CLASSIFICATION OF COMPACT ANCIENT SOLUTIONS
TO THE RICCI FLOW ON SURFACES

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Abstract. We consider an ancient solution $g(\cdot,t)$ of the Ricci flow on a compact surface that exists for $t \in (-\infty,T)$ and becomes spherical at time $t = T$. We prove that the metric $g(\cdot,t)$ is either a family of contracting spheres, which is a type I ancient solution, or a Rosenau solution, which is a type II ancient solution.

1. Introduction

We consider an ancient solution of the Ricci flow
\[
\frac{\partial g_{ij}}{\partial t} = -2 R_{ij}
\] (1.1)
on a compact two-dimensional surface that exists for time $t \in (-\infty,T)$. In two dimensions we have $R_{ij} = \frac{1}{2} R g_{ij}$, where $R$ is the scalar curvature of the surface. Moreover, on an ancient non-flat solution we have $R > 0$. Such a solution becomes singular at some finite time $T$, namely its scalar curvature is blowing up at $t = T$. It is well known ([1], [5]) that the surface also extincts at $T$ and it becomes spherical, which means that after a normalization, the normalized flow converges to a spherical metric.

We can parametrize the Ricci flow by the limiting sphere at time $T$, that is, we can write
\[ g(\cdot,t) = u(\cdot,t) \, ds^2_p. \]
The spherical metric can be written as
\[ ds^2_p = d\psi^2 + \cos^2 \psi \, d\theta^2. \] (1.2)
where $\psi, \theta$ denote the global coordinates on the sphere. An easy computation shows that ([1]) is equivalent to the following evolution equation for the

* : Partially supported by NSF grant 0604657.
conformal factor $u(\cdot, t)$, namely
\[ u_t = \Delta_{S^2} \log u - 2, \quad \text{on } S^2 \times (-\infty, T) \quad (1.3) \]
where $\Delta_{S^2}$ denotes the Laplacian on the sphere. Let us recall, for future references, that the only nonzero Christoffel symbols for the spherical metric (1.2) are
\[ \Gamma^2_{12} = \Gamma^2_{21} = -\tan \psi, \quad \Gamma^1_{22} = \frac{\sin 2\psi}{2}, \quad \Gamma^2_{22} = -\tan \psi. \]
It follows that for any function $f$ on the sphere we have
\[ \Delta_{S^2} f = f_{\psi\psi} - \tan \psi f_\psi + \sec^2 \psi (f_{\theta\theta} + \tan \psi f_\theta) \]
which, in the case of a radially symmetric function $f = f(\psi)$, becomes only
\[ \Delta_{S^2} f = f_{\psi\psi} - \tan \psi f_\psi. \]

We next introduce Mercator’s projection, where
\[ \cosh x = \sec \psi \quad \text{and} \quad \sinh x = \tan \psi \quad (1.4) \]
to project the sphere $S^2$ onto the cylinder, where $(x, \theta)$ denote the cylindrical coordinates in $\mathbb{R}^2$. We can also express
\[ g(\cdot, t) = U(\cdot, t) ds^2, \quad \text{where } ds^2 = dx^2 + d\theta^2 \]
where $U(x, \theta, t)$ is a conformal factor on $\mathbb{R}^2$. A simple computation shows that $U$ satisfies the logarithmic fast-diffusion equation
\[ U_t = \Delta_c \log U, \quad \text{on } \mathbb{R} \times [0, 2\pi] \times (-\infty, T) \quad (1.5) \]
where $\Delta_c$ is the cylindrical Laplacian, that is
\[ \Delta_c f = f_{xx} + f_{\theta\theta} \]
for a function $f$ defined on $\mathbb{R} \times [0, 2\pi]$. It follows that $U$ is given in terms of $u$ by
\[ U(x, \theta, t) = u(\psi, \theta, t) \cos^2 \psi, \quad \cos \psi = (\cosh x)^{-1}. \quad (1.6) \]

We will assume, throughout this paper, that $g = u ds^2_\psi$ is an ancient solution to the Ricci flow (1.3) on the sphere which becomes extinct at time $T = 0$. It is natural to consider the pressure function $v = u^{-1}$ which evolves by
\[ v_t = v^2 (\Delta_{S^2} \log v + 2), \quad \text{on } S^2 \times (-\infty, 0) \quad (1.7) \]
or, after expanding the laplacian of log \( v \),

\[
v_t = v \Delta_{S^2} v - |\nabla_{S^2} v|^2 + 2v^2, \quad \text{on } S^2 \times (-\infty, 0).
\]  

(1.8)

**Definition 1.1.** We will say that an ancient solution to the Ricci flow (1.1) on a compact surface \( M \) is of type I, if it satisfies

\[
\limsup_{t \to -\infty} \left( |t| \max_M R(\cdot, t) \right) < \infty.
\]

A solution which is not of type I, will be called of type II.

Explicit examples of ancient solutions to the Ricci flow in two dimensions are:

i. **The contracting spheres**
   They are described on \( S^2 \) by a pressure \( v_S \) that is given by
   \[
   v_S(\psi, t) = \frac{1}{2(-t)}
   \]  
   (1.9)
   and they are examples of ancient type I shrinking Ricci solitons.

ii. **The Rosenau solutions**
   They were first discovered by P. Rosenau ([12]) and are described on \( S^2 \) by a pressure \( v_R \) that has the form
   \[
   v_R(\psi, t) = a(t) - b(t) \sin^2 \psi
   \]  
   (1.10)
   with \( a(t) = -\mu \coth(2\mu t) \), \( b(t) = -\mu \tanh(2\mu t) \), for some \( \mu > 0 \). These solutions are particularly interesting because they are not solitons. We can visualize them as two cigars "glued" together to form a compact solution to the Ricci flow. They are type II ancient solutions.

Our goal in this paper is to prove the following classification result:

**Theorem 1.2.** Let \( v \) be an ancient compact solution to the Ricci flow (1.8). Then, the solution \( v \) is either one of the contracting spheres or one of the Rosenau solutions.

**Remark 1.3.** The classification of two-dimensional, complete, non-compact ancient solutions of the Ricci flow was recently given in [4] (see also in [7], [13]). The result in Theorem 1.2 together with the results in [4] and [13] provide a complete classification of all ancient two-dimensional complete solutions to the Ricci flow.
The outline of the paper is as follows:

i. In section 2 we will show a’ priori derivative estimates on any ancient solution \( v \) of (1.8), which hold uniformly in time, up to \( t = -\infty \). These estimates will play a crucial role throughout the rest of the paper.

ii. In section 3 we will introduce a suitable Lyapunov functional for our flow and we will use it to show that the solution \( v(\cdot, t) \) of (1.8) converges, as \( t \to -\infty \), in the \( C^{1,\alpha} \) norm, to a steady state \( \tilde{v} \).

iii. Section 4 will be devoted to the classification of all backward limits \( \tilde{\tilde{v}} \).

We will show that there is a parametrization of the flow by a sphere in which \( \tilde{v} = C \cos^2 \psi \), for some \( C \geq 0 \) (\( \psi, \theta \) are the global coordinates on \( S^2 \)).

iv. In section 5 we will show that if \( C = 0 \), then the solution \( v \) is one of the contracting spheres.

v. In section 6 we will show that all ancient compact solutions of the Ricci flow in two dimensions are radially symmetric.

vi. Finally, in section 7 we will show that any radially symmetric solution \( v \) of (1.8) for which \( C > 0 \) must be one of the Rosenau solutions.

### 2. A’ priori estimates

We will assume, throughout this section, that \( v \) is an ancient solution of the Ricci flow (1.8) on \( S^2 \times (-\infty, 0) \) which becomes extinct at \( T = 0 \). We fix \( t_0 < 0 \). We will establish a’priori derivative estimates on \( v \) which hold uniformly on \( S^2 \times (-\infty, t_0] \). We will denote by \( C \) various constants which may vary from line to line but they are always independent of time \( t \).

Since our solution is ancient, the scalar curvature \( R = v_t/v \) is strictly positive. This in particular implies that \( v_t > 0 \). Hence, we have the bound

\[
v(\cdot, t) \leq C, \quad \text{on} \ (-\infty, 0).
\] (2.1)

However, the backward limit \( \tilde{v} = \lim_{t \to -\infty} v(\cdot, t) \), which exists because of the inequality \( v_t > 0 \), may vanish at some points on \( S^2 \). This actually happens on our model, the Rosenau solution. As a result, the equation (1.8) fails to be uniformly parabolic near those points, as \( t \to -\infty \), and the standard parabolic and elliptic derivative estimates fail as well. Nevertheless, it is essential for our classification result, to establish a priori derivative estimates which hold uniformly in time, as \( t \to -\infty \).
We recall that on our ancient solution, the Harnack estimate for the scalar curvature shown in [5] takes the form

\[ R_t \geq \frac{|\nabla R|^2}{R} \]

which in particular implies that \( R_t \geq 0 \). Hence, we also have

\[ R(\cdot, t) \leq C \]

on \(( -\infty, t_0 ]\) (2.2) because \( R(\cdot, t) \leq R(\cdot,t_0) \). We also know that the evolution equation for \( R \) is

\[ R_t = \Delta_g R + R^2. \]

If we express \( g = v^{-1} ds_p^2 \), in which case \( \Delta_g = v \Delta_{S^2} \) and \( |\nabla \cdot |^2_g = v|\nabla \cdot |^2_{S^2} \), we can rewrite the Harnack estimate for \( R \) as

\[ v \Delta_{S^2} R + R^2 \geq \frac{v |\nabla_{S^2} R|^2}{R} \]

or, equivalently

\[ \Delta_{S^2} R + \frac{R^2}{v} \geq \frac{|\nabla_{S^2} R|^2}{R}. \]

The pressure \( v \) satisfies the elliptic equation

\[ v \Delta_{S^2} v - |\nabla_{S^2} v|^2 + 2v^2 = R v. \]

We will next use this equation and the bounds (2.1) and (2.2) to establish uniform first and second order derivative estimates on \( v \).

**Lemma 2.1.** There exists a uniform constant \( C \), independent of time, so that

\[ \sup_{S^2} (|\Delta_{S^2} v| + \frac{|\nabla_{S^2} v|^2}{v}) \leq C, \quad \text{for all } t \leq t_0 < 0. \]

**Proof.** To simplify the notation we will set \( \Delta := \Delta_{S^2} \) and \( \nabla := \nabla_{S^2} \). We first differentiate (2.3) twice to compute the equation for \( \Delta v \). After some direct calculations we find that

\[ \Delta (\Delta v) + \frac{(\Delta v)^2 - 2 |\nabla^2 v|^2}{v} + 4 \Delta v + 2 \frac{|\nabla v|^2}{v} = \frac{\Delta (Rv)}{v} \]

which implies that

\[ \Delta (\Delta v + 4v) \geq -2 \frac{|\nabla v|^2}{v} + \Delta R + 2 \frac{\nabla R \cdot \nabla v}{v} + \frac{R \Delta v}{v} \]

(2.5) since by the trace formula we have

\[ (\Delta v)^2 \leq 2 |\nabla^2 v|^2. \]
By (2.4) we also have
\[ \Delta v = \frac{|\nabla v|^2}{v} - 2v + R \] (2.6)
and if we use it to replace \( \Delta v \) from the last term on the right hand side of (2.5) we obtain
\[
\Delta (\Delta v + 4v) \geq -2 \frac{|\nabla v|^2}{v} + \Delta R + 2 \frac{ \nabla R \cdot \nabla v}{v} + \frac{R}{v} \left( \frac{|\nabla v|^2}{v} - 2v + R \right)
\]
\[
= -2 \frac{|\nabla v|^2}{v} + (\Delta R + \frac{R^2}{v}) + 2 \frac{ \nabla R \cdot \nabla v}{v} + \frac{R|\nabla v|^2}{v^2} - 2R.
\]
By the Harnack estimate (2.3) we get
\[
\Delta (\Delta v + 4v) \geq -2 \frac{|\nabla v|^2}{v} + \frac{(|\nabla R|^2 + 2 \nabla R \cdot \nabla v + R^2 |\nabla v|^2)}{v^2} - 2R
\]
\[
= -2 \frac{|\nabla v|^2}{v} + \frac{1}{R} \left( |\nabla R|^2 + 2 \nabla R \cdot \nabla v + R^2 |\nabla v|^2 \right) - 2R
\]
\[
= -2 \frac{|\nabla v|^2}{v} + \frac{1}{R} \left( |\nabla R + R \frac{\nabla v}{v}|^2 \right) - 2R.
\]
Since \( R > 0 \) we conclude the estimate
\[ \Delta (\Delta v + 4v) \geq -2 \frac{|\nabla v|^2}{v} - 2R. \] (2.7)
If we multiply (2.6) by \( M = 2 \) and add it to (2.7) we get
\[ \Delta (\Delta v + 6v) \geq -4v \geq -C \]
for a uniform constant \( C \) (independent of time). By (2.6) we also have
\[ \Delta v \geq -2v + R > -\bar{C} \] (2.8)
and therefore
\[ X := \Delta v + \bar{C} + 6v > 0 \]
and
\[ \Delta X \geq -C. \] (2.9)
Standard Moser iteration applied to (2.9) yields the bound
\[ \sup X \leq C_1 \int_{S^2} X \, da + C_2. \] (2.10)
Observe that
\[ \int_{S^2} X \, da = \int_{S^2} (\Delta v + \bar{C} + 6v) \, da = \int_{S^2} (\bar{C} + 6v) \, da \leq C. \]
The last estimate combined with (2.10) yields to the bound
\[ \Delta v \leq C. \]
This together with (2.8) imply
\[ \sup_{S^2} |\Delta v| (\cdot, t) \leq C, \quad \text{for all } t \leq t_0 < 0 \] (2.11)
for \( C \) is a uniform constant. Since
\[ \frac{|\nabla v|^2}{v} = \Delta v + 2v - R \]
the estimate (2.11) and the upper bound \( R \leq C \), readily imply the bound
\[ \sup_{S^2} \frac{|\nabla v|^2}{v} \leq C, \quad \text{for all } t \in (-\infty, t_0]. \] (2.12)

As a consequence of the previous lemma we have:

**Corollary 2.2.** For any \( p \geq 1 \), we have
\[ \|v(\cdot, t)\|_{W^{2,p}(S^2)} \leq C(p), \quad \text{for all } t \in (-\infty, t_0]. \] (2.13)
It follows that for any \( \alpha < 1 \), we have
\[ \|v(\cdot, t)\|_{C^{1,\alpha}(S^2)} \leq C(\alpha), \quad \text{for all } t \in (-\infty, t_0]. \] (2.14)

**Proof.** Since \( \Delta_{S^2} v = f \) in \( S^2 \), with \( f \in L^\infty \), standard \( W^{2,p} \) estimates for Laplace’s equation imply that \( v \in W^{2,p}(S^2) \) for all \( p \geq 1 \). Hence, (2.14) follows by the Sobolev embedding theorem. \( \square \)

We will now use the estimates proven above to improve the regularity of the function \( v \).

**Lemma 2.3.** For every \( 0 < \alpha < 1 \), there is a uniform constant \( C(\alpha) \) so that
\[ \|\nabla_{S^2} v(\cdot, t)\|_{C^{1,\alpha}(S^2)} \leq C(\alpha) \quad \text{for all } t \leq t_0 < 0. \] (2.15)

**Proof.** To simplify the notation we will set \( \Delta := \Delta_{S^2} \) and \( \nabla := \nabla_{S^2} \). A direct computation shows that \( |\nabla v|^2 \) satisfies the evolution equation
\[ \frac{\partial}{\partial t} |\nabla v|^2 = v \Delta |\nabla v|^2 - 2v |\nabla^2 v|^2 + 2v |\nabla v|^2 + 2|\nabla v|^2 \Delta v - 2\nabla (|\nabla v|^2) \cdot \nabla v. \]
On the other hand, differentiating the equation \( v_t = R v \) gives
\[ \frac{\partial}{\partial t} |\nabla v|^2 = 2 \nabla (R v) \cdot \nabla v. \]
Combining the above gives that
\[ \Delta (|\nabla v|^2) = f \] (2.16)
with \( f \) given by
\[
f = 2|\nabla^2 v|^2 - 2|\nabla v|^2 - 2|\nabla v|^2 \Delta v + \frac{2\nabla(|\nabla v|^2) \cdot \nabla v}{v} + \frac{2\nabla(Rv) \cdot \nabla v}{v}. \tag{2.17}
\]

We will show that for every \( p \geq 1 \), we have
\[
\|f(\cdot, t)\|_{L^p(S^2)} \leq C(p), \quad \text{for all } t \leq t_0 < 0 \tag{2.18}
\]
with \( C(p) \) independent of \( t \). We will denote in the sequel by \( C(p) \) various constants that are independent of \( t \). We begin by recalling that by (2.13), we have
\[
\|\nabla^2 v(\cdot, t)\|_{L^p} \leq C(p), \quad \text{for all } t \leq t_0 < 0.
\]
Also, by Lemma 2.1, we have
\[
\|2|\nabla v|^2 \Delta v\|_{L^p(S^2)} + \|\nabla v|^2\|_{L^p(S^2)} \leq C(p), \quad \text{for all } t \leq t_0 < 0.
\]
Since
\[
|\nabla(|\nabla v|^2) \cdot \nabla v| \leq |\nabla^2 v| |\nabla v|^2
\]
by the previous estimates, we have
\[
\|\nabla(|\nabla v|^2) \cdot \nabla v\|_{L^p(S^2)} \leq C(p), \quad \text{for all } t \leq t_0 < 0.
\]
We also have
\[
|\nabla(Rv) \cdot \nabla v| \leq R\frac{|\nabla v|^2}{v} + |\nabla R| |\nabla v|
\leq C + (\sqrt{v}|\nabla R|) \left( \frac{|\nabla v|}{\sqrt{v}} \right) \leq C
\]
for all \( t \leq t_0 < 0 \), since \( \sqrt{v}|\nabla R| = |\nabla R|_{g(t)} \leq C \). We can now conclude that (2.18) holds, for \( p \geq 1 \). Standard elliptic regularity estimates applied on (2.16) imply the bound
\[
\|\nabla v|^2\|_{W^{2,p}} \leq C(p), \quad \text{for all } t \leq t_0 < 0.
\]
Since the previous estimate holds for any \( p \geq 1 \), by the Sobolev embedding theorem, we conclude (2.15).

\[\square\]

**Lemma 2.4.** For every \( 0 < \alpha < 1 \), there is a uniform constant \( C(\alpha) \), so that
\[
\|v \nabla^3 v\|_{C^{0,\alpha}(S^2)} + \|\sqrt{v} \nabla^2 v\|_{C^{0,\alpha}(S^2)} \leq C(\alpha)
\]
for all \( t \leq t_0 < 0 \).
Proof. To simplify the notation we will set $\Delta := \Delta_{S^2}$ and $\nabla := \nabla_{S^2}$. To prove the estimate on $\|\sqrt{v} \nabla^2 v\|_{C^{0,\alpha}(S^2)}$ we observe that we can rewrite (1.8) in the form

$$\Delta v^{3/2} = \frac{9|\nabla v|^2}{4\sqrt{v}} - 3v^{3/2} + \frac{3}{2} R\sqrt{v}. \quad (2.19)$$

We claim that the right hand side of the previous identity has uniformly in time bounded $C^{0,\alpha}$-norm, for any $0 < \alpha < 1$. To see that, observe that for every $p \geq 1$, we have

$$\|\nabla\left(\frac{|\nabla v|^2}{\sqrt{v}}\right)\|_{L^p(S^2)} \leq \|\nabla^2 v\|_{L^p(S^2)} \left\|\frac{|\nabla v|}{\sqrt{v}}\right\|_{L^\infty(S^2)} + \left\|\frac{|\nabla v|^2}{v}\right\|_{L^\infty(S^2)} \leq C(p)$$

and also

$$\|\nabla(R\sqrt{v})\|_{L^\infty(S^2)} \leq C$$

since

$$|\nabla(R\sqrt{v})| \leq |\nabla R|\sqrt{v} + R\left|\frac{|\nabla v|}{\sqrt{v}}\right| \leq |\nabla R|_{H^1(S^2)} + C \leq \tilde{C}.$$ 

All of the above inequalities hold uniformly on $t \leq t_0 < 0$. By the Sobolev embedding theorem we conclude that the right hand side of (2.19) has uniformly bounded $C^{0,\alpha}$ norm, for any $\alpha < 1$. Standard elliptic regularity theory applied to (2.19) implies that

$$\|v^{3/2}\|_{C^{2,\alpha}(S^2)} \leq C(\alpha)$$

which in particular yields to the estimate

$$\|\sqrt{v} \nabla^2 v\|_{C^{0,\alpha}(S^2)} \leq C(\alpha), \quad \text{for all } t \leq t_0 < 0$$

since

$$\nabla \nabla v^{3/2} = \frac{3}{4} \frac{|\nabla v|^2}{\sqrt{v}} + \frac{3}{2} \sqrt{v} \nabla^2 v$$

and the first term on the right hand side is in $C^{0,\alpha}$ by (2.20).

To prove the second estimate, we now rewrite (1.8) as

$$\Delta v^2 = 4 |\nabla v|^2 - 4v^2 + 2R v. \quad (2.21)$$

Lemma 2.3 implies that $4 |\nabla v|^2 - 4v^2$ has uniformly in time bounded $C^{1,\alpha}$ norm. We claim the same is true for the term $Rv$. To see that, we differentiate it twice and use the inequality

$$|\nabla^2 (R v)| \leq |\nabla^2 v| R + |\nabla^2 R| v + 2|\nabla R| |\nabla v|.$$
By Lemmas 2.1 and 2.3 and the bounds
\[ v |\nabla^2 R| = |\nabla^2 R|_g \leq C, \quad \sqrt{v} |\nabla R| = |\nabla R|_g \leq C, \quad R \leq C \]
we conclude that for all \( p \geq 1 \), we have
\[ \|\nabla^2 (Rv)\|_{L^p(S^2)} \leq C(p), \quad \text{for all } t \leq t_0 < 0 \]
The Sobolev embedding theorem now implies that \( \|Rv\|_{C^{1,\alpha}(S^2)} \) is uniformly bounded in time, for every \( \alpha < 1 \). Standard elliptic theory applied to (2.21) yields to the bound
\[ \|v^2\|_{C^{3,\alpha}} \leq C(\alpha), \quad \text{for all } t \leq t_0 < 0, \]
which in particular implies that
\[ \|v^3\|_{C^{0,\alpha}(S^2)} \leq C(\alpha). \]

Recall that in the case of a spherical metric, given by \( ds^2_p = d\psi^2 + \cos^2 \psi d\theta^2 \), we have
\[ \nabla_1 \nabla_1 f = \frac{\partial^2 f}{\partial \psi^2} - \Gamma_{11}^k \frac{\partial f}{\partial \psi} = f_{\psi\psi} \]
and
\[ \nabla_2 \nabla_2 f = \frac{\partial^2 f}{\partial \theta^2} - \Gamma_{22}^k \frac{\partial f}{\partial \psi} = f_{\theta\theta} - f_{\psi} \tan \psi + f_{\theta} \tan \psi. \]
Using the above notation, we will next show the following bound.

**Proposition 2.5.** There is a uniform constant \( C \), so that
\[ |\nabla_1 \nabla_1 v| + \sec^2 \psi |\nabla_2 \nabla_2 v| \leq C \]
for all \( t \leq t_0 < 0 \).

**Proof.** To simplify the notation we set \( \Delta := \Delta_{S^2} \) and \( \nabla := \nabla_{S^2} \). We begin by differentiating the equation
\[ v \Delta v - |\nabla v|^2 + 2v^2 = Rv \quad (2.22) \]
to get
\[ v\nabla_i \Delta v + \nabla_i v \Delta v - 2\nabla_i \nabla_j v \nabla_j v + 4v\nabla_i v = \nabla_i (Rv). \quad (2.23) \]
Since
\[ \nabla_i \Delta v = \Delta \nabla_i v - \frac{1}{2} \nabla_i v \]
we obtain that
\[
\Delta \nabla_i v = \frac{1}{v} \left( \nabla_i (Rv) - \nabla_i v \Delta v + 2 \nabla_i v \nabla_j v \nabla_j v - \frac{7}{2} v \nabla_i v \right). \tag{2.24}
\]

If we differentiate (2.23), using that
\[
\nabla_k \nabla_i \Delta v = \Delta (\nabla_k \nabla_i v) - 2 \left( \nabla_i \nabla_k v - \frac{1}{2} \Delta v \, g_{ki} \right)
\]
we obtain that
\[
\nabla_k \nabla_i (Rv) = v \Delta (\nabla_k \nabla_i v) - 2 \nabla_k \nabla_i v + v \Delta v \, g_{ik} + \nabla_k v \nabla_i \Delta v + \nabla_i v \nabla_k \Delta v - 2 \nabla_k \nabla_i \nabla_j v \nabla_j v - 2 \nabla_i \nabla_j v \nabla_j \nabla_k v + 4 \nabla_k v \nabla_i v + 4 \nabla_i v \nabla_k v. \tag{2.25}
\]

Take \( i = k = 1 \) and consider the maximum point of \( \nabla_1 \nabla_1 v \). We may assume that at the maximum point the matrix \( \nabla^2 v \) is diagonal. Hence, at that point we have \( \Delta (\nabla_1 \nabla_1 v) \leq 0 \) and
\[
\nabla_1 \nabla_j v \nabla_j \nabla_1 v = (\nabla_1 \nabla_1 v)^2.
\]

By (2.25) we then obtain
\[
2(\nabla_1 \nabla_1 v)^2 \leq |\nabla_1 \nabla_1 (Rv)| - 2 v \nabla_1 \nabla_1 v + v \Delta v \, g_{11} + 2 \nabla_1 v \nabla_1 \Delta v + \nabla_1 v \nabla_1 \Delta v - 2 \nabla_1 \nabla_1 v \nabla_j v + 4 |\nabla_1 v|^2 + 4 \nabla_1 \nabla_1 v. \tag{2.26}
\]

We will now estimate the terms on the right hand side of (2.26) at the maximum point of \( \nabla_1 \nabla_1 v \). We begin by estimating the term \( \nabla_1 v \nabla_1 \Delta v \). By switching the order of differentiation, using also (2.24), we get
\[
|\nabla_1 v \nabla_1 \Delta v| = |\nabla_1 v \Delta \nabla_1 v - \frac{1}{2} (\nabla_1 v)^2| \\
\leq C + \frac{|\nabla_1 v|}{v} |\nabla_1 (Rv) - \nabla_1 v \Delta v + 2 \nabla_1 v \nabla_j v \nabla_j v - \frac{7}{2} v \nabla_1 v| \\
= C + \frac{|\nabla_1 v|}{v} |\nabla_1 (Rv) - \nabla_1 v \Delta v + 2 \nabla_1 \nabla_1 v \nabla_1 v| \leq C + \epsilon |\nabla_1 \nabla_1 v|^2
\]
where we have used that \( |\nabla v| \leq C \) for all \( t \leq t_0 < 0 \) (c.f. Lemma 2.1). To estimate the terms above, we begin by observing that
\[
\frac{|\nabla_1 v|}{v} |\nabla_1 \nabla_1 v| \leq |\nabla_1 \nabla_1 v| \frac{|\nabla_1 v|^2}{v} \leq C |\nabla_1 \nabla_1 v| \leq C + \epsilon |\nabla_1 \nabla_1 v|^2
\]
by the interpolation inequality and the estimate \( |\nabla v|^2 \leq C \) shown in Lemma 2.1. In addition, we have
\[
\frac{|\nabla_1 v|}{v} |\nabla_1 (Rv)| \leq \frac{|\nabla_1 v|}{\sqrt{v} \sqrt{\nabla_1 v}} (\sqrt{v} |\nabla R|) + R \frac{|\nabla_1 v|^2}{v} \leq C
\]
and
\[ \frac{|\nabla v|}{v} |\nabla_1 v \Delta v| \leq |\Delta v| \frac{|\nabla v|^2}{v} \leq C \]
where we have used that \( R + |\nabla R|_g + |\Delta v| \leq C \), for all \( t \leq t_0 < 0 \). Combining
the above yields to the bound
\[ |\nabla_1 v \nabla_1 \Delta v| \leq \epsilon |\nabla_1 \nabla_1|^2 + C\epsilon, \quad \text{for all } t \leq t_0 < 0 \tag{2.27} \]
where \( C\epsilon \) is a uniform positive constant and \( \epsilon > 0 \) is a small real number.
Moreover, after switching the order of differentiation, we get
\[ |\nabla_j v \nabla_1 \nabla_1 \nabla_j v| = |\nabla_j v||\nabla_1 \nabla_1 v| + \frac{1}{2}(\nabla_j v)^2 \]
\[ \leq \frac{1}{2} |\nabla v|^2 \leq C \tag{2.28} \]
since \( \nabla_j \nabla_1 \nabla_1 v = 0 \), for \( j \in \{1, 2\} \), at the maximum point of \( \nabla_1 \nabla_1 v \). Also,
\[ |\nabla_1 \nabla_1 (Rv)| \leq v |\nabla_1 \nabla_1 R| + R |\nabla_1 \nabla_1 v| + 2 |\nabla_1 R||\nabla_1 v| \]
\[ \leq |\nabla^2 R|_g + C |\nabla_1 \nabla_1 v| + |\nabla R|_g \frac{|\nabla v|}{\sqrt{v}} \]
\[ \leq C + \epsilon |\nabla_1 \nabla_1 v|^2. \tag{2.29} \]
Combining the estimates (2.27), (2.28), (2.29) with (2.26) and choosing \( \epsilon > 0 \)
sufficiently small, we finally obtain the bound
\[ 2(\nabla_1 \nabla_1 v)^2 \leq (\nabla_1 \nabla_1 v)^2 + C \]
which implies the estimate
\[ |\nabla_1 \nabla_1 v| \leq C, \quad \text{for all } t \leq t_0 < 0. \tag{2.30} \]
Since \( \Delta v = g^{ij} \nabla_i \nabla_j v = \nabla_1 \nabla_1 v + \sec^2 \psi \nabla_2 \nabla_2 v \) and \( |\Delta v| \leq C \), by (2.30) we
also obtain the estimate
\[ \sec^2 \psi |\nabla_2 \nabla_2 v| \leq C, \quad \text{for all } t \leq t_0 < 0. \]
This finishes the proof of the Proposition. \( \square \)

**Lemma 2.6.** There is a uniform constant \( C \), independent of time, such that
\[ \sup_{S^2} |v_\theta \sec^2 \psi| \leq C, \quad \text{for all } t \leq t_0 < 0. \]
Proof. To simplify the notation we will set \( \Delta := \Delta_{S^2} \) and \( \nabla := \nabla_{S^2} \). Recall that \( \Delta v = v_{\psi\psi} - \tan \psi v_\psi + \sec^2 \psi (v_{\theta\theta} + \tan \psi v_\theta) \). Hence, we can rewrite equation (2.22) as

\[
|\sec \psi (v_{\theta\theta} + \tan \psi v_\theta)| = \left| \cos \psi \left( \frac{\nabla |v|^2}{v} - 2v + R \right) - \cos \psi (v_{\psi\psi} - \tan \psi v_\psi) \right|.
\]

By Proposition 2.5 and our earlier estimates the right hand side of the previous identity is uniformly bounded in time. Fix \( \psi \in (-\frac{\pi}{2}, \frac{\pi}{2}) \) and consider the maximum (minimum) of \( v_\theta(\psi, \cdot, t) \) as a function of \( \theta \). At such a point we have \( v_{\theta\theta} = 0 \) and therefore from the previous equation we get the bound

\[
|\tan \psi \sec \psi v_\theta| \leq C, \quad \text{for all } t \leq t_0 < 0.
\]

We conclude that for \( \delta > 0 \) small, we have

\[
|v_\theta \sec^2 \psi| \leq C, \quad \text{for } (\theta, \psi) \in [0, 2\pi] \times (-\frac{\pi}{2}, -\delta) \cup (\delta, \frac{\pi}{2})
\]

for all \( t \leq t_0 < 0 \). Notice that, due to our earlier estimates, we immediately have that

\[
|v_\theta \sec^2 \psi| \leq C, \quad \text{for } (\theta, \psi) \in [0, 2\pi] \times (-\delta, \delta)
\]

for all \( t \leq t_0 < 0 \). This finishes the proof of our estimate. \( \square \)

**Lemma 2.7.** There is a uniform constant \( C \) so that

\[
|v_\psi \tan \psi| \leq C, \quad \text{for all } t \leq t_0 < 0.
\]

**Proof.** To simplify the notation we will set \( \Delta := \Delta_{S^2} \) and \( \nabla := \nabla_{S^2} \). We start with the following two claims which justify the well posedness of certain quantities at each time slice \( t < 0 \).

**Claim 2.8.** For every \( t < 0 \) there is a constant \( C(t) \) so that \( \sup_{S^2} |v_\psi \tan \psi| \leq C(t) \).

**Proof.** If we integrate

\[
\Delta v = v_{\psi\psi} - v_\psi \tan \psi + \sec^2 \psi (v_{\theta\theta} + v_\theta \tan \psi) \quad (2.31)
\]

in \( \theta \in [0, 2\pi] \), using the uniform in time bound \( |\Delta v| \leq C \), we get

\[
\left| \int_0^{2\pi} v_{\psi\psi} d\theta - \tan \psi \int_0^{2\pi} v_\psi d\theta \right| \leq C. \quad (2.32)
\]

Consider

\[
H(\psi, t) := \sec \psi \int_0^{2\pi} v_\psi d\theta
\]
a radial function on \( S^2 \). For \( \psi \in (-\delta, \delta) \), where \( \delta > 0 \) is a fixed small real number, we have that

\[
|H(\psi, t)| \leq C,
\]

where \( C \) is a uniform constant, independent of time. If \( \psi \in [-\pi/2, -\delta] \cup [\delta, \pi/2] \), by Proposition 2.5 and (2.32), we have

\[
|H(\psi, t)| \leq C
\]

where \( C > 0 \) is a time independent constant. Notice also that Lemma 2.6 implies that

\[
v_{\theta \psi}(\pi/2, \theta, t) = v_{\theta \psi}(-\pi/2, \theta, t) = 0
\]  

(2.33)

Indeed, if there were some \( \theta, t \) for which that were not true (say for \( \psi = \pi/2 \)), then by using the Taylor expansion for \( v_{\theta}(\cdot, \theta, t) \) as a function of \( \psi \) and (2.33) we would have

\[
v_{\theta}(\psi, \theta, t) = v_{\theta \psi}(\pi/2, \theta, t) (\psi - \pi/2) + O((\psi - \pi/2)^2), \quad \text{for } \psi \sim \pi/2
\]

where \( v_{\theta \psi}(\pi/2, \theta, t) \neq 0 \). Since \( \psi - \pi/2 \sim \cos \psi \) as \( \psi \sim \pi/2 \), Lemma 2.6 yields to a contradiction, unless \( v_{\theta \psi}(\pi/2, \theta, t) = 0 \).

Also, by the Taylor’s expansion around \( \pi/2 \) (similarly around \( -\pi/2 \)) we have

\[
v_{\theta \psi}(\psi, \theta, t) = v_{\theta \psi \psi}(\pi/2, \theta, t) (\psi - \pi/2) + o(\psi - \pi/2), \quad \text{for } \psi \sim \pi/2.
\]

Since at each time slice we are on a smooth compact surface, it follows that

\[
|v_{\theta \psi}| \leq C(t) |\psi - \pi/2| \leq \bar{C}(t) \cos \psi
\]  

(2.34)

when \( |\psi - \pi/2| < \delta \), where \( \delta > 0 \) is a small, time independent number and for every \( \theta \in [0, 2\pi] \).

Let us now use the above to prove our bound \( |v_{\psi} \tan \psi| \leq C(t) \). For \( \psi \) in the interval \( (-\pi/2 + \delta, \pi/2 - \delta) \) we have our bound, since \( |v_{\psi} \sec \psi| \leq C \) for \( t \leq t_0 < 0 \), for a uniform constant \( C \). Assume that \( \psi \in [-\pi/2, -\pi/2 + \delta] \cup [\pi/2 - \delta, \pi/2] \).

We have shown above that for every \( \psi \), we have

\[
|\sec \psi \int_0^{2\pi} v_\psi d\theta| \leq C, \quad \text{for } t \leq t_0 < 0.
\]
By the integral mean value theorem, for every $t, \psi$ there exists $\theta_0$ (that depends on $t, \psi$) so that

$$|v_\psi(\psi, \theta_0, t) \sec \psi| \leq C$$

for a uniform constant $C$. By (2.34) we have

$$|v_\psi(\psi, \theta, t) - v_\psi(\psi, \theta_0, t)| \sec \psi = |v_{\psi\theta}(\psi, \theta', t)||\theta - \theta_0| \sec \psi \leq C(t)$$

for every $\theta \in [0, 2\pi]$ ($\theta'$ is in between $\theta$ and $\theta_0$). We conclude the bound

$$|v_\psi(\psi, \theta, t)| \sec \psi \leq C(t)$$

which finishes the proof of Claim 2.8. □

We would like to say that at the maximum (minimum) of $v_\psi \sec \psi$, considered as a function on a sphere, its first derivative vanishes. Hence, we need the following claim.

**Claim 2.9.** At each time slice $t \leq t_0 < 0$, we have

$$\sup_{S^2} \left( |(v_\psi \sec \psi)_\theta| + |(v_\psi \sec \psi)_\psi| \right) \leq C(t).$$

**Proof.** By Proposition 2.5, Claim 2.8 (2.31) and the bound $|\Delta v| \leq C$, we have

$$\sec^2 \psi |v_{\theta\theta} + v_\theta \tan \psi| \leq C(t).$$

Since at the maximum (minimum) in $\theta$ of $\sec^3 \psi v_\theta$ we have $v_{\theta\theta} = 0$, similarly as in Lemma 2.6 we obtain the bound

$$\sec^3 \psi |v_\theta| \leq C(t), \quad \text{on } S^2.$$

This together with developing $v_\theta$ in Taylor series around $\psi = \frac{\pi}{2}$ (or $\psi = -\frac{\pi}{2}$), similarly as in the proof of Claim 2.8, yield to

$$v_{\theta\psi}(\frac{\pi}{2}, \theta, t) = v_{\theta\psi}(\frac{\pi}{2}, \theta, t) = 0 \quad (2.35)$$

for every $t < t_0$ and every $\theta$. This implies the bounds

$$|(v_\psi \sec^2 \psi)_\theta| = |v_{\psi\theta} \sec^2 \psi| \leq C(t) \quad \text{and} \quad |v_{\psi\psi\theta}| \sec \psi \leq C(t) \quad (2.36)$$

which can be seen by expanding $v_{\psi\theta}$ in Taylor series around $\psi = \frac{\pi}{2}$ (or $\psi = -\frac{\pi}{2}$) and using (2.35). We have shown, in particular that $|(v_\psi \sec \psi)_\theta| \leq C(t)$. 
It remains to show that \(|(v_\psi \sec \psi)_\psi| \leq C(t)|. To this end, we first integrate (2.31) in \(\theta \in [0, 2\pi]\) and then differentiate it in \(\psi\) to get

\[
\left| \tan \psi \int_0^{2\pi} v_\psi d\theta \right|_\psi = \left| \int_0^{2\pi} v_{\psi\psi} d\theta - \int_0^{2\pi} (\Delta v)_\psi d\theta \right| \leq C(t).
\]

The last bound holds because \(v\) is a smooth function on the sphere, at each time slice \(t\). Since we have the estimate we want away from \(\frac{\pi}{2}\) and \(-\frac{\pi}{2}\), we only need to establish the estimate in the neighborhood of those two points. Keeping that in mind, the previous estimate implies that

\[
\left| \sec \psi \int_0^{2\pi} v_\psi d\theta \right|_\psi \leq C(t).
\]  

(2.37)

By the integral mean value theorem, we have

\[
\frac{1}{2\pi} \left( \sec \psi \int_0^{2\pi} v_\psi d\theta \right)_\psi = \frac{1}{2\pi} \sec \psi \tan \psi \int_0^{2\pi} v_\psi d\theta + \frac{1}{2\pi} \sec \psi \int_0^{2\pi} v_{\psi\psi} d\theta = \sec \psi \tan \psi v_\psi(\psi, \theta_1, t) + \sec \psi v_{\psi\psi}(\psi, \theta_2, t)
\]

where \(\theta_1, \theta_2\) depend on \(\psi\) and \(t\). Since

\[
(\sec \psi v_\psi)_\psi = \sec \psi \tan \psi v_\psi + \sec \psi v_{\psi\psi}
\]

the previous equality implies that (for some \(\theta_i\) in between \(\theta\) and \(\theta_i\) and \(i \in \{1, 2\}\))

\[
\left| \frac{1}{2\pi} \left( \sec \psi \int_0^{2\pi} v_\psi d\theta \right)_\psi - (\sec \psi v_\psi)_\psi \right| \leq 2\pi \left( |\sec \psi \tan \psi| |v_\psi(\psi, \theta_1', t)| + \sec \psi |v_{\psi\psi}(\psi, \theta_2', t)| \right) \leq C(t)
\]

where we have used (2.36). This together with (2.37) and (2.36) imply the claim.

To finish the proof of Lemma 2.7 consider the maximum (minimum) of \(v_\psi \sec \psi\) on \(S^2\) at each time-slice \(t < 0\). By Claim 2.8 and Claim 2.9 this function and its first derivatives are well defined on \(S^2\) at each time-slice. At the maximum (minimum) point we have

\[
v_{\psi\psi} \sec \psi + v_\psi \sec \psi \tan \psi = 0
\]

which by Proposition 2.5 implies

\[
|v_\psi \tan \psi| \leq |v_{\psi\psi}| \leq C, \quad \text{for } t \leq t_0 < 0
\]
at the maximum (minimum) point of $v \sec \psi$, where $C$ is a uniform constant. Since we worry only when the maximum or the minimum occur close to $\frac{\pi}{2}$ and $-\frac{\pi}{2}$ (otherwise we have the Lemma 2.7 due to our earlier uniform derivative estimates) we conclude from the previous bound that

$$\sup_{S^2} |v \psi \tan \psi| \leq C, \quad \text{for } t \leq t_0 < 0$$

where $C$ is a uniform constant. This finishes the proof of our lemma. □

Once we have Lemma 2.7 we can improve the estimate on $v_\theta$ shown in Lemma 2.6.

**Lemma 2.10.** There exists a uniform in time constant $C$, so that

$$\sup_{S^2} |v_\theta| \leq C \cos^3 \psi, \quad \text{for } t \leq t_0 < 0.$$  \hfill (2.38)

**Proof.** Since

$$|\sec^2 \psi v_{\psi\theta} + \sec^2 \psi \tan \psi v_\theta| = |\Delta v - v_{\psi\psi} + v_\psi \tan \psi|$$

by Proposition 2.5, Lemma 2.7 and the estimate $|\Delta v| \leq C$, it follows that

$$|\sec^2 \psi v_{\psi\theta} + \sec^2 \psi \tan \psi v_\theta| \leq C, \quad t \leq t_0 < 0.$$  \hfill (2.38)

Fix $\psi \in (-\frac{\pi}{2}, \frac{\pi}{2})$ and look at the maximum (minimum) of $v_\theta(\psi, \theta, t)$ as a function of $\theta$. At that point $v_{\theta\theta} = 0$ and we have

$$|\sec^2 \psi \tan \psi v_\theta| \leq C.$$

When $\psi$ is away from zero (around which we have the estimate (2.38) anyway), this also implies (2.38) with a possibly different uniform constant $C$. □

**Remark 2.11.** In the case where $v = v(\psi, t)$ is radially symmetric we can improve estimates shown in Lemma 2.4. The reason is that in this case the $\text{Hess}(v)$ is given by the matrix

$$\begin{pmatrix}
v_{\psi\psi} & 0 \\
0 & -\frac{\sin 2\psi}{2} v_\psi
\end{pmatrix}$$

and therefore

$$|\text{Hess}(v)|_{g(t)} = v_{\psi\psi}^2 + \sec^4 \psi \left(\frac{\sin 2\psi}{2}\right)^2 v_\psi^2 = v_{\psi\psi}^2 + \tan^2 \psi v_\psi^2 \leq C$$

for a uniform constant $C$ and all $t \leq t_0 < 0$.  \hfill (2.38)
From the above remark and the estimates shown previously we obtain the following result which will be used in section 7.

**Corollary 2.12.** Let $v$ be a radial solution of equation (1.8) on $S^2 \times (-\infty, 0)$. Then, there is a uniform constant $C$ so that

$$
\|\sqrt{v} \nabla^2_{S^2} v\|_{C^{0,1}} \leq C \quad \text{and} \quad \|\sqrt{v} \nabla^3_{S^2} v\|_{C^{0}} \leq C
$$

for all $t \leq t_0 < 0$.

**Proof.** Fix $t < t_0 < 0$. As in the proof of Lemma 2.4, we rewrite (1.8) in the form

$$
\Delta v^{3/2} = \frac{9|\nabla v|^2}{4\sqrt{v}} - 3v^{3/2} + \frac{3}{2} R\sqrt{v}.
$$

By Remark 2.11 we have

$$
\left| \nabla \left( \frac{|\nabla v|^2}{\sqrt{v}} \right) \right| \leq C \left| \nabla^2 v \right| \frac{|\nabla v|}{\sqrt{v}} + \frac{1}{2} \left( \frac{|\nabla v|}{\sqrt{v}} \right)^3 \leq C
$$

thus

$$
\left\| \frac{|\nabla v|^2}{\sqrt{v}} \right\|_{C^{0,1}(S^2)} \leq C
$$

for uniform in time constant $C$. Also, similarly as in the proof of Lemma 2.4 we have $|\nabla (R\sqrt{v})| \leq C$ and $|\nabla v^{3/2}| \leq C$.

The above estimates imply the right hand side of (2.19) has uniformly in time bounded $C^{0,1}$ norm and standard elliptic regularity theory yields

$$
\|v^{3/2}\|_{C^{2,1}(S^2)} \leq C
$$

which readily implies the bound

$$
\|\left( \sqrt{v} \nabla^2 v \right)\|_{C^{0,1}(S^2)} \leq C.
$$

We conclude that

$$
|\nabla (\sqrt{v} \nabla^2 v)| \leq C
$$

that is,

$$
|\sqrt{v} \nabla^3 v| \leq C + \frac{|\nabla v|}{2\sqrt{v}} |\nabla^2 v| \leq C
$$

almost everywhere on $S^2$. Since $\sqrt{v} \nabla^3 v$ is a continuous function on $S^2$ at each time $t$, we obtain the estimate

$$
\|\sqrt{v} \nabla^3 v\|_{L^\infty} \leq C.
$$

Since the constants $C$ are independent on $t$, our proof is complete. $\Box$
3. Lyapunov functional and convergence

We introduce the Lyapunov functional
\[
J(t) = \int_{S^2} \left( \frac{\|\nabla_S v\|^2}{v} - 4v \right) \, da. \tag{3.1}
\]
We will show next that \( J(t) \) is non-decreasing and bounded. In the sequel, we will use these properties and the a priori estimates shown in the previous section to prove that \( v(\cdot, t) \) converges, as \( t \to -\infty \), in the \( C^{1,\alpha} \) norm, to a steady state solution \( \tilde{v} \).

\textbf{Lemma 3.1.} The Lyapunov functional \( J(t) \) is monotone under \( (1.8) \) and in particular, we have
\[
\frac{d}{dt} J(t) = -2 \int_{S^2} \frac{v^2}{v^2} \, da - \int_{S^2} \frac{\|\nabla_S v\|^2}{v} v_t \, da \tag{3.2}
\]
Proof. To simplify the notation we will set \( \Delta := \Delta_{S^2} \) and \( \nabla := \nabla_{S^2} \). Equation \( (1.8) \) and a direct calculation show:
\[
\frac{d}{dt} \int_{S^2} \frac{\|\nabla v\|^2}{v} \, da = 2 \int_{S^2} \frac{\nabla v \nabla v_t}{v} \, da - \int_{S^2} \frac{\|\nabla v\|^2}{v^2} v_t \, da
= -2 \int_{S^2} \frac{\Delta v}{v} v_t \, da + \int_{S^2} \frac{\|\nabla v\|^2}{v^2} v_t \, da
= -2 \int_{S^2} \left( \frac{v_t}{v^2} + \frac{\|\nabla v\|^2}{v^2} - 2 \right) v_t \, da + \int_{S^2} \frac{\|\nabla v\|^2}{v^2} v_t \, da
= -2 \int_{S^2} \frac{v_t^2}{v^2} \, da - \int_{S^2} \frac{\|\nabla v\|^2}{v} v_t \, da + 4 \int_{S^2} v_t \, da.
\]
We then conclude that
\[
\frac{d}{dt} \int_{S^2} \left( \frac{\|\nabla v\|^2}{v} - 4v \right) \, da = -2 \int_{S^2} \frac{v_t^2}{v^2} \, da - \int_{S^2} \frac{\|\nabla v\|^2}{v} v_t \, da \tag{3.3}
\]
that is,
\[
\frac{d}{dt} J(t) = -2 \int_{S^2} \frac{v_t^2}{v^2} \, da - \int_{S^2} \frac{\|\nabla v\|^2}{v} v_t \, da \tag{3.4}
\]
where both terms on the right hand side of \( (3.4) \) are nonnegative, since on an ancient solution of \( (1.8) \) we have \( v_t \geq 0 \).

□

As an immediate consequence of the estimate in Lemma 2.1 and the inequality
\[
v \Delta v - \|\nabla v\|^2 + 2 v^2 \geq 0
\]
we have:
Lemma 3.2. There exists a uniform constant $C$ so that $-C \leq J(t) \leq 0$ for all $t \in (-\infty, t_0]$ and $t_0 < 0$.

We will next use Lemma 3.1 to show that on our ancient solution the backward in time limit of the scalar curvature $R$ is zero. We recall that the scalar curvature $R$ is given, in terms of the pressure function $v$, by $R = \frac{v_t}{v}$. We also recall that on an ancient solution we have $R > 0$ and by the Harnack estimate on the curvature $R$, the inequality $R_t \geq 0$. Hence, the point-wise limit

$$ \tilde{R} = \lim_{t \to -\infty} R(\cdot, t) $$

exists.

Lemma 3.3. On an ancient solution $v$ of equation (1.8), we have $\tilde{R} = 0$ a.e. on $S^2$.

Proof. It is enough to show that

$$ \int_{S^2} \tilde{R}^2 \, da = 0. $$

Indeed, assume the opposite, namely that $\int_{S^2} \tilde{R}^2 \, da := c > 0$. Then, since $R_t \geq 0$ we will have that $\int_{S^2} R^2(\cdot, t) \, da \geq c$, i.e.

$$ \int_{S^2} \frac{v_t^2}{v^2} \, da \geq c. $$

Integrating (3.2) in time while using the above inequality and the fact that $v_t \geq 0$, gives

$$ J(t_2) - J(t_1) \leq - \int_{t_1}^{t_2} \int_{S^2} \frac{v_t^2}{v^2} \, da \leq -c(t_2 - t_1) $$

for every $-\infty < t_1 < t_2 < t_0 < 0$. This obviously contradicts the uniform bound $-C \leq J(t) \leq 0$, shown in Lemma 3.2. $\square$

We will now combine the a priori estimates of the previous section with Lemma 3.3 to prove the following convergence result.

Proposition 3.4. The solution $v(\cdot, t)$ of (1.8) converges, as $t \to -\infty$, to a limit $\tilde{v} \in C^{1,\alpha}(S^2)$, for any $\alpha < 1$. Moreover, $\|\tilde{v} \nabla_{S^2} \tilde{v}\|_{C^0(S^2)} < \infty$, for all $\alpha < 1$, and $\tilde{v}$ satisfies the steady state equation

$$ \tilde{v} \Delta_{S^2} \tilde{v} - |\nabla_{S^2} \tilde{v}|^2 + 2 \tilde{v}^2 = 0. \quad (3.5) $$
Proof. Since $v_t \geq 0$ and $v > 0$, the pointwise limit

$$\tilde{v} := \lim_{t \to -\infty} v(\cdot, t)$$

exists. Lemmas 2.3 and 2.4 ensure that for every $\alpha < 1$ and every sequence $t_i \to -\infty$, along a subsequence still denoted by $t_i$, we have $v(\cdot, t_i) \overset{C^{1,\alpha}(S^2)}{\to} \tilde{v}(\cdot)$ and $v \nabla^2_{S^2} v(\cdot, t_i) \overset{C^{\alpha}(S^2)}{\to} \tilde{v} \nabla^2_{S^2} \tilde{v}$. By the uniqueness of the limit, $\tilde{v} = \tilde{v}$, which means that for every $\alpha < 1$, we have

$$v(\cdot, t) \overset{C^{1,\alpha}(S^2)}{\to} \tilde{v} \quad \text{and} \quad (v \nabla^2_{S^2} v)(\cdot, t) \overset{C^{\alpha}(S^2)}{\to} \tilde{v} \nabla^2_{S^2} \tilde{v}, \quad \text{as} \quad t \to -\infty.$$ 

We can now let $t \to -\infty$ in equation

$$v \Delta_{S^2} v - |\nabla_{S^2} v|^2 + 2 v^2 = R v$$

and use Lemma 3.3 to conclude that $\tilde{v}$ satisfies equation (3.5). 

□

4. The backward limit

Our goal in this section is to classify the backward limit $\tilde{v} = \lim_{t \to -\infty} v(\cdot, t)$, which has been shown to exist in the previous section.

Theorem 4.1. There exists a conformal change of $S^2$ in which the limit

$$\tilde{v}(\psi, \theta) := \lim_{t \to -\infty} v(\psi, \theta, t) = C \cos^2 \psi$$

for some constant $C \geq 0$, where $\psi, \theta$ are global coordinates on a conformally changed sphere. Moreover, the convergence is in the $C^{1,\alpha}$-norm on $S^2$ and it is smooth, uniformly on every compact subset of $S^2 \setminus \{S, N\}$, where $S, N$ are the south and the north pole on $S^2$ (the points that correspond to $\psi = \frac{\pi}{2}$ and $\psi = -\frac{\pi}{2}$).

We have shown in the previous section that $v(\cdot, t) \overset{C^{1,\alpha}(S^2)}{\to} \tilde{v}$, for any $\alpha \in (0, 1)$, where $\tilde{v}$ satisfies the steady state equation

$$\tilde{v} \Delta \tilde{v} - |\nabla \tilde{v}|^2 + 2 \tilde{v}^2 = 0.$$ 

To classify the backward limit $\tilde{v}$, we will need the following lemma, which constitutes the main step in the proof of Theorem 4.1.

Proposition 4.2. The limit $\tilde{v}$ is either identically equal to zero, or it has at most two zeros.
We will outline the main steps in the proof of Proposition 4.2. Their detailed proofs will be given below. Recall that \( \tilde{R}(x) := \lim_{t \to -\infty} R(x,t) = 0 \)
a almost everywhere, otherwise \( \tilde{R} \geq 0 \) and \( R_t \geq 0 \). Let
\[
Z := \{ x \in S^2 \mid \tilde{R}(x) > 0 \}.
\]
Using a covering argument and the fact that the total curvature of our evolving metric is equal to \( \int_{S^2} R \, da = 8\pi < \infty \), we will show that there are at most finitely many points in \( Z \), call them \( \{ p_1, \ldots, p_N \} \), so that
(i) if \( Z = \emptyset \), then \( \tilde{v} \equiv 0 \), and
(ii) if \( Z \neq \emptyset \), on every compact set \( K \subset S^2 \setminus \{ p_1, \ldots, p_N \} \), \( u(\cdot, t) \) is uniformly bounded by a constant that may depend on \( K \), but is time independent. This implies that \( \tilde{v} > 0 \) on \( S^2 \setminus \{ p_1, \ldots, p_N \} \).

In the case (ii) we will consider a dilated sequence of solutions around each of the points \( p_i \) and show that it converges to a cigar soliton. This will imply that for a neighbourhood around each of them we have that
\[
\int_{U} R \, da \approx 4\pi. \tag{4.1}
\]
Since the total curvature \( \int_{S^2} R \, da = 8\pi \), we will conclude that there can be at most two curvature concentration points. This readily implies \( \tilde{v} \) has at most two zeros.

We will denote in the sequel by \( \Delta_{\delta}(x) \) a ball of radius \( \delta \), centered at \( x \), computed in the standard spherical metric.

**Lemma 4.3.** Either \( \tilde{v} \equiv 0 \) or there are at most finitely many points \( p_1, \ldots, p_N \) so that for every \( \delta > 0 \) there exists a \( C(\delta) > 0 \) such that for all \( t < 0 \) we have
\[
\sup_{\Delta_{\delta/4}(x)} u(\cdot, t) \leq C(\delta), \quad \text{for all } x \in S^2 \setminus \bigcup_{i=1}^{N} \Delta_{2\delta}(p_i). \tag{4.2}
\]
Moreover, \( \tilde{R}(x) = 0 \) for every \( x \in S^2 \setminus \{ p_1, \ldots, p_N \} \).

**Proof.** We will prove the Lemma in a few steps. For each \( t < 0 \) and \( r > 0 \), we define the set
\[
L_t^r := \{ x \in S^2 \mid \int_{\Delta_{2r}(x)} (Ru)(\cdot, t) \, da \geq \epsilon \}.
\]
Note that we can choose a cover of \( L_t^r \) by finitely many balls \( \{ \Delta_{2r}(p_{jr}^t) \}_{j=1}^{N_t^2} \) so that the balls \( \Delta_{r}(p_{jr}^t) \) are all disjoint and \( p_{jr}^t \in L_t^r \).
Step 1. For every $\epsilon > 0$ there exists a uniform constant $N$ so that $N^t \leq N$, for all $t < 0$ and all $r > 0$.

Proof. Since $\Delta_r(x)$ are balls in the spherical metric, there are uniform constants $C_1, C_2$ so that

$$C_1 r^2 \leq \text{area}(\Delta_r(x)) \leq C_2 r^2$$

which implies that there is a uniform upper bound on the number of disjoint balls of radius $r$ contained in a ball of radius $4r$. Then, since $p^t_j \in L^t_r$ and $\int_{S^2} Ru\,da = 8\pi$, we have

$$N^t \epsilon \leq \sum_{j=1}^{N^2} \int_{\Delta_r(p^t_j)} (Ru)(\cdot, t)\,da \leq m \int_{S^2} (Ru)(\cdot, t)\,da = 8\pi m$$

for a uniform constant $m$, which yields to the uniform upper bound on $N^t$.

Assume that $\tilde{v}$ is not identically equal to zero and that it has at least three different zeros (otherwise we are done). We can perform a conformal change of coordinates (note that our evolution equations are invariant under conformal changes) in which we bring two of the zeros of $\tilde{v}$ to the poles of $S^2$. In other words, if $\psi, \theta$ are global coordinates on $S^2$, two of the zeros of $\tilde{v}$ will correspond to $\psi = \frac{\pi}{2}$ and $\psi = -\frac{\pi}{2}$. Denote by $w$ the pressure function $v$ in cylindrical coordinates, namely

$$w(x, \theta, t) = v(\psi, \theta, t) \sec^2 \psi, \quad \sec \psi = \cosh x$$

and by $U$ the conformal factor in cylindrical coordinates, namely

$$U(x, \theta, t) = \frac{1}{w(x, \theta, t)} = \frac{u(\psi, \theta, t)}{\cosh^2 x}$$

where $u$ is the conformal factor of our evolving metric on $S^2$. The function $U$ satisfies the equation

$$U_t = \Delta_c \log U = -RU$$

where $\Delta_c$ denotes the cylindrical Laplacian.

Step 2. Either $\tilde{v} \equiv 0$ or, for every compact set $K \subset \mathbb{R} \times [0, 2\pi]$, there exists a uniform constant $C(K)$ so that

$$\int_0^{2\pi} (\log U(x, \theta, t))^+\,d\theta \leq C(K), \quad \text{for all } t < 0.$$
Proof. Let

\[ F(x, t) = \frac{1}{2\pi} \int_0^{2\pi} \log U(x, \theta, t) \, d\theta. \]

**Claim 4.4.** \(|F_x| \leq 2\), for all \(x \in \mathbb{R}\) and all \(t < 0\).

**Proof.** We begin by integrating in \(\theta\) the equation

\[ \Delta c \log U = (\log U)_{xx} + (\log U)_{\theta\theta} = -RU \leq 0 \]

to obtain (since \(U\) is periodic in \(\theta\)) that \(F_{xx} \leq 0\), which implies that \(F_x\) is decreasing in \(x\) and therefore

\[ \lim_{x \to \infty} F_x \leq F(x, t) \leq \lim_{x \to -\infty} F_x. \]

At each time-slice \(t\) we have

\[ \log U(x, \theta, t) = \log u(\psi, \theta, t) - 2 \log \cosh x \]

which implies that

\[ (\log U)_x = (\log u)_\psi \cos \psi - 2 \tanh x \]

since \(d\psi/dx = \cos \psi\). We recall that \(|(\log u)_\psi| \leq C(t)\), as each time \(t\). Hence, as \(x \to -\infty\) (or equivalently \(\psi \to -\frac{\pi}{2}\)), we have

\[ \lim_{x \to -\infty} F_x = \lim_{x \to -\infty} \frac{1}{2\pi} \int_0^{2\pi} (\log U)_x(x, \theta, t) \, d\theta = 2. \]

Similarly,

\[ \lim_{x \to \infty} F_x = \lim_{x \to \infty} \frac{1}{2\pi} \int_0^{2\pi} (\log U)_x(x, \theta, t) \, d\theta = -2 \]

and therefore

\[ |F_x| \leq 2, \quad \text{for all } x \text{ and all } t \leq t_0 < 0. \]

\[ \square \]

**Claim 4.5.** There exist \(C > 0, x_0 \in \mathbb{R}\) so that

\[ F(x_0, t) := \frac{1}{2\pi} \int_0^{2\pi} \log U(x_0, \theta, t) \, d\theta \leq C, \quad \text{for all } t \leq t_0 < 0. \]

**Proof.** We begin by choosing \((x_0, \theta_0)\) so that

\[ U(x_0, \theta_0, t) \leq C_0, \quad \text{for all } t < 0. \]

(4.7)

If there were no such a point, that would mean that for all points \((x, \theta)\) we would have \(\lim_{t \to -\infty} U(x, \theta, t) = \infty\) (remember that \(U_t \leq 0\) which would
imply that the backward limit $\tilde{v} = 0$ and would end the proof of Theorem 4.1 for a constant $C = 0$.

We will prove that for this point $x_0$ the statement of the claim holds. We argue by contradiction. We assume that there is a sequence of times $t_k \to -\infty$ so that

$$F(x_0, t) = \frac{1}{2\pi} \int_0^{2\pi} \log U(x_0, \theta, t_k) d\theta \geq 2\tilde{C}, \quad \text{for all } k$$

for $\tilde{C}$ a sufficiently large constant (to be chosen in the sequel). By Claim 4.4 for $x \in (x_0 - \pi, x_0 + \pi)$ we have

$$\frac{1}{2\pi} \int_0^{2\pi} \log U(x, \theta, t_k) d\theta \geq \frac{1}{2\pi} \int_0^{2\pi} \log U(x_0, \theta, t_k) d\theta - 4\pi \geq 2\tilde{C} - 4\pi \geq \tilde{C}$$

if $\tilde{C} \geq 4\pi$. Hence, by integrating in $x$, we have

$$\frac{1}{4\pi^2} \int_{x_0 - \pi}^{x_0 + \pi} \int_0^{2\pi} \log U(x, \theta, t_k) d\theta dx \geq \tilde{C}. \quad (4.8)$$

Notice that, since $u_t(\psi, \theta, t) \leq 0$, we have

$$u(\cdot, t) \geq C_1, \quad \text{for all } t \leq t_0 < 0$$

for a uniform constant $C_1$. Therefore

$$U(x, \theta, t) = \frac{u(\psi, \theta, t)}{\cosh^2 x} \geq \frac{C_1}{\cosh^2 x} \geq C(K)$$

for all $(x, \theta) \in K$, for any compact set $K \subset \mathbb{R} \times [0, 2\pi]$. Hence, there is a uniform constant $C_2$, independent of time, dependent on the compact set $K$, so that

$$\log U(x, \theta, t) + C_2 > 0, \quad \text{for all } (x, \theta) \in K \text{ and } t \leq t_0 < 0. \quad (4.9)$$

Let

$$Q = \{(x, \theta)| x \in [x_0 - \pi, x_0 + \pi], \theta \in [\theta_0 - \pi, \theta_0 + \pi]\}$$

and $B = B_{\sqrt{2\pi}}(x_0, \theta_0)$ be the euclidean ball of radius $\sqrt{2\pi}$, centered at $(x_0, \theta_0)$. Then, $Q \subset B$. If we understand $\log U$ as a function which is defined on $\mathbb{R}^2$ and is $2\pi$ periodic in the $\theta$ direction, then since

$$\Delta_c(\log U + C_2) \leq 0$$
by the mean value property and (4.8), we have
\[
\log U(x_0, \theta_0, t_k) + C_2 \geq \frac{1}{|B|} \int_B (\log U(\cdot, t_k) + C_2) \geq \frac{1}{|B|} \int_B (\log U(\cdot, t_k) + C_2)
\]
\[
= \frac{1}{|B|} \int_{x_0+\pi}^{2\pi} \int_0^{2\pi} (\log U(\cdot, t_k) + C_2)
\]
\[
\geq \frac{1}{2\pi^2} (4\pi^2 \bar{C} + 4\pi^2 C_2) = 2(\bar{C} + C_2)
\]
which implies that
\[
U(x_0, \theta_0, t_k) \geq e^{2\bar{C}}.
\]
If \(\bar{C}\) sufficiently large so that \(e^{2\bar{C}} \geq 2C_0\) where \(C_0\) is taken from (4.7), we contradict (4.7), finishing the proof of our claim. □

Claims 4.4 and 4.5 imply the bound
\[
F(x,t) := \frac{1}{2\pi} \int_{-\pi}^\pi \log U(x,\theta,t) d\theta \leq C(K)
\]
for any compact set \(K \subset \mathbb{R} \times [-\pi,\pi]\). By (4.9) we have
\[
\int_{-\pi}^\pi (\log U(x,\theta,t))^+ d\theta \leq \bar{C}(K) \quad (4.10)
\]
for all \(t \leq t_0 < 0\) and all \(x\) so that \(\{x\} \times [0,2\pi] \subset K\). □

**Step 3.** There is an \(\epsilon_0 > 0\) so that for every compact set \(K \subset \mathbb{R} \times [0,2\pi]\), if \(\epsilon < \epsilon_0\) and \(\int_K (Ru) (\cdot, t) da < \epsilon\), then
\[
\sup_{\tilde{K}} u(\cdot, t) \leq \bar{C}(K), \quad \text{for all } \tilde{K} \subset \subset K. \quad (4.11)
\]

**Proof.** We have shown in Step 2 the bound \(\int_0^{2\pi} (\log U(x,\theta,t))^+ d\theta \leq C(K)\) for all \((x,\theta) \in K\) and \(t \leq t_0 < 0\). Since \(U(x,\theta,t)\) satisfies \(\Delta_c \log U = -RU\), we can apply the same arguments as in the proof of Lemma 2.3 in [4]. Notice that in (4.11) we do not have a dependence on time (as in [4]) because our smallness assumption (ii) is time independent. □

We will now conclude the proof of Lemma 4.3. Note that the estimate (4.11) can be rewritten on a sphere as
\[
\sup_{\tilde{K}} u(\cdot, t) \leq C(\tilde{K}) \quad (4.12)
\]
where \(\tilde{K} \subset \subset S^2 \setminus \{\text{poles of } S^2\}\) corresponds to a compact set \(\tilde{K}\) from (4.11) under Mercator’s projection.
Choose $\epsilon_0$ from Step 3. By Step 1, for every $r > 0$ and $t < 0$ there are $N$ points $p_{1r}, \ldots, p_{Nr}$ so that the balls $\{\Delta_2r(p_{jr})\}_{j=1}^N$ cover the set $L_t$ and for every point $x \in S^2 \setminus (\bigcup_{j=1}^N \Delta_2r(p_{jr}))$, we have

$$\int_{\Delta_2r(x)} (Ru)(\cdot, t) \, da < \epsilon_0.$$  

It follows from Step 3 that

$$\sup_{\Delta_r(x)} u(\cdot, t) \leq C(r) \quad (4.13)$$

for every $x \in S^2 \setminus (\bigcup_{j=1}^N \Delta_2r(p_{jr}))$ and all $t < 0$.

Now fix a monotone sequence $t_k \to -\infty$. For a sequence of radii $r_k \to 0$ we have (passing to a subsequence) that $p_{jr_k}^t \to p_j$ as $k \to \infty$, for $N$ points $p_j \in S^2$ (these points may not be distinct).

Let $x \in S^2 \setminus (\bigcup_{j=1}^N \Delta_{\delta/2}(p_j))$. We may choose $k$ sufficiently large, so that $p_{jr_k}^t$ is sufficiently close to $p_j$, for all $1 \leq j \leq N$ and therefore $x \in S^2 \setminus (\bigcup_{j=1}^N \Delta_{\delta/2}(p_{jr_k}^t))$. It follows by (4.13) with $r = \delta/4$, that

$$\sup_{\Delta_{\delta/4}(x)} u(x, t_k) \leq C(\delta)$$

for all $x \in S^2 \setminus (\bigcup_{j=1}^N \Delta_{\delta/2}(p_j))$ and all $k$ sufficiently large. Recalling that $u_t \leq 0$, we conclude from the above that (4.2) holds for all $t < 0$.

This estimate implies $u(\cdot, t)$ is uniformly bounded from below on every compact subset of $S^2 \setminus \{p_1, \ldots, p_N\}$, which means that the equation (1.8) is uniformly parabolic there and the standard parabolic estimates imply that $R(\cdot, t)$ converges to zero uniformly on those sets away from $\{p_1, \ldots, p_N\}$. In other words, $\tilde{R} = 0$ on $S^2 \setminus \{p_1, \ldots, p_N\}$. This finishes the proof of Lemma 4.3. 

Let $p_1, \ldots, p_N$ be the points as chosen in Lemma 4.3.

**Corollary 4.6.** For every $1 \leq j \leq N$ we have that $\lim_{t \to -\infty} u(p_j, t) = +\infty$.

**Proof.** If $\tilde{R}(p_j) = \gamma > 0$, then since $R_t \geq 0$ we have $R(p_j, t) \geq \gamma$ for all $t < 0$. Since $u_t = -R u$, at the particular point $p_j$ we have $(\ln u)_t(p_j, t) \leq -\gamma$ and therefore

$$u(p_j, t) \geq u(p_j, t_0) e^{\gamma(t_0 - t)} = c_j \, e^{\gamma |t|},$$

which implies the $\lim_{t \to -\infty} u(p_j, t) = \infty$. 

If $\tilde{R}(p_j) = 0$, assume the limit $\lim_{t \to -\infty} u(p_j, t) \leq C < \infty$, otherwise we are done. Our construction of $p_j$ implies that for a sequence $r_k \to 0$ there are points $p_{jk}^r$ so that

$$\int_{\Delta_{r_k}(p_{jk}^r)} (Ru)(\cdot, t_k) \, da \geq \epsilon_0. \quad (4.14)$$

On the other hand, since $u_t \leq 0$ we have $c \leq u(p_j, t) \leq C$, which implies that $C^{-1} \leq v(p_j, t) \leq c^{-1}$, for all $t \leq t_0 < 0$. Since $v(\cdot, t)$ converges as $t \to -\infty$ to $\tilde{v}$ in $C^{1,\alpha}$ norm, there exists $\delta > 0$ so that $c/2 \leq u(z, t) \leq 2C$, for $z \in \Delta_\delta(p_j) \subset S^2$ and all $t \leq t_0 < 0$. It follows that our equation is uniformly parabolic on $\Delta_\delta(p_j)$ and therefore, by standard parabolic estimates, $R(\cdot, t) \to 0$, as $t \to -\infty$ uniformly on $\Delta_{\delta/2}(p_j)$ (since $R(\cdot, t) \to 0$ a.e. as $t \to -\infty$). This obviously contradicts (4.14), finishing the proof.

We will now conclude the proof of Proposition 4.2 based on Lemma 4.3 and Corollary 4.6.

**Proof of Proposition 4.2.** Assume that $\tilde{v}$ is not identically equal to zero, and let $p_1, \ldots, p_N$ be the points from Lemma 4.3. Denote by $2\rho$ the minimum distance between any of those points. We will show that for any $p_j$, $j = 1, \ldots, N$, and any $\epsilon > 0$, there exists a $t_0 = t_0(\epsilon)$ such that

$$\int_{\Delta_\rho(p_j)} (Ru)(\cdot, t) \, da \geq 4\pi - \epsilon, \quad \text{for all } t \leq t_0$$

which will readily imply that there are only two of them, since we have

$$\int_{S^2} (Ru)(\cdot, t) \, da = 8\pi, \quad \text{for all } t \leq t_0.$$ 

Consider any of these points, say $p_1$, and perform a stereographic projection to map $S^2$ onto $\mathbb{R}^2$ so that $p_1$ is mapped to the origin in $\mathbb{R}^2$. Denote by $\tilde{u}$ the conformal factor of our evolving metric on the plane and by $\tilde{R}$ its curvature.

It is well known that $\tilde{u}$ satisfies the equation

$$\tilde{u}_t = \Delta \log \tilde{u} \quad (4.15)$$

on $\mathbb{R}^2 \times (-\infty, 0)$. Moreover, by Corollary 4.6 we have

$$\lim_{t \to -\infty} \tilde{u}(0, t) = +\infty.$$
Each of the other points $p_j$ are mapped to points $\bar{p}_j \in \mathbb{R}^2 \cup \{\infty\}$. Denote by $2\bar{\rho}$ the minimum distance between zero and the points $\bar{p}_j$. It will be sufficient to show that for any $\epsilon > 0$, there exists a $t_0 = t_0(\epsilon)$ such that

$$\int_{B_{\bar{\rho}}(0)} (\bar{R} \bar{u})(x, t) \, dx \geq 4\pi - \epsilon, \quad \text{for all } t \leq t_0.$$  \hspace{1cm} (4.16)

To this end, we will rescale $\bar{u}$ and show that the rescaled solution converges to a cigar solution of the Ricci flow.

Fix a sequence $t_k \to -\infty$. Since $0$ may not need to be a local maximum for $\bar{u}(\cdot, t_k)$ and we will need to rescale $\bar{u}$ around local maximum points, we let $x_k$ be such that $u(x_k, t_k) := \max_{B_{\bar{\rho}}(0)} u(\cdot, t_k)$.

Since $u(x_k, t_k) \geq u(0, t_k)$, we have $\lim_{t_k \to -\infty} u(x_k, t_k) = \infty$. It follows that $\lim_{k \to \infty} x_k = 0$, otherwise a subsequence $x_{k_l}$ would converge to a point $x_\infty \in B_{\bar{\rho}}(0) \setminus \{0\}$ and would contradict Lemma 4.3.

Set $\alpha_k = u(x_k, t_k)^{-1}$ and recall that $\alpha_k \to 0$. In particular, we may assume that $\alpha_k < 1/4$. Since $x_k \to 0$, we may also assume that $|x_k| < \frac{\bar{\rho}}{2}$.

Consider the rescaled solutions of equation (4.15) defined by

$$\bar{u}_k(x, t) = \alpha_k \bar{u}(x_k + \sqrt{\alpha_k} x, t + t_k), \quad x \in B_{\bar{\rho}}(0), \quad -\infty < t < -t_k.$$  

We observe that for $|x| < \rho$, we have $|x_k + \sqrt{\alpha_k} x| < \rho$, hence $\bar{u}_k(\cdot, 0) \leq 1$, from the definition of $x_k$ and $\alpha_k$. Moreover, since $\alpha_k \to 0$, for every $M > 0$, there is a $k_0 = k_0(M)$ such that $|x_k + \sqrt{\alpha_k} x| < \rho$ for all $|x| \leq M$, which shows that $\bar{u}_k(\cdot, 0)$ is uniformly bounded on any compact set of $\mathbb{R}^2$. By standard estimates on equation (4.15), it follows that the sequence of solutions $\{\bar{u}_k\}$ is uniformly bounded on compact subsets of $\mathbb{R}^2 \times \mathbb{R}$, hence equicontinuous. We conclude, that passing to a subsequence, which we still denote by $\bar{u}_k$, we have that $\bar{u}_k \to \bar{u}$ uniformly on compact subsets of $\mathbb{R}^2 \times \mathbb{R}$ and that $\bar{u}$ is an eternal solution of (4.15) (i.e. it is defined on $\mathbb{R}^2 \times (-\infty, \infty)$). Moreover, $\bar{u}(0, 0) = \lim_{k \to -\infty} \bar{u}(x_k, 0) = 1$. It follows, from the results in [4], that $\bar{u}$ is a cigar solution which in particular implies that

$$\int_{\mathbb{R}^2} \bar{R} \bar{u} \, dx = 4\pi, \quad \text{for all } t \in (-\infty, \infty)$$

where $\bar{R}$ denotes the curvature of $\bar{u}$. 
From here the conclusion of our proposition is straightforward. For any \( \epsilon > 0 \), choose \( M > 0 \) sufficiently large so that
\[
\int_{|x| \leq M} (\tilde{R} \tilde{u})(x,0) \, dx > 4\pi - 2\epsilon.
\]
From the uniform convergence of \( \bar{u}_k(\cdot,0) \to \tilde{u}(\cdot,0) \) on \( |x| \leq M \), which also implies uniform convergence of the corresponding scalar curvatures, it follows that for \( k \) sufficiently large
\[
\int_{|x| \leq M} (\tilde{R}_k \bar{u}_k)(x,0) \, dx > 4\pi - \epsilon
\]
or equivalently
\[
\int_{|y| \leq \sqrt{\alpha_k} M} (\tilde{R} \bar{u})(x_k + y, t_k) \, dy > 4\pi - \epsilon.
\]
Recalling that \( \alpha_k \to 0 \) and \( x_k \to 0 \), the above readily implies that (4.16) holds for \( t = t_k \). Since \( t_k \) is an arbitrary sequence, this finishes the proof of the proposition.

\[\square\]

Assume from now on that the backward limit \( \tilde{v} \) is not identically zero. We have just shown in Proposition 4.2 that \( \tilde{v} \) has at most two zeros. Choose a conformal change of \( S^2 \) which brings those two zeros to two antipodal poles on \( S^2 \). Let \( \psi, \theta \) be global coordinates on \( S^2 \), where \( \psi = \frac{\pi}{2} \) and \( \psi = -\frac{\pi}{2} \) correspond to the poles (denote them by \( S \) and \( N \)). Observe that equation (1.8) is strictly parabolic away from the poles, uniformly as \( t \to -\infty \). It follows that the convergence \( v(\cdot, t) \to \tilde{v} \), as \( t \to -\infty \), is smooth on compact subsets of \( S^2 \setminus \{S,N\} \). Perform the Mercator’s transformation (4.4) with respect to the poles \( S,N \) at which \( \tilde{v} \) is potentially zero.

Since \( w(x, \theta, t) = v(\psi, \theta, t) \cosh^2 x \), we conclude that
\[
\lim_{t \to -\infty} w(x, \theta, t) = \tilde{w}(x, \theta) := \tilde{v}(\psi, \theta) \cosh^2 x > 0
\]
and the convergence is smooth on compact subsets of \( \mathbb{R} \times [0, 2\pi] \). The limit \( \tilde{w} \) satisfies
\[
\tilde{w} \Delta \tilde{w} - |\nabla \tilde{w}|^2 = 0
\]
or (since \( \tilde{w}(x, \theta) > 0 \) on \( \mathbb{R} \times [0, 2\pi] \)) equivalently,
\[
\Delta_c \log \tilde{w} = 0 \tag{4.17}
\]
where $\Delta_c$ is the cylindrical laplacian on $\mathbb{R}^2$. To finish the proof of Theorem 4.1 we need to classify the solutions of the steady state equation (4.17) that come as limits of ancient solutions $w(\cdot, t)$.

**Proof of Theorem 4.1.** Denote by

$$W(x) := \int_0^{2\pi} \log \tilde{w}(x, \theta) \, d\theta.$$ 

Keeping in mind that $\Delta_c \log \tilde{w} = (\log \tilde{w})_{xx} + (\log \tilde{w})_{\theta\theta}$, if we integrate (4.17) in $\theta$ we obtain

$$W_{xx} = 0$$

which implies that

$$W(x) = \int_0^{2\pi} \log \tilde{w}(x, \theta) \, d\theta = C_1 + C_2 x$$  \hspace{1cm} (4.18)

for some constants $C_1, C_2$. Denote by $f := \log \tilde{w}$. We can view $f$ as a function on $\mathbb{R}^2$, after extending it in the $\theta$ direction so that it remains $2\pi$-periodic.

Since $\tilde{w}(x, \theta) = \tilde{v}(\psi, \theta) \cosh^2 x$ and $\tilde{v} = \lim_{t \to -\infty} v(\cdot, t) \leq C$, it follows that there are uniform constants $\tilde{C}_1, \tilde{C}_2 > 0$ so that

$$f \leq \tilde{C}_1 + \tilde{C}_2 |x|. \hspace{1cm} (4.19)$$

We define the function $h(x, \theta) = \tilde{C}_1 + \tilde{C}_2 |x| - \log \tilde{w} > 0$, where $\tilde{C}_1, \tilde{C}_2$ are taken from (4.19). Since $\Delta_c h = 0$ on $|x| > 0$, if $(x, \theta)$ is an arbitrary point with $|x| \geq 2\pi$, by the mean value theorem applied to $h(x, \theta)$ we have

$$h(x, \theta) = \frac{1}{|B(x, \theta)(\pi)|} \int_{B(x, \theta)(\pi)} h(x', \theta) \, dx' \, d\theta$$

where $B(x, \theta)(\pi)$ is a ball centered at $(x, \theta)$ and of radius $\pi$. If

$$Q(x, \theta) := \{(x', \theta') | |x - x'| \leq \pi, |\theta - \theta'| \leq \pi\}$$

then $B(x, \theta)(\pi) \subset Q(x, \theta)$ and therefore

$$h(x, \theta) \leq C \int_{Q(x, \theta)} h(x', \theta') \, dx' \, d\theta'$$

$$= C \int_{x - \pi}^{x + \pi} \int_0^{2\pi} (\tilde{C}_1 + \tilde{C}_2 x') \, d\theta' \, dx' - C \int_{x - \pi}^{x + \pi} \int_0^{2\pi} \log \tilde{w} \, d\theta' \, dx'$$

$$\leq \tilde{C}_1 + \tilde{C}_2 |x|$$
where we have used that $\int_0^{2\pi} \log \tilde{w} \, d\theta = C_1 + C_2 x$, by (4.18). This implies the bound
\[
f \geq -C_3 - C_4 |x|, \quad \text{for } |x| \geq 2\pi \quad (4.20)
\]
for some uniform constants $C_3, C_4$. Combining the (4.19) and (4.20) yields to the estimate
\[
|f| \leq A + B |x| \leq A + B r, \quad \text{for } r = |x| > 1
\]
where $r = \sqrt{x^2 + \theta^2}$ and $A, B$ are some uniform constants.

To finish our argument, we will need to classify all $2\pi$-periodic functions $f$ on $\mathbb{R}^2$ so that
\[
\Delta_c f = 0 \quad \text{and} \quad f = O(r), \quad \text{for } r >> 1.
\]
By the result of Li and Tam in [11], the space $\mathcal{H}_2$ of harmonic functions with linear growth in $\mathbb{R}^2$ is at most 3-dimensional. The harmonic polynomials $\{1, x, \theta\}$ are linearly independent, hence they form a basis of $\mathcal{H}_2$. It follows that
\[
f(x, \theta) = C_1 + C_2 x + C_3 \theta.
\]
Since $f$ is a $2\pi$-periodic function in $\theta$, we must have $C_3 = 0$ which implies that
\[
\tilde{w}(x, \theta) = Ae^{Bx}
\]
for some constants $A \geq 0$ and $B$. Since we have assumed that the function $\tilde{w}$ is not identically zero, we have $A > 0$.

By our previous estimates and the inequality $w_t \leq 0$, we have the bound
\[
C_1(K) \leq w(x, \theta, t) \leq C_2(K), \quad \text{for } |x| \leq K \text{ and all } t \leq t_0 < 0 \quad (4.21)
\]
on any compact subset $K$ of $\mathbb{R} \times [0, 2\pi]$. Hence, by the standard parabolic estimates $w(x, \theta, t)$ converges to $\tilde{w}(x, \theta)$, uniformly on compact subsets of $\mathbb{R} \times [0, 2\pi]$ in the $C^\infty$ norm. In particular, it follows that the metric $g(x, t) = w(x, t)^{-1} \, ds^2$ converges to the metric $\tilde{g}(x) = \tilde{w}(x)^{-1} \, ds^2$ in $C^\infty$ norm. Observe that the limiting metric
\[
\tilde{g}(x, \theta) = \tilde{w}^{-1} \, ds^2 = A^{-1} e^{-Bx} \, ds^2
\]
is not complete unless $B = 0$.

On the other hand, (4.21) implies the bound
\[
inj_{rad}(g(\cdot, t)) \geq \delta > 0, \quad \text{for all } t \leq t_0 < 0,
\]
where the \( \text{injrad}_0(g(\cdot, t)) \) is the injectivity radius at \( x = 0 \) with respect to the metric \( g(\cdot, t) \). Since \( 0 < R(\cdot, t) \leq C \) for all \( t \leq t_0 < 0 \), by Hamilton’s compactness theorem for the Ricci flow (c.f. in [9]), there exists a subsequence \( t_k \rightarrow -\infty \) so that \((\mathbb{R}^2, g(t_k), 0)\) converges as \( t_k \rightarrow -\infty \) to a complete metric \( g_\infty \) on \( \mathbb{R}^2 \). This contradicts the incompleteness of \( \bar{g} \) and implies that \( B = 0 \) unless \( A = 0 \). We conclude that \( \tilde{w}(x, t) = A \).

If we go back to a sphere \( S^2 \) via Mercator’s transformation, this means that
\[
\lim_{t \rightarrow -\infty} v(\psi, \theta, t) = C \cos^2 \psi
\]
for some constant \( C \geq 0 \). Moreover, the convergence is smooth on compact subsets of \( S^2 \setminus \{S, N\} \). This finishes the proof of our proposition. \( \square \)

5. The case when \( \tilde{v} \equiv 0 \)

Throughout this section we will assume that the backward limit
\[
\tilde{v} := \lim_{t \rightarrow -\infty} v(\cdot, t) \equiv 0.
\]
Our goal is to show that in this case the ancient solution \( v \) must be a family of contracting spheres, as stated in the following proposition.

**Proposition 5.1.** *If the backward limit \( \tilde{v} \equiv 0 \), then*
\[
v(\cdot, t) = \frac{1}{(-2t)}
\]
*that is, our ancient solution is a family of contracting spheres.*

To prove the proposition we will use an isoperimetric estimate for the Ricci flow which was proven by R. Hamilton in [8]. Let \( M \) be any compact surface. Any simple closed curve \( \gamma \) on \( M \) of length \( L(\gamma) \) divides the compact surface \( M \) into two regions with areas \( A_1(\gamma) \) and \( A_2(\gamma) \). We define the isoperimetric ratio as in [2], namely
\[
I = \frac{1}{4\pi} \inf_{\gamma} L^2(\gamma) \left( \frac{1}{A_1(\gamma)} + \frac{1}{A_2(\gamma)} \right).
\]
It is well known that \( I \leq 1 \) always, and that \( I \equiv 1 \) if and only if the surface \( M \) is a sphere.

We will briefly outline the proof of Proposition 5.1 whose steps will be proven in detail afterwards. We consider our evolving surfaces at each time \( t < 0 \), and define the isoperimetric ratio \( I(t) \) as above. Our goal is to show
that our assumption (5.1) implies that \( I(t) \equiv 1 \), which forces \((M, g(t))\) to be a family of contracting spheres. We will argue by contradiction and assume that \( I(t_0) < 1 \), for some \( t_0 < 0 \). In that case we will show that there exists a sequence \( t_k \to -\infty \) and closed curves \( \beta_k \) on \( S^2 \) so that simultaneously we have

\[
L_{S^2}(\beta_k) \geq \delta > 0 \quad \text{and} \quad L_{g(t_k)}(\beta_k) \leq C, \quad \forall k
\]

(5.3)

where \( L_{S^2} \) and \( L_{g(t_k)} \) denote the length of a curve in the round metric on \( S^2 \) and in the metric \( g(t_k) \), respectively. This clearly contradicts the fact that \( u(\cdot, t_k) \to \infty \), uniformly in \( S^2 \) (implied by (5.1)) and finishes the proof.

We will now outline how we will find the curves \( \beta_k \). For each \( t < t_0 \), let \( \gamma_t \) be a curve for which the isoperimetric ratio \( I(t) \) is achieved.

i. If \( I(t_0) < 1 \), for some \( t_0 < 0 \), we will show that \( I(t) \leq C |t| \), for \( t < t_0 \). We will use that to show \( L_{g(t)}(\gamma_t) \leq C \), for all \( t < t_0 \).

ii. For any sequence \( t_k \to -\infty \) and \( p_k \in \gamma_{t_k} \), we will show that there exists a subsequence such that \((M, g(t_k), p_k)\) converges to \((M_\infty, g_\infty, p_\infty)\), where \( M_\infty = S^1 \times \mathbb{R} \) and \( \gamma_\infty := \lim_{k \to -\infty} \gamma_k \) is a closed geodesic on \( M_\infty \), one of the cross circles of \( S^1 \times \mathbb{R} \).

iii. Let \( t_k \) be as above. If \( A_1(t_k), A_2(t_k) \) are the areas of the two regions into which \( \gamma_{t_k} \) divides \( S^2 \), we show that both of them are comparable to \( \tau_k = -t_k \).

iv. We show that the maximal distances from \( \gamma_k \) to the points of the two regions of areas \( A_1(t_k), A_2(t_k) \) respectively are both of length comparable to \( \tau_k \).

v. The curves \( \gamma_k \) do not necessarily satisfy (5.3). However, we use them and (ii) to define a foliation \( \{\beta_k\} \) of our surfaces \((M, g(t_k))\) and we choose the curve \( \beta_k \) from this foliation that splits \( S^2 \) into two parts of equal areas with respect to the round metric. We prove that this is the curve that satisfies (5.3) by using that \( I_{S^2} = 1 \), the Bishop Gromov volume comparison principle, (iii) and (iv).

**Lemma 5.2.** If \( I(t_0) < 1 \), for some \( t_0 < 0 \), then there exist positive constants \( C_1, C_2 \) so that

\[
I(t) \leq \frac{C_1}{|t| + C_2}, \quad \text{for all} \ t < t_0.
\]
Moreover, if $\gamma_t$ is the curve at which the infimum in (5.2) is attained then,

$$L(t) := L(\gamma_t) \leq C, \quad \text{for all } t < 0.$$ 

Proof. Let $t < t_0$, with $t_0$ as in the statement of the lemma. It has been shown in [8] that

$$I'(t) \geq \frac{4\pi (A_1^2 + A_2^2)}{A_1 A_2 (A_1 + A_2)} I(1 - I^2).$$

Since $A_1 + A_2 = 8\pi |t|$ and $A_1^2 + A_2^2 \geq 2A_1 A_2$, we conclude the differential inequality

$$I'(t) \geq \frac{1}{|t|} I(1 - I^2).$$

Since $I(t_0) < 1$, the above inequality implies the bound

$$I(t) \leq \frac{C_1}{|t| + C_2}, \quad \text{for all } t < t_0$$

for uniform in time constants $C_1$ and $C_2$. Using that $\frac{1}{A_1} + \frac{1}{A_2} \geq \frac{1}{4\pi |t|}$, we will conclude that that the length $L(t)$ of a curve $\gamma_t$ at which the isoperimetric ratio is attained satisfies $L(t) \leq C$, for all $t < t_0$.

$\square$ 

We also have the following estimate from below on the length $L(t)$ of the curve at which the infimum in (5.2) is attained.

Lemma 5.3. There is a uniform constant $c > 0$, independent of time so that

$$L(t) \geq c, \quad \text{for all } t \leq t_0 < 0.$$ 

Proof. Recall that for $t_0 < 0$, the scalar curvature $R$ satisfies $0 < R(\cdot, t) \leq C$, for all $t \leq t_0$. The Klingenberg injectivity radius estimate for even dimensional manifolds implies the bound

$$\text{injrad}(g(t)) \geq \frac{c}{\sqrt{R_{\text{max}}}} \geq \delta > 0, \quad \text{for all } t \leq t_0 < 0$$

(5.4) 

for a uniform in time constant $c > 0$. We will prove the Lemma by contradiction. Assume that there is a sequence $t_i \to -\infty$, so that $L_i := L(t_i) \to 0$, as $i \to \infty$, and denote by $\gamma_{t_i}$ a curve at which the isoperimetric ratio is attained, i.e. $L(t_i) = L(\gamma_{t_i})$. 

}\end{document}
Define a new sequence of re-scaled Ricci flows, \( g_i(t) := L_i^{-2} g(t_i + L_i^2 t) \) and take a sequence of points \( p_i \in \gamma_{t_i} \). The bound (5.4) implies a lower bound on the injectivity radius at \( p_i \) with respect to metric \( g_i \), namely
\[
injrad_{g_i}(p_i) = \frac{\text{injrad}_{g(t_i)}(p_i)}{L_i^2} \geq \frac{\delta}{L_i^2} \to \infty, \quad \text{as } i \to \infty. \tag{5.5}
\]
Also, since \( R_i(\cdot, t) = L_i^2 R(\cdot, t_i + L_i^2 t) \leq C L_i^2 \) and \( L_i \to 0 \), we get
\[
\max R_i(\cdot, t) \to 0, \quad \text{as } i \to \infty. \tag{5.6}
\]
Hamilton’s compactness theorem (c.f. in [9]) implies, passing to a subsequence, the pointed smooth convergence of \((M, g_i(0), p_i)\) to a complete manifold \((M_\infty, g_\infty, p_\infty)\), which is, due to (5.5) and (5.6), a standard plane. Moreover,
\[
I(t_i) = L_i^2 \left( \frac{1}{A_1(t_i)} + \frac{1}{A_2(t_i)} \right) = \frac{1}{A_1(g_i(0))} + \frac{1}{A_2(g_i(0))}
\]
where \( A_1(g_i(0)) \) and \( A_2(g_i(0)) \) are the areas inside and outside the curve \( \gamma_{t_i} \), respectively, both computed with respect to metric \( g_i(0) \). Since \( g_i(0) \) converges to the euclidean metric and \( \gamma_{t_i} \) converges to a curve of length 1, it follows that \( \lim_{i \to \infty} A_1(g_i(0)) = \alpha > 0 \) and \( \lim_{i \to \infty} A_2(g_i(0)) = \infty \), which implies that
\[
\lim_{i \to \infty} I(t_i) \geq \delta > 0
\]
and obviously contradicts Lemma 5.2.

We recall that each time \( t \), a curve \( \gamma_t \) at which the isoperimetric ratio is achieved splits the surface into two regions of areas \( A_1(t) \) and \( A_2(t) \). Lemma 5.3 yields to the following conclusion.

**Corollary 5.4.** There are uniform constants \( c > 0 \) and \( C > 0 \) so that
\[
c |t| \leq A_1(t) \leq C |t| \quad \text{and} \quad c |t| \leq A_2(t) \leq C |t|
\]
for all \( t < t_0 < 0 \).

**Proof.** It is well known that the total area of our evolving surface is \( A(t) = 8\pi |t| \). Hence, \( A_1(t) \leq 8\pi |t| \) and \( A_2(t) \leq 8\pi |t| \). On the other hand, by lemmas 5.2 and 5.3 we have
\[
\frac{c}{A_j(t)} \leq \frac{L^2(t)}{A_j(t)} \leq I(t) \leq \frac{C}{|t|}, \quad j = 1, 2
\]
for all \( t < t_0 \), which shows that \( A_j(t) \geq c |t|, j = 1, 2 \), for a uniform constant \( c > 0 \), therefore proving the corollary. \( \square \)

We will fix in the sequel a sequence \( t_k \to -\infty \). Let \( \gamma_k \) be, as before, a curve at which the isoperimetric ratio is achieved. From now on we will refer to \( \gamma_k \) as an isoperimetric curve at time \( t_k \). To simplify the notation, we will set \( A_1 := A_1(t_k), A_2 := A_2(t_k) \) and \( L_k = L(t_k) \). It follows from Corollary 5.4 that
\[
\lim_{k \to \infty} A_1 = +\infty \quad \text{and} \quad \lim_{k \to \infty} A_2 = +\infty. \quad (5.7)
\]

Pick a sequence of points \( p_k \in \gamma_k \) and look at the pointed sequence of solutions \((M, g(t_k + t), p_k)\). Since the curvature is uniformly bounded and since the injectivity radius at \( p_k \) is uniformly bounded from below, by Hamilton’s compactness theorem we can find a subsequence of pointed solutions that converge, in the Cheeger-Gromov sense, to a complete smooth solution \((M_\infty, g_\infty, p_\infty)\). This means that for every compact set \( K \subset M_\infty \) there are compact sets \( K_k \subset M \) and diffeomorphisms \( \phi_k : K \to K_k \) so that \( \phi_k^* g(t_k) \) converges to \( g_\infty \). From Lemma 5.2, \( L(t_k) \leq C \), for all \( k \), and therefore our curves \( \gamma_k \) converge to a curve \( \gamma_\infty \) (this convergence is induced by the manifold convergence) which by \( (5.7) \) has the property that it splits \( M_\infty \) into two parts (call them \( M_{1\infty} \) and \( M_{2\infty} \)), each of which has infinite area. It follows that we can choose points \( x_j \in M_{1\infty} \) and \( y_j \in M_{2\infty} \) so that \( \text{dist}_{g_\infty}(x_j, p_\infty) = \text{dist}_{g_\infty}(p_\infty, y_j) = \rho_j \), where \( \rho_j \) is an arbitrary sequence so that \( \rho_j \to \infty \). Since \((M_\infty, g_\infty)\) is complete, there exists a minimal geodesic \( \beta_j \) from \( x_j \) to \( y_j \). This geodesic \( \beta_j \) intersects \( \gamma_\infty \) at some point \( q_j \). Since \( q_j \in \gamma_\infty \) and \( \gamma_\infty \) is a closed curve of finite length, the set \( \{q_j\} \) is compact and therefore there is a subsequence so that \( q_j \to q_\infty \in \gamma_\infty \). This implies that there is a subsequence of geodesics \( \{\beta_j\} \) so that, as \( j \to \infty \), it converges to a minimal geodesic \( \beta_\infty : (-\infty, \infty) \to M_\infty \) (minimal geodesic means a globally distance minimizing geodesic). It follows that our limiting manifold \( M_\infty \) contains a straight line. Since the curvature of \( M_\infty \) is zero, by the splitting theorem our manifold splits off a line and therefore is diffeomorphic to the cylinder \( S^1 \times \mathbb{R} \).

We next observe that the limiting curve \( \gamma_\infty \) is a geodesic, as shown in the following lemma.

**Lemma 5.5.** The geodesic curvature \( \kappa \) of the curve \( \gamma_\infty \) is zero.
Proof. As in \cite{[8]}, at each time $t < t_0 < 0$ we start with the isoperimetric curve $\gamma_t$ and we construct the one-parameter family of parallel curves $\gamma_t^r$ at distance $r$ from $\gamma_t$ on either side. We take $r > 0$ when the curve moves from the region of area $A_1(t)$ to the region of area $A_2(t)$, and $r < 0$ when it moves the other way. We then regard $L, A_1, A_2$ and

$$I = I(\gamma_t^r) = L^2(\gamma_t^r) \left( \frac{1}{A_1(\gamma_t^r)} + \frac{1}{A_2(\gamma_t^r)} \right)$$

as functions of $r$ and $t$. By the computation in \cite{[8]} we have

$$\frac{\partial A_1}{\partial r} = L, \quad \frac{\partial A_2}{\partial r} = -L, \quad \frac{dL}{dr} = \int \kappa ds = \kappa L$$

where $\kappa$ is the geodesic curvature of the curve $\gamma_t^r$. By a standard variational argument $\kappa$ is constant on $\gamma_t$. If $A := A_1 + A_2$ is the total surface area, we have

$$\log I = 2 \log L + \log A - \log A_1 - \log A_2.$$ 

Since $\frac{\partial I}{\partial r} |_{r=0} = 0$, we conclude that

$$0 = 2 \frac{\partial L}{L} + \frac{1}{A} \frac{\partial A}{\partial r} - \frac{1}{A_1} \frac{\partial A_1}{\partial r} - \frac{1}{A_2} \frac{\partial A_2}{\partial r} = 2 \kappa L - \frac{1}{A_1} L + \frac{1}{A_2} L$$

which leads to

$$\kappa = \frac{L}{2} \left( \frac{1}{A_1} - \frac{1}{A_2} \right).$$

By lemmas \ref{5.2} and \ref{5.3} and \eqref{5.7} we conclude that

$$\kappa_\infty := \lim_{t \to -\infty} \kappa = \lim_{t \to -\infty} \frac{L}{2} \left( \frac{1}{A_1} - \frac{1}{A_2} \right) = 0$$

that is the geodesic curvature $\kappa_\infty$ of the limiting curve $\gamma_\infty$ is zero. \hfill \Box

We have just shown that our limiting manifold is a cylinder $M_\infty = S^1 \times \mathbb{R}$ and $\gamma_\infty$ is a closed geodesic on $M_\infty$. Hence, $\gamma_\infty$ is one of the cross circles of $M_\infty$.

We have the following picture assuming that the radius of $\gamma_\infty$ is 1.
Assume that we have a foliation of our limiting cylinder $M_\infty$ by circles $\beta_w$, where $|w|$ is the distance from $\beta_w$ to $\gamma_\infty$, taking $w > 0$ if $\beta_w$ lies on the upper side of the cylinder and $w < 0$ if $\beta_w$ lies on its lower side. Denote by $\beta^k_w$ the curve on $M$ such that $\phi^k_\gamma \beta^k_w = \beta_w$.

One of the properties of the cylinder is that for every $\delta > 0$ there is a $w_0 > 0$ so that for every $|w| \geq w_0$ we have

$$|\sup_{x \in \beta_w, y \in \gamma_\infty} \text{dist}(x, y) - \inf_{x \in \beta_w, y \in \gamma_\infty} \text{dist}(x, y)| \leq \sqrt{w^2 + \pi^2} - w \leq \frac{C}{|w|} \leq \frac{C}{w_0} < \frac{\delta}{2}$$

where the distance is computed in the cylindrical metric on $M_\infty$.

Since for every sequence $t_k \to -\infty$, there exists a subsequence for which we have uniform convergence of our metrics $\{g(t_k)\}$ on bounded sets around the points $p_k \in \gamma_{t_k}$, the previous observation implies the following claim which will be used frequently from now on.

**Claim 5.6.** For every sequence $t_k \to -\infty$ and every $\delta > 0$ there exists $k_0$ and $w$ so that for $k \geq k_0$,

$$|\sup_{x \in \beta_w^{t_k}, y \in \gamma(t_k)} \text{dist}_{g(t_k)}(x, y) - \inf_{x \in \beta_w^{t_k}, y \in \gamma(t_k)} \text{dist}_{g(t_k)}(x, y)| < \delta.$$

The variant of the Bishop-Gromov volume comparison principle (since $R \geq 0$) implies the following area comparison of the annuli, for each $t < 0$,

$$\frac{\text{area}(b_1 \leq s \leq b_2)}{\text{area}(a_1 \leq s \leq a_2)} \leq \frac{b_2^2 - b_1^2}{a_2^2 - a_1^2} \quad (5.8)$$
where \(a_1 \leq a_2 \leq b_1 \leq b_2\) and \(s\) is the distance from a fixed point on \((M, g(t))\), computed with respect to the metric \(g(t)\). We are going to use this fact in the following lemma.

For each \(k\), \(\gamma_{tk}\) splits our manifold in two parts, call them \(M_{1k}\) and \(M_{2k}\) with areas \(A_{1k}\) and \(A_{2k}\) respectively. Choose points \(x_k \in M_{1k}\) and \(y_k \in M_{2k}\) so that \(\text{dist}_{g(tk)}(x_k, \gamma_{tk}) = \max_{z \in M_{1k}} \text{dist}_{g(tk)}(x_k, z) =: \rho_k\) and \(\text{dist}_{g(tk)}(y_k, \gamma_{tk}) = \max_{z \in M_{2k}} \text{dist}_{g(tk)}(y_k, z) =: \sigma_k\).

By the definition of \(\sigma_k\) and \(\rho_k\) and from the convergence of \((M, g(tk), p_k)\) to an infinite cylinder, we have
\[
\lim_{k \to \infty} \sigma_k = +\infty \quad \text{and} \quad \lim_{k \to \infty} \rho_k = +\infty.
\]

**Lemma 5.7.** There are uniform constants \(k_0 > 0\) and \(c > 0\) so that the
\[
\text{area} \left( B_{\rho_k}(x_k) \right) \geq c \rho_k \quad \text{and} \quad \text{area} \left( B_{\sigma_k}(y_k) \right) \geq c \sigma_k, \quad \text{for all } k \geq k_0
\]
where both the distance and the area are computed with respect to the metric \(g(tk)\).

**Proof.** We take \(a_1 = 0\), \(a_2 = b_1 = \sigma_k \geq 1\) and \(b_2 = \sigma_k + 1\) in (5.8). Then, if \(s\) is the distance from \(y_k\) computed with respect to \(g(tk)\), we have
\[
\frac{\text{area} \left( \sigma_k \leq s \leq \sigma_k + 1 \right)}{\text{area} \left( 0 \leq s \leq \sigma_k \right)} \leq \frac{(\sigma_k + 1)^2 - \sigma_k^2}{\sigma_k^2} \leq \frac{3}{\sigma_k^2}.
\]
Hence,
\[
\text{area} \left( B_{\sigma_k}(y_k) \right) \geq \frac{\sigma_k}{3} \text{area} \left( \sigma_k \leq s \leq \sigma_k + 1 \right).
\]

**Claim 5.8.** There are uniform constants \(c > 0\) and \(k_1\) so that for \(k \geq k_1\),
\[
\text{area} \left( \sigma_k \leq s \leq \sigma_k + 1 \right) \geq c.
\]

**Proof.** Denote by \(U_k := \{z \mid \sigma_k \leq s \leq \sigma_k + 1\}\). We consider the set
\[
V_k := \{z \mid \text{dist}_{tk}(z, \gamma_k) \leq \frac{1}{2}, \overline{y_k z} \cap \gamma_k \neq \emptyset\}
\]
where \(\overline{y_k z}\) denotes a geodesic connecting the points \(y_k\) and \(z\). It is enough to show that \(V_k \subset U_k\), for \(k\) sufficiently large, and that \(\text{area}(V_k) \geq c > 0\). To
prove that $V_k \subset U_k$, take $z \in V_k$ and let $w_k \in \gamma_k$ be such that $\text{dist}_{t_k}(z, w_k) = \text{dist}_{t_k}(z, \gamma_k) \leq \frac{1}{2}$. If $q_k := \gamma_k \cap \overline{yw}$, then
\[
\sigma_k = \text{dist}_{t_k}(y_k, \gamma_k) \leq \text{dist}_{t_k}(y_k, q_k) \leq \text{dist}_{t_k}(z, y_k)
\]
which implies that $\sigma_k \leq \text{dist}_{t_k}(z, y_k)$. On the other hand, by Claim 5.6 we have $\text{dist}_{t_k}(w_k, y_k) \leq \sigma_k + \frac{1}{2}$, for $k$ sufficiently large. Hence
\[
\text{dist}_{t_k}(z, y_k) \leq \text{dist}_{t_k}(y_k, w_k) + \text{dist}_{t_k}(w_k, z) \leq \sigma_k + \frac{1}{2} + \frac{1}{2} < \sigma_k + 1
\]
for $k$ sufficiently large. This proves that $V_k \subset U_k$ and hence
\[
\text{area}(U_k) \geq \text{area}(V_k).
\]
To estimate area $(V_k)$ from below, we recall that for $p_k \in \gamma_k$, we have pointed convergence of $(M, g(t_k), p_k)$ to a cylinder which is uniform on compact sets around $p_k$. To use this we need to show that there is a constant $C > 0$, for which
\[
V_k \subset B_{t_k}(p_k, C), \quad \text{for all } k \geq k_0.
\]
Let $z \in V_k$ and let $q_k \in \overline{yw} \cap \gamma_k$. Then by Claim 5.6 for $k$ sufficiently large, we have
\[
\sigma_k - 1 \leq \text{dist}_{g(t_k)}(y_k, q_k) \leq \sigma_k + 1. \tag{5.10}
\]
We also have
\[
\text{dist}_{g(t_k)}(p_k, z) \leq \text{dist}_{g(t_k)}(p_k, q_k) + \text{dist}_{g(t_k)}(q_k, z).
\]
Since, $z \in V_k \subset U_k$ and (5.11) holds, we get
\[
\text{dist}_{g(t_k)}(q_k, z) \leq \text{dist}_{g(t_k)}(y_k, z) - \text{dist}_{g(t_k)}(y_k, q_k) \leq \sigma_k + 1 - \sigma_k + 1 = 2
\]
which combined with $\text{dist}_{g(t_k)}(p_k, q_k) \leq L(\gamma_k) \leq C$ gives us the bound
\[
\text{dist}_{g(t_k)}(p_k, z) \leq C.
\]
This guarantees that, as $k \to \infty$, $V_k$ converges to a part of the cylinder $S^1 \times \mathbb{R}$, while $(M, g(t_k), p_k) \to (S^1 \times \mathbb{R}, g_\infty, p_\infty)$ and $g_\infty$ is the cylindrical metric. Recall that $\gamma_{t_k} \to \gamma_\infty$ and $\gamma_\infty$ is one of the cross circles on $S^1 \times \mathbb{R}$. It follows that $V_k$ converges as $k \to \infty$ to the upper or lower part of the set
\[
\{z \in S^1 \times \mathbb{R} \mid \text{dist}_{g_\infty}(z, \gamma_\infty) \leq \frac{1}{2}\}
\]
with respect to $\gamma_\infty$. This implies that
\[
c \leq \text{area}(\{\sigma_k \leq s \leq \sigma_k + 1\}) \leq C, \quad \text{for } k \geq k_0, \tag{5.11}
\]
for some uniform constants $c, C > 0$ finishing the proof of the claim. \qed
The claim together with (5.9) yield to the estimate
\[
\text{area (} B_{\sigma_k}(y_k) \text{)} \geq c \sigma_k
\]
when \( k \) is sufficiently large. Similarly, \( \text{area (} B_{\rho_k}(x_k) \text{)} \geq c \rho_k \), for \( k \) sufficiently large. This concludes the proof of the proposition. \( \square \)

Let us denote briefly by \( A_{\sigma_k} := \text{area (} B_{\sigma_k}(y_k) \text{)} \) and \( A_{\rho_k} := \text{area (} B_{\rho_k}(x_k) \text{)} \).

**Lemma 5.9.** There exist a number \( k_0 \) and constants \( c_1 > 0, c_2 > 0 \), so that
\[
c_1 \tau_k \leq A_{\rho_k} \leq c_2 \tau_k \quad \text{and} \quad c_1 \tau_k \leq A_{\sigma_k} \leq c_2 \tau_k,
\]
for all \( k \geq k_0 \) where \( \tau_k = -t_k \).

**Proof.** Notice that
\[
A_{\rho_k} + A_{\sigma_k} \leq 2A(t_k) = 16\pi \tau_k \quad (5.12)
\]
since \( A(t_k) = 8\pi \tau_k \) is the total surface area. Hence,
\[
A_{\rho_k} \leq C \tau_k \quad \text{and} \quad A_{\sigma_k} \leq C \tau_k. \quad (5.13)
\]
To establish the bounds from below, we will use Lemma 5.7 and show that there is a uniform constant \( c \) so that
\[
\sigma_k \geq c \tau_k \quad \text{and} \quad \rho_k \geq c \tau_k, \quad \text{for all} \quad k \geq k_0. \quad (5.14)
\]

**Claim 5.10.** There are uniform constants \( c > 0 \) and \( C < \infty \), so that
\[
c \rho_k \leq \sigma_k \leq C \rho_k.
\]

**Proof.** Recall that \( \sigma_k = \text{dist}_g(t_k)(y_k, \gamma_k) \). By our choice of points \( x_k, y_k \) and the figure we have that the diam \( (S^2, g(t_k)) \leq \sigma_k + \rho_k + 1 \) for \( k \geq k_0 \), sufficiently large. We also have that the subset of \( S^2 \) that corresponds to area \( A_1(t_k) \) contains a ball \( B_{\sigma_k}(y_k) \). By Corollary 5.4 and the comparison inequality (5.8), we have
\[
c \leq \frac{A_2(t_k)}{A_1(t_k)} \leq \frac{\text{area}(B_{\rho_k + \sigma_k + 1}(y_k) \setminus B_{\sigma_k}(y_k))}{\text{area}(B_{\sigma_k}(y_k))}
\]
\[
= \frac{\text{area}(\sigma_k \leq s \leq \rho_k + \sigma_k + 1)}{\text{area}(0 \leq s \leq \sigma_k)} \leq \frac{(\rho_k + \sigma_k + 1)^2 - \sigma_k^2}{\sigma_k^2}.
\]
Using the previous inequality we obtain the bound
\[
c \sigma_k^2 \leq 2 \rho_k - 2 \sigma_k^2 - 2 \rho_k \sigma_k - 1 \leq \rho_k^2.
\]
If there is a uniform constant $c$ so that $\sigma_k \leq c \rho_k$ we are done. If not, then $\rho_k << \sigma_k$ for $k >> 1$, and from the inequality above we get

$$\frac{c}{2} \sigma_k^2 \leq \rho_k^2,$$

for $k >> 1$.

In any case there are $k_1$ and $C_1 > 0$ so that

$$\rho_k \leq C_1 \sigma_k,$$

for $k \geq k_1$. (5.15)

By a similar analysis as above there are $k_2 \geq k_1$ and $C_2 > 0$ such that

$$\sigma_k \leq C_2 \rho_k,$$

for $k \geq k_1$. (5.16)

We will now conclude the proof of Lemma 5.9. By Lemma 5.7 and (5.12) it follows that

$$\rho_k + \sigma_k \leq C \tau_k,$$

for $k >> 1$.

By (5.15) and (5.16) it follows that

$$\rho_k \leq C \tau_k \quad \text{and} \quad \sigma_k \leq C \tau_k,$$

for $k >> 1$.

Moreover, by (5.8), we have

$$\frac{A_2(t_k)}{\text{area}(\sigma_k - 1 \leq s \leq \sigma_k)} \leq \frac{\text{area}(B_{\rho_k + \sigma_k + 1}(y_k) \setminus B_{\sigma_k}(y_k))}{\text{area}(\sigma_k - 1 \leq s \leq \sigma_k)} \leq \frac{(\rho_k + \sigma_k + 1)^2 - \sigma_k^2}{\sigma_k^2 - (\sigma_k - 1)^2} \leq \frac{(\rho_k + \sigma_k + 1)^2 - \sigma_k^2}{2\sigma_k - 1} \leq C \rho_k$$

(5.17)

where we have used (5.15) and (5.16). The same analysis that yielded to (5.11) can be applied again to conclude that

$$\text{area}(\sigma_k - 1 \leq s \leq \sigma_k) \leq C.$$

This together with Corollary 5.4 and (5.17) imply

$$\rho_k \geq c \tau_k,$$

for $k \geq k_0$.

Claim 5.10 implies the same conclusion about $\sigma_k$. This is sufficient to conclude the proof of Lemma 5.9 as we have explained at the beginning of it.

We will now finish the proof of Proposition 5.1.
Proof of Proposition 5.1. If the isoperimetric constant \( I(t) \equiv 1 \), it follows by a well known result that our solution is a family of contracting spheres. Hence we will assume that \( I(t_0) < 1 \), for some \( t_0 < 0 \), which implies all the results in this section are applicable. We will show that this contradicts the fact that \( \lim_{t \to -\infty} v(\cdot, t) = 0 \), uniformly on \( S^2 \).

As explained at the beginning of this section, it suffices to find positive constants \( \delta, C \) and curves \( \beta_k \), so that
\[
L_{S^2}(\beta_k) \geq \delta > 0 \quad \text{and} \quad L_k(\beta_k) \leq C < \infty \tag{5.18}
\]
where \( L_{S^2} \) denotes the length of a curve computed in the round spherical metric and \( L_k \) denotes the length of a curve computed in the metric \( g(t_k) \).

If we manage to find those curves \( \beta_k \) that would imply
\[
C \geq L_k(\beta_k) = \int_{\beta_k} \sqrt{u(t_k)} \, ds \geq M L_{S^2}(\beta_k) \geq M \delta, \quad \text{for } k \geq k_0
\]
where \( M > 0 \) is an arbitrary big constant and \( k_0 \) is sufficiently large so that \( \sqrt{u(t_k)} \geq M \), for \( k \geq k_0 \), uniformly on \( S^2 \) (which is justified by the fact \( v(\cdot, t) \) converges uniformly to zero on \( S^2 \), in \( C^{1,\alpha} \) norm). The last estimate is impossible, when \( M \) is taken larger than \( C/\delta \), hence finishing the proof of our proposition.

We will now prove (5.18). Our isoperimetric curves \( \gamma_{t_k} \) have the property that \( L_k(\gamma_{t_k}) \leq C \) for all \( k \), but we do not know whether \( L_{S^2}(\gamma_{t_k}) \geq \delta > 0 \), uniformly in \( k \). For each \( k \), we will choose the curve \( \beta_k \) which will satisfy (5.18), from a constructed family of curves \( \{\beta_k^\alpha\} \) that foliate our solution \((M, g(t_k))\). Define the foliation of \((M, g(t_k))\) by the curves \( \{\beta_k^\alpha\} \) so that for every \( \alpha \) and every \( x \in \beta_k^\alpha \), \( \text{dist}_{t_k}(x, y_k) = \alpha \). Choose a curve \( \beta_k \) from that foliation so that the corresponding curve \( \tilde{\beta}_k \) on \( S^2 \) splits \( S^2 \) in two parts of equal areas, where the area is computed with respect to the round metric.

Since the isoperimetric constant for the sphere \( I_{S^2} = 1 \), that is
\[
1 \leq L_{S^2}(\tilde{\beta}_k) \left( \frac{1}{A_1} + \frac{1}{A_2} \right) = L_{S^2}(\tilde{\beta}_k) \frac{4}{A_{S^2}}
\]
we have
\[
L_{S^2}(\tilde{\beta}_k) \geq \delta > 0, \quad \text{for all } k.
\]

To finish the proof of the proposition we will now show that there exists a uniform constant \( C \) so that \( L_k(\beta_k) \leq C \), for all \( k \).
To this end, we observe first that the area element of $g(t_k)$, when computed in polar coordinates, is

$$da_k = J_k(r, \theta) \, r \, dr \, d\theta$$

where $J_k(r, \theta)$ is the Jacobian and $r$ is the radial distance from $y_k$. The length of $\beta^k_r$ is given by

$$L^r_k = \int_0^{2\pi} J_k(r, \theta) \, r \, d\theta$$

which implies that

$$\frac{L^r_k}{r} = \int_0^{2\pi} J_k(r, \theta) \, d\theta.$$ 

By the Jacobian comparison theorem, for each fixed $\theta$, we have

$$\frac{J'_k(r, \theta)}{J_k(r, \theta)} \leq \frac{J'_a(r, \theta)}{J_a(r, \theta)}$$

(5.19)

where the derivative is in the $r$ direction and $J_a(r, \theta)$ denotes the Jacobian for the model space and $a$ refers to a lower bound on Ricci curvature (the model space is a simply connected space of constant sectional curvature equal to $a$). In our case $a = 0$ (since $R \geq 0$) and the model space is the euclidean plane, which implies that the right hand side of (5.19) is zero and therefore $J_k(r, \theta)$ decreases in $r$. Hence $L^r_k/r$ decreases in $r$.

In the proof of Lemma 5.9 we showed that there are uniform constants $C_1, C_2$ so that

$$C_1 \tau_k \leq \rho_k \leq C_2 \tau_k \quad \text{and} \quad c_1 \tau_k \leq \sigma_k \leq C_2 \tau_k.$$ 

We have shown that $\gamma_{t_k} \to \gamma_\infty$ and $\gamma_\infty$ is a circle in $M_\infty$. Let $y_k, p_k$ be the points which we have chosen previously. We may assume that $\text{dist}_{t_k}(y_k, p_k) = \sigma_k$. Choose a curve $\tilde{\gamma}_k \in M$ so that $p_k \in \tilde{\gamma}_k$ and that for every $x \in \tilde{\gamma}_k$ we have the $\text{dist}_{t_k}(y_k, x) = \sigma_k$. Observe that for every $x \in \gamma_k$, by the figure we have

$$\sigma_k \leq \text{dist}_{t_k}(x, y_k) \leq \sigma_k + \frac{C}{\sigma_k},$$

for sufficiently big $k$. For $x \in \gamma_k$ let $z = \tilde{\gamma}_k \cap \overline{y_k x}$. Then

$$\text{dist}_{t_k}(x, \tilde{\gamma}_k) \leq \text{dist}_{t_k}(x, z) \leq \text{dist}_{t_k}(y_k, x) - \text{dist}(y_k, z) \leq \sigma_k + \frac{C}{\sigma_k} - \sigma_k = \frac{C}{\sigma_k}.$$ 

This implies that the curves $\tilde{\gamma}_k$ converge to $\gamma_\infty$ as $k \to \infty$. Moreover, this also implies the curve $\tilde{\gamma}_k$ is at distance $\sigma_k = O(\tau_k)$ from $y_k$ and if $s_k = \text{dist}_{t_k}(\beta_k, y_k)$, then $s_k = O(\tau_k)$ and we also know $L_k(\tilde{\gamma}_k) \leq C$, for all
We may assume $s_k \geq \sigma_k$ for infinitely many $k$, otherwise we can consider point $x_k$ instead of $y_k$ and do the same analysis as above but with respect to $x_k$. Since $J_k(r, \theta)$ decreases in $r$ we have

$$\frac{L^k_{s_k}}{s_k} \leq \frac{L^k_{\sigma_k}}{\sigma_k}$$

that is

$$L_k(\beta_k) = L^k_{s_k} \leq \frac{s_k}{\sigma_k} L^k_{\sigma_k} = \frac{s_k}{\sigma_k} L_k(\bar{\gamma}_k) \leq C,$$

finishing the proof of (5.18) and the proposition.

\[\square\]

6. Radial symmetry of an ancient compact solution

Let $g(\cdot, t)$ be a compact ancient solution to the Ricci flow in two dimensions which becomes extinct at time $t = 0$. We denote, as in the previous sections, by $\psi, \theta$ the global coordinates on $S^2$ by which we parametrize equation (1.8) (they are chosen as in Theorem 4.1, so that the zeros of the backward limit $\bar{\gamma}$ correspond to the poles of $S^2$). We will show in this section that our ancient solution needs to be radially symmetric with respect to those coordinates.

Assuming that $g = u ds^2_p$, where $ds^2_p$ is the spherical metric on the limiting sphere, we have shown in section 4 that the pressure $v = u^{-1}$ satisfies

$$\lim_{t \to -\infty} v(\psi, \theta, t) = \bar{v}(\psi) := C \cos^2 \psi$$

for $C \geq 0$, which in particular shows that the backward limit is radially symmetric, that is,

$$\bar{v}_\theta = 0.$$

Since we have discussed the case $C = 0$ in the previous section, we will assume throughout this section that $C > 0$.

**Theorem 6.1.** The solution $g(\cdot, t) = u(\cdot, t) ds^2_p$, where $ds^2_p$ denotes the spherical metric, is radially symmetric, that is, $u_\theta(\cdot, t) \equiv 0$ for all $t < 0$.

Before we continue, let us recall our notation. We denoted by $u(\psi, \theta, t)$ the conformal factor of our evolving metric on $S^2$, by $U(x, \theta, t)$ the conformal
factor in cylindrical coordinates, by \( v(\psi, \theta, t) = \frac{1}{u(\psi, \theta, t)} \) the pressure function on \( S^2 \) and by \( w(x, \theta, t) = \frac{1}{V(x, \theta, t)} \) the pressure function in cylindrical coordinates. We recall that we have

\[
v(\psi, \theta, t) = w(x, \theta, t) \cos^2 \psi, \quad \text{with} \quad \cos \psi = (\cosh x)^{-1}.
\]

Let us also denote by \( \bar{u}(r, \theta, t) \) the conformal factor of the metric in polar coordinates on \( \mathbb{R}^2 \). The relation is as follows:

\[
U(x, \theta, t) = r^2 \bar{u}(r, \theta, t), \quad \text{with} \quad x = \log r.
\]

It easily follows that \( \bar{u} \) satisfies the equation

\[
\bar{u}_t = \Delta \log \bar{u}, \quad \text{on} \quad \mathbb{R}^2 \times (-\infty, 0)
\]

where \( \Delta \) is the Laplacian on \( \mathbb{R}^2 \) expressed in polar coordinates. If we differentiate the previous equation in \( \theta \) we get

\[
(\bar{u}_\theta)_t = \Delta (\log \bar{u})_\theta = \Delta \left( \frac{\bar{u}_\theta}{\bar{u}} \right).
\] (6.1)

Setting

\[
I(t) := \int_{\mathbb{R}^2} |\bar{u}_\theta| = \int_{\mathbb{R} \times [0, 2\pi]} |u_\theta| dx d\theta
\]

we will show in the sequel that \( I(t) \) is well defined and decreasing in time. Moreover, we will show that since the backward limit \( \bar{v} \) is radially symmetric, we have

\[
\lim_{t \to -\infty} I(t) = 0.
\]

Since \( I(t) \geq 0 \) we will conclude that \( I(t) \equiv 0 \), which will prove that \( u_\theta \equiv 0 \), therefore \( u \) is radially symmetric.

**Proposition 6.2.** For every \( t < 0 \), we have

\[
\int_{\mathbb{R}^2} |\bar{u}_\theta| \leq C(t).
\]

Moreover,

\[
\frac{d}{dt} \int_{\mathbb{R}^2} |\bar{u}_\theta| \leq 0.
\]

**Proof.** We recall Kato’s inequality, namely \( \Delta f^+ \geq \chi_{f > 0} \Delta f \), in the distributional sense. Writing \( |f| = f^+ + f^- \), it follows that \( \Delta(|f|) \geq \text{sign} f \Delta f \).

Applying this inequality to the function \( \bar{u}_\theta/u \) gives us the inequality

\[
\Delta \left( \frac{|\bar{u}_\theta|}{\bar{u}} \right) \geq \text{sign}(\bar{u}_\theta) \Delta \left( \frac{\bar{u}_\theta}{\bar{u}} \right).
\]
Multiplying (6.1) by \( \text{sign}(\bar{u}_\theta) \) and using the previous inequality we obtain

\[
|\bar{u}_\theta|_t \leq \Delta\left(\frac{|\bar{u}_\theta|}{\bar{u}}\right).
\] (6.2)

**Claim 6.3.** For every \( t \leq t_0 \) there is a constant \( C(t) \), so that

\[
\int_{\mathbb{R}^2} |\bar{u}_\theta| \leq C(t).
\]

**Proof.** Since \( w(x, \theta, t) = v(\psi, \theta, t) \cosh^2 x \) and since \( v \) is a positive function on \( S^2 \), for every time \( t \) there is a constant \( c(t) \) so that

\[
w(x, \theta, t) \geq c(t) \cosh^2 x.
\] (6.3)

Moreover,

\[
|w_\theta| = |v_\theta| \cosh^2 x \leq C \cosh x.
\]

Hence, similarly as before, using also (6.3), (6.4) and the change of coordinates, \( x = \log r \), we obtain

\[
\int_{\mathbb{R}^2} |\bar{u}_\theta| = \int_{0}^{\infty} \int_{0}^{2\pi} |\bar{u}_\theta| r \, d\theta \, dr = \int_{-\infty}^{\infty} \int_{0}^{2\pi} \frac{|U_\theta|}{r^2} r \, d\theta \, dx
\]

\[
= \int_{\mathbb{R} \times [0,2\pi]} |U_\theta| \, dx \, d\theta = \int_{\mathbb{R} \times [0,2\pi]} \frac{|w_\theta|}{w^2} \, dx \, d\theta
\]

\[
\leq C(t) \int_{\mathbb{R} \times [0,2\pi]} \frac{\cosh^2 x}{\cosh^4 x} \, dx \, d\theta
\]

\[
= C(t) \int_{\mathbb{R} \times [0,2\pi]} \frac{dx \, d\theta}{\cosh^2 x} \leq \tilde{C}(t) < \infty
\]

which finishes the proof of the claim. \( \square \)

**Claim 6.4.** For every \( t \leq t_0 < 0 \) there is a constant \( C(t) < \infty \) so that

\[
\int_{\mathbb{R}^2} \frac{|\bar{u}_\theta|}{\bar{u}} \leq C(t).
\]

**Proof.** We begin by noticing that by Lemma 2.10, we have

\[
|w_\theta| = \frac{|v_\theta|}{\cos^2 \psi} \leq C \cos \psi = \frac{C}{\cosh x}.
\]

Hence, similarly as before, using also (6.3), we have

\[
\int_{\mathbb{R}^2} \frac{|\bar{u}_\theta|}{\bar{u}} = \int_{0}^{\infty} \int_{0}^{2\pi} \frac{|\bar{u}_\theta|}{\bar{u}} \, d\theta \, dr = \int_{-\infty}^{\infty} \int_{0}^{2\pi} \frac{|U_\theta|}{U} r \, d\theta \, dx
\]

\[
= \int_{-\infty}^{\infty} \int_{0}^{2\pi} \frac{|w_\theta|}{w} e^{2x} \, d\theta \, dx \leq C \int_{\mathbb{R} \times [0,2\pi]} \frac{e^{2x}}{w \cosh x} \, d\theta \, dx
\]

\[
\leq C(t) \int_{\mathbb{R} \times [0,2\pi]} \frac{e^{2x}}{\cosh^3 x} \, d\theta \, dx \leq \tilde{C}(t) < \infty.
\]
We will now conclude the proof of our proposition. We multiply (6.2) by a cut off function $\eta$ so that $\text{supp} \eta \subset B_{2\rho} := \{(r,\theta)|\ r < 2\rho\}, \eta \equiv 1$ for $r \leq \rho$, $|\Delta \eta| \leq C/\rho^2$ and then integrate the equation over $\mathbb{R}^2$ to get

$$\frac{d}{dt} \int_{\mathbb{R}^2} |\bar{u}_\theta| \eta \leq \int_{\mathbb{R}^2} \Delta (|\bar{u}_\theta|/\bar{u}) \eta = \int_{\mathbb{R}^2} \frac{|\bar{u}_\theta|}{\bar{u}} \Delta \eta \leq \frac{C}{\rho^2} \|\bar{u}_\theta \|_{L^1(\mathbb{R}^2)} \leq \frac{C(t)}{\rho^2}$$

where we have used Claim 6.4. If we let $\rho \to \infty$ in the previous estimate we get

$$\frac{d}{dt} \int_{\mathbb{R}^2} |\bar{u}_\theta| \leq 0.$$ 

□

Once we have Proposition 6.2 we can proceed with the proof of Theorem 6.1.

**Proof of Theorem 6.1** Proposition 6.2 implies that the integral

$$I(t) := \int_{\mathbb{R}^2} |\bar{u}_\theta| = \int_{\mathbb{R} \times [0,2\pi]} |U_\theta| \, dx \, d\theta$$

is decreasing in time. If we manage to show that the $\lim_{t \to -\infty} I(t) = 0$, since $I(t) \geq 0$, this will imply that $I(t) \equiv 0$, for all $t < 0$.

To this end, we recall that

$$\lim_{t \to -\infty} v(\psi,\theta, t) = C \cos^2 \psi, \ C > 0 \quad \text{and} \quad \lim_{t \to -\infty} v_\theta(\psi, \theta, t) = 0$$

which is equivalent to

$$\lim_{t \to -\infty} w(x, \theta, t) = C > 0 \quad \text{and} \quad \lim_{t \to -\infty} w_\theta(x, \theta, t) = 0.$$ 

Since $w_t \geq 0$, this means that $w(x, \theta, t) \geq C$ for all $t < 0$ and therefore using Lemma 2.10 we have

$$|U_\theta| = \frac{|v_\theta|}{w^2} \leq \frac{|v_\theta|}{C^2} \leq \frac{\bar{C}}{\cosh x} \cos \psi = \frac{\bar{C}}{\cosh x}$$ (6.5)

for all $t \leq t_0 < 0$ and with $\bar{C}$ a uniform, time independent positive constant. Since $(\cosh x)^{-1} \in L^1(\mathbb{R} \times [0,2\pi])$, by Lebesgue’s dominated convergence theorem we conclude that

$$\lim_{t \to -\infty} \int_{\mathbb{R} \times [0,2\pi]} |U_\theta| \, dx \, d\theta = \int_{\mathbb{R} \times [0,2\pi]} \lim_{t \to -\infty} |U_\theta| \, dx \, d\theta = 0$$
since \( \lim_{t \to -\infty} |U_\theta| = \lim_{t \to -\infty} \frac{|w_\theta|}{w} = 0 \). Hence, \( I(t) \equiv 0 \) for all \( t < 0 \) and therefore
\[
u_\theta(x, \theta, t) \equiv 0
\]
which implies that \( u(x, \theta, t) \) is independent of \( \theta \), that is, our solution \( u \) is radially symmetric. \( \square \)

7. **Classification of radial ancient solutions in the case \( \tilde{v} \neq 0 \)**

In the previous section we have shown that our ancient solution is radially symmetric with respect to the coordinates chosen in Theorem 4.1. In this section we will prove Theorem 1.2, assuming that the solution \( v \) of (1.1) is radial, hence \( v = v(\psi, t) \). Since the case \( \tilde{v} \equiv 0 \) has been discussed in section 5, we will assume \( \tilde{v} \neq 0 \), which by Proposition 4.1 means \( \tilde{v} = C \cos^2 \psi \), or equivalently \( \tilde{w} = \lim_{t \to -\infty} w(\cdot, t) = C \), for \( C > 0 \).

Let us briefly outline the main steps of the proof of Theorem 1.2.

i. We define the positive quantity \( F(x, t) = (w_{xxx} - 4w_x)^2 \) for which it turns out that the \( \sup_R F(\cdot, t) \) decreases in time. Our goal is to show that the \( \lim_{t \to -\infty} \sup_R F(\cdot, t) = 0 \), since this together with the monotonicity yields to that \( F \equiv 0 \). We know that \( F(\cdot, t) \) converges to zero pointwise, however we will need to control \( F(\cdot, t) \) near \( |x| \to \infty \) and uniformly in \( t \) to obtain the desired result.

ii. To obtain the results discussed above, we show the bound \( F(\cdot, t) \leq \frac{C}{\sqrt{-t}} \), for \( t \leq t_0 < 0 \) and a uniform constant \( C \). We do so by comparing \( F \) (which is a subsolution to a heat equation where the laplacian is computed with respect to the changing metric \( g(t) \)) with the fundamental solution to the heat equation on \( \mathbb{R}^2 \), centered at an arbitrary point \( x_0 \in \mathbb{R} \), namely with \( \phi(x, t) = \frac{1}{\sqrt{4\pi t}} e^{\frac{|x-x_0|^2}{4t}} \), for \( b \) chosen appropriately. We will do the comparison on the region
\[
S_\tau := \{(x, \theta) \, | \, |x - x_0| \leq b\sqrt{\tau \log \tau} \}, \quad \tau = -t.
\]

iii. In order to compare our subsolution to the fundamental solution, we need to show that our metric \( g = w^{-1} (dx^2 + d\theta^2) \) is uniformly equivalent to the cylindrical metric \( ds^2 = dx^2 + d\theta^2 \) in \( S_\tau \), where the comparison is being performed. We show that \( \sup_{S_\tau} R(\cdot, t) \) decays at a rate at most \( \tau^{-\alpha} \) for any \( \alpha \in (0, 1) \). We conclude the bound \( C_1 \leq w(\cdot, t) \leq C_2(x_0) \), which implies that all the derivatives of \( w \) decay nicely on \( S_\tau \), as \( \tau \to \infty \).
iv. The bound on $F$ which was discussed above implies that $\lim_{t \to -\infty} F(\cdot, t) = 0$, uniformly on $\mathbb{R}$. Hence $F(\cdot, t) \equiv 0$, for all $t < 0$, i.e. $w_{xx} - 4w \equiv 0$.

It is now straightforward to conclude that our condition $w$ must be a Rosenau solution.

We will next proceed to the detailed proof of Theorem 1.2. To define our quantity $F$, we will use cylindrical coordinates. To change to cylindrical coordinates we use as before Mercator’s projection (1.4) and denote by $w(x,t)$ the function defined by

$$w(x,t) = v(\psi,t) \cosh^2 x, \quad \cosh x = \sec \psi.$$  

The function $w(x,t)$ solves the equation

$$w_t = w w_{xx} - w_x^2, \quad \text{for } (x,t) \in \mathbb{R} \times (-\infty, 0). \quad (7.1)$$

Notice that the Rosenau solution given by (1.10), when expressed in cylindrical coordinates, it takes the form

$$w_R(x,t) = c(t) + d(t) \cosh 2x$$

for some functions of time $c(t)$ and $b(t)$. Define

$$Q(\cdot, t) := w_{xx}(\cdot, t) - 4w(\cdot, t)$$

and observe that on both, the contracting spheres and the Rosenau solution, $Q$ is just a function of time and therefore $Q_x(\cdot, t) \equiv 0$. This motivates our definition

$$F(\cdot, t) := Q_x^2(\cdot, t)$$

as our good monotone quantity. Indeed, we have:

**Lemma 7.1.** The function $F := Q_x^2$ satisfies the differential inequality

$$F_t \leq w F_{xx}. \quad (7.2)$$

**Proof.** We begin by differentiating (7.10) in $x$ once, to get

$$(w_x)_t = w w_{xxx} - w_x w_{xx}$$

and then once again, to also get

$$(w_{xx})_t = w w_{xxxx} - w_{xx}^2.$$ 

Hence

$$Q_t = w Q_{xx} - w_{xx}^2 + 4w_x^2.$$
Furthermore,
\[(Q_x)_t = w Q_{xxx} + w_x Q_{xx} - 2w_{xx} Q_x\]
which implies that
\[(Q^2_x)_t = w (Q^2_x)_{xx} - 2w Q^2_{xx} + 2w_x Q_x Q_{xx} - 4w_{xx} Q^2_x.\]
By the interpolation inequality and the fact that
\[w_{xx} = \frac{w^2_x + Rw}{w} \geq \frac{w^2}{w}\]
since \(R > 0\), we obtain
\[(Q^2_x)_t = w (Q^2_x)_{xx} - 2w Q^2_{xx} + 2\sqrt{w} Q_{xx} \left( \frac{w_x Q_x}{\sqrt{w}} \right) - 4w_{xx} Q^2_x \leq w (Q^2_x)_{xx} - 4w_{xx} Q^2_x \]
which proves (7.2). \(\square\)

Our goal is to utilize the maximum principle on (7.2) to conclude that \(F \equiv 0\). To this end we will need some a priori bounds on \(F\).

**Lemma 7.2.** The quantity \(F := Q^2_x\) is uniformly bounded in time, that is there exists a constant \(C\) such that
\[F(\cdot, t) \leq C, \quad \text{for all } t \leq t_0. \tag{7.4}\]
Moreover, at each time-slice \(t\), \(F\) is a smooth function with all the derivatives uniformly bounded in space.

**Proof.** We will prove the lemma by considering \(Q_x\) to be a function on the sphere, that is we define \(G(\psi, t)\) to be equal to \(Q_x(x, t)\) after the transformation
\[w(x, t) = v(\psi, t) \sec^2 \psi \quad \text{and} \quad \frac{d\psi}{dx} = \frac{1}{\cosh x} = \cos \psi.\]
Remembering that \(Q_x = w_{xxx} - 4w_x\), we compute
\[G = \left[ (v \sec^2 \psi)_{\psi} \cos \psi \right]_{\psi} \cos \psi - 4(v \sec^2 \psi)_{\psi} \cos \psi \]
\[= (\cos \psi + 2 \sec \psi + \sin \psi \tan \psi) v_{\psi} + 3 \sin \psi v_{\psi} + \cos \psi v_{\psi} \psi \psi (7.5)\]
We see from the above expression that
\[|G(\psi, t)| \leq C(\delta), \quad \text{for } \psi \in \left[ -\frac{\pi}{2} + \delta, \frac{\pi}{2} - \delta \right] \]
with \( C(\delta) \) independent of time. To also see that \( G(\psi, t) \) is bounded near the poles \( \psi = \pm \pi/2 \), we recall the bounds in Proposition 2.5 which readily imply that the first four terms on the right hand side of the last equation are bounded there, independently of time. To see that the last term is bounded as well, we observe that since \( v_t \geq 0 \), by Theorem 4.1 we have
\[
v(\psi, t) \geq \lim_{t \to -\infty} v(\psi, t) = C \cos^2 \psi
\]
for some constant \( C > 0 \) (the case \( C = 0 \) has been discussed in section 5). Hence, by Corollary 2.12 we obtain
\[
\cos \psi v_{\psi \psi} \leq \sqrt{\frac{\nu}{C}} v_{\psi \psi} \leq C.
\]
We conclude that \( G(\psi, t) \) is bounded near the poles as well, independently of time. This proves the bound in (7.4).

Also, from (7.5) it follows that \( F(\cdot, t) := Q^2_x(\cdot, t) \) has bounded higher order derivatives, at each time-slice \( t \).

By Lemma 7.2 the sup \( F(\cdot, t) \) is finite for every \( t \). From the evolution equation for \( F(\cdot, t) \) we immediately obtain the following lemma.

**Lemma 7.3.** The sup \( F(\cdot, t) \) decreases in time.

*Proof.* Using that \( d\psi/dx = \cos \psi \), we can consider (7.3) as an equation on \( S^2 \). Hence
\[
\frac{d}{dt} F \leq w [F_{\psi} \cos \psi]_{\psi} \cos \psi \leq w F_{\psi \psi} \cos^2 \psi - w F_{\psi} \sin \psi \cos \psi = w \cos^2 \psi (F_{\psi \psi} - F_{\psi} \tan \psi) = v \Delta_{S^2} F.
\]

We have seen in Lemma 7.2 that for each \( t \), \( F(\psi, t) \) is a smooth function on \( S^2 \). Hence, by the maximum principle applied to (7.6) we easily obtain that \( F_{\text{max}}(t) \) is decreasing in time, or equivalently, sup \( Q^2_x \) decreases in time. \( \square \)

By Theorem 4.1 \( w(\cdot, t) \) converges, as \( t \to -\infty \), uniformly on compact subsets of \( \mathbb{R} \), to a constant \( C \geq 0 \) which we assume to be positive. Since \( w_t \geq 0 \), we have \( w(\cdot, t) \geq C \), for all \( t \). It follows by standard parabolic estimates applied to (7.10), that the derivatives of \( w \) converge, as \( t \to -\infty \), uniformly on compact subsets of \( \mathbb{R} \) to zero. Hence, \( F \equiv 0 \) and \( F(\cdot, t) \xrightarrow{t \to -\infty} 0 \).
uniformly on compact subsets of $\mathbb{R}$. We would like to show that actually the convergence is uniform on $\mathbb{R}$. This will follow from the following proposition.

**Proposition 7.4.** We have

$$F(\cdot,t) \leq \frac{1}{\sqrt{-t}}, \quad \text{for } t \leq t_0 < 0. \quad (7.7)$$

To prove the Proposition we will compare $F(\cdot,t)$ with the fundamental solution to the heat equation centered at $x_0$, namely with $\phi(x,t) = \frac{1}{\sqrt{-t}} e^{\frac{|x-x_0|^2}{4t}}$. To do so we will first need some extra estimates, which will be shown in the next three lemmas. Fix $x_0 \in \mathbb{R}$ to be an arbitrary point.

**Lemma 7.5.** There exists a uniform constant in time $C(x_0)$, so that the scalar curvature $R(x_0,t)$ satisfies the estimate

$$R(x_0,t) \leq \frac{C(x_0)}{(-t)}, \quad \text{for } t \leq t_0 < 0.$$

**Proof.** Notice that

$$R(x_0,t) = R(\psi_0,t) = (\log v)_t(\psi_0,t)$$

where $x_0 \in \mathbb{R}$ and $\psi_0 \in (-\frac{x}{2}, \frac{x}{2})$ are related via Mercator’s transformation. Take a sequence of points $t_i = -2^i \leq t_0$. Then,

$$\int_{t_{i+1}}^{t_i} R(x_0,s) \, ds = \log v(\psi_0,t_i) - \log v(\psi_0,t_{i+1})$$

which, since $0 < c_1(x_0) \leq v(x_0,t) \leq c_2(x_0)$ for all $t < t_0$, implies the bound

$$R(x_0,\tau_i) \leq \frac{C(x_0)}{2^i}$$

for some $\tau_i \in (t_{i+1}, t_i)$. By the Harnack estimate for ancient solutions to the Ricci flow, $R_t \geq 0$. Using that property, for any $t \in [\tau_{i+1}, \tau_i]$, we have

$$R(x_0,t) \leq R(x_0,\tau_i) \leq \frac{C(x_0)}{2^i} \leq \frac{C(x_0)}{(-t)}.$$

\[\square\]

Let $x_0 \in \mathbb{R}$ be arbitrary and let $S_\tau$ be as before. Then we have the following lemma.

**Lemma 7.6.** For every $\alpha > 0$ there exist $b > 0$, independent of $x_0$ and $C(x_0) > 0$, so that

$$\sup_{S_\tau} R(\cdot,t) \leq \frac{C(x_0)}{\tau^\alpha}, \quad \text{for } t \leq t_0 < 0. \quad (7.8)$$
Proof. To simplify the notation, set \( x_\tau = x_0 \pm b \sqrt{\tau \log \tau} \). Let \( x \in S_\tau \). We will use the Harnack estimate for the curvature \( R \) shown in [6], which in our case implies the bound

\[
R(x, t) \leq C R(x_0, \tau^{t/2}) e^{\frac{2\log^2(x_0, x)}{\sqrt{\tau \log \tau}}}
\]

for a uniform constant \( C \). Using that \( w_t \geq 0 \) and \( \lim_{t \to -\infty} w(x, t) = C > 0 \), which implies \( w(x, t) \geq C > 0 \), we can estimate the distance at time \( t \), as follows:

\[
\text{dist}_t(x_0, x) \leq \int_{x_0}^{x} \frac{dx}{\sqrt{w(x, t)}} = \int C |x_\tau - x_0| \leq C b \tau \log \tau
\]

where the integral above is taken over a straight euclidean line connecting the points \( x_0 \) and \( x_\tau \). Using Lemma 7.5, we obtain the bound

\[
R(x_\tau, t) \leq \frac{C(x_0)}{\tau^{\frac{t}{2}}} C_2 b^2
\]

where \( C \) is a uniform constant, independent of time and \( x_0 \). Choosing \( b \) so that \( 1 - C_2 b^2 = \alpha \in (0, 1) \) yields to the lemma. \( \square \)

Remark 7.7. From the proof of Lemma 7.6, the estimate (7.8) holds also on a ball \( B_{g(t)}(x_0, b \tau \log \tau) \), where the ball is taken in metric \( g(t) \).

Let \( b \) be as in Lemma 7.6.

Lemma 7.8. For every \( x_0 \in \mathbb{R} \) there is a uniform constant \( C(x_0) \) so that

\[
w(x, t) \leq C(x_0), \quad \text{on } S_\tau
\]

for every \( t \leq t_0 < 0 \). Moreover, \( F \to 0 \) as \( t \to -\infty \) uniformly on \( S_\tau \).

Proof. We will use Shi’s local derivative estimates to prove the Lemma. For every \( t < t_0 \) consider \( \tilde{g}(x, s) = g(x, 2t + s) \) for \( s \in [0, -t] \) and \( x \in S_\tau \). By Lemma 7.6 we have

\[
\max_{[-2t, -t] \times S_\tau} \tilde{R}(x, s) \leq \frac{C}{\tau^{\alpha}}
\]

where \( C \) depends on \( x_0 \) but does not depend on \( \tau \) and \( \tilde{R} \) is the scalar curvature of \( \tilde{g} \). Since Shi’s estimates are local in nature, to get derivative estimates on all of \( S_\tau \) we can initially take slightly bigger \( b \) in a definition of \( S_\tau \) if necessary. From Remark 7.7 and by Shi’s local derivative estimates, there is a constant \( C_m \) so that for every \( (x, \theta) \in S_\tau \), we have
\[ \sup_{S_r} |\nabla^m \tilde{R}|_g^2 (\cdot, s) = \sup_{S_r} |\nabla^m R|^2_{g(s+2t)} (\cdot, s + 2t) \leq \frac{C_m}{s^{\alpha(2+m)}}, \]

for all \( s \in (0, -t] \). In particular, if \( s \in [-t/2, -t] \) this yields

\[ \sup_{S_r} |\nabla^m R|_{g(t')} (\cdot, t') \leq \frac{C}{(-t')^{\alpha(1+m/2)}}, \]

for all \( t' \in [-3t/2, -t] \), where \( C \) is a universal constant. In particular it holds for \( t' = t \). Note that the derivative and the norm in the previous estimate are computed with respect to the metric \( g(\cdot, t) \). Since \( c_1 \leq w(x, t) \) for all \( x \) and since the previous analysis has been done for arbitrary \( t \leq t_0 < 0 \), we get

\[ \sup_{S_r} |\nabla^m \ln w (x, t')| \leq \frac{C}{\tau^{\alpha(1+m/2)}} \]  

(7.11)

The number \( \alpha \) can be taken to be any positive number less than one, \( C \) is a universal constant depending only on \( x_0 \) which is the center of the set \( S_r \) and the derivative and the norm in (7.11) are taken with respect to the standard cylindrical metric on \( \mathbb{R}^2 \) (in our case of a rotationally symmetric solution \( \nabla \) is just a derivative with respect to the \( x \) coordinate).

We will now conclude the proof of the lemma. The equation \( w_t = R w \), yields to

\[ \frac{\partial}{\partial t} \ln w = R, \]

and therefore for \( \tau = -t \) implies

\[ \frac{\partial}{\partial \tau} \nabla^m \ln w = -\nabla^m R. \]

If we integrate this over \([\tau, \tau']\), using (7.11) we get

\[ |\nabla^m \ln w(x, \tau') - \nabla^m \ln w(x, \tau)| \leq \int_{\tau}^{\tau'} |\nabla^m R| ds \leq \int_{\tau}^{\tau'} \frac{C ds}{s^{\alpha(1+m/2)}} = \frac{C}{\alpha(m/2 + 1) - 1} \left( \frac{1}{\tau^{\alpha(m/2+1)-1}} - \frac{1}{\tau^{\alpha(m/2+1)-1}} \right) \]

(7.12)

for every \( x \in S_r \). We choose \( \alpha \in (\frac{1}{2}, 1) \) so that \( \alpha(m/2 + 1) > 1 \) for every \( m \geq 1 \). Letting \( \tau' \to \infty \) in (7.12) and using that the \( \lim_{\tau' \to \infty} \nabla^m \ln w(x, \tau') = 0 \) we get

\[ \sup_{S_r} |\nabla^m \ln w| \leq \frac{C_m}{\tau^{\alpha(m/2+1)-1}}. \]

(7.13)

This in particular implies that for every \( x \in S_r \), we have

\[ |\ln w(x, t) - \ln w(x_0, t)| \leq \sup_{S_r} |(\ln w)_x| |x - x_0| \leq \frac{C}{\tau^{\frac{1}{2}}} \sqrt{\tau \log \tau} \leq \tilde{C}. \]
and therefore
\[ \sup_{S_{\tau}} w(x, t) \leq C(x_0), \quad \text{for all } t \leq t_0 < 0. \]

Finally, by (7.13) and the previous estimate we obtain the bound
\[ \sup_{S_{\tau}} |\nabla^m w| \leq \frac{C_m}{\tau^{\alpha(m/2+1)-1}} \]
which implies \( F \to 0 \) uniformly as \( t \to -\infty \) on \( S_{\tau} \).

We will now finish the proof of Proposition 7.4.

**Proof of Proposition 7.4.** We will compare \( F(\cdot, t) \) with the fundamental solution to the heat equation centered at \( x_0 \). More precisely, we set
\[ \phi^\epsilon(x, t) = \frac{1}{\sqrt{-t}} e^{\frac{|x-x_0|^2}{b^2(-t)}} + \epsilon \]
where \( b \) is taken from Lemma 7.6. An easy computation shows that
\[ \phi^\epsilon_t = \frac{b^2}{4} \phi^\epsilon_{xx}. \]

Let \( C(x_0) \) be the constant defined in 7.10. By (7.10), since \( \phi^\epsilon_{xx} \geq 0 \), we have
\[ \frac{\partial \phi^\epsilon}{\partial t} = \frac{b^2}{4C(x_0)} C(x_0) \phi^\epsilon_{xx} \geq \bar{C}(x_0) w \phi^\epsilon_{xx} \]
for \( |x-x_0| \leq b\sqrt{\tau \log \tau} \), where \( \bar{C}(x_0) = \frac{b^2}{4C(x_0)} \). Define
\[ \tilde{F}(x, t) = F(x, \bar{C}(x_0) t). \]

It follows from (7.2) that \( \tilde{F} \) satisfies the differential inequality
\[ \frac{\partial \tilde{F}}{\partial t} \leq \bar{C}(x_0) w \tilde{F}_{xx}. \]

By Lemma 7.2 we have that \( \tilde{F} \leq C \), for a uniform constant \( C \). Hence
\[ \tilde{F}(x, t) \leq C = \phi^\epsilon(x, t), \quad \text{for } |x-x_0| = \frac{b}{\sqrt{2}} \sqrt{\tau \log \tau + 2 \log(C-\epsilon)}. \]

By Lemma 7.8, for every \( \epsilon > 0 \) there is a \( t_\epsilon \) so that for any \( t \) for which \( \bar{C}(x_0) t \leq t_\epsilon \), we have
\[ \tilde{F}(x, t) < \epsilon \leq \phi^\epsilon(x, t), \quad \text{on } S_{\tau} \]
and in particular (for \( \tau \) sufficiently large) on
\[ |x-x_0| \leq \frac{b}{\sqrt{2}} \sqrt{\tau \log \tau + 2 \log(C-\epsilon)}. \]
This together with (7.16), equations (7.14), (7.15) and the comparison principle yield to the inequality
\[ \tilde{F}(x, t) \leq \phi'(x, t) \]
for all \( |x - x_0| \leq b \sqrt{2} \sqrt{\log \tau + 2 \log(C - \varepsilon)} \) and all \( t \leq t_0 \). In particular, the estimate holds for \( x_0 \), that is, after passing to the limit \( \varepsilon \to 0 \), we have
\[ F(x_0, t) \leq \frac{\sqrt{C(x_0)}}{\sqrt{-t}}, \quad \text{for } t \leq \tilde{C}(x_0) t_0 \]
where \( \tilde{C}(x_0) = \frac{b^2}{C(x_0)} \). Recall that \( b \) does not depend on \( x_0 \) and that \( C(x_0) \) is the constant from Lemma 7.8 and therefore it can be taken bigger than 1. Hence, \( \tilde{C}(x_0) \leq A \), where \( A \) is a uniform constant, independent of \( x_0 \). We conclude that
\[ F(x_0, s) \leq \frac{\sqrt{A}}{\sqrt{-t}}, \quad \text{for } t \leq A t_0 \]
since \( A t_0 \leq \tilde{C}(x_0) t_0 \) (recall that \( t_0 < 0 \)). The estimate (7.7) now readily follows.

As an easy consequence of Lemma 7.3 and Proposition 7.4 we have the following Corollary.

**Corollary 7.9.** \( F(\cdot, t) \equiv 0 \), for all \( t < 0 \).

**Proof.** Proposition 7.4 implies that \( F(\cdot, t) \) converges to zero, as \( t \to -\infty \), uniformly on \( \mathbb{R} \). Hence
\[ \lim_{t \to -\infty} \sup F(\cdot, t) = \sup_{t \to -\infty} \lim F(\cdot, t) = 0. \]
Since \( F \geq 0 \), by Lemma 7.3 we conclude that
\[ F(\cdot, t) \equiv 0, \quad \text{for } t < 0. \]

**Proof of Theorem 1.2 - Radial Case.** Corollary 7.9 shows that
\[ w_{xx} - 4w = c(t), \quad \text{for every } t < 0. \]
Solving the above ODE gives us the solution
\[ w(x, t) = a(t) e^{2x} + b(t) e^{-2x} + d(t) \]
with \( d(t) = -c(t)/4 \). When we plug it in equation (7.11) we get the following system of ODE’s:

\[
\begin{align*}
    a'(t) &= 4a(t)d(t) \\
    b'(t) &= 4b(t)d(t) \\
    d'(t) &= 16a(t)b(t).
\end{align*}
\]

Since \( w(x, t) > 0 \), we obviously have \( a(t) > 0 \) and \( b(t) > 0 \). Hence, the first two equations imply that

\[
(\log a(t))' = (\log b(t))'
\]

which shows that

\[
b(t) = \lambda^2 a(t)
\]

for a constant \( \lambda > 0 \). Since \( b(t) = \lambda^2 a(t) \), we may express \( w(x, t) \) as

\[
w(x, t) = \lambda \left( \frac{a(t)}{\lambda} e^{2x} + \lambda a(t) e^{-2x} \right) + d(t) \\
= \lambda a(t) \left( e^{2(x-x_0)} + e^{-2(x-x_0)} \right) + d(t) \\
= 2\lambda a(t) \cosh(2(x-x_0)) + d(t)
\]

with \( \lambda = e^{2x_0} \). Hence

\[
w(x, t) = \tilde{a}(t) \cosh(2(x-x_0)) + d(t)
\]

with \( \tilde{a}(t) = 2\lambda a(t) \). The functions \( \tilde{a}(t) \) and \( d(t) \) satisfy the system

\[
\begin{align*}
    \tilde{a}'(t) &= 4\tilde{a}(t)d(t) \\
    d'(t) &= 4\tilde{a}^2(t).
\end{align*}
\]

Solving this system gives us

\[
\tilde{a}(t) = -\mu \operatorname{csch}(4\mu t) \quad \text{and} \quad d(t) = -\mu \coth(4\mu t)
\]

for a positive constant \( \mu > 0 \). Combining the above, we conclude that

\[
w(x, t) = -\mu [\operatorname{csch}(4\mu t) \cosh(2(x-x_0)) + \coth(4\mu t)].
\]

Assume for simplicity that \( x_0 = 0 \). Then, we obtain the solution

\[
w(x, t) = -\mu [\operatorname{csch}(4\mu t) \cosh(2x) + \coth(4\mu t)].
\]

Let us now express \( w \) on a sphere. The corresponding pressure function \( v \) on the sphere is defined by

\[
v(\psi, t) = (\cosh x)^{-2} w(x, t), \quad \sec \psi = \cosh x.
\]
Using the formulas $\cosh(2x) = 2\cosh^2 x - 1$ and $\coth(4\mu t) - \text{csch}(4\mu t) = \tanh(2\mu t)$ we have

$$v(\psi, t) = -\mu (\cosh x)^{-2} \{\text{csch}(4\mu t) [2\cosh^2 x - 1] + \coth(4\mu t)\}$$

$$= -\mu \{2\text{csch}(4\mu t) + [\coth(4\mu t) - \text{csch}(4\mu t)] \cos^2 \psi\}$$

$$= -\mu \{2\text{csch}(4\mu t) + \tanh(2\mu t) (1 - \sin^2 \psi)\}$$

$$= -\mu \{[2\text{csch}(4\mu t) + \tanh(2\mu t)] - \tanh(2\mu t) \sin^2 \psi\}$$

Finally, since $2\text{csch}(4\mu t) + \tanh(2\mu t) = \coth(2\mu t)$ we conclude that

$$v(\psi, t) = -\mu \coth(2\mu t) + \mu \tanh(2\mu t) \sin^2 \psi$$

and the proof of our theorem (in the radial case) is now complete.

□

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CLASSIFICATION OF COMPACT ANCIENT SOLUTIONS TO THE RICCI FLOW ON SURFACES

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