Environmental decoherence appears to be the biggest obstacle for successful construction of quantum mind theories. Nevertheless, the quantum physicist Henry Stapp promoted the view that the mind could utilize quantum Zeno effect to influence brain dynamics and that the efficacy of such mental efforts would not be undermined by environmental decoherence of the brain. To address the physical plausibility of Stapp’s claim, we modeled the brain using quantum tunneling of an electron in a multiple-well structure such as the voltage sensor in neuronal ion channels and performed Monte Carlo simulations of quantum Zeno effect exerted by the mind upon the brain in the presence or absence of environmental decoherence. The simulations unambiguously showed that the quantum Zeno effect breaks down for timescales greater than the brain decoherence time. To generalize the Monte Carlo simulation results for any \( n \)-level quantum system, we further analyzed the change of brain entropy due to the mind probing actions and proved a theorem according to which local projections cannot decrease the von Neumann entropy of the unconditional brain density matrix. The latter theorem establishes that Stapp’s model is physically implausible but leaves a door open for future development of quantum mind theories provided the brain has a decoherence-free subspace.

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1. **Introduction**

The mainstream view in cognitive neurosciences identifies mind states with physical states realized within the brain. Major evidence for such mind-brain identity thesis comes from the clinical examination of patients with brain trauma in which loss of certain cognitive abilities occurs, and from the ability to elicit subjective experiences by direct electric stimulation of the brain cortex. If, however, one further postulates that the brain states obey the deterministic laws of classical mechanics, several counterintuitive results would follow. For example, the intuitively evident propositions that we have a free will allowing us to make choices, or that our subjective experiences can have a causal influence upon the brain and the sur-
rounding physical world, would turn out to be nothing but illusions [10]. Defending such a viewpoint seems to be possible, because the physical reality may not conform to our expectations of what the physical reality should be. Nevertheless, an increasing number of scientists think that we could construct a better theory of mind if we take into account quantum physics [11, 12, 13, 14, 15, 16].

One of the most elaborate proposals for a quantum theory of mind is due to Henry Stapp, who suggested that (1) the mind could influence the dynamics of the brain using quantum Zeno effect [17, 18, 19, 20, 21], and (2) the efficacy of the mental efforts would not be undermined by environmentally induced decoherence of the brain:

The quantum Zeno effect is itself a decoherence effect, and it is not diminished by environmental decoherence. Thus the decoherence argument against using quantum mechanics to explain the influence of conscious thought upon brain activity is nullified [21].

If correct, Stapp’s model could provide a scientific basis for the existence of free will [17]. Also, it would explain how mental efforts causally affect and restructure the organization of the brain in health or psychiatric disease [19]. Finally, it would establish that environmental decoherence is not an obstacle for the construction of quantum theories of mind [20, 21].

In a previous work [22], we have argued that if Stapp’s model worked in the presence of environmental decoherence, then mind efforts would have been capable of exerting paranormal effects upon nearby physical measuring devices. Our argument was based on a theorem stating that if mind efforts operate only upon the brain density matrix $\hat{\rho}$ using projection operators, then the von Neumann entropy production cannot be negative [22]. Stapp agreed that the proof of the particular theorem is correct [23], but argued that it presents no harm for his model because: (1) the studied two-level model system (polarization of a photon) is too simple to represent a human brain; (2) the studied quantum Zeno effect was based on no collapse version of quantum mechanics, which lacks an essential ingredient of Stapp’s model, namely the wavefunction collapse following the mind probing action; and (3) the mind action onto the brain density matrix does not need to slow down the environmental decoherence in order to exert quantum Zeno effect upon the brain [23].

In this work, to address the physical plausibility of Stapp’s model, we performed computer simulations of quantum Zeno effect exerted by the mind upon a model quantum brain in the presence or absence of environmental decoherence. The simulations were meticulously constructed according to the postulates in Stapp’s model (§2). The biological implementation was based on detailed molecular structural data of voltage-gated ion channel function in brain cortical neurons (§3). To address Stapp’s claim that the wavefunction collapse is a necessary ingredient for the model to work, we performed Monte Carlo simulations in the absence (§4) or
presence (§5) of environmental decoherence. The results from the Monte Carlo simulations unambiguously show that the quantum Zeno effect breaks down and the brain behaves as a ‘random telegraph’ for timescales greater than the decoherence time of the brain. The point at which the breakdown of quantum Zeno effect occurs is assessed by the statistical average outcome of many such Monte Carlo simulation trials. Because the statistical average is indistinguishable from the brain dynamics simulated within no collapse version of quantum mechanics (see §4 and §5) one can study the quantum Zeno effect in the brain using unconditional density matrices only, as done in Ref. [22] without the necessity to explicitly consider collapses and conditional density matrices. In §6 we prove a theorem according to which operating only locally on the brain with projection operators cannot decrease the von Neumann entropy of the unconditional brain density matrix. This allows us to generalize the results from the Monte Carlo simulations to any $n$-level quantum system in §7 and show that the failure of Stapp’s model is due to basic property (concavity) of the von Neumann entropy of quantum systems, rather than our failure to simulate the complexity of the real brain.

In conclusion, the present work shows that the breakdown of quantum Zeno effect looks differently in Stapp’s model (‘random telegraph’) compared with no collapse version of quantum mechanics (‘smear of probabilities’). This difference, however, is not enough to make Stapp’s model plausible solution to the problem of environmental decoherence. Instead, the future development of quantum theories of mind needs to consider seriously the increase of quantum entropy due to the inevitable coupling between the brain and its physical environment.

2. Stapp’s model

Stapp describes the interaction between the mind and the brain with the use of three basic processes 1, 2 and 3, attributed to John von Neumann [24]. In modern quantum mechanical terminology these processes can be referred to as (1) projective measurement, (2) unitary evolution and (3) wavefunction collapse. Interestingly, Stapp always discusses these processes in the order 2, 1, 3 as they appear in his model of mind-brain interaction.

2.1. Process 2

The brain is considered to be an $n$-level quantum system whose states belong to the Hilbert space $\mathcal{H}$. Unless the brain interacts with the mind or the surrounding environment, the brain density matrix $\hat{\rho}$ evolves according to the Schrödinger equation:

$$i\hbar \frac{\partial}{\partial t} \hat{\rho} = [\hat{H}, \hat{\rho}]$$

(1)

where the brackets denote a commutator. If the Hamiltonian $\hat{H}$ is time-independent, the solution of the Schrödinger equation is given by:
\[
\hat{\rho}(t) = e^{-i\hat{H}t/\hbar}\hat{\rho}(0)e^{i\hat{H}t/\hbar}
\]

According to Stapp, Process 2 generates “a cloud of possible worlds, instead of the one world we actually experience” [19] or “a smear of classically alternative possibilities” [20]. Furthermore, Stapp claims that “the automatic mechanical Process 2 evolution generates this smearing, and is in principle unable to resolve or remove it” [20].

2.2. Process 1

The mind is able to perform repeated projective measurements upon the brain using a freely chosen set of projection operators \(\{\hat{P}_1, \hat{P}_2, \ldots, \hat{P}_n\}\), which are mutually orthogonal \(\hat{P}_i\hat{P}_j = \delta_{ij}\hat{P}_j\) and complete to identity \(\sum_j \hat{P}_j = \hat{I}\). After each projective measurement the brain density matrix undergoes non-unitary transition:

\[
\hat{\rho}(t) \rightarrow \sum_j \hat{P}_j \hat{\rho}(t) \hat{P}_j
\]

According to Stapp the “Process 1 action extracts from [the] jumbled mass of possibilities a particular [set] of alternative possibilities” among which only one is going to be actualized by the Nature [18].

2.3. Process 3

The actualization of only one possibility, from the set of available possibilities, is done by the Nature. Colloquially, this process is referred to as “reduction of the wave packet” or “collapse of the wave function”. Within the more general density matrix formalism, Process 3 is described by a non-unitary transition that converts the unconditional density matrix into conditional one (Lüders rule) [25,26]:

\[
\sum_j \hat{P}_j \hat{\rho}(t) \hat{P}_j \rightarrow \frac{\hat{P}_k \hat{\rho}(t) \hat{P}_k}{\text{Tr} \left[ \hat{P}_k \hat{\rho}(t) \right]}
\]

where \(\hat{P}_k\) is a particular projector from the set \(\{\hat{P}_1, \hat{P}_2, \ldots, \hat{P}_n\}\) selected by the Nature and \(\text{Tr} \left[ \hat{P}_k \hat{\rho}(t) \right]\) is the probability for the state to collapse to that particular state.

Here we remark that throughout this work we use the terms projective measurements, local projective measurements and local projections in the precise mathematical sense of Process 1 given by Eq. 3. In no collapse models of quantum mechanics the entanglement between the measured system and the measuring apparatus results in the transition given by Eq. 3 and this is all there is in a physical measurement. In collapse models of quantum mechanics the measurement is completed only after the
entangled state between the measured system and the measuring apparatus reduces to a single outcome conditionally purifying the state and disentangling the measured system from the measuring apparatus according to Eq. 4. With our particular choice of terminology we allow the usage of the term *measurement* when discussing both collapse and no collapse models of quantum mechanics and avoid trivial identification of *measurement* with *wavefunction collapse* (for a general introduction to quantum measurement see Ref. [26]).

3. Quantum tunneling in a multiple-well as a quantum brain model

Biological implementation of Stapp’s model should ultimately take into account the basic electrophysiological processes that occur within neurons in the human cerebral cortex [27]. Each neuron is composed of three different compartments: dendrites, soma and axon (Fig. 1). Typically, the dendrites receive electric inputs, the soma integrates the dendritic inputs, and the axon outputs electric spikes that subsequently affect the electric properties of dendrites of target neurons. At the molecular level, the electric processes in neurons are regulated by opening and closing of voltage-gated ion channels that are inserted in the plasma membrane.

Three families of voltage-gated ion channels are most abundant and physiologically important in neurons: sodium (Nav), potassium (Kv) and calcium (Cav) ion channels. All the three families of ion channels share a common evolutionary conserved structure (Fig. 1). Each channel is formed by a pore forming α-subunit. The α-subunit of sodium (Nav) and calcium (Cav) channels is composed of four protein domains I-IV, each of which contains 6 transmembrane α-helices. There is a minor difference in the structure of the potassium (Kv) channels in which the protein domains I-IV are disconnected from each other giving rise to four α-subunits instead of a single one. Structurally, the channel pore is formed by four protein loops (P) located between the 5th and the 6th α-helices of the protein domains I-IV. The voltage-sensing is performed by a charged 4th α-helix of each domain [28,29,30,31,32,33,34].

Macroscopic electric currents in neurons produced by voltage-gated ion channels flow continuously across the plasma membrane depending on the transmembrane voltage and the maximal channel conductance density. The electric conductance through each individual channel, however, can take only 2 discrete values: in the open conformation the channel has a certain characteristic single channel conductance, whereas in the closed conformation the conductance is zero. Single-channel recordings have shown that at a given transmembrane voltage each voltage-gated ion channel undergoes stochastic transitions between open and closed states characterized by a certain probability for the given channel type to be in the open conformation [35]. For transmembrane voltages that are far away from the threshold for generation of electric spike, the behavior of brain neurons is insensitive to the small stochastic fluctuations in the potential due to single-channel conformational transitions. For transmembrane voltages near the threshold of −55 mV, however,
opening or closing of a single channel can affect the generation of an electric spike. In the cerebral cortex of humans there are $\approx 1.6 \times 10^{10}$ neurons and these neurons can fire spikes with frequencies of $\approx 40$ Hz. Therefore, it is expected that each second in the human cerebral cortex there are thousands of neurons that are sensitive to the opening or closing of a single channel. Firing or not firing of these neurons that are near the voltage threshold may have huge impact on cognitive processes due to the highly nonlinear character of the cortical neuronal networks.

Because the opening and closing of the voltage-gated ion channels is a stochastic process controlled by electron motion in the charged 4th $\alpha$-helix of each channel domain, it is biologically feasible to assume that Stapp’s quantum Zeno model could be implemented via frequent measurements on the position of an electron in the voltage-sensing 4th $\alpha$-helix of ion channels. Here, we would like to underline the facts that the quantum state of the brain is generally unobservable, and that there is an upper bound on the classical information that can be extracted in a quantum measurement as shown by Alexander Holevo. That is why the measured brain
observable in the model, namely the position of the electron inside the voltage sensor of a neuronal ion channel, has been chosen to be physiologically paramount for the processing of information by real brains. In addition, the quantum tunneling of an electron in a multiple-well structure has already been considered to be an important toy model for studying quantum Zeno effect\textsuperscript{38,39}. For the subsequent numerical simulations we decided to use a symmetric triple-well potential\textsuperscript{40}, however it should be noted that the general algorithm for the simulations could be easily applied for more complex $n$-level systems and/or asymmetric potentials (see the Appendix).

Let us suppose that the normalized position states of the electron in each of the wells are $|A\rangle$, $|B\rangle$ and $|C\rangle$, there is no offset energy between different wells, and the tunneling matrix elements between the states $|A\rangle$ and $|B\rangle$ or $|B\rangle$ and $|C\rangle$ are equal $\kappa_{12} = \kappa_{23}$. The Hamiltonian of the system in position basis is\textsuperscript{40}:

$$\hat{H} = \begin{pmatrix}
0 & -\kappa_{12} & 0 \\
-\kappa_{12} & 0 & -\kappa_{23} \\
0 & -\kappa_{23} & 0
\end{pmatrix} \quad (5)$$

The energy eigenstates of the system are the eigenstates of the Hamiltonian. If, under suitable choice of units, we set $\kappa_{12} = \kappa_{23} = \frac{1}{\sqrt{2}}$, the energy eigenstates can be written as:

$$|E_{\pm 1}\rangle = \frac{1}{2} |A\rangle \mp \frac{1}{\sqrt{2}} |B\rangle + \frac{1}{2} |C\rangle \quad (6)$$

$$|E_0\rangle = \frac{1}{\sqrt{2}} |A\rangle - \frac{1}{\sqrt{2}} |C\rangle \quad (7)$$

with eigenvalues $E_{\pm 1} = \pm 1$ and $E_0 = 0$.

With the use of the projectors $\hat{P}_J = |J\rangle \langle J|$, $J \in \{A, B, C\}$, we can express the probability to find the electron in well $J$ at time $t$ as:

$$p_J(t) = \text{Tr} \left[ \hat{P}_J e^{-i\hat{H}t/\hbar} \hat{\rho}(0) e^{i\hat{H}t/\hbar} \right] \quad (8)$$

If at $t = 0$ the electron is located in well $A$, the initial density matrix of the system is:

$$\hat{\rho}(0) = \begin{pmatrix}
1 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{pmatrix} \quad (9)$$

and the probabilities for detection in each of the three wells at time $t$ calculated from Eq. 8 are:
Fig. 2. Quantum tunneling of the brain state in a triple well potential simulated for a period of time $t = 3\pi/\hbar$. At times $t = 2k\pi/\hbar$, $k = 0, 1, 2, \ldots$, the brain state is localized in well $A$, whereas at times $t = (2k + 1)\pi/\hbar$, the brain state is localized in well $C$. If unperturbed, the quantum system in a multiple potential well tunnels coherently forth-and-back from well $A$ to wells $B$ and $C$ and returns periodically to its initial state. The probability $|\psi|^2$ is normalized so that $\int |\psi|^2 dx = 1$.

$$p_A(t) = \frac{1}{4} \left[ \cos \left( \frac{t}{\hbar} \right) + 1 \right]^2$$

$$p_B(t) = \frac{1}{2} \left[ \sin \left( \frac{t}{\hbar} \right) \right]^2$$

$$p_C(t) = \frac{1}{4} \left[ \cos \left( \frac{t}{\hbar} \right) - 1 \right]^2$$

The electron tunnels coherently forth-and-back from well $A$ to wells $B$ and $C$ as illustrated in Fig. 2. Such behavior is consistent with previous works and shows that the quantum system in a multiple potential well, if unperturbed, is able to return periodically to its initial state.

4. Monte Carlo simulation of quantum Zeno effect

If the electron in the voltage-sensing 4th $\alpha$-helix of an ion channel is initially in an eigenstate of the position operator, it would be possible to achieve quantum Zeno effect using projective measurements in a position basis. Let us suppose that the electron is initially in well $A$ and we perform repeated projective measurements with the projectors $\hat{P}_J$ at regular time intervals $\xi$. Between the projective measurements the electron state evolves coherently according to the Schrödinger equation (Process 2). After each measurement the density matrix $\hat{\rho}$ of the electron in the quantum brain is diagonalized in the position basis (Process 1). To achieve “reduction of the wave packet” (Process 3), we implement weighted Random Choice in Wolfram’s
Fig. 3. Quantum Zeno effect in a triple well potential simulated for a period of time $t = 3\pi/\hbar$. Mind efforts perform projective position measurements of the brain state at times separated by time interval $\xi = \pi/8\hbar$ resulting in suppressed evolution of the brain state. (a) The result from a single Monte Carlo simulation trial successfully achieving quantum Zeno effect manifested as suppressed decay of the initial state $|A\rangle$. The quantum state of the brain starts in well $A$ and remains there for a period of time equal to 19 $\xi$-steps before jumping to well $B$. (b) Statistical average of multiple Monte Carlo trials reproduces the probabilities contained in the unconditional density matrix of the system. This result can also be interpreted as the state of the multiverse in no-collapse models of quantum mechanics. Red arrow indicates the time point at which the quantum Zeno effect breaks down—the probability for the electron not to be in well $A$ is at least equal to the probability to be in well $A$. The probability $|\psi|^2$ is normalized so that $\int |\psi|^2 dx = 1$.

*Mathematica* 9, where one of the pure state density matrices $\hat{\rho}_A = |A\rangle\langle A|$, $\hat{\rho}_B = |B\rangle\langle B|$ or $\hat{\rho}_C = |C\rangle\langle C|$ is randomly chosen with corresponding weights $p_A(\xi)$, $p_B(\xi)$ and $p_C(\xi)$ and the state of the quantum brain is updated accordingly. In the limit $\xi \to 0$, the electron in the quantum brain stays with probability of 1 in its initial state. A single trial from the Monte Carlo simulation with a non-zero $\xi = \pi/8\hbar$ is shown in Fig. 3a. The quantum state of the brain remains in the well $A$ for a period of time equal to 19 $\xi$-steps before jumping to well $B$, where it stays until the end of the simulation. We note that while running multiple Monte Carlo trials the majority of them achieve the intended quantum Zeno effect, still there are trials in which the state jumps early to well $B$ and stays there instead of well $A$. The results show that in the absence of environmental decoherence the quantum Zeno effect indeed could be achieved with a certain efficiency that is proportional to the average time for which the decay of the initial state is suppressed. The time point at which the quantum Zeno effect breaks down is best visualized on a plot showing the statistical average of multiple Monte Carlo trials (Fig. 3b). The breakdown of the quantum Zeno effect could be understood as the drop of the unconditional probability for the system to be in the initial state (in this case well $A$) to a value $\leq \frac{1}{2}$. Since the efficiency of any scheme attempting to achieve quantum Zeno effect is estimated by calculation of unconditional density matrices and unconditional probabilities, it is clear that the presence of wavefunction collapses (reductions) is irrelevant and cannot help repairing a faulty quantum Zeno scheme.
Now, we are ready to address the main claim made by Stapp, whether mind efforts could achieve quantum Zeno effect upon the brain even in the presence of environmentally induced decoherence of the brain. In general, the effect of environmental decoherence is to diagonalize the density matrix $\hat{\rho}$ in a certain basis (so called pointer basis) depending on the interaction Hamiltonian $\hat{H}_{\text{int}}$ that describes the coupling between the brain and its environment. Because Stapp claims that his model is robust against the effects of environmental decoherence, we could choose $\hat{H}_{\text{int}}$ such that it diagonalizes the density matrix of the brain $\hat{\rho}$ in a basis different from the position basis. Here, we take the pointer basis to be the energy basis -- that is we assume that the brain undergoes dephasing in the process of its interaction with the environment \[41\]. Alternatively, identical results will be obtained if we start from a brain that is in an eigenstate of the energy basis, require the mind to exert quantum Zeno effect in the energy basis, and consider environmental decoherence in position basis.

If $\tau$ is the decoherence time of the brain \[43\], the action of the environmental decoherence could be modeled using non-unitary transition:

$$\hat{\rho}(\tau) \rightarrow \sum_j \hat{P}_{Ej} \hat{\rho}(\tau) \hat{P}_{Ej} \quad (13)$$

where the projectors $\hat{P}_{Ej} = |E_j\rangle\langle E_j|$, $j \in \{\pm 1, 0\}$. Next, to achieve maximal efficiency of mind efforts, we let the mind perform repeated projective measurements with the projectors $\hat{P}_J$ at regular time intervals $\xi \rightarrow 0$. The weights in the Random Choice function at times separated by intervals $\tau$ are as follows: if $\hat{\rho}(\tau) = \hat{\rho}_A$ or $\hat{\rho}(\tau) = \hat{\rho}_C$ then $p_A(\tau) = \frac{3}{8}$, $p_B(\tau) = \frac{1}{4}$, $p_C(\tau) = \frac{3}{8}$ and if $\hat{\rho}(\tau) = \hat{\rho}_B$ then $p_A(\tau) = \frac{1}{4}$, $p_B(\tau) = \frac{1}{2}$, $p_C(\tau) = \frac{1}{4}$. These weights could be calculated from an initial position eigenstate that undergoes a projective measurement in the energy basis using the projectors $\hat{P}_{Ej}$, followed by a projective measurement into the position basis using the projectors $\hat{P}_J$. A typical result from the Monte Carlo simulation of mind efforts attempting to achieve quantum Zeno effect in the presence of environmental decoherence is shown in Fig. 4a. For timescales greater than the decoherence time $\tau$, the quantum state of the brain randomly jumps between the different wells -- a behavior previously described as 'random telegraph' \[43\]. The statistical average of multiple Monte Carlo trials shown in Fig. 4b does not show the individual random walks produced by the collapses. Nevertheless the probability distribution in Fig. 4b unambiguously demonstrates that the breakdown of the quantum Zeno effect occurs at a timescale comparable with the decoherence time $\tau$. For $t \geq \tau$ the probability for the quantum brain to be in any of the potential wells becomes equal to $\frac{1}{3}$ -- a state described by unconditional density matrix with maximal von Neumann entropy. Realization of a quantum Zeno effect-like walk in which the state stays in well $A$ for $t = 24\tau$ will occur with probability of $(\frac{1}{3})^{23} = 1.06 \times 10^{-11}$.

The above results affirm that to study the quantum Zeno effect in the brain, one does not really need to model the collapse (Process 3) as claimed by Stapp.
Fig. 4. Decoherence induced breakdown of quantum Zeno effect simulated for a period of time \( t = 24\tau \) during which mind efforts perform projective position measurements of the brain at times separated by time interval \( \xi \to 0 \). (a) The result from a single typical Monte Carlo simulation trial. Decoherence induces random jumps of the quantum brain state within the triple well potential—the brain behaves as a ‘random telegraph’. The quantum Zeno effect does not persist for timescales greater than the decoherence time \( \tau \). (b) Statistical average of multiple Monte Carlo trials reproduces the probabilities contained in the unconditional density matrix of the system. This result can also be interpreted as the state of the multiverse in no-collapse models of quantum mechanics. Red arrow indicates the time point at which the quantum Zeno effect breaks down—the probability for the electron not to be in well A is at least equal to the probability to be in well A. The probability \(|\psi|^2\) is normalized so that \( \int |\psi|^2 dx = 1 \).

because in order to produce the unconditional density matrix one has to average over the results from individual collapses. If the mind action is not followed by a collapse, and provided that the mind probing actions and environmental decoherence do not occur in the same basis (that is the two bases do not have shared basis vectors), for time \( t \geq \tau \) the unconditional density matrix of the brain will tend to one with maximal von Neumann entropy (in which the probability to find the brain in any state is \( \frac{1}{n} \)). The result from a simulation in which the collapse (Process 3) is omitted is identical with the plot shown in Fig. 4b, and thus provides all the necessary information needed for one to find the time point at which the unconditional probability for the system to be in the initial state drops to a value \( \leq \frac{1}{2} \).

6. Local projections and brain density matrix

The Monte Carlo simulations of the \( n = 3 \) level quantum system could be easily performed on a personal computer. Repeating the algorithm for large \( n \), however, would need the processing power of a supercomputer and substantial financial and time investment. It is thus desirable to prove as a theorem that the Monte Carlo results obtained for \( n = 3 \) would hold as well for any \( n \). To achieve this we will estimate the change of the von Neumann entropy of the unconditional brain density matrix under local projections.
Definition 1. The von Neumann entropy of a quantum-mechanical system described by a density matrix $\hat{\rho}$ is:

$$S(\hat{\rho}) = -\text{Tr}(\hat{\rho} \ln \hat{\rho}) = -\sum_i \lambda_i \ln \lambda_i$$  \hspace{1cm} (14)

Theorem 2. The von Neumann entropy is invariant under unitary evolution $^{44,45,46}$:

$$S(\hat{U}\hat{\rho}\hat{U}^\dagger) = S(\hat{\rho})$$  \hspace{1cm} (15)

Theorem 3. The von Neumann entropy is a concave functional $^{44,45,46,47,48}$. If $p_1, p_2, \ldots, p_n \geq 0$ and $\sum_i p_i = 1$, then:

$$S\left(\sum_i p_i \hat{\rho}_i\right) \geq \sum_i p_i S(\hat{\rho}_i)$$  \hspace{1cm} (16)

We have argued previously that if the mind efforts were to act only locally at the brain using projection operators, then such action cannot decrease the von Neumann entropy of the brain density matrix $^{22}$. Because decoherence increases the von Neumann entropy over time, it would follow as a corollary that quantum Zeno effect cannot be achieved via local projections in the presence of environmental decoherence. Our previous argument, however, utilized a two-level approximation and a pure initial state. Stapp objected that the argument is based on an improper extrapolation of a theorem valid for a two-level system to the much more complicated $n$-level system of the real brain $^{23}$. Here, we prove a generalization of our previous theorem, which is valid for any $n$-level quantum system and any purity of the initial density matrix $\hat{\rho}$ of that system.

Theorem 4. Local projective measurements upon a quantum system $Q$ using a freely chosen set of projection operators $\{\hat{P}_1, \hat{P}_2, \ldots, \hat{P}_n\}$, which are mutually orthogonal $\hat{P}_i \hat{P}_j = \delta_{ij} \hat{P}_j$ and complete to identity $\sum_j \hat{P}_j = \hat{I}$ cannot decrease the von Neumann entropy of the unconditional density matrix $\hat{\rho}$ of the system $Q$. Moreover, the entropy cannot be decreased for any subspace of the density matrix $\hat{\rho}$.

Proof: Represent the initial density matrix $\hat{\rho}_0$ in the basis in which the projection operators $\{\hat{P}_1, \hat{P}_2, \ldots, \hat{P}_n\}$ are expressed in their simplest form (a single unit on the diagonal with zeros elsewhere). If $\hat{\rho}_0$ is diagonal in that basis, it will remain unchanged by the action of the projectors and the entropy will stay the same. In general, however, $\hat{\rho}_0$ will not be diagonal:
The action of the projectors (see Stapp’s Process 1 given by Eq. 3) will be to kill all off-diagonal entries:

\[ \hat{\rho}_0 = \begin{pmatrix}
a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\
a_{21}^* & a_{22} & a_{23} & \cdots & a_{2n} \\
a_{31}^* & a_{23}^* & a_{33} & \cdots & a_{3n} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
a_{1n}^* & a_{2n}^* & a_{3n}^* & \cdots & a_{nn}
\end{pmatrix} \quad (17) \]

To show that the entropy of \( \hat{\rho}_n \) is larger than the entropy of \( \hat{\rho}_0 \), we construct a chain of inequalities. First, note that we can evolve unitarily \( \hat{\rho}_0 \) in two different ways and sum the results to kill all off-diagonal entries in the first row and column:

\[ \sum_j \hat{P}_j \hat{\rho}_j(t) \hat{P}_j = \hat{\rho}_n = \begin{pmatrix}
a_{11} & 0 & 0 & \cdots & 0 \\
0 & a_{22} & 0 & \cdots & 0 \\
0 & 0 & a_{33} & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & a_{nn}
\end{pmatrix} \quad (18) \]

Using Theorems 2 and 3, we obtain \( S(\hat{\rho}_1) \geq S(\hat{\rho}_0) \). Next, we can evolve unitarily \( \hat{\rho}_1 \) in two different ways and sum the results to kill its off-diagonal entries in the second row and column:

\[ \hat{\rho}_1 = \begin{pmatrix}
a_{11} & 0 & 0 & \cdots & 0 \\
0 & a_{22} & a_{23} & \cdots & a_{2n} \\
0 & a_{23}^* & a_{33} & \cdots & a_{3n} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & a_{2n}^* & a_{3n}^* & \cdots & a_{nn}
\end{pmatrix} \]

\[ = \frac{1}{2} \begin{pmatrix}
-1 & 0 & 0 & \cdots & 0 \\
0 & 1 & 0 & \cdots & 0 \\
0 & 0 & 1 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & 1
\end{pmatrix} \hat{\rho}_0 \begin{pmatrix}
-1 & 0 & 0 & \cdots & 0 \\
0 & 1 & 0 & \cdots & 0 \\
0 & 0 & 1 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & 1
\end{pmatrix} + \frac{1}{2} \hat{\rho}_0 \hat{I} \hat{\rho}_0 \hat{I} \quad (19) \]
\[ \hat{\rho}_2 = \begin{pmatrix} a_{11} & 0 & 0 & \cdots & 0 \\ 0 & a_{22} & 0 & \cdots & 0 \\ 0 & 0 & a_{33} & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & a_{n3} & \cdots & a_{nn} \end{pmatrix} \]

\[ = \frac{1}{2} \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & -1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix} \hat{\rho}_1 \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & -1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix} + \frac{1}{2} \hat{I} \hat{\rho}_1 \hat{I} \] (20)

Again, using Theorems 2 and 3, we obtain \( S(\hat{\rho}_2) \geq S(\hat{\rho}_1) \). Repeating the construction \( n \) times gives us the bound

\[ S(\hat{\rho}_n) \geq S(\hat{\rho}_{n-1}) \geq \cdots \geq S(\hat{\rho}_2) \geq S(\hat{\rho}_1) \geq S(\hat{\rho}_0) \] (21)

The above construction can be applied as well for every subspace of the density matrix from which follows that there is no subspace in which the change of the entropy is negative after projective measurement.

7. Implications for Stapp’s model

Wavefunction collapses (Process 3) produce pure states that are described by conditional density matrices with zero quantum entropy. Conditional density matrices however cannot assess whether a given quantum Zeno scheme is plausible or not – for that one needs unconditional probabilities. A classical example nicely illustrates this point: suppose that one buys a lottery ticket that has a 1 in a million chance to win the jackpot. It is wrong to claim that buying the lottery ticket is an efficient or plausible way to become a millionaire based on a conditional reasoning such as ‘if you happen to win the lottery then there is an absolute certainty (probability of 1) that you will become a millionaire’. Instead, the efficiency of buying the lottery ticket is 1 in a million based on the unconditional probability for the event. Thus, the main result provided by Theorem 4 concerning the entropy of the unconditional density matrix of the brain is exactly tailored to assess the plausibility of Stapp’s quantum Zeno scheme.

The implications of Theorem 4 for Stapp’s model become clearer if we recognize that the enviromental decoherence is just a form of quantum Zeno effect in a so-called pointer basis. If the decoherence pointer basis happens to be the one in which the mind action is intended, then the decoherence alone can achieve the quantum Zeno effect and the mind action will be irrelevant or redundant. Thus, in order for mind’s action to be functionally meaningful it needs to achieve something different that is not already achieved by the brain environment.
First, suppose that the brain does not possess *decoherence-free subspace*. Trivially, quantum Zeno effect can be achieved for states located within decoherence-free subspace because the environmental action for these states is absent. However, Stapp explicitly states that quantum Zeno effect could work even if the brain coherence is destroyed due to environmental interaction, hence regardless the lack of decoherence-free subspace. This is also evident in Stapp’s own illustrations showing that decoherence reduces the density matrix of the brain to nearly diagonal form in the position basis. Since the density matrix $\hat{\rho}$ is a Hermitian operator, it always has a representation basis in which it is diagonal. If only environmental interaction is considered, for times larger than the decoherence time $\tau$ one may identify the basis in which $\hat{\rho}$ is diagonal with the pointer basis of environmental decoherence.

Second, suppose that the mind intends to keep the probability $p(a_1, t_0) > \frac{1}{2}$ of a given brain state $|a_1\rangle\langle a_1|$ as high as possible at a later time $t > t_0$ in the presence of environmental decoherence. Breakdown of quantum Zeno effect will be signaled by $p(a_1, t_0) \leq \frac{1}{2}$ because at this point the probability for the brain *not to be* in the given state is at least equally, if not more, probable than being in the given state.

We can now show that if the vector $|a_1\rangle$ does not belong to the pointer basis, the repeated local projective measurements of the brain by the mind using the projector $|a_1\rangle\langle a_1|$ will always have probability $p(a_1, t)$ that is lower or equal to the probability $p'(a_1, t)$ for the case in which the mind does not perform any measurement.

Consider an initial brain density matrix that is diagonalized in the pointer basis. Let the mind perform two projective measurements of the brain at times $t_1$ and $t_2$ using the same projector $|a_1\rangle\langle a_1|$ (as part of a complete set of projectors) in an attempt to keep the brain in this state and preserve the probability $p(a_1, t) > \frac{1}{2}$ for as long period of time as possible. From Eq. 3 it follows that at both times $t_1$ and $t_2$ the density matrix $\hat{\rho}$ will be diagonal in the basis chosen by the mind. From Theorem 4 follows that $S[\hat{\rho}(t_2)] \geq S[\hat{\rho}(t_1)]$ and similar inequalities hold for all subspaces of $\hat{\rho}$. Here we remind that the von Neumann entropy is directly related to the eigenvalues of the density matrix and that the eigenvalues are exhibited on the main diagonal of the density matrix when the matrix is diagonal. Thus, using the facts that the matrices are diagonal, the function $x \ln x$ is concave in the interval $[0, 1]$, and $p(a_1, t_0) > \frac{1}{2}$, we obtain $p(a_1, t_2) \leq p(a_1, t_1)$. Because repeated local projections instead of increasing the probability of $|a_1\rangle\langle a_1|$ over the initial $p(a_1, t_0)$ can only speed up the probability decay, it follows that the mind efforts can only lead to extra decay of probability in addition to the decay due to environmental decoherence. Indeed, if there are no shared basis vectors between the pointer basis and the basis chosen by the mind for the projective measurements, for $t \geq \tau$ the entropy of the brain density matrix will tend to the maximal one:
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\[ S(\tilde{\rho})_{\text{max}} = S \begin{bmatrix} \frac{1}{n} & 0 & \cdots & 0 \\ 0 & \frac{1}{n} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \frac{1}{n} \end{bmatrix} = \ln n \] (22)

If the basis chosen by the mind happens to be mutually unbiased with the decoherence pointer basis (that is the inner product of any two vectors from the two bases is equal to \( \frac{1}{n} \)), then the decay of the initial probability from \( p(a_1, t_0) \) to \( \frac{1}{n} \) will be fastest and will occur immediately after the first projective measurement performed by the mind.

The above result is devastating for Stapp’s model because instead of achieving quantum Zeno effect, the mind efforts can only speed up the decay of probability for the brain to be in a given state. In other words, in the presence of environmental decoherence the best strategy of the mind is to \textit{withhold} performing any projective measurements. Because Theorem 4 follows from the standard Hilbert space formalism of quantum mechanics, any amendment done to Stapp’s model will necessarily result in a new physical theory that is inconsistent with quantum mechanics.

8. Discussion

The linear unitary time evolution of quantum systems given by the Schrödinger equation (Eq. 2) preserves inner products. This implies, however, that quantum systems prepared in a superposition of states could interact with macroscopic objects such as cats forcing them into a superposition of mutually exclusive alternatives such as “dead cat” or “alive cat” \cite{52}. To explain the apparent lack of macroscopic superpositions in the surrounding world, two different approaches have been proposed. The first approach is to counterintuitively accept the linear unitary time evolution as fundamental, and exclude the collapse postulate from the list of fundamental quantum mechanical axioms. In such no collapse models of quantum mechanics all possible outcomes do actually get realized in different universes consistently with the Born rule, and the sum of all decoherent universes forms a multiverse \cite{53,54,55}. The second approach is to accept the collapse postulate as a description of an objective physical process and introduce nonunitary time evolution of quantum systems \cite{56,57,58}. In this work, we have considered both types of approaches. First, the results obtained for the unconditional density matrix of the brain could be directly interpreted as describing the state of the multiverse in no collapse models of quantum mechanics. Second, Monte Carlo simulations of collapse models of quantum mechanics were performed using the Lüders rule (Eq. 4) that provides an effective mathematical description of the objective collapse regardless of the exact nature of the underlying physical process. Both collapse and no collapse approaches are restrained by the quantum informational theorem \cite{1} because, if the probabilities for all collapse outcomes are consistent with the Born rule, one could statistically average
over all possible outcomes obtained from Eq. (4) to get the very same unconditional density matrix that is predicted by no collapse models.

The mathematical description of the basic postulates entering into Stapp’s model appears to be similar to previous works on foundations of quantum mechanics. However, Stapp surprisingly claimed that his model is robust against environmental decoherence and that mind efforts could exert quantum Zeno effect upon the brain for timescales longer than the decoherence time of the brain $\tau$. Stapp attributed the putative success of his model to objective collapses (Process 3) resulting from the mind probing action. To directly test this claim, we performed Monte Carlo simulations of quantum Zeno effect in the brain in the presence or absence of environmental decoherence. The results from the simulations unambiguously show that if the environmental decoherence happens to be in a basis different from the one chosen by the mind, the quantum Zeno effect is lost for timescales greater than the decoherence time $\tau$. Furthermore, the conditional purification of the brain density matrix due to objective collapses cannot remedy the faulty quantum Zeno effect attempted by the mind.

Because every attempt to computationally simulate the brain could be objected on the grounds that it is too simplistic to capture the full complexity of the real brain, we have shown that the breakdown of quantum Zeno effect is due to fundamental property (concavity) of quantum entropy. Namely, repeated projective measurements cannot decrease the von Neumann entropy $S(\hat{\rho})$ of the unconditional density matrix of the measured system but may only increase it. The increase of the entropy is particularly pronounced in the case when the measured system is subject to environmental decoherence in a basis different from the one of the performed projective measurements. If the two bases are mutually unbiased the entropy reaches the maximum of $\ln n$ within a period of time equal to the decoherence time $\tau$. Only in the special case when both the mind and the environment perform projective measurements upon the brain in the same basis, one could expect to observe quantum Zeno effect for timescales larger than the decoherence time $\tau$. Such scenario, however, makes the mind efforts useless from a functional viewpoint, because the action of the mind becomes redundant with the action of the environment. In addition, there could be no free will if the mind cannot choose the basis in which the projective measurement is performed. Since the efficacy of mind efforts to exert quantum Zeno effect upon the brain depend strongly on the basis in which the environmental decoherence occurs, Stapp’s model does not appear to be physically plausible.

Theorem 4 sets an important constraint on the future development of quantum theories of mind and shows that quantum Zeno effect cannot work if the environmental decoherence affects the whole Hilbert space of the brain. In the presence of decoherence-free subspace, however, one could use quantum Zeno effect to combat some of the negative effects of decoherence on probability decay using unitary operations that first ‘hide’ the initial state in the decoherence-free subspace and at a later time restore the state of interest using inverse unitary operations.
The latter admittedly speculative possibility brings us back to the problem of decoherence albeit in a mitigated form: one needs to explain not why environmental decoherence does not affect the whole brain, but why environmental decoherence does not affect a subspace of the brain. Since such an explanation is likely dependent on the specific architecture of the brain, to find it a tighter interdisciplinary collaboration and further research crossing the boundaries of quantum physics and molecular neuroscience would be needed.

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Appendix A. Algorithm for the Monte Carlo simulations

The main steps in the Monte Carlo simulations of Stapp’s model, that could be used for any $n$-level system and any multiple-well potential, are:

1. Express the Hamiltonian $\hat{H}$ of the system in a matrix form using the position basis.
2. Calculate the (energy) eigenvectors $|E\rangle$ and eigenvalues $E$ of the Hamiltonian $\hat{H}$.
3. Write the projectors in the position basis $\hat{P}_J = |J\rangle\langle J|$ (trivial) and the projectors in energy basis $\hat{P}_E = |E\rangle\langle E|$.
4. Between projective measurements evolve the density matrix $\hat{\rho}$ using the Schrödinger equation (Eq. 2).
5. At times separated by time steps $\xi$ apply the projectors $\hat{P}_E \hat{P}_J$ according to Eq. 3 and discontinuously update the density matrix to one of the pure density matrices $|J\rangle\langle J|$ using weighted Random Choice function with corresponding weight $p_J(\xi)$.
6. Plot the results graphically.

Even though it is straightforward to implement the algorithm to more complex $n$-level systems, there will be several drawbacks. First, the visual comprehension of the plotted results in larger number of dimensions $n$ becomes difficult and the plots would lose their didactic utility in explaining why objective wavefunction collapses cannot help Stapp’s quantum Zeno mind-brain model in the presence of environmental decoherence. Second, for large $n$ one needs to numerically approximate the energy eigenvectors and eigenvalues instead of providing concise analytical expressions as in the case of symmetric 3-level system. The resulting lengthy numerical formulas will distract the reader rather than highlight the conceptual issues at stake. Third, any computationally accessible $n$ would inevitably be considered too small to account for the complexity of the real brain. To avoid that problem, we have proven a quantum informational theorem that allows us to apply the results from

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the simulations of the symmetric 3-level system to an arbitrary $n$-level quantum system.

1. C. V. Borst, *The Mind-Brain Identity Theory* (Macmillan Press, London, 1982).
2. P. S. Churchland, *Neurophilosophy: Toward a Unified Science of the Mind-Brain* (MIT Press, Cambridge, Massachusetts, 1989).
3. P. S. Churchland, *Brain-Wise: Studies in Neurophilosophy* (The MIT Press, Cambridge, Massachusetts, 2002).
4. V. S. Ramachandran, *The Tell-Tale Brain: A Neuroscientist’s Quest for What Makes Us Human* (W. W. Norton & Company, New York, 2010).
5. W. H. Dobelle, S. S. Stensaas, M. G. Mladejovsky, and J. B. Smith, *Annals of Otology, Rhinology, and Laryngology* **82**, 445 (1973). doi:10.1177/000348947308200404
6. W. H. Dobelle and M. G. Mladejovsky, *Journal of Physiology* **243**, 553 (1974).
7. W. H. Dobelle, M. G. Mladejovsky, J. R. Evans, T. S. Roberts, and J. P. Girvin, *Nature* **259**, 111 (1976). doi:10.1038/259111a0
8. W. H. Dobelle, D. O. Quest, J. L. Antunes, T. S. Roberts, and J. P. Girvin, *Neurosurgery* **5**, 521 (1979). doi:10.1097/00006123-197910000-00022
9. W. H. Dobelle, *ASAIO Journal* **46**, 3 (2000). doi:10.1097/00002480-200001000-00002
10. D. M. Wegner, *The Illusion of Conscious Will* (The MIT Press, Cambridge, Massachusetts, 2002).
11. E. Schrödinger, *What Is Life?* (Cambridge University Press, Cambridge, 1967).
12. D. Georgiev and J. F. Glazebrook, *Informatica (Slovenia)* **30**, 221 (2006).
13. D. Georgiev, *Axiomathes* **23**, 683 (2013). doi:10.1007/s10516-012-9204-1
14. F. Beck and J. C. Eccles, *Proceedings of the National Academy of Sciences* **89**, 11357 (1992).
15. M. Jibu and K. Yasue, *Quantum Brain Dynamics and Consciousness: An introduction* (John Benjamins, Amsterdam, Philadelphia, 1995).
16. R. Penrose, * Shadows of the Mind: A Search for the Missing Science of Consciousness* (Vintage, London, 2005).
17. H. P. Stapp, *Foundations of Physics* **31**, 1465 (2001). doi:10.1023/A:1012682413597
18. H. P. Stapp, *Journal of Consciousness Studies* **12**, 43 (2005).
19. J. M. Schwartz, H. P. Stapp, and M. Beaueregard, *Philosophical Transactions of the Royal Society of London B* **360**, 1309 (2005). doi:10.1098/rstb.2004.1598
20. H. P. Stapp, *Mindful Universe: Quantum Mechanics and the Participating Observer* (Springer, Berlin, 2007).
21. H. P. Stapp, *Mind & Matter* **5**, 83 (2007).
22. D. Georgiev, *NeuroQuantology* **10**, 374 (2012). doi:10.14704/nq.2012.10.3.552
23. H. P. Stapp, *NeuroQuantology* **10**, 601 (2012). doi:10.14704/nq.2012.10.4.619
24. J. von Neumann, *Mathematical Foundations of Quantum Mechanics* (Princeton University Press, Princeton, 1955).
25. G. Lüders, *Annalen der Physik* **443**, 322 (1951). doi:10.1002/andp.19504430321
26. B. E. Y. Svensson, *Quanta* **2**, 18 (2013). doi:10.12743/quanta.v2i1.12
27. E. R. Kandel, J. H. Schwartz, T. M. Jessell, S. A. Siegelbaum, and A. J. Hudspeth, *Principles of Neural Science* (McGraw-Hill Professional, New York, 2012).
28. F. Bezanilla, *Physiological Reviews* **80**, 555 (2000).
29. F. Bezanilla, *Trends in Biochemical Sciences* **30**, 166 (2005). doi:10.1016/j.tibs.2005.02.006
30. D. J. S. Elliott, E. J. Neale, Q. Aziz, J. P. Dunham, T. S. Munsey, M. Hunter, and A. Sivaprasadaraao, *EMBO Journal* **23**, 4717 (2004). doi:10.1038/sj.emboj.7600484
31. V. K. Gribkoff and L. K. Kaczmarek, *Structure, Function and Modulation of Neuronal Voltage-Gated Ion Channels* (Wiley, Hoboken, New Jersey, 2009).

32. V. Yarov-Yarovoy et al., *Proceedings of the National Academy of Sciences* **109**, E93 (2012). doi:10.1073/pnas.1118434109

33. M. O. Jensen, V. Jogini, D. W. Borhani, A. E. Leffler, R. O. Dror, and D. E. Shaw, *Science* **336**, 229 (2012). doi:10.1038/sci.201118434109

34. Q. Li et al., *Nature Structural & Molecular Biology* **21**, 244 (2014). doi:10.1038/nsmb.2768

35. B. Sakmann and E. Neher, *Single-Channel Recording* (Springer, New York, 1995).

36. F. A. C. Azevedo, L. R. B. Carvalho, L. T. Grinberg, J. M. Farfel, R. E. L. Ferretti, R. E. P. Leite, W. J. Filho, R. Lent, and S. Herculano-Houzel, *Journal of Comparative Neurology* **513**, 532 (2009). doi:10.1002/cne.21974

37. A. S. Holevo, *Problems of Information Transmission* **9**, 177 (1973).

38. M. J. Gagen, H. M. Wiseman, and G. J. Milburn, *Physical Review A* **48**, 132 (1993). doi:10.1103/PhysRevA.48.132

39. T. P. Altenmüller and A. Schenzle, *Physical Review A* **49**, 2016 (1994). doi:10.1103/PhysRevA.49.2016

40. J. H. Cole, A. D. Greentree, L. C. L. Hollenberg, and S. Das Sarma, *Physical Review B* **77**, 235418 (2008). doi:10.1103/PhysRevB.77.235418

41. A. Venugopalan, *Pramana* **51**, 625 (1998). doi:10.1007/BF02827454

42. E. H. Lieb, *Bulletin of the American Mathematical Society* **81**, 1 (1975).

43. E. Joos, *Reviews of Modern Physics* **50**, 221 (1978). doi:10.1103/RevModPhys.50.221

44. I. Bengtsson and K. Zyczkowski, *Geometry of Quantum States. An Introduction to Quantum Entanglement* (Cambridge University Press, Cambridge, 2006).

45. M. Ohya and N. Watanabe, *Entropy* **12**, 1194 (2010). doi:10.3390/e12051194

46. M. Delbrück and G. Molière, *Statistische Quantenmechanik und Thermodynamik* (Verlag der Akademie der Wissenschaften, Berlin, 1936).

47. E. H. Lieb, *Zeitschrift für Physik B* **59**, 223 (1985). doi:10.1007/BF01725541

48. E. Joos, *Physics & Philosophy* **2**, 10 (2007).

49. E. Schrödinger, *Nature* **23**, 807 (1935). doi:10.1007/bf01491891

50. H. Everett III, *The Everett Interpretation of Quantum Mechanics: Collected Works 1955-1980 with Commentary, J. A. Barrett and P. Byrne (editors)* (Princeton University Press, Princeton, 2012).

51. H.-D. Zeh, *Foundations of Physics* **44**, 557 (2014). doi:10.1007/s10701-013-9770-0

52. G. C. Ghirardi, A. Rimini, T. Weber, *Physical Review D* **34**, 470 (1986). doi:10.1103/PhysRevD.34.470

53. D. A. Lidar, I.-L. Chuang, and K. B. Whaley, *Physical Review Letters* **81**, 2594 (1998). doi:10.1103/PhysRevLett.81.2594

54. Y.-S. Kim, J.-C. Lee, O. Kwon, and Y.-H. Kim, *Nature Physics* **8**, 117 (2012). doi:10.1038/nphys2178