Weak Complicial Sets
A Simplicial Weak ω-Category Theory
Part II: Nerves of Complicial Gray-Categories

Dominic Verity

To Ross Street on the occasion of his 60th birthday.

Abstract. This paper continues the development of a simplicial theory of weak ω-categories, by studying categories which are enriched in weak complicial sets. These complicial Gray-categories generalise both the Kan complex enriched categories of homotopy theory and the 3-categorical Gray-categories of weak 3-category theory. We derive a simplicial nerve construction, which is closely related to Cordier and Porter’s homotopy coherent nerve, and show that this faithfully represents complicial Gray-categories as weak complicial sets. The category of weak complicial sets may itself be canonically enriched to a complicial Gray-category whose homsets are higher generalisations of the bicategory of homomorphisms, strong transformations and modifications. By applying our nerve construction to this structure, we demonstrate that the totality of all (small) weak complicial sets and their structural morphisms at higher dimensions form a richly structured (large) weak complicial set.

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1. Introduction

This paper is the second in a series of works on the simplicial weak ω-category theory of weak complicial sets. In the first of these [12] we studied the fundamental homotopy theory of such structures by constructing a canonical model structure on the category of stratified simplicial sets whose fibrant objects are these weak complicial sets. On the way we developed a range of technical tools for studying such objects, including an equivalence based characterisation of outer horn filling, Gray tensor products and associated complicial closures, join and décalage constructions and so forth. We also showed that weak complicial sets usefully subsume and generalise a range of common weakened and higher categorical notions, including Kan complexes, Joyal’s quasi-categories and strict ω-categories.

Having proposed a candidate simplicial weak ω-category theory we might now wish to establish its credentials from a number of perspectives. From below, we should be interested in establishing the bounds of its expressiveness by demonstrating that it may be used to faithfully represent a range of common weak ω-categorical notions and their “strictifications”. At the very least, we should be prepared to demonstrate how the well understood low dimensional cases may be represented in this way. We might also wish to undertake a similar program with regard to higher strict structures such as the classical Gray-categories, that is to say categories enriched in strict ω-categories with respect to their Gray tensor product. From above, we must also have regard to ensuring that the devices we define carry enough structure to allow us to prove strong structural theorems. Category theorists often code this intuition into coherence theorems demonstrating that our weakened structures may be rectified and replaced by equivalent strict structures of a certain class.

Of course the completion of such an investigation would not, in itself, establish the full utility of our theory. For example, this purely structural study might completely circumvent the development of the actual category theory of these structures. However, if we are to use these objects in practice they must come equipped with all of the basic constructions of (n-)category theory, such as the (generalised) span construction. We should also, at the very least, be able to establish homotopical versions of all of the standard results, constructions and structures of classical category theory, such as discrete and categorical fibrations, Yoneda’s lemma, adjunctions, limits and colimits and so forth. This latter project is the subject of another companion paper [13], wherein we represent weak complicial sets as certain kinds of complicially enriched quasi-categories. This provides us with a natural context in which to generalise traditional category theory to a kind of homotopy coherent quasi-category theory within the Quillen model category of weak complicial sets itself.

Our far more modest goal here is simply to take the first steps toward implementing the coherence and representation program of the paragraph before last. The completion of that program will have to wait for later papers in this series. Herein we generalise the Gray-categories of weak 3-category theory to complicial Gray-categories, which are simply categories enriched in weak complicial sets with respect to the Gray tensor product of stratified sets. Examples of such structures include classical Gray-categories and the category of weak complicial sets itself. Then we extend the homotopy coherent nerve construction of Cordier and Porter [2] to provide a nerve functor which faithfully represents complicial Gray-categories as
weak complicial sets. Finally we apply this to demonstrate that the totality of all (small) weak complicial sets and their structural morphisms may be gathered together to form a richly structured (large) weak complicial set.

In later work, our eventual goal will be to prove a coherence theorem, which we might state informally as:

_Every weak complicial set satisfying certain mild equivalence replication conditions on its thin simplices is homotopy equivalent to the nerve of some complicial Gray-category whose homsets also satisfy that condition._

It is these structures amongst the weak complicial sets which we might truly call our simplicial weak \( \omega \)-categories.

2. Background Definitions and Notational Conventions

This paper should be read in conjunction with [12]. In particular, we will rely upon that work to describe the notational conventions used here and to provide a foundational introduction to the homotopy theory of weak complicial sets. However, as a warm up we recall a few basic definitions and pivotal results here:

RECALL 1 (simplicial and stratified sets). As usual we let \( \Delta_+ \) denote the (skeletal) category of finite ordinals and order preserving maps between them and use the notation \( \Delta \) to denote its full subcategory of non-zero ordinals. Following tradition we let \([n]\) denote the ordinal \(n + 1\) as an object of \(\Delta_+\) and refer to arrows of \(\Delta_+\) as simplicial operators which are denoted using Greek letters \(\alpha, \beta, \ldots\). Then the category \(\text{Simp}\) of simplicial sets and simplicial maps between them is simply the functor category \([\Delta^{\text{op}}, \text{Set}]\), where Set denotes the (large) category of all (small) sets and functions between them. We shall assume that the reader is familiar the basic homotopy theory of simplicial sets as expounded in [4] or [8]. We shall make repeated use of the classical nerve construction \(N : \text{Cat} \rightarrow \text{Simp}\) which associates with each (small) category \(C\) a simplicial set \(N(C)\) whose \(n\)-simplices correspond to composable paths of arrow of \(C\) of length \(n\).

A stratification on a simplicial set \(X\) is a subset \(tX\) of its simplices satisfying the conditions that

- no 0-simplex of \(X\) is in \(tX\), and
- all of the degenerate simplices of \(X\) are in \(tX\).

A stratified set is a pair \((X, tX)\) consisting of a simplicial set \(X\) and a chosen stratification \(tX\) the elements of which we call thin simplices. In practice, we will elect to notionally confuse stratified sets with their underlying simplicial sets \(X, Y, Z, \ldots\) and uniformly adopt the notation \(tX, tY, tZ, \ldots\) for corresponding sets of thin simplices. A stratified map \(f : X \rightarrow Y\) is simply a simplicial map of underlying simplicial sets which preserves thinness in the sense that for all \(x \in tX\) we have \(f(x) \in tY\). Identities and composites of stratified maps are clearly stratified maps, from which it follows that we have a category \(\text{Strat}\) of stratified sets and maps.

We refer the reader to [14] and [12] for a full development of the theory of stratified sets. In particular, we should draw attention to definition 8 of the latter which defines what it means to be a stratified subset and identifies two important classes of such subsets, the regular and entire ones.

RECALL 2 (elementary operators). We will use the following standard notation and nomenclature throughout:
The injective maps in \( \Delta^+ \) are referred to as face operators. For each \( j \in [n] \) we use the \( \delta^j_i : [n-1] \rightarrow [n] \) to denote the elementary face operator distinguished by the fact that its image does not contain the integer \( j \).

The surjective maps in \( \Delta^- \) are referred to as degeneracy operators. For each \( j \in [n] \) we use \( \sigma^n_j : [n+1] \rightarrow [n] \) to denote the elementary degeneracy operator determined by the property that two integers in its domain map to the integer \( j \) in its codomain.

For each \( i \in [n] \) the operator \( \varepsilon^n_i : [0] \rightarrow [n] \) given by \( \varepsilon^n_i(0) = i \) is called the \( i \)th vertex operator of \([n]\).

We also use the notations \( \eta^n : [n] \rightarrow [0] \) and \( \iota^n : [-1] \rightarrow [n] \) to denote the unique such simplicial operators.

Unless doing so would introduce an ambiguity, we will tend to reduce notational clutter by dropping the superscripts of these elementary operators.

**Notation 3.** We also use the following notation for the simplices of the standard simplex \( \Delta[1] \):
- \( 0' : [r] \rightarrow [1] \) is the operator which maps each \( i \in [r] \) to \( 0 \in [1] \).
- \( 1' : [r] \rightarrow [1] \) is the operator which maps each \( i \in [r] \) to \( 1 \in [1] \).
- \( \rho^r_i : [r] \rightarrow [1] \) (\( 1 \leq i \leq r \)) is the operator defined by

\[
\rho^r_i(j) = \begin{cases} 
0 & \text{if } j < i, \\
1 & \text{if } j \geq i.
\end{cases}
\]

Later on it will become convenient to index the \( r \)-simplices of \( \Delta[1] \) using the doubly pointed set \( [r] \defeq \{-, +, 1, 2, \ldots, r\} \), by letting \( \rho^- = 0' \), \( \rho^+ = 1' \) and defining \( \rho'_i \) as above for an arbitrary integer (non-point) in \([r]\).

**Recall 4 (anodyne extensions and weak complicial sets).** We refer the reader to notations 2 and 9 of [12] for the definitions of the following stratified sets:
- the standard \( n \)-simplex \( \Delta[n] \), its boundary \( \partial \Delta[n] \) and the standard thin \( n \)-simplex \( \Delta[n]^- \),
- the \( k \)-complicial \( n \)-simplex \( \Delta^k[n] \) and its variants \( \Delta^k[n]' \) and \( \Delta^k[n]'' \), and
- the \((n-1)\)-dimensional \( k \)-complicial horn \( \Lambda^k[n] \).

The set of *elementary anodyne extensions* in \( \text{Strat} \) consists of two families of subset inclusions:
- \( \Lambda^k[n] \subseteq \Delta^k[n] \) for \( n = 1, 2, \ldots \) and \( k \in [n] \), these are called complicial horn extensions, and
- \( \Delta^k[n]' \subseteq \Delta^k[n]'' \) for \( n = 2, 3, \ldots \) and \( k \in [n] \), these are called complicial thinness extensions.

We classify these elementary anodyne extensions into two sub-classes, the *inner* ones for which the index \( k \) satisfies \( 0 < k < n \) and the remaining *left* and *right* *outer* ones for which \( k = 0 \) or \( k = n \) respectively. We say that a stratified inclusion (monomorphism) \( e : U \hookrightarrow V \in \text{Strat} \) is an (*inner*) anodyne extension if it is in the cellular completion (that is the completion under pushouts and transfinite composition) of the set of elementary (inner) anodyne extensions.

A stratified set \( A \) is said to be a

- *weak inner complicial set* if it has the right lifting property (RLP) with respect to all inner elementary anodyne extensions.
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- A weak complicial set if it has the RLP with respect to all elementary anodyne extensions.

Informally we might simply say that a weak complicial set has fillers for all complicial horns. We also say that a stratified map \( p : X \rightarrow Y \) is a (inner) complicial fibration if it has the RLP with respect to elementary (inner) anodyne extensions.

**Recall 5** (Gray tensor products). The category \( \text{Strat} \) of stratified sets supports three related tensor products:

1. **The Gray tensor product** \( \otimes : \text{Strat} \times \text{Strat} \rightarrow \text{Strat} \): This is defined as the cartesian product construction in \( \text{Strat} \). Its closure is denoted \( \text{hom}(X,Y) \) and is often referred to as the *stratified set of strong transformations* (cf. definition 54 of [12]).

2. **The lax Gray tensor product** \( \otimes : \text{Strat} \times \text{Strat} \rightarrow \text{Strat} \): This acts like the cartesian product on the underlying simplicial sets of stratified sets \( X \) and \( Y \) but which enjoys a stratification which is a subset of that of \( X \otimes Y \) (cf. definition 56 of loc. cit.). This tensor makes \( \text{Strat} \) into a monoidal category but it is neither symmetric nor does it have a corresponding closure.

3. **The lax Gray pre-tensor product** \( \boxtimes : \text{Strat} \times \text{Strat} \rightarrow \text{Strat} \): This acts like the cartesian product on the underlying simplicial sets and enjoys a stratification which is a subset of that of the lax Gray tensor \( X \otimes Y \) (cf. definition 59 of loc. cit.). Unlike \( \otimes \), this pre-tensor \( \boxtimes \) has left and right closures \( \text{lax}_l(X,Y) \) and \( \text{lax}_r(X,Y) \), called the *stratified sets of left and right lax transformations*, but it is not associative so fails to make \( \text{Strat} \) into a monoidal category. However, the tensors \( \otimes \) and \( \boxtimes \) are indistinguishable from the point of view of weak (inner) complicial sets, in the sense that each inclusion \( X \boxtimes Y \rightarrow X \otimes Y \) is an inner anodyne extension (cf. observation 60 of loc. cit.).

These tensors generalise the Gray tensor products of 2-category theory to the current context. In particular, it is shown in [14] that the lax Gray tensor \( \otimes \) coincides with the Gray tensor product of \( \omega \)-categories of Steiner [9] or Crans [3] under Street’s \( \omega \)-categorical nerve construction [10].

**Recall 6** (the complicial model structure). The category \( \text{Strat} \) supports a Quillen model structure, called the *complicial model structure*, under which:

- A stratified map \( w : X \rightarrow Y \) is a weak equivalence if and only if the stratified map \( \text{hom}(w,A) : \text{hom}(Y,A) \rightarrow \text{hom}(X,A) \) is a homotopy equivalence, in the sense of definition 79 of [12], for each weak complicial set \( A \).

- The cofibrations are precisely the inclusions of stratified sets, thus ensuring that all stratified sets are cofibrant.

- The fibrant objects are the weak complicial sets.

- An arrow \( p : A \rightarrow B \) between weak complicial sets is a fibration, called a *completely complicial fibration*, if and only if it is a complicial fibration (cf. lemma 99 of loc. cit.). All completely complicial fibrations are actually complicial fibrations but the reverse implication does not necessarily hold in general.

- An inclusion \( e : U \rightarrow V \) is a trivial cofibration, called a *complicial cofibration*, if and only if it has the left lifting property (LLP) with respect to
all complicial fibrations between weak complicial sets (cf. corollary 100 of loc. cit.). In particular, all anodyne extensions are complicial cofibrations. Details of the construction of this model structure may be found in section 6 of loc. cit.

3. Complicial Gray-Categories

We start with a definition, which we claim is the appropriate generalisation of the tricategorical notion of Gray-category to our current context.

**Definition 7 (Strat⊛-categories and Strat⊗-categories).** A Strat⊛-category is a category which is enriched over the category of stratified set Strat with respect to the Gray tensor product ⊛. Correspondingly, a Strat⊗-category is a category which is enriched over the category Strat with respect to the lax Gray tensor product ⊗. We will adopt the notations Strat⊛-Cat and Strat⊗-Cat to denote the corresponding (huge) categories of (large) enriched categories and enriched functors between them.

**Observation 8.** Of course, the lax Gray tensor ⊗ is not symmetric and so the reader may be a little concerned that the basic theory of enriched categories, as expounded in Kelly [7] say, might fail to carry over in whole or part. However, it is well known that almost all of the fundamental definitions of enriched category theory may be recast in this context and that basic results regarding limits, colimits, Yoneda’s lemma and the like have direct analogues in this setting. Indeed, in our case the presence of a dual operation (−)◦: Strat → Strat which is well behaved with respect to ⊗ (cf. observations 23 and 58 of [12]) allows almost all of the symmetrical theory to be generalised with little alteration. Although we will need no more than the (obvious) basic definitions here, the interested reader might like to consult Street’s recently reprinted paper [11] for an excellent introduction to the basic theory of enriched categories in an even more general setting, that of enrichment over a bicategory.

**Definition 9 (complicial Gray-category).** A complicial Gray-category (or usually just a Gray-category) is a Strat⊛-category in which each homset is a weak complicial set. A Gray-functor is defined to be a Strat⊛-enriched functor between Gray-categories. Let Gray denote the (huge) category of all (possibly large) Gray-categories and Gray-functors between them.

**Example 10 (the Gray-category of weak complicial sets).** We may canonically enrich Strat with respect to its Gray tensor ⊗, to obtain an enriched category denoted Strat⊛, by taking hom(X, Y) to be the stratified homset between the stratified sets X and Y (cf. Kelly [7] for the details). Now theorem 70 of [12] tells us that its enriched full subcategory Wcs of weak complicial sets is actually a Gray-category in the sense of the last definition.

**Example 11 (complicial Gray-categories generalise classical ones).** A classical Gray-category is simply a category enriched over the category ω-Cat of (strict) ω-categories with respect to its Gray tensor product ⊗. For example, if we restrict attention to those classical Gray-categories whose homsets are 2-categories then we obtain the kind of 3-dimensional Gray-category discussed by Gordon, Power and Street in [5].

For most purposes, it is easier to present such classical Gray-categories in terms of the lax Gray tensor ⊗ on ω-Cat. Recall, from say Crans [3] or Steiner [9], that if...
\[ \mathbb{B} \] and \[ \mathbb{C} \] are \( \omega \)-categories then their lax Gray tensor \( \mathbb{B} \otimes \mathbb{C} \) is an \( \omega \)-category generated by cells \([\beta, \gamma] \) with \( \beta \in \mathbb{B} \) and \( \gamma \in \mathbb{C} \) subject to certain relations which ensure that this cell “looks like” a geometric product of globs. We may describe the Gray tensor \( \otimes \) as the “quotient” of the lax Gray tensor obtained by making each \([\beta, \gamma] \) into an \( \omega \)-categorical \((r + s)\)-equivalence whenever \( \beta \) is an \( r \)-cell, \( \gamma \) is an \( s \)-cell and \( r, s > 0 \). It follows, therefore, that a classical Gray-category \( \mathcal{G} \) may be presented as a category enriched over the category \( \omega \text{-Cat} \) with respect to its lax Gray tensor product \( \otimes \) in which each composition operation \( \circ : \mathcal{G}(b, c) \otimes \mathcal{G}(a, b) \rightarrow \mathcal{G}(a, c) \) carries the generators identified in the last sentence to \( \omega \)-categorical \((r + s)\)-equivalences.

Now theorem 255 of [14] tells us that \( \mathcal{F}_\omega : \text{Strat} \rightarrow \omega \text{-Cat} \), the left adjoint to Street’s nerve functor \( \mathcal{N}_\omega : \omega \text{-Cat} \rightarrow \text{Strat} \), is strongly monoidal with respect to the lax Gray tensor products on those categories, so it follows that \( \mathcal{N}_\omega \) is itself monoidal. The last paragraph tells us that any classical Gray-category \( \mathcal{G} \) may be considered to be enriched with respect to the lax Gray tensor product, so it follows that if we apply \( \mathcal{N}_\omega \) to the homsets of \( \mathcal{G} \) we obtain a category enriched in (strict) complicial sets with respect to the lax Gray tensor \( \otimes \) on \( \text{Strat} \).

Furthermore, it is easy to extend the calculations of loc. cit. to show that the Gray tensor product \( \mathcal{F}_\omega(X) \otimes \mathcal{F}_\omega(Y) \) may be constructed from the lax Gray tensor \( \mathcal{F}_\omega(X) \otimes \mathcal{F}_\omega(Y) \cong \mathcal{F}_\omega(X \otimes Y) \) by making each of its generators of the form \([x, \pi_1^{r_1}, y, \pi_2^{r_2}]\) (cf. observation 246 of loc. cit.) with \( r, s > 0 \) into an \( \omega \)-categorical \((r + s)\)-equivalence. Dualising this result using the adjunction \( \mathcal{F}_\omega \dashv \mathcal{N}_\omega \) and the fact that \( \mathcal{N}_\omega \) is full and faithful (cf. theorem 266 of loc. cit.), we find that the composition operation of \( \mathcal{G} \) satisfies the condition described two paragraphs ago if and only if the compositions \( \circ : \mathcal{N}_\omega(\mathcal{G}(b, c)) \otimes \mathcal{N}_\omega(\mathcal{G}(a, b)) \rightarrow \mathcal{N}_\omega(\mathcal{G}(a, c)) \) carry each simplex of the form \((g \cdot \pi_1^{r_1}, f \cdot \pi_2^{r_2})\) with \( r, s > 0 \) to a \((r + s)\)-simplex in \( \mathcal{N}_\omega(\mathcal{G}(a, c)) \) which is thin in the equivalence stratified version of Street’s nerve \( \mathcal{N}_\omega^e(\mathcal{G}(a, c)) \) discussed in example 17 of [12].

Notice, however, that the simplices \((g \cdot \pi_1^{r_1}, f \cdot \pi_2^{r_2})\) identified in the last paragraph are simply those that are made thin in constructing the pre-tensor \( \mathcal{N}_\omega(\mathcal{G}(b, c)) \sqcup \mathcal{N}_\omega(\mathcal{G}(a, b)) \) in observation 61 of loc. cit. Consequently, if \( \mathcal{G} \) is a classical Gray-category the associated composition operations of the last paragraph actually provide stratified maps \( \circ : \mathcal{N}_\omega(\mathcal{G}(b, c)) \sqcup \mathcal{N}_\omega(\mathcal{G}(a, b)) \rightarrow \mathcal{N}_\omega^e(\mathcal{G}(a, c)) \). Now, on consulting the definitions of the pre-tensors \( \boxtimes \) and \( \sqcup \), we find that the only \( n \)-simplices which are thin in \( \mathcal{N}_\omega^e(\mathcal{G}(b, c)) \sqcup \mathcal{N}_\omega^e(\mathcal{G}(a, b)) \) but are not already thin in \( \mathcal{N}_\omega(\mathcal{G}(b, c)) \sqcup \mathcal{N}_\omega(\mathcal{G}(a, b)) \) are of the form \((g \cdot \eta^n, f)\) with \( f \) a thin \( n \)-simplex in \( \mathcal{N}_\omega^e(\mathcal{G}(a, b)) \) or \((g, f \cdot \eta^n)\) with \( g \) a thin \( n \)-simplex in \( \mathcal{N}_\omega^e(\mathcal{G}(b, c)) \). In the first of these cases, we know that \( g \) simply corresponds to a 0-cell of \( \mathcal{G}(b, c) \) and it is easily shown that the composite \((g \cdot \eta^n) \circ f \) in \( \mathcal{N}_\omega^e(\mathcal{G}(a, c)) \) is simply the thin \( n \)-simplex obtained by applying the stratified map \( \mathcal{N}_\omega^e(\mathcal{G}(a, c)) \rightarrow \mathcal{N}_\omega^e(\mathcal{G}(a, b)) \sim \mathcal{N}_\omega^e(\mathcal{G}(c)) \) to the thin \( n \)-simplex \( f \) in \( \mathcal{N}_\omega^e(\mathcal{G}(a, b)) \). Similarly, in the second case, the composite \( g \circ (f \cdot \eta^n) \) is simply the thin \( n \)-simplex \( \mathcal{N}_\omega^e(\mathcal{G}(b, c)) \) and it follows therefore that our composition operation actually provides us with a stratified map \( \circ : \mathcal{N}_\omega^e(\mathcal{G}(b, c)) \sqcup \mathcal{N}_\omega^e(\mathcal{G}(a, b)) \rightarrow \mathcal{N}_\omega^e(\mathcal{G}(a, c)) \).

Finally we know, from example 17 of loc. cit., that \( \mathcal{N}_\omega^e(\mathcal{G}(a, c)) \) is a weak complicial set and, from observation 61 of the same paper, that the entire inclusion \( \mathcal{N}_\omega^e(\mathcal{G}(b, c)) \sqcup \mathcal{N}_\omega^e(\mathcal{G}(a, b)) \rightarrow \mathcal{N}_\omega^e(\mathcal{G}(a, c)) \) is an inner anodyne extension. It follows that we have lifts \( \circ : \mathcal{N}_\omega^e(\mathcal{G}(b, c)) \sqcup \mathcal{N}_\omega^e(\mathcal{G}(a, b)) \rightarrow \mathcal{N}_\omega^e(\mathcal{G}(a, c)) \) providing composition operations which make the weak complicial sets \( \mathcal{N}_\omega^e(\mathcal{G}(a, b)) \)
into the homsets of a complicial Gray-category. Finally it is clear that this construction faithfully represents classical Gray-categories as complicial ones.

**Observation 12 (size).** In what follows, we shall work somewhat informally with categorical structures that naturally live in different universes of sets. To formalise things the reader might like to think in terms of three fixed and nested Grothendieck universes of sets, whose denizens are called small, large and huge sets respectively. For example, in [12] our stratified sets were generally drawn from the world of small sets and so the categories Strat and Wcs were large and locally small (small homsets). Our new category Gray is a huge structure.

This hierarchy all works very well, until we wish to start talking about the nerves of Gray-categories. These latter structures are large so their nerves should also be large structures and thus are not strictly speaking members of the category Strat of small stratified sets. One solution to this problem might be to adopt a separate notation STRAT for the category of large stratified sets, which can then act as a carrier for the nerves of large Gray-categories. This, however, simply serves to complicate our notation and confuse matters.

We instead take the approach of overloading notations such as Strat to denote a structure inhabiting whichever universe is necessary for the current argument. This then allows us to write $N: \text{Gray} \to \text{Strat}$ for the nerve construction we will discuss and to infer that in this case Strat denotes the category of large stratified sets.

### 4. The Classical Theory of Homotopy Coherent Nerves

Now we recall (a suitable version of) the homotopy coherent nerve construction for categories enriched in simplicial sets (called simplicially enriched categories). Much of the work presented in this section is due to Cordier [1] and Cordier and Porter [2].

**Definition 13 (the homotopy coherent $\omega$-path).** We define a locally ordered category $\overline{S}$ whose objects are the integers and whose homsets are ordered cubes, that is to say that they are all suitable powers

$$[1]^n \overset{\text{def}}{=} [1] \times [1] \times \cdots \times [1]$$

of the two point ordinal $[1]$. We recall briefly that, as an iterated product of ordered sets, such powers are ordered “pointwise”, that is to say $(a_n, a_{n-1}, \ldots, a_1) \leq (b_n, b_{n-1}, \ldots, b_1)$ iff $a_i \leq b_i$ for all $i = 1, 2, \ldots, n$. We choose to index the ordinates of our tuples from right to left, a convention we adopt in order to simplify our notation later on.

For integers $r < s$ our homset $\overline{S}(r, s)$ will be defined to be isomorphic to $[1]^{(s-r-1)}$, however for reasons of combinatorial convenience we actually elect to specify it as a subset of $[1]^{(s-t)}$:

$$\overline{S}(r, s) = \begin{cases} \emptyset & \text{if } s < r, \\ \{()\} = [1]^0 & \text{if } s = r, \\ \{(a_s, a_{s-1}, \ldots, a_{r+1}) \in [1]^{(s-r)} \mid a_s = 0\} & \text{if } s > r. \end{cases}$$

A consequence of this convention is that while $\overline{S}(r, r)$ and $\overline{S}(r, r+1)$ are both isomorphic to the single point set, their respective elements are clearly distinguished.
in intention. Notice also that we adopt the convention of indexing the elements of an arrow in $\mathcal{S}(r,s)$ using the integers $r+1, r+2, \ldots, s$, which further simplifies and illuminates many of the arguments we make later on.

The composition of $\mathcal{S}$ is now a matter of mere concatenation, in other words if $\vec{a} = (a_s, a_{s-1}, \ldots, a_{r+1})$ is an element of $\mathcal{S}(r,s)$ and $\vec{b} = (b_t, b_{t-1}, \ldots, b_{s+1})$ is an element of $\mathcal{S}(s,t)$ then their composite is defined by

$$(b_t, b_{t-1}, \ldots, b_{s+1}) \circ (a_s, a_{s-1}, \ldots, a_{r+1}) \overset{\text{def}}{=} (b_t, b_{t+1}, a_s, \ldots, a_{r+1})$$

which is clearly an element of $\mathcal{S}(r,t)$. Of course this operation is associative and it preserves the orderings of the homsets since they are defined pointwise. It also immediately illuminates why we defined $\mathcal{S}(r,r)$ to contain only the unique 0-tuple $()$, since this is the identity for concatenation of tuples. It follows therefore that $\mathcal{S}$ is a well-defined locally ordered category.

Our real interest, however, is in defining an associated simplicially enriched category $\mathcal{S}$ by applying the classical categorical nerve functor $N: \text{Cat} \rightarrow \text{Simp}$ (cf. observation 4 of [12]) to each of the homsets of $\mathcal{S}$. Abstractly we find that this process gives us a category enriched over $\text{Simp}$ simply because we know that $N$ preserves products (it is a right adjoint). More concretely, however, the $l$-simplices of $\mathcal{S}(r,s)$ are ordered sequences of tuples $\vec{a}_0 \leq \vec{a}_1 \leq \cdots \leq \vec{a}_l$ of length $(l+1)$ in $\mathcal{S}(r,s)$ and these are composed pointwise, in other words if $\vec{b}_0 \leq \vec{b}_1 \leq \cdots \leq \vec{b}_l$ is a $l$-simplex of $\mathcal{S}(s,t)$ then their composite is $\vec{b}_0 \circ \vec{a}_0 \leq \vec{b}_1 \circ \vec{a}_1 \leq \cdots \leq \vec{b}_l \circ \vec{a}_l$.

We might call $\mathcal{S}$ the generic homotopy coherent $\omega$-path, since simplicial functors out of it and into any other simplicially enriched category $\mathcal{E}$ correspond to weakened functors from the ordered set $\mathcal{N}$ into $\mathcal{E}$ which preserve composition up to coherent homotopy. It was this point of view which originally motivated Cordier to study structures of this kind [1].

Before moving on we will also adopt the notation $\mathcal{S}[n]$ (resp. $\mathcal{S}[n]$) for the (enriched) full subcategory of $\mathcal{S}$ (resp. $\mathcal{S}$) whose objects are the elements of the ordinal $[n]$, which we might think of as being the generic homotopy coherent $n$-path.

**Observation 14** ($\mathcal{S}$ as a free structure). We may also describe $\mathcal{S}$ as a freely generated locally ordered category. To do so, observe that if we have an arrow $\vec{a} = (a_s, \ldots, a_{r+1})$ in $\mathcal{S}(r,s)$ then we may find some index $r < k < s$ for which $a_k = 0$ if and only if we may decompose $\vec{a}$ as a composite of two non-identity arrows $(a_k, \ldots, a_{r+1})$ in $\mathcal{S}(r,k)$ and $(a_k, \ldots, a_{k+1})$ in $\mathcal{S}(k,s)$. Consequently there is precisely one arrow in $\mathcal{S}(r,s)$ which is not decomposable in this way, that being the one consisting of a sequence of $(s-r-1)$ copies of 1 followed by a single 0 and for which we adopt the denotation $\langle r,s \rangle$. Furthermore any $\vec{a} \in \mathcal{S}(r,s)$ may be “split at its 0s” to express it uniquely as a composite of indecomposable arrows. More precisely if $r_1 < r_2 < \cdots < r_k = s$ enumerate the indices at which the $\vec{a}$ has a ordinate with value 0 then we may uniquely express it as the composite $\langle r_{k-1}, r_k \rangle \circ \cdots \circ \langle r_1, r_2 \rangle \circ \langle r, r_1 \rangle$.

This certainly demonstrates that $\mathcal{S}$ is freely generated as a mere category by its indecomposable arrows, indeed we’ve shown that it is the free category associated with the (reflexive) graph underlying the ordered set $\mathcal{N}$ (qua category). We can go further, however, and describe its local orders in a similar fashion, by observing that if $\vec{a} \leq \vec{b}$ in $\mathcal{S}(r,s)$ then $\vec{a}$ has a 0 wherever $\vec{b}$ has one and thus that the splitting process described above (at least) splits the former wherever it splits the
latter. Consequently any such inequality may be obtained by using $\circ$ to compose inequalities of the form $\langle r_{k-1}, r_k \rangle \circ \cdots \circ \langle r_1, r_2 \rangle \circ \langle r_0, r_1 \rangle \leq \langle r_0, r_k \rangle$ which in turn may be expressed canonically in terms of the most primitive inequalities of the form $\langle s, t \rangle \circ \langle r, s \rangle \leq \langle r, t \rangle$ (for $r < s < t$).

Summarising all of this we see that, as a locally ordered category, $\mathbb{S}$ is freely generated by its indecomposable arrows $\langle r, s \rangle$ subject to the primitive inequalities $\langle s, t \rangle \circ \langle r, s \rangle \leq \langle r, t \rangle$. In other words, if $\mathcal{E}$ is any other locally ordered category then to (uniquely) define a locally ordered functor $f: \mathbb{S} \rightarrow \mathcal{E}$ it is enough to specify its action $f(r)$ on objects and $f(r, s)$ on indecomposable arrows and to check that $f(r) = \text{dom}(f(r, s))$ and $f(s) = \text{cod}(f(r, s))$ in $\mathcal{E}$ (for all $r < s$) and that $f(s, t) \circ f(r, s) \leq f(r, t)$ in the ordered set $\mathcal{E}(f(r), f(t))$ (for all $r < s < t$).

Observation 15. We may now construct a functor from $\Delta$ to the category of locally ordered categories, which carries $[n]$ to $\mathbb{S}[n]$. To define its action on simplicial operators $\alpha: [n] \rightarrow [m]$ we see from the last observation that it is enough to specify how the order enriched functor $\mathbb{S}(\alpha): \mathbb{S}[n] \rightarrow \mathbb{S}[m]$ should act on indecomposable arrows $\langle r, s \rangle$ in $\mathbb{S}[m]$. This, however, is easily done by letting

$$\mathbb{S}(\alpha)(r, s) \overset{\text{def}}{=} \langle \alpha(r), \alpha(s) \rangle \tag{1}$$

under the convention that the notation $(t, t)$, which becomes an issue if $\alpha(r) = \alpha(s)$, is taken to mean the identity 0-tuple $()$ in $\mathbb{S}(t, t)$. The verifications of the conditions given in the final paragraph of the last observation are now a trivial matter, thus giving the required functor. Furthermore, if $\beta: [m] \rightarrow [p]$ is a second simplicial operator than we may show that the functors $\mathbb{S}(\beta) \circ \mathbb{S}(\alpha)$ and $\mathbb{S}(\beta \circ \alpha)$ are identical by checking that they coincide on indecomposables, simply using the definition in (1), and then appealing to the uniqueness clause of the last observation.

Again, our intent here is actually to use the functor of the last paragraph to construct a functor $\mathcal{S}$ from $\Delta$ to the category of simplicially enriched categories and simplicial functors. The we may now do simply by applying the categorical nerve construction $N: \text{Cat} \longrightarrow \text{Simp}$ to the homsets and maps of homsets that arise when we apply the functor $\mathbb{S}$. Explicitly, this functor maps $[n]$ to the simplicially enriched category $\mathbb{S}[n]$ of observation 13 and it maps a simplicial operator $\alpha: [n] \rightarrow [m]$ to a simplicial functor $\mathcal{S}(\alpha): \mathbb{S}[n] \rightarrow \mathbb{S}[m]$ which acts like $\alpha$ on objects and applies $\mathcal{S}(\alpha)$ pointwise to each $k$-simplex $\vec{a}_0 \preceq \vec{a}_1 \preceq \cdots \preceq \vec{a}_k$ in $\mathbb{S}(r, s) = \mathbb{S}[n](r, s)$ to give $\mathcal{S}(\alpha)(\vec{a}_0) \leq \mathcal{S}(\alpha)(\vec{a}_1) \leq \cdots \leq \mathcal{S}(\alpha)(\vec{a}_k)$ in $\mathcal{S}(\alpha(r), \alpha(s)) = \mathbb{S}[m](\alpha(r), \alpha(s))$.

Recall 16 (homotopy coherent nerve). We may apply Kan’s construction [6] to the functor $\mathcal{S}$ of the last observation and form an adjoint pair of functors:

$$\text{Simp-Cat} \xrightarrow{\text{F}_{hc}} \mathcal{S} \xleftarrow{\text{N}_{hc}} \text{Simp}$$

The functor $\text{N}_{hc}$ is customarily called the homotopy coherent nerve construction. It carries a simplicially enriched category $\mathcal{E}$ to a simplicial set whose $n$-simplices are simplicial functors $f: \mathbb{S}[n] \rightarrow \mathcal{E}$ and whose simplicial action is given by precomposition $f \cdot \alpha \overset{\text{def}}{=} f \circ \mathbb{S}(\alpha)$. Furthermore, its left adjoint may be expressed as the colimit $\text{F}_{hc}(X) \cong \text{colim}(X, \mathcal{S})$ of the diagram $\mathbb{S}$ in $\text{Simp-Cat}$ weighted by the presheaf $X: \Delta^{op} \rightarrow \text{Set}$ (cf. Kelly [7]).
Observation 17 (horns in homotopy coherent nerves). In [2] Cordier and Porter demonstrate that the homotopy coherent nerve of a simplicially enriched category whose homsets are all Kan complexes is actually a quasi-category (weak Kan complex). We will prove a generalisation of this result for Gray-categories below. For now however it will be useful to provide an explicit description of the inner horns \( \Lambda^k[n] \rightarrow N_{hc}(E) \) of a homotopy coherent nerve. Applying the adjunction \( F_{hc} \dashv N_{hc} \) we see that these correspond to simplicial functors \( \hat{h} : F_{hc}(\Lambda^k[n]) \rightarrow E \) so this task immediately reduces to that of providing an explicit description of the simplicially enriched category \( F_{hc}(\Lambda^k[n]) \). Indeed, we will actually show that this bears a natural description as a subcategory of \( F_{hc}(\Delta[n]) \cong S[n] \).

Observation 18 (\( S \) as a free structure). Forgetting, for the moment, the simplicial set structures on the homsets of \( S \) we obtain a category which may again be described as the free category on some reflexive graph. In particular, following the pattern laid down in observation 14, we consider an \( l \)-arrow \( \hat{a}_0 \leq \hat{a}_1 \leq \cdots \leq \hat{a}_t \) of \( S \) and observe that if its topmost tuple \( \hat{a}_t \) has a 0 at some index then each \( \hat{a}_i \) must also have a 0 at that index, since they are each less than or equal to the former in the pointwise ordering. It follows that each tuple in \( \hat{a}_0 \leq \hat{a}_1 \leq \cdots \leq \hat{a}_t \) “may be split at the 0s of \( \hat{a}_t \)” to express this \( l \)-arrow as a unique (pointwise) composite of indecomposable arrows, these being those arrows of \( S \) whose topmost tuple is an indecomposable arrow \( \langle r, s \rangle \) of \( S \). Furthermore, if \( \alpha : [n] \rightarrow [m] \) is a simplicial operator then by definition the simplicial functor \( S(\alpha) : S[n] \rightarrow S[m] \) carries an indecomposable \( l \)-arrow with topmost tuple \( \langle r, s \rangle \) to an \( l \)-arrow with topmost tuple \( \langle \alpha(r), \alpha(s) \rangle \), which is thus either indecomposable itself or an identity (when \( \alpha(r) = \alpha(s) \)). Consequently, if we regard \( S \) as a functor from \( \Delta \) to \( \text{Cat} \), then we may factor it (up to isomorphism) through the free category functor \( F : \text{Grph} \rightarrow \text{Cat} \), from reflexive graphs to categories, by restricting the image \( S[n] \) of each object \( n \in \Delta \) to its subgraph of indecomposables to obtain a functor \( \hat{S} : \Delta \rightarrow \text{Grph} \).

The utility of this observation lies in the fact that it enables us to evaluate the colimits used to define \( F_{hc} \) in observation 16. First we may happily calculate without worrying about the simplicial structure of homsets, since the forgetful functor \( \text{Simp-Cat} \rightarrow \text{Cat} \) preserves and reflects (indeed creates) the colimits of \( \text{Simp-Cat} \). Then we may construct the colimit \( F_{hc} \cong \text{colim}(X, S) \) in \( \text{Cat} \) by evaluating the corresponding colimit \( \text{colim}(X, \hat{S}) \) in \( \text{Grph} \), where such things are constructed as in \( \text{Set} \), and then applying the free category functor.

Observation 19. For instance, the last observation allows us to show that \( F_{hc} \) carries inclusions of simplicial sets to inclusions of simplicially enriched categories. To do so first observe that the functor \( \hat{S}(\delta_i) : \hat{S}[n-1] \rightarrow \hat{S}[n] \) of locally ordered categories is injective and that its image contains the arrow \( \hat{a} \in \hat{S}(r, s) \) if and only if \( s < i \) or \( i < r \) or \( (r < i < s \) and \( a_i = 1 \)). This latter fact may be ascertained simply by checking that it holds for indecomposables \( (r, s) \) and extending to all arrows using the decomposition at 0s result. It follows immediately that \( \hat{S}(\delta_i) : \hat{S}[n-1] \rightarrow \hat{S}[n] \) in \( \text{Grph} \) is also injective and that the indecomposable \( l \)-arrow \( \hat{a}_0 \leq \hat{a}_1 \leq \cdots \leq \hat{a}_t \) is in its image iff its bottommost tuple \( \hat{a}_0 \), and thus each one of its members, satisfies the condition of the last sentence. Applying this characterisation it is also clear
that for each $i < j$ we have a pullback

$$
\begin{array}{ccc}
\hat{S}[n-2] & \subseteq & \hat{S}[n] \\
\downarrow \hat{S}(\delta_i) & & \downarrow \hat{S}(\delta_i) \\
\hat{S}[n-1] & \subseteq & \hat{S}[n-1] \\
\end{array}
$$

(2)

in $\text{Grph}$. So consider the “wide pushout” diagram in simplicial sets consisting of $(n+1)$ copies of $\Delta[n-1]$ indexed by elements of $[n]$ and $(n+1)n/2$ copies of $\Delta[n-2]$ indexed by pairs $i, j \in [n]$ with $i < j$ each of which comes equipped with a pair of simplicial maps into two of the copies of $\Delta[n-1]$ as depicted below.

$$
\begin{array}{ccc}
\Delta[n-2]_{i,j} & \subseteq & \Delta[n-1]_i \\
\downarrow \Delta(\delta_{i,j}) & & \downarrow \Delta(\delta_{i}) \\
\Delta[n-1]_j & \subseteq & \Delta[n-1]_i \\
\end{array}
$$

(3)

The dotted inclusions in this picture provide a cone under our diagram and each of the diamonds above is a pullback in $\text{Simp}$. Therefore, since the colimits of $\text{Simp}$ are calculated as in $\text{Set}$, it is easily seen that the colimit of our wide pushout diagram is isomorphic to the simplicial set obtained by taking the union of the images of the dotted inclusions in $\Delta[n]$, which is simply the boundary $\partial \Delta[n]$.

Now recall that colimits weighted by representables are easily calculated by “evaluation”, in other words we have a natural isomorphism $\text{colim}(\Delta[n], S) \cong S[n]$. It follows that if we apply the functor $\text{colim}(-, \hat{S})$ to the diamond in display (3) then we obtain the one in display (2). Furthermore the weighted colimit construction preserves colimits in each variable, so in particular the functor $\text{colim}(-, \hat{S})$ preserves the colimit of our wide pushout $\partial \Delta[n]$. However, arguing just as we did before, using the fact that colimits in $\text{Grph}$ are calculated as in $\text{Set}$, we see that the pullbacks of display (2) allow us to show that the colimit of this wide pushout in $\text{Grph}$ is also isomorphic to the subgraph of $\hat{S}[n]$ obtained by taking the union of the images of the inclusions $\hat{S}(\delta_i): \hat{S}[n-1] \rightarrow \hat{S}[n]$. In other words, we have demonstrated that the induced map

$$
\text{colim}(\subseteq_s, \hat{S}): \text{colim}(\partial \Delta[n], \hat{S}) \rightarrow \text{colim}(\Delta[n], \hat{S})
$$

(4)

restricts to an isomorphism between its codomain and the subgraph of $\hat{S}[n] \rightarrow \text{colim}(\Delta[n], \hat{S})$ identified in the last sentence.

However, we also know that every inclusion $i: X \subseteq Y$ of simplicial sets may be constructed as a transfinite composite of pushouts of the boundary inclusions $\subseteq_s: \partial \Delta[n] \rightarrow \Delta[n]$. So applying the colimit preservation property of $\text{colim}(-, \hat{S})$ again we see that the induced map $\text{colim}(i, \hat{S}): \text{colim}(X, \hat{S}) \rightarrow \text{colim}(Y, \hat{S})$ may be constructed as a transfinite composite of pushouts of the inclusions of display (4), and is thus itself an inclusion since these operations clearly preserve inclusions in $\text{Set}$ and therefore also do so in $\text{Grph}$. Finally, applying the free category functor, which carries inclusions of graphs to inclusions of categories, and applying observation 18 we find that the simplicial functor $F_{hc}(i): F_{hc}(X) \rightarrow F_{hc}(Y)$ is also an inclusion.
of simplicially enriched categories. In particular, the simplicial functor

$$F_{hc}(\subseteq): F_{hc}(\Lambda^k[n]) \longrightarrow F_{hc}(\Delta[n]) \cong S[n]$$

is an inclusion, as suggested above, and its image $\mathbb{H}^k[n]$, called the homotopy coherent $k$-horn, can be characterised as being the smallest subcategory of $S[n]$ containing the images of $S(\delta_i): S[n-1] \subseteq S[n]$ for each $i \in [n] \setminus \{k\}$.

5. Nerves of $\mathbf{Strat}_S$-Categories

We seek to extend this classical material and apply it to the problem of providing a well-behaved nerve construction which carries Gray-categories to weak complicial sets. To do so we first provide the homsets of $S$ with an appropriate stratification.

Observation 20. It will be convenient for what follows to slightly rephrase the definition of $S$ given in definition 13. To do so recall that the categorical nerve construction $N: \mathbf{Cat} \longrightarrow \mathbf{Simp}$ preserves products and carries each ordinal $[n]$ to the standard simplex $\Delta[n]$. Furthermore, we know that the homset $S(r, s)$ $(r \leq s)$ is defined to be a subset of the iterated product $[1]^{(s-r)}$, so it follows that we can think of $S(r, s)$, its nerve, as the simplicial subset of $N([1]^{(s-r)}) \cong \Delta[1]^{(s-r)}$ of those simplices $\vec{\alpha} = (\alpha_s, ..., \alpha_{r+1})$ for which $\alpha_s$ is the constant 0 operator defined in notation 3. Under this representation the composition operation is again given by concatenation of tuples of operators. More abstractly, it will be useful to recast this by saying that composition is a restriction of the canonical associativity isomorphism of the monoidal category $(\mathbf{Simp}, \times, \Delta[0])$ depicted in the following square:

$$\begin{array}{ccc}
S(s,t) \times S(r,s) & \longrightarrow & S(r,t) \\
\subseteq & & \subseteq \\
\Delta[1]^{(t-s)} \times \Delta[1]^{(s-r)} & \longrightarrow & \Delta[1]^{(t-r)}
\end{array}$$

Of course, it is immediate that an arrow $\vec{\alpha}$ in $S(r, s)$ is decomposable iff there exists some integer $k$ with $r < k < s$ and for which $\alpha_k$ is equal to the constant operator 0 of notation 3.

We will also find it useful to identify, in these terms, the arrows of $S[n]$ which are in the inner homotopy coherent horn $\mathbb{H}^k[n]$ of observation 19. Recasting the characterisation derived in the first paragraph of that observation, we find that our arrow $\vec{\alpha}$ of $S[n](r, s)$ is in the image of a face inclusion $S(\delta_i): S[n-1] \subseteq S[n]$ iff $s < i$ or $i < r$ or $(r < i < s$ and $\alpha_i = 1)$. In the case of $S(\delta_0)$ this is simply the full subcategory of $S[n]$ on the objects $1, 2, ..., n$, whereas for $S(\delta_n)$ it is the full subcategory on objects $0, 1, ..., n-1$. This immediately implies that $\mathbb{H}^k[n]$ completely contains all homsets of $S[n]$ except for $S[n](0,n)$. Furthermore, any decomposable arrow of $S[n](0,n)$ factors into a composite of arrows in $S[n](0,r)$ and $S[n](r,n)$ for some $0 < r < n$, but the former homset is in the image of $S(\delta_n)$ and the latter is in the image of $S(\delta_0)$ so it follows that their composite is in the subcategory generated by the union of these images. Combining this observation with our characterisation of decomposable arrows and throwing in the images of $S(\delta_i)$ for the remaining faces with $i \neq k$, we find that the arrow $\vec{\alpha}$ of $S(0,n)$ is in $\mathbb{H}^k[n](0,n)$ iff there exists some integer $i$ with $0 < i < n$ for which either $\alpha_i = 0$ or $(i \neq k$ and $\alpha_i = 1)$. 

After a few moments reflection, it becomes clear that this latter characterisation may itself be re-expressed in terms of the corner product $\times_c$ associated with the product of simplicial sets as defined in observation 116 of [12]. To be precise, the simplicial map which “forgets the leftmost ordinate” provides us with an isomorphism between $S[n](0, n)$ and $\Delta[1]^{(n-1)}$ the codomain of the following iterated corner product
\[(\partial\Delta[1] \triangleleft \Delta[1]) \times_c (\Lambda^1[1] \triangleleft \Delta[1]) \times_c (\partial\Delta[1] \triangleleft \Delta[1]) \times_c (\Lambda^1[1] \triangleleft \Delta[1]) \times_c \cdots)
\]and this restricts to provide us with an isomorphism between its domain and $\mathbb{H}^k[n](0, n)$.

**Observation 21** (the nerve of homotopy coherent paths). Consider the functor $\text{th}_0: \text{Simp} \to \text{Strat}$ which stratifies each simplicial set using the maximal stratification. This is right adjoint to the stratification forgetting functor and thus preserves all limits, so in particular it provides us with a strict monoidal functor from $(\text{Simp}, \times, \Delta[0])$ to $(\text{Strat}, \circ, \Delta[0])$. Consequently, we may apply this trivialisation locally to the homsets of simplicially enriched categories to obtain a functor $\text{th}_0: \text{Simp-Cat} \to \text{Strat}_{hc-Cat}$. Composing this with the functor $\mathcal{S}: \Delta \to \text{Simp-Cat}$ of observation 15 we obtain a functor which we shall also denote by $\mathcal{S}$ and from which we may derive an adjunction
\[
\text{Strat}_{hc-Cat} \quad \downarrow \quad \text{Simp}
\]
by Kan’s construction. The simplicial set $H_{hc}(\mathcal{E})$ is called the *nerve of homotopy coherent paths*.

Now suppose that $\mathcal{E}$ is a $\text{Strat}_{hc}$-category, then applying the analysis of horns in homotopy coherent nerves, which we commenced in observation 17, it becomes clear that the nerve $N_{hc}(\mathcal{E})$ is a quasi-category iff $\mathcal{E}$ has the RLP with respect to each inner homotopy coherent horn inclusion $\mathbb{H}^k[n] \to S[n]$, where again $\mathbb{H}^k[n]$ is stratified by applying $\text{th}_0$ to its homsets. In turn, this reduces to showing that each homset $\mathcal{E}(c, e')$ of $\mathcal{E}$ has the RLP with respect to $\mathbb{H}^k[n](0, n) \triangleleft S[n](0, n)$, the only homset inclusion at which $\mathbb{H}^k[n]$ and $S[n]$ differ. Returning to observation 20 we see that this latter inclusion is isomorphic to the one obtained by applying $\text{th}_0$ to the the corner product in display (7) and appealing to the monoidal properties of $\text{th}_0$ we see that this is equal to the iterated corner tensor
\[(\partial\Delta[1] \triangleleft \Delta[1]) \odot_c (\Lambda^1[1] \triangleleft \Delta[1]) \odot_c (\partial\Delta[1] \triangleleft \Delta[1]) \odot_c \cdots)
\]in $\text{Strat}$. Finally, the inner factor $\Lambda^1[1] \triangleleft \Delta[1]$ of this corner tensor is an elementary anodyne extension and its remaining factors are all inclusions so we may apply corollary 71 of [12] to show that this inclusion enjoys the LLP with respect to all weak complicial sets, and thus in particular that it does so with respect to the homsets of all Gray-categories. It follows therefore that the nerve $N_{hc}(\mathcal{G})$ is a quasi-category whenever $\mathcal{G}$ is a Gray-category.

**Observation 22.** The result of the last observation is somewhat uninspiring, since all we have done is to reprove Cordier and Porter’s analysis of homotopy coherent nerves [2]. It is important to note that the nerve $N_{hc}(\mathcal{G})$ takes no account whatsoever of the non-thin simplices in the homsets of $\mathcal{G}$. To rectify this deficiency
we propose to stratify \( S \) more carefully as a \( \text{Strat}_\otimes \)-category, by replacing the use of cartesian products of simplicial sets in observation 20 by lax Gray tensor products of the corresponding stratified sets and insisting that \( S(r, s) \) is given a stratification which makes it into a regular subset of the \((s - r)\)-fold tensor power of \( \Delta[1] \). By doing so in the next few pages we will construct a stratified nerve which \textit{faithfully} represents Gray-categories as weak complicial sets.

\textbf{Definition 23.} Let \( S_\otimes \) denote the \( \text{Strat}_\otimes \)-category whose underlying simplicially enriched category, obtained by forgetting stratifications on homsets, is \( S \) and for which each homset \( S_\otimes(r, s) \) \((r \leq s)\) is a regular subset of the iterated tensor:

\[
\Delta[1]^{\otimes(s-r)} \overset{\text{def}}{=} \underbrace{\Delta[1] \otimes [1] \otimes \cdots \otimes \Delta[1]}_{(s-r) \text{ factors}}
\]

\textbf{Observation 24.} This definition depends upon observation 58 of [12], wherein we find that the monoidal structure associated with the tensor \( \otimes \) is completely determined by the fact that the stratification forgetting functor is a strict monoidal functor from \((\text{Strat}_\otimes, \otimes, \Delta[0])\) to \((\text{Simp}, \times, \Delta[0])\). First of all, this ensures that if we forget the stratifications of the homsets of any \( \text{Strat}_\otimes \)-category we do actually obtain a genuine simplicially enriched category. It also ensures that \( S_\otimes \) is completely determined by the given conditions, since the stratified maps that make up its structure are themselves determined by their underlying simplicial maps which we’ve specified must be the corresponding components of the structure of \( S \). Indeed, in order to demonstrate the existence of \( S_\otimes \) all we need do is show that the compositions of \( S \) respect the stratifications we’ve specified for our homsets.

To that end, consider the stratified isomorphism at the bottom of the following square

\[
\begin{array}{ccc}
S_\otimes(s, t) \otimes S_\otimes(r, s) & \overset{\circ}{\longrightarrow} & S_\otimes(r, t) \\
\subseteq_r & & \subseteq_r \\
\Delta[1]^{\otimes(t-s)} \otimes \Delta[1]^{\otimes(s-r)} & \cong & \Delta[1]^{\otimes(t-r)}
\end{array}
\]

obtained from the monoidal structure associated with \( \otimes \). Since the stratification functor is strictly monoidal we know that the simplicial map underlying this isomorphism is simply the corresponding isomorphism for \( \times \) as depicted in display (6).

So we may define the composition of \( S_\otimes \) to be the restriction to regular subsets depicted and immediately infer that its underlying simplicial map is the composition of \( S \) as postulated.

Notice that the lax Gray tensor \( \otimes \) preserves regularity, in the sense that if \( f \) and \( g \) are regular stratified maps then so is \( f \otimes g \) (see observation 132 of [14]). It follows that the map \( \Delta(e_0^r) \otimes \Delta[1]^{\otimes(s-r-1)} \Delta[1]^{\otimes(s-r-1)} \rightarrow \Delta[1]^{\otimes(s-r)} \) is a regular inclusion and that it restricts to a canonical isomorphism between \( S_\otimes(r, s) \) and \( \Delta[1]^{\otimes(s-r-1)} \).

\textbf{Observation 25.} To show that the functor \( S : \Delta \longrightarrow \text{Simp}_\otimes \text{Cat} \) also lifts to a functor \( S_\otimes : \Delta \longrightarrow \text{Strat}_\otimes \text{Cat} \) we again apply the methodology of the last observation.

In other words, we first show that we may describe some aspect of the structure of \( S \) abstractly in terms of the monoidal structure of \((\text{Simp}, \times, \Delta[0])\), then make the analogous construction for \( S_\otimes \) using the monoidal structure of \((\text{Strat}_\otimes, \otimes, \Delta[0])\) and finally appeal to the to the strong monoidality of the forgetful functor to show that the former simplicial structure underlies the newly constructed stratified structure.
To simplify matters, observe that it is enough to analyse the simplicial functors obtained by applying $S$ to each of the elementary face and degeneracy operators, since they generate $\Delta$. So consider these in turn:

- $S(\delta_k) : S[n-1] \rightarrow S[n]$. This acts on objects $r \in [n-1]$ simply by applying $\delta_k$ and we may ascertain its action on homsets by first considering how the corresponding functor $\mathbb{S}(\delta_k)$ acts on an indecomposable $(r,s)$, as we did in observation 19. On doing so we find that

$$
\mathbb{S}(\delta_k)(r,s) = \begin{cases} 
(r,s) & \text{if } s < k, \\
(r+1,s+1) & \text{if } k \leq r, \\
(r,s+1) & \text{if } r < k \leq s.
\end{cases}
$$

where we might think of the second case as shifting our indecomposable left to make space for a new symbol at index $k$ below it and the third case as the actual insertion of a new 1 symbol at index $k$ in our indecomposable. Extending this to all arrows in $S(r,s)$ we find that this latter description also succinctly summarises the action of $\mathbb{S}(\delta_k)$ and immediately provides the following case-wise description of the related simplicial functor $S(\delta_k)$ on a homset $S(r,s)$:

- $s < k$ wherein it is simply the identity on $S(r,s)$,
- $k \leq r$ when it is the canonical isomorphism between $S(r,s)$ and $S(r+1,s+1)$,
- $r < k \leq s$ in which case it is the simplicial map which carries a simplex $(\alpha_s, \ldots, \alpha_{r+1})$ of $S(r,s)$ to the simplex $(\alpha_s, \ldots, \alpha_k, 1, \alpha_{k-1}, \ldots, \alpha_{r+1})$ in $S(r,s+1)$, or in other words we can picture it as the restriction:

$$
\begin{array}{ccc}
S(r,s) & \xrightarrow{\mathbb{S}(\delta_k)} & S(r,s+1) \\
\Delta[1] & \xrightarrow{\Delta[1]^{(s-r)}} & \Delta[1]^{(s-r+1)}
\end{array}
$$

- $S(\sigma_k) : S[n+1] \rightarrow S[n]$. Again this acts on objects $r \in [n+1]$ by applying $\sigma_k$ and we may again determine its action on homsets by considering how the corresponding functor $\mathbb{S}(\sigma_k)$ acts on an indecomposable $(r,s)$:

$$
\mathbb{S}(\sigma_k)(r,s) = \begin{cases} 
(r,s) & \text{if } s < k + 1, \\
(r-1,s-1) & \text{if } k + 1 \leq r, \\
(r,s-1) & \text{if } r < k + 1 \leq s.
\end{cases}
$$

This time we cannot simply interpret this as the mere removal of a symbol at index $k + 1$, since if $k + 1 = s$ this would result in the removal of a terminating 0 to give a tuple which is not an element of $\mathbb{S}(r,s-1)$. A moment’s reflection, however, reveals that we may instead summarise the action of $\mathbb{S}(\sigma_k)$ on a general arrow $(a_s, a_{s-1}, \ldots, a_{r+1})$ of $\mathbb{S}(r,s)$ by saying that it drops the rightmost ordinate $a_{r+1}$ if $r = k$ and otherwise replaces the ordinates $a_k$ and $a_{k+1}$ by a single entry obtained by taking the minimum $\min(a_k, a_{k+1})$ of these two values if $r < k < s$. We may now apply this observation to obtain a case-wise description of $S(\sigma_k)$ on a homset $S(r,s)$:

- $s \leq k$ wherein it is simply the identity on $S(r,s)$,
- $k < r$ when it is the canonical isomorphism between $S(r,s)$ and $S(r-1,s-1)$,
- $k = r$ in which case it is the map which drops the initial ordinate of each simplex $(\alpha_s, \ldots, \alpha_{r+1})$ in $S(r,s)$, or in other words we can picture it as the
of simplices in an iterated product $\Delta[1]^{(s-r)}$ of simplices in a strict interval. Our primary use for these intervals is that they index the ordinates $S(r, s)$, thereby showing that $S(r, s) = \{\alpha_k, \alpha_{k+1}\}$ of each simplex $(\alpha_s, ..., \alpha_{r+1})$ in $S(r, s)$ by their pointwise minimum $\min(\alpha_k, \alpha_{k+1})$, or in other words we can picture it as the restriction:

$$
\begin{align*}
\mathbb{S}(r, s) & \xrightarrow{S(\sigma_k)} \mathbb{S}(r, s-1) \\
\subseteq & \xleftarrow{\sigma_k} \\
\Delta[1]^{(s-r)} & \xrightarrow{\Delta[1]^{(s-r-1)} \times \Delta(\sigma_0)} \Delta[1]^{(s-r-1)}
\end{align*}
$$

$r < k < s$ in which case it is the map which replaces ordinates $\alpha_k$ and $\alpha_{k+1}$ of each simplex $(\alpha_s, ..., \alpha_{r+1})$ in $S(r, s)$ by their pointwise minimum $\min(\alpha_k, \alpha_{k+1})$, or in other words we can picture it as the restriction:

$$
\begin{align*}
\mathbb{S}(r, s) & \xrightarrow{S(\sigma_k)} \mathbb{S}(r, s-1) \\
\subseteq & \xleftarrow{\sigma_k} \\
\Delta[1]^{(s-r)} & \xrightarrow{\Delta[1]^{(s-r-1)} \times \Delta[1]^{(s-r-1)}} \Delta[1]^{(s-r-1)}
\end{align*}
$$

Consequently, in each of these cases we may apply the argument outlined in the first paragraph, thereby showing that $S(\delta_k)$ and $S(\sigma_k)$ lift to $\text{Strat}_{\mathbb{S}}$-enriched functors $\mathbb{S}(\delta_k): \mathbb{S}[n-1] \to \mathbb{S}[n]$ and $\mathbb{S}(\sigma_k): \mathbb{S}[n+1] \to \mathbb{S}[n]$, and thus completing the demonstration that the functor $S$ lifts as promised.

**Notation 26.** Let $(r, s)$ denote $\{i \in \mathbb{N} \mid r < i < s\}$ the strict interval of integers between $r$ and $s$ and let $(r, s)$ denote $\{i \in \mathbb{N} \mid r < i \leq s\}$ the corresponding half strict interval. Our primary use for these intervals is that they index the ordinates of simplices in an iterated product $\Delta[1]^n$ or a homset $\mathbb{S}_\otimes(r, s)$.

It is clear that the $m$-simplices of $\Delta[1]^{(s-r)}$ correspond to arbitrary functions $w: (r, s) \to [m]$, simply because the operators of definition 3 enumerate all of the $m$-simplices of $\Delta[1]$ and are indexed by $[m]$. The simplex $\vec{\rho}_w$ associated with such a $w$ is simply the $(s-r)$-tuple $(\rho_{w(r)}^m, \rho_{w(s-1)}^m, ..., \rho_{w(r+1)}^m)$. We will, if necessary, implicitly extend a function $w: (r, s) \to [m]$ to a function with domain $(r, s)$ by letting $w(s) = -$, under which convention the $m$-simplex $\vec{\rho}_w$ becomes an element of the homset $\mathbb{S}_\otimes(r, s)$. Furthermore, we say that such a function is:

(i) a *partial bijection* if for each integer $i \in [m]$ there exists a unique integer $j$ in its domain such that $w(j) = i$.

(ii) *order reversing* if whenever $i, j$ are integers in the domain of $w$ with $i \leq j$ and for which $f(i)$ and $f(j)$ are integers in $[m]$ then $f(j) \leq f(i)$.

Notice that these definitions impose no conditions at those integers in the domain of $w$ which actually map to one of the points $- +$ in its codomain.

**Observation 27** (simplicial cubes). In order to understand the homsets $\mathbb{S}(r, s)$ more thoroughly, we should rehearse some basic facts about the structure of the simplicial cube $\Delta[1]^n$ and apply them to analysing our stratified cube $\Delta[1]^{\otimes n}$. Firstly, it is clear that the $m$-simplex $\vec{\rho}_w \in \Delta[1]^n$ associated with a function $w: [0, n] \to [m]$ is degenerate iff there is some integer $k \in [m-1]$ such that $\rho_{w(i)}(k) = \rho_{w(i)}(k+1)$ for all $i = 1, 2, ..., n$ and this happens iff $\rho_{k+1}$ is not numbered amongst its ordinates, since this is the only such operator which maps $k$ and $k+1$ to different elements of $[1]$. In other words, the simplex corresponding to $w$ is degenerate iff $w$ is not surjective onto the integers (non-points) of $[m]$. This immediately implies that all
of the simplices of $\Delta[1]^n$ of dimension greater than $n$ are degenerate and that its non-degenerate $n$-simplices correspond to partial bijections $w: (0, n) \longrightarrow [n]$.

Extending this analysis to study the stratification of $\Delta[1]^{\otimes n}$, we may show that if $w: (0, n) \longrightarrow [m]$ is a partial bijection then the corresponding $m$-simplex $\bar{\rho}_w$ is non-thin if and only if $w$ is order reversing. First observe that we only need demonstrate that the result holds when $n = m$, since if $m < n$ then $\bar{\rho}_w$ is a special simplex in some $m$-dimensional cubical face of $\Delta[1]^{\otimes n}$ isomorphic to $\Delta\otimes^m$. To prove the “only if” part, we will assume, for a contradiction, that there exists some pair of integers $i < j$ in $(0, n]$ for which $w(i) < w(j)$. Of course, we know that $\Delta[1]^{\otimes n}$ is isomorphic to $\Delta[1]^{\otimes(n-i)} \otimes \Delta[1]^{\otimes i}$ and if we express $\bar{\rho}_w$ as a simplex in the latter set it becomes a pair with first component $(\rho^a_w(n), \ldots, \rho^a_w(i+1))$ and second component $(\rho^a_w(i), \ldots, \rho^a_w(1))$. Since $w$ is a partial bijection we know that $\rho^a_w(i)$ does not occur as an ordinate of the first of these components, (since it occurs in the second) and thus that it is degenerate at $w(i) - 1$ by the argument of the last paragraph. Dually we see that the second component is degenerate at $w(j) - 1$. Now we may apply lemma 129 of [14] to show that our pair $((\rho^a_w(n), \ldots, \rho^a_w(i+1)), (\rho^a_w(i), \ldots, \rho^a_w(1)))$ is thin in $\Delta[1]^{\otimes(n-i)} \otimes \Delta[1]^{\otimes i}$ and thus that $\bar{\rho}_w$ is thin in $\Delta[1]^{\otimes n}$, thereby providing the desired contradiction.

For the implication in the “if” direction we define an order preserving function $\overline{c}^n: [1]^n \longrightarrow [n]$ by letting

$$c^n(a_n, a_{n-1}, \ldots, a_1) \overset{\text{def}}{=} \min\{(n) \cup \{n - i \mid i \in (0, n] \land a_i = 0\}\}$$

from which we construct a simplicial map $c^n: \Delta[1]^n \rightarrow \Delta[n]$ by applying the categorical nerve construction $N: \text{Cat} \rightarrow \text{Simp}$. It is the case that this is the underlying simplicial map of a stratified map $c^n: \Delta[1]^{\otimes n} \rightarrow \Delta[n]$, which fact we prove via a simple induction which re-expresses $\overline{c}^n$ as a composite

$$[1]^n \xrightarrow{\overline{c}^{n-1}} [1] \times [n-1] \xrightarrow{d^n} [n]$$

wherein the order preserving map $d^n$ is given by:

$$d^n(i, j) = \begin{cases} 0 & \text{if } i = 0, \\ j+1 & \text{if } i = 1. \end{cases}$$

Applying the nerve construction to this composite we obtain a pair of simplicial maps which compose to give $c^n$. Considering these factors in turn, we know that $\overline{c}^{n-1}$ underlies a suitable stratified map, by the induction hypothesis, so the first of these $\Delta[1] \times \overline{c}^{n-1}$ is the underlying simplicial map of the stratified map $\Delta[1] \otimes \overline{c}^{n-1}: \Delta[1]^{\otimes n} \rightarrow \Delta[1] \otimes \Delta[n-1]$. To show that the second factor also underlies a suitable stratified map, we may appeal corollary 130 of loc. cit. to find that a simplex $(\alpha, \beta) \in \Delta[1] \otimes \Delta[n-1]$ is thin if and only if there exists a $k \leq l$ such that $\alpha(k) = \alpha(k+1)$ and $\beta(l) = \beta(l+1)$. If that thin simplex is also non-degenerate then it follows that we have $0 \leq k < l$ and $\alpha(l) < \alpha(l + 1)$, so $\alpha(l - 1) = \alpha(l) = 0$ and it follows that $\overline{c}^n$ maps both of the vertices $(\alpha(l - 1), \beta(l - 1))$ and $(\alpha(l), \beta(l))$ to the vertex 0 of $\Delta[n]$. Consequently, $d^n$ maps our thin simplex $(\alpha, \beta)$ to a simplex in $\Delta[n]$ which is degenerate at $(l - 1)$, and is thus thin, from which it follows that $d^n$ also underlies a stratified map $d^n: \Delta[1] \otimes \Delta[n-1] \rightarrow \Delta[n]$. So we find that the composite of these factors is a stratified map $c^n: \Delta[1]^{\otimes n} \rightarrow \Delta[n]$ and we may now prove that the $n$-simplex $\bar{\rho}_w$ is non-thin in $\Delta[1]^{\otimes n}$ simply by observing that
we obtain an adjoint pair:

\[
C^n \text{ maps it to the unique non-degenerate } n\text{-simplex } \text{id}_{[n]} \text{ in } \Delta[n] \text{ which is non-thin in there.}
\]

**Definition 28.** We shall say that a function \( w: (r, s) \rightarrow [m] \) is special if it is partially bijective and order reversing. Consulting observation 27 we see that these conditions ensure that the corresponding special simplex \( \bar{\rho}_w \) is a non-thin \( m \)-simplex of \( \mathcal{S}_\otimes(r, s) \). Furthermore, this special simplex is an indecomposable arrow of \( \mathcal{S}_\otimes(r, s) \) if and only if \( w(i) \neq -i \) for all integers \( i \in (r, s) \), in which case we shall say that \( w \) itself is indecomposable. We reserve the notation \( s_n: (0, n + 1) \rightarrow [n] \) to denote the unique such special function, given by \( s_n(i) = n - i + 1 \).

**Definition 29** (the nerve of a Gray-category). Applying Kan’s construction to the functor \( \mathcal{S}_\otimes: \Delta \rightarrow \mathbf{Strat}_\otimes\mathbf{-Cat} \) we obtain an adjoint pair:

\[
\begin{array}{ccc}
\mathbf{Strat}_\otimes\mathbf{-Cat} & \xrightarrow{F} & \mathbf{Simp} \\
\xrightarrow{\perp} & & \xleftarrow{N}
\end{array}
\]

To stratify the nerve of a \( \mathbf{Strat}_\otimes \)-category \( \mathcal{E} \) above dimension 1 we specify that an \( n \)-simplex \( f: \mathcal{S}_\otimes[n] \rightarrow \mathcal{E} \) (with \( n > 1 \)) will be taken to be thin in the nerve \( N(\mathcal{E}) \) iff it maps the unique non-degenerate \((n - 1)\)-simplex of the homset \( \mathcal{S}_\otimes(0, n) \) to a thin simplex in the stratified set \( \mathcal{E}(f(0), f(n)) \). We can then specify that a 1-simplex will be thin in \( N(\mathcal{E}) \) iff it has an equivalence inverse in the sense of theorem 51 of [12] with respect to the given stratification at dimension 2. It is now a matter of routine verification to check that this stratification is well defined and functorial.

Now we know that the family of entire inclusions \( X \otimes Y \xhookrightarrow{\xi} X \otimes Y \) make the identity on \( \mathbf{Strat} \) into a monoidal functor from \( (\mathbf{Strat}, \otimes, \Delta[0]) \) to \( (\mathbf{Strat}, \otimes, \Delta[0]) \). This allows us to regard every \( \mathbf{Strat}_\otimes \)-category as a \( \mathbf{Strat}_\otimes \)-category, and it follows that we may apply the nerve construction above to any Gray-category. We shall show that the nerve of a Gray-category \( \mathcal{G} \) is always a weak complicial set, for which we only need check that \( N(\mathcal{G}) \) has fillers for inner horns and then apply theorem 51 of [12].

**Lemma 30.** The nerve functor \( N: \mathbf{Strat}_\otimes\mathbf{-Cat} \rightarrow \mathbf{Simp} \) is faithful.

**Proof.** To prove this result we start by defining a locally ordered category \( \Sigma[n] \) which has two objects 0 and 1 and homsets \( \Sigma[n](0, 1) \overset{\text{def}}{=} [n] \) and \( \Sigma[n](0, 0) = \Sigma[n](1, 1) \overset{\text{def}}{=} [0] \) under the obvious (trivial) composition. Now we may define a local order preserving functor \( f^n: \mathcal{S}[n + 1] \rightarrow \Sigma[n] \) which maps the objects 0, 1, ..., \( n \) to 0, the object \( n + 1 \) to 1 and acts on an arrow \( a \) in the homset \( \mathcal{S}(r, s) \) as follows:

\[
f^n(a_1, ..., a_{r+1}) = \begin{cases} 
0 & \text{if } s \leq n, \\
0 & \text{if } r, s = n + 1, \\
\min(\{n\} \cup \{n - i \mid i \in (r, n] \land a_i = 0\}) & \text{otherwise}.
\end{cases}
\]

To check that this is indeed functorial, simply observe that if \( \bar{a} \in \mathcal{S}(s, n) \) and \( \bar{b} \in \mathcal{S}(r, s) \) then their composite \( \bar{a} \circ \bar{b} \) has a \( 0 \) at index \( s \) which ensures that the minima used to define \( f^n(\bar{a}) = f^n(\bar{a}) \circ f^n(\bar{b}) \) and \( f^n(\bar{a} \circ \bar{b}) \) coincide. Applying the categorical nerve construction \( N: \mathbf{Cat} \rightarrow \mathbf{Simp} \) to homsets we obtain a corresponding simplicial functor \( f^n: \mathcal{S}[n + 1] \rightarrow \Sigma\Delta[n] \), where the latter category has \( \Delta[n] \) as its only non-trivial homset \( \Sigma\Delta[n](0, 1) \). However, by construction the action of \( f^n \) on the homset \( \mathcal{S}(0, n) \) is simply (isomorphic to) the simplicial
map \( c^n : \Delta[1]^n \rightarrow \Delta[n] \) of observation 27, which we know to underlie a suitably stratified map thereby demonstrating that \( f^n \) actually provides a Strat\(_\otimes\)-functor \( f^n : \mathcal{S}_\otimes[n+1] \rightarrow \Sigma \Delta[n] \).

Now observe that if \( \mathcal{E} \) is a Strat\(_\otimes\)-category then (by Yoneda’s lemma) a Strat\(_\otimes\)-functor \( g : \Sigma \Delta[n] \rightarrow \mathcal{E} \) corresponds uniquely to an \( n \)-arrow in some homset of \( \mathcal{E} \). Furthermore if we compose this enriched functor with \( f^n : \mathcal{S}_\otimes[n+1] \rightarrow \Sigma \Delta[n] \) we obtain an \((n+1)\)-simplex \( g \circ f^n \) of \( \text{N}(\mathcal{E}) \) from which we may regain the \( n \)-arrow that \( g \) corresponds to by applying \( g \circ f^n \) to the unique special \( n \)-simplex \( \tilde{\rho}_{s_n} \) in \( \mathcal{S}_\otimes(0,n+1) \cong \Delta[1]^\otimes n \). This fact is enough to show that \( \text{N} \) is faithful, since if \( u, v : \mathcal{C} \rightarrow \mathcal{D} \) are two Strat\(_\otimes\)-functors for which \( \text{N}(u) = \text{N}(v) \) and we consider an arbitrary \( n \)-arrow \( x \) of \( \mathcal{C} \) whose corresponding enriched functor we denote by \( x^n : \Sigma \Delta[n] \rightarrow \mathcal{C} \) and for which we know that \( u(x) = u(\tilde{x}^n \circ f^n(\tilde{\rho}_{s_n})) = \text{N}(u)(\tilde{x}^n \circ f^n) = \text{N}(v)(\tilde{x}^n \circ f^n(\tilde{\rho}_{s_n})) = v(\tilde{x}^n \circ f^n(\tilde{\rho}_{s_n})) = v(x) \). Consequently we have succeeded in demonstrating that \( u \) and \( v \) act identically on all arrows of \( \mathcal{C} \) and thus that they are identical as enriched functors.

6. Nerves of Gray Categories as Weak Complicial Sets

Observation 31 (compilial simplices in nerves of Strat\(_\otimes\)-categories). By definition, an inner complicial simplex \( \Delta^k[n] \rightarrow \text{N}(\mathcal{E}) \) in the nerve of a Strat\(_\otimes\)-category \( \mathcal{E} \) corresponds to an enriched functor \( f : \mathcal{S}_\otimes[n] \rightarrow \mathcal{E} \) satisfying an appropriate thinness property. To be precise, if \( \alpha : [m] \rightarrow [n] \) is a \( k \)-admissible operator then the composite \( f \circ \mathcal{S}_\otimes(\alpha) : \mathcal{S}_\otimes[m] \rightarrow \mathcal{E} \) must map the unique special \((m-1)\)-simplex of \( \mathcal{S}_\otimes(0,m) \) to a thin simplex in the homset \( \mathcal{E}(f(\alpha(0)), f(\alpha(m))) \).

Arguing as we did in observation 25, we may summarise the action of \( \mathcal{S}_\otimes(\alpha) \), for any face operator \( \alpha : [m] \rightarrow [n] \), as a process of “inserting the constant operator 1 at indices in the set \([n] \setminus \text{im}(\alpha)^\circ \)”. More formally, if we define an associated indecomposable special function \( \hat{\alpha} : (\alpha(0), \alpha(m)) \rightarrow [m-1] \) by

\[
\hat{\alpha}(i) = \begin{cases} 
  j & \text{if } j \in [m-1] \text{ and } \alpha(j) = i, \\
  + & \text{otherwise.}
\end{cases}
\]

then \( \mathcal{S}_\otimes(\alpha) \) acts on the unique special \((m-1)\)-simplex of \( \mathcal{S}_\otimes(0,m) \) to map it to the corresponding special simplex \( \tilde{\rho}_{\hat{\alpha}} \) in \( \mathcal{S}_\otimes(\alpha(0), \alpha(m)) \). Observe also that any indecomposable special function \( w : (r, s) \rightarrow [m-1] \) gives rise to a face operator \( \hat{w} : [m] \rightarrow [n] \) given by

\[
\hat{w}(j) = \begin{cases} 
  r & \text{if } j = 0, \\
  i & \text{if } w(i) = j \\
  s & \text{if } j = m
\end{cases}
\]

and that these two constructions are mutually inverse. Furthermore, the face operator \( \hat{w} \) is \( k \)-admissible if and only if \( k \in (r,s) \) and \( w(i) \) is an integer (not a point) for all \( i \in \{ k-1, k, k+1 \} \cap (r,s) \), in which case we simply say that \( w \) itself is \( k \)-admissible. Finally we see that an enriched functor \( f : \mathcal{S}_\otimes[n] \rightarrow \mathcal{E} \) corresponds to an inner \( k \)-complicial \( n \)-simplex in \( \text{N}(\mathcal{E}) \) if and only if it maps the simplex \( \tilde{\rho}_{\hat{w}} \) to a thin simplex in \( \mathcal{E} \) for each \( k \)-admissible function \( w \) as above.

Observation 32 (compilial simplices in nerves of Gray-categories). Now if \( \mathcal{G} \) is actually a Gray-category then we find that any enriched functor \( f : \mathcal{S}_\otimes[n] \rightarrow \mathcal{G} \) which corresponds to a \( k \)-complicial simplex, as in the last observation, actually
maps many of the decomposable simplices of $\mathbb{S}_\otimes[n]$ to thin simplices in the homsets of $\mathcal{G}$. In particular if each component of the composite $\bar{\rho}_2 \circ \bar{\rho}_1$ in $\mathbb{S}_\otimes$ maps to a thin arrow in $\mathcal{G}$ then the pair $(f(\bar{\rho}_2), f(\bar{\rho}_1))$ is a thin simplex in $\mathcal{G}(f(r), f(s)) \otimes \mathcal{G}(f(s), f(t))$ which must thus map to a thin simplex $f(\bar{\rho}_2) \circ f(\bar{\rho}_1) = f(\bar{\rho}_2 \circ \bar{\rho}_1)$ in $\mathcal{G}(f(r), f(t))$. This is not generally the case for general $\text{Strat}_\otimes$-categories, which prevents their nerves from being weak complicial sets even if their homsets happen to be such.

Applying this to the result of the last observation, we find that any $k$-admissible simplex $f: \mathbb{S}_\otimes[n] \to \mathcal{G}$ also maps the special simplex $\bar{\rho}_w$ associated with the function $w: (r, s) \to [m - 1]$ to a thin simplex in a homset of $\mathcal{G}$ if:

**case (i)** There are integers $i, j$ and $l$ in $(r, s)$ with $i < l < j$ such that $w(l) = -$ and both of $w(i)$ and $w(j)$ are integers (not points).

**Proof.** Observe that the condition $w(l) = -$ implies that $\bar{\rho}_w$ may be decomposed at index $l$ to express it as a composite $\bar{\rho}_2 \circ \bar{\rho}_1$ where $\bar{\rho}_1 \in \mathbb{S}_\otimes(r, l)$ has the operator $\rho_{w(i)}$ as its $i^{th}$ ordinate, which is non-constant since $w(i)$ is an integer, and similarly $\bar{\rho}_2 \in \mathbb{S}_\otimes(l, s)$ has the non-constant $\rho_{w(j)}$ as its $(j - 1)^{th}$ ordinate. However $w$ is partially bijective, so we know that each non-constant $\rho$ operator occurs exactly once as an ordinate of $\bar{\rho}_w$ and it follows that $\bar{\rho}_1$ does not have $\rho_{w(i)}$ as an ordinate and thus that it must be degenerate at $w(j) - 1$. Dually $\bar{\rho}_2$ is degenerate at $w(i) - 1$, so both of $f(\bar{\rho}_1)$ and $f(\bar{\rho}_2)$ are degenerate, and thus thin, simplices of their respective homsets and thus we find that $f(\bar{\rho}_w)$, their composite, is also thin in $\mathcal{G}(r, s)$. \hfill $\Box$

**case (ii)** We have $k \in (r, s)$ and $w(k)$ is an integer and $w(i)$ is an integer or the point $-$ (in other words is not the point $+$) for any $i \in \{k - 1, k + 1\} \cap (r, s)$.

**Proof.** If $\bar{\rho}_w$ is not already indecomposable then we may decompose it as $\bar{\rho}_3 \circ \bar{\rho}_2 \circ \bar{\rho}_1$ in such a way that $\bar{\rho}_2$ is indecomposable with and ordinate indexed by $k$ and in which either of the other two factors could be identities. Now we have two possibilities, one of $\bar{\rho}_1$ or $\bar{\rho}_3$ has a non-constant ordinate, in which case we are reduced to the case presented above, or they are both degenerate and $\bar{\rho}_2$ is an indecomposable special simplex in some homset $\mathbb{S}_\otimes(r', s')$. Now it is clear that the doubly pointed morphism $w': (r', s') \to [m - 1]$ corresponding to $\bar{\rho}_2$ is simply obtained by restricting $w$ to $(r', s')$ and that the conditions given ensure that this $w'$ is $k$-admissible. So we know that $f(\bar{\rho}_2)$ is thin in its homset, since $\bar{\rho}_2$ is $k$-admissible, and that $f(\bar{\rho}_1)$ and $f(\bar{\rho}_3)$ are degenerate in their homsets, so it follows that their composite $f(\bar{\rho}_w)$ is thin in its homset as postulated. \hfill $\Box$

**Observation 33.** Given this series of reductions, we are finally able to recast the task of proving that $\text{N}(\mathcal{G})$ is a weak complicial set as a demonstration that the members of a certain set of inclusions are complicial cofibrations.

Firstly our analysis of horns in homotopy coherent nerves, which culminated in observation 21, told us that in order to fill an inner horn in the nerve of homotopy coherent paths $\text{N}_{hc}(\mathcal{G})$ it is enough to demonstrate that each inclusion $\subset_r : \mathbb{H}^k[n](0, n) \to \mathbb{S}[n](0, n)$ is an anodyne extension. In the current case, it follows that all we need do is re-stratify $\mathbb{S}[n](0, n)$ and $\mathbb{H}^k[n](0, n)$ in line with the analysis of complicial simplices in $\text{N}(\mathcal{G})$ provided by observation 32 and demonstrate that the corresponding regular inclusion is a complicial cofibration. Well in fact that isn’t quite enough on its own, since this time the stratification of $\text{N}(\mathcal{G})$ is non-trivial at all dimensions so we will also need to ensure that the corresponding
result holds for elementary inner thinness extensions. This however requires only a minor modification of the argument for horns.

**Notation 34.** So to fix our notation we define the following stratified sets:

- **$C_k^n$** ($n \geq 2$, $1 \leq k \leq n$) by starting with the stratified cube $\Delta[1] \otimes^n$ and making thin those simplices $\bar{\rho}_w$ for which:
  - (i) there are integers $i, j, l \in (0, n]$ with $i < l < j$ such that $w(l) = -$ and both of $w(i)$ and $w(j)$ are integers,
  - (ii) the value $w(k)$ is an integer and $w(i)$ is an integer or the point $-$ for each $i \in \{k - 1, k + 1\} \cap (0, n]$.
- **$H_n^k$** is the regular subset of $C_k^n$ whose underlying simplicial set is the domain of the corner product depicted in display (7).
- **$w_i^n$** : $(0, n) \rightarrow [n - 1]$ ($1 \leq i \leq n$) is the unique order reversing partial bijection for which $w_i(i) = +$. As ever we will habitually drop the superscript in this notation if it may be inferred from the context.
- **$\check{C}_n^k$** is obtained from $C_n^k$ by making the special $(n - 1)$-simplices $\bar{\rho}_w$, thin for all $i \in \{k - 1, k + 1\} \cap (0, n]$ and $\check{C}_n^k$ is obtained from that by making $\bar{\rho}_{w_k}$ thin as well.

**Observation 35.** Of course, observation 32 tells us that $C_n^{k-1}$ is simply the set obtained by taking $S_\otimes(0, n)$ and making thin those simplices that would be mapped to thin simplices by any enriched functor $f : S_\otimes[n] \rightarrow \mathcal{G}$ that corresponds to a $k$-complicial $n$-simplex in the nerve of the Gray-category $\mathcal{G}$. Furthermore, $H_n^k$ is the stratified set with the corresponding property for $k$-complicial horns in $N(\mathcal{G})$. It follows, therefore, that we may demonstrate that $N(\mathcal{G})$ has fillers for such horns by showing that the inclusion $H_n^k \subseteq C_n^k$ is a complicial cofibration for all $n \geq 2$ and $1 \leq k \leq n$. Correspondingly, we may demonstrate the thinness extension property required of a weak inner complicial set by showing that each inclusion $\check{C}_n^k \subseteq \check{C}_n^k$ is a complicial cofibration.

**Lemma 36.** If $k > 1$ then the Gray pre-tensor $C_n^{k-1} \otimes \Delta[1]$ is an entire subset of $C_n^{k+1}$ and the inclusion $C_n^{n-1} \otimes \Delta[1] \subseteq C_n^{n+1}$ is an anodyne extension. Dually if $k < n - 1$ then the corresponding result relates $\Delta[1] \otimes C_n^{k-1}$ and $C_n^{k}$.

**Proof.** By definition we know that $\Delta[1] \otimes (n-1) \otimes \Delta[1]$ is an entire subset of $C_n^{n-1} \otimes \Delta[1]$ and, consulting definition 59 of [12], we see that the first of the criteria listed there to define the stratifications of these pre-tensors depends only on underlying simplicial sets which in this case are identical. So applying the second of those criteria we see that an $m$-simplex $\bar{\rho}_w = (\bar{\rho}, \rho_{w(1)})$ is thin in the latter set and non-thin in the former if and only if either

- $w(1)$ is equal to $-$ or $+$ and $\bar{\rho}$ is special and satisfies one of the criteria for thinness given in the definition of $C_n^{n-1}$, or
- $w(1)$ is equal to $m$ (thus ensuring that $\rho_{w(1)} = \pi_2^{m-1,1}$), the $(m - 1)$-simplex $\bar{\rho} \cdot \pi_1^{m-1,1}$ is special and satisfies one of the criteria given in the definition of $C_n^{n-1}$ and $(\bar{\rho} \cdot \pi_1^{m-1,1}) \cdot \pi_1^{m-1,1} = \bar{\rho}$ with all other possibilities for the application of that thinness criterion turning out to be degenerate simplices.

In the first case, if $w' : (0, n) \rightarrow [m]$ is the unique special function with $\bar{\rho} = \bar{\rho}_{w'}$ then it is clear that $w(i + 1) = w'(i)$ and so $w$ is also special since $w(1) = -$.
In the second case, if we now take \( w' \) to be the special function corresponding to the simplex \( \hat{\rho} \cdot \pi^{m-1,1}_1 \) we may replace the other equality given there to \( \hat{\rho} \cdot \pi^{m-1,1}_1 = \hat{\rho} \) from which we may again infer easily that \( w(i+1) = w'(i) \) since \( \rho_t^{m-1} \circ \pi^{m-1,1}_1 = \rho_t^{m-1} \) for all \( i \in (0, m - 1) \). So we see again that \( w \) is special since \( w' \) maps to integers less than \( w(1) = m \). Now it is a routine matter, in either case, to check that \( w' \) satisfies one of the thinness conditions on \( C_{k-1}^{n} \) given in observation 34 if and only if \( w \) satisfies the corresponding thinness condition on \( C_k^{n+1} \). Consequently, since we know that \( \Delta[1] \otimes (n-1) \otimes \Delta[1] \) is also an entire subset of \( C_k^{n+1} \) we have certainly shown that \( C_k^{n-1} \otimes \Delta[1] \) is an entire subset of \( C_k^{n+1} \). Indeed what we have actually shown is that \( C_k^{n+1} = (C_k^{n-1} \otimes \Delta[1]) \cup \Delta[1] \otimes n \) and thus that we have a pushout square:

\[
\begin{array}{c}
\Delta[1] \otimes (n-1) \otimes \Delta[1] \xrightarrow{\ell_x} \Delta[1] \otimes n \\
\Uparrow \quad \Uparrow \\
C_k^{n-1} \otimes \Delta[1] \xrightarrow{\ell_x} C_k^{n+1}
\end{array}
\]

Now observation 60 of [12], which collects lemma 139 of [14], tells us that the upper horizontal here is an anodyne extension so it follows that its lower horizontal is also an anodyne extension, as a pushout of such, as required.

Notice that the condition that \( k > 1 \) is vital to the argument of the last paragraph, since it ensures that the “right hand flank” of the pivotal ordinate \( k \) is protected from the insertion of a + which might disrupt condition (ii) of observation 34. In the case \( k = 1 \) this result does not hold since, for instance, \( (\rho_1, \rho_2, ..., \rho_{n-1}) \) is thin in \( C_0^{1} \) but \( (\rho_1, ..., \rho_{n-1}, +) \) is not thin in \( C_0^{2} \).

**Corollary 37.** If each of the inclusions \( H_1^k \subseteq C_1^k \), \( H_2^k \subseteq C_2^k \) and \( H_3^k \subseteq C_3^k \) are complicial cofibrations then so are the inclusions \( H_n^k \subseteq C_n^k \)

and \( \bar{C}_n^k \subseteq \bar{C}_n^k \) for \( n \geq 2 \) and \( 0 < k < n \).

**Proof.** For \( n \geq 3 \) and \( 1 < k < n \) consider the following commutative square

\[
\begin{array}{c}
(H_n^{k-1} \otimes \Delta[1]) \cup (C_n^{k-1} \otimes \partial \Delta[1]) \xrightarrow{\ell_x} C_n^{k-1} \otimes \Delta[1] \\
\Uparrow \quad \Uparrow \\
H_n^{k+1} \otimes \Delta[1] \xrightarrow{\ell_x} \Delta[1] \\
\end{array}
\]

in which the right hand vertical is an anodyne extension by the last lemma as, quite clearly, is the left hand vertical by a minor modification of the proof of that same lemma. So these verticals are both complicial cofibrations and we may apply the 2-of-3 property to show that the upper horizontal here is a complicial cofibration if and only if its lower horizontal is such. Observe, however, that the upper horizontal is actually the corner pre-tensor

\[
(H_n^{k-1} \subseteq C_n^{k-1}) \otimes_{c} (\partial \Delta[1] \subseteq \Delta[1])
\]

which we know is a complicial cofibration whenever \( H_n^{k-1} \subseteq C_n^{k-1} \) is such, by theorem 103. Applying this result inductively we have shown that if \( k > 1 \) then

\[
H_n^{k+1} \subseteq C_n^{k+1}
\]

is a complicial cofibration whenever \( H_{n-k+1} \subseteq C_{n-k+1} \) is such. Furthermore, arguing dually, we see that if \( k < n - 1 \) then \( H_n \subseteq C_n \) is a complicial cofibration whenever \( H_{k+1} \subseteq C_{k+1} \) is such. So we may
• use the first of these reductions to show that the inclusion \( H^n \subseteq C^n \) is a complicial cofibration under the assumption that \( H^2 \subseteq C^2 \) is such,
• use its dual to show that the inclusion \( H^1 \subseteq C^1 \) is a complicial cofibration under the assumption that \( H^2 \subseteq C^2 \) is such, and
• use each reduction in turn to show that the inclusion \( H^k \subseteq C^k \) with \( 1 < k < n \) is a complicial cofibration assuming that \( H^1 \subseteq C^1 \) is such, thereby demonstrating the desired result for all such inclusions.

To demonstrate the corresponding result for the inclusion \( \hat{C}^k \subseteq \hat{C}^k \) start by observing that the \((n-1)\)-simplex \( \hat{w}_i \in C_n^k \) corresponding to some function \( w: [0, n] \rightarrow [n-1] \) is in \( H_n^k \) if and only if there is an \( i \in (0, n] \) for which \( w(i) = - \) or \( i \neq k \) and \( w(i) = + \). Consequently, by observation 27, it follows that if such an \((n-1)\)-simplex is non-degenerate then \( w \) must be a partial bijection and if it is non-thin then \( w \) must also be order reversing and thus special. Consulting the definition of \( C_n^k \) we see that it only non-thin special \((n-1)\)-simplices are the \( \hat{w}_i \) for each \( i \in \{k-1, k, k+1\} \cap (1, n) \) and that \( \hat{w}_i \) is not a simplex of \( H_n^k \). It follows, therefore, that \( \hat{C}^k = C_n^k \cup \text{th}_{n-2}(H_n^k) \) and we may thus apply corollary 102 of [12] to the complicial cofibration \( H_n^k \subseteq C^k_n \) of the last paragraph to demonstrate that \( \hat{C}^k = \text{th}_{n-2}(C^k_n) \) is a complicial cofibration. Now it is the case that if the composite of two entire inclusions is a complicial cofibration then so is each inclusion separately, which fact we leave to the reader to demonstrate using the characterisation given in corollary 100 of loc. cit. Applying this to the inclusion of the sentence before last, which factors as a composite of the entire inclusions \( \hat{C}^k \subseteq \hat{C}^k \) and \( \hat{C}^k \subseteq \text{th}_{n-2}(C^k_n) \), we obtain the desired result. \( \square \)

**Lemma 38.** The inclusions \( H^1 \subseteq C^1, H^2 \subseteq C^2, H^3 \subseteq C^3 \) are indeed complicial cofibrations.

**Proof.** The easiest way to visualise the calculations involved here is diagrammatically. The first two inclusions are duals of each other and the first may be pictured as follows

\[
\begin{array}{ccc}
(0, 1) & \sim & (1, 1) \\
\downarrow & & \downarrow \\
(0, 0) & \sim & (0, 1)
\end{array}
\]

wherein the solid arrows represent the 1-simplices of \( C^2 \) which are in \( H^2 \) and the dotted ones represent 1-simplices which will be obtained by filling some 2-simplex. The left hand vertical 1-simplex is labelled with a \( \sim \) symbol because it is thin in \( C^1 \) by the virtue of satisfying the first condition in the definition of the stratification of \( C^3 \) given in notation 34. We may decompose the regular subset inclusion \( H^2 \subseteq C^1 \) by defining the intermediate regular subset

\[U \overset{\text{def}}{=} H^2 \cup \{(0, 0) \lessdot (0, 1) \lessdot (1, 1)\}^*\]

where the notation \( \{-\}^* \) denotes the regular subset generated by the given set of simplices, and show that:

• The inclusion \( H^3 \subseteq U \) is anodyne since it is a pushout of the horn extension \( \Lambda^1[2] \subseteq \Delta^1[2] \) along the evident stratified map mapping its domain to the 1-complicial horn on the vertices \( (0, 0), (0, 1) \) and \( (1, 1) \).
• The inclusion \( U \xrightarrow{\sim} C_1^2 \) is anodyne since it is a pushout of the horn extension \( \Lambda^0[2] \xrightarrow{\sim} \Delta^1[2] \) along the evident stratified map mapping its domain to the 1-complicial horn on the vertices \((0,0), (1,0)\) and \((1,1)\).

It follows therefore that their composite \( H_2 \xrightarrow{\sim} C_2^1 \) is an anodyne extension, and thus is a complicial cofibration as required.

The proof for the inclusion \( H_2^3 \xrightarrow{\sim} C_3^2 \) is only slightly more involved, again we assist the reader with a diagram

\[
\begin{align*}
(0,1,1) & \xrightarrow{\rho_{w_3}} (1,1,1) \\
(0,0,0) & \sim (1,0,0) \\
(0,1,1) & \xrightarrow{\rho_{w_1}} (1,1,0) \\
(0,0,1) & \sim (0,1,0)
\end{align*}
\]

depicting the set \( H_2^3 \). Here the labels in the square faces and on one of the arrows provide information about the corresponding special 2-simplices. Those labelled with \( \sim \) symbols are thin in \( H_2^3 \), with the one in the left hand diagram satisfying the first condition in the definition of the stratification of \( C_2^3 \) given in notation 34 and those in the right hand diagram all satisfying the second condition given there. The remaining labels simply identify the corresponding special simplex, with the one in the upper-right most square being prefixed with a \( ? \) since that is the square 2-face which is not present in the set \( H_2^3 \).

This time we actually need to consider the entire superset \( \hat{C}_2^3 \) of \( C_2^2 \) constructed by making the 2-simplex \((0,0,0) \prec (0,1,0) \prec (1,1,1)\) thin and construct a decomposition of the inclusion \( H_2^3 \xrightarrow{\sim} \hat{C}_2^3 \) consisting of an increasing tower of regular subsets of \( \hat{C}_2^3 \) given by:

\[
\begin{align*}
V_1 & \overset{\text{def}}{=} H_2^3 \cup \{(0,0,0) \prec (0,1,1) \prec (1,1,1)\}^* \\
V_2 & \overset{\text{def}}{=} V_1 \cup \{(0,0,0) \prec (0,0,1) \prec (0,1,1) \prec (1,1,1)\}^* \\
V_3 & \overset{\text{def}}{=} V_2 \cup \{(0,0,0) \prec (0,0,1) \prec (0,1,1) \prec (1,1,1)\}^* \\
V_4 & \overset{\text{def}}{=} V_3 \cup \{(0,0,0) \prec (1,0,0) \prec (1,0,1) \prec (1,1,1)\}^* \\
V_5 & \overset{\text{def}}{=} V_4 \cup \{(0,0,0) \prec (1,0,0) \prec (1,0,0) \prec (1,1,1)\}^* \\
V_6 & \overset{\text{def}}{=} V_5 \cup \{(0,1,0) \prec (0,1,1) \prec (0,1,1)\}^* \\
V_7 & \overset{\text{def}}{=} V_6 \cup \{(0,0,0) \prec (0,1,0) \prec (0,1,1) \prec (1,1,1)\}^* \\
\hat{C}_2^3 & = \overset{\text{def}}{=} V_7 \cup \{(0,0,0) \prec (0,1,0) \prec (1,1,0) \prec (1,1,1)\}^*
\end{align*}
\]

Now we find that

• the inclusions \( H_2^3 \xrightarrow{\sim} V_1 \) and \( V_5 \xrightarrow{\sim} V_6 \) may be constructed as evident pushouts of the 1-complicial horn \( \Lambda^1[2] \xrightarrow{\sim} \Delta^1[2] \),

• the inclusions \( V_1 \xrightarrow{\sim} V_2 \) and \( V_6 \xrightarrow{\sim} V_7 \) may be constructed as evident pushouts of the thin 2-complicial horn \( \Lambda^2[3]'' \xrightarrow{\sim} \Delta^2[3]'' \),

• the inclusion \( V_5 \xrightarrow{\sim} V_4 \) may be constructed as a pushout of the 2-complicial horn \( \Lambda^2[3] \xrightarrow{\sim} \Delta^2[3] \).
• the inclusions $V_3 \subseteq V_4$, $V_4 \subseteq V_5$ and $V_5 \subseteq V_6$ may be constructed as pushouts of the thin 1-complicial horn $\Lambda^1[3] \subseteq \Delta^1[3]$, and
• the inclusion $V_7 \subseteq \hat{C}_3^2$ may be constructed as a pushout of the 0-complicial horn $\Lambda^0[3] \subseteq \Delta^0[3]$.

as the reader may readily verify, so it follows that their composite $H^2 \subseteq \hat{C}_3^2$ is an anodyne extension. It is also easily demonstrated that the entire inclusion $\hat{C}_3^2 \subseteq \hat{C}_3^2 \hat{C}_3^2$ is an anodyne extension, since it may be constructed as a pushout of the elementary thinness extension $\Delta^2[3] \subseteq \Delta^2[3]''$ along the evident map of its domain onto the simplex $(0,0,0) < (0,0,1) < (0,1,1) < (1,1,1)$. So these inclusions are both complicial cofibrations, as are all anodyne extensions, so we may apply the 2-of-3 property to show that the inclusion $H^2 \subseteq \hat{C}_3^2$ is also a complicial cofibration as required.

**DEFINITION 39.** An enriched functor $f : \mathcal{G} \rightarrow \mathcal{H}$ of Gray-categories is said to be a *local complicial fibration* if each stratified map $f : \mathcal{G}(a,b) \rightarrow \mathcal{H}(f(a), f(b))$ of homsets is a complicial fibration.

**THEOREM 40.** If $\mathcal{G}$ is a Gray-category then its nerve $N(\mathcal{G})$ is a weak complicial set. Furthermore, if $f : \mathcal{G} \rightarrow \mathcal{H}$ is a functor of Gray-categories which is a local complicial fibration then the corresponding stratified map $N(f) : N(\mathcal{G}) \rightarrow N(\mathcal{H})$ is an inner complicial fibration.

**PROOF.** This is now just a simple matter of following the proof outline given in observation 35 using the results established in corollary 37 and lemma 38 to demonstrate that $N(\mathcal{G})$ is (almost) a weak inner complicial set whenever $\mathcal{G}$ is a Gray-category. Now we may apply theorem 51 of [12] to show that $N(\mathcal{G})$ is actually a weak complicial set as postulated. The remainder is a triviality, since it is clear that the results of corollary 37 and lemma 38 also show that we may lift inner horns and thinness extensions along $N(f) : N(\mathcal{G}) \rightarrow N(\mathcal{H})$ so long as its actions on homsets has the RLP with respect to the inclusions featured in those lemmata.

**OBSERVATION 41** (the large universe of weak complicial sets). We have now succeeded in showing that the weak complicial sets themselves are the 0-simplices of a large and richly structured weak complicial set. This is simply constructed by taking the nerve of the Gray-category $W_{cs}$ of weak complicial sets discussed in example 10. It is worth observing that we may easily show that this is not $n$-trivial for any $n \in \mathbb{N}$.

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Centre of Australian Category Theory, Macquarie University, NSW 2109, Australia

E-mail address: domv@ics.mq.edu.au