Topological classification of defects in non-Hermitian systems

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We classify topological defects in non-Hermitian systems with point gap, real line gap and imaginary line gap for all the Bernard-LeClair classes in all dimensions. The defect Hamiltonian $H(k, r)$ is described by a non-Hermitian Hamiltonian with spatially modulated adiabatical parameter $r$ surrounding the defect and belongs to any of 38 symmetry classes of general no-Hermitian systems. While the classification of defects in Hermitian systems has been explored in the context of standard ten-fold Altland-Zirnbauer symmetry classes, a complete understanding of the role of the general non-Hermitian symmetries on the topological defects and their associated classification are still lacking. By continuous transformation and homeomorphic mapping, these non-Hermitian defect systems can be mapped to topologically equivalent Hermitian systems with associated symmetries, and we get the topological classification by classifying the corresponding Hermitian Hamiltonians. We discuss some non-trivial classes with point gap according to our classification table, and give explicitly the topological invariants for these classes. By studying some lattice or continuous models, we find the correspondence between zero modes at the topological defect and the topological number in our studied models.

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I. INTRODUCTION

Topological band theory has achieved great success in the past decades$^{1-10}$ and topological classification according to standard ten-fold Altland-Zirnbauer (AZ) symmetry classes, defined by time-reversal symmetry, particle-hole symmetry and their combination, has been well established$^{11-19}$. In the scheme of AZ classification, a topological phase can be characterized by either $\mathbb{Z}$ or $\mathbb{Z}_2$ number. Two well-known examples are Haldane model and condensed matter systems$^{20-23}$. Systematic classifications of topological defects in insulators and superconductors have been carried out for AZ symmetry classes$^{24}$. Recent experimental studies of non-Hermitian properties in optic systems, electrical systems and open quantum systems$^{27-34}$ have stimulated the development of non-Hermitian physics$^{35-38}$. Motivated by these progresses, the scope of topological phase of matter has also been extended to non-Hermitian systems$^{39-64}$. It has been unveiled that non-Hermitian systems exhibit many different properties from the Hermitian systems, e.g., biorthonormal eigenvectors$^{56}$, the existence of exception points$^{65-68}$, unusual bulk-edge correspondence$^{53-56}$ and emergence of non-Hermitian skin effect in non-reciprocal systems$^{55,56,69-73}$. Effects of non-Hermiticity on defect states are also studied for some specific non-Hermitian topological models$^{74-77}$. Except the fundamental interest of understanding these differences, novel topological properties of non-Hermitian systems may bring potential application in topological lasers and high-sensitive sensors. Besides extensive studies on various topological non-Hermitian models$^{46,47,57-62}$, classification of non-Hermitian topological phases has been carried out by different groups$^{52,78-80}$. In a recent work, Gong et. al went an important step for the classification of non-Hermitian phases with non-spatial symmetries$^{52}$ by considering only time-reversal, pseudo particle-hole and sublattice symmetries. After that, the topological classification of non-Hermitian systems in the AZ classes with an additional reflection symmetry was given by us$^{80}$. In general, non-Hermitian systems have more types of fundamental non-spatial symmetries than their Hermitian counterparts, i.e., there are totally 38 different classes according to Bernard-LeClair (BL) symmetry classes$^{81}$, which can be viewed as a natural non-Hermitian generalization of the ten AZ random matrix classes$^{81,82}$. Very recently, the full classifications for non-Hermitian topological band systems are obtained by Sato et.al$^{79}$ and Zhou et.al$^{78}$. So far, the topological classification of non-Hermitian systems focused on the band systems, classification of topological defects in these systems is still lack. Aiming to fill this gap, in this work we make a full classification of topological defects for non-Hermitian systems. Due to the complex nature of spectra, non-Hermitian systems have the point-like and line-like gaps, which corresponds to different complex-spectral-flattening procedure$^{79}$. For both types of complex-energy gaps, we can transform the classification problem of non-Hermitian Hamiltonian to the classification problem of Hermitian Hamiltonian by continuous transformation and homeomorphic mapping. Then we classify the Hermitian Hamiltonian by using the K theory and Clifford algebra$^{12,18,19}$. And we also con-
struct the topological number of topological defects for some BL classes. Then we study some continuous models and lattice models, and find that there is a correspondence between defect’s topological number and gapless modes at the defect in these models.

The paper is organized as follows. In section II, we first give an introductory description of symmetry of non-Hermitian system in the AZ and BL classes, and also a brief introduction of different gap types. In section III, we complete topological classification of defects in non-Hermitian systems with point gap, real line gap and imaginary line gap, respectively, for all the Bernard-LeClair classes. In Section IV, we construct topological numbers of topological defects for some Bernard-LeClair classes. In Section V, we study several example systems. A summary is given in the last section.

II. BRIEF REVIEW OF NON-HERMITIAN SYMMETRIES AND GAP TYPES

Before presenting our approach of topological classification of defects in non-Hermitian systems, we first review briefly symmetries of non-Hermitian systems in the AZ and BL classes and the definitions of point gap, real and line gaps. An introduction to these notations and definitions is necessary for our further classification of defects.

A. Altland-Zirnbauer Class

For Hermitian topological systems, the topological classification according time-reversal symmetry, particle-hole symmetry and chiral symmetry is called the AZ symmetry class, which leads to the standard 10-fold symmetry classification. It is known that AZ classes are not able to fully describe the internal symmetries of non-Hermitian systems, which belong to more general BL classes. Nevertheless, the AZ symmetry classes still attract particular attention in the non-Hermitian classification as it is a natural generalization of the Hermitian classification. For a non-Hermitian Hamiltonian in momentum space, the time-reversal symmetry is defined by

\[ T = U_T K, \quad TH(-k) = H(k)T, \]  

where \( K \) is a complex conjugation operator and \( U_K \) is a unitary matrix. Similarly, the particle-hole symmetry is defined by

\[ A = U_A \mathcal{R}, \quad AH(-k) = -H(k)A, \]  

where \( \mathcal{R} \) is a transport operator and \( U_A \) is a unitary matrix. Given that \( T \) is the time-reversal operator and \( A \) the particle-hole operator, \( \Gamma = TA \) is the chiral symmetry operator. Similar to the Hermitian case, AZ classes are constructed by \( T \), \( A \) and \( \Gamma \) (see Table II).

B. Bernard-LeClair Class

Bernard and LeClair made a full classification for non-Hermitian random matrix. And it is the basic building block of non-Hermitian symmetries. In general, there are four type of symmetries in non-Hermitian case denoted by \( K \), \( Q \), \( C \) and \( P \). And they are defined by

\[ H = kH^*k^{-1}, \quad kk^* = \epsilon_k I, \quad K \text{ sym.} \]  
\[ H = qH^1q^{-1}, \quad q^2 = I, \quad Q \text{ sym.} \]  
\[ H = \epsilon_c cH^T c^{-1}, \quad cc^* = \eta_c I, \quad C \text{ sym.} \]  
\[ H = -pHp^{-1}, \quad p^2 = I, \quad P \text{ sym.} \]

where \( k, q, c \) and \( p \) are unitary matrices and \( \epsilon_k, \epsilon_c, \eta_c = \pm 1 \). The four unitary matrices satisfy:

\[ c = \epsilon_{pc} p_c p^T, \quad k = \epsilon_{pk} p_k p^T, \quad c = \epsilon_{qc} q_c q^T, \quad p = \epsilon_{pq} q_p q^\dagger, \]  

with \( \epsilon_{pc}, \epsilon_{pk}, \epsilon_{qc}, \epsilon_{pq} = \pm 1 \). We can construct 63 symmetry classes by these symmetries and the sign of \( \epsilon_k, \epsilon_c, \eta_c, \epsilon_{pc}, \epsilon_{pk}, \epsilon_{qc} \) and \( \epsilon_{pq} \). The 63 symmetry classes reduce to 38 topological classes up to a redefinition of equivalence (see Table III).

C. Point gap, real line gap and imaginary line gap

In general, an energy gap in the band theory means a forbidden energy region with no occupancy of states. For the non-Hermitian systems, the definition of energy gap is nontrivial as the spectrum becomes complex. According to Kawabata et. al, non-Hermitian systems should have two different types of complex-energy gaps,
i.e., the point-like and line-like gaps. Here we follow the definitions of Kawabata et. al. Consider the complex energy plane, if a system has a point gap, it means that the band spectra can’t cross the zero point (i.e., \( \forall k, \det(H(k)) \neq 0 \)) as schematically shown in Fig.(1a). If a system has a real line gap, it means that the bands energy can’t cross the imaginary axis (i.e., \( \forall j, k, \text{Im}(E_j(k)) \neq 0 \)) as shown in Fig.(1b). If a system has an imaginary line gap, it means that the bands energy can’t cross the imaginary axis (i.e., \( \forall j, k, \text{Im}(E_j(k)) \neq 0 \)) as shown in Fig.(1c). Based on the three different constraints, we can get three different classifications.

| D=0 | d=1 | d=2 |
|-----|-----|-----|
| \( \tilde{H}(k, r) \) | \( \tilde{H}(k, r) \) | |

**FIG. 2:** Schematic diagram of topological defects. A topological defect in \( d \) dimension is surrounded by a \( D \)-dimensional surface \( S^D \). For point defects, \( d - D = 1 \), for line defects, \( d - D = 2 \).

### III. TOPOLOGICAL CLASSIFICATION

A non-Hermitian periodic system with topological defects is described by \( H(k, r) \), where \( k \) is defined in a \( d \)-dimensional Brillouin zone \( T^d \), and \( r \) is defined on a \( D \)-dimensional surface \( S^D \) surrounding the defect (Fig.2). The defect Hamiltonian is a band Hamiltonian slowly modulated by a parameter \( r \), which includes spatial coordinates and/or a temporal parameter and changes slow enough so that the bulk system separated far from the defect core still can be characterized by momentum \( k \).

#### A. Point gap classification

If the system has a point gap, we can map the non-Hermitian Hamiltonian to a Hermitian one by the doubling process\(^{52,79,83}\).

\[
\tilde{H}(k, r) = \begin{bmatrix}
0 & H(k, r) \\
H(k, r)^\dagger & 0
\end{bmatrix}.
\]

The doubled Hamiltonian \( \tilde{H}(k, r) \) fulfills an enforced additional chiral symmetry:

\[
\Sigma \tilde{H}(k, r) = -\tilde{H}(k, r) \Sigma,
\]

where \( \Sigma = \sigma_z \otimes 1 \) and \( \Sigma^2 = 1 \). And the constraint on \( \det(H(k, r)) \neq 0 \) is equivalent to \( \det(\tilde{H}(k, r)) \neq 0 \). Then \( H(k, r) \) is homeomorphic to \( \tilde{H}(k, r) \), and it is equivalent to classify Hermitian Hamiltonian \( \tilde{H}(k, r) \) with chiral symmetry. If \( H(k, r) \) has \( T, A, \Gamma, K, Q, P \) and \( C \) symmetries, \( \tilde{H}(k, r) \) has the corresponding symmetries:

\[
\tilde{T} \tilde{H}(k, r) = \tilde{H}(-k, r) \tilde{T},
\]

\[
\tilde{A} \tilde{H}(k, r) = -\tilde{H}(-k, r) \tilde{A},
\]

\[
\tilde{K} \tilde{H}(k, r) = \tilde{H}(-k, r) \tilde{K},
\]

\[
\tilde{Q} \tilde{H}(k, r) = \tilde{H}(k, r) \tilde{Q},
\]

\[
\tilde{P} \tilde{H}(k, r) = -\tilde{H}(k, r) \tilde{P},
\]

\[
\tilde{C} \tilde{H}(k, r) = \epsilon_c \tilde{H}(k, r) \tilde{C},
\]

with \( \tilde{T} = \sigma_0 \otimes U_T K, \tilde{A} = \sigma_z \otimes U_A K, \tilde{K} = \sigma_0 \otimes k K, \tilde{Q} = \sigma_x \otimes q, \tilde{P} = \sigma_0 \otimes p \), and \( \tilde{C} = \sigma_x \otimes c K \).

Generally, we can represent the defect Hamiltonian as:

\[
\tilde{H}(k, r) = \tilde{\gamma}_0 + k_1 \tilde{\gamma}_1^k + ... + k_d \tilde{\gamma}_d^k + r \tilde{\gamma}_j^k + ... + r_D \tilde{\gamma}_D^k,
\]

where \( \tilde{\gamma}_0, \tilde{\gamma}_i^k(i = 1, ..., d) \) and \( \tilde{\gamma}_j^k(j = 1, ..., D) \) anticommute with each other and their squares equal to the identity operator. Using the commutation relations of symmetry operators and Hamiltonian, we can construct the Clifford algebra’s extension for each symmetry class. Then we can get the classifying space of the mass term \( (\gamma_0) \) by the correspondence between the Clifford algebra’s extension and space of the mass term (Table 1).

| Clifford algebra’s extension | Space of the mass term |
|-----------------------------|------------------------|
| \( C_i \rightarrow C_{i+2} \) | \( R_{0,p-q} \rightarrow R_{0,p-q+8} \) |
| \( C_{i,p,q} \rightarrow C_{i+1,p,q+1} \) | \( R_{0,p-q} \rightarrow R_{0,p-q+10} \) |

For convenience, here we briefly introduce Clifford algebra. There are two types of Clifford algebra, i.e., the complex Clifford algebra and real Clifford algebra. Complex Clifford algebra is defined in the complex domain, and its generators \( \{ \gamma_1, ..., \gamma_n \} \) satisfy that \( \{ \gamma_i, \gamma_j \} = 2 \delta_{ij} \).
Complex Clifford algebra can be represented as $\mathbb{C}l_n$. On the other hand, real Clifford algebra is defined in the real domain, and its generators $\{\gamma_1, ..., \gamma_n, \gamma_1^+, ..., \gamma_n^+\}$ anticommute with each other. While $\gamma_i$ $(i = 1, ..., n)$ are squared to $-1$, $\gamma_i^+$ $(i = 1, ..., n/2)$ are squared to $1$. The real Clifford algebra can be represented as $\mathbb{C}l_{n1,n2}$.

Once we know the space of the mass term, we can get the topological classification by calculating the zero-order homotopy group of the space of the mass term. For the case of point gap, we give the classification according to the symmetry classes labeled by both standard AZ classes and BL classes, despite that the former ones are the subclasses of the latter ones. For convenience, when we discuss the AZ classes, we also label the corresponding BL classes simultaneously. Firstly, we consider the complex classes:

**Class A (Non):** The generators of this class are $\{\tilde{\gamma}_0, \tilde{\gamma}_1^k, ..., \tilde{\gamma}_d^k, \tilde{\gamma}_1^r, ..., \tilde{\gamma}_d^r, \Sigma, \bar{T}, J\}$, where $J$ is the imaginary unit. The Clifford algebra’s extension of this class is $\{\tilde{\gamma}_0, \tilde{\gamma}_1^k, ..., \tilde{\gamma}_d^k, J\tilde{\gamma}_1^r, ..., J\tilde{\gamma}_d^r, J\Sigma, J\bar{T}, JT\}$ $\rightarrow$ $\{J\tilde{\gamma}_0, J\tilde{\gamma}_1^k, ..., J\tilde{\gamma}_d^k, J\tilde{\gamma}_1^r, ..., J\tilde{\gamma}_d^r, J\Sigma, J\bar{T}, JT\} = CL_{d+1,d+2} \rightarrow CL_{d+2,d+2}$. The space of the mass term is $R_{1-d}$.

**Class C (C4):** The generators of this class are $\{\tilde{\gamma}_0, \tilde{\gamma}_1^k, ..., \tilde{\gamma}_d^k, \tilde{\gamma}_1^r, ..., \tilde{\gamma}_d^r, \Sigma, \bar{T}, J\}$. The Clifford algebra’s extension of this class is $\{\tilde{\gamma}_0, \tilde{\gamma}_1^k, ..., \tilde{\gamma}_d^k, \tilde{\gamma}_1^r, ..., \tilde{\gamma}_d^r, \Sigma, \bar{T}, J\}$ $\rightarrow$ $\{\tilde{\gamma}_0, \tilde{\gamma}_1^k, ..., \tilde{\gamma}_d^k, \tilde{\gamma}_1^r, ..., \tilde{\gamma}_d^r, \Sigma, \bar{T}, J\} = CL_{d+1,d+1} \rightarrow CL_{d+2,d+2}$. The space of the mass term is $R_{1-d}$.

**Class BDI (QC5):** The generators of this class are $\{\tilde{\gamma}_0, \tilde{\gamma}_1^k, ..., \tilde{\gamma}_d^k, \tilde{\gamma}_1^r, ..., \tilde{\gamma}_d^r, \Sigma, J\bar{T}, J\}$, where $J$ is the imaginary unit. The Clifford algebra’s extension of this class is $\{\tilde{\gamma}_0, \tilde{\gamma}_1^k, ..., \tilde{\gamma}_d^k, J\tilde{\gamma}_1^r, ..., J\tilde{\gamma}_d^r, J\Sigma, J\bar{T}, JT\}$ $\rightarrow$ $\{J\tilde{\gamma}_0, J\tilde{\gamma}_1^k, ..., J\tilde{\gamma}_d^k, J\tilde{\gamma}_1^r, ..., J\tilde{\gamma}_d^r, J\Sigma, J\bar{T}, JT\} = CL_{d+1,d+2} \rightarrow CL_{d+2,d+2}$. The space of the mass term is $R_{1-d}$.

**Class AI (K1):** The generators of this class are $\{\gamma_0, \gamma_1^k, ..., \gamma_d^k, \gamma_1^r, ..., \gamma_d^r, \Sigma, J\}$, where $J$ is the imaginary unit. The Clifford algebra’s extension of this class is $\{\gamma_0, \gamma_1^k, ..., \gamma_d^k, \gamma_1^r, ..., \gamma_d^r, \Sigma, J\}$ $\rightarrow$ $\{\gamma_0, \gamma_1^k, ..., \gamma_d^k, \gamma_1^r, ..., \gamma_d^r, \Sigma, J\} = CL_{d+1,d+1} \rightarrow CL_{d+2,d+2}$. The space of the mass term is $R_{1-d}$.

**Class AIII (Q):** The generators of this class are $\{\gamma_0, \gamma_1^k, ..., \gamma_d^k, \gamma_1^r, ..., \gamma_d^r, \Sigma, \bar{T}, J\}$. The Clifford algebra’s extension of this class is $\{\gamma_0, \gamma_1^k, ..., \gamma_d^k, \gamma_1^r, ..., \gamma_d^r, \Sigma, \bar{T}, J\}$ $\rightarrow$ $\{\gamma_0, \gamma_1^k, ..., \gamma_d^k, \gamma_1^r, ..., \gamma_d^r, \Sigma, \bar{T}, J\} = CL_{d+1,d+2} \rightarrow CL_{d+2,d+3}$. The space of the mass term is $R_{1-d}$.

**Class All (Q):** The generators of this class are $\{\gamma_0, \gamma_1^k, ..., \gamma_d^k, \gamma_1^r, ..., \gamma_d^r, \Sigma, \bar{T}, J\}$. The Clifford algebra’s extension of this class is $\{\gamma_0, \gamma_1^k, ..., \gamma_d^k, \gamma_1^r, ..., \gamma_d^r, \Sigma, \bar{T}, J\}$ $\rightarrow$ $\{\gamma_0, \gamma_1^k, ..., \gamma_d^k, \gamma_1^r, ..., \gamma_d^r, \Sigma, \bar{T}, J\} = CL_{d+1,d+1} \rightarrow CL_{d+2,d+2}$. The space of the mass term is $R_{1-d}$.
B. Real line gap classification

If a non-Hermitian Hamiltonian has a real line gap, it has been demonstrated that the Hamiltonian can continuously transform to a Hermitian Hamiltonian $H$ while keeping its symmetry and line gap\(^{79}\). Then the non-Hermitian AZ class classification is same as the Hermitian AZ class classification. To classify 38 BL classes, the $K$, $Q$, $C$ and $P$ symmetries reduce to:

\[
\begin{align*}
H &= kH^*k^{-1}, \quad kk^* = \epsilon_k \mathbb{I} \quad & K \text{ sym.} \\
H &= qHq^{-1}, \quad q^2 = \mathbb{I} \quad & Q \text{ sym.} \\
H &= \epsilon_c cH^*c^{-1}, \quad cc^* = \eta_c \mathbb{I} \quad & C \text{ sym.} \\
H &= -pHp^{-1}, \quad p^2 = \mathbb{I} \quad & P \text{ sym.}
\end{align*}
\]

Here $H$ is a Hermitian Hamiltonian. The classification problem reduces to a Hermitian Hamiltonian classification problem. And we can classify each class by Clifford algebra (See Appendix A). The results are listed in Table IV.

C. Imaginary line gap classification

If a non-Hermitian Hamiltonian has an imaginary line gap, the Hamiltonian can continuously transform to an anti-Hermitian Hamiltonian $iH^{79}$, where $H$ is a Hermitian Hamiltonian. To classify 38 BL classes, the $K$, $Q$, $C$ and $P$ symmetries reduce to:

\[
\begin{align*}
H &= -kH^*k^{-1}, \quad kk^* = \epsilon_k \mathbb{I} \quad & K \text{ sym.} \quad (22) \\
H &= -qHq^{-1}, \quad q^2 = \mathbb{I} \quad & Q \text{ sym.} \quad (23) \\
H &= \epsilon_c cH^*c^{-1}, \quad cc^* = \eta_c \mathbb{I} \quad & C \text{ sym.} \quad (24) \\
H &= -pHp^{-1}, \quad p^2 = \mathbb{I} \quad & P \text{ sym.} \quad (25)
\end{align*}
\]

The classification problem reduces to a Hermitian Hamiltonian classification problem. And we can classify each class by Clifford algebra (See Appendix B). The results are listed in Table V.

IV. CONSTRUCTION OF EXPLICIT TOPOLOGICAL INVARIANTS FOR POINT DEFECTS

Although the periodic tables of topological classification are given, they do not provide explicit forms of topological invariants. In this section, we discuss the construction of explicit topological invariants for point gap systems by considering some specific examples. To begin with, we note that the open boundary condition in one dimension is the simplest example of topological defects. For this case, our topological classification of topological defects ($d = 1, D = 0$) is consistent with classification of one-dimensional gapped system ($d = 1$). Next we discuss several two-dimensional and three-dimensional examples.

Class A (Non), Class AI (K1) and Class AII (K2). Given the Hamiltonian $H(k, r)$, which describes a point defect in $d$ dimensions and is a function of $d$ momentum variables and $D = d - 1$ position variables. For any invertible Hamiltonian, it has a polar decomposition $H = UP$, where $U$ is a unitary matrix and $P$ is a positive definite Hermitian matrix. Such a decomposition is unique, and the Hamiltonian $H(k, r)$ can be continuously deformed to $U(k, r)$ with the same symmetry constraint\(^{78,79}\). Since $H(k, r)$ is topologically equivalent to $U(k, r)$, the $\mathbb{Z}$ topological number is defined as:

\[
n = \frac{(d-1)!}{(2d-1)!(2\pi i)^d} \int_{T^d \times S^{d-1}} \text{Tr}[(UdU^\dagger)^{2d-1}], \quad (26)
\]

which is the winding number associated with the homotopy group $\pi_{2d-1}[U(n)] = \mathbb{Z}^{26}$.

Class P, Class PK1 and Class PK2. Suppose that the Hamiltonian is $H(k, r)$ and the system has $P$ symmetry described by $p = \sigma_z$. The polar decomposition is $H = UP$. Consider the symmetry constraint $\sigma_z U = -U \sigma_z^{78}$, and $U$ can be represented as

\[
U(k, r) = \begin{bmatrix} 0 & U_1(k, r) \\ U_2(k, r) & 0 \end{bmatrix}.
\]

Then we can define the $\mathbb{Z} \oplus \mathbb{Z}$ topological number as

\[
n_i = \frac{(d-1)!}{(2d-1)!(2\pi i)^d} \int_{T^d \times S^{d-1}} \text{Tr}[(U_1dU_1^\dagger)^{2d-1}], \quad (28)
\]

where $i = 1$ and 2.

Class PK3. Suppose that the Hamiltonian of a point defect is $H(k, r)$, and the system has both $P$ and $K$ symmetries described by $p = \sigma_z$ and $k = \sigma_x$, respectively. The polar decomposition is $H = UP$. Consider the symmetry constraint $\sigma_x U(k, r) = -U(k, r) \sigma_x$ and $\sigma_z U^*(-k, r) = U(k, r) \sigma_x$, and $U$ can be represented as

\[
U(k, r) = \begin{bmatrix} 0 & U_1(k, r) \\ U_1^*(-k, r) & 0 \end{bmatrix}.
\]

Then we can define the $\mathbb{Z}$ topological number as

\[
n = \frac{(d-1)!}{(2d-1)!(2\pi i)^d} \int_{T^d \times S^{d-1}} \text{Tr}[(U_1dU_1^\dagger)^{2d-1}]. \quad (30)
\]
TABLE III: Periodic table of point gap classification of topological defects in non-Hermitian systems. The rows correspond to the different BL symmetry classes, while the columns depend on $\delta = d - D$.

| BL | Gen. Rel. | C0 | C1 | 2 | 3 | 4 | 5 | 6 | 7 |
|----|-----------|----|----|---|---|---|---|---|---|
| Non | $\eta_k = 1$ | $R_1$ | $Z_2$ | $Z$ | 0 | 0 | 0 | $Z$ | 0 | 0 |
| P | $\eta_k = -1$ | $R_5$ | 0 | $Z \oplus Z$ | 0 | 0 | $Z \oplus Z$ | 0 | 0 | $Z \oplus Z$ |
| Q | $\eta_k = 1$ | $R_7$ | 0 | 0 | 0 | $Z_2$ | $Z_2$ | $Z_2$ | $Z_2$ | $Z_2$ |
| K1 | $\eta_k = -1$ | $R_3$ | 0 | 0 | 0 | $Z_2$ | $Z_2$ | $Z$ | 0 | 0 |
| C1 | $\epsilon_c = 1$, $\eta_c = 1$ | $R_3$ | 0 | 0 | 0 | $Z_2$ | $Z_2$ | $Z_2$ | $Z_2$ | $Z_2$ |
| C2 | $\epsilon_c = 1$, $\eta_c = -1$ | $R_3$ | 0 | 0 | 0 | $Z_2$ | $Z_2$ | $Z_2$ | $Z_2$ | $Z_2$ |
| C3 | $\epsilon_c = 1$, $\eta_c = 1$ | $R_3$ | 0 | 0 | 0 | $Z_2$ | $Z_2$ | $Z_2$ | $Z_2$ | $Z_2$ |
| C4 | $\epsilon_c = 1$, $\eta_c = -1$ | $R_3$ | 0 | 0 | 0 | $Z_2$ | $Z_2$ | $Z_2$ | $Z_2$ | $Z_2$ |
| PQC1 | $\epsilon_{pq} = 1$ | $C_5$ | 0 | 0 | 0 | $Z$ | 0 | 0 | $Z$ | 0 |
| PQC2 | $\epsilon_{pq} = -1$ | $C_5$ | $Z \oplus Z$ | 0 | 0 | 0 | $Z \oplus Z$ | 0 | 0 | $Z \oplus Z$ |
| PQC3 | $\epsilon_{pq} = 1$ | $R_5$ | 0 | $Z \oplus Z$ | 0 | 0 | $Z \oplus Z$ | 0 | 0 | $Z \oplus Z$ |
| PQC4 | $\epsilon_{pq} = -1$ | $C_5$ | 0 | 0 | 0 | $Z$ | 0 | 0 | $Z$ | 0 |
| PQC5 | $\epsilon_{pq} = 1$ | $C_5$ | 0 | 0 | 0 | $Z \oplus Z$ | 0 | 0 | $Z \oplus Z$ | 0 | 0 | $Z \oplus Z$ |
| PQC6 | $\epsilon_{pq} = -1$ | $C_5$ | $Z \oplus Z$ | 0 | 0 | 0 | $Z \oplus Z$ | 0 | 0 | $Z \oplus Z$ |
| PQC7 | $\epsilon_{pq} = 1$ | $C_5$ | 0 | 0 | 0 | $Z$ | 0 | 0 | $Z$ | 0 |
| PQC8 | $\epsilon_{pq} = -1$ | $C_5$ | 0 | 0 | 0 | $Z \oplus Z$ | 0 | 0 | $Z \oplus Z$ | 0 | 0 | $Z \oplus Z$ |
| PQC9 | $\epsilon_{pq} = 1$ | $C_5$ | 0 | 0 | 0 | $Z$ | 0 | 0 | $Z$ | 0 |
| PQC10 | $\epsilon_{pq} = -1$ | $C_5$ | 0 | 0 | 0 | $Z \oplus Z$ | 0 | 0 | $Z \oplus Z$ | 0 | 0 | $Z \oplus Z$ |
| PQC11 | $\epsilon_{pq} = 1$ | $C_5$ | 0 | 0 | 0 | $Z$ | 0 | 0 | $Z$ | 0 |
| PQC12 | $\epsilon_{pq} = -1$ | $C_5$ | 0 | 0 | 0 | $Z \oplus Z$ | 0 | 0 | $Z \oplus Z$ | 0 | 0 | $Z \oplus Z$ |
### TABLE IV: Periodic table of real line gap classification of topological defects in non-Hermitian systems. The rows correspond to the different BL symmetry classes, while the columns depend on $\delta = d - D$.

| BL | Gen. Rel. | C1 | C2 | C3 | C4 | C5 | C6 | C7 | C8 |
|----|-----------|----|----|----|----|----|----|----|----|
| Non | $C_0$ | Z 0 Z 0 Z 0 Z 0 Z 0 | Z 0 Z 0 Z 0 Z 0 Z 0 Z 0 |
| P  | $C_0$ | Z 0 Z 0 Z 0 Z 0 Z 0 Z 0 |
| Q  | $C_0^1$ | Z Z 0 0 0 Z Z 0 0 0 Z Z 0 0 0 Z Z 0 0 0|
| R1 | $\eta_k = 1$ | $R_0$ | Z 0 0 0 Z 0 Z 0 Z 0 |
| R2 | $\eta_k = 1$ | $R_0$ | Z 0 0 0 Z 0 Z 0 Z 0 |
| C1 | $\epsilon_c = 1$, $\eta_c = 1$ | $R_2$ | Z 0 Z 0 Z 0 Z 0 Z 0 |
| C2 | $\epsilon_c = 1$, $\eta_c = 1$ | $R_2$ | Z 0 Z 0 Z 0 Z 0 Z 0 |
| C3 | $\epsilon_c = 1$, $\eta_c = 1$ | $R_4$ | Z 0 Z 0 Z 0 Z 0 Z 0 |
| C4 | $\epsilon_c = 1$, $\eta_c = 1$ | $R_6$ | Z 0 Z 0 Z 0 Z 0 Z 0 |
| PQ1 | $\epsilon_{pq} = 1$ | $C_0^2$ | Z Z 0 Z 0 Z 0 Z 0 |
| PQ2 | $\epsilon_{pq} = 1$ | $C_0$ | Z Z 0 Z 0 Z 0 Z 0 |
| PK1 | $\eta_k = 1$, $\epsilon_{pk} = 1$ | $R_2$ | Z 0 Z 0 Z 0 Z 0 Z 0 |
| PK2 | $\eta_k = 1$, $\epsilon_{pk} = 1$ | $R_2$ | Z 0 Z 0 Z 0 Z 0 Z 0 |
| PK3 | $\eta_k = 1$, $\epsilon_{pk} = 1$ | $R_7$ | Z 0 Z 0 Z 0 Z 0 Z 0 |
| PC1 | $\epsilon_c = 1$, $\eta_c = 1$, $\epsilon_{pc} = 1$ | $R_1$ | Z 0 Z 0 Z 0 Z 0 Z 0 |
| PC2 | $\epsilon_c = 1$, $\eta_c = 1$, $\epsilon_{pc} = 1$ | $R_7$ | Z 0 Z 0 Z 0 Z 0 Z 0 |
| PC3 | $\epsilon_c = 1$, $\eta_c = 1$, $\epsilon_{pc} = 1$ | $R_5$ | Z 0 Z 0 Z 0 Z 0 Z 0 |
| PC4 | $\epsilon_c = 1$, $\eta_c = 1$, $\epsilon_{pc} = 1$ | $R_5$ | Z 0 Z 0 Z 0 Z 0 Z 0 |
| PC5 | $\epsilon_c = 1$, $\eta_c = 1$, $\epsilon_{pc} = 1$ | $C_0$ | Z Z 0 Z 0 Z 0 Z 0 |
| PC6 | $\epsilon_c = 1$, $\eta_c = 1$, $\epsilon_{pc} = 1$ | $R_2^7$ | Z Z 0 Z 0 Z 0 Z 0 |
| PC7 | $\epsilon_c = 1$, $\eta_c = 1$, $\epsilon_{pc} = 1$ | $R_2^7$ | Z Z 0 Z 0 Z 0 Z 0 |
| PC8 | $\epsilon_c = 1$, $\eta_c = 1$, $\epsilon_{pc} = 1$ | $C_1$ | Z Z 0 Z 0 Z 0 Z 0 |
| PC9 | $\epsilon_c = 1$, $\eta_c = 1$, $\epsilon_{pc} = 1$ | $R_6$ | Z 0 Z 0 Z 0 Z 0 Z 0 |
| PC10 | $\epsilon_c = 1$, $\eta_c = 1$, $\epsilon_{pc} = 1$ | $R_6$ | Z 0 Z 0 Z 0 Z 0 Z 0 |
| PC11 | $\epsilon_c = 1$, $\eta_c = 1$, $\epsilon_{pc} = 1$ | $R_4$ | Z 0 Z 0 Z 0 Z 0 Z 0 |
| PC12 | $\epsilon_c = 1$, $\eta_c = 1$, $\epsilon_{pc} = 1$ | $R_4$ | Z 0 Z 0 Z 0 Z 0 Z 0 |
TABLE V: Periodic table of imaginary line gap classification of topological defects in non-Hermitian systems. The rows correspond to the different BL symmetry classes, while the columns depend on $\delta = d - D$.

| B.L. | Gen. Rel. | C1 | δ = 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
|------|-----------|----|-------|---|---|---|---|---|---|---|
| Non  |           | C0 | Z     | 0 | Z | 0 | Z | 0 | Z | 0 |
| P    |           | C1 | Z     | 0 | Z | 0 | Z | 0 | Z | 0 |
| Q    |           | C1 | Z     | 0 | Z | 0 | Z | 0 | Z | 0 |
| K1   | $\eta_k = 1$ | R2 | Z     | Z | Z | 0 | Z | 0 | Z | 0 |
| K2   | $\eta_k = -1$ | R3 | Z     | Z | Z | 0 | Z | 0 | Z | 0 |
| C1   | $\epsilon_c = 1$, $\eta_c = 1$ | R3 | Z     | Z | Z | 0 | Z | 0 | Z | 0 |
| C2   | $\epsilon_c = 1$, $\eta_c = 1$ | R3 | Z     | Z | Z | 0 | Z | 0 | Z | 0 |
| C3   | $\epsilon_c = -1$, $\eta_c = 1$ | R3 | Z     | Z | Z | 0 | Z | 0 | Z | 0 |
| C4   | $\epsilon_c = -1$, $\eta_c = 1$ | R3 | Z     | Z | Z | 0 | Z | 0 | Z | 0 |
| PQ1  | $\epsilon_{pq} = 1$ | C2 | Z     | Z | Z | 0 | Z | 0 | Z | 0 |
| PQ2  | $\epsilon_{pq} = 1$ | C2 | Z     | Z | Z | 0 | Z | 0 | Z | 0 |
| PK1  | $\eta_k = 1$, $\epsilon_{pk} = 1$ | R1 | Z     | Z | Z | 0 | Z | 0 | Z | 0 |
| PK2  | $\eta_k = 1$, $\epsilon_{pk} = 1$ | R1 | Z     | Z | Z | 0 | Z | 0 | Z | 0 |
| PC1  | $\epsilon_c = 1$, $\eta_c = 1$, $\epsilon_{pc} = 1$ | R1 | Z     | Z | Z | 0 | Z | 0 | Z | 0 |
| PC2  | $\epsilon_c = -1$, $\eta_c = 1$, $\epsilon_{pc} = 1$ | R1 | Z     | Z | Z | 0 | Z | 0 | Z | 0 |
| PC3  | $\epsilon_c = 1$, $\eta_c = 1$, $\epsilon_{pc} = 1$ | R1 | Z     | Z | Z | 0 | Z | 0 | Z | 0 |
| PC4  | $\epsilon_c = -1$, $\eta_c = 1$, $\epsilon_{pc} = 1$ | R1 | Z     | Z | Z | 0 | Z | 0 | Z | 0 |
| QC1  | $\epsilon_c = 1$, $\eta_c = 1$, $\epsilon_{qc} = 1$ | R1 | Z     | Z | Z | 0 | Z | 0 | Z | 0 |
| QC2  | $\epsilon_c = -1$, $\eta_c = 1$, $\epsilon_{qc} = 1$ | R1 | Z     | Z | Z | 0 | Z | 0 | Z | 0 |
| QC3  | $\epsilon_c = 1$, $\eta_c = -1$, $\epsilon_{qc} = 1$ | R1 | Z     | Z | Z | 0 | Z | 0 | Z | 0 |
| QC4  | $\epsilon_c = -1$, $\eta_c = -1$, $\epsilon_{qc} = 1$ | R1 | Z     | Z | Z | 0 | Z | 0 | Z | 0 |
| QC5  | $\epsilon_c = 1$, $\eta_c = 1$, $\epsilon_{qc} = 1$ | R1 | Z     | Z | Z | 0 | Z | 0 | Z | 0 |
| QC6  | $\epsilon_c = -1$, $\eta_c = 1$, $\epsilon_{qc} = 1$ | R1 | Z     | Z | Z | 0 | Z | 0 | Z | 0 |
| QC7  | $\epsilon_c = 1$, $\eta_c = 1$, $\epsilon_{qc} = 1$ | R1 | Z     | Z | Z | 0 | Z | 0 | Z | 0 |
| QC8  | $\epsilon_c = -1$, $\eta_c = 1$, $\epsilon_{qc} = 1$ | R1 | Z     | Z | Z | 0 | Z | 0 | Z | 0 |
| PQC1 | $\epsilon_c = 1$, $\eta_c = 1$, $\epsilon_{pq} = 1$, $\epsilon_{pc} = 1$ | R1 | Z     | Z | Z | 0 | Z | 0 | Z | 0 |
| PQC2 | $\epsilon_c = -1$, $\eta_c = 1$, $\epsilon_{pq} = 1$, $\epsilon_{pc} = 1$ | R1 | Z     | Z | Z | 0 | Z | 0 | Z | 0 |
| PQC3 | $\epsilon_c = 1$, $\eta_c = 1$, $\epsilon_{pq} = 1$, $\epsilon_{pc} = 1$ | R1 | Z     | Z | Z | 0 | Z | 0 | Z | 0 |
| PQC4 | $\epsilon_c = -1$, $\eta_c = 1$, $\epsilon_{pq} = 1$, $\epsilon_{pc} = 1$ | R1 | Z     | Z | Z | 0 | Z | 0 | Z | 0 |
| PQC5 | $\epsilon_c = 1$, $\eta_c = 1$, $\epsilon_{pq} = 1$, $\epsilon_{pc} = 1$ | R1 | Z     | Z | Z | 0 | Z | 0 | Z | 0 |
| PQC6 | $\epsilon_c = -1$, $\eta_c = 1$, $\epsilon_{pq} = 1$, $\epsilon_{pc} = 1$ | R1 | Z     | Z | Z | 0 | Z | 0 | Z | 0 |
| PQC7 | $\epsilon_c = 1$, $\eta_c = 1$, $\epsilon_{pq} = 1$, $\epsilon_{pc} = 1$ | R1 | Z     | Z | Z | 0 | Z | 0 | Z | 0 |
| PQC8 | $\epsilon_c = -1$, $\eta_c = 1$, $\epsilon_{pq} = 1$, $\epsilon_{pc} = 1$ | R1 | Z     | Z | Z | 0 | Z | 0 | Z | 0 |
| PQC9 | $\epsilon_c = 1$, $\eta_c = 1$, $\epsilon_{pq} = 1$, $\epsilon_{pc} = 1$ | R1 | Z     | Z | Z | 0 | Z | 0 | Z | 0 |
| PQC10 | $\epsilon_c = -1$, $\eta_c = 1$, $\epsilon_{pq} = 1$, $\epsilon_{pc} = 1$ | R1 | Z     | Z | Z | 0 | Z | 0 | Z | 0 |
| PQC11 | $\epsilon_c = 1$, $\eta_c = 1$, $\epsilon_{pq} = 1$, $\epsilon_{pc} = 1$ | R1 | Z     | Z | Z | 0 | Z | 0 | Z | 0 |
| PQC12 | $\epsilon_c = -1$, $\eta_c = 1$, $\epsilon_{pq} = 1$, $\epsilon_{pc} = 1$ | R1 | Z     | Z | Z | 0 | Z | 0 | Z | 0 |
Then we can define the Z topological number by Eq.(30).

Class PC1. The Hamiltonian of a point defect is \( H(k, r) \) with the P and Q symmetries described by \( p = q = \sigma_z \). Consider the symmetry constraints \( \sigma_z H(k, r) = -H(k, r) \sigma_z \) and \( \sigma_z H^\dagger(k, r) = H(k, r) \sigma_z \) as well as the polar decomposition \( H = UP \), and thus \( U \) can be written as

\[
H(k, r) = \begin{bmatrix}
0 & U_1(k, r) \\
-U_1^\dagger(-k, r) & 0
\end{bmatrix}.
\]  

Then we can define the Z topological number by Eq.(30).

Class PC3. The Hamiltonian of a point defect is \( H(k, r) \) with the P symmetry described by \( p = \sigma_z \) and \( C \) symmetry described by \( c = \sigma_0 \) as well as the polar decomposition \( H = UP \). Consider the symmetry constraints \( \sigma_z U(k, r) = -U(k, r) \sigma_z \) and \( U^\dagger(-k, r) = U(k, r) \), and thus \( U \) has the following form:

\[
U(k, r) = \begin{bmatrix}
0 & U_1(k, r) \\
U_1^\dagger(-k, r) & 0
\end{bmatrix}.
\]  

Then we can define the Z topological number by Eq.(30).

Class PQ1 and Class QC5. If we chose \( q = 1 \), the Hamiltonian reduces to a Hermitian Hamiltonian. And the non-Hermitian classes of PQ1 and QC5 reduce to Hermitian classes of BDI and D. So we can define the Z and \( Z_2 \) topological numbers for classes of PC1 and QC5 the same as the Hermitian classes of BDI and D, which is consistent with our classification.

V. EXAMPLES OF POINT-DEFECT MODELS

To gain an intuitive understanding of nontrivial topological defects in non-Hermitian systems, we construct some examples with a point gap. Through the study of some concrete models, we discuss the correspondence between topological number and zero modes at the defect.

A. Point defects in 1D PC1 class

Consider the one-dimensional (1D) lattice models described by:

\[
H_1 = \sum_n [t_1(a_n^\dagger b_n + b_n^\dagger a_n) + t_2 e^{i\theta_1}(b_n^\dagger a_{n+1} + a_n^\dagger b_{n+1})]
\]  

\[
H_2 = \sum_n [t_3(a_n^\dagger b_n + b_n^\dagger a_n) + t_4 e^{i\theta_2}(b_n^\dagger a_{n+1} + a_n^\dagger b_{n+1})].
\]  

These two Hamiltonians have the same forms but with different parameters. If we consider the periodic boundary condition, both of them can be represented in the momentum space after a Fourier transformation. The corresponding Hamiltonians in the \( k \) space are

\[
H_1(k) = \begin{bmatrix}
0 & t_1 + t_2 e^{i\theta_1 + ik} \\
t_1 + t_2 e^{i\theta_1 - ik} & 0
\end{bmatrix},
\]  

\[
H_2(k) = \begin{bmatrix}
0 & t_3 + t_4 e^{i\theta_2 + ik} \\
t_3 + t_4 e^{i\theta_2 - ik} & 0
\end{bmatrix}.
\]

Both the Hamiltonians belong to the PC1 Class as they have \( P \) and \( C \) symmetries with \( p = \sigma_z \) and \( c = \sigma_0 \). The topological properties of the systems are characterized by the \( Z \) topological numbers, i.e., \( n_1 = \frac{1}{2\pi} \int d\theta_1 \int d\theta_2 \ln(t_1 + t_2 e^{i\theta_1 - ik}) \) and \( n_2 = \frac{1}{2\pi} \int d\theta_1 \int d\theta_2 \ln(t_3 + t_4 e^{i\theta_2 - ik}) \) for \( H_1 \) and \( H_2 \), respectively.

FIG. 3: Parameters are set as: \( t = t_2 = t_3 = 1, t_4 = 0.5, \theta_1 = \pi/3 \) and \( \theta_2 = \pi/4 \). The number of cells for \( H_1 \) is 100, and the number of cells for \( H_2 \) is also 100. (a) Schematic diagram, the left part represents \( H_1 \) and the right part represents \( H_2 \). (b) The topological number \( W \) versus \( t_1 \). (c) Energy spectrum as a function of \( t_1 \). (d) The spatial distributions for zero modes of the system with \( t_1 = 0.5 \). There is a zero mode at the place of each connected point (defect). Now we connect \( H_1 \) and \( H_2 \) together end to end to form a new Hamiltonian \( H \). As schematically displayed in Fig.(3a), \( H_1 \) and \( H_2 \) are coupled together with a coupling constant \( t \). Each connected point can be viewed as a topological defect. The topological invariant for the defect is characterized by \( W = n_2 - n_1 \). In Fig.(3b), we display \( W \) versus \( t_1 \) by fixing \( t = t_2 = t_3 = 1, t_4 = 0.5, \)
where \( \hat{k} \) represents the momentum operator (\( \hat{k} = -i\partial_x \) in the coordinate representation), \( \alpha \) is a real constant and \( m(x) \) is a function of \( x \), which changes sign as \( x \) crosses the zero point. Here we take \( |\alpha| \ll |m(x)| \) with \( x \in S^0 \), where \( S^0 \) is the 0D loop around the defect. When \( \alpha = 0 \), this model reduces to the Jackiw-Rebbi model \[^{20,47}\]. The Hamiltonian has \( P \) and \( C \) symmetries with \( \epsilon_c = -1, \epsilon_p = 1 \), \( c = \sigma_z \) and \( p = \sigma_y \), and belongs to the \( PC1 \) class. The topological defect is characterized by a \( Z \) topological number. When \( \alpha \) continuously changes from 0 to nonzero, \( det(H) \neq 0 \) on \( S^0 \) because \( |\alpha| \ll |m(x)| \) on \( S^0 \). Then the topological number is the same as that of Jackiw-Rebbi model, i.e.,

\[
W = \frac{1}{2}[\text{sgn}(m(x_1)) - \text{sgn}(m(x_2))],
\]

which is a special form of Jackiw-Rebbi Hamiltonian and has a topologically protected zero mode at the defect. Denote the zero mode state of \( H' \) as \( \Phi_0'(x) = \langle x |\Phi_0'(x) \rangle \), then \( H \) has a topologically protected zero mode \( \Phi_0(x) = e^{i\alpha\partial_x/b}\Phi_0'(x) \) at the defect. Our results show that there is a correspondence between zero modes of defect and the topological number of defect.

For the non-Hermitian continuous model described by

\[
H(k, x) = (-i\partial_x + i\beta)\sigma_z + m(x)\sigma_x,
\]

we can make a similarity transformation \( H' = e^{-\beta x}He^{\beta x} \) and get

\[
H'(\hat{k}, x) = -i\partial_x\sigma_z + m(x)\sigma_x,
\]

which is a standard Jackiw-Rebbi Hamiltonian. Similarly, we conclude that \( H \) has a topologically protected zero mode \( \Phi_0(x) = e^{i\beta k}\Phi_0'(x) \), where \( \Phi_0'(x) \) is the zero mode wavefunction of \( H' \).

### B. Point defects in 2D \( PC1 \) class

Consider a two-dimensional (2D) Hamiltonian described by

\[
H(\hat{k}, \hat{r}) = v\hat{k}_x\gamma_1 + v\hat{k}_y\gamma_2 + (\Delta_1 + i\alpha_1)\gamma_3 + (\Delta_2 + i\alpha_2)\gamma_4 = \begin{pmatrix} \sigma \cdot k & \Delta + i\alpha \\ \Delta^* - \sigma \cdot k & \Delta^* + i\alpha \end{pmatrix},
\]

where \((\gamma_1, \gamma_2, \gamma_3, \gamma_4) = (\tau_x\sigma_x, \tau_y\sigma_y, \tau_x, -\tau_y)\), \( k = (\hat{k}, \hat{b}) \) and \( \sigma = (\sigma_x, \sigma_y) \). \( \sigma_{x,y,z} \) and \( \tau_{x,y,z} \) are Pauli matrices in the spin and particle-hole spaces. Here \( \Delta_1 \) and \( \Delta_2 \) are functions of \( \hat{r} = (\hat{x}, \hat{y}) \), \( \alpha_1 \) and \( \alpha_2 \) are real constants, \( \Delta_1 + i\Delta_2 = \Delta = |\Delta|e^{i\theta} \) and \( \alpha_1 + i\alpha_2 = \alpha \). We take \( |\alpha_1| \ll |\Delta_1|, |\alpha_2| \ll |\Delta_2| \) on \( S^1 \), where \( S^1 \) is the loop around the defect. When \( \alpha_1 = \alpha_2 = 0 \), this model reduces to the Jackiw-Rossi model \[^{21,26}\]. The Hamiltonian has \( P \) and \( C \) symmetries with \( \epsilon_c = -1, \epsilon_p = 1 \), \( c = \tau_y\sigma_y \) and \( p = \tau_x\sigma_z \), and thus belongs to the \( PC1 \) class. The topological defect is characterized by a \( Z \) topological number. When \( \alpha_1 \) and \( \alpha_2 \) change continuous change from 0 to nonzero, the \( det(U(\hat{k}, \hat{r})) \neq 0 \) on \( S^1 \) in Eq.(32) because \( |\alpha_1| \ll |\Delta_1| \) and \( |\alpha_2| \ll |\Delta_2| \) on \( S^1 \). Then the topological number is same as that of Jackiw-Rossi model with

\[
n = \frac{1}{2\pi} \int_{S^1} d\phi.
\]

When \( \Delta_1 = b\hat{x} \) and \( \Delta_2 = b\hat{y} \) \((b \text{ is a real constant})\), in momentum representation, the Hamiltonian takes the following form:

\[
H(\hat{k}, i\partial_x) = v\gamma_1\hat{k}_x + v\gamma_2\hat{k}_y + \gamma_3(b\partial_x + i\alpha_1) + \gamma_4(b\partial_y + i\alpha_2).
\]

Define \( H' = e^{\alpha_1(\partial_x + b\partial_y)/b}\Phi_0(\hat{r}) \) at the defect.

which is a special form of Jackiw-Rossi Hamiltonian and has a topologically protected zero mode at the defect. Denote the zero mode state of \( H' \) as \( \Phi_0'(r) = \langle r |\Phi_0'(r) \rangle \), then \( H \) has a topologically protected zero mode \( \Phi_0(r) = e^{i\alpha_1(\partial_x + b\partial_y)/b}\Phi_0'(r) \) at the defect. When we take \( \Delta_1 = b(\hat{x}^2 - \hat{y}^2) \) and \( \Delta_2 = 2b\hat{x}\hat{y} \), the corresponding topological number is given by \( n = 2 \). We can prove that there are two topologically protected zero modes at the defect by a similar method. In the momentum space, we can write the Hamiltonian explicitly as

\[
H(\hat{k}, i\partial_x) = v\gamma_1\hat{k}_x + v\gamma_2\hat{k}_y + \gamma_3[b(-\partial_k^2 + \partial_y^2) + i\alpha_1] + \gamma_4(-2b\partial_k\partial_y + i\alpha_2).
\]

Define \( H' = e^{\beta(\partial_k + \partial_y)/2}\Phi_0(\hat{r}) \) at the defect. When \( \beta_1 \) and \( \beta_2 \) fulfill that: \( -b\beta_1^2 + b\beta_2^2 + \alpha_1 = 0 \) and \( -2b\beta_1\beta_2 + \alpha_2 = 0 \), then \( H' \) is given by

\[
H'(\hat{k}, \hat{r}) = v\gamma_1\hat{k}_x + v\gamma_2\hat{k}_y + \gamma_3(b(\hat{x}^2 - \hat{y}^2) + \gamma_42b\hat{x}\hat{y}.
\]
The Hermitian Hamiltonian $H'$ has two zero modes, then $H$ also has two zero modes at the defects. The study can be directly extended to situations with $n > 2$. There is a correspondence between the zero modes of defects and the topological number in the 2D $PC_1$ class.

For the non-Hermitian continuous model described by

$$H(k, r) = \gamma_1(-iv\partial_x + i\beta_1) + \gamma_2(-iv\partial_y + i\beta_2) + b\Delta_1\gamma_3 + b\Delta_2\gamma_4, \quad (50)$$

where $\Delta_1$ and $\Delta_2$ are functions of $r$, $\Delta = \Delta_1 + i\Delta_2$ has non-trivial winding on $S^1$, and $S^1$ is the loop around the defect. We can make a similarity transformation $H' = e^{-\frac{\Delta_1}{\Delta} + \frac{\Delta_2}{\Delta} \gamma_4} H e^{\frac{\Delta_1}{\Delta} + \frac{\Delta_2}{\Delta} \gamma_4}$ and get

$$H'(\hat{k}, \hat{r}) = v^k_1\gamma_1 + v^k_2\gamma_2 + b\Delta_1\gamma_3 + b\Delta_2\gamma_4\gamma_4, \quad (51)$$

which is a standard Jackiw-Rossi Hamiltonian. Similarly, we conclude that $H$ has a topologically protected zero mode $\Phi_0(r) = e^{\frac{\Delta_1}{\Delta} + \frac{\Delta_2}{\Delta} \gamma_4} \Phi'_0(r)$, where $\Phi'_0(r)$ is the zero mode wavefunction of $H'$.

C. Point defects in 2D C3 class

Consider the Hamiltonian:

$$H(\hat{k}, \hat{r}) = v^k_1\gamma_1 + v^k_2\gamma_2 + (\Delta_1 + i\alpha_1)\gamma_3 + (\Delta_2 + i\alpha_2)\gamma_4 + h\sigma_z - \mu\tau_z$$

$$= \left[ \begin{array}{cc} \sigma \cdot k + \sigma_z h - \mu & \Delta + i\alpha \\ \Delta^* + i\alpha^* & -\sigma \cdot k + \sigma_z h + \mu \end{array} \right], \quad (52)$$

where the first four terms are the same as Eq.(44), $h$ is a magnetic field and $\mu$ is a chemical potential. When $\alpha_1 = \alpha_2 = 0$, this model is Fu-Kane model$^{84-86}$. The Hamiltonian has $C$ symmetries with $\epsilon_1 = -1$, $\eta_1 = 1$ and $c = \tau_0\sigma_y$ and belongs to the $C3$ class. The $h$ and $\mu$ terms couple the zero modes that we discussed in the above section. The topological properties of the system are characterized by a $Z_2$ topological number. Consider $\alpha_1$ and $\alpha_2$ continuously changing from 0 to nonzero. In this process, $\det(H(k, r)) \neq 0$ on $S^1$ (loop round the defect) because $|\alpha_1| \ll |\Delta_1|$ and $|\alpha_2| \ll |\Delta_2|$ on $S^1$. Then the topological number is the same as the Hermitian case$^{84,86}$ given by

$$n = \frac{1}{2\pi} \int_{S^1} d\phi \mod 2. \quad (53)$$

When $\Delta_1 = b\dot{x}$ and $\Delta_2 = b\dot{y}$ ($b$ is a real constant), the topological invariant is given by $n = 1$. Next, we demonstrate the existence of a zero mode at the defect. In momentum representation, the Hamiltonian is written as

$$H(k, i\partial_k) = v^k_1k_x + v^k_2k_y + \gamma_3(bi\partial_{kx} + i\alpha_1) + \gamma_4(bi\partial_{ky} + i\alpha_2) + h\sigma_z - \mu\tau_z. \quad (54)$$

Define $H' = e^{\frac{\alpha_1}{\Delta_1} + \frac{\alpha_2}{\Delta_2} \gamma_4} H e^{-\frac{\alpha_1}{\Delta_1} + \frac{\alpha_2}{\Delta_2} \gamma_4}$, then $H'$ is given by

$$H'(\hat{k}, \hat{r}) = v\gamma_1\hat{k}_x + v\gamma_2\hat{k}_y + \gamma_3b\hat{k}_y + \gamma_4b\hat{k}_x + h\sigma_z - \mu\tau_z, \quad (55)$$

which is a special form of Fu-Kane Hamiltonian and has a topological protected zero mode at the defect. Denote the zero mode state of the Fu-Kane Hamiltonian $H'$ as $\Phi_1'(r) = (r|\Phi'_1)$, then $H$ has a topologically protected zero mode $\Phi_1(r) = e^{\frac{\alpha_1}{\Delta_1} + \frac{\alpha_2}{\Delta_2} \gamma_4}\Phi'_1(r)$ at the defect. When we choose $\Delta_1 = b(\dot{x}^2 - \dot{y}^2)$ and $\Delta_2 = 2b\dot{x}\dot{y}$, the corresponding topological number is $n = 0$. And we can demonstrate that there is no topologically protected zero mode at the defect by a similar method. There is a correspondence between the zero mode of defect and the topological number in the 2D $C3$ class.

For the non-Hermitian continuous model described by

$$H(k, r) = v\gamma_1\hat{k}_x + v\gamma_2\hat{k}_y + \gamma_4(bi\partial_{kx} + i\alpha_1) + \gamma_5(bi\partial_{ky} + i\alpha_2) + h\sigma_z - \mu\tau_z, \quad (56)$$

where $\Delta_1$ and $\Delta_2$ are functions of $r$, $\Delta = \Delta_1 + i\Delta_2$ has odd winding number on $S^1$, and $S^1$ is the loop around the defect. We can prove that there is topologically protected zero mode by a similar method.

D. Point defects in 2D PQC1 and QC5 class

Consider the Hamiltonian Eq.(44) with $\alpha_1 = \alpha_2 = 0$, i.e., the Jackiw-Rossi model. This model has one more symmetry $Q$ than Eq.(44) with $q = 1$. According to the BL classification, the Jackiw-Rossi model belongs to the PQC1 class, and the Hermitian model has a correspondence between zero modes and $Z$ topological number. Similarly, consider the Hamiltonian Eq.(52) with $\alpha_1 = \alpha_2 = 0$, i.e., the Fu-Kane model. This model has one more symmetry $Q$ than Eq.(52) with $q = 1$. According to the BL classification, the Fu-Kane model belongs to the QC5 class, and the Hermitian model has a correspondence between zero modes and $Z$ topological number.

VI. SUMMARY

In summary, we have studied topological defects in non-Hermitian systems with point gap, real line gap and imaginary line gap and made a topological classification for all the 38 BL classes in all dimensions. The periodic classification tables of point defects are summarized in table II for the AZ classes and in table III for the BL classes with point gap, respectively, and periodical classification tables of real and imaginary line defects are summarized in table IV and V, respectively. By considering some concrete examples of point defects, we constructed explicitly the topological invariants. Through the study of some
concrete models, we calculated explicitly the topological invariants and discussed the correspondence between topological invariants and zero modes at the defect.

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VII. APPENDIX

A. Real line gap classification

According to the discussion in the main text, the classification is equivalent to the classification of a Hermitian Hamiltonian with corresponding symmetries:

\[ H = kH^* k^{-1}, \quad k k^* = \varepsilon_k \mathbb{I} \quad K \text{ sym.} \]  
\[ H = qHq^{-1}, \quad q^2 = \mathbb{I} \quad Q \text{ sym.} \]  
\[ H = \varepsilon_c H_c^* \varepsilon_c^{-1}, \quad c c^* = \eta_c \mathbb{I} \quad C \text{ sym.} \]  
\[ H = -pHp^{-1}, \quad p^2 = \mathbb{I} \quad P \text{ sym.} \]  

We can use Clifford algebra to represent the Hamiltonian:

\[ H(k, r) = \gamma_0 + k_1 \gamma_1 + \ldots + k_d \gamma_d + r_1 \gamma_1 + \ldots + r_d \gamma_d. \]  

Define \( K = k \mathbb{K}, \quad Q = q, \quad C = c \mathbb{K} \) and \( P = p, \) where \( \mathbb{K} \) is a complex conjugate operator. Then we can get the space of mass term by constructing the Clifford algebra's extension. The correspondence between Clifford algebra's extension and space of mass term was listed in Table I and Table VI. And we get the topological classification by calculating the homotopy group of the mass term. The following are the topological classification for some BL classes.

**Class Non:** The generators are \( \{\gamma_0, \gamma_1, \gamma_2, \ldots, \gamma_d, \gamma_1^*, \gamma_2^*, \ldots, \gamma_D^*\} \).

\[ \{\gamma_0, \gamma_1^k, \gamma_2^k, \ldots, \gamma_d^k, \gamma_1^r, \gamma_2^r, \ldots, \gamma_D^r\} = Cl_{d+D} \to Cl_{d+D+1}. \]

The space of mass term is \( Cl_d = C - (d - D) = C - \delta. \) The topological classification is determined by \( \pi_0(C - \delta) = 0 (Z) \) for even (odd) \( \delta. \)

**Class P:** The generators are \( \{\gamma_0, \gamma_1^k, \gamma_2^k, \ldots, \gamma_d^k, \gamma_1^r, \gamma_2^r, \ldots, \gamma_D^r, P\} \).

Clifford algebra's extension is \( \{\gamma_0, \gamma_1^k, \gamma_2^k, \ldots, \gamma_d^k, \gamma_1^r, \gamma_2^r, \ldots, \gamma_D^r, P\} \to \{\gamma_0, \gamma_1^k, \gamma_2^k, \ldots, \gamma_d^k, \gamma_1^r, \gamma_2^r, \ldots, \gamma_D^r, P\} = Cl_{d+D} \to Cl_{d+D+2}. \)

The space of mass term is \( C_{d+D+1} = C_{d+D+1} = C_{d+D+1} + C_{d+D+1} \). The topological classification is determined by \( \pi_0(C_{d+D+1}) = 0 (Z) \) for even (odd) \( \delta. \)

**Class Q:** The generators are \( \{\gamma_0, \gamma_1^k, \gamma_2^k, \ldots, \gamma_d^k, \gamma_1^r, \gamma_2^r, \ldots, \gamma_D^r, Q\} \).

Clifford algebra's extension is \( \{\gamma_0, \gamma_1^k, \gamma_2^k, \ldots, \gamma_d^k, \gamma_1^r, \gamma_2^r, \ldots, \gamma_D^r\} \to \{\gamma_0, \gamma_1^k, \gamma_2^k, \ldots, \gamma_d^k, \gamma_1^r, \gamma_2^r, \ldots, \gamma_D^r, Q\} = Cl_{d+D} \otimes Cl_1 \to Cl_{d+D+1} \otimes Cl_1 \).

The space of mass term is \( C_{d+D} \times C_{d+D} = C_{d+D} \times C_{d+D} \). The topological classification is determined by \( \pi_0(C_{d+D+1}) = Z \oplus Z \oplus (0 \oplus 0) \) for even (odd) \( \delta. \)

**Class C1:** The generators are \( \{\gamma_0, \gamma_1^k, \gamma_2^k, \ldots, \gamma_d^k, \gamma_1^r, \gamma_2^r, \ldots, \gamma_D^r, C, J\} \).

Clifford algebra's extension is \( \{\gamma_0, \gamma_1^k, \gamma_2^k, \ldots, \gamma_d^k, \gamma_1^r, \gamma_2^r, \ldots, \gamma_D^r, C, J\} \to \{\gamma_0, \gamma_1^k, \gamma_2^k, \ldots, \gamma_d^k, \gamma_1^r, \gamma_2^r, \ldots, \gamma_D^r, C, J\} = Cl_{d+D} \to Cl_{d+D+2} \) for \( C_{d+D+2} \) of mass term is \( R_{d+D} = R_{d+D} \).

**Class PQ1:** The generators are \( \{\gamma_0, \gamma_1^k, \gamma_2^k, \ldots, \gamma_d^k, \gamma_1^r, \gamma_2^r, \ldots, \gamma_D^r, P, Q\} \).

Clifford algebra's extension is \( \{\gamma_0, \gamma_1^k, \gamma_2^k, \ldots, \gamma_d^k, \gamma_1^r, \gamma_2^r, \ldots, \gamma_D^r, P\} \to \{\gamma_0, \gamma_1^k, \gamma_2^k, \ldots, \gamma_d^k, \gamma_1^r, \gamma_2^r, \ldots, \gamma_D^r, P\} \oplus \{Q\} = Cl_{d+D+1} \otimes Cl_1 \to Cl_{d+D+2} \otimes Cl_1 \). The space of mass term is \( C_{d+D+1} \times C_{d+D+1} = C_{d+D+1} \times C_{d+D+1} \). The topological classification is determined by \( \pi_0(C_{d+D+1} \times C_{d+D+1}) = Z \oplus Z \oplus (0 \oplus 0) \) for even (odd) \( \delta. \)

**Class PK1:** The generators are \( \{\gamma_0, \gamma_1^k, \gamma_2^k, \ldots, \gamma_d^k, \gamma_1^r, \gamma_2^r, \ldots, \gamma_D^r, P, K, J\} \).

Clifford algebra's extension is \( \{\gamma_0, \gamma_1^k, \gamma_2^k, \ldots, \gamma_d^k, \gamma_1^r, \gamma_2^r, \ldots, \gamma_D^r, P\} \to \{\gamma_0, \gamma_1^k, \gamma_2^k, \ldots, \gamma_d^k, \gamma_1^r, \gamma_2^r, \ldots, \gamma_D^r, P, K, J\} = Cl_{d+D+2} \to Cl_{d+D+2} \) for mass term is \( R_{d+D+1} = R_{d+D+1} \).

**Class PC1:** The generators are \( \{\gamma_0, \gamma_1^k, \gamma_2^k, \ldots, \gamma_d^k, \gamma_1^r, \gamma_2^r, \ldots, \gamma_D^r, P, C, J\} \).

Clifford algebra's extension is \( \{\gamma_0, \gamma_1^k, \gamma_2^k, \ldots, \gamma_d^k, \gamma_1^r, \gamma_2^r, \ldots, \gamma_D^r, P\} \to \{\gamma_0, \gamma_1^k, \gamma_2^k, \ldots, \gamma_d^k, \gamma_1^r, \gamma_2^r, \ldots, \gamma_D^r, P, C, J\} = Cl_{d+D+2} \to Cl_{d+D+2} \) for mass term is \( R_{d+D+1} = R_{d+D+1} \).

**Class QC1:** The generators are \( \{\gamma_0, \gamma_1^k, \gamma_2^k, \ldots, \gamma_d^k, \gamma_1^r, \gamma_2^r, \ldots, \gamma_D^r, Q, C, J\} \).

Clifford algebra's extension is \( \{\gamma_0, \gamma_1^k, \gamma_2^k, \ldots, \gamma_d^k, \gamma_1^r, \gamma_2^r, \ldots, \gamma_D^r, Q\} \to \{\gamma_0, \gamma_1^k, \gamma_2^k, \ldots, \gamma_d^k, \gamma_1^r, \gamma_2^r, \ldots, \gamma_D^r, Q, C, J\} \to \{\gamma_0, \gamma_1^k, \gamma_2^k, \ldots, \gamma_d^k, \gamma_1^r, \gamma_2^r, \ldots, \gamma_D^r, Q, C, J\} \to \{Q\} = Cl_{d+D+2} \otimes Cl_1 \to Cl_{d+D+2} \otimes Cl_1 \). The space of mass term is \( R_{d+D} = R_{d+D} \times R_{d+D} \times R_{d+D} \times R_{d+D} \times R_{d+D} \).
We can use Clifford algebra to represent the Hamiltonian:

\[ \{ J_{\gamma_0}, \gamma_1^k, \gamma_2^k, \gamma_3^k, J_{\gamma_1}, J_{\gamma_2}, J_{\gamma_3}, J_{\gamma_D}, JP, C, JC \} \otimes \{ Q \} = Cl_{D+1,d+2} \times Cl_{0,1} \rightarrow Cl_{D+2,d+2} \times Cl_{0,1}. \]

The space of mass term is \( R_{D-d+1} \times R_{D-d+1} = R_{1-d} \times R_{1-d}. \)

**B. Imaginary line gap classification**

According to the discussion in the main text, the classification is equivalent to the classification of a Hermitian Hamiltonian with corresponding symmetries:

\[ H(-k,r) = -kH(k,r)^{p-1}, k^s = c_k I \text{ Sym. (62)} \]
\[ H(k,r) = -qH(k,r)q^{p-1}, q^2 = \mathbb{I} \text{ Sym. (63)} \]
\[ H(-k,r) = \epsilon_c cH(k,r)\epsilon c^{-1}, \epsilon c^* = \eta_c I \text{ Sym. (64)} \]
\[ H(k,r) = -pH(k,r)p^{1}, p^2 = \mathbb{I} \text{ Sym. (65)} \]

We can use Clifford algebra to represent the Hamiltonian:

\[ H(k,r) = \gamma_0 + k_1 \gamma_1 + \ldots + k_d \gamma_d + r_1 \gamma_1 + \ldots + r_D \gamma_D. \]

Define \( K = kK, Q = q, C = cK \) and \( P = p \), where \( K \) is the complex conjugate operator. Then we can get the space of mass term by constructing the Clifford algebra's extension. And we get the topological classification by calculating the homotopy group of the mass term. The \( C \) and \( P \) give the same symmetry constraint for the topological equivalent Hamiltonian in real-line-gap systems and imaginary-line-gap systems. Then the two systems have the same topological classification for BL classes \( \text{Non, P, C1 - 4} \) and \( \text{PC1 - 4} \). The following are the topological classification for some BL classes.

**Class Q:** The generators are \( \{ \gamma_0, \gamma_1^k, \gamma_2^k, \ldots, \gamma_D, Q \} \). Clifford algebra's extension is \( \{ \gamma_0, \gamma_1^k, \gamma_2^k, \ldots, \gamma_D, Q \} \rightarrow Cl_{D+1} \rightarrow Cl_{D+2,d+2}. \) The space of mass term is \( C_{D+2,d+2} = C_{D+1,d+1} = 1. \) The topological classification is determined by \( \eta_0(C_{D+1}) = 0 (Z) \) for even (odd) \( \delta. \)

**Class K1:** The generators are \( \{ \gamma_0, \gamma_1^k, \gamma_2^k, \ldots, \gamma_D, K, J \} \). Clifford algebra's extension is \( \{ \gamma_0, \gamma_1^k, \gamma_2^k, \ldots, \gamma_D, K, J \} \rightarrow Cl_{D+1,d+2} \rightarrow Cl_{D+2,d+2} \times Cl_{0,1}. \) The space of mass term is \( R_{D-d+1} \times R_{D-d+1} = R_{1-d} \times R_{1-d}. \)

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