Magnetism in strongly interacting one-dimensional quantum mixtures

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We consider two species of bosons in one dimension near the Tonks-Girardeau limit of infinite interactions. For the case of equal masses and equal intraspecies interactions, the system can be mapped to a $S = 1/2$ XXZ Heisenberg spin chain, thus allowing one to access different magnetic phases. Using a powerful ansatz developed for the two-component Fermi system, we elucidate the evolution from few to many particles for the experimentally relevant case of an external harmonic confinement. In the few-body limit, we already find clear evidence of both ferromagnetic and antiferromagnetic spin correlations as the ratio of intraspecies and interspecies interactions is varied. Furthermore, we observe the rapid emergence of symmetry-broken magnetic ground states as the particle number is increased. We therefore demonstrate that systems containing only a few bosons are an ideal setting in which to realize the highly sought-after itinerant ferromagnetic phase.

Quantum magnetism is ubiquitous in nature and plays a central role in important phenomena such as high-temperature superconductivity [1]. Furthermore, it underpins the technological advances in data storage [2], and it promises a new generation of spintronic devices, where the spin of the electron rather than its charge is used to transfer information. However, despite its ubiquity, magnetic phenomena are often difficult to characterise and treat theoretically; for instance, itinerant ferromagnetism of delocalized fermions requires strong interactions and is thus not completely understood [3].

One can gain insight into magnetic phases by considering cleaner, more idealised versions of the phenomena. In particular, an important question is whether or not ferromagnetism can exist without an underlying lattice [4]. This possibility was investigated experimentally in atomic Fermi gases [5, 6], but the system proved to be unstable towards fermion pairing rather than magnetism. More recently, it has been proposed that itinerant ferromagnetism can be realised in a strongly interacting one-dimensional (1D) Fermi gas [7]. However, in this case, one cannot access the ferromagnetic phase without explicitly breaking the spin symmetry with an external field [8], and this potentially complicates the observation of ferromagnetism.

In this Letter, we show that both itinerant ferromagnetism and antiferromagnetism can be investigated with a 1D two-component mixture of bosons [9]. For the case of equal masses and equal intraspecies interactions ($g_{\uparrow\uparrow} = g_{\downarrow\downarrow}$), the Bose-Bose mixture may be regarded as a pseudo-spin $S = 1/2$ system, and it can be mapped to a 1D spin chain in the limit of strong interactions [10, 11]. In particular, for infinite intraspecies interactions, the system is formally equivalent to a 1D Fermi gas. However, in contrast to the Fermi case, one can break the $SU(2)$ symmetry and access different magnetic phases by varying the ratio of intraspecies and interspecies interactions (see Fig. 1). Moreover, the ferromagnetic state should be stable, unlike in higher dimensions [12–14].

Experimentally, 1D quantum gases have been successfully realized with strongly interacting bosons [15, 16], two species of fermions [17–20], and, more recently, fermions with $SU(n)$ symmetry, where the number of spin components can be tuned from $n = 2$ to $6$ [21]. All these 1D experiments feature an underlying harmonic trapping potential, which has enabled the study of the evolution from few to many particles [18, 20, 22]. However, it is theoretically challenging to treat 1D interacting particles in a harmonic trap, since the problem cannot be solved exactly in general [23]. Only a small number of trapped particles can be treated exactly numeri-
The Hamiltonian is \( H \) for a pseudo spin with pseudo spin \( \psi \) ranged, and parametrized respectively by \( \phi \), intraspecies interactions are assumed to be short-ranged, and parametrized respectively by \( g_{\uparrow \downarrow} \) and \( g_{\nu \nu} \).

The Hamiltonian is

\[
H = \sum_{\nu} \int dx \left[ \frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2} + \frac{1}{2} m \omega^2 x^2 \right] \psi_\nu(x) + \frac{1}{2} \sum_{\nu, \nu'} g_{\nu \nu'} \int dx \psi_\nu^\dagger(x) \psi_\nu^\dagger(x) \psi_{\nu'}(x) \psi_{\nu'}(x),
\]

where \( \psi_\nu^\dagger \) and \( \psi_\nu \), respectively, create and annihilate a boson with pseudo spin \( \nu \). In the following we use harmonic oscillator units, where \( \hbar = m = \omega = 1 \). Moreover, we consider the symmetric case where the two intraspecies interactions are of equal strength and the number of particles in each spin state is the same, i.e., \( g_{\uparrow \downarrow} = g_{\downarrow \uparrow} \) and \( N_\uparrow = N_\downarrow \equiv N/2 \).

In the Tonks-Girardeau limit \( |g_{\nu \nu}| \to \infty \), the particles become impenetrable and retain their particular ordering. Thus, even in the absence of an underlying lattice, one can consider the system as a discrete chain of length \( N \), where the site index defines the position along the ordered chain. Perturbing away from this limit allows nearest neighbors to exchange position and one can describe the system using a spin-chain model [11, 28, 29, 33–35]. To make this connection evident, it is convenient to replace the inter- and intraspecies interactions \( g_{\uparrow \downarrow} \) and \( g_{\uparrow \downarrow} = g_{\downarrow \uparrow} \) by effective parameters \( g > 0 \) and \( \phi \), such that the entire phase diagram is traversed by changing the “angle” \( \phi \), as depicted in Fig. 1. These are related to the original interactions by \( \sin \phi = g/g_{\uparrow \downarrow} \) and \( \cos \phi = g/g_{\uparrow \downarrow} - 2g/g_{\uparrow \downarrow} \). To linear order in \( 1/g \), the original Hamiltonian may then be written as [35, 36]:

\[
H = \varepsilon_0 + (\Delta \varepsilon + \mathcal{H})/g.
\]

Here, \( \varepsilon_0 \) is the energy of \( N \) identical fermions, and \( \mathcal{H} \) is an effective Heisenberg XXZ Hamiltonian:

\[
\mathcal{H} = \sum_{i=1}^{N-1} \eta_i \left[ J^{\perp} \left( \sigma_i^x \sigma_{i+1}^x + \sigma_i^y \sigma_{i+1}^y \right) + J^z \sigma_i^z \sigma_{i+1}^z \right],
\]

where \( J^{\perp} \equiv \sin \phi, J^z \equiv \cos \phi, \) and \( \sigma_i \) are the Pauli spin matrices at position \( i \). The constant \( \Delta \varepsilon = (J^z - 2J^{\perp}) \sum_{i=1}^{N-1} \eta_i \) ensures that only identical (distinguishable) bosons interact via \( g_{\uparrow \downarrow} \). Note that, apart from parity, the properties of the eigenstates of \( \mathcal{H} \) are invariant under the transformation \( \phi \to -\phi \) [37].

The energy cost of exchanging opposite spins at positions \( i \) and \( i + 1 \) depends on the underlying external confinement, an effect encapsulated in the coefficients \( \eta_i \) of Eq. (3). These must in general be calculated numerically, but a key simplification is that they are independent of the interactions and thus one need only determine them for a single \( \phi \). In particular, for infinite intraspecies interaction \( (\phi = \pi/4, 5\pi/4) \), the system is equivalent to a two-component Fermi gas, where the \( \eta_i \) have already been investigated for \( N \leq 9 \) [28, 29, 33]. We have recently shown for the harmonically trapped Fermi gas that \( \eta_i \) is well approximated by the analytic expression \( \eta(N - i) \eta(N) \) [33]. Referring to the Supplemental Material [37], we find that this approach also yields near exact wavefunctions for any \( \phi \). We will, thus, use this ansatz to consider larger numbers of particles.

Phase diagram. — To gain insight into the behavior of the trapped boson system, it is instructive to first consider the uniform XXZ model \((g = 1, \) where the magnetic phases are well characterized [38, 39]. In the thermodynamic limit, the local density approximation (LDA) shows that the topology of the uniform phase diagram is the same as that of the trapped system. As displayed in Fig. 1, we expect to have fully saturated ferromagnetism (FM) of the Ising type when \( J^z < -|J^{\perp}| \), i.e., when \( 3\pi/4 < \phi < 5\pi/4 \) [38]. Indeed, the first-order transition at \( J^z = -J^{\perp} < 0 \) corresponds to \( g_{\uparrow \downarrow} = g_{\nu \nu} > 0 \), the criterion for phase separation in a weakly interacting two-component Bose gas [39]. However, as we discuss below, the interface between \( \uparrow \) and \( \downarrow \) bosons is sharper in the strong-coupling limit, being of order the interparticle spacing. The other first-order FM transition, \( J^z = J^{\perp} < 0 \), has no weak-coupling analogue since it occurs on the metastable “upper branch” for attractive interactions (or “super-Tonks” regime).

When \( |J^{\perp}| = |J^z| \), \( \mathcal{H} \) becomes equivalent to the XXX Heisenberg model that can be realized with fermions, e.g., \( \phi = 3\pi/4 \) can be mapped to the upper branch of the attractive Fermi gas [28, 35], which is the regime where itinerant ferromagnetism exists [7]. Once \( 1/g_{\nu \nu} < 0 \), we will also have Ising antiferromagnetism (AFM) for \( -\pi/4 < \phi < \pi/4 \), but the phase transition to this state is of the Kosterlitz-Thouless type in the uniform case [38].

Ground-state energy. — We now turn to the question of how the few-body system approaches the many-body
Within the entire region $3\pi/4 \leq \phi \leq 5\pi/4$, we find that the ground-state energy in the limit $N \rightarrow \infty$ corresponds to that of spin-polarized bosons, where it follows from Eq. (5) that $C_{\text{gs}} = \frac{128^{N/2}}{45\pi^2} \cos(3\pi/4)$. For fixed spin populations $N_\uparrow = N_\downarrow$, this result physically corresponds to phase separation of $\uparrow$ and $\downarrow$ particles, where we can neglect the energy of the interface in the thermodynamic limit since it scales like $1/N$ compared to the total energy $E_0$. The formation of spin-polarized domains is a signature of ferromagnetism \cite{45}, and the emergence of kinks in the energy at $\phi = 3\pi/4$ and $5\pi/4$ clearly indicates a first-order transition to the FM phase (Fig. 2).

While there are no sharp features in the energy at the AFM phase boundaries, the fact that $C_{\text{gs}} = 0$ at $\phi = 0$ is a direct consequence of classical Néel ordering, where identical spins are staggered and thus experience no intraspecies interaction.

**Magnetic phases.** — A clear signature of magnetic order can be extracted from the degeneracy of the ground state. A two-fold degenerate ground state is a necessary condition for Ising magnetic order, and thus its emergence in the few-body system is a precursor for symmetry-broken magnetic states as $N \rightarrow \infty$. As shown in Fig. 3(a), the two lowest energy states become rapidly degenerate with increasing $N$ in the FM regime. In the few-body system, these two states have opposite parity, but as they become degenerate they can be combined to form states that break spin symmetry and parity. Physically, this degeneracy emerges once the tunnelling between different symmetry broken states is suppressed for large $N$. A similar situation occurs in the AFM region, but the degeneracy manifests more slowly. Therefore, very few particles are necessary to observe strong magnetic correlations and a large breaking of the $Z_2$ symmetry in the FM region, while larger systems are needed to observe similar effects in the AFM region.

We can further characterize the different phases using the short-range correlation function $\chi_i = \langle \sigma^z_i \sigma_{i+1}^z \rangle$ in the ground state. Referring to the insets of Fig. 3(b), we see that $\chi_i$ is always close to $-1$ around $\phi = 0$, signifying AFM correlations. Likewise, for the FM phase, we obtain $\chi_i \approx 1$ throughout the spin chain, excluding the interface between the spin-polarized domains. For strong interactions, this domain wall is sharp, decaying exponentially with distance even in the few-body system.

To obtain a more global description of magnetic correlations, we take the trap-averaged function $\tilde{\chi}(\phi) = \frac{1 + \sum_i \chi_i}{N-2}$, normalized such that $\tilde{\chi}(0) = -1$ and $\tilde{\chi}(\pi) = 1$. At the FM boundaries, the ground state is a Heisenberg ferromagnet with total spin $S = N/2$ oriented in the $x$-$y$ plane. Here, the wavefunction is obtained by applying the total spin lowering operator $N/2$ times to the spin-polarized system $|\uparrow \uparrow \ldots \uparrow \rangle$. Thus, by symmetry, $\tilde{\chi} = \chi_i = -1$. With increasing $N$, we find that the averaged short-range correlations approach a rectangular
function in the FM region, where \(1 - \tilde{\chi} \simeq (\pi - \phi)^2/N\) around \(\phi = \pi\) for \(N \gg 1\). This is consistent with the appearance of classical Ising FM along the \(z\) direction.

By contrast, we see that quantum fluctuations reduce the Ising AFM correlations around \(\phi = 0\), with \(1 + \tilde{\chi} \simeq \phi^2\) in the limit \(N \to \infty\). Furthermore, \(\tilde{\chi}\) evolves smoothly with \(\phi\) from the AFM phase to the disordered “spin liquid” occupying the region \(\pi/4 < |\phi| < 3\pi/4\). The disordered liquid also features \(\tilde{\chi} < 0\) since its spin correlations resemble those in a Fermi system. Indeed, for \(\phi = \pm \pi/2\), the spin model can be formally mapped onto free fermions using the Jordan-Wigner transformation [39]. For the uniform case, the free-fermion problem can be solved analytically and we find \(\tilde{\chi} = -4/\pi^2\), which compares well with the large \(N\) limit of the trapped system [see Fig. 3(b)].

Spin imbalance. — When \(N_{\uparrow} \neq N_{\downarrow}\), magnetic correlations strongly influence the behavior of the trapped system. In the limit of a single impurity \(N_{\downarrow} = 1\), we find that the type of correlations determines the position of the \(\downarrow\) impurity: in the AFM regime, the impurity sits at the trap center [33], while FM correlations confine it to the edge. More generally, we find in the few-body system that the fully spin-polarized state always has the lowest energy within the FM regime, as expected, while the ground state for all other \(\phi\) is unpolarized. In practice, one can observe this by adding a small spin-interconversion coupling [36]. To quantify this behavior, we determine the energy cost of flipping a spin in the ground state [Fig. 3(c)]. For the fully polarized FM ground state, we see that the “spin gap” \(\Delta E\) always vanishes at the FM phase boundaries and rapidly converges to a universal curve with increasing \(N\) [37]. This further supports the idea that FM can be realized in a few-boson system. On the other hand, for the unpolarized system we find that the AFM spin gap remains finite even in the disordered phase, illustrating the different nature of the spin excitations. In both cases, we find that the behavior in the trap is qualitatively different from that in the uniform system, since the flipped spins predominantly reside at the trap edges and, hence, the spin gaps may not be obtained via the LDA.

Concluding remarks. — The physics described in this Letter may be realized in a two-component Bose gas close to overlapping 1D inter- and intraspecies resonances, which obviously necessitates some fine tuning. However, we emphasize that the 1D Bose gas is substantially more tunable than its 3D counterpart. Firstly, the magnetic field need not be set exactly to any of the 3D resonances, since part of the fine tuning can be done by varying the strength of the transverse confinement [46], and this may even be done in a species-selective manner. Secondly, we propose to employ any two hyperfine spin states, as spin-changing collisions will be suppressed in the strongly interacting “fermionized” limit. Thus we anticipate that the strongly-coupled two-component Bose gas could be investigated in current experimental setups.

We have argued that clear signatures of a first-order quantum phase transition to a ferromagnetic state – the analogue of itinerant electron Stoner ferromagnetism – appear already in the few-body limit. Furthermore, we emphasize that the few-body scenario is likely to be ideal for observing magnetic phases since the condition of staying in the strongly interacting regime, \(g/\sqrt{N} \gg 1\), becomes increasingly difficult to satisfy as \(N\) increases.
Smaller values of $N$ also enlarge the relative energy spacing between eigenstates, thus making it easier to tune interactions adiabatically and access stable magnetic ground states that are robust with respect to thermal fluctuations. Hence, the harmonically trapped and strongly interacting few-body Bose gas appears ideally suited for the realisation of quantum magnetic phases.

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Supplemental Material: Magnetism in strongly interacting 1D quantum mixtures

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SYMMETRIES OF THE HAMILTONIAN

As discussed in the manuscript, when we are in the Tonks-Girardeau limit \(|g_{\nu\nu'}| \to \infty\), the system may be described in terms of an effective XXZ Hamiltonian,

\[ \mathcal{H} = \sum_{i=1}^{N-1} \eta_i \left[ J^+ \left( \sigma_i^x \sigma_{i+1}^x + \sigma_i^y \sigma_{i+1}^y \right) + J^z \sigma_i^z \sigma_{i+1}^z \right] , \tag{S1} \]

where the parameters are defined as in the main text. For even particle number \(N\) the coefficients are exact. For odd particle number \(N\), populations are balanced \((N_\uparrow = N_\downarrow)\), one can show that this spin-chain Hamiltonian satisfies \(\mathcal{H}(\phi) = U \mathcal{H}(\phi) U^\dagger\), with the unitary operator \(U = \prod_{i=1}^{N/2} \sigma_{2i}^z\). Thus, with the exception of parity, we expect the properties of the eigenstates to be invariant under \(\phi \mapsto -\phi\). The parity operator \(P = \prod_{i=1}^{N/2} \sigma_{2i}^x\), so that for even \(N/2\) we have \([U, P] = 0\) and the ground state always has the same parity. For odd \(N/2\), instead \([U, P] = 0\), and consequently the ground state changes parity at \(\phi = 0\) and \(\pi\). On the other hand, the total energy is not invariant under the unitary transformation due to the presence of the constant \(\Delta \varepsilon(\phi)\) in the strong-coupling Hamiltonian, Eq. (2).

PARAMETERS OF THE MODEL

The parameters \(\eta_i\) which yield the values of the spin-exchange coefficients of the spin-chain Hamiltonian (S1) were computed in Refs. [28, 29, 33]. For completeness, we list these in Table S1 up to \(N = 9\).

| \(N\) | 2 \(\sqrt{1/2\pi}\) | 3 \(27/(16\sqrt{2\pi})\) | 4 \(1.08303\) | 5 \(1.25109\) | 6 \(1.40370\) | 7 \(1.54440\) | 8 \(1.67556\) | 9 \(1.70016\) |
|---|---|---|---|---|---|---|---|---|
| \(\eta_1\) | \(\sqrt{1/2\pi}\) | \(0.893823\) | \(1.17325\) | \(1.58860\) | \(2.17856\) | \(3.17251\) | \(3.60715\) |
| \(\eta_2\) | \(27/(16\sqrt{2\pi})\) | \(1.08303\) | \(1.58860\) | \(2.17856\) | \(3.17251\) | |
| \(\eta_3\) | 2 | 4 | 5 | 6 | 7 | 8 | 9 |
| \(\eta_4\) | 3 | 27/(16\sqrt{2\pi}) | 1.08303 | 1.25109 | 1.40370 | 1.54440 | 1.67556 |
| \(\eta_5\) | 4 | \(27/(16\sqrt{2\pi})\) | \(1.08303\) | \(1.25109\) | \(1.40370\) | \(1.54440\) | |

TABLE S1. Distinct parameters of the spin-chain Hamiltonian. The full set can be obtained using \(\eta_{N-1} = \eta_i\). For \(N \leq 3\) the coefficients are exact. For \(N = 4\) the coefficients may be obtained analytically [33], however the resulting expressions are cumbersome and are not repeated here.

In the manuscript, instead of using the numerically exact coefficients in Table S1 we use an approximate model derived in Ref. [33], where \(\eta_i \equiv i(N-i)\eta(N)\). Note that \(\eta(N)\) only sets the absolute value of the energy and does not affect the eigenstates or the energy ratios for each \(N\). We fix \(\eta(N)\) as following: At \(\phi = \pi\), corresponding to infinite interspecies interactions, we know the exact ground state which is fully phase separated. Thus, we may evaluate the ground state energy to find

\[ E_0 = \varepsilon_0 - \frac{4}{g_{\nu\nu'}} \left[ -\eta_{N/2} + \sum_{i=1}^{N-1} \eta_i \right] \tag{S2} \]

Requiring that we reproduce the exact ground state energy at \(\phi = \pi\) thus results in the simple expression for \(\eta(N)\):

\[ \eta(N) = \frac{\sum_{i=1}^{N-1} \eta_i - \eta_{N/2}}{N^3/6 - N^2/4 - N/6} \tag{S3} \]

with \(\eta_i\) those of Table S1. This procedure yields \(\eta(2) = \sqrt{1/2\pi}\), \(\eta(4) = 0.297941\), \(\eta(6) = 0.246318\), and \(\eta(8) = 0.214440\). For larger \(N\) the exact coefficients are not known, and we instead extrapolate \(\eta\) towards the thermodynamic
FIG. S1. $\bar{\eta}(N)$ as a function of $1/N$. For $N = 2, 4, 6, 8$ we use the exact result, indicated by dots, while for larger $N$ we approximate it by the dashed line, as outlined in the text [see Eq. (S6)]. The dotted line corresponds to $\bar{\eta}(N)$ calculated from the large $N$ limit, Eq. (S5).

limit $N \to \infty$. In this limit, the domain wall may be ignored, and the energy to leading non-trivial order is that of a Lieb-Liniger gas, Eq. (5) in the main text:

$$E_{\text{LL}} = \frac{N^2}{2} - \frac{128\sqrt{2}}{45\pi^2} \frac{N^{5/2}}{g_{\nu\nu}}. \quad (S4)$$

This result was obtained in Ref. [42] by applying the local density approximation to the uniform space Bethe ansatz result. Hence, in the limit $N \to \infty$, the function $\bar{\eta}(N)$ is, to leading order,

$$\bar{\eta}_{\text{LL}}(N) = \frac{64\sqrt{2}}{15\pi^2} \frac{1}{\sqrt{N}}. \quad (S5)$$

To go to finite $N$, we choose the simplest possible extrapolation, requiring that the $O(N^{3/2})$ correction to Eq. (S5) produces the correct $\bar{\eta}(8)$. The result for $N \geq 8$ is

$$\bar{\eta}(N) \approx \bar{\eta}_{\text{LL}}(N) \left(1 - 0.063343/N\right), \quad (S6)$$

which is shown in Fig. S1.

ACCURACY OF THE STRONG-COUPLED ANSATZ

The high accuracy of the strong-coupling ansatz developed in Ref. [33] and used here to determine the spin-exchange coefficients beyond the few-body limit may be confirmed by computing the overlap between the eigenstates $|\tilde{\psi}\rangle$ of the approximate model and the eigenstates $|\psi\rangle$ obtained in the exact solution of the problem. This we have computed for balanced systems containing at most 8 particles, analogously to what was done in our earlier work [33]. As may be seen in Fig. S2, the wavefunction overlaps for both the ground and first excited state (in the upper branch) remain extremely close to 1 for all ratios of intra- and interspecies interaction strengths. The largest deviations (of at most a few parts in $10^4$) are found near the boundaries of the FM region.

SPIN GAPS

The stability of the system in the presence of a small inter-conversion coupling may be analyzed in terms of the “spin gap”, i.e., the energy difference between the ground state of a system of $N$ particles, and the configuration in which one of the spins is flipped. In the FM phase ($3\pi/4 < \phi < 5\pi/4$), the ground state corresponds to a fully polarized gas of $N$ identical bosons, and we thus define the FM spin gap to be $\Delta E_{\text{FM}} = E_0(N - 1, 1) - E_0(N, 0)$, where $E_0(N_\uparrow, N_\downarrow)$ is the ground state energy of a gas composed of $N_\uparrow$ spin-up particles and $N_\downarrow$ spin-down ones. For
FIG. S2. Wavefunction overlaps between exact and approximate ground (solid) and first excited (dashed) states for \(N_t = N_\downarrow = 2, 3,\) and 4 particles.

all other values of \(\phi,\) including those in the AFM phase, the ground state corresponds to a balanced system with \(N_t = N_\downarrow.\) Thus the spin gap in this region is instead defined as \(\Delta E_{\text{AFM}} = E_0(N/2 - 1, N/2 + 1) - E_0(N/2, N/2).\)

At the classical Ising points \(\phi = 0, \pi,\) we know the ground states exactly and thus we also know the spin gap, which at both points is \(\Delta E = 2\bar{\eta}(N)(N - 1)/g.\) This follows from the observation that in both cases the energy cost associated with flipping a spin is smallest at the trap edge. We then apply perturbation theory to estimate the behaviour of the spin gap around these points. In the AFM regime, we have for \(|\phi| \ll 1,\)

\[
\Delta E_{\text{AFM}} \approx 2 \frac{\bar{\eta}(N)}{g} [N - 1 - (N + 3)\phi^2] \tag{S7}
\]

\[
\overset{N\to\infty}{\to} \frac{2N}{g} \bar{\eta}_{\text{LL}}(N)(1 - \phi^2), \tag{S8}
\]

where in Eq. (S7) we have kept the leading order correction in \(N.\) We display the analytic result, Eq. (S8), in Fig. 3(c) to demonstrate the manner in which the AFM spin gap approaches the thermodynamic limit.

We may perform a similar perturbative expansion around the FM Ising point, but in this case we find that we can describe the whole FM region for large \(N\) with the following analytic form:

\[
\Delta E_{\text{FM}} \approx 2(N - 1) \frac{\bar{\eta}_{\text{LL}}(N)}{g} \sqrt{\cos^2 \phi - \sin^2 \phi}, \tag{S9}
\]

as shown in Fig. 3(c).