Circular symmetry in the Hitchin system

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Abstract
To study circularly symmetric field configurations in the $SU(2)$ Hitchin system an $SO(2)$ symmetry, $[J_3, \phi] = 0$ and $[J_3, A_\pm] = \pm A_\pm$, is imposed on the Higgs scalar $\phi$ and the gauge fields $A_\pm$ of the system, respectively, where $J_3$ is a sum of the third components of the orbital angular momenta and the generators of the $SU(2)$. The circular symmetry and the equation $\bar{D}\phi = 0$ yield constant, generally nonzero, vacuum expectation values for $\text{Tr}(\phi^2)$. The equation $4F_{zz} = [\phi, \phi^*]$ yields a system of differential equations which govern the circularly symmetric field configurations and an exact solution to these equations in a pure gauge form with nontrivial Higgs scalar is obtained.

Keywords: Hitchin system, circular symmetry, self-duality, vortex

1. Introduction

The BPST instantons \cite{1} in the Euclidean four-dimensional space $\mathbb{R}^4$ and the BPS monopoles \cite{2} in the three-dimensional space $\mathbb{R}^3$ are well described and constructed through the ADHM \cite{3} and Nahm \cite{4, 5} constructions, respectively. However, for the Hitchin system \cite{6, 7} which is defined on the two-dimensional space $\mathbb{R}^2$, physically interesting solutions have not been known so far. Although the system has a noble property, $i.e.$, the codimension of the space $\mathbb{R}^2$ in $\mathbb{R}^4$ is also two and the system is at a fixed point of the reciprocity \cite{3, 9}, the analysis on the system is not yet sufficient both from the mathematical and physical points of view.

It is well known \cite{10} for the spherically symmetric 'tHooft-Polyakov magnetic monopole \cite{11, 12} that the Higgs scalar $\phi$ and the gauge potentials $A_j$...
are subject to the conditions
\[ [J_j, \phi] = 0, \quad [J_j, A_k] = i\varepsilon_{jkl}A_l, \]  
where \( j, k \) and \( l \) run from 1 to 3 and \( J_j = L_j + T_j \) are the sum of the orbital angular momenta \( L_j \) and the generators \( T_j \) of \( SU(2) \) which is locally isomorphic to \( SO(3) \). The Higgs scalar \( \phi \) and the gauge potentials \( A_j \) transform as a scalar and a vector under the subgroup \( SO(3)_{L+T} \) of \( SO(3)_L \otimes SO(3)_T \), respectively. The aim of this letter is to study circularly symmetric field configurations in the \( SU(2) \) Hitchin system and to examine their properties. For this end we will impose an \( SO(2)_J \) symmetry
\[ [J_3, \phi] = 0, \quad [J_3, A_{\pm}] = \pm A_{\pm}, \]  
on Hitchin’s scalar \( \phi \) and the gauge potentials \( A_{\pm} = A_1 \pm iA_2 \), respectively. The conditions (2) are obtained from (1) by restricting the range of indices to \( j = 3 \) and \( k, l = 1, 2 \) and the \( \pm \) signs represent the helicity of \( A_{\pm} \). Here we have defined \( J_3 := L_3 + iT_3 \), where the imaginary unit \( i \) is due to our convention used in this letter. We will show that the circular symmetry (2) and the equation \( \bar{D}\phi = 0 \) of the Hitchin system yield constant \( Tr(\phi^2) \), whose value is determined by the asymptotic values of the Higgs scalar at the infinity and generally nonzero. This circumvents Derrick’s theorem13\] or a dimensional argument \[14\] about the nonexistence of finite action configurations in the system. The interpretation of these circularly symmetric configurations as topologically stable Abrikosov-Nielsen-Olesen vortices \[15\] fails because the surface term in the action integral vanishes for these configurations. Finally an exact solution in a pure gauge form with nontrivial Higgs scalar will be presented. Although the field strength has a pure gauge form \( F_{z\bar{z}} = 0 \), the solution is not a genuine pure gauge one, because the relevant gauge transformation is singular.

The letter is organized as follows: In Section 2 we will briefly summarize the Hitchin system. In Section 3 we will impose the circular symmetry on the Hitchin system and see that \( Tr(\phi^2) \) is a constant. In Section 4 we will derive differential equations for circularly symmetric configurations. In Section 5 an exact solution in a pure gauge form with nontrivial Higgs scalar will be presented. Section 6 will be devoted to conclusions and outlook.

2. The Hitchin system

We here briefly summarize the Hitchin system \[6, 7\]. This system is derived by imposing a two-dimensional translational invariance on the four-
dimensional self-dual (SD) Yang-Mills equations $F = \ast F$. Assuming the system to be invariant under two translations $\partial_3 = \partial_4 = 0$, we have a complementary two-dimensional $x_1, x_2$ space and obtain two Higgs scalars $\phi_1 = A_3$ and $\phi_2 = A_4$ defined on this plane. In our convention forms are anti-hermetian; a 1-form $A$, for example, is defined by

$$A = \sum_{a=1}^{3} \sum_{\mu=1}^{4} T_a A^a_{\mu} dx^\mu,$$

where $T_a = \sigma_a$ are anti-hermetian $SU(2)$ generators and $\sigma_a$ are the Pauli matrices. The SD Yang-Mills equations $F = \ast F$ reduce to

$$4 F_{\bar{z} z} = [\phi, \phi^*],$$

$$\bar{D}\phi = [\nabla_1 + i \nabla_2, \phi] = 0, \quad D\phi^* = [\nabla_1 - i \nabla_2, \phi^*] = 0,$$

where $z = x_1 + i x_2$, and $\nabla_j = \partial_j + A_j$ are covariant derivatives with respect to the two-dimensional gauge potentials. The complex Higgs scalar $\phi$ and its conjugate $\phi^*$ are defined by $\phi = \phi_1 - i \phi_2$ and $\phi^* = \phi_1 + i \phi_2$, respectively.

Note that $\phi^\dagger = -\phi^*$ and $(\bar{D}\phi)^\dagger = -D\phi^*$, where the symbol $\dagger$ represents the hermitian conjugation and the minus signs in the right hand sides are due to anti-hermiticity of our $T_j$.

3. Circularly symmetric field configurations

Let us consider hereafter the case of plus helicity in the condition (2). Using the polar coordinates $(\rho, \theta)$ on $\mathbb{R}^2$ with $\rho = \sqrt{x_1^2 + x_2^2}$ and $\theta$ the polar angle, we have $L_3 = -i \partial_\theta$ and the $\theta$ dependence of $\phi$ and $A_+$ can be determined from (2) as follows

$$\phi = -i \left( \begin{array}{cc} h & f_2 e^{-i\theta} \\
 f_1 e^{i\theta} & -h \end{array} \right), \quad A_+ = -i \left( \begin{array}{cc} a & be^{-i\theta} \\
 ce^{i\theta} & -a \end{array} \right) e^{i\theta},$$

where $f_j$ ($j = 1, 2$), $h$, $a$, $b$ and $c$ are functions of $\rho$ only. Note that $A_+$ has the same $\theta$ dependence as $\phi$ except for the overall extra factor $e^{i\theta}$; this is because $\phi$ and $A_+$ are transformed as scalar and vector, respectively, under the $SO(2)_{J_3}$ rotation. The Higgs scalar $\phi = \phi(\rho, \theta)$ of (5) can be expressed as

$$\phi = g(\theta) \phi_0 g(\theta)^{-1},$$

where $\phi_0 = \phi|_{\theta=0}$ and $g(\theta) = \exp(T_3 \theta) \in SU(2)$. We can remove the factor $g(\theta)$ of (6) by singular gauge transformation $g(\theta)^{-1}$, but this gives rise to a two-dimensional delta function singularity $2\pi \delta^2(x) T_3$ [16], a Dirac string term [17], to $F_{\bar{z} z}$. We will discuss this point again later.
Since $\phi$ is traceless, we have from (6)

$$\phi^2 = -\det\phi \cdot 1_2 = -\det\phi_0 \cdot 1_2.$$  \hspace{1cm} (7)

The equation $\bar{D}\phi = 0$ of (4) implies $\bar{D}(\phi^2) = 0$, which yields $\partial_z\det\phi_0 = 0$ because of (7). In the polar coordinates, this equation is expressed as $e^{i\theta}(\partial_\rho + \frac{i}{\rho}\partial_\theta)\det\phi_0 = 0$ and we have $\partial_\rho\det\phi_0 = 0$. Therefore we see that $\det\phi = f_1f_2 + h^2$ is a constant, whose value is determined by the asymptotic values of $f_1$, $f_2$ and $h$ at the infinity and is generally nonzero. The significant point here is that the constant valuedness of $\text{Tr}(\phi^2) = -2(f_1f_2 + h^2)$ is a consequence of the dynamics and symmetry of the system. This situation is quite different from other typical models in mathematical physics, for example, in non-linear sigma models, the constraint on the scalar fields $|\phi| = 1$ is imposed by hand, or implemented through the Fayet-Iliopoulos term, which will introduce additional parameters into the models.

4. Differential equations for circularly symmetric configurations

Substituting (5) for $\phi$ and $A_+$ of $\bar{D}\phi = 0$ of (1) we have

$$i\left(\frac{df_1}{d\rho} - \frac{f_1}{\rho}\right) + 2(ch - af_1) = 0,$$  \hspace{1cm} (8)

$$i\left(\frac{df_2}{d\rho} + \frac{f_2}{\rho}\right) + 2(af_2 - bh) = 0,$$  \hspace{1cm} (9)

$$i\frac{dh}{d\rho} + bf_1 - cf_2 = 0,$$  \hspace{1cm} (10)

from which we again see that $\det\phi = f_1f_2 + h^2$ is a constant. Substituting (4) for (3) we have the following system of differential equations

$$-2\left(\frac{d}{d\rho} + \frac{1}{\rho}\right)\text{Im}(a) + |c|^2 - |b|^2 = |f_2|^2 - |f_1|^2,$$  \hspace{1cm} (11)

$$i\left(\frac{d}{d\rho} + \frac{2}{\rho}\right)c - i\frac{db}{d\rho} + 2(\bar{b}a - \bar{a}c) = 2(\bar{h}f_1 - \bar{f}_2h),$$  \hspace{1cm} (12)

which stem from the diagonal and off-diagonal parts of (3), respectively.

The linear equations (8), (9) and (10) for $a$, $b$ and $c$ are underdetermined system, because the gauge potential $A_+$ of the equation $\bar{D}\phi = 2\partial_z\phi + \bar{D}\phi_0 = 0$ is a constant, whose value is determined by the asymptotic values of $f_1$, $f_2$ and $h$ at the infinity and is generally nonzero.
\[ [A_+, \phi] = 0 \] of (4) is not unique. We observe that the general solution is given by

\[ A_+ = \lambda \phi + \frac{1}{2h} \left( \begin{array}{cc} 0 & \tilde{f}_2 \\ -\tilde{f}_1 e^{2i\theta} & 0 \end{array} \right), \]

where \( \lambda \) is an arbitrary complex scalar and \( \tilde{f}_1 \) and \( \tilde{f}_2 \) are defined by

\[ \tilde{f}_1 = f'_1 - \frac{f_1}{\rho}, \quad \tilde{f}_2 = f'_2 + \frac{f_2}{\rho}, \]

respectively, in which the prime stands for the differentiation with respect to \( \rho \). Similar circumstances occurred in a study of the \( SU(3) \) magnetic monopoles by Corrigan et al. [10]. They solved the equations \( D_\mu \phi = 0 \) for the gauge potentials \( A_\mu \) and obtained asymptotic field strengths \( F_{\mu\nu} \) at infinities.

A convenient choice of \( \lambda \), or gauge, in (13) is given by \( \lambda = \frac{i\tilde{f}_1}{2h} e^{i\theta} \), which corresponds to \( c = 0 \). In this case, \( A_+ \) becomes an upper triangular matrix in the Atiyah and Ward form [18]

\[ A_+ = \frac{1}{2f_1} \left( \begin{array}{cc} \tilde{f}_1 e^{i\theta} & -2h' \\ 0 & -\tilde{f}_1 e^{i\theta} \end{array} \right). \]

From this we see \( a = \frac{i\tilde{f}_1}{2f_1} \) and \( b = -\frac{ih'}{f_1} \) and the equations (8) - (12) reduce to the equations

\[ f_1 f_2 + h^2 = C^2, \]

\[ \left( \frac{|f_1'|}{f_1} \right)' + \frac{1}{\rho} \frac{|f_1'|}{f_1} + \frac{|h'|}{f_1} = |f_1|^2 - |f_2|^2, \]

\[ \left( \frac{h'}{f_1} \right)' + \frac{1}{\rho} \frac{h'}{f_1} - \frac{\tilde{f}_1 h'}{|f_1|^2} = 2(\tilde{f}_1 h - \tilde{h} f_2), \]

where \( C \) is a constant. Note that only the absolute value of \( f_1 \) appears in (17). If we expand \( f_j \ (j = 1, 2) \) and \( h \) in power series of \( \rho \) around \( \rho = 0 \)

\[ f_j = f_j^{(0)} + f_j^{(1)} \rho + f_j^{(2)} \rho^2 + \cdots, \quad h = h^{(0)} + h^{(1)} \rho + h^{(2)} \rho^2 + \cdots, \]

we see \( f_j^{(0)} = f_j(0) = 0 \) from the single-valuedness of \( f_j \) at \( \rho = 0 \). We also have \( h^{(1)} = 0 \) from (16) when \( C = \pm h^{(0)} \neq 0 \). Using (19) in (17) and (18) we will obtain a formal power series solution with coefficients fulfilling some relations between them. We have also asymptotic solution \( f_1 \to f_1^\infty, \ f_2 \to f_2^\infty \) and \( h \to h^\infty \) at the infinity \( \rho \to \infty \) where \( f_1^\infty, f_2^\infty \) and \( h^\infty \) are constants satisfying the condition \( f_1^\infty f_2^\infty + (h^\infty)^2 = C^2 \).
5. An exact solution in a pure gauge form with nontrivial Higgs scalar

We here consider a special case such that \( f_1, f_2 \) and \( h \) are all real and the relation \( f_1 = f_2 (\equiv f) \) holds at some interval of \( \rho \). In this case, since the right-hand sides of (17) and (18) vanish, we have \( F_{zz} = 0 \). Putting \( t = \ln \rho \) and \( \ln(\frac{dh}{dt}/f) = -u \) in (17) and (18) we can derive a one-dimensional Liouville equation

\[
\frac{d^2u}{dt^2} = e^{-2u}
\]

with a minus sign in the exponent. A general solution to (20) is given by

\[
u = \ln \left\{ \nu_0 - \frac{1}{2} \cosh \nu (t - t_0) \right\},
\]

where \( \nu_0 \) and \( t_0 \) are constants of integration and \( f \) and \( h \) are given by

\[
n \rho_0 \alpha \frac{f_2}{\rho_0^2 + \rho_2^2}, \quad h = C \rho_0 \beta \frac{f_2}{\rho_0^2 + \rho_2^2},
\]

(21)

with \( \rho_0 = e^{t_0} \). Note that \( f \) and \( h \) of (21) is a parameterization of an ellipse \( f^2/\nu^2 + h^2 = C^2 \). Without loss of generality we can restrict ourselves to the case \( \nu > 0 \), because changing the sign \( \nu \rightarrow -\nu \) in (21) yields \( f \rightarrow -f \) and \( h \rightarrow -h \) which do not alter the equations (16)-(18).

If we apply the solution (21) to the whole interval \( 0 \leq \rho < \infty \) we have a solution in a pure gauge form with nontrivial Higgs scalar, in which \( f \) tends to zero both as \( \rho \rightarrow 0 \) and as \( \rho \rightarrow \infty \), and \( h \) connects smoothly two boundary values \( \pm C \) at the infinity \( \rho \rightarrow \infty \) and at the origin \( \rho = 0 \), respectively. Although the field strength in this case has a pure gauge form \( F_{zz} = 0 \), the solution (21) is not a genuine pure gauge one, because the relevant gauge transformation is singular. In fact [16], after performing the singular gauge transformation \( g = g(\theta)^{-1} = \exp(-T_3\theta) \), we have a nonzero field strength

\[
n F_{12}^g = g(\theta)^{-1}F_{12}g(\theta) + g(\theta)^{-1}\partial_1 \partial_2 g(\theta) = 2\pi\delta^2(x)T_3,
\]

(22)

which corresponds to a Dirac string term [17]. We can show (22) by explicit calculation. The transformed potentials are given by

\[
n A_1^g = \frac{h'}{f_1}(T_1 \sin \theta + T_2 \cos \theta) - \frac{f}{f_1}T_3 \sin \theta + T_3 \partial_1 \theta,
\]

(23)

\[
n A_2^g = -\frac{h'}{f_1}(T_1 \cos \theta - T_2 \sin \theta) + \frac{f}{f_1}T_3 \cos \theta + T_3 \partial_2 \theta,
\]

(24)
and we have the field strength
\[ F_{12}^{g} = \partial_{1} A_{2}^{g} - \partial_{2} A_{1}^{g} + [A_{1}^{g}, A_{2}^{g}] \]
\[ = -\left\{ \left( \frac{h'}{f_{1}} \right)' + \frac{1}{\rho} \frac{h'}{f_{1}} - \frac{f'_{1} h'}{f_{1}^{2}} \right\} T_{1} + \left\{ \left( \frac{f'_{1}}{f_{1}} \right)' + \frac{1}{\rho} \frac{f'_{1}}{f_{1}} + \left( \frac{h'}{f_{1}} \right)^{2} \right\} T_{3} + T_{3}[\partial_{1}, \partial_{2}]\theta \]
\[ = T_{3} \nabla \cdot \frac{\vec{D}}{\rho^{2}} = 2\pi \delta^{2}(x) T_{3}, \] (25)

where equations (17) and (18) have been used in the last line and then we have obtained (22). In this singular gauge the Higgs scalar has no \( \theta \) dependence: \( \phi' = g^{-1} \phi g = \phi_{0} \) where \( \phi_{0} = \phi|_{\theta=0} \) of (6).

6. Conclusions and outlook

We have found that the equation \( \bar{D}\phi = 0 \) together with the circular symmetry (2) yields a constant, generally nonzero, vacuum expectation value for \( \text{Tr}(\phi^{2}) \). We have also seen that the equation \( 4F_{zz} = [\phi, \phi^{*}] \) yields a system of differential equations and from which an exact solution in a pure gauge form with nontrivial Higgs scalar is obtained. Physical meaning of this solution is not clear and further study is needed, which will be discussed in a separate paper.

We finally discuss the surface terms in the action integrals which usually emerge when considering the SD, or the BPS, conditions. After a dimensional reduction from the four-dimensional Yang-Mills action, we consider the following action integral defined over a disk \( D = \{(\rho, \theta) | 0 \leq \rho \leq R, 0 \leq \theta \leq 2\pi\} \)

\[ S = \frac{1}{2} \text{Tr} \int_{D} \{(F_{12})^{2} + \bar{D}\phi_{1} D\phi_{1} + \bar{D}\phi_{2} D\phi_{2} + [\phi_{1}, \phi_{2}]^{2}\} d^{2}x \]
\[ = \frac{1}{16i} \text{Tr} \int_{D} \{(4F_{zz} - [\phi, \phi^{*}])^{2} + 4(\bar{D}\phi)^{\dagger} \bar{D}\phi\} dz d\bar{z} + \text{Tr} \int_{D} \partial_{j} J_{j} d^{2}x, \] (26)

where \( J_{j} = -\varepsilon_{jk} \text{Tr} \phi_{1} D_{k} \phi_{2} \) and \( D_{k} \phi_{2} = \partial_{k} \phi_{2} + [A_{k}, \phi_{2}] \). The total divergence term in the last expression can be rewritten as a line integral along the boundary \( \partial D = S_{R}^{1} \). For the circularly symmetric configurations (5), we have

\[ \text{Tr} \int_{D} \partial_{j} J_{j} d^{2}x = \text{Tr} \int_{S_{R}^{1}} (J_{1} \cos \theta + J_{2} \sin \theta) \rho d\theta \]
\[ = -2\pi (|f_{1}|^{2} + |f_{2}|^{2} + 2|h|^{2}) \rho|_{0}^{R}, \] (27)
which vanishes \[1, 13\] as \( R \to \infty \) due to boundary conditions discussed in Section 4. Since the boundary term \( (27) \) vanishes the interpretation of these configurations as topologically stable Abrikosov-Nielsen-Olesen vortices \(15\) fails.

Group theoretically, however, we expect the existence of topologically stable solitons. For the 'tHooft-Polyakov magnetic monopole, since the isotropy subgroup of the Higgs scalar \( \phi \) is given by \( \{ \exp(\phi \alpha) | \alpha \in \mathbb{R} \} \cong SO(2) \), we have the second homotopy group \( \pi_2(SO(3)/SO(2)) \cong \pi_2(S^2) \cong \mathbb{Z} \); this gives topologically stable magnetic monopoles. As for the Hitchin system, if \( \phi_1 \) and \( \phi_2 \) are not parallel nor anti parallel \(15\) then the isotropy subgroup of \( \phi = \phi_1 - i\phi_2 \) is trivial and in this generic case we have the first homotopy group \( \pi_1(SO(3)) \cong \pi_1(SU(2)/\mathbb{Z}_2) \cong \mathbb{Z}_2 \). Note that \( \exp(\phi \alpha) \) does not belong to \( SO(3) \) but to \( SL(2, \mathbb{C}) \). From this consideration, we expect the existence of topologically stable solitons in the circularly symmetric configurations in the Hitchin system. For example, we can extend the conditions \( (2) \) to more general ones

\[
[J_3, \phi] = l\phi, \quad [J_3, A_{\pm}] = \pm(l + 1)A_{\pm}.
\] (28)

Solving these we have \( \phi = e^{il\theta}\phi^{(0)} \) and \( A_{\pm} = e^{il\theta}A_{\pm}^{(0)} \), where \( \phi^{(0)} \) and \( A_{\pm}^{(0)} \) are the \( \phi \) and the \( A_{\pm} \) of \( (3) \), respectively. For \( l = 0 \), the highest symmetry, we have seen that the surface term \( (27) \) vanishes. For \( l \neq 0 \) we expect the existence of a field configuration with non-zero surface term, which will be examined elsewhere.

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