COHOMOLOGY AND DEFORMATION THEORY OF MONOIDAL
2-CATEGORIES I

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Abstract. In this paper we define a cohomology for an arbitrary $K$-linear semistrict semi-
groupal 2-category $(\mathcal{C}, \otimes)$ (called for short a Gray semigroup) and show that its first order
(unitary) deformations, up to the suitable notion of equivalence, are in bijection with the ele-
ments of the second cohomology group. Fundamental to the construction is a double complex,
similar to the Gerstenhaber-Schack double complex for bialgebras, the role of the multiplica-
tion and the comultiplication being now played by the composition and the tensor product of
1-morphisms. We also identify the cohomologies describing separately the deformations of the
tensor product, the associator and the pentagonator. To obtain the above results, a cohomol-
ogy theory for an arbitrary $K$-linear (unitary) pseudofunctor is introduced describing its purely
pseudofunctorial deformations, and generalizing Yetter's cohomology for semigroupal functors.
The corresponding higher order obstructions will be considered in detail in a future paper.

1. Introduction

This is the first of two papers where we intend to give a cohomological description of the
infinitesimal deformations of a monoidal 2-category.

Roughly speaking, a monoidal 2-category is a 2-category equipped with a binary operation,
usually called the tensor product, defined at the three levels existing in any 2-category, i.e., ob-
jects, 1-morphisms and 2-morphisms, and which is associative and with a unit up to suitable 2-
isomorphisms. Actually, in this paper we will consider the more general structure of a semigroupal
2-category $\mathcal{C}$, namely, a 2-category with a tensor product as above but which is only associative (up
to a suitable 2-isomorphism) with no unit. More explicitly, we show that the first order (unitary)
deformations of such an object $\mathcal{C}$ can be identified with the elements of some cohomology group
associated to $\mathcal{C}$. The generalization to the case of monoidal categories and the question of the
obstructions will be treated in a future paper.

This work is an extension to the context of 2-categories of the the ory developped by Crane and
Yetter for semigroupal categories [6] and by Yetter for braided monoidal categories [38], [39], which
are in turn an extension to the context of (1-)categories of Gers tenhaber’s work on deformations
of algebras [13], [14], later generalized to the case of Hopf algebras by Gerstenhaber and Schack
[16] (see also [15], [17]). These classical works should be viewed as the corresponding theories in
the so-called 0-dimensional algebra setting [1], which is the algebra in the context of sets. The
situation can be schematically represented as in the table below. This table is the $K$-linear version
of the first two rows in the table of $k$-tuply monoidal $n$-categories of Baez and Dolan; see [1],
Table I. The $n$ here denotes the “dimensionality” of the algebraic framework we work with. So,
dimension $n$ corresponds to work in the context of $n$-categories, a natural generalization of the
notion of 2-category where we also have 3-morphisms between the 2-morphisms, and so on, until
$n$-morphisms between $(n-1)$-morphisms.

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|     | \( n=0 \)                  | \( n=1 \)                  | \( n=2 \)                  |
|-----|----------------------------|----------------------------|----------------------------|
| \( k=0 \) | \( K \)-vector space | \( K \)-linear category | \( K \)-linear 2-category |
| \( k=1 \) | \( K \)-algebra \((\text{Gerstenhaber’s work})\) | \( K \)-linear monoidal category \((\text{Crane-Yetter’s work})\) | \( K \)-linear monoidal 2-category |

Notice that going from the top to the bottom row along the diagonal corresponds to taking the one object case (for example, a \( K \)-linear category of only one object is exactly the same thing as a \( K \)-algebra). For a more expanded explanation of this table, see the reference above.

This idea of generalizing Gerstenhaber’s work to 1- and 2-categories comes from the important notion of categorification, which in the table corresponds to moving to the right. It first appears in the work by Crane and Frenkel \((\text{Gerstenhaber’s work})\) on Hopf categories, although it seems it was already present in Grothendieck’s thought. A Hopf category is an analog of a Hopf algebra where the underlying \( K \)-vector space has been substituted by a special kind of \( K \)-linear category, usually called a 2-vector space over \( K \). For a more precise definition, the reader is referred to Neuchl’s thesis \(\text{(28)}\). The basic idea behind the notion of categorification is that constructions from usual algebra can be translated to the level of categories and to higher levels, the \( n \)-categories, for \( n \geq 1 \), making one essential step: to substitute equations for isomorphisms. The price to pay is that it is necessary to simultaneously impose equations on these isomorphisms, which are the so-called coherence relations. These ideas have been developed by different authors, in particular, Crane and Yetter \(\text{(7)}\) and in the more general language of \( n \)-categories, Baez and Dolan (see, for ex., \(\text{(2)}\)).

Apart from its own interest, our motivation for studying deformations of monoidal 2-categories has to be found in its potential applications to the construction of interesting four dimensional Topological Quantum Field Theories (briefly, TQFT’s). Indeed, in \(\text{(5)}\), Crane and Frenkel suggest that Hopf categories may be used to construct four dimensional TQFT’s, in a similar way as three dimensional TQFT’s can be constructed from Hopf algebras (see, for ex., \(\text{(23)}\), \(\text{(12)}\)). Now, it is well-known that three dimensional TQFT’s can also be obtained using the monoidal categories of representations of suitable Hopf algebras (see, for ex., \(\text{(25)}\), \(\text{(37)}\), \(\text{(3)}\)). This clearly suggests the possibility that four dimensional TQFT’s could be obtained from the category of representations of suitable Hopf categories, which will be, going up in the categorification process, some kind of monoidal 2-categories (actually, Neuchl \(\text{(28)}\) has proved that the 2-category of representations of a Hopf category is indeed monoidal). That idea has been made explicit by Mackaay \(\text{(24)}\), who develops a method to construct invariants of piecewise linear four manifolds from a special kind of monoidal 2-categories he calls spherical 2-categories. His construction parallels that of Barrett and Westbury \(\text{(3)}\) for three manifolds. This explains the interest of monoidal 2-categories in the construction of four dimensional TQFT’s. But, why are we interested in their deformations? The answer is again an expected analogy between the cases of dimension three and four. In dimension three, we can get a state sum invariant of a piecewise linear three-manifold using irreducible representations of an arbitrary semisimple Lie algebra. The method comes from the classical work of Ponzano-Regge \(\text{(29)}\). The problem is that the sum turns out to be infinite. Progress was made possible only when the corresponding quantum group was discovered, which is a deformation (as a braided bialgebra; see \(\text{(21)}\)) of the classical universal envelopping algebra of the Lie algebra. Using the representations of the quantum group at a root of unity instead of those of the classical version, the state sum invariant becomes a convergent sum. That’s what Turaev and Viro do in their paper \(\text{(35)}\). The reader can also find more details, for example, in the book by Carter, Flath and Saito \(\text{(11)}\). The hope is that a similar situation reproduces in dimension four. So, instead of having a Lie algebra or, equivalently, its universal envelopping algebra, which has a natural structure of a (trivially braided)
Hopf algebra, we should now have a Hopf category, and instead of having the monoidal category of representations of the Lie algebra, we should have the monoidal 2-category of representations of the Hopf category. Via the reconstruction theorems of the Tannaka-Krein type, the deformations of the universal enveloping algebra of the Lie algebra correspond to deformations of its category of representations. Similarly, deformations of the Hopf category should correspond to deformations of its 2-category of representations. Therefore, we are indeed led in this way to consider the theory of deformations for monoidal 2-categories. In the above mentioned paper [5], Crane and Frenkel already outline a method for constructing interesting Hopf categories out of the quantum groups and their canonical bases. A difficult point is to find the analog of the quantum groups in this new framework, which could be called 2-quantum groups and which would correspond to non trivial deformations of these Hopf categories.

Let’s say a few words about what it means to give a cohomological description, in the sense of Gerstenhaber, of the theory of deformations of a semigroupal 2-category. Although we will be thinking of this case, the situation is similar in all of the above mentioned settings. Given an arbitrary 2-category \( \mathcal{C} \), it will be possible to define more than one semigroupal structure on it. So, we can think of a space \( X(\mathcal{C}) \) whose points are in 1-1 correspondence with all such possible semigroupal structures on \( \mathcal{C} \), up to a suitable notion of equivalence. The ultimate goal should be to have a description of such a moduli space \( X(\mathcal{C}) \) in terms, for example, of a suitable parametrization of its points. However, this is difficult. The idea is then to focus the attention on one particular point in that space and to study the corresponding “tangent space”. That’s why we speak of infinitesimal deformations of the (reference) semigroupal 2-category. Clearly, the first point is how to formalize that idea of a tangent space, because a priori we have no differentiable manifold structure on \( X(\mathcal{C}) \). In the sequel, we will see how to do this. We will need to assume some \( K \)-linear structure on the 2-category, for some commutative unitary ring \( K \), and to have some local \( K \)-algebra extending \( K \), and over which the deformations will take place. In the classical algebra setting, this is accomplished by considering, instead of the original \( K \)-algebra \( A \), its \( K[[h]] \)-linear extension \( A[[h]] \) (see [13]). According to Gerstenhaber’s foundational work, to give a cohomological description of such infinitesimal deformations amounts then to find a suitable cohomology \( H^*(\mathcal{C}) \) such that the so-called first order deformations (with respect to a formal deformation parameter) are classified, up to equivalence, by the elements of one of the cohomology groups \( H^n(\mathcal{C}) \), for some \( n \). But this is only the first point. According to Gerstenhaber, a nice cohomological description is required to further satisfy the property that the obstructions to extending such a first order deformation to higher order deformations or even to formal series deformations also live in some of the groups \( H^n(\mathcal{C}) \). In the 0-dimensional setting of algebras, it turns out that the corresponding obstructions are described by a graded Lie algebra structure on the cochain complex governing the deformations [14], and, after Gerstenhaber, this should be a basic principle of any obstruction theory. As mentioned before, however, in this paper we will not consider the question of higher order obstructions, whose treatment is deferred to a future paper. Therefore, the goal of the present work is to just develop the first of the above points, i.e., to identify the first order deformations of a semigroupal 2-category with the cocycles of a suitable cohomology theory.

An important point is how the infinitesimal deformations of a semigroupal 2-category are defined. In the classical algebra setting [13,14], recall that the deformation consists of taking a new (deformed) product \( \mu_h \) of the form

\[ \mu_h(a, a') = \mu(a, a') + \mu_1(a, a')h + \mu_2(a, a')h^2 + \cdots \]

where \( \mu : A \times A \to A \) denotes the original (undeformed) product and the \( \mu_i : A \times A \to A \), \( i \geq 1 \), are suitable \( K \)-bilinear maps such that \( \mu_h \) is indeed associative and with unit. In the category setting, this should correspond to considering a new (deformed) tensor product \( \otimes_h \).
between morphisms of the form

\[ f \otimes_h g = f \otimes g + (f \otimes_1 g)h + (f \otimes_2 g)h^2 + \cdots \]

where \( \otimes = \otimes_{(X,Y),(X',Y')} : C \times C((X,Y),(X',Y')) \to C(X \otimes Y, X' \otimes Y') \) corresponds to the original tensor product and the \( \otimes_i = (\otimes_i)_{(X,Y),(X',Y')} : C \times C((X,Y),(X',Y')) \to C(X \otimes Y, X' \otimes Y'), \)

\( i \geq 1, \) are suitable \( K \)-bilinear functors. In the category setting, however, we should further consider possible deformations of the structural isomorphisms taking account of the associativity and unit character of the deformed tensor product, i.e., we should consider, for example, a new (deformed) associator \( a_h \) of the form

\[ (a_h)_{X,Y,Z} = a_{X,Y,Z} + a^{(1)}_{X,Y,Z}h + a^{(2)}_{X,Y,Z}h^2 + \cdots \]

for suitable morphisms \( a^{(i)}_{X,Y,Z} : X \otimes (Y \otimes Z) \to (X \otimes Y) \otimes Z. \) Now, in the definition of an infinitesimal deformation of a monoidal category \( C, \) as given by Crane and Yetter \[3\], the only structure susceptible of being deformed is that defined by these structural isomorphisms \[1\]. In other words, the tensor product \( \otimes : C \times C \to C \) is assumed to remain the same (except for a trivial linear extension). Apart from the fact that this clearly simplifies the theory, there is another reason that may induce to adopt this point of view. Indeed, in his milestone paper \[9\], Drinfeld proved that the category of representations of the quantum group \( U_h(g) \) associated to a simple Lie algebra \( g, \) which corresponds to a certain deformation in the above generic sense of the category of representations of \( U(g), \) is in fact equivalent to the category of representations of \( U(g)[[h]] \) but with a non trivially deformed associator (see also \[21\]). Hence, at least in this case, it is enough to consider those deformations where only the isomorphisms included in the monoidal structure are deformed, keeping the tensor product undeformed.

In defining the infinitesimal deformations of a semigroupal 2-category we will adopt the same point of view as Crane and Yetter. So, an infinitesimal deformation of a semigroupal 2-category will be defined in such a way that the only things susceptible of deformation are the 2-isomorphisms defining the semigroupal structure on the 2-category. Contrary to the case of monoidal categories, however, this involves many things. So, among the 2-isomorphisms susceptible of deformation, we can distinguish three groups: (1) the 2-isomorphisms included in the tensor product, coming from the weakening of the definition of the tensor product as a bifunctor, (2) the 2-isomorphisms included in the associator, and coming from the weakening of the naturality of the maps \( a_{X,Y,Z}, \) and (3) the 2-isomorphisms included in the so called pentagonator, coming from the weakening of the pentagon axiom on the associator. In a generic infinitesimal deformation, all of them will be deformed.

The outline of the paper is as follows. In Section 2 we recall the basic definitions from bicategory theory, together with the corresponding strictification theorem (MacLane-Pare’s theorem). In Section 3 we generalize to (unitary) pseudofunctors Epstein’s coherence theorem for semigroupal functors \[10\] and introduce the analog of Crane-Yetter’s “padding” composition operators \[3\] in this setting. They are essential in the development of the theory. Section 4 is devoted to reviewing in detail the definition of a semigroupal 2-category, giving a formulation adapted to our purposes, and we also give an explicit definition of the corresponding notion of morphism, deduced from the notion of morphisms between tricategories as it appears in the paper by Gordon, Power and Street \[8\]. In Section 5, we give the precise definition of deformation of a semigroupal 2-category we will work with, together with the notion of equivalence of deformations. For later use, we also define in this section the notion of purely pseudofunctorial infinitesimal deformation of a pseudofunctor. In Section 6, and using results from Section 3, we develop a cohomology theory for the

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1 As shown by Yetter \[2\], it is enough to only consider deformations of the associator, since they already induce a deformation of the unital structure.
purely pseudofunctorial infinitesimal deformations of a pseudofunctor, which partially generalizes Yetter’s theory for monoidal functors. The cohomological description of the deformations of a semigroupal 2-category is then initiated in Section 7, where we consider the particular case of the pentagonator-deformations, i.e., those deformations where only the pentagonator is deformed, all the other structural 2-isomorphisms remaining undeformed. The next section is devoted to determine a cohomological description of the infinitesimal deformations involving both the tensor product and the associator. We do that in the special case where the deformations are unitary, i.e., such that the structural 2-isomorphisms \( \otimes_0(X,Y) \) remain undeformed. They will be called unitary (tensorator,associator)-deformations. We also identify cohomologies describing the deformations separately of both structures. For the sake of simplicity, in this section we restrict ourselves to the case of a Gray semigroup. Finally, in Section 9 we show how the cohomologies in Sections 7 and 8 fit together to give a cohomology which describes the generic (unitary) deformations.

2. Basic concepts from bicategory theory

2.1. Recall that a bicategory, also called lax or weak 2-category, and first defined by Bénabou, can be obtained from a category after doing the following two steps: (1) enrich the sets of morphisms with the category of small categories in the sense of Kelly, and (2) weaken the associativity and unit axioms on the composition by substituting 2-isomorphisms for the equations, with the consequent introduction of coherence relations on these 2-isomorphisms, as explained in the introduction. When we do that, we obtain the following definition.

**Definition 2.1.** A bicategory \( \mathcal{C} \) consists of:

- A class \( |\mathcal{C}| \) of objects or 0-cells.
- For any ordered pair of objects \( X,Y \in |\mathcal{C}| \), a small category \( \mathcal{C}(X,Y) \).

The objects of \( \mathcal{C}(X,Y) \), denoted by \( f : X \to Y \), are called 1-morphisms or 1-cells, and its morphisms, denoted by \( \tau : f \Rightarrow f' \), are called 2-morphisms or 2-cells. Remark that, included in this data \( \mathcal{C}(X,Y) \), there is a distinguished identity 2-morphism \( 1_f : f \Rightarrow f \) for any 1-morphism \( f : X \to Y \), and an associative composition between 2-morphisms, called vertical composition. Given 1-morphisms \( f,f',f'' : X \to Y \), the vertical composite of two 2-morphisms \( \tau : f \Rightarrow f' \) and \( \tau' : f' \Rightarrow f'' \) will be denoted by \( \tau \cdot \tau' \).

- For any ordered triple of objects \( X,Y,Z \in |\mathcal{C}| \), a functor
  \[ c_{X,Y,Z} : \mathcal{C}(X,Y) \times \mathcal{C}(Y,Z) \to \mathcal{C}(X,Z) \]

These functors provide us not only the composition \( c_{X,Y,Z}(f,g) := g \circ f \) of two 1-morphisms \( f : X \to Y \) and \( g : Y \to Z \), but also a second composition between 2-morphisms, called the horizontal composition, which involves three objects. If \( f,f' : X \to Y \) and \( g,g' : Y \to Z \), the horizontal composition \( c_{X,Y,Z}(\tau,\eta) \) between the two 2-morphisms \( \tau : f \Rightarrow f' \) and \( \eta : g \Rightarrow g' \) will be denoted by \( \eta \circ \tau \). In a general bicategory, this composition may be nonassociative.

- For any object \( X \in |\mathcal{C}| \), a distinguished 1-morphism \( id_X \in \mathcal{C}(X,X) \).
- For any objects \( X,Y,Z,T \in |\mathcal{C}| \) and any composable 1-morphisms \( f : X \to Y, g : Y \to Z \), \( h : Z \to T \), a 2-isomorphism \( \alpha_{h,g,f} : h \circ (g \circ f) \Rightarrow (h \circ g) \circ f \), called the associator or associativity constraint on \( f,g,h \).
- For any 1-morphism \( f : X \to Y \), two 2-isomorphisms \( \lambda_f : id_Y \circ f \Rightarrow f \) and \( \rho_f : f \circ id_X \Rightarrow f \), called the left and right unit constraints on \( f \), respectively.

Moreover, these data must satisfy the following axioms:

1. The \( \alpha_{h,g,f} \) are natural in \( f,g,h \) and the \( \lambda_f \) and \( \rho_f \) natural in \( f \).
2. The associator \( \alpha = \{ \alpha_{h,g,f} \} \) is such that the following diagram commutes:
3. The left and right unit constraints \( \lambda = \{ \lambda_f \} \) and \( \rho = \{ \rho_f \} \) make commutative the following diagram:

\[
\begin{align*}
(g \circ \text{id}_Y) \circ f & \xrightarrow{\alpha_{g, \text{id}_Y, f}} g \circ (\text{id}_Y \circ f) \\
& \downarrow \rho \circ \alpha_f \\
& g \circ f
\end{align*}
\]

When all the associators \( \alpha_{h,g,f} \) and left and right unit constraints \( \lambda_f, \rho_f \) are identities, which in particular means that the composition of 1-morphisms is strictly associative and the identity 1-morphisms are strict units, we will speak of a 2-category.

The reader should check that a bicategory \( \mathcal{C} \) with only one object corresponds exactly to the notion of a monoidal category. If \( X \) is the only object of \( \mathcal{C} \), the monoidal category is \( \mathcal{C}(X, X) \) with the composition functor as tensor product. This fact will be used repeatedly in what follows. We also leave to the reader to check that in a 2-category horizontal composition is strictly associative and that the identity 2-morphisms of the identity 1-morphisms act as strict units with respect to horizontal composition. Both facts will also be frequently used.

Given two bicategories \( \mathcal{B} \) and \( \mathcal{C} \), their cartesian product is defined as the bicategory \( \mathcal{B} \times \mathcal{C} \) such that

\[
|\mathcal{B} \times \mathcal{C}| := |\mathcal{B}| \times |\mathcal{C}|
\]

\[(\mathcal{B} \times \mathcal{C})((X,Y),(X',Y')) := \mathcal{B}(X, X') \times \mathcal{C}(Y, Y') \]

\[c_{(X,Y),(X',Y'),X'',Y''} := (c_{X,X',X''} \times c_{Y,Y',Y''}) \circ P_{23}\]

with identity 1-morphisms \( \text{id}_{(X,Y)} = (\text{id}_X, \text{id}_Y) \) and whose structural 2-isomorphisms \( \alpha_{(f'',g''),(f',g'),(f,g)} \), \( \lambda_{(f,g)} \) and \( \rho_{(f,g)} \) are componentwise given by those of \( \mathcal{B} \) and \( \mathcal{C} \) (\( P_{23} \) denotes the functor which permutes factors 2 and 3). The same construction obviously extends to a finite number of bicategories.

We will mainly work with 2-categories. This means no loss of generality because of the following strictification theorem for bicategories, due to MacLane and Pare \[27\] (see also \[30\], §1.3):

**Theorem 2.2.** Any bicategory is biequivalent (in the sense defined below) to a 2-category.

Diagrammatically, a 2-category differs from a category in that it has vertices (the objects) and edges (the 1-morphisms) but also faces between pairs of edges. In other words, while a category can be represented as a 1-dimensional cellular complex, a 2-category is a 2-dimensional cellular complex. As a consequence, when working with 2-categories, a generic diagram will be a three dimensional one, with a new “pasting” game where both vertical and horizontal compositions are combined. We will find some examples in the sequel. Another significant difference is that in 2-categories (and in bicategories in general), we have 1-isomorphisms (i.e., invertible 1-morphisms), but also equivalences, i.e., 1-morphisms which are invertible only up to a 2-isomorphism. This leads
Moreover, this data must satisfy the following conditions:

2.2. We will also need the corresponding notion of morphism between bicategories. There are in the literature various versions and names for this notion. Following Gray I will call them pseudofunctors, although our definition differs slightly from that of Gray.

**Definition 2.3.** If $\mathcal{C}$ and $\mathcal{D}$ are two bicategories, a pseudofunctor from $\mathcal{C}$ to $\mathcal{D}$ is any quadruple $\hat{\mathcal{F}} = (|\mathcal{F}|, \mathcal{F}_*, \hat{\mathcal{F}}_*, \mathcal{F}_0)$, where

- $|\mathcal{F}| : |\mathcal{C}| \rightarrow |\mathcal{D}|$ is an object map (the image $|\mathcal{F}|(X)$ of $X \in |\mathcal{C}|$ will be denoted by $\mathcal{F}(X)$);
- $\mathcal{F}_* = \{ \mathcal{F}_{X,Y} : \mathcal{C}(X,Y) \rightarrow \mathcal{D}(\mathcal{F}(X), \mathcal{F}(Y)) \}$ is a collection of functors, indexed by ordered pairs of objects $X, Y \in |\mathcal{C}|$;
- $\hat{\mathcal{F}}_* = \{ \hat{\mathcal{F}}_{X,Y,Z} : c_{\mathcal{F}(X),\mathcal{F}(Y),\mathcal{F}(Z)} \circ (\mathcal{F}_{X,Y} \times \mathcal{F}_{Y,Z}) \Rightarrow \mathcal{F}_{X,Z} \circ c_{\mathcal{X},\mathcal{Y},\mathcal{Z}} \}$ is a family of natural isomorphisms, indexed by triples of objects $X, Y, Z \in |\mathcal{C}|$. Explicitly, this corresponds to having, for all composable 1-morphisms $X \overset{f}{\rightarrow} Y \overset{g}{\rightarrow} Z$, a 2-isomorphism

$$\hat{\mathcal{F}}_{X,Y,Z}(g,f) : \mathcal{F}_{Y,Z}(g) \circ \mathcal{F}_{X,Y}(f) \Rightarrow \mathcal{F}_{X,Z}(g \circ f)$$

natural in $(f,g)$, and
- $\mathcal{F}_0 = \{ \mathcal{F}_0(X) : \mathcal{F}_{X,X}(id_X) \Rightarrow id_{\mathcal{F}(X)} \}$ is a collection of 2-isomorphisms, indexed by objects $X \in |\mathcal{C}|$.

Moreover, this data must satisfy the following conditions:

1. **(hexagonal axiom)** for all composable 1-morphisms $X \overset{f}{\rightarrow} Y \overset{g}{\rightarrow} Z \overset{h}{\rightarrow} T$, it commutes

$$\xymatrix{ \mathcal{F}(h) \circ (\mathcal{F}(g) \circ \mathcal{F}(f)) \ar[r]^{\alpha_{\mathcal{F}(h),\mathcal{F}(g),\mathcal{F}(f)}} \ar[d]_{(\mathcal{F}(h) \circ \mathcal{F}(g)) \circ \mathcal{F}(f)} & \mathcal{F}(h) \circ (\mathcal{F}(g \circ f)) \ar[r]_{\mathcal{F}(\alpha_{h,g,f})} & \mathcal{F}(h \circ (g \circ f)) \\
(\mathcal{F}(h) \circ \mathcal{F}(g)) \circ \mathcal{F}(f) \ar[r]_{\mathcal{F}(h \circ g) \circ \mathcal{F}(f)} & \mathcal{F}(h \circ g) \circ \mathcal{F}(f) \ar[r]_{\mathcal{F}(h \circ g \circ f)} & \mathcal{F}((h \circ g) \circ f) 
}$$

2. **(triangular axioms)** for any 1-morphism $f : X \rightarrow Y$, the following diagrams commute:

$$\xymatrix{ \mathcal{F}(f) \circ id_{\mathcal{F}(X)} \ar[r]^{\nu_{\mathcal{F}(f)}} \ar[d]_{id_{\mathcal{F}(f)}} & \mathcal{F}(f) \circ id_{\mathcal{F}(X)} \ar[r]_{\mathcal{F}(\nu_{f})} & \mathcal{F}(f \circ id_X) \\
& \mathcal{F}(f) \ar[ru]_{\mu_{\mathcal{F}(f)}} & 
}$$

$$\xymatrix{ id_{\mathcal{F}(Y)} \circ \mathcal{F}(f) \ar[r]_{\lambda_{\mathcal{F}(f)}} \ar[d]_{id_{\mathcal{F}(Y)}} & \mathcal{F}(id_Y) \circ \mathcal{F}(f) \ar[r]_{\mathcal{F}(\lambda_f)} & \mathcal{F}(id_Y \circ f) \\
& \mathcal{F}(f) \ar[ru]_{\mu_{\mathcal{F}(f)}} & 
}$$

(here, and from now on, we just write $\hat{\mathcal{F}}(g, f)$ and $\mathcal{F}(f)$, the indexing objects being omitted for short).

The $\hat{\mathcal{F}}(g, f)$ and $\mathcal{F}_0(X)$, for all objects $X$ and composable 1-morphisms $f, g$, will be called the structural 2-isomorphisms of $\mathcal{F}$, and the whole set will be called the pseudofunctorial structure on $\mathcal{F}$. When they are all identities, which in particular means that the functors $\mathcal{F}_{X,Y}$ preserve
the composition of 1-morphisms and the identity 1-morphisms, the pseudofunctor will be called a 2-functor. When only the \( F_0(X) \) are identities, we will call it a unitary pseudofunctor.

If no confusion arises, a pseudofunctor \( F = (|\mathcal{F}|, \mathcal{F}_*, \hat{\mathcal{F}}_*, F_0) \) from \( \mathcal{C} \) to \( \mathcal{D} \) will be denoted by \( \mathcal{F} : \mathcal{C} \to \mathcal{D} \) or simply by \( \mathcal{F} \).

**Remark 2.4.** The only difference between this definition and the one by Gray in [18] is our assumption that all structural 2-morphisms \( \hat{\mathcal{F}}(g, f) \) and \( F_0(X) \) are actually 2-isomorphisms.

From the above definition, it follows immediately that a pseudofunctor between one object bicategories amounts to a monoidal functor between the corresponding monoidal categories, as the reader should check.

Given two pseudofunctors \( \mathcal{F} : \mathcal{B} \to \mathcal{C} \) and \( \mathcal{G} : \mathcal{C} \to \mathcal{D} \), the composite pseudofunctor \( \mathcal{G} \circ \mathcal{F} \) is defined by

\[
|\mathcal{G} \circ \mathcal{F}| = |\mathcal{G}| \circ |\mathcal{F}|
\]

\[
(\mathcal{G} \circ \mathcal{F})_{X,Y} = \mathcal{G}_{\mathcal{F}(X), \mathcal{F}(Y)} \circ \mathcal{F}_{X,Y}
\]

\[
(\mathcal{G} \circ \mathcal{F})(g, f) = \mathcal{G}(\hat{\mathcal{F}}(g, f)) \cdot \mathcal{G}(\mathcal{F}(g)) \cdot \mathcal{F}(f)
\]

\[
(\mathcal{G} \circ \mathcal{F})_0(X) = \mathcal{G}_0(\mathcal{F}(X)) \cdot \mathcal{F}(F_0(X))
\]

The direct product of pseudofunctors can also be defined, whose source and target bicategories are the corresponding product bicategories. We leave to the reader to write out the explicit definition. Finally, let’s recall that a pseudofunctor \( \mathcal{F} : \mathcal{C} \to \mathcal{D} \) is called a biequivalence if for any object \( Y \in |\mathcal{D}| \), there exists an object \( X \in |\mathcal{C}| \) whose image \( \mathcal{F}(X) \) is equivalent to \( Y \), and for any pair of objects \( X, X' \in |\mathcal{C}| \) the functor \( \mathcal{F}_{X,X'} \) is an equivalence of categories.

### 2.3. As in the case of categories, there is a notion of morphism between pseudofunctors, which I will call pseudonatural transformations.

**Definition 2.5.** Let \( \mathcal{C} \) and \( \mathcal{D} \) be two bicategories, and \( \mathcal{F}, \mathcal{G} : \mathcal{C} \to \mathcal{D} \) two pseudofunctors. Then a pseudonatural transformation from \( \mathcal{F} \) to \( \mathcal{G} \) is any pair \( \xi = (\xi_*, \hat{\xi}_*) \), where

- \( \xi_* = \{\xi_X : \mathcal{F}(X) \to \mathcal{G}(X)\} \) is a collection of 1-morphisms, indexed by objects \( X \in |\mathcal{C}| \);
- \( \hat{\xi}_* = \{\hat{\xi}_{X,Y} : c_{\mathcal{F}(X), \mathcal{G}(X), \mathcal{G}(Y)}(\xi_X, -) \circ \mathcal{G}_{X,Y} \Rightarrow c_{\mathcal{D}(X), \mathcal{F}(Y), \mathcal{G}(Y)}(\hat{\xi}_Y, \mathcal{F}(\hat{\xi}_X)) \} \) is a family of natural isomorphisms, indexed by pairs of objects \( X,Y \in |\mathcal{C}| \). Explicitly, this means having for any 1-morphism \( f : X \to Y \) a 2-isomorphism \( \xi_{X,Y}(f) : \mathcal{G}_{X,Y}(f) \circ \xi_X \Rightarrow \xi_Y \circ \mathcal{F}_{X,Y}(f) \), natural in \( f \).

Moreover, this data must satisfy the conditions

1. for all composable 1-morphisms \( X \xrightarrow{f} Y \xrightarrow{g} Z \), the following diagram commutes

\[
\begin{array}{ccc}
\mathcal{G}(g) \circ (\mathcal{G}(f) \circ \xi_X) & \xrightarrow{\alpha_{\mathcal{G}(g), \mathcal{G}(f), \xi_X}} & \mathcal{G}(g) \circ (\xi_Y \circ \mathcal{F}(f)) \\
(\mathcal{G}(g) \circ \hat{\mathcal{G}}(f)) \circ \xi_X & \xrightarrow{\hat{\xi}_X} & \xi_Z \circ (\mathcal{F}(g) \circ \mathcal{F}(f))
\end{array}
\]

\[
\begin{array}{ccc}
\xi_Z \circ (\mathcal{F}(g) \circ \mathcal{F}(f)) & \xrightarrow{\alpha_{\xi_Z, \mathcal{F}(g), \mathcal{F}(f)}} & (\xi_Z \circ \mathcal{F}(g)) \circ \mathcal{F}(f) \\
(\mathcal{G}(g) \circ \mathcal{F}(f)) \circ \xi_X & \xrightarrow{\xi_{X,Y}(f) \circ 1_{\xi_X}} & (\mathcal{G}(g) \circ \mathcal{F}(f)) \circ \xi_X
\end{array}
\]
2. for all objects $X$, the following diagram commutes

\[
\begin{array}{ccc}
\xi_X \circ \text{id}_{F(X)} & \xrightarrow{\rho_{\xi_X}} & \xi_X \\
1_{\xi_X} \circ \phi_0(X) & & \downarrow \xi^{-1}_X \circ \text{id}_G(X) \circ \xi_X \\
\xi_X \circ \text{F}(\text{id}_X) & \xrightarrow{\xi_1} & \text{G}(\text{id}_X) \circ \xi_X \\
\end{array}
\]

(for short again, here and from now on, we will omit the indexing objects in $\xi_{X,Y}(f)$).

When all the 2-isomorphisms $\xi(f)$ are identities, $\xi$ will be called a 2-natural transformation.

On the other hand, given pseudofunctors $F, G, H : \mathcal{C} \to \mathcal{D}$, recall that the vertical composite of $\xi : F \Rightarrow G$ and $\zeta : G \Rightarrow H$, denoted $\zeta \cdot \xi$, is defined by

\[
(\zeta \cdot \xi)_X = \zeta_X \circ \xi_X
\]

\[
\tilde{\zeta} \cdot \tilde{\xi}(f) = a^D_{\zeta_Y, \xi_Y, F(f)} \cdot (1_{\zeta_Y} \circ \tilde{\xi}(f)) \cdot (a^{-1}_{\zeta_Y, \tilde{G}(f), \xi_X}) \cdot (\tilde{\zeta}(f) \circ 1_{\xi_X}) \cdot a^D_{\zeta_X, \xi_X}
\]

On the other hand, given pseudofunctors $F, F' : \mathcal{B} \to \mathcal{C}$ and $G : \mathcal{C} \to \mathcal{D}$, and a pseudonatural transformation $\xi : F \Rightarrow F'$, the horizontal composition $1_{\text{id}_{F}} \circ \xi$ is defined by

\[
1_{\text{id}_{F}} \circ \xi(f) = \tilde{G}^{-1}(\xi_Y, F(f)) \cdot \tilde{G}(\tilde{\xi}(f)) \cdot \text{id}_{F'(f), \xi_X}
\]

Finally, the horizontal composition $\zeta \circ 1_{\text{id}_{F}}$, where $\zeta : G \Rightarrow G' : \mathcal{C} \to \mathcal{D}$ is any pseudonatural transformation and $F : \mathcal{B} \to \mathcal{C}$ any pseudofunctor, is given by

\[
(\zeta \circ 1_{\text{id}_{F}})_X = \zeta_{F(X)} \\
\zeta \circ 1_{\text{id}_{F}}(f) = \tilde{\zeta}(F(f))
\]

2.4. Let’s finish this section by recalling that, in the context of bicategories, there is still a notion of morphism between two pseudonatural transformations, usually called a modification, and which has no analog in the category setting.

**Definition 2.7.** Let $\mathcal{C}$ and $\mathcal{D}$ be two bicategories, $F, G : \mathcal{C} \to \mathcal{D}$ two pseudofunctors and $\xi, \zeta : F \Rightarrow G$ two pseudonatural transformations. Then, a modification from $\xi$ to $\zeta$ is any family of 2-morphisms $n = \{n_X : \xi_X \Rightarrow \zeta_X\}$, indexed by the objects of $\mathcal{C}$, such that for any 1-morphism $f : X \to Y$ in $\mathcal{C}$, it holds

\[
\tilde{\zeta}(f) \cdot (1_{G(f)} \circ n_X) = (n_Y \circ 1_{F(f)}) \cdot \tilde{\xi}(f).
\]

This condition expresses the fact that the 2-morphisms $n_X$ are natural in $X$. A modification from $\xi$ to $\zeta$ will be denoted by $n : \xi \Rightarrow \zeta$ or simply by $n$ if no confusion arises.

A family of 2-morphisms as above which not necessarily satisfy the previous naturality condition will be called a pseudomodification from $\xi$ to $\zeta$. This more general notion will be needed later.
3. Coherence and padding operators for unitary pseudofunctors

3.1. Before giving the definition of a semigroupal 2-category and the corresponding notion of morphism, we consider in this section a coherence theorem for unitary pseudofunctors which generalizes to the many objects setting Epstein’s coherence theorem for semigroupal functors \[10\]. Such a coherence result allows us to introduce the analog of Crane-Yetter’s “padding” composition operators \[\hat{A}\] in this setting. As the reader will realize later, these results are essential in what follows. So, they are first used in Section 6 to associate a cochain complex to a unitary pseudofunctor describing its purely pseudofunctorial infinitesimal deformations, and which is a key ingredient in the definition of the double complex of a \(K\)-linear Gray semigroup introduced in Section 8. The coherence result is also needed to prove that this double complex is indeed a double complex.

3.2. Recall that in \[10\], for any pair of semigroupal categories \((\mathcal{B}, \otimes, a)\) and \((\hat{\mathcal{B}}, \hat{\otimes}, \hat{a})\) and a semigroupal functor \((G, 
abla)\), the author defines the \(G\)-iterates (of multiplicity \(n, n \geq 1\)) as the set of all functors \(\mathcal{B}^n \to \hat{\mathcal{B}}\) that can be obtained as compositions of product functors \(G^i : \mathcal{B}^i \to \hat{\mathcal{B}}^i\) (\(i \leq n\)) and \(j\)-iterates (\(j \leq n\)) of the tensor products \(\otimes\) and \(\hat{\otimes}\) in \(\mathcal{B}\) and \(\hat{\mathcal{B}}\), respectively, which are functors \(\mathcal{B}^j \to \mathcal{B}\) or \(\hat{\mathcal{B}}^j \to \hat{\mathcal{B}}\). So, a generic \(G\)-iterate (of multiplicity \(n\)) will apply the object \((A_1, \ldots, A_n)\) of \(\mathcal{B}^n\) to an object of \(\hat{\mathcal{B}}\) of the form

\[
G(A_1 \otimes \cdots \otimes A_{i_1}) \hat{\otimes} G(A_{i_1+1} \otimes \cdots \otimes A_{i_2}) \hat{\otimes} \cdots \hat{\otimes} G(A_{i_r} \otimes \cdots \otimes A_n),
\]

with a suitable parenthesization of the \(A_i\)'s inside each group and of the \(\hat{\otimes}\)-factors that we omit because it will depend on the \(\otimes\)- and \(\hat{\otimes}\)-iterates used. Let \(\hat{\text{Cat}}(\mathcal{G}, \hat{\otimes}, \hat{\Delta})\) be the category whose objects are all these \(G\)-iterates and whose morphisms are all the natural transformations between them. Then, the structural natural isomorphisms \(\hat{G}, a, \hat{a}\) define a subcategory \(\hat{\text{Cat}}(\mathcal{G}, \otimes, \hat{\Delta}, \hat{G}, a, \hat{a})\) with the same objects as \(\hat{\text{Cat}}(\mathcal{G}, \otimes, \hat{\Delta})\) but whose morphisms are only those induced by these natural isomorphisms \(\hat{G}, a, \hat{a}\), which are called the canonical ones. More precisely, a canonical morphism is any morphism obtained as a compositions of expansions of instances of \(\hat{G}, a, \hat{a}\) or its inverses, where by an expansion of a morphism \(f\) one means any morphism obtained from \(f\) by tensorially multiplying it by identity morphisms. For example, a canonical morphism from \(\hat{G}(\langle A \otimes B \rangle \otimes \langle C \otimes D \rangle)\) to \(\hat{G}(\langle A \otimes B \rangle \otimes C) \hat{\otimes} G(D)\) is

\[
(\hat{G}(A \otimes B, C) \hat{\otimes} \text{id}_{G(D)}) \circ \hat{G}((A \otimes B, G(C), G(D)) \circ (\text{id}_{G(A \otimes B)} \hat{\otimes} \hat{G}(C, D)^{-1}) \circ \hat{G}(A \otimes B, C \otimes D)^{-1}.
\]

A priori, there are other canonical isomorphisms between the same two objects, as the reader should check. Epstein’s coherence theorem states that, when \(\hat{G}, a, \hat{a}\) satisfy the appropriate coherence relations, for any two objects of \(\hat{\text{Cat}}(\mathcal{G}, \otimes, \hat{\Delta}, \hat{G}, a, \hat{a})\) there is at most one morphism (actually an isomorphism).

\[2\]The term semigroupal applied to categories means a category with a tensor product and a coherent associator, but without unit constraints; when applied to functors, it means a functor \(F\) between semigroupal (or monoidal) categories together with a coherent natural isomorphism \(\hat{F} : \otimes \circ (F \times F) \Rightarrow F \circ \otimes\), but without the isomorphism \(F_0 : F(I) \to I\). Note also that Epstein actually considers semigroupal categories equipped with a symmetry.
Let’s consider now the case of a unitary pseudofunctor \((|\mathcal{G}|, G_*, \hat{G}_*)\) between two bicategories \(\mathcal{B}, \hat{\mathcal{B}}\). To emphasize the similarity between both situations, we present here the “conversion table”:

\[
\begin{align*}
\mathcal{B} & \leftrightarrow \mathcal{B}_* = \{\mathcal{B}(X,Y)\}_{X,Y} \\
\hat{\mathcal{B}} & \leftrightarrow \hat{\mathcal{B}}_* = \{\hat{\mathcal{B}}(U,V)\}_{U,V} \\
\otimes & \leftrightarrow c_* = \{c_{X,Y,Z}\}_{X,Y,Z} \\
\hat{\otimes} & \leftrightarrow \hat{c}_* = \{\hat{c}_{U,V,W}\}_{U,V,W}
\end{align*}
\]

Looking at this table, we see that to go to the bicategory-pseudofunctor setting simply means substituting any thing in the left-hand side by a family of things of exactly the same type and indexed by objects of the appropriate bicategory. One can now proceed in the same way as Epstein substituting any thing in the left-hand side by a family of things of exactly the same type and indexed by objects of the appropriate bicategory.

\[
G : \mathcal{B} \rightarrow \hat{\mathcal{B}} \leftrightarrow G_* = \{\mathcal{G}_{X,Y} : \mathcal{B}(X,Y) \rightarrow \hat{\mathcal{B}}(\mathcal{G}(X), \mathcal{G}(Y))\}_{X,Y}
\]

\[
\hat{G} = \{\hat{G}(A,B)\}_{A,B} \leftrightarrow \hat{G}_* = \{\hat{G}_{X,Y,Z} = \{\hat{G}(g,f)\}_{g,f}\}_{X,Y,Z}
\]

Formally, the proof is the same as that of Epstein, but ignoring the permutations which appear in his paper because we do not consider commutativity constraints. The main difference is that we work simultaneously with various functors and natural isomorphisms.

**Theorem 3.1.** Let \(\mathcal{B}, \hat{\mathcal{B}}\) be two bicategories and let \((|\mathcal{G}|, G_*, \hat{G}_*)\) be a unitary pseudofunctor between them. Then for any pair of objects of \(\text{Cat}(|\mathcal{G}|, G_*, c_*, \hat{c}_*, \hat{G}_*, \alpha_*, \hat{\alpha}_*)\) there is at most one morphism.

**Proof.** Formally, the proof is the same as that of Epstein, but ignoring the permutations which appear in his paper because we do not consider commutativity constraints. The main difference is that we work simultaneously with various functors and natural isomorphisms.

**Remark 3.2.** This coherence theorem already appears in a different formulation in [6], §1.6.

3.3. The previous result allows us to introduce the analog of Crane-Yetter’s “padding” composition operators \(\hat{\mathcal{B}}\) in the context of a unitary pseudofunctor \(\mathcal{G}\) between two bicategories \(\mathcal{B}\) and \(\mathcal{B}\). The main difference is that now we have a whole collection of such padding operators, indexed by pairs of objects of the target bicategory \(\hat{\mathcal{B}}\). So, given two such objects \(U,V\), the situation
is that depicted in Fig. 1. We have a sequence $\tau_1, \ldots, \tau_n$ of 2-morphisms in $\mathcal{B}$ such that the source 1-morphism of $\tau_{i+1}$ is canonically 2-isomorphic to the target 1-morphism of $\tau_i$ (i.e., they are 2-isomorphic through a composition of expansions of the structural 2-isomorphisms coming from $\mathcal{G}, \mathcal{B}, \hat{\mathcal{B}}$). Then, define

$$\lceil \tau_n \cdot \tau_{n-1} \cdot \cdots \cdot \tau_1 \rceil_{U,V} := \beta_n \cdot \tau_n \cdot \beta_{n-1} \cdot \tau_{n-1} \cdot \cdots \beta_1 \cdot \tau_1 \cdot \beta_0,$$

where the $\beta_i$'s are the canonical 2-isomorphisms between the target of $\tau_i$ and the source of $\tau_{i+1}$, $\beta_0$ is the canonical 2-isomorphism whose source 1-morphism has no identity composition factors and it is completely right-parenthesized and free from images of composite morphisms under $\mathcal{G}$, and $\beta_n$ is the canonical 2-isomorphism whose target 1-morphism has no identity composition factors and it is completely left-parenthesized and free from compositions both of whose factors are images under $\mathcal{G}$. Note that these are the padding operators when one chooses as “references” the $\mathcal{G}$-iterates $e^{(n)} \circ \mathcal{G}^{(n)}$ and $\mathcal{G} \circ^{(n)} c$, where $e^{(n)}$ denotes the appropriate iterate of the composition functors of $\hat{\mathcal{B}}$ for the resulting composition to be completely right-parenthesized, $(n)c$ the same thing but using the composition functors of $\mathcal{B}$ and so that the resulting composition is completely left-parenthesized, and $\mathcal{G}^{(n)}$ denotes the appropriate $\mathcal{G}$-iterate (probably with some factor equal to an identity functor). Other choices of references are also possible. That the above 2-morphism is well defined is a consequence of the previous coherence theorem.

**Example 3.3.** Let $\mathcal{G} = ([\mathcal{G}], \mathcal{G}_*, \hat{\mathcal{G}}_*)$ be a unitary pseudofunctor between two bicategories $\mathcal{B}$ and $\hat{\mathcal{B}}$. Let $X, Y, Z, T$ be objects of $\mathcal{B}$ and let us consider 1-morphisms $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} T$. Taking $U = \mathcal{G}(X)$ and $V = \mathcal{G}(T)$, we have $[1_{\mathcal{G}(h)} \circ \hat{\mathcal{G}}(g,f)]|_{\mathcal{G}(X),\mathcal{G}(T)} = \mathcal{G}(\alpha_{h,g,f}) \cdot \hat{\mathcal{G}}(h,g \circ f)(1_{\mathcal{G}(h)} \circ \hat{\mathcal{G}}(g,f))$.

**4. Semigroupal 2-categories and their morphisms**
4.1. From now on, and unless otherwise indicated, all bicategories will be assumed to be 2-categories. This assumption does not imply loss of generality due to MacLane-Pare’s strictification theorem for bicategories (see Theorem 2.2).

4.2. The objects of our interest are the semigroupal 2-categories. Recall that semigroupal 2-category is a monoidal 2-category without the unit object for the tensor product, and hence without the structural 1- and 2-isomorphisms related to the unital structure.

A standard reference on monoidal 2-categories is the paper by Kapranov-Voevodsky [20]. In that paper, however, they give an unraveled definition which involves many data and an even greater number of axioms. To make things more intelligible, it is worth to point out that a semigroupal 2-category is just the categorification of the definition of a semigroupal category. This naturally leads to the following definition (except for the $K_5$ coherence condition on the pentagonator).

**Definition 4.1.** A semigroupal 2-category consists of the following data SBDi and axiom SBA:

- **SBD1:** A 2-category $\mathcal{C}$.
- **SBD2:** A pseudofunctor $\otimes : \mathcal{C} \times \mathcal{C} \rightarrow \mathcal{C}$, called the tensor product.
- **SBD3:** A pseudonatural isomorphism $a : (\otimes(3)) \Rightarrow (\otimes(3) \otimes) : \mathcal{C} \times \mathcal{C} \times \mathcal{C} \rightarrow \mathcal{C}$, called the associator, where $\otimes(3)$ and $(3) \otimes$ denote, respectively, the composite pseudofunctors $\otimes \circ (id_{\mathcal{C}} \times \otimes)$ and $\otimes \circ (\otimes \times id_{\mathcal{C}})$.
- **SBD4:** An invertible modification $\pi : a(4) \Rightarrow (4) a : \otimes(4) \Rightarrow (4) \otimes : \mathcal{C} \times \mathcal{C} \times \mathcal{C} \times \mathcal{C} \rightarrow \mathcal{C}$, called the pentagonator, where $\otimes(4)$ and $(4) \otimes$ denote, respectively, the composite pseudofunctors $\otimes \circ (id_{\mathcal{C}} \times \otimes) \circ (id_{\mathcal{C}} \times id_{\mathcal{C}} \times \otimes)$ and $\otimes \circ (\otimes \times id_{\mathcal{C}}) \circ (\otimes \times id_{\mathcal{C}} \times id_{\mathcal{C}})$, and $a(4)$, $(4) a$ are the pseudonatural isomorphisms

$$a(4) = (1_\otimes \circ (a \times 1_id)) \cdot (a \circ 1_{id \times \otimes \times id}) \cdot (1_\otimes \circ (1_id \times a))$$
$$a(4) a = (a \circ 1_{\otimes \times id \times id}) \cdot (a \circ 1_{id \times id \times \otimes})$$

(here, $id$ denotes the identity 2-functor of $\mathcal{C}$). See Fig. 2.

- **SBA:** The data $(\otimes, a, \pi)$ is such that the equality in Fig. 3 holds (to simplify notation, the tensor product of objects or 1-morphisms is denoted by simple juxtaposition, and the identity 1-morphisms are represented by the corresponding objects; for more details about the notations in this Figure, see the next Proposition). This condition will be called the $K_5$ coherence relation (the name comes from the fact that the two pastings in Fig. 3 respresent together a realization of the $K_5$ Stasheff polytope; see [23]).

A semigroupal 2-category will be denoted by $(\mathcal{C}, \otimes, a, \pi)$ and the triple $(\otimes, a, \pi)$ will be called a semigroupal structure on the 2-category $\mathcal{C}$.
Figure 3. $K_5$ coherence relation on the pentagonator
For convenience in what follows, we give an explicit description of the structural 1- and 2-

isomorphisms involved in the previous definition, as well as the whole list of equations they must

satisfy.

**Proposition 4.2.** Let \( \mathcal{C} \) be a 2-category. Then, a semigroupal structure \((\otimes, a, \pi)\) on \( \mathcal{C} \) consists of the following data:

- \( \otimes: \) An object \( X \otimes Y \) for any object \( (X, Y) \) of \( \mathcal{C} \times \mathcal{C} \).
- \( \pi: \) A collection of functors \( \otimes_{(X,Y)}(X',Y'): \mathcal{C}(X,X') \times \mathcal{C}(Y,Y') \to \mathcal{C}(X \otimes Y, X' \otimes Y') \) for all \( (X, Y), (X', Y') \) objects of \( \mathcal{C} \times \mathcal{C} \).
- \( \delta: \) As usual, the image of the 1-morphism \((f, g): (X, Y) \to (X', Y')\) and the 2-morphism \((\tau, \sigma): (f, g) \to (\tilde{f}, \tilde{g}): (X, Y) \to (X', Y')\) by this functor \( \otimes_{(X,Y)(X',Y')} \) will be denoted by \( f \otimes g \) and \( \tau \otimes \sigma \), respectively.
- \( \phi: \) A collection of 2-isomorphisms \( \hat{\phi}((f', g'), (f, g)): (f' \otimes g') \circ (f \otimes g) \to (f' \circ f) \otimes (g' \circ g) \) for all composable 1-morphisms \((f, g): (X, Y) \to (X', Y')\) and \((f', g'): (X', Y') \to (X'', Y'')\) of \( \mathcal{C} \times \mathcal{C} \).
- \( \gamma: \) A collection of 2-isomorphisms \( \otimes_0((X, Y): id_X \otimes id_Y \Rightarrow id_{X \otimes Y}) \) for all objects \( (X, Y) \) of \( \mathcal{C} \times \mathcal{C} \).

Moreover, all the above 1- and 2-isomorphisms must satisfy the following equations:

**A\(\hat{\otimes}\)1:** For all 2-morphisms \((\tau, \sigma): (f, g) \Rightarrow (\tilde{f}, \tilde{g}): (X, Y) \to (X', Y')\) and \((\tau', \sigma'): (f', g') \Rightarrow (\tilde{f}', \tilde{g}'): (X', Y') \to (X'', Y'')\) of \( \mathcal{C} \times \mathcal{C} \)

\[
((\tau' \circ \tau) \otimes (\sigma' \circ \sigma)) \cdot \hat{\otimes}((f', g'), (f, g)) \cdot \hat{\otimes}((\tilde{f}', \tilde{g}'), (\tilde{f}, \tilde{g})) = \hat{\otimes}((\tilde{f}', \tilde{g}'), ((\tau' \otimes \sigma') \circ (\tau \otimes \sigma)) \cdot \hat{\otimes}((f', g'), (f, g))
\]

\( \hat{\otimes}((f''', g'''), (f' \circ f' \circ g' \circ g)) \cdot \hat{\otimes}((f''', g''')) = \hat{\otimes}((f''', g'''), (f', g') \circ \hat{\otimes}((f'', g''), (f'', g'')) \circ 1_{f \otimes g})
\]

**A\(\hat{\otimes}\)2:** For any 1-morphism \((f, g): (X, Y) \to (X', Y')\) of \( \mathcal{C} \times \mathcal{C} \)

\[
\hat{\otimes}((id_{X'}, id_{Y'}), (f, g)) = \otimes_0((X', Y') \circ 1_{f \otimes g}) \cdot \hat{\otimes}((f, g), (id_X, id_Y)) = 1_{f \otimes g} \circ \otimes_0((X, Y)
\]

**A\(\hat{\alpha}\)1:** For all 2-morphisms \((\tau, \sigma, \eta): (f, g, h) \Rightarrow (\tilde{f}, \tilde{g}, \tilde{h}): (X, Y, Z) \to (X', Y', Z')\) of \( \mathcal{C}^3 \)

\[
(1_{a_{X', Y', Z'}} \circ (\tau \otimes (\sigma \otimes \eta))) \cdot \hat{\alpha}(f, g, h) = \hat{\alpha}(\tilde{f}, \tilde{g}, \tilde{h}) \cdot (((\tau \otimes \sigma) \otimes \eta) \circ 1_{a_{X,Y,Z}})
\]

**A\(\hat{\alpha}\)2:** For all composable 1-morphisms \((X, Y, Z) \xrightarrow{(f, g, h)} (X', Y', Z') \xrightarrow{(f', g', h')} (X'', Y'', Z'')\) of \( \mathcal{C} \times \mathcal{C} \times \mathcal{C} \)

\[
\hat{\alpha}(f' \circ f, g' \circ g, h' \circ h) \cdot \hat{\otimes}((\hat{\otimes}((f', g'), (f, g)) \circ 1_{h \otimes h})) \circ 1_{a_{X,Y,Z}} =
\]

\[
(1_{a_{X'', Y'', Z''}} \circ (1_{f \circ f} \circ \hat{\otimes}((g', h')(g, h))) \circ (1_{a_{X', Y', Z'}} \circ \hat{\otimes}((f', g' \otimes h'), (f, g \otimes h))) \circ (\hat{\alpha}(f', g', h') \circ 1_{f \otimes (g \otimes h)}) \cdot (1_{(f' \otimes g') \otimes h'} \circ \hat{\alpha}(f, g, h))
\]
\(\textbf{A}\hat{\textbf{a}}3:\) For all objects \((X,Y,Z)\) of \(\mathcal{C} \times \mathcal{C} \times \mathcal{C}\)
\[
\hat{a}(id_X, id_Y, id_Z) = (1_{a_{X,Y,Z}} \circ \hat{\otimes}(X,Y,Z) \cdot (1_{id_X} \otimes \hat{\otimes}(Y,Z))^{-1}) \cdot
\]
\[
\cdot (\hat{\otimes}(X \otimes Y, Z) \cdot (1_{X,Y,Z}) \circ 1_{a_{X,Y,Z}})
\]
\(\textbf{A}\hat{\textbf{a}}1:\) For any 1-morphism \((f,g,h): (X,Y,Z,T) \to (X',Y',Z',T')\) of \(\mathcal{C}^4\)
\[
(1_{a_{X',Y',Z',T'}} \circ \hat{a}(f,g,h,k)) \cdot (\hat{a}(f \otimes g, h,k) \circ 1_{a_{X,Y,Z,T}}) \cdot (1_{((f \otimes g) \otimes h) \otimes k} \circ \pi_{X,Y,Z,T}) =
\]
\[
= (\pi_{X',Y',Z',T'} \circ 1_\{(f \otimes (g \otimes (h \otimes k)))\}) \cdot
\]
\[
\cdot (1_{a_{X',Y',Z',T'}} \circ \hat{\otimes}((id_{X'}, a_{Y',Z',T'}), (f,g \otimes (h \otimes k)))^{-1}) \cdot
\]
\[
\cdot (1_{a_{X',Y',Z',T'}} \circ \hat{\otimes}((f,g \otimes (h \otimes k)), (id_{X}, a_{Y,Z,T}))) \cdot
\]
\[
\cdot (\hat{\otimes}((f,g,h) \otimes k) \circ 1_{a_{X,Y,Z,T}}) \cdot
\]
\[
\cdot (\hat{\otimes}((f \otimes g,h) \otimes k)) \circ 1_{a_{X,Y,Z,T} \otimes (id_X \otimes a_{Y,Z,T})})
\]
\(\textbf{A}\pi2:\) For any object \((X,Y,Z,T,U)\) of \(\mathcal{C} \times \mathcal{C} \times \mathcal{C} \times \mathcal{C} \times \mathcal{C} \times \mathcal{C}\)
\[
(\pi_{X,Y,Z,T,U} \circ 1_{a_{X,Y,Z,T}}) \cdot
\]
\[
\cdot (1_{a_{X,Y,Z,T} \otimes id_U} \circ \hat{\otimes}((id_{X,Y,Z,T} \otimes id_U) \circ (id_X \otimes a_{Y,Z,T})))^{-1}) \cdot
\]
\[
\cdot (1_{a_{X,Y,Z,T} \otimes id_U} \circ \hat{\otimes}((id_{X,Y,Z,T} \otimes id_U) \circ (id_X \otimes a_{Y,Z,T}))) \cdot
\]
\[
\cdot ((\pi_{X,Y,Z,T} \otimes 1_{id_U}) \circ \hat{\otimes}((id_{X,Y,Z,T} \otimes 1_{id_U}) \circ (id_X \otimes a_{Y,Z,T}))) \circ
\]
\[
\cdot 1_{id_X \otimes \hat{\otimes}((id_{X,Y,Z,T} \otimes (id_X \otimes a_{Y,Z,T})))}
\]

We will refer to the previous equations as the structural equations of a semigroupial 2-category. Notice that, in Equation (A\pi2), both terms \(1_{id_X \otimes \hat{\otimes}((f',g') \otimes (f,g))}\) and the two axioms on the pseudofunctorial structure which appear in the definition of pseudofunctor. Similarly, equations (A\hat{\textbf{a}}i) correspond to the naturality of the 2-isomorphisms \(\hat{\otimes}((f',g'),(f,g))\) and the two axioms on the pseudofunctorial structure which appear in the definition of pseudonatural transformation. On the other hand, Equation (A\pi1) corresponds to the naturality condition on the pentagonator \(\pi_{X,Y,Z,T}\) in \((X,Y,Z,T)\), namely,
\[
(\hat{a}(f,g,h,k) \cdot 1_{((f \otimes g) \otimes h) \otimes k} \circ \pi_{X,Y,Z,T}) = (\pi_{X',Y',Z',T'} \circ 1_\{(f \otimes (g \otimes (h \otimes k)))\}) \cdot \hat{a}(f,g,h,k)
\]

\textbf{Proof.} Equations (A\hat{\textbf{a}}i) correspond to the naturality of the 2-isomorphisms \(\hat{\otimes}((f',g'),(f,g))\) and the two axioms on the pseudofunctorial structure which appear in the definition of pseudofunctor. Similarly, equations (A\hat{\textbf{a}}i) correspond to the naturality of the 2-isomorphisms \(\hat{\otimes}((f,g,h))\) and the axioms appearing in the definition of pseudonatural transformation. On the other hand, Equation (A\pi1) corresponds to the naturality condition on the pentagonator \(\pi_{X,Y,Z,T}\) in \((X,Y,Z,T)\), namely,
after making explicit the 2-isomorphisms \((4)\alpha(f, g, h, k)\) and \(\hat{\alpha}(f, g, h, k)\) using the definitions in Section 2. Finally, Equation (Ar2) is the algebraic expression of the \(K_5\) coherence relation.

**Definition 4.3.** A semigroupal 2-category \((\mathcal{C}, \otimes, a, \pi)\) is called **strict** when all the above structural isomorphisms are identities (notice that this is not possible for an arbitrary tensor product \(\otimes\); for example, it must satisfy that \(X \otimes (Y \otimes Z) = (X \otimes Y) \otimes Z\), etc.).

**Remark 4.4.** Let us remark that, apart from the structural 1- and 2-isomorphisms related to the unital structure that do not appear in our definition above, Kapranov-Voevodsky’s definition of a monoidal 2-category (see [20]) includes a different collection of structural 2-isomorphisms. So, instead of our \(\hat{\otimes}((f', g'), (f, g)) : (f' \otimes g') \circ (f \otimes g) \Rightarrow (f' \circ f) \otimes (g' \circ g)\), they introduce the two sets of 2-isomorphisms \(\otimes_{f', f, Y} : (f' \otimes Y) \circ (f \otimes Y) \Rightarrow (f' \circ f) \otimes Y\) and \(\otimes_{X, g, g'} : (X \otimes g') \circ (X \otimes g) \Rightarrow X \otimes (g' \circ g)\), together with the basic 2-isomorphisms \(\otimes_{f,g} : (f \otimes Y') \circ (X \otimes g) \Rightarrow (X \otimes g) \circ (f \otimes Y')\).

Similarly, instead of our \(\hat{\alpha}(f, g, h) : ((f \otimes g) \circ h) \circ a_{X,Y,Z} \Rightarrow a_{X,Y,Z} \circ (f \otimes (g \circ h))\), they use 2-isomorphisms \(\alpha_{f,Y,Z} : ((f \otimes Y) \otimes Z) \circ a_{X,Y,Z} \Rightarrow a_{X,Y',Z} \circ (f \otimes (Y \otimes Z))\) and the similarly defined \(a_{X,Y,Z}, a_{X,Y,h}\). This obviously implies a different set of axioms. However, both formulations are equivalent, and correspond to the two possible ways of defining a “bipseudofunctor” directly as a pseudofunctor of two variables or as two collections of pseudofunctors of one variable. Although it is possible to work with Kapranov-Voevodsky’s 2-isomorphisms, the cohomological nature of the axioms is much more clear when working with those of the previous proposition. Let us further remark that the special case where the pseudofunctor \(\otimes : \mathcal{C}^2 \rightarrow \mathcal{C}\) in our definition is cubical (see the definition below) corresponds, in the Kapranov-Voevodsky’s formulation, to the notion of a quasifunctor of 2-variables introduced by Gray in [18], p.56. The equivalence between both notions, cubical pseudofunctor and quasifunctor of 2-variables, is in fact the content of Proposition I.4.8. in Gray’s book. We do not enter into the details of the equivalence, but let us mention that, in terms of our \(\hat{\otimes}((f', g'), (f, g))\), the above Kapranov-Voevodsky’s 2-isomorphisms \(\otimes_{f,g}\) correspond to

\[
\otimes_{f,g} = \hat{\otimes}((id_{Y'}, g), (f, id_{Y}))^{-1} \cdot \hat{\otimes}((f, id_{Y'}), (id_{X}, g))
\]

and conversely, our \(\hat{\otimes}((f', g'), (f, g))\) are given by

\[
\hat{\otimes}((f', g'), (f, g)) = 1_{f' \otimes Y'} \circ (\otimes_{f,g})^{-1} \circ 1_{X \otimes g}
\]

The reader may also check that our structural equation (Ar2) exactly corresponds to the axiom Kapranov and Voevodsky denote by \((\rightarrow \otimes \rightarrow \otimes \bullet)\) (see Fig. [3]) together with two more similar axioms.

### 4.3. A fundamental fact in the theory of semigroupal 2-categories is the corresponding **strictification theorem**, due to Gordon-Power-Street [30]. In fact, they proved a much more general strictification theorem, valid for an arbitrary tricategory (i.e., the categorification of the notion of a bicategory). In the same way as a monoidal category just corresponds to a bicategory of only one object, a monoidal 2-category is just a tricategory of only one object\(^3\). Now, contrary to the case of bicategories, not all tricategories are equivalent to the corresponding 3-categories (the reader may figure out the precise definition of such objects). Indeed, some of the structural 3-isomorphisms can not be strictified in general, i.e., made equal to identities. In our case, this means that an arbitrary semigroupal 2-category is in some sense equivalent to a particular kind of semigroupal 2-categories, which, following Day and Street [4], we will call **Gray semigroups**\(^4\) and which are not the strict semigroupal 2-categories. Since this theorem plays an essential role in what follows, allowing us to greatly simplify the theory, we review here the precise definitions.

---

3 Strictly speaking, tricategories of one object correspond to the more general notion of a monoidal bicategory.

4 Actually, they use the name **Gray monoid**, because they consider monoidal 2-categories.
**Definition 4.5.** Let $\mathcal{C}$ be any 2-category. A pseudofunctor $\mathcal{F} : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ is called **cubical** if its structural 2-isomorphisms $\hat{\mathcal{F}}((f', g'), (f, g)) : \mathcal{F}(f', g') \circ \mathcal{F}(f, g) \Rightarrow \mathcal{F}(f' \circ f, g' \circ g)$ and $\mathcal{F}_0(X, Y) : \mathcal{F}(\text{id}_X, \text{id}_Y) \Rightarrow \text{id}_{\mathcal{F}(X, Y)}$ are such that:

1. The $\hat{\mathcal{F}}((f', \text{id}_Y), (f, g))$ and $\hat{\mathcal{F}}((f, g'), (\text{id}_X, g))$ are all identity 2-morphisms.
2. $\mathcal{F}$ is a unitary pseudofunctor, i.e., for all $(X, Y)$, $\mathcal{F}_0(X, Y) = 1_{\mathcal{F}(X, Y)}$.

Notice that our definition here differs from that in [30], p.31, in that we explicitly require the pseudofunctor to be unitary. Indeed, although the authors say that this condition follows from the cubical condition of $\mathcal{F}$, it seems that this is not the case, and the assumption must be included in the definition.

**Definition 4.6.** A **cubical semigroupal 2-category** is any semigroupal 2-category $(\mathcal{C}, \otimes, a, \pi)$ such that the tensor product $\otimes$ is a cubical pseudofunctor. A cubical semigroupal 2-category will be called a **Gray semigroup** whenever its structural 2-isomorphisms included in the associator $a$ and the pentagonator $\pi$ are all identities.

**Remark 4.7.** The analogous notions in the more general context of tricategories are respectively called **cubical tricategories** and **Gray categories** in [31].

A Gray semigroup will be simply denoted by $(\mathcal{C}, \otimes)$, the $a$ and $\pi$ being trivial. We leave to the reader to make explicit this definition. Notice that the set of structural 2-isomorphisms reduces in this case to the $\hat{\otimes}((f', g'), (f, g))$, most of which are moreover trivial by the cubical condition. This is the reason a Gray semigroup is usually described in terms of Kapranov-Voevodsky’s 2-isomorphisms $\otimes_{f,g}$. It is worth to point out that not every cubical pseudofunctor $\otimes$ defines a structure of Gray semigroup on a 2-category.

The fundamental strictification theorem for semigroupal 2-categories can now be stated as follows:

---

5 We would like to thank J. Power and R. Street for the emails interchanged about this point.
Theorem 4.8. \( ([30]) \) Every semigroupal 2-category is equivalent (in a sense we do not make precise) to a Gray semigroup.

After reading the next section, where the notion of a morphism between semigroupal 2-categories is defined, the reader may figure out by himself the sense in which this equivalence should be understood.

4.4. Let us finish this section by giving the corresponding notion of morphisms between semigroupal 2-categories, which will be needed in the next section in order to define equivalence of deformations. In the case of Gray semigroups, this definition appears, for example, in [8], Def.2 (in fact, they define morphism between Gray monoids). Our definition below follows from the general definition of morphism between tricategories which appears in [30] when restricted to the one object case (and forgetting the unital structure).

Definition 4.9. Let \((\mathcal{C}, \otimes, a, \pi)\) and \((\mathcal{C}', \otimes', a', \pi')\) be semigroupal 2-categories. A semigroupal pseudofunctor from \(\mathcal{C}\) to \(\mathcal{C}'\) is a pseudofunctor \(F: \mathcal{C} \to \mathcal{C}'\) together with the following data \(\text{SPDi}\) and axiom \(\text{SPA}\):

\[ \text{SPD1:} \quad \text{A pseudonatural isomorphism } \psi : \otimes' \circ (F \times F) \Rightarrow F \circ \otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}'. \]

\[ \text{SPD2:} \quad \text{An invertible modification } \omega : (1_F \circ a) \cdot (\psi \circ 1_{id\mathcal{C} \times \otimes}) \cdot (1_{\otimes'} \circ (1_F \times \psi)) \Rightarrow (\psi \circ 1_{\otimes \times id\mathcal{C}}) \cdot (1_{\otimes'} \circ (\psi \circ 1_{1_F \times \otimes})) \cdot (a' \circ 1_{F \times F \times F}) \] (see Fig. 5)

\[ \text{SPA:} \quad \text{The pair } (\psi, \omega) \text{ is such that the equation in Fig. 5 holds (to simplify notation, the tensor product of objects and 1-morphisms is again denoted by simple juxtaposition and the identity 1-morphisms are represented by the corresponding objects; furthermore, the action of the pseudofunctor on objects, 1-morphisms or 2-morphisms is again denoted by the symbols } [-], \text{ so that, for example, } \psi_{X,Y,Z}([Z][T]) \text{ denotes the 1-morphism } \psi_{X \otimes Y, Z} \otimes' (F(id_Z) \otimes' F(id_T))). \]

For more details about the notations appearing in this Figure, see the next Proposition).

A semigroupal pseudofunctor will be denoted by the triple \((F, \psi, \omega)\) and the pair \((\psi, \omega)\) will be called a semigroupal structure on \(F\).

Observe that the above definition indeed corresponds to categorifying the definition of a semigroupal functor between semigroupal categories: the axiom on the semigroupal structure is substituted for the modification \(\omega\), which in turn must satisfy the additional coherence relation (SPA).

A more explicit description of a semigroupal structure \((\psi, \omega)\) on \(F\), with the whole list of equations on the structural 1- and 2-isomorphisms, is as follows:

Proposition 4.10. Let \((\mathcal{C}, \otimes, a, \pi)\) and \((\mathcal{C}', \otimes', a', \pi')\) be semigroupal 2-categories and \(F: \mathcal{C} \to \mathcal{C}'\) a pseudofunctor. Then, a semigroupal structure \((\psi, \omega)\) on \(F\) consists of:
Figure 6. The coherence relation (SPA)
ψ: A collection of 1-isomorphisms $\psi_{X,Y} : \mathcal{F}(X) \otimes \mathcal{F}(Y) \to \mathcal{F}(X \otimes Y)$ for all objects $(X,Y)$ of $\mathcal{C} \times \mathcal{C}$.

$\hat{\psi}$: A collection of 2-isomorphisms $\hat{\psi}(f,g) : \mathcal{F}(f \otimes g) \circ \psi_{X,Y} \to \psi_{X',Y'} \circ (\mathcal{F}(f) \otimes' \mathcal{F}(g))$ for all 1-morphisms $(f,g) : (X,Y) \to (X',Y')$ of $\mathcal{C} \times \mathcal{C}$.

ω: A collection of 2-isomorphisms $\omega_{X,Y,Z} : \mathcal{F}(a_{X,Y,Z}) \circ \psi_{X,Y \otimes Z} \circ (\text{id}_{\mathcal{F}(X)} \otimes' \psi_{Y,Z}) \to \psi_{X \otimes Y,Z} \circ (\psi_{X,Y} \otimes' \text{id}_{\mathcal{F}(Z)}) \circ a'_{\mathcal{F}(X) \otimes \mathcal{F}(Y), \mathcal{F}(Z)}$ for all objects $(X,Y,Z)$ of $\mathcal{C} \times \mathcal{C} \times \mathcal{C}$.

Moreover, these data must satisfy the following equations:

A$\hat{\psi}1$: For all 2-morphisms $(\tau, \sigma) : (f,g) \to (\tilde{f}, \tilde{g}) : (X,Y) \to (X',Y')$ of $\mathcal{C} \times \mathcal{C}$

$$(1_{\psi_{X',Y'}} \circ (\mathcal{F}(\tau) \otimes' \mathcal{F}(\sigma))) \cdot \hat{\psi}(f,g) = \hat{\psi}(\tilde{f}, \tilde{g}) \cdot (\mathcal{F}(\tau \otimes \sigma) \circ 1_{\psi_{X,Y}}).$$

A$\hat{\psi}2$: For all composable 1-morphisms $(X,Y) \xrightarrow{(f,g)} (X',Y') \xrightarrow{(f',g')} (X'',Y'')$ of $\mathcal{C} \times \mathcal{C}$

$$\hat{\psi}(f' \circ f, g' \circ g) \cdot ([\mathcal{F}(\hat{\psi}(f',g'), (f,g))] \cdot \hat{\psi}(f' \otimes g', f \otimes g) \circ 1_{\psi_{X,Y}}) =$$

$$= (1_{\psi_{X'',Y''}} \circ [(\hat{\psi}(f', f) \otimes' \hat{\psi}(g', g')) \cdot \hat{\psi}'(\mathcal{F}(f', f), \mathcal{F}(g', g)), (\mathcal{F}(f), \mathcal{F}(g))]) \cdot$$

$$\cdot (\hat{\psi}(f', g') \circ 1_{\mathcal{F}(f \otimes g)}, (f', g') \circ 1_{\mathcal{F}(f \otimes g)} \circ \hat{\psi}(f,g)).$$

Aω1: For all 1-morphisms $(f,g,h) : (X,Y,Z) \to (X',Y',Z')$ of $\mathcal{C} \times \mathcal{C} \times \mathcal{C}$

$$(\omega_{X',Y',Z'} \circ 1_{\mathcal{F}(f) \otimes' \mathcal{F}(g) \otimes' \mathcal{F}(h)}) \cdot$$

$$\cdot (1_{\mathcal{F}(\text{id}_X, \text{id}_Y)} \circ [\hat{\psi}(\mathcal{F}(X), \mathcal{F}(Y)) \cdot (\mathcal{F}(0) \otimes' \mathcal{F}(0)(Y))]^{-1} \cdot$$

$$\cdot ([\mathcal{F}(\text{id}_X \otimes \text{id}_Y) \cdot \mathcal{F}(\otimes_0(X,Y))] \circ 1_{\psi_{X,Y,Y'}}.\right.$$
\(A_0\): For all objects \((X,Y,Z,T)\) of \(C \times C \times C \times C\)
\[
\left(1_{(X \otimes Y) @ Z,T} \circ (\psi_{X \otimes Y,Z,T} \circ F(id_T)) \circ \tilde{\alpha}(\psi_{X,Y,T}, F(id_Y), F(id_Z))^{-1} \circ 1_{(X \otimes Y) @ Z,T} \right) \circ \left(1_{(X \otimes Y,Z,T)} \circ (\psi_{X \otimes Y,Z,T} \circ F(id_T)) \circ \tilde{\alpha}(\psi_{X,Y,T}, F(id_Y), F(id_Z))^{-1} \right)
\]
\[
\left(1_{(a X \otimes a Y) @ a Z,T} \circ (\psi_{a X \otimes a Y,Z,T} \circ F(id_T)) \circ \tilde{\alpha}(\psi_{X,Y,T}, a F(id_Y), a F(id_Z))^{-1} \circ 1_{(a X \otimes a Y) @ a Z,T} \right) \circ \left(1_{(a X \otimes a Y,Z,T)} \circ (\psi_{a X \otimes a Y,Z,T} \circ F(id_T)) \circ \tilde{\alpha}(\psi_{X,Y,T}, a F(id_Y), a F(id_Z))^{-1} \circ 1_{(a X \otimes a Y,Z,T)} \right)
\]
\[
\left(\tilde{\alpha}(\psi_{X,Y,Z,T}) \circ (\psi_{X,Y,Z,T} \circ F(id_T)) \circ \tilde{\alpha}(\psi_{X,Y,T}, F(id_Y), F(id_Z))^{-1} \circ (\psi_{X,Y,Z,T} \circ F(id_T)) \circ \tilde{\alpha}(\psi_{X,Y,T}, F(id_Y), F(id_Z))^{-1} \right)
\]
(To simplify notation, \(F(-)\) is denoted in some places by \([-\])).

In the last equation (\(A_0\)), the term \(\chi_{X,Y,Z,T}\) denotes the 2-isomorphism
\[
\chi_{X,Y,Z,T} = \tilde{\alpha}(\psi_{X,Y,Z,T}) \circ (\psi_{X,Y,Z,T} \circ F(id_T))^{-1} \circ (\psi_{X,Y,Z,T} \circ F(id_T))^{-1} \circ (\psi_{X,Y,Z,T} \circ F(id_T))\]

\[
\cdot \left(1_{(a X \otimes a Y,Z,T)} \circ (\psi_{a X \otimes a Y,Z,T} \circ F(id_T)) \circ (\psi_{a X \otimes a Y,Z,T} \circ F(id_T))^{-1} \circ (\psi_{a X \otimes a Y,Z,T} \circ F(id_T))^{-1} \circ (\psi_{a X \otimes a Y,Z,T} \circ F(id_T))\right)
\]

Notice that, in the particular case of a Gray semigroup, they just reduce to Kapranov-Voevodsky’s 2-isomorphisms \(\psi_{X,Y,Z,T}\), as it appears in Day-Street’s definition mentioned above. Observe also that all the terms \(\tilde{\alpha}(\psi_{X,Y,Z,T})\) are pastings of the corresponding 2-isomorphisms with the appropriate structural 2-isomorphisms from \(\tilde{\alpha}, \tilde{\alpha}\). We leave to the reader to find out the explicit formulas.

**Proof.** The Proposition again follows from the definitions in Section 2. In particular, Equation (\(A_0\)) corresponds to the naturality condition on \(\omega_{X,Y,Z}\) in the object \((X,Y,Z)\), and (\(A_0\)) is the algebraic expression of the coherence relation (SPA).

Later on, we will need this Proposition in the very special case where \(C' = C\) (but with different semigroupal structures \((\otimes, a, \pi)\) and \((\otimes', a', \pi')\) and \(F = id_C\), the identity 2-functor of \(C\).

5. Deformations of Pseudofunctors and Semigroupal 2-Categories

5.1. In this section we formalize the ideas outlined in the introduction, i.e., we “linearize” the problem of deforming a semigroupal 2-category \((C, \otimes, a, \pi)\). To do that, we will need to assume that \(C\) has some \(K\)-linear structure, for some commutative ring with unit \(K\). Before that, however, we introduce the notion of a purely pseudufactorial infinitesimal deformation of a \((K\text{-linear})\) pseudofunctor, a notion which appears later in Section 8 when we study the deformations of the tensor product in a semigroupal 2-category. The corresponding notions of equivalent deformations are also introduced, and they are made explicit in the case of first order deformations for later use.
5.2. Recall that a category $\mathcal{C}$ is called $K$-linear when all its hom-sets $\mathcal{C}(A, B)$, $A, B \in |\mathcal{C}|$, are $K$-modules and all composition maps are $K$-bilinear. On the other hand, a $K$-linear functor between two $K$-linear categories $\mathcal{C}, \mathcal{D}$ is any functor $F : \mathcal{C} \rightarrow \mathcal{D}$ such that all maps $F_{A,B} : \mathcal{C}(A, B) \rightarrow \mathcal{D}(F(A), F(B))$, $A, B \in |\mathcal{C}|$, are $K$-linear. The analogous definitions for 2-categories are as follows.

**Definition 5.1.** Let $K$ a commutative ring with unit. A $K$-linear 2-category is a 2-category $\mathcal{C}$ such that all its hom-categories $\mathcal{C}(X,Y)$ are $K$-linear, and all the composition functors $c_{X,Y,Z} : \mathcal{C}(X,Y) \times \mathcal{C}(Y,Z) \rightarrow \mathcal{C}(X,Z)$ are $K$-bilinear. Given $K$-linear 2-categories $\mathcal{C}$ and $\mathcal{D}$, a $K$-linear pseudofunctor between them is any pseudofunctor $F : \mathcal{C} \rightarrow \mathcal{D}$ such that all functors $F_{X,Y} : \mathcal{C}(X,Y) \rightarrow \mathcal{D}(F(X), F(Y))$, $X, Y \in |\mathcal{C}|$, are $K$-linear.

Notice that, according to this definition, we only have a structure of $K$-module on the sets of 2-morphisms. This will mean that, in our definition of deformation below, all structural 1-morphisms will remain undeformed, and the only thing susceptible to be deformed will be the 2-morphisms.

The following result brings together some easy facts about $K$-linear 2-categories and pseudofunctors whose proof is left to the reader.

**Proposition 5.2.** Let $\mathcal{B}$, $\mathcal{C}$ and $\mathcal{D}$ be $K$-linear 2-categories, and $F, F' : \mathcal{B} \rightarrow \mathcal{C}$ and $G : \mathcal{C} \rightarrow \mathcal{D}$ $K$-linear pseudofunctors. Then:

(i) The product 2-category $\mathcal{B} \times \mathcal{C}$ is $K$-linear.

(ii) The composition pseudofunctor $G \circ F : \mathcal{B} \rightarrow \mathcal{D}$ is $K$-linear.

(iii) For any $\xi, \zeta : \mathcal{F} \Rightarrow \mathcal{F}'$, the set $\text{Mod}(\xi, \zeta)$ (resp. $\text{PseudMod}(\xi, \zeta)$) of modifications (resp. pseudomodifications) between $\xi$ and $\zeta$ is a $K$-vector space.

Our main objects of interest are the $K$-linear semigroupal 2-categories, defined as follows:

**Definition 5.3.** A $K$-linear semigroupal 2-category is a semigroupal 2-category $(\mathcal{C}, \otimes, a, \pi)$ such that both $\mathcal{C}$ and $\otimes$ are $K$-linear.

5.3. Fundamental for the definitions of infinitesimal deformation given later are the notions of $R$-linear extension for $K$-linear (semigroupal) 2-categories and $K$-linear pseudofunctors, for any $K$-algebra $R$. As the reader will see, in the first case it provides us with the necessary “tangent space” at the point of $X(\mathcal{C})$ defined by the semigroupal structure in question.

**Definition 5.4.** Let $\mathcal{C}$ be a $K$-linear 2-category. Given a $K$-algebra $R$, the $R$-linear extension of $\mathcal{C}$ is the $R$-linear 2-category $\mathcal{C}^0_R$ defined as follows: (1) its objects and 1-morphisms are the same as in $\mathcal{C}$, (2) its sets of 2-morphisms are given by $(\mathcal{C}^0_R(X, Y))(f, f') := (\mathcal{C}(X,Y))(f, f') \otimes_K R$, (3) the vertical composition is defined by $(\tau \otimes r) \cdot (\bar{\tau} \otimes \bar{r}) := (\tau \cdot \bar{\tau}) \otimes (r \bar{r})$ and by linear extension, (4) the composition functors $c_{X,Y,Z}^R \equiv \circ_R : \mathcal{C}^0_R(X, Y) \times \mathcal{C}^0_R(Y, Z) \rightarrow \mathcal{C}^0_R(X, Z)$ are defined on 1-morphisms as in $\mathcal{C}$ and on 2-morphisms by $(\eta \otimes s) \circ_R (\tau \otimes r) := (\eta \circ \tau) \otimes (rs)$ and by linear extension, and (5) the identity 1-morphisms are the same as in $\mathcal{C}$.

The reader may easily check that these data indeed define an $R$-linear 2-category. The reason to add the zero superscript in $\mathcal{C}_R^0$ will be soon understood.

**Remark 5.5.** If $K$ is a topological ring and $R$ is an $m$-adically complete local $K$-algebra (for example, $R = K[[h]]$), it can be defined the $m$-adically complete $R$-linear extension of $\mathcal{C}$. This extension is the starting point for the definition of the $m$-adically complete infinitesimal deformations. In this work, however, we are mainly interested in the non topological case, and we leave to the reader to figure out the corresponding definitions in this topological setting.

We are specially interested in the case $R = K[\epsilon]/<\epsilon^{n+1}>$. The corresponding $R$-linear extension will be denoted by $\mathcal{C}^0_R(n)$. In this case, a generic 2-morphism $\tau_\epsilon : f \Rightarrow f' : X \rightarrow Y$ in the linear extension can be written in the form

$$\tau_\epsilon = \tau_0 + \tau_1 \epsilon + \cdots + \tau_n \epsilon^n \quad (5.1)$$
where \( \tau_0, \ldots, \tau_n \in \mathcal{C}(X,Y)(f,f') \).

The above definition is part of a functor of extension of scalars for \( K \)-linear 2-categories, a fact which allows us to further introduce the required \( R \)-linear extension of a \( K \)-linear pseudofunctor.

**Proposition 5.6.** Let \( \mathcal{C}, \mathcal{D} \) be two \( K \)-linear 2-categories and \( R \) a \( K \)-algebra. Then, any \( K \)-linear pseudofunctor \( F = ([F], F_*, \hat{F}_*, F_0) \) from \( \mathcal{C} \) to \( \mathcal{D} \) extends to an \( R \)-linear pseudofunctor \( F^0_0 = ([F^0_0], (F_*)^0_0, (\hat{F}_*)^0_0, (F_0)^0_0) \) from \( \mathcal{C}^0 \) to \( \mathcal{D}^0 \). Furthermore, if \( \xi = (\xi_*, \hat{\xi}_*) \) is a pseudonatural transformation between two \( K \)-linear pseudofunctors \( F \) and \( G \), it extends to a pseudonatural transformation \( \xi^0_R = ((\xi_*)^0_R, (\hat{\xi}_*)^0_R) \) between the \( R \)-linear extensions \( F^0_0 \) and \( G^0_0 \), and the same thing for modifications between pseudonatural transformations.

**Proof.** Take \( |F|^0_R = |F| \), and for any pair of objects \( X, Y \in |\mathcal{C}| \), define the functor \( (\mathcal{F}_X, Y)^0_R \) as follows: \( (\mathcal{F}_X, Y)^0_R(f) = \mathcal{F}_X, Y(f) \) for all 1-morphisms \( f : X \to Y \), and on 2-morphisms, take \( (\mathcal{F}_X, Y)^0_R(\tau \otimes r) = \mathcal{F}_X, Y(\tau) \otimes r \) and extend by linearity. Finally, define a pseudofunctorial structure on \( F^0_0 \) by taking \( (\hat{\mathcal{F}}^0_R)(g, f) = \hat{\mathcal{F}}(g, f) \otimes 1 \) and \( (\mathcal{F}^0_0)(X) = \mathcal{F}_0(X) \otimes 1 \) for all objects \( X, Y, Z \) and 1-morphisms \( f, g \). The rest of the proposition is proved similarly and is left to the reader.

As a by-product, we obtain the notion of \( R \)-linear extension for \( K \)-linear semigroupal 2-categories. Indeed, we have:

**Corollary 5.7.** Let \( (\mathcal{C}, \otimes, a, \pi) \) be a \( K \)-linear semigroupal 2-category. Then, for any \( K \)-algebra \( R \), the extension \( \mathcal{C}^0_R \) inherits a structure \( (\otimes^0_R, a^0_R, \pi^0_R) \) of \( R \)-linear semigroupal 2-category.

**Proof.** Indeed, to give a semigroupal structure on a 2-category means to give a pseudofunctor, a pseudonatural isomorphism and a modification, and all of them can be extended according to the previous Proposition. We leave to the reader to check that this extensions satisfy the appropriate axioms.

### 5.4

We can now define the corresponding notions of infinitesimal deformation. Let us begin with the case of a \( K \)-linear pseudofunctor. According to Proposition 5.6, given such a pseudofunctor \( F : \mathcal{C} \to \mathcal{D} \), we have a “copy” of it \( F^0_R : \mathcal{C}^0_R \to \mathcal{D}^0_R \) in the “category of \( R \)-linear pseudofunctors”.

The reason to consider such a copy is that the infinitesimal deformations of \( F \) will actually be, strictly speaking, deformations of that copy, for some local \( K \)-algebra \( R \). The copy itself will be called the **null deformation of \( F \) over \( R \)**. A generic deformation is then defined as follows.

**Definition 5.8.** Let \( \mathcal{C}, \mathcal{D} \) be two \( K \)-linear 2-categories, and \( F = ([F], F_*, \hat{F}_*, F_0) \) a \( K \)-linear pseudofunctor between them. Given a local \( K \)-algebra \( R \), a **purely pseudofunctorial infinitesimal deformation of \( F \) over \( R \)** is the pair \( ([F]^0_R, (F_*)^0_R) \) of Proposition 5.6 equipped with a pseudofunctorial structure \( (\mathcal{F}^0_R(\mathcal{F})^0_R, (F_0)^0_R) \) which reduces mod. \( m \) to that of the null deformation. When \( R = K[\epsilon] \), the corresponding deformations are called **purely pseudofunctorial \( n \)-th-order deformations** of \( F \).

The terms “purely pseudofunctorial” in this definition refer to the fact that the only deformed thing is the pseudofunctorial structure of \( F^0_R \), the source and target 2-categories remaining undeformed, in the sense that they are simply substituted for the corresponding \( R \)-linear extensions.

For example, it is easy to see that to give a purely pseudofunctorial \( n \)-th-order deformation of \( F \) simply amounts to give new families of 2-isomorphisms of the form

\[
\begin{align*}
\hat{F}_{e}(g, f) &= \hat{F}(g, f) + \hat{F}^{(1)}(g, f)e + \cdots + \hat{F}^{(n)}(g, f)e^n \\
(F_0)(X) &= F_0(X) + F_0^{(1)}(X)e + \cdots + F_0^{(n)}(X)e^n
\end{align*}
\]

where \( \hat{F}^{(i)}(g, f) : F(g \circ f) \Rightarrow F(g \circ f) \) and \( F^{(i)}_0(X) : F(id_X) \Rightarrow id_{F(X)} \), for all \( i = 1, \ldots, n \), are suitable 2-morphisms in \( \mathcal{D} \) such that the above 2-isomorphisms indeed define a pseudofunctorial structure on the pair \( ([F]^0_R, (F_*)^0_R) \). To emphasize that, a purely pseudofunctorial \( n \)-th-order
deformation will be denoted by the pair \( \{ \hat{\mathcal{F}}^{(i)} \}_{i=1,...,n}, \{ \hat{\mathcal{F}}_0^{(i)} \}_{i=1,...,n} \). In particular, when all these 2-morphisms are zero, we recover the null deformation \( \mathcal{F}_R^0 \).

We are only interested in the equivalence classes of such purely pseudofunctorial infinitesimal deformations, two such deformations being considered equivalent in the following sense:

**Definition 5.9.** Let \( \mathcal{F} = (|\mathcal{F}|, \mathcal{F}_*, \hat{\mathcal{F}}_*, \mathcal{F}_0) \) be a \( K \)-linear pseudofunctor. Then, given two purely pseudofunctorial infinitesimal deformations \( \mathcal{F}_R = (|\mathcal{F}_R|, (\mathcal{F}_R)_0) \) and \( \mathcal{F}'_R = ((\hat{\mathcal{F}}'_*)_R, (\mathcal{F}'_0)_R) \), they are called equivalent if there exists a pseudonatural isomorphism \( \xi : \mathcal{F}_R \Rightarrow \mathcal{F}'_R \) such that

1. \( \xi_X = id_{\mathcal{F}(X)} \) for all objects \( X \in |\mathcal{C}| \), and
2. \( \hat{\xi}(f) = 1_{\mathcal{F}(f)} \pmod{m} \), for all 1-morphisms \( f \) of \( \mathcal{C} \).

For later use, let us make explicit what this definition means in the case of first order deformations.

**Proposition 5.10.** Two purely pseudofunctorial first order deformations \( \mathcal{F}_*, \mathcal{F}'_* \) of a \( K \)-linear pseudofunctor \( \mathcal{F} : \mathcal{C} \rightarrow \mathcal{D} \), defined by 2-morphisms \( (\hat{\mathcal{F}}^{(1)}(g,f), \mathcal{F}_0^{(1)}(X)) \) and \( ((\hat{\mathcal{F}}^{(1)})'(g,f), (\mathcal{F}'_0^{(1)})'(X)) \), respectively, are equivalent if and only if there exists 2-morphisms \( \tilde{\xi}^{(1)}(f) : \mathcal{F}(f) \Rightarrow \mathcal{F}'(f) \) for all 1-morphisms \( f \) of \( \mathcal{C} \), satisfying the following conditions:

1. They are natural in \( f \).
2. For all composable 1-morphisms \( X \xrightarrow{f} Y \xrightarrow{g} Z \), it holds
   \[
   (\hat{\mathcal{F}}^{(1)})(g,f) - \hat{\mathcal{F}}^{(1)}(g,f) = \hat{\mathcal{F}}^{(0)}(g,f) \cdot (1_{\mathcal{F}(g)} \circ \tilde{\xi}^{(1)}(f)) - \hat{\xi}^{(1)}(g \circ f) \cdot \hat{\mathcal{F}}^{(0)}(g,f) + \hat{\mathcal{F}}^{(0)}(g,f) \cdot (\tilde{\xi}^{(1)}(g) \circ 1_{\mathcal{F}(f)})
   \]
3. For all objects \( X \) of \( \mathcal{C} \), it holds
   \[
   (\mathcal{F}_0^{(1)})'(X) - \mathcal{F}_0^{(1)}(X) = \mathcal{F}_0(X) \cdot \tilde{\xi}^{(1)}(id_X)
   \]

**Proof.** Indeed, let us go back to the definition of pseudonatural transformation (see Definition 5.5) and take \( \mathcal{F} = \mathcal{F}_*, \mathcal{G} = \mathcal{F}'_* \), and \( \xi \) defined by \( \xi_X = id_{\mathcal{F}(X)} \) and \( \tilde{\xi}(f) = 1_{\mathcal{F}(f)} + \tilde{\xi}^{(1)}(f)\epsilon \). The conditions above follow then by writing out the first order terms in \( \epsilon \) in every condition satisfied by \( \xi \). \( \square \)

5.5. Let us consider now the deformations of a \( K \)-linear semigroupal 2-category \( (\mathcal{C}, \otimes, a, \pi) \). As in the case of a pseudofunctor, the first thing we need is its “copy” \( (\mathcal{C}_R^0, \otimes_R^0, a_R, \pi_R) \) in the “category of \( R \)-linear semigroupal 2-categories” (see Corollary 5.7). It will be called the null deformation of \( (\mathcal{C}, \otimes, a, \pi) \) over \( R \). A generic infinitesimal deformation is then a deformation of that copy. More precisely:

**Definition 5.11.** Let \( (\mathcal{C}, \otimes, a, \pi) \) be a \( K \)-linear semigroupal 2-category, and let \( R \) be a local \( K \)-algebra, with maximal ideal \( m \). An infinitesimal deformation of \( (\mathcal{C}, \otimes, a, \pi) \) over \( R \) is the \( R \)-linear extension 2-category \( \mathcal{C}_R^0 \) equipped with a semigroupal structure \( (\otimes_R^0, a_R, \pi_R) \) which reduces \( m \) to that of the null deformation. More explicitly, \( \otimes_R^0, a_R, \pi_R \) must be such that: (1) \( \otimes_R \) only differs from \( \otimes_R^0 \) in the pseudofunctorial structure, (2) the 1-isomorphisms \( (a_R)_X,Y,Z \) coincide with those of \( a_R^0 \), and (3) all structural 2-isomorphisms \( \otimes_R((f',g'),(f,g)), (\otimes_0)_R(X,Y), a_R(f,g,h), (\pi_R)_X,Y,Z,T \) reduce \( m \) to those of the null deformation.

When \( R = K[\epsilon]/\langle \epsilon^{n+1} \rangle \), the corresponding infinitesimal deformations are called \( n \)-th-order deformations of \( \mathcal{C} \).

**Remark 5.12.** Since the associator as well as the left and right unit constraints on the composition of 1-morphisms (which we are assuming trivial) both live in the “deformable world” of 2-morphisms, it would be possible to modify the above definitions of infinitesimal deformation in such a way that also the bicategory structure of \( \mathcal{C} \) is deformed. In the case of a pseudofunctor, this will lead us to
For example, according to the above definition, an arbitrary \( n^{th} \)-order deformation of \((\mathcal{C}, \otimes, a, \pi)\) amounts to a new set of structural 2-isomorphism \( \tilde{\psi}_e \), \( (\otimes_0)e, \tilde{\alpha}_e, \pi_e \) of the form in Eq. (5.3) with the zero order term equal to the original 2-isomorphism, i.e.,

\[
\tilde{\psi}_e((f', g'), (f, g)) = \tilde{\psi}(f', g'), (f, g) + \otimes_1((f', g'), (f, g))\epsilon + \cdots + \otimes_n((f', g'), (f, g))\epsilon^n
\]

\[
(\otimes_0)e(X, Y) = \otimes_0(X, Y) + \otimes_1(X, Y)\epsilon + \cdots + \otimes_n(X, Y)\epsilon^n
\]

\[
\tilde{\alpha}_e(f, g, h) = \tilde{\alpha}(f, g, h) + \tilde{\alpha}_1(f, g, h)\epsilon + \cdots + \tilde{\alpha}_n(f, g, h)\epsilon^n
\]

\[
(\pi)_eX,Y,Z,T = \pi X,Y,Z,T + (\pi_1)X,Y,Z,T\epsilon + \cdots + (\pi_n)X,Y,Z,T\epsilon^n
\]

where \( \otimes(i) \), \( (f', g'), (f, g) \), \( \otimes(i)(X, Y) \), \( \tilde{\alpha}(i)(f, g, h) \), and \( (\pi(i))X,Y,Z,T \), for all \( i = 1, \ldots, n \), are suitable 2-morphisms in \( \mathcal{C} \) with the same source and target 1-morphisms as the corresponding undeformed 2-isomorphisms and such that the whole set of new 2-isomorphisms satisfy all the necessary equations to define a semigroupal structure on \( \mathcal{C}_{(\epsilon)} \). Such a \( n^{th} \)-order deformation will be denoted by \( \{ (\otimes(i))_1, \{ \otimes(i)_1 \}, \{ \tilde{\alpha}(i)_1 \}, \{ \pi(i)_1 \} \} \). In particular, when all these 2-morphisms are zero we again recover the null deformation.

As in the case of pseudofunctors, we are only interested in the equivalence classes of infinitesimal deformations.

**Definition 5.13.** Let \((\mathcal{C}, \otimes, a, \pi)\) be a \( K \)-linear semigroupal 2-category. Two infinitesimal deformations over \( (\otimes, a, \pi_R) \) and \( (\otimes_R', a_R', \pi_R') \) are called equivalent if the identity 2-functor \( \text{id}_{\mathcal{C}_R} : (\mathcal{C}_{(\epsilon)}^R, \otimes_R, a_R, \pi_R) \to (\mathcal{C}_{(\epsilon)}^R, \otimes_R', a_R', \pi_R') \) admits a semigroupal structure \((\psi, \omega)\) such that:

1. \( \psi_{X,Y} = \text{id}_{X\otimes Y} \) for all objects \( X, Y \);
2. \( \tilde{\psi}(f, g) = 1_{f\otimes g} \) (mod. \( m \)), for all 1-morphisms \( f, g \), and
3. \( \omega_{X,Y,Z} = (\otimes_0(X \otimes Y, Z)^{-1} \circ \psi_{X,Y,Z}) : \psi_{X,Y,Z} \otimes_0(X, Y \otimes Z) \) (mod. \( m \)), for all objects \( X, Y, Z \).

The deformations will be called \( \omega \)-equivalent when there exists a semigroupal structure \((\psi, \omega)\) satisfying the first and third conditions above and such that \( \tilde{\psi}(f, g) = 1_{f\otimes g} \) for all \( f, g \) (not only mod. \( m \)). Similarly, the deformations will be called \( \psi \)-equivalent when there exists a semigroupal structure \((\psi, \omega)\) satisfying the first and second conditions above and such that \( \omega_{X,Y,Z} = (\otimes_0(X \otimes Y, Z)^{-1} \circ \psi_{X,Y,Z}) : \psi_{X,Y,Z} \otimes_0(X, Y \otimes Z) \) for all \( X, Y, Z \) (not only mod. \( m \)).

Let us also make explicit for its later use what this definition means in the case of first order deformations. To simplify equations, however, let us assume, without loss of generality by Theorem 1.3, that the undeformed tensor product \( \otimes \) is unitary, i.e., that all 2-isomorphisms \( \otimes_0(X, Y) \) are identities (see Definition 2.3). Notice, however, that the deformed tensor product may no longer be unitary.

**Proposition 5.14.** Let \((\mathcal{C}, \otimes, a, \pi)\) be a \( K \)-linear semigroupal 2-category, with \( \otimes \) a unitary tensor product. Let consider two first order deformations defined by 2-morphisms \( (\tilde{\psi}^{(1)}, \tilde{\alpha}^{(1)}, \pi^{(1)}) \) and \( (\tilde{\psi}'^{(1)}, \tilde{\alpha}'^{(1)}, \pi'^{(1)}) \). Then, they are equivalent if and only there exists 2-morphisms \( \tilde{\psi}^{(1)}(f, g) : f \otimes g \Longrightarrow f \otimes g \) and \( \tilde{\psi}'^{(1)}(X,Y,Z) : a_{X,Y,Z} \Longrightarrow a_{X,Y,Z} \), for all objects \( X, Y, Z \) and 1-morphisms \( f, g \) of \( \mathcal{C} \), such that the following equations hold:

**E1:** For all 2-morphisms \( (\tau, \sigma) : f, g \to (\tilde{f}, \tilde{g}) \) of \( \mathcal{C} \)

\[
(\tau \otimes \sigma) \cdot \tilde{\psi}^{(1)}(f, g) = \tilde{\psi}^{(1)}(f, g) \cdot (\tau \otimes \sigma)
\]
\textbf{Ex\hat{\psi}2:} For all composable 1-morphisms \((X,Y) \xrightarrow{(f,g)} (X',Y') \xrightarrow{(f',g')} (X'',Y'')\) of \(\mathcal{C}\):

\[
\hat{\psi}^{(1)}(f' \circ f, g' \circ g) \cdot \hat{\otimes}((f',g'),(f,g)) + \hat{\otimes}^{(1)}(f',g'),(f,g)) = \\
= (\hat{\otimes}^{(1)})'((f',g'),(f,g)) + \hat{\otimes}((f',g'),(f,g)) \cdot (\hat{\psi}^{(1)}(f' \circ f, g' \circ g) \circ 1_{f \otimes g}) + \\
+ \hat{\otimes}((f',g'),(f,g)) \cdot (1_{f' \otimes g'} \circ \hat{\psi}^{(1)}(f, g))
\]

\textbf{Ex\hat{\psi}3:} For all objects \((X,Y)\) of \(\mathcal{C} \times \mathcal{C}\):

\[
\hat{\psi}^{(1)}(id_X, id_Y) = \hat{\otimes}^{(1)}(X,Y) - (\hat{\otimes}^{(1)})'(X,Y)
\]

\textbf{Ex\omega1:} For all 1-morphisms \((f,g,h) : (X,Y,Z) \xrightarrow{} (X',Y',Z')\) of \(\mathcal{C} \times \mathcal{C} \times \mathcal{C}\):

\[
(a^{(1)})'(f,g,h) - \hat{a}(f,g,h) \cdot ((\hat{\otimes}^{(1)})'(id_{X' \otimes Y'}, id_{Z'}), (f \otimes g, h)) \circ 1_{a_{X,Y,Z}} + \\
+ \hat{a}(f,g,h) \cdot ((\hat{\psi}^{(1)}(f,g) \otimes 1_h) \circ 1_{a_{X,Y,Z}}) + \\
+ \hat{a}(f,g,h) \cdot ((\hat{\otimes}^{(1)})'(f \otimes g, h), (id_{X' \otimes Y'} \circ id_{Z'})) \circ 1_{a_{X,Y,Z}} + \\
+ \hat{a}(f,g,h) \cdot (\hat{\psi}^{(1)}(f \otimes g, h) \circ 1_{a_{X,Y,Z}}) + \\
+ \hat{a}(f,g,h) \cdot (1_{f \otimes (g \otimes h)} \circ (\omega^{(1)})_{X,Y,Z}) = \\
= ((\omega^{(1)})_{X',Y',Z', 1_{f \otimes (g \otimes h)}}) \circ \hat{a}(f,g,h) - \\
- (1_{a_{X',Y',Z'}} \circ (\hat{\omega}^{(1)})'(id_{X' \otimes Y'}, id_{Z'}), (f, g \otimes h))) \cdot \hat{a}(f,g,h) + \\
+ (1_{a_{X',Y',Z'}} \circ (1_{f'} \otimes \hat{\psi}^{(1)}(g,h))) \cdot \hat{a}(f,g,h) + \\
+ (1_{a_{X',Y',Z'}} \circ (\hat{\psi}^{(1)}(f,g \otimes h))) \cdot \hat{a}(f,g,h) + \\
+ (1_{a_{X',Y',Z'}} \circ (\hat{\psi}^{(1)}(f, g \otimes h))) \cdot \hat{a}(f,g,h) + \hat{a}(f,g,h)
\]
**Ew2:** For all objects \((X, Y, Z, T)\) of \(\mathcal{C}^4\)

\[
(\pi^{(1)})'_{X,Y,Z,T} + \\
\quad + \pi_{X,Y,Z,T} \cdot ((\otimes^{(1)}_0)'((X \otimes Y) \otimes Z, T) \circ 1_{(a_{X,Y,Z,T})_0,\psi_{X,Y,Z,T}}(id_X \otimes ay_Z, T)) + \\
\quad + \pi_{X,Y,Z,T} \cdot ((\omega^{(1)})_{X,Y,Z,T} \circ (\otimes^{(1)}_0)'(X \otimes (Y \otimes Z), T) \circ 1_{a_{X,Y,Z,T}(id_X \otimes ay_Z, T)}) + \\
\quad + \pi_{X,Y,Z,T} \cdot (1_{a_{X,Y,Z,T}} \circ (\otimes^{(1)}_0)'(1_{X,Y,Z,T} \circ (id_X \otimes ay_Z, T))) - \\
\quad - \pi_{X,Y,Z,T} \cdot (1_{a_{X,Y,Z,T}} \circ (\otimes^{(1)}_0)'(id_X \circ (X, Y \otimes Z, T)) \circ 1_{id_X \otimes ay_Z, T}) + \\
\quad + \pi_{X,Y,Z,T} \cdot (1_{a_{X,Y,Z,T}} \circ (\otimes^{(1)}_0)'(1_{X,Y,Z,T} \circ (id_X \otimes (\omega^{(1)})_{X,Y,Z,T}))) + \\
\quad + \pi_{X,Y,Z,T} \cdot (1_{a_{X,Y,Z,T}} \circ (\otimes^{(1)}_0)'(X \otimes (Y \otimes Z) T)) + \\
\quad + \pi_{X,Y,Z,T} \cdot (1_{a_{X,Y,Z,T}} \circ (\otimes^{(1)}_0)'(id_X \circ (X, Y \otimes Z, T))) = \\
\quad - (\omega^{(1)}_{X,Y,Z,T} \circ 1_{a_{X,Y,Z,T}}) \cdot \pi_{X,Y,Z,T} + \\
\quad + ((\omega^{(1)})_{X \otimes Y, Z, T} \circ 1_{a_{X,Y,Z,T}}) \cdot \pi_{X,Y,Z,T} - \\
\quad - (1_{a_{X,Y,Z,T}} \circ (\otimes^{(1)}_0)'(id_X \otimes Y, id_Z, T)) \circ 1_{a_{X,Y,Z,T}} \cdot \pi_{X,Y,Z,T} - \\
\quad - (1_{a_{X,Y,Z,T}} \circ (\otimes^{(1)}_0)'(id_X \otimes Y, id_Z, T) \circ 1_{a_{X,Y,Z,T}} \cdot \pi_{X,Y,Z,T} + \\
\quad + (1_{a_{X,Y,Z,T}} \circ (\otimes^{(1)}_0)'(id_X \otimes Y, id_Z, T) \circ 1_{a_{X,Y,Z,T}}) \cdot \pi_{X,Y,Z,T} - \\
\quad - (1_{a_{X,Y,Z,T}} \circ (\otimes^{(1)}_0)'(id_X \otimes Y, id_Z, T) \circ 1_{a_{X,Y,Z,T}} \cdot \pi_{X,Y,Z,T} + \\
\quad + (1_{a_{X,Y,Z,T}} \circ (\otimes^{(1)}_0)'(id_X \otimes Y, id_Z, T) \circ 1_{a_{X,Y,Z,T}}) \cdot \pi_{X,Y,Z,T} + \\
\quad + (\pi^{(1)}_{X,Y,Z,T}) \cdot \pi_{X,Y,Z,T} + \\
\quad + (\omega^{(1)}_{X,Y,Z,T}) \cdot \pi_{X,Y,Z,T} + \\
\quad + (\pi^{(1)}_{X,Y,Z,T}) \cdot \pi_{X,Y,Z,T} + \\
\quad + (\pi^{(1)}_{X,Y,Z,T}) \cdot \pi_{X,Y,Z,T}
\]

**Proof.** The proof is a long but straightforward computation of the first order term in each of the conditions in Proposition 4.10 when \(\mathcal{F}\) is taken equal to the identity 2-functor \(id_{\mathcal{C}^0_{(1)}}\), the semigroupal structures \((\otimes, a, \pi)\) and \((\otimes', a', \pi')\) are those of the first order deformations, and the \(\psi\) and \(\omega\) are of the form

\[
\psi_{X,Y} = id_{X \otimes Y}
\]

\[
\hat{\psi}(f, g) = 1_{f \otimes g} + \hat{\omega}^{(1)}(f, g)
\]

\[
\omega_{X,Y,Z} = 1_{a_{X,Y,Z}} + \omega^{(1)}_{X,Y,Z}
\]

Notice that the zero order term of \(\omega\) is trivial because we are assuming \(\otimes\) is unitary. \(\square\)

### 6. Cohomology of a Unitary Pseudofunctor

6.1. This section contains preliminary results that will be used in Section 8 to construct the cochain complex which describes the simultaneous deformations of both the tensor product and the associator in a \(K\)-linear semigroupal 2-category. More explicitly, we associate a cohomology to an arbitrary \(K\)-linear unitary pseudofunctor and prove that this cohomology describes its purely pseudofunctorial infinitesimal deformations in the sense of Gerstenhaber. The main idea is to use
the fact mentioned in Section 2 that a pseudofunctor between one object bicategories corresponds to the notion of a monoidal functor. In this sense, our results generalize the cohomology theory for monoidal functors described by Yetter in [38]. Let us remark that our restriction to the case of unitary pseudofunctors implies no loss of generality for our purposes, because, as indicated before, the results obtained here will just be used to study the deformations of the tensor product and the associator in a $K$-linear semigroupal 2-category. Now, by Theorem 13, the undeformed semigroupal 2-category may be assumed to be a Gray semigroup and, hence, such that the original tensor product is indeed a unitary pseudofunctor.

6.2. Let us consider a $K$-linear unitary pseudofunctor $\mathcal{F} = ([F], F^\ast, \hat{F}_\ast)$ between $K$-linear 2-categories $\mathcal{C}$ and $\mathcal{D}$. Since we assume $\mathcal{C}$ is a 2-category, for each $n \geq 2$ and each ordered family $X_0, \ldots, X_n$ of $n+1$ objects of $\mathcal{C}$, we have a uniquely induced composition functor $c_{X_0,\ldots,X_n} \in \mathcal{C}(X_0, X_1) \times \cdots \times \mathcal{C}(X_{n-2}, X_{n-1}) \rightarrow \mathcal{C}(X_0, X_n)$, obtained by applying the appropriate elementary composition functors $c_{X,Y,Z}$ in any order [8]. In the same way, we have the induced composition functors $D_{X_0,\ldots,X_n} \in \mathcal{D}(X_0, X_1)$ for all $X_0, \ldots, X_n$. Then, given $X_0, \ldots, X_n$, let's consider the functors

$$C_{X_0,\ldots,X_n}, FC_{X_0,\ldots,X_n} : \mathcal{C}(X_0, X_n) \times \mathcal{C}(X_{n-2}, X_{n-1}) \times \cdots \times \mathcal{C}(X_0, X_1) \rightarrow \mathcal{D}(X_0, X_n)$$

defined by

$$CF_{X_0,\ldots,X_n} := D_{X_0,\ldots,X_n} \circ (F_{X_0,\ldots,X_n} \times \cdots \times F_{X_0, X_1})$$

$$FC_{X_0,\ldots,X_n} := F_{X_0, X_n} \circ c_{X_0,\ldots,X_n}$$

When $n = 1$, let $CF_{X_0, X_1} := F_{X_0, X_1} = : FC_{X_0, X_1}$.

We now define the vector spaces $X^n(\mathcal{F})$ of the cochain complex we are looking for as follows:

$$X^n(\mathcal{F}) := \left\{ \prod_{i=0}^n X_{i,n} \in \mathcal{C} \vert \text{Nat}(CF_{X_0,\ldots,X_n}, FC_{X_0,\ldots,X_n}) \right\} n \geq 1$$

otherwise

Notice that they are indeed vector spaces over $K$ because we are assuming that the target 2-category $\mathcal{D}$ is $K$-linear. According to this definition, a generic element $\phi \in X^n(\mathcal{F})$, $n \geq 1$, is of the form $\phi = (\phi_{X_0,\ldots,X_n}, \phi_{X_0,\ldots,X_n})$, with $\phi_{X_0,\ldots,X_n} = \{ \phi(f_0, \ldots, f_{n-1}) \vert f_i \in \mathcal{C}(X_{i-1}, X_i), i = 0, \ldots, n-1 \}$ and

$$\phi(f_0, \ldots, f_{n-1}) : F_{f_0} \circ F_{f_1} \circ \cdots \circ F_{f_{n-1}}$$

a 2-morphism natural in $(f_0, \ldots, f_{n-1})$. On the other hand, the “padding” composition operators introduced in Section 3 allows us to define coboundary maps $\delta : X^{n-1}(\mathcal{F}) \rightarrow X^n(\mathcal{F})$, for all $n \geq 2$, in the usual way. So, if $\phi \in X^{n-1}(\mathcal{F})$, $\delta \phi \in X^n(\mathcal{F})$ is given by

$$(\delta \phi)(f_0, f_1, \ldots, f_{n-1}) = [\phi(f_0, f_1 \circ \cdots \circ f_{n-1})]_{F(X_0) \circ F(X_n)} +$$

$$+ \sum_{i=1}^{n-1} (-1)^i [\phi(f_0, \ldots, f_{i-1} \circ f_i \circ \cdots \circ f_{n-1})]_{F(X_0) \circ F(X_n)} +$$

$$+ (-1)^n [\phi(f_0, \ldots, f_{n-2} \circ 1_{f_{n-1}})_{F(X_0) \circ F(X_n)}$$

for all $f_i \in \mathcal{C}(X_{i-1}, X_i)$, $i = 0, \ldots, n-1$.

**Proposition 6.1.** For any $K$-linear unitary pseudofunctor $\mathcal{F} = ([F], F^\ast, \hat{F}_\ast)$, the pair $(X^\ast(\mathcal{F}), \delta)$ is a cochain complex.

---

6Here, we think of the elementary composition functors $c_{X,Y,Z}$ as defined on the product category $\mathcal{C}(Y,Z) \times \mathcal{C}(X,Y)$, instead of $\mathcal{C}(X,Y) \times \mathcal{C}(Y,Z)$. Hence, they differ from those appearing in Definition 4.3 by a permutation functor.
The other terms are similarly worked. Finally, to prove that \( \delta \) is the same as doing nothing. So, let’s provisionally each other. Reinserting now the padding operators in each summand of this formal expression, \( \delta \) and \( \phi \) in its arguments and the interchange law gives that

\[
\mathcal{F}(\tau_0 \circ \cdots \circ \tau_{n-1}) \cdot [1_{\mathcal{F}(f_0)} \circ \phi(f_1, \ldots, f_{n-1})]_{\mathcal{F}(X_0), \mathcal{F}(X_n)} =
\]

\[
= \mathcal{F}(\tau_0 \circ \cdots \circ \tau_{n-1}) \cdot \hat{\mathcal{F}}(f_0, f_1 \circ \cdots \circ f_{n-1} \circ (1_{\mathcal{F}(f_0)} \circ \phi(f_1, \ldots, f_{n-1}))
\]

Now, for any 2-morphism \((\tau_0, \ldots, \tau_{n-1})(f_0, \ldots, f_{n-1}) \Rightarrow (f'_0, \ldots, f'_{n-1})\), the naturality of \( \hat{\mathcal{F}} \) and \( \phi \) in its arguments and the interchange law gives that

\[
\mathcal{F}(\tau_0 \circ \cdots \circ \tau_{n-1}) \cdot [1_{\mathcal{F}(f_0)} \circ \phi(f_1, \ldots, f_{n-1})]_{\mathcal{F}(X_0), \mathcal{F}(X_n)} =
\]

\[
\sum_{i=1}^{n} (-1)^i \left( [1_{\mathcal{F}} \circ \phi](f_0, \ldots, f_{n}) \right)
\]

Now, it is easy to check that the horizontal compositions of the 2-morphisms \(1_{\mathcal{F}(f_0)}\) and \(1_{\mathcal{F}(f_n)}\) in the first and last terms commute with the padding. Our assertion follows then from the obvious fact that taking a padding of a padding is the same as doing nothing. So, let’s provisionally forget the extra padding operators in the computation of \( \delta^2 \) and use the same argument which shows the \( \delta \) in the bar resolution satisfies \( \delta^2 = 0 \) to deduce that the terms formally cancel out each other. Reinserting now the padding operators in each summand of this formal expression, corresponding terms still cancel out each other because, by the coherence theorem, their paddings will also coincide.

This complex will be called the purely pseudofunctorial deformation complex of \( \mathcal{F} \), and the corresponding cohomology will be denoted by \( H^\bullet(\mathcal{F}) \). Notice that the dependence of this cohomology on the structural 2-isomorphisms \( \hat{\mathcal{F}}_\ast \) of \( \mathcal{F} \) is entirely encoded in the padding operators involved in the definition of \( \delta \).

6.3. Let us suppose that both \( C \) and \( D \) have only one object. Let us denote by \( X \) the only object of \( C \), so that the \( \mathcal{F}(X) \) will be the only object of \( D \). If we denote the (unique) composition functor \( c_{X,X,X} : C(X,X) \times C(X,X) \to C(X,X) \) in \( C \) by \( \otimes^C \) and in the same way denote by \( \otimes^D \) the (unique) composition functor in \( D \), the purely pseudofunctorial deformation complex of \( \mathcal{F} \) clearly reduces to

\[
X^n(\mathcal{F}) := \begin{cases} 
\text{Nat}((\otimes^D)^n \circ \mathcal{F}, \mathcal{F} \circ (\otimes^C)^n) & n \geq 1 \\
0 & \text{otherwise}
\end{cases}
\]

which is exactly the cochain complex associated by Yetter to a semigroupal functor. We have then the following generalization of Yetter’s result:
Theorem 6.2. The equivalences classes of purely pseudofunctorial first order deformations of a K-linear unitary pseudofunctor $F = (|F|, F, \hat{F})$ are in bijection with the elements of $H^2(\mathcal{F})$.

Proof. Let us consider 2-isomorphisms $\hat{F}_c(g, f) = \hat{F}(g, f) + \hat{F}^{(1)}(g, f)\epsilon$ and $(\mathcal{F}_0)c(X) = 1_{|\mathcal{F}|(X)} + \mathcal{F}_0^{(1)}(X)c$, with $\hat{F}^{(1)}(g, f) : F(g) \circ F(f) \Rightarrow F(g \circ f)$ and $\mathcal{F}_0^{(1)} : F(id_X) \Rightarrow id_{|\mathcal{F}|(X)}$. We want to find the necessary and sufficient conditions on the $\hat{F}^{(1)}(g, f)$ and $\mathcal{F}_0^{(1)}(X, Y)$ for these 2-isomorphisms to define a purely pseudofunctorial first order deformation of $F$. Let us first observe the following, which in particular shows that such a first order deformation of a unitary pseudofunctor $F$ is completely determined by the 2-morphisms $\hat{F}^{(1)}(g, f)$.

Lemma 6.3. Let $F = (|F|, F, \hat{F})$ be a K-linear unitary pseudofunctor between K-linear 2-categories $\mathcal{C}$ and $\mathcal{D}$, and let's consider a purely pseudofunctorial first order deformation given by 2-morphisms $\hat{F}^{(1)}(g, f)$ and $\mathcal{F}_0^{(1)}(X)$ (the deformation need not be unitary). Then, for all objects $X$ of $\mathcal{C}$, we have:

(i) $\hat{F}(id_X, id_X) = 1_{|\mathcal{F}|(X)}$.
(ii) $\mathcal{F}_0^{(1)}(X) = \hat{F}^{(1)}(id_X, id_X)$.

Proof. (i) For any pseudofunctor between 2-categories, it directly follows from the axioms that $\hat{F}(f, id_X) = 1_{F(f)} \circ \mathcal{F}_0(X)$, for all 1-morphisms $f$. In particular, this is true when $f = id_X$. Now, if $F$ is unitary, we have $F(id_X) = id_{|\mathcal{F}|(X)}$, and since in any 2-category identity 2-morphisms of an identity 1-morphism are units with respect to horizontal composition, we get $\hat{F}(id_X, id_X) = \mathcal{F}_0(X) = 1_{id_{|\mathcal{F}|(X)}}$.

(ii) The same argument as before shows that $\mathcal{F}_c(id_X) = \hat{F}(id_X, id_X)$. Notice that, although $F_0$ is no longer unitary, it still holds that $\mathcal{F}_c(id_X) = id_{\mathcal{F}_c(X)}$, which is the only thing needed to show the previous equality. The desired result follows then by taking the first order terms in $c$. □

Let us now prove the proposition. According to the lemma and the definition of a pseudofunctor, the 2-morphisms $\hat{F}^{(1)}(g, f)$ and $\mathcal{F}_0^{(1)}(X)$ above define a purely pseudofunctorial first order deformation of $F$ if and only if: (1) $\mathcal{F}_0^{(1)}(X) = \hat{F}^{(1)}(id_X, id_X)$, and (2) the $\hat{F}^{(1)}(g, f)$ are such that $\hat{F}_c(g, f)$ is natural in $(g, f)$ and satisfies the hexagonal and triangular axioms in Definition 2.3.

Since $\hat{F}(g, f)$ is natural in $g, f$ by hypothesis, naturality in $g, f$ of $\hat{F}_c(g, f)$ amounts to the naturality of $\hat{F}^{(1)}(g, f)$ in $g, f$. Hence the $\hat{F}^{(1)}(g, f)$ define an element $\hat{F}^{(1)} \in X^2(\mathcal{F})$. On the other hand, the hexagonal axiom on $\mathcal{F}_c$ gives the following condition on $\hat{F}^{(1)}$: for all composable morphisms $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} T$

$$\hat{F}(h, g \circ f) \cdot (1_{\mathcal{F}(h)} \circ \hat{F}^{(1)}(g, f))) + \hat{F}^{(1)}(h, g \circ f) \cdot (1_{\mathcal{F}(h)} \circ \hat{F}(g, f)) = \hat{F}^{(1)}(h \circ g, f) \cdot (\mathcal{F}(h, g) \circ 1_{\mathcal{F}(f)}) + \hat{F}(h \circ g, f) \cdot (\hat{F}^{(1)}(h, g) \circ 1_{\mathcal{F}(f)})$$

It is easily seen that this condition exactly corresponds to the fact that $\delta(\hat{F}^{(1)}) = 0$. Hence, $\hat{F}^{(1)}$ is a 2-cocycle of the complex $X^*(\mathcal{F})$. As regards the triangular axioms, notice that they imply no additional conditions on $\hat{F}^{(1)}$. For example, since $\mathcal{F}_0^{(1)}(X) = \hat{F}^{(1)}(id_X, id_X)$, the first of these triangular axioms gives the condition

$$\hat{F}^{(1)}(f, id_X) = 1_{\mathcal{F}(f)} \circ \hat{F}^{(1)}(id_X, id_X)$$

for all 1-morphisms $f : X \rightarrow Y$. Now, this condition is nothing more than the condition $\delta(\hat{F}^{(1)})(f, id_X, id_X) = 0$, as the reader may easily check.

7 This result should be viewed as an analog of Yetter’s result that a semigroupal deformation of a monoidal functor uniquely extends to a monoidal deformation. See [4], Theorem 17.2.
Suppose now that the 2-morphisms \((\hat{F}^{(1)})'(g,f)\) define another purely pseudofunctorial first order deformation of \(F\) equivalent to the previous one. We need to show that \(\hat{F}^{(1)}\) and \((\hat{F}^{(1)})'\) are cohomologous 2-cocycles. Now, from Proposition 5.10 and by definition of \(\delta\), it follows immediately that both deformations are equivalent if and only if there exists \(\xi^{(1)} \in X^1(F)\) such that

\[
(\hat{F}^{(1)})' - \hat{F}^{(1)} = \delta(\xi^{(1)}),
\]
as required. Let’s remark that the third condition in Proposition 5.10 is again superfluous. Indeed, just take \(f = g = id_X\) in the second condition and use the previous lemma to conclude that

\[
(\hat{F}^{(1)})'_0(X) - \hat{F}^{(1)}_0(X) = (\hat{F}^{(1)})'(id_X, id_X) - \hat{F}^{(1)}(id_X, id_X) = \xi^{(1)}(id_X) = \mathcal{F}_0(X) \cdot \xi^{(1)}(id_X)
\]

\(\square\)

7. Cohomology theory for the deformations of the pentagonator

7.1. In this section we initiate the study of the infinitesimal deformations of a \(K\)-linear semigroupal 2-category \((\mathcal{C}, \otimes, a, \pi)\). Notice first of all that, according to Definition 5.11, in a generic infinitesimal deformation of \(\mathcal{C}\) all structural 2-isomorphisms \(\otimes((f', g'), (f, g)), \otimes_0(X, Y), \tilde{a}(f, g, h), \pi_{X, Y, Z, T}\) will be deformed. Now, instead of treating directly such a generic deformation from the outset, we will proceed in three steps. So, in this section we consider those infinitesimal deformations where only the pentagonator is deformed, the tensor product and the associator remaining undeformed. They will be called infinitesimal pentagonator-deformations. In the following section we will treat the case where both the tensor product and the associator are simultaneously deformed, although under the assumption that the tensor product remains unitary, even after the deformation. These deformations will be called infinitesimal unitary (tensorator, associator)-deformations. We will obtain in this way two different cohomologies that separately describe the deformations of both parts of the semigroupal structure. Section 9 is devoted to see how both cohomologies fit together in a global cohomology describing the generic infinitesimal unitary deformations.

7.2. Let us consider an arbitrary \(K\)-linear semigroupal 2-category \((\mathcal{C}, \otimes, a, \pi)\). Recall from Section 4 that, given the data \((\mathcal{C}, \otimes, a)\), a pentagonator \(\pi\) is defined as a modification between two induced pseudonatural transformations \(a^{(4)}, \tilde{a}^{(4)} : \otimes^{(4)} \Longrightarrow \otimes^{(4)}\) which satisfies the \(K_5\) coherence relation.

More generally, given the pseudofunctor \(\otimes : \mathcal{C} \times \mathcal{C} \longrightarrow \mathcal{C}\), we can consider the induced pseudofunctors \(\otimes^{(n)}_{\ast}, (\otimes^{(n)}) : \mathcal{C} \times \cdots \times \mathcal{C} \longrightarrow \mathcal{C}, n \geq 3\), defined by

\[
\otimes^{(n)} = \otimes \circ (id_{\mathcal{C}} \times \otimes) \circ (id_{\mathcal{C}} \times id_{\mathcal{C}} \times \otimes) \circ \cdots \circ (id_{\mathcal{C}} \times \cdots \times id_{\mathcal{C}} \times \otimes)
\]

\[
(\otimes^{(n)}) = \otimes \circ (\otimes \times id_{\mathcal{C}}) \circ (\otimes \times id_{\mathcal{C}} \times id_{\mathcal{C}}) \circ \cdots \circ (\otimes \times \cdots \times id_{\mathcal{C}} \times \otimes)
\]

These are just two examples of a lot of induced tensor products of multiplicity \(n\). In the same way, we can generalize the induced pseudonatural transformations \(a^{(4)}, \tilde{a}^{(4)}\) to suitable pseudonatural transformations \(a^{(n)}, (\otimes^{(n)}_{\ast} : (\otimes^{(n)}) \Longrightarrow (\otimes^{(n)})\), for all \(n \geq 4\). Here, we also have many possible choices, because there are many possible \(a\)-paths (i.e., paths constructed as compositions of expansions of instances of the 1-isomorphisms \(a_{X,Y,Z}\) from the completely right-parenthesized one \(\otimes^{(n)}(X_1, \ldots, X_n)\) to the completely left-parenthesized one \((\otimes^{(n)}(X_1, \ldots, X_n)\). In the case \(n = 4\), the 1-isomorphisms of \(a^{(4)}\) and \(\tilde{a}^{(4)}\) are defined by taking the extremal paths, i.e., those characterized by the fact that, in each step, always the most internal parenthesis or the most external parenthesis, respectively, is moved. This leads us to introduce the following generalization.

Definition 7.1. Given 1-isomorphisms \(a_{X,Y,Z} : X \otimes (Y \otimes Z) \longrightarrow (X \otimes Y) \otimes Z\) for all objects \((X, Y, Z)\) of \(\mathcal{C}^3\), let \(a^{(n)}_{X_1, \ldots, X_n}, (\otimes^{(n)}_{\ast})_{a_{X_1, \ldots, X_n}}, n \geq 4\), be the \(a\)-paths from \(\otimes^{(n)}(X_1, \ldots, X_n)\) to
\begin{align*}
\mathsf{(n)} \otimes (X_1, \ldots, X_n) &\text{ induced by the } a_{X,Y,Z} \text{ and corresponding to always moving the most internal parenthesis and the most external parenthesis, respectively.}
\end{align*}

It is possible to give a more explicit description of these 1-isomorphisms. Indeed, the objects \( \mathsf{(n)} \otimes (X_1, \ldots, X_n) \) and \( \mathsf{(n)} \otimes (X_1, \ldots, X_n) \) can be graphically represented as in Fig. 7. Then, the a-path \( a_{X_1, \ldots, X_n} \) corresponds to moving to the left all the legs associated to the objects \( X_2, \ldots, X_{n-1} \), starting with \( X_{n-1} \) and so on until \( X_2 \), while the path \( \mathsf{(n)} a_{X_1, \ldots, X_n} \) corresponds to doing the same thing but starting with \( X_2 \) and so on until \( X_{n-1} \). Using this graphical presentation, we obtain the following description of both paths:

**Lemma 7.2.** For any \( n \geq 4 \) and any objects \( X_1, \ldots, X_n \), we have

\[
a_{X_1, \ldots, X_n}^{(n)} = \prod_{i=3}^{n} ((a_{X_1, \cdots, X_{n-2}, X_{n-1}, X_i, X_{i+1} \otimes id_{X_{i+2}}}) \otimes id_{X_1}) \circ id_{X_1} \times a_{X_2, \ldots, X_n}^{(n-1)}
\]

\[
a_{X_1, \ldots, X_n}^{(n)} = \prod_{i=2}^{n-1} a_{X_1, \ldots, X_{i-1}, X_{i+1}, \cdots, X_n}^{(n-i) \otimes (n-1) (X_1, \ldots, X_{i-2}, \ldots)}
\]

\( (n) \otimes (X_1, \ldots, X_n) \) intended to mean the composition of the tensor multiplications of \( id_{X_1} \) by each one of the composition factors defining \( a_{X_2, \ldots, X_n}^{(n-1)} \).

For example, when \( n = 5 \), the reader may easily check that one recovers the a-paths that appear in Fig. 8 defining the common boundary of the polytope.

By definition, \( a^{(n)} \circ a^{(n)} : \otimes (n) \otimes (n) \) are the pseudonatural isomorphisms (induced by \( a \)) whose 1-isomorphisms are precisely the above 1-morphisms \( a_{X_1, \ldots, X_n}^{(n)} \), \( a_{X_1, \ldots, X_n}^{(n)} \). So, from the formulas defining the vertical composition of 2-morphisms and the horizontal compositions of the form \( 1 \circ 1_F \) (see Section 2), it is clear that \( \mathsf{(n)} a \) is the pseudonatural isomorphism given by the pasting

\[
\mathsf{(n)} a = \prod_{i=2}^{n-1} (a \circ 1_{X_1, \ldots, X_{n-i-2}, \ldots} \otimes id_{X_{n-i-1}})
\]

the product here denoting vertical composition of pseudonatural transformations. The formula giving the pasting that defines \( a^{(n)} \) is a bit more complicated and is omitted because it is not relevant in what follows. In Fig. 8, however, both pastings are explicitly represented in the case \( n = 5 \). This defines the pseudonatural isomorphisms \( a^{(n)} \), \( \mathsf{(n)} a \) for all \( n \geq 4 \). When \( n = 1, 2, 3 \), let

\begin{figure}
\centering
\includegraphics[width=\textwidth]{figure7.png}
\caption{A graphical representation of the objects \( \otimes (n)(X_1, \ldots, X_n) \) and \( \mathsf{(n)} \otimes (X_1, \ldots, X_n) \).
\end{figure}
Figure 8. Pastings defining the pseudonatural isomorphisms $^{(5)}a$ and $a^{(5)}$

us take $\otimes^{(2)} = (2) \otimes = \otimes$, $\otimes^{(1)} = (1) \otimes = \text{id}_\mathcal{C}$ and define

$$a^{(3)} = (3) a = a$$

$$a^{(2)} = (2) a = 1 \otimes$$

$$a^{(1)} = (1) a = 1 \text{id}_\mathcal{C}$$
7.3. We can now define the cochain complex we are looking for. So, for all \( n \in \mathbb{N} \), let’s denote by \( X^n_{\text{pent}}(\mathcal{C}) \) the following vector spaces over \( K \):

\[
X^n_{\text{pent}}(\mathcal{C}) = \begin{cases} 
PseudMod(a^{(n+1)}, \sigma)(n+1) & \text{if } n \geq 0 \\ 0 & \text{otherwise} \end{cases}
\]

where PseudMod\((a^{(n+1)}, \sigma)(n+1)\) denotes the set of pseudomodifications from \( a^{(n+1)} \) to \( \sigma \) (see Definition 2.7). They are indeed vector spaces over \( K \) because of Proposition 5.2. As for the coboundary operator \( \delta_{\text{pent}} : X^{n+1}_{\text{pent}}(\mathcal{C}) \rightarrow X^n_{\text{pent}}(\mathcal{C}) \), we would like to take the usual formula, i.e.,

\[
(\delta_{\text{pent}}(n))_{X_0, \ldots, X_n} \approx 1_{id_{X_n}} \otimes n_{X_0, \ldots, X_n} + \sum_{i=1}^{n} (-1)^i n_{X_0, \ldots, X_{i-1} \otimes X_i, \ldots, X_n} + (-1)^{n+1} n_{X_0, \ldots, X_{n-1} \otimes 1_{id_{X_n}}}
\]

But the 2-morphisms corresponding to each of the terms in this sum are not 2-cells from \( a^{(n+1)}_{X_0, \ldots, X_n} \) to \( (n+1)_{a_{X_0, \ldots, X_n}} \), as required. If \( \sigma : f \Rightarrow f' \) denotes any one of these 2-morphisms, the situation is like that in Fig. 3. Note, however, that \( f, f' \) are always \( a \)-paths, because \( n \) is a modification from \( a^{(n)} \) to \( \sigma(n) \). The claim is that, once more, there exist suitable analogs of Crane-Yetter’s “padding” operators in this 2-dimensional setting of pastings that give sense to the previous definition. Behind these new “padding” operators there is again a coherence theorem, which in this case can be stated as follows:

**Theorem 7.3.** Let \((\mathcal{C}, \otimes, a, \pi)\) be a semigroupal 2-category, and let \( U, V \) be any two objects of \( \mathcal{C} \) both obtained as a certain tensor product of the objects \( X_1, \ldots, X_n \). Then, given any two \( a \)-paths from \( U \) to \( V \), there is a unique 2-isomorphism between them constructed as a pasting of instances of the structural 2-isomorphisms of \( \mathcal{C} \) and identity 2-morphisms (of expansions of instances) of the structural 1-isomorphisms.

**Proof.** Although this is a particular consequence of the strictification theorem for semigroupal 2-categories (see Theorem 4.8), let’s give a direct and somewhat more appealing argument using the Stasheff polyhedra [23]. Recall that in [23], the author introduces, for each \( n \geq 2 \), a polyhedron \( K_n \) whose vertices are in bijection with all possible parenthesizations of a word \( x_1 x_2 \cdots x_n \) of length \( n \) and whose edges all correspond to moves of the type \(-(-)-\) to \( (-)-\), where \(-\) stands for a letter or a block of letters. Stasheff shows that \( K_n \) is homeomorphic to the \((n-2)\)-dimensional ball \( D^{n-2} \). In particular, \( K_5 \) is a homeomorphic image of the 3-ball whose faces (six pentagons and three quadrilaterals) are those represented in Fig. 3. On the other hand, notice that the \((n-3)\)-dimensional faces of \( K_n \) constituting its boundary \( \partial K_n \) correspond to all meaningful ways of inserting one pair of parentheses \( x_1 x_2 \cdots (x_k \cdots x_{k+s-1}) \cdots x_n \), where \( 2 \leq s \leq n-1 \) and \( 1 \leq k \leq n-s+1 \) (in particular, \( K_n \) has \( n(n-1)/2 - 1 \) such faces). Since the next insertion of parentheses must be either within the block \((x_k \cdots x_{k+s-1})\) or treating this block as a unit,
this face can be thought of as a homeomorphic image of $K_{n-s+1} \times K_s$. All these faces are not disjoint, but intersect along their boundaries in such a way that the “edges” so formed correspond to inserting two pairs of parentheses in the word $x_1 \cdots x_n$. This allows one to construct the $K_n$, for all $n \geq 2$, by induction: $K_2$ is a point, and given $K_2, \ldots, K_{n-1}$, the next one $K_n$ is defined as the cone on $\partial K_n$, whence $\partial K_n$ is a quotient of the form $\partial K_n := \left( \prod_{s,k}(K_{n+s-1} \times K_s)_k \right) / \sim$

with $2 \leq s \leq n-1$ and $1 \leq k \leq n-s+1$.

MacLane’s classical coherence theorem for semigroupal categories (see, for example, [26]) is nothing more than an algebraic interpretation of the fact that the 2-dimensional skeleton of $K_n$, for all $n \geq 4$, is a union of homeomorphic copies of $K_4$ or $K_3 \times K_3$ (see [24]), a copy of $K_4$ corresponding to an instance of the Stasheff pentagon axiom and a copy of $K_3 \times K_3$ corresponding to an instance of a naturality square of $a_{X,Y,Z}$ applied to a morphism which is itself some $a_{X',Y',Z'}$. In the same way, the above coherence result we want to prove is a consequence of the following fact about these polyhedra:

**Lemma 7.4.** For all $n \geq 5$, the 3-dimensional skeleton of $K_n$ is a union of homeomorphic copies of $K_5$, $K_3 \times K_4$ or $K_3 \times K_3 \times K_3$.

**Proof.** Indeed, the 3-cells of $K_n$ correspond to all ways of inserting $n-5$ pairs of parenthesis in the word $x_1 \cdots x_n$. Now, as we have seen before, the insertion of the first pair gives an $(n-3)$-cell of $K_n$ homeomorphic to a suitable product $K_{s_1} \times K_{s_2'}$. Similarly, the insertion of the second pair corresponds to an $(n-4)$-cell of $K_n$ homeomorphic to some product $K_{s_1'} \times K_{s_2} \times K_{s_3''}$, because it is obtained by substituting one of the previous factors $K_{s_i}^j$ for one of its faces, etc. We conclude that all 3-cells of $K_n$ will be homeomorphic images of a suitable product $K_{s_1} \times \cdots \times K_{s_{n-4}}$. Furthermore, since $K_{s_i}$ is of dimension $s_i - 2$, it must be $(s_1 - 2) + \cdots + (s_{n-4} - 2) = 3$. This, together with the fact that $s_i \geq 2$ for all $i = 1, \ldots, n-3$, implies that at most three of the $s_i$ can be greater than 2. In other words, any 3-cell of $K_n$ is homeomorphic to a product of $n-7$ copies of $K_2$ by $K_3 \times K_3 \times K_3$, by $K_2 \times K_3 \times K_4$ or by $K_2 \times K_2 \times K_5$. \[\square\]

To prove the proposition using this lemma, let’s consider the $(\otimes, a, \pi)$-realization of $K_n$ associated to the objects $X_1, \ldots, X_n$, defined as follows: (1) as vertices, it has all possible tensor products of $X_1, \ldots, X_n$, with all possible parenthesizations, (2) as edges, it has expansions of instances of the structural 1-isomorphisms $a_{X,Y,Z}$, and (3) as 2-faces, instances of the structural 2-isomorphisms $\otimes((f', g'), (f, g)), \otimes_0(X, Y), \tilde{a}(f, g, h)$ and $\pi_{X,Y,Z,T}$. Observe that, here, the $X, Y, Z, T$ are all objects obtained as tensor products of the $X_1, \ldots, X_n$, and that the $f, g, h, f', g'$ are all identity 1-morphisms or instances of the 1-isomorphisms $a_{X,Y,Z}$. Any a-path from $U$ to $V$ is then a path in this realization of $K_n$, and the 2-isomorphisms mentioned in the proposition between two such paths correspond to 2-faces between them in this realization. Now, two such 2-faces are equal whenever the 3-cell diagram they define is commutative. But, according to the previous lemma, any 3-cell in $K_n$ is a union of 3-cells of the types $K_5$, $K_3 \times K_4$ and $K_3 \times K_3 \times K_3$. The proof of the proposition then finishes by checking that the 3-dimensional diagrams corresponding to these three possible types of 3-cells are just realizations of $K_5$, pentagonal prisms corresponding to the naturality of the pentagonator in any of the variables and instances of the cube in Fig. [4]. All of them commutative by hypothesis. \[\square\]

This unique 2-isomorphism will be called the canonical 2-isomorphism between both $\alpha$-paths. Using them, we can extend any $\sigma : f \Rightarrow f'$ above to a 2-morphism from $a_{X_0,\ldots,X_n}^{(n+1)}$ to $a_{X_0,\ldots,X_n}^{(n+1)}$ as follows. Since the source and target objects $Y, Y'$ of $f$ and $f'$ are canonically isomorphic.
to the reference objects $\otimes^{(n+1)}(X_0,\ldots,X_n)$ and $(n+1) \otimes (X_0,\ldots,X_n)$, we can choose a-paths $g : \otimes^{(n+1)}(X_0,\ldots,X_n) \to Y$ and $g' : Y' \to (n+1) \otimes (X_0,\ldots,X_n)$, represented in Fig. 3 by dashed arrows. Now, by the previous coherence theorem, there are unique canonical 2-isomorphisms $\gamma_{g,g'} : \delta^{(n+1)}_{X_0,\ldots,X_n} \to g' \circ f \circ g$ and $\gamma'_{g,g'} : g' \circ f \circ g \Rightarrow (n+1) a_{X_0,\ldots,X_n}$. The desired extension of $\sigma$ is then the pasting

$$[[\sigma]]_{g,g'} := \gamma'_{g,g'} \cdot (1_{g'} \circ \sigma \circ 1_g) \cdot \gamma_{g,g'}.$$  

Since the 2-morphism $\sigma$ will not generally be a pasting of the structural 2-isomorphisms of $\mathcal{C}$, this extension may a priori depend on the paths $g,g'$. The next result shows that this is not the case.

**Proposition 7.5.** In the above notations, the extension $[[\sigma]]_{g,g'}$ is independent of the chosen canonical 1-isomorphisms $g,g'$. In particular, there is a unique extension of $\sigma$ by canonical 2-isomorphisms.

**Proof.** Let $\hat{g}, \hat{g}'$ be any other choice. Then, we have the two extensions

$$[[\sigma]]_{\hat{g},\hat{g}'} = \gamma'_{\hat{g},\hat{g}'} \cdot (1_{\hat{g}'} \circ \sigma \circ 1_{\hat{g}}) \cdot \gamma_{\hat{g},\hat{g}'}$$

$$[[\sigma]]_{\hat{g},\hat{g}'} = \gamma'_{\hat{g},\hat{g}'} \cdot (1_{\hat{g}'} \circ \sigma \circ 1_{\hat{g}}) \cdot \gamma_{\hat{g},\hat{g}'}$$

Now, since $g, \hat{g}$ are both canonical 1-isomorphisms between the same vertices, coherence theorem implies that there exists a unique canonical 2-isomorphism $\tau : g \Rightarrow \hat{g}$. By the same reason, we also have a unique canonical 2-isomorphism $\tau' : g' \Rightarrow \hat{g}'$. Hence, the pastings

$$(\tau' \circ 1_f \circ \tau) \cdot \gamma_{g,g'}$$

$$\gamma'_{g,g'} \cdot ((\tau')^{-1} \circ 1_f \circ \tau^{-1})$$

define canonical 2-isomorphisms from $a_{X_0,\ldots,X_n}$ to $g' \circ f \circ \hat{g}$ and from $g' \circ f' \circ \hat{g}$ to $(n+1)a_{X_0,\ldots,X_n}$, respectively. By unicity, we must have

$$\gamma_{\hat{g},\hat{g}'} = (\tau' \circ 1_f \circ \tau) \cdot \gamma_{g,g'}$$

$$\gamma'_{\hat{g},\hat{g}'} = \gamma'_{g,g'} \cdot ((\tau')^{-1} \circ 1_f \circ \tau^{-1})$$

Hence, applying the interchange law, we obtain that

$$[[\sigma]]_{\hat{g},\hat{g}'} = \gamma'_{g,g'} \cdot ((\tau')^{-1} \circ 1_f \circ \tau^{-1}) \cdot (1_{g'} \circ \sigma \circ 1_{\hat{g}}) \cdot (\tau' \circ 1_f \circ \tau) \cdot \gamma_{g,g'}$$

$$= \gamma'_{g,g'} \cdot (1_{g'} \circ \sigma \circ 1_{\hat{g}}) \cdot \gamma_{g,g'}$$

$$= [[\sigma]]_{g,g'}.$$

\[\square\]

**Corollary 7.6.** Let us consider a 2-morphism $\sigma : f \Rightarrow f'$, where $f,f'$ are some a-paths between suitable parenthesizations of the tensor product of $X_0,\ldots,X_n$. Then, there exists a unique extension of $\sigma$ by canonical 2-isomorphisms to a 2-morphism between the reference $a_{X_0,\ldots,X_n}$ and $(n+1)a_{X_0,\ldots,X_n}$.

Let us denote by $[[\sigma]]$ this unique extension of $\sigma$ by canonical 2-isomorphisms. The $[[\sigma]]$ are, then, the analogs of the “padding” operators in this 2-dimensional setting (for the chosen reference 1-morphisms). Notice that they should be strictly denoted by $[[\sigma]]_{X_0,\ldots,X_n}$, because there is such an operator for every ordered set of objects $(X_0,\ldots,X_n)$.

We can now define the coboundary operator $\delta_{pent} : \wedge^{n-1}(C) \to \wedge^n(pent)(C)$ by

$$(\delta_{pent}(n))_{X_0,\ldots,X_n} = [[1_{id_{X_0}} \otimes n_{X_1,\ldots,X_n}]]$$

$$+ \sum_{i=1}^n (-1)^i [[n_{X_0,\ldots,X_{i-1} \otimes X_i,\ldots,X_n}]] + (-1)^{n+1}[[n_{X_0,\ldots,X_{n-1} \otimes 1_{id_{X_n}}}]].$$
Using similar arguments to those made to prove Proposition 6.1, it can be shown that

**Proposition 7.7.** For any $K$-linear semigroupal 2-category $(\mathcal{C}, \otimes, a, \pi)$, the pair $(\tilde{X}^\bullet_{\text{pent}}(\mathcal{C}), \delta_{\text{pent}})$ is a cochain complex.

This complex will be called the general pentagonator-deformation complex of $(\mathcal{C}, \otimes, a, \pi)$, and the corresponding cohomology groups will be denoted by $\tilde{H}^\bullet_{\text{pent}}(\mathcal{C})$, the semigroupal structure $(\otimes, a, \pi)$ being omitted for the sake of simplicity. Note that the dependence on the pentagonator $\pi$ comes exclusively through the “padding” operators $[[-]]$. Although this complex and its cohomology will be relevant in the sequel, it is not the right complex describing the infinitesimal pentagonator-deformations. Indeed, we need to take the following subcomplex:

**Proposition 7.8.** The vector subspaces $\text{Mod}(a^{(n)}_{X_0}, a^{(n)}_{X_n}) \subset \tilde{X}^{n-1}_{\text{pent}}(\mathcal{C})$ define a subcomplex of the general pentagonator-deformation complex of $\mathcal{C}$.

**Proof.** We only need to see that the naturality of the $n_{X_1, \ldots, X_n}$ in $(X_1, \ldots, X_n)$ implies that of the $(\delta_{\text{pent}}(n))_{X_0, \ldots, X_n}$ in $(X_0, \ldots, X_n)$. Let us consider for example the first term $[[1_{id_{X_0}} \otimes n_{X_1, \ldots, X_n}]]$. The naturality in $(X_1, \ldots, X_n)$ of $n_{X_1, \ldots, X_n}$ implies that of $1_{id_{X_0}} \otimes n_{X_1, \ldots, X_n}$ in $(X_0, \ldots, X_n)$. So, the situation is like that in Fig. 10 with two cylinders, one inside the other. We already know that the inner one commutes, and we want to see that the same is true for the outer one. Let’s think of this outer cylinder without the inner one as being decomposed in its upper and lower halves. Each one of these halves is itself a cylinder. Now, both bases of any one of these cylinders will correspond to modifications between $a^{(n+1)}$ or $a^{(n+1)}_\pi$ and some other induced pseudonatural isomorphism. Indeed, they are nothing more than the canonical 2-isomorphisms of the previous coherence theorem between the corresponding $a$-paths. But these 2-isomorphisms are pastings of 2-isomorphisms all natural in $(X_0, \ldots, X_n)$. It then follows that both halves also commute.

This subcomplex will be denoted by $X^\bullet_{\text{pent}}(\mathcal{C})$ and called the pentagonator-deformation complex of $\mathcal{C}$. If we denote its cohomology by $H^\bullet_{\text{pent}}(\mathcal{C})$, we have the following:

**Theorem 7.9.** For any $K$-linear semigroupal 2-category $(\mathcal{C}, \otimes, a, \pi)$, the $\omega$-equivalence classes of its first order pentagonator-deformations are in bijection with the elements of $H^3_{\text{pent}}(\mathcal{C})$. 

**Figure 10.** Naturality of the first term in $\delta_{\text{pent}}(n)$.
Proof. Let’s consider 2-isomorphisms of the form
\[ \hat{\otimes}_\epsilon((f', g'), (f, g)) = \hat{\otimes}((f', g'), (f, g)) \]
\[ (\otimes_0)\epsilon(X, Y) = (\otimes_0)(X, Y) \]
\[ \hat{a}_\epsilon(f, g, h) = \hat{a}(f, g, h) \]
\[ (\pi_\epsilon)_{X,Y,Z,T} = \pi_{X,Y,Z,T} + (\pi^{(1)})_{X,Y,Z,T,\epsilon} \]
From Proposition 4.2, it is easy to check that they define a semigroupal structure on \( C_0^{(1)} \) (hence, a first order pentagonator-deformation of \( C \)) if and only if the following conditions are satisfied (each condition is identified by the structural equation it comes from):

**A\pi 1:** The \( \pi^{(1)}_{X,Y,Z,T} \) are natural in \((X, Y, Z, T)\), i.e., they define an element \( \pi^{(1)} \in X^3_{pent}(C) \).

**A\pi 2:** \( \delta_{pent}(\pi^{(1)}) = 0 \).

The remaining structural equations are clearly superfluous in this case. This proves that first order pentagonator-deformations of \( C \) indeed correspond to 3-cocycles of \( X^3_{pent}(C) \).

Let’s suppose now that two 3-cocycles \( \pi^{(1)} \) and \( \pi^{(1)'} \) define \( \omega \)-equivalent first order pentagonator-deformations. We need to see that they are cohomologous cocycles. Indeed, from Proposition 5.14, it easily follows that both deformations are \( \omega \)-equivalent if and only if there exists 2-morphisms \( (\omega^{(1)})_{X,Y,Z} : a_{X,Y,Z} \Rightarrow a_{X,Y,Z} \), hence, and element \( \omega^{(1)} \in X^2_{pent}(C) \), such that (each condition is again identified with the corresponding condition in Proposition 5.14 it comes from)

**E\omega 1:** The \( (\omega^{(1)})_{X,Y,Z} \) are natural in \((X, Y, Z)\), i.e., \( \omega^{(1)} \in X^2_{pent}(C) \).

**E\omega 2:** \( (\pi^{(1)'} - \pi^{(1)}) = -\delta_{pent}(\omega^{(1)}) \)

(the remaining conditions \( (E\hat{\psi}1) - (E\hat{\psi}3) \) in Proposition 5.14 are clearly empty in this case). Hence, both 3-cocycles are indeed cohomologous. \( \square \)

8. Cohomology theory for the unitary deformations of the tensor product and the associator

8.1. As already indicated in the previous section, in this section we give a cohomological description of the infinitesimal unitary \( \omega \) (tensorator, associator)-deformations. To do that, we will make the simplifying assumption that the undeformed semigroupal 2-category is actually a Gray semigroup, since otherwise the theory becomes extremely cumbersome. This means, however, no loss of generality because of Theorem 4.8.

The situation we will encounter for these deformations closely resembles the cohomology theory discovered by Gerstenhaber and Schack to describe the infinitesimal deformations of a bialgebra, and later extended by Crane and Yetter to the case of a bitensor category (i.e., the categorification of a bialgebra). So, we associate a double complex to any \( K \)-linear Gray semigroup and prove that the second cohomology group of the corresponding total complex provides us with the desired description of the simultaneous first order unitary deformations of both the tensor product and the associator. As we will see, the role played by the multiplication and comultiplication in the bialgebra case corresponds in our case to the tensor product and composition of 1-morphisms. Furthermore, from this double complex we will easily get cohomologies describing the (unitary) deformations of the tensor product and the associator separately. Roughly, they are respectively related to the rows and the columns of the double complex of \( (C, \otimes) \), in much the same way as in the classical bialgebra case.

*Recall that the term *unitary* applied to an infinitesimal deformation always means that the deformed tensor product is supposed to be still unitary, i.e., such that the 2-isomorphisms \( \otimes_0(X, Y) \) remain trivial even after the deformation. At the moment of writing, the author doesn’t know how to take into account the non-trivial deformations of these 2-isomorphisms.
8.2. Let \((\mathcal{C}, \otimes)\) be a \(K\)-linear Gray semigroup. In particular, \(\otimes: \mathcal{C} \times \mathcal{C} \rightarrow \mathcal{C}\) is a cubical pseudofunctor. Recall, however, that not all cubical pseudofunctors \(\otimes\) will provide the 2-category \(\mathcal{C}\) with the structure of a Gray semigroup. More explicitly, the non-trivial 2-isomorphisms \(\hat{\otimes}((f', g'), (f, g))\) must additionally satisfy the equation
\[
\hat{\otimes}((f', g'), (f, g)) \otimes 1_{\text{hom}} = \hat{\otimes}((f' \otimes g', h'), (f \otimes g, h)) = (1_{f \circ f} \otimes \hat{\otimes}((g', h'), (g, h))) \hat{\otimes}((f', g' \otimes h'), (f, g \otimes h))
\]
coming from the structural condition \((A\tilde{\alpha}2)\) in Proposition 4.2 when the associator is trivial (the reader may easily check that the remaining structural equations \((A\hat{\alpha}3)\) and \((A\pi1) - (A\pi2)\) give no additional conditions on \(\otimes\)).

Recall from the previous section that, for all \(n \geq 1\), we introduced pseudofunctors \(\otimes^{(n)}, (n) \otimes: \mathcal{C} \times \cdots \times \mathcal{C} \rightarrow \mathcal{C}\). Then, because of the above additional equation on \(\otimes\), in a Gray semigroup we have the following:

**Proposition 8.1.** Let \((\mathcal{C}, \otimes)\) be a Gray semigroup. Then, for all \(n \geq 1\), we have the equality of pseudofunctors
\[
(n)\otimes = \otimes^{(n)} := \otimes(n).
\]
Moreover, \(\otimes(n)\) is unitary.

**Proof.** For \(n = 1, 2\) it is obvious. Let’s consider the case \(n \geq 3\). Since the structural isomorphisms \(a_{X,Y,Z}, \hat{a}(f, g, h)\) and \(\otimes_0(X, Y)\) are identities, it is clear that we only need to prove that
\[
\otimes^{(n)}((f'_1, \ldots, f'_n), (f_1, \ldots, f_n)) = \otimes^{(n)}((f'_1, \ldots, f'_n), (f_1, \ldots, f_n))
\]
The proof is by induction on \(n\). The case \(n = 3\) is nothing more than the previously mentioned additional equation, as the reader may easily check. Let \(n > 3\). By definition of \((n)\otimes\) and using the induction hypothesis, we have
\[
\otimes^{(n)}((f'_1, \ldots, f'_n), (f_1, \ldots, f_n)) = \otimes((f'_1, f'_2), (f_1, f_2)) \otimes 1_{(f'_3 \circ f_3) \circ \cdots \circ (f'_n \circ f_n)},
\]
\[
= \otimes((f'_1, f'_2), (f_1, f_2)) \otimes 1_{(f'_3 \circ f_3) \circ \cdots \circ (f'_n \circ f_n)},
\]
\[
\cdot \otimes^{(n-1)}((f'_1, f'_2, \ldots, f'_n), (f_1 \circ f_2, \ldots, f_n))
\]
Now, from the definition of \(\otimes^{(n-1)}\) and using the equality \((\tau' \cdot \tau) \otimes (\sigma' \cdot \sigma) = (\tau' \otimes \sigma') \cdot (\tau \otimes \sigma)\), it follows that
\[
\otimes^{(n-1)}((f'_1 \otimes f'_2, \ldots, f'_n), (f_1 \otimes f_2, \ldots, f_n)) = (1_{(f'_1 \otimes f'_2) \circ (f_1 \otimes f_2)} \otimes \otimes^{(n-2)}((f'_3, \ldots, f'_n), (f_3, \ldots, f_n)))
\]
\[
\cdot \otimes((f'_1 \otimes f'_2, f'_3 \otimes \cdots \otimes f'_n), (f_1 \otimes f_2, f_3 \otimes \cdots \otimes f_n))
\]
Therefore, we have
\[
\otimes^{(n)}((f'_1 \otimes f'_2, \ldots, f'_n), (f_1 \otimes f_2, \ldots, f_n)) = (\otimes((f'_1, f'_2), (f_1, f_2)) \otimes 1_{(f'_3 \circ f_3) \circ \cdots \circ (f'_n \circ f_n)}),
\]
\[
\cdot (1_{(f'_1 \otimes f'_2) \circ (f_1 \otimes f_2)} \otimes \otimes^{(n-2)}((f'_3, \ldots, f'_n), (f_3, \ldots, f_n)));
\]
\[
\cdot \otimes((f'_1 \otimes f'_2, f'_3 \otimes \cdots \otimes f'_n), (f_1 \otimes f_2, f_3 \otimes \cdots \otimes f_n))
\]
\[
= (1_{(f'_1 \circ f_1) \otimes (f'_2 \circ f_2)} \otimes \otimes^{(n-2)}((f'_3, \ldots, f'_n), (f_3, \ldots, f_n)))
\]
\[
\cdot \otimes((f'_1 \otimes f'_2, f'_3 \otimes \cdots \otimes f'_n), (f_1 \otimes f_2, f_3 \otimes \cdots \otimes f_n))
\]
The proof finishes by applying \((A\tilde{\alpha}2)\) to the last two factors. \(\square\)
operators of 1-morphisms. In Figure 11, with $m_1 K$ for these the elements $\phi$ one in the case of a Gray semigroup).

\[
\begin{array}{c}
\delta_v \downarrow \\
X^{0,2}(C) \longrightarrow X^{1,2}(C) \longrightarrow X^{2,2}(C) \longrightarrow \cdots \\
\delta_v \downarrow \\
X^{0,1}(C) \longrightarrow X^{1,1}(C) \longrightarrow X^{2,1}(C) \longrightarrow \cdots \\
\delta_v \downarrow \\
X^{0,0}(C) \delta_h \longrightarrow X^{1,0}(C) \delta_h \longrightarrow X^{2,0}(C) \delta_h \longrightarrow \cdots 
\end{array}
\]

**Figure 11.** Arrangement of the vector spaces $X^m(\otimes(n)) = X^{m-1,n-1}(C)$

8.3. Since all the pseudofunctors $\otimes(n)$ are unitary, we have for each of them the corresponding cochain complex $X^\bullet(\otimes(n))$, $n \geq 1$, describing their purely pseudofunctorial deformations (see Section 6). More precisely, if $m \leq 0$, $X^m(\otimes(n)) = 0$, while for all $m \geq 1$, it is

$$X^m(\otimes(n)) = \prod_{(X_0^0, \ldots, X_0^n), \ldots, (X_m^m, \ldots, X_m^n) \in [C^\otimes]} \text{Nat}(C \otimes (n)(X_0^0), \ldots, (X_m^m), \otimes(n)C(X_0^0), \ldots, (X_m^m))$$

To simplify, we write $(X_i^i)$ instead of $(X_i^0, \ldots, X_i^n)$. Here, $C \otimes(n)(X_i^i), \ldots, (X_m^m)$ and $\otimes(n)C(X_i^0), \ldots, (X_m^m)$ denote the functors

$$\mathcal{C}^n((X_0^{m-1}, \ldots, X_n^{m-1}), (X_0^m, \ldots, X_n^m)) \times \cdots \times \mathcal{C}^n((X_0^0, \ldots, X_n^0), (X_1^1, \ldots, X_n^1))$$

$$\longrightarrow \mathcal{C}(X_0^0 \otimes \cdots \otimes X_0^m, X_1^m \otimes \cdots \otimes X_n^m)$$

which apply the composable 1-morphisms

$$(f_0^1, \ldots, f_0^n) : (X_0^{m-1-i}, \ldots, X_n^{m-1-i}) \longrightarrow (X_1^0, \ldots, X_n^m), \quad i = 0, \ldots, m - 1$$

to $(f_0^1 \otimes \cdots \otimes f_0^n) \circ \cdots \circ (f_1^0 \circ \cdots \circ f_0^n)$ and $(f_0^1 \circ \cdots \circ f_0^n) \circ \cdots \circ (f_1^0 \circ \cdots \circ f_0^n)$, respectively.

Hence, a generic element $\phi \in X^m(\otimes(n))$ is a collection of 2-morphisms

$$\phi((f_0^1, \ldots, f_0^n), \ldots, (f_m^1, \cdots, f_m^n)) : (f_0^1 \circ \cdots \circ f_0^n) \circ \cdots \circ (f_m^1 \circ \cdots \circ f_m^n)$$

$$\Longrightarrow (f_0^1 \circ \cdots \circ f_m^1) \circ \cdots \circ (f_0^n \circ \cdots \circ f_m^n)$$

natural in the $(f_0^1, \ldots, f_0^n)$. In particular, notice that the structural 2-isomorphisms $\hat{\otimes}((f', g'), (f, g))$ define an element $\otimes \in X^2(\otimes(2))$, while the $\hat{\alpha}(f, g, h)$ define an element $\alpha \in X^1(\otimes(3))$ (the trivial one in the case of a Gray semigroup).

Instead of $X^m(\otimes(n))$, let’s use the more suggestive notation $X^{m-1,n-1}(C, \otimes)$, or just $X^{m-1,n-1}(C)$, for these $K$-vector spaces (the change of indices is for later convenience). They can be arranged as in Figure 11 with $m - 1 \geq 0$ and $n - 1 \geq 0$ being the row and column index, respectively. Since the elements $\phi \in X^{m-1,0}(C)$ are of the form $\phi((f_0, \ldots, f_m) : f_0 \cdots f_{m-1} \Longrightarrow f_0 \cdots f_{m-1}$, while those $\phi \in X^{0,n}(C)$ are of the form $\phi(f_0^1, \ldots, f_n) : f_0^1 \cdots f_n \Longrightarrow f_0^1 \cdots f_n$, we can think of the rows and columns as related to the composition and the tensor product, respectively, of 1-morphisms.

Arranged in this way, each row corresponds to the cochain complexes $X^\bullet(\otimes(n))$, the coboundary operators $\delta_h : X^{m-1,n-1}(C) \longrightarrow X^{m,n-1}(C)$, $m \geq 1$, being those defined in the previous section.
Namely, if $\phi \in X^{m-1,n-1}(\mathcal{C})$, then

\[
(\delta_h(\phi))(f_0^{0}, \ldots, f_0^{n}), \ldots, (f_{m-1}^{0}, \ldots, f_{m-1}^{n}) = [1_{f_0^{0} \circ \cdots \circ f_0^{n}} \circ \phi(f_1^{0}, \ldots, f_1^{n}), \ldots, (f_m^{0}, \ldots, f_m^{n})] \\
+ \sum_{i=1}^{m} (-1)^i [\phi(f_0^{0}, \ldots, f_0^{n}), \ldots, (f_{i-1}^{0} \circ f_i^{0}, \ldots, f_{i-1}^{n} \circ f_i^{n}), \ldots, (f_m^{0}, \ldots, f_m^{n})] + \\
+ (-1)^{m+1} [\phi(f_0^{0}, \ldots, f_0^{n}), \ldots, (f_{m-1}^{0}, \ldots, f_{m-1}^{n}) \circ 1_{f_0^{0} \circ \cdots \circ f_0^{n}}]
\]

The claim is that it is possible to define vertical coboundary maps $\delta_v : X^{m-1,n-1}(\mathcal{C}) \rightarrow X^{m-1,n}(\mathcal{C})$, for all $m \geq 1$, making each column a cochain complex and in such a way that the whole set of vector spaces and maps define a double complex. Indeed, if $\phi \in X^{m-1,n-1}(\mathcal{C})$, let’s define

\[
(\delta_v(\phi))(f_0^{0}, \ldots, f_0^{n}), \ldots, (f_{m-1}^{0}, \ldots, f_{m-1}^{n}) = [1_{f_0^{0} \circ \cdots \circ f_0^{n}} \circ \phi(f_1^{0}, \ldots, f_1^{n}), \ldots, (f_m^{0}, \ldots, f_m^{n})] \\
+ \sum_{i=1}^{n} (-1)^i [\phi(f_0^{0}, \ldots, f_0^{i-1} \circ f_0^{n}), \ldots, (f_{m-1}^{0}, \ldots, f_{m-1}^{i-1} \circ f_{m-1}^{n}), \ldots, (f_m^{0}, \ldots, f_m^{n})] + \\
+ (-1)^{n+1} [\phi(f_0^{0}, \ldots, f_0^{n-1}), \ldots, (f_{m-1}^{0}, \ldots, f_{m-1}^{n-1}) \circ 1_{f_0^{0} \circ \cdots \circ f_0^{n}}]
\]

Using arguments similar to those used to prove previous results of the same kind, one shows the following (once more, the coherence theorem for unitary pseudofunctors takes account of the padding operators):

**Proposition 8.2.** For all $m \geq 1$, the pair $(X^{\bullet;m}(\mathcal{C}), \delta_v)$ is a cochain complex.

Actually, as indicated before, we have the following stronger result, which is fundamental in our theory:

**Theorem 8.3.** The $K$-vector spaces $X^{\bullet;m}(\mathcal{C})$ together with the above maps $\delta_h : X^{\bullet;m}(\mathcal{C}) \rightarrow X^{\bullet+1;m}(\mathcal{C})$ and $\delta_v : X^{\bullet;m}(\mathcal{C}) \rightarrow X^{\bullet+1;m}(\mathcal{C})$ define a double complex.

**Proof.** It remains to prove that both coboundary maps $\delta_h$ and $\delta_v$ commute. Let’s consider an element $\phi \in X^{m-2,n-1}(\mathcal{C})$, $m \geq 2$, $n \geq 1$, with $\phi = \{\phi((f_1^{0}, \ldots, f_1^{n}), \ldots, (f_{m-1}^{0}, \ldots, f_{m-1}^{n}))\}$. Then, the reader may easily check that
\[
(\delta_h(\delta_v(\phi))(f_0^n, \ldots, f_0^n), \ldots, (f_{m-1}^n, \ldots, f_{m-1}^n)) = \\
\quad \left[1 f_0^n \otimes \cdots \otimes f_0^n \circ \left[1 f_0^n \otimes \cdots \otimes f_{m-1}^n \otimes \phi((f_1^n, \ldots, f_1^n), \ldots, (f_{m-1}^n, \ldots, f_{m-1}^n))\right]\right] \\
\quad + \sum_{i=1}^n (-1)^i \left[1 f_0^n \otimes \cdots \otimes f_0^n \circ \left[\phi((f_1^n, \ldots, f_1^n), \ldots, (f_{m-1}^n, \ldots, f_{m-1}^n)) \otimes 1 f_0^n \otimes \cdots \otimes f_{m-1}^n\right]\right] \\
\quad + \sum_{i=1}^{m-1} (-1)^{i+n+1} \left[1 f_0^n \otimes \cdots \otimes f_{m-2}^n \otimes \phi((f_1^n, \ldots, f_0^n), \ldots, (f_{m-2}^n, \ldots, f_{m-2}^n)) \circ 1 f_{m-1}^n \otimes \cdots \otimes f_{m-1}^n\right] \\
\quad + \sum_{i=1}^n (-1)^{m+i} \left[1 f_0^n \otimes \cdots \otimes f_{m-2}^n \otimes \phi((f_0^n, \ldots, f_0^n), f_{m-2}^n, \ldots, f_{m-2}^n) \circ 1 f_{m-1}^n \otimes \cdots \otimes f_{m-1}^n\right] \\
\quad + (-1)^{n+i+1} \left[1 f_0^n \otimes \cdots \otimes f_{m-2}^n \otimes \phi((f_0^n, \ldots, f_0^n), f_{m-2}^n, \ldots, f_{m-2}^n) \circ 1 f_{m-1}^n \otimes \cdots \otimes f_{m-1}^n\right] \\
\quad \left(\delta_h(\delta_v(\phi))(f_0^n, \ldots, f_0^n), \ldots, (f_{m-1}^n, \ldots, f_{m-1}^n)\right)
\]

while
\[(\delta_{\phi}(\delta_h(\phi))((f_0^0, \ldots, f_0^n), \ldots, (f_{m-1}^0, \ldots, f_{m-1}^n)) =
\]
\[[1_{f_0^0 \circ \cdots \circ f_0^m} \circ [1_{f_1^0 \circ \cdots \circ f_1^m} \circ \phi(f_1^0, \ldots, f_1^n), \ldots, (f_{m-1}^0, \ldots, f_{m-1}^n))] +
\sum_{i=1}^{m-1} (-1)^i[1_{f_0^0 \circ \cdots \circ f_0^m} \circ [\phi((f_0^0, \ldots, f_0^n), \ldots, (f_{i-1}^0 \circ f_i^1, \ldots, f_{i-1}^n \circ f_i^n), \ldots, (f_{m-1}^0, \ldots, f_{m-1}^n))] +
(-1)^m[1_{f_0^0 \circ \cdots \circ f_0^m} \circ [\phi((f_0^0, \ldots, f_0^n), \ldots, (f_{m-2}^0, \ldots, f_{m-2}^n) \circ 1_{f_{m-1}^0 \circ \cdots \circ f_{m-1}^n}) +
\sum_{i=1}^{n} (-1)^i[1_{f_0^0 \circ \cdots \circ f_0^m} \circ \phi(f_1^0, \ldots, f_1^{i-1} \circ f_1^1, \ldots, f_1^n), \ldots, (f_{m-1}^0, \ldots, f_{m-1}^n \circ f_{m-1}^1, \ldots, f_{m-1}^n))] +
\sum_{i=1}^{n} \sum_{j=1}^{m-1} (-1)^{i+j}[\phi((f_0^0, \ldots, f_0^{i-1} \circ f_0^1, \ldots, f_0^n), \ldots, (f_{m-2}^0, \ldots, f_{m-2}^1 \circ f_{m-2}^n), \ldots, (f_{m-1}^0, \ldots, f_{m-1}^n \circ f_{m-1}^1, \ldots, f_{m-1}^n))] +
\sum_{i=1}^{n} \sum_{j=1}^{m-1} (-1)^{i+j}[\phi((f_0^0, \ldots, f_0^{i-1} \circ f_0^1, \ldots, f_0^n), \ldots, (f_{m-2}^0, \ldots, f_{m-2}^1 \circ f_{m-2}^n), \ldots, (f_{m-1}^0, \ldots, f_{m-1}^n \circ f_{m-1}^1, \ldots, f_{m-1}^n))] +
\sum_{i=1}^{m-1} (-1)^{i+n}[1_{f_0^0 \circ \cdots \circ f_0^m} \circ [\phi((f_0^0, \ldots, f_0^{i-1} \circ f_0^1, \ldots, f_0^n), \ldots, (f_{m-2}^0, \ldots, f_{m-2}^1 \circ f_{m-2}^n), \ldots, (f_{m-1}^0, \ldots, f_{m-1}^n \circ f_{m-1}^1, \ldots, f_{m-1}^n))] +
\sum_{i=1}^{m-1} (-1)^{i+n}[1_{f_0^0 \circ \cdots \circ f_0^m} \circ [\phi((f_0^0, \ldots, f_0^{i-1} \circ f_0^1, \ldots, f_0^n), \ldots, (f_{m-2}^0, \ldots, f_{m-2}^1 \circ f_{m-2}^n), \ldots, (f_{m-1}^0, \ldots, f_{m-1}^n \circ f_{m-1}^1, \ldots, f_{m-1}^n))] +
1_{f_0^0 \circ \cdots \circ f_0^m} \circ [1_{f_1^0 \circ \cdots \circ f_1^m} \circ \phi(f_1^1, \ldots, f_1^n), \ldots, (f_{m-1}^0, \ldots, f_{m-1}^n)]\]

Notice that in both expressions there are nine terms. Now, recall that taking the padding \([-]\) of some 2-morphism \((-\)) simply means to take a vertical composition of \((-\)) with the appropriate expansions of the 2-isomorphisms \(\hat{\otimes}(-,-)\). It then follows by the interchange law that

\[[1_{f_0^0 \circ \cdots \circ f_0^m} \circ [-]] = [[1_{f_0^0 \circ \cdots \circ f_0^m} \circ (-)]] = [1_{f_0^0 \circ \cdots \circ f_0^m} \circ (-)]\]

This proves the equality between the second term in the expression of \(\delta_h(\delta_i(\phi))\) and the fourth term in the expression of \(\delta_i(\delta_h(\phi))\). The same argument shows the equality between the eighth and sixth terms in the first and second expression, respectively. On the other hand, we can also conclude that the first term in \(\delta_i(\delta_h(\phi))\) is the padding of

\[1_{f_0^0 \circ \cdots \circ f_0^m} \circ (1_{f_1^0 \circ \cdots \circ f_1^m} \circ \phi((f_1^1, \ldots, f_1^n), \ldots, (f_{m-1}^0, \ldots, f_{m-1}^n)))\]

But by the naturality of \(\hat{\otimes}(-,-)\) (equation \((A \hat{\otimes} 1)\)), this is the same as

\[\hat{\otimes}(-,-)^{-1} \cdot (1_{f_0^0 \circ \cdots \circ f_0^m} \circ (1_{f_1^0 \circ \cdots \circ f_1^m} \circ \phi((f_1^1, \ldots, f_1^n), \ldots, (f_{m-1}^0, \ldots, f_{m-1}^n)))) \cdot \hat{\otimes}(\hat{\otimes}(-,-))\]

whose padding clearly coincides with that of

\[1_{f_0^0 \circ \cdots \circ f_0^m} \circ (1_{f_1^0 \circ \cdots \circ f_1^m} \circ \phi((f_1^1, \ldots, f_1^n), \ldots, (f_{m-1}^0, \ldots, f_{m-1}^n)))\]

Using now that \((\hat{\tau} \cdot \tau) \otimes (\hat{\sigma} \cdot \sigma) = (\hat{\tau} \otimes \hat{\sigma}) \cdot (\tau \otimes \sigma)\), we obtain that both first terms also coincide. Similar arguments can be made to show the equality between both last terms and between the terms: third and seventh, fourth and second, sixth and eighth and seventh and third in the first and second expressions, respectively. Hence, it only remains to prove the equality between both fifth terms in each expression, and this easily follows from the naturality of \(\phi\) in its arguments applied to the 2-morphism

\[1_{f_0^0 \circ \cdots \circ f_0^m} \circ \cdots \circ (1_{f_{i+1}^{n-1}} \circ (f_{j-1}^{i-1} \circ f_{j-1}^1, f_{j-1}^j)) \cdots \circ (1_{f_{m-1}^0} \circ f_{m-1}^n) \circ \cdots \circ 1_{f_{m-1}^0 \circ \cdots \circ f_{m-1}^n}\]
This double complex \((X^{\bullet,\bullet}(\mathcal{C}), \delta_h, \delta_v)\) will be called the extended double complex of the Gray semigroup \((\mathcal{C}, \otimes)\). We are actually interested in the double complex obtained after deleting the bottom row \(X^{m,0}(\mathcal{C})\). It will be called the double complex of \((\mathcal{C}, \otimes)\). Furthermore, for our purposes, we need to take a subcomplex of this double complex. This is related to the fact that we only consider infinitesimal unitary deformations.

**Definition 8.4.** An element \(\phi \in X^{m-1,n-1}(\mathcal{C})\) will be called special if whenever \((f_1^i, \ldots, f_n^i) = (id, \ldots, id)\) for some \(i \in \{0, \ldots, m-1\}\), it holds

\[
\phi((f_0^i, \ldots, f_0^m), \ldots, (f_{m-1}^i, \ldots, f_{m-1}^m)) = 0
\]

The set of \(\phi \in X^{m-1,n-1}(\mathcal{C}), m, n \geq 1\), which are special clearly define a vector subspace, which will be denoted by \(X^{m-1,n-1}_s(\mathcal{C})\). We have then the following:

**Proposition 8.5.** The vector subspaces \(X^{m,n}_s(\mathcal{C}), m, n \geq 0\), define a subcomplex \(X^{\bullet,\bullet}_s(\mathcal{C})\) of the extended double complex of \((\mathcal{C}, \otimes)\).

**Proof.** We need to see that both coboundary operators \(\delta_h\) and \(\delta_v\) preserve the special elements. Indeed, let \(\phi \in X^{m-1,n-1}_s(\mathcal{C})\). Then, from the above expression of \((\delta_h(\phi))\), it is clear that when \((\langle f_1^i, \ldots, f_n^i \rangle = (id, \ldots, id)\) for some \(i \in \{0, \ldots, m-1\}\), all terms are zero except the \((i + 1)th\) term, and the \((i + 2)th\) terms, which are equal but of opposite sign (recall that, \(\otimes\) being unitary, the tensor product of identity 1-morphisms is always an identity 1-morphism, and that the identity 2-morphism of an identity 1-morphism is a unit with respect to horizontal composition). Hence, \(\delta_h(\phi)\) is special. On the other hand, if \(\phi\) is special and \((\langle f_1^i, \ldots, f_n^i \rangle = (id, \ldots, id)\) for some \(i \in \{0, \ldots, m-1\}\), all terms in \(\delta_v(\phi)\) are clearly zero, so that \(\delta_v(\phi)\) is also special. \(\Box\)

The double complex defined by the vector subspaces \(X^{m,n}_s(\mathcal{C}), m, n \geq 0\), and the corresponding restrictions of both \(\delta_h\) and \(\delta_v\) will be called the special extended double complex of \((\mathcal{C}, \otimes)\), or just the special double complex of \((\mathcal{C}, \otimes)\), when the bottom row is deleted.

8.4. Let \(X^{\bullet,\bullet}_{\text{tens,ass}}(\mathcal{C})\) denote the total complex associated to the special double complex \((X^{\bullet,\bullet}_s(\mathcal{C}), \delta_h, \delta_v)\) of \((\mathcal{C}, \otimes)\). It will be called the unitary (tensorator, associator)-deformation complex of \((\mathcal{C}, \otimes)\). By definition, it is the cochain complex with vector spaces

\[
X^q_{\text{tens,ass}}(\mathcal{C}) = \bigoplus_{m+n=q \atop m \geq 0, n \geq 1} X^{m,n}_s(\mathcal{C}), \quad q \geq 1
\]

and coboundary operators \(\delta_{\text{tens,ass}} : X^q_{\text{tens,ass}}(\mathcal{C}) \to X^{q+1}_{\text{tens,ass}}(\mathcal{C})\) given by

\[
\delta_{\text{tens,ass}} = \bigoplus_{m+n=q \atop m \geq 0, n \geq 1} ((-1)^n \delta_h + \delta_v)
\]

The corresponding cohomology groups will be denoted by \(H^\bullet_{\text{tens,ass}}(\mathcal{C})\). The reason we choose the above name for this total complex is the following theorem:

**Theorem 8.6.** If \((\mathcal{C}, \otimes)\) is a \(K\)-linear Gray semigroup, the \(\psi\)-equivalence classes of its first order unitary (tensorator, associator)-deformations are in bijection with the elements of the group \(H^2_{\text{tens,ass}}(\mathcal{C})\).
Proof. Let’s consider 2-isomorphisms
\[ \tilde{\Psi}_\epsilon((f', g'), (f, g)) = \tilde{\Theta}((f', g'), (f, g)) + \tilde{\Theta}^{(1)}((f', g'), (f, g)) \epsilon \]
\[ (\otimes_0)_*(X, Y) = 1_{id_X \otimes Y} \]
\[ \tilde{\alpha}(f, g, h) = 1_{f \otimes g \otimes h} + \tilde{\Theta}^{(1)}(f, g, h) \epsilon \]
\[ (\pi_\epsilon)_X, Y, Z, T = 1_{id_X \otimes Y \otimes Z \otimes T} \]
Substituting these 2-isomorphisms in the structural equations in Proposition 1.2 and computing the first order term in \( \epsilon \), it turns out that they define a first order unitary (tensorator, associator)-deformation of \( \mathfrak{C} \) if and only if the 2-morphisms \( \tilde{\Theta}^{(1)}((f', g'), (f, g)) \) satisfy the following conditions:

A\( \tilde{\Theta}^1 \): The \( \tilde{\Theta}^{(1)}((f', g'), (f, g)) \) are natural in \( (f', g'), (f, g) \), i.e., they define an element
\[ \tilde{\Theta}^{(1)} \in X_{\otimes(2)}^{1,1}(\mathfrak{C}) = X_{\otimes(2)}^{2} \]
A\( \tilde{\Theta}^2 \): For all composable 1-morphisms \( (f'', g''), (f', g'), (f, g) \)
\[ \tilde{\Theta}^{(1)}((f'', g''), (f' \circ f, g' \circ g)) \cdot (1_{f'' \otimes g''} \circ \tilde{\Theta}^{(1)}((f', g'), (f, g))) = \]
\[ \tilde{\Theta}((f'', g''), (f' \circ f, g' \circ g)) \cdot (1_{f'' \otimes g''} \circ \tilde{\Theta}^{(1)}((f', g'), (f, g))) = \]
\[ = \tilde{\Theta}^{(1)}((f'' \circ f', g'' \circ g'), (f, g)) \cdot (1_{f'' \otimes g''} \circ \tilde{\Theta}^{(1)}((f', g'), (f, g))) + \]
\[ \tilde{\Theta}((f'' \circ f', g'' \circ g'), (f, g)) \cdot (\tilde{\Theta}^{(1)}((f'', g''), (f', g')) \circ 1_{f \otimes g}) + \]
It is easy to check that this is exactly the condition \( \delta_h(\tilde{\Theta}^{(1)}) = 0 \).

A\( \tilde{\Theta}^3 \): For all 1-morphisms \( (f, g) : (X, Y) \longrightarrow (X', Y') \), it holds
\[ \tilde{\Theta}^{(1)}((id_{X'}, id_{Y'}), (f, g)) = \tilde{\Theta}^{(1)}((f, g), (id_X, id_Y)) = 0 \]
i.e., \( \tilde{\Theta}^{(1)} \in X_{\otimes(2)}^{2,1}(\mathfrak{C}) \subset X_{\otimes(2)}^{2,1}(\mathfrak{C}) \).

A\( \tilde{\alpha}^1 \): The \( \tilde{\alpha}^{(1)}(f, g, h) \) are natural in \( (f, g, h) \), i.e., they define an element
\[ \tilde{\alpha}^{(1)} \in X_{\otimes(3)}^{0,2}(\mathfrak{C}) = X_{\otimes(3)}^{1} \]
A\( \tilde{\alpha}^2 \): For all 1-morphisms \( (f, g, h) \)
\[ (\tilde{\Theta}^{(1)}((f', g'), (f, g)) \otimes 1_{h', h}) \cdot \tilde{\Theta}((f' \otimes g', h'), (f \otimes g, h)) = \]
\[ (\tilde{\Theta}((f', g'), (f, g)) \otimes 1_{h', h}) \cdot \tilde{\Theta}^{(1)}((f' \otimes g', h'), (f \otimes g, h)) = \]
\[ \tilde{\alpha}^{(1)}(f' \circ f, g' \circ g, h' \circ h) \cdot \tilde{\Theta}((f', g'), (f, g)) \otimes 1_{h', h}) \cdot \tilde{\Theta}((f' \otimes g', h'), (f \otimes g, h)) = \]
\[ = (1_{f' \circ f} \otimes \tilde{\Theta}^{(1)}((g', h'), (g', h') \otimes (f, g, h))) + \tilde{\Theta}((f', g' \otimes h'), (f \otimes g, h)) + \]
\[ + (1_{f' \circ f} \otimes \tilde{\Theta}^{(1)}((g', h'), (g', h') \otimes (f, g, h))) + \tilde{\Theta}((f', g' \otimes h'), (f \otimes g, h)) + \]
\[ + (1_{f' \circ f} \otimes \tilde{\Theta}^{(1)}((g', h'), (g', h') \otimes (f, g, h))) + \tilde{\Theta}((f', g' \otimes h'), (f \otimes g, h)) + \]
\[ + (1_{f' \circ f} \otimes \tilde{\Theta}^{(1)}((g', h'), (g', h') \otimes (f, g, h))) + \tilde{\Theta}((f', g' \otimes h'), (f \otimes g, h)) + \]
\[ + (1_{f' \circ f} \otimes \tilde{\Theta}^{(1)}((g', h'), (g', h') \otimes (f, g, h))) + \tilde{\Theta}((f', g' \otimes h'), (f \otimes g, h)) + \]
It is easy to check that this is exactly the condition \( \delta_h(\tilde{\Theta}^{(1)}) + \delta_h(\tilde{\alpha}^{(1)}) = 0 \).

A\( \tilde{\alpha}^3 \): For all objects \( (X, Y, Z) \), it holds
\[ \tilde{\alpha}^{(1)}(id_X, id_Y, id_Z) = 0 \]
i.e., \( \tilde{\alpha}^{(1)} \in X_{\otimes(3)}^{0,2}(\mathfrak{C}) \subset X_{\otimes(3)}^{2,1}(\mathfrak{C}) \).
A\pi 1: For all 1-morphisms \((f,g,h,k)\)
\[
\hat{a}^{(1)}(f,g,h \otimes k) + \hat{a}^{(1)}(f \otimes g,h,k) = \\
= -\hat{\circ}^{(1)}((id_{Y'},id_{Z',Z} \otimes \epsilon_T'),(f,g \otimes h \otimes k)) + 1_f \otimes \hat{a}^{(1)}(g,h,k) \\
+ \hat{\circ}^{(1)}((f,g \otimes h \otimes k),(id_{X',id_{Z} \otimes T}) + \hat{a}^{(1)}(f,g \otimes h,k) \\
- \hat{\circ}^{(1)}((id_{Y' \otimes Z',Z} \otimes \epsilon_T'),(f \otimes g \otimes h,k)) + \hat{a}^{(1)}(f,g,h) \otimes 1_k \\
+ \hat{\circ}^{(1)}((f \otimes g \otimes h,k),(id_{X' \otimes Y' \otimes Z} \otimes id_T))
\]

Since the terms in \(\hat{\circ}^{(1)}\) are all zero by condition \((A \hat{\otimes} 3)\), this exactly corresponds to the condition \(\delta_v(\hat{a}^{(1)}) = 0\).

As the reader may easily check, the structural equation \((A \hat{\pi} 2)\) gives no additional conditions for this kind of deformation. Now, \((A \hat{\otimes} 1), (A \hat{\otimes} 3)\) together with \((A \hat{\pi} 1), (A \hat{\pi} 3)\) say that \((\hat{a}^{(1)}, \hat{\circ}^{(1)})\) is \(X^{2}_{\text{tens, ass}}(\mathcal{C})\). On the other hand, we have
\[
\delta_{\text{tens, ass}}(\hat{a}^{(1)}, \hat{\circ}^{(1)}) = (\delta_v(\hat{a}^{(1)}), \delta_h(\hat{a}^{(1)}) + \delta_v(\hat{\circ}^{(1)}), -\delta_h(\hat{\circ}^{(1)}))
\]

Hence, \((A \hat{\otimes} 2), (A \hat{\pi} 2)\) and \((A \pi 1)\) together say that \((\hat{a}^{(1)}, \hat{\circ}^{(1)})\) is a 2-cocycle.

Let us now suppose that \((\hat{a}^{(1)}'), (\hat{\circ}^{(1)})'\) is another 2-cocycle defining a \(\psi\)-equivalent first order unitary (tens, ass)-deformation of \(\mathcal{C}\). We need to see that both 2-cocycles are actually cohomologous. Indeed, from Proposition \(5.14\), it immediately follows that they are \(\psi\)-equivalent deformations if and only if there exists 2-morphisms \(\hat{\psi}^{(1)}(f,g) : f \otimes g \Longrightarrow f \otimes g\) such that

\begin{align*}
\text{E} \hat{\psi} 1: & \quad \text{The } \hat{\psi}^{(1)}(f,g) \text{ are natural in } (f,g), \text{i.e., they define an element } \hat{\psi}^{(1)} \in X^{0,1}_{\mathcal{C}} \\
\text{E} \hat{\psi} 2: & \quad (\hat{\circ}^{(1)})' - \hat{\circ}^{(1)} = -\delta_h(\hat{\psi}^{(1)}). \\
\text{E} \hat{\psi} 3: & \quad \hat{\psi}^{(1)} \text{ is special.} \\
\text{E} \omega 1: & \quad (\hat{a}^{(1)}')' - \hat{a}^{(1)} = \delta_v(\hat{\psi}^{(1)}). 
\end{align*}

(in this case, the condition coming from equation \((E \omega 2)\) is empty). The first and third conditions together say that \(\hat{\psi}^{(1)} \in X^{0,1}_{\mathcal{C}} = X^1_{\text{tens, ass}}(\mathcal{C})\), while the second and fourth express nothing more than the fact that
\[
((\hat{a}^{(1)}'), (\hat{\circ}^{(1)})') - (\hat{a}^{(1)}, \hat{\circ}^{(1)}) = \delta_{\text{tens, ass}}(\hat{\psi}^{(1)})
\]
as required. \(\square\)

8.5. With the above results, it is easy to obtain a cochain complex whose cohomology describes the infinitesimal associator-deformations of \((\mathcal{C}, \otimes)\), i.e., those deformations where only the associator is deformed. Indeed, such a deformation is given by 2-isomorphisms of the form
\[
\hat{\circ}_\epsilon((f',g'),(f,g)) = \hat{\circ}((f',g'),(f,g)) \\
(\otimes_0)_\epsilon(X,Y) = 1_{id_X \otimes Y} \\
\hat{a}_\epsilon(f,g,h) = 1_{f \otimes g \otimes h} + \hat{a}^{(1)}(f,g,h) \epsilon \\
(\pi_\epsilon)(X,Y,Z,T) = 1_{id_{X' \otimes Y' \otimes Z' \otimes T}}
\]
According to the proof of the previous theorem, they define a first-order associator-deformation of \(\mathcal{C}\) if and only if the \(\hat{a}^{(1)}(f,g,h)\) define an element \(\hat{a}^{(1)} \in X^{0,2}_{\mathcal{C}}\) which moreover satisfies that
\[
\delta_v(\hat{a}^{(1)}) = 0 = \delta_h(\hat{a}^{(1)})
\]
The first equality just says that $\hat{\alpha}^{(1)}$ is a 2-cocycle of the cochain complex $(X_s^{0,*}(\mathcal{C}), \delta_v)$, while the second one serves to define a subcomplex of this complex. More explicitly, let’s define

$$X_{ass}^n(\mathcal{C}) := \text{Ker}(\delta_h : X_s^{0,n}(\mathcal{C}) \rightarrow X_s^{1,n}(\mathcal{C})) \quad n \geq 0$$

Since $X_s^{*,*}(\mathcal{C})$ is a double complex, its horizontal coboundary map $\delta_h$ is a morphism of complexes. But the kernel of a morphism of complexes is a subcomplex. Hence, the subspaces $X_{ass}^n(\mathcal{C})$, $n \geq 0$, together with the corresponding restriction of $\delta_v$, which will be denoted by $\delta_{ass}$, define a cochain complex. Let us call it the \emph{associator-deformation complex} of the Gray semigroup $(\mathcal{C}, \otimes)$. Then, if $H_{ass}^*(\mathcal{C})$ denote the corresponding cohomology groups, the following result is an immediate consequence of the previous theorem:

**Theorem 8.7.** If $(\mathcal{C}, \otimes)$ is a $K$-linear Gray semigroup, the $\psi$-equivalence classes of its first order associator-deformations are in bijection with the elements of $H_{ass}^2(\mathcal{C})$.

**Proof.** Notice that the $\hat{\psi}^{(1)} \in X_s^{0,2}(\mathcal{C})$ must be such that $\delta_h(\hat{\psi}^{(1)}) = -((\hat{\psi}^{(1)})') + \hat{\psi}^{(1)} = 0$, i.e., $\hat{\psi}^{(1)} \in X_{ass}^3(\mathcal{C})$, as required. \hfill $\square$

8.6. In a similar way, we can easily get a cochain complex describing the infinitesimal unitary tensorator-deformations of $\mathcal{C}$, i.e., those deformations where only the tensor product is deformed, and in such a way that it remains unitary. Indeed, such a deformation is given by 2-isomorphisms of the form

$$\hat{\alpha}_c((f', g'), (f, g)) = \hat{\alpha}((f', g'), (f, g)) + \hat{\psi}^{(1)}((f', g'), (f, g))\epsilon$$

$$\hat{\alpha}_c(X, Y) = 1_{id_{X \otimes Y}}$$

$$\hat{\alpha}_c(f, g, h) = 1_{f \otimes g \otimes h}$$

$$\hat{\alpha}_c(X, Y, Z, T) = 1_{(id_{X \otimes Y} \otimes id_{Z \otimes T})}$$

Again going back to the proof of Theorem 8.7, it immediately follows that they define a first-order unitary tensorator-deformation of $\mathcal{C}$ if and only if the $\hat{\psi}^{(1)}((f', g'), (f, g))$ define an element $\hat{\psi}^{(1)} \in X_{s,ass}^3(\mathcal{C})$ which moreover satisfies that

$$\delta_h(\hat{\psi}^{(1)}) = 0 = \delta_v(\hat{\psi}^{(1)})$$

Now, recall that $X_{s,ass}^n(\mathcal{C}) = X_{s}^{n,1}(\otimes)$, the special $n$-cochains of the purely pseudofunctorial deformation complex of the (unitary) pseudofunctor $\otimes$ (see Section 6). Then, the first equality just says that $\hat{\psi}^{(1)}$ is a 2-cocycle of this cochain complex $X_{s}^{*,*}(\otimes)$, while the second one serves to define a subcomplex of this complex. More explicitly, let’s define

$$X_{tens}^n(\mathcal{C}) := \text{Ker}(\delta_v : X_s^{n,1}(\otimes) \rightarrow X_s^{n,1}(\mathcal{C})) \quad n \geq 0$$

The same argument as before shows that these subspaces define a subcomplex. Let us call it the \emph{unitary tensorator-deformation complex} of the Gray semigroup $(\mathcal{C}, \otimes)$. Then, if $H_{tens}^*(\mathcal{C})$ denote the corresponding cohomology groups, the following result is an immediate consequence of the previous theorem:

**Theorem 8.8.** If $(\mathcal{C}, \otimes)$ is a $K$-linear Gray semigroup, the $\psi$-equivalence classes of its first order unitary tensorator-deformations are in bijection with the elements of $H_{tens}^2(\mathcal{C})$.

9. Cohomology theory for the generic unitary deformations

9.1. In Sections 7 and 8 we have constructed complexes $X_{pent}^*(\mathcal{C})$ and $X_{tens,ass}^*(\mathcal{C})$ whose cohomologies separately describe the infinitesimal unitary deformations of the pentagonator, on the one hand, and the tensor product and the associator, on the other, of a $K$-linear Gray semigroup.
(\mathcal{C}, \otimes) (actually, of an arbitrary K-linear semigroupal 2-category in the case of the deformations of the pentagonator). The goal of this section is to obtain a cohomology describing the simultaneous unitary deformations of both sets of structural 2-isomorphisms. To do that, we will need to go back to the bigger cochain complex $X_{pent}^\bullet(\mathcal{C}) \supseteq X^\bullet(\mathcal{C})$ introduced in Section 7. The appropriate cohomology for the generic unitary deformations turns out to be the total complex of a modified version of the double complex of $(\mathcal{C}, \otimes)$ introduced in the previous section. The modification consists of the substitution of the first column $X^1(\mathcal{C})$ in this double complex by a suitable cone of that column and the general pentagonator-deformation complex $\tilde{X}_{pent}^\bullet(\mathcal{C})$.

9.2. Recall that given two cochain complexes $(A^\bullet, \delta_A)$ and $(B^\bullet, \delta_B)$ and a morphism of complexes $\varphi : (A^\bullet, \delta_A) \rightarrow (B^\bullet, \delta_B)$, the cone of $A^\bullet$ and $B^\bullet$ over $\varphi$ is the cochain complex $(C^\bullet(\varphi(A^\bullet, B^\bullet), \delta_C)$ defined by

$$C^n(\varphi(A^\bullet, B^\bullet)) = B^n \oplus A^{n+1}$$

and

$$\delta_C(b, a) = (\delta_B(b) + \varphi(a) - \delta_A(a)), \quad (b, a) \in B^n \oplus A^{n+1}.$$ 

The minus sign in $\delta_A$ is to ensure that $\delta_C \circ \delta_C = 0$. For more details see, for example, Weibel\[36].

9.3. Let $(\mathcal{C}, \otimes)$ be a K-linear Gray semigroup, and let’s consider the general pentagonator-deformation complex $\tilde{X}_{pent}^\bullet(\mathcal{C})$ defined in Section 7. The associator $a$ of $\mathcal{C}$ being trivial, this complex reduces to

$$\tilde{X}_{pent}^{n-1}(\mathcal{C}) = \text{PseudMod}(1_n, 1_n), \quad n \geq 1$$

where $1_n$ is the pseudonatural isomorphism $1_n : \otimes(n) \Rightarrow \otimes(n)$ whose structural 1- and 2-isomorphisms are all identities, while the coboundary is given by

$$(\delta_{pent}(n))x_0 \ldots x_n = 1_{id_{x_0}} \otimes nx_1 \ldots x_n + \sum_{i=1}^{n} (-1)^{i} nx_{0} \ldots x_{i-1} \otimes x_{i} \ldots x_n + (-1)^{n+1} nx_0 \ldots x_{n-1} \otimes 1_{id_{x_n}}$$

(in this case, the padding operators are superfluous). Notice that a generic element $n \in \tilde{X}_{pent}^{n-1}(\mathcal{C})$, $n \geq 1$, is just a collection of 2-morphisms $n_{x_1, \ldots, x_n} : id_{x_1} \otimes \ldots \otimes x_n \Rightarrow id_{x_1} \otimes \ldots \otimes x_n$, for all objects $(X_1, \ldots, X_n)$ of $\mathcal{C}^n$.

Let us consider, on the other hand, the complex $(X^{0,n}(\mathcal{C}), \delta_n)$ corresponding to the first column of the extended double complex of $(\mathcal{C}, \otimes)$. We can define K-linear maps $\varphi : \tilde{X}_{pent}^{n}(\mathcal{C}) \rightarrow X^{0,n-1}(\mathcal{C})$, $n \geq 1$ by

$$(\varphi(n))(f_1, \ldots, f_n) = n_{x_1' \ldots, x_n'} \circ (f_1 \otimes \ldots \otimes f_n) - (f_1 \otimes \ldots \otimes f_n) \circ n_{x_1' \ldots, x_n}$$

for all 1-morphisms $(f_1, \ldots, f_n) : (X_1, \ldots, X_n) \rightarrow (X_1', \ldots, X_n')$ of $\mathcal{C}^n$.

**Proposition 9.1.** The above maps $\varphi_* : \tilde{X}_{pent}^\bullet(\mathcal{C}) \rightarrow X^{0,\bullet}(\mathcal{C})$ define a morphism of cochain complexes.
Proof. Let \( n \in \tilde{X}^{n-1}_{\text{pent}}(\mathcal{C}) \). Then, we have
\[
(\delta_v(\varphi(n))(f_0, \ldots, f_n) = 1_{f_0} \otimes (\varphi(n))(f_1, \ldots, f_n) + \\
\sum_{i=1}^{n} (-1)^i(\varphi(n))(f_0, \ldots, f_{i-1} \otimes f_i, \ldots, f_n) + \\
(-1)^{n+1}(\varphi(n))(f_0, \ldots, f_{n-1}) \otimes 1_{f_n} \\
= -1_{f_0} \otimes (1_{f_1} \otimes \cdots \otimes f_n \circ n_{X_1, \ldots, X_n}) + \\
1_{f_0} \otimes (n_{X_1', \ldots, X_n'} \circ 1_{f_1} \otimes \cdots \otimes f_n) - \\
\sum_{i=1}^{n} (-1)^i1_{f_0} \otimes \cdots \otimes f_n \circ n_{X_0, \ldots, X_{i-1} \otimes X_i, \ldots, X_n} + \\
\sum_{i=1}^{n} (-1)^i n_{X_0', \ldots, X_{i-1}', X_i', \ldots, X_n'} \otimes 1_{f_0} \otimes \cdots \otimes f_n - \\
(-1)^{n+1}(1_{f_0} \otimes \cdots \otimes f_{n-1} \circ n_{X_0, \ldots, X_{n-1}}) \otimes 1_{f_n} + \\
+ (-1)^{n+1}(n_{X_0', \ldots, X_{n-1}'} \otimes 1_{f_0} \otimes \cdots \otimes f_{n-1}) \otimes 1_{f_n}
\]
(the reader may easily check that the padding operators appearing in the definition of \( \delta_v \) are indeed trivial in this case). Now, since \( \mathcal{C} \) is a Gray semigroup, it is \( \oplus((f_0, f_1 \otimes \cdots \otimes f_n), (id_{X_0}, id_{X_1 \otimes \cdots \otimes X_n})) = \oplus((id_{X_0}, id_{X_1 \otimes \cdots \otimes X_n}), (f_0, f_1 \otimes \cdots \otimes f_n)) = 1_{f_0} \otimes \cdots \otimes f_n \). Therefore, using Equation \((A \otimes 1)\), we get
\[
1_{f_0} \otimes (1_{f_1} \otimes \cdots \otimes f_n \circ n_{X_1, \ldots, X_n}) = 1_{f_0} \otimes \cdots \otimes f_n \circ (1_{id_{X_0}} \otimes n_{X_1, \ldots, X_n}) \\
1_{f_0} \otimes (n_{X_1', \ldots, X_n'} \circ 1_{f_1} \otimes \cdots \otimes f_n) = (1_{id_{X_0}} \otimes n_{X_1', \ldots, X_n'}) \circ 1_{f_0} \otimes \cdots \otimes f_n
\]
The last two terms are treated similarly. It follows immediately that \( \delta_v(\varphi(n)) = \varphi(\delta_{\text{pent}}(n)) \), as required.

Remark 9.2. Notice that the pentagonator-deformation subcomplex \( X^\bullet_{\text{pent}}(\mathcal{C}) \subseteq \tilde{X}^\bullet_{\text{pent}}(\mathcal{C}) \) is nothing more that the kernel of this morphism \( \varphi \).

Associated to this cochain map, there is the corresponding cone complex, which will be denoted by \( (X^\bullet_{\text{pent}, \text{ass}}(\mathcal{C}), \delta_{\text{pent}, \text{ass}}) \). By definition
\[
X^n_{\text{pent}, \text{ass}}(\mathcal{C}) = X^{0,n}(\mathcal{C}) \oplus \tilde{X}^{n+1}_{\text{pent}}(\mathcal{C}), \quad n \geq 0
\]
with coboundary map \( \delta_{\text{pent}, \text{ass}} : X^n_{\text{pent}, \text{ass}}(\mathcal{C}) \longrightarrow X^{n+1}_{\text{pent}, \text{ass}}(\mathcal{C}) \) given by
\[
\delta_{\text{pent}, \text{ass}}(\phi, n) = (\delta_v(\phi) + \varphi(n), -\delta_{\text{pent}}(n))
\]
for all \((\phi, n) \in X^n_{\text{pent}, \text{ass}}(\mathcal{C})\). We have the following modified version of the extended double complex of \( (\mathcal{C}, \otimes) \):

**Proposition 9.3.** Let’s substitute the first column \((X^0\bullet(\mathcal{C}), \delta_v)\) of the extended double complex \((X^\bullet\bullet(\mathcal{C}), \delta_h, \delta_v)\) of \((\mathcal{C}, \otimes)\) for the previous cone complex \((X^\bullet\bullet_{\text{ass}}(\mathcal{C}), \delta_{\text{pent}, \text{ass}})\), and the coboundary maps \( \delta_h : X^{0,n}(\mathcal{C}) \longrightarrow X^{1,n}(\mathcal{C}), n \geq 0 \), for the maps \( \delta'_h : X^n_{\text{pent}, \text{ass}}(\mathcal{C}) \longrightarrow X^{1,n}(\mathcal{C}) \) given by the projection to \( X^{0,n}(\mathcal{C}) \) followed by \( \delta_h \). Then, the resulting collection of \( K \)-vector spaces and linear maps is a double complex (see Fig. 7).

Proof. By the way the \( \delta'_h \) are defined, it is clear that the new rows are cochain complexes. So, we only need to see that the squares on the left of Fig. still commute. Let \((\phi, n) \in X^{n-1}_{\text{pent}, \text{ass}}(\mathcal{C})\).
Since the coboundary maps $\delta_h, \delta_v$ commute, we have

$$
\delta_h'(\delta_{\text{pent,ass}}(\phi, n)) = \delta_h'(\delta_v(\phi) + \varphi(n), -\delta_{\text{pent}}(n))
= \delta_h(\delta_v(\phi)) + \delta_h(\varphi(n))
= \delta_v(\delta_h(\phi)) + \delta_h(\varphi(n))
= \delta_v(\delta_h'(\phi, n)) + \delta_h(\varphi(n))
$$

Therefore, the proof reduces to show that the term $\delta_h(\varphi(n))$ is zero for all $n \in \tilde{X}_{\text{pent}}^n(\mathcal{C})$. Now, by definition of $\delta_h$ and $\varphi$, we have

$$
\delta_h(\varphi(n))(f_0', \ldots, f_n')(f_0, \ldots, f_n) = [1_{f_0' \otimes \cdots \otimes f_n'} \circ (\varphi(n))(f_0, \ldots, f_n)]
- [(\varphi(n))(f_0' \circ f_0, \ldots, f_n' \circ f_n)]
+ [(\varphi(n))(f_0', \ldots, f_n') \circ 1_{f_0 \otimes \cdots \otimes f_n}]
= -[1_{f_0' \otimes \cdots \otimes f_n'} \circ (1_{f_0 \otimes \cdots \otimes f_n} \circ n_{X_0' \otimes \cdots \otimes X_n})]
+ [1_{f_0' \otimes \cdots \otimes f_n'} \circ (n_{X_0' \otimes \cdots \otimes X_n} \circ 1_{f_0' \otimes \cdots \otimes f_n})]
+ [1_{f_0 \otimes \cdots \otimes f_n} \circ n_{X_0' \otimes \cdots \otimes X_n}]
- n_{X_0' \otimes \cdots \otimes X_n} \circ 1_{f_0' \otimes \cdots \otimes f_n}
- [(1_{f_0' \otimes \cdots \otimes f_n'} \circ n_{X_0' \otimes \cdots \otimes X_n} \circ 1_{f_0' \otimes \cdots \otimes f_n})]
+ [(n_{X_0' \otimes \cdots \otimes X_n} \circ 1_{f_0' \otimes \cdots \otimes f_n} \circ 1_{f_0' \otimes \cdots \otimes f_n})]
$$

Since $\mathcal{C}$ is a 2-category, the second and fifth terms clearly cancel out each other. On the other hand, using again that $\mathcal{C}$ is a 2-category and the interchange law and the naturality of the 2-morphisms
\[ [1, f'_0 \otimes \cdots \otimes f'_n, \circ (\otimes f_0 \otimes \cdots \otimes f_n) \circ n_{X_0 \otimes \cdots \otimes X_n}] = \]
\[ \text{Therefore, the first and third term above also cancel out each other. The same thing can be shown similarly for the fourth and sixth terms.} \]

This new double complex will be called the modified extended double complex of \((\mathcal{C}, \otimes)\), and denoted by \(X_{\text{mod}}^{\bullet, \bullet}(\mathcal{C})\). So

\[ X_{\text{mod}}^{m,n}(\mathcal{C}) = \begin{cases} X_{\text{mod}}^{0, n}(\mathcal{C}) \oplus \widetilde{X}_{\text{pent}}^{n+1}(\mathcal{C}) & \text{if } m = 0, n \geq 0 \\ X_{\text{mod}}^{m,n}(\mathcal{C}) & \text{if } m \geq 1, n \geq 0 \end{cases} \]

For short, the corresponding horizontal and vertical coboundary maps will be denoted by \(\delta'_v\) and \(\delta'_h\), respectively. But remind that \(\delta'_v = \delta_v\) except for the first column, where \(\delta'_v = \delta_{\text{pent,ass}}\), and that \(\delta'_h = \delta_h\) except for \(m = 0\), where it is the projection to the first component followed by \(\delta_h\).

9.4. Let us proceed now as in Section 8 and consider the double complex obtained from \(X_{\text{mod}}^{\bullet, \bullet}(\mathcal{C})\) after deleting the first row. It will be the modified double complex of \((\mathcal{C}, \otimes)\). Furthermore, let’s take the subcomplex \(X_{\text{mod},s}^{\bullet, \bullet}(\mathcal{C})\) of this modified double complex corresponding to the special elements.

These are defined in the same way as in Section 8 for all \(m \geq 1, n \geq 1\), while \((\phi, n) \in X_{\text{mod}}^{0, n}(\mathcal{C}), n \geq 1\), is called special whenever \(\phi\) is special. We leave to the reader to check that special elements are preserved by the coboundary maps \(\delta'_v, \delta'_h\), so that \(X_{\text{mod},s}^{\bullet, \bullet}(\mathcal{C})\) is indeed a double complex (the modified special double complex of \((\mathcal{C}, \otimes)\)). Then, we can consider the associated total complex, which will be denoted by \(X_{\text{unit}}^{\bullet}(\mathcal{C})\), and called the unitary deformation complex of the Gray semigroup \((\mathcal{C}, \otimes)\). By definition

\[ X_{\text{unit}}^{q}(\mathcal{C}) = \bigoplus_{m + n = q} X_{\text{mod},s}^{m,n}(\mathcal{C}), \quad q \geq 1 \]

and the coboundary operator \(\delta_{\text{unit}} : X_{\text{unit}}^{q}(\mathcal{C}) \rightarrow X_{\text{unit}}^{q+1}(\mathcal{C})\) is

\[ \delta_{\text{unit}} = \bigoplus_{m + n = q} ((-1)^n \delta'_h + \delta'_v), \quad q \geq 1 \]

If \(H_{\text{unit}}^{n}(\mathcal{C})\) denote its cohomology groups, we have then the following final theorem, which says that this is the right cochain complex describing the generic unitary deformations:

**Theorem 9.4.** Given a \(K\)-linear Gray semigroup \((\mathcal{C}, \otimes)\), the equivalence classes of its first order unitary deformations are in bijection with the elements of \(H_{\text{unit}}^{2}(\mathcal{C})\).
Proof. Let’s consider 2-isomorphisms of the form

\[ \hat{\otimes}((f', g'), (f, g)) = \otimes((f', g'), (f, g)) + \overset{(1)}{\otimes}((f', g'), (f, g)) \]

\[ (\otimes_0)_\epsilon(X, Y) = 1_{id_X \otimes Y} \]

\[ \tilde{a}_\epsilon(f, g, h) = 1_{f \otimes g \otimes h} + \overset{(1)}{a}(f, g, h) \epsilon \]

\[ (\pi_\epsilon)_X Y Z T = 1_{id_X \otimes Y \otimes Z \otimes T} + (\overset{(1)}{\pi})(X, Y, Z, T) \epsilon \]

with \( \otimes((f', g'), (f, g)) : (f' \otimes g') \circ (f \otimes g) \Longrightarrow (f' \circ f) \otimes (g' \circ g) \), \( \overset{(1)}{a}(f, g, h) : f \otimes g \otimes h \Longrightarrow f \otimes g \otimes h \) and \( (\overset{(1)}{\pi})(X, Y, Z, T) : id_X \otimes Y \otimes Z \otimes T \Longrightarrow id_X \otimes Y \otimes Z \otimes T \). In particular, the \( (\overset{(1)}{\pi})(X, Y, Z, T) \) clearly define an element \( \overset{(1)}{\pi} \in \overset{3}{X}_{pent}(\mathcal{C}) \subseteq X_s^{0,2}(\mathcal{C}) \oplus X_s^{3}(\mathcal{C}) = X_{mod,s}^{0,2}(\mathcal{C}) \subseteq X_{unit}^{2}(\mathcal{C}) \)

Then, applying Proposition 5.11, it follows that the above 2-isomorphisms define a semigroupal structure on \( \overset{(1)}{X}(\mathcal{C}) \) (hence, a first order unitary deformation of \( \mathcal{C} \)) if and only if:

A\( \hat{\otimes}1 \): The \( \overset{(1)}{\otimes}((f', g'), (f, g)) \) define an element \( \overset{(1)}{\otimes} \in X^{1,1}(\mathcal{C}) = X_{mod}^{1,1}(\mathcal{C}) \).

A\( \hat{\otimes}2 \): \( \overset{(1)}{\otimes} \) is such that \( \delta_h(\overset{(1)}{\otimes}) = 0 \).

A\( \hat{\otimes}3 \): \( \overset{(1)}{\otimes} \) is special.

A\( \tilde{a}1 \): The \( \overset{(1)}{a}(f, g, h) \) define an element \( \overset{(1)}{a} \in X^{0,2}(\mathcal{C}) \subset X^{0,2}(\mathcal{C}) \oplus \overset{3}{X}_{pent}(\mathcal{C}) = X_{mod}^{0,2}(\mathcal{C}) \).

A\( \tilde{a}2 \): \( \overset{(1)}{a} \) and \( \overset{(1)}{\otimes} \) are such that \( \delta_h(\overset{(1)}{a}) + \delta_\epsilon(\overset{(1)}{\otimes}) = 0 \).

A\( \tilde{a}3 \): \( \overset{(1)}{a} \) is special.

A\( \pi1 \): \( \overset{(1)}{a} \) and \( \overset{(1)}{\pi} \) are such that \( \delta_\epsilon(\overset{(1)}{a}) + \varphi(\overset{(1)}{\pi}) = 0 \).

A\( \pi2 \): \( \overset{(1)}{\pi} \) is such that \( \delta_{pent}(\overset{(1)}{\pi}) = 0 \).

Now, \( \overset{(1)}{\pi} \in \overset{3}{X}_{pent}(\mathcal{C}) \) together with (A\( \hat{\otimes}1 \)), (A\( \hat{\otimes}3 \)) and (A\( \tilde{a}1 \)), (A\( \tilde{a}3 \)) say that

\[ ((\overset{(1)}{a} (\overset{(1)}{\pi} (\overset{(1)}{\pi} (f', g'), (f, g)) \in (X^{0,2}(\mathcal{C}) \oplus \overset{3}{X}_{pent}(\mathcal{C})) \oplus X^{1,1}(\mathcal{C}) = X_{unit}^{2}(\mathcal{C}) \]

On the other hand, we have

\[ \delta_{unit}((\overset{(1)}{a} (\overset{(1)}{\pi} (\overset{(1)}{\pi} (f', g'), (f, g)) = ((\delta_\epsilon(\overset{(1)}{a}) + \varphi(\overset{(1)}{\pi}), -\delta_{pent}(\overset{(1)}{\pi})), \delta_h(\overset{(1)}{a}) + \delta_\epsilon(\overset{(1)}{\otimes}) , -\delta_h(\overset{(1)}{\otimes})) \]

Hence, (A\( \pi1 \)), (A\( \pi2 \)), (A\( \tilde{a}2 \)) and (A\( \tilde{a}2 \)) together exactly say that \( ((\overset{(1)}{a} (\overset{(1)}{\pi} (\overset{(1)}{\pi} (f', g'), (f, g)) \) is a 2-cocycle.

Let’s consider \( ((\overset{(1)}{a}' (\overset{(1)}{\pi}' (\overset{(1)}{\pi}' f', g'), (f, g)) \) another 2-cocycle defining an equivalent first order unitary deformation of \( \mathcal{C} \). Then, by Proposition 5.14 applied to our situation, there exists 2-morphisms \( \overset{(1)}{\psi}(f, g) : f \otimes g \Longrightarrow f \circ g \) and \( \overset{(1)}{\pi} : id_X \otimes Y \otimes Z \Longrightarrow id_X \otimes Y \otimes Z \) (in particular, \( \overset{(1)}{\pi} \in \overset{2}{X}_{pent}(\mathcal{C}) \)) such that

E\( \psi1 \): The \( \overset{(1)}{\psi}(f, g) \) define an element \( \overset{(1)}{\psi} \in X^{0,1}(\mathcal{C}) \).

E\( \psi2 \): \( \overset{(1)}{\psi}' - \overset{(1)}{\psi} = -\delta_h(\overset{(1)}{\psi}) \).

E\( \psi3 \): \( \overset{(1)}{\psi} \) is special.

E\( \omega1 \): \( \overset{(1)}{\omega}' - \overset{(1)}{\omega} = \delta_\epsilon(\overset{(1)}{\psi}) + \varphi(\overset{(1)}{\omega}) \)

E\( \omega2 \): \( \overset{(1)}{\omega}' - \overset{(1)}{\omega} = -\delta_{pent}(\overset{(1)}{\omega}) \)

The first and third conditions together say that \( (\overset{(1)}{\psi}, \overset{(1)}{\omega}) \in X^{0,1}(\mathcal{C}) \oplus \overset{2}{X}_{pent}(\mathcal{C}) = X_{mod,s}^{0,1}(\mathcal{C}) = X_{unit}^{1}(\mathcal{C}) \).

On the other hand, the reader may check that

\[ \delta_{unit}(\overset{(1)}{\psi}, \overset{(1)}{\omega}) = ((\delta_\epsilon(\overset{(1)}{\psi}) + \varphi(\overset{(1)}{\omega}), -\delta_{pent}(\overset{(1)}{\omega})), -\delta_h(\overset{(1)}{\psi})) \]

so that the remaining conditions just say that

\[ (((\overset{(1)}{a}' (\overset{(1)}{\pi}' (\overset{(1)}{\pi}' f', g'), (f, g)) \), \overset{(1)}{\omega}' - ((\overset{(1)}{a} (\overset{(1)}{\pi} (\overset{(1)}{\pi} (f', g'), (f, g)) \), \overset{(1)}{\omega}) = \delta_{unit}(\overset{(1)}{\psi}, \overset{(1)}{\omega}) \]

Hence, both 2-cocycles are cohomologous, as required. \( \square \)
10. Concluding remarks

The present work is not intended to give the complete picture of the theory of infinitesimal deformations of a monoidal 2-category. Indeed, various points still remain for future work. Among them, let us mention the following:

1: There is the all-important question of the higher-order obstructions. This turned out to be the most difficult point in the cohomological deformation theory for monoidal categories first initiated by Crane and Yetter [6] and further developed by Yetter [39]. We guess that our cohomological description also fits nicely into the general picture established by Gerstenhaber for a good deformation theory [14]. But as already mentioned, this is left for a future work.

2: As stated at the beginning of this work, the ultimate goal should be to get a cohomological description of the infinitesimal deformations of a monoidal 2-category. Hence, it also deserves further work the question of how to take into account the additional unital structure in the whole theory. In the case of monoidal categories, Yetter [39] has shown that the deformations of this additional structure are already determined by those of the semigroupal structure. It seems possible that the same situation reproduces in our framework.

3: Finally, in the present work we have restricted our attention to those infinitesimal deformations of the semigroupal 2-category \((\mathcal{C}, \otimes, a, \pi)\) such that the bicategory structure of \(\mathcal{C}\) remains undeformed. But, as pointed out previously, this structure can also be deformed. As regards this point, notice that the elements \(\phi \in X^{2,0}(\mathcal{C})\) of our extended double complex are of the form
\[
\phi(h, g, f) : h \circ g \circ f \Rightarrow h \circ g \circ f
\]
(we are thinking of a Gray semigroup, so that parenthesis are not needed here). This suggests that the possible deformations of the bicategory structure of \(\mathcal{C}\) may be related to such elements \(\phi \in X^{2,0}(\mathcal{C})\). In this sense, the situation can once more resemble that encountered in the deformation theory of a bialgebra. Indeed, it can be shown (see [17], [31], [32]) that the right cochain complex describing the deformations of a bialgebra as a quasibialgebra (i.e., coassociative only up to conjugation) is precisely that associated to the double complex obtained after adding the bottom row of the full Gerstenhaber-Schack complex, which had to be deleted to study the deformations in the bialgebra setting. In our case, the weakening of the coassociativity condition should correspond to the weakening of the 2-category condition \(\alpha_{h,g,f} = 1_{h \circ g \circ f}\). It seems possible, then, that taking into account the deleted bottom row in our double complex is just the only step needed to consider these more general deformations of \(\mathcal{C}\).

Another important point not addressed in this paper, and which we are currently working on, is the question of examples. In particular, a simple example of a \(K\)-linear Gray semigroup where our theory can be applied is that introduced by Mackaay [25], denoted by \(\mathcal{N}(G, H, K^*)\), and associated to a pair of finite groups \(G, H\) (with \(H\) an abelian group); it includes as a special case the (semistrict version of) the monoidal 2-category of 2-vector spaces. More interesting examples, however, are expected to come from the 2-categories of representations of the Hopf categories associated to quantum groups, whose construction was sketched by Crane and Frenkel [5].

Finally, let us finish by mentioning the interest our work may have for homotopy theory. Indeed, since Grothendieck [13], it was suspected that homotopy \(n\)-types were somewhat equivalent to certain algebraic structures called weak \(n\)-groupoids, which should be a particular kind of weak \(n\)-categories characterized by the fact that all morphisms are invertible up to suitable equivalence. Recently, Tamsamani [34] realized this idea, giving a precise definition of a weak \(n\)-groupoid, for any non-negative integer \(n\), together with a suitable notion of equivalence, and showing that the equivalence classes of weak \(n\)-groupoids bijectively correspond to homotopy classes of \(n\)-anticonnected CW-complexes. Since weak 3-groupoids with one object should correspond to a special type of
semigroupal 2-categories, it naturally raises the question about the meaning our cohomology theory has in this topological setting. Such a relation between certain monoidal 2-categories and homotopy 3-types has recently been discussed by Mackaay \[25\], who conjectures that the classification of semi-weak monoidal 2-category structures on the above mentioned 2-category \(N(G, H, K^*)\) boils down to the classification up to homotopy equivalence of connected 3-anticonnected (>1-simple) CW-complexes \(X\) with \(\pi_1(X) = G, \pi_2(X) = H\) and \(\pi_3(X) = K^*\). Via Postnikov’s theory, this leads him to conjecture a bijection between the equivalence classes of semi-weak monoidal 2-category structures on \(N(G, H, K^*)\) and pairs of cohomology classes \(\alpha \in H^3(BG, H)\) and \(\beta \in H^4(W(\alpha), K^*)\). \(W(\alpha)\) denotes a certain CW-complex induced by \(\alpha\); see \[25\].

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References

1. J. Baez and J. Dolan, Higher dimensional algebra and topological quantum field theory, J. Math. Phys. 36 (1995), 6073–6105.
2. , Categorification, in: Higher Category Theory (M. Kapranov E. Getzler, ed.), American Mathematical Society, 1998, A.M.S. Contemporary Mathematics, vol. 230, pp. 1–36.
3. J. Barrett and B. Westbury, Invariants of piecewise-linear 3-manifolds, Trans. Amer. Math. Soc. 348 (1996), 3997–4022.
4. J. Bénabou, Introduction to bicategories, Lecture Notes in Mathematics, vol. 47, Springer-Verlag, 1967.
5. L. Crane and I. Frenkel, Four dimensional topological quantum field theory, hopf categories, and the canonical bases, J. Math. Phys. 35 (1994), 5136–5154.
6. L. Crane and D. Yetter, Deformations of (bij)tensor categories, Cahier de Topologie et Géometrie Différentielle Catégorique 39 (1998), 163–180.
7. , Examples of categorification, Cahiers Topologie Géometrie Différentielle Catégorique 39 (1998), 3–25.
8. B. Day and R. Street, Monoidal bicategories and hopf algebroids, Adv. Math. 129 (1997), 99–157.
9. V.G. Drinfeld, On quasitriangular quasi-hopf algebras and a group closely connected with \(gal(\mathbb{Q}/\mathbb{Q})\), Leningrad Math. J. 2 (1991), 829–860.
10. D. Epstein, Functors between tensored categories, Invent. Math. 1 (1966), 221–228.
11. S. Carter; D. Flath and M. Saito, The classical and quantum 6j-symbols, Mathematical Notes, vol. 43, Princeton University Press, 1995.
12. S. Chung; M. Fukuma and A. Shapere, The structure of topological field theories in three dimensions, Int. J. Mod. Phys. A 9 (1994), 1305–1360.
13. M. Gerstenhaber, The cohomology of an associative ring, AAnn. of Math. 78 (1963), 267–288.
14. , On the deformations of rings and algebras, Ann. of Math. 79 (1964), 59–103.
15. M. Gerstenhaber and S. Schack, Algebraic cohomology and deformation theory, Deformation Theory of Algebras and Structures and Applications (M. Gerstenhaber M. Hazewinkel, ed.), Kluwer Academic Publishers, 1988, pp. 11–264.
16. , Bialgebra cohomology, deformations and quantum groups, Proc. Natl. Acad. Sci. USA 87 (1990), 478–481.
17. , Algebras, bialgebras, quantum groups, and algebraic deformations, Contemporary Mathematics 134 (1992), 51–92.
18. J. Gray, Formal category theory: Adjontness for 2-categories, Lecture Notes in Mathematics, vol. 391, Springer-Verlag, 1974.
19. A. Grothendieck, Pursuing stacks, unpublished manuscript, distributed from UCNW, Bangor, UK, 1983.
20. M. Kapranov and V. Voevodsky, *2-categories and zamo|odchikov tetrahedra equations*, Proc. Sympos. Pure Math., vol. 56(2), American Mathematical Society, 1994, pp. 177–260.
21. C. Kassel, *Quantum groups*, Graduate Texts in Mathematics, vol. 155, Springer-Verlag, 1995.
22. G.M. Kelly, *Basic concepts of enriched category theory*, LMS Lecture Notes, Cambridge University Press, 1982.
23. G. Kuperberg, *Involuntary hopf algebras and 3-manifold invariants*, Internat. J. Math. 2 (1991), 41–66.
24. M. Mackaay, *Spherical 2-categories and 4-manifold invariants*, Adv. Math. 143 (1999), 288–348.
25. __________, *Finite groups, spherical 2-categories, and 4-manifold invariants*, Adv. Math. 153 (2000), 353–390.
26. S. MacLane, *Categories for the working mathematician*, second ed., Graduate Texts in Mathematics, vol. 5, Springer-Verlag, 1998.
27. S. MacLane and P. Pare, *Coherence in bicategories and indexed categories*, J. Pure Appl. Algebra 37 (1985), 59–80.
28. M. Neuchl, *Representation theory of hopf categories*, PhD. Thesis, University of Munich, 1997.
29. G. Ponzano and T. Regge, *Semiclassical limits of racah coefficients*, in: Spectroscopic and group theoretical methods in Physics (F. Bloch, ed.), North Holland, 1968.
30. R. Gordon; A. Powers and R. Street, *Coherence for tricategories*, Mem. Amer. Math. Soc. 117 (1995).
31. S. Shnider, *Deformation theory for bialgebras and quasibialgebras*, Contemp. Math. 134 (1992), 259–296.
32. S Shnider and S. Sternberg, *Quantum groups: from coalgebras to drinfeld algebras*, International Press, 1993.
33. J.D. Stasheff, *Homotopy associativity of h-spaces. 1*, Trans. Amer. Math. Soc. 108 (1963), 275–292.
34. Z. Tamsamani, *Sur des notions de n-categorie et n-groupoide non-strictes via des ensembles multi-simpliciaux, K-Theory* 16 (1999), 51–99.
35. V. Turaev and O. Viro, *State sum invariants of 3-manifolds and quantum 6j-symbols*, Topology 31 (1992), 865–902.
36. C.A. Weibel, *An introduction to homological algebra*, Cambridge University Press, 1994.
37. D. Yetter, *State-sum invariants of 3-manifolds associated to artinian semisimple tortile categories*, Topology Appli. 58 (1994), 47–80.
38. __________, *Braided deformations of monoidal categories and vassiliev invariants*, in: Higher Category Theory (M. Kapranov E. Getzler, ed.), American Mathematical Society, 1998, A.M.S. Contemporary Mathematics, vol. 230, pp. 117–134.
39. __________, *Functorial knot theory*, Series on Knots and Everything, vol. 26, World Scientific, 2001.