NONCOMMUTATIVE GEOMETRY, CONFORMAL GEOMETRY, AND THE LOCAL EQUIVARIANT INDEX THEOREM.

RAPHAËL PONGE AND HANG WANG

Abstract. We prove a local index formula in conformal geometry by computing the Connes-Chern character for the conformal Dirac (twisted) spectral triple recently constructed by Connes-Moscovici. Following an observation of Moscovici, the computation reduces to the computation of the CM cocycle of an equivariant Dirac (ordinary) spectral triple. This computation is obtained as a straightforward consequence of a new proof of the local equivariant index theorem of Patodi, Donnelly-Patodi and Gilkey. This proof is obtained by combining Getzler’s rescaling with an equivariant version of Greiner’s approach to the heat kernel asymptotic. It is believed that this approach should hold in various other geometric settings. On the way we give a geometric description of the index map of a twisted spectral in terms of (twisted) connections on finitely generated projective modules.

1. Introduction

Motivated by type-III geometric situations in which an arbitrary group of diffeomorphisms acts on a manifold, Connes-Moscovici [CM2] introduced the notion of a twisted spectral triple. This is a modification of the definition of an ordinary spectral triple \((\mathcal{A}, \mathcal{H}, D)\), where the boundedness condition on commutators \([D, a], a \in \mathcal{A}\), is replaced by the boundedness of twisted commutators defined in terms of an automorphism \(\sigma\) of the algebra \(\mathcal{A}\). Examples include conformal deformation of spectral triples ([CM2]), Dolbeaut spectral triple over the noncommutative torus ([CT], see also [FK]), and the conformal Dirac spectral triple \((C^{\infty}(M) \rtimes G, L^2_\gamma(M, \mathcal{S}), D_\gamma)_\sigma\), where \(D_\gamma\) is the Dirac operator acting on spinors and \(G\) is a group of conformal diffeomorphisms ([CM2]). (We refer to Section 3 for a review of these definitions and examples.)

As shown by Connes-Moscovici [CM2], the datum of a twisted spectral \((\mathcal{A}, \mathcal{H}, D)_\sigma\) gives rise to an index map \(\text{ind}_{D, \sigma} : K_0(\mathcal{A}) \to \mathbb{Z}\), where \(K_0(\mathcal{A})\) is the K-theory of \(\mathcal{A}\). Furthermore, this is computed by pairing \(K_0(\mathcal{A})\) with a Connes-Chern character that lies in ordinary cyclic cohomology (see [CM2]). The question that naturally arises is whether the framework for the local index formula in noncommutative geometry of Connes-Moscovici [CM1] can be extended to the setting of twisted spectral triples, i.e., whether the Connes-Chern character can be represented by a version of the CM cocycle for twisted spectral triples.

Moscovici [Mo2] devised an Ansatz for a local index formula for twisted spectral triples and showed the Ansatz is verified in the case of an ordinary spectral triple twisted by scaling automorphisms. An example of such a twisted spectral triple is given by a conformal Dirac spectral triple \((C^{\infty}(S^n) \rtimes G, L^2_\gamma(M, \mathcal{S}), D_\gamma)_\sigma\) associated to the round sphere \(S^n\) and a group \(G\) of similarities (i.e., a parabolic subgroup of \(\text{PO}(n + 1, 1)\) fixing a point). Whether Moscovici’s Ansatz holds for other twisted spectral triples remains an open question to date.

By the Ferrand-Obata theorem, the group of conformal diffeomorphisms of a compact manifold \(M^n\) is compact, unless \(M^n\) is conformally equivalent to the round sphere \(S^n\). Using this fact Moscovici [Mo2, Remark 3.8] observed that, in the non-conformally-flat case, the conformal Dirac spectral triple \((C^{\infty}(M) \rtimes G, L^2_\gamma(M, \mathcal{S}), D_\gamma)_\sigma\) is equivalent to the conformal deformation of an (ordinary) equivariant Dirac spectral triple \((C^{\infty}(M) \rtimes G, L^2_\gamma(M, \mathcal{S}), D_\gamma)\), where \(\gamma\) is a \(G\)-invariant metric. As a result, the Connes-Chern character of \((C^{\infty}(M) \rtimes G, L^2_\gamma(M, \mathcal{S}), D_\gamma)\) is represented by the CM cocycle of that equivariant Dirac spectral triple.

R.P. was partially supported by Research Resettlement Fund of Seoul National University (South Korea).
The aim of this paper is threefold. First, after a review of the main definition and properties of twisted spectral triples, we give a geometric definition of the index map of a twisted spectral triple in terms of twisted connections on finitely generated projective modules (Proposition 4.3). This description parallels the description for ordinary spectral triples given in [Mo1].

Second, in the non-conformally-flat case, we compute the Connes-Chern character of the conformal Dirac spectral triple \((C^\infty(M) \rtimes G, L^2(M,g), D_g)\) in terms of universal polynomials of the curvatures of \(M\) (see Theorem 6.6 for the precise statement). This is done by using Moscovici’s observation and computing the CM cocycle of the equivariant Dirac spectral triple. In particular, at level of Hochschild cohomology, the Connes-Chern character agrees with Connes’s fundamental class \([M/G]\).

The third aim of this paper is to give a new proof of the local equivariant index theorem which, as an immediate byproduct, yields an elementary calculation of the CM cocycle of an equivariant Dirac spectral triple. Recall that, given a compact spin Riemannian manifold \((M^n, g)\) (\(n\) even) and a smooth isometry \(\phi\) preserving the spin structure and acting on \(L^2\)-spinors by the unitary operator \(U_\phi\), the local equivariant index theorem establishes that, for all \(f \in C^\infty(M)\),

\[
\text{Str} \left[ fe^{-t D_g U_\phi} \right] = \int_{M^\phi} f \omega + O(t) \quad \text{as } t \to 0^+,
\]

where \(M^\phi\) is the fixed-point set of \(\phi\) and the form \(\omega\) is a universal polynomial in \(\phi'\) and the curvatures of \(M^\phi\) and its normal bundle (see Section 9 for the precise statement). This result implies the equivariant index theorem of Atiyah-Segal-Singer \([AS, ASi2]\), which is a fundamental generalization of Lefschetz’s fixed-point formula to elliptic complexes and isometries with non-isolated fixed-points.

The original proofs of the local equivariant index theorem by Patodi, Donnelly-Patodi and Gilkey involved Riemannian invariant theory. We refer to [Bi, BV, LYZ, LM] for more analytical treatments. Our approach is an equivariant version of the approach of [Po1] to the proof of the local index theorem and the computation of the CM cocycle of a (non-equivariant) Dirac spectral triple. Namely, it combines the rescaling of Getzler [Ge2] with an equivariant version of the approach to the heat kernel asymptotic of Greiner [Gr].

In order to compute the CM cocycle of an equivariant Dirac spectral triple we really need a differentiable version of the local equivariant index theorem, which is, a version of the asymptotic (1.1), where the function \(f\) is replaced by a differential operator. As it is based on the representation of the heat kernel as the kernel of a Volterra \(\Psi\)DO, Greiner’s approach to the heat kernel asymptotic immediately produces differentiable heat kernel asymptotics. Furthermore, these asymptotics are straightforward consequences of Taylor’s formula and elementary properties of Volterra \(\Psi\)DOs.

There is no difficulty to extend Greiner’s approach to the equivariant setting by working in tubular coordinates (see Section 7). In the equivariant setting too, differentiable asymptotics are produced as consequences of Taylor’s formula and elementary properties of Volterra \(\Psi\)DOs. In particular, no use is made of the stationary phase method.

Once the differentiable equivariant heat kernel asymptotics are established, we may apply the approach of [Po1]. As observed [Po1], the rescaling of Getzler [Ge2] naturally defines a filtration on Volterra \(\Psi\)DOs. Thereby this defines a new notion of order for these operators, which is called Getzler order. The convergence of the supertrace stated in (1.1) then follows from elementary considerations on Getzler orders of Volterra \(\Psi\)DOs (see Lemma 9.12).

Notice that in the proof of the local equivariant index theorem there is a tension between the tubular coordinates in which the equivariant heat kernel asymptotics are derived and the normal coordinates in which the Getzler’s rescaling is performed. In our approach, this tension is taken care of by means of an elementary application of the change of variable formula for symbols of pseudodifferential operators.
The arguments of our approach are fairly general and produce a differentiable version of the local equivariant index theorem. As a result, a straightforward elaboration of those arguments enables us to compute the CM cocycle of an equivariant Dirac spectral triple (see Section 9). It is believed that this approach to the local equivariant index theorem and the computation of the CM cocycle of an equivariant Dirac spectral triple could be used in various geometric situations. Therefore, this should be a useful tool to reformulate the equivariant index theorem and the Lefschetz fixed-point formula in various new geometric settings.

The paper is organized as follows. In Section 2, we review the local index formula in non-commutative geometry. In Section 3, we review some important examples of twisted spectral triple, including the conformal Dirac spectral triple. In Section 4, we give a geometric description of the index map of a twisted spectral triple. In Section 5, we review the construction of the Connes-Chern character of a twisted spectral triple. In Section 6, we compute the Connes-Chern character of the conformal Dirac spectral triple. In Section 7, we review the Volterra calculus and the pseudodifferential representation of the heat kernel. In Section 8, we derive equivariant heat kernel asymptotics. In Section 9, we prove the local equivariant index theorem and complete our computation of the Connes-Chern character of the conformal Dirac spectral triple.

2. Spectral Triples and Connes-Chern Character

In this section, we recall the framework for the Connes-Chern character and CM cocycle of an ordinary spectral triple.

**Definition 2.1.** A spectral triple \((A, \mathcal{H}, D)\) consists of the following data:

1. A \(\mathbb{Z}_2\)-graded Hilbert space \(\mathcal{H} = \mathcal{H}^+ \oplus \mathcal{H}^-\).
2. An involutive unital algebra \(A\) represented by bounded operators on \(\mathcal{H}\) preserving its \(\mathbb{Z}_2\)-grading.
3. A self-adjoint unbounded operator \(D\) on \(\mathcal{H}\) such that
   1. \(D\) maps \(\text{dom } D \cap \mathcal{H}^\pm\) to \(\mathcal{H}^\pm\).
   2. The resolvent \((D + i)^{-1}\) is a compact operator.
   3. \(a(\text{dom } D) \subseteq \text{dom } D\) and \([D, a]\) is bounded for all \(a \in A\).

In the sequel, we shall further assume that the algebra \(A\) is closed under holomorphic functional calculus. The paradigm of a spectral triple is given by a Dirac spectral triple,

\[
\left( \mathcal{C}^\infty(M), L^2(M, \mathcal{S}), \slashed{D}_g \right),
\]

where \((M^n, g)\) is a compact spin Riemannian manifold \((n \text{ even})\) and \(\slashed{D}_g\) is its Dirac operator acting on the spinor bundle \(\mathcal{S}\).

The datum of a spectral triple \((A, \mathcal{H}, D)\) defines an additive index map,

\[
\text{ind}_D : K_0(A) \to \mathbb{Z},
\]

\[
\text{ind}_{D}[e] := \text{ind}_D e, \quad \forall e \in M_2(A), \quad e^2 = e^* = e,
\]

where \(D_e\) is the operator \(e(D \otimes 1) : e(\text{dom } D)^q \to e\mathcal{H}^q\). This is a selfadjoint Fredholm operator. With respect to the orthogonal splitting \(e\mathcal{H}^q = e(\mathcal{H}^+)^q \oplus e(\mathcal{H}^-)^q\) it takes the form,

\[
D_e = \begin{pmatrix}
0 & D_e^- \\
D_e^+ & 0
\end{pmatrix}, \quad D_e^\pm : e(\text{dom } D \cap \mathcal{H}^\pm)^q \to e(\mathcal{H}^\pm)^q.
\]

We then define the index \(\text{ind } D_e\) to be the usual Fredholm index \(\text{ind } D_e^+\). This is an invariant of the \(K\)-theory class of \(e\). Moreover, the selfadjointness of \(D_e\) implies that \((D_e^+)^* = D_e^-\). Thus,

\[
\text{ind } D_e = \text{ind } D_e^+ = - \text{ind } D_e^- = \dim \ker D_e^+ - \dim \ker D_e^-.
\]

Notice also that the index map can be equivalently described in terms of connections on finitely projective modules (see [Mo1]).

The index map is computed by pairing the \(K\)-theory of \(A\) with a cyclic cohomology class \(\text{Ch}(A, D)\), called the Connes-Chern character \((\text{Co2, Co3})\). (We refer to \[Co3\] for background on cyclic cohomology and its pairing with \(K\)-theory.)
In order to define the Connes-Chern character of the spectral triple \((\mathcal{A}, \mathcal{H}, D)\), we need to further assume that it is \(p^+\)-summable for some \(p \geq 1\), i.e.,
\[
\mu_k(D^{-1}) = O(k^{-\frac{p}{2}}) \quad \text{as } k \to \infty,
\]
where \(\mu_k(D^{-1})\) is the \((k+1)\)-th eigenvalue of \(|D|^{-1}\) counted with multiplicity.

Given any integer \(k > \frac{p-1}{2}\), the Connes-Chern character \(\text{Ch}(\mathcal{A}, D)\) is represented by the cyclic cocycle,
\[
\tau_k^D(a^0, \ldots, a^{2k}) := \frac{1}{2 (2k)!} \text{Str} \left\{ D^{-1}[D, a^0] \cdots D^{-1}[D, a^{2k}] \right\}, \quad a^j \in \mathcal{A},
\]
where \(\text{Str}\) is the supertrace on \(\mathcal{L}^1(\mathcal{H})\), i.e., \(\text{Str}[T] := \text{Tr}[\gamma T]\), where \(\gamma := \text{id}_{\mathcal{H}^+} \text{id}_{\mathcal{H}^-}\) is the \(\mathbb{Z}_2\)-grading operator. Moreover, the class of \(\tau_k^D\) in the periodic cyclic cohomology \(\text{HP}^{ev}(\mathcal{A})\) is independent of the value of \(k > \frac{p-1}{2}\).

The definition (2.3) of the cocycle \(\tau_{2k}\) involves the usual (super)trace, which is not a local functional. As a result this cocycle is difficult to compute in practice (see, e.g., [BF]). To remedy this, Connes-Moscovici [CM1] constructed a (periodic) representative of the Connes-Chern character, the so-called CM cocycle, whose components are given by formulas that are local in the sense that they involve an analogue of the noncommutative residue trace of Guillemin [Gu] and Wodzicki [Wo]. We shall now review the main facts of the construction of the CM cocycle in [CM1].

Consider the unbounded derivation of \(\mathcal{L}(\mathcal{H})\) defined by
\[
\delta(T) := \|[D], T\|, \quad \text{dom } \delta := \{ T \in \mathcal{L}(\mathcal{H}); \|[D], T\| \in \mathcal{L}(\mathcal{H}) \}.
\]
The spectral triple \((\mathcal{A}, \mathcal{H}, D)\) is said to be regular when \(a\) and \([D, a]\) are contained in \(\bigcup_{j \geq 0} \text{dom } \delta^j\) for all \(a \in \mathcal{A}\). Assuming \((\mathcal{A}, \mathcal{H}, D)\) to be regular, we denote by \(\mathcal{B}\) the subalgebra of \(\mathcal{L}(\mathcal{H})\) generated by the grading operator \(\gamma\) and the operators \(\delta^j(a)\) and \(\delta^j([D, a])\) where \(a \in \mathcal{A}, j \geq 0\).

In addition, we say that \((\mathcal{A}, \mathcal{H}, D)\) has a simple and discrete dimension spectrum when there exists a discrete subset \(\Sigma \subset \mathbb{C}\) such that, for every \(b \in \mathcal{B}\), the zeta function \(\zeta_b(z) := \text{Tr}[b[D]^{-z}]\) has a meromorphic extension to \(\mathbb{C}\) in such way to be holomorphic outside \(\Sigma\) and to have at worst simple pole singularities on \(\Sigma\).

From now on we assume that \((\mathcal{A}, \mathcal{H}, D)\) is regular and has a simple and discrete dimension spectrum. This enables us to construct a class of pseudo differential operators for the spectral triple \((\mathcal{A}, \mathcal{H}, D)\) as follows. Let \(\Psi^+_B(A), q \in \mathbb{C}\), be the space of unbounded operators \(P\) on \(\mathcal{H}\) such that the domain of \(P\) contains \(\cap_{s \in \mathbb{R}} \text{dom } |D|^s\) and \(P\) has an asymptotic expansion of the form,
\[
P \simeq \sum_{j \geq 0} b_j D^{q-j}, \quad b_j \in \mathcal{B},
\]
in the sense that, for all \(N \in \mathbb{N}\) and \(s \in \mathbb{R}\),
\[
|D|^{s-q+N} \left( P - \sum_{j < N} b_j D^{q-j} \right) |D|^{-s} \in \mathcal{L}(\mathcal{H}).
\]

As shown in [CM1] it holds that \(\Psi^+_B(A) \Psi^+_B(A) \subset \Psi^+_B(A)\), so that \(\Psi^+_B(A) = \cup_{q \in \mathbb{C}} \Psi^+_B(A)\) is an algebra. In addition, for all \(P \in \Psi^+_B(A), q \in \mathbb{C}\), the function \(z \mapsto \text{Tr}[P|D|^{-z}]\) has a meromorphic extension to the entire complex plane with at worst simple pole singularities contained in \(q + \Sigma\). We then set
\[
\int P := \text{Res}_{z=0} \text{Tr} \left[ P|D|^{-2z} \right] \quad \forall P \in \Psi^+_B(A).
\]
As it turns out, this defines a linear trace on the algebra \(\Psi^+_B(A)\) (see [CM1]). Notice that the residual trace \(\int\) is local in the sense that it vanishes on all operators \(P \in \Psi^+_B(A)\) with \(\Re q < -p\), since those operators are trace-class.

For instance, a Dirac spectral triple \((C^\infty(M), L^2(M, \mathcal{S}), \Phi_g)\) is \(n^+\)-summable and regular (with \(n = \dim M\)), and it has a discrete dimension spectrum contained in \(\{ k \in \mathbb{N}; k \leq n \}\). Moreover,
each space $\Psi_q, q \in \mathbb{C}$, is contained in the space of classical $\Psi$DOs of order $q$ and the residual trace $\hat{f}$ agrees with the noncommutative residue trace of Guillemin [Gu] and Wodzicki [Wo].

In the sequel, for $P \in \Psi_D(A)$ we denote by $P^j$, $j \geq 0$, the $j$-th iterated commutator of $P$ with $D^2$, that is,

$$P^j = [D^2, [D^2, \cdots [D^2, P] \cdots]].$$

**Theorem 2.2** ([CM1]). Suppose that the spectral triple $(A, \mathcal{H}, D)$ is $p^+$-summable, regular and has a simple and discrete dimension spectrum.

1. The following formulas define an even periodic cyclic cocycle $\varphi^{CM} = (\varphi_{2k})_{k \geq 0}$ on the algebra $A$:

$$\varphi_0(a^0) = \text{Res}_{z=0} \Gamma(z) \text{Str} \left[a^0(|D|^{-2} + \Pi_0)\right] \quad (k=0),$$

$$\varphi_{2k}(a^0, \cdots, a^{2k}) = \sum_{\alpha} c_{k,\alpha} \int \gamma a_0 [D, a^1]^{|\alpha_1|} \cdots [D, a^{2k}]^{|\alpha_{2k}|} |D|^{-2(\alpha|k)} \quad (k \geq 1),$$

where $\Pi_0$ is the orthogonal projection onto $\ker D$ and

$$c_{k,\alpha} := \frac{(-1)^{|\alpha|} \Gamma(|\alpha| + k)}{\alpha_1(\alpha_1 + 1) \cdots (\alpha_1 + \cdots + \alpha_{2k} + 2k)}.$$

2. The CM cocycle $\varphi^{CM}$ represents the Connes-Chern character $\text{Ch}(A, D)$ in periodic cyclic cohomology, and hence

$$\text{ind}_D(e) = \langle \varphi^{CM}, e \rangle \quad \forall e \in K_0(A).$$

**Example 2.3.** For a Dirac spectral triple $\left(C^\infty(M), L^2_g(M, S), \mathcal{D}_g\right)$, the CM cocycle $\varphi^{CM} = (\varphi_{2k})_{k \geq 0}$ is given by

$$\varphi_{2k}(f^0, \cdots, f^{2k}) = \frac{(2i\pi)^{-k}}{(2k)!} \int_M \cdots \int f^0 df^1 \wedge \cdots \wedge df^{2k} \wedge \hat{A}(R^{TM}), \quad f^j \in C^\infty(M),$$

where $\hat{A}(R^{TM}) := \text{det}^{\frac{1}{2}} \left(\frac{R^{TM}}{\text{snh}(R^{TM}/2)}\right)$ is the $\hat{A}$-form of the Riemann curvature of $(M, g)$ (see [CM] Remark II.1), [Po1]). As a result, the index formula (2.12) gives back the index theorem for Dirac operators of Atiyah-Singer ([AS1], [AS2]).

### 3. Twisted Spectral Triples. Examples

In this section, we review some main definitions and examples regarding twisted spectral triples.

Twisted spectral triples were introduced in [CM2]. Their definition is similar to that of an ordinary spectral triple, except for some “twist” given by the condition (3)(c) below.

**Definition 3.1.** A twisted spectral triple $(A, \mathcal{H}, D, \sigma)$ consists of the following:

1. A $\mathbb{Z}_2$-graded Hilbert space $\mathcal{H} = \mathcal{H}^+ \oplus \mathcal{H}^-$.  
2. An involutive unital algebra $A$ represented by bounded operators on $\mathcal{H}$ preserving its $\mathbb{Z}_2$-grading and equipped with a $*$-automorphism $\sigma : A \to A$.

3. A selfadjoint unbounded operator $D$ on $\mathcal{H}$ such that
   
   (a) $D$ maps $\text{dom } D \cap \mathcal{H}^\pm$ to $\mathcal{H}^\mp$.
   
   (b) The resolvent $(D + i)^{-1}$ is a compact operator.
   
   (c) $\sigma(|\text{dom } D|) \subset \text{dom } D$ and $[D, a]_\sigma := Da - \sigma(a)D$ is bounded for all $a \in A$.

An important class of examples of twisted spectral triples arises from the conformal deformation of an ordinary spectral triple $(A, \mathcal{H}, D)$. Let $h$ be a selfadjoint element of $A$ such that $e^{ih} \in A$ (this condition is automatically satisfied if $A$ is closed under holomorphic functional calculus).

If we think of $D$ as providing us with the inverse of the metric of $(A, \mathcal{H}, D)$, then it stems for reason to define a conformal deformation of this metric as being provided by the operator,

$$D_h := e^{-h}De^{-\frac{h}{2}}.$$
As it turns out, \((A, \mathcal{H}, D_h)\) is not a spectral triple in general, but it can be turned into a twisted spectral triple. Namely, we have

**Proposition 3.2** ([CM2]). Let \(\sigma_h : A \to A\) be the automorphism defined by

\[
\sigma_h(a) := e^{-h}ae^h \quad \forall a \in A.
\]

Then \((A, \mathcal{H}, D_h, \sigma_h)\) is a twisted spectral triple.

We also can obtain a twisted spectral triple by twisting \((A, \mathcal{H}, D)\) by scaling automorphisms ([Mo2]). A scaling automorphism of \((A, \mathcal{H}, D)\) is a unitary operator \(U \in \mathcal{L}(\mathcal{H})\) such that

\[
UAU^* = A \quad \text{and} \quad UD^* = \lambda(U)D \quad \text{with} \quad \lambda(U) > 0.
\]

Denote by \(G\) the group of scaling automorphisms of the spectral triple \((A, \mathcal{H}, D)\). Observe that the map \(U \to \lambda(U)\) is a character of \(G\). We refer to Remark 3.6 for geometric examples of scaling automorphisms.

In the sequel, we denote by \(A \times G\) the (discrete) crossed-algebra of \(A\) and \(G\) and we represent it as the sub-algebra of \(\mathcal{L}(\mathcal{H})\) generated by operators of the form \(au\) with \(a \in A\) and \(U \in G\).

**Proposition 3.3** ([Mo2]). Let \(\sigma : A \times G \to A \times G\) be the automorphism defined by

\[
\sigma(aU) := \lambda(U)^{-1}aU \quad \forall a \in A \quad \forall U \in G.
\]

Then \((A \times G, \mathcal{H}, D, \sigma)\) is a twisted spectral triple.

Another interesting example of twisted spectral is the twisted spectral triple of Connes-Tretkoff [CT] over the noncommutative torus \(A_\theta, \theta \in \mathbb{R} \setminus \mathbb{Q}\) (see also [FK]). Recall that \(A_\theta\) is the algebra,

\[
A_\theta = \left\{ \sum a_{m,n}U^mV^n : (a_{m,n}) \in \mathcal{S}(\mathbb{Z}^2) \right\},
\]

where \(U\) and \(V\) are unitaries of \(L^2(S^1)\) such that \(VU = e^{2\pi i \theta}UV\) and \(\mathcal{S}(\mathbb{Z}^2)\) is the space of rapid decay sequences \((a_{m,n})_{m,n} \in \mathbb{Z}\) with complex entries.

The (anti-)Cauchy-Riemann operator \(\partial : A_\theta \to A_\theta\) is given by

\[
\partial = \partial_1 + i\partial_2,
\]

where \(\partial_j : A_\theta \to A_\theta, j = 1, 2\), are the canonical derivations defined by

\[
\delta_1(U) = U, \quad \delta_2(V) = V, \quad \delta_1(V) = \delta_2(U) = 0.
\]

In addition, we denote by \(\tau : A_\theta \to \mathbb{C}\) the unique normalized trace of \(A_\theta\), i.e.,

\[
\tau \left( \sum a_{m,n}U^mV^n \right) = a_{00}.
\]

Let \(A_{1,0}^\theta\) be the subspace of \(A_\theta\) spanned by “holomorphic 1-forms" \(a\partial b\), where \(a\) and \(b\) range over \(A_\theta\). We denote by \(\mathcal{H}_{1,0}\) the Hilbert space obtained as the completion of \(A_{1,0}^\theta\) with respect to the inner product

\[
\langle a_{1}\partial b_1, a_{2}\partial b_2 \rangle = \tau (a_{1}^*a_{2} (\partial b_1)(\partial b_2)^*) = a_{1}^*, b_{2}^* \in A_\theta.
\]

Let \(h \in A_\theta, h^* = h\), and let \(\varphi : A_\theta \to \mathbb{C}\) be the functional defined by

\[
\varphi(a) := \tau (ae^{-2h}) \quad \forall a \in A_\theta.
\]

We denote by \(\mathcal{H}_{\varphi}\) be the Hilbert space obtained as the completion of \(A_\theta\) with respect to the inner product,

\[
\langle a, b \rangle_{\varphi} := \varphi (b^*a) = \tau (b^*ae^{-2h}), \quad a, b \in A_\theta.
\]

In addition, we let \(\partial_{\varphi} : \text{dom } \partial_{\varphi} \subset \mathcal{H}_{\varphi} \to \mathcal{H}_{1,0}\) be the closed extension of \(\partial\) with respect to the inner products of \(\mathcal{H}_{\varphi}\) and \(\mathcal{H}_{1,0}\). We denote by \(\partial_{\varphi}^*\) its adjoint; this an operator from \(\text{dom } \partial_{\varphi} \subset \mathcal{H}_{1,0}\) to \(\mathcal{H}_{\varphi}\). Then on the Hilbert space \(\mathcal{H} := \mathcal{H}_{\varphi} \oplus \mathcal{H}_{1,0}\) we can form the twisted Dolbeault-Dirac operator,

\[
D_{\varphi} = \begin{pmatrix} 0 & \partial_{\varphi}^* \\ \partial_{\varphi} & 0 \end{pmatrix}, \quad \text{dom } D = \text{dom } \partial_{\varphi} \oplus \text{dom } \partial_{\varphi}^*.
\]
Let $A_\theta^\text{op}$ be the opposite algebra of $A_\theta$, i.e., the algebra $A_\theta$ with product $a \cdot^\text{op} b := ba, a, b \in A_\theta$. We equip $A_\theta$ with the automorphism $\sigma_h : A_\theta^\text{op} \rightarrow A_\theta^\text{op}$ defined by

$$\sigma_h(a) := e^{-h}ae^h \quad \forall a \in A_\theta^\text{op}.$$ 

Furthermore, we represent the elements of $A_\theta^\text{op}$ as bounded operators on $\mathcal{H}$ by means of the representation $\pi^\text{op}_h : A_\theta^\text{op} \rightarrow \mathcal{L}(\mathcal{H})$ defined by

$$\pi^\text{op}_h(a)\xi := \xi\sigma(a) \quad \forall a \in A_\theta^\text{op} \forall \xi \in \mathcal{H}.$$

**Proposition 3.4** ([C1]). The triple $(A_\theta^\text{op}, \mathcal{H}, D_\theta)_{\pi^\text{op}_h}$ is a twisted spectral triple.

The main focus of this paper is the twisted spectral triple in conformal geometry constructed by Connes-Moscovici [CM2]. The remainder of this section is devoted to a review of its construction.

Let $(M^n, g)$ be a compact spin oriented Riemannian manifold ($n$ even). We shall denote by $\mathcal{D}_g : C^\infty(M, \mathbb{S}) \rightarrow C^\infty(M, \mathbb{S})$ its Dirac operator acting on the sections of the spinor bundle $\mathbb{S} = \mathbb{S}^+ \oplus \mathbb{S}^-$. We also denote by $L^2_\mathcal{D}(M, \mathbb{S})$ the corresponding Hilbert space of $L^2$-spinors.

Let $G$ be the identity component of the group of conformal diffeomorphisms of $M$ that preserves the orientation and the spin structure. If $\phi : M \rightarrow M$ is such a diffeomorphism, then there is a unique function $h_\phi \in C^\infty(M, \mathbb{R})$ such that

$$\phi_* g = e^{2h_\phi} g. \quad (3.3)$$

In addition, $\phi$ uniquely lifts to a unitary vector bundle isomorphism $\phi^\mathcal{H} : \mathcal{H} \rightarrow \mathcal{H}, \mathcal{S}$, i.e., a unitary section of $\text{Hom}(\mathbb{S}, \phi_* \mathcal{S})$ (see [BG]). We then let $U_\phi : L^2_\mathcal{D}(M, \mathbb{S}) \rightarrow L^2_\mathcal{D}(M, \mathbb{S})$ be the bounded operator defined by

$$U_\phi u(x) = \phi^\mathcal{H} (u \circ \phi^{-1}(x)) \quad \forall u \in L^2_\mathcal{D}(M, \mathbb{S}) \forall x \in M. \quad (3.4)$$

The map $\phi \rightarrow U_\phi$ is a representation of $G$ in $L^2_\mathcal{D}(M, \mathbb{S})$, but this is not a unitary representation. In order to get a unitary representation we need to take into account the Jacobian $|\phi'(x)|$ of $\phi \in G$. This is achieved by considering the unitary operator $V_\phi : L^2_\mathcal{D}(M, \mathbb{S}) \rightarrow L^2_\mathcal{D}(M, \mathbb{S})$ defined by

$$V_\phi = e^{\frac{i}{2} h_\phi} U_\phi, \quad \phi \in G. \quad (3.5)$$

Then $\phi \rightarrow V_\phi$ is a unitary representation of $G$ in $L^2_\mathcal{D}(M, \mathbb{S})$. This enables us to realize the crossed-product algebra $\mathcal{A} := C^\infty(M) \rtimes G$ as the sub-algebra of $\mathcal{L}(L^2_\mathcal{D}(M, \mathbb{S}))$ generated by operators of the form $fV_\phi$ with $f \in C^\infty(M)$ and $\phi \in G$.

The conformal invariance of the Dirac operator ([BG]) implies that

$$V_\phi \mathcal{D}_g V_\phi^* = e^{-\frac{h_\phi}{2}} \mathcal{D}_g e^{\frac{h_\phi}{2}} \quad \forall \phi \in G.$$

In addition, consider the automorphism $\sigma : C^\infty(M) \rtimes G \rightarrow C^\infty(M) \rtimes G$ defined by

$$\sigma(fV_\phi) := e^{hs} fV_\phi \quad \forall f \in C^\infty(M) \forall \phi \in G. \quad (3.6)$$

**Proposition 3.5** ([CM2]). The triple $\left(C^\infty(M) \rtimes G, L^2_\mathcal{D}(M, \mathbb{S}), \mathcal{D}_g, \sigma \right)$ is a twisted spectral triple.

In the sequel, we shall refer to $\left(C^\infty(M) \rtimes G, L^2_\mathcal{D}(M, \mathbb{S}), \mathcal{D}_g, \sigma \right)$ as the conformal Dirac spectral triple.

**Remark 3.6.** Suppose now that $(M, g)$ is the round sphere $(\mathbb{S}^n, g_0)$. Then $G$ agrees with the identity connected component $\text{PO}(n + 1, 1)_0$ acting by Möbius transformations. If we restrict ourselves to the parabolic subgroup $P \subset G$ fixing the North Pole, then $P$ is a group of similarities and acts on the spectral triple $\left(C^\infty(\mathbb{S}^n), L^2_\mathcal{D}(\mathbb{S}^n, \mathbb{S}), \mathcal{D}_g \right)$ by scaling automorphisms.
4. THE INDEX MAP OF A TWISTED SPECTRAL TRIPLE

In this section, we give a geometric description of the index map of a twisted spectral triple in terms of twisted connections on finitely generated projective modules.

Let \((A, \mathcal{H}, D)_\sigma\) be a twisted spectral triple. As observed in [CM1], the datum of \((A, \mathcal{H}, D)_\sigma\) gives rise to a well-defined additive index map,

\[
\ind_{D,\sigma} : K_0(A) \to \mathbb{Z},
\]

(4.1)

where \(D_{e,\sigma}\) is the operator \(\sigma(e)(D \otimes 1_q) : e(\text{dom } D)^q \to \sigma(e)\mathcal{H}^q\). The operator \(D_{e,\sigma}\) is Fredholm, and with respect to the splittings \(e\mathcal{H}^q = (e\mathcal{H}^+)^q \oplus (e\mathcal{H}^-)^q\) and \(\sigma(e)e\mathcal{H}^q = \sigma(e)(e\mathcal{H}^+)^q \oplus \sigma(e)(e\mathcal{H}^-)^q\) it takes the form,

\[
D_{e,\sigma} = \begin{pmatrix}
0 & D_{e,\sigma}^- \\
D_{e,\sigma}^+ & 0
\end{pmatrix}, \quad D_{e,\sigma}^\pm : e(\text{dom } D \cap \mathcal{H}^\pm)^q \to \sigma(e)(\mathcal{H}^\mp)^q.
\]

In general, \(D_{e,\sigma}\) is not selfadjoint (unless \(\sigma(e) = e^*\)), so we define its index by

\[
\ind D_{e,\sigma} := \frac{1}{2} (\ind D_{e,\sigma}^+ - \ind D_{e,\sigma}^-),
\]

where \(\ind D_{e,\sigma}^\pm\) is the usual Fredholm index of \(D_{e,\sigma}^\pm\). In view of (2.1), when \(\sigma = \text{id}\) this definition of the index map agrees with that given in Section 2.

We shall now give a more geometric description of the above index map (compare [Mo1]). Let \(\mathcal{E}\) be a finitely generated projective right-module, i.e., \(\mathcal{E} = \mathcal{E}^0\) with \(e \in M_q(A)\), \(e^2 = e\). Set \(\mathcal{E}^\sigma := \sigma(e)\mathcal{E}^0\) and let \(\mathcal{E}^\sigma : \mathcal{E} \to \mathcal{E}^\sigma\) be the \(A\)-module map defined by

\[
\sigma^\mathcal{E}(\xi) = (\sigma(\xi_j)) \quad \forall \xi = (\xi_j) \in \mathcal{E}.
\]

Notice that both \(\mathcal{E}\) and \(\mathcal{E}^\sigma\) inherit a Hermitian structure from the canonical Hermitian structure of \(\mathcal{E}^0\) defined by

\[
\langle \xi, \eta \rangle = \sum \xi_j^* \eta_j \quad \text{for all } \xi = (\xi_j) \text{ and } \eta = (\eta_j) \text{ in } \mathcal{E}^0.
\]

Following [CM1] we consider the space of twisted 1-forms,

\[
\Omega^1_{D,\sigma} = \{ \Sigma a_i[D, b_i]_\sigma : a_i, b_i \in A \}.
\]

This is naturally an \((A, A)\)-bimodule, since

\[
a^2(a_1[D, b_1]_\sigma)b^2 = a^2 a_1^1[D, b_1 b_2^1]_\sigma - a^2 a_1^1 \sigma(b^1)[D, b_2^2]_\sigma \quad \forall a_j, b_i \in A.
\]

In addition, consider the linear map \(d_\sigma : A \to \Omega^1_{D,\sigma}\) defined by

\[
d_\sigma a := [D, a]_\sigma \quad \forall a \in A.
\]

This is a \(\sigma\)-derivation, in the sense that

\[
d_\sigma(ab) = (d_\sigma a)b + \sigma(a)d_\sigma b \quad \forall a, b \in A.
\]

**Definition 4.1.** A \(\sigma\)-connection on \(\mathcal{E}\) is a \(C\)-linear map \(\nabla : \mathcal{E} \to \mathcal{E}^\sigma \otimes_A \Omega^1_{D,\sigma}\) such that

\[
\nabla(\xi a) = \nabla(\xi) a + \sigma^\mathcal{E}(\xi) \otimes d_\sigma a \quad \forall \xi \in \mathcal{E} \forall a \in A.
\]

An example of \(\sigma\)-connection is the Grassmannian \(\sigma\)-connection \(\nabla_0\) defined by

\[
\nabla_0 \xi = \sigma(e)(d_\sigma \xi_j) \quad \forall \xi = (\xi_j) \in \mathcal{E}.
\]

Moreover, the space of \(\sigma\)-connections is an affine space modeled on \(\text{Hom}_A(\mathcal{E}, \mathcal{E}^\sigma)\).

Let \(\nabla\) be a \(\sigma\)-connection on \(\mathcal{E}\). Using \(\nabla\) we can twist \(D\) into a new operator as follows. Recall that \(A\) naturally acts on \(\mathcal{H}\) and this action preserves dom \(D\). We denote by \(\mathcal{E} \otimes_A \mathcal{H}\) the Hilbert space obtained by completing the algebraic tensor product with respect to the inner product,

\[
\langle \xi_1 \otimes \eta_1, \xi_2 \otimes \eta_2 \rangle = \langle \eta_1, \xi_1 \xi_2 \eta_2 \rangle \quad \forall \xi_j, \eta_j \in \mathcal{H}.
\]

We similarly define the Hilbert spaces \(\mathcal{E}^\sigma \otimes_A \mathcal{H}\) and \(\mathcal{E} \otimes_A \text{dom } D\), where dom \(D\) is equipped with its (Hilbertian) graph norm \(\|\xi\|_D := \sqrt{\|\xi\|^2 + \|D\xi\|^2}\), \(\xi \in \text{dom } D\).
Definition 4.2. The operator $D_{\mathcal{E},\nabla} : \text{dom } D \otimes_A \mathcal{H} \to \mathcal{E}^{\sigma} \otimes_A \mathcal{H}$ is defined by

$$D_{\mathcal{E},\nabla}(\xi \otimes \eta) := \sigma^\nabla(\xi) \otimes D\eta + (\nabla\xi)\eta \quad \forall \xi \in \mathcal{E}, \forall \eta \in \mathcal{H},$$

where $(\nabla\xi)\eta$ has the following meaning: if $\nabla\xi = \sum \xi_j \otimes \omega_j \in \mathcal{E}^{\sigma} \otimes \Omega^1_{D,\sigma}$, then

$$\nabla\xi \eta := \sum \xi_j \otimes \omega_j(\eta) \in \mathcal{E}^{\sigma} \otimes_A \mathcal{H}.$$

In case $\nabla$ is the Grassmannian $\sigma$-connection $\nabla_0$, the operator $D_{\mathcal{E},\nabla_0}$ agrees with the operator $D_\sigma$ considered in (3.1). Recall that this operator is Fredholm. Moreover, if $\nabla_1$ and $\nabla_2$ are two $\sigma$-connections, then they differ by an element $T \in \text{Hom}_A(\mathcal{E},\mathcal{E}^{\sigma})$, and hence $D_{\mathcal{E},\nabla_1}$ and $D_{\mathcal{E},\nabla_2}$ differ by $T \otimes 1_\mathcal{H}$ which is a bounded operator from $\mathcal{E} \otimes_A \mathcal{H}$ to $\mathcal{E}^{\sigma} \otimes_A \mathcal{H}$. It then follows that all the operators $D_{\mathcal{E},\nabla}$ are Fredholm.

In addition, with respect to the splitting $\mathcal{E} \otimes_A \mathcal{H} = (\mathcal{E} \otimes_A \mathcal{H}^+) \oplus (\mathcal{E} \otimes_A \mathcal{H}^-)$ and the similar splitting for $\mathcal{E} \otimes_A \mathcal{H}$, the operator $D_{\mathcal{E},\nabla}$ takes the form,

$$D_{\mathcal{E},\nabla} = \begin{pmatrix} 0 & D_{\mathcal{E},\nabla}^- \\ D_{\mathcal{E},\nabla}^+ & 0 \end{pmatrix},$$

where $D_{\mathcal{E},\nabla}^\pm$ maps $\mathcal{E} \otimes_A (\text{dom } D \cap \mathcal{H}^\pm)$ to $\mathcal{E}^{\sigma} \otimes_A \mathcal{H}^\pm$. We then define the index of $D_{\mathcal{E},\nabla}$ as

$$\text{ind } D_{\mathcal{E},\nabla} = \frac{1}{2} \left( \text{ind } D_{\mathcal{E},\nabla}^+ - \text{ind } D_{\mathcal{E},\nabla}^- \right).$$

where $\text{ind } D_{\mathcal{E},\nabla}^\pm$ is the usual Fredholm index of $D_{\mathcal{E},\nabla}^\pm$.

When $\nabla$ is the Grassmannian $\sigma$-connection we recover the index (4.3). Moreover, as $D_{\mathcal{E},\nabla}^\pm$ depends on the datum of the $\sigma$-connection $\nabla$ only up to a bounded operator from $\mathcal{E} \otimes_A \mathcal{H}^\pm$ to $\mathcal{E}^{\sigma} \otimes_A \mathcal{H}^\pm$, its Fredholm index is actually independent of that datum. Therefore, we arrive at the following statement.

Proposition 4.3. For any $\sigma$-connection on $\mathcal{E}$,

$$\text{ind } D_{\sigma}[\mathcal{E}] = \text{ind } D_{\mathcal{E},\nabla}^\nabla.$$

5. The Connes-Chern Character of a Twisted Spectral Triple

In this section, we recall the construction of the Connes-Chern character of a twisted spectral triple.

As for ordinary spectral triples, the index map of the twisted triple $(\mathcal{A}, \mathcal{H}, D)_\sigma$ can be computed by pairing $K_0(\mathcal{A})$ with some cyclic cohomology class. More precisely, we have

Theorem 5.1 (CM2). Assume that $(\mathcal{A}, \mathcal{H}, D)_\sigma$ is $p^+$-summable in the sense of (2.2).

1. For any integer $k > \frac{1}{2}(p - 1)$, the following formula defines a cyclic cocycle on $\mathcal{A}$,

$$\tau_{2k,\sigma}^D(a^0, \cdots, a^{2k}) := \frac{1}{2} \frac{k!}{(2k)!} \text{Str } \left\{ D^{-1}[D, a^0]_\sigma \cdots D^{-1}[D, a^{2k}]_\sigma \right\}, \quad a^j \in \mathcal{A}.$$

2. The class of $\tau_{2k,\sigma}^D$ in the periodic cyclic cohomology $\text{HP}^{cn}(\mathcal{A})$ is independent of $k$.

3. For all $e \in K_0(\mathcal{A})$,

$$\text{ind } D_{\sigma}[e] = \langle \tau_{2k,\sigma}^D, e \rangle.$$

Definition 5.2. The class of $\tau_{2k,\sigma}^D$ in $\text{HP}^{cn}(\mathcal{A})$ is denoted $\text{Ch}(\mathcal{A}, D)_\sigma$ and is called the Connes-Chern character of $(\mathcal{A}, \mathcal{H}, D)_\sigma$.

For instance, suppose that $(\mathcal{A}, \mathcal{H}, D)_\sigma$ is the conformal deformation of some $p^+$-summable ordinary spectral triple $(\mathcal{A}, \mathcal{H}, D)$, i.e., $(\mathcal{A}, \mathcal{H}, D)_\sigma = (\mathcal{A}, \mathcal{H}, D_h)_{\sigma_h}$ for some $h \in \mathcal{A}$, $h^* = h$, where $D_h = e^{-h^2} D e^{-\frac{h}{2}}$ and $\sigma_h(a) = e^{-h}ae^{h}$. Then we can check that

$$\tau_{2k,\sigma_h}^D(a^0, \cdots, a^{2k}) = \frac{1}{2} \frac{k!}{(2k)!} \text{Str } \left\{ D^{-1}[D, \sigma_h(a^0)]_\sigma \cdots D^{-1}[D, \sigma_h(a^{2k})_\sigma] \right\}$$

$$= \tau_{2k}^D(\sigma_\frac{h}{2}(a^0), \cdots, \sigma_\frac{h}{2}(a^{2k})).$$
where \( \tau^g_2 \) is the cocycle (2.3) that defines the ordinary Connes-Chern character of \((\mathcal{A}, \mathcal{H}, D)\). This shows that \( \tau^D_n \) and \( \tau^g_2 \) are homotopically equivalent, and hence define the same class in the cyclic cohomology. Therefore, we obtain

**Proposition 5.3 (CM2).** For all \( h \in \mathcal{A} \), \( h^* = h \), we have

\[
\text{Ch}(A, e^{-\frac{i}{2} D} e^{-\frac{i}{2}})_{\tau^g} = \text{Ch}(A, D) \in \text{HP}^{\epsilon\tau}(\mathcal{A}).
\]

The natural question that arises is to find a local representative for the Connes-Chern character \( \text{Ch}(A, D)_{\tau} \), i.e., an analogue of the CM cocycle (2.9) (see [CM2]). Moscovici [Mo2] devised an Ansatz for such a local representative and proved that the Ansatz is verified in the case of an ordinary spectral triple \((\mathcal{A}, \mathcal{H}, D)\) twisted by scaling automorphisms (cf. Proposition 3.3), provided that \((\mathcal{A}, \mathcal{H}, D)\) is regular and has simple and discrete dimension spectrum. To date this is the only example of twisted spectral triple known to verify Moscovici’s Ansatz.

6. THE CONNES-CHERN CHARACTER OF THE CONFORMAL DIRAC SPECTRAL TRIPLE

Our aim in this section is to give a geometric expression for the Connes-Chern character of the conformal Dirac spectral triple, the construction of which was recalled in Section 3.

Throughout this section we shall use the same notation as in Section 3. In particular, \((M^n, g)\) is a closed spin oriented Riemannian manifold (n even) with Dirac operator \(D_g : C^\infty(M, \mathcal{S}) \to C^\infty(M, \mathcal{S})\), where \(\mathcal{S} = \mathcal{S}^+ \oplus \mathcal{S}^-\) is the spinor bundle of \(M\). We also denote by \(L^2(M, \mathcal{S})\) the corresponding Hilbert space of \(L^2\)-spinors. In addition, \(G\) is the identity component of the group of conformal diffeomorphisms of \(M\) that preserve the orientation and the spin structure.

Using the automorphism \(\sigma\) of \(\mathcal{A}\) defined by (3.6), Proposition 3.3 asserts that \((\mathcal{A}, L^2(M, \mathcal{S}), D_g)_{\sigma}\) is a twisted spectral triple.

By Ferrand-Obata Theorem [Fe] (see also [Sc]) there are two main possibilities for \(G\):

(a) \(M\) is conformally equivalent to the sphere \(S^n\) and \(G\) is isomorphic to \(\text{PSO}(n + 1, 1)\).

(b) \(M\) is not conformally flat and \(G\) is compact with respect to the compact-open topology.

In this paper, we shall focus on the non-(conformally-)flat case exclusively. Henceforth we assume throughout the rest of this section that \((M^n, g)\) is closed and not conformally equivalent to \(S^n\). Notice that, as \(M\) is compact, this means that, either \(M\) is not simply connected, or its Weyl curvature tensor of \(g\) is not identically zero (see [Ku]).

As pointed out by Moscovici [Mo2], Remark 3.8], in the non conformally flat case the conformal Dirac spectral triple is unitarily equivalent to the conformal perturbation of an equivariant Dirac spectral triple, and hence the Connes-Chern character of the conformal spectral triple is represented by the CM cocycle of the equivariant Dirac spectral triple. We shall describe this equivalence in full details and use it to compute the Connes-Chern character of the conformal Dirac spectral triple.

As the group \(G\) is compact, it admits a Haar measure \(d\lambda(\phi)\), using which we can exhibit a \(G\)-invariant metric \(\bar{g}\) in the conformal class of \(g\). Namely,

\[
\bar{g} := \int_G \phi_* g \, d\lambda(\phi) = \left( \int_M e^{2h \phi} \, d\lambda(\phi) \right) g = e^{2h} g,
\]

where \(h := \frac{1}{2} \log \left( \int_M e^{2h \phi} \, d\lambda(\phi) \right)\). If \(\phi \in G\), then the equality \(\phi_* \bar{g} = \bar{g}\) implies that

\[
\phi_* g = \phi_* (e^{2h} \bar{g}) = e^{2h} \bar{g} = e^{2h} g^{-1} = e^{2h} g^{-1} - 2h g.
\]

Comparing this to (3.8) then shows that

\[
h_\phi = h \circ \phi^{-1} - h \quad \forall \phi \in G.
\]

For \(\phi \in G\) we denote by \(U_\phi : L^2(M, \mathcal{S}) \to L^2(M, \mathcal{S})\) the operator defined by (3.4) using the metric \(\bar{g}\). As the metric \(\bar{g}\) is \(G\)-invariant, this operator is actually unitary, and hence \(\phi \to U_\phi\) is a unitary representation of \(G\) on \(L^2(M, \mathcal{S})\). This enables us to represent the crossed-product algebra \(C^\infty(M) \rtimes G\) in \(L^2(M, \mathcal{S})\) as the subalgebra of \(\mathcal{L}(L^2(M, \mathcal{S}))\) generated by operators of the form \(f U_\phi\) with \(f \in C^\infty(M)\) and \(\phi \in G\).
Let $\mathcal{D}_g : L^2_0(M, \mathcal{S}) \to L^2(M, \mathcal{S})$ be the Dirac operator associated to the metric $\bar{g}$. The $G$-invariance of $\bar{g}$ and the fact that $G$ preserves the spin structure imply that

\begin{equation}
[\mathcal{D}_g, U_{\phi}] = 0 \quad \forall \phi \in G.
\end{equation}

Combining this property with the fact that $(C^\infty(M), L^2_0(M, \mathcal{S}), \mathcal{D}_g)$ is an (ordinary) spectral triple, we can easily check that $(C^\infty(M) \rtimes G, L^2_0(M, \mathcal{S}), \mathcal{D}_g)$ too is a spectral triple. Both spectral triples are $n^+$-sumnable. Furthermore, we have

**Proposition 6.1.** The spectral triple $(C^\infty(M) \rtimes G, L^2_0(M, \mathcal{S}), \mathcal{D}_g)$ is regular and has simple and discrete dimension spectrum.

**Proof.** Consider the derivation $\delta(T) := [\mathcal{D}_g, T]$ as defined in (6.3). Let $f \in C^\infty(M)$ and $\phi \in G$. As (6.3) shows that $\mathcal{D}_g$ commutes with $U_{\phi}$, we see that, for all $j \in \mathbb{N}$,

\begin{equation}
\delta^j(f U_{\phi}) = \delta^j(f) U_{\phi} \quad \text{and} \quad \delta^j([\mathcal{D}_g, f U_{\phi}]) = \delta^j([\mathcal{D}_g, f]) U_{\phi} = \delta^j(c(df)) U_{\phi},
\end{equation}

where $c(df)$ is the action on $\mathcal{S}$ of the differential $df$ by Clifford multiplication; this is a section of $\text{End}_G \mathcal{S}$.

Notice that $\delta^j(f)$ and $\delta^j(c(df))$ are contained in the algebra $\Psi^0(M, \mathcal{S})$ of zeroth order $\Psi$DOs on $M$ acting on the sections of $\mathcal{S}$. Therefore (6.4) shows that $\delta^j(f U_{\phi})$ and $\delta^j([\mathcal{D}_g, f U_{\phi}])$ are bounded operators. Thus $f U_{\phi}$ and $[\mathcal{D}_g, f U_{\phi}]$ are contained in $\text{dom} \delta^j$ for all $j \in \mathbb{N}$. This proves that the spectral triple $(C^\infty(M) \rtimes G, L^2_0(M, \mathcal{S}), \mathcal{D}_g)$ is regular.

Let us denote by $\mathcal{B}$ the sub-algebra of $\mathcal{L}(L^2_0(M, \mathcal{S}))$ generated by the grading operator $\gamma := id_{\mathcal{S}} + id_{\mathcal{S}^\perp}$ and the operators $\delta^j(f U_{\phi})$ and $\delta^j([\mathcal{D}_g, f U_{\phi}])$ as above. It follows from (6.4) and the previous discussion that $\mathcal{B}$ is spanned by operators of the form,

$$P_0 U_{\phi_0} P_1 U_{\phi_1} \cdots P_h U_{\phi_h}, \quad P_j \in \Psi^0(M, \mathcal{S}), \quad \phi_j \in G.$$ 

In fact, $U_{\phi_0} P_{j+1} U_{\phi_j} U_{\phi_{j+1}} = (\phi_j)_{P_{j+1} U_{\phi_j} U_{\phi_{j+1}}} = (\phi_j)_{P_{j+1} U_{\phi_{j+1}}} = U_{\phi_{j+1}}$, and $\phi_j \in G$. That is, the algebra $\mathcal{B}$ is contained in the crossed-product algebra $\Psi^0(M, \mathcal{S}) \rtimes G$.

If $P \in \Psi^0(M, \mathcal{S})$ and $\phi \in G$, then the result of [Da] shows that the function $\text{Tr} \left[ P U_{\phi} \mathcal{D}_g^{1/2} \right]$ has a meromorphic extension to $\mathbb{C}$ with at worst simple pole singularities on $\Sigma := \{ k \in \mathbb{Z}; \ k \leq n \}$. This shows that the spectral triple $(C^\infty(M) \rtimes G, L^2_0(M, \mathcal{S}), \mathcal{D}_g)$ has a discrete and simple dimension spectrum. The proof is thus complete. \hfill \Box

**Remark 6.2.** Consider the space $\Psi^q_\mathcal{D}(C^\infty(M))$, $q \in \mathbb{C}$, as defined in (2.5)–(2.4). By arguing as in the proof above, it can be shown that $\Psi^q_\mathcal{D}(C^\infty(M))$ is contained in the crossed-product $\Psi^q(M, \mathcal{S}) \rtimes G$, where $\Psi^q(M, \mathcal{S})$ is the space of $\Psi$DOs of order $q$ on $M$ acting on the sections of $\mathcal{S}$. Moreover, the residual trace (2.6) on $\Psi^q_\mathcal{D}(C^\infty(M))$ agrees with the noncommutative residue trace on $\Psi^q(M, \mathcal{S}) \rtimes G$ constructed in [Da].

As the function $h$ in (6.11) is real-valued, and hence is a selfadjoint element of $C^\infty(M) \rtimes G$, we can form the conformally deformed twisted spectral triple,

$$(C^\infty(M) \rtimes G, L^2_0(M, \mathcal{S}), e^{-\frac{1}{2} \mathcal{D}_g} e^{-\frac{1}{2} h})_{\sigma_h},$$

where the automorphism $\sigma_h$ is defined as in (5.2).

Observe that, as $\bar{g} = e^{2h} g$ the multiplication operator by $e^{\frac{1}{2} nh}$ gives rise to a unitary operator from $L^2_0(M, \mathcal{S})$ to $L^2_0(M, \mathcal{S})$, since, for all $u \in L^2_0(M, \mathcal{S})$,

$$\int_M |u(x)|^2 \text{vol}_g(x) = \int_M |e^{\frac{1}{2} nh(x)} u(x)|^2 \text{vol}_g(x) = \int_M |e^{\frac{1}{2} nh(x)} u(x)|^2 \text{vol}_g(x),$$

where $| \cdot |$ is the Hermitian metric of $\mathcal{S}$. Notice also that the conformal invariance of the Dirac operator (HR) implies that

$$\mathcal{D}_g = \mathcal{D}_g e^{-\frac{1}{2} (\alpha + 1) h} e^{-\frac{1}{2} (\alpha - 1) h} = e^{-\frac{1}{2} nh} \mathcal{D}_g e^{-\frac{1}{2} h} e^{\frac{1}{2} nh}. $$
Let $\phi \in G$. Combining the very definitions (3.3)–(3.4) of $U_\phi$ and $V_\phi$ with (6.2) we see that

$$V_\phi = e^{\frac{i}{\hbar}h}U_\phi = e^{\frac{i}{\hbar}(\hbar \phi - h)}U_\phi = e^{-\frac{i}{\hbar}h}U_\phi e^{\frac{i}{\hbar}h}.$$

Using the definitions of the automorphisms $\sigma_\hbar$ and $\sigma$ in (3.2) and (3.6) we also get

$$\sigma(V_\phi) = e^{h_\hbar}V_\phi = e^{\frac{i}{\hbar}(h + 1)}h \phi U_\phi = e^{-\frac{i}{\hbar}(h + 1)}h \phi U_\phi e^{\frac{i}{\hbar}(h + 1)}h = e^{-\frac{i}{\hbar}h} \sigma(h)(U_\phi)e^{\frac{i}{\hbar}h}. $$

This implies that the multiplication operator by $e^{\frac{i}{\hbar}h}$ intertwines the representations of $C^\infty(M) \rtimes G$ in $L^2_\hbar(M, \mathcal{S})$ and $L^2_\hbar(M, \mathcal{S})$, and under this intertwining the automorphism $\sigma$ agrees with $\sigma_\hbar$. Therefore, we obtain

**Proposition 6.3.** The multiplication operator by $e^{\frac{i}{\hbar}h}$ gives rise to a unitary equivalence,

$$\left( C^\infty(M) \rtimes G, L^2_\hbar(M, \mathcal{S}), \mathcal{D}_g \right)_\sigma \simeq \left( C^\infty(M) \rtimes G, L^2_\hbar(M, \mathcal{S}), e^{-\frac{i}{\hbar}h} \mathcal{D}_g e^{-\frac{i}{\hbar}h} \right)_{\sigma_\hbar}.$$ 

This implies that the twisted spectral triples $(C^\infty(M) \rtimes G, L^2_\hbar(M, \mathcal{S}), \mathcal{D}_g)_\sigma$ and $(C^\infty(M) \rtimes G, L^2_\hbar(M, \mathcal{S}), e^{-\frac{i}{\hbar}h} \mathcal{D}_g e^{-\frac{i}{\hbar}h})_{\sigma_\hbar}$ have the same Connes-Chern character. As it follows from Proposition 5.3 that the latter twisted spectral triple has the same Connes-Chern character as the ordinary spectral triple $(C^\infty(M) \rtimes G, L^2_\hbar(M, \mathcal{S}), \mathcal{D}_g)$, we deduce that so does the Connes-Chern character of $(C^\infty(M) \rtimes G, L^2_\hbar(M, \mathcal{S}), \mathcal{D}_g)_\sigma$. Moreover, thanks to Proposition 6.1 the spectral triple $(C^\infty(M) \rtimes G, L^2_\hbar(M, \mathcal{S}), \mathcal{D}_g)$ satisfies the assumptions of Theorem 2.2. That is, its Connes-Chern character is represented by the CM cocycle $\varphi^{\text{CM}}$ defined by (2.9)–(2.10). Therefore, we obtain

**Proposition 6.4.** The Connes-Chern character of $(C^\infty(M) \rtimes G, L^2_\hbar(M, \mathcal{S}), \mathcal{D}_g)_\sigma$ is represented by the CM cocycle of $(C^\infty(M) \rtimes G, L^2_\hbar(M, \mathcal{S}), \mathcal{D}_g)_\sigma$.

We are thus reduced to determining the CM cocycle of the equivariant Dirac spectral triple $(C^\infty(M) \rtimes G, L^2_\hbar(M, \mathcal{S}), \mathcal{D}_g)$. To this end we need to introduce some notation.

Let $\phi \in G$ and denote by $M^\phi$ its fixed-point set. Since $\phi$ preserves the orientation and the metric $g$, it is a disconnected union of submanifolds of even dimension $a = 0, 2, \cdots, n$ (see Section 5). Therefore, we may treat $M^\phi$ as if it were a manifold.

We let $\mathcal{N}^\phi = (TM^\phi)^\perp$ be the normal bundle of $M^\phi$, which we regard as a vector bundle over $M^\phi$. We denote by $\phi^\mathcal{N}$ the isometric vector bundle isomorphism induces on $\mathcal{N}^\phi$ by $\phi$. Notice that the eigenvalues of $\phi^\mathcal{N}$ are either $-1$ (which has even multiplicity), or complex conjugates $e^{\pm i\theta}$, $\theta \in (0, \pi)$, with same multiplicity (see Section 8).

Let $R^{TM^\phi}$ be the curvature of $(M, g)$, seen as a section of $\Lambda^2 T^* M \otimes \text{End}(TM)$. As the Levi-Civita connection $\nabla^{TM^\phi}$ is preserved by $\phi$, it preserves the splitting $TM_{M^\phi} = TM^\phi \oplus \mathcal{N}^\phi$ over $M^\phi$, and hence it induces connections $\nabla^{TM^\phi}$ and $\nabla^{\mathcal{N}^\phi}$ on $TM^\phi$ and $\mathcal{N}^\phi$, so that

$$\nabla^{TM^\phi}_{|M^\phi} = \nabla^{TM^\phi} \oplus \nabla^{\mathcal{N}^\phi} \quad \text{on } M^\phi.$$ 

Notice that $\nabla^{TM^\phi}$ is the Levi-Civita connection of $TM^\phi$.

Let $R^{TM^\phi}$ and $R^{\mathcal{N}^\phi}$ be the respective curvatures of $\nabla^{TM^\phi}$ and $\nabla^{\mathcal{N}^\phi}$. Define

$$\tilde{A}(R^{TM^\phi}) := \text{det}^{\frac{1}{2}} \left( \frac{R^{TM^\phi}}{\sinh(R^{TM^\phi}/2)} \right) \quad \text{and} \quad \nu_\phi(R^{\mathcal{N}^\phi}) := \text{det}^{\frac{1}{2}} \left( 1 - \phi^\mathcal{N} e^{-R^{\mathcal{N}^\phi}} \right),$$

where $\text{det}^{\frac{1}{2}} \left( 1 - \phi^\mathcal{N} e^{-R^{\mathcal{N}^\phi}} \right)$ is defined in the same way as in [BGV] Section 6.3.

In the sequel, if $f$ is a smooth function on $M$ we shall denote by $d'f$ the component of the differential $df$ in $T^* M_{M^\phi}^\phi$, i.e., $d'f = df_{|TM^\phi}$ in $M^\phi$. In addition, we shall orient $M^\phi$ like in [BGV] Prop. 6.14, so that the $\phi^\mathcal{N}$ gives rise to a section of $\mathcal{N}^\phi$ which is positive with respect to the orientation of $\mathcal{N}^\phi$ defined by the orientations of $M$ and $M^\phi$.

The following is the key technical result in the computation of the Connes-Chern character of the twisted spectral triple $(C^\infty(M) \rtimes G, L^2_\hbar(M, \mathcal{S}), \mathcal{D}_g)_\sigma$. 


Proposition 6.5. Let \( \phi \in G \) and consider a differential operator of the form,

\[
P_{k, \alpha} = f^0[\partial_g, f^1[\cdot]_{\omega_1}, \ldots, [\partial_g, f^{2k}][\cdot]_{\omega_{2k}}],
\]

where the notation is the same as in (2.8). Then, as \( t \to 0^+ \),

\[
(6.6) \quad \text{Str} \left[ P_{k, \alpha} e^{-\frac{it}{2} U_\phi} \right] = \begin{cases} \left( -i \right)^\frac{d}{2} t^{-k} \sum_a (2\pi)^{-\frac{d}{2}} \int_{M^a} \omega_k + O(t^{-k+1}) & \text{if } \alpha = 0, \\ O \left( t^{-(|\alpha|+k)+1} \right) & \text{if } \alpha \neq 0, \end{cases}
\]

where we have set

\[
\omega_k := \hat{A}(R^{TM^a}) \wedge \nu_{\phi} \left( R^{V^a} \right) \wedge f^0 d f^1 \wedge \cdots \wedge d f^{2k}.
\]

The asymptotics (6.6) are proved in [CH]; see Corollary 3.16 of [CH] for the case \( \alpha = 0 \) and Theorem 2 of [CH] for the case \( \alpha \neq 0 \). The approach in [CH] uses an equivariant version of the Clifford asymptotic pseudodifferential calculus of [Yu]. This equivariant Clifford asymptotic pseudodifferential calculus was developed in [LYZ] to give a new proof of the local equivariant index theorem ([Pa], [DP], [Gi]).

In Section 9, we will give a new, and fairly elementary, proof of the local equivariant index theorem. As an immediate by-product of this proof, we will get a proof of the local index theorem ([Pa], [DP], [Gi]).

Using the fact that \( U_\phi \) is the orthogonal projection onto \( \ker \partial_g \) and \( \phi_{k, \alpha} \) is defined as in (2.11), we see that

\[
(6.7) \quad \text{var} \left( f^0 U_{\phi_0}, \ldots, f^{2k} U_{\phi_{2k}} \right) = \frac{(-i)^\frac{d}{2}}{(2k)!} \sum_a (2\pi)^{-\frac{d}{2}} \int_{M^a} \hat{A}(R^{TM^a}) \wedge \nu_{\phi_a} \left( R^{V^a} \right) \wedge f^0 d f^1 \wedge \cdots \wedge d f^{2k},
\]

where \( \phi_a := \phi_0 \circ \cdots \circ \phi_{2k} \) and \( \tilde{f}^j := f^j \circ \phi_0^{-1} \circ \cdots \circ \phi_{j-1}^{-1} \).

Proof. It follows from Proposition 6.3 that the Connes-Chern character \( \text{Ch}(C^\infty(M) \times G, \partial_g)_\sigma \) is represented by the CM cocycle \( \varphi_{CM} = (\varphi_{2k}) \) of the spectral triple \( (C^\infty(M) \times G, L^2(M, \mathbb{F}), \partial_g) \).

By (2.9) - (2.11), the components \( \varphi_{2k} \) are given by

\[
(6.8) \quad \varphi_0(f^0 U_{\phi_0}) = \text{Res}_{z=0} \Gamma(z) \text{Str} \left[ f^0 U_{\phi_0} \langle |\partial_g|^{-2z} + \Pi_0 \right],
\]

\[
(6.9) \quad \varphi_{2k}(f^0 U_{\phi_0}, \ldots, f^{2k} U_{\phi_{2k}}) = \sum_{\alpha} c_{k, \alpha} \int \gamma f^0 U_{\phi_0} \left[ \partial_g, f^1 U_{\phi_1} \right]^{[\alpha_1]} \cdots \left[ \partial_g, f^{2k} U_{\phi_{2k}} \right]^{[\alpha_{2k}]} \langle |\partial_g|^{-2(|\alpha|+k)} \rangle \quad (k \geq 1),
\]

where \( \Pi_0 \) is the orthogonal projection onto \( \ker \partial_g \) and \( c_{k, \alpha} \) is defined as in (2.11).

Using the fact that \( U_{\phi_0} \) commutes with \( \partial_g \) (cf. Eq. (6.3)) we see that

\[
(6.10) \quad \varphi_0(f^0 U_{\phi_0}) = \text{Res}_{z=0} \Gamma(z) \text{Str} \left[ f^0 \langle |\partial_g|^{-2z} U_{\phi_0} \right] + \text{Str} \left[ f^0 \Pi_0 U_{\phi_0} \right].
\]

Likewise, for \( k \geq 1 \), using (6.3) and arguing as in the proof of Proposition 6.1 we deduce that

\[
f^0 U_{\phi_0} \left[ \partial_g, f^1 U_{\phi_1} \right]^{[\alpha_1]} \cdots \left[ \partial_g, f^{2k} U_{\phi_{2k}} \right]^{[\alpha_{2k}]} \langle |\partial_g|^{-2(|\alpha|+k)} \rangle
\]

\[
= f^0 \left[ \partial_g, f^1 \right]^{[\alpha_1]} \cdots \left[ \partial_g, f^{2k} \right]^{[\alpha_{2k}]} \langle |\partial_g|^{-2(|\alpha|+k)} U_{\phi(k)} \rangle.
\]
where \( \phi(k) := \phi_0 \circ \cdots \circ \phi_{2k} \) and \( \tilde{f} : = f^3 \circ \phi_0^{-1} \circ \cdots \circ \phi_{j-1}^{-1} \). Set \( P_{k,\alpha} = f^0[\partial_g f^1]_{[\alpha_1]} \cdots [\partial_g f^{2k}]_{[\alpha_{2k}]} \). Then from (2.27) we get

\[
\varphi_{2k}(f^0 U_{\phi_0}, \cdots, f^{2k} U_{\phi_{2k}}) = \sum \epsilon_{k,\alpha} \text{Res}_{z=0} \text{Str} \left[ P_{k,\alpha} [\partial_g]^{-2[\alpha]} \phi \langle \partial_g \rangle^{-2z} \right] \\
= \sum \epsilon_{k,\alpha} \Gamma(\alpha + 1) \text{Res}_{z=\alpha+k} \Gamma(z) \text{Str} \left[ P_{k,\alpha} [\partial_g]^{-2z} U_{\phi(k)} \right].
\]

(6.11)

By Mellin’s formula \( \Gamma(z)[\partial_g]^{-2z} = \int_0^\infty t^{z-1} (1 - \Pi_0) e^{-t \partial_g^2} dt \), so we see that, for \( \Re z > 1 \),

\[
\Gamma(z) \text{Str} \left[ P_{k,\alpha} [\partial_g]^{-2z} U_{\phi(k)} \right] = \int_0^\infty t^{z-1} \text{Str} \left[ P_{k,\alpha} (1 - \Pi_0) e^{-t \partial_g^2} U_{\phi(k)} \right] dt,
\]

with the convention that \( P_{k,\alpha} = f^0 \) and \( \phi(k) = \phi_0 \) when \( k = 0 \) and \( \alpha = 0 \). In other words, \( \Gamma(z) \text{Str} \left[ P_{k,\alpha} [\partial_g]^{-2z} U_{\phi(k)} \right] \) is the Mellin transform of the function,

\[
\theta_{\alpha,k}(t) = \text{Str} \left[ P_{k,\alpha} (1 - \Pi_0) e^{-t \partial_g^2} U_{\phi(k)} \right] = \text{Str} \left[ P_{k,\alpha} e^{-t \partial_g^2} U_{\phi(k)} \right] - \text{Str} \left[ P_{k,\alpha} \Pi_0 U_{\phi(k)} \right], \quad t > 0.
\]

The poles of the Mellin transform of a function \( \theta(t) \), \( t > 0 \), are intimately related to the behavior of \( \theta(t) \) as \( t \to 0^+ \) (see, e.g., [GS, Proposition 5.1]). In particular, the residue at \( z = \alpha + k \) of \( \Gamma(z) \text{Str} \left[ P_{k,\alpha} [\partial_g]^{-2z} U_{\phi(k)} \right] \) is equal to the coefficient of \( t^{-(\alpha+k)} \) in the asymptotic of \( \theta_{\alpha,k}(t) \) as \( t \to 0 \). Therefore, using Proposition 6.5 we deduce that

\[
\text{Res}_{z=0} \Gamma(z) \text{Str} \left[ f^0 [\partial_g]^{-2z} U_{\phi_0} \right] = (-i)^{\frac{n}{2}} \sum_a (2\pi)^{-\frac{n}{2}} \int_{M_a^{\phi_0}} \omega_a - \text{Str} \left[ f^0 \Pi_0 U_{\phi_0} \right],
\]

(6.12)

\[
\text{Res}_{z=k} \Gamma(z) \text{Str} \left[ P_{k,0} [\partial_g]^{-2z} U_{\phi(k)} \right] = (-i)^{\frac{n}{2}} \sum_a (2\pi)^{-\frac{n}{2}} \int_{M_a^{\phi(k)}} \omega_a \quad (k \geq 1),
\]

(6.13)

\[
\text{Res}_{z=\alpha+k} \Gamma(z) \text{Str} \left[ P_{k,\alpha} [\partial_g]^{-2z} U_{\phi(k)} \right] = 0 \quad (k \geq 1, \alpha \neq 0),
\]

(6.14)

where \( \omega_a := \tilde{A}(R T M_a^{\phi(k)}) \wedge \nu_{\phi(k)} \left( R^N_{\phi(k)} \right) \wedge f^0 d\tilde{f}^1 \wedge \cdots \wedge d\tilde{f}^{2k} \).

Combining (6.12) with (6.8) and (6.10) gives

\[
\varphi_0(f^0 U_{\phi_0}) = (-i)^{\frac{n}{2}} \sum_a (2\pi)^{-\frac{n}{2}} \int_{M_a^{\phi_0}} \omega_a.
\]

Similarly, for \( k \geq 1 \), by combining (6.13)–(6.14) with (6.9) and (6.11) we get

\[
\varphi_{2k}(f^0 U_{\phi_0}, \cdots, f^{2k} U_{\phi_{2k}}) = c_{k,\alpha} \Gamma(k)^{-1} \text{Res}_{z=k} \Gamma(z) \text{Str} \left[ P_{k,\alpha} [\partial_g]^{-2z} U_{\phi(k)} \right] \\
= (-i)^{\frac{n}{2}} \frac{1}{(2k)!} \sum_a (2\pi)^{-\frac{n}{2}} \int_{M_a^{\phi(k)}} \omega_a.
\]

The proof is complete. \( \square \)

To understand the formula (6.7) it is worth looking at the top-degree component \( \varphi_n \). Observe that for \( k = \frac{1}{2} n \) the r.h.s. of (6.7) reduces to an integral over \( M_n^{\phi(n)} \) and this submanifold is empty unless \( \phi(n) = \text{id} \). Thus,

\[
\varphi_n(f^0 U_{\phi_0}, \cdots, f^n U_{\phi_n}) = \begin{cases} 
\frac{(2\pi)^{-\frac{n}{2}}}{n!} \int_M f^0 \tilde{f}^1 \wedge \cdots \wedge \tilde{f}^n & \text{if } \phi_0 \circ \cdots \circ \phi_n = \text{id}, \\
0 & \text{otherwise}.
\end{cases}
\]

That is, \( \varphi_n \) agrees with the transverse fundamental cyclic cocycle introduced by Connes [Co1].

In addition, the proof of the 2nd part of Theorem 2.2 amounts to show that the cocycle \( \tau_p^D \) in [LS] and the CM cocycle are cohomologous in periodic cyclic cohomology (assuming the spectral triple to be \( p \)-summable with \( p \) even). The proof of this result actually shows that \( \tau_p^D \) and the cocycle \( \varphi_p \) differ by a Hochschild coboundary (see [Hi, Lemma 7.8 and Appendix C]). Therefore, we arrive at the following statement (compare [Mo2, Proposition 3.7]).
Proposition 6.7. In Hochschild cohomology, the Connes-Chern character $\text{Ch}(C^\infty(M) \times G, \mathcal{D}_g)_\sigma$ agrees with Connes’ transverse fundamental class $[M/G]$.

7. Volterra Pseudodifferential Calculus and Heat Kernels

In this section, we recall the main definitions and properties of the Volterra pseudodifferential calculus and its relationship with the heat kernel of an elliptic operator. The pseudodifferential representation of the heat kernel appeared in [GR], but some of the ideas can be traced back to Hadamard [HA]. The presentation here follows closely that of [BGS].

Let $(M^s, g)$ be a compact Riemannian manifold and $E$ a Hermitian vector bundle over $M$. The metrics of $M$ and $E$ naturally define a continuous inner product on the space $L^2(M, E)$ of the $L^2$-sections of $E$. In addition, we let $L : C^\infty(M, E) \to C^\infty(M, E)$ be a selfadjoint 2nd order differential operator whose principal symbol is positive-definite. In particular, $L$ is elliptic.

The operator $L$ generates a continuous heat semigroup $[0, \infty) \ni t \mapsto e^{-tL} \in \mathcal{L}(L^2(M, E))$. Standard ellipticity theory shows that the heat semigroup further induces a continuous semigroup $[0, \infty) \ni t \mapsto e^{-tL} \in \mathcal{L}(C^\infty(M, E))$. In particular, for all $u \in C^\infty(M, E)$, as $t \to 0^+$

\begin{equation}
(7.1)
\quad e^{-tL}u \longrightarrow u \quad \text{and} \quad \frac{d}{dt}e^{-tL}u \longrightarrow -Lu \quad \text{in} \quad C^\infty(M, E).
\end{equation}

In the sequel, we shall make some notation abuse by also denoting by $E$ the vector bundle over $M \times \mathbb{R}$ whose fiber at $(x, t) \in M \times \mathbb{R}$ is $E_x$, i.e., the pullback by the projection $(x, t) \mapsto x$. The heat operator $L + \partial_t$ then acts on the sections of this vector bundle over $M \times \mathbb{R}$.

As it is well known the heat semigroup enables us to invert heat operator $L + \partial_t$. More precisely, the continuity of the heat semi-group ensures us that we define a continuous operator $Q_0$ from $C^\infty_c(M \times \mathbb{R}, E)$ to $C^\infty(M \times \mathbb{R}, E)$ by

\begin{equation}
(7.2)
\quad Q_0u(x, s) := \int_0^\infty e^{-tL}u(x, s - t)dt \quad \forall u \in C^\infty_c(M \times \mathbb{R}, E).
\end{equation}

Furthermore, using (7.1) we obtain

Proposition 7.1 ([GR BGS]). For all $u \in C^\infty_c(M \times \mathbb{R}, E)$,

\begin{equation}
(7.3)
\quad Q_0(L + \partial_t)u = (L + \partial_t)Q_0u = u.
\end{equation}

In other words, the operator $Q_0$ inverts the heat operator $L + \partial_t$ on smooth sections of $E$ over $M \times \mathbb{R}$.

Let us denote by $E \boxtimes E^*$ the vector bundle over $M \times M \times \mathbb{R}$ whose fiber at $(x, y, t) \in M \times M \times \mathbb{R}$ is $\text{Hom}(E_x, E_y)$. We define the heat kernel $k_t(x, y)$, $t > 0$, as the smooth section of $E \boxtimes E^*$ over $M \times M \times (0, \infty)$ such that

\begin{equation}
(7.4)
\quad e^{-tL}u(x) = \int_M k_t(x, y)u(y)|dy| \quad \forall u \in L^2(M, E),
\end{equation}

where $|dy|$ is the Riemannian density defined by $g$ on $M$. That is, $k_t(x, y)|dy|$ is the Schwartz kernel of $e^{-tL}$.

The operator $Q_0$ is intimately related to the heat kernel. Indeed, let $k_{Q_0}(x, y, t) \in C^\infty(M_x \times \mathbb{R}_t, E) \boxtimes \mathcal{D}'(M_y \times \mathbb{R}_t, E)$ be the kernel of $Q_0$, i.e.,

\begin{equation}
(7.5)
\quad k_{Q_0}(x, y, t) = \begin{cases} k_{s-t}(x, y) & \text{for } s - t > 0, \\ 0 & \text{for } s - t < 0. \end{cases}
\end{equation}

Thus $Q_0$ has the Volterra property in the following sense.

Definition 7.2 ([FR]). A continuous linear operator $Q : C^\infty_c(M \times \mathbb{R}, E) \to C^\infty(M \times \mathbb{R}, E)$ satisfies the Volterra property when there is $K_Q(x, y, t) \in C^\infty(M \times \mathbb{R}, E) \boxtimes \mathcal{D}'(M, E)$ such that

\begin{enumerate}
\item $Q$ has kernel $k_Q(x, y, t) = K_Q(x, y, s - t)$.
\item $K_Q(x, y, t) = 0$ on the region $t < 0$.
\end{enumerate}
Remark 7.3. The property (i) means that $Q$ is time-translation invariant and $K_Q(x, t, y, 0) = k_Q(x, t, y, 0)$. The property (ii) implies that the value of $Qu(x, t)$ at a given time $t = t_0$ do not depend on the values of $u(x, t)$ at future times $t > t_0$, i.e., $Q$ satisfies the causality principle.

The Volterra ΨDO calculus aims at constructing a class of ΨDOs which is a natural receptacle for the inverse of the heat operator. The idea is to modify the classical ΨDO calculus in order to take into account:

(i) The aforementioned Volterra property.
(ii) The parabolic homogeneity of the heat operator $L + \partial_t$, i.e., the homogeneity with respect to the dilations,

$$\lambda.(\xi, \tau) := (\lambda \xi, \lambda^2 \tau) \quad \forall (\xi, \tau) \in \mathbb{R}^{n+1} \forall \lambda \in \mathbb{R}^*.$$

In the sequel, for $G \in S'(\mathbb{R}^{n+1})$ and $\lambda \neq 0$, we denote by $G_\lambda$ the distribution in $S'(\mathbb{R}^{n+1})$ defined by

$$\langle G_\lambda(\xi, \tau), u(\xi, \tau) \rangle := |\lambda|^{-(n+2)} \langle G(\xi, \tau), u(\lambda^{-1} \xi, \lambda^{-2} \tau) \rangle \quad \forall u \in S(\mathbb{R}^{n+1}).$$

Definition 7.4. A distribution $G \in S'(\mathbb{R}^{n+1})$ is (parabolic) homogeneous of degree $m$, $m \in \mathbb{Z}$, when

$$G_\lambda = \lambda^m G \quad \forall \lambda \in \mathbb{R} \setminus \{0\}.$$

In addition, we denote by $\mathbb{C}_-$ the complex halfplane $\{3 \tau < 0\}$ with closure $\overline{\mathbb{C}_-}$.

Lemma 7.5 ([BCS Prop. 1.9]). Let $q(\xi, \tau) \in C^\infty(\mathbb{R}^n \times \mathbb{R}) \setminus \{0\}$ be a parabolic homogeneous symbol of degree $m$ such that

(i) $q(\xi, \tau)$ extends to a continuous function on $(\mathbb{R}^n \times \mathbb{C}_-) \setminus \{0\}$ in such way to be holomorphic w.r.t. the variable $\tau$ when restricted to $\mathbb{C}_-$.

Then there is a unique $G \in S'(\mathbb{R}^{n+1})$ agreeing with $q$ on $\mathbb{R}^{n+1} \setminus \{0\}$ and such that

(ii) $G$ is homogeneous of degree $m$.

(iii) The inverse Fourier transform $\hat{G}(x, t)$ vanishes for $t < 0$.

Remark 7.6 (See [BCS]). The homogeneity of $q$ and $G$ implies that $G$ has the following homogeneity property:

$$\hat{G}_\lambda = |\lambda|^{-(n+2)} \lambda^{-m} G \quad \forall \lambda \in \mathbb{R}^*.$$

Let $U$ be an open subset of $\mathbb{R}^n$. We define Volterra symbols and Volterra ΨDOs on $U \times \mathbb{R}^{n+1} \setminus \{0\}$ as follows.

Definition 7.7. $S^m_V(U \times \mathbb{R}^{n+1})$, $m \in \mathbb{Z}$, consists of smooth functions $q(x, \xi, \tau)$ on $U \times \mathbb{R}^n \times \mathbb{R}$ with an asymptotic expansion $q(x, \xi, \tau) \sim \sum_{j \geq 0} q_m(x, \xi, \tau)$, where

- $q(x, \xi, \tau) \in C^\infty(U \times [\mathbb{R}^n \times \mathbb{R}] \setminus \{0\})$ is a homogeneous Volterra symbol of degree $l$, i.e. $q$ is parabolic homogeneous of degree $l$ and satisfies the property (i) in Lemma 7.5 with respect to the last $n+1$ variables.

- The sign $\sim$ means that, for all compacts $K \subset U$, integers $N$ and $k$ and multi-orders $\alpha$ and $\beta$, there is a constant $C_{NK\alpha\beta k} > 0$ such that

$$|D_\xi^a D_\tau^\beta q_m(x, \xi, \tau)| \leq C_{NK\alpha\beta k} (|\xi| + |\tau|)^{1/2} q_m(x, \xi, \tau) \quad \forall (x, \xi, \tau) \in K \times \mathbb{R}^n \times \mathbb{R} \text{ with } |\xi| + |\tau|^{1/2} \geq 1.$$

In the sequel, for a symbol $q(x, \xi, \tau) \in S^m_V(U \times \mathbb{R})$ we denote by $q(x, D_x, D_t)$ the operator from $C^\infty_c(U \times \mathbb{R})$ to $C^\infty(U \times \mathbb{R})$ defined by

$$q(x, D_x, D_t)u(x, t) := (2\pi)^{-(n+1)} \int \int e^{i(x+\xi+x\tau)} q(x, \xi, \tau) \tilde{u}(\xi, \tau) d\xi dt \quad \forall u \in C^\infty_c(U \times \mathbb{R}).$$

Definition 7.8. $\Psi^m_V(U \times \mathbb{R})$, $m \in \mathbb{Z}$, consists of continuous linear operators $Q$ from $C^\infty_c(U_x \times \mathbb{R}_x)$ to $C^\infty(U_x \times \mathbb{R}_x)$ such that
(i) $Q$ has the Volterra property in the sense of Definition 7.2.
(ii) $Q$ can be put in the form,
\begin{equation}
Q = q(x, D_x, D_t) + R,
\end{equation}
for some symbol $q(x, \xi, \tau) \in S^m_v(U \times \mathbb{R})$ and some smoothing operator $R$.

If $Q \in \Psi_v^m(U \times \mathbb{R})$, then there is a unique $K_Q(x, y, t) \in C^\infty(U, D'(U \times \mathbb{R}))$ such that
\[ Q u(x, s) = (K_Q(x, y, s - t), u(y, t)) \quad \forall u \in C^\infty(U \times \mathbb{R}). \]

In fact, if we put $Q$ in the form \((7.8)\) and we denote by $k_R(x, s, y, t)$ the Schwartz kernel of the smoothing operator $R$ given by \((7.4)\), then
\[ K_Q(x, y, t) = \hat{q}(x, x - y, t) + k_R(x, 0, y, t). \]

By abuse of language, we shall call $K_Q(x, y, t)$ the kernel of $Q$ (although the actual Schwartz kernel is $k_Q(x, s, y, t) := K_Q(x, y, s - t)$).

Example 7.9. Let $P$ be a differential operator of order $2$ on $U$ with principal symbol $p_2(x, \xi)$. Then the operator $P + \partial_t$ is a Volterra $\Psi DO$ of order $2$ with principal symbol $p_2(x, \xi) + i\tau$. In particular, if $p_2(x, \xi) > 0$ for all $(x, \xi) \in U \times (\mathbb{R}^n \setminus 0)$, then $p_2(x, \xi) + i\tau \neq 0$ for all $(x, \xi, \tau) \in U \times [(\mathbb{R}^n \times \mathbb{C}^\times \setminus 0)]$.

Other examples of Volterra $\Psi DO$s are given by the following.

Definition 7.10. Let $q_m(x, \xi, \tau) \in C^\infty(U \times (\mathbb{R}^{n+1} \setminus 0))$ be a homogeneous Volterra symbol of order $m$ and let $G_m(x, \xi, \tau) \in C^\infty(U, S'(\mathbb{R}^{n+1}))$ denote its unique homogeneous extension given by Lemma 7.2. Then
- $\hat{q}_m(x, y, t)$ is the inverse Fourier transform of $G_m(x, \xi, \tau)$ w.r.t. the last $n + 1$ variables.
- The operator $q_m(x, D_x, D_t) : C_c^\infty(U \times \mathbb{R}) \to C^\infty(U \times \mathbb{R})$ is defined by
\begin{equation}
q_m(x, D_x, D_t) u(x, s) := \langle \hat{q}_m(x, x - y, s - t), u(y, t) \rangle \quad \forall u \in C_c^\infty(U \times \mathbb{R}).
\end{equation}

Remark 7.11. It follows from the proof of [BGS, Prop. 1.9] that the homogeneous extension $G_m(x, \xi, \tau)$ depends smoothly on $x$, i.e., it belongs to $C^\infty(U, S'(\mathbb{R}^{n+1}))$.

Lemma 7.12. The operator $q_m(x, D_x, D_t)$ is a Volterra $\Psi DO$ of order $m$ with symbol $q \sim q_m$.

Sketch of Proof. Set $Q = q_m(x, D_x, D_t)$. Since $\hat{q}_m(x, y, t)$ belongs to $C^\infty(U, S'(\mathbb{R}^{n+1}))$, it follows from \((7.2)\) that the operator $q_m(x, D_x, D_t)$ is continuous and satisfies the Volterra property.

Denote by $G_m(x, \xi, \tau)$ the unique homogeneous extension of $q_m(x, \xi, \tau)$ given by Lemma 7.2.

In addition, let $\varphi \in C^\infty_c(\mathbb{R}^{n+1})$ be such that $\varphi(\xi, \tau) = 1$ near $(\xi, \tau) = (0, 0)$. Then the symbol $\varphi G_m(x, \xi, \tau) := (1 - \varphi(\xi, \tau)) q_m(x, \xi, \tau)$ lies in $S^0_v(U \times \mathbb{R}^{n+1})$ and we have
\[ K_Q(x, y, t) = (\varphi G_m)^\vee(x, y, t) + (\varphi G_m)^\vee(x, y, t). \]

Observe that $(\varphi G_m)^\vee(x, y, t)$ is smooth since this is the inverse Fourier transform of a compactly supported function. Thus $Q$ agrees with $\tilde{q}_m(x, D_x, D_t)$ up to a smoothing operator, and hence is a Volterra $\Psi DO$ of order $m$. Furthermore, it has symbol $\tilde{q}_m \sim q_m$. The lemma is proved. \qed

Proposition 7.13. \textbf{[Gr, Pd, BGS].} The following properties hold.

1. Pseudolocality. For any $Q \in \Psi_v^m(U \times \mathbb{R})$, the kernel $K_Q(x, y, t)$ is smooth on $\{ (x, y, t) \in M \times M \times \mathbb{R}; x \neq y \text{ or } t \neq 0 \}$.

2. Proper Support. For any $Q \in \Psi_v^m(U \times \mathbb{R})$ there exists $Q' \in \Psi_v^m(U \times \mathbb{R})$ such that $Q'$ is properly supported and $Q - Q'$ is a smoothing operator.

3. Composition. Let $Q_j \in \Psi_v^{m_j}(U \times \mathbb{R})$, $j = 1, 2$, have symbol $q_j$ and suppose that $Q_1$ or $Q_2$ is properly supported. Then $Q_1 Q_2$ lies in $\Psi_v^{m_1 + m_2}(U \times \mathbb{R})$ and has symbol $q_1 \# q_2 \sim \sum_j \frac{1}{m_j} \partial_j^m q_j \partial_j^m q_2$.

4. Parametrices. Any $Q \in \Psi_v^m(U \times \mathbb{R})$ admits a parametrix in $\Psi_v^{-m}(U \times \mathbb{R})$ if and only if its principal symbol is nowhere vanishing on $U \times [(\mathbb{R}^n \times \mathbb{C}^\times \setminus 0)]$.

5. Diffeomorphism Invariance. Let $\phi$ be a diffeomorphism from $U$ onto an open subset $V$ of $\mathbb{R}^n$. Then for any $Q \in \Psi_v^m(U \times \mathbb{R})$ the operator $(\phi \circ \text{id}_U), Q$ is contained in $\Psi_v^m(V \times \mathbb{R})$. 17
Remark 7.14. Most properties of Volterra ΨDOs can be proved in the same way as with classical ΨDOs or by observing that Volterra ΨDOs are ΨDOs of type \((\frac{1}{2}, 0)\) in the sense of [HS]. One important exception is the asymptotic completeness, i.e., given homogeneous Volterra symbols \(q_{m-j}\) of degree \(m-j\), \(j = 0, 1, \ldots\), there is a Volterra ΨDOs with symbol \(q \sim \sum q_{m-j}\). This property is a crucial ingredient in the parametrix construction. The point is that the Volterra property is not preserved by the multiplications by cut-off functions involved in the standard proof of the asymptotic completeness for classical ΨDOs (see [BGS] for a discussion on this point).

As usual with ΨDOs, the asymptotic expansion (7.4) for the symbol of a ΨDO can be translated in terms of an asymptotic expansion for the kernel of the ΨDO. For Volterra ΨDOs we have:

**Proposition 7.15 ([Gr, Pi] BGS).** Let \(Q \in \Psi^m_\omega(U \times \mathbb{R})\) and let \(q \sim \sum_{j \geq 0} q_{m-j}\) be its symbol. Then, for all \(N \in \mathbb{N}_0\), there is \(J \in \mathbb{N}\) such that

\[
K_Q(x, y, t) = \sum_{j \leq J} q_{m-j}(x, x - y, t) \mod C^N(U \times U \times \mathbb{R}).
\]

**Sketch of Proof.** As Volterra ΨDOs are ΨDOs of type \((\frac{1}{2}, 0)\), the kernel of a Volterra ΨDO of order \(\leq -(n + 2 + 2N)\) is \(C^N\) (see [HS]). Let us choose \(J\) so that \(m - J \leq -(n + 2 + 2N)\), then \(Q - \sum_{j \leq J} q_{m-j}(x, D_x, D_t)\) is a Volterra ΨDOs with symbol \(q' \sim \sum_{j \geq J+1} q_{m-j}\), and hence it has order \(m - J - 1 \leq -(n + 2 + 2N)\). Therefore, its kernel is \(C^N\). This proves the result. \(\square\)

The invariance property in Proposition 7.13 enables us to define Volterra ΨDOs on the manifold \(M \times \mathbb{R}\) and acting on the sections of the vector bundle \(E\) (seen as a vector bundle over \(M \times \mathbb{R}\)). All the aforementioned properties hold verbatim in this context. We shall denote by \(\Psi^m_\omega(M \times \mathbb{R}, E)\) the space of Volterra ΨDOs of order \(m\) on \(M \times \mathbb{R}\).

If \(Q \in \Psi^m_\omega(M \times \mathbb{R}, E)\), then there is a unique \(K_Q(x, y, t) \in C^\infty(M \times \mathbb{R}) \otimes \mathcal{D}'(M, E)\) such that

\[
Q u(x, s) = \langle K_Q(x, y, s-t), u(y, t) \rangle \quad \forall u \in C^\infty_c(M \times \mathbb{R}, E).
\]

We shall refer to \(K_Q(x, y, t)\) as the kernel of \(Q\).

Proposition 7.13 ensures us that \(K_Q(x, y, t)\) is smooth for \(t \neq 0\). Therefore, on \(M \times M \times \mathbb{R}^+\) we may regard \(K_Q(x, y, t)\) as a smooth function section of \(E \boxtimes E^*\) over \(M \times M \times \mathbb{R}^+\) such that

\[
\langle K_Q(x, y, t), u(y, t) \rangle = \int_{M \times \mathbb{R}} K_Q(x, y, t) u(y, t) dy dt \quad \forall u \in C^\infty_c(M \times \mathbb{R}^+, E),
\]

where in the l.h.s. \(K_Q(x, y, t)\) is meant as an element of \(C^\infty(M \times \mathbb{R}) \otimes \mathcal{D}'(M, E)\) and in the r.h.s. it is meant as a smooth section of \(E \boxtimes E^*\).

It follows from Example 7.9 and Proposition 7.13 that the heat operator \((L + \partial_t)^{-1}\) admits a (properly supported) parametrix in \(\Psi^2(M \times \mathbb{R}, E)\). Comparing such a parametrix with the inverse \(Q_0 = (L + \partial_t)^{-1}\) defined by (7.2) and using (7.5) we obtain

**Proposition 7.16 ([Gr, Pi], BGS pp. 363-362]).** The operator \((L + \partial_t)^{-1}\) defined by (7.2) is a Volterra ΨDO of order \(-2\). Moreover,

\[
k_t(x, y) = K_{(L+\partial_t)^{-1}}(x, y, t) \quad \forall t > 0.
\]

This result provides us with a representation of the heat kernel as the (Volterra) kernel of a Volterra ΨDO. Combining it with (7.10) enables us to get a precise description of the asymptotic of \(k_t(x, x)\) as \(t \to 0^+\) (see [Gr, HBCS]; see also the next section).

More generally, we have

**Proposition 7.17.** Let \(P : C^\infty(M, E) \to C^\infty(M, E)\) be a differential operator of order \(m\). For \(t > 0\) denote by \(h_t(x, y)\) the kernel of \(Pe^{-tL}\) defined as in (7.4). Then

\[
h_t(x, y) = K_{P(L+\partial_t)^{-1}}(x, y, t) \quad \forall t > 0.
\]

**Proof.** We have \(h_t(x, y) = P \ast k_t(x, y) = P \ast K_{(L+\partial_t)^{-1}}(x, y, t) = K_{P(L+\partial_t)^{-1}}(x, y, t)\). \(\square\)

**Remark 7.18.** The operator \(P(L + \partial_t)^{-1}\) is a Volterra ΨDO of order \(m - 2\).
8. Equivariant Heat Kernel Asymptotics

In this section we keep the same notation as in the previous section. In addition, we let \( \phi : M \to M \) be an isometric diffeomorphism of \((M,g)\) which lifts to a unitary vector bundle isomorphism \( \phi^E : E \to \phi_* E \), i.e., a unitary section of \( \text{Hom}(E,\phi_* E) \). Then \( \phi \) defines a unitary operator \( U_\phi : L^2(M,E) \to L^2(M,E) \) by

\[
U_\phi u(x) = \phi^E (\phi^{-1}(x)) u (\phi^{-1}(x)) \quad \forall u \in L^2(M,E).
\]

Our aim in this section is to derive short-time asymptotic for equivariant traces \( \text{Tr} [P e^{-tL} U_\phi] \), where \( P : C^\infty(M,E) \to C^\infty(M,E) \) is any differential operator.

For \( t > 0 \) denote by \( h_t(x,y) \) the kernel of \( P e^{-tL} \) as defined in (7.4). Observe that the kernel of \( P e^{-tL} U_\phi \) is \( h_t(x,\phi(y)) \phi^E(x) \), and hence

\[
(8.1) \quad \text{Tr} [P e^{-tL} U_\phi] = \int_M \text{tr}_E [h_t(x,\phi(x)) \phi^E(x)] |dx| = \int_M \text{tr}_E [\phi^E(x) h_t(x,\phi(x))] |dx|.
\]

We are thus led to look for the short-time behavior of \( h_t(x,\phi(x)) \). Since by Proposition 7.14 we can represent \( h_t(x,y) \) as the kernel of a Volterra PDO, we shall more generally study the small time behavior of \( K_Q(x,\phi(x),t) \), where \( Q \in \Psi^m_v(M \times \mathbb{R}) \), \( m \in \mathbb{Z} \).

In the sequel, we denote by \( M^\phi \) the fixed-point set of \( \phi \), and for \( a = 0, \ldots, n \), we let \( M^\phi_a \) be the subset of \( M^\phi \) consisting fixed-point \( x \) at which \( \phi'(x) - 1 \) has rank \( n - a \), i.e., the eigenvalue 1 of \( \phi'(x) \) has multiplicity \( a \). Therefore, we have the disjoint-sum decomposition,

\[
M^\phi = \bigsqcup_{0 \leq a \leq n} M^\phi_a.
\]

In addition, we pick some \( \epsilon_0 \in (0,\rho_0) \), where \( \rho_0 \) is the injectivity radius of \((M,g)\).

Let \( x_0 \) be a point in some component \( M^\phi_a \). Denote by \( B_{\epsilon_0}(x_0) \) the ball of radius \( \epsilon_0 \) around the origin in \( T_{x_0}M \). Then \( \exp_{x_0} \) induces a diffeomorphism from \( B_{\epsilon_0}(x_0) \) onto an open neighborhood \( U_{\epsilon_0} \) of \( x_0 \) in \( M \). Moreover, as \( \phi \) is an isometry, for all \( x \in B_{\epsilon_0}(x_0) \), we have

\[
(8.2) \quad \phi (\exp_{x_0}(X)) = \exp_{\phi(x_0)} (\phi'(x_0)X) = \exp_{x_0} (\phi'(x_0)X).
\]

Thus under \( \exp_{x_0}|_{B_{\epsilon_0}(x_0)} \) the diffeomorphism \( \phi \) corresponds to \( \phi'(x_0) \), and hence \( M^\phi \cap U_{\epsilon_0} \) is identified with \( B_{\epsilon_0}(x_0) := B_{\epsilon_0}(x_0) \cap \ker(\phi'(x_0) - 1) \). Incidentally, the tangent bundle \( TM^\phi \mid_{M^\phi \cap U_{\epsilon_0}} \) and the normal bundle \( \left( TM^\phi \right)^\perp \mid_{M^\phi \cap U_{\epsilon_0}} \) are identified with \( B_{\epsilon_0}(x_0) \times \ker(\phi'(x_0) - 1) \) and \( B_{\epsilon_0}(x_0) \times \ker(\phi'(x_0) - 1)^\perp \) respectively. Notice also that when \( k = 0 \) this shows that \( x_0 \) is an isolated fixed-point.

It follows from this that each component \( M^\phi_a \) is a (closed) submanifold of dimension \( a \) of \( M \) and over \( M^\phi_a \) the set \( N^\phi := \cup_{a \in M^\phi} \ker(\phi'(x) - 1) \) can be organized as a smooth vector bundle. We denote by \( \pi : N^\phi \to M^\phi \) the corresponding canonical map. We shall refer to \( N^\phi \) as the normal bundle of \( M^\phi \). Notice that \( \phi' \) induces (over each component \( M^\phi_a \)) an isometric vector bundle isomorphism of \( N^\phi \) on itself.

As it is well known, using \( N^\phi \) we can construct a tubular neighborhood of \( M^\phi \) as follows. Let \( N^\phi(x_0) \) be the ball bundle of \( N^\phi \) of radius \( \epsilon_0 \) around the zero-section. Then the map \( N^\phi(x_0) \ni X \mapsto \exp_{x_0}(X) \) is a homeomorphism from \( N^\phi(x_0) \) onto an open tubular neighborhood \( V_{\epsilon_0} \) of \( M^\phi \) in \( M \). Moreover, if \( M^\phi_a \) is a connected component of \( M^\phi \), then this map induces a diffeomorphism from \( N^\phi(x_0)|_{M^\phi_a} \) onto its image.

Let us fix some \( \epsilon \in (0,\epsilon_0) \) and let \((x,t) \in M^\phi \times (0,\infty) \). Observe that, in view of (8.2), for all \( v \in N^\phi_\epsilon(x) \), we have

\[
K_Q (\exp_x v, \exp_x (\phi'(x)v), t) = K_Q (\exp_x v, \phi(\exp_x v), t).
\]

For \( x \in M^\phi \) and \( t > 0 \) set

\[
I_Q(x,t) := \phi^E(x)^{-1} \int_{N^\phi_\epsilon(x)} \phi^E(\exp_x v) K_Q (\exp_x v, \exp_x (\phi'(x)v), t) \, dv.
\]

This defines a smooth section of \( \text{End} \, E \) over \( M^\phi \times (0,\infty) \), since \( \phi^E(x) \in \text{End} \, E_x \) for all \( x \in M^\phi \).
In the sequel, we say that a function \( f(t) \) is \( O(t^\infty) \) as \( t \to 0^+ \) when \( f(t) \) is \( O(t^N) \) for all \( N \in \mathbb{N} \).

**Lemma 8.1.** As \( t \to 0^+ \),
\[
\int_M \text{tr}_E \left[ \phi^E(x)K_Q(x,\phi(x),t) \right] \, dx = \int_{M^\phi} \text{tr}_E \left[ \phi^E(x)I_Q(x,t) \right] \, dx + O(t^\infty).
\]

**Proof.** If we regard \( K_Q(x,y,t) \) as a distributional section of \( E \otimes E^* \) over \( M \times M \times \mathbb{R} \), then Proposition 7.14 tells us that \( K_Q(x,y,t) \) is smooth on \( \{(x,y,t) \in M \times M \times \mathbb{R} ; \, x \neq y \} \). Incidentally, \( K_Q(x,\phi(x),t) \) is smooth on \( (M \setminus M^\phi) \times \mathbb{R} \). Let \( N \in \mathbb{N} \). Since \( K_Q(x,y,t) = 0 \) for \( t < 0 \), we see that \( \partial_t^N K_Q(x,\phi(x),0) = 0 \) for all \( x \in M \setminus M^\phi \). The Taylor formula at \( t = 0 \) then implies that, uniformly on compact subsets of \( M \setminus M^\phi \),
\[
K_Q(x,\phi(x),t) = O(t^N) \quad \text{as } t \to 0^+.
\]

As \( M \) is compact and \( V_c \) is an open neighborhood of \( M^\phi \), the complement \( M \setminus V_c \) is a compact subset of \( M \setminus M^\phi \). Thus,
\[
\int_M \text{tr}_E \left[ \phi^E(x)K_Q(x,\phi(x),t) \right] \, dx = \int_{V_c} \text{tr}_E \left[ \phi^E(x)K_Q(x,\phi(x),t) \right] \, dx + O(t^N)
\]
\[
= \int_{M^\phi} \left( \int_{N^\phi(c)} \text{tr}_E \left[ \phi^E(\exp_y(v))K_Q(\exp_y(v),\phi(\exp_y(v)),t) \right] \, dv \right) \, dx + O(t^N)
\]
\[
= \int_{M^\phi} \text{tr}_E \left[ \phi^E(x)I_Q(x,t) \right] \, dx + O(t^N).
\]
This proves the lemma. \( \square \)

Thanks to this lemma we are led to study the small-time behavior of \( I_Q(x,t) \). Notice this is a purely local issue and \( I_Q(x,t) \) depends on \( \epsilon \) only up to \( O(t^\infty) \) near \( t = 0 \). Therefore, upon choosing \( \epsilon_0 \) small enough so that there is a local trivialization of \( E \) over the tubular neighborhood \( V_{\epsilon_0} \), we may assume that \( E \) is a trivial vector bundle.

Given a fixed-point \( x_0 \) in a component \( M^\phi_0 \), consider some local coordinates \( x = (x^1, \ldots, x^a) \) around \( x_0 \). Setting \( b = n - a \), we may further assume that over the range of the domain of the local coordinates there is an orthonormal frame \( e_1(x), \cdots, e_b(x) \) of \( N^\phi \). This defines fiber coordinates \( v = (v^1, \cdots, v^b) \). Composing with the map \( N^\phi(\epsilon_0) \ni (x,v) \rightarrow \exp_x v \) we then get local coordinates \( x^1, \ldots, x^a, v^1, \cdots, v^b \) for \( M \) near the fixed-point \( x_0 \). We shall refer to this type of coordinates as **tubular coordinates**.

Let \( q(x,v; \xi,\nu;\tau) := \sum_{j \geq 0} q_{m-j}(x,v;\xi,\nu;\tau) \) be the symbol \( Q \) in these tubular coordinates. We denote by \( K_Q(x,v;\nu;\tau) \) the kernel of \( Q \) in these coordinates. In the local coordinates \( x^1, \ldots, x^a \) we have
\[
I_Q(x,t) = \int_{|v|<\epsilon} \phi^E(x,0)^{-1}\phi^E(x,v)K_Q(x,v;\phi(x)v) \, dv,
\]
where \( \phi^E(x,v) \) is \( \phi^E \) in the tubular coordinates \( (x,v) \).

In the sequel we denote by \( U \) the range of the coordinates \( x = (x^1, \ldots, x^a) \) and by \( B(\epsilon_0) \) (resp., \( B(\epsilon) \)) the open ball about the origin in \( \mathbb{R}^b \) with radius \( \epsilon_0 \) (resp., \( \epsilon \)). Notice that the range of \( v = (v^1, \cdots, v^b) \) is \( B(\epsilon_0) \). In addition, for \( j = 0,1,\ldots \) we set
\[
q_{m-j}^E(x,v;\xi,\nu;\tau) := \phi^E(x,0)^{-1}\phi^E(x,v)q_{m-j}(x,v;\xi,\nu;\tau).
\]

**Lemma 8.2.** As \( t \to 0^+ \) and uniformly on compact subsets of \( U \),
\[
I_Q(x,t) \sim \sum_{j \geq 0} \int_{|v|<\epsilon} \left( q_{m-j}^E \right)^v(x,v;0,1-\phi(x)v) \, dv.
\]

**Proof.** Let \( N \in \mathbb{N}_0 \). By Proposition 7.14 there is \( J \in \mathbb{N} \) such that \( K_Q - \sum_{j \leq J} q_{m-j} \) is \( C^N \). Set
\[
R_N(x,v,t) := K_Q(x,v;\phi(x)v) - \sum_{j \leq J} q_{m-j}(x,v;0,1-\phi(x)v) \, dt.
\]
Then \( R_N(x, v, t) \) is \( C^N \) on \( U \times B(\epsilon_0) \times \mathbb{R} \). Moreover \( R_N(x, v, t) = 0 \) for \( t < 0 \), since \( K_Q(x, v, y, w; t) \) and all the \( \tilde{q}_{m-j}(x, v, y, w; t) \) vanish for \( t < 0 \). This implies that \( \partial^j R_N(x, v, 0) = 0 \) for all \( j \leq N \). Applying Taylor’s formula at \( t = 0 \) to \( R_N(x, v, t) \) then shows that, as \( t \to 0^+ \) and uniformly on compact subsets of \( U \times B(\epsilon_0) \), the function \( R_N(x, v, t) \) is \( O(t^N) \), that is,

\[
K_Q(x, v; x, \phi'(x); t) = \sum_{j \leq J} \tilde{q}_{m-j}(x, v; 0, (1 - \phi'(x))v; t) + O(t^N).
\]

Therefore, uniformly on compact subsets of \( U \),

\[
I_Q(x, t) = \sum_{j \leq J} \int_{|v| < \epsilon} (\tilde{q}_{m-j}^E(v)) (x, v; 0, (1 - \phi'(x))v; t) dv + O(t^N).
\]

This gives the lemma. \( \square \)

**Lemma 8.3.** As \( t \to 0^+ \) and uniformly on compact subsets of \( U \),

\[
(8.5) \quad \int_{|v| < \epsilon} (\tilde{q}_{m-j}^E(v)) (x, v; 0, (1 - \phi'(x))v; t) dv \\
\sim \sum_{|a| + j + m + n \text{ even}} t^{-(m+n+1)/2} \int_{\mathbb{R}^d} \frac{v^a}{\alpha!} (\partial_c^a \tilde{q}_{m-j}^E(v)) (x, 0; 0, (1 - \phi'(x))v; 1) dv.
\]

**Proof.** Let \( h(x, v, w, t) \) be the function on \( U \times B(\epsilon_0) \times \mathbb{R} \) defined by

\[
h(x, v, w, t) := (\tilde{q}_{m-j}^E(v)) (x, v; 0, (1 - \phi'(x))v; t) dv.
\]

We observe that \( h(x, v, w, t) \) is smooth on \( U \times B(\epsilon_0) \times \mathbb{R} \) and vanishes for \( t < 0 \). Moreover, the homogeneity of \( \tilde{q}_{m-j} \) in the sense of \( \tilde{Z}_L \) implies that

\[
(8.6) \quad h(x, v, w, \lambda t^2) = |\lambda|^{-(n+2)} \lambda^{-m} h(x, v, w, t) \quad \forall \lambda \in \mathbb{R}^*.
\]

Setting \( k = j - (m + n + 2) \), this implies that, for all \( t > 0 \),

\[
(8.7) \quad \int_{|v| < \epsilon} h(x, v, w, t) dv = t^k \int_{B(\epsilon)} h(x, \sqrt{t} v, \sqrt{t} v, t) dv = t^{k+\frac{1}{2}} \int_{B(\epsilon)} h(x, \sqrt{t} v, v, 1) dv.
\]

Let \( N \in \mathbb{N} \). By Taylor’s formula,

\[
(8.8) \quad h(x, \sqrt{t} v, v, 1) = \sum_{|a|<N} \frac{v^a}{\alpha!} \partial_c^a h(x, 0, v, 1) + t^{\frac{1}{2}} R_N(x, \sqrt{t} v, v),
\]

where \( R_N(x, v, w) \) is the function on \( U \times B(\epsilon_0) \times \mathbb{R}^d \) defined by

\[
R_N(x, v, w) = \sum_{|a|<N} \int_0^1 (1 - s)^{N-1} w^a \partial_c^a h(x, s v, w, 1) ds.
\]

Let \( K \) be a compact subset of \( U \). As \( w^a \partial_c^a h(x, v, w, t) \) is smooth on \( U \times B(\epsilon_0) \times \mathbb{R} \) and vanishes for \( t < 0 \), we see that \( w^a \partial_c^a h(x, v, w, 0) = 0 \) for all \( l \in \mathbb{N}_0 \). Therefore, using once more Taylor’s formula around \( t = 0 \) shows that, for any \( l \in \mathbb{N}_0 \), there is a constant \( C_{K\alpha l} > 0 \) such that

\[
|w^a \partial_c^a h(x, v, w, t)| \leq C_{K\alpha l} |t|^l \quad \forall (x, v, w, t) \in K \times B(\epsilon) \times \mathbb{R}^{l+1} \times (0, 1).
\]

In addition, the homogeneity of \( h(x, v, w, t) \) implies that, when \( w \neq 0 \),

\[
w^a \partial_c h(x, v, w, 1) = w^a |w|^b \partial_c h(x, v, |w|^{-1} w, |w|^{-2}).
\]

Thus,

\[
|w^a \partial_c^a h(x, v, w, 1)| \leq C_{K\alpha l} |w|^{l+|a|-2l} \quad \forall (x, v, w) \in K \times B(\epsilon) \times (\mathbb{R}^d \setminus 0).
\]

The above estimate shows that \( w^a \partial_c h(x, v, w, 1) \) has rapid decay in \( w \) uniformly with respect to \( x \) and \( v \), as \( x \) ranges over \( K \) and \( v \) ranges over \( B(\epsilon) \). Incidentally, both \( w^a \partial_c h(x, w, w, 1) \) and \( R_N(x, v, w) \) are uniformly bounded on \( K \times B(\epsilon) \times \mathbb{R}^d \). It then follows that there is a constant \( C_{KN} > 0 \) such that

\[
|R_N(x, v, t)| \leq C_{KN} \quad \forall (x, v) \in K \times B(\epsilon).
\]

\[\square\]
Therefore, integrating both sides of (8.3) with respect to \( v \) over \( B \left( \frac{Q}{2} \right) \) we see that, as \( t \to 0^+ \) and uniformly on \( K \),

\[
\int_{B \left( \frac{Q}{2} \right)} h(x, \sqrt{t}v, v, 1) dv = \sum_{|\alpha| < N} t^{\frac{|\alpha|}{2}} \int_{B \left( \frac{Q}{2} \right)} \frac{v^\alpha}{\alpha!} \partial_\alpha^v h(x, 0, v, 1) dv + O \left( t^{\frac{|\alpha|}{2}} \right).
\]

Together with (8.7) this proves that, as \( t \to 0^+ \) and uniformly on \( K \),

\[
(8.9) \quad \int_{|v| < \varepsilon} h(x, v, v, t) dv \sim \sum_{|\alpha| < N} t^{\frac{|\alpha|}{2}} \int_{B \left( \frac{Q}{2} \right)} \frac{v^\alpha}{\alpha!} \partial_\alpha^v h(x, 0, v, 1) dv.
\]

We observe that \( k + b = j - (m + a + 2) \). Moreover, as mentioned above, the function \( v^\alpha \partial_\alpha^v h(x, 0, v, 1) \) has rapid decay uniformly with respect to \( x \), as \( x \) ranges over \( K \). Therefore, as \( t \to 0^+ \) and uniformly on \( K \),

\[
(8.10) \quad \int_{B \left( \frac{Q}{2} \right)} v^\alpha \partial_\alpha^v h(x, 0, v, 1) dv = \int_{\mathbb{R}^b} v^\alpha \partial_\alpha^v h(x, 0, v, 1) dv + O(t^\infty).
\]

In addition, the homogeneity property (8.6) for \( \lambda = -1 \) gives

\[
\int_{\mathbb{R}^b} v^\alpha \partial_\alpha^v h(x, 0, v, 1) dv = \int_{\mathbb{R}^b} (v)^{\alpha} \partial_\alpha^v h(x, 0, v, 1) dv = (-1)^{|\alpha|+j-m} \int_{\mathbb{R}^b} v^\alpha \partial_\alpha^v h(x, 0, v, 1) dv.
\]

Thus \( \int_{\mathbb{R}^b} v^\alpha \partial_\alpha^v h(x, 0, v, 1) dv = 0 \) whenever \( |\alpha| + j - m \) is odd. Combining this with (8.9) and (8.10) shows that, as \( t \to 0^+ \) and uniformly on \( K \),

\[
\int_{|v| < \varepsilon} h(x, v, v, t) dv \sim \sum_{|\alpha| + j - m \text{ even}} t^{\frac{|\alpha|}{2}+j-1} \int_{B \left( \frac{Q}{2} \right)} \frac{v^\alpha}{\alpha!} \partial_\alpha^v h(x, 0, v, 1) dv.
\]

This proves the lemma. \( \square \)

Combining Lemma 8.2 and Lemma 8.3 we see that, as \( t \to 0^+ \) and uniformly on compact subsets of \( U \),

\[
(8.11) \quad I_Q(x, t) \sim \sum_{|\alpha| + j - m \text{ even}} t^{\frac{|\alpha|}{2}+j-1} \int_{\mathbb{R}^b} \frac{v^\alpha}{\alpha!} \left( \partial_\alpha^v q^E \right)^\lor (x, 0; 0, (1 - \phi'(x))v; 1) dv.
\]

If \( |\alpha| + j - m \) is even, then \( \frac{(m + a + 2) + j + |\alpha|}{2} \) is an integer and is greater than \(-\frac{m+a}{2} - 1\), i.e., it is greater than or equal to \(-\frac{m+a}{2} - 1\). We actually have an equality when \( j = |\alpha| = 0 \) and \( m \) is even and when \( |\alpha| + j = 1 \) and \( m \) is odd. Therefore, grouping together all the terms with same powers of \( t \), we can rewrite the above asymptotic in the form,

\[
I_Q(x, t) \sim \sum_{j \geq 0} t^{-\left( \frac{|\alpha|}{2}+1 \right)+1} I_Q^{(j)}(x),
\]

where

\[
(8.12) \quad I_Q^{(j)}(x) := \sum_{|\alpha| \leq m - 2j + |\alpha|} t^{\frac{|\alpha|}{2}+j} \int_{\mathbb{R}^b} \partial_\alpha^v q^E (x, 0; 0, (1 - \phi'(x))v; 1) dv.
\]

Therefore, we obtain

**Proposition 8.4.** Let \( Q \in \Psi^m_v(M \times \mathbb{R}, E), m \in \mathbb{Z} \). Uniformly on each component \( M^\phi_a \),

\[
(8.13) \quad I_Q(x, t) \sim \sum_{j \geq 0} t^{-\left( \frac{|\alpha|}{2}+1 \right)+j} I_Q^{(j)}(x) \quad \text{as } t \to 0^+,
\]

where \( I_Q^{(j)}(x) \) is the section of \( \text{End} E \) over \( M^\phi_a \) defined by (8.12) in terms of the symbol of \( Q \) in local tubular coordinates over which \( E \) is trivial.

22
Remark 8.5. On $M_\phi^n$ the leading term in (8.13) is $t^{-(\frac{m}{2}+1)} I_Q^{(0)}(x)$, where $I_Q^{(0)}(x)$ depends only on the principal symbol of $Q$. Namely,

$$I_Q^{(0)}(x) = \int_{\mathbb{R}^n} (\bar{d}_m)^\top (x,0;0,(1-\phi'(x))v;1)dv = |1-\phi'(x)|^{-1} \int_{\mathbb{R}^n} \bar{q}_m(x,0;0,v;1)dv.$$

Remark 8.6. The asymptotic (8.13) is expressed in terms of the symbol of $Q$ in tubular coordinates. However, we usually start with a symbol in some local coordinates before passing to tubular coordinates. We determine the symbol in the tubular coordinates by applying the change of variable formula for symbols for the change of variable $\psi(x,v) = \exp_v(v)$ (see, e.g., [10]).

We are now in a position to prove the main result of this section.

**Proposition 8.7.** Let $P : C^\infty(M,E) \to C^\infty(M,E)$ be a differential operator of order $m$.

1. Uniformly on each component $M_\phi^n$,

$$I_{P(L+\partial_t)}(x,t) \sim \sum_{j \geq 0} t^{-\left(\frac{m}{2}+\frac{|\phi|}{2}\right)+j} I_{P(L+\partial_t)}^{(j)}(x) \quad \text{as } t \to 0^+,$$

where $I_{P(L+\partial_t)}^{(j)}(x)$ is the section of $\text{End } E$ over $M_\phi$ defined by (8.12) in terms of the symbol of $P(L+\partial_t)^{-1}$ in any tubular coordinates over which $E$ is trivial.

2. As $t \to 0^+$,

$$\text{Tr} \left[ P e^{-tL} U_\phi \right] = \int_{M^n} \text{tr}_E \left[ \phi^E(x) I_{P(L+\partial_t)^{-1}}(x,t) \right] dx + O(t^\infty)$$

$$\sim \sum_{0 \leq a \leq n} \sum_{j \geq 0} t^{-\left(\frac{m}{2}+\frac{|\phi|}{2}\right)+j} \int_{M_\phi^n} \text{tr}_E \left[ \phi^E(x) I_{P(L+\partial_t)^{-1}}^{(j)}(x) \right] dx.$$

**Proof.** The first part is an immediate consequence of Proposition 8.4 since $P(L+\partial_t)^{-1}$ is a Volterra $\Psi DO$ of order $m - 2$. Combining Proposition 8.4 and 8.5 shows that

$$\text{Tr} \left[ P e^{-tL} U_\phi \right] = \int_{M^n} \text{tr}_E \left[ \phi^E(x) K_{P(L+\partial_t)^{-1}}(x,\phi(x),t) \right].$$

The 2nd part then follows from Lemma 8.1 and the first part. The proof is complete. \qed

**Remark 8.8.** The formula (8.12) expresses the coefficients $I_{P(L+\partial_t)^{-1}}^{(j)}(x)$ in terms of the homogeneous components of the symbol of $P(L+\partial_t)^{-1}$. Therefore, in order to compute them at a given point $x_0 \in M_\phi$, we may replace $P(L+\partial_t)^{-1}$ by $PQ$, where $Q$ is Volterra $\Psi DO$ parametrix of $L+\partial_t$ defined near $x_0$. In particular,

$$I_{P(L+\partial_t)}^{(j)}(x_0,t) = I_{PQ}(x_0,t) + O(t^\infty).$$

9. The Local Equivariant Index Theorem

In this section, we shall give a new proof of the local equivariant index theorem of Patodi [Pa], Donnelly-Patodi [DP] and Gilkey [Gi]. As an immediate by-product of this proof, we will get a proof of Proposition 8.5 which is the key technical result in the computation of the Connes-Chern character of the conformal Dirac twisted spectral triple in Section 8.

Let $(M^n, g)$ be an even dimensional compact spin oriented Riemannian manifold. As before we denote by $\mathcal{D}_g : C^\infty(M,\mathcal{S}) \to C^\infty(M,\mathcal{S})$ the Dirac operator acting on the spinor bundle $\mathcal{S} = \mathcal{S}^+ \oplus \mathcal{S}^-$. In addition, we let $\phi : M \to M$ be a smooth isometry of $(M,g)$ preserving the orientation and the spin structure and which lies in the identity component of the group of all such diffeomorphisms. Then $\phi \in G$ has a unique lift to a unitary bundle isomorphism $\phi^\mathcal{S} : \mathcal{S} \to \phi_*\mathcal{S}$.

As in the previous section, we shall denote by $M^\phi$ the fixed-point set of $\phi$ and by $N^\phi$ the normal bundle of $M^\phi$. We also denote by $R^{TM^\phi}$ and $R^N$ the respective curvatures of the connections on $M^\phi$ and $N^\phi$ induced by the Levi-Civita connection of $M$. Moreover, we orient $M^\phi$ and $N^\phi$ in the same way as in Section 8. In particular, $\phi^\mathcal{S}$ defines a positive section of $\Lambda^{n-a}(N^\phi)^+$ over each component $M_\phi^n$. 

23
Our aim is to give a new proof of the following.

**Theorem 9.1 (Local Equivariant Index Theorem).** Let \( f \in C^\infty(M) \). As \( t \to 0^+ \),

\[
\text{Str} \left[ e^{-i\phi^2} U_\phi \right] = (-i)^{\frac{n}{2}} \sum_a (2\pi)^{-\frac{n}{2}} \int_{\mathcal{M}_a^g} f(x) \hat{A}(R^{T M^g}) \wedge \nu_\phi \left( R^{N^g} \right) + O(t),
\]

where \( \hat{A}(R^{T M^g}) \) and \( \nu_\phi \left( R^{N^g} \right) \) are defined as in (6.3).

This result is originally due to Patodi [P], Donnelly-Patodi [DP] and Gilkey [G]. Their arguments partly involved Riemannian invariant theory. More analytical proofs were later provided by Bismut [B], Berline-Vergne [BV] [BGV] and Lafferty-Yu-Zhang [LYZ]. We also mention that Liu-Ma [LM] proved a version of this result for families of Dirac operators.

Let us briefly recall how the local equivariant index theorem implies the equivariant index theorem of Atiyah-Segal-Singer [AS, ASi2]. The equivariant index of \( D \) by Bismut [Bi], Berline-Vergne [BV, BGV] and Lafferty-Yu-Zhang [LYZ]. We also mention that arguments partly involved Riemannian invariant theory. More analytical proofs were later provided by Bismut [B], Berline-Vergne [BV] [BGV] and Lafferty-Yu-Zhang [LYZ]. We also mention that Liu-Ma [LM] proved a version of this result for families of Dirac operators.

**Theorem 9.2 (Equivariant Index Theorem [AS, ASi2]).** We have

\[
\text{ind} D_\phi(\phi) := \text{Tr} U_\phi|_{\ker D^+_\phi} - \text{Tr} U_\phi|_{\ker D^-_\phi}.
\]

By the equivariant version of the McKean-Singer formula (see, e.g., [BGV Prop. 6.3]), it holds that

\[
\text{ind} D_\phi(\phi) = \text{Str} \left[ e^{-i\phi^2} U_\phi \right] \forall t > 0.
\]

Therefore, from the local equivariant index theorem we obtain

**Theorem 9.3 (Local Equivariant Index Theorem, Pointwise Version).** Let \( x_0 \in M^\phi \). Then, as \( t \to 0^+ \), we have

\[
\text{Str} \left[ \phi^g(x_0) I_{(\mathcal{B}_a^g + \partial_0)^{-1}}(x_0, t) \right] = (-i)^{\frac{n}{2}} (2\pi)^{-\frac{n}{2}} \int_{\mathcal{M}_a^g} \hat{A}(R^{T M^g}) \wedge \nu_\phi \left( R^{N^g} \right) \bigg|^{(0)}(x_0) + O(t).
\]

This result is actually an equivalent reformulation of Theorem 9.1 for we have

**Lemma 9.4.** Theorem 9.1 and Theorem 9.3 are equivalent.

**Proof.** For \( j = 0, 1, \ldots \) and \( x \in M^\phi \) set \( A_j(x) = \text{Str} \left[ \phi^g(x) f^{(j)}_{(\mathcal{B}_a^g + \partial_0)^{-1}}(x) \right] \). It follows from Proposition 8.7 that, as \( t \to 0^+ \), it holds that

- For all \( f \in C^\infty(M) \),

\[
\text{Str} \left[ e^{-i\phi^2} U_\phi \right] \sim \sum_{0 \leq a \leq n} \sum_{j \geq 0} t^{-\frac{n}{2} + j} \int_{\mathcal{M}_a^g} f(x) A_j(x) dx.
\]
- For all $x \in M^{\phi}_a$, $a = 0, 2, \cdots, n$,

$$\text{Str} \left[ \phi^g \left( x \right) I_{(\partial^2 + \partial_v)^{-1}} \left( x, t \right) \right] \sim \sum_{j \geq 0} t^{-\frac{\delta}{2} - j} A_j(x).$$

It then follows that both asymptotics (9.1) and (9.2) are equivalent to the following: for all $x \in M^{\phi}_a$, $a = 0, 2, \ldots, n$, it holds that

$$A_0(x) = \cdots = A_{\frac{\delta}{2} - 1}(x) = 0 \quad \text{and} \quad A_{\frac{\delta}{2}}(x) = (-i)^{\frac{\delta}{2}}(2\pi)^{-\frac{\delta}{2}} \left| \hat{A}(R^{TM^a}) \wedge \nu_0 \left( R^{N^a} \right) \right|^{(a)}(x).$$

Whence the lemma. \(\square\)

We are thus reduced to proving the asymptotic (9.2) for any given point $x_0 \in M^{\phi}_a$. In view of Remark 8.8, given any Volterra $\Psi$DO parametrix $Q$ defined near $x_0$, we have

$$(9.3) \quad I_{(\partial^2 + \partial_v)^{-1}}(x_0, t) = I_Q(x_0, t) + O(t^\infty).$$

As a result we may replace the Dirac operator $\partial_\gamma$ by any differential operator that agrees with $\partial_\gamma$ in any given local chart near $x_0$. In other words, this enables us to localize the problem and replace $\partial_\gamma$ by an operator on $\mathbb{R}^n$ and acting on the trivial bundle with fiber $S_n$, the spinor space of $\mathbb{R}^n$.

We proceed as follows. Let $e_1, \ldots, e_n$ be an oriented orthonormal basis of $T_{x_0}M$ such that $e_1, \ldots, e_a$ span $T_{x_0}M^\phi$ and $e_{a+1}, \ldots, e_n$ span $N^\phi_{x_0}$. This provides us with normal coordinates $(x^1, \cdots, x^n) \to \exp_{x_0}(x^1 e_1 + \cdots + x^n e_n)$. Moreover using parallel translation enables us to construct a synchronous local oriented tangent frame $e_1(x), \ldots, e_n(x)$ such that $e_1(x), \ldots, e_a(x)$ form an oriented frame of $TM^a$ and $e_{a+1}(x), \ldots, e_n(x)$ form an (oriented) frame $N^\phi$ (when both frames are restricted to $M^\phi$). This gives rise to trivializations of the tangent and spinor bundles. Using these coordinates and trivialization, we let $\partial$ be a Dirac operator on $\mathbb{R}^n$ acting on the trivial bundle with fiber $S_n$ associated to a metric which agrees with the metric $g$ near $x = 0$. Incidentally, $\partial$ agrees with $\partial_\gamma$ near $x = 0$.

Notice that $e_j(x) = \partial_j$ at $x = 0$. Moreover, the coefficients $g_{ij}(x)$ of the metric and the coefficients $\omega_{ijk} := \langle \nabla^T Me_k, e_i \rangle$ of the Levi-Civita connection satisfy

$$(9.4) \quad g_{ij}(x) = \delta_{ij} + O(|x|^2), \quad \omega_{ijk}(x) = -\frac{1}{2} R_{ijk} x^j + O(|x|^2),$$

where $R_{ijkl} := (R^{TM}(0) (\partial_i, \partial_j) (\partial_k, \partial_l))$ are the coefficients of the curvature tensor at $x = 0$ (see, e.g., [BGV, Chap. 1]).

In order to simplify notation we shall denote by $\phi'$ the endomorphism $\phi'(0)$ of $\mathbb{R}^n$. We shall use similar notation for $\phi'(0)$ and $\phi^g(0)$, where the former is regarded as the element of $\text{SO}(b)$ such that

$$\phi' = \begin{pmatrix} 0 & \phi' \\ -\phi' & 0 \end{pmatrix}.$$ 

Let $\Lambda(n) = \Lambda^+_b \mathbb{R}^n$ be the complexified exterior algebra of $\mathbb{R}^n$. We shall use the following gradings on $\Lambda(n)$,

$$\Lambda(n) = \bigoplus_{1 \leq j \leq n} \Lambda^j(n) = \bigoplus_{1 \leq k \leq n} \Lambda^{k, l}(n),$$

where $\Lambda^j(n)$ is the space of forms of degree $j$ and $\Lambda^{k, l}(n)$ is the span of forms $dx^{i_1} \wedge \cdots \wedge dx^{i_k}$ with $1 \leq i_1 < \cdots < i_k \leq a$ and $a+1 \leq i_{k+1} < \cdots < i_{k+l} \leq n$. Given a form $\omega \in \Lambda(n)$ we shall denote by $\omega^{(j)}$ (resp., $\omega^{(k, l)}$) its component in $\Lambda^j(n)$ (resp., $\Lambda^{k, l}(n)$).

Let $\text{Cl}(n)$ be the complexified Clifford algebra of $\mathbb{R}^n$ (seen as a subalgebra of $\text{End}(\Lambda(n))$) and denote by $c : \Lambda(n) \to \text{Cl}(n)$ the linear isomorphism given by Clifford multiplication. Composing with the spinor representation $\text{Cl}(n) \to \text{End} S_n$ (which is an algebra isomorphism since $n$ is even), we get a linear isomorphism $c : \Lambda(n) \to \text{End} S_n$. We denote by $\sigma : \text{End} S_n \to \Lambda(n)$ its inverse.
Recall that, although $c$ and $\sigma$ are not isomorphisms of algebras, we observe that if $\omega_j \in \Lambda^{k_j,l_j}(n)$, $j = 1, 2$, then
\begin{equation}
\sigma[a(\omega_1)c(\omega_2)] = \omega_1 \wedge \omega_2 \mod \bigoplus_{(k,l) \in K} \Lambda^{k,l}(n),
\end{equation}
where $K$ consists of all pairs $(k,l)$ such that, either $k \leq k_1 + k_2 - 2$ and $l \leq l_1 + l_2$, or $k \leq k_1 + k_2$ and $l \leq l_1 + l_2 - 2$.

In the sequel, for a form $\omega \in \Lambda(n)$, we shall simply denote by $|\omega|^{(a,0)}$ the Berezin integral $|\omega(x,0)|^{(a,0)}$ of its component $\omega(x,0)$ in $\Lambda^{a,0}(n)$. That is, $|\omega|^{(a,0)}$ is the inner product of $\omega$ with $dx^1 \wedge \cdots \wedge dx^n$.

**Lemma 9.5.** Let $A \in \text{End} \mathfrak{g}$. Then
\[
\text{Str}[\phi^A] = (-2i)^{q_2} \cdot 2^{q_2} \det \frac{1}{2} (1 - \phi^A) |\sigma(A)|^{(a,0)}.
\]

**Proof.** It follows from [Ge2] Thm. 1.8 (see also [BV] Prop. 3.21) that
\begin{equation}
\text{Str}[\phi^A] dx^1 \wedge \cdots \wedge dx^n = (-2i)^{q_2} \sigma[\phi^A]|^{(a,0)}.
\end{equation}

As $\phi^N$ is an element of $\text{SO}(b)$ there is an oriented orthonormal basis $\{v_{a+1}, \ldots, v_n\}$ of $\{0\}^a \times \mathbb{R}^b$ such for $j = \frac{a}{2} + 1, \ldots, \frac{n}{2}$ the subspace $\text{Span}\{v_{a+1}, v_{a+2}\}$ is invariant under $\phi^N$ and the matrix of $\phi^N$ with respect to the basis $\{v_{a+1}, v_{a+2}\}$ is a rotation matrix of the form,
\[
\begin{pmatrix}
\cos \theta_j & -\sin \theta_j \\
\sin \theta_j & \cos \theta_j
\end{pmatrix} = \exp \begin{pmatrix} 0 & -\theta_j \\ \theta_j & 0 \end{pmatrix}, \quad 0 < \theta_j \leq \pi.
\]

Using [BV] Eqs. (3.4)–(3.5)] we then see that
\begin{equation}
\phi^A = \prod_{1 < j \leq \frac{n}{2}} \left( \cos \frac{\theta_j}{2} + \sin \frac{\theta_j}{2} c(v_{a+1})c(v_{a+2}) \right),
\end{equation}
where $\{v_{a+1}, \ldots, v_n\}$ is the basis of $\Lambda^{0,1}(n)$ that is dual to $\{v_{a+1}, \ldots, v_n\}$.

It follows from (9.7) that $\sigma(\phi^A)$ is an element of $\Lambda^{b,0}(n)$ and we have
\[
\sigma[\phi^A]|^{(a,0)} = \prod_{1 < j \leq \frac{n}{2}} \sin \frac{\theta_j}{2} v_{a+1} \wedge \cdots \wedge v_{a+n} = 2^{-\frac{n}{2}} \det \frac{1}{2} (1 - \phi^N) dx^{a+1} \wedge \cdots \wedge dx^n,
\]
where we have used the equality $2 \sin^2 \theta = \frac{1 - \cos \theta}{\sin \theta}$. Combining this with (9.6) and using (9.5) we deduce that
\[
\text{Str}[\phi^A] dx^1 \wedge \cdots \wedge dx^n = (-2i)^{q_2} \cdot 2^{q_2} \det \frac{1}{2} (1 - \phi^N) dx^{a+1} \wedge \cdots \wedge dx^n \wedge |\sigma[A]|^{(a,0)}.
\]

Contracting both sides of the equality with $dx^1 \wedge \cdots \wedge dx^n$ then proves the lemma.

Let $Q \in \Psi^* \mathbb{R}^n \times \mathbb{R}$ be a parametrix of $\mathfrak{g} + \partial_t$. Using (9.3) and Lemma 9.3 we get
\begin{equation}
\text{Str} \left[ \phi^A(x_0) I(\partial_x, x_0) \right] = \text{Str} \left[ \phi^A I(0, t) \right] + O(t^\infty)
\end{equation}
\[
= (-2i)^{q_2} 2^{q_2} \cdot 2^{q_2} \det \frac{1}{2} (1 - \phi^N) |\sigma[I(0, t)]|^{(a,0)} + O(t^\infty).
\]

We shall determine the small-time behavior of $|\sigma[I(0, t)]|^{(a,0)}$ by using considerations on Getzler orders of Volterra $\Psi$DOs in the sense of [Po1]. This notion is intimately related to the rescaling of Getzler [Ge2], which is motivated by the following assignment of degrees:
\begin{equation}
\text{deg } \partial_j = \text{deg } c(dx^j) = 1, \quad \text{deg } \partial_t = 2, \quad \text{deg } x^j = -1.
\end{equation}

As observed in [Po1] this degree assignment gives rise to a new filtration on Volterra $\Psi$DOs.

Let $Q \in \Psi^m \mathbb{R}^n \times \mathbb{R}$ have symbol $q(x, \xi, \tau) \sim \sum_{r \geq m} q_{m-r}(x, \xi, \tau)$. Taking components in each subspace $\Lambda^J T^*_x \mathbb{R}^n$ and using Taylor expansions at $x = 0$ we get asymptotic expansions of symbols,
\begin{equation}
\sigma[q(x, \xi, \tau)] \sim \sum c\sigma[q_{m-r}(x, \xi, \tau)]^{(j)} \sim \sum \frac{x^a}{a!} \sigma[\partial^a_x \partial^b_\xi q_{m-r}(0, \xi, \tau)]^{(j)},
\end{equation}
The last asymptotic is meant in the following sense: for \( j = 0, \ldots, n \) and all \( N \in \mathbb{N} \), as \( x \to 0 \) and \( |\xi| + |\tau|^2 \to \infty \), we have

\[
(9.11) \quad \sigma[q(x, \xi, \tau)]^{(j)} - \sum_{r+|\alpha|=N+j} \frac{\alpha^0}{\alpha!} \sigma[\partial^a_x \partial^b_v q_{m-r}(0, \xi, \tau)]^{(j)} = O \left( \|\xi, \tau\|^{m'} \left( |x| + \|\xi, \tau\|^{-1} \right)^N \right),
\]

where we have set \( \|\xi, \tau\| := |\xi| + |\tau|^2 \), and there are similar asymptotics for all \( \partial^a_x \partial^b_v \)-derivatives (upon replacing the exponent \( m' \) by \( m' - |\beta| - k \)).

In addition, the degrees’ assignment \((9.9)\) leads us to define (Getzler-)rescaling operators \( \delta^*_\lambda \), \( \lambda \in \mathbb{R} \), on Volterra symbols with differential-form coefficients by

\[
\delta^*_\lambda q(x, \xi, \tau) := \lambda^j q(\lambda^{-1} x, \lambda \xi, \lambda^2 \tau) \quad \forall q \in S^*_n(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}) \otimes \Lambda^j(n).
\]

Notice that in \((9.10)\) each symbol \( x^a \partial^a_v \sigma[q_{m-r}(0, \xi, \tau)]^{(j)} \) is homogeneous of degree \( \mu := m' - r + j - |\alpha| \) with respect to this rescaling. We shall say that such a symbol is \textit{Getzler homogeneous} of degree \( \mu \).

Moreover, the asymptotics \((9.10)\) imply that, in the sense of \((9.11)\), we have

\[
(9.12) \quad q_{(\mu)}(x, \xi, \tau) := \sum_{m' - r + j - |\alpha| = \mu} \frac{\alpha^0}{\alpha!} \sigma[\partial^a_x \partial^b_v q_{m-r}(0, \xi, \tau)]^{(j)},
\]

and \( m \) is the greatest integer \( \mu \) such that \( q_{(\mu)} \neq 0 \).

Alternatively, in terms of the rescaling operators \( \delta^*_\lambda \), for all \( (x, \xi, \tau) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R} \), \( (\xi, \tau) \neq 0 \), we have

\[
\delta^*_\lambda \sigma[q(x, \xi, \tau)] \sim \sum_{\mu \leq m} \lambda^\mu q_{(\mu)}(x, \xi, \tau) \quad \text{as } \lambda \to 0.
\]

We observe that the homogeneous symbols \( q_{(\mu)}(x, \xi, \tau) \) are uniquely determined by the above asymptotic. In particular, the leading Getzler-homogeneous symbol \( q_{(m)}(x, \xi, \tau) \) is uniquely determined by

\[
(9.13) \quad \delta^*_\lambda \sigma[q(x, \xi, \tau)] = \lambda^m q_{(m)}(x, \xi, \tau) + O(\lambda^{m-1}).
\]

**Definition 9.6 (Pol).** Using \((9.8)\) we make the following definitions:

1. The integer \( m \) is called the \textit{Getzler order} of \( Q \).
2. The symbol \( q_{(m)} \) is called the \textit{principal Getzler-homogeneous symbol} of \( Q \).
3. The operator \( Q_{(m)} = q_{(m)}(x, D_x, D_t) \) is called the \textit{model operator} of \( Q \).

**Remark 9.7.** As the Getzler-homogeneous symbol \( q_{(m)}(x, \xi, \tau) \) is linear combination of homogeneous Volterra symbols with coefficients in \( \Lambda(n) \), we may define the operator \( q_{(m)}(x, D_x, D_t) \) as in \((7.4)\). Notice also that \( Q_{(m)} \) is an element of \( \Psi^*_n(\mathbb{R}^n \times \mathbb{R}) \otimes \Lambda(n) \), rather than an actual operator.

**Remark 9.8.** As the symbol \( \sigma[\partial^a_x \partial^b_v q_{m-r}(0, \xi, \tau)]^{(j)} \) is Getzler-homogeneous degree \( m' - r + j - |\alpha| \leq m + n \), we see that the Getzler order of \( Q \) is always \( \leq m + n \).

**Example 9.9.** It follows from \((9.3)\) spinor covariant derivative \( \nabla^S_i := \partial_i + \frac{1}{4} \omega_{ijkl}(x)c(e^k)c(e^l) \) has Getzler order 1 and its model operator is

\[
(9.14) \quad \nabla^S_{(1)} := \partial_i - \frac{1}{4} R_{ij} x^j, \quad \text{where } R_{ij} := \sum_{k<l} R^T_{ijkl}(0) dx^k \wedge dx^l.
\]

In the sequel, we shall often look at symbols or operators up to terms that have lower Getzler order. For this reason, it would be convenient to use the notation \( O_{G}(m) \) to denote any remainder term (symbol or operator) of Getzler order \( \leq m \).

Notice that in view of \((9.13)\) a Volterra symbol \( q \in S^*_n(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}) \otimes \Lambda^*(n) \) is \( O_{G}(m) \) if and only if, for all \( (x, \xi, \tau) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R} \), \( (\xi, \tau) \neq 0 \),

\[
(9.15) \quad \delta^*_\lambda q(x, \xi, \tau) = O(\lambda^m) \quad \text{as } \lambda \to 0.
\]
Moreover, two Volterra symbols have Getzler order \( m \) and same leading Getzler-homogeneous symbol if and only if they agree modulo \( O_G(m-1) \).

**Lemma 9.10** ([Po1].) For \( j = 1, 2 \) let \( Q_j \in \Psi^*_c(\mathbb{R}^n \times \mathbb{R}, S) \) have Getzler order \( m_j \) and model operator \( Q_{(m_j)} \) and assume that either \( Q_1 \) or \( Q_2 \) is properly supported. Then

\[
\sigma [Q_1, Q_2] = Q_{(m_1)} Q_{(m_2)} + O_G(m_1 + m_2 - 1).
\]

**Example 9.11.** Let \( D_i \) be the Christoffel symbols of the metric and \( k \) is the scalar curvature. Therefore, combining Lemma 9.10 with (9.14) we see that \( D_i \) has Getzler order 2 and its model operator is the harmonic oscillator,

\[
H_R := -\sum_{i=1}^n (\partial_i - \frac{1}{4} R_{ij} x^j)^2.
\]

In the sequel, it would be convenient to introduce the variables \( x' = (x_1, \ldots, x^n) \) and \( x'' = (x_1', \ldots, x^n') \), so that \( x = (x', x'') \). When using these variables we shall denote by \( q(x', x''; \xi', \xi''; \tau) \) and \( K_Q(x', x''; y'', t) \) the symbol and kernel of any given \( Q \in \Psi^*_c(\mathbb{R}^n \times \mathbb{R}, S) \otimes \Lambda(n) \). We then define

\[
I_Q(x', t) := \int_{\mathbb{R}^n} K_Q(x', x''; 0, \phi t x''; t) dx'', \quad x' \in \mathbb{R}^n.
\]

**Lemma 9.12.** Let \( Q \in \Psi^*_c(\mathbb{R}^n \times \mathbb{R}, S) \) have Getzler order \( m \) and model operator \( Q_{(m)} \). Then, as \( t \to 0^+ \),

\[
\begin{align*}
(1) \quad & \sigma[I_Q(0, t)][(j)] = O(t^{m-j+1}) \quad \text{if } m-j \text{ is odd,} \\
(2) \quad & \sigma[I_Q(0, t)][(j)] = t^{m-j} I_{Q_{(m)}}(0, 1)^{(j)} + O(t^{m-j-1}) \quad \text{if } m-j \text{ is even.}
\end{align*}
\]

In particular, for \( m = -2 \) and \( j = a \) we get

\[
\sigma[I_Q(0, t)][(a, 0)] = I_{Q_{(-2)}}(0, 1)^{(a, 0)} + O(t).
\]

**Proof.** Let \( q(x, \xi, \tau) \sim \sum_{k \leq m} q_k(x, \xi, \tau) \) be the symbol of \( Q \) and denote by \( q_{(m)}(x, \xi, \tau) \) its principal Getzler homogeneous symbol. Recall that Proposition 8.4 provides us with an asymptotic for \( I_Q(x, t) \) in terms of the symbol of \( Q \) in the tubular coordinates. We shall use the tubular coordinates \( (x', v) \in \mathbb{R}^n \times \mathbb{R}^b \) given by the change of variable,

\[
x = \psi(x', v) := \exp_{x'} (v_1 c_{a+1}(x') + \cdots + v_b c_{an}(x')) , \quad (x', v) \in \mathbb{R}^n \times \mathbb{R}^b,
\]

where on the far right-hand side we have identified \( x' \) with \( (x', 0) \in \mathbb{R}^n \). Notice that, as the original coordinates are normal coordinates, for all \( v \in \mathbb{R}^b \), we have

\[
\psi(0, v) = \exp_{x'} (v_1 \partial_{a+1} + \cdots + v_b \partial_n) = (0, v).
\]

Furthermore, in the sequel, upon identifying \( \mathbb{R}^n \) and \( \mathbb{R}^n \times \mathbb{R}^b \), it will be convenient to regard functions on \( \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R} \) as functions on \( \mathbb{R}^n \times \mathbb{R}^b \times \mathbb{R}^b \times \mathbb{R} \).

Let \( \tilde{q}(x', v; \xi', v; \tau) \sim \sum_{k \leq m'} \tilde{q}_k(x', v; \xi', v; \tau) \) be the symbol of \( Q \) in the tubular coordinates, i.e., \( \tilde{q}(x', v; \xi', v; \tau) \) is the symbol of \( \psi^* Q \). As in the tubular coordinates, the derivative \( \phi^* \) is constant along the fibers of \( N^\phi \), we see that \( \phi^* \) too is fiberwise constant. Incidentally, in the notation of (8.3) the symbols \( \tilde{q}_k \) and \( \tilde{q}_k \) agree for all \( k \leq m' \). Bearing this mind, Proposition 8.4 shows that, as \( t \to 0^+ \),

\[
\sigma[I_Q(0, t)][(j)] \sim \sum_{\alpha = \sum_{k \leq m'} k \leq m'} I^{(j)} \sum_{l = 0}^{|\alpha| - (k + 1 + 2)} \int_{\mathbb{R}^b} v^l \alpha! \left( \hat{\partial}_{\xi'} \sigma[\tilde{q}_k]^{(j)} \right)^v (0, 0; 0, (1 - \phi^v(0)v; 1) dv.
\]
Lemma 9.13. Let $\beta, \gamma$ be smooth functions such that $a_{\beta, \gamma}(x) = 1$ when $\beta = \gamma = 0$. Thus, (9.23) $\sigma[Q(0, t)]^{(j)} \sim \sum_{[\alpha] - \ell + [\beta] - [\gamma] \text{ even}} \sum_{t \leq m, 2[\gamma] \leq |\beta|} I^{(j)}_{\alpha \beta \gamma},$

where the $a_{\beta, \gamma}(x', v)$ are some smooth functions such that $a_{\beta, \gamma}(x) = 1$ when $\beta = \gamma = 0$. Therefore, the change of variable formula for symbols (see Thm. 18.1.17) gives

Using (9.21), the change of variable formula for symbols gives

$$\hat{q}_k(0, v; \xi', \nu; \tau) = \sum_{l - |\beta| + |\gamma| = k} a_{\alpha \beta}(0, v) \xi' D_{\xi}^{l} q_k (0, v; \xi', \nu; \tau)$$

where the $a_{\beta, \gamma}(x', v)$ are some smooth functions such that $a_{\beta, \gamma}(x) = 1$ when $\beta = \gamma = 0$. Thus, (9.23) $\sigma[Q(0, t)]^{(j)} \sim \sum_{[\alpha] - \ell + [\beta] - [\gamma] \text{ even}} \sum_{t \leq m, 2[\gamma] \leq |\beta|} I^{(j)}_{\alpha \beta \gamma},$

where $I^{(j)}_{\alpha \beta \gamma} := \int_{\mathbb{R}^v} a_{\alpha \beta}(0, v) \frac{e^a}{v^a} \left( \partial^a_{\nu} \sigma[\xi D_{\xi}^{m} q_k(0)]^{(j)} \right)^{\nu} (0, 0; 0, (1 - \phi^N(0))v; 1) dv.$

Notice that the symbol $v^a \partial^a_{\nu} \sigma[\xi D_{\xi}^{m} q_k(0)]^{(j)}(0, \xi', \nu; \tau)$ is Getzler homogeneous of degree $l + j - |\alpha|$. Therefore, it must be zero if $l + j - |\alpha| > m$, since otherwise $Q$ would have Getzler order $> m$. This implies that in (9.23) the odd terms in $(j)$ contain only integer powers of $t$ (non-negative or negative). Therefore, from the observations above we deduce that if $m - j$ is odd, then all the (non-zero) terms in (9.23) are $O(t^{l(j-m-a+1)})$, and hence

$$\sigma[Q(0, t)]^{(j)} = O(t^{l(j-m+1)}).$$

Likewise, if $m - j$ is even, then all the terms in (9.23) with $l + j - |\alpha| \neq m$ or with $l - |\alpha| = m - j$ and $(\beta, \gamma) \neq (0, 0)$ are $O(t^{l(j-m-a)})$. Thus, (9.24) $\sigma[Q(0, t)]^{(j)} = t^{l(j-m+1)} \sum_{l - |\alpha| = m - j} I^{(j)}_{\alpha \beta \gamma} + O(t^{l(j-m+1)})$.

To complete the proof it remains to identify the coefficient of $t^{l(j-m+1)}$ in (9.24) with $I_{Q(0, t)}(0, 1)^{(j)}$. To this end observe that the formula (9.112) for $q_{(m)}$ at $x' = 0$ gives

$$q_{(m)}(0, v; \xi, \nu; \tau)^{(j)} = \sum_{k + j - |\alpha| = m} \frac{e^a}{a^a} \left( \partial^a_{\nu} \sigma[\xi D_{\xi}^{m} q_k(0)]^{(j)} \right) (0, 0; 0, (1 - \phi^N(0))v; 1) dv.$$

Thus,

$$I_{Q(0, t)}(0, 1)^{(j)} = \sum_{k - |\alpha| = m - j} \int_{\mathbb{R}^v} \frac{v^a}{a^a} \left( \partial^a_{\nu} \sigma[\xi D_{\xi}^{m} q_k(0)]^{(j)} \right)^{\nu} (0, 0; 0, (1 - \phi^N(0))v; 1) dv,$$

This completes the proof. \hfill \Box

In the sequel, we shall use the following “curvature forms”:

$$R' := (R_{ij})_{1 \leq i, j \leq a} \quad \text{and} \quad R'' := (R_{a+i,a+j})_{1 \leq i, j \leq b}.$$

Notice that the components in $A^{\alpha \beta}(m)$ of $R'$ and $R''$ are $RTMF^a(0)$ and $R^{N^a}(0)$ respectively.

Lemma 9.13. Let $Q \in \Psi^{-2}(R^n \times \mathbb{R}, S^m)$ be a parametrix for $E^2 + \partial_t$. Then

1. $Q$ has Getzler order $-2$ and its model operator is $(HR + \partial_t)^{-1}$.
2. For all $t > 0$,

$$I_{(H_{m} + \partial_t)^{-1}}(0, t) = \frac{(4\pi t)^\frac{1}{2}}{\det \frac{1}{2} \left( 1 - \phi^N \right)} \det \frac{1}{2} \left( tR'/2 \sinh(tR'/2) \right) \det \frac{1}{2} \left( 1 - \phi^N e^{-tR''} \right).$$
Proof. The first part is contained in [Pao] Lemma 5. The formula for \( I_{(H_R + \partial_t)^{-1}}(0,t) \) is obtained exactly like in [LM] p. 459. For reader’s convenience we mention the main details of the computation.

The kernel of \( (H_R + \partial_t)^{-1} \) can be determined from the arguments of [Ge2]. More precisely, let \( A \in \mathfrak{so}_n(\mathbb{R}) \) and set \( B = A^t A \). Consider the harmonic oscillators,

\[
H_A := - \sum_{1 \leq i \leq n} (\partial_i + \sqrt{-1} A_{ij} x^j)^2 \quad \text{and} \quad H_B := - \sum_{1 \leq i \leq n} \partial_i^2 + \frac{1}{4} (Bx, x).
\]

In particular substituting \( A = \frac{1}{2} \sqrt{-1} R \) in the formula for \( H_A \) gives \( H_R \). In addition, define

\[
X := \sqrt{-1} \sum_{i,j} A_{ij} x^i \partial_j = \sqrt{-1} \sum_{i<j} A_{ij} (x^i \partial_j - x^j \partial_i).
\]

Notice that \( H_A = H_B + X \). Observe also that, as \( X \) is linear combination of the infinitesimal rotations \( x^i \partial_i - x^j \partial_i \), the \( O(n) \)-invariance of \( H_B \) implies that \( [H_B, X] = 0 \). Thus,

\[
e^{-tH_A} = e^{-tX} e^{-tH_B} \quad \forall t \geq 0.
\]

The heat kernel of \( H_B \) is determined by Melcher’s formula in its version of [Ge2]. We get

\[
(9.27) \quad K_{(H_B + \partial_t)^{-1}}(x,y,t) = (4\pi t)^{-\frac{n}{2}} \det \left( \frac{t \sqrt{B}}{\sinh(t \sqrt{B})} \right) \exp \left( -\frac{1}{4t} \Theta_B(x,y,t) \right), \quad t > 0,
\]

\[
\Theta_B(x,y,t) := \left\langle \frac{t \sqrt{B}}{\tanh(t \sqrt{B})} x, x \right\rangle + \left\langle \frac{t \sqrt{B}}{\tanh(t \sqrt{B})} y, y \right\rangle - 2 \left\langle \frac{t \sqrt{B}}{\sinh(t \sqrt{B})} x, y \right\rangle,
\]

where \( \sqrt{B} \) is any square root of \( B \) (e.g., \( \sqrt{B} = \sqrt{-1} A \)). Notice that the r.h.s. of (9.27) is actually an analytic function of \( (\sqrt{B})^2 \).

Observe that for \( t \in \mathbb{R} \) the matrix \( e^{-t\sqrt{-1}A} \) is an element of \( O(n) \), since in a suitable orthonormal basis it can be written as a block diagonal of \( 2 \times 2 \) rotation matrices (with purely imaginary angles). Moreover, the family of operators \( u \rightarrow u(e^{-t\sqrt{-1}A}) \), \( t \in \mathbb{R} \), is a one-parameter group of operators on \( L^2(\mathbb{R}^n) \) with infinitesimal generator \( X \), so it agrees with \( e^{-tX} \) for \( t > 0 \). Combining this with (9.26) and (9.27) then gives

\[
(9.28) \quad K_{(H_A + \partial_t)^{-1}}(x,y,t) = (4\pi t)^{-\frac{n}{2}} \det \left( \frac{t \sqrt{B}}{\sinh(t \sqrt{B})} \right) \exp \left( -\frac{1}{4t} \Theta_A(x,y,t) \right),
\]

\[
\Theta_A(x,y,t) := \left\langle \frac{t \sqrt{B}}{\tanh(t \sqrt{B})} x, x \right\rangle + \left\langle \frac{t \sqrt{B}}{\tanh(t \sqrt{B})} y, y \right\rangle - 2 \left\langle \frac{t \sqrt{B}}{\sinh(t \sqrt{B})} e^{-t\sqrt{-1}A} x, y \right\rangle,
\]

where we have used the fact that \( e^{-t\sqrt{-1}A} \) is an orthogonal matrix. Substituting \( A = \frac{1}{2} \sqrt{-1} R \) and \( \sqrt{B} = \frac{1}{2} R \) then gives the kernel of \( (H_R + \partial_t)^{-1} \). We obtain

\[
(9.29) \quad I_{(H_R + \partial_t)^{-1}}(0,t) = (4\pi t)^{-\frac{n}{2}} \det \left( \frac{t R^2/2}{\sinh(t R^2/2)} \right) \int_{\mathbb{R}^n} \exp \left( -\frac{1}{4t} \Theta(v, t) \right) dv,
\]

where \( \Theta(v, t) := \Theta_R(v, \phi^N v, t) \). Set \( \omega = \frac{1}{2} t R'' \). As \( [\phi^N, \omega] = 0 \), we see that

\[
\Theta(v, t) = \left\langle \frac{\omega}{\tanh \omega} v, v \right\rangle + \left\langle \frac{\omega}{\tanh \omega} \phi^N v, \phi^N v \right\rangle - 2 \left\langle \frac{\omega}{\sinh \omega} e^{\omega} v, e^{\omega} v \right\rangle = 2 \left\langle \frac{\omega}{\sinh \omega} \left( \cosh \omega - \left( \phi^N \right)^{-1} e^{\omega} \right) v, v \right\rangle.
\]
Observe that
\[(\cosh \mathcal{A} - (\phi^N)^{-1} e^{\mathcal{A}}) + (\cosh \mathcal{A} - (\phi^N)^{-1} e^{\mathcal{A}})^T = e^{\mathcal{A}} + e^{-\mathcal{A}} - (\phi^N)^{-1} e^{\mathcal{A}} - \phi^N e^{-\mathcal{A}} \]
\[= e^{\mathcal{A}} \left( 1 - (\phi^N)^{-1} \right) (1 - \phi^N e^{-2\mathcal{A}}). \]
Therefore, using the formula for the integral of a Gaussian function and its extension to Gaussian functions associated to form-valued symmetric matrices, we get
\[\int_{\mathbb{R}^n} \exp \left( -\frac{1}{4t} \Theta(v, t) \right) dv = \int_{\mathbb{R}^n} \exp \left( -\frac{1}{4t} \left( \frac{\mathcal{A}}{\sinh \mathcal{A}} \right) \left( 1 - (\phi^N)^{-1} \right) (1 - \phi^N e^{-2\mathcal{A}})v, v \right) \right) dv \]
\[= (4\pi)^{\frac{n}{2}} \det^{-\frac{n}{2}} \left( \frac{\mathcal{A}}{\sinh \mathcal{A}} \right) \det^{-\frac{n}{2}} \left( e^{\mathcal{A}} \left( 1 - (\phi^N)^{-1} \right) \right) \det^{-\frac{n}{2}} (1 - \phi^N e^{-2\mathcal{A}}). \]
Observe that \(\det^{-\frac{n}{2}} \left( e^{\mathcal{A}} \left( 1 - (\phi^N)^{-1} \right) \right) = \det^{-\frac{n}{2}} (1 - \phi^N)\), so using (9.29) we get
\[I_{(H_R + \partial_t)_{-1}}(0, t) = (4\pi)^{-\frac{n}{2}} \det^{-\frac{n}{2}} (1 - \phi^N) \det^{-\frac{n}{2}} \left( \frac{tR'/2}{\sinh(tR'/2)} \right) \det^{-\frac{n}{2}} (1 - \phi^N e^{-tR'}) \]
This proves (9.26) and completes the proof. \(\square\)

Let \(Q \in \Psi^{-2}(\mathbb{R}^n \times \mathbb{R}, \mathcal{S}_n)\) be a parametrix for \(\mathcal{D}^2 + \partial_t\). The first part of Lemma 9.13 says that \(Q\) has Getzler order \(-2\) and its model operator is \((H_R + \partial_t)^{-1}\). Therefore, using (9.8) and Lemma 9.5 we get
\[\text{Str} \left[ \phi^q(x_0) I_{(\phi_t + \partial_t)_{-1}}(x_0, t) \right] = (-2i)^{\frac{n}{2}} 2^{-\frac{n}{2}} \det^{-\frac{n}{2}} (1 - \phi^N) \left| \sigma[IQ(0, t)] \right|^{(a, 0)} + O(t^{\infty}) \]
\[= (-2i)^{\frac{n}{2}} 2^{-\frac{n}{2}} \det^{-\frac{n}{2}} (1 - \phi^N) \left| I_{(H_R + \partial_t)_{-1}}(0, 1) \right|^{(a, 0)}. \]
(9.30)
As the components in \(\Lambda^*, 0(n)\) of the curvatures \(R'\) and \(R''\) are \(R^{TM^q}(0)\) and \(R^{N^q}(0)\) respectively, from (9.22) we get
\[I_{(H_R + \partial_t)_{-1}}(0, 1)^{(a, 0)} = \left( \frac{4\pi}{\det^\frac{n}{2} (1 - \phi^N)} \right) \left\{ \det^\frac{n}{2} \left( \frac{R^{TM^q}(0)/2}{\sinh(R^{TM^q}(0)/2)} \right) \det^{-\frac{n}{2}} \left( 1 - \phi^N e^{-tR^{N^q}(0)} \right) \right\}^{(a, 0)} \]
\[= \frac{4\pi}{\det^\frac{n}{2} (1 - \phi^N)} \left[ \hat{A}(R^{TM^q}(0)) \wedge \nu_\phi \left( R^{N^q}(0) \right) \right]^{(a, 0)}. \]
Combining this with (9.30) then gives
\[\text{Str} \left[ \phi^q(x_0) I_{(\phi_t + \partial_t)_{-1}}(x_0, t) \right] = (-i)^{\frac{n}{2}} (2\pi)^{-\frac{n}{2}} \hat{A}(R^{TM^q}(0)) \wedge \nu_\phi \left( R^{N^q}(0) \right) \]
\[\quad + O(t), \quad \text{This proves (9.22)} \text{ and completes the proof of the local equivariant index theorem.}\]
Let us now indicate how the previous arguments enables us to prove Proposition 6.5
Proof of Proposition 6.5 Let \(f^0, f^1, \ldots, f^k\) be smooth functions on \(M\) and set
\[P_{k, \alpha} := f^0 \mathcal{D} g, f^1 |^{[\alpha]} \ldots, f^k |^{[\alpha_2]}, \]
where the notation is the same as in (2.8). We would like to prove an asymptotic of the form (6.6) for \(\text{Str} \left[ P_{k, \alpha} e^{-\mathcal{D}^2 U_\phi} \right] \) as \(t \to 0^+\). Since Proposition 8.7 provides us with full asymptotics for \(\text{Str} \left[ P_{k, \alpha} e^{-\mathcal{D}^2 U_\phi} \right] \) and \(I_{P_{k, \alpha}(\mathcal{D}^2 + \partial_t)}(x, t)\), arguing as in the proof of Lemma 9.4 shows that in order to prove (6.6) it is enough to show that (9.31)
\[\text{Str} \left[ \phi^q(x_0) I_{P_{k, \alpha}(\mathcal{D}^2 + \partial_t)_{-1}}(x_0, t) \right] = \begin{cases} (-i)^{\frac{n}{2}} t^{-k} (2\pi)^{-\frac{n}{2}} |\omega_k(x_0)|^{(\alpha)} + O(t^{-k+1}) & \text{if } \alpha = 0, \\ O(t^{-(\alpha+k)+1}) & \text{if } \alpha \neq 0, \end{cases} \]
where \(\omega_k := \hat{A}(R^{TM^q}) \wedge \nu_\phi \left( R^{N^q} \right) \wedge f^0 df^1 \wedge \cdots \wedge df^{2k}.\)
Let $Q \in \Psi^{-2}(\mathbb{R}^n \times \mathbb{R}, S^1)$ be a Volterra parametrix for $\partial^2 + \partial_t$. Then, exactly like in (9.3) and (9.8), we have

\begin{equation}
\text{Str} \left[ \phi^S(x_0)I_{P_{k,\alpha}(\mathbb{R}^n_+ + \partial_t)}^{-1}(x_0,t) \right] = \text{Str} \left[ \phi^S I_{P_{k,\alpha}Q}(0,t) \right] + O(t^\infty)
\end{equation}

(9.32)

\begin{equation}
= (-2i)^{k/2} \xi^\alpha \sigma(I_{P_{k,\alpha}Q}(0,t)) |-(\alpha,0)| + O(t^\infty).
\end{equation}

(9.33)

Notice that $P_{k,\alpha} = f^0(0f(1)[\alpha_1] \cdots c(df^{(2k)}[\alpha_{2k}])$. Assume that $\alpha = 0$. Then $P_{k,0}$ has Getzler order $2k$ and model operator $\Pi_{2k} := f^0(0df(1)) \wedge \cdots \wedge df^{(2k)}(0)$. Thus by Lemma 9.10 and Lemma 9.13 the operator $P_{k,0}Q$ has Getzler order $2k - 2$ and model operator $\Pi_{2k}(H_R + \partial_t)^{-1}$. Applying Lemma 9.5 we then see that, as $t \to 0^+$,

\[
\sigma(I_{P_{k,\alpha}Q}(0,t)) |-(\alpha,0)| = t^{-k} \Pi_{2k}(H_R + \partial_t)^{-1}(0,1) |-(\alpha,0)| + O(t^{-(k+1)}).
\]

Combining this with (9.33) and the formula (9.25) for $I(\partial_t) \wedge I_{P_{k,\alpha}Q}(0,t)$ we obtain

\[
\text{Str} \left[ \phi^S(x_0)I_{P_{k,\alpha}(\mathbb{R}^n_+ + \partial_t)}^{-1}(x_0,t) \right] = \left( -i \right)^{k/2} t^{-k} (2\pi)^{-\frac{k}{2}} \left| \alpha_k(0) \right| + O(t^{-(k+1)}),
\]

which is the asymptotic (9.31) in the case $\alpha = 0$.

Suppose that $\alpha \neq 0$. Then Lemma 9.10 implies that

\[
\sigma(P_{k,\alpha}) = f^0(0f(1)[\alpha_1] \cdots c(df^{(2k)}[\alpha_{2k}]) + O(2k + 2|\alpha| - 1) = O(2k + 2|\alpha| - 1).
\]

Thus, $P_{k,\alpha}$ has Getzler order $\leq 2k + 2|\alpha| - 1$, and hence $P_{k,\alpha}Q$ has Getzler order $\leq 2k + 2|\alpha| - 3$ by Lemma 9.11. It then follows from Lemma 9.5 that $\sigma(I_{P_{k,\alpha}Q}(0,t)) |-(\alpha,0)| = O(t^{-(|\alpha|+k)+1})$, and so using (9.33) we immediately see that

\[
\text{Str} \left[ \phi^S(x_0)I_{P_{k,\alpha}(\mathbb{R}^n_+ + \partial_t)}^{-1}(x_0,t) \right] = O \left( t^{-(|\alpha|+k)+1} \right).
\]

This completes the proofs of (9.31) and Proposition 9.5. \hfill \Box

Acknowledgements

The authors would like to thank Xiaonan Ma and Bai-Ling Wang for useful discussions related to the subject matter of this paper. They also would like to thank the following institutions for their hospitality during the duration of the preparation of this manuscript: Mathematical Science Center of Tsinghua University, Research Institute of Mathematical Sciences of Kyoto University, and University Paris 6 (RP), Seoul National University (HW), Australian National University, Chern Institute of Mathematics of Nankai University, and Fudan University (RP+HW).

References

[AS] Atiyah, M., Segal, G.: The index of elliptic operators. II. Ann. of Math. (2) 87 (1968), 531–545.

[ASi1] Atiyah, M., Singer, I.: The index of elliptic operators. I. Ann. of Math. (2) 87 (1968), 484–530.

[ASi2] Atiyah, M., Singer, I.: The index of elliptic operators. III. Ann. of Math. (2) 87 (1968), 546–604.

[BGS] Beals, R., Greiner, P., Stanton, N.: The heat equation on a CR manifold. J. Differential Geom. 20 (1984), 343–387.

[BV] Berline, N.; Vergne, M.: A computation of the equivariant index of the Dirac operator. Bull. Soc. Math. France 113 (1985), 305–345.

[BGV] Berline, N.; Getzler, E.; Vergne, M.: Heat kernels and Dirac operators. Springer-Verlag, Berlin, 1992.

[Bi] Bismut, J.-M.: The Atiyah-Singer theorems: a probabilistic approach. II. The Lefschetz fixed-point formulas. J. Funct. Anal. 57 (1984), 329–348.

[BF] Block, J.; Fox, J.: Asymptotic pseudodifferential operators and index theory. Geometric and topological invariants of elliptic operators (Brunswick, ME, 1988), 132, Contemp. Math., 105, Amer. Math. Soc. Providence, RI, 1990.

[BG] Bourguignon, J.-P.; Gauduchon, P.: Spinors, opérateurs de Dirac et variations de métriques. Comm. Math. Phys. 144 no. 3, (1992), 581–599.

[CH] Chern, S., Hu, X.: Equivariant Chern character for the invariant Dirac operator. Michigan Math J. 44 (1997), 451–473.

[Col] Connes, A.: Cyclic cohomology and the transverse fundamental class of a foliation. Geometric methods in operator algebras (Kyoto, 1983), pp. 52–144, Pitman Res. Notes in Math. 123, Longman, Harlow (1986).
[Co2] Connes, A.: Noncommutative differential geometry. Inst. Hautes Études Sci. Publ. Math. 62 (1985), 257–360.

[Co3] Connes, A.: Noncommutative geometry. Academic Press, San Diego, 1994.

[CM1] Connes, A., Moscovici, H.: The local index formula in noncommutative geometry. Geom. Funct. Anal. 5 (1995), 174–243.

[CM2] Connes, A., Moscovici, H.: Type III and spectral triples. Traces in Geometry, Number Theory and Quantum Fields, Aspects of Mathematics E38, Vieweg Verlag 2008, 57–71.

[CT] Connes, A.; Tretkoff, P.: The Gauss-Bonnet theorem for the noncommutative two torus. Noncommutative geometry, arithmetic, and related topics, pp. 141–158, Johns Hopkins Univ. Press, Baltimore, MD, 2011.

[Da] Dave, S.: An equivariant noncommutative residue. E-print, arXiv, math.AP/0610371, 17 pages.

[DP] Donnelly, H.; Patodi, V. K.: Spectrum and the fixed point set of isometries. Topology 16 (1977), 1–11.

[FK] Fatehiadeh, F.; Khalkhali, M.: The Gauss-Bonnet Theorem for Noncommutative Two Tori With a General Conformal Structure. J. Noncommut. Geom. 6, 457–480 (2012).

[Fe] Ferrand, J.: The action of conformal transformations on a Riemannian manifold. Math. Ann. 304 (1996), 277–291.

[Ge1] Getzler, E.: Pseudodifferential operators on supermanifolds and the Atiyah-Singer index theorem. Comm. Math. Phys. 92, 163–178 (1983).

[Ge2] Getzler, E.: A short proof of the local Atiyah-Singer index theorem. Topology 25 (1986), 111–117.

[Gi] Gilkey, P. B.: Lefschetz fixed point formulas and the heat equation. Partial differential equations and geometry (Park City, 1977), ed. C. Byrnes, Lecture notes in pure and applied math, vol. 48, Marcel Dekker, 1979, pp. 91–147.

[Gr] Greiner, P.: An asymptotic expansion for the heat equation. Arch. Rational Mech. Anal. 41 (1971), 163–218.

[GS] Grubb, G.; Seeley, R.: Zeta and eta functions for Atiyah-Patodi-Singer operators. J. Geom. Anal. 6 no. 1, (1996), 31–77.

[Gu] Guillenin, V.: A new proof of Weyl’s formula on the asymptotic distribution of eigenvalues. Adv. in Math. 55 (1985), 131–160.

[Ha] Hadamard, J.: Lectures on Cauchy’s problem in linear partial differential equations. Dover Publications, 1953.

[Hi] Higson, N.: The residue index theorem of Connes and Moscovici. Surveys in Noncommutative Geometry, pp. 71–126, Clay Mathematics Proceedings 6, AMS, Providence, 2006.

[Hitchin, N.: Harmonic spinors. Adv. Math. 14 (1974), 1–55.

[Hö] Hörmander, L.: The analysis of linear partial differential operators. III. Pseudo-differential operators. Grundlehren der Mathematischen Wissenschaften, 274. Springer, Berlin, 1994.

[Ke] Kupper, N.: Conformally flat spaces in the large. Ann. of Math. 50 (1949), 916–924.

[Lafferty, J.; Yu, Y.; Zhang, W.: A direct geometric proof of the Lefschetz fixed point formulas. Trans. Amer. Math. Soc. 329 (1992), 571–583.

[LM] Liu, K., Ma, X.: On family rigidity theorems. I. Duke Math. J. 102, no. 3 (2000), 451–474.

[Moi] Moscovici, H.: Eigenvalue inequalities and Poincaré duality in noncommutative geometry. Comm. Math. Phys. 184 (1997), no. 3, 619–628.

[Moo1] Moscovici, H.: Local index formula and twisted spectral triples. Quanta of maths, pp. 465–500, Clay Math. Proc., 11, Amer. Math. Soc. Providence, RI, 2010.

[Pa] Patodi, V.K.: Holomorphic Lefschetz fixed point formula. Bull. Amer. Math. Soc. 79 (1973), 825–828.

[Pit] Pirion, A.: Une classe d’opérateurs pseudo-différentiels du type de Volterra. Ann. Inst. Fourier 20 (1970), 77–94.

[Pol] Ponge, R.: A new short proof of the local index formula and some of its applications. Comm. Math. Phys. 241 (2003), 215–234.

[Po2] Ponge, R.: On the asymptotic completeness of the Volterra calculus. J. Anal. Math. 94 (2004), 249–263.

[Schoen, R.: On the conformal and CR automorphisms groups. Geom. Funct. Anal. 5 (1995), 464–481.

[Wodzicki, M.: Local invariants of spectral asymmetry. Invent. Math. 75 (1984), 143–177.

[Yu] Yu, Y. L.: Local index theorem for Dirac operator. Acta Math. Sinica (N.S.) 3 (2) (1987), 152–169.