'(x) \geq 0 \), \( W'(x) \geq 0 \) for \( x \in \mathbb{R} \), then

\[
\int_0^T \int_{\mathbb{R}} \left\{ 5h'(|u|^2) + 2h''(|u|^2)|u|^2h'(|u|^2)|u|^2 + 1 \right\} [\partial_x |u(t, x)|^2]^2 dx dt \\
+ \int_0^T \int_{\mathbb{R}} [G(|u(t, x)|^2) - F(|u(t, x)|^2)|u(t, x)|^2]|u(t, x)|^2 dx dt \\
\leq C \sup_{t \in [0,T]} \|u(t, x)\|^2_{L_x^1} \\
\tag{5.31}
\]

**Remark 4.1.** If \( h(|u|^2) \equiv 0 \), our results meet with those of [8, 9]. If \( h(|u|^2) = |u|^2 \), our results meet with those of [28]. If \( h(|u|^2) = |u|^{2\alpha} \), under the assumptions of Theorem 4, we have

\[
\|D^{-\frac{N-1}{2}} \left( |u(x)|^2 + |u(t, x)|^{2\alpha+1} \right) \|^2_{L^2_t L^2_x} \leq C \sup_{t \in [0,T]} \|u(t, x)\|^2_{H^{1/2}_x}.
\]

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