Explicit finite difference schemes for solving the one – dimensional tsunami wave propagation equations

D Deleanu and C L Dumitrache

Constanta Maritime University, Department of General Engineering Sciences, Mircea cel Batran street, No. 104, 900663, Romania.

Abstract. Because the tsunami wavelength is much longer than the sea depth, the propagation of this powerful destructive wave can be modelled by the non-linear shallow water equations (SWEs). To solve accurately these hyperbolic partial differential equations, one can resort to a variety of numerical techniques, including finite difference method (FDM) or finite volume method (FVM). In this paper, we have selected FDM, as a well verified instrument on other fluid flow problems, to estimate the tsunami wave propagation in one dimension. The ability of four explicit scheme (Forward Euler, Lax Friedrichs, Mac Cormack and Richtmyer) to solve SWEs was tested on three different test problems. Starting from the idea that a tsunami may begin as a sudden rise of a column of water, the first test represents a square or a sinusoidal pulse which breaks up into two identical waves moving in opposing directions on a constant depth seabed. In the other two test problems we simulate a tsunami wave approaching the shore on a variable seabed depth having either a steady or a highly variable slope. The numerical results allow us to assert that, in general, the used explicit schemes lead to a correct description of a tsunami wave propagation, both offshore and near the coastline. The exception is represented by the Forward Euler scheme that, due to its instability, was only used in the first test.

1. Introduction

Tsunamis are giant waves produced mainly by earthquakes, volcanic eruptions under the sea or submarine landslides. The consequence of such an event consists in a vertical movement of the sea floor associated to a displacement of a huge water mass. From the place where this mass is displaced, waves travel outward in all directions at a speed of 800 – 900 km/h and a height of a few tens of centimetres. Because of their wavelengths, of the order of several hundred kilometres, tsunamis are considered to be shallow – water waves. For this kind of waves, the travel speed is equal to the square root of the product of the gravitational acceleration and the water’s depth. Thus, when the tsunami approaches the shore, its speed diminishes to 50 - 100 km/h. But the change of total energy remains constant, so the height of the wave grows to 3 – 4 m or, under certain conditions, to 15 – 30 m. In the same time, the wavelength decreases to 10 – 20 km.

Figure 1. Tsunami wave main characteristics [2].
When the tsunami touches the coast line, it may appear as a rising tide, a series of breaking waves or a bare (a step-like wave with a steep breaking front). The topography of the coast line can have an important effect on the size of the tsunami wave. Usually, there are more than one wave and the succeeding one may be larger than the one before [1 - 4]. Figure 1 captures the main features of a tsunami wave.

The most used variant to describe the tsunami propagation is the nonlinear shallow water model that neglects the shoaling effect of the shore and the Coriolis’ effect due to the Earth rotation. Far from the tsunami’s generation place and close to the coast line, the wave propagation may be expressed in one dimension (1D). In this case, the shallow water equations (SWEs) form a system of two coupled hyperbolic partial derivative equations, with time \( t \) and distance \( x \) along the flow as independent variables, and flow depth \( h = h(t, x) \) and flow velocity \( v = v(t, x) \) as dependent variables. This system, known also as Saint – Venant equations, reads as

\[
\begin{align*}
\frac{\partial h}{\partial t} + \frac{\partial}{\partial x} (h v) &= 0 \\
\frac{\partial}{\partial t} (h v) + \frac{\partial}{\partial x} \left( h v^2 + \frac{gh^2}{2} \right) &= -gh \frac{dB}{dx}
\end{align*}
\]

where \( g \) is the gravitational acceleration and \( B(x) \) the bed height (see figure 2).

![Shallow water main dimensions.](image)

For numerical purposes, the system (1) is rewritten in vectorial form as

\[
\frac{\partial \mathbf{U}}{\partial t} + \frac{\partial \mathbf{F} (\mathbf{U})}{\partial x} = \mathbf{R} (x, \mathbf{U})
\]

where

\[
\mathbf{U} = \begin{pmatrix} h \\ hv \end{pmatrix}, \quad \mathbf{F} (\mathbf{U}) = \begin{pmatrix} hv \\ hv^2 + \frac{gh^2}{2} \end{pmatrix}, \quad \mathbf{R} (x, \mathbf{U}) = \begin{pmatrix} 0 \\ -ghB'(x) \end{pmatrix}
\]

It can be solved numerically by different techniques, including the method of characteristics, finite difference methods (FDMs) or finite volume methods. Of these methods, we have selected here the FDMs, which can further be classified as explicit or implicit techniques, each of them having distinct numerical properties.

In the following section, we will summarize the explicit numerical schemes we used in the paper to study the propagation of a tsunami wave.

2. Explicit finite difference numerical schemes

In an explicit scheme, the dependant variables’ values at the next time level are determined from an explicit formula involving data from previous time levels. Before applying such a scheme, one needs to define a mesh as well as initial and boundary conditions [5 – 8].
Let us assume that we want to solve the system (2) on the finite domain $x_{\text{min}} \leq x \leq x_{\text{max}}$ and $0 \leq t \leq t_{\text{max}}$. A suitable mesh can be a fixed and equidistant one,

$$x_{\text{min}} = x_0 < x_1 < x_2 < \ldots < x_I = x_{\text{max}}, \quad 0 = t_0 < t_1 < t_2 < \ldots < t_N = t_{\text{max}},$$

with $x_{i+1} - x_i = \Delta x$, $i = 0, 1, \ldots, I - 1$ and $t_{n+1} - t_n = \Delta t$, $n = 0, 1, \ldots, N - 1$. The exact values $h(x, t)$ and $v(x, t)$ will be approximated on this grid by $h^n_i \approx h(i \Delta x, n \Delta t)$ and $v^n_i \approx v(i \Delta x, n \Delta t)$. Initial conditions refer to the flow depth and velocity at $t = 0$, $h_0(x) = h(x, 0)$ and $v_0(x) = v(x, 0)$. Boundary conditions are required at $x = x_{\text{min}}$ and $x = x_{\text{max}}$, as well as ghost values at left and/or right ends of the computational domain.

The basic idea of a FDM is to replace the partial derivatives with finite difference approximations, e.g.

$$\frac{\partial U}{\partial t} = \frac{U^{n+1}_i - U^n_i}{\Delta t}, \quad \frac{\partial U}{\partial x} = \frac{U^{n+1}_{i+1} - U^n_{i-1}}{2\Delta x},$$

which is a forward time centred space approximation.

The obtained scheme must be conservative, otherwise it will be unstable or the numerical results will be extremely inaccurate. As a result of applying more explicit FD schemes to the system case (2), we will report in the paper the results obtained with four of them, namely:

a) **Forward time - centred space scheme (FTCS scheme)**

$$U_i^{n+1} = U_i^n - \frac{r}{2}(F_{i+1}^n - F^n_{i-1}) + r R^n_i$$  \hspace{1cm} (3)

b) **Lax - Friedrichs scheme (LF scheme)**

$$U_i^{n+1} = \frac{1}{2}(U_{i+1}^n + U_{i-1}^n) - \frac{r}{2}(F^n_{i+1} - F^n_{i-1}) + r R^n_i$$  \hspace{1cm} (4)

c) **Mac - Cormack scheme (MC scheme)**

$$U_i^{n+1} = \frac{1}{2}(U_i^n + U_i^{1(1)}) - \frac{r}{2}(F^{1(1)}_{i+1} - F^{1(1)}_{i-1}) + \frac{r}{2} R^{1(1)}_i$$  \hspace{1cm} (5)

where

$$U_i^{1(1)} = U_i^n - r(F^n_{i+1} - F^n_{i-1}) + r R^n_i.$$  

d) **Richtmyer scheme (RM scheme)**

$$U_i^{n+1} = U_i^n - r(F_{i+1/2}^* - F_{i-1/2}^*) + r R_i^n$$  \hspace{1cm} (6)

where

$$F_{i+1/2}^* = F(U_{i+1/2}^{n+1}) , \quad U_{i+1/2}^{n+1/2} = \frac{1}{2}(U_i^n + U_{i+1}^n) - \frac{r}{2}(F(U_{i+1}^n) - F(U_i^n)) + r R_i^n.$$  

In the above relations,

$$r = \frac{\Delta t}{\Delta x}, \quad U^n_i = \left( \begin{array}{c} h^n_i \\ h^n_i \cdot v^n_i \end{array} \right), \quad F^n_i = \left( \begin{array}{c} h^n_i \\ h^n_i \cdot v^n_i \end{array} \right), \quad R^n_i = \left( \begin{array}{c} 0 \\ -g h^n_i (B_i - B_{i-1}) \end{array} \right).$$

These explicit schemes require a stability restriction, called Courant condition, which establishes the maximum allowable time step, $\Delta t$. This is
\[ \Delta t \leq \frac{\Delta x}{\max \left( v + \sqrt{gh}, v - \sqrt{gh} \right)} \]  

In practice, one can choose a constant \( \Delta t \), or we can calculate it with (7) for each time level. The Courant condition may lead to low values of \( \Delta t \), making the scheme very inefficient from computational time point of view.

3. Test problems

In this section, we will test the ability of the four explicit FD schemes to correctly describe the propagation of a tsunami wave. Three test problems were chosen for this purpose.

Having in mind that a tsunami can start with a sudden rise of a column of water, the first test represents a square or a sinusoidal pulse which breaks up into two identical waves moving in opposite directions on a constant depth seabed (meaning \( B'(x) = 0 \)). The system (2) was integrated on the computational domain \((x,t) \in [0,1] \times [0,0.2]\) using the initial conditions

\[ h(x,0) = \begin{cases} 5.2, & \text{if } 0.1 \leq x \leq 0.2 \\ 5, & \text{otherwise} \end{cases}, \quad v(x,0) = 0 \]  

or

\[ h(x,0) = \begin{cases} 5 + 0.2 \sin \frac{\pi}{0.1}(x - 0.1), & \text{if } 0.1 \leq x \leq 0.2 \\ 5, & \text{otherwise} \end{cases}, \quad v(x,0) = 0 \]

We check through this example if the schemes are able to preserve the wave amplitude.

In the second test problem we simulate a tsunami wave which approaches the shore on a variable seabed depth having a steady slope \( s \), meaning \( B(x) = sx \). The computational domain has realistic dimensions, \((x,t) \in [0,L] \times [0,T]\), with \( L = 1.296 \text{ km} \) and \( T = 60,000 \text{ s} \). This time, the initial disturbance is thought like an incoming sinusoidal wave which “enters” in the computational grid on its left side. The initial conditions are \( \begin{align*} h(x,0) &= H_0 = 6.15 \text{ m} \quad \text{and} \quad v(x,0) = 0, \end{align*} \) while the boundary conditions read as

\[ h(0,t) = 64.5 + 3 \sin \left[ \pi \left( \frac{4t}{86400} - \frac{1}{2} \right) \right], \quad v(0,t) = \sqrt{g \left( \sqrt{h(0,t)} - \sqrt{H_0} \right)}h(L,t) = H_0, v(L,t) = 0 \]  

We expect the wave amplitude to increase with time while its wavelength to decrease.

The last test is the toughest. The sea depth is again generally decreasing with its proximity to the shore, but intermediate climbs and descends are included in the seabed profile. This is given by

\[ B(x) = 10 + \frac{40x}{L} - 10 \sin \left[ \pi \left( \frac{4x}{L} + \frac{1}{2} \right) \right] \]

The other conditions are the same as those used in the second test problem.

4. Numerical results

In this part, we will apply the four explicit FD schemes presented in Section 2 to the three test problem proposed in Section 3. We try to identify the scheme that most accurately describes the propagation of a tsunami wave, as observed in real cases and briefly discussed in Section 1.
4.1. Test problem no. 1
For this application, no source term is present (\( R = 0 \)). The Courant condition (7) gives very low and variable time step values (\( \Delta t \approx 10^{-18} \)) for FTCS scheme. We fixed \( \Delta t = 10^{-5} \) and obtained the results shown in figure 3. Here, we can see that the scheme suffered strongly from oscillations in a short amount of time. The same result is recorded, sooner or later, if we change the time step or the parameters that define the water column (height, shape, position on the grid or undisturbed water column). This situation is maintained for the other test problems, so no additional results provided by this scheme will be reported.

![Figure 3](image1.png)

**Figure 3.** The flow’s depth \( h \) and speed \( v \) given by FTCS scheme for an initial sinusoidal pulse and a constant water depth

![Figure 4](image2.png)

**Figure 4.** The flow’s depth \( h \) and speed \( v \) given by Lax-Friedrichs scheme for an initial rectangular pulse and a constant water depth

![Figure 5](image3.png)

**Figure 5.** The flow’s depth \( h \) and speed \( v \) given by Lax-Friedrichs scheme for an initial sinusoidal pulse and a constant water depth
Lax-Friedrichs scheme is more stable than FCTS scheme. If the Courant condition is used to set the time step, then the two columns of water retain their shape and amplitude over time, as proved in figures 4 and 5. The resulted time step was $\Delta t = 1.387 \cdot 10^{-4}$.

If the sea depth is increased, the LF scheme retains its ability to accurately describe the wave motion. Contrary, if the sea depth is decreased (for example to 1), then at least three things happen. First, the total wave decreases so that a larger time $T$ is required for covering the distance $L = 1$. Second, if the time step is obtained from (7), once the wave advances towards $x = 1$, it turns into triangular wave with a slightly decreasing amplitude over time, as illustrated in figure 6.

Figure 6. One of the effects of decreasing the sea depth on the flow’s depth $h$ and speed $v$ if the Lax - Friedrichs scheme is used. The initial sinusoidal pulse transforms into a triangular one. The Courant condition is maintained.

Third, if a smaller time step is used, the wave retains the original sinusoidal shape but the dissipation effect becomes more pronounced, as shown in figure 7.

Figure 7. One of the effects of decreasing the sea depth on the flow’s depth $h$ and speed $v$ if the Lax - Friedrichs scheme is used. The initial sinusoidal pulse is dissipated over time. The Courant condition is replaced by choosing a fixed and less time step.

For this test problem, the other two schemes behave almost similarly, so we only present the results obtained with the Mac Cormack method. If the sea depth is large compared to the height of the water column, then the two semi – columns move left – right with a constant amplitude over time (see figure 8 for $H_0 = 5$).

Lowering the depth of the water to $H_0 = 1$, in addition to the triangulation effect observed in the LF scheme, there are some oscillations in the extreme points of the column. These are more noticeable as the time step is chosen further than the one calculated with relation (7). Figure 9 shows this significant deviation from the initial waveform for $\Delta t = 10^{-4}$ (the Courant condition provides the value $\Delta t = 2.81 \cdot 10^{-4}$).
Figure 8. The flow’s depth $h$ and speed $v$ given by Mac Cormack scheme for an initial sinusoidal pulse and a constant water depth

Figure 9. The emergence and development of oscillations in applying the Mac Cormack scheme to test problem no. 1 in the case of low water depth

LF and MC schemes lead to very close results. Thus, figure 10 shows the $h(x, 0.1)$ and $v(x, 0.1)$ curves. With the exception of the beginning and ending zones of the nonzero parts, the difference between the curves is almost unnoticed.

Figure 10. Mac Cormack versus Lax Friedrichs scheme for an initial sinusoidal pulse and a constant water depth

4.2. Test problem no. 2

The inputs of this test problem were chosen so that the mesh to fit a little over a full sinusoid from the incoming wave. The step sizes $\Delta x = 324$ m and $\Delta t = 10$ s were chosen. For a slope $s = 40/L$ the sea calm water shrinks from 61.5 m to 21.5 m. In $T = 60,000$ s, the incoming wave traverses almost the entire distance $L = 1,296$ km and its peak is 68.266 m above the reference level. With time running out,
the wave front becomes steeper. Even if $v$ grows, the total wave speed $v + \sqrt{gh}$ decreases when the wave approaches the shore (see figure 11, for Mac Cormack scheme).

If the slope is steeper (e.g. $s = 60.5/L$), then the wave becomes slower and its peak reaches 68.731 m above the reference base, as shown in figure 12. After $T = 60,000$ s the wave front is practically vertical. Because the scheme cannot describe a multi-modal function, the results obtained for a higher $T$ will be affected by large errors.

**Figure 11.** The depth $h + B$ and speed $v$ given by Mac Cormack scheme for an incoming wave moving on an inclined seabed having the constant slope $s = 40/L$, with $L = 1,296$ km.

**Figure 12.** The depth $h + B$ and speed $v$ given by Mac Cormack scheme for an incoming wave moving on an inclined seabed having the constant slope $s = 60.5/L$.

The other two schemes, LF and RM, behave similarly and lead to outputs that are in the 0.1 % range relative to those yielded by the MC scheme, as illustrated in figure 13 for the slope $s = 40/L$. Thus, MC scheme gives $(h + B)_{\text{max}} = 68.266m$, obtained for $x = 976.2$ km, while LF yields $(h + B)_{\text{max}} = 68.210m$ at $x = 974.6$ km. The relative difference is just 0.08%.

**Figure 13.** Mac Cormack versus Lax Friedrichs scheme for an incoming wave moving on an inclined seabed having the constant slope $s = 40/L$.
4.3. Test problem no. 3

The variable water depth and seabed cause some notable changes in the wave profile and speed distribution, as shown in figure 14. If we look at figure 15, where the wave motion is followed at the same scale as the changing of the seabed profile, we notice that the ups and downs of the latter produce similar effects on the curves $h + B$ and $v$ as well as distortions of these (especially at the speed distribution). The local slopes are steeper than those in test problem no 2, with obvious consequences on the changes in wave height and front shape. In in the previous test the wave was above the reference level of the still water with the approaching shoreline, now one observes a water hole behind the wave.

Figure 14. The depth $h + B$ and speed $v$ given by Mac - Cormack scheme for an incoming wave moving on an inclined seabed having the variable slope

Figure 15. The seabed profile and the wave represented at the same scale

The results provided by LF and RM schemes are in a very good agreement with those presented above, the differences being again in the limit of $0.1 – 0.2 \%$. If we extend the study interval to 75,000 s, the wave front (which was already vertical at 60,000 s) moves towards the shore as a wall. Some oscillations appear at the top of this “wall” when applying the MC and RM schemes, as illustrated in figure 16.

Figure 16. Mac Cormack versus Lax Friedrichs scheme for an incoming wave moving on an inclined seabed having the variable slope
5. Conclusions
This study showed that by manipulating explicit finite difference schemes one can successfully provide a right description of a tsunami wave propagation, both offshore and near the coastline, no matter if the seabed has a constant slope or significant climbs and descends are included in it. With the exception of the Forward time – Centered space scheme, that suffered badly from oscillations soon enough after the movement study’s start, the other used schemes (Lax Friedrichs, Mac Cormack and Richtmyer) produced accurate numerical outputs, the relative differences between the results achieved with them not exceeding 0.1 – 0.2%. An advantage offered by explicit schemes is the relative simplicity of their implementation in a programming environment such as Matlab. On the other hand, these schemes are not unconditionally stable and require a restriction on the maximum allowable time step with immediate consequences on the computational time. However, for the problem analyzed in the paper this maximum time step was large enough to allow for a rapid “tracking” of the tsunami wave in its movement.

6. References
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