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Amorphous topological superconductivity in a Shiba glass

Kim Pöyhönen, Isac Sahlberg, Alex Westström & Teemu Ojanen

Topological states of matter support quantised nondissipative responses and exotic quantum particles that cannot be accessed in common materials. The exceptional properties and application potential of topological materials have triggered a large-scale search for new realisations. Breaking away from the popular trend focusing almost exclusively on crystalline symmetries, we introduce the Shiba glass as a platform for amorphous topological quantum matter. This system consists of an ensemble of randomly distributed magnetic atoms on a superconducting surface. We show that subgap Yu-Shiba-Rusinov states on the magnetic moments form a topological superconducting phase at critical density despite a complete absence of spatial order. Experimental signatures of the amorphous topological state can be obtained by scanning tunnelling microscopy measurements probing the topological edge mode. Our discovery demonstrates the physical feasibility of amorphous topological quantum matter, presenting a concrete route to fabricating new topological systems from non-topological materials with random dopants.

1Department of Applied Physics (LTL), Aalto University, P. O. Box 15100, FI-00076 AALTO, Finland. Correspondence and requests for materials should be addressed to T.O. (email: teemuo@boojum.hut.fi)
Topological states are characterised by integer-valued invariants, \(^1, 2\) that remain robust in the presence of imperfections. While topological properties can be studied independently of local order, spatial symmetries play a central role in virtually all material realisations. This is emphasised by the fact that the theoretical search for new topological materials extensively employs band structures and reciprocal space. While topological states are generically robust to disorder which breaks spatial symmetries, this is typically established by treating the disorder as an additional feature in a well-defined clean system. Even topological Anderson insulators, \(^3, 4\), where moderate disorder actually gives rise to nontrivial topological properties, rely crucially on a specific band structure of the clean system. The concept of disorder, almost by definition, implies the existence of an underlying ordered reference state.

The role of spatial symmetries in topological materials raises the question of how much spatial order is necessary for topological states to persist. In addition to the fundamental interest, possible realisations have far-reaching practical implications. The search for topological states has already moved beyond the elementary Note 1. Physical intuition can be obtained by considering the special case of fully out-of-plane ferromagnetic spins, where the model reduces to experimental evidence\(^19\). More recently, ferromagnetic 2D lattices have emerged as a promising platform for chiral superconductivity\(^22, 23\) with a rich topological phase diagram\(^24, 25\). Classical magnetic moments embedded in a gapped s-wave superconductor give rise to Yu–Shiba–Rusinov (YSR) subgap states\(^26\), localised subgap states which decay algebraically for distances smaller than the superconducting coherence length. In 2D superconductors, such as layered systems, thin films and surfaces, the decay of the wavefunctions from the deep-lying impurity has a functional form \(e^{-\sqrt{\Delta/k_F r}}\), where \(\xi\) and \(k_F\) are the superconducting coherence length and the Fermi wave vector of the underlying bulk. The Shiba glass results from a hybridisation of randomly distributed YSR states. To model the system, we consider deep-lying YSR states with energies \(\epsilon_0\) located in the vicinity of the gap centre \(\epsilon_0/\Delta \ll 1\), where \(\Delta\) is the pairing gap in the bulk. The energy of a single YSR state is given by \(\epsilon_0 = \Delta \frac{1-\alpha}{1+\alpha}\) where \(\alpha = \pi J\mathcal{N}\) is a dimensionless impurity strength, \(J\) is the magnetic coupling, \(S\) is the magnitude of the magnetic moment and \(\mathcal{N}\) is the spin-averaged density of states at the Fermi level. The deep-impurity assumption translates to \(|1-\alpha| \ll 1\) and the energy of an impurity state is given by \(\epsilon_0 = \Delta (1-\alpha)\). As outlined in the Methods section, the low-energy properties of the coupled impurity moments are modelled by a tight-binding Bogoliubov-de Gennes Hamiltonian\(^24\)

\[
H_{mn} = \left( \frac{\hbar}{\Delta_{mn}} \frac{\Delta_{mn}}{(\Delta_{mn})^*} - \hbar^*_{mn} \right),
\]

which describes a long-range hopping between YSR states centred at random positions \(r_m\). The entries \(h_{mn}\) are for arbitrary configuration of magnetic moments is lengthy and given in Supplementary Note 1. Physical intuition can be obtained by considering the special case of fully out-of-plane ferromagnetic spins, where the model reduces to

\[
h_{mn} = \begin{cases} \\
\frac{\epsilon_0}{2} \left[ I_+^m(r_m) + I_-^m(r_m) \right] & m = n \\
\frac{\alpha}{2} \left[ I_+^m(r_m) - I_-^m(r_m) \right] & m \neq n
\end{cases},
\]

\[
\Delta_{mn} = \begin{cases} \\
\frac{\alpha}{2} \left[ I_+^m(r_m) - I_-^m(r_m) \right] & m = n \\
0 & m \neq n
\end{cases}.
\]

In the above expression \(r_m = |x_m - r_n|\), and \(x_m\) are components of \(r_m - r_n\). The hopping elements are expressed in terms of the functions

\[
I_\pm^m(r) = \frac{\mathcal{N}_\pm}{\pi} \Im \left[ I_n(k_F^m r + i\xi) \pm H_{-1} (k_F^n r + i\xi) \right],
\]

\[
I_\pm^m(r) = \frac{\mathcal{N}_\pm}{\pi} \Im \left[ I_0(k_F^m r + i\xi) \pm iH_0 (k_F^n r + i\xi) \right],
\]

where \(J_n\) and \(H_n\) are Bessel and Struve functions of order \(n\). The Rashba spin-orbit coupling induces two helical Fermi surfaces with density of states \(\mathcal{N}_\pm = \mathcal{N} \left( 1 \mp \lambda/\sqrt{1 + \lambda^2} \right)\) and Fermi
wavenumber $k_F^2 = k_F \left( \sqrt{1 + \lambda^2} \mp \lambda \right)$, where $\lambda = a_0/(h\nu_F)$ is the dimensionless Rashba coupling and $k_F, \nu_F$ the Fermi wavenumber and velocity in the absence of spin-orbit coupling. The Rashba coupling also slightly modifies the superconducting coherence length $\xi = (h\nu_F/\Delta) \sqrt{1 + \lambda^2}$. For ferromagnetic textures, the pairing term $\Delta_0$ vanishes with vanishing Rashba coupling $\alpha_0 = 0$. The low-energy Hamiltonian (1) describes an odd-parity pairing $\Delta_{\mu\nu} = -\Delta_{\nu\mu}$, which is a long-range hopping variant of a $p_x + ip_y$ superconductivity. In Eq. (1) the hopping and pairing functions decay as $f(r) \propto e^{-\lambda r^2}$ and display oscillations at wave vectors $k_F^2$.

**Physical properties of the Shiba glass.** The spectrum and the topological phase diagram of a finite system can be calculated by diagonalising the effective Hamiltonian (1) for spatially uncorrelated random positions of magnetic moments. After deriving the finite-size properties, we discuss the extrapolation to the thermodynamic limit. For 2D time-reversal breaking topological superconductors, the relevant topological index classifying the state is the Chern number. We will evaluate Chern numbers by employing the real-space approach of Eq. (5).

By evaluating the Chern number, we uncover the topological phase diagram of finite Shiba glass systems which can be seen in Fig. 2a. For sufficiently high densities, a ferromagnetically ordered system is generally in a topological phase with Chern number $|C| = 1$. For the employed parameters, the critical density $\rho_c$ corresponds to the characteristic length scale $\tau_c = \rho_c^{-1/2} \approx k_F^{-1}$. For lower densities ($\tau \gg k_F^{-1}$), the system is in general topologically trivial and gapless; rare configurations can manage to enter a topological phase but do not survive disorder averaging. The pattern persists even when the directions of the local spins deviate from the perfect ferromagnetic configuration; in Fig. 2b we plot the phase diagram for spin configurations drawn from a thermal distribution where the angles $\theta_i$ between the moments and the surface normal are determined by the Boltzmann weights $e^{-\beta E_z \cos \theta_i}$. This situation corresponds to an ensemble of decoupled spins at Zeeman field $E_z$ polarising the moments perpendicular to the plane and disordered by thermal fluctuations at inverse temperature $\beta$. Alternatively, the situation can be regarded as a magnetic disorder where the disorder is parametrised by the thermal distribution and $\beta E_z$ instead of some other random distribution. For $\beta E_z = 10$, as indicated by Fig. 2b, the phase diagram remains qualitatively unchanged when compared to that for the completely polarised case. The robustness to moment disorder is not an artefact of the thermal distribution, and we discover qualitatively similar results for other disorder averages exhibiting comparable polarisation.

The physical consequences of the topological nature of the Shiba glass are illustrated in Fig. 2c, d. The first one shows that the local density of states (LDOS) is concentrated on the sample edges. This is a consequence of a topological edge mode enclosing...
a finite system and is directly observable as discussed below. In Fig. 2d we have plotted the thermal conductance of finite systems coupled to external leads, as detailed in Supplementary Note 2. In the topological phase, the system exhibits a quantised thermal conductance which is a direct consequence of the nontrivial topology. The quantised conductance is effected by the edge modes despite the system being highly irregular in real space. In finite-size systems, for parameters close to the phase boundary, the quantised conductance plateau is destroyed and the system exhibits a mobility gap instead of an energy gap, the system exhibits a quantised thermal conductance in the trivial phase indicates that the low-energy states there extend over the sample. The behaviour and exact phase transition point depends on the system parameters, though the overall trend of a topological phase at high densities remains. In Fig. 2 we have used parameters with high Rashba splitting $\lambda$ and low value of $k_F\xi$ as appropriate for a proximity-superconducting 2D semiconductor; a phase diagram for parameters more appropriate for metals are presented in Supplementary Fig. 2b, also indicating a transition to a topological phase at sufficiently high densities.

Now we turn to discuss the features seen when increasing the system size. First of all, in the thermodynamic limit the Shiba glass phase is gapless. While this is a generic feature of a superconductor with magnetic impurities, a qualitatively new mechanism for low-energy excitations arises in the topological phase. These emerge from rare fluctuations that leave a substantial area where magnetic moments are sparse. As depicted in Fig. 1, these empty antipuddles give rise to low-energy modes which are reminiscent of the gapless edge states circulating around a hole punched in a gapped topological phase. While the probability of formation of antipuddles is exponentially suppressed as a function of their size and their effect is relatively unimportant in finite systems with high density, in infinite systems antipuddles give rise to a tail down to zero energy in the DOS. The antipuddle mechanism provides a simple physical argument why the energy gap must scale to zero in the DOS. The antipuddle mechanism is effective in the case of defects which are rare and effectively decoupled, thus they cannot destroy the conductance quantisation. Behaviour, in sufficiently dense systems $\pi/4 < k_Fr < 3\pi/4$ the interaction is effectively ferromagnetic. Therefore, in the large part of the topological region $k_Fr \lesssim 1$, this mechanism favours a ferromagnetic ordering polarising the system. In addition, an anisotropic crystal field splitting $D_{2g}$ and an external Zeeman field $BS_z$ would drive the system towards an out-of-plane polarisation.

The studied Shiba glass system could be realised by decorating an effective 2D or a layered 3D superconductor with magnetic atoms or molecules. Considering the requirement $k_Fr \lesssim 1$, dilute electron systems such as proximity-superconducting $2d$ semiconductors with Rashba spin-orbit coupling are promising candidate systems. Another candidate system is the layered superconductor $\text{NbSe}_2$, where $2d$ YSR states $^{29}$ and their coupling have been observed recently. The most direct experimental probe is provided by STM measurement of the LDOS. As shown above, in the topological phase the Shiba glass system exhibits a significant concentration of the subgap LDOS at the sample boundaries, which can be directly observed by STM. This signal is clearly detectable at temperatures below the mobility gap scale which can be of the order of $k_BT = 0.1\Delta - 0.3\Delta$ as shown in Fig. 2d.

In summary, we introduced the Shiba glass as a platform for amorphous topological superconductivity and elucidated the general properties of such systems. Our results illustrate the physical feasibility of amorphous topological quantum materials and provide a concrete prescription to experimentally realise and observe them. Our discovery motivates expanding the search for topological materials beyond crystalline systems and paves the way for fabricating topological matter from nontopological materials with random dopants.

**Methods**

**Real-space evaluation of the topological invariant.** To find the topological phase diagram, we need to evaluate topological invariants in real space. The relevant topological index for 2D systems with broken time-reversal symmetry is the Chern number. This is generally obtained in $k$-space, but there are various methods of computing it in real space as well $^{11,12}$. A comparison shows that these methods are generally of similar computational efficiency and yield the same values for the topological invariant.

The real-space Chern number method of ref. $^{35}$ proceeds by defining the coupling matrices $C_{\alpha,\beta} + 1$, with elements

$$C_{\alpha,\beta} = \langle \psi^\alpha | \sigma(q_{\alpha} - q_{\beta}) | \psi^\beta \rangle,$$

where $\mathbf{R}$ is the position operator, $q_{\alpha} = \pi(\delta_{\alpha,1} + \delta_{\alpha,2} - \delta_{\alpha,3})$ for $\alpha = 0, \ldots, 3$, and $\psi^\alpha$ are the eigenfunctions of the system with periodic boundary conditions. By use of these matrices, the Chern number is then obtained through the equation

$$C = \frac{1}{8\pi} \sum_{\alpha,\beta} \text{arg}(C_{\alpha,\beta}),$$

with $\lambda_\alpha$ being the complex eigenvalues of the matrix $C_{01}C_{12}C_{23}C_{30}$.

**Data availability.** Data sharing not applicable to this article as no datasets were generated or analysed during the current study.

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