ON CAUCHY-SZEGÖ KERNEL FOR QUATERNIONIC SIEGEL
UPPER HALF SPACE

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Abstract. The work is dedicated to the construction of the Cauchy-Szegő kernel for
the Cauchy-Szegő projection integral operator from the space of \( L^2 \)-integrable functions
defined on the boundary of the quaternionic Siegel upper half space to the space of
boundary values of the quaternionic regular functions of the Hardy space over the
quaternionic Siegel upper half space.

1. Introduction

It is a well known fact that the unit disc (or 2 dimensional ball) is bi-holomorphically
equivalent to the upper half space of the complex plane by Cayley transform. The abelian
group \((\mathbb{R},+)\) acts as translations parallel to the boundary in the upper half plane and
can be extended to the boundary. Since the action of the group \((\mathbb{R},+)\) is transitive on
the boundary, the boundary can be identified with the group by its action on the origin.
Passing to the two dimensional complex plane one obtains that 4 dimensional real open
ball is bi-holomorphically equivalent to the Siegel upper half space by 2 dimensional
Cayley transform. The abelian group \((\mathbb{R},+)\) is replaced by non-abelian Heisenberg group,
that is a subgroup of the group of automorphisms of the Siegel upper half space and
also can be extended to the transitive action on the boundary. It allows us to identify
the points on the boundary of the Siegel upper half space with the Heisenberg group.
This construction can be generalized to the \(n\)-dimensional complex space. Moreover, if
we change 2 dimensional complex space by 2 dimensional quaternionic space, then the
corresponding Cayley transform maps 8 dimensional real open ball to the quaternionic
Siegel upper half space and it extends to the boundary. The analogue of the Heisenberg
group is the, so called, quaternionic Heisenberg group and it forms a subgroup of the
group of automorphisms of the quaternionic Siegel upper half space. Extending the action
of the quaternionic Heisenberg group to the boundary of the Siegel upper half space and
taking into account its transitive action, one realizes the boundary as a group. As in the

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case of complex variables, the latter situation can be generalized to the multidimensional quaternionic space.

The classical Hardy space $H^2(\mathbb{R}_+^2)$ consists of holomorphic functions on the upper half plane $\mathbb{R}_+^2$ such that

$$\sup_{y>0} \int_{-\infty}^{+\infty} |f(x+iy)|^2 dy < +\infty.$$  

Standard argument shows that such functions have boundary values in $L^2(\mathbb{R})$. (See e.g. Chapter 3 in [16] and Chapter 2 in [13]). The set of all boundary values forms a closed subspace of $L^2(\mathbb{R})$ and the Cauchy-Szegö integral is the projection operator from $L^2(\mathbb{R})$ to this closed subspace. The Cauchy-Szegö integral is written as a convolution with the Cauchy-Szegö kernel, that in the same time is the reproducing kernel for the functions from the Hardy space $H^2(\mathbb{R}_+^2)$. Following this line, in the books [13] and [14] the construction of the Cauchy-Szegö kernel was realized as a kernel of the projection operator from $L^2(\partial\mathcal{U}_n)$ space of functions on the boundary $\partial\mathcal{U}_n$ of the Siegel upper half space to the space of boundary values of the functions from Hardy space $H^2(\mathcal{U}_n)$ over the Siegel upper half space. The projection operator is given as a convolution with respect to the Heisenberg group product and has the reproducing property. In the present paper, we present analogue of this construction for the quaternionic regular functions, the quaternionic Siegel upper half space and the quaternionic Heisenberg group. We compute the Cauchy-Szegö kernel explicitly for any dimension $n$. The construction is much more complicated than in the case $\mathbb{C}^2$.

We denote by $\mathbb{H}$ the space of quaternionic numbers $q = x_1 + x_2i + x_3j + x_4k$. We write $\text{Re}\mathbb{H}$ for one dimensional subspace of $\mathbb{H}$ spanned by 1 and $\text{Im}\mathbb{H}$ for 3 dimensional subspace of $\mathbb{H}$ spanned by $\{i, j, k\}$. The $n$-dimensional quaternionic space $\mathbb{H}^n$ is the collection of n-tuples $(q_1, \ldots, q_n)$, $q_i \in \mathbb{H}$. For l-th coordinate of a point $q = (q_1, \ldots, q_n) \in \mathbb{H}^n$ we write

$$q_l = x_{4l-3} + x_{4l-2}i + x_{4l-1}j + x_{4l}k, \quad l = 1, \ldots, n.$$  

For a domain $D \subset \mathbb{H}^n$, a $C^1$-smooth function $f = f_1 + if_2 + jf_3 + kf_4: D \to \mathbb{H}$ is called (left ) regular on $D$ if it satisfies the Cauchy-Fueter equations

$$\overline{\partial}_{q_l} f(q) = 0, \quad l = 1, \ldots, n, \quad q \in D,$$

where

$$\overline{\partial}_{q_l} = \partial_{x_{4l-3}} + i\partial_{x_{4l-2}} + j\partial_{x_{4l-1}} + k\partial_{x_{4l}}.$$  

Recently, people are interested in developing a theory for the regular functions of several quaternionic variables, as the counterpart of the theory of several complex variables for holomorphic functions (see [1], [2], [3], [5], [9], [10], [17], [18], [19] and references therein).

The quaternionic Siegel upper half space is

$$\mathcal{U}_n := \{q = (q_1, \ldots, q_n) = (q_1, q') \in \mathbb{H}^n \mid \text{Re} q_1 > |q'|^2\},$$
where we denoted \( q' = (q_2, \ldots, q_n) \in \mathbb{H}^{n-1} \). Its boundary \( \partial U_n \) is a quadratic hypersurface defined by equation

\[
\text{Re} \ q_1 = |q'|^2. \tag{1.5}
\]

Notice that the quaternionic space \( \mathbb{H}^n \) is isomorphic to \( \mathbb{R}^{4n} \) as a vector space and the pure imaginary quaternions \( \text{Im} \, \mathbb{H} \) are isomorphic to \( \mathbb{R}^3 \). The quaternion **Heisenberg group** \( qH^{n-1} \) is the space \( \mathbb{R}^{4n-1} = \mathbb{R}^3 \times \mathbb{R}^{4(n-1)} \), that is isomorphic to \( \text{Im} \, \mathbb{H} \times \mathbb{H}^{n-1} \), furnished with the non-commutative product

\[
p \cdot q = (w, p') \cdot (v, q') = (w + v + 2 \text{Im}(p', q'), p' + q') , \tag{1.6}
\]

where \( p = (w, p') \), \( q = (v, q') \in \text{Im} \, \mathbb{H} \times \mathbb{H}^{n-1} \), and \( \langle \cdot, \cdot \rangle \) is the inner product defined in (2.3) on \( \mathbb{H}^{n-1} \).

The **projection**

\[
\pi : \partial U_n \to \text{Im} \, \mathbb{H} \times \mathbb{H}^{n-1} ,
\]

\[
(|q'|^2 + x_2 i + x_3 j + x_4 k, q') \mapsto (x_2 i + x_3 j + x_4 k, q').
\]

identifies the boundary of the quaternionic Siegel upper half space \( \partial U_n = \{(q_1, q') \in U_n | \text{Re} \ q_1 = |q'|^2 \} \) with the quaternionic Heisenberg group \( qH^{n-1} \). Let \( d\beta(\cdot) \) be the Lebesgue measure on \( \partial U_n \) obtained by pulling back by the projection \( \pi \) (1.7) the Haar measure on the group \( qH^{n-1} \).

For any function \( F : U_n \to \mathbb{H} \), we write \( F_\varepsilon \) for its “vertical translate”. We mean that the vertical direction is given by the positive direction of \( \text{Re} \ q_1 \): \( F_\varepsilon (q) = F(q + \varepsilon e) \), where \( e = (1, 0, 0, \ldots, 0) \). If \( \varepsilon > 0 \), then \( F_\varepsilon \) is defined in a neighborhood of \( \partial U_n \). In particular, \( F_\varepsilon \) is defined on \( \partial U_n \). The **Hardy space** \( \mathcal{H}^2(U_n) \) consists of all regular functions \( F \) on \( U_n \), for which

\[
\sup_{\varepsilon > 0} \int_{\partial U_n} |F_\varepsilon(q)|^2 d\beta(q) < \infty . \tag{1.8}
\]

The norm \( \| F \|_{\mathcal{H}^2(U_n)} \) of \( F \) is then the square root of the left-hand side of (1.8). A function \( F \in \mathcal{H}^2(U_n) \) has boundary value \( F^b \) that belongs to \( L^2(\partial U) \) by Theorem 1.1.

Now we can state the main result of the paper.

**Theorem 1.1.** The **Cauchy-Szegő kernel** is given by

\[
S(q, p) = s \left( q_1 + \overline{p}_1 - 2 \sum_{k=2}^n \overline{p}_k q_k' \right) , \tag{1.9}
\]

for \( p = (p_1, p') = (p_1, \ldots, p_n) \in U_n \), \( q = (q_1, q') = (q_1, \ldots, q_n) \in U_n \), where

\[
s(\sigma) = c_n \frac{\partial^{2n}}{\partial x^{2n}} |\sigma|^{\frac{1}{4}}, \quad \sigma = x_1 + x_2 i + x_3 j + x_4 k \in \mathbb{H} . \tag{1.10}
\]

Here

\[
c_n = \frac{1}{2^{2n+3} \pi^{2n+1} ((2n)!)^{\frac{1}{2}} K(n) (n+2)(2n+3)} . \tag{1.11}
\]
where the constant

\[ K(n) = \sum_{k=0}^{2n} \alpha_k \sum_{l=0}^{k} C_k^l \sum_{m=0}^{l} (-1)^{k+m} C_l^m \]

\[ \sum_{s=0}^{k-2m} \frac{C_k^s}{2^{k-2m-s+1}} \frac{(-1)^s (2(k - 2m - s + 1))!}{(k - 2m - s + 1)!(4n + 5 + k - 2m - s)!} \]

depends only on the dimension \( n \), and \( \alpha_k = \frac{(2n+1-k)(2n+2-k)(4n+3+k)}{6} \).

The Cauchy-Szegő kernel satisfies the reproducing property in the following sense

\[ F(q) = \int_{\partial U_n} S(q, Q) F^b(Q) d\beta(Q), \quad q \in U_n, \]

whenever \( F \in H^2(U_n) \) and \( F^b \) its boundary value on \( \partial U_n \).

The paper is organized as follows. In Section 2 we recall the structure of quaternion numbers and the Siegel upper half space, mentioning some invariance properties. In Section 3 we study regular functions in domains of multidimensional quaternionic space. In Section 4 we discuss the boundary value of regular functions in the Siegel upper half space \( U_n \) and invariance properties of the Hardy space \( H^2(U_n) \) over \( U_n \). The main part of Section 4 is devoted to determining the Cauchy-Szegő kernel \( S \) and the proof of Theorem 1.1.

2. The quaternionic Siegel upper half space

2.1. Right quaternionic vector space. The space \( \mathbb{H} \) of quaternionic numbers forms a division algebra with respect to the coordinate addition and the quaternion multiplication

\[ \sigma q = (x_1 + ix_2 + jx_3 + kx_4)(\sigma_1 + i\sigma_2 + j\sigma_3 + k\sigma_4) \]

\[ = \sigma_1 x_1 - \sigma_2 x_2 - \sigma_3 x_3 - \sigma_4 x_4 + (\sigma_2 x_1 + \sigma_1 x_2 + \sigma_4 x_3 - \sigma_3 x_4)i + (\sigma_3 x_1 - \sigma_4 x_2 + \sigma_1 x_3 + \sigma_2 x_4)j + (\sigma_4 x_1 + \sigma_3 x_2 - \sigma_2 x_3 + \sigma_1 x_4)k, \]

for \( q, \sigma \in \mathbb{H} \). Denote by \( \text{Re} q = x_1 \) the real part of \( q \) and by \( \text{Im} q \) the imaginary part of \( q \) that is a three dimensional vector \( \vec{q} = (x_2, x_3, x_4) \).

The conjugate \( \overline{q} \) of a quaternion \( q = x_1 + x_2i + x_3j + x_4k \) is defined by \( \overline{q} = x_1 - x_2i - x_3j - x_4k \) and the norm is \( |q|^2 = \overline{q}q \). The conjugation inverses the product of quaternion numbers in the following sense \( \overline{\sigma q} = \overline{\sigma} \cdot \overline{q} \) for any \( \sigma, \tau \in \mathbb{H} \). As a vector space, \( \mathbb{H} \) is isomorphic to \( \mathbb{R}^4 \).

Since the quaternionic algebra \( \mathbb{H} \) is associative, although it is not commutative, there is a natural notion of a vector space over \( \mathbb{H} \), and many definitions and propositions for real or complex linear algebra also hold for quaternionic linear spaces, see \([4, 12, 20]\). Let us recall here some definitions and basic properties of vector spaces over \( \mathbb{H} \).

A right quaternionic vector space is a set \( V \) with addition \(+\) : \( V \times V \rightarrow V \) and right scalar multiplication \( V \times \mathbb{H} \rightarrow V, (v, \sigma) \mapsto v\sigma \), where \( V \) is an abelian group with respect to the addition, and the right scalar multiplication satisfies the following axioms:
(1) \((v + w)\sigma = v\sigma + w\sigma\),
(2) \(v(\sigma_1 + \sigma_2) = v\sigma_1 + v\sigma_2\),
(3) \(v(\sigma_1\sigma_2) = (v\sigma_1)\sigma_2\),
(4) \(v1 = v\),

for any \(v, w \in V\) and \(\sigma, \sigma_1, \sigma_2 \in \mathbb{H}\).

A hyperhermitian semilinear form on a right quaternionic vector space \(V\) is a map \(a: V \times V \to \mathbb{H}\) satisfying the following properties:

1. \(a\) is additive with respect to each argument,
2. \(a(q, q')\sigma = a(q, q')\sigma\) for any \(q, q' \in V\) and any \(\sigma \in \mathbb{H}\),
3. \(a(q, q') = \sigma a(q, q')\).

Properties (2) and (3) imply that \(a\) is conjugate right linear with respect to the first argument: \(a(q\sigma, q') = \overline{a}(q, q')\).

A quaternionic \((n \times n)\)-matrix \(A\) is called hyperhermitian if \(A^* = A\), where \((A^*)_jk := \overline{A}_{kj}\). For instance, for \(q = (q_1, \ldots, q_n), p = (p_1, \ldots, p_n) \in \mathbb{H}^n\), set \(a(q, p) = \sum_{i,j} \overline{p}_i A_{ij} p_j\). Then \(a(\cdot, \cdot)\) defines a hyperhermitian semilinear form on \(\mathbb{H}^n\).

A hyperhermitian semilinear form \(a(\cdot, \cdot)\) is called positive definite if \(a(v, v) \geq 0\) for any \(v \in V\), and \(a(v, v) = 0\) if and only if \(v = 0\). A positive definite hyperhermitian semilinear form \(a(\cdot, \cdot)\) on a right quaternionic vector space is called an inner product and will be denoted from now on by \(\langle v, w \rangle := a(v, w)\).

Now set

\[
(2.2) \quad \|v\| := \langle v, v \rangle^\frac{1}{2}, \quad \text{and} \quad \rho(v, w) = \|v - w\|.
\]

The value \(\|v\|\) is called the norm of \(v \in V\) and \(\rho(v, w)\) is a distance between \(v\) and \(w\) on \(V\).

To show that \(\rho(\cdot, \cdot)\) is a distance, we need the quaternion version of the Cauchy-Schwarz inequality: \(0 \leq \langle v, w \rangle \leq \|v\|\|w\|\), that follows from the observation

\[
0 \leq \langle v - w\sigma, v - w\sigma \rangle = \langle v, v \rangle - \sigma \langle w, v \rangle - \langle v, w \rangle \sigma + |\sigma|^2 \langle w, w \rangle.
\]

Write \(\langle v, w \rangle = r\xi\) for a unit quaternion \(\xi\) and \(r \geq 0\), and choose \(\sigma = t\overline{\xi}\), \(t \in \mathbb{R}\). Then we find that \(0 \leq \|v\|^2 - 2rt + t^2\|w\|^2\) for any \(t\). The Cauchy-Schwarz inequality follows. This makes \(V\) as a space of homogeneous type.

If \(\rho(\cdot, \cdot)\) is a complete distance, we call \((V, \langle \cdot, \cdot \rangle)\) a right quaternionic Hilbert space.

**Proposition 2.1.** (The quaternion version of Riesz’s representation theorem) Suppose that \((V, \langle \cdot, \cdot \rangle)\) is a right quaternionic Hilbert space and \(h: V \to \mathbb{H}\) is a bounded right quaternionic linear functional: \(h\) is additive and \(h(\sigma v) = h(v)\sigma\) for any \(v \in V\) and \(\sigma \in \mathbb{H}\). Then there exists a unique element \(v_h \in V\) such that

\[
h(v) = \langle v_h, v \rangle, \quad \text{for any} \quad v \in V.
\]

**Proof.** Let \(M = \ker h\), where \(\ker h\) is the kernel of the linear functional \(h\). Then \(M\) is a closed subspace because \(h\) is continuous. Moreover, \(M\) is a right quaternionic linear space since \(h\) is. Set \(M^\perp := \{v \in V | \langle w, v \rangle = \langle v, w \rangle = 0\ \text{for any} \ w \in M\}\). If \(h\) is
non-vanishing, then $M \neq V$ and so $M^\perp \neq \{0\}$. Thus, there exists an element $v_0 \in M^\perp$ such that $h(v_0) = 1$. Now $h(v - v_0h(v)) = h(v) - h(v_0)h(v) = 0$ for any $v \in V$, i.e., $v - v_0h(v) \in M$. So

$$0 = \langle v_0, v - v_0h(v) \rangle = \langle v_0, v \rangle - \|v_0\|^2h(v).$$

Namely, we can choose $v_0 = h^{-1}v_0\|v_0\|^2$. The uniqueness is easily follows from the positive definiteness of the product.

At the end of the subsection we notice that the space $\mathbb{H}^n$ is a complete right quaternionic Hilbert space endowed with the inner product

$$\langle p, q \rangle = \sum_{l=1}^{n} \overline{p}_l q_l, \quad p = (p_1, \ldots, p_n), \quad q = (q_1, \ldots, q_n) \in \mathbb{H}^n.$$ 

2.2. The quaternionic Siegel upper half space and the quaternionic Heisenberg group. The next step is to present transformations acting on the Siegel upper half space.

A quaternionic $(n \times n)$-matrix $a = (a_{jk})$ acts on $\mathbb{H}^n$ on left as follows:

$$q \mapsto aq, \quad (aq)_j = \sum_{k=1}^{n} a_{jk} q_k$$

for $q = (q_1, \ldots, q_n)^t$, where the upper index $^t$ denotes the transposition of the vector. Note that the transformation in (2.4) commutes with right multiplication by $i_\beta$ ($i_1 = 1, i_2 = i, i_3 = j, i_4 = k$), i.e.

$$(aq)i_\beta = a(qi_\beta).$$

Namely, $a$ transforms a right quaternionic line to a right quaternionic line. Here the right quaternionic line through the origin and the point $q = (q_1, \ldots, q_n)^t$ we mean the set $\{(q_1\sigma, \ldots, q_n\sigma)|\sigma \in \mathbb{H}\}$. The group $\text{GL}(n, \mathbb{H})$ is isomorphic to the group of all linear transformations of $\mathbb{R}^{4n}$ commuting with $i_3$, while the compact Lie group $\text{Sp}(n)$ consists of orthogonal transformations of $\mathbb{R}^{4n}$ commuting with $i_3$.

**Proposition 2.2.** The Siegel upper half space $U_n$ is invariant under the following transformations.

1. Translates:

$$\tau_p : (q_1, q') \mapsto (q_1 + p_1 + 2\langle p', q' \rangle, q' + p').$$

for $p = (p_1, p') = (p_1, \ldots, p_n) \in \partial U_n$, where $p' = (p_2, \ldots, p_n) \in \mathbb{H}^{n-1}$.

2. Rotations:

$$R_a : (q_1, q') \mapsto (q_1, aq')$$

for $a \in \text{Sp}(n - 1)$, and

$$R_\sigma : (q_1, q') \mapsto (\overline{q_1}\sigma q', q'\sigma)$$

for $\sigma \in \mathbb{H}$ with $|\sigma| = 1$. 

\text{□}
(3) dilations:

\( \delta_r : (q_1, q') \mapsto (r^2 q_1, rq') , \quad r > 0. \)

All the maps are extended to the boundary \( \partial \mathcal{U}_n \) and transform the boundary \( \partial \mathcal{U}_n \) to itself. Moreover, all the maps transform the hypersurface \( \partial \mathcal{U}_n + \varepsilon e \) to itself for each \( \varepsilon > 0 \).

**Proof.** The formula (2.5) follows from

\[
\operatorname{Re}(q_1 + p_1 + 2(p', q')) - |q' + p'|^2
\]

(2.9)

\[= \operatorname{Re} q_1 + \operatorname{Re} p_1 + 2\operatorname{Re}(p', q') - (|q'|^2 + |p'|^2 + 2\operatorname{Re}(p', q'))
\]

\[= \operatorname{Re}(q_1) - |q'|^2 > 0
\]

by \( \operatorname{Re} p_1 = |p'|^2 \).

The rotations (2.6) obviously map \( \mathcal{U}_n \) to itself. For rotations (2.7), note that

\[ q_1^2 = -1 \quad \text{if and only if} \quad x_1 = 0 \text{ and } x_2^2 + x_3^2 + x_4^2 = 1
\]

for a quaternion number \( q_1 = x_1 + ix_2 + jx_3 + kx_4 \). This is because of

\[ q_1^2 = x_1^2 + 2x_1(ix_2 + jx_3 + kx_4) + (ix_2 + jx_3 + kx_4)^2
\]

and

\[ (ix_2 + jx_3 + kx_4)^2 = -|ix_2 + jx_3 + kx_4|^2 = -x_2^2 - x_3^2 - x_4^2.
\]

Since

\[ \overline{\sigma} q_1 \sigma = x_1 + \overline{\sigma}(ix_2 + jx_3 + kx_4)\sigma,
\]

(2.12)

and

\[ \overline{\sigma}(ix_2 + jx_3 + kx_4)\sigma \overline{\sigma}(ix_2 + jx_3 + kx_4)\sigma = -x_2^2 - x_3^2 - x_4^2,
\]

by (2.11), we see that the second term in the right hand side of (2.12) is imaginary by using (2.10). Consequently, \( \operatorname{Re}(\overline{\sigma} q_1 \sigma) = \operatorname{Re}(q_1) \) and so

\[ \operatorname{Re}(\overline{\sigma} q_1 \sigma) - |q'\sigma|^2 = \operatorname{Re}(q_1) - |q'|^2.
\]

The invariance of the hypersurface \( \partial \mathcal{U}_n + \varepsilon e \) under the maps \( \tau_p \) and \( R_\sigma \) follows from (2.9) and (2.13). The other statements are obvious. The result follows. \( \square \)

The total group of rotations for \( \mathcal{U}_n \) is \( \text{Sp}(n-1)\text{Sp}(1) \) with \( \text{Sp}(1) \cong \{ \sigma \in \mathbb{H} \mid |\sigma| = 1 \} \).

**Remark 2.1.** Translate \( \tau_p \) can be viewed as an action of the quaternionic Heisenberg group \( qH^{n-1} \) on the quaternionic Siegel upper half space \( \mathcal{U}_n \). Let \( p = (v, p') \in qH^{n-1} \), then the translates (2.5) can be written as

\[ \tau_p : (q_1, q') \mapsto (q_1 + |p'|^2 + v + 2\langle p', q' \rangle, q' + p').
\]

It is obviously extended to the boundary \( \partial \mathcal{U}_n \). It is easy to see that the action on \( \partial \mathcal{U}_n \) is transitive, for calculation see also [6]. Therefore, we can identify points in \( qH^{n-1} \) with points in \( \partial \mathcal{U}_n \) by the result of the translates by \( \tau_p \) of the origin \( (0, 0) \).
3. Regular functions on the quaternionic Siegel upper half space

In the present Section we show the invariance of the regularity under linear transformations in Proposition 2.2.

**Proposition 3.1.** Let \( f: D \rightarrow \mathbb{H} \) be \( C^1 \)-smooth function, where \( D \) is a domain in \( \mathbb{H}^n \).

(1) Define the pull-back function \( \widehat{f} \) of \( f \) under the mapping \( q \rightarrow Q = aq \) for \( a = (a_{jk}) \in \text{GL}(n, \mathbb{H}) \) by \( \widehat{f}(q) := f(aq) \). Then we have

\[
\partial_{q_j} \widehat{f}(q) = \sum_{k=1}^{n} a_{kj} \partial_{Q_k} f(Q) \big|_{Q=aq} .
\]

(2) Define the pull-back function \( \tilde{f} \) of \( f \) under the mapping \( q \rightarrow Q = q \sigma \) for \( \sigma \in \mathbb{H} \) by \( \tilde{f}(q) := f(q_1 \sigma, \ldots, q_n \sigma) \). Then

\[
\partial_{q_l} \tilde{f}(q) = \partial_{Q_l} f(Q) \big|_{Q=q \sigma} , \quad l = 1, \ldots, n.
\]

**Proof.** The proof of the first statement can be found in [20, Proposition 3.1].

The second statement is analogous to the formula of one quaternionic variable. Write the \( l \)-th coordinate of \( q = (q_1, \ldots, q_n) \) as the quaternionic number \( q_l = x_1 + ix_2 + jx_3 + kx_4 \in \mathbb{H} \), and define the associated real vector \( q_l^R := (x_1, x_2, x_3, x_4)^t \) in \( \mathbb{R}^4 \). Then for the product

\[
q_l \sigma = (x_1 + ix_2 + jx_3 + kx_4)(\sigma_1 + i\sigma_2 + j\sigma_3 + k\sigma_4)
= \sigma_1 x_1 - \sigma_2 x_2 - \sigma_3 x_3 - \sigma_4 x_4
+ (\sigma_2 x_1 + \sigma_1 x_2 + \sigma_4 x_3 - \sigma_3 x_4)i
+ (\sigma_3 x_1 - \sigma_4 x_2 + \sigma_1 x_3 + \sigma_2 x_4)j
+ (\sigma_4 x_1 + \sigma_3 x_2 - \sigma_2 x_3 + \sigma_1 x_4)k
\]

we define the associated matrix

\[
\tilde{\sigma}^R := \begin{pmatrix}
\sigma_1 & -\sigma_2 & -\sigma_3 & -\sigma_4 \\
\sigma_2 & \sigma_1 & \sigma_4 & -\sigma_3 \\
-\sigma_3 & -\sigma_4 & \sigma_1 & \sigma_2 \\
-\sigma_4 & \sigma_3 & -\sigma_2 & \sigma_1
\end{pmatrix}.
\]

Thus (3.3) can be written as

\[
(q_l \sigma)^R = \tilde{\sigma}^R q_l^R,
\]

for \( \tilde{\sigma}^R \) given by (3.4). It follows from (3.4) that \( \tilde{\sigma}^R = (\tilde{\sigma}^R)^t \), where \( \tilde{\sigma} \) is the conjugate of \( \sigma \).
Denote \((y_1, \ldots, y_4)^t = \sigma^R(x_1, \ldots, x_4)^t\), i.e. \(y_k = \sum_{k=1}^4 \sigma^R_{kj}x_j, \ k = 1, \ldots, 4\). Since \(\partial_{x_j}[f(\ldots, \sigma^R_{kj}q_{j_1}, \ldots)] = \sum_{k=1}^4 \sigma^R_{kj}\partial_{y_k}f(\ldots, y_\ldots, \ldots)\), we find that
\[
\overline{\partial}_q[f(\ldots, \sigma^R_{kj}q_{j_1}, \ldots)] = \sum_{j=1}^4 i_j\partial_{x_j}[f(\ldots, \sigma^R_{kj}q_{j_1}, \ldots)]
= \sum_{j,k=1}^4 i_j\sigma^R_{kj}\partial_{y_k}f(\ldots, y_\ldots, \ldots)
= \overline{\partial}_q^a(\sigma f)(\ldots, Q_1, \ldots),
\]
by \(\left(\sum_{j=1}^4 i_j\partial_{y_j}\right)\overline{\sigma} = \sum_{j,k=1}^4 i_j\sigma^R_{kj}\partial_{y_k}\) and (3.5).

Corollary 3.1. If \(f\) is regular, then \(\hat{f} = f(aq)\) for some \(a \in \text{GL}(n, \mathbb{H})\) and \(\tilde{f} = f(q\sigma)\) for some \(\sigma \in \mathbb{H}\) are both regular.

Corollary 3.2. The space of all regular functions on \(U_n\) is invariant under the transformations defined in Propositions 2.2. Namely, if \(f\) is regular on the Siegel upper half space \(U_n\), then the functions \(f(\tau_p(q))\), \(p \in \partial U_n\); \(f(R_a(q))\), \(a \in \text{Sp}(n-1)\); \(\sigma f(R_a(q))\) for some \(\sigma \in \mathbb{H}\) with \(|\sigma| = 1\), and \(f(\delta_s(q))\) are all regular on \(U_n\).

Proof. The translate \(\tau_p\) in (2.5) can be represented as a composition of the linear transformation given by the quaternionic matrix
\[
\begin{bmatrix}
1 & 2p' \\
0 & I_{n-1}
\end{bmatrix},
\]
and the Euclidean translate \((q_1, q') \mapsto (q_1 + p_1, q' + p')\). The first transformation preserves the regularity of a function by Proposition 3.1 while the later one obviously preserves the regularity of a function since the Cauchy-Fueter operators are of constant coefficients.

The equation
\[
\overline{\partial}_q[\sigma f(q\sigma)] = \overline{\partial}_Q[\overline{\sigma}f(Q)]|_{Q=q\sigma} = |\sigma|^2\overline{\partial}_Q f(Q)|_{Q=q\sigma} = 0,
\]
follows from Proposition 3.1(2) and shows that \(\sigma f(\overline{\sigma}q_1\sigma, q'\sigma)\) is regular.

The rest of the corollary is obvious. \(\square\)

4. Hardy space \(H^2(U_n)\)

This section is devoted to the properties of Hardy space on \(U_n\). The identification of the quaternionic Heisenberg group and the boundary of the quaternionic Siegel upper half space allows to define the Lebesgue measure \(d\beta(\cdot)\) on \(\partial U_n\) by pulling back by the projection \(\pi\) (1.7) the Haar measure on \(qH^{n-1}\). The latter measure, in its term, is a pull back of the Lebesgue measure \(d\mu(\cdot) = dx\, dq'\) from \(\mathbb{R}^3 \times \mathbb{R}^{4(n-1)}\). Let \(L^2(\partial U_n)\) denote the space of all \(\mathbb{H}\)-valued functions which are square integrable with respect to the measure
\(d\beta\). It is easy to see by definition that \(L^2(\partial \mathcal{U}_n)\) is a right quaternionic Hilbert space with the following inner product:

\[
(4.1) \quad \langle f, g \rangle_{L^2} = \int_{\partial \mathcal{U}_n} \overline{f(q)} g(q) d\beta(q).
\]

A function \(F \in H^2(\mathcal{U}_n)\) has boundary value \(F^b\) that belongs to \(L^2(\partial \mathcal{U})\) in the following sense.

**Theorem 4.1.** Suppose that \(F \in H^2(\mathcal{U}_n)\). Then

1. There exists a function \(F^b \in L^2(\partial \mathcal{U}_n)\) such that \(F(q + \varepsilon e)_{|\partial \mathcal{U}_n} \to F^b(q)\) as \(\varepsilon \to 0\) in \(L^2(\partial \mathcal{U}_n)\) norm.
2. \(\|F^b\|_{L^2(\partial \mathcal{U}_n)} = \|F\|_{H^2(\mathcal{U}_n)}\).
3. The space of all boundary values forms a closed subspace of the space \(L^2(\partial \mathcal{U}_n)\).

**Proof.** This theorem was proved in [8, Theorem 4.2] for \(n = 2\). The arguments work for an arbitrary \(n\) if we consider the following slice functions. Let \(H^2(\mathbb{R}^n_+)\) be the classical Hardy space, that is the set of all harmonic functions \(u: \mathbb{R}^n_+ \to \mathbb{R}\) such that

\[
\sup_{t > 0} \|u(t, \cdot)\|_{L^2(\mathbb{R}^n_+)} < \infty.
\]

Assume that \(F = F_1 + iF_2 + jF_3 + kF_4 \in H^2(\mathcal{U}_n)\). Then the slice function \(f_j(q_1) := F_j(q_1 + |q'|^2, q')\) is harmonic by (4.3), and belongs to \(H^2(\mathbb{R}^n_+)\) for each \(j = 1, \ldots, 4\) and any fixed \(q' \in \mathbb{H}^{n-1}\). We omit further details. \(\square\)

**Proposition 4.1.** The Hardy space \(H^2(\mathcal{U}_n)\) is a right quaternionic Hilbert space under the inner product \(\langle F, G \rangle = \langle F^b, G^b \rangle_{L^2(\partial \mathcal{U}_n)}\).

**Proof.** Since the Cauchy-Fueter operator \(\overline{\partial}_{q_1}\) in (1.3) is right quaternionic linear, i.e., for a fixed \(\sigma\)

\[
\overline{\partial}_{q_1}(f(q)\sigma) = (\overline{\partial}_{q_1}f(q))\sigma,
\]

we see that \(f(q)\sigma\) is regular if \(f(q)\) is. Thus, the Hardy space \(H^2(\mathcal{U}_n)\) is a right quaternionic vector space.

Set

\[
(4.2) \quad \partial_{q_{n+1}} f := \overline{\partial}_{q_{n+1}} f = \partial_{x_{4n+1}} f - \partial_{x_{4n+2}} f_1 - \partial_{x_{4n+3}} f j - \partial_{x_{4n+4}} f k.
\]

It is straightforward to see that

\[
(4.3) \quad 0 = \partial_{q_{n+1}} \overline{\partial}_{q_{n+1}} f = (\partial^2_{x_{4n+1}} + \partial^2_{x_{4n+2}} + \partial^2_{x_{4n+3}} + \partial^2_{x_{4n+4}}) f.
\]

Consequently, \(f_1, \ldots, f_4\) are harmonic on the line \(\{(0, \ldots, q_1, \ldots, 0)\}\) and so \(f_1, \ldots, f_4\) are harmonic on \(\mathbb{H}^n\). Thus for \(q \in \mathcal{U}_n\),

\[
f_j(q) = \frac{1}{|B|} \int_B f_j(p) dV(p), \quad j = 1, 2, 3, 4,
\]

where \(B\) is a small ball centered at \(q\) and contained in \(\mathcal{U}_n\), from which we see that

\[
(4.4) \quad |f(q)| \leq \frac{1}{|B|} \int_B |f(p)| dV(p) \leq \left(\frac{1}{|B|} \int_B |f(p)|^2 dV(p)\right)^{\frac{1}{2}}.
\]
There exist $a, b > 0$ such that $B \subset \mathcal{U}_{n,a,b} := \{ q \in \mathcal{U}_n \mid a < \text{Re } q_1 - |q|^2 < b \}$, and so

$$
|f(q)|^2 \leq \frac{1}{|B|} \int_{\mathcal{U}_{n,a,b}} |f(x_1, \ldots, x_{4n})|^2 dx_1 \cdots dx_{4n}
$$

(4.5)

$$
\leq \frac{1}{|B|} \int_{(a,b) \times \mathbb{R}^{4n-1}} \left| f \left( x_1 + \sum_{j=5}^{4n} |x_j|^2, x_2, \ldots, x_{4n} \right) \right|^2 dx_1 dx_2 \cdots dx_{4n}
$$

$$
\leq \frac{1}{|B|} \int_a^b dx_1 \int_{\partial \mathcal{U}_n} |f(p + x_1 e)|^2 d\beta(p) \leq c \|f\|^2_{H^2(\mathcal{U}_n)},
$$

where $c = (b - a)/|B|$ is a positive constant depending on $q$, and independent of the functions $f \in H^2(\mathcal{U}_n)$. Here we have used the coordinates transformation $(x_1, \ldots, x_{4n}) \rightarrow (x_1 + \sum_{j=5}^{4n} |x_j|^2, x_2, \ldots, x_{4n})$, whose Jacobian is identity.

To prove the completeness, we suppose that a Cauchy sequence $\{f^{(k)}\}$ in the Hardy space $H^2(\mathcal{U}_n)$ is given. We need to show that some subsequence converges to an element in $H^2(\mathcal{U}_n)$. Apply the estimate (4.5) to regular functions $f^{(k)} - f^{(l)}$ to get that for any compact subset $K \subset \mathcal{U}_n$ and $q \in K$,

$$
|f^{(k)}(q) - f^{(l)}(q)| \leq c_K \|f^{(k)} - f^{(l)}\|_{H^2(\mathcal{U}_n)},
$$

(4.6)

where $c_K$ is a positive constant only depending on $K$. It means that the sequence $\{f^{(k)}\}$ converges uniformly on any compact subset of $\mathcal{U}_n$. Denote by $f$ the limit. Recall the well known estimate

$$
\|u\|_{C^1(B(q, r))} \leq C_r \|u\|_{C^0(B(q, 2r))}
$$

for any harmonic function $u$ defined on the ball $B(q, 2r)$, where $C_r$ is a positive constant only depending on $r$ and the dimension, and independent of a function $u$ (see e.g., pp. 307-312 in [15]). Now apply the estimate (4.7) to each component of regular function $f = f_1 + if_2 + jf_3 + kf_4$, which is harmonic. By the argument of finite covering and estimate (4.5), we easily see that

$$
\|f\|_{C^1(K)} \leq C'_K \|f\|_{H^2(\mathcal{U}_n)}
$$

for some constant $C'_K$ only depending on $K$. It follows that $|\partial_{x_j} f^{(k)}(q) - \partial_{x_j} f^{(l)}(q)| \leq C'_K \|f^{(k)} - f^{(l)}\|_{H^2(\mathcal{U}_n)}$ for $q \in K$, $j = 1, \ldots, 4n$. Consequently, the limit function $f$ is also $C^1$ and $\lim_{k \to \infty} \partial_{x_j} f^{(k)}(q) = \partial_{x_j} f(q)$. Thus, $\partial_{\bar{q}} f(q) = \lim_{k \to \infty} \partial_{\bar{q}} f^{(k)}(q) = 0$. Namely, the limit function $f$ is regular.

Since on the compact subset $K_{R, \varepsilon} := \partial \mathcal{U}_n \cap B(0, R) + \varepsilon e$ for fixed $R, \varepsilon > 0$, the sequence $\{f^{(k)}\}$ is uniformly convergent, we find that

$$
\int_{\partial \mathcal{U}_n \cap B(0, R)} |f_\varepsilon(q)|^2 d\beta(q) = \int_{K_{R, \varepsilon}} |f(q)|^2 d\beta(q)
$$

$$
= \lim_{k \to \infty} \int_{K_{R, \varepsilon}} |f^{(k)}(q)|^2 d\beta(q) \leq \sup_k \|f^{(k)}\|_{H^2(\mathcal{U}_n)} < \infty.
$$
Consequently, \( f_\varepsilon \) is square integrable on \( \partial \mathcal{U}_n \) for every \( \varepsilon > 0 \), and \( \int_{\partial \mathcal{U}_n} |f_\varepsilon(q)|^2 d\beta(q) \leq \sup \|f^{(k)}\|_{H^2(\mathcal{U}_n)}. \) Thus \( f \in H^2(\mathcal{U}_n). \) \( \square \)

**Proposition 4.2.** The Hardy space \( H^2(\mathcal{U}_n) \) is invariant under the transformations of Proposition 2.2.

**Proof.** Since the regularity property and the hypersurface \( \partial \mathcal{U}_n + \varepsilon e \) for each \( \varepsilon > 0 \) are invariant under these transformations by Corollary 3.2 and the measure \( d\beta \) either invariant or has a finite distortion, the proof follows. \( \square \)

5. The Cauchy-Szegő kernel

In this section we introduce the notion of the Cauchy-Szegő kernel for the projection operator from the space \( L^2(\partial \mathcal{U}) \) to the space of the boundary values of function from Hardy space \( H^2(\mathcal{U}_n) \). We study its properties, particularly showing that it is invariant under translates, rotations and dilations defined in Proposition 2.2 and, finally, we present the formula for the Cauchy-Szegő kernel.

5.1. Existence and characterization of the Cauchy-Szegő kernel.

**Theorem 5.1.** The Cauchy-Szegő kernel \( S(q,p) \) is a unique \( \mathbb{H} \)-valued function, defined on \( \mathcal{U}_n \times \mathcal{U}_n \) satisfying the following conditions.

1. For each \( p \in \mathcal{U}_n \), the function \( q \mapsto S(q,p) \) is regular for \( q \in \mathcal{U}_n \), and belongs to \( H^2(\mathcal{U}_n) \). This allows to define the boundary value \( S^b(q,p) \) for each \( p \in \mathcal{U}_n \) and for almost all \( q \in \partial \mathcal{U}_n \).
2. The kernel \( S \) is symmetric: \( S(q,p) = S(p,q) \) for each \( (q,p) \in \mathcal{U}_n \times \mathcal{U}_n \). The symmetry permits to extend the definition of \( S(q,p) \) so that for each \( q \in \mathcal{U}_n \), the function \( S_0(q,p) \) is defined for almost every \( p \in \partial \mathcal{U}_n \) (here we use the subscript \( b \) to indicate the boundary value with respect to the second argument).
3. The kernel \( S \) satisfies the reproducing property in the following sense

\[
F(q) = \int_{\partial \mathcal{U}_n} S_b(q,Q)F^b(Q)d\beta(Q), \quad q \in \mathcal{U}_n,
\]

whenever \( F \in H^2(\mathcal{U}_n) \).

**Proof.** We must show that the Hardy space \( H^2(\mathcal{U}_n) \) is nontrivial first. Otherwise the Cauchy-Szegő kernel vanishes. We claim that the function \( s(\bar{q}_1, -2 \sum_{k=2}^{n} \bar{g}'_k') \) for fixed \( (p_1, \ldots, p_n) \in \mathcal{U}_n \), with \( s(\cdot) \) given by (1.10), is in the Hardy space \( H^2(\mathcal{U}_n) \).

Apply the Laplace operator

\[
\overline{\partial}_{q_1} \partial_{q_1} = \partial^2_{x_1} + \partial^2_{x_2} + \partial^2_{x_3} + \partial^2_{x_4}
\]

to the harmonic function \( \frac{1}{|q_1|} \) on \( \mathbb{H} \setminus \{0\} \) to see that \( \partial_{q_1} \frac{1}{|q_1|} = -\frac{2\bar{q}_1}{|q_1|^3} = h(q_1) \) is a regular function on \( \mathbb{H} \setminus \{0\} \), which is homogeneous of degree \(-3\). Since \( \frac{2\bar{q}_1}{|q_1|^3} \) commutes with \( \overline{\partial}_{q_1} \), the function \( s(q_1) = c_{q_1} \frac{2\bar{q}_1}{|q_1|^3} \) in (1.10) is regular on \( \mathbb{H} \setminus \{0\} \). Consequently, \( s(q_1 + x_1) \) for fixed \( x_1 > 0 \) ia also regular on \( \mathbb{H} \setminus \{-x_1\} \), and so \( \tilde{s}(q_1, \ldots, q_n) := s(q_1 + x_0) \) is regular.
By definition (5.5), \(\varepsilon\) taking Theorem 4.1. Moreover, we have

\[ F \rightarrow F(q). \]

It is bounded by estimate (4.5). Apply the quaternion version of Riesz’s representation theorem to see that there exists an element, denoted by \(K\), which is obviously integrable with respect to the measure \(d\beta\). Namely, \(\tilde{s}(\cdot)\) is in Hardy space \(H^2(\mathcal{U}_n)\), and so is \(\tilde{s}(\tau_{p^{-1}}(q))\) by the invariance of the Hardy space under the translates in Proposition 4.2. The claim is proved.

Now for fixed \(q \in \mathcal{U}_n\), define a quaternion-valued right linear functional

\[ F \mapsto F(q). \]

(5.3)

For fixed \(p \in \mathcal{U}_n\), applying (5.4) to \(K(\cdot, q)\) and \(K(\cdot, q)\), we see that

\[ F(q) = \int_{\partial \mathcal{U}_n} \overline{K_b(Q, q)} F_b(Q) d\beta(Q). \]

(5.4)

Denote \(S_b(q, p) := \overline{K(p, q)}\) for \((q, p) \in \mathcal{U}_n \times \mathcal{U}_n\). Then \(S_b(q, p) = K(p, q)\) is regular in \(q\), and \(S_b(q, p) = K(p, q) = S(p, q)\). The function \(S\) has the boundary values as in Theorem 4.1. Moreover, we have

\[ S_b(Q, p) = \overline{S_b(p, Q)} \]

(5.5)

for \(p \in \mathcal{U}_n\), \(Q \in \partial \mathcal{U}_n\), which follows from the symmetry \(S(q + \varepsilon e, p) = \overline{S(p, q + \varepsilon e)}\) by taking \(\varepsilon \rightarrow 0^+\).

To show the uniqueness, suppose that \(\tilde{S}(\cdot, \cdot)\) is another function satisfying Theorem 5.1. By definition \(\tilde{S}(\cdot, q) \in H^2(\mathcal{U}_n)\) for any fixed \(q \in \mathcal{U}_n\). Choose an arbitrary \(p \in \mathcal{U}_n\) and apply the reproducing formula (5.1) of \(S(\cdot, \cdot)\) and \(\tilde{S}(\cdot, \cdot)\) to get

\[ \tilde{S}(p, q) = \int_{\partial \mathcal{U}_n} S_b(p, Q) \tilde{S}_b(Q, q) d\beta(Q) = \int_{\partial \mathcal{U}_n} \overline{\tilde{S}_b(Q, q)} S_b(p, Q) d\beta(Q) \]

\[ = \int_{\partial \mathcal{U}_n} \tilde{S}_b(q, Q) \overline{\tilde{S}(q, p)} d\beta(Q) = S(q, p). \]
In the third identity, we used the equation (5.5) for $S(\cdot, \cdot)$ and $\tilde{S}(\cdot, \cdot)$. The theorem is proved. □

The function $S(q, p)$ is conjugate right regular in variables $p = (p_1, \ldots, p_n)$:

$$\partial_p S(q, p) = \overline{\partial_p K(p, q)} = 0.$$

5.2. Invariance of the Cauchy-Szegő kernel. Since the Siegel upper half space possesses some invariance properties, it is expected that the Cauchy-Szegő kernel also inherits them. Namely, the following proposition is true.

**Proposition 5.1.** The Cauchy - Szegő kernel has following invariance properties.

$$S(\tau_p(q), \tau_p(Q)) = S(q, Q),$$

$$S(R_\mathbf{a}(q), R_\mathbf{a}(Q)) = S(q, Q),$$

$$\sigma S(R_\mathbf{a}(q), R_\mathbf{a}(Q))\overline{\sigma} = S(q, Q),$$

$$S(\delta_r(q), \delta_r(Q))r^{4n+6} = S(q, Q).$$

(5.6)

where $p \in \partial U_n$, $\mathbf{a} \in \text{Sp}(n-1)$, $\sigma \in \mathbb{H}$ with $|\sigma| = 1$ and $r > 0$.

**Proof.** Note that the measure $d\beta(Q)$ is invariant under the translate $\tau_p$. If $F \in H^2(U_n)$, then $F(\tau_{-p}(q)) \in H^2(U_n)$ by Proposition 4.2. We get

$$F(\tau_{-p}(q)) = \int_{\partial U_n} S_b(q, \tau_p(Q))F^b(\tau_{-p}(Q))d\beta(Q) = \int_{\partial U_n} S_b(q, \tau_p(Q))F^b(Q)d\beta(Q),$$

and by substituting $\tau_{-p}(q) \mapsto q$, we obtain

$$F(q) = \int_{\partial U_n} S_b(\tau_p(q), \tau_p(Q))F^b(Q)d\beta(Q).$$

We conclude that the function $S(\tau_p(q), \tau_p(Q))$ is also regular in $q$ by Corollary 3.2 and it is symmetric. The first identity in (5.6) follows by the uniqueness in Theorem 5.1.

It follows from $|\xi\sigma| = |\sigma\xi| = |\xi|$ for any quaternionic number $\xi \in \mathbb{H}$ that $(q_1, q') \mapsto (q_1\sigma, q')$ and $(q_1, q') \mapsto (\overline{q_1}, q')$ are both orthogonal maps, so is their composition $R_\sigma: (q_1, q') \mapsto (\overline{q_1}, q')$. If $F \in H^2(U_n)$, then $\sigma^{-1}F(R_{\sigma^{-1}}(q))$ is regular by Corollary 3.2 and is in $H^2(U_n)$ by definition. Therefore,

$$\sigma^{-1}F(R_{\sigma^{-1}}(q)) = \int_{\partial U_n} S_b(q, \tau_p(Q))\sigma^{-1}F^b(R_{\sigma^{-1}}(Q))d\beta(Q)$$

$$= \int_{\partial U_n} S_b(q, R_\sigma(Q))\sigma^{-1}F^b(Q)d\beta(Q),$$

since $d\beta$ is invariant under the orthogonal transformation $R_\sigma$. Substituting $R_{\sigma^{-1}}(q) \mapsto q$ and multiplying by $\sigma$ on both sides, we get

$$F(q) = \int_{\partial U_n} \sigma S_b(R_\sigma(q), R_\sigma(Q))\overline{\sigma}F^b(Q)d\beta(Q).$$

The function $\sigma S(R_\sigma(q), R_\sigma(Q))\overline{\sigma}$ is also regular in $q$ by Corollary 3.2 again and it is symmetric. The third identity in (5.6) follows by the uniqueness in Theorem 5.1.

The second and the fourth identities are proved by the similar arguments. □
5.3. Determination of the Cauchy-Szegő kernel. It is sufficient to show $S_b(q, 0) = s(q_1)$. This is because

$$S_b(q, p) = S_b(p^{-1} \cdot q, 0) = s \left( q_1 - p_1 - 2 \sum_{k=2}^{n} p_k q'_k \right)$$

for $p = (p_1, \ldots, p_l) \in \partial U_n$, $q \in U_n$. Taking conjugate in both sides of (5.7), we see that

$$S^b(q, p) = s \left( q_1 - p_1 - 2 \sum_{k=2}^{n} p_k q'_k \right)$$

holds for $p \in U_n$ and $q \in \partial U_n$ by the symmetry of the Cauchy-Szegő kernel $S(q, p)$ in Theorem 5.1. Now we fix a point $(p_1, \ldots, p_n) \in U_n$. In the proof of Theorem 5.1, we have seen that $s(q_1 - p_1 - 2 \sum_{k=2}^{n} p_k q'_k)$ is in the Hardy space $H^2(U_n)$. As elements of the Hardy space $H^2(U_n)$, $S(\cdot, p)$ and $s(q_1 - p_1 - 2 \sum_{k=2}^{n} p_k q'_k)$ coincide on the boundary $\partial U_n$. They must coincide on the whole $U_n$ by the uniqueness of the Cauchy-Szegő kernel following from the reproducing property (5.1).

Since by

$$0 = \sum_{l=2}^{n} \partial_{\bar{q}} u(q_1, q') = \sum_{l=2}^{n} (\partial_{q_{4l-3}}^2 + \partial_{q_{4l-2}}^2 + \partial_{q_{4l-1}}^2 + \partial_{q_{4l}}^2) u(q_1, q'),$$

where $u(q) = S_b(q, 0)$, each component of $u(q_1, \cdot)$ is a harmonic function on the ball $\{q' \in \mathbb{H}^{n-1} \mid |q'| < q_1 \}$ for fixed $q_1$ with $\Re q_1 > 0$. On the other hand,

$$s_b((q_1, aq'), 0) = s_b((q_1, q'), 0) \quad \text{for} \quad q \in U_n,$$

by Proposition 5.1. Since $\text{Sp}(n-1)$ acts on the sphere $\{q' \in \mathbb{H}^{n-1} \mid |q'| = R \}$ transitively, where $R < \Re q_1$, we see that $S_b((q_1, q'), 0)$ is constant on the sphere. Applying the maximum principle to each component of $S_b((q_1, q'), 0)$ as a harmonic function in $q'$, we conclude that $S_b((q_1, q'), 0)$ is constant on the ball $\{q' \in \mathbb{H}^{n-1} \mid |q'| < q_1 \}$, and so $S_b((q_1, q'), 0) = S_b((q_1, 0), 0)$. Denote $s(q_1) := S_b((q_1, 0), 0)$, an $\mathbb{H}$-valued function defined on the half space $\mathbb{H}^n_+ = \{q_1 \in \mathbb{H} \mid \Re q_1 > 0\}$.

By the third identity in (5.6) we have $\sigma s_b((\bar{\sigma} q_1, 0), 0) \sigma = s_b((q_1, 0), 0)$. More precisely,

$$s(\bar{\sigma} q_1, \sigma) = \bar{\sigma} s(q_1) \sigma,$$

for any $\sigma \in \mathbb{H}$ with $|\sigma| = 1$, and similarly

$$s(r q_1) = r^{-2n-3} s(q_1).$$

by the fourth identity in (5.6) and $\delta_r : (q_1, 0) \mapsto (r^2 q_1, 0)$.

Take $q_1 = x_1 \in \mathbb{R}$ in (5.11) to get $s(x_1) = \bar{\sigma} s(x_1) \sigma$. Write $s(x_1) = \xi_1 + \xi_2 i + \xi_3 j + \xi_4 k$ and choose $\sigma = i$. Then $\xi_1 + \xi_2 i + \xi_3 j + \xi_4 k = i(\xi_1 + \xi_2 i + \xi_3 j + \xi_4 k) i = \xi_1 + \xi_2 i - \xi_3 j - \xi_4 k$, and so $\xi_3 = \xi_4 = 0$. Similarly, $\xi_2 = 0$ by choosing $\sigma = j$. Thus, (5.11) implies that $s(x_1)$ must be real.
Note that
\begin{equation}
(5.13) \quad \overline{\sigma}(x_1 + ix_2)\sigma = x_1 + x_2[(2y_2^2 - 1)i + 2y_2y_3j + 2y_2y_4k],
\end{equation}
if \(\sigma = y_2i + y_3j + y_4k\) with \(|\sigma| = 1\). It easily follows from (5.13) that the orbit of \(x_1 + ix_2\)
under the adjoint action of unit quaternions is the 2-dimensional sphere
\[\{x_1 + \xi_2i + \xi_3j + \xi_4k; \xi_2^2 + \xi_3^2 + \xi_4^2 = x_2^2\}.
\]
Hence \(s(q_1)\) is determined by its values on \(\mathbb{R}^2_+ = \{(x_1, x_2) \in \mathbb{R}^2; x_1 > 0\}\) by (5.11). The homogeneous degree of \(s\) in (5.12) implies that \(s(q_1)\) is determined by its values in the semicircle \((x_1, x_2) \in \mathbb{R}^2; x_1 > 0, x_1^2 + x_2^2 = 1\). At last, the Cauchy-Fueter equation for \(s\) gives four ordinary differential equations for four components of \(s\) along the semicircle. These ODEs together with the value \(s(1)\) uniquely determine the function \(s\).

**Proposition 5.2.** On the half space \(\mathbb{R}^4_+ = \{q_1 \in \mathbb{H} \mid \Re q_1 > 0\}\), there exists a unique regular function up to a real constant satisfying (5.11)-(5.12).

**Proof.** Since the conjugation action of unit quaternions leaves the function \(s\) invariant, see (5.11), its infinitesimal action coincides. From one side, choose \(\sigma_t = \cos t + \sin t\) for small \(t\). Then
\begin{equation}
\overline{\sigma_t}q_1\sigma_t = q_1 - tj(x_1 + x_2i + x_3j + x_4k) + t(x_1 + x_2i + x_3j + x_4k) + O(t^2)
= q_1 + 2t(-x_2i + x_4k) + O(t^2),
\end{equation}
where \(q_1 = x_1 + x_2i + x_3j + x_4k\), from which we get
\[
\frac{d}{dt} \bigg|_{t=0} s(\overline{\sigma_t}q_1\sigma_t) = -2x_4\partial_{x_2}s(q_1) + 2x_2\partial_{x_4}s(q_1).
\]
From the other side, taking derivatives of \(\overline{\sigma_t}s(q_1)\sigma_t\) with respect to \(t\) at 0 we get
\begin{equation}
(5.14) \quad -2x_3\partial_{x_2}s(q_1) + 2x_2\partial_{x_4}s(q_1) = -js(q_1) + s(q_1)j.
\end{equation}
Similarly, choosing \(\sigma_t = \cos t + \sin t\) for small \(t\), we find that
\begin{equation}
(5.15) \quad 2x_3\partial_{x_2}s(q_1) - 2x_2\partial_{x_3}s(q_1) = -ks(q_1) + s(q_1)k.
\end{equation}

The homogeneity of degree \(-2n - 3\) of the function \(s\) in (5.12) implies the Euler equation for \(s\):
\begin{equation}
(5.16) \quad x_1\partial_{x_1}s(q_1) + x_2\partial_{x_2}s(q_1) + x_3\partial_{x_3}s(q_1) + x_4\partial_{x_4}s(q_1) = -(2n + 3)s(q_1).
\end{equation}
Restricting to \(q_1 = x_1 + x_2i \in \mathbb{R}^2_+\), i.e. \(x_3 = x_4 = 0\), we obtain
\[2x_2\partial_{x_4}s(q_1) = -js(q_1) + s(q_1)j, \quad -2x_2\partial_{x_3}s(q_1) = -ks(q_1) + s(q_1)k.
\]
Substitute it into the Cauchy-Fueter equation
\[\partial_{x_1}s(q_1) + i\partial_{x_2}s(q_1) + j\partial_{x_3}s(q_1) + k\partial_{x_4}s(q_1) = 0\]
to deduce
\begin{equation}
(5.17) \quad 2x_2\partial_{x_1}s(q_1) + 2x_2i\partial_{x_2}s(q_1) = -2is(q_1) + js(q_1)k - ks(q_1)j.
\end{equation}
Write \( s(x_1 + ix_2) = f_1 + f_2i + f_3j + f_4k \) on \( \mathbb{R}^4_+ \). Then, the equation (5.17) is equivalent to

\[
\begin{align*}
x_2(\partial_{x_1} f_1 - \partial_{x_2} f_2) &= 2f_2, \\
x_2(\partial_{x_1} f_2 + \partial_{x_2} f_1) &= 0, \\
x_2(\partial_{x_1} f_3 - \partial_{x_2} f_4) &= f_4, \\
x_2(\partial_{x_1} f_4 + \partial_{x_2} f_3) &= -f_3.
\end{align*}
\]

(5.18)

Euler’s equation (5.16) implies

\[
x_1 \partial_{x_1} f_k + x_2 \partial_{x_2} f_k = -(2n + 3)f_k, \quad k = 1, 2, 3, 4,
\]

on \( \mathbb{R}^2_+ \). Now we have four real functions \( f_1, f_2, f_3, f_4 \) on the upper half plane \( \mathbb{R}^2_+ \) satisfying 8 equations in (5.18)-(5.19) with conditions \( f_2(x_1, 0) = f_3(x_1, 0) = f_4(x_1, 0) = 0 \) and \( f_1(x_1, 0) \) is real.

On the semicircle \( \{(x_1, x_2) \in \mathbb{R}^2 \mid x_1 > 0, x_1^2 + x_2^2 = 1\} \) we can separate the system (5.18) into two parts, one of which depends on \((x_1, x_2)\) and another one depends on \((x_3, x_4)\) variables. Take the sum of the first identity in (5.18), multiplying by \( x_2 \), and the second one multiplying by \(-x_1\) to get

\[
x_2(x_2 \partial_{x_1} - x_1 \partial_{x_2}) f_1 = x_2(2f_2 + x_1 \partial_{x_1} f_2 + x_2 \partial_{x_2} f_2) = -(2n + 1)x_2 f_2.
\]

(5.20)

Now set \( x_1 = \cos \theta, \ x_2 = \sin \theta, \ \theta \in (-\pi, \pi), \) and \( g_j(\theta) := f_j(\cos \theta, \sin \theta, 0, 0) \). The equality (5.20) implies

\[
g_1'(\theta) = (2n + 1)g_2.
\]

(5.21)

Similarly, we have

\[
\begin{align*}
x_2(x_2 \partial_{x_1} - x_1 \partial_{x_2}) f_2 &= 2x_1 f_2 + (2n + 3)x_2 f_1, \\
x_2(x_2 \partial_{x_1} - x_1 \partial_{x_2}) f_3 &= x_1 f_3 - (2n + 2)x_2 f_4, \\
x_2(x_2 \partial_{x_1} - x_1 \partial_{x_2}) f_4 &= x_1 f_4 + (2n + 2)x_2 f_3,
\end{align*}
\]

and so

\[
\begin{align*}
sin \theta g_2'(\theta) &= -2g_2 \cos \theta - (2n + 3)g_1 \sin \theta, \\
sin \theta g_3'(\theta) &= -g_3 \cos \theta + 2(n + 1)g_4 \sin \theta, \\
sin \theta g_4'(\theta) &= -g_4 \cos \theta - 2(n + 1)g_3 \sin \theta.
\end{align*}
\]

(5.22)

We obtain four real functions \( g_1, g_2, g_3, g_4 \) on \((-\pi, \pi)\) satisfying 4 ordinary differential equations (5.21)-(5.22) under the condition

\[
g_1(0) \in \mathbb{R}^1, \quad g_2(0) = g_3(0) = g_4(0) = 0.
\]

(5.23)

To see that \( g_3 \) and \( g_4 \) vanishing, note that \( s \) is real analytic since it is harmonic. So the functions \( g_j, \ j = 3, 4, \) are real analytic in \( \theta \). Inductively, we can assume \( g_j(\theta) = \)
Corollary 5.1. The function \( s(q_1) \) is given by

\[
(5.24) \quad c_n \frac{\partial^{2n}}{\partial x_1^{2n}} |q_1|^4
\]

for some real constant \( c_n \).

Proof. In the proof of Theorem 5.1 we have seen that \( h(q_1) = \partial_q \frac{1}{|q_1|^2} = -\frac{\sigma q_1}{|q_1|^4} \) is a regular function on \( \mathbb{H} \setminus \{0\} \), which obviously satisfies the invariance (5.11), and so is the function (5.24). The conjugation action \( \overline{\sigma q_1} \sigma \), fixing \( x_1 \) for any \( \sigma \in \mathbb{H} \) with \( |\sigma| = 1 \), implies

\[
\left( \frac{\partial^{2n}}{\partial x_1^{2n}} \right) (\overline{\sigma q_1} \sigma) = \frac{\partial^{2n}}{\partial x_1^{2n}} [h(\overline{\sigma q_1} \sigma)] = \frac{\partial^{2n}}{\partial x_1^{2n}} [\overline{\sigma} h(q_1) \sigma] = \overline{\sigma} \frac{\partial^{2n}}{\partial x_1^{2n}} h(q_1) \sigma,
\]

i.e., (5.24) satisfies the invariance (5.11). The function, defined by (5.24), is homogeneous of degree \(-2n - 3\). So \( s \) is given by (5.24) by the uniqueness in Proposition 5.2.

We verify now that \( s(x_1 + ix_2) \) satisfies (5.18). Write \( s(x_1 + ix_2) = f_1 + f_2i + f_3j + f_4k \) on \( \mathbb{R}^4_+ \). Then, \( f_3 \equiv 0, f_4 \equiv 0 \) and

\[
(5.25) \quad f_1 = \frac{\partial^{2n}}{\partial x_1^{2n}} \frac{x_1}{(x_1^2 + x_2^2)^2}, \quad f_2 = \frac{\partial^{2n}}{\partial x_1^{2n}} \frac{-x_2}{(x_1^2 + x_2^2)^2},
\]

up to a constant \( c_n \). Functions \( f_j \)'s satisfy (5.18)-(5.19). Note that

\[
(5.26) \quad \partial_{x_1} \left( \frac{-x_2}{(x_1^2 + x_2^2)^2} \right) + \partial_{x_2} \left( \frac{x_1}{(x_1^2 + x_2^2)^2} \right) = 0.
\]

Taking derivatives \( \frac{\partial^{2n}}{\partial x_1^{2n}} \) and multiplying by \( x_2 \) both sides of (5.26), one obtains the second equation in (5.18). Note that

\[
(5.27) \quad \partial_{x_1} \left( \frac{x_1}{(x_1^2 + x_2^2)^2} \right) - \partial_{x_2} \left( \frac{-x_2}{(x_1^2 + x_2^2)^2} \right) = \frac{-2}{(x_1^2 + x_2^2)^2}.
\]

Taking derivatives \( \frac{\partial^{2n}}{\partial x_1^{2n}} \) and multiplying by \( x_2 \) on both sides of (5.27), we get the first equation in (5.18). \( \square \)

5.4. Calculation of the constant for the Cauchy-Szegö operator.

Theorem 5.2. The constant \( c_n \) in the function

\[
(5.28) \quad s(q_1) = c_n \frac{\partial^{2n}}{\partial x_1^{2n}} \bar{q}_1, \quad q_1 = x_1 + x_2i + x_3j + x_4k
\]

is given by (1.11).
Proof. To calculate the constant $c_n$, we choose
\[ F(q) = F((q_1, q')) = c_n^{-1} S(e, q), \quad e = (1, 0, \ldots, 0). \]

Then
\[ F(q) = \frac{\partial^{2n}}{\partial y_1^{2n}} \frac{1 + \bar{q}_1 - 2(0, q')}{|1 + \bar{q}_1 - 2(0, q')|^4} = \frac{\partial^{2n}}{\partial y_1^{2n}} \frac{1 + q_1}{|1 + q_1|^4} \quad \text{with} \quad y_1 = 1 + x_1. \]

First we calculate the value $F(e)$. We obtain
\[ F(e) = \frac{\partial^{2n}}{\partial x_1^{2n}} \frac{1}{(1 + x_1)^3} \bigg|_{x_1=1} = (-3)(-4) \cdots (-3 - (2n - 1))(1 + x_1)^{-3-2n} \frac{(-2)}{(-2)} \bigg|_{x_1=1} \]
\[ = \frac{(-1)^{2(n+1)}}{2} (2n + 2)! 2^{-2n-3} = \frac{(2n + 2)!}{2^{2n+4}}. \]

On the other hand,
\[ F(e) = \int_{\partial \mathcal{U}_n} S(e, Q) F^h(Q) d\beta(Q) = c_n^{-1} \int_{\partial \mathcal{U}_n} S(e, Q) S^h(e, Q) d\beta(Q) \]
\[ = c_n \int_{\mathbb{H}^n} \int_{\mathbb{R}^3} \left| \frac{\partial^{2n}}{\partial x_1^{2n}} \frac{1 + \bar{q}_1}{|1 + q_1|^4} \right|^2 dq \, dx_2 dx_3 dx_4, \]

where $q_1 = |q'|^2 + x_2 i + x_3 j + x_4 k$, since $Q$ is in $\partial \mathcal{U}_n$.

We start from calculating the derivative $\frac{\partial^{2n}}{\partial x_1^{2n}} \frac{\bar{p}}{|p|^4}$. We have
\[ \frac{\bar{p}}{|p|^4} = \frac{\bar{p}^{-1}}{|p|^2} \frac{p^{-1}}{p^{-2}} = \frac{p^{-1}}{p^{-2}} \quad \text{by} \quad \frac{\bar{p}}{|p|^2} = p^{-1}. \]

Thus
\[ \frac{\partial^{2n}}{\partial x_1^{2n}} \frac{\bar{p}^{-1} p^{-2}}{p^{-2}} = \sum_{k=0}^{2n} C_{2n}^k \frac{\partial^k}{\partial x_1^k} \frac{\bar{p}^{-1}}{\partial x_1^{-k} p^{-2}}. \]

Since
\[ \frac{\partial^k}{\partial x_1^k} \frac{p^{-1}}{p^{-2}} = (-1)(-2) \cdots (-1 - (k - 1)) p^{-1-k} = (-1)^k k! p^{-1-k} \]

and
\[ \frac{\partial^{2n-k}}{\partial x_1^{2n-k}} p^{-2} = (-2)(-3) \cdots (-2 - (2n - k - 1)) p^{-2-2n+k} = (-1)^{2n-k} (2n - k + 1)! p^{-2-2n+k}, \]
substituting them in (5.28), we get
\[ \frac{\partial^{2n}}{\partial x_1^{2n}}p^{-1}p^{-2} = \sum_{k=0}^{2n} \frac{(2n)!}{k!(2n-k)!}(-1)^k k! p^{-1-k} (-1)^{2n-k} (2n-k+1)! p^{-2-n+k} \]
\[ = (2n)! \sum_{k=0}^{2n} (2n-k+1) p^{-1-k} p^{-2-n+k} \]
\[ = (2n)! \sum_{k=0}^{2n} (2n-k+1) \frac{p^{-k} p^k}{p^2} p^{-2n-1}. \]

Taking the absolute value, we get
\[ \left| \frac{\partial^{2n}}{\partial x_1^{2n}}p^{-1}p^{-2} \right|^2 = \left( \frac{(2n)!}{[p]^{4n+6}} \right)^2 \sum_{k=0}^{2n} (2n-k+1) \left( \frac{p^2}{p^2} \right)^k, \quad p = 1 + q_1. \]

Now we concentrate in calculating the absolute value of the latter sum. Observe that it is square of the length of the sum of quaternions obtained by rotation on the same angle. We denote by \( \frac{(1+q_1)^2}{|1+q_1|^2} = s = \text{Re } s + i s_2 + j s_3 + k s_4 \), where
\[ \text{Re } s = \frac{(1 + x_1)^2 - (x_2^2 + x_3^2 + x_4^2)}{(1 + x_1)^2 + x_2^2 + x_3^2 + x_4^2}, \quad i s_2 + j s_3 + k s_4 = \frac{2(1 + x_1)(x_2 i + x_3 j + x_4 k)}{(1 + x_1)^2 + x_2^2 + x_3^2 + x_4^2}. \]

Note that for any quaternion \( s \), written as \( s = \text{Re}(s) + \vec{v} \), we have \( \text{Re}(s) = \|s\| \cos \theta \) and \( \vec{v} = \|s\| \frac{\vec{v}}{\|\vec{v}\|} \sin \theta \), because of \( \text{Re}(s)^2 + \|\vec{v}\|^2 = \|s\|^2 \left( \cos^2 \theta + \|\vec{v}\| \sin^2 \theta \right) = \|s\|^2. \)

(See e.g. (12).) Since \( s \) is a unit quaternion it can be also written as
\[ s = e^{\hat{n} \theta}, \quad \text{with } \cos \theta = \text{Re}(s), \]
and the unite vector \( \hat{n} = \frac{2(1+x_1)(x_2 i + x_3 j + x_4 k)}{|2(1+x_1)(x_2 i + x_3 j + x_4 k)|} \), where \( (x_2, x_3, x_4) \) denotes the vector in \( \mathbb{R}^3 \). Moreover, \( s^k = e^{\hat{n} k \theta} \). Thus
\[ \left( \sum_{k=0}^{2n} (2n-k+1) e^{\hat{n} k \theta} \right)^2 = \left( (2n+1) + 2n e^{\hat{n} \theta} + (2n-1) e^{\hat{n} 2 \theta} + \ldots + 2 e^{\hat{n}(2n-1) \theta} + e^{\hat{n} 2n \theta} \right)^2 \]
\[ = \sum_{k=0}^{2n} (2n+1-k)^2 \]
\[ + 2 \left( (2n+1)2n + 2n(2n-1) + \ldots + 3 \cdot 2 + 2 \cdot 1 \right) \cos \theta \]
\[ + 2 \left( (2n+1)(2n-1) + 2n(2n-2) + \ldots + 4 \cdot 2 + 3 \cdot 1 \right) \cos(2\theta) \]
\[ + \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \]
\[ + 2 \left( (2n+1)2 + 2n \cdot 1 \right) \cos((2n-1)\theta) \]
\[ + 2 \left( (2n+1) \cdot 1 \right) \cos(2n\theta). \]
We calculate by using the auxiliary formulas
\[
\alpha_0 = \sum_{j=0}^{2n}(2n+1-j)^2 = \sum_{j=1}^{2n+1} j^2 = \frac{(n+1)(2n+1)(4n+3)}{3},
\]
\[
\alpha_1 = 2 \sum_{j=1}^{2n} j(j+1) = 2 \frac{(n+1)(2n+1)(4n)}{3},
\]
\[
\alpha_2 = 2 \sum_{j=1}^{2n-1} j(j+2) = 2 \frac{(n)(2n-1)(4n+5)}{3},
\]
\[
\cdots \cdots , \quad \alpha_{2n-1} = 2(6n+2), \quad \alpha_{2n} = 2(2n+1).
\]
In general
\[
(5.30) \quad \alpha_k = \sum_{j=1}^{2n+1-k} j(j+k) = \frac{(2n+1-k)(2n+2-k)(4n+3+k)}{6}.
\]

Conclude that
\[
\left| \sum_{k=0}^{2n} (2n-k+1)e^{ik\theta} \right|^2 = \sum_{k=0}^{2n} \alpha_k \cos(k\theta).
\]

Summarizing all that we did, we come to calculation of the following integral
\[
F(e) = c_n ((2n)!)^2 \sum_{k=0}^{2n} \alpha_k \int_{H^n} dq \int_{[0,\pi] \times [0,2\pi]} \sin \psi \, d\psi \, d\phi \int_{0}^{\infty} \frac{r^2 \cos(k\theta)}{(1 + |q|^2)^2 + r^2} 4n+6 dr,
\]
where \( r^2 = x_2^2 + x_3^2 + x_4^2 \) and \( \cos \theta = \frac{(1+|q|^2)^2-r^2}{(1+|q|^2)^2+r^2} \). Recall the formula
\[
\cos(k\theta) = \sum_{l=0}^{k} \left( C_{2l}^k \left( \sum_{m=0}^{l} (-1)^{m} C_{l}^{m} \cos^{k-2m} \theta \right) \right),
\]
where \( C_s^t = \binom{t}{s(t-s)!}, \ s, t \in \{0,1,2,\ldots\} \) and it is vanish otherwise. That leads to calculations of
\[
F(e) = c_n 4\pi ((2n)!)^2 \sum_{k=0}^{2n} \alpha_k \int_{H^n} dq \sum_{l=0}^{k} C_{2l}^k \sum_{m=0}^{l} (-1)^{m} C_{l}^{m} \int_{H^n} dq \int_{0}^{\infty} \frac{r^2 \cos^{k-2m} \theta}{((1 + |q|^2)^2 + r^2)} 4n+6 dr.
\]
Substituting the value of \( \cos \theta \) and using the notation \( d = k - 2m \), we concentrate on the calculations of the integrals of type
\[
I_{n,d}(w) = \int_{0}^{\infty} \frac{r^2}{(w^2 + r^2)^{4n+6}} \left( \frac{w^2 - r^2}{w^2 + r^2} \right)^d dr \quad \text{with} \quad w^2 = (1 + |q|^2)^2.
\]
Changing variable \( \frac{w^2}{w^2} = t \), we write the integral in the form
\[
I_{n,d}(w) = \frac{(-1)^d}{2w^{8n+9}} \int_{0}^{\infty} t^{1/2} (1+t)^{4n-6-d} (t-1)^d dt.
\]
Now we can apply the formula (see [11])
\[
(5.31) \quad \int_{0}^{\infty} t^{\lambda-1}(1+t)^{-\mu+\nu}(t+\beta)^{-\nu} dt = B(\mu-\lambda,\lambda)_{2}F_{1}(\nu,\mu-\lambda,\mu;1-\beta),
\]
where \( \Re \mu > \Re \lambda > 0 \), with 
\[
\beta = -1, \quad \lambda = \frac{3}{2}, \quad \nu = -d, \quad \mu = 4n + 6, \quad \mu \geq 10 > \frac{3}{2} = \lambda > 0.
\]

Then
\[
B(\mu - \lambda, \lambda) = B(4n + 6 - 3/2, 3/2) = \frac{\Gamma(4n + 9/2)\Gamma(3/2)}{\Gamma(4n + 6)} = \frac{(8n + 8)! \pi}{2^{8n+9}(4n + 4)!(4n + 5)!},
\]
where we used \( B(a,b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}, \quad \Gamma(1/2) = \sqrt{\pi}, \quad \Gamma(n) = (n-1)!, \quad \Gamma\left(\frac{1}{2} + n\right) = \frac{(2n)!}{4^n n!} \sqrt{\pi}. \)

For hypergeometric function 
\[
_2F_1(\nu, \mu - \lambda, \mu; 1 - \beta) = _2F_1(-d, 4n + 9/2, 4n + 6; 2)
\]
we apply the formula
\[
_2F_1(-d, a + 1 + b + d, a + 1; x) = \frac{d!}{(a + 1)_d} P_{d}^{a,b}(1 - 2x),
\]
where
\[
(a)_d = \begin{cases} 
1 & \text{if } d = 0 \\
(a+1)(a+2)\ldots(a+d-1) & \text{if } d > 0
\end{cases}
\]
with \( a + 1 = 4n + 6 \) and \( b = -d - 3/2 \), and obtain
\[
_2F_1(-d, 4n + 9/2, 4n + 6; 2) = \frac{d!}{(4n + 6)_d} P_{d}^{4n+5,-d-3/2}(-3).
\]

To calculate the value of the Jacobi polynomial \( P_{d}^{4n+5,-d-3/2}(-3) \), we use the formula

\[
P_{d}^{a,b}(x) = \sum_{s=0}^{\infty} \binom{d + a}{s} \binom{d + b}{d - s} \left(\frac{x-1}{2}\right)^{d-s} \left(\frac{x+1}{2}\right)^{s},
\]
where \( s \geq 0 \) and \( d - s \geq 0 \), and for integer \( s \)
\[
\binom{z}{s} = \frac{\Gamma(z+1)}{\Gamma(s+1)\Gamma(z-s+1)} \quad \text{with} \quad \binom{z}{s} = 0 \quad \text{for} \quad s < 0.
\]

Observe that the terms
\[
\binom{d + b}{d - s} = \binom{-3/2}{d - s}
\]
vanish for \( d - s < 0 \), that allows to conclude that the series in the Jacobi polynomial \( P_{d}^{4n+5,-d-3/2} \) has only finite number of terms and reduces to the sum from \( s = 0 \) to \( s = d \).

Now we calculate each term in the sum
\[
P_{d}^{4n+5,-d-3/2}(-3) = \sum_{s=0}^{d} \binom{4n + 5 + d}{s} \binom{-3/2}{d - s} (-1)^{d} d^{d-s}.
\]
Replacing $d$ by $k - 2m$ and substituting the value of $I_{n,k-2m}(w)$ into $F(e)$, we get

\[
F(e) = \sum_{k=0}^{n-m} \alpha_k \sum_{l=0}^{k} \sum_{m=0}^{l} (-1)^{k-m} C^m_l \sum_{s=0}^{k-2m} \frac{C^s_{k-2m}}{2^{k-2m-s+1}} \frac{(-1)^s (2(k - 2m - s + 1))}{(k - 2m + s + 1)!} \int_{\mathbb{H}^n} \frac{dq'}{(1 + |q'|^2)^{4n+9/2}}.
\]

In order to finish the calculations, we need to evaluate the integral

\[
\int_{\mathbb{H}^n} \frac{dq'}{(1 + |q'|^2)^{4n+9/2}} = \int_{S^{4n-1}} dV^{4n-1} \int_0^{r^{4n-2}} \frac{r^{4n-2} dr}{(1 + r^2)^{4n+9/2}}.
\]

It is well known that the volume of the sphere $S^{4n-1}$ is

\[
\int_{S^{4n-1}} dV^{4n-1} = \frac{\pi^{2n-1/2}}{\Gamma(2n + 1/2)}.
\]

Making use the substitution $t = r^2$ we obtain

\[
\int_0^{\infty} \frac{r^{4n-2} dr}{(1 + r^2)^{4n+9/2}} = \frac{1}{2} \int_0^{\infty} \frac{t^{2n-3/2} dt}{(1 + t)^{4n+9/2}} = \frac{\Gamma(2n - 1/2)\Gamma(2n + 5)}{2\Gamma(4n + 9/2)}.
\]
Multiplying two latter expressions, we find
\[
\int_{\mathbb{H}^n} \frac{dq'}{(1 + |q'|^2)^{4n+9/2}} = 2^{8n+8} \pi^{2n-1} \frac{(2n + 4)! (4n + 4)!}{4n - 1 (8n + 8)!}.
\]
Substituting the value of the integral \(\int_{\mathbb{H}^n} \frac{dq'}{(1 + |q'|^2)^{4n+9/2}}\) to the expression for \(F(e)\) we get
\[
F(e) = c_n \frac{\pi^{2n+1} ((2n)!)^2 (2n + 4)!}{4n - 1} K(n),
\]
where the constant \(K(n)\) is given by (1.12).

Corollary 5.2. Particularly, the constant in the Cauchy - Szegö kernel for low dimensions are equal to \(c_1 = \frac{6237}{872\pi^3}\) and \(c_2 = \frac{11486475}{193472\pi^5}\).

Proof. Case \(n = 1\). From one hand \(F(e) = \frac{\pi}{8}\). From the other hand we have to calculate the integral
\[
F(e) = c_1 4\pi (2!)^2 \left( \int_{\mathbb{H}^1} \frac{dq'}{(1 + |q'|^2)^{4+9/2}} \frac{1}{2} \int_0^\infty \frac{t^{1/2}}{(1 + t)^{10}} \sum_{k=0}^{2} \alpha_k \cos(k\theta) \, dt \right).
\]
We know that
\[
\int_{\mathbb{H}^1} \frac{dq'}{(1 + |q'|^2)^{4+9/2}} = \frac{4\pi 6!8!}{3 \cdot 16!}
\]
by (5.36), and
\[
\sum_{k=0}^{2} \alpha_k \cos(k\theta) = 14 + 8 \cos \theta + 3 \cos(2\theta) = 11 + 8 \cos \theta + 6 \cos^2 \theta.
\]
by (5.30). Then
\[
\frac{1}{2} \int_0^\infty \frac{t^{1/2}}{(1 + t)^{10}} \sum_{k=0}^{2} \alpha_k \cos(k\theta) \, dt
\]
\[
= \frac{B \left( \frac{47}{2}, \frac{3}{2} \right)}{2} \left( 112F_1(0, 17/2, 10; 2) - 82F_1(-1, 17/2, 10; 2) + 62F_1(-2, 17/2, 10; 2) \right)
\]
\[
= \frac{\pi 16!}{21889!} (11 + 8 \frac{7}{10} + 6 \frac{59}{110}) = \frac{\pi 16!}{21889!} \frac{218}{11}.
\]
It gives
\[
F(e) = c_1 4\pi (2!)^2 \frac{2^{16}\pi 6!8! \pi 16!}{3 \cdot 16!} \frac{218}{21889!} = c_1 \pi^3 \frac{109}{2079},
\]
and, finally, \(c_1 = \frac{6237}{872\pi^3}\).

Case \(n = 2\). Again, from one hand we get \(F(e) = \frac{45}{16}\). From the other hand we get
\[
\int_{\mathbb{H}^2} \frac{dq'}{(1 + |q'|^2)^{3+9/2}} = \frac{2^{24}\pi^3 8!12!}{7 \cdot 24!},
\]
\[
\alpha_0 = 55, \quad \alpha_1 = 40, \quad \alpha_2 = 26, \quad \alpha_3 = 14, \quad \alpha_4 = 5,
\]
\[
\cos(2\theta) = 2 \cos^2 \theta - 1, \quad \cos(3\theta) = 4 \cos^3 \theta - 3 \cos \theta, \quad \cos(4\theta) = 8 \cos^4 \theta - 8 \cos^2 \theta + 1,
\]
\[ \sum_{k=0}^{2} \alpha_k \cos(k\theta) = 34 - 2 \cos \theta + 12 \cos^2 \theta + 56 \cos^3 \theta + 40 \cos^4 \theta, \]

\[ \frac{1}{2} \int_0^{\infty} t^{1/2} \frac{\sum_{k=0}^{4} \alpha_k \cos(k\theta)}{(1 + t)^{14}} dt = \frac{B(25/2, 3/2)}{2} \left( 34_2 F_1(0, 25/2, 14; 2) + 2_2 F_1(-1, 25/2, 114; 2) \right) \]

\[ + 12_2 F_1(-2, 25/2, 14; 2) - 56_2 F_1(-3, 25/2, 14; 2) + 40_2 F_1(-4, 25/2, 14; 2) \]

\[ = \frac{\pi 24!}{2^{28} 12! 13!} \left( 34 - \frac{2}{14} + 12 \cdot \frac{9}{14} + \frac{56}{224} + 40 \cdot \frac{1763}{3808} \right) = \frac{\pi 24!}{2^{28} 12! 13!} 3023. \]

Thus

\[ F(e) = c_2 4 \pi (4!)^2 \frac{\pi 24! 3023}{7 \cdot 24!} \frac{2^{24} \pi^3 12!}{2^{28} 12! 13!} = c_2 \pi^5 \frac{12092}{255255}, \]

that leads to \( c_2 = \frac{11486475}{1934725} \) and the proof of the corollary is therefore complete. \( \Box \)

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