A review on minimax rates in change point detection and localisation

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Abstract

This paper reviews recent developments in fundamental limits and optimal algorithms for change point analysis. We focus on minimax optimal rates in change point detection and localisation, in both parametric and nonparametric models. We start with the univariate mean change point analysis problem and review the state-of-the-art results in the literature. We then move on to more complex data types and investigate general principles behind the optimal procedures that lead to minimax rate-optimal results.

1 Introduction

Change point analysis, as a statistics research area, can be traced back to the World War II. Wallis (1980) provided a detailed account on how a request from Navy became the prologue of sequential analysis, which can be regarded as a sibling of change point analysis. The timely demands from manufactory sector during the war boosted the developments of sequential analysis and therefore change point analysis. The lasting demands from the post-war manufactory sector were a continuing source of fuel fanning the developments of change point analysis in the second half of the 20th century. In recent years, change point analysis is receiving an unprecedented interest due to the advancement in data collecting, storing and analysing. We have witnessed the use of change point analysis methods in a wide range of application areas, including finance (e.g. Aggarwal et al., 1999; Andreou and Ghysels, 2002; Ross, 2013), economy (e.g. Fernandez, 2006), neuroscience (e.g. Chamroukhi et al., 2013; Lindquist et al., 2007; Robinson et al., 2010), climatology (e.g. Elsner et al., 2004), biology (e.g. Erdman and Emerson, 2008; Kwon et al., 2008; Lio and Vannucci, 2000; Oliver et al., 2004; Picard et al., 2011; Plummer and Chen, 2014; Shen and Zhang, 2012), chemistry (e.g. Ferreira et al., 2017), medical sciences (e.g. Han et al., 2014; Henderson and Matthews, 1993; Huang et al., 2013; McLain and Albert, 2014), clinical trials (e.g. Koziol and Wu, 1996), transport (e.g. Hsu, 1979), oceanography (e.g. Killick et al., 2010), environmental science (e.g. Whitcher et al., 2002), to name but a few.

Generally speaking, change point analysis is concerned with piecewise-stationary time series data and aims to break them down into stationary pieces. To be specific, for a length-$T$ time series, we assume that there exists a strictly increasing sequence of unknown time points $\{\eta_k\}_{k=1}^K \subset \{2, \ldots, T\}$, namely change points, with $K \geq 1$, satisfying that

$$X_t \sim P_t, \quad t \in \mathbb{N}$$
and
\[ P_t \neq P_{t-1} \quad \text{if and only if} \quad t \in \{\eta_k\}_{k=1}^K, \]
where \( P_t \)’s are distributions. The data \( X_t \)’s can be scalars, vectors, matrices, networks, functionals, etc. Given such data, our goal is to estimate the change points accurately.

The problem can be further characterised by two additional parameters – the minimal spacing \( \Delta \) and the minimal jump size \( \eta \), which are defined as follows:
\[
\Delta = \min_{k=1, \ldots, K+1} (\eta_k - \eta_{k-1}) \quad \text{and} \quad \kappa = \min_{k=1, \ldots, K} \kappa_k = \min_{k=1, \ldots, K} \left\| P_{\eta_k} - P_{\eta_k-1} \right\|_*,
\]
where \( \| \cdot \|_* \) is a certain distance, \( \eta_0 = 1 \) and \( \eta_{K+1} = T + 1 \). Throughout this survey, we let
\[
\kappa \sqrt{\Delta}
\]
be a form of signal-to-noise ratio indicating the fundamental difficulty of the problems. We remark that this quantity is called differently in different literature, for instance in Verzelen et al. (2020) it is called energy.

Given data \( \{X_t\}_{t=1}^T \), we seek estimators \( \{\hat{\eta}_k\}_{k=1, \ldots, \hat{K}} \) satisfying that with probability tending to 1 as \( T \to \infty \), the following holds:
\[
\hat{K} = K \quad \text{and} \quad \lim_{T \to \infty} \frac{\max_{k=1, \ldots, K} \epsilon_k}{\Delta} = \lim_{T \to \infty} \frac{\max_{k=1, \ldots, K} |\hat{\eta}_k - \eta_k|}{\Delta} = 0. \tag{1}
\]
We refer to \( \epsilon_k \) the individual localisation error of \( \eta_k \), \( \epsilon = \max_{k=1, \ldots, K} |\hat{\eta}_k - \eta_k| \) as the localisation error and \( \Delta^{-1} \max_{k=1, \ldots, K} |\hat{\eta}_k - \eta_k| \) as the localisation rate. For change point estimators satisfying (1), we call them consistent change point estimators.

1.1 What we will cover in this survey

In this survey, we focus on understanding the minimax rates of change point detection and localisation. These two goals are tightly intertwined. We endeavour on distinguishing these two concepts in this survey. Roughly speaking, these two can be regarded as the condition on consistent estimation and the optimal estimation errors.

Rigorously speaking, when \( K \geq 1 \), the fundamental limit in terms of detection can be presented as a phase transition phenomenon. One would like to show that in the low signal-to-noise ratio regime
\[
\kappa \sqrt{\Delta} \lesssim \text{a detection lower bound}, 
\]
no algorithm is guaranteed to provide consistent change point estimators; and in the high signal-to-noise ratio regime
\[
\kappa \sqrt{\Delta} \gtrsim \text{a detection upper bound},
\]
we would like to review some computationally-efficient algorithms which can provide consistent change point estimators.

On the other hand, the fundamental limit in terms of localisation is that
\[
\inf_{\hat{\eta}} \sup_{P} \mathbb{E}_P \{d_H(\{\hat{\eta}_k\}, \{\eta_k\}) \} \gtrsim \text{optimal localisation error}. \tag{4}
\]
The infimum is taken over all possible estimators of the change points, i.e. all measurable functions of data. The supremum is across all possible distributions with signal-to-noise ratios at least higher than optimal localisation conditions. The distance \( d_H(\cdot, \cdot) \) is the two-sided Hausdorff distance, i.e. for any subset \( S_1, S_2 \subset \mathbb{Z} \),

\[
d_H(S_1, S_2) = \max \left\{ \max_{s_1 \in S_1} \min_{s_2 \in S_2} |s_1 - s_2|, \max_{s_2 \in S_2} \min_{s_1 \in S_1} |s_1 - s_2| \right\},
\]

with the convention that

\[
d_H(S_1, S_2) = \begin{cases} 
\infty, & S_1 = \emptyset \neq S_2 \text{ or } S_2 = \emptyset \neq S_1, \\
0, & S_1 = S_2 = \emptyset.
\end{cases}
\]

The localisation task is to seek change point estimators achieving the optimal localisation rate. We, ideally, wish for:

(i) the detection lower and upper bounds in (2) and (3) coincide in terms of rates;

(ii) the matched detection upper and lower bound coincides with the optimal localisation condition imposed on the distributions considered in (4); and

(iii) there exists a computationally-efficient algorithm which can provide estimators with localisation errors matching that in (4).

In various problems we will cover in this survey, these three goals are achievable simultaneously in some settings, but not all. We will provide in-depth discussions, including open problems and our conjectures. In this survey, when talking about optimality, we allow for logarithmic gaps. When there exist logarithmic gaps, we do not distinguish the term “nearly-optimal” and “optimal”.

So far, we characterise the changes occurring with an unspecified distance \( \| \cdot \|_* \) between distinct underlying distributions. In this survey, we will cover both parametric and nonparametric models. In parametric models, we will cover univariate mean changes, univariate polynomial coefficients changes, high-dimensional covariance changes, high-dimensional sparse network changes and high-dimensional linear regression coefficient changes. Absolute values, \( \ell_2 \)-norms of vectors, operator norms and Frobenius norms of matrices, are used as examples of \( \| \cdot \|_* \). In nonparametric models, regarding the distance \( \| \cdot \|_* \), we will cover the univariate Kolmogorov–Smirnov distance, multivariate supreme norm and a general reproducing kernel Hilbert space distance.

We will use the univariate mean change problem as the blueprint, studying the fundamental limits of the detection and localisation problems and analysing two types of popular computationally-efficient and statistically-optimal methods. For all the other aforementioned problems, we will present the information-theoretic lower bounds and an algorithm providing the state-of-art theoretical results.

A summary of these limits can be found in Table 1. For detailed parameter definitions, see corresponding sections. In Table 1, detection lower bounds correspond to the detection boundaries in the sense of (2), consistence upper bounds correspond to the detection boundaries in the sense of (3) and localisation lower bounds correspond to the localisation errors in (4). The optimality upper bounds and localisation upper bounds are the state-of-the art results in the literature. They are the conditions for computationally-efficient algorithms achieving optimal localisation rates and the localisation errors they achieve.
Table 1: Summary.

| Model                        | Detection lower bound | Consistence upper bound | Optimality upper bound | Localisation lower bound | Localisation upper bound | Section |
|------------------------------|-----------------------|-------------------------|------------------------|--------------------------|--------------------------|---------|
| Univariate mean              | \( \kappa \sqrt{\Delta} \leq \sigma \log^{1/2} T \) | \( \kappa \sqrt{\Delta} \leq \sigma \log^{1/2+\xi} T \) | \( \kappa \sqrt{\Delta} \geq \sigma \log^{1/2+\xi} T \) | \( \frac{\sigma^2}{\kappa^2} \) | \( \frac{\sigma^2 \log(T)}{\kappa^2} \) | 2       |
| Univariate polynomials       | \( \kappa \Delta^{r+1/2} \leq \sigma T^r \log^{1/2} (T) \) | \( \kappa \Delta^{r+1/2} \geq \sqrt{\kappa} \sigma T^r \log^{1/2+\xi} (T) \) | \( \kappa \Delta^{r+1/2} \geq \sqrt{\kappa} \sigma T^r \log^{1/2+\xi} (T) \) | \( \left( \frac{\sigma^2}{\kappa^2 T} \right)^{\frac{1}{r+1}} \) | \( \left( \frac{\sigma^2 \log(T)}{\kappa^2 T} \right)^{\frac{1}{r+1}} \) | 3       |
| High-dim covariance          | \( \kappa \sqrt{\Delta} \leq \frac{1}{\sigma^2 \sqrt{p}} \) | \( \kappa \sqrt{\Delta} \geq \sigma^2 \sqrt{\log^{1/2+\xi} (T)} \) | \( \kappa \sqrt{\Delta} \geq \sigma^2 \sqrt{\log^{1/2+\xi} (T)} \) | \( \frac{\sigma^4}{\kappa^2} \) | \( \frac{\sigma^4 \log(T)}{\kappa^2} \) | 4.1     |
| Dynamic networks             | \( \sqrt{\kappa} \sigma \log^{1/2+\xi} (T) \) | \( \sqrt{\kappa} \sigma \log^{1/2+\xi} (T) \) | \( \sqrt{\kappa} \sigma \log^{1/2+\xi} (T) \) | \( \frac{1}{\kappa \sigma n^2 \rho} \) | \( \frac{\log^2(T)}{\kappa \sigma n^2 \rho} \) | 4.2     |
| High-dim linear regression   | \( \kappa \sqrt{\Delta} \leq \sigma \epsilon \sqrt{d_0 \log^{1/2+\xi} (T)} \) | \( \kappa \sqrt{\Delta} \geq \sigma \epsilon \sqrt{K \log^{1/2+\xi} (T)} \) | \( \kappa \sqrt{\Delta} \geq \sigma \epsilon \sqrt{K \log^{1/2+\xi} (T)} \) | \( \frac{d_0 \sigma^2}{\kappa^2} \) | \( \frac{d_0 \sigma^2 \log(T \vee p)}{\kappa^2} \) | 4.3     |
| Univariate nonparametric     | \( \kappa \sqrt{\Delta} \leq 1 \) | \( \kappa \sqrt{\Delta} \geq \log^{1/2+\xi} (T) \) | \( \kappa \sqrt{\Delta} \geq \log^{1/2+\xi} (T) \) | \( \kappa^{-2} \) | \( \frac{\log(T)}{\kappa^2} \) | 5.1     |
| Multivariate nonparametric   | \( \kappa^{p/2+1} \sqrt{\Delta} \leq 1 \) | \( \kappa^{p/2+1} \sqrt{\Delta} \geq \log^{1/2+\xi} (T) \) | \( \kappa^{p/2+1} \sqrt{\Delta} \geq \log^{1/2+\xi} (T) \) | \( \kappa^{-(p+2)} \) | \( \frac{\log(T)}{\kappa^{p+2}} \) | 5.2     |
| An RKHS example              | \( \kappa \sqrt{\Delta} \leq \log^{1/2} (T) \) | \( \kappa \sqrt{\Delta} \geq \log^{1/2+\xi} (T) \) | \( \kappa \sqrt{\Delta} \geq \log^{1/2+\xi} (T) \) | \( \kappa^{-2} \) | \( \frac{\log(T)}{\kappa^2} \) | 5.3     |
1.2 What we will not cover in this survey

After more than seven decades of developing, change point analysis has become an extremely fruitful area with numerous papers from a wide range of areas. Even within the statistics community, change point analysis has been studied extensively from many different angles. There are a few important topics we do not intend to discuss in depth in this survey. We briefly mention them here.

**Online change point analysis.** In this survey, we will only focus on offline change point analysis, i.e. given data \( \{X_t\}_{t=1}^T \), we retrospectively seek change points in \( \{1, \ldots, T\} \). Another important area of change point analysis is called online/sequential change point analysis, in which one is making sequential decisions on whether a change point has occurred while collecting data.

The statistical problems associated with online change point detection include minimising the detection delay, e.g. upper bounding \( (\hat{\eta} - \eta)_+ \), while controlling false positives, e.g. upper bounding the probability of \( \hat{\eta} < \eta \). There is a vast body of existing literature on this topic, including Moustakides (1986), Ritov (1990), Lorden (1971), Lai (1981), Lai (1995), Lai (1998), Lai (2001), Lai and Xing (2010), Chu et al. (1996), Aue and Horváth (2004), Aue et al. (2009c), Kirch (2008), Huskova and Kirch (2012), Mei (2010), Huskova et al. (2010), Hlávka et al. (2016), Desobry et al. (2005), Fearnhead and Liu (2007), He et al. (2018), Kirch and Weber (2018), Kurt et al. (2018), Chen (2019b), Dette and Gösmann (2019), Gösmann et al. (2019), Dette and Gösmann (2019), Keshavarz et al. (2018), Chen et al. (2020), Siegmund (2013), Tartakovsky et al. (2014), Namoano et al. (2019), Maillard (2019) and Yu et al. (2020), among others.

**Testing.** Testing and estimation are two indispensable pillars in statistical problems. In Section 1.1, we mentioned that detection and localisation are two different estimation tasks in change point analysis. The testing aspects in change point analysis focus on the Type-I and -II errors controls on testing the presence of change points, and also include the limiting distributions of change point estimators, constructing confidence intervals of change points, etc. As a statistical problem, testing is generally easier than estimation, in terms of the fundamental limits. Quite the contrary, the study of the fundamental limits of the testing problem is lagged behind. The literature on different aspects of testing includes Yao and Au (1989), Frick et al. (2014), Enikeeva et al. (2019), Vanegas et al. (2019), Dette and Kutta (2019), Dette et al. (2018a), Akashi et al. (2018), Dette et al. (2018c), Aue et al. (2018), Aue and Horváth (2013), Robbins et al. (2011), Liu et al. (2019), Stoehr et al. (2020), Kirch et al. (2015), Jewell et al. (2019), Chen (2019a), Jirak (2015), Chu and Chen (2019) and Verzelen et al. (2020), among others.

**Computation.** In this survey, for each problem, we will only present one or two polynomial-time algorithms which provide nearly-optimal results. In practice, especially for high-dimensional data, it is crucial to improve the computational efficiency without sacrificing too much statistical accuracy. There is a line of attack on improving the computational time of the methods we will introduce later in this survey. These works include Romano et al. (2020), Hocking et al. (2020), Tickle et al. (2020), Haynes et al. (2017a), Hocking et al. (2017), Maidstone et al. (2017), Haynes et al. (2017b), Killick et al. (2012), Rigail (2010), Kovács et al. (2020a) and Kovács et al. (2020b), among others.

**Tuning parameter selection.** For all the methods studied in this survey, the theoretical results rely on some properly chosen tuning parameters. This is always an important but hard-to-address problem in statistics. Generally speaking, some papers use information-type criteria pioneered in Yao (1988), and others use data-driven methods (e.g. Padilla et al., 2019b;
2 Univariate mean changes: a blueprint

2.1 Setup and overview

Arguably, the simplest and best-studied change point analysis problem is the univariate time series with piecewise-constant mean and independent sub-Gaussian noise. We formalise the problem below.

**Assumption 1.** Let \( \{X_t\}_{t=1}^T \subset \mathbb{R} \) be independent sub-Gaussian random variables with continuous density such that \( \mathbb{E}(X_t) = f_t \) and \( \|X_t\|_{\psi_2} \leq \sigma \) for all \( t \in \{1, \ldots, T\} \).

Let \( \{\eta_k\}_{k=0}^{K+1} \subset \{1, \ldots, T + 1\} \) be a collection of change points such that \( 1 = \eta_0 < \eta_1 < \ldots < \eta_K \leq T < \eta_{K+1} = T + 1 \) and \( f_t \neq f_{t-1} \), if and only if \( t \in \{\eta_k\}_{k=1}^K \).

Assume the minimal spacing \( \Delta \) and the jump size \( \kappa \) are defined to be

\[
\Delta = \min_{k=1, \ldots, K+1} \{\eta_k - \eta_{k-1}\} > 0,
\]

and

\[
\kappa = \min_{k=1, \ldots, K} \kappa_k = \min_{k=1, \ldots, K} |f_{\eta_k} - f_{\eta_{k-1}}| > 0.
\]

Note that the \( \|\cdot\|_{\psi_2} \) is the Orlicz-\( \psi_2 \)-norm or the sub-Gaussian norm, defined as, for any random variable \( X \),

\[
\|X\|_{\psi_2} = \inf \{t > 0 : \mathbb{E}\{\exp(X^2/t^2)\} \leq 2\}.
\]

The condition on the continuous density is merely to impose uniqueness of the estimators. We impose it here just for simplicity.

As for the problem detailed in Assumption 1, the detection lower bound is \( \sigma \log^{1/2}(T) \) and the localisation lower bound is \( \sigma^2 \kappa^{-2} \). These two results are in Lemmas 1 and 4, respectively. To match these lower bounds, we will show two nearly-optimal polynomial-time algorithms.

2.2 Detection boundary

The detection boundary \( \sigma \log^{1/2}(T) \) has been established in various different papers, including Chan and Walther (2013), Frick et al. (2014), Dümbgen and Spokoiny (2001), Dümbgen and Walther (2008), Li et al. (2017), Jeng et al. (2012), Enikeeva et al. (2018) and Wang et al. (2020a). We formalise the result below.

**Lemma 1** (Lemma 1 in Wang et al., 2020a). Let \( \{X_t\}_{t=1}^T \) be a time series satisfying Assumption 1. Let \( P_{\Delta,\sigma}^T \) denote the corresponding joint distribution. For any \( 0 < c < 1 \), consider the class of distributions

\[
P^T = \left\{ P_{\kappa,\Delta,\sigma}^T : \Delta = \min \left\{ \left\lfloor \frac{\log(T)}{\kappa^2 / \sigma^2} \right\rfloor, \left\lfloor \frac{T}{4} \right\rfloor \right\} \right\}.
\]

Then, there exists an \( T(c) \), which depends on \( c \), such that, for all \( T \) larger than \( T(c) \),

\[
\inf_{\{\hat{\eta}\}} \sup_{P \in P^T} \mathbb{E}_P \{d_H(\{\hat{\eta}\}, \{\eta(P)\})\} \geq \frac{\Delta}{2},
\]
where the infimum is over all estimators \( \{ \hat{\eta} \} \) of the change point locations and \( \{ \eta(P) \} \) is the set of locations of the change points of \( P \in \mathcal{P}^T \).

For the localisation rate achieved in Lemma 1, it holds that
\[
\frac{d_H(\{ \hat{\eta} \}, \{ \eta \})}{\Delta} \geq \frac{1}{2},
\]
which does not vanish. Corresponding to (1), Lemma 1 shows that if the signal-to-noise ratio is in the regime
\[
\kappa \sqrt{\Delta} \lesssim \sigma \log^{1/2}(T),
\]
then no algorithm is guaranteed to provide consistent estimators.

To complete the phase transition phenomenon, one needs to provide computationally-efficient algorithms, which provide consistent change point estimators in the regime
\[
\kappa \sqrt{\Delta} \gtrsim \sigma \log^{1/2}(T).
\]

As for change point analysis, there are two main types of algorithms, which we will refer to as penalisation-based estimators and scan-statistics-type estimators. The penalisation-based estimators are based on a penalised objective function. The objective function is often a goodness-of-fit criterion and the penalty is usually imposed on the number of change points, to avoid overfitting. The scan-statistics-type estimators are obtained by adopting a certain statistic, scanning through most if not all the data to evaluate each time point’s potential of being a change point. Each of these two types of estimation contains a variety of methods, and each of these two has a representative method being nearly-optimal in the sense we focus on in this paper.

### 2.2.1 Penalisation-based estimators

Let \( \mathcal{P} \) be any interval partition of \( \{1, \ldots, T\} \), i.e. a collection of \( |\mathcal{P}| \geq 1 \) disjoint interval subsets of \( \{1, \ldots, T\} \) in the form of
\[
\mathcal{P} = \{ \{1, \ldots, i_1 - 1\}, \{i_1, \ldots, i_2 - 1\}, \ldots, \{i_{|\mathcal{P}|-1}, \ldots, i_{|\mathcal{P}|} - 1\} \},
\]
for some integers \( 1 < i_1 < \cdots < i_{|\mathcal{P}|-1} \leq T < i_{|\mathcal{P}|} = T + 1 \). For a positive tuning parameter \( \lambda > 0 \) and data \( \{X_t\}_{t=1}^T \), let
\[
\hat{P}(\lambda) \in \arg \min_{\mathcal{P}} G(\mathcal{P}, \{X_t\}_{t=1}^T, \lambda),
\]
where the minimum ranges over all interval partitions of \( \{1, \ldots, T\} \) and, for any such partition \( \mathcal{P} \),
\[
G(\mathcal{P}, \{X_t\}_{t=1}^T, \lambda) = \sum_{I \in \mathcal{P}} H(I) + \lambda|\mathcal{P}| = \sum_{I \in \mathcal{P}} \|X_I - \Pi_I X_I\|^2 + \lambda|\mathcal{P}|,
\]
where \( \Pi_I \) is the projection matrix of the subspace spanned by an all-one vector, i.e.
\[
\Pi_I = I_I^\top \left( I_I^\top I_I \right)^{-1} I_I^\top, \quad 1_I = (1, \ldots, 1)^\top \in \mathbb{R}^{|I|}
\]
and \( X_I = (X_i, i \in I)^\top \in \mathbb{R}^{|I|} \). In fact, for any interval \( I \subset \{1, \ldots, T\} \),
\[
\Pi_I X_I = |I|^{-1} \sum_{i \in I} X_i.
\]
We adopt the seemingly unnecessary notation (7) to be consistent with that in Section 3.

The optimization problem (5) is known as the minimal partition problem and can be solved using dynamic programming in polynomial time (e.g. Algorithm 1 in Friedrich et al., 2008). The change point estimator resulting from the solution to (5) is simply obtained from taking all the left endpoints of the intervals \( I \in \hat{P} \), while letting \( \tilde{\eta}_0 = 1 \). For completeness, we include the algorithm in Algorithm 1. As we have emphasised in Section 1.2, the computational issue is not covered in this paper. The computational cost of Algorithm 1 is of order \( O(T^2 \text{Cost}(T)) \), where \( \text{Cost}(T) \) is the computational cost of calculating the function \( H(I) \), with an interval \( I \) of length \( T \).

There exist more efficient variants of Algorithm 1 in solving the optimisation problem (5), including Killick et al. (2012).

Algorithm 1 Penalised dynamic programming.

**INPUT:** Data \( \{X(t)\}_{t=1}^T \), tuning parameter \( \lambda > 0 \).

\[(B, s, t, \text{FLAG}) \leftarrow (\emptyset, 0, 2, 0)\]

while \( s < T - 3 \) do

\[s \leftarrow s + 1\]

while \( t < T \) and \( \text{FLAG} = 0 \) do

\[t \leftarrow t + 1\]

if \( \min_{l \in \{s+1, \ldots, t-1\}} \{H([s, l]) + H([l+1, t])\} + \lambda < H([s, t]) \) then

\[s \leftarrow \min \{l \in s + 1, \ldots, t - 1 : H([s, l]) + H([l+1, t]) + \lambda < H([s, t])\}\]

\[B \leftarrow B \cup \{s\}\]

\[\text{FLAG} \leftarrow 1\]

end if

end while

end while

**OUTPUT:** The set of estimated change points \( B \).

**Theorem 2** (Theorem 3 in Wang et al., 2020a). Let \( \{X_i\}_{i=1}^T \) satisfy Assumption 1. Assume that there exists a sufficiently large absolute constant \( C_{\text{SNR}} > 0 \) such that for any \( \xi > 0 \),

\[
\kappa \sqrt{\Delta}/\sigma \geq C_{\text{SNR}} \sqrt{\log 1+\xi(T)}.
\]

For any \( \lambda > 0 \), let \( \{\tilde{\eta}_k\}_{k=1}^{\hat{K}} \) be the output of Algorithm 1 with function \( H(\cdot) \) defined in (6). We have that, for any choice of \( c > 0 \), there exists a constant \( C_\lambda > 0 \), which depends on \( c \) such that, for \( \lambda = C_\lambda \sigma^2 \log(T) \), it holds that

\[
\mathbb{P}\{\hat{K} = K \text{ and } \epsilon_k = |\tilde{\eta}_k(\lambda) - \eta_k| \leq C_\epsilon \sigma^2 \log(T)/\kappa_k^2, \forall k \in \{1, \ldots, K\}\} \geq 1 - T^{-c},
\]

where \( C_\epsilon > 0 \) is a constant depending on \( C_\lambda \) and \( C_{\text{SNR}} \).

Theorem 2 shows that in the regime \( \kappa \sqrt{\Delta} \geq \sigma \log^{1/2+\xi}(T) \), the outputs of Algorithm 1 with (6) are consistent. To be specific, in this signal-to-noise ratio regime,

\[
\lim_{T \to \infty} \frac{\max_{k=1, \ldots, K} \epsilon_k}{\Delta} \lesssim \lim_{T \to \infty} \frac{\max_{k=1, \ldots, K} \sigma^2 \log(T) / \kappa_k^2}{\Delta} \lesssim \lim_{T \to \infty} \log^{-\xi}(T) = 0.
\]
This also explains the role of $\xi$. It is introduced merely for mathematical purposes on enforcing the vanishing ratio and for notational simplicity. The term $\log^\xi(T)$ can be replaced by any diverging sequence $a_T$.

Theorem 2 and Lemma 1 together show a phase transition phenomenon that:

- in the low signal-to-noise ratio regime
  \[ \kappa \sqrt{\Delta} \lesssim \sigma \log^{1/2}(T), \]
  no algorithm is guaranteed to be consistent; and

- in the high signal-to-noise ratio regime
  \[ \kappa \sqrt{\Delta} \lesssim \sigma \log^{1/2+\xi}(T), \quad \forall \xi > 0, \]
  we have a computationally-efficient algorithm which achieves a consistent change point estimation.

We remark that there are other types of penalisations. In (6), the penalty is imposed on the number of change points and is equivalent to an $\ell_0$ penalty. It is natural to replace the $\ell_0$ penalty with an $\ell_1$ penalty and ends up with a fused Lasso (Tibshirani et al., 2005) or a trend filtering (e.g. Tibshirani, 2014) problem. There have indeed been works analysing change points using $\ell_1$ penalties due to its computational efficiency. It is known that in terms of change point detection and localisation, $\ell_1$ penalisation based methods are sub-optimal (Lin et al., 2016), but the estimators can be improved with proper post-processing (e.g. Zhang, 2019; Hyun et al., 2018).

### 2.2.2 Scan-statistics-type estimators

Arguably, the most popular statistic used in change point analysis is the cumulative sum (CUSUM, Page, 1954) statistic, which was proposed as an extension of the sequential probability ratio test statistics (Wald, 1945).

**Definition 1** (CUSUM statistics). For a sequence $\{X_t\}_{t=1}^T$, any integer triplet $(s, t, e)$, $0 \leq s < t < e \leq T$, let the CUSUM statistic be

\[
\widetilde{X}_{s,e}^t = \sqrt{\frac{e-t}{(e-s)(t-s)}} \sum_{i=s+1}^{t} X_i - \sqrt{\frac{t-s}{(e-s)(e-t)}} \sum_{i=t+1}^{e} X_i.
\]

We will encounter multiple versions of Definition 1 in the rest of this survey. The original CUSUM statistic is restricted to the case that $X_t$’s are scalars, but they will be allowed to be in different spaces in this survey.

The CUSUM statistic is originated from a log-likelihood ratio test statistic. We elaborate this from the example below.

**Example 1.** Let $\{X_t\}_{t=1}^T$ be a sequence of independent Gaussian random variables with unknown mean $\mu_t$ and known variance $\sigma^2$. For any $t \in \{1, \ldots, T-1\}$, let

$H_{1,t} : \mu_1 = \cdots = \mu_t \neq \mu_{t+1} = \cdots = \mu_T$. 

We want to test

$H_0 : \mu_1 = \cdots = \mu_T$ vs. $H_1 = \cup_{t=1}^{T-1} H_{1,t}$.  

(8)
For any fixed \( t \in \{1, \ldots, T - 1\} \), define
\[
X_1 = \frac{1}{t} \sum_{i=1}^{t} X_i, \quad X_2 = \frac{1}{T-t} \sum_{i=t+1}^{T} X_i \quad \text{and} \quad X = \frac{1}{T} \sum_{i=1}^{T} X_i.
\]

The generalised likelihood ratio test statistic of the problem in Example 1 is that
\[
T_t = \log \left[ \prod_{i=1}^{t} \frac{1}{\sqrt{2\pi\sigma^2}} \exp \left\{ -\frac{(X_i - \bar{X}_1)^2}{2\sigma^2} \right\} \prod_{i=t+1}^{T} \frac{1}{\sqrt{2\pi\sigma^2}} \exp \left\{ -\frac{(X_i - \bar{X}_2)^2}{2\sigma^2} \right\} \right]
= \frac{1}{2\sigma^2} \left\{ \sum_{i=1}^{T} (X_i - \bar{X})^2 - \sum_{i=1}^{t} (X_i - \bar{X}_1)^2 - \sum_{i=t+1}^{T} (X_i - \bar{X}_2)^2 \right\}
= \frac{1}{2\sigma^2} \left\{ \sqrt{\frac{T-t}{Tt}} \sum_{i=1}^{t} X_i - \sqrt{\frac{t}{T(T-t)}} \sum_{i=t+1}^{T} X_i \right\}^2
= \frac{1}{2\sigma^2} \left( \tilde{X}_0^{0,T} \right)^2.
\]

Then (8) can be conducted based on
\[
\max_{t=1,\ldots,T-1} T_t = \frac{1}{2\sigma^2} \max_{t=1,\ldots,T-1} \left| \tilde{X}_0^{0,T} \right|^2,
\]
which is equivalent to the use of CUSUM statistics in change point detection.

In fact, CUSUM statistics can be used for change point detection in a number of ways. Arguably, the most popular and standard method is the binary segmentation (e.g. Scott and Knott, 1974; Venkatraman, 1992; Vostrikova, 1981). The key idea is to find
\[
\hat{t} \in \max_{t=1,\ldots,T-1} \left| \tilde{X}_t^{0,T} \right|.
\]

For a pre-specified threshold \( \tau \), if \( |\tilde{X}_t^{0,T}| \geq \tau \), then we declare \( \hat{t} \) to be a change point estimator and the procedure is conducted on the intervals \((0, \hat{t})\) and \([\hat{t}, T]\) respectively. The procedure is terminated if there is no more change point estimator declared, or if the resulting interval is too narrow. Binary segmentation is a computationally-efficient algorithm, but sub-optimal. The sub-optimality can be intuitively explained as follows. When there are potentially multiple change points, the consecutive change points may cancel out each other (e.g. Fryzlewicz, 2014).

In order to improve the theoretical guarantees of CUSUM-based algorithms, especially to tackle the multiple change points scenario, a large number of variants have been proposed, including Fryzlewicz (2014), Kovács et al. (2020a), Kovács et al. (2020b), Anastasiou and Fryzlewicz (2019), Baranowski et al. (2016), among others.

We use the wild binary segmentation (Fryzlewicz, 2014) as an example to illustrate how a CUSUM-based method can achieve optimality.

**Theorem 3** (Theorem 4 in Wang et al., 2020a). *Assume that the inputs of Algorithm 2 are as follows.*

- The sequence \( \{X_t\}_{t=1}^{T} \) satisfies Assumption 1. In addition, assume that there exists a sufficiently large absolute constant \( C_{\text{SNR}} > 0 \) such that for any \( \xi > 0 \),
\[
\kappa \sqrt{\Delta} / \sigma \geq C_{\text{SNR}} \sqrt{\log^{1+\xi} (T)}.
\]
Algorithm 2 Wild Binary Segmentation. WBS\(((s, e), \{(\alpha_m, \beta_m)\}_{m=1}^{M}, \tau)\)

**INPUT:** Independent samples \(\{X_t\}_{t=1}^{T}\), collection of intervals \(\{(\alpha_m, \beta_m)\}_{m=1}^{M}\), tuning parameter \(\tau > 0\).

for \(m = 1, \ldots, M\) do

\((s_m, e_m) \leftarrow (s, e) \cap [\alpha_m, \beta_m]\)

if \(e_m - s_m > 1\) then

\(b_m \leftarrow \arg \max_{t=s_m+1, \ldots, e_m-1} |\tilde{X}_{t}|_{b_m, e_m}\)

\(a_m \leftarrow |\tilde{X}_{b_m, e_m}|\)

else

\(a_m \leftarrow -1\)

end if

end for

\(m^* \leftarrow \arg \max_{m=1, \ldots, M} a_m\)

if \(a_{m^*} > \tau\) then

add \(b_{m^*}\) to the set of estimated change points

WBS\(((s, b_{m^*}), \{(\alpha_m, \beta_m)\}_{m=1}^{M}, \tau)\)

WBS\(((b_{m^*} + 1, e), \{(\alpha_m, \beta_m)\}_{m=1}^{M}, \tau)\)

end if

**OUTPUT:** The set of estimated change points.

- The collection of intervals \(\{(\alpha_m, \beta_m)\}_{m=1}^{M} \subseteq \{1, \ldots, T\}\), whose endpoints are drawn independently and uniformly from \(\{1, \ldots, T\}\), satisfy
  \[
  \max_{m=1, \ldots, M} (\beta_m - \alpha_m) \leq C_R \Delta,
  \]
  almost surely, for an absolute constant \(C_R > 1\).

- The tuning parameters \(\tau\) satisfies
  \[
  c_{\tau,1} \sigma \sqrt{\log(T)} < \tau < c_{\tau,2} \kappa \sqrt{\Delta},
  \]
  where \(c_{\tau,1}, c_{\tau,2} > 0\) are sufficiently large and small absolute constants.

Let \(\hat{\eta}_k\) be the corresponding output of Algorithm 2. It holds that

\[
\mathbb{P}\left\{\hat{K} = K \quad \text{and} \quad \epsilon_k = |\hat{\eta}_k - \eta_k| \leq C c \sigma^2 \log(T) \kappa_k^{-2}, \forall k \in \{1, \ldots, K\}\right\} \geq 1 - T^{-c} - \exp\left\{\log\left(\frac{T}{\Delta}\right) - \frac{M \Delta^2}{16 T^2}\right\},
\]

where \(C, c > 0\) are absolute constants.

Following the same discussions after Theorem 2, provided \(\log(T/\Delta) \lesssim M \Delta^2 T^{-2}\), Theorem 3 shows that Algorithm 2 provides consistent change point estimation under a nearly-optimal signal-to-noise ratio regime. The key to the success of Algorithm 2 is the usage of random intervals, but in order to achieve the optimality, in Theorem 3, the lengths of the random intervals are at
most of the order of the minimal spacing. This is of course not practical, but essential in deriving the optimality. Similar treatments can be found in other forms, such as the parameter $\beta$ used in Wang and Samworth (2018). If we relax the condition that $C_R$ being an absolute constant, then $C_R \leq T/\Delta$. This results in an inflation in the required signal-to-noise ratio and the resulting localisation rate. To be specific, one would require

$$\kappa\sqrt{\Delta}/\sigma \geq C_{\text{SNR}} T\sqrt{\log^{1+\xi}(T)}$$

and have the localisation rate being

$$C_\epsilon\sigma^2\log(T)\kappa^{-2}T^2/\Delta^2.$$  

As we have pointed out, the sub-optimality of the binary segmentation roots in the multiple change point scenario. Note that the detection upper bound is $\kappa\sqrt{\Delta} \propto \sigma \log^{1/2+\xi}(T)$. For simplicity, we let $\kappa, \sigma \propto 1$ and $\xi = 1/2$, then this means $\Delta$ can be as small as $\log^{1/2}(T)$. In this case, the number of change points can be as many as $T/\Delta \propto T \log^{-2}(T)$, which diverges as $T$ grows unbounded. Most if not all of the variants of the binary segmentation works on how to narrow the focus to intervals containing only finite number of true change points. Different variants use different additional parameters to guarantee this for theoretical purposes. To the best of our knowledge, there is no algorithm can deal with this issue satisfactorily both theoretically and practically. For example, the WBS-type methods require this additional constant $C_R$ in the upper bound on the lengths of random intervals. The narrowest-over-threshold method (Baranowski et al., 2016) is shown to be too sensitive to tuning parameters in numerical experiments. The optimistic search strategy (Kovács et al., 2020b) works under a stronger condition on the minimal spacing for the multiple change points scenario.

The CUSUM statistics essentially can be regarded as differences between weighted sample means. The weights play the role of variance stabilisation. We remark that there are other types of scan statistics, including those used in Cribben and Yu (2017), Liu et al. (2018) and Niu and Zhang (2012).

### 2.3 Optimal localisation rate

As for the localisation, we have the following minimax lower bound.

**Lemma 4** (Lemma 2 in Wang et al., 2020a). Assume that the sequence $\{X_t\}_{t=1}^T$ satisfies Assumption 1. Let $P_{\kappa,\Delta,\sigma}^T$ denote the corresponding joint distribution. Consider the class of distributions

$$Q^T = \left\{ P_{\kappa,\Delta,\sigma}^T : \Delta < T/2, \kappa\sqrt{\Delta}/\sigma \geq \zeta_T \right\},$$

for any sequence $\{\zeta_T\}$ such that $\lim_{T \to \infty} \zeta_T = \infty$. Then, for all $T$ large enough, it holds that

$$\inf_{\hat{\eta}} \sup_{P \in Q^T} \mathbb{E}_P(|\hat{\eta} - \eta(P)|) \geq \max \left\{ 1, \frac{1}{2} \frac{\sigma^2}{\kappa^2} e^{-2} \right\},$$

where the infimum is over all estimators $\hat{\eta}$ of the change point location and $\eta(P)$ denotes the change point location of $P \in Q^T$.

Lemma 4 shows the minimax lower bound on the localisation error is of order $\sigma^2\kappa^{-2}$. The localisation errors achieved by the estimators from Algorithms 1 and 2 can both be nearly optimal, off by logarithmic factors, under suitable conditions.
2.4 Conclusions

The univariate piecewise constant change point detection and localisation are the blueprints for more complicated situations. The optimal localisation rate is achievable under the nearly minimax optimal signal-to-noise ratio regime. Recalling Section 1.1, in the univariate mean change point problem, all three optimality goals are achieved, saving for logarithmic factors. However, this phenomenon is not always true and we will see later.

We remark on some comparisons between Algorithms 1 and 2.

- There is only one tuning parameter $\lambda$ in Algorithm 1, but there are in fact two in Algorithm 2, $\tau$ and $C_R$ (involved in the upper bound on the random interval lengths, see Theorem 3). Since all tuning parameters need to be specified in practice, Algorithm 1 is superior than Algorithm 2 in this aspect.

- The worst-case computational costs for Algorithms 1 and 2 are of order $O(T^2)$ and $O(T^3)$, respectively.

- Beyond univariate mean change point problems, one key in the construction of the optimisation problem in Algorithm 1 is a proper choice of the cost function $H(\cdot)$ in (6). This is not a problem in many cases, but might be a problem in nonparametric cases, which we will discuss in Section 5.4. In this case, Algorithm 2 may enjoy some flexibility in constructing a corresponding CUSUM statistic according to the choice of $\| \cdot \|_*$ in the model assumption.

In this section so far, we have only studied the situations where the data are independent and identically distributed between two consecutive change points. There have been some works on the possible relaxations.

- Temporal dependence. There are two popular ways to impose temporal dependence: one is through the noise sequence and the other is to assume the data are from some time series models.

As for the former, if one aims for fixed sample results, the results can easily be extended by using concentration inequalities developed for dependent data (e.g. Delyon, 2009). This method is adopted in Padilla et al. (2019a), among others. Alternatively, if only asymptotic results are required, then one can just assume that the noise sequence is uncorrelated. A scaled sum of the noise sequence can be shown to follow a Brownian bridge in the limit (e.g. Aue et al., 2008; Lavielle, 1999).

As for the latter, the existing literature handles correlated data includes Wang et al. (2020b), Wang et al. (2019b), Dette et al. (2018a), Akashi et al. (2018), Dette et al. (2018c) and Aue et al. (2009b), among others.

- Robust estimators. If the data between two consecutive change points are not necessarily identically distributed, then certain forms of robust estimation is required. Works along this line include Fearnhead and Rigaill (2019), Pein et al. (2015) and Yu and Chen (2019), among others.
3 Extension 1: Piecewise polynomials

3.1 Overview

The studies on the univariate piecewise constant change point detection problem lays the foundation for studying more complicated problems. One direction to generalise the results we discussed in Section 2 is to piecewise polynomials with any arbitrary but fixed orders. In this section, we are concerned with the model, for each \( t \in \{1, \ldots, T\} \),

\[
X_t = \theta_t + \varepsilon_t = f(t/T) + \varepsilon_t,
\]

where \( f(\cdot) \) is an unknown function belonging to the class \( \mathcal{F}^{r,K} \), defined as

\[
\mathcal{F}^{r,K} = \left\{ f(\cdot) : [0, 1] \to \mathbb{R} : f \text{ has } K + 1 \text{ pieces and each piece is a right-continuous with left limit polynomial of order at most } r \right\},
\]

The first task we have is to quantify the changes at every change point.

**Definition 2.** Let \( f(\cdot) \in \mathcal{F}^{r,K} \), \( \{s_k\}_{k=1}^{K} \) be the collection of all the change points of \( f(\cdot) \), and \( s_0 = 0, s_{K+1} = 1 \). For any \( k \in \{1, \ldots, K\} \), let \( f_{[s_{k-1}, s_{k+1})}(\cdot) : [s_{k-1}, s_{k+1}) \to \mathbb{R} \) be the restriction of \( f(\cdot) : [0, 1] \to \mathbb{R} \) on \( [s_{k-1}, s_{k+1}) \). Define the reparameterisation of \( f_{[s_{k-1}, s_{k+1})}(\cdot) \) as

\[
f(x) = \begin{cases} 
\sum_{l=0}^{r} a_l (x - s_k)^l, & x \in [s_{k-1}, s_k), \\
\sum_{l=0}^{r} b_l (x - s_k)^l, & x \in [s_k, s_{k+1}), 
\end{cases}
\]

where \( \{a_l, b_l\}_{l=0}^{r} \subset \mathbb{R} \). Define the jump associated with the change point \( s_k \) as

\[
\kappa_k = |a_{r_k} - b_{r_k}| > 0,
\]

where

\[
r_k = \min\{l = 0, \ldots, r : a_l \neq b_l\}.
\]

**Assumption 2.** Assume that the data \( \{X_t\}_{t=1}^{T} \) are generated from (9), where \( f(\cdot) \) belongs to \( \mathcal{F}^{r,K} \) defined in (10) and \( \varepsilon_i \)'s are independent zero mean sub-Gaussian random variables with \( \max_{i=1}^{n} \|\varepsilon_i\|_{\psi_2} \leq \sigma^2 \).

Let \( \theta = (\theta_t)_{t=1}^{T} \), with \( \theta_t = f(t/T) \), be the discretised \( f(\cdot) \) on the grid of \( \{1/T, 2/T, \ldots, 1\} \). We denote the collection of all change points of \( \theta \) to be \( \{\eta_1, \ldots, \eta_K\} \), satisfying

\[
\Delta = \min_{k \in \{1, \ldots, K+1\}} (\eta_k - \eta_{k-1}) > 0
\]

where \( \eta_0 = 1 \) and \( \eta_{K+1} = T + 1 \).

In addition, for any \( k \in \{1, \ldots, K\} \), let

\[
\kappa = \min_{k=1, \ldots, K} \kappa_k > 0,
\]

where \( \kappa_k \) is defined in Definition 2.
Comparing to Assumption 1, in Assumption 2 we can see that the underlying signals are allowed to be any arbitrary but fixed order of polynomials, instead of just constants. Besides this apparent difference, we would like to highlight a few more.

- With the sample size $T$, the time scales in Assumptions 1 and 2 are $O(1)$ and $O(1/T)$, respectively. These two are in fact equivalent, but we adopt the two different scales to follow the suit in the existing literature. Estimating piecewise polynomial signals has a rich body of literature, including Shen et al. (2020), Mammen and van de Geer (1997), Tibshirani (2014), Rudin et al. (1992), Zhang (2002) and Chatterjee et al. (2015), among others.

- The jumps in Assumption 1 are characterised by the mean changes, which are natural due to the piecewise-constant features. For two different at-most-order-$r$ polynomials, they are specified by two coefficient vectors. There are different ways to measure the difference between two different coefficient vectors. We characterise the distance in Assumption 2 (c) – this provides the sharpest localisation rates.

3.2 Consistent localisation

Recall that the penalised estimator we studied in Section 2.2.1 is a penalised sum of residuals. The residuals are defined to be the residuals after projecting data onto the space spanned by the all one vector. To be specific, for the interval $I$ and its corresponding data vector $X_I = (X_i, i \in I)^\top$, the projection matrix is defined to be $\Pi_{I,0} = \Pi_I = I_{|I|}(I_{|I|})^{-1}1_{|I|}$. We add the extra subscript 0 in $\Pi_I$, since constants are order-0 polynomials. When we move from piecewise constant signals to piecewise polynomial signals, one can generalise the projection matrix correspondingly.

Let $I = [s,e] \subset \{1, \ldots, T\}$, $r \in \mathbb{N}$ and

$$U_{I,r} = \begin{pmatrix} 1 & s/T & \cdots & (s/T)^r \\ \vdots & \vdots & \ddots & \vdots \\ 1 & e/T & \cdots & (e/T)^r \end{pmatrix} \in \mathbb{R}^{(e-s+1) \times (r+1)}.$$  

(13)

We define

$$\Pi_{I,r} = U_{I,r}(U_{I,r}^\top U_{I,r})^{-1}U_{I,r}^\top$$

(14)

to be the order-$r$ polynomial projection matrix. The change point estimators are the output of Algorithm 1 with

$$H(I) = \|X_I - \Pi_{I,r}X_I\|^2.$$  

(15)

The theoretical guarantees of the outputs are given below.

**Theorem 5.** Let $\{X_t\}_{t=1}^T$ satisfy Assumption 2. In addition, assume there exists a large enough constant $C_{\text{SNR}} > 0$ and any $\xi > 0$, such that

$$\min_{k=1,\ldots,K} \frac{k^2 \Delta_{2r_k+1}}{\sigma^2 T^{2r_k}} \geq C_{\text{SNR}} \log^{1+\xi}(T).$$  

(16)

With $\lambda = C_\lambda K\sigma^2 \log(T)$, let $\{\hat{\eta}_k\}_{k=1, \ldots, \hat{K}}$ be the collection of change point estimators from Algorithm 1, with $H(\cdot)$ defined in (15) and $C_\lambda > 0$ being an absolute constant, satisfy that

$$\mathbb{P}\left\{ \hat{K} = K, \forall k \in \{1, \ldots, K\}, |\hat{\eta}_k - \eta_k| \leq \left[ \frac{C_e KT^{2r_k} \sigma^2 \log(T)}{k^2} \right]^{1/(2r_k+1)} \right\} > 1 - T^{-c},$$

(17)

where $c, C_e > 0$ are absolute constant.
The assumption (16) and the localisation error in (17) show that the penalised estimator is consistent. To be specific,

\[
\frac{1}{\Delta} \max_{k=1,\ldots,K} \left[ \frac{C_k KT^2 r_k \sigma^2 \log(T)}{\kappa^2_k} \right]^{1/(2r_k+1)} \lesssim \max_{k=1,\ldots,K} \left[ \frac{K \sigma^2 \log(T) T^{2r_k}}{\kappa^2_k \Delta^{2r_k+1}} \right]^{1/(2r_k+1)} \lesssim \log^{-1/(2r_k+1)}(T) \to 0,
\]
as \(T\) grows unbounded.

Note that in (17), the larger \(r_k\) is, the larger the localisation error is. This explains our choice of distribution difference in Definition 2. Choosing the smallest order with different coefficients yields sharpest localisation errors.

### 3.3 Optimal localisation

The signal-to-noise ratio and the localisation error lower bounds are presented in Lemmas 6 and 7, respectively.

**Lemma 6.** Under Assumption 2, assume that there exists one and only one change point and \(d_1 = r\). Let \(P_{\kappa,\Delta,\sigma, r, T}\) denote the joint distribution of the data. For a small enough \(c_1 > 0\), consider the class

\[
P^T = \left\{ P_{\kappa,\Delta,\sigma, r, T} : \Delta = \min \left\{ \left[ \frac{c_1 T^{2r}}{\kappa^2 \sigma^2} \right]^{1/(2r+1)} , T/3 \right\} \right\}.
\]

Then we have

\[
\inf \sup_{\hat{\eta}} \mathbb{E}_P(\|\hat{\eta} - \eta(P)\|) \geq c,
\]

where \(\eta(P)\) is the location of the change point for distribution \(P\), the minimum is taken over all the measurable functions of the data and \(0 < c < 1\) is an absolute constant depending on \(c_1\).

**Lemma 7.** Under Assumption 2, assume that there exists one and only one change point and \(d_1 = r\). Let \(P_{\kappa,\Delta,\sigma, r, T}\) denote the joint distribution of the data. Consider the class

\[
Q^T = \left\{ P_{\kappa,\Delta,\sigma, r, T} : \Delta < T/2, \kappa^2 \Delta^{2r+1} \geq \sigma^2 T^{2r} \zeta_T \right\},
\]

for any diverging sequence \(\{\zeta_T\}\). Then for all \(T\) large enough, it holds

\[
\inf \sup_{\hat{\eta}} \mathbb{E}_P(\|\hat{\eta} - \eta(P)\|) \geq \max \left\{ 1, \left[ \frac{c \sigma^2}{T \kappa^2} \right]^{1/(2r+1)} \right\},
\]

where \(\eta(P)\) is the location of the change point for distribution \(P\), the minimum is taken over all the measurable functions of the data and \(0 < c < 1\) is an absolute constant.

We can see from Lemmas 6 and 7 that both the signal-to-noise ratio condition and the localisation error we have in Theorem 5 are off by a logarithmic factor and another factor of \(K\). Since \(K\) is allowed to diverge, Theorem 5 is sub-optimal in detection and localisation. We conjecture that this sub-optimality is due to an artefact of the proof and the optimisation problem we study in Theorem 5 should have been nearly-optimal only off by a logarithmic factor. Having said this, in order to improve, we present a refinement step.
Theorem 8. Under all the assumptions in Theorem 5, let \( \{\nu_k\}_{k=1}^K \) satisfy
\[
T \max_{k=1,\ldots,K} |\nu_k - \eta_k| \leq \Delta/5.
\] (18)

For each \( k \in \{1,\ldots,K\} \), define
\[
s_k = T\nu_{k-1}/2 + T\nu_k/2, \quad e_k = T\nu_k/2 + T\nu_{k+1}/2 \quad \text{and} \quad I_k = (s_k, e_k),
\]
with \( \nu_0 = 1/T \) and \( \nu_{K+1} = 1 + 1/T \). For \( k \in \{1,\ldots,K\} \), we let
\[
\bar{\eta}_k = \min_{i \in I_k} \left( \|X_{[s_k,t)} - \Pi_{[s_k,t),r}X_{[s_k,t)}\|^2 + \|X_{[t,e_k)} - \Pi_{[t,e_k),r}X_{[t,e_k)}\|^2 \right),
\]
where \( \Pi_{\cdot,r} \) is defined in (13) and (14). Then we have
\[
P \left\{ \forall k \in \{1,\ldots,K\}, |\bar{\eta}_k - \eta_k| \leq \left[ \frac{C_c\sigma^2T^{2r_k} \log(T)}{\kappa_k^2} \right]^{1/(2r_k+1)} \right\} > 1 - T^{-c},
\]
where \( c, C_c > 0 \) are absolute constant.

Theorem 8 guarantees that, under (18), one can achieve nearly optimal localisation errors. Since Theorem 5 shows the outputs of Algorithm 1 with (15) satisfy (18) with large probability, Theorem 8 can be used as a second step to refine the outputs of Algorithm 1.

As for the signal-to-noise ratio, there is still a gap of order \( K \) between the detection upper and lower bounds, therefore we are not able to fulfill the three goals listed in Section 1.1. We conjecture that this gap is due to a loose upper bound and the loose upper bound is due to a loose control of some cross terms. We explain it below.

Lemma 9. Let \( I_1 \) and \( I_2 \) denote any two disjoint intervals of \( \{1,\ldots,T\} \) and \( I = I_1 \cup I_2 \). For any sequences \( \{X_i\}_{i=1/T,2/T,\ldots,1} \subset \mathbb{R} \), it holds that
\[
\|X_I - \Pi_{I,r}X_I\|^2 = \|X_{I_1} - \Pi_{I_1,r}X_{I_1}\|^2 + \|X_{I_2} - \Pi_{I_2,r}X_{I_2}\|^2 + Q(I_1, I_2, \{X\}, r),
\]
where
\[
Q(I_1, I_2, \{X\}, r) = \left\{X_{I_1}^T U_{I_1,r} U_{I_1,r}^{-1} - X_{I_2}^T U_{I_2,r} (U_{I_2,r} U_{I_2,r})^{-1} \right\}\left\{\frac{U_{I_1,r} U_{I_1,r}^{-1}}{I_1} + \frac{U_{I_2,r} U_{I_2,r}^{-1}}{I_2}\right\}^{-1}
\times \left\{\frac{U_{I_1,r} U_{I_1,r}^{-1}}{I_1} X_{I_1} - \frac{U_{I_2,r} U_{I_2,r}^{-1}}{I_2} X_{I_2}\right\}
\]
and \( U_{\cdot,r} \) is defined in (13).

The key to provide localisation errors is to lower bound the cross term \( Q(I_1, I_2, \{X\}, r) \), when partitioning \( I \) into \( I_1 \) and \( I_2 \) provides a good estimator of a change point; and to upper bound \( Q(I_1, I_2, \{X\}, r) \), if partitioning \( I \) into \( I_1 \) and \( I_2 \) leads to over-partitioning.

Note that when \( r = 0 \), i.e. in the piecewise constant case,
\[
Q(I_1, I_2, \{\mathbb{E}(X)\}, 0) = \frac{|I_1||I_2|}{|I_1| + |I_2|} \left| \frac{|I_1|^{-1} \sum_{i \in I_1} \mathbb{E}(X_i) - |I_2|^{-1} \sum_{i \in I_2} \mathbb{E}(X_i)}{} \right|^2.
\]

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In addition, it holds that
\[
\frac{\min\{|I_1|, |I_2|\}}{2} \geq \frac{|I_1||I_2|}{2\max\{|I_1|, |I_2|\}} \leq \frac{|I_1||I_2|}{|I_1| + |I_2|} \leq \min\{|I_1|, |I_2|\}.
\]

If \( I_1 = [s, \eta) \) and \( I_2 = [\eta, e) \), and if \( \eta \) is the only true change point in \( I_1 \cup I_2 \), then we have
\[
Q(I_1, I_2, \{E(X)\}, 0) \propto \min\{|I_1|, |I_2|\} \kappa^2.
\]

As for general \( r \) and for the case discussed above, we have that
\[
Q(I_1, I_2, \{E(X)\}, r) \geq \frac{\min\{|I_1|, |I_2|\}}{2} \kappa^2,
\]
but lacks an upper bound of the same order. The lack of such an upper bound directly resulted in the term \( K \) in (16). We remark that, when \( r = 1 \), a similar estimator is studied in Fearnhead et al. (2019), where \( K \) is assumed to be an absolute constant. In general, an ideal solution to match the three goals in Section 1.1 is yet known. We conjecture the ideal solution is reachable if one can provide an upper bound of order \( \min\{|I_1|, |I_2|\} \kappa^2 \) in (19).

### 3.4 Conclusions

The parting words of this section are about the refinement step we discussed in Theorem 8. The refinement idea will appear again in more complicated situations, for instance the high-dimensional graphon in Theorem 16, the high-dimensional linear regression in Theorem 20 and Wang et al. (2019a), in the high-dimensional vector autoregressive models in Wang et al. (2019b).

The motivation of adopting an additional step is that directly localising multiple change points may lead to a consistent but not necessarily optimal localisation error rates. An additional step is useful if at least one of the following situations holds.

- The sub-optimality of the directly localising multiple change points is due to the multiple change points. This is the case we have in Theorem 5. The additional step works in the interval contains, with large probability, one and only one true change point, and therefore improves the rate with respect to the number of change points.

- The sub-optimality of the directly localising multiple change points is due to the choice of estimators of the underlying distributions, and there exists better estimators of the underlying distributions. This is the case we have in Theorems 16 and 20.

Although we emphasised at the beginning that this survey is not covering in-depth results and discussions on the computational aspect, this is a serious issue especially in the more complicated data type scenarios. In order to save computational costs, in the multiple change points case, one may first use some computationally-cheaper estimation in either the \( H(\cdot) \) function in (6) or the CUSUM statistic defined in Definition 1. Then in the refinement step, since the optimisation in each working interval is independent and there is only one change point in each interval, one may want to adopt some computationally-more-expensive estimator to yield better estimation.
4 Extension 2: High-dimensional problems

As we have mentioned in Section 1, the change point analysis is by no means restricted to detecting changes in a sequence of univariate data. In this section, we consider three high-dimensional extensions, the data of which are sequences of high-dimensional vectors, high-dimensional matrices and high-dimensional regression coefficients, respectively. For each of these three scenarios, we use one type of change point problem to illustrate, and we conclude this section with other problems studied in the existing literature, in addition to some insights on how the high-dimensionality affects the difficulties of the problems.

4.1 Covariance changes

The first high-dimensional case we consider is a sequence of high-dimensional random vectors, the covariances of which are piecewise constant. The model is detailed below.

**Assumption 3.** Let \( \{X_t\}_{t=1}^T \subset \mathbb{R}^p \) be independent, zero mean random vectors such that \( \mathbb{E}(X_tX_t^\top) = \Sigma_t \) and \( \|X_t\|_{\psi_2} \leq \sigma \) for all \( t = 1, \ldots, T \), where \( \sigma > 0 \). Let \( \{\eta_0, \ldots, \eta_{K+1}\} \subset \{1, \ldots, T+1\} \) be a strictly increasing subsequence of change points such that \( \eta_0 = 1 \), \( \eta_{K+1} = T+1 \) and \( \Sigma_t \neq \Sigma_{t-1} \) if and only if \( t \in \{\eta_1, \ldots, \eta_K\} \). The minimal spacing between jumps is defined to be

\[
\Delta = \min_{k=1,\ldots,K+1} \{\eta_k - \eta_{k-1}\} > 0,
\]

and the magnitude of changes is

\[
\kappa = \min_{k=1,\ldots,K} \kappa_k = \min_{k=1,\ldots,K} \|\Sigma_{\eta_k} - \Sigma_{\eta_{k-1}}\|_{\text{op}} > 0.
\]

Note that the \( \|\|_{\psi_2} \) is the Orlicz-\( \psi_2 \)-norm or the sub-Gaussian norm, defined as, for any random vector \( X \in \mathbb{R}^p \),

\[
\|X\|_{\psi_2} = \sup_{v \in \mathbb{R}^p, \|v\|=1} \|X^\top v\|_{\psi_2}.
\]

The parameters \( \sigma \) and \( \kappa \) reflect the magnitudes of noise and signal, respectively. However they are not variation independent, as they satisfy the inequality \(\kappa \leq \sigma^2/4\), due to the following derivations:

\[
\kappa \leq \max_{k=1}^K \|\Sigma_{\eta_k} - \Sigma_{\eta_{k-1}}\|_{\text{op}} \leq 2 \max_{t=1}^T \|\Sigma_t\|_{\text{op}} = 2 \max_{t=1}^T \sup_{v \in \mathbb{R}^p, \|v\|=1} \mathbb{E}[(v^\top X_t)^2] \leq 4 \max_{t=1}^T \|X_t\|_{\psi_2}^2 \leq 4\sigma^2.
\]

This is a trademark of the covariance change point problems and provides extra difficulties as opposed to mean change point problems – the larger the jumps are, the larger the variances are.

To thoroughly understand the difficulties of high-dimensional covariance change point problems, we provide the minimax lower bounds on the detection and localisation errors, which are collected in Lemmas 10 and 11, respectively.

**Lemma 10** (Lemma 3 in Wang et al., 2017). Under Assumption 3, assume that there is one and only one change point. Let \( P_{\kappa,\Delta,\sigma,p,T} \) denote the joint distribution of the data. Consider the class of distributions

\[
P^T = \left\{ P_{\kappa,\Delta,\sigma,p,T} : \Delta \leq \min \left\{ \frac{2\sigma^4 p}{33\kappa^2}, \frac{T}{3} \right\}, \kappa \leq \sigma^2/4 \right\}.
\]
We have that,
\[
    \inf_{\tilde{\eta}} \sup_{P \in \mathcal{P}} \mathbb{E}_P(|\tilde{\eta} - \eta(P)|) \geq \Delta/2,
\]
where \(\eta(P)\) is the location of the change point of distribution \(P\) and the infimum is over all estimators of the change point.

**Lemma 11** (Lemma 4 in Wang et al., 2017). Under Assumption 3, assume that there is one and only one change point. Let \(P_{\kappa, \Delta, \sigma, p, T}\) denote the joint distribution of the data. Consider the class of distributions
\[
    \mathcal{Q}^T = \{ P_{\kappa, \Delta, \sigma, p, T} : \Delta \kappa^2 \geq p \log(T) \sigma^4, \kappa \leq \sigma^2/4, 4 \leq \Delta \leq 4/5(T - 1) \}.
\]
Then,
\[
    \inf_{\tilde{\eta}} \sup_{P \in \mathcal{Q}} \mathbb{E}_P(|\tilde{\eta} - \eta(P)|) \geq \frac{\sigma^4}{20 \kappa^2},
\]
where \(\eta(P)\) is the location of the change point of distribution \(P\) and the infimum is over all estimators of the change point.

Lemmas 10 and 11 show that in the low signal-to-noise ratio regime \(\kappa^2 \Delta \lesssim p \sigma^4\), no algorithm is guaranteed to produce consistent change point estimators. The optimal localisation error in the high signal-to-noise ratio regime \(\kappa^2 \Delta \gtrsim p \sigma^4 \log(T)\) is \(\sigma^4 \kappa^{-2}\).

A scan-statistics-based algorithm is able to achieve the near-optimality in the sense of detection boundary and localisation error. The detailed algorithm is given in Algorithm 4, with a subroutine specified in Algorithm 3 and theoretical guarantees available in Theorem 12. The quantity \(\tilde{Y}_{s_m,e_m}(u_m)\) is a CUSUM statistic defined in Definition 1.

**Algorithm 3** Principal Component Estimation PC((\(X_t\))\(_{t=1}^T\), \((\alpha_m, \beta_m)\)\(_{m=1}^M\))

**INPUT:** \(\{X_t\}_{t=1}^T\), \((\alpha_m, \beta_m)\)\(_{m=1}^M\)

**for** \(m = 1, \ldots, M\) **do**

**if** \(\beta_m - \alpha_m > 2p \log(T) + 1\) **then**

\(d_m \leftarrow \arg \max_{[\alpha_m + p \log(T)] \leq t \leq [\beta_m - p \log(T)]} \| S_{\alpha_m, \beta_m} \|_{op} \)

\(u_m \leftarrow \arg \max_{\|v\|=1} |v^T S_{d_m} \tilde{\eta}_{\alpha_m, \beta_m} v| \)

**else**

\(u_m \leftarrow 0\)

**end if**

**end for**

**OUTPUT:** \(\{u_m\}_{m=1}^M\).

The requirement of two independent samples in Algorithm 4 can be achieved by splitting data into even and odd indices subsets.

**Theorem 12** (Theorem 2 in Wang et al., 2017). Let Assumption 3 hold and let \((\alpha_m, \beta_m)\)\(_{m=1}^M \subset (0, T)\) be a collection of intervals whose endpoints are drawn independently and uniformly from \(\{1, \ldots, T\}\) and such that \(\max_{1 \leq m \leq M} (\beta_m - \alpha_m) \leq C_R \Delta\) for an absolute constant \(C_R > 0\). In
Algorithm 4: Wild Binary Segmentation through Independent Projection.

**INPUT:** Two independent samples, \( \{W_t\}_{t=1}^T \) and \( \{X_t\}_{t=1}^T \), and the threshold parameter \( \tau > 0 \).

\[
u_m \leftarrow \text{PC}(\{W_t\}_{t=1}^T, \{(\alpha_m, \beta_m)\}_{m=1}^M)
\]

for \( t \in \{s, \ldots, e\} \) do
  for \( m = 1, \ldots, M \) do
    \[
    Y_t(u_m) \leftarrow (u_m^\top X_t)^2
    \]
  end for
end for

for \( m = 1, \ldots, M \) do
  \[
  (s_m, e_m) \leftarrow [s, e] \cap [\alpha_m, \beta_m]
  \]
  if \( e_m - s_m \geq 2 \log(T) + 1 \) then
    \[
    b_m \leftarrow \arg \max_{s_m + \log(T) \leq t \leq e_m - \log(T)} |\tilde{Y}_{s_m, e_m}^m(u_m)|
    \]
  else
    \[
    a_m \leftarrow -1
    \]
  end if
end for

\[
m^* \leftarrow \arg \max_{m=1, \ldots, M} a_m
\]

if \( a_{m^*} > \tau \) then
  add \( b_{m^*} \) to the set of estimated change points
  WBSIP(\{X_t, W_t\}_{t=1}^T, (s, b_{m^*}), \{(\alpha_m, \beta_m)\}_{m=1}^M, \tau)
  WBSIP(\{X_t, W_t\}_{t=1}^T, (b_{m^*} + 1, e), \{(\alpha_m, \beta_m)\}_{m=1}^M, \tau)
end if

**OUTPUT:** The set of estimated change points.

**Supplementary Content:**

addition, assume that for any \( \xi > 0 \), there exists a sufficiently large absolute constant \( C > 0 \) such that

\[
\Delta \kappa^2 \geq Cp \log^{1+\xi}(T) \sigma^4.
\]

Suppose there exist sufficiently small constant \( c_2 > 0 \) and sufficiently large constant \( c_3 > 0 \) such that the input parameter \( \tau \) satisfy

\[
c_3 \sigma^2 \sqrt{\log(T)} < \tau < c_2 \kappa \sqrt{\Delta}.
\]

Then the collection of the estimated change points \( \{\hat{\eta}_k\}_{k=1}^{\hat{K}} \) returned by Algorithm 4 with input parameters of \( (0, T), \{(\alpha_m, \beta_m)\}_{m=1}^M \) and \( \tau \), satisfies

\[
P\left\{ \hat{K} = K \quad \text{and} \quad |\eta_k - \hat{\eta}_k| \leq C_1 \sigma^4 \log(T) \kappa^{-2}, \forall k \in \{1, \ldots, K\} \right\}
\geq 1 - 4MT^{2-c} - 4 \times 9^pT^{3-cp} \exp\{\log(T/\Delta) - M\Delta/(4C_RT)\},
\]

for some absolute constants \( c > 3 \) and \( C_1 > 0 \).

Provided that \( \log(T/\Delta) \lesssim M\Delta/T \), we see from Theorem 12 that Algorithm 4 is nearly optimal in terms of both detection and localisation, despite that the dimension \( p \) is allowed to grow.
unbounded as the sample size diverges. This means all three goals we listed in Section 1.1 are achieved. It might come as a surprise that the optimal localisation error rate is not a function of the dimension $p$, and we are actually able to achieve it under the minimal conditions. We will come back to discuss this phenomenon in Section 4.4, together with more high-dimensional cases.

Finally, the covariance change point analysis has also been studied in different settings over the years, including Inclan and Tiao (1994), Gombay et al. (1996), Dette et al. (2018b), Avanesov and Buzun (2016), Birke and Dette (2005) and Aue et al. (2009b), among others.

4.2 Graphon changes
4.2.1 Overview

Instead of obtaining a random vector at every time, the random objects obtained can be in the form of random matrices. With the surging of network data, we use a dynamic networks model as an example and study the graphon changes in this subsection. The detailed model assumptions are collected in Assumption 4, with a general definition in Definition 3.

**Definition 3 (Inhomogeneous Bernoulli networks).** A network with node set \( \{1, \ldots, n\} \) is an inhomogeneous Bernoulli network if its adjacency matrix \( A \in \mathbb{R}^{n \times n} \) satisfies

\[
A_{ij} = A_{ji} = \begin{cases} 
1, & \text{nodes } i \text{ and } j \text{ are connected by an edge,} \\
0, & \text{otherwise}; 
\end{cases}
\]

and \( \{A_{ij}, i < j\} \) are independent Bernoulli random variables with \( \mathbb{E}(A_{ij}) = \Theta_{ij} \).

**Assumption 4.** Let \( \{X_t\}_{t=1}^T \subset \mathbb{R}^{n \times n} \) be a collection of adjacency matrices of independent inhomogeneous Bernoulli networks with means \( \{\Theta_t\}_{t=1}^T \) satisfying the following properties.

The sparsity parameter

\[
\rho = \max_{t=1, \ldots, T} \|\Theta_t\|_\infty
\]

is such that \( \rho n \geq \log(n) \), where \( \| \cdot \|_\infty \) denotes the entrywise maximum norm of a matrix.

There exists a sequence \( \eta_0 < \eta_1 < \ldots < \eta_{K+1} \) of time points, called change points, with \( \eta_0 = 1 \) and \( \eta_{K+1} = T + 1 \), such that

\[
\Theta_t \neq \Theta_{t-1}, \quad \text{if and only if} \quad t \in \{\eta_k\}_{k=1}^K.
\]

The minimal spacing between two consecutive change points satisfies

\[
\min_{k=1, \ldots, K+1} \{\eta_k - \eta_{k-1}\} = \Delta > 0.
\]

The magnitudes of the changes in the data generating distribution are such that

\[
\|\Theta(\eta_k) - \Theta(\eta_k - 1)\|_F = \kappa_k, \quad k = 1, \ldots, K,
\]

where \( \| \cdot \|_F \) denotes the Frobenius norm of a matrix. Let

\[
\kappa_0 = \frac{\kappa}{n\rho} = \frac{\min_{k=1, \ldots, K} \kappa_k}{n\rho}.
\]
For dynamic networks, the data are a sequence of adjacency matrices and their distributions are determined by their graphons, i.e. the expectations of the adjacency matrices, if we assume the networks are inhomogeneous Bernoulli networks defined in Definition 3. Then in terms of characterising jumps, it would be natural to seek a certain matrix norm. In Section 4.1, the matrix operator norm is adopted, and in Assumption 4, the matrix Frobenius norm is summoned. One could argue that as for network models, the Frobenius norm is able to capture a richer collection of changes. In this survey, we would pay more attention on how different choices of norms affect the difficulty of the problems. We will come back to this in Section 4.4.

The difficulty of the graphon change point analysis is explained in Lemmas 13 and 14, on the minimax lower bounds on detection and localisation errors, respectively.

Lemma 13 (Lemma 1 in Wang et al., 2018). Let \( \{X_t\}_{t=1}^T \) be a sequence of independent inhomogeneous Bernoulli networks satisfying Assumption 4 with \( K = 1 \). Let \( P^{T}_{\kappa_0,\Delta,n,\rho} \) denote the corresponding joint distribution. For \( \zeta \leq 1/33 \), consider the class of distributions

\[
\mathcal{P}^T = \left\{ P^{T}_{\kappa_0,\Delta,n,\rho} : \Delta = \min \left\{ \left\lfloor \frac{\log(T)\zeta}{n\rho\kappa_0^2} \right\rfloor, \lfloor T/3 \rfloor \right\}, \rho \leq 1/2, \kappa_0 \leq 1 \right\}.
\]

For each \( P \in \mathcal{P}^T \), let \( \eta(P) \in \{1, \ldots, T\} \) denote the location of the corresponding change point. It holds that

\[
\inf_{\hat{\eta}} \sup_{P \in \mathcal{P}^T} \mathbb{E}_P(|\hat{\eta} - \eta(P)|) \geq 3\Delta/4,
\]

where the infimum is over all the possible estimators of the change point location.

Lemma 14 (Lemma 2 in Wang et al., 2018). Let \( \{X_t\}_{t=1}^T \) be a sequence of independent inhomogeneous Bernoulli networks satisfying Assumption 4 with \( K = 1 \). Let \( P^{T}_{\kappa_0,\Delta,n,\rho} \) denote the corresponding joint distribution. Consider the class of distributions

\[
\mathcal{Q}^T = \left\{ P^{T}_{\kappa_0,\Delta,n,\rho} : \kappa_0 \leq 1/2, \rho \leq 1/2 \right\}.
\]

It holds that

\[
\inf_{\hat{\eta}} \sup_{P \in \mathcal{Q}^T} \mathbb{E}_P(|\hat{\eta} - \eta(P)|) \geq \max\{c\kappa_0^{-2}n^{-2}\rho, 1/2\},
\]

where \( \eta(P) \) denotes the location of the corresponding change point and the infimum is over all the possible estimators of the change point location.

Lemmas 13 and 14 show that in the low signal-to-noise ratio regime \( \kappa_0 \sqrt{\Delta} \lesssim (n\rho)^{-1/2} \log^{1/2}(T) \), no algorithm is guaranteed to be consistent, and the localisation error is lower bounded by \( \rho \kappa_0^{-2}n^{-2} \). To match these lower bounds, we provide two sets of algorithms.

- Algorithm 5 is theoretically supported by Theorem 15, showing that there exists a computationally-efficient method providing consistent change point estimators, with a nearly optimal signal-to-noise ratio condition.
- Algorithm 7, with a subroutine in Algorithm 6, is theoretically supported by Theorem 16, showing that under a stronger condition, there exists a method providing nearly optimal localisation errors.
Algorithm 5 Network Binary Segmentation. \( \text{NBS}((s,e), \{(\alpha_m, \beta_m)\}_{m=1}^M, \tau_1) \)

**INPUT:** Two independent samples \( \{X_t\}_{t=1}^T, \{W_t\}_{t=1}^T \in \mathbb{R}^{n \times n}, \tau_1 \).

for \( m = 1, \ldots, M \) do

\[
\begin{align*}
[s'_m, e'_m] &\leftarrow [s, e] \cap [\alpha_m, \beta_m] \\
(s_m, e_m) &\leftarrow [s'_m + 64^{-1}(e'_m - s'_m), e'_m - 64^{-1}(e'_m - s'_m)]
\end{align*}
\]

if \( e_m - s_m \geq 1 \) then

\[
\begin{align*}
b_m &\leftarrow \arg \max_{s_m + 1, \ldots, e_m - 1} (\tilde{X}_{s_m, e_m}, \tilde{W}_{s_m, e_m}) \\
a_m &\leftarrow (X_{s_m, e_m}, W_{s_m, e_m})
\end{align*}
\]

else

\[
a_m &\leftarrow -1
\]

end if

end for

\( m^* \leftarrow \arg \max_{m=1, \ldots, M} a_m \)

if \( a_{m^*} > \tau_1 \) then

add \( b_{m^*} \) to the set of estimated change points

\[
\text{NBS}((s, b_{m^*}), \{(\alpha_m, \beta_m)\}_{m=1}^M, \tau_1)
\]

\[
\text{NBS}(b_{m^*} + 1, e, \{(\alpha_m, \beta_m)\}_{m=1}^M, \tau_1)
\]

end if

**OUTPUT:** The set of estimated change points.

### 4.2.2 Consistent localisation

We study a scan-statistics-based algorithm, using a network CUSUM statistic, i.e. Definition 1 with \( X_t \)'s being adjacency matrices.

**Theorem 15** (Theorem 1 in Wang et al., 2018). Let Assumption 4 hold and assume that there exists a constant \( C_\alpha > 0 \) such that, for some \( \xi > 0 \),

\[
\kappa_0 \sqrt{n} \rho \geq C_\alpha \sqrt{\frac{1}{n \Delta}} \log^{1+\xi}(T).
\]

Let \( \{(\alpha_m, \beta_m)\}_{m=1}^M \subset (0, T) \) be a collection of intervals whose end points are drawn independently and uniformly from \( [1, \ldots, T] \) and such that \( \max_{m=1, \ldots, M}(\beta_m - \alpha_m) \leq C_R \Delta \), for an absolute constant \( C_R > 0 \).

Suppose that there exists sufficiently small \( 0 < c_2 < 1 \) such that the input parameter \( \tau \) of Algorithm 5 satisfy

\[
C_\alpha \rho n \log^{3/2}(T) < \tau < C_2 \kappa_0^2 n^2 \rho^2 \Delta.
\]

Then the collection of the estimated change points \( \mathcal{B} = \{\hat{\eta}_k\}_{k=1}^K \) returned by Algorithm 5 with input parameters \( (0, T), \{(\alpha_m, \beta_m)\}_{m=1}^M \) and \( \tau \) is such that

\[
\mathbb{P} \left\{ \hat{K} = K \quad \text{and} \quad |\hat{\eta}_k - \eta_k| \leq C_{\epsilon} \log(T) \left( \frac{\sqrt{\Delta}}{\kappa_k} + n \rho \log^{1/2}(T) \frac{1}{\kappa_k^2} \right), \forall k \in \{1, \ldots, K\} \right\}
\]

\[
\geq 1 - \exp \left( \log \frac{T}{\Delta} - M \frac{\Delta^2}{16T^2} \right) - T^{-c},
\]

for some absolute constants \( C_{\epsilon}, c > 0 \) and any \( n, T \geq 2 \).
We remark that, in terms of $\kappa_0$, Theorem 15 shows the localisation error is of order
\[
\log(T) \left( \frac{\sqrt{\Delta}}{\kappa_0 \rho} + \frac{\log^{1/2}(T)}{\kappa_0^2 \rho} \right).
\]

Therefore, provided that $\log(T/\Delta) \lesssim M \Delta^2 T^{-2}$, Theorem 15 shows that Algorithm 5 achieves consistent localisation in the regime that
\[
\kappa_0 \sqrt{\rho \Delta} \gtrsim \log^{1+\xi}(T),
\]
for any $\xi > 0$. The role of $\xi$ is the same as that in Section 2. Together with Lemma 13, we know that it is nearly optimal, save a logarithmic factor. However, the localisation error achieved in Theorem 15 is sub-optimal given Lemma 14. Two natural questions await: (1) why is it sub-optimal? (2) how can it be improved?

The sub-optimality is rooted in the high-dimensionality. Algorithm 5 in fact only takes weighted sample mean of matrix inner products, which are merely $\ell_2$-norms of vectorised matrices. Even for a network of a moderately-high dimension, its vectorised version is of very high dimension. It is well-understood that merely taking sample means does not lead to good estimation in high-dimensional statistics. What we learn from Theorem 15 is that, if the goal is to localise change points consistently, then one could sacrifice some accuracy in estimating the underlying high-dimensional distributions. However, if one wishes for more accurate, say optimal change point localisation, then this sacrificed accuracy is probably to be blamed.

4.2.3 Optimal localisation

In order to improve localisation, as we discussed before, one needs to provide more accurate estimation of the underlying distributions. Just like other problems in high-dimensional statistics, some form of sparsity condition is needed. In the context of networks, a natural choice of the sparsity is the low rank assumption.

**Assumption 5.** Let $\{\Theta(t)\}_{t=1}^T$ be defined as in Assumption 4. For some $0 < r \leq n$,
\[
\max_{k=1,\ldots,K} \text{rank} (\Theta(\eta_k) - \Theta(\eta_{k-1})) \leq r.
\]

With the additional low rank assumption Assumption 5, we will show that the localisation errors can be improved.

**Theorem 16** (Theorem 2 in Wang et al., 2018). Let Assumptions 4 and 5 hold. Assume that for any $\xi > 0$, there exists an absolute constant $C_\alpha > 0$ such that
\[
\kappa_0 \sqrt{\rho \Delta} \geq C_\alpha \frac{\log^{1+\xi}(T)}{\sqrt{\Delta}} \sqrt{n}.
\]

Let $B = \{\nu_k\}_{k=1}^K \subset \{1, \ldots, T\}$ be a collection of time points. Suppose that
\[
\max_{k=1,\ldots,K} |\nu_k - \eta_k| < \Delta/6.
\]
\begin{algorithm}[H]
\caption{USVT($A, \tau_2, \tau_3$)}
\textbf{INPUT:} Symmetric matrix $A \in \mathbb{R}^{n \times n}$, $\tau_2, \tau_3 > 0$.
\begin{itemize}
    \item $(\kappa_i(A), v_i) \leftarrow$ the $i$th eigen-pair of $A$, with $|\kappa_1(A)| \geq \cdots |\kappa_n(A)|$
    \item $A' \leftarrow \sum_{i, \kappa_i(A) \geq \tau_2} \kappa_i(A) v_i v_i^\top$
\end{itemize}
USVT($A, \tau_2, \tau_3$) $\leftarrow$ ($A''_{ij}$) with
\begin{align*}
(A'')_{ij} &\left\{ \begin{array}{ll}
    (A')_{ij}, & \text{if } |(A'_{ij})| \leq \tau_3 \\
    \text{sign}((A')_{ij}) \tau_3, & \text{if } |(A'_{ij})| > \tau_3
    \end{array} \right.
\end{align*}
\textbf{OUTPUT:} USVT($A, \tau_2, \tau_3$).
\end{algorithm}

\begin{algorithm}[H]
\caption{Local Refinement}
\textbf{INPUT:} \{X(t)\}_{t=1}^T, \{W(t)\}_{t=1}^T \in \mathbb{R}^{n \times n}$, $\tau_2, \tau_3$, $\{\nu_k\}_{k=1}^K \subset \{2, \ldots, T\}$, $\nu_0 = 0$, $\nu_{K+1} = T+1$.
\begin{algorithmic}
    \For{$k = 1, \ldots, K$}
        \State $[s, e] \leftarrow \left[ 2^{-1}(\nu_{k-1} + \nu_k), 2^{-1}(\nu_k + \nu_k) \right]$ \\
        \State $\tilde{\Delta}_k \leftarrow \sqrt{\frac{e-s}{4}}$ \\
        \State $\tilde{\Theta}_k \leftarrow \text{USVT}(B^{s,e}(\nu_k), \tau_2, \tau_3 \tilde{\Delta}_k)$ \\
        \State $b_k \leftarrow \arg \max_{s \leq t \leq e} \{A^{s,e}(t), \Theta_k\}$
    \EndFor
\end{algorithmic}
\textbf{OUTPUT:} $\{b_k\}_{k=1}^K$.
\end{algorithm}

For a large enough absolute constant $C_a > 0$ suppose that
\begin{equation*}
    \tau_2 = (3/4)(C \sqrt{n} \rho + C_a \log(T)) \quad \text{and} \quad \tau_3 = \rho,
\end{equation*}
where $C > 64 \times 2^{1/4e^2}$ and $C_a > 12$. Then the outputs of Algorithm 7 with input parameters of \((0, T)\), $\{\nu_k\}_{k=1}^K$, $\tau_2$ and $\tau_3$ satisfy
\begin{equation*}
    \mathbb{P}\left\{ \max_{k=1,\ldots,K} |\eta_k - \tilde{\eta}_k| \leq C_2 \log^2(T) \kappa_0^{-2} n^{-2} \rho^{-1} \right\} \geq 1 - T^{-c},
\end{equation*}
where $C_2, c > 0$ are absolute constants.

Algorithm 7 can be seen as a refinement of a set of initial estimators, as we discussed in Section 3.4. The refinement is conducted based on a better estimation of the underlying graphons, by using the universal singular value thresholding (USVT, Algorithm 6) method developed in Chatterjee (2015). Note that the output of Algorithm 5 satisfies the condition on the initial estimators, detailed in (20), and the signal-to-noise ratio condition required in Theorem 16 is stronger than that in Theorem 15. This means Algorithm 7 can be used as a second step after Algorithm 5, and the final outputs are nearly optimal in terms of the localisation errors.

Like we discussed in Section 3.4, one may directly integrate the USVT estimation in the main algorithm Algorithm 5, but since the computation costs of conducting the singular value decomposition is of order $O(n^2)$ and the WBS procedure itself is of order $O(n^3)$. If we directly adopting Algorithm 6 in Algorithm 5, then the computational cost is $O(n^3)$. On the contrary, we use a sample mean in Algorithm 5 and use the USVT as a refinement, then the computational cost is of order $O(n^4) + O(n^3) = O(n^4)$. 

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4.2.4 Conclusions

The natural question is what happens in the regime
\[ \sqrt{\frac{1}{n\Delta}} \log^{1+\xi}(T) \lesssim \kappa_0 \sqrt{\rho} \lesssim \sqrt{\frac{r}{n\Delta}} \log^{1+\xi}(T). \]

The complete answer is yet known and we will provide some partial answers in line with other high-dimensional statistics problems.

If we replace the subroutine Algorithm 7, which is a polynomial-time algorithm, with an NP-hard graphon-based algorithm (see, e.g. Pensky, 2016; Gao et al., 2015), then we will be able to produce a nearly optimal localisation rate in the regime
\[ \kappa_0 \sqrt{\rho} \gtrsim \sqrt{\frac{1+r^2/n}{n\Delta}} \log^{1+\xi}(T). \]

This means that (i) in the very sparse regime, i.e. \( r \lesssim \sqrt{n} \), the condition required by NP-hard algorithms is nearly optimal, save for a logarithmic factor; (ii) in the moderately sparse regime, i.e. \( \sqrt{n} \lesssim r \lesssim n \), there is a gap between statistical and computational limits; (iii) in the very dense regime, i.e. \( r \asymp n \), NP-hard algorithms are not gaining over polynomial methods. These observations is consistent with similar phenomena observed in other statistical problems, see e.g. Zhang et al. (2012), Loh and Wainwright (2013), to name but a few.

Finally, the graphon change point analysis has also been studied in different settings over the years, including Zhao et al. (2019), Liu et al. (2018), Cribben and Yu (2017) and Bhattacharjee et al. (2018), among others.

4.3 Regression coefficients changes

4.3.1 Overview

In this case, we consider at every time point \( t \in \{1, \ldots, T\} \), \( (x_t, y_t) \in \mathbb{R}^p \times \mathbb{R} \) is collected, where \( y_t \)'s are response variables and \( x_t \)'s are high-dimensional covariates. In the change point analysis context, we assume the regression coefficients are piecewise constant. The detailed assumptions are collected below.

**Assumption 6.** Let the data be \( \{(x_t, y_t)\}_{t=1}^T \subset \mathbb{R}^p \times \mathbb{R} \), satisfying
\[ y_t = x_t^\top \beta_t^* + \varepsilon_t, \]
where \( \beta_t^* \in \mathbb{R}^p \) is the unknown coefficient vector, \( x_t \)'s are independent and identically distributed, and \( \varepsilon_t \)'s are independent centred sub-Gaussian random variables with parameters \( \sigma_t^2 \leq \sigma_\varepsilon^2 \) and independent of \{\( x_t \)\}.

In addition, there exists a collection of change points \( \{\eta_k\}_{k=0}^{K+1} \subset \{1, \ldots, T+1\} \) with \( \eta_0 = 1 \) and \( \eta_{K+1} = T+1 \) such that \( \beta_t^* \neq \beta_{t-1}^* \), if and only if \( t \in \{\eta_k\}_{k=1}^K \).

**Assumption 7.** Consider the model defined in Assumption 6, where \( x_t \)'s are centred sub-Gaussian random vectors with \( \mathbb{E}(x_t x_t^\top) = \Sigma \). We impose the following additional assumptions.

There exists a subset \( S \subset \{1, \ldots, p\} \) such that
\[ \beta_t^*(j) = 0, \quad t = 1, \ldots, T, \quad j \in S^c = \{1, \ldots, p\} \setminus S. \]
Let \( d_0 = |S| \).

For some absolute constant \( C_\beta > 0 \), \( \max_{t=1,...,T} \| \hat{\beta}_t^* \|_\infty \leq C_\beta \).

We have that
\[
\Delta_{\min}(\Sigma) = c^2_{T} > 0 \quad \text{and} \quad \max_{j=1,...,p} (\Sigma)_{jj} = C^2_{\epsilon} > 0.
\]

Let \( \kappa \) and \( \Delta \) be the minimal jump size and minimal spacing defined as follows, respectively,
\[
\kappa = \min_{k=1,...,K} \kappa_k = \min_{k=1,...,K} \| \beta^*_t - \beta^*_t \| \quad \text{and} \quad \Delta = \min_{k=1,...,K+1} (\eta_k - \eta_{k-1}).
\]

The difficulty of this problem is characterised in Lemmas 17 and 18, on the minimax lower bounds on detection and localisation, respectively.

**Lemma 17** (Lemma 3 in Rinaldo et al., 2020). Let \( \{(x_t, y_t)\}_{t=1}^T \subset \mathbb{R}^p \times \mathbb{R} \) satisfy Assumptions 6 and 7, with \( K = 1 \). In addition, assume \( x_t \stackrel{iid}{\sim} \mathcal{N}(0, I_p) \) and \( \varepsilon_t \stackrel{iid}{\sim} \mathcal{N}(0, \sigma^2_\varepsilon) \). Let \( P^T_{\kappa, \Delta, \sigma, c \varepsilon} \) be the corresponding joint distribution. For any \( 0 < c < \frac{2}{\inf \kappa \varepsilon} \), consider the class of distributions
\[
P^T = \{ P^T_{\kappa, \Delta, \sigma, c \varepsilon} : \Delta = \min \{ \lfloor c d_0 \sigma^2_\varepsilon \kappa^{-2} \rfloor, \lfloor T/4 \rfloor \}, 2 c d_0 \max\{d_0, 2\} \leq \Delta \}.
\]

There exists a \( T(c) \), which depends on \( c \), such that for all \( T \geq T(c) \),
\[
\inf_{\hat{\eta}} \sup_{P \in \mathbb{P}^T} \mathbb{E}_P(|\hat{\eta} - \eta(P)|) \geq \Delta,
\]
where \( \eta(P) \) is the location of the change point of distribution \( P \) and the infimum is over all estimators of the change point.

**Lemma 18** (Lemma 4 in Rinaldo et al., 2020). Let \( \{(x_t, y_t)\}_{t=1}^T \subset \mathbb{R}^p \times \mathbb{R} \) satisfy Assumptions 6 and 7, with \( K = 1 \). In addition, assume \( x_t \stackrel{iid}{\sim} \mathcal{N}(0, I_p) \) and \( \varepsilon_t \stackrel{iid}{\sim} \mathcal{N}(0, \sigma^2_\varepsilon) \). Let \( P^T_{\kappa, \Delta, \sigma, c \varepsilon} \) be the corresponding joint distribution. For any diverging sequence \( \zeta_T \), consider the class of distributions
\[
\mathbb{Q}^T = \{ P^T_{\kappa, \Delta, \sigma, c \varepsilon} : \Delta = \min \{ \lfloor \zeta_T d_0 \sigma^2_\varepsilon \kappa^{-2} \rfloor, \lfloor T/4 \rfloor \} \}.
\]

Then
\[
\inf_{\hat{\eta}} \sup_{P \in \mathbb{Q}^T} \mathbb{E}_P(|\hat{\eta} - \eta(P)|) \geq \frac{c d_0 \sigma^2_\varepsilon}{\kappa^2},
\]
where \( \eta(P) \) is the location of the change point of distribution \( P \), the infimum is over all estimators of the change point and \( c > 0 \) is an absolute constant.

Lemmas 17 and 18 show that in the low signal-to-noise ratio regime \( \kappa \sqrt{\Delta} \leq \sigma_\varepsilon \sqrt{d_0} \), no algorithm is guaranteed to be consistent, and the localisation error lower bound is \( d_0 \sigma^2_\varepsilon \kappa^{-2} \).

**4.3.2 Consistent localisation**

In order to estimate the change points, we adopt Algorithm 1. To be specific, for any interval \( I \subset \{1, \ldots, T\} \), let
\[
H(I) = \sum_{t \in I} (y_t - x_t^\top \hat{\beta}^*_t)^2,
\]
with
\[
\widehat{\beta}^* = \arg \min_{v \in \mathbb{R}^p} \left\{ \sum_{t \in I} (y_t - x_t^\top v)^2 + \gamma \max\{|I|, \log(n \lor p)\} \|v\|_1 \right\},
\]
(22)
where \(\| \cdot \|_1\) denotes the vector \(\ell_1\)-norm. With this construction, the loss function is the sum of residual squares, with a Lasso estimator of the coefficients. The theoretical guarantees of the output of Algorithm 1 with (21) and (22) are given below.

**Theorem 19** (Theorem 1 in Rinaldo et al., 2020). Let Assumptions 6 and 7 hold. Assume for any \(\xi > 0\), there exists an absolute constant \(C_{\text{SNR}} > 0\) such that
\[
\Delta \kappa^2 \geq C_{\text{SNR}} d_0^2 K \sigma_z^2 \log (1 + \xi (T \lor p)).
\]
(23)
Let \(\{\hat{\eta}_k\}_{k=1}^K\) be the output of Algorithm 1, with the objective function defined in obtained as solution to the dynamic programming optimisation problem given in (21) and (22) and with tuning parameters
\[
\gamma = C_\gamma \sigma_z \sqrt{d_0 \log (T \lor p)} \quad \text{and} \quad \lambda = C_\lambda \sigma_z^2 (K + 1) d_0^2 \log (T \lor p).
\]
It holds that
\[
\mathbb{P}\left\{ \hat{K} = K, \max_{k=1, \ldots, K} |\hat{\eta}_k - \eta_k| \leq \frac{K C_\varepsilon d_0^2 \sigma_z^2 \log (T \lor p)}{\kappa^2} \right\} \geq 1 - C (T \lor p)^{-c},
\]
where \(C_\lambda, C_\gamma, C_\varepsilon, C, c > 0\) are absolute constants depending only on \(C_\beta, C_x\) and \(c_x\).

In view of Lemmas 17 and 18, we can see that Theorem 19 requires a stronger signal-to-noise ratio and achieves a sub-optimal localisation error. We will improve the localisation in the sequel. As for the signal-to-noise ratio condition, we remark that if one further assumes \(\kappa = O(1)\) and \(K = O(1)\), then one can replace (23) with
\[
\Delta \kappa^2 \geq C_{\text{SNR}} d_0^2 \sigma_z^2 \log (1 + \xi (T \lor p)),
\]
and Theorem 19 still holds. This shows the nearly-optimality of Algorithm 1 in terms of the signal-to-noise ratio condition. However, without the extra condition that \(\kappa = O(1)\) and \(K = O(1)\), it remains an open problem in deriving a consistent change point estimator under minimal conditions.

### 4.3.3 Optimal localisation

So far we have already used the refinement idea twice. In Section 3, a refinement step is to improve the localisation rate so it is not a function of \(K\), the number of change points. In Section 4.2, a refinement step is to improve the localisation rate by providing a better estimation of the underlying high-dimensional objects. Comparing the localisation error in Theorem 19 and the minimax lower bound in Lemma 18, we see that a refinement should ideally eliminate the dependence on \(K\) and improve from \(d_0^2\) to \(d_0\). This suggests that a refinement should not only work in the intervals containing one and only one true change point, but also need to provide better estimation of the underlying distributions.
Algorithm 8 Local refinement. LR(\{(x_t, y_t)\}_{t=1}^T, \{\tilde{\eta}_k\}_{k=1}^\tilde{K}, \zeta)

**INPUT:** Data \{(x_t, y_t)\}_{t=1}^T, a collection of time points \{\tilde{\eta}_k\}_{k=1}^\tilde{K}, tuning parameter \(\zeta > 0\).

\((\tilde{\eta}_0, \tilde{\eta}_{\tilde{K}+1}) \leftarrow (0, T)\)

for \(k = 1, \ldots, \tilde{K}\) do

\((s_k, e_k) \leftarrow (\tilde{\eta}_{k-1}/3 + 2\tilde{\eta}_k/3, 2\tilde{\eta}_k/3 + \tilde{\eta}_{k+1}/3)\)

\[
\left(\hat{\beta}_1, \hat{\beta}_2, \hat{\eta}_k\right) \leftarrow \underset{\eta \in \{s_k+1, \ldots, e_k-1\}}{\text{arg min}} \frac{1}{\beta_1, \beta_2 \in \mathbb{R}^p} \left\{ \sum_{t=s_k+1}^{\eta} \|y_t - \beta_1 x_t\|^2 + \sum_{t=\eta+1}^{e_k} \|y_t - \beta_2 x_t\|^2 + \zeta \sum_{i=1}^{p} \left( (\eta - s_k)(\beta_1)_i^2 + (e_k - \eta)(\beta_2)_i^2 \right) \right\}
\]

end for

**OUTPUT:** The set of estimated change points \{\hat{\eta}_k\}_{k=1}^\tilde{K}

**Theorem 20** (Corollary 2 in Rinaldo et al., 2020). Assume the same conditions of Theorem 19.

Let \{\tilde{\eta}_k\}_{k=1}^{\tilde{K}} be a set of time points satisfying

\[
\max_{k=1,\ldots,K} |\tilde{\eta}_k - \eta_k| \leq \Delta/7.
\]

Let \{\hat{\eta}_k\}_{k=1}^{\tilde{K}} be the change point estimators generated from Algorithm 8 with \{\tilde{\eta}_k\}_{k=1}^{\tilde{K}} and

\[
\zeta = C_\zeta \sqrt{\log(T \lor p)}
\]

as inputs. Then,

\[
P\left\{ \tilde{K} = K, \max_{k=1,\ldots,K} |\hat{\eta}_k - \eta_k| \leq \frac{C_\epsilon d_0 \log(T \lor p)}{K^2} \right\} \geq 1 - T^{-c},
\]

where \(C_\zeta, C_\epsilon, c > 0\) are absolute constants depending only on \(C_\beta, M\) and \(c_x\).

Theorem 20 shows that if Algorithm 8 is adopted as a refinement step of Algorithm 1 with (21) and (22), then the corresponding localisation error is nearly-optimal, off by a logarithmic factor. The near optimality is achieved due to two key ingredients.

- The condition on the initial estimators (25) provides the opportunity that we are able to work in intervals containing one and one true change point. This eliminates the dependence on \(K\) in the localisation error.

- In Algorithm 8, a group lasso estimation is adopted in (24). Since we have already guaranteed that there is only one change point in the working interval, the group lasso penalty captures this feature and returns better estimation, with a higher computational cost.

In this section, we only reviewed the coefficients change in linear regression models. In fact, similar techniques can also be extended to other forms of regression problems, including (vector)
autoregressive models, self-exciting Poisson processes, and other time series models. We conclude this subsection with a list of existing literature on different aspects of different regression coefficient change point problems. These papers include Aue et al. (2006), Wang et al. (2019a), Wang et al. (2020b), Safikhani and Shojaie (2020), Leonardi and Bühlmann (2016) and others.

4.4 Conclusions

The three high-dimensional cases we reviewed here are representative.

- In Section 4.1, despite the high-dimensionality, we are able to find a polynomial-time algorithm achieves nearly optimal localisation rates under nearly optimal signal-to-noise conditions, both off by logarithmic factors. In other words, all three goals we listed in Section 1.1 are achieved.

- In Section 4.2, we reviewed a case exhibiting statistical and computational tradeoffs. In terms of the three goals we listed in Section 1.1, only the third one is achieved.

- In Section 4.3, we show that under some mild conditions and a nearly optimal signal-to-noise ratio condition, a penalisation-based method is able to provide consistent change point estimators, based on which, a refinement can improve the localisation error to be nearly optimal. In terms of the three goals we listed in Section 1.1, only the third one is achieved. With some mild extra conditions, all three goals can be achieved.

Recall that in the univariate mean change point analysis in Section 2, there exist nearly-optimal polynomial-time methods, in terms of both detection and localisation. The high-dimensionality obviously increases difficulties. The reason that we can achieve the near optimality without any additional steps or conditions in the covariance change point problem in Section 4.1, is largely due to the matrix operator norm used in the model assumption. The operator norm plays the role of dimension reduction – it essentially means all the useful information lies in the largest eigenvalue, despite the high-dimensionality of the data. To elaborate, if instead of the operator norm, we use the entry-wise maximum norm, then we can still achieve the near-optimality despite the high-dimensionality. However, if we use the Frobenius norm instead of the operator norm, then we will essentially meet the difficulty as that in Section 4.2.

In Section 4.2, the matrix Frobenius norm is adopted in defining the distributional differences. Different from the matrix operator norm, the Frobenius norm is not helping at all in terms of dimension reduction. In other words, the jump size κ defined thereof is allowed to vary in (0, nρ). The high-dimensional nature of the problem leads to the gaps in Section 4.2.

We conclude this section with a list of papers working on change point detection in other types of high-dimensional data.

- High-dimensional mean change points: Aston and Kirch (2014), Barigozzi et al. (2016), Cho (2015), Cho and Fryzlewicz (2015), Horváth and Hušková (2012), Wang and Samworth (2018) and Jirak (2015), among others.

- Graphical models and high-dimensional precision matrices change points: Amini and Nguyen (2013), Gibberd and Nelson (2017), Gibberd and Nelson (2014), Gibberd and Roy (2017), Keshavarz et al. (2018) and Londschien et al. (2019), among others.
5 Extension 3: Nonparametric models

In this section, we discuss nonparametric models. We will build up from a univariate case, then a multivariate case and conclude with a reproducing kernel Hilbert space case. Regarding the distances used to characterise the distribution differences, we will cover three different distances. As for the algorithms, we will study both the penalisation-based and scan-statistics-based methods.

5.1 Univariate

Different from the cases studied in Sections 2 and 3, in this subsection, the differences of the distributions are characterised by the Kolmogorov–Smirnov distance.

**Assumption 8.** Let \(\{X_t\}_{t=1}^T \subset \mathbb{R}\) be a collection of independent random variables such that \(X_t \sim F_t\), where \(F_t\)’s are cumulative distribution functions (CDFs). Let \(\{\eta_k\}_{k=0}^{K+1} \subset \{1, \ldots, T+1\}\) be a collection of change points with \(1 = \eta_0 < \eta_1 < \ldots < \eta_K \leq T < \eta_{K+1} = T+1\) such that

\[
F_t \neq F_{t-1}, \quad \text{if and only if } t \in \{\eta_1, \ldots, \eta_K\}.
\]

Define the minimal spacing \(\Delta\) and the jump size \(\kappa\) as

\[
\Delta = \min_{k=1,\ldots,K+1} \{\eta_k - \eta_{k-1}\} > 0
\]

and

\[
\kappa = \min_{k=1,\ldots,K} \kappa_k = \min_{k=1,\ldots,K} \sup_{z \in \mathbb{R}} \left| F_{\eta_k}(z) - F_{\eta_{k-1}}(z) \right| > 0.
\]  \(\text{(26)}\)

The difficulty of this problem is characterised in Lemmas 21 and 22, focusing on the minimax lower bounds on detection and localisation, respectively.

**Lemma 21** (Lemma 3 in Padilla et al., 2019b). Let \(\{X_t\}_{t=1}^T\) be a time series satisfying Assumption 8 with one and only one change point. Let \(P_{T,\Delta}\) denote the corresponding joint distribution. For any \(0 < \zeta < 1/\sqrt{2}\), denote

\[
\mathcal{P}^T = \left\{ P_{T,\Delta} : \Delta = \min \left\{ \frac{\zeta^2}{\kappa^2}, \left\lfloor \frac{T}{3} \right\rfloor \right\} \right\}.
\]

Let \(\hat{\eta}\) and \(\eta(P)\) be an estimator and the true change point, respectively. It holds that

\[
\inf_{\hat{\eta}} \sup_{P \in \mathcal{P}^T} \mathbb{E}_P(|\hat{\eta} - \eta(P)|) \geq (1 - 2\zeta^2)\Delta,
\]

where the infimum is over all possible estimators of the change point location.
Lemma 22 (Lemma 4 in Padilla et al., 2019b). Let \( \{X_t\}_{t=1}^T \) be a time series satisfying Assumption 8 with one and only one change point. Let \( P^T_{\kappa, \Delta} \) denote the corresponding joint distribution. Consider the class of distributions

\[
Q^T = \left\{ P^T_{\kappa, \Delta} : \Delta < T/2, \kappa < 1/2, \kappa \sqrt{\Delta} \geq \zeta_T \right\},
\]

for any sequence \( \{\zeta_T\} \) such that \( \lim_{T \to \infty} \zeta_T = \infty \). Let \( \hat{\eta} \) and \( \eta(P) \) be an estimator and the true change point, respectively. Then, for all \( T \) large enough, it holds that

\[
\inf_{\hat{\eta}} \sup_{P \in Q^T} \mathbb{E}_P(\left| \hat{\eta} - \eta(P) \right|) \geq \max \left\{ 1, \frac{1}{2} \frac{1}{\kappa^2} e^{-2} \right\},
\]

where the infimum is over all possible estimators of the change point locations.

Lemmas 21 and 22 show that in the low signal-to-noise ratio regime \( \kappa \sqrt{\Delta} \ll 1 \), no algorithm is guaranteed to provide consistent change point estimators, and the minimax lower bounds on the localisation is \( \kappa^{-2} \).

We will demonstrate how a scan-statistics-based method is able to reach near optimality in the sense of both detection and localisation. Based on Assumption 8, we tailor the CUSUM statistics defined in Definition 1 to incorporate the Kolmogorov–Smirnov distance. Definition 4 in fact replaces \( X_t \)'s in Definition 1 with indicator functions \( \mathbb{1}\{X_t \leq s\} \), \( t \in \{1, \ldots, T\} \), \( s \in \mathbb{R} \).

Definition 4. For any integer triplet \((s, t, e)\), \(0 \leq s < t < e \leq T\), define

\[
D^t_{s,e} = \sup_{z \in \mathbb{R}} \left| \sqrt{\frac{(t-s)(e-t)}{e-s}} \left\{ \hat{F}_{(s+1):z}(z) - \hat{F}_{(t+1):z}(z) \right\} \right|,
\]

where for all integer pair \((s, e)\), \(1 \leq s < e \leq T\) and any \( z \in \mathbb{R} \),

\[
\hat{F}_{s,e}(z) = \frac{1}{e-s} \sum_{t=s}^{e} \mathbb{1}\{X_t \leq z\}.
\]

With the Kolmogorov–Smirnov version of the CUSUM statistics, we can adapt Algorithm 2 by replacing the CUSUM statistics there with the one in Definition 4. To be specific, given data \( \{X_t\}_{t=1}^T \) and any integer triplet \((s, t, e)\), \(0 \leq s < t < e \leq T\), we let

\[
\bar{D}^t_{s,e} = \max_{i=1, \ldots, T} \left| \sqrt{\frac{(t-s)(e-t)}{e-s}} \left\{ \hat{F}_{(s+1):z}(X_i) - \hat{F}_{(t+1):z}(X_i) \right\} \right|,
\]

i.e. the supreme is taken on the support of all data points. The theoretical guarantee is give below.

Theorem 23 (Theorem 2 in Padilla et al., 2019b). Let the CUSUM statistics used in Algorithm 2 be (27). Assume the inputs of Algorithm 2 are as follows:

- the sequence \( \{X_t\}_{t=1}^T \) satisfies Assumption 8 and there exists a constant \( C_{SNR} > 0 \) such that
  \[ \kappa \sqrt{\Delta} > C_{SNR} \sqrt{\log(T)}; \]
• the collection of intervals \( \{(\alpha_m, \beta_m)\}_{m=1}^M \subset \{1, \ldots, T\} \), with endpoints drawn independently and uniformly from \( \{1, \ldots, T\} \), satisfy \( \max_{m=1, \ldots, M} (\beta_m - \alpha_m) \leq C_R \Delta \), almost surely, for an absolute constant \( C_R > 1 \); and

• the tuning parameter \( \tau \) satisfies \( c_{\tau, 1} \sqrt{\log(T)} \leq \tau \leq c_{\tau, 2} \kappa \Delta^{1/2} \), where \( c_{\tau, 1}, c_{\tau, 2} > 0 \) are constants.

Let \( \{\hat{\eta}_k\}_{k=1}^K \) be the corresponding output. Then

\[
P\left\{ \hat{K} = K \quad \text{and} \quad |\hat{\eta}_k - \eta_k| = \epsilon_k \leq C \epsilon_k \kappa^{-2} \log(T), \forall k = 1, \ldots, K \right\} 
\geq 1 - 24 \log(T) T^{-4} - \frac{48}{\log(T) \Delta} - \exp\left\{ \log\left( \frac{T}{\Delta} \right) - \frac{M \Delta^2}{16 T^2} \right\},
\]

where \( C \epsilon > 0 \) is an absolute constant.

Theorem 23 shows that there exists a CUSUM-based algorithm which is nearly optimal in terms of both detection and localisation, save for logarithmic factors.

It is interesting to compare the results we obtained here and those in Section 2. Comparing the signal-to-noise ratio conditions on consistent localisations

\[
\kappa \sqrt{\Delta} \gtrsim \log(T) \quad \text{and} \quad \kappa \sqrt{\Delta} \gtrsim \sigma \log(T),
\]

and the two localisation errors

\[
\kappa^{-2} \quad \text{and} \quad \kappa^{-2} \sigma^2,
\]

we see that the univariate nonparametric change point detection problem with Kolmogorov–Smirnov distance, can be seen as a univariate mean change point detection problem with \( \sigma \asymp O(1) \). This connection is due to the definition of empirical distribution functions used in Definition 4. Each observation is turned to an indicator variable, which is a Bernoulli random variable with variance upper bounded by 1.

Regarding the comparisons with Section 2, another remark is in Section 2, it is known that a minimax lower bound on detection is

\[
\kappa \sqrt{\Delta} \lesssim \sigma \sqrt{\log(T)},
\]

while in this subsection, the state-of-the-art result on the lower bound is

\[
\kappa \sqrt{\Delta} \lesssim 1,
\]

which leaves a gap of order \( \log^{1/2+\xi}(T) \) between the lower and upper bounds. It remains an open question on how to close this gap even further. We conjecture that the gap is due to a loose lower bound.

5.2 Multivariate

In the second nonparametric example, we study a sequence of random vectors and the distance used to define the distribution jumps is chosen to be the supreme norm of a function. The detailed model assumption is provided below.
Assumption 9. Let \( \{X_t\}_{t=1}^T \subset \mathbb{R}^p \) be a sequence of independent random vectors with unknown distributions \( \{P_t\}_{t=1}^T \) such that, for an unknown sequence of change points \( \{\eta_k\}_{k=1}^K \subset \{2, \ldots, T\} \) with \( 1 = \eta_0 < \eta_1 < \ldots < \eta_K \leq T < \eta_{K+1} = T + 1 \), we have
\[
P_t \neq P_{t-1} \quad \text{if and only if} \quad t \in \{\eta_1, \ldots, \eta_K\}.
\]
Assume that, for each \( t = 1, \ldots, T \), the distribution \( P_t \) has a bounded Lebesgue density function \( f_t : \mathbb{R}^p \to \mathbb{R} \) such that
\[
\max_{t=1, \ldots, T} \|f_t(s_1) - f_t(s_2)\| \leq C_{\text{Lip}} \|s_1 - s_2\|, \quad \text{for all} \ s_1, s_2 \in \mathcal{X},
\]
where \( \mathcal{X} \subset \mathbb{R}^p \) is the union of the supports of all the density functions \( f_t \), \( \| \cdot \| \) represents the \( \ell_2 \)-norm, and \( C_{\text{Lip}} > 0 \) is an absolute constant. We let
\[
\Delta = \min_{k=1, \ldots, K+1} \{\eta_k - \eta_{k-1}\}
\]
denote the minimal spacing between any two consecutive change points, and let
\[
\kappa = \min_{k=1, \ldots, K} \kappa_k = \min_{k=1, \ldots, K} \sup_{z \in \mathbb{R}^p} |f_{\eta_k}(z) - f_{\eta_k-1}(z)| = \|f_{\eta_k} - f_{\eta_k-1}\|_{\infty} > 0
\]
be the minimal jump size.

As usual, we first study the minimax lower bounds determining the difficulties of the problem.

Lemma 24 (Lemma 2 in Padilla et al., 2019c). Let \( \{X_t\}_{t=1}^T \) be a sequence of random vectors satisfying Assumption 9 with one and only one change point and let \( P_{\kappa, \Delta}^T \) denote the corresponding joint distribution. Then, there exist universal positive constants \( C_1, C_2 \) and \( c < \log(2) \) such that, for all \( T \) large enough,
\[
\inf_{\hat{\eta}} \sup_{P \in \mathcal{P}^T} \mathbb{E}_P(\|\hat{\eta} - \eta(P)\|) \geq \Delta/4,
\]
where
\[
\mathcal{P}^T = \{P_{\kappa, \Delta}^T : \Delta < T/2, \kappa < C_1, \kappa^{p+2} \Delta \leq c, C_{\text{Lip}} \leq C_2\},
\]
the quantity \( \eta(P) \) denotes the true change point location of \( P \in \mathcal{P}^T \) and the infimum is over all possible estimators of the change point location.

Lemma 25 (Lemma 3 in Padilla et al., 2019c). Let \( \{X_t\}_{t=1}^T \) be a sequence of random vectors satisfying Assumption 9 with one and only one change point and let \( P_{\kappa, \Delta}^T \) denote the corresponding joint distribution. Then, there exist universal positive constants \( C_1 \) and \( C_2 \) such that, for any sequence \( \{\zeta_T\} \) satisfying \( \lim_{T \to \infty} \zeta_T = \infty \),
\[
\inf_{\hat{\eta}} \sup_{P \in \mathcal{Q}} \mathbb{E}_P(\|\hat{\eta} - \eta(P)\|) \geq \max \left\{ 1, \frac{1}{4} \left[ \frac{1}{V_p^{p/2}} \right] e^{-2} \right\},
\]
where \( V_p = \pi^{p/2} (\Gamma(p/2 + 1))^{-1} \) is the volume of a unit ball in \( \mathbb{R}^p \),
\[
\mathcal{Q}^T = \{P_{\kappa, \Delta}^T : \Delta < T/2, \kappa < C_1, \kappa^{p+2} V_p^2 \Delta \geq \zeta_T, C_{\text{Lip}} \leq C_2\},
\]
the quantity \( \eta(P) \) denotes the true change point location of \( P \in \mathcal{Q}^T \) and the infimum is over all possible estimators of the change point location.
Lemmas 24 and 25 show that in the low signal-to-noise ratio regime where $\kappa^{p+2} \Delta \lesssim 1$, no algorithm is guaranteed to be consistent and a minimax lower bound on the localisation error is of order $\kappa^{-(p+2)}$.

In order to match these lower bound, we use the CUSUM-based methods again here and summon Algorithm 2 by adjusting the CUSUM statistics. Following the same routine, we first define the corresponding CUSUM statistics. Definition 5 replaces the data $X_t$’s in Definition 1 with a kernel function.

**Definition 5.** Let $\{X_t\}_{t=1}^T$ be a sample in $\mathbb{R}^p$. For any integer triplet $(s, t, e)$ satisfying $0 \leq s < t < e \leq T$ and any $x \in \mathbb{R}^p$, the multivariate nonparametric CUSUM statistic is defined as the function

$$x \in \mathbb{R}^p \mapsto \tilde{Y}^{s,e}_t(x) = \sqrt{(t-s)(e-t)} \left\{ \hat{f}_{s+1,t,h}(x) - \hat{f}_{t+1,e,h}(x) \right\},$$

where

$$\hat{f}_{s,e,h}(x) = \frac{h^{-p}}{e-s} \sum_{i=s+1}^e k \left( \frac{x - X_i}{h} \right)$$

and $k(\cdot)$ is a kernel function (see e.g. Parzen, 1962). In addition, define

$$\tilde{Y}^{s,e}_t = \max_{i=1,\ldots,T} |\tilde{Y}^{s,e}_t(X_i)|.$$

Note that in Definition 5, the CUSUM statistics is based on a kernel estimator of underlying densities. The theoretical guarantees of Algorithm 2 with Definition 5 are presented in Theorem 26, with additional assumptions collected in Assumption 10.

**Assumption 10.** Let $k : \mathbb{R}^p \to \mathbb{R}$ be a kernel function with $\|k\|_\infty, \|k\|_2 < \infty$ such that,

(i) the class of functions

$$\mathcal{F}_{k,[l,\infty)} = \left\{ k \left( \frac{x}{h} \right) : x \in \mathcal{X}, h \geq l \right\}$$

from $\mathbb{R}^p$ to $\mathbb{R}$ is separable in $L_\infty(\mathbb{R}^p)$, and is a uniformly bounded VC-class with dimension $\nu$, i.e. there exist positive numbers $A$ and $\nu$ such that, for every positive measure $Q$ on $\mathbb{R}^p$ and for every $u \in (0, \|k\|_\infty)$, it holds that

$$\mathcal{N}(\mathcal{F}_{k,[l,\infty)}, L_2(Q), u) \leq \left( \frac{A \|k\|_\infty}{u} \right)^\nu;$$

(ii) for a fixed $m > 0$,

$$\int_0^\infty t^{p-1} \sup_{\|x\| \geq t} |k(x)|^m dt < \infty.$$

(iii) there exists a constant $C_k > 0$ such that

$$\int_{\mathbb{R}^p} k(z) \|z\| dz \leq C_k.$$
Theorem 26 (Theorem 1 in Padilla et al., 2019c). Assume that the sequence $\{X_t\}_{t=1}^T$ satisfies the model described in Assumption 9 and assume that for a given $\xi > 0$, there exists an absolute constant $C_{\text{SNR}} > 0$ such that

$$\kappa^{p+2}\Delta > C_{\text{SNR}} \log^{1+\xi}(T).$$

Let $k(\cdot)$ be a kernel function satisfying Assumption 10. Then, there exist positive universal constants $C_R$, $c_{r,1}$, $c_{r,2}$ and $c_\eta$, such that if Algorithm 2 is applied to the sequence $\{X_t\}_{t=1}^T$ using the CUSUM statistics defined in Definition 5, any collection $\{(\alpha_m, \beta_m)\}_{m=1}^M \subset \{1, \ldots, T\}$ of random time intervals with endpoints drawn independently and uniformly from $\{1, \ldots, T\}$ with $\max_{m=1, \ldots, M}(\beta_m - \alpha_m) \leq C_R \Delta$, almost surely, tuning parameter $\tau$ satisfying

$$c_{r,1} \max \left\{ h^{-2p/2} \log^{1/2}(T), h\Delta^{1/2} \right\} \leq \tau \leq c_{r,2} \kappa \Delta^{1/2},$$

and bandwidth $h$ given by $h = c_h \kappa$, then the resulting change point estimator $\{\hat{\eta}_k\}_{k=1}^K$ satisfies

$$\mathbb{P} \left\{ \hat{K} = K \right\} \quad \text{and} \quad \epsilon_k = |\hat{\eta}_k - \eta_k| \leq C_\epsilon \kappa^{-2} \kappa^{-p} \log(T), \forall k = 1, \ldots, K$$

$$\geq 1 - 3T^{-c} - \exp \left\{ \log \left( \frac{T}{\Delta} \right) - \frac{M \Delta}{4C_R T} \right\},$$

for universal positive constants $C_\epsilon$ and $c$.

Theorem 26 shows that a CUSUM-based method is nearly optimal in terms of both detection and localisation.

Recall in Section 4.2 we remark that if the goal is to estimate change points, then we can sacrifice some accuracy in estimating the underlying distributions. We actually have similar observations here. In the density estimation literature, with the Lipschitz condition imposed on the densities, the optimal bandwidth rate is $h_1 \asymp \{\log(\Delta)/\Delta\}^{1/(p+2)}$, with the goal of estimating the underlying densities. In Theorem 26, we see that in order to obtain optimal change point estimation, the bandwidth is required to be $h_{\text{opt}} \asymp \kappa$. In the situations where $\kappa \gtrsim \{\log(\Delta)/\Delta\}^{1/(p+2)}$, using a bandwidth $h_{\text{opt}}$ lead to larger bias in estimating the densities, but is required if the goal is to estimate change points optimally.

We reviewed the state-of-the-art results in this subsection. The dimensionality $p$ is considered as an absolute constant. It remains an open problem if we allow $p$ to diverge, what the minimax rates are in terms of both detection and localisation.

5.3 A reproducing kernel Hilbert space

So far the random objects we considered are in Euclidean spaces and the optimal methods we present are both scan-statistics-based methods. In this section, we consider general $X$-valued random objects, where $X$ is an arbitrary (measurable) space, and consider a penalisation-based-method. The change points are defined to be the change points in a reproducing kernel Hilbert space, which is induced by a certain kernel. The detailed assumptions are collected below.

Assumption 11. Let $\{X_t\}_{t=1}^T \subset X$ be a sequence of independent random objects with unknown distributions $\{P_t\}_{t=1}^T$, where $X$ is a measurable space.

Let $k : X \times X \rightarrow \mathbb{R}$ be a positive semidefinite kernel. Let $H$ be the reproducing kernel Hilbert space associated with the kernel $k$, together with the canonical feature map $\Phi : X \rightarrow H$, satisfying
Φ(\(x\)) = \(k(\cdot, x)\), \(x \in X\). Assume \(\mathcal{H}\) is separable. For any \(t \in \{1, \ldots, T\}\), define \(Y_t = \Phi(X_t) \in \mathcal{H}\), let \(\mu^*_t\) be the Bochner integral (e.g. Ganiev, 2013) of \(Y_t\) and \(\varepsilon_t = Y_t - \mu^*_t\). Assume that there exists a positive absolute constant \(V\) such that

\[
\max_{t=1, \ldots, T} \mathbb{E}\left(\|\varepsilon_t\|_\mathcal{H}^2\right) \leq V.
\]

Let \(\{\eta_k\}_{k=0}^{K+1} \subset \{1, \ldots, T + 1\}\) be a strictly increasing sequence, with \(\eta_0 = 1\) and \(\eta_{K+1} = T + 1\), satisfying

\[
\mu^*_t \neq \mu^*_{t-1} \quad \text{if and only if} \quad t \in \{\eta_k\}_{k=1}^K.
\]

Let

\[
\kappa = \min_{k=1, \ldots, K} \|\mu^*_{\eta_k} - \mu^*_{\eta_k-1}\|_\mathcal{H} \quad \text{and} \quad \Delta = \min_{k=1, \ldots, K+1} (\eta_k - \eta_{k-1}),
\]

where \(\|\cdot\|_\mathcal{H}\) is the norm of \(\mathcal{H}\).

In order to handle random objects in general space, Assumption 11 adopts a kernel function to turn the general space into univariate random variables. Recall in Section 1, the change points are defined to be the change points of the sequence \(\{P_t\}\), but the change points in Assumption 11 are defined to be the change points of \(\{\mu^*_t\}\). Note that, if the kernel \(k(\cdot, \cdot)\) is a characteristic kernel, then \(X_t\) and \(X_{t+1}\) have the same distribution if and only if \(\mu^*_t = \mu^*_{t+1}\). In this case, the set of the change points of \(\{\mu^*_t\}\) is identical to that of the change points of \(\{P_t\}\). In general, this one-to-one correspondence might not hold.

To match the lower bounds, we resort to the penalisation-based method in Algorithm 1 with the function \(H(I)\) defined to be

\[
H(I) = \sum_{t \in I} k(X_t, X_t) - \frac{1}{|I|} \sum_{t \in I} \sum_{s \in I} k(X_t, X_s).
\]

When \(X = \mathbb{R}\) and \(k(\cdot, \cdot)\) is the linear kernel, then (28) is exactly (6).

**Theorem 27** (Theorem 3.1 in Garreau and Arlot, 2018). Let \(\{X_t\}_{t=1}^T\) satisfy Assumption 11. Assume that there exists a positive absolute constant \(M\) such that

\[
k(X_t, X_t) \leq M^2 < \infty, \quad \forall t \in \{1, \ldots, T\}.
\]

Assume that there exists a sufficiently large absolute constant \(C_{\text{SNR}} > 0\) such that for any \(\xi > 0\),

\[
\kappa \sqrt{\Delta} \geq C_{\text{SNR}} K \sqrt{\log^{1+\xi}(T)}.
\]

Let \(\{\tilde{\eta}_k\}_{k=1}^{\tilde{K}}\) be the output of Algorithm 1 with the loss function defined in (28). We have that, for an absolute constant \(C > 0\), define \(\lambda = C K^2 \log(T)\). It holds that

\[
\mathbb{P}\left\{\tilde{K} = K; \ |\eta_k - \tilde{\eta}_k| \leq C_{\varepsilon} K \log(T) \kappa^{-2}, \text{ for all } k\right\} \geq 1 - T^{-c},
\]

where \(C_{\varepsilon}, c > 0\) are absolute constants.
Note that the localisation error obtained in Theorem 27 is linear in $K$, the number of true change points. In order to match the minimax lower bound $\kappa^{-2}$, we can adopt the refinement idea again.

**Corollary 28.** Under the same settings and conditions in Theorem 27, let $\{\nu_k\}_{k=1}^K$ be a set of initial change point estimators satisfying $\max_{k=1,\ldots,K}|\nu_k - \eta_k| \leq \Delta/5$. For each $k \in \{1, \ldots, K\}$, define

$$s_k = \nu_k - 1/2 + \nu_k / 2, \quad e_k = \nu_k / 2 + \nu_k + 1/2$$

and $I_k = [s_k, e_k]$, with $\nu_0 = 1$ and $\nu_{K+1} = T + 1$. For $k \in \{1, \ldots, K\}$, we let

$$\tilde{\eta}_k = \min_{t \in \{s_k + 1, \ldots, e_k - 1\}} \left\{ \sum_{i \in I_k} k(X_i, X_i) - \frac{1}{t - s_k} \sum_{i \in [s_k, t]} \sum_{j \in [s_k, t]} k(X_i, X_j) - \frac{1}{e_k - t} \sum_{i \in (t, e_k]} \sum_{j \in (t, e_k]} k(X_i, X_j) \right\}.$$ 

Then we have

$$\mathbb{P}\{\hat{K} = K; \quad |\eta_k - \tilde{\eta}_k| \leq C \log(T) \kappa^{-2}, \quad \text{for all } k\} \geq 1 - T^{-c},$$

where $C, c > 0$ are absolute constants.

Corollary 28 is straightforward based on the observation that there is one and only one change point in each interval $I_k$, $k = 1, \ldots, K$. Together with Theorem 27, it shows that the *de facto* $K$ is exactly one in each working interval $I_k$, and the results hold.

Considering Corollary 28 as a refinement step of Algorithm 1, we show that a penalisation-based-method is able to achieve the near optimality in localisation error, off by a logarithmic factor. It remains an open question that in terms of the signal-to-noise ratio, if one can weaken the condition from $\kappa \sqrt{\Delta} \geq C_{\text{SNR}} \sqrt{\log(1 + \xi)}$ to $\kappa \sqrt{\Delta} \geq C_{\text{SNR}} \sqrt{\log(1 + \xi)}(T)$.

A more interesting open problem in this problem is how one can achieve the nonparametric rate in the reproducing kernel Hilbert space change point analysis. To explain this, we remark that despite the great flexibility in terms of $X$ we reviewed in this subsection, the rates achieved are the same as the rates in Section 2, which deals with a parametric problem. This is because the *de facto* complexity of the space is hidden in the conditions that $V$ and $M$ are absolute constants. A more thorough result, which remains open, should involve the complexity of the reproducing kernel Hilbert space. For instance, one should consider the Rademacher complexity (e.g. Mendelson, 2002; Bartlett et al., 2005) and the covering number of the reproducing kernel Hilbert space unit-ball in $\ell_2$-norm. If the reproducing kernel Hilbert space is a Sobolev space $W^{\alpha, 2}$, $\alpha > 1/2$, we conjecture that the detection boundary should be of order $O(T^{-\alpha/(2\alpha + 1)})$ and the localisation rate should be of order $O(T^{-2\alpha/(2\alpha + 1)})$. However, these remain as open problems in this area.

### 5.4 Conclusions

In this section, we reviewed three different nonparametric change point detection problems. We would like to mention that the key to deploy Algorithm 1 is to define a suitable loss function. It is the sum of residual squares in Sections 2 and 3, and a kernel version of the sum of residual squares.
in Section 5.3. The key for Algorithm 1 to execute in polynomial time is that the loss function is separable in terms of the intervals. To be specific, it should be of the form

\[ G(\mathcal{P}, \{X_t\}, \lambda) = \sum_{I \in \mathcal{P}} H(I) + \lambda |\mathcal{P}| \]  

(30)

and \( H(I) \) is solely a function relying on the data in the interval \( I \). This suggests that it is not clear how one can directly apply Algorithm 1 to the problems studied in Sections 5.1 and 5.2. For instance, in Section 5.1, we see the jump is defined in (26) and it is the Kolmogorov–Smirnov distance between two different distributions. To estimate the Kolmogorov–Smirnov distance, one needs to know where on the support the difference is taken to be largest, namely

\[ z^* \in \arg \max_{z \in \mathbb{R}} |F_1(z) - F_2(z)|. \]

Back to (30), in order to estimate the change points, a certain form of the loss function is inevitably a function of \( z^* \), which is not solely determined by one interval.

The reason that Algorithm 1 is applicable in Section 5.3 is due to the construction of the reproducing kernel Hilbert space, which transforms the change points of the distributions of data, to the change points in the Bochner integrals. This transformation to a certain extent turns a general nonparametric problem to a parametric one, which echoes the discussions at the end of Section 5.3.

Finally, we mention a list of papers working on different aspects in this area: Hawkins and Deng (2010), Haynes et al. (2017b), Itoh and Kurths (2010), Matteson and James (2014), Vanegas et al. (2019), Zou et al. (2014), Harchaoui and Cappé (2007), Garreau and Arlot (2018), Celisse et al. (2018) and Arlot et al. (2019), among others.

6 Conclusions

In this survey, we covered a range of change point analysis problems, focusing on the minimax rates of detection and localisation, with an emphasis on distinguishing these two ideas. The univariate mean change point detection problem lays down the foundation in terms of minimax lower bounds and two types of popular methods which are nearly optimal. For more complicated cases we covered in this survey, we have reviewed different situations, in some cases we can show the near optimality in terms of both detection and localisation, in some cases we show that an extra refinement step can reach the near optimality in localisation but the detection conditions remain sub-optimal, and in some cases we show that the near optimality is reachable under some extra conditions.

There are still many open questions in the change point analysis area. Throughout the survey, we have identified a few. In addition, we would like to reiterate that the minimax results we reviewed in this paper are all on offline change point detection and localisation. The minimax rates of online change point detection and localisation, and minimax rates of both online and offline change point testing remain largely unknown. The optimality we achieved in this survey are all off by logarithmic factors. It would also be interesting to further refine the results improving the results. Efforts along this line include some results in Verzelen et al. (2020).
References

Aggarwal, R., Inclan, C. and Leal, R. (1999). Volatility in emerging stock markets. *The Journal of Financial and Quantitative Analysis*, 34 33–55. URL http://www.jstor.org/stable/2676245.

Akashi, F., Dette, H. and Liu, Y. (2018). Change-point detection in autoregressive models with no moment assumptions. *Journal of Time Series Analysis*, 39 763–786.

Amini, A. A. and Nguyen, X. (2013). Sequential detection of multiple change points in networks: a graphical model approach. *IEEE transactions on information theory*, 59 5824–5841.

Anastasiou, A. and Fryzlewicz, P. (2019). Detecting multiple generalized change-points by isolating single ones. *arXiv preprint arXiv:1901.10852*.

Andreou, E. and Ghysels, E. (2002). Detecting multiple breaks in financial market volatility dynamics. *Journal of Applied Econometrics*, 17 579–600. URL http://onlinelibrary.wiley.com/doi/10.1002/jae.684/abstract.

Arlot, S., Celisse, A. and Harchaoui, Z. (2019). A kernel multiple change-point algorithm via model selection. *Journal of Machine Learning Research*, 20 1–56.

Aston, J. A. and Kirch, C. (2011). Power analysis for functional change point detection. In *Recent Advances in Functional Data Analysis and Related Topics*. Springer, 23–26.

Aston, J. A. and Kirch, C. (2012). Detecting and estimating changes in dependent functional data. *Journal of Multivariate Analysis*, 109 204–220.

Aston, J. A. D. and Kirch, C. (2014). Efficiency of change point tests in high dimensional settings. *arXiv preprint arXiv: 1409.1771*.

Aue, A., Gabrys, R., Horváth, L. and Kokoszka, P. (2009a). Estimation of a change-point in the mean function of functional data. *Journal of Multivariate Analysis*, 100 2254–2269.

Aue, A., Hörmann, S., Horváth, L. and Reimherr, M. (2009b). Break detection in the covariance structure of multivariate time series models. *The Annals of Statistics*, 37 4046–4087.

Aue, A. and Horváth, L. (2004). Delay time in sequential detection of change. *Statistics & Probability Letters*, 67 221–231.

Aue, A. and Horváth, L. (2013). Structural breaks in time series. *Journal of Time Series Analysis*, 34 1–16.

Aue, A., Horváth, L., Hušková, M. and Kokoszka, P. (2006). Change-point monitoring in linear models. *The Econometrics Journal*, 9 373–403.

Aue, A., Horvath, L., Hušková, M. and Kokoszka, P. (2008). Testing for changes in polynomial regression. *Bernoulli*, 14 637–660.

Aue, A., Horváth, L. and Reimherr, M. L. (2009c). Delay times of sequential procedures for multiple time series regression models. *Journal of Econometrics*, 149 174–190.
Aue, A., Rice, G. and Sönmez, O. (2018). Detecting and dating structural breaks in functional data without dimension reduction. *Journal of the Royal Statistical Society: Series B (Statistical Methodology)*, 80, 509–529.

Avanesov, V. and Buzun, N. (2016). Change-point detection in high-dimensional covariance structure. *arXiv preprint arXiv:1610.03783*.

Baranowski, R., Chen, Y. and Fryzlewicz, P. (2016). Narrowest-over-threshold detection of multiple change-points and change-point-like features. *arXiv preprint arXiv:1609.00293*.

Barigozzi, M., Cho, H. and Fryzlewicz, P. (2016). Simultaneous multiple change-point and factor analysis for high-dimensional time series. *arXiv preprint arXiv: 1612.06928*.

Bartlett, P. L., Bousquet, O. and Mendelson, S. (2005). Local rademacher complexities. *The Annals of Statistics*, 33, 1497–1537.

Berkes, I., Gabrys, R., Horváth, L. and Kokoszka, P. (2009). Detecting changes in the mean of functional observations. *Journal of the Royal Statistical Society: Series B (Statistical Methodology)*, 71, 927–946.

Bhattacharjee, M., Banerjee, M. and Michailidis, G. (2018). Change point estimation in a dynamic stochastic block model. *arXiv preprint arXiv:1812.03090*.

Birke, M. and Dette, H. (2005). A note on testing the covariance matrix for large dimension. *Statistics and Probability Letters*, 74, 281–289.

Celisse, A., Marot, G., Pierre-Jean, M. and Rigaill, G. (2018). New efficient algorithms for multiple change-point detection with reproducing kernels. *Computational Statistics & Data Analysis*, 128, 200–220.

Chamroukhi, F., Mohammed, S., Trabelsi, D., Oukhellou, L. and Amirat, Y. (2013). Joint segmentation of multivariate time series with hidden process regression for human activity recognition. *Neurocomputing*, 120, 633 – 644. URL http://www.sciencedirect.com/science/article/pii/S0925231213004086.

Chan, H. P. and Walther, G. (2013). Detection with the scan and the average likelihood ratio. *Statistica Sinica*, 1, 409–428.

Chatterjee, S. (2015). Matrix estimation by universal singular value thresholding. *The Annals of Statistics*, 43, 177–214.

Chatterjee, S., Guntuboyina, A. and Sen, B. (2015). On risk bounds in isotonic and other shape restricted regression problems. *The Annals of Statistics*, 43, 1774–1800.

Chen, H. (2019a). Change-point detection for multivariate and non-euclidean data with local dependency. *arXiv preprint arXiv:1903.01598*.

Chen, H. (2019b). Sequential change-point detection based on nearest neighbors. *The Annals of Statistics*, 47, 1381–1407.
Chen, Y., Wang, T. and Samworth, R. J. (2020). High-dimensional, multiscale online change-point detection. *arXiv preprint arXiv:2003.03668.*

Chiou, J.-M., Chen, Y.-T. and Hsing, T. (2019). Identifying multiple changes for a functional data sequence with application to freeway traffic segmentation. *The Annals of Applied Statistics, 13* 1430–1463.

Cho, H. (2015). Change-point detection in panel data via double cusum statistic. *Electronic Journal of Statistics* in press.

Cho, H. and Fryzlewicz, P. (2015). Multiple change-point detection for high-dimensional time series via Sparsified Binary Segmentation. *Journal of the Royal Statistical Society: Series B (Statistical Methodology), 77* 475–507.

Chu, C.-S. J., Stinchcombe, M. and White, H. (1996). Monitoring structural change. *Econometrica: Journal of the Econometric Society* 1045–1065.

Chu, L. and Chen, H. (2019). Asymptotic distribution-free change-point detection for multivariate and non-euclidean data. *The Annals of Statistics, 47* 382–414.

Cribben, I. and Yu, Y. (2017). Estimating whole-brain dynamics by using spectral clustering. *Journal of the Royal Statistical Society: Series C (Applied Statistics), 66* 607–627.

Delyon, B. (2009). Exponential inequalities for sums of weakly dependent variables. *Electronic Journal of Probability, 14* 752–779.

Desobry, F., Davy, M. and Doncarli, C. (2005). An online kernel change detection algorithm. *IEEE Trans. Signal Processing, 53* 2961–2974.

Dette, H., Eckle, T. and Vetter, M. (2018a). Multiscale change point detection for dependent data. *Scandinavian Journal of Statistics.*

Dette, H. and Gösmann, J. (2019). A likelihood ratio approach to sequential change point detection for a general class of parameters. *Journal of the American Statistical Association* 1–17.

Dette, H. and Kutta, T. (2019). Detecting structural breaks in eigensystems of functional time series. *arXiv preprint arXiv:1911.07580.*

Dette, H., Pan, G. M. and Yang, Q. (2018b). Estimating a change point in a sequence of very high-dimensional covariance matrices. *arXiv preprint.*

Dette, H., Wu, W. and Zhou, Z. (2018c). Change point analysis of correlation in non-stationary time series. *arXiv preprint arXiv:1801.10478.*

Dümbgen, L. and Spokoiny, V. G. (2001). Multiscale testing of qualitative hypotheses. *Annals of Statistics* 124–152.

Dümbgen, L. and Walther, G. (2008). Multiscale inference about a density. *The Annals of Statistics, 36* 1758–1785.
Elsner, J. B., Xu, F. N. and Jagger, T. H. (2004). Detecting shifts in hurricane rates using a markov chain monte carlo approach. *Journal of Climate*, **17** 2652–2666. URL http://dx.doi.org/10.1175/1520-0442(2004)017<2652:DSIHRU>2.0.CO;2.

Enikeeva, F., Munk, A., Pohllmann, M. and Werner, F. (2019). Bump detection in the presence of dependency: Does it ease or does it load? *arXiv preprint arXiv:1906.08017*.

Enikeeva, F., Munk, A. and Werner, F. (2018). Bump detection in heterogeneous gaussian regression. *Bernoulli*, **24** 1266–1306.

Erdman, C. and Emerson, J. W. (2008). A fast bayesian change point analysis for the segmentation of microarray data. *Bioinformatics*, **24** 2143–2148. URL http://bioinformatics.oxfordjournals.org/content/24/19/2143.short.

Fearnhead, P. and Liu, Z. (2007). On-line inference for multiple changepoint problems. *Journal of the Royal Statistical Society: Series B (Statistical Methodology)*, **69** 589–605.

Fearnhead, P., Maidstone, R. and Letchford, A. (2019). Detecting changes in slope with an l0 penalty. *Journal of Computational and Graphical Statistics*, **28** 265–275.

Fearnhead, P. and Rigaill, G. (2019). Changepoint detection in the presence of outliers. *Journal of the American Statistical Association*, **114** 169–183.

Fernandez, V. (2006). The impact of major global events on volatility shifts: Evidence from the asian crisis and 9/11. *Economic Systems*, **30** 79–97. URL http://www.sciencedirect.com/science/article/pii/S0939362506000094.

Ferreira, C. S., Zeller, C. B., Mimura, A. M. and Silva, J. C. (2017). Partially linear models and their applications to change point detection of chemical process data. *Journal of Applied Statistics*, **44** 2125–2141.

Frick, K., Munk, A. and Sieling, H. (2014). Multiscale change point inference. *Journal of the Royal Statistical Society: Series B: Statistical Methodology* 495–580.

Friedrich, F., Kempe, A., Liebscher, V. and Winkler, G. (2008). Complexity penalized m-estimation: Fast computation. *Journal of Computational and Graphical Statistics*, **17** 201–204.

Fryzlewicz, P. (2014). Wild binary segmentation for multiple change-point detection. *The Annals of Statistics*, **42** 2243–2281.

Ganiev, I. (2013). The bochner integral for measurable sections and its properties. *Annals of Functional Analysis*, **4** 1–10.

Gao, C., Lu, Y. and Zhou, H. H. (2015). Rate-optimal graphon estimation. *The Annals of Statistics*, **43** 2624–2652.

Garreau, D. and Arlot, S. (2018). Consistent change-point detection with kernels. *Electronic Journal of Statistics*, **12** 4440–4486.

Gibberd, A. J. and Nelson, J. D. (2014). High dimensional changepoint detection with a dynamic graphical lasso. In *2014 IEEE International Conference on Acoustics, Speech and Signal Processing (ICASSP)*. IEEE, 2684–2688.
Gibberd, A. J. and Nelson, J. D. (2017). Regularized estimation of piecewise constant gaussian graphical models: The group-fused graphical lasso. *Journal of Computational and Graphical Statistics*, **26** 623–634.

Gibberd, A. J. and Roy, S. (2017). Multiple changepoint estimation in high-dimensional gaussian graphical models. *arXiv preprint arXiv:1712.05786*.

Gombay, E., Horváth, L. and Hušková, M. (1996). Estimators and tests for change in variances. *Statistics and Risk Modeling*, **14** 145–160.

Gösmann, J., Kley, T. and Dette, H. (2019). A new approach for open-end sequential change point monitoring. *arXiv preprint arXiv:1906.03225*.

Han, S. W., Mesquita, R. C., Busch, T. M. and Putt, M. E. (2014). A method for choosing the smoothing parameter in a semi-parametric model for detecting changepoints in blood flow. *Journal of Applied Statistics*, **41** 26–45. URL http://www.tandfonline.com/doi/abs/10.1080/02664763.2013.830085.

Harchaoui, Z. and Cappé, O. (2007). Retrospective multiple change-point estimation with kernels. In *2007 IEEE/SP 14th Workshop on Statistical Signal Processing*. IEEE, 768–772.

Hawkins, D. M. and Deng, Q. (2010). A nonparametric change-point control chart. *Journal of Quality Technology*, **42** 165–173.

Haynes, K., Eckley, I. A. and Fearnhead, P. (2017a). Computationally efficient changepoint detection for a range of penalties. *Journal of Computational and Graphical Statistics*, **26** 134–143.

Haynes, K., Fearnhead, P. and Eckley, I. A. (2017b). A computationally efficient nonparametric approach for changepoint detection. *Statistics and Computing*, **27** 1293–1305.

He, X., Xie, Y., Wu, S.-M. and Lin, F.-C. (2018). Sequential graph scanning statistic for change-point detection. In *2018 52nd Asilomar Conference on Signals, Systems, and Computers*. IEEE, 1317–1321.

Henderson, R. and Matthews, J. N. S. (1993). An investigation of changepoints in the annual number of cases of haemolytic uraemic syndrome. *Applied Statistics*, **42** 461–471. URL http://www.jstor.org/stable/2986325.

Hlávka, Z., Hušková, M., Kirch, C. and Meintanis, S. G. (2016). Bootstrap procedures for online monitoring of changes in autoregressive models. *Communications in Statistics-Simulation and Computation*, **45** 2471–2490.

Hocking, T. D., Rigaill, G., Fearnhead, P. and Bourque, G. (2017). A log-linear time algorithm for constrained changepoint detection. *arXiv preprint arXiv:1703.03352*.

Hocking, T. D., Rigaill, G., Fearnhead, P. and Bourque, G. (2020). Constrained dynamic programming and supervised penalty learning algorithms for peak detection in genomic data. *Journal of Machine Learning Research*.

Horváth, L. and Hušková, M. (2012). Change-point detection in panel data. *Journal of Time Series Analysis*, **33** 631–648.
HSU, D. A. (1979). Detecting shifts of parameter in gamma sequences with applications to stock price and air traffic flow analysis. *Journal of the American Statistical Association*, **74** 31–40. URL http://www.jstor.org/stable/2286717.

HUANG, Y., DAGNE, G. A. and PARK, J.-G. (2013). Segmental modeling of changing immunologic response for cd4 data with skewness, missingness and dropout. *Journal of Applied Statistics*, **40** 2244–2258. URL http://www.tandfonline.com/doi/abs/10.1080/02664763.2013.809569.

HUŠKOVÁ, M. and KIRCH, C. (2012). Bootstrapping sequential change-point tests for linear regression. *Metrika*, **75** 673–708.

HUŠKOVÁ, M., KIRCH, C. and MEINTANIS, S. G. (2010). Fourier methods for sequential change point analysis in autoregressive models. In *Proceedings of COMPSTAT’2010*. Springer, 501–508.

HYUN, S., G’SELL, M. and TIBSHIRANI, R. J. (2018). Exact post-selection inference for the generalized lasso path. *Electronic Journal of Statistics*, **12** 1053–1097.

INCLAN, C. and TIAO, G. C. (1994). Use of cumulative sums of squares for retrospective detection of changes of variance. *Journal of the American Statistical Association*, **89** 913–923.

ITOH, N. and KURTHS, J. (2010). Change-point detection of climate time series by nonparametric method. In *Proceedings of the world congress on engineering and computer science*, vol. 1. Citeseer, 445–448.

JENG, X. J., CAI, T. T. and LI, H. (2012). Simultaneous discovery of rare and common segment variants. *Biometrika*, **100** 157–172.

JEWELL, S., FEARNHEAD, P. and WITTEN, D. (2019). Testing for a change in mean after change-point detection. *arXiv preprint arXiv:1910.04291*.

JIAO, S., FROSTIG, R. D. and OMBAO, H. (2020). Break point detection for functional covariance. *arXiv preprint arXiv:2006.13887*.

JIRAK, M. (2015). Uniform change point tests in high dimension. *The Annals of Statistics*, **43** 2451–2483.

KESHAVARZ, H., MICHAELIDIS, G. and ATCHADÉ, Y. (2018). Sequential change-point detection in high-dimensional gaussian graphical models. *arXiv preprint arXiv:1806.07870*.

KILlick, R., ECKLEY, I. A., JONATHAN, P. and EWANS, K. (2010). Detection of changes in the characteristics of oceanographic time-series using statistical change point analysis. *Ocean Engineering*, **37** 1120–1126. URL http://www.sciencedirect.com/science/article/pii/S0029801810001162.

KILlick, R., Fearnhead, P. and Eckley, I. A. (2012). Optimal detection of changepoints with a linear computational cost. *Journal of the American Statistical Association*, **107** 1590–1598.

KIRCH, C. (2008). Bootstrapping sequential change-point tests. *Sequential Analysis*, **27** 330–349.

KIRCH, C., MUHSAL, B. and OMBAO, H. (2015). Detection of changes in multivariate time series with application to eeg data. *Journal of the American Statistical Association*, **110** 1197–1216.
KIRCH, C. and WEBER, S. (2018). Modified sequential change point procedures based on estimating functions. *Electronic Journal of Statistics*, 12 1579–1613.

KOVÁCS, S., LI, H., BÜHLMANN, P. and MUNK, A. (2020a). Seeded binary segmentation: A general methodology for fast and optimal change point detection. *arXiv preprint arXiv:2002.06633*.

KOVÁCS, S., LI, H., HAUBNER, L., MUNK, A. and BÜHLMANN, P. (2020b). Optimistic search strategy: Change point detection for large-scale data via adaptive logarithmic queries. *arXiv preprint arXiv:2010.10194*.

KOZIOL, J. and WU, S. (1996). A review of nonparametric tests for changepoint problems, with application to a recombinant drug therapy clinical trial. *J Biopharm Stat.*, 6 425–441.

KURT, M. N., YILMAZ, Y. and WANG, X. (2018). Real-time nonparametric anomaly detection in high-dimensional settings. *arXiv preprint arXiv:1809.05250*.

KWOON, D., VANNUCCI, M., SONG, J. J., JEONG, J. and PFIEFFER, R. M. (2008). A novel wavelet-based thresholding method for the pre-processing of mass spectrometry data that accounts for heterogeneous noise. *Proteomics*, 8 3019–3029. URL http://www.ncbi.nlm.nih.gov/pmc/articles/PMC2855839/.

LAI, T. L. (1981). Asymptotic optimality of invariant sequential probability ratio tests. *The Annals of Statistics* 318–333.

LAI, T. L. (1995). Sequential changepoint detection in quality control and dynamical systems. *Journal of the Royal Statistical Society: Series B (Methodological)*, 57 613–644.

LAI, T. L. (1998). Information bounds and quick detection of parameter changes in stochastic systems. *IEEE Transactions on Information Theory*, 44 2917–2929.

LAI, T. L. (2001). Sequential analysis: some classical problems and new challenges. *Statistica Sinica* 303–351.

LAI, T. L. and XING, H. (2010). Sequential change-point detection when the pre-and post-change parameters are unknown. *Sequential analysis*, 29 162–175.

LAVIELLE, M. (1999). Detection of multiple changes in a sequence of dependent variables. *Stochastic Processes and their Applications*, 83 79–102.

LEONARDI, F. and BÜHLMANN, P. (2016). Computationally efficient change point detection for high-dimensional regression. *arXiv preprint arXiv:1601.03704*.

LI, H., GUO, Q. and MUNK, A. (2017). Multiscale change-point segmentation: Beyond step functions. *arXiv preprint arXiv: 1708.03942*.

LI, X. and GHOSAL, S. (2018). Bayesian change point detection for functional data. *arXiv preprint arXiv:1808.01236*.

LIN, K., SHARPACK, J., RINALDO, A. and TIBSHIRANI, R. J. (2016). Approximate recovery in changepoint problems, from ℓ2 estimation error rates. *arXiv preprint arXiv:1606.06746*.
LINDQUIST, M. A., WAUGH, C. and WAGER, T. D. (2007). Modeling state-related fmri activity using change-point theory. *NeuroImage, 35* 1125–1141. URL 10.1016/j.neuroimage.2007.01.004.

LIO, P. and VANNUCCI, M. (2000). Wavelet change-point prediction of transmembrane proteins. *Bioinformatics, 16* 376–382. URL http://bioinformatics.oxfordjournals.org/content/16/4/376.abstract.

LIU, F., CHOI, D., XIE, L. and ROEDER, K. (2018). Global spectral clustering in dynamic networks. *Proceedings of the National Academy of Sciences of the United States of America.*

LIU, H., GAO, C. and SAMWORTH, R. J. (2019). Minimax rates in sparse, high-dimensional changepoint detection. *arXiv preprint arXiv:1907.10012.*

LIU, Y.-W. and CHEN, H. (2020). A fast and efficient change-point detection framework for modern data. *arXiv preprint arXiv:2006.13450.*

LOH, P.-L. and WAINWRIGHT, M. J. (2013). Regularized m-estimators with nonconvexity: Statistical and algorithmic theory for local optima. In *Advances in Neural Information Processing Systems.* 476–484.

LONDSCHIEN, M., KOVÁCS, S. and BÜHLMANN, P. (2019). Change point detection for graphical models in presence of missing values. *arXiv preprint arXiv:1907.05409.*

LORDEN, G. (1971). Procedures for reacting to a change in distribution. *The Annals of Mathematical Statistics, 42* 1897–1908.

MAIDSTONE, R., HOCKING, T., RIGAILL, G. and FEARNHEAD, P. (2017). On optimal multiple changepoint algorithms for large data. *Statistics and Computing, 27* 519–533.

MAILLARD, O.-A. (2019). Sequential change-point detection: Laplace concentration of scan statistics and non-asymptotic delay bounds. In *Algorithmic Learning Theory.* 610–632.

MAMMEN, E. and VAN DE GEER, S. (1997). Locally adaptive regression splines. *The Annals of Statistics, 25* 387–413.

MATTESON, D. S. and JAMES, N. A. (2014). A nonparametric approach for multiple change point analysis of multivariate data. *Journal of the American Statistical Association, 109* 334–345.

MCLAINE, A. C. and ALBERT, P. S. (2014). Modeling longitudinal data with a random change point and no time-zero: Applications to inference and prediction of the labor curve. *Biometrics, 70* 1052–1060. URL http://onlinelibrary.wiley.com/doi/10.1111/biom.12218/abstract?campaign=woearlyview.

MEI, Y. (2010). Efficient scalable schemes for monitoring a large number of data streams. *Biometrika, 97* 419–433.

MENDELSOHN, S. (2002). Geometric parameters of kernel machines. In *International Conference on Computational Learning Theory.* Springer, 29–43.
Moustakides, G. V. (1986). Optimal stopping times for detecting changes in distributions. *The Annals of Statistics, 14* 1379–1387.

Namoano, B., Starr, A., Emmanouilidis, C. and Cristobal, R. C. (2019). Online change detection techniques in time series: An overview. In *2019 IEEE International Conference on Prognostics and Health Management (ICPHM)*. IEEE, 1–10.

Niu, Y. S. and Zhang, H. (2012). The screening and ranking algorithm to detect dna copy number variations. *The annals of applied statistics, 6* 1306.

Oliver, J. L., Carpena, P., Hackenberg, M. and Bernaola-Galvan, P. (2004). Isofinder: computational prediction of isochores in genome sequences. *Nucleic Acid Research, 32* W287–W292. URL http://www.ncbi.nlm.nih.gov/pmc/articles/PMC441537/.

Padilla, O. H. M., Yu, Y. and Priebe, C. E. (2019a). Change point localization in dependent dynamic nonparametric random dot product graphs. *arXiv preprint arXiv:1911.07494*.

Padilla, O. H. M., Yu, Y., Wang, D. and Rinaldo, A. (2019b). Optimal nonparametric change point detection and localization. *arXiv preprint arXiv:1905.10019*.

Padilla, O. H. M., Yu, Y., Wang, D. and Rinaldo, A. (2019c). Optimal nonparametric multivariate change point detection and localization. *arXiv preprint arXiv:1910.13289*.

Page, E. S. (1954). Continuous inspection schemes. *Biometrika, 41* 100–115.

Parzen, E. (1962). On estimation of a probability density function and mode. *The annals of mathematical statistics, 33* 1065–1076.

Pein, F., Sieling, H. and Munk, A. (2015). Heterogeneous change point inference. *arXiv preprint arXiv:1505.04898*.

Pensky, M. (2016). Dynamic network models and graphon estimation. *arXiv preprint arXiv:1607.00673*.

Picard, F., Lebarbier, M., Hoebeke, M., Rigaill, G., Thiam, B. and Robin, S. (2011). Joint segmentation, calling and normalization of multiple cgh profiles. *Biostatistics, 12* 413–428. URL http://pbil.univ-lyon1.fr/members/ftpicas/franckpicard_fichiers/pdf/PLH11.pdf.

Plummer, P. J. and Chen, J. (2014). A bayesian approach for locating change points in a compound poisson process with application to detecting dna copy number variations. *Journal of Applied Statistics, 41* 423–438. URL http://www.tandfonline.com/doi/abs/10.1080/02664763.2013.840272.

Rigaill, G. (2010). Pruned dynamic programming for optimal multiple change-point detection. *arXiv preprint arXiv:1004.0887, 17*.

Rinaldo, A., Wang, D., Wen, Q., Willett, R. and Yu, Y. (2020). Localizing changes in high-dimensional regression models. *arXiv preprint arXiv:2010.10410*.

Ritov, Y. (1990). Decision theoretic optimality of the cusum procedure. *The Annals of Statistics* 1464–1469.
Robbins, M., Gallagher, C., Lund, R. and Aue, A. (2011). Mean shift testing in correlated data. *Journal of Time Series Analysis, 32* 498–511.

Robinson, L. F., Wager, T. D. and Lindquist, M. A. (2010). Change point estimation in multi-subject fmri studies. *NeuroImage, 49* 1581–1592. URL http://www.sciencedirect.com/science/article/pii/S1053811909009641.

Romano, G., Rigaill, G., Runge, V. and Fearnhead, P. (2020). Detecting abrupt changes in the presence of local fluctuations and autocorrelated noise. *arXiv preprint arXiv:2005.01379*.

Ross, G. J. (2013). Modelling financial volatility in the presence of abrupt changes. *Physica A: Statistical Mechanics and its Applications, 392* 350 – 360. URL http://www.sciencedirect.com/science/article/pii/S0378437112008084.

Rudin, L. I., Osher, S. and Fatemi, E. (1992). Nonlinear total variation based noise removal algorithms. *Physica D: nonlinear phenomena, 60* 259–268.

Safikhani, A. and Shojaie, A. (2020). Joint structural break detection and parameter estimation in high-dimensional non-stationary var models. *Journal of the American Statistical Association* 1–26.

Scott, A. J. and Knott, M. (1974). A cluster analysis method for grouping means in the analysis of variance. *Biometrics* 507–512.

Shen, J. J. and Zhang, N. R. (2012). Change-point model on nonhomogeneous poisson processes with application in copy number profiling by next-generation dna sequencing. *Annals of Applied Statistics, 6* 476–496. URL http://projecteuclid.org/euclid.aoas/1339419604.

Shen, Y., Han, Q. and Han, F. (2020). On a phase transition in general order spline regression. *arXiv preprint arXiv:2004.10922*.

Siegmund, D. (2013). *Sequential analysis: tests and confidence intervals*. Springer Science & Business Media.

Stoehr, C., Aston, J. A. and Kirch, C. (2020). Detecting changes in the covariance structure of functional time series with application to fmri data. *Econometrics and Statistics*.

Tartakovsky, A., Nikiforov, I. and Basseville, M. (2014). *Sequential analysis: Hypothesis testing and changepoint detection*. Chapman and Hall/CRC.

Tibshirani, R., Saunders, M., Rosset, S., Zhu, J. and Knight, K. (2005). Sparsity and smoothness via the fused lasso. *Journal of the Royal Statistical Society: Series B (Statistical Methodology), 67* 91–108.

Tibshirani, R. J. (2014). Adaptive piecewise polynomial estimation via trend filtering. *The Annals of Statistics, 42* 285–323.

Tickle, S., Eckley, I., Fearnhead, P. and Haynes, K. (2020). Parallelization of a common changepoint detection method. *Journal of Computational and Graphical Statistics, 29* 149–161.
Vanegas, L. J., Behr, M. and Munk, A. (2019). Multiscale quantile segmentation. arXiv preprint arXiv:1902.09321.

Venkatraman, E. S. (1992). Consistency results in multiple change-point problems. Ph.D. thesis, Stanford University.

Verzele, N., Fromont, M., Lerasle, M. and Reynaud-Bouret, P. (2020). Optimal change-point detection and localization. arXiv preprint arXiv:2010.11470.

Vostrikova, L. (1981). Detection of the disorder in multidimensional random-processes. Doklady Akademii Nauk SSSR, 259 270–274.

Wald, A. (1945). Sequential tests of statistical hypotheses. The Annals of Mathematical Statistics, 16 117–186.

Wallis, W. A. (1980). The statistical research group, 1942–1945. Journal of the American Statistical Association, 75 320–330.

Wang, D., Lin, K. and Willett, R. (2019a). Statistically and computationally efficient change point localization in regression settings. arXiv preprint arXiv:1906.11364.

Wang, D., Yu, Y. and Rinaldo, A. (2017). Optimal covariance change point localization in high dimension. arXiv preprint arXiv:1712.09912.

Wang, D., Yu, Y. and Rinaldo, A. (2018). Optimal change point detection and localization in sparse dynamic networks. arXiv preprint arXiv:1809.09602.

Wang, D., Yu, Y. and Rinaldo, A. (2020a). Univariate mean change point detection: Penalization, cusum and optimality. Electronic Journal of Statistics, 14 1917–1961.

Wang, D., Yu, Y., Rinaldo, A. and Willett, R. (2019b). Localizing changes in high-dimensional vector autoregressive processes. arXiv preprint arXiv:1909.06359.

Wang, D., Yu, Y. and Willett, R. (2020b). Detecting abrupt changes in high-dimensional self-exciting poisson processes. arXiv preprint arXiv:2006.03572.

Wang, T. and Samworth, R. J. (2018). High-dimensional changepoint estimation via sparse projection. Journal of the Royal Statistical Society: Series B (Statistical Methodology).

Whitcher, B., Byers, S. D., Guttorm, P. and Percival, D. B. (2002). Testing for homogeneity of variance in time series: Long memory, wavelets and the nile river. Water Resources Research, 38 12–1–12–16. URL http://www.agu.org/pubs/crossref/2002/2001WR000509.shtml.

Yao, Y.-C. (1988). Estimating the number of change-points via schwarz’criterion. Statistics & Probability Letters, 6 181–189.

Yao, Y.-C. and Au, S.-T. (1989). Least-squares estimation of a step function. Sankhyā: The Indian Journal of Statistics, Series A 370–381.

Yu, M. and Chen, X. (2019). A robust bootstrap change point test for high-dimensional location parameter. arXiv preprint arXiv:1904.03372.
Yu, Y., Padilla, O. H. M., Wang, D. and Rinaldo, A. (2020). A note on online change point detection. *arXiv preprint arXiv:2006.03283.*

Zhang, C.-H. (2002). Risk bounds in isotonic regression. *The Annals of Statistics, 30* 528–555.

Zhang, T. (2019). Element-wise estimation error of a total variation regularized estimator for change point detection. *arXiv preprint arXiv:1901.00914.*

Zhang, Y., Wainwright, M. J. and Duchi, J. C. (2012). Communication-efficient algorithms for statistical optimization. In *Advances in Neural Information Processing Systems.* 1502–1510.

Zhao, Z., Chen, L. and Lin, L. (2019). Change-point detection in dynamic networks via graphon estimation. *arXiv preprint arXiv:1908.01823.*

Zou, C., Yin, G., Feng, L. and Wang, Z. (2014). Nonparametric maximum likelihood approach to multiple change-point problems. *The Annals of Statistics, 42* 970–1002.