THE NEUMANN PROBLEM FOR STOCHASTIC CONSERVATION LAWS

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Abstract. The paper establishes the well-posedness of the Neumann problem for stochastic conservation laws with multiplicative noise. First we work with the usual concept of stochastic kinetic solution introduced by Debussche and Vovelle [Journal of Functional Analysis 259 (2010), 1014–1042] and establish the well-posedness assuming that the noise is compactly supported in the interior of the space domain. Then we introduce the concept of limit noise class kinetic solutions and extend the well-posedness result for general non-compactly supported noises.

1. Introduction

We consider the following initial boundary value problem for a stochastic conservation law, on a bounded smooth domain \( \mathcal{O} \subset \mathbb{R}^d \),
\[
(1.1) \quad du + \nabla \cdot A(u) \, dt = \Phi(u) \, dW, \quad (t, x) \in (0, \infty) \times \mathcal{O},
\]
\[
(1.2) \quad u(0, x) = u_0(x), \quad x \in \mathcal{O},
\]
\[
(1.3) \quad A(u(t, x)) \cdot \nu(x) = 0, \quad (t, x) \in (0, \infty) \times \partial \mathcal{O}.
\]
Here \( A \in C^3(\mathbb{R}; \mathbb{R}^d) \) is the flux function and \( \nu \) is the normal vector to \( \partial \mathcal{O} \). Let \( (\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_t)) \) be a stochastic basis, where \( (\Omega, \mathcal{F}, \mathbb{P}) \) is a probability space and \( (\mathcal{F}_t) \) is a complete filtration. We use the same framework as in [13]. We assume that \( W \) is a cylindrical Wiener process: \( W = \sum_{k \geq 1} \beta_k e_k \), where \( \beta_k \) are independent brownian processes and \( (e_k)_{k \geq 1} \) is a complete orthonormal basis in a Hilbert space \( \mathbb{U} \). For each \( u \in \mathbb{R} \), \( \Phi(u) : \Omega \to L^2(\mathcal{O}) \) is defined by \( \Phi(u) e_k = g_k(\cdot, u) \), where \( g_k(\cdot, u) \) is a regular function on \( \mathcal{O} \). More specifically, we assume that, for some bounded open set \( \mathcal{V} \), with \( \mathcal{V} \subset \mathcal{O} \), for some \( M > 0 \), \( g_k \in C_c(\mathcal{V} \times (-M, M)) \), with the bounds
\[
(1.4) \quad |g_k(x, 0)| + |\nabla_x g_k(x, \xi)| + |\partial_\xi g_k(x, \xi)| \leq \alpha_k, \quad \forall x \in \mathcal{O}, \xi \in \mathbb{R},
\]
where \( (\alpha_k)_{k \geq 1} \) is a sequence of positive numbers satisfying \( D := 4 \sum_{k \geq 1} \alpha_k^2 < \infty \).
Observe that (1.4) implies
\[
(1.5) \quad G^2(x, u) = \sum_{k \geq 1} |g_k(x, u)|^2 \leq D(1 + |u|^2),
\]
\[
(1.6) \quad \sum_{k \geq 1} |g_k(x, u) - g_k(y, v)|^2 \leq D(|x - y|^2 + |u - v|^2),
\]

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for all \( x, y \in \mathcal{O}, u, v \in \mathbb{R} \).

The conditions on \( \Phi \) imply that \( \Phi : L^2(\mathcal{O}) \to L_2(\mathcal{H}; L^2(\mathcal{O})) \), where the latter denotes the space of Hilbert-Schmidt operators from \( \mathcal{H} \) to \( L^2(\mathcal{O}) \).

In particular, given a predictable process \( u \in L^2(\Omega \times [0, T]; L^2(\mathcal{O})) \), the stochastic integral is a well defined process taking values in \( L^2(\mathcal{O}) \).

Also, for each \( u \in \mathbb{R}, \Phi(u) : \mathcal{H} \to L^2(\mathcal{O}) \) is Hilbert-Schmidt since \( \|g_k(\cdot, u)\|_{L^2(\mathcal{O})} \leq |\mathcal{O}|^{1/2}\|g_k(\cdot, u)\|_{L^2(\mathcal{O})} \) thus

\[
\sum_{k \geq 1} \|g_k(\cdot, u)\|_{L^2(\mathcal{O})}^2 \leq D(1 + |u|^2).
\]

Since, clearly, the series defining \( W \) does not converge in \( \mathcal{H} \), in order to have \( W \) properly defined as a Hilbert space valued Wiener process, one usually introduces an auxiliary space \( \mathcal{H}_0 \supset \mathcal{H} \), such as

\[
\mathcal{H}_0 = \{ v = \sum_{k \geq 1} a_k e_k : \sum_{k \geq 1} \frac{a_k^2}{k^2} < \infty \},
\]

endowed with the norm

\[
|v|_{\mathcal{H}_0}^2 = \sum_{k \geq 1} \frac{a_k^2}{k^2}, \quad v = \sum_{k \geq 1} a_k e_k.
\]

In this way, one may check that the trajectories of \( W \) are \( \mathbb{P} \)-a.s. in \( C([0, T], \mathcal{H}_0) \) (see [12]).

For simplicity, we will assume that \( u_0 \) is independent of \( \omega \in \Omega \), that is, \( u_0 \in L^\infty(\mathcal{O}) \). More precisely, we assume that \( u_0 \in L^\infty(\mathcal{O}) \), and there exists an interval \([a, b]\), with \((-M, M) \subset [a, b]\), such that

\[
a \leq u_0(x) \leq b, \ a.e. \ x \in \mathcal{O}.
\]

We also assume that

\[
A(a) = A(b) = 0.
\]

The extension to the case where \( u_0 \in L^\infty(\Omega, \mathcal{I}_0, \mathbb{P}; L^\infty(\mathcal{O})) \) is straightforward and comes down to taking expectation wherever an integral involving \( u_0 \) is present.

We also need to impose a non-degeneracy condition on the symbol (cf. [18])

\[\mathcal{L}(i\tau, i\kappa, \xi) := i(\tau + a(\xi) \cdot \kappa),\]

\( \tau \in \mathbb{R}, \kappa \in \mathbb{R}^d \), and \( a(\xi) = A(\xi) \). For \( \kappa = (\kappa_1, \cdots, \kappa_d) \in \mathbb{R}^d \), let \( |\kappa|^2 = \kappa_1^2 + \kappa_2^2 + \cdots + \kappa_d^2 \). For \( J, \delta > 0 \), let

\[\Omega_{\mathcal{L}}(\tau, \kappa; \delta) := \{ \xi \in [a, b] : |\mathcal{L}(i\tau, i\kappa, \xi)| \leq \delta \} \]

We suppose there exist \( \alpha \in (0, 1) \) such that

\[
\sup_{\tau \in \mathbb{R}, \kappa \in \mathbb{R}^d, \|\kappa\| = 1} |\Omega_{\mathcal{L}}(\tau, \kappa, \delta)| \lesssim \delta^\alpha,
\]

where we employ the usual notation \( x \lesssim y \), if \( x \leq Cy \), for some absolute constant \( C > 0 \), and \( x \sim y \), if \( x \lesssim y \) and \( y \lesssim x \).

Since we expect solutions of (1.1)-(1.3) to be bounded from above and from below by \( b \) and \( a \), respectively (see Theorem 3.3), we only need to impose the nondegeneracy assumptions (1.9) for \( \xi \in [a, b] \).
Examples of flux functions $A(u)$ satisfying (1.8) and (1.9) are given by (cf. [40])
\[
A(u) = \left( \frac{1}{(l_1 + 1)}(u - a)^{l_1} + (u - b)^{l_1 + 1}, \cdots, \frac{1}{(l_d + 1)}(u - a)^{l_d} + (u - b)^{l_d + 1} \right),
\]
where, $l_i \in \mathbb{N}$, $l_i \neq l_j$, if $i \neq j$, $i, j = 1, \cdots, d$, as it is not difficult to check.
Note that, in this case, for each $\kappa \in \mathbb{R}^d$ with $|\kappa| = 1$, the sup of $|\Omega(\tau, \kappa, \delta)|$, for $\tau \in \mathbb{R}$, will be assumed when $-\tau \pm \delta$ is a critical value of $a(\xi) \cdot \kappa$. Moreover, if $l_{i_0} = \max\{l_1, \cdots, l_d\}$, then it is not difficult to see that the sup of $|\Omega(\tau, \kappa, \delta)|$, for $\tau \in \mathbb{R}$ and $\kappa \in \mathbb{R}^d$, $|\kappa| = 1$, will be assumed for $\kappa = e_{i_0}$, the $i_0$-th element of the canonical basis, and it is achieved for $-\tau \pm \delta$ running along the local extremes of $a_{i_0}(\xi)$, in the interval $[a, b]$, with $+$ or $-$ depending on whether it is a maximum or a minimum, respectively, and so, condition (1.9) is satisfied for $\alpha = 1/l_{i_0}$.

Evidently, (1.9) implies the following weaker condition: For $(\tau, \kappa) \in \mathbb{R}^{d+1}$, $(\tau, \kappa) \neq 0$,
\[
(1.10) \quad |\{\xi \in [a, b] : \tau + a(\xi) \cdot \kappa = 0\}| = 0.
\]

1.1. Definitions and Main Theorem.

**Definition 1.1** (Kinetic measure). As in [13], we call a map $m$ from $\Omega$ to the set of non-negative finite measures over $\Omega \times [0, T] \times \mathbb{R}$ a kinetic measure if

(1) $m$ is measurable, in the sense that for each $\phi \in C_b(\Omega \times [0, T] \times \mathbb{R})$, $(m, \phi) : \Omega \to \mathbb{R}$ is measurable,

(2) $m$ vanishes for large $\xi$, that is, if $B_R^c = \{\xi \in \mathbb{R} : |\xi| \geq R\}$, then
\[
\lim_{R \to \infty} \mathbb{E} m(\Omega \times [0, T] \times B_R^c) = 0,
\]

(3) for all $\phi \in C_b(\Omega \times \mathbb{R})$, the process
\[
t \mapsto \int_{\Omega \times [0, t] \times \mathbb{R}} \phi(x, \xi) \, dm(x, s, \xi)
\]
is predictable.

**Definition 1.2** (Kinetic solution). Let $u_0 \in L^\infty(\Omega)$. A measurable function $u : \Omega \times \Omega \times [0, T] \to \mathbb{R}$ is said to be a kinetic solution to (1.1)–(1.3) with initial datum $u_0$ if $(u(t))$ is predictable, $u \in L^\infty(\Omega \times [0, T] \times \Omega)$ and there exists a kinetic measure $m$ such that $f := 1_{u > \xi}$ satisfies: for all $\varphi \in C_c^\infty(\Omega \times [0, T] \times \mathbb{R})$,
\[
(1.12) \quad \int_0^T \langle f(t), \partial_t \varphi(t) \rangle \, dt + \langle f_0, \varphi(0) \rangle + \int_0^T \langle f(t), a(\xi) \cdot \nabla \varphi(t) \rangle \, dt
\]
\[= -\sum_{k \geq 1} \int_0^T \int_{\Omega} g_k(x, u(t, x)) \varphi(t, x, u(t, x)) \, dx \, d\beta_k(t)
\]
\[= -\frac{1}{2} \int_0^T \int_{\Omega} \partial_t \varphi(t, x, u(t, x)) G^2(t, x, u(t, x)) \, dx \, dt + m(\partial_t \varphi),
\]
a.s., where $f_0 = 1_{u_0(x) \geq \xi}$, $G^2 := \sum_{k=1}^\infty |g_k|^2$ and $a(\xi) := A'\xi)$. Concerning the Neumann condition (1.3), we ask that for all $\psi \in C_c^\infty((0, T) \times \mathbb{R}^d)$ we have a.s.
\[
(1.13) \quad \int_0^T \int_{\Omega} \psi_t + A(u) \cdot \nabla \psi \, dx \, dt + \sum_{k \geq 1} \int_0^T \int_{\Omega} g_k(x, u(t, x)) \psi(t, x) \, dx \, d\beta_k(t) = 0.
\]

We now state the main result of this paper.
**Theorem 1.1.** Let $u_0 \in L^\infty(\mathcal{O})$ satisfying $a \leq u_0(x) \leq b$ a.e. in $\mathcal{O}$. Assume that conditions (1.4)–(1.9) are satisfied. Then there is a unique kinetic solution to (1.1)–(1.3).

The proof of Theorem 1.1 is given along the remaining sections.

Before we pass to a description of earlier works and an overview of the paper we state the definition of weak entropy solution and the equivalence between this concept and the one of kinetic solution.

**Definition 1.3 (Weak entropy solution).** Let $u_0 \in L^\infty(\mathcal{O})$. A bounded measurable function $u \in L^\infty(\Omega \times [0,T] \times \mathcal{O})$ is said to be a weak entropy solution to (1.1)–(1.3) if $(u(t))$ is an adapted $L^2(\mathcal{O})$-valued process, and for all convex $\eta \in C^2(\mathbb{R})$, for all non-negative $\varphi \in C^1_c([0,T] \times \mathcal{O})$,

$$\int_0^T \langle \eta(u(t)), \partial_t \varphi(t) \rangle dt + \langle \eta(u_0), \varphi(0) \rangle + \int_0^T \langle q(u(t)), \nabla \varphi \rangle dt,$$

$$\geq - \sum_{k \geq 1} \int_0^T \langle g_k(\cdot, u(t)) \eta'(u(t)), \varphi \rangle d\beta_k(t) - \frac{1}{2} \int_0^T \langle G^2(\cdot, u(t)) \eta''(u(t)), \varphi \rangle dt,$$

a.s. where $q(u) = \int_0^u a(\xi)\eta'(\xi) d\xi$ and $\langle \cdot, \cdot \rangle$ represents the inner product of $L^2(\mathcal{O})$. Also, $u$ must satisfy (1.13).

The following proposition is proven exactly as proposition 15 of [13] with minor adaptations, and we refer to the latter for its proof.

**Proposition 1.1.** Let $u_0 \in L^\infty(\mathcal{O})$. For a measurable function $u : \Omega \times [0,T] \times \mathcal{O} \rightarrow \mathbb{R}$ it is equivalent to be a kinetic solution of (1.1)–(1.3) and a weak entropy solution of (1.1)–(1.3).

### 1.2. Earlier works and overview of the paper.

In the deterministic case, that is, in the absence of the stochastic term $\Phi(u) dW$, the system (1.1)–(1.3) is a well-known model for many natural phenomena, such as the sedimentation of suspensions in closed vessels, the dispersal of a single species of animals in a finite territory, etc. (see, e.g., [9] and the references therein). One may thus introduce such random perturbation to take into account uncertainties and fluctuations arising in these applications.

The deterministic counterpart of (1.1)–(1.3) has long been addressed. First, Karlsen, Lie and Risebro [25] constructed a weak solution to (1.1)–(1.3) in one spatial dimension via the front-tracking method, whose uniqueness was established only in the class of solutions obtained by the front-tracking approximations. Later on, Bürger, Frid and Karlsen [9] adopting a natural definition of entropy solution showed the existence and uniqueness of such solutions in arbitrary space dimensions. Their argument runs along three basic steps: (i) the well known vanishing viscosity method, which provides approximate solutions by parabolic equations; (ii) the averaging lemma by Lions, Perthame and Tadmor [35], which guarantees the pre-compactness of the approximate solutions; (iii) the strong trace property by Vasseur [43], which enables one to verify the uniqueness and continuous dependence of the initial data at once. See also [1] and [17] for related generalized problems.

On the other hand, stochastic conservation laws have a recent yet intense history. For the sake of examples, we mention Kim [28] for the first result of existence
and uniqueness of entropy solutions of the Cauchy problem for a one-dimensional stochastic conservation law, in the additive case, that is, $\Phi$ does not depend on $u$. Feng and Nualart [16], where a notion of strong entropy solution is introduced, which is more restrictive than that of entropy solution, and for which the uniqueness is established in the class of entropy solutions in any space dimension, in the multiplicative case, i.e., $\Phi$ depending on $u$; existence of such strong entropy solutions is proven only in the one-dimensional case. Debussche and Vovelle [13], where the definition of kinetic solution of a stochastic conservation law is introduced and the well-posedness is established in the periodic context in any space dimension. Bauzet, Vallet and Wittbold [2], where the existence and uniqueness of entropy solutions for the general Cauchy problem is proved in any space dimension (see also, [11], [26]). Concerning boundary value problems, Valet and Wittbold [42], in the additive case, and Bauzet, Vallet and Wittbold [3], in the multiplicative case, obtain existence and uniqueness of entropy solutions to the homogeneous Dirichlet problem, i.e., null boundary condition. See also [29] where the notion of renormalized kinetic solution is introduced to provide the well-posedness of the non-homogeneous Dirichlet problem.

The great challenge in these works is that, in virtue of Itô formula, one is prevented to use the usual Kruzhov’s entropies and thus needs to elaborate some more involving adaptations. We would like to especially highlight the paper [13], for its stochastic kinetic formulation provide elegant proofs to deep theorems, such as the comparison principle. The methods therein were later extended to degenerated parabolic problems by Debussche, Hofmanová and Vovelle [14] and Gess and Hofmanová [18].

The stochastic Neumann problem (1.1)–(1.3) is investigated here for the first time. In the present work, we state the definition of entropy solution to the initial-boundary value problem, and prove its well-posedness under some appropriate hypotheses. Our approach in this paper is a combination of both the reasonings of [9] and [13]. Such synthesis however is not simple, as several difficulties arise from the distinction of their contexts, [9] in the deterministic setting, and [13] in the periodic stochastic case.

To begin with, the parabolic approximation,

\begin{align}
&\frac{du^\varepsilon}{dt} + \nabla \cdot A(u^\varepsilon) dt = \varepsilon \Delta u^\varepsilon dt + \Phi(x, u^\varepsilon) dW \quad \text{in } \{t > 0\} \times \mathcal{O} \\
&u^\varepsilon(0, x) = u_0(x) \quad \text{on } \{t = 0\} \times \mathcal{O} \\
&A(u^\varepsilon(t, x)) \cdot \nu(x) = \varepsilon \frac{\partial u^\varepsilon}{\partial \nu}(t, x) \quad \text{on } \{t > 0\} \times \partial \mathcal{O},
\end{align}

already poses a curious difficulty. While for the deterministic problem one could always refer to the classic book of Ladyzhenskaya et al. [33], here, it is not possible to do so. Indeed, the presence of the stochastic term in (1.15) prevents us to have regularity in the time variable better than Hölder continuity and, in particular, classical solutions are not available here. So here we need to come back to the basic Duhamel formula and application of Banach fixed point theorem scheme. But then the nonlinear boundary condition (1.17) becomes a challenging difficulty to overcome.

Nevertheless, methods for dealing with stochastic parabolic equations were previously introduced by Gyöngy and Rovira [21], for problems with homogeneous Dirichlet conditions, and by Hofmanová [23] in the periodic setting. However, they
are not applicable here due to the presence of the nonlinear boundary condition (1.8) which imposes the need of subtle deviations from the usual Duhamel plus Banach fixed point approach.

Namely, we define the mapping

$$K(v(t)) = S(t)u_0 - \int_0^t S(t-s)\nabla \cdot A(v(s)) \, ds + \int_0^t \Phi(v(s)) \, dW(s) + w^\nu(t),$$

where $S(t)$ is the semigroup generated by the operator $A = \Delta$ defined on $D(A) = \{f \in H^2(O); \frac{\partial f}{\partial \nu} = 0 \text{ on } \partial O \text{ in the sense of traces}\}$, and $w^\nu$ is the weak solution of

$$\begin{align*}
\partial_t w^\nu &= \varepsilon \Delta w^\nu & \text{in } \{t > 0\} \times O \\
w^\nu(0, x) &= 0 & \text{on } \{t = 0\} \times O \\
A(v(t, x)) \cdot \nu(x) &= \frac{\partial w^\nu}{\partial \nu}(t, x) & \text{on } \{t > 0\} \times \partial O.
\end{align*}$$

By establishing some subtle regularization properties of $S(t)$, we are able to establish relative entropy identities based on the Itô formula for the approximate solutions $K(v(t))$. Using the latter, we can show that $K$ is a contraction on a suitable Sobolev space, demonstrating both existence and uniqueness of solutions to (1.15)–(1.17).

The vanishing viscosity passage $\varepsilon \to 0$ also has its nontrivial issues. The argument of [13], based on the comparison principle, does not seem to work here, as their uniform estimates do not hold near the boundary. Fortunately, we have at our disposal the novel stochastic averaging lemma of [18] which then allows us to obtain regularity in the space variable enough for applying a compactness scheme, similar to the well known Aubin-Lions lemma (see, e.g., [34]), consisting in a combination of Prokhorov theorem, Skorokhod representation theorem and Gyöngy and Krylov [20] criterion for convergence in probability, as done by Hofmanová in [22]. In particular, the mentioned criterion for convergence in probability in [20] amounts to a simple but very useful observation which asserts, roughly speaking, that, given a weakly compact sequence of distributions $\phi^\alpha$, if for all pairs of subsequences $(\phi^\alpha_k, \phi^\alpha_j)$ there exists a further subsequence weakly converging to a distribution concentrated at the diagonal, then the entire sequence converges in probability. The condition of concentration on the diagonal in general is verified by the fact that the limit of each of the subsequences of the pair satisfies a certain equation for which uniqueness has been proven. Therefore, one needs to establish the uniqueness for (1.1)–(1.3), which is achieved by combining the doubling of variables technique in [13] with the aforementioned strong trace theorem in [43].

Concerning the relation between the noise and the boundary of the domain we adopt the following approach. For most of the paper we develop our analysis assuming that the support of the noise is compact lying in the interior of the domain. Then at the end we extend the well-posedness result to general noises by introducing the concept of limit noise class kinetic solution.

The rest of this paper is organized as follows. In Section 2, we recall the Strong Trace theorem of [43]. In Section 3, we prove a comparison principle for kinetic solutions to (1.1)–(1.3), from which the uniqueness of such solutions follows. In Section 4 we deal with the parabolic approximation (1.15)–(1.17), detailing each step we have delineated above. In Section 5, we analyse the passage to the limit $\varepsilon \to 0$ and show the desired convergence. Finally, in Section 6 we introduce the
concept of limit noise class kinetic solutions and extend the well-posedness result for noises not supported in the interior of the domain. We have also included an Appendix concerning the smoothing effects of the propagator $S(t)$, which play a central role in the analysis developed in Section 4.

2. Strong Trace

In this section we recall a strong trace theorem proved by Vasseur in [43], that will be used later on in this paper.

**Definition 2.1.** Let $\mathcal{U} \subset \mathbb{R}^d$ be an open set. We say that $\partial \mathcal{U}$ is a Lipschitz deformable boundary if the following hold:

(i) For each $x \in \partial \mathcal{U}$, there exist $r > 0$ and a Lipschitz mapping $\gamma : \mathbb{R}^{d-1} \to \mathbb{R}$ such that, upon relabeling, reorienting and translation,

$$\mathcal{U} \cap Q(x,r) = \{ y \in \mathbb{R}^{d-1} : \gamma(y_1, \ldots, y_{d-1}) < y_d \} \cap Q(x,r),$$

where $Q(x,r) = \{ y \in \mathbb{R}^d : |y_i - x_i| \leq r, i = 1, \ldots, d \}$. We denote by $\hat{\gamma}$ the map $\hat{y} \mapsto (\hat{y}, \gamma(\hat{y}))$, $\hat{y} = (y_1, \ldots, y_{d-1})$.

(ii) There exists a map $\Psi : [0,1] \times \partial \mathcal{U} \to \bar{\mathcal{U}}$ such that $\Psi$ is a bi-Lipschitz homeomorphism over its image and $\Psi(0,x) = x$, for all $x \in \partial \mathcal{U}$. For $s \in [0,1]$, we denote by $\Psi_s$ the mapping from $\partial O$ to $\bar{\mathcal{U}}$ given by $\Psi_s(x) = \Psi(s,x)$, and set $\partial \mathcal{U}_s := \Psi_s(\partial \mathcal{U})$. We call such map a Lipschitz deformation for $\partial \mathcal{U}$.

**Definition 2.2.** Let $\mathcal{U} \subset \mathbb{R}^d$ be an open set with a Lipschitz deformable boundary and $\Psi : [0,1] \times \partial \mathcal{U} \to \bar{\mathcal{U}}$ a Lipschitz deformation for $\partial \mathcal{U}$. The Lipschitz deformation is said to be regular over $\Gamma \subset \partial \mathcal{U}$, if $D\Psi_s \to \text{Id}$, as $s \to 0$, in $L^1(\Gamma, \mathcal{H}^{d-1})$. It is simply said to be regular if it is regular over $\partial \mathcal{U}$.

We recall the following result about the existence of strong traces by Vasseur [43] (see [37], for less stringent assumptions on the flux function).

**Theorem 2.1** ([43]). Let $\mathcal{U}$ be a smooth bounded domain and $\Psi : [0,1] \times \partial \mathcal{U} \to \bar{\mathcal{U}}$ a regular Lipschitz deformation for $\partial \mathcal{U}$. Let $A \in C^3(\mathbb{R}, \mathbb{R}^d)$ satisfy (1.10) and $u(t,x)$ be an entropy solution of

$$u_t + \nabla \cdot A(u) = 0,$$

in $(0,T) \times \mathcal{U}$, that is, for all $C^2$ convex function $\eta$ and $q$ such that $q' = \eta' A'$,

$$\eta(u_t) + \nabla \cdot q(u) \leq 0,$$

in the sense of distributions in $(0,T) \times \mathcal{U}$. Then, there exists $u_b \in L^\infty((0,T) \times \partial \mathcal{U})$ such that

$$\lim_{s \to 0} \int_0^T \int_{\partial \mathcal{U}} |u(t,\Psi(s,x)) - u_b(t,x)| d\mathcal{H}^{d-1}(x)dt = 0.$$

We observe that, if $u$ is a kinetic solution of (1.1)-(1.3) and $\mathcal{U}$ is a smooth bounded domain such that for all $\xi \in \mathbb{R}$, $\mathcal{U} \subset \mathcal{O} \setminus \{\text{supp} g_k(\cdot, \xi)\}$, for all $k \geq 1$, then $u(t,x)$ is a.s. an entropy solution of (2.1) and Theorem 3.1 applies. This is particularly interesting if we choose $\mathcal{U}$ such that $\partial \mathcal{U} \cap \partial \mathcal{O} = \partial \mathcal{O}$. We can then conclude that the kinetic solution of (1.1)-(1.3) is endowed with a strong trace on $(0,T) \times \partial \mathcal{O}$. 
3. Doubling of Variables, Kato-Kruzhkov Inequality, Comparison Principle and Uniqueness

We first state the following extension of proposition 9 of [13]. In the latter, the statement of the proposition is preceded by an auxiliary result establishing the existence \( P \)-a.s. of the right and left weak limits of \( f(t, \cdot, \cdot) = 1_{u(t, \cdot) > \xi} \) in the sense of distributions. A similar fact holds here also, with exactly the same proof. Namely, there are \( f^\pm(t, \cdot, \cdot) \) which coincide with \( f(t, \cdot, \cdot) \) for a.a. \( t \in [0, T] \), such that

\[
(f(t_* \varepsilon), \varphi) \to (f^\pm, \varphi),
\]

for all \( \varphi \in C^1_c(O \times \mathbb{R}) \), and all \( t_* \in (0, T) \); for \( t_* = 0 \), only the right limit, and \( t_*= T \), only the left limit.

**Proposition 3.1 (Doubling of variables).** Let \( f_i = 1_{u_i > \xi} \), where \( u_i \) is a kinetic solution of (1.1)–(1.3), \( i = 1, 2 \). Set \( f_2 = 1 - f_1 \). Then, for \( 0 \leq t \leq T \), and non-negative test functions \( \rho \in C^\infty_c(\mathbb{R}^d) \), \( \phi \in C^\infty_c(O) \), \( \psi \in C^\infty_c(\mathbb{R}) \), with \( \text{supp} \rho \subset B(0; r) \) with \( r > 0 \) sufficiently small, we have

\[
(3.1) \quad E \int_{O^2} \int_{\mathbb{R}^2} \rho(x-y)\psi(\xi - \zeta)\phi((x+y)/2)f^\pm_1(t, x, \xi)f^\pm_2(t, y, \zeta) \, d\xi \, d\zeta \, dx \, dy
\]

\[
\leq E \int_{O^2} \int_{\mathbb{R}^2} \rho(x-y)\psi(\xi - \zeta)\phi((x+y)/2)f_{1,0}(x, \xi)f_{2,0}(y, \zeta) \, d\xi \, d\zeta \, dx \, dy \, + I_\rho + I_\phi + I_\psi,
\]

where

\[
I_\rho = E \int_0^t \int_{O^2} \int_{\mathbb{R}^2} f_1(s, x, \xi)f_2(s, y, \zeta)(a(\xi) - a(\zeta))\phi((x+y)/2)\psi(\xi - \zeta) \, d\xi \, d\zeta \,
\]

\[
\cdot \nabla_x \rho(x-y) \, dx \, dy \, ds,
\]

\[
I_\phi = \frac{1}{2} E \int_0^t \int_{O^2} \int_{\mathbb{R}^2} f_1(s, x, \xi)f_2(s, y, \zeta)(a(\xi) + a(\zeta))\rho(x-y)\psi(\xi - \zeta) \, d\xi \, d\zeta \,
\]

\[
\cdot \nabla_x \phi((x+y)/2) \, dx \, ds,
\]

and

\[
I_\psi = \frac{1}{2} \int_{O^2} \rho(x-y)\phi((x+y)/2)E \int_0^t \int_{\mathbb{R}^2} \psi(\xi - \zeta) \sum_{k \geq 1} |g_k(x, \xi) - g_k(y, \zeta)|^2 \, d\nu^{1}_{s,x} \otimes \nu^{2}_{s,y}(\xi, \zeta) \, dx \, dy \, ds,
\]

where \( \nu^{i}_{s,x} = \delta_{u_i = \xi}, \ i = 1, 2 \).

**Proof.** The proof of the proposition follows the lines of the one in [13]. The introduction of the new test function serves to keep things far from the boundary, since here we are no longer in the periodic context but the treatment is similar.

Denote \( \langle \cdot, \cdot \rangle_{L^2} \), the scalar product in \( L^2(O_x \times O_y \times \mathbb{R}_x \times \mathbb{R}_y) \). Set \( G_1^1(x, \xi) = \sum_{k \geq 1} |g_k(x, \xi)|^2 \) and \( G_2^2(x, \zeta) = \sum_{k \geq 1} |g_k(x, \zeta)|^2 \). Let \( \varphi_1 \in C^\infty_c(O_x \times \mathbb{R}_x) \) and \( \varphi_2 \in C^\infty_c(O_y \times \mathbb{R}_y) \). By (1.11) we have

\[
\langle f^+_1, \varphi_1 \rangle = \langle m^1_1, \partial_t \varphi_1 \rangle([0, t]) + F_1(t),
\]
with

\[ F_1(t) = \sum_{k \leq 1} \int_0^t \int_{\mathbb{R}} g_{k,1} \varphi_1 dv_{x,s}^1(\xi) d\beta_k(s), \]

and

\[ \langle m_1^+, \partial_\xi \varphi_1 \rangle ([0, t]) = (f_{1,0}, \varphi_1) \delta_0([0, t]) + \int_0^t \langle f_1, a \cdot \nabla \varphi_1 \rangle ds \]

\[ + \frac{1}{2} \int_0^t \int_{\mathbb{R}} \partial_\xi \varphi_1 G_1^2 dv_{x,s}^1(\xi) dx ds - \langle m_1, \partial_\xi \varphi_1 \rangle ([0, t]). \]

Notice that, \( \langle m_1^+, \partial_\xi \varphi_1 \rangle ([0, t]) = \langle f_{1,0}, \varphi_1 \rangle. \) Similarly,

\[ \langle \mathcal{F}_2^+, \varphi_2 \rangle = \langle m_2^+, \partial_\xi \varphi_2 \rangle ([0, t]) + F_2(t), \]

with

\[ \mathcal{F}_2(t) = -\sum_{k \leq 1} \int_0^t \int_{\mathbb{R}} g_{k,2} \varphi_2 dv_{y,s}^2(\zeta) d\beta_k(s), \]

and

\[ \langle m_2^+, \partial_\xi \varphi_2 \rangle ([0, t]) = \langle \mathcal{F}_{2,0}, \varphi_2 \rangle \delta_0([0, t]) + \int_0^t \langle \mathcal{F}_2, a \cdot \nabla \varphi_2 \rangle ds \]

\[ - \frac{1}{2} \int_0^t \int_{\mathbb{R}} \partial_\xi \varphi_2 G_2^2 dv_{y,s}^2(\zeta) dy ds + \langle m_2, \partial_\xi \varphi_2 \rangle ([0, t]). \]

We can also get \( \langle m_2^+, \partial_\xi \varphi_2 \rangle ([0, t]) = \langle \mathcal{F}_{2,0}, \varphi_2 \rangle. \) Now let \( \alpha(x, \xi, y, \zeta) = \varphi_1(x, \xi) \varphi_2(y, \zeta). \)

Using the Itô formula for \( F_1(t) \mathcal{F}_2(t), \) integration by parts for functions of bounded variation (see e.g., [39], chapter 0) for

\[ \langle m_1^+, \partial_\xi \varphi_1 \rangle ([0, t]) \langle m_2^+, \partial_\xi \varphi_2 \rangle ([0, t]), \]

which gives

\[ \langle m_1^+, \partial_\xi \varphi_1 \rangle ([0, t]) \langle m_2^+, \partial_\xi \varphi_2 \rangle ([0, t]) \]

\[ = \langle m_1^+, \partial_\xi \varphi_1 \rangle ([0, t]) \langle m_2^+, \partial_\xi \varphi_2 \rangle ([0, t]) + \int_{(0,t]} \langle m_1^+, \partial_\xi \varphi_1 \rangle ([0, s]) \langle m_2^+, \partial_\xi \varphi_2 \rangle (s) \]

\[ + \int_{(0,t]} \langle m_2^+, \partial_\xi \varphi_2 \rangle ([0, s]) \langle m_1^+, \partial_\xi \varphi_1 \rangle (s) \]

and the formula

\[ \langle m_1^+, \partial_\xi \varphi_1 \rangle ([0, t]) \mathcal{F}_2(t) = \int_0^t \langle m_1^+, \partial_\xi \varphi_1 \rangle ([0, s]) d\mathcal{F}_2(s) + \int_0^t \mathcal{F}_2(s) \langle m_1^+, \partial_\xi \varphi_1 \rangle (ds) \]

which is clear because \( \mathcal{F}_2 \) is continuous, and a similar formula for

\[ \langle m_2^+, \partial_\xi \varphi_2 \rangle ([0, t]) F_1(t), \]

we have

\[ \langle f_1^+(t), \varphi_1 \rangle \langle \mathcal{F}_2^+(t), \varphi_2 \rangle = \langle f_1^+(t) \mathcal{F}_2^+(t), \alpha \rangle_{L^2}. \]
which satisfies

\[
E\langle f_1^+(t)\mathcal{F}_2^+(t), \alpha \rangle_{L^2} = \langle f_{1,0} \mathcal{F}_{2,0}, \alpha \rangle_{L^2} + I_1 + I_\psi,
\]

where\( G_{1,2}(x, \xi; y, \zeta) := \sum_{k \geq 1} g_k(x, \xi)g_k(y, \zeta) \). By density, we can take \( \alpha = \rho \phi \psi \), with \( \rho = \rho(x-y) \), \( \psi = \psi(|\xi-\zeta|) \) and \( \phi = \phi((x+y)/2) \). Since \( \alpha \geq 0 \) and by the identity

\[
\partial_\xi \alpha = -\partial_\zeta \alpha
\]

the last term in (3.2) satisfies

\[
E \int_0^t \int_{\mathcal{O}^2} \int_{\mathbb{R}^2} f_1^- (s) \partial_\xi \alpha dm_2(y, s, \xi) d\xi dx
\]

\[
= -E \int_0^t \int_{\mathcal{O}^2} \int_{\mathbb{R}^2} f_1^- (s) \partial_\zeta \alpha dm_2(y, s, \zeta) d\xi dx
\]

\[
= -E \int_0^t \int_{\mathcal{O}^2} \int_{\mathbb{R}^2} \alpha d\nu_{x,s}^1 (\xi) dm_2(y, s, \xi) dx \leq 0
\]

Similarly,

\[
- E \int_0^t \int_{\mathcal{O}^2} \int_{\mathbb{R}^2} \mathcal{F}_2^+ (s) \partial_\xi \alpha dm_1(x, s, \xi) d\zeta dy
\]

\[
= -E \int_0^t \int_{\mathcal{O}^2} \int_{\mathbb{R}^2} \alpha d\nu_{y,s}^2 (\zeta) dm_1(x, s, \xi) dy \leq 0
\]

Thus, we have

\[
(3.3) \quad E\langle f_1^+(t)\mathcal{F}_2^+(t), \alpha \rangle_{L^2} \leq \langle f_{1,0} \mathcal{F}_{2,0}, \alpha \rangle_{L^2} + I_1 + I_\psi,
\]

where

\[
I_1 = - E \int_0^t \int_{\mathcal{O}^2} \int_{\mathbb{R}^2} f_1 \mathcal{F}_2 (a(\xi) \cdot \nabla x + a(\zeta) \cdot \nabla y) \alpha d\xi d\zeta dx dy ds
\]

\[
= I_\rho + I_\phi
\]
\[
I_\psi = \frac{1}{2} \mathbb{E} \int_0^t \int_{\Omega^2} \int_{\mathbb{R}^2} \partial_t \alpha f_2(s) G_1^2 \, d\nu_{x,s}(\xi) d\zeta dxdyds \\
- \frac{1}{2} \mathbb{E} \int_0^t \int_{\Omega^2} \int_{\mathbb{R}^2} \partial_t \alpha f_1(s) G_2^2 \, d\nu_{y,s}(\zeta) d\xi dydxds \\
- \mathbb{E} \int_0^t \int_{\Omega^2} \int_{\mathbb{R}^2} \partial_x \alpha G_1^2 \, d\nu_{x,s}(\xi) d\nu_{y,s}(\zeta) dxdyds.
\]

Integration by part in \( I_\psi \), we obtain
\[
I_\psi = \frac{1}{2} \mathbb{E} \int_0^t \int_{\Omega^2} \int_{\mathbb{R}^2} \alpha (G_1^2 + G_2^2 - 2G_{1,2}) \, d\nu_{x,s}(\xi) d\zeta dxdyds \\
= \frac{1}{2} \mathbb{E} \int_0^t \int_{\Omega^2} \int_{\mathbb{R}^2} \alpha \sum_{k \geq 0} |g_k(x,\xi) - g_k(y,\zeta)|^2 \, d\nu_{x,s}(\xi) d\nu_{y,s}(\zeta) dxdyds.
\]

So (3.3) is exactly (3.1). In the case of \( f_1^-, f_2^- \), we take \( t_n \uparrow t \), write (1.11) for \( f_1^+(t_n) \), \( f_2^+(t_n) \) and let \( n \to \infty \). □

As a consequence of Proposition 3.1, we have the Kato-Kruzhkov inequality.

**Theorem 3.1.** Let \( u_1, u_2 \) be kinetic solutions of (1.1)-(1.3). Then, for a.e. \( 0 \leq t \leq T \) and \( 0 \leq \phi \in C^\infty_c(\Omega) \), we have

\[
\mathbb{E} \int_\Omega (u_1(t,x) - u_2(t,x))^+ \phi(x) \, dx \\
\leq -\mathbb{E} \int_\Omega \text{sgn}(u_1 - u_2)^+ (A(u_1) - A(u_2)) \cdot \nabla \phi(x) \, dx \\
+ \int_\Omega (u_{10}(x) - u_{20}(x))^+ \phi(x) \, dx.
\]

**Proof.** This is similar to the proof of the comparison principle in [13].

Let \( \rho \in C^\infty_c(\Omega), \psi \in C^\infty_c(\mathbb{R}) \) be symmetric nonnegative functions such that \( \int_\Omega \rho = 1, \int_\mathbb{R} \psi = 1 \) and \( \text{supp} \psi \subseteq (-1,1) \). Define

\[
\rho_\varepsilon = \frac{1}{\varepsilon} \rho\left(\frac{x}{\varepsilon}\right), \quad \psi_\delta = \frac{1}{\delta} \psi\left(\frac{\xi}{\delta}\right).
\]

Then

\[
\mathbb{E} \int_\Omega \int_\mathbb{R} f_1^+(x,t,\xi) \mathcal{F}_2^+(x,t,\xi) \phi(x) d\xi dx \\
= \mathbb{E} \int_\Omega \int_\mathbb{R} \rho_\varepsilon(x-y) \psi_\delta(\xi - \zeta) \phi((x+y)/2) f_1^+(x,t,\xi) \mathcal{F}_2^+(y,t,\zeta) d\xi d\zeta dxdy + \omega_t(\varepsilon, \delta),
\]

where \( \lim_{\varepsilon,\delta} \omega_t(\varepsilon, \delta) = 0 \). Using (3.1), we now consider the terms \( I_\rho, I_\psi \) and \( I_\phi \).
For the term $I_\psi$, due to (1.6) one has

$$I_\psi \leq CE \int_0^t \int_{\mathbb{R}^2} \phi((x+y)/2) \rho_\epsilon(x-y)|x-y|^2 \int_{\mathbb{R}^2} \psi_\beta(\xi - \zeta) dv_{\epsilon,s}(\xi) dv_{\epsilon,s}(\zeta) dxdyds$$

$$+ CE \int_0^t \int_{\mathbb{R}^2} \phi((x+y)/2) \rho_\epsilon(x-y)$$

$$\times \int_{\mathbb{R}^2} \psi_\beta(\xi - \zeta) |\xi - \zeta|^2 dv_{\epsilon,s}(\xi) dv_{\epsilon,s}(\zeta) dxdyds$$

$$\leq Ct\delta^{-1}\varepsilon^2 + Ct\delta.$$

Next we consider $I_\phi$, one has

$$I_\phi = E \int_0^t \int_{\mathbb{R}^2} f_1(s, x, \xi) f_2(s, y, \xi) a(\xi) d\xi \cdot \nabla \phi(x) dxdyds + \theta_1(\varepsilon, \delta)$$

$$= -E \int_0^t \int_{\mathbb{R}^2} \text{sgn}(u_1 - u_2)_+ (A(u_1) - A(u_2)) \cdot \nabla \phi(x) dxdyds + \theta_1(\varepsilon, \delta),$$

where $\lim_{\varepsilon, \delta} \theta_1(\varepsilon, \delta) = 0$.

Finally we think about the term $I_\rho$, since $a'$ is locally bounded, setting $\|a'||_{\infty, \delta} := \|a'||_{L_\infty(-2\delta - N, 2\delta + N)}$, where $N \geq \max\{\|u_1\|_{\infty}, \|u_2\|_{\infty}\}$, we deduce

$$I_\rho \leq \|a'||_{\infty, \delta} E \int_0^t \int_{\mathbb{R}^2} f_1 f_2 |\xi - \zeta| \phi((x+y)/2) |\psi_\beta(\xi - \zeta)| d\xi d\zeta$$

$$\times |\nabla \rho_\epsilon(x-y)| dxdyds$$

$$\leq Ct\varepsilon^{-1} \delta \|a'||_{\infty, \delta}.$$

In conclusion, we deduce for all $t \in [0, T]$

$$E \int_{\mathcal{O}} \int_{\mathbb{R}} f_1^\pm(x, t, \xi) f_2^\pm(x, t, \xi) \phi(x) d\xi dx$$

$$\leq E \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \rho_\epsilon(x-y) \psi_\beta(\xi - \zeta) \phi((x+y)/2) f_1 f_2^\pm d\xi d\zeta dxdy$$

$$- E \int_0^t \int_{\mathcal{O}} \text{sgn}(u_1 - u_2)_+ (A(u_1) - A(u_2)) \cdot \nabla \phi(x) dxdyds + \theta_1(\varepsilon, \delta)$$

$$+ \omega_1(\varepsilon, \delta) + C t\delta^{-1}\varepsilon^2 + C t\delta + C t\varepsilon^{-1} \delta \|a'||_{\infty, \delta}.$$

Taking $\delta = \varepsilon^\beta$ with $\beta \in (1, 2)$ and letting $\varepsilon \to 0$, and observing that the left-hand side of the above inequality coincides with the left-hand side of (3.4), for a.e. $t \in (0, T)$, the desired estimate (3.4) follows.

As a direct consequence of Theorem 3.1 and Theorem 2.1 we have the following.

**Theorem 3.2 (Comparison Principle).** Let $u_1, u_2$ be kinetic solutions of (1.1)–(1.3). Then for a.e. $t \in (0, T)$,

$$E \int_{\mathcal{O}} (u_1(t, x) - u_2(t, x))^+ dx \leq \int_{\mathcal{O}} (u_{10}(x) - u_{20})^+ dx.$$

**Proof.** Let $\Psi : \partial \mathcal{O} \times [0, 1] \to \mathcal{O}$ be a regular deformation for $\partial \mathcal{O}$ and let $h : \mathcal{O} \to [0, 1]$ be defined by $h(x) = s$, if $x \in \partial \mathcal{O}_s$, $h(x) = 1$, if $x \in \mathcal{O} \setminus \Psi(\partial \mathcal{O} \times [0, 1])$, and $h(x) = 0$,
for \( x \notin \Omega \).

\[(3.6) \quad \varphi_\rho(x) := \min\{1, \frac{1}{\rho}h(x)\}.\]

The inequality (3.4) easily extends to \( \phi \) Lipschitz vanishing on \( \partial \Omega \). So we take \( \phi = \varphi_\rho \) in (3.4). We then make \( \rho \to 0 \) and observe that the first integral on the right-hand side of (3.4) vanishes when \( \rho \to 0 \) because of the strong trace property in (2.3). Indeed, we see that

\[
\nabla \varphi_\rho(x) = \begin{cases} \frac{1}{\rho}C(\Psi_{h(x)}(x))\nu(\Psi_{h(x)}(x)), & \text{for } x \in \Psi([0, \rho] \times \partial \Omega), \\ 0, & \text{otherwise}, \end{cases}
\]

for a smooth function \( C(y) \). But, from the regularity of the deformation, we deduce that \( \nu(\Psi_s(x)) \to \nu(x) \) in \( L^1(\partial \Omega) \), as \( s \to 0 \). Then, since the noise is compactly supported in the interior of \( \Omega \), Theorem 2.1 implies that the integral

\[
\int_0^t \int_\Omega \text{sgn}(u_1 - u_2)_+ (\mathbf{A}(u_1) - \mathbf{A}(u_2)) \cdot \nabla \phi(x) \, dx \, ds
\]

vanishes a.s. as \( \rho \to 0 \). Hence, by the dominated convergence theorem, we conclude that the first integral on the right-hand side of (3.4) vanishes as \( \rho \to 0 \), while \( \varphi_\rho \to 1 \) in \( \Omega \).

We remark that the a.s. continuity of the trajectories of a kinetic solution follows exactly as in [13]. In particular, in the statements of Theorem 3.1 and Theorem 3.2, the conclusion holds a.s. for all \( t \in [0, T] \).

We conclude this section by establishing a maximum principle for the kinetic solution of (1.1)–(1.3).

**Theorem 3.3** (Maximum Principle). Let \( u \) be a kinetic solution of (1.1)–(1.3), with \( u_0 \) satisfying (1.7). Then, a.s., \( a \leq u(t, x) \leq b \), a.e. \((t, x) \in (0, T) \times \Omega \).

**Proof.** It suffices to observe that under the hypothesis (1.8) and the fact that \( g_k(x, u) \) vanishes for \( u \notin (-M, M) \), \( k \in \mathbb{N} \), the functions \( v_1 \equiv a \) and \( v_2 \equiv b \) are kinetic solutions of (1.1)–(1.3). Therefore, applying (3.5) first with \( u_1 = a \), \( u_2 = u \) and then with \( u_1 = u \) and \( u_2 = b \), we get the desired result.

We observe here that Proposition 3.1 may be easily extended to the case where \( u_1 \) and \( u_2 \) satisfy equations with different stochastic terms \( \Phi(u) \, dW(t) \) and \( \hat{\Phi}(u) \, dW(t) \), respectively, the only change being that the integral \( I_\psi \) in the statement now becomes

\[
I_\psi = \frac{1}{2} \int_{\partial \Omega} \rho(x - y)\phi((x + y)/2)\mathbb{E} \int_0^t \int_{\mathbb{R}^2} \psi(\xi - \zeta) \sum_{k \geq 1} |g_k(x, \xi) - \tilde{g}_k(y, \zeta)|^2 \, du_{x,y}^1 \otimes \nu_{x,y}^2(\xi, \zeta) \, dx \, dy \, ds,
\]

where \( g_k(\cdot, \xi) = \Phi(\xi) e_k, \tilde{g}_k(\cdot, \xi), \hat{g}_k(\cdot, \xi) = \hat{\Phi}(\xi) e_k \), \( k \in \mathbb{N} \). Therefore, we have the following straightforward extensions to Theorem 3.1 and Theorem 3.2 for the case in which the stochastic terms of the equations satisfied by \( u_1 \) and \( u_2 \) are distinct, \( \Phi(u) \, dW(t), \hat{\Phi}(u) \, dW(t) \), respectively.
Theorem 3.4. Let $u_1, u_2$ be kinetic solutions of (1.1)–(1.3), with distinct stochastic terms $\Phi(u)\, dW(t)$, $\hat{\Phi}(u)\, dW(t)$, respectively. Then, for $0 \leq t \leq T$ and $0 \leq \phi \in C^\infty_c(O)$, we have

\begin{equation}
E \int_O (u_1(t,x) - u_2(t,x))_+ \phi(x) \, dx 
\end{equation}

\begin{equation}
\leq -E \int_0^t \int_O \text{sgn}(u_1 - u_2)_+ (A(u_1) - A(u_2)) \cdot \nabla \phi(x) \, dx \, ds 
+ \|\phi\|_\infty t \sup_{\xi \in [-M,M]} \sum_{k \geq 1} \|g_k(\cdot, \xi) - \hat{g}_k(\cdot, \xi)\|_{L^2(O)}^2 
+ \int_O (u_{10}(x) - u_{20}(x))_+ \phi(x) \, dx.
\end{equation}

Theorem 3.5 (Comparison Principle, different noises). Let $u_1, u_2$ be kinetic solutions of (1.1)–(1.3) with distinct stochastic terms $\Phi(u)\, dW(t)$, $\hat{\Phi}(u)\, dW(t)$, respectively. Then

\begin{equation}
E \int_O (u_1(t,x) - u_2(t,x))_+ \, dx \leq \int_O (u_{10}(x) - u_{20})_+ \, dx 
+ t \sup_{\xi \in [-M,M]} \sum_{k \geq 1} \|g_k(\cdot, \xi) - \hat{g}_k(\cdot, \xi)\|_{L^2(O)}^2.
\end{equation}

4. Existence: The parabolic approximation

For the existence of a kinetic solution to problem (1.1)–(1.3) we will perform the following steps. First, we establish the existence of the parabolic approximation and its kinetic formulation. Second, we prove a spatial regularity for the parabolic approximation which is independent of the vanishing artificial viscosity. Third, using the regularity obtained in the second, we show that the sequence of parabolic approximate solutions is compact in $L^1_{loc}$.

We consider the following parabolic approximation of problem (1.1)–(1.3),

\begin{align}
du^\varepsilon + \nabla \cdot A^\varepsilon(u^\varepsilon) \, dt - \varepsilon \Delta u^\varepsilon \, dt &= \Phi^\varepsilon(u^\varepsilon) \, dW(t), \quad t > 0, \quad x \in O, \\
u^\varepsilon(0,x) &= u_0^\varepsilon(x), \quad x \in O, \\
(A(u^\varepsilon(t,x)) - \varepsilon \nabla u^\varepsilon(t,x)) \cdot \nu(x) &= 0, \quad t > 0, \quad x \in \partial O,
\end{align}

where $u_0^\varepsilon$ is a smooth approximation of $u_0$, $u_0^\varepsilon \in L^\infty(C^\infty_c(O))$, $a \leq u_0^\varepsilon \leq b$, $\Phi^\varepsilon$ is a suitable Lipschitz approximation of $\Phi$ satisfying (1.5) and (1.6) uniformly, with $g_k^\varepsilon$ and $G^\varepsilon$ as in the case $\varepsilon = 0$, $g_k^\varepsilon$ smooth with compact support contained in $V \times (-M, M)$. Moreover, $g_k^\varepsilon \equiv 0$ for $k \geq 1/\varepsilon$. Finally, $A^\varepsilon \in C^2(R; R^d)$, $A^\varepsilon(u) = A(u)$, for $u \in [a, b]$, and, setting $a^\varepsilon = (A^\varepsilon)'$, we assume that $a^\varepsilon \in L^\infty(R; R^d)$. The justification for the latter assumption is the fact that, by Theorem 3.3, any solution of (1.1)–(1.3) takes values in the interval $[a, b]$ and so, since our goal is to use the solution of (4.1)–(4.3) as an approximation as $\varepsilon \to 0$ to a kinetic solution of (1.1)–(1.3), we may modify $A$ out of $[a, b]$ as we wish. In particular, we may assume that $\text{supp } A^\varepsilon \subseteq [a-1, b+1]$, so that $A$ has a primitive which is bounded uniformly in $\varepsilon$.

Theorem 4.1. There exists a unique solution of (4.1)–(4.3), $u^\varepsilon \in C([0,T]; L^2(O \times O)) \cap L^2(O \times [0,T]; H^1(O))$, for any $\varepsilon > 0$. Moreover, $u^\varepsilon$ satisfies the following
energy estimate

\[ (4.4) \quad \mathbb{E} \sup_{0 \leq t \leq T} \| u^\varepsilon(t) \|^2_{L^2(\Omega)} + 2\varepsilon \mathbb{E} \int_0^T \| \nabla u^\varepsilon(s) \|^2_{L^2(\Omega)} ds \leq C(T)(\| u_0^\varepsilon \|^2_{L^2(\Omega)} + 1). \]

The plan of the proof is to apply Banach’s fixed point theorem. Let \( E := L^2(\Omega; C([0, T]; L^2(\Omega))) \cap L^2(\Omega \times [0, T]; H^1(\Omega)) \). Here, we consider \( \Omega \times [0, T] \) endowed with the \( \sigma \)-algebra of the predictable sets, that is, the \( \sigma \)-algebra generated by the sets of the form \( \{ 0 \} \times A_0 \), \( \{ s, t \} \times A_s \), \( A_0 \in \mathcal{F}_0 \), \( A_s \in \mathcal{F}_s \).

To begin with we endow \( E \) with the following standard norm

\[ (4.5) \quad \| v \|^2_E := \| v \|^2_{L^2(\Omega; C([0, T]; L^2(\Omega)))} + \| \nabla v \|^2_{L^2(\Omega \times [0, T]; H^1(\Omega))}. \]

Later on we will introduce another equivalent norm for \( E \) for the purpose of proving the contraction property of the mapping \( K \) defined subsequently.

Let us define

\[ (4.6) \quad K(v)(t) = S(t)u_0 - \int_0^t S(t-s)\nabla \cdot A(v(s)) \, ds + \int_0^t S(t-s)\Phi(v)\,dW(s) + w^v(t), \]

where \( S(t) \) is the semigroup generated by the problem

\[ (4.7) \quad \begin{cases} w_t - \varepsilon \Delta w = 0, & t > 0, \ x \in \mathcal{O}, \\ w(0, x) = u_0(x), & x \in \mathcal{O}, \\ \varepsilon \nabla w(t, x) \cdot \nu(x) = 0, & t > 0, \ x \in \mathcal{O}, \end{cases} \]

and \( w^v(t) \) is the solution of

\[ (4.8) \quad \begin{cases} w^v_t - \varepsilon \Delta w^v = 0, & t > 0, \ x \in \mathcal{O}, \\ w^v(0, x) = 0, & x \in \mathcal{O}, \\ \varepsilon \nabla w^v \cdot \nu(x) = A(v(t)) \cdot \nu(x), & t > 0, \ x \in \mathcal{O}, \end{cases} \]

and we have dropped the \( \varepsilon \) for simplicity of notation.

The energy estimate for the heat equation with null Neumann condition gives

\[ (4.9) \quad \frac{1}{2} \| S(t)w_0 \|^2_2 + \varepsilon \int_0^t \| \nabla_x S(s)w_0 \|^2_2 \, ds = \frac{1}{2} \| w_0 \|^2_2. \]

**Lemma 4.1.** If \( v \in E \), then \( K(v) \in E \). Moreover, if \( v_k \to v \) in \( L^2(\Omega \times [0, T]; H^1(\mathcal{O})) \), then \( K(v_k) \to K(v) \) in \( E \).

**Proof.** We write \( K(v) = S(t)u_0 + K_1(v) + K_2(v) + w^v \), where

\[ K_1(v)(t) = \int_0^t S(t-s)\nabla \cdot A(v(s)) \, ds \]

\[ K_2(v)(t) = \int_0^t S(t-s)\Phi(v(s)) \, dW(s) \]

Concerning, \( S(t)u_0 \), (4.9) trivially gives

\[ (4.10) \quad \| S(t)h \|^2_2 \leq \| h \|^2_2, \]

so

\[ (4.11) \quad \left\| \int_0^t S(t-s)h(s) \, ds \right\|^2_2 \leq T \int_0^T |h(s)|^2 \, ds. \]

We denote

\[ \nabla_x S(t)h := \nabla_x (S(t)h). \]
We have, also from (4.9),

\begin{equation}
\int_0^T \|\nabla_x S(t) h\|_2^2 dt \leq \frac{1}{2\varepsilon} \|h\|_2^2,
\end{equation}

Therefore, we have

\begin{equation}
\|S(t)u_0\|_2^2 \leq \|u_0\|_2^2,
\end{equation}

\begin{equation}
\int_0^T \|S(t)u_0\|_2^2 dt \leq T \|u_0\|_2^2,
\end{equation}

\begin{equation}
\int_0^T \|\nabla_x S(t)u_0\|_2^2 dt \leq \frac{1}{2\varepsilon} \|u_0\|_2^2.
\end{equation}

In sum, we have

\begin{equation}
\|S(t)u_0\|_\varepsilon \leq C(T) \|u_0\|_{L^2(\mathcal{O})},
\end{equation}

where, throughout this proof \(C(T)\) is a positive constant depending only on \(T\) and the data of the problem (4.1)–(4.3).

Concerning \(K_2(v)\), again directly from (4.9), we get

\begin{equation}
\left\| \int_0^t S(t-s)\nabla_x \cdot A(v(s)) \, ds \right\|_2^2 \leq CT \int_0^T \|\nabla_x v(t)\|_2^2 dt,
\end{equation}

\begin{equation}
\int_0^T \left\| \int_0^t S(t-s)\nabla_x \cdot A(v(s)) \, ds \right\|_2^2 dt \leq CT^2 \int_0^T \|\nabla_x v(t)\|_2^2 dt,
\end{equation}

where, throughout this proof, \(C > 0\) is a constant only depending on \(\varepsilon\) and the given functions on (4.1) whose value may change from one line to the next.

Observe now that Cauchy-Schwarz and (4.9) and Fubini yield

\begin{equation}
\frac{1}{2\varepsilon} T \int_0^T \|h(s)\|_2^2 ds
\end{equation}

Therefore, we get

\begin{equation}
\int_0^T \left\| \int_0^t \nabla_x S(t-s) \nabla \cdot A(v(s)) \, ds \right\|_2^2 dt \leq CT \int_0^T \|\nabla_x v(t)\|_2^2 dt.
\end{equation}

In sum, we have

\begin{equation}
\|K_1(v)\|_\varepsilon \leq C(T) \|v\|_{L^2(\Omega \times [0,T];H^1(\mathcal{O}))}.
\end{equation}

Similarly, we obtain for \(v_1, v_2 \in \mathcal{E}\),

\begin{equation}
\|K_1(v_1) - K_1(v_2)\|_\varepsilon \leq C(T) \|A(v_1) - A(v_2)\|_{L^2(\Omega \times [0,T];H^1(\mathcal{O}))}.
\end{equation}

Relatively to (4.20), we remark that the mapping \(v \mapsto A(v)\) is continuous from \(L^2(\Omega \times [0,T]; H^1(\mathcal{O}))\) to \(L^2(\Omega \times [0,T]; H^1(\mathcal{O}))\) as can be easily verified.
Concerning $K_2(v)$, we need to apply the important maximal estimate for stochastic convolution (see [41, 30, 31]; see also [12, 8, 23]). Using the mentioned inequality, we have

\[
\mathbb{E} \sup_{0 \leq t \leq T} \left\| \int_0^t S(t-s)\Phi(v(s))dW(s) \right\|_2^2 \\
\leq C\mathbb{E} \sum_{1 \leq k \leq N} \int_0^T \|g_k(\cdot, v(s))\|_2^2 ds \\
\leq C\int_0^T (1 + \|v(s)\|_2^2) ds.
\]

On the other hand,

\[
\int_0^T \mathbb{E} \left\| \int_0^t S(t-s)\Phi(v(s))dW(s) \right\|_2^2 dt \\
\leq \mathbb{E} \int_0^T \sum_{1 \leq k \leq N} \int_0^t \|S(t-s)g_k(\cdot, v(s))\|_2^2 ds dt \\
\leq CT\mathbb{E} \int_0^T (1 + \|v(s)\|_2^2) ds,
\]

\[
\int_0^T \mathbb{E} \left\| \int_0^t \nabla_x S(t-s)\Phi(v(s))dW(s) \right\|_2^2 dt \\
\leq C\mathbb{E} \int_0^T \int_0^t \sum_{1 \leq k \leq N} \|\nabla_x S(t-s)g_k(\cdot, v(s))\|_2^2 ds dt \\
\leq \frac{C}{2\varepsilon} \mathbb{E} \int_0^T \sum_{1 \leq k \leq N} \|g_k(\cdot, v(t))\|_2^2 dt \\
\leq \frac{C}{2\varepsilon} \mathbb{E} \int_0^T (1 + \|v(t)\|_2^2) dt.
\]

In particular, we also have

\[
\|K_2(v)\|_\varepsilon \leq C(T)(1 + \|v\|_{L^2(\Omega \times [0,T]; H^1(\Omega))}),
\]

and for $v_1, v_2 \in \mathcal{E}$, observing that $\|g_k(\cdot, v_1(t)) - g_k(\cdot, v_2(t))\|_2 \leq C\|v_1(t) - v_2(t)\|_2$, we get

\[
\|K_2(v_1) - K_2(v_2)\|_\varepsilon \leq C(T)\|v_1 - v_2\|_{L^2(\Omega \times [0,T]; H^1(\Omega))}.
\]

Finally, for $w^w$ we have the following. First, assume that $v \in C^\infty_c((0,T) \times \tilde{\mathcal{O}})$, in which case also $\mathbf{A}^\varepsilon(v) \in C^\infty_c((0,T) \times \tilde{\mathcal{O}})$ by the hypotheses on $\mathbf{A}^\varepsilon(v)$. In this case, the problem has a classical smooth solution. Multiply the equation for $w^w$ by $w^w$,
integrate in $\mathcal{O}$ to get
\[
\frac{1}{2} \frac{d}{ds} \|w^v(s)\|_2^2 + \varepsilon \|\nabla w^v(s)\|_2^2 = \int_{\partial\mathcal{O}} w^v(s, \omega) A(v(s, \omega)) \cdot \nu dH^{d-1}(\omega)
\]
\[
\leq C \|w^v(s)\|_{L^2(\partial\mathcal{O})} \|A(v(s))\|_{L^2(\partial\mathcal{O})}
\]
\[
\leq C \|w^v(s)\|_{H^1} \|A(v(s))\|_{H^1}
\]
\[
\leq \frac{\varepsilon}{2} \|w^v(s)\|_{H^1}^2 + C \|A(v(s))\|_{H^1}^2
\]
where we used the trace theorem for the Sobolev space $H^1(\mathcal{O})$ and standard estimates. Integrating in $t$ we get
\[
\frac{1}{2} \|w^v(t)\|_2^2 + \varepsilon \int_0^t \|\nabla w^v(s)\|_2^2 ds \leq \frac{\varepsilon}{2} \|w^v(0)\|_{H^1}^2 + C \int_0^t \|A(v(s))\|_{H^1}^2 ds.
\]
So, using Grönwall, we get
\[
\frac{1}{2} \|w^v(t)\|_2^2 + \varepsilon \int_0^t \|\nabla w^v(s)\|_2^2 ds \leq C(T) \int_0^t \|A(v(s))\|_{H^1}^2 ds.
\]
Similarly, for $v^1, v^2 \in C_c^\infty((0, T) \times \overline{\mathcal{O}})$, we get
\[
\frac{1}{2} \|w^{v_1}(t) - w^{v_2}(t)\|_2^2 + \varepsilon \int_0^t \|\nabla w^{v_1}(s) - \nabla w^{v_2}(s)\|_2^2 ds
\]
\[
\leq C(T) \int_0^t \|v_1(s) - v_2(s)\|_{H^1}^2 ds.
\]
For the latter inequality we used that $\|A(v_1) - A(v_2)\|_{L^2(\partial\mathcal{O})} \leq \text{Lip}(A) \|v_1 - v_2\|_{L^2(\mathcal{O})} \leq C \|v_1 - v_2\|_{H^1(\mathcal{O})}$. Now, given $v \in L^2(\Omega \times [0, T]; H^1(\mathcal{O}))$, a.s. given $\omega \in \Omega$, we can approximate $v(\omega)$ by a sequence $v_k \in C_c((0, T) \times \overline{\mathcal{O}})$, and from (4.25) we see that $w^{v_k}$ converges in $C([0, T]; L^2(\mathcal{O})) \cap L^2(0, T; H^1(\mathcal{O}))$ to a certain $w^v$. Also, given any $\varphi \in H^1(\mathcal{O})$, for each $k \in \mathbb{N}$, we get
\[
\int_{\mathcal{O}} w^{v_k}(t, x) \varphi(x) dx + \varepsilon \int_0^t \int_{\mathcal{O}} \nabla w^{v_k}(s, x) \cdot \nabla \varphi(x) dx ds
\]
\[
= \int_0^t \int_{\partial\mathcal{O}} \varphi(y) A(v_k) \cdot \nu(y) dH^{d-1}(y) ds
\]
So, making $k \to \infty$, observing that $v_k \to v$ and $w^{v_k} \to w^v$ in $L^2((0, T) \times \partial\mathcal{O})$ by the continuity of the trace operator from $L^2(0, T; H^1(\mathcal{O}))$ to $L^2((0, T) \times \partial\mathcal{O})$, we get that $w^v$ satisfies (4.8) in the following weak sense: for all $\varphi \in H^1(\mathcal{O})$ we have
\[
\int_{\mathcal{O}} w^v(t, x) \varphi(x) dx + \varepsilon \int_0^t \int_{\mathcal{O}} \nabla w^v(s, x) \cdot \nabla \varphi(x) dx ds
\]
\[
= \int_0^t \int_{\partial\mathcal{O}} \varphi(y) A(v) \cdot \nu(y) dH^{d-1}(y) ds.
\]
By passing to the limit we also see that (4.25) and (4.26) are satisfied for any $v \in L^2(0, T; H^1(\mathcal{O}))$. Thus, from (4.25) and (4.26) we get
\[
\|w^v\|_{H^1} \leq C(T) \|v\|_{L^2(\Omega \times [0, T]; H^1(\mathcal{O}))},
\]
and
\[
\|w^{v_1} - w^{v_2}\|_{H^1} \leq C(T) \|v_1 - v_2\|_{L^2(\Omega \times [0, T]; H^1(\mathcal{O}))},
\]
for \( v, v_1, v_2 \in L^2(\Omega \times [0, T]; H^1(\mathcal{O})) \).

Now, putting together the inequalities (4.14), (4.19), (4.20), (4.23), (4.24), (4.28), (4.29) the proof of the lemma is finished.

\( \square \)

Now we apply Propositions A.2–A.5 in Section 6 to analyze the map \( K(v) \).

**Proposition 4.1.** Assume \( v \in L^\infty(\Omega; C^\infty_c((0, T) \times \overline{\mathcal{O}})) \). We have the following:

(i) \( K(v) \in \mathcal{E}_* \), where

\[
\mathcal{E}_* := L^2(\Omega; C([0, T]; L^2(\mathcal{O}))) \cap L^2(\Omega \times [0, T]; H^2(\mathcal{O})).
\]

(ii) \( K(v) \) satisfies the following initial-boundary value problem for a stochastic equation with coefficients taking values in \( L^2(\mathcal{O}) \)

\[
\begin{cases}
    dK(v)(t) - \varepsilon \Delta K(v)(t) dt = -\nabla \cdot A(v(t)) dt + \Phi(v(t)) dW(t) \\
    K(v)(0) = u_0, \\
    \varepsilon \partial_v K(v)(t)|_{\partial \mathcal{O}} = A(v(t)) \cdot v.
\end{cases}
\]

(iii) More generally, for all \( v \in \mathcal{E} \), given \( \varphi \in C^\infty(\overline{\mathcal{O}}) \), almost surely we have

\[
\int_\mathcal{O} K(v(t)) \varphi(x) dx + \varepsilon \int_0^t \int_\mathcal{O} \nabla K(v) \cdot \nabla \varphi(x) dx dt = \int_\mathcal{O} u_0(x) \varphi(x) dx + \int_0^t \int_\mathcal{O} A(v(s)) \cdot \nabla \varphi(x) dx dt + \int_0^t \int_\mathcal{O} \varphi(x) \Phi(v(s)) dx dW(s).
\]

**Proof.** From the assumption on \( v \) it follows from what was seen in the proof of Lemma 4.1 that \( w^v \in \mathcal{E}_* \). Also, from the smoothness of \( u_0 \) it follows immediately \( S(t)u_0 \in \mathcal{E}_* \). Concerning \( K_1(v) \) and \( K_2(v) \) the fact that \( K_1(v), K_2(v) \in \mathcal{E}_* \) follows from Lemma 4.1 and Propositions A.2, A.4 applied to the operator defined in (A.6), taking into account Proposition A.5, which concludes the proof.

Concerning the proof of (4.32), first, since \( S(t)u_0 \) is a classical solution of the heat equation, we clearly have

\[
\int_\mathcal{O} \varphi(x) S(t)u_0 dx + \int_0^t \int_\mathcal{O} \nabla S(s) u_0 \cdot \nabla \varphi dx ds = \int_\mathcal{O} \varphi(x) u_0(x) dx.
\]

Now, by the definition of \( S(t) \) we see that \( K_1(v) \) is a solution of the problem

\[
\begin{cases}
    \ddot{w} - \Delta w = \nabla_x \cdot A(v(t)), \\
    \varepsilon \partial_x \dot{w} = 0, \\
    \dot{w}(0, x) = 0.
\end{cases}
\]

Therefore, we have

\[
\begin{align*}
\int_\mathcal{O} K_1(v(t)) \varphi(x) dx + \int_0^t \int_\mathcal{O} \nabla K_1(v(s)) \cdot \nabla \varphi(x) dx ds & = \int_0^t \int_\mathcal{O} \varphi \nabla \cdot A(v(s)) dx ds \\
& = -\int_0^t \int_\mathcal{O} \nabla \varphi(x) \cdot A(v(s)) dx ds + \int_0^t \int_{\partial \mathcal{O}} \varphi(y) A(v(s)) \cdot \nu(y) d\mathcal{H}^{d-1}(y) ds.
\end{align*}
\]
As for $K_2(v)$ we have the following. First, we observe that $S(t-s)\Phi(v(s))$ solves the problem
\[
\begin{cases}
z_t - \Delta z = 0, \\
\varepsilon \partial_n z = 0, \\
z(s,x) = \Phi(v(s)),
\end{cases}
\]
and so
\[
\int_{\Omega} \varphi(x) S(t-s) \Phi(v(s)) \, dx + \int_{s}^{t} \int_{\Omega} \nabla \varphi(x) \cdot \nabla S(t-\tau) \Phi(v(s)) \, dx \, d\tau \\
= \int_{\Omega} \varphi(x) \Phi(v(s)) \, dx.
\]
Integrating in $dW(s)$ from 0 to $t$, using the stochastic Fubini theorem (see, e.g., [12], [19]), we obtain
\[
\int_{\Omega} \varphi(x) K_2(v(t)) \, dx + \int_{0}^{t} \int_{s}^{t} \int_{\Omega} \nabla \varphi(x) \cdot \nabla S(t-\tau) \Phi(v(s)) \, dx \, d\tau \, dW(s) \\
= \int_{0}^{t} \int_{\Omega} \varphi(x) \Phi(v(s)) \, dx \, dW(s).
\]
Now, again by the stochastic Fubini theorem, we have
\[
\int_{0}^{t} \int_{s}^{t} \int_{\Omega} \nabla \varphi(x) \cdot \nabla S(t-\tau) \Phi(v(s)) \, dx \, d\tau \, dW(s) \\
= \int_{0}^{t} \int_{\Omega} \nabla \varphi(x) \cdot \nabla \left( \int_{0}^{s} S(t-\tau) \Phi(v(\tau)) \, dW(\tau) \right) \, dx \, ds \\
= \int_{0}^{t} \int_{\Omega} \nabla \varphi(x) \cdot \nabla K_2(v(s)) \, dx \, ds,
\]
and so we get
\[
\int_{\Omega} K_2(v(t)) \varphi(x) \, dx + \int_{0}^{t} \int_{\Omega} \nabla K_2(v(s)) \cdot \nabla \varphi(x) \, dx \, ds \\
= \int_{0}^{t} \int_{\Omega} \varphi(x) \Phi(v(s)) \, dx \, dW(s).
\]
Now, putting together the identities obtained for $S(t)u_0$, $K_1(v)$, $K_2(v)$ and recalling (4.27), we obtain that $K(v)$ satisfies (4.32). Since, for $v \in L^\infty(\Omega; C_c^\infty((0, T) \times \bar{\Omega}))$, $K(v) \in \mathcal{E}$, the above integral equation easily implies (4.31). Finally, given $v \in \mathcal{E}$, we may approximate it in $L^2(\Omega \times [0, T]; H^1(\Omega))$ by a sequence $\{v_k\} \subset L^\infty(\Omega; C_c^\infty((0, T) \times \bar{\Omega}))$, write the integral identity (4.32) with $v_k$ instead of $v$ and pass to the limit in $L^2(\Omega \times [0, T]; H^1(\Omega))$ when $k \to \infty$. Using the continuity property for $K(v)$ in the statement of Lemma 4.1, we obtain (4.32) for $v \in \mathcal{E}$, which finishes the proof.

The following proposition is a decisive step in the proof of the contraction property of $K$ on $\mathcal{E}$ endowed with a suitable norm.
Proposition 4.2. Given $\psi \in C^1(\Omega)$, $\eta \in C^2(\mathbb{R})$ with $\eta'' \in L^\infty$, $v^1, v^2 \in \mathcal{E}$, it holds almost surely and for every $0 \leq t \leq T$, the equalities

$$
\int_0^t \eta(K(v^1)(t,x))\psi(x) dx = \int_0^t \eta(u_0(x))\psi(x) dx
$$

$$
- \varepsilon \int_0^t \int_0^x \eta''(K(v^1)(s,x))|\nabla K(v^1)(s,x)|^2 \psi(x) dx ds
$$

$$
+ \int_0^t \int_0^x \eta''(K(v^1)(s,x))\nabla K(v^1)(s,x) \cdot A(v^1(s,x))\psi(x) dx ds
$$

$$
- \int_0^t \int_0^x \eta'(K(v^1)(s,x))(\varepsilon \nabla K(v^1)(s,x) - A(v^1(s,x))) \cdot \nabla \psi(x) dx ds
$$

$$
+ \int_0^t \int_0^x \eta'(K(v^1)(s,x))\Phi(v^1(s))\psi(x) dx dW(s)
$$

(4.33)

$$
= \frac{1}{2} \int_0^t \int_0^x \eta''(K(v^1)(s,x)) G^2(x,v^1(s,x))\psi(x) dx ds
$$

where $G(x,v)$ is as in (1.5), and

$$
\int_0^t \eta(K(v^1)(t,x) - K(v^2)(t,x))\psi(x) dx
$$

$$
= -\varepsilon \int_0^t \int_0^x \eta''(K(v^1)(s,x) - K(v^2)(s,x))
$$

$$
|\nabla K(v^1)(s,x) - \nabla K(v^2)(s,x)|^2 \psi(x) dx ds
$$

$$
+ \int_0^t \int_0^x \eta''(K(v^1)(s,x) - K(v^2)(s,x))(\nabla K(v^1)(s,x) - \nabla K(v^2)(s,x))
$$

$$
\cdot (A(v^1(s,x)) - A(v^2(s,x)))\psi(x) dx ds
$$

$$
- \int_0^t \int_0^x \eta'(K(v^1)(s,x) - K(v^2)(s,x))(\varepsilon \nabla K(v^1)(s,x) - \varepsilon \nabla K(v^2)(s,x))
$$

$$
- A(v^1(s,x)) + A(v^2(s,x))) \cdot \nabla \psi(x) dx ds
$$

$$
+ \int_0^t \int_0^x \eta'(K(v^1)(s,x) - K(v^2)(s,x))\Phi(v^1(s)) - \Phi(v^2(s))) dx dW(s)
$$

$$
+ \frac{1}{2} \int_0^t \int_0^x \eta''(K(v^1)(s,x) - K(v^2)(s,x))
$$

(4.34)

$$
\sum_{k=1}^\infty \left| g_k(x,v^1(s,x)) - g_k(x,v^2(s,x)) \right|^2 \psi(x) dx ds.
$$

Proof. Let us assume initially that $v^1$ and $v^2 \in L^\infty(\Omega; C^\infty_c((0,T) \times \overline{\Omega}))$. Then, by the previous proposition, $u^1(t) = K(v^1)$ lies in $C((0,T]; L^2(\Omega \times \Omega)) \cap L^2(\Omega \times [0,T]; H^2(\Omega))$, so that one may write in $L^2(\Omega)$ that almost surely and for every $0 \leq t \leq T$,

(4.35)

$$
u^1(t,x) = u_0(x) + \varepsilon \int_0^t \Delta u^1(s,x) ds - \int_0^t \text{div}_x A(v^1(s,x)) ds
$$

$$
+ \int_0^t \Phi(x,v^1(s,x)) dW(s)
$$
we have by parts, to get of (4.38) cancel each other, to finally obtain (4.33).

Then we use (4.36), from which the fourth and seventh term on the right-hand side (4.38) with

Hence, by Itô formula (see, e.g., [12]), almost surely and and for all \(0 \leq t \leq T\), we have

\[
\eta(u^1(t, x)) = \eta(u_0(x)) + \varepsilon \int_0^t \eta'(u^1(s, x)) \Delta u^1(s, x) \, ds
- \int_0^t \eta'(u^1(s, x)) \nabla \cdot A(v^1(s, x)) \, ds
+ \int_0^t \psi(x) \eta''(u^1(s, x)) \Phi(v^1(s, x)) \, dW(s)
+ \frac{1}{2} \int_0^t \psi(x) \eta''(u^1(s, x)) G^2(x, v^1(s, x)) \, ds
\]

We now multiply equation (4.37) by \(\psi \in C^1(\bar{O})\), integrate in \(x \in \mathcal{O}\), use integration by parts, to get

\[
\int_{\mathcal{O}} \psi(x) \eta(u^1(t, x)) \, dx = \int_{\mathcal{O}} \psi(x) \eta(u_0(x)) \, dx
- \varepsilon \int_0^t \int_{\mathcal{O}} \psi(x) \eta''(u^1(s, x)) |\nabla u^1(s, x)|^2 \, dx \, ds
- \varepsilon \int_0^t \int_{\mathcal{O}} \eta'(u^1(s, x)) \nabla \psi(x) \cdot \nabla u^1(s, x) \, dx \, ds
+ \varepsilon \int_0^t \int_{\partial \mathcal{O}} \psi(y) \eta''(u^1(s, y)) \partial_{
u} u^1(s, y) \, d\mathcal{H}^{d-1}(y) \, ds
+ \int_0^t \int_{\mathcal{O}} \psi(x) \eta''(u^1(s, x)) \nabla u^1(s, x) \cdot A(v^1(s, x)) \, dx \, ds
+ \int_0^t \int_{\mathcal{O}} \eta'(u^1(s, x)) \nabla \psi(x) \cdot A(v^1(s, x)) \, dx \, ds
- \int_0^t \int_{\partial \mathcal{O}} \psi(y) \eta''(u^1(s, y)) A(v^1(s, y)) \cdot \nu(y) \, d\mathcal{H}^{d-1}(y) \, ds
+ \int_0^t \int_{\mathcal{O}} \psi(x) \eta''(u^1(s, x)) \Phi(v^1(s, x)) \, dx \, dW(s)
+ \frac{1}{2} \int_0^t \int_{\mathcal{O}} \psi(x) \eta''(u^1(s, x)) G^2(x, v^1(s, x)) \, dx \, ds
\]

Then we use (4.36), from which the fourth and seventh term on the right-hand side of (4.38) cancel each other, to finally obtain (4.33).

As for (4.34), it may be obtained by a totally similar argument.

In order to obtain both identities for \(v^1, v^2 \in \mathcal{E}\), we approximate them in \(L^2(\Omega \times [0, T]; H^1(O))\) by sequences \(v^1_k, v^2_k \in L^\infty(\Omega; C^\infty_c((0, T) \times \bar{O}))\), and pass to the limit in the identities (4.33), (4.34) for \(v^1_k, v^2_k\) using the continuity property of \(K(v)\) stated in Lemma 4.1. Since \(v'' \in (C \cap L^\infty)(-\infty, \infty)\), all terms on the equations are preserved on the limit, so (4.33) and (4.34) are proven.

\[\square\]
Let us now define a new norm for $\mathcal{E}$, equivalent to norm $\| \cdot \|_{\mathcal{E}}$ defined in (4.5). We set

\begin{equation}
\| u \|_{\mathcal{E}}^2 = \sup_{0 \leq t \leq T} e^{-C_* t/\alpha} \mathbb{E} \left\{ \sup_{0 \leq s \leq t} \| u(s) \|_{L^2(\Omega)}^2 + \varepsilon \int_0^t \| \nabla u(s) \|_{L^2(\Omega)}^2 \, ds \right\},
\end{equation}

where $C_* > 0$ and $0 < \alpha < 1$ can be arbitrarily chosen.

As a corollary of Proposition 4.2, let us prove that $K(v)$ is a contraction on $\mathcal{E}$ endowed with the norm $\| \cdot \|_{\mathcal{E}}$, with $C_*$ suitably chosen.

**Proposition 4.3.** If $\mathcal{E}$ is endowed with the norm $\| \cdot \|_{\mathcal{E}}$, with $C_*$ suitably chosen, then $K : \mathcal{E} \to \mathcal{E}$ is a contraction.

**Proof.** Given $v^1$ and $v^2$ in $\mathcal{E}$, let us apply (4.34) with $\psi(x) \equiv 1$ and $\eta(s) = s^2/2$ to obtain

\begin{align*}
\frac{1}{2} \mathbb{E} \sup_{0 \leq t \leq T} \| K(v^1)(t) - K(v^2)(t) \|_{L^2(\Omega)}^2 + \varepsilon \mathbb{E} \int_0^T \| \nabla K(v^1)(s) - \nabla K(v^2)(s) \|^2_{L^2(\Omega)} \, ds \\
\leq \mathbb{E} \int_0^T \int_\Omega | \nabla K(v^1)(s,x) - \nabla K(v^2)(s,x) | | A(v^1(s,x)) - A(v^2(s,x)) | \, dx \, ds \\
+ \varepsilon \sup_{0 \leq t \leq T} \left| \sum_{k \geq 1} \int_0^t \int_\Omega (K(v^1)(s,x) - K(v^2)(s,x))(g_k(v^1(s,x)) - g_k(v^2(s,x))) \, dx \, d\beta_k(s) \right| \\
+ \frac{1}{2} \sum_{k \geq 1} \mathbb{E} \int_0^T \int_\Omega | g_k(x,v^1(s,x)) - g_k(x,v^2(s,x)) |^2 \, dx \, ds \\
\leq \frac{\varepsilon \mathbb{E}}{2} \int_0^T \| \nabla K(v^1)(s) - \nabla K(v^2)(s) \|_{L^2(\Omega)}^2 \, ds + C \mathbb{E} \int_0^T \| v^1(s) - v^2(s) \|_{L^2(\Omega)}^2 \, ds \\
+ CE \left( \int_0^T \frac{1}{2} \sum_{k \geq 1} \left( \int_\Omega (K(v^1)(s,x) - K(v^2)(s,x))(g_k(v^1(s,x)) - g_k(v^2(s,x))) \, dx \right)^2 \, ds \right)^{1/2} \\
+ CE \int_0^T \| K(v^1(s)) - K(v^2(s)) \|^2_{L^2(\Omega)} \, ds
\end{align*}
where we have used the Burkholder-Davis-Gundy inequality (see, e.g., [36]). So that the right-hand side of the above inequality may be estimated as

\[
\leq \frac{\varepsilon}{2} \mathbb{E} \int_0^T \|\nabla K(v^1)(s) - \nabla K(v^2)(s)\|_{L^2(O)}^2 \, ds + CE \int_0^T \|v_1(s) - v_2(s)\|_{L^2(O)}^2 \, ds
\]

\[+ CE \left( \int_0^T \|K(v^1)(s) - K(v^2)(s)\|_{L^2(O)}^2 \sum_{k \geq 1} \|g_k(v^1(s)) - g_k(v^2(s))\|_{L^2(O)}^2 \, ds \right)^{1/2} \]

\[+ CE \int_0^T \|K(v_1(s)) - K(v_2(s))\|_{L^2(O)}^2 \, ds \]

\[\leq \frac{\varepsilon}{2} \mathbb{E} \int_0^T \|\nabla K(v^1)(s) - \nabla K(v^2)(s)\|_{L^2(O)}^2 \, ds + CE \int_0^T \|v_1(s) - v_2(s)\|_{L^2(O)}^2 \, ds \]

\[+ CE \left( \sup_{0 \leq t \leq T} \|K(v^1)(t) - K(v^2)(t)\|_{L^2(O)}^2 \right)^{1/2} \left( \int_0^T \|v^1(s) - v^2(s)\|_{L^2(O)}^2 \, ds \right)^{1/2} \]

\[+ CE \int_0^T \|K(v_1(s)) - K(v_2(s))\|_{L^2(O)}^2 \, ds, \]

from which it follows

\[E \sup_{0 \leq t \leq T} \|K(v^1)(t) - K(v^2)(t)\|_{L^2(O)}^2 + \varepsilon E \int_0^T \|\nabla K(v^1)(s) - \nabla K(v^2)(s)\|_{L^2(O)}^2 \, ds \]

\[\leq CE \int_0^T \|v_1(s) - v_2(s)\|_{L^2(O)}^2 \, ds + CE \int_0^T \|K(v_1(s)) - K(v_2(s))\|_{L^2(O)}^2 \, ds. \]

Applying Grönwall inequality we get

\[E \sup_{0 \leq t \leq T} \|K(v^1)(t) - K(v^2)(t)\|_{L^2(O)}^2 + \varepsilon E \int_0^T \|\nabla K(v^1)(s) - \nabla K(v^2)(s)\|_{L^2(O)}^2 \, ds \]

\[\leq C^*_s \mathbb{E} \int_0^T \|v^1(s) - v^2(s)\|_{L^2(O)}^2 \, ds, \]

for some constant \(C_s > 0\), depending only on the data of the problem (4.1)–(4.3). Now, take (4.40) with \(t\) instead of \(T\), multiply both sides of it by \(e^{-C_s \cdot t/\alpha}\), take the \(\sup_{0 \leq t \leq T}\)–majorizing the resulting right-hand side by

\[C_s \sup_{0 \leq t \leq T} e^{-C_s \cdot t/\alpha} \int_0^t e^{C_s \cdot s/\alpha} e^{-C_s \cdot s/\alpha} E \sup_{0 \leq \tau \leq s} \|v^1(\tau) - v^2(\tau)\|_{L^2(O)}^2 \, ds \]

\[\leq C_s \sup_{0 \leq t \leq T} e^{-C_s \cdot t/\alpha} \|v^1 - v^2\|_{L^2(O)}^2 \leq \alpha^{1/2} \|v^1 - v^2\|_{L^2(O)}, \]

and then taking the \(\sup_{0 \leq t \leq T}\) on the resulting left-hand side we deduce that

\[\|K(v^1) - K(v^2)\|_{L^2(O)} \leq \alpha^{1/2} \|v^1 - v^2\|_{L^2(O)}. \]

Since \(0 < \alpha < 1\) we have the desired conclusion. \(\square\)
Conclusion of the proof of Theorem 4.1. Proposition 4.3 implies the existence of a unique fixed point \( u^\varepsilon \) for the operator \( K : \mathcal{E} \to \mathcal{E} \). In particular, from Proposition 4.1, \( u^\varepsilon \) satisfies almost surely, for all \( \varphi \in C^\infty(\bar{\Omega}) \),

\[
\begin{align*}
(4.41) \quad & \int_\Omega u^\varepsilon(t)\varphi(x) \, dx + \varepsilon \int_0^t \int_\Omega \nabla u^\varepsilon(s,x) \cdot \nabla \varphi(x) \, dx \, ds = \int_\Omega u_0(x)\varphi(x) \, dx \\
& \quad + \int_0^t \int_\Omega A(u^\varepsilon(s)) \cdot \nabla \varphi(x) \, dx \, ds + \int_0^t \int_\Omega \varphi(x)\Phi(u^\varepsilon(s)) \, dx \, dW(s).
\end{align*}
\]

This means that \( u^\varepsilon \) is a solution to the initial-boundary value problem (4.1)–(4.3). Now, from (4.41), it is easy to deduce that, for \( \varphi \in C^\infty([0,T] \times \Omega) \), we have

\[
\begin{align*}
(4.42) \quad & \int_\Omega u^\varepsilon(t)\phi(t,x) \, dx - \int_0^t \int_\Omega u^\varepsilon(s,x)\phi_s(s,x) \, dx \, ds \\
& \quad + \varepsilon \int_0^t \int_\Omega \nabla u^\varepsilon(s,x) \cdot \nabla \phi(s,x) \, dx \, ds \\
& \quad = \int_\Omega u_0(x)\phi(0,x) \, dx + \int_0^t \int_\Omega A(u^\varepsilon(s)) \cdot \nabla \phi(s,x) \, dx \, ds \\
& \quad \quad + \int_0^t \int_\Omega \phi(s,x)\Phi(u^\varepsilon(s)) \, dx \, dW(s),
\end{align*}
\]

which is another equivalent way to formulate the fact that \( u(t,x) \) is a solution of (4.1)–(4.3).

Now, suppose \( \bar{u} \in \mathcal{E} \) is another solution of (4.1)–(4.3), that is, if (4.42) is satisfied with \( \bar{u} \) instead of \( u^\varepsilon \). Then, for a given \( t \in (0,T] \), we take in (4.42), with \( \bar{u} \) instead of \( u^\varepsilon \), \( \phi(s,x) = S(t-s)\varphi(x) \), with \( \varphi \in C^\infty_c(\Omega) \), and use the symmetry of \( S(t-s) \) as an operator on \( L^2(\Omega) \), to get

\[
\begin{align*}
(4.43) \quad & \int_\Omega \bar{u}(t)\varphi(x) \, dx - \int_0^t \int_\Omega \bar{u}(s,x)\partial_sS(t-s)\varphi(x) \, dx \, ds \\
& \quad - \varepsilon \int_0^t \int_\Omega \bar{u}(s,x) \cdot \Delta S(t-s)\varphi(x) \, dx \, ds \\
& \quad = \int_\Omega S(t)u_0(x)\varphi(x) \, dx - \int_0^t \int_\Omega S(t-s)\nabla \cdot A(\bar{u}(s))\varphi(x) \, dx \, ds \\
& \quad + \int_{\partial\Omega} S(t-s)\varphi(y)A(\bar{u}(s,y)) \cdot \nu(y) \, dS(y) + \int_0^t \int_{\Omega} S(t-s)\Phi(\bar{u}(s))\varphi(x) \, dx \, dW(s).
\end{align*}
\]

Thus, using the fact that \( \partial_t S(t-s)\varphi = \varepsilon \Delta S(t-s)\varphi \), we deduce

\[
\begin{align*}
(4.44) \quad & \int_\Omega \bar{u}(t)\varphi(x) \, dx = \int_\Omega S(t)u_0(x)\varphi(x) \, dx \\
& \quad - \int_0^t \int_\Omega S(t-s)\nabla \cdot A(\bar{u}(s))\varphi(x) \, dx \, ds \\
& \quad + \int_{\partial\Omega} S(t-s)\varphi(y)A(\bar{u}(s,y)) \cdot \nu(y) \, dS(y) + \int_0^t \int_{\Omega} S(t-s)\Phi(\bar{u}(s))\varphi(x) \, dx \, dW(s).
\end{align*}
\]
Now, from (4.27), similarly, we deduce that \( w^\bar{u} \) satisfies

\[
\int_{\mathcal{O}} w^\bar{u}(t) \phi(t,x) \, dx - \int_{0}^{t} \int_{\mathcal{O}} w^\bar{u}(s,x) \phi_s(s,x) \, dx \, ds
+ \varepsilon \int_{0}^{t} \int_{\mathcal{O}} \nabla w^\bar{u}(s,x) \cdot \nabla \phi(s,x) \, dx \, ds
= \int_{0}^{t} \int_{\partial \mathcal{O}} \phi(s,y) A(\bar{u}(s,y)) \cdot \nu(y) \, d\mathcal{H}^{d-1}(y) \, ds
\]

Again taking \( \phi(s,x) = S(t-s) \varphi(x) \) and using that \( \partial_t S(t-s) \varphi = \varepsilon \Delta S(t-s) \varphi \) we get

\[
\int_{\mathcal{O}} w^\bar{u}(t) \varphi(x) \, dx = \int_{0}^{t} \int_{\partial \mathcal{O}} S(t-s) \varphi(y) A(\bar{u}(s,y)) \cdot \nu(y) \, d\mathcal{H}^{d-1}(y) \, ds.
\]

Then, using (4.46) into (4.44)

\[
\int_{\mathcal{O}} \bar{u}(t) \varphi(x) \, dx = \int_{\mathcal{O}} S(t) u_0(x) \varphi(x) \, dx
- \int_{0}^{t} \int_{\mathcal{O}} S(t-s) \nabla \cdot A(\bar{u}(s)) \varphi(x) \, dx \, ds
+ \int_{0}^{t} \int_{\mathcal{O}} S(t-s) \Phi(\bar{u}(s)) \varphi(x) \, dx \, dW(s)
+ \int_{\mathcal{O}} w^\bar{u}(t) \varphi(x) \, dx.
\]

Since \( \varphi \in C^{\infty}_c(\mathcal{O}) \) is arbitrary we conclude that \( \bar{u} \) satisfies \( K(\bar{u}) = \bar{u} \), that is, \( \bar{u} \) is also a fixed point of \( K \) and so, by the uniqueness of the fixed point in Banach’s theorem, we have \( \bar{u} = u^\varepsilon \).

Finally, regarding the energy estimate (4.4), from identity (4.33) of Proposition 4.2 applied to the fixed point \( u^\varepsilon \) with \( \eta(s) = s^2/2 \) and proceeding as in the proof of Proposition 4.3 we have that

\[
\frac{1}{2} \mathbb{E} \sup_{0 \leq s \leq t} \| u^\varepsilon(t) \|^2_{L^2(\mathcal{O})} + \varepsilon \mathbb{E} \int_{0}^{t} \| \nabla u^\varepsilon(s) \|^2_{L^2(\mathcal{O})} \, ds
\]

\[
\leq \frac{1}{2} \| u_0 \|^2_{L^2(\mathcal{O})} + \mathbb{E} \left| \int_{0}^{t} \int_{\mathcal{O}} A(u^\varepsilon(s,x)) \cdot \nabla u^\varepsilon(s,x) \, dx \, ds \right|
+ \frac{1}{4} \mathbb{E} \sup_{0 \leq s \leq t} \| u^\varepsilon(t) \|^2_{L^2(\mathcal{O})} + C \mathbb{E} \int_{0}^{t} \| u^\varepsilon(s) \|^2_{L^2(\mathcal{O})} \, ds
\]

\[
\leq \frac{1}{2} \| u_0 \|^2_{L^2(\mathcal{O})} + C \max_{\lambda \in \mathbb{R}} | \hat{A}(\lambda) |
+ \frac{1}{4} \mathbb{E} \sup_{0 \leq s \leq t} \| u^\varepsilon(s) \|^2_{L^2(\mathcal{O})} + C \int_{0}^{t} \mathbb{E} \sup_{0 \leq r \leq s} \| u^\varepsilon(r) \|^2_{L^2(\mathcal{O})} \, ds,
\]

where \( \hat{A}'(u) = A(u) \). We recall that by our assumptions on the approximate flux function \( \mathbf{A} \) is a bounded function. Then, using Gronwall’s inequality, we obtain (4.4). Let us point out that the uniform boundedness of \( \hat{A} \) is not essential as \( u^\varepsilon \) satisfies a maximum principle (see Theorem 4.2 below).

For future reference we state here the following direct consequence of Proposition 4.2.
Lemma 4.2 (Entropy identity). Let \( u^\varepsilon \in E \) be the solution of (4.1)–(4.3). For all \( \eta \in C^2(\mathbb{R}) \), for all \( \psi \in C^2_c(\mathcal{O}) \), for all \( 0 \leq s \leq t \leq T \),

\[
\langle \eta(u^\varepsilon(t)), \psi \rangle - \langle \eta(u(s)), \psi \rangle = -\varepsilon \int_s^t \langle \eta''(u^\varepsilon(r))|\nabla u^\varepsilon(r)|^2, \psi \rangle dr + \int_s^t \langle q(u^\varepsilon(r)), \nabla \psi \rangle dr
- \varepsilon \int_s^t \langle \nabla \eta(u^\varepsilon(r)), \nabla \psi \rangle dr + \sum_{k \geq 1} \int_s^t \langle g_k^\varepsilon(\cdot, u^\varepsilon(r))\eta'(u^\varepsilon(r)), \psi \rangle d\beta_k(r)
+ \frac{1}{2} \int_s^t \langle G^\varepsilon(\cdot, u^\varepsilon(r))\eta''(u^\varepsilon), \psi \rangle dr,
\]
a.s., where \( q(u) = \int_0^u a(\xi)\eta'(\xi) d\xi \). Moreover, if \( \phi \in C^2_c([0,T) \times \mathbb{R}^d) \), we have

\[
\int_0^T \int_\Omega \phi'(t,x)\phi_t(t,x) dx dt + \int_0^T \int_\Omega A(u^\varepsilon(t)) \cdot \nabla \phi(t,x) dx dt
+ \int_0^T \int_\Omega \phi(t,x)\Phi(u^\varepsilon(t)) dx dW(t) = \varepsilon \int_0^T \int_\Omega \nabla u^\varepsilon(t,x) \cdot \nabla \phi(t,x) dx dt.
\]

Proof. Relation (4.48) follows immediately from (4.33), using integration by parts, since \( u^\varepsilon \) is the fixed point of \( K \). As to relation (4.49), it follows from (4.42) by taking a test function in \( \phi \in C^2_c([0,T) \times \mathbb{R}^d) \) and evaluating it at \( t = T \).

We close this section with the following maximum principle for the parabolic approximation.

Theorem 4.2 (Maximum Principle for the parabolic approximation). Let \( u^\varepsilon \) be the solution of (4.1)–(4.3). Then, a.s., \( a \leq u^\varepsilon(t,x) \leq b \), a.e. \( (t,x) \in (0,T) \times \mathcal{O} \).

Proof. We take in (4.33) \( \psi \equiv 1 \) and \( \eta(u^\varepsilon) = \frac{1}{2}|u - b|^2_{\delta,+} \), where \( |u - b|_{\delta,+} \) is a \( C^2 \) convex approximation of \( |u - b|_{\delta,+} \), the latter being the positive part of \( u - b \), such that \( u \mapsto |u - b|_{\delta,+} \) is monotone nondecreasing, \( |u - b|_{\delta,+} = 1 \), for \( u > b + \delta \), and \( |u - b|_{\delta,+} = 0 \), for \( u \leq b \). Then, after sending \( \delta \to 0 \), we obtain a.s.

\[
\frac{1}{2} \int_\mathcal{O} [u^\varepsilon(t,x) - b]^2 dx =
- \varepsilon \int_0^t \int_\mathcal{O} 1_{u^\varepsilon(s,x) > b} |\nabla u^\varepsilon(s,x)|^2 dx ds
+ \int_0^t \int_\mathcal{O} 1_{u^\varepsilon(s,x) > b} \nabla u^\varepsilon(s,x) \cdot A(u^\varepsilon(s,x)) dx ds
+ \int_0^t \int_\mathcal{O} [u(s,x) - b]_+ \Phi(u^\varepsilon(s)) dx dW(s)
+ \frac{1}{2} \int_0^t \int_\mathcal{O} 1_{u^\varepsilon(s,x) > b} G^2(x, u^\varepsilon(s,x)) dx ds.
\]
Now, by virtue of (1.8), Young’s inequality with $\varepsilon$ yields

$$
\int_0^t \int_0^1 u_\varepsilon^\ast(s,x) > b \nabla u_\varepsilon^\ast(s,x) \cdot A(u_\varepsilon^\ast(s,x)) \, dx \, ds \\
\leq \varepsilon \int_0^t \int_0^1 u_\varepsilon^\ast(s,x) > b \nabla u_\varepsilon^\ast(s,x) \| A(u_\varepsilon^\ast(s,x)) \|_{Lip} \, dx \, ds + C \varepsilon \| \nabla u_\varepsilon^\ast \|_{Lip} \int_0^t \int_0^1 \left| u_\varepsilon^\ast(t,x) - b \right|^2 \, dx \, ds.
$$

On the other hand, by assumption, $g_k(\cdot, \xi) = 0$ for any $\xi > b$ so that the last two integrals on the right hand side of (4.50) are equal to zero.

Thus, by Gronwall’s inequality we conclude that a.s.

$$
\frac{1}{2} \int_0^1 \left| u_\varepsilon^\ast(t,x) - b \right|^2 \, dx = 0,
$$

for a.e. $t \in (0, T)$.

Similarly, if $[u - a]_-$ denotes the negative part of $u - a$, in the same way we can also prove that a.s.

$$
\frac{1}{2} \int_0^1 \left| u_\varepsilon^\ast(t,x) - a \right|^2 \, dx = 0,
$$

for a.e. $t \in (0, T)$, which implies the result.

\[\square\]

5. Existence: The vanishing viscosity limit

In this section we prove the convergence of the parabolic approximation (4.1)–(4.3) when $\varepsilon \to 0$ to the unique solution of (1.1)–(1.3).

5.1. Kinetic formulation for the parabolic approximation. The following proposition is essentially established in [13] in the periodic case and it can be proved as stated here in exactly the same way.

**Proposition 5.1.** Let $u_0^\ast \in C_c^\infty(\mathcal{O})$ and let $u_\varepsilon^\ast$ be the solution of (4.1)–(4.3). Then $f_\varepsilon = 1_{u_\varepsilon^\ast > \xi}$ satisfies: for all $\varphi \in C_c(\overline{\mathcal{O}} \times [0, T] \times \mathbb{R})$,

\begin{equation}
\begin{aligned}
\int_0^T \langle f_\varepsilon(t), \partial_t \varphi(t) \rangle \, dt + \langle f_0, \varphi(0) \rangle &+ \int_0^T \langle f_\varepsilon(t), a(\xi) \cdot \nabla \varphi(t) - \varepsilon \Delta \varphi(t) \rangle \, dt \\
&= -\sum_{k \geq 1} \int_0^T \int_{\mathcal{O}} \int_{\mathbb{R}} g_k^\varepsilon(x, \xi) \varphi(t, x, \xi) \, d\nu_{t,x}^\varepsilon(\xi) \, dx \, d\beta_k(t) \\
&\quad - \frac{1}{2} \int_0^T \int_{\mathcal{O}} \int_{\mathbb{R}} \partial_\xi \varphi(t, x, \xi) G_2^\varepsilon(x, \xi) \, d\nu_{t,x}^\varepsilon(\xi) \, dx \, dt + m_\varepsilon(\partial_\xi \varphi),
\end{aligned}
\end{equation}

a.s., where $f_0(\xi) = 1_{u_0 > \xi}$, $\nu_{t,x}^\varepsilon = \delta_{u_\varepsilon^\ast(t,x) = \xi}$, and, for $\phi \in C_b(\overline{\mathcal{O}} \times [0, T] \times \mathbb{R})$,

\begin{equation}
m_\varepsilon(\phi) = \int_{\mathcal{O} \times [0, T] \times \mathbb{R}} \phi(t, x, u_\varepsilon^\ast(t,x)) \, \varepsilon |\nabla u_\varepsilon^\ast|^2 \, dx \, dt,
\end{equation}

so $m_\varepsilon = \varepsilon |\nabla u_\varepsilon^\ast|^2 \delta_{u_\varepsilon^\ast = \xi}$. 

5.2. Local uniform space regularity of the parabolic approximation. By Proposition 5.1 we have that \( \chi^\varepsilon := 1_{u^\varepsilon} - 1_{0^\varepsilon} \) satisfies the parabolic stochastic equation

\[
\partial_t \chi^\varepsilon + a(\xi) \cdot \nabla \chi^\varepsilon - \varepsilon \Delta \chi^\varepsilon = \partial_t q - \sum_{k=1}^\infty (\partial_k \chi) g_k \beta_k + \sum_{k=1}^\infty \delta_0 g_k \beta_k,
\]

where \( q = m^\varepsilon - \frac{1}{2} G^2 \delta_{u^\varepsilon} \) and \( m^\varepsilon \) is given by (5.2).

Next, we state a local version of the corollary 3.3 in [18]. For that we multiply (5.3) by \( \varphi \in C_0^\infty([0,T] \times \mathcal{O}) \) to get

\[
\partial_t (\varphi \chi^\varepsilon) + a(\xi) \cdot \nabla (\varphi \chi^\varepsilon) - \varepsilon \Delta (\varphi \chi^\varepsilon)
= \partial_t q^\varepsilon - \sum_{k=1}^\infty (\partial_k (\varphi \chi^\varepsilon)) g_k \beta_k + \sum_{k=1}^\infty \varphi \delta_0 g_k \beta_k + \chi^\varepsilon (\partial_t \varphi + a^\varepsilon(\xi) \cdot \nabla \varphi - \varepsilon \Delta \varphi),
\]

where \( q^\varepsilon = \varphi m^\varepsilon - \varphi \frac{1}{2} G^2 \delta_{u^\varepsilon} + \varepsilon \nabla \varphi \cdot \nabla u^\varepsilon 1_{u^\varepsilon > \beta} \). We observe that \( q^\varepsilon \) is also a.s. a finite measure on \([0,T] \times \mathcal{O} \times \mathbb{R} \) with total variation uniformly bounded with respect to \( \varepsilon \), by the energy estimate (4.4) and the maximum principle from Theorem 4.2.

We first remark that the condition (1.9) implies the non-degeneracy condition of [18], namely, for some

\[
\omega (J, \delta) \lesssim \left( \frac{\delta}{J^\beta} \right)^\alpha \quad \forall \delta > 0, \forall J \gtrsim 1,
\]

and

\[
\omega (J; \delta) := \sup_{\tau \in \mathbb{R}, n \in \mathbb{Z}^d, \xi \in [a,b]} |\mathcal{L}_\xi(\tau, in; \xi)| \lesssim J^\beta,
\]

with \( \beta = 1 \), where,

\[
\omega (J; \delta) := \sup_{\tau \in \mathbb{R}, n \in \mathbb{Z}^d, \xi \in [a,b]} |\Omega_\xi(\tau, n, \delta)|,
\]

and \( \mathcal{L}_\xi(\tau, in; \xi) = \partial_\tau \mathcal{L}(\tau, in; \xi) \). Indeed, we note that if \( n \sim J \) then

\[
\Omega_\xi(\tau, n, \delta) \subset \Omega_\xi\left(\frac{\tau}{|n|}, \frac{n}{|n|}, \frac{C \delta}{J} \right),
\]

for some \( C > 0 \). Therefore, from (1.9) we conclude that (5.5) is satisfied with \( \beta = 1 \). Moreover, as \( \mathcal{L}_\xi(\tau, in; \xi) = ia^\varepsilon(\xi) \cdot n \) we see that (5.6) is satisfied trivially with \( \beta = 1 \) as well.

Concerning the symbol of the kinetic parabolic approximation

\[
\mathcal{L}^\varepsilon(\tau, in, \xi) := i(\tau + a(\xi) \cdot n) + \varepsilon |n|^2,
\]

for \( J, \delta > 0 \), let

\[
\Omega_{\mathcal{L}^\varepsilon}(\tau, n; \delta) := \{ \xi \in [a, b] : |\mathcal{L}^\varepsilon(\tau, in, \xi)| \leq \delta \},
\]

\[
\omega_{\mathcal{L}^\varepsilon}(J; \delta) := \sup_{\tau \in \mathbb{R}, n \in \mathbb{Z}^d, \xi \in [a,b]} |\Omega_{\mathcal{L}^\varepsilon}(\tau, n, \delta)|
\]

and \( \mathcal{L}^\varepsilon := \partial_\tau \mathcal{L}^\varepsilon \). As in [18], we note that, for some \( C > 0 \),

\[
\{ \xi \in [a, b] : |\mathcal{L}^\varepsilon(\tau, in, \xi)| \leq \delta \} \subset \{ \xi \in [a, b] : |\mathcal{L}(\tau, in, \xi)| \leq C \delta \}
\]

which, combined with (5.5), implies

\[
\omega_{\mathcal{L}^\varepsilon}(J; \delta) \lesssim \omega (J, C \delta) \lesssim \left( \frac{\delta}{J} \right)^\alpha \quad \forall \delta > 0, \forall J \gtrsim 1.
\]
Further, \( L^2(\tau, \nu, \zeta) = L_2(\tau, \nu, \zeta) \), and thus

\[
\sup_{r \in \mathbb{R}, n \in \mathbb{Z}^d, \zeta \in [a,b]} |L_2^\zeta(\tau, \nu, \zeta)| \leq \sup_{r \in \mathbb{R}, n \in \mathbb{Z}^d} |L_2^\zeta(\tau, \nu, \zeta)| \leq J.
\]

Therefore, \( L^2 \) satisfies the nondegeneracy conditions (5.5) and (5.6) uniformly in \( \varepsilon \), with \( \beta = 1 \).

Next, we may choose \( \varphi \) in equation (5.4) as \( \varphi(t, x) = \phi(t) \psi(x) \), where \( \psi \in C^\infty_c(O) \) and \( \phi = \phi^\lambda \in C^\infty_c([0, \infty)) \) is a cut-off in time such that \( 0 \leq \phi \leq 1 \), \( \phi \equiv 1 \) on \( [0, T - \lambda) \), \( \phi \equiv 0 \) on \( [T, \infty) \) and \( |\partial_t \phi| \leq \frac{1}{\lambda} \) for some \( \lambda \in (0, 1) \).

Note that this localization reduces the problem of the regularity of averages of \( \psi \chi^\varepsilon \) to the periodic case treated in [18]. The only difference from the kinetic equation studied in [18] (equation (3.3) from that paper) is the appearance of the term \( \chi^\varepsilon(\cdot, \cdot) \cdot \nabla \varphi - \varepsilon \Delta \varphi \), which may me treated exactly as the term \( \chi \partial_t \phi \) in their argument.

Thus, the averaging techniques from the proof of corollary 3.3 in [18] may be applied to equation (5.4) and, eventually sending \( \lambda \) to zero, we obtain the following result.

**Theorem 5.1.** Suppose (1.9) is satisfied. Let \( u^\varepsilon \) be the kinetic solution of (4.1)–(4.3) and \( \psi \in C^\infty_c(O) \). Then

\[
\|\psi u^\varepsilon\|_{L^r(O \times [0, T]; W^{s,r}(O))} \leq C_\psi(\|u_0\|_{L^3}^3 + 1),
\]

uniformly in \( \varepsilon > 0 \), with \( s < \frac{\alpha^2}{\theta(1 + 2\alpha)} - \frac{1}{r} > \frac{1 - \theta}{2} + \theta = \frac{2\alpha}{4 + 2\alpha} \). In particular,

\[
\|u^\varepsilon\|_{L^r(O \times [0, T]; W^{s,r}(O))} \leq C_\psi(\|u_0\|_{L^3}^3 + 1),
\]

for any \( O_0 \subset O \), uniformly in \( \varepsilon \).

### 5.3. Compactness argument.

The general lines of the compactness argument described here are motivated by the compactness argument put forth in [22].

**Proposition 5.2.** For all \( \lambda \in (0, 1/2) \), there exists a constant \( C > 0 \) such that for all \( \varepsilon \in (0, 1) \)

\[
\mathbb{E}\|u^\varepsilon\|_{C^\lambda([0, T]; H^{-1}(O))} \leq C.
\]

**Proof.** Recall that, due to (4.4), the set \( \{u^\varepsilon: \varepsilon \in (0, 1)\} \) is bounded in

\[
L^2(\Omega; L^2(0, T; H^{-1}(O))).
\]

By Theorem 4.2, we may take \( A^\varepsilon = A \), which is Lipschitz. We then have, in particular, that

\[
\{\text{div}(A(u^\varepsilon))\}, \quad \{\varepsilon \Delta u^\varepsilon\}
\]

are bounded in \( L^2(\Omega, L^2(0, T; H^{-1}(O))) \), and consequently

\[
\mathbb{E}\|u^\varepsilon - \int_0^T \Phi^\delta(u^\varepsilon) dW\|_{C^{1/2}([0, T]; H^{-1}(O))} \leq C.
\]

Moreover, for all \( \lambda \in (0, 1/2) \), paths of the above stochastic integral are \( \lambda \)-Hölder continuous \( L^2(O) \)-valued functions and

\[
\mathbb{E}\|\int_0^T \Phi^\delta(u^\varepsilon) dW\|_{C^\lambda([0, T]; L^2(O))} \leq C.
\]
Indeed, this is a consequence of the Kolmogorov continuity theorem (see, e.g., [12]) since the following uniform estimate is true. Let \( a > 2, s, t \in [0, T] \),
\[
E \left\| \int_s^t \Phi^\varepsilon(u^\varepsilon) \, dW \right\|_a^a \leq C \left( \int_s^t \left\| \Phi^\varepsilon(u^\varepsilon) \right\|_{L^2(U;L^2(O))}^2 \, dr \right)^{a/2}
\leq C|t-s|^{a/2} E \int_s^t \left( \sum_{k \geq 1} \left\| g_k^\varepsilon(u^\varepsilon) \right\|_{L^2(O)}^2 \right)^{a/2} \, dr
\leq C|t-s|^{a/2} \left( 1 + E \sup_{0 \leq t \leq T} \left\| u^\varepsilon(t) \right\|_{L^2(O)}^a \right)
\leq C|t-s|^{a/2} \left( 1 + E \| u_0 \|_{L^2(O)}^a \right),
\]
where we have made use of Burkholder inequality, (1.5) and (4.4).

Observe that, since, in Theorem 5.1, \( 1 < r < 2 \), from Proposition 5.2 it also follows that \( E \| u^\varepsilon \|_{C^\lambda([0,T];W^{1,r}(O))} \leq C \), for some \( C > 0 \) independent of \( \varepsilon \).

Let us define the path space
\[
\mathcal{X}_u = L^r(0, T; L^r(O)) \cap C([0,T]; W^{-2,r}(O)).
\]

Let us denote by \( \mu_{u^\varepsilon} \) the law of \( u^\varepsilon \) on \( \mathcal{X}_u, \varepsilon \in (0,1) \).

**Proposition 5.3.** The set \( \{ \mu_{u^\varepsilon} : \varepsilon \in (0,1) \} \) is tight and, therefore, relatively weakly compact in \( \mathcal{X}_u \).

**Proof.** Let \( O_1 \subseteq \cdots \subseteq O_3 \subseteq \cdots \subseteq O \) be a sequence of nonempty smooth open sets such that \( \bigcup_{n=1}^\infty O_n = \overline{O} \). Given \( R > 0 \), let us consider the set
\[
K_R = \{ u \in L^\infty((0,T) \times O) \cap L^r(0,T; W_{locc}^{1,r}(O)) \cap C^\lambda([0,T]; W^{-1,r}(O)) : \nabla \}
\]
\[
\| u \|_{L^\infty((0,T) \times O)} \leq R, \quad \| u \|_{C^\lambda([0,T]; W^{-1,r}(O))} \leq R,
\]
and \( \| u \|_{L^r(0,T; W^{1,r}(O_n))} \leq 2^n (C_{O_n} + 1) R, \forall n \geq 1 \},
\]
where the constants \( C_{O_n} \) are given by (5.8).

We assert that \( K_R \) is a relatively compact subset of \( \mathcal{X}_u \). Indeed, let \( \{ \psi_k \}_{k \in \mathbb{N}} \) be a sequence in \( K_R \). Taking a countable dense set in \((0,T)\) consisting of Lebesgue points of the elements of the sequence, as Banach space valued functions, from the boundedness in \( L^\infty(O) \), we may extract a subsequence (not relabeled) which strongly converges in \( W^{-1,r}(O) \) at each point of the dense set. Hence, by the boundedness in \( C^\lambda([0,T]; W^{-1,r}(O)) \), the subsequence strongly converges in \( W^{-1,r}(O) \) at all points of \([0,T]\), and so, by dominated convergence, it strongly converges in \( L^r(0,T; W^{-1,r}(O)) \). On the other hand, if \( O_0 \) is any smooth open subset of \( O \) with \( \overline{O}_0 \subseteq O \), by interpolation we have (see, e.g., [5])
\[
\| \varphi \|_{L^r(0,T; W^2,r(O_0))} \leq \| \varphi \|_{L^r(0,T; W^{1,r}(O_0))}^{1/(1+s)} \| \varphi \|_{L^r(0,T; W^{2,s,r}(O_0))}^{1-1/(1+s)}.
\]
Then, taking \( \varphi = (-\Delta)^{-1} \psi_k \), where by \( -\Delta \) we mean the minus Laplacian operator with 0 Dirichlet condition on \( \partial O_0 \), we conclude that it strongly converges in \( L^r(0,T; L^r(O_0)) \), using that \( (-\Delta)^{-1} \) isomorphically takes \( L^r(0,T; L^r(O_0)) \) onto \( L^r(0,T; W^{2,r} \cap W^{1,r}_0(O_0)) \).

Applying the above argument repeatedly for \( O_1, O_2, \) etc. in the place of \( O_0 \), by a diagonal argument we find a subsequence that converges in \( L^r(0,T; L^r_{locc}(O)) \). And since this sequence is uniformly bounded in \( L^\infty((0,T) \times O) \), it converges in \( L^r(0,T; L^r(O)) \), by the dominated convergence theorem.
Finally, using the embedding
\[ C^\lambda([0, T]; W^{-1, r}(\mathcal{O})) \xrightarrow{\mathcal{C}} C([0, T]; W^{-2, r}(\mathcal{O})), \]
we conclude that (the subsequence of) \( \{ \psi_k \}_{k \in \mathbb{N}} \) is convergent in \( \mathcal{X}_u \) by possibly passing to a further subsequence, thus proving that \( K_R \) is relatively compact in \( \mathcal{X}_u \).

Now, concerning the tightness of \( \{ \mu_{\psi} : \varepsilon \in (0, 1) \} \), we see that
\[
\mu_{\psi} (K^C_R) \leq \mathbb{P} (\| u \|_{L^\infty([0, T] \times \mathcal{O})} > R) + \sum_{n=1}^{\infty} \mathbb{P} (\| u \|_{L^\infty([0, T]; W^{-1, r}(\mathcal{O}))} > R) + \frac{1}{2^n (C_{\mathcal{O}} + 1) R} \leq \frac{C}{R}.
\]

Passing to a weakly convergent subsequence \( \mu^n = \mu_{\psi_n} \), and denoting the limit law by \( \mu \), we now apply the Skorokhod representation theorem (see, e.g., [6]) to infer the following result.

**Proposition 5.4.** There exists a probability space \((\bar{\Omega}, \bar{\mathcal{F}}, \bar{\mathbb{P}})\) with a sequence of \( \mathcal{X}_u \)-valued random variables \( \bar{u}^n, n \in \mathbb{N} \), and \( \bar{u} \) such that:

- (i) the laws of \( \bar{u}^n \) and \( \bar{u} \) under \( \bar{\mathbb{P}} \) coincide with \( \mu^n \) and \( \mu \), respectively,
- (ii) \( \bar{u}^n \) converges \( \bar{\mathbb{P}} \)-almost surely to \( \bar{u} \) in the topology of \( \mathcal{X}_u \).

Now, let us define \( \bar{\Omega} = \bar{\Omega} \times \mathbb{R} \), and \( \bar{\mathbb{P}} = \bar{\mathbb{P}} \times \mathbb{P} \), the product measure. Also, let \( \bar{\mathcal{F}} \) be the \( \sigma \)-algebra generated by \( \bar{\mathcal{F}} \times \mathcal{F} \). We also extend \( \bar{u}^n, \bar{u} \) and \( W : \Omega \to C([0, T]; \mathcal{U}_0) \) to \( \bar{\Omega} \) by simply setting \( \bar{u}^n (\bar{\omega}, \omega) := \bar{u}^n (\bar{\omega}) \), \( \bar{u} (\bar{\omega}, \omega) := \bar{u} (\bar{\omega}) \), and \( \bar{W} (\bar{\omega}, \omega) := W (\omega) \).

We have \( (\bar{u}, \bar{W}) \in \mathcal{X}_u \times C([0, T]; \mathcal{U}_0) \subset C([0, T], W^{-2, r}(\mathcal{O})) \times C([0, T], \mathcal{U}_0) \), with continuous inclusion. On the other hand, given any Banach space \( E \), and \( t \in [0, T] \), the operator \( \rho_t : C([0, T]; E) \to E \), with \( \rho_t k = k(t) \), is continuous. So, let \( (\mathcal{F}_t) \) be the \( \bar{\mathbb{P}} \)-augmented canonical filtration of the process \( (\rho_t \bar{u}, \rho_t \bar{W}), t \in [0, T] \), that is
\[
\mathcal{F}_t = \sigma \left( \rho_t \bar{u}, \rho_t \bar{W} : s \in [0, t] \right) \cup \{ N \in \bar{\mathcal{F}} : \bar{\mathbb{P}} (N) = 0 \}.
\]
We observe that, by the maximum principle, \( \bar{u}^n \) is uniformly bounded in \( L^\infty(\bar{\Omega} \times [0, T] \times \mathcal{O}) \), and so \( \bar{u}^n \rightharpoonup \bar{u} \) in the weak star topology of \( L^\infty(\bar{\Omega} \times [0, T] \times \mathcal{O}) \). In particular, \( \bar{u} \) is predictable.
We notice that the process \( \tilde{W} \) is a \((\tilde{F}_t)\)-cylindrical Wiener process, that is, \( \tilde{W} = \sum_{k \geq 1} \tilde{\beta}_k \epsilon_k \), where \( \{\tilde{\beta}_k\}_{k \geq 1} \), with \( \tilde{\beta}_k : \tilde{\Omega} \times [0, T] \to \mathbb{R} \), is the collection of mutually independent real-valued \((\tilde{F}_t)\)-processes given by \( \tilde{\beta}_k(\tilde{\omega}, \omega, t) = \beta_k(\omega, t) \). In particular, a.s. in \( \tilde{\Omega} \), \( \tilde{W} \in C([0, T], \mathcal{M}_0) \).

### 5.3.1. Identification of the limit

We say that \((\tilde{\Omega}, \tilde{F}, (\tilde{\mathcal{F}}_t), \tilde{\mathbb{P})}, \tilde{W}, \tilde{u})\) is a martingale weak entropy solution to (1.1)–(1.3) if \( \tilde{u} \) satisfies Definition 1.3, with \((\Omega, \mathbb{P})\) instead of \((\tilde{\Omega}, \tilde{\mathbb{P}})\), and \( \tilde{W} \) instead of \( W \).

**Proposition 5.5.** \((\tilde{\Omega}, \tilde{F}, (\tilde{\mathcal{F}}_t), \tilde{\mathbb{P})}, \tilde{W}, \tilde{u})\) is a martingale weak entropy solution to (1.1)–(1.3).

**Proof.**

Given a convex \( \eta \in C^2(\mathbb{R}) \) and a test function \( 0 \leq \psi \in C_c^\infty(\mathcal{O}) \) let us define for all \( s \leq t \in [0, T] \)

\[
M^n_s(t) = \langle \eta(u^n(t)), \psi \rangle - \langle \eta(u^n_0), \psi \rangle - \int_0^t \langle q(u^n(s)), \nabla \psi \rangle \, ds \\
+ \varepsilon_n \int_0^t \langle \nabla \eta(u^n(s)), \nabla \psi \rangle \, ds + \varepsilon_n \int_0^t \langle \eta''(u^n(s))|\nabla u^n(s)|^2, \psi \rangle \, ds \\
- \frac{1}{2} \int_0^t \langle G^{n\alpha}(\cdot, u^n(s))\eta''(u^n(s)), \psi \rangle \, ds, \quad n \in \mathbb{N},
\]

\[
\tilde{M}^n_s(t) = \langle \eta(\tilde{u}^n(t)), \psi \rangle - \langle \eta(\tilde{u}^n_0), \psi \rangle - \int_0^t \langle q(\tilde{u}^n(s)), \nabla \psi \rangle \, ds \\
+ \varepsilon_n \int_0^t \langle \nabla \eta(\tilde{u}^n(s)), \nabla \psi \rangle \, ds + \varepsilon_n \int_0^t \langle \eta''(\tilde{u}^n(s))|\nabla \tilde{u}^n(s)|^2, \psi \rangle \, ds \\
- \frac{1}{2} \int_0^t \langle G^{n\alpha}(\cdot, \tilde{u}^n(s))\eta''(\tilde{u}^n(s)), \psi \rangle \, ds, \quad n \in \mathbb{N},
\]

\[
\tilde{M}_\eta(t) = \langle \eta(\tilde{u}(t)), \psi \rangle - \langle \eta(\tilde{u}_0), \psi \rangle - \int_0^t \langle q(\tilde{u}(s)), \nabla \psi \rangle \, ds \\
+ \langle \tilde{\mu}_\eta, \chi_{[0,t]} \psi \rangle - \frac{1}{2} \int_0^t \langle G^2(\cdot, \tilde{u}(s))\eta''(\tilde{u}), \psi \rangle \, ds
\]

where \( \tilde{\mu}_\eta \) is the limit, by passing to a subsequence if necessary, of

\[
\eta''(\tilde{u}^\varepsilon)|\nabla \tilde{u}^\varepsilon|^2
\]

in \( L^2_w(\tilde{\Omega}; \mathcal{M}_b([0, T] \times \mathcal{O})) \), the space of the weak star measurable mappings \( m : \tilde{\Omega} \to \mathcal{M}_b([0, T] \times \mathcal{O}) \), such that \( \mathbb{E}[m] \mathcal{M}_b < \infty \), where \( \mathcal{M}_b([0, T] \times \mathcal{O}) \) is the space of bounded Radon measures over \([0, T] \times \mathcal{O}\). Indeed, we have

\[
\mathbb{E} \left| \varepsilon_n \int_0^T \int_{\mathcal{O}} \eta''(\tilde{u}^\varepsilon) |\nabla \tilde{u}^\varepsilon|^2 \, dt \, dx \right|^2 = \mathbb{E} \left| \varepsilon_n \int_0^T \int_{\mathcal{O}} \eta''(u^\varepsilon) |\nabla u^\varepsilon|^2 \, dt \, dx \right|^2 \leq C,
\]

which follows from the entropy identity (4.48) with \( \psi \equiv 1 \), \( t = T \), \( s = 0 \), by taking the square, then the expectation, using Itô isometry and making trivial estimates.
Let $D \subset [0, T]$ be a subset of full measure such that $\tilde{u}^n(t) \to \tilde{u}(t)$ in $L^p_{loc}(\tilde{\Omega} \times \mathcal{O})$ and $\tilde{P}(\tilde{\mu}_n((\tau = t) \times \mathcal{O})) = 0$, for $t \in D$. We claim that the processes

$$
(5.9) \quad \tilde{M}_n, \quad \tilde{M}^2_n = \sum_{k \geq 1} \int_0^t \langle g_k(\tilde{u})\eta'(\tilde{u}), \varphi \rangle \, dr, \quad \tilde{M}_n\tilde{\beta}_k = \int_0^t \langle g_k(\tilde{u})\eta'(\tilde{u}), \varphi \rangle \, dr,
$$

are $(\mathcal{F}_t)$-martingales indexed by $t \in D$.

Indeed, for all $n \in \mathbb{N}$, the process

$$
M^n_n = \int_0^t \langle \eta'(u^n)\Phi^n(u^n), dW(s), \psi \rangle = \sum_{k \geq 1} \int_0^t \langle \eta'(u^n)g_k^n(u^n), \psi \rangle \, d\beta_k(s)
$$

is a square integrable $(\mathcal{F}_t)$-martingale by (1.5) and (4.48). Denoting by $\langle \cdot, \cdot \rangle$ the quadratic variation, by the Doob-Meyer decomposition (see, e.g., [27]) we then have that

$$
(5.10) \quad (M^n_n)^2 - \sum_{k \geq 1} \int_0^t \langle \eta'(u^n)g_k^n(u^n), \psi \rangle^2 \, ds, \quad M^n_n\beta_k = \int_0^t \langle \eta'(u^n)g_k^n(u^n), \psi \rangle \, ds
$$

are $(\mathcal{F}_t)$-martingales, since

$$
\langle M^n_n \rangle = \sum_{k \geq 1} \int_0^t \langle \eta'(u^n)g_k^n(u^n), \psi \rangle^2 \, ds, \quad \langle M^n_n, \beta_k \rangle = \int_0^t \langle \eta'(u^n)g_k^n(u^n), \psi \rangle \, ds.
$$

Let $\gamma : W^{-2, r}(\mathcal{O}) \times \mathcal{U}_0 \to [0, 1]$ be arbitrarily given. From what we have just seen and the equality of laws, we have

$$
(5.11) \quad \mathbb{E}_\gamma(\rho \tilde{u}^n, \rho \tilde{W})[\tilde{M}^n_n(t) - \tilde{M}^n_n(s)] = \mathbb{E}_\gamma(\rho \tilde{u}^n, \rho \tilde{W})[M^n_n(t) - M^n_n(s)] = 0,
$$

$$
(5.12) \quad \mathbb{E}_\gamma(\rho \tilde{u}^n, \rho \tilde{W})[\tilde{M}^n_n(t)\tilde{\beta}_k(t) - \tilde{M}^n_n(s)\tilde{\beta}_k(s) - \int_0^t \langle \eta'(u^n)g_k^n(u^n), \varphi \rangle \, dr] = \mathbb{E}_\gamma(\rho \tilde{u}^n, \rho \tilde{W})[M^n_n(t)\beta_k(t) - M^n_n(s)\beta_k(s) - \int_0^t \langle \eta'(u^n)g_k(u^n), \varphi \rangle \, dr] = 0.
$$

Then, for $s, t \in D$ the expectations in (5.10)-(5.12) converge by the Vitali convergence theorem, since all terms are uniformly integrable and converge $\tilde{P}$-a.s. by
Proposition 5.4. Hence

\[ \bar{\mathbb{E}}\gamma(\rho_s \bar{u}, \rho_s \bar{W}) \left[ \bar{M}_\eta(t) - \bar{M}_\eta(s) \right] = 0, \]

\[ \bar{\mathbb{E}}\gamma(\rho_s \bar{u}, \rho_s \bar{W}) \left[ \bar{M}_\eta^2(t) - \bar{M}_\eta^2(s) - \sum_{k \geq 1} \int_s^t (\eta'(\bar{u})g_k(\bar{u}), \psi)^2 dr \right] = 0, \]

\[ \bar{\mathbb{E}}\gamma(\rho_s \bar{u}, \rho_s \bar{W}) \left[ \bar{M}_\eta^2(t) - \bar{M}_\eta^2(s) - \int_s^t (\eta'(\bar{u})g_k(\bar{u}), \psi) dr \right] = 0, \]

which gives the \( (\bar{\mathcal{F}}_t) \)-martingale property for \( t, s \in \mathcal{D} \).

If all the processes in (5.9) were continuous-time martingales then we would have

\[ \langle \bar{\mathbb{M}}_\eta - \int_0^t (\eta'(\bar{u})\Phi(\bar{u}) d\bar{W}, \psi) \rangle = 0, \]

which implies the equality \( \bar{M}_\eta = \int_0^t (\eta'(\bar{u})\Phi(\bar{u}) d\bar{W}, \psi) \), and so

\[ \langle \eta(\bar{u}(t)), \psi \rangle - \langle \eta(\bar{u}(s)), \psi \rangle = \int_0^t \langle q(\bar{u}(s)), \nabla \psi \rangle ds \]

\[ - \langle \bar{\mu}_\eta, \chi_{[0,t]} \psi \rangle + \frac{1}{2} \int_0^t \langle G^2(\cdot, \bar{u}(s))\eta''(\bar{u}), \psi \rangle ds + \int_0^t \langle \eta'(\bar{u})\Phi(\bar{u}) d\bar{W}, \psi \rangle. \]

In the case we are dealing here, that is, of martingales indexed by \( t \in \mathcal{D} \), we employ proposition A.1 in [22] to conclude, from (5.13), the validity of (5.15) for all \( \psi \in C^\infty_c(\mathcal{O}) \), \( t \in \mathcal{D}, \bar{\mathbb{P}} \)-a.s. In particular, for all \( s, t \in \mathcal{D} \) we have

\[ \langle \eta(\bar{u}(t)), \psi \rangle - \langle \eta(\bar{u}(s)), \psi \rangle = \int_s^t \langle q(\bar{u}(r)), \nabla \psi \rangle dr \]

\[ - \langle \bar{\mu}_\eta, \chi_{[s,t]} \psi \rangle + \frac{1}{2} \int_s^t \langle G^2(\cdot, \bar{u}(r))\eta''(\bar{u}), \psi \rangle dr + \int_s^t \langle \eta'(\bar{u})\Phi(\bar{u}) d\bar{W}, \psi \rangle, \]

and so, for all \( \theta \in C^\infty_c([0, T]) \) we get

\[ \int_0^T \langle \eta(\bar{u}(t)), \theta_t \psi \rangle dt + \langle \eta(u_0), \theta(0) \psi \rangle - \int_0^T \theta(t) \langle q(\bar{u}(t)), \nabla \psi \rangle dt \]

\[ = \langle \bar{\mu}_\eta, \theta \psi \rangle - \frac{1}{2} \int_0^T \langle G^2(\cdot, \bar{u}(r))\eta''(\bar{u}), \theta \psi \rangle dr - \int_0^T \langle \eta'(\bar{u})\Phi(\bar{u}) d\bar{W}, \theta \psi \rangle, \]

from which, by density, (1.14) follows, with \( \bar{\Omega}, \bar{\mathbb{W}}, \bar{\mathbb{P}} \) instead of \( \Omega, \mathbb{W}, \mathbb{P} \).
The same argument just used for the martingales $M^n_\eta$, $\tilde{M}^n_\eta$, $\tilde{M}_\eta$, can be similarly applied to the martingales

$$N^n(t) = \langle u_n(t), \psi \rangle - \langle u_n^0, \psi \rangle - \int_0^t \langle \mathbf{A}(u_n(s)), \nabla \psi \rangle \, ds$$

$$+ \varepsilon_n \int_0^t \langle \nabla u_n(s), \nabla \psi \rangle \, ds$$

(5.16)

$$\tilde{N}^n(t) = \langle \tilde{u}_n(t), \psi \rangle - \langle \tilde{u}_n^0, \psi \rangle - \int_0^t \langle \mathbf{A}(\tilde{u}_n(s)), \nabla \psi \rangle \, ds$$

$$+ \varepsilon_n \int_0^t \langle \nabla \tilde{u}_n(s), \nabla \psi \rangle \, ds$$

(5.17)

$$\tilde{N}(t) = \langle \tilde{u}(t), \psi \rangle - \langle \tilde{u}_0, \psi \rangle - \int_0^t \langle \mathbf{A}(\tilde{u}(s)), \nabla \psi \rangle \, ds$$

(5.18)

where now $\psi \in C^\infty(\mathbb{R}^d)$ and $\langle \cdot, \cdot \rangle$ keeps denoting the inner product in $L^2(\mathcal{O})$, which then leads us to

$$\tilde{N}(t) = \int_0^t \langle \Phi(\tilde{u}(s)) \, d\tilde{W}(s), \psi \rangle,$$

for $t \in \mathcal{D}$. From this, similarly to what was just done for $\tilde{M}_\eta$, we arrive at (1.13)

This concludes the proof.

Conclusion of the existence part of Theorem 1.1. The conclusion of the proof of the existence part of Theorem 1.1 follows the same lines in subsection 4.5 of [22], that is, we apply the Gyöngy and Krylov’s criterion for convergence in probability [20]. The latter states that a sequence $u^n$ of random variables assuming values in a complete metric space $\mathcal{X}$ converges in probability if and only if given any pair of subsequences $(u_{nk}, u_{mk})$ the corresponding sequence of joint laws $\{\mu_{nk, mk}\}$, by passing to a subsequence if necessary, converges weakly to a probability measure $\mu$ satisfying $\mu((x, y) \in \mathcal{X} \times \mathcal{X}; x = y) = 1$. The way to use this criterion is to proceed as above but using a pair of subsequences instead of just a single subsequence. So, one proves the tightness of the joint laws of a pair of subsequences, then one proves that each of them converge to a martingale weak entropy solution of (1.1)–(1.3) for the same probability space $(\tilde{\Omega}, \tilde{\mathbb{P}})$ and the same Wiener process $\tilde{W}$. Then we use the equivalence between weak entropy and kinetic solutions and the uniqueness of kinetic solutions to conclude that the joint laws converge to a measure concentrated on the diagonal of the cartesian product $\mathcal{X}_u \times \mathcal{X}_u$. Hence, the sequence converges in probability, which implies the convergence a.e. in $\Omega \times [0, T] \times \mathcal{O}$ and the limit is a kinetic solution of (1.1)–(1.3).

\[\square\]

6. Stochastic term with non-compact support

Here we consider the case where the stochastic term $\Phi(u) \, dW(t)$, defined as before by $\Phi(u)\varepsilon_k = g_k(\cdot, u)$, $k \in \mathbb{N}$, where now the functions $g_k \in C_c(\mathcal{O} \times (-M, M))$, $k \in \mathbb{N}$, do not need to have compact support in the space variable in $\mathcal{O}$, but still satisfy (1.4). Except for this we assume that all other conditions in Section 1 still hold.
Definition 6.1. We say that $P$ is the predictable $\in_k$ the problem (1.1)–(1.3) with non-compact noise, that is, $g_k \in C_c(\bar{\Omega} \times (-M, M))$, $k \in \mathbb{N}$, if there are $\Phi^\alpha : L^2(\mathcal{O}) \to L^2(\mathcal{O})$, $\alpha \in \mathbb{N}$, with $\Phi^\alpha(u)e_k = g_k^\alpha(\cdot, u)$, with $g_k^\alpha \in C_c(\mathcal{V}^\alpha \times (-M, M))$, for an open subset $\mathcal{V}^\alpha$ with $\mathcal{V}^\alpha \subset \subset \mathcal{O}$, and

$$
\lim_{\alpha \to \infty} \sup_{\xi \in (-M, M)} \sum_{k \geq 1} \|g_k^\alpha(\cdot, \xi) - g_k(\cdot, \xi)\|_{L^2(\mathcal{O})}^2 = 0,
$$

such that $u$ is the limit in $L^1(\Omega \times [0, T] \times \mathcal{O})$ of the kinetic solutions $u^\alpha$ of (1.1)–(1.3) with stochastic term $\Phi^\alpha(u) \, dW(t)$.

We remark that, since the concept of kinetic solution in Definition 1.2 does not need the assumption of the compactness of the support of the functions $g_k(\cdot, u)$, $k \in \mathbb{N}$, it applies also to more general noises with $g_k \in C_c(\bar{\Omega} \times (-M, M))$. In particular, it is immediate to verify that limit noise class kinetic solutions are kinetic solutions in the sense of Definition 1.2.

Theorem 6.1. Let $u_0 \in L^\infty(\mathcal{O})$ satisfying $a \leq u_0(x) \leq b$ a.e. in $\mathcal{O}$. Assume that conditions (1.4)–(1.9) are satisfied with $g_k \in C_c(\bar{\Omega} \times (-M, M))$, $k \in \mathbb{N}$. Then there is a unique limit noise class kinetic solution to (1.1)–(1.3).

Proof. Concerning the existence, consider a sequence $\varphi^\alpha$, $\alpha \in \mathbb{N}$, of functions in $C_c^\infty(\mathcal{O})$ with supp $\varphi^\alpha \subset \mathcal{V}^\alpha$, $\mathcal{V}^\alpha \subset \subset \mathcal{O}$, $0 \leq \varphi^\alpha \leq 1$, and $\varphi^\alpha(x) \to 1$ as $\alpha \to 1$, for all $x \in \mathcal{O}$. For each $\alpha \in \mathbb{N}$, we define $g_k^\alpha(x, u) = \varphi^\alpha(x)g_k(x, u)$, $k \in \mathbb{N}$. Dominated convergence implies that (6.1) is satisfied. For each $\alpha \in \mathbb{N}$, Theorem 1.1 establishes the existence and uniqueness of the kinetic solution $u^\alpha(t, x)$ of (1.1)–(1.3). Moreover, from Theorem 3.5, for $\alpha, \beta \in \mathbb{N}$, we have

$$
\int_{\mathcal{O}} |u^\alpha(t, x) - u^\beta(t, x)| \, dx \leq 2t \sup_{\xi \in [-M, M]} \sum_{k \geq 1} \|g_k^\alpha(\cdot, \xi) - g_k^\beta(\cdot, \xi)\|_{L^2(\mathcal{O})}^2.
$$

Therefore, the sequence $u^\alpha(t, x)$ is a Cauchy sequence in $L^1(\Omega \times [0, T] \times \mathcal{O})$, and so $u^\alpha \to u$ in $L^1(\Omega \times [0, T] \times \mathcal{O})$ for some $u \in L^\infty(\Omega \times [0, T] \times \mathcal{O}) \cap L^2(\Omega \times [0, T])$, $\mathcal{P} \vdash L^2(\mathcal{O})$, which then proves the existence of a limit noise class kinetic solution of (1.1)–(1.3).

Concerning the uniqueness, suppose there were two limit class kinetic solutions of problem (1.1)–(1.3), $u, \bar{u}$, which are limits in $L^1(\Omega \times [0, T] \times \mathcal{O})$ of subsequences $u^\alpha, u^\beta$ of (1.1)–(1.3), with noise components $g_k^\alpha(x, u) \in C_c(\mathcal{V}^\alpha \times (-M, M))$, $g_k^\beta(x, u) \in C_c(\mathcal{V}^\beta \times (-M, M))$, $\mathcal{V}^\alpha, \mathcal{V}^\beta \subset \subset \mathcal{O}$, respectively, with $u^\alpha \to u$, and $u^\beta \to \bar{u}$, in $L^1(\Omega \times [0, T] \times \mathcal{O})$. Again, by Theorem 3.5, we have

$$
\int_{\mathcal{O}} |u^\alpha(t, x) - u^\beta(t, x)| \, dx \leq 2t \sup_{\xi \in [-M, M]} \sum_{k \geq 1} \|g_k^\alpha(\cdot, \xi) - g_k^\beta(\cdot, \xi)\|_{L^2(\mathcal{O})}^2.
$$

Hence, since both $\{g_k^\alpha\}_{k \in \mathbb{N}}$ and $\{g_k^\beta\}_{k \in \mathbb{N}}$ satisfy (6.1), we deduce that $|u^\alpha - u^\beta|$ converges to 0 in $L^1(\Omega \times [0, T] \times \mathcal{O})$ as $\alpha, \beta \to \infty$. Since $u^\alpha \to u$ and $u^\beta \to \bar{u}$, we deduce that $u = \bar{u}$, which proves the uniqueness.

Appendix A. On convolutions of semigroups in Hilbert spaces

First of all, let us recall the well known spectral theorem in its multiplicative operator form, whose statement, exactly as given in [38], we recall here.
Proposition A.1. Let $A$ be a self-adjoint operator on a separable Hilbert space $H$ with domain $D(A)$. Then there is a measure space $(M, \mu)$ with $\mu$ a finite measure, a unitary operator $U : H \to L^2(M, d\mu)$, and a real-valued function $f$ on $M$ which is finite a.e. so that

1. $\psi \in D(A)$ if and only if $f(\cdot)(U\psi)(\cdot) \in L^2(M, d\mu)$;
2. If $\varphi \in U(D(A))$, then $(UAU^{-1}\varphi)(m) = f(m)\varphi(m)$.

In the remainder of this section, we will preserve the notations and assumptions of this spectral theorem. Also we will assume that the operator $A$ is nonnegative, which allows us to characterize its generated semigroup $S(t) = \exp -tA$ by means of the operational calculus simply as

$$(U \exp\{-tA\}\psi)(m) = \exp\{-tf(m)\}(U\psi)(m).$$

Furthermore, it allows us to characterize the spaces

$$H^\alpha_A := D(A^\alpha) = U^{-1}(L^2(M, (1 + f(m))^{2\alpha} d\mu)) =: U^{-1}(V^\alpha_f),$$

for $\alpha \geq 0$. For $\alpha < 0$, we set $H^\alpha_A = (H^{-\alpha}_A)^*$, which may still naturally be identified with $V^\alpha_f = L^2(M, (1 + f(m))^{2\alpha} d\mu)$. Of course, then $(I + A)^\beta$ defines a linear isometry between $H^\alpha_A$ and $H^{-\alpha}_A$. Observe that $S(t)$ is still a contraction semi-group on the spaces $H^\alpha_A$.

Let $T > 0$. For any $-\infty < \alpha < \infty$, we may define the Duhamel convolution operator

$$(\mathcal{I}h)(t) = \int_0^t S(t - s)h(s)ds$$

for $h \in L^2(0, T; H^\alpha_A)$. Clearly, if $h \in C([0, T]; H^\alpha_A)$,

$$\sup_{t \in [0, T]} \|\mathcal{I}h(t)\|_{H^\alpha_A}^2 \leq T^2 \sup_{t \in [0, T]} \|h(t)\|_{H^\alpha_A}^2. \tag{A.1}$$

Also clearly, one has that

$$\int_0^T \|\mathcal{I}h(s)\|_{H^\alpha_A}^2 ds \leq \frac{T^2}{2} \int_0^T \|h(s)\|_{H^\alpha_A}^2 ds. \tag{A.2}$$

However, one may say more regarding the regularization provided by the operator $\mathcal{I}$.

Proposition A.2. In the notations above, $\mathcal{I}$ maps $L^2(0, T; H^\alpha_A)$ into $L^2(0, T; H^{\alpha+1}_A)$ and

$$\int_0^T \|\mathcal{I}h(s)\|_{H^{\alpha+1}_A}^2 ds \leq C \int_0^T \|h(s)\|_{H^\alpha_A}^2 ds, \tag{A.3}$$

for some absolute constant $C$ depending only on $T$.

Proof. By the remarks above, we see that it suffices to analyse the case $\alpha = 0$. Since

$$\int_0^T \|\mathcal{I}h(s)\|_{H^0_A}^2 ds \leq 2 \left( \int_0^T \|\mathcal{I}h(s)\|_{H^0_A}^2 ds + \int_0^T \|A(\mathcal{I}h)(s)\|_{H^0_A}^2 ds \right),$$

and the first term was already estimated in (A.2), we may concentrate ourselves only on estimating the second one. Applying the spectral theorem, Proposition A.1,
its operational calculus and Cauchy-Schwarz inequality, we get
\[
\int_0^T \|A(Ih)(t)\|^2_{H^1} dt \\
= \int_0^T \int_M f(m)^2 \left| \int_0^t \exp\{- (t - s) f(m)\} \{Uh(s)\}(m) ds \right|^2 d\mu(m) dt \\
= \int_0^T \int_M \left| \int_0^t \exp\{- (t - s) f(m)\} f(m) (Uh(s))(m) ds \right|^2 d\mu(m) dt \\
\leq \int_0^T \int_M \left| \int_0^t \exp\{- (t - s) f(m)\} f(m) \{Uh(s)\}(m) ds \right|^2 d\mu(m) dt.
\]
For \( h \in L^2(0, T; H) \), \( Uh \in L^2(0, T; M) \), and thus Tonelli’s theorem yields
\[
\int_0^T \|A(Ih)(t)\|^2_{H^1} dt \\
\leq \int_M \int_0^T \int_0^t \exp\{- (t - s) f(m)\} f(m) \{Uh(s)\}(m)^2 ds dt d\mu(m) \\
= \int_M \int_0^T \int_s^T \exp\{- (t - s) f(m)\} f(m) \{Uh(s)\}(m)^2 dt ds d\mu(m) \\
\leq \int_M \int_0^T |\{Uh(s)\}(m)|^2 ds d\mu(m) \\
= \|h\|^2_{L^2(0, T; H)}.
\]
This shows the validity of the inequality. \( \square \)

Our second inequality will be for stochastic convolutions. So, let us first fix some notations and additional hypothesis.

As in Section 1, let \( (\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P}) \) be a stochastic basis with a complete and right-continuous filtration. Moreover, let \( \mathcal{P} \) be the predictable \( \sigma \)-algebra on \( \Omega \times [0, T] \) associated to \( (\mathcal{F}_t)_{t \geq 0} \) and \( W \) be a cylindrical Wiener process, i.e.,
\[
W(t) = \sum_{k=1}^{\infty} \beta_k(t)e_k
\]
where the \( \beta_k \)'s are mutually independent real-valued standard Wiener processes relative to \( (\mathcal{F}_t)_{t \geq 0} \), and \( (e_k) \) is an orthonormal basis of another separable Hilbert space \( \mathcal{H} \).

Let \( T > 0 \) and \( -\infty < \alpha < \infty \). Under these conditions, we may introduce the stochastic Duhamel operator
\[
(I_W \Psi)(t) = \int_0^t S(t - s) \Psi(s) dW(s)
\]
for predictable processes \( \Psi \in L^2(\Omega \times [0, T]; L_2(\mathcal{H}; H^\alpha_A)) \). Concerning properties of \( I_W \), we have the following result.

**Proposition A.3.** In the notations above, \( I_W \) maps \( L^2(\Omega \times [0, T]; L_2(\mathcal{H}; H^\alpha_A)) \) into \( L^2(\Omega \times [0, T]; H^{\alpha + 1/2}_A) \) and
\[
\|I_W \Psi\|_{L^2(\Omega \times [0, T]; H^{\alpha + 1/2}_A)} \leq C \|\Psi\|_{L^2(\Omega; L^2(0, T; L_2(\mathcal{H}; H^\alpha_A)))},
\]
for some \( C > 0 \) depending only on \( T \).
Proof. The verification of (A.5) is similar to that of (A.3), but here we have to use also Itô isometry. For this reason, the smoothing effect is weaker. Writing

$$\int_0^T \|I_W \Phi(t)\|_{H_{A}^{1/2}}^2 dt$$

$$= \int_0^T \left\| \int_0^t S(t-s)\Psi(s) dW(s) \right\|_{H_{A}^{1/2}}^2 dt$$

$$= \int_0^T \int_0^t \|S(t-s)\Psi(s)\|_{L_2(\Omega;H_{A}^{1/2})}^2 ds dt$$

$$= \sum_{k=1}^{\infty} \int_0^T \int_0^t \|S(t-s)\psi_k(s)\|_{H_{A}^{1/2}}^2 ds dt$$

$$= \sum_{k=1}^{\infty} \int_0^T \int_0^1 e^{-2(t-s)f(m)}(1 + f(m))^{2\alpha+1}(U\psi_k(s))(m)^2 d\mu(m) ds dt$$

$$\leq C \sum_{k=1}^{\infty} \int_0^T (1 + f(m))^{2\alpha}(U\psi_k(s))(m)^2 ds d\mu(m)$$

$$\leq C \int_0^T \|\Phi(s)\|_{L_2(\Omega;H_{A}^{1/2})}^2 ds,$$

hence the proposition. \qed

In this paper, we use this proposition as follows. Let \( \Phi : H \to L(\Omega; H) \) be continuous. If, for any \( k \in \mathbb{N} \), \( g_k : H \to H \) is given by \( \Phi(h)e_k = g_k(h) \), assume that each \( g_k : H_{A}^{1/2} \to H_{A}^{1/2} \) is continuous and that there exist constants \( \gamma_k > 0 \), such that

$$\|g_k(h)\|_H \leq \gamma_k(1 + \|h\|_H),$$

$$\|A^{1/2}g_k(h)\|_H \leq \gamma_k(1 + \|A^{1/2}h\|_H),$$

and

$$\sum_{k=1}^{\infty} \gamma_k^2 = D < \infty.$$ 

**Proposition A.4.** Under the hypothesis above, if \( u \in L^2(\Omega \times [0, T]; H_{A}^{1/2}) \) is predictable, then

$$I_W \Phi(u) \in L^2(\Omega; L^2(0, T; H_{A}^{1/2})),$$

and

(A.5) $$\|I_W \Phi(u)\|_{L^2(\Omega \times [0, T]; H_{A}^{1/2})} \leq C (1 + \|u\|_{L^2(\Omega \times [0, T]; H_{A}^{1/2})}),$$

where \( C \) only depends on \( T \) and \( D \).
Proof. We just need to verify that \( \Psi = \Phi(u) \) as in the statemente of Proposition A.3 with \( \alpha = 1/2 \). Since \( \| h \|_{L^2(\Omega)}^2 \leq C(\| h \|_H^2 + \| A^{1/2} h \|_H^2) \), we have that

\[
\| \Phi(u) \|_{L^2(\Omega; L^2(0,T;L^2(\Omega,H^1_\nu)))}^2 \\
\leq C(\| \Phi(u) \|_{L^2(\Omega; L^2(0,T;L^2(\Omega,H^1_\nu)))}^2 + \| A^{1/2} \Phi(u) \|_{L^2(\Omega; L^2(0,T;L^2(\Omega,H^1_\nu)))}^2) \\
= C \sum_{k=1}^\infty (\| g_k(u) \|_{L^2(\Omega; L^2(0,T;H^1_\nu)))}^2 + \| A^{1/2} g_k(u) \|_{L^2(\Omega; L^2(0,T;H^1_\nu)))}^2) \\
\leq C \sum_{k=1}^\infty C_k^2 (1 + \| u \|_{L^2(\Omega; L^2(0,T;H^1_\nu)))}^2 + \| A^{1/2} u \|_{L^2(\Omega; L^2(0,T;H^1_\nu)))}^2) \\
\leq CD (1 + \| u \|_{L^2(\Omega; L^2(0,T;H^1_\nu)))}^2).
\]

Therefore, as the argument above also shows the predicability of \( \Phi(u) \), the desired result is now a direct consequence of Proposition A.3.

Now, in order to apply the above abstract theory, let us fix

\[ H = L^2(\Omega), \]

and, denoting as usual \( H^k(\Omega) \) the \( k \)-th order Sobolev space,

\begin{align}
D(A) &= \left\{ u \in H^2(\Omega); \frac{\partial u}{\partial \nu} = 0 \text{ on } \partial \Omega \text{ (in the sense of traces in } H^1(\Omega)) \right\}, \\
Au &= -\Delta u, \ \text{for } u \in D(A).
\end{align}

Proposition A.5. The operator \( A : D(A) \rightarrow H \) defined in (A.6) is self-adjoint.

Proof. First, it is immediate to see that

\begin{equation}
(A + I)u = \| u \|_{H^1(\Omega)}^2,
\end{equation}

for \( u \in D(A) \), where \( \langle \cdot, \cdot \rangle \) and \( \| \cdot \| \) are the inner product and norm of \( H = L^2(\Omega) \). We denote \( H_1 = H^1(\Omega), \ H_{-1} = (H_1)^* \).

The inclusion \( H_1 \subset H \) is clearly compact and dense. From the latter it follows that the inclusion \( H \subset H_{-1} \) is also dense, So \( H_1, H, H_{-1} \) form what is then called a triplet of Hilbert spaces (see, e.g., [7]).

We claim that \( D(A) \) is dense in \( H_1 \).

Indeed, since \( D := C^\infty(\Omega) \) is dense in \( H_1 \), it suffices to show that we can arbitrarily approximate a function in \( D \) by functions in \( D(A) \). Let \( s_0 > 0 \) be such that \( \Psi(s, y) := y - sv(y) \) is one to one from \( \partial \Omega \) into \( \bar{\Omega} \), for \( 0 \leq s \leq s_0 \), where we keep denoting \( \nu \) as the unity outward normal to \( \partial \Omega \). Given any \( \varphi \in D \), for \( 0 < s_0 < s_* \), define \( \tilde{\varphi}^{s_0}(x) = \varphi(x) \), for \( x \in \Omega \setminus L_{s_0} \), where \( L_{s_0} = \Psi([0, s_0] \times \partial \Omega) \). For \( x \in L_{s_0} \), \( x = y - sv(y) \), for a unique pair \( (s, y) \), \( 0 \leq s \leq s_0 \), \( y \in \partial \Omega \), and we define

\[
\tilde{\varphi}^{s_0}(y - sv(y)) = \varphi(y - s_0 \nu(y)) + (s - s_0) \frac{d}{ds} \varphi(y - sv(y))(s = s_0) \\
\frac{1}{2}(s - s_0)^2 \left[ \frac{d^2}{ds^2} \varphi(y + sv(y))(s = s_0) + \frac{1}{6}(s - s_0)^3 C(s_0), \right.
\]

where \( C(s_0) \) is some constant. Here, we have used the fact that \( \varphi \) is smooth and \( \varphi \) is compactly supported in \( \Omega \).
we have \( C > 0 \) where the assertion is proven.

So, if we define

\[
\varphi^{s_0} := \tilde{\varphi}^{s_0} - \int_{\partial \mathcal{O}} \tilde{\varphi}^{s_0}(x) \, dx,
\]

we have \( \varphi^{s_0} \in H_2 \). Moreover, it is easy to check that

\[
\begin{align*}
\| \varphi - \varphi^{s_0} \|_{H_1} & \leq Cs_0,
\end{align*}
\]

where \( C > 0 \) does not depend on \( s_0 \), and since \( s_0 \) may be made arbitrarily small, the assertion is proven.

Now, we may extend (A.7) to \( u \in H_1 \) to get

\[
(A + I)u, u)_{H^{-1}, H_1} = \| u \|^2_{H^1(\mathcal{O})},
\]

which is the same as

\[
(A + I)u, u)_{H^{-1}, H_1} = \| u \|^2_{H_1},
\]

where we use the fact that \( Au \in H^{-1} \) if \( u \in H_1 \) which easily follows from \( |(Au, v)| = |(\nabla u, \nabla v)| \leq \|u\|_{H^1} \|v\|_{H_1} \) for \( v \in H_1 \). From (A.9) it follows that the image of the extension \( A + I : H_1 \rightarrow H^{-1} \) is dense in \( H^{-1} \). Indeed, otherwise there would be \( 0 \neq u \in H_1 \), such that \( (\ell, u)_{H^{-1}, H_1} = 0 \) for all \( \ell \in H^{-1} \), which contradicts (A.9). It also follows immediately

\[
\| (A + I)u \|_{H^{-1}} = \| u \|_{H_1},
\]

which implies that \( A + I \) is injective as an operator from \( H_1 \) to \( H^{-1} \) and so since, as an operator from \( H_1 \) to \( H^{-1} \), it is continuous, we have that its image is \( H^{-1} \). On the other hand, from (A.10) it follows that \( A + I : H_1 \rightarrow H^{-1} \) has a bounded inverse \( (A + I)^{-1} : H^{-1} \rightarrow H_1 \). Now, restricting \( (A + I)^{-1} \) to \( H \) and using the compactness of the embedding \( H_1 \subset H \) we see that \( (A + I)^{-1} |_H : H \rightarrow H \) is a compact operator. Now, from (A.8) and polarity identity it follows that for all \( u, v \in H_1 \) we have

\[
(A + I)u, v)_{H^{-1}, H_1} = (u, v)_{H^1(\mathcal{O})}.
\]

Therefore, \( (I + A)^{-1} \) is also self-adjoint since

\[
(I + A)^{-1} v_1, v_2 = (I + A)^{-1} v_1, (I + A)(I + A)^{-1} v_2 = (v_1, (I + A)^{-1} v_2).
\]

Hence, by the spectral theorem for compact self-adjoint operators (see, e.g., [38]) we get that there exists an orthonormal basis of \( H \), \( \{ \phi_k \}_{k=1}^{\infty} \) and a sequence \( \{ \lambda_k \} \) such that

\[
(I + A)^{-1} \phi_k = \lambda_k \phi_k,
\]

and so we get

\[
A \phi_k = \left( \frac{1}{\lambda_k} - 1 \right) \phi_k.
\]

We claim that \( \{ \phi_k \}_{k=1}^{\infty} \subset D(A) \).
Indeed, by the regularity theory for elliptic operators we have that $\phi_k \in H^2(\mathcal{O})$, $k \in \mathbb{N}$. Now, using (A.11) with $u = \phi_k$ and $v = h(1 - \varphi_\rho) - \int_\mathcal{O} h(1 - \varphi_\rho) \, dx$ with $h \in C^1(\mathcal{O})$ and $\varphi_\rho$ as in (3.6), after making $\rho \to 0$ we get

$$0 = \int_{\partial\mathcal{O}} h\phi_k \, dS,$$

for all $h \in C^1(\partial\mathcal{O})$, which implies that $\partial_n \phi_k \equiv 0$, for all $k \in \mathbb{N}$, and so $\{\phi_k\}_{k=1}^\infty \subset D(A)$.

Now we claim that $A : D(A) \to H = L^2(\mathcal{O})$ is closed. Indeed, let $u_k \in D(A)$, $u_k \to u$ in $H$ and $Au_k \to \eta$ in $H$. By the regularity theory for elliptic operators (see, e.g., [15]), $u_k$ converges in $H^2(\mathcal{O})$ and so $u \in H^2(\mathcal{O})$. Moreover, since $\partial_n u_k = 0$ on $\partial\mathcal{O}$, for all $k \in \mathbb{N}$, from the continuity of the trace operator from $H^1(\mathcal{O})$ to $L^2(\partial\mathcal{O})$, we conclude that $\partial_n u \equiv 0$ a.e. on $\partial\mathcal{O}$.

Hence, since $A$ is closed, symmetric and $A : D(A) \to H$ is a bijective, $I + A$ is self-adjoint and so is $A$.

Moreover, defining the operator $U : H \to \ell^2(\mathbb{N}) = L^2(\mathbb{N}; \#)$, where $\#$ is the counting measure, given by

$$Uv = \sum_{k=1}^\infty \langle v, \phi_k \rangle \phi_k.$$ 

We see that $U$ is unitary and $UAU^{-1} = T$, with $T : D(T) \subset \ell^2(\mathbb{N}) \to \ell^2(\mathbb{N})$ given by $T(a_k)_{k=1}^\infty = ((\frac{1}{\lambda_k} - 1)a_k)_{k=1}^\infty$, with $\lambda_k = \lambda_{k+1}$ for $k \in \mathbb{N}$,

$$D(T) = \{(a_k)_{k=1}^\infty \in \ell^2(\mathbb{N}) : ((\frac{1}{\lambda_k} - 1)a_k)_{k=1}^\infty \in \ell^2(\mathbb{N})\},$$

that is, $T$ is a multiplication operator by a (measurable) real function on $\ell^2(\mathbb{N})$. Hence $A$ is unitarily equivalent to the so defined multiplication operator $T : D(T) \subset \ell^2(\mathbb{N}) \to \ell^2(\mathbb{N})$.\hfill\Box
Now we show that, for $A$ defined by (A.6), $H_1^{1/2} = D((I + A)^{1/2}) = H^1(O)$. Indeed, if $u \in D(A)$, we have

$$
\| (I + A)^{1/2} u \|_{L^2(O)}^2 = (u + Au, u)_{L^2(O)}
$$

$$
= (u + Au, u)_{L^2(O)}
$$

$$
= - \int_O (u + \Delta u) u \, dx
$$

$$
= \int_O (|u|^2 + |\nabla u|^2) \, dx
$$

$$
= \| u \|_{H^1(O)}^2.
$$

Since $D(A)$ is dense in $H_1^{1/2}$ we may extend this equality to $u \in H_1^{1/2}$. On the other hand, we have shown in the proof of Proposition A.5 that $D(A)$ is dense in $H^1(O)$, so the inclusion $H_1^{1/2} \subset H^1(O)$ is isometric and dense, which implies that, in fact, we have the equality

(A.12) $D((I + A)^{1/2}) = H^1(O)$.

APPENDIX B. ACKNOWLEDGEMENTS

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