Cosmological Limits on Slightly Skew Stresses

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Abstract

We present an analysis of the cosmological evolution of matter sources with small anisotropic pressures. This includes electric and magnetic fields, collisionless relativistic particles, gravitons, antisymmetric axion fields in low-energy string cosmologies, spatial curvature anisotropies, and stresses arising from simple topological defects. We calculate their evolution during the radiation and dust eras of an almost isotropic universe. In many interesting cases the evolution displays a special critical behaviour created by the non-linear evolution of the pressure and expansion anisotropies. The isotropy of the microwave background is used to place strong limits of order $\Omega_{a0} \leq 5 \times 10^{-6} \Delta (1 + z_{rec})^{-\Delta}$ on the possible contribution of these matter sources to the total density of the universe, where $1 \leq \Delta \leq 3$ characterises the anisotropic stress. The present abundance of an anisotropic stress which becomes non-relativistic at a characteristic low-energy scale is also calculated. We explain why the limits obtained from primordial nucleosynthesis are generally weaker than those imposed by the microwave background isotropy. The effect of inflation on these stresses is also calculated.
1 Introduction

One of the key problems of cosmology is deciding which matter fields are present in the Universe. The most notable uncertainties govern the presence or absence of influential matter fields which might contribute to the total density of the Universe today, and thereby offer a resolution of the 'dark matter' problem or provide an effective cosmological constant or vacuum stress which alters the significance of the evidence governing the process of large-scale structure formation. Other matter fields, like magnetic fields or topological defects, can have a profound influence upon the evolution and properties of galaxies. In the early universe the identity of matter fields is even more uncertain. String theories are populated by large numbers of high-mass fields but they will not survive to influence the late-time evolution of the Universe significantly if inflation takes place at high energies. Inflation, of course, involves its own material uncertainties: it requires the existence of a weakly-coupled scalar field which evolves slowly during the early universe. In all these cases the principal problems are ones of fundamental physics - do the postulated matter fields exist or not?, what are the strengths of their couplings to themselves and to other fields? Relic densities residing in isotropic or scalar stresses are the simplest to determine. They can be calculated from a semi-analytic solution of the Boltzmann equation if their interaction strengths, lifetimes and masses are known. Limits on their masses, lifetimes, and interaction cross-sections are obtained by requiring that they do not contribute significantly more than the closure density to the Universe today, or produce too many decay photons in a particular portion of the electromagnetic spectrum.

In this paper we are going to consider the behaviour of anisotropic stresses, of which electric and magnetic fields are the simplest examples. We shall not assume that inflation has taken place, although we will also calculate the effects of inflation on their present abundances. The isotropy of the microwave background requires any such anisotropies to be extremely small but we shall find that their evolution is subtle because of the coupling between pressure anisotropies and the expansion dynamics during the radiation era. Anisotropic stresses have many possible sources — besides cosmological magnetic and electric fields, examples are provided by populations of collisionless particles like gravitons, photons, or relativistic neutrinos; long-wavelength gravitational waves; Yang-Mills fields in low-energy string theory; and topological defects like
cosmic strings and domain walls [15]-[17]. Our experience with high-energy physics theories also alerts us to the possibility that a (more) final theory of high-energy physics will contain many other matter fields, some of which may well exert anisotropic stresses in the universe.

Cosmological observations of the isotropy of the microwave background tell us that the Universe is expanding almost isotropically. Therefore we can expect to employ the high isotropy of the microwave background to constrain the possible density of any relic anisotropic matter fields in the Universe. We shall see that a very general analysis is possible, largely independent of the specific identity of the physical field in question, which leads to strong limits on the existence of homogeneous matter fields with anisotropic pressures, irrespective of initial conditions in many cases. This analysis exhibits a number of interesting features of the general behaviour of small deviations from isotropy in expanding universes in the presence and absence of a period of inflation.

In section 2 we set up the cosmological evolution equations for a universe containing a perfect fluid and a general form of anisotropic stress. Under the assumption that the anisotropy is small these equations can be solved. They reveal two distinctive forms of evolution in which the abundance of the anisotropic fluid is determined linearly or non-linearly, respectively. The evolution can be parametrised in terms of a single parameter which characterises the pressure anisotropy of the anisotropic fluid. In section 3 we give a number of specific examples of anisotropic fluids and explain how this analysis also allows us to understand the evolution of the most general anisotropic universes with 3-curvature anisotropy with respect to the simplest examples which possess isotropic 3-curvature. In section 4 we derive limits on the abundances of anisotropic stresses in the Universe today by using the COBE four-year anisotropy data [18]. We also consider the case of an anisotropic stress which becomes non-relativistic after the end of the radiation era (analogous to a light neutrino species becoming effectively massive). In section 5 we consider the effect of a period of de Sitter or power-law inflation on the abundances of anisotropic stresses surviving the early universe, and in section 6 we compare the constraints we have obtained on anisotropic stresses from COBE with those that can be derived by considering their effects on the primordial nucleosynthesis of helium-4. The effect of the pressure anisotropy is to slow the decay of shear anisotropies to such an extent (logarithmic decay in time during the radiation era) that in the general case the COBE limits are far stronger than those provided by nucleosynthesis.
2 The Cosmological Evolution of Skew Stresses

Consider an anisotropic universe with metric [19]

\[ ds^2 = dt^2 - a^2(t)dx^2 - b^2(t)dy^2 - c^2(t)dz^2. \]  

(1)

If we define the expansion rates by

\[ \alpha = \frac{\dot{a}}{a}; \beta = \frac{\dot{b}}{b}; \theta = \frac{\dot{c}}{c}, \]

where overdot denotes \( d/dt \), then the Einstein equations are

\[ \dot{\alpha} + \alpha^2 + \alpha(\beta + \theta) = -8\pi G(T_{11}^1 - \frac{1}{2}T), \]

(2)

\[ \dot{\beta} + \beta^2 + \beta(\alpha + \theta) = -8\pi G(T_{22}^2 - \frac{1}{2}T), \]

(3)

\[ \dot{\theta} + \theta^2 + \theta(\alpha + \beta) = -8\pi G(T_{33}^3 - \frac{1}{2}T), \]

(4)

\[ \frac{\ddot{a}}{a} + \frac{\ddot{b}}{b} + \frac{\ddot{c}}{c} = -8\pi G(T_{00}^0 - \frac{1}{2}T), \]

(5)

where \( T_{ab} \) is the energy-momentum tensor and henceforth we set \( 8\pi G = 1 \).

We define the mean Hubble expansion rate, \( H \), by

\[ H = \frac{\alpha + \beta + \theta}{3}, \]

(6)

and the two relative shear anisotropy parameters by

\[ R = \frac{\alpha - \beta}{H} \quad \text{and} \quad S = \frac{\alpha - \theta}{H}. \]

(7)

When \( R = S = 0 \) the universe will be the isotropic flat Friedmann universe.

We are interested in studying the behaviour of a non-interacting combination of a perfect fluid and a fluid with an anisotropic pressure distribution. Thus the energy-momentum tensor for the universe is a sum of two parts:

\[ T_{ab}^u \equiv t_{ab}^u + s_{ab}^u, \]

(8)

where \( t_{ab}^u \) is the energy-momentum tensor of the perfect fluid, so
$t^{a}_{b} = \text{diag}(\rho_{*}, -p_{*}, -p_{*}, -p_{*})$ \hspace{1cm} (9)

with equation of state

\[ p_{*} = (\gamma - 1)\rho_{*}, \] \hspace{1cm} (10)

and $s^{a}_{b}$ is the energy-momentum tensor of a fluid with density $\rho$ and principal pressures $p_{1}, p_{2},$ and $p_{3}$, so

\[ s^{a}_{b} = \text{diag}(\rho, -p_{1}, -p_{2}, -p_{3}). \] \hspace{1cm} (11)

We shall assume that the principal pressures of the anisotropic fluid are each proportional to its density, $\rho$, so that we have

\[ (p_{1}, p_{2}, p_{3}) \equiv (\lambda \rho, \nu \rho, \mu \rho), \] \hspace{1cm} (12)

where $\lambda, \nu,$ and $\mu$ are constants. It will be useful to define the sum of the pressure parameters by

\[ \Delta \equiv \lambda + \mu + \nu. \] \hspace{1cm} (13)

In the special case of an axisymmetric metric, (1), two of these three constants would be equal. The ratio of the densities of the two fluids is defined by

\[ Q \equiv \frac{\rho}{\rho_{*}}. \] \hspace{1cm} (14)

The two stresses $t^{a}_{b}$ and $s^{a}_{b}$ are assumed to be separately conserved, so

\[ \nabla^{b}t^{a}_{b} = 0 = \nabla^{b}s^{a}_{b}, \] \hspace{1cm} (15)

the $a = 0$ components of which, using (10)-(12), give two conservation equations,

\begin{align*}
\dot{\rho}_{*} + 3H\gamma\rho_{*} & = 0, \quad (16) \\
\dot{\rho} + \rho[\alpha(\lambda + 1) + \beta(\mu + 1) + \theta(\nu + 1)] & = 0. \quad (17)
\end{align*}

Hence, the perfect fluid density falls in proportion to the comoving volume, as

\[ \rho_{*} \propto \frac{1}{(abc)^{\gamma}}, \] \hspace{1cm} (18)
and the anisotropic fluid density falls as
\[ \rho \propto \frac{1}{a^{\lambda+1}b^{\nu+1}c^{\mu+1}}. \]  
(19)

One can see from these two equations that \( Q \propto a^{\gamma-\lambda-1}b^{\gamma-\nu-1}c^{\gamma-\mu-1} \) admits a variety of possible time-evolutions according to the values of the isotropic and anisotropic pressures. The evolution equations become,

\[
\begin{align*}
\dot{\alpha} + \alpha^2 + \alpha(\beta + \theta) &= \frac{1}{2} [\rho + p_1 - p_2 - p_3 + (2 - \gamma)\rho_*] \quad (20) \\
\dot{\beta} + \beta^2 + \beta(\alpha + \theta) &= \frac{1}{2} [\rho + p_3 - p_1 - p_2 + (2 - \gamma)\rho_*] \quad (21) \\
\dot{\theta} + \theta^2 + \theta(\alpha + \beta) &= \frac{1}{2} [\rho + p_2 - p_3 - p_1 + (2 - \gamma)\rho_*] \quad (22)
\end{align*}
\]

Substituting \( H, R \) and \( S \) for \( \alpha, \beta, \) and \( \theta \), from (7) and using (20)-(22) and (5), we have two propagation equations for the anisotropies,

\[
\begin{align*}
H\dot{R} + R\dot{H} + 3RH^2 &= p_1 - p_3 = (\lambda - \mu)Q\rho_*; \quad (23) \\
H\dot{S} + S\dot{H} + 3SH^2 &= p_1 - p_2 = (\lambda - \nu)Q\rho_*; \quad (24)
\end{align*}
\]

and the conservation equations (16)-(17) combine to give

\[
\dot{Q} = \frac{qH}{3} [9(\gamma - 1) - 3(\lambda + \nu + \mu) + R(2\mu - \nu - \lambda) + S(2\nu - \lambda - \mu)] \quad (25)
\]

The three equations (23)-(25) completely determine the evolution when the anisotropy is small. We are interested in the solutions of these equations in the case where the anisotropy is realistically small. Thus we assume that \( H \) and \( \rho_* \) take the values they would have in an isotropic Friedmann universe containing isotropic density \( \rho_* \) (ie. \( S = R = Q = 0 \)), so

\[
H = \frac{2}{3\gamma t} \text{ and } \rho_* = \frac{4}{3\gamma^2 t^2}. \quad (26)
\]

This is consistent with the (0) – equation, (3). The three equations (23)-(24) then completely determine the evolution of \( R, S, \) and \( Q \). They reduce to
\[ \dot{R} + \frac{R}{\gamma t} (2 - \gamma) = \frac{2Q(\lambda - \mu)}{\gamma t}, \]  
(27)

\[ \dot{S} + \frac{S}{\gamma t} (2 - \gamma) = \frac{2Q(\lambda - \nu)}{\gamma t}, \]  
(28)

\[ \dot{Q} = \frac{2Q}{9\gamma t} \left[ 9(\gamma - 1) - 3\Delta + R(2\mu - \nu - \lambda) + S(2\nu - \lambda - \mu) \right]. \]  
(29)

Axisymmetric solutions exist when \( S = 0 \) and \( \lambda = \nu \). No essential simplification arises by imposing axial symmetry and so we shall treat the general case with \( S \neq 0 \).

The system of equations (27)-(30) gives rise to a characteristic pattern of cosmological evolution when stresses with anisotropic pressures are present. Equation (30) reveals that there is a critical condition which, if satisfied, makes the problem a second-order stability problem; that is, if we linearised the equations about the isotropic solution \( (R = S = 0) \) with zero anisotropic stress density \( (Q = 0) \) we would find a zero eigenvalue associated with the evolution of \( Q(t) \). This critical condition is

\[ \text{Criticality condition : } 3(\gamma - 1) = \Delta. \]  
(30)

When \( 3\gamma \leq 3 + \Delta \) the shear distortion variables \( R \) and \( S \) relax towards their attractor where \( \dot{R} = \dot{S} = 0 \) as \( t \to \infty \), and so we have

\[ R = \frac{2Q(\lambda - \mu)}{2 - \gamma} + \delta_1 t^{\frac{2-\gamma}{\gamma}} \to \frac{2Q(\lambda - \mu)}{2 - \gamma}, \]  
(31)

\[ S = \frac{2Q(\lambda - \nu)}{2 - \gamma} + \delta_2 t^{\frac{2-\gamma}{\gamma}} \to \frac{2Q(\lambda - \nu)}{2 - \gamma}. \]  
(32)

with \( \delta_1 \) and \( \delta_2 \) constants. The \( \delta_i t^{\frac{2-\gamma}{\gamma}} \) terms are the contributions from the isotropic part of the 3-curvature. They fall off more rapidly than the part of the shear that is driven by the anisotropic pressure (assuming \( \lambda \neq \mu \neq \nu \)). As \( t \) grows, the \( \delta \) terms become negligible if \( 3\gamma > 2\Delta \) (and recall that the isotropy is stable so long as \( 3\gamma \leq 3 + \Delta \)). Thus, \( R \) and \( S \) become proportional to \( Q(t) \), which is determined by
\[
\dot{Q} = \frac{2Q}{9\gamma t}\{9(\gamma - 1) - 3\Delta + \frac{2Q}{(2 - \gamma)}[(\lambda - \mu)(2\mu - \nu - \lambda) + (\lambda - \nu)(2\nu - \lambda - \mu)]\}. 
\]

(33)

When the evolution is not critical \((3(\gamma - 1) \neq \Delta)\) the evolution of \(Q(t)\) is determined by the terms linear in \(Q\) on the right-hand side of (33); eqns. (31)-(32) still hold, but now we have

\[
Q(t) = Q_0 t^k; \ k \neq 0, 
\]

(34)

\[
k = \frac{2}{3\gamma}[3(\gamma - 1) - \Delta]. 
\]

(35)

Our assumption, eq.(26), that the evolution of \(\rho_*\) and \(H\) follow their values in an isotropic Friedmann universe will only be consistent at large times if \(k \leq 0\), that is if

\[
-3 \leq 3(\gamma - 1) \leq \Delta \leq 3. 
\]

(36)

The right-hand side of this inequality derives from the causality conditions for signal propagation in the \(i = 1, 2, 3\) directions \((p_i \leq \rho \text{ so } \lambda \leq 1, \mu \leq 1, \nu \leq 1)\); the left-hand side arises from \(p_i \geq -\rho\), which ensures the stability of the vacuum. When \(k > 0\) the anisotropic stress redshifts away more slowly than the isotropic perfect fluid on the average (over directions) and comes to dominate the expansion dynamics, making them completely anisotropic. We do not live in such a universe. By contrast, when \(k < 0\), the isotropic stresses redshift away the slowest and increasingly dominate the dynamics, so the gravitational effect of the anisotropic stresses steadily diminishes. In the critical case the average stress energy of the anisotropic stress is counter-balanced by the isotropising effect of the perfect fluid and the shear evolution is determined by the second-order effects of the pressure anisotropy. This effect can be seen in the study of free neutrinos by Doroshkevich, Zeldovich and Novikov [8] and in the study of axisymmetric magnetic fields by Zeldovich [20].

When the evolution is critical, (30) holds, \(k = 0\), and the evolution of \(Q(t)\) is decided at second-order in \(Q\) by

\[
\dot{Q} = \frac{4Q^2}{9\gamma(2 - \gamma)t}\{(\lambda - \mu)(2\mu - \nu - \lambda) + (\lambda - \nu)(2\nu - \lambda - \mu)\}. 
\]

(37)
Hence,

\[ Q(t) = \frac{Q_0}{1 + AQ_0 \ln \left( \frac{t}{t_1} \right)}, \quad (38) \]

\[ A \equiv -\frac{4}{9\gamma(2-\gamma)} \{ (\lambda - \mu)(2\mu - \nu - \lambda) + (\lambda - \nu)(2\nu - \lambda - \mu) \}, \quad (39) \]

where \( Q_0 \) and \( t_1 \) are constants. For physically realistic stresses we have \( A > 0 \), and so as \( t \to \infty \) the ratio of the energy densities approaches the attractor

\[ Q \to \frac{1}{A \ln \left( \frac{t}{t_1} \right)}, \quad (40) \]

while the associated shear distortions approach the values given by eqns. (31)-(32). Since the values of \( R \) and \( S \) at the epoch of last scattering of the microwave background radiation determine the observed temperature anisotropy we will be able to constrain the allowed value of \( Q \) at the present time by placing bounds on \( R \) and \( S \) at last scattering and then evolving the bounds forward to the present day.

We can also calculate the asymptotic forms for the expansion scale factors of (1). In the critical cases, as \( t \to \infty \), they evolve towards

\[ a(t) \propto t^{\frac{2}{3\gamma}} \{ \ln t \}^m, \quad (41) \]

\[ b(t) \propto t^{\frac{2}{3\gamma}} \{ \ln t \}^n, \quad (42) \]

\[ c(t) \propto t^{\frac{2}{3\gamma}} \{ \ln t \}^w, \quad (43) \]

where the constants \( m, n, w \) are defined by

\[ m = \frac{4(2\lambda - \mu - \nu)}{9\gamma A(2-\gamma)}, \quad (44) \]

\[ n = \frac{4(2\mu - \nu - \lambda)}{9\gamma A(2-\gamma)}, \quad (45) \]
\[ w = 4(2\nu - \mu - \lambda) \frac{9}{9\gamma A(2 - \gamma)}. \]  

(46)

Some specific cases will be considered below. Note that the asymptotic form for the scale factors, (41)-(43) is consistent with the principal approximation imposed by eq. (26).

In the non-critical cases (with \( k < 0 \)), as \( t \to \infty \), the scale factors evolve to first order as

\[
\begin{align*}
\text{a}(t) & \propto t^{\frac{2}{3}\gamma} \{1 - \text{V}_a t^k\}, \\
\text{b}(t) & \propto t^{\frac{2}{3}\gamma} \{1 - \text{V}_b t^k\}, \\
\text{c}(t) & \propto t^{\frac{2}{3}\gamma} \{1 - \text{V}_c t^k\},
\end{align*}
\]

(47)  
(48)  
(49)

where the constants \( \text{V}_a, \text{V}_b, \) and \( \text{V}_c \) are defined by,

\[
\{\text{V}_a, \text{V}_b, \text{V}_c\} \equiv \frac{-4q_0}{9k\gamma(2 - \gamma)} \times \{\mu + \nu - 2\lambda, \lambda + \mu - 2\nu, \lambda + \nu - 2\mu\}. \]

(50)

These asymptotes reveal the different behaviour in the critical cases which produces the logarithmic decay of the shear. In the non-critical cases the anisotropic perturbations to the scale factors decay as power laws in time since \( k < 0 \).

3 Some Particular Skew Stresses

There are many examples of anisotropic stresses to which the analysis of the last section might be applied. We shall consider some of the most interesting. In each case we can determine the characteristics of the 'critical' state in which the evolution of the anisotropy is determined by the non-linear coupling with the anisotropic stress. In section 4 we will go on to calculate the observed microwave background temperature anisotropy. The most familiar example of an anisotropic stress is that of an electromagnetic field.
3.1 Magnetic or Electric fields

In this case the anisotropic energy-momentum tensor $s^b_a$ of eqn. (11) has a simple form [20], [22], [24]. For example, for a pure magnetic field of strength $B$ directed along the $z$-axis, we have $T^0_0 = T^3_3 = -T^1_1 = -T^2_2 = B^2/8\pi$ and so this corresponds to the choice

\[ \lambda = \nu = -\mu = 1 \]  

Hence,

\[ \Delta(magnetic) = \Delta(electric) = 1 \]  

Therefore, from eqn. (30), we see that the criticality condition is obeyed when the background universe is radiation dominated. Thus, in the standard model of the early universe, the evolution of magnetic (or electric) fields will exhibit the non-linear logarithmic decay found in equations (37)-(40). This was first pointed out by Zeldovich [20], and can be identified in the calculations of Shikin [21] and Collins [22]. Barrow, Ferreira, and Silk [23] used the COBE data set [18] to place constraints on the allowed strength of any cosmological magnetic field.

As a specific example of a critical case, a radiation dominated universe ($\gamma = 4/3$) containing a magnetic field aligned along the $z$-axis, $\lambda = \nu = -\mu = 1$, gives $A = 4$ in (39) and hence

\[ a(t) \propto b(t) \propto t^{\frac{1}{2}} \{ \ln t \}^{\frac{1}{2}} \]  

\[ c(t) \propto t^{\frac{1}{2}} \{ \ln t \}^{-\frac{1}{2}} \]  

Notice that the volume expansion goes as in the isotropic radiation universe, $abc \propto t^{\frac{1}{2}}$ in accord with the principal approximation stipulated by (20).

As an example of a non-critical case consider a pure magnetic field aligned along the $z$-axis of a dust universe ($\gamma = 1$). We have $k = -2/3$, and so at late times

\[ a(t) \propto b(t) \propto t^{\frac{2}{3}} \{ 1 - \frac{4Q_0}{3t^{\frac{2}{3}}} + \ldots \}, \]  

\[ c(t) \propto t^{\frac{2}{3}} \{ 1 + \frac{8Q_0}{3t^{\frac{2}{3}}} + \ldots \} \]
This behaviour can be seen in the exact magnetic dust solutions of Thorne [24] and Doroshkevich [25]. Other studies of the evolution of cosmological magnetic fields in anisotropic universes can be found in refs. [28].

The axion field in low-energy string theory [13], [14] also creates stresses of this characteristic form and a source-free magnetic field has been used to study the possibility of dimensional reduction in cosmologies with additional spatial dimensions near the Planck time by Linde and Zelnikov [26] and Yearsley and Barrow [27].

3.2 General Trace-free stresses

The magnetic field case is just one of a whole class of skew fields which exhibit non-linear evolution during the radiation era. Any anisotropic stress which has a trace-free energy-momentum tensor, $s^b_a$, will have

$$\Delta = 1$$

and will exhibit critical evolution in the presence of isotropic black body radiation with $\gamma = 4/3$. This case includes free-streaming gravitons produced at $t_{\text{pl}} \sim 10^{-43}s$ which are collisionless at $t > 10t_{\text{pl}}$ because of the weakness of the gravitational interaction mediating graviton scatterings [6]. It is also likely to include all asymptotically-free particles at energies exceeding $\sim 10^{15}GeV$ when interparticle scatterings, decays and inverse decays have interaction rates slower than the expansion rate of the universe, $H$. In all cases defined by $\Delta = 1$ the shear to Hubble expansion rate decays only logarithmically during the radiation era,

$$R \propto S \propto \frac{1}{\ln \left( \frac{t}{t_1} \right)}.$$ (56)

In the dust era the critical condition will not continue to be met for trace-free fields. Although the evolution of $Q$, $R$, and $S$ is now determined at linear order in (31), (32), and (40), there is still a significant slowing of the decay of isotropy by the anisotropic stresses compared to the case where anisotropic stresses are absent. We see that when $\gamma = 1$ we have

$$\frac{\rho}{\rho_*} \propto R \propto S \propto \frac{1}{t^{2/3}}.$$ (57)
compared to a decay of $R \propto S \propto t^{-1}$ when anisotropic stresses are absent ($Q = 0$) or ignored. The study by Maartens et al [29] of the evolution of anisotropies in a dust-dominated universe containing possible anisotropic stresses at second order (described by a tensor $\pi^b_a$ in reference [28] which is equivalent to the $s^b_a$ used here) displays this same slowing of the shear decay (the discussion of ref. [41] omits this consideration).

In general, we can see that if the criticality condition is satisfied in the radiation era, when $\gamma = 4/3$, then it cannot be satisfied during the dust era that follows, when $\gamma = 1$. However, an interesting situation can arise if the source of the trace-free stress is a population of particles which are relativistic above some energy (so $\Delta = 1$ there) and non-relativistic when the universe cools below this energy scale (so $\Delta = 0$ there). Thus there can be a change in the value of $\Delta$ with time. We shall examine this case in section 4.

Some matter fields with anisotropic pressures, like Yang-Mills fields [12], correspond to an anisotropic fluid with time-dependent $\lambda, \mu, \nu$. But if the evolution is slow enough, it is well-approximated by the model with constant values of $\lambda, \mu, \nu$ used here.

3.3 Long-wavelength gravitational waves

In the last section we considered the evolution of anisotropic stresses in the simplest flat ($\Omega_0 = 1$) anisotropic cosmological model of Bianchi type I with the metric (1). The most general anisotropic universes of this curvature are of Bianchi type $VII_0$ and they have an Einstein tensor that can be decomposed into a sum of two pieces: one corresponding to the Einstein tensor for the simple type I geometry, the other to a piece that describes additional long-wavelength gravitational waves [10], [11], [30]. These waves create anisotropies in the 3-curvature of the universe in addition to the simple expansion-rate anisotropies present in the Bianchi I universe (which has isotropic 3-curvature). However, the contribution by the long-wavelength gravitational waves can be moved to the other side of the Einstein equations and reinterpreted as an additional ’effective’ energy-momentum tensor describing a ’fluid’ of gravitational waves. It has vanishing trace. This decomposition means that any general flat (or open) Bianchi type universe of type $VII_0$ (or $VII_h$) containing an isotropic perfect fluid, (10) behaves like a Bianchi type I (or type V) universe containing that fluid plus an additional traceless anisotropic fluid. The parameters $\lambda, \mu, \nu$ are approximately constant when the anisotropies are small. Hence the general evolution
of anisotropic universes with anisotropic curvature containing isotropic ra-
diation will exhibit the same characteristic logarithmic decay of the shear
anisotropy given by eqns. (31), (32), and (40) during the radiation era.
This behaviour appears in Collins and Hawking [31], Doroshkevich et al [32],
and other authors, [33], [34]. During the dust era any anisotropic curvature
modes will behave like trace-free stresses and, although the evolution will no
longer be critical, the shear anisotropy will fall more slowly \( R \propto S \propto t^{-2/3} \)
than in simple isotropic universes with isotropic curvature and no anisotropic
stresses (where \( R \propto S \propto t^{1-2\gamma} \)).

3.4 Strings and Walls

Topological defects provide another class of anisotropic energy-momentum
tensors which fall into the category of stress modelled by (12) for part of their
evolution. The energy-momentum tensors of string sources were also consid-
ered more generally by Stachel [16], Marder and Israel [17]. The specific de-
scription of line stresses created by topological defects is reviewed in ref. [15].
An infinite string with mass per unit length \( \mu \) extending in the x-direction
contributes an anisotropic stress-tensor \( s^b_a = \mu \delta(z) \delta(y) \text{diag}(1,1,0,0) \); that
is

\[
\lambda = -1, \ \mu = \nu = 0 \rightarrow \Delta(\text{string}) = -1. \tag{58}
\]

Their evolution could therefore only be critical in a universe containing
a perfect fluid with equation of state \( p_* = -\rho_*/3 \). It is worth noting that
this corresponds to the evolution of the curvature term in the Friedmann
equation when the universe is open. Thus we would expect the string stress
to evolve critically during the late curvature-dominated stage \( 1 + z < \Omega_0^{-1} \)
of an open universe, during a curvature-dominated pre-inflationary phase, or
during any period when quantum matter fields with \( \gamma = 1/3 \) dominate the
expansion.

For slow-moving infinite planar domain walls of constant surface density
in the \( x - y \) plane, the stress corresponds \( s^b_a = \eta(z) \text{diag}(1,1,1,0) \), where
\( \eta(z) \) is a local bell-shaped curve centered on \( z = 0 \) [15], and corresponds to
the choice

\[
\lambda = \nu = -1, \ \mu = 0 \rightarrow \Delta(\text{wall}) = -2. \tag{59}
\]
The wall stress would only be critical if the equation of state of the background universe is \( p_\ast = -2\rho_\ast/3 \).

These descriptions do not include the complicated effects of the nonlinear evolution of a population of open strings and loops which must include intersections, kinks, gravitational collapse and gravitational radiation.

### 3.5 General classification and energy conditions

When evaluating the form of the residual anisotropy and energy density created by anisotropic cosmological stresses it is most convenient to classify stresses by the value of \( \Delta \). We can circumscribe the likely range of realistic \( \Delta \) values by considering some general restrictions on the energy-momentum tensor. The dominant energy conditions [35] require us to impose the physical limits

\[
|\lambda| \leq 1, \quad |\mu| \leq 1, \quad |\nu| \leq 1.
\]

Hence we have the bounds

\[
-3 \leq \Delta \leq 3.
\]

If the strong-energy condition [35] were imposed on the energy-momentum tensor \( s^a_b \) then we would have a stronger restriction

\[
\Delta \geq -1.
\]

This condition is violated by string and wall stresses and necessarily by any isotropic stress which drives inflation \( 0 \leq \gamma < 2/3 \) since \( \Delta(\text{wall}) = -2 \) and \( \Delta(\text{string}) = -1 \). We see from (34)-(35) that there can only be approach to isotropy at late times in the radiation era if \( \Delta \geq 1 \). When \( 0 < \Delta < 1 \) the expansion approaches isotropy during the dust era but not during the radiation era. When \( \Delta \leq 0 \) isotropy is unstable during both the dust and radiation eras.

### 4 Microwave background limits

Let us consider the simplest realistic case in which the Universe is radiation dominated until \( 1 + z_{eq} = 2.4 \times 10^4 \Omega_0 h_0^2 \) and then dust dominated thereafter; here; \( \Omega_0 \leq 1 \) is the cosmological density parameter and \( h_0 \) is the present
value of the Hubble constant in units of $100Kms^{-1}Mpc^{-1}$. We shall assume that the microwave background was last scattered at a redshift $z_{rec}$, where, in the absence of reheating of the cosmic medium,

$$1 + z_{rec} = 1100.$$  \hspace{1cm} (63)

If there is reionization of the universe then last scattering can be delayed until $1 + z_{rec} = 39(\Omega_{b0}h_0)^{-1}$ for $\Omega_{b0}z_{rec} << 1$, where $\Omega_{b0}$ is the present baryon density parameter [36].

The microwave background temperature anisotropy is determined by the values of shear distortions $R$ and $S$ at the redshift $z_{rec}$. The evolution of the photon temperature in the $x, y,$ and $z$ directions is given by

$$T_x = T_0 \frac{a_0}{a(t)} = T_0 \exp\{-\int \alpha dt\},$$ \hspace{1cm} (64)

$$T_y = T_0 \frac{b_0}{b(t)} = T_0 \exp\{-\int \beta dt\},$$ \hspace{1cm} (65)

$$T_z = T_0 \frac{c_0}{c(t)} = T_0 \exp\{-\int \theta dt\}.$$. \hspace{1cm} (66)

If we define the temperature anisotropy by

$$\frac{\delta T}{T} = \frac{(T_x - T_y) + (T_x - T_z)}{T_0}$$ \hspace{1cm} (67)

then, for small anisotropies (so $\exp\{-\int \alpha dt\} \simeq 1 - \int \alpha dt$ etc), using the definitions of (6) in (67), we have

$$\frac{\delta T}{T} = -H \int (R + S) dt$$ \hspace{1cm} (68)

Since recombination always occurs in the dust era, $H = 2/3t$, and the observed microwave background anisotropy will be

$$\frac{\delta T}{T} = \left[ \frac{R + S}{\Delta} \right]_{rec} \times f(\vartheta, \phi, \Omega_0).$$ \hspace{1cm} (69)

where $f \sim O(1)$ is a pattern factor taking into account non-gaussian statistical factors and the possible pattern structure created by complicated forms of anisotropy in the general case [13], [23], [37]. Typically, in the most general homogeneous flat universes the pattern combines a distorted quadrupole
with a spiral geodesic motion. In ref. [23] a detailed discussion of the sampling statistics of the microwave background temperature distribution and the non-gaussian nature of the anisotropic pattern was given. This leads to bounds on \( f \) in flat and open universes of

\[
0.6 < f < 2.2.
\]

The discussion above shows that the \( \Delta \geq 1 \) case is the realistic one which allows evolution towards isotropy at late times. The observed anisotropy is therefore given in terms of the present value of the density ratio, \( Q(t_0) \), by

\[
\frac{\delta T}{T} = \frac{2Q(t_0)f(2\lambda - \mu - \nu)(1+z_{\text{rec}})^\Delta}{\Delta}
\]

(71)

If we take the COBE 4-year data set to impose a limit of \( \delta T/T \leq 10^{-5} \) on contributions by the anisotropic fluid, and use (70), then the present density parameter of the anisotropic fluid, \( \Omega_{a0} \), is limited by

\[
\Omega_{a0} \leq \frac{8.3 \times 10^{-6} \Delta}{(2\lambda - \mu - \nu)(1+z_{\text{rec}})^\Delta} = \left( \frac{\Delta}{2\lambda - \mu - \nu} \right) \left( \frac{1100}{1+z_{\text{rec}}} \right)^\Delta \times 10^{-5.08 - 3.04\Delta}
\]

(72)

Since \( 1 \leq \Delta \leq 3 \) for realistic fluids we can examine the extremes of this limit which is strongest when there is no reionization of the Universe at \( z << 1100 \) because reionization allows a longer period of power-law decay of \( R, S, \) and \( Q \), hence a smaller residual effect on the microwave background isotropy. In the two extreme cases we have limits of

\[
\Delta = 1 : \Omega_{a0} \leq 3.7 \times 10^{-9} \times \left( \frac{1100}{1+z_{\text{rec}}} \right)
\]

(73)

\[
\Delta = 3 : \Omega_{a0} \leq 9.5 \times 10^{-15} \times \left( \frac{1100}{1+z_{\text{rec}}} \right)^3
\]

(74)

The \( \Delta = 1 \) case with the choice of (64) corresponds to limits on the present cosmological energy density allowed in magnetic (or electric fields) which was studied in [20] and in [23] where the most general evolution of anisotropy was included. Note that these limits are far stronger than those that are generally obtained for isotropic forms of dark matter by imposition of astronomical limits on the maximum total density, the age of the Universe, or the deceleration parameter [1], [4].
4.1 Evolution of anisotropic dark matter with a characteristic energy scale

We are familiar with the standard picture of the cosmological evolution of light \((m \ll 1\text{MeV})\) weakly interacting particles [1], [4]. When \(T > m\) they behave like massless particles with their number density similar to that in photons, up to statistical weight factors. Their energy density redshifts away like \((1 + z)^4\) until the temperature falls to the value of their rest mass. Thereafter they behave like non-relativistic massive particles and their energy density redshifts away more slowly, as \((1 + z)^3\). We can consider an analogous scenario for anisotropic stresses. Suppose we have an anisotropic fluid which behaves relativistically until the temperature falls to some value \(T_+ < 10^4 K \approx 1\text{eV}\) and then behaves non-relativistically at lower temperatures. This situation corresponds to following the evolution with \(\Delta = 1\) for \(T \geq T_+\) and then with \(\Delta = 0\) for \(T < T_+\). The evolution is complicated by the fact that it will be critical during the radiation era but non-critical during the first stage of the dust era when the temperature exceeds \(T_+\) when \(Q\) will decay in accord with (35), before becoming critical again during the remainder of the dust era until the present when \(Q\) will decay logarithmically in accord with (40). In this case the final density of the anisotropic dark matter is constrained by the microwave background isotropy to have

\[
\Omega_{a0} \leq 1.2 f^{-1} \times 10^{-4} \left( \frac{1100}{1 + z_{\text{rec}}} \right) \times \frac{1}{A \ln(1 + z_+)} \tag{75}
\]

where

\[
A \equiv \frac{4}{9} \left\{ (\mu - \lambda)(2\mu - \nu - \lambda) + (\nu - \lambda)(2\nu - \lambda - \mu) \right\} \tag{76}
\]

For example, if the anisotropic matter has a characteristic energy scale \(m\) then, since \(m = T_+ = 2.4 \times 10^{-4}(1+z_+)\text{eV}\), we have (picking \(\lambda = \nu = -\mu = 1\) so \(A = 32/9\), with (70), that,

\[
\Omega_{a0} \leq 5.7 \times 10^{-5} \left( \frac{1100}{1 + z_{\text{rec}}} \right) \times \frac{1}{(8.3 + \ln M_{\text{eV}})} \tag{77}
\]

and there is a very weak dependence on the mass scale \(m_{\text{eV}} \equiv (m/1\text{eV})\). For \(m > 1\text{eV}\) we have, roughly, that

\[
\Omega_{a0} \leq 6.9 \times 10^{-6} \left( \frac{1100}{1 + z_{\text{rec}}} \right) \tag{78}
\]
Therefore these fields are always constrained by the microwave background anisotropy to be a negligible contributor to the total density of the Universe.

## 5 The Effects of Inflation

So far we have ignored the consequences of any period of inflation in the very early universe [3]. The equations (27)-(31) hold for the case of generalised (power-law) inflation with $0 < \gamma < 2/3$, [37]. However, for de Sitter inflation with $\gamma = 0$, the isotropic density drives the inflation with $\rho_* = 3H_0^2 = \text{constant}$, and they are changed to

$$
\dot{R} + 3RH_0 = 3H_0Q(\lambda - \mu),
$$

$$
\dot{S} + 3SH_0 = 3H_0Q(\lambda - \nu),
$$

$$
\dot{Q} = \frac{QH_0}{3}[-9 - 3\Delta + R(2\mu - \nu - \lambda) + S(2\nu - \lambda - \mu)].
$$

This system can only be critical for the evolution of $Q$ only if $\Delta = -3$ but this cannot occur for an anisotropic fluid subject to (60). As $t \to \infty$ we have

$$
R \to q(\lambda - \mu) + \delta_1 \exp(-3H_0t)
$$

$$
S \to q(\lambda - \nu) + \delta_2 \exp(-3H_0t)
$$

$$
Q \to Q_0 \exp[-(3 + \Delta)H_0t]
$$

Thus, since $\Delta + 3 > 0$ the contribution made to the shear by the anisotropic pressure decays. Moreover, we see that the contribution by the isotropic curvature ($\delta$) terms to the shear dominates the contribution by the pressure anisotropy at late times if $\Delta > 0$. In both cases the shear anisotropy decays away exponentially fast. This is in accord with the expectations of the cosmic no hair theorems [38] because the anisotropic stresses considered here obey the strong energy condition when $\Delta > -3$. Thus if $N$ e-folds of de Sitter inflation occur in the very early universe, then the values of the shear distortion parameters, $R$ and $S$, are each depleted by a factor $\exp(-3N)$ whilst the relative density in the anisotropic fluid, $q$, is reduced by a factor $\exp\{- (3 + \Delta)N\}$. 

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Similarly, in the case of power-law inflation \(0 < \gamma < 2/3\), the isotropic curvature mode will dominate the late time evolution of the shear during the inflationary phase if

\[
\gamma < \frac{2\Delta}{3}. \tag{85}
\]

If anisotropic stresses can be generated at the end of inflation then the analysis of the previous sections applies.

6 Primordial Nucleosynthesis

It is instructive to consider the question of whether primordial nucleosynthesis can provide stronger limits than microwave background isotropy on the densities of anisotropic stresses in the Universe. The key issue that decides this is whether the microwave background constraint on the expansion anisotropy created by an anisotropic energy density, which is of order \(10^{-5}\) and imposed at a redshift \(z_{\text{rec}} \sim 10^{3}\) or lower (in the event of reionization), is stronger than a limit of order \(10^{-1}\) on changes to the mean expansion rate imposed at the epoch of neutron-proton freeze-out, \(z_{f} \sim 10^{10}\). The issue is decided by knowing how fast the anisotropic stresses decay with time between \(z_{f}\) and \(z_{\text{rec}}\).

In the simplest anisotropic universes, which have isotropic 3-curvature, the shear anisotropy falls like the that of the \(\delta\) mode in eqns. (31)-(32). Consider first the simple isotropic curvature case with no anisotropic stress, so \(Q = 0\). The evolution of the shear to Hubble rate parameters, \(R\) and \(S\), during the dust and radiation era of a universe with \(\Omega_0 = 1\) is

\[
z > z_{\text{eq}}: R \propto S \propto t^{-\frac{1}{2}} \propto 1 + z \tag{86}
\]

\[
z < z_{\text{eq}}: R \propto S \propto t^{-1} \propto (1 + z)^{\frac{3}{2}} \tag{87}
\]

Hence, if nucleosynthesis gives an upper limit of \(\sim 0.2\) (roughly equivalent to adding one neutrino type to the standard three-neutrino model) on the values of \(|R|\) and \(|S|\) at the redshift of neutron-proton freeze-out, \(z_{f}\), this corresponds to an upper limit at the time of last scattering of 

\[
0.2(1 + z_{f})^{-1}(1 + z_{\text{eq}})^{-\frac{1}{2}}(1 + z_{\text{rec}})^{\frac{3}{2}},
\]

so we have
Since the microwave background gives a limit of $\delta T/T \sim (R + S)_{rec} \leq 10^{-5}$,[18], we see that the nucleosynthesis limit is much stronger when the 3-curvature is isotropic, as first pointed out by Barrow [39]; nor are conceivable improvements in microwave receiver sensitivity ever likely to close to gap of $10^4$ that exists between the nucleosynthesis and microwave limits on the isotropic 3-curvature $\delta$ modes. These effects of these simple anisotropy modes on nucleosynthesis were considered in the papers of Hawking and Tayler [40] and Thorne [24].

However, the situation changes when we consider the most general forms of anisotropy with anisotropic curvature or when matter is present with anisotropic pressures. If the evolution will be critical during the radiation era the anisotropy falls off so slowly during the period before $z_{eq}$ that the microwave background limits become stronger than the nucleosynthesis limits. This effect was considered in both the dust and radiation eras by Barrow [39] in the evolution of Bianchi type $VII$ universes close to isotropy but the observational limits were 100 times weaker then. For example, in the interesting case where $\Delta = 1$ the evolution of anisotropies follows the form, (56)-(57),

\[ z > z_{eq} : R \propto S \propto \frac{1}{\ln(z_{eq}/z)}, \] (89)
\[ z < z_{eq} : R \propto S \propto 1 + z. \] (90)

Hence, the nucleosynthesis limit translates into an upper limit on $R$ and $S$ at the redshift of last scattering of only

\[ R_{rec} \sim S_{rec} < 7.1 \times 10^{-4}(\Omega_0 h_0^2)^{-1} \times \left(\frac{1 + z_{rec}}{1100}\right). \] (91)

We see that this is never stronger than the microwave background limit of $\leq 10^{-5}$ for any possible redshift of last scattering. Alternatively, we might restate this result as follows: it is possible for anisotropic fluids to create a measurable temperature anisotropy in the microwave background radiation without having any significant effect upon the primordial nucleosynthesis of helium-4.
7 Discussion

We have set up the cosmological evolution equations for a very general class of anisotropic stresses in the presence of an isotropic perfect fluid. Such anisotropic stresses encompass a wide range of physically interesting cases, including those of cosmological electric and magnetic fields, a variety of topological defects, gravitons, and other populations of collisionless particles. When the universe is close to isotropy the mean expansion rate behaves as in the isotropic Friedmann universe to leading order, but the density of the anisotropic stresses is coupled to the expansion anisotropy in an interesting way. It was found that there are two possible forms for the evolution. In the critical case the density of anisotropic matter is determined by its non-linear coupling to the expansion anisotropy and density of anisotropic stress falls only logarithmically relative to the isotropic background density. By contrast, in the non-critical case the density falls as a power-law in time relative to the background density. In all cases the microwave background anisotropy can be used to place a limit on the density of matter that could be residing in the universe today in forms with anisotropic pressures. These limits arise because their evolution approaches an asymptotic attractor in which the temperature anisotropy produced in the microwave background by the anisotropic stresses is simply related to their density relative to the isotropic background density. The limits obtained on the possible cosmological density of matter of this form are far stronger than conventional limits on isotropic forms of dark matter because they make use of the microwave isotropy limits ($\delta T/T \leq 10^{-5}$) rather than the far weaker limits from observations of the Hubble flow and age of the Universe ($\Omega_0 < O(1)$). We went on to explain why the limits that can be impose upon anisotropic stresses by the microwave background isotropy measurements are in general stronger than those arising from the limits on the primordial nucleosynthesis of helium-4.

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