The $\mathcal{A}$-Stokes approximation for non-stationary problems

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Abstract

Let $\mathcal{A}$ be an elliptic tensor. A function $v \in L^1(I; \text{LD}_{\text{div}}(B))$ is a solution to the non-stationary $\mathcal{A}$-Stokes problem iff

$$\int_Q v \cdot \partial_t \varphi \, dx \, dt - \int_Q \mathcal{A}(\varepsilon(v), \varepsilon(\varphi)) \, dx = 0 \quad \forall \varphi \in C^\infty_0(Q),$$

(0.1)

where $Q := I \times B, B \subset \mathbb{R}^d$ bounded. If the l.h.s. is not zero but small we talk about almost solutions. We present an approximation result in the fashion of the $\mathcal{A}$-caloric approximation for the non-stationary $\mathcal{A}$-Stokes problem. Precisely, we show that every almost solution $v \in L^p(I; W^{1,p}_{\text{div}}(B)), 1 < p < \infty$, to (0.1) can be approximated by a solution to (0.1) in the $L^s(I; W^{1,s}_{\text{div}}(B))$-sense for all $s < p$. So, we extend the stationary $\mathcal{A}$-Stokes approximation from [BrDF] to parabolic problems.

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1. Introduction

A very crucial tool in the analysis of nonlinear partial differential equations is a comparison with solutions to linear problems. This powerful idea firstly appears in the work of De Giorgi [De1] in the context of minimal surfaces. Roughly speaking we consider the nonlinear problem locally as a perturbation of a linear partial differential equation and try to transfer qualitative properties from the linear theory. Solutions to linear problems typically are smooth so we are hoping to approximate solutions to nonlinear problems by solutions to linear ones in an appropriate way in order to establish regularity properties (of the solutions to nonlinear equations).

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Let $\mathcal{A} : \mathbb{R}^{d \times D} \to \mathbb{R}^{d \times D}$ be an elliptic tensor, i.e.

$$\mathcal{A}(\xi, \xi) := \mathcal{A}\xi : \xi \geq \lambda |\xi|^2 \quad \forall \xi \in \mathbb{R}^{d \times D},$$

with $\lambda > 0$ and $B \subset \mathbb{R}^d$ open and bounded with Lipschitz boundary (for instance a ball). We call a function $u \in W^{1,1}(\Omega)$ with

$$\int_B \mathcal{A}(\nabla u, \nabla \varphi) \, dx = 0 \quad \forall \varphi \in C^\infty_0(B)$$

(1.1)

an $\mathcal{A}$-harmonic on $B$. On the other, if we have for some $\delta \ll 1$

$$\left| \int_B \mathcal{A} (\nabla u, \nabla \varphi) \, dx \right| \leq \delta \quad \forall \varphi \in C^\infty_0(B)$$

(1.2)

i.e. the integral $\int_B \mathcal{A}(\nabla u, \nabla \varphi) \, dx$ is small (compared to some norms of $u$ and $\varphi$) we call $u$ almost $\mathcal{A}$-harmonic. De Giorgi’s observation in [De1] was the fact that almost harmonic functions can be approximated by harmonic ones. Precisely, if $u \in W^{1,2}(\Omega)$ is almost harmonic it can be approximated with respect to the $L^2(\Omega)$-norm. Since this usually does not suffice to show regularity one needs in addition a Caccioppoli inequality. It bounds the $L^2(\Omega)$-norm of the gradient in terms of the $L^2(B)$-norm of the function itself. Both together - harmonic approximation and Caccioppoli-inequality - finally yield local $C^{1,\alpha}$-estimates.

Since the pioneering work of De Giorgi a lot of improvement and generalizations have been done (see [DM2] for an overview). For instance the $p$-harmonic approximation was introduced in [DM1] and gives a nonlinear variant considering the $p$-Laplace equation

$$\int_B |\nabla u|^{p-2} \nabla u : \nabla \varphi \, dx = 0 \quad \forall \varphi \in C^\infty_0(\Omega).$$

Here almost solutions in $W^{1,p}(B)$ can be approximated by solutions in the $L^p(B)$-sense. This technique has been improved in [DSV] where an approximation in $W^{1,s}(B)$ for all $s < p$ is possible. Moreover, it applies also to the more general setting of Orlicz spaces. A crucial tool for the approximation result in [DSV] is the Lipschitz truncation method (originally developed in [AF]; it allows to approximate a Sobolev function by a Lipschitz continuous function in a way that they are equal on a large set whose size can be controlled). Based on the $\mathcal{A}$-harmonic approximation and its generalizations a lot of (partial-) regularity results for nonlinear PDE’s have been shown. A nice overview about regularity regularity and irregularity for elliptic problems is given in [Mi].

Let us turn to fluid mechanics. The $p$-Stokes problem - describing the slow stationary flow of a Non-Newtonian fluid [AM, BirAH] - reads as follows: for a given volume force $f: \Omega \to \mathbb{R}^d$ find $(v, \pi)$ such that

$$\begin{cases}
\text{div } S(\varepsilon(v)) = \nabla \pi - f & \text{in } B, \\
\text{div } v = 0 & \text{in } B, \\
v = v_0 & \text{on } \partial B.
\end{cases}$$

(1.3)
Here the nonlinear tensor \( S \) satisfies the \( p \)-growth condition

\[
\lambda (1 + |\varepsilon|^2)^{\frac{p-2}{2}} |\tau|^2 \leq D_S(\varepsilon)(\tau, \tau) \leq \Lambda (1 + |\varepsilon|^2)^{\frac{p-2}{2}} |\tau|^2
\]

for \( \varepsilon, \tau \in \mathbb{S}^d \). The problem firstly appears in the mathematical literature in the work of Ladyshenskaya and Lions (see [La1]-[La3] and [Li]). Regularity results are shown in [MNRR], [Fu2], [KMS], [NW], [DK] and others. In contrast to classical problems in nonlinear PDEs like the \( p \)-Laplace equation we have only control over the symmetric part \( \varepsilon(v) := \frac{1}{2}(\nabla v + \nabla v^T) \) of the gradient and more important we have the side condition \( \text{div} \ v = 0 \). A corresponding approximation theory is developed in [BrDF] and approximates an almost solutions \( v \in W_{0,\text{div}}^1(B) \) to the \( A \)-Stokes problem by a solution in the \( W^1_s \)-sense for all \( s < p \). Let us consider the function spaces

\[
LD(B) := \{ u \in L^1(B) : \varepsilon(u) \in L^1(B) \}, \\
LD_0(B) := \{ u \in LD(B) : u|_{\partial B} = 0 \}, \\
LD_{\text{div}}(B) := \{ u \in LD(B) : \text{div} u = 0 \}, \\
LD_{0,\text{div}}(B) := LD_0(B) \cap LD_{\text{div}}(B),
\]

where \( u|_{\partial B} \) has to be understood in the \( H^{d-1}(\partial B) \)-sense (see [ST]). A function \( v \in LD_{\text{div}}(B) \) is a solution to the \( A \)-Stokes problem iff

\[
\int_B A(\varepsilon(v), \varepsilon(\varphi)) \, dx = 0 \quad \forall \varphi \in C_0^{\infty}(B).
\]  

It is an almost solution if we have

\[
\left| \int_B A(\varepsilon(v), \varepsilon(\varphi)) \, dx \right| \leq \delta \int_B |\varepsilon(v)| \, dx \|\varepsilon(\varphi)\|_{\infty} \quad \forall \varphi \in C_0^{\infty}(B)
\]

with some \( \delta \ll 1 \). Note that the formulation above is the weakest possible as it only requires \( v \in LD(B) \). The approximation result in [BrDF] is based on the solenoidal Lipschitz truncation combined with results developed in [DieLSV] for the \( A \)-harmonic approximation in Orlicz spaces.

In order to study regularity properties of nonlinear parabolic equations Duzaar and Mingione [DuzM2] introduce the \( A \)-caloric approximation which compares almost solutions to the \( A \)-heat equation with its solutions. We call a function \( u \in L^1(I; W^{1,1}(B)) \) with

\[
\int_Q u \cdot \partial_t \varphi \, dx \, dt - \int_Q A(\nabla u, \nabla \varphi) \, dx \, dt = 0 \quad \forall \varphi \in C_0^{\infty}(Q)
\]

\( A \)-caloric on \( Q := I \times B \), where \( I \subset \mathbb{R} \) is a bounded interval. If the left hand side is small we talk about an almost \( A \)-caloric function. In [DuzM2] it is shown that every almost \( A \)-caloric function \( u \in L^p(I; W^{1,p}(B)) \) can be approximated by a \( A \)-caloric function in the parabolic \( L^p \)-sense. This is used to establish partial regularity results for nonlinear parabolic systems (see [DuMiSt] for an
overview). Despite the elliptic setting there is not so much literature available. The aim of the present paper is to develop an approximation theory for non-stationary problems in fluid mechanics in the fashion of the $A$-caloric approximation. Let us be a little bit more precise: A function $\mathbf{v} \in L^1(I; LD_{\text{div}}(B))$ is a solution to the non-stationary $A$-Stokes problem iff
\[
\int_Q \mathbf{v} \cdot \partial_t \phi \, dx \, dt - \int_Q A(\varepsilon(\mathbf{v}), \varepsilon(\phi)) \, dx = 0 \quad \forall \phi \in C_0^{\infty}(Q). \quad (1.7)
\]
It is an almost solution if we have
\[
\left| \int_Q \mathbf{v} \cdot \partial_t \phi \, dx \, dt - \int_Q A(\varepsilon(\mathbf{v}), \varepsilon(\phi)) \, dx \right| \leq \delta \int_Q |\varepsilon(\mathbf{v})| \, dx \|\varepsilon(\phi)\|_\infty \quad (1.8)
\]
for all $\phi \in C_0^{\infty}(Q)$ with some $\delta \ll 1$. The main results of this paper (see Theorem 4.2 in section 4) states that every almost solution $\mathbf{v} \in L^p(I; W^{1,p}_{0,\text{div}}(B))$ to the non-stationary $A$-Stokes problem can be approximated by a solution in the $L^s(I; W^{1,s}(B))$-sense for all $s < p$. So, it extend the result from [BrDF] to non-stationary flows. Again we are able to work with the weakest formulation of almost solutions. The main tool is the solenoidal Lipschitz truncation for parabolic PDE’s which was recently developed in [BrDS]. We present a version of it which is appropriate for our purposes in section 3. In section 2 we present an $L^q$-theory for the non-stationary $A$-Stokes problem in divergence form. It might not be surprising for experts but it is hard to find a reference in literature.
2. $L^q$-theory for the $A$-Stokes system

The aim of this section is to present regularity results for the (non-stationary) $A$-Stokes system depending on the right hand side (in divergence form). Let us fix for this section a bounded domain $\Omega \subset \mathbb{R}^d$ with $C^2$-boundary and a time interval $(0, T)$. The $A$-Stokes problem (in the pressure-free formulation) with right hand side $f \in L^1(\Omega)$ reads as: find $v \in LD_0(B)$ such that

$$\int_{\Omega} A(\varepsilon(v), \varepsilon(\varphi)) \, dx = \int_{\Omega} f \cdot \varphi \, dx \quad \text{for all } \varphi \in C_0^{\infty}(\Omega). \quad (2.1)$$

The right hand side can also be given in divergence form, i.e.

$$\int_{\Omega} A(\varepsilon(v), \varepsilon(\varphi)) \, dx = \int_{\Omega} F : \nabla \varphi \, dx \quad \text{for all } \varphi \in C_0^{\infty}(\Omega) \quad (2.2)$$

for $F \in L^1(B)$. For certain purposes it is convenient to discuss the problem with a fixed divergence. To be precise for $g \in L^1_0(\Omega)$, where

$$L^q_0(\Omega) := \left\{ u \in L^q(\Omega) : \int_{\Omega} f \, dx = 0 \right\},$$

we are seeking for a function $v \in LD_0(B)$ with $\text{div} \, v = g$ satisfying (2.1) or (2.2). We have the following $L^q$-estimates.

**Lemma 2.1.** Let $\Omega \subset \mathbb{R}^d$ be a bounded $C^2$-domain and $1 < q < \infty$.

a) Let $f \in L^q(\Omega)$ and $g \in W^{1,q}(\Omega)$ with $\int_{\Omega} g \, dx = 0$. Then there is a unique solution $w \in W^{2,q} \cap W_0^{1,q}(\Omega)$ to (2.1) such that $\text{div} \, v = g$ and

$$\int_B |\nabla^2 w|^q \, dx \leq c \int_{\Omega} |f|^q \, dx + c \int_{\Omega} |\nabla g|^q \, dx,$$

where $c$ only depends on $A$ and $q$.

b) Let $F \in L^q(\Omega)$ and $g \in L^q_0(\Omega)$. Then there is a unique solution $w \in W_0^{1,q}(B)$ to (2.2) such that $\text{div} \, v = g$ and

$$\int_{\Omega} |\nabla w|^q \, dx \leq c \int_{\Omega} |F|^q \, dx + c \int_{\Omega} |g|^q \, dx,$$

where $c$ only depends on $A$ and $q$.

In case $A = I$ both parts follow from [AmrG1], Thm 4.1. However, the main tool in [AmrG1] is the theory from [AgmDN1, AgmDN2] where very general linear systems are investigated. Hence it is clear that the results also hold in case of an arbitrary elliptic tensor $A$.

Now we turn to the parabolic problem and the first result is a local $L^q$-estimate for weak solutions. In case of the $A$-heat system this follows from the continuity of the corresponding semigroup (see [Sh]). It is also non for the non-stationary Stokes-system (see [Sa] and [Gi]) but not in our setting.
THEOREM 2.2. Let \( f \in L^q(Q_0) \) for some \( q > 2 \), where \( Q_0 := (0, T) \times \mathbb{R}^d \) and let \( v \in L^{1}(0, T; LD_{\text{div}}(\mathbb{R}^d)) \) be a weak solution to
\[
\int_{Q_0} v \cdot \partial_t \phi \, dt - \int_{Q_0} A(\varepsilon(v), \varepsilon(\phi)) \, dx = \int_{Q_0} f \cdot \phi \, dx \, dt \tag{2.3}
\]
for all \( \phi \in C_0^{\infty}([0, T) \times \mathbb{R}^d) \). Then we have \( \nabla^2 v \in L^q_{\text{loc}}(Q_0) \) and there holds
\[
\int_0^T \int_B |\nabla^2 v|^q \, dx \, dt \leq c \int_{Q_0} |f|^q \, dx \, dt
\]
for all balls \( B \subset \mathbb{R}^d \).

Proof. The main ingredient is the proof of the following auxiliary result which has been used in a similar version in [ByWa].

We say that \( Q' = I' \times B' \subset \mathbb{R} \times \mathbb{R}^3 \) is a parabolic cylinder if \( r_{I'} = \alpha r_{I}^2 \). For \( \kappa > 0 \) we define the scaled cylinder \( \kappa Q' := (\kappa I') \times (\kappa B') \). By \( Q \) we denote the set of all parabolic cylinders. We define the parabolic maximal operators \( M \) and \( M_s \) for \( s \in [1, \infty) \) by
\[
(Mf)(t, x) := \sup_{Q' \in (t, x) \in Q} \int_{Q'} |f(\tau, y)| \, d\tau \, dy,
\]
\[
M_s f(t, x) := (M(|f|^s(t, x)))^{\frac{1}{s}}
\]
It is standard [Str] that for all \( q \in (s, \infty] \)
\[
\|M_s f\|_{L^q(\mathbb{R}^{n+1})} \leq c \|f\|_{L^s(\mathbb{R}^{n+1})}. \tag{2.4}
\]
i) We start with interior estimates. Let \( Q_r := Q_r(x_0, t_0) := (t_0 - r^2, t_0 + r^2) \times B_r(x_0) \) be a parabolic cube such that \( 4Q_r \subset Q_0 \) then there holds\(\)
\[
L^{d+1}(Q_r \cap \{M(|f|^2) > N^2_1\}) \geq \varepsilon L^{d+1}(Q_r)
\]
\[
\Rightarrow Q_r \subset \{M(|\nabla^2 v|^2) > 1\} \cap \{M(|f|^2) > \delta^2\}. \tag{2.5}
\]
In fact, we will establish (2.5) by showing
\[
Q_1 \cap \{M(|\nabla^2 v|^2) \leq 1\} \cap \{M(|f|^2) \leq \delta^2\} \neq \emptyset
\]
\[
\Rightarrow L^{d+1}(Q_1 \cap \{M(|\nabla^2 v|^2) > N_1^2\}) < \varepsilon L^{d+1}(Q_1). \tag{2.6}
\]
In order to show (2.6) we compare \( v \) with a solution to a homogeneous problem (with the same boundary data) which is smooth in the interior. So let us define \( h \) as the unique solution to
\[
\begin{cases}
\partial_t h - \text{div} A(\varepsilon(h)) = \nabla \pi_h & \text{in } Q_4, \\
\text{div} h = 0 & \text{in } Q_4, \\
h = v & \text{on } I_4 \times \partial B_4, \\
h(0, \cdot) = v(0, \cdot) & \text{in } B_4.
\end{cases} \tag{2.7}
\]
\[\text{Since } f \in L^2(0, T; L^2_{\text{loc}}(\mathbb{R}^d)) \text{ the standard interior regularity theory implies } \nabla^2 v \in L^2(0, T; L^2_{\text{loc}}(\mathbb{R}^d)) \text{ and } \partial_t v \in L^2(0, T; L^2_{\text{loc}}(\mathbb{R}^d)).\]
We test the difference of both equations with \( \mathbf{v} - \mathbf{h} \). This yields by the ellipticity of \( \mathcal{A} \)

\[
\sup_{t \in \mathcal{I}} \int_{B} |\mathbf{v}(t) - \mathbf{h}(t)|^2 \, dx + \int_{Q} |\mathbf{v}(t) - \mathbf{h}(t)|^2 \, dx \, dt \\
\leq c \int_{Q} |\mathbf{f}|^2 \, dx \, dt + c \int_{Q} |\mathbf{v} - \mathbf{h}|^2 \, dx \, dt.
\]

An application of Korn’s inequality and Gronwall’s lemma implies

\[
\sup_{t \in \mathcal{I}} \int_{B} |\mathbf{v}(t) - \mathbf{h}(t)|^2 \, dx + \int_{Q} |\nabla \mathbf{v} - \nabla \mathbf{h}|^2 \, dx \, dt \leq c \int_{Q} |\mathbf{f}|^2 \, dx \, dt.
\]

(2.8)

Now we choose a cut off function \( \eta \in C_{0}^{\infty}(B) \) with \( 0 \leq \eta \leq 1 \) and \( \eta \equiv 1 \) on \( B \). We test the equation for \( \mathbf{v} - \mathbf{h} \) with \( \text{curl}(\eta^2 \text{curl}(\mathbf{h} - \mathbf{h})) \); cf. [DK]. We gain

\[
\sup_{t \in \mathcal{I}} \int_{B} \eta^2 |\text{curl}(\mathbf{v} - \mathbf{h})|^2 \, dx + \int_{Q} \eta^2 |\nabla \mathbf{v} - \nabla \mathbf{h}|^2 \, dx \, dt \\
\leq c \int_{Q} f \cdot \text{curl}(\eta^2 \text{curl}(\mathbf{h} - \mathbf{h})) \, dx \, dt \\
+ c \int_{Q} |\nabla \mathbf{v} - \nabla \mathbf{h}|^2 \, dx \, dt.
\]

Estimating the term involving \( \mathbf{f} \) by

\[
\int_{Q} f \cdot \text{curl}(\eta^2 \text{curl}(\mathbf{v} - \mathbf{h})) \, dx \, dt \\
\leq c(\kappa) \int_{Q} |\mathbf{f}|^2 \, dx \, dt + \kappa \int_{Q} |\nabla \mathbf{v} - \nabla \mathbf{h}|^2 \, dx \, dt + c \int_{Q} |\nabla \mathbf{v} - \nabla \mathbf{h}|^2 \, dx \, dt,
\]

where \( \kappa > 0 \) is arbitrary, and using the inequality \( |\nabla^2 \mathbf{u}| \leq c |\nabla \mathbf{v}(\mathbf{u})| \) as well as (2.8) shows

\[
\sup_{t \in \mathcal{I}} \int_{B} |\text{curl}(\mathbf{v} - \mathbf{h})(t)|^2 \, dx + \int_{\mathcal{I}} \int_{B} |\nabla^2 \mathbf{v} - \nabla^2 \mathbf{h}|^2 \, dx \, dt \leq c \int_{Q} |\mathbf{f}|^2 \, dx \, dt.
\]

(2.9)

Now, let us assume that (2.6) holds. Then there is a point \( (t_0, x_0) \in Q_1 \) such that

\[
\int_{\mathcal{Q}_{r}(t_0, x_0)} |\nabla^2 \mathbf{v}|^2 \, dx \, dt \leq 1, \quad \int_{\mathcal{Q}_{r}(t_0, x_0)} |\mathbf{f}|^2 \, dx \, dt \leq \delta^2 \quad \forall r > 0. \quad (2.10)
\]

Since \( Q_4 \subset Q_6(t_0, x_0) \) we have

\[
\int_{Q_4} |\nabla^2 \mathbf{v}|^2 \, dx \, dt \leq c, \quad \int_{Q_4} |\mathbf{f}|^2 \, dx \, dt \leq c \delta^2. \quad (2.11)
\]
As \( h \) is smooth we know that
\[
N_0^2 := \sup_{Q_3} |\nabla^2 h| < \infty
\]  
(2.12)
and can conclude (see [ByWa], formula (3.34))
\[
Q_1 \cap \{M(|\nabla^2 v|^2) > N_0^2\} \subset Q_1 \cap \{M(\chi_{Q_3}|\nabla^2 v - \nabla^2 h|^2) > N_0^2\}
\]  
(2.13)
if we set \( N_1^2 := \max\{4N_0^2, 2^{d+2}\} \). To establish (2.13) suppose that
\[
(t, x) \in Q_1 \cap \{M(\chi_{Q_3}|\nabla^2 v - \nabla^2 h|^2) > N_0^2\}.
\]  
(2.14)
If \( r \leq 2 \) we have \( Q_r(t, x) \subset Q_3 \) and gain by (2.12)
\[
\int_{Q_r(t, x)} |\nabla^2 v|^2 \, dx \, dt \leq 2 \int_{Q_r(t, x)} \chi_{Q_3}|\nabla^2 v - \nabla^2 h|^2 \, dx \, dt + 2 \int_{Q_r(t, x)} \chi_{Q_3}|\nabla^2 h|^2 \, dx \, dt \leq 4N_0^2.
\]
If \( r \geq 2 \) we have by (2.10)
\[
\int_{Q_r(t, x)} |\nabla^2 v|^2 \, dx \, dt \leq 2^{d+2} \int_{Q_{2r}(t_0, x_0)} |\nabla^2 v|^2 \, dx \, dt \leq 2^{d+2}.
\]
Combining the both cases yields (2.13) which implies together with the continuity of the maximal function on \( L^2 \), (2.9) and (2.11)
\[
L^{d+1}(Q_1 \cap \{M(|\nabla^2 v|^2) > N_0^2\}) \leq c \frac{c}{N_0^2} \int_{Q_3} |\nabla^2 v - \nabla^2 h|^2 \, dx \, dt \leq c \frac{c}{N_0^2} \delta^2 = \varepsilon L^{d+1}(Q_1).
\]
So we have shown (2.6) which yields (2.5) by a scaling argument.
We gain from Vitali’s covering Theorem and (2.6)
\[
L^{d+1}(Q_r \cap \{M(|\nabla^2 v|^2) > N_1^2\}) \leq 5^{d+1} \varepsilon L^{d+1}(Q_r \cap \{M(|\nabla^2 v|^2) > 1\} \cap \{M(|f|^2) > \delta^2\})
\]  
\[
\leq 5^{d+1} \varepsilon \left( L^{d+1}(Q_r \cap \{M(|\nabla^2 v|^2) > 1\}) + L^{d+1}(Q_r \cap \{M(|f|^2) > \delta^2\}) \right).
\]
Multiplying the equation for \( v \) by small number \( \varrho = \varrho(||f||_q, ||\nabla^2 v||_2) \) we can assume that
\[
L^{d+1}(Q_r \cap \{M(|\nabla^2 v|^2) > N_1^2\}) < \varepsilon
\]  
(2.15)
By induction we can establish that for \( \epsilon := (\frac{10}{9})^{d+2} \epsilon \)
\[
L^{d+1}(Q_r \cap \{\mathcal{M}(|\nabla^2 \mathbf{v}|^2) > N_1^{2k}\})
\]
\[
\leq \epsilon^k L^{d+1}(Q_r \cap \{\mathcal{M}(|\nabla^2 \mathbf{v}|^2) > 1\}) + c \sum_{i=1}^{k} \epsilon^i L^{d+1}(Q_r \cap \{\mathcal{M}(|\mathbf{f}|^2) > \delta^2 N_1^{2(k-i)}\}).
\]

Since \( \mathbf{f} \in L^q(Q_0) \) and \( q > 2 \) we have \( \mathcal{M}(|\mathbf{f}|^2) \in L^{q/2}(Q_0) \) and hence
\[
\int_{Q_r} |\nabla^2 \mathbf{v}|^q \, dx \, dt \leq \int_{Q_r} |\mathcal{M}(|\nabla^2 \mathbf{v}|^2)|^{q/2} \, dx \, dt
\]
\[
\leq c \sum_{k=1}^{\infty} (N_1^{q})^{2k} L^{d+1}(Q_r \cap \{\mathcal{M}(|\nabla^2 \mathbf{v}|^2) > N_1^{2k}\})
\]
\[
\leq c \sum_{k=1}^{\infty} (N_1^{q})^{2k} \epsilon^k L^{d+1}(Q_r \cap \{\mathcal{M}(|\nabla^2 \mathbf{v}|^2) > 1\})
\]
\[
+ c \sum_{k=1}^{\infty} (N_1^{q})^{2k} \sum_{i=1}^{k} \epsilon^i L^{d+1}(Q_r \cap \{\mathcal{M}(|\mathbf{f}|^2) > \delta^2 N_1^{2(k-i)}\})
\]
\[
\leq c \sum_{k=1}^{\infty} (\epsilon N_1^{q})^{2k} \left(1 + \int_{Q_r} |\mathcal{M}(|\mathbf{f}|^2)|^{q/2} \, dx \, dt\right).
\]

If we choose \( \epsilon N_1^{q} < 1 \) the sum is converging and we have \( \nabla^2 \mathbf{v} \in L^q(Q_r) \).
Since the mapping \( \mathbf{f} \mapsto \nabla^2 \mathbf{v} \) is linear we gain the desired estimate
\[
\int_{Q_r} |\nabla^2 \mathbf{v}|^q \, dx \, dt \leq c \int_{Q_0} |\mathbf{f}|^q \, dx \, dt.
\]

ii) Now let \( Q_1 \) be a cube such that \( 4Q_1 \cap (-\infty, 0] \times \mathbb{R}^d \neq \emptyset \). Moreover, assume that \( Q_1 \cap Q_0 \neq \emptyset \). We consider the solution \( \tilde{\mathbf{h}} \) to
\[
\begin{cases}
\partial_t \tilde{\mathbf{h}} - \text{div} \mathcal{A}(\epsilon(\tilde{\mathbf{h}})) = \nabla \pi_{\tilde{\mathbf{h}}} & \text{in } \tilde{Q}_4, \\
\text{div} \mathbf{h} = 0 & \text{in } Q_4, \\
\tilde{\mathbf{h}} = \mathbf{v} & \text{on } I_4 \times \partial B_4, \\
\tilde{\mathbf{h}}(0, \cdot) = 0 & \text{in } B_4,
\end{cases}
\]
where \( I_m := I_m \cap (0, T) \) and \( \tilde{Q}_4 := \tilde{I}_m \times B_m \). We can establish a variant of (2.16) on \( \tilde{Q}_4 \). Now we have \( \sup_{Q_1} |\nabla^2 \tilde{\mathbf{h}}|^2 < \infty \) due to the smooth initial datum of \( \tilde{\mathbf{h}} \) (recall that \( \mathbf{v}(0, \cdot) = 0 \text{ a.e.} \)). So we can finish the proof as before and gain \( \nabla^2 \mathbf{v} \in L^q(Q_1) \). This implies the claim.

iii) The situation \( 4Q_1 \cap [T, \infty) \times \mathbb{R}^d \neq \emptyset \) is uncritical again and we can assume that ii) and iii) do not occur for the same cube (by choosing sufficiently small cubes)

Covering the set \((0, T) \times B\) by smaller cubes and combing i)-iii) yield the desired estimate.
Corollary 2.3. Under the assumptions of Theorem 2.2 we have for all balls $B \subset \mathbb{R}^d$ the following estimates.

a) There is $c$ independent of $T$ such that

$$
\int_0^T \int_B \left( \left\| \frac{\mathbf{v}}{T} \right\|^q + \left\| \frac{\nabla \mathbf{v}}{\sqrt{T}} \right\|^q + \left\| \nabla^2 \mathbf{v} \right\|^q \right) \, dx \, dt \leq c \int_{Q_0} |\mathbf{f}|^q \, dx \, dt.
$$

b) We have $\partial_t \mathbf{v} \in L^q(0, T; L^q_{\text{loc}}(\mathbb{R}^d))$ together with

$$
\int_0^T \int_B |\partial_t \mathbf{v}|^q \, dx \, dt \leq c \int_{Q_0} |\mathbf{f}|^q \, dx \, dt,
$$

where $c$ does not depend on $T$.

c) There is $\pi \in L^q((0, T), W^{1,q}_{\text{loc}}(\mathbb{R}^d))$ such that

$$
\int_{Q_0} \mathbf{v} \cdot \partial_t \varphi \, dx \, dt - \int_{Q_0} A(\varepsilon(\mathbf{v}), \varepsilon(\varphi)) \, dx = \int_{Q_0} \pi \, \text{div} \, \varphi \, dx \, dt + \int_{Q_0} \mathbf{f} \cdot \varphi \, dx \, dt
$$

for all $\varphi \in C_0^\infty(0, T) \times \mathbb{R}^d$.

d) There is $c$ independent of $T$ such that

$$
\int_0^T \int_B \left( \left\| \frac{\pi}{\sqrt{T}} \right\|^q + \left\| \nabla \pi \right\|^q \right) \, dx \, dt \leq c \int_{Q_0} |\mathbf{f}|^q \, dx \, dt.
$$

Proof. The estimate in a) is a simple scaling argument. Having a solution $\mathbf{v}$ defined on $(0, T) \times \mathbb{R}^d$ we gain a solution $\tilde{\mathbf{v}}$ on $(0, 1) \times \mathbb{R}^d$ by setting

$$
\tilde{\mathbf{v}}(s, x) := \frac{1}{T} \mathbf{v}(Ts, \sqrt{T}x).
$$

Now we apply Theorem 2.2 to $\tilde{\mathbf{v}}$. The constant which appears is independent of $T$. Transforming back to $\mathbf{v}$ yields the claimed inequality.

b) For $\varphi \in C_0^\infty(Q_0)$ with $\varphi(t, x) = \tau(t) \psi(x)$ where $\varphi \in C_0^\infty(G)$ ($G \in \mathbb{R}^d$ a bounded Lipschitz domain) we have

$$
\int_{Q_0} \mathbf{v} \cdot \partial_t \varphi \, dx \, dt = \int_0^T \partial_t \tau \int_{\mathbb{R}^d} \mathbf{v} \cdot (\psi_{\text{div}} + \nabla \psi) \, dx \, dt
$$

$$
= \int_0^T \partial_t \tau \int_{\mathbb{R}^d} \mathbf{v} \cdot \psi_{\text{div}} \, dx \, dt
$$

$$
= \int_{Q_0} \mathbf{v} \cdot \partial_t \psi_{\text{div}} \, dx \, dt
$$
\[
\phi_{\text{div}} := \phi - \nabla \Delta_{-1}^{-1} \div \phi \quad \text{and} \quad \Psi := \Delta_{-1}^{-1} \div \phi
\]

Here we took into account that \( \Psi|_{\partial G} = 0 \). We proceed by

\[
\int_{Q_0} v \cdot \partial_t \phi \, dx \, dt = - \int_0^T \int_G \div (A(\varepsilon(v))) \cdot \phi_{\text{div}} \, dx \, dt
\]

\[
\leq c \left( \int_0^T \int_G |\nabla^2 v|^q \, dx \, dt \right)^{\frac{1}{q}} \left( \int_0^T \int_G |\phi_{\text{div}}|^{q'} \, dx \, dt \right)^{\frac{1}{q'}}
\]

\[
\leq c \left( \int_{Q_0} |f|^q \, dx \, dt \right)^{\frac{1}{q}} \left( \int_0^T \int_G |\phi|^{q'} \, dx \, dt \right)^{\frac{1}{q'}}.
\]

In the last step we used the estimate from Theorem 2.2 and continuity of \( \nabla \Delta_{-1}^{-1} \div \) on \( L^q(G) \). Duality implies \( \partial_t v \in L^q(0, T; L^q_{\text{loc}}(\mathbb{R}^d)) \) and we can introduce the pressure function \( \pi \in L^q(0, T; L^q_{\text{loc}}(\mathbb{R}^d)) \) as claimed in b) by De Rahm’s Theorem. Using the equation for \( v \) and the estimates in a) and b) we gain

\[
\int_Q |\nabla \pi|^q \, dx \, dt \leq c \int_{Q_0} |f|^q \, dx \, dt.
\]

The estimate for \( \pi \) in d) follows again by scaling.

**Corollary 2.4.** Let \( f \in L^q(Q_0^+) \) for some \( q > 2 \) where \( Q_0^+ := (0, T) \times \mathbb{R}^d_+ \), \( \mathbb{R}^d_+ = \mathbb{R}^d \cap [x_d > 0] \), and let \( v \in L^1((0, T); L^1(\mathbb{R}^d_+)) \) with \( v|_{x_d=0} = 0 \) be a weak solution to

\[
\int_{Q_0^+} v \cdot \partial_t \phi \, dx \, dt - \int_{Q_0^+} A(\varepsilon(v), \varepsilon(\phi)) \, dx = \int_{Q_0^+} f \cdot \phi \, dx \, dt \tag{2.17}
\]

for all \( \phi \in C_0^\infty(\mathbb{R}^d_+ \times [0, T]) \). Then the results from Theorem 2.2 and Corollary 2.3 hold for \( v \) for all half balls \( B^+(z) \subset \mathbb{R}^d \) with \( z_d = 0 \).

**Proof.** We will show a variant of the \( L^q \)-estimate from Theorem 2.2 on half balls, i.e.

\[
\int_0^T \int_{B^+} |\nabla^2 v|^q \, dx \, dt \, dt c \leq \int_0^T \int_{4B^+} |f|^q \, dx \, dt. \tag{2.18}
\]

From this we can follow estimates in the fashion of Corollary 2.3 as done there. In order to establish (2.18) we will proceed as in the proof of Theorem 2.2 replacing all balls with half ball. So let \( Q_1 \subset \mathbb{R}^{d+1} \) such that \( 4Q_1 \subset Q_0 \) (the other situation can be shown along the modifications indicated at the end of the
proof of Theorem 2.2. Moreover, assume that \( Q_1 = I_1 \times B_1(z) \) where \( z_d = 0 \).

We compare \( v \) with the unique solution \( h^+ \) to

\[
\begin{cases}
\partial_t h^+ - \text{div} \mathcal{A}(\varepsilon(h^+)) = \nabla \pi_{h^+} & \text{in } Q_1^+, \\
\text{div} h^+ = 0 & \text{in } Q_1^+, \\
h^+ = v & \text{on } I_4 \times \partial B_4^+, \\
h^+(0, \cdot) = v(0, \cdot) & \text{in } B_4^+.
\end{cases}
\]

(2.19)

We gain a version of the estimate (2.8) on half-balls. In fact, there holds

\[
\sup_{t \in I_4} \int_{B_4^+} |v - h|^2 \, dx + \int_{Q_4^+} |\nabla v - \nabla h|^2 \, dx \, dt \leq c \int_{Q_4^+} |f|^2 \, dx \, dt.
\]

(2.20)

Unfortunately, we cannot insert curl(\( v - h \)) as \( \partial_d (v - h) \) is not necessarily zero on \( |x_d| = 0 \). So we need some more subtle arguments. First we insert \( \partial_t (v - h) \) which yields

\[
\int_{Q_4^+} |\partial_t (v - h)|^2 \, dx + \sup_{t \in I_4} \int_{B_4^+} |\nabla v - \nabla h|^2 \, dx \leq c \int_{Q_4^+} |f|^2 \, dx \, dt.
\]

(2.21)

We can introduce the pressure terms \( \pi_v, \pi_{h^+} \in L^q(I_4, L_0^q(B_4^+)) \) in the equations for \( v \) and \( h^+ \) and show

\[
\int_{Q_4^+} |\pi_v - \pi_{h^+}|^2 \, dx \leq c \int_{Q_4^+} |f|^2 \, dx \, dt.
\]

(2.22)

Estimate (2.22) is a consequence of the continuity of Bog : \( L_0^q(B_4^+) \to W_{0,1}^{1,2}(B_4^+) \) (see Bog) and (2.21): for \( \varphi \in C_0^\infty(Q_4^+) \) there holds

\[
\int_{Q_4^+} (\pi_v - \pi_{h^+}) \varphi \, dx \, dt = \int_{Q_4^+} (\pi_v - \pi_{h^+}) \text{div} \text{Bog}(\varphi - \varphi_{B_4^+}) \, dx \, dt
\]

\[
= \int_{Q_4^+} \mathcal{A}(\varepsilon(v - h^+), \varepsilon(\text{Bog}(\varphi - \varphi_{B_4^+}))) \, dx \, dt
\]

\[
- \int_{Q_4^+} \partial_t (v - h^+) \cdot \text{Bog}(\varphi - \varphi_{B_4^+}) \, dx \, dt
\]

\[
\leq c \Big( \|\nabla (v - h^+)\|_2 + \|\partial_t (v - h^+)\|_2 \Big) \|\nabla \text{Bog}(\varphi - (\varphi)_{B_4^+})\|_2
\]

\[
\leq c \left( \int_{Q_4^+} |f|^2 \, dx \, dt \right)^{\frac{1}{2}} \left( \int_{Q_4^+} |\varphi|^2 \, dx \, dt \right)^{\frac{1}{2}}.
\]

Now we insert \( \partial_\gamma (\eta^2 \partial_\gamma (v - h)) \) for \( \gamma \in \{1, \ldots, d-1\} \), where \( \eta \in C_0^\infty(B_4) \) with \( 0 \leq \eta \leq 1 \) and \( \eta \equiv 1 \) on \( B_3 \). This yields together with (2.20)-(2.22)

\[
\int_{Q_4^+} |\nabla (v - h)|^2 \, dx \leq c \int_{Q_4^+} |f|^2 \, dx \, dt,
\]

(2.23)
where $\tilde{\nabla} := (\partial_1, \ldots, \partial_{d-1})$. Finally, the only term which is missing is $\partial_2^2 (v - h)$. On account of $\text{div}(v - h) = 0$ we gain (cf. BKR)

$$|\partial_2^2 (v - h)| \leq c \left( |\tilde{\nabla}(\pi_v - \pi_h^+)| + |\tilde{\nabla}(v - h)| + |f| \right). \quad (2.24)$$

So we have to estimate derivatives of the pressure. In fact we have

$$\int_{Q_3^+} |\tilde{\nabla}(\pi_v - \pi_h^+)|^2 \, dx \leq c \int_{Q_3^+} |f|^2 \, dx \, dt. \quad (2.25)$$

We can show this similarly to the proof of (2.22) replacing $\varphi$ by $\partial_2 \varphi$ and using (2.23). Combining (2.23)-(2.25) implies

$$\int_{Q_3^+} |\nabla^2 v - \nabla^2 h|^2 \, dx \leq c \int_{Q_3^+} |f|^2 \, dx \, dt.$$

Moreover, we know $\sup_{Q_3^+} |\nabla^2 h^+|^2 < \infty$. Note that $h^+ = 0$ on $Q_3 \cap [x_d = 0]$. This allows to show $\nabla^2 v \in L^q(Q_1^+)$ as in the proof of Theorem 2.2.

**THEOREM 2.5.** Let $Q := (0, T) \times \Omega$ with a bounded domain $\Omega \subset \mathbb{R}^d$ having a $C^2$-boundary. Let $f \in L^q(Q)$ for some $q > 2$, where $q \in (1, \infty)$. Then there is a unique weak solution $v \in L^\infty(0, T; L^2(B)) \cap L^q(0, T; W_0^{1,q}(\Omega))$ to

$$\int_{Q} v \cdot \partial_t \varphi \, dx \, dt - \int_{Q} A(\varepsilon(v), \varepsilon(\varphi)) \, dx = \int_{Q} f \cdot \varphi \, dx \, dt \quad (2.26)$$

for all $\varphi \in C_0^{\infty}((0, T) \times \Omega)$ such that $\nabla^2 v \in L^q(Q)$. Moreover, we have

$$\int_{Q} |\nabla^2 v|^q \, dx \, dt \leq c \int_{Q} |f|^q \, dx \, dt.$$

**Proof.** Due to the local $L^q$-theory for the whole space problem and the half-space problem which follow from Corollary 2.3 and Corollary 2.4 (with the right scaling in $T$) the proof follows exactly as in [So], Thm. 4.1, in the case $A = I$. Note that $L^q$-estimates for the stationary problem on bounded domains with given divergence are stated in Lemma 2.1.

In order to treat problems with right hand side in divergence form we consider the $A$-Stokes operator

$$\mathcal{A}_q := -P_q \text{div} A(\varepsilon(\cdot)),$$

where $P_q$ is the Helmholtz projection from $L^q(B)$ into $L^q_{\text{div}}(B)$. The latter one is defined by

$$L^q_{\text{div}}(B) := C_0^{\infty}(B)^d_{\|v\|}.$$
The Helmholtz-projection $P_q u$ of a function $u \in L^q(B)$ can be defined as $P_q u := u - \nabla h$, where $h$ is the solution to the Neumann-problem
\[
\begin{align*}
\Delta h &= \text{div } u \quad \text{on } B, \\
N_B \cdot (\nabla h - u) &= 0 \quad \text{on } \partial B.
\end{align*}
\]

The $\mathcal{A}$-Stokes operator $\mathcal{A}_q$ enjoys the same properties than the Stokes operator $\mathcal{A}_q$ (see for instance [GalSS]).

For the $\mathcal{A}$-Stokes operator it holds $\mathcal{D}(\mathcal{A}_q) = W^{1,q}_0 \cap W^{2,q}(\Omega)$ and
\[
\begin{align*}
\|u\|_{2,q} &\leq c_1 \|\mathcal{A}_q u\|_q \leq c_2 \|u\|_{2,q}, \quad u \in \mathcal{D}(\mathcal{A}_q), \tag{2.27} \\
\int \mathcal{A}_q u \cdot w \, dx &= \int \mathcal{A}_q w \, dx \quad u \in \mathcal{D}(\mathcal{A}_q), \quad w \in \mathcal{D}(\mathcal{A}_q'). \tag{2.28}
\end{align*}
\]

Inequality (2.27) is a consequence of Lemma [2.1]a) and the continuity of $P_q$.

Finally, the inverse operator $\mathcal{A}_q^{-\frac{1}{2}} : L^q_\text{div}(\Omega) \to W^{1,q}_0(\Omega)$ is defined and it holds
\[
\begin{align*}
\|\nabla \mathcal{A}_q^{-\frac{1}{2}} u\|_q &\leq c \|u\|, \quad u \in \mathcal{D}(\mathcal{A}_q^{-\frac{1}{2}}), \tag{2.31} \\
\int \mathcal{A}_q^{-\frac{1}{2}} u \cdot w \, dx &= \int u \cdot \mathcal{A}_q^{-\frac{1}{2}} w \, dx \quad u \in \mathcal{D}(\mathcal{A}_q^{-\frac{1}{2}}), \quad w \in \mathcal{D}(\mathcal{A}_q^{-\frac{1}{2}}). \tag{2.32}
\end{align*}
\]

From the definition of the square root of an positive self-adjoint operator follows also that
\[
\begin{align*}
\mathcal{A}_q^{\frac{1}{2}} : W^{2,q'} \cap W^{1,q'}_0(\Omega) &\to W^{1,q'}_0(\Omega), \\
\mathcal{A}_q^{-\frac{1}{2}} : W^{1,q'}_0(\Omega) &\to W^{2,q'} \cap W^{1,q'}_0(\Omega),
\end{align*}
\]

with
\[
\begin{align*}
\|\nabla \mathcal{A}_q^{\frac{1}{2}} u\|_q &\leq c \|u\|_{2,q}, \quad u \in W^{2,q'} \cap W^{1,q'}_0(\Omega), \tag{2.33} \\
\|\nabla \mathcal{A}_q^{-\frac{1}{2}} u\|_q &\leq c \|u\|_q, \quad u \in W^{1,q'}_0(\Omega). \tag{2.34}
\end{align*}
\]

Finally we state the main result of this section.

**THEOREM 2.6.** Let $Q := (0, T) \times \Omega$ with a bounded domain $\Omega \subset \mathbb{R}^d$ having a $C^2$-boundary. Let $F \in L^q(Q)$, where $q \in (1, \infty)$. There is a unique solution $w \in L^q(0, T; W^{1,q}_0(\Omega))$ to
\[
\int_Q w \cdot \partial_t \varphi \, dx \, dt - \int_Q A(\varepsilon(w), \varepsilon(\varphi)) \, dx \, dt = \int_Q F : \nabla \varphi \quad \tag{2.35}
\]
for all $\varphi \in C_{0,\text{div}}^\infty ([0,T) \times \Omega)$. Moreover we have
\[
\int_Q |\nabla w|^q \, dx \, dt \leq c \int_Q |F|^q \, dx \, dt,
\]
where $c$ only depends on $A$ and $q$.

Proof. Let us first assume that $q > 2$. Then Theorem 2.5 applies. We set $f := \mathcal{A}_q^{-\frac{1}{2}} \text{div} F$ which is defined via the duality
\[
\int_{\Omega} \mathcal{A}_q^{-\frac{1}{2}} \text{div} F : \varphi \, dx = \int_{\Omega} F : \nabla \mathcal{A}_q^{-\frac{1}{2}} \varphi \, dx, \quad \varphi \in C_{0,\text{div}}^\infty (\Omega),
\]
using (2.32). So we gain $f \in L^q(\Omega)$ with
\[
\|f\|_q \leq c \|F\|_q. \tag{2.36}
\]
We define $\tilde{w} \in L^q(0,T;W_{0,\text{div}}^{1,q}(\Omega))$ as the unique solution to
\[
\int_Q \tilde{w} : \partial_t \varphi \, dx \, dt - \int_Q \mathcal{A}(\varepsilon(\tilde{w}), \varepsilon(\varphi)) \, dx \, dt = \int_Q f : \varphi \, dx \, dt \tag{2.37}
\]
for all $\varphi \in C_{0,\text{div}}^\infty ([0,T) \times \Omega)$. Theorem 2.5 yields $\tilde{w} \in L^q(0,T;W^{2,q}(\Omega))$ and
\[
\|\tilde{w}\|_{2,q} \leq c \|f\|_q. \tag{2.38}
\]
We want to return to the original problem and set $w := \mathcal{A}_q^{\frac{1}{2}} \tilde{w}$ thus $w \in L^q(0,T;W_{0,\text{div}}^{1,q}(\Omega))$. Since $\mathcal{A}_q^{\frac{1}{2}} : W_{0,\text{div}}^{1,q'} \cap W^{2,q'}(\Omega) \to W_{0,\text{div}}^{1,q'}(\Omega)$ we can replace $\varphi$ by $\mathcal{A}_q^{\frac{1}{2}} \varphi$ in (2.37). This implies using (2.30) and the definition of $f$
\[
\int_Q w : \partial_t \varphi \, dx \, dt + \int_Q \text{div} \mathcal{A}(\varepsilon(w)) : \mathcal{A}_q^{\frac{1}{2}} \varphi \, dx \, dt = \int_Q F : \nabla \varphi
\]
for all $\varphi \in C_{0,\text{div}}^\infty (Q)$. On account of $\mathcal{A}_q^{\frac{1}{2}} \varphi \in W_{0,\text{div}}^{1,q'}(\Omega)$ and $\mathcal{A}_q^{\frac{1}{2}} \tilde{w} \in W_{0,\text{div}}^{1,q}(\Omega)$ we gain due to (2.30)
\[
\int_Q \text{div} \mathcal{A}(\varepsilon(w)) : \mathcal{A}_q^{\frac{1}{2}} \varphi \, dx \, dt = \int_Q \mathcal{A}_q^{\frac{1}{2}} \tilde{w} : \mathcal{A}_q^{\frac{1}{2}} \varphi \, dx \, dt = \int_Q \mathcal{A}_q^{\frac{1}{2}} \tilde{w} : \mathcal{A}_q^{\frac{1}{2}} \varphi \, dx \, dt
\]
\[
= \int_Q \text{div} \mathcal{A}(\varepsilon(\varphi)) \, dx \, dt = - \int_Q \mathcal{A}(\varepsilon(w), \varepsilon(\varphi)) \, dx \, dt
\]
using (2.30) and $w \in W_{0,\text{div}}^{1,q}(\Omega)$. This shows that $w$ is the unique solution to (1.2). Moreover, we obtain the desired regularity estimate via
\[
\int_Q |\nabla w|^q \, dx \, dt \leq c \int_Q |\mathcal{A}_q^{\frac{1}{2}} w|^q \, dx \, dt = c \int_Q |\mathcal{A}_q^{\frac{1}{2}} \tilde{w}|^q \, dx \, dt
\]
\[
\int_Q |\nabla^2 \tilde{w}|^q \leq c \int_Q |f|^q \, dx \, dt \\
\leq c \int_Q |F|^q \, dx \, dt
\]

as a consequence of (2.29), the definition of \( w \), (2.27), (2.38), and (2.36). A simple scaling argument shows that the inequality is independent of the diameter of \( I \) and \( B \). So we have shown the claim for \( q > 2 \).

The case \( q = 2 \) follows easily from a priori estimates and Korn’s inequality. So let us assume that \( q < 2 \). Duality arguments show that

\[
\int_Q |\nabla w|^q \, dx \, dt = \sup_{G \in L^{q'}(Q)} \left[ \int_Q \nabla w : G \, dx \, dt - \int_Q |G|^q \, dx \, dt \right].
\]

For a given \( G \in L^{q'}(Q) \) let \( z_G \) be the unique \( L^{q'}(0, T; W^{-1,q'}_0(\Omega)) \)-solution to

\[
-\int_Q z_G : \partial_t \xi \, dx \, dt + \int_Q A(\varepsilon(z), \varepsilon(\xi)) \, dx \, dt = \int_Q G : \nabla \xi \, dx \, dt
\]

for all \( \xi \in C_0^{\infty}(Q) \). Its existence together with the estimate

\[
\int_Q |\nabla z_G|^q \, dx \, dt \leq c \int_Q |G|^q \, dx \, dt
\]

follows from the first part of the proof as \( q' > 2 \). This, the density of \( C_0^{\infty}(Q) \) and \( \partial_t z \in L^{q'}(0, T; W^{-1,q'}_0(\Omega)) \) yield

\[
\int_Q |\nabla w|^q \, dx \, dt \\
\leq c \sup_{G \in L^{q'}(Q)} \left[ \int_Q A_0(\varepsilon(w), \varepsilon(z_G)) \, dx \, dt + \int_Q \partial_t z_G : u \, dx \, dt - \int_Q |\nabla z_G|^{q'} \, dx \, dt \right] \\
\leq c \sup_{\xi \in C_0^{\infty}(Q)} \left[ \int_Q A_0(\varepsilon(w), \varepsilon(\xi)) \, dx \, dt - \int_Q u : \partial_t \xi \, dx \, dt - \int_Q |\nabla \xi|^{q'} \, dx \, dt \right].
\]

The equation for \( w \) and Young’s inequality imply

\[
\int_Q |\nabla w|^q \, dx \, dt \leq c \sup_{\xi \in C_0^{\infty}(Q)} \left[ \int_Q F : \nabla \xi \, dx \, dt - \int_Q |\nabla \xi|^{q'} \, dx \, dt \right] \\
\leq c \int_Q |F|^q \, dx \, dt
\]

and hence the claim. \( \square \)
3. Solenoidal Lipschitz truncation

The purpose of the Lipschitz truncation technique is to approximate a Sobolev function \(u \in W^{1,p}\) by \(\lambda\)-Lipschitz functions \(u_\lambda\) that coincides with \(u\) up to a set of small measure. The functions \(u_\lambda\) are constructed nonlinearly by modifying \(u\) on the level set of the Hardy-Littlewood maximal function of the gradient \(\nabla u\).

This idea goes back to Acerbi and Fusco \([AF]\). Lipschitz truncations are used in various areas of analysis: calculus of variations, in the existence theory of partial differential equations, and in the regularity theory. We refer to [DMS] for a longer list of references. The Lipschitz truncation in the context of parabolic PDE’s can be found in [KinL] and [DRW]. The solenoidal Lipschitz truncation in the non-stationary setting is introduced in [BrDS]. We present a version of it which is appropriate for our purposes. For notational simplicity we assume \(d = 3\) and refer to [BrDS] (remark 2.1) for the higher dimensional case.

**THEOREM 3.1.** Let \(u \in L^\sigma(I; W^{1,\sigma}_{\text{div}}(B)) \cap L^\infty(I; L^\sigma(B))\) with \(\partial_t u = \text{div} H\) in \(D''_{\text{div}}(Q)\) with \(H \in L^\sigma(Q)\). Then for every \(m_0 \gg 1\) and \(\gamma > 0\) there exist \(\lambda \in [2^{m_0\gamma}, 2^{2m_0\gamma}]\) and a function \(u_\lambda\) with the following properties

(a) It holds \(u_\lambda \in L^\infty(I, W^{1,\infty}_{-1,\sigma}(B))\) with \(\|\nabla u_\lambda\|_\sigma \leq c\lambda\).

(b) We have

\[\chi^\sigma \frac{Q \cap \{u_\lambda \neq u\}}{|Q|} \leq \frac{c}{m_0} \left( \int_Q r_B^- |u|^\sigma + |\nabla u|^\sigma \ dx \ dt + \int_Q |H|^\sigma \ dx \ dt \right)\]

(c) It holds

\[\int_Q |u_\lambda|^\sigma \ dx \ dt \leq c \left( \int_Q |u|^\sigma + \int_Q r_B^- |H|^\sigma \ dx \ dt \right),\]

\[\int_Q |\nabla u_\lambda|^\sigma \ dx \ dt \leq c \left( \int_Q r_B^- |u|^\sigma + |\nabla u|^\sigma \ dx \ dt + \int_Q |H|^\sigma \ dx \ dt \right)\]

(d) We have \(\partial_t (u - u_\lambda) \in L^\sigma(\frac{1}{2}I, W^{-1,\sigma}(\frac{1}{2}B))\) and

\[-\int_{\frac{1}{2}Q} (u - u_\lambda) \cdot \partial_t \varphi \ dx \ dt \leq c(\kappa) \int_{\frac{1}{2}Q} \chi_{\sigma} |\nabla \varphi|^\sigma \ dx \ dt + \kappa \left( \int_Q r_B^- |u|^\sigma + |\nabla u|^\sigma + |H|^\sigma \ dx \ dt \right)\]

for all \(\varphi \in C_0^\infty(\frac{1}{2}Q)\) and all \(\kappa > 0\).
Now, we define pointwise in time \( w := \text{curl}^{-1}(\tilde{u}) = \text{curl}^{-1} (\gamma u - \text{Bog}_{B_0 \setminus \frac{1}{2} B_0} (\nabla \gamma \cdot u)). \)

It follows from Lemma 2.1 in [BrDS] and continuity properties of Bog that

\[
\begin{align*}
\int_{Q} |\nabla w|^q \, dx \, dt &\leq c \int_{Q} |u|^q \, dx \, dt, \\
\int_{Q} |\nabla^2 w|^q \, dx \, dt &\leq c \left( \int_{Q} r_B^{-q} |u|^q \, dx \, dt + \int_{Q} |\nabla u|^q \, dx \, dt \right).
\end{align*}
\]

(3.2)

For a ball \( B' \subset \mathbb{R}^3 \) and a function \( f \in L^s(B') \) we define \( \Delta_B^{-2} \Delta f \) as the weak solution \( F \in W^{2,s}_0(B') \) of

\[
\int_{B'} \Delta F \Delta \varphi \, dx = \int_{B'} f \Delta \varphi \, dx \quad \text{for all } \varphi \in C_0^\infty(B').
\]

(3.3)

Then \( f - \Delta(\Delta_B^{-2} \Delta f) \) is harmonic on \( B' \). We define \( z(t) := z_{\frac{1}{2}Q_0}(t) = \Delta \Delta_B^{-2} \Delta w(t) \) for \( t \in \frac{1}{2} I_0 \), then

\[
\int_{Q_0} z \cdot \partial_t \Delta \psi \, dx \, dt = \int_{Q_0} w \cdot \partial_t \Delta \psi \, dx \, dt = - \int_{Q_0} H : \nabla^2 \psi \, dx \, dt,
\]

(3.4)

for all \( \psi \in C_0^\infty(\frac{1}{2}Q_0) \). Since \( \Delta_B^{-2} \Delta w(t) \in W^{2,s}_0(\frac{1}{2} B_0) \), we can extend it by zero to a function from \( W^{2,s}(\mathbb{R}^3) \). In this sense it is natural to extend \( z(t) \) by zero to

\[C_{0,0}^\infty \] is the subspace of \( C_0^\infty \) whose elements have mean value zero.
a function $L^s(\mathbb{R}^3)$. As a consequence of [BrDS, Lemma 2.3] and (3.2) we have
\[
\int_{\frac{1}{2}Q} |\nabla z|^\sigma \, dx \, dt \leq c \int_{Q} |u|^\sigma \, dx \, dt, \\
\int_{\frac{1}{2}Q} |\nabla^2 z|^\sigma \, dx \, dt \leq c \left( \int_{\frac{1}{2}Q} |r_B^{-\sigma}| |u|^\sigma \, dx \, dt + \int_{Q} |\nabla u|^\sigma \, dx \, dt \right), \\
\int_{\frac{1}{2}Q} |\partial_t z|^\sigma \, dx \, dt \leq c \int_{Q} |H|^\sigma \, dx \, dt 
\] (3.5)

For $\lambda, \alpha > 0$ and $\sigma > 1$ we define
\[
O_\lambda := \{ M_\sigma (\chi_{\frac{1}{4}Q_0} |\nabla^2 z|) > \lambda \} \cup \{ \alpha M_\sigma (\chi_{\frac{1}{4}Q_0} |\partial_t z|) > \lambda \}. 
\] (3.6)

We decompose $O_\lambda$ into a family of parabolic Whitney cubes $(Q_i)_{i \in \mathbb{N}}$ and consider a decomposition of unity $(\varphi_i)_{i \in \mathbb{N}}$ with respect to it as done in [BrDS] after (2.11). We define
\[
I := \{ i : Q_i \cap \frac{1}{2}Q_0 \neq \emptyset \}.
\]

Taking $\lambda$ large enough, the continuity of the maximal function (2.4) implies $Q_i \subset \frac{1}{2}Q_0$ (and $Q_j \subset \frac{1}{2}Q_0$ for $j \in A_i$) for all $i \in I$. For each $i \in I$ we define local approximation $z_i$ for $z$ on $Q_i$ by
\[
z_i := \Pi_{1,B_i}^0(z), 
\] (3.7)

where $\Pi_{1,B_i}^0(z)$ is the first order averaged Taylor polynomial [BrenS, DieR2] with respect to space and $\Pi_{0,B_i}^I$ is the zero order averaged Taylor polynomial in time. We set
\[
z_\lambda^\alpha = \begin{cases} 
z & \text{on } \frac{1}{2}Q_0 \setminus O_\lambda, \\
\sum_{i \in I} \varphi_i z_i & \text{on } \frac{1}{2}Q_0 \cap O_\lambda. 
\end{cases} 
\] (3.8)

We apply the arguments used in the proof of Theorem [BrDS] (Thm. 2.2) to the constant sequence $u$ with the choice $\alpha = 1$. So we have
\[
u_\lambda := \text{curl}(\zeta z_\lambda) + \text{curl}(\zeta (w - z)), 
\] (3.9)

where $\zeta \in C^\infty_c(\frac{1}{5}Q)$ with $\chi_{\frac{1}{5}Q} \leq \zeta \leq \chi_{\frac{1}{4}Q}$. This means on $\frac{1}{5}Q$ it holds
\[
u_\lambda = u + \text{curl}(z_\lambda - z).
\]

We immediately obtain the claim of (a). As a consequence of the Lemmas 2.1,
2.4 and 2.9 in [BrDS] we gain the inequalities

\[
\int_{\frac{1}{2}Q} |\nabla z_\lambda|^\delta \, dx \, dt \leq c \left( \int_{\frac{1}{2}Q} |u|^\delta + \int_{\frac{1}{2}Q} r_B^{-\sigma} |\partial_t z|^\sigma \, dx \, dt \right),
\]

\[
\int_{\frac{1}{2}Q} |\nabla^2 z_\lambda|^\sigma \, dx \, dt \leq c \left( \int_{\frac{1}{2}Q} r_B^{-\sigma} |u|^\sigma + |\nabla u|^\sigma \, dx \, dt + \int_{\frac{1}{2}Q} |\partial_t z|^\sigma \, dx \, dt \right),
\]

\[
\int_{\frac{1}{2}Q} |\partial_t z_\lambda|^\sigma \, dx \, dt \leq c \left( \int_{\frac{1}{2}Q} r_B^{-\sigma} |u|^\sigma + |\nabla u|^\sigma \, dx \, dt + \int_{\frac{1}{2}Q} |\partial_t z|^\sigma \, dx \, dt \right),
\]

which imply the estimates

\[
\int_{Q} |u_\lambda|^\sigma \, dx \, dt \leq c \left( \int_{Q} |u|^\sigma + \int_{Q} r_B^{-\sigma} |\partial_t z|^\sigma \, dx \, dt \right),
\]

\[
\int_{Q} |\nabla u_\lambda|^\sigma \, dx \, dt \leq c \left( \int_{Q} r_B^{-\sigma} |u|^\sigma + |\nabla u|^\sigma \, dx \, dt + \int_{Q} |\partial_t z|^\sigma \, dx \, dt \right).
\]

Finally we can replace \( \partial_t z \) by \( H \) on account of (3.5).

It remains to find good levels. So we set

\[
g := M_s(\chi_{\frac{1}{2}Q} |\nabla^2 z|) + M_s(\chi_{\frac{1}{2}Q} |\partial_t z|)
\]

and gain from the continuity of \( M_s \) and (3.5)

\[
\int_{\mathbb{R}^{d+1}} |g|^\sigma \, dx \leq c \left( \int_{\frac{1}{2}Q} |\nabla^2 z|^\sigma \, dx \, dt + \int_{\frac{1}{2}Q} |\partial_t z|^\sigma \, dx \, dt \right)
\]

\[
\leq c \left( \int_{Q} r_B^{-\sigma} |u|^\sigma + |\nabla u|^\sigma \, dx \, dt + \int_{Q} |H|^\sigma \, dx \, dt \right).
\]

Furthermore, it holds for every \( m_0 \in \mathbb{N} \) and every \( \gamma > 0 \)

\[
\int_{\mathbb{R}^{d+1}} |g|^\sigma \, dx = \int_{\mathbb{R}^{d+1}} \int_{0}^{\infty} \sigma t^{\sigma-1} \chi_{\{|g|>t\}} \, dt \, dx \\
\geq \int_{\mathbb{R}^{d+1}} \sum_{m=m_0}^{2m_0-1} \sigma (2^m \gamma)^\sigma \chi_{\{|g|>2^m+1\}} \, dx.
\]

So, there is \( m_1 \in \{m_0, ..., 2m_0 - 1\} \) such that

\[
\int_{\mathbb{R}^{d+1}} (2^{m_1} \gamma)^\sigma \chi_{\{|g|>2^{m_1}+1\}} \, dx \leq \frac{c}{s m_0} \int_{\mathbb{R}^{d+1}} |g|^\sigma \, dx.
\]

Setting \( \lambda = \gamma 2^{m_1+1} \) we obtain

\[
\lambda^\sigma |\frac{1}{2}Q \cap \{|g| > \lambda\}| \leq \frac{c}{m_0} \int_{\mathbb{R}^{d+1}} |g|^\sigma \, dx.
\]

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Combining this with (3.11) gives the estimate in b) due to the definition of \( O_\lambda \). We have \( u_\lambda - u = \text{curl}(z_\lambda - z) \) on \( \frac{1}{8}Q \) such that (3.5) and (3.10) imply
\[
\partial_t(u_\lambda - u) \in L^\sigma\left(\frac{1}{8}I, W^{-1,\sigma'}(\frac{1}{8}B)\right)
\]
Moreover, we gain for \( \varphi \in C_0^\infty(\frac{1}{8}Q) \)
\[
- \int_{\frac{1}{8}Q} (u - u_\lambda) \cdot \partial_t \varphi \, dx \, dt = \int_{\frac{1}{8}Q} \chi_{O_\lambda} \partial_t(z - z_\lambda) \cdot \text{curl} \varphi \, dx \, dt 
\leq c(\kappa) \int_{\frac{1}{8}Q} \chi_{O_\lambda} |\nabla \varphi|^{\sigma'} \, dx \, dt + \kappa \left( \int_{\frac{1}{8}Q} |\partial_t(z - z_\lambda)|^{\sigma} \, dx \, dt \right)
\]
as a consequence of Young’s inequality. Applying (3.5) and (3.10) yields
\[
\int_{\frac{1}{8}Q} |\partial_t(z_\lambda - z)|^{\sigma} \, dx \, dt \leq c \left( \int_{\frac{1}{8}Q} r_B^{-\sigma} |u|^{\sigma} + |\nabla u|^{\sigma} \, dx \, dt + \int_{\frac{1}{8}Q} |H|^{\sigma} \, dx \, dt \right).
\]
So we have shown the estimate claimed in (d) on \( \frac{1}{8}Q \).
A simple scaling argument allows us to obtain the desired estimates on \( \frac{1}{2}Q \). □
4. \textbf{A-Stokes approximation - evolutionary case}

For a function \( w \in L^1(Q) \) with \( \partial_t w \in L^{q'}(J; W^{-1,q'}_\text{div}(B)) \) we introduce the unique function \( H_w \in L^q_\infty(Q) \) with

\[
\int_Q w \cdot \partial_t \varphi \, dx \, dt = \int_Q H_w : \nabla \varphi \, dx \, dt
\]

for all \( \varphi \in C_0^{\infty}(Q) \). We begin with a variational inequality for the non-stationary \( \mathcal{A}\text{-Stokes system} \).

\textbf{Lemma 4.1.} For all balls \( B \subset \mathbb{R}^d \), all bounded intervals \( J = (t_0, t_1) \) and \( u \in C_0^0([0,T]; L^1(B)) \cap L^{q'}(J; W^{1,q'}_\text{div}(B)) \) with \( u(0, \cdot) = 0 \) a.e. it holds for \( Q = J \times B \)

\[
\int_Q |\nabla u|^q \, dx \, dt \leq c \sup_{\xi \in C_0^{\infty}(Q)} \left[ \int_Q (\mathcal{A}( \varepsilon(u), \varepsilon(\xi)) - u \cdot \partial_t \xi) \, dx \, dt \right. \\
- \left. \int_Q (|\nabla \xi|^q + |H_\xi|^q) \, dx \, dt \right].
\]

where \( c \) only depends on \( \mathcal{A} \) and \( d \).

\textit{Proof.} Duality arguments show that

\[
\int_Q |\nabla u|^q \, dx \, dt = \sup_{G \in L^{q'}(Q)} \left[ \int_Q \nabla u : G \, dx \, dt - \int_Q |G|^{q'} \, dx \, dt \right].
\]

For a given \( G \in L^{q'}(Q) \) let \( z_G \) be the unique \( L^{q'}(J; W^{1,q'}_\text{div}) \)-solution to

\[
- \int_Q z \cdot \partial_t \xi \, dx \, dt + \int_Q \mathcal{A}(z, \varepsilon(\xi)) \, dx \, dt = \int_Q G : \nabla \xi \, dx \, dt
\]

for all \( \xi \in C_0^{\infty}([t_0, t_1] \times B) \). We also have that \( \partial_t z_G \in L^{q'}(J; W^{-1,q'}_\text{div}(B)) \).

Due to Theorem 2.6 this solution satisfies

\[
\int_Q |\nabla z_G|^{q'} \, dx \, dt + \int_Q |H_{z_G}|^{q'} \, dx \, dt \leq c \int_Q |G|^{q'} \, dx \, dt.
\]

In other words, the mapping \( L^{q'}(B) \ni G \mapsto z_G \in L^{q'}(J; W^{1,q'}_\text{div}(B)) \) is continuous. This and the density of \( C_0^{\infty}(Q) \) gives due to \( u(0, \cdot) = 0 \)

\[
\int_Q |\nabla u|^q \, dx \, dt \\
\leq c \sup_{G \in L^{q'}(Q)} \left[ \int_Q \mathcal{A}(z_G, \varepsilon(\xi)) \, dx \, dt + \int_Q \partial_t z_G \cdot u \, dx \, dt \right]
\]
\[
- \int_Q \left( |\nabla \mathbf{H}^\prime| + |\mathbf{H}_{zG}^\prime| \right) dx dt \\
\leq c \sup_{\xi \in C_\infty_0(Q)} \left[ \int_Q \mathcal{A}(\varepsilon(u), \varepsilon(\xi)) dx dt - \int_Q u \cdot \partial_t \xi dx dt \\
- \int_Q \left( |\nabla \xi^\prime| + |\mathbf{H}_{\xi}^\prime| \right) dx dt \right]
\]

which yields the claim. \(\blacksquare\)

Let us now state the \(A\)-Stokes approximation. In the following let \(B\) be a ball with radius \(r\) and \(J\) an interval with length \(2r^2\). Let \(\tilde{Q}\) denote either \(Q = J \times B\) or \(2Q\). We use similar notations for \(\tilde{J}\) and \(\tilde{B}\).

**THEOREM 4.2.** Let \(v \in L^{qs}(2\tilde{J}; W_0^{1,qs}(2\tilde{B}))\), \(q, s > 1\), be an almost \(A\)-Stokes solution in the sense that

\[
\left| \int_{2\tilde{Q}} v \cdot \partial_t \xi dx dt - \int_{2\tilde{Q}} \mathcal{A}(\varepsilon(v), \varepsilon(\xi)) dx dt \right| \leq \delta \int_{2\tilde{Q}} |\varepsilon(v)| dx dt \|\nabla \xi\|_\infty
\]

(4.1)

for all \(\xi \in C_\infty_0(2Q)\) and some small \(\delta > 0\). Then the unique solution \(w \in L^q(J; W_0^{1,q}(B))\) to

\[
\int_Q w \cdot \partial_t \xi dx dt - \int_Q \mathcal{A}(\varepsilon(w), \varepsilon(\xi)) dx dt
= \int_Q v \cdot \partial_t \xi dx dt - \int_Q \mathcal{A}(\varepsilon(v), \varepsilon(\xi)) dx dt
\]

(4.2)

for all \(\xi \in C_\infty_0([t_0, t_1] \times B)\) satisfies

\[
\int_{2\tilde{Q}} \frac{|w|^q}{r} dx dt + \int_{2\tilde{Q}} |\nabla w|^q dx dt \leq \kappa \left( \int_{2\tilde{Q}} |\nabla v|^{qs} dx dt \right)^{\frac{1}{s}}.
\]

It holds \(\kappa = \kappa(q, s, \delta)\) and \(\lim_{\delta \to 0} \kappa(q, s, \delta) = 0\). The function \(h := v - w\) is called the \(A\)-Stokes approximation of \(v\).

**Remark 4.3.** From the proof of Theorem 4.2 we gain the following stability result choosing \(p = qs = q\).

\[
\int_{2\tilde{Q}} \frac{|w|^p}{r} dx dt + \int_{2\tilde{Q}} |\nabla w|^p dx dt \leq c \int_{2\tilde{Q}} |\nabla v|^p dx dt.
\]

Indeed \(\kappa\) stays bounded if \(s \to 1\).
Proof. Let \( w \) be defined as in \( (4.2) \). Combining Poincaré’s inequality with Lemma 4.1 and (4.2) shows

\[
\int_Q \left| \nabla w \right|^q \, dx \, dt + \int_Q |\nabla w|^q \, dx \, dt \leq c \sup_{\xi \in C^\infty_0(\Omega)} \left[ \int_Q A(\varepsilon(v), \varepsilon(\xi)) \, dx \, dt - \int_Q v \cdot \partial_t \xi \, dx \, dt \right. \\
\left. \quad - \int_Q \left( \nabla \xi \right)^q + \left( |H_\xi|^q \right) \right] \, dx \, dt. \tag{4.3}
\]

In the following let us fix \( \xi \in C^\infty_0(\Omega) \). Let

\[
\gamma := \left( \int_Q |\nabla \xi|^q \, dx \, dt + \int_Q |H_\xi|^q \, dx \, dt \right)^{\frac{1}{q'}}
\]

and \( m_0 \in \mathbb{N}, m_0 \gg 1 \). Due to Theorem 3.1 (d) implies that \( \partial_t \xi \) can be extended by 0 to \( 4Q \) by the properties of \( \text{Bog}_B \) (since \( H_\xi \) can be extended as well). Theorem 3.1 (d) implies that \( \partial_t (\xi - \xi_\lambda) \in L^{q'}(2J, W^{-1,q'}(2B)) \) and

\[
\int_{2J} \langle \partial_t (\xi - \xi_\lambda), \varphi \rangle \, dt \leq c(\kappa) \int_{2Q} \chi_{(\xi \neq \xi_\lambda)} |\nabla \varphi|^q \, dx \, dt + \kappa \left( \int_Q |\nabla \xi|^{q'} + |H_\xi|^{q'} \right) \tag{4.8}
\]

for all \( \varphi \in W_0^{1,q}(2Q) \). We calculate for \( \eta \in C^\infty_0(2Q) \) with \( \eta \equiv 1 \) on \( Q \), \( |\nabla^k \eta| \leq cr^{-k} \) and \( |\partial_t \nabla^{k-1} \eta| \leq cr^{-(k+1)} \) (\( k = 1, 2 \))

\[
\int_Q A(\varepsilon(v), \varepsilon(\xi)) \, dx \, dt - \int_Q v \cdot \partial_t \xi \, dx \, dt
\]
= 2^{d+2} \int_2^{2Q} A(\varepsilon, \varepsilon(\eta \xi - \operatorname{Bog}_2B/\partial v(\nabla \eta \xi))) \, d\eta \, dt - \int_2^{2Q} \varepsilon \cdot \partial_t (\eta \xi - ...) \, d\eta \, dt

= 2^{d+2} \left( \int_2^{2Q} A(\varepsilon, \varepsilon(\eta \xi - \operatorname{Bog}_2B/\partial v(\nabla \eta \xi))) \, d\eta \, dt + \int_2^{2Q} \partial_t \varepsilon \cdot (\eta \xi - ...) \, d\eta \, dt \right)

+ 2^{d+2} \int_2^{2Q} A(\varepsilon, \varepsilon(\eta \xi - \xi)) - \operatorname{Bog}_2B(\nabla \eta(\xi - \xi))) \, d\eta \, dt

+ 2^{d+2} \int_2^{2Q} \partial_t \varepsilon \cdot (\eta(\xi - \xi)) - \operatorname{Bog}_2B(\nabla \eta(\xi - \xi))) \, d\eta \, dt

= 2^{d+2}(I + II + III).

Note that the time-derivative of \( v \) exists in the \( W_{\text{div}}^{-1,\infty} \)-sense as a consequence of (4.6). Therefore all terms are well-defined by the properties of \( \xi \). We have the following inequality on account of (4.6), (4.7) and the continuity properties of \( \operatorname{Bog}(\xi := \xi - \xi) \):

\[
\int_2^{2Q} |\nabla \Psi|^q \, dx \, dt := \int_2^{2Q} |\nabla (\eta \tilde{\xi}) - \nabla \operatorname{Bog}_2B(\nabla \eta \tilde{\xi})|^q \, dx \, dt
\]

\[
\leq c \int_2^{2Q} |\nabla \xi|^q \, dx \, dt + c \int_2^{2Q} \tilde{\xi}_\xi |^q \, dx \, dt
\]

\[
\leq c \int_2^{2Q} |\nabla \xi|^q \, dx \, dt + c \int_2^{2Q} \tilde{\xi}_\xi |^q \, dx \, dt + c \int_2^{2Q} |H_\xi|^q \, dx \, dt
\]

\[
\leq c \int_2^{2Q} |\nabla \xi|^q \, dx \, dt + c \int_2^{2Q} |H_\xi|^q \, dx \, dt.
\]

Young’s inequality for an appropriate choice of \( \varepsilon > 0 \) and Poincaré’s inequality imply together with (4.6) and (4.7)

\[
II \leq c(\varepsilon) \int_2^{2Q} |\varepsilon(v)|^q \chi(\xi \neq \xi) \, dx \, dt + \varepsilon \int_2^{2Q} |\nabla \Psi|^q \, dx \, dt
\]

\[
\leq c \int_2^{2Q} |\varepsilon(v)|^q \chi(\xi \neq \xi) \, dx \, dt + \frac{1}{3} \int_2^{2Q} |\nabla \xi|^q + |H_\xi|^q \, dx \, dt
\]

\[
=: II_1 + II_2,
\]

where \( c \) depends on \( |A|, q \) and \( q' \). With Hölder’s inequality we gain

\[
II_1 \leq c \left( \int_2^{2Q} |\nabla v|^{q*} \, dx \, dt \right)^{\frac{q}{q'}} \left( \frac{L^{d+1}(2Q \cap \{ \xi \neq \xi \} \cup |Q| \, dx \, dt \right)^{1-\frac{q}{q'}}.
\]
If follows from (4.5), by the choice of $\gamma$ and $\lambda \geq \gamma$ that
\[
\frac{L^{d+1} (2Q \cap \{ \xi_\lambda \neq \xi \})}{|Q|} \leq \frac{c \gamma'}{m_0 \lambda q'} \leq \frac{c}{m_0}.
\] (4.10)
Thus
\[
II_1 \leq c \left( \int_{2Q} |\nabla v|^{q_s} \, dx \, dt \right)^{\frac{1}{q_s}} \left( \frac{c}{m_0} \right)^{1-\frac{1}{q_s}}.
\]
We choose $m_0$ so large such that
\[
II_1 \leq \frac{\kappa}{3} \left( \int_{2Q} |\nabla v|^{q_s} \, dx \, dt \right)^{\frac{1}{q_s}}.
\]

Since $\partial_t (\xi - \xi_\lambda) \in L^{q'} (2J, W^{-1,q'}(2B))$ we can write $III$ as
\[
III = \int_{2Q} v \cdot \partial_t \eta(\xi - \xi_\lambda) \, dx \, dt + \int_{2Q} \eta v \cdot \partial_t (\xi - \xi_\lambda) \, dx \, dt
\]
\[
- \int_{2Q} v \cdot \text{Bog}_{2B \setminus B}(\partial_t \eta(\xi - \xi_\lambda)) \, dx \, dt
\]
\[
- \int_{2Q} \text{Bog}^*_{2B \setminus B}(v) \nabla \eta \cdot \partial_t (\xi - \xi_\lambda) \, dx \, dt
\]
\[=: III_1 + III_2 + III_3 + III_4.
\]
The formulation in (4.11) does not change if we subtract terms which are constant in space from $v$. So we can assume
\[
\int_{2B \setminus B} v(t) \, dx = 0 \quad \text{for a.e. } t \in 2J.
\] (4.11)

Therefore $\text{Bog}^*_{2B \setminus B}(v)$ is well-defined (in the sense of $L^2$-duality). We consider the four terms separately and obtain for the first one
\[
III_1 \leq c \int_{2I \setminus B} \chi(\xi_\lambda \neq \xi) \left| \frac{v}{r} \right| \left| \frac{\xi - \xi_\lambda}{r} \right| \, dx \, dt
\]
\[
\leq c(\varepsilon) \int_{2I \setminus B} \left| \frac{v}{r} \right|^q \chi(\xi_\lambda \neq \xi) \, dx \, dt + c \int_{2Q} \left| \frac{\xi - \xi_\lambda}{r} \right|^{q'} \, dx \, dt
\]
\[=: c(\varepsilon)III_{11} + \varepsilon III_{12}.
\]
Poincaré’s inequality and Young’s inequality yield

\[
III_{11} \leq c \left( \int_{2Q} |\nabla v|^{qs} \, dx \, dt \right)^{\frac{1}{s}} \left( \frac{\mathcal{L}^{d+1}(2Q \cap \{\xi_\lambda \neq \xi\})}{|Q|} \right)^{1-\frac{1}{s}}
\]

Arguing as done for the term \(II_1\) yields

\[
III_{11} \leq \frac{C}{12} \left( \int_{2Q} |\nabla v|^{qs} \, dx \, dt \right)^{\frac{1}{s}}.
\]

Moreover, we gain from (4.16) and Poincaré’s inequality

\[
III_{12} \leq c \int_{Q} |\xi'|^{q'} \, dx \, dt + c \int_{Q} |H_\xi|^{q'} \, dx \, dt
\]

\[
\leq c \int_{Q} |\nabla \xi'|^{q'} \, dx \, dt + c \int_{Q} |H_\xi|^{q'} \, dx \, dt
\]

and finally

\[
III_1 \leq \frac{C}{12} \left( \int_{2Q} |\nabla v|^{qs} \, dx \, dt \right)^{\frac{1}{s}} + \frac{C}{12} \left( \int_{Q} |\nabla \xi'|^{q'} + |H_\xi|^{q'} \, dx \, dt \right).
\]

As a consequence of (4.8), (4.11) and Poincaré’s inequality we obtain similarly

\[
III_2 \leq c(\varepsilon) \int_{2Q} \chi_{\{\xi \neq \xi_\lambda\}} |\nabla (\eta v)|^{q'} \, dx \, dt + \varepsilon \left( \int_{Q} |\nabla \xi'|^{q'} + |H_\xi|^{q'} \, dx \, dt \right)
\]

\[
\leq \frac{C}{12} \left( \int_{2Q} |\nabla v|^{qs} \, dx \, dt \right)^{\frac{1}{s}} + \frac{C}{12} \left( \int_{Q} |\nabla \xi'|^{q'} + |H_\xi|^{q'} \, dx \, dt \right).
\]

Taking into account continuity properties of the Bogovskiĭ-operator we can estimate \(III_3\) via (recall (4.11))

\[
III_3 \leq \left( \int_{2Q} |\nabla v|^{qs} \, dx \, dt \right)^{\frac{1}{2q'}} \left( \int_{2Q} \mathcal{I}^{2(qs)'} \left( \|\partial_k \nabla \eta(\xi - \xi_\lambda)\|^{(qs)'_{\|}} dx \, dt \right) \right)^{\frac{1}{(qs)'}}
\]

\[
\leq c \left( \int_{2Q} |\nabla v|^{qs} \, dx \, dt \right)^{\frac{1}{2q'}} \left( \int_{2Q} \mathcal{I}^{2(qs)'} \left( \|\partial_k \nabla \eta(\xi - \xi_\lambda)\|^{(qs)'_{\|}} dx \, dt \right) \right)^{\frac{1}{(qs)'}}
\]

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\[\begin{align*}
&\leq c\left(\int_{2Q} |\nabla v|^{qs} \, dx \, dt\right)^{\frac{1}{q'}} \left(\int_{2Q} \chi(\xi \notin \xi_\lambda) \frac{|\xi - \xi_\lambda|}{r}^{(qs)'} \, dx \, dt\right)^{\frac{1}{(qs)'}}.
\end{align*}\]

We gain from Young’s inequality

\[III_3 \leq \frac{\varepsilon q}{12} \left(\int_{2Q} |\nabla v|^{qs} \, dx \, dt\right)^{\frac{1}{q'}} + c\varepsilon^{-q'} \left(\int_{2Q} \chi(\xi \notin \xi_\lambda) \frac{|\xi - \xi_\lambda|}{r}^{(qs)'} \, dx \, dt\right)^{\frac{1}{(qs)'}}\]

\[=: \frac{\varepsilon q}{12} III_{31} + c\varepsilon^{-q'} III_{32}.
\]

It holds due to Hölder’s inequality, (4.11), (4.5) and (4.6) for \(m_0\) large enough

\[III_{32} \leq \left(\frac{\varepsilon^{d+1} (2Q \cap \{\xi_\lambda \neq \xi\})}{|Q|}\right)^{1-\frac{1}{q'}} \left(\int_{2Q} |\xi - \xi_\lambda| \frac{q'}{r} \, dx \, dt\right)\]

\[\leq \frac{\kappa}{12c} \left(\int_{Q} |\nabla \xi|^{q'} + |H_\xi|^{q'} \, dx \, dt\right).
\]

Choosing \(\varepsilon := \kappa^{1/q'}\) implies

\[III_3 \leq \frac{\kappa}{12} \left(\int_{2Q} |\nabla v|^{qs} \, dx \, dt\right)^{\frac{1}{q'}} + \frac{1}{12} \left(\int_{Q} |\nabla \xi|^{q'} + |H_\xi|^{q'} \, dx \, dt\right).
\]

By (4.8) and (4.10) we have for \(m_0\) large enough

\[III_4 \leq c \int_{2Q} \chi(\xi \notin \xi_\lambda) |\nabla(\nabla \eta \text{Bog}_{2B \setminus B}(v))|^{q'} \, dx \, dt + \frac{1}{12} \left(\int_{Q} |\nabla \xi|^{q'} + |H_\xi|^{q'} \, dx \, dt\right)\]

\[\leq \varepsilon \left(\int_{2Q} |\nabla(\nabla \eta \text{Bog}_{2B \setminus B}(v))|^{q}\, dx \, dt\right)^{\frac{1}{q'}} + \frac{1}{12} \left(\int_{Q} |\nabla \xi|^{q'} + |H_\xi|^{q'} \, dx \, dt\right)\]

\[=: \varepsilon III_{41} + \frac{1}{2} III_{42}.
\]

Continuity properties of \(\text{Bog}_{2B \setminus B}\) and (4.11) yield

\[III_{41} \leq c \left(\int_{2Q} \text{Bog}_{2B \setminus B}(v) \frac{1}{r^q} \, dx \, dt + \int_{2Q} |\nabla \text{Bog}_{2B \setminus B}(v) \frac{1}{r} | \, dx \, dt\right)^{\frac{1}{q}}\]

\[\leq c \left(\int_{2Q} |\nabla \text{Bog}_{2B \setminus B}(v) \frac{1}{r} | \, dx \, dt\right)^{\frac{1}{q}} \leq c \left(\int_{212B \setminus B} |\nabla v|^{q} \, dx \, dt\right)^{\frac{1}{q}}\]

\[\leq c \left(\int_{2Q} |\nabla v|^{q} \, dx \, dt\right)^{\frac{1}{q}}.
\]
and hence for $\varepsilon := \kappa/12c$

$$III_4 \leq \frac{\kappa}{12} \left( \int_{2Q} |\nabla v|^{qs} \, dx \, dt \right)^{\frac{1}{qs}} + \frac{1}{12} \left( \int_{Q} |\nabla \xi|^{q'} + |H_\xi|^{q'} \, dx \, dt \right).$$

Plugging the estimates for $III_1$-$III_4$ together we see

$$III \leq \frac{\kappa}{3} \left( \int_{2Q} |\nabla v|^{qs} \, dx \, dt \right)^{\frac{1}{qs}} + \frac{1}{3} \left( \int_{Q} |\nabla \xi|^{q'} + |H_\xi|^{q'} \, dx \, dt \right).$$

Since $v$ is an almost $A$-Stokes solution and $\|\nabla \xi_\lambda\|_{\infty} \leq c \lambda \leq c 2^{m_0} \gamma$ we have

$$|I| \leq \delta \int_{2Q} |\nabla v| \, dx \, dt \|\nabla \xi_\lambda\|_{\infty, 2Q} \leq \delta \left( \int_{2Q} |\nabla v|^{qs} \, dx \, dt \right)^{\frac{1}{qs}} c^{2m_0} \gamma.$$

We apply Young's inequality and Jensen's inequality to gain

$$|I| \leq \delta^{2m_0} c \left( \int_{2Q} |\nabla v|^{qs} \, dx \, dt \right)^{\frac{1}{qs}} + \delta^{2m_0} c^{q'}$$

$$\leq \delta^{2m_0} c \left( \int_{2Q} |\nabla v|^{qs} \, dx \, dt \right)^{\frac{1}{qs}} + \delta^{2m_0} c \left( \int_{Q} |\nabla \xi|^{q'} \, dx \, dt + \int_{Q} |H_\xi|^{q'} \, dx \, dt \right).$$

Now, we choose $\delta > 0$ so small such that $\delta^{2m_0} c \leq \kappa/3$. Thus

$$|I| \leq \frac{\kappa}{3} \left( \int_{2Q} |\nabla v|^{qs} \, dx \, dt \right)^{\frac{1}{qs}} + \frac{1}{3} \left( \int_{Q} |\nabla \xi|^{q'} \, dx \, dt + \int_{Q} |H_\xi|^{q'} \, dx \, dt \right).$$

Combining the estimates for $I$, $II$ and $III$ we have established

$$\int_{2Q} A(\varepsilon(v), \varepsilon(\xi)) \, dx \, dt - \int_{Q} v \cdot \partial_t \xi \, dx \, dt$$

$$\leq \kappa \left( \int_{2Q} |\nabla v|^{qs} \, dx \, dt \right)^{\frac{1}{qs}} + \int_{Q} |\nabla \xi|^{q'} \, dx \, dt + \int_{Q} |H_\xi|^{q'} \, dx \, dt.$$

Inserting this in (4.3) shows the claim.
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