New variational and multisymplectic formulations of the Euler-Poincaré equation on the Virasoro-Bott group using the inverse map

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Abstract

We derive a new variational principle, leading to a new momentum map and a new multisymplectic formulation for a family of Euler–Poincaré equations defined on the Virasoro-Bott group, by using the inverse map (also called ‘back-to-labels’ map). This family contains as special cases the well-known Korteweg-de Vries, Camassa-Holm, and Hunter-Saxton soliton equations. In the conclusion section, we sketch opportunities for future work that would apply the new Clebsch momentum map with 2-cocycles derived here to investigate a new type of interplay among nonlinearity, dispersion and noise.

1 Introduction

The family of equations

\[
\alpha(u_t + 3uu_x) - \beta(u_{xxx} + 2u_xu_{xx} + uu_{xxx}) + au_{xxx} = 0,
\]

where \(a, \alpha, \beta\) are real nonnegative parameters, was introduced in \[33\] as the geodesic flow dynamics associated to a variety of right-invariant metrics on the Virasoro-Bott group (see also \[34\], \[42\]). Various well-known completely integrable soliton equations are special cases of \((1.1)\). For example, when \(\alpha = 1\) and \(\beta = 0\), then equation \((1.1)\) specialises to the Korteweg-de Vries equation \((35), (18)\)

\[
u_t + 3uu_x + au_{xxx} = 0;
\]

whereas for \(\alpha = \beta = 1\) one obtains the Camassa-Holm equation \((9), (10), (24)\)

\[
u_t - u_{xxx} + 3uu_x - 2u_xu_{xx} - uu_{xxx} + au_{xxx} = 0;
\]
and for $\alpha = 0$ and $\beta = 1$ one finds the Hunter-Saxton equation \([30], \ [31]\),
\[
    u_{xxt} + 2u_x u_{xx} + uu_{xxx} - au_{xxx} = 0.
\] (1.4)

The aim of this paper is to derive a new canonical variational principle for the family of equations \((1.1)\), and thereby determine its new multisymplectic formulation. By doing so, we obtain unified variational and multisymplectic characterizations of all three of the well-known integrable soliton equations, KdV, CH, and HS, which are subcases of the general family of equations in \((1.1)\).

Variational principles have proved extremely useful in the study of nonlinear evolution PDEs. For instance, they often provide physical insights into the problem being considered; facilitate discovery of conserved quantities by relating them to symmetries via Noether’s theorem; allow one to determine approximate solutions to PDEs by minimizing the action functional over a class of test functions (see, e.g., \([12]\)); and provide a way to construct a class of numerical methods called variational integrators (see \([39]\), \([40]\)). A canonical variational principle for the KdV equation expressed in terms of the velocity potential was first proposed by Whitham \([52]\); see also \([12], \ [13], \ [31], \ [38]\). In fact, there is an infinite family of such Lagrangians, as shown by Nutku \([45]\). Two canonical variational principles for the dispersionless CH equation \((a = 0)\) were introduced in \([15]\) and \([36]\). Two variational structures are also known for the HS equation with \(a = 0\) (see \([1], \ [30], \ [31]\)).

Multisymplectic structures of Hamiltonian PDEs were first considered by Bridges \([6]\) as a natural generalization of the symplectic structure of Hamiltonian ODEs. Among other applications, the multisymplectic formalism is useful for, e.g., the stability analysis of water waves (see \([6], \ [7]\)) and construction of a class of numerical methods known as multisymplectic integrators (see \([8], \ [39]\)). It has been observed in the literature that, as for symplectic integrators for Hamiltonian ODEs, multisymplectic integrators demonstrate superior performance in capturing long time dynamics of PDEs (see \([14]\)). To the best of our knowledge, only one multisymplectic formulation of the KdV equation has been considered so far (see \([7], \ [55]\)). Four different multisymplectic formulations are known for the dispersionless CH equation (see \([11], \ [15], \ [36]\)). Two multisymplectic structures for the HS equation with \(a = 0\) were described in \([43]\).

**Main content** The main content of the remainder of this paper is, as follows.

In Section 2 we review the Euler-Poincaré theory on the Virasoro-Bott group and then construct a new canonical variational principle in terms of the inverse map. The main result of this section is Theorem 2.2 which provides the Clebsch variational principle for Euler-Poincaré equations on the dual of the Virasoro-Bott algebra. The variational equations yield the new Clebsch momentum map $T^*\Diff(S^1) \to \mathfrak{vir}^*$ in \((2.18)\) associated with particle relabeling by cotangent-lifted right actions of $\Diff(S^1)$ that include the Bott 2-cocycle. Section 2 also introduces the simplified Clebsch variational principle \((2.25)\), which yields the special family of equations in \((1.1)\) as its Euler-Lagrange equation.

In Section 3 we use the Clebsch representation based on the inverse flow map to derive the multisymplectic form formula associated with our variational principle. We then deduce a new multisymplectic formulation of the family of equations \((1.1)\). The main result of this section is Theorem 3.1 which derives the multisymplectic formulation based on the inverse flow map.
Section 4 contains the summary of the present work and a discussion of several new directions for research that it reveals.

2 The inverse map and Clebsch representation

Equation (1.1) was first introduced in the Lie-Poisson context (see [33], [34], [42]). In this section we take the Lagrangian point of view, instead, and formulate (1.1) as an Euler-Poincaré equation on the Virasoro-Bott group. For this, we construct a canonical variational principle that will later allow us to determine a multisymplectic formulation of (1.1).

2.1 Euler-Poincaré equation on the Virasoro-Bott group

Let $S^1 = \mathbb{R}/2\pi\mathbb{Z} = \{\theta \in [0, 2\pi)\}$ denote the circle group, and let Diff$(S^1)$ be the diffeomorphism group of $S^1$. The tangent bundles can be identified as $T S^1 = S^1 \times \mathbb{R}$ and $T\text{Diff}(S^1) = \text{Diff}(S^1) \times \mathcal{X}(S^1)$, where $\mathcal{X}(S^1) = \{\chi : S^1 \rightarrow \mathbb{R}\}$ is the set of all smooth vector fields on $S^1$. In particular, the Lie algebra of $S^1$ is $\mathbb{R}$, and the Lie algebra of Diff$(S^1)$ is $\mathcal{X}(S^1)$. The Virasoro-Bott group is the central extension $\text{Diff}(S^1) = \text{Diff}(S^1) \times S^1$ with the group operation

$$(\psi_1, \theta_1) \cdot (\psi_2, \theta_2) = (\psi_1 \circ \psi_2, B(\psi_1, \psi_2) + \theta_1 + \theta_2),$$

(2.1)

where the 2-cocycle $B(\psi_1, \psi_2)$ is given by

$$B(\psi_1, \psi_2) = \frac{1}{2} \int_{S^1} \log \frac{\partial (\psi_1 \circ \psi_2)}{\partial x} d\log \frac{\partial \psi_2}{\partial x}.$$ 

(2.2)

The tangent bundle of the Virasoro-Bott group is $T\text{Diff}(S^1) = \text{Diff}(S^1) \times \mathcal{X}(S^1) \times \mathbb{R}$. The Virasoro algebra $\mathfrak{vir}$ is the Lie algebra of the Virasoro-Bott group and it can be identified as $\mathfrak{vir} = \mathcal{X}(S^1) \times \mathbb{R}$. The Lie algebra bracket (or adjoint action) on $\mathfrak{vir}$ is given by

$$\text{ad}_{(u,a)}(v,b) = [(u,a), (v,b)] = (-uv_x + u_x v, \int_{S^1} u_x v_{xx} dx).$$

(2.3)

for $(u,a), (v,b) \in \mathfrak{vir}$. One identifies the dual of $\mathfrak{vir}$ with itself, for the $L^2$ inner product

$$\langle (u,a), (v,b) \rangle = ab + \int_{S^1} uv dx.$$ 

(2.4)

With respect to this inner product, the coadjoint action $\text{ad}_{(u,a)}^*: \mathfrak{vir} \rightarrow \mathfrak{vir}$ can be represented as

$$\text{ad}_{(u,a)}^*(v,b) = (2vu_x + uv_x + bu_{xxx}, 0).$$

(2.5)

For more information about the Virasoro-Bott group and the Virasoro algebra we refer the reader to [34] and [38].

Suppose a Lagrangian system is defined on $T\text{Diff}(S^1)$ by specifying the right-invariant Lagrangian $L : T\text{Diff}(S^1) \rightarrow \mathbb{R}$. Rather then on the full tangent bundle, the dynamics of such a system can be analyzed on the Lie algebra $\mathfrak{vir}$ via the process called Euler-Poincaré reduction (see
where \( R \) denotes right translation on the Virasoro-Bott group and \( TR \) its tangent lift (see \([28, 38]\)). We consider the reduced Lagrangian \( \ell : \text{vir} \rightarrow \mathbb{R} \) defined by \( \ell(u, a) = L(\text{id}, 0, u, a) \) and the reduced variational principle

\[
\delta \int_{t_1}^{t_2} \ell(u(t), a(t)) \, dt = 0, \tag{2.6}
\]

using variations of the form \( \delta(u, a) = \frac{d}{dt}(v, b) - [(u, a), (v, b)] \), where \( (v(t), b(t)) \) vanish at the endpoints, and the time derivative of a Virasoro algebra-valued function of time is understood as \( \frac{d}{dt}(v(t), b(t)) = \left( \frac{dv}{dt}(\cdot, t), \frac{db}{dt}(t) \right) \). This variational principle leads to the Euler-Poincaré equation,

\[
\frac{d}{dt} \frac{\delta \ell}{\delta(u, a)} + \text{ad}^*_R(\cdot, a) \frac{\delta \ell}{\delta(u, a)} = 0, \tag{2.7}
\]

where the variational derivatives and the coadjoint action are computed with respect to the inner product \((2.4)\). Below, we demonstrate that \((1.1)\) can be written as an Euler-Poincaré equation.

**Theorem 2.1.** Let the reduced Lagrangian be defined as

\[
\ell(u, a) = \frac{1}{2} a^2 + \frac{1}{2} \int_{S^1} (\alpha u^2 + \beta u_x^2) \, dx, \tag{2.8}
\]

where \( \alpha, \beta \geq 0 \). Then the corresponding Euler-Poincaré equations take the form

\[
\alpha \left( u_t + 3u u_x \right) - \beta \left( u_{xxx} + 2u_x u_{xx} + uu_{xxx} \right) + \alpha u u_{xxt} = 0, \quad \text{and} \quad \frac{da}{d\ell} = 0. \tag{2.9}
\]

**Proof.** The case \( \alpha = 1 \) and \( \beta = 0 \) is shown in \([38]\). The case \( \alpha, \beta \geq 0 \) is a straightforward generalization. \( \square \)

The first equation in \((2.9)\) is equivalent to \((1.1)\); since the second equation in \((2.9)\) implies \( a = \text{const.} \)

### 2.2 Reconstruction equations and the inverse map

A solution \((u(t), a(t))\) of \((2.7)\) describes the evolution of the (right-invariant) Lagrangian system in the Virasoro algebra, denoted \( \text{vir} \). One can reconstruct the evolution on the whole Virasoro-Bott group by finding a curve \((\psi(t), \theta(t)) \in \text{Diff}(S^1)\) which right-translates its tangent vector back to \((u(t), a(t))\), i.e., in short-hand notation \((u(t), a(t)) = (\psi(t), \theta(t)) \cdot (\psi(t), \theta(t))^{-1}\). More precisely,

\[
(u(t), a(t)) = (\text{id}, 0, u(t), a(t)) = T_{(\psi(t), \theta(t))} R_{(\psi^{-1}(t), -\theta(t))} \cdot (\psi(t), \theta(t), \dot{\psi}(t), \dot{\theta}(t)), \tag{2.10}
\]

where \( R \) denotes right translation on the Virasoro-Bott group and \( TR \) its tangent lift (see \([28, 38]\)). By using \((2.1)\) and \((2.2)\), we obtain the reconstruction equations

\[
u(t) = \dot{\psi}(t) \circ \psi^{-1}(t),
\]

\[
a(t) = \dot{\theta}(t) + \frac{d}{ds} \bigg|_{s=t} B(\psi(s), \psi^{-1}(t)). \tag{2.11}
\]
In the context of fluid dynamics, a time-dependent diffeomorphism \( \psi(t) \in \text{Diff}(S^1) \) maps a given reference configuration to the fluid domain at each instant of time, i.e., \( \psi(X, t) \) represents the position at time \( t \) of the fluid particle labeled by \( X \). On the other hand, the inverse map \( l(t) = \psi^{-1}(t) \) maps from the current configuration of the fluid to the reference configuration, i.e., \( l(x, t) \) is the label of the fluid particle occupying the position \( x \) at time \( t \). The Eulerian velocity field \( u(x, t) \) gives the velocity of the fluid particle that occupies the position \( x \) at time \( t \), i.e., \( \dot{\psi}(X, t) = u(\psi(X, t), t) \). This is precisely the meaning of the first of the reconstruction equations in (2.11). It will be convenient for us to rewrite the reconstruction equations in terms of the inverse map. One can check via a straightforward calculation that the first equation in (2.11) is equivalent to

\[
\left. \frac{d}{ds} \right|_{s=t} B(\psi(s), \psi^{-1}(s)) = \frac{1}{2} \int_{S^1} \frac{\partial (\dot{\psi}(t) \circ \psi^{-1}(t))}{\partial x} d\log \frac{\partial \psi^{-1}(t)}{\partial x} \\
= \frac{1}{2} \int_{S^1} u_x d\log l_x \\
= \frac{1}{2} \int_{S^1} \frac{u_x l_{xx}}{l_x} dx,
\]

where in deriving the second equality we have used the first reconstruction equation in (2.11) and the definition of the inverse map. Therefore, the reconstruction equations in terms of the inverse map take the forms

\[
l_t + u l_x = 0, \quad \text{and} \quad a(t) = \dot{\theta}(t) + \frac{1}{2} \int_{S^1} \frac{u_x l_{xx}}{l_x} dx.
\]

Thus, given a solution \((u(t), a(t))\) of (2.7), one can easily solve (2.14) for \( l(x, t) \) and \( \theta(t) \).

### 2.3 Clebsch variational principle

#### 2.3.1 General reduced Lagrangian

As discussed in Section 2.1 Equation (1.1) has an underlying variational structure. However, the Euler-Poincaré variational principle (2.6) imposes constraints on the variations of the functions \( u \) and \( a \), which may be inconvenient in some applications, for instance, when one is interested in deriving variational integrators, or determining the underlying multisymplectic structure, as is our goal in this work. One can circumvent this issue by considering an augmented action functional which includes the reconstruction equations as constraints. This idea was formalized in the context of variational Lie group integrators in back-to-back papers in [5] and [14]. The idea of using the inverse map \( l(x, t) \) (also called ‘back-to-labels’ map) and the advection condition (2.12) appeared in [25, 26], and was later used in [15] to construct multisymplectic formulations of a class of fluid dynamics equations. We extend these ideas to systems defined on the Virasoro-Bott group.
The Clebsch variational principle (also sometimes called the Hamilton-Pontryagin principle) enforces stationarity of the action \( S = \int \ell(u,a) \, dt \) under the constraint that the reconstruction equations (2.14) are satisfied. We define the augmented action functional

\[
S[u,a,l,\theta,\pi,\lambda] = \int_{t_1}^{t_2} \ell(u,a) \, dt + \int_{t_1}^{t_2} \int_{S^1} \pi(l_t + ul_x) \, dx \, dt + \int_{t_1}^{t_2} \lambda \left( \dot{\theta} - a + \frac{1}{2} \int_{S^1} \frac{u_x l_{xx}}{l_x} \, dx \right) \, dt, \tag{2.15}
\]

where \( \pi = \pi(x,t) \) and \( \lambda = \lambda(t) \) are Lagrange multipliers, and consider the variational principle

\[
\delta S = 0, \tag{2.16}
\]

with respect to arbitrary variations \( \delta u, \delta a, \delta \pi, \delta \lambda \), and vanishing endpoint variations \( \delta l \) and \( \delta \theta \), i.e.,

\( \delta l(x,t_1) = \delta l(x,t_2) = \delta \theta(t_1) = \delta \theta(t_2) = 0 \).

The resulting variational equations are

\[
\begin{align*}
\delta \theta : & \quad \dot{\lambda} = 0, \tag{2.17a} \\
\delta a : & \quad \frac{\partial \ell(u,a)}{\partial a} = \lambda, \tag{2.17b} \\
\delta \lambda : & \quad \dot{\theta} = a - \frac{1}{2} \int_{S^1} \frac{u_x l_{xx}}{l_x} \, dx, \tag{2.17c} \\
\delta u : & \quad \frac{\delta \ell(u,a)}{\delta u} = -\pi l_x + \frac{\lambda}{2} \frac{\partial}{\partial x} l_{xx}, \tag{2.17d} \\
\delta \pi : & \quad l_t + ul_x = 0, \tag{2.17e} \\
\delta l : & \quad \pi l_t + \frac{\partial}{\partial x} \left( \pi u - \frac{\lambda}{2} \frac{u_{xx}}{l_x} \right) = 0. \tag{2.17f}
\end{align*}
\]

Consequently, we obtain the components of the Clebsch momentum map, given by

\[
\frac{\delta \ell}{\delta(u,a)} = \left( \frac{\delta \ell}{\delta u}, \frac{\delta \ell}{\delta a} \right) = \left( -\pi l_x + \frac{\lambda}{2} \frac{\partial}{\partial x} l_{xx}, \lambda \right). \tag{2.18}
\]

Remark. Equation (2.18) is the Clebsch momentum map \( T^*\text{Diff}(S^1) \to \text{vir}^* \) associated with particle relabeling by cotangent-lifted right actions of \( \text{Diff}(S^1) \) that include the Bott 2-cocycle in equation (2.2).

We will now show that the dynamics generated by the system (2.17) is equivalent to the dynamics generated by the Euler-Poincaré equation (2.7).

**Theorem 2.2.** Suppose the functions \( u(x,t), a(t), l(x,t), \theta(t), \pi(x,t), \) and \( \lambda(t) \) satisfy the Euler-Lagrange equations (2.17). Then the functions \( u(x,t) \) and \( a(t) \) satisfy the Euler-Poincaré equation (2.7).

**Proof.** Let \( (w, c) \) be an arbitrary element of the Virasoro algebra \( \text{vir} \). Let us calculate

\[
\left\langle \frac{d}{dt} \frac{\delta \ell}{\delta(u,a)}, (w,c) \right\rangle = \int_{S^1} \left( \frac{\partial}{\partial t} \frac{\delta \ell}{\delta u} \right) \cdot w \, dx + \left( \frac{\partial}{\partial t} \frac{\delta \ell}{\delta a} \right) \cdot c, \tag{2.19}
\]
where the inner product $\langle \cdot , \cdot \rangle$ was defined in (2.4). By using (2.17a), (2.17b), and (2.17c), we further have

\[
\frac{d}{dt} \delta \ell \Big|_{\delta(u,a), (w,c)} = \int_{S^1} \left[ \frac{1}{2} \lambda \frac{\partial}{\partial x} l_x^2 - \pi l_x \right] \cdot w \, dx
\]

\[
= \int_{S^1} \left[ \frac{1}{2} \lambda \frac{\partial}{\partial x} l_x^2 - \pi l_x \right] \cdot w \, dx.
\]

(2.20)

We now use (2.17e) and (2.17f) to eliminate the time derivatives in the integrand, which yields

\[
\frac{d}{dt} \delta \ell \Big|_{\delta(u,a), (w,c)} = \int_{S^1} \left[ \frac{1}{2} \lambda \frac{\partial}{\partial x} l_x^2 - \pi l_x \right] \cdot w \, dx.
\]

(2.21)

Note that the expression $A$ contains the functions $u, l, \pi, \lambda$, and their spatial derivatives. On the other hand we have

\[
\frac{d}{dt} \delta \ell \Big|_{\delta(u,a), (w,c)} = \int_{S^1} \left[ \frac{1}{2} \lambda \frac{\partial}{\partial x} l_x^2 - \pi l_x \right] \cdot w \, dx.
\]

(2.22)

where in the first equality we used (2.5), and in the second equality we used (2.17b) and (2.17d). Note that the expression $B$ contains the functions $u, l, \pi, \lambda$, and their spatial derivatives. After rather tedious, albeit straightforward algebraic manipulations we find that $A + B = 0$. Therefore, we have that for all $(w,c) \in \text{vir}

\[
\frac{d}{dt} \delta \ell \Big|_{\delta(u,a), (w,c)} + \text{ad}_{(u,a)}^* \frac{\delta \ell}{\delta (u,a)} (w,c) = 0,
\]

(2.23)

which completes the proof, since the inner product is nondegenerate.

\[\square\]

2.3.2 Separable reduced Lagrangian

The variational principle (2.16) simplifies significantly when one considers separable Lagrangians of the form

\[
\ell(u,a) = \frac{1}{2} a^2 + \bar{\ell}(u).
\]

(2.24)

In that case Equations (2.17a) and (2.17b) imply $\lambda = a = \text{const}$. Treating $a$ as a constant, we can eliminate the variables $\theta$ and $\lambda$ from the action functional (2.15). Consider the action functional
\[ S[u, l, \pi] = \int_{t_1}^{t_2} \left( \hat{\ell}(u) + \frac{a}{2} \int_{S^1} u_x l_{xx} \, dx \right) \, dt + \int_{t_1}^{t_2} \int_{S^1} \pi (l_t + ul_x) \, dx \, dt. \] (2.25)

The stationarity condition \( \delta S = 0 \) with respect to arbitrary variations \( \delta u, \delta \pi, \) and vanishing endpoint variations \( \delta l, \) yields the variational equations

\[ \delta u : \quad \frac{\delta \hat{\ell}(u)}{\delta u} = -\pi l_x + \frac{a}{2} \frac{\partial}{\partial x} l_{xx}, \] (2.26a)

\[ \delta \pi : \quad l_t + ul_x = 0, \quad \frac{\partial}{\partial x} l_{xx} \]

\[ \delta l : \quad \pi_t + \frac{\partial}{\partial x} \left( \pi u - \frac{a}{2} u_{xx} \right) = 0. \] (2.26b)

It is straightforward to see that the system (2.17) reduces to (2.26) for Lagrangians of the form (2.24).

**Remark.** The action functional (2.25) provides a new variational formulation for Equation (1.1) when the Lagrangian (2.8) is considered. For \( a = 0 \) this action functional reduces to the action functional for the dispersionless CH equation \((\alpha = \beta = 1) \) introduced in [15] and one of the action functionals for the HS equation \((\alpha = 0 \text{ and } \beta = 1) \) described in [31]. For \( \alpha = 1 \) and \( \beta = a = 0 \) we also obtain a variational principle for the inviscid Burgers’ equation.

### 3 Inverse map multisymplectic formulation

The action functional and variational principle introduced in Section 2.3.2 allow the identification and analysis of a new multisymplectic formulation of the family of equations (1.1). Multisymplectic geometry provides a covariant formalism for the study of field theories in which time and space are treated on equal footing. Multisymplectic formalism is useful for, e.g., the stability analysis of water waves (see [6], [7]) or construction of structure-preserving numerical algorithms (see [8], [39]). The multisymplectic form formula was first proved by Marsden & Patrick & Shkoller [39] and provides an intrinsic and covariant description of the conservation of symplecticity law, first introduced by Bridges [6] in the context of multisymplectic Hamiltonian PDEs. In Section 3.1 we review the multisymplectic geometry formalism and derive the multisymplectic form formula associated with (2.25). We further make a connection with Bridges’ approach to multisymplecticity in Section 3.2 and determine a multisymplectic Hamiltonian form of the Euler-Lagrange equations (2.26).

#### 3.1 Multisymplectic form formula and conservation of symplecticity

The multisymplectic form formula is the multisymplectic counterpart of the fact that in finite-dimensional mechanics, the flow of a mechanical system consists of symplectic maps. It was first proved for first-order field theories in [39], and later generalized to second-order field theories in [36]. Since the field theory described by the action functional (2.25) with the Lagrangian (2.8) is second-order, we follow the theory developed in [36]. For the convenience of the reader, below we briefly review multisymplectic geometry and jet bundle formalism necessary for our discussion.
Let \( X = S^1 \times \mathbb{R} \) represent spacetime and denote the local coordinates by \((x^\mu) = (x^1, x^0)\), where \( x^1 \equiv x \) is the spatial coordinate and \( x^0 \equiv t \) is time. Define the configuration fiber bundle \( \tau_{XY} : Y \rightarrow X \) as \( Y = X \times S^1 \times \mathbb{R} \times \mathbb{R} \). Denote the fiber coordinates by \((y^A) = (y^1, y^2, y^3)\) with \( y^1 \equiv l \), \( y^2 \equiv u \), and \( y^3 \equiv \pi \). Physical fields are sections of the configuration bundle, that is, continuous maps \( \phi : X \rightarrow Y \) such that \( \tau_{XY} \circ \phi = \text{id}_X \). In the coordinates \((x^\mu, y^A)\) a field \( \phi \) is represented as \( \phi(x, t) = (x^\mu, \phi^A(x^\mu)) = (x, t, l(x, t), u(x, t), \pi(x, t)) \).

For a \( k \)-th order field theory, the evolution of the field takes place on the \( k \)-th jet bundle \( J^kY \). The first jet bundle \( J^1Y \) is the affine bundle over \( Y \) with the fibers \( J^1_yY \) defined as

\[
J^1_yY = \left\{ \vartheta : T_{(x,t)}X \rightarrow T_yY \mid T\tau_{XY} \circ \vartheta = \text{id}_{T_{(x,t)}X} \right\}
\]

for \( y \in Y_{(x,t)} \), where the linear maps \( \vartheta \) represent the tangent mappings \( T_{(x,t)}\phi \) for local sections \( \phi \) such that \( \phi(x, t) = y \). The local coordinates \((x^\mu, y^A)\) on \( Y \) induce the coordinates \((x^\mu, y^A, v^A)\) on \( J^1Y \). Intuitively, the first jet bundle consists of the configuration bundle \( Y \), and of the first partial derivatives of the field variables with respect to the independent variables. We can think of \( J^1Y \) as a fiber bundle over \( X \). Given a section \( \phi : X \rightarrow Y \), we can define its first jet prolongation

\[
j^1\phi : X \ni (x, t) \rightarrow T_{(x,t)}\phi \in J^1Y,
\]

in coordinates given by

\[
j^1\phi(x^\mu) = \left( x^\mu, \phi^A(x^\nu), \frac{\partial \phi^A(x^\nu)}{\partial x^\mu} \right),
\]

which is a section of the fiber bundle \( J^1Y \) over \( X \). For higher-order field theories we consider higher-order jet bundles, defined iteratively by \( J^{k+1}Y = J^1(J^kY) \). We denote the local coordinates on \( J^2Y \) by \((x^\mu, y^A, v^A_\mu, w^A_{\mu\nu})\). The second jet prolongation \( j^2\phi : X \rightarrow J^2Y \) is given in coordinates by

\[
j^2\phi(x^\mu) = (x^\mu, \phi^A, \partial \phi^A / \partial x^\mu, \partial^2 \phi^A / \partial x^\mu \partial x^\nu). \]

Let \((x^\mu, y^A, v^A_\mu, w^A_{\mu\nu}, s^A_{\mu\nu\rho})\) denote the coordinates on \( J^3Y \). The third jet prolongation \( j^3\phi \) is defined similar to \( j^1\phi \) and \( j^2\phi \). For more information about the geometry of jet bundles see [17] and [21].

In the jet bundle formalism introduced above, the action functional \((\ref{2.25})\) with the reduced Lagrangian \((\ref{2.28})\) can be written as

\[
S[\phi] = \int_{\mathcal{U}} \mathcal{L}(j^2\phi) \, d^2x,
\]

where \( \mathcal{U} = S^1 \times [t_1, t_2] \), \( d^2x = dx \wedge dt \), and the Lagrangian density \( \mathcal{L} : J^2Y \rightarrow \mathbb{R} \) is

\[
\mathcal{L}(x^\mu, y^A, v^A_\mu, w^A_{\mu\nu}) = \frac{\alpha}{2} (y^2)^2 + \frac{\beta}{2} (v^2)^2 + \frac{\alpha}{2} \frac{v_1^2 w_1}{v_1^1} + y^3 (v_0 + y^2 v_1^1).
\]

Hamilton’s variational principle seeks fields \( \phi(x, t) \) that extremize \( S \), that is,

\[
\frac{d}{d\lambda} \bigg|_{\lambda=0} S[\eta^j_Y \circ \phi] = 0,
\]

for all \( \eta^j_Y \) that keep the boundary conditions on \( \partial \mathcal{U} \) fixed, where \( \eta^j_Y : Y \rightarrow Y \) is the flow of a vertical vector field \( V \) on \( Y \). This leads to the Euler-Lagrange equations.
The multisymplectic structure is defined on \( J^3Y \) (see [36]). Given the Lagrangian density \( \mathcal{L} \) one can define the Cartan 2-form \( \Theta_\mathcal{L} \) on \( J^3Y \), in local coordinates given by

\[
\Theta_\mathcal{L} = \left( \frac{\partial \mathcal{L}}{\partial v^A_\mu} - D_\nu \left( \frac{\partial \mathcal{L}}{\partial w^A_{\mu\nu}} \right) \right) dy^A \wedge dx_\mu + \frac{\partial \mathcal{L}}{\partial w^A_{\nu\mu}} dv^A_\mu \wedge dx_\mu \\
+ \left( \mathcal{L} - \frac{\partial \mathcal{L}}{\partial v^A_\mu} v^A_\mu + D_\nu \left( \frac{\partial \mathcal{L}}{\partial w^A_{\mu\nu}} \right) v^A_\mu - \frac{\partial \mathcal{L}}{\partial w^A_{\nu\mu}} w^A_{\nu\mu} \right) dx^2,
\]

where \( dx_\mu = \partial_\mu x^dx \), i.e., \( dx_0 = -dx \) and \( dx_1 = dt \), and the formal partial derivative in the direction \( x^\nu \) of a function \( f : J^2Y \rightarrow \mathbb{R} \) is defined in coordinates as

\[
D_\nu f = \frac{\partial f}{\partial x^\nu} + \frac{\partial f}{\partial y^A} v^A_\nu + \frac{\partial f}{\delta v^A_\mu} w^A_{\mu\nu} + \frac{\partial f}{\delta w^A_{\sigma\mu}} w^A_{\sigma\mu}.
\]
on \( Y \) such that \((x, t) \rightarrow \eta^j_v \circ \phi(x, t)\) is also a solution, where \( \eta^j_v \) is the flow of \( V \). The multisymplectic form formula for second-order field theories (see [36]) states that if \( \phi \in \mathcal{P} \) then for all \( V \) and \( W \) in \( \mathcal{F} \),

\[
\int_{\partial \mathcal{U}} (j^3 \phi)^* (j^3 V \wedge j^3 W \wedge \Omega_3) = 0,
\]

where \((j^3 \phi)^*\) denotes the pull-back by the mapping \( j^3 \phi \), and \( j^3 V \) is the third jet prolongation of \( V \), that is, the vector field on \( J^3 Y \) whose flow is the third jet prolongation of the flow \( \eta^j_v \) for \( V \), i.e.,

\[
j^3 V = \frac{d}{dt} \bigg|_{t=0} j^3 \eta^j_v.
\]

Consider two arbitrary first variation vector fields \( V, W \), in the local coordinates \((x^j, y^a)\) represented by \((V^\mu(x^j, y^a), V^A(x^j, y^a))\) and \((\xi^\nu(x^j, y^a), \xi^A(x^j, y^a))\), respectively. Let us work out the form of the formula (3.11) for \( \tau_{XY}\)-vertical first variations, i.e., \( V^\mu(x^j, y^a) = W^\mu(x^j, y^a) = 0 \). Denote the components of \( j^3 V \) as \((0, V^A, V^A, V_{\mu}^A, V_{\mu\nu}^A)\), and similarly for \( j^3 W \). The multisymplectic form formula then becomes

\[
\int_{\partial \mathcal{U}} -F(x, t) \, dx + G(x, t) \, dt = 0,
\]

with

\[
F(x, t) = -W^1 V^3 + W^3 V^1,
\]

\[
G(x, t) = -\pi(W^1 V^2 - W^2 V^1) - u(W^1 V^3 - W^3 V^1) - \frac{a}{2l_x^2}(W^1 V^1 - W^1 V^1)
\]

\[
+ \frac{a}{2l_x}(W^1 V^2_{11} - W^2_{11} V^1) - \beta(W^2 V^2_{11} - W^2_{11} V^2) + \frac{a}{2l_x^2}(W^2 V^1 - W^1 V^1)
\]

\[
- \frac{a}{2l_x}(W^2 V^1_{11} - W^1_{11} V^2) - \frac{a}{2l_x}(W^1 V^2 - W^2 V^1),
\]

where the vector components are evaluated at \( j^3 \phi(x, t) \). By applying Stokes’ theorem and using the fact that \( \mathcal{U} \) is arbitrary, the multisymplectic form formula (3.14) can be rewritten equivalently as the conservation law

\[
\frac{\partial}{\partial t} F(x, t) + \frac{\partial}{\partial x} G(x, t) = 0.
\]

This kind of a conservation law was first considered by Bridges [6]. In Section 3.2 we make a further connection with Bridges’ theory and find a multisymplectic PDE form of the Euler-Lagrange equations (2.2).

### 3.2 Multisymplectic Hamiltonian PDE formulation

Bridges [6] introduced the notion of multisymplecticity by generalizing the notion of Hamiltonian systems to Partial Differential Equations (PDEs). A multisymplectic structure \((\mathcal{M}, \omega, \kappa)\) consists of the phase space \( \mathcal{M} = \mathbb{R}^n \), and pre-symplectic 2-forms \( \omega \) and \( \kappa \), where pre-symplectic means that
the 2-forms are closed, but not necessarily nondegenerate. A multisymplectic Hamiltonian system is a PDE of the form

$$M(z)z_t + K(z)z_x = \nabla H(z),$$  \hspace{1cm} (3.17)

where $z : X \ni (x,t) \mapsto z(x,t) \in \mathcal{M}$ is a function of the spacetime variables $x$ and $t$, $H : \mathcal{M} \to \mathbb{R}$ is the Hamiltonian, and $M(z)$, $K(z)$ are $n \times n$ antisymmetric matrices defined by

$$\omega(W, V) \equiv \langle M(z)W, V \rangle_{\mathcal{M}}, \quad \kappa(W, \nabla) \equiv \langle K(z)W, \nabla \rangle_{\mathcal{M}},$$  \hspace{1cm} (3.18)

where $V, W$ are arbitrary vector fields on $\mathcal{M}$, and $\langle \cdot, \cdot \rangle_{\mathcal{M}}$ is the standard Euclidean inner product on $\mathcal{M} = \mathbb{R}^n$.

We will use the multisymplectic form formula (3.14) to deduce the multisymplectic Hamiltonian PDE form (3.17) of the Euler-Lagrange equations (2.26). We note that for $a > 0$ the vector components that appear in (3.15) only correspond to the 7 coordinate directions $y^1$, $y^2$, $v^1$, $v^2$, $w^1_1$, $w^2_1$, $w^2_{11}$ on $J^3Y$. We will therefore consider $\mathcal{M} = \mathbb{R}^7$ and denote the coordinates on $\mathcal{M}$ as $(l, u, \pi, \Delta, \Theta, \Xi, \Pi)$. Define the projection map

$$\mathcal{F} \mathcal{L} : J^3Y \ni (x^\mu, y^A, v^A_\mu, w^A_\mu, s^A_{\mu\nu\sigma}) \mapsto (y^1, y^2, v^1, v^2, w^1_1, w^2_1, w^2_{11}) \in \mathcal{M}.$$  \hspace{1cm} (3.19)

The suitable entries for the matrices $M(z)$ and $K(z)$ can be read off from (3.15) as

$$M = \begin{pmatrix}
0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}, \quad K(z) = \begin{pmatrix}
0 & \pi & u & a \Pi & 0 & 0 & -a \\
-\pi & 0 & 0 & -a \Pi & \beta & a \Xi & 0 \\
u & 0 & 0 & 0 & 0 & 0 & 0 \\
a \Pi & -a \Xi & 0 & 0 & a \Pi & 0 & 0 \\
a \Pi & 0 & -a \Xi & 0 & 0 & 0 & 0 \\
a \Pi & 0 & 0 & 0 & 0 & 0 & 0 \\
a \Pi & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}.$$  \hspace{1cm} (3.20)

With that choice, we have $F(x, t) = \omega(W, V)$ and $G(x, t) = \kappa(W, \nabla)$, where $W = T\mathcal{F} \mathcal{L} \cdot j^3W$ and $\nabla = T\mathcal{F} \mathcal{L} \cdot j^3V$. The Hamiltonian $H$ can be read off from the $dx \wedge dt$ term in (3.10) as

$$H(z) = \frac{a}{2} u^2 - \frac{\beta}{2} \Theta^2 - \frac{a}{2} \frac{\Theta \Xi}{\Delta} + \frac{a}{2} \Pi.$$  \hspace{1cm} (3.21)

Below we show that the Euler-Lagrange equations (2.26) indeed can be given the multisymplectic structure (3.17).

**Theorem 3.1.** Suppose $a > 0$. Then the Euler-Lagrange equations (2.26) with the Lagrangian (2.8) are equivalent to the multisymplectic Hamiltonian system (3.17) with the matrices (3.20) and the Hamiltonian (3.21). That is, if $\phi(x, t) = (x, t, l(x, t), u(x, t), \pi(x, t))$ is a solution of (2.26), then $z(x, t) = \mathcal{F} \mathcal{L} \circ j^3 \phi(x, t)$ is a solution of (3.17). Conversely, if $z(x, t)$ is a solution of (3.17), then $\phi(x, t) = (x, t, z_1(x, t), z_2(x, t), z_3(x, t)) = (x, t, l(x, t), u(x, t), \pi(x, t))$ is a solution of (2.26).

**Proof.** Substituting (3.20) and (3.21) in (3.17) yields the system of equations
\[\pi_t + \pi u_x + u \pi_x + \frac{a}{2} \frac{\Pi}{\Delta^2} \Delta x - \frac{a}{2\Delta} \Pi_x = 0, \quad (3.22a)\]

\[-\pi l_x - \frac{a}{2} \frac{\Xi}{\Delta^2} \Delta x + \beta \Theta_x + \frac{a}{2\Delta} \Xi_x = \alpha u, \quad (3.22b)\]

\[-l_t - u l_x = 0, \quad (3.22c)\]

\[-\frac{a}{2} \frac{\Pi}{\Delta^2} l_x + \frac{a}{2} \frac{\Xi}{\Delta^2} u_x + \frac{a}{2\Delta} \Theta x = \frac{a}{2} \frac{\Theta}{\Delta^2}, \quad (3.22d)\]

\[-\beta u_x - \frac{a}{2\Delta} \Delta x = -\beta \Theta - \frac{a}{2\Delta}, \quad (3.22e)\]

\[-\frac{a}{2} \frac{\Theta}{\Delta^2} u_x = -\frac{a}{2\Delta}, \quad (3.22f)\]

\[-\frac{a}{2\Delta} l_x = \frac{a}{2}. \quad (3.22g)\]

Equation (3.22a) implies \(\Delta = l_x\) and Equation (3.22b) implies \(\Theta = u_x\). Then, Equations (3.22c) and (3.22d) imply \(\Xi = l_{xx}\) and \(\Pi = u_{xx}\), respectively. By substituting these identities in the remaining equations (3.22a)-(3.22c), we obtain a system equivalent to (2.26), which completes the proof.

Bridges [6] showed that the conservation of symplecticity law

\[\frac{\partial}{\partial t}\omega(W, V) + \frac{\partial}{\partial x} \kappa(W, V) = 0 \quad (3.23)\]

is satisfied for solutions \(z(x, t)\) of (3.17), where \(W, V\) are arbitrary first variations of \(z(x, t)\). This is an equivalent statement of (3.16), since if \(W\) and \(V\) are first variations for (3.7), then \(W = TF_L \cdot j^3 W\) and \(V = TF_L \cdot j^3 V\) are first variations for (3.17).

**Remark.** Equations (3.17), (3.20), and (3.21) provide a new multisymplectic formulation for the family of equations (1.1) with \(a > 0\). For \(a = 0\) several special cases can be obtained. If \(\beta > 0\), then Equations (3.22d), (3.22e), and (3.22g) become trivial, and it is enough to consider the variables \(z = (l, u, \pi, \Theta)\). The matrices \(M\) and \(K\) then take the form

\[
M = \begin{pmatrix}
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 \\
-1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
\end{pmatrix}, \quad K(z) = \begin{pmatrix}
0 & \pi & u & 0 \\
-\pi & 0 & 0 & \beta \\
-\pi & 0 & 0 & 0 \\
0 & -\beta & 0 & 0 \\
\end{pmatrix}, \quad (3.24)
\]

and the Hamiltonian becomes \(H(z) = \frac{a}{2} u^2 - \frac{\beta}{2} \Theta^2\). For \(\alpha = \beta = 1\) this reproduces the multisymplectic structure for the dispersionless CH equation found in [15], and for \(\alpha = 0, \beta = 1\) we obtain a new multisymplectic formulation of the HS equation with \(a = 0\). If in addition \(\beta = 0\), then Equation (3.22e) also becomes trivial, and a further simplification is possible: we consider the variables \(z = (l, u, \pi)\) with the matrices

\[
M = \begin{pmatrix}
0 & 0 & 1 \\
0 & 0 & 0 \\
-1 & 0 & 0 \\
\end{pmatrix}, \quad K(z) = \begin{pmatrix}
0 & \pi & u \\
-\pi & 0 & 0 \\
-\pi & 0 & 0 \\
\end{pmatrix}, \quad (3.25)
\]
and the Hamiltonian \( H(z) = \frac{\alpha}{2} u^2 \). This final simplification provides a multisymplectic formulation for the inviscid Burgers’ equation.

4 Summary, open problems and opportunities for future work

In this paper, we have introduced a new type of Clebsch representation that extends the momentum map formulation for fluid dynamics introduced in Holm, Kupershmidt & Levermore [25, 26] based on the inverse flow map to the case when the group action governing Lagrangian fluid paths includes the Bott 2-cocycle in equation (2.2). Physically, this means that linear dispersion with third order spatial derivatives can be included, as required for investigating the multisymplectic structures of the Korteweg-de Vries, Camassa-Holm, and Hunter-Saxton equations. Moreover, the multisymplectic form formula was shown to persist and was derived explicitly for this important class of equations, by using our new type of Clebsch representation, identified in equation (2.18) as the momentum map associated with particle relabeling with group actions which include the Bott 2-cocycle. In addition, symplecticity was found to be conserved in this new class of flows. Consequently, new types of structure-preserving numerics for soliton equations with linear dispersion can now be developed.

Multisymplectic integrators are methods that preserve a discrete version of the symplectic conservation law (3.23). There is numerical evidence that these schemes locally conserve energy and momentum remarkably well (see, e.g., [3], [4], [8], [11], [13], [51], [53], [54], [55]), which is a much stronger property than merely global conservation over the whole spatial domain (see [41]). Variational integrators are based on discrete variational principles, which provide a natural framework for the discretization of Lagrangian systems (see, e.g., [37], [39], [40], [46], [48], [49], [50]). A discrete action functional can be obtained by discretizing the functional (2.25) on a spacetime mesh. A variational numerical scheme is then derived by extremizing the discrete action with respect to the discrete set of the values of the fields \( l, u, \) and \( \pi \). Variational integrators satisfy a discrete version of the multisymplectic form formula (3.12), and are therefore multisymplectic. Moreover, in the presence of a symmetry, they satisfy a discrete version of Noether’s theorem, as a consequence of which many of the conservation laws of the continuous system persist. These directions will be explored in future work. They are beyond the scope of the present derivation and formulation.

Furthermore, the new Clebsch momentum map with the Bott 2-cocycle in equation (2.18) represents an opportunity to extend the approach in [22] of using Clebsch variational principles for introducing noise into continuum mechanics. The new Clebsch momentum map (2.18) will enable us to investigate a new type of interplay among nonlinearity and noise that also includes stochastic linear dispersion. This interplay introduces a class of dynamical problems addressing ‘wobbling’ solitons governed by SPDEs with stochastic mass/label transport. Consider a stochastic deformation of the Euler-Poincaré equation (2.7) such that the coadjoint action \( \text{ad}^* \) is taken with respect to the perturbed Virasoro algebra element \( (u+\xi(x)\circ W(t),a+\eta\circ W(t)) \) rather than \( (u,a) \), where \( W(t) \) denotes the white noise, and the prescribed function \( \xi(x) \) and element \( \eta \in \mathbb{R} \) represent the spatial correlations of the noise in the advection velocity and the intensity of the noise in \( a \), respectively. Because of the definition of the coadjoint action (2.5), the perturbation of \( a \) does not have any effect on the equation and can be omitted. The stochastic Euler-Poincaré equation will therefore take the form

\[
\frac{d}{dt} \frac{\delta \ell}{\delta (u,a)} + \text{ad}^*_{(u,a)} \frac{\delta \ell}{\delta (u,a)} \delta l + \text{ad}^*_{(\xi,a)} \frac{\delta \ell}{\delta (u,a)} \circ dW(t) = 0, 
\tag{4.1}
\]

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where $W(t)$ is the standard Wiener process, $d$ denotes stochastic differential with respect to the time variable $t$, and $\circ$ denotes Stratonovich time integration. For the Lagrangian (2.8), we obtain a stochastic deformation of the family of equations (1.1) as

$$
d(\alpha u - \beta u_{xx}) + \left[3\alpha uu_x - \beta(2u_x u_{xx} + uu_{xxx}) + au_{xxxx}\right]dt + \left[\alpha(2\xi_x u + \xi u_x) - \beta(2\xi_x u_{xx} + \xi u_{xxx}) + a\xi_{xxx}\right] \circ dW(t) = 0. \tag{4.2}$$

This kind of stochastic deformation has been proposed for the dispersionless Camassa-Holm equation ($\alpha = \beta = 1$ and $a = 0$ in the equation above), electromagnetic field equations, and various fluid dynamics equations (see [13], [16], [17], [19], [22], [23], [29]). This approach retains many properties of the unperturbed equations, such as the peaked soliton solutions of the Camassa-Holm equation, and the Kelvin circulation theorem for fluid dynamics. Moreover, for certain functional forms of $\xi(x)$, the introduction of this type of coadjoint transport noise can preserve the deterministic isospectral problem, while introducing stochasticity into the evolution equations for the corresponding eigenfunctions. This stochastic process preserves certain aspects of the inverse scattering methods for determining the soliton solutions of SPDEs, as discussed in [17]. The results presented in [16], [17], [29] suggest that for smooth initial conditions and for a proper class of the correlation functions $\xi(x)$, the solutions of (4.2) are likely to retain their spatial regularity. For instance, in the case of spatially uniform noise, with $\xi(x) = \gamma = \text{const}$, if $u(x, t)$ is a solution of (1.1), then $u(x - \gamma W(t), t)$ is a solution of (4.2), which can be easily verified by a direct substitution.

Under this regularity hypothesis, solutions of Equation (4.2) are seen to be critical points of the action functional

$$S[u, l, \pi] = \int_{t_1}^{t_2} \ell(u) dt + \frac{a}{2} \int_{S^1} \int_{t_1}^{t_2} \left(\frac{2 u_x l_{xx}}{l_x} dt + \frac{\xi_{xx} l_{xx}}{l_x} \circ dW(t)\right) dx$$

$$+ \int_{S^1} \int_{t_1}^{t_2} \pi \left(\circ dl + ul_x dt + \xi(x) l_x \circ dW(t)\right) dx, \tag{4.3}$$

which is a stochastic deformation of (2.25), in which the velocity field $u$ in the reconstruction equations (2.14) is replaced with $u + \xi(x) \circ W(t)$. By following the reasoning presented in Section 3.1 and ignoring the analytical difficulties arising from introducing stochastic integrals, direct calculation shows that a stochastic version of the multisymplectic form formula (3.12) holds, and the corresponding stochastic conservation of symplecticity law can be written heuristically as

$$dF(x, t) + \frac{\partial}{\partial x} G(x, t) dt + \frac{\partial}{\partial x} \overline{G}(x, t) \circ dW(t) = 0, \tag{4.4}$$

where $F(x, t)$ and $G(x, t)$ have been defined in (3.13), and the function $\overline{G}(x, t)$ is given by

$$\overline{G}(x, t) = -\xi(x)(W^1 V^3 - W^3 V^1) - \frac{a}{2} \frac{\xi_{xx}(x)}{l_x^2} (W^1 V^1 - W^1 V^1). \tag{4.5}$$

Rigorous derivations and proofs of all these heuristic formulas for the effects of introducing noise this way will be subjects of future work. Of course, it would also be of interest to construct the corresponding stochastic variational and multisymplectic integrators for these investigations in future work.
Finally, it is worth pointing out that Equation (4.2) can be put into Lie-Poisson Hamiltonian form as coadjoint motion,

\[ dm = -J \left( \frac{\delta h}{\delta m} \, dt + \frac{\delta \overline{h}}{\delta m} \circ dW(t) \right), \]

where \( m = \alpha u - \beta u_{xx} \) is the momentum map, \( J = \partial_x m + m \partial_x + a \partial_{xxx} \) is the Lie-Poisson Hamiltonian operator, and the Hamiltonians are \( h(m) = \frac{1}{2} \int_{S^1} (\alpha u^2 + \beta u_x^2) \, dx \) and \( \overline{h}(m) = \int_{S^1} \xi(x) \, m \, dx \), for which \( \delta h/\delta m = u \) and \( \delta \overline{h}/\delta m = \xi(x) \), respectively. We remark that the stochastic KdV form of Equation (4.2) with \( \alpha = 1 \) and \( \beta = 0 \), expressed here as a member of the class of stochastic Hamiltonian PDEs in (4.6), has also appeared in [2], as

\[ du = -(\partial_x u + u \partial_x + a \partial_{xxx})(u \, dt + \xi(x) \circ dW(t)). \]

This new form of the stochastic KdV equation reveals that the class of stochastic Hamiltonian PDEs in (4.6) involves the interplay between stochastic nonlinear transport and stochastic linear dispersion. The investigation of the dynamical effects arising from these two quite different stochastic mechanisms in the contexts of the KdV and CH equations will be yet another subject of future work.

Acknowledgments

We are very grateful to Alexis Arnaudon and Nader Ganaba for many useful comments, references and stimulating discussions during the present work. During this work, the authors were partially supported by the European Research Council Advanced Grant 267382 FCCA and the UK EPSRC Grant EP/N023781/1 held by DH.

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