Second order deformations of associative submanifolds in nearly parallel $G_2$-manifolds

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Abstract

Associative submanifolds $A$ in nearly parallel $G_2$-manifolds $Y$ are minimal 3-submanifolds in spin 7-manifolds with a real Killing spinor. The Riemannian cone over $Y$ has the holonomy group contained in Spin(7) and the Riemannian cone over $A$ is a Cayley submanifold.

Infinitesimal deformations of associative submanifolds were considered by the author [11]. This paper is a continuation of the work. We give a necessary and sufficient condition for an infinitesimal associative deformation to be integrable (unobstructed) to second order explicitly. As an application, we show that the infinitesimal deformations of a homogeneous associative submanifold in the 7-sphere given by Lotay [16], which he called $A_3$, are unobstructed to second order.

1 Introduction

Associative submanifolds $A$ in nearly parallel $G_2$-manifolds are minimal 3-submanifolds in spin 7-manifolds $Y$ with a real Killing spinor. The Riemannian cone over $Y$ has the holonomy group contained in Spin(7) and the Riemannian cone over $A$ is a Cayley submanifold. There are many examples of associative submanifolds. For example, special Legendrian submanifolds and invariant submanifolds in the sense of [3, Section 8.1] in Sasaki-Einstein manifolds are associative. Lagrangian submanifolds in the sine cones of nearly Kähler 6-manifolds are also associative ([16, Propositions 3.8, 3.9 and 3.10]).

We are interested in deformations of associative submanifolds in nearly parallel $G_2$-manifolds. Since associative deformations are equivalent to Cayley cone deformations, it may help to develop the deformation theory of a Cayley submanifold with conical singularities. This study can also be regarded as an analogous study of associative submanifolds in torsion-free $G_2$-manifolds.

The standard 7-sphere $S^7$ has a natural nearly parallel $G_2$-structure. Lotay [16] studied associative submanifolds in $S^7$ intensively. In particular, he classified homogeneous associative submanifolds ([16 Theorem 1.1]), in which he gave the first explicit homogeneous example which does not arise from other geometries. He called it $A_3$. This is the only known example of this property up to the Spin(7)-action. Hence $A_3$ is a very mysterious example. It would be very interesting to see whether it is possible to obtain other new associative submanifolds not arising from other geometries by deforming it.

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It is known that the expected dimension of the moduli space of associative submanifolds is 0. However, there are many examples which have nontrivial deformations as pointed out in [16 Theorem 1.3]. In [11], the author studied infinitesimal associative deformations of homogeneous associative submanifolds in $S^7$. Infinitesimal associative deformations of other homogeneous examples than $A_3$ are unobstructed (namely, they extend to actual deformations) or reduced to the Lagrangian deformation problems in a totally geodesic $S^6$ ([11 Theorems 1.1 and 1.2]). However, we did not know whether infinitesimal associative deformations of $A_3$ are unobstructed or not ([11 Theorem 1.1]). The associative submanifold $A_3$ does not arise from other known geometries so its deformations are more complicated.

In this paper, we study second order deformations of associative submanifolds. Second order deformations of other geometric objects are considered by many people. For example, see [5, 13, 20]. We give a necessary and sufficient condition for an infinitesimal associative deformation to be integrable (unobstructed) to second order explicitly (Lemma 3.6 and Proposition 3.8). As an application, we obtain the following.

**Theorem 1.1.** All of the infinitesimal deformations of the associative submanifold $A_3$ defined by (4.2) in $S^7$ are unobstructed to second order.

As stated above, the expected dimension of the moduli space of associative submanifolds is 0. Thus we will expect that an associative submanifold does not admit associative deformations generically. Theorem 1.1 is an unexpected result because it implies that infinitesimal associative deformations of $A_3$ might extend to actual deformations. (For example, by the action of some group.) Unfortunately, we have no idea currently.

If all infinitesimal associative deformations of $A_3$ are unobstructed, we will be able to know the type of singularities of Cayley submanifolds in some cases. Namely, as in [15 Theorem 1.1], we can expect that if a Cayley integral current has a multiplicity one tangent cone of the form $\mathbb{R}_{>0} \times A_3$ with isolated singularity at an interior point $p$, then it has a conical singularity at $p$. Moreover, as in [15 Theorem 1.3], it might be useful to construct Cayley submanifolds with conical singularities in compact manifolds with Spin(7) holonomy.

**Remark 1.2.** In [12], the author classified homogeneous associative submanifolds and studied their associative deformations in the squashed 7-sphere, which is a 7-sphere with another nearly parallel $G_2$-structure. In this case, all of homogeneous associative submanifolds arise from pseudoholomorphic curves of the nearly Kähler $\mathbb{C}P^3$. Thus the deformation problems are easier and all infinitesimal associative deformations of homogeneous associative submanifolds in the squashed $S^7$ are unobstructed ([12 Theorem 1.6]).

This paper is organized as follows. In Section 2, we review the fundamental facts of $G_2$ and Spin(7) geometry. In Section 3, we recall the infinitesimal deformations of associative submanifolds and consider their second order deformations. We give a necessary and sufficient condition for an infinitesimal associative deformation to be integrable (unobstructed) to second order (Lemma 3.6) and describe it explicitly (Proposition 3.8). In Section 4, we prove Theorem 1.1 by using Proposition 3.8 and the Clebsch-Gordan decomposition. We also describe the trivial deformations (deformations given by the Spin(7)-action) of $A_3$ explicitly.
Notation: Let \((M, g)\) be a Riemannian manifold. We denote by \(i(\cdot)\) the interior product. For a tangent vector \(v \in TM\), define a cotangent vector \(v^\flat \in T^*M\) by \(v^\flat = g(v, \cdot)\). For a cotangent vector \(\alpha \in T^*M\), define a tangent vector \(\alpha^\flat \in TM\) by \(\alpha = g(\alpha^\flat, \cdot)\). For a vector bundle \(E\) over \(M\), we denote by \(C^\infty(M,E)\) the space of all smooth sections of \(E \to M\).

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2 \(G_2\) and \(\text{Spin}(7)\) geometry

First, we review the fundamental facts of \(G_2\) and \(\text{Spin}(7)\) geometry.

**Definition 2.1.** Define a 3-form \(\varphi_0\) on \(\mathbb{R}^7\) by

\[
\varphi_0 = dx_{123} + dx_1(dx_{45} + dx_{67}) + dx_2(dx_{46} - dx_7) - dx_3(dx_{47} + dx_{56}),
\]

where \((x_1, \cdots, x_7)\) is the standard coordinate system on \(\mathbb{R}^7\) and wedge signs are omitted. The Hodge dual of \(\varphi_0\) is given by

\[
\ast\varphi_0 = dx_{4567} + dx_{23}(dx_{67} + dx_{45}) + dx_{13}(dx_{57} - dx_{46}) - dx_{12}(dx_{56} + dx_{47}).
\]

Decompose \(\mathbb{R}^8 = \mathbb{R} \oplus \mathbb{R}^7\) and denote by \(x_0\) the coordinate on \(\mathbb{R}\). Define a self-dual 4-form \(\Phi_0\) on \(\mathbb{R}^8\) by

\[
\Phi_0 = dx_0 \wedge \varphi_0 + \ast\varphi_0.
\]

Identifying \(\mathbb{R}^8 \cong \mathbb{C}^4\) via

\[
\mathbb{R}^8 \ni (x_0, \cdots, x_7) \mapsto (x_0 + ix_1, x_2 + ix_3, x_4 + ix_5, x_6 + ix_7) =: (z_1, z_2, z_3, z_4) \in \mathbb{C}^4,
\]

\(\Phi_0\) is described as

\[
\Phi_0 = \frac{1}{2} \omega_0 \wedge \omega_0 + \text{Re}\Omega_0,
\]

where \(\omega_0 = \frac{i}{2} \sum_{j=1}^{4} dz_j \wedge dz_{\overline{j}}\) and \(\Omega_0 = dz_{1234}\) are the standard Kähler form and the holomorphic volume form on \(\mathbb{C}^4\), respectively.

The stabilizers of \(\varphi_0\) and \(\Phi_0\) are the Lie groups \(G_2\) and \(\text{Spin}(7)\), respectively:

\(G_2 = \{g \in GL(7, \mathbb{R}); g^* \varphi_0 = \varphi_0\}\), \quad \(\text{Spin}(7) = \{g \in GL(8, \mathbb{R}); g^* \Phi_0 = \Phi_0\}\).

The Lie group \(G_2\) fixes the standard metric \(g_0 = \sum_{i=1}^{7} (dx_i)^2\) and the orientation on \(\mathbb{R}^7\). They are uniquely determined by \(\varphi_0\) via

\[
6g_0(v_1, v_2)\text{vol}_{g_0} = i(v_1)\varphi_0 \wedge i(v_2)\varphi_0 \wedge \varphi_0,
\]

where \(\text{vol}_{g_0}\) is a volume form of \(g_0\) and \(v_i \in T(\mathbb{R}^7)\).

Similarly, \(\text{Spin}(7)\) fixes the standard metric \(h_0 = \sum_{i=0}^{7} (dx_i)^2\) and the orientation on \(\mathbb{R}^8\). We have the following identities:

\[
\Phi_0^2 = 14\text{vol}_{h_0}, \quad (i(w_2)i(w_1)\Phi_0)^2 \wedge \Phi_0 = 6\|w_1 \wedge w_2\|_{h_0}^2\text{vol}_{h_0},
\]

where \(\text{vol}_{h_0}\) is a volume form of \(h_0\) and \(w_i \in T(\mathbb{R}^8)\).
Definition 2.2. Let $M^7$ be an oriented 7-manifold and $\varphi$ be a 3-form on $M^7$. A 3-form $\varphi$ is called a $G_2$-structure on $M^7$ if for each $p \in M^7$, there exists an oriented isomorphism between $T_pM^7$ and $\mathbb{R}^7$ identifying $\varphi_p$ with $\varphi_0$. From Definition 2.2, $\varphi$ induces the metric $g$ and the volume form on $M^7$. Similarly, for an oriented 8-manifold with a 4-form $\Phi$, we can define a Spin(7)-structure by $\Phi_0$.

Lemma 2.3. A $G_2$-structure $\varphi$ is called torsion-free if $d\varphi = d\star\varphi = 0$. A Spin(7)-structure $\Phi$ is called torsion-free if $d\Phi = 0$. It is well-known that a $G_2$- or Spin(7)-structure is torsion-free if and only if the holonomy group is contained in $G_2$ or Spin(7). This is also equivalent to saying that $\varphi$ or $\Phi$ is parallel with respect to the Levi-Civita connection of the induced metric.

Definition 2.4. Let $(M^7, \varphi, g)$ be a manifold with a $G_2$-structure. Let $\nabla$ be the Levi-Civita connection of $g$. A $G_2$-structure $\varphi$ is called a nearly parallel $G_2$-structure if one of the following equivalent conditions is satisfied.

1. $d\varphi = 4\star\varphi$,
2. $\nabla\varphi = \frac{1}{2}d\varphi$,
3. $\nabla\varphi = \star d\varphi$,
4. $\nabla_v(\star\varphi) = -v^\flat \wedge \varphi$ for any $v \in TM$,
5. $i(v)\nabla_v\varphi = 0$ for any $v \in TM$,
6. The Riemannian cone $C(M) = \mathbb{R}_{>0} \times M$ admits a torsion-free Spin(7)-structure $\Phi = r^3dr \wedge \varphi + r^4\star\varphi$ with the induced cone metric $g = dr^2 + r^2g$.

We call a manifold with a nearly parallel $G_2$-structure a nearly parallel $G_2$-manifold for short.

Definition 2.5. Let $(M^7, \varphi, g)$ be a manifold with a $G_2$-structure. Define the cross product $\cdot \times \cdot : TM \times TM \to TM$ and a tangent bundle valued 3-form $\chi \in \Omega^3(M, TM)$ by

$$g(x \times y, z) = \varphi(x, y, z), \quad g(\chi(x, y, z), w) = \star \varphi(x, y, z, w)$$

for $x, y, z, w \in TM$. They are related via

$$\chi(x, y, z) = -x \times (y \times z) - g(x, y)z + g(x, z)y.$$  \hspace{1cm} (2.4)

Next, we summarize the facts about submanifolds in $G_2$ and Spin(7) settings. Let $M^7$ be a manifold with a $G_2$-structure $\varphi$ and the induced metric $g$.

Lemma 2.6. For every oriented $k$-dimensional subspace $V^k \subset T_pM^7$ where $p \in M^7$ and $k = 3, 4$, we have $\varphi|_{V^3} \leq \text{vol}_{V^3}$, $\star\varphi|_{V^4} \leq \text{vol}_{V^4}$. An oriented 3-submanifold $L^3 \subset M^7$ is called associative if $\varphi|_{TL^3} = \text{vol}_{L^3}$, which is equivalent to $\chi|_{TL^3} = 0$ and $\varphi|_{TL^3} > 0$. An oriented 4-submanifold $L^4 \subset M^7$ is called coassociative if $\star\varphi|_{TL^4} = \text{vol}_{L^4}$, which is equivalent to $\varphi|_{TL^4} = 0$ and $\star\varphi|_{TL^4} > 0$. 

\hspace{1cm}
Associative submanifolds have the following good properties with respect to the cross product.

Lemma 2.7. Let \( L^3 \subset M^7 \) be an associative submanifold and \( \nu \rightarrow L \) be the normal bundle of \( L^3 \) in \( M^7 \). Then we have

\[
TL \times TL \subset TL, \quad TL \times \nu \subset \nu, \quad \nu \times \nu \subset TL.
\]

Here, the left hand sides are the spaces given by the cross product of elements of \( TL \) or \( \nu \).

Definition 2.8. Let \( X \) be a manifold with a \( \text{Spin}(7) \)-structure \( \Phi \). Then for every oriented 4-dimensional subspace \( W \subset T_x X \) where \( x \in X \), we have \( \Phi|_W \leq \text{vol}_W \). An oriented 4-submanifold \( N \subset X \) is called Cayley if \( \Phi|_T N = \text{vol}_N \).

Definition 2.9. If a \( G_2 \)-structure is torsion-free, \( \varphi \) and \( *\varphi \) define calibrations. Hence compact (co)associative submanifolds are volume minimizing in their homology classes, and hence minimal. We also know that any (not necessarily compact) (co)associative submanifolds are minimal. Similar statement holds for Cayley submanifolds in a manifold with a torsion-free \( \text{Spin}(7) \)-structure.

Lemma 2.10. Let \( (M^7, \varphi, g) \) be a nearly parallel \( G_2 \)-manifold. Then there are no coassociative submanifolds in \( M \) (Lemma 3.2). An oriented 3-dimensional submanifold \( L \subset M \) is associative if and only if \( C(L) = \mathbb{R}^> \times L \subset \mathbb{R}^> \times M = C(Y) \) is Cayley. In particular, \( L \) is minimal.

3 Deformations of associative submanifolds

3.1 Infinitesimal deformations of associative submanifolds

First, we describe the infinitesimal deformation space explicitly again. The arguments here are based on [7, Section 2], [8, Section 6.1], [13, Section 3.1].

Let \( (M^7, \varphi, g) \) be a manifold with a \( G_2 \)-structure and let \( L^3 \subset M^7 \) be a compact associative submanifold. Let \( \nu \rightarrow L \) be the normal bundle of \( L^3 \) in \( M^7 \). By the tubular neighborhood theorem there exists a neighborhood of \( L \) in \( M \) which is identified with an open neighborhood \( T \subset \nu \) of the zero section by the exponential map. Set

\[
C^\infty(L, T) = \{ v \in C^\infty(L, \nu); v_x \in T \text{ for any } x \in L \}.
\]

The exponential map induces the embedding \( \exp_V : L \rightarrow M \) by \( \exp_V(x) = \exp_x(V_x) \) for \( V \in C^\infty(L, T) \) and \( x \in L \). Let \( \mathcal{P}_V : TM|_L \rightarrow TM|_{\exp_V(L)} \) for \( V \in C^\infty(L, T) \)

be the isomorphism given by the parallel transport along the geodesic \( [0,1] \ni t \mapsto \exp_t(V_x) \in M \), where \( x \in L \), with respect to the Levi-Civita connection of \( g \). Let \( \perp : TM|_L = TL \oplus \nu \rightarrow \nu \) be the orthogonal projection and \( \nu_V \subset TM|_{\exp_V(L)} \) be the normal bundle of \( \exp_V(L) \). Consider the orthogonal projection

\[
\perp|_{\mathcal{P}_V^{-1}(\nu_V)} : \mathcal{P}_V^{-1}(\nu_V) \rightarrow \nu.
\]
The condition for this map to be an isomorphism is open and it is an isomorphism for \( V = 0 \). Thus shrinking \( T \) if necessary, we may assume that

\[
\phi_V : C^\infty(L,\nu) \to C^\infty(L,\nu), \quad \phi_V(W) = (P_V^{-1}(W))\perp
\]

is an isomorphism for \( V \in C^\infty(L, T) \). Then define the first order differential operator \( F : C^\infty(L, T) \to C^\infty(L, \nu) \) by

\[
F(V) = \phi_V \left( (\exp^*_V \chi)(e_1, e_2, e_3) \right),
\]

where \( \{e_1, e_2, e_3\} \) is a local oriented orthonormal frame of \( TL \). Then \( \exp_V(L) \subset M \) is associative if and only if \( F(V) = 0 \). Thus a neighborhood of \( L \) in the moduli space of associative submanifolds is identified with that of 0 in \( F^{-1}(0) \) (in the \( C^1 \) sense).

Set

\[
D = (dF)_0 : C^\infty(L, \nu) \to C^\infty(L, \nu),
\]

which is the linearization of \( F \) at 0. The operator \( D \) is computed as follows.

**Proposition 3.1 ([7] Section 5, [6] Theorem 2.1).** Let \((M^7, \varphi, g)\) be a manifold with a \( G_2 \)-structure and let \( L^3 \subset M^7 \) be a compact associative submanifold. The operator \( D \) above is given by

\[
DV = \sum_{i=1}^{3} e_i \times \nabla^\perp_{e_i} V + \left( (\nabla V \ast \varphi)(e_1, e_2, e_3, \cdot) \right)_{i=0}^1,
\]

where \( \{e_1, e_2, e_3\} \) is a local oriented orthonormal frame of \( TL \) satisfying \( e_i = e_{i+1} \times e_{i+2} \) for \( i \in \mathbb{Z}/3 \), \( \nabla^\perp \) is the connection on the normal bundle \( \nu \) induced by the Levi-Civita connection \( \nabla \) of \((M, g)\).

**Proof.** For simplicity, we write \( \exp_V = \iota_t \). Then

\[
DV = \frac{d}{dt} \left( P_{\iota_t}^{-1}(\iota_t^* \chi)(e_1, e_2, e_3) \right)_{|t=0}^\perp = \left( \nabla^\varphi_{\iota_t}(\iota_t^* \chi)(e_1, e_2, e_3) \right)_{|t=0}^\perp,
\]

where \( \nabla^\varphi_{\iota_t} \) is the covariant derivative along the geodesic \([0, 1] \ni t \mapsto \exp_x(tV_x) \in M \), where \( x \in L \), induced from the Levi-Civita connection of \( g \). Let \( \{\eta_j\}_{j=1}^7 \) be a local orthonormal frame of \( TM \). Then we have

\[
\chi = -\sum_{j=1}^7 i(\eta_j) \ast \varphi \otimes \eta_j.
\]

We further compute

\[
DV = -\sum_j \left( \nabla^\varphi_{\iota_t}((\ast \varphi \circ \iota_t)(\eta_j \circ \iota_t, (\iota_t)_* e_1, (\iota_t)_* e_2, (\iota_t)_* e_3)\eta_j \circ \iota_t) \right)_{|t=0}^\perp
\]

\[
= -\sum_j \left( (\nabla V \ast \varphi)(\eta_j, e_1, e_2, e_3)\eta_j + \sum_{i \in \mathbb{Z}/3} \ast \varphi(\eta_j, \nabla^\varphi_{\iota_t}(\iota_t)_* e_i)_{|t=0, e_i+1, e_i+2} \eta_j \right)_{|t=0}^\perp,
\]

where we use \( \ast \varphi(e_1, e_2, e_3, \cdot) = 0 \) since \( L \) is associative. Note that \( \nabla^\varphi_{\iota_t}(\iota_t)_* e_i \) is the restriction of the covariant derivative \( \nabla^\varphi_{\iota_t}(\iota_t)_* e_i \) along the map \( t : L \times [0, 1] \ni
(x, t) \mapsto \iota_t(x) \in M$. Then the standard equations of the covariant derivative along the map imply that

$$\nabla_{\frac{d}{dt}}(\iota_t)_* e_i \bigg|_{t=0} = \nabla_{e_i}(\iota_t)_* \left( \frac{d}{dt} \right) \bigg|_{t=0} = \nabla_{e_i} V,$$

$$\ast \varphi(\nabla_{e_i} V, e_{i+1}, e_{i+2}, \eta_j) = \ast \varphi(\nabla_{e_i} V, e_{i+1}, e_{i+2}, \eta_j) = 0 \quad (\text{Lemma 3.3}).$$

By \cite{4, Section 4}, we have an endomorphism $\varphi: \mathfrak{gl}(\mathfrak{m}) \to \mathfrak{gl}(\mathfrak{m})$ given by $\varphi = \sum_{i=1}^{3} e_i \otimes \nabla_{e_i} V$, where we use the fact that $L$ is associative, \cite{2,4} and $e_i = e_{i+1} \times e_{i+2}$. Then we obtain the statement. \hfill \Box

We can also describe the last term of $DV$ as follows.

**Lemma 3.2.** By \cite{4, Section 4}, we have an endomorphism $T \in C^\infty(M, \text{End}(TM))$ given by

$$\nabla_v \varphi = i(T(v)) \ast \varphi$$

for any $v \in TM$. Then we have

$$((\nabla V \ast \varphi)(e_1, e_2, e_3, \cdot))^\sharp = (T(V))^\perp.$$

**Proof.** We easily see that $\nabla_v \ast \varphi = (\nabla \varphi) = -(T(v))^\flat \wedge \varphi$. Then

$$((\nabla V \ast \varphi)(e_1, e_2, e_3, \cdot))^\sharp = - (T(v))^\flat \wedge \varphi(e_1, e_2, e_3, \cdot))^\sharp = \varphi(e_1, e_2, e_3) T(v) - \sum_{i \in \mathbb{Z}/3} g(T(v), e_i) \varphi(e_{i+1}, e_{i+2}, \cdot)^\sharp$$

$$= (T(v))^\perp,$$

where we use $\varphi(e_1, e_2, e_3) = 1$ and $\varphi(e_{i+1}, e_{i+2}, \cdot)^\sharp = e_{i+1} \times e_{i+2} = e_i$. \hfill \Box

Using this lemma, we see the following.

**Lemma 3.3.** If $d \ast \varphi = 0$, $D$ is self-adjoint.

**Proof.** For any normal vector fields $V, W \in C^\infty(L, \nu)$, we compute

$$g(DV, W) = g\left( \sum_{i=1}^{3} e_i \times \nabla_{e_i} V + T(V), W \right)$$

$$= \sum_{i=1}^{3} g(e_i \times \nabla_{e_i} V, W) + g(T(V), W),$$

$$\sum_{i=1}^{3} g(e_i \times \nabla_{e_i} V, W) = - \sum_{i=1}^{3} \varphi(V, e_i, W)$$

$$= \sum_{i=1}^{3} (-e_i(\varphi(V, e_i, W)) + (\nabla_{e_i} \varphi)(V, e_i, W) + \varphi(V, \nabla_{e_i} e_i, W) + \varphi(V, e_i, \nabla_{e_i} W)).$$
Define a 1-form $\alpha$ on $L$ by $\alpha = \varphi(V, \cdot, W)$. Then

$$d^* \alpha + \sum_{i=1}^{3} (\nabla_{e_i} \varphi)(V, e_i, W) + g(V, e_i \times \nabla_{e_i} W)$$

$$= d^* \alpha + \sum_{i=1}^{3} \ast \varphi(T(e_i), V, e_i, W) + g(V, e_i \times \nabla_{e_i} W).$$

By (2.4), it follows that

$$\ast \varphi(T(e_i), V, e_i, W) = g(\chi(e_i, V, W), T(e_i)).$$

By Lemma 2.7, $e_i \times (V \times W)$ is a (local) tangent vector field to $L$. Then

$$\sum_{i=1}^{3} \ast \varphi(T(e_i), V, e_i, W) = \sum_{i,j=1}^{3} g(T(e_i), e_j) \ast \varphi(e_j, V, e_i, W).$$

Hence we obtain

$$g(DV, W) = g(V, DW) + g(T(V), W) - g(V, T(W)) + \sum_{i,j=1}^{3} g(T(e_i), e_j) \ast \varphi(e_j, V, e_i, W) + d^* \alpha.$$

Then we see that $D$ is self-adjoint if $T$ is symmetric. In terms of [10, Section 2.5], this is the case $\nabla \varphi \in W_1 \oplus W_{27}$, which is equivalent to $d^* \varphi = 0$ by [10] Table 2.1.

**Remark 3.4.** If a $G_2$-structure is torsion-free, we have $T = 0$, and hence $((\nabla V \ast \varphi)(e_1, e_2, e_3, \cdot))^2 = 0$. If a $G_2$-structure is nearly parallel $G_2$, we have $T = \text{id}_{TM}$ and $((\nabla V \ast \varphi)(e_1, e_2, e_3, \cdot))^2 = V$. We can also deduce this by Definition 2.4 ([11, Lemma 3.5]). In these cases, $D$ is self-adjoint as stated in Lemma 3.3.

We easily see that the operator $D$ is elliptic, and hence Fredholm. Since $L$ is 3-dimensional, the index of $D$ is 0. Thus if $D$ is surjective, the moduli space of associative submanifolds is 0-dimensional. See [7, Proposition 2.2].

To understand the moduli space of associative submanifolds more, we consider their second order deformations in the next subsection.

### 3.2 Second order deformations of associative submanifolds

Use the notation in Section 3.1. The principal task in deformation theory is to integrate given infinitesimal (first order) deformations $V \in \ker D$. Namely, to find a one-parameter family $\{V(t)\} \subset C^\infty(L, \nu)$ such that

$$F(V(t)) = 0 \quad \text{and} \quad \frac{d}{dt}V(t) \bigg|_{t=0} = V.$$

In general, this is not possible. In this subsection, we define the second order deformations of associative submanifolds and give a necessary and sufficient condition for an infinitesimal associative deformation to be integrable (unobstructed) to second order.
Definition 3.5. Let $M^7$ be a manifold with a $G_2$-structure and $L^3 \subset M^7$ be a compact associative submanifold. An infinitesimal associative deformation $V_1 \in \ker D \subset C^\infty(L, \nu)$ is said to be unobstructed to second order if there exists $V_2 \in C^\infty(L, \nu)$ such that

$$\frac{d^2}{dt^2} F(tV_1 + \frac{1}{2} t^2 V_2) \bigg|_{t=0} = 0.$$ 

In other words, $tV_1 + \frac{1}{2} t^2 V_2$ gives an associative submanifold up to terms of the order $o(t^2)$.

We easily compute

$$\frac{d^2}{dt^2} F(tV_1 + \frac{1}{2} t^2 V_2) \bigg|_{t=0} = \frac{d^2}{dt^2} F(tV_1) \bigg|_{t=0} + D(V_2).$$

Since $D$ is elliptic and $L$ is compact, we have an orthogonal decomposition $C^\infty(L, \nu) = \text{Im} D \oplus \text{Coker} D$ with respect to the $L^2$ inner product. Then we obtain the following.

Lemma 3.6. Let $\pi : C^\infty(L, \nu) \rightarrow \text{Coker} D$ be an orthogonal projection with respect to the $L^2$ inner product. Then an infinitesimal deformation $V_1 \in \ker D$ is unobstructed to second order if and only if

$$\pi \left( \frac{d^2}{dt^2} F(tV_1) \bigg|_{t=0}, W \right)_{L^2} = 0 \text{ for any } W \in \text{Coker} D. \quad (3.3)$$

In other words, we have

$$\left. \frac{d^2}{dt^2} F(tV_1) \bigg|_{t=0} \right. , W \right)_{L^2} = 0$$

for the second order deformations. It is explicitly computed as follows.

Proposition 3.8. Use the notation in Proposition 3.4. For $V \in \ker D$, we have

$$\left. \frac{d^2}{dt^2} F(tV) \right|_{t=0} = ((\nabla_V \nabla \varphi)(V))(e_1, e_2, e_3, \cdot)^2$$

$$+ \sum_{i \in \mathbb{Z}/3} \left( (\nabla_V \ast \varphi)(\nabla^\perp_{e_i} V, e_{i+1}, e_{i+2}, \cdot)^2 \right)^\perp$$

$$+ \sum_{i=1}^3 e_i \times (R(V, e_i) V)^\perp + 2 \sum_{i,j=1}^3 g(V, II(e_i, e_j)) e_i \times \nabla^\perp_{e_j} V,$$
where $R$ is the curvature tensor of $(M,g)$ and $\Pi$ is the second fundamental form of $L$ in $M$.

If a $G_2$-structure $\varphi$ is torsion-free or nearly parallel $G_2$, we have

$$
\frac{d^2}{dt^2} F(tV)\bigg|_{t=0} = \sum_{i=1}^{3} e_i \times (R(V,e_i)V) + 2 \sum_{i,j=1}^{3} g(V,\Pi(e_i,e_j)) e_i \times \nabla_{e_i} V.
$$

**Proof.** Use the notation in the proof of Proposition 3.1. Setting $\chi_j = -i(\eta_j) \ast \varphi$, we have $\chi = \sum_{j=1}^{3} \chi_j \otimes \eta_j$. Then

$$
\frac{d}{dt} F(tV) = \left( P_{tV}^{-1} \nabla_{\pi} \left( (t^* \chi_j)(e_1,e_2,e_3) \right) \right) + \nabla_{\pi} \left( (t^* \chi_j)(e_1,e_2,e_3) \right)
$$

$$
\frac{d^2}{dt^2} F(tV)\bigg|_{t=0} = \sum_{j} \left( \nabla_{\pi} \nabla_{\pi} (t^* \chi_j(e_1,e_2,e_3) \eta_j \circ t) \right) + \nabla_{\pi} (t^* \chi_j(e_1,e_2,e_3)) t=0 \nabla V \eta_j
$$

$$
+ \chi_j(e_1,e_2,e_3) \nabla_{\pi} (t^* \chi_j(e_1,e_2,e_3)) t=0 \nabla V \eta_j.
$$

Since $\frac{d^2}{dt^2} t^* \chi_j(e_1,e_2,e_3) t=0 = g(DV,\eta_j)$ by the proof of Proposition 3.1 and $\chi_j(e_1,e_2,e_3) = 0$, we only have to compute $\frac{d^2}{dt^2} t^* \chi_j(e_1,e_2,e_3) t=0$. Then

$$
\frac{d^2}{dt^2} t^* \chi_j(e_1,e_2,e_3) t=0 = - \frac{d^2}{dt^2} \left( (\ast \varphi \circ t) (\eta_j \circ t, (t_i) \ast e_1, (t_i) \ast e_2, (t_i) \ast e_3) \right) t=0
$$

$$
- \frac{d}{dt} \left( (\nabla_{\pi} \ast \varphi \circ t) (\eta_j \circ t, (t_i) \ast e_1, (t_i) \ast e_2, (t_i) \ast e_3) \right)
$$

$$
+ \sum_{i \in \mathbb{Z}/3} \left( \ast \varphi \circ t \ast (t_i) \ast \nabla_{\pi} \ast (t_i) \ast (t_i) \ast e_1, (t_i) \ast e_1, (t_i) \ast e_1, (t_i) \ast e_1 \right) \right) t=0
$$

$$
= - \left( \nabla_{\pi} \ast \varphi \ast (t) (t_i) \ast (t_i) \ast e_1, (t_i) \ast e_1, (t_i) \ast e_1 \right) t=0 (\eta_j, e_1, e_2, e_3)
$$

$$
- 2 \sum_{i \in \mathbb{Z}/3} \left( \ast \varphi \ast \left( (t_i) \ast (t_i) \ast (t_i) \ast e_1, (t_i) \ast e_1, (t_i) \ast e_1 \right) \right) t=0 (\eta_j, e_1, e_2, e_3)
$$

$$
- \left( (t_i) \ast (t_i) \ast (t_i) \ast e_1, (t_i) \ast e_1, (t_i) \ast e_1 \right) t=0 (\eta_j, e_1, e_2, e_3)
$$

$$
- 2 \sum_{i \in \mathbb{Z}/3} \ast \varphi \ast \left( (t_i) \ast (t_i) \ast e_1, (t_i) \ast e_1, (t_i) \ast e_1 \right) t=0 (\eta_j, e_1, e_2, e_3)
$$

$$
- \left( (t_i) \ast (t_i) \ast (t_i) \ast e_1, (t_i) \ast e_1, (t_i) \ast e_1 \right) t=0 (\eta_j, e_1, e_2, e_3)
$$

$$
- 2 \sum_{i \in \mathbb{Z}/3} \ast \varphi \ast \left( (t_i) \ast e_1, (t_i) \ast e_1, (t_i) \ast e_1 \right) t=0 (\eta_j, e_1, e_2, e_3).
$$

By the same argument as in the proof of Proposition 3.1, we have
\[ *\varphi(e_1, e_2, e_3, \cdot) = 0, \]
\[ \nabla \phi(t_\tau^*) e_i |_{t=0} = \nabla e_i V, \]
\[ \nabla \phi \nabla \phi(t_\tau^*) e_i |_{t=0} = R(V, e_i)V + \nabla e_i \nabla \phi(t_\tau^*) \left( \frac{d}{dt} \right) |_{t=0} = R(V, e_i)V, \]
where we use \( \nabla \phi(t_\tau^*) \left( \frac{d}{dt} \right) = 0 \) because \( t_\tau = \exp(tV) \) is a geodesic. By the definition of the induced connection, we have
\[ \nabla \phi \nabla \phi \nabla \phi \nabla \phi(t_\tau^*) e_i |_{t=0} = (\nabla \phi(V) * \varphi) \nabla e_i V, \]
where we use \( \nabla \phi \left( \frac{d}{dt} \right) = 0. \) Moreover, by the proof of Proposition 3.1 we have
\[ (\nabla \phi(V) * \varphi)(\nabla \phi(V) \eta_j, e_1, e_2, e_3) + \sum_{i \in \mathbb{Z}/3} *\varphi(\nabla \phi(V) \eta_j, \nabla \phi(t_\tau^*) e_i |_{t=0}, e_{i+1}, e_{i+2}) = -g(DV, \nabla \phi(V)). \]
Thus it follows that
\[ \frac{d^2}{dt^2} \chi_j(e_1, e_2, e_3) |_{t=0} = -((\nabla \phi(V) * \varphi)(V))(\eta_j, e_1, e_2, e_3) \]
\[ - 2 \sum_{i \in \mathbb{Z}/3} (\nabla \phi(V) * \varphi)(\eta_j, \nabla e_i V, e_{i+1}, e_{i+2}) \]
\[ - \sum_{i \in \mathbb{Z}/3} *\varphi(\eta_j, R(V, e_i)V, e_{i+1}, e_{i+2}) \]
\[ - 2 \sum_{i \in \mathbb{Z}/3} *\varphi(\eta_j, \nabla e_i V, \nabla e_{i+1} V, e_{i+2}) + 2g(DV, \nabla \phi(V)). \]
(3.5)
Hence from (2.4), (3.4) and (3.5), we obtain
\[ \frac{d^2}{dt^2} F(tV) |_{t=0} = ((\nabla \phi(V) * \varphi)(V))(e_1, e_2, e_3) \]
\[ + 2 \sum_{i \in \mathbb{Z}/3} ((\nabla \phi(V) * \varphi)(\nabla e_i V, e_{i+1}, e_{i+2}, \cdot))^\perp \]
\[ + \sum_{i=1}^3 e_i \times (R(V, e_i)V)^\perp \]
\[ + 2 \sum_{i \in \mathbb{Z}/3} \chi(e_i V, \nabla e_{i+1} V, e_{i+2})^\perp \]
\[ + 2 \sum_j g(DV, \nabla \phi(V) \eta_j)^\perp + 2 \sum_j g(DV, \eta_j)(\nabla \phi(V) \eta_j)^\perp. \] (3.6)
Hence we obtain \[ \chi(\nabla_e V, \nabla_{e_i+1} V, e_{i+2}) \perp. \]

Let \( T : TM_L \rightarrow TL \) be the projection. Since \( L \) is associative, we have

\[ \chi(\nabla_e V, \nabla_{e_i+1} V, e_{i+2}) \]

\[ = \chi(\nabla^\top_e V, \nabla_{e_i+1} V, e_{i+2}) \perp + \chi(\nabla^\perp_e V, \nabla_{e_i+1} V, e_{i+2}) \perp. \]

The first term is computed as

\[ \chi(\nabla^\top_e V, \nabla_{e_i+1} V, e_{i+2}) = -\sum_{j=0}^2 g(V, \Pi(e_i, e_{i+j})) \chi(e_{i+j}, \nabla^\perp_{e_i+1} V, e_{i+2}) \]

\[ = -\sum_{j=0}^2 g(V, \Pi(e_i, e_{i+j})) \nabla^\perp_{e_i+1} V \times (e_{i+j} \times e_{i+2}) \]

\[ = -g(V, \Pi(e_i, e_i)) e_{i+1} \times \nabla^\perp_{e_i+1} V + g(V, \Pi(e_i, e_{i+1})) e_i \times \nabla^\perp_{e_i+1} V. \]

The second term is computed as

\[ \chi(\nabla^\perp_e V, \nabla_{e_i+1} V, e_{i+2}) = -\sum_{j=0}^2 g(V, \Pi(e_i+1, e_{i+j})) \chi(\nabla^\perp_{e_i} V, e_{i+j}, e_{i+2}) \]

\[ \sum_{j=0}^2 g(V, \Pi(e_{i+1}, e_{i+j})) \nabla^\perp_{e_i} V \times (e_{i+j} \times e_{i+2}) \]

\[ = g(V, \Pi(e_{i+1}, e_i)) e_{i+1} \times \nabla^\perp_{e_i} V - g(V, \Pi(e_{i+1}, e_{i+1})) e_i \times \nabla^\perp_{e_i} V. \]

The third term is computed as

\[ \chi(\nabla^\perp_e V, \nabla^\perp_{e_i+1} V, e_{i+2}) = \chi(e_{i+2}, \nabla^\perp_{e_{i+1}} V, \nabla^\perp_{e_i+1} V) \]

\[ = -e_{i+2} \times (\nabla^\perp_{e_i} V \times \nabla^\perp_{e_i+1} V), \]

which is a section of \( TL \) by Lemma 2.7. Then

\[ \chi(\nabla^\perp_{e_i} V, \nabla^\perp_{e_i+1} V, e_{i+2}) \perp = 0. \]

Hence we obtain

\[ \sum_{i \in \mathbb{Z}/3} \chi(\nabla_e V, \nabla_{e_i+1} V, e_{i+2}) \perp \]

\[ = -\sum_{i \in \mathbb{Z}/3} \{ g(V, \Pi(e_{i+1}, e_i)) + g(V, \Pi(e_{i+2}, e_i)) \} e_i \times \nabla^\perp_{e_i} V \]

\[ + \sum_{i \in \mathbb{Z}/3} g(V, \Pi(e_i, e_{i+1})) \left( e_i \times \nabla^\perp_{e_i+1} V + e_{i+1} \times \nabla^\perp_{e_i} V \right) \]

\[ = \sum_{i,j=1}^3 g(V, \Pi(e_i, e_j)) e_i \times \nabla^\perp_{e_j} V - \left( \sum_{i=1}^3 g(V, \Pi(e_i, e_i)) \right) \left( \sum_{j=1}^3 e_j \times \nabla^\perp_{e_j} V \right). \]

(3.7)

Thus using the equation

\[ \sum_{i \in \mathbb{Z}/3} (\nabla^\top V \circ \phi)(\nabla_{e_i} V, e_{i+1}, e_{i+2}, \cdot) \]

\[ = \sum_{i \in \mathbb{Z}/3} (\nabla^\top V \circ \phi)(\nabla^\perp_{e_i} V, e_{i+1}, e_{i+2}, \cdot) - \left( \sum_{i=1}^3 g(V, \Pi(e_i, e_i)) \right) (\nabla V \circ \phi)(e_1, e_2, e_3, \cdot), \]

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we obtain from Proposition 3.1, (3.6) and (3.7)

\[
\frac{d^2}{dt^2} F(tV) \bigg|_{t=0} = ((\nabla_V \nabla \ast \phi)(V))(e_1, e_2, e_3, \cdot) \\
+ 2 \sum_{i \in \mathbb{Z}/3} ((\nabla_V \ast \phi)(\nabla_{e_i}^\perp V, e_{i+1}, e_{i+2}, \cdot))^\perp \\
+ \sum_{i=1}^3 e_i \times (R(V, e_i)V)^\perp + 2 \sum_{i,j=1}^3 g(V, \Pi(e_i, e_j))e_i \times \nabla_{e_j}^\perp V \\
+ 2 \sum_j g(DV, \nabla_V \eta_j)\eta_j^\perp + 2 \sum_j g(DV, \nabla_V \eta_j)(\nabla_V \eta_j)^\perp \\
- 2 \left( \sum_{i=1}^3 g(V, \Pi(e_i, e_i)) \right) DV,
\]

which implies the first equation of Proposition 3.8.

If a $G_2$-structure $\phi$ is torsion-free, the second equation of Proposition 3.8 is obvious. If $\phi$ is nearly parallel $G_2$, we have by Definition 2.4

\[
((\nabla_V \nabla \ast \phi)(V))(e_1, e_2, e_3, \cdot) = 0.
\]

Similarly, we have by Definition 2.4

\[
((\nabla_V \ast \phi)(\nabla_{e_i}^\perp V, e_{i+1}, e_{i+2}, \cdot))^\perp = -g(V, \nabla_{e_i}^\perp V)e_i,
\]

where we use $\phi(e_{i+1}, e_{i+2}, \cdot) = e_{i+1} \times e_{i+2} = e_i$. Hence

\[
\sum_{i \in \mathbb{Z}/3} ((\nabla_V \ast \phi)(\nabla_{e_i}^\perp V, e_{i+1}, e_{i+2}, \cdot))^\perp = 0.
\]

Thus we obtain the second equation of Proposition 3.8.

\[
\text{Remark 3.9. Using the endomorphism } T \text{ given by (3.3), we have}
\]

\[
((\nabla_V \nabla \ast \phi)(V))(e_1, e_2, e_3, \cdot)^\perp + 2 \sum_{i \in \mathbb{Z}/3} ((\nabla_V \ast \phi)(\nabla_{e_i}^\perp V, e_{i+1}, e_{i+2}, \cdot))^\perp \\
= ((\nabla_V T)(V))^\perp + T(V)^\perp \times T(V)^\perp - 2\nabla_{T(V)^\perp}^\perp V
\]

by a direct computation. If $\phi$ is torsion-free ($T = 0$) these terms obviously vanish. If $\phi$ is nearly parallel $G_2$ ($T = \text{id}_{TM}$), these terms vanish again because $\nabla \text{id}_{TM} = 0$ and $(\text{id}_{TM}(V))^\perp = 0$ for a normal vector field $V$. 

\]

\[13\]
4 Associative submanifolds in $S^7$

In this section, we give a proof of Theorem 1.1. The standard 7-sphere $S^7$ has a natural nearly parallel $G_2$-structure ([10, Section 2]). Homogeneous associative submanifolds in $S^7$ are classified by Lotay ([16, Theorem 1.1]). As noted in the introduction, there is a mysterious homogeneous example called $A_3$ which does not arise from other geometries.

First, we summarize the facts for $A_3$ from [11, Example 6.3, Section 6.3.3].

Define $\rho_3 : SU(2) \leftrightarrow SU(4)$ by

$$\rho_3\left(\begin{pmatrix} a & -b \\ b & \pi \end{pmatrix}\right) = \begin{pmatrix}
a^3 & -\sqrt{3}a^2b \\
\sqrt{3}ab & b^3
\end{pmatrix}
$$

where $a, b \in \mathbb{C}$ such that $|a|^2 + |b|^2 = 1$. This is an irreducible $SU(2)$-action on $\mathbb{C}^3$. By using the notation of Appendix A, $\rho_3$ is the matrix representation of $\rho_3 : SU(2) \to GL(V_3) \cong GL(4, \mathbb{C})$ with respect to the basis $\{e_0^{(3)}, \ldots, e_3^{(3)}\}$. Then

$$A_3 = \rho_3(SU(2)) \cdot \frac{1}{\sqrt{2}}(0, 1, i, 0) \cong SU(2)$$

is an associative submanifold in $S^7$.

Define the basis of the Lie algebra $su(2)$ of $SU(2)$ by

$$E_1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad E_2 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \quad E_3 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$$

which satisfies the relation $[E_i, E_{i+1}] = 2E_{i+2}$ for $i \in \mathbb{Z}/3$. Denote by $e_1, e_2, e_3$ the left invariant vector fields on $SU(2) \cong A_3$ induced by $\frac{1}{\sqrt{7}}E_1, \frac{1}{\sqrt{3}}E_2, E_3$, respectively. Then they define a global orthonormal frame of $TA_3$. Explicitly, we have at $p_0 = \frac{1}{\sqrt{2}}(0, 1, i, 0)$

$$e_1 = \frac{1}{\sqrt{14}}(\sqrt{3}, 2i, -2, -\sqrt{3}i), \quad e_2 = \frac{1}{\sqrt{14}}(\sqrt{3}i, -2, 2i, -\sqrt{3}), \quad e_3 = \frac{1}{\sqrt{2}}(0, i, 1, 0),$$

and $(e_i)_{\rho_3(g) \cdot p_0} = \rho_3(g) \cdot (e_i)_{p_0}$ for $g \in SU(2)$. Set

$$\eta_1_{p_0} = \frac{1}{\sqrt{2}}(i, 0, 0, 1), \quad \eta_3_{p_0} = \frac{1}{\sqrt{42}}(2\sqrt{3}i, -2, 3i, 2\sqrt{3}),$$

$$\eta_2_{p_0} = \frac{1}{\sqrt{2}}(-1, 0, 0, -i), \quad \eta_4_{p_0} = \frac{1}{\sqrt{42}}(-3, 3i, -2, \sqrt{3}i),$$

which is an orthonormal basis of the normal bundle at $p_0$. Setting $(\eta_j)_{\rho_3(g) \cdot p_0} = \rho_3(g) \cdot (\eta_j)_{p_0}$ for $g \in SU(2)$, we obtain an orthonormal frame $\{\eta_j\}_{j=1}^4$ of the normal bundle $\nu$. 

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4.1 Second order deformations of $A_3$

Now, we consider the second order deformations of $A_3$. First, we describe the second derivative of the deformation map in a normal direction $V \in C^\infty(A_3, \nu)$ explicitly using Proposition 3.8. Since $S^7$ with the round metric $\langle \cdot, \cdot \rangle$ has constant sectional curvature 1, we have

$$R(x, y)z = \langle y, z \rangle x - \langle x, z \rangle y \quad \text{for } x, y, z \in TS^7,$$

which implies that

$$(R(V, e_i)V)^\perp = 0.$$  

Then by Proposition 3.8 it follows that

$$\frac{d^2}{dt^2} F(tV) \bigg|_{t=0} = 2 \sum_{i,j=1}^{3} \langle V, \Pi(e_i, e_j) \rangle e_i \times \nabla_{e_j} V.$$

We will compute this. By [11, Lemma 6.20] and its proof, we have the following.

**Lemma 4.1.**

$$\left(\nabla_{e_i} \eta_j\right)_{1 \leq i \leq 3, 1 \leq j \leq 4} = \frac{3}{7} \left( \begin{array}{cccc} -\eta_4 & -\eta_3 & \eta_2 & \eta_1 \\ \eta_3 & -\eta_4 & -\eta_1 & \eta_2 \\ 7\eta_2 & -7\eta_1 & -5\eta_4 & 5\eta_3 \end{array} \right),$$

$$\left( e_i \times \eta_j \right)_{1 \leq i \leq 3, 1 \leq j \leq 4} = \left( \begin{array}{cccc} \eta_4 & \eta_3 & -\eta_2 & -\eta_1 \\ -\eta_3 & \eta_4 & -\eta_1 & -\eta_2 \\ \eta_2 & -\eta_1 & \eta_4 & -\eta_3 \end{array} \right),$$

$$\left( \Pi(e_i, e_j) \right)_{1 \leq i, j \leq 3} = \frac{2\sqrt{3}}{7} \left( \begin{array}{ccc} -\eta_4 & -\eta_3 & 2\eta_1 \\ \eta_2 & -\eta_1 & 2\eta_4 \\ -2\eta_3 & 2\eta_4 & 0 \end{array} \right).$$

Then $\frac{d^2}{dt^2} F(tV) \bigg|_{t=0}$ is described explicitly as follows.

**Lemma 4.2.** Set

$$V = \sum_{j=1}^{4} V_j \eta_j \in \ker D, \quad \frac{d^2}{dt^2} F(tV) \bigg|_{t=0} = \sum_{j=1}^{4} F_j \eta_j,$$

where $V_j, F_j \in C^\infty(A_3)$ are smooth functions on $A_3$. Denoting $V_1 = V_1 + iV_2$, $V_2 = V_3 - iV_4$, we have

$$F_1 + iF_2 = \frac{4\sqrt{3}}{7} \left\{ -(ie_1 + e_2)(V_1 V_2) + \bar{V}_2(-ie_1 + e_2)V_1 + \left( ie_3 - \frac{24}{7} \right) (V_2^2) \right\},$$

$$F_3 - iF_4 = \frac{4\sqrt{3}}{7} \left\{ \bar{V}_1(-ie_1 + e_2)V_1 + \frac{1}{2} (ie_1 + e_2)(V_2^2) + \bar{V}_2 \left( 2ie_3 - \frac{48}{7} \right) V_1 - (-ie_1 + e_2)V_2 \right\} .$$
Proof. By the third equation of Lemma 4.1 we have
\[
\sum_{i,j=1}^{3} (V, \Pi(e_i, e_j)) e_i \times \nabla_{e_j}^\perp V
\]
\[
= \frac{2\sqrt{3}}{7} V_1 (e_1 \times \nabla_{e_1}^\perp V - e_2 \times \nabla_{e_2}^\perp V) + \frac{2\sqrt{3}}{7} V_2 (e_1 \times \nabla_{e_2}^\perp V + e_2 \times \nabla_{e_1}^\perp V)
\]
\[
- \frac{4\sqrt{3}}{7} V_3 (e_1 \times \nabla_{e_3}^\perp V + e_3 \times \nabla_{e_1}^\perp V) + \frac{4\sqrt{3}}{7} V_4 (e_2 \times \nabla_{e_3}^\perp V + e_3 \times \nabla_{e_2}^\perp V).
\]

By the first and the second equations of Lemma 4.1 we have
\[
\nabla_{e_1}^\perp V = \sum_{j=4}^{7} e_1(V_j) \eta_j + \frac{3}{7} (-V_1 \eta_4 - V_2 \eta_3 + V_3 \eta_2 + V_4 \eta_1),
\]
\[
\nabla_{e_2}^\perp V = \sum_{j=4}^{7} e_2(V_j) \eta_j + \frac{3}{7} (V_1 \eta_3 - V_2 \eta_4 - V_3 \eta_1 + V_4 \eta_2),
\]
\[
\nabla_{e_3}^\perp V = \sum_{j=4}^{7} e_3(V_j) \eta_j + \frac{3}{7} (7V_1 \eta_2 - 7V_2 \eta_1 - 5V_3 \eta_4 + 5V_4 \eta_3).
\]

Then by the second equation of Lemma 4.1 and a straightforward computation, we obtain
\[
\left. \frac{d^2}{dt^2} F(tV) \right|_{t=0}
\]
\[
= \frac{4\sqrt{3}}{7} V_1 \left\{ (-e_1(V_4) - e_2(V_3)) \eta_1 + (-e_1(V_3) + e_2(V_4)) \eta_2
\right.
\]
\[
+ (e_1(V_2) + e_2(V_1)) \eta_3 + (e_1(V_1) - e_2(V_2)) \eta_4
\]
\[
+ \frac{4\sqrt{3}}{7} V_2 \left\{ (-e_2(V_4) + e_1(V_3)) \eta_1 + (-e_2(V_3) - e_1(V_4)) \eta_2
\right.
\]
\[
+ (e_2(V_2) - e_1(V_1)) \eta_3 + (e_2(V_1) + e_1(V_2)) \eta_4
\]
\[
- \frac{8\sqrt{3}}{7} V_3 \left\{ (-e_3(V_4) - e_1(V_2)) \eta_1 + \left( -e_3(V_3) + e_1(V_1) - \frac{12}{7} V_4 \right) \eta_2
\right.
\]
\[
+ \left( e_3(V_2) - e_1(V_3) + \frac{24}{7} V_1 \right) \eta_3 + \left( e_3(V_1) + e_1(V_3) - \frac{24}{7} V_2 \right) \eta_4
\]
\[
+ \frac{8\sqrt{3}}{7} V_4 \left\{ \left( e_3(V_3) - e_2(V_2) + \frac{12}{7} V_4 \right) \eta_1 + \left( -e_3(V_4) + e_2(V_1) + \frac{12}{7} V_3 \right) \eta_2
\right.
\]
\[
+ \left( -e_3(V_1) - e_2(V_4) + \frac{24}{7} V_2 \right) \eta_3 + \left( e_3(V_2) + e_2(V_3) + \frac{24}{7} V_1 \right) \eta_4 \}.
\]
Hence

\[
F_1 + iF_2 = \frac{4\sqrt{3}}{7} V_1 (-e_1(V_4 + iV_3) - e_2(V_3 - iV_4)) \\
+ \frac{4\sqrt{3}}{7} V_2 (-e_2(V_4 + iV_3) + e_1(V_3 - iV_4)) \\
- \frac{8\sqrt{3}}{7} V_3 \left( -e_3(V_4 + iV_3) + e_1(iV_1 - V_2) + \frac{12}{7} (V_3 - iV_4) \right) \\
+ \frac{8\sqrt{3}}{7} V_4 \left( e_3(V_3 - iV_4) + e_2(iV_1 - V_2) + \frac{12}{7} (V_4 + iV_3) \right) \\
= \frac{4\sqrt{3}}{7} V_1 (-ie_1 - e_2)(V_3 - iV_4) + \frac{4\sqrt{3}}{7} V_2(e_1 - ie_2)(V_3 - iV_4) \\
- \frac{8\sqrt{3}}{7} V_3 \left( ie_1(V_1 + iV_2) + \left( -ie_3 + \frac{12}{7} \right) (V_3 - iV_4) \right) \\
+ \frac{8\sqrt{3}}{7} V_4 \left( ie_2(V_1 + iV_2) + \left( e_3 + \frac{12}{7} i \right) (V_3 - iV_4) \right),
\]

\[
F_3 - iF_4 = \frac{4\sqrt{3}}{7} V_1 (e_1(V_2 - iV_1) + e_2(V_1 + iV_2)) \\
+ \frac{4\sqrt{3}}{7} V_2 (e_2(V_2 - iV_1) - e_1(V_1 + iV_2)) \\
- \frac{8\sqrt{3}}{7} V_3 \left( e_3(V_2 - iV_1) - e_1(V_4 + iV_3) + \frac{24}{7} (V_1 + iV_2) \right) \\
+ \frac{8\sqrt{3}}{7} V_4 \left( -e_3(V_1 + iV_2) - e_2(V_4 + iV_3) + \frac{24}{7} (V_2 - iV_1) \right) \\
= \frac{4\sqrt{3}}{7} V_1 (-ie_1 + e_2)(V_1 + iV_2) + \frac{4\sqrt{3}}{7} V_2(-e_1 - ie_2)(V_1 + iV_2) \\
- \frac{8\sqrt{3}}{7} V_3 \left( \left( -ie_3 + \frac{24}{7} \right) (V_1 + iV_2) - ie_1(V_3 - iV_4) \right) \\
+ \frac{8\sqrt{3}}{7} V_4 \left( \left( -e_3 - \frac{24}{7} i \right) (V_1 + iV_2) - ie_2(V_3 - iV_4) \right).
\]

Using \(2i(-V_3e_1 + V_4e_2) = -V_2(ie_1 + e_2) + V_2(-ie_1 + e_2)\), we obtain the statement. \(\square\)

By \(11\) (6.24),(6.25)] and the proof of \(11\) Proposition 6.22, we know the following about \(\ker D\), where \(D\) is given in Proposition 6.11. Note that \(D\) in this paper corresponds to \(D + \text{id}_\nu\) in \(11\).

**Lemma 4.3.** For \(V = \sum_{j=1}^4 V_j \eta_j \in C^\infty(L, \nu)\), set \(V_1 = V_1 + iV_2\) and \(V_2 = V_3 - iV_4\). Then \(DV = 0\) is equivalent to

\[
\left( ie_3 - \frac{8}{7} \right) V_1 + (-ie_1 + e_2) V_2 = 0,
\]

\[
-(ie_1 + e_2) V_1 + (-ie_3 + 4) V_2 = 0. \quad (4.5)
\]

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By using the notation in Appendix A, elements of \( \ker D \) are explicitly described as

\[
\mathcal{V}_1 = -i \sqrt{\frac{7}{10}} \langle \rho_0(\cdot) e_5^{(6)}, u_1 \rangle - 2i \sqrt{\frac{7}{6}} \langle \rho_4(\cdot) e_3^{(4)}, u_2 \rangle,
\]
\[
\mathcal{V}_2 = \langle \rho_0(\cdot) e_5^{(6)}, u_1 \rangle + \langle \rho_4(\cdot) e_2^{(4)}, u_2 \rangle + \langle \rho_4(\cdot) e_4^{(4)}, u_3 \rangle
\]

for \( u_1 \in \mathcal{V}_6, u_2, u_3 \in \mathcal{V}_4 \).

**Lemma 4.4.** For \( V, W \in \ker D \), the \( L^2 \) inner product of \( \frac{d^2}{dt^2} F(tV) \bigg|_{t=0} \) and \( W \) is given by

\[
\left\langle \frac{d^2}{dt^2} F(tV) \bigg|_{t=0}, W \right\rangle_{L^2} = \frac{4 \sqrt{3}}{7} \text{Re} \left( I(V, W) + I(V + W, V) - I(V, V) - I(W, V) \right).
\]

Here,

\[
I(V, W) = \int_{SU(2)} \left( \mathcal{V}_1 \mathcal{V}_2 \cdot (-ie_1 + e_2) \mathcal{W}_1 + \frac{1}{2} \mathcal{V}_2 \cdot (3ie_3 - 8) \mathcal{W}_1 \right) dg,
\]

where \( V = \sum_{j=1}^{4} V_j \eta_j, W = \sum_{j=1}^{4} W_j \eta_j, \mathcal{V}_1 = V_1 + iV_2, \mathcal{V}_2 = V_3 - iV_4, \mathcal{W}_1 = W_1 + iW_2 \) and \( \mathcal{W}_2 = W_3 - iW_4 \).

**Proof.** Use the notation in Lemma 4.2. First note that

\[
\left\langle \frac{d^2}{dt^2} F(tV) \bigg|_{t=0}, W \right\rangle_{L^2} = \text{Re} \int_{SU(2)} ((F_1 + iF_2) \cdot \mathcal{W}_1 + (F_3 - iF_4) \cdot \mathcal{W}_2) dg.
\]

By using Lemma A.2 we can integrate by parts to obtain

\[
- \int_{SU(2)} (ie_1 + e_2)(\mathcal{V}_1 \mathcal{V}_2) \cdot \mathcal{W}_1 dg = \int_{SU(2)} \mathcal{V}_1 \mathcal{V}_2 \cdot (-ie_1 + e_2) \mathcal{W}_1 dg,
\]

\[
\int_{SU(2)} \left( \left( ie_3 - \frac{24}{7} \right) \mathcal{V}_2 \cdot \mathcal{W}_1 + \frac{1}{2} (ie_1 + e_2)(\mathcal{V}_2^2) \cdot \mathcal{W}_2 \right) dg
\]

\[
= \int_{SU(2)} \mathcal{V}_2 \cdot \left( \left( ie_3 - \frac{24}{7} \right) \mathcal{W}_1 - \frac{1}{2} (ie_1 + e_2) \mathcal{W}_2 \right) dg = \frac{1}{2} \int_{SU(2)} \mathcal{V}_2 \cdot (3ie_3 - 8) \mathcal{W}_1 dg.
\]

We also have

\[
\mathcal{V}_2 \left( 2ie_3 - \frac{48}{7} \right) \mathcal{V}_1 - (-ie_1 + e_2) \mathcal{V}_2 \mathcal{V}_2 \cdot (3ie_3 - 8) \mathcal{V}_1.
\]

Thus it follows that

\[
\left\langle \frac{d^2}{dt^2} F(tV) \bigg|_{t=0}, W \right\rangle_{L^2} = \frac{4 \sqrt{3}}{7} \text{Re} \int_{SU(2)} \left( \mathcal{V}_1 \mathcal{V}_2 \cdot (-ie_1 + e_2) \mathcal{W}_1 + \frac{1}{2} \mathcal{V}_2 \cdot (3ie_3 - 8) \mathcal{W}_1 \right.
\]

\[
\left. + (\mathcal{V}_2 \mathcal{W}_1 + \mathcal{V}_1 \mathcal{W}_2) \cdot (-ie_1 + e_2) \mathcal{V}_1 + \mathcal{V}_2 \mathcal{W}_2 \cdot (3ie_3 - 8) \mathcal{V}_1 \right) dg.
\]

From the equations \( \mathcal{V}_2 \mathcal{W}_1 + \mathcal{V}_1 \mathcal{W}_2 = (\mathcal{V}_1 + \mathcal{W}_1)(\mathcal{V}_2 + \mathcal{W}_2) - (\mathcal{V}_1 \mathcal{V}_2 + \mathcal{W}_1 \mathcal{W}_2) \) and \( 2\mathcal{V}_2 \mathcal{W}_2 = (\mathcal{V}_2 + \mathcal{W}_2)^2 - \mathcal{V}_2^2 - \mathcal{W}_2^2 \), the proof is done.
Thus we only have to calculate $I(V, W)$ for any $V, W \in \ker D$ to compute
\[
\left< \frac{d}{dt} F(tV) \right|_{t=0}, W \right>_{L^2}.
\] In fact, we have the following.

**Lemma 4.5.** For $V, W \in \ker D$, we have
\[
I(V, W) = 0.
\]

**Proof.** For $V = \sum_{j=1}^4 V_j \eta_j$ and $W = \sum_{j=1}^4 W_j \eta_j$, set $\mathcal{V}_1 = V_1 + i V_2$, $\mathcal{V}_2 = V_3 - i V_4$, $\mathcal{W}_1 = W_1 + i W_2$ and $\mathcal{W}_2 = W_3 - i W_4$. By Lemma 4.3, we may assume that $\mathcal{V}_1, \mathcal{V}_2$ are given by (4.6) for $u_1 \in V_0, u_2, u_3 \in V_4$ and $\mathcal{W}_1, \mathcal{W}_2$ are given by the right hand side of (4.6), where we replace $u_j$ with $w_j$ for $j = 1, 2, 3$ and $w_1 \in V_0, w_2, w_3 \in V_4$.

By (A.3) and $\{e_1, e_2, e_3\} = \{E_1/\sqrt{7}, E_2/\sqrt{7}, E_3\}$, note that
\[
(-ie_1 + e_2)\mathcal{W}_1 = 2\sqrt{\frac{3}{5}} \langle \rho_0(\cdot)v_6^{(6)}, w_1 \rangle + \frac{8}{\sqrt{6}} \langle \rho_4(\cdot)v_4^{(4)}, w_2 \rangle,
\]
\[
(3ie_3 - 8)\mathcal{W}_1 = -4i \sqrt{\frac{7}{10}} \langle \rho_0(\cdot)v_6^{(6)}, w_1 \rangle + 4i \sqrt{\frac{7}{6}} \langle \rho_4(\cdot)v_4^{(4)}, w_2 \rangle.
\]
Then by Lemmas 3.3 and 3.4, we compute
\[
I(V, W) = \int_{\text{SU}(2)} \left( \mathcal{V}_1 \mathcal{V}_2 \cdot (\rho_4(\cdot)v_4^{(4)}, u_2) \langle \rho_4(\cdot)v_4^{(4)}, u_3 \rangle - \frac{1}{2} \mathcal{V}_2 \cdot (3ie_3 - 8)\mathcal{W}_1 \right) \frac{d}{dt} F(tV) \left|_{t=0} \right. \, dg
\]
\[
= \int_{\text{SU}(2)} \left( -2i \sqrt{\frac{7}{6}} \langle \rho_4(g)v_4^{(4)}, u_2 \rangle \langle \rho_4(g)v_4^{(4)}, u_3 \rangle \cdot 2 \sqrt{\frac{3}{5}} \langle \rho_0(g)v_6^{(6)}, w_1 \rangle \right.
\]
\[
- i \sqrt{\frac{7}{10}} \langle \rho_0(g)v_6^{(6)}, u_1 \rangle \langle \rho_4(g)v_2^{(4)}, u_2 \rangle \cdot \frac{8}{\sqrt{6}} \langle \rho_4(g)v_4^{(4)}, w_2 \rangle
\]
\[
- 2i \sqrt{\frac{7}{6}} \langle \rho_4(g)v_3^{(4)}, u_2 \rangle \langle \rho_0(g)v_6^{(6)}, u_1 \rangle \cdot \frac{8}{\sqrt{6}} \langle \rho_4(g)v_4^{(4)}, w_2 \rangle
\]
\[
+ 4i \sqrt{\frac{7}{10}} \langle \rho_4(g)v_2^{(4)}, u_2 \rangle \langle \rho_4(g)v_4^{(4)}, u_3 \rangle \cdot \langle \rho_0(g)v_6^{(6)}, w_1 \rangle
\]
\[
- 4i \sqrt{\frac{7}{6}} \langle \rho_0(g)v_6^{(6)}, u_1 \rangle \langle \rho_4(g)v_2^{(4)}, u_2 \rangle \langle \rho_4(g)v_2^{(4)}, u_3 \rangle \langle \rho_0(g)v_6^{(6)}, w_1 \rangle.
\]

(A.3)
\[
\int_{\text{SU}(2)} \left( -4i \sqrt{\frac{7}{10}} \langle \rho_4(g)v_2^{(4)}, u_2 \rangle \langle \rho_4(g)v_2^{(4)}, u_3 \rangle \langle \rho_0(g)v_6^{(6)}, w_1 \rangle
\]
\[
+ i \sqrt{\frac{7}{10}} \langle \rho_4(g)v_2^{(4)}, u_2 \rangle \langle \rho_4(g)v_2^{(4)}, u_3 \rangle \langle \rho_0(g)v_6^{(6)}, u_1 \rangle
\]
\[
- 2i \sqrt{\frac{7}{6}} \langle \rho_4(g)v_2^{(4)}, u_2 \rangle \langle \rho_4(g)v_2^{(4)}, u_3 \rangle \langle \rho_0(g)v_6^{(6)}, u_1 \rangle
\]
\[
+ 4i \sqrt{\frac{7}{10}} \langle \rho_4(g)v_2^{(4)}, u_2 \rangle \langle \rho_4(g)v_2^{(4)}, u_3 \rangle \langle \rho_0(g)v_6^{(6)}, w_1 \rangle
\]
\[
+ 4i \sqrt{\frac{7}{6}} \langle \rho_4(g)v_2^{(4)}, u_2 \rangle \langle \rho_4(g)v_2^{(4)}, u_3 \rangle \langle \rho_0(g)v_6^{(6)}, u_1 \rangle \right) \, dg.
\]
By Lemmas 3.2 and B.5, we further compute
\[ \frac{1}{7} \left( -4i \sqrt{\frac{7}{10}} (v_3^{(4)} \otimes v_4^{(4)}, \alpha_{4,4,1}(v_6^{(6)})) \cdot \langle u_2 \otimes u_3, \alpha_{4,4,1}(w_1) \rangle + i \sqrt{\frac{7}{10}} \cdot \frac{8}{\sqrt{6}} (v_2^{(4)} \otimes v_0^{(4)}, \alpha_{4,4,1}(v_1^{(6)})) \cdot \langle u_2 \otimes w_2^{+}, \alpha_{4,4,1}(u_1^{+}) \rangle - 2i \sqrt{\frac{7}{6}} \cdot \frac{8}{\sqrt{6}} (v_3^{(4)} \otimes v_0^{(4)}, \alpha_{4,4,1}(v_2^{(6)})) \cdot \langle u_2 \otimes w_2^{+}, \alpha_{4,4,1}(u_1^{+}) \rangle + 4i \sqrt{\frac{7}{10}} (v_2^{(4)} \otimes v_4^{(4)}, \alpha_{4,4,1}(v_6^{(6)})) \cdot \langle u_2 \otimes u_3, \alpha_{4,4,1}(w_1) \rangle + 4i \sqrt{\frac{7}{6}} (v_2^{(4)} \otimes v_1^{(4)}, \alpha_{4,4,1}(v_2^{(6)})) \cdot \langle u_2 \otimes w_2^{+}, \alpha_{4,4,1}(u_1^{+}) \rangle \right) \]
\[ = \frac{1}{7} \left( 4i \sqrt{\frac{7}{10}} \sqrt{\frac{4}{3}} (-24 \sqrt{5} + 24 \sqrt{5}) \cdot \langle u_2 \otimes u_3, \alpha_{4,4,1}(w_1) \rangle + 4i \sqrt{\frac{7}{6}} \sqrt{\frac{4}{3}} \left( \frac{2}{\sqrt{10}} (-24 \sqrt{5}) + \frac{4}{\sqrt{6}} \cdot 24 \sqrt{3} - 24 \sqrt{2} \right) \cdot \langle u_2 \otimes w_2^{+}, \alpha_{4,4,1}(u_1^{+}) \rangle \right) = 0. \]

Theorem 1.1 follows from these lemmas.

**Proof of Theorem 1.1.** Recall that $D$ given in Proposition 3.1 is self-adjoint by Lemma 3.3. Then by Lemma 3.6, we only have to show that $\langle \frac{d}{dt} F(tV), W \rangle_{L^2} = 0$ for any $V, W \in \ker D$. This equation is satisfied by Lemmas 4.4 and 4.5.

### 4.2 Deformations of $A_3$ arising from Spin(7)

To see whether infinitesimal associative deformations of $A_3$ extend to actual deformations, it would be important to understand the trivial deformations (deformations given by the Spin(7)-action) of $A_3$. Since $A_3 \cong \text{SU}(2)$, the dimension of the subgroup of Spin(7) preserving $A_3$ is at least 3. We show that it is 4-dimensional. More precisely, we have the following.

**Lemma 4.6.** Use the notation in (4.1), (4.4), Lemmas C.2 and C.3. Set $p_0 = \frac{1}{\sqrt{2}} (0, 1, i, 0)$. Then we have
\[ \{ X \in \text{spin}(7); \langle X \cdot \rho_3(g) \cdot p_0, (\eta_i)_{\rho_3(g) \cdot p_0} \rangle = 0 \text{ for any } g \in \text{SU}(2) \text{ and } i = 1, \ldots, 4 \} = W_{1}^{\text{spin}(7)} \oplus W_{3}^{\text{spin}(7)}. \]

**Proof.** Since the left hand side is SU(2)-invariant, it is a direct sum of $W_{k}^{\text{spin}(7)}$ or $W_{l}^{\text{su}(4)}$. Thus we only have to see whether an element in $W_{k}^{\text{spin}(7)}$ or $W_{l}^{\text{su}(4)}$ is contained in the left hand side.
By definition, $W^{\text{spin}(4)}_3$ is contained in the left hand side. Via the identification of $\mathbb{C}^4 \cong \mathbb{R}^8$ given by (2.1), we see that

$$(\rho_3(g^{-1})H_0\rho_3(g)) \cdot p_0 = H_0 \cdot p_0 = \frac{1}{\sqrt{2}} t(0,0,0,1,1,0,0,0) = (\rho_3)_*(E_3) \cdot p_0$$

for any $g \in \text{SU}(2)$. Hence $W^{\text{spin}(4)}_1$ is contained in the left hand side.

For $X = \begin{pmatrix} 0 & i & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & 0 & -i & 0 \end{pmatrix} \in W^{\text{su}(4)}_5$, $Y = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \in W^{\text{su}(4)}_7$, and $Z = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \in W^{\text{spin}(7)}_5$, we have

$$(X \cdot p_0, (\eta_1)_{p_0}) = 1, (Y \cdot p_0, (\eta_1)_{p_0}) = 1 \text{ and } (Z \cdot p_0, (\eta_1)_{p_0}) = 2/\sqrt{7}. \text{ Note that}$$

$$p_0 = \frac{1}{\sqrt{2}} t(0,0,1,0,0,1,0,0) \text{ and } (\eta_1)_{p_0} = \frac{1}{\sqrt{42}} t(-2\sqrt{3},0,0,3,-3,0,0,2\sqrt{3}).$$

Hence $W^{\text{su}(4)}_5$, $W^{\text{su}(4)}_7$ and $W^{\text{spin}(7)}_5$ are not contained in the left hand side. \(\square\)

Hence by Lemma 4.6 we see that the space of trivial deformations of $A_3$ is isomorphic to

$$\text{spin}(7)/(W^{\text{spin}(7)}_1 \oplus W^{\text{su}(4)}_3) \cong W^{\text{spin}(7)}_5 \oplus W^{\text{su}(4)}_5 \oplus W^{\text{su}(4)}_7,$$

which is a 17-dimensional subspace of the 34-dimensional space $\ker D$.

**Remark 4.7.** Use the notation in [A.1], [A.2], [C.1] and [C.2]. By tedious calculations, we can describe elements of $\ker D$ given by $\text{spin}(7)/(W^{\text{spin}(7)}_1 \oplus W^{\text{su}(4)}_3) \cong W^{\text{spin}(7)}_5 \oplus W^{\text{su}(4)}_5 \oplus W^{\text{su}(4)}_7$. Elements in $\ker D$ are of the form (4.7).

In the following table, each space in the left-hand side corresponds to the elements in $\ker D$ given by the right-hand side.

| $\text{ker } D$ | $W^{\text{su}(4)}_3$ | $W^{\text{su}(4)}_5$ | $W^{\text{spin}(7)}_5$ |
|-----------------|-----------------|-----------------|-----------------|
| $u_1 \in (1-j)V_6$, $u_2 = u_3 = 0.$ | $u_1 = 0$, $u_2 \in (1-j)V_4$, $u_3 = (2\sqrt{6}/3) \cdot u_5^3.$ | $u_1 = 0$, $u_2 \in (1+j)V_4$, $u_3 = (2\sqrt{6}/3) \cdot u_5^3.$ |

**Appendix**

**A Representations of SU(2)**

In this section, we summarize the results about representations of SU(2). First, we recall the $\mathbb{C}$-irreducible representations of SU(2).

Let $V_n$ be a $\mathbb{C}$-vector space of all complex homogeneous polynomials with two variables $z_1, z_2$ of degree $n$, where $n \geq 0$, and define the representation
\( \rho_n : \text{SU}(2) \rightarrow \text{GL}(V_n) \) as

\[
\left( \rho_n \left( \begin{array}{cc} a & -\bar{b} \\ b & \bar{a} \end{array} \right) \right) f(z_1, z_2) = f \left( (z_1, z_2) \left( \begin{array}{cc} a & -\bar{b} \\ b & \bar{a} \end{array} \right) \right).
\]

Define the Hermitian inner product \( \langle , \rangle \) of \( V_n \) such that

\[
\left\{ v_k^{(n)} = z_1^{n-k} \bar{z}_2^k / \sqrt{k!(n-k)!} \right\}_{0 \leq k \leq n}
\]

is a unitary basis of \( V_n \). Denoting by \( \text{SU}(2) \) the set of all equivalence classes of finite dimensional irreducible representations of \( \text{SU}(2) \), we know that \( \text{SU}(2) = \{ (V_n, \rho_n); n \geq 0 \} \). Then every \( \mathbb{C} \)-valued continuous function on \( \text{SU}(2) \) is uniformly approximated by the \( \mathbb{C} \)-linear combination of

\[
\left\{ (\rho_n(f))^{(n)}_{i, j}; n \geq 0, \ 0 \leq i, j \leq n \right\},
\]

which are mutually orthogonal with respect to the \( L^2 \) inner product.

Next, we review the \( \mathbb{R} \)-irreducible representations of \( \text{SU}(2) \) by [17, Section 2]. A more general reference of this topic is [21].

Define the map \( j : V_n \rightarrow V_n \) by

\[
(jf)(z_1, z_2) = f(-z_2, z_1), \quad (A.1)
\]

which is a \( \mathbb{C} \)-antilinear \( \text{SU}(2) \)-equivariant map satisfying \( j^2 = (-1)^n \). This map \( j \) is called a structure map ([17, Section 2]).

When \( n \) is even, we have \( j^2 = 1 \) and \( V_n \) decomposes into two mutually equivalent real irreducible representations: \( V_n = (1+j)V_n \oplus (1-j)V_n \). When \( n \) is odd, \( V_n \) is also irreducible as a real representation.

All of the real irreducible representations are given in this way, and hence their dimensions are given by \( 4m \) or \( 2n + 1 \) for \( m, n \geq 0 \). Denote by \( W_k \), where \( k \in 4 \mathbb{Z} \cup (2 \mathbb{Z} + 1) \), the \( k \)-dimensional \( \mathbb{R} \)-irreducible representation of \( \text{SU}(2) \). It follows that

\[
V_{2m+1} = W_{4m+4}, \quad V_{2m} = W_{2m+1} \oplus W_{2m+1} \quad \text{for } m \geq 0. \quad (A.2)
\]

The characters \( \chi_{V_n} \) of \( V_n \) are determined by the values on the maximal torus \( \{ h_a = \left( \begin{array}{cc} a & 0 \\ 0 & \bar{a} \end{array} \right); a \in \mathbb{C}, |a| = 1 \} \) of \( \text{SU}(2) \). It is well-known that

\[
\chi_{V_n}(h_a) = \sum_{k=0}^{n} a^{2k-n} = \frac{a^{n+1} - a^{-(n+1)}}{a - a^{-1}}.
\]

By (A.2), the characters \( \chi_{W_k} \) of \( W_k \) on the maximal torus are given by

\[
\chi_{W_{4m+4}}(h_a) = 2 \chi_{V_{2m+1}}(h_a) = 2 \sum_{k=0}^{2m+1} a^{2k-(2m+1)} = \frac{2(a^{2m+2} - a^{-(2m+2)})}{a - a^{-1}},
\]

\[
\chi_{W_{2m+1}}(h_a) = \chi_{V_{2m}}(h_a) = \sum_{k=0}^{2m} a^{2k-2m} = \frac{a^{2m+1} - a^{-(2m+1)}}{a - a^{-1}}. \quad (A.3)
\]

Finally, we summarize technical lemmas.
Lemma A.1 ([11] Lemma 6.9). For \( u = \sum_{i=0}^{n} C_{i}v_{i}^{(n)} \in V_{n} \), set
\[
  u^{*} = ju = \sum_{l=0}^{n} (-1)^{n-l}C_{n-l}v_{l}^{(n)} \in V_{n}.
\]
Then for any \( n \geq 0, 0 \leq k \leq n, u \in V_{n} \), we have
\[
  \langle \rho_{(\cdot)} v_{k}^{(n)}, u \rangle = (-1)^{k} \langle \rho_{(\cdot)} v_{n-k}^{(n)}, u^{*} \rangle. \tag{A.4}
\]
Let \( \{E_{1}, E_{2}, E_{3}\} \) be the basis of the Lie algebra \( \mathfrak{su}(2) \) of \( SU(2) \) given by (4.3). Identify \( E_{i} \in \mathfrak{su}(2) \) with the left invariant differential operator on \( SU(2) \). Then
\[
  (-iE_{1} + E_{2})(\rho_{(\cdot)} v_{k}^{(n)}, u) = \begin{cases} 
    2i\sqrt{(k+1)(n-k)}\langle \rho_{(\cdot)} v_{k+1}^{(n)}, u \rangle, & (k < n) \\
    0, & (k = n)
  \end{cases}
\]
\[
  (iE_{1} + E_{2})(\rho_{(\cdot)} v_{k}^{(n)}, u) = \begin{cases} 
    2i\sqrt{k(n-k+1)}\langle \rho_{(\cdot)} v_{k-1}^{(n)}, u \rangle, & (k > 0) \\
    0, & (k = 0)
  \end{cases}
\]
\[
  iE_{3}(\rho_{(\cdot)} v_{k}^{(n)}, u) = (-n+2k)\langle \rho_{(\cdot)} v_{k}^{(n)}, u \rangle. \tag{A.5}
\]
Since the Haar measure is \( SU(2) \)-invariant, we have the following.

**Lemma A.2.** For any \( X \in \mathfrak{su}(2) \) and a smooth function \( f \) on \( SU(2) \), we have
\[
  \int_{SU(2)} X(f)(g) dg = 0.
\]

### B Clebsch-Gordan decomposition

Use the notation in Section A. In the computation in Section A we need the irreducible decomposition of \( V_{m} \otimes V_{n} \) for \( m, n \geq 0 \). This is well-known as the Clebsch-Gordan decomposition:
\[
  V_{m} \otimes V_{n} = \bigoplus_{h=0}^{\min\{m,n\}} V_{m+n-2h}.
\]
Identify \( V_{m} \otimes V_{n} \) with the vector subspace of polynomials in \((z_{1}, z_{2}, w_{1}, w_{2})\) consisting of homogeneous polynomials of degree \( m \) in \((z_{1}, z_{2})\) and of degree \( n \) in \((w_{1}, w_{2})\). Then the inclusion \( V_{m+n-2h} \rightarrow V_{m} \otimes V_{n} \) is explicitly given as follows.

**Lemma B.1 ([22] p.46, [2] Section 2.1.2).** For \( 0 \leq h \leq \min\{m,n\} \), define the map
\[
  \alpha_{m,n,h} : V_{m+n-2h} \rightarrow V_{m} \otimes V_{n}
\]
by
\[
  \alpha_{m,n,h}(f(z_{1}, z_{2})) = \sqrt{c_{m,n,h}}(z_{1}w_{2} - z_{2}w_{1})^{h} \left( w_{1} \frac{\partial}{\partial z_{1}} + w_{2} \frac{\partial}{\partial z_{2}} \right)^{n-h} (f(z_{1}, z_{2})),
\]
where \( c_{m,n,h} > 0 \) is given in [2] Section 2.2.2. Then the map \( \alpha_{m,n,h} \) is \( SU(2) \) equivariant and isometric.
Denote by \( \rho_{m,n} \) the induced representation of SU(2) on \( V_m \otimes V_n \). Since we know that
\[
\langle \rho_m(g)u_m, u'_m \rangle \langle \rho_n(g)u_n, u'_n \rangle = \langle \rho_{m,n}(g)(u_m \otimes u_n), u'_m \otimes u'_n \rangle
\]
for \( u_m, u'_m \in V_m, u_n, u'_n \in V_n \) and \( g \in \text{SU}(2) \), we have the following by Lemma B.1 and the Schur orthogonality relations.

**Lemma B.2.** Set \( r = m + n - 2h \). Then we have
\[
\int_{\text{SU}(2)} \langle \rho_m(g)u_m, u'_m \rangle \langle \rho_n(g)u_n, u'_n \rangle \langle \rho_r(g)u_r, u'_r \rangle dg = \frac{1}{r + 1} \langle u_m \otimes u_n, \alpha_{m,n,h}(u_r) \rangle \langle u'_m \otimes u'_n, \alpha_{m,n,h}(u'_r) \rangle
\]
for \( u_r, u'_r \in V_r \).

The next lemma is very useful for the computation in Section 3.

**Lemma B.3.**
\[
\int_{\text{SU}(2)} \langle \rho_m(g)v^{(m)}_a, u'_m \rangle \langle \rho_n(g)v^{(n)}_b, u'_n \rangle \langle \rho_r(g)v^{(r)}_c, u'_r \rangle dg = 0
\]
for any \( u'_m \in V_m, u_n' \in V_n, u'_r \in V_r \) if
\[
a + b \neq c + h \left( = c + \frac{m + n - r}{2} \right).
\]

**Proof.** We compute
\[
\left( \sqrt{(r-c)!/\sqrt{c_{m,n,h}}(v^{(r)}_c)} \right) = (z_1w_2 - z_2w_1)^h \left( w_1 \frac{\partial}{\partial z_1} + w_2 \frac{\partial}{\partial z_2} \right)^{n-h} (z_1^{r-c}z_2^c)
\]
\[
= \sum_{i=0}^{h} \sum_{j=0}^{n-h} \binom{h}{i} \binom{n-h}{j} (z_1w_2)^i (-z_2w_1)^{h-i} w_1^{n-h-j} \left( \frac{\partial}{\partial z_1} \right)^i \left( \frac{\partial}{\partial z_2} \right)^{n-h-j} (z_1^{r-c}z_2^c)
\]
\[
\in \text{span} \left\{ v^{(m)}_{i-h+j} \otimes v^{(n)}_{i+h-j} : 0 \leq i \leq h, 0 \leq j \leq n-h \right\}
\]
\[
\in \text{span} \left\{ v^{(m)}_d \otimes v^{(n)}_c : d + e = c + h \right\},
\]
which gives the proof. \( \square \)

In this paper, the case of \( (m, n, h) = (4, 4, 1) \) or \( (6, 6, 3) \) is important. Recall that the character of the induced representation on the second symmetric power \( S^2(V_n) \) is given by \( (\chi_{V_n}(g)^2 + \chi_{V_n}(g^2))/2 \). (For example, see Exercise 2.2.) By computing the character of \( S^2(V_4) \) and \( S^2(V_6) \), we see that
\[
S^2(V_4) = V_8 \oplus V_4 \oplus V_0, \quad S^2(V_6) = V_{12} \oplus V_8 \oplus V_4 \oplus V_0.
\]
Thus we have \( \alpha_{4,4,1}(V_6) \subset \Lambda^2 V_4, \alpha_{6,6,3}(V_6) \subset \Lambda^2 V_6 \) and we obtain the following.
Lemma B.4. Suppose that \( m = 4 \) or 6 and \( u_m, \hat{u}_m, u'_m, \hat{u}'_m \in V_m \). If \( u_m = \hat{u}_m \) or \( u'_m = \hat{u}'_m \), we have

\[
\int_{SU(2)} \langle \rho_m(g)u_m, u'_m \rangle \langle \rho_m(g)\hat{u}_m, \hat{u}'_m \rangle \langle \rho_6(g)v_6, v'_6 \rangle \, dg = 0
\]

for any \( v_6, v'_6 \in V_6 \).

The next lemma is straightforward and we omit the proof.

Lemma B.5.

\[
\alpha_{4,4,1}(v_0^{(6)}) = \sqrt{c_{4,4,1} \cdot 24} v_0^{(4)} \wedge v_1^{(4)}, \\
\alpha_{4,4,1}(v_1^{(6)}) = \sqrt{c_{4,4,1} \cdot 24} v_1^{(4)} \wedge v_2^{(4)}, \\
\alpha_{4,4,1}(v_2^{(6)}) = \sqrt{c_{4,4,1} \cdot 24} (v_1^{(4)} \wedge v_3^{(4)} + \sqrt{2} v_2^{(4)} \wedge v_3^{(4)}), \\
\alpha_{4,4,1}(v_3^{(6)}) = \sqrt{c_{4,4,1} \cdot 24} (v_3^{(4)} \wedge v_4^{(4)} + \sqrt{2} v_4^{(4)} \wedge v_3^{(4)}), \\
\alpha_{4,4,1}(v_4^{(6)}) = \sqrt{c_{4,4,1} \cdot 24} v_4^{(4)} \wedge v_5^{(4)}, \\
\alpha_{4,4,1}(v_5^{(6)}) = \sqrt{c_{4,4,1} \cdot 24} v_5^{(4)} \wedge v_4^{(4)}. \]

C Irreducible decomposition of \( \text{spin}(7) \)

In this section, we give an irreducible decomposition of the Lie algebra \( \text{spin}(7) \) of Spin(7) under an SU(2)-action. First, we study the \( \mathfrak{su}(4) \subset \text{spin}(7) \) case.

Lemma C.1. Use the notation in Appendix A. Let SU(2) act on \( \mathfrak{su}(4) \) by the composition of \( p_3 : SU(2) \hookrightarrow SU(4) \) given by (4.7) and the adjoint action of SU(4) on \( \mathfrak{su}(4) \). Then we have

\[
\mathfrak{su}(4) \cong W_3 \oplus W_5 \oplus W_7.
\]

More explicitly, \( W_k \) corresponds to the \( k \)-dimensional SU(2)-invariant subspace \( W_k^{\mathfrak{su}(4)} \) of \( \mathfrak{su}(4) \), where \( k = 3, 5, 7 \), given by

\[
W_3^{\mathfrak{su}(4)} = (\rho_3), \mathfrak{su}(2) = \left\{ \begin{pmatrix} 3ia & \sqrt{3}z & 0 & 0 \\ -\sqrt{3}z & ia & 2z & 0 \\ 0 & -2z & -ia & \sqrt{3}z \\ 0 & 0 & -\sqrt{3}z & -3ia \end{pmatrix} ; z \in \mathbb{C}, a \in \mathbb{R} \right\},
\]

\[
W_5^{\mathfrak{su}(4)} = \left\{ \begin{pmatrix} ia & z & w & 0 \\ -\bar{z} & -ia & 0 & w \\ -\bar{w} & 0 & -ia & -\bar{z} \\ 0 & -\bar{w} & \bar{z} & ia \end{pmatrix} ; z, w \in \mathbb{C}, a \in \mathbb{R} \right\},
\]

\[
W_7^{\mathfrak{su}(4)} = \left\{ \begin{pmatrix} ia & z_1 & z_2 & z_3 \\ -\bar{z}_1 & -3ia & -\sqrt{3}z_1 & -z_2 \\ -\bar{z}_2 & \sqrt{3}z_1 & 3ia & z_1 \\ -\bar{z}_3 & \bar{z}_2 & -\bar{z}_1 & -ia \end{pmatrix} ; z_1, z_2, z_3 \in \mathbb{C}, a \in \mathbb{R} \right\}.
\]
Proof. First, we compute the character of this representation on the maximal torus by using \(\text{SU}(2)\)-rotation and \(\text{SU}(3)\). Then we see that it is given by \(\chi W_5 + \chi W_5 + \chi W_5\). This is a straightforward computation, so we omit it. Hence we obtain the first statement.

We easily see that the three spaces above are invariant by the adjoint action of \((\rho_3)_\ast\text{su}(2)\). Hence the three spaces above are \text{SU}(2)-invariant and the proof is done. \(\Box\)

The explicit description of \(\text{spin}(7)\) is given in [16, Proposition 4.2]. It is straightforward to deduce the following so we omit the proof.

**Lemma C.2.** We have

\[
\text{spin}(7) = \text{su}(4) \oplus W_1^{\text{spin}(7)} \oplus W_5^{\text{spin}(7)},
\]

where

\[
W_1^{\text{spin}(7)} = \mathbb{R} H_0, \quad H_0 = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\
-1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\end{pmatrix},
\]

\[
W_5^{\text{spin}(7)} = \left\{ \begin{pmatrix}
0 & 0 & -a_1 & -a_2 & -a_3 & -a_4 & -a_5 & 0 \\
0 & 0 & a_1 & a_2 & a_3 & a_4 & a_5 & 0 \\
a_1 & a_2 & 0 & 0 & 0 & -a_3 & -a_5 & -a_4 \\
a_2 & -a_1 & 0 & 0 & -a_3 & 0 & a_4 & a_3 \\
a_3 & a_4 & 0 & a_5 & 0 & 0 & a_1 & -a_2 \\
a_4 & -a_3 & a_5 & 0 & 0 & 0 & -a_2 & -a_1 \\
a_5 & a_3 & -a_4 & -a_1 & a_2 & a_1 & 0 & 0 \\
a_5 & 0 & -a_4 & -a_3 & a_2 & a_1 & 0 & 0
\end{pmatrix} : a_1, \ldots, a_5 \in \mathbb{R} \right\},
\]

By using the notation in Appendix A, \(H_0\) is the structure map \(j : V_3 \to V_3\) given in [4.1] with respect to the basis \(\{v_0^{(3)}, v_0^{(3)}, \ldots, v_3^{(3)}\} \).

**Lemma C.3.** Use the notation in Appendix A. Let \(\text{SU}(2)\) act on \(\text{spin}(7)\) by the composition of \(\rho_3 : \text{SU}(2) \hookrightarrow \text{SU}(4) \subset \text{Spin}(7)\) given by [4.7] and the adjoint action of \(\text{Spin}(7)\) on \(\text{spin}(7)\). Then we have

\[
\text{spin}(7) \cong (W_1 \oplus W_5) \oplus (W_3 \oplus W_5 \oplus W_7).
\]

The subspaces \(W_1\) and the first \(W_5\) correspond to \(W_1^{\text{spin}(7)}\) and \(W_5^{\text{spin}(7)}\) in Lemma 4.2 respectively. The subspace \(W_3 \oplus W_5 \oplus W_7\) corresponds to \(\text{su}(4)\), whose irreducible decomposition is given in Lemma 4.1.

Proof. By Lemma 4.2 we only have to prove that \(H_0\) is invariant under the \(\text{SU}(2)\)-action and \(W_5^{\text{spin}(7)}\) is an irreducible 5-dimensional representation of \(\text{SU}(2)\).

Since \(H_0\) is the structure map, it is invariant under the \(\text{SU}(2)\)-action. We easily see that \(W_5^{\text{spin}(7)}\) is invariant by the adjoint action of \((\rho_3)_\ast\text{su}(2)\). Hence it is \(\text{SU}(2)\)-invariant. As in the proof of Lemma 4.1 we can compute the character of the \(\text{SU}(2)\)-representation on \(W_5^{\text{spin}(7)}\) and it is equal to \(\chi W_5\). Hence the proof is done. \(\Box\)
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