Being Fast Means Being Chatty: The Local Information Cost of Graph Spanners

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We introduce a new measure for quantifying the amount of information that the nodes in a network need to learn to jointly solve a graph problem. We show that the local information cost (LIC) presents a natural lower bound on the communication complexity of distributed algorithms. For the synchronous CONGEST-KT model, where each node has initial knowledge of its neighbors’ IDs, we prove that \( \Omega\left(\frac{\text{LIC}(P)}{\log \tau \log n}\right) \) bits are required for solving a graph problem \( P \) with a \( \tau \)-round algorithm that errs with probability at most \( \gamma \). Our result is the first lower bound that yields a general trade-off between communication and time for graph problems in the CONGEST-KT model.

We demonstrate how to apply the local information cost by deriving a lower bound on the communication complexity of computing a \((2t - 1)\)-spanner that consists of at most \( \frac{1}{2} \log \log n\) edges, where \( t \) = \( \Theta\left(\frac{1}{\gamma} n^{1/2}\right) \). Our main result is that any \( O(\text{poly}(n))\)-time algorithm must send at least \( \tilde{\Omega}\left(\frac{1}{2} n^{1/2}\right) \) bits in the CONGEST model under the KT assumption. Previously, only a trivial lower bound of \( \tilde{\Omega}\left(\frac{1}{n}\right) \) bits was known for this problem; in fact, this is the first nontrivial lower bound on the communication complexity of a sparse subgraph problem in this setting.

A consequence of our lower bound is that achieving both time- and communication-optimality is impossible when designing a distributed spanner algorithm. In light of the work of King, Kutten, and Thorup (PODC 2015), this shows that computing a minimum spanning tree can be done significantly faster than finding a spanner when considering algorithms with \( \tilde{O}(n) \) communication complexity. Our result also implies time complexity lower bounds for constructing a spanner in the node-congested clique of Augustine et al. (2019) and in the push-pull gossip model with limited bandwidth.

1 INTRODUCTION

Designing distributed algorithms that are fast and communication-efficient is crucial for many applications. Modern-day examples include building large-scale networks of resource-restricted devices or processing massive data sets in a distributed system. When analyzing the performance of distributed algorithms, communication-efficiency is usually quantified by the message complexity, i.e., the total number of messages sent by the algorithm, or the communication complexity, which refers to the total number of bits sent throughout the execution. During the past decade, there has been significant interest in obtaining communication-efficient algorithms for solving fundamental graph problems in the message passing setting.

Due to [26], it is known that for very basic problems such as single-source broadcast and constructing a spanning tree, \( \Omega(m) \) messages (and bits) are required in an \( n \)-node graph \( G \) that has \( m \) edges, assuming that nodes are initially unaware of the IDs of their neighbors and messages are addressed using port numbers rather than specific IDs. This model is called the clean network model [36] or port numbering model [40] (if nodes do not have IDs), and several time- and communication-optimal algorithms have been obtained that match the \( \Omega(m) \) barrier, e.g., for minimum spanning trees (MST) [16, 34], approximate single-source shortest paths [22], and leader election [26].

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Since the clean network model does not capture the more realistic setting of nowadays IP networks where nodes are aware of the IDs of their peers, there has been a growing interest in studying communication-efficiency under the KT$_1$ assumption [3, 36], where nodes have unique IDs of length $\Theta(\log n)$ and each node knows all the IDs of its neighbors in $G$ from the start. [24] were the first to present a minimum spanning tree (MST) algorithm that runs in $\tilde{O}(n)$ rounds and sends $\tilde{O}(n)$ bits in the CONGEST-KT$_1$ model [36] under the KT$_1$ assumption by using linear graph sketching techniques. More recently, new distributed algorithms have been proposed for several graph problems in this setting; for instance, algorithms for single-source broadcast [20], graph verification problems [21], and computing an MST in the asynchronous setting [28, 30] have been obtained that achieve a message complexity of $O(n)$.

A somewhat counterintuitive feature of the KT$_1$ assumption is that it allows solving any graph problem using only $\tilde{O}(n)$ bits of communication by leveraging silence to convey information, albeit at the cost of increasing the running time to an exponential number of rounds: If we first construct a rooted spanning tree using the algorithm of [24], we can then employ a simple time-encoding scheme to collect the entire graph topology at the root who locally solves the problem and, subsequently, the result be can be disseminated to all nodes again by using time-encoding with $\tilde{O}(n)$ bits.\footnote{For completeness, we provide a more detailed description of this time-encoding procedure in Appendix B.} Consequently, any lower bound on the required communication must be conditional on the algorithm terminating in a sufficiently small number of rounds; note that this stands in contrast to the clean network model where the $\Omega(m)$ barrier holds for all algorithms that terminate in a finite number of rounds. Hence, it is not too surprising that the stronger guarantees provided by the KT$_1$ assumption make it significantly harder to show meaningful lower bounds on the required communication of graph algorithms. To the best of our knowledge, to this date there are no lower bounds known for sparse subgraph problems such as constructing a minimum spanning tree, single-source broadcast or finding a spanner, apart from the immediate observation that a constant fraction of the nodes need to send at least one bit.

While the above-mentioned work assumes the synchronous model, where nodes can send messages of $O(\log n)$ bits over each link in each round, communication-efficiency has also been studied in the asynchronous CONGEST-KT$_1$ model [29, 30], which, analogously to its synchronous counterpart, assumes that a node can send a message of size $O(\log n)$ whenever it is scheduled to take a step. Even though time-encoding is not possible in the asynchronous model due to the absence of lock-step synchronicity, the state of the art is that non-trivial lower bounds on the required communication for these types of graph problems are still lacking even in the asynchronous setting.

In this paper, we introduce a new way to quantify the communication complexity of distributed graph problems, and we apply our method to obtain lower bounds for constructing a spanner with a distributed algorithm. Given an unweighted graph $G$, a multiplicative $k$-spanner [3, 37] is a subgraph $S \subseteq G$, such that the distance in $S$ between any pair of vertices is at most $k$ times the distance in the original graph $G$. Fast distributed algorithms for constructing a $(2t - 1)$-spanner with only $O(n^{1+1/t})$ edges in expectation and $O(n^{1+1/t} \log n)$ edges (whp\footnote{With high probability, i.e., with probability at least $1 - \frac{1}{n}$}) are well known, e.g., see [7]. Moreover, this size-stretch ratio is believed to be tight due to the girth conjecture of Erdős [18], namely that there exist graphs with $\Omega(n^{1+1/t})$ edges and girth $2t + 1$. We point out that all existing distributed spanner algorithms for the CONGEST model send
at least $\Omega(t \cdot m)$ bits in the worst case on a graph having $m$ edges. Recently, [8] showed that it is possible to obtain a spanner in constant rounds with $O(n^{1+\varepsilon})$ edges and constant stretch in the LOCAL model by sending only $O(n^{1+\varepsilon})$ messages. Note, however, that there is no bandwidth restriction in the LOCAL model and hence their result does not imply an upper bound on the communication complexity. Since a spanner can be used to solve other problems such as single-source broadcast with $o(m)$ bits of communication by using the spanner as a communication backbone, we know from the existing lower bounds (see [26]) that, in the clean network model, constructing a spanner is subject to the above-mentioned barrier of $\Omega(m)$ on the message complexity; the $\Omega(m)$ lower bound for spanners was also pointed out explicitly by [15]. In other words, the existing distributed spanner algorithms are near-optimal with respect to time, as well as communication complexity in the clean network model, for small values of $t$. So far, however, a lower bound on the communication complexity of graph spanners under the KT$_1$ assumption has been elusive. Our work is a first step towards resolving this open question.

It does not seem viable to directly obtain a communication complexity bound for spanners under the KT$_1$ assumption via the standard route of a reduction from 2-party communication complexity. First off, much of the work in this setting (e.g. [19]) assumes that the edges are adversarially partitioned between Alice and Bob, which makes it challenging to apply these results to the KT$_1$ setting, where edges are always shared between nodes that are neighbors.

On the other hand, if we consider the vertex-partition model in the 2-party setting, where the vertices and their incident edges are distributed between Alice and Bob (with shared edges being duplicated), it is possible to simulate the 3-spanner algorithm of [7] by using only $O(n \log n)$ bits of communication: First, Alice and Bob locally sample the clusters required by the first phase of the algorithm as follows: They each add all nodes with degree at most $\sqrt{n}$ to their set of clusters. Then, they both locally sample each of their of the remaining vertices independently with probability $\Theta(\log n/\sqrt{n})$. This ensures that every node that has degree $> \sqrt{n}$ has a cluster neighbor whp. Alice and Bob then exchange their computed cluster assignment using $O(n \log n)$ bits of communication. In the second phase of Baswana/Sen, for each vertex, we need to add an edge to each neighboring cluster. Since Alice and Bob both know the cluster membership of all nodes, they can simulate this step without further communication by using the simple rule$^3$ that, for each vertex $v$ and each adjacent cluster $c$, we add a spanner edge to the neighbor of $v$ that has the smallest ID among all neighbors that are members of $c$.

1.1 Our Results

In this work, we take a step towards a systematic approach to proving lower bounds on the required communication for sparse subgraph problems under the KT$_1$ assumption. Our main results are as follows:

- In Section 3, we define the local information cost (LIC$_\gamma(P)$) of solving a problem $P$ with error at most $\gamma$ and show that it represents a lower bound on the communication complexity in the asynchronous message passing model, as well as a lower bound of $\Omega\left(\frac{\text{LIC}_\gamma(P)}{\log r \log n}\right)$ bits for $r$-round algorithms in the synchronous CONGEST-KT$_1$ model. To the best of our knowledge, this is the first lower bound that provides a general trade-off between communication and time in this setting. As the local information cost can be characterized for any graph problem $P$, we believe these results to be of independent

$^3$A similar rule was used to simulate the algorithm of [7] in the LCA model, see [35].
interest. Moreover, we show that the local information cost implies a time complexity lower bound of 
\( \Omega \left( \frac{\text{LIC}_r(P)}{n \log^3 n} \right) \) rounds in the node-congested [1], and \( \Omega \left( \frac{\text{LIC}_r(P)}{n \log^3 n} \right) \) rounds in the push-pull gossip model.

- We show that constructing a \((2t - 1)\)-spanner with \( O(n^{1+\epsilon/2}) \) edges, for any \( \epsilon \leq \frac{1}{64t^2} \), has a high local information cost, resulting in a communication complexity of \( \Omega \left( \frac{1}{t^2 \log n} n^{1+1/2t} \right) \) bits in the CONGEST-KT, model, for any algorithm that terminates in \( O(\text{poly}(n)) \) rounds (Theorem 7.2), which is the first nontrivial lower bound for a sparse subgraph problem in this setting. This reveals a sharp contrast to the known fast \((2t - 1)\)-spanner algorithms such as [7], which takes only \( O(t^2) \) rounds and hence does not depend on \( n \) at all, but sends \( \tilde{\Omega}(t \cdot m) \) bits. Interestingly, our lower bound result holds even in the synchronous congested clique model, thus showing that the availability of additional communication links does not help in achieving simultaneous time- and message-optimality. We obtain these results by proving a lower bound on the local information cost for constructing a spanner in the asynchronous KT clique. In the proof, we use tools from information theory to quantify the information that many nodes need to learn about their incident edges and then apply known synchronization techniques.

- As a consequence of the above, we obtain Corollary 7.3, which states that it is impossible to obtain a spanner algorithm that is both time- and communication optimal.

- Our technique also implies a time complexity lower bound of \( \Omega \left( \frac{n^{1/2t}}{t^2 \log^3 n} \right) \) rounds for constructing a \((2t - 1)\)-spanner in the node-congested clique model of [1] (see Theorem 7.4). Similarly, we obtain a lower bound of \( \Omega \left( \frac{n^{1/2t}}{t^2 \log^3 n} \right) \) rounds in the push-pull gossip model with restricted bandwidth (see Theorem 7.5).

### 1.2 Technical Approach

To obtain a lower bound on the local information cost of computing a \((2t - 1)\)-spanner, we consider a graph \( G \) on two sets of nodes \( U \) and \( V \), each of size \( n/2 \). The topology of \( G \) consists of some static edges and a smaller number of randomly sampled edges. The static edges are given by connecting \( U \) and \( V \) by many disjoint copies of the complete bipartite graph which we call the regions of \( G \). In contrast, the random edges are independently sampled only on \( U \) and loosely interconnect the regions on \( U \)’s side (see Figure 1 on Page 25 where static edges are blue and random edges are red). The intuition behind this construction is that, from the point of view of a node \( u \in U \), most of its incident static edges cannot be part of a \((2t - 1)\)-spanner, whereas most of \( u \)’s incident random edges are crucial for maintaining the stretch bound. That is, if a random edge \((u, v)\) is removed and \( u \) and \( v \) are not part of the same region of \( G \), then it is unlikely that there is a path of length at most \((2t - 1)\) from \( u \) to \( v \), due to the topological structure of \( G \).

It is of course possible (and even likely when considering [7]!) that some nodes include most of their incident edges as part of the spanner. Nevertheless, most nodes in \( U \) need to restrict their output to roughly their (expected) number of incident random edges to guarantee the bound on the size of the spanner. For the rest of this overview, we focus on this set of sparse nodes. While the lower bound graph construction assures us that the crucial random edges must be part of the spanner, we also need to quantify the information that \( u \) needs to receive in order to distinguish its static edges from its random ones. Initially, \( u \)’s knowledge is limited to its local view of the graph topology, consisting only of its incident edges and the IDs of its
neighboring nodes. Assuming the algorithm terminates correctly, the (initially) high entropy regarding which of \( u \)'s incident edges are crucial random edges must be reduced significantly over the course of the algorithm.

In more detail, there are two bounds on entropy quantities that we need to compute w.r.t. to the incident critical edges of \( u \): the “initial” entropy conditioned on \( u \)'s initial state and the “final” entropy conditioned on \( u \)'s output. Both of these quantities depend on the number of incident critical edges of \( u \), called \( Y \), which itself is a random variable. We point out that it is unclear whether it is possible to show a strong concentration bound on \( Y \) due to dependencies arising between incident edges.\(^4\) Even though we could try to use a looser concentration bound on \( Y \) to separately obtain a lower bound on the initial entropy and an upper bound on the final entropy, the difference between these two quantities turned out to be too sensitive for this approach to yield a meaningful bound on the information learned by \( u \).

Instead we compute the entropy bounds in terms of the expected number of incident critical edges. A technical complication emanates from the fact that these expected values of \( Y \) are computed over distinct probability spaces, as we condition on additional knowledge when computing the upper bound on the final entropy. Consequently, we cannot directly use the linearity property of expectation to bound the difference between initial and final entropy. To circumvent this obstacle, we first need to lift the expected value of \( Y \) (conditioned on \( u \)'s initial knowledge) to the probability space where we also condition on \( u \)'s output. We show that this changes the expected values by at most \( 1 - o(1) \), which is sufficient to obtain the sought bound on the local information cost.

### 1.3 Other Measures of Information Cost

Several ways of measuring the learned information that is revealed when executing an algorithm have been defined in the information and communication complexity literature. Closely related to our notion of local information cost is the *internal information cost* that has been studied extensively in the 2-player model of communication complexity (see [4, 6, 11]): Consider inputs \( X \) and \( Y \) of Alice and Bob respectively that are sampled according to the joint distribution \( \mu \) and let \( \Pi \) denote the transcript of a 2-party protocol. The *internal information cost with respect to \( \mu \)* is defined as \( I[X : \Pi \mid Y] + I[Y : \Pi \mid X] \). This corresponds to the expected amount of information that Alice learns about the input of Bob (and vice versa) from the transcript of the protocol.

In [25], the authors generalize the *external information cost* of [6] to the multiparty setting in the shared blackboard model, which measures the amount of information that an external observer learns about the players’ inputs. More recently, [9] defined the *switched information cost* for the multiparty number-in-hand message passing model, which, in a sense, combines internal and external information cost. Their setting is similar to the asynchronous clique with the main difference being that players communicate only through a coordinator.

\(^4\)For instance, knowing that \((u, v)\) is critical, may increase the probability that some other edge \((u, w)\) is critical.
2 PRELIMINARIES

2.1 Distributed Computing Models

For the main technical part of the paper, we consider an asynchronous message passing model, where \( n \) nodes can communicate by sending messages across point-to-point links. The set of nodes together with the available communication links form a clique, which we call the communication network. As we are interested in studying graph problems in this setting, we consider a graph \( G \) with \( m \) edges as the input, which is a spanning subgraph of the clique. Thus, each node of the communication network is associated with one vertex from \( G \) and its incident edges. We equip the nodes with unique identifiers (ID) chosen from an integer range of small polynomial size such that an ID can be represented using \( \Theta(\log n) \) bits. Throughout this work we consider the KT\(_1\) assumption introduced by [3]. In our setting, the KT\(_1\) assumption means that each node starts out knowing its own ID in addition to the IDs of all other nodes. In particular, a node \( u \) is aware of the IDs of its neighbors in \( G \). We point out that the KT\(_1\) assumption has been used by several recent works to obtain sublinear (in the number of edges) bounds on the communication complexity for various graph problems (e.g., [3, 20, 21, 24, 28–30]).

In the asynchronous model, a node only takes steps whenever it is given its turn by the (adversarial) scheduler, who controls the speed at which messages travel across links with the restriction that each message sent takes at most one unit of time to be delivered to its destination. Whenever a node takes a step, it can process all received messages, perform some local computation including accessing a private source of random bits, and send (possibly distinct) messages to an arbitrary subset of its peers. We assume that each message contains the ID of the sender in addition to the actual payload and hence the smallest message size is \( \Theta(\log n) \) bits. We say that an algorithm \( \mathcal{A} \) errs with probability at most \( \gamma \), if, for any given graph \( G \), the execution of \( \mathcal{A} \) has probability at least \( 1 - \gamma \) to yield a correct output at each node.

In Sections 3.1 and 7, we also consider the congested clique [27] and the synchronous CONGEST-KT\(_1\) model [36], where the computation is structured in rounds, and nodes can send at most \( O(\log n) \) bits over each communication link per round. The former model allows all-to-all communication, analogously to the asynchronous clique described above, whereas the CONGEST-KT\(_1\) model restricts the communication to the edges of \( G \).

2.2 Time and Message Complexity

The message complexity of a distributed algorithm \( \mathcal{A} \) is the maximum number of messages sent by the nodes when executing \( \mathcal{A} \). The communication complexity of \( \mathcal{A} \), on the other hand, takes into account the maximum number of bits sent in any run of \( \mathcal{A} \). As these two quantities are within logarithmic factors of each other in the CONGEST model, we will mostly state our bounds in terms of communication complexity. We assume that each message contains the ID of the sender; hence the smallest message size is \( \Omega(\log n) \) in the asynchronous model and, as explained above, \( \Theta(\log n) \) in the CONGEST model. When considering the synchronous CONGEST-KT\(_1\) model in Section 7, we are interested in algorithms that run for a maximum number of \( \tau \) rounds, and we define \( \tau \) to be their time complexity.
3 LOCAL INFORMATION COST

When solving a sparse subgraph construction problem \( P \), each node must output a set of incident edges such that the union of the edges satisfies the properties of \( P \), and the output of any two adjacent nodes must be consistent. In contrast to verification problems such as graph connectivity, it is possible that the nodes’ output for solving a subgraph construction problem on a given graph is not unique. For instance, when constructing a \( k \)-spanner, the local output of a node is the set of its incident edges that are part of the spanner with the requirement that the total number of output edges is sufficiently small and that their union guarantees a stretch of at most \( k \).

As mentioned in Section 2.1, we consider an asynchronous clique as the underlying communication network and we assume that we sample the graph \( G \) according to some distribution \( G \). Given \( G \), each node \( u_i \) observes its initial local state represented by a random variable \( \mathcal{B} \). We point out that \( \mathcal{B}_i \) and \( \mathcal{B}_j \) are not necessarily independent for nodes \( u_i \) and \( u_j \). At the very least, \( \mathcal{B}_i \) contains \( u_i \)'s ID as well as the IDs of its neighbors in \( G \), according to the KT\(_1\) assumption. For technical reasons, it sometimes makes sense to reveal additional information about \( G \) to \( u_i \) (as being part of \( \mathcal{B}_i \)); we will do so in Section 6. Clearly, this can only help the algorithm and hence strengthens the lower bound.

**Definition 3.1 (Local Information Cost (LIC)).** For an algorithm \( \mathcal{A} \) that errs with probability at most \( \gamma \), let random variable \( \Pi_i \) denote the transcript of the messages received by node \( u_i \). We use \( \text{LIC}_G(\mathcal{A}) \) to denote the local information cost of algorithm \( \mathcal{A} \) under distribution \( G \), and define

\[
\text{LIC}_G(\mathcal{A}) = \sum_{i=1}^{n} I[\Pi_i : G \mid X_i].
\]  

The local information cost for solving problem \( P \) with error at most \( \gamma \) is defined as

\[
\text{LIC}_\gamma(P) = \inf_{\mathcal{A}: \gamma\text{-error}} \max_G \text{LIC}_G(\mathcal{A}).
\]  

That is, we obtain \( \text{LIC}_\gamma(P) \) by considering the best-performing \( \gamma \)-error algorithm (if one exists) and the worst-case input graph distribution \( G \) with respect to \( \mathcal{A} \). Lemma 3.2 (proved in Appendix C) confirms that there always exists a worst-case distribution \( G \), which justifies using the maximum instead of the supremum in (2).

**Lemma 3.2.** Consider any \( \gamma \)-error algorithm \( \mathcal{A} \) as stated in Definition 3.1. Then, \( \max_G (\text{LIC}_G(\mathcal{A})) = \sup_G (\text{LIC}_G(\mathcal{A})) \).

We emphasize that, in our definition of \( \text{LIC}_G(\mathcal{A}) \), the node inputs \( X_i \) and \( X_j \) are not necessarily independent for nodes in \( G \); for instance, if \( u_i \) and \( u_j \) have a common neighbor, its ID will show up in both \( X_i \) and \( X_j \). Consequently, if \( u_i \) has neighbors \( u_1, \ldots, u_l \), then \( u_i \)'s input \( X_i \) is fully determined by its neighbors’ inputs \( X_1, \ldots, X_l \). This is a crucial feature of the KT\(_1\) assumption and a difference to the edge-partition multiparty number-in-hand model of communication complexity (c.f. [38, 41]), where the input distribution of the edges between the players can be chosen independently from the graph topology.

3.1 Lower Bounds in Distributed Computing via Local Information Cost

We now show that the local information cost presents a lower bound on the communication complexity of distributed graph algorithms. We first show the result for the asynchronous KT\(_1\) clique.
Lemma 3.3. The communication complexity of solving problem $P$ in the asynchronous $K_{T_1}$ model with error at most $\gamma$, is at least $\text{LIC}_{\gamma}(P)$.

Proof. Define the communication complexity $CC(\mathcal{A})$ of algorithm $\mathcal{A}$ to be the maximum number of bits sent in any run of the $\gamma$-error algorithm $\mathcal{A}$, and recall that the communication complexity $CC_{\gamma}(P)$ of solving problem $P$ with error $\gamma$ is defined as

$$CC_{\gamma}(P) = \min_{\mathcal{A}: \gamma\text{-error}} CC(\mathcal{A}).$$

Let $\mathcal{A}$ be a communication-optimal $\gamma$-error algorithm for $P$, i.e., $CC(\mathcal{A}) = CC_{\gamma}(P)$. For each node $u_i$, let random variable $L_i$ be the length of the transcript received by $u_i$ during the run of algorithm $\mathcal{A}$. Notice that $L_i$ is a random variable that may vary from run to run, which is different from the worst case transcript length, usually denoted by $|\Pi_i|$ in the literature (e.g. [5]). Moreover, let $G$ be sampled according to $\mathcal{G}$, which is the distribution that maximizes (2) with respect to $\mathcal{A}$. We have

$$\mathbb{E}[L_i] \geq H[\Pi_i] \quad \text{(by Lemma A.5)}$$

$$\geq H[\Pi_i \mid X_i] \quad \text{(by (A.3))}$$

$$\geq I[\Pi_i : G \mid X_i], \quad \text{(3)}$$

where the last step follows from Lemma A.4. Let random variable $M$ be the number of bits sent by algorithm $\mathcal{A}$, i.e., $M = \sum_{u_i \in G} L_i$. Since $\mathcal{A}$ is communication-optimal w.r.t. $P$, we have

$$CC_{\gamma}(P) = CC(\mathcal{A}) \geq \mathbb{E}[M] = \sum_{u_i \in G} \mathbb{E}[L_i]$$

$$\geq \sum_{u_i \in G} I[\Pi_i : G \mid X_i] \quad \text{(by (3))}$$

$$= \text{LIC}_{\mathcal{G}}(\mathcal{A}) \quad \text{(by (1))}$$

$$= \max_{\mathcal{G}} \text{LIC}_{\mathcal{G}}(\mathcal{A}) \quad \text{(since $\mathcal{G}$ maximizes (2))}$$

$$\geq \inf_{\mathcal{A}: \gamma\text{-error}} \max_{\mathcal{G}} \text{LIC}_{\mathcal{G}}(\mathcal{A})$$

$$= \text{LIC}_{\gamma}(P). \quad \text{(by (2))}$$

□

Lemma 3.4. The communication complexity of solving problem $P$ in at most $\tau$ rounds with probability at least $1 - \gamma$ is $\Omega \left( \frac{\text{LIC}_{\gamma}(P)}{\log \tau \log n} \right)$ bits. This holds in the synchronous $\text{CONGEST KT}_{\tau}$ model as well as in the congested clique.

Proof. Several synchronizers have been proposed in the literature (see [2, 36]). As we consider a clique communication topology, we will use the $\sigma$-synchronizer of [33]. Given a synchronous algorithm with communication complexity $C_S$ and round complexity $T_S$ for some problem $P$, the $\sigma$-synchronizer yields an algorithm in the asynchronous clique with a communication complexity of $CC_{\gamma}(P) = O(C_S \log(T_S) \log n)$ bits (see Theorem 1 in [33]) by exploiting the clique topology and compressing silent rounds. Applying
Lemma 3.3, we know that
\[ \text{LIC}_G(P) \leq \text{CC}_G(P) = O(C_S \log T_S \log n) \]
and hence
\[ C_S = \Omega \left( \frac{\text{LIC}_G(P)}{\log T_S \log n} \right). \]
This shows the result for the congested clique. Since the CONGEST-KT$_1$ model can be simulated in the congested clique by simply ignoring communication links that are not part of the graph, the same lower bound also holds in the former model. \( \square \)

We now move on to models where a high local information cost implies a lower bound on the time complexity, due to limitations imposed on the communication capabilities of the nodes.

**Lemma 3.5.** Solving problem $P$ with probability at least $1 - \gamma$ in the node-congested clique requires $\Omega \left( \frac{\text{LIC}_G(P)}{n \log^5 n} \right)$ rounds.

**Proof.** Assume towards a contradiction that there exists an algorithm $A$ that solves $P$ whp in $\tau = o \left( \frac{\text{LIC}_G(P)}{n \log^5 n} \right)$ rounds in the node-congested clique. During each of the at most $\tau$ rounds of execution, a node can send at most $O(\log n)$ messages of $O(\log n)$ bits according to the specification of the node-congested clique [1], i.e., the total number of bits sent per round is $O(n \log^2 n)$. Now suppose we execute the presumed algorithm $A$ in the KT$_1$ congested clique. Since $\text{LIC}_G(P) = \sum_{u_i} I[G : \Pi_i | X_i] \leq n \text{H}[G] = O(n^2)$, it follows that $\tau = O(\text{poly}(n))$. Therefore, the communication complexity $\text{CC}(A)$ of $A$ is
\[ \text{CC}(A) = O(n \log^3 n) \cdot \tau = o \left( \frac{\text{LIC}_G(P)}{\log^2 n} \right) = o \left( \frac{\text{LIC}_G(P)}{\log \tau \log n} \right), \]
which is a contradiction to Lemma 3.4. \( \square \)

Finally, we also provide a way to obtain lower bounds in the gossip model when the link bandwidth is limited to $O(\log n)$ bits. Observing that per round at most $O(n \log n)$ bits are being sent, a similar argument as in the proof of Lemma 3.5 shows the following:

**Lemma 3.6.** Consider a gossip model (as described in [10]) where, in each round, each node can initiate a message exchange with a single neighbor in graph $G$, but, in contrast to [10], assume that the bandwidth of each link is limited to $O(\log n)$ bits per round. Solving problem $P$ with probability at least $1 - \gamma$ requires $\Omega \left( \frac{\text{LIC}_G(P)}{n \log^5 n} \right)$ rounds.

4 THE LOWER BOUND GRAPH

Let $k = 2t - 1$ for some integer $t \geq 2$. In this section, we describe the lower bound graph distribution $G_k$ on which computing a multiplicative $k$-spanner with a distributed algorithm incurs a high local information cost.

Consider $n$ vertices and divide them into two sets, each of size $n/2$, called $U = \{u_1, \ldots, u_{n/2}\}$ and $V = \{v_1, \ldots, u_{n/2}\}$. To equip each node with a unique ID, we fix the enumeration $u_1, \ldots, u_{n/2}, v_1, \ldots, v_{n/2}$ and choose a permutation of $[1, n]$ uniformly at random as the ID assignment.

---

5To simplify the presentation, we assume that $n/2, n^{\frac{1}{3t}}, \frac{1}{2} n^{1-\frac{1}{3t}} \cdot \frac{1}{2}, n^{rac{1}{3t}}$ are integers.
We will classify edges into blue edges and red edges. Note that we only introduce this coloring for the purpose of our analysis, i.e., the edge colors are not part of the nodes’ input. We partition the vertices into subgraphs of size \(2n^{\frac{1}{2+k+1}}\) that we call regions. Each region consists of \(n^{\frac{1}{2+k+1}}\) nodes each from \(U\) and \(V\) and we form a complete bipartite graph of blue edges between the vertices from \(U\) and \(V\). For instance, the first region consists of the vertices \(u_1, \ldots, u_{n^{2/(k+1)+1/4k^2}}\) and \(v_1, \ldots, v_{n^{2/(k+1)+1/4k^2}}\) and the blue edges between them. Note that each vertex \(u_i\) is in exactly one region, denoted by \(P(u_i)\).

Next, we add red edges by creating a random graph on the subgraph \(G[U]\) (induced by the nodes in \(U\)) according to the Erdős-Rényi model. That is, we sample each one of the \(|U|(|U| - 1)\) possible edges independently with probability \(\frac{3}{2n^1-1/(k+1)}\). Figure 1 (on Page 25) depicts an example of a lower bound graph instance.

Let \(X\) be the number of red edges incident to \(u_i\). According to the random graph model, we have

\[
E[X] = (|U| - 1) \left( \frac{3}{2n^{1-1/(k+1)}} \right) = \frac{3}{4}n^{\frac{1}{k+1}} - o(1).
\]

The following lemma shows that all degrees are likely to be concentrated around the mean.

**Lemma 4.1.** Let \(D \subseteq U\) be the subset of nodes whose number of incident red edges is in the range \(\left[ \frac{3}{4}n^{\frac{1}{k+1}} - \Theta \left( n^{\frac{1}{k+1}} \sqrt{\log n} \right), \frac{3}{4}n^{\frac{1}{k+1}} + \Theta \left( n^{\frac{1}{k+1}} \sqrt{\log n} \right) \right]\). Then, with high probability, \(D = U\).

**Proof.** Consider some \(u \in U\) and let random variable \(X\) be the number of its incident red edges. The proof uses standard Chernoff bounds (Theorems 4.4 and 4.5 in [31]). We only show concentration for the lower tail and omit the upper tail argument as it is similar.

Let \(\gamma = 4\sqrt{\log(n)/n^{\frac{1}{k+1}}}\). Recalling (4) and applying Theorem 4.5 in [31] reveals that

\[
\Pr \left[ X \leq (1 - \gamma) E[X] \right] \leq \exp \left( -\frac{\gamma^2 E[X]}{2} \right) \leq \exp \left( -\frac{6 \log(n) \left( n^{\frac{1}{k+1}} - o(1) \right)}{n^{\frac{1}{k+1}}} \right) \leq \frac{1}{n^2}
\]

By a similar argument we can use a Chernoff bound (Theorem 4.4 in [31]) to obtain concentration for the upper bound of the range. The lemma follows by taking a union bound over the \(n/2\) nodes in \(U\). \(\square\)

## 5 Reachability and Critical Edges

**Critical Edges.** Suppose that we sample \(G\) according to \(G_k\). This results in each node \(u \in U\) having \(\Theta \left( n^{\frac{1}{k+1}} \right)\) incident red edges in expectation. A crucial property of our lower bound graph is that some of these red edges are likely to be part of any \(k\)-spanner. We say that a red edge \((u, v)\) is critical if any cycle that contains both \(u\) and \(v\) has length at least \(k + 2\). See Figure 1 for an example of a critical edge. From this definition, we immediately have the following property:

**Lemma 5.1.** Any \(k\)-spanner of \(G\) includes all critical edges.

### 5.1 Traversal Sequences

Consider a critical edge \((u, v)\) in \(G\), where \(u, v \in U\), and let \(H\) be the graph obtained by removing \((u, v)\). There is a nonzero probability that \(v\) is reachable from \(u\) in \(H\) by traversing some sequence of red and blue edges. This motivates us to consider a traversal sequence (starting from \(u\)), which is a \(k\)-length sequence...
of edge colors that specify, for each step, whether we follow the red or blue edges. Formally, a traversal sequence $T$ is a $k$-length string where each character is chosen from the alphabet $\{R, B\}$; $R$ and $B$ stand for colors red and blue, respectively. We use the notation $T[i]$ to refer to the $i$-th character of $T$, and $T[i, j]$ to identify the traversal subsequence $T[i] \ldots T[j]$. We say that the $i$-th step is a $B$-step if $T[i] = B$, and define $R$-step similarly.

A traversal sequence $T$ induces a reachable set of nodes, denoted by $\mathcal{R}(T)$, which is determined by the subset of all nodes in the $k$-hop neighborhood of $u$ that are reachable from $u$ in $H$ by following all possible paths along edges with the colors in the same order as specified by $T$. We use $\mathcal{R}_i(T)$ to refer to the reachable set of the traversal subsequence $T[1, i]$.

For instance, assume that $k = 5$ and consider the traversal sequence $T = RBBRR$. To obtain the reachable set $\mathcal{R}(T)$, we conceptually build a tree of $k + 1$ levels, which corresponds to the view tree of node $u$. We first add $u$ as the root on level 0. To obtain level 1, we add as child each neighbor $w$ of $u$ that is reachable by traversing a red edge. Similarly, for each such added $w$, we add the nodes reachable across a blue edge incident to $w$ as its children on level 2 and so forth. Note that we only add nodes that were not already included in the tree. We continue this process until we have added all nodes on all levels up to and including level $k = 5$. Consequently, a reachable set has the following property:

**Lemma 5.2.** Node $v$ is reachable from $u$ in at most $k$ hops in $H$ if there exists a traversal sequence $T$ of length $k$ such that $v \in \mathcal{R}(T)$.

### 5.2 A Bound on the Number of Critical Edges

In this section, we prove the following result:

**Lemma 5.3.** Every $u \in D$ is incident to at least $\frac{3}{2} n^{1/k^2} \left(1 - 2n^{-\frac{1}{k^2}}\right) - o(1)$ critical edges in expectation.

Fix some $u \in U$ and an arbitrary red edge $(u, v)$ to some other $v \in U$. Recall that if $(u, v)$ is critical then, if we remove $(u, v)$, reaching $v$ from $u$ requires more than $k$ hops. Thus, we will analyze the reachability of $v$ from $u$ on the graph $H = G \setminus \{(u, v)\}$.

**High-Level Overview of the Proof of Lemma 5.3:** Lemma 5.2 provides us with the following strategy for bounding the probability that $v$ is reachable from $u$ in $k$ steps: We first identify the type of traversal sequence that exhibits the largest growth with respect to the reachable set of nodes. Below we show that sequences that follow a certain structure, i.e., are “$B$-maximal”, dominate all other sequences regarding the probability of leading to a reachable set that contains $v$. Intuitively speaking, a sequence is $B$-maximal if, starting from nodes in $U$, it alternates crossing the blue edges twice (this will extend the reach to all nodes in $U$ that are inside the regions of the currently reachable nodes) with a single hop over red edges (which may lead to nodes outside of these regions). Since this type of traversal sequences reach the most number of nodes, we pessimistically assume that all traversal sequences are $B$-maximal and, having obtained an upper bound of roughly $\frac{1}{2^k n^{1/k^2}}$ on the probability of reaching $v$ with a $B$-maximal traversal sequence, we can take a union bound over the total number of traversal sequences. This shows that, at least a $\left(1 - n^{-\frac{1}{k^2}}\right)$-fraction of the incident red edges of $u$ are critical in expectation. We postpone the detailed proof to Appendix D.
6 THE LOCAL INFORMATION COST OF SPANNERS

In this section, we will use an information-theoretic approach to bound the local information cost of computing a spanner. In more detail, we will prove the following result:

**Theorem 6.1.** Consider the problem of constructing a \((2t - 1)\)-spanner that, with high probability, has at most \(O(n^{1 + \frac{1}{2t} + \frac{1}{16(2t-1)^2}})\) edges, for any positive integer \(t\) such that \(2 \leq t = O\left(\frac{\log n/\log \log n}{1/3}\right)\). Then,

\[
\text{LiC}_{1/n}(2t - 1\text{-spanner}) = \Omega \left(\frac{1}{t} n^{1 + \frac{1}{2t} \log \log n}\right).
\]

To put Theorem 6.1 into perspective, note that the distributed \((2t - 1)\)-spanner algorithm of Baswana & Sen [7] outputs at most \(O(n^{1 + \frac{1}{2} \log n}) = O(n^{1 + \frac{1}{2} \log n/\log n})\) edges with high probability. For \(t = o\left(\sqrt{\log n/\log \log n}\right)\), this amounts to strictly less than \(n^{1 + \frac{1}{2t} + \frac{1}{16(2t-1)^2}}\) edges.

Consider any \((2t - 1)\)-spanner algorithm \(
\mathcal{A}\n\) that achieves the properties stated in the theorem’s premise, and define

\[
k = 2t - 1 = \Omega \left(\frac{\log n}{\log \log n}\right)^{1/3}.
\]

Using (5), we can rewrite the bound on the number of edges in Theorem 6.1 as \(O\left(n^{1 + \frac{2}{k^2} + \frac{1}{8k^2}}\right)\), which we will use throughout this section. In parts of our analysis, in particular Lemma 6.7, we will focus on the nodes where the output is sparse, in the sense that each of them outputs at most \(O(n^{\frac{2}{k^2} + \frac{1}{8k^2}})\) spanner edges. In particular, we will prove that these nodes learn a significant amount of information about which of their incident edges are critical. We start by showing that this set (which may depend on the private randomness of the nodes) is likely to contain all but \(o(n)\) nodes of \(U\).

**Lemma 6.2.** Let \(S \subseteq U\) be the subset of nodes such that each \(u \in S\) outputs at most \(n^{\frac{2}{k^2} + \frac{1}{8k^2}}\) edges. If the algorithm terminates correctly, then

\[
|S| \geq |U| - \frac{n}{\log n} = \frac{n}{2} - \frac{n}{\log n}.
\]

**Proof.** Assume towards a contradiction that \(|U \setminus S| \geq \frac{n}{\log n}\). This means that a set \(B\) of at least \(\frac{n}{\log n}\) nodes in \(U\) output at least \(n^{\frac{2}{k^2} + \frac{1}{8k^2}}\) spanner edges each. Consequently, the algorithm outputs at least

\[
|B| \geq \Omega \left(n^{1 + \frac{2}{k^2} + \frac{1}{8k^2}}\right)\log n
\]

edges in total. From (5), we know that \(k = O\left(\left(\frac{\log n}{\log \log n}\right)^{1/3}\right)\) and hence \(k \leq \sqrt{\frac{\log n}{24 \log \log n}}\) for sufficiently large \(n\), which implies that

\[
\frac{1}{8k^2} - \frac{\log n}{\log \log n} \geq \frac{1}{12k^2}.
\]

It follows that the right-hand side of (6) is at least \(\Omega \left(n^{1 + \frac{2}{k^2} + \frac{1}{8k^2}}\right)\), thus exceeding the assumed bound on the size of the spanner stipulated by Theorem 6.1 and resulting in a contradiction. □

**Indicator Random Variables and Notation.** Throughout this section, we use capitals to denote random variables and corresponding lowercase characters for values. To shorten the notation, we will sometimes
abbreviate the event "\(X = x\)" by simply writing "\(x\)" for random variables \(X\) and a value \(x\). For the indicator random variables defined above, we use shorthands such as \(I_{D,Y_i} = 1\) to refer to the event \(I_D = I_{Y_i} = 1\). When computing expected values, we sometimes use the subscript notation \(E_X\) to clarify that the expectation is taken over the distribution of random variable \(X\).

We make use of the following indicator random variables (RV):

- We define \(I_D = 1\) if and only if \(U = D\), which happens with high probability (see Lemma 4.1).
- Let \(Y_i\) be the number of critical edges incident to \(u_i\). We define \(I_{Y_i} = 1\) if and only if \(Y_i \geq \frac{1}{16} n^{\frac{1}{2\pi} + \frac{1}{3\pi}}\) (see Lemma 5.3 below).
- We use the indicator random variable \(I_{S_i}\) that is 1 if and only if \(u_i \in S\) and \(|S| \geq \frac{n}{2} - \frac{n}{\log n}\) (notice this is the same bound as in Lemma 6.2). The reason for introducing \(I_{S_i}\) will become clearer in the proof of Lemma 6.6.

We first prove a concentration bound on the number of critical edges by leveraging Lemma 5.3 and the upper bound on the incident red edges stated in Lemma 4.1.

**Lemma 6.3.** Recall that \(I_{Y_i} = 1\) if and only if \(u_i\) has at least \(\frac{1}{16} n^{\frac{1}{2\pi}}\) incident critical edges. It holds that

\[
\Pr[I_{Y_i} = 1 \mid I_D = 1] \geq 1 - O\left(n^{-\frac{1}{2\pi}}\right)
\]

**Proof.** Let \(Y_i\) be the number of critical edges incident to \(u_i\). From Lemma 5.3, we know that

\[
\mathbb{E}[Y_i \mid I_D = 1] \geq \frac{2}{3} \left(n^{\frac{1}{2\pi}} - 2n^{\frac{1}{2\pi} - \frac{1}{3\pi}}\right) - o(1).
\]

(7)

We know that \(Y_i\) cannot exceed the number of red edges incident to \(u_i\) and, conditioned on \(I_D = 1\) (i.e. \(u_i \in D\)), Lemma 4.1 tells us that we can set the upper bound to be

\[
B = \frac{3}{4} n^{\frac{1}{2\pi}} + c_1 n^{\frac{1}{2\pi} + \frac{1}{3\pi}} \sqrt{\log n},
\]

(8)

for some suitable constant \(c_1 > 0\). Consider the random variable \(B - Y\), which is always positive due to the conditioning on \(I_D = 1\). By Markov’s inequality (c.f. Theorem 3.1 in [31]), it follows that, for any \(a < B\),

\[
\Pr[Y_i \leq a \mid I_D = 1] = \Pr[B - Y_i \geq B - a \mid I_D = 1] \leq \frac{B - \mathbb{E}[Y_i \mid I_D = 1]}{B - a}.
\]

Choosing \(a = \frac{1}{16} n^{\frac{1}{2\pi}}\), and plugging (7) and (8) into this concentration bound yields

\[
\Pr\left[Y_i \leq \frac{1}{16} n^{\frac{1}{2\pi}} \mid I_D = 1\right] \leq \frac{\left(\frac{3}{4} n^{\frac{1}{2\pi}} + c_1 n^{\frac{1}{2\pi} + \frac{1}{3\pi}} \sqrt{\log n}\right) - \frac{3}{4} \left(n^{\frac{1}{2\pi}} - 2n^{\frac{1}{2\pi} - \frac{1}{3\pi}}\right) + o(1)}{\left(\frac{3}{4} n^{\frac{1}{2\pi}} + c_1 n^{\frac{1}{2\pi} + \frac{1}{3\pi}} \sqrt{\log n}\right) - \frac{1}{16} n^{\frac{1}{2\pi}}}
\]

\[
\leq \frac{c_1 n^{\frac{1}{2\pi} + \frac{1}{3\pi}} \sqrt{\log n} + \frac{3}{2} n^{\frac{1}{2\pi} - \frac{1}{3\pi}} + o(1)}{\frac{11}{16} n^{\frac{1}{2\pi}} + c_1 n^{\frac{1}{2\pi} + \frac{1}{3\pi}} \sqrt{\log n}}.
\]

\[\text{[For instance, the expression } \mathbb{H}[C_i \mid F, y_i]\text{ is the same as } \mathbb{H}[C_i \mid F, Y_i = y_i].\]
Observe that $n^{\frac{1}{2(k+1)}} \sqrt{\log n} = n^{\frac{1}{2(k+1)}} + \frac{\sqrt{\log n}}{\log n} \leq n^{\frac{1}{2(k+1)}} + \frac{1}{k^2}$, for any $k \geq 3$. Moreover, $n^{\frac{1}{2(k+1)}} + \frac{1}{2k} \leq n^{\frac{1}{2(k+1)}} - \frac{1}{k^3}$, and therefore

$$
\Pr \left[ Y_i \leq \frac{1}{16} n^{\frac{k}{k+1}} \mid I_D = 1 \right] \leq \frac{\frac{3}{2} c_1 n^{\frac{1}{k+1}} - \frac{1}{k^3} + o(1)}{\frac{11}{16} n^{\frac{1}{k+1}} + c_1 n^{\frac{1}{2(k+1)}} \sqrt{\log n}} \\
\leq \frac{\frac{3}{2} c_1 n^{\frac{1}{k+1}} - \frac{1}{k^3} + o(1)}{\frac{11}{16} n^{\frac{1}{k+1}}} \\
= O \left( n^{-\frac{1}{k^3}} \right).
$$

The Initial Knowledge of Nodes. We consider the following information to be part of $u_i$’s input $X_i$: Due to the KT$_1$ assumption, $u_i$ knows the random variable $Z_i$, which contains the list of IDs of its neighbors in $G$ including its own ID, as well as the IDs of all other nodes. Notice that $Z_i$ does not contain any other information about the network at large. In addition, node $u_i$ also knows the number of incident critical edges to $u_i$, which we denote by the random variable $Y_i$. Note that this is not part of the KT$_1$ assumption but extra knowledge given to $u_i$ for free. Since $Y_i$ fully determines the indicator random variable $I_{Y_i}$ defined above, it follows that $u_i$ also has knowledge of $I_{Y_i}$. Finally, we assume that $u_i$ knows $I_D$, i.e., whether $U = D$.

We make use of the following simple conditioning property of mutual information:

**Lemma 6.4.** Let $X, Y, Z_1,$ and $Z_2$ be discrete random variables. Then, for any $z$ in the support of $Z_1$, it holds that

$$
I \left[ X : Y \mid Z_1, Z_2 \right] \geq \Pr[Z_1 = z] \cdot I \left[ X : Y \mid Z_1 = z, Z_2 \right].
$$

**Proof.**

$$
I \left[ X : Y \mid Z_1, Z_2 \right] = \sum_{z_1} \sum_{z_2} \Pr[z_1, z_2] \cdot I \left[ X : Y \mid Z_1 = z_1, Z_2 = z_2 \right] \\
= \sum_{z_1} \Pr[z_1] \sum_{z_2} \Pr[z_2 \mid z_1] \cdot I \left[ X : Y \mid Z_1 = z_1, Z_2 = z_2 \right] \\
\geq \Pr[z] \cdot I \left[ X : Y \mid Z_1 = z, Z_2 \right],
$$

for any $z$ in the support of $Z_1$. □

According to the definition of the local information cost in (1) on Page 7, we have

$$
\text{LIC}_G(F) \geq \sum_{u_i \in U} I \left[ G : \Pi_i \mid X_i \right] \quad (\text{since } U \subseteq V(G))
$$

$$
= \sum_{u_i \in U} I \left[ G : \Pi_i \mid Y_i, Z_i, I_D, I_{Y_i} \right] \quad (\text{since } X_i = (Y_i, Z_i, I_D, I_{Y_i}))
$$

$$
\geq \Pr[I_D = 1] \sum_{u_i \in U} I \left[ G : \Pi_i \mid Y_i, Z_i, I_{Y_i}, I_D = 1 \right] \quad (\text{by Lemma 6.4})
$$

$$
\geq \Pr[I_D = 1] \sum_{u_i \in U} \Pr[I_{Y_i} = 1 \mid I_D = 1] \cdot I \left[ G : \Pi_i \mid Y_i, Z_i, I_D, I_{Y_i} = 1 \right], \quad (9)
$$
where, in the last step, we have applied Lemma 6.4 to each term of the sum. Without conditioning on the initial states of the nodes, it holds for all \( u_i, u_j \in U \) that

\[
\Pr[I_{Y_i} = 1 \mid I_D = 1] = \Pr[I_{Y_j} = 1 \mid I_D = 1] \geq 1 - O\left( n^{-\frac{k}{3}} \right),
\]

where the inequality follows from Lemma 6.3. Furthermore, we know from Lemma 4.1 that \( \Pr[I_D = 1] \geq 1 - \frac{1}{n} \), and hence \( \Pr[I_D = 1] \Pr[I_{Y_i} = 1 \mid I_D = 1] \geq \left( 1 - O\left( n^{-\frac{k}{3}} \right) \right) (1 - \frac{1}{n}) \geq \frac{1}{2} \), for sufficiently large \( n \), since we assume that \( k = O \left( (\log n / \log \log n)^{\frac{1}{3}} \right) \). Applying these observations to (9), we get

\[
\text{LIC}_G(\mathcal{A}) \geq \frac{1}{2} \sum_{u_i \in U} \left[ \frac{H[C_i \mid Y_i, Z_i, I_{D,Y_i} = 1]}{\sum_{u_i \in U} H[C_i \mid \Pi_i, Y_i, Z_i, I_{D,Y_i} = 1]} - H[C_i \mid \Pi_i, Y_i, Z_i, I_{D,Y_i} = 1] \right].
\]

For each \( u_i \in U \), its incident critical edges, given by the random variable \( C_i \), are determined by the graph \( G \). This means that \( \Pi_i \rightarrow G \rightarrow C_i \) forms a Markov chain and hence the data processing inequality (see Lemma A.6) implies that \( H[C_i \mid \Pi_i, Y_i, Z_i, I_{D,Y_i} = 1] \geq H[C_i \mid Y_i, Z_i, I_{D,Y_i} = 1] \). Moreover, we can write the mutual information between \( C_i \) and \( \Pi_i \) in terms of the conditional entropies (see (21) in Appendix A), yielding

\[
\text{LIC}_G(\mathcal{A}) \geq \frac{1}{2} \sum_{u_i \in U} \left[ H[C_i \mid Y_i, Z_i, I_{D,Y_i} = 1] - H[C_i \mid \Pi_i, Y_i, Z_i, I_{D,Y_i} = 1] \right].
\]  

(10)

We will proceed by analyzing the entropy terms on the right-hand side in (10). Bounding the "remaining" entropy \( H[C_i \mid \Pi_i, Y_i, Z_i, I_{D,Y_i} = 1] \), i.e., after \( u_i \) has received all messages, will require us to reason about the number of spanner edges output by \( u_i \). We give more power to the algorithm by revealing to each \( u_i \) whether \( u_i \in S \) and \( |S| \geq \frac{n}{2} - \frac{n}{\log n} \), which is captured by the indicator random variable \( I_{S_i} \) defined above. In more detail, we modify the algorithm by prepending the value of \( I_{S_i} \) to the transcript \( \Pi_i \), as the very first bit of each node \( u_i \in U \); this increases \( \text{LIC}_G(\mathcal{A}) \) by \( O(n) \) bits and hence does not change the asymptotic bound in Theorem 6.1.

The conditioning on \( I_{S_i} \) introduces a technical challenge, as we would like to compute the difference in expectation between the two entropy terms, but we are computing the initial entropy of the critical edges (i.e. \( H[C_i \mid Y_i, Z_i, I_{D,Y_i} = 1] \)) conditioned on \( I_{D,Y_i} = 1 \), whereas, for the remaining entropy, we also condition on \( I_{S_i} = 1 \). To this end, we need the following helper lemma that we will use in the proof of Lemma 6.6 to switch to the probability space where we condition on all three indicator random variables \( I_{D,S_i,Y_i} = 1 \).

**Lemma 6.5.** \( \Pr[I_{S_i} = 1 \mid I_{D,Y_i} = 1] \geq 1 - O\left( \frac{1}{\log n} \right) \).

**Proof.** We start our analysis by first deriving a lower bound on \( \Pr[I_{S_i} = 1] \). Without conditioning on the initial states, \( \Pr[u_i \in S] = \Pr[u_j \in S] \), for all \( u_i, u_j \in U \), and hence it follows that

\[
\Pr\left[ u_i \in S \mid |S| \geq \frac{n}{2} - \frac{n}{\log n} \right] \geq \frac{n}{2} - \frac{n}{\log n} = 1 - O\left( \frac{1}{\log n} \right).
\]

By assumption, the algorithm succeeds with probability at least \( 1 - \frac{1}{n} \) and, if it does so, then it also holds that \( |S| \geq \frac{n}{2} - \frac{n}{\log n} \) according to Lemma 6.2. This means that

\[
\Pr[I_{S_i} = 1] = \Pr \left[ u_i \in S \mid |S| \geq \frac{n}{2} - \frac{n}{\log n} \right] \Pr \left[ |S| \geq \frac{n}{2} - \frac{n}{\log n} \right] \geq 1 - O\left( \frac{1}{\log n} \right) - \frac{1}{n} = 1 - O\left( \frac{1}{\log n} \right).
\]
We now return to bounding $\Pr[I_{S_i}=1 \mid I_{D,Y_i}=1]$. We have

$$1 - O\left(\frac{1}{\log n}\right) \leq \Pr[I_{S_i}=1] \leq \Pr[I_{S_i}=1 \mid I_{D,Y_i}=1] \Pr[I_{D,Y_i}=1] + \Pr[I_{S_i}=1 \mid \neg I_{D,Y_i}=1] \Pr[\neg I_{D,Y_i}=1],$$

and hence

$$\Pr[I_{S_i}=1 \mid I_{D,Y_i}=1] \geq 1 - O\left(\frac{1}{\log n}\right) - \Pr[\neg I_{D,Y_i}=1].$$

To complete the proof of the lemma, we need to argue that $\Pr[\neg I_{D,Y_i}=1] = O\left(\frac{1}{\log n}\right)$. We have

$$\Pr[\neg I_{D,Y_i}=1] \leq \Pr[I_D=0] + \Pr[I_{Y_i}=0 \mid I_D=1] \Pr[I_D=1]$$

$$\leq \Pr[I_{Y_i}=0 \mid I_D=1] + O\left(\frac{1}{n}\right) \quad \text{(by Lemma 4.1)}$$

$$= O\left(\frac{1}{\log n}\right). \quad \text{(by Lemma 6.3)}$$

□

**Lemma 6.6.** Let $C_i$ denote the critical edges incident to $u_i$. It holds that

$$H[C_i \mid Y_i, Z_i, I_{D,Y_i}=1] \geq \left(\frac{2}{k+1} + \frac{1}{4k^2}\right) (\log_2(n) - O(1)) \cdot E[Y_i \mid I_{D,S_i,Y_i}=1] - E[Y_i \log_2 Y_i \mid I_{D,S_i,Y_i}=1].$$

**Proof.** By the definition of conditional entropy, we have

$$H[C_i \mid Y_i, Z_i, I_{D,Y_i}=1] = \sum_{y,z} \Pr[y, z \mid I_{D,Y_i}=1] H[C_i \mid Y_i=y, Z_i=z, I_{D,Y_i}=1], \quad (11)$$

where $y$ and $z$ are such that $\Pr[Y_i=y, Z_i=z \mid I_{D,Y_i}=1] > 0$.

We will derive a bound on $H[C_i \mid Y_i=y, Z_i=z, I_{D,Y_i}=1]$. Recall from Section 4 that the assignment of the random IDs to the nodes is done independently of the sampling of the red edges (which determines $I_D$ and $I_{Y_i}$), and hence the assignment of the IDs to the nodes in the neighborhood of $u_i$ (given by $z$) is independent of $I_D$ and $I_{Y_i}$. In other words, the ID assignments given to $u_i$’s neighbors which is known to $u_i$ does not reveal any information about which edges are blue and which ones are red, let alone which edges are critical. More formally, for each neighbor $w \in U \cup V$ of $u_i$, the edge $(u_i, w)$ is red with some probability $p$ (independent of $w$’s ID), and critical with some probability $p' \leq p$, where this event is also independent of $w$’s ID. Therefore, if we consider any two subsets $c$ and $c'$ of exactly $y$ nodes chosen from the neighborhood $z$ of $u_i$ that are identified by their (unique) IDs, it follows from the above that $\Pr[C_i=c \mid y, z, I_{D,Y_i}=1] = \Pr[C_i=c' \mid y, z, I_{D,Y_i}=1]$.

To obtain a lower bound on $u_i$’s degree, we recall that $u_i$ has $\frac{n^{2\gamma} + \frac{1}{4k^2}}{y}$ incident (blue) edges, and hence

$$\Pr[C_i=c \mid y, z, I_{D,Y_i}=1] \leq 1/\left(\frac{n^{2\gamma} + \frac{1}{4k^2}}{y}\right). \quad (12)$$
We combine these observations to obtain that
\[ H[C_i \mid y, z, I_{D,Y_i} = 1] = \sum_c \Pr[C_i = c \mid y, z, I_{D,Y_i} = 1] \log_2 \left( \frac{\sum_c \Pr[c \mid y, z, I_{D,Y_i} = 1]}{y} \right) \]
\[
\geq \sum_c \Pr[c \mid y, z, I_{D,Y_i} = 1] \log_2 \left( \frac{n^{\frac{2}{k+1} + \frac{1}{4k^2}}}{y} \right) \quad \text{(by (12))}
\]
\[
= \log_2 \left( \frac{n^{\frac{2}{k+1} + \frac{1}{4k^2}}}{y} \right) \sum_c \Pr[c \mid y, z, I_{D,Y_i} = 1] = \log_2 \left( \frac{n^{\frac{2}{k+1} + \frac{1}{4k^2}}}{y} \right) y
\]
\[
\geq \log_2 \left( \frac{n^{\frac{2}{k+1} + \frac{1}{4k^2}}}{y} \right)^{\frac{2}{k+1} + \frac{1}{4k^2}} \quad \text{(since } \sum_c \Pr[c \mid y, z, I_{D,Y_i} = 1] = 1)\]
\[
= \left( \frac{2}{k+1} + \frac{1}{4k^2} \right) y \log_2 n - y \log_2 y.
\]

Applying this bound to (11), we get
\[ H[C_i \mid Y_i, Z_i, I_{D,Y_i} = 1] \geq \sum_{y,z} \Pr[y, z \mid I_{D,Y_i} = 1] \left( \frac{2}{k+1} + \frac{1}{4k^2} \right) \log_2 n - y \log_2 y \]
\[ \geq \sum_y \Pr[y \mid I_{D,Y_i} = 1] \left( \frac{2}{k+1} + \frac{1}{4k^2} \right) \log_2 n - y \log_2 y \]
\[ = \sum_y \Pr[y \mid I_{D,Y_i} = 1] \left( \frac{2}{k+1} + \frac{1}{4k^2} \right) \log_2 n - y \log_2 y \] (13)

The final expression is equivalent to \( \mathbb{E}_{Y_i} \left[ \left( \frac{2}{k+1} + \frac{1}{4k^2} \right) Y_i \log_2 n - Y_i \log_2 Y_i \mid I_{D,Y_i} = 1 \right] \).

As explained earlier, we need to compute the expectation conditioned on \( I_{D,S_i,Y_i} = 1 \). We will use Lemma 6.5 to lift (13) to this probability space. For any \( y \) in the support of \( Y_i \), we have
\[ \Pr[y \mid I_{D,Y_i} = 1] \geq \Pr[y \mid I_{D,S_i,Y_i} = 1] \Pr[I_{S_i} = 1 \mid I_{D,Y_i} = 1] \]
\[ \geq \left( 1 - O \left( \frac{1}{\log n} \right) \right) \Pr[y \mid I_{D,S_i,Y_i} = 1]. \quad \text{(by Lemma 6.5)}
\]

Returning to (13), we conclude that
\[ H[C_i \mid Y_i, Z_i, I_{D,Y_i} = 1] \geq \left( 1 - O \left( \frac{1}{\log n} \right) \right) \sum_y \Pr[y \mid I_{D,S_i,Y_i} = 1] \left( \frac{2}{k+1} + \frac{1}{4k^2} \right) \log_2 n - y \log_2 y \]
\[ = \left( 1 - O \left( \frac{1}{\log n} \right) \right) \mathbb{E}_{Y_i} \left[ \left( \frac{2}{k+1} + \frac{1}{4k^2} \right) Y_i \log_2 n - Y_i \log_2 Y_i \mid I_{D,S_i,Y_i} = 1 \right].
\]

The lemma follows by linearity of expectation. \( \Box \)

**Lemma 6.7.**
\[ H[C_i \mid \Pi_i, Y_i, Z_i, I_{D,Y_i} = 1] \leq \left( \frac{2}{k+1} + \frac{1}{8k^2} \right) \log_2 n + \log_2 e \mathbb{E} \left[ Y_i \mid I_{D,S_i,Y_i} = 1 \right]
\[ + O \left( \frac{3}{k+1} \right) \mathbb{E} \left[ Y_i \mid I_{D,Y_i} = 1, I_{S_i} = 0 \right] - \mathbb{E} \left[ Y_i \log_2 (Y_i) \mid I_{D,S_i,Y_i} = 1 \right] \]
Proof. As outlined above, we can deduce $I_S$ from the transcript because the first bit that $u_i$ receives is the value of $I_{S_i}$. Moreover, the set of incident spanner edges $F_i$ that are finally output by $u_i$ is determined by the transcript $\Pi_i$ and the initial state of $u_i$ as given by $Y_i, Z_i$ (including $u_i$’s private random bits). Hence,

$$H[C_i \mid \Pi_i, Y_i, Z_i, I_{D,Y} = 1] \leq H[C_i \mid F_i, I_{S_i}, Y_i, Z_i, I_{D,Y} = 1] \leq H[C_i \mid F_i, I_{S_i}, Y_i, I_{D,Y} = 1]$$ (by Lemma A.3)

$$= \sum_{b \in \{0,1\}} \sum_{f,y} Pr[F_i = f, I_{S_i} = b, Y_i = y \mid I_{D,Y} = 1] H[C_i \mid F_i = f, I_{S_i} = b, Y_i = y, I_{D,Y} = 1]$$

$$= Pr[I_{S_i} = 1 \mid I_{D,Y} = 1] \sum_{f,y} Pr[f, y \mid I_{D,S_i,Y_i} = 1] H[C_i \mid f, y, I_{D,S_i,Y_i} = 1]$$

$$+ Pr[I_{S_i} = 0 \mid I_{D,Y} = 1] \sum_{f,y} Pr[f, y \mid I_{D,Y} = 1, I_{S_i} = 0] H[C_i \mid f, y, I_{D,Y} = 1, I_{S_i} = 0]$$

$$\leq \sum_{f,y} Pr[f, y \mid I_{D,S_i,Y_i} = 1] H[C_i \mid f, y, I_{D,S_i,Y_i} = 1]$$

$$+ O\left(\frac{1}{\log n}\right) \sum_{f,y} Pr[f, y \mid I_{D,Y} = 1, I_{S_i} = 0] H[C_i \mid f, y, I_{D,Y} = 1, I_{S_i} = 0]$$ (14)

where the last inequality holds because Lemma 6.5 implies that $Pr[I_{S_i} = 0 \mid I_{D,Y} = 1] = O(1/\log n)$.

We will separately bound the sums on the right-hand side. Fix any $f$ and $y$. To obtain an upper bound on $H[C_i \mid f, y, I_{D,S_i,Y_i} = 1]$, recall that the conditioning on $I_{S_i} = 1$ says that $u_i$’s output $f$ contains at most $n^{\frac{2}{k+1} + \frac{1}{sk^2}}$ spanner edges (see Lemma 6.2) and, we know from Lemma 5.1 that every critical edge must be part of the spanner, which tells us that the $y$ critical edges incident to $u_i$ are part of $u_i$’s output. The total number of ways we can choose a subset $c$ of size $y$ from the edges in $f$ is

$$\binom{|f|}{y} \leq \left(\frac{n^{\frac{2}{k+1} + \frac{1}{sk^2}}}{y}\right).$$

Moreover, the (remaining) entropy $H[C_i \mid f, y, I_{D,S_i,Y_i} = 1]$ is maximized if the conditional distribution of $C_i$ is uniform over all subsets $c$ of size $y$. Combining these observations, we get

$$H[C_i \mid f, y, I_{D,S_i,Y_i} = 1] = \sum_c Pr[C_i = c \mid f, y, I_{D,S_i,Y_i} = 1] \log_2 \left(\frac{1}{Pr[c \mid f, y, I_{D,S_i,Y_i} = 1]}\right)$$

$$\leq \sum_c Pr[C_i = c \mid f, y, I_{D,S_i,Y_i} = 1] \log_2 \left(\frac{n^{\frac{2}{k+1} + \frac{1}{sk^2}}}{y}\right)$$

$$= \log_2 \left(\frac{n^{\frac{2}{k+1} + \frac{1}{sk^2}}}{y}\right)$$ (since $\sum_c Pr[C_i = c \mid \ldots] = 1$)

$$\leq \log_2 \left(\frac{e n^{\frac{2}{k+1} + \frac{1}{sk^2}}}{y}\right)^y$$ (using $\left(\frac{n}{k}\right) \leq \left(\frac{en}{k}\right)^k$)

$$= y \left(\frac{2}{k+1} + \frac{1}{8k^2}\right) \log_2 n + \log_2 \left(\frac{e}{y}\right).$$ (15)
Next, we derive an upper bound on the entropy term in the second sum of (14), i.e., \( H \left[ C_i \mid f, y, I_{D,Y_i} = 1, I_{S_i} = 0 \right] \).

Since \( I_{S_i} = 0 \), we cannot make any assumptions on the number of edges that are included in the output of \( u_i \); in fact, it may happen that \( u_i \) simply outputs all incident edges and so the entropy of \( C_i \) may not decrease at all. Since we condition on \( I_D = 1 \), we know that \( u_i \in D \) and, by Lemma 4.1, the maximum number of blue and red edges incident to \( u_i \) is bounded by

\[
d_{\text{max}} = n \frac{k}{2} + \frac{1}{8k^2} + n \frac{k}{2k^2} \leq n \frac{k}{2k^2},
\]

for sufficiently large \( n \). The conditioning on \( Y_i = y \) ensures that \( u_i \) has exactly \( y \) critical edges, which means that the entropy is maximized if the distribution of \( C_i \) over the at most \( d_{\text{max}} \) edges incident to \( u_i \) is uniform with probability \( 1/\binom{d_{\text{max}}}{y} \). That is,

\[
H \left[ C_i \mid f, y, I_{D,Y_i} = 1, I_{S_i} = 0 \right] = \sum_c \Pr \left[ c \mid f, y, I_{D,Y_i} = 1, I_{S_i} = 0 \right] \log_2 \left( \frac{1}{\Pr \left[ c \mid f, y, I_{D,Y_i} = 1, I_{S_i} = 0 \right]} \right)
\leq \sum_c \Pr \left[ c \mid f, y, I_{D,Y_i} = 1, I_{S_i} = 0 \right] \log_2 \left( \frac{d_{\text{max}}}{y} \right)
\leq y \log_2 d_{\text{max}} \sum_c \Pr \left[ c \mid f, y, I_{D,Y_i} = 1, I_{S_i} = 0 \right]
\leq \frac{3y}{k+1} \log_2 n,
\]

where the last step follows from (16) and the fact that \( \sum_c \Pr \left[ c \mid \ldots \right] = 1 \).

We now combine the bounds that we obtained in (15) and (17) to obtain an upper bound on the expression \( H \left[ C_i \mid \Pi_i, Y_i, Z_i, I_{D,Y_i} = 1 \right] \). From (14), it follows that

\[
H \left[ C_i \mid \Pi_i, Y_i, Z_i, I_{D,Y_i} = 1 \right] \leq \sum_{f,y} \Pr \left[ f, y \mid I_{D,S_i,Y_i} = 1 \right] y \left( \left( \frac{2}{k+1} + \frac{1}{8k^2} \right) \log_2 n + \log_2 \left( \frac{e}{y} \right) \right)
+ O \left( \frac{1}{\log n} \right) \sum_{f,y} \Pr \left[ f, y \mid I_{D,Y_i} = 1, I_{S_i} = 0 \right] \frac{3y}{k+1} \log_2 n
\leq \mathbb{E}_{f_i,y_i} \left[ Y_i \left( \left( \frac{2}{k+1} + \frac{1}{8k^2} \right) \log_2 n + \log_2 \left( \frac{e}{Y_i} \right) \right) \mid I_{D,S_i,Y_i} = 1 \right]
+ O \left( \frac{1}{\log n} \right) \mathbb{E}_{f_i,Y_i} \left[ \frac{3Y_i}{k+1} \log_2 n \mid I_{D,Y_i} = 1, I_{S_i} = 0 \right]
= \left( \frac{2}{k+1} + \frac{1}{8k^2} \right) \log_2 n + \log_2 \left( \frac{e}{Y_i} \right) \mathbb{E}_{Y_i} \left[ Y_i \mid I_{D,S_i,Y_i} = 1 \right]
+ O \left( \frac{3}{k+1} \right) \mathbb{E}_{Y_i} \left[ Y_i \mid I_{D,Y_i} = 1, I_{S_i} = 0 \right] - \mathbb{E}_{Y_i} \left[ Y_i \log_2 \left( Y_i \right) \mid I_{D,S_i,Y_i} = 1 \right].
\]
Equipped with Lemmas 6.6 and 6.7, we can continue our derivation on LIC$G(\mathcal{A})$. By applying these bounds to the two entropy terms on the right-hand side of (10), we get
\[
\text{LIC}_G(\mathcal{A}) \geq \frac{1}{2} \sum_{u_i \in U} \left( \left( \frac{1}{4k^2} - \frac{1}{8k^2} \right) \log_2 n - O \left( \frac{2}{k+1} + \frac{1}{4k^2} \right) - \log_2 e \right) \mathbb{E} \left[ Y_i \mid I_{D,S_i,Y_i} = 1 \right]
- O \left( \frac{3}{k+1} \right) \mathbb{E} \left[ Y_i \mid I_{D,Y_i} = 1, I_{S_i} = 0 \right]
\]
\[
\geq \frac{1}{2} \sum_{u_i \in U} \left( \frac{1}{16k^2} \log_2 n \cdot \mathbb{E} \left[ Y_i \mid I_{D,S_i,Y_i} = 1 \right] - O \left( \frac{3}{k+1} \right) \mathbb{E} \left[ Y_i \mid I_{D,Y_i} = 1, I_{S_i} = 0 \right] \right),
\]
since $\frac{1}{16k^2} \log_2 n \geq O \left( \frac{2}{k+1} + \frac{1}{4k^2} \right) + \log_2 e$ for sufficiently large $n$. The conditioning on $I_{Y_i} = 1$ tells us that $Y_i \geq \frac{1}{16n^{n/16}}$ and hence also $\mathbb{E} \left[ Y_i \mid I_{D,S_i,Y_i} = 1 \right] \geq \frac{1}{16n^{n/16}}$. On the other hand, the conditioning on $I_D = 1$ guarantees that the number of red edges incident to $u_i$ is at most $n^{n/16}$ (see Lemma 4.1) and hence the same bound holds for the number of critical edges, which guarantees that $\mathbb{E} \left[ Y_i \mid I_{D,Y_i} = 1, I_{S_i} = 0 \right] \leq n^{n/16}$. From this we conclude that
\[
\text{LIC}_G(\mathcal{A}) \geq \frac{|U|}{2} \left( \frac{1}{16k^2} n^{n/16} \log_2 n - O \left( n^{n/16} \right) \right)
= \Omega \left( \frac{1}{k^2} n^{n^{1/16}} \log n \right) \quad \text{(since } |U| = \Omega(n) \text{ and } k = O \left( (\log n/\log \log n)^{1/3} \right) \text{)}
= \Omega \left( \frac{1}{k^2} n^{n^{1/16}} \log n \right). \quad \text{(by (5))}
\]
This completes the proof of Theorem 6.1.

7 LOWER BOUNDS FOR DISTRIBUTED SPANNER ALGORITHMS

In this section, we derive communication and time lower bounds from Theorem 6.1. We first apply Lemma 3.3 to obtain the claimed bound on the communication complexity in the asynchronous model:

**Theorem 7.1.** Any algorithm that, with high probability, constructs a $(2t-1)$-spanner with $O \left( n^{1+1/4t-1/(16t-1)^2} \right)$ edges, for $2 \leq t \leq O \left( (\log(n)/\log \log n)^{1/3} \right)$, has a communication complexity of $\Omega \left( \frac{1}{t^2} n^{1+1/4t} \log n \right)$ bits in the asynchronous message passing clique under the KT$_1$ assumption.

From Lemma 3.4, we get a similar result for the synchronous CONGEST-KT$_1$ model and the congested clique. We do not explicitly state our bounds in terms of the message complexity, as this changes the result only by a logarithmic factor.

**Theorem 7.2.** Consider the synchronous KT$_1$ congested clique model and any $\tau = O \left( \text{poly}(n) \right)$. Any $\tau$-round algorithm that, with high probability, outputs a $(2t-1)$-spanner with at most $O \left( n^{1+1/4t-1/(16t-1)^2} \right)$ edges sends at least $\Omega \left( \frac{1}{\tau^2 \log n} \cdot n^{1+1/4t} \right)$ bits in the worst case, for $2 \leq t \leq O \left( (\log(n)/\log \log n)^{1/3} \right)$. The same result holds in the CONGEST-KT$_1$ model.

When considering the CONGEST-KT$_1$ model and $t = O(1)$, Theorem 7.2 implies that, for any $(2t-1)$-spanner algorithm that succeeds with high probability and takes a polynomial number of rounds, there
exists a graph where at least $\Omega(n^{1+\epsilon})$ bits are sent, for some constant $\epsilon > 0$. On the other hand, we know from [7] that the time complexity of $(2t-1)$-spanners is $O(t^2)$ rounds and, according to [14], $\Omega(t)$ is a lower bound even in the more powerful LOCAL model, which means that a time-optimal algorithm does not need to depend on $n$ at all. Furthermore, by leveraging the time-encoding trick mentioned in Section 1 (and described in more detail in Appendix B) it is possible to send only $\tilde{\Theta}(n)$ bits at the cost of a larger running time, which matches the trivial lower bound on the communication complexity of $\Omega(n)$ bits up to polylogarithmic factors. Combining these observations we have the following:

**Corollary 7.3.** There is no $(2t-1)$-spanner algorithm in the CONGEST-KT$_1$ model (or the congested clique) that outputs at most $O\left(n^{1+\frac{1}{t^2}+\frac{1}{16(2t-1)^2}}\right)$ edges with high probability and simultaneously achieves optimal time and optimal communication complexity.

Notice that Corollary 7.3 reveals a gap between constructing a spanner and the problem of finding a minimum spanning tree (MST) in the congested clique, as the work of [23] shows that it is possible to solve the latter in $O(\text{polylog}(n))$ time while sending only $\tilde{O}(n)$ bits.

We now turn our attention towards time complexity in the node-congested clique and the gossip model. Applying Lemma 3.5 reveals that constructing a spanner is harder than MST in the node-congested clique model, for which there is a $O(\text{polylog}(n))$ time algorithm (see [1]):

**Theorem 7.4.** Consider the node-congested clique model of [1]. Constructing a $(2t-1)$-spanner that, with high probability, has $O\left(n^{1+\frac{1}{t^2}+\frac{1}{16(2t-1)^2}}\right)$ edges requires $\Omega\left(\frac{1}{t^2\log n} n^{\frac{1}{3t}}\right)$ rounds, for $2 \leq t \leq O\left((\log(n)/\log \log n)^{1/3}\right)$.

From Lemma 3.6 we get a similar result for the gossip model:

**Theorem 7.5.** Consider the push-pull gossip model where the link bandwidth is limited to $O(\log n)$ bits. Constructing a $(2t-1)$-spanner that, with high probability, has $O\left(n^{1+\frac{1}{t^2}+\frac{1}{16(2t-1)^2}}\right)$ edges requires $\Omega\left(\frac{1}{t^2\log^2 n^{\frac{1}{3t}}}\right)$ rounds, for $2 \leq t \leq O\left((\log(n)/\log \log n)^{1/3}\right)$.

### 8. Future Work and Open Problems

We conclude by listing some interesting open problems and possible future research directions.

We have argued in Section 7 that it is impossible to construct a spanner in a way that is both time- and communication-optimal. However, this question is still unresolved for the basic problem of constructing minimum spanning trees:

**Open Problem 1.** Is it possible to construct a minimum spanning tree in $\tilde{O}(D+\sqrt{n})$ time and $O(n \text{ polylog}(n))$ communication complexity in the synchronous CONGEST-KT$_1$ model?

We have shown in Section 3 that the local information cost LIC$_{\gamma}(P)$ presents a lower bound on the communication complexity CC$_{\gamma}(P)$ of solving $P$ in the asynchronous message passing model with error at most $\gamma$. A natural question is whether this relationship is tight (up to polylogarithmic factors) for all graph problems.

**Open Problem 2.** Does it hold that $\text{CC}_{\gamma}(P) = \tilde{\Theta}(\text{LIC}_{\gamma}(P))$, for all graph problems $P$?
In this work, we have studied communication complexity lower bounds for multiplicative \((2t - 1)\)-spanners. To the best of our knowledge, all existing distributed spanner algorithms send at least \(\Omega(tm)\) bits and hence it is still an open question whether the \(\Omega(m)\) barrier known to hold for the clean network model (see Section 1) also holds under the KT\(_1\) assumption. In fact, this question is still open even if we allow a polynomial number of rounds:

**Open Problem 3.** Does there exist a \((2t - 1)\)-spanner algorithm that sends \(\tilde{O}\left(n^{1+\frac{1}{2t}}\right)\) bits and runs in \(O(poly(n))\) rounds in the synchronous CONGEST-KT\(_1\) model?

Finally, there are many fundamental graph problems, for which we are still very much in the dark regarding their communication complexity under the KT\(_1\) assumption.

**Open Problem 4.** What is the local information cost of other graph problems such as neighborhood covers [36], hopsets [12], and constructing a distributed routing scheme with small-size routing tables [17]?

**REFERENCES**

[1] John Augustine, Mohsen Ghaffari, Robert Gmyr, Kristian Hinnenthal, Christian Scheideler, Fabian Kuhn, and Jason Li. 2019. Distributed Computation in Node-Capacitated Networks. In The 31st ACM on Symposium on Parallelism in Algorithms and Architectures, SPAA 2019, Phoenix, AZ, USA, June 22-24, 2019, Christian Scheideler and Petra Berenbrink (Eds.). ACM, 69–79. https://doi.org/10.1145/3323165.3323195

[2] Baruch Awerbuch. 1985. Complexity of Network Synchronization. J. ACM 32, 4 (1985), 804–823. https://doi.org/10.1145/4221.4227

[3] Baruch Awerbuch, Oded Goldreich, David Peleg, and Ronen Vainish. 1990. A Trade-Off between Information and Communication in Broadcast Protocols. J. ACM 37, 2 (1990), 238–256. https://doi.org/10.1145/77600.77618

[4] Ziv Bar-Yossef, Thathachar S Jayram, Ravi Kumar, and D Sivakumar. 2004. An information statistics approach to data stream and communication complexity. J. Comput. System Sci. 68, 4 (2004), 702–732.

[5] Ziv Bar-Yossef, T. S. Jayram, Ravi Kumar, and D. Sivakumar. 2004. An information statistics approach to data stream and communication complexity. J. Comput. Syst. Sci. 68, 4 (2004), 702–732. https://doi.org/10.1016/j.jcss.2003.11.006

[6] Boaz Barak, Mark Braverman, Xi Chen, and Anup Rao. 2013. How to Compress Interactive Communication. SIAM J. Comput. 42, 3 (2013), 1327–1363. https://doi.org/10.1137/100811969

[7] Surender Baswana and Sandeep Sen. 2007. A simple and linear time randomized algorithm for computing sparse spanners in weighted graphs. Random Struct. Algorithms 30, 4 (2007), 532–563. https://doi.org/10.1002/rsa.20130

[8] Shimon Bitton, Yuval Emek, Taisuke Izumi, and Shay Kutten. 2019. Message Reduction in the LOCAL Model Is a Free Lunch. In 33rd International Symposium on Distributed Computing, DISC 2019, October 14-18, 2019, Budapest, Hungary. 7:1–7:15. https://doi.org/10.4230/LIPIcs.DISC.2019.7

[9] Mark Braverman, Faith Ellen, Rotem Oshman, Toniann Pitassi, and Vinod Vaikuntanathan. 2013. A Tight Bound for Set Disjointness in the Message-Passing Model. In 54th Annual IEEE Symposium on Foundations of Computer Science, FOCS 2013, 26-29 October, 2013, Berkeley, CA, USA. IEEE Computer Society, 668–677. https://doi.org/10.1109/FOCS.2013.77

[10] Keren Censor-Hillel, Bernhard Haeupler, Jonathan A. Kelner, and Petar Maymounkov. 2012. Global computation in a poorly connected world: fast rumor spreading with no dependence on conductance. In Proceedings of the 44th Symposium on Theory of Computing Conference, STOC 2012, New York, NY, USA, May 19 - 22, 2012, Howard J. Karloff and Toniann Pitassi (Eds.). ACM, 961–970. https://doi.org/10.1145/2213977.2214064

[11] Amit Chakrabarti, Yaojun Shi, Anthony Wirth, and Andrew Yao. 2001. Informational complexity and the direct sum problem for simultaneous message complexity. In Proceedings 42nd IEEE Symposium on Foundations of Computer Science. IEEE, 270–278.

[12] Edith Cohen. 2000. Polylog-time and near-linear work approximation scheme for undirected shortest paths. J. ACM 47, 1 (2000), 132–166. https://doi.org/10.1145/331605.331610

[13] T. Cover and J.A. Thomas. 2006. Elements of Information Theory, second edition. Wiley.
[14] Bilel Derbel, Cyril Gavoille, David Peleg, and Laurent Viennot. 2008. On the locality of distributed sparse spanner construction. In *Proceedings of the twenty-seventh ACM symposium on Principles of distributed computing*, 273–282.

[15] Michael Elkin. 2007. A near-optimal distributed fully dynamic algorithm for maintaining sparse spanners. In *Proceedings of the Twenty-Sixth Annual ACM Symposium on Principles of Distributed Computing, PODC 2007, Portland, Oregon, USA, August 12-15, 2007*, Indranil Gupta and Roger Wattenhofer (Eds.). ACM, 185–194. https://doi.org/10.1145/1281100.1281128

[16] Michael Elkin. 2017. A Simple Deterministic Distributed MST Algorithm, with Near-Optimal Time and Message Complexities. In *Proceedings of the ACM Symposium on Principles of Distributed Computing, PODC 2017, Washington, DC, USA, July 25-27, 2017*, Elad Michael Schiller and Alexander A. Schwarzmann (Eds.). ACM, 157–163. https://doi.org/10.1145/3087801.3087823

[17] Michael Elkin and Ofer Neiman. 2018. On efficient distributed construction of near optimal routing schemes. *Distributed Comput.* 31, 2 (2018), 119–137. https://doi.org/10.1007/s00446-017-0304-4

[18] Paul Erdös. 1963. Extremal problems in graph theory. In *Proc. Symp. Theory of Graphs and its Applications*. 2936.

[19] Manuel Fernandez, David P. Woodruff, and Taisuke Yasuda. 2020. Graph Spanners in the Message-Passing Model. In *11th Innovations in Theoretical Computer Science Conference, ITCS 2020, January 12-14, 2020, Seattle, Washington, USA (LIPIcs)*, Thomas Vidick (Ed.), Vol. 151. Schloss Dagstuhl - Leibniz-Zentrum für Informatik, 77:1–77:18. https://doi.org/10.4230/LIPIcs.ITCS.2020.77

[20] Mohsen Ghaffari and Fabian Kuhn. 2018. Distributed MST and Broadcast with Fewer Messages, and Faster Gossiping. In *32nd International Symposium on Distributed Computing, DISC 2018, New Orleans, LA, USA, October 15-19, 2018 (LIPIcs)*, Vol. 121. Schloss Dagstuhl - Leibniz-Zentrum für Informatik, 30:1–30:12. https://doi.org/10.4230/LIPIcs.DISC.2018.30

[21] Robert Gmyr and Gopal Pandurangan. 2018. Time-Message Trade-Offs in Distributed Algorithms. In *32nd International Symposium on Distributed Computing, DISC 2018, New Orleans, LA, USA, October 15-19, 2018 (LIPIcs)*, Vol. 121. Schloss Dagstuhl - Leibniz-Zentrum für Informatik, 32:1–32:18. https://doi.org/10.4230/LIPIcs.DISC.2018.32

[22] Bernhard Haeupler, D. Ellis Hershkowitz, and David Wajc. 2018. Round- and Message-Optimal Distributed Graph Algorithms. In *Proceedings of the 2018 ACM Symposium on Principles of Distributed Computing, PODC 2018, Egham, United Kingdom, July 23-27, 2018*. 119–128. https://doi.org/10.1145/3212734.3212737

[23] James W Hegeman, Gopal Pandurangan, Sriram V Pemmaraju, Vivek B Sardeshmukh, and Michele Squizzato. 2015. Toward optimal bounds in the congested clique: Graph connectivity and MST. In *Proceedings of the 2015 ACM Symposium on Principles of Distributed Computing*. 91–100.

[24] Valerie King, Shay Kutten, and Mikkel Thorup. 2015. Construction and Impromptu Repair of an MST in a Distributed Network with o(m) Communication. In *Proceedings of the 2015 ACM Symposium on Principles of Distributed Computing, PODC 2015, Donostia-San Sebastián, Spain, July 21 - 23, 2015*, Chryssis Georgiou and Paul G. Spirakis (Eds.). ACM, 71–80. https://doi.org/10.1145/2767386.2767405

[25] Gillat Kol, Rotem Oshman, and Dafna Sadeh. 2017. Interactive Compression for Multi-Party Protocol. In *31st International Symposium on Distributed Computing, DISC 2017, October 16-20, 2017, Vienna, Austria (LIPIcs)*, Andréa W. Richa (Ed.), Vol. 91. Schloss Dagstuhl - Leibniz-Zentrum für Informatik, 31:1–31:15. https://doi.org/10.4230/LIPIcs.DISC.2017.31

[26] Shay Kutten, Gopal Pandurangan, David Peleg, Peter Robinson, and Amitabh Trehan. 2015. On the Complexity of Universal Leader Election. *J. ACM* 62, 1 (2015), 7:1–7:27. https://doi.org/10.1145/2699440

[27] Zvi Lotker, Boaz Patt-Shamir, and David Peleg. 2006. Distributed MST for constant diameter graphs. *Distributed Computing* 18, 6 (2006), 453–460. https://doi.org/10.1007/s00446-005-0127-6

[28] Ali Mashreghi and Valerie King. 2006. Distributed MST for constant diameter graphs. *Distributed Computing* 18, 6 (2006), 453–460. https://doi.org/10.1007/s00446-005-0127-6

[29] Ali Mashreghi and Valerie King. 2018. Broadcast and Minimum Spanning Tree with o(m) Messages in the Asynchronous CONGEST Model. In *32nd International Symposium on Distributed Computing, DISC 2018, New Orleans, LA, USA, October 15-19, 2018 (LIPIcs)*, Vol. 121. Schloss Dagstuhl - Leibniz-Zentrum für Informatik, 37:1–37:17. https://doi.org/10.4230/LIPIcs.DISC.2018.37

[30] Ali Mashreghi and Valerie King. 2019. Brief Announcement: Faster Asynchronous MST and Low Diameter Tree Construction with Sublinear Communication. In *33rd International Symposium on Distributed Computing, DISC 2019, October 14-18, 2019, Budapest, Hungary (LIPIcs)*, Jukka Suomela (Ed.), Vol. 146. Schloss Dagstuhl - Leibniz-Zentrum für Informatik, 49:1–49:3.
Here we restate some basic facts (without proofs) that we use throughout the paper. More details can be found in [13].

Let $X$, $Y$, and $Z$ be discrete random variables.

**Definition A.1.** The entropy of $X$ is defined as

$$H[X] = \sum_x \Pr[X=x] \log_2 \left( \frac{1}{\Pr[X=x]} \right).$$

The conditional entropy of $X$ conditioned on $Y$ is given by

$$H[X \mid Y] = \mathbb{E}_y[H[X \mid Y=y]] = \sum_y \Pr[Y=y] H[X \mid Y=y].$$

**Definition A.2.** Let $X$, $Y$, and $Z$ be discrete random variables. The conditional mutual information is defined as

$$I[X : Y \mid Z] = \mathbb{E}_z[I[X : Y \mid Z=z]] = H[X \mid Z] - H[X \mid Y, Z].$$

**Lemma A.3.** $H[X \mid Y, Z] \leq H[X \mid Y]$. 

Fig. 1. An instance of the lower bound graph sampled according to $G_k$ with randomly assigned node IDs. The region of $u_1$ consists of nodes $u_1, \ldots, u_4$, $v_1, \ldots, v_4$. Assuming that $k = 5$, the edge $\left(u_1, u_{i+1}\right)$ is critical as any other path from $u_1$ to $u_{i+1}$ has length greater than 5. On the other hand, the edges $\left(u_1, u_5\right)$ and $\left(u_1, u_{10}\right)$ are not critical as they are both reachable by traversal sequences of length (at most) $k$: after removing $\left(u_1, u_5\right)$, node $u_5$ is reachable via the traversal sequence $RBBR$. Similarly, if we discard $\left(u_1, u_{10}\right)$, then $u_{10}$ is still in the reachable set $R(RRBB)$.

**Lemma A.4.** $I[X : Y | Z] \leq H[X | Z]$.

**Lemma A.5** (Theorem 6.1 in [39]). Every encoding of a random variable $X$ has expected length at least $H(X)$. 
We describe a folklore technique for obtaining a message-optimal algorithm for any problem $P$ in the CONGEST-KT$_1$ model. We first construct a spanning tree by using the algorithm of [24], which takes time $\tilde{O}(n)$ and requires sending $\tilde{O}(n)$ bits. Then, we elect a leader on this tree, which requires $\tilde{O}(n)$ bits using the algorithm of [26]. Subsequently, the leader $u_*$ serves as the root of the spanning tree $T$ and every node knows its distance in $T$ from $u_*$. Assuming an ID range of size $n^c$, there are at most $t = 2\binom{2}{n} \cdot (n^c)$ possible $n$-node graphs where a subset of $n$ IDs is chosen and assigned to the nodes by selecting one of the possible $n!$ permutations. Let $E$ be some arbitrary enumeration of these possibilities and let $E_i$ refer to the $i$-th item in the order stipulated by $E$. We split the computation into iterations of $t$ rounds with the goal of performing a convergecast. Let $d$ be the maximum distance of a node from the root $u_*$ in $T$, which can be known by all nodes in $O(D)$ additional rounds.

In the first iteration, each leaf at distance $d$ sends exactly 1 bit to its parent in $T$ at round $k$ if its local neighborhood corresponds to $E_k$. Similarly, in iteration $i > 1$, every node at distance $d - i + 1$, sends 1 bit in round $k'$ such that $E_k'$ corresponds to the subgraph consisting of $u'$'s neighborhood as well as the topology information received from its children in the previous iterations. Proceeding in this manner ensures that a node can convey all information about the network topology that it knows about to its parent by sending only a single bit during one of the rounds in this iteration while remaining silent in all others. After $d$ iterations, the entire topological information is received by the root $u_*$, which is again represented by some $E_{k''}$. Subsequently we again use $d$ iterations of $t$ rounds similarly to the mechanism described above to disseminate $E_{k''}$ starting from $u_*$ to all nodes in the network. Finally, each node locally computes its output from $E_{k''}$ according to the solution to problem $P$.

### B. A SIMPLE TIME-ENCODING MECHANISM FOR OBTAINING OPTIMAL COMMUNICATION COMPLEXITY IN THE CONGEST-KT$_1$ MODEL

We describe a folklore technique for obtaining a message-optimal algorithm for any problem $P$ in the CONGEST-KT$_1$ model. We first construct a spanning tree by using the algorithm of [24], which takes time $\tilde{O}(n)$ and requires sending $\tilde{O}(n)$ bits. Then, we elect a leader on this tree, which requires $\tilde{O}(n)$ bits using the algorithm of [26]. Subsequently, the leader $u_*$ serves as the root of the spanning tree $T$ and every node knows its distance in $T$ from $u_*$. Assuming an ID range of size $n^c$, there are at most $t = 2\binom{2}{n} \cdot (n^c)$ possible $n$-node graphs where a subset of $n$ IDs is chosen and assigned to the nodes by selecting one of the possible $n!$ permutations. Let $E$ be some arbitrary enumeration of these possibilities and let $E_i$ refer to the $i$-th item in the order stipulated by $E$. We split the computation into iterations of $t$ rounds with the goal of performing a convergecast. Let $d$ be the maximum distance of a node from the root $u_*$ in $T$, which can be known by all nodes in $O(D)$ additional rounds.

In the first iteration, each leaf at distance $d$ sends exactly 1 bit to its parent in $T$ at round $k$ if its local neighborhood corresponds to $E_k$. Similarly, in iteration $i > 1$, every node at distance $d - i + 1$, sends 1 bit in round $k'$ such that $E_k'$ corresponds to the subgraph consisting of $u'$'s neighborhood as well as the topology information received from its children in the previous iterations. Proceeding in this manner ensures that a node can convey all information about the network topology that it knows about to its parent by sending only a single bit during one of the rounds in this iteration while remaining silent in all others. After $d$ iterations, the entire topological information is received by the root $u_*$, which is again represented by some $E_{k''}$. Subsequently we again use $d$ iterations of $t$ rounds similarly to the mechanism described above to disseminate $E_{k''}$ starting from $u_*$ to all nodes in the network. Finally, each node locally computes its output from $E_{k''}$ according to the solution to problem $P$.

### C. PROOF OF LEMMA 3.2

**Lemma 3.2 (restated).** Consider any $\gamma$-error algorithm $\mathcal{A}$ as stated in Definition 3.1. Then, $\max_{\mathcal{G}} \text{LIC}_G(\mathcal{A}) = \sup_{\mathcal{G}} \text{LIC}_G(\mathcal{A})$.

**Proof.** Let $S$ be the set of all graphs of $n$ nodes where each node has a unique ID and let $s = |S|$. Consider the set $\mathcal{D}$ of all probability distributions $\mathcal{G}$ over graphs in $S$. We will first show that $\mathcal{D}$ is compact by showing that it is bounded and closed. If we represent each element of $a \in \mathcal{D}$ as a vector of length $s$, where entry $a[i]$ is the probability of sampling the $i$-th graph (assuming some fixed ordering) of $S$, it follows that $\mathcal{D}$ corresponds to a bounded subset of $\mathbb{R}^s$.

To see that $\mathcal{D}$ is closed (i.e. contains all limit points), consider the map $f : \mathbb{R}^s \to \mathbb{R}$ where

$$f(a) = \sum_{i=1}^{s} a[i].$$

**LEMMA A.6 (Data Processing Inequality).** If random variables $X$, $Y$, and $Z$ form the Markov chain $X \to Y \to Z$, i.e., the conditional distribution of $Z$ depends only on $Y$ and is conditionally independent of $X$, then

$$I[X : Y] \geq I[X : Z].$$
Since \( f \) is continuous and the set \( \{1\} \) is closed in \( \mathbb{R} \), it follows that \( f^{-1}(\{1\}) \) is closed too. Moreover, the hypercube \([0,1]^s\) is closed and hence
\[
[0,1]^s \cap f^{-1}(\{1\}) = \mathcal{D}
\]
is closed too. Thus we have shown that \( \mathcal{D} \) is compact.

Consider the function \( g : \mathcal{D} \to \mathbb{R} \) defined as \( g(\mathcal{G}) = \text{LIC}_G(\mathcal{A}) = \sum_{i=1}^{n} I[\Pi_i : G \mid X_i] \). Since \( g \) is continuous and we have shown that \( \mathcal{D} \) is compact, it holds that \( g(\mathcal{D}) \) is compact too (see [32]). The lemma follows since supremum and maximum coincide in compact spaces. \( \square \)

**D PROOF OF LEMMA 5.3**

**Lemma 5.3 (restated).** Every \( u \in \mathcal{D} \) is incident to at least \( \frac{3}{4} n^{\frac{1}{2s}} \left( 1 - 2n^{-\frac{1}{2s}} \right) - o(1) \) critical edges in expectation.

As outlined in Section 5, we will focus on traversal sequences that exhibit the largest growth possible. We will use the conservative upper bound that each step of the traversal sequence reaches the maximum number of nodes possible, with the underlying assumption that \( U = D \), which holds with high probability (see Lemma 4.1).

**Property 1.** Let \( T \) be a \( k \)-length \((k \geq 3)\) traversal sequence \( T \). For each step \( i \in [1, k] \) it holds that
\[
|\mathcal{R}_{i+1}(T)| = \begin{cases} 
|\mathcal{R}_i(T)| n^{\frac{1}{2s}} & \text{if } T[i] = R, \\
|\mathcal{R}_i(T)| n^{\frac{1}{2s} + \frac{1}{2s^2}} & \text{if } T[i] = T[i+1] = B.
\end{cases}
\]

The next lemma will be instrumental for weeding out slowly growing traversal sequences. Lemma D.1 confirms the (intuitively obvious) fact that a traversal sequence is wasteful in terms of making progress towards reaching as many nodes as possible, if it contains a subsequence of 3 or more consecutive B-steps.

**Lemma D.1.** Consider any traversal sequence \( T \), for which there exists an index \( i \) and an integer \( \ell \geq 2 \), such that \( T[i] = \cdots = T[i+\ell] = B \); assume that \( i \) is the smallest index for which this holds. Define traversal sequence \( T' \), where
\[
T'[j] = \begin{cases} 
R & \text{if } j \in [i+2, i+\ell]; \\
T[j] & \text{otherwise}.
\end{cases}
\]

Then, it holds that \( \mathcal{R}(T) \subseteq \mathcal{R}(T') \).

**Proof.** By assumption \( \mathcal{R}_{i+1}(T) = \mathcal{R}_{i+1}(T') \), i.e., \( T \) and \( T' \) have reached the same set of nodes before taking the \((i+2)\)-th step. Consider the set of regions
\[
A = \{ P(w) : w \in \mathcal{R}_i(T) \}.
\]
Recalling that \( i \) was chosen to be the smallest index for which the premise of the lemma holds, it follows that either \( i = 1 \) or \( T[i-1] = R \). Moreover, the number of B steps prior to step \( i \) must have been even. To see why this is the case, note that when we traverse the blue edges for the first time, we reach nodes in \( V \) which do not have any red edges, and hence the next step must be B. The same reasoning holds for the \( m \)-th B-step, where \( m \) is odd.
Thus it follows for both $T$ and $T'$ that, after the $(i+1)$-th step, we are located in $U$. In the next step in $T'$, we traverse red edges, which potentially can reach nodes in new regions that are not already in set $A$. In sequence $T$, in contrast, we again take a step along the blue edges which must lead to nodes that are in regions already contained in $A$ (and hence already in $R_{i+1}(T)$), which does not increase the reachable set. We continue to apply this argument to all indices $i+2, \ldots, i+\ell$ and conclude that $R_{i+\ell}(T') \supseteq R_{i+\ell}(T)$.

Since the color sequences of $T$ and $T'$ are the same in the remaining indices $i+\ell+1, \ldots, k$, recalling Property 1 tells us that a similar invariant continues to hold until the end of the sequence, and thus $R(T') \supseteq R(T)$.

A consequence of Lemma D.1 is that we only consider traversal sequences that do not contain subsequences of 3 or more consecutive $B$-steps in the rest of the proof, as our goal is to bound the probability of reaching $v$ assuming that the sequence that attains the largest reachable set.

**Property 2.** For any traversal sequence $T$, it holds that each $B$-step is adjacent to exactly one other $B$-step.

For the remainder of the proof, we will silently assume that all traversal sequences satisfy Properties 1 and 2. As we are trying to show an upper bound on the number of nodes reached by any traversal sequence, this assumption only strengthens our result.

We say that a traversal sequence $T$ is $B$-maximal if we cannot obtain a traversal sequence $T'$ from $T$ by replacing two adjacent $R$-steps with a $BB$-pair without violating Property 2. Lemma D.2 below shows that whenever we encounter a traversal sequences $T$ that is not $B$-maximal, there is another traversal sequence that contains an additional pair of $B$-steps and that reaches a larger set of nodes.

**Lemma D.2.** Consider a traversal sequence $T$ that contains at least one $BB$-pair (i.e. two consecutive occurrences of $B$). Let $i$ be the position of the first $BB$-pair in $T$. Define $T'$ to be the traversal sequence obtained from $T$ by replacing the $B$-character at positions $i$ and $i+1$ with $R$. Then $|R(T)| > |R(T')|$.

**Proof.** Recall that $R_{i-1}(T)$ is the reachable set after steps $T[1] \ldots T[j-1]$. Let $\alpha = |R_{i-1}(T)|$. Since $T$ and $T'$ are identical up to including index $i-1$, we also have $\alpha = |R_{i-1}(T')|$. Then, after following only red edges for two consecutive steps, as required by $T'$, Property 1 implies that

$$|R_{i+2}(T')| = \alpha \left(1 + \frac{1}{n^{1/3}}\right)^2 = \alpha n^{2/3}.$$

On the other hand, if we consider $T$ and take two hops following only blue edges, it again follows from Property 1 that

$$|R_{i+2}(T)| = \alpha n^{2/3} + \frac{1}{d^2},$$

and hence $|R_{i+2}(T)| > |R_{i+2}(T')|$.

To see that this inequality continues to hold for the remaining indices, we use the fact that $T$ and $T'$ perform the same sequence of colors from that point onward in conjunction with Property 1.

Next, we will show that even if we expand the reachability set by following a $B$-maximal traversal sequence, the resulting size still falls short of containing a constant fraction of the nodes in $U$.

**Lemma D.3.** If $T$ is $B$-maximal, then it holds that

$$|R(T)| \leq \frac{n^{1-\frac{1}{2k}}}{2^k}.$$
PROOF. We consider 3 separate cases in our proof depending on the value of \( k \mod 3 \).

First, consider the case where \( k \mod 3 = 0 \). Let \( T \) be a \( B \)-maximal traversal sequence. Property 2 together with the \( B \)-maximality of \( T \) tells us that the remaining sequence is fully determined by whether the first character is \( B \) or \( R \). More specifically, the only two possibilities for \( T \) are either

\[
T = \overbrace{BBR BBR \ldots BBR}^{k/3\text{-times}} \quad \text{or} \quad T = \overbrace{RBB RBB \ldots RBB}^{k/3\text{-times}}.
\]

Thus, we have exactly \( k/3 \) triples of the form \( BBR \) or \( RBB \) in \( T \). In either case, we have \( k/3 \) \( BB \)-pairs and \( R \)-steps. Thus, applying Property 1 for \( R \) and \( BB \) exactly \( k/3 \) times we obtain the size of the reachable set as

\[
|\mathcal{R}(T)| = n^\frac{k}{3}\left(\frac{\log_2 n}{k^2}\right)^{\frac{k}{3}}.
\]  

(22)

Our goal is to show that this is at most

\[
\frac{n^{1-\frac{1}{3k}}}{2^k} \leq n^{1-\frac{k}{k^2}} \leq n^{1-\frac{k}{k^3} - \frac{1}{k^2}},
\]

where the last inequality follows from (5), i.e., \( k \leq \log_2 n \). We point out that the \( \frac{1}{2^k} \) factor will be needed when taking a union bound later on. Comparing the exponents in (22) and (23), we observe that

\[
|\mathcal{R}(T)| \leq n^{1-\frac{1}{3k} - \frac{1}{k^2}}
\]

if

\[
\frac{k}{3}\left(\frac{3}{k+1} + \frac{1}{4k^2}\right) \leq 1 - \frac{1}{k^3} - \frac{1}{k^2}.
\]

(24)

Multiplying by the common denominator, this is equivalent to

\[
\frac{-11k^3 + 13k^2 + 24k + 12}{12k^3(k+1)} \leq 0.
\]

Consider the function \( f(k) = 11k^3 - 13k^2 - 24k - 12 \). We need to show that \( f(k) \geq 0 \) for all \( k \geq 3 \), which implies (24). By factorizing, we obtain \( f(k) = k(k+1)(11k - 24) - 12 \geq 0 \), for all \( k \geq 3 \), as required.

Next, we consider the case \( k \mod 3 = 1 \). Since \( (k-1) \mod 3 = 0 \), we can apply the reasoning from the previous case to observe that there must be exactly \( \ell = \frac{k-1}{3} \) \( BB \)-pairs and \( R \)-steps in the \( (k-1) \)-length subsequence \( T' = T[1,k-1] \). We say that an \( R \)-step \( \rho \) at index \( i \) is free in \( T \), if modifying \( T \) by left-shifting \( T[i], \ldots, T[k] \) by one position and moving \( \rho \) to \( T[k] \) results in a sequence that still satisfies Property 2.

Let \( b_1, \ldots, b_\ell \) be the list of \( BB \)-pairs in their order of occurrence in \( T' \). Property 2 tells us that there is at least one \( R \)-step in between \( b_i \) and \( b_{i+1} \) (\( i \in [1, \ell - 1] \)) and thus there is at most \( \ell - (\ell - 1) = 1 \) free \( R \)-step in \( T' \). It follows that there is no \( RRR \)-triple in \( T' \) and, consequently, when considering the \( k \)-length sequence \( T \), we cannot add another \( BB \)-pair nor a singleton \( B \)-step, due to Property 2. It follows that there are exactly \( \frac{2k-2}{3} \) \( B \)-steps in \( T \), and we must have \( \left(\frac{k-1}{3} + 1\right) R \)-steps in \( T \). To obtain the size of the reachable step of \( T \), we apply Property 1 to each one of the \( k \) steps, which yields

\[
|\mathcal{R}(T)| = n^{\frac{k+1}{3}\left(\frac{\log_2 n}{k^3} + \frac{1}{4k^2}\right)^{\frac{1}{3}}},
\]
Analogously to the first case, we compare \( |\mathcal{R}(T)| \) with (23), and hence the condition that we need to check becomes

\[
\frac{k - 1}{3} \left( \frac{3}{k + 1} + \frac{1}{4k^2} \right) + \frac{1}{k + 1} \leq 1 - \frac{1}{k^3} - \frac{1}{k^2},
\]

and, by multiplying with the common denominator, this holds if \( \frac{k^3 - 12k^2 - 2k + 12}{k^3(k + 1)} \geq 0 \), which is equivalent to \( k(k + 1)(11k - 23) \geq 12 \). This holds for all \( k \geq 3 \).

The final case that we need to analyze is \( k \ mod \ 3 = 2 \). From the first case above, namely where \( k \) divides \( 3 \), we know that \( T \) is exactly one step shorter than a sequence \( T' \) of length \( k + 1 \) that contains exactly \( (k + 1)/3 \) \( BB \)-pairs and \( R \)-steps each, and we have previously argued that only one of the \( R \)-steps is free. Thus, the number of \( BB \)-pairs in \( T \) and \( T' \) is the same, and hence the number of \( R \)-steps in \( T \) is

\[
k - \frac{2(k + 1)}{3} = k + 1 - 1.
\]

Similarly as in the previous cases, we apply Property 1 to each of the steps in \( T \) and compare the resulting size of the reachable set with the bound in (23). We obtain the condition that needs to hold as

\[
n \leq k \left( \frac{3}{k + 1} + \frac{1}{4k^2} \right) + \frac{1}{k + 1} \leq n^{1 - \frac{1}{k^3} - \frac{1}{k^2}}.
\]

Similarly as in the previous two cases, after comparing the exponents, we obtain the condition \( \frac{k^3 + 3}{3} \left( \frac{2}{k + 1} + \frac{1}{4k^2} \right) + \left( \frac{k^3}{3} - 1 \right) \frac{1}{k + 1} \leq 1 - \frac{1}{k^3} - \frac{1}{k^2} \). Multiplying by the common denominator, this holds if \( \frac{k^3 - (14k^2)/11 - (25k)/11 - 12/11}{k^3(k + 1)} \geq 0 \) and, equivalently, \( k(k + 1)(11k - 25) \geq 12 \). This clearly holds for \( k \geq 3 \) and completes the proof. \( \square \)

Armed with Lemma D.3, we can now complete the proof of Lemma 5.3. Consider any traversal sequence \( T \). If \( T \) is not \( B \)-maximal, we iteratively apply Lemma D.2 to obtain a \( B \)-maximal traversal sequence that reaches at least as many nodes as \( T \). Thus, to obtain an upper bound on the number of nodes in the reachable sets of the possible traversal sequences, we can pessimistically assume that each traversal sequence is \( B \)-maximal. Applying Lemma D.3, it follows that

\[
|\mathcal{R}(T)| \leq n^{1 - \frac{1}{k^3}}.
\]

Recall that we have analyzed the reachability sets in the graph \( H \), which does not contain the edge \( (u, v) \). Moreover, \( (u, v) \) is sampled independently from the other edges according to our lower bound graph distribution \( \mathcal{G}_k \) (see Sec. 4). It follows that, for each \( T \), node \( v \) is independent of \( \mathcal{R}(T) \) in \( U \) and hence the probability that \( \mathcal{R}(T) \) contains \( v \) is at most \( \frac{|\mathcal{R}(T)|}{n^2} \). By taking a union bound over the (at most) \( 2^k \) possible traversal sequences and using (25), we obtain

\[
\Pr[\text{not critical}] = \Pr[\exists T: v \in \mathcal{R}(T)] \leq 2^{k + 1} \frac{|\mathcal{R}(T)|}{n} \leq 2n^{1 - \frac{1}{k^3}}.
\]

Let \( Y \) be the random variable denoting the number of critical edges incident to \( u \) and let \( X \) be the number of red edges incident to \( u \). From (26), we get

\[
\mathbf{E}[Y | u \in D, X] > X \left( 1 - 2n^{-\frac{1}{k^3}} \right).
\]
Therefore,

\[
\mathbb{E} [Y \mid u \in D] = \mathbb{E} \left[ \mathbb{E} [Y \mid u \in D, X] \mid u \in D \right] \\
\geq \left( 1 - 2n^{-\frac{1}{k^3}} \right) \mathbb{E} [X \mid u \in D] \\
\geq \left( 1 - 2n^{-\frac{1}{k^3}} \right) \left( \frac{3}{4} n^{\frac{1}{k^3}} - o(1) \right) \quad \text{(by (4))} \\
\geq \frac{3}{4} \left( n^{\frac{1}{k^3}} - 2n^{\frac{1}{k^3}} - \frac{1}{k^3}\right) - o(1).
\]

This completes the proof of Lemma 5.3.