On cyclotomic cosets and code constructions

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Abstract

New properties of $q$-ary cyclotomic cosets modulo $n = q^m - 1$, where $q \geq 3$ is a prime power, are investigated in this paper. Based on these properties, the dimension as well as bounds for the designed distance of some families of classical cyclic codes can be computed. As an application, new families of nonbinary Calderbank-Shor-Steane (CSS) quantum codes as well as new families of convolutional codes are constructed in this work. These new CSS codes have parameters better than the ones available in the literature. The convolutional codes constructed here have free distance greater than the ones available in the literature.

keywords: cyclotomic cosets; BCH codes; CSS construction

1 Introduction

Properties of cyclotomic cosets are extensively investigated in the literature in order to obtain the dimension as well as lower bounds for the minimum distance of cyclic codes [23, 22, 24, 30, 33, 34]. Such properties were useful to derive efficient quantum codes [6, 31, 5, 32, 12, 2, 3, 13, 14, 15, 20]. In [24], the authors explored properties of binary cyclotomic cosets to compute the ones containing two consecutive integers. In [30, 33, 34], properties of $q$-ary cyclotomic cosets ($q$-cosets for short) modulo $q^m - 1$ were investigated. In [2, 3], the authors established properties on $q$-cosets modulo $n$, where gcd$(n, q) = 1$, to compute the exact dimension of BCH codes of small designed distance, providing new families of quantum codes. Additionally, they employed such properties to show necessary and sufficient conditions for dual containing (Euclidean and Hermitian) BCH codes. Recently, in [13, 15, 20], the author has investigated properties of $q$-cosets as well as properties of $q^2$-cosets in order to construct several new families of good quantum BCH codes.

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Motivated by all these works, we show new properties of $q$-cosets modulo $n = q^m - 1$, where $q \geq 3$ is a prime power. Since the nonbinary case has received less attention in the literature, in this paper we deal with nonbinary alphabets. As was said previously, these properties allow us to compute the dimension and bounds for the designed distance of some families of cyclic codes. Since the true dimension and minimum distance of BCH codes are not known in general, this paper contributes to this research. As an application of these results, we construct families of new Calderbank-Shor-Steane (CSS) quantum codes (i.e., CSS codes with new parameters; codes with parameters not known in the literature) as well as new families of convolutional codes. These new CSS codes have parameters given by

- $[[q^2 - 1, q^2 - 4c + 5, d \geq c]]_q$, where $2 \leq c \leq q$ and $q \geq 3$ is a prime power;
- $[[n, n - 2m(c - 2) - m/2 - 1, d \geq c]]_q$, where $n = q^m - 1$, $q \geq 3$ is a prime power, $2 \leq c \leq q$ and $m \geq 2$ is even;
- $[[n, n - m(2c - 3) - 1, d \geq c]]_q$, where $n = q^m - 1$, $q \geq 3$ is a prime power, $m \geq 2$ and $2 \leq c \leq q$.

The new convolutional codes constructed here have parameters

- $(n, n - 2q + 1, 2q - 3; 1, d_{\text{free}} \geq 2q + 1)_q$, where $q \geq 4$ is a prime power and $n = q^2 - 1$;
- $(n, n - 2q, 2q - 4; 1, d_{\text{free}} \geq 2q + 1)_q$, where $q \geq 4$ is a prime power and $n = q^2 - 1$;
- $(n, n - 2[q + i], 2[q - 2 - i]; 1, d_{\text{free}} \geq 2q + 1)_q$, where $1 \leq i \leq q - 3$, $q \geq 4$ is a prime power and $n = q^2 - 1$;
- $(n, n - 2q + 1, 1; 1, d_{\text{free}} \geq q + 2)_q$, $q \geq 4$ is a prime power and $n = q^2 - 1$;
- $(n, n - 2q + 1, 2i + 1; 1, d_{\text{free}} \geq q + i + 3)_q$, $1 \leq i \leq q - 3$, $q \geq 4$ is a prime power and $n = q^2 - 1$.

The paper is organized as follows. In Section 2, we review some basic concepts on $q$-cosets and cyclic codes. In Section 3, we present new results and properties of $q$-cosets. In Section 4, by applying some properties of $q$-cosets developed in the previous section, we compute the dimension and lower bounds for the minimum distance of some families of classical cyclic codes. Further, we utilize these cyclic codes to construct new good quantum codes by applying the CSS construction. In Section 5, we utilize the classical cyclic codes constructed in Section 3 to derive new families of convolutional codes with greater free distance. Section 6 is devoted to compare the new code parameters with the ones available in the literature and, in Section 7, a summary of the paper is given.
2 Background

In this section, we review the basic concepts utilized in this paper. For more details we refer to [23].

Notation. In this paper $\mathbb{Z}$ denotes the ring of integers, $q \geq 3$ denotes a prime power, $\mathbb{F}_q$ is the finite field with $q$ elements, $\alpha$ denotes a primitive element of $\mathbb{F}_{q^m}$, $M^{(i)}(x)$ denotes the minimal polynomial of $\alpha^i \in \mathbb{F}_{q^m}$ and $C^\perp$ denotes the Euclidean dual of a code $C$. We always assume that a $q$-coset is considered modulo $n = q^m - 1$. The notation $m = \text{ord}_n(q)$ denotes the multiplicative order of $q$ modulo $n$.

Recall that a $q$-coset modulo $n = q^m - 1$ containing an element $s$ is defined by $C_s = \{ s, sq, sq^2, sq^3, \ldots, sq^{m-1} \}$, where $m_s$ is the smallest positive integer such that $sq^m \equiv s \mod n$. In this case, $s$ is the smallest positive integer of the coset. The notation $C[a]$ means a $q$-coset containing $a$, where $a$ is not necessarily the smallest integer in such coset.

Theorem 2.1 (BCH bound) Let $C$ be a cyclic code with generator polynomial $g(x)$ such that, for some integers $b \geq 0$, $\delta \geq 1$, and for $\alpha \in \mathbb{F}_{q^m}$, we have $g(\alpha^b) = g(\alpha^{b+1}) = \ldots = g(\alpha^{b+\delta-2}) = 0$, that is, the code has a sequence of $\delta - 1$ consecutive powers of $\alpha$ as zeros. Then the minimum distance of $C$ is, at least, $\delta$.

A cyclic code of length $n$ over $\mathbb{F}_q$ is a BCH code with designed distance $\delta$ if, for some integer $b \geq 0$, we have $g(x) = 1. \text{c.m.} \{ M^{(b)}(x), M^{(b+1)}(x), \ldots, M^{(b+\delta-2)}(x) \}$, i.e., $g(x)$ is the monic polynomial of smallest degree over $\mathbb{F}_q$ having $\alpha^b, \alpha^{b+1}, \ldots, \alpha^{b+\delta-2}$ as zeros. From the BCH bound, the minimum distance of BCH codes is greater than or equal to their designed distance $\delta$. If $C$ is a BCH code then a parity check matrix is given by

$$H_C = \begin{bmatrix}
1 & \alpha^b & \alpha^{2b} & \ldots & \alpha^{(n-1)b} \\
1 & \alpha^{(b+1)} & \alpha^{2(b+1)} & \ldots & \alpha^{(n-1)(b+1)} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
1 & \alpha^{(b+\delta-2)} & \ldots & \ldots & \alpha^{(n-1)(b+\delta-2)}
\end{bmatrix},$$

by expanding each entry as a column vector with respect to some $\mathbb{F}_q$-basis $B$ of $\mathbb{F}_{q^m}$ over $\mathbb{F}_q$, where $m = \text{ord}_n(q)$, after removing any linearly dependent rows. The rows of the resulting matrix over $\mathbb{F}_q$ are the parity checks satisfied by $C$.

Let $B = \{ b_1, \ldots, b_m \}$ be a basis of $\mathbb{F}_{q^m}$ over $\mathbb{F}_q$. If $u = (u_1, \ldots, u_n) \in \mathbb{F}_{q^m}^n$ then one can write the vectors $u_i, 1 \leq i \leq n$, as linear combinations of the elements of $B$, i.e., $u_i = u_{i1} b_1 + \ldots + u_{im} b_m$. Let $u^{(j)} = (u_{i1}, \ldots, u_{mj}) \in \mathbb{F}_q^m$, where $1 \leq j \leq m$. Then, if $v \in \mathbb{F}_{q^n}$, one has $v \cdot u = 0$ if and only if $v \cdot u^{(j)} = 0$ for all $1 \leq j \leq m$. 

3
3 New properties of cosets

In this section we explore the structure of \( q \)-cosets in order to obtain new properties of them. As it is well known, the knowledge of the structure (cardinality, disjoint cosets and so on) of cyclotomic cosets provide us conditions to compute the dimension and the (lower bounds for) minimum distance of (classical) cyclic codes. These two parameters of cyclic codes are not known in general in the literature. Therefore, we utilize our new properties of \( q \)-cosets to compute these two parameters of some families of (classical) cyclic codes. Further, we use these classical cyclic codes to construct new quantum codes by applying the CSS construction. Theorem 3.1 is the first result of this section.

**Theorem 3.1** Let \( q \) be an odd prime power and \( C_s \) be a \( q \)-coset. Then \( s \) is even if and only if \( \forall t \in C_s \), \( t \) is even.

**Proof:** Suppose first \( s = 2k \), where \( k \in \mathbb{Z} \), and let \( t \) be an element of the coset \( C_s \) without considering the modulo operation. Then \( t = 2kq^l \), where \( 0 \leq l \leq m_s - 1 \). Applying the division with remainder for \( t \) and \( q^m - 1 \) one has \( 2kq^l = (q^m - 1)a + r \), where \( r \in \mathbb{Z} \), \( 0 \leq r < q^m - 1 \); so \( r = 2kq^l - (q^m - 1)a \). Since \( q^m - 1 \) is even, \( r \) is also even.

Conversely, suppose that each \( t \), where \( t \in C_s \), (considering the modulo operation) is of the form \( t = 2k \), \( k \in \mathbb{Z} \). Then by applying again the division with remainder for \( sq^l \) and \( q^m - 1 \) one obtains \( sq^l = (q^m - 1)a + t \), where \( 0 \leq t < q^m - 1 \) is even. Since \( t \) and \( q^m - 1 \) are even also is \( sq^l \), and because \( q^l \) is odd it follows that \( s \) is even, as required. The proof is complete. \( \square \)

As direct consequences of Theorem 3.1 we present straightforward corollaries (Corollaries 3.1 and 3.2).

**Corollary 3.1** There are no consecutive integers belonging to the same \( q \)-coset modulo \( n = q^m - 1 \), where \( q \) is an odd prime power.

**Remark 3.1** Note that in the binary case there exists at least one coset containing two consecutive elements, namely \( C_1 \).

**Corollary 3.2** Suppose that \( C_x \) and \( C_y \) are two \( q \)-cosets, where \( q \) is an odd prime power. Assume also that \( a \in C_x \) and \( b \in C_y \). If \( a \not\equiv b \mod 2 \), then \( C_x \neq C_y \).

In Theorem 3.2 we introduce the positive integers \( L_s \) in order to compute the minimum absolute value of the difference between elements in the same \( q \)-coset. This fact will be utilized in the computation of the maximum designed distance of the corresponding cyclic code (see Corollary 3.3).

**Theorem 3.2** Let \( q \geq 3 \) be a prime power and \( C_s \) be a \( q \)-coset with representative \( s \). Define \( L_s = \min \{| [sq^j]_n - [sq^l]_n | : 0 \leq j, l \leq m_s - 1, j \neq l \} \), where \( | \cdot |_n \) denotes the absolute value function and \( [\cdot]_n \) denotes the remainder modulo \( n = q^m - 1 \). Then one has \( L_s \geq q - 1 \) for all \( s \), where \( s \) runs through the coset representatives. Moreover, there exists at least one \( q \)-coset \( C_{s^*} \) such that \( L_{s^*} = q - 1 \).
Proof: Let \( C_s \) be a \( q \)-coset and assume that there exist integers such that \( 0 \leq j, l \leq m_s - 1 \). Assume without loss of generality that \( l > j \). Applying the division with remainder for \( sq^l \) and \( n, sq^j \) and \( n \) one obtains \( sq^l = an + [sq^l]_n \) and \( sq^j = bn + [sq^j]_n \), where \( n, a, b \in \mathbb{Z} \), so \( sq^l(q^{j-l} - 1) = -sq^j = (a-b)n + ([sq^l]_n - [sq^j]_n) \). Since \( q - 1 \) divides \( q^{j-l} - 1 \) and also divides \( n \), it follows that \( (q-1)([sq^l]_n - [sq^j]_n) \), hence \( L_s \geq q - 1 \). To finish the proof, it suffices to consider the coset \( C_1 \) and its element \( s^* = 1 \in C_1 \); we have \( L_{s^*} = q - 1 \). Now the result follows.

Corollary 3.3 Let \( q \geq 3 \) be a prime power. If \( C \) is a \( q \)-ary cyclic code of length \( n = q^m - 1 \), \( m \geq 2 \), whose defining set \( Z \) is the union of \( c \) cosets \( C_{s+1}, C_{s+2}, \ldots, C_{s+c}, \) where \( c \geq 1 \) is an integer, \( s \geq 0 \) is an integer and \( 1 \leq s + c \leq q - 2 \), then \( \delta \leq c + 2 \), where \( \delta \) is the designed distance of \( C \). In particular, if \( Z \) consists of only one \( q \)-coset then \( \delta = 2 \).

Proof: It suffices to note that \( Z \) contains at most a sequence of \( c+1 \) consecutive integers and the result follows.

Now, we define the concept of complementary \( q \)-coset and show some interesting properties of it.

Definition 3.1 Let \( q \) be a prime power. Given a \( q \)-coset \( C_s = \{s, qs, q^2s, q^3s, \ldots, q^{m_s-1}s\} \), a complementary coset of \( C_s \) is a \( q \)-coset given by \( C_r = \{r, qr, q^2r, q^3r, \ldots, q^{m_r-1}r\} \), containing an element \( q^l r \), where \( 0 \leq l \leq m_r - 1 \), such that \( s + q^l r \equiv 0 \mod n \), where \( n = q^m - 1 \).

Proposition 3.1 establishes some properties of complementary \( q \)-cosets:

Proposition 3.1 Let \( C_s = \{s,qs,q^2s,q^3s,\ldots,q^{m_s-1}s\} \) be a \( q \)-coset modulo \( n = q^m - 1 \). Then the following results hold:

(i) For each given \( q \)-coset \( C_s \), there exists only one complementary coset of \( C_s \), denoted by \( \overline{C_s} \).

(ii) The \( q \)-coset and its complementary coset have the same cardinality.

(iii) Define the operation \( C_s \oplus \overline{C_r} = C_{[s+qr]} \) \( (q^l r \) is given in Definition 3.1); then one has \( C_s \oplus \overline{C_s} = \{0\} \).

(iv) If \( C_r \) is the complementary coset of \( C_s \) then \( L_s = L_r \).

(v) \( \overline{C_s} = C_s, \)

Proof: (i) Suppose that \( C_s \) is a \( q \)-coset and that \( C_{r_1} = \{r_1, qr_1, q^2r_1, q^3r_1, \ldots, q^{m_{r_1}-1}r_1\} \) and \( C_{r_2} = \{r_2, qr_2, q^2r_2, q^3r_2, \ldots, q^{m_{r_2}-1}r_2\} \) are two complementary \( q \)-cosets of \( C_s \) with representatives \( r_1 \) and \( r_2 \), respectively. From definition, there exist two elements \( q^l r_1, 0 \leq l \leq m_{r_1} - 1 \), and \( q^l r_2, 0 \leq l \leq m_{r_2} - 1 \) such that \( s + q^l r_1 \equiv 0 \mod n \) and \( s + q^l r_2 \equiv 0 \mod n \). Thus \( q^l r_1 \equiv q^l r_2 \mod n \), so \( C_{r_1} = C_{r_2} \).

(ii) Let \( C_s \) be the coset containing \( s \) with cardinality \( m_s \). Let \( C_{[n-s]} \) be the \( q \)-coset containing \( l = n - s \) of cardinality \( m_l \), given by \( C_{[n]} = \{(n-s), (n-s)q, (n-s)q^2, \ldots, (n-s)q^{m_l-1}\} \). It is clear that \( C_{[n]} \) is the complementary
Lemma 3.2. A cyclic code contains its Euclidean dual.

Proposition 3.2. A cyclic code of length \(n\) over \(\mathbb{F}_q\) with defining set \(Z\) contains its Euclidean dual code if and only if \(Z \cap Z^{-1} = \emptyset\), where \(Z^{-1} = \{-z \mod n : z \in Z\}\).

The next proposition characterizes Euclidean self-orthogonal cyclic codes in terms of complementary \(q\)-cosets:

Lemma 3.1. \(n\) is a prime power. A cyclic code of length \(n\) over \(\mathbb{F}_q\) with defining set \(Z\) contains its Euclidean dual code if and only if \(\bigcup_{i=1}^{q-1} \mathbb{C}_{r_i} \cap Z = \emptyset\).

Proof: Since the \(q\)-coset \(\mathbb{C}_{-j}\) is the complementary coset of \(\mathbb{C}_j\), the result follows.

Let us now consider the following result shown in [3]:

Lemma 3.2. Let \(n \geq 1\) be an integer and \(q\) be a power of a prime such that \(\gcd(n, q) = 1\) and \(q^{\lfloor m/2 \rfloor} < n \leq q^m - 1\), where \(m = \text{ord}_n(q)\).

(a) The \(q\)-coset \(\mathbb{C}_x = \{q^j \mod n : 0 \leq j < m\}\) has cardinality \(m\) for all \(x\) in the range \(1 \leq x \leq q^{\lfloor m/2 \rfloor}/(q^m - 1)\).

(b) If \(x\) and \(y\) are distinct integers in the range \(1 \leq x, y \leq \min\{\lfloor nq^{\lfloor m/2 \rfloor}/(q^m - 1) - 1\}, n - 1\}\) such that \(x, y \not\equiv 0 \mod q\), then the \(q\)-cosets of \(x\) and \(y\) (modulo \(n\)) are disjoint.

In Theorem 3.3 the upper bound for the number of disjoint \(q\)-cosets modulo \(n = q^m - 1\) is improved, where \(m\) is an even integer. More specifically, if \(m\) is...
even, we show that the number of disjoint $q$-cosets is greater than the number of disjoint cosets presented in Lemma 3.2 Item (b). This fact is useful to compute the dimension of BCH codes whose defining set contains such $q$-cosets.

**Theorem 3.3** Let $n = q^m - 1$, where $q$ is a prime power and $m$ is even. If $x$ and $y$ are distinct integers in the range $1 \leq x, y \leq 2q^{m/2}$, such that $x, y \not\equiv 0 \pmod{q}$, then the $q$-cosets of $x$ and $y$ modulo $n$ are disjoint.

**Proof:** Recall the following result shown in (K8 Theorem 2.3)): Let $n = q^m - 1$, where $q$ is a prime power and $m$ is even. Let $s^* = \min \{ t : t \in \mathbb{C}_x \}$ be the minimum coset representative. If $0 \leq s \leq T$, where $T := 2q^{m/2}$, and $q \mid s$ then $s = s^*$, and $T$ is the greatest value having this property.

From hypothesis, the inequalities $0 \leq s \leq T := 2q^{m/2}$ hold. Thus, for every $0 \leq x, y \leq T := 2q^{m/2}$ such that $x, y \not\equiv 0 \pmod{q}$, it follows that the minimum coset representatives for $\mathbb{C}_y$ are $x$ and $y$, respectively. Since distinct minimum coset representatives belong to disjoint $q$-cosets, $\mathbb{C}_x$ and $\mathbb{C}_y$ are disjoint, as required. We are done. □

**Lemma 3.3** Let $n = q^m - 1$, where $q \geq 3$ is a prime power and $c$ be a positive integer. If the inequality $cq + 1 < \lfloor q^{m/2} \rfloor - 1$ holds then the $c$ $q$-cosets given by $\mathbb{C}_{q+1}, \mathbb{C}_{2q+1}, \mathbb{C}_{3q+1}, \ldots, \mathbb{C}_{cq+1}$ are mutually disjoint and each of them has $m$ elements. Moreover, each of them are disjoint of the $q$-cosets $\mathbb{C}_1, \mathbb{C}_2, \ldots, \mathbb{C}_c$.

**Proof:** Apply Lemma 3.2 □

Combining Lemma 3.3 and Theorem 3.4 we can construct more families of cyclic codes.

**Theorem 3.4** Let $q \geq 3$ be a prime power and $n = q^m - 1$, with $m \geq 2$, and assume that $cq + 1 < \lfloor q^{m/2} \rfloor - 1$. Then the last elements in the $c$ cosets given by $\mathbb{C}_{q+1}, \mathbb{C}_{2q+1}, \mathbb{C}_{3q+1}, \ldots, \mathbb{C}_{cq+1}$, form a sequence of $c$ consecutive integers.

**Proof:** Consider the $c$ $q$-cosets given by $\mathbb{C}_{q+1}, \mathbb{C}_{2q+1}, \mathbb{C}_{3q+1}, \ldots, \mathbb{C}_{cq+1}$. From Lemma 3.3 these $q$-cosets have cardinality $m$. Let $\mathbb{C}_x$ and $\mathbb{C}_{s+q}$ be two of them. Let $u$ and $v$ be the last elements in $\mathbb{C}_x$ and $\mathbb{C}_{s+q}$, respectively, where $u$ and $v$ are integers considered without using the modulo $n$ operation. Let $t = m - 1$; then $u = sq^{m-1} = sq^t$ and $v = (s + q)q^{m-1} = (s + q)q^t$. Since $v = sq^t + q^{t+1}$, it follows that $v \equiv sq^t + 1 \pmod{n}$, i.e., $v \equiv u + 1 \pmod{n}$. Applying the division with remainder for $v$ and $n$ and for $u + 1$ and $n$, there exist integers $a, b, r_1$ and $r_2$, where $0 \leq r_1, r_2 < n$ such that $v = an + r_1$; $u + 1 = bn + r_2$. Since $v \equiv u + 1 \pmod{n}$, it follows that $r_1 = r_2$. Since the $q$-cosets $\mathbb{C}_{q+1}, \mathbb{C}_{2q+1}, \mathbb{C}_{3q+1}, \ldots, \mathbb{C}_{cq+1}$ have cardinality $m \geq 2$, it follows that $r_1 = r_2 \neq 0$. If $v^* = r_1 = r_2$, one has $v = an + v^*$ and $u + 1 = bn + v^*$, where $1 \leq v^* < n$. Let $u^*$ be the remainder of $u$ modulo $n$. Since $u = bn + v^* - 1$, where $0 \leq v^* - 1 < n$, it follows that $v^* = u^* + 1$, as required. The proof is complete. □
4 New quantum codes

In this section we apply some results of Section 3 in order to construct CSS codes with parameters shown the Introduction. We note that constructions of quantum codes derived from classical ones by computing the generator or parity check matrices of the latter codes, in several cases, does not provide families of codes but only codes with specific parameters. This is one advantage of our constructions presented here. Let us recall the well known CSS quantum code construction:

Lemma 4.1 [25] [6] [12] Let $C_1$ and $C_2$ denote two classical linear codes with parameters $[n, k_1, d_1]_q$ and $[n, k_2, d_2]_q$, respectively, such that $C_2 \subset C_1$. Then there exists an $[[n, K = k_1 - k_2, D]]_q$ quantum code where $D = \min\{\text{wt}(c) : c \in (C_1 \setminus C_2) \cup (C_2 \setminus C_1^\perp)\}$.

In order to proceed further we establish Lemma 4.2. Note that in Lemma 4.2 the structure and the cardinality of some $q$-cosets are computed. These results allow us to compute the dimension and lower bounds for the minimum distance of the corresponding families of cyclic codes derived from such $q$-cosets.

Lemma 4.2 Let $q \geq 3$ be a prime power and $n = q^2 - 1$. Consider the $(2q - 2)$ $q$-cosets modulo $n$ given by $C_0 = \{0\}, C_1 = \{1, q\}, C_2 = \{2, 2q\}, C_3 = \{3, 3q\}, \ldots, C_{q-2} = \{q - 2, (q - 2)q\}, C_{q+1} = \{q + 1\}, C_{q+2} = \{q + 2, 1 + 2q\}, \ldots, C_{2q-1} = \{2q - 1, 1 + (q - 1)q\}$. Then, these $q$-cosets are disjoint.

In addition, with exception of the $q$-cosets $C_0$ and $C_{q+1}$, that contain only one element, all of them have exactly two elements.

Proof: It is easy to show that the inequalities $q^2 - 1 > 1 + (q - 1)q$ and $q^2 - 1 > (q - 2)q$ are true. It is clear that $q$-cosets $C_0$ and $C_{q+1}$ contain only one element. Next we show that the remaining $q$-cosets have cardinality two. If $l = lq$, where $2 \leq l \leq q - 1$ is an integer, since $l = lq < q^2 - 1$ we obtain $q = 1$, a contradiction since $q$ is a prime power. Assume that $q + l = 1 + lq$, where $2 \leq l \leq q - 1$. Then one has $l - 1 = q(l - 1)$. Since $q + l = 1 + lq < q^2 - 1$ and $l - 1 \neq 0$, one obtains $q = 1$, a contradiction.

Since these $q$-cosets have different smallest representatives, it follows that they are mutually disjoint.

In Theorem 4.1 we construct new families of good nonbinary CSS codes of length $q^2 - 1$.

Theorem 4.1 Let $q \geq 3$ be a prime power and let $n = q^2 - 1$. Then, there exist quantum codes with parameters $[[q^2 - 1, q^2 - 4q + 5, d \geq q]]_q$.

Proof: Let $C_1$ be the classical BCH code generated by $g_1(x)$, that is the product of the minimal polynomials $M^{(0)}(x)M^{(1)}(x) \ldots M^{(q-2)}(x)$, and let $C_2$ be the cyclic code generated by $g_2(x)$, that is the product of the minimal polynomials $M^{(i)}(x)$, where $M^{(i)}(x)$ are the minimal polynomials of $\alpha^i$ such that $i \notin \{q +
1, q + 2, \ldots, 2q - 1). We know the minimum distance of the code $C_1$ is greater than or equal to $q$ since its defining set contains the sequence of $q - 1$ consecutive integers given by 0, 1, \ldots, q-2. From the BCH bound, $C_1$ has minimum distance $d_1 \geq q$. Similarly, the defining set of $C$ generated by the polynomial $h(x) = \frac{x^n - 1}{g(x)}$ contains the sequence of $q - 1$ consecutive integers given by $q+1, q+2, \ldots, 2q-1$ so, from the BCH bound, $C$ also has minimum distance greater than or equal to $q$. Since the code $C_2^+$ is equivalent to $C$, then it follows that $C_2^+$ also has minimum distance greater than or equal to $q$. Therefore, the resulting CSS code has minimum distance $d \geq q$.

We know the defining set $Z_1$ of $C_1$ has $q - 1$ disjoint $q$-cosets. Moreover, from Lemma 4.2, all of them (except coset $C_0$) have two elements. Thus, $C_1$ has dimension $k_1 = q^2 - 2q + 2$. Similarly, the dimension of $C_2$ equals $k_2 = 2q - 3$, so $k_1 - k_2 = q^2 - 4q + 5$. Applying the CSS construction to the codes $C_1$ and $C_2$, we can get a CSS code with parameters $[[q^2 - 1, q^2 - 4q + 5, d \geq q]]_q$. The proof is complete.

We illustrate Theorem 4.1 by means of a graphical scheme:

$$
\begin{array}{c}
\bigcirc C_1 \\
\bigcirc C_0 C_1 C_2 \ldots C_{q-2}
\end{array}
\begin{array}{c}
\bigcirc C_2 \\
\bigcirc C_{q+1} C_{q+2} \ldots C_{2q-1}
\end{array}
\begin{array}{c}
C_r C_r \ldots C_r
\end{array}
\begin{array}{c}
\bigcirc C_2
\end{array}
$$

The union of the $q$-cosets $C_0, C_1, \ldots, C_{q-2}$ is the defining set of code $C_1$; the union of the $q$-cosets $C_0, C_1, \ldots, C_{q-2}, C_{r_1}, \ldots, C_{r_j}$ is the defining set of $C_2$, where $C_{r_1}, \ldots, C_{r_j}$ are the remaining $q$-cosets in order to complete the set of all $q$-cosets; and the union of the $q$-cosets $C_{q+1}, C_{q+2}, \ldots, C_{2q-1}$ is the defining set of $C$.

Proceeding similarly as in the proof of Theorem 4.1, we can also generate new families of quantum codes by means of Corollary 4.1.

**Corollary 4.1** There exist quantum codes with parameters $[[q^2 - 1, q^2 - 4c + 5, d \geq c]]_q$, where $c < q$, and $q \geq 3$ is a prime power.

**Proof:** Let $C_1$ and $C_2$, respectively, be the BCH codes generated by the product of the minimal polynomials $C_1 = \langle M^{(0)}(x), M^{(1)}(x), M^{(2)}(x) \ldots M^{(c-2)}(x) \rangle$ and $C_2 = \langle \prod M^{(i)}(x) \rangle$, where $M^{(i)}(x)$ are all minimal polynomials of $a^i$ such that $i \notin \{q + 1, q + 2, \ldots, q + (c - 1)\}$. Proceeding similarly as in the proof of Theorem 4.1, new families of quantum codes with good parameters $[[q^2 - 1, q^2 - 4c + 5, d \geq c]]_q$ are constructed. The proof is complete. \qed
Example 4.1 Applying Corollary 4.1, one can get quantum codes with parameters $[[15, 9, d \geq 3]], [[15, 5, d \geq 4]], [[24, 18, d \geq 3]]$, and $[[24, 14, d \geq 4]]$.

Since we improved the upper bound for the number of disjoint $q$-cosets (see Theorem 3.3), we are able to construct new families of CSS codes. Theorem 4.2, the main result of this subsection, asserts the existence of such codes. Note that Theorem 4.1 is a particular case of Theorem 4.2.

Theorem 4.2 Let $n = q^m - 1$, where $q \geq 3$ is a prime power and $m \geq 2$ is an even integer. Then there exist quantum codes whose parameters are given by $[[n, n - 2m(c - 2) - m/2 - 1, d \geq c]]_q$, where $2 \leq c \leq q$.

Proof: Recall the following result shown in [34]: $|C_s| = m$ for all $0 < s < T := 2q^{m/2}$ except $|C_{q^{m/2} + 1}| = m/2$ when $m$ is even.

Let $C_1$ be the BCH code generated by the product of the minimal polynomials $M^{(0)}(x)M^{(1)}(x)\ldots M^{(c-2)}(x)$ and $C_2$ be the cyclic code generated by $g_2(x) = \prod_i M^{(i)}(x)$, where $M^{(i)}(x)$ are the minimal polynomials of $\alpha^i$ such that $i \not\in \{q^{m/2} + 1, q^{m/2} + 2, \ldots, q^{m/2} + c - 1\}$. From the BCH bound, the minimum distance $d_1$ of $C_1$ satisfies $d_1 \geq c$ since its defining set contains the sequence $0, 1, \ldots, c - 2$ of consecutive integers. Similarly, the minimum distance of $C_2$ is also greater than or equal to $c$, because $C_2^{(i)}$ is equivalent to code $C = \langle (x^n - 1)/g_2(x) \rangle$ and $C$ contains the sequence $q^{m/2} + 1, q^{m/2} + 2, \ldots, q^{m/2} + c - 1$ of consecutive integers. The resulting CSS code has minimum distance $d \geq c$. From construction we have $C_2 \subseteq C_1$. The dimension of $C_1$ is given by $k_1 = n - m(c - 2) - 1$. Applying Theorem 3.3, since $q^{m/2} + c - 1 < T := 2q^{m/2}$ and because the corresponding $q$-cosets are mutually disjoint, it follows that $C_2$ has dimension $k_2 = m(c - 2) + m/2$; so $k_1 - k_2 = n - 2m(c - 2) - m/2 - 1$. Then there exists an $[[n, n - 2m(c - 2) - m/2 - 1, d \geq c]]_q$ quantum code, as required. □

Applying Theorem 4.1 given in the following, one can also construct good quantum codes:

Theorem 4.3 Let $n = q^m - 1$, where $q \geq 3$ is a prime power and $m \geq 2$. Then there exist quantum codes with parameters $[[n, n - m(2c - 3) - 1, d \geq c]]_q$, where $2 \leq c \leq q$ and $(c - 1)q + 1 < \lceil q^{m/2} \rceil - 1$.

Proof: Let $C_1$ be the cyclic code generated by $M^{(0)}(x)M^{(1)}(x)\ldots M^{(c-2)}(x)$, $2 \leq c \leq q$, and $C_2$ generated by $\prod_i M^{(i)}(x)$, where $i \not\in \{q + 1, 2q + 1, \ldots, (c - 1)q + 1\}$. Applying Lemma 3.3 and Theorem 3.4 and proceeding similarly as in the proof of Theorem 4.1 the result follows. □
In this section, we apply the cyclic codes constructed in Section 4 to derive new families of convolutional codes with great free distance.

The theory of convolutional codes is well investigated in the literature [7, 21, 26, 11, 27, 28, 9, 8, 16, 17, 18, 19]. We assume the reader is familiar with the theory of convolutional codes (see [11] for more details). Recall that a polynomial encoder matrix \( G(D) = (g_{ij}) \in \mathbb{F}_q[D]^{k \times n} \) is called basic if \( G(D) \) has a polynomial right inverse. A basic generator matrix is called reduced (or minimal [29, 10]) if the overall constraint length \( \gamma = \sum_{i=1}^{k} \gamma_i \), where \( \gamma_i = \max_{1 \leq j \leq n} \{ \deg g_{ij} \} \), has the smallest value among all basic generator matrices. In this case, the smallest overall constraint length \( \gamma \) is called the degree of the code.

**Definition 5.1** [4] A rate \( k/n \) convolutional code \( C \) with parameters \( (n, k, \gamma; \mu, d_f) \) is a submodule of \( \mathbb{F}_q[D]^n \) generated by a reduced basic matrix \( G(D) = (g_{ij}) \in \mathbb{F}_q[D]^{k \times n} \), i.e., \( C = \{ u(D)G(D) | u(D) \in \mathbb{F}_q[D]^k \} \), where \( n \) is the code length, \( k \) is the code dimension, \( \gamma = \sum_{i=1}^{k} \gamma_i \) is the degree, \( \mu = \max_{1 \leq i \leq k} \{ \gamma_i \} \) is the memory and \( d_f = \min \{ \wt(v(D)) : v(D) \in C, v(D) \neq 0 \} \) is the free distance of the code.

Recall that the Euclidean inner product of two \( n \)-tuples \( u(D) = \sum_i u_i D^i \) and \( v(D) = \sum_j v_j D^j \) in \( \mathbb{F}_q[D]^n \) is defined as \( \langle u(D) | v(D) \rangle = \sum_{i,j} u_i \cdot v_j \). If \( C \) is a convolutional code then we define its Euclidean dual code as \( C^\perp = \{ u(D) \in \mathbb{F}_q[D]^n : \langle u(D) | v(D) \rangle = 0 \text{ for all } v(D) \in C \} \).

Let \( C \) an \( [n, k, d]_q \) block code with parity check matrix \( H \). We split \( H \) into \( \mu + 1 \) disjoint submatrices \( H_i \) such that

\[
H = \begin{bmatrix}
H_0 \\
H_1 \\
\vdots \\
H_\mu
\end{bmatrix},
\]

where each \( H_i \) has \( n \) columns, obtaining the polynomial matrix

\[
G(D) = \tilde{H}_0 + \tilde{H}_1 D + \tilde{H}_2 D^2 + \ldots + \tilde{H}_\mu D^\mu,
\]

where the matrices \( \tilde{H}_i \), for all \( 1 \leq i \leq \mu \), are derived from the respective matrices \( H_i \) by adding zero-rows at the bottom in such a way that the matrix \( H_i \) has \( \kappa \) rows in total, where \( \kappa \) is the maximal number of rows among the matrices \( H_i \). The matrix \( G(D) \) generates a convolutional code. Note that \( \mu \) is the memory of the resulting convolutional code generated by \( G(D) \). Let \( \text{rk} A \) denote the rank of the matrix \( A \).
Theorem 5.1 \footnote{Theorem 3] Let $C \subseteq \mathbb{F}_q^n$ be an $[n, k, d]_q$ linear code with parity check $H \in \mathbb{F}_q^{(n-k) \times n}$, partitioned into submatrices $H_0, H_1, \ldots, H_\mu$, as above such that $\kappa = \text{rk} \, H_0$ and $\text{rk} \, H_i \leq \kappa$ for $1 \leq i \leq \mu$. Let $G(D)$ be the polynomial matrix given above. Then the following conditions hold:

(a) The matrix $G(D)$ is a reduced basic generator matrix;

(b) Let $V$ be the convolutional code generated by $G(D)$ and $V^\perp$ its Euclidean dual code. If $d_f$ and $d^\perp$ denote the free distances of $V$ and $V^\perp$, respectively, $d_i$ denote the minimum distance of the code $C_i = \{v \in \mathbb{F}_q^n : vH_i^\top = 0\}$ and $d^\perp$ is the minimum distance of $C^\perp$, then one has $\min\{d_0 + d_\mu, d\} \leq d^\perp \leq d$ and $d_f \geq d^\perp$.

In Theorem 5.2, the first result of this section, we construct new convolutional codes:

Theorem 5.2 Assume that $q \geq 4$ is a prime power and $n = q^2 - 1$. Then there exists a convolutional code with parameters $(n, n - 2q + 1, 2q - 3; 1, d_{\text{free}} \geq 2q + 1)_q$.

Proof: The $q$-coset $C_{q-1}$ has two elements and it is disjoint from all $q$-cosets given in Lemma 4.2. Let $C$ be the BCH code generated by $g(x)$, that is the product of the minimal polynomials $M^{(0)}(x)M^{(1)}(x) \cdots M^{(q-2)}(x)M^{(q-1)}(x)M^{(q+1)}(x) \cdots M^{(2q-1)}(x)$. A parity check matrix of $C$ is obtained from the matrix

$$
H = \begin{bmatrix}
1 & \alpha^{(0)} & \alpha^{(0)} & \cdots & \alpha^{(0)} \\
1 & \alpha^{(1)} & \alpha^{(2)} & \cdots & \alpha^{(n-1)} \\
\vdots & \vdots & \vdots & \cdots & \vdots \\
1 & \alpha^{(q-1)} & \cdots & \cdots & \alpha^{(n-1)(q-1)} \\
1 & \alpha^{(q+1)} & \cdots & \cdots & \alpha^{(n-1)(q+1)} \\
\vdots & \vdots & \vdots & \cdots & \vdots \\
1 & \alpha^{(2q-1)} & \cdots & \cdots & \alpha^{(n-1)(2q-1)}
\end{bmatrix},
$$

by expanding each entry as a column vector with respect to some $\mathbb{F}_q$-basis $\mathcal{B}$ of $\mathbb{F}_{q^2}$. Note that since $\text{ord}_n(q) = 2$, each entry contains 2 rows. This new matrix $H_C$ is a parity check matrix of $C$ and it has $4q - 2$ rows. Since $C$ has dimension $k = n - \deg g(x)$, i.e., $k = n - 4q + 4$, it follows that $H_C$ has rank $4q - 4$; $C$ has parameters $[n, n - 4q + 4, d \geq 2q + 1]_q$.

We next assume that $C_0$ is the BCH code generated by $M^{(0)}(x)M^{(1)}(x) \cdots M^{(q-2)}(x)M^{(q-1)}(x)$. $C_0$ has a parity check matrix derived from the matrix

$$
H_0 = \begin{bmatrix}
1 & \alpha^{(0)} & \alpha^{(0)} & \cdots & \alpha^{(0)} \\
1 & \alpha^{(1)} & \alpha^{(2)} & \cdots & \alpha^{(n-1)} \\
\vdots & \vdots & \vdots & \cdots & \vdots \\
1 & \alpha^{(q-1)} & \cdots & \cdots & \alpha^{(n-1)(q-1)}
\end{bmatrix},
$$

by expanding each entry as a 2-column vector with respect to $\mathcal{B}$. This new matrix is denoted by $H_{C_0}$ (note that $H_{C_0}$ is also a submatrix of $H_C$). The matrix $H_{C_0}$ has rank $2q - 1$ and the code $C_0$ has parameters $[n, n - 2q + 1, d_0 \geq q + 2]_q$. 

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Finally, let $C_1$ be the BCH code generated by $M(q+1)(x)M(q+2)(x)\cdots M(2q-1)(x)$. $C_1$ has parameters $[n, n-2q+3, d_1 \geq q]_q$. A parity check matrix $H_{C_1}$ of $C_1$ is given by expanding each entry of the matrix

\[
H_1 = \begin{pmatrix}
1 & \alpha(q+1) & \cdots & \alpha^{(n-1)(q+1)} \\
1 & \alpha(q+2) & \cdots & \alpha^{(n-1)(q+2)} \\
\vdots & \vdots & \ddots & \vdots \\
1 & \alpha^{(2q-1)} & \cdots & \alpha^{(n-1)(2q-1)}
\end{pmatrix},
\]

with respect to $B$. Since $C_1$ has dimension $n-2q+3$, $H_{C_1}$ has rank $2q-3$ ($H_{C_1}$ is also a submatrix of $H_C$).

We next construct a convolutional code $V$ generated by the matrix $G(D) = \tilde{H}_{C_0} + H_{C_1}D$, where $H_{C_0} = H_{C_0}$ and $H_{C_1}$ is obtained from $H_{C_1}$ by adding zero-rows at the bottom such that $\tilde{H}_{C_1}$ has the number of rows of $H_{C_0}$ in total. According to Theorem 5.1 Item (a), $G(D)$ is reduced and basic. We know that $\text{rk} H_{C_0} \geq \text{rk} H_{C_1}$. By construction, $V$ is a unit-memory convolutional code of dimension $2q-1$ and degree $\delta_V = 2q-3$. The Euclidean dual $V^\perp$ of the convolutional code $V$ has dimension $n-2q+1$ and degree $2q-3$. From Theorem 5.1 Item (b), the free distance $d^f$ of $V^\perp$ is bounded by $\min\{d_0 + d_1, d\} \leq d^f \leq d$, so $d^f \geq 2q + 1$. Hence, the convolutional code $V^\perp$ has parameters $(n, n-2q+1, 2q-3; 1, d^f \geq 2q+1)_q$. Now the result follows.

**Theorem 5.3** Let $q \geq 4$ be a prime power and $n = q^2 - 1$. Then there exists an $(n, n-2q, 2q-4; 1, d_{\text{free}} \geq 2q+1)_q$ convolutional code.

**Proof:** Let $C$ be the BCH code generated by $M^{(0)}(x)M^{(1)}(x)\cdots M^{(q-2)}(x)M^{(q-1)}(x)M^{(q+1)}(x)\cdots M^{(2q-1)}(x)$, given in the proof of Theorem 5.2. Suppose that $C_0$ is the BCH code generated by $M^{(0)}(x)M^{(1)}(x)\cdots M^{(q-2)}(x)M^{(q-1)}(x)M^{(q+1)}(x)$ and assume that $C_1$ is the BCH code generated by $M^{(q+2)}(x)\cdots M^{(2q-1)}(x)$. Proceeding similarly as in the proof of Theorem 5.2, the result follows.

**Theorem 5.4** Let $q \geq 4$ be a prime power and $n = q^2 - 1$. Then there exists a convolutional code with parameters $(n, n-2q+i, 2q-2-i; 1, d_{\text{free}} \geq 2q+1)_q$, where $1 \leq i \leq q-3$.

**Proof:** Let $C$ be the $[n, n-4q+4, d \geq 2q+1]_q$ BCH code generated by $M^{(0)}(x)M^{(1)}(x)\cdots M^{(q-2)}(x)M^{(q-1)}(x)M^{(q+1)}(x)\cdots M^{(2q-1)}(x)$, with parity check matrix $H_C$ of rank $4q - 4$. Suppose that $C_0$ is the BCH code generated by $M^{(0)}(x)M^{(1)}(x)\cdots M^{(q-2)}(x)M^{(q-1)}(x)M^{(q+1)}(x)\cdots M^{(q+1+i)}(x)$, where $1 \leq i \leq q-3$, with parity check matrix $H_{C_0}$ as per Theorem 5.2. $C_0$ has parameters $[n, n-2q-2i, d_0 \geq q+i+3]_q$ and $H_{C_0}$ has rank $2(q+i)$. Let $C_1$ be the BCH code generated by $M^{(q+2+i)}(x)\cdots M^{(2q-1)}(x)$, with parity check matrix $H_{C_1}$ as per Theorem 5.2. The code $C_1$ has parameters
Remark 5.1 Proof: (classical) convolutional codes constructed here. In this section we compare the parameters of the new CSS codes with the parameters of the best CSS codes shown in [2, 3] and we also exhibit some new families of convolutional codes as well. These ideas can be explored in future works.

In the proofs of Theorems 4.2 and 4.3, can be utilized similarly to derive novel other parameters can be constructed. Moreover, the classical codes constructed similarly as in the proof of Theorem 5.2, the result follows. □

Theorem 5.5 Assume that \( q \geq 4 \) is a prime power and \( n = q^2 - 1 \). Then there exists an \( (n, n - 2q + 1, 2i + 1; 1, d_{\text{free}} \geq q + i + 3)_q \) convolutional code, where \( 1 \leq i \leq q - 3 \).

Proof: Let \( C \) be the BCH code generated by \( M(0)(x)M(1)(x) \cdots M(q-2)(x) \) \( M(q-1)(x)M(q+1)(x)M(q+2)(x) \cdots M(q+i)(x) \), where \( 1 \leq i \leq q - 3 \) with parity check matrix \( H_C \) of rank \( 2q + 2i \); \( C \) has parameters \( [n, n - 2q - 2i; d \geq q + 3 + i]_q \). Let \( C_0 \) be the BCH code generated by \( M(0)(x)M(1)(x) \cdots M(q-2)(x)M(q-1)(x) \).

Let \( C_1 \) be the BCH code generated by \( M(q+1)(x)M(q+2)(x) \cdots M(q+i)(x) \). \( C_1 \) has parameters \( [n, n - 2i - 1; d_1 \geq i + 2]_q \); \( H_{C_1} \) has rank \( 2i + 1 \). Proceedings similarly as in the proof of Theorem 5.2, the result follows. □

Theorem 5.6 Assume that \( q \geq 4 \) is a prime power and \( n = q^2 - 1 \). Then there exists a convolutional codes with parameters \( (n, n - 2q + 1, 1; 1, d_{\text{free}} \geq q + 2)_q \).

Proof: Similar to that of Theorem 5.5. □

Remark 5.1 It is interesting to note that by applying the same construction method shown in this section, several new families of convolutional codes with other parameters can be constructed. Moreover, the classical codes constructed in the proofs of Theorems [3.2] and [4.3] can be utilized similarly to derive novel families of convolutional codes as well. These ideas can be explored in future works.

6 Code Comparisons

In this section we compare the parameters of the new CSS codes with the parameters of the best CSS codes shown in [2, 3] and we also exhibit some new (classical) convolutional codes constructed here.

The parameters \( [n', k', d']_q = [n, n - 2m([(\delta - 1)(1 - 1/q)])], d \geq \delta]_q \) displayed in Tables [1] and [2] are the parameters of the codes shown in [2, 3]. In Table [1] the parameters \( [n, k, d \geq c]_q \) assume the values \( [(q^2 - 1, q^2 - 4c + 5, d \geq c)]_q \), where \( 2 \leq c \leq q \) and \( q \geq 3 \) is a prime power.

As can be seen in Tables [1] and [2], the new CSS codes have parameters better than the ones available in [2, 3]. More precisely, fixing \( n \) and \( d \), the new codes achieve greater values of the number of qudits than the codes shown in [2, 3].
Table 1: Code Comparisons

| New CSS codes | Best CSS Codes in [24, 3] |
|----------------|---------------------------|
| $[n, k, d \geq c]_q$ | $[n', k', d']_q$ |
| $[24, 18, d \geq 3]_5$ | $[24, 16, d \geq 3]_4$ |
| $24, 10, d \geq 5$ | — |
| $[48, 42, d \geq 3]_7$ | $[48, 40, d \geq 3]_7$ |
| $[48, 38, d \geq 4]_7$ | $[48, 36, d \geq 4]_7$ |
| $[48, 34, d \geq 5]_7$ | $[48, 32, d \geq 5]_7$ |
| $[48, 30, d \geq 6]_7$ | $[48, 28, d \geq 6]_7$ |
| $[48, 26, d \geq 7]_7$ | — |
| $[63, 57, d \geq 3]_8$ | $[63, 55, d \geq 3]_8$ |
| $[63, 53, d \geq 4]_8$ | $[63, 51, d \geq 4]_8$ |
| $[63, 49, d \geq 5]_8$ | $[63, 47, d \geq 5]_8$ |
| $[63, 45, d \geq 6]_8$ | $[63, 43, d \geq 6]_8$ |
| $[63, 41, d \geq 7]_8$ | $[63, 39, d \geq 7]_8$ |
| $[80, 54, d \geq 8]_9$ | $[80, 52, d \geq 8]_9$ |
| $[80, 50, d \geq 9]_9$ | — |
| $[120, 114, d \geq 3]_{11}$ | $[120, 112, d \geq 3]_{11}$ |
| $[120, 106, d \geq 5]_{11}$ | $[120, 104, d \geq 5]_{11}$ |
| $[120, 98, d \geq 7]_{11}$ | $[120, 96, d \geq 7]_{11}$ |
| $[120, 90, d \geq 9]_{11}$ | $[120, 88, d \geq 9]_{11}$ |
| $[120, 82, d \geq 11]_{11}$ | — |
| $[168, 162, d \geq 3]_{13}$ | $[168, 160, d \geq 3]_{13}$ |
| $[168, 154, d \geq 5]_{13}$ | $[168, 152, d \geq 5]_{13}$ |
| $[168, 146, d \geq 7]_{13}$ | $[168, 144, d \geq 7]_{13}$ |
| $[168, 138, d \geq 9]_{13}$ | $[168, 136, d \geq 9]_{13}$ |
| $[168, 130, d \geq 11]_{13}$ | $[168, 128, d \geq 11]_{13}$ |
| $[168, 122, d \geq 13]_{13}$ | — |
| New CSS codes | CSS Codes in $[2, 3]$ |
|---------------|-----------------------|
| $[n, n - 2m(c - 2) - m/2 - 1, d \geq c]_q$ | $[n', k', d]_q$ |
| $[15, 9, d \geq 3]_4$ | $[15, 7, d \geq 3]_4$ |
| $[15, 5, d \geq 4]_4$ | $[15, 3, d \geq 4]_4$ |
| $[24, 18, d \geq 3]_5$ | $[24, 16, d \geq 3]_7$ |
| $[24, 14, d \geq 4]_5$ | $[24, 12, d \geq 4]_7$ |
| $[24, 10, d \geq 5]_5$ | $[24, 8, d \geq 5]_7$ |
| $[63, 57, d \geq 3]_8$ | $[63, 55, d \geq 3]_8$ |
| $[63, 53, d \geq 4]_8$ | $[63, 51, d \geq 4]_8$ |
| $[63, 49, d \geq 5]_8$ | $[63, 47, d \geq 5]_8$ |
| $[63, 45, d \geq 6]_8$ | $[63, 43, d \geq 6]_8$ |
| $[63, 41, d \geq 7]_8$ | $[63, 39, d \geq 7]_8$ |
| $[63, 37, d \geq 8]_8$ | $[63, 35, d \geq 8]_8$ |
| $[255, 244, d \geq 3]_4$ | $[255, 239, d \geq 3]_4$ |
| $[255, 236, d \geq 4]_4$ | $[255, 231, d \geq 4]_4$ |
| $[624, 613, d \geq 3]_5$ | $[624, 608, d \geq 3]_5$ |
| $[624, 605, d \geq 4]_5$ | $[624, 600, d \geq 4]_5$ |
| $[624, 597, d \geq 5]_5$ | $[624, 592, d \geq 5]_5$ |
| $[[n, n - m(2c - 3) - 1, d \geq c]_q, 2 \leq c \leq q]_q$ | $[[n', k', d]_q$ |
Now, we address the comparison of the new convolutional codes with the ones available in literature. The new convolutional codes constructed here have great free distance. Note that the (classical) convolutional codes constructed in [1, 16, 17] do not attain the free distance of the codes constructed in the present paper. Additionally, we did not have seen in literature convolutional codes (having corresponding $n$ and $k$) with minimum distances as great as the ones presented here. Because of this fact, it is difficult to compare the new code parameters with the ones available in literature. Therefore, we only exhibit, in Table 3 the parameters of some convolutional codes constructed here.

7 Summary

In this paper we have shown new properties on $q$-cosets modulo $n = q^m - 1$, where $q \geq 3$ is a prime power. Since the dimension and minimum distance of BCH codes are not known, these properties are important because they can be utilized to compute the dimension and bounds for the designed distance of some families of cyclic codes. Applying some of these properties, we have constructed classical cyclic codes which were utilized in the algebraic construction of new families of quantum codes by means of the CSS construction. Additionally, new families of convolutional codes have also been presented in this paper. These new quantum CSS codes have parameters better than the ones available in the literature. The new convolutional codes have free distance greater than the ones available in the literature.

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References

[1] S. A. Aly, M. Grassl, A. Klappenecker, M. Rötteler, P. K. Sarvepalli. Quantum convolutional BCH codes. In Proc. Canadian Workshop on Information Theory (CWIT), pp.180–183, 2007.

[2] S. A. Aly, A. Klappenecker, and P. K. Sarvepalli. Primitive quantum BCH codes over finite fields. In Proc. Int. Symp. Inform. Theory (ISIT), pp.1114–1118, 2006.

[3] S. A. Aly, A. Klappenecker, and P. K. Sarvepalli. On quantum and classical BCH codes. IEEE Trans. Inform. Theory, 53(3):1183–1188, 2007.

[4] S. A. Aly, A. Klappenecker, P. K. Sarvepalli. Quantum convolutional codes derived from Reed-Solomon and Reed-Muller codes. e-print arXiv:quant-ph/0701037, 2007.
### Table 3: New codes

| New convolutional codes |
|-------------------------|
| \((n, n - 2q + 1, 2q - 3; 1, d_{\text{free}} \geq 2q + 1)_q, n = q^2 - 1, q \geq 4\) |
| \(15, 8, 5; 1, d_{\text{free}} \geq 9\)_4 |
| \(24, 15, 7; 1, d_{\text{free}} \geq 11\)_5 |
| \(48, 35, 11; 1, d_{\text{free}} \geq 15\)_7 |
| \(63, 48, 13; 1, d_{\text{free}} \geq 17\)_8 |
| \(80, 63, 15; 1, d_{\text{free}} \geq 19\)_9 |
| \(120, 99, 19; 1, d_{\text{free}} \geq 23\)_11 |
| \(168, 143, 23; 1, d_{\text{free}} \geq 27\)_13 |
| \(255, 224, 29; 1, d_{\text{free}} \geq 33\)_16 |

| \((n, n - 2q, 2q - 4; 1, d_{\text{free}} \geq 2q + 1)_q, n = q^2 - 1, q \geq 4\) |
| \(15, 7, 4; 1, d_{\text{free}} \geq 9\)_4 |
| \(24, 14, 6; 1, d_{\text{free}} \geq 11\)_5 |
| \(120, 98, 18; 1, d_{\text{free}} \geq 23\)_11 |
| \(168, 142, 22; 1, d_{\text{free}} \geq 27\)_13 |
| \(255, 223, 28; 1, d_{\text{free}} \geq 33\)_16 |

| \((n, n - 2(q + i), 2(q - 2 - i); 1, d_{\text{free}} \geq 2q + 1)_q, 1 \leq i \leq q - 3, n = q^2 - 1, q \geq 4\) |
| \(15, 5, 2; 1, d_{\text{free}} \geq 9\)_4 |
| \(24, 12, 4; 1, d_{\text{free}} \geq 11\)_5 |
| \(24, 10, 2; 1, d_{\text{free}} \geq 11\)_5 |
| \(48, 32, 8; 1, d_{\text{free}} \geq 15\)_7 |
| \(48, 30, 6; 1, d_{\text{free}} \geq 15\)_7 |
| \(48, 28, 4; 1, d_{\text{free}} \geq 15\)_7 |
| \(48, 26, 2; 1, d_{\text{free}} \geq 15\)_7 |
| \(255, 221, 26; 1, d_{\text{free}} \geq 33\)_16 |
| \(255, 219, 24; 1, d_{\text{free}} \geq 33\)_16 |
| \(255, 213, 18; 1, d_{\text{free}} \geq 33\)_16 |
| \(255, 209, 14; 1, d_{\text{free}} \geq 33\)_16 |
| \(255, 203, 8; 1, d_{\text{free}} \geq 33\)_16 |
| \(255, 197, 2; 1, d_{\text{free}} \geq 33\)_16 |

| \((n, n - 2q + 1, 2i + 1; 1, d_{\text{free}} \geq q + i + 3)_q, 1 \leq i \leq q - 3, q \geq 4\) and \(n = q^2 - 1\) |
| \(15, 8, 3; 1, d_{\text{free}} \geq 8\)_4 |
| \(24, 15, 3; 1, d_{\text{free}} \geq 9\)_5 |
| \(24, 15, 5; 1, d_{\text{free}} \geq 10\)_5 |
| \(48, 35, 3; 1, d_{\text{free}} \geq 11\)_7 |
| \(48, 35, 5; 1, d_{\text{free}} \geq 12\)_7 |
| \(48, 35, 7; 1, d_{\text{free}} \geq 13\)_7 |
| \(48, 35, 9; 1, d_{\text{free}} \geq 14\)_7 |
[5] A. Ashikhmin and E. Knill. Non-binary quantum stabilizer codes. *IEEE Trans. Inform. Theory*, 47(7):3065–3072, 2001.

[6] A. R. Calderbank, E. M. Rains, P. W. Shor, and N. J. A. Sloane. Quantum error correction via codes over $GF(4)$. *IEEE Trans. Inform. Theory*, 44(4):1369–1387, 1998.

[7] G. D. Forney Jr. Convolutional codes I: algebraic structure. *IEEE Trans. Inform. Theory*, 16(6):720–738, November 1970.

[8] H. Gluesing-Luerssen, J. Rosenthal and R. Smarandache. Strongly MDS convolutional codes. *IEEE Trans. Inform. Theory*, 52:584–598, 2006.

[9] H. Gluesing-Luerssen, W. Schmale. Distance bounds for convolutional codes and some optimal codes. e-print arXiv:math/0305135, 2003.

[10] H. Gluesing-Luerssen and F-L Tsang. A matrix ring description for cyclic convolutional codes. *Advances in Math. Communications*, 2(1):55–81, 2008.

[11] R. Johannesson and K. S. Zigangirov. *Fundamentals of Convolutional Coding*. Digital and Mobile Communication, Wiley-IEEE Press, 1999.

[12] A. Ketkar, A. Kllapenecker, S. Kumar, and P. K. Sarvepalli. Nonbinary stabilizer codes over finite fields. *IEEE Trans. Inform. Theory*, 52(11):4892–4914, 2006.

[13] G. G. La Guardia. Constructions of new families of nonbinary quantum codes. *Phys. Rev. A*, 80(4):042331(1–11), 2009.

[14] G. G. La Guardia and Reginaldo Palazzo Jr.. Constructions of new families of nonbinary CSS codes. *Discrete Math.*, 310:2935–2945, 2010.

[15] G. G. La Guardia. New quantum MDS codes. *IEEE Trans. Inform. Theory*, 57(8):5551–5554, 2011.

[16] G. G. La Guardia. On nonbinary quantum convolutional BCH codes. *Quantum Inform. Computation*, 12(9-10):0820–0842, 2012.

[17] G. G. La Guardia. Nonbinary convolutional codes derived from group character codes. *Discrete Math.*, 313:2730–2736, 2013.

[18] G. G. La Guardia. On classical and quantum MDS-convolutional BCH codes. *IEEE Trans. Inform. Theory*, 60(1):304–312, 2014.

[19] G. G. La Guardia. On negacyclic MDS-convolutional codes. *Linear Alg. Applications*, 448:85–96, 2014.

[20] G. G. La Guardia. On the construction of nonbinary quantum BCH codes. *IEEE Trans. Inform. Theory*, 60(3):1528–1535, 2014.

[21] L. N. Lee. Short unit-memory byte-oriented binary convolutional codes having maximum free distance. *IEEE Trans. Inform. Theory*, 22:349–352, 1976.

[22] R. Lidl and H. Niederreiter. *Finite Fields*. Cambridge Univ.Press, 1997.

[23] F. J. MacWilliams and N. J. A. Sloane. *The Theory of Error-Correcting Codes*. North-Holland, 1977.

[24] D. M. Mandelbaum. Two applications of cyclotomic cosets to certain BCH codes. *IEEE Trans. Inform. Theory*, 26(6):737–738, 1980.

[25] M. A. Nielsen and I. L. Chuang. *Quantum Computation and Quantum Information*. Cambridge University Press, 2000.
[26] Ph. Piret. *Convolutional Codes: An Algebraic Approach*. Cambridge, Massachusetts: The MIT Press, 1988.

[27] J. Rosenthal and R. Smarandache. Maximum distance separable convolutional codes. *Applicable Algebra in Eng. Comm. Comput.*, 10:15–32, 1998.

[28] J. Rosenthal and E. V. York. BCH convolutional codes. *IEEE Trans. Inform. Theory*, 45(6):1833–1844, 1999.

[29] R. Smarandache, H. G.-Luerssen, J. Rosenthal. Constructions of MDS-convolutional codes. *IEEE Trans. Inform. Theory*, 47(5):2045–2049, 2001.

[30] A. Sharma, G. K. Bakshi, V. C. Dumir, and M. Raka. Cyclotomic number and primitive idempotents in the ring $GF(q)/(x^{5^n} − 1)$. *Finite Fields and Their Applications*, 10:653–673, 2004.

[31] A. M. Steane. Enlargement of Calderbank-Shor-Steane quantum codes. *IEEE Trans. Inform. Theory*, 45(7):2492–2495, 1999.

[32] L. Xiaoyan. Quantum cyclic and constacyclic codes. *IEEE Trans. Inform. Theory*, 50(3):547–549, 2004.

[33] D.-W. Yue and G.-Z Feng. Minimum cyclotomic coset representatives and their applications to BCH codes and Goppa codes. *IEEE Trans. Inform. Theory*, 46(7):2625–2628, 2000.

[34] D.-W. Yue and Z.-M. Hu. On the dimension and minimum distance of BCH codes over $GF(q)$. *Chin. J. Electron.*, 18:263–269, 1996.