Toda Lattice Hierarchy and the Topological Description of the $c = 1$ String Theory

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Abstract

The Toda lattice hierarchy is discussed in connection with the topological description of the $c = 1$ string theory compactified at the self-dual radius. It is shown that when special constraints are imposed on the Toda hierarchy, it reproduces known results of the $c = 1$ string theory, in particular the $W_{1+\infty}$ relations among tachyon correlation functions. These constraints are the analogues of string equations in the topological minimal theories. We also point out that at $c = 1$ the Landau-Ginzburg superpotential becomes simply a $U(1)$ current operator.
Recently a topological description of the \( c = 1 \) string theory has been proposed in \[1, 2\] using the Landau-Ginzburg formulation. In these articles tachyon correlation functions of the \( c = 1 \) theory compactified at the self-dual radius are shown to be reproduced by the topological Landau-Ginzburg theory with a superpotential of the form \( W = 1/x + \cdots \). This suggests that the \( c = 1 \) theory may be identified as the \( k = -3 \) extension of the topological minimal theories with the superpotential \( W = x^{k+2} + \cdots \) as discussed in \[3, 4\].

In refs. \[1, 2\] positive momentum tachyons \( T_n(n = 1, 2, \cdots) \) with momenta \( n \) (in the unit of \( 1/\sqrt{2} \)) are regarded as primary fields and \( T_1 \) is identified as the puncture operator. On the other hand negative momentum tachyons \( \bar{T}_n(n = 1, 2, \cdots) \) with momenta \(-n\) are regarded as the descendants of the cosmological constant operator \( T_0 \). Some irregular behaviors of the \( N \)-point tachyon correlation functions \[5\] are interpreted as being due to the existence of contact terms among the gravitational descendants.

On the other hand, using the results of the S-matrix calculation of the \( c = 1 \) matrix model \[6, 7, 8\] it was shown \[9\] that the free energy of the \( c = 1 \) theory at the self-dual radius is a tau function of the Toda lattice hierarchy \[10\]. Thus the Toda lattice hierarchy may be considered as the underlying integrable structure in the \( c = 1 \) theory. In this article we would like to discuss in some detail how the Toda lattice hierarchy fits into the Landau-Ginzburg description of the \( c = 1 \) model.

We first point out that the integrable structure of the \( c = 1 \) string theory is described by the Toda lattice hierarchy when special constraints are imposed on the Toda system (see eqs.(24),(25) below). These constraints are the analogues of the string equations in the minimal theories which select special solutions of the KP hierarchy. Our conditions reproduce the \( W_{1+\infty} \) constraints discovered in \[3, 4\]. It turns out that the constrained Toda hierarchies are described in terms of a pair of Lax-like operators \( L, W \) which are the analogues of the \( P, Q \) operators \[11\] of the \( c < 1 \) minimal theories. In the Landau-Ginzburg description one may regard \( W \) as the superpotential and \( L(W) \) as the Landau-Ginzburg field. \( W \) has an expansion in terms of \( L \) and has the form of a \( U(1) \) current \( W = \partial \phi / \partial L \) where \( \phi(L) \) is a free scalar field \( \phi = \mu \log L + \sum t_n L^n - \sum \partial / n \partial t_n L^{-n} \). Here \( t_n \) is the coupling constant of the positive tachyon field \( T_n \) and \( \mu \) is the cosmological constant (\( \partial / \partial t_n \) acts on the partition function and is replaced by the 1-point function \( <T_n> \) at genus=0). Similarly the operator \( L \) is expanded in terms of \( W \) and has the form of a \( U(1) \) current. From these expressions \( W_{1+\infty} \) symmetry follows
immediately. We note that in the $c = 1$ theory $L$ and $W$ plays a completely symmetric role unlike the $P$ and $Q$ operators in the minimal theories. We also note that the free boson $\phi$ has a zero mode whose eigenvalue is the cosmological constant $\mu$. In the case of minimal theories the free boson was twisted and devoid of the zero mode \[12, 13\].

Let us first recall the structure of the Toda lattice hierarchy \[10\]. In this paper we essentially restrict ourselves to the genus-zero case and consider the dispersionless limit of the Toda hierarchy \[14\]. One introduces two pairs of Lax-like operators, $L, M$ and $\bar{L}, \bar{M}$,

\[
L = x + \sum_{i=0}^{\infty} u_i x^{-i},
\]

\[
M = \mu + \sum_{n=1}^{\infty} n t_n L^n + \sum_{n=1}^{\infty} v_n L^{-n},
\]

\[
\bar{L} = \sum_{i=1}^{\infty} \bar{u}_i x^i,
\]

\[
\bar{M} = \mu + \sum_{n=1}^{\infty} n \bar{t}_n \bar{L}^n + \sum_{n=1}^{\infty} \bar{v}_n \bar{L}^{-n}.
\]

They obey the Poisson bracket relations

\[
\{L, M\} = L,
\]

\[
\{\bar{L}, \bar{M}\} = \bar{L},
\]

where the Poisson bracket is defined by

\[
\{A(x, \mu), B(x, \mu)\} = x \left( \frac{\partial A}{\partial x} \frac{\partial B}{\partial \mu} - \frac{\partial A}{\partial \mu} \frac{\partial B}{\partial x} \right).
\]

Time evolutions of the operators are given by

\[
\frac{\partial}{\partial t_n} \{H_n, \} = \{H_n, \} , \quad H_n = (L^n)_+ , \quad n = 1, 2, \cdots,
\]

\[
\frac{\partial}{\partial \bar{t}_n} \{\bar{H}_n, \} = -\{\bar{H}_n, \} , \quad \bar{H}_n = (\bar{L}^{-n})_- , \quad n = 1, 2, \cdots,
\]

where $**$ stands for $L, M, \bar{L}$ and $\bar{M}$. $(**)_+((**)_-)$ means to take terms with non-negative (negative) powers of $x$ in the series $(**)$.

It is easy to check that the flows (8),(9) commute with each other.
The operator $M$ is reexpressed as

$$M = K(\mu + \sum_{n=1}^{\infty} nt_n x^n) K^{-1} ,$$

(10)

if one introduces the “dressing operator” $K$ defined by

$$L = K x K^{-1} .$$

(11)

In fact one can check

$$\{L, M\} = K \{x, \mu + \sum n t_n x^n\} K^{-1} = K x K^{-1} = L .$$

(12)

Furthermore, using $\partial K/\partial t_m = H_m K - K x^m, \partial K/\partial \bar{t}_m = -\bar{H}_m K$, one finds

$$\frac{\partial M}{\partial t_m} = \{H_m, M\} - K \{x^m, \mu + \sum n t_n x^n\} K^{-1} + K m x^m K^{-1}$$

(13)

$$= \{H_m, M\} , \quad m = 1, 2, \ldots$$

(14)

$$\frac{\partial M}{\partial \bar{t}_m} = -\{\bar{H}_m, M\} , \quad m = 1, 2, \ldots .$$

(15)

Defining the coefficient functions $\{v_n\}$ by the expansion

$$K \mu K^{-1} = \mu + \sum_{n=1}^{\infty} v_n L^{-n}$$

(16)

we recover the formula eq.(2). Eq.(4) may be analyzed in a similar manner. Eqs.(1)-(4) appear to be a pair of KP systems, however, the presence of the zero-mode $\mu$ in $M, \bar{M}$ and the extra factor of $x$ in the RHS of eq.(7) make an important distinction between Toda and KP systems.

It is known that the coefficient functions $\{v_n\} (\{\bar{v}_n\})$ are $\partial/\partial t_n (\partial/\partial \bar{t}_n)$ derivatives of the (logarithm of) tau function and hence are the positive (negative) tachyon one-point functions \[14\]

$$v_n = \frac{1}{Z} \frac{\partial Z}{\partial t_n} = <T_n> , \quad \bar{v}_n = \frac{1}{Z} \frac{\partial Z}{\partial \bar{t}_n} = <\bar{T}_n> .$$

(17)

On the other hand, from the Poisson bracket relations (5)(6) the coefficient functions $\{u_n\}, \{\bar{u}_n\}$ are expressed in terms of $\{v_n\}, \{\bar{v}_n\}$ and their
\[\mu\text{-derivatives. Then the flow equations (8)(9) lead to the following relations}
\]
\[
\frac{\partial v_n}{\partial t_1} = \left< T_1 T_n \right> = \oint L^n dx , \tag{18}
\]
\[
\frac{\partial \bar{v}_n}{\partial t_1} = \left< T_1 \bar{T}_n \right> = \oint \bar{L}^{-n} dx . \tag{19}
\]

Also
\[
\frac{\partial v_n}{\partial \mu} = \left< T_0 T_n \right> = \oint x^{-1} L^n dx , \tag{20}
\]
\[
\frac{\partial \bar{v}_n}{\partial \mu} = \left< T_0 \bar{T}_n \right> = \oint x^{-1} \bar{L}^{-n} dx . \tag{21}
\]

hold. We have obtained formulas for the \(\bar{t}_1\)-derivatives
\[
\frac{\partial v_n}{\partial \bar{t}_1} = \left< \bar{T}_1 T_n \right> = \frac{1}{\bar{u}_1} \oint x^{-2} L^n dx , \tag{22}
\]
\[
\frac{\partial \bar{v}_n}{\partial \bar{t}_1} = \left< \bar{T}_1 \bar{T}_n \right> = \frac{1}{\bar{u}_1} \oint x^{-2} \bar{L}^{-n} dx . \tag{23}
\]

When we switch off the couplings \(\{\bar{t}_n\}\{\{t_n\}\},\) functions \(\{u_n\}, \{v_n\}\) \(\{\bar{u}_n\}, \{\bar{v}_n\}\) vanish. Thus we may regard \(L\) as being the deformation of the variable \(x\) in the presence of nonzero descendant couplings. We identify \(L\) as the Landau-Ginzburg field in the large phase space. This is reminiscent of the work [15] where the operator \(Q\) is regarded as the Landau-Ginzburg field in the large phase space of minimal theories.

Let us next introduce the following constraints on the Lax-like operators
\[
ML^{-1} = L^{-1} , \tag{24}
\]
\[
\bar{M} \bar{L} = L . \tag{25}
\]

It turns out that these conditions are compatible with the Toda system (1)-(9) and reproduce known results of the \(c = 1\) string theory compactified at the \(SU(2)\) radius. We note that (24),(25) eliminate two out of four operators \(L, M, \bar{L}, \bar{M}\). Let us define
\[
W \equiv ML^{-1} = \mu L^{-1} + \sum_{n=1} nt_n L^{n-1} + \sum_{n=1} v_n L^{-n-1} \tag{26}
\]
which obeys

$$\{L, W\} = 1.$$  \hfill (27)

We now describe the Toda hierarchy under the constraints (24), (25) by a pair of operators $L$ and $W$. Let us first note that since $\bar{L}$ has a Taylor expansion in $x$ starting from $x^1$ (eq.(3), eq.(24) implies the following expansion of $W$

$$W = a_{-1}x^{-1} + a_0 + \sum_{n=1} a_n x^n.$$  \hfill (28)

Thus the operator $W$ does not contain terms $x^m$ with powers $m \leq -2$. On the other hand, $L$ has the expansion eq.(1) and hence $L^{-n-1}(n = 1, 2 \cdots)$ contain only terms $x^m$ with $m \leq -2$. Therefore the last sum in eq.(26) actually does not contribute and (26) may be reexpressed as

$$W = a_{-1}x^{-1} + \sum n t_n (L^{n-1})_+ ,$$  \hfill (29)

$$a_{-1} = \mu + \sum n t_n (L^{n-1})_{-1} .$$  \hfill (30)

Here ($\ast\ast)_n$ denotes the coefficient of the $x^n$ term in $\ast\ast$.

On the other hand, the constraint (25) leads to an expansion of $L$ in terms of $W$

$$L = \mu W^{-1} + \sum n \bar{t}_n W^{n-1} + \sum \bar{v}_n W^{-n-1}.$$  \hfill (31)

Again by comparing the $x$-expansion of both sides of eq.(31) it may be reexpressed as

$$L = u_{-1}x + u_0 + \sum n \bar{t}_n (W^{n-1})_+ ,$$  \hfill (32)

$$u_{-1} = \mu(a_{-1})^{-1} + \sum n \bar{t}_n (W^{n-1})_{-1} ,$$  \hfill (33)

$$u_0 = \sum n \bar{t}_n (W^{n-1})_0.$$  \hfill (34)

We shall show below (see eq.(41)) that $u_{-1} = 1$ and hence (32) is in fact in accord with the expansion (1). One may also prove that (34) holds.

On the other hand, the $W_{1+\infty}$ constraints. Using (31) we can express the coefficient function $\bar{v}_n$ by a residue integral

$$\bar{v}_n = \oint L W^n dW = \frac{1}{n+1} \oint W^{n+1} dL , \quad n = 1, 2, \cdots .$$  \hfill (35)
(The direction of the integration contour flips under the change of variable and this cancels the \(-\) sign coming from the partial integration). Similarly from (26) we have

\[ v_n = \frac{1}{n+1} \oint L_n^{n+1} dW , \quad n = 1, 2, \cdots . \] (36)

If we substitute (26) into the RHS of (35), we obtain a relation representing \( \bar{v}_n \) in terms of \( \{v_m\}, \{t_m\} \). These are the (zero-genus version of) \( W_{1+\infty} \) constraints [8]. Eq.(35) was postulated in [1] based on an analogy with the representation of 1-point functions in the minimal theories by means of period integrals [16]. Eq.(36) gives the \( \bar{W}_{1+\infty} \) constraints.

Now we prove \( u_{-1} = 1 \). First by putting \( n = 1 \) in (35),(36) we have

\[ \bar{v}_1 = \mu t_1 + \sum_{n=2} \bar{t}_n v_{n-1} , \] (37)

\[ v_1 = \mu \bar{t}_1 + \sum_{n=2} \bar{t}_n \bar{v}_{n-1} . \] (38)

It turns out that these formulas correspond to the puncture equation in the case of minimal models. Taking the \( t_1 \)-derivative of (37) and using (18), one finds

\[ \frac{\partial \bar{v}_1}{\partial t_1} = \mu + \sum_{n=2} \bar{t}_n \frac{\partial v_{n-1}}{\partial t_1} = \mu + \sum_{n=2} \bar{t}_n (L^{n-1})_{-1} \]

\[ = a_{-1} . \] (39)

On the other hand eq.(23) reads (note \( a_{-1} = \bar{u}_1^{-1} \))

\[ (W^{n-1})_{+1} = \frac{1}{a_{-1}} \frac{\partial \bar{v}_{n-1}}{\partial \bar{t}_1} . \] (40)

Therefore

\[ u_{-1} = \mu (a_{-1})^{-1} + \sum n \bar{t}_n (W^{n-1})_{+1} \]

\[ = \mu (a_{-1})^{-1} + (a_{-1})^{-1} \frac{\partial}{\partial \bar{t}_1} \sum n \bar{t}_n \bar{v}_{n-1} \]

\[ = (a_{-1})^{-1} (\mu + \frac{\partial}{\partial \bar{t}_1} (v_1 - \mu \bar{t}_1)) = (\frac{\partial \bar{v}_1}{\partial \bar{t}_1})^{-1} \frac{\partial v_1}{\partial \bar{t}_1} = 1 \] (41)
where we used (38) in going from the 2nd to the 3rd line. Thus in fact 
\( u_{-1} = 1 \) and the constraints (24)(25) are fully consistent with the Toda lattice hierarchy.

Let us now derive the analogue of the puncture equation (genus-zero version of the string equation). We first note that formulas (29)(32) may be expressed in an appealing manner as a sum of the Hamiltonians (8)(9)

\[
W = \bar{H}_1 + \sum nt_n H_{n-1} , \\
L = H_1 + \sum n\bar{t}_n \bar{H}_{n-1} .
\]

If we evaluate \( \{L, W\} = 1 \) using these formulas, we obtain puncture equations for our system

\[
\frac{\partial W}{\partial t_1} - \sum nt_n \frac{\partial W}{\partial t_{n-1}} = 1 , \\
\frac{\partial L}{\partial \bar{t}_1} - \sum nt_n \frac{\partial L}{\partial \bar{t}_{n-1}} = 1 .
\]

It is possible to show that these formulas are equivalent to the \( W^{(2)} \)-constraints eqs.(37)(38). We may, for instance, multiply (45) by \( mL^{m-1}x^{-1} \) and take the residue in \( x \) on both sides of the equation. Then by using the relation (20) we obtain

\[
\frac{\partial}{\partial \mu} \frac{\partial v_m}{\partial t_1} = \frac{\partial}{\partial \mu} \left( \sum nt_n \frac{\partial v_m}{\partial t_{n-1}} + mv_{m-1} \right)
\]

(we formally set \( v_0 = \mu \)). Integrating over \( \mu \) (we assume the absence of the integration constant) we obtain

\[
\frac{\partial v_m}{\partial t_1} = \sum nt_n \frac{\partial v_m}{\partial t_{n-1}} + mv_{m-1} = \frac{\partial}{\partial t_m} \left( \sum nt_nv_{n-1} \right) .
\]

It is easy to see that this equation agrees with the \( \partial/\partial t_m \)-derivative of (37). Hence \( W^{(2)} \)-constraint corresponds to the puncture \( (L_{-1}) \) equation in the minimal models. In fact the identification \( T_1 = \)puncture operator was made previously in \cite{17} via the analysis of this constraint.

Let us next discuss some formal aspects of the \( W_{1+\infty} \) symmetry. We first note that since the coefficient functions \( \{v_n\}, \{\bar{v}_n\} \) are 1-point functions, we
may replace them by the derivatives \( \{ \partial / \partial t_n \} , \{ \partial / \partial \bar{t}_n \} \) acting on the partition function. Thus we can rewrite the superpotential as the \( U(1) \) current

\[
W = \partial \phi , \quad \phi = \mu \log L + \sum t_n L^n - \sum \frac{1}{n} \frac{\partial}{\partial t_n} L^{-n} .
\]  

(48)

The \( W_{1+\infty} \) relation (35) may then be rewritten as

\[
\frac{1}{Z} \frac{\partial Z}{\partial t_n} = \frac{1}{n + 1} \frac{1}{Z} \oint dL (\partial \phi)^{n+1} Z .
\]

(49)

(49) reduces to (35) at genus \( g = 0 \) since genus-zero contributions come only when the derivative \( \{ \partial / \partial t_n \} \) hits the partition function (the genus-dependence of our system will be recovered if we rescale the derivative term in (48) as \(-\lambda^2 \sum 1/n \partial / \partial t_n L^{-n} \) and expand the free energy as \( \log Z = \sum_g \lambda^{2g-2} F_g \) where \( \lambda \) is the genus-expansion parameter and \( F_g \) is the genus-\( g \) free-energy). The \( n \)-th power of the \( U(1) \) current \( (\partial \phi)^n \) describes the spin-\( n \) field \( W(n) \) of the \( W_{1+\infty} \) algebra. The contour integral \( \oint dL (\partial \phi)^{n+1} \) extracts the mode \( W_{n+1} \). In Eq.(49) the \( W_{1+\infty} \) symmetry of the theory is seen explicitly. Thus it seems natural to regard the superpotential \( W \) as the \( U(1) \) current operator in the \( c = 1 \) theory. We clearly see how the Landau-Ginzburg description emerges at the \( g = 0 \) limit where the \( U(1) \) operator is replaced by its expectation value.

One of the basic differences between the Toda and KP systems is in the symmetry between their Lax-like operators. In the case of the KP hierarchy the \( Q \) operator can be expanded in terms of \( P \), however, there does not seem to be an inverse expansion of \( P \) in terms of \( Q \). On the other hand, in the Toda theory there exists a complete symmetry between \( L \) and \( W \) operators and they are mutually expanded into each other. This symmetry controls the Toda system in an efficient manner and reduces it almost to that of free fields in the target space. In this respect, despite the presence of an infinity of primary fields, the \( c = 1 \) theory is somewhat simpler than the minimal theories.

In this paper we have restricted ourselves to the \( c = 1 \) string theory at the self-dual radius. It will be extremely interesting to study the theory in different backgrounds, for instance, the 2-dimensional black hole (for a recent attempt, see \[ IR \]). It is a challenging problem to see if other reductions of the Toda theory are possible than the case treated in this paper.
During the completion of this manuscript a new preprint by K.Takasaki has appeared. This paper discusses subjects closely related to ours.

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