Strong chromatic index of $k$-degenerate graphs

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Abstract

A strong edge coloring of a graph $G$ is a proper edge coloring in which every color class is an induced matching. The strong chromatic index $\chi_s'(G)$ of a graph $G$ is the minimum number of colors in a strong edge coloring of $G$. In this note, we improve a result by Dębski et al. [Strong chromatic index of sparse graphs, arXiv:1301.1992v1] and show that the strong chromatic index of a $k$-degenerate graph $G$ is at most $(4k-2)\Delta(G) - 2k^2 + 1$. As a direct consequence, the strong chromatic index of a 2-degenerate graph $G$ is at most $6\Delta(G) - 7$, which improves the upper bound $10\Delta(G) - 10$ by Chang and Narayanan [Strong chromatic index of 2-degenerate graphs, J. Graph Theory 73 (2013) (2) 119–126]. For a special subclass of 2-degenerate graphs, we obtain a better upper bound, namely if $G$ is a graph such that all of its $3^*$-vertices induce a forest, then $\chi_s'(G) \leq 4\Delta(G) - 3$; as a corollary, every minimally 2-connected graph $G$ has strong chromatic index at most $4\Delta(G) - 3$. Moreover, all the results in this note are best possible in some sense.

1 Introduction

A strong edge coloring of a graph $G$ is a proper edge coloring in which every color class is an induced matching. That is, an edge coloring is strong if for each edge $uv$, the color of $uv$ is distinct from the colors of the edges (other than $uv$) incident with $N_G(u) \cup N_G(v)$. The strong chromatic index $\chi_s'(G)$ of a graph $G$ is the minimum number of colors in a strong edge coloring of $G$. The degree of a vertex $v$ in $G$, denoted by $\deg(v)$, is the number of incident edges of $v$ in $G$. A vertex of degree $k$, at most $k$ and at least $k$ are called a $k$-vertex, $k^*$-vertex and $k^+$-vertex, respectively. The graph $(\emptyset, \emptyset)$ is an empty graph, and $(V, \emptyset)$ is an edgeless graph. We denote the minimum and maximum degrees of vertices in $G$ by $\delta(G)$ and $\Delta(G)$, respectively.

In 1985, Erdős and Nešetřil [5] constructed graphs with strong chromatic index $\frac{5}{2}\Delta^2$ when $\Delta$ is even, $\frac{1}{2}(5\Delta^2 - 2\Delta + 1)$ when $\Delta$ is odd. Inspired by their construction, they proposed the following strong edge coloring conjecture.

Conjecture 1. If $G$ is a graph with maximum degree $\Delta$, then

$$\chi_s'(G) \leq \begin{cases} \frac{5}{2}\Delta^2, & \text{if } \Delta \text{ is even;} \\ \frac{1}{2}(5\Delta^2 - 2\Delta + 1), & \text{if } \Delta \text{ is odd.} \end{cases}$$

A graph is $k$-degenerate if every subgraph has a vertex of degree at most $k$. Chang and Narayanan [3] showed the strong chromatic index of a 2-degenerate graph $G$ is at most $10\Delta(G) - 10$. Recently, Luo and Yu [6] improved the upper bound to $8\Delta(G) - 4$. For general $k$-degenerate graphs, Dębski et al. [4] presented an upper bound $(4k-1)\cdot\Delta(G) - k(2k+1) + 1$. Very recently, Yu [7] obtained an improved upper bound $(4k-2)\cdot\Delta(G) - 2k^2 + k + 1$. In this note, we use the method developed in [4] and improve the upper bound to $(4k-2)\cdot\Delta(G) - 2k^2 + 1$. In particular, when $G$ is a 2-degenerate graph, the strong chromatic index is at most $6\Delta(G) - 7$, which improves the upper bound $10\Delta(G) - 10$ by Chang and Narayanan [3]. In addition, we show that if $G$ is a graph such that all of its $3^*$-vertices induce a forest, then $\chi_s'(G) \leq 4\Delta(G) - 3$.

2 Results

Lemma 1 (Chang and Narayanan [3]). If $G$ is a $k$-degenerate graph with at least one edge, then there exists a vertex $w$ such that at least $\max\{1, \deg(w) - k\}$ of its neighbors are $k^*$-vertices.

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Theorem 2.1. If $G$ is a $k$-degenerate graph with maximum degree $\Delta$ and $k \leq \Delta$, then $\chi'_k(G) \leq (4k - 2)\Delta - 2k^2 + 1$.

Proof. By the definition of $k$-degenerate graph, every subgraph of $G$ is also a $k$-degenerate graph. We want to obtain a sequence $\Lambda_1, \ldots, \Lambda_m$ of subsets of edges as follows. Let $\Lambda_0 = \emptyset$. Suppose that $\Lambda_{i-1}$ is well-defined, let $G_i$ be the graph $G - \left( \Lambda_0 \cup \cdots \cup \Lambda_{i-1} \right)$. Notice that $G_i$ is an edgeless graph or a $k$-degenerate graph with at least one edge. Denote the degree of $v$ in $G_i$ by $\deg(v)$. If $G_i$ has at least one edge, then we choose a vertex $w_i$ of $G_i$ as described in Lemma 1, and let

$$\Lambda_i = \{ w, v \mid w, v \in E(G_i) \text{ and } \deg(w) \leq k \}.$$ 

Lemma 1 guarantees $\Lambda_i \neq \emptyset$, and this process terminates with a subset $\Lambda_m$. Note that the subgraph induced by $\Lambda_i$ is a star with center $w_i$, so we call $w_i$ the center of $\Lambda_i$.

Claim 1. If $i \neq j$, then $\Lambda_i$ and $\Lambda_j$ have distinct centers.

Proof of the Claim. Suppose to the contrary that there exists $j < i$ such that $\Lambda_j$ and $\Lambda_i$ have the same center $w$. Let $dv$ be an edge in $\Lambda_i$. From the construction, the vertex $v$ is the center of $\Lambda_j$, thus $\deg_{G_j}(v) \leq k$ and $\deg_{G_j}(v) \leq \deg_{G_j}(w) \leq k$. The fact $w \notin \Lambda_j$ implies that $\deg_{G_j}(v) > k$. Since $w \in \Lambda_i$, it follows that $\deg_{G_i}(v) \leq k$, and then there exists $j < t < i$ such that $v$ is the center of $\Lambda_t$. But $w \notin \Lambda_t$, which leads to a contradiction that $\deg_{G_t}(v) \leq k < \deg_{G_i}(w)$. This completes the proof of the claim. □

We color the edges from $\Lambda_{k-1}$ to $\Lambda_1$, we remind the readers this is the reverse order of the ordinary sequence.

In the following, we want to give an algorithm to obtain a strong edge coloring of $G$. First, we can color the edges in $\Lambda_m$ with distinct colors. Now, we consider the edge $w_i\mu_i$, where $w_i \in \Lambda_i$. We want to assign a color to $w_i\mu_i$ such that the resulting coloring is still a partial strong edge coloring. In order to guarantee the resulting coloring is a partial strong edge coloring in each step, any two edges which are incident with $N_G(w_i) \cup N_G(\mu_i)$ must receive distinct colors.

Now, we compute the number of colored edges in $G_i$, which are incident with $N_G(w_i) \cup N_G(\mu_i)$. Let $X_i = N_G(w_i) \setminus N_G(\mu_i)$, let $x_1 = |X_i|$, let $y_i = |\{ v \mid w_i \in E(G_i) \text{ and } \deg_{G_i}(v) \leq k \}|$, and let $z_i = |\{ v \mid w_i \in E(G_i) \text{ and } \deg_{G_i}(v) \geq k \}|$. Suppose that $x_i > 0$ and let $u$ be an arbitrary vertex in $X_i$. By the claim, the edge $uw_i$ is in some $\Lambda_i$ with center $w_i$, thus $\deg_{G_i}(w_i) \leq k$ and $w_i$ is incident with at most $k$ colored edges. Therefore, if $x_i \neq 0$, then the number of colored edges which are incident with $N_G(w_i) \setminus \{ w_i \}$ is at most

$$t_i = \frac{(x_i + y_i - 1) \cdot k + z_i \cdot \Delta}{\Delta} \leq (k - 1)k + (k - 2)(\Delta - k),$$

if $x_i = 0$, then the number of colored edges which are incident with $N_G(w_i) \setminus \{ w_i \}$ is at most

$$t_i = \frac{(y_i - 1) \cdot k + z_i \cdot \Delta}{\Delta} \leq (\Delta - 1)k + (\Delta - k) \cdot z_i \leq k - 2$$

if $x_i = 0$. Let $p_i = |N_G(\mu_i) \setminus N_G(w_i)|$, and let $q_i = |N_G(\mu_i)|$. If $\deg(\mu_i) > k$, then $p_i > 0$ and there exists some $s$ with $s < i$ such that $w_i$ is the center of $\Lambda_s$. It follows that $\deg_{G_{s+1}}(v) \leq k$ and for every edge $uw_i$ in $\Lambda_s$, the vertex $u$ is incident with at most $k - 1$ colored edges. Therefore, if $\deg(\mu_i) > k$, then the number of colored edges which are incident with $N_G(w_i) \setminus \{ w_i \}$ is at most

$$t_i = \frac{(k - 1) \cdot (\deg(\mu_i) - \deg_{G_{s+1}}(w_i)) + (\deg_{G_{s+1}}(w_i) - 1) \cdot \Delta}{\Delta} \leq 2k\Delta - k^2 - k.$$ 

if $\deg(\mu_i) \leq k$, then the number of colored edges which are incident with $N_G(w_i) \setminus \{ w_i \}$ is at most $(k - 1) \cdot \Delta$.

Hence, the number of colored edges incident with $N_G(w_i) \cup N_G(\mu_i)$ is at most

$$\max \{ k - 2 \Delta + k, 2k\Delta - k^2 - k \} + \max \{ 2(k - 1)\Delta + k(1 - k), (k - 1)\Delta \}$$

$$= (2\Delta - k^2 - k) + 2(k - 1)\Delta + k(1 - k)$$

$$= (4k - 2)\Delta - 2k^2.$$
Thus, there are at least one available color for \( w, v \). When all the edges are colored, we obtain a strong edge coloring of \( G \).

\[ \square \]

**Corollary 1.** If \( G \) is a 2-degenerate graph with maximum degree at least two, then \( \chi'_s(G) \leq 6\Delta(G) - 7 \).

**Remark 1.** The strong chromatic index of a 5-cycle is five, so it achieves the upper bound in Corollary 1, thus the obtained upper bound is best possible in some sense.

Next, we investigate a class of graphs whose all \( 3^+ \)-vertices induce a forest.

**Lemma 2.** If \( G \) is a graph with at least one edge such that all the \( 3^+ \)-vertices induce a forest, then there exists a vertex \( w \) such that at least \( \max\{1, \deg_G(v) - 1\} \) of its neighbors are \( 2^+ \)-vertices.

**Proof.** Let \( A \) be the set of \( 2^- \)-vertices. It is obvious that \( A \neq \emptyset \), otherwise every vertex has degree at least three, and then \( G \) contains cycles. Now, consider the graph \( G - A \). If \( G - A \) is an empty graph, then every nonisolated vertex satisfies the desired condition. So we may assume that the \( G - A \) has at least one vertex. Since the graph \( G - A \) is a forest, it follows that there exists at least one \( 1^- \)-vertex in \( G - A \), and every such vertex satisfies the desired condition. \[ \square \]

**Theorem 2.2.** If \( G \) is a graph such that all of its \( 3^+ \)-vertices induce a forest, then \( \chi'_s(G) \leq 4\Delta(G) - 3 \).

**Proof.** The proof is analogous to that in Theorem 2.1, so we omit it. \[ \square \]

A 2-connected graph \( G \) is **minimally 2-connected** if \( G - e \) is not 2-connected for each edge \( e \) in \( G \).

**Theorem 2.3 ([2]).** The minimum degree of a minimally 2-connected graph is two, and the subgraph induced by all the \( 3^+ \)-vertices is a forest.

**Corollary 2.** If \( G \) is a minimally 2-connected graph, then \( \chi'_s(G) \leq 4\Delta(G) - 3 \).

**Remark 2.** Once again the strong chromatic index of a 5-cycle achieves the upper bound in Theorem 2.2, so the upper bound is best possible in some sense.

A graph is chordless if every cycle is an induced cycle. It is easy to show that a 2-connected graph is chordless if and only if it is minimally 2-connected. Every chordless graph is a 2-degenerate graph. In [3], Chang and Narayanan proved that the strong chromatic index of a chordless graph is at most \( 8\Delta(G) - 6 \), and Dębski et al. [4] improved the upper bound to \( 4\Delta(G) - 3 \), but I doubt that their proofs are correct since they incorrectly used a lemma (see [3, Lemma 7]). Recently, Narayanan (private communication, Mar. 2014) has confirmed the mistake in the proof of the result \( 8\Delta(G) - 6 \). For more detailed discussion on the strong chromatic index of chordless graphs, we refer the reader to [1].

**Remark 3.** All the proofs used the greedy algorithm, thus all the results are true for the list version of strong edge coloring.

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**References**

[1] M. Basavaraju and M. C. Francis, Strong chromatic index of chordless graphs, eprint arXiv:1305.2009v2 (2013).

[2] B. Bollobás, Extremal graph theory, Dover Publications, Inc., Mineola, NY, 2004. xx+488 pp.

[3] G. J. Chang and N. Narayanan, Strong chromatic index of 2-degenerate graphs, J. Graph Theory 73 (2013) (2) 119–126.

[4] M. Dębski, J. Grytczuk and M. Śleszyńska-Nowak, Strong chromatic index of sparse graphs, eprint arXiv:1301.1992v1 (2013).

[5] P. Erdős, Problems and results in combinatorial analysis and graph theory, Discrete Math. 72 (1988) (1-3) 81–92.

[6] R. Luo and G. Yu, A note on strong edge-colorings of 2-degenerate graphs, eprint arXiv:1212.6092v1 (2012).

[7] G. Yu, Strong edge-colorings for \( k \)-degenerate graphs, eprint arXiv:1212.6093v3 (2013).