COUNTING ESSENTIAL SURFACES IN A CLOSED
HYPERBOLIC THREE MANIFOLD

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Abstract. Let $M^3$ be a closed hyperbolic three manifold. We show that the number of genus $g$ surface subgroups of $\pi_1(M^3)$ grows like $g^{2g}$.

1. Introduction

Let $M^3$ be a closed hyperbolic 3-manifold and let $S_g$ denote a closed surface of genus $g$. Given a continuous mapping $f : S_g \to M^3$ we let $f_* : \pi_1(S_g) \to \pi_1(M^3)$ denote the induced homomorphism.

Definition 1.1. We say that $G < \pi_1(M^3)$ is a surface subgroup of genus $g \geq 2$ is there exists a continuous map $f : S_g \to M^3$ such that the induced homomorphism $f_*$ is injective and $f_*(\pi_1(S_g)) = G$. Moreover, the subsurface $f(S_g) \subset M^3$ is said to be an essential subsurface.

Recently, we showed [4] that every closed hyperbolic 3-manifold $M^3$ contains an essential subsurface and consequently $\pi_1(M^3)$ contains a surface subgroup. It is therefore natural to consider the question: How many conjugacy classes of surface subgroups of genus $g$ there are in $\pi_1(M^3)$? This has already been considered by Masters [5], and our approach to this question builds on our previous work and improves on the work by Masters.

Let $s_2(M^3, g)$ denote the number of conjugacy classes of surface subgroups of genus at most $g$. We say that two surface subgroups $G_1$ and $G_2$ of $\pi_1(M^3)$ are commensurable if $G_1 \cap G_2$ has a finite index in both $G_1$ and $G_2$. Let $s_1(M^3, g)$ denote the number surface subgroups of genus at most $g$, modulo the equivalence relation of commensurability. Then clearly $s_1(M^3, g) \leq s_2(M^3, g)$. The main result of this paper is the following theorem.

Theorem 1.1. Let $M^3$ be a closed hyperbolic 3-manifold. There exist two constants $c_1, c_2 > 0$ such that

$$(c_1 g)^{2g} \leq s_1(M^3, g) \leq s_2(M^3, g) \leq (c_2 g)^{2g},$$

for $g$ large enough. The constant $c_2$ depends only on the injectivity radius of $M^3$.

In fact, Masters shows that

$$s_2(g, M^3) < g^{c_2 g}$$

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for some $c_2 \equiv c_2(M^3)$, and likewise for some $c_1 \equiv c_1(M^3)$

$$g^{f_1g} < s_1(g, M^3)$$

when $M^3$ has a self-transverse totally geodesic subsurface. We follow Masters’ approach to the upper bound, improving it from $g^{f_2g}$ to $(c_2g)^{2g}$ by more carefully counting the number of suitable triangulations of a genus $g$ surface. Using our previous work [4] we replace Masters’ conditional lower bound with an unconditional one, and we improve it from $g^{f_2g}$ to $(c_1g)^{2g}$ with the work of Muller and Puchta [5] counting number of maximal surface subgroups of a given surface group. We then make new subgroup from old in the spirit of Masters’ construction, but taking the nearly geodesic subgroup from [4] as our starting point.

The above theorem enables us to determine the order of the number of surface subgroups up to genus $g$. We have the following corollary.

**Corollary 1.1.** We have

$$\lim_{g \to \infty} \frac{\log s_1(M^3, g)}{2g \log g} = \lim_{g \to \infty} \frac{\log s_2(M^3, g)}{2g \log g} = 1.$$

We make the following conjecture.

**Conjecture 1.1.** For a given closed hyperbolic 3-manifold $M^3$, there exists a constant $c(M) > 0$ such that

$$\lim_{g \to \infty} \frac{1}{g} 2^{g/s_i(M^3, g)} = c(M), \ i = 1, 2.$$  

2. **The upper bound**

Fix a closed hyperbolic 3-manifold $M^3$. In this section we prove the upper bound in Theorem 1.1, that is we show

(1)

$$s_2(M^3, g) \leq (c_2g)^{2g},$$

for some constant $c_2 > 0$.

2.1. **Genus $g$ triangulations.** We have the following definition.

**Definition 2.1.** Let $S_g$ denote a closed surface of genus $g$. We say that a connected graph $\tau$ is a triangulation of genus $g$ if it can be embedded into the surface $S_g$ such that every component of the set $S_g \setminus \tau$ is a triangle. The set of genus $g$ triangulations is denoted by $T(g)$. We say that $\tau \in T(k, g) \subset T(g)$ if:

- each vertex of $\tau$ has the degree at most $k$,
- the graph $\tau$ has at most $kg$ vertices and edges.

We observe that any given genus $g$ triangulation $\tau$, can be in a unique way (up to a homeomorphism of $S_g$) be embedded in $S_g$.

We say that Riemann surface is $s$-thick is its injectivity radius is bounded below by $s > 0$. Every thick Riemann surface has a good triangulation.
Lemma 2.1. Let $S$ be an $s$-thick Riemann surface of genus $g \geq 2$. Then there exists $k = k(s) > 0$ and a triangulation $\tau \in T(k, g)$ that embeds in $S$, such that

1. Every edge of $\tau$ is a geodesic arc of length at most $s$,
2. The triangulation $\tau$ has at most $kg$ vertices and edges,
3. The degree of each vertex is at most $k$.

Proof. Choose a a maximal collection of disjoint open balls in $S$ of radius $s/4$. Let $V$ denote the set of centers of the balls from the collection. We may assume that no four points from $V$ lie on a round circle (we always reduce the radius of the balls by a small amount and move them into a general position). We construct the Delaunay triangulation associated to the set $V$ as follows. We connect two points from $V$ with the shortest geodesic arc between them, providing they belong to the boundary of a closed ball in $S$ that does not contain any other point from $S$. This gives an embedded graph $\tau$. Since no four points from $V$ lie on the same circle the graph $\tau$ is a triangulation. It is elementary to check that $\tau$ has the stated properties, and we leave it to the reader. \qed

Given any injective immersion of $g : S_g \to M^3$, we can find a genus $g$ hyperbolic surface $S$, and a map $f : S \to M^3$ homotopic to $g$, such that $f(S)$ is a pleated surface. Then $f$ does not increase the hyperbolic distance. Let $s$ denote the injectivity radius of $M^3$. It follows that the injectivity radius of $S$ is bounded below by $s$. We choose a triangulation $\tau(S)$ of $S$ that satisfies the conditions in Lemma 2.1.

Let $C = \{C_1, ..., C_m\}$ be a finite collection of balls of radius $s/4$ that covers $M^3$. We may assume that $C$ is a minimal collection, that is, if we remove a ball from $C$, the new collection of balls does not cover $M^3$. Let $f_i : S_i \to M^3$, $i = 1, 2$, be two pleated maps, and denote by $\tau(S_1)$ and $\tau(S_2)$ the corresponding triangulations of genus $g$ surfaces $S_1$ and $S_2$. If the genus $g$ triangulations $\tau(S_1)$ and $\tau(S_2)$ are identical, there exists a homeomorphism $h : S_1 \to S_2$ such that $h(\tau(S_1)) = \tau(S_2)$. Assume in addition that for every vertex $v$ of $\tau(S_1)$, the points $f_1(v)$ and $f_2(h(v))$ belong to the same ball $C_i \in C$. Then by Lemma 2.4 in [5], the maps $f_1$ and $f_2$ are homotopic.

Since the set $C$ has $m$ elements, there are at most $m$ ways of mapping a given vertex of $\tau$ to the set $C$. Choose a vertex $v_1$ of $\tau$ and choose an image of $v_1$ in $C$, say $v_1$ is mapped to $C_1$. Let $v_1$ be a vertex of $\tau$, such that $v_0$ and $v_1$ are the endpoints of the same edge. Since each edge of $\tau$ has the length at most $s$, and the balls from $C$ have the radius $s/4$. Since $f$ does not increase the distance, and $C$ is a minimal cover of $M^3$, it follows that $v_1$ can be mapped to at most $K$ elements of $C$, where $K$ is a constant that depends only on $s$. Repeating this analysis yields the following estimate:

\[
\tilde{s}_2(M^3, g) \leq mK^{g-1}|T(k, g)|,
\]
where \( \tilde{s}_2(M^3, g) \) denotes the number of conjugacy classes of surface subgroups of genus equal to \( g \).

Let \( \nu(k, n) \) denote the set of all graphs on \( n \) vertices so that each vertex has the degree at most \( k \). Then \(|T(k, g)| \leq |\nu(k, kg)|\).

Remark. Observing the estimate \(|\nu(k, n)| \leq n^{kn}\), Masters showed \( \tilde{s}_2(M^3, g) \leq g^{Dg} \), for some constant \( D > 0 \). However, the set \( \nu(k, kg) \) has many more elements than the set \( T(k, g) \).

The following lemma will be proved in the next subsection.

**Lemma 2.2.** There exists a constant \( C > 0 \) that depends only on \( k \), such that for \( g \) large we have

\[ |T(k, g)| \leq (C g)^{2g}. \]

Given this lemma we now prove estimate (1). It follows from the Lemma 2.2 that for every \( g \) large we have

\[ |T(k, g)| \leq (C g)^{2g}. \]

Combining this with (2) we get

\[ \tilde{s}_2(M^3, g) \leq m K^{k g - 1} (C g)^{2g} \leq (C_1 g)^g, \]

holds for every \( g \geq 2 \), for some constant \( C_1 \). Then

\[ s_2(M^3, g) = \sum_{r=2}^{g} \tilde{s}_2(M^3, r) = \sum_{r=2}^{g} (C_1 r)^{2r} \leq (c_2 g)^{2g}, \]

for some constant \( c_2 \). This proves the estimate (1).

**2.2. The proof of Lemma 2.2.** Fix a triangulation \( \tau \in T(k, g) \) and denote the set of oriented edges by \( E(\tau) \). Let \( QE(\tau) \) denote the vector space of all formal sums (with rational coefficients) of edges from \( E(\tau) \).

Choose a spanning tree \( T \) (a spanning tree of a connected graph is a connected tree that contains all of its vertices) for \( \tau \). Let \( H_1(S_g) \) denote the first homology with rational coefficients of the surface \( S_g \). We define the linear map \( \phi : QE(\tau) \to H_1(S_g) \) as follows. Let \( e \in (E(\tau) \setminus T) \). Then the union \( e \cup T \) is homotopic (on \( S_g \)) to a unique (up to homotopy) simple closed curve \( \gamma_e \subset S_g \). We let \( \phi(e) \) denote the homology class of the curve \( \gamma_e \) in \( H_1(S_g) \). We extend the map \( \phi \) to \( QE(\tau) \) by linearity.
Denote the kernel of $\phi$ by $K(\phi)$ and set

$$H_1(\tau, T) = \frac{\mathbb{Q}E(\tau)}{K(\phi)}.$$ 

Then the quotient map (also denoted by) $\phi : H_1(\tau, T) \to H_1(S_g)$ is injective, and in fact it an isomorphism. Since $\tau$ is a genus $g$ triangulation, the embedding of the triangulation $\tau$ to $S_g$ induces the surjective map of the fundamental group of $\tau$ to the fundamental group of $S_g$. Then the induced map $\phi$ between the corresponding homology groups is injective.

Let $e_1, ..., e_{2g} \in E(\tau)$ denote a set of $2g$ edges whose equivalence classes generate $H_1(\tau, T)$.

Lemma 2.3. Let $X = T \cup \{e_1, ..., e_{2g}\}$. Then every component of the set $S_g \setminus X$ is simply connected.

Proof. The set $X$ is connected (since it contains the spanning tree $T$, and the tree $T$ contains all the vertices). Suppose that there exists a component of the set $S_g \setminus X$ that is not simply connected. Then there exists a simple closed curve $\gamma \subset S_g$ that is not homotopic to a point, and such that

$$\gamma \cap X = \emptyset.$$ 

If $\gamma$ is a non-separating curve then the homology class of $\gamma$ is non-trivial in $H_1(S_g)$. Therefore, there exists a non-separating simple closed $\alpha \subset S_g$ that intersects the curve $\gamma$ exactly once. Let $q_1, ..., q_{2g} \in \mathbb{Q}$ be such that

$$\phi(q_1e_1 + ... + q_{2g}e_{2g}) = [\alpha],$$

where $[\alpha] \in H_1(S_g)$ denotes the homology class of $\alpha$. Since the intersection pairing between $[\alpha]$ and $[\gamma]$ is non-zero, and $\phi(e_1), ..., \phi(e_{2g})$ is a basis for $H_1(S_g)$, we conclude that for some $i \in \{1, ..., 2g\}$, the curve $\gamma$ intersects $e_i \cup T$, which is a contradiction.

Suppose that $\gamma$ is a separating curve and denote by $A_1$ and $A_2$ the two components of the set $S_g \setminus \gamma$. The set $X$ is connected, and by the assumption it does not intersect $\gamma$. This implies that $X$ is contained in one of the two sub-surfaces $A_i$, say $X \subset A_1$. Then $X \cap A_2 = \emptyset$.

Since $\gamma$ is not homotopic to a point, each $A_i$ is a non-planar surface with one boundary component. Therefore, the subsurface $A_2$ contains a non-separating simple closed curve $\gamma_2$. Then $\gamma_2$ is a non-separating simple closed curve in $S_g$ by the above argument we have that $\gamma_2$ intersects the set $X$. This is a contradiction since $X \cap A_2 = \emptyset$.

□

Let $P_1, ..., P_l$ denote the components of the set $S_g \setminus X$. Each $P_i$ is a polygon and we let $m_i$ denote the number of sides of the polygon $P_i$. Since each edge in $X$ can appear as a side in at most two such polygons, we have the inequality
(3) \[ \sum_{i=1}^{l} m_i \leq 2kg, \]

since by definition the triangulation \( \tau \) has at most \( kg \) edges.

We proceed to prove Lemma 2.2. We can obtain every triangulation \( \tau \in T(k,g) \) as follows. We first choose a spanning tree \( T \), which is a tree that has at most \( kg \) vertices. Then to the tree \( T \) we add \( 2g \) edges \( e_1, \ldots, e_{2g} \) in an arbitrary way. After adding the edges, at each vertex of the graph \( T \cup \{e_1, \ldots, e_{2g}\} \) we choose a cyclic ordering. We thicken the edges of the graph \( T \cup \{e_1, \ldots, e_{2g}\} \) to obtain the ribbon graph and the corresponding surface \( R \) with boundary (if this surface does not have genus \( g \) we discard this graph). The boundary components of the surface \( R \) are polygonal curves \( P_i, i = 1, \ldots, l \), made out of the edges from \( T \cup \{e_1, \ldots, e_{2g}\} \). We then choose a triangulation of each polygon \( P_i \).

It follows from this description that we can bound the number of triangulations from \( T(k,g) \) by \( |T(k,g)| \leq abcd \), where

\[
\begin{align*}
  a &= \text{number of unlabelled trees } T \text{ with } n \leq kg \text{ vertices}, \\
  b &= \text{number of ways of adding } 2g \text{ unlabelled edges } e_1, \ldots, e_{2g} \text{ to } T, \\\n  c &= \text{number of cyclic orderings of edges of } T \cup \{e_1, \ldots, e_{2g}\}, \\
  d &= \text{number of triangulations of the polygons } P_i. 
\end{align*}
\]

Let \( t(n) \) denote the number of different unlabelled trees on \( n \) vertices. By \([1]\) we have \( t(n) \leq C12^n \), for some universal constant \( C > 0 \). It follows that \( a \leq 2C12^{kg} \). The tree \( T \) has at most \( kg \) edges, so there are at most \((kg)^2\) ways of adding a labelled edge to \( T \). All together there are at most \((kg)^{4g}\) ways of adding a labelled collection of \( 2g \) edges to \( T \). To obtain the number of ways of adding unlabelled collection of \( 2g \) edges we need to divide this number by \((2g)!\). This yields the estimate

\[
b \leq \frac{(kg)^{4g}}{(2g)!} < (k^2g)^{2g},
\]

for \( g \) large.

Since each vertex of \( \tau \) has the degree at most \( k \), and \( \tau \) has at most \( kg \) edges, we obtain the estimate

\[
c \leq (k!)^{kg}.
\]

Let \( p(m) \) denote the number of triangulations of a polygon with \( m \) sides. Then \( p(m) \) is the \((m - 2)\)-th Catalan number and we have \( p(m) < 2^{2m} \). As
above, let \( P_1, \ldots, P_l \) denote the polygons that we need to triangulate and let \( m_i \) denote the number of sides of the polygon \( P_i \). Then

\[
d \leq \max \Pi_{i=1}^l p(m_i) \leq \max \leq 4^{m_1 + \ldots + m_l},
\]

where the maximum is taken over all possible vectors \((m_1, \ldots, m_l)\), \(1 \leq l \leq 2kg\), such that \( m_1 + \ldots + m_l \leq 2kg\) (see estimate (3) above). But since \( m_1 + \ldots + m_l \leq 2kg\) we have \( d \leq 4^{2kg}\).

Putting the estimates for \( a, b, c, d \) together we prove the lemma.

**Remark.** If we are given a tree on a surface \( S \), along with \( 2g \) edges connecting the vertices of the tree (and satisfying the hypothesis of Lemma 2.3) and a map of the resulting graph into \( M^3 \), the we can determine the map of \( S \) into \( M^3 \), up to homotopy. Thus we need only bound \( |\mathcal{T}'(k, g)| \), where \( \mathcal{T}'(k, g) \) is the set of trees of size at most \( kg \), with \( 2g \) more edges added; we observe that \( |\mathcal{T}'(k, g)| < ab \).

3. Quasifuchsian representations of surface groups

3.1. Generalized pants decomposition and the Complex Fenchel-Nielsen coordinates. For background on complex Fenchel-Nielsen coordinates see [8], [3], [7], [4]. The exposition and notation we use here is in line with Section 2 in [4].

Let \( X \) a compact topological surface (possibly with boundary) and let \( \rho : \pi_1(X) \to \text{PSL}(2, \mathbb{C}) \) be a representation (a homomorphism). We say that \( \rho \) is a \( K \)-quasifuchsian representation if the group \( \rho(\pi_1(X)) \) is \( K \)-quasifuchsian, in which case we can equip \( X \) with a complex structure \( X = \mathbb{H}^2/F \), for some Fuchsian group \( F \), such that \( f_* = \rho \circ \iota \). Here \( \iota : F \to \pi_1(X) \) is an isomorphism, and \( f_* : F \to fFf^{-1} \) is the conjugation homomorphism, induced by an equivariant \( K \)-quasiconformal map \( f : \partial \mathbb{H}^3 \to \partial \mathbb{H}^3 \).

We will also say that a quasisymmetric map \( f : \partial \mathbb{H}^2 \to \partial \mathbb{H}^3 \) is \( K \)-quasiconformal if it has a \( K \)-quasiconformal extension to \( \partial \mathbb{H}^3 \).

By \( \Pi \) we denote a topological pair of pants with cuffs \( C_i \), \( i = 1, 2, 3 \). Recall that to every representation \( \rho : \pi_1(\Pi) \to \text{PSL}(2, \mathbb{C}) \), we associate the three half lengths \( \text{hl}(C_i) \in \mathbb{C}_+ / 2i\pi\mathbb{Z} \), where \( \mathbb{C}_+ = \{ z \in \mathbb{C} : \text{Re}(z) > 0 \} \). If \( \rho \) is quasifuchsian then it is uniquely determined by the half lengths. The conjugacy class \( [\rho] \) of a quasifuchsian representation \( \rho \) is called a skew pair of pants.

We let \( \Pi \) and \( \Pi' \) denote two pairs of pants and let \( \rho : \pi_1(\Pi) \to \text{PSL}(2, \mathbb{C}) \) and \( \rho' : \pi_1(\Pi') \to \text{PSL}(2, \mathbb{C}) \) denote two representations. Suppose that for some \( c_1 \in \pi_1(\Pi) \) and \( c'_1 \in \pi_1(\Pi') \), that belong to the conjugacy classes of \( C_1 \) and \( C'_1 \) respectively, we have \( \rho(c_1) = \rho'(c'_1) \), and \( \text{hl}(C_1) = \text{hl}(C'_1) \). By \( s(C) \in \mathbb{C} / \text{hl}(C)\mathbb{Z} + 2i\pi\mathbb{Z} \) we denote the reduced twist-bend parameter, which measures how the two skew pairs of pants \([\rho]\) and \([\rho']\) align together along the axis of the loxodromic transformation \( \rho(c_1) = \rho'(c'_1) \).
A pair \((\tilde{\Pi}, \chi)\) is a generalized pair of pants if \(\tilde{\Pi}\) is a compact surface with boundary and \(\chi\) is a finite degree covering map \(\chi : \tilde{\Pi} \to \Pi\), where \(\Pi\) is a pair of pants. (We will also call \(\tilde{\Pi}\) a generalized pair of pants if \(\chi\) is understood.) By \(\chi_* : \pi_1(\tilde{\Pi}) \to \pi_1(\Pi)\) we denote an induced homomorphism.

**Definition 3.1.** Let \((\tilde{\Pi}, \chi)\) be a generalized pair of pants and 

\[
\tilde{\rho} : \pi_1(\tilde{\Pi}) \to \text{PSL}(2, \mathbb{C}),
\]

be a representation. We say that \(\tilde{\rho}\) is admissible with respect to \(\chi\) if it factors through \(\chi_*\), that is there exists \(\rho : \pi_1(\Pi) \to \text{PSL}(2, \mathbb{C})\) such that \(\tilde{\rho} = \rho \circ \chi_*\).

Let \(\tilde{C}_j\), \(j = 1, \ldots, k\), denote the cuffs (the boundary curves) of the surface \(\tilde{\Pi}\), and let \(C_1, C_2, C_3\) continue to denote the cuffs of \(\Pi\). Then \(\chi\) maps each \(\tilde{C}_j\) onto some \(C_i\) with some degree \(m_j \in \mathbb{N}\). We say that such a curve \(\tilde{C}_j\) is a *degree \(m_j\) curve.* For every admissible \(\tilde{\rho}\) we define the half length \(\text{hl}(\tilde{C}_j)\) as \(\text{hl}(\tilde{C}_j) = \text{hl}(C_i)\). Let \(\tilde{c}_j \in \pi_1(\tilde{\Pi})\) be in the conjugacy class that corresponds to the cuff \(\tilde{C}_j\). Then

\[
1(\tilde{\rho}(c_i)) = 2m_j \text{hl}(C_i) \pmod{2\pi i \mathbb{Z})}.
\]

Let \(S\) be an oriented closed topological surface with a generalized pants decomposition. By this we mean that we are given a collection \(\mathcal{C}\) of disjoint simple closed curves on \(S\), such that for every component \(\tilde{\Pi}\) of \(S \setminus \mathcal{C}\) there is an associated finite cover \(\chi : \tilde{\Pi} \to \Pi\). Let

\[
\tilde{\rho} : \pi_1(S) \to \text{PSL}(2, \mathbb{C})
\]

be a representation. We make the following assumptions on \(\rho\):

- Given a curve \(C \in \mathcal{C}\) there exists two (not necessarily different) generalized pairs of pants \(\tilde{\Pi}_1\) and \(\tilde{\Pi}_2\) that both contain \(C\) as a cuff, and that lie on different sides of \(C\). Let \(\chi_1 : \tilde{\Pi}_1 \to \Pi_1\) and \(\chi_2 : \tilde{\Pi}_2 \to \Pi_2\) be the corresponding finite covers, where \(\Pi_1\) and \(\Pi_2\) are two pairs of pants. We assume that the restrictions of \(\chi_1\) and \(\chi_2\) on the curve \(C\) are of the same degree.
- For every generalized pair of pants \(\tilde{\Pi}\) from the above decomposition of \(S\), the restriction \(\rho : \pi_1(\tilde{\Pi}) \to \text{PSL}(2, \mathbb{C})\) is admissible with respect to the covering map \(\chi : \tilde{\Pi} \to \Pi\) (in the sense of Definition 3.1).
- For every \(C \in \mathcal{C}\), the half lengths of \(C\) coming from the representations \(\rho : \pi_1(\tilde{\Pi}_1) \to \text{PSL}(2, \mathbb{C})\) and \(\rho : \pi_1(\tilde{\Pi}_2) \to \text{PSL}(2, \mathbb{C})\) are one and the same.

Continuing with the above notation, let \(C_i \subset \Pi_i\) denote the cuff such that \(\chi_i(C) = C_i\). Let \(\rho_i : \pi_1(\Pi_i) \to \text{PSL}(2, \mathbb{C})\), \(i = 1, 2\), be the representations such that the restriction of \(\rho\) to \(\pi_1(\tilde{\Pi}_i)\) is equal to \(\rho_i \circ (\chi_i)_*\). We define the reduced twist-bend parameter \(s(C)\) associated to \(\rho\) to be equal to the reduced twist-bend parameter for the representations \(\rho_1\) and \(\rho_2\).
So given a closed surface $S$ with a generalized pants decomposition $\mathcal{C}$, and a representation $\rho: \pi_1(S) \to \text{PSL}(2, \mathbb{C})$, we have defined the parameters $\mathbf{hl}(C) \in \mathbb{C}_+/2k\pi \mathbb{Z}$ and $s(C) \in \mathbb{C}/(\mathbf{hl}(C)\mathbb{Z} + 2\pi \mathbb{Z})$. The collection of pairs $(\mathbf{hl}(C), s(C))$, $C \in \mathcal{C}$, is called the reduced Fenchel-Nielsen coordinates. We observe that a representation $\rho: \pi_1(S) \to \text{PSL}(2, \mathbb{C})$ is Fuchsian if and only if all the coordinates $(\mathbf{hl}(C), s(C))$ are real.

The following elementary proposition (see [4]) states that although a representation $\rho: \pi_1(S) \to \text{PSL}(2, \mathbb{C})$ is not uniquely determined by its reduced Fenchel-Nielsen coordinates, it can be in a unique way embedded in a holomorphic family of representations.

**Proposition 3.1.** Fix a closed topological surface $S$ with a generalized pants decomposition $\mathcal{C}$. Let $z \in \mathbb{C}_+$ and $w \in \mathbb{C}$ denote complex parameters. Then there exists a holomorphic (in $(z, w)$) family of representations

$$\rho_{z,w}: \pi_1(S) \to \text{PSL}(2, \mathbb{C}),$$

such that $\mathbf{hl}(C) = z(C)$, $(\text{mod}(2\pi i \mathbb{Z}))$ and $s(C) = w(C)$, $(\text{mod}(\mathbf{hl}(C)\mathbb{Z} + 2\pi i \mathbb{Z}))$. Moreover, for any $(z_0, w_0) \in \mathbb{C}_+ \times \mathbb{C}$, the family of representations $\rho_{z,w}$ is uniquely determined by the representation $\rho_{z_0,w_0}$.

The representation $\rho_{z,w}$ is Fuchsian if and only if both $z$ and $w$ are real, that is $z \in \mathbb{R}_+$ and $w \in \mathbb{R}$. In this case the group $\rho_{z,w}(\pi_1(S))$ is of course discrete. Moreover, in [3] it has been proved that all quasifuchsian representations (up to conjugation in $\text{PSL}(2, \mathbb{C})$) of $\pi_1(S)$ correspond to some neighborhood of the set $\mathbb{R}_+ \times \mathbb{R}$ But in general, little is known for which choice of parameters $z, w$ the group $\rho_{z,w}(\pi_1(S))$ will be discrete. In the next subsection we prove the following result in this direction. Start with a nearly Fuchsian group $G < \text{PSL}(2, \mathbb{C})$. We obtain a new group $G_1 < \text{PSL}(2, \mathbb{C})$ from $G$ by bending (by some definite angles) along some sparse equivariant collection of geodesics whose endpoints are in the limit set of $G$. Then the new group $G_1$ is also quasifuchsian (although it is not nearly Fuchsian anymore).

### 3.2. Small deformations of a sparsely bent pleated surface.

We let $S$ continue to denote a closed surface with a generalized pants decomposition $\mathcal{C}$, and we fix a holomorphic family of representations $\rho_{z,w}$ as in Proposition 3.1. We set $G(z, w) = \rho_{z,w}(\pi_1(S))$.

Let $\mathcal{C}_0 \subset \mathcal{C}$ denote a sub-collection of curves. For $z \in \mathbb{R}_+$ and $w \in \mathbb{R}$, we let $S_{z,w}$ denote the Riemann surface isomorphic to $\mathbb{H}^2/G(z, w)$, and on $S_{z,w}$ we identify the curves from $\mathcal{C}$ with the corresponding geodesics representatives. By $\mathcal{K}(S_{z,w})$ we denote the largest number so that the collection of collars (of width $\mathcal{K}(S_{z,w})$) around the curves from $\mathcal{C}_0$ is disjoint on $S_{z,w}$. For each $C \in \mathcal{C}_0$, we choose a number $-\frac{3}{4}\pi < \theta_C < \frac{3}{4}\pi$ (for each curve $C \in (\mathcal{C} \setminus \mathcal{C}_0)$ we set $\theta_C = 0$).

The purpose of this subsection is to prove the following theorem.
**Theorem 3.1.** There exist constants $K > 1$ and $C > 0$ such that the following holds. Let $z_0 \in \mathbb{R}^C_*$ and $w_0 \in \mathbb{R}^C$, and $z_1 \in \mathbb{C}^C_*$ and $w_1 \in \mathbb{C}^C$ be such that the representation $\rho = \rho_{z_1,w_1} \circ \rho_{z_0,w_0}^{-1} : G(z_0,w_0) \to G(z_1,w_1)$, is $K$-quasifuchsian. Set $z_2 = z_1$ and $w_2 = w_1 + i\theta C$. If $K(S_{z_0,w_0}) \geq C$, then the representation $\rho_{z_2,w_2} : \pi_1(S) \to \text{PSL}(2,\mathbb{C})$ is $K_1$-quasifuchsian, where $K_1$ depends only on $K$ and $C$.

The following lemma is elementary.

**Lemma 3.1.** Let $0 \leq \theta_0 < \pi$ and $B_0 \geq 1$. There exist constants $L(\theta_0, B_0) > 0$ and $C(\theta_0, B_0) > 0$ such that the following holds. Let $I \subset \mathbb{R}$ be an interval that is partitioned into intervals $I_j$, $j = 1, \ldots, k$. Let $\psi : I \to \mathbb{H}^3$ be a continuous map, such that $\psi$ maps each $I_j$ onto a geodesic segment and the restriction of $\psi$ on $I_j$ is $B_0$-bilipschitz. Assume in addition that the bending angle between two consecutive geodesic intervals $\psi(I_j)$ and $\psi(I_{j+1})$ is at most $\theta_0$. If the length of every $I_j$ is at least $C(\theta_0, B_0)$ then $\psi$ is $L(\theta_0, B_0)$-bilipschitz.

Let $\psi : I \to \mathbb{H}^3$ be a $C^1$ map, where $I \subset \mathbb{R}$ is a closed interval. For $x \in I$ let $v(x) \in T^1 I$ denote the unit vector that points toward $+\infty$. Let $\delta > 0$. We say that the map $\psi$ is $\delta$-nearly geodesic if for every $x, y \in I$ such that $x < y \leq x + 1$, we have that the angle between the vector $\psi_*(v(x))$ and the oriented geodesic segment from $\psi(x)$ to $\psi(y)$ is at most $\delta$.

Clearly, every $0$-nearly geodesic map is an isometry, and a sequence of $\delta_n$-nearly geodesic maps converges (uniformly on compact sets) in the $C^1$ sense to an isometry, when $\delta_n \to 0$. The following lemma is a generalization of the previous one.

**Lemma 3.2.** There exist universal constants $L, C, \delta > 0$, such that the following holds. Suppose that $I$ is partitioned into intervals $I_j$, $j = 1, \ldots, k$, and let $\psi : I \to \mathbb{H}^3$ be a continuous map, whose restriction on every closed sub-interval $I_j$ is $C^1$ and $\delta$-nearly geodesic. Assume that the bending angle between two consecutive curves $\psi(I_j)$ and $\psi(I_{j+1})$ is at most $\frac{\pi}{2}$ (by the bending angle between two $C^1$ curves we mean the appropriate angle determined by the two tangent vectors at the point where the two curves meet). If the length of every $I_j$ is at least $C$ then $\psi$ is $L$-bilipschitz.

**Proof.** Choose any two numbers $\frac{\pi}{4} < \theta_0 < \pi$ and $B_0 > 1$. Assuming that $C > C(\theta_0, B_0)$ we can partition each $I_j$ into sub-intervals of length between $C(\theta_0, B_0)$ and $2C(\theta_0, B_0)$. Replacing each $I_j$ with these new intervals we obtain the new partition of $I$ into intervals $J_i$, where each $J_i$ has the length between $C(\theta_0, B_0)$ and $2C(\theta_0, B_0)$. Let $\psi : I \to \mathbb{H}^3$ be the continuous map that agrees with $\psi$ at the endpoints of all intervals $J_i$, and such that the restriction of $\psi$ to each $J_i$ maps $J_i$ onto a geodesic segment in $\mathbb{H}^3$, and is affine (the map $\psi$ either stretches or contracts distances by a constant factor on a given $J_i$).

Next, since we have the upper bound $2C(\theta_0, B_0)$ on the length of each interval $J_i$, we can choose $\delta > 0$ small enough such that the bending angle
between two consecutive geodesic segments $\phi(J_i)$ and $\phi(J_{i+1})$ is at most $\theta_0$. Also, by choosing $\delta$ small we can arrange that the map $\phi \circ \psi^{-1}$ is 2-bilipschitz (the same statement holds if we replace 2 by any other number greater than 1). By the previous lemma the map $\phi$ is $L(\theta_0, B_0)$-bilipschitz. Then the map $\psi$ is $2L(\theta_0, B_0)$-bilipschitz. We take $L = 2L(\theta_0, B_0)$, and $C = C(\theta_0, B_0)$, and the lemma is proved.

We are now ready to prove Theorem 3.1.

Proof. Recall that $f : \partial \mathbb{H}^2 \to \partial \mathbb{H}^3$ is a $K$-quasiconformal map that conjugates $G(z_0, w_0)$ to $G(z_1, w_1)$. Let $\tilde{f} : \mathbb{H}^2 \to \mathbb{H}^3$ denote the Douady-Earle extension of $f$. Then $\tilde{f}$ is $\delta$-nearly geodesic (this means that the restriction of $\tilde{f}$ to every geodesic segment is $\delta$-nearly geodesic in the sense of the above definition) for some $\delta = \delta(K)$, and $\delta(K) \to 0$, when $K \to 1$.

If we assume that $K(S_{z_0,w_0})$ is large enough, by adjusting $\tilde{f}$, we can arrange that $\tilde{f}$ is then $C^\infty$ mapping that maps the geodesics in $\mathbb{H}^2$ that are lifts of the geodesics from $C_0$ onto the corresponding geodesics in $\mathbb{H}^3$, and ensure that $\tilde{f}$ is $2\delta$-nearly geodesic. Moreover, we can arrange that $\tilde{f}$ is conformal at every point of every geodesic $\gamma$ that is a lift of a curve from $C_0$.

We construct the map $\tilde{g} : \mathbb{H}^2 \to \mathbb{H}^3$ that conjugates $G(z_0, w_0)$ to $G(z_1, w_1)$ as follows. Let $M$ be a component of the set $S_{z_0,w_0} \setminus C_0$, and let $\tilde{M} \subset \mathbb{H}^2$ denote its universal cover, that is $\tilde{M}$ is an ideal polygon with infinitely many sides in $\mathbb{H}^2$, whose sides are lifts of the geodesics from $C_0$ that bound $M$. We set $\tilde{g} = \tilde{f}$ on $\tilde{M}$.

Let $M_1 \subset \mathbb{H}^2$ be the universal cover of some other component $M_1$ of the set $S_{z_0,w_0} \setminus C_0$. Let $\gamma$ denote a lift of a geodesic $C \subset C_0$, and assume that the polygons $\tilde{M}$ and $\tilde{M}_1$ are glued to each other along $\gamma$ (that is, $C$ is in the boundary of both $M$ and $M_1$). Let $R(\theta_C) \in PSL(2, \mathbb{C})$, denote the rotation about $\tilde{g}(\gamma)$ for the angle $\theta_C$. We define $\tilde{g}$ on $\tilde{M}_1$ by letting $\tilde{g} = R(\theta_C) \circ \tilde{f}$. We then define $\tilde{g}$ inductively on the rest of $\mathbb{H}^2$.

Clearly $\tilde{g}$ conjugates $G(z_0, w_0)$ to $G(z, w)$. Let $x \in \gamma$, and $v(x)$ a non-zero vector that is orthogonal to $\gamma$. Since $|\theta_C| \leq \frac{\pi}{4}$, and since $\tilde{f}$ is differentiable at $x$, it follows that the bending angle between the vectors $\tilde{g}_* (v(x))$ and $\tilde{g}_*(u(x))$ is at most $\frac{3}{4}\pi$. If $u(x)$ is any other vector at $x$, since $\tilde{f}$ is conformal at $x$, it follows that the bending angle between the vectors $\tilde{g}_* (u(x))$ and $\tilde{g}_*(-u(x))$ is at most as big as the bending angle between the vectors $\tilde{g}_*(v(x))$ and $\tilde{g}_*(-v(x))$. Therefore, the restriction of the map $\tilde{g}$ on every geodesic segment satisfies the assumptions of Lemma 3.2. It follows that $\tilde{g}$ is $L$-bilipschitz, where $L$ depends only on $K$ and $C$. Therefore the representation $\rho_{z_2,w_2} : \pi_1(S) \to PSL(2, \mathbb{C})$ is $K_1$-quasifuchsian, where $K_1$ depends only on $K$ and $C$.

3.3. Convex hulls and pleated surfaces. In this subsection we digress from the notions of generalized pants decompositions and Fenchel-Nielsen
coordinates, to prove a preliminary lemma about hyperbolic convex hulls of quasicircles.

Let $\lambda$ be a discrete geodesic lamination in $\mathbb{H}^2$, and let $K(\lambda)$ denote the largest number such that for every small $\epsilon > 0$, the collection of collars (crescent in $\mathbb{H}^2$) of width $K(\lambda) - \epsilon$ around the leafs of $\lambda$ is disjoint in $\mathbb{H}^2$. Let $\mu$ denote a real valued measure on $\lambda$. By $\iota_{\lambda, \mu} = \iota : \mathbb{H}^2 \to \mathbb{H}^3$, we denote the corresponding pleating map. As usual, by $\iota(\lambda)$ we denote the collection of geodesics in $\mathbb{H}^3$ that are images of geodesics from $\lambda$ under $\iota$. If the map $\iota$ is $L$-bilipschitz then $\iota$ extends continuously to a $K$-quasiconformal map $f : \partial \mathbb{H}^2 \to \partial \mathbb{H}^3$, for some $K = K(L)$. In this case, let $W \subset \mathbb{H}^3$ denote the convex hull of the quasicircle $\iota(\partial \mathbb{H}^2)$. The convex hull $W$ has two boundary components which we denote by $\partial_1 W$ and $\partial_2 W$. We prove the following lemma.

**Lemma 3.3.** There exist universal constants $C_1, \delta_1 > 0$, with the following properties. Assume that $K(\lambda) > C_1$, and that $\frac{\pi}{4} \leq |\mu(l)| \leq \frac{3\pi}{4}$, for every $l \in \lambda$. Then for every geodesic $\gamma \subset W$ the following holds:

1. If $\gamma \in \iota(\lambda)$, then for every point $p \in \gamma$, the inequality
   \[ \max_{i=1,2} d(p, \partial_i W) > \delta_1 \]
   holds,
2. If $\gamma$ does not belong to $\iota(\lambda)$, then for some point $p \in \gamma$, the inequality
   \[ \max_{i=1,2} d(p, \partial_i W) < \frac{\delta_1}{4} \]
   holds.

Compare this lemma with Lemma 4.2 in [5].

**Proof.** It follows from Lemma 3.1 that for $C_1$ large enough, the pleating map $\iota$ is $L$-bilipschitz for some universal constant $L > 1$. Observe that $\iota(\mathbb{H}^2) \subset W$. Moreover, there is a constant $M_0 > 0$, that depends only on $L$, such that for every $p \in W$ we have $d(p, \iota(\mathbb{H}^2)) < M_0$

We choose $\delta_1 > 0$ as follows. Let $P_0$ be the pleated surface in $\mathbb{H}^3$ that has a single bending line $\gamma_0$, and with the bending angle equal to $\frac{\pi}{4}$. Then $P_0$ is bounded by a quasicircle at $\partial \mathbb{H}^3$. Denote by $W_0$ the convex hull of this quasicircle and let $\partial_i(W_0)$, $i = 1, 2$, denote the two boundary components of $W_0$. Then there exists $\delta_1 > 0$ such that for every point $p \in \gamma_0$, we have $\max_{i=1,2} d(p, \partial_i W_0) > 2\delta_1$. Observe that $\gamma_0$ belongs to exactly one of the convex hull boundaries $\partial_1 W_0$ and $\partial_2 W_0$, so one of the numbers $d(p, \partial_1 W_0)$ and $d(p, \partial_2 W_0)$ is zero and the other one is larger than $2\delta_1$.

Assume that the first statement of the lemma is false. Then there exists a sequence of measured laminations $(\lambda_n, \mu_n)$ with the property $K(\lambda_n) \to \infty$, and there are geodesics $l_n \in \lambda_n$, and points $p_n \in \gamma_n = \iota_n(l_n)$, such that the inequality

\[ \max_{i=1,2} d(p_n, \partial_i W_n) \leq \delta_1, \]

holds. We may assume that $p_n = p$, and $\gamma_n = \gamma$, for every $n$, where $p$ and $\gamma$ are fixed. Since $\iota_n$ is $L$-bilipschitz, after passing to a subsequence
if necessary, the sequence $\tau_n$ converges (uniformly on compact sets) to a pleating map $\iota_\infty$. The pleating map $\iota_\infty$ corresponds to the pleating surface $P_\infty$, that has a single bending line $\gamma_\infty$, with the bending angle at least $\frac{\pi}{4}$. Then $W_n$ converges to $W_\infty$ uniformly on compact sets in $\mathbb{H}^3$, where $W_\infty$ is the convex hull of the quasicircle that bounds $P_\infty$. It follows that $d(p_n, \partial_i W_n) \to d(p, \partial_i W_\infty)$. We may assume that $\gamma_\infty = \gamma_0$, where $\gamma_0$ is the bending line of the pleated surface $P_0$ defined above. Then we have

$$\max_{i=1,2} d(p, \partial_i W_n) \geq \frac{\delta_1}{3},$$

holds for $n$ large enough. By the previous discussion, there exists a sequence of points $p_n \in \gamma_n$, such that $d(p_n, \iota_n(\lambda_n)) > K(\lambda_n).

Let $q_n \in \iota_n(\mathbb{H}^2)$ be points such that $d(p_n, q_n) < M_0$, where $M_0$ is the constant defined at the beginning of the proof. Let $z_n \in \mathbb{H}^2$, such that $q_n = \iota(z_n)$. We may assume that $z_n = 0$ and $q_n = q$, for some point $q$ that we fix. Then $p_n \to p$, where $d(p, q) \leq M_0$. Moreover, since $K(\lambda_n) \to \infty$, the pleating maps $\iota(\lambda_n)$ converge to an isometry uniformly on compact sets in $\mathbb{H}^2$. In particular, the sequence of convex hulls $W_n$ converges to a geodesic plane uniformly on compact sets, and therefore $d(p_n, \partial_i W_n) \to 0$. But this contradicts (5), and thus we have completed the proof of the lemma.

3.4. $(\epsilon, R)$ skew pants. We let $S$ continue to denote a closed surface with a generalized pants decomposition $C$, and we fix a holomorphic representations $\rho_{z,w}$ as in Proposition 3.1.

Let $C_0 \subset C$ denote a sub-collection of curves, and for each $C \in C_0$ we choose a number $-\frac{3}{4}\pi < \theta_C < \frac{3}{4}\pi$ (for each curve $C \in (C \setminus C_0)$ we set $\theta_C = 0$).

For $C \in C$, let $\zeta_C, \eta_C \in \mathbb{D}$, where $\mathbb{D}$ denotes the unit disc in the complex plane. Let $\tau \in \mathbb{D}$ denote a complex parameter and let $t \in \{0,1\}$. Fix $R > 1$, and let $z : \mathbb{D} \to \mathbb{C}_+^C$ and $w : \mathbb{D} \to \mathbb{C}^C$ be the mappings given by

$$z(C)(\tau) = \frac{R}{2} + \frac{\tau \zeta_C}{2},$$

and

$$w(C)(\tau, t) = 1 + it \theta_C + \frac{\tau \eta_C}{R}.$$
holomorphic in $\tau$ and $t$. Note that $\rho_{\tau,t}$ depends on $R$, $\zeta_C$, $\eta_C$ and $\theta_C$, but we suppress this.

The representation $\rho_{0,0}$ is Fuchsian. Let $S_0$ denote the Riemann surface isomorphic to the quotient $\mathbb{H}^2/\rho_{0,0}(\pi_1(S))$ (we also equip $S_0$ with the corresponding hyperbolic metric). Let $K(\rho_{0,0})$ denote the largest number so that the collection of collars (of width $K(\rho_{0,0})$) around the curves from $C_0$ is disjoint on $S_0$.

The representation $\rho_{0,1}$ is not Fuchsian (unless $\theta(C_0) = 0$), and the following proposition gives a sufficient condition for it to be quasifuchsian.

We adopt the following notation. Let $G(\tau, t) = \rho_{\tau,t}(\pi_1(S))$. If $G(\tau,t)$ is a quasifuchsian group we let $f_{\tau,t} : \partial \mathbb{H}^2 \to \partial \mathbb{H}^3$, denote the quasiconformal map that conjugates $G(0,0)$ to $G(\tau,t)$. The following theorem is a generalization of Theorem 2.2 from [4] (see Theorem 3.4 below). Assuming the above notation, we have:

**Theorem 3.2.** There exist universal constants $\hat{R}, \hat{\epsilon}, M > 0$, such that the following holds. If $K(\rho_{0,0}) > M$, then for every $R \geq \hat{R}$ and $|\tau| < \hat{\epsilon}$, and any choice of constants $\eta_C, \zeta_C \in \mathbb{D}$, and $-\frac{3}{4} < \theta_C < \frac{3}{4}$, for $C \in C_0$, the group $G(\tau,1)$ is quasifuchsian and the induced quasiconformal map $f_{\tau,1} \circ f_{0,1}$ (that conjugates $G(0,1)$ to $G(\tau,1)$), is $K(\tau)$-quasiconformal, where

$$K(\tau) = \frac{\hat{\epsilon} + |\tau|}{\epsilon - |\tau|}.$$ 

Let $C_0(\tau,t)$ denote the collection of axes of elements of the form $\rho_{\tau,t}(c)$, where $c \in \pi_1(S)$ and $c$ belongs to the conjugacy class of some curve $C \in C_0$. Then by definition, the set $C_0(\tau,t)$ is invariant under the group $G(\tau,1)$. Next, we prove that $C_0(\tau,1) \cap \partial W$ is invariant under any Möbius transformation from $\text{PSL}(2,\mathbb{C})$ that preserves the limit set of $G(\tau,1)$. The following theorem is the main result of this section.

**Theorem 3.3.** There exist constants $\hat{\epsilon}_1, M_1 > 0$, with the following properties. Assume that $K(\rho_{0,0}) > M_1$ and let $|\tau| < \hat{\epsilon}_1$. If $T \in \text{PSL}(2,\mathbb{C})$, is a Möbius transformation that preserves the limit set of $G(\tau,1)$, then the set of geodesics $C_0(\tau,1)$ is invariant under $T$.

Compare this theorem with Lemma 4.2 in [5].

**Proof.** Let $W(\tau,t)$ denote the convex hull of the limit set of $G(\tau,t)$. It follows from Lemma 3.3 that for $K(\rho_{0,0})$ large enough, the following holds

1. For every $\gamma \in C_0(0,1)$ and $p \in \gamma$, the inequality $\max_{i=1,2} \, d(p, \partial_i W(0,t)) > \delta_1$ holds,

2. For every $\gamma \subset W(0,1)$ the inequality, there exists $p \in \gamma$ such that $\max_{i=1,2} \, d(p, \partial_i W(0,1)) < \delta_1$.

Then by Theorem 3.2 we can choose $\hat{\epsilon}_1$ small enough so that for $|\tau| < \hat{\epsilon}_1$, the constant $K(\tau)$ (from Theorem 3.2) is close enough to 1, so that the following holds:
(1) For every $\gamma \in C_0(\tau, 1)$ and $p \in \gamma$, the inequality $\max_{i=1,2} d(p, \partial_i W(0, t)) > \frac{4\delta}{5}$ holds.

(2) For every $\gamma \subset W(0, 1)$ the inequality, there exists $p \in \gamma$ such that $\max_{i=1,2} d(p, \partial_i W(0, 1)) < \frac{2\delta}{3}$.

Then any Möbius transformation $A \in \text{PSL}(2, \mathbb{C})$ that preserves $W(\tau, 1)$ will also preserve the set $C(\tau, 1)$. This proves the theorem.

$\square$

3.5 A proof of Theorem 3.2. We need to prove that $G(\tau, 1)$ is a quasifuchsian group. The last estimate in Theorem 3.2 then follows from the fact that a holomorphic map from the unit disc into the Teichmüller space of a Riemann surface is a contraction with respect to the hyperbolic metric on the unit disc and the Teichmüller metric.

Recall Theorem 2.2 from [4].

Theorem 3.4. There exist universal constants $\hat{R}, \hat{\epsilon}$, such that the following holds. For every $R \geq \hat{R}$ and $|\tau| < \hat{\epsilon}$, and any choice of constants $\eta_C, \zeta_C \in \mathbb{D}$, the group $G(\tau, 0)$ is quasifuchsian, and the induced quasiconformal map $f_{\tau,0}$ that conjugates $G(0,0)$ to $G(\tau,0)$, is $K(\tau)$-quasiconformal, where

$$K(\tau) = \frac{\hat{\epsilon} + |\tau|}{\epsilon - |\tau|}.$$ 

The group $G(\tau, 1)$ is obtained from the group $G(\tau, 0)$, by bending along the lifts of curves $C \in C_0$, for the angle $\theta_C$. It follows from Theorem 3.1 that the group $G(\tau, 1)$ is quasifuchsian if $K(\rho_{0,0}) > C$, and if the map $f_{\tau,0}$ is $K$-quasiconformal, where $K$ is close enough to 1. But it follows from Theorem 3.4 that for $|\tau|$ small enough this will be the case. This proves Theorem 3.2.

4. The lower bound

4.1 Amalgamating two representations. Let $S$ denote a closed surfaces with generalized pants decompositions $C$, and let $\rho : \pi_1(S) \to \text{PSL}(2, \mathbb{C})$ denote an admissible (in sense of Definition 3.1) representation with the reduced Fenchel-Nielsen coordinates satisfying the inequalities

$$|h(C) - \frac{R}{2}| \leq \epsilon,$$

and

$$|s(C) - 1| \leq \frac{\epsilon}{R},$$

for some $\epsilon, R > 0$, and $C \in C$. We say that such a representation is $(\epsilon, R)$-good.

Let $M^3$ denote a closed hyperbolic manifold such that $M^3 = \mathbb{H}^3/\Gamma$ for some Kleinian group $\Gamma$. In [4] we proved that one can find many $(\epsilon, R)$-good representations $\rho : \pi_1(S) \to \Gamma$, for a given $\epsilon > 0$ and $R$ large enough. Moreover, if $A \in \Gamma$ has the translation length $l(A)$ satisfying the inequality $|l(A) - R| \leq \frac{\epsilon}{2}$, then we can find such $\rho$ so that $A$ is in the image of $\rho$. From
now on we assume that such $A \in \Gamma$ is primitive, that is $A$ is not equal to an integer power of another element of $\Gamma$.

In particular, it follows from Section 4 of [4] (the statements about the equidistribution of $(\epsilon, R)$-good pairs of skew pants around a given closed curve in $M^3$ whose length is $\epsilon$ close to $R$) that we can find two $(\epsilon, R)$-good representations $\rho(i) : \pi_1(S(i)) \to \Gamma$, $i = 1, 2$, where $S(1)$ and $S(2)$ are two closed surfaces with pants decompositions $C(i)$, and two pairs of pants $\Pi^+_i$ and $\Pi^-_i$ with the following properties:

- There are two oriented, degree one curves $C(i) \in C(i)$, and $c(i) \in \pi_1(S(i))$ in the conjugacy classes of $C(1)$ and $C(2)$ respectively, such that $\rho(1)(C(1)) = \rho(2)(C(2)) = [A]$, where $[A]$ is the conjugacy class of a given primitive element $A \in \Gamma$, whose translation length $l(A)$ satisfies the inequality $|l(A) - R| \leq \frac{\epsilon}{2}$.

- Let $\gamma$ denote the closed geodesic corresponding to $A$. There exist two pairs of skew pants $\Pi^+_i$ and $\Pi^-_i$ in $\rho(i)(\pi_1(S(i)))$ such that $\gamma$ is positively oriented boundary component of $\Pi^+_i$ and negatively oriented for $\Pi^-_i$, and recalling the notation from [4] we have the inequality

$$ (6) \quad |\text{foot}_\gamma(\Pi^+_i) - \text{foot}_\gamma(\Pi^-_i) - \frac{\pi}{2}| \leq \frac{\epsilon}{R}. $$

After replacing $S(1)$ and $S(2)$ with appropriate finite degree covers if necessary, we may assume in addition to the above two conditions the following also hold:

- The curves $C(1)$ and $C(2)$ are non-separating simple closed curves in $S(1)$ and $S(2)$ respectively,
- The surfaces $S(1)$ and $S(2)$ have the same genus,
- By Proposition 3.1 the representation $\rho(i)$ can be embedded in the holomorphic family of representations $\rho_{\tau,t}(i)$. We may assume that $\mathcal{K}(\rho_{0,0}(S(i))) > C_1$, $i = 1, 2$, where $C_1$ is the constant from Theorem 3.3.

We now fix such two representations $\rho(1)$ and $\rho(2)$, surfaces $S(1)$ and $S(2)$, and the two oriented curves $C(1)$ and $C(2)$ (we also fix the corresponding primitive element $A \in \Gamma$).

Let $i \in \{1, 2\}$. For $n > 1$, let $S_n(1)$ and $S_n(2)$ denote two primitive degree $n$ covers of $S(1)$ and $S(2)$ respectively (a finite cover of a surface is primitive if it does not factor through an intermediate cover), such that for some $1 \leq k \leq (n - 1)$, the curves $C(1)$ and $C(2)$ have two degree $k$ lifts $C_n(1)$ and $C_n(2)$. Then $C_n(1)$ and $C_n(2)$ are two oriented, non-separating simple closed curves in $S_n(1)$ and $S_n(2)$ respectively. We then have the two induced representations $\rho_n(i) : \pi_1(S_n(i)) \to \Gamma$, that also satisfy the above five conditions, except that

$$ \rho_n(1)(\pi_1(S_n(1))) \cap \rho_n(2)(\pi_1(S_n(2))) = \{A^k\}. $$
We amalgamate them as follows. Cut the surface $S_n(i)$ along $C_n(i)$, to get two topological surfaces $\overline{S}_n(i)$, $i = 1, 2$, each having two boundary components $C^1_n(i)$ and $C^2_n(i)$. We glue together the surfaces $\overline{S}_n(1)$ and $\overline{S}_n(2)$ by gluing $C^1_n(1)$ to $C^1_n(2)$, $j = 1, 2$, and obtain a closed topological surface $S_n$ (this is well defined up to a twist by $\mathfrak{R}(\mathfrak{I}(A))$ which has a period $k$). The surface $S_n$ has the induced generalized pants decomposition $\mathcal{C}_n$. The pair of curves $C^1_n(1)$ and $C^1_n(2)$ that were glued together produce a closed curve $C^1_n$ in $S_n$. Similarly, the pair of curves $C^2_n(1)$ and $C^2_n(2)$ that were glued together produce a closed curve $C^2_n$ in $S_n$. We set $\mathcal{C}_{0,n} = \{C^1_n, C^2_n\}$.

Then there is the induced representation $\rho_n : \pi_1(S_n) \to \Gamma$. We orient the curves $C^1_n$ and $C^2_n$ such that for any choice of $c_i \in \pi_1(S_n)$, where $c_i$ is in the conjugacy class of $C^i_n$, we have that both $\rho_n(c_1)$ and $\rho_n(c_2)$ are in the conjugacy class of $A^k$ in $\Gamma$.

The representation $\rho_n$ has the reduced Fenchel-Nielsen coordinates satisfying the relations

$$|h l(C) - \frac{R}{2}| \leq \epsilon,$$

and

$$|s(C) - 1| \leq \frac{\epsilon}{R},$$

if $C$ does not belong to $\mathcal{C}_{0,n}$, and

$$|s(C) - (1 + i \frac{\pi}{2})| \leq \frac{\epsilon}{R},$$

if $C \in \mathcal{C}_{0,n}$.

It follows from Theorem 3.2 that for $\epsilon$ small enough and $R$ large enough, the group $\rho_n(\pi_1(S_n))$ is quasifuchsian. In the remainder of this subsection we prove that the group $\rho_n(\pi_1(S_n))$ is a maximal subgroup of $\Gamma$.

First we prove a preliminary lemma. Let $\overline{S}$ be a surface with boundary components $C_+$ and $C_-$, oriented so that $\overline{S}$ is on the left of $C_+$ and the right of $C_-$. We say that $f : \overline{S} \to M^3$ is rejoinable if the restrictions of $f$ to $C^+$ and $C_-$ respectively are freely homotopic in $M^3$. We say $(f, \overline{S})$ is geodesically rejoinable if $f|_{C_+}$ and $f|_{C_-}$ map to the same closed geodesic in $M^3$. In this case we say a rejoining of $(f, \overline{S})$ is a homeomorphism $h : C_+ \to C_-$ such that $f \circ h = f$, and we say $(f, \overline{S}/h)$ is $\overline{S}$ rejoined by $h$.

**Lemma 4.1.** If $(f, \overline{S})$, and $(g, \overline{T})$ are (geodesically) rejoinable surfaces, and $\pi : \overline{S} \to \overline{T}$ is a finite cover such that $g \circ \pi$ is homotopic to $f$, then for any rejoining $h$ of $(f, \overline{S})$ we can find a rejoining $k$ of $(g, \overline{T})$ such that $(f, \overline{S})$ rejoined by $h$ covers $(g, \overline{T})$ rejoined by $k$.

**Proof.** Left to the reader. $\square$

The following theorem is a corollary of Theorem 3.2. We adopt the following definition. Let $f : S \to M^3$ be a quasifuchsian map, and let $\mathcal{C}_0$ denote a collection of disjoint simple closed curves on $S$. We say that $f$ is bent along
Theorem 4.1. Let $S$ be a closed surface. Suppose that $f : S \to M^3$ is a $\pi_1$-injective and quasifuchsian, and $C_0$ is a collection of disjoint simple closed curves on $S$, such that $f$ is bent along each curve of $C_0$ and nearly locally isometric on $S \setminus C_0$. Suppose that $f = g \circ \pi$, where $\pi : S \to S$ is a covering, and $g : S \to M^3$ is $\pi_1$-injective and quasifuchsian. Then we can find a collection of simple closed curves $\tilde{C}_0$ on $Q$ such that $C_0 = \pi^{-1}(\tilde{C}_0)$.

Proof. We get a discrete lamination $\tilde{C}_0$ on $H^2$, which we push forward by $\tilde{f} = \tilde{g}$ to $H^3$. We find a homomorphism $\sigma : \text{Deck}(H^2/Q) \to \Gamma$ such that $\tilde{f}(\gamma(x)) = \sigma(\gamma)(\tilde{f}(x))$ for every $x \in H^2$ and $\gamma \in \text{Deck}(H^2/Q)$.

We let $G = \sigma(\text{Deck}(H^2/Q))$, and $H = \sigma(\text{Deck}(H^2/S)) < G$. Then $|G : H| < \infty$, and $G$ and $H$ are quasifuchsian groups, and they have the same limit set, so by Theorem 4.1 every element of $G$ maps $\tilde{g}(\tilde{C}_0)$ to itself. Hence $\text{Deck}(H^2/Q)$ maps $\tilde{C}_0$ to itself, so $\tilde{C}_0$ is a lift of $C_0$ on $Q$, and hence $C_0$ is.

Theorem 4.2. The quasifuchsian group $\rho_n(\pi_1(S_n)) < \Gamma$ is a maximal surface subgroup of $\Gamma$, that is, if $\rho_n(\pi_1(S_n)) < G$ for a surface subgroup $G < \Gamma$, then $G = \rho_n(\pi_1(S_n))$.

Proof. For simplicity let $G_n = \rho_n(\pi_1(S_n))$ and $G(1) = \rho(1)(\pi_1(S(1)))$. Also set $G_n(1) = \rho_n(\pi_1(S_n(1)))$, where we consider $\pi_1(S_n(1))$ as a subgroup of $\pi_1(S_n)$.

Let $f_n : S_n \to M^3$ denote the continuous map that corresponds to the representation $\rho_n$. We claim that $f_n : S_n \to M^3$ is primitive. If not, we can find a Riemann surface $Q$ and $\pi : S_n \to Q$ and $g : Q \to M^3$ such that $g \circ \pi = f_n$ and $d > 1$ where $d$ is the degree of the cover $\pi$. We recall that $f_n$ is bent along $C_n^1$ and $C_n^2$, and nearly isometric on the complement. So by Theorem 4.1 $\{\tilde{C}_n^1, C_n^2\}$ are the lifts by $\pi$ of some set $C_Q$ of simple closed curves on $Q$. So $|C_Q| = 1$ or $|C_Q| = 2$.

If $|C_Q| = 2$, then each component of $S_n \setminus \cup C_n^i$ maps by degree $d$ to a component of $Q \setminus C_Q$. We can then write $Q \setminus C_Q = \overline{C}(1) \cup \overline{C}(2)$ such that $\pi : S_n(i) \to \overline{C}(i)$ is a degree $d$ cover, and then by Lemma 4.1 we can rejoin the boundary curves of $\overline{C}(1)$ to form $Q'(1)$ such that $S_n(1)$ is a cover of $Q'(1)$. But then we get a subgroup $G_{Q'}$ of $G_n(1)$ ($G_{Q'} = \pi_1(Q'(1))$), and $G_n(1) < G_{Q'} \cap G(1) < G(1)$, where both inclusions are proper. The first inclusion is proper because $A \hat{\pi} \in G_{Q'} \cap G(1) \setminus G_n(1)$, and the second is proper because $k < n$. This contradicts the assumption on the maximality of $G_n(1)$.

If $|C_Q| = 1$, we let $C_Q = \{C_Q\}$. First suppose that $C_Q$ is non-separating. Then writing $Q \setminus C_Q = \overline{Q}$ we find that $S_n(1)$ and $S_n(2)$ are both degree $d$ covers of $\overline{Q}$. But then we can reassemble $\overline{Q}$ to make $Q'$ (by Lemma 4.1).
such that $S_n(1)$ is a degree $\frac{d}{2}$ cover of $Q'$, when $\frac{d}{2} \leq k$. Then we arrive at a contradiction by the same reasoning as before.

Finally, suppose that $C_Q$ is separating. Then we can write $Q \setminus C_Q = Q(1) \cup Q(2)$ so that the restriction of $\pi$ to $S_n(i)$ is a cover of $Q(i)$. Then the conjugacy classes for $C_{1n}$ and $C_{2n}$, oriented as curves covered by the axis of $A$, are both in $[A^{2k}]$, but $C_{1n}$ and $C_{2n}$ both cover $C_Q$ with opposite orientations, so the conjugacy class for $C_Q$ must be both $[A^l]$ and $[A^{-l}]$, where $l = \frac{2k}{d}$. But then $B^{-1}A^lB = A^{-l}$ for some $B \in \Gamma$, which means that $B$ preserves the axis of $A$ and reverses its orientation; such $B$ would have a fixed point in $\mathbb{H}^3$, which is a contradiction.

\[\square\]

4.2. The lower bound. We now proceed to prove the lower bound

\[(7) \quad (c_1g)^{2g} \leq s_1(M^3, g),\]

for $g$ large enough, from Theorem 1.1.

By the above theorem the representation $\rho_n : \pi_1(S_n) \to \Gamma$, is maximal. It remains to count the number of such representations. Let $g_n$ denote the genus of the surface $S_n$. If $g_0$ denotes the genus of the surfaces $S(1)$ and $S(2)$, we have

\[g_n = n(2g_0 - 1).\]

Given a closed surface $S_0$, Let $m_n(S_0)$ denote the number of maximal degree $n$ covers of $S_0$. Let $C_0$ denote a simple closed and non-separating curve in $S_0$. For $1 \leq k \leq n$, by $m_n(S_0, C_0, k)$ we denote the number of maximal $n$ degree covers of $S_0$ such that the curve $C_0$ has at least one lift of degree $k$. Clearly the number $m_n(S_0, C_0, k)$ does not depend on the choice of the simple closed and non-non-separating curve $C_0$, so we sometimes write $m_n(S_0, k) = m_n(S_0, C_0, k)$.

**Theorem 4.3.** Let $g_0$ denote the genus of $S_0$. Then for $n$ large we have:

\[m_n(S_0) = (n!)^{g_0-2}(1 + o(1)),\]

where $o(1) \to 0$ when $n \to \infty$. Moreover, for some $1 \leq k \leq (n - 1)$, $k = k(n, g_0)$, we have

\[m_n(S_0, k) > ((n - 1)!)^{g_0-2}(1 + o(1)).\]

**Proof.** The first equality directly follows from Corollary 3 and the formula in Section 4.4 in [6], which shows that a random finite cover of a closed surface is maximal. It remains to prove the second inequality.

Since

\[\sum_{k=1}^{n} m_n(S_0, k) \geq m_n(S_0),\]
it follows that for some \(1 \leq k \leq n\), the second inequality in the statement of the theorem holds. The following lemma implies that this inequality holds for some \(1 \leq k \leq (n - 1)\).

**Lemma 4.2.** The inequality \(m_n(S_0, 1) \geq m_n(S_0, n)\), holds for every \(n\).

**Proof.** Let \(C_0\) and \(D_0\) be two simple closed and non-separating curves on \(S_0\), that intersect exactly once. Let \(S_n\) be a degree \(n\) cover of \(S_0\), such that the curve \(C_0\) has a degree \(n\) lift which we denote by \(C_n\). Then \(C_n\) is the only lift of \(C_0\). We show that in this case, every lift of the curve \(D_0\) is a degree one lift. Let \(\tilde{S}_0 = S_0 \setminus C_0\) and \(\tilde{S}_n = S_n \setminus C_n\), denote the two surfaces each having exactly two boundary components. Then \(\tilde{S}_n\) covers \(\tilde{S}_0\), because \(C_n\) is the only lift of \(C_0\) to \(S_n\). After removing the curve \(C_0\) from \(S_0\), the closed curve \(D_0\) becomes an interval \(I_0 \subset \tilde{S}_0\), whose endpoints lie on different boundary components of \(\tilde{S}_0\). Therefore, every lift of \(I_0\) to \(\tilde{S}_n\) is a degree one lift. This proves the statement.

Restricting to the cases when \(S_n\) is a maximal cover, yields the inequality \(m_n(S_0, C_0, n) \leq m_n(S_0, D_0, 1)\). Since \(m_n(S_0, C_0, k) = m_n(S_0, D_0, k) = m_n(S_0, k)\), it follows that \(m_n(S_0, 1) \geq m_n(S_0, n)\), and we have proved the lemma.

\(\square\)

This proves the theorem.

\(\square\)

Now fix a large \(n\) and choose \(1 \leq k \leq (n - 1)\) so that the second inequality in Theorem 4.3 holds. We then amalgamate any two maximal covers \(S_n(1)\) and \(S_n(2)\) along the curves \(C_n(1)\) and \(C_n(2)\) that are both \(k\) degree lifts of the curves \(C(1)\) and \(C(2)\) respectively (there may be more than one such \(k\) degree lift, but we choose arbitrarily). Then the corresponding group \(\rho_n(\pi_1(S_n)) < \Gamma\) is maximal surface subgroup of \(\Gamma\). Combining the above formula for \(g_n\) with the Theorem 4.3, we derive the estimate \(\Box\) for some \(c_1 > 0\).

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