Study of the attractor structure of an agent-based sociological model

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Abstract. The Sznajd model is a sociophysics model that is based in the Potts model, and used for describing opinion propagation in a society. It employs an agent-based approach and interaction rules favouring pairs of agreeing agents. It has been successfully employed in modeling some properties and scale features of both proportional and majority elections (see for instance the works of A. T. Bernardes and R. N. Costa Filho), but its stationary states are always consensus states. In order to explain more complicated behaviours, we have modified the bounded confidence idea (introduced before in other opinion models, like the Deffuant model), with the introduction of prejudices and biases (we called this modification confidence rules), and have adapted it to the discrete Sznajd model. This generalized Sznajd model is able to reproduce almost all of the previous versions of the Sznajd model, by using appropriate choices of parameters. We solved the attractor structure of the resulting model in a mean-field approach and made Monte Carlo simulations in a Barabási-Albert network. These simulations show great similarities with the mean-field, for the tested cases of 3 and 4 opinions. The dynamical systems approach that we devised allows for a deeper understanding of the potential of the Sznajd model as an opinion propagation model and can be easily extended to other models, like the voter model. Our modification of the bounded confidence rule can also be readily applied to other opinion propagation models.

1. Introduction and Model Description
In the last decade, econophysics and sociophysics models have attracted the attention of physicists, due to the current availability of great amounts of data and computing power. In this context, the Sznajd model simulates the propagation of opinions, using the idea that bigger groups of agreeing people have a stronger convincing power. The society is modeled as a network with \( N \) nodes, each node in the network is an agent (person), each edge is a social connection (friendship, marriage, acquaintances, etc.) and each agent has an integer between 1 and \( M \), representing its opinion \( \sigma \) (i.e. \( \sigma \in \{1, 2, \ldots, M\} \)). At each time step, a pair of connected nodes, \( i \) and \( j \), are chosen at random (first one of the nodes is chosen and then one of its neighbours). If they agree (that is, if \( \sigma_i = \sigma_j \)) a neighbour \( k \) of \( j \) is chosen and convinced to opinion \( \sigma_j = \sigma_i \) with some probability. Otherwise (\( \sigma_i \neq \sigma_j \)), nothing happens. The idea is that the first step represents a conversation between two people that know each other and discuss some issue. If they disagree, none manages to convince the other. But, if they agree, they may set to convince another person, that one of them knows.

In the original model the conviction is always effective, meaning the pair always convinces the isolated site with probability 1. This leads to a dynamic where the only stationary states...
are consensus states (all the sites have the same opinion). Some modifications of the original model have been suggested in the literature in order to create stable coexistence states, like the Bounded Confidence rule [3] which says that all opinions are nuances between 2 extremes and opinion changes are not abrupt. More precisely, in a discrete opinion model, it means that agents with opinion $\sigma$ can only interact with agents that have opinion $\sigma \pm 1$. Another modification that has been used is to add an undecided state to agents [1].

In a recent work [4], we generalized the idea of Bounded Confidence, introducing a modification to the dynamics of the Sznajd Model that we called Confidence Rules. In this generalized version of the model, the probability that a pair of agreeing sites with opinion $\sigma$ convinces a site with opinion $\sigma'$ is $p_{\sigma'\rightarrow\sigma}$, that is a parameter of the model. The reason why this probability should depend on both opinions is that, usually an opinion includes prejudices about differing points of view. The set of probabilities $p_{\sigma\rightarrow\sigma'}$ will be referred to as the Confidence Rule.

We can recover the original Sznajd model and some of its modifications in the following limits:

- The usual Sznajd model is the case $p_{\sigma'\rightarrow\sigma} = 1$ for all $\sigma \neq \sigma'$.
- If an opinion $\sigma$ has $p_{\sigma'\rightarrow\sigma} = 0$ and $p_{\sigma\rightarrow\sigma'} = p \neq 0$, for all $\sigma' \neq \sigma$, then $\sigma$ behaves analogously to an undecided state.
- Usual bounded confidence is the case $p_{\sigma'\rightarrow\sigma} = 1$ if $\sigma' = \sigma \pm 1$ and $p_{\sigma'\rightarrow\sigma} = 0$ otherwise.

Note that $p_{\sigma\rightarrow\sigma}$ has no relevance whatsoever for any of the opinions $\sigma$.

### 2. Mean-Field Attractor Structure

The mean field version can be obtained from the master equation of the model defined in a complete network. When this network is large, this equation reduces to the flow

$$\dot{\eta}_\sigma = \sum_{\sigma'} \left( \eta^2_\sigma p_{\sigma'\rightarrow\sigma} - \eta_\sigma \eta^2_{\sigma'} p_{\sigma\rightarrow\sigma'} \right), \quad (1)$$

where $\eta_\sigma$ is the expected value of the fraction of sites holding opinion $\sigma$ and one time unit corresponds to $N$ simulation time steps, where $N$ is the number of nodes. Since $\sum_{\sigma=1}^{M} \eta_\sigma = 1$, the phase space of this flow can be represented by an $(M - 1)$-simplex (that is embedded in an $M$ dimensional space for convenience), where the vertices correspond to consensus states and the other states are convex combinations of the vertices, with coefficients $\eta_\sigma$.

![Figure 1: The 2 dimensional phase space (gray) for 3 opinions embedded in $\mathbb{R}^3$ ($\eta_1 + \eta_2 + \eta_3 = 1$; $\eta_1, \eta_2, \eta_3 \geq 0$)](image-url)
The fixed points are given by
\[ \eta_\sigma = 0 \text{ or } \sum_{\sigma'} \left( \eta_\sigma \eta_{\sigma'} p_{\sigma' \rightarrow \sigma} - \eta_\sigma^2 p_{\sigma \rightarrow \sigma'} \right) = 0 \]
for each opinion \( \sigma \). Let \( \vec{F} \) be the time derivative of \( \vec{\eta} \equiv (\eta_1, \eta_2, \ldots, \eta_M) \), \( \vec{F} = \dot{\vec{\eta}} \) and \( J \) be the jacobian of \( \vec{F} \). If \( J^* \) is this jacobian, evaluated at a fixed point \( \vec{\eta}^* \), then the analysis of the eigenvalues yields the following conclusions:

- If all opinions survive, \( J^* \) is positive semidefinite, but all eigenvalues with null real part are either due to embedding artifacts or to degeneracies (when the model can be split into simpler, non-interacting models), playing no role in the linear stability analysis. Hence, these fixed points have no stable manifolds. Also, with the exception of the case where the opinions are all non-interacting (when \( J^* = 0 \)), these fixed points have at least one unstable manifold.

- If there is a set \( \Omega \) of opinions that don’t survive, then there exists a permutation that leads \( J^* \) to
\[
\begin{bmatrix}
J^*_\Delta & M \\
0 & J^*_\Omega
\end{bmatrix},
\]
where \( J^*_\Delta \) (\( \Delta \equiv \bar{\Omega} \), the set of opinions not in \( \Omega \)) is the jacobian restricted to the surviving opinions and \( J^*_\Omega \) is the jacobian restricted to opinions in \( \Omega \) (note that \( M \) is not relevant to the eigenvalues of \( J^* \)). We can use for the block \( J^*_\Delta \) the former conclusion (for a coexistence fixed point in the model restricted to the surviving opinions). The block \( J^*_\Omega \) is a real, diagonal and negative semidefinite matrix. An analysis beyond linear stability shows that the null eigenvalues in \( J^*_\Omega \) correspond to unstable manifolds of the fixed point. Moreover, we can associate each of these eigenvalues to an opinion in \( \Omega \); the associated eigenvalue is negative iff the confidence rule is such that the opinion in \( \Omega \) can be convinced by some of the surviving opinions.

One interesting aspect of these results is that the only property of the parameters \( p_{\sigma \rightarrow \sigma'} \) relevant to the qualitative attractor structure is which of them are null and which are positive. The system has \( M(M - 1) \) of these probability parameters (\( p_{\sigma \rightarrow \sigma} = 0 \) in what follows), where \( M \) is the number of opinions and \( p_{\sigma \rightarrow \sigma'} \in [0, 1] \). Also, the parameters \( p_{\sigma \rightarrow \sigma'} \) can be thought as the elements of the adjacency matrix of a weighted graph (where an arrow \( (\sigma \rightarrow \sigma') \) means that the opinion \( \sigma \) can be convinced to opinion \( \sigma' \)), that will also be called confidence rule (as it is just a representation of the parameters that constitute the confidence rule) and denoted \( R \).

This graph helps to schematize the interactions among the different opinions. We can assign a skeleton \( Skel(G) \) to a weighted graph \( G \), which is the directed graph made of the same nodes as the weighted graph and all the arrows that have a non-zero weight (Figure 2). We can also assign, to each group of nodes \( \Delta \) in the confidence rule, the manifold \( M_\Delta \), where only the opinions in \( \Delta \) survive. In this way, the flow restricted to \( M_\Delta \) is equivalent to the model with a confidence rule \( R_\Delta \), the subgraph of \( R \) induced by \( \Delta \).

Our results for the mean field model can then be summed up as

- The stability properties of a fixed point depend only on \( Skel(R) \) and on which opinions survive in the fixed point. Particularly, if 2 interacting opinions survive, then the fixed point is always Lyapunov unstable.

- The manifold \( M_\Delta \) is an attractor, where all of its points are fixed points iff the opinions in \( \Delta \) are non-interacting, and every opinion not in \( \Delta \) can be convinced by at least one opinion in it.

A phase portrait that helps to illustrate the linear analysis results for \( M = 3 \) can be found in figure 3.
Figure 2: A confidence rule for 5 opinions and its skeleton. Here \( p_{1\rightarrow 2} = 0.2, p_{1\rightarrow 4} = 0.1, p_{1\rightarrow 5} = 0.3, p_{3\rightarrow 1} = 1, p_{3\rightarrow 2} = 0.8, p_{3\rightarrow 4} = 1, p_{4\rightarrow 3} = 0.2, p_{5\rightarrow 4} = 0.5 \) and \( p_{\sigma\rightarrow \sigma'} = 0 \) otherwise. The arrow \((1 \rightarrow 3)\) does not appear in the skeleton because it has weight 0, even though it was represented as part of the directed graph.

Figure 3: Phase portrait for a three opinions case. The arrows in the skeleton give the direction of the trajectories along the borders. The circles indicate unstable nodes, the diamonds indicate saddle points and the squares indicate stable nodes. The parameters \( p_{\sigma\rightarrow \sigma'} \) are either 0 or 1.

3. Simulations
Simulations of the model in a Barabási-Albert (BA) network show a very similar phase portrait (Figure 4 shows a comparison between the two) for 3 opinions and the attractor structure agrees with what was found in the mean-field analysis. The “phase space” in this case is actually a projection to the \((M-1)\)-simplex, where the states are described by \( \eta_\sigma \). As such, some odd phenomena are possible, as the crossing of ‘trajectories’ seen in figure 4. Moreover, the derivation of equation 1 makes it clear that the variables in the flow are actually the expected values for the simulation (in a complete network). So, to draw these ‘stochastic trajectories’, we made an average over many simulations and the initial conditions were enforced only by changing the probabilities that a site would start with a given opinion.
Figure 4 shows that for 3 opinions the fixed points found in the mean-field approximation give the “fixed points” for the simulated model with good accuracy and the trajectories also show similar patterns. For 4 opinions, the qualitative attractor structure remains the same but the trajectories are different, especially in the bulk of the simplex. Figure 5 shows time series for a 4 opinions case.

Figure 4: Comparison between a confidence rule simulated in a BA network (10^5 sites) (left) and its mean-field counterpart (right). The marks in the phase space to the left indicate the locations of the fixed points of the mean-field case. The parameters $p_{\sigma \rightarrow \sigma'}$ are either 0 or 1.

These correspondences are also network-dependent. For a square lattice and at least 4 opinions, the attractor structure (for the usual bounded confidence case) is completely different [5]. In a general network, the stationary states are characterized by contiguous domains of sites with the same opinion, arranged in such a way that domains with interacting opinions don’t touch each other (isolated sites may violate this, depending on the confidence rule adopted).
Figure 5: Time series for $\eta_1 + \eta_3$ (top) and $\eta_3$ (bottom). The mean field results imply that the only attractor is a coexistence of opinions 1 and 3, even though opinion 3 has no convincing power. This happens because opinion 1 always extinguishes opinions 2 and 4, before they can extinguish opinion 3. Each time step corresponds to $10^5$ (network size) steps. The parameters $p_{\sigma \to \sigma'}$ are either 0 or 1.

4. Conclusions
We have shown that the Sznajd model can be extended to include more complex interactions among opinions, mimicking biases and prejudices. We have found out most of the relevant qualitative features for the mean-field case of this generalized version of the model, through a typical dynamical systems approach. The simulations in a Barabási-Albert network show great similarities with the mean-field case (even quantitative similarities for 3 opinions). We believe this connection may be related to the small-world properties of BA networks, as they make it
more difficult the formation of isolated domains.

The dynamical systems approach we used for the mean-field analysis can be generalized to other similar models, like the voter model as well as more complex variants of the Szna\j d model. Our generalization of the bounded confidence rule can be readily applied to other discrete opinion models or adapted to continuous opinion models.

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