ON THE ZASSENHAUS CONJECTURE FOR CERTAIN CYCLIC-BY-NILPOTENT GROUPS.

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Abstract. Hans Zassenhaus conjectured that every torsion unit of the integral group ring of a finite group \( G \) is conjugate within the rational group algebra to an element of the form \( \pm g \) with \( g \in G \). This conjecture has been disproved recently for metabelian groups, by Eisele and Margolis, but before it had been proved for many classes of solvable groups, as for example nilpotent groups, cyclic-by-abelian groups and groups having a cyclic Sylow subgroup with abelian complement. However, it is not known whether the conjecture holds for supersolvable groups. This paper is a contribution to this question. More precisely, we study the conjecture for the class of cyclic-by-nilpotent groups with special attention to the class of cyclic-by-Hamiltonian groups. We prove the conjecture for cyclic-by-\( p \)-groups and for a large class of cyclic-by-Hamiltonian groups.

1. Introduction

In this paper, \( G \) is a finite group and \( \mathbb{Z}G \) denotes the group ring of \( G \) with coefficients in \( \mathbb{Z} \). The study of the group of units of \( \mathbb{Z}G \) has been an active field of research since the seminal work of Graham Higman [Hig40]. One of the main aims in this area, which is partially motivated by the Isomorphism Problem, is the description of the torsion units of \( \mathbb{Z}G \). Let \( V(\mathbb{Z}G) \) denote the group formed by the units of \( \mathbb{Z}G \) of augmentation 1. Then the group of units of \( \mathbb{Z}G \) is \( \pm V(\mathbb{Z}G) \) and therefore we can restrict our attention to the group \( V(\mathbb{Z}G) \). In the 1970’s, Hans Zassenhaus conjectured that every torsion element \( u \) of \( V(\mathbb{Z}G) \) is conjugate within the rational group algebra \( \mathbb{Q}G \) to an element \( g \) of \( G \) [Zas74]. In such case one says that \( u \) and \( g \) are rationally conjugate. Eisele and Margolis have shown recently a counterexample to the Zassenhaus Conjecture [EM17]. However the Zassenhaus Conjecture has been confirmed for large families of solvable groups. Therefore the Zassenhaus Conjecture is not the “correct solution” to the original problem of describing the torsion units of \( \mathbb{Z}G \), yet it provides a right answer for many solvable groups and some non-solvable groups (see for [MRT18] for a recent survey). A big step in finding the “correct solution” would be to classify the finite groups for which the Zassenhaus Conjecture holds. Obtaining such a classification looks as an impossible task, but proving or disproving the Zassenhaus Conjecture for large classes of groups is a way to “converge to the correct answer”. For example, the counterexample of Eisele and Margolis shows that the Zassenhaus Conjecture fails for metabelian groups but the conjecture holds for nilpotent groups [Wei91], cyclic-by-abelian groups [CMdR13].
and for groups having a Sylow subgroup with abelian complement [Her06]. Some of the ideas that led to the discovery of the Eisele-Margolis counterexample do not work to disprove the Zassenhaus Conjecture for supersolvable groups (see [MR18a]). At least the positive results known so far, suggests that the Zassenhaus Conjecture might hold for cyclic-by-nilpotent groups. The aim of this paper is to make some contributions in that direction. Actually, our original motivation was to study the Zassenhaus Conjecture for cyclic-by-Hamiltonian groups, a question posed by Kimmerle to us in a private conversation.

Another aim of this paper is to consider a special case of the Zassenhaus Conjecture which appeared as Research Problem 35 in [Seh93]. If \( N \) is a normal subgroup of \( G \) then \( V(ZG, N) \) denotes the group of units of \( ZG \) mapped to 1 by the natural homomorphism \( ZG \to Z(G/N) \). Sehgal’s Problem asks whether every torsion element of \( V(ZG, N) \) is rationally conjugate to an element of \( G \) provided \( N \) is nilpotent. Actually, the torsion unit in the counterexample of Eisele and Margolis is a negative solution to Sehgal’s Problem for some group \( G \) and an normal subgroup \( N \) of \( G \) with \( N \) and \( G/N \) abelian. This raises the question of when Sehgal’s Problem has a positive solution. We do not know the answer even in the case where \( N \) and \( G/N \) are abelian.

Our main results are the following:

**Theorem 1.1.** If \( G \) is a cyclic-by-p-group (i.e. \( G \) has a cyclic normal subgroup of prime power index) then every torsion unit of \( V(ZG) \) is rationally conjugate to an element of \( G \).

**Theorem 1.2.** If \( G \) is a finite group with a cyclic normal subgroup \( A \) such that the factor group \( G/A \) is Hamiltonian and the Sylow 2-group of \( G/A \) has order 8 then every torsion unit of \( V(ZG) \) is rationally conjugate to an element of \( G \).

**Theorem 1.3.** Let \( N \) be a nilpotent subgroup of \( G \) containing \( G' \) and suppose that \( N_p' \) is abelian and \( G_p' \) is cyclic for some prime \( p \). Then every torsion element of \( V(ZG, N) \) is rationally conjugate to an element of \( N \).

By the classification of Hamiltonian groups, Kimmerle’s question asks whether the Zassenhaus Conjecture holds for a finite group \( G \) having a normal cyclic subgroup \( A \) such that \( G/A \cong Q_8 \times E \times B \), with \( Q_8 \) the quaternion group of order 8, \( E \) an elementary abelian 2-group and \( B \) an abelian group of odd order. Theorem 1.1 and Theorem 1.2 answer Kimmerle’s question for the cases where \( B = 1 \) or \( E = 1 \). Unfortunately we have not been able to avoid the latest hypothesis. Nevertheless, the results of Section 6 give strong conditions for a minimal cyclic-by-Hamiltonian counterexample to the Zassenhaus Conjecture.

Theorem 1.3 which will be used in the proofs of Theorem 1.1 and Theorem 1.2 implies that in all the negative solutions to Sehgal’s Problem, with \( N \) and \( G/N \) abelian, \( G' \) has at least two non-cyclic Sylow subgroups. This seems to be quite sharp because, by a result of Cliff and Weiss [CW00], if \( N \) has at most one non-cyclic Sylow subgroup then Sehgal’s Problem has a positive solution for every \( G \), while for some of the counterexample in [EM17] one has \( G' = N \cong C_p^2 \times C_q^2 \).

The paper is structured as follows. In Section 2 we establish the basic notation and collect some known facts about the Zassenhaus Conjecture which will be used in the subsequent sections. For example, we recall the role of partial augmentations which gives the necessary background to prove Theorem 1.3 by combining results by Hertweck, Margolis and del Río. Section 3 starts proving general results for
cyclic-by-nilpotent groups and finishes with the proof of Theorem 1.1. Section 3 revisits an equality from [CMdR83] concerning partial augmentations, which plays an important role in proving, in Section 4, some features of an hypothetical minimal cyclic-by-Hamiltonian counterexample to the Zassenhaus Conjecture, and hence in the proof of Theorem 1.2 which is given at the end of the paper.

2. Notation and Preliminaries

The cardinality of a set $X$ is denoted by $|X|$. Let $\varphi$ denote the Euler’s totient function. For every integer $n$, we let $\zeta_n$ denote a fixed complex primitive root of unity of order $n$ and $C_n$ denotes an arbitrary cyclic group of order $n$. If $F/K$ is an extension of number fields then $\text{tr}_{F/K}$ denotes the trace map of $F$ over $K$.

All throughout $G$ is a finite group, $\text{Cl}(G)$ denotes the set of conjugacy classes of $G$, $Z(G)$ denotes the center of $G$, $G'$ stands for the commutator subgroup of $G$, $\text{Exp}(G)$ for the exponent of $G$ and $\text{Soc}(G)$ for the socle of $G$, i.e. the subgroup generated by the minimal (non-trivial) normal subgroups of $G$. If $g, h \in G$, then $|g|$ denotes the order of $g$, $g^h = h^{-1}gh$, $(g, h) = g^{-1}h^{-1}gh$ and $g^G$ denotes the conjugacy class of $g$ in $G$. If $X \subseteq G$, then $\langle X \rangle$ denotes the subgroup generated by $X$, $C_G(X)$ denotes the centralizer of $X$ in $G$, $N_G(X)$ denotes the normalizer of $X$ in $G$ and for each $g \in G$ we denote $(g, X) = \{(g, x) : x \in X\}$.

If $g$ is an element of order $n$ in a group then the proper powers of $g$ are the elements of the form $g^d$ with $d \mid n$ and $d \neq 1$. If $\pi$ is a set of prime integers then $g_\pi$ and $g_\pi'$ denote the $\pi$ and $\pi'$ parts of $g$, respectively. The notation $G_\pi$ (respectively, $G_{\pi'}$) refers to a Hall $\pi$-subgroup (respectively, Hall $\pi'$-subgroup) of $G$. Furthermore, if $p$ is a prime integer then $g_p$, $g_p'$, $G_p$ and $G_p'$ are abbreviations of $g_{\{p\}}$, $g_{\{p\}'}$, $G_{\{p\}}$ and $G_{\{p\}'}$, respectively. All the finite groups $G$ appearing in this paper are solvable, so the existence of $G_\pi$ and $G_{\pi'}$ is warranted and these subgroups are unique up to conjugation in $G$ (see e.g. [Rob82 9.1.7]). Moreover, if $G$ is nilpotent then $G_\pi$ and $G_{\pi'}$ are unique.

If $R$ is a ring and $N$ is a normal subgroup of $G$ then the natural map $G \to G/N$ extends to a ring homomorphism $\omega_N : R(G) \to R(G/N)$. The augmentation map of $RG$ is $\omega = \omega_G$. Furthermore, we denote by $V(RG, N)$ the group of units of $RG$ mapped to the identity via $\omega_N$.

If $a = \sum_{x \in G} a_x x \in RG$, with $a_x \in R$ for every $x \in G$, and $X$ is a subset of $G$ then we denote

$$\varepsilon_X(a) = \sum_{x \in X} a_x.$$  

If $g \in G$ then the partial augmentation of $a$ at $g$ is $\varepsilon_{g^G}(a)$. For brevity, when $G$ is our target group, the partial augmentation of $a$ at $g$ is simply denoted $\varepsilon_g(a)$. The following formula, for $N$ a normal subgroup of $G$, is easy to check

$$\varepsilon_{\omega_N(g^GN)}(\omega_N(a)) = \varepsilon_{g^GN}(a) = \sum_{X \in \text{Cl}(G), X \subseteq g^GN} \varepsilon_X(a),$$

(2.1)

where $g^GN = \{xn : x \in g^G, n \in N\}$.

Our main tool is the following well known result.

Proposition 2.1. [MRSW87] Let $G$ be finite group and let $u$ be a torsion element of $V(ZG)$. Then $u$ is rationally conjugate to an element of $G$ if and only if $\varepsilon_g(u^d) \geq 0$ for every $d \mid n$ and every $g \in G$. 


We can now give the proof of Theorem 1.3. Let $c \in N$ with $\omega_{G'}(c) = \omega_{G'}(b)$. By the main result of [Mar17] there is $c \in N$ such that for every prime $q$ we have that $u_q$ is conjugate in $Z_qG$ to $c_q$. Then, by [Her08] Lemma 2.2, if $\varepsilon_x(u) \neq 0$ then $x_q$ is conjugate to $c_q$ in $G$ and, in particular, $x \in N$. Thus, if $\varepsilon_x(u) \neq 0$ then $\omega_{G'}(x_q) = \omega_{G'}(c_q) = \omega_{G'}(u_q) = \omega_{G'}(b_q)$ for every prime integer $q$, and hence $xG' = cG' = bG'$. This proves that if $\varepsilon_x(u) \neq 0$ then $x \in bG'$.

Let $K$ be a subgroup of $N_{p'}$, which is maximal with respect to the following condition: $K \cap G'_{p'} = 1$. Then $N/K$ is cyclic. By [MR18b] Proposition 1.1], $0 \leq \sum_{k \in K} |C_G(xk)|\varepsilon_{xk}(u)$ for every $x \in N$, however, by the previous paragraph, if $\varepsilon_{xk}(u) \neq 0$ then $xk \in bG' = xG'$. Hence $k \in K \cap G' \cap N_{p'} = K \cap G'_{p'} = 1$. Therefore $0 \leq \sum_{k \in K} |C_G(xk)|\varepsilon_{xk}(u) = |C_G(x)|\varepsilon_x(u)$. This proves that $\varepsilon_x(u) \geq 0$ for every $x \in G$. Moreover, by Proposition 3.1 and Theorem 3.3 of [MR17], we have $\varepsilon_g(u^d) \geq 0$ for every $g \in G$ and every $d \mid n$. Then $u$ is rationally conjugate to an element of $G$, by Proposition 2.4. Since $u \in V(ZG, N)$, necessarily $g \in N$. 

Another important tool is the following result [Her08] Remark 2.4.

**Lemma 2.2.** Let $u$ be a torsion element of $V(ZG)$ and let $g \in G$. If $\varepsilon_g(u) \neq 0$ then the order of $g$ divides the order of $u$.

From now on we use (ZC) as an abbreviation of “the Zassenhaus Conjecture”.

**Remark 2.3.** In several arguments we will assume that (ZC) holds for proper quotients of $G$ or for proper powers of a particular torsion element $u$ of $V(ZG)$.

1. In the first case, by [2.1], we have $\varepsilon_{gN}(u) \geq 0$ for every $g \in G$ and every $1 \neq N \trianglelefteq G$.

2. In the second case, by Proposition 2.4, to prove that $u$ is rationally conjugate to an element of $G$ it suffices to show that $\varepsilon_g(u) \geq 0$ for every $g \in G$.

3. **The Zassenhaus Conjecture holds for cyclic-by-p-groups**

In this section we prove Theorem 1.1 whose statement is precisely the title of the section. The first part of the section considers the following broader setting rather than that of Theorem 1.1 because some of the lemmas will be used also in subsequent sections.

Throughout $G$ is a finite group and $A$ is normal cyclic subgroup of $G$ such that $G/A$ is nilpotent.

As $A$ is normal in $G$, so is $C_G(A)$. Moreover, as $G/A$ is nilpotent, so is $C_G(A)/A$. Furthermore, since $A$ is central in $C_G(A)$, the latter is nilpotent. Finally, as $\text{Aut}(A)$ is abelian we conclude that $G/C_G(A)$ is abelian and hence $G' \subseteq C_G(A)$, i.e. $(G', A) = 1$. We record these for future use:

**Lemma 3.1.** $C_G(A)$ is a normal nilpotent subgroup of $G$ containing $G'$. Thus, $(G', A) = 1$.

The following lemma shows more restrictions on a minimal cyclic-by-nilpotent counterexample to (ZC) and on its possible negative partial augmentations.
Lemma 3.2. Suppose that (ZC) holds for every proper quotient of $G$ and let $u$ be a torsion unit of $V(ZG)$. Then

(1) If one of the following conditions hold for $x \in G$ then $\varepsilon_x(u) \geq 0$.
   (a) $x \notin C_G(A)$.
   (b) $(x,G)$ contain a non-trivial normal subgroup of $G$.

(2) If $u$ is not rationally conjugate to an element of $G$ then every prime divisor of the order of $A$ divides the order of $u$.

Proof. Let $x \in G \setminus C_G(A)$ and take $N = \langle (a,x^g) : a \in A, g \in G \rangle$. Then $N$ is a non-trivial normal subgroup of $G$ contained in $G'$. By Remark 2.3.(1) we have $\varepsilon_{x^g}N(u) \geq 0$.

On the other hand, if $g, h \in G$ and $a \in A$ then, using $(G', A) = 1$ (Lemma 3.1), we have

$$((a,x^g)x^h) x^g = a^{-1} a^{x^h} x^g = a^{-1} a^{x^h} (x^h)^{-1} g (x^h)^{-1} x^h$$
$$= a^{-1} a^{x^h} (h^{-1} g, (x^h)^{-1}) x^h = a^{x^h} (h^{-1} g, (x^h)^{-1}) a^{-1} x^h = a^{x^h} x^g (a^{x^h})^{-1}$$

This proves that $x^G N = x^G$. Therefore $\varepsilon_x(u) = \varepsilon_{x^g}N(u) \geq 0$, as desired.

Let $N$ be a non-trivial normal subgroup of $G$ contained in $(x,G)$. Then for every $n \in N$ and $h \in G$ there is $g \in G$ with $n^{h^{-1}} = (x,g)$. Thus $x^h n = (x^{(h^{-1})} n^h = (x(g,x^{-1} g)^{-1}) h = x^g h \in x^G$. Therefore $\varepsilon_x(u) = \varepsilon_{x^g}A_p(u) \geq 0$, by Remark 2.3.(1).

We may assume without loss of generality that the order, say $n$, of $u$ is minimal among the torsion units which are not rationally conjugate to any element of $G$.

By Proposition 2.1 and (1), we have that $x \notin C_G(A)$. By means of contradiction, let $p$ be a prime divisor of the order of $A$ not dividing $m$. By Lemma 2.2 $\varepsilon_x(u) = 0$ for every $g \in G$ with $g_p \neq 1$ and, in particular, $x_p = 1$. However, $A_p$ is a non-trivial normal subgroup of $A$ and hence $\varepsilon_x(u) = \varepsilon_{x^g}A_p(u) \geq 0$ by (2.1)

Remark 2.3.(1). Hence $\varepsilon_x(u) > 0$ for some $g \in x^G A_p$. This leads to a contradiction since $(x,A) = 1$ and hence $g_p \neq x_p = 1$.

Lemma 3.3. Suppose that $A$ is a Hall subgroup of $G$. Then

(1) $C_G(A) = C_G(Soc(A))$.

(2) If $Z(G)_p = 1$ for some prime integer $p$ then the Sylow $p$-subgroups of $G$ are abelian.

Proof. By assumption $G = A \rtimes N$ where $A$ and $N$ are Hall subgroups of $G$ and $N$ is nilpotent.

If $p$ and $q$ are prime integers with $p \mid |A|$ and $q \mid |N|$ then $q \neq p$. This implies that $C_N(A_p) = C_{N_q}(Soc(A_p))$. Thus

$$C_N(A) = \prod_{q \mid |N|} \bigcap_{p \mid |A|} C_{N_q}(A_p) = \prod_{q \mid |N|} \bigcap_{p \mid |A|} C_{N_q}(Soc(A)) = C_N(Soc(A))$$

and therefore $C_G(A) = A \rtimes C_N(A) = A \rtimes C_N(Soc(A)) = C_G(Soc(A))$.

Suppose that one Sylow $p$-subgroup of $G$ is non-abelian. Then $N_p$ is non-abelian. Consider the homomorphism $\alpha : N_p \rightarrow Aut(A)$ mapping $g \in N_p$ to the automorphism of $A$ given by conjugation by $g$. Since $Aut(A)$ is abelian, $\ker(\alpha)$ is a non-trivial normal subgroup of $N_p$. As $N$ is nilpotent, $1 \neq \ker(\alpha) \cap Z(N_p) \subseteq Z(G)_p$.

□
Proof of Theorem 1.1. Assume that \( p \) is a prime integer, \( G \) is a finite group and \( A \) is a normal cyclic subgroup of \( G \) such that the factor group \( G/A \) is a \( p \)-group. We may assume without loss of generality that \( A \) is a Hall \( p' \)-subgroup. Then \( G = A \times P \) where \( P \) is a Sylow \( p \)-subgroup of \( G \). Furthermore, by means of contradiction, we assume that \( G \) is a counterexample of minimal order to the theorem and that \( u \) is a torsion unit of \( V(\mathbb{Z}G) \) of minimal order which is not rationally conjugate to any element of \( G \). Hence the Zassenhaus Conjecture holds for every proper quotient of \( G \). By Proposition 2.1 there exists \( x \in G \) such that \( \varepsilon_x(u) < 0 \). As \( C_G(A) \) has at most one non-cyclic Sylow subgroup, by Theorem 1.3, \( u \notin V(\mathbb{Z}G, C_G(A)) \) i.e. \( \omega_{C_G(A)}(u) \neq 1 \). Furthermore, by Lemma 3.2 (1a) it follows that \( x \) lies in \( C_G(A) \).

Set \( m = |u| \). By Lemma 3.2 (2), \( m \) is multiple of every prime dividing the order of \( A \). Moreover, \( p \) divides \( m \), since \( \omega_{C_G(A)}(u) \neq 1 \). Thus every prime divisor of the order of \( G \) divides \( m \). It is well-known that (ZC) holds for cyclic-by-abelian groups [CMdR13]. Thus \( P \) is not abelian. By Lemma 3.3 (2) there exists a central element \( z \) of \( G \) of order \( p \). Then, by the induction hypothesis \( \omega(z)(u) \) is conjugate to \( \omega(z)(g) \) in the units of \( \mathbb{Q}(G/z) \), for some \( g \in G \). As \( 1 \neq \omega_{C_G(A)}(u) = \omega_{C_G(A)/z}(\omega(z)(u)) \) we have \( \omega(z)(g) \notin C_G(A)/z \). Hence \( g \notin C_G(A) \). On the other hand, by the above paragraph the order of \( \omega(z)(u) \), \( \omega(z)(g) \) and \( g \) are divisible by all the prime divisors of the order of \( A \). Then \( Soc(A) \subseteq \langle g^p \rangle \subseteq \langle g \rangle \) and hence \( C_G(g) \subseteq C_G(g^p) \subseteq C_G(Soc(A)) = C_G(A) \), by Proposition 3.3 (1). Thus, \( g \) lies in \( C_G(A) \), a contradiction. This finishes the proof of Theorem 1.1.

4. A formula for partial augmentations

In this section we revisit a formula from [CMdR13].

For any abelian normal subgroup \( N \) of \( G \) let

\[
K_N = \left\{ K \leq N : \frac{N}{K} \text{ is cyclic and } K \text{ does not contain any non-trivial normal subgroup of } G \right\}.
\]

For every \( K \in K_N \), we select a linear character \( \psi_K \) of \( N \) with kernel \( K \) and we fix a representation \( \rho_K \) of \( G \) affording the induced character \( \psi^G_K \). Observe that if \( K_1 \) and \( K_2 \) are conjugate in \( G \), then \( \psi^G_{K_1} = \psi^G_{K_2} \) and therefore we may assume \( \rho_{K_1} = \rho_{K_2} \).

Given \( g, h \in G \) and a subgroup \( K \) of \( G \) let

\[
X_{K,g,h} = \{ t \in G : g^t \in hK \}
\]

and

\[
Y_{K,g,h} = \{ k \in K : k = (g^h, h^{-1}x) \text{ for some } x \in G \}.
\]

The following lemma will be used in Section 5.

Lemma 4.1. Let \( g \in G \), \( x \in X_{K,g,h} \). Then \(|X_{K,g,h}| \) is a multiple of \(|C_G(g)|\) and \(|X_{K,g,h}| \leq |C_G(g)| |Y_{K,g,h}| \).

Proof. Clearly \( C_G(g)X_{K,g,h} \subseteq X_{K,g,h} \) and therefore \( X_{K,g,h} \) is a union of right \( C_G(g) \)-cosets and in particular \(|X_{K,g,h}| \) is a multiple of \(|C_G(g)|\).

Let \( x, y \in X_{K,g,h} \). Then

\[
(g^x, x^{-1}y) = x^{-1}g^{-1}xy^{-1}gy = (g^x)^{-1}y^p \in K.
\]
Therefore \((g^z, x^{-1}y) \in Y_{K,g,x}\). Moreover, if \(z\) is another element of \(X_{K,g,h}\) then \((g^z, x^{-1}y) = (g^z, x^{-1}z)\) if and only if \(g^z = g^2\) if and only if \(yz^{-1} \in C_G(g)\). This proves that \(yC_G(g) \mapsto (g, xy^{-1})\) defines an injective map from the set of right \(C_G(g)\)-cosets contained in \(X_{K,g,h}\) to \(Y_{K,g,x}\). Thus \(|X_{K,g,h}| \leq |C_G(g)| |Y_{K,g,x}|\). □

For a square matrix \(U\) with entries in \(\mathbb{C}\) and \(\alpha \in \mathbb{C}\), let \(\mu_U(\alpha)\) denote the multiplicity of \(\alpha\) as eigenvalue of \(U\).

**Proposition 4.2.** Let \(G\) be a finite group such that \((ZC)\) holds for every proper quotient of \(G\). Let \(N\) be an abelian normal subgroup of \(G\), let \(\mathbb{K} = \mathbb{K}_N\) and let \(\mathcal{K}\) be a set of representatives of the \(G\)-conjugacy classes of the elements of \(\mathcal{K}\). Let \(u\) be an element of order \(m\) in \(V(\mathbb{Z}G) \setminus V(\mathbb{Z}G,N)\) such that every proper power of \(u\) is rationally conjugate to an element of \(G\). Let \(x \in N\) with \(x^m = 1\) and let \(f\) be a positive integer such that \(u^f\) is rationally conjugate to an element \(\gamma\) of \(N\). Then

\[
\sum_{K \in \mathcal{K}} \varphi(\{N : K\}) \mu_{\rho_K(u)}(\psi_K(x)) = \frac{\varphi(m)}{m} |C_G(x)| \varepsilon_x(u) + \sum_{K \in \mathcal{K}} \sum_{\varphi([N : K])} |N_G(K)| \sum_{z \in \gamma^G} |X_{K,x^f,z}|.
\]

(4.2)

**Proof.** By [CMdR13, Lemma 3.1] we have

\[
\sum_{K \in \mathcal{K}} \varphi(\{N : K\}) \mu_{\rho_K(u)}(\psi_K(x)) = \frac{\varphi(m)}{m} |C_G(x)| \varepsilon_x(u) + \frac{1}{f} \sum_{K \in \mathcal{K}} \varphi(\{N : K\}) \mu_{\rho_K(u^f)}(\psi_K(x^f)).
\]

(4.3)

Choose a transversal \(T\) of \(N\) in \(G\). Observe that \(N \subseteq C_G(\gamma)\), as \(N\) is abelian. Thus \(|\{gN \in G/N : x^f \in \gamma^G\}| = |C_G(\gamma) : N| |x^f K \cap \gamma^G|\). As \(u^f\) is conjugate to \(\gamma\) in \(QG, \rho_K(u^f), \rho_K(\gamma)\) and \(\text{diag}(\psi_K(\gamma^g) : g \in T)\) are conjugated as complex matrices. Thus

\[
\mu_{\rho_K(u^f)}(\psi_K(x^f)) = |\{gN \in G/N : \psi_K(x^f) = \psi_K(\gamma^g)\}| = |\{gN \in G/N : \gamma^g \in x^f K\}| = |C_G(\gamma) : N| |x^f K \cap \gamma^G|.
\]

Therefore

\[
\sum_{K \in \mathcal{K}} \varphi(\{N : K\}) \mu_{\rho_K(u^f)}(\psi_K(x^f)) = \sum_{K \in \mathcal{K}} \sum_{\varphi([N : K])} \frac{\varphi([N : K])}{|N_G(K)|} \mu_{\rho_K(u^f)}(\psi_K(x^f)) = |C_G(\gamma) : N| \sum_{K \in \mathcal{K}} \varphi(\{N : K\}) |x^f K \cap \gamma^G|.
\]

(4.4)

On the other hand,

\[
\sum_{t \in G} |(x^f)^t K \cap \gamma^G| = |\{(t, z) \in G \times \gamma^G : (x^f)^t \in zK\}| = \sum_{z \in \gamma^G} |X_{K,x^f,z}|.
\]

(4.5)

Then the lemma follows from (4.3), (4.4) and (4.5). □
5. On the Zassenhaus Conjecture for cyclic-by-Hamiltonian groups

In this section we investigate the Zassenhaus Conjecture for cyclic-by-Hamiltonian groups. In the first part of the section we do not consider any additional hypothesis but after Lemma 5.3 we consider a hypothetical minimal counterexample to (ZC) in the class of cyclic-by-Hamiltonian groups and prove some features of it. This is used to prove Theorem 1.2 at the end of the section.

So throughout

\[ G \] is a finite group and \( A \) is a cyclic normal subgroup of \( G \) such that \( G/A \) is Hamiltonian.

We start with a description \( \mathbb{K}_N \) for \( N \) an abelian subgroup of \( G \) containing \( A \). The following lemma generalizes Remark 3.2 in [CMdR13].

Lemma 5.1. Let \( G \) be a finite group containing a normal cyclic subgroup \( A \) such that \( G/A \) is nilpotent. Let \( N \) be an abelian normal subgroup of \( G \) containing \( A \). Then

\[ \mathbb{K}_N = \{ K \leq N : K \cap A = K \cap Z(G) = 1 \text{ and } N/K \text{ is cyclic} \} \]

and for every \( K \in \mathbb{K}_N \) we have

\[ |\mathbb{K}_N| \leq |K| = \frac{|N|}{\text{Exp}(N)}. \]

Proof. Every subgroup of \( A \) and every subgroup of \( Z(G) \) is normal in \( G \). Thus, if \( K \in \mathbb{K}_N \) then \( K \cap A = K \cap Z(G) = 1 \). This proves one inclusion of the first equality. Conversely, let \( K \) be a subgroup of \( N \) containing a non-trivial normal subgroup \( X \) of \( G \) and such that \( K \cap A = 1 \). Let us use bar notation for reduction modulo \( A \). Observe that \( \overline{X} \cap Z(\overline{G}) \neq 1 \) since \( \overline{X} \) is a non-trivial normal subgroup of \( \overline{G} \), and the latter is nilpotent. Let \( \overline{x} \) be a non-trivial element of \( \overline{X} \cap Z(\overline{G}) \). Then \( (x, g) \in A \) for every \( g \in G \). On the other hand, as \( X \) is a normal subgroup of \( G \), \( (x, g) \in A \cap X \subseteq A \cap K = 1 \). Thus \( x \) is a central element of \( G \) and therefore \( 1 \neq x \in K \cap Z(G) \). This proves the first equality.

Let \( L_N = \{ L \subseteq N : [N : L] = \text{Exp}(N) \text{ and } N/L \text{ is cyclic} \} \). To finish the proof we will show that \( \mathbb{K}_N \subseteq L_N \) and \( |L_N| \leq \frac{|N|}{\text{Exp}(N)} \). Indeed, suppose that \( K \in \mathbb{K}_N \). Then \( [N : K] \leq \text{Exp}(N) \). This is a consequence of the well known fact that there is an involution \( \alpha \) of the lattice of subgroups of \( N \) such that \( N/H \cong \alpha(H) \) for every \( H \leq N \). In particular, if \( K \in \mathbb{K}_N \) then \( \alpha(K) \) is a cyclic subgroup of \( N \) of order \( [N : K] \) and hence \( [N : K] \leq \text{Exp}(N) \). Suppose that \( [N : K] < \text{Exp}(N) \) and let \( X = N^{[N : K]} \). Then \( 1 \neq X \subseteq K \). As \( N \) is normal in \( G \) and \( X \) is a characteristic subgroup of \( N \) we deduce that \( X \) is normal in \( G \). Thus \( K \) contains a non-trivial normal subgroup of \( G \), contradicting the hypothesis. Thus \( [N : K] = \text{Exp}(N) \), i.e. \( K \in L_N \). Using the involution \( \alpha \) we see that the cardinality of \( L_N \) coincides with the cardinality of the set \( Y_N \) of cyclic subgroups of \( N \) of order \( \text{Exp}(N) \). So it remains to prove that \( |Y_N| \leq \frac{|N|}{\text{Exp}(N)} \). For that we may assume without loss of generality that \( N \) is a \( p \)-group because if \( N = P \times Q \) with \( P \) and \( Q \) of coprime order then the map \( (K, L) \mapsto K \times L \) defines a bijection from \( Y_P \times Y_Q \) to \( Y_N \). So suppose that \( N \) is a \( p \)-group and let \( p^r \) be the exponent of \( N \). Then \( N = P \times Q \) with \( P \cong C_{p^l} \), for some \( l \geq 1 \) and \( Q \) of exponent smaller than \( p^r \). Then the elements of \( N \) generating a cyclic subgroup of order strictly smaller than \( p^r \) are precisely the elements of \( P^{p^l} \times Q \). Therefore \( N \) has exactly \( |Q| - p^{(p^{l-1})} \) elements of order \( p^r \) and hence \( |Y_N| = \frac{|N| - p^{(p^{l-1})} \cdot |Q|}{(p-1)p^{l-1}} = \frac{|N| - p^{(p^{l-1})} \cdot |Q|}{p^r} \leq \frac{|N|}{p^r} \), because \( p^{l-1} \leq 1 \). □
We also fix a generator $a$ of $A$ and set

$$n = |A|, \quad C = C_G(A), \quad D = Z(C).$$

Furthermore, we assume that $G$ is not cyclic-by-abelian, for otherwise (ZC) holds for $G$ by the main result of [CMdR13]. As a consequence, $n$ must be even. Further, note that $G' \subseteq C$ by Lemma 3.1. Combining this with the fact that $G/A$ is Hamiltonian we have $G' \subseteq A \times \langle \nu \rangle$ with $\nu$ an element of $G' \setminus A$ of order 2. Since $G'_2$ is cyclic and $D$ is abelian, Theorem 1.3 implies that

**Proposition 5.2.** Every torsion unit of $V(ZG, D)$ is rationally conjugate to an element of $D$.

We now deal with some partial augmentations.

**Lemma 5.3.** Suppose that (ZC) holds for proper quotients of $G$. If $u$ is a torsion unit of $V(ZG)$ and $g \in G \setminus D$, then $\varepsilon_g(u) \geq 0$.

**Proof.** Let $C_1 = C_G(A, \nu)$. Let $x, y \in G$ such that $(x, y) \notin A$. As $G/A$ is Hamiltonian, $(A, \nu) = \langle A, (x, y) \rangle = \langle A, x_2 \rangle \subseteq \langle A, x^2 \rangle$. Therefore $\nu \in \langle x^2, A \rangle$ and hence, if moreover $x \in C$ then $x \in C_1$.

Claim: If there is $x \in G$ such that $\langle (g, x) \rangle$ is non-trivial and normal in $G$ and $(g, x, g) = (g, G, x, G) = 1$ then \( \varepsilon_g(u) \geq 0 \).

Indeed, by

$$\langle (g, x) \rangle^h = (g, x)g(h, x) = g(h, x)g(x^2, h) = g(x^2, h) = g(h, x)g(x^2, h) = (g, h)x = g^h x$$

we have $g^o N = g^C$ for $N = \langle (g, x) \rangle$. Therefore $\varepsilon_g(u) = \varepsilon_{gC}(u) \geq 0$, as desired.

Suppose that $g \in G \setminus D$. If $g \notin C$ then $\varepsilon_g(u) \geq 0$ by Lemma 3.2. If $g \in C$ then $\varepsilon_g(u) = 0$ by Lemma 3.2. Thus we may assume that $g \in C$. If $(g, \nu) \neq 1$ then $(g, \nu) = a^n$ and hence $\langle a^n \rangle$ is a non-trivial normal subgroup of $G$ contained in $(g, G)$. Then $\varepsilon_g(u) \geq 0$ by Lemma 3.2. Thus we may assume that $(g, \nu) = 1$. Suppose that $(g, y^2) \neq 1$ for some $y \in C$. Then $y^2 \in C_1$ and $(g, y^2) = (g, y)(y, y)^{\nu} \in A$, since $G' : G' \cap A \leq 2$. Thus $g$ and $x = y^2$ satisfy the conditions of the claim and hence $\varepsilon_g(u) \geq 0$. Thus as well as $(g, \nu) = 1$ we may also assume that $(g, x^2) = 1$ for every $x \in C$. Therefore $C/C_G(g)$ is an elementary abelian 2-group and hence $|g^C|$ is a power of 2. Moreover, $(g, x)(g, x)^{-1} = 1$, i.e. $(g, x)^x = (g, x)^{-1}$ for every $x \in C$. If $(g, x) \in A$ then $(g, x) = (g, x)^x = (g, x)^{-1}$ and hence $(g, x) \in \langle a^2 \rangle$. Therefore, if $(g, x) \in A \setminus \{1\}$ then $(g, x) = 1$ is a non-trivial normal subgroup of $G$ contained in $(g, G)$. In such case $\varepsilon_g(u) \geq 0$, by Lemma 3.2. Thus $\varepsilon_g(u) = \varepsilon_{gC}(u) \geq 0$. Therefore we may assume that $(g, x) \notin A$ for every $x \in C \setminus C_G(g)$.

Then for every $x \in C \setminus C_G(g)$ we have $(g, x) = a^i \nu$ for some $i$ and hence $a^i \nu^x = (g, x)^x = (g, x)^{-1} = a^{-i} \nu$. Therefore $\nu^x = \nu$ and $a^i = a^{-i}$. Thus $a^i \in \langle a^2 \rangle$ and $(g, x) \in \{\nu, a^2 \nu\}$. By the first paragraph of the proof we have $C \setminus C_G(g) \subseteq C_1$ and $g^C \subseteq \{g, vg, a^2 \nu g\}$. As $|g^C|$ is a power of 2 we deduce that $g^C$ is either $\{g, vg\}$ or $\{a^2 \nu g\}$. Replacing $\nu$ by $a^2 \nu$ one may assume that $g^C = \{g, vg\}$. Fix $x \in C$ with $g^x = vg$. Moreover, as $1 \in C_1 \cap C_G(g)$, we have that $C \setminus C_G(g)$ is properly included in $C_1$ and hence $C_1$ is a subgroup of $C$ with $|C_1| > |C|$. Thus $C = C_1$.

As $G/(A, \nu)$ is abelian, for every $h \in G$ we have $g^h = ug$ and $x^h = vx$ with $u, v \in (A, \nu)$. Then $(u, g) = (u, x) = (v, g) = (v, x) = (u, v) = 1$ and therefore $\nu^h = (g^h, x^h) = (ug, vx) = (g, x) = \nu$. Thus $\nu = (g, x)$ is central in $G$ and hence $g$ and $x$ satisfies the conditions of the claim, and once more $\varepsilon_g(u) \geq 0$. \qed
In the remainder we suppose that (ZC) holds for proper quotients of $G$ but not for $G$. Therefore, there is an element $u \in ZG$ such that $u$ is not rationally conjugate to any element of $G$ and we assume that the order of $u$ is minimal with this property.

We fix the following notation

$$m = |u|, \quad f = |\omega_D(u)| \quad K = K_D.$$  

By Proposition 5.2, $f > 1$ and by Proposition 2.1 and Lemma 5.3 there are $\delta, \gamma \in D$, with $\varepsilon_\delta(u) < 0$ and $u^f$ rationally conjugate to $\gamma$.

**Lemma 5.4.** The orders of $D$ and $C$ and the exponent of $D$ are divisible by the same primes and every primes dividing $|D|$ divides $m$.

**Proof.** As $C$ is nilpotent and $D = Z(C)$, we have that $|C|$, $|D|$ and $\text{Exp}(D)$ are divisible by the same primes. Let $p$ be a prime divisor of $C$. Then $A$ is contained in a cyclic subgroup $B$ of order divisible by $p$. As $G/A$ is Hamiltonian, $B$ is normal in $G$. Applying Lemma 3.2 to $B$ we conclude that $p$ divides $m$.

$$\Box$$

Write $f = f'f''$ with $f'$ and $f''$ positive integers so that $\gcd(f'', |D|) = 1$ and the prime divisors of $f'$ divides $|D|$. Then $m = f''m'$ with $m'$ having the same prime divisors as $\text{Exp}(D)$, by Lemma 5.3. In particular,

$$\phi(\text{Exp}(D)) \cdot \frac{\phi(m')}{m'} = \frac{\phi(m')}{m'}.$$

By (1.2) we have that

$$\frac{|C_G(y) : D|}{f} \sum_{K \in K} \frac{\phi([D : K])}{|N_G(K)|} \sum_{z \in y^G} |X_{K,\delta', z}| = \sum_{K \in K} \phi([D : K]) \mu_{p_K(u)}(\psi_K(\delta))$$

$$- \frac{\phi(m')}{m'} |C_G(\delta)| \varepsilon_\delta(u) > 0,$$

which implies that $K \neq \emptyset$. Moreover, if $x$ is an element of $K \cap (A \times \langle \nu \rangle)$ with $K \in K$ then $x^2 \in K \cap A = 1$ and $x \neq a^{\frac{1}{2}}$. Also $K \cap (A \times \langle \nu \rangle) \neq 1$ because $D/K$ is cyclic. This proves the following:

(5.7) For every $K \in K$, we have $K \cap G' = K \cap (A \times \langle \nu \rangle) = \begin{cases} \langle \nu \rangle, \\ \langle a^{\frac{1}{2}} \nu \rangle. \end{cases}$

It follows that $\nu \notin Z(G)$ and hence $\nu^G = \{ \nu, a^{\frac{1}{2}} \nu \}$. Moreover, for every $g, h \in G$ we have $Y_{K, g, h} \subseteq K \cap G'$ and hence $|Y_{K, g, h}| \leq 2$. Thus, by Lemma 1.1, $|X_{K, g, h}|$ is either 0, $|C_G(g)|$ or $2|C_G(g)|$. In particular

(5.8) $|X_{K, \delta', z}| \leq 2|C_G(\delta')|$, for every $z \in G$.

Furthermore, if $(\delta', G) \cap \nu^G = \emptyset$ then $Y_{K, \delta', z} \subseteq K \cap A = 1$. If $2 \mid f$ then $(\delta', G) \subseteq A$ and therefore

(5.9) if $2 \mid f$ or $(\delta', G) \cap \nu^G = \emptyset$ then $|X_{K, \delta', z}| \leq |C_G(\delta')|$, for every $z \in G$. 

Combining the last two inequalities we obtain
\[
|X_{K,\delta^f,z}| \leq \frac{2}{\gcd(2, f')} |C_G(\delta^f)| = \frac{2}{\gcd(2, f')} |C_G(\delta^f)|. 
\]

Let \( B \) be the subgroup of \( A \) of order \( \gcd(f, n) (= \gcd(f', n)) \). As \( \delta \in D \), we have \( \langle \delta^f \rangle = \langle \delta'^f \rangle \) and therefore \( C_G(\delta^f) = C_G(\delta'^f) \). Thus, if \( g \in C_G(\delta^f) \) then \( (\delta, g)^{f'} = (\delta'^f, g) = 1 \) and so \( (\delta, g) \in B \times \langle \nu \rangle \). Hence \( (\delta, C_G(\delta^f)) \subseteq B \times \langle \nu \rangle \) and if \( f \) is odd then \( (\delta, C_G(\delta^f)) \subseteq B \subseteq A \). This proves that:
\[
|C_G(\delta^f) : C_G(\delta)| \leq \gcd(2, f) \ gcd(f, n) = \gcd(2, f') \ gcd(f', n).
\]

Also, if \( \gcd(f, n) \neq 1 \) then, by Lemma 3.1, \( B \not\subseteq (\delta, C_G(\delta^f)) \). Thus,
\[
(5.12) \quad \text{if } \gcd(f, n) \neq 1 \text{ then } |C_G(\delta^f) : C_G(\delta)| < \gcd(2, f) \ gcd(f, n).
\]

Further, by Lemma 5.1 for every \( K \in \mathbb{K} \) we have \( [D : K] = \operatorname{Exp}(D) \). Combining this with (5.11), and then using \( D \subseteq C_G(\gamma) \) and the inequality \( |\mathbb{K}| \leq \frac{|D|}{\operatorname{Exp}(D)} \) from Lemma 5.1 and (5.10) we obtain
\[
\begin{align*}
&\frac{|C_G(\gamma) : D|}{f} \sum_{K \in \mathbb{K}} \frac{\varphi([D : K])}{|N_G(K)|} \sum_{z \in \gamma^d} |X_{K,\delta^f,z}| \\
&\leq \frac{2}{f} \frac{|C_G(\gamma) : D|}{\gcd(2, f')} \frac{\varphi(\operatorname{Exp}(D))}{|\mathbb{K}|} \frac{|C_G(\delta^f)|}{|D|} \sum_{K \in \mathbb{K}} \frac{1}{|N_G(K)|} \\
&= \frac{2}{f} \frac{|C_G(\gamma) : D|}{\gcd(2, f')} \frac{\varphi(\operatorname{Exp}(D))}{|\mathbb{K}|} \frac{|G : C_G(\gamma)|}{|D|} |C_G(\delta^f)| \sum_{K \in \mathbb{K}} \frac{1}{|N_G(K)|} \\
&\leq \frac{2}{f} \frac{\varphi(\operatorname{Exp}(D))}{|\mathbb{K}|} \frac{|C_G(\delta^f)|}{|D|} \sum_{K \in \mathbb{K}} |G : N_G(K)| \\
&= \frac{2}{f} \frac{\varphi(\operatorname{Exp}(D))}{|\mathbb{K}|} \frac{|C_G(\delta^f)|}{|D|} \\
&= \frac{2}{f} \frac{\varphi(m')}{|\mathbb{K}|} |C_G(\delta^f)| \\
&= \frac{2}{f'} \frac{\varphi(f'') \varphi(m')}{f' m'} |C_G(\delta^f)| \\
&= \frac{2}{f'} \frac{\varphi(m)}{m} |C_G(\delta^f)|.
\end{align*}
\]

On the other hand, the left side of the equality (12) is positive and \( \varepsilon_\delta(u) \leq -1 \). This implies that
\[
\frac{|C_G(\gamma) : D|}{f} \sum_{K \in \mathbb{K}} \frac{\varphi([D : K])}{|N_G(K)|} \sum_{z \in \gamma^d} |X_{K,\delta^f,z}| \geq \frac{-\varphi(m)}{m} |C_G(\delta)| |N_G(\delta)| \\
\geq \frac{\varphi(m)}{m} |C_G(\delta)|.
\]
the main result of [CMdR13] applies, or holds for every proper quotient \( H \subseteq G \). Thus (ZC) holds for \( u \) ample to the theorem with minimal order and this finishes the proof of (3).

Proof of Theorem 1.2. By means of contradiction we assume that \( f \neq 1 \). To prove (2), we deduce that \( f' = \gcd(2, f') \varphi(f') \). As \( f' \) divides \( f \), we deduce that \( f' \) divides \( f' \). In particular, \( \varphi(f') \leq \gcd(f', n) \). The latter implies that if \( f' \) divides \( f' \) then \( \gcd(f', n) = f' \). In this case \( f' \) is even and hence \( 2 \mid \gcd(f', n) \). Thus the second inequality in (5.15) is strict, by (5.12), and then \( f' \neq \gcd(f', n) = f' \), a contradiction. Therefore

\[
\varphi(f') \leq 2, \quad f' \mid n \quad \text{and} \quad f' \neq 1 \text{ then } f'' \leq 2.
\]

We finish our analyses with the following lemma.

Lemma 5.5. (1) If \( f \) is even then \( f'' = 1 \), \((\delta, G) \subseteq A\) and \((\delta, C_G(\delta)) \not\subseteq A\). (2) If \( f \) is odd then \((\delta, C_G(\delta)) \subseteq B\) and \((\delta, G) \cap \nu G \neq \emptyset\). Furthermore either \( f = f'' = 1 \) or \( f' = 1 \).

(3) \((\delta, G) \not\subseteq A\).

(4) \( G' \subseteq A \times \langle \nu \rangle \subseteq D\) and \( 4 \nmid f \).

Proof. We first observe that by Lemma 3.2 (1b), \( B \times \langle \nu \rangle \not\subseteq (\delta, C_G(\delta)) \) and if \( f' \neq 1 \) then \( B \not\subseteq (\delta, C_G(\delta)) \).

(1) If \( f \) is even then \((\delta, G) \subseteq A\) and \( f' \neq 1 \). Then \( f'' \leq 2 \) and, as \( \gcd(f', f'') = 1 \), we deduce that \( f'' = 1 \). Thus \( |B| = f' \neq 1 \) and hence \( B \not\subseteq (\delta, C_G(\delta)) \) by Lemma 3.2 (1b). By (5.15)

\[
f \leq [C_G(\delta) : C_G(\delta)] = |\delta C_G(\delta)| = |(\delta, C_G(\delta))|.
\]

Therefore \((\delta, C_G(\delta)) \not\subseteq A\) for otherwise \((\delta, G)\) contains \( B\).

(2) If \( f \) is odd then \((\delta, C_G(\delta)) \subseteq B\). If \( f' = 1 \) then \( f'' \neq 1 \) and hence \( \varphi(f) = \varphi(f') = 2 \). Then \( f'' = 3 \). If \( f' \neq 1 \) then, by (5.12) and (5.14) we have \( f' \varphi(f') < f' \) and hence \( \varphi(f') = 1 \). As \( f'' \) is odd we deduce that \( f'' = 1 \).

We now prove that \((\delta, G) \cap \nu G \neq \emptyset\). Otherwise \([X_{K, \delta, \nu, z}] \leq [C_G(\delta)]\) for every \( K \subseteq K \) and \( z \in G \), by (5.10). Using this in the calculations in (5.13) we get \( f' \varphi(f') \leq [C_G(\delta) : C_G(\delta)] \leq f' \). Therefore \( \varphi(f') = 1 \) and \([C_G(\delta) : C_G(\delta)] = f' \).

As \( f \) is odd this implies that \( f'' = 1 \). Therefore \( 1 \neq f = \gcd(f', n) \) and hence \( f = [C_G(\delta) : C_G(\delta)] < f' \), by (5.12). This yields a contradiction and finishes the proof of (2).

(3) If \( f \) is even then \((\delta, G) \) follows from (1). So, if \((\delta, G) \subseteq A\) then \( f \) is odd. Moreover, for every \( g \in G \) we have \((\delta, g) \subseteq A\) in contradiction with (2). This finishes the proof of (3).

(4) By (3), \( G' \subseteq A \times \langle \nu \rangle \subseteq (A, (\delta, G)) \subseteq D\). Then as \( f \) divides \([G : D]\), it also divides \([G : G']\) and the latter is not multiple of 4. So \( 4 \nmid f \).

We are ready for the

Proof of Theorem 1.2. By means of contradiction we assume that \( G \) is a counterexample to the theorem with minimal order and \( u \) is a torsion element of \( V(\mathbb{Z}G) \). Then (ZC) holds for every proper quotient \( H \) because \( H \) is either cyclic-by-abelian, and hence the main result of [CMdR13] applies, or \( H \) satisfies the hypothesis of the theorem, and hence (ZC) holds for \( H \) by the minimality of the order of \( G \). So all the assumptions made above in this section are satisfied and we adopt the above notation: \( \delta, \nu, \gamma, n, m, f, K \). We also let \( n_2 \) denote the order of \( A_2 \).
As the 2-subgroup of $G/A$ is isomorphic to $Q_8$, $G$ has a Sylow 2-subgroup $G_2 = \langle A_2, x, y \rangle$ such that, using bar notation for reduction modulo $A_2$, we have $(\overline{A}_2, \overline{y}) \cong Q_8$.

We now prove the following

Claim. $(D, G) \cap K = 1$ for every $K \in \mathbb{K}$.

If $D_2 \subseteq \langle A_2, (x, y) \rangle$ then $(G, D) \subseteq A$ and hence $(G, D) \cap K \subseteq A \cap K = 1$, as desired. So we suppose that $D_2 \not\subseteq \langle A_2, (x, y) \rangle$. Then we may assume that $D_2 = \langle A_2, x \rangle$. As $G$ is not cyclic-by-abelian, $D_2$ is not cyclic and hence $x^4 \in A_2^\delta$.

Thus the exponent of $D_2$ is either $n_2$ or $2n_2$. Moreover we may assume that one of the following conditions hold:

(A) $\text{Exp}(D_2) = 2n_2$, $D_2 = \langle x \rangle \times \langle \nu \rangle$ with $|\nu| = 2n_2$, 8 \text{ and } A_2 = \langle x^2 \nu \rangle$.

(B) $\text{Exp}(D_2) = n_2$, $D_2 = \langle a_2 \rangle \times \langle x \rangle$ with $x^2 = \nu$, 4 \text{ and } x^4 = x^{-1}a_2 \overline{x}^{-1}$.

We prove that indeed one may assume that one of these conditions hold and at the same time we complete the proof of the claim.

If $\text{Exp}(D_2) = 2n_2$ then $D_2 = \langle x \rangle \times \langle b \rangle$ with $|\nu| = 2n_2$ and $|b| = 2$. As $G$ is not cyclic-by-abelian, $\langle x \rangle$ is not normal in $G$ and hence $x^y = b^{-1}(a^y b)^i$ with $i$ odd. Moreover, $y^2 \in \langle x^2, A \rangle$ and hence $(x, y^2) = 1$. Then $2n_2 \mid 4(i - 1)$ and hence one may assume that $i = 1 + 2n_2$. Then $4 \mid n$. Actually, $8 \mid n$ for otherwise $n_2 = 4$ and $(x, y) = x^2b \in A$. Therefore $(x, y) = a^\overline{x}b \not\in A$ and we may assume that $\nu = b$.

Suppose that $(z, g)$ is a non-trivial element of $K$ with $z \in D, g \in G$ and $K \in \mathbb{K}$. Write $z = x^r b^s$. If $r$ is even then $(z, g) \in A \cap K = 1$, contradicting the hypothesis. Thus $r$ is odd and $(z, g) = x^r b^s c$ with $c \in \langle a^\overline{x} \rangle$. Then $(z, g)^2 = x^{rn} \neq 1$ in contradiction with $|G' \cap K| = 2$.

Suppose now that $\text{Exp}(D_2) = n_2$. Then one may assume that $D_2 = \langle a_2 \rangle \times \langle x \rangle$ with $|\nu| = 4$ and $x^y = x^{-1}a_2^j$ for some integer $i$. If $a_2^j = 1$ then $y$ commutes with $x^2$ and hence the socle of $D_2$ is central in $G$ which contradicts that $\nu$ is a non-zero central element of $D_2$. Therefore $4 \mid n, i = \pm \frac{n}{2}$ and $a_2^j = a_2^i$ with $j \equiv 1 \mod \frac{n}{2}$.

Then one may assume that $\nu = x^2$. As in the previous case, suppose that $(z, g)$ is a non-trivial element of $K$ with $z \in D, g \in G$ and $K \in \mathbb{K}$. Write $z = a_2^s x^r$. If $s$ is even then $(z, g) \in A \cap K = 1$, contradicting the hypothesis. Otherwise $(z, g) = a_2^j x^r c \overline{x}^{zn} a_2^s$ and again $(z, g)^2 = a_2^r x^{zn} \neq 1$.

This finishes the proof of the Claim.

By the claim we have in particular that $(\delta^f, G) \cap \nu^G = \emptyset$. Hence $f$ is even by Lemma 5.3 2 and then $\langle \delta, C_G(\delta^f) \rangle \not\subseteq A$ by Lemma 5.3 1. This implies that $D_2 \not\subseteq \langle A_2, (x, y) \rangle$ and therefore one of the cases (A) or (B) of the proof of the claim holds and we may assume that $\nu = b$ in case (A) and $\nu = x^2$ in case (B).

Let $g \in C_G(\delta^f)$ with $(\delta, g) \not\subseteq A$, since $(\delta_2, g) \in A$ and $1 = (\delta^f_2, g) = (\delta_2, y)^f$, we have $(\delta_2, g) = (\delta_2, y) \not\in A$. Thus $(\delta_2, c)^2 = 1$, as $4 \nmid f$, by Lemma 5.3 3. Hence $(\delta_2, c)$ is an element of order 2 which is not in $A_2$. Assume that (A) holds and let $\delta_2 = x^r b^s$ for some integers $r$ and $s$. Then $(\delta_2, y) = x^{r+n} b^s$. If $r$ is even then this element belongs to $A_2$ and otherwise it has order 4. In both cases we obtain a contradiction. However, if (B) holds then $\delta_2 = a^r x^s$ for some integers $r$ and $s$. Then $(\delta_2, c) = a^{r+s} x^{s} b^{2s}$. If $s$ is even then this element belongs to $A_2$ and otherwise its order is 4, again a contradiction. So the proof Theorem 1.2 is finished. \hfill \Box

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