ON THE GRAVITATIONAL REDSHIFTS OF SPECTRAL LINES
A CRITICO-HISTORICAL STUDY

ANGELO LOINGER AND TIZIANA MARSICO

Abstract. A beautiful, detailed computation by Whittaker has enabled us to prove in a rigorous way that the gravitationally redshifted frequency of a monochromatic e.m. wave sent forth at the surface of a celestial body is propagated unaltered from the emitting source to terrestrial observers. We remark that in the customary treatments only qualitative and inaccurate justifications of this fact are given.

Summary – 1 Aim of the paper. – 2. Observational and experimental data. – 3. The usual deduction of the gravitational redshift, and its weak point. – 4. Electromagnetic equations in an arbitrary gravitational field. – 5. Electromagnetic equations in the field of a gravitating body. – 6. Spherical e.m. waves starting from a gravitating body. – 7. Waves of very high frequency. – 7bis. The eikonal equation as the equation of the characteristics of Maxwell’s wave equations. – 8. Waves of arbitrary frequency. – Appendix: Geometrical optics in GR – Maxwellian and Einsteinian characteristic surfaces. –

PACS 04.20 – General relativity.

1. – As it is well known, there are three kinds of redshift: purely gravitational, cosmological, generated by a Doppler effect. In this paper we shall clarify a subtle point of the usual treatments of the first instance.

2. – For the observational data on the astrophysical gravitational redshifts, see the recent article by Pasquini et alii [1]. We limit ourselves to emphasize that evident gravitational redshifts have been detected in spectral lines of X-ray bursts on neutron-star surfaces [2].

Interesting terrestrial experiments, based on the Principle of Equivalence, have been made by Pound et alii in 1960 and in 1964 [3].
3. – For the theoretical definition of the gravitational redshift, it is suitable to consider static gravitational potentials $g_{jk}$, $(j, k = 0, 1, 2, 3)$, referred to static coordinate systems.

Let $\nu$ be the frequency of a monochromatic radiation sent forth by a given atom (at rest) at some point $P$ on the surface of the celestial body, which generates the field $g_{jk}$. We have:

\begin{equation}
\frac{1}{c} ds = \sqrt{g_{00}(P)} \, dt_P;
\end{equation}

for the radiation of frequency $\nu'$, emitted by an identical atom at a point $P'$ on the Earth, we can write analogously:

\begin{equation}
\frac{1}{c} ds = \sqrt{g_{00}(P')} \, dt_{P'};
\end{equation}

from which:

\begin{equation}
\nu = \nu' \frac{\sqrt{g_{00}(P)}}{\sqrt{g_{00}(P')}};
\end{equation}

since $\nu/\nu' = dt_P/dt_{P'}$. We see that $\nu < \nu'$, because $g_{00}(P) < g_{00}(P')$: gravitational redshift.

We recall that the wavelength $\lambda$ and the frequency $\nu$ satisfy the relation $\lambda \nu = V(t)$, if $V(t)$ is the propagation velocity of the e.m. wave through the pseudo-Riemannian spacetime. At the arrival on Earth, $V(t) \approx c$.

In a Newtonian approximation, we have $g_{00} = 1 - 2U/c^2$, where $U$ is the Newtonian potential which satisfies the equation $\nabla^2 U = -4\pi G \mu$, if $\mu$ is the mass density. Accordingly, $\sqrt{g_{00}} \approx 1 - U/c^2$, and we get:

\begin{equation}
\frac{\nu}{\nu'} = \frac{1 - U(P)/c^2}{1 - U(P')/c^2};
\end{equation}

from which:

\begin{equation}
\frac{\nu - \nu'}{\nu'} \approx \frac{U(P')}{c^2} - \frac{U(P)}{c^2}.
\end{equation}

It is customary to assert – as a consequence of generic, or seemingly intuitive, arguments – that the frequency $\nu$ of the considered radiation arrives unaltered at the terrestrial observers. Only Weyl [4] and v. Laue [5] remark, but without an explicit proof, that by virtue of Maxwell equations the e.m. monochromatic waves are propagated in a pseudo-Riemannian spacetime without any modification of their frequencies.

By making use of a result by Whittaker [6], we shall demonstrate with an accurate computation that effectively the frequency $\nu$ of the considered wave traverses unchanged the distance between the celestial body and the Earth.
4. With Whittaker [6], we consider the ideal case in which the gravitational action of the e.m. field $F_{jk}$, $(j, k = 0, 1, 2, 3)$, – action which is in general small – can be ignored. If $\Phi_j$ is the e.m. 4-potential, we have:

\[ F_{jk} = \Phi_{j;k} - \Phi_{k;j} = \Phi_{j,k} - \Phi_{k,j}, \]
where a colon and a comma denote respectively a covariant and an ordinary derivative. If $j_k$ is the 4-current, Maxwell equations tell us that

\[ g^{jk} \Phi_{n;k} = j_n, \quad (n = 0, 1, 2, 3), \]

if $R_{jk}$ is the contracted curvature tensor. The deduction of eqs. (4) implies the use of covariant Lorentz condition $g^{jk} \Phi_{j;k} = 0$.

We shall consider the gravitational field about a gravitating centre with a given mass $M$, and e.m. fields acting in the spatial region in which Ricci tensor $R_{jk}$ is equal to zero. Accordingly, eqs. (4) become:

\[ \Box \Phi_n = j_n; \quad (n = 0, 1, 2, 3), \]

if $\Box \equiv g^{jk} (\ldots ; j,k)$.

The equations for $F_{jk}$, which correspond to eqs. (4) for $\Phi_n$, are [7]:

\[ \Box F_{pq} + R_{mpq} F_{mp} - 2R_{pqlm} F_{ml} = j_p - j_q; \]

and for the region in which $R_{jk} = 0$:

\[ \Box F_{pq} - 2R_{pqlm} F_{ml} = j_p - j_q. \]

5. [6] starts from the metric of the Schwarzschild manifold as described by the standard (Hilbert-Droste-Weyl) form of the interval $ds$ (where $\alpha \equiv 2GM/c^2$):

\[ ds^2 = \left( \frac{r - \alpha}{r} \right) c^2 dt^2 - \left( \frac{r}{r - \alpha} \right) dr^2 - r^2 (d\vartheta^2 + \sin^2 \vartheta d\varphi^2). \]

He calculates the covariant derivatives $\Phi_{n;j}$ and $\Phi_{n;j,k}$; the substitution of their values in eqs. [5] gives a system of four partial differential equations for determining $\Phi_0$, $\Phi_1$, $\Phi_2$, $\Phi_3$. (see eqs. (21), (22), (23), (24) of [6]).

6. We consider systems of e.m. spherical waves with the centre at the centre of the gravitating spherical body, and having origin (or ending up) at the surface $(r > \alpha)$ of this body. With Whittaker, we specialize his eqs. (21)–(22)–(23)–(24) for the instance $\Phi_0 = \Phi_1 = \Phi_2 = 0$, $\Phi_3 \neq 0$. Also the 4-current $j_n$ is zero, except at the origin of the waves. In this way, only eq. (24) of [6] survives, in the following simpler form:
We require a $\Phi_3$ of the following kind (Whittaker denotes with $p$ the pulsation $\omega = 2\pi\nu$):
\[
\exp(i\omega t) \times \text{(a function of $r$ only)} \times \text{(a function of $\vartheta$ only)} - \text{say},
\]
\[
\Phi_3 = \exp(i\omega t) f(r) g(\vartheta).
\]

By substituting (8) in (7), we obtain:
\[
-\omega^2 \frac{r}{c^2(r-\alpha)} \frac{\partial^2 \Phi_3}{\partial t^2} - \frac{r-\alpha}{r^2} \frac{\partial^2 \Phi_3}{\partial r^2} - \frac{\alpha}{r} \frac{\partial \Phi_3}{\partial r} + \frac{1}{r^2} \cot \vartheta \frac{\partial \Phi_3}{\partial \vartheta} = 0.
\]

With a classic procedure, Whittaker finds that
\[
g(\vartheta) = \sin^2 \vartheta \frac{dP_n(\cos \vartheta)}{d(\cos \vartheta)},
\]
and $f(r)$ satisfies the equation
\[
f(r) = 0 \text{ (11)}
\]
where $n = 0, 1, 2, \ldots$; eq. (11) is similar to Mathieu’s equation [8], its solution $f(r)$ will be calculated in sects. 7. and 8.

7. - When the e.m. waves are of a very high frequency, so that the pulsation $\omega$ is very large, we can neglect in eq. (11) the term $-n(n+1)/r^2$ in comparison with $\omega^2 r/c^2(r-\alpha)$, and thus eq. (11) becomes
\[
r - \alpha \frac{\partial^2 f}{\partial r^2} + \frac{\alpha}{r^2} \frac{\partial f}{\partial r} + \left\{ \frac{\omega^2 r}{c^2(r-\alpha)} - \frac{n(n+1)}{r^2} \right\} f = 0 \text{ (12)}
\]

the solution is:
\[
f = A \exp(i\omega r/c) \cdot (r-\alpha)^{i\omega/c} + B \exp(-i\omega r/c) \cdot (r-\alpha)^{-i\omega/c} \text{ (13)}
\]
with $A$, $B$ arbitrary constants. Therefore we can write:
\[
\Phi_3 = \exp(i\omega t) \cdot \exp(\pm i\omega r/c)(r-\alpha)^{i\omega/c} \cdot \sin^2 \vartheta \frac{dP_n(\cos \vartheta)}{d(\cos \vartheta)} \text{ (14)}
\]
the signs plus and minus denote, respectively, convergent or divergent waves. If we write eq. (14) in the form
\[
\Phi_3 = [\text{a function of } (t \pm h(r))] \times [\text{a function of } \vartheta] \text{ (14')}.
\]
we see that the wave velocity at the point \((r, \theta)\) is \(1/(dh/dr)\); now, eq. \((14)\) tells us that \(h(r) = (r/c) + (\alpha/c) \ln |r - \alpha|\), from which \(dh/dr = r/c(r - \alpha)\); accordingly:

\[
\left| \frac{dr}{dt} \right| = c \left( 1 - \frac{\alpha}{r} \right).
\]

This value coincides with the velocity of the light-rays as null geodesics of metric \((6)\); indeed, \(ds^2 = 0\) gives:

\[
\frac{r - \alpha}{r} \frac{dr^2}{dt^2} - \frac{r}{c^2(r - \alpha)} dr^2 = 0,
\]

from which eq. \((15)\). As we shall see in the sequel, this coincidence is not casual.

7bis. — We recall that, both in the pre-relativistic and in the relativistic physics, the notion of geometrical optics is susceptible of two different interpretations. First interpretation: the eikonal equation represents the approximation of the wave equation for a very high frequency; the general-relativistic proof of this result is due to v. Laue \[9\]. The procedure of previous section 7 applies essentially this viewpoint. Second interpretation: the eikonal equation gives the propagation law of the wave-fronts, without any reference to wave frequencies; the general-relativistic proof of this result is due to Whittaker \[10\].

He started from eqs. \((4)\), written in the following form:

\[
g^{jk} \frac{\partial^2 \Phi_n}{\partial x^j \partial x^k} + \text{(terms not involving second derivatives of the } \Phi_n \text{'s)} = 0.
\]

His purpose was to find the characteristic surfaces (that are discontinuity – or singular – surfaces) of eqs. \((4)\). Now, formula \((17)\) recalls that the characteristics of a partial differential equation of the second order depend only on the terms involving the second derivatives of the solutions. Eqs. \((4)\) are of the hyperbolic kind, and therefore the differential equation of the characteristics (see eq. \((18)\), infra) gives also the law of motion of the wave-fronts; it coincides with the eikonal equation of the first interpretation.

The theory of the characteristics tells us that the functions \(z(x^0, x^1, x^2, x^3)\) describing the characteristic surfaces \(z = 0\) of \((17)\) are the solutions of the following differential equation of the first order:

\[
g^{jk} \frac{\partial z}{\partial x^j} \frac{\partial z}{\partial x^k} = 0.
\]

The characteristics of \((18)\), or bicharacteristics of \((17)\), are the light-rays, i.e. the null geodesics of the manifold, whose interval is given by \(ds^2 = g_{jk} dx^j dx^k\). Thus, starting from Maxwell electrodynamics, Whittaker found again two fundamental properties of the Einsteinian gravitation theory.
This result has a great conceptual meaning, and an important consequence, as it was emphasized by Levi-Civita [11] – see the Appendix.

Let us observe that in the first interpretation the light-rays are represented by very narrow beams of light of very high frequency – i.e., by physical objects –, while in the second interpretation they are represented by geometrical trajectories, without any limitation to the frequency values. Both e.m. interpretations and general relativity give consistently the value (15) for the velocity of the light-rays; the wavelength $\lambda$ does not remain unchanged as the frequency $\nu$, but is proportional to velocity. The light-rays are repulsed by the gravitating centre; $|dr/dt| = 0$ for $r = \alpha$, and is equal to $c$ for $r = \infty$.

A last remark. It is very easy to see that the characteristics of eqs. (15) coincide with the characteristics of eqs. (11), and are solutions of eq. (18).

8. – For the solution $f(r)$ of eq. (11) when the frequency of the e.m. waves is arbitrary, Whittaker – see §11 of [6] – starts from the following series:

\[
f(r) = \exp(\pm i\omega r/c) \cdot (r - \alpha)^{\pm i\omega\alpha/c} \left\{ 1 + \frac{h_1(r)}{\omega} + \frac{h_2(r)}{\omega^2} + \frac{h_3(r)}{\omega^3} + \ldots \right\} ;
\]

then, with some passages, he obtains from eq. (11) that

\[
h_{s+1}(r) = \frac{1}{2} ic \left\{ 1 + \frac{\alpha}{r} \right\} \frac{dh_s(r)}{dr} + n(n-1) \int_r^\infty \frac{h_s(r)dr}{r^2} ,
\]

from which:

\[
h_1(r) = \frac{n(n+1)ic}{2r} ; \quad h_2(r) = -\frac{n(n+1)c^2}{4} \left\{ \frac{(n+2)(n-1)}{2r^2} + \frac{\alpha}{r^3} \right\} , \text{etc. ;}
\]

with this determination of the series of eq. (19), the e.m. potential $\Phi_3$ of eq. (8) is completely known:

\[
\Phi_3(r, \vartheta) = \exp(i\omega t) f(r) \sin^2 \vartheta \cdot \frac{dP_n(\cos \vartheta)}{d(\cos \vartheta)} .
\]

We see that also the e.m. waves of arbitrary frequencies are propagated in a pseudo-Riemannian manifold with unaltered frequencies, and with the velocity given by eq. (15).

It is interesting to remark that a computation concerning spherical e.m. waves only dependent on the radial coordinate $r$, would give immediately the above result, since the solution $f(r)$ of

\[
\frac{r - \alpha}{r} \frac{d^2f}{dr^2} + \frac{1}{r^2} \frac{df}{dr} + \frac{\omega^2 r}{c^2(r - \alpha)} f = 0 ,
\]

is simply the $f(r)$ of eq. (13).
We give here a résumé of the formalism and of the main properties that characterize the geometrical optics in GR.

§1. – Let us consider the spacetime interval $d\mathbf{s}$ in a generic pseudo-Riemannian manifold:

$$d\mathbf{s}^2 = g_{jk}(x^0, x^1, x^2, x^3) \, dx^j \, dx^k, \quad (j,k = 0,1,2,3),$$

where $g_{00} > 0$, and $g_{\alpha\beta} \, dx^\alpha \, dx^\beta$, $(\alpha, \beta = 0,1,2,3)$, is negative definite.

Two basic equations ($\sigma$ is an affine parameter, and $x \equiv (x^0, x^1, x^2, x^3)$):

$$\begin{align*}
L & := g_{jk} \frac{dx^j}{d\sigma} \frac{dx^k}{d\sigma} = 0, \quad (\text{Lagrange-Monge}) ; \\
H & := \frac{1}{2} g^{jk} \frac{\partial z(x)}{\partial x^j} \frac{\partial z(x)}{\partial x^k} = 0, \quad (\text{Hamilton-Jacobi}) ;
\end{align*}$$

Lagrange equations ($\dot{x}^j := \frac{dx^j}{d\sigma}$)

$$\frac{\partial L}{\partial x^j} - \frac{d}{d\sigma} \left( \frac{\partial L}{\partial \dot{x}^j} \right) = 0$$

give the characteristic lines of (A.2), which coincide with the null geodesics.

Hamilton equations ($p_j := \frac{\partial z(x)}{\partial x^j}$)

$$\dot{x}^j = \frac{\partial H}{\partial p_j} ; \quad \dot{p}_j = - \frac{\partial H}{\partial x^j}$$

give the characteristic lines of (A.3), which coincide with the null geodesics.

There is a unique physical interpretation of these results, that holds both for $g_{jk}$’s dependent on, and for $g_{jk}$’s independent of, the time coordinate $x^0$: the null geodesics represent the light-rays, equation $z(x) = 0$ represents the wave-front of an electromagnetic wave. Remark that no use of Einstein field equations has been made.

§2. – In 1930 Levi-Civita [1] discovered that the differential equation of the characteristic surfaces of Einstein field equations coincides with eq. (A.3). He concluded immediately that a metric tensor with an undulatory nature is simply the support of an e.m. disturbance – in accord with the fact that the metric tensor “is” the spacetime. This conclusion was corroborated by Whittaker’s proof [10] that eq. (A.3) is also the differential equation of the characteristic surfaces of Maxwell’s wave equations. On the other hand, from the mathematical standpoint the equality
\[ g^{jk} \left[ z(x) \right] \frac{\partial z(x)}{\partial x^j} \frac{\partial z(x)}{\partial x^k} = 0 \]

is not an equation, but a trivial identity.

Levi-Civita’s interpretation is reinforced by the following facts:

i) No motion of masses generates undulatory \( g_{jk} \)’s \([2]\);

ii) For \( G \rightarrow 0 \), eq. (A.1) gives Minkowski’s interval, which admits only an e.m. interpretation;

iii) The absence of a set of privileged reference systems in GR tells us that the wave nature of a given \( g_{jk} \) can be always destroyed by a suitable change of spacetime coordinates;

iv) Last but not least: as Levi-Civita observed, an undulatory \( g_{jk} \), which is solution of Einstein equations \( R_{jk} = 0 \), does not have a true energy-tensor.

An inappropriate analogical comparison with Maxwell electrodynamics has generated the conviction that undulating \( g_{jk} \)’s have a physical reality.

§3. – Let us write Einstein equations in a reference system of harmonic coordinates \( y^0, y^1, y^2, y^3 \); we get:

\[ \frac{1}{2} g^{mn} \frac{\partial^2 g^{jk}}{\partial y^m \partial y^n} - g^{mr} g^{ns} \Gamma^j_{rs} \Gamma^k_{mn} = \kappa \left( T^{jk} - \frac{1}{2} g^{jk} T \right) ; \]

the general theory of the characteristics affirms that the differential equation of the characteristic surfaces depends only on the terms containing the derivatives of the highest order – in our case, the d’Alembertian terms of eqs. (A.7), from which eq. (A.3).

Now, we could obtain this same equation for any field, say \( U^{jk} \ldots \), whose equations of motion contain the same d’Alembertian terms of Einstein equations. this means that eq. (A.3) must satisfy the condition of describing the characteristics of a field different from the metric field \( g^{jk} \) – otherwise, it becomes a void identity.

References
[1] L. Pasquini et alii, arXiv:1011.4635 [astro.ph.SR] 21 Nov 2010; and references therein.
[2] J. Cottam et alii, Nature, 420 (2002) 51.
[3] R.V. Pound and G.A. Rebka, Jr., Phys. Rev. Lett., 4 (1960) 337; R.V. Pound and J.L. Snider, Phys. Rev. Lett., 13 (1964) 539. – See also C. Møller, The Theory of Relativity (Clarendon Press, Oxford) 1972, pp. 487-488.
[4] H. Weyl, Raum-Zeit-Materie, Siebente Auflage (Springer-Verlag, Berlin, etc.) 1988, p.244; at p.322 Weyl gives a formula for a gravitational redshift concerning arbitrary motions of source and observer. We remark, however, the superfluity of this result, because – as our Author emphasizes (see p.268) –: “Wie die beiden Körper sich auch bewegen mögen, immer kann ich durch Einführung eines geeigneten Koordinatensystems die beide zusammen auf Ruhe transformieren.”
ON THE GRAVITATIONAL REDSHIFTS OF SPECTRAL LINES

[5] M. v. Laue, *Die Relativit"atstheorie, Zweiter Band* (Friedr. Vieweg und Sohn, Braunschweig) 1956, p.111.
[6] E.T. Whittaker, *Proc. Roy. Soc. London*, 116 (1927) 720.
[7] A.S. Eddington, *The Mathematical Theory of Relativity*, Second Edition (Cambridge University Press, Cambridge) 1960, p.176.
[8] See E.T. Whittaker and G.N. Watson, *A Course of Modern Analysis - etc.*, Fourth Edition - reprinted (The University Press, Cambridge) 1958, Chapt. XIX.
[9] M. v. Laue, *Phys. Zeits.*, 21 (1920) 659 – and the book quoted in [5], sect. 36.
[10] E.T. Whittaker, *Proc. Cambridge Phil. Soc.*, 24/I (1927) 32.
[11] T. Levi-Civita, *Rend. Acc. Lincei*, s.6", 11 (1930) 3; Idem, *ibid.*, 113. – See also: A. Loinger, *arXiv:physics/0609161* (September 19th, 2006) – in Appendix: Some passages from the above memoirs by Levi-Civita.
[12] A. Loinger, *arXiv:physics/0606019* (June 2nd, 2006); Idem, *arXiv:0804.3991* [physics.gen-ph] 24 Apr 2008; Idem, *arXiv:1006.3844* [physics.gen-ph] 19 Jun 2010.

A.L. – Dipartimento di Fisica, Universit`a di Milano, Via Celoria, 16 - 20133 Milano (ITALY)
T.M. – Liceo Classico “G. Berchet”, Via della Commenda, 26 - 20122 Milano (ITALY)

*E-mail address: angelo.loinger@mi.infn.it*
*E-mail address: martiz64@libero.it*