THE SUPPORT OF THE LOGARITHMIC
EQUILIBRIUM MEASURE ON SETS OF REVOLUTION
IN $\mathbb{R}^3$

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ABSTRACT. For surfaces of revolution $B$ in $\mathbb{R}^3$, we investigate the limit distribution of minimum energy point masses on $B$ that interact according to the logarithmic potential $\log(1/r)$, where $r$ is the Euclidean distance between points. We show that such limit distributions are supported only on the “out-most” portion of the surface (e.g., for a torus, only on that portion of the surface with positive curvature). Our analysis proceeds by reducing the problem to the complex plane where a non-singular potential kernel arises whose level lines are ellipses.

1. Introduction

For a collection of $N(\geq 2)$ distinct points $\omega_N := \{x_1, \ldots, x_N\} \subset \mathbb{R}^3$ and $s > 0$, the Riesz $s$-energy of $\omega_N$ is defined by

$$E_s(\omega_N) := \sum_{1 \leq i \neq j \leq N} k_s(x_i, x_j) = \sum_{i=1}^{N} \sum_{j=1 \atop j \neq i}^{N} k_s(x_i, x_j),$$

where, for $x, y \in \mathbb{R}^3$, $k_s(x, y) := 1/|x-y|^s$. As $s \to 0$, it is easily verified that

$$(k_s(x, y) - 1)/s \to \log(1/|x-y|)$$

and so it is natural to define $k_0(x, y) := \log(1/|x-y|)$. For a compact set $B \subset \mathbb{R}^3$ and $s \geq 0$, the $N$-point $s$-energy of $B$ is defined by

$$E_s(B, N) := \inf \{ E_s(\omega_N) \mid \omega_N \subset B, |\omega_N| = N \},$$

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where \(|X|\) denotes the cardinality of a set \(X\). Note that the logarithmic \((s = 0)\) minimum energy problem is equivalent to the maximization of the product

\[
\prod_{1 \leq i \neq j \leq N} |x_i - x_j|,
\]

and that for planar sets, such optimal points are known as Fekete points.

(The fast generation of near optimal logarithmic energy points for the sphere \(S^2\) is the focus of one of S. Smale’s “mathematical problems for the next century”; see [14].)

If \(0 \leq s < \dim B\) (the Hausdorff dimension of \(B\)), the limit distribution (as \(N \to \infty\)) of optimal \(N\)-point configurations is given by the equilibrium measure \(\lambda_{s,B}\) that minimizes the continuous energy integral

\[
I_s(\mu) := \int_B \int_B k_s(x, y) \, d\mu(x) \, d\mu(y)
\]

over the class \(\mathcal{M}(B)\) of (Radon) probability measures \(\mu\) supported on \(B\). In addition, the asymptotic order of the Riesz \(s\)-energy is \(N^2\); more precisely we have \(E_s(B, N)/N^2 \to I_s(\lambda_{s,B})\) as \(N \to \infty\) (cf. [11, Section II.3.12]). In the case when \(B = S^2\), the unit sphere in \(\mathbb{R}^3\), the equilibrium measure is simply the normalized surface area measure. If \(s \geq \dim B\), then \(I_s(\mu) = \infty\) for every \(\mu \in \mathcal{M}(B)\) and potential theoretic methods cannot be used. However, it was recently shown in [7] that when \(B\) is a \(d\)-rectifiable manifold of positive \(d\)-dimensional Hausdorff measure and \(s \geq d\), optimal \(N\)-point configurations are uniformly distributed (as \(N \to \infty\)) on \(B\) with respect to \(d\)-dimensional Hausdorff measure restricted to \(B\). (The assertion for the case \(s = d\)
further requires that $B$ be a subset of a $C^1$ manifold.) For further extensions of these results, see [3]. Related results and applications appear in [5] (coding theory), [13] (cubature on the sphere), and [1] (finite normalized tight frames).

In Figure 1 we show near optimal Riesz $s$-energy configurations for the values of $s = 0, 1,$ and 2 for $N = 1000$ points restricted to live on the torus $B$ obtained by revolving the circle of radius 1 and center $(3, 0)$ about the $y$-axis. (For recent results on the disclinations of minimal energy points on toroidal surfaces, see [4].) The somewhat surprising observation that there are no points on the “inner” part of the torus in the case $s = 0$ (and, in fact, as well for $s$ near 0) is what motivated us to investigate the support of the logarithmic equilibrium measure $\lambda_{0,B}$. In this paper we show that, in fact, this is a general phenomenon for optimal logarithmic energy configurations of points restricted to sets of revolution in $\mathbb{R}^3$ (see Figure 2).

2. Preliminaries

In this paper we focus on the logarithmic kernel $k_0$. Let $B \subset \mathbb{R}^3$ be compact. As in the previous section, the logarithmic energy of a measure $\mu \in \mathcal{M}(B)$ is given by

$$I_0(\mu) = \int_{B \times B} \log \frac{1}{|p-q|} d\mu(p) d\mu(q)$$

and the corresponding potential $U^\mu$ is defined by

$$U^\mu(p) := \int_B \log \frac{1}{|p-q|} d\mu(q) \quad (p \in \mathbb{R}^3).$$
Let $V_B := \inf_{\mu \in \mathcal{M}(B)} I_0(\mu)$. The logarithmic capacity of $B$, denoted by $\text{cap}(B)$, is $\exp(-V_B)$. A condition $C(p)$ is said to hold quasi-everywhere on $B$ if it holds for all $p \in B$ except for a subset of logarithmic capacity zero.\(^1\) If $\text{cap}(B) > 0$, then there is a unique probability measure $\mu_B \in \mathcal{M}(B)$ (called the equilibrium measure on $B$) such that $I_0(\mu_B) = V_B$ (this is implicit in the references [11, 12]). Furthermore, the equality $U_{\mu_B}(p) = V_B$ holds quasi-everywhere on the support of $\mu_B$ and $U_{\mu_B}(p) \geq V_B$ quasi-everywhere on $B$.

We now turn our attention to sets of revolution in $\mathbb{R}^3$. Let $\mathbb{R}_+ := [0, \infty)$ and, for $t \in [0, 2\pi)$, let $\sigma_t : \mathbb{R}^3 \to \mathbb{R}^3$ denote the rotation about the $y$-axis through an angle $t$:

$$\sigma_t(x, y, \zeta) = (x \cos t - \zeta \sin t, y, x \sin t + \zeta \cos t).$$

For a compact set $A$ contained in the right half-plane $H^+ := \mathbb{R}_+ \times \mathbb{R}$, let $\Gamma(A) \subset \mathbb{R}^3$ be the set obtained by revolving $A$ around the $y$-axis, that is,

$$\Gamma(A) := \{\sigma_t(x, y, 0) \mid (x, y) \in A, 0 \leq t < 2\pi\}.$$

We say that $A \subset H^+$ is non-degenerate if $\text{cap}(\Gamma(A))$ is positive. For example, if $A$ contains at least one point not on the $y$-axis, then $A$ is non-degenerate.

### 3. Reduction to the $xy$-plane

A Borel measure $\tilde{\nu} \in \mathcal{M}(\mathbb{R}^3)$ is rotationally symmetric about the $y$-axis if $\tilde{\nu} = \tilde{\nu} \circ \sigma_t$ for all $t \in [0, 2\pi)$. If $\tilde{\nu}$ is rotationally symmetric about the $y$-axis, then $d\tilde{\nu} = \frac{1}{2\pi} dt d\nu$, where $\nu := \tilde{\nu} \circ \Gamma \in \mathcal{M}(H^+)$ and $dt$ denotes Lebesgue measure on $[0, 2\pi)$. Identifying points $z, w \in H^+$ as complex numbers $z = x + iy = (x, y, 0)$ and $w = u + iv = (u, v, 0)$ we have

$$I_0(\tilde{\nu}) = \iint_{\mathbb{R}^3 \times \mathbb{R}^3} \log\frac{1}{|p - q|} d\tilde{\nu}(p) d\tilde{\nu}(q)$$

$$= \iint_{H^+ \times H^+} K(z, w) d\nu(z) d\nu(w)$$

$$=: J(\nu),$$

where

$$K(z, w) := \frac{1}{2\pi} \int_0^{2\pi} \log\frac{1}{|\sigma_t(z) - w|} dt.$$

\(^1\)The logarithmic capacity of a Borel set $E$ is the sup of the capacities of its compact subsets. Any set that is contained in a Borel set of capacity zero is said to have capacity zero.
Notice that
\[ |\sigma_t(z) - w|^2 = (x \cos t - u)^2 + (y - v)^2 + x^2 \sin^2 t \]
\[ = x^2 + u^2 + (y - v)^2 - 2xu \cos t. \]

Let \( w_* := -u + iv = -\overline{w} \) denote the reflection of \( w \) in the \( y \)-axis. Then, using (6) and the formula
\[ \frac{1}{2\pi} \int_0^{2\pi} \log(a + b \cos t) \, dt = \log \frac{a + \sqrt{a^2 - b^2}}{2} \]
with \( a = (y - v)^2 + x^2 + u^2 \) and \( b = -2xu \), we obtain
\[ K(z, w) = -\frac{1}{2} \log \frac{a + \sqrt{a^2 - b^2}}{2} = \log \frac{2}{|z - w| + |z - w_*|}, \]
where we have used
\[ 2 \left(a + \sqrt{a^2 - b^2}\right) = \left(\sqrt{a + b} + \sqrt{a - b}\right)^2 = (|z - w| + |z - w_*|)^2. \]

3.1. Equilibrium measure \( \lambda_A \in \mathcal{M}(A) \). For a non-degenerate compact set \( A \subset H^+ \), the uniqueness of the equilibrium measure \( \mu_{\Gamma(A)} \) and the symmetry of the revolved set \( \Gamma(A) \) imply that \( \mu_{\Gamma(A)} \) is rotationally symmetric about the \( y \)-axis and so \( d\mu_{\Gamma(A)} = \frac{1}{2\pi} dt d\lambda_A \), where for any Borel set \( B \subset H^+ \)
\[ \lambda_A(B) := \mu_{\Gamma(A)}(\Gamma(B)). \]
Furthermore, if \( \nu \in \mathcal{M}(A) \), then \( d\bar{\nu} := \frac{1}{2\pi} dt d\nu \) is rotationally symmetric about the \( y \)-axis and so we have
\[ J(\lambda_A) \geq \inf_{\nu \in \mathcal{M}(A)} J(\nu) = \inf_{\nu \in \mathcal{M}(A)} I_0(\bar{\nu}) \geq I_0(\mu_{\Gamma(A)}) = J(\lambda_A), \]
which leads to the following proposition.

**Proposition 1.** Suppose \( A \) is a non-degenerate compact set in \( H^+ \) and let \( \lambda_A \in \mathcal{M}(A) \) be defined by (8). Then \( \lambda_A \) is the unique measure in \( \mathcal{M}(A) \) that minimizes \( J(\nu) \) over all measures \( \nu \in \mathcal{M}(A) \). That is, \( \lambda_A \) is the equilibrium measure for the kernel \( K \) and set \( A \).

For \( \nu \in \mathcal{M}(A) \), we define the \((K-)\)potential \( W^\nu \) by
\[ W^\nu(z) := \int_A K(z, w) \, d\nu(w) \]
\[ = \int_A \log \frac{2}{|z - w| + |z - w_*|} \, d\nu(w) \quad (z \in H^+). \]
Then, for \( z = (x, y, 0) \in H^+ \), we have

\[
U_{\mu_G(A)}(z) = \int_{\Gamma(A)} \log \frac{1}{|z - q|} \, d\mu_G(q) \\
= \frac{1}{2\pi} \int_A \int_0^{2\pi} \log \frac{1}{|z - \sigma_t(w)|} \, dt \, d\lambda_A(w) \\
= \int_A K(z, w) \, d\lambda_A(w) = W^{\lambda_A}(z).
\]

From the properties of \( U_{\mu_G(A)} \), we then infer the following lemma.

**Lemma 2.** Suppose \( A \) is a non-empty compact set in the interior of \( H^+ \). Let \( \lambda_A \) be the equilibrium measure for \( A \) with respect to the kernel \( K \). Then the potential \( W^{\lambda_A} \) satisfies

\[
W^{\lambda_A}(z) = J(\lambda_A) \quad \text{for} \quad z \quad \text{in the support of} \quad \lambda_A \quad \text{and} \quad W^{\lambda_A}(z) \geq J(\lambda_A) \quad \text{for} \quad z \in A.
\]

**Remark:** In Lemma 2 we no longer need a quasi-everywhere exceptional set, since each point of \( A \) generates a circle in \( \mathbb{R}^3 \) with positive logarithmic capacity.

**Figure 3.** Level curves for \( K(z, w) \) for \( w \) a fixed point on the unit circle centered at \( (2,0) \).

### 3.2. Properties of \( K \).

Let \( s(z, w) := |z - w| + |z - w_*| \). Then \( K(z, w) = -\log(s(z, w)/2) \) and so, for fixed \( w \in H^+ \), the level sets of \( K(\cdot, w) \) are ellipses with foci \( w \) and \( w_* \) as shown in Figure 3. Since the foci have the same imaginary part \( v = \text{Im}[w] = \text{Im}[w_*] \), it follows from geometrical considerations that \( K(\cdot, w) \) is strictly decreasing along horizontal rays \( [iy, \infty + iy] \) for \( y \neq v \). Along the horizontal ray
[iv, ∞+iv), we have that $K(\cdot, w)$ is constant on the line segment [iv, w] and strictly decreasing on the ray [w, ∞+iv).

Furthermore, $K$ is clearly continuous at any $(z, w) \in H^+ \times H^+$ unless $z = w = iy$ for some $y \in \mathbb{R}$. Since $|z - w_*| = |(z - w_*)_*| = |w - z|$, it follows that $K$ is symmetric, that is, $K(z, w) = K(w, z)$ for $z, w \in H^+$.

We summarize these properties of $K$ in the following lemma.

**Lemma 3.** The kernel $K : H^+ \times H^+ \to \mathbb{R}$ in (7) has the following properties:

(a) $K$ is symmetric: $K(z, w) = K(w, z)$ for $w, z \in H^+$.

(b) $K$ is continuous at all points $(z, w) \in H^+ \times H^+$ except points $(z, z)$ such that Re$(z) = 0$.

(c) Let $u \geq 0$ and $y \neq v \in \mathbb{R}$ be fixed. Then $K(x + iy, u + iv)$ is a strictly decreasing function of $x$ for $x \in [0, \infty)$. Furthermore, $K(x + iy, u + iy)$ is constant for $x \in [0, u]$ and is strictly decreasing for $x \in [u, \infty)$.

The following lemma is then a consequence of Lemma 3.

**Lemma 4.** Suppose $\nu \in \mathcal{M}(A)$ is not a point mass (that is, the support of $\nu$ contains at least two points). Then the potential $W^\nu(x)$ is strictly decreasing along the horizontal rays $[iy, \infty + iy)$ for all $y \in \mathbb{R}$.

If $A$ is a non-degenerate compact set in $H^+$, let $P(A)$ denote the projection of the set $A$ onto the $y$-axis and for $y \in P(A)$, define $x_A(y) = \max\{x \mid (x, y) \in A\}$. We then let $A_+$ denote the “right-most” portion of $A$, that is,

$$A_+ := \{(x_A(y), y) \mid y \in P(A)\}.$$ 

Using Lemmas 2 and 3 we then obtain the following result.

**Theorem 5.** Suppose $A$ is a compact set in $H^+$ such that $A_+$ is contained in the interior of $H^+$. Then the support of the equilibrium measure $\lambda_A \in \mathcal{M}(A)$ is contained in $A_+$.

4. **Convexity**

Recall that a function $f : [a, b] \to \mathbb{R}$ is strictly convex on $[a, b]$ if $f(\theta x + (1 - \theta)y) < \theta f(x) + (1 - \theta)f(y)$ for all $a \leq x < y \leq b$ and $0 < \theta < 1$.

**Theorem 6.** Suppose $A$ is a compact set in $H^+$ such that $A_+$ is contained in the interior of $H^+$ and $\gamma : [a, b] \to H^+$ is continuous. Further suppose that

(a) $A_+ \subset \gamma^* := \{\gamma(s) \mid a \leq s \leq b\}$ and
(b) $K(\gamma(\cdot), \gamma(s))$ is a strictly convex function on the intervals $[a, s]$ and $[s, b]$ for each fixed $s \in [a, b]$.

Then there is some closed interval $I \subset [a, b]$ such that $\text{supp} \lambda_A = \gamma(I) \cap A_+$.

Proof. Suppose $A$ and $\gamma$ satisfy (a) and (b). From Theorem 5 we have $\text{supp} \lambda_A \subseteq \gamma^*$. Let $t_1 := \min_{a \leq t \leq b} \{ t \mid \gamma(t) \in \text{supp} \lambda_A \}$ and $t_2 := \max_{a \leq t \leq b} \{ t \mid \gamma(t) \in \text{supp} \lambda_A \}$. Suppose that $G$ is an open interval in $I := [t_1, t_2]$ such that $\gamma(G) \cap \text{supp} \lambda_A = \emptyset$. Then $W^\lambda_A \circ \gamma$ is strictly convex on $G$ and $W^\lambda_A(z) = J(\lambda_A)$ for $z \in \text{supp} \lambda_A$ and so we have $W^\lambda_A(\gamma(t)) < J(\lambda_A)$ for $t \in G$. Hence, Lemma 2 implies that $\gamma(G) \cap A = \emptyset$ which then implies $\text{supp} \lambda_A = \gamma(I) \cap A_+$. \hfill \Box

We next consider several examples where we can verify that the hypotheses of Theorem 6 hold. In these examples, $\gamma$ is a smooth curve, but note that $A_+$ is only required to be a compact subset of $\gamma^*$. For example, $A_+$ may be a Cantor subset of $\gamma^*$.

We first consider a case where we can completely specify the support of $\lambda_A$.

Corollary 7. Suppose $A$ is a non-degenerate compact subset in $H^+$ such that $A_+$ is contained in a vertical line segment $[R + ci, R + di]$ for some $R > 0$. Then $\text{supp} \lambda_A = A_+$.

Proof. Consider the parametrization $\gamma(t) = R + it$, $c \leq t \leq d$, of the line segment $[R + ci, R + di]$. For $s, t \in [c, d]$, $s \neq t$, direct calculation shows $K(\gamma(t), \gamma(s)) = -\log(|s - t| + \sqrt{4R^2 + (s - t)^2}) + \log 2$ and

$\begin{align*}
(10) \quad \frac{d}{dt} K(\gamma(t), \gamma(s)) &= \frac{\text{sgn}(s - t)}{\sqrt{4R^2 + (s - t)^2}}, \\
(11) \quad \frac{d^2}{dt^2} K(\gamma(t), \gamma(s)) &= \frac{|s - t|}{(4R^2 + (s - t)^2)^{3/2}}.
\end{align*}$

Then (11) shows that condition (b) of Theorem 6 holds and therefore there is some interval $I = [t_1, t_2]$ such that $\text{supp} \lambda_A = \gamma(I) \cap A_+$. Furthermore, from (10) we see that $W^\lambda_A(R + it)$ is strictly increasing on $(-\infty, t_1]$ and is strictly decreasing on $[t_2, \infty)$. By Lemma 2 we can take $I = [c, d]$ and so $\text{supp} \lambda_A = A_+$. \hfill \Box

Even in the case when $A$ is a circle in $H^+$ (so that $\Gamma(A)$ is a torus in $\mathbb{R}^3$), it is difficult to directly verify the hypothesis (b) of Theorem 6. We next develop sufficient conditions for (b) that, at least in the case $A$ is a circle, are relatively simple to verify.
For $w \in H^+$ and $t \in [a,b]$, let $r_w(t) := |\gamma(t) - w|$, and $s_w(t) := r_w(t) + r_w(t)$. Assuming $\gamma$ is twice differentiable at $t$ we have

$$
\frac{d^2}{dt^2} K(\gamma(t), w) = \frac{-s''_w(t)s_w(t) + s'_w(t)^2}{s_w(t)^2} \quad (t \in [a,b]).
$$

Then for fixed $w$, we have that $K(\gamma(t), w)$ is strictly convex on any interval where $s''_w < 0$. Let $u_w(t)$ denote the unit vector $(\gamma(t) - w)/r_w(t)$. Differentiating the dot product $r_w(t)^2 = (\gamma(t) - w) \cdot (\gamma(t) - w)$ we obtain

$$
\begin{align*}
\frac{d}{dt} r_w(t) &= \gamma'(t) \cdot u_w(t), \\
\frac{d}{dt} u_w(t) &= (\gamma'(t) - (\gamma'(t) \cdot u_w(t))u_w(t)) / r_w(t), \quad \text{and} \\
\frac{d}{dt} r_w(t) &= \gamma''(t) \cdot u_w(t) + (|\gamma'(t)|^2 - (\gamma'(t) \cdot u_w(t))^2) / r_w(t).
\end{align*}
$$

In the event that $\gamma$ is parametrized by arclength the above equations can be simplified. In this case $|\gamma'(t)| = 1$. We further assume that $\gamma''(t) \neq 0$ for any $t \in [a,b]$. Then $T(t) = \gamma'(t)$ denotes the unit tangent vector, $\kappa(t) = |T'(t)|$ denotes the curvature, and $N(t) = T'(t)/|T'(t)| = \gamma''(t)/\kappa(t)$ denotes the unit normal vector to the curve $\gamma$ for $t \in [a,b]$. Substituting these expressions into (13) and (14) we obtain

$$
\begin{align*}
r''_w(t) &= \gamma''(t) \cdot u_w(t) + \gamma'(t) \cdot u'_w(t) \\
&= (N(t) \cdot u_w(t)) \left[ \kappa(t) + \frac{N(t) \cdot u_w(t)}{r_w(t)} \right].
\end{align*}
$$

From this last representation deduce the following.

**Lemma 8.** Let $\gamma : [a,b] \to H^+$ be a twice differentiable curve such that $|\gamma'(t)| = 1$ and $\gamma''(t) \neq 0$ for all $t \in [a,b]$. Suppose that for all $s,t \in [a,b]$, $s \neq t$, and $w \in \{\gamma(s), \gamma(s)\}$ we have

$$
N(t) \cdot u_w(t) < 0 \quad \text{and} \quad \left[ \kappa(t) + \frac{N(t) \cdot u_w(t)}{r_w(t)} \right] > 0.
$$

Then $\gamma$ satisfies hypothesis (b) of Theorem 7.

We now apply Lemma 8 to the case when $A_+$ is a subset of a circle.

**Corollary 9.** Suppose $C \subset \mathbb{C}$ is a circle of radius $r > 0$ and center $a$ with Re$[a] > 0$ and suppose $A$ is a compact set in $H^+$ such that $A_+ \subset C_+$. Then $\operatorname{supp} \lambda_A = A^\theta_+ := A_+ \cap \{a + re^{it} \mid |t| \leq \theta\}$ for some $\theta \in [0, \pi/2]$. In particular, if $A_+$ is a circular arc contained in $C_+$, then so is $\operatorname{supp} \lambda_A$; consequently, $\operatorname{supp} \mu_{\Gamma(A)}$ is connected.

**Remark:** In the case when $\Gamma(A)$ is a torus (that is, if $A = C$), it follows from Corollary 9 that $\operatorname{supp} \mu_{\Gamma(A)}$ is a connected strip of $\Gamma(A)$ of the form $\Gamma(C^\theta_+)$ for some $\theta \in [0, \pi/2]$.
Proof. Without loss of generality we may assume that $C$ has radius $r = 1$ and center $a = R$ for some $R > 0$. We then consider the parametrization of $C$ given by $\gamma(t) := a + e^{it}$ for $t \in [-\pi/2, \pi/2]$. By direct calculation (assisted by Mathematica) we find, for $w = \gamma(s)$,

$$N(t) \cdot u_w(t) = -\left| \frac{s - t}{2} \right| \text{ and } \left[ \kappa(t) + \frac{N(t) \cdot u_w(t)}{r_w(t)} \right] = \frac{1}{2},$$

and for $w = \gamma(s)_*$ we find

$$N(t) \cdot u_w(t) = -\frac{2R \cos t + \cos(s + t) + 1}{\sqrt{(2R + \cos s + \cos t)^2 + (\sin s - \sin t)^2}}$$

and

$$\left[ \kappa(t) + \frac{N(t) \cdot u_w(t)}{r_w(t)} \right] = \frac{1}{2} + \frac{2R(R + \cos s)}{(2R + \cos s + \cos t)^2 + (\sin s - \sin t)^2}.$$

Then it is easy to verify that the inequalities (17) hold for both $w = \gamma(s)$ and for $w = \gamma(s)_*$ for all $s, t \in [-\pi/2, \pi/2]$ with $s \neq t$. \qed

5. Kernel in limit $R \to \infty$

One might well conjecture looking at Figure 1 and in light of Theorem 5 or Corollary 7 that for the case of the circle $A = \{z \mid |z - R| = 1\}$, $R > 0$, the support of $\lambda_A$ is the right-half circle $A_+$, or equivalently, that the support of the equilibrium measure on the torus $\Gamma(A)$ is the portion of its surface with positive curvature. However, as we see in the limiting case $R \to \infty$, this is not correct.

Define the kernels $K_R : H^+ \times H^+ \to \mathbb{R}$, $R > 0$, and $K_\infty : H^+ \times H^+ \to \mathbb{R}$ by

$$(18) \quad K_R(z, w) := 2R(K(R + z, R + w) + \log R),$$

$$(19) \quad K_\infty(z, w) := -(\text{Re}[z - w_\ast] + |z - w|).$$

Using

$$\frac{|z - w| + |2R + z - w_\ast|}{2R} = 1 + \frac{\text{Re}[z - w_\ast] + |z - w|}{2R} + O(R^{-2})$$

we obtain

$$K_R(z, w) = -2R \log \frac{|z - w| + |2R + z - w_\ast|}{2R}$$

$$= -2R \log \left( 1 + \frac{\text{Re}[z - w_\ast] + |z - w|}{2R} + O(R^{-2}) \right)$$

$$= -(\text{Re}[z - w_\ast] + |z - w|) + O(R^{-1})$$
and hence
\[ \lim_{R \to \infty} K_R(z, w) = K_\infty(z, w), \]
where the convergence is uniform on compact subsets of \( H^+ \times H^+ \). We let \( J_{K_R}(\mu) \) and \( J_{K_\infty}(\mu) \) denote the associated energy integrals defined for compactly supported measures \( \mu \in \mathcal{M}(H^+) \).

From the definition of \( K_R \) we see that the equilibrium measure \( \lambda_A^R \) on a compact set \( A \subset H^+ \) with respect to the kernel \( K_R \) is equal to \( \lambda_{A+R}(\cdot + R) \), that is, \( \lambda_A^R(B) = \lambda_{A+R}(B + R) \) where, for a set \( B \subset H^+ \) and \( R > 0 \), \( B + R \) denotes the translate \( \{ b + R \mid b \in B \} \).

5.1. The existence and uniqueness of an equilibrium measure
for \( K_\infty \). The weak-star compactness of \( \mathcal{M}(A) \) and the continuity of \( J_{K_\infty} \) imply the existence of a measure \( \lambda_A^\infty \in \mathcal{M}(A) \) such that \( J_{K_\infty}(\lambda_A^\infty) = \inf_{\mu \in \mathcal{M}(A)} J_{K_\infty}(\mu) \).

We follow arguments developed in [2] to prove the uniqueness of \( \lambda_A^\infty \). First, note that \( K_\infty(z, w) = -k_1(z, w) - k_2(z, w) \) where \( k_1(z, w) := |z - w| \) and \( k_2(z, w) = \text{Re}[z] + \text{Re}[w] \) and so
\[ J_{K_\infty}(\mu) = -I_1^*(\mu) - I_2^*(\mu), \]
where \( I_1^* \) and \( I_2^* \) are the energy integrals associated with the kernels \( k_1 \) and \( k_2 \), respectively. We need the following lemma of Frostman ([6], also see [2, Lemma 1]).

**Lemma 10.** Suppose \( \nu \) is a compactly supported signed Borel measure on \( H^+ \) such that \( \int d\nu = 0 \) and \( I_1^*(\nu) \geq 0 \). Then \( \nu \equiv 0 \).

For compactly supported Borel measures \( \mu \) and \( \nu \) on \( H^+ \), let
\[ J_{K_\infty}(\mu, \nu) := \iint K_\infty(z, w) \, d\mu(z) \, d\nu(w). \]

**Lemma 11.** Suppose \( A \) is a compact set in \( H^+ \) and \( \mu^* \in \mathcal{M}(A) \) satisfies \( J_{K_\infty}(\mu^*) = \inf_{\mu \in \mathcal{M}(A)} J_{K_\infty}(\mu) \). For any signed Borel measure \( \nu \) with support contained in \( A \) such that \( \nu(A) = \int_A d\nu = 0 \) and \( \mu^* + \nu \geq 0 \), we have \( J_{K_\infty}(\mu^*, \nu) \geq 0 \).

**Proof.** With \( \nu \) and \( \mu^* \) as above, we have \( \mu^* + \epsilon \nu \in \mathcal{M}(A) \) for \( 0 \leq \epsilon \leq 1 \) and so
\[ J_{K_\infty}(\mu^*) \leq J_{K_\infty}(\mu^* + \epsilon \nu) = J_{K_\infty}(\mu^*) + 2\epsilon J_{K_\infty}(\mu^*, \nu) + \epsilon^2 J_{K_\infty}(\nu). \]
Since (20) holds for all \( 0 \leq \epsilon \leq 1 \), then \( J_{K_\infty}(\mu^*, \nu) \geq 0 \). \( \square \)

**Theorem 12.** Suppose \( A \) is a compact set in the interior of \( H^+ \). There is a unique equilibrium measure \( \lambda_A^\infty \) minimizing \( J_{K_\infty}(\mu) \) over all \( \mu \in \mathcal{M}(A) \). The support of \( \lambda_A^\infty \) is contained in \( A_+ \). Furthermore, \( \lambda_A^R \) converges weak-star to \( \lambda_A^\infty \) as \( R \to \infty \).
Remark: Recall that $\lambda^R_A$ converges weak-star to $\lambda^\infty_A$ (and we write $\lambda^R_A \rightharpoonup \lambda^\infty_A$) as $R \to \infty$ means that

$$\lim_{R \to \infty} \int_A f \ d\lambda^R_A = \int_A f \ d\lambda^\infty_A$$

for any function $f$ continuous on $A$.

Proof. Suppose $\mu^*$ and $\tilde{\mu}^*$ are measures in $\mathcal{M}(A)$ such that $J_{K_\infty}(\mu^*) = J_{K_\infty}(\tilde{\mu}^*) = \inf_{\mu \in \mathcal{M}(A)} J_{K_\infty}(\mu)$. Then $\nu := \tilde{\mu}^* - \mu^*$ satisfies the hypotheses of Lemma 11 and thus $J_{K_\infty}(\mu^*, \nu) \geq 0$. On the other hand,

$$J_{K_\infty}(\tilde{\mu}^*) = J_{K_\infty}(\mu^* + \nu) = J_{K_\infty}(\mu^*) + 2 J_{K_\infty}(\mu^*, \nu) + J_{K_\infty}(\nu),$$

which, since $J_{K_\infty}(\mu^*) = J_{K_\infty}(\tilde{\mu}^*)$, implies that $J_{K_\infty}(\nu) = -2 J_{K_\infty}(\mu^*, \nu) \leq 0$. Now, $J_{K_\infty}(\nu) = -I^*_1(\nu) - I^*_2(\nu) = -I^*_1(\nu)$ since

$$I^*_1(\nu) = \iint (\Re[z] + \Re[w]) \ d\nu(z) \ d\nu(w) = 0.$$

Hence, $I^*_1(\nu) = -J_{K_\infty}(\nu) = 2 J_{K_\infty}(\mu^*, \nu) \geq 0$ and so, by Lemma 10, it follows that $\nu \equiv 0$ and thus $\mu^* = \tilde{\mu}^*$.

The fact that $\text{supp} \lambda^\infty_A \subset A_+$ follows from the observation that $K_\infty(z, w)$ is strictly decreasing for $z$ varying along all horizontal rays $[iy, \infty + iy]$ for $y \neq \Im[w]$ and along the ray $[w, \infty + iv]$ for $v = \Im[w]$, and is constant along the line segment $[iv, w]$.

The weak-star convergence of $\lambda^R_A$ to $\lambda^\infty_A$ follows from the weak-star compactness of $\mathcal{M}(A)$ and the uniqueness of the equilibrium measure $\lambda^\infty_A$. 

Remarks:

1. The level sets of $K_\infty(\cdot, w)$ are parabolas with focus $w$ and directrix $x = a$ for $a > \Re[w]$ (in the case $a = \Re[w]$, the level set is the line segment $[iv, w]$ where $v = \Im[w]$). Notice that these parabolas can also be viewed as arising from the elliptical level curves illustrated in Figure 3 by letting the real part of the focus $w_*$ tend to $-\infty$.

2. One may also consider $K_\infty(z, w)$ on $\mathbb{C} \times \mathbb{C}$ rather than $H^+ \times H^+$ (in effect, the line $\Re[z] = -\infty$ may be considered the axis of rotation).

Let $W^\mu_\infty$ denote the potential for a measure $\mu \in \mathcal{M}(H^+)$ and kernel $K_\infty$:

$$W^\mu_\infty(z) = \int_A K_\infty(z, w) \ d\mu(w) \quad (z \in H^+).$$

Then $W^\mu_\infty$ is continuous on $H^+$. Furthermore, if $W^\mu_\infty(z)$ is not constant for $z \in \text{supp} \mu$, then one may construct a signed Borel measure $\nu$ with
support contained in $A$ such that $\nu(A) = \int_A d\nu = 0$, $\mu + \nu \geq 0$, and such that $J_{K_\infty}(\mu, \nu) < 0$ (cf. [2]). Lemma [1] then implies that $J_{K_\infty}(\mu, \nu)$ cannot be minimal, which gives the following result.

**Lemma 13.** The equilibrium potential $W_\infty^{\lambda_\infty^A}$ satisfies

$$W_\infty^{\lambda_\infty^A}(z) \geq J_{K_\infty}(\lambda_\infty^A) \quad (z \in A)$$

with equality if $z \in \text{supp } \lambda_\infty^A$.

5.2. **Properties of the equilibrium measure for a circle.** We next consider the support of the $K_\infty$-equilibrium measure in the case that $A_+$ is contained in the right-half of a circular arc (as in Corollary 9). Recall that if $C$ is the circle with center $a$ and radius $r$ and $B \subset C$, we define $B^\theta := B \cap \{a + r e^{i\theta} \mid -\theta \leq \theta \leq \theta\}$.

**Theorem 14.** Suppose $C \subset \mathbb{C}$ is a circle of radius $r > 0$ and center $a$ with $\Re[a] > 0$ and suppose $A$ is a non-empty compact set in $H^+$ such that $A_+ \subset C_+$. Then $\text{supp } \lambda_\infty^A = A^\theta_+$ for some $\theta \in [0, \pi/2]$.

Furthermore, if $A_+$ is also symmetric about the line $y = \Im[a]$ and $A_+^{\pi/3}$ is non-empty, then $\text{supp } \lambda_\infty^A = A^\theta_+$ for some $\theta \in [0, \pi/3]$. Moreover, if $A_+$ is also symmetric about the line $y = \Im[a]$ and $A_+^{\pi/3}$ is empty, then $\lambda_\infty^A = (\delta_{a+\zeta} + \delta_{a+\overline{\zeta}})/2$ where $\zeta := re^{i\theta_m}$ and $\theta_m := \min\{\theta \geq 0 \mid a + re^{i\theta} \in A_+\}$.

**Proof.** Without loss of generality we may assume that $C$ has radius $r = 1$ and center $a = 0$. We then consider the parametrization of $C$ given by $\gamma(t) := e^{it}$ for $-\pi/2 \leq t \leq \pi/2$. Then, using $|e^{it} - e^{i\pi}| = 2|\sin((s - t)/2)|$, we find

$$K_\infty(\gamma(t), \gamma(s)) = -\cos(t) - \cos(s) - 2\left|\sin\left(\frac{s-t}{2}\right)\right| \quad (s, t \in [-\pi/2, \pi/2]).$$

Differentiating twice with respect to $s$ we obtain

$$\frac{\partial^2}{\partial t^2} K_\infty(\gamma(t), \gamma(s)) = \frac{1}{2} \left|\sin\left(\frac{s-t}{2}\right)\right| + \cos(t)$$

which is positive for $-\pi/2 < s, t < \pi/2$. Then (as in the proof of Corollary 9) it follows that $\text{supp } \lambda_\infty^A = A^\theta_+$ for some $\theta \in [0, \pi/2]$.

Now suppose $A_+$ is symmetric about the $x$-axis. Then the uniqueness of $\lambda_\infty^A$ shows that $\lambda_\infty^A$ is also symmetric about the $x$-axis, that is, $d\lambda_\infty^A(w) = d\lambda_\infty^A(\overline{w})$ for $w \in H^+$. Thus we have

$$W_\infty^{\lambda_\infty^A}(\gamma(t)) = \int_{A_+} K_\infty^s(z, w) d\lambda_\infty^A(w),$$
where

\[ K^s_\infty(z, w) := (K_\infty(z, w) + K_\infty(z, \overline{w})) / 2 \quad (z, w \in H^+). \]

Then we have

\[
K^s_\infty(\gamma(t), \gamma(s)) = \begin{cases} 
- \cos(s) - \cos(t) - 2 \cos(s/2) \sin(t/2), & 0 \leq s < t \leq \pi/2, \\
- \cos(s) - \cos(t) - 2 \cos(t/2) \sin(s/2), & 0 \leq t < s \leq \pi/2.
\end{cases}
\]

and differentiating with respect to \(t\) we obtain

\[
\frac{\partial}{\partial t} K^s_\infty(\gamma(t), \gamma(s)) = \begin{cases} 
\sin(t) - \cos(s/2) \cos(t/2), & 0 \leq s < t \leq \pi/2, \\
\sin(t) + \sin(s/2) \sin(t/2), & 0 \leq t < s \leq \pi/2.
\end{cases}
\]

We claim that

\[
(23) \quad \frac{\partial}{\partial t} K^s_\infty(\gamma(t), \gamma(s)) > 0 \quad (-\pi/2 \leq s \leq \pi/2, t > \pi/3).
\]

Clearly (23) holds in the second case of (22) when \(0 < t < s \leq \pi/2\). If \(\pi/3 < t \leq \pi/2\) and \(0 \leq s < t\), then using the first case of (22),

\[
\sin(t) - \cos(s/2) \cos(t/2) = \cos(t/2) (2 \sin(t/2) - \cos(s/2))
\]

and \(2 \sin(t/2) - \cos(s/2) \geq 2 \sin(t/2) - 1 > 0\) for this range of \(s\) and \(t\), we see that (23) holds in this case as well. Hence, we have

\[
\frac{d}{dt} W^{\lambda^\infty_A}_{\infty}(\gamma(t)) = \int_{A_+} \frac{\partial}{\partial t} K^s_\infty(\gamma(t), w) d\lambda^\infty_A(w) > 0 \quad (t > \pi/3).
\]

Thus, in light of Lemma 13, we have \(\text{supp} \lambda^\infty_A \subset A^{\pi/3}_+\) if \(A^{\pi/3}_+ \neq \emptyset\), while if \(A^{\pi/3}_+ = \emptyset\), then \(\lambda^\infty_A = (\delta_{a+\zeta} + \delta_{a+\overline{\zeta}})/2\).

5.3. The vertical line segment. In this section we consider sets \(A \subset H^+\) such that \(A_+\) is contained in a vertical line segment \([a + ic, a + id]\) and further suppose the endpoints \(a + ic\) and \(a + id\) are in \(A_+\). Then

\[
K_\infty(a + it, a + is) = -2a - |t - s| \quad (s, t \in [c, d])
\]

which falls into the class of kernels studied in [2] and it follows from results there that \(\lambda^\infty_A = (\delta_{a+ic} + \delta_{a+id})/2\) where \(\delta_w\) denotes the unit point mass at \(w\). In particular, for the “infinite washer” in \(\mathbb{R}^3\) obtained by rotating \([a + ic, a + id]\) about the \(y\)-axis and letting \(a \to \infty\), the support of the equilibrium measure degenerates to two circles. We contrast this with the finite \(R\) case where, by Corollary 7 we have \(\text{supp} \lambda^R_A = A_+\).
6. Discrete Minimum Energy Problems on $A \subset H^+$

Suppose $A \subset H^+$ is compact, $k : A \times A \to \mathbb{R}_+$ is continuous and nonnegative, and that there is a unique equilibrium measure $\lambda_{k,A}$ minimizing the $k$-energy

$$I_k(\mu) := \int \int_{A \times A} k(x, y) \, d\mu(x) \, d\mu(y)$$

over measures $\mu \in \mathcal{M}(A)$. In this case we say that $k$ is a continuous admissible kernel on $A$. In particular, we have in mind the reduced kernel $K$ as defined in (5) or the limiting kernel $K_\infty$ as defined in (19).

We consider the following discrete minimum $k$-energy problem. The arguments in this section closely follow those in [11, pp. 160–162]; however, the continuity of $k$ here allows for some simplification. For a collection of $N \geq 2$ distinct points $\omega_N := \{x_1, \ldots, x_N\} \subset A$, let

$$E_k(\omega_N) := \sum_{1 \leq i \neq j \leq N} k(x_i, x_j) = \sum_{i=1}^{N} \sum_{j=1}^{N} k(x_i, x_j),$$

and

(24) \quad $E_k(A, N) := \inf \{ E_k(\omega_N) \mid \omega_N \subset A, |\omega_N| = N \}$.

Since

(25) \quad $E_k(A, N) \leq \sum_{1 \leq i \neq j \leq N} k(x_i, x_j)$

for any configuration of $N$ points $\{x_1, \ldots, x_N\} \subset A$, integrating (25) with respect to $d\lambda_{k,A}(x_1)d\lambda_{k,A}(x_2) \cdots d\lambda_{k,A}(x_N)$ we find $E_k(A, N) \leq N(N-1)I_k(\lambda_{k,A})$ and so we have

(26) \quad $\frac{E_k(A, N)}{N(N-1)} \leq I_k(\lambda_{k,A}) \quad (N \geq 2)$.

On the other hand, the compactness of $A$ and continuity of $k$ imply that for each $N \geq 2$ there exists some optimal $k$-energy configuration $\omega_N^* \subset A$ such that $E_k(\omega_N^*) = E_k(A, N)$. Let $\lambda_{A,N} = \frac{1}{N} \sum_{x \in \omega_N^*} \delta_x \in \mathcal{M}(A)$ (where $\delta_x$ denotes the unit point mass at $x$). Then

(27) \quad $I_k(\lambda_{k,A}) \leq I_k(\lambda_{A,N}) = \frac{E_k(A, N)}{N} + \frac{\sum_{i=1}^{N} k(x_i, x_i)}{N^2} \quad (N \geq 2)$.

Combining (25) and (27) we have

(28) \quad $\frac{E_k(A, N)}{N(N-1)} \leq I_k(\lambda_{k,A}) \leq I_k(\lambda_{A,N}) \leq \frac{E_k(A, N)}{N^2} + \frac{\|k\|_A}{N} \quad (N \geq 2)$,
where \( \|k\|_A := \sup_{z \in A} k(z, z) \). Since \( \mathcal{E}_k(A, N)/N^2 \leq I_k(\lambda_{k,A}) < \infty \), the inequalities in [28] show that there is some constant \( C \) such that
\[
0 \leq I_k(\lambda_{A,N}) - I_k(\lambda_{k,A}) \leq C/N \quad \text{for } N \geq 2, \text{ and so}
\]
(29)\[
I_k(\lambda_{A,N}) \to I_k(\lambda_{k,A}) \text{ as } N \to \infty.
\]
If \( \mu^* \) is a weak-star limit point of the sequence \( \{\lambda_{A,N}\} \), then (29) shows that \( I_k(\mu^*) = I_k(\lambda_{k,A}) \) and so \( \mu^* = \lambda_{k,A} \). By the weak-star compactness of \( \mathcal{M}(A) \), any subsequence of \( \{\lambda_{A,N}\} \) must contain a weak-star convergent subsequence. Hence, we have the following result.

**Proposition 15.** Suppose \( A \) is a compact set in \( H^+ \) and that \( k : A \times A \to \mathbb{R}_+ \) is a continuous admissible kernel on \( A \). For \( N \geq 2 \), let \( \omega_{N}^* \) be an optimal \( k \)-energy configuration of \( N \) points \( \{x_1, x_2, \ldots, x_N\} \subset A \). Then \( \frac{1}{N} \sum_{i=1}^{N} \delta_{x_i} \rightharpoonup \lambda_{k,A} \) as \( N \to \infty \).

Figure 4 shows (near) optimal \( K \)-energy configurations for \( N = 30 \) points restricted to various ellipses in \( H^+ \).

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Figure 4. Near optimal $K$-energy configurations ($N = 30$ points) on various ellipses in $H^+$. 
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