Entanglement is one of the most important properties in quantum mechanics, and provides us with fruitful applications, as one can see in quantum communication protocols such as quantum teleportation and quantum key distribution.

There are two kinds of entangled states. One is the distillable entangled (DE) state, and the other is the bound entangled (BE) state. While, from several copies of DE states, some pure entanglement can be distilled by local quantum operations and classical communication (LOCC), one cannot extract any pure entanglement from BE states by LOCC. Nevertheless, it has been shown that any BE states are useful in quantum information processing protocols such as quantum teleportation and quantum key distribution.

There is another important property in quantum mechanics, called nonlocality, which can be seen from violation of some conditions that are satisfied by any local variable theory. The conditions are known as Bell inequalities. Since Werner [9] discovered DE states which can be described by a local hidden variable model, there have been a lot of research works about the following question: Does there exist a BE state violating a Bell inequality?

Since there is no BE state in two-qubit system [10], the answer is trivially “No”, and it was known that if a three-qubit state violates a specific form of the Bell inequality then it is distillable [11]. However, Dürr [12] found that for $N \geq 8$ there exist $N$-qubit BE states which violate a Bell inequality, and his result was recently improved by showing that there exists an $N$-qubit bound entangled state violating the Bell inequality if and only if $N \geq 6$ [13]. On the other hand, it has been also shown that the states which Dürr considered violate Bell inequalities different from the inequality for $N \geq 6$. In this paper, by employing different forms of Bell inequalities, in particular, a specific form of Bell inequalities with $M$ settings of the measuring apparatus for sufficiently large $M$, we prove that there exists an $N$-qubit bound entangled state violating the $M$-setting Bell inequality if and only if $N \geq 4$.

PACS numbers: 03.67.Mn 03.65.Ud, 42.50.Dv

Multpartite bound entanglement and multi-setting Bell inequalities

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(Dated: April 30, 2009)

Dirr [Phys. Rev. Lett. 87, 230402 (2001)] constructed $N$-qubit bound entangled states which violate a Bell inequality for $N \geq 8$, and his result was recently improved by showing that there exists an $N$-qubit bound entangled state violating the Bell inequality if and only if $N \geq 6$ [Phys. Rev. A 79, 032309 (2009)]. On the other hand, it has been also shown that the states which Dürr considered violate Bell inequalities different from the inequality for $N \geq 6$. In this paper, by employing different forms of Bell inequalities, in particular, a specific form of Bell inequalities with $M$ settings of the measuring apparatus for sufficiently large $M$, we prove that there exists an $N$-qubit bound entangled state violating the $M$-setting Bell inequality if and only if $N \geq 4$.

Distillable bipartite splits. Therefore, for sufficiently large $M$, we conclude that at least one $N$-qubit BE state violates the $M$-setting Bell inequality if and only if $N \geq 4$.

We first consider a Bell inequality with $M$ settings on the $N$-qubit system, proposed in [16]. Let $B_N^M$ be the Bell operator defined as

$$B_N^M = \sum_{m_1,\ldots,m_N=0}^{M-1} c_{m_1,\ldots,m_N} \sigma_{m_1} \otimes \cdots \otimes \sigma_{m_N},$$

where $\sigma_{m_n} = \sigma_x \cos(\phi_{m_n}) + \sigma_y \sin(\phi_{m_n})$, and the coefficients $c_{m_1,\ldots,m_N}$ are in a form

$$c_{m_1,\ldots,m_N} = \frac{\sin^N(\pi/2M)}{\cos(\pi/2M)} \cos \left( \sum_{j=1}^{N} \phi_{m_j} \right)$$

with the angles given by

$$\phi_{m_n} = \frac{\pi}{2M} m_n + \frac{\pi}{2MN} \eta.$$

In Eq. (4), the number $\eta = 1, 2$ is fixed for a given experimental situation, that is,

$$\eta = [M + 1]_2 [N]_2 + 1,$$

where $[x]_2$ stands for $x$ modulo 2. Then the $M$-setting Bell inequality is as follows:

$$| \text{tr} \left( B_N^M \rho \right) | \leq 1.$$

It was shown [16] that the Bell operator $B_N^M$ has only two eigenvalues $\pm (M^N/2) \sin^N(\pi/2M)/\cos(\pi/2M)$, and is essentially equivalent to

$$B_N^M = \frac{M^N \sin^N(\pi/2M)}{2\cos(\pi/2M)} \left( |\Psi_0^+\rangle\langle\Psi_0^+| - |\Psi_0^-\rangle\langle\Psi_0^-| \right),$$

where $\Psi_{0\pm} = (|1\rangle \pm |0\rangle)/\sqrt{2}$.
where $|\Psi_0^\pm\rangle$ are $N$-qubit maximally entangled states defined as
\[ |\Psi_0^\pm\rangle = \frac{1}{\sqrt{2}} (|00\ldots0\rangle \pm |11\ldots1\rangle). \] (8)

Hence, we here deal with the $M$-setting Bell inequality with respect to the Bell operator in (7), $|\text{tr} (B_M^N \rho)| \leq 1$.

In order to obtain our results, we now consider the family of $N$-qubit states $\rho_N$ presented in [17,18].

\[ \rho_N = \sum_{\sigma=\pm} \lambda_0^\sigma |\Psi_0^\sigma\rangle \langle \Psi_0^\sigma| \]
\[ + \sum_{j=1}^{2^{N-1}-1} \lambda_j (|\Psi_j^+\rangle \langle \Psi_j^+| + |\Psi_j^-\rangle \langle \Psi_j^-|), \] (9)

where
\[ |\Psi_j^\pm\rangle = \frac{1}{\sqrt{2}} (|j\rangle|0\rangle \pm |2^{N-1}-j\rangle|1\rangle), \] (10)

and $\lambda_j^+ + \lambda_j^- = 2 \sum_{j=1}^{2^{N-1}-1} \lambda_j = 1$. We remark that any arbitrary $N$-qubit state can be depolarized to a state in this family [17]. In other words, by the depolarizing process, any $N$-qubit state $\rho$ can be transformed into one in the family of $\rho_N$ with

\[ \lambda_j^+ = \langle \Psi_j^+| \rho \rangle |\Psi_0^+\rangle = \langle \Psi_0^+| \rho_N |\Psi_0^+\rangle, \]
\[ 2 \lambda_j = \langle \Psi_j^+| \rho \rangle |\Psi_j^+\rangle + \langle \Psi_j^-| \rho \rangle |\Psi_j^-\rangle = \langle \Psi_j^+| \rho_N |\Psi_j^+\rangle + \langle \Psi_j^-| \rho_N |\Psi_j^-\rangle. \] (11)

For each $0 < j < 2^{N-1}$, let $P_j$ be the bipartite split such that the coefficient of $2^{N-i-1}$ in the binary representation of $j$ is zero if and only if party $i$ belongs to the same set as the last party. Then the following proposition about bipartite distillability of the states $\rho_N$ has been known by Dür and Cirac [18].

**Proposition 1.** $\rho_N$ is distillable for the bipartite split $P_j$ if and only if $2 \lambda_j < \Delta \equiv \lambda_0^+ - \lambda_0^-$. By exploiting the proof of Lemma 2 in Ref. [13] and Proposition [11] we can obtain the following key lemma for our results.

**Lemma 2.** If
\[ \Delta > \frac{2 \cos(\pi/2M)}{M^N \sin^N(\pi/2M)} \] (12)

then there exist at least $[2^{N-1} - \frac{M^N \sin^N(\pi/2M)}{2 \cos(\pi/2M)} + 1]$ distillable bipartite splits in $\rho_N$.

**Proof.** Let $m$ be the number of distillable bipartite splits, $P_{j_1}, P_{j_2}, \ldots, P_{j_m}$. Suppose that $m \leq 2^{N-1} - \frac{M^N \sin^N(\pi/2M)}{2 \cos(\pi/2M)}$. Then we readily obtain the following inequality:
\[ 1 - \Delta \geq 2 \sum_{j=1}^{2^{N-1}-1} \lambda_j \]
\[ = 2(\lambda_{j_1} + \lambda_{j_2} + \cdots + \lambda_{j_m}) + 2 \sum_{j \notin \{j_1, \ldots, j_m\}} \lambda_j \]
\[ \geq 2(\lambda_{j_1} + \lambda_{j_2} + \cdots + \lambda_{j_m}) + (2^{N-1} - 1 - m) \Delta. \] (13)

It follows that
\[ 1 \geq 2(\lambda_{j_1} + \lambda_{j_2} + \cdots + \lambda_{j_m}) + (2^{N-1} - m) \Delta \]
\[ > 2(\lambda_{j_1} + \lambda_{j_2} + \cdots + \lambda_{j_m}) + \frac{(2^{N-1} - m)}{\frac{M^N \sin^N(\pi/2M)}{2 \cos(\pi/2M)}} \]
\[ \geq 2(\lambda_{j_1} + \lambda_{j_2} + \cdots + \lambda_{j_m}) + 1. \] (14)

The inequality (14) leads to a contradiction. Therefore, we can conclude that $m > 2^{N-1} - \frac{M^N \sin^N(\pi/2M)}{2 \cos(\pi/2M)}$. \(\square\)

Then, by equalities in (11), we obtain the following equalities:
\[ \frac{2 \cos(\pi/2M)}{M^N \sin^N(\pi/2M)} \text{tr} (B_M^N \rho) = \langle \Psi_0^+| \rho |\Psi_0^+\rangle - \langle \Psi_0^-| \rho |\Psi_0^-\rangle \]
\[ = \langle \Psi_0^+| \rho_N |\Psi_0^+\rangle - \langle \Psi_0^-| \rho_N |\Psi_0^-\rangle = \lambda_0^+ - \lambda_0^- = \Delta. \] (15)

where $\rho$ is a given arbitrary $N$-qubit state, and $\rho_N$ is the state transformed from $\rho$ by the depolarizing process. Hence, we have the following theorem by Lemma 2.

**Theorem 3.** For all the $N$-qubit states $\rho$ violating the $M$-setting Bell inequality with respect to the Bell operator in (7), there exist at least $[2^{N-1} - \frac{M^N \sin^N(\pi/2M)}{2 \cos(\pi/2M)} + 1]$ distillable bipartite splits.

**Theorem 4.** A maximally entangled pair between particles $i$ and $j$ can be distilled from $\rho_N$ if and only if all possible bipartite splits of $\rho_N$ where the particles $i$ and $j$ belong to different parties, have NPT.

By Theorem 3 and Proposition 4 we can prove the following theorem.

**Theorem 5.** For sufficiently large $M$, there exists at least one $N$-qubit BE state violating the $M$-setting Bell inequality with respect to the Bell operator in (7) if and only if $N \geq 4$. 


Proof. We note that the number of total bipartite splits is $2^{N-1} - 1$, and that the number of all distillable bipartite splits is at least $[2^{N-1} - \frac{M^N \sin(N\pi/2M)}{2 \cos(N\pi/2M)} + 1]$ by Theorem 3.

We first assume that $N = 3$. Then it follows from Theorem 3 that all bipartite splits are distillable, and so have NPT. By Proposition 4, a maximally entangled state can be distilled between any particles $i$ and $j$.

Conversely, if $N \geq 4$ then the $N$-qubit state $\rho_N$ presented in Ref. [13] violates the $M$-setting Bell inequality for sufficiently large $M$, as follows: The $N$-qubit state $\rho_N$ is defined as

$$\rho_N = \frac{1}{N-1} \sum_{j \in J_N} (|\Psi_j^+\rangle\langle \Psi_j^+| + |\Psi_j^-\rangle\langle \Psi_j^-|),$$

where $J_N = \{3, 6, \ldots, 3 \cdot 2^{N-3}\}$. Then, for $M \geq 6$, we can readily obtain that

$$\text{tr} (B^M_N \rho_N) = \frac{M^N \sin(N\pi/2M)}{2(N-1) \cos(N\pi/2M)} > 1$$

if and only if $N \geq 4$, as shown in Fig. 1. Therefore, the state $\rho_N$ violates the $M$-setting Bell inequality with respect to the Bell operator in (1).

Furthermore, by the same reason as that in Ref. [13], it can be shown that the $N$-qubit state $\rho_N$ is undistillable, and hence there exists an $N$-qubit BE state $\rho_N$ violating the $M$-setting Bell inequality if $N \geq 4$ for sufficiently large $M$.

Remark that $\text{tr} (B^M_N \rho_N)$ in the inequality (17) increases as $M$ tends to infinity, but if $M \geq 6$ then we have the same result as in Theorem 3 for any $M$-setting Bell inequality, since

$$\lim_{M \to \infty} \text{tr} (B^M_N \rho_N) = \frac{\pi^N}{2^{N+1}(N-1)} > 1$$

if and only if $N \geq 4$.

In conclusion, by employing a specific form of Bell inequalities with $M$ settings of the measuring apparatus for sufficiently large $M$, we have shown that if any $N$-qubit state violates the inequality then there exist at least $[2^{N-1} - \frac{M^N \sin(N\pi/2M)}{2 \cos(N\pi/2M)} + 1]$ distillable bipartite splits, and have concluded that there exists an $N$-qubit BE state violating the $M$-setting Bell inequality if and only if $N \geq 4$.

This work improves the previous results [12, 13, 14, 15] related to the multipartite BE states and Bell inequalities, even though it has been already known that there exists a four-qubit BE state, the so-called Smolin state [16], violating some other Bell inequality [18].

Furthermore, our technique in this paper can be also applied to the positive partial transpose inequality, $|\text{tr} (P_N \rho)| \leq 1$ with $P_N = 2^{N-1} (|\Psi_0^+\rangle\langle \Psi_0^+| - |\Psi_0^-\rangle\langle \Psi_0^-|)$, which was proposed in [21], and so we can construct a three-qubit BE state $\rho_3'$ violating the inequality as in Fig. 2:

$$\rho_3' = \frac{1}{3} |\Psi_0^+\rangle\langle \Psi_0^+| + \frac{1}{6} \sum_{j \in \{1, 3\}} (|\Psi_j^+\rangle\langle \Psi_j^+| + |\Psi_j^-\rangle\langle \Psi_j^-|),$$

since $|\text{tr} (P_3 \rho_3')| = 4/3 > 1$. Hence, the result in [21] can be enhanced as well.

This work was supported by the IT R&D program of MKE/IITA (2008-F-035-02, Development of key technologies for commercial quantum cryptography communication system). D.P.C. was supported by the Korea Science and Engineering Foundation (KOSEF) grant funded by the Korea government (MOST) (No. R01-2006-000-10698-0), and S.L. was supported by the Korea
Research Foundation Grant funded by the Korean Government (MOEHRD, Basic Research Promotion Fund) (KRF-2007-331-C00049).

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