RIBBONLENGTH OF TORUS KNOTS

BROOKE BRENNA, THOMAS W. MATTMAN, ROBERTO RAYA, AND DAN TATING

Abstract. Using Kauffman’s model of flat knotted ribbons, we demonstrate how all regular polygons of at least seven sides can be realised by ribbon constructions of torus knots. We calculate length to width ratios for these constructions thereby bounding the Ribbonlength of the knots. In particular, we give evidence that the closed (respectively, truncation) Ribbonlength of a \((q + 1, q)\) torus knot is \((2q + 1) \cot(\pi/(2q + 1))\) (resp., \(2q \cot(\pi/(2q + 1))\)). Using these calculations, we provide the bounds \(c_1 \leq 2/\pi\) and \(c_2 \geq 5/3\cot\pi/5\) for the constants \(c_1\) and \(c_2\) that relate Ribbonlength \(R(K)\) and crossing number \(C(K)\) in a conjecture of Kusner: \(c_1C(K) \leq R(K) \leq c_2C(K)\).

1. Introduction

In [2], Kauffman introduces a model for knots presented as flat knotted ribbons and gives constructions of the trefoil and figure eight knots. He defines the (closed) Ribbonlength of a knot to be the smallest length to width ratio possible among the ways of forming the knot as a closed loop of ribbon. A truncation presentation of a knot is one formed from a length of ribbon such that the ends of the ribbon lie flush with segments in its edges. The minimum length to width ratio over such truncation presentations is the truncation Ribbonlength.

In her Master’s thesis, DeMaranville [1] showed how to build regular polygons by tying torus knots with ribbon. In the current paper, we summarise some of the constructions of [1] and use them to bound the Ribbonlength of certain torus knots. In particular, we conjecture that

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the closed (respectively, truncation) Ribbonlength of a \((q + 1, q)\) torus knot is \((2q + 1) \cot(\pi/(2q + 1))\) (resp., \(2q \cot(\pi/(2q + 1))\)). We illustrate how to construct regular polygons by forming \((q + 1, q)\), \((p, 2)\), \((2q + 2, q)\), and \((2q + 4, q)\) torus knots with ribbons. In fact, these families include all regular polygons except triangles, squares, and hexagons.

Kauffman \([2]\) reports that Kusner conjectures a linear relationship between Ribbonlength and crossing number:

\[
(1) \quad c_1 \text{Crossing}(K) \leq \text{Ribbonlength}(K) \leq c_2 \text{Crossing}(K)
\]

Using estimates of the Ribbonlength for torus knots, we deduce bounds on the constants \(c_1\) and \(c_2\). For truncated knots, we conclude \(c_1 \leq 4/\pi\) while for closed knots we have \(c_1 \leq 2/\pi\). For \(c_2\), we cannot improve on the bounds that follow from Kauffman’s \([2]\) study. The closed trefoil shows \(c_2 \geq \frac{5}{\pi} \cot(\pi/5)\) while the truncated figure eight knot yields \(c_2 \geq \frac{(3 + \sqrt{2})}{2}\).

The paper is structured as follows. After this introduction we devote one section to each of the following families of torus knots: \((q + 1, q)\), \((p, 2)\), \((2q + 2, q)\), and \((2q + 4, q)\). In each case we determine the length to width ratio of the construction. Together, these families yield all regular \(n\)-gons for \(n > 6\) as well as the regular pentagon. Next, in Section 6, we refine our estimates of the Ribbonlength for the \((5, 2)\) and \((7, 2)\) torus knots and show how the \(7_4\) knot (which is not torus) can be tied as a rectangle. In the final section we compile the length to width ratios of the various families of torus knots in order to obtain bounds on the constants \(c_1\) and \(c_2\) of Equation 1.

2. \((q, q + 1)\) TORSUS KNOTS

In \([2]\), Kauffman uses the pentagon formed when a trefoil knot (a \((3, 2)\) torus knot) is tied with a ribbon to find the (conjectured) Ribbonlength of that knot. DeMaranville \([1]\) observed that a \((4, 3)\) torus knot forms a heptagon and, in general, a \((q + 1, q)\) torus knot yields a \(2q + 1\)-gon. We will use this observation to calculate the Ribbonlength of these knots.

To illustrate the geometry of the \((q, q + 1)\) torus knots, we begin with the truncated \((4, 3)\) torus knot. As in Figure \([1]\) when the ribbon is folded to make a regular heptagon, the fold lines segment the ribbon into six congruent isosceles trapezoids. By circumscribing the heptagon, we recognize the top and base of each trapezoid as chords of a circle (Figure \([2]\)). We will use elementary geometry to find those lengths.
Figure 1. The (4, 3) torus knot forms a heptagon.

Figure 2. The circumscribed heptagon.

In Figure 2, C is the centre of the circle. Since \( \angle GCE = \pi - \pi/7 \), the remaining angles in the isosceles triangle \( \Delta GCE \) are each \( \frac{1}{2}(\pi/7) \). By the Law of Sines, the length \( b \) of the base of the trapezoid is related to the radius \( r \) by \( b = r \sin(\angle GCE)/\sin(\angle CEG) \). Since \( \Delta C E J \) is isosceles, we can find \( r \) in terms of \( l \), the length of the sides of the heptagon: \( r = l/(2 \sin(\frac{1}{2}(2\pi/7))) \). Thus,

\[
b = \frac{l \sin(\pi - \pi/7)}{2 \sin(\pi/7) \sin(\frac{1}{2}\pi/7)} \]

\[
= \frac{l}{2 \sin(\frac{1}{2}\pi/7)}
\]
Similarly, the top of the trapezoid has length

\[
t = \frac{l \sin(\pi - 3\pi/7)}{2 \sin(\pi/7) \sin(4\pi/7)} = \frac{2l \sin(\frac{\pi}{2} \pi/7) \cos(\frac{3\pi}{2} \pi/7)}{2 \sin(\pi/7) \sin(\frac{3\pi}{2} \pi/7)} = \frac{l \cos(\frac{3\pi}{2} \pi/7)}{\sin(\pi/7)}
\]

The length of the ribbon is \( L = 3(b + t) \). In order to compare it with the width \( w \), note that \( w = l \sin(3\pi/7) = l \sin(\pi/2 - \frac{1}{2} \pi/7) = l \cos(\frac{1}{2} \pi/7) \). So, the length to width ratio of the \((4, 3)\) torus knot is

\[
L/w = \frac{3l}{l \cos(\frac{1}{2} \pi/7)} \left[ \frac{1}{2 \sin(\frac{1}{2} \pi/7)} + \frac{\cos(\frac{3\pi}{2} \pi/7)}{\sin(\pi/7)} \right] = \frac{3}{\cos(\frac{1}{2} \pi/7)} \left[ \frac{\cos(\frac{1}{2} \pi/7)}{2 \sin(\frac{1}{2} \pi/7) \cos(\frac{3\pi}{2} \pi/7)} + \frac{\cos(\frac{3\pi}{2} \pi/7)}{\sin(\pi/7)} \right] = \frac{3(2 \cos((\frac{1}{2} \pi/7 + \frac{3}{2} \pi/7)/2) \cos((\frac{1}{2} \pi/7 - \frac{3}{2} \pi/7)/2))}{\cos(\frac{1}{2} \pi/7) \sin(\pi/7)} = \frac{6 \cos(\pi/7) \cos(\frac{3\pi}{2} \pi/7)}{\cos(\frac{1}{2} \pi/7) \sin(\pi/7)} = 6 \cot(\pi/7)
\]

We conjecture that \( 6 \cot(\pi/7) \) is the Ribbonlength of this knot; we expect that there is no way to tie this knot with a ribbon that will result in a smaller length to width ratio.

DeMaranville observed that the pattern of pentagon formed by a \((3, 2)\) torus and heptagon by a \((4, 3)\) torus persists. In general, a \((q + 1, q)\) torus knot can be tied with ribbon to form a regular \((2q + 1)\)-gon. The calculation above also generalises. The ribbon will be segmented into \(2q\) isosceles trapezoids with tops and bottoms of length \( t = \frac{l \cos(\frac{1}{2} \pi/(2q+1))}{\sin(\pi/(2q+1))} \) and \( b = \frac{l}{2 \sin(\frac{1}{2} \pi/(2q+1))} \) respectively. Comparing the length \( q(b + t) \) with the width \( w = l \cos(\frac{1}{2} \pi/(2q + 1)) \) we arrive at our conjectured truncation Ribbonlength for a \((q + 1, q)\) torus knot:
For the closed knot, we need one additional trapezoid. Measuring the length along the centre of the ribbon, we conjecture that the closed Ribbonlength of a \((q + 1, q)\) torus knot is \((2q + 1) \cot(\pi/(2q + 1))\).

### 3. \((p, 2)\) Torus Knots

In this section we investigate \((p, 2)\) torus knots \((p \geq 7, \text{odd})\). DeMaranville [1] proved that knots in this family also form regular polygons when tied with ribbon (see Figure 3). Note that there is a “hole” in the centre of the polygon that is in the shape of a smaller concentric regular \(p\)-gon.

As illustrated in Figure 3 in tying this knot, the ribbon is segmented into \(p\) congruent isosceles trapezoids. Three of the sides of the trapezoid are equal to the side \(l\) of the regular \(p\)-gon. The base of the trapezoid is again a chord of the circumscribing circle of length \(b = l(1 + 2 \cos(2\pi/p))\). The length of the ribbon (measured along its centre line) is \(L = \frac{2}{p}(l + b)\) while the width is \(w = l \sin(2\pi/p)\). The closed length to width ratio for these knots is therefore

\[
\frac{L}{w} = \frac{2pl(1 + \cos(2\pi/p))}{2l \sin(2\pi/p)} = \frac{p(1 + \cos^2(\pi/p) - \sin^2(\pi/p))}{2 \sin(\pi/p) \cos(\pi/p)} = \frac{p}{2} \frac{2 \cos^2(\pi/p)}{\sin(\pi/p) \cos(\pi/p)} = \frac{p}{2} \frac{2 \cos(\pi/p)}{\sin(\pi/p)} = p \cot(\pi/p).
\]

We anticipate that this length to width ratio is not the Ribbonlength for these knots. For example, in Section 6 we discuss another method for tying the \((7, 2)\) torus knot that leads to a smaller length to width

![Figure 3. The \((7, 2)\) torus knot.](image)
ratio than the value $7 \cot(\pi/7)$ just obtained. On the other hand, the ratio of $p \cot(\pi/p)$ is an upper bound for the Ribbonlength of knots in this family.

4. $(2q + 1, q)$ Torus Knots

Unlike the previous two families, the $(2q + 1, q)$ knots ($q > 1$) form “pinwheels” rather than polygons when tied with ribbon (Figure 4). However, like the other families, the ribbon is segmented into congruent isosceles trapezoids. Each trapezoid has a top and base of length $t = 2w \cot(\pi/(2q + 1))$ and $b = 2w(\cot(\pi/(2q + 1)) + \tan(\pi/2(2q + 1))) = 2w/\sin(\pi/(2q + 1))$ respectively, where $w$ is the width of the ribbon. The length to width ratio is, therefore,

$$L/w = \frac{2q + 1}{2} \frac{(b + t)}{w} \quad = \quad (2q + 1)(\cot(\pi/(2q + 1)) + \csc(\pi/(2q + 1))) \quad = \quad (2q + 1) \cot(\pi/(2(2q + 1)))$$

Again, while this provides an upper bound for the Ribbonlength, we anticipate that the Ribbonlength for these knots is not realised by this construction. In particular, we demonstrate below (Section 6) how the $(5, 2)$ torus knot can be tied with a smaller length to width ratio than the value $5 \cot(\pi/10)$ just obtained.

5. Torus Knots That Form Even-Sided Polygons

The regular polygons formed by $(q + 1, q)$ and $(p, 2)$ torus knots are odd-sided. In this section we examine the $(2q + 2, q)$ and $(2q + 4, q)$ (where $q > 1$, odd) torus knots which result in $2q + 2$- and $2q + 4$-sided regular polygons [1], see Figure 5. With the exception of the $(8, 3)$ torus knot, these are similar to the $(p, 2)$ examples above in that there is a
hole in centre of the polygon. They are also similar in that the length to width ratio is \( n \cot(\pi/n) \) where \( n = 2q + 2 \) or \( 2q + 4 \) is the number of sides of the polygon. Due to the polygonal hole in the centre, we anticipate that this length to width ratio is not the Ribbonlength. The only exception is the \((8, 3)\) torus knot which is formed with no gap in the centre (Figure 5). We conjecture that the Ribbonlength of that knot is \( 8 \cot(\pi/8) \).

The \((q + 1, q)\) (for \( q \geq 3 \)) and \((p, 2)\) (for \( p \geq 7 \)) torus knots give us two different ways to construct each odd-sided polygon of seven or more sides. In addition, the \((3, 2)\) torus knot results in a regular pentagon. On the other hand, the \((2q + 2, q)\) and \((2q + 4, q)\) knots (for \( q \geq 3 \)) yield all the even sided regular polygons of eight or more sides. Together these families account for all regular \( n \)-gons of seven or more sides as well as the regular pentagon.

6. **The knot 7/4 and the (5, 2) and (7, 2) torus knots**

In this section, we demonstrate “shorter” ways of tying the \((5, 2)\) and \((7, 2)\) torus knots. This suggests that the length to width ratios for the \((p, 2)\) and \((2q+1, q)\) knots above are likely not the Ribbonlengths.
of those knots. In addition, we look at how the $7_4$ knot results in a rectangle.

6.1. **The $(5, 2)$ torus knot.** We will tie a $(5, 2)$ torus knot so as to obtain a smaller length to width ratio than the value $5 \cot(\pi/10) \approx 15.4$ obtained above. In Figure 6 we illustrate how to tie this knot to form a pentagon much like the one given by the trefoil knot. Two sides of the pentagon are used twice and we replace those sides by two parallel fold lines separated by a small distance $\varepsilon$. Although the three lines joining the doubled sides should coincide, we have displaced them in the figure to show how those edges cross over one another.

Much like the trefoil knot, this way of folding $(5, 2)$ will segment the ribbon into 7 trapezoids (as compared to the 5 obtained for the closed trefoil). Therefore, this method of folding the $(5, 2)$ torus will result in a length to width ratio of approximately $7 \cot(\pi/5)$ or 9.6.

6.2. **The $(7, 2)$ torus knot.** The $(7, 2)$ torus knot can also be tied to realise a length to width ratio smaller than the value of $7 \cot(\pi/7) \approx 14.5$ mention above. As in Figure 7, this knot also results in a pentagon-like shape. (Again, coincident edges in the knot have been displaced in order to show crossings.) This time the ribbon is segmented into 9 trapezoids resulting in a length to width ratio of approximately $9 \cot(\pi/5)$ or 12.4.

6.3. **The $7_4$ knot.** Although it is not a torus knot, the $7_4$ knot can be folded to give a nice geometric shape, a $2 \times 3$ rectangle [1], see Figure 8. The length to width ratio of this pattern is 24, as may easily
be verified. However, this is likely not the Ribbonlength for this knot. Informal experiments suggest a Ribbonlength of less than 20.

7. Ribbonlength and crossing number

In [2], Kauffman reports on Kusner’s conjecture of a relationship between Ribbonlength and crossing number:

\[ c_1 \text{Crossing}(K) \leq \text{Ribbonlength}(K) \leq c_2 \text{Crossing}(K) \]

As the crossing number of a \((p, q)\) torus knot is \(\min\{p(q-1), q(p-1)\}\), we can use our calculations of length to width ratios above to bound the constants \(c_1\) and \(c_2\). To this end, for each family of torus knots discussed above, we divide the length to width ratio by the crossing number.
7.1. \((q + 1, q)\) torus knots. Dividing the conjectured Ribbonlength by the crossing number \(q^2 - 1\) yields \(\frac{2q}{q^2 - 1} \cot(\pi/(2q + 1))\) for the truncation presentation and \(\frac{2q + 1}{q^2 - 1} \cot(\pi/(2q + 1))\) for the closed knot. As \(q\) increases, the value of this ratio decreases, approaching \(4/\pi\) in the limit. The largest values are obtained with the trefoil knot \((3, 2)\), \(\frac{\pi}{5} \cot(\pi/5) \approx 1.83\) and \(\frac{\pi}{5} \cot(\pi/5) \approx 2.29\) for the truncated and closed knot respectively.

7.2. \((p, 2)\) torus knots. Dividing the length to width ratio by the crossing number \(p\), we have \(\cot(\pi/p)\). Notice that this tends to infinity as \(p\) increases. We take this as further evidence that this configuration does not realise the Ribbonlength for these knots. Since \(p \geq 7\), the smallest value we obtain is \(\cot(\pi/7) \approx 2.07\) for the closed \((7, 2)\) torus knot.

7.3. \((2q + 1, q)\) torus knots. Dividing the length to width ratio by the crossing number \((2q + 1)(q - 1)\), we have \(\cot(\pi/(2q + 2))/(q - 1)\) which again tends to \(4/\pi\). The largest value obtained in this family is \(\cot(\pi/10) \approx 3.1\) for the \((5, 2)\) torus knot.

7.4. Even-sided polygon knots, the figure eight knot, and 7
d. For the \((2q + 2, q)\) torus knots, dividing the length to width ratio by the crossing number \((2q + 2)(q - 1)\) yields \(\cot(\pi/(2q + 2))/(q - 1)\) which tends to \(2/\pi\) as \(q\) goes to infinity. The \((8, 3)\) knot gives the biggest ratio in this family: \(\cot(\pi/8)/2 \approx 1.2\). The crossing number of the \((2q + 4, q)\) knots is \((2q + 4)(q - 1)\). The resulting ratio \(\cot(\pi/(2q + 4))/(q - 1)\) again tends to \(2/\pi\) as \(q\) goes to infinity. The largest value in this family is the \((10, 3)\) knot for which \(\cot(\pi/10)/2 \approx 1.5\).

For the 7
d knot, on dividing the length to width ratio 24 by the crossing number 7, we obtain \(24/7 = 3\frac{3}{7}\). In [2], the Ribbonlength of the (truncated) figure eight knot is conjectured to be \(6 + 2\sqrt{2}\). Dividing by the 4 crossings, we have \((3 + \sqrt{2})/2 \approx 2.2\).

7.5. Bounding \(c_1\) and \(c_2\). Each of the length to width ratios we have calculated above serves as an upper bound for the Ribbonlength of the corresponding knot. Therefore, the comparisons of these ratios with crossing number we have made in this section all serve to bound the constant \(c_1\) of Equation \[\text{1}\] above. For closed knots, we have the bound \(c_1 \leq 2/\pi\) realised by torus knots of the form \((2q + 2, q)\) and \((2q + 4, q)\) in the limit as \(q\) goes to infinity. For truncated knots, the best bound is \(c_1 \leq 4/\pi\) realised by \((q + 1, q)\) torus knots as \(q\) tends to infinity.

In order to estimate \(c_2\), we restrict attention to those knots where we believe we know the Ribbonlength, namely the \((q + 1, q)\) torus knots,
the knot (8, 3) and the figure eight knot \[2]. For truncated knots the figure eight knot results in the bound \( c_2 \geq (3 + \sqrt{2})/2 \). For closed knots, the trefoil yields the bound \( c_2 \geq \frac{5}{3} \cot(\pi/5) \).

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Department of Mathematics and Statistics, California State University, Chico, Chico CA 95929-0525, USA

E-mail address: bbrennan@rbuhsd.k12.ca.us  
E-mail address: tmattman@csuchico.edu  
E-mail address: rraya@mail.csuchico.edu  
E-mail address: dtating@tco.net