Traveling wave solutions in a model for social outbursts in a tension-inhibitive regime

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Abstract
In this work, we investigate the existence of nonmonotone traveling wave solutions to a reaction-diffusion system modeling social outbursts, such as rioting activity, originally proposed in Berestycki et al (Netw Heterog Media. 2015;10(3):443–475). The model consists of two scalar values, the level of unrest \(u\) and a tension field \(v\). A key component of the model is a bandwagon effect in the unrest, provided the tension is sufficiently high. We focus on the so-called tension-inhibitive regime, characterized by the fact that the level of unrest has a negative feedback on the tension. This regime has been shown to be physically relevant for the spatiotemporal spread of the 2005 French riots. We use Geometric Singular Perturbation Theory to study the existence of such solutions in two situations. The first is when both \(u\) and \(v\) diffuse at a very small rate. Here, the time scale over which the bandwagon effect is observed plays a key role. The second case we consider is when the tension diffuses at a much slower rate than the level of unrest. In this case, we are able to deduce that the driving dynamics are modeled by the well-known Fisher–Kolmogorov-Petrovsky-Piskunov (KPP) equation.

KEYWORDS
Fisher equation, geometric singular perturbation theory, KPP equation, riots, traveling front
1 | INTRODUCTION

Civil unrest, protests, and rioting are tools that populations use to express objection or dissent toward an idea or action, usually political. These outbursts of social activity have been ubiquitous in time and space and, in many cases, have changed the course of history. From the religious protest in the early 16th century to the recent George Floyd protests, which have engulfed the United States, these outbursts of activity amplify in time and have an underlying field of “tension” driving them. In Ref. 2, the authors introduce a reaction-diffusion model for the dynamics of rioting activity (or unrest) and social tension, motivated by the 2005 French riots. The model assumes a bandwagon effect on the level of unrest that turns on when the social tension is above a certain threshold value. Moreover, this model assumes a nearest-neighbor spread, in other words the spatial contagion is local and modeled by the classical diffusion operator. Some robust features observed in these social outbursts are the temporal up-and-down dynamics and, in cases such as the 2005 French riots or the Velvet Revolution of 2018 in Armenia, the spatial spread of the activity. These features have been observed in the data and can be expressed mathematically as the existence of traveling wave solutions.

The system introduced in Ref. 2 models the spatiotemporal dynamics of the “level of unrest,” denoted by \( u \), and the “social tension,” denoted by \( v \) (see Section 1.1 for a detailed explanation of the model). The unknowns are coupled via a parameter, denoted by \( p \), which leads to two regimes of interest. In the case when \( p < 0 \), known as the social-enhancing regime, the level of unrest has a positive feedback on the social tension. This is based on the assumption that the active participation of individuals in a protest or riot consequent increases the social tension. On the other hand, when \( p > 0 \), known as the tension inhibitive, the level of unrest has a negative feedback on the social tension. This regime is relevant when the active participation of an individual in a riot or protest helps the system release energy, thus reducing the social tension. The case when \( p = 0 \) decouples the two unknowns. The details of this coupling are explained in Section 1.1.

Note that the tension-enhancing regime leads to a monotone system where classical techniques can provide significant insight into the model dynamics, such as the existence and stability of traveling wave solutions. However, in this regime, the traveling waves are monotone. For riots, such as the 2005 French riots, this is a huge limitation because the monotone traveling wave solutions do not present the temporal up-and-down dynamic feature observed in actual data. This implies that the physical regime is the tension-inhibitive case, where the system loses monotonicity. Of course, here classical maximum principles are no longer available, making it a more challenging case to analyze. However, this case does lead to the existence of nonmonotone traveling wave solutions, which were explored numerically in Ref. 5. The motivation of this work is the need to provide a rigorous analysis of nonmonotone traveling wave solutions observed.

Specifically, we prove the existence of traveling wave solutions in two subregimes of the tension-inhibitive case, which we recall is the physical regime. The main tool used here is Geometric Singular Perturbation theory. We first consider the regime when the spatial spread of the level of unrest and the social tension are small. In this case, the parameter that sets the time scale over which the bandwagon effect would be observed, denoted by \( \omega \), plays a key role in the analysis. Specifically, we consider the singular limits as \( \omega \to 0 \) and \( \omega \to \infty \) to find the appropriate heteroclinic orbits. We then use the theory of rotated vector fields for the intermediate values of \( \omega \). We see in Section 3 that in the limit as \( \omega \to 0 \) the dynamics of the system are driven by the dynamics of \( u \) and evolve slowly along the \( v \)-nullcline (see Figure 2). Recall that the time scale over which the bandwagon effect is observed is given by \( \frac{1}{\omega} \), which goes to \( \infty \) as \( \omega \to 0 \). Thus, we expect that the dynamics of the level of unrest to dominate here. On the other hand, as \( \omega \to \infty \), the dynamics of the system are driven by the dynamics of \( v \) and evolve slowly along the \( u \)-nullcline (see Figure 3).
Of course, here the time scale \( \frac{1}{\omega} \to 0 \) as \( \omega \to \infty \) and the dynamics of the system are driven by the social tension.

The second case we consider is when the social tension diffuses at a much slower rate than the level of unrest. Interestingly, the dynamics here can be reduced to a Fisher–KPP type equation for the level of unrest. The case \( p = 0 \) was analyzed in Ref. 5 and decouples the dynamics between the level of unrest and social tension. In this case, the equation for the level of unrest also reduced to a Fisher–KPP equation with the social tension being equal to one. The situation here is a bit different as \( v \) is a function of \( u \), specifically \( v = (1 + u)^p \). Fisher–KPP equations have been found to model a wide range of biological phenomena, ranging from its original application in population genetics 9 to population dynamics in ecology 10 and wound healing. 11 Moreover, these types of equations are understood well from a mathematical point of view, see for example, Refs. 12–14. Due to its ubiquity, the Fisher–KPP equation seems to be as fundamental to biology, ecology, and sociology, as the Navier–Stokes equation is to physics. A recent example that supports this is due to Berestycki et al 15 who studied a classical epidemic Susceptible-Infected-Recovered (SIR) model with diffusion and with an additional compartment of infected individuals traveling on a line with fast diffusion. Interestingly, a classical transformation reduces the proposed model to a Fisher–KPP type equation. This provides evidence that these seemingly different models, with very different source terms, are fundamentally related. Our work provides additional evidence that the Fisher–KPP equation is fundamental in social applications.

**Outline**: We present the model and background information in Section 1.1. In Section 2, we discuss the type of solutions that we seek and the model formulation that we use for each of the two cases to be considered. In Section 3, we discuss the vanishing diffusion limit case. In Section 4, we consider the reduction of the model of study to the Fisher–KPP equation and prove the existence of traveling wave solutions. We conclude with some numerical experiments in Section 5.

1.1 | The model

Much research has led to the belief that certain external events are responsible for initiating a period of unrest, 16 the so-called *triggering events*. However, one must also take into account long-established frustrations, which can play a role in the intensity and duration of these social outbursts. 17 This leads to a dynamic tension field, which is also important to understand as it is coupled with the level of unrest. The system proposed in Ref. 2 involves the coupling of an explicit variable, representing the level of activity (or level of unrest), and an underlying social tension field. The level of unrest measures the amount of activity at any given time and location, and it can be measured explicitly, that is, the number of people participating in a protest. On the other hand, the social tension is an implicit variable, that provides a measure of long established frustrations, for example, due to lack of employment opportunities, certain political decisions, racial inequality, and so forth. The levels of unrest and social tension at location \( x \) and time \( \tau \) are represented by \( u(x, \tau) \) and \( v(x, \tau) \), respectively, and they satisfy the system:

\[
\begin{align*}
    u_\tau &= d_1 \Delta u + r(v)G(u) - \omega u, \\
    v_\tau &= d_2 \Delta v + 1 - h(u)v
\end{align*}
\]  

satisfied for \( \tau > 0 \) and \( x \in \mathbb{R}^n \) and with nonnegative initial data.
It is assumed that there is a bandwagon effect in the level of unrest, in other words there is a self-excitement in the system up to a carrying capacity. A natural function to model this effect is a KPP-type function. Here we use the prototypical logistic function, $G = u(1 - u)$, for concreteness. This effect is assumed to be negligible until the social tension $v$ is sufficiently large. This switch mechanism is described by the sigmoid-type function $r$. The effect that $u$ has on $v$ is modeled by the function $h(u) : [0, \infty) \to (0, \infty)$, and is either monotone increasing or decreasing. The monotonicity of $h$ determines whether (1) is of cooperative or activator-inhibitor type. For this reason, we refer to (1) in the case when $h$ is decreasing as a tension-enhancing system and in the case when $h$ is increasing as a tension-inhibitive system. The specific functions considered are given by:

$$r(v) = \frac{\Gamma}{1 + e^{-\beta(v-\alpha)}}$$ and $$h(u) = \theta(1 + u)^p.$$ 

The model also assumed a nearest-neighbor contagion that is modeled by the diffusion terms $d_1 \Delta u$ and $d_2 \Delta v$. Note that $p < 0$ corresponds to the tension-enhancing case and $p > 0$ to the tension-inhibitive case. Throughout the remainder of the paper, we make the assumption that $\alpha = \theta = 1$ and that $d_1, d_2, p, \Gamma, \beta$ are positive parameters. In particular, we will be working in the tension-inhibitive case. We remark that there are different functions $G, r,$ and $h$ than can be used to describe similar effects. The functions used here are those chosen in Ref. 2 for the purpose of concreteness, but our results can be generalized. However, we feel that the generalizations would come at the expense of technicalities with little payoff.

## 2 CONSTANT STATES AND TRAVELING WAVE SOLUTIONS

Our interest lies in studying planar traveling wave solutions and thus we can safely consider the one-dimensional version of (1). To study the two distinct parameter regimes discussed above: (i) $d_1, d_2$ small and (ii) $d_2 \ll d_1$, we view system (1) from different angles. In the former case, we rename $\Gamma/\omega = \gamma$ and recast (1) as:

$$\begin{cases} u_\tau = d_1 u_{xx} + \omega \left( \frac{\gamma}{1 + e^{-\beta(v-\alpha)}} u(1 - u) - u \right), \\
 v_\tau = d_2 v_{xx} + 1 - (1 + u)^p v. \end{cases} \tag{2}$$

For the latter case, with abuse of notation, we replace the time variable $\tau$ with $\omega \tau$ and spatial variable $x$ with $\sqrt{\omega} x$ and get an equivalent system:

$$\begin{cases} u_\tau = d_1 u_{xx} + \frac{\gamma}{1 + e^{-\beta(v-\alpha)}} u(1 - u) - u, \\
 v_\tau = d_2 v_{xx} + \frac{1}{\omega}(1 - (1 + u)^p v). \end{cases} \tag{3}$$

To find the constant states of (1) (equivalently of (2) and (3)), we solve the system of algebraic equations:

$$\frac{\gamma}{1 + e^{-\beta(v-\alpha)}} u(1 - u) - u = 0, \quad 1 - (1 + u)^p v = 0.$$
As illustrated in Figure 1, there are two physically relevant constant states: \( A(0, 1) \) and \( B(\bar{u}, \bar{v}) \), where \( \bar{u}, \bar{v} > 0 \). More precisely, \( \bar{u} \) is defined as the solution of the transcendental equation:

\[
\gamma - 1 - \gamma u = e^{-\beta \left( \frac{1}{(1+u)^p} - 1 \right)} \tag{4}
\]

and then

\[
\bar{v} = \frac{1}{(1 + \bar{u})^p}. \tag{5}
\]

The constant state \( A(0, 1) \) is the \textit{relaxed} state with no activity and \( B(\bar{u}, \bar{v}) \) is the \textit{excited} state with a positive level of activity. To study traveling wave solutions, it is convenient to introduce a moving coordinate frame \( \xi = x - ct \), where \( c \) is the propagating speed of the front. Note that due to the symmetry \((c, \xi) \leftrightarrow (-c, -\xi)\), it is enough to consider \( c > 0 \). In the new variable \( \xi = x - ct \), the system given by (1) reads as follows:

\[
\begin{cases}
    u_\tau = d_1 u_\xi + cu_\xi + r(v)u(1-u) - \omega u, \\
    v_\tau = d_2 v_\xi + cv_\xi + 1 - h(u)v
\end{cases} \tag{6}
\]

for \( \xi \in \mathbb{R} \). Traveling wave solutions do not change their profile in time, so the corresponding traveling wave ordinary differential equation (ODE) system to (1) is given by

\[
\begin{cases}
    0 = d_1 u_\xi + cu_\xi + r(v)u(1-u) - \omega u, \quad \text{for } \xi \in \mathbb{R}, \\
    0 = d_2 v_\xi + cv_\xi + 1 - h(u)v, \quad \text{for } \xi \in \mathbb{R}, \\
    (u(-\infty), v(-\infty)) = B(\bar{u}, \bar{v}) \quad \text{and} \quad (u(\infty), v(\infty)) = A(0, 1),
\end{cases} \tag{7}
\]

where we have used the notation \( u(\pm \infty) = \lim_{\xi \to \pm \infty} u(x) \) and \( v(\pm \infty) = \lim_{\xi \to \pm \infty} v(x) \).
3 | VANISHING DIFFUSION LIMIT

In this section, we consider the case when \( d_1, d_2 \ll 1 \). Here we study the traveling wave ODE system corresponding to (2), which reads as:

\[
\begin{align*}
0 &= d_1 u_{\xi \xi} + cu_{\xi} + \omega \left( \frac{\gamma}{1 + e^{-\beta (v - 1)}} u (1 - u) - u \right), \quad \text{for } \xi \in \mathbb{R}, \\
0 &= d_2 v_{\xi \xi} + cv_{\xi} + 1 - (1 + u)^p v, \quad \text{for } \xi \in \mathbb{R}, \\
(u(-\infty), v(-\infty)) &= B(\bar{u}, \bar{v}) \quad \text{and} \quad (u(\infty), v(\infty)) = A(0, 1). 
\end{align*}
\]  

(8)

We will consider (8) as a singular perturbation of a related vanishing diffusion limit. In what follows, as we perform a variety of change of variables to (8), we will not include the conditions at \( \pm \infty \) for notational simplicity, but we remind the reader that the solutions to the new systems are subject to the same limiting conditions. To reflect that both diffusion coefficients \( d_1 \) and \( d_2 \) are small and comparable parameters, we introduce the following notation:

\[
d_1 = \varepsilon, \quad \text{where } 0 < \varepsilon \ll 1 \quad \text{and} \quad d_2 = \mu d_1, \quad \text{where } 0 < \mu = O(1).  
\]  

(9)

The corresponding version of (6) and (8) are as follows:

\[
\begin{align*}
\frac{du}{d\xi} &= \varepsilon u_{\xi} + cu_{\xi} + \omega \left( \frac{\gamma}{1 + e^{-\beta (v - 1)}} u (1 - u) - u \right), \\
\frac{dv}{d\xi} &= \varepsilon \mu v_{\xi} + cv_{\xi} + 1 - (1 + u)^p v,
\end{align*}
\]  

(10)

and

\[
\begin{align*}
0 &= \varepsilon u_{\xi \xi} + cu_{\xi} + \omega \left( \frac{\gamma}{1 + e^{-\beta (v - 1)}} u (1 - u) - u \right), \\
0 &= \varepsilon \mu v_{\xi \xi} + cv_{\xi} + 1 - (1 + u)^p v.
\end{align*}
\]  

(11)

To prove the existence of a traveling wave solution \((u, v, c)\) which satisfies (7), we use Applied Dynamical Systems techniques. More precisely, when \( \varepsilon \ll 1 \) the dynamical system associated to the ODE system (11) is a singular perturbation of a lower-dimensional dynamical system, therefore it is natural to use Geometric Singular Perturbation theory. We seek traveling fronts of Equation (10) as heteroclinic orbits for the first-order system:

\[
\begin{align*}
\frac{du_1}{d\xi} &= u_2, \\
\varepsilon \frac{du_2}{d\xi} &= -cu_2 - \omega \left( \frac{\gamma}{1 + e^{-\beta (v_1 - 1)}} u_1 (1 - u_1) - u_1 \right), \\
\frac{dv_1}{d\xi} &= v_2, \\
\varepsilon \mu \frac{dv_2}{d\xi} &= -cv_2 - 1 + (1 + u_1)^p v_1.
\end{align*}
\]  

(12)
We call system (12) a slow system, as opposed to the fast system that is obtained from (12) through the scaling $\zeta = \xi / \epsilon$:

\[
\begin{align*}
\frac{du_1}{d\zeta} &= \epsilon u_2, \\
\frac{du_2}{d\zeta} &= -cu_2 - \omega \left( \frac{\gamma}{1 + e^{-\beta(v_1 - 1)}} u_1(1 - u_1) - u_1 \right), \\
\frac{dv_1}{d\zeta} &= \epsilon v_2, \\
\mu \frac{dv_2}{d\zeta} &= -cv_2 - 1 + (1 + u_1)^p v_1.
\end{align*}
\] (13)

We next consider the limit of the systems (12) and (13) as $\epsilon \to 0$. Since $\mu = O(1)$, then $\mu \epsilon \to 0$ as well. In this limit, system (12) produces a description of the set that the solution belongs to

\[
M_0 = \left\{ (u_1, u_2, v_1, v_2) | u_2 = -\frac{\omega}{c} \left( \frac{\gamma}{1 + e^{-\beta(v_1 - 1)}} u_1(1 - u_1) - u_1 \right), v_2 = \frac{1}{c}(-1 + (1 + u_1)^p v_1) \right\}.
\] (14)

On $M_0$, the dynamics of the slow variables $u_1$ and $v_1$ are given by

\[
\begin{align*}
\frac{du_1}{d\zeta} &= -\frac{\omega}{c} \left( \frac{\gamma}{1 + e^{-\beta(v_1 - 1)}} u_1(1 - u_1) - u_1 \right), \\
\frac{dv_1}{d\zeta} &= -\frac{1}{\epsilon}(1 - (1 + u_1)^p v_1).
\end{align*}
\] (15)

The set $M_0$ also serves as a set of equilibrium points for (13) with $\epsilon = 0$,

\[
\begin{align*}
\frac{du_1}{d\zeta} &= 0, \\
\frac{du_2}{d\zeta} &= -cu_2 - \omega \left( \frac{\gamma}{1 + e^{-\beta(v_1 - 1)}} u_1(1 - u_1) - u_1 \right), \\
\frac{dv_1}{d\zeta} &= 0, \\
\mu \frac{dv_2}{d\zeta} &= -cv_2 - 1 + (1 + u_1)^p v_1.
\end{align*}
\] (16)

The linearization of (16) about any point of the set $M_0$, defined in (14), has two zero eigenvalues and two eigenvalues equal to $-c$. Therefore, the set $M_0$ is a normally hyperbolic and an attracting set. By the Fenichel's invariant manifold theory, there exists an $\epsilon$-order perturbation of $M_0$, which is an invariant manifold for (12), equivalently for (13):

\[
M_\epsilon = \left\{ (u_1, u_2, v_1, v_2) | u_2 = -\frac{\omega}{c} \left( \frac{\gamma u_1(1 - u_1)}{1 + e^{-\beta(v_1 - 1)}} - u_1 \right) + O(\epsilon), v_2 = \frac{1}{c}(-1 + (1 + u_1)^p v_1) + O(\epsilon) \right\}.
\] (17)
On that manifold, the flow generated by (12) is then an $\varepsilon$-order perturbation of the flow (15),

$$\begin{align*}
\frac{du_1}{d\xi} &= -\frac{\omega}{\epsilon} \left( \frac{\gamma}{1+e^{-\beta(u_1-1)}} u_1(1 - u_1) - u_1 \right) + O(\varepsilon), \\
\frac{dv_1}{d\xi} &= -\frac{1}{\epsilon} (1 - (1 + u_1)^p v_1) + O(\varepsilon),
\end{align*}$$

so the slow dynamics of (12) is restricted to the two-dimensional set (17). The nullclines of the planar system (15) are given by

$$u_1 = 1 - \frac{1}{\gamma} (1 + e^{-\beta(v_1-1)}), \quad u_1 = 0, \quad v_1 = \frac{1}{(u_1 + 1)^p}.$$  

Note that there are no equilibrium solutions in the open first quadrant when $\gamma \leq 2$, therefore we will only consider the case when $\gamma > 2$. When $\gamma > 2$, there are two relevant equilibria: $A(0, 1)$ and $B(\bar{u}, \bar{v})$, where the components of $B$ are described in (4)–(5). In [Ref. 5, theorem 2.1], it is proved that in the system:

$$\begin{align*}
\frac{du}{d\xi} &= \gamma \left( \frac{\gamma}{1+e^{-\beta(u-1)}} u(1 - u) - u \right), \\
\frac{dv}{d\xi} &= 1 - (1 + u)^p v \tag{20}
\end{align*}$$

the nontrivial steady state $(\bar{u}, \bar{v})$ with positive components is globally stable in the open first quadrant. The system (20) is a scaled version of (15) with reversed dynamics. The global stability of $(\bar{u}, \bar{v})$ in (20) implies global stability of the corresponding equilibrium $(\bar{u}, \bar{v})$ in (15) in reversed “time” $\xi$.

The linearization of the vector field generated by (15) at the equilibrium $A$ has the eigenvalues $\lambda_1 = 1$ and $\lambda_2 = \frac{2-\gamma}{2} \omega$, so $A$ is a saddle when $\gamma > 2$ and it is a node when $\gamma < 2$. The global stability of $B = (\bar{u}, \bar{v})$ in reversed “time” $\xi$ implies that for $\gamma > 2$ the equilibria $A$ and $B$ are connected along the stable manifold of the saddle $A$. We give a detailed geometric description of the structure of this orbit below.

For brevity, we introduce the following notation:

$$\begin{align*}
f_1(u_1, v_1) &= -\left( \frac{\gamma}{1+e^{-\beta(v_1-1)}} u_1(1 - u_1) - u_1 \right), \\
f_2(u_1, v_1) &= -(1 - (1 + u_1)^p v_1), \tag{21}
\end{align*}$$

thus (15) now reads as follows:

$$\begin{align*}
\frac{du_1}{dz} &= \frac{\omega}{\epsilon} f_1(u_1, v_1), \\
\frac{dv_1}{dz} &= \frac{1}{\epsilon} f_2(u_1, v_1). \tag{22}
\end{align*}$$
The eigenvalues of the linearization of (15) at $B$ are as follows:

$$
\lambda_{1,2}(B) = \frac{1}{2c} \left( f_{2v_1}(B) + \omega f_{1u_1}(B) \pm \sqrt{(f_{2v_1}(B) - \omega f_{1u_1}(B))^2 + 4\omega f_{1v_1}(B)f_{2u_1}(B)} \right),
$$

(23)

where

$$
\begin{aligned}
  f_{1u_1}(B) &= \frac{\gamma - 1 - e^{-\beta(\hat{u}_1 - 1)}}{1 + e^{-\beta(\hat{u}_1 - 1)}} = \frac{\hat{u}_1}{1 - \hat{u}_1}, \\
  f_{1v_1}(B) &= -\beta \hat{u}_1 e^{-\beta(\hat{u}_1 - 1)} = -\frac{\beta}{1 - \hat{u}_1}, \\
  f_{2u_1}(B) &= p(1 + \hat{u}_1)^{p-1} \hat{u}_1 = \frac{p}{1 + \hat{u}_1}, \\
  f_{2v_1}(B) &= (1 + \hat{u}_1)^p.
\end{aligned}
$$

(24)

Since $\gamma > 2$ and $\hat{u}_1 > 0$, it is easy to see that $f_{1u_1}(B) > 0$, $f_{2u_1}(B) > 0$, $f_{2v_1}(B) > 0$, and $f_{1v_1}(B) < 0$, and so the equilibrium $B$ is an unstable node. The eigenvalues (23) may be real or complex depending on the parameters of the system. Note that

$$
f_{2v_1}(B) + \omega f_{1u_1}(B) > 0
$$

(25)

and the expression under the root sign in (23) becomes zero at the points:

$$
\omega_1 = \frac{f_{1u} f_{2v} - f_{1v} f_{2u} - 2\sqrt{-f_{1u} f_{1v} f_{2v} + f_{1v}^2 f_{2u}^2}}{f_{1u}^2},
$$

$$
\omega_2 = \frac{f_{1u} f_{2v} - f_{1v} f_{2u} + 2\sqrt{-f_{1u} f_{1v} f_{2v} + f_{1v}^2 f_{2u}^2}}{f_{1u}^2}.
$$

(26)

From $f_{1v_1}(B) < 0$, it follows that $\omega_2 > 0$ and $\omega_2 > \omega_1$. Since for small $\omega$ both eigenvalues $\lambda_{1,2}(B)$ are positive, then $\omega_1 > 0$. Therefore, $\lambda_{1,2}(B)$ are:

- positive for $\omega \in (0, \omega_1) \cup (\omega_1, \infty)$;
- complex with positive real part for $\omega \in (\omega_1, \omega_2)$.

To analyze the dynamics of the system (15), we consider separately the cases when $\omega \ll 1$ and $\omega \gg 1$, and then discuss the situation of the intermediate values of $\omega$. In the first case, when $\omega \ll 1$ the following theorem holds.

**Theorem 1.** Assume that $\gamma > 2$, $\mu > 0$, and $c > 0$ are fixed parameters. Assume also that $0 < \epsilon \ll \omega$.

There exists an $\omega_0 > 0$ such that for any $0 < \omega < \omega_0$, there is $\epsilon_0 = \epsilon(\omega) > 0$ such that for any $\epsilon < \epsilon_0$ in the system (12) and, equivalently, for the system (13), there exists a heteroclinic orbit connecting $(0,0,1,0)$ and $(\hat{u},0,\hat{v},0)$. Thus, for (10) there exists a translationally invariant family of fronts that have the constant states $A(0,1)$ and $B(\hat{u},\hat{v})$ as their rest states.
Proof. Let us consider system (15) along with a rescaled version of it, in terms of the variable \( \eta = \omega \xi \),

\[
\begin{align*}
\frac{du_1}{d\eta} &= -\frac{1}{c} \left( \gamma \frac{1}{1 + e^{-\beta(v_1 - 1)}} u_1 (1 - u_1) - u_1 \right), \\
\omega \frac{dv_1}{d\eta} &= -\frac{1}{c} (1 - (1 + u_1)^p v_1).
\end{align*}
\] (27)

When \( \omega = 0 \), the system (15) becomes

\[
\begin{align*}
\frac{du_1}{d\xi} &= 0, \\
\frac{dv_1}{d\xi} &= -\frac{1}{c} (1 - (1 + u_1)^p v_1).
\end{align*}
\] (28)

On the other hand, when we set \( \omega = 0 \) in (27)

\[
\begin{align*}
\frac{du_1}{d\eta} &= -\frac{1}{c} \left( \gamma \frac{1}{1 + e^{-\beta(v_1 - 1)}} u_1 (1 - u_1) - u_1 \right), \\
0 &= -\frac{1}{c} (1 - (1 + u_1)^p v_1),
\end{align*}
\] (29)

we obtain the manifold to which the solution of this reduced system belongs:

\[
\left\{ (u_1, v_1) : v_1 = \frac{1}{(1 + u_1)^p} \right\}
\] (30)

and the reduced flow on this manifold:

\[
\frac{du_1}{d\eta} = -\frac{1}{c} u_1 \left( \gamma \frac{1}{1 + e^{-\beta(1 + u_1)^p - 1}} (1 - u_1) - 1 \right).
\] (31)

Equation (31) has two equilibrium points: \( \tilde{A} = 0 \) and \( \tilde{B} = \bar{u}_1 \). It is easy to see that the linearization of (28) about any point \( (u_1, v_1) \) of (30) has a positive eigenvalue \( p (1 + u_1)^p - 1 / c \) and a zero eigenvalue, so the set (30) is normally hyperbolic and repelling.

The linearization of (31) about \( \tilde{A} \) has a negative eigenvalue, while the linearization of (31) about \( u_1 = \bar{u}_1 \) has a positive eigenvalue, so \( \tilde{A} \) is a stable node and \( \tilde{B} \) is an unstable node. Therefore, there is an asymptotic connection from \( \tilde{B} \) at \(-\infty\) to \( \tilde{A} \) at \( \infty \). Within the one-dimensional slow manifold (30), this intersection is transversal by the dimension counting. Since the slow manifold (30) is normally hyperbolic, by Fenichel's invariant manifold theory, it persists when a sufficiently small \( \omega \) is introduced, that is, there is an invariant manifold in (15) which is also normally repelling and is an \( \omega \)-order perturbation of (15):

\[
v_1 = \frac{1}{(1 + u_1)^p} + O(\omega),
\] (32)
on which the flow is an $\omega$-order perturbation of (31) given by

$$
\frac{du_1}{d\eta} = -\frac{1}{c}u_1\left(\frac{\gamma}{1 + e^{-\beta(1 + u_1)^p}(1 - u_1) - 1}\right) + O(\omega). \quad (33)
$$

Since the set is repelling, the stable manifold of the saddle (0,1) must stay on the manifold. This stable manifold then intersects with the unstable manifold of the equilibrium $(\bar{u}, \bar{v})$; thus, forming a heteroclinic orbit along the set (32). In the two-dimensional phase space, the intersection of the one-dimensional stable manifold of the saddle (0,1) with the two-dimensional unstable manifold of the node $(\bar{u}, \bar{v})$ is transversal by the dimension counting.

This geometric construction of a heteroclinic orbit is performed on the slow manifold $M_0$ of the system (16), which was shown above to be normally hyperbolic and attracting. For a sufficiently small $\varepsilon > 0$, the slow manifold $M_0$ perturbs to an attracting, two-dimensional invariant set $M_\varepsilon$. Since $M_0$ is attracting, the two-dimensional unstable manifold of equilibrium $(\bar{u}, 0, \bar{v}, 0)$ is confined to $M_0$, and thus any orbit that follows this manifold is also confined to $M_0$. Therefore, within $M_\varepsilon$, the intersection of two-dimensional unstable manifold of equilibrium $(\bar{u}, 0, \bar{v}, 0)$ and the one-dimensional slow stable manifold of the equilibrium (0,0,1,0) persists, forming a “slow” heteroclinic orbit. ■

In the case of $\omega \gg 1$, the following theorem holds.

**Theorem 2.** Assume that $\gamma > 2$, $\mu > 0$, and $c > 0$ are fixed parameters. Also assume that $0 < \varepsilon \ll 1/\omega$. There is $\omega_0 \gg 1$ such that for any $\omega > \omega_0$, there exists $\varepsilon_0 = \varepsilon(\omega) > 0$ such that for any $\varepsilon < \varepsilon_0$ in the system (12) and, equivalently, for the system (13), there exists a heteroclinic orbit connecting (0,0,1,0) and $(\bar{u}, 0, \bar{v}, 0)$. Thus, for (10) there exists a translationally invariant family of fronts that have the constant states $A(0, 1)$ and $B(\bar{u}, \bar{v})$ as their rest states.

**Proof.** We denote $\delta = \frac{1}{\omega}$ and rewrite (15) as follows:

$$
\begin{cases}
\delta \frac{du_1}{d\xi} = -\frac{1}{c}u_1\left(\frac{\gamma}{1 + e^{-\beta(1 + u_1)^p}} (1 - u_1) - 1\right), \\
\frac{dv_1}{d\xi} = -\frac{1}{c}(1 - (1 + u_1)^p v_1).
\end{cases} \quad (34)
$$

We then introduce $z = \xi / \delta$ and rewrite (34) as

$$
\begin{cases}
\frac{du_1}{dz} = -\frac{1}{c}u_1\left(\frac{\gamma}{1 + e^{-\beta(1 + u_1)^p}} (1 - u_1) - 1\right), \\
\frac{dv_1}{dz} = \frac{\delta}{c}(-1 + (1 + u_1)^p v_1).
\end{cases} \quad (35)
$$

When $\delta = 0$, the system (35) reads as:

$$
\begin{cases}
\frac{du_1}{dz} = -\frac{1}{c}u_1\left(\frac{\gamma}{1 + e^{-\beta(1 + u_1)^p}} (1 - u_1) - 1\right), \\
\frac{dv_1}{dz} = 0.
\end{cases} \quad (36)
$$
The singular limit when: (left panel) $\omega = 0$ and (right panel) $0 < \omega \ll 1$

The slow manifold for this system, which is also the set of equilibrium points for (36), consists of two one-dimensional sets: a line $S^1_0 = \{(u_1, v_1) : u_1 = 0\}$ and a curve

$$S^2_0 = \left\{(u_1, v_1) : \frac{\gamma}{1 + e^{-\beta(v_1-1)}}(1 - u_1) - 1 = 0\right\}.$$  

Linearizing about points from each set, we see that $S^1_0$ is normally attracting and $S^2_0$ is normally repelling. Each point of $S^1_0$, including $v_1 = 1$, has a one-dimensional, linear stable manifold. The stable manifold of $S^2_0$ is an open subset of the phase space of the $(u_1, v_1)$-plane.

The reduced flow on $S^1_0$ is given by:

$$\frac{dv_1}{dz} = -\frac{1}{c}(1 - v_1),$$  

which has exactly one equilibrium, $v_1 = 1$, that corresponds to the equilibrium $(0,1)$ in the system (35). Within the set $S^1_0$, this equilibrium of (37) is repelling. For sufficiently small $\delta$, the unstable manifold of the whole set $S^1_0$ perturbs to the two-dimensional unstable manifold of $(0,1)$. The reduced system on $S^2_0$ is given by the equation:

$$\frac{dv_1}{dz} = \frac{\delta}{c}\left(-1 + \left(2 - \frac{1 + e^{-\beta(v_1-1)}}{\gamma}\right)^p v_1\right).$$  

Equation (38) has a single equilibrium at $v_1 = \bar{v}$, which corresponds to the equilibrium $(v_1, u_1)$ in (35). Within $S^2_0$, this equilibrium is attracting. For sufficiently small $\delta$, the stable manifold of $v_1 = \bar{v}$ perturbs to the one-dimensional stable manifold of $(\bar{v}, \bar{u})$ in (35).

By the dimension counting, the stable manifold of $S^1_0$ intersects the one-dimensional stable manifold of $(\bar{v}, \bar{u})$ transversally; therefore, for sufficiently small $\delta$ in (35), the unstable manifold of $(0,1)$ and the stable manifold of $(\bar{v}, \bar{u})$ intersect, thus forming a heteroclinic orbit, which is a perturbation of the singular orbit depicted in Figure 2.

The same argument given in case when $\omega \ll 1$ then shows that this heteroclinic orbit persists for the system (12) or, equivalently, (13) with sufficiently small values of $\epsilon$.  

A portion of $u_1 = 0$ is attractive when $\gamma - 1 - e^{-\beta(v_1 - 1)} > 0$: the singular limit when $\omega = \infty$ (left panel); the connecting orbit when $\omega \gg 1$ (right panel).

The heteroclinic orbits in the system \eqref{system} at intermediate values of $\omega$ may be traced as continuous deformations of the orbits in singular cases, according to the theory of rotated vector fields. Figure 4 illustrates the heteroclinic orbit. We consider the angle between the $u$-axis and the vector given by the right-hand side of \eqref{system}: $\Phi(u, v) = \tan^{-1} \left(\frac{f_2(u, v)}{\omega f_1(u, v)}\right)$. It is easy to see that

$$\frac{\partial \Phi}{\partial \omega} = \frac{-f_1(u, v)f_2(u, v)}{\omega^2 f_1^2(u, v) + f_2^2(u, v)}.$$

In the region above both nullclines of \eqref{system}, $f_1 < 0$ and $f_2 > 0$. Therefore, $\frac{\partial \Phi}{\partial \omega} > 0$, as $\omega$ decreases from infinity to zero, the segment of the stable manifold $W^s(A)$ of the saddle $A$ in the described region rotates monotonically (Ref. [8, section 2]), clockwise from its limiting position of the singular orbit corresponding to $\omega = \infty$ ($\delta = 0$) to its position of the singular orbit when $\omega = 0$. While in the region above the both nullclines, $W^s(A)$ for each value of $\omega$ does not cross any of its positions for other values of $\omega$. We point out that when $\omega \ll 1$, the vector field points vertically up along the nullcline $v_1 = \frac{1}{(u_1 + 1)^{\rho}}$ in the region above the nullcline $u_1 = 1 - \frac{1}{\gamma}(1 + e^{-\beta(v_1 - 1)})$. 
This implies that the orbits which are small perturbations of the singular orbit with \( \omega = 0 \) stay above \( v_1 = \frac{1}{(u_1+1)^p} \) as they never can cross this nullcline. On the other hand, the vector field allows the orbits to cross the nullcline \( u_1 = 1 - \frac{1}{\gamma}(1 + e^{-\beta(v_1-1)}) \) in the region above the nullcline \( u_1 = \frac{1}{(u_1+1)^p} \). This implies that the orbits for the intermediate values of \( \omega = O(1) \) may be characterized by the point of intersection of \( W^s(A) \) with \( u_1 = 1 - \frac{1}{\gamma}(1 + e^{-\beta(v_1-1)}) \), which moves down the nullcline monotonically. It follows from Ref. [5, theorem 2.1] that for any \( \omega \) in the system (15) there is an orbit that follows \( W^s(A) \) and connects the equilibrium \( A \) to the equilibrium \( B \). The intersection of one-dimensional stable manifold \( W^s(A) \) with the two-dimensional unstable manifold \( W^u(B) \) in the two-dimensional phase space is transversal by the dimension counting, therefore will persist as a solution of the system (12), or, equivalently, the system (13) with sufficiently small \( \epsilon \).

4 | REDUCTION TO THE KPP EQUATION

In this regime, we consider the partial differential equation (PDE) system (3) under the assumption that \( d_1 = O(1) \) and \( d_2 \ll 1 \). To make this more definitive, we set \( d_1 = 1 \) and \( d_2 = \epsilon \ll 1 \). In a moving coordinate frame \( \xi = x - ct \), system (3) reads as:

\[
\begin{align*}
    u_\tau &= u_{\xi\xi} + cu_\xi + \frac{\gamma}{1 + e^{-\beta(v-1)}} u(1-u) - u, \\
    v_\tau &= \epsilon v_{\xi\xi} + cu_\xi + \frac{1}{\omega} (1 - (1+u)^p v)
\end{align*}
\]

for \( \xi \in \mathbb{R} \) and \( \tau > 0 \).

**Theorem 3.** Assume that \( \epsilon \ll \omega \) in (39). Also assume that \( \gamma > 2 \) is fixed, and parameters \( \beta \) and \( p > 0 \) are such that

\[
\frac{d^2}{du^2} \left( \frac{(1-u)u}{1 + e^{-\beta(\frac{1}{1+u})^p}} \right) < 0, \text{ for } 0 < u < \bar{u}(\gamma, \beta, p),
\]

where \( \bar{u} \) is the solution of Equation (4). For every fixed value of \( c \geq \sqrt{2(\gamma - 2)} \), there exists \( \epsilon_0 > 0 \) such that for any \( \epsilon < \epsilon_0 \) in (39) there is \( \omega_0 = \omega_0(\epsilon) > 0 \) such that for every \( 0 < \omega < \omega_0 \) there exists a translationally invariant family of fronts in (39) that have the equilibria \( A(0,1) \) and \( B(\bar{u}, \bar{v}) \) as rest states. As \( \epsilon \to 0 \) each front converges to a front in

\[
\begin{align*}
    u_\tau &= u_{\xi\xi} + cu_\xi + \frac{\gamma}{1 + e^{-\beta(v-1)}} u(1-u) - u, \\
    v_\tau &= cu_\xi + \frac{1}{\omega} (1 - (1+u)^p v),
\end{align*}
\]

that moves with the same velocity.
Proof. The proof of this theorem is based on the geometric construction of a heteroclinic orbit in the associated dynamical system, which is corresponding to the front. The traveling wave ODE for the system (39) is

\[
\begin{align*}
0 &= u_{\xi\xi} + cu_{\xi} + \frac{\gamma}{1 + e^{-\beta(v-1)}} u(1 - u) - u, \\
0 &= \epsilon v_{\xi\xi} + cv_{\xi} + \frac{1}{\omega}(1 - (1 + u)^p) v).
\end{align*}
\]

(41)

We rewrite (41) as a dynamical system:

\[
\begin{align*}
\frac{du_1}{d\xi} &= u_2, \\
\frac{du_2}{d\xi} &= u_1 - cu_2 - \frac{\gamma}{1 + e^{-\beta(v_1-1)}} u_1(1 - u_1), \\
\frac{dv_1}{d\xi} &= v_2, \\
\frac{dv_2}{d\xi} &= -c v_2 + \frac{1}{\omega}((1 + u_1)^p v_1 - 1).
\end{align*}
\]

(42)

We also consider an equivalent system that captures the fast dynamics by setting \(\zeta = \xi / \epsilon\),

\[
\begin{align*}
\frac{du_1}{d\zeta} &= \epsilon u_2, \\
\frac{du_2}{d\zeta} &= \epsilon \left( u_1 - cu_2 - \frac{\gamma}{1 + e^{-\beta(v_1-1)}} u_1(1 - u_1) \right), \\
\frac{dv_1}{d\zeta} &= \epsilon v_2, \\
\frac{dv_2}{d\zeta} &= -c v_2 + \frac{1}{\omega}((1 + u_1)^p v_1 - 1).
\end{align*}
\]

(43)

We study the singular limit of (42) when \(\epsilon \to 0\), thus obtaining an algebraic description of the slow manifold on which the solution of the limiting system exists on the following three-dimensional set:

\[
M_{\epsilon=0,\omega} = \left\{ (u_1, u_2, v_1, v_2) | v_2 = \frac{1}{c\omega}((1 + u_1)^p v_1 - 1) \right\}
\]

(44)

with the flow given by:

\[
\begin{align*}
\frac{du_1}{d\xi} &= u_2, \\
\frac{du_2}{d\xi} &= u_1 - cu_2 - \frac{\gamma}{1 + e^{-\beta(v_1-1)}} u_1(1 - u_1), \\
\omega \frac{dv_1}{d\xi} &= \frac{1}{c}((1 + u_1)^p v_1 - 1),
\end{align*}
\]

(45)
or in a variable $z = \omega \xi$:

$$
\begin{align*}
\frac{du_1}{dz} &= \omega u_z, \\
\frac{du_2}{dz} &= \omega \left( u_1 - cu_2 - \frac{\gamma}{1+e^{-\beta(u_1-1)}} u_1(1-u_1) \right), \\
\frac{dv_1}{dz} &= \frac{1}{c} \left( 1 + u_1 \right)^p v_1 - 1.
\end{align*}
$$

(46)

On the other hand, $M_{\epsilon=0,\omega}$ is a set equilibria for (43) with $\epsilon = 0$:

$$
\begin{align*}
\frac{du_1}{d\xi} &= 0, \\
\frac{du_2}{d\xi} &= 0, \\
\frac{dv_1}{d\xi} &= 0, \\
\frac{dv_2}{d\xi} &= -cv_2 + \frac{1}{\omega} \left( 1 + u_1 \right)^p v_1 - 1.
\end{align*}
$$

(47)

The linearization of the system (47) about any point of $M_{\epsilon=0,\omega}$ has three zero eigenvalues and a negative eigenvalue $-c$, therefore $M_{\epsilon=0,\omega}$ is normally hyperbolic. For sufficiently small $\epsilon$, by Fenichel’s invariant manifold theory, there exists an invariant, normally attracting manifold $M_{\epsilon,\omega}$ in the system (43), which is an $O(\epsilon)$-order perturbation of $M_{\epsilon=0,\omega}$, where

$$
M_{\epsilon,\omega} = \left\{ (u_1,u_2,v_1,v_2) \middle| v_2 = \frac{1}{c\omega} \left( 1 + u_1 \right)^p v_1 - 1 \right\} + O(\epsilon).
$$

(48)

The flow generated by (43) on $M_{\epsilon,\omega}$ is an $O(\epsilon)$-order perturbation of the flow on $M_0$:

$$
\begin{align*}
\frac{du_1}{d\xi} &= u_2, \\
\frac{du_2}{d\xi} &= u_1 - cu_2 - \frac{\gamma}{1+e^{-\beta(u_1-1)}} u_1(1-u_1), \\
\omega \frac{dv_1}{d\xi} &= \frac{1}{c} \left( 1 + u_1 \right)^p v_1 - 1 + O(\epsilon).
\end{align*}
$$

(49)

Our further analysis is based on considering another singular limit in (45) as $\omega \to 0$. Taking this limit, we obtain a description of a two-dimensional slow manifold:

$$
M_{\epsilon=0,\omega=0} = \left\{ (u_1,u_2,v_1) \middle| v_1 = \frac{1}{(1 + u_1)^p} \right\}
$$

(50)

to which the solutions of the limiting system must belong. With $\omega = 0$ the system (46) reads as:

$$
\begin{align*}
\frac{du_1}{dz} &= 0, \\
\frac{du_2}{dz} &= 0, \\
\frac{dv_1}{dz} &= \frac{1}{c} \left( 1 + u_1 \right)^p v_1 - 1.
\end{align*}
$$

(51)
The linearization of (51) about any point \((\bar{u}_1, \bar{v}_1)\) of the set \(M_{\varepsilon=0, \omega=0}\) has two zero eigenvalues and a positive eigenvalue \(\frac{1}{c}(1 + \bar{u}_1)^p\), therefore \(M_{\varepsilon=0, \omega=0}\) is repelling. The dynamics on \(M_{\varepsilon=0, \omega=0}\) is given by:

\[
\begin{cases}
\frac{du_1}{d\xi} = u_2, \\
\frac{du_2}{d\xi} = u_1 - cu_2 - \frac{\gamma}{1 + e^{-\beta(u_1 - 1)}} u_1(1 - u_1),
\end{cases}
\]

or, equivalently by,

\[
0 = \frac{d^2u_1}{d\xi^2} + c \frac{du_1}{d\xi} - u_1 + \frac{\gamma}{1 + e^{-\beta(1+u_1)^p-1}} u_1(1 - u_1).
\]

Recall that in the original variables \(u_1 = u\), so the latter equation is a traveling wave equation for the scalar partial differential equation:

\[
u_t = u_{xx} - u + \frac{\gamma}{1 + e^{-\beta(1+u)^p-1}} u(1 - u).
\]

Equation (54) is a PDE of a Fisher–KPP type,\(^9,14\) at least, for some parameter regimes. To streamline the current proof, we describe these regimes later in this section.

The existence of fronts is well known for the Fisher–KPP equation. In particular, it is proved by a trapping region argument that for \(c \geq 2\sqrt{f''(0)} = \sqrt{2(\gamma - 2)}\) there is a heteroclinic orbit that converges to its asymptotic limits in a monotone way and that is a representation of a monotone front. These heteroclinic orbits are formed by the intersection of the one-dimensional unstable manifold of the equilibrium at \((\bar{u}, 0)\) and the two-dimensional stable manifold of the equilibrium \((0, 0)\) in the two-dimensional phase space. By dimension counting, this intersection is transversal.

Since the set \(M_{\varepsilon=0, \omega=0}\) described in (50) is normally hyperbolic, by Fenichel’s theory there is an invariant manifold of (46) which is an \(O(\omega)\)-order perturbation \(M_{\varepsilon=0, \omega}\) of \(M_{\varepsilon=0, \omega=0}\) which is also normally repelling. The flow on that two-dimensional manifold \(M_{\varepsilon=0, \omega}\) is an \(O(\omega)\)-order perturbation of the flow given by (16).

In the perturbed system (45), or equivalently (46), with a sufficiently small \(\omega > 0\), the equilibrium \((0,0,1)\) is a saddle with two-dimensional stable manifold and one-dimensional unstable manifold. To show that, we linearize (46) about the equilibrium \((0,0,1)\):

\[
\begin{cases}
\frac{du}{dz} = \omega u_1, \\
\frac{du_1}{dz} = -\omega \left(\frac{\gamma}{2} - 1\right) u - \omega cu_1, \\
\frac{dv}{dz} = \frac{p}{c} u + \frac{1}{c} v,
\end{cases}
\]

and calculate the eigenvalues of the linear operator defined by the right-hand side of this system. For \(\gamma > 2\), it has two negative eigenvalues \((-\omega c \pm \sqrt{\omega^2 c^2 - 2\omega^2(\gamma - 2)})/2\) and a positive eigenvalue \(\frac{1}{c}\). On the other hand, the eigenvalues of the linearization of (46) about the equilibrium \((\bar{u}, 0, \bar{v})\) can be deduced from the slow–fast structure of the system (46). Since the slow manifold...
is normally repelling and this equilibrium on the slow manifold is a saddle, then, for small \( \omega \), this equilibrium will have two positive eigenvalues and one negative eigenvalue. Thus, the equilibrium \((\bar{u}, 0, \bar{v})\) has a one-dimensional stable manifold and a two-dimensional unstable manifold.

Any solution of \((46)\) approaching \((1, 0, 0)\) does so while staying on the set \(M_{\varepsilon=0,\omega}\) since this set is repelling. The solution that belongs to \(M_{\varepsilon=0,\omega}\) and leaves \((\bar{u}, 0, \bar{v})\) must follow the direction within the two-dimensional unstable manifold \(W^{u}(\bar{u}, 0, \bar{v})\) that is aligned with \((50)\). Indeed, one of the unstable eigendirections of \((\bar{u}, 0, \bar{v})\) is transversal to \(M_{\varepsilon=0,\omega}\), so the intersection of \(W^{u}(\bar{u}, 0, \bar{v})\) with the set \((50)\) is one-dimensional. We further consider the intersection of this one-dimensional set with the two-dimensional stable manifold \(W^{s}(1; 0; 0)\) and notice that it is by dimension counting transversal. Thus, for a sufficiently small \(\omega > 0\), this intersection persists as a transversal intersection and thus, a heteroclinic orbit for \((45)\), or equivalently \((46)\), is formed.

We now recall that the set \(M_{\varepsilon=0,\omega}\) given by \((44)\) is normally hyperbolic and attracting. The normal hyperbolicity of \(M_{\varepsilon=0,\omega}\) implies that in the full system \((42)\), there exists an invariant manifold \(M_{\varepsilon,\omega}\) which is an \(O(\varepsilon)\)-order perturbation of \(M_{\varepsilon=0,\omega}\) and as such converges to \(M_{\varepsilon=0,\omega}\) in the limit \(\varepsilon \to 0\). For sufficiently small \(\varepsilon\), it is also normally attracting and the flow generated by \((42)\) on \(M_{\varepsilon,\omega}\) is an \(O(\varepsilon)\)-order perturbation of the limiting flow generated by the system \((46)\).

We claim that there exists a heteroclinic orbit of \((42)\) that asymptotically connects equilibria \((0, 0, 1, 0)\) and \((\bar{u}, 0, \bar{v}, 0)\) and which is an \(O(\varepsilon)\)-order perturbation of the heteroclinic orbit that exists on \(M_{\varepsilon=0,\omega}\). According to Ref. 18, any invariant set for the system \((42)\) that is sufficiently close to \(M_{\varepsilon=0,\omega}\) is located on \(M_{\varepsilon,\omega}\). Therefore, both equilibria \((0, 0, 1, 0)\) and \((\bar{u}, 0, \bar{v}, 0)\) belong to \(M_{\varepsilon,\omega}\). Because \(M_{\varepsilon=0,\omega}\) is normally attracting, the two-dimensional unstable manifold of \((\bar{u}, 0, \bar{v}, 0)\) must stay on \(M_{\varepsilon,\omega}\), and so does any orbit that follows that unstable manifold. On the other hand, the intersection of the three-dimensional stable manifold of \((0, 0, 1, 0)\) with \(M_{\varepsilon,\omega}\) is two-dimensional. When \(\varepsilon = 0\), these two two-dimensional sets intersect transversally within the three-dimensional set, and therefore, the intersection persists when a perturbation with a sufficiently small \(\varepsilon\) is introduced.

We complete the proof of Theorem 3 by showing that parameter regimes exist such that Equation \((54)\) is a PDE of a Fisher–KPP type.\(^9\) The Fisher–KPP type equations are PDEs of the form

\[
u_t = \nu_{xx} + f(\nu),
\]

where \(f\) satisfies the following conditions: there are two equilibrium points for the equation, say 0 and \(a\), so \(f(0) = 0, f(a) = 0, f'(0) > 0, f'(a) < 0, f''(u) < 0, \) for \(0 < u < a\). In Equation \((54)\), we have

\[
f(\nu) = -u \left(1 - \frac{Y}{1 + e^{-\frac{1}{(1+u)^p} - 1}}(1 - u)\right),
\]

so \(f(0) = 0, f(\bar{u}) = 0,\) and

\[
f'(0) = -\left(1 - \frac{Y}{2}\right) > 0, \text{ when } Y > 2.
\]
For any $\gamma > 0, \beta > 0$, since
\[
\frac{\gamma}{1+e^{\beta u}} = \frac{1}{1-u} \quad \text{and} \quad \bar{u} < 1,
\]
\[
f'(\bar{u}) = -\bar{u}\left(\frac{1}{1-\bar{u}} + \frac{p\beta}{(1 + \bar{u})^{p+1}}\right) < 0.
\] (56)

Below we show that there are values of $\beta$ and $p$ such that $f''(u) < 0$ for $0 < u < \bar{u}$. To show that, we introduce, for $\beta > 0$, a function:
\[
h(u) = \frac{1}{1 + e^{-\beta \left(\frac{1}{(1+u)^p-1}\right)}}.
\]
The function $h$ is decreasing since:
\[
h'(u) = \frac{-1}{\left(1 + e^{-\beta \left(\frac{1}{(1+u)^p-1}\right)}\right)^2 \beta p e^{-\beta \left(\frac{1}{(1+u)^p-1}\right)} < 0}
\]
and convex since:
\[
h''(u) = \frac{e^{-\beta \left(\frac{1}{(1+u)^p-1}\right)}}{\left(1 + e^{-\beta \left(\frac{1}{(1+u)^p-1}\right)}\right)^2 (1 + u)^{p+2}} \beta p \left(\frac{2 e^{-\beta \left(\frac{1}{(1+u)^p-1}\right)}}{1 + e^{-\beta \left(\frac{1}{(1+u)^p-1}\right)}} - 1\right) + (p + 1) \geq 0.
\]
Here, we took into account the fact that:
\[
\frac{1}{2} \leq \frac{e^{-\beta \left(\frac{1}{(1+u)^p-1}\right)}}{1 + e^{-\beta \left(\frac{1}{(1+u)^p-1}\right)}} < 1, \quad \text{for} \quad u \geq 0.
\]
We next investigate the convexity of the function $f$. We want to find parameter regimes when
\[
f'''(u) = -2\gamma h(u) + 2\gamma(1 - 2u)h'(u) + \gamma u(1 - u)h''(u) < 0.
\]
A straightforward calculation of the derivative and estimates on some terms show that

\[
\frac{1}{\gamma} \left( 1 + e^{-\gamma \left( \frac{1}{1+u} - 1 \right)} \right)^2 \frac{1}{e^{-\gamma \left( \frac{1}{1+u} - 1 \right)}} f''(u) = -2 \frac{1}{e^{-\gamma \left( \frac{1}{1+u} - 1 \right)}} e^{-\gamma \left( \frac{1}{1+u} - 1 \right)}
\]

\[+ \frac{\beta p}{(1 + u)^{p+2}} \left( -2(1 - 2u)(1 + u) + u(1 - u) \frac{\beta p}{(1 + u)^p} e^{-\gamma \left( \frac{1}{1+u} - 1 \right)} - 1 + u(1 - u)(\beta p + p + 1) \right)\]

\[\leq -2 \frac{1}{e^{-\gamma \left( \frac{1}{1+u} - 1 \right)}} \frac{\beta p}{(1 + u)^{p+2}} (-2(1 - 2u)(1 + u) + u(1 - u)(\beta p + p + 1))\]

\[\leq \beta p(3 - (\beta + 1)p)u^2 + (3 + (\beta + 1)p)u - 2(1 + \beta p).
\]

We next show that \( p \) and \( \beta \) exist such that the upper bound obtained above, which is a quadratic expression in \( u \), is negative for \( u \in (0, \bar{u}) \subset (0,1) \).

First, we observe that if \( 3 - (\beta + 1)p > 0 \), then

\[
\beta p(3 - (\beta + 1)p)u^2 + (3 + (\beta + 1)p)u - 2(1 + \beta p) < 0,
\]

when \( \rho_- < u < \rho_+ \), where

\[
\rho_{\pm} = \frac{-3 + (\beta + 1)p \pm \sqrt{(3 + (\beta + 1)p)^2 + 8\beta p(3 - (\beta + 1)p)(1 + \beta p)}}{2\beta p(3 - (\beta + 1)p)}.
\]

Therefore, we want to guarantee that \( (0, \bar{u}) \subset (0,1) \subset (\rho_-, \rho_+) \). A sufficient condition for this inclusion is

\[
\frac{-(3 + (\beta + 1)p) + \sqrt{(3 + (\beta + 1)p)^2 + 8\beta p(3 - (\beta + 1)p)(1 + \beta p)}}{2\beta p(3 - (\beta + 1)p)} \geq 1,
\]

which leads to the expression

\[-\beta(\beta + 1)p^2 + 2(\beta + 1)p + 1 \leq 0.
\]

We consider this condition as quadratic in \( p \). Its roots are given by

\[
\frac{(2\beta + 1) \pm \sqrt{(2\beta + 1)^2 + 4\beta(\beta + 1)}}{2\beta(\beta + 1)}.
\]

so the inequality (57) occurs when

\[
p \geq \frac{(2\beta + 1) + \sqrt{(2\beta + 1)^2 + 4\beta(\beta + 1)}}{2\beta(\beta + 1)}.
\]
The region in $\beta p$-plane described by (65) where the KPP dynamics is guaranteed to be prevalent in the system (41), relative to the curve $p = 3/(\beta + 1)$.

From (4), we get then the following sufficient condition:

$$
\frac{(2\beta + 1) + \sqrt{(2\beta + 1)^2 + 4\beta(\beta + 1)}}{2\beta(\beta + 1)} \leq p < \frac{3}{\beta + 1}.
$$

The region above is not empty if $\beta \geq 2$, as illustrated in Figure 5.

When $3 - (\beta + 1)p < 0$, the inequality (57) holds for $0 < u < \bar{u}$ when either the quadratic expression in (57) has no roots or when the smallest root is larger than 1, since the coefficient of $u^2$ is negative and if the roots (58) are real then they are nonnegative. The first case occurs when the following holds:

$$
(3 + (\beta + 1)p)^2 + 8\beta p(3 - (\beta + 1)p)(1 + \beta p) < 0
$$

and the second holds when:

$$
-(3 + (\beta + 1)p) - \sqrt{(3 + (\beta + 1)p)^2 + 8\beta p(3 - (\beta + 1)p)(1 + \beta p)}
$$

These two regions are complementary to each other in the intersection of

$$
p \geq \frac{(2\beta + 1) + \sqrt{(2\beta + 1)^2 + 4\beta(\beta + 1)}}{2\beta(\beta + 1)} \quad \text{and} \quad p > \frac{3}{\beta + 1}.
$$

Combining this region with the region described in (61), we conclude that the region where (57) holds for $0 < u \leq 1$ and therefore for $0 < u < \bar{u}$ is

$$
p \geq \frac{(2\beta + 1) + \sqrt{(2\beta + 1)^2 + 4\beta(\beta + 1)}}{2\beta(\beta + 1)}.
$$
In Figure 5, the region in (65) is depicted as the union of the regions described by (61) and (64).

In conclusion, we have proved the following statement.

**Proposition 1.** For any $\gamma > 2$, if $\beta > 0$ and $p > 0$ satisfy (65), then Equation (54) is a Fisher–KPP type equation.

On the other hand, we note that since $f''(0) = -\gamma(\beta p + 2)/2 < 0$, then $f''(u) < 0$ for sufficiently small positive values of $u$. Moreover, the solution $\tilde{u}$ of Equation (4) is a locally increasing function $\tilde{u}(\cdot)$ of $\gamma$, while $\tilde{u}(2) = 0$. It is easy to see that if $\tilde{u}$ is sufficiently small, then $f'(u) < 0$ for $0 < u < \tilde{u}$. Therefore, the following statement holds.

**Proposition 2.** For any fixed $\beta > 0$ and $p > 0$, there exists $\gamma_0 = \gamma_0(\beta, p) > 2$ such that for any $2 < \gamma \leq \gamma_0$ Equation (54) is a Fisher–KPP type equation.

**Remark 1.** We point out that the conditions on $p$ and $\beta$ described above are sufficient but not necessary. For any particular $p$ and $\beta$ outside of these intervals, one would have to check if the second derivative of the function

$$f(u) = -u\left(1 - \frac{\gamma}{1 + e^{-\beta(\frac{1}{1+u})p-1}}(1-u)\right)$$

is negative for $0 < u < \tilde{u}$.

**Remark 2.** The reduction of (39) to Equation (54) holds for $p < 0$ as well. Proposition 2 indicates that as long as $\beta p + 1 > 0$ and $\gamma$ is close to 2, Equation (54) is a Fisher–KPP type equation since...
the conditions on $f'$ and $f''$ are satisfied. Therefore, the Fisher–KPP dynamics are important in the case of $p < 0$ as well.

5 | NUMERICAL RESULTS

In this section, we describe some numerical results for the computation of the traveling wave fronts. To compute an approximation to the traveling front on a sufficiently larger finite domain, we used the Crank–Nicolson method, which is an implicit finite difference method that is
second-order accurate in both time and space. The approximation is typically found by discretizing the system (10) on a finite domain \([0, L]\), with zero Neumann boundary conditions. The idea is that if the domain is large enough, the boundary will have no effect on the actual front. We can think of recouping the traveling front on \(\mathbb{R}\) in the limit \(L \to \infty\) (scaling \([0, L] \to [-L, L]\)). To obtain the appropriate solution on the finite domain, a decreasing exponential function is used as the initial condition for \(u\) and a constant function is used as an initial condition for \(v\). More precisely, we consider

\[
\begin{aligned}
u_t &= d_2 \nu_{xx} + cv_x + g(u, v), \\
\nu(0, \tau) &= 0, \quad \nu(L, \tau) = 0,
\end{aligned}
\]

where

\[
f(u, v) = \frac{\omega \gamma u(1 - u)}{1 + e^{-\beta(v - \alpha)}} - \omega u \quad \text{and} \quad g(u, v) = 1 - (1 + u)^p v.
\]

The discretized scheme of the problem has the following form:

\[
\begin{aligned}
M_u u^{l+1} &= N_u u^l + 4f(u^l, v^l)\lambda(\Delta \xi)^2, \\
M_v v^{l+1} &= N_v v^l + 4g(u^l, v^l)\lambda(\Delta \xi)^2,
\end{aligned}
\]

where \(l\) represents the number of time steps, \(\Delta \tau\) represents the size of each time steps, \(u^l\) and \(v^l\) represent vectors of \(u\) and \(v\) at each point of the domain at time step \(l\),

\[
M_u = \begin{pmatrix}
4(1 + \lambda_u) & -\lambda_u(2 + c\Delta \xi) & 0 & \cdots & 0 \\
-\lambda_u(2 - c\Delta \xi) & 4(1 + \lambda_u) & -\lambda_u(2 + c\Delta \xi) & \cdots & 0 \\
0 & \cdots & -\lambda_u(2 - c\Delta \xi) & 4(1 + \lambda_u) & -\lambda_u(2 + c\Delta \xi) \\
0 & \cdots & 0 & -\lambda_u(2 - c\Delta \xi) & 4(1 + \lambda_u)
\end{pmatrix},
\]

where \(\lambda_u = d_1 \frac{\Delta \tau}{(\Delta \xi)^2}\). The matrix \(N_u\) has an analogous definition. The matrices \(M_u\) and \(N_v\) are built similarly, but with \(\lambda_v = d_2 \frac{\Delta \tau}{(\Delta \xi)^2}\). We note that the Neumann boundary conditions are incorporated in the matrices. To find the solution, we solve the discretized system for \(u^{l+1}\) and \(v^{l+1}\) at each time step.

Depending on the values of parameters \(\gamma, \beta, p,\) and \(\omega\), we observed both monotone and non-monotone fronts. The figures below depict typical shapes of the fronts solution. In these calculations, \(\alpha = 1\) and the diffusion constants are \(d_1 = 0.001\) and \(d_2 = 0.002\).
We illustrate small perturbations of the case $\omega = 0$ in Figure 6 where we set $\omega = 0.01$. Figure 7 corresponds to a relatively large value $\omega = 100$. In both cases, we set $c = 2$.

To illustrate the fronts described in Section 4, we take $d_1 = 1$, $d_2 = 0.0001$. The simulations produce a monotone profile illustrated on Figure 8.

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