Biinvariant functions on the group of transformations leaving a measure quasiinvariant

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Let $G_{\text{ms}}$ be the group of transformations of a Lebesgue space leaving the measure quasiinvariant, let $A_{\text{ms}}$ be its subgroup consisting of transformations preserving the measure. We describe canonical forms of double cosets of $G_{\text{ms}}$ by the subgroup $A_{\text{ms}}$ and show that all continuous $A_{\text{ms}}$-biinvariant functions on $G_{\text{ms}}$ are functionals on of the distribution of a Radon–Nikodym derivative.

1 Statements

1.1. The group $G_{\text{ms}}$. By $\mathbb{R}^\times$ we denote the multiplicative group of positive reals. By $t$ we denote the coordinate on $\mathbb{R}^\times$.

Let $M$ be a Lebesgue space (see [1]) with a continuous probabilistic measure $\mu$ (recall that any such space is equivalent to the segment $[0,1]$). Denote by $A_{\text{ms}} = A_{\text{ms}}(M)$ the group of all transformations (defined up to a.s.) preserving the measure $\mu$. By $G_{\text{ms}} = G_{\text{ms}}(M)$ we denote the group of transformations (defined up to a.s.) leaving the measure $\mu$ quasiinvariant.

The group $A_{\text{ms}}$ was widely discussed in connection with ergodic theory, the group $G_{\text{ms}}$, which is a topic of the present note, only occasionally was mentioned in the literature. However, it is an interesting object from the point of view of representations of infinite-dimensional groups ("large groups" in the terminology of A.M. Vershik), see [2], [3].

1.2. The topology on $G_{\text{ms}}$. A separable topology on $G_{\text{ms}}$ was defined in [1] 17.46, [5], §4.5 by different ways. One of the purposes of the present note is to show that these ways are equivalent.

The first way is following. Let $A, B \subset M$ be measurable subsets. For $g \in G_{\text{ms}}$ we define the distribution

$$\kappa[g; A, B]$$

of the Radon–Nikodym derivative $g'$ on the set $A \cap g^{-1}(B)$. We say that a sequence $g_j \in G_{\text{ms}}$ converges to $g$, if for any measurable sets $A, B$ we have the following weak convergences of measures on $\mathbb{R}^\times$

$$\kappa[g_j; A, B] \to \kappa[g; A, B], \quad t\kappa[g_j; A, B] \to t\kappa[g; A, B]. \quad (1.1)$$

**Remark 1.** Point out evident identities:

$$\int_{\mathbb{R}^\times} \kappa[g; A, M](t) = \mu(A), \quad \int_{\mathbb{R}^\times} t\kappa[g; A, M](t) = \mu(gA). \quad (1.2)$$

**Remark 2.** Consider a measurable finite partition

$$\mathcal{H} : M = M^1 \cup M^2 \cup \ldots
\[1\]Supported by the grant FWF, P25142.
of the space $M$. This gives us a matrix $S_{\alpha\beta}[g; h] := \mathcal{X}[g; M^\alpha, M^\beta]$, composed of measures on $\mathbb{R}^\times$. If a partition $\mathfrak{p}$ is a refinement of $\mathfrak{h}$, we write $\mathfrak{h} \leq \mathfrak{p}$. Consider a sequence of partitions $\mathfrak{h}_1 \leq \mathfrak{h}_2 \leq \ldots$, generating the $\sigma$-algebra of the space $M$. A convergence $g_j \to g$ is equivalent to an element-wise convergence in the sense of all matrices $S[g_j; h_n] \to S[g; h_n]$.

**Proposition 1.1** The group $G_{ms}$ is a Polish group with respect to this topology, i.e., $G_{ms}$ is a separable topological group complete with respect to the two-side uniform structure and homeomorphic to a complete metric space.

Let $1 \leq p \leq \infty$, $s \in \mathbb{R}$. The group $G_{ms}$ acts in the space $L^p(M)$ by isometric transformations according to the formula

$$T_1/p+is f(x) = f(g(x))g'(x)^{1/p+is}. $$

On the space $\mathcal{B}(V)$ of operators of a Banach space $V$ we define in the usual way (see, e.g., [7], VI.1) the strong and weak topologies. Also, on the set $\mathcal{GL}(V)$ of invertible operators we introduce a bi-strong topology, $A_j$ converges to $A$, if $A_j \to A$ and $A_j^{-1} \to A^{-1}$ strongly. The embedding $T_1/p+is : G_{ms} \to \mathcal{B}(L^p)$ induces a certain topology on $G_{ms}$ from any operator topology on $\mathcal{B}(V)$ or $\mathcal{GL}(V)$.

**Proposition 1.2** a) Let $1 < p < \infty$, $s \in \mathbb{R}$. A topology on $G_{ms}$ induced from any of three topologies (strong, weak, bi-strong) coincides with the topology defined above.

b) Let $p = 1$, $s \in \mathbb{R}$. A topology on $G_{ms}$ induced from strong or bi-strong topology coincides with the topology defined above.

c) Let $1 \leq p \leq \infty$, $s \in \mathbb{R}$. Then the image of $G_{ms} \mathcal{GL}(L^p(M))$ is closed in the bi-strong topology.

Point out that the coincidence of topologies is not surprising. It is known that two different Polish topologies on a group can not determine the same Borel structure, see [4], 12.24. There are also theorems about automatic continuity of homomorphisms, see [4], 9.10.

**1.3. Double cosets $A_{ms} \setminus G_{ms}/A_{ms}$. Canonical forms.** We reformulate the problem of description of double cosets $A_{ms} \setminus G_{ms}/A_{ms}$ in the following way. Let $(P, \pi)$, $(R, \rho)$ be Lebesgue spaces with continuous probabilistic measures. Denote by $G_{ms}(P, R)$ the space of all bijections $g : P \to R$ (defined up to a.s.),

\footnote{As $h_n$ we can take a partition of the segment $M = [0, 1]$ into $2^n$ pieces of type $[k2^{-n}, (k + 1)2^{-n})$}

\footnote{A metric is compatible with the topology of the group, but not with its algebraic structure; in particular a metric is not assumed to be invariant. A completeness of a group in the sense of two-side uniform structure (in Raikov's sense) is defined (for metrizable groups) in the following way. Let double sequences $g_jg_j^{-1}$ and $g_j^{-1}g_j$ converge to $1$ as $i, j \to \infty$. Then $g_j$ has a limit in the group. This definition is not equivalent to the definition of Bourbaki [9], III.3.3, who requires a completeness with respect to both one-side uniform structures. The group $G_{ms}$ is not complete in the sense of Bourbaki.}
such that images and preimages of sets of zero measure have zero measure. We wish to describe such bijections up to the equivalence
\[ g \sim u \cdot g \cdot v, \quad \text{where } v \in \text{Ams}(P), \ u \in \text{Ams}(R) \] (1.3)
(clearly, such classes are in-to-one correspondence with double cosets \( \text{Ams} \setminus \text{Gms}/\text{Ams} \)).

**Lemma 1.3** Two elements \( g_1, g_2 \in \text{Gms}(P,R) \) are contained in one class if and only if the Radon–Nikodym derivatives \( g'_1, g'_2 : P \to \mathbb{R} \) are equivalent with respect to the action of the group \( \text{Ams}(P) \), i.e., \( g'_2(m) = g'_1(hm) \), where \( h \) is an element of \( \text{Ams}(P) \).

An evident invariant of this action is the distribution \( \nu \) of the Radon–Nikodym derivative \( g' \) of the map \( g \),
\[ \int_{\mathbb{R}^s} d\nu(t) = 1, \quad \int_{\mathbb{R}^s} t \, d\nu(t) = 1. \] (1.4)
This invariant is not exhaust, the problem is reduced to the Rokhlin theorem \([10]\) on metric classification of functions, see discussion below, §3.2. The final answer is following.

Consider a countable number of copies \( \mathbb{R}^1 \times \mathbb{R}, \mathbb{R}^2 \times \mathbb{R}, \ldots \) of half-line \( \mathbb{R}^s \). Consider one more copy \( \mathbb{R}^\infty \times [0,1] \). Consider the disjoint union
\[ \mathcal{L} := \mathbb{R}^1 \times \mathbb{R}^2 \times \mathbb{R}^3 \times \cdots \times (\mathbb{R}^\infty \times [0,1]). \]

Let \( \nu_1, \nu_2, \ldots, \nu_\infty \) be a family of measure on \( \mathbb{R}^s \) satisfying the following conditions
1. \( \nu_1, \nu_2, \ldots \) are continuous (but \( \nu_\infty \) admits atoms).
2. \( \nu_1 \geq \nu_2 \geq \ldots \)
3. The measure \( \nu := \nu_1 + \nu_2 + \cdots + \nu_\infty \) satisfies (1.4).

Equip each \( \mathbb{R}^s_j \) with the measure \( \nu_j \), equip \( \mathbb{R}^\infty \times [0,1] \) with the measure \( \nu_\infty \times dx \), where \( dx \) is the Lebesgue measure on the segment. Denote the resulting measure space by \( \mathcal{L}[\nu_1, \nu_2, \ldots; \nu_\infty] \).

Consider the same measure on \( \mathcal{L} \) multiplied by \( t \), we denote the resulting measure space by \( \mathcal{L}_s[\nu_1, \nu_2, \ldots; \nu_\infty] \).

Consider the identity map
\[ \text{id} : \mathcal{L}[\nu_1, \nu_2, \ldots; \nu_\infty] \to \mathcal{L}_s[\nu_1, \nu_2, \ldots; \nu_\infty] \] (1.5)
Evidently, the distribution of the Radon–Nikodym derivative of the map \( \text{id} \) coincides with \( \nu \).

**Proposition 1.4** Any equivalence class (1.3) contains a unique representative of the type (1.5).

Denote the double coset containing this representative by \( S[\nu_1, \nu_2, \ldots; \nu_\infty] \).

1.4. On closures of double cosets.
Theorem 1.5 Let a measure $\nu$ on $\mathbb{R}^\times$ satisfy (1.4), let $\nu = \nu^c + \nu^d$ be its decomposition into continuous and discrete parts. Then the closure of the double coset $S[\nu^c, 0, 0, \ldots; \nu^d]$ contains all double cosets $S[\nu_1, \nu_2, \nu_3, \ldots; \nu^c + \nu^d]$ with $\nu_1 + \nu_2 + \cdots + \nu^c = \nu^c$.

1.5. Hausdorff quotient. Consider the space $\mathcal{M}$ of all measures $\nu$ on $\mathbb{R}^\times$ satisfying (1.4). Say that $\nu_j \in \mathcal{M}$ converges to $\nu$ if $\nu_j \to \nu$ and $t\nu_j \to t\nu$ weakly.

Consider a map $\Phi : Gms \to \mathcal{M}$ that for any $g$ assigns the distribution of its Radon–Nikodym derivative (i.e., $\Phi(g) = \kappa(g; M, M)$). In virtue of Theorem 1.5, preimages of points $\nu \in \mathcal{M}$ are closures of double cosets $S[\nu^c, 0, 0, \ldots; \nu^d]$.

Theorem 1.6 Let $f$ be a continuous map of $Gms$ to a metric space $T$, moreover, $f$ let be constant on double cosets. Then $f$ has the form $f = q \circ \Phi$, where $q : \mathcal{M} \to T$ is a continuous map.

1.6. A continuous section $\mathcal{M} \to Gms$. We say that a function $h : [0, 1] \to [0, 1]$ is contained in the class $\mathcal{G}$, if

- $h$ is downward convex;
- $h(0) = 0$, $h(1) = 1$, and $h(x) > 0$ for $x > 0$.

Any such function is an element of the group $Gms([0, 1])$.

Proposition 1.7 Let $\nu \in \mathcal{M}$. Then there is a unique function $\psi : [0, 1] \to [0, 1]$ of the class $\mathcal{G}$ such that the distribution of the derivative $\psi'$ is $\nu$. Moreover, the map $\nu \mapsto \psi$ is a continuous map $\mathcal{M} \to Gms$.

1.7. A more general statement. Consider a finite or countable measurable partition of our measure space $M = \bigsqcup_j M_j$. Denote by $K$ the direct product $K = Ams(M_1) \times Ams(M_2) \times \ldots$. Consider the double cosets $K \setminus Gms / K$. Assign to each $g \in Gms$ the matrix $\kappa_{ij} = \kappa(g; M_i, M_j)$ composed of measures on $\mathbb{R}^\times$. Denote by $S$ the set of matrices that can be obtained in this way, i.e.,

$$
\sum_j \int_{\mathbb{R}^\times} d\kappa_{ij}(t) = \mu(M_i), \quad \sum_i \int_{\mathbb{R}^\times} t\kappa d\mu_{ij}(t) = \mu(M_j).
$$

 Equip $S$ with element-wise convergence (1.1). Denote by $\Psi$ the natural map $Gms \to S$.

Theorem 1.8 Let $f$ be a continuous map from $Gms$ to a metric space $T$. Then there exists a continuous map $q : S \to T$, such that $f = q \circ \Psi$.

Point out that this statement was actually used in [5], [2].

1.8. The structure of the note. The statements about topology on $Gms$ are proved in §2, about double cosets in §3. Theorem 1.6 follows from Theorem 1.5. However, as the referee pointed out, the first statement is simpler than the second (and it is more important). Therefore in the beginning of §3 we present a separate proof of Theorem 1.6.
2 The topology on the group Gms

Below we prove Propositions 1.1 and 1.2. The main auxiliary statement is Lemma 2.4. The remaining lemmas are proved in a straightforward way.

Notation:
• $\delta_a$ is a probabilistic atomic measure $\mathbb{R}^\times$ supported by a point $a$.
• $\{\cdot, \cdot\}_{pq}$ is the natural pairing of $L^p$ and $L^q$, where $1/p + 1/q = 1$;
• $\chi_A$ is the indicator function of a set $A \subset M$, i.e., $\chi_A(x) = 1$ for $x \in A$ and $\chi_A(x) = 0$ for $x \notin A$.

2.1. Preliminary remarks on the spaces $L^p$.

1) Recall (see [11], § 3.3) that for $p \neq 2$ the group of isometries $\text{Isom}(L^p(M))$ consists of operators of the form

$$R(g, \sigma)f(x) = \sigma(x)f(g(x))g'(x)^{1/p}, \quad (2.1)$$

where $g \in Gms$, and $\sigma : M \to \mathbb{C}$ is a function whose absolute value equals 1.

2) For $1 < p < \infty$ the space $L^p$ is uniformly convex (see [12], § 26.7), therefore the restrictions of the strong and weak topologies to the unit sphere coincide. Therefore on the group of isometries $\text{Isom}(L^p(M))$ the weak and strong operator topologies coincide.

3) Recall that for separable Banach spaces (in particular, for $L^p$ with $p \neq \infty$) the group of all isometries equipped with bi-strong topology is a Polish group, see [4], 9.B9.

2.2. Preliminary remarks on the group Gms.

1) The invariance of the topology. Equip Gms with topology from Subsection 1.2. The product in Gms is separately continuous (this is a special case of Theorem 5.9 from [13]). In particular, this implies that the topology on Gms is invariant with respect to left and right shifts.

The map $g \mapsto g^{-1}$ is continuous. Indeed,

$$\varkappa[g^{-1}; B, A](t) = t^{-1}\varkappa[g; A, B](t^{-1}),$$

and this map transpose the convergences (1.1).

2) Separability of Gms. For a measure $\varkappa[g; A, B]$ consider the characteristic function

$$\chi(z) = \int_{\mathbb{R}^\times} t^zd\varkappa[g; A, B](t), \quad (2.2)$$

continuous in the strip $0 \leq \text{Re} z \leq 1$ and holomorphic in the open strip. The convergence of measures $\varkappa$ is equivalent to point-wise convergence of characteristic functions uniform in each rectangle

$$0 \leq \text{Re} z \leq 1, \quad -N \leq \text{Im} z \leq N,$$

[13], Propositions 4.4-4.5. This convergence is separable. Next, by Remark 2 of [14] it suffices to verify the convergence of measures $\varkappa[g_j; A, B] \to \varkappa[g; A, B]$ for an appropriate countable set of pairs measurable subsets $(A, B)$.

3) The action on Boolean algebra of sets.
Lemma 2.1 Let \( g_j \to g \) in Gms. Then for any measurable set \( A \subset M \) we have
\[
\mu(g_j A \triangle g A) \to 0. \tag{2.3}
\]

Proof. By the invariance of the topology it suffices to consider \( g = 1 \). Then
\[
\mu(g_j A \cap A) = \int_{\mathbb{R}^*} \delta([g_j^{-1}; A; A])(t) \to \int_{\mathbb{R}^*} \delta([A; A])(t) = \mu(A);
\]
\[
\mu(g_j A) = \int_{\mathbb{R}^*} t \delta([g_j; A; M])(t) \to \int_{\mathbb{R}^*} t \delta_0(t) = \mu(A).
\]
Comparing two rows we get the desired statement. \( \square \)

Remark. The opposite is false. Let \( M = [0, 1] \),
\[
g_j(x) = x + \frac{1}{2\pi n} \sin(2\pi nx).
\]
Then for any \( A \subset [0, 1] \) we have \( \mu(g_j(A) \triangle A) \to \mu(A) \). But there is no convergence \( g_j \to 1 \) in Gms; \( T_1(g_j) \) converges weakly to \( 1 \) in \( L^1 \), but there is no strong convergence. \( \square \)

4) The continuity of representations \( T_{1/p+is} \).

Lemma 2.2 For \( p < \infty \) the homomorphisms \( T_{1/p+is} : \text{Gms} \to \text{Isom}(L^p) \) are continuous with respect to the weak topology \( \text{Isom}(L^p) \).

Proof. Let \( g_j \to g \) in Gms. Consider 'matrix elements'
\[
\{T_{1/p+is}(g_j)\chi_A, \chi_B\}_{pq} = \int_{A \cap g_j^{-1}B} g_j'(x)^{1/p+is} \mu(x) = \int_{\mathbb{R}^*} t^{1/p+is} \delta([g_j; A; B])(t)
\]
Weak convergence of measures \( [\mathbf{1}] \) implies the convergence of characteristic functions \( \mathbf{2}[\mathbf{2}] \), our expression tends to
\[
\int_{\mathbb{R}^*} t^{1/p+is} \delta([g; A; B])(t) = \int_{A \cap g^{-1}B} g'(x)^{1/p+is} \mu(x) = \{T_{1/p+is}(g)\chi_A, \chi_B\}_{pq},
\]
as required. \( \square \)

Thus, for \( 1 < p < \infty \) the maps \( T_{1/p+is} : \text{Gms} \to \text{Isom}(L^p) \) are continuous with respect to the strong (=weak) topology. Keeping in mind the continuity of the map \( g \mapsto g^{-1} \), we get that the maps \( T_{1/p+is} \) are continuous with respect to the bi-strong topology.

The case \( L^1 \) must be considered separately.

Lemma 2.3 Let \( g_j \to g \). Then \( T_{1+is}(g_j) \in \text{Isom}(L^1) \) strongly converges to \( T_{1+is}(g) \).
Proof. Without loss of generality, we can set $g = 1$. It suffices to verify the convergence $\|T_{1+is}(g_j)\chi_A - \chi_A\| \to 0$ for any measurable $A$. This equals

$$
\int_M |\chi_A(g_jx)g'(x)^{1+is} - \chi_A(x)| \, d\mu(x) = \\
= \int_{A \cap g_j^{-1}A} |g'(x)^{1+is} - 1| \, d\mu(x) + \int_{A \setminus g_j^{-1}A} g'(x) \, d\mu(x) = \\
= \int_{\mathbb{R}^n} |t^{1+is} - 1| \, d\kappa(g_j; A, A) + \mu(A \setminus g_j^{-1}A) + \mu(A \setminus g_jA). \tag{2.4}
$$

The second and the third summands tend to 0 by Lemma 2.1, measures $\kappa[\ldots]$ and $t\kappa[\ldots]$ converge weakly to $\mu(A)\delta_0$, therefore the integral tends to 0. □

2.3. The coincidence of topologies and the continuity of the multiplication.

Lemma 2.4 Let $1 < p < \infty$. Let $T_{1/p+is}(g_j)$ weakly converge to 1 in $\text{Isom}(L_p)$. Then $g_j$ converges to 1 in $Gms$.

Proof. Step 1. Now it will be proved that $g_j'$ converges to 1 in $L^1(M)$. For this purpose, we notice that the following sequence of matrix elements must converge to 1:

$$
\{T_{1/p+is}(g_j) 1, 1\}_{pq} = \int_M g_j'(x)^{1/p+is} \, d\mu(x) = \int_{\mathbb{R}^n} t^{1/p+is} \, d\kappa(g_j; M, M)(t). \tag{2.5}
$$

Estimate the integrand:

$$
\text{Re} t^{1/p+is} \leq t^{1/p} \leq \frac{1}{q} + \frac{t}{p}.
$$

The second inequality means that the graph of upward convex function is lower than the tangent line at $t = 1$. From another hand:

$$
\int_{\mathbb{R}^n} \left( \frac{1}{q} + \frac{t}{p} \right) \, d\kappa(g_j; M, M)(t) = \\
= \frac{1}{q} \int_{\mathbb{R}^n} d\kappa(g_j; M, M)(t) + \frac{1}{p} \int_{\mathbb{R}^n} t \, d\kappa(g_j; M, M)(t) = \frac{1}{q} + \frac{1}{p} = 1.
$$

Look to a deviation of integral (2.5) from 1. The same reasoning with tangent line allows to estimate the difference $\frac{1}{q} + \frac{t}{p} - t^{1/p}$. For any $\varepsilon > 0$ there is $\sigma > 0$ such that

$$
\frac{1}{q} + \frac{t}{p} - t^{1/p} > \begin{cases} 
\sigma & \text{for } t < 1 - \varepsilon; \\
\sigma t & \text{for } t > 1 + \varepsilon.
\end{cases}
$$

Therefore

$$
1 - \text{Re}\{T_{1/p+is}(g_j) 1, 1\}_{pq} > \sigma \int_0^{1-\varepsilon} d\kappa(g_j; M, M)(t) + \sigma \int_{1+\varepsilon}^\infty t \, d\kappa(g_j; M, M)(t).
$$
This must tend to 0, therefore $\kappa[g_j; M, M]$ and $t \cdot \kappa[g_j; M, M]$ tend to $\delta_0$ weakly. This implies the convergence $g_j' \to 1$ in the sense of $L^1$.

The remaining part of the proof is more-or-less automatic.

Step 2. Let $z$ be contained in the strip $0 \leq \Re z \leq 1$. Let us show that $(g_j')^z$ tends to 1 in the sense of $L^1$. Let $\|g' - 1\|_{L^1(M)} < \varepsilon$. Then there is an uniform estimate

$$\| (g_j')^z - 1 \|_{L^1(M)} < \psi_z(\varepsilon),$$

where $\psi_z(\varepsilon)$ tends to 0 as $\varepsilon$ tends to 0. For this aim it is sufficient to notice that

$$|a^z - 1| = \begin{cases} |z| (a - 1) & \text{for } a > 1; \\ |z| 2^{-\Re z + 1} |a - 1| & \text{for } 1/2 \leq a \leq 1; \\ 2 & \text{for } 0 < a < 1/2, \end{cases}$$

moreover, $g' < 1/2$ can be only on the set of measure $\leq 2\varepsilon$.

In particular, for any subset $C \subset M$ we have

$$\left| \int_C g'(x)^z dx - \mu(C) \right| \leq \psi_z(\varepsilon). \quad (2.6)$$

Step 3. Now we use convergence of matrix elements:

$$\left\{ T_{1/p+is}(g_j) \chi_A, \chi_B \right\}_{pq} = \int_{A \cap g_j^{-1}B} g_j'(x)^{1/p+is} d\mu(x) \to \left\{ \chi_A, \chi_B \right\}_{pq} = \mu(A \cap B).$$

By (2.6), we have convergence

$$\int_{A \cap g_j^{-1}B} g_j'(x)^{1/p+is} d\mu(x) - \mu(A \cap g_j^{-1}B) \to 0.$$

Comparing two last convergences we get $\mu(A \cap g_j^{-1}B) \to \mu(A \cap B)$.

Step 4. By the convergence $(g_j')^z$ in $L^1(M)$, we have

$$\int t^z d\kappa[g_j; A, B](t) = \int_{A \cap g_j^{-1}B} g'(x)^z dx \to \mu(A \cap B)$$

for each $z$; the point-wise convergence of characteristic functions implies weak converges to measures (see [13]), in our case, to $\mu(A \cap B) \delta_0$. □

Thus the topology on $G_{ms}$ is induced from the strong operator topology of the spaces $L^p$. In separable Banach spaces the multiplication is continuous in the strong topology on bounded sets. Therefore, the multiplication in $G_{ms}$ is continuous.

Lemma 2.5 Let operators $T_{1+is}(g_j)$ converge to 1 in the strong operator topology of spaces $L^1$. Then $g_j \to 1$ in $G_{ms}$. 

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Proof. In (2.4) the first row must tend to zero. Therefore all summands of the last row tend 0, in particular the first one. This implies weak convergences of measures \( \varepsilon [g_j; A, A] \) and \( tx[g_j; A, A] \) to \( \mu(A)\delta_1 \). Comparing this with (1.2), we get convergences \( \varepsilon [g_j; A, M \setminus A] \) and \( tx[g_j; A, M \setminus A] \) to 0. Now it is easy to derive the convergence of \( g_j \to g \) in Gms. \( \Box \)

2.4. The completeness of Gms. The group of isometries of a separable Banach space is a Polish group with respect to the bi-strong topology (\cite{4}, 9.3.9). Let \( p \neq 1, 2, \infty \), \( s = 0 \). Then the isometries \( T_{1/p}(g) \) are precisely isometries (2.1) that send the cone of non-negative functions to itself. Obviously, the set of operators sending this cone to itself is weakly closed. Therefore, Gms is a closed subgroup in the group of all isometries and therefore it is complete.

2.5. Bi-strong closeness of the image. The group Gms is closed in the group Isom\((L^p)\), since it is complete with respect of the induced topology.

It is noteworthy that the group Isom\((L^p)\) is not strongly closed in the space of bounded operators in \( L^p \). The images of the groups Ams and Gms also are not closed.

Example. Let \( p \neq \infty \). Consider an operator in \( L^p \) of the form

\[
Rf(x) = \begin{cases} 
  f(2x), & \text{for } 0 \leq x \leq 1/2; \\
  f(2x - 1), & \text{for } 1/2 < x \leq 1;
\end{cases}
\]

For any function \( f \) we have \( \|Rf\| = \|f\| \). However, this operator is not invertible. For the sequence \( g_n \in Ams \) from Fig. 1 we have the strong convergence \( T_{1/p}(g_n) \to R \).

\[ \Box \]

Weak closures for some subgroups Gms are discussed in [2], [3].

3 Double cosets

3.1. Proof of Theorem 1.6. Denote by \( G^0 \subset Gms \) the group of transformations whose Radon–Nikodym derivative has only finite number of values. Obviously,

• The subgroup \( G^0 \) is dense in Gms.

• Double cosets \( Ams \backslash G^0 / Ams \) are completely determined by the distribution of the Radon–Nikodym derivative.
Consider a measure \( \nu \in \mathcal{M} \). Consider a sequence of discrete measures \( \nu_N \in \mathcal{M} \) convergent to \( \nu \) and having the following property: Fix \( N \) and cut the semi-axis \( t > 0 \) into pieces of length \( 2^{-N} \). For any \( j \in \mathbb{N} \) we require the following coincidence of measures of semi-intervals

\[
\int_{\frac{j-1}{2^N} < t \leq \frac{j}{2^N}} d\nu(t) = \int_{\frac{j-1}{2^N} < t \leq \frac{j}{2^N}} d\nu_N(t),
\]

\[
\int_{\frac{j}{2^N} < t \leq \frac{j+1}{2^N}} t \cdot d\nu(t) = \int_{\frac{j}{2^N} < t \leq \frac{j+1}{2^N}} t \cdot d\nu_N(t).
\]

Consider \( g \in \text{Gms} \) whose distribution of the Radon–Nikodym derivative equals \( \nu \). Consider a sequence \( g_N \in G^0 \) convergent to \( g \) such that a distribution of the Radon–Nikodym derivative of \( g_N \) is \( \nu_N \). For this, we fix \( N \) and for each \( j \) consider the subset \( A_j \subset M \), where the Radon–Nikodym derivative satisfies

\[
\frac{j-1}{2^N} < g'(x) \leq \frac{j}{2^N}.
\]

Set \( B_j = g(A) \). Consider an arbitrary map \( g_N \in G^0 \) such that \( g_N \) sends \( A_j \) to \( B_j \) and the distribution of the Radon–Nikodym derivative of \( g_N \) coincides with the restriction of the measure \( \nu_N \) of the semi-interval \( \left( \frac{j-1}{2^N}, \frac{j}{2^N} \right) \). It is easy to see that the sequence \( g_N \) converges to \( g \).

Now, let \( f \) be a continuous function on \( \text{Gms} \) constant on double cosets. \( g \) and \( h \in \text{Gms} \) have same distribution of Radon–Nikodym derivatives. Then \( g_N \) and \( h_N \) are contained in the same double coset, wherefore \( f(g_N) = f(h_N) \). By continuity of \( f \) we get \( f(g) = f(h) \).

To avoid a proof of the continuity the map \( q \) (see the statement of the theorem), we refer to Proposition 1.7 (which is proved below independently of the previous considerations).

### 3.2. Proof of Proposition 1.7

Let \( M \simeq [0, 1] \) be a Lebesgue space. Invariants of measurable functions \( f : M \to \mathbb{R} \) with respect to the action of \( \text{Ams}(M) \) were described by Rokhlin in [10]. To any function \( f \) he assigns its distribution function \( F(y) \), i.e., the measure of the set \( M_y \subset M \) determined by the inequality \( f(x) < y \). Also he assigns to \( f \) a sequence of functions \( F_1, F_2, \ldots, F_n \), where \( F_n(y) \) is the supremum of measures of all sets \( A \subset M_y \), on which \( f \) takes each value \( \leq n \) times. These data satisfy the following conditions:

- the function \( F \) satisfies the usual properties of distribution functions: \( F \) is a left-continuous non-decreasing function, \( \lim_{y \to -\infty} f(y) = 0 \), \( \lim_{y \to +\infty} f(y) = 1 \);
- \( F_n \) are non-decreasing functions;
- \( 0 \leq F_1(y) \leq F_2(y) \leq \ldots \leq F(y) \);
- \( F_k(y) - 2F_{k+1}(y) + F_{k+2}(y) \geq 0 \) for all \( k \).

According [10], a function \( f \) determined up to the action of the group \( \text{Ams} \) is uniquely defined by the invariants \( F_1, F_2, \ldots, F \). Moreover, for any collection of functions \( F_1, F_2, \ldots, F \) with above listed properties there exists \( f \), whose invariants coincide with \( F_1, F_2, \ldots, F \).

Now we will describe canonical forms of functions \( f \) under the action of the group \( \text{Ams} \). Consider a collection of continuous measures \( \nu_1 \leq \nu_2 \leq \ldots \) on \( \mathbb{R} \).
and the measure $\nu_\infty$ on $\mathbb{R}$ such that $\nu_1(\mathbb{R}) + \nu_2(\mathbb{R}) + \cdots + \nu_\infty(\mathbb{R}) = 1$. Denote by $t$ the coordinate on $\mathbb{R}$. Consider the disjoint union of the spaces with measures

$$
\mathcal{L} = \left( (\mathbb{R}, \nu_1) \coprod (\mathbb{R}, \nu_2) \coprod \cdots \right) \coprod (\mathbb{R} \times [0,1], \nu_\infty \times ds),
$$

where $ds$ is the Lebesgue measure on the segment $[0,1]$. Consider the function $f$ on $\mathcal{L}$ that equals to $t$ on each copy of $\mathbb{R}$ and equals to $t$ on $\mathbb{R} \times [0,1]$. The invariants of this function are

$$
F_\nu(y) = \sum_{j \leq n} \nu_j(-\infty, y), \quad F(y) = \sum_{1 \leq j < \infty} \nu_j(-\infty, y) + \nu_\infty(-\infty, y)
$$

It can be readily seen that measures $\nu_1, \nu_2, \ldots, \nu_\infty$ admit a reconstruction from the invariants $F_1, F_2, \ldots, F$. Moreover any admissible collection of invariants corresponds to a certain collection of measures $\nu_1, \nu_2, \ldots, \nu_\infty$.

Now consider an element $g \in \mathrm{Gms}(P, R)$. Reduce the derivative $g' : P \rightarrow \mathbb{R}^\times$ to the canonical form by a multiplication $g \mapsto gh$, where $h \in \mathrm{Ams}$. Since $g'(x) > 0$, all the measures $\nu_j, \nu$ are supported by the half-line $t > 0$. The integral of $g'$ is $1$, therefore

$$
\sum_j \int t \, d\nu_j(t) + \int t \, d\nu_\infty(t) = 1. \tag{3.2}
$$

Now we assume $P = \mathcal{L}$, see (3.1). Let $\mathcal{L}_s$ be obtained from $\mathcal{L}$ by a multiplication of the measure by $t$. In virtue of (3.2), this measure must be probabilistic. The map $g : \mathcal{L} \rightarrow R$ can be regarded as a map $g_s : \mathcal{L}_s \rightarrow R$. Since $g' = t$, for any measurable set $B \subset \mathcal{L}$ the measure of $B$ in $\mathcal{L}_s$ coincides with the measure $g(B)$. Therefore $g_s : \mathcal{L}_s \rightarrow R$ preserves measure.

Thus $g$ is reduced to the canonical form.

### 3.3. Splitting of measures.

We start a proof of Theorem 1.5. Modify the notation for $\mathcal{L}[\nu_1, \nu_2, \ldots; \nu_\infty], \mathcal{L}_s[\nu_1, \nu_2, \ldots; \nu_\infty]$ and $S[\nu_1, \nu_2, \ldots; \nu_\infty]$ from (1.3). Now it is convenient to reject the condition $\nu_1 \geq \nu_2 \geq \ldots$. Also, we weaken condition (1.3) and set

$$
\int_{\mathbb{R}} t \, d\nu(t) < \infty, \quad \int_{\mathbb{R}} t \, d\nu(t) < \infty. \tag{3.3}
$$

Let $\nu$ be a continuous measure on $\mathbb{R}^\times$ satisfying (3.3). Consider the space $\mathcal{L}[\nu, 0, 0, \ldots; 0]$. Represent $\nu$ as a sum $\nu = \nu_1 + \nu_2$.

**Lemma 3.1** The closure of the class $S[\nu, 0, 0, \ldots; 0]$ contains $S[\nu_1, \nu_2, 0, \ldots; 0]$.

**Proof.** Denote

$$
\mathcal{L} := \mathcal{L}[\nu_1, 0, \ldots; 0], \quad \mathcal{L}' := \mathcal{L}[\nu_1, \nu_2, 0, \ldots; 0].
$$

The same measure spaces with the measure multiplied by $t$ we denote as $\mathcal{L}_s, \mathcal{L}_s'$.

Now we will construct two sequences of measure preserving bijections

$$
\phi_n : \mathcal{L} \rightarrow \mathcal{L}', \quad \psi_n : \mathcal{L}_s \rightarrow \mathcal{L}_s'.
$$

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Cut \((\mathbb{R},\nu)\) by \(2^n\) intervals \(C_0,\ldots,C_{2^n-1}\) by points

\[ a_k = \frac{k2^{-n}}{1-k2^{-n}}, \quad k = 1, 2, \ldots, 2^n - 1. \]

Denote this partition\(^4\) by \(\mathfrak{h}_n\).

**Lemma 3.2** There exists a sub-interval \(B_k \subset C_k\) such that \(\nu(B_k) = \nu_1(C_k)\), 
\((t \cdot \nu)(B_k) = (t \cdot \nu_1)(C_k)\).

**Proof.** We have \(C_k = [a_k, a_{k+1}]\). Consider segments \([a_k, u], [v, a_{k+1}] \subset [a_k, a_{k+1}]\) such that \(\nu'[a_k, u] = \nu'[v, a_{k+1}] = \nu_1[C_k]\). For any \(z \in [a_k, v]\) there exists \(z^o \leq a_{k+1}\) such that \(\nu'[z, z^o] = \nu_1[C_k]\). It is easy to see that 
\[(t \cdot \nu)[a_k, u] \leq \nu_1[C_k], \quad (t \nu)[v, a_{k+1}] \geq (t \cdot \nu_1)[C_k].\]

Form continuity reasoning there exists \([z, z^o]\) satisfying the desired property. \(\square\)

For each \(k\) consider arbitrary measure preserving maps 
\((B_k, \nu) \rightarrow (C_k, \nu_1), \quad (C_k \setminus B_k, \nu_2) \rightarrow (C_k, \nu_2)\).

This produces a map \(\varphi_n\) (see. Fig.2). To obtain \(\psi_n\) we take arbitrary measure preserving maps 
\((B_k, t \cdot \nu) \rightarrow (C_k, t \cdot \nu_1), \quad (C_k \setminus B_k, t \cdot \nu_2) \rightarrow (C_k, t \cdot \nu_2)\).

Consider a map

\[ \theta_n : \psi_n \circ \text{id} \circ \varphi^{-1}_n : \mathcal{L}' \rightarrow \mathcal{L}'_n. \]

The space \(\mathcal{L}'\) consists of two copies \(\mathbb{R}_1^+, \mathbb{R}_2^+\) of the half-line \(\mathbb{R}^+\), each copy is cutted into segments \(C_k\). The map \(\theta_n\) send each copy of a segment \(C_k \subset \mathbb{R}_1^+, C_k \subset \mathbb{R}_2^+\) to itself, moreover the Radon–Nikodym derivative of \(\theta_n\) takes values \(C_k\) in limits \([a_k, a_{k+1}]\).

It is easy to see that the sequence \(\theta_n\) converges to the map \(\text{id} : \mathcal{L}' \rightarrow \mathcal{L}'_n\). \(\square\)

---

\(^4\)The only necessary for us property of partition is the following: a diameter of a partition on any finite interval \((0, M]\) tends to 0 as \(n \to \infty\).
3.4. The spreading of measures. Denote
\[ L := L[\nu, 0, \ldots, 0], \quad L'' = L[0, 0, \ldots; \nu]. \]
Let \( L_*, L_*'' \) be the same measure spaces with the measure multiplied by \( t \). We construct a sequence of measure preserving bijections
\[ \xi_n : L \to L'', \quad \zeta_n : L_* \to L_*''. \]
For this aim, consider the same partitions \( \eta_n \) of the space \( (R^\times, \nu) \). Consider arbitrary measure preserving maps
\[ (C_k, \nu) \to (C_k \times [0, 1], \nu \times dx), \quad (C_k, t\nu) \to (C_k \times [0, 1], (t\nu) \times dx) \]
This gives us the maps \( \xi_n, \zeta_n \). Consider the map
\[ u_n = \zeta_n \circ \text{id} \circ \xi_n^{-1} : L'' \to L''. \]
The map \( u_n \) sends each \( C_k \times [0, 1] \) to itself, its Radon–Nikodym derivative on \( C_k \times [0, 1] \) varies in the limits \( [a_{k-1}, a_k] \). Passing to a limit as \( n \to \infty \), we get the identity map \( L'' \to L''_* \).

3.5. Proof of Theorem 1.5 \( \nu \in M \). Without loss of generality, we can assume that \( \nu \) is continuous. Expand \( \nu = \nu_1 + \nu_2 + \cdots + \nu_\infty \). Set
\[ \mathcal{L}^k = L[\nu_1, \ldots, \nu_k, \sum_{j=k+1}^{\infty} \nu_j, 0, 0, \ldots; \nu_\infty], \quad \mathcal{L}^\infty := L[\nu_1, \nu_2, \ldots; \nu_\infty]. \]

Let \( \mathcal{L}^k_*, \mathcal{L}^\infty_* \) be the same measure spaces with measures multiplied by \( t \). Let \( \text{id}^k : \mathcal{L}^k \to \mathcal{L}^k_*, \text{id}^\infty : \mathcal{L}^\infty \to \mathcal{L}^\infty_* \) denote the identical maps.

Iterating arguments of the two previous subsections, we obtain that the closure of \( S[\nu, 0, \ldots, 0] \) contains elements \( \text{id}^k \) for any finite \( k \). Consider a map \( \alpha_k : \mathcal{L}^k \to \mathcal{L}^\infty \) constructed in the following way. It is identical on \( \mathbb{R}^\times_1, \ldots, \mathbb{R}^\times_k \) send the semi-line \( \mathbb{R}^\times_{k+1} \) to \( \bigcup_{j \geq k+1} \mathbb{R}_j \) preserving the measure. In the same way we construct a map \( \beta_k : \mathcal{L}^k_* \to \mathcal{L}^\infty_* \). It is easy to see that the sequence
\[ \chi_k := \beta_k \circ \text{id}_k \circ \alpha_k^{-1} : \mathcal{L}^\infty \to \mathcal{L}^\infty_* \]
converges to \( \text{id}^\infty \).

3.6. Construction of the function \( \psi \). Here we obtain the continuous section \( M \to \text{Gms} \). Consider the distribution function \( z = F(y) \) of the measure \( \nu \) and the inverse function \( y = G(z) \). If \( y_0 \) is a discontinuity point of \( F \), we set \( G(z) = y_0 \) on the segment \( [F(y_0) - 0), F(y_0) + y_0) \). If \( F \) takes some value \( z_0 \) on a segment of nonzero length, then \( G(z_0) \) is not defined. Further, we set \( \psi(x) = \int_0^x G(z) \, dz \).

\( ^5 \)Recall that any two Lebesgue spaces with continuous probabilistic measures are equivalent, see e.g., [1].
Figure 3: To proof of Proposition 1.7: \((u, v) = H(s), (u_j, v_j) = H_j(s)\). We mark an interval of possible values of \(\psi_j(u)\).

3.7. Proof of Proposition 1.7. Let \(\nu_j\) converges to \(\nu\) in \(\mathcal{M}\), \(y = \psi_j(x)\), \(y = \psi(x)\) be the corresponding maps \([0, 1] \to [0, 1]\). We must prove that \(\psi_j\) converges to \(\psi\) in \(\mathcal{G}_{ms}\).

1) Let \(\nu \in \mathcal{M}\). Consider the map \(\mathbb{R}^\infty \to [0, 1] \times [0, 1]\) given by the formula

\[
H : s \mapsto \left(\nu([0, s]), (t \cdot \nu)(0, s)\right).
\]

It easy to see that we get the graph of the functions \(\psi\), from which we remove all straight segments. The convergence \(\nu_j \to \nu\) means the point-wise convergence of the maps \(H_j(s) \to H(s)\). From this it is easy to derive that \(\psi_j\) converges to \(\psi\) point-wise (See Fig. 3). In virtue of monotonicity and continuity of our functions, the point-wise convergence implies the uniform convergence.

2) Let us show that derivatives \(\psi'_j\) converge \(\psi'\) a.s. Take a point \(a\), where all derivatives \(\psi'_j(a), \psi'(a)\) are defined. Let \(\ell_j, \ell\) - be tangent lines to graphs of \(\psi_j, \psi\) at \(a\). Suppose that \(\psi'_j(a)\) does not converge to \(\psi'(a)\). Choose a subsequence \(\psi'_{n_k}(a)\) convergent to \(\alpha \neq \psi'(a)\). Consider the limit line \(\ell_n\), i.e.,

\[
\ell^\alpha : \; y = \alpha(x - a) + \psi(a)
\]

It is easy to see (for more details, see [14], Addendum, §6) that the graph \(y = \psi(x)\) is located upper this line. I.e., \(\ell^\alpha\) is the second supporting line at \(a\) (the first one was the tangent line), this contradicts to the existence of \(\psi'(a)\).

3) Now we prove a weak convergence of operators \(T_{1/2}(\psi_j) \in L^2[0, 1]\). Let \(f, h\) be continuous functions. We must check that the following expressions approach zero

\[
\left| \int_0^1 f(\psi_j(x))\psi'_j(x)^{1/2}h(x) \, dx - \int_0^1 f(\psi(x))\psi'(x)^{1/2}h(x) \, dx \right| \leq \int_0^1 \left| f(\psi_j(x)) - f(\psi(x)) \right| \psi'_j(x)^{1/2}h(x) \, dx + \int_0^1 \left| f(\psi(x))(\psi'_j(x)^{1/2} - \psi'(x)^{1/2})h(x) \right| \, dx
\]

\[\leq \int_0^1 \left| f(\psi_j(x)) - f(\psi(x)) \right| \psi'_j(x)^{1/2}h(x) \, dx + \int_0^1 \left| f(\psi(x))(\psi'_j(x)^{1/2} - \psi'(x)^{1/2})h(x) \right| \, dx\]
In (3.4) the convergence $f(\psi_j(x)) \to f(\psi(x))$ is uniform and
\[
\int_0^1 \psi_j^{1/2}(x) \leq \int_0^1 (\psi_j^{1/2})^2(x) = 1.
\]
By the Fatou Lemma, (3.4) tends to zero. Further notice that for functions $\psi \in G$ we have a priori estimation
\[
\psi'(x) \leq \frac{1 - \psi(x)}{1 - x} \leq \frac{1}{1 - x}.
\]
Hence the convergence in the integral (3.5) is dominated on each segment $[0, 1 - \varepsilon]$. This implies that integrals $\int_0^{1-\varepsilon} (\ldots)$ approach zero. Further, denote $C = \left(\max |f(x)| \cdot \max |g(x)|\right)$,
\[
\int_{1-\varepsilon}^1 (\ldots) \leq C \int_{1-\varepsilon}^1 (\psi_j'(x) + \psi'(x))^2 dx = \varepsilon C \left[(1 - \psi_j(1 - \varepsilon)) + (1 - \psi(1 - \varepsilon))\right]
\]
and this value is small for small $\varepsilon$. □

3.8. **Proof of Theorem 1.8.** Cut $M$ into pieces $A_{ij} := M_i \cap g^{-1}M_j$, and also into pieces $B_{ij} = gM_{ij} = g(M_i) \cap M_j$. We get a collection of maps $A_{ij} \to B_{ij}$. Now the question is reduced to a canonical form of each map.

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