ON INDECOMPOSABLE MODULES OVER THE VIRASORO ALGEBRA

Yucai Su

Department of Applied Mathematics, Shanghai Jiaotong University, Shanghai 200030, China

Abstract It is proved that an indecomposable Harish-Chandra module over the Virasoro algebra must be (i) a uniformly bounded module, or (ii) a module in Category $\mathcal{O}$, or (iii) a module in Category $\mathcal{O}^-$, or (iv) a module which contains the trivial module as one of its composition factors.

Keywords: Virasoro algebra, intermediate series, uniformly bounded, Category $\mathcal{O}$.

The Virasoro algebra $\text{Vir}$, as the universal central extension of the infinite dimensional complex Lie algebra of the linear differential operators $\{t^{i+1} \frac{d}{dt} \mid i \in \mathbb{Z}\}$, of interest to both physicists and mathematicians [1-9], is the Lie algebra with basis $\{L_i, c \mid i \in \mathbb{Z}\}$ such that $[L_i, L_j] = (j - i)L_{i+j} + \frac{i^3 - i}{12} \delta_{i,-j} c$, $[L_i, c] = 0, i, j \in \mathbb{Z}$.

Set $\text{Vir}_\pm = \bigoplus_{i \in \mathbb{Z} \pm \{0\}} L_i$, $\text{Vir}_0 = G L_0 \oplus c$, $\text{Vir}_{[i,j]} = \bigoplus_{i \leq k \leq j} L_k$, $i, j \in \mathbb{Z}$ and $\text{Vir}_{[i,\infty)} = \bigoplus_{k \leq i} L_k$. Then $\text{Vir}$ has the triangular decomposition $\text{Vir} = \text{Vir}_- \oplus \text{Vir}_0 \oplus \text{Vir}_+$ and we have the universal enveloping algebra decomposition $U(\text{Vir}) = U(\text{Vir}_-)U(\text{Vir}_0)U(\text{Vir}_+)$. Consider a Harish-Chandra $\text{Vir}$-module $V$, i.e., a module with finite dimensional weight space decomposition:

$$V = \bigoplus_{\lambda \in \mathfrak{g}'} V_\lambda, \ V_\lambda = \{v \in V \mid L_0v = \lambda v\}, \dim V_\lambda < \infty, \ \lambda \in \mathfrak{g}'.$$  \hspace{1cm} (1)

Since the central element $c$ acts as a scalar on any indecomposable module, we shall always suppose $cV = hV$ for some $h \in \mathfrak{g}'$ (and $h = 0$ when $V$ is uniformly bounded).

Throughout this paper, we shall always suppose that $V$ is a $\text{Vir}$-module with decomposition (1).

Definition 1. A module $V$ is said to be (i) a module of the intermediate series [5] if it is indecomposable and $\dim V_\lambda \leq 1$ for all $\lambda \in \mathfrak{g}'$; (ii) a uniformly bounded module if there exists $N > 0$ such that $\dim V_\lambda \leq N$ for all $\lambda \in \mathfrak{g}'$; (iii) a highest or lowest weight module if it is generated by a highest or lowest weight vector $v$ with $L_i v = 0$ for all $i > 0$ or $i < 0$ respectively; (iv) a module in Category $\mathcal{O}$ or $\mathcal{O}^-$ if there exists $\lambda_0 \in \mathfrak{g}'$ such that for $\lambda \in \mathfrak{g}'$ with $V_\lambda \neq 0$ one has $\lambda \leq \lambda_0$ or $\lambda_0 \leq \lambda$ respectively (here and below a partial order is defined on $\mathfrak{g}'$ by: $\lambda \leq \mu \Leftrightarrow \lambda - \mu \in \mathbb{R}_-$).

A vector $v \neq 0$ is said to be primitive, anti-primitive, strongly primitive or strongly anti-primitive if $v \notin U(\text{Vir})\text{Vir}_+ v$, $v \notin U(\text{Vir})\text{Vir}_- v$, $V_+ v = 0$ or $V_- v = 0$ respectively. 

\footnote{This work is supported by a Fund from National Education Department of China.}
Highest weight modules are examples of modules in Category $\mathcal{O}$. However a module in Category $\mathcal{O}$ may have infinite number of composition factors and may not be generated by any finite number of primitive vectors. Simple modules over the Virasoro algebra have been studied in [1,3,5-9]. In particular, we have

**Theorem 2.** A simple module over the Virasoro algebra must be (i) a module of the intermediate series, or (ii) a highest weight module, or (iii) a lowest weight module.

Thus simple modules are, in this sense, well known. This theorem was conjectured in [3,4] and proved in [8] and also partially in [1,7,9], and was later generalized to the super-Virasoro algebras in [10] and the higher rank Virasoro algebras (in some sense) in [11].

A module of the intermediate series [5] must be one of $A_{a,b}, A(a), B(a)$, $a, b \in \mathbb{C}$, or one of their quotient submodules, where $A_{a,b}, A(a), B(a)$ all have a basis \( \{ x_k \mid k \in \mathbb{Z} \} \) such that $c_c$ acts trivially and

\[
A_{a,b} : \quad L_i x_k = (a + k + bi)x_{i+k},
\]

\[
A(a) : \quad L_i x_k = (i + k)x_{i+k}, \quad k \neq 0, \quad L_i x_0 = i(i + a)x_i, \quad (2)
\]

\[
B(a) : \quad L_i x_k = k x_{i+k}, \quad k \neq -i, \quad L_i x_{-i} = -i(i + a)x_0,
\]

for $i, k \in \mathbb{Z}$. We have

(i) $A_{a,b}$ is simple $\iff a \notin \mathbb{Z}$ or $a \in \mathbb{Z}, b \neq 0, 1$, and (ii) $A_{a,1} \cong A_{a,0}$ if $a \notin \mathbb{Z}$. \( (3) \)

Our main result is the following.

**Theorem 3.** An indecomposable module over the Virasoro algebra must be (i) a uniformly bounded module, or (ii) a module in Category $\mathcal{O}$, or (iii) a module in Category $\mathcal{O}^-$, or (iv) a module which contains the trivial module $V(0)$ as one of its composition factors.

Theorem 3 shows that if an indecomposable module does not contain $V(0)$ as a composition factor, then its composition factors are all of the same type: (i) modules of the intermediate series, (ii) highest weight modules, (iii) lowest weight modules. However, if an indecomposable module contains the composition factor $V(0)$, then it can have all types of composition factors. As an example, let $V'$ be the Verma module $M(0)$ [2] generated by the highest weight vector $v'_0$ with weight 0 such that $cv'_0 = 0$, let $V''$ be the anti-Verma module $M_+(0)$ generated by the lowest weight vector $v''_0$ with weight 0 such that $cv''_0 = 0$, let $V'''$ be the module of the intermediate series of type $A(a)$ in (2), and finally let $V$ be the submodule of $V' \oplus V'' \oplus V'''$ generated by $v_0 = v'_0 + v''_0 + x_0$, then $V$ is an indecomposable module which contains all types of composition factors such that $V(0)$ is the top composition factor. This example shows how one can get a module of type (iv) in Theorem 3 and how such a module can be like.

## 1 Proof of Theorem 3

Let $V$ be an indecomposable module over $\text{Vir}$. We can therefore suppose

\[
V = \sum_{k \in \mathbb{Z}} V_{k+a} \text{ for some } a \in \mathbb{C}'.
\]
We always suppose that $0 \leq \text{Re}(a) < 1$.

From now to Lemma 8, we suppose that $V$ is any module as in (4) such that the trivial module $V(0)$ is not a composition factor of $V$.

**Lemma 4.** Let $0 \neq v \in V$, $i \in \mathbb{Z}_+$. If $\text{Vir}_{[i, \infty)}v = 0$, then the submodule $V' = U(\text{Vir})v$ has a highest weight vector (which is clearly strongly primitive in $V$).

**Proof.** The proof is exactly similar to that of Lemma 3.1 in [10] (see also Lemma 1.6 in [8]).

**Lemma 5.** For any $i \in \mathbb{Z}$, there exist only a finite number of primitive (anti-primitive) vectors with weights $a + j$ such that $j \geq i$ (respectively $j \leq i$).

**Proof.** Suppose there are infinite number of primitive vectors $v_k$ with weights $\lambda_k = a + j_k$, $j_k > i$, $k = 1, 2, \cdots$. Let $U^{(k)}$ be the composition factors corresponding to $v_k$. Then

$$\dim V_{a+i-i_0} \geq \sum_{k \geq 1} \dim U^{(k)}_{a+i-i_0}, \text{ where } i_0 = 0 \text{ or } 1. \quad (5)$$

For any $k \geq 1$, since $U^{(k)} \cong V(\lambda_k)$ is a nontrivial simple highest weight $\text{Vir}$-module and $\lambda_k > a + i$, we always have either $\dim U^{(k)}_{a+i} \geq 1$ or $\dim U^{(k)}_{a+i-1} \geq 1$; if not, then $L_{-1}U^{(k)}_{a+i+1} = L_{-2}U^{(k)}_{a+i+1} = 0$ and since $\text{Vir}_{-}$ is generated by $L_{-1}, L_{-2}$, we obtain that $\text{Vir}_{-}U^{(k)}_{a+i+1} = 0$ and that $U^{(k)}$ has a lowest weight, this is impossible. This shows that the right-hand side of (5) is the infinity for at least one $i_0 = 0$ or $1$, this is a contradiction to (1).

**Lemma 6.** Any submodule $U$ generated by a primitive vector $v_\lambda$ of weight $\lambda$ is in Category $O$.

**Proof.** Suppose conversely that $U$ is not in Category $O$. If $U$ has strongly primitive vectors, let $W$ be sum of all submodules $U(v_\mu)$ of $U$ generated by strongly primitive vectors $v_\mu$. Since each $U(v_\mu)$ is a highest weight module, by Lemma 5, $W$ must be in Category $O$. Thus $U/W$ is still not in Category $O$. For a given $k \in \mathbb{Z}_+$, since the number of the primitive vectors with weights larger than $\lambda - k$ is finite, by considering the module $U/W$ instead of $U$ and if necessary repeating the above process, we can suppose that in $U/W$, there is no strongly primitive vector with weight $\mu > \lambda - k$, and that $U/W$ is not in Category $O$. Now by Lemma 4, for any $i \in \mathbb{Z}_+, 0 \neq x \in (U/W)_{\lambda-k}$, we have $\text{Vir}_{[i, \infty)}x \neq 0$. Let $S$ be the Lie subalgebra of $\text{Vir}$ generated by $\text{Vir}_{[k,k+1]}$, then clearly $S \supset \text{Vir}_{[i, \infty)}$ for some $i$ (for any integer $m$ large enough, it can be written as $m = xk + y(k+1), 1 \leq x, y \in \mathbb{Z}$, so $L_m$ can be generated by $L_k, L_{k+1}$). Thus $\text{Vir}_{[k,k+1]}x \neq 0$, i.e., $\ker(L_k|_{(U/W)_{\lambda-k}}) \cap \ker(L_{k+1}|_{(U/W)_{\lambda-k}}) = 0$. That is, $L_k|_{(U/W)_{\lambda-k}} \oplus L_{k+1}|_{(U/W)_{\lambda-k}}$: $(U/W)_{\lambda-k} \to (U/W)_{\lambda} \oplus (U/W)_{\lambda+1}$ is injective, and so $\dim (U/W)_{\lambda+k} \leq \dim (U/W)_{\lambda} + \dim (U/W)_{\lambda+1}$. In particular,

$$\dim V(\lambda)_{\lambda-k} \leq \dim (U/W)_{\lambda-k} \leq \dim (U/W)_{\lambda} + \dim (U/W)_{\lambda+1} \leq \dim U_{\lambda} + \dim U_{\lambda+1}. \quad (6)$$

for $k \in \mathbb{Z}_+$, where $V(\lambda)$ is the simple highest weight module with highest weight $\lambda$, which is the top composition factor of $U$. The right-hand side of (6) is a fixed number. This is impossible since every nontrivial simple highest weight module $V(\lambda)$ is not uniformly bounded [1,3].

Exactly analogous to Lemma 6, we have
Lemma 7. Any submodule $U'$ generated by an anti-primitive vector is in Category $\mathcal{O}^-$. 

Lemma 8. Let $U''$ be an indecomposable submodule generated by $x \in V$, where $x$ corresponds to a basis element of $A_{a,b}$ in (2), such that the top composition factor of $U''$ has type $A_{a,b}$, then $U''$ is uniformly bounded.

Proof. If $U''$ does not have primitive vectors or anti-primitive vectors, then using similar arguments in the proof of Lemma 6, we can show that $U''$ is uniformly bounded. Now suppose that $U''$ has, say, a primitive vector. Let $W', W''$ be the submodules of $U''$ generated by all primitive vectors and by all anti-primitive vectors respectively. Then by Lemmas 5-7, $W' \cap W'' = \{0\}$ since the only nonzero module both in Category $\mathcal{O}$ and in Category $\mathcal{O}^-$ is the trivial module. Let $W = U''/W''$. Then $W$ is still indecomposable, generated by $x$, having no anti-primitive vectors, and $W/W'$ has no primitive or anti-primitive vectors, thus $W/W'$ is uniformly bounded. Hence by [7], there exists $N \in \mathbb{Z}_+$ such that $\dim(W/W')_\lambda \leq N$ for all $\lambda \in a + \mathbb{Z}$.

For any $\lambda \in a + \mathbb{Z}$, choose a basis $B_\lambda = C_\lambda \cup D_\lambda$ of $W_\lambda$ such that $C_\lambda$ is a basis of $W'_\lambda$ and $D_\lambda$ corresponds to a basis of $(W/W')_\lambda$. In the dual space $W_\lambda^*$ of $W_\lambda$, take the dual basis $B_\lambda^* = C_\lambda^* \cup D_\lambda^*$ of $B_\lambda$. For any $v = \sum_{u \in B_\lambda} c_u u \in W_\lambda$, $c_u \in \mathbb{C}$, writing as a linear combination of elements of $B_\lambda$, we define the dual element $v^* = \sum_{u \in B_\lambda^*} c_u u^*$, the same combination of elements of $B_\lambda^*$. Now in the dual space $W^* = \bigoplus_{\lambda \in a + \mathbb{Z}} W_\lambda^*$ of $W$, we define a Vir-module structure as follows:

$$\langle L_i u^*_\mu, v_\nu \rangle = \langle u^*_\mu, L_{-i} v_\nu \rangle,$$

for $i \in \mathbb{Z}$, $u_\mu \in W_\mu$, $v_\nu \in W_\nu$, $\mu, \nu \in a + \mathbb{Z}$.

It is straightforward to verify that (7) defines a Vir-module $W^*$.

Claim 1. For any $0 \neq u^*_\lambda \in W^*_\lambda$, there exist $i \in \mathbb{Z}_+ \{0\}$ and some $g^* \in U(Vir)$, such that $0 \neq g^* u^*_\lambda \in W^*_{i+\lambda}$.

Suppose $u^*_\lambda \neq 0$, i.e., $u_\lambda = \sum_{u \in B_\lambda} c_u u$ with $c_{u_0} \neq 0$ for some $u_0 \in B_\lambda$. Since $u_0$ is generated by $x$, which corresponds to a basis element of the top composition factor of type $A_{a,b}$, there exist $\eta > \lambda$ and some element $v_\eta \in W_\eta$ such that $u_0$ is also generated by $v_\eta$, i.e., there exists $g \in U(Vir)_{-i}$, where $i = \eta - \lambda \in \mathbb{Z}_+ \{0\}$, such that $u_0 = g v_\eta$. Then

$$0 \neq c_{u_0} = \langle u^*_\lambda, u_0 \rangle = \langle u^*_\lambda, g v_\eta \rangle = \langle \omega(g) u^*_\lambda, v_\eta \rangle,$$

where $\omega$ is the anti-involution of $U(Vir)$ defined by $\omega(L_j) = L_{-j}$ for all $j \in \mathbb{Z}$, and where the last equality follows from definition (7). This shows that $g^* u^*_\lambda \neq 0$ for $g^* = \omega(g) \in U(Vir)_i$.

Claim 1 in particular shows that $W^*$ has no strongly primitive vectors. Then as in the proof of Lemma 6, $W^*$ is uniformly bounded. This contradicts that $W$ is not uniformly bounded.

Proof of Theorem 3. Suppose that $V$ is not of type (iv). Let $W, W', W''$ be the submodules of $V$ generated respectively by all modules $U, U', U''$ in Lemmas 6-8. Then clearly, $W, W', W''$ are disjoint with each other and $V = W \oplus W' \oplus W''$. Since $V$ is indecomposable, $V$ must be one of $W, W', W''$. 


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