An extension of the Duistermaat-Singer Theorem
to the semi-classical Weyl algebra

Yves Colin de Verdière∗

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Abstract
Motivated by many recent works (by L. Charles, V. Guillemin, T. Paul,
J. Sjöstrand, A. Uribe, San Vũ Ngọc, S. Zelditch and others) on the semi-
classical Birkhoff normal forms, we investigate the structure of the group
of automorphisms of the graded semi-classical Weyl algebra. The answer is
quite similar to the Theorem of Duistermaat and Singer for the usual alge-
bra of pseudo-differential operators where all automorphisms are given by
conjugation by an elliptic Fourier Integral Operator (a FIO). Here what re-
places general non-linear symplectic diffeomorphisms is just linear complex
symplectic maps, because everything is localized at a single point.1

1 The result
Let \( W = W_0 \oplus W_1 \oplus \cdots \) be the semi-classical graded Weyl algebra (see Section 2
for a definition) on \( \mathbb{R}^{2d} \). Let us define \( X_j := W_j \oplus W_{j+1} \oplus \cdots \). We want to prove
the:

Theorem 1.1 There exists an exact sequence of groups

\[ 0 \to \mathcal{I} \to \text{Aut}(W) \to \text{Sympl}_\mathbb{C}(2d) \to 0 \]

where

- \( \text{Sympl}_\mathbb{C}(2d) \) is the group of linear symplectic transformations of \( \mathbb{C}^{2d} = \mathbb{R}^{2d} \otimes \mathbb{C} \)

∗Institut Fourier, Unité mixte de recherche CNRS-UJF 5582, BP 74, 38402-Saint Martin
d’hères Cedex (France); yves.colin-de-verdiere@ujf-grenoble.fr

1Thanks to Louis Boutet de Monvel for his comments and suggestions, in particular the
proofs of Lemmas 6.1 and 6.2. As he said, the main result is not surprising!
• \( \text{Aut}(W) \) is the group of automorphisms \( \Phi \) of the semi-classical graded\(^2\) Weyl algebra preserving \( \hbar \)

• \( \mathcal{I} \) is the group of “inner” automorphisms \( \Phi_S \) of the form \( \Phi_S = \exp(i\text{ad}S/\hbar) \), i.e. \( \Phi_S(w) = \exp(iS/\hbar) \ast w \ast \exp(-iS/\hbar) \) as a formal power series, with \( S \in X_3 \)

• The arrow \( \rightarrow \) is just given from the action of the automorphism \( \Phi \) on \( W_1 = (\mathbb{R}^{2d})^* \otimes \mathbb{C} \) modulo \( X_2 \)

The proof follows [3] and also the semi-classical version of it by H. Christianson [2]. This result is implicitly stated in Fedosov’s book [4] in Chapter 5, but it could be nice to have a direct proof in a simpler context. The result is a consequence of Lemmas 4.1, 6.1 and 6.2.

2 The Weyl algebra

The elements of the “Weyl algebra” are the formal power series in \( \hbar \) and \( (x, \xi) \)

\[ W = \bigoplus_{n=0}^{\infty} W_n \]

where \( W_n \) is the space of complex valued homogeneous polynomials in \( z = (x, \xi) \) and \( \hbar \) of total degree \( n \) where the degree of \( \hbar^j z^\alpha \) is \( 2j + |\alpha| \). The Moyal \( \ast \)–product

\[
 a \ast b := \sum_{j=0}^{\infty} \frac{1}{j!} \left( \frac{\hbar}{2i} \right)^j a \left( \sum_{p=1}^{d} \bar{\partial}_{\xi_p} \bar{\partial}_x p - \bar{\partial}_x p \bar{\partial}_{\xi_p} \right)^j b = ab + \frac{\hbar}{2i} \{a, b\} + \cdots
\]

(where \( \{a, b\} \) is the Poisson bracket of \( a \) and \( b \) ) gives to \( W \) the structure of a graded algebra: we have \( W_m \ast W_n \subset W_{m+n} \) and hence, for the brackets, \( \frac{\hbar}{i}[W_m, W_n]^* \subset W_{m+n-2} \).

The previous grading of \( W \) is obtained by looking at the action of \( W \) on the (graded) vector space \( S \) of symplectic spinors (see [3]): if \( F = \sum_{j=0}^{\infty} \hbar^j F_j(X) \) with \( F_j \in S(\mathbb{R}) \), we define \( f_\hbar(x) = \hbar^{-d/2} F(x/\hbar) \) whose micro-support is the origin. \( W \) acts on \( S \) in a graded way as differential operators of infinite degree: if \( w \in W \), \( w.f = \text{OP}_{\hbar}(w)(f) \).

3 A remark

We assumed in Theorem [1.1] that \( \hbar \) is fixed by the automorphism. If not, the symplectic group has to be replaced by the homogeneous symplectic group: the group of the linear automorphisms \( M \) of \( (\mathbb{R}^{2d}, \omega) \) which satisfies \( M^* \omega = c \omega \). We then have to take into account a multiplication of \( \hbar \) by \( c \). For \( c = -1 \), it is a semi-classical version of the transmission property of Louis Boutet de Monvel.

\(^2\)“graded” means that \( \Phi(W_n) \subset W_n \oplus W_{n+1} \oplus \cdots \)
4 Surjectivity of the arrow $\rightarrow_2$

Lemma 4.1 The arrow $\rightarrow_2$ is surjective.

Proof.–

Let us give $\chi \in \text{Sympl}_C(2d)$. The map $a \rightarrow a \circ \chi$ is an automorphism of the Weyl algebra: the Moyal formula is given only in terms of the Poisson bracket.

5 The principal symbols

Let $\Phi$ be an automorphism of $W$. Then $\Phi$ induces a linear automorphism $\Phi_n$ of $W_n$: if $w = w_n + r$ with $w_n \in W_n$ and $r \in X_{n+1}$ and $\Phi(w) = w'_n + r'$ with $w'_n \in W_n$ and $r' \in X_{n+1}$, $\Phi_n(w_n) := w'_n$ is independent of $r$. The polynomial $w_n$ is the principal symbol of $w \in X_n$. We have $\Phi_{m+n}(w_m \ast w_n) = \Phi_m(w_m) \ast \Phi_n(w_n)$. Hence $\Phi_n$ is determined by $\Phi_1$ because the algebra $W$ is generated by $W_1$ and $\hbar$. The linear map $\Phi_1$ is an automorphism of the complexified dual of $\mathbb{R}^{2d}$. Let us show that it preserves the Poisson bracket and hence is the adjoint of a linear symplectic mapping of $\mathbb{C}^{2d}$. We have:

$$\Phi([w, w']^*) = [\Phi(w), \Phi(w')]^* .$$

By looking at principal symbols, for $w, w' \in X_1$, we get

$$\{\Phi_1(w), \Phi_1(w')\} = \{w, w'\} .$$

6 Inner automorphisms

The kernel of $\rightarrow_2$ is the group of automorphisms $\Phi$ which satisfy $\Phi_n = \text{Id}$ for all $n$, i.e. for any $w_n \in W_n$

$$\Phi(w_n) = w_n \text{ mod } X_{n+1} .$$

The following fact is certainly well known:

Lemma 6.1 If $\Phi \in \ker(\rightarrow_2)$, $\Phi = \exp(D)$ where $D : W_n \rightarrow X_{n+1}$ is a derivation of $W$.

Proof [following a suggestion of Louis Boutet de Monvel]– We define $\Phi^s$ for $s \in \mathbb{Z}$. Let $\Phi^s_{p,n} : W_n \rightarrow W_{n+p}$ be the degree $(n + p)$ component of $(\Phi^s)_n : W_n \rightarrow X_n$. Then $\Phi^s_{p,n}$ is polynomial w.r. to $s$. This allows to extend $\Phi^s$ to $s \in \mathbb{R}$ as a 1-parameter group of automorphisms. We put $D = \frac{d}{ds} (\Phi^s)|_{s=0}$. We have $\Phi^s = \exp(sD)$. We deduce that $D$ is a derivation.

We need to show the:
Lemma 6.2 Every derivation $D$ of $W$ sending $W_1$ into $X_2$ is an inner derivation of the form

$$Dw = \frac{i}{\hbar} [S, w]$$

with $S \in X_3$.

Proof [following a suggestion of Louis Boutet de Monvel]–

Let $(\zeta_k)$ the basis of $W_1$ dual to the canonical basis $(z_k)$ for the star bracket, i.e. satisfying $[\zeta_k, z_l] = \frac{\hbar}{i} \delta_{k,l}$. We have $[\zeta_k, w] = \frac{\hbar}{i} \frac{\partial w}{\partial z_k}$. Put $y_k = D\zeta_k \in X_2$. As the brackets $[\zeta_k, \zeta_l]$ are constants, we have: $[D\zeta_k, \zeta_l] + [\zeta_k, D\zeta_l] = 0$, or $\partial y_k/\partial z_l = \partial y_l/\partial z_k$. There exists an unique $S$ vanishing at 0 so that:

$$[S, \zeta_k] = - \frac{\partial S}{\partial z_k} = \frac{\hbar}{i} y_k.$$ 

Hence $D = (i/\hbar)[S, \cdot]$. Because $y_k \in X_2$, $S$ is in $X_3$. □

7 An homomorphism from the group $G$ of elliptic FIO’s whose associated canonical transformation fixes the origin into $\text{Aut}(W)$

Each $w \in W$ is the Taylor expansion of a Weyl symbol $a \equiv \sum_{j=0}^{\infty} \hbar^j a_j(x, \xi)$ of a pseudo-differential operator $\hat{a}$. Let us give an elliptic Fourier Integral Operator $U$ associated to a canonical transformation $\chi$ fixing the origin. The map $\hat{a} \rightarrow U^{-1} \hat{a} U$ induces a map $F$ from $S^0$ into $S^0$ which is an automorphism of algebra (for the Moyal product).

Lemma 7.1 The Taylor expansion of $F(a)$ only depends on the Taylor expansion of $a$.

This is clear from the explicit computation and the stationary phase expansions.

As a consequence, $F$ induces an automorphism $F_0$ of the Weyl algebra graded by powers of $\hbar$.

Lemma 7.2 $F_0$ is an automorphism of the algebra $W$ graded as in Section 2.

Proof.–

We have to check the $F_0(W_n) \subset X_n$. Because $F_0$ preserves the $\star$—product, it is enough to check that $F_0(W_1) \subset X_1$. It only means that the (usual) principal symbol of $F(a)$ vanishes at the origin if $a$ does. It is consequence of Egorov Theorem.
Summarizing, we have constructed a group morphism $\alpha$ from the group $G$ of elliptic FIO’s whose associated canonical transformation fixes the origin in the group $\Aut(W)$.

**Definition 7.1** An automorphism $\Phi$ of $W$ is said to be real ($\Phi \in \Aut_{\mathbb{R}}(W)$) if the mapping $\Phi \mod(\hbar W)$ is real.

**Theorem 7.1** The image of the group $G$ by the homomorphism $\alpha$ is $\Aut_{\mathbb{R}}(W)$. In particular, any $\Phi \in \Aut_{\mathbb{R}}(W)$ can be “extended” to a semi-classical Fourier Integral Operator.

**Proof.**–

The image of $\alpha$ is in the sub-group $\Aut_{\mathbb{R}}(w)$ because the canonical transformation $\chi$ is real.

We have still to prove that the image of $\alpha$ is $\Aut_{\mathbb{R}}(W)$. Using the Theorem [1.1] and the metaplectic representation, it is enough to check that the automorphism $\exp(i\text{ad}S/\hbar)$ comes from an FIO. Let $H$ be a symbol whose Taylor expansion is $S$ (the principal symbol of $H$ is a real valued Hamiltonian). The OIF $U = \exp\left(i\hat{H}/\hbar\right)$ will do the job.

□

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