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A CURIOUS POLYNOMIAL INTERPOLATION OF CARLITZ-RIORDAN’S \(q\)-BALLOT NUMBERS

FRÉDÉRIC CHAPOTON AND JIANG ZENG

ABSTRACT. We study a polynomial sequence \(C_n(x|q)\) defined as a solution of a \(q\)-difference equation. This sequence, evaluated at \(q\)-integers, interpolates Carlitz-Riordan’s \(q\)-ballot numbers. In the basis given by some kind of \(q\)-binomial coefficients, the coefficients are again some \(q\)-ballot numbers. We obtain in a combinatorial way another curious recurrence relation for these polynomials.

1. INTRODUCTION

This paper was motivated by a previous work of the first author on flows on rooted trees [Cha13b], where the well-known Catalan numbers and the closely related ballot numbers played an important role. In fact, one can easily introduce one more parameter \(q\) in this work, and then Catalan numbers and ballots numbers get replaced by their \(q\)-analogues introduced a long time ago by Carlitz-Riordan [CR64], see also [Car72, FH85].

These \(q\)-Catalan numbers have been recently considered by many people, see for example [Cig05, BF07, Hag08, BPT12], including some work by Reineke [Rein05] on moduli space of quiver representations.

Inspired by an analogy with another work of the first author on rooted trees [Cha13a], it is natural to try to interpolate the \(q\)-ballot numbers. In the present article, we prove that this is possible and study the interpolating polynomials.

Our main object of study is a sequence of polynomials in \(x\) with coefficients in \(\mathbb{Q}(q)\), defined by the \(q\)-difference equation:

\[
\Delta_q C_{n+1}(x|q) = qC_n(q^2x + q + 1|q),
\]

where \(\Delta_q f(x) = (f(1 + qx) - f(x))/(1 + (q - 1)x)\) is the Hahn operator.

After reading a previous version of this paper, Johann Cigler has kindly brought the two related references [Cig97, Cig98] to our attention, where a sequence of more general polynomials \(G_n(x, r)\) was introduced through a \(q\)-difference operator for \(q\)-integer \(x\) and positive integer \(r\). Comparing these two sequences one has

\[
G_n(qx + 1, 2) = C_{n+1}(x|q).
\]

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In the next section we recall classical material on Carlitz-Riordan’s \(q\)-analogue for Catalan and ballot numbers and define our polynomials. In the third section, we evaluate our polynomials at \(q\)-integers in terms of \(q\)-ballot numbers and prove a product formula when \(q = 1\). In the fourth section, we find their expansion in a basis made of a kind of \(q\)-binomial coefficients and obtain another recurrence for these polynomials. This recurrence is not usual even in the special \(x = q = 1\) case and we have only a combinatorial proof in the general case. We conclude the paper with some open problems.

Nota Bene: Figures are best viewed in color.

2. Carlitz-Riordan’s \(q\)-ballot numbers

Recall that the Catalan numbers \(C_n = \frac{1}{n+1} \binom{2n}{n}\) may be defined as solutions to

\[
C_{n+1} = \sum_{k=0}^{n} C_k C_{n-k}, \quad (n \geq 0), \quad C_0 = 1.
\]

The first values are

| \(n\) | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
|------|---|---|---|---|---|---|---|---|---|
| \(C_n\) | 1 | 1 | 2 | 5 | 14 | 42 | 132 | 429 | 1430 |

It is well known that \(C_n\) is the number of lattice paths from \((0, 0)\) to \((n, n)\) with steps \((1, 0)\) and \((0, 1)\), which do not pass above the line \(y = x\). As a natural generalization, one considers the set \(\mathcal{P}(n, k)\) of lattice paths from \((0, 0)\) to \((n + 1, k)\) with steps \((1, 0)\) and \((0, 1)\), such that the last step is \((1, 0)\) and they never rise above the line \(y = x\). Let \(f(n, k)\) be the cardinality of \(\mathcal{P}(n, k)\). The first values of \(f(n, k)\) are given in Table 1. These numbers are called ballot numbers and have a long history in the literature of combinatorial theory. Moreover, one (see [Com74]) has the explicit formula

\[
f(n, k) = \frac{n-k+1}{n+1} \binom{n+k}{k}, \quad (n \geq k \geq 0).
\]

Carlitz and Riordan [CR64] introduced the following \(q\)-analogue of these numbers

\[
f(n, k|q) = \sum_{\gamma \in \mathcal{P}(n,k)} q^{A(\gamma)},
\]

where \(A(\gamma)\) is the area under the path (and above the \(x\)-axis). The first values of \(f(n, k|q)\) are given in Table 2. Furthermore, Carlitz [Car72] uses a variety of elegant techniques to derive several basic properties of the \(f(n, k|q)\), among which the following is the basic recurrence relation

\[
f(n, k|q) = qf(n, k-1|q) + q^k f(n-1, k|q) \quad (n, k \geq 0),
\]
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| n \ k | 0  | 1  | 2  | 3  | 4  | 5  | 6  |
|-------|----|----|----|----|----|----|----|
| 0     |    | 1  |    |    |    |    |    |
| 1     |    | 1  |    |    |    |    |    |
| 2     | 1  | 2  |    |    |    |    |    |
| 3     | 1  | 3  | 5  | 5  |    |    |    |
| 4     | 1  | 4  | 9  | 14 |    |    |    |
| 5     | 1  | 5  | 14 | 28 | 42 | 42 |    |
| 6     | 1  | 6  | 20 | 48 | 90 | 132| 132|

Table 1. The first values of ballot numbers $f(n,k)$

| n \ k | 0  | 1  | 2  | 3  | 4  |
|-------|----|----|----|----|----|
| 0     |    | 1  |    |    |    |
| 1     | 1  |    |    |    |    |
| 2     | 1  | $q+q^2$ | $q^2+q^3$ | $q^3+q^4+2q^5+q^6$ |
| 3     | 1  | $q+q^2+q^3$ | $q^2+q^3+2q^4+q^5$ | $q^3+q^4+2q^5+q^6$ |
| 4     | 1  | $q+q^2+q^3+q^4$ | $q^2+q^3+2q^4+2q^5+2q^6+q^7$ | $q^3Y$ | $q^4Y$ |

Table 2. The first values of $q$-ballot numbers $f(n,k|q)$ with $Y = q^6 + 3q^5 + 3q^4 + 3q^3 + 2q^2 + q + 1$

where $f(n,k|q) = 0$ if $n < k$ and $f(0,0|q) = 1$.

It is also easy to see that the polynomial $f(n,k|q)$ is of degree $kn - k(k-1)/2$ and satisfies the equation $f(n,n|q) = qf(n,n-1|q)$. If one defines the $q$-Catalan numbers by

\[(2.5) \quad C_{n+1}(q) = \sum_{k=0}^{n} f(n,k|q) = q^{-n-1}f(n+1,n+1|q) \quad (n \geq 0),\]

then, one obtains the following analogue of (2.1) for the Catalan numbers

\[(2.6) \quad C_{n+1}(q) = \sum_{i=0}^{n} C_i(q)C_{n-i}(q)q^{(i+1)(n-i)},\]

where $C_0(q) = 1$. Setting $\tilde{C}_n(q) = q^{\binom{n}{2}}C_n(q^{-1})$, one has a simpler $q$-analog of (2.1)

\[(2.7) \quad \tilde{C}_{n+1}(q) = \sum_{i=0}^{n} q^i \tilde{C}_i(q)\tilde{C}_{n-i}(q).\]
The first values are $C_1(q) = 1$, $C_2(q) = 1 + q$, $C_3(q) = 1 + q + 2q^2 + q^3$ and
\[ C_4(q) = 1 + q + 2q^2 + 3q^3 + 3q^4 + 3q^5 + q^6. \]

No explicit formula is known for Carlitz-Riordan’s $q$-Catalan numbers. However, Andrews [And75] proved the following recurrence formula
\begin{equation}
C_n(q) = \frac{q^n}{[n+1]_q} \left[ \begin{array}{c}
2n \\
n
\end{array} \right]_q + q \sum_{j=0}^{n-1} (1 - q^{n-j})q^{(n+1-j)j} \left[ \begin{array}{c}
2j + 1 \\
j
\end{array} \right]_q C_{n-j}(q),
\end{equation}

where $[x]_q = \frac{q^x - 1}{q - 1}$.

Recall that the $q$-shifted factorial $(x; q)_n$ is defined by
\[ (x; q)_n = (1 - x)(1 - xq) \cdots (1 - xq^{n-1}) \quad (n \geq 1) \quad \text{and} \quad (x; q)_0 = 1. \]

The two kinds of $q$-binomial coefficients are defined by
\[ \left[ \begin{array}{c}
k \\
n
\end{array} \right]_q := \frac{(q; q)_n}{(q; q)_k(q; q)_{n-k}}, \quad \left( \begin{array}{c}
x \\
k
\end{array} \right)_q := \frac{x(x-1) \cdots (x-[k-1]_q)}{[k]_q!}, \]

with $(x)_0 = 1$. Note that
\[ \left( \begin{array}{c}
[n]_q \\
k
\end{array} \right)_q = q^k \left[ \begin{array}{c}
k \\
n
\end{array} \right]_q, \quad \left( \begin{array}{c}
-k \\
k
\end{array} \right)_q = (-1)^k q^{-kn} \left[ \begin{array}{c}
k + n - 1 \\
k
\end{array} \right]_q. \]

The $q$-derivative operator $D_q$ and Hahn operator $\Delta_q$ are defined by
\begin{equation}
D_q f(x) = \frac{f(qx) - f(x)}{(q-1)x} \quad \text{and} \quad \Delta_q f(x) = \frac{f(1 + qx) - f(x)}{1 + (q-1)x}. \end{equation}

**Definition 1.** The sequence of polynomials \( \{C_n(x|q)\}_{n \geq 1} \) is defined by the $q$-difference equation (1.1) or equivalently
\begin{equation}
C_{n+1}(x|q) - C_{n+1}(q^{-1}x - q^{-1}|q) = C_n(qx + 1|q) \quad (n \geq 1),
\end{equation}

with the initial condition $C_1(x|q) = 1$ and $C_n(-\frac{1}{q}|q) = 0$ for $n \geq 2$. 
For example, we have

\[ C_2(x|q) = 1 + q \binom{x}{1}_q, \]
\[ C_3(x|q) = (1 + q) + (q + q^2 + q^3) \binom{x}{1}_q + q^4 \binom{x}{2}_q, \]
\[ C_4(x|q) = (q^3 + q^2 + 2q + 1) + (q^6 + q^5 + 2q^4 + 2q^3 + 2q^2 + q) \binom{x}{1}_q \]
\[ + (q^9 + q^8 + q^7 + q^6 + q^5)q^{-1} \binom{x}{2}_q + q^9 \binom{x}{3}_q. \]

It is clear that \( C_n(x|q) \) is a polynomial in \( \mathbb{Q}(q)[x] \) of degree \( n - 1 \) for \( n \geq 1 \).

### 3. Some preliminary results

We first show that the evaluation of the polynomials \( C_n(x|q) \) at \( q \)-integers is always a polynomial in \( \mathbb{N}[q] \). Note that formulae (3.1) and (3.4) were implicitly given in [Cig97].

**Proposition 2.** When \( x = [k]_q \) we have

\[ C_{n+1}([k]_q|q) = q^{kn+\frac{n(n+1)}{2}} f(k+n,n|q^{-1}) \quad (n, k \geq 0). \]

**Proof.** When \( x = [k]_q \) Eq. (2.10) becomes

\[ C_{n+1}([k]_q|q) = q^k C_n([k+1]_q|q) + C_{n+1}([k-1]_q|q). \]

It is easy to see that (3.1) is equivalent to (2.4). \( \square \)

**Corollary 3.** We have

\[ C_{n+1}(0|q) = C_n(1|q) \quad \text{and} \quad C_{n+1}(1|q) = \tilde{C}_{n+1}(q). \]

**Proof.** Letting \( x = 0 \) in (2.10) we get \( C_{n+1}(0|q) = C_n(1|q) \). Letting \( k = 1 \) in (3.1) we have

\[ C_{n+1}(1|q) = q^{n+\frac{n(n+1)}{2}} f(1+n,n|q^{-1}) = q^{n+1+\frac{n(n+1)}{2}} f(n+1,n+1|q^{-1}) = q^{\frac{n(n+1)}{2}} C_{n+1}(q^{-1}), \]

which is equal to \( \tilde{C}_{n+1}(q) \) by definition. \( \square \)
The shifted factorial is defined by
\((x)_0 = 1\) and \((x)_n = x(x+1) \cdots (x+n-1), \ n = 1, 2, 3, \ldots,\)
and \((x)_{-n} = 1/(x-n)_n.\)

**Proposition 4.** When \(q = 1\) we have the explicit formula
\[
C_{n+1}(x|1) = \frac{(x+1)(x+n+2)_{n-1}}{n!} = \frac{x+1}{x+n} \binom{x+2n}{n} \quad (n \geq 0).
\]

**Proof.** When \(q = 1\) the equation (2.10) reduces to
\[
C_{n+1}(x|1) = C_{n+1}(x-1|1) + C_n(x+1|1).
\]
Since \(C_{n+1}(x|1)\) is a polynomial in \(x\) of degree \(n\), it suffices to prove that the right-hand side of (3.4) satisfy (3.5) for \(x\) being positive integers \(k\). By Proposition 2 and (2.2) it suffices to check the following identity
\[
\frac{k+1}{k+1+n} \binom{k+2n}{n} = \frac{k}{k+n} \binom{k-1+2n}{n} + \frac{k+2}{k+1+n} \binom{k+2n-1}{n-1}.
\]
This is straightforward. \(\square\)

To motivate our result in the next section we first prove two \(q\)-versions of a folklore result on the polynomials which take integral values on integers (see [St86, p. 38]).

Introduce the polynomials \(p_k(x)\) by
\[
p_0(x) = 1 \quad \text{and} \quad p_k(x) = (-1)^k q^{-\binom{k}{2}} \frac{(x-1)(x-q) \cdots (x-q^{k-1})}{(q; q)_k}, \quad k \geq 1.
\]
So \(p_k(q^n) = [n]_{k,q}\) for \(n \in \mathbb{N}\).

**Proposition 5.** The following statements hold true.

(i) The polynomial \(f(x)\) of degree \(k\) assumes values in \(\mathbb{Z}[q]\) at \(x = 1, q, \ldots, q^k\) if and only if
\[
f(x) = c_0 + c_1 p_1(x) + \cdots + c_k p_k(x),
\]
where \(c_j = q^{\binom{j}{2}} (1-q)^2 \Delta_{q}^j f(1)\) are polynomials in \(\mathbb{Z}[q]\) for \(0 \leq j \leq k\).

(ii) The polynomial \(\tilde{f}(x)\) of degree \(k\) assumes values in \(\mathbb{Z}[q]\) at \(x = 0, [1]_q, \ldots, [k]_q\) if and only if
\[
\tilde{f}(x) = \sum_{j=0}^{k} \tilde{c}_j q^{-\binom{j}{2}} \binom{x}{j},
\]
where \(\tilde{c}_j = q^{\binom{j}{2}} \Delta_{q}^j \tilde{f}(0)\) are polynomials in \(\mathbb{Z}[q]\) for \(0 \leq j \leq k\).
Proof. Clearly we can expand any polynomial \( f(x) \) of degree \( k \) in the basis \( \{ p_j(x) \}_{0 \leq j \leq k} \) as in (3.7). Besides, it is easy to see that

\[
\mathcal{D}_q p_k(x) = \frac{q^{1-k}}{1-q} p_{k-1}(x) \implies \mathcal{D}_q^j p_k(x) = \frac{q^{(j+1)-jk}}{(1-q)^j} p_{k-j}(x).
\]

(3.9)

Hence, applying \( \mathcal{D}_q^j \) to the two sides of (3.7) we obtain

\[
\mathcal{D}_q^j f(1) = c_j q^{-\binom{j}{2}} \frac{1}{(1-q)^j} \implies c_j = q^{\binom{j}{2}} (1-q)^j \mathcal{D}_q^j f(1).
\]

Since \( \mathcal{D}_q^j f(1) \) involves only the values of \( f(x) \) at \( x = 0, [1], \ldots, [k]_q \) for \( 0 \leq j \leq k \), the result follows. In the same manner, since

\[
\Delta_q \left( \begin{array}{c} x \\ k \end{array} \right)_q = \left( \begin{array}{c} x \\ k-1 \end{array} \right)_q,
\]

we obtain the expansion (3.8).

**Remark.** (1) We can also derive (3.8) from (3.7) as follows. Let \( y = \frac{x-1}{q} \). For any polynomial \( f(x) \) define \( \tilde{f}(y) = f(1 + (q-1)y) \). Since \( q^n = 1 + (q-1)[n]_q \), it is clear that \( f(q^n) \in \mathbb{Z}[q] \iff \tilde{f}([n]_q) \in \mathbb{Z}[q] \). Writing \( 1 + qx - [j]_q = q(x - [j-1]_q) \) we see that

\[
\tilde{f}_j(x) = q^{-\binom{j}{2}} \left( \begin{array}{c} x \\ j \end{array} \right)_q.
\]

The expansion (3.8) follows from (3.7) immediately.

(2) When \( \tilde{f}(x) = x^n \), it is known (see, for example, [Ze06]) that

\[
\tilde{c}_k = \Delta_q 0^n = [k]_q! S_q(n, k),
\]

where \( [k]_q! = [1]_q \cdots [k]_q \) and \( S_q(n, k) \) are classical \( q \)-Stirling numbers of the second kind defined by

\[
S_q(n, k) = S_q(n-1, k-1) + [k]_q S_q(n-1, k) \quad \text{for} \quad n \geq k \geq 1,
\]

with \( S_q(n, 0) = S_q(0, k) = 0 \) except \( S_q(0, 0) = 1 \).

(3) The two formulas (3.7) and (3.8) are special cases of the Newton interpolation formula, namely, for any polynomial \( f \) of degree less than or equal to \( n \) one has

\[
f(x) = \sum_{k=0}^n \left( \sum_{j=0}^k \frac{f(b_j)}{\prod_{r=0,r \neq j}^k (b_j - b_r)} \right) (x - b_0) \cdots (x - b_{k-1}),
\]

(3.10)
where \(b_0, b_1, \ldots, b_{n-1}\) are distinct complex numbers. Some recent applications of (3.10) in the computation of moments of Askey-Wilson polynomials are given in [GITZ].

4. Main results

In the light of Propositions 2 and 5 it is natural to consider the expansion of \(C_{n+1}(x|q)\) and \(C_{n+1}(qx+1|q)\) in the basis \(\left[\frac{f}{j}\right]_q\) \((j \geq 0)\). It turns out that the coefficients in such expansions are Carlitz-Riordan’s \(q\)-ballot numbers. Note that formula (4.2) was implicitly given in [Cig97].

**Theorem 6.** For \(n \geq 0\) we have

\[
(4.1) \quad C_{n+1}(x|q) = \sum_{j=0}^{n} f(n+j, n-j|q^{-1})q^{j n + \frac{1}{2}(n-j)(n+j+1)} \left[\frac{x}{j}\right]_q,
\]

\[
(4.2) \quad C_n(qx+1|q) = \sum_{j=0}^{n-1} f(n+j, n-1-j|q^{-1})q^{j n + \frac{1}{2}(n+1)-\frac{1}{2}(j+1)(j+2)} \left[\frac{x}{j}\right]_q.
\]

**Proof.** It is sufficient to prove the theorem for \(x = [k]_q\) with \(k = 0, 1, \ldots, n\). By Proposition 2 the two equations (4.1) and (4.2) are equivalent to

\[
(4.3) \quad f(k+n, n|q^{-1}) = \sum_{j=0}^{k} f(n+j, n-j|q^{-1})q^{j n - kn - j} \left[\frac{k}{j}\right]_q,
\]

\[
(4.4) \quad f(k+n, n-1|q^{-1}) = \sum_{j=0}^{k} f(n+j, n-j-1|q^{-1})q^{j n - kn - 2j + k} \left[\frac{k}{j}\right]_q,
\]

for \(n \geq k \geq j\). Replacing \(q\) by \(1/q\) and using \(\left[\frac{k}{j}\right]_{q^{-1}} = q^{-j(k-j)} \left[\frac{k}{j}\right]_{q}\) we get

\[
(4.5) \quad f(n+k, n|q) = \sum_{j=0}^{k} f(n+j, n-j|q)q^{(n-j)(k-j) + j} \left[\frac{k}{j}\right]_q,
\]

\[
(4.6) \quad f(k+n, n-1|q) = \sum_{j=0}^{k} f(n+j, n-j-1|q)q^{(n-j-1)(k-j) + j} \left[\frac{k}{j}\right]_q.
\]

We only prove (4.5). By definition, the left-hand side \(f(n+k, n|q)\) is the enumerative polynomial of lattice paths from \((0,0)\) to \((n+k+1, n)\) with \((1,0)\) as the last step. Each such path \(\gamma\) must cross the line \(y = -x + 2n\). Suppose it crosses this line at the point \((n+j, n-j), 0 \leq j \leq k\). Then the path corresponds to a unique pair \((\gamma_1, \gamma_2)\), where \(\gamma_1\) is a path from \((0,0)\) to \((n+j, n-j)\) and \(\gamma_2\) is a path from \((n+j, n-j)\)
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\[ n \quad n \quad n \quad n + j \quad n - j \]

**Figure 1.** The decomposition \( \gamma \mapsto (\gamma_1, \gamma_2) \)

to \((n + k, n)\). It is clear that the area under the path \(\gamma\) is equal to \(S_1 + S_2 + S_3 + j\), where

- \(S_1\) is the area under the path \(\gamma_1'\), which is obtained from \(\gamma_1\) plus the last step \((n + j, n - j) \to (n + j + 1, n - j)\);
- \(S_2\) is the area under the path \(\gamma_2\) and above the line \(y = n - j\);
- \(S_3\) is the area of the rectangle delimited by the four lines \(y = 0, y = n - j, x = n + j + 1 \text{ and } x = n + k + 1\), i.e., \((n - j)(k - j)\).

This decomposition is depicted in Figure 1. Clearly, summing over all such lattice paths gives the summand on the right-hand side of (4.5). This completes the proof.

\[ \square \]

**Remark.** When \(q = 1\), by (2.2), the above theorem implies that

\[
\begin{align*}
\frac{x + 1}{x + n + 1} \left( \frac{x + 2n}{n} \right) &= \sum_{j=0}^{n} \frac{2j + 1}{n + j + 1} \left( \frac{2n}{n - j} \right) \left( \frac{x}{j} \right), \\
\frac{x + 2}{x + n + 1} \left( \frac{x + 2n - 1}{n - 1} \right) &= \sum_{j=0}^{n-1} \frac{2j + 2}{n + j + 1} \left( \frac{2n - 1}{n - j - 1} \right) \left( \frac{x}{j} \right).
\end{align*}
\]

(4.7) \quad (4.8)

Note that the two sequences

\[ \{f(n + j, n - j | 1)\} \quad \text{and} \quad \{f(n + j, n - j - 1 | 1)\} \quad (0 \leq j \leq n) \]

correspond, respectively, to the \((2n - 1)\)-th and \(2n\)-th anti-diagonal coefficients of the triangle \(\{f(n, k)\}_{0 \leq k \leq n}\), see Table 1.
Theorem 7. The polynomials $C_n(x|q)$ satisfy $C_1(x|q) = 1$ and
\[
[n]_q C_{n+1}(x|q) = ([2n - 1]_q + xq^{2n-1})C_n(x|q)
+ \sum_{j=0}^{n-2} [n - j - 1]_q \tilde{C}_j(q)C_{n-j}(x|q)q^{2j+1}.
\]

Proof. Since $C_{n+1}(x|q)$ is a polynomial in $x$ of degree $n$, it suffices to prove (4.9) for $x = [k]_q$, where $k$ is any positive integer, namely,
\[
[n]_q C_{n+1}([k]_q|q) = [2n + k - 1]_q C_n([k]_q|q)
+ \sum_{j=0}^{n-2} [n - j - 1]_q \tilde{C}_j(q)C_{n-j}([k]_q|q)q^{2j+1}.
\]

Let $m \geq n$ and
\[
\tilde{f}(m, n|q) = q^{(m-n)n + \binom{n+1}{2}} f(m, n|q^{-1}).
\]
In view of the definition (2.3) it is clear that
\[
\tilde{f}(m, n|q) = \sum_{\gamma \in P(m,n)} q^{A' (\gamma)},
\]
where $A'(\gamma)$ denotes the area above the path $\gamma$ and under the line $y = x$ and $y = n$. Since
\[
\tilde{C}_j(q) = q^{(j)} C_j(q^{-1}) = q^{(j+1)} f(j, j|q) = \tilde{f}(j, j|q),
\]
using Proposition 2 and (4.11) with $m = k + n$, we can rewrite (4.10) as
\[
[n]_q \tilde{f}(m, n|q) = [n + m - 1]_q \tilde{f}(m - 1, n - 1|q)
+ \sum_{j=0}^{n-2} q^j [n - j - 1]_q \tilde{f}(j, j|q)q^{j+1} [n - j - 1]_q \tilde{f}(m - j - 1, n - j - 1|q).
\]
A pointed lattice path is a pair $(\alpha, \gamma)$ such that $\alpha \in \{(1,1), \ldots, (n,n)\}$ and $\gamma \in P(m,n)$. If $\alpha = (i, i)$ we call $i$ the height of $\alpha$ and write $h(\alpha) = i$. Let $P^*(m,n)$ be the set of all such pointed lattice paths. It is clear that the left-hand side of (4.12) has the following interpretation
\[
[n]_q \tilde{f}(m, n|q) = \sum_{(\alpha, \gamma) \in P^*(m,n)} q^{h(\alpha) + A'(\gamma)}.
\]
Now, we compute the above enumerative polynomial of $P^*(m,n)$ in another way in order to obtain the right-hand side of (4.12). We distinguish two cases.
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Figure 2. \((\alpha, \gamma) \rightarrow (\gamma_1, (\alpha', \gamma_2))\)

- Let \(P_1^*(m, n, j)\) be the set of all pointed lattice paths \((\alpha, \gamma)\) in \(P^*(m, n)\) such that \(h(\alpha) \in \{j+2, \ldots, n\}\), where \(j\) is the smallest integer such that \((j+1, j) \rightarrow (j+1, j+1)\) is a step of \(\gamma\), i.e., the first step of \(\gamma\) touching the line \(y = x\). If \((\alpha, \gamma) \in P_1^*(m, n, j)\), then we have the correspondence \((\alpha, \gamma) \rightarrow (\gamma_1, (\alpha', \gamma_2))\), where \(\gamma_1\) is a lattice path from \((0, 0)\) to \((j+1, j+1)\) which touches the line \(y = x\) only at the two extremities, and \((\alpha', \gamma_2)\) is a pointed lattice path from \((0, 0)\) to \((m - j, n - j - 1)\) with \(h(\alpha') = h(\alpha) - j - 1\). This decomposition is depicted in Figure 2.

Thus the corresponding enumerative polynomial of such paths for the fixed \(j\) is

\[
\sum_{(\alpha, \gamma) \in P_1^*(m, n, j)} q^{h(\alpha) + A(\gamma)} = q^j \tilde{f}(j, j | q) \cdot q^{j+1}[n - j - 1] \tilde{f}(m - j - 1, n - j - 1 | q).
\]

Summing over all \(j\) \((0 \leq j \leq n - 2)\) we obtain the second term on the right-hand side of (4.12).

- Let \(P_2^*(m, n)\) be the set of all pointed lattice paths \((\alpha, \gamma)\) in \(P^*(m, n)\) such that \(h(\alpha) \in \{1, \ldots, n\}\) and \(h(\alpha) \leq j + 1\) where \(j\) (if any) is the smallest integer such that \((j+1, j) \rightarrow (j+1, j+1)\) is a step of \(\gamma\), i.e., the first step of \(\gamma\) touching the line \(y = x\). If \((\alpha, \gamma) \in P_2^*(m, n)\), where \(\gamma = (p_0, \ldots, p_{m+n+1})\) with \(p_0 = (0, 0)\) and \(p_{m+n+1} = (m + n + 1, n)\), we can associate a pair \((i, \gamma')\) where \(\gamma' \in P(m-1, n-1)\) is obtained from \(\gamma\) by deleting the vertical step \((x, h(\alpha) - 1) \rightarrow (x, h(\alpha))\) and the first horizontal step \((0, 0) \rightarrow (1, 0)\), i.e.,

\[
\gamma' = (p'_0, \ldots, p'_i, p'_{i+2}, \ldots, p'_{n+m+1})
\]

where \(i = x + h(\alpha) - 1\), \(p'_k = p_k - (1, 0)\) if \(k = 1, \ldots, i\) and \(p'_k = p_k - (0, 1)\) if \(k = i + 2, \ldots, m + n + 1\). It is easy to see that the mapping \((\alpha, \gamma) \mapsto (i, \gamma')\) is a bijection, which is depicted in Figure 3.
Since \( 1 \leq x \leq m \) and \( 0 \leq h(\alpha) - 1 \leq n - 1 \) we have \( i \in \{1, \ldots, m + n - 1\} \). As \( A'(\gamma) = x - 1 + A'(\gamma') \) we have

\[
h(\alpha) - 1 + A'(\gamma) = i - 1 + A'(\gamma').
\]

It follows that

\[
\sum_{(\alpha, \gamma) \in P_2(m, n)} q^{h(\alpha) + A'(\gamma)} = \sum_{i=1}^{m+n-1} q^{i-1} \sum_{\gamma'} q^{A'(\gamma')} = [n + m - 1]_q \tilde{f}(m - 1, n - 1|q).
\]

Summing up the two cases we obtain the right-hand side of (4.12). \qed

When \( q = 1 \) we have an alternative proof of Theorem 7.

Another proof of the \( q = 1 \) case. When \( q = 1 \) Eq. (4.9) reduces to

\[
(4.14) \quad nC_{n+1}(x|1) = (2n - 1 + x)C_n(x|1) + \sum_{j=0}^{n-2} (n - j - 1)C_jC_{n-j}(x|1) \quad (n \geq 2).
\]

This yields immediately \( C_1(x|1) = 1, \ C_2(x|1) = x + 1, \ C_3(x|1) = (x + 1)(x + 4)/2, \) in accordance with the formula (3.4). For \( n \geq 3 \), letting \( k = n - j - 3, \ N = n - 3 \) and \( z = x + 3 \), by (3.4), the recurrence (4.14) is equivalent to the following identity

\[
\frac{(z + N + 2)_N}{N!} = \sum_{k=0}^{N} 4^{N-k} \binom{3/2}{N-k}(z+k)_k \binom{3}{N-k} \quad (N \geq 0).
\]
Notice that we can rewrite the right-hand side as

\[
\frac{(3/2)_N}{4^N} \sum_{k=0}^{N} \frac{(-2 - N)_k((z + 1)/2)_k(z/2)_k}{k!(-1/2 - N)_k(z)_k}
= \frac{(3/2)_N}{(3)_N} 4^N \left( \sum_{k=0}^{N} \frac{(-2 - N)_k((z + 1)/2)_k(z/2)_k}{k!(-1/2 - N)_k(z)_k} \right).
\]

Invoking Pfaff-Saalschütz formula \[AAR99, \text{Theorem 2.2.6}\] we obtain

\[
3F2 \left( -2 - N, \frac{z + 1}{2}, \frac{z}{2}; 1 \right) = \frac{(z/2)_N + 1}{(z)_N + 1}.
\]

Substituting this in the previous expression yields \(\frac{z + N + 2}{N!}\) after simplification. \(\square\)

When \(x = 1\) Eq. (4.14) reduced to the following identity for Catalan numbers:

\[
(4.15) \quad nC_{n+1} = 2nC_n + \sum_{j=0}^{n-2} (n - j - 1)C_jC_{n-j}.
\]

5. Concluding remarks

We conclude this paper with a few open problems. By Theorem 6 it is clear that \(C_{n+1}(x|q)\) is a polynomial in \(x\) of degree \(n\) with leading coefficient \(q^n/[1][2]_q \cdots [n]_q\).

**Conjecture 8.** One can write \(C_n(x|q)\) as an irreducible fraction

\[
P_n(x|q) = \frac{P_n(x|q)}{[1]_q \cdots [n-1]_q},
\]

where \(P_n\) has only positive coefficients.

This has been checked up to \(n = 27\). The similar conjecture is true when \(q = 1\) by Proposition 4.

Finally, the Newton polytope of the numerator of \(C_n(x|q)\) seems to have a nice shape. This is illustrated in Figure 4, where the horizontal axis is associated with powers of \(q\) and the vertical axis with powers of \(x\). The slopes of the upper part seems to be given in general by the odd integers 1, 3, \ldots, 2n − 3.
Figure 4. Newton polytope of the numerator of $C_n(x|q)$ for $n = 6$

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