REEB ORBITS AND THE MINIMAL DISCREPANCY OF AN ISOLATED SINGULARITY

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Abstract. Let $A$ be an affine variety inside a complex $N$ dimensional vector space which either has an isolated singularity at the origin or is smooth at the origin. The intersection of $A$ with a very small sphere turns out to be a contact manifold called the link of $A$. Any contact manifold contactomorphic to the link of $A$ is said to be Milnor fillable by $A$. If the first Chern class of our link is torsion then we can assign an invariant of our singularity called the minimal discrepancy, which is an important invariant in birational geometry. We define an invariant of the link up to contactomorphism using Conley-Zehnder indices of Reeb orbits and then we relate this invariant with the minimal discrepancy. As a result we show that the standard contact 5 dimensional sphere has a unique Milnor filling up to normalization proving a conjecture by Seidel.

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1. Introduction

Suppose we have an irreducible affine variety $A \subset \mathbb{C}^N$ of complex dimension $n$ which is either smooth at 0 or has an isolated singularity at 0. For any $\epsilon > 0$ small enough we have that $L_A := A \cap S_\epsilon$ is a differentiable manifold of real dimension $2n-1$ and such a manifold is an invariant of the germ of $A$ at 0. We call $L_A$ the link of $A$. The simplest example is when $A$ is smooth at 0 in which case $L_A$ is diffeomorphic to a sphere. Many people have studied the relationship between the algebraic properties of $A$ at 0 and the topology of $L_A$. Such results go back to [Hee]. There have been particularly powerful results when $\dim \mathbb{C}A = 2$ but there are far less powerful results in higher dimensions. For instance, lets start with the following definition inspired by [Hee][Page 236 (French Translation)]: A singularity is topologically smooth if its link is diffeomorphic to a sphere. Mumford in [Mum61] showed that every normal topologically smooth singularity of complex dimension 2 is in fact smooth. But in complex dimension 3 or higher there are many examples of isolated normal singularities which are topologically smooth but not smooth at 0 such as $\{x^2 + y^2 + z^2 + w^3 = 0\}$ (see for
instance [Kwo13] [Theorem 3.13]. Normality of this singularity follows from Serre's criterion for normality [Eis95] [Theorem 18.15]).

Having said this, one can put additional structure on the link. Let $J_0 : TC^N \to TC^N$ be the standard complex structure on $\mathbb{C}^N$ viewed as an automorphism of the tangent bundle whose square is $-\text{id}$. Then $\xi_A := TL_A \cap J_0(TL_A) \subset TL_A$ is a contact structure for $\epsilon$ small enough and $(L_A, \xi_A)$ is an invariant of the germ of $A$ at 0 up to contactomorphism (see [Var80]). One can also view $\xi_A$ as the kernel of $\sum_{j=1}^N x_j dy_j - y_j dx_j \mid_{L_A}$ where $z_j = x_j + iy_j$ are coordinates for $\mathbb{C}^N$. A contact manifold $(C, \xi)$ is said to be \textit{Milnor fillable} if it is contactomorphic to $(L_A, \xi_A)$ for some $A$. An example of a Milnor fillable contact structure is the standard contact sphere $(S^{2n-1}, \xi_{\text{std}})$ which is defined to be the link of $\mathbb{C}^n$ (i.e. $S^{2n-1}$ is the unit sphere in $\mathbb{C}^n$ and $\xi_{\text{std}}$ is the unique hyperplane distribution which is $J_0$ invariant).

In [Ust99] it was shown, for each $m > 0$, that there are infinitely many examples of isolated singularities whose links are diffeomorphic to $S^{4m+1}$ but not contactomorphic to each other. Hence $(L_A, \xi_A)$ is a stronger invariant than $L_A$ on its own. Building on the work of [Ust99], [Kwo13] systematically investigated the links of weighted homogenous hypersurface singularities $\{\sum z_j^{k_j} = 0\}$. In particular using indices of Reeb orbits, [Kwo13] [Theorem 6.3] (along with its proof) tells us when $\sum_j k_j$ is greater than 1 or not just from $(L_A, \xi_A)$. Such a result is significant because Reid in [Rei79] [Proposition 4.3] showed that such a singularity is canonical at 0 if and only if $\sum_j k_j$ is greater than 1 (see [Rei79] [Section 1] for a definition of canonical singularity).

For certain singularities called $Q$ Gorenstein singularities, one can define an invariant taking values in $Q$ called the \textit{minimal discrepancy} (see [Amb06]). We write $\text{md}(A, 0)$ for the minimal discrepancy of $A$ at 0. All isolated complete intersection singularities of complex dimension 2 or higher are $Q$ Gorenstein, and it turns out that canonical singularities are characterized among $Q$ Gorenstein singularities to be the ones with non-negative minimal discrepancy. Hence there is a direct relationship between the result in [Kwo13] mentioned earlier and minimal discrepancy. Minimal discrepancy can be defined for a larger class of singularities called numerically $Q$ Gorenstein singularities in [BdFU13]. An isolated singularity is numerically $Q$ Gorenstein if $c_1(\xi_A)$ is torsion in $H^2(L_A, \mathbb{Z})$ (see Section 2). Singularities with positive minimal discrepancy are called \textit{terminal singularities} and have special importance in the minimal model program ([KM08]). See [BdFU13] [Corollary 5.17] for a proof that positive minimal discrepancy is equivalent to being terminal. In fact minimal discrepancy itself has a special importance in the minimal model program ([Sho88]).

Now let $(C, \xi)$ be a contact manifold of dimension $2n-1$ with $H^1(C; Q) = 0$ and $c_1(\xi; Q) = 0 \in H^2(C, Q)$. Let $\alpha$ be a contact form with $\ker(\alpha) = \xi$. To any Reeb orbit $\gamma : \mathbb{R}/L\mathbb{Z} \to C$ of $\alpha$, we have an associated index $\text{CZ}(\gamma) \in Q$ called the \textit{Conley-Zehnder index}. This index will be defined in Section 3.1. Let $\phi_t : C \to C$, $t \in \mathbb{R}$ be the Reeb flow of $\alpha$. The differential $D\phi_t : TC \to TC$ of the Reeb flow preserves $\xi$ and so for $p \in C$ let $D^\prime_p\phi_t : \xi_p \to \xi_{\phi_t(p)}$ be the restriction of this differential to the contact distribution. The \textit{minimal discrepancy} $\text{md}(C, \xi)$ of $(C, \xi)$ is defined to be:

$$\sup_\alpha \inf_\gamma (\text{CZ}(\gamma) - \frac{1}{2} \dim \ker(D^\prime_{\gamma(0)}\phi_L - \text{id}) + (n - 3))$$

where the supremum is taken over all 1-forms $\alpha$ with $\ker(\alpha) = \xi$ and the infimum is taken over all Reeb orbits $\gamma$ of $\alpha$. Note that because the Reeb orbit has length $L$, our linearized map $D^\prime_{\gamma(0)}\phi_L$ sends $\xi_{\gamma(0)}$ to itself because $\xi_{\gamma(L)} = \xi_{\gamma(0)}$.

Our main theorem is:
**Theorem 1.1.** Let \( A \) have a normal isolated singularity at 0 or be smooth at 0 with \( H^1(L_A; \mathbb{Q}) = 0 \) and \( c_1(\xi_A; \mathbb{Q}) = 0 \) then:

- If \( \text{md}(A, 0) \geq 0 \) then \( 2\text{md}(A, 0) = \text{md}(L_A, \xi) \).
- If \( \text{md}(A, 0) < 0 \) then \( \text{md}(L_A, \xi) < 0 \).

This Theorem will follow from Theorems 4.11 and 6.1. Our main theorem works for any normal isolated singularity, even if it cannot be smoothed. We have the following corollary, proving a conjecture by Seidel \[\text{Sei07}\] [Lecture 6].

**Corollary 1.2.** Suppose that \( A \) is normal and that \((L_A, \xi)\) is contactomorphic to the link of \( \mathbb{C}^3 \) (i.e. the standard contact sphere \((S^5, \xi_{\text{std}})\)), then \( A \) is smooth at 0.

The above corollary says that the standard contact 5 dimensional sphere has a unique Milnor filling up to normalization. This generalizes the previously stated theorem by Mumford because the three sphere has a unique strongly fillable contact structure (see \[\text{Eli90}, \text{Gro85}\]) and because Milnor fillable contact structures are strongly fillable by resolving the singularity (Lemma 4.12). In fact every oriented three manifold admits at most one Milnor fillable contact structure up to orientation preserving diffeomorphism \[\text{CNPP06}\].

The above corollary is a direct consequence of the following Conjecture by Shokurov proven for complex dimension 3 in \[\text{Amb99}, \text{Mar96}\].

**Conjecture 1.3.** (Shokurov \[\text{Sho}\] [Conjecture 2]).

Suppose \( A \) is normal and numerically \( \mathbb{Q} \) Gorenstein with \( \text{md}(A, 0) = n - 1 \) then \( A \) is smooth at 0.

Shokurov has the stronger condition that \( A \) is \( \mathbb{Q} \) Gorenstein, but \[\text{BdPFU13}\] [Corollary 5.17] ensures that any numerically Gorenstein singularity with minimal discrepancy \( > -1 \) is in fact \( \mathbb{Q} \) Gorenstein.

As a result we have the following corollary:

**Corollary 1.4.** Assuming that Conjecture 1.3 is true, \( A \) is normal and \((L_A, \xi)\) is contactomorphic to the standard contact sphere of any dimension greater than 1 then \( A \) is smooth at 0.

In other words, Shokurov’s conjecture combined with Theorem 1.1 implies that the standard contact sphere has a unique Milnor filling up to normalization.

We also have the following corollary:

**Corollary 1.5.** Suppose that \( A, A' \) are normal affine varieties with isolated singularities at 0 or smooth at 0 and whose links are contactomorphic to each other. If \( H^1(L_A; \mathbb{Q}) = 0 \) and \( c_1(\xi_A; \mathbb{Q}) = 0 \) then \( A \) is terminal (resp. canonical) at 0 if and only if \( A' \) is terminal (resp. canonical) at 0. I.e. the property of being terminal or canonical is an invariant of the link.

We will now wildly speculate on the relationship between the main result of this paper and other results concerning the arc space. The arc space was introduced by Nash in \[\text{Nas95}\]. Let \( \text{Arc}(A) \) be the space of germs of holomorphic disks mapping to \( A \) and \( \text{Arc}(A, 0) \subset \text{Arc}(A) \) the subspace of such disks passing through our singularity. Very roughly, \[\text{EMY03}\] [Theorem 2.6] relates the codimension of \( \text{Arc}(A, 0) \) inside \( \text{Arc}(A) \) with the minimal discrepancy. Morally, one might imagine as one of these holomorphic disks in \( \text{Arc}(A, 0) \) approaches the origin, it converges to some Reeb orbit, and that the component of \( \text{Arc}(A, 0) \) of highest codimension finds the lowest index Reeb orbit. Hence one can ask, what is the relationship between the space of pseudo holomorphic curves on the symplectization of this link (such as those curves encoded by symplectic field theory) and the arc space?
We can also ask other questions. For instance Minimal discrepancy is also defined for non-isolated singularities and more generally for log pairs, and it would be interesting to see if there is some way of characterizing the minimal discrepancy of such objects using contact geometry.

This paper is structured as follows: In Section 2 we give two definitions of numerically $\mathbb{Q}$ Gorenstein. One is algebraic and the other is topological. Then we define the minimal discrepancy. In Section 3 we define the Conley-Zehnder index for a Reeb orbit, define minimal discrepancy of a contact manifold and then we relate the Conley-Zehnder indices of degenerate orbits with indices of non-degenerate orbits coming from perturbations of these degenerate orbits. In Section 4 we show that a resolution of $A$ admits a nice symplectic structure and the boundary of a neighborhood of the exceptional divisors is a contact manifold contactomorphic to $(L_A,\xi_A)$ and admitting a contact form with nice families of Reeb orbits. As a result we prove the inequality $2\text{md}(A,0) \leq \text{md}(L_A,\xi_A)$. In Section 5 we show how to define genus 0 Gromov Witten invariants for certain open symplectic manifolds (i.e. we count compact curves in some compact subset of such manifolds). We also prove some important technical Lemmas involving these open manifolds. In Section 6 we use results from the previous section to show $2\text{md}(A,0) \geq \text{md}(L_A,\xi_A)$ if $\text{md}(A,0) \geq 0$ and $\text{md}(L_A,\xi_A) < 0$ if $\text{md}(A,0) < 0$. This is done by partially compactifying some resolution of $A$ and then using Gromov Witten invariants along with a neck stretching argument to find Reeb orbits of the appropriate index. Appendix A reviews neck stretching and proves a compactness result when the contact structure is degenerate. Appendix B proves a maximum principle for stable Hamiltonian structures which is a key argument enabling us to show that we can define Gromov Witten invariants for some partial compactification of a resolution of $A$.

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2. Minimal Discrepancy of Isolated Singularities Definition

The main ideas in this section come from [BdFFU13]. Let $A$ be a singularity which is either isolated at 0 or smooth at 0. First of all, we will give two definitions of a numerically $\mathbb{Q}$ Gorenstein singularity. One definition will be algebraic involving $\mathbb{Q}$ Cartier divisors, and the other will be topological involving the first Chern class of our contact structure $\xi_A$.

We will start with the algebraic definition and then we will give the topological one and prove they are equivalent. Start with some resolution $\pi: \tilde{A} \to A$ so that the preimage of 0 is a union of smooth normal crossing divisors $E_i$ and so that $\pi$ is an isomorphism away from these divisors (If $A$ is smooth we blow up at least once, so $\pi$ is never an isomorphism). Let $K_{\tilde{A}}$ be the canonical bundle of $\tilde{A}$ which we will view as some a $\mathbb{Q}$ Cartier divisor. We say that $A$ is numerically $\mathbb{Q}$ Gorenstein if there exists a $\mathbb{Q}$ Cartier divisor $K_{\tilde{A}/A}^{\text{num}} := \sum_j a_j E_j$ with the property that $C \cdot (K_{\tilde{A}/A}^{\text{num}} - K_{\tilde{A}}) = 0$ for any curve $C \subset \pi^{-1}(0)$. By the negativity Lemma [KM98][8.39] one can show that the coefficients $a_j$ are unique (see [BdFFU13][Proposition 5.3]). Here $a_j \in \mathbb{Q}$ is called the discrepancy of $E_j$.

We will now give a topological characterization of being numerically $\mathbb{Q}$ Gorenstein for isolated singularities (which we regard as the alternate topological definition).
Lemma 2.1. We have that $A$ is numerically $\mathbb{Q}$ Gorenstein if and only if $c_1(\xi_A; \mathbb{Q}) = 0 \in H^2(L_A; \mathbb{Q})$.

Proof. of Lemma 2.1. We let $\tilde{A}_\epsilon := \pi^{-1}(B^{2n}_\epsilon)$ where $B^{2n}_\epsilon \subset \mathbb{C}^n$ is the closed $\epsilon$ ball. For $\epsilon$ small enough we have that $\cup_i E_i \hookrightarrow \tilde{A}_\epsilon$ is a homotopy equivalence. We have that $\partial \tilde{A}_\epsilon = \pi^{-1}(L_A) \cong L_A$. We will write $L_A$ instead of $\pi^{-1}(L_A)$ by abuse of notation.

Note that $H_{2n-2}(\tilde{A}_\epsilon, \mathbb{Q}) \cong H^2(\tilde{A}_\epsilon, L_A, \mathbb{Q})$ and $H_{2n-2}(\tilde{A}_\epsilon, L_A, \mathbb{Q}) \cong H^2(\tilde{A}_\epsilon, \mathbb{Q})$ by Lefschetz duality and that $H_{2n-2}(\tilde{A}_\epsilon)$ is freely generated by the classes $[E_i]$. For a class $x$ we write $LD(x)$ for its Lefschetz dual.

Suppose first that $A$ is numerically $\mathbb{Q}$ Gorenstein. Then there exists a $\mathbb{Q}$ Cartier divisor $\sum_j a_j E_j$ with the property that $C \cdot (\sum_j a_j E_j - K_{\tilde{A}}) = 0$ for any curve $C \subset \pi^{-1}(0)$. We have that $[K_{\tilde{A}}]$ and $\sum_j a_j [E_j]$ are classes in $H_{2n-2}(\tilde{A}_\epsilon, L_A, \mathbb{Q})$. Because $\cup_i E_i \hookrightarrow \tilde{A}_\epsilon$ is a homotopy equivalence, we get by [BdFFF13] [Lemma 5.13] that $\sum_j a_j LD([E_j]) - LD([K_{\tilde{A}}]) = 0 \in H^2(\cup E_i; \mathbb{Q}) = H^2(\tilde{A}_\epsilon, \mathbb{Q})$. If $\nu : H^2(\tilde{A}_\epsilon, \mathbb{Q}) \to H^2(L_A, \mathbb{Q})$ is the natural restriction map then $\nu(LD([E_i])) = 0$. Hence $\nu(LD([K_{\tilde{A}}])) = \nu(LD([K_{\tilde{A}}] - \sum_j a_j[E_j])) = 0$. Now $\nu(LD([K_{\tilde{A}}])) = c_1(T^* \tilde{A}_{\epsilon|L_A}; \mathbb{Q}) = -c_1(\xi_A; \mathbb{Q})$ and so we get $c_1(\xi_A; \mathbb{Q}) = 0$ which implies $c_1(\xi_A) \in H^2(L_A, \mathbb{Z})$ is torsion.

Conversely suppose that $c_1(\xi_A)$ is torsion. Then $c_1(T^* \tilde{A}_\epsilon|L_A; \mathbb{Q}) = -c_1(\xi_A; \mathbb{Q}) = 0 \in H^2(L_A; \mathbb{Q})$. Consider the long exact sequence

$$\xymatrix{ \to H^2(\tilde{A}_\epsilon, L_A, \mathbb{Q}) \ar[r]^-{\nu_2} & H^2(\tilde{A}_\epsilon, \mathbb{Q}) \ar[r]^-{\nu_1} & H^2(L_A, \mathbb{Q}) \to }.$$ 

Because $\nu_2(c_1(T^* \tilde{A}_\epsilon; \mathbb{Q})) = 0$, there exists some class $c_1(T^* \tilde{A}_\epsilon, L_A; \mathbb{Q}) \in H^2(\tilde{A}_\epsilon, L_A, \mathbb{Q})$ satisfying $\nu_1(c_1(T^* \tilde{A}_\epsilon, L_A; \mathbb{Q})) = c_1(T^* \tilde{A}_\epsilon; \mathbb{Q})$. There exists constants $a_i \in \mathbb{Q}$ so that

$$LD(c_1(T^* \tilde{A}_\epsilon, L_A; \mathbb{Q})) = \sum_j a_j [E_j] \in H_{2n-2}(\tilde{A}_\epsilon, \mathbb{Q}).$$

For any curve $C$ in $\cup_i E_i$, we have $C \cdot (\sum_j a_j E_j) = c_1(T^* \tilde{A}_\epsilon, L_A; \mathbb{Q})(C) = c_1(T^* \tilde{A}_\epsilon; \mathbb{Q})(C) = C \cdot K_{\tilde{A}}$. Define $K_{\tilde{A}/A}$ via $\sum_j a_j E_j$. Then $C \cdot (K_{\tilde{A}/A} - K_{\tilde{A}}) = 0$ for any curve $C \subset \pi^{-1}(0)$ which implies that $A$ is numerically $\mathbb{Q}$ Gorenstein. \hfill $\square$

The proof of the above lemma gives us a way of computing discrepancies for numerically $\mathbb{Q}$ Gorenstein singularities in a topological way. Suppose that $A$ is numerically $\mathbb{Q}$ Gorenstein at 0. For simplicity we will assume that $H^1(L_A; \mathbb{Q}) = 0$. We let $\tilde{A}_\epsilon$, $L_A = \pi^{-1}(L_A)$ be as in the proof of the above lemma. We have a long exact sequence:

$$0 = H^1(L_A, \mathbb{Q}) \to H^2(\tilde{A}_\epsilon, L_A, \mathbb{Q}) \to H^2(\tilde{A}_\epsilon, \mathbb{Q}) \to H^2(L_A, \mathbb{Q}).$$

Now $c_1(\tilde{A}_\epsilon) \in H^2(\tilde{A}_\epsilon, \mathbb{Q})$ maps to zero in this long exact sequence since $c_1(T^* \tilde{A}_\epsilon|L_A) = c_1(\xi_A)$ is torsion and so there is a unique element $c_1(\tilde{A}_\epsilon, L_A; \mathbb{Q}) \in H^2(\tilde{A}_\epsilon, L_A, \mathbb{Q})$ called the relative first Chern class. Because $\cup E_i \hookrightarrow \tilde{A}_\epsilon$ is a homotopy equivalence, we get that $H^2(\tilde{A}_\epsilon, L_A, \mathbb{Q}) = H_{2n-2}(\tilde{A}_\epsilon, \mathbb{Q})$ is freely generated by classes $[E_i]$ and so $c_1(\tilde{A}_\epsilon, L_A) = \sum_i a_i [S_i]$ for some unique $a_1, \ldots, a_t \in \mathbb{Q}$. Here $a_i$ is the discrepancy of $E_i$. Note that we do not really need the condition $H^1(L_A; \mathbb{Q}) = 0$ here by the negativity lemma [KM98] [8.39], but we place it here anyway for simplicity and also because we need such a condition when dealing with the Conley-Zehnder index later.

The minimal discrepancy is the infimum over all resolutions $\pi$ of $a_j$. One can calculate the minimal discrepancy from a single resolution by defining it to be $\min_j(a_j)$ if $\min_j(a_j) \geq -1$
and $-\infty$ otherwise. This follows from the fact any two resolutions of $A$ at 0 are related by a sequence of blowups and blowdowns along smooth subvarieties and the fact that the discrepancy of the blowup of some smooth variety along some smooth subvariety of complex codimension $k$ is $k - 1$. In order to calculate the minimal discrepancy of $A$ when $A$ is smooth at 0, you have to blow up at least once, which gives a minimal discrepancy of $n - 1$ where $n = \dim_{\mathbb{C}} A$.

3. THE CONLEY-ZEHNDER INDEX

In this section we will define the Conley-Zehnder index of a Reeb orbit. We will then define what a pseudo Morse Bott family of Reeb orbits is and then prove some result relating the Conley-Zehnder index of degenerate orbits with the index of non-degenerate orbits coming from perturbations of such degenerate orbits.

3.1. Definition of Conley-Zehnder Index. We will first give a definition of the Conley-Zehnder index of a path of symplectic matrices in terms of the Maslov index before we define it for Reeb orbits. This will be useful later on. Let $\mathcal{L}$ be the set of Lagrangian vector subspaces of a symplectic vector space $W'$. Fix some $L \in \mathcal{L}$. To any smooth path $\Lambda : [a, b] \to \mathcal{L}$ we can assign an index $\text{Mas}_{W', L}(\Lambda) \in \frac{1}{2}\mathbb{Z}$ (see [RS93]). We will just write $\text{Mas}(\Lambda) = \text{Mas}_{W', L}(\Lambda)$ when the context is clear.

Let $W$ be a symplectic vector space and $Sp(2n)$ the space of linear symplectomorphisms of $W$. For any path $A : [a, b] \to Sp(2n)$ we can define an index $\text{CZ}(A(t)) \in \frac{1}{2}\mathbb{Z}$ as follows: Let $W \times W$ be the product symplectic vector space with symplectic form $(-\omega_W, \omega_W)$ where $W$ is equal to $W$ and let $\Delta$ be the diagonal Lagrangian. We define $\text{CZ}(A) := \text{Mas}_{W \times W, \Delta}(\Gamma(A(t)))$ where $\Gamma(A(t))$ is the path of Lagrangians given by the graph of $A(t)$ in $W \times W$ viewed as a map $A(t) : W \to W$. This is an invariant of $A(t)$ up to homotopies fixing its endpoints.

We will now define the Conley-Zehnder index for Reeb orbits. Let $(C, \xi)$ be a contact manifold of dimension $2n - 1$ and let $\alpha$ be a contact form with $\ker(\alpha) = \xi$. The contact hyperplane distribution $\xi$ on $C$ has a natural symplectic structure given by restricting $d\alpha$ and so if we choose a compatible almost complex structure on this contact hyperplane distribution then it has a natural $U(n - 1)$ structure. Because this bundle is a complex bundle we can take its highest exterior power. Such a bundle is called the anticanonical bundle of $C$. The dual of such a bundle is called the canonical bundle. From now on we will assume that $c_1(\xi; \mathbb{Q}) = 0 \in H^2(C; \mathbb{Q})$. This means there is some number $N_{c_1} \in \mathbb{N}$ so that $N_{c_1}c_1(\xi) = 0 \in H^2(C; \mathbb{Z})$. This means that we can trivialize the $N_{c_1}$th power of the anticanonical bundle.

Let $\tau : (\kappa^*)^{\otimes N_{c_1}} \to C \times \mathbb{C}$ be a choice of such a trivialization where $\kappa^*$ is the anticanonical bundle. Let $\gamma : \mathbb{R}/L\mathbb{Z} \to C$ be a Reeb orbit of $\alpha$. Let $D'\phi_t : \xi \to \xi$ be the restriction of the linearization $D\phi_t : TC \to TC$ of $\phi_t$. Choose a trivialization of $\gamma^* \oplus_{j=1}^{N_{c_1}} \xi$ as a complex vector bundle so that its highest exterior power coincides with our trivialization $\tau$. Such a choice is unique up to homotopy. Let $\phi_t : C \to C$ be the flow of the Reeb vector field of $C$. Let $D'\phi_t : \xi \to \xi$ be the restriction of the linearization $D\phi_t : TC \to TC$ of the Reeb flow $\phi_t$ of $\alpha$. Then using the above trivialization along $\gamma$ we have that $\oplus_{j=1}^{N_{c_1}} D'\phi_t$ gives us a family of symplectic matrices in $\mathbb{R}^{(2n-2)N_{c_1}}$ parameterized by $\mathbb{R}/L\mathbb{Z}$. We view this as a path parameterized by $[0, L]$ under the surjection $[0, L] \to \mathbb{R}/L\mathbb{Z}$ and hence we can define $\text{CZ}_\tau(\gamma) \in \frac{1}{2N_{c_1}}\mathbb{Z} \subset \mathbb{Q}$ to be the Conley-Zehnder index of such a path of matrices divided by $N_{c_1}$.

This index only depends on $\tau$ up to homotopy. Note that if $H^1(C, \mathbb{Q}) = 0$ then $\tau$ is unique up to homotopy and in fact our indices also do not depend on $N_{c_1}$ either. Here is the reason
why: If we have some other choice of trivialization \( \tau' : (\kappa^*)^{\otimes N_{c_1}} \rightarrow C \times \mathbb{C} \) where \( N'_{c_1} \in \mathbb{N} \) then \( \tau \) and \( \tau' \) induce natural trivializations \( \tau^{\otimes N_{c_1}} \) and \( (\tau')^{\otimes N_{c_1}} \) of \( (\kappa^*)^{\otimes N_{c_1} N'_{c_1}} \). Because Conley-Zehnder index is additive under direct sum we then get that \( CZ_\tau(\gamma) = CZ_{\tau'}^{\otimes N_{c_1}}(\gamma) \) and similarly \( CZ_{\tau'}(\gamma) = CZ_{\tau^{\otimes N_{c_1}}}(\gamma) \). Now \( \tau^{\otimes N_{c_1}} \circ (\tau')^{\otimes N_{c_1}})^{-1} \) is a bundle morphism between trivial bundles and hence a section of the trivial bundle \( C \times S^1 \) which is equivalent to a smooth map from \( C \rightarrow S^1 \). Because the pullback of \( d\theta \) via \( C \rightarrow S^1 \) is exact as \( H^1(C, \mathbb{R}) = 0 \) we get that such a smooth map is homotopic to the constant map and this means that \( \tau^{\otimes N_{c_1}} \) is homotopic to \( \tau'^{\otimes N_{c_1}} \) through trivializations. Hence \( CZ_{\tau^{\otimes N_{c_1}}}(\gamma) = CZ_{\tau^{\otimes N_{c_1}}}(\gamma) \). And so \( CZ_\tau(\gamma) = CZ_{\tau'}(\gamma) \). Summarizing we get that if \( H^1(C, \mathbb{Q}) = 0 \) then \( CZ_\tau(\gamma) \) does not depend on \( \tau \) or \( N_{c_1} \). We will write \( CZ(\gamma) = CZ_\tau(\gamma) \) in this case or when the context is clear.

3.2. Pseudo Morse Bott families. We will be looking at families of Reeb orbits of \( (C, \alpha) \) that are not necessarily Morse Bott but have very similar properties. Suppose \( B_T \) is a subset of \( C \) satisfying:

1. There is a Reeb orbit of length \( T \) passing through every point of \( B_T \) and this Reeb orbit is contained in \( B_T \).
2. There is a neighborhood \( N_{B_T} \) containing \( B_T \) and a constant \( \delta > 0 \) so that any Reeb orbit with length in \([T - \delta, T + \delta] \) starting inside \( N_{B_T} \) starts inside \( B_T \).

Then we say \( B_T \) is an isolated family of Reeb orbits of length \( T \). We will say that \( N_{B_T} \) is an isolating neighborhood for \( B_T \). Note that the union of two isolated families of Reeb orbits with the same length \( T \) is an isolated family of Reeb orbits of length \( T \).

A typical example of an isolated family of length \( T \) is a Morse Bott submanifold which is a submanifold \( B \) so that every Reeb orbit starting in \( B \) is contained in \( B \) and so that if \( \phi_T : C \rightarrow C \) is the time \( T \) flow of the Reeb vector field then \( \ker(D\phi_T - \text{id}) = TB \) at any point of \( B \). We will be dealing with something more general than this. We are interested in indices of Reeb orbits, and so from now on we will assume that there is some fixed trivialization of the canonical bundle of \((C, \alpha)\).

**Definition 3.1.** A pseudo Morse Bott family is an isolated family of Reeb orbits \( B_T \) of length \( T \) with the additional property that \( B_T \) is path connected and for each point \( p \in B_T \) we have \( \text{Size}(B_T) := \dim \ker(D\phi_T - \text{id}) \) is constant along \( B_T \) (Recall, \( D\phi_t : \ker(\alpha) \rightarrow \ker(\alpha) \) is the restriction of the linearization of \( \phi_t \) to \( \ker(\alpha) \)).

We now need a Lemma enabling us to define an index for \( B_T \). Let \( S_k \subset \text{Sp}(2n) \) be the set of symplectic matrices \( A \) with rank \( \ker(A - \text{id}) = k \). Let \( o_1(t), o_2(t) \) be two paths in \( \text{Sp}(2n) \). We say that \( o_1, o_2 \) are stratum homotopic if there is a smooth family of paths \( \psi_s : [0, 1] \rightarrow \text{Sp}(2n) \) where \( \psi_s(0) \in S_{k_1}, \psi_s(1) \in S_{k_2} \) for all \( s \) and some fixed \( k_1, k_2 \) and \( \psi_0, \psi_1 \) are homotopic to \( o_1, o_2 \) respectively relative to their endpoints.

**Lemma 3.2.** If \( o_1, o_2 \) are stratum homotopic then they have the same Conley-Zehnder indices.

*Proof.* of Lemma 3.2 This follows directly from [RS93][Theorem 2.4]. \( \square \)

By Lemma 3.2 we have that the Conley Zehnder index of the length \( T \) orbits starting in \( B_T \) are all the same because \( B_T \) is path connected. Hence we define the Conley Zehnder index of \( B_T \), \( CZ(B_T) \), to be the Conley Zehnder index of one of its length \( T \) Reeb orbits.
### 3.3. Perturbations of Contact Forms and Indices

The following Lemmas enable us to calculate the indices of Reeb orbits coming from perturbations of degenerate Reeb orbits. We need a linear algebra lemma and corollary first.

Let $W'$ be a symplectic vector space and $\mathcal{L}$ the set of linear Lagrangians inside $W'$. Let $L \in \mathcal{L}$ be a fixed Lagrangian. Define $\mathcal{L}_k$ to be the set of Lagrangians in $W'$ whose intersection with our fixed Lagrangian $L$ has dimension $k$.

**Lemma 3.3.** Fix $\Lambda_0 \in \mathcal{L}_k$. Then for a sufficiently small neighborhood $\mathcal{N}_{\Lambda_0}$ of $\Lambda_0$ we have that any path $\Lambda : [a, b] \to \mathcal{N}_{\Lambda_0}$ with $\Lambda(0) = \Lambda$ has Maslov index in $[-\frac{k}{2}, \frac{k}{2}]$.

**Proof.** of Lemma 3.3 First of all we can identify our symplectic vector space $W'$ with a subspace of $L$ so that $(L \cap \Lambda_0) \oplus Z = L$. We get that $A|_Z$ is non-degenerate as a quadratic form. We choose $\mathcal{N}_{\Lambda_0}$ small enough so that every element of this set can also be expressed as the graph of a symmetric matrix whose restriction to $Z$ is non-degenerate. If $\Lambda(t)$ is a smooth path in $\mathcal{N}_{\Lambda_0}$ starting at $\Lambda_0$ then this is represented by a smooth family of symmetric matrices $A_t$ with $A_0 = A$ whose restriction to $Z$ is non-degenerate. By the localization axiom from [RS03] Theorem 2.3 we then get that the Maslov index is $\frac{1}{2}(\text{sign}(A_1) - \text{sign}(A))$. Here sign means the sign of the symmetric matrix as a quadratic form. Now $\text{sign}(A_t) = \text{sign}(A_t|_Z) + \text{sign}(A_t|_{L \cap \Lambda_0})$. Because $A_t|_Z$ is a smooth family of non-degenerate quadratic forms, they have the same sign. Also $\text{sign}(A)|_{L \cap \Lambda_0} = 0$. Hence

$$\text{Mas}(\Lambda(t)) = \frac{1}{2}(\text{sign}(A_1) - \text{sign}(A)) = \frac{1}{2}\text{sign}(A_1)|_{L \cap \Lambda_0}.$$ 

Because the dimension of $L \cap \Lambda_0$ is $k$ this means that $\text{Mas}(\Lambda(t)) \in [-\frac{k}{2}, \frac{k}{2}]$. \hfill \Box

As a result we have the following direct corollary:

**Corollary 3.4.** Let $B \in S_k$. Then for a sufficiently small neighborhood $\mathcal{N}_B$ of $B$ we have that any path $\alpha(t) \in \mathcal{N}_B$ with $\alpha(0) = B$ has Conley-Zehnder index in $[-\frac{k}{2}, \frac{k}{2}]$.

The following lemma tells us that if we perturb a pseudo Morse Bott family then all the nearby Reeb orbits have a bound on their Conley-Zehnder indices. Recall that $D\phi_t$ is the linearization of the Reeb flow $\phi_t$ of $\alpha$. Let $\xi := \ker(\alpha)$ be the contact hyperplane distribution. Because $D\phi_t$ preserves $\xi$ we will write $D\phi_t : \xi \to \xi$ to be the restriction of $D\phi_t$ to the hyperplane distribution. We choose a trivialization $\tau$ of $\otimes_{j=1}^{N_i} \kappa^*$ where $\kappa^*$ is the highest exterior power of $\xi$.

**Lemma 3.5.** Let $\gamma$ be any Reeb orbit of $\alpha$ of length $T$ and define $K := \dim \ker(D'\phi_T(\gamma(0)) - id)$. Fix some metric on $C$. There are constants $\delta_1, \delta_2 > 0$ and a neighborhood $\mathcal{N}$ of $\gamma(0)$ so that for any contact form $\alpha'$ with $|\alpha - \alpha'|_{C^2} < \delta_1$ and any Reeb orbit $\gamma'$ of $\alpha'$ starting in $N$ of length in $[T - \delta_2, T + \delta_2]$ we have $\text{CZ}(\gamma') \in [\text{CZ}(\gamma) - \frac{1}{2}K, \text{CZ}(\gamma) + \frac{1}{2}K]$.

**Proof.** of Lemma 3.5 Choose a sequence of contact forms $\alpha_i$, $C^2$ converging to $\alpha$, and a sequence of Reeb orbits $\gamma_i$ of length $T_i$ converging to $T$ with the property that $\gamma_i(0)$
converges to $\gamma(0)$. By Gray’s stability theorem we can assume that $\ker(\alpha_i) = \ker(\alpha) = \xi$. We wish to show that $\mathrm{CZ}(\gamma_i) \in [\mathrm{CZ}(\gamma) - \frac{1}{2}K, \mathrm{CZ}(\gamma) + \frac{1}{2}K]$ for sufficiently large $i$. Let $D_t(\cdot) : T_{\gamma_i(0)} \to T_{\gamma_i(t)}$ be the linearization of the Reeb flow of $\gamma_i$ from $\gamma_i(0)$ to $\gamma_i(t)$. Similarly define $D_\infty(t)$ for $\gamma(t)$. Because these linearizations preserve the contact form $\theta$, the Conley-Zehnder index of $\gamma(t)$ will actually view them as symplectic linear maps: $D_t'(\cdot) : \ker(\alpha_i'(\gamma_i(0)) = \ker(\alpha'(\gamma_i(t))$ and $D_\infty'(\cdot) : \ker(\alpha_\infty(0)) = \ker(\alpha(\gamma(0))$. Choose trivializations $\tau_i : \gamma^*_i + \bigoplus_{j=1}^{N_{c_1}} \xi \to (\mathbb{R}^n/T\mathbb{Z}) \times \mathbb{R}^{(2n-2)N_{c_1}}$ and $\tau_\infty : \gamma^* + \bigoplus_{j=1}^{N_{c_1}} \xi \to (\mathbb{R}^n/T\mathbb{Z}) \times \mathbb{R}^{(2n-2)N_{c_1}}$ whose highest exterior power agrees with our chosen trivialization $\tau$. This means that $\bigoplus_{j=1}^{N_{c_1}} D'_i(t)$ and $\bigoplus_{j=1}^{N_{c_1}} D'_\infty(t)$ are represented by paths of symplectic matrices in $\text{Sp}(2n-2N_{c_1})$ respectively, which we write as $D'_i(t)$ and $D'_\infty(t)$ so that $D'_i(t), t \in [0, T]$ converges to $\gamma(0)$ through $C^\infty$ converges (after linearly rescaling the $t$ coordinate) to $D'_\infty(t)$, $t \in [0, T]$. Note that $\dim \ker(D_i'(T_i) - \text{id}) = N_{c_1} \dim \ker(D_i'(T_i) - \text{id})$ and that $\mathrm{CZ}(D_i') = N_{c_1} \mathrm{CZ}(\gamma_i)$ for all $i \in \mathbb{N} \cup \{\infty\}$.

By Corollary 3.4 there is a small neighborhood $\mathcal{N}$ of $\mathrm{D}_\infty'(T)$ in $\text{Sp}(2n-2N_{c_1})$ so that every path of symplectic matrices in $\mathcal{N}$ starting at $\gamma(T)$ has Conley-Zehnder index in the interval $[-\frac{1}{2}N_{c_1}K, \frac{1}{2}N_{c_1}K]$. We can also assume that $\mathcal{N}$ is contractible. For $i$ large enough we have that $D_i'(T_i) \in \mathcal{N}$ and also that $D_i'(t)$ is homotopic to $D_\infty'(t)$ through paths whose starting point is fixed and whose endpoint is contained in $\mathcal{N}$. Let $p_i(t) \in \mathcal{N}$ be a path starting at $D_i'(T_i)$ and ending at $D_\infty'(T)$. The concatenation of $D_i'(t)$ and $p_i(t)$ is a path homotopic to $D_\infty'(t)$ through paths with fixed endpoints $D_\infty'(0), D_\infty'(T)$ and so have the same Conley-Zehnder index. Using the fact that the Conley-Zehnder index of $p_i(t)$ is in $[-\frac{1}{2}N_{c_1}K, \frac{1}{2}N_{c_1}K]$ and the fact that Conley-Zehnder index is additive under path concatenation (see [RS93]) we get that the Conley-Zehnder index of $D_i'(t)$ is in $\mathrm{CZ}([D_\infty', -\frac{1}{2}N_{c_1}K, \mathrm{CZ}(D_\infty') + \frac{1}{2}N_{c_1}K]$. Now $\mathrm{CZ}(\gamma_i) = \frac{1}{N_{c_1}} \mathrm{CZ}(D_i')$ and $\mathrm{CZ}(\gamma) = \frac{1}{N_{c_1}} \mathrm{CZ}(D_\infty')$ and hence $\mathrm{CZ}(\gamma_i) \in [\mathrm{CZ}(\gamma) - \frac{1}{2}K, \mathrm{CZ}(\gamma) + \frac{1}{2}K]$ for $i$ sufficiently large.

4. Neighbourhoods of Symplectic Submanifolds with Contact Boundary

Let $(M, \omega)$ be a symplectic manifold with boundary and let $S_1, \ldots, S_l$ be real codimension 2 submanifolds that intersect transversally and are disjoint from the boundary of $M$. For $I \subset \{1, \ldots, l\}$ we define $S_I := \bigcap_{i \in I} S_i$. We say that $S_1, \ldots, S_l$ are positively intersecting if:

1. $S_I$ is a symplectic submanifold for each $I \subset \{1, \ldots, l\}$.
2. For each $I, J \subset \{1, \ldots, l\}$ with $S_{I \cup J} \neq 0$ let $N_1$ be the symplectic normal bundle for $S_{I \cup J}$ in $S_I$ and $N_2$ the symplectic normal bundle for $S_{I \cup J}$ in $S_J$. Because $TS_{I \cup J}$, $N_1$, $N_2$ have natural orientations, we require that the orientation of $N_1 \oplus N_2 \oplus TS_{I \cup J}$ agrees with the natural symplectic orientation on $TM|_{S_{I \cup J}}$ for all $I, J$ with $S_{I \cup J} \neq 0$.

We will assume for simplicity that $M$ is a compact manifold with boundary homotopic to $\bigcup_i S_i$. Because the results in this section are local near $\bigcup_i S_i$ this assumption does not really matter.

Now suppose $\theta$ is a 1-form on $M \setminus \bigcup_i S_i$ so that $d\theta = \omega$. Let $\theta_e$ be a 1 form on $M$ equal to $\theta$ near $\partial M$ but equal to 0 near $\bigcup_i S_i$. Then $\omega - d\theta_e$ represents an element of $H^2(M, \partial M; \mathbb{R})$ (the chain complex for this cohomology group consists of de Rham forms whose restriction to $\partial M$ is 0). Such a class only depends on $\theta$. By Lefschetz duality we have $H^2(M, \partial M; \mathbb{R}) = H_{2n-2}(M; \mathbb{R})$ and because $M$ is homotopic to $\bigcup_i S_i$ we get that $[\omega]$ is Lefschetz dual to $-\sum_i \lambda_i [S_i] \in H_{2n-2}(M; \mathbb{R})$. The constant $\lambda_i$ is called the wrapping number of $\theta$ around $S_i$. It turns out that this definition is identical to the one in [MCL12, Section 5.2] modulo rescaling by 2$\pi$. We will always assume that $\lambda_i > 0$ are positive wrapping numbers.
Throughout this section, the following examples should be kept in mind: If \( A \subset \mathbb{C}^N \) is an affine variety with an isolated singularity at zero, then we can intersect it with a small closed ball \( B_\delta \). We resolve \( A \) at 0 by blowing up along smooth subvarieties and take the preimage \( \widetilde{A}_\delta \) of \( B_\delta \) under this resolution map. Later on (see Lemma 4.12) we will show that such a resolution has a compatible symplectic form and the exceptional divisors will be positively intersecting symplectic submanifolds of \( \widetilde{A}_\delta \) and we will also show that there is a 1-form \( \theta \) as above giving these submanifolds positive wrapping numbers for an appropriate resolution.

4.1. Boundary Construction and Uniqueness. Let \( (M, \omega), S_1, \ldots, S_l \) and \( \theta \) be as above. Also suppose we have neighborhoods \( N_i \) of \( S_i \) and smooth projection maps \( p_i : N_i \to S_i \) whose fibers are diffeomorphic to disks transversally intersecting \( S_i \) at 0 and admitting some connection rotating these fibers. This means that there is a natural radial coordinate \( r_i : N_i \to [0, \delta_i] \) whose zero set is \( S_i \) for some small \( \delta_i > 0 \). Let \( \rho : [0, \delta_i) \to [0, 1] \) be a smooth function so that \( \rho(x) = x^2 \) near 0 and \( \rho(x) = 1 \) near \( \delta_i \) with \( \rho' \geq 0 \). We define \( \rho(r_i) \) to be \( \rho \circ r_i \) inside \( N_i \) and 1 elsewhere. A smooth function \( f : M \to \mathbb{R} \) is said to be compatible with \( \cup_i S_i \) if it is equal to \( \sum_i \log(\rho(r_i))) + \tau \) for some chosen coordinates \( r_i \), choice of bump function \( \rho \) as above and some smooth function \( \tau : M \to \mathbb{R} \).

The main proposition of this section is:

**Proposition 4.1.** For any function \( f \) compatible with \( \cup_i S_i \), there exists a smooth function \( g \) on \( M \setminus \cup_i S_i \) so that \( df(X_{\theta+dg}) > 0 \) in some small neighborhood of \( \cup_i S_i \).

We also have a parameterized version of this theorem: Suppose that \( \omega_t \) is a smooth family of symplectic forms parameterized by \( t \) in some compact manifold and \( \theta_t \) a smooth family of 1-forms with \( d\theta_t = \omega_t \) making \( S_1, \ldots, S_l \) positively intersecting with positive wrapping numbers. Also for any 1-form \( \eta \) we define \( X^t_\eta \) to be the \( \omega_t \) dual of \( \eta \).

**Proposition 4.2.** For any smooth family of functions \( f_t \) compatible with \( \cup_i S_i \), there is a smooth family of functions \( g_t \) on \( M \setminus \cup_i S_i \) so that \( df_t(X_{\theta_t+dg_t}) > 0 \) near \( \cup_i S_i \).

The proof of this proposition is exactly the same as the proof of Proposition 4.1 except that we now have everything parameterized by \( t \).

We have the following corollary:

**Corollary 4.3.** Let \( f_0, f_1 : M \setminus \cup_i S_i \to \mathbb{R} \) be compatible with \( \cup_i S_i \) and let \( g_0, g_1 : M \setminus \cup_i S_i \to \mathbb{R} \) be such that for \( j = 0, 1 \), \( df_j(X^t_{\theta_j+dg_j}) > 0 \) near \( \cup_i S_i \). Then for all sufficiently negative \( l \) we have that \( (f_0^{-1}(l), \theta_0 + dg_0|_{f_0^{-1}(l)}) \) is contactomorphic to \( (f_1^{-1}(l), \theta_1 + dg_1|_{f_1^{-1}(l)}) \).

This means that we can associate a canonical contact manifold to the deformation class of \( (M, \theta, \cup_i S_i) \).

**Definition 4.4.** Such a contact structure will be called the link of \( (M, \theta, \cup_i S_i) \).

**Proof.** of Corollary 4.3 It is fairly straightforward to construct a smooth family \( f_t : M \setminus \cup_i S_i \to \mathbb{R} \) of functions compatible with \( \cup_i S_i \) joining \( f_0 \) and \( f_1 \). By Proposition 4.1 there is a smooth family of functions \( g_t \) so that \( df_t(X^t_{\theta_t+dg_t}) > 0 \) near \( \cup_i S_i \). Hence by Gray’s stability theorem we have for all sufficiently negative \( l \) that \( (f_0^{-1}(l), \theta_0 + dg_0|_{f_0^{-1}(l)}) \) is contactomorphic to \( (f_1^{-1}(l), \theta_1 + dg_1|_{f_1^{-1}(l)}) \). Also for all sufficiently negative \( l \), \( t \in [0, 1] \) and \( j = 0, 1 \) we have \( (f_1^{-1}(l), \theta_j + (1-t)dg_j + tdg_j|_{f_1^{-1}(l)}) \) are all contact manifolds and so are all contactomorphic if \( j \) is fixed. Hence \( (f_0^{-1}(l), \theta_0 + dg_0|_{f_0^{-1}(l)}) \) is contactomorphic to \( (f_1^{-1}(l), \theta_1 + dg_1|_{f_1^{-1}(l)}) \).

To prove Proposition 4.1 we need the following Lemma:
**Lemma 4.5.** Let $\pi : U \rightarrow S$ be a smooth fibration where $U$ is a symplectic manifold whose fibers are all symplectomorphic to a ball of small radius in $\mathbb{R}^{2n}$ and so that the natural symplectic connection has structure group in $U(n)$. Hence we can view $S$ as a submanifold of $U$ given by the 0 section. We will assume that $S$ is diffeomorphic to a ball. Suppose that $S'_1, \ldots, S'_n$ are real codimension 2 positively intersecting symplectic submanifolds of $U$ so that $\cap_i S'_i = S$. Let $\theta'$ be a 1-form on $U \setminus \cup_i S'_i$ so that $d\theta' = \omega_U$ is the symplectic form on $U$ and so that $S'_1, \ldots, S'_n$ all have positive wrapping numbers with respect to $\theta'$. Then (after shrinking the fibers of $\pi$) there is a function $g'$ so that for all smooth functions $f : U \setminus \cup_i S'_i \rightarrow \mathbb{R}$ compatible with $\cup_i S'_i$ we have that $df(X_{\theta'+g'}) > cf\|\theta' + dg'\|\|df\|$ near $\cup_i S'_i$ where $cf > 0$ is a constant depending on $f$. Also $\|\theta' + g'\| < \|df\|$ near $\cup_i S'_i$ for some smooth function $b$ compatible with $\cup_i S'_i$.

Here $\|\cdot\|$ is some choice of metric.

**Proof.** of Lemma 4.5 After shrinking the fibers of $U$ by reducing the radius we will assume that each submanifold $S'_i$ intersects each fiber of $\pi$ transversally. Let $\lambda_1, \ldots, \lambda_n > 0$ be the wrapping numbers of $\theta'$ around $S'_1, \ldots, S'_n$ respectively. For the moment we will fix some $1 \leq i \leq n$. Again after shrinking the fibers of $U$ we have smooth functions $x_1, y_1, \ldots, x_n, y_n$ in $U$ so that $S'_i = \{x_1, y_1 = 0\}$ and so that the restriction of these functions to each fiber of $\pi$ gives us a symplectic coordinate system centered at 0. Note that these coordinates are dependent on $i$. We also have polar coordinates $r, \vartheta$ depending on $x_1, y_1$ only so that $x_1 = r \cos(\vartheta)$ and $y_1 = r \sin(\vartheta)$. The universal cover $\tilde{U}_i$ of $U \setminus S'_i$ admits a fibration $\tilde{\pi}_i : \tilde{U}_i \rightarrow S$ equal to the covering map composed with $\pi$. Also $\tilde{U}_i$ has functions $\tilde{x}_1, \tilde{y}_1, \tilde{x}_2, \tilde{y}_2, \ldots, \tilde{x}_n, \tilde{y}_n$ which are pullbacks of $r^2, \vartheta, x_2, y_2, \ldots, x_n, y_n$ respectively. Each fiber of $\tilde{\pi}_i$ is the universal cover of each fiber of $\pi|_{U \setminus S'_i}$. The coordinates $(\tilde{x}_1, \tilde{y}_1, \tilde{x}_2, \tilde{y}_2, \ldots, \tilde{x}_n, \tilde{y}_n)$ naturally identify the fibers of $\tilde{\pi}_i$ with an open subset of $\mathbb{R}^{2n}$ hence $\tilde{\pi}_i : \tilde{U}_i \rightarrow S$ enlarges to a smooth fibration $\tilde{\pi}'_i : \tilde{U}'_i \rightarrow S$ whose fibers are diffeomorphic to $\mathbb{R}^{2n}$ with standard coordinates $(\tilde{x}_1, \tilde{y}_1, \tilde{x}_2, \tilde{y}_2, \ldots, \tilde{x}_n, \tilde{y}_n)$ but the symplectic form does not necessarily extend. Having said that the restriction of the symplectic form to each fiber of $\tilde{\pi}'_i$ extends to $\sum_i d\tilde{x}_i \wedge d\tilde{y}_i$ on the fibers of $\tilde{\pi}'_i$. The group of deck transformations $\mathbb{Z}$ acts on each fiber of $\tilde{\pi}_i$ by sending the coordinate $(\tilde{x}_1, \tilde{y}_1, \tilde{x}_2, \tilde{y}_2, \ldots, \tilde{x}_n)$ to $(\tilde{x}_1, \tilde{y}_1 + 2\pi k, \tilde{x}_2, \tilde{y}_2, \ldots, \tilde{x}_n)$ for each $k \in \mathbb{Z}$. This group action extends to one on $\tilde{U}'_i$.

Note that if we have submanifold $\tilde{V} \subset \tilde{U}'_i$ which intersects each fiber of $\tilde{\pi}'_i$ transversally and whose intersection with each fiber is an affine linear subspace of $\mathbb{R}^{2n}$ in the coordinates $(\tilde{x}_1, \tilde{y}_1, \tilde{x}_2, \tilde{y}_2, \ldots, \tilde{x}_n, \tilde{y}_n)$ where the restriction of $\tilde{y}_1$ to this submanifold is constant and invariant under translations in the $x_1$ direction, then it maps via the covering map to a fiberwise linear submanifold $V$ of $U$ in the coordinates $x_1, y_1, \ldots, x_n, y_n$. We can view the tangent space at 0 of the intersection of $\cap_{j \neq i} S'_{j}$ with a fiber as a subspace of this fiber linear in the coordinates $(x_1, y_1, \ldots, x_n, y_n)$. Let $L_{\theta'} \subset \tilde{U}'_i$ be the unique submanifold of $\tilde{U}'_i$ so that:

1. it’s intersection with each fiber $F = (\tilde{\pi}'_i)^{-1}(q)$ is a line $L_{\theta'}^F$ passing through the point $(0, \theta', 0, 0, \ldots, 0)$.
2. $\tilde{y}_1$ restricted to $L_{\theta'}^F$ is constant for all $F$.
3. The projection of $L_{\theta'}^F \cap \tilde{U}_i$ to $U$ is contained in the tangent space at zero of $\pi^{-1}(q) \cap \cap_{j \neq i} S'_j$.

If $q$ is a 1-form in $\tilde{U}'_i$ then define $X^x_q$ to be the unique vector field tangent to the fibers $(\tilde{\pi}'_i)^{-1}(p)$ of $\tilde{\pi}'_i$ so that $q|_{(\tilde{\pi}'_i)^{-1}(p)} = i_x^q(\sum_j d\tilde{x}_j \wedge d\tilde{y}_j)$.

There is a 1-form $\tilde{q}_i$ in $\tilde{U}'_i$ with the following properties:
(1) The restriction of $\tilde{q}_i$ to each fiber is closed.
(2) Inside each fiber, $X^\nu_{\tilde{q}_i}$ is tangent to the line $L^\nu$ at the point $(0, \nu', 0, 0, \ldots, 0)$ and pointing in the direction in which $\tilde{x}_i$ is increasing.
(3) $d\tilde{q}_i(X^\nu_{\tilde{q}_i}) > 0$ along $\nu := \cap_j\{x_j = 0\} \cap \cap_j \tilde{y}_j = 0$ and the integral from 0 to 1 of $\tilde{q}_i$ along each line $\nu \cap (\tilde{\pi}_i)^{-1}(p)$ is $\lambda_i$.
(4) $\tilde{q}_i$ is invariant under the $\mathbb{Z}$ action near $\nu$.

Because the fibers of $\tilde{\pi}_i$ are contractible we have that $\tilde{q}_i$ is fiberwise exact and so there is a smooth function $g_i$ whose differential restricted to each fiber is $\tilde{q}_i$. Let $\tilde{q}_i' = dg_i$. This is a closed 1-form with exactly the same properties as $\tilde{q}_i$. After shrinking $U$, we have that $\tilde{q}_i'$ descends to a fiberwise closed 1 form $q_i$ on $U \setminus S'_i$.

Because $dq_i = 0$ we have, after possibly shrinking $U$, that the symplectic form $\omega_U$ is equal to the exterior derivative of $\theta'' := \theta_1 + \sum_i q_i$ where $\theta_1$ is a 1-form whose norm is bounded. This is equal to $\theta' + dg$ near the zero section for some smooth function $g : U \setminus \cup_i S_i' \to \mathbb{R}$ because both $\theta'$ and $\theta''$ have the same wrapping numbers. We now wish to show that $df(X_{\sum_i q_i}) > c \sum_i q_i ||df||$ near $S$ for each $f$ compatible with $\cup_i S_i$. Because the norm of $\theta_1$ is bounded it is sufficient to show that $df(X_{\sum_i q_i}) > c ||\sum_i q_i|| ||df||$ for some constant $c$ near $S$. By definition there are smooth functions $x_1^{\nu'}, y_1^{\nu'}, \ldots, x_n^{\nu'}, y_n^{\nu'}$ whose set is $S$ and whose restriction to each fiber is a coordinate system with the property that $S_i' = \{x_i^{\nu'}, y_i^{\nu'} = 0\}$ (again after shrinking the fibers of $U$) and so that $f = \sum_i \log((x_i^{\nu'})^2 + (y_i^{\nu'})^2) + \tau$ near 0 where $\tau$ is smooth. For any sequence of points $p_k \in U \setminus \cup_j S_j$ tending to $p \in S$ we have that (after passing to a subsequence) $X_{q_i'}/\langle q_i \rangle$ at $p_k$ tends to a vector $v$ transverse to $S_i'$ because each such vector $v$ is transverse to $S_i'$ and because $S_i' = \{x_i = y_i = 0\}$. Also for any sequence of points $p_k \in U \setminus \cup_j S_j$ tending to 0 and any $i_1 \neq i_2$ we have that (after passing to a subsequence) $X_{q_{i_1'}}/\langle q_{i_1} \rangle$ at $p_k$ tends to a vector $v$ tangent to $S_{i_2}$. This implies that both $dx_{i_2}(X_{q_{i_1}}/\langle q_{i_2} \rangle)$ and $dy_{i_2}(X_{q_{i_1}}/\langle q_{i_2} \rangle)$ tend to 0 and hence

$$d \log((x_{i_2}^{\nu'})^2 + (y_{i_2}^{\nu'})^2)(X_{q_{i_1}}) \overline{q_{i_2}} \overline{d \log((x_{i_2}^{\nu'})^2 + (y_{i_2}^{\nu'})^2)}$$

$$\overline{q_{i_2}} \overline{d \log((x_{i_2}^{\nu'})^2 + (y_{i_2}^{\nu'})^2)}$$

tends to 0. Putting everything together we get that

$$d \log((x_i^{\nu'})^2 + (y_i^{\nu'})^2)(\sum_j X_{q_j}) > c' \sum_j ||q_j|| ||d \log((x_i^{\nu'})^2 + (y_i^{\nu'})^2)||$$

for some constant $c'$. Hence

$$d \left( \sum_i \log((x_i^{\nu'})^2 + (y_i^{\nu'})^2) \right) \left( \sum_j X_{q_j} \right) > c' \sum_i ||q_i|| ||d \log((x_i^{\nu'})^2 + (y_i^{\nu'})^2)||$$

for some constant $c' > 0$. Now there are constants $c_1, c_2$ so that $c_1 ||q_i|| < ||d \log((x_i^{\nu'})^2 + (y_i^{\nu'})^2)|| < c_2 ||q_i||$. Hence $\sum_i ||q_i|| ||d \log((x_i^{\nu'})^2 + (y_i^{\nu'})^2)||$ is greater than some constant times $\sum_i ||d \log((x_i^{\nu'})^2 + (y_i^{\nu'})^2)||^2$ which in turn is greater than a constant times $\sum_i ||\sum_i q_i || ||d \log((x_i^{\nu'})^2 + (y_i^{\nu'})^2)||$. This implies that there is a constant $c'' > 0$ so that:

$$d \left( \sum_i \log((x_i^{\nu'})^2 + (y_i^{\nu'})^2) \right) \left( \sum_j X_{q_j} \right) > c' \sum_i ||q_i|| \sum_i ||d \log((x_i^{\nu'})^2 + (y_i^{\nu'})^2)||.$$
for each $i$, we get our bound $df(X_{g_i}) > c \| \sum_i q_i \| || df ||$ for some $c > 0$. Hence $df(X_{\theta + db}) > c_f || \theta' + db || || df ||$ near $S$ for some $c_f > 0$. Because $\theta' + db = \theta_1 + \sum_i q_i$ near $S$ and that $|| \theta' ||$ is bounded near 0, there exists some function $b$ of the form $C \sum \log((x_i')^2 + (y_i'))^2$ where $C$ is some constant so that $|| \theta' + db || < || db ||$.

Proof of Proposition 4.1. Fix some metric $|| \cdot ||$ on $M$. For each $I \subset \{1, \ldots, l \}$ define $S_I := \cap_{i \in I} S_i$ and also choose open sets $V_{I,1}, \ldots, V_{I,k_I}$ of $M$ so that:

1. $\cup_{I,j} V_{I,j}$ contains $\cup_{i \in I} S_i$.
2. $W_{I,j} := V_{I,j} \cap S_I$ is diffeomorphic to an open ball.
3. There are smooth fibrations $\pi_{I,j} : V_{I,j} \to W_{I,j}$ whose fibers are symplectomorphic to small open balls in $\mathbb{C}^{|I|}$ and so that the natural symplectic connection has parallel transport maps in $U(|I|)$.

Choose some total order $\leq$ on the set of pairs $(I,j)$ where $I \subset \{1, \ldots, l \}$ and $1 \leq j \leq k_I$. We write $(I', j') \prec (I, j)$ if $(I', j') \neq (I, j)$ and $(I', j') \not\sim (I, j)$. We will also choose slightly smaller subsets $V'_{I,j}$ whose closure is contained in $V_{I,j}$ but which still cover $\cup_i S_i$ and we define $(W')_{I,j} := V'_{I,j} \cap S_I$. Suppose (inductively) we have constructed some function $g'$ so that $df(X_{\theta + dg'}) > c_f || \theta + dg' || || df ||$ and $|| \theta + dg' || < || db' ||$ in some small open set $N$ containing $\cup_{(I,j) \leq (I,j)} W^{I,j}$ where $b'$ is some function compatible with $\cup_i S_i$. We wish to prove the same thing in some open set containing $\cup_{(I,j) \leq (I,j)} W^{I,j}$. By Lemma 4.5 there is a smooth function $g'' : M \setminus \cup_{i \in I} S_i \to \mathbb{R}$ so that $df(X_{\theta + dg'+dg''}) > c_f || \theta + dg' + dg'' || || df ||$ in some neighborhood of $W^{I,j}_j$. Also there is a smooth function $b$ compatible with $\cup_i S_i$ so that $|| \theta + dg' + dg'' || < || db ||$. Let $\rho : M \to \mathbb{R}$ be a bump function equal to 0 outside $V_{I,j}$ and equal to 1 in $V'_{I,j}$. Define $g := g' + \rho g''$. Inside $N \cap V_{I,j}$, we have $|| \theta + dg' || < || db' ||$ and $|| \theta + dg' + dg'' || < || db ||$ and so $|| db'' || < || d\beta ||$ in $N \cap V_{I,j}$ for some function $\beta$ compatible with $\cup_i S_i$. This means that $|g''|$ is bounded above by some function $\nu$ so that $-\nu$ is compatible with $\cup_i S_i$ inside $N \cap V_{I,j}$. Now $X_{\theta + dg} = X_1 (1 - \rho) (\theta + dg') + X_2 \rho (\theta + dg' + dg'') + g'' X_2$.

Because $|| g'' X_2 ||_{|db''|}$ is bounded inside $N \cap V_{I,j}$ we get $df(X_{\theta + dg}) > c'' || df || || \theta + dg ||$ for some $c'' > 0$ in some open set containing $\cup_{(I,j) \leq (I,j)} W^{I,j}_j$. Also by construction we have $|| \theta + dg ||$ is bounded above by $d\beta''$ in this same open set where $\beta''$ is compatible with $\cup_i S_i$. Hence by induction we have shown $df(X_{\theta + dg}) > 0$ near $\cup_i S_i$.

The proof of Proposition 4.2 is identical to the proof of Proposition 4.1 but now $\theta$, $\pi_{I,j} : V_{I,j} \to W_{I,j}$, $f, g, g', g'', \rho$ and $N$ all smoothly depend on the parameter $t$.

4.2. Constructing a Specific Contact Form on Our Boundary. Let $M$ be a symplectic manifold with symplectic form $\omega$ and let $S_1, \ldots, S_l$ be real codimension two positively intersecting symplectic submanifolds so that the inclusion map $\cup_i S_i \to M$ is a homotopy equivalence. We will now assume that $H^1(M | M \setminus \cup_i S_i, \mathbb{Q}) = 0$, $c_1(TM | M \setminus \cup_i S_i)$ is torsion in $H^2(M \setminus \cup_i S_i, \mathbb{Z})$ and that $S_i$ is connected for each $i$. There is a long exact sequence

$$0 = H^1(M \setminus \cup_i S_i, \mathbb{Q}) \to H^2(M, M | M \setminus \cup_i S_i, \mathbb{Q}) \to H^2(M, \mathbb{Q}) \to H^2(M \setminus \cup_i S_i, \mathbb{Q}).$$
Now $c_1(TM) \in H^2(M, \mathbb{Q})$ maps to zero in this long exact sequence and so there is a unique element $c_1(M, M \setminus \bigcup S_i) \in H^2(M, M \setminus \bigcup S_i, \mathbb{Q})$ called the relative first Chern class. Because $\bigcup S_i \to M$ is a homotopy equivalence, we get that $H^2(M, M|\bigcup S_i, \mathbb{Q}) = H_{2n-2}(M, \mathbb{Q})$ is generated by classes $[S_i]$ and so $c_1(M, M|\bigcup S_i) = -\sum a_i[S_i]$ for some $a_1, \ldots, a_l \in \mathbb{Q}$ (note we have a minus sign because we are dealing with Chern classes of tangent bundles instead of cotangent bundles). We call $a_i$ the discrepancy. The minimal discrepancy is defined to be $\min a_i$ if $\min a_i \geq -1$ and $-\infty$ otherwise. For $I \subset \{1, \ldots, l\}$ we write $S_I := \bigcap_i S_i$.

Let $\theta$ be a 1-form on $M \setminus \bigcup S_i$ so that $d\theta = \omega$. From now on we will assume that:

1. $S_1, \ldots, S_l$ have positive wrapping numbers $\lambda_1, \ldots, \lambda_l > 0$ with respect to $\theta$.
2. $S_1, \ldots, S_l$ are symplectically orthogonal submanifolds (i.e. for each $i \neq j$ we have that the symplectic normal bundle of $S_i$ along $S_i \cap S_j$ is contained in $TS_J$).

**Theorem 4.6.** Let $\epsilon > 0$ be small. There exist functions $f : M \setminus \bigcup S_i \to \mathbb{R}$, $g : M \setminus \bigcup S_i \to \mathbb{R}$, a contact submanifold $C \subset M$ with contact form $\alpha = \theta'|_C$ where $\theta' = \theta + dg$ satisfying the following properties:

1. We have that $f$ is compatible with $\bigcup S_i$ and $df(X_{\theta'}) > 0$ inside $N \setminus \bigcup S_i$ where $N$ is a small neighborhood of $\bigcup S_i$. Also $C \subset N$ and is transverse to $X_{\theta'}$ where $X_{\theta'}$ points outwards along $C$ away from $\bigcup S_i$.
2. Let $V_S = \mathbb{N}\langle S_1, \ldots, S_l \rangle$ be the free commutative monoid generated by $S_1, \ldots, S_l$. For each non-zero element of $V_S$ given by $V := \sum_{j \in I_V} d_j S_j$ we have a pseudo Morse Bott family $R_V$ whose Conley-Zehnder index is given by $2\sum_{i \in I_V} (a_i + 1)d_i + \frac{1-|I_V|}{2}$. The length of the respective Reeb orbits minus $\sum_{i \in I} d_i(2\pi \epsilon^2 + \lambda_i)$ has absolute value less than $\epsilon\left(\sum_{i \in I} d_i\right)$. Also there exists a disk bounding each Reeb orbit in $R_V$ whose intersection with $S_i$ is $d_i$. We require that every Reeb orbit sits inside some family $R_V$.
3. Let $B^{2n}_\nu \subset \mathbb{C}^{2n}$ be the symplectic ball of radius $\nu$. For each $i$, there is some $\epsilon'_M > \epsilon > \epsilon_M$, a subset $W_i \subset M \setminus \bigcup_{j \neq i} S_j$ symplectomorphic to $B^{2n-2}_{\epsilon'_M} \times B^2_{\epsilon_M}$ with the property that $W_i \cap S_i = B^{2n-2}_{\epsilon'_M} \times \{0\} \subset S_i \setminus \bigcup_{j \neq i} S_j$. The restriction of $\theta'$ to $B^{2n-2}_{\epsilon'_M} \times B^2_{\epsilon_M}$ is $pr_1^*\beta + pr_2^*(r^2_i d\varphi_i)$ where $\beta$ is a 1 form, $(r_i, \varphi_i)$ are polar coordinates on $B^2_{\epsilon_M}$ and $pr_1$ and $pr_2$ are the projections to $B^{2n-2}_{\epsilon'_M}$ and $B^2_{\epsilon_M}$ respectively. Also $C \cap W_i = B^{2n-2}_{\epsilon'_M} \times \partial B^2_{\epsilon_M}$.

**Definition 4.7.** We will say that $f, g, C$ is a nice contact neighborhood of size $\epsilon$ for $S_1, \ldots, S_n$ if it has the properties described in Theorem 4.6 above.

We need some preliminary lemmas before we prove this theorem (Lemmas 4.8, 4.9 and 4.10). Before we state these Lemmas, we construct nice fibrations around each $S_i$. Lemma 4.8 will give us a function $g : M \setminus \bigcup S_i \to \mathbb{R}$ so that $\theta' := \theta + dg$ behaves well with respect to these fibrations. In the proof of Theorem 4.6 we construct some nice Hamiltonian $H : M \to \mathbb{R}$ so that $C := H^{-1}(\delta)$ for $\delta > 0$ small will be a contact manifold with contact form $\theta + dg$. Note that there is a one to one correspondence between Hamiltonian orbits of $H$ inside $H^{-1}(\delta)$ and Reeb orbits of $C$. In order to calculate the indices of the Reeb orbits of $C$, we need a relationship between these indices and the Conley-Zehnder indices of the respective Hamiltonian orbits of $H$ inside $H^{-1}(\delta)$. This is the reason why we have Lemmas 4.9 and 4.10. Lemma 4.9 is an easier case of Lemma 4.10 and is used to prove Lemma 4.10.
For $I \subset \{1, \cdots, l\}$ we write $S_I := \cup_{i \in I} S_i$. By [McL12, Lemma 5.14] we have that for each $I \subset \{1, \cdots, l\}$ there are open neighborhoods $U_I$ of $S_I$, and smooth fibrations $\pi_I : U_I \to S_I$ satisfying the following properties:

1. $U_{I \cup J} = U_I \cap U_J$.
2. Each fiber of $\pi_I$ is a symplectic submanifold symplectomorphic to $\prod_{i \in I} \mathbb{D}^i_\epsilon$ where $\mathbb{D}^i_\epsilon \subset \mathbb{C}$ is the open disk of radius $\epsilon$ labeled by $i \in I$. Here $\epsilon$ is a fixed constant that can be made as small as we like. For $J \subset I$, the fibers of $\pi_I \mid U_J$ are contained in the fibers $\prod_{i \in I} \mathbb{D}^i_\epsilon$ of $\pi_I$ and are of the form $\prod_{i \in J} \mathbb{D}^i_\epsilon \times \prod_{i \in I \setminus J}(z_i)$ for points $z_i \in \mathbb{D}^i_\epsilon$.

Also for all $I \subset \{1, \cdots, l\}$ we have $\pi_I(U_I \setminus \cup_{i \in I} U_{\{i\}})$ is a compact subset of $I_Y$.

3. The set of vectors in $U_I$ which are symplectically orthogonal to the fibers of $\pi_I$ induce a connection whose structure group is $U(1)^{|I|}$ where each $U(1)$ in this product rotates the disk $\mathbb{D}^i_\epsilon$ in the product $\prod_{i \in I} \mathbb{D}^i_\epsilon$.

We define $\pi_i, U_i$ as $\pi_{\{i\}}$ and $U_{\{i\}}$ respectively. Because the structure group of $\pi_i$ is $U(1)$ we have have function $r_i$ given by the radial coordinate in the fibers which generates the $U(1)$ action rotating the fibers.

We define $\rho : [0, \epsilon^2) \to \mathbb{R}$ to be equal to 1 near 0 and equal to 0 near $\epsilon^2$ and define $\rho(r_i) : M \setminus S_i \to \mathbb{R}$ to be 0 outside $U_i$. Now on each fiber we have an angle coordinate $\vartheta_i$ which is only well defined up to adding a constant, and so $d\vartheta_i$ is well defined on each fiber. By abuse of notation we will let $d\vartheta_i$ be a 1-form on $U_i \setminus S_i$ whose restriction to each fiber is $d\vartheta_i$ with the additional property that $d\vartheta_i(X) = 0$ for any vector $X$ symplectically orthogonal to the fibers of $\pi_i$. Note this 1-form may not be closed.

**Lemma 4.8.** There exists a function $g : M \to \mathbb{R}$ with the property that $\theta + dg$ restricted to a fiber $\prod_{i \in I} \mathbb{D}^i_\epsilon$ of $\pi_I$ is equal to $\prod_{i \in I}(r_i^2 + \frac{1}{2\pi}\lambda_i)d\vartheta_i$. We also have that the norm with respect to some metric on $M$ of $\theta + dg - \sum_i \rho(r_i^2)\frac{1}{2\pi}\lambda_i d\vartheta_i$ is bounded.

**Proof.** of Lemma 4.8 Choose a total ordering $\preceq$ on the set of subsets $I \subset \{1, \cdots, l\}$ so that if $|I| > |J|$ then $J \preceq I$. We write $J < I$ if $J \preceq I$ and $J \neq I$. We proceed by induction on this total ordering. Suppose that on some neighborhood of $\cup_{J < I} S_J$ there is a smooth function $g' : M \setminus U_i S_i$ so that for each $J < I$, $\theta + dg'$ restricted to a fiber $\prod_{i \in J} \mathbb{D}^i_\epsilon$ of $\pi_J$ is equal to $\prod_{i \in J}(r_i^2 + \frac{1}{2\pi}\lambda_i)d\vartheta_i$.

For each $p \in S_I$ there is a function $f_p$ so that $(\theta + dg + df_p)_{|\pi_I^{-1}(p)} = \prod_{i \in I}(r_i^2 + \frac{1}{2\pi}\lambda_i)d\vartheta_i$ near $S_I$ using the definition of wrapping number. We can ensure that $f_p$ varies smoothly as $p$ varies and that $f_p$ is compactly supported in $U_I \setminus \cup_{j \neq I} S_J$. We can extend $f_p$ to some smooth function on the whole of $M$. Hence our induction step is done and we have proved the first part of the lemma.

We now need to show that the norm of $\theta + dg - \sum_i \rho(r_i^2)\frac{1}{2\pi}\lambda_i d\vartheta_i$ is bounded. Let $p \in U_i S_i$. Let $I \subset \{1, \cdots, l\}$ be the set of $i$ with $S_i$ containing $p$. Choose a 1-form $\beta$ near $p$ with the property that $d\beta = \omega$ and with the property that the restriction of $\beta$ to each fiber of $\pi_I$ is $\sum_{i \in I} r_i^2 d\vartheta_i$ near $S_I$. Near $p$ we have $\theta + dg - \sum_i \rho(r_i^2)\frac{1}{2\pi}\lambda_i d\vartheta_i - \beta$ is a closed 1-form because $\rho(r_i^2) = 1$ near $p$ for $i \in I$. Hence it is equal to $dg_p$ near $p$ for some function $g_p : M \setminus U_i S_i \to \mathbb{R}$. Also by construction its restriction to the fibers of $\pi_I$ vanish which means that $dg_p$ restricted to the fibers of $\pi_I$ vanish which means that $g_p$ is constant along the fibers of $\pi_I$ and hence $g_p$ smoothly extends over $\cup_i S_i$ near $p$. Putting all of this together we get that the norm of $\theta + dg - \sum_i \rho(r_i^2)\frac{1}{2\pi}\lambda_i d\vartheta_i - \beta$ is bounded near $p$. Because the norm of $\beta$ and $\rho(r_i^2)$ is also bounded we then get that the norm of $\theta + dg - \sum_i \rho(r_i^2)\frac{1}{2\pi}\lambda_i d\vartheta_i$ is bounded near $p$. Because $\cup_i S_i$ is compact we then get that the norm of $\theta + dg - \sum_i \rho(r_i^2)\frac{1}{2\pi}\lambda_i d\vartheta_i$ is bounded. \qed
Recall that for a Hamiltonian $H$ on a symplectic manifold $(M, \omega)$ we define $X_H$ to be the unique vector field satisfying $i_{X_H} \omega = dH$. To any 1-periodic orbit of $X_H$, if we have a trivialization of some power $\kappa^{\otimes N_{c_1}}$ of the canonical bundle of $M$ near this orbit then we can define its Conley-Zehnder index. This is defined in exactly the same way as for Reeb orbits except we now look at the linearization of $X_H$ viewed as a map from $TM$ to $TM$ and then look at the diagonal action of this linearization on $\bigoplus_{j=1}^{N_{c_1}} TM \to \bigoplus_{j=1}^{N_{c_1}} TM$.

**Lemma 4.9.** Let $(C, \xi)$ be a contact manifold with contact form $\alpha$ (ker$(\alpha) = \xi$) and $h : \mathbb{R} \to \mathbb{R}$ a function with $h' < 0, h'' > 0$ and $h'(1) = -1$. Let $\tilde{C} := C \times \mathbb{R}$ be the symplectization of $C$ with symplectic form $d(e^r \alpha)$ where $r$ parameterizes $\mathbb{R}$. Let $\gamma(t)$ be a Reeb orbit of $\alpha$ of length $L$ with a choice of symplectic trivialization of $\bigoplus_{j=1}^{N_{c_1}} TM$ along this orbit. This choice of trivialization induces a choice of trivialization of $\gamma^* \bigoplus_{j=1}^{N_{c_1}} \xi$ in a natural way. Then the Hamiltonian $Lh(e^r)$ has a 1-periodic orbit equal to $\gamma(Lt)$ inside $C \times \{0\} = C$ and its Conley-Zehnder index is equal to $\text{CZ}(\gamma) - \frac{1}{2}$.

**Proof.** of Lemma 4.9 We identify $C$ with $C \times \{0\}$ in the natural way. Let $H'' := Lh(e^r)$ and let $\gamma_{\alpha}(t) := \gamma(Lt)$ be the Hamiltonian 1-periodic orbit of $H''$. We will define $\text{CZ}'(\gamma_{\alpha})$ to be the Conley-Zehnder index of the orbit $\gamma_{\alpha}$. By abuse of notation we define $\alpha$ to be a 1-form on $\mathbb{R} \times C$ by pulling back $\alpha$ along the natural projection $\mathbb{R} \times C \to C$. Let $R$ be the vector field tangent to $C \times \{r\}$ and equal to the Reeb flow of $\alpha|_{C \times \{r\}}$ for each $r$. Along $\gamma$ we have that the symplectic vector bundle $T\tilde{C}$ splits into two symplectically orthogonal subspaces $\xi \oplus \xi^\perp$ where $\xi$ is the contact distribution and $\xi^\perp$ is the symplectic vector space spanned by $X_\alpha$ and $R$. The vectors $(X_\alpha, R)$ form a symplectic basis and so give us a symplectic trivialization of $\xi^\perp$ along $\gamma$ and hence because $\bigoplus_{j=1}^{N_{c_1}} T\tilde{C}$ also has a trivialization along $\gamma$ we also get a trivialization of $\bigoplus_{j=1}^{N_{c_1}} \xi$ along $\gamma$. The linearization of the flow of $R$ along $\gamma$ gives us a sequence of linear maps $\xi_{\gamma}(0) \to \xi_{\gamma}(t)$. Using the trivialization of $\gamma^* \bigoplus_{j=1}^{N_{c_1}} \xi$ we get that this linearization can be viewed as a family of symplectic matrices $S_t$ in $\mathbb{C}^{(n-1)N_{c_1}}$. The Conley-Zehnder index of $S_{t}$ is $N_{c_1}$ times the Conley-Zehnder index of $\gamma$. The linearization of the flow of $X_{H''}$ along $\gamma$ preserves our splitting $\xi \oplus \xi^\perp$ and its restriction to $\xi$ is equal to the linearization of the flow of $R$ after a linear reparameterization of time. Also the linearization of the flow of $X_{H''}$ preserves $R$ and sends $X_\alpha$ to $X_{\alpha} - a \xi$ where $a > 0$ and $t$ is time. Hence using our trivializations of $\gamma^* \bigoplus_{j=1}^{N_{c_1}} \xi$ and $\gamma^* \xi^\perp$ we get a family of symplectic matrices $S_t \bigoplus_{j=1}^{N_{c_1}} Q_t$ on $\mathbb{C}^{(n-1)N_{c_1}} \bigoplus_{j=1}^{N_{c_1}} \mathbb{C}, t \in [0, 1]$ where the first factor comes from a trivialization of $\gamma^* \bigoplus_{j=1}^{N_{c_1}} \xi$ and the remaining $N_{c_1}$ factors come from our trivialization of $\xi^\perp$. The family of symplectic matrices $Q_t$ is equal to \[ \begin{pmatrix} 1 & 0 \\ -ta & 1 \end{pmatrix} \] and has Conley-Zehnder index equal to $-\frac{1}{2}$ (this is calculated using the normalization property from [Gut13] [Theorem 55]). Also the Conley-Zehnder index of $S_{Lt}, t \in [0, 1]$ is the Conley-Zehnder index of $S_t, t \in [0, L]$. So the Conley-Zehnder index of $S_{Lt} \bigoplus_{j=1}^{N_{c_1}} Q_t$ is $\text{CZ}(S_{Lt}) - \frac{1}{2}N_{c_1}$. Hence $\text{CZ}'(\gamma_{\alpha}) = \text{CZ}(\gamma) - \frac{1}{2}$. \[ \square \]

If we have some Hamiltonian $H'$ and we have some set of fixed points $B$ of $\phi^1_{H'}$ (the time 1-flow of $X_{H'}$) then we say that such a set of fixed points is isolated if any sufficiently close fixed point is contained in $B$. If $B$ is a path connected topological space and the canonical bundle of our symplectic manifold has a choice of trivialization then by Lemma 3.2 we have that every such Hamiltonian orbit has the same Conley-Zehnder index and we will write
Lemma 4.10. Let $(M',\omega')$ be an exact symplectic manifold with primitive $d\beta' = \omega'$ and a choice of trivialization of the $N_{c_1}$-th tensor power of its canonical bundle. Let $K$ be a Hamiltonian with the property that $b := -i_{X_{\beta'}}dK > 0$. This means $C_r := K^{-1}(r)$ is a contact manifold with contact form $\alpha_r := \beta'|_{C_r}$. Let $B \subset M'$ be a connected submanifold transverse to $C_r$ for each $r$ so that $B_r := C_r \cap B$ is a Morse Bott submanifold of length $L_r$ where $L_r$ smoothly depends on $r$. Suppose that $b = L_0$ along $B_0$ and that $db(V) < \frac{d[L_{r=0}]}{dr}|_{r=0}$ along $B_0$ where $V$ is a vector field tangent to $B$ satisfying $dK(V) = -1$. Then $B_0$ is Morse Bott for $K$ and $\text{CZ}(B_0,K) = \text{CZ}(B_0,\alpha_0) - \frac{1}{2}$.

Proof. Lemma 4.10. We will deform $\beta'$ and $K$ through appropriate forms, and then use Lemmas 3.2 and 4.9 for our Conley-Zehnder index calculations. This will be done in 3 steps.

Step 1. In this step we will deform $\beta'$ through certain primitives of symplectic forms. Because $b = -i_{X_{\beta'}}dK = i_{\nu_{\beta'}} \beta'$ and $b = L_0$ along $B_0$ we have that for each Reeb orbit $\gamma(t), t \in [0,L_0]$ of $C_0$ starting in $B_0$, there is a 1-periodic Hamiltonian orbit $\gamma(L_0t)$ of $K$. By considering the flow of $\frac{1}{\nu_{\beta'}}X_{\beta'}$ we have that a neighborhood of $C_0$ is identified with $C_0 \times (-\epsilon', \epsilon')$ satisfying:

1. If $r : C_0 \times (-\epsilon', \epsilon') \rightarrow (-\epsilon', \epsilon')$ is the natural projection map then $K = -r$.
2. $X_{\beta'} = b \frac{\partial}{\partial r}$.

Let $\Phi : [0,1] \times C_0 \times (-\epsilon', \epsilon') \rightarrow [0,1] \times C_0 \times (-\epsilon', \epsilon')$ be a smooth map sending $(w, y, r)$ to $(y, (1-w)r)$. Let $\pi_{(-\epsilon', \epsilon')} : [0,1] \times C_0 \times (-\epsilon', \epsilon') \rightarrow C_0 \times (-\epsilon', \epsilon')$ be the natural projection.

Define $\beta := \Phi^* \pi_{(-\epsilon', \epsilon')} \beta'$. Let $\tau_{w} : C_0 \times (-\epsilon', \epsilon') \rightarrow [0,1] \times C_0 \times (-\epsilon', \epsilon')$ send $(y, r)$ to $(w, y, r)$. Let $\pi_{(-\epsilon', \epsilon')} : C_0 \times (-\epsilon', \epsilon') \rightarrow C_0$ be the natural projection. Let $\eta : [0,1] \times (-\epsilon', \epsilon') \rightarrow \mathbb{R}$ be a smooth function satisfying:

1. $\frac{\partial}{\partial r}\eta(w, r) \geq 0$.
2. $\eta(0, r) = 1$, $\eta(1, r) = e^r$ and $\eta(w, 0) = 1$ for all $w, r$.

Define $\beta_w := \nu_{(-\epsilon', \epsilon')} \eta_{w} \beta'$.

For $r$ near 0 we have that the restriction of $\beta'_w$ to $C_r$ is a contact form. So after shrinking $\epsilon'$ we can assume that the restriction of $\beta'_w$ to $C_r$ is a contact form for all $r$. We will write $b$ as a function $b(y, r)$ of two variables $(y, r) \in C_0 \times (-\epsilon', \epsilon')$. Let $R_w$ be a vector field tangent to $C_r$ and equal to the Reeb vector field of $\beta'_w|_{C_r}$ for each $r$. Note that $\zeta_w : = \pi_{(-\epsilon', \epsilon')} \circ \Phi \circ \tau_w$ sends $(y, r)$ to $(y, (1-w)r)$. Because $\frac{\partial}{\partial r} = \frac{\partial}{\partial w} \frac{\partial}{\partial r}$ and $\zeta_w \frac{\partial}{\partial r} - \frac{1}{(1-w)} \frac{\partial}{\partial w}$ we have $\zeta^*_w X_{\beta} = \frac{\partial}{\partial w} (1) = \frac{\partial}{\partial w} b(y, (1-w)r)$. Hence $\nu_{(-\epsilon', \epsilon')} \frac{\partial}{\partial w} \nu_{(-\epsilon', \epsilon')} = 0$. So near $C_0$ we then get that $i_{R_w} \frac{\partial}{\partial w} \nu_{(-\epsilon', \epsilon')} > 0$. This means that $\nu_{(-\epsilon', \epsilon')} \beta'_w$ is a symplectic form after shrinking $\epsilon'$.

Step 2. In this step we construct a smooth family of Hamiltonians depending on $w$ so that $B_0$ is a Morse Bott submanifold using the symplectic form $\nu_{(-\epsilon', \epsilon')} \beta'_w$. Let $B^w := \zeta_w^{-1}(B)$. We
have that \( B^w|_{C_r} \) is a Morse Bott submanifold of \((C_r, \beta^r_w)\) whose length is a smooth function \( L(w, r) \) of \( w \) and \( r \).

Let \( q_\lambda : \mathbb{R} \to \mathbb{R} \) be a smooth family of functions with the property that \( q_\lambda(0) = 0, q'_\lambda(0) = 1 \) and \( q''_\lambda(0) = \lambda \). We will also assume that \( q_0 = \text{id} \). For \( 0 < \eta < 1 \) let \( \rho_\eta : [0, 1] \to [0, 1] \) be a bump function with the property that \( \rho_\eta(x) = 0 \) near \( 0 \) and \( \rho_\eta(x) = 1 \) inside \([\eta, 1]\). Define \( K_{w, \eta, \lambda} := q_{\lambda \rho_\eta(w)} \circ K \). Here \( \frac{1}{\lambda} \leq \eta \leq 1 \). For any 1-form \( \alpha \) we define \( X^{\alpha}_{w} \) to be the \( d\beta^l_{w} \) dual of \( \alpha \). We define \( s_w := i_{X^{\alpha}_{w}K_{w, \eta, \lambda}} \beta^l_{w} \). Here \( s_w \) depends on \( \lambda \) and \( \eta \) but we suppress this notation for simplicity. We have that \( s_w = (q'_{\lambda \rho_\eta(w)} \circ K) b_w \) where \( b_w := i_{X^{\eta \rho_\eta, \lambda}_{w} K_{w, \eta, \lambda}} \). Let \( V_w \) be a smooth family of vector fields parameterized by \( w \in [0, 1] \) with the property that \( V_w \) is tangent to \( B^w \) with \( dK(V_w) = -1 \) which means that \( dr(V_w) = 1 \). We will also assume that \( V_0|_B = V \).

Now
\[
ds_w(V_w) = (q'_{\lambda \rho_\eta(w)} \circ K) db_w(V_w) + (q''_{\lambda \rho_\eta(w)} \circ K) dK(V_w)b_w = (q'_{\lambda \rho_\eta(w)} \circ K) db_w(V_w) - (q''_{\lambda \rho_\eta(w)} \circ K) b_w.
\]

Because \( db_0(V_0) < \frac{\partial L(0, r)}{\partial r}|_{r=0} \) and \( q''_{\lambda \rho_\eta(w)} > 0 \) we get for \( \eta > 0 \) small enough that \( ds_w(V_w) < \frac{\partial L(0, r)}{\partial r}|_{r=0} \) along \( C_0 \) for \( w < \eta \). Such a bound is true for any \( \lambda \) as long as \( \eta \leq 1 \). Now we fix such a small \( \eta \) and increase \( \lambda \). Because \( q''_{\lambda \rho_\eta(w)} > 0 \) and \( q''_{\lambda \rho_\eta(w)}(0) = \lambda \) for \( w \in [\eta, 1] \) we get for \( \lambda \) sufficiently large that \( ds_w(V_w) < \frac{\partial L(0, r)}{\partial r}|_{r=0} \) along \( C_0 \) for \( w \in [\eta, 1] \).

Hence \( ds_w(V_w) < \frac{\partial L(0, r)}{\partial r}|_{r=0} \) along \( C_0 \) for \( \frac{1}{\lambda} \leq \eta \leq 1 \).

Let \( p \) be a point in \( B_0 \) and let \( D^w_p : TM_p \to TM_p \) be the linearization of the time 1 flow of \( X^w_{K_{w, \eta, \lambda}} \). Because \( X^w_{K_{w, \eta, \lambda}} = s_w R_w \) and \( R_w = R \) along \( C_0 \) we get that any vector \( W \) at \( p \) tangent to \( C_0 \) but not tangent to \( B_0 \) satisfies \( D^w_p(W) \neq W + lR_w \) for any \( l \in \mathbb{R} \). The inequality \( ds_w(V_w) < \frac{\partial L(0, r)}{\partial r}|_{r=0} \) combined with the fact that the Reeb flow \( R_w \) of \( \beta^l_w \) is tangent to \( B^w \) ensures that \( D^w_p(V_w) = V_w - \kappa(p) R_w \) at \( p \in B_0 \) for some positive smooth function \( \kappa : B_0 \to \mathbb{R} \). Any vector \( W^l \) in \( M \) at \( p \in B_0 \) is equal to a unique sum \( W + kV_w \) for some \( k \in \mathbb{R} \) where \( W \) is tangent to \( C_0 \) which means \( D^w_p(W^l) = D^w_p(W) + kV_w - k\kappa(p) R_w \).

Now suppose \( W^l \) is not tangent to \( B_0 \). This means \( k \neq 0 \) or \( W \neq 0 \). Suppose \( W \neq 0 \). If \( W \) is not tangent to \( B_0 \) then \( D^w_p(W) \neq W + lR_w \) for any \( l \in \mathbb{R} \) and so \( W + kV_w - D^w_p(W^l) = W - D^w_p(W) + k\kappa(p) R_w \) is non-zero. If \( W \) is tangent to \( B_0 \) then \( k \neq 0 \) and \( W = D^w_p(W) \) which implies that \( W + kV_w - D^w_p(W^l) = k\kappa(p) V_w \neq 0 \). Putting everything together we get \( \ker(D^w_p - \text{id}) = TB_0 \) along \( B_0 \) for all \( w \).

**Step 3.** In this step we finish the proof by calculating Conley-Zehnder indices. Lemma 5.2 combined with the fact that \( K = K_{0, \eta, \lambda} \) implies that
\[
\text{CZ}(B_0, K, d\beta^l) = \text{CZ}(B_0, K_{0, \eta, \lambda}, d\beta^l_0) = \text{CZ}(B_0, K_{1, \eta, \lambda}, d\beta^l_1)
\]
By Lemma 4.9 the Conley-Zehnder index of \( B_0 \) with respect to the Hamiltonian \( K_{1, \eta, \lambda} \) and the symplectic form \( \beta^l_1 \) is \( \text{CZ}(B_0, \alpha_0) - \frac{1}{2} \). Putting everything together tells us that \( \text{CZ}(B_0, K) = \text{CZ}(B_0, \alpha_0) - \frac{1}{2} \) which gives us our result.

**Proof.** of Theorem 4.6 We will prove this Theorem in 6 steps. In the first step, we will construct our contact manifold and functions \( f \) and \( g \). In the second step, we will find our family of Reeb orbits \( R_V \), and then we will show that they are pseudo Morse Bott. In the third step we will calculate their indices using Lemma 4.10. The second and third steps are the longest and most technical steps. In Step 4 we estimate the length of these families of Reeb orbits. In Step 5 we construct the submanifolds \( W_i \) as described in the statement of...
this Theorem. Finally in Step 6 we show that \( df(X_{\theta'}) > 0 \) in the region bounded by \( C_\delta \). Steps 4, 5, 6 are very short.

**Step 1:** In this Step we will construct our contact manifold and functions \( f \) and \( g \). We define \( q : [0, \varepsilon^2] \to \mathbb{R} \) to be a smooth function so that:

1. There is some \( \varepsilon_q \in (0, \varepsilon) \) with \( q(x) = 0 \) if and only if \( x \in [\varepsilon_q^2, \varepsilon^2] \). Also \( q(x) = 1 - x^2 \) near \( x = 0 \).
2. We also assume that the derivative of \( q \) is non-positive and that it is strictly negative when \( q(x) \) is positive and \( x \neq 0 \).
3. There is a unique point \( x \) with \( q''(x) = 0 \) and \( q(x) \neq 0 \).

![Diagram](image)

We define \( H := \sum_i q(r_i^2) \).

Define \( g : M \setminus \cup_i S_i \to \mathbb{R} \) as in Lemma 4.8 and define \( \theta' := \theta + dg \). Our contact manifold will be \( C_\delta := H^{-1}(\delta) \) with contact form \( \alpha_\delta := \theta'|_{C_\delta} \) for \( \delta \) sufficiently small. Let \( \nu : (0, \varepsilon) \to \mathbb{R} \) be a smooth function equal to 0 near \( \varepsilon^2 \) and satisfying \( \nu(x) = \log(x) \) for \( x \leq \varepsilon_q^2 \). Define \( \nu(r_i^2) \) to be 0 outside \( U_i \). Our function \( f : M \setminus \cup_i S_i \to \mathbb{R} \) will be defined as \( \sum_i \nu(r_i^2) \).

**Step 2:** In this step we will find our family of Reeb orbits \( R_\delta \), and then we will show that they are pseudo Morse Bott. For each \( i \in \{1, \cdots, l\} \) define \( h_i : [0, \varepsilon^2] \to \mathbb{R} \) by \( h_i(x) := -q'(x)(x + \frac{1}{2x}\lambda_i) \). We can extend the function \( h_i(r_i^2) : U_i \to \mathbb{R} \) to a function \( M \setminus \cup_i S_i \to \mathbb{R} \) by defining it to be 0 outside \( U_i \). Because the derivative of \( \log(h_i(x)) \) tends to \( -\infty \) as we approach \( \varepsilon_q \) from below we get that \( \frac{h_i'}{h_i} \) tends to \( -\infty \) as we approach \( \varepsilon_q \) from below. Similarly by looking at \( \log(-q') \) we have that \( \frac{q''}{q} \) tends to \( -\infty \) as we approach \( \varepsilon_q \) from below. We choose \( \delta \) small enough so that:

1. For each \( i \in I \) we have \( q'(r_i^2) < 0, q''(r_i^2) > 0, h_i'(r_i^2) < 0, -\frac{q''(r_i^2)}{q'(r_i^2)} > n \) and \( r_i > \frac{\varepsilon_q}{2} \) along \( H^{-1}(\delta) \cap U_i \).
2. For each \( I \subset \{1, \cdots, l\} \) we require \( \sum_{j \in I} h_j'(r_j^2) > -1 \) along \( H^{-1}(\delta) \cap U_I \).

The Reeb flow \( R_\delta \) of \( \alpha_\delta \) is equal to \( \frac{1}{b}X_H \) where \( b = i_{X_H} \theta' \). Now let's look inside the region \( U_I \) for some \( I \subset \{1, \cdots, l\} \). Define \( H_I : S_I \setminus \cup_{j \notin I} S_j \to \mathbb{R} \) by \( \sum_{j \notin I} q(r_j^2) \). Inside \( (H_I \circ \pi_I)^{-1}(0) \) we have that \( H \) is a function of \( (r_i^2)_{i \in I} \) given by \( \sum_{i \in I} q(r_i^2) \) and so inside \( C_\delta \) the coordinates \( r_i \) are subject to the conditions \( \sum_{i \in I} q(r_i^2) = \delta \). Note that if \( (r_i, \vartheta_i) \) are polar coordinates for \( \mathbb{D}_i \) then \( \theta' \) restricted to a fiber \( \prod_{i \in I} \mathbb{D}_i \) is \( \sum_{i \in I} (r_i^2 + \frac{1}{2r_i^2}\lambda_i)d\vartheta_i \). Because \( X_H \) is tangent
to the fibers of $\pi_I$ inside $(H_I \circ \pi_I)^{-1}(0)$ we then get that $X_H = -\sum_{i \in I} q'_I(r_i^2) \frac{\partial}{\partial \theta_i}$. Hence $b = i \chi_H \theta' = \sum_{i \in I} h_i(r_i^2)$ inside $(H_I \circ \pi_I)^{-1}(0)$.

Let $F_I$ be the set of $|I|$ tuples $(d_i)_{i \in I} \in F_I$. Define $O_{a,d} := (H_I \circ \pi_I)^{-1}(0) \cap \cap_{i \in I} \{q_i(r_i^2) = -ad_i\}$ and $O_d := \cup_{a>0} O_{a,d}$. Near $C_S$ we have that $q_i(r_i^2)$ can be expressed as a smooth function of $q_i(r_i^2)$ with non-zero derivative and so $O_d$ is a submanifold near $C_S$ which intersects $C_S$ transversally. We define $O_d := O_d \cap C_S$. Now every point on $O_d$ is the starting point of an orbit of $H$ contained inside a fiber $\pi_I^{-1}(p)$ representing the homology class $(d_i)_{i \in I} \in \prod_{i \in I} \mathbb{Z} \cong H_1(\pi_I^{-1}(p) \setminus \cup_i S_i)$. Let $l_d$ be the length of such a family of Reeb orbits. To connect these computations back to the statement of the Theorem we should note that if $M = \sum_{i \in I} d_i E_i$ is an element of our monoid $S_T$ then $R_V := O_d^\delta$.

We will now show that this is a pseudo Morse Bott family of Reeb orbits. Now the set $O_d^\delta$ is a manifold with corners diffeomorphic to a torus bundle over the manifold with corners $H_T^{-1}(0)$ where the tori have dimension $|I|$. Let $\phi^R_i : C_S \to C_S$ be the flow of the Reeb vector field $R_S$ of $\alpha_S$. Let $D_i : T(C_S)_{p_0} \to T(C_S)_{\phi^R_i(p)}$ be the linearization of $\phi^R_i$ and similarly let $D'_i : TM_{p_0} \to TM_{\phi^R_i(p)}$ the linearization of $\phi^H_i$. We will show that $\ker(D_{l_d,\delta}(p) - \text{id}) = TO_d^\delta$. One can think of this as a Morse Bott manifold with corners. Because $X_H = b R_S$ and $b$ is constant along $O_d^\delta$ we have for a vector $V$ tangent to $C_S$ at $p \in O_d^\delta$ that $D_i(V) = D'_i V - \int_0^1 ((\phi^R_i)^* dB)(V) ds R_\delta$ for $V$. We have

$$D'_i \left( \frac{\partial}{\partial r_j} \right) = \frac{\partial}{\partial r_j} + t q''(r_j^2) \frac{\partial}{\partial \theta_j}$$

and $D_i'$ is the identity on any other vector symplectically orthogonal to the fibers of $\pi_I$ and also on $\frac{\partial}{\partial \theta_j}$ for each $j$.

Let $V$ be a vector in $TC_S$ at a point $p \in O_d^\delta$. We have

$$D_{l_d,\delta}(V) = D'_{l_d,\delta}(V) - \int_0^{l_d,\delta} ((\phi^R_i)^* dB)(V) ds R_\delta$$

$$= V + \frac{l_d,\delta}{b(p)} \sum_{j \in I} q''(r_j) (i_V dr_j^2) \frac{\partial}{\partial \theta_j}$$

$$- \int_0^{l_d,\delta} ((\phi^R_i)^* dB)(V) ds R_\delta.$$ 

Now $V = W + \sum_{j \in I} \alpha_j \frac{\partial}{\partial \theta_j} + \beta_j \frac{\partial}{\partial r_j}$ for some $\alpha_j, \beta_j \in \mathbb{R}$ where $W$ is symplectically orthogonal to the fibers of $\pi_I$. Because $db$ does not change as we flow along $O_d$, we have that $\int_0^{l_d,\delta} ((\phi^R_i)^* dB)(V) ds = l_d,\delta \sum_{j \in I} \beta_j h'_i(r_j^2)$. Also, $R_\delta = \frac{1}{b} \sum_{i \in I} q'_i(r_i^2) \frac{\partial}{\partial \theta_i}$. Hence

$$D_{l_d,\delta}(V) = W + \sum_{j \in I} \alpha_j \frac{\partial}{\partial \theta_j} + \beta_j \frac{\partial}{\partial r_j}$$

$$+ \frac{l_d,\delta}{b(p)} \sum_{i \in I} \left( q''(r_i) \beta_i - \left( \sum_{j \in I} \beta_j h'_i(r_j^2) \right) q'(r_i) \right) \frac{\partial}{\partial \theta_i}.$$ 

If $V$ is not tangent to $O_d^\delta$ then $dr_k(V) = \beta_k \neq 0$ for some $k$ where $\beta_k$ satisfies $|\beta_k| \geq |\beta_i|$ for all $i$. Conditions (1) and (2) above then imply $\left| \left( \sum_{j \in I} \beta_j h'_i(r_j^2) \right) q'(r_k) \right| < |\beta_k q''(r_k)|$. This implies $D_{l_d,\delta}(V) \neq V$. Hence $O_d^\delta$ is pseudo Morse Bott.
Step 3: We now need to calculate the Conley-Zehnder index of $O_d^\delta$. Recall $b = i_{X_H} \theta' = \sum_{i \in I} h_i(r_i^2)$ inside $(H \circ \pi_t)^{-1}(0)$. Let $W'$ be a vector in $TM|_{O_d^\delta}$ tangent to $O_d$ but transverse to $O_d^\delta$ and satisfying $dH(W') = -1$. We have that $db(W') = \sum_{i \in I} h'_i(r_i^2)d(r_i^2)(W')$. Now $(r_i)_{i \in I}$ restricted to $O_d^\delta$ is a constant $(c_i)_{i \in I}$. Hence $b$ is a constant $b_0 = \sum_{i \in I} h_i(c_i^2) \in \mathbb{R}$ along $O_d^\delta$.

Recall that each orbit of $O_d^\delta$ is contained in some fiber $\pi_I^{-1}(q)$ and the restriction of $\theta$ to this fiber is $\sum_{i \in I}(r_i^2 + \frac{1}{2\pi} \lambda_i) d\theta_i$. So we have that the length $l^\delta$ of our family of Reeb orbits $O_d^\delta$ is $\sum_{j \in I} d_j(2\pi r_j^2 + \lambda_j)$. Hence we have

$$\frac{dl^\delta}{ds} = \sum_{j \in I} 2d_j \pi d r_j^2(W').$$

Let

$$K := \frac{1}{b_0} l^\delta H = \frac{\sum_{j \in I} d_j(2\pi c_j^2 + \lambda_j)}{\sum_{i \in I} h_i(c_i^2)} H$$

where $c$ is some fixed point in $O_d^\delta$. We have rescaled $K$ here so that $O_d^\delta$ becomes an isolated family of time 1 orbits. We have by condition (1) that

$$d(i_K \theta)(W') = \frac{\sum_{j \in I} d_j(2\pi c_j^2 + \lambda_j)}{\sum_{i \in I} h_i(c_i^2)} \sum_{i \in I} h'_i(c_i^2)d(r_i^2)(W')$$

$$< 0 < \sum_{j \in I} 2d_j \pi d r_j^2(W') = \frac{dl^\delta}{ds}.$$ 

So by Lemma 4.10 we get $\text{CZ}(O_d^\delta, K) = \text{CZ}(O_d^\delta) - \frac{1}{2}$. Hence $\text{CZ}(O_d^\delta) = \text{CZ}(O_d^\delta, K) + \frac{1}{2}$. We will now calculate $\text{CZ}(O_d^\delta, K)$. Because $K$ is a constant multiple of $H$, this is the same as $\text{CZ}(O_d^\delta, H)$ where we are looking at orbits of $X_H$ of length $\frac{l^\delta}{b_0}$. Let $p \in O_d^\delta$. The tangent space $TM$ at $p$ splits as $A \oplus \oplus_{i \in I} B_i$ where $B_i$ consists of the tangent space to

$$\mathbb{D}_i^\delta = \mathbb{D}_i^\delta \times \prod_{j \in I \setminus \{i\}} \{r_j(p)\} \subset \prod_{j \in I} \mathbb{D}_j^\delta = \pi_I^{-1}(\pi_I(p))$$

and $A$ is the subspace symplectically orthogonal to the fiber of $\pi_I$ through $p$. The Hamiltonian flow of $H$ preserves this splitting and is the identity on $A$ and is equal to the differential of the flow of $q(r_i^2)$ on $\mathbb{D}_i^\delta$ on $B_i$. So we have $\text{CZ}(O_d^\delta, H, d\theta') = \sum_{i \in I} \left(2(a_i + 1)d_i - \frac{1}{2}\right)$. Hence the index of our Reeb orbit is: $2\left(\sum_{i \in I}(a_i + 1)d_i\right) + \frac{1-|I|}{2}$. To connect these computations back to the statement of the Theorem we have $R_V = O_d^\delta$ for $V = \sum_{i \in I} d_i E_i$. The size of $R_V$ is $2n - |I| - 1$ and its index is the index of $O_d^\delta$ which is $2\left(\sum_{i \in I}(a_i + 1)d_i\right) + \frac{1-|I|}{2}$.

Step 4: Now we need to estimate the length of each pseudo Morse Bott family of Reeb orbits. In our case we need to calculate the length of $O_d^\delta$ for $d = (d_i)_{i \in I}$. This pseudo Morse Bott family corresponds to some element $V = \sum_{i \in I} d_i S_i$ in our monoid which is our family of Reeb orbits. The length of $O_d^\delta$ is $\sum_{i \in I} d_i(2\pi r_i^2 + \lambda_i)$. For $0 < \delta \ll \epsilon$ small enough we then get that the length of $O_d^\delta$ minus $\sum_{i \in I} d_i(2\pi c_i^2 + \lambda_i)$ is less than $\epsilon \sum_{i \in I} d_i$.

Step 5: We now need to construct our submanifolds $W_i$ as described in the statement of this theorem. Let $p \in S_i$ be a point in $H_i^{-1}(0) \subset S_i$. We let $\epsilon_M := q^{-1}(\delta)$. For $\epsilon > 0$ small enough there is some $\epsilon_M' > \epsilon$ so that $B_{\epsilon_M'^{-2}}$ symplectically embeds in $H_i^{-1}(0)$ centered at $p$. 

Now $\pi_i^{-1}(B^{2n-2}_{\epsilon_i'})$ is a symplectic fibration with fibers symplectomorphic to $B^2$. Inside this region, we can deform our fibration $\pi_i$ (along with our Hamiltonian $H$) near $\pi_i^{-1}(B^{2n-2}_{\epsilon_i'})$ so that $\pi_i^{-1}(B^{2n-2}_{\epsilon_i'})$ becomes symplectomorphic to $B^{2n-2}_{\epsilon_i'} \times B^2$ and so that $\theta'$ becomes a product $\pi_i'(F + r_2^2 d\theta)$ in this region. We can also assume that $r_i$ and $d\theta_i$ become polar coordinates for $B^2$. For $\delta$ small enough, this does not change the properties of $(H^{-1}(\delta), \theta'_{H^{-1}(\delta)})$. Our region $W_i'$ is now $B^{2n-2}_{\epsilon_i'} \times B^2_{\epsilon_i'}$.  

Step 6. Here we show that $df(X_{\theta'}) > 0$ in the region bounded by $C_{\delta}$. We have inside $(H_1 \circ \pi_i)^{-1}(0)$ that $X_{\theta'} = X_1 + X_2$ where $X_1$ is tangent to the fibers $\prod_{i \in I} D_i$ of $\pi_i$ and $X_2$ is symplectically orthogonal to these fibers. Because the restriction of $\theta'$ to the fiber is $\sum_{i \in I} (r_i^2 + \frac{1}{\pi_i} \lambda_i) d\theta_i$ we get that $X_1 = \sum_{i \in I} (r_i^2 + \frac{1}{\pi_i} \lambda_i) \partial_{\theta_i}$. Because $f = \sum_{i \in I} \nu(r_i^2)$, and $df(\frac{\partial}{\partial r}) > 0$ in the region $\cap_{i \in I} \{ r_i^2 \leq \epsilon_q \} \cap (H_1 \circ \pi_i)^{-1}(0)$, we have that $df(X_{\theta'}) > 0$ in the region bounded by $C_{\delta}$.

This completes the theorem. \hfill \Box

4.3. Bounding the Minimal Discrepancy of a Singularity from Above. In this subsection we will use the results from the last two subsections to give an upper bound for the minimal discrepancy of an isolated singularity in terms of indices of Reeb orbits of its link. The main theorem of this section is the following:

**Theorem 4.11.** Let $A \subset \mathbb{C}^N$ be a smooth affine variety which is either smooth at zero or has an isolated singularity at zero. Suppose that its link $(L_A, \xi_A)$ satisfies $H^1(L_A; \mathbb{Q}) = 0$ and $c_1(\xi_A; \mathbb{Q}) = 0 \in H^2(L_A; \mathbb{Q})$. Then $2md(A, 0) \leq md(L_A, \xi_A)$.

We will need a preliminary lemma before we prove the above theorem. This lemma will also be useful in Section 6.2 where we bound minimal discrepancy from below. Let $A$ be as in Theorem 4.11 above and let $A_{\delta}$ be its intersection with a small ball of radius $\delta$. We resolve $A$ at 0 by blowing up along smooth loci by [Hir64] and take the preimage $\tilde{A}_{\delta}$ of $A_{\delta}$ under this resolution map. We suppose that the exceptional divisors $E_1, \ldots, E_l$ are smooth normal crossing.

**Lemma 4.12.** There exists a symplectic form $\omega_A$ on $\tilde{A}_{\delta}$ compatible with the natural complex structure which means that $E_1, \ldots, E_l$ are positively intersecting. Also there exists a 1-form $\theta_A$ on $\tilde{A}_{\delta} \setminus \cup_i E_i$ with $d\theta_A = \omega_A$ and so that $E_1, \ldots, E_l$ have positive wrapping numbers with respect to $\theta_A$. There also exists a smooth function $f_A : \tilde{A}_{\delta} \setminus \cup_i E_i \to \mathbb{R}$ compatible with $\cup_i E_i$ with the property that $df(X_{\theta_A}) > 0$ near $\cup_i E_i$. There exists $L \geq 0$ so that for all $l \leq L$ we have $(f_A^{-1}(l), \theta_A|_{f_A^{-1}(l)})$ is contactomorphic to $(L_A, \alpha_A)$.

**Proof.** of Lemma 4.12. Because our resolution is obtained by blowing up along smooth loci starting from $A \subset \mathbb{C}^N$, we can also blow up $\mathbb{C}^N$ along the same loci giving us $\mathbb{C}^N$. Let $E'_1, \ldots, E'_l \subset \mathbb{C}^N$ be the corresponding exceptional divisors with $E'_l \cap \tilde{A}_{\delta} = E_l$. We can reorder them so that the preimage of the exceptional divisor of the $i$th blowup of $\mathbb{C}^N$ is $E'_i$. Hence for positive integers $\nu_1 \gg \cdots \gg \nu_l > 0$ we have $- \sum_{i \in I} \nu_i E'_i$ is ample in some neighborhood of $\cup_i E_i$. For $\delta > 0$ small enough we have $- \sum_{i \in I} \nu_i E_i$ is ample in $\tilde{A}_{\delta}$. This means that if $L \to \tilde{A}_{\delta}$ is the line bundle associated to $- \sum_{i \in I} \nu_i E_i$ then it has a Hermitian metric $\| \cdot \|$ with positive curvature $F$ and a non-zero meromorphic section $s$ with poles $\sum_{i \in I} \nu_i E_i$. Our symplectic form is $\omega_A := -2\pi i F$ and $\theta_A = -d^c \log(\|s\|)$. Our function $f_A$...
is log(||s||). The wrapping numbers of $E_1, \ldots, E_l$ with respect to $\theta_A$ are positive numbers $2\pi \nu_1, \ldots, 2\pi \nu_l$ respectively.

Let $z_1, \ldots, z_N$ be the natural coordinates on $\mathbb{C}^N$. Define $\rho : \tilde{A}_\delta \setminus \cup_i E_i \to \mathbb{R}$ to be the pullback of $\sum_{i \in I} |z_i|^2$. For all small enough $\eta > 0$ we have that $(\rho^{-1}(\eta), -d^c\rho(\eta)|_{\rho^{-1}(\eta)})$ is contactomorphic to $(L_A, \xi_A)$ by definition.

For $t \in [0, 1]$ define $\rho_t := (1 - t)p - t \log(||s||)$. For each $p \in \cup_i E_i$ choose local holomorphic coordinates $z_1', \ldots, z_N'$ so that $\cup_i E_i = \{ \prod_{i=1}^k |z_i'|^2 = 0 \}$. Let $R$ be the radial vector field emanating from 0 with respect to this coordinate system. Now $\rho = s \prod_{i=1}^k |z_i'|^{2q_i}$ for some integers $q_1, \ldots, q_k > 0$ and some non-zero holomorphic function $s$. Hence $dp(R) > 0$ near $p$.

Also with respect to some trivialization of $L$ near $p$ we have $||.|| = e^{t||.||}$ for some smooth function $\mu$ defined near $p$. Hence $\log(||s||) = \mu + \sum_{i=1}^k k_i \log(|z_i'|)$ for positive integers $k_1, \ldots, k_n$. Again this implies that $d(\log(||s||))(R) > 0$ near $p$. Hence $dp(R) > 0$ near $p$. Because $\cup_i E_i$ is compact we then get $dp_t \neq 0$ near $\cup_i E_i$. Hence there is a smooth function $\lambda : [0, 1] \to \mathbb{R}$ satisfying:

1. For any $l \leq L := \lambda(1)$ we have that $l$ is a regular value of $\rho_1 = \log(||s||)$.
2. We have $\lambda(0) > 0$ and for any $l \in (0, \lambda(0))$, $l$ is a regular value of $\rho_0 = \rho$.
3. $\lambda(t)$ is a regular value of $\rho_t$ for all $t \in [0, 1]$.

Hence by Gray’s stability theorem we have that $(\rho_0^{-1}(\lambda(0)), -d^c\rho_0|_{\rho_0^{-1}(\lambda(0))})$ is contactomorphic to $(\rho_1^{-1}(\lambda(1)), -d^c\rho_1|_{\rho_1^{-1}(\lambda(1))})$. Hence $(\rho_1^{-1}(\lambda(1)), -d^c\rho_1|_{\rho_1^{-1}(\lambda(1))})$ is contactomorphic to $(L_A, \alpha_A)$ and $(f_A^{-1}(l), \theta_A|_{f_A^{-1}(l)})$ for $l \leq L$. Hence $(f_A^{-1}(l), \theta_A|_{f_A^{-1}(l)})$ is contactomorphic to $(L_A, \alpha_A)$ for all $l \leq L$.

**Proof.** of Theorem 4.11 Let $\omega_A, \tilde{A}_\delta, E_1, \ldots, E_l, \theta_A, f_A : \tilde{A}_\delta \setminus \cup_i E_i \to \mathbb{R}$, $L \geq 0$ and $(L_A, \alpha_A)$ be from Lemma 4.12. Let $a_1, \ldots, a_l \in \mathbb{Q}$ be the discrepancies of $E_1, \ldots, E_l$ respectively. Now $f_A$ is compatible with $\cup_i E_i$ and so by Corollary 4.3 and Theorem 4.6 we have that $(f_A^{-1}(l), \theta_A|_{f_A^{-1}(l)})$ admits a compatible contact form $\beta$ so that every Reeb orbit sits inside a pseudo Morse Bott family of size $2n - |I_V| - 1$ and Conley-Zehnder index $2 \sum_{i \in I_V} (a_i + 1)d_i + \frac{1 - |I_V|}{2}$, where $I_V \subset \{ 1, \ldots, l \}$ is a subset of size at most $n = \dim_c(A)$ and $d_i$ are positive integers labeled by $i \in I_V$. The minimal discrepancy $\text{md}(\beta)$ of this contact form is the infimum over all $I_V$ and $d_i$ of

$$2 \sum_{i \in I_V} (a_i + 1)d_i + \frac{1 - |I_V|}{2} - \frac{1}{2}(2n - |I_V| - 1) + n - 3 = 2 \sum_{i \in I_V} (a_i + 1)d_i - 2.$$

Now if $a_i < -1$ then both $\text{md}(A, 0)$ and $\text{md}(\beta)$ are $-\infty$ and so $2\text{md}(A, 0) \geq \text{md}(L_A, \xi_A)$ in this case. Otherwise the infimum over all $I_V$ and $d_i$ of $2 \sum_{i \in I_V} (a_i + 1)d_i - 2$ is $2\text{inf}(a_i)$. So $2\text{md}(A, 0) = 2\text{inf}(a_i) = \text{md}(\beta) \leq \text{md}(L_A, \xi_A)$.

$$\square$$

5. Gromov Witten Invariants on Open Symplectic Manifolds

Genus 0 Gromov Witten invariants for general symplectic manifolds have now been defined in many different ways: [FO99], [CM07], [Hof11] and [LT98]. Earlier work for special symplectic manifolds such as projective varieties of complex dimension 3 or less are done in
The results apply as the main application of this paper is in complex dimension 3, although there are results in higher dimensions too. These invariants can also be defined in a purely algebraic way [LT98a, BF97] and [Beh97] but we will not use these theories here. In all of these cases Gromov Witten invariants are defined for closed manifolds. The symplectic manifolds in our case are open but all the holomorphic curves stay inside a fixed compact subset which is not a problem (with the possible exception of [CM07] which is reliant on Donaldson hypersurface techniques).

Let \((S, \omega_S)\) be a (possibly non-compact) symplectic manifold, \(J\) be a family of compatible almost complex structures in \(S\) and \([A] \in H_2(S)\) so that:

1. \(J\) is non-empty and connected, i.e. for any \(J_1, J_2 \in J\) there exists a smooth path of almost complex structures in \(J\) joining \(J_1\) and \(J_2\).
2. There is a relatively compact open subset \(U_S\) of \(S\) so that for every \(J \in \mathcal{J}\), every compact genus 0 nodal \(J\) holomorphic curve representing \([A]\) is contained in \(U_S\).
3. \(c_1(TS)([A]) + (n - 3) = 0\).

We will say that \((S, [A], J)\) is a \textit{GW triple} if it satisfies the above properties.

We have an invariant \(GW_0(S, [A], J) \in \mathbb{Q}\) which ‘counts’ genus 0 nodal \(J\) holomorphic curves representing \([A]\). This has the property that if our invariant satisfies \(GW_0(S, [A], J) \neq 0\), then there exists a compact nodal \(J\) holomorphic curve representing \([A]\). If \(J' \in \mathcal{J}\) then there is a smooth path of almost complex structures in \(J\) joining \(J\) and \(J'\). This implies that \(GW_0(S, [A], J) = GW_0(S, [A], J')\). So from now on we will define \(GW_0(S, [A], J) := GW_0(S, [A], J)\) for some \(J \in \mathcal{J}\).

Sometimes we need to perturb \(J\) by a small amount. For instance when defining the invariant \(GW_0(S, [A], J)\) one quite often needs to perturb \(J\) to some \(C^\infty\) generic \(J\) before counting the holomorphic curves. In this case we do the following. Choose a relatively compact open set \(U\) containing \(U_S\). By a Gromov compactness argument we have that for any \(J'\) sufficiently \(C^\infty\) close to \(J \in \mathcal{J}\) we either have that every \(J'\) holomorphic curve is contained in \(U_S\) or some part of the curve maps outside \(U\). This means that \((U, [A], J')\) is a GW triple and so we define \(GW_0(U, [A], J') := GW_0(U, [A], J')\). Note that this count is independent of the choice of open set \(U\) because if we had some other set \(V\) then we can make \(J'\) sufficiently close to \(J\) to ensure that all holomorphic curves are either inside \(U_S\) or have some point outside \(U \cup V\). We also have the following similar lemma.

**Lemma 5.1.** Let \((S, [A], J)\) be a GW triple. Suppose that \(S'\) is a large relatively compact open subset of \(S\) containing the closure of our relatively compact open subset \(U_S\) with the additional property that there is a unique homology class \([A']\) mapping to \([A]\) under the inclusion map. Then there is an open subset \(\mathcal{J}'\) of the space of compatible almost complex structures with respect to the \(C^\infty\) topology so that \(\mathcal{J}'\) contains \(J\) and so that: \((S', [A'], J')\) is a GW triple with \(GW_0(S, [A], J) = GW_0(S', [A'], J')\). We can ensure that any genus 0 nodal \(J'\) holomorphic curve for \(J' \in \mathcal{J}'\) whose image is contained in \(S'\) has its image contained in \(U_S\).

**Proof.** Suppose that the Theorem above is false. This means that there is a \(J \in \mathcal{J}\) and a sequence of compatible almost complex structures \(J_i\) \(C^\infty\) converging to \(J\) and a sequence of \(J_i\) holomorphic curves \(u_i\) mapping to \(S'\) but not to \(U_S\) representing \([A']\). By a Gromov compactness argument, a subsequence then converges to a \(J\) holomorphic curve mapping to the closure of \(S'\). Because \((S, [A], J)\) is a GW triple we then get that such a limit curve is contained in \(U_S\). But this means that \(u_i\) maps to \(U_S\) for \(i\) large enough which is a contradiction. Hence \((S', [A'], J)\) is a GW triple. The reason why \(GW_0(S, [A], J) = GW_0(S', [A'], J)\) is because we can count our curves with respect to some almost complex structure \(J \in \mathcal{J} \subset \mathcal{J}'\).
Suppose we have a family of GW triples $((S_i, [A]), \mathcal{J}_t)$ labeled by $t \in [0, 1]$ such that:

1. For each $t \in [0, 1]$ the associated symplectic form is equal to $\omega_{S_i, t}$ and this smoothly varies with $t$.
2. There is a smooth family of almost complex structures $J_t$, $t \in [0, 1]$ such that $J_t \in \mathcal{J}_t$ for all $t \in [0, 1]$.
3. There is a relatively compact open subset $U'_S$ with the property that any $J_t$ holomorphic curve representing $[A]$ is contained in $U'_S$.

We will say such a family is called a smooth deformation of GW triples. Or we will say that $(S, [A], \mathcal{J}_0)$ is deformation equivalent to $(S, [A], \mathcal{J}_1)$.

**Lemma 5.2.** We have that $GW_0(S, [A], \mathcal{J}_0) = GW_0(S, [A], \mathcal{J}_1)$.

The proof of this lemma is basically the same as the proof that Gromov Witten invariants do not change when deforming the symplectic and almost complex structure. Our deformation is $(\omega_{S, t}, J_t)$. The only difference in our argument is that we are in an open symplectic manifold but this is OK as all the $J_t$ holomorphic curves stay inside a fixed relatively compact open subset. Again sometimes we would like to perturb $J_t$ slightly in which case we fix a relatively compact open subset containing $U'_S$ and count curves inside this subset using the same ideas from Lemma 5.1.

Lemmas 5.3 and 5.4 below are important tools which will be used later to give a lower bound on minimal discrepancy. Lemma 5.3 gives us some sufficient conditions for a symplectic manifold, a homology class and a family of almost complex structures to be a GW triple. Lemma 5.4 uses a neck stretching argument to find Reeb orbits with an appropriate upper bound on their Conley-Zehnder index. We refer the reader to Appendix A (Section 7) for definitions concerning neck stretching along stable Hamiltonian structures.

**Lemma 5.3.** Suppose that $S$ is a symplectic manifold, $C \subset S$ a compact stable Hamiltonian hypersurface, $J_i$ a sequence of almost complex structures compatible with the symplectic form $\omega_S$ and $[A]$ a homology class in $S$ satisfying:

1. We have $S \setminus C$ is a disjoint union $\hat{S}_+ \cup \hat{S}_-$ where the closure of $\hat{S}_\pm$ inside $S$, which is equal to $\hat{S}_\pm \cup C \subset S$, is a stable Hamiltonian cobordism $S_{\pm}$. Here $C$ has a standard neighborhood $(-\epsilon_h, \epsilon_h) \times C$ symplectomorphic to its symplectization and we require that $[0, \epsilon_h] \times C \subset S_+$ and $(-\epsilon_h, 0] \times C \subset S_-$ which means that $S_+$ has a negative boundary and no positive boundary and $S_-$ has a positive boundary but no negative boundary. We will assume that $S_-$ is compact but $S_+$ may not be compact.
2. $J_i$ is a sequence of almost complex structures stretching the neck along $C$ with respect to $(-\epsilon_h, \epsilon_h) \times C$ so that $J_i|_{\hat{S}_\pm}$ converges in $C^\infty_{loc}$ inside $\hat{S}_\pm$ to an almost complex structure $J_\pm$ compatible with the completion of $S_\pm$ after neck stretching along $C$.
3. There exists a properly embedded $J$ holomorphic real codimension 2 submanifold $E$ in $\hat{S}_+$ satisfying $[A].E = 1$ and whose closure in $S$ does not intersect $C$. Also there is a relatively compact open subset $U_S \subset S$ so that any finite energy proper $J_+\text{ holomorphic curve in } \hat{S}_+$ intersecting $E$ with multiplicity 1 is contained inside $U_S \cap \hat{S}_+$.
4. $c_1([A]) + n - 3 = 0$ and $S$ admits a Morse function with finitely many critical points.

Let $\omega_t$ in $S$ ($t \in [0, 1]$) be a smooth family of symplectic forms agreeing with $\omega_S$ inside $(-\epsilon_h, \epsilon_h) \times C \cup \hat{S}_\pm$ and define $\mathcal{J}_{i, \omega_t}$ to be the set of compatible almost complex structures $J_{i, t}$ agreeing with $J_t$ inside $(-\epsilon_h, \epsilon_h) \times C \cup \hat{S}_\pm$. Then if $S'$ is some relatively compact open subset containing $U_S \cup S_- \cup ([0, \epsilon_h] \times C)$ and homotopic to $S$ then there exists an $i_0$ so that $(S', [A], \mathcal{J}_{i, \omega_t})$ is a smooth deformation of GW triples for all $i \geq i_0$. 


To provide some context for this lemma, we will apply it in the proof of Theorem 6.1 later on in the following way: Let \( \tilde{A} \) be a resolution of our singularity \( A \) and let \( \tilde{A}_\delta \) be the preimage of a small ball under the resolution map. The manifold \( S \) in this proof will be some partial compactification of \( \tilde{A}_\delta \). The homology class \([A]\) will be Lefschetz dual to the exceptional divisor in this resolution with the lowest discrepancy (roughly). The contact manifold \( C \) will be our link \( \partial \tilde{A}_\delta \) inside \( S \) and this divides \( S \) into \( \tilde{A}_\delta \) (representing \( S_- \)) and the extra non-compact piece \( S_+ \). The submanifold \( E \) is morally some ‘divisor’ that compactifies \( \tilde{A}_\delta \). The reason why we need a well defined GW triple is so that we can apply a neck stretching argument along \( C = \partial \tilde{A}_\delta \) so that we can find a Reeb orbit of an appropriate index in order to bound the minimal discrepancy of \( A \) from below.

**Proof.** of Lemma 5.4. Let \( U'_S := \hat{S}_- \cup (-\epsilon, \epsilon) \) \( U_S \). We will show that for \( i \) large enough, every genus 0 nodal \( J \) holomorphic curve in \( S' \) representing \([A]\) for \( J \in J_{i, \omega} \) is contained inside the relatively compact open set \( U'_S \). Suppose for a contradiction that we have a sequence \( t_i \in [0, 1] \), a sequence \( J'_{i,t} \in J_{i, \omega} \) and a sequence of genus 0 \( J'_{i,t} \) holomorphic curves \( u_i : \Sigma \to S' \) representing \([A]\) not contained in \( U'_S \) but all contained in some much larger relatively compact set \( S \). After passing to a subsequence we can assume that \( t_i \) converges to \( t_\infty \) for some \( t_\infty \in [0, 1] \) and hence \( \omega_{t_i} C'^\infty \) converges to \( \omega_{t_\infty} \). Because \( J'_{i,t} \) stretch the neck along \( C \) and converge in \( C_{k,\infty} \) to \( J_+ \) inside \( \hat{S}_+ \) as \( i \to \infty \) we have by Proposition 7.2 a finite energy \( J_+ \) holomorphic curve in \( \hat{S}_+ \) intersecting \( Q \) with multiplicity 1. Such a curve is contained inside \( U_S \) by assumption. Proposition 7.2 then tells us that for some \( i \) large enough, the image of \( u_i \) is contained in \( U'_S \) which gives us a contradiction. Hence for \( i \) large enough we have that the image of every \( J'_{i,t} \) holomorphic curve is contained in the relatively compact open set \( U'_S \). Hence \((S', [A], J'_{i,t})\) is a smooth deformation of GW triples.

We are interested in indices of Reeb orbits and the following lemma gives us an upper bound for the Conley-Zehnder index of a Reeb orbit. We give a slightly different (but equivalent) definition of the relative first Chern class here. Let \( M \) be any symplectic manifold with boundary \( \partial M \) and \( J \) a compatible almost complex structure. We will assume that \( \partial M \) is compact but \( M \) may not be compact. Now let \( K \) be the canonical bundle of \( M \) (i.e. the highest exterior power of \( T^* M \) viewed as a complex vector bundle) and let \( \tau \) be a trivialization of \( K^{\otimes N_{c_1}} \) on \( \partial M \) for some \( N_{c_1} \in \mathbb{N} \). Using the trivialization \( \tau \) we have a complex line bundle \( K^{\otimes N_{c_1}} \) in the quotient \( M/\partial M \). We define the relative first Chern class \( c_1(K; \mathbb{Q}) \in H^2(M, \partial M; \mathbb{Q}) \) to be \( \frac{1}{N_{c_1}} c_1(K; \mathbb{Q}) \in H^2(M/\partial M; \mathbb{Q}) = H^2(M, \partial M; \mathbb{Q}) \).

**Lemma 5.4.** Suppose that \((S, [A], J)\) is a GW triple and \( C' \subset S \) a contact hypersurface with the following properties:

1. The hypersurface \( C' \) splits \( S \) into a two stable Hamiltonian cobordisms \( S_+, S_- \) where \( \partial_+ S_+ = C' \) and \( \partial_- S_- = C' \). We will also assume that there is a compact codimension 2 submanifold \( Q \) of \( S_- \) so that \([A], Q \neq 0 \).
2. The contact form \( \alpha' \) on \( C' \) only has non-degenerate Reeb orbits. We will suppose that \( C' \) has a natural trivialization of the \( N_{c_1} \)th tensor power of its canonical bundle so that we can define Conley-Zehnder indices for Reeb orbits. This also implies that the \( N_{c_1} \)th tensor power of the canonical bundle of \( S_+ \) restricted to \( \partial_- S_+ \) also has a chosen trivialization.
3. We have two properly embedded codimension 2 symplectic submanifolds \( D_\infty, E \) of \( \hat{S}_+ \) which intersect transversally and are holomorphic with respect to some \( J \in \mathfrak{J} \) and
disjoint from some neighborhood $C' \times (-\epsilon_h, \epsilon_h)$ of $C'$ given by the symplectization. We will assume that $E$ is compact. We require $[A] \cdot E = 1$, $[A] \cdot D_{\infty} = 0$ and $S_{\infty} \setminus (D_{\infty} \cup E)$ deformation retracts onto $C' = \partial S_{\infty}$.

(4) For any compatible almost complex structure equal $J$ equal to some $J_1 \in \mathcal{J}$ outside a small (fixed) neighborhood of the closure of $S_- \cup \{(0, \epsilon_h) \times C'\}$, we have $J \in \mathcal{J}$. We will assume that $\mathcal{J}$ is open in the $C^\infty$ topology. Finally we assume $GW(S, [A], \mathcal{J}) \neq 0$.

Let $a, b$ the unique rational numbers with the property that $aE + b D_{\infty}$ is Lefschetz dual to the relative first Chern class of $S_+$. Then there exists a Reeb orbit $R$ of $(C', \alpha')$ of Conley-Zehnder index less than or equal to $n - 3 - 2a$ if $n - 3 \geq a$ and less than $3 - n$ otherwise, where $n = \frac{1}{2} \dim_{S}(S)$.

Now $\alpha'$ extends to a 1-form $\theta'$ on $S_+ \setminus (D_{\infty} \cup E)$ with $d\theta' = \omega_S$ where $\omega_S$ is the symplectic form on $S$. The length of $R$ is $\leq -\alpha'$ where $\alpha'$ is the wrapping number of $\theta'$ around $E$.

Proof. of Lemma 5.4 We stretch the neck along $C'$ using almost complex structures $J_i$ from $\mathcal{J}$. We will also assume that $J_i$ is cylindrical near $C'$. This can be done by property (4) from above. We can also ensure that $J_i|_{\hat{S}_+}$ converges in $C^\infty_{\text{loc}}$ to an almost complex structure $J_+$ compatible with the completion of $S_+$. We will also assume that $J_+$ makes $E$ and $D_{\infty}$ holomorphic.

Because $GW(S, [A], \mathcal{J}) \neq 0$ we have a $J_i$ holomorphic curve from a genus 0 connected nodal curve representing $[A]$. By SFT compactness [BEH+03] we get a connected $J_+$ holomorphic curve $u_\infty : \Sigma_\infty \rightarrow \hat{S}_+$ which is contained in compact subset of $S$ (not $\hat{S}_+$) which intersects $E$ once and does not intersect $D_{\infty}$ by positivity of intersection. This is the top level of our holomorphic building. The holomorphic curve $\Sigma_\infty$ has irreducible components $\Sigma_1^\infty, \Sigma_2^\infty, \ldots$. Exactly one of these components intersects $E$ with multiplicity 1 after mapping them to $\hat{S}_+$ under $u_\infty$. After relabeling such components we can assume that this component is $\Sigma_1^\infty$. Also by positivity of intersection we have that no irreducible component intersects $D_{\infty}$ after mapping them to $\hat{S}_+$ by $u_\infty$.

Because the image of $u_\infty|_{\Sigma_2^\infty}$ is contained in the region $\hat{S}_+ \setminus (D_{\infty} \cup E)$ where $\omega_S$ is exact, we have by the maximum principle in [AS10] Lemma 7.2 that such a curve does not exist. Hence $\Sigma_2^\infty$ does not exist and so $\Sigma_\infty$ has exactly one irreducible component.

Now we need to show that $\Sigma_1^\infty$ converges to a Reeb orbit. The results in [BEH+03] tell us that in fact $u_i$ converges to a holomorphic building. The lowest level of such a holomorphic building is contained in $S_-$. Such a lowest level is non-empty because $[A], Q \neq 0$ where $Q$ is a compact submanifold of $\hat{S}_+$. Now because our holomorphic building is connected we get that the top level of our holomorphic building $u' = u_\infty$ must converge to a Reeb orbit. Hence $u_\infty$ is an irreducible $J_+$ holomorphic curve intersecting $E$ once and $D_{\infty}$ 0 times and converging to at least one Reeb orbit in $C' = \partial S_+$.

By property (4), we can assume that $J_+$ is $C^\infty$ generic among almost complex structures equal to $J_+$ outside a compact subset of $\hat{S}_+$ and making $D_{\infty}$ and $E$ holomorphic. In particular we can assume by [Dra04] that any irreducible $J_+$ holomorphic curve intersecting $E$ with multiplicity 1 and with finite energy is regular by perturbing $J_+$ near $E$ (but not along $E$ or $D_{\infty}$). This is because it is somewhere injective near $E$ as its intersection multiplicity with $E$ is 1. In particular we can assume that $u_\infty$ is regular. Now [BEH+03] [Proposition 5.6] tells us that $\Sigma_\infty$ compactifies to a surface with boundary $\Sigma_\infty$ and $u_\infty$ extends continuously to a map $\overline{u}_\infty : \overline{\Sigma_\infty} \rightarrow S_+$ so that $\overline{u}_\infty(\partial \Sigma_\infty)$ is a union of Reeb orbits $R_1, \ldots, R_l$.

We now need to compute the sum of the Conley-Zehnder indices of these orbits using [Dra04]. The orientation of the boundary of $\Sigma_\infty$ coming from the inward normal is equal to the natural orientation of the Reeb orbits $R_1, \ldots, R_l$ (i.e. these orbits are negative ends).
Now \( \overline{p}_{\infty} \) intersects \( E \) once and \( D_\infty \) \( 0 \) times and so let \( p \in \Sigma_\infty \) be the unique point satisfying \( \{ p \} = \overline{p}_{\infty}^{-1}(E \cup D_\infty) \). Let \( K_+ \) be the canonical bundle of \( \hat{S}_+ \). Because \( S_+ \setminus (E \cup D_\infty) \) deformation retracts onto \( C' = \partial_- S_+ \) we have a trivialization of \( K_+^{\otimes N_\tau} |_{\partial_- S_+} \) coming from our trivialization of \( K_+^{\otimes N_\tau} |_{\partial_- S_+} \) and this gives us a trivialization

\[
\tau : C \times (\Sigma_\infty \setminus \{ p \}) \to \overline{p}_{\infty}^{-1} K_+^{\otimes N_\tau} |_{\Sigma_\infty \setminus \{ p \}}.
\]

Using our trivialization \( \tau \), any smooth section \( \sigma \) of \( \overline{p}_{\infty}^{-1} K_+^{\otimes N_\tau} |_{\Sigma_\infty \setminus \{ p \}} \) is given by a function \( \tau^{-1} \circ \sigma : \Sigma_\infty \setminus \{ p \} \to C \). Because \( aE + bD_\infty \) is Lefschetz dual to the relative first Chern class of \( S_+ \), we can choose such a section \( \sigma \) so that \( \tau^{-1} \circ \sigma \) is equal to \( z^{N_\tau} \) near \( p \) where \( z \) is a local holomorphic coordinate chart around \( p \) and is non-zero away from \( p \) and constant near \( \partial \Sigma_\infty \). This means we can view \( \tau^{-1} \circ \sigma \) as a map from \( \Sigma_\infty \setminus \{ p \} \) to \( \mathbb{C}^* \). Because \( \Sigma_\infty \) is homotopic to a wedge of circles we can choose some trivialization \( \tau' \) of \( u_\tau^\ast K_+ \) (i.e. \( \tau' : C \times \Sigma_\infty \to u_\tau^\ast K_+ \) is a complex bundle isomorphism). Unlike \( \tau \) our choice is not unique up to homotopy. Using our trivializations \( \tau \) and \( \tau' \otimes N_\tau \) respectively we can get trivializations of the \( N_\tau \)th tensor power of the canonical bundle along \( R_1, \ldots, R_l \) and this means we get two Conley-Zehnder indices \( CZ_\tau(R_j) \) and \( CZ_{\tau'}(R_j) = CZ_{\tau \otimes N_\tau}(R_j) \). Now \( T := ((\tau' \otimes N_\tau)^{-1} \circ \tau)|_{\partial \Sigma_\infty} \) is an automorphism of trivial bundles and so we can view \( T \) as a map \( T : \partial \Sigma_\infty \to \mathbb{C}^* \). Because \( \partial \Sigma_\infty \) is a union of \( l \) oriented circles, this means that \( T \) represents \( l \) elements of \( \pi_1(\mathbb{C}^*) = \mathbb{Z} \) given by \( q_1, \ldots, q_l \in \mathbb{Z} \) (from now on we will give \( \partial \Sigma_\infty \) the orientation coming from the inward normal, which is the same orientation as the Reeb orbits). By the properties of the Conley-Zehnder index this implies that: 

\[
-2q_j + N_\tau CZ_\tau(R_j) = N_\tau CZ_{\tau'}(R_j) = \text{the catenation axiom in } [\text{Gut13}] \text{[Theorem 55]}
\]

(remember that the Conley-Zehnder index is calculated from a trivialization of some multiple of the tangent bundle and not the cotangent bundle and so the dual of our morphism \( T \) sends the trivialization of the \( N_\tau \)th tensor power of the \textit{anti} canonical bundle induced by the dual of \( \tau' \) to the trivialization induced by the dual of the \( N_\tau \)th tensor power of \( \tau \) which is represented by \(-2q_j\)). Also because our section \( \sigma \) is constant along \( \partial \Sigma_\infty \) with respect to our trivialization \( \tau \) we get that \( (\tau' \otimes N_\tau)^{-1} \circ \sigma|_{\partial \Sigma_\infty} : \partial \Sigma_\infty \to \mathbb{C}^* \) is homotopic to \( T \). This implies that \( \sum_j q_j = N_\tau a \). Hence \(-2a + \sum_{j=1}^l CZ_{\tau'}(R_j) = \sum_{j=1}^l CZ_\tau(R_j) \).

By the remark after Corollary 2 in [Dra04], we get that \((n-3)(2-l) - \sum_{j=1}^l CZ_\tau(R_j) \geq 0 \). Hence \((n-3)(2-l) - 2a - \sum_{j=1}^l CZ_\tau(R_j) \geq 0 \). This implies: \( \sum_{j=1}^l (CZ_{\tau'}(R_j) + (n-3)) \leq 2(n-3) - 2a \). If \( 2(n-3) - 2a < 0 \) then there exists a \( j \) so that \( CZ_{\tau'}(R_j) + (n-3) < 0 \). And so \( CZ_{\tau'}(R_j) < 3 - n \) in this case. If \( 2(n-3) - 2a \geq 0 \) then there exists a \( j \) so that \( CZ_{\tau'}(R_j) + (n-3) \leq 2(n-3) - 2a \). And so: \( CZ_{\tau'}(R_j) \leq n-3 - 2a \) if \( n-3 \geq a \) and less than \( 3-n \) otherwise. The bound on the length of \( R_j \) comes from the fact that \( \int_{\Sigma_\infty} u_\tau^\ast \omega_S \geq 0 \) and that \(-a' - \sum_i \text{length}(R_i) = \int_{\Sigma_\infty} u_\tau^\ast \omega_S \) by Stokes' theorem. This gives us our result. \( \square \)

6. Bounding Minimal Discrepancy from Below

6.1. Bounding Minimal Discrepancy of Positively Intersecting Submanifolds from Below. Let \((M, \omega)\) be a compact symplectic manifold with boundary and let \(S_1, \ldots, S_l\) be symplectically orthogonal submanifolds of the interior so that \( \omega = d\theta \) outside \( S_1 \cup \cdots \cup S_l \) and so that \( \cup_i S_i \hookrightarrow M \) is a homotopy equivalence. We will assume that the wrapping numbers \( \lambda_1, \ldots, \lambda_l \) of \( \theta \) around \( S_1, \ldots, S_l \) are positive. We will also assume that there is a trivialization of the \( N_\tau \)th power of the canonical bundle \( \wedge^\tau T^*(M \setminus \cup_i S_i) \). Because we have such a trivialization we have that the relative first Chern class is Lefschetz dual to
Let \( (C, \alpha) \) be contactomorphic to the link of \( S_1, \ldots, S_l \). Then there exists a Reeb orbit \( \gamma \) of \( \alpha \) so that

\[
\text{CZ}(\gamma) - \frac{1}{2} \dim \ker (D\gamma - \text{id}) + (n - 3) \leq 2 \min_i (a_i) \quad \text{if } \min_i (a_i) \geq 0.
\]

\[
\text{CZ}(\gamma) - \frac{1}{2} \dim \ker (D\gamma - \text{id}) + (n - 3) < 0 \quad \text{if } \min_i (a_i) < 0.
\]

where \( D\gamma : \ker (\alpha) |_{\gamma(0)} \to \ker (\alpha) |_{\gamma(0)} \) is the linearized return map of the Reeb flow along the length of \( \gamma \) restricted to the contact hyperplane distribution.

In the next section we will apply this to isolated singularities. The symplectic manifold \( M \) should be thought of as a neighborhood of the exceptional divisors of a resolution of an isolated singularity \( S_i \) and should be thought of as an exceptional divisor.

**Proof.** We will prove this theorem in 5 steps. In the first step we deform our symplectic form to a better one so that the problem is slightly easier. In the second step, we construct some appropriate contact and stable Hamiltonian submanifolds of \( M \) deformation equivalent to \( (C, \alpha) \) and we will also partially compactify \( M \) to a symplectic manifold \( S \). In the third step we show that a certain class of holomorphic curves inside \( S \) stay inside a compact subset. In the fourth step we will construct a certain family of GW triples using \( S \) and then finally in the last step we use Lemma 5.4 to find Reeb orbits of \( (C, \alpha) \) of the appropriate index.

**Step 1.** In this step we will construct a symplectic manifold and a codimension 0 submanifold whose boundary is contactomorphic to \( (C, \alpha) \) and whose Reeb flow has nice properties. By [McLi12] (Theorem 5.3) we can deform \( S_1, \ldots, S_l \) through positively intersecting submanifolds so that \( S_1, \ldots, S_l \) becomes symplectically orthogonal. By Corollary 4.3 we have that the link of this new set of symplectically orthogonal divisors is contactomorphic to \( (C, \alpha) \) and so from now on we will assume that \( S_i \) are symplectically orthogonal.

By Theorem 4.6 we have for each \( \epsilon > 0 \), there exist functions \( f : M \setminus \bigcup S_i \to \mathbb{R} \), \( g : M \setminus \bigcup S_i \to \mathbb{R} \), a contact submanifold \( C_1 \subset M \) with contact form \( \alpha_1 = \theta' |_{C_1} \) where \( \theta' = \theta + dg \) satisfying the following properties:

1. We have that \( f \) is compatible with \( \bigcup S_i \) and \( df(X_{\theta'}) > 0 \) inside \( N \setminus \bigcup S_i \) where \( N \) is a small neighborhood of \( \bigcup S_i \). Also \( C_1 \) is a subset of \( N \) and is transverse to \( X_{\theta'} \) where \( X_{\theta'} \) points outward along \( C_1 \) away from \( \bigcup S_i \). This condition ensures that \( (C_1, \alpha_1) \) is contactomorphic to the link of \( \bigcup S_i \) and hence is contactomorphic to \( (C, \alpha) \).

2. Let \( V_S = N(S_1, \ldots, S_l) \) be the free commutative monoid generated by \( S_1, \ldots, S_l \). For each non-zero element of \( V_S \) given by \( V := \sum_{j \in I_V} d_j S_j \) we have a pseudo Morse Bott family \( R_V \) whose Conley-Zehnder index is given by \( 2 \sum_{i \in I_V} (a_i + 1) d_i - \frac{1}{2} |I_V| \) where we have \( \text{Size}(R_V) = 2n - |I_V| - 1 \). The length of the respective Reeb orbits minus \( \sum_{i \in I} d_i (2\pi^2 + \lambda_i) \) has absolute value less than \( \epsilon \left( \sum_{i \in I} d_i \right) \). Also there exists a disk bounding each Reeb orbit in \( R_V \) whose intersection with \( S_i \) is \( d_i \). We require that every Reeb orbit sits inside some family \( R_V \).

3. Let \( B^{2n}_\nu \subset C^{2n} \) be the symplectic ball of radius \( \nu \). For each \( i \), there is some \( \epsilon'_M > \epsilon > \epsilon_M \), a subset \( W_i \subset M \setminus \bigcup_{j \neq i} S_j \) symplectomorphic to \( B^{2n-2}_\theta \times B^2_{\epsilon'_M} \) with the property...
that \( W_i \cap S_i = B^{2n-2}_{\epsilon_i^M} \times \{0\} \subset S_i \setminus \cup_{j \neq i} S_j \). The restriction of \( \theta' \) to \( B^{2n-2}_{\epsilon_M} \times B^2_{\epsilon_M} \) is \( \text{pr}_1^* \beta + \text{pr}_2^*(r^2 d\theta) \) where \( \text{pr}_1 \) and \( \text{pr}_2 \) are the projections to \( B^{2n-2}_{\epsilon_M} \) and \( B^2_{\epsilon_M} \), respectively, \( \beta \) is a 1 form and \((r_i, \theta_i)\) are polar coordinates. Also \( C \cap W_i = B^{2n-2}_{\epsilon_M} \times \partial B^2_{\epsilon_M} \).

Because \( N_c, c_1(\ker(\alpha_1)) = 0 \) for some \( N_c \in \mathbb{N} \) we get that \( a_1, \ldots, a_t \in \frac{1}{2N_c} \mathbb{Z} \). Let

\[
\mu = \max(-\frac{1}{2N_c}, \min_i(a_i)).
\]

After reordering the submanifolds \( S_i \) we will assume that \( S_1 \) has the property that \( a_1 \leq \mu \) and \( \lambda_i = \min_i(\{a_i, \mu\}) \). We will choose \( \epsilon > 0 \) small enough so that every pseudo Morse Bott family \( R_V \) of Reeb orbits described as above of length less than \( 2\pi \epsilon^2 + \lambda_1 \) satisfies

\[
CZ(R_V) - \frac{1}{2} |I_V| > \max(2(\frac{1}{2N_c} + 1), 2(a_1 + 1)).
\]

Because \( CZ(R_V) - \frac{1}{2} |I_V| \) is an even multiple of \( \frac{1}{N_c} \) we get that:

\[
CZ(R_V) - \frac{1}{2} |I_V| > \max\left(2\left(\frac{1}{2N_c} + 1\right), \frac{1}{2N_c} 2(a_1 + 1)\right) = \max(2 - \frac{1}{2N_c}, 2a_1 + 2)
\]

Hence:

\[
CZ(R_V) - \frac{1}{2} \text{Size}(R_V) = CZ(R_V) - \frac{1}{2}(2n - 2 + (1 - |I_V|)) \geq \max(3 - n - \frac{1}{2N_c}, 2a_1 + 3 - n).
\]

Because \( C_1 \) separates \( M \) into two pieces we let \( M_1 \subset M \) be the unique codimension 0 submanifold containing \( \cup_i S_i \) whose boundary is equal to \( C_1 \). This is a cobordism of stable Hamiltonian structures with \( \partial_-.M_1 = \emptyset \) and \( \partial_+M_1 = C_1 \).

**Step 2.** We will now construct an appropriate symplectic manifold containing \( M_1 \) along with some natural compatible almost complex structures on this manifold. Define \( \omega'_+ := d\theta'|_{C_1} \). By flowing backwards along \( X_{\theta'} \) we will assume that a neighborhood of \( \partial_+M_1 = C_1 \) is diffeomorphic to \( (-\epsilon, 0] \times C_1 \) with \( \theta' = e^r\alpha_1 \) where \( r \) parameterizes \( (-\epsilon, 0] \). Define \( C_2 := \{r = -\frac{1}{2}\} \).

Now let \( S^2_\epsilon \) be the two sphere of area \( 2\pi \epsilon^2 \) and let \( B^2_\epsilon \subset S^2_\epsilon \) be some symplectic embedding of the ball into the sphere and let \( q_\infty \in S^2_\epsilon \setminus B^2_\epsilon \) be the unique point in the complement of this embedding. We define \((S', \omega_{S'})\) to be the symplectic manifold given by the interior of the union of \( M_1 \) with \( B^{2n-2}_{\epsilon_M} \times S^2_{\epsilon} \) along \( W_1 = B^{2n-2}_{\epsilon_M} \times B^2_{\epsilon_M} \) where \( W_1 \subset M_1 \) is the submanifold described earlier. We define \((S, \omega_{S})\) to be the symplectic blowup of \((S', \omega_{S'})\) along \( \{0\} \times \{q_\infty\} \in B^{2n-2}_{\epsilon_M} \times S^2_{\epsilon} \). We let \( Bl : S \to S' \) be the blowdown map which is a diffeomorphism away from some symplectic submanifold \( E \) (the exceptional divisor) and a symplectomorphism outside some very small open subset containing \( E \). Hence because \( q_\infty \) is disjoint from \( M_1 \), we have that \( M_1 \) is naturally a submanifold of \( S \). We require that this blow up should be small enough so that the restriction of \( \omega_{S} \) to \( M_1 \) is \( \omega_{S'} \).

We will now put a stable Hamiltonian structure on \( C_1 \). Define \( \epsilon_M' = \frac{1}{2}(\epsilon + \epsilon'_M) \). Define \( \alpha_+ \in \Omega^1(C_1) \) to be \( \alpha_1 \) outside \( B^{2n-2}_{\epsilon_M'} \times \partial B^2_{\epsilon_M} \subset C_1 \) and equal to \((r^2 + \frac{1}{2\pi} \lambda_1)\theta \) inside \( B^{2n-2}_{\epsilon_M'} \times \partial B^2_{\epsilon_M} \subset C_1 \). We do this by choosing some bump function \( \rho \) on \( B^{2n-1}_{\epsilon_M} \) equal to 0 near its boundary and equal to 1 along \( B^{2n-1}_{\epsilon_M} \) and then defining \( \alpha_+ \) to be \( (\text{pr}_1^*(1 - \rho))\alpha_1 + (\text{pr}_1^*\rho)(r^2 + \frac{1}{2\pi} \lambda_1)\theta \) where \( \text{pr}_1 : B^{2n-2}_{\epsilon_M'} \times B^2_{\epsilon_M} \to B^{2n-2}_{\epsilon_M'} \) is the natural projection map. Then \((\omega_{S'}, \alpha_+)\) is a Stable Hamiltonian structure on \( C_1 \) with exactly the same Reeb flow as \( \alpha_1 \).
restriction of $\alpha_+$ to \( \{a\} \times \partial B^2_{\epsilon M} \) is \((r^2 + \frac{1}{2\pi} \lambda_1) d\theta\) for each $a \in B^2_{\epsilon M}$. Because \( (\omega'_+, \alpha_+) \) is a contact structure away from our pseudo Morse Bott family $R_{S_1}$ we can perturb $\alpha_+$ outside $R_{S_1}$ and hence also $\omega'_+ = d\alpha_+$ by a $C^\infty$ small amount so that all Reeb orbits of $\alpha_+$ whose length is less than $2\pi (\epsilon_M + \lambda_1)$ are non-degenerate (note that these orbits are disjoint from $R_{S_1}$). By Lemma 8.3.5 we have that the indices of any of these non-degenerate Reeb orbits is strictly greater than $\max(3 - n - 2N_{\epsilon_1}, 2a_1 + 3 - n)$ from the inequality in Step 1.

A small neighborhood of $C_1$ inside $S$ is symplectomorphic to \((-\eta, \eta) \times C_1\) with $\omega_S = \omega'_+ + d(r_1 \alpha_+)$ where $\eta > 0$ is small and $r_1$ parameterizes $(-\eta, \eta)$. Let $J_1$ be a sequence of almost complex structures on $S$ compatible with $\omega_S$ which stretch the neck along $C_1$ with respect to $(-\eta, \eta) \times C_1$. Now $C_1$ splits $S$ into two regions $S_+$ and $S_-$ where $S_+, S_-$ are cobordisms of stable Hamiltonian structures with $\cup_i S_i \subset S_-, \partial_+ S_+ = \partial_- S_- = \emptyset$ and $\partial_+ S_+ = \partial_- S_- = C_1$. Let $S'_+ := \text{Bl}(S_+) \subset S'$ where $\text{Bl} : S \to S'$ is our blowdown map. This is also a cobordism of stable Hamiltonian structures. We will assume that $J_1|_{S'_+}$ converges in $C^\infty_{\text{loc}}$ to some almost complex structure $J_+$ compatible with the completion of $S_+$. We will choose $J_+$ so that the blowdown map $\text{Bl}|_{S'_+}$ is a $(J_+, J'_+)$ holomorphic map where $J'_+$ is compatible with the completion of $S'_+$. We can assume that $D'_\infty := B^2_{\epsilon M} \times \{q_\infty\}$ is $J'_+$ holomorphic and that the preimage of $D_\infty$ under the blowdown map is a union of transversally intersecting $J_+$ holomorphic submanifolds $E \cup D_\infty$ where $D_\infty$ the proper transform of $D'_\infty$ (proper transform here means we take the closure of the preimage of $D'_\infty \backslash \{q_\infty\}$ under the blowdown map). Because $(\omega'_+, \alpha_+)$ is a specific stable Hamiltonian structure in the region $B^2_{\epsilon M} \times S^2_\epsilon$ we can assume (maybe after deforming our neighborhood $(-\eta, \eta) \times C_1$) that $J'_+$ is a product almost complex structure $(J_B^{2n-2}, J_{S_\infty,S^2})$ on $B^2_{\epsilon M} \times (S^2_\epsilon \backslash B^2_{\epsilon M})$ where $J_B^{2n-2}$ is the standard complex structure on $B^2_{\epsilon M} \subset \mathbb{C}^{n-1}$ and where $J_{S_\infty,S^2}$ is some compatible complex structure on $S^2_\epsilon \backslash B^2_{\epsilon M}$.

Step 3. In this step we will now show that any finite energy proper $J_+$ holomorphic curve intersecting $E$ once and $D_\infty$ 0 times is contained inside a fixed compact subset of $S_+$. In fact we will show that it is contained inside the interior of $\text{Bl}^{-1}(B^2_{\epsilon M} \times S^2_\epsilon)$ where $\text{Bl} : S \to S'$ is the blowdown map. The reason why we want to do this is that in Step 4 we create an appropriate family of GW triples using Lemma 8.3.5 combined with results from this step.

Let $u : \Sigma \to S'_+$ be such a curve. Let $u' : \Sigma \to S'_+ \backslash D'_\infty$ be the composition of $u$ with the blowdown map. We will first show $\Sigma$ is irreducible. Let $\Sigma_1, \Sigma_2, \cdots$ be the irreducible components of $\Sigma$. Exactly one of these components intersects $D'_\infty$ after mapping them with $u'$ by positivity of intersection. We will assume after reordering these components that $\Sigma_1$ intersects $D'_\infty$. If $\Sigma_2$ exists then it must map to the stable Hamiltonian cobordism $S'_+ \backslash D'_\infty$. Because $\theta'$ evaluated on the Reeb vector field along $\partial_- S'_+$ is 1 and because $\theta'$ extends to a 1 form on $S'_+ \backslash D'_\infty$ whose exterior derivative is the symplectic form we then get that $\Sigma_2$ cannot exist by Proposition 8.4. Hence $\Sigma$ is irreducible.

Now $(\text{pr}_1 \circ u')|_{\text{pr}_1^{-1}(B^2_{\epsilon M} \backslash S^2_\epsilon)} : \text{pr}_1^{-1}(B^2_{\epsilon M} \backslash S^2_\epsilon) \to B^2_{\epsilon M} \backslash S^2_\epsilon$ is a proper $J_B^{2n-2}$ holomorphic map and so is either the constant map, or a map whose energy is greater than or equal to $2\pi (\epsilon''_M)^2$ by the monotonicity formula. Let $\sigma_i \in \Sigma$ be a sequence of points so that $u'(\sigma_i)$ gets arbitrarily close to $\partial_- S'_+$ . Now Lemma 7.3 tells us that after passing to a subsequence, $u'(\sigma_i)$ converges to a point on a Reeb orbit of length at most $2\pi (\epsilon)^2 + \lambda_1$ because the intersection of $u'$ with $D'_\infty$ is 1 and the wrapping number of $\theta'$ around $D'_\infty$ is $-2\pi (\epsilon)^2 - \lambda_1$ after extending $\theta'$ in some way to $S'_+ \backslash D'_\infty$ so that $d\theta'$ is the symplectic form on $S'_+ \backslash D'_\infty$. 
We have two cases: Case 1: This Reeb orbit is contained in $R_{S_1}$ for some sequence of points $\sigma_i$ as described above. Case 2: No such Reeb orbit is contained in $R_{S_1}$ for any sequence of points. We will in fact show that this case cannot occur using a proof by contradiction.

Case 1: Suppose this Reeb orbit is contained in $R_{S_1}$. The second half of Lemma [7.3] tells us $\int_{S_l}(u)^*\omega_{S_l} \leq 2\pi(\epsilon)^2 + \lambda_1 - (2\pi(\epsilon_M)^2 + \lambda_1) < 2\pi(\epsilon_M)^2$. The energy argument above then tells us that the image of $u'$ under $pr_1$ is contained inside $B^{2n-2}_{\epsilon_M} \subset B^{2n-2}_{\epsilon_M}$. Hence the image of $u'$ is contained inside $B^{2n-2}_{\epsilon_M} \times S^2_l$ assuming that $u'(\sigma_i)$ converges to a Reeb orbit in $R_{S_1}$.

Case 2: This case involves a lot more work. Here we suppose (for a contradiction) that every such sequence $\sigma_i$ converges to a Reeb orbit not in $R_{S_1}$. All such Reeb orbits are non-degenerate by construction. Lemma [7.3] tells us that $\Sigma$ is biholomorphic to $\mathbb{P}^1 \backslash \{w_1, \ldots, w_l\}$ where $w_1, \ldots, w_l$ are $l$ distinct points. By [BEH+03], these punctures converge to non-degenerate Reeb orbits $R_1, \ldots, R_l$. We have that $u$ is somewhere injective because it intersects our holomorphic submanifold $E$ with multiplicity 1 and because $\Sigma$ is irreducible. By perturbing $J_+$ appropriately we can ensure that $J_+$ has the property that every somewhere injective $J_+$ holomorphic curve not contained in $E \cup D_\infty \cup (B^{2n-2}_{\epsilon_M} \times (S^2 \setminus B^{2n}_{\epsilon_M}))$ and whose domain is a punctured sphere is regular by [Dra04]. We can still ensure that all the properties of $J_+$ hold. The point here is that to achieve regularity we only need to perturb $J_+$ outside $E \cup D_\infty \cup (B^{2n-2}_{\epsilon_M} \times (S^2 \setminus B^{2n}_{\epsilon_M}))$ as every somewhere injective $J_+$ holomorphic curve not contained in $E \cup D_\infty \cup (B^{2n-2}_{\epsilon_M} \times (S^2 \setminus B^{2n}_{\epsilon_M}))$ has a somewhere injective point outside this region.

Let $K$ be the canonical bundle of $S$ and $K'$ the canonical bundle of $S'$. Because $H_1(C_1; \mathbb{Q}) = 0$ and $N_{c_1}(T S|_{C_1}) = 0$ we have a canonical trivialization $\tau : C_1 \times \mathbb{C} \to K^\otimes N_{c_1}|_{C_1} = K'^\otimes N_{c_1}|_{C_1}$ of $K^\otimes N_{c_1}$ and $K'^\otimes N_{c_1}$ along $C_1 = \partial S_+ = \partial S'_+$. Also because $Q := S'_+ \cap (B^{2n-2}_{\epsilon_M} \times S^2)$ is contractible we also have a canonical trivialization $\tau'_+$ of $K'^\otimes N_{c_1}$ in this region. Similarly we have a trivialization $\tau'_-$ of $K'^\otimes N_{c_1}$ inside $M_1 \cap (B^{2n-2}_{\epsilon_M} \times S^2)$. Now $\partial Q$ is homotopic to a circle (oriented to coincide with the boundary of $\{0\} \times B^{2}_{\epsilon_M}$) and so $\tau'^{-1} \circ \tau'_+$ is represented map $a$ from $S^4$ to $S^1 = U(1)$. The degree of this map is $-2N_{c_1}$ because the Chern number of the canonical bundle of $S^2$ is $-2$ and the normal bundle of $\{0\} \times S^2_l$ inside $S'$ is trivial as a complex vector bundle. Hence the degree of $\tau'^{-1} \circ \tau'_+$ is $2N_{c_1}$. Also $\tau'^{-1} \circ \tau$ has degree $a_1 N_{c_1}$ by the definition of discrepancy. Hence $\tau'^{-1} \circ \tau$ has degree $(2 + a_1) N_{c_1}$. The trivialization $\tau'_+$ also gives a trivialization $\tau'_+$ of $K^\otimes N_{c_1}$ along $S_+ \setminus E$ because $E$ is a biholomorphism onto its image away from $E$. Because $E$ is the exceptional divisor of a blowup we can construct a section $\sigma$ which is constant with respect to $\tau_+$ near $\partial S_+$ and non-zero away from a small neighborhood of $E$ and whose zero set is homologous to $(n - 1)N_{c_1}|_{S_+}$ near $E$. All of this implies that there is a section $\sigma_1$ of $K^\otimes N_{c_1}|_{S_+}$ which is constant with respect to the trivialization $\tau$ along $\partial S_+$ and whose zero set is homologous to $(-2 - a_1) N_{c_1} D_\infty + (2n - 1 - 2 - a_1) N_{c_1}|_{E} = (-2 - a_1) N_{c_1} D_\infty + (2 + a_1 - (n - 3)) N_{c_1}|_{E}$ (note that $2 + a_1$ now becomes $-2 - a_1$ in our calculations because if we have a disk inside $(B^{2n-2}_{\epsilon_M} \times S^2) \cap S_+$ intersecting $D_\infty$ positively once with boundary $\{0\} \times \partial B^{2}_{\epsilon_M}$ then its boundary orientation is the opposite of $\{0\} \times \partial B^{2}_{\epsilon_M}$). This means that the Chern number of $u^*K^\otimes N_{c_1}$ with respect to the pullback of the trivialization $\tau$ is $(-a_1 + (n - 3)) N_{c_1}$. Hence the Chern number of the pullback via $u$ of the $N_{c_1}$th tensor power of the anti-canonical bundle is $(a_1 - (n - 3)) N_{c_1}$. 
Now choose some trivialization $\tau''$ of the pullback of the anti-canonical bundle along $u$. By the statement after Corollary 2 in [Dra01] we get that $(n - 3)(2 - l) - \sum_{i=1}^{l} CZ_{\tau''}(R_i) \geq 0$. Now $N_{c_1} \sum_{i=1}^{l} CZ_{\tau''}(R_i) = N_{c_1} \sum_{i=1}^{l} CZ(R_i) - 2(a_1 - (n - 3)) N_{c_1}$ by the above Chern number calculation. Hence $(n - 3)(2 - l) + 2a_1 - 2(n - 3) - \sum_{i=1}^{l} CZ(R_i) \geq 0$. Hence $\sum_{i}(CZ(R_i) + (n - 3)) \leq 2a_1$. Now if $a_1 \geq 0$ then $CZ(R_i) + (n - 3) \leq 2a_1$ for some $i$. If $a_1 < 0$, we get $CZ(R_i) + (n - 3) < 0$ for some $i$. Because $CZ(R_i) + n - 3$ is a multiple of $\frac{1}{2N_{c_1}}$ we get that $CZ(R_i) + (n - 3) \leq -\frac{1}{2N_{c_1}}$ if $a_1 < 0$. Hence $CZ(R_i) + (n - 3) \leq \max(-\frac{1}{2N_{c_1}},2a_1)$, so $CZ(R_i') \leq \max(-\frac{1}{2N_{c_1}} - (n - 3),2a_1 - (n - 3)) = \max(3 - n - \frac{1}{2N_{c_1}},2a_1 + 3 - n))$. But this contradicts the earlier fact: $CZ(R_i) > \max(3 - n - \frac{1}{2N_{c_1}},2a_1 + 3 - n)$. Hence $u$ is contained inside $B_{\epsilon^2_M}^{2n-2} \times B_{\epsilon^i_M}^2$.

**Step 4.** In this step we construct an appropriate family of symplectic structures and almost complex structures on $S$ giving us a family of GW triples. We will use Lemma 5.3 combined with Step 3 to do this.

We now have two regions containing $C_1$. One is $(-\epsilon,0) \times C_1$ with $\theta' = e^\epsilon \alpha_1$ and the other is $(-\eta,\eta) \times C_1$ with $\omega_S = \omega^S_+ + d(r_\alpha \alpha_+)$. To avoid confusion we will write the second neighborhood $(-\epsilon,0) \times C_1$ as $(-\frac{\epsilon}{\eta},\frac{\epsilon}{\eta}) \times C_2$ with $C_2 = \{ r = -\frac{\epsilon}{\eta} \}$ and $\theta' = e^\epsilon \alpha_2$ where $\alpha_2 = \theta'|_{C_2}$, and $r_2$ parameterizes the interval $(-\frac{\epsilon}{\eta},\frac{\epsilon}{\eta})$. We will assume that $\eta$ is small enough so that $(-\eta,\eta) \times C_1$ is disjoint from $[-\frac{\epsilon}{\eta},\frac{\epsilon}{\eta}] \times C_2$. We will assume that $J_i$ is cylindrical inside $[-\frac{\epsilon}{\eta},\frac{\epsilon}{\eta}] \times C_2$ and also translation invariant in this region with respect to the coordinate $r_2$. We can also assume that $J_i = J_j$ in this region for all $i,j$. Define $J_{C_2}$ to be the restriction of $J_i$ to the contact distribution on $C_2$ and $\hat{J}_{C_2}$ the respective cylindrical almost complex structure on the symplectization $\mathbb{R} \times C_2$.

We need to perform a certain symplectic dilation along $C_2$ parameterized smoothly by some $R \in \mathbb{R}$. Now $C_2$ splits $S$ into two manifolds with boundary $S_{2,+}$ and $S_{2,-}$ where $S_{2,-}$ is the region containing our exceptional divisors. We have that $S_{2,+}, S_{2,-}$ are cobordisms of stable Hamiltonian structures with $\partial S_{2,+} = \partial S_{2,-} = \emptyset$ and $\partial_- S_{2,+} = \partial_+ S_{2,-} = C_2$.

Choose a smooth family of orientation preserving diffeomorphisms $\phi_R : [-R,R] \rightarrow [-\frac{\epsilon}{\eta},\frac{\epsilon}{\eta}]$ parameterized by $R$ so that $\phi'_R = 1$ near $\pm \frac{\epsilon}{\eta}$. We can also define a smooth family of 1-forms $\theta_R$ where $\theta_R|_{(S_{2,-})\times \{0\} \times C_2} = \theta'$, $\theta_R|_{(S_{2,+})\times \{0\} \times C_2} = e^{2R+4\epsilon}\theta'$, and $\theta_R = (\theta_R, \text{id}_{C_2}, e^{4+R-4\epsilon})$ inside $[-\frac{\epsilon}{\eta},\frac{\epsilon}{\eta}] \times C_2$ where $\alpha_2 = \theta'|_{C_2}$ and $r$ parameterizes $[-R,R]$. We let $\omega_R := d\theta_R$. This extends to a symplectic form on the whole of $S'$. We let $\beta^i_{R,i}$ be the set of almost complex structures compatible with $\omega_R$ and equal to $J_i$ in the region $S_+ \cup ((-\eta,\eta) \times C_1)$. Choose $\epsilon_{\max} > \epsilon$ with the property that there is some embedding $i : C \hookrightarrow (-\frac{\epsilon}{\eta},\frac{\epsilon}{\eta}) \times C_2 \subset S'$ with the property that $i^* \omega_{\max} = \alpha$ and so that $i$ is isotopic to $C_1$ through contact embeddings. We define our homology class $[A]$ to be represented by the proper transform of $\{0\} \times S^2_\epsilon$ (i.e. represented by the closure of the preimage of $\emptyset$ \times $(S^2_\epsilon \setminus \phi S)$ inside $S$). We have $c_1([A]) + n - 3 = 0$. So by Lemma 5.3 combined with Step 3 we get for $i$ large enough that $(S, [A], \beta^i_{R,i})$ is a smooth family of GW triples parameterized by $R \in [-\frac{\epsilon}{\eta},\epsilon_{\max}]$ (maybe after shrinking $S$ very slightly).

We now need to calculate $\text{GW}_0(S,[A],\beta^i_{R,i})$. Define $\epsilon_{\min} := \frac{\epsilon}{\eta}$. Now $\theta_{\epsilon_{\min}} = \theta'$. We have $J_i \in \beta^i_{R,i}$. We have that any $J_i$ holomorphic curve representing $[A]$ has energy less than $\epsilon_{\max}^i$ with respect to $\theta'$. If $u$ is such a curve then $\text{pr}_1 \circ \text{Bl} \circ u|_{B_{\epsilon^i_M}^{2n-2} \times S^2_\epsilon}$ must have image equal to a point by the monotonicity formula. Hence every such curve is contained inside $\text{Bl}^{-1}(B_{\epsilon^i_M}^{2n-2} \times S^2_\epsilon)$, the image of such a curve must be contained inside $\{0\} \times S^2_\epsilon$ and so
there is a unique such curve up to reparameterization. The almost complex structure here is integrable near the image of this curve and the normal bundle of this curve is \( \prod_{i=1}^{n-1} \Theta(-1) \) and so the curve is regular. Hence \( GW_0(S, [A], J_1, R_{\text{min}}) = GW_0(S, [A], J_1) = 1. \) By Lemma 5.3 we then get \( GW_0(S, [A], J_1, R) = 1 \) for each \( R \leq R_{\text{max}}. \)

**Step 5.** Here we will perturb \( \alpha \) slightly to a contact manifold \((C, \alpha')\) whose Reeb orbits have indices related to those of \((C, \alpha)\) and then use Lemma 5.4 to find Reeb orbits of \((C, \alpha')\) of the appropriate index which in turn gives us our result.

Let \( d := \inf(CZ(\gamma) - \frac{1}{2} \dim \ker(D_\gamma - \text{id})) \) where our infimum runs over all Reeb orbits \( \gamma \) of \( \alpha \) and where \( D_\gamma : \ker(\delta)_{\gamma(0)} \to \ker(\delta)_{\gamma(0)} \) is the linearized return map of the Reeb flow along \( \gamma \). If there are no Reeb orbits we define \( d := \infty \). If \( d = -\infty \) then we are done, so from now an assume that \( d \neq -\infty \). Let \( \alpha' \) be a \( C^\infty \) small perturbation so that all the Reeb orbits are nondegenerate. By a repeated use of Lemma 3.5 we have that all orbits of \( \alpha' \) whose length is \( \leq e^{2R_{\text{max}} - \frac{1}{2} (2\pi(\epsilon)^2 + \lambda_1)} \) have Conley-Zehnder index \( \geq d \) so long as \( \alpha' \) is sufficiently \( C^\infty \) close to \( \alpha \). We wish to show that \( d \leq 2a_1 - (n - 3) \) if \( a_1 \geq 0 \) or \( d < - (n - 3) \) if \( a_1 \leq 0 \). It is therefore sufficient to show that there is a Reeb orbit of \( \alpha' \) of length \( \leq e^{2R_{\text{max}} - \frac{1}{2} (2\pi(\epsilon)^2 + \lambda_1)} \) and index \( \leq 2a_1 - (n - 3) \) if \( a_1 > 0 \) or index \( < 3 - n \) if \( a_1 < 0 \).

We now have an embedding \( \iota' : C \hookrightarrow [-\frac{1}{2}, \frac{1}{2}] \times C_2 \) which is \( C^\infty \) close to \( \iota \) with the property that \((\iota')^* \theta_{R_{\text{max}}} = \alpha'\). We have that \((\iota')^{*}(C)\) splits \( S \) into two symplectic cobordisms \( S_{+,t'}, S_{-,t'} \) where \( \partial_+ S_{+,t'} = \partial_- S_{-,t'} = \iota'(C) \) and \( \partial_+ S_{+,t'} = \partial_- S_{-,t'} = 0 \). Now because \((\iota')^{*}(C)\) is homotopic to \( C \) through stable Hamiltonian hypersurfaces, we get that the \( N_{t^1} \) th tensor power of the canonical bundle has a natural trivialization along \((\iota')^{*}(C)\) whose relative first Chern class on \( S_{+,t'} \) is Poincaré dual to \((-2 - a_1)D_{\infty} + (-a_1 + (n - 3))E\) as calculated above. Hence by Lemma 5.4 using our symplectic form \( \omega_{R_{\text{max}}} \), cobordisms \( S_{+,t'}, S_{-,t'} \) and \( D_{\infty} \), \( E \) as above we get that \((C, \alpha')\) admits a Reeb orbit of length \( \leq e^{2R_{\text{max}} - \frac{1}{2} (2\pi(\epsilon)^2 + \lambda_1)} \) (i.e. minus the wrapping number of \( \theta_{R_{\text{max}}} \) around \( E \)) and whose index is less than or equal to \( n - 3 - 2(-a_1 + (n - 3)) \) if \( n - 3 \geq (a_1 + (n - 3)) \) and less than \( 3 - n \) otherwise. Hence \( d \) is less than or equal to \( 2a_1 - (n - 3) \) if \( a_1 > 0 \) and less than \( 3 - n \) otherwise which in turn implies that there is a Reeb orbit \( \gamma \) of \( \alpha \) satisfying:

\[
\text{CZ}(\gamma) - \frac{1}{2} \dim \ker(D_\gamma - \text{id}) + (n - 3) \leq 2a_1
\]

if \( a_1 \geq 0 \) and

\[
\text{CZ}(\gamma) - \frac{1}{2} \dim \ker(D_\gamma - \text{id}) + (n - 3) < 0
\]

otherwise. Because \( a_1 \leq \max(-\frac{1}{2N_{t^1}}, \min_i a_i) \) we get

\[
\text{CZ}(\gamma) - \frac{1}{2} \dim \ker(D_\gamma - \text{id}) + (n - 3) \leq \min_i a_i
\]

if \( \min_i a_i \geq 0 \) and

\[
\text{CZ}(\gamma) - \frac{1}{2} \dim \ker(D_\gamma - \text{id}) + (n - 3) < 0
\]

otherwise. This proves our Theorem. \( \Box \)

6.2. **Bounding Minimal Discrepancy of a Singularity from Below.** Let \( A \subset \mathbb{C}^n \) be an affine variety of dimension \( n \) with an isolated singularity at \( 0 \) or which is smooth at \( 0 \). Recall that the link of \( A \) is a manifold \( L_A \) with a contact hyperplane distribution \( \xi_A \).

**Theorem 6.2.** Suppose \( L_A \) satisfies \( H^1(L_A; \mathbb{Q}) = 0 \) and \( c_1(\xi_A; \mathbb{Q}) = 0 \in H^2(L_A; \mathbb{Q}) \). Then

- \( \text{md}(L_A, \xi_A) \leq \min_i (a_i) \) if \( \min_i (a_i) \geq 0 \) and
where $V, W \in \text{almost complex structure } \hat{J}$.

The discrepancies defined in section 2. Our result then follows from Theorem 6.1.

Although we will not need this condition here. A \textit{stable Hamiltonian structure} is a cylindrical almost complex structure, i.e., for all non-zero $V, W \in \ker(\alpha_h)$ we have $\omega_h(V, JW) = \omega_h(V, W)$. We can define an almost complex structure $\tilde{J}_C$ on $\mathbb{R} \times C$ in the following way: For vectors of the form $(0, V)$ where $V \in \ker(\alpha_h)$ it is defined by $\tilde{J}_C(V) := J_C(V)$. Also $\tilde{J}_C(\frac{d}{\partial r_C}) = R$ and $\tilde{J}_C(R) = -\frac{\partial}{\partial r_C}$ where $r_C$ parameterizes $\mathbb{R}$. We say that $\tilde{J}_C$ is a \textit{cylindrical almost complex structure} associated to $J_C$. A \textit{symplectization} of $(\omega_h, \alpha_h)$ is the product $(\epsilon_h, \epsilon_h) \times C$ for $\epsilon_h > 0$ small with symplectic form $\tilde{\omega}_h := \omega_h + r_C d\alpha_h + r_C \omega_h$. Here by abuse of notation we have identified $\alpha_h$ with $\pi_C^* \alpha_h$ where $\pi_C : (\epsilon_h, \epsilon_h) \times C \to C$ is the natural projection. Also $\epsilon_h$ has to be sufficiently small to ensure that our symplectic form is non-degenerate. If $C \subset M$ is a subset of a symplectic manifold $(M, \omega)$ then we say that it is a \textit{stable Hamiltonian hypersurface} if $\omega|_C = \omega_h$. A neighborhood of $C$ is symplectomorphic to its symplectization $(\epsilon_h, \epsilon_h) \times C$. We will call such a neighborhood a \textit{standard neighbourhood}.

An \textit{almost complex structure} $\tilde{J}_C$ is said to be \textit{compatible with the symplectization} $(\epsilon_h, \epsilon_h) \times C$ if

1. it is compatible with the symplectic form $\tilde{\omega}_h$,

Proof of Theorem 6.2. Let $A_\delta$ be the intersection of $A$ with a small ball of radius $\delta$. We resolve $A$ at 0 by blowing up along smooth subvarieties by [Hir63] and take the preimage $\tilde{A}_\delta$ of $A_\delta$ under this resolution map. We suppose that the exceptional divisors $E_1, \cdots, E_l$ are smooth normal crossing.

By Lemma 4.12 there exists a symplectic form $\omega_A$ on $\tilde{A}_\delta$ and a 1-form $\theta_A$ on $\tilde{A}_\delta \setminus \cup_i E_i$ so that:

1. $\omega_A$ is compatible with the natural complex structure which means that $E_1, \cdots, E_l$ are positively intersecting.
2. The wrapping number of $\theta_A$ around $E_i$ is positive for each $i$.
3. There also exists a smooth function $f_A : \tilde{A}_\delta \setminus \cup_i E_i \to \mathbb{R}$ compatible with $\cup_i E_i$ with the property that $df(X_{\theta_A}) > 0$ near $\cup_i E_i$ and so that $(f_A^{-1}(l), \theta_A)|_{f^{-1}(l)}$ is contactomorphic to $(L_A, \xi_A)$ for all sufficiently negative $l$.

The discrepancies $a_i$ of $E_i$ defined at the start of this section (Section 6) coincide with the discrepancies defined in section 2. Our result then follows from Theorem 6.1.

7. \textbf{Appendix A: Stable Hamiltonian Cobordisms}

Most material from this section is taken from [BEH03] and from [CV10]. We need to cover this material as we have to deal with certain compactness results involving holomorphic curves in stable Hamiltonian structures whose associated Reeb flow is not necessarily Morse Bott. Having said that the structures we are interested in are pseudo Morse Bott submanifolds although we will not need this condition here. A \textit{stable Hamiltonian structure} on a manifold $C$ of dimension $2n-1$ is a pair $(\omega_h, \alpha_h)$ where $\omega_h$ is a closed 2-form and $\alpha_h$ is a 1-form with the property that $\alpha_h \wedge \omega_h^{n-1}$ is a volume form. We also require that $\ker(\omega_h) \subset \ker(d\alpha_h)$. Here for any differential form $\gamma$, $\ker(\gamma)$ means the set of vectors $V$ so that $i_V \gamma = 0$. From this we can construct the \textit{Reeb vector field} $R$ which is the unique vector field $R$ on $C$ satisfying $i_R \omega_h = 0$ and $i_R \alpha_h = 1$. The condition $R \in \ker(\omega_h) \subset \ker(\alpha_h)$ and $i_R \alpha_h = 1$ ensure that the flow of $R$ preserves $\omega_h$ and $\alpha_h$. An example of a stable Hamiltonian structure is a contact structure $\lambda$ where $\alpha_h = \lambda$ and $\omega_h = d\lambda$.

An almost complex structure $J_C$ on the hyperplane bundle $\ker(\alpha_h)$ is \textit{compatible} with $(\omega_h, \alpha_h)$ if it is compatible with the symplectic structure $\omega_h|_{\ker(\alpha_h)}$ (i.e. for all non-zero $V, W \in \ker(\alpha_h)$ we have $\omega_h(V, J_V) > 0$ and $\omega_h(JV, JW) = \omega_h(V, W)$). We can define an almost complex structure $\tilde{J}_C$ on $\mathbb{R} \times C$ in the following way: For vectors of the form $(0, V)$ where $V \in \ker(\alpha_h)$ it is defined by $\tilde{J}_C(V) := J_C(V)$. Also $\tilde{J}_C(\frac{d}{\partial r_C}) = R$ and $\tilde{J}_C(R) = -\frac{\partial}{\partial r_C}$ where $r_C$ parameterizes $\mathbb{R}$. We say that $\tilde{J}_C$ is a \textit{cylindrical almost complex structure} associated to $J_C$. A \textit{symplectization} of $(\omega_h, \alpha_h)$ is the product $(\epsilon_h, \epsilon_h) \times C$ for $\epsilon_h > 0$ small with symplectic form $\tilde{\omega}_h := \omega_h + r_C d\alpha_h + r_C \omega_h$. Here by abuse of notation we have identified $\alpha_h$ with $\pi_C^* \alpha_h$ where $\pi_C : (\epsilon_h, \epsilon_h) \times C \to C$ is the natural projection. Also $\epsilon_h$ has to be sufficiently small to ensure that our symplectic form is non-degenerate. If $C \subset M$ is a subset of a symplectic manifold $(M, \omega)$ then we say that it is a \textit{stable Hamiltonian hypersurface} if $\omega|_C = \omega_h$. A neighborhood of $C$ is symplectomorphic to its symplectization $(\epsilon_h, \epsilon_h) \times C$. We will call such a neighborhood a \textit{standard neighbourhood}.

An almost complex structure $\tilde{J}_C$ is said to be \textit{compatible with the symplectization} $(\epsilon_h, \epsilon_h) \times C$ if

1. it is compatible with the symplectic form $\tilde{\omega}_h$,
If \( \tilde{J}_C \) is only defined on \( I \times C \) where \( I \subset (\varepsilon_h, \varepsilon_h) \) then we also say it is compatible with the partial symplectization \( I \times C \) if it satisfies the above properties. The smooth positive function \( f \) may only have domain given by our subset \( I \subset (\varepsilon_h, \varepsilon_h) \).

Let \((M, \omega)\) be a symplectic manifold whose boundary is a disjoint union \( \partial_- M \sqcup \partial_+ M \). Suppose that we have a stable Hamiltonian structure \((\omega^+_h, \alpha^+_h)\) on \( \partial_+ M \). We say that \((M, \omega)\) is a stable Hamiltonian cobordism from \( \partial_- M \) to \( \partial_+ M \) if there is a neighborhood of \( \partial_+ M \) diffeomorphic to \((-\varepsilon_h, 0] \times \partial_+ M \) with \( \omega = \omega_h + r_+ d\alpha^+_h + dr_+ \wedge \alpha_h \) and a neighborhood of \( \partial_- M \) diffeomorphic to \([0, \varepsilon_h) \times \partial_+ M \) with \( \omega = \omega_h + r_- d\alpha^+_h + dr_- \wedge \alpha_h \). Here \( \varepsilon_h > 0 \) is a small constant and \( r_+ \) parameterizes \((-\varepsilon_h, 0] \) and \( r_- \) parameterizes \([0, \varepsilon_h) \). We will call the neighbourhoods \((-\varepsilon_h, 0] \times \partial_+ M \) and \([0, \varepsilon_h) \times \partial_- M \) the positive and negative cylindrical ends of \( M \).

An almost complex structure \( J \) is compatible with \( M \) if it is compatible with the symplectic form and it is equal to almost complex structures \( \tilde{J}_\pm \) compatible with the partial symplectizations \([0, \varepsilon_h) \times \partial_- M \) and \((-\varepsilon_h, 0] \times \partial_+ M \) on the cylindrical ends. An almost complex structure \( J \) defined on the interior of \( M \) is said to be compatible with the completion of \( M \) if it is compatible with the symplectic form and there are smooth maps \( \phi_+: (-\varepsilon_h, 0) \to \mathbb{R}, \phi_-: (0, \varepsilon_h) \to \mathbb{R} \) so that:

1. On the cylindrical ends \( J \) is equal to almost complex structures \( \tilde{J}_\pm \) compatible with the partial symplectizations \((0, \varepsilon_h) \times \partial_- M \) and \((-\varepsilon_h, 0) \times \partial_+ M \).
2. \( \phi_+ \) is an orientation preserving diffeomorphism onto its image \((0, \infty) \) and \( \phi_- \) is an orientation preserving diffeomorphism onto its image \((-\infty, 0) \).
3. There is a compatible almost complex structure \( J_+ \) (resp. \( J_- \)) on \( \ker(\alpha_+) \) (resp. \( \ker(\alpha_-) \)) with the property that \((s_\tau \circ \phi_+, \text{id}), \tilde{J}_+ \) (resp. \((s_\tau \circ \phi_-, \text{id}), \tilde{J}_- \)) converges in \( C^\infty_{\text{loc}} \) to \( \tilde{J}_+ \) in \( \mathbb{R} \times \partial_+ M \) as \( r \to -\infty \) (resp. \( r \to \infty \)) where \( s_\tau : \mathbb{R} \to \mathbb{R} \) sends \( x \) to \( x + r \).

Let \( u : \Sigma \to \mathbb{R} \times C \) be any smooth map where \( \Sigma \) is some surface. We define \( E_{\omega_h}(u) \) to be \( \int_{\Sigma}(\pi_C \circ u)^* \omega_h \) where \( \pi_C \) is the projection \( \mathbb{R} \times C \to C \). Let \( r \) be the coordinate parameterizing \( \mathbb{R} \) in \( \mathbb{R} \times C \). We define \( E_{\omega_h}(u) := \sup_{\phi \in \mathcal{C}} \int_{\Sigma}(\phi \circ r \circ u)^*(dr \wedge d\alpha_h) \) where \( \mathcal{C} \) is the set of compactly supported smooth maps \( \phi : \mathbb{R} \to \mathbb{R} \) whose integral is \( 1 \). We define \( E_C(u) := E_{\omega_h}(u) + E_{\alpha_h}(u) \). Now suppose \( J_M \) is an almost complex structure compatible with the completion of \( M \) and \( u' : \Sigma' \to M \) a \( J_M \) holomorphic curve where \( \tilde{M} \) is the interior of \( M \). This means that there are smooth maps \( \phi_+: (-\varepsilon_h, 0) \to \mathbb{R}, \phi_-: (0, \varepsilon_h) \to \mathbb{R} \) satisfying the conditions [2], [3] above. Define \( N_+M := (-\varepsilon_h, 0) \times \partial_+ M \) and \( N_-M := (0, \varepsilon_h) \times \partial_- M \). We define \( E_{\text{int}}(u') := \int_{(u')^{-1}(\{0\} \times \partial_+ M \cup \{\varepsilon_h\} \times \partial_- M \}) \omega_M \). Let \( u'_- := u'|_{(u')^{-1}(\{0\} \times \partial_- M \)} \). We define \( E_{\text{int}}(u') := E_{\partial_- M}(\phi_- \circ \text{id}_{\partial_+ M} \circ u'_-) \). We have a similar definition of \( E_{\text{int}}(u') \). We define \( E_{\text{int}}(u') := E_{\text{int}}(u') + E_-(u') + E_+(u') \) and we will call this the energy of \( u' \).

We now need to understand stretching the neck. Suppose we have a symplectic manifold \((S, \omega_S)\). Let \( C \subset S \) be a stable Hamiltonian hypersurface with stable Hamiltonian structure \((\omega_h, \alpha_h)\). A small neighborhood of \( C \) is symplectomorphic to the symplectization \((-\varepsilon_h, \varepsilon_h) \times C \) for small enough \( \varepsilon_h \). We define \( S_{\text{split}} \) to be the manifold with boundary obtained from \( S \) by gluing \( S \setminus C \) with the disjoint union \([0, \varepsilon_h) \times C \sqcup (-\varepsilon_h, 0] \times C \) along \((0, \varepsilon_h) \times C \sqcup (-\varepsilon_h, 0] \times C \subset S \setminus C \). Here \( S_{\text{split}} \) is a cobordism of stable Hamiltonian structures with \( \partial_- S_{\text{split}} = \partial_+ S_{\text{split}} = C \). Note that the interior \( \tilde{S}_{\text{split}} \) of \( S_{\text{split}} \) is diffeomorphic in a canonical way to \( S \setminus C \). It is possible
to choose a sequence of almost complex structures $J_i$ on $S$ compatible with the symplectic form, a compatible almost complex structure $J_C$ on $S$ preserving diffeomorphisms $\phi_i : (-\epsilon_h, \epsilon_h) \rightarrow (-D_i, D_i)$ so that:

1. $\phi_i(0) = 0$ and $\phi_i' = 1$ near $-D_i$ and $D_i$. Also $D_i$ is strictly increasing and $D_i \rightarrow \infty$ as $i \rightarrow \infty$.
2. $J_i \mid (-\epsilon_h, \epsilon_h) \times C$ is compatible with the symplectization $(-\epsilon_h, \epsilon_h) \times C$.
3. $(\phi_i, \iota_{DC})_\ast J_i$ viewed as an almost complex structure in $(-D_i, D_i) \times C \subset \mathbb{R} \times C$ converges in $C^\infty_{\text{loc}}$ to $\tilde{J}_C$ as $i \rightarrow \infty$. In the region $(-h^{\frac{\epsilon}{2}}, h^{\frac{\epsilon}{2}}) \times C$ we will assume that the restriction of $J_i$ to $\ker(\pi^C \ast \alpha_h) \cap \ker(dr_C)$ where $\pi_C$ is the projection to $C$ is uniformly convergent.
4. There is an almost complex structure $J_{\text{split}}$ compatible with the completion of $S_{\text{split}}$ and an open neighborhood $NC$ of $C$ so that $J_i \mid NC \times C$ converges in $C^\infty_{\text{loc}}$ to $J_{\text{split}} \mid NC \setminus C$.

The family $J_i$ of almost complex structures stretch the neck along $C$ if they satisfy the above properties. We will give such a construction now. Note that we do not have any constraints on $J_i$ away from $(-\epsilon_h, \epsilon_h) \times C$ although when we prove a compactness result later we will assume that $J_i$ converges $C^\infty_{\text{loc}}$ to an almost complex structure in one region inside $S$ but there will be no constraints in other regions of $S$. We need to do this in order to prove certain symplectic manifolds are in fact GW triples.

Suppose we have a compatible almost complex structure $J_S$ on $S$ which is compatible with the symplectization $(-\epsilon_h, \epsilon_h) \times C$. Then we can construct a sequence of almost complex structures $J^i_S$ stretching the neck along $C$ as follows: Let $J_C$ be an almost complex structure on $\ker(\alpha_h)$ given by $J_C \mid TC \cap \ker(\alpha_h)$ and let $D_i > \max(h^{\frac{\epsilon}{2}}, \epsilon_h, 1)$ be a sequence satisfying $D_{i+1} > 3D_i$. We choose a sequence of diffeomorphisms $\phi_i : (-\epsilon_h, \epsilon_h) \rightarrow (-D_i, D_i)$ satisfying:

1. $\frac{\partial \phi_i(x)}{\partial x} = 1$ near $\pm D_i$.
2. $\phi_i(x) = \frac{D_i x}{\epsilon_h}$ for $x \in \left[ -\frac{\epsilon_h}{3D_i}, \frac{\epsilon_h}{3D_i} \right]$.
3. $\phi_i(x) = \phi_{i-1}(x) + D_i - D_{i-1}$ for $x \geq \frac{\epsilon_h}{2D_i}$ and $\phi_i(x) = \phi_{i-1}(x) - D_i + D_{i-1}$ for $x \leq -\frac{\epsilon_h}{2D_i}$.

By the definition of $J_S$ we have a function $f_S : (-\epsilon_h, \epsilon_h) \rightarrow (0, \infty)$ so that if $R$ is the Reeb vector field of $(\omega_h, \alpha_h)$ which we extend in the natural way to $(-\epsilon_h, \epsilon_h) \times C$ then $J_S(\frac{\partial}{\partial r_C}) = f_S(r_C) R$ and $J_S(R) = -\frac{1}{f_S(r_C)} \frac{\partial}{\partial r_C}$ where $r_C$ is the natural coordinate parameterizing $(-\epsilon_h, \epsilon_h)$. Let $\pi_C : (-\epsilon_h, \epsilon_h) \times C \rightarrow C$ be the natural projection map. Inside $(-\epsilon_h, \epsilon_h) \times C$ we define $J_i(V) := J_S(V)$ for $V \in \ker(\pi^C \ast \alpha_h) \cap \ker(dr_C)$. Let $\rho : (-\epsilon_h, \epsilon_h) \rightarrow (0, \infty)$ be a smooth function with $\rho(x) = f_S(x)$ for $|x| \leq \frac{\epsilon_h}{2}$ and $\rho(x) = 1$ for $x$ near $\pm \epsilon_h$. Also we define $J_i(\frac{\partial}{\partial r_C}) := \frac{f_S(r_C) \phi'_{r_C}}{\rho(r_C)} R$ and $J_i(R) := -\frac{\rho(r_C)}{f_S(r_C) \phi'(r_C)} \frac{\partial}{\partial r_C}$. We also define $J_i$ to be any compatible almost complex structure on $S \setminus (-\epsilon_h, \epsilon_h) \times C$. We define a diffeomorphism $\phi_+ : (0, \epsilon_h) \rightarrow (0, -\infty)$ so that for $x \in \left[ -\frac{\epsilon_h}{2D_i+1}, \frac{\epsilon_h}{2D_i+1} \right]$, $\phi_+(x) := \phi_i(x) - D_i$. We also assume that $\phi_+(x) = \phi_0 - D_0$ for $x > \frac{\epsilon_h}{2D_0}$. Similarly we define a diffeomorphism $\phi_- : (-\epsilon_h, 0) \rightarrow (0, \infty)$ so that for $x \in \left[ -\frac{\epsilon_h}{2D_i+1}, \frac{\epsilon_h}{2D_i+1} \right]$, $\phi_-(x) := \phi_i(x) + D_i$. Again we define $\phi_-(x) := \phi_0 + D_0$ for $x < -\frac{\epsilon_h}{2D_0}$. We construct an almost complex structure $J_S^{\text{split}}$ on $S \setminus C$ so that $J_S^{\text{split}} = J_S$ on $S \setminus (-\epsilon_h, \epsilon_h) \times C$ and in the region $S \setminus (-\frac{\epsilon_h}{2D_i}, \frac{\epsilon_h}{2D_i})$ it is equal to $J_i$. This definition makes sense because $J_i = J_k$ in this region for all $k \geq i$. Under the identification $S \setminus C = \hat{S}_0^{\text{split}}$ we get that $J_S^{\text{split}}$ is compatible with the completion of $S^{\text{split}}$. The sequence of almost complex structures $J_i$ is a sequence stretching the neck along $C$ using the functions $\phi_i$ and if we want we can choose $J_i$ outside $(-\epsilon_h, \epsilon_h) \times C$ so that $J_i C^\infty_{\text{loc}}$ converges in $S \setminus C$ to $J_S^{\text{split}}$. 
The main goal of this next section is to prove a compactness result coming from neck stretching. First we need a compactness result from [Fis11] which we state here as a Theorem (the compactness result from [Fis11] is much stronger but we do not need the full force of the theorem here). Let \( \Omega \) be a non-degenerate (not necessarily closed) two form on a manifold \( B \) and \( J \) an almost complex structure. We say that \( (\Omega, J) \) is an almost Hermitian structure if \( J \) is compatible with \( \Omega \). A nodal curve with boundary is given by a closed subset in the Euclidean topology of a complex algebraic curve with nodal singularities with the property that the boundary (i.e. the closure minus the interior) is a 1-dimensional submanifold of the smooth part of this complex curve. A \( J \) holomorphic map from a nodal curve \( \Sigma \) with boundary is a continuous map \( \Sigma \to B \) which is smooth and \( J \) holomorphic away from these singularities.

Suppose we have a sequence of almost Hermitian structures \((\Omega_i, J_i)\) \( C^\infty \) converging to \((\Omega_\infty, J_\infty)\) in a manifold \( B \). Suppose that we have:

1. a sequence of \( J_i \) holomorphic curves \( u_i : \Sigma_i \to B \) and a nodal \( J_\infty \) holomorphic curve \( u_\infty : \Sigma_\infty \to B \).
2. a smooth surface \( \tilde{\Sigma} \) and smooth embeddings \( v_i : \tilde{\Sigma} \to \Sigma_i \), a continuous surjection \( v_\infty : \tilde{\Sigma} \to \Sigma_\infty \).
3. a compact set \( K \subset B \).

so that

1. If \( \tilde{\Sigma}' := \tilde{\Sigma} \setminus v_\infty^{-1}(\Sigma^{\text{sing}_\infty}) \) where \( \Sigma^{\text{sing}_\infty} \) is the set of nodal points of \( \Sigma_\infty \) then \( v_\infty^{\text{non-sing}} := v_\infty|_{\tilde{\Sigma}'} \) is a diffeomorphism onto its image \( \Sigma_\infty \setminus \Sigma^{\text{sing}_\infty} \).
2. \( u_i \circ v_i \) \( C^0 \) converges to \( u_\infty \circ v_\infty \) and \( u_i \circ v_i|_{\tilde{\Sigma}'} \) \( C^\infty_{\text{loc}} \) converges to \( u_\infty \circ v_\infty|_{\tilde{\Sigma}'} \).
3. \( u_i \circ v_i \) and \( u_\infty \circ v_\infty \) map the boundary of \( \tilde{\Sigma} \) to \( B \setminus K \)

then we say that \( u_i \) converges in a Gromov sense to \( u_\infty \) near \( K \).

The following theorem is proven by Fish in [Fis11].

**Theorem 7.1.** Let \( B \) be a compact manifold with boundary, \((\Omega_i, J_i)\) a sequence of almost Hermitian structures \( C^\infty \) converging to a compatible almost Hermitian structure \((\Omega_\infty, J_\infty)\). Let \( u_i : \Sigma_i \to B \) be a sequence of smooth genus 0 \( J_i \) holomorphic curves whose boundary maps to \( \partial B \) with \( \int_{\Sigma_i} u_i^* \Omega_i \) bounded above by a fixed constant independent of \( i \) and let \( K \) be a compact subset of the interior of \( B \). Then after passing to a subsequence there is a genus 0 compact nodal \( J_\infty \) holomorphic curve \( u_\infty : \Sigma_\infty \to B \) so that \( u_i \) converges in a Gromov sense to \( u_\infty \) near \( K \).

The following result is basically a much weaker version of the main result of [BEH+03] except that we allow our stable Hamiltonian structure to be degenerate. Basically this compactness result only remembers the top level of the holomorphic building.

**Proposition 7.2.** Let \((S, \omega_S)\) be a connected symplectic manifold and \( C \) a stable Hamiltonian hypersurface in \( S \) which means that its symplectization \((-\epsilon_h, \epsilon_h) \times C \) symplectically embeds in \( S \) identifying \( C \) with \( \{0\} \times C \). We suppose that \( S \setminus C \) is a disjoint union \( \hat{S}_+ \sqcup \hat{S}_- \) where \( \hat{S}_+ \) contains \((0, \epsilon_h) \times C \). We define \( S_+ \) to be the union of \( \hat{S}_+ \) and \((0, \epsilon_h) \times C \) along \((0, \epsilon_h) \times C \) which is a stable Hamiltonian coisotropic with \( \partial_- S_+ = C \). Let \( J_i' \) be a sequence of almost complex structures stretching the neck along \( C \) using our symplectization \((-\epsilon_h, \epsilon_h) \times C \) with the additional property that \( J_i'|_{\hat{S}_+} \) converges in \( C^\infty_{\text{loc}} \) to an almost complex structure \( J_+ \) compatible with the completion of \( S_+ \). Let \( \omega_i \) be a sequence of symplectic structures in the same cohomology class equal to \( \omega_S \) inside \((-\epsilon_h, \epsilon_h) \times C \cup \hat{S}_+ \) \( C^\infty \) converging to \( \omega_S \) and \( J_i \)
a sequence of \( \omega_i \) compatible almost complex structures equal to \( J_i \) inside \((-\epsilon_h, \epsilon_h) \times C \cup \hat{S}_+\). Let \( u_i : \mathbb{P}^1 \to S \) be a sequence of genus zero \( J_i \) holomorphic curves so that:

1. their images stay inside a fixed compact set \( \kappa \subset S \).
2. \( \int_{\mathbb{P}^1} u_i^* \omega \) is bounded above by a fixed constant \( E \).

Then there exists a proper \( J_+ \) holomorphic curve \( u_\infty : \Sigma_\infty \to \hat{S}_+ \) so that:

1. \( \Sigma_\infty \) is a genus 0 nodal \( J_+ \) holomorphic curve without boundary mapping to \( \kappa \cap \hat{S}_+ \).
2. If \( N \subset S \) is a neighborhood of the image of \( u_\infty \) inside \( S \) and \( \epsilon_h > \eta > 0 \) then for all \( i \), the image of \( u_i \) is contained in \( \hat{S}_- \cup ([0, \eta) \times C) \cup N \) for some \( j \geq i \).
3. Its energy satisfies \( E_{\hat{S}_+}(u_\infty) < \infty \) and \( \int_{\Sigma_\infty} u_\infty^* \omega_S \leq E \).

Suppose \( Q \) is a properly embedded codimension 2 submanifold \( Q \subset \hat{S}_+ \) whose closure in \( S \) is disjoint from \( C \). Then each \( u_i \) for \( i \geq 1 \) has a positive intersection number with \( Q \) if and only if \( u_\infty \) also has a positive intersection number. If \( (u_i)_*((\mathbb{P}^1)) \cdot [Q] = 1 \) for all \( i \) then the intersection number of \( u_\infty \) with \( Q \) is also 1.

**Proof.** of Proposition 7.2. We proceed in 3 steps. In the first step we use the compactness result Theorem 7.1 to construct our holomorphic curve \( u_\infty \). In the second step we show that the energy of \( u_\infty \) is bounded. In the third step we prove the remaining parts of the Proposition.

**Step 1:** Define \( S^k_k := \hat{S}_+ \setminus (0,\frac{\epsilon_k}{k}) \times C \subset S \). We define \( \Sigma^i_{k,+} := u_i^{-1}(S^k_k) \). Maybe after perturbing \( \epsilon_h \) slightly, we can assume that \( \Sigma^i_{k,+} \) is a manifold with boundary. For each \( k \) we have by Theorem 7.1 that a subsequence \( u_{k,j} \) of \( u_i|_{\Sigma^i_{k+1,+}} : \Sigma^i_{k+1,+} \to S^k_{k+1} \) converges in a Gromov sense as \( j \to \infty \) to a \( J_+ \) holomorphic nodal curve \( u_{k,\infty} : \Sigma_{k,\infty} \to S^k_{k+1} \) near \( S^k_k \). By an inductive argument we can assume \( \{i_k,j\} : \{i,\infty\} \subset \{i,\infty\} \) for all \( k \leq k' \). We then get that the diagonal sequence \( u_{i,j} \) has the property that for each \( k \in \mathbb{N} \), \( u_{i,j} : \Sigma^i_{k,+} \to S^k_{k+1} \) converges in a Gromov sense to \( u_{k,\infty} \) near \( S^k_k \). After replacing \( u_i \) with a subsequence we will assume that \( u_i = u_{i,j,k} \) for all \( j \). The curve \( u_{k,\infty} \) might have multiply covered components and so let \( \overline{u}_{k,\infty} : \Sigma_{k,\infty} \to S^k_{k+1} \) be the respective holomorphic curve without multiply covered components whose image is the same. Let \( \overline{\Sigma}_{k,\infty}^{\text{non-sing}} \subset \Sigma_{k,\infty} \) be the maximal open subset of the set of nonsingular points with the property that \( v_k := \overline{u}_{k,\infty} |_{\overline{\Sigma}_{k,\infty}^{\text{non-sing}}} \) is a diffeomorphism onto its image. Such an open set has a finite complement. For \( k \leq k' \) we have that the image of \( v_k \) is contained in the image of \( v_{k'} \). Hence we can construct a new Riemann surface \( \Sigma_{k,\infty}^{\text{non-sing}} \) without boundary by gluing \( \overline{\Sigma}_{k,\infty}^{\text{non-sing}} \) to \( \overline{\Sigma}_{k',\infty}^{\text{non-sing}} \) using the maps \( v_k^{-1} \circ v_{k'} \) for all \( k \leq k' \). The maps \( v_k \) also glue and give us a \( J_+ \) holomorphic map \( v : \Sigma_{k,\infty}^{\text{non-sing}} \to \hat{S}_+ \). By using the removable singularity theorem we then get \( \Sigma_{k,\infty}^{\text{non-sing}} \) extends to a nodal Riemann curve \( \Sigma_\infty^{\text{non-sing}} \) containing \( \Sigma_{k,\infty}^{\text{non-sing}} \) and a \( J_+ \) holomorphic map \( u_\infty : \Sigma_\infty \to \hat{S}_+ \) extending \( v \). Such a map is proper.

**Step 2:** We now need to prove our energy bounds. The bound \( \int_{\Sigma_\infty} u_\infty^* \omega_S \leq E \) follows from the fact that \( J_+ \) is compatible with \( \omega_i \) and the fact that \( \int_{\mathbb{P}^1} u_i^* \omega \leq E \) for all \( i \in \mathbb{N} \). We now need to show \( E_{\hat{S}_+}(u_\infty) < \infty \). Let \( C_{(a,b)} := (a,b) \times C \subset (0,\epsilon_h) \times C \subset S \) where \( 0 < a < b < \epsilon_h \) and let \( C_a := \{a\} \times C \). Let \( \Sigma^i_{(a,b)} := u_i^{-1}(C_{(a,b)}) \) where \( 0 < a < b < \epsilon_h \) and \( i \in \mathbb{N} \cup \{\infty\} \). We define \( \Sigma^i_{a} := u_i^{-1}(C_a) \).
From now on in \( i \in \mathbb{N} \) (i.e. \( i \neq \infty \)). We let \( r_C \) be the variable parameterizing \((-\epsilon_h, \epsilon_h)\) in \((-\epsilon_h, \epsilon_h) \times C\). Let \( \delta > 0 \) be small. Let \( \psi_\delta : (-\epsilon_h, \epsilon_h) \to \mathbb{R} \) be a smooth function whose derivative is non-negative and so that \( \psi(x) = x \) for \( x \notin (-\delta, \delta) \). Then we define \( \omega_i, \psi_\delta \) to be equal to \( \omega_i \) outside \((-\epsilon_h, \epsilon_h) \times C\) and equal to \( \omega_i + \psi_\delta \) inside \((-\epsilon_h, \epsilon_h) \times C\). This is in the same cohomology class as \( \omega_i \). For \( i \) large enough and for \( \delta > 0 \) small enough we then have that \( \omega_i, \psi_\delta \in \mathcal{V}, J_i \mathcal{V} \geq 0 \) for all vectors \( V \). The reason why this is true is because the restriction of \( \omega_i \) to the bundle \( \ker(d\gamma) \cap \ker(\alpha_h) \) is uniformly convergent to some symplectic form with compatible almost complex structure on this bundle in the region \((-\frac{\epsilon_h}{2}, \frac{\epsilon_h}{2}) \times C\).

From now on we fix \( \delta > 0 \) small enough and pass to a subsequence of \( u_i \)'s to ensure the semi-positivity condition above holds for all \( i \in \mathbb{N} \). Then \( \int_{\Sigma}^i u_i^* \omega_i, \psi_\delta \geq 0 \) and \( \int_{\bar{\Sigma}}^i u_i^* \omega_i, \psi_\delta \geq 0 \) for all \( 0 \leq a \leq b \leq \delta \), and so combined with \( \leq \int_{\bar{\Sigma}}^i u_i^* \omega_i, \psi_\delta \leq E \) we get \( 0 \leq \int_{\bar{\Sigma}}^i u_i^* \omega_i, \psi_\delta \leq E \).

Now let \( \Psi_1 : (-\epsilon_h, \epsilon_h) \to \mathbb{R} \) be a smooth function with \( \Psi_1(x) = x \) for \( x \notin (-\delta, \delta) \), \( \Psi_1 \geq 0 \) and \( \Psi_1(x) = 0 \) for \( x \in \left(\frac{\delta}{2}, \frac{\delta}{2}\right) \). We then have \( 0 \leq \int_{\Sigma}^i u_i^* \omega_i, \psi_1 \leq E \) for all \( 0 \leq a \leq \delta/2 \) and so \( 0 \leq \int_{\Sigma}^i u_i^* \omega_i \leq E \). In a similar way, by considering the function \( \Psi_2 : (-\epsilon_h, \epsilon_h) \to \mathbb{R} \) with \( \Psi_2(x) = x \) for \( x \notin (-\delta, \delta) \), \( \Psi_2 \geq 0 \) and \( \Psi_2(x) = \frac{\delta}{2} \) for \( x \in \left(\frac{\delta}{2}, \frac{\delta}{2}\right) \) we can show \( 0 \leq \int_{\Sigma}^i u_i^* (\omega_i + \frac{\delta}{2} \delta \alpha_h) \leq E \) for all \( 0 \leq a \leq \delta/2 \). The inequalities \( 0 \leq \int_{\Sigma}^i u_i^* \omega_i \leq E \) and \( 0 \leq \int_{\Sigma}^i u_i^* \left(\omega_i + \frac{\delta}{2} \delta \alpha_h\right) \leq E \) then tell us that \( \int_{\Sigma}^i u_i^* \delta \alpha_h \leq \frac{2E}{\delta} \) for all \( a \).

Because the curves \( u_i \) converge to \( u_\infty \) uniformly near \( C_\delta/2 \) we can assume that \( \int_{\Sigma}^i u_i^* \alpha_h \leq K_1 \) for some \( K_1 > 0 \) independent of \( i \). By Stokes' theorem we have

\[
\int_{\Sigma}^i u_i^* \delta \alpha_h = \int_{\Sigma}^i u_i^* \alpha_h - \int_{\Sigma}^i u_i^* \alpha_h
\]

Hence we get \( \int_{\Sigma}^i u_i^* \alpha_h \leq \frac{E}{\delta} + K_1 \) for all \( i \in \mathbb{N} \). This implies that \( \int_{\Sigma}^i u_i^* \alpha_h \leq \frac{E}{\delta} + K_1 \) which in turn implies that \( E_{S_+}(u_\infty) \) is finite.

**Step 3:** We will now prove the remaining parts of this Proposition. The set \( \kappa \setminus (\hat{S}_- \cup (0, \delta) \times C) \) is a compact subset of \( \hat{S}_+ \). Now the \( u_i \)'s converge in a Gromov sense to some holomorphic curve near this compact subset whose image is contained in the image of \( u_\infty \). So for \( i \) large enough, we have that the image of \( u_i \) is contained in \( \hat{S}_- \cup (0, \delta) \times C \cup N \).

Suppose \( Q \) is a properly embedded codimension 2 submanifold \( Q \subset \hat{S}_+ \) whose closure in \( S \) disjoint from \( C \). Then each \( u_i \) for \( i \gg 1 \) has a positive intersection number with \( Q \) if and only if \( u_\infty \) also has a positive intersection number due to the fact that \( u_i \) converges in a Gromov sense to a curve whose image is equal to the image of \( u_\infty \) near \( \kappa \cap Q \).

Now suppose \( (u_i)_\ast ([\mathbb{P}^1]) \cdot [Q] = 1 \) for all \( i \). Then if (for a contradiction), \( u_\infty \) has intersection greater than 1 with \( Q \) then \( u_i \) also has an intersection number greater than 1 because \( u_i \) converges in a Gromov sense to a \( J_+ \) holomorphic curve whose image is some branched cover of \( u_\infty \) near \( \kappa \cap Q \). Hence the intersection number of \( u_\infty \) with \( Q \) is also 1. \( \square \)

Now we need a result telling us how a holomorphic curve inside the interior of a cobordism of stable Hamiltonian structures behaves near the boundary. Again this result is weaker than the result in \cite{BEH+03} except that we allow possibly degenerate stable Hamiltonian structures.
Lemma 7.3. Suppose we have a symplectic cobordism of stable Hamiltonian structures $(M, \omega_M)$ with $\partial_+ M = \emptyset$ and let $J$ be an almost complex structure compatible with the completion of $M$. Let $u : \Sigma \to M$ be a proper $J$ holomorphic curve with finite energy. Then for any sequence $\sigma_i \in \Sigma$ with $u(\sigma_i)$ converging to a point on some Reeb orbit $R$. If $\Sigma$ is smooth and connected of genus $0$ then $\Sigma$ is biholomorphic to $\mathbb{P}^1$ minus a finite number of points.

Let $Q$ be a properly embedded $J$ holomorphic hypersurface $Q$ of $M$ not intersecting $\partial_- M$ and let $\theta_M$ be a $1$-form on $M \setminus Q$ satisfying $d\theta_M = \omega_M|_{M \setminus Q}$ and $i_X(\theta_M|_{\partial_+ M}) = 1$ where $X$ is the Reeb vector field on $\partial_-$ $M$. Let $\eta$ be the wrapping number of $\theta_M$ around $Q$ and let $\int(Q \cdot u)$ be the intersection number between $u$ and $Q$. Then the length of $R$ is bounded above by $-(\int(Q \cdot u)\eta - \int_\Sigma u^* \omega_M)$.

There are similar results when $\partial_- M$ is non-empty but we decided to omit this so the statement and the proof become less cluttered. Note that if the Reeb orbit $R$ from this Lemma is non-degenerate then by [BEH+03] Lemma 5.1 there is some holomorphic subset $(-\infty, 1] \times S^1 \subset \Sigma$ with coordinates $(s, t)$ so that this region extends continuously to $[-\infty, 1] \times S^1$ with $u(-\infty, t)$.

Proof. of Lemma 7.3 We prove this in 3 steps. In Step 1 we find our Reeb orbit. In Step 2 we construct our bound for the length of our Reeb orbit $R$. In Step 3 we show that the domain of our curve is $\mathbb{P}^1$ minus a finite number of points.

Step 1: Because $\partial_- M$ is compact we can, after passing to a subsequence, ensure that $u(\sigma_i)$ converges to some point $x \in \partial_- M$. Because $J$ is compatible with the completion of $M$ we have smooth embeddings $\Phi_z : (0, \epsilon_h) \times \partial_- M \to \mathbb{R} \times \partial_- M$ defined by a pair $(\tau \circ \phi_z, id_{\partial_- M})$. Here $\phi_z : (0, \epsilon_h) \to (-\infty, 0)$ is a diffeomorphism, $\tau$ is the translation map sending $x \in \mathbb{R}$ to $x + r$ with the property that there is a cylindrical almost complex structure $\tilde{J}_- \subset \mathbb{R} \times \partial_- M$ so that $(\Phi_z)_* J$ converges in $C^\infty$ to $\tilde{J}_-$ as $r$ tends to $\infty$ for some $J_-$ compatible with the stable Hamiltonian structure on $\partial_- M$. After passing to a subsequence again we will assume that $u(\sigma_i) = (\phi_z^{-1}(x_i)) \in (0, \epsilon_h) \times \partial_- M$ for some $x_i \in (-\infty, 0)$, $y_i \in \partial_- M$ where $x_i \to -\infty$ and $y_i \to y_\infty \in \partial_- M$.

Fix a small constant $\Delta > 0$ and consider the interval $I_i := [x_i - \Delta, x_i + \Delta] \times \partial_- M$. Let $T_i : I_i \to [-\Delta, \Delta] \times \partial_- M$ be the natural translation map and define $J_i$ to be the pushforward $((T_i)_*(\Phi_\partial M)_* J)|_{[-\Delta, \Delta] \times \partial_- M}$. These almost complex structures $C^\infty$ converge to $\tilde{J}_-$. If $\pi_{\partial_- M} : \mathbb{R} \times \partial_- M \to \partial_- M$ is the natural projection map then by abuse of notation we write $\alpha_- = \pi_{\partial_- M} \alpha_-$ and $\omega_h = \pi_{\partial_- M} \omega_h$ where $(\omega_h, \alpha_-)$ is the stable Hamiltonian structure on $\partial_- M$. We define $\Omega_i := ((T_i)_*(\Phi_\partial M)_* J) + e'dr \wedge \alpha_-$ where $r$ is the coordinate parameterizing the $\mathbb{R}$ factor in $\mathbb{R} \times \partial_- M$. Here $\Omega_i$ may not be closed but it is compatible with $J_i$. We have that $(\Omega_i, J_i) C^\infty$ converges to an almost Hermitian structure $(\Omega_\infty, \tilde{J}_-)$ where $\Omega_\infty = \omega_h + e'dr \wedge \alpha_-$. Define $\Sigma_i := u_i^{-1}(\phi_i^{-1}(I_i))$. Because $E_M(u)$ is bounded we then get that $\int_{\Sigma_i} u^* \Phi_\partial M T_i \Omega_i$ is bounded by a constant independent of $i$. Hence by Theorem 7.1 we have after passing to a subsequence that $T_i \circ \Phi_\partial M \circ u_i$ converges in a Gromov sense to a $\tilde{J}_-$ holomorphic curve $u_\infty : \Sigma \to [-\Delta, \Delta] \times \partial_- M$ near $[-\frac{\Delta}{2}, \frac{\Delta}{2}] \times \partial_- M$.

Let $r_C$ be the coordinate parameterizing $(0, \epsilon_h)$ in $(0, \epsilon_h) \times \partial_- M$. For $r_C$ small enough we get that $r_C d\alpha_- (V, JV) \leq B_1 r_C ||V||^2$ for any non-zero vector tangent to $\ker(\alpha_h)$ where $|| \cdot ||$ is the natural metric for some constant $B_1 > 0$. Using this fact combined with the fact that $J$ is compatible with our stable Hamiltonian cobordism and the fact that $\int_{\Sigma_i} u^* \omega$ is finite, we have that $\int_{\Sigma_i} u^* \Phi_\partial M T_i \omega_h$ tends to $0$. Hence because $u_i$ Gromov converges to $u_\infty$ we then get
that $\int_{\Sigma_{\infty}} u_{\infty}^* \omega_h^- = 0$. Because $\tilde{J}_-$ is cylindrical this means that the projection of the image of $u_{\infty}$ under $\pi_{\partial_M}$ is contained in a Reeb flow line of $(\omega_h^-, \alpha_-)$. For generic small $\delta > 0$, the intersection $u_{\infty}(\Sigma_{\infty}) \cap \{\delta\} \times \partial_\infty M$ is transverse and hence is a one dimensional manifold $\Sigma$ with the property that $\pi_{\partial_\infty}(A)$ is contained in a Reeb flow line. Also the tangent space of $A$ maps to a non-trivial 1-dimensional subspace of $T\partial_\infty M$ under the map $\pi_{\partial_\infty}$. Hence $\pi_{\partial_\infty}(A)$ is a union of Reeb orbits. Hence $u_i(\sigma_i)$ converges to a point in $\pi_{\partial_\infty}(A)$ which contains some Reeb orbit $R$.

Step 2: Now let $Q$ be as in the statement of this Lemma. We will now find an upper bound for the length of our Reeb orbit $R$. Let $A_i := (T_i \circ \Phi_0 \circ u_i)^{-1}(\{\delta\} \times \partial_\infty M)$ for some generic small $\delta > 0$ and for all $i \in \mathbb{N}$. Define $A_{\infty} := u_{\infty}(\{\delta\} \times \partial_\infty M)$. We have that $\int_{A_i} u_i^* \Phi_i^* \alpha_-$ converges to $\int_{A_i} u_i^* \theta_M$ as $i$ tends to infinity. Let $q_1, \ldots, q_l$ be the set of points in $\Sigma$ intersecting $Q$ and let $l_1, \ldots, l_i$ be small loops around these points oriented positively. Now if the intersection multiplicity between $u$ and $Q$ at $q_j$ is $i_j$ then $\int u_i^* \theta_M$ converges to $i_j \eta$ as $l_j$ gets smaller and smaller. Hence $\sum_{j} \int_{l_j} u_i^* \theta_M$ converges to $\int_{\Sigma} (Q, u) \eta$ as the loops $l_j$ shrink. Stokes' theorem tells us that

$$- \sum_{j} \int_{l_j} u_i^* \theta_M - \int_{A_i} u_i^* \theta_M \leq \int_{\Sigma} \omega_M.$$

Hence as $i$ tends to infinity and as the loops $l_j$ get smaller in length around $q_i$, we get $\int_{A_i} u_i^* \theta_M$ converges to some limit $L \leq -((Q, u) \eta) - \int_{\Sigma} \omega_M$. Also $\int_{A_i} u_i^* \theta_M$ converges to $\int_{A_{\infty}} u_{\infty}^* \pi_{\partial_\infty}(\theta_M|_{\partial_\infty M})$. Here $\int_{A_{\infty}} u_{\infty}^* \pi_{\partial_\infty}(\theta_M|_{\partial_\infty M}) = \int_{A_{\infty}} u_{\infty}^* \alpha_-$ because $iR\alpha_- = iR\theta_M$ and so putting everything together we get $\int_{A_{\infty}} u_{\infty}^* \alpha_- \leq -((Q, u) \eta) - \int_{\Sigma} \omega_M$. Because $\pi_{\partial_\infty}(A_{\infty})$ is a union of Reeb orbits we then get that the sum of their lengths is bounded above by $-((Q, u) \eta) - \int_{\Sigma} \omega_M$ and in particular this is an upper bound for the length of $R$.

Step 3: We now need to show that $\Sigma$ is biholomorphic to $\mathbb{P}^1$ minus a finite number of points. We will first show that there exists some $\delta_\Sigma > 0$ and an increasing sequence of compact codimension 0 submanifolds $\kappa_i$ whose union is $\Sigma$ so that we can holomorphically embed the annulus $[-\delta_\Sigma, \delta_\Sigma] \times S^1$ into each connected component of $\Sigma \setminus \kappa_i$ and so that $\partial \kappa_i$ has a bounded number of connected components independent of $i$. Standard Riemann surface theory then implies that $\Sigma$ is biholomorphic to $\mathbb{P}^1$ minus finitely many points. The point here is that the bounded number of connected components condition combined with the fact that we are in genus 0 implies that the topology of $\Sigma$ is bounded and hence can be embedded as an open subset of $\mathbb{P}^1$. The annulus embedding condition then implies that the complement of $\Sigma$ in $\mathbb{P}^1$ has no accumulation points.

Let $I'_i := [-\Delta - i, \Delta - i] \times \partial_\infty M \subset \mathbb{R} \times \partial_\infty M$ and let $T'_i : [-\Delta - i, \Delta - i] \times \partial_\infty M \to [-\Delta, \Delta] \times \partial_\infty M$ be the natural translation map. Let $\Sigma'_i := ((T'_i \circ \Phi_0 \circ u)^{-1})([-\Delta, \Delta] \times \partial_\infty M)$. Now a similar Gromov compactness argument as above tells us that after passing to a subsequence we have that $(T'_i \circ \Phi_0 \circ u|_{\Sigma'_i})$ converges in a Gromov sense to some $\tilde{J}_-$ holomorphic curve $v : \Sigma'_\infty \to [-\Delta, \Delta] \times \partial_\infty M$ near $[-\frac{\Delta}{2}, \frac{\Delta}{2}] \times \partial_\infty M$. Now the annulus $[-\delta_\Sigma, \delta_\Sigma] \times \partial_\infty M$ holomorphically embeds into $v^{-1}((-\frac{\Delta}{2}, \frac{\Delta}{2}) \times \partial_\infty M)$ for some $\delta_\Sigma > 0$. By the nature of Gromov convergence this means such an annulus holomorphically embeds into the interior of $\Sigma'_i$ for $i$ large enough. This means that such an annulus is embedded into $\Sigma$ minus the compact subset $\kappa_i := (\Phi_0 \circ u)^{-1}((-\frac{\Delta}{2}, 0)) \cup u^{-1}(M \setminus ((0, \epsilon_i) \times \partial_\infty M))$. We can choose $\Delta$ generically so that $(T'_i \circ \Phi_0 \circ u)^{-1}((-\frac{\Delta}{2}) \times \partial_\infty M)$ is a submanifold of $\Sigma$. Such a submanifold is equal $\partial \kappa_i$. Again by Gromov convergence this sequence of submanifolds converges to some compact
submanifold of $\Sigma'_\infty$ hence the number of connected components of $\partial \kappa_i$ is bounded. Hence $\Sigma$ is biholomorphic to $\mathbb{P}^1$ minus finitely many points.

\[
\square
\]

8. Appendix B: A Maximum Principle

Let $M$ be a cobordism of stable Hamiltonian structures where $\partial_- M$ has a stable Hamiltonian structure $(\omega_h, \alpha_h)$ and suppose we have a 1 form $\theta$ on $M$ satisfying:

1. $d\theta = \omega_M$. In particular $d\theta|_{\partial_- M} = \omega_h$.
2. We have $X_\theta$ points inwards along $\partial_- M$.
3. $\theta(R) = \alpha_h(R)$ where $R$ is the Reeb vector field of $(\omega_h, \alpha_h)$.

Proposition 8.1. Let $J$ be an almost complex structure compatible with the completion of $M$. Every properly embedded $J$ holomorphic curve has at least one positive point (i.e. there is a sequence of points on this curve converging to $\partial_+ M$). In particular $\partial_+ M \neq \emptyset$.

The proof of this proposition borrows its main idea from [AST0][Lemma 7.2].

Proof. of Proposition 8.1 Let $u : \Sigma \to M$ be a properly embedded $J$ holomorphic curve with no positive ends. We will create a contradiction by showing that $\int_{\Sigma} u^* \omega_M = 0$. A neighborhood of $\partial_- M$ is symplectomorphic to $(0, \epsilon_h) \times \partial_- M$ with $\omega_M = \omega_h + r_M d\alpha_h + dr_M \wedge \alpha_h$ where $r_M$ parameterizes $[0, \epsilon_h)$. We have a diffeomorphism $\phi : (0, \epsilon_h) \to (-\infty, 0)$ and a cylindrical almost complex structure $\tilde{h}$ on $\mathbb{R} \times \partial_- M$ so that $(s_r \circ \phi, id)_* J$ converges in $C^\infty$ to $\tilde{h}$ in $\mathbb{R} \times \partial_- M$.

Now consider a sequence $r_i < -1$ tending to $-\infty$ and let $I_i$ be the region $\phi^{-1}([r_i - 1, r_i + 1]) \times \partial_- M$ near $\partial_- M$. Let $\Sigma_i := u^{-1}(I_i)$ where we choose $r_i$ generically so $\Sigma_i$ is a manifold with boundary. We let $u_i : \Sigma_i \to I_0 \times \partial_- M$ be the sequence of maps $u|_{\Sigma_i}$ composed with the natural translation map from $I_i$ to $I_0$. Let $I_i' := \phi^{-1}([r_i - \frac{1}{2}, r_i + \frac{1}{2}]) \times \partial_- M$.

By Theorem 7.1 we get after passing to a subsequence a genus 0 nodal $\tilde{h}$ holomorphic curve $u : \Sigma_\infty \to I_0$ and genus zero compact curve $\overset{\sim}{\Sigma}$ with boundary, smooth embeddings $v_i : \overset{\sim}{\Sigma} \to \Sigma_i$ and a continuous surjection $v_\infty : \overset{\sim}{\Sigma} \to \Sigma_\infty$ such that:

1. Let $\Sigma_\infty' := \Sigma_\infty \setminus \Sigma_\infty^\text{sing}$ where $\Sigma_\infty^\text{sing}$ is the preimage under $v_\infty$ of the nodal points of $\Sigma_\infty$.
2. Then $v_\infty|_{\Sigma_\infty'}$ is a diffeomorphism onto its image.
3. We have $u_i \circ v_i \ C^0$ converges to $u_\infty \circ v_\infty$ and $u_i \circ v_i|_{\overset{\sim}{\Sigma}}$ $C^\infty$ converges to $u_\infty \circ v_\infty|_{\overset{\sim}{\Sigma}}$.
4. The $u_i \circ v_i$ and $u_\infty \circ v_\infty$ map the boundary of $\Sigma$ outside $I'_0$.

Let $B_\infty := (u_\infty \circ v_\infty)^{-1}(\{\delta\} \times \partial_- M)$ where $\delta$ is small and generic making $B_\infty$ into a manifold. This has a natural orientation coming from the outward normal of the complex surface $(u_\infty \circ v_\infty)^{-1}(\delta, \delta + \delta')$ for some even smaller $\delta' > 0$. Basically by Stokes' theorem we have $\int_{\Sigma} u^* \omega_M = \int_{B_\infty} (u_\infty \circ v_\infty)^* \pi_{\partial M}^* (\theta|_{\partial_+ M})$ where $\pi_{\partial M}$ is the natural projection to $\partial_+ M$. Because $\int_{\Sigma_\infty} u_\infty^* \omega_h = 0$ we have that the tangent space $T\Sigma_\infty$ gets mapped via $(u_\infty \circ v_\infty)$ to the kernal of $\omega_h$. Because $\theta$ restricted to the kernal of $\omega_h$ is equal to $\alpha_h$ restricted to the kernal of $\omega_h$ we then get:

$$\int_{\Sigma} u^* \omega_M = \int_{B_\infty} (u_\infty \circ v_\infty)^* \alpha_h.$$ 

This is equal to:

$$\int_{B_\infty} \alpha_h \circ d(u_\infty \circ v_\infty) = \int_{B_\infty} \alpha_h \circ \tilde{h} \circ d(u_\infty \circ v_\infty) \circ j$$
where $j$ is the pullback of the complex structure on $\Sigma_{\infty}$ via $v_{\infty}$. Now $\alpha_h \circ \hat{J}_h = dr_M$ and $j(\zeta)$ points outwards along the surface $(u_{\infty} \circ v_{\infty})^{-1}(\delta, \delta + \delta') \times \partial_{-}M$ where $\zeta$ is tangent to $B_{\infty}$ and respecting its orientation. This implies that $\int_{\Sigma} u^{*}\omega_M$ is non-positive. This gives us our contradiction.

\[\square\]

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