Article

Unification Theories: New Results and Examples

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Abstract: This paper is a continuation of a previous article that appeared in AXIOMS in 2018. A Euler’s formula for hyperbolic functions is considered a consequence of a unifying point of view. Then, the unification of Jordan, Lie, and associative algebras is revisited. We also explain that derivations and co-derivations can be unified. Finally, we consider a “modified” Yang–Baxter type equation, which unifies several problems in mathematics.

Keywords: Euler’s formula; hyperbolic functions; Yang–Baxter equation; Jordan algebras; Lie algebras; associative algebras; UJLA structures; (co)derivation

MSC: 17C05; 17C50; 16T15; 16T25; 17B01; 17B40; 15A18; 11J81

1. Introduction

Voted the most famous formula by undergraduate students, the Euler’s identity states that $e^{\pi i} + 1 = 0$. This is a particular case of the Euler’s–De Moivre formula:

$$\cos x + i \sin x = e^{ix} \quad \forall x \in \mathbb{R}, \quad (1)$$

and, for hyperbolic functions, we have an analogous formula:

$$\cosh x + J \sinh x = e^{xJ} \quad \forall x \in \mathbb{C}, \quad (2)$$

where we consider the matrices

$$J = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \quad (3)$$

$$I = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad (4)$$

$$I' = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \quad (5)$$

In fact, $R(x) = \cosh(x)I + \sinh(x)J = \cosh x + J \sinh x = e^{xJ}$ also satisfies the equation

$$(R \otimes I')(x) \circ (I' \otimes R)(x + y) \circ (R \otimes I')(y) = (I' \otimes R)(y) \circ (R \otimes I')(x + y) \circ (I' \otimes R)(x) \quad (6)$$
called the colored Yang–Baxter equation. This fact follows easily from \( f^{12} \circ f^{23} = f^{23} \circ f^{12} \) and \( x f^{12} + (x + y) f^{23} + y f^{12} = y f^{23} + (x + y) f^{12} + x f^{23} \), and it shows that the formulas (1) and (2) are related.

While we do not know a remarkable identity related to (2), let us recall an interesting inequality from a previous paper: \( |\varepsilon - \pi| > \epsilon \). There is an open problem to find the matrix version of this inequality.

The above analysis is a consequence of a unifying point of view from previous papers ([1,2]).

In the remainder of this paper, we first consider the unification of the Jordan, Lie, and associative algebras. In Section 3, we explain that derivations and co-derivations can be unified. We suggest applications in differential geometry. Finally, we consider a “modified” Yang–Baxter equation which unifies the problem of the three matrices, generalized eigenvalue problems, and the Yang–Baxter matrix equation. There are several versions of the Yang–Baxter equation (see, for example, [3,4]) presented throughout this paper.

We work over the field \( k \), and the tensor products are defined over \( k \).

### 2. Weak Ujla Structures, Dual Structures, Unification

**Definition 1.** (Ref. [5]) Given a vector space \( V \), with a linear map \( \eta : V \otimes V \to V \), \( \eta(a \otimes b) = ab \), the couple \((V, \eta)\) is called a “weak Ujla structure” if the product \( ab = \eta(a \otimes b) \) satisfies the identity

\[
(ab)c + (bc)a + (ca)b = a(bc) + b(ca) + c(ab) \quad \forall a, b, c \in V.
\]  

**Definition 2.** Given a vector space \( V \), with a linear map \( \Delta : V \to V \otimes V \), the couple \((V, \Delta)\) is called a “weak co-Ujla structure” if this co-product satisfies the identity

\[
(Id + S^2) \circ (\Delta \otimes 1) \circ \Delta = (Id + S^2) \circ (1 \otimes \Delta) \circ \Delta
\]  

where \( S : V \otimes V \to V \otimes V \), \( a \otimes b \circ c \mapsto b \otimes c \circ a \), \( I : V \to V \), \( a \mapsto a \) and \( Id : V \otimes V \to V \otimes V \), \( a \otimes b \circ c \mapsto a \otimes b \circ c \).

**Definition 3.** Given a vector space \( V \), with a linear map \( \phi : V \otimes V \to V \otimes V \), the couple \((V, \phi)\) is called a “weak (co)Ujla structure” if the map \( \phi \) satisfies the identity

\[
(Id + S^2) \circ \phi^{12} \circ \phi^{23} \circ \phi^{12} \circ (Id + S^2) = (Id + S^2) \circ \phi^{23} \circ \phi^{12} \circ \phi^{23} \circ (Id + S^2)
\]  

where \( \phi^{12} = \phi \otimes 1 \), \( \phi^{23} = 1 \otimes \phi \), \( Id : V \otimes V \to V \otimes V \), \( a \otimes b \circ c \mapsto a \otimes b \circ c \) and \( I : V \to V \), \( a \mapsto a \).

**Theorem 1.** Let \((V, \eta)\) be a weak Ujla structure with the unity \( 1 \in V \). Let \( \phi : V \otimes V \to V \otimes V \), \( a \otimes b \mapsto ab \otimes 1 \). Then, \((V, \phi)\) is a “weak (co)Ujla structure”.

**Proof.** \((Id + S^2) \circ \phi^{23} \circ \phi^{12} \circ \phi^{23} \circ (Id + S^2)(a \otimes b \circ c) = (Id + S^2) \circ \phi^{23} \circ \phi^{12} \circ \phi^{23}(a \otimes b \circ c + b \otimes c \circ a + c \otimes a \circ b) = (Id + S^2) \circ \phi^{23} \circ \phi^{12}(a \otimes bc \otimes 1 + b \otimes ca \otimes 1 + c \otimes ab \otimes 1) = (Id + S^2) \circ \phi^{23}(a(bc) \otimes 1 \otimes 1 + b(ca) \otimes 1 \otimes 1 + c(ab) \otimes 1 \otimes 1) = (Id + S^2)(a(bc) \otimes 1 \otimes 1 + b(ca) \otimes 1 \otimes 1 + c(ab) \otimes 1 \otimes 1) \circ 1 \otimes 1 + 1 \otimes 1 \otimes a(bc) + 1 \otimes 1 \otimes 1 \circ c(ab) + 1 \otimes 1 \otimes 1 \circ a(bc) + 1 \otimes 1 \otimes 1 \circ b(ca) + 1 \otimes 1 \otimes 1 \circ c(ab) + 1 \otimes 1 \otimes 1 \circ b(ca)

Similarly,

\((Id + S^2) \circ \phi^{12} \circ \phi^{23} \circ \phi^{12} \circ (Id + S^2)(a \otimes b \circ c) = (Id + S^2) \circ \phi^{12} \circ \phi^{23} \circ \phi^{23}(a \otimes b \circ c + b \otimes c \circ a + c \otimes a \circ b) = (ab)c \otimes 1 \otimes 1 + (bc)a \otimes 1 \otimes 1 + (ca)b \otimes 1 \otimes 1 + 1 \otimes 1 \otimes (ab)c + 1 \otimes 1 \otimes (bc)a + 1 \otimes 1 \otimes (ca)b \otimes 1 \).

We now use the axiom of the “weak Ujla structure”.
Theorem 2. Let \((V, \Delta)\) be a weak co-UJLA structure with the co-unity \(\epsilon : V \to k\). Let \(\phi = \Delta \otimes \epsilon : V \otimes V \to V \otimes V\). Then, \((V, \phi)\) is a “weak (co)UJLA structure”.

Proof. The proof is dual to the above proof. We refer to [6–8] for a similar approach.

A direct proof should use the property of the co-unity: \((\epsilon \otimes I) \circ \Delta = I = (I \otimes \epsilon) \circ \Delta\). After computing

\[
\phi^{12} \circ \phi^{23} \circ \phi^{12}(a \otimes b \otimes c) = \epsilon(b) \epsilon(c)(a_1 \otimes (a_2)_2 \otimes a_2)
\]

and

\[
\phi^{23} \circ \phi^{12} \circ \phi^{23}(a \otimes b \otimes c) = \epsilon(b) \epsilon(c)a_1 \otimes (a_2)_1 \otimes (a_2)_2.
\]

one just checks that the properties of the linear map \(Id + S + S^2\) will help to obtain the desired result.

Theorem 3. Let \((V, \eta)\) be a weak UJLA structure with the unity \(1 \in V\). Let \(\phi : V \otimes V \to V \otimes V, a \otimes b \mapsto ab \otimes 1 + 1 \otimes ab - a \otimes b\). Then, \((V, \phi)\) is a “weak (co)UJLA structure”.

Proof. One can formulate a direct proof, similar to the proof of Theorem 1. Alternatively, one could use the calculations from [7] and the axiom of the “weak UJLA structure”.

3. Unification of (Co)Derivations and Applications

Definition 4. Given a vector space \(V\), a linear map \(d : V \to V\), and a linear map \(\phi : V \otimes V \to V \otimes V\), with the properties

\[
\phi^{12} \circ \phi^{23} \circ \phi^{12} = \phi^{23} \circ \phi^{12} \circ \phi^{23} \tag{10}
\]

\[
\phi \circ \phi = Id, \tag{11}
\]

the triple \((V, d, \phi)\) is called a “generalized derivation” if the maps \(d\) and \(\phi\) satisfy the identity

\[
\phi \circ (d \otimes I + I \otimes d) = (d \otimes I + I \otimes d) \circ \phi.
\]

Here, we have used our usual notation: \(\phi^{12} = \phi \otimes I, \phi^{23} = I \otimes \phi\), \(Id : V \otimes V \to V \otimes V, a \otimes b \mapsto a \otimes b\) and \(I : V \to V, a \mapsto a\).

Theorem 4. If \(A\) is an associative algebra and \(d : A \to A\) is a derivation, and \(\phi : A \otimes A \to A \otimes A, a \otimes b \mapsto ab \otimes 1 + 1 \otimes ab - a \otimes b\), then \((A, d, \phi)\) is a “generalized derivation”.

Proof. According to [7], \(\phi\) verifies conditions (10) and (11). Recall now that \(d(ab) = d(a)b + ad(b)\) \(\forall a, b \in A, d(1_A) = 0\).

\[
(d \otimes I + I \otimes d) \circ \phi(a \otimes b) = (d \otimes I + I \otimes d)(ab \otimes 1 + 1 \otimes ab - a \otimes b) = d(ab) \otimes 1 - d(a) \otimes b + 1 \otimes d(ab) - a \otimes d(b).
\]

\[
\phi \circ (d \otimes I + I \otimes d)(a \otimes b) = \phi(d(a)b + a \otimes d(b)) = d(a)b \otimes 1 + 1 \otimes d(a)b - d(a) \otimes b + ad(b) \otimes 1 + 1 \otimes ad(b) - a \otimes d(b).
\]

Theorem 5. If \((C, \Delta, \epsilon)\) is a co-algebra, \(d : C \to C\) is a co-derivation, and \(\psi = \Delta \otimes \epsilon + \epsilon \otimes \Delta - Id : C \otimes C \to C \otimes C, c \otimes d \mapsto \epsilon(d)c_1 \otimes c_2 + \epsilon(c)d_1 \otimes d_2 - c \otimes d\), then \((C, d, \psi)\) is a “generalized derivation”.

(We use the sigma notation for co-algebras.)

Proof. The proof is dual to the above proof.

According to [7], \(\psi\) verifies conditions (10) and (11). From the definition of the co-derivation, we have \(\epsilon(d(c)) = 0\) and \(\Delta(d(c)) = d(c_1) \otimes c_2 + c_1 \otimes d(c_2) \forall c \in C\).

\[
\psi \circ (d \otimes I + I \otimes d)(c \otimes a) = \epsilon(a)d(c_1) \otimes d(c_2) - d(c) \otimes a + \epsilon(c)d_1 \otimes d(a) - c \otimes d(a),
\]

\[
(d \otimes I + I \otimes d) \circ \psi(c \otimes a) = \epsilon(a)d(c_1) \otimes c_2 + \epsilon(c)d_1 \otimes a_2 - d(c) \otimes a + \epsilon(a)c_1 \otimes d(c_2) + \epsilon(c)a_1 \otimes d(a_2) - c \otimes d(a).
\]

The statement follows from the main property of the co-derivative.
Definition 5. Given an associative algebra $A$ with a derivation $d : A \rightarrow A$, $M$ an $A$-bimodule and $D : M \rightarrow M$ with the properties

$$D(am) = d(a)m + aD(m) \quad D(ma) = D(m)a + md(a) \quad \forall a \in A, \forall m \in M,$$

the quadruple $(A, d, M, D)$ is called a “module derivation”.

Remark 1. A “module derivation” is a module over an algebra with a derivation. It can be related to the co-variant derivative from differential geometry. Definition 5 also requires us to check that the formulas for $D$ are well-defined.

Note that there are some similar constructions and results in [9] (see Theorems 1.27 and 1.40).

Theorem 6. In the above case, $A \oplus M$ becomes an algebra, and $\delta : A \oplus M \rightarrow A \oplus M$, $(a \oplus m) \mapsto (d(a) \oplus D(m))$ is a derivation of this algebra.

Proof. We just need to check that $\delta((a \oplus m)(b \oplus n)) = \delta((ab \oplus an + mb)) = d(ab) \oplus D(an + mb)$ equals $\delta((a \oplus m)(b \oplus n)) = (d(a) \oplus D(m))(b \oplus n) + (a \oplus m)(d(b) \oplus D(n)) = (d(a)b + d(a)n + D(m)b) + (ad(b) \oplus aD(n) + md(b))$. \[\square\]

Remark 2. A dual statement with a co-derivation and a co-module over that co-algebra can be given.

Remark 3. The above theorem leads to the unification of module derivation and co-module derivation.

4. Modified Yang–Baxter Equation

For $A \in M_n(\mathbb{C})$ and $D \in M_n(\mathbb{C})$, a diagonal matrix, we propose the problem of finding $X \in M_n(\mathbb{C})$, such that

$$AXA + XAX = D. \quad (12)$$

This is an intermediate step to other “modified” versions of the Yang–Baxter equation (see, for example, [10]).

Remark 4. Equation (12) is related to the problem of the three matrices. This problem is about the properties of the eigenvalues of the matrices $A$, $B$ and $C$, where $A + B = C$. A good reference is the paper [11]. Note that if $A$ is “small” then $D - AXA$ could be regarded as a deformation of $D$.

Remark 5. Equation (12) can be interpreted as a “generalized eigenvalue problem” (see, for example, [12]).

Remark 6. Equation (12) is a type of Yang–Baxter matrix equation (see, for example, [13,14]) if $D = O_n$ and $X = -Y$.

Remark 7. For $A \in M_2(\mathbb{C})$, a matrix with trace -1 and

$$D = -\begin{pmatrix} \det(A) & 0 \\ 0 & \det(A) \end{pmatrix}, \quad (13)$$

Equation (12) has the solution $X = I'$.

Remark 8. There are several methods to solve (12). For example, for $A^3 = I_n$, one could search for solutions of the following type: $X = aI_n + bA + \gamma A^2$. Now, (12) implies that $(2\alpha \beta + \gamma^2 + \alpha)A^2 + (\alpha^2 + 2\beta \gamma + \gamma)A + (2\alpha \gamma + \beta^2 + \beta)I_n - D = 0$.

It can be shown that we can produce a large class of solutions in this way, if $D$ is of a certain type.
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