Estimating Subagging by cross-validation

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Abstract

In this article, we derive concentration inequalities for the cross-validation estimate of the generalization error for subagged estimators, both for classification and regressor. General loss functions and class of predictors with both finite and infinite VC-dimension are considered. We slightly generalize the formalism introduced by [DUD03] to cover a large variety of cross-validation procedures including leave-one-out cross-validation, $k$-fold cross-validation, hold-out cross-validation (or split sample), and the leave-$\nu$-out cross-validation.

An interesting consequence is that the probability upper bound is bounded by the minimum of a Hoeffding-type bound and a Vapnik-type bounds, and thus is smaller than 1 even for small learning set. Finally, we give a simple rule on how to subbag the predictor.

Keywords: Cross-validation, generalization error, concentration inequality, optimal splitting, resampling.
1 Introduction and motivation

One of the main issues of pattern recognition is to create a predictor (a regressor or a classifier) which takes observable inputs in order to predict the unknown nature of an output. Typical applications range from predicting the figures of a digitalized zip code to predicting the chance of survival from clinical measurements. Formally, a predictor $\phi$ is a measurable map from some measurable space $\mathcal{X}$ to some measurable space $\mathcal{Y}$. When $\mathcal{Y}$ is a countable set (respectively $\mathbb{R}^n$), the predictor is called a classifier (respectively a regressor). The strategy of Machine Learning consists in building a learning algorithm $\Phi$ from both a set of examples and a class of methods. Typical class of methods are empirical risk minimization or $k$-nearest neighbors rules. The set of examples consists in the measurement of $n$ observations $(x_i, y_i)_{1 \leq i \leq n}$. Thus, formally, $\Phi$ is a measurable map from $\mathcal{X} \times \mathcal{Y}$ to $\mathcal{Y}$. One of the main issues of Statistical Learning is to analyze the performance of a learning machine in a probabilistic setting. $(x_i, y_i)_{1 \leq i \leq n}$ are supposed to be observations from $n$ independent and identically distributed (i.i.d.) random variables $(X_i, Y_i)_{1 \leq i \leq n}$ with distribution $\mathbb{P}$. $(X_i, Y_i)_{1 \leq i \leq n}$ is denoted $D_n$ in the following and called the learning set. In order to analyze the performance, it is usual to consider the conditional risk of a machine learning $\Phi$ denoted $\tilde{R}_n$, so called the generalization error. It is defined by the conditional expectation of $L(Y, \Phi(X, D_n))$ given $D_n$ where $(X, Y) \sim \mathbb{P}$ is a random variable independent of $D_n$, i.e. $\tilde{R}_n := \mathbb{E}_{X,Y}(L(Y, \Phi(X, D_n))|D_n)$ with $L$ a cost function from $\mathcal{Y}^2 \rightarrow \mathbb{R}_+$. Notice that $\tilde{R}_n$ is a random variable measurable with respect to $D_n$.

Bagging, to be defined formally below, is a procedure building an estimator by a resample and combine technique. Bagging [bootstrap aggregating] was introduced by [?] to reduce the variance of a predictor. From an original estimator, a bagged regressor is produced by averaging several replicates trained on bootstrap samples, a bagged classifier is produced by voting at the majority. It is one of the recent and successful computationally intensive methods for improving unstable estimation or classification schemes. It is extremely useful for large, high dimensional data set problems where finding a good model or classifier in one step is impossible because of the complexity and scale of the problem. Regarding prediction error, the method often compares favorably with the original predictor, and also, in situations with substantial noise, with other ensemble methods such as boosting or randomization. Hence it is very important to understand the reasons for its successes, and also for its occasional failures. However, even if it has attracted much attention and is frequently applied, important questions remain unanswered theoretically. In this article, we study a variant of bagging called Subagging [Subsample aggregating] that has appeared in [?] and [?]. It is more accessible for analysis and has also substantial computational advantages. The subagged estimator will be denoted by $\Phi^B_n(X, D_n)$ or $\Phi^B_n(X)$ in the following.

Important questions are: *Is the generalization error of a subagged predictor lower than the original predictor, i.e $\tilde{R}_n(\Phi^B_n) \leq \tilde{R}_n(\Phi)$? The distribution $\mathbb{P}$ of the generating process being unknown, can we estimate the generalization error of a subagged predictor?* Our strategy is the following: after briefly emphasizing the difficulty to provide a general answer to the first question, we will concentrate on the second question. To estimate the generalization error of a subagged predictor, we propose to use an adapted cross-validation estimator denoted by $\tilde{R}_{CV}^B_n(\Phi)$.

[?] aggregates regression trees to build random forest and calls this process bagging. [?] prove that the bagged functional is always smooth in some sense. [?] also show that bagging can increase both bias and variance. [?] prove that (in the limit of infinite samples) bagging reduces the variance of non-linear components of the Taylor decomposition while leaving the linear part unaffected. [?] consider non-differentiable and discontinuous predictors and concentrate on the asymptotic smoothing effect of bagging on neighborhood of discontinuities of decision surfaces. [?] brings new argument to explain bagging effect: bagging’s improvement/deteriorations are explained by the goodness/badness of highly influential examples. [?] prove the effect of bagging on the stability of a learning method and derive non asymptotic bounds for the approximation error of the bagging predictor. An interesting asymptotic result was derived in [?]: asymptotically, bagging of weak predictors can produce a strong learner,
namely the bayes classifier. However, a general answer to the following non-asymptotic question $\tilde{R}_n(\Phi_B) \leq \tilde{R}_n(\Phi)$? seems hard to reach in a general framework. Using Gauss-Markov theorem, \cite{va} shows that both bagged and unbaggd predictor are unbiased, thus the variance of the unbagged predictor is lower than the variance of the bagged one. \cite{va} exhibit general quadratic statistics for which the bagged predictor increase both variance and bias. Thus, we propose to estimate directly the generalization error of the subagged predictor by an adapted cross-validation procedure. The latter is inspired by \cite{ka}, who proposed to use the left-out example of the bootstrap samples.

In the general setting, the cross-validation procedures include leave-one-out cross-validation, $k$-fold cross-validation, hold-out cross-validation (or split sample), leave-$\nu$-out cross-validation (or Monte Carlo cross-validation or bootstrap cross-validation). With the exception of \cite{be}, theoretical investigations of multifold cross-validation procedures have first concentrated on linear models (\cite{de} ; \cite{sh} ; \cite{zh}). Results of \cite{gi} and \cite{gy} are discussed in Section 3. The first finite sample results are due to Wagner and Devroye \cite{de} and concern $k$-local rules algorithms under leave-one-out and hold-out cross-validation. More recently, \cite{ho} \cite{ho2} derived finite sample results for $\nu$-out cross-validation, $k$-fold cross-validation, and leave-one-out cross-validation for ERM over a class of predictors with finite VC-dimension in the realisable case (the generalization error is equal to zero). \cite{bl} have emphasized when $k$-fold can beat $\nu$-out cross-validation in the particular case of $k$-fold predictor. \cite{ka} has extended such results in the case of stable algorithms for the leave-one-out cross-validation procedure. \cite{ke} also derived results for hold-out cross-validation for ERM, but their arguments rely on the traditional notion of VC-dimension. In the particular case of ERM over a class of predictors with finite VC-dimension but with general cross-validation procedures, \cite{ka} derived probability upper bounds. \cite{ka} derived upper bounds for general cross-validation estimate of the generalization error of stable predictors that do make reference to VC-dimension. However, these bounds obtained are called "sanity check bounds" since they are not better than classical Vapnik-Chernovenkis's bounds.

We introduce our main result for symmetric cross-validation procedures (i.e. the probability for an observation to be in the test set is independent of its index) in the special case of empirical risk minimization (ERM). We divide the learning sample into two samples: the training sample and the test sample, to be defined below. We denote by $p_n$ the percentage of elements in the test sample. Suppose that $\mathcal{H}$ holds, to be defined below. Suppose also that $\phi_n$ is an empirical risk minimizer. Then, we have for all $\varepsilon > 0$,

$$
\Pr(\tilde{R}_n(\Phi_B) - \tilde{R}_n^{\text{CV}} \geq \varepsilon) \leq \min(B_{\text{ERM}}(n, p_n, \varepsilon), V_{\text{ERM}}(n, p_n, \varepsilon)) < 1,
$$

with

- $B_{\text{ERM}}(n, p_n, \varepsilon) = \min((2np_n+1)^{4\varepsilon/p_n} \exp(-n\varepsilon^2), (2n(1-p_n)+1)^{4\varepsilon/p_n} \exp(-n\varepsilon^2/9))$
- $V_{\text{ERM}}(n, p_n, \varepsilon) = \exp(-2np_n\varepsilon^2)$.

The term $B(n, p_n, \varepsilon)$ is a Vapnik-Chernovenkis-type bound controlled by the size of the training sample $n(1-p_n)$ whereas the term $V(n, p_n, \varepsilon)$ is the minimum between a Hoeffding-type term controlled by the size of the test sample $np_n$, a polynomial term controlled by the size of the training sample. This bound can be interpreted as a quantitative answer to a trade-off issue. As the percentage of observations in the test sample $p_n$ increases, the term $V(n, p_n, \varepsilon)$ decreases but the term $B(n, p_n, \varepsilon)$ increases. Other similar bounds are derived for infinite VC-dimension machine learning in the stability framework.

The main interest of the previous results is in the following

- our bounds are valid for machine learning with both finite and infinite VC-dimension. In the latter, it is sufficient that the machine learning satisfies some stability property as introduced in chapter 2. As a motivation, we quote the following list of algorithms satisfying stability properties: regularization networks, ERM, k-nearest rules, boosting.
our bounds are strictly less than 1 for any size of learning set. Thus it is also valid for small samples.

Using these probability bounds, we can then deduce that the expectation of the difference between the generalization error and the cross-validation estimate

\[
\mathbb{E}_{\mathcal{D}_n} \hat{R}_n(\Phi_n^B) - \hat{R}_{CV}^{Out} \leq \min(\sqrt{1/np_n}, 0) \sqrt{\frac{V_c(\ln(n(1-p_n)) + 2)}{n(1-p_n)}}.
\]

Eventually, we define a splitting rule on how to chose the percentage of elements \( p_n^* \) in the test sample in order to get both a low generalization error together with a good approximation rate. We derive for this optimal choice of \( p^* \) a bound of the form

\[
\Pr(\hat{R}_n(\Phi_n^B) - \hat{R}_{CV}^{Out}(p_n^*) \geq \varepsilon) = O_n((n + 1)^8V_c\exp(-2n(\varepsilon - 2\sqrt{2V_c^{1/2}\ln(n)/n})^2/(1 - \exp(-2\varepsilon^2))).
\]

The paper is organized as follows. We detail the main cross-validation procedures and we summarize the previous results for the estimation of generalization error. In Section 3, we introduce the main notations and definitions. Finally, in Section 4, we introduce our results, in terms of concentration inequalities.

2 Main notations

In the following, we follow the notations of cross-validation introduced in [?].

We will consider the following shorter notations inspired by the literature on empirical processes. In the sequel, we will denote \( \mathcal{Z} := \mathcal{X} \times \mathcal{Y} \), and \( (Z_i)_{1 \leq i \leq n} := ((X_i, Y_i))_{1 \leq i \leq n} \) the learning set. For a given loss function \( L \) and a given class of predictors \( \mathcal{G} \), we define a new class \( \mathcal{F} \) of functions from \( \mathcal{Z} \) to \( \mathbb{R}_+ \) by \( \mathcal{F} := \{\psi \in \mathbb{R}_+^\mathcal{Z} | \psi(Z) = L(Y, \phi(X)), \phi \in \mathcal{G}\} \). For a machine learning \( \Phi \), we have the natural definition \( \Psi(Z, \mathcal{D}_n) = L(Y, \Phi(X, \mathcal{D}_n)) \). With these notations, the conditional risk \( \tilde{R}_n \) is the expectation of \( \Psi(Z, \mathcal{D}_n) \) with respect to \( \mathbb{P} \) conditionally on \( \mathcal{D}_n \): \( \tilde{R}_n := \mathbb{E}_Z[\Psi(Z, \mathcal{D}_n) | \mathcal{D}_n] \) with \( Z \sim \mathbb{P} \) independent of \( \mathcal{D}_n \). In the following, if there is no ambiguity, we will also allow the following notation \( \psi(X, \mathcal{D}_n) \) instead of \( \Psi(X, \mathcal{D}_n) \).

To define the accurate type of cross-validation procedure, we introduce binary vectors. Let \( V_n = (V_{n,i})_{1 \leq i \leq n} \) be a vector of size \( n \). \( V_n \) is a binary vector if for all \( 1 \leq i \leq n, V_{n,i} \in \{0, 1\} \) and if \( \sum_{i=1}^{n} V_{n,i} \neq 0 \). Consequently, we can define the subsample associated with it: \( \mathcal{D}_{V_n} := \{Z_i \in \mathcal{D}_n | V_{n,i} = 1, 1 \leq i \leq n\} \). We define a weighted empirical measure on \( \mathcal{Z} \)

\[
\mathbb{P}_{n,V_n} := \frac{1}{\sum_{i=1}^{n} V_{n,i}} \sum_{i=1}^{n} V_{n,i} \delta_{Z_i},
\]

with \( \delta_{Z} \) the Dirac measure at \( \{Z_i\} \). We also define a weighted empirical error \( \mathbb{P}_{n,V_n,\psi} \) where \( \mathbb{P}_{n,V_n,\psi} \) stands for the usual notation of the expectation of \( \psi \) with respect to \( \mathbb{P}_{n,V_n} \). For \( \mathbb{P}_{n,1_n} \), with \( 1_n \) the binary vector of size \( n \) with 1 at every coordinate, we will use the traditional notation \( \mathbb{P}_n \). For a predictor trained on a subsample, we define

\[
\psi_{V_n}(.) := \Psi(., \mathcal{D}_{V_n}).
\]

With the previous notations, notice that the predictor trained on the learning set \( \psi(., \mathcal{D}_n) \) can be denoted by \( \psi_{1_n}(.) \). We will allow the simpler notation \( \psi_n(.) \). The learning set is divided into two disjoint sets: the training set of size \( n(1 - p_n) \) and the test set of size \( np_n \), where \( p_n \) is the percentage of elements in the test set. To represent the training set, we define \( V_{n}^{tr} \) a random binary vector of size
\( n \) independent of \( D_n \). \( V_n^{tr} \) is called the training vector. We define the test vector by \( V_n^{ts} := 1_n - V_n^{tr} \) to represent the test set.

The distribution of \( V_n^{tr} \) characterizes all the subagging procedures described in the previous section. Using our notations, we can now define the bagged predictor.

**Definition 1 (Subagged regressor)** The subagged predictor built from \( \phi_n \) denoted \( \phi_n^B \) is defined by:

\[
\phi_n^B(\cdot) := E_{V_n^{tr}} \phi_{V_n^{tr}}(\cdot).
\]

In the case of classifiers, the bagging rule corresponds to the vote by majority. We suppose in this case that \( Y = \{1, \ldots, M\} \).

**Definition 2 (Subagged classifier)** Cross-validated subagged classifiers of \( \phi_n^B \) defined by:

\[
\phi_n^B(X) := \arg \min_{k \in \{1, \ldots, M\}} E_{V_n^{tr}} L(k, \Phi(X, D_n^{tr})).
\]

We can now define the cross-validation estimator.

**Definition 3 (Cross-validated subagged estimator)** Cross-validated subagged estimates of \( \phi_n^B \) denoted can be defined in two different ways by:

\[
\hat{R}_{CV}^{Out}(\Phi_n^B) := E_{V_n^{tr}} P_{n, V_n^{tr}}(\psi_{V_n^{tr}})
\]

and

\[
\hat{R}_{CV}^{In}(\Phi_n^B) := E_{V_n^{tr}} P_{n, V_n^{tr}}(\psi_{V_n^{tr}})
\]

**Remark 4** Recall that \( E_{V_n^{tr}} P_{n, V_n^{tr}}(\psi_{V_n^{tr}}) \) is the conditional expectation of \( P_{n, V_n^{tr}}(\psi_{V_n^{tr}}) \) with respect to the random vector \( V_n^{tr} \) given \( D_n \).

**Remark 5** The cross-validated subagged estimate differs from the usual cross-validation estimate of \( \hat{R}_{CV}^{Out}(\Phi_n^B) \) which is equal to \( E_{U_n^{tr}} P_{n, U_n^{tr}}(\psi_{U_n^{tr}}) \) with \( U_n^{tr} \) the training vector as defined in chapter 1.

We will give here a few examples of distributions of \( V_n^{tr} \) to show we retrieve subagging procedures described previously. Suppose \( n/k \) is an integer. The \( k \)-fold subagging procedure divides the data into \( k \) equally sized folds. It then produces a predictor by training on \( k - 1 \) folds. This is repeated for each fold, and the trained predictors are averaged to form the subagged predictor.

**Example 6 (\( k \)-fold cross-validation)**

\[
\Pr(V_n^{tr} = (\underbrace{0, \ldots, 0}_{n/k\ observations}, \underbrace{1, \ldots, 1}_{n(1-1/k)\ observations})) = \frac{1}{k}
\]

\[
\Pr(V_n^{tr} = (\underbrace{1, \ldots, 1}_{n/k\ observations}, \underbrace{0, \ldots, 0}_{n/k\ observations}, \underbrace{1, \ldots, 1}_{n(1-2/k)\ observations})) = \frac{1}{k}
\]

\[
\ldots
\]

\[
\Pr(V_n^{tr} = (\underbrace{1, \ldots, 1}_{n(1-1/k)\ observations}, \underbrace{0, \ldots, 0}_{n/k\ observations})) = \frac{1}{k}
\]

We provide another popular example: the leave-one-out cross-validation. In leave-one-out cross-validation, a single sample of size \( n \) is used. Each member of the sample in turn is removed, the full modeling method is applied to the remaining \( n - 1 \) members, and the fitted model is applied to the hold-back member.
Example 7 (leave-one-out cross-validation)

\[
\begin{align*}
Pr(V^r_n = (0, 1, \ldots, 1)) &= \frac{1}{n} \\
Pr(V^r_n = (1, 0, 1, \ldots, 1)) &= \frac{1}{n} \\
\cdots \\
Pr(V^r_n = (1, \ldots, 1, 0)) &= \frac{1}{n}.
\end{align*}
\]

3 Results for the cross-validated subagged regressor

3.1 VC Framework

3.1.1 Notations and definition

We denote by \( R_{\text{opt}} \) the minimal generalization error attained among the class of predictors \( C \), \( R_{\text{opt}} = \inf_{\phi \in C} R(\phi) \). In the sequel, we suppose that \( \phi_n \) belongs to some \( C \). Notice that \( R_{\text{opt}} \) is a parameter of the unknown distribution \( P(X,Y) \) whereas \( \tilde{R}_n \) is a random variable.

At last, recall the definitions of:

Definition 8 (Shatter coefficients) Let \( A \) be a collection of measurable sets. For \((z_1,\ldots,z_n) \in \{\mathbb{R}^d\}^n\), let \( N_A(z_1,\ldots,z_n) \) be the number of different sets in \( \{\{z_1,\ldots,z_n\} \cap A; A \in A\} \)

The \( n \)-shatter coefficient of \( A \) is

\[
S(A, n) = \max_{(z_1,\ldots,z_n) \in \{\mathbb{R}^d\}^n} N_A(z_1,\ldots,z_n)
\]

That is, the shatter coefficient is the maximal number of different subsets of \( n \) points that can be picked out by the class of sets \( A \).

and

Definition 9 (VC dimension) Let \( A \) be a collection of sets with \( A \geq 2 \). The largest integer \( k \geq 1 \) for which \( S(A,k) = 2^k \) is denoted by \( V_C \), and it is called the Vapnik-Chernovenkis dimension (or VC dimension) of the class \( A \). If \( S(A,n) = 2^n \) for all \( n \), then by definition \( V_C = \infty \).

A class of predictors \( C \) is said to have a finite VC-dimension \( V_C \) if the dimension of the collection of sets \( \{A_{\phi,t} : \phi \in C, t \in [0,1]\} \) is equal to \( V_C \), where \( A_{\phi,t} = \{(x,y)/L(y,\phi(x)) > t\} \).

3.1.2 Results

In the sequel, we suppose that the cross-validation is symmetric (i.e. \( Pr(V_{n,i} = 1) \) is independent of \( i \)) and the number of elements in the training set is constant and equal to \( np_n \), that the training sample and the test sample are disjoint and that the number of observations in the training sample and in the test sample are respectively \( n(1-p_n) \) and \( np_n \). Moreover, we suppose also that \( \phi_n \) belongs to a class of predictor with finite VC-dimension. Suppose also that \( L \) is bounded in the following way: \( L(Y,\phi(X)) \leq C(h(Y,\phi(X)) \) with \( C \) convex function -bounded itself by 1 on the support of \( h(Y,\phi_{\text{TRn}}(X)) \) for simplicity-, and \( h \) such that for any \( 0 < \lambda < 1 \), we have \( h(y, \lambda \phi(x_1) + (1-\lambda)\phi(x_2)) \leq \lambda h(y, \phi(x_1)) + (1-\lambda)h(y, \phi(x_2)) \). We will also suppose that the predictors are symmetric according to the training sample, i.e. the predictor does not depend on the order of the observations in \( D_n \). We denote these hypotheses by \( \mathcal{H} \).
To prove our result, we proceed now in two steps. For this, we consider

Thus, we obtain: \( \Pr(\tilde{R}_n(\Phi_n^B) - \hat{R}^\text{Out}_{CV} \geq \varepsilon) \leq \min(B(n,p_n,\varepsilon), V(n,p_n,\varepsilon)) \) with \( \varepsilon > 0 \). The term \( B(n,p_n,\varepsilon) \) is a Vapnik-Chervonenkis-type bound whereas the term \( V(n,p_n,\varepsilon) \) is a Hoeffding-type term controlled by the size of the test sample \( np_n \). This bound can be interpreted as a quantitative answer to a trade-off question. As the percentage of observations in the test sample \( p_n \) increases, the \( V(n,p_n,\varepsilon) \) term decreases but the \( B(n,p_n,\varepsilon) \) term increases.

**Theorem 11 (Absolute error for symmetric cross-validation)** Suppose that \( \mathcal{H} \) holds. Then, we have for all \( \varepsilon > 0 \),

\[
\Pr(\tilde{R}_n(\Phi_n^B) - \hat{R}^\text{Out}_{CV} \geq \varepsilon) \leq \min(B_{\text{sym}}(n,p_n,\varepsilon), V_{\text{sym}}(n,p_n,\varepsilon)) < 1
\]

with

- \( B_{\text{sym}}(n,p_n,\varepsilon) = (2np_n + 1)^{4V/p_n} e^{-n \varepsilon^2} \)
- \( V_{\text{sym}}(n,p_n,\varepsilon) = \exp(-2np_n \varepsilon^2) \).

**Remark 12** We do not require \( \phi_n \) to be an empirical risk minimizer.

**Proof.**

We have \( \tilde{R}_n(\Phi_n^B) = \mathbb{P}_{\psi_n^B} = \mathbb{P}L(Y, \mathbb{E}_{V^*} \phi_{V^*}(X)) \). Since \( C \) is a convex function -bounded itself by 1 on the support of \( h(Y, \phi_{V^*}(X)) \), and \( h \) linear in the second variable, we get

\[
\tilde{R}_n(\Phi_n^B) \leq \mathbb{P}C(h(Y, \mathbb{E}_{V^*} \phi_{V^*}(X))) \leq \mathbb{E}_{V^*} \mathbb{P}C(h(Y, \phi_{V^*}(X))
\]

Then, we split according to \( \mathbb{E}_{V^*} \mathbb{P}_{n,V^*} C(h(Y, \phi_{V^*}(X))) \):

\[
\tilde{R}_n(\Phi_n^B) \leq \mathbb{E}_{V^*} \mathbb{P}_{n,V^*} C(h(Y, \phi_{V^*}(X))) + \mathbb{E}_{V^*} (\mathbb{P} - \mathbb{P}_{n,V^*}) C(h(Y, \phi_{V^*}(X))
\]

\[
= \hat{R}^\text{Out}_{CV} + \mathbb{E}_{V^*} (\mathbb{P} - \mathbb{P}_{n,V^*}) C(h(Y, \phi_{V^*}(X))
\]

Thus, we obtain: \( \Pr(\tilde{R}_n(\psi_n^B) - \hat{R}^\text{Out}_{CV} \geq \varepsilon) \leq \Pr(\mathbb{E}_{V^*} (\mathbb{P} - \mathbb{P}_{n,V^*}) C(h(Y, \phi_{V^*}(X)) \geq \varepsilon) \).

To prove our result, we proceed now in two steps. For this, we consider

\[
\mathbb{E}_{V^*} (\mathbb{P}_{n,V^*} C(h(Y, \phi_{V^*}(X)) - \mathbb{P}C(h(Y, \phi_{V^*}(X)))
\]

in two different ways

1. using conditional Hoeffding’s inequality,
2. using Vapnik-Chernovenkis-type inequality to bound the supremum over a class.

1. First, by conditional Hoeffding arguments (for a proof, see e.g. chapter 1),

\[
\Pr(\tilde{R}_n(\Phi_n^B) - \hat{R}^\text{Out}_{CV} \geq \varepsilon) \leq \exp(-2np_n \varepsilon^2).
\]
2. Secondly, we derive the bound:
\[
\Pr(\tilde{R}_n(B) - \hat{R}_{CV}^{{\text{out}}} \geq \varepsilon) \leq \Pr(\mathbb{E}_{\nu_{\varepsilon}^H}(\mathbb{P} - \mathbb{P}_{n, \nu_{\varepsilon}^H})C(h(Y, \phi_{\nu_{\varepsilon}^H}(X)) \geq \varepsilon) \\
\leq \Pr(\mathbb{E}_{\nu_{\varepsilon}^H} \sup_{\phi \in \mathcal{C}}(\mathbb{P} - \mathbb{P}_{n, \nu_{\varepsilon}^H})C(h(Y, \phi(X)) \geq \varepsilon).
\]

Recall a useful lemma (for the proof, see Appendices).

**Lemma 13** Under the assumptions \(\mathcal{H}\), we have for all, \(\varepsilon > 0\),
\[
\Pr(\mathbb{E}_{\nu_{\varepsilon}^H} \sup_{\phi \in \mathcal{C}}(\mathbb{P} - \mathbb{P}_{n, \nu_{\varepsilon}^H})C(h(Y, \phi(X)) \geq \varepsilon) \leq (S(2np_n, C))^{4/p_ne^{-n\varepsilon^2}}.
\]
and we also have (for the proof, see e.g. [DGL96]): \(\forall n, S(n, C) \leq (n + 1)^{V_c}\).

Thus, it follows that \(\Pr(\tilde{R}_n(B) - \hat{R}_{CV}^{{\text{out}}} \geq \varepsilon) \leq (2np_n + 1)^{\frac{4V_c}{p_n}e^{-n\varepsilon^2}}\).

Putting altogether, we get \(\Pr(\tilde{R}_n(B) - \hat{R}_{CV}^{{\text{out}}} \geq \varepsilon) \leq \min(\exp(-2np_n\varepsilon^2), (2np_n + 1)^{\frac{4V_c}{p_n}e^{-n\varepsilon^2}})\).

\[\square\]

**Theorem 14** (Absolute error for symmetric cross-validation) Suppose that \(\mathcal{H}\) holds. Then, we have for all \(\varepsilon > 0\),
\[
\Pr(\tilde{R}_n(B) - \hat{R}_{CV}^{{\text{in}}} \geq \varepsilon) \leq \min(B_{\text{sym}}(n, p_n, \varepsilon), V_{\text{sym}}(n, p_n, \varepsilon)) < 1
\]

with
- \(B_{\text{sym}}(n, p_n, \varepsilon) = (2n(1 - p_n) + 1)^{\frac{4V_c}{p_n}e^{-n\varepsilon^2}}\)
- \(V_{\text{sym}}(n, p_n, \varepsilon) = \exp(-2np_n\varepsilon^2)\).

**Proof.**

We proceed as previously: \(\tilde{R}_n(B) = P\Phi_n = P\mathbb{L}(Y, \mathbb{E}_{\nu_{\varepsilon}^H} \phi_{\nu_{\varepsilon}^H}(X)) \leq \mathbb{P}(h(Y, \mathbb{E}_{\nu_{\varepsilon}^H} \phi_{\nu_{\varepsilon}^H}(X)) \leq \mathbb{E}_{\nu_{\varepsilon}^H} \mathbb{P}(h(Y, \phi_{\nu_{\varepsilon}^H}(X))

We then split this quantity according to \(\mathbb{E}_{\nu_{\varepsilon}^H} \mathbb{P}_{n, \nu_{\varepsilon}^H}(h(Y, \phi_{\nu_{\varepsilon}^H}(X))

\[
\tilde{R}_n(B) \leq \mathbb{E}_{\nu_{\varepsilon}^H} \mathbb{P}_{n, \nu_{\varepsilon}^H}(h(Y, \phi_{\nu_{\varepsilon}^H}(X)) + \mathbb{E}_{\nu_{\varepsilon}^H}(\mathbb{P} - \mathbb{P}_{n, \nu_{\varepsilon}^H})C(h(Y, \phi_{\nu_{\varepsilon}^H}(X))
\]

Thus, we get
\[
\Pr(\tilde{R}_n(B) - \hat{R}_{CV}^{{\text{in}}} \geq \varepsilon) \leq \Pr(\mathbb{E}_{\nu_{\varepsilon}^H}(\mathbb{P} - \mathbb{P}_{n, \nu_{\varepsilon}^H})C(h(Y, \phi_{\nu_{\varepsilon}^H}(X)) \geq \varepsilon)
\]
\[
\leq \Pr(\mathbb{E}_{\nu_{\varepsilon}^H} \sup_{\phi \in \mathcal{C}}(\mathbb{P} - \mathbb{P}_{n, \nu_{\varepsilon}^H})C(h(Y, \phi(X)) \geq \varepsilon).
\]

Recall two useful results (for the proof, see e.g. chapter 1)

**Lemma 15** Under the assumptions \(\mathcal{H}\), we have for all \(\varepsilon > 0\),
\[
\Pr(\mathbb{E}_{\nu_{\varepsilon}^H} \sup_{\phi \in \mathcal{C}}(\mathbb{P}(\phi) - \mathbb{P}_{n, \nu_{\varepsilon}^H}(\phi)) \geq \varepsilon) \leq (S(2n(1 - p_n), C))^{4/(1-p_n)e^{-n(1-p_n)e^2}}.
\]
In the special case of empirical risk minimization, we can obtain a stronger result.

**Theorem 16 (Absolute error for symmetric cross-validation)** Suppose that \( \mathcal{H} \) holds. Suppose also that \( \phi_n \) is based on empirical risk minimization. But instead of minimizing \( \tilde{R}_n(\phi) \), we suppose \( \phi_n \) minimizes \( \frac{1}{n} \sum_{i=1}^n C(h(Y_i, \phi(X_i)) \). For simplicity, we suppose the infimum is attained i.e. \( \phi_n = \arg\min_{\phi \in \mathcal{C}} \frac{1}{n} \sum_{i=1}^n C(h(Y_i, \phi(X_i)) \). Then, we have for all \( \varepsilon > 0 \),

\[
\Pr(\tilde{R}_n(\Phi_n^B) - \tilde{R}_{CV}^B \geq \varepsilon) \leq \min(\text{B}_{\text{ERM}}(n, p_n, \varepsilon), \text{V}_{\text{ERM}}(n, p_n, \varepsilon)) < 1,
\]

with

- \( \text{B}_{\text{ERM}}(n, p_n, \varepsilon) = \min((2np_n + 1)^{4V \epsilon/ p_n} \exp(-n\varepsilon^2), (2n(1 - p_n) + 1)^{4V \epsilon/ p_n} \exp(-n\varepsilon^2/9)) \)
- \( \text{V}_{\text{ERM}}(n, p_n, \varepsilon) = \exp(-2np_n \varepsilon^2) \).

**Remark 17**

1. The assumption \( \phi_n = \arg\min_{\phi \in \mathcal{C}} \frac{1}{n} \sum_{i=1}^n C(h(Y_i, \phi(X_i)) \) is not so restrictive, since in practice in order to numerically minimize \( \frac{1}{n} \sum_{i=1}^n L(Y_i, \phi(X_i)) \), one looks for \( C \) convex such that for all \( x, y \), \( L(y, \phi(x)) \leq C(h(y, \phi(x)) \).

2. Thanks to the Hoeffding’s part, the bound is always smaller than 1, so it remains valid for small samples. For bigger samples, we will prefer the Vapnik-Chervonenkis’s part.

**Proof.**

Applying the previous result, we have \( \Pr(\tilde{R}_n(\Phi_n^B) - \tilde{R}_{CV}^B \geq \varepsilon) \leq \min(\exp(-2np_n \varepsilon^2), (2np_n + 1)^{4V \epsilon/ p_n} \exp(-n\varepsilon^2)) \).

Recall that \( \tilde{R}_n(\Phi_n^B) - \tilde{R}_{CV}^B \leq \mathbb{E}_{\mathcal{V}^n} (\mathbb{P}(C(h(Y, \phi_{\mathcal{V}^n}(X)) - \mathbb{P}_{n, \mathcal{V}^n}(C(h(Y, \phi_{\mathcal{V}^n}(X)))). \)

We need the following lemma (for a proof, see chapter 1): \( \mathbb{E}_{\mathcal{V}^n} \mathbb{P}_{n, \mathcal{V}^n} C(h(Y, \phi_{\mathcal{V}^n}(X)) \geq \mathbb{P}_{n} C(h(Y, \phi_{\mathcal{V}^n}(X)) \)

since \( \phi_n = \arg\min_{\phi \in \mathcal{C}} \frac{1}{n} \sum_{i=1}^n C(h(Y_i, \phi(X_i)) \).

Denote \( \psi(Z) := C(h(Y, \phi(X))) \) with \( Z := (X, Y) \). We have the following natural notation \( \psi_{\mathcal{V}^n}(Z) := C(h(Y, \phi_{\mathcal{V}^n}(X))) \).

We thus get

\[
\Pr(\tilde{R}_n(\Phi_n^B) - \tilde{R}_{CV}^B \geq 3\varepsilon) \leq \Pr(\mathbb{E}_{\mathcal{V}^n} (\mathbb{P}_{n, \mathcal{V}^n} \psi_{\mathcal{V}^n} - \mathbb{P}_{n} \psi_{\mathcal{V}^n}) \geq 3\varepsilon) \leq \Pr(\mathbb{E}_{\mathcal{V}^n} (\mathbb{P}_{\psi_{\mathcal{V}^n}} - \mathbb{P}_{n} \psi_{\mathcal{V}^n}) \geq 3\varepsilon)
\]

and by splitting according to \( \mathbb{P}\psi_{\text{opt}} \), we have:

\[
\Pr(\tilde{R}_n(\Phi_n^B) - \tilde{R}_{CV}^B \geq 3\varepsilon) \leq \Pr(\mathbb{E}_{\mathcal{V}^n} (\mathbb{P}_{\psi_{\mathcal{V}^n}} - \mathbb{P}_{n, \mathcal{V}^n} \psi_{\mathcal{V}^n} + \mathbb{P}_{n, \mathcal{V}^n} \psi_{\mathcal{V}^n} - \mathbb{P}_{\psi_{\text{opt}}} + \mathbb{P}_{\psi_{\text{opt}} - \mathbb{P}_{n} \psi_{\mathcal{V}^n}) \geq 3\varepsilon)
\]

\[
\leq \Pr(\mathbb{E}_{\mathcal{V}^n} \sup_{\psi \in \mathcal{F}} (\mathbb{P}_{\psi} - \mathbb{P}_{n, \mathcal{V}^n} \psi) \geq \varepsilon) + \Pr(\sup_{\psi \in \mathcal{F}} (\mathbb{P}_{n, \mathcal{V}^n} \psi - \mathbb{P}_{\psi}) \geq \varepsilon)
\]

+ \( \Pr(\sup_{\psi \in \mathcal{F}} (\mathbb{P}_{\psi} - \mathbb{P}_{n} \psi) \geq \varepsilon). \)

Recall the following lemma (for the proof, see e.g. chapter 1),
Lemma 18 Under the assumption of Proposition ??, we have for all \( \varepsilon > 0 \),
\[
\Pr \left( \mathbb{E}_{\psi \in \mathcal{F}} \sup_{\psi \in \mathcal{F}} (\mathbb{P}_{n,V_{tsn}} \psi - \mathbb{P}_n) \geq \varepsilon \right) \leq (S(2n(1 - p_n), \mathcal{C})) \frac{4}{1 - p_n} e^{-n \varepsilon^2}
\]
and symmetrically
\[
\Pr \left( \mathbb{E}_{\psi \in \mathcal{F}} \sup_{\psi \in \mathcal{F}} (\mathbb{P}_n - \mathbb{P}_{n,V_{tsn}} \psi) \geq \varepsilon \right) \leq (S(2n(1 - p_n), \mathcal{C})) \frac{4}{1 - p_n} e^{-n \varepsilon^2}.
\]

Then, we get
\[
\Pr(\tilde{R}_n(\psi^B_n) - \hat{R}^{Out}_{CV} \geq 3\varepsilon) \leq 2(2n(1 - p_n) + 1)^4 \frac{4V_c}{p_n} e^{-n \varepsilon^2}.
\]

This implies in turn that
\[
\Pr(\tilde{R}_n(\psi^B_n) - \hat{R}^{Out}_{CV} \geq \varepsilon) \leq (2n(1 - p_n) + 1)^4 \frac{4V_c}{p_n} e^{-n \varepsilon^2} \exp(-n\varepsilon^2/9).
\]

Putting altogether, we get
\[
\Pr(\tilde{R}_n(\psi^B_n) - \hat{R}^{Out}_{CV} \geq \varepsilon) \leq \min(B_k(n, p_n, \varepsilon), V_k(n, p_n, \varepsilon))
\]

\[\text{with}
\]
\[\begin{align*}
& B_k(n, p_n, \varepsilon) = (2n/k + 1)^4 \frac{4V_c}{p_n} e^{-n \varepsilon^2} \\
& V_k(n, p_n, \varepsilon) = \min \left( \exp(-2n/p_n \varepsilon^2), (2n/p_n + 1)^{4V_c/p_n} e^{-n \varepsilon^2}, \left(2n/p_n + 1\right)^{4V_c/p_n} \exp(-n\varepsilon^2/9) \right)
\end{align*}\]

\[\square\]

Theorem 19 Suppose that \( \mathcal{H} \) holds. Suppose also and that \( n/k \) is an integer. Then, we have also for all \( \varepsilon > 0 \),
\[
\Pr(\tilde{R}_n(\psi^B_n) - \hat{R}^{Out}_{CV} \geq \varepsilon) \leq \min(B_k(n, p_n, \varepsilon), V_k(n, p_n, \varepsilon))
\]

\[\text{with}
\]
\[\begin{align*}
& B_k(n, p_n, \varepsilon) = \left(2n/k + 1\right)^4 \frac{4V_c}{p_n} e^{-n \varepsilon^2} \\
& V_k(n, p_n, \varepsilon) = \min \left( \exp(-2n/k \varepsilon^2), 2 \frac{\varepsilon^2}{64(\sqrt{V_c} \ln(2(2n/k + 1)) + 2)} \right)
\end{align*}\]

\[\text{Proof.}\]
\[\text{The proofs starts as previously. We have}
\]
\[
\Pr(\hat{R}^{Out}_{CV} - \tilde{R}_n(\psi^B_n) \geq \varepsilon) \leq \Pr(\mathbb{E}_{\psi \in \mathcal{F}} (\mathbb{P}_{n,V_{tsn}} \psi - \mathbb{P}_n) \geq \varepsilon) \leq \exp(-2n/p_n \varepsilon^2)
\]

but we also have
\[
\Pr(\hat{R}^{Out}_{CV} - \tilde{R}_n(\psi^B_n) \geq \varepsilon) \leq \Pr(\mathbb{E}_{\psi \in \mathcal{F}} (\sup_{\psi \in \mathcal{F}} (\mathbb{P}_{n,V_{tsn}} \psi - \mathbb{P}_n) \geq \varepsilon) \leq 2 \frac{\varepsilon^2}{64(\sqrt{V_c} \ln(2(2n/p_n + 1)) + 2)} \).
\]

\[\text{according to chapter 1.}\]

\[\square\]

Following the previous results, we can obtain results for the expectation of the difference \( \tilde{R}_n(\psi^B_n) - \hat{R}^{Out}_{CV} \).
Theorem 20 \textbf{(L1 error)} Suppose that \( \mathcal{H} \) holds. Suppose also and that \( n/k \) is an integer. Then, we have also for all \( \varepsilon > 0 \),
\[
\mathbb{E}_{\mathcal{D}_n} \left( \hat{R}_n(\psi_n^B) - \hat{R}^{\text{Out}}_{CV} \right) \leq \sqrt{1/np_n}
\]
Furthermore, suppose also that \( \phi_n \) is based on empirical risk minimization. But instead of minimizing \( \hat{R}_n(\phi) \), we suppose \( \phi_n \) minimizes \( \frac{1}{n} \sum_{i=1}^{n} C(h(Y_i, \phi(X_i))) \). For simplicity, we suppose the infimum is attained i.e. \( \phi_n = \arg \min_{\phi \in \mathcal{C}} \frac{1}{n} \sum_{i=1}^{n} C(h(Y_i, \phi(X_i))) \). Then, we have,
\[
\mathbb{E}_{\mathcal{D}_n} \left( \hat{R}_n(\psi_n^B) - \hat{R}^{\text{Out}}_{CV} \right) \leq \min \left( \sqrt{1/np_n}, 6 \sqrt{\frac{V_C \left( \ln(n(1-p_n)) + 2 \right)}{n(1-p_n)}} \right)
\]

Proof.
We just need to apply the previous results together with the following useful lemma (for a proof, see e.g. [DGL96]):

Lemma 21 Let \( X \) be a nonnegative random variable. Let \( K, C \) nonnegative real such that \( C \geq 1 \). Suppose that for all \( \varepsilon > 0 \), \( P(X \geq \varepsilon) \leq C \exp(-K\varepsilon^2) \). Then, we have
\[
\mathbb{E}X \leq \sqrt{\ln(C) + 2/K}.
\]

\( \square \)

3.2 Stability framework

3.2.1 Introduction to stability

To avoid the traditional analysis in the VC framework, notions of stability have been intensively worked through in the late 90’s [KEA93, BE01, BE02, KUT02, and KUNIY02]. The object of stability framework is the learning algorithm rather than the space of classifiers. The learning algorithm is a map (effective procedure) from data sets to classifiers. An algorithm is stable at a learning set \( \mathcal{D}_n \) if changing one point in \( \mathcal{D}_n \) yields only a small change in the output hypothesis. Several different notions of algorithmic stability are described. The attraction of such an approach is that it avoids the traditional notion of VC-dimension, and allows to focus on a wider class of learning algorithms than empirical risk minimization. For example, this approach provides generalization error bounds for regularization-based learning algorithms that have been difficult to analyze within the VC framework such as boosting. If a map is stable, exponential bounds on generalization error may be obtained. As a motivation, we quote the following list of algorithms satisfying stability properties: regularization networks, ERM, k-nearest rules, boosting.

3.2.2 Definitions and notations of stability

The basic idea is that an algorithm is stable at a training set \( \mathcal{D}_n \) if changing one point in \( \mathcal{D}_n \) yields only a small change in the output hypothesis. Formally, a learning algorithm maps a weighted training set into a predictor space. Thus, stability can be translated into a Lipschitz condition for this mapping with high probability. To be more formal, following [?], we define a distance between two weighted empirical errors:

Definition 22 \textbf{(Total variation)} Let \( \mathbb{P}_{n,V_n} \) and \( \mathbb{P}_{n,U_n} \) be two empirical measures on \( \mathcal{Z} \) with respect to the binary vectors \( V_n \) and \( U_n \). We do not assume their support to be equal. The distance between them is defined as their total variation:
\[
\|\mathbb{P}_{n,U_n} - \mathbb{P}_{n,V_n}\| = \sup_{A \in \mathcal{P}(\mathcal{Z})} |(\mathbb{P}_{n,U_n} - \mathbb{P}_{n,V_n})(A)|.
\]
Example 23  In the case of leave-one-out (i.e. \( \sum_{i=1}^n U_{n,i} = n - 1 \)), we have:

\[
||P_{n,U_n} - P_n|| = \frac{2}{n}.
\]

In the case of leave-\(\nu\)-out, we get:

\[
||P_{n,U_n} - P_n|| = \frac{2\nu}{n}.
\]

At least, we need a distance \(d\) on the set \(\mathcal{F}\). Let us quote three important examples. Let \(\psi_1, \psi_2 \in \mathcal{F}\). The uniform distance is defined by: \(d_\infty(\psi_1, \psi_2) = \sup_{z \in Z} |\psi_1(z) - \psi_2(z)|\), the \(L_1\)-distance by: \(d_1(\psi_1, \psi_2) = \mathbb{E}(|\psi_1 - \psi_2|)\), the error-distance \(d_e(\psi_1, \psi_2) = \mathbb{P}(\psi_1 - \psi_2)\). It is important to notice that what matters here is not an absolute distance between the original class of predictors \(\mathcal{G}\) seen as functions but the distance with the respect to the loss or/and the distribution \(P\). In particular, for the \(L_1\)-distance, we do not care about the behavior of the original predictors \(\phi_1\) and \(\phi_2\) outside the support of \(P\). At last, notice that we always have \(d_e \leq d_1 \leq d_\infty\).

We are now in position to define the different notions of stability of a learning algorithm which cover notions introduced by [KUNIY02]. We begin with the notion of weak stability. In essence, it says that for any given resampling vectors, the distance between two predictors is controlled with high probability by the distance between the resampling vectors. As a motivation, notice that algorithms such as Adaboost ([KUNIY02]) satisfies this property. With the previous notations, we have:

Definition 24 (Weak stability) Let \(D_n = (Z_i)_{1 \leq i \leq n}\) be a learning set. Let \(\lambda, (\delta_{n,p_n})_{n,p_n}\) be nonnegative real numbers. A learning algorithm \(\Psi\) is said to be weak \((\lambda, (\delta_{n,p_n})_{n,p_n}, d)\) stable if for any training vector \(U_n\) whose sum is equal to \(n(1 - p_n)\):

\[
\mathbb{P}(d(\psi_{U_n}, \psi_n) \geq \lambda ||P_{n,U_n} - P_n||) \leq \delta_{n,p_n}.
\]

Notice that in the former definition \(\mathbb{P}\) stands for \(\mathbb{P}^{\otimes n}\). Indeed, \(\psi_n\) is trained with \(n\) observations, drawn independently from \(P\). A stronger notion is to consider \(\psi_n\) trained with \(n - 1\) observations drawn independently from \(P\) and an additional general observation \(z\). We consider the stronger notion of strong stability. As a motivation, notice that algorithms such as Empirical Risk Minimization with finite VC dimension ([KUNIY02]) satisfies this property.

Definition 25 (Strong stability) Let \(z \in Z\). Let \(D_n = D_{n-1} \cup \{z\}\) be a learning set. Let \(\lambda, (\delta_{n,p_n})_{n,p_n}\) be nonnegative real numbers. A learning algorithm \(\Psi\) is said to be strong \((\lambda, (\delta_{n,p_n})_{n,p_n}, d)\) stable if for any training vector \(U_n\) whose sum is equal to \(n(1 - p_n)\):

\[
\mathbb{P}(d(\psi_{U_n}, \psi_n) \geq \lambda ||P_{n,U_n} - P_n||) \leq \delta_{n,p_n}.
\]

What we have in mind for classical algorithms is \(\delta_{n,p_n} = O_n(p_n \exp(-n(1 - p_n)))\). We can state the last definition in other words. Let \(V_{n}^{tr}\) be a training vector with distribution \(Q\) such that the number of elements in the training set is constant and equal to \(n(1 - p_n)\). Notice then that the former definition also implies that \(\sup_{U_n \in \text{support}(Q)} \mathbb{P}(d(\psi_{U_n}, \psi_n) \geq \lambda ||P_{n,U_n} - P_n||) \leq \delta_{n,p_n}\), where \(\text{support}(Q)\) stands for the support of \(Q\). The previous notion stands for any \(U_n\) having the same support of \(Q\). A stronger hypothesis would be that the previous probability stands uniformly over \(U_n\) in \(\text{support}(Q)\). This leads formally to the notion of cross-validation stability. To be more accurate:

Definition 26 (Cross-validation weak stability) Let \(D_n = (Z_i)_{1 \leq i \leq n}\) a learning set. Let \(V_{n}^{tr}\) a training vector with distribution \(Q\). Let \(\lambda, (\delta_{n,p_n})_{n,p_n}\) be nonnegative real numbers. A learning algorithm \(\Psi\) is said to be weak \((\lambda, (\delta_{n,p_n})_{n,p_n}, d, Q)\) stable if it is weak \((\lambda, (\delta_{n,p_n})_{n,p_n}, d)\) stable and if:

\[
\mathbb{P}(\sup_{U_n \in \text{support}(Q)} \frac{d(\psi_{U_n}, \psi_n)}{||P_{n,U_n} - P_n||} \geq \lambda) \leq \delta_{n,p_n}.
\]
As before, we also define the following stronger notion:

**Definition 27 (Cross-validation strong stability)** Let \( z \in Z \). Let \( D_n = D_{n-1} \cup \{z\} \) a learning set. Let \( V_n^{tr} \) a cross-validation vector with distribution \( Q \). A learning algorithm \( \Psi \) is said to be strongly \((\lambda, (\delta_{n,p_n})_{n,p_n}, d, Q)\) stable if it is strong \((\lambda, (\delta_{n,p_n})_{n,p_n}, d, Q)\) stable and if:

\[
\Pr \left( \sup_{U_n \in \text{support}(Q)} \frac{d(\psi_{U_n}, \psi_n)}{||P_n,U_n - P_n||} \geq \lambda \right) \leq \delta_{n,p_n}.
\]

**Remark 28** If the cardinal of the support of \( Q \) is denoted \( \kappa(n) \), then a learning algorithm which is weak \((\lambda, (\delta_{n,p_n})_{n,p_n}, d, Q)\)-stable is also strong \((\lambda, (\kappa(n)\delta_{n,p_n})_{n,p_n}, d, Q)\)-stable.

As seen in the following table, we retrieve with those notations the different notions of stability introduced by [DEWA79], [KEA95] and also [BE01], [KUNIY02].

| stability distance | \( d_\infty \) | \( d_1 \) | \( d_e \) |
|-------------------|----------------|-------------------|
| Weak              | weak \((\lambda, \delta)\) hypothesis stability | weak \((\lambda, \delta)\) \(L_1\) stability | weak \((\lambda, \delta)\) error stability |
| Strong            | strong \((\lambda, \delta)\) hypothesis stability | strong \((\lambda, \delta)\) \(L_1\) stability | strong \((\lambda, \delta)\) error stability |
| Sure Stability    | uniform stability | [DEWA79] | [DEWA79] |

To motivate this approach, we also quote a list of class of predictors satisfying the previous stability conditions.

| stability distance | \( d_\infty \) | \( d_1 \) | \( d_e \) |
|-------------------|----------------|-------------------|
| Weak              | Adaboost ([KUNIY02]) | -ERM ([KUNIY02]) | Bayesian algorithm [KEA95] |
| Strong            | 13k-nearest rule | [KUNIY02] | [KEA95] |
| Uniform           | Regularization networks | | |

We recall the main notations and definitions:

| Name                  | Notation | Definition |
|-----------------------|----------|------------|
| Risk or generalization error | \( R_n \) | \( E_P[L(Y, \phi(X, D_n)) \mid D_n] \) |
| Resubstitution error   | \( \hat{R}_n \) | \( \frac{1}{n} \sum_{i=1}^{n} L(Y_i, \phi_n(X_i, D_n)) \) |
| Cross-validation error | \( \hat{R}_{CV} \) | \( E_{V^{tr}} P_{n,V^{tr}} \psi_{V^{tr}} \) |

**Table 1:** Main notations

### 3.2.3 Main results

Let \( D_n \) be a learning set of size \( n \). Let \( V_n^{tr} \sim Q \) be a training vector independent of \( D_n \) such that the cross-validation is symmetric and the number of elements in the training set is constant and equal to \( np_n \). Let \( d \) be a distance among \( d_\infty, d_1, d_\infty \). At last, we suppose that the loss function \( L \) is bounded by 1. We derive the following general results that stands for general cross-validation procedures and stable algorithms.
Theorem 29 (Cross-validation Strong stability) Suppose that $\mathcal{H}$ holds. Let $\Psi$ a machine learning which is strong $(\lambda, (\delta_n, p_n)_{n,p_n}, Q)$ stable with respect to the distance $d$. Then, for all $\varepsilon \geq 0$, we have:

$$\Pr(\tilde{R}_n(\Phi_n^B) - \tilde{R}_n^{Out} \geq \varepsilon) \leq \exp(-2np_n\varepsilon^2)$$

Furthermore, if $d$ is the uniform distance $d_{\infty}$, then we have for all $\alpha > 0$:

$$\Pr(\tilde{R}_n(\Phi_n^B) - \tilde{R}_n^{Out} \geq \varepsilon) \leq \min(\exp(-2np_n\varepsilon^2), 2(\exp(-\frac{\varepsilon^2}{8n(8\lambda p_n + \alpha)^2}) + \frac{n}{\alpha} \delta_{n,p_n}))$$

Thus, if we choose $\alpha = 8\lambda p_n$,

$$\Pr(\tilde{R}_n(\Phi_n^B) - \tilde{R}_n^{Out} \geq \varepsilon) \leq \min(\exp(-2np_n\varepsilon^2), 2(\exp(-\frac{\varepsilon^2}{8(16\lambda)^2 np_n^2}) + \frac{n}{8\lambda p_n} \delta_{n,p_n}))$$

Proof.

On the one hand, we have as before by conditional Hoeffding’s inequality (for a proof, see e.g. chapter 1):

$$\Pr(\tilde{R}_n(\Phi_n^B) - \tilde{R}_n^{Out} \geq \varepsilon) \leq \Pr(\mathbb{E}_{V_n^r}(P_{V_n^r} - P_n, V_n^r, \psi_{V_n^r}) \geq \varepsilon) \leq \exp(-2np_n\varepsilon^2)$$

On the other hand, notice that $\mathbb{P}^n(\psi_{V_n^r} - P_n, V_n^r, \psi_{V_n^r}) = 0$

Denote $f(Z_1, Z_2, \ldots, Z_n) := \mathbb{E}_{V_n^r}(\psi_{V_n^r} - P_n, V_n^r, \psi_{V_n^r})$. Let $z \in \mathcal{Z}$. Now denote:

$$B := \left\{ \sup_{U_n \in \text{support}(Q)} \frac{d(U_n, \psi_{n+1})}{\|P_n - \psi_{n+1}\|} \geq \lambda \right\}$$

with $\psi_{n+1}$ trained on $D_{n+1} = \{Z_1, \ldots, Z_{t-1}, Z_t, Z_{t+1}, \ldots, Z_n, z\}$. Under our assumptions, we have $\Pr(B) \leq \delta_{n+1, p_n+1}$.

We want to show that with high probability there exist constants $c_i$ such that for all $i \in \{1, \ldots, n\}$, for all $z \in \mathcal{Z}$,

$$\Delta_i := |f(Z_1, \ldots, Z_i, \ldots, Z_n) - f(Z_1, \ldots, Z_{i-1}, z, Z_{i+1}, \ldots, Z_n)| \leq c_i$$

Notice that:

$$|\Delta_i| = |\mathbb{E}_{V_n^r}(\psi_{V_n^r} - P_n, V_n^r, \psi_{V_n^r}) - (\mathbb{E}_{V_n^r}\mathbb{P}_{V_n^r} - P_n, V_n^r, \psi_{V_n^r})|$$

$$\leq |\mathbb{E}_{V_n^r}\mathbb{P}(\psi_{V_n^r} - \psi'_{V_n^r})| + |\mathbb{E}_{V_n^r}(P_{V_n^r} - P_n, V_n^r, \psi'_{V_n^r})|$$

with $P_{n, V_n^r}$ the weighted empirical measure on the sample

$$\mathcal{E}_n = \{Z_1, \ldots, Z_{t-1}, z, Z_{t+1}, \ldots, Z_n\}$$

and $\psi'_{V_n^r}$ the predictor trained on $\mathcal{E}_{V_n^r}$.

So, first, let us bound the first term, $|\mathbb{E}_{V_n^r}\mathbb{P}(\psi_{V_n^r} - \psi'_{V_n^r})| \leq \mathbb{E}_{V_n^r}\mathbb{P}(\psi_{V_n^r} - \psi_{n+1}) + \mathbb{E}_{V_n^r}\mathbb{P}(\psi_{n+1} - \psi'_{V_n^r})$. Thus, on $B^c$, we have $|\mathbb{E}_{V_n^r}\mathbb{P}(\psi_{V_n^r} - \psi'_{V_n^r})| \leq \frac{4\lambda}{n+1}$.

To upper bound the second term, notice that:
We want to show that for all $\psi$, $|\mathbb{P}_n(V_{n,i}^t) - \psi_{n,i}^t| \leq 1/np_n$ thus $|\mathbb{E}_n(V_{n,i}^t) - \psi_{n,i}^t)|V_{n,i}^t = 1 \times p_n| \leq 1/n$

We still have to bound $|\mathbb{E}_n(V_{n,i}^t) - \psi_{n,i}^t)|V_{n,i}^t = 1$ in the special case of the most stable kind of stability namely the uniform stability.

On $B^c$, we get $d_\infty(\psi_{n,i}^t, \psi_{n+1}^t) \leq d_\infty(\psi_{n,i}^t, \psi_{n+1}^t) + d_\infty(\psi_{n+1}^t, \psi_{n,i}^t) \leq 4\lambda p_n$.

Thus, on $B^c$, we derive

$$\mathbb{E}_n(V_{n,i}^t)(d_\infty(\psi_{n,i}^t, \psi_{n+1}^t)|V_{n,i}^t = 1 \leq 4\lambda p_n.$$ 

Putting all together, with probability at least $1 - \delta_{n,p_n}$, we get

$$\sup_{1 \leq i \leq n, z \in Z}|f(Z_1, \ldots, Z_i, \ldots, Z_n) - f(Z_1, \ldots, z, \ldots, Z_n)| \leq \frac{4\lambda}{n+1} + 4\lambda p_n(1 - p_n) \leq 8\lambda p_n.$$

Applying theorem ??, we obtain that for all $\varepsilon \geq 0$:

$$\Pr(\mathbb{E}_n(V_{n,i}^t)(\mathbb{P}_n(V_{n,i}^t) - \mathbb{P}_n(V_{n,i}^t)) \geq \varepsilon) \leq 2(\exp(-\frac{\varepsilon^2}{8n(8\lambda p_n + \alpha)^2}) + \delta_{n,p_n})$$

$$\leq 2(\exp(-\frac{\varepsilon^2}{8(16\lambda^2)^2np_n^2}) + \frac{n\delta_{n,p_n}}{8\lambda p_n})$$

by taking $\alpha = 8\lambda p_n$.

\square

**Theorem 30 (Cross-validation Weak stability)** Suppose that $\mathcal{H}$ holds. Let $\Psi$ be a machine learning which is weak $(\lambda, (\delta_{n,p_n}), (\varepsilon))$ stable with respect to the distance $d$. Then, for all $\varepsilon \geq 0$, we have

$$\Pr(\tilde{R}_n(\Phi_n^o) - \tilde{R}^o_{CV} \geq \varepsilon) \leq \exp(-2np_n\varepsilon^2).$$

Furthermore, if the distance is the uniform distance $d_\infty$, we have for all $\varepsilon \geq 0$:

$$\Pr(\tilde{R}_n(\Phi_n^o) - \tilde{R}^o_{CV} \geq \varepsilon) \leq \min(\exp(-2np_n\varepsilon^2), 2p_n(\exp(-\frac{n\varepsilon^2}{10(\delta_{n,p_n})^2}) + \frac{n\delta_{n,p_n}}{8\lambda p_n}))$$

\section*{Proof}

Denote $f(Z_1, Z_2, \ldots, Z_n) := \tilde{R}^o_{CV} - \tilde{R}_n$ and $B := \{\sup_{U_n \in \text{support}(\mathcal{Q})} d(U_n, U_n) \geq \lambda\}$ with $\psi_{n+1}$ trained on $\mathcal{D}_{n+1} = \{Z_1, \ldots, Z_{n+1}, \ldots, Z_n, Z'_i\}$.

We want to show that for all $i$, there exists constant $c_i$ such $|\Delta_i| := |f(Z_1, \ldots, Z_i, \ldots, Z_n) - f(Z_1, \ldots, Z'_i, \ldots, Z_n)| \leq c_i$ with high probability where $Z_1, \ldots, Z_i, \ldots, Z_n, Z'_i$ are i.i.d. variables.

$$|\Delta_i| = |\mathbb{E}_{n,i}^t(\mathbb{P}_n(V_{n,i}^t) - \psi_{n,i}^t)| - (\mathbb{E}_{n,i}^t(\mathbb{P}_n(V_{n,i}^t) - \psi_{n,i}^t))|\leq |\mathbb{E}_{n,i}^t(\mathbb{P}_n(V_{n,i}^t) - \psi_{n,i}^t)| + |\mathbb{E}_{n,i}^t(\mathbb{P}_n(V_{n,i}^t) - \psi_{n,i}^t)|.$$

with $\mathbb{P}_n', \mathbb{P}_n(V_{n,i}^t)$ the weighted empirical measures of the sample $\mathcal{D}_n = \{Z_1, \ldots, Z_i, \ldots, Z_n\}$ and $\psi_i$ the predictor built on $\mathcal{D}_n'$. 

15
Theorem 31

In the case of classification, we can bound the excess risk by

\[ R \]

Following the previous results, we can obtain results for the expectation of the difference

\[ - \]

On \( B^C \), we have \[ | \mathbb{E}_{V_n^t} \mathbb{P}(\psi_{V_n^t} - \psi_{V_n^t}^t) | \leq \frac{\lambda}{n+1} \]

To upper bound the second term, notice that:

\[ | \mathbb{E}_{V_n^t} (P_{n,V_n^t}^t) \psi_{V_n^t} - \mathbb{E}_{V_n^t} (P_{n,V_n^t}^t) \psi_{V_n^t}^t | = | \mathbb{E}_{V_n^t} (P_{n,V_n^t}^t (\psi_{V_n^t} - \psi_{V_n^t}^t), V_n^{tr} = 1) \times (1 - p_n) + \mathbb{E}_{V_n^t} ((P_{n,V_n^t}^t - P_{n,V_n^t}^t) \psi_{V_n^t}^t, V_n^{ts} = 1) \times p_n |. \]

We always have for all \( \psi \), \( | (P_{n,V_n^t}^t - P_{n,V_n^t}^t) \psi | \leq 1/np_n \) thus we get

\[ | \mathbb{E}_{V_n^t} ((P_{n,V_n^t}^t - P_{n,V_n^t}^t) \psi_{V_n^t}, V_n^{ts} = 1) \times p_n | \leq 1/n. \]

We still have to bound \( | \mathbb{E}_{V_n^t} (P_{n,V_n^t}^t (\psi_{V_n^t} - \psi_{V_n^t}^t), V_n^{tr} = 1) | \leq \mathbb{E}_{V_n^t} (d_\infty (\psi_{V_n^t}, \psi_{V_n^t}^t), V_n^{tr} = 1) \) in the special of the uniform stability.

On \( B^C \), we derive \( d_\infty (\psi_{V_n^t}, \psi_{V_n^t}^t) \leq d_\infty (\psi_{V_n^t}, \psi_{V_n^t}^t) + d_\infty (\psi_{V_n^t}, \psi_{V_n^t}^t) \leq 4\lambda p_n \), thus on \( B^C \)

\[ \mathbb{E}_{V_n^t} (d_\infty (\psi_{V_n^t}, \psi_{V_n^t}^t), V_n^{tr} = 1) \leq 4\lambda p_n. \]

Putting all together, with probability at least 1 - \( \delta_{n,p_n} \),

\[ | f(Z_1, \ldots, Z_i, \ldots, Z_n) - f(Z_1, \ldots, Z_i', \ldots, Z_n) | \leq 8\lambda p_n. \]

□

Following the previous results, we can obtain results for the expectation of the difference \( \hat{R}_n(\Phi^B_n) - \hat{R}^{Out}_{CV} \).

**Theorem 31** 
In the case of classification, we can bound the excess risk by

\[ \mathbb{E}_{D_n} (\hat{R}_n(\Phi^B_n) - \hat{R}^{Out}_{CV}) \leq \sqrt{1/np_n} \]

Furthermore, if \( d \) is the uniform distance \( d_\infty \), then we have for all \( \alpha > 0 \):

\[ \mathbb{E}_{D_n} (\hat{R}_n(\Phi^B_n) - \hat{R}^{Out}_{CV}) \leq \min(\sqrt{1/np_n}, \sqrt{163^2n\alpha p_n} + \frac{n}{4\lambda p_n} \delta_{n,p_n}) \]

Similar results can be derived in the context of the weak stability.

**Proof**

It is sufficient to apply the previous probability upper bounds together with the lemma[21]

□

4 Results for the cross-validated subagged classification

In the case of subagging of classifiers (i.e. the majority vote), we can obtain the following results:

**Theorem 32** For any subagged classifier, we can bound the excess risk.

\[ \Pr(\hat{R}_n(\Phi^B_n) - \frac{1}{2}\hat{R}^{Out}_{CV} \geq \varepsilon) \leq \exp(-8np_n\varepsilon^2/9) \]

and also

\[ \Pr(\hat{R}_n(\Phi^B_n) - \hat{R}^{Maj}_{CV} \geq \varepsilon) \leq t \exp(-2np_n\varepsilon^2/9) \]
where \( N \) denotes the total number of training vectors in the cross-validation and \( l \) denotes \([(N-1)/2]+1\) that is the strict majority of the subbaged classifiers and \( \hat{R}_{CV}^{\text{Maj}} \) the cross-validated estimate of this majority.

Furthermore, in the particular case of binary classification we also have

\[
\Pr(\hat{R}_n(\Phi^B) - (\hat{R}_{CV}^\text{Out}/2 - 1/2)) \leq -\varepsilon \leq \exp(-2np_n\varepsilon^2/9)
\]

and

\[
\Pr(\hat{R}_n(\Phi^B) - (l\hat{R}_{CV}^{\text{Maj}} - l + 1) \leq -\varepsilon \leq l\exp(-2np_n\varepsilon^2)
\]

**Proof.**

We consider a ghost sample i.i.d. of size \( m \): \((X'_1, Y'_1), \ldots, (X'_m, Y'_m)\). Denote \( \eta_i := L(Y'_i, \phi^B_i(X'_i)) \).

Then \( c^B_m := \frac{1}{m} \sum_{i=1}^{m} \eta_i \) corresponds to the average number of mistakes of \( \phi^B \) on the ghost sample. In the same way, \( e^r_m := \frac{1}{m} \sum_{i=1}^{m} L(Y'_i, \phi_{V^{tr}}(X'_i)) \) (respectively \( e^a_m := \mathbb{E}_{V^{tr}}[\frac{1}{m} \sum_{i=1}^{m} L(Y'_i, \phi_{V^{tr}}(X'_i))] \)) is the average number of the mistakes of \( \phi_{V^{tr}} \) (respectively the weighted average number of mistakes of the family of predictors \( \phi_{V^{tr}} \)).

Denote by

1. \( L_1 := \hat{R}_n(\Phi^B) - \frac{1}{2}\hat{R}_{CV}^\text{Out} \)
2. \( L_2 := \hat{R}_n(\Phi^B) - c^B_m \)
3. \( L_3 := e^r_m - e^a_m/2 \)
4. \( L_4 := \frac{1}{2}[e^r_m - \mathbb{E}_{X,Y} \mathbb{E}_{V^{tr}} L(Y, \phi_{V^{tr}}(X))] \)
5. \( L_5 := \frac{1}{2}[\mathbb{E}_{X,Y} \mathbb{E}_{V^{tr}} L(Y, \phi_{V^{tr}}(X)) - \hat{R}_{CV}^\text{Out}] \)

We have

\[
\Pr(L_1 \geq 3\varepsilon) \leq \Pr(L_2 \geq \varepsilon) + \Pr(L_3 \geq 0) + \Pr(L_4 \geq \varepsilon) + \Pr(L_5 \geq \varepsilon)
\]

By Hoeffding’s inequality, we have:

\[
\Pr(L_2 \geq \varepsilon) \leq \exp(-2m\varepsilon^2).
\]

and also \( \Pr(L_4 \geq \varepsilon) \leq \exp(-2m(2\varepsilon)^2) \)

By conditionnal Hoeffding’s inequality (for a proof, see e.g. [?]), we deduce

\[
\Pr(L_5 \geq \varepsilon) \leq \exp(-2np_n(2\varepsilon)^2)
\]

By conditionnal Hoeffding’s inequality, we also have

\[
\Pr(e^a_m - \mathbb{E}_{X,Y} \mathbb{E}_{V^{tr}} L(Y, \phi_{V^{tr}}(X)) \geq \varepsilon) \leq \exp(-2m\varepsilon^2).
\]

since for fixed \( v^{tr}_n \)

\[
\Pr(\frac{1}{m} \sum_{i=1}^{m} L(Y'_i, \phi_{v^{tr}}(X'_i)) - \mathbb{E}_{X,Y} L(Y, \phi_{v^{tr}}(X)) \geq \varepsilon) \leq \exp(-2m\varepsilon^2)
\]

We suppose here that \( \Pr(V^{tr}_n = v_n) \) are rational numbers whose smallest multiplicator is denoted by \( N \). Thus \( e^a_m \) can be seen as a simple average number of mistakes of a family of predictors \( (\phi_j)_{1 \leq j \leq N} \) on the ghost sample.
First notice, that if \( e_m^a \) is small then \( e_m^B \) must be small either. Indeed, we have

\[
e_m^a = \frac{1}{N} \sum_{j=1}^{N} \sum_{i=1}^{m} L(Y'_i, \phi_j(X'_i)) = \frac{1}{N} \sum_{1 \leq j \leq N, 1 \leq i \leq m} \epsilon_{i,j}
\]

with \( \epsilon_{i,j} := L(Y'_i, \phi_j(X'_i)) \in \{0, 1\} \). We thus deduce that the total number of mistakes on the ghost sample of the family of predictors \( (\phi_j)_{1 \leq j \leq N} \) is equal to \( N e_m^a \). Notice that if the number of mistakes of the family \( (\phi_j)_{1 \leq j \leq N} \) on the \( i \)-th observation is less than \( \lfloor (N-1)/2 \rfloor \) (i.e. \( \sum_{j=1}^{N} \epsilon_{i,j} \leq \lfloor (N-1)/2 \rfloor \)) then it means that a strict majority of predictors have classified correctly \( Y'_i \), which in turn tells us that a strict majority of predictors have the same output \( Y'_i = \phi_j(X'_i) \). We thus have \( \phi_n^B(X'_i) = Y'_i \) which implies \( \eta_j = L(Y'_i, \phi_n^B(X'_i)) = 0 \).

Denoting by \( \kappa = n e_m^B \) the number of mistakes of the subbaged classifier on the ghost sample, we necessarily have

\[
\sum_{i=1}^{m} \sum_{j=1}^{N} \epsilon_{i,j} \geq \kappa(\lfloor (N-1)/2 \rfloor + 1) = \kappa(\lfloor (N+1)/2 \rfloor).
\]

It follows that

\[
e_m^B \leq \frac{N}{\lfloor (N+1)/2 \rfloor} e_m^a < e_m^a/2.
\]

Thus \( \Pr(L_B \geq 0) = 0 \)

We conclude \( \Pr(\tilde{R}_m(\Phi_n^B) - \frac{1}{2} \hat{R}_{CV}^B \geq 3\epsilon) \leq \exp(-2n_\kappa(2\epsilon)^2) + \exp(-2m(2\epsilon)^2) + \exp(-2m\epsilon^2) \).

If we let \( m \to \infty \),

\[
\Pr(\tilde{R}_m(\Phi_n^B) - \frac{1}{2} \hat{R}_{CV}^B \geq \epsilon) \leq \exp(-8n_\kappa \epsilon^2/9)
\]

Notice that in the particular case of the binary classification, we have by symmetry, \( 1 - e_m^B \leq \frac{N}{\lfloor (N+1)/2 \rfloor} (1 - e_m^a) \), which gives

\[
\frac{N}{\lfloor N/2 + 1 \rfloor} e_m^a - \frac{N}{\lfloor N/2 + 1 \rfloor} \leq e_m^B
\]

and eventually \( e_m^B \geq \frac{N}{\lfloor N/2 + 1 \rfloor} e_m^a - 1/2 \geq e_m^a - 1/2 \)

Thus, for binary classification, we can even obtain an probability upper bound for \( \Pr(|\tilde{R}_m(\Phi_n^B) - \frac{1}{2} \hat{R}_{CV}^B| \geq \epsilon) \) not only for \( \Pr(\tilde{R}_m(\Phi_n^B) - \frac{1}{2} \hat{R}_{CV}^B \geq \epsilon) \). Indeed, denote by

1. \( L'_1 := \tilde{R}_m(\Phi_n^B) - \frac{N}{\lfloor N/2 + 1 \rfloor} \hat{R}_{CV}^B - 1/2 \)
2. \( L'_2 := \tilde{R}_m(\Phi_n^B) - e_m^B \)
3. \( L'_3 := e_m^B - (\frac{N}{\lfloor N/2 + 1 \rfloor} e_m^a - 1/2) \)
4. \( L'_4 := (\frac{N}{\lfloor N/2 + 1 \rfloor} e_m^a - 1/2) - (\frac{N}{\lfloor N/2 + 1 \rfloor} E_{X,Y} E_{V_n} L(Y, \phi_{V_n}^B(X)) - 1/2) \)
5. \( L'_5 := (\frac{N}{\lfloor N/2 + 1 \rfloor} E_{X,Y} E_{V_n} L(Y, \phi_{V_n}^B(X)) - 1/2) - (\frac{N}{\lfloor N/2 + 1 \rfloor} \hat{R}_{CV}^B - 1/2) \)

We get
\[
\Pr(L' \leq -3\varepsilon) \leq \Pr(L' \leq -\varepsilon) + \Pr(L' < 0) + \Pr(L' \leq -\varepsilon) + \Pr(L' \leq -3\varepsilon) \leq \exp(-2m\varepsilon^2) + 0 + \exp(-2m(\frac{N}{N/2 + 1})^2) + \exp(-2np_n(\frac{N}{N/2 + 1})^2)
\]

Taking \(m \to \infty\), and noticing that \(N/\lfloor N/2 + 1 \rfloor > 1\)

\[
\Pr(\tilde{R}_n(\Phi^B_n) - (\tilde{R}^{Out}_{CV}/2 - 1/2) \leq -\varepsilon) \leq \Pr(\tilde{R}_n(\Phi^B_n) - (\tilde{R}^{Out}_{CV}/2 - 1/2) \leq -\varepsilon) \leq \Pr(L' \leq -\varepsilon) \leq \exp(-2np_n\varepsilon^2/9)
\]

For binary classification, we can eventually obtain that

\[
\Pr(\tilde{R}_n(\Phi^B_n) - (\tilde{R}^{Out}_{CV}/2 - 1/2) \geq \varepsilon) \leq \exp(-8np_n\varepsilon^2/9) + \exp(-2np_n\varepsilon^2/9) \leq 2\exp(-2np_n\varepsilon^2/9)
\]

Denote by \(\epsilon_j := \frac{1}{N} \sum_{i=1}^{N} \epsilon_{i,j}\) the average number of mistakes by predictors \(j\) on the ghost sample. We can order them by increasing order: \(\epsilon_{(1)},...,\epsilon_{(N)}\). Let \(l := \lfloor N/2 + 1 \rfloor\) be the strict majority. An interesting case is when we know that a strict majority of classifiers are very good. Denote by

\[
e^G_m := \frac{1}{l} \sum_{j=1}^{l} \epsilon_{(j)}
\]

their global average error of the first \(l\) best classifiers on the ghost sample.

In the same way, denote by \(\mu_j := E_{X,Y} L(Y, \phi_j(X))\) the risk of the \(j\)-th classifier. We introduce now a cross-validation estimate of the average risk \(\frac{1}{l} \sum_{j=1}^{l} \mu_{(j)}\) of the \(l\) best classifiers: \(\hat{R}^{Maj}_{CV}\). For this, recall that each \(\phi_j\) corresponds to some \(\phi_{i,j}\), thus we can define an out sample error for the predictor \(j : \bar{r}_j := P_{n,v_{\tilde{v}}} (L(Y, \phi_j(X)))\). And we define \(\hat{R}^{Maj}_{CV} := \frac{1}{l} \sum_{j=1}^{l} \bar{r}_j\)

1. \(R_1 := \tilde{R}_n(\Phi^B_n) - l\hat{R}^{Maj}_{CV}\)
2. \(R_2 := \tilde{R}_n(\Phi^B_n) - e^B_m\)
3. \(R_3 := e^B_m - l\epsilon^G_m\)
4. \(R_4 := l(e^G_m - \frac{1}{l} \sum_{j=1}^{l} \mu_{(j)})\)
5. \(R_5 := l(\frac{1}{l} \sum_{j=1}^{l} \mu_{(j)} - \hat{R}^{Maj}_{CV})\)

We have

\[
\Pr(R_1 \geq 3\varepsilon) \leq \Pr(R_2 \geq \varepsilon) + \Pr(R_3 > 0) + \Pr(R_4 \geq \varepsilon) + \Pr(R_5 \geq \varepsilon)
\]

By Hoeffding’s inequality, we have:

\[
\Pr(R_2 \geq \varepsilon) \leq \exp(-2m\varepsilon^2).
\]

We also derive

\[
\Pr(R_4 \geq \varepsilon) = \Pr(e^G_m - \frac{1}{l} \sum_{j=1}^{l} \mu_{(j)} \geq \varepsilon/l) = \Pr(\sum_{j=1}^{l} \mu_{(j)} - \sum_{j=1}^{l} \mu_{(j)} \geq \varepsilon)
\]
There exist permutations $\sigma$ and $\sigma'$ such that $\epsilon(j) = \epsilon_{\sigma(j)}$ and $\mu(j) = \mu_{\sigma'(j)}$. Thus, we get

$$\Pr(R_4 \geq \varepsilon) \leq \Pr(\sum_{j=1}^{l} \epsilon_{\sigma(j)} - \mu_{\sigma'(j)} \geq \varepsilon)$$

$$\leq \Pr(\sum_{j=1}^{l} \epsilon_{\sigma'(j)} - \mu_{\sigma'(j)} \geq \varepsilon)$$

by definition of $\epsilon(j)$. It follows that

$$\Pr(R_4 \geq \varepsilon) \leq \sum_{j=1}^{l} \Pr(\epsilon_{\sigma'(j)} - \mu_{\sigma'(j)} \geq \varepsilon)$$

$$\leq l \exp(-2m\varepsilon^2).$$

In the same way, we deduce $\Pr(R_5 \geq \varepsilon) \leq l \exp(-2np_n\varepsilon^2)$. By conditional Hoeffding’s inequality (for a proof, see e.g. [?]), we deduce $\Pr(L_5 \geq \varepsilon) \leq \exp(-2np_n(2\varepsilon)^2)$ and also for a fixed $v^r_n$

$$\Pr(|e_{m}^{v^r_n} - E_{X,Y}L(Y, \phi_{v^r_n}(X))| \geq \varepsilon) \leq 2\exp(-2m\varepsilon^2).$$

By conditional Hoeffding’s inequality (for a proof, see e.g. [?]), we also have

$$\Pr(|e_{m}^{a} - E_{X,Y}E_{v^r_n}L(Y, \phi_{v^r_n}(X))| \geq \varepsilon) \leq 2\exp(-2m\varepsilon^2).$$

Notice that if all the $l$ best classifiers classify correctly the $i$-th observation (i.e. $\epsilon_{i,(j)} = 0$ for all $j \in \{1, \ldots, M\}$), then the subbaged classification classifies also correctly. Thus $\eta_i = 0$. Let $\kappa$ be the number of mistakes of the subbaged classifier on the ghost sample and let $x$ the number of observations correctly classified by all the $l$ classifiers. Then we obtain that the number of correctly classified observations by the subbagging is greater than $x$, i.e. $m - \kappa \geq x$. On the other hand, there is at least one predictor that makes a mistake on each of the remaining $m - x$ observations. Thus $m - x$ is less that the total number of mistakes made by the $l$ best classifiers

$$(m - x) \leq ml\epsilon_m^G.$$

From which, it follows that

$$e_m^B \leq l\epsilon_m^G.$$

Thus $\Pr(R_3 > 0) = 0$.

Putting altogether, we have

$$\Pr(\tilde{R}_n(\Phi_n^B) - l\tilde{R}_n^{Maj} \geq 3\varepsilon) \leq \exp(-2m\varepsilon^2) + l \exp(-2m\varepsilon^2) + l \exp(-2np_n\varepsilon^2).$$

If we let $m \to \infty$, $\Pr(\tilde{R}_n(\Phi_n^B) - l\tilde{R}_n^{Maj} \geq \varepsilon) \leq l \exp(-2np_n\varepsilon^2/9)$.

Once again, in the particular case of binary classification, we have by symmetry $1 - e_m^B \leq l(1 - e_m^G)$ which leads to

$$e_m^B \geq 1 - l(1 - e_m^G).$$

In the same way, we have a symmetrical result for binary classification:

$$\Pr(\tilde{R}_n(\Phi_n^B) - (l\tilde{R}_n^{Maj} - l + 1) \leq -3\varepsilon) \leq \exp(-2m\varepsilon^2) + l \exp(-2m\varepsilon^2) + l \exp(-2np_n\varepsilon^2)$$

$$\leq l \exp(-2np_n\varepsilon^2).$$
which gives \( \Pr(|\tilde{R}_n(\Phi_n^B) - (l\hat{R}_CV^Maj - l + 1)| \geq \varepsilon) \leq 2l \exp(-2np_n\varepsilon^2/9). \)

\[ \square \]

In the case of subagging of classifiers (i.e. the majority vote) whose VC dimension is finite, we can obtain a stronger result:

**Theorem 33** Suppose \( \mathcal{H} \) holds and that the machine learning is based on empirical risk minimization. We can bound the excess risk.

\[
\Pr(\tilde{R}_n(\Phi_n^B) - \frac{1}{2}\hat{R}_CV^Out \geq \varepsilon) \leq \min(\exp(-8np_n\varepsilon^2/9), (2n(1 - p_n) + 1)^{4V_C/(1-p_n)}e^{-4n(1-p_n)\varepsilon^2}).
\]

and also

\[
\Pr(\tilde{R}_n(\Phi_n^B) - l\hat{R}_CV^Maj \geq \varepsilon) \leq l \exp(-2np_n\varepsilon^2/9)
\]

with the \( l := \lfloor (N-1)/2 \rfloor + 1 \) the strict majority of the subagged classifiers and \( \hat{R}_CV^Maj \) the cross-validated estimate of this majority.

Furthermore, in the particular case of binary classification we also have

\[
\Pr(\tilde{R}_n(\Phi_n^B) - (\hat{R}_CV^Out/2 - 1/2) \leq -\varepsilon) \leq \min(\exp(-2np_n\varepsilon^2/9), (2n(1 - p_n) + 1)^{4V_C/(1-p_n)}e^{-4n(1-p_n)\varepsilon^2})
\]

and

\[
\Pr(\tilde{R}_n(\Phi_n^B) - (l\hat{R}_CV^Maj - l + 1) \leq -\varepsilon) \leq l \exp(-2np_n\varepsilon^2)
\]

**Proof.**

We use again the lemma (for a proof, see chapter 1): \( \hat{R}_CV^Out \geq \mathbb{P}_n L(Y, \phi_n(X)) \) since

\[
\phi_n = \arg\min_{\phi \in \mathcal{C}} \frac{1}{n} \sum_{i=1}^{n} L(Y_i, \phi(X_i)).
\]

Following the last proof, we can bound \( L_5 \) in another way.

\[
\Pr(L_5 \geq 3\varepsilon) \leq \Pr(\mathbb{E}_{\mathcal{V}_n^T}[E_{X,Y} L(Y, \phi_{\mathcal{V}_n^T}(X)) - \mathbb{P}_n L(Y, \phi_n(X))] \geq 6\varepsilon) = \Pr(\mathbb{E}_{\mathcal{V}_n^T}[E_{X,Y} L(Y, \phi_{\mathcal{V}_n^T}(X)) - \mathbb{P}_n L(Y, \phi_n(X))] \geq 6\varepsilon) \]

Then as in proof, we split according to \( \mathbb{P} L(Y, \phi_{opt}(X)) \) and we obtain by lemma 21

\[
\Pr(L_5 \geq \varepsilon) \leq (2n(1 - p_n) + 1)^{4V_C/(1-p_n)}e^{n(1-p_n)(2\varepsilon)^2}
\]

\[ \square \]
5 Results for the subagged predictor selection

The remaining important question is: in practice, how should we choose \( p_n \)? We give a hint for this question.

First, suppose that the final user wants to have an accuracy equal to a certain level \( \eta \).

Then we need to provide him a rule to chose an optimal \( p_n^* \) and to upper bound the probability of excess risk \( \Pr(\tilde{R}_n(\phi_n^{B,p_n^*}) - \hat{R}_{CV}^o(p_n^*) \geq \eta) \). Previous bounds tell us that for any fixed \( p_n \), \( \Pr(\tilde{R}_n(\phi_n^B) - \hat{R}_{CV}^o(p_n) \geq \varepsilon) \leq \min(B(n, p_n, \varepsilon), V(n, p_n, \varepsilon)) \). Notice that \( \min(B(n, p_n, \varepsilon), V(n, p_n, \varepsilon)) \) seen as a function of \( \varepsilon \) is a continuous non-increasing function. Thus, we can define an inverse denoted by \( f \).

The previous probability bound becomes for any \( p_n \): \( \Pr(\tilde{R}_n(\phi_n^B) - \hat{R}_{CV}^o(p_n) \geq f(n, p_n, \delta) \leq \delta \).

For each \( k \), define \( \delta_{n,k} \) by \( f(n, k/n, \delta_{n,k}) = \eta \), i.e. \( \delta_{n,k} = \min(B(n, k/n, \eta), V(n, k/n, \eta)) \). Denote \( k^* = \arg\min_{k \in \{1 \ldots n-1\}} \hat{R}_{CV}^o(k/n) + f(n, k/n, \delta_{n,k}) \) and denote by \( p_n^* := k^*/n \). Thus, we obtain:

**Theorem 34 (Subbaging selection)** Suppose that \( \mathcal{H} \) holds. Suppose also that \( \phi_n \) is based on empirical risk minimization. But instead of minimizing \( \tilde{R}_n(\phi_n) \), we suppose \( \phi_n \) minimizes \( \frac{1}{n} \sum_{i=1}^n C(h(Y_i, \phi(X_i))) \). For simplicity, we suppose the infimum is attained i.e. \( \phi_n = \arg\min_{\phi \in C} \frac{1}{n} \sum_{i=1}^n C(h(Y_i, \phi(X_i))) \). In this context, we have:

- if \( \delta \geq \delta_n \)

\[
f(n, p_n, \delta) = \sqrt{\frac{\ln(1/\delta)}{2np_n}}
\]

- and if \( \delta < \delta_n \)

\[
f(n, p_n, \delta) = 3 \sqrt{\frac{4\sqrt{\ln(2n(1-p_n)+1)}}{n} (1-p_n) + \ln(1/\delta)}
\]

with \( \delta_n := (2n(1-p_n)+1)^{-\frac{4\sqrt{n\ln(2n)}/\ln(n)/n}{\ln(n)/n}} \).

Furthermore, we have for all \( \varepsilon > 0 \):

\[
\Pr(\tilde{R}_n(\phi_n^{B,p_n^*}) - \hat{R}_{CV}^o(p_n^*) \geq \varepsilon) = O_n((n+1)^{8V_n} \exp \left( -\frac{2n\varepsilon - 2\sqrt{2V_n/3} \sqrt{\ln(n)/n^2}}{1 - \exp(-2\varepsilon^2)} \right)).
\]

**Proof**

We have:

\[
\Pr(\tilde{R}_n(\phi_n^{B,p_n^*}) - \hat{R}_{CV}^o(p_n^*) \geq \eta) = \Pr(\tilde{R}_n(\phi_n^{B,p_n^*}) - \hat{R}_{CV}^o(p_n^*) \geq f(n, p_n^*, \delta_{n,k^*}))
\]

\[
\leq \sum_{k \in \{1 \ldots n-1\}} \Pr(\tilde{R}_n(\phi_n^{B,p_k}) - \hat{R}_{CV}^o(p_k) \geq f(n, k/n, \delta_{n,k}))
\]

It follows that:

\[
\Pr(\tilde{R}_n(\phi_n^{B,p_n^*}) - \hat{R}_{CV}^o(p_n^*) \geq \eta) \leq \sum_{k \in \{1 \ldots n-1\}} \Pr(\tilde{R}_n(\phi_n^{B,p_k}) - \hat{R}_{CV}^o(p_k) \geq \eta)
\]

\[
\leq \sum_{k \in \{1 \ldots n-1\}} \min(B(n, k/n, \eta), V(n, k/n, \eta)).
\]

Thus, using previous bounds we get:
In the case of ERM algorithm, 

\[
\Pr(\tilde{R}_n(\phi^*_n) - \hat{R}^\mathrm{Out}_{CV}(p^*_n) \geq \eta) \leq \min_{k_0 \in \{1, \ldots, n-1\}} \left( \sum_{k=1}^{k_0-1} (2n(1 - k/n) + 1)^{4\tilde{C}/(1-k/n)} \exp(-2n\eta^2) + \sum_{k=k_0}^{n-1} \exp(-2k\eta^2) \right) \leq \min_{k_0 \in \{1, \ldots, n-1\}} \left( k_0(2n + 1)^{4\tilde{C}/(1-k_0/n)} \exp(-2n\eta^2) + \exp(-2k_0\eta^2) \frac{1 - \exp(-2k\eta^2))^{n-k_0}}{1 - \exp(-2\eta^2)} \right) \leq \min_{k_0 \in \{1, \ldots, n-1\}} \left( (2n + 1)^{4\tilde{C}/(1-k_0/n)} \alpha^n + \frac{\alpha^{k_0}}{1 - \alpha} \right) \text{ with } \alpha := \exp(-2\eta^2) \\
\]

We look for \(k_0\) in \(\{(1 - z_n)n, 0 < z_n < 1 \text{ and } z_n \to n\infty 0\}\)

\[
\Pr(\tilde{R}_n(\phi^*_n) - \hat{R}^\mathrm{Out}_{CV}(p^*_n) \geq \eta) \leq \min_{z_n} \left( (2n + 1)^{4\tilde{C}/z_n} \alpha^n + \frac{\alpha^{(1-z_n)n}}{1 - \alpha} \right) \\
\]

We look for \(z_n\) such that \((2n + 1)^{4\tilde{C}/z_n} \sim_{n\to\infty} \frac{1}{\alpha^{z_n/n}}\). It is thus equivalent to: 

\[-n \ln(\alpha) z_n^2 - \ln(1 - \alpha) z_n - 4\tilde{C} \ln(2n + 1) = 0 \]

We have \(\Delta = \ln(1 - \alpha)^2 - 16\tilde{C} \ln(2n + 1) n \ln(\alpha) > 0 \text{ since } |\alpha| < 1\)

Since \(0 < z_n < 1\), we have necesserally \(z_n\) the non negative root of the previous equation which leads to:

\[
z_n = \frac{\ln(1 - \alpha) + \sqrt{\ln(1 - \alpha)^2 - 16\tilde{C} \ln(2n + 1) n \ln(\alpha)}}{-2n \ln(\alpha)} \\
\sim \frac{4\tilde{C}^{1/2} \ln((1/\alpha)^{1/2})}{\ln(n)^{1/2} \sqrt{n \ln(n)}} \\
\sim 2\sqrt{2}\tilde{V}_{\tilde{C}}^{1/2} \sqrt{\frac{\ln(n)}{n}} \\
\]

We can inject \(z_n\) in \((2n + 1)^{4\tilde{C}/z_n} \alpha^n + \frac{\alpha^{(1-z_n)n}}{1 - \alpha}\) and we find that

\[
\Pr(\tilde{R}_n(\phi^*_n) - \hat{R}^\mathrm{Out}_{CV}(p^*_n) \geq \eta) = O_n((n + 1)^{8\tilde{C}} \exp(-2n(\eta - 2\sqrt{2}\tilde{V}_{\tilde{C}}^{1/2} \sqrt{\ln(n)/n})))/(1 - \exp(-2\eta^2)) \\
\]

Let us now find the expression of \(f\) the inverse of \(\min_{\varepsilon}(B(n, p_n, \varepsilon), V(n, p_n, \varepsilon))\) with

- \(B(n, p_n, \varepsilon) = \min((2n(1 - p_n) + 1)^{4\tilde{C}/p_n} \exp(-n\varepsilon^2/9))\)
- \(V(n, p_n, \varepsilon) = \exp(-2np_n\varepsilon^2)\)

In the case of ERM algorithm,

\[
\exp(-2np_n\varepsilon^2) \leq (2n(1 - p_n) + 1)^{4\tilde{C}/p_n} \exp(-n\varepsilon^2/9) \\
\]

if and only if \(-2np_n\varepsilon^2 \leq \frac{4\tilde{C}}{1 - p_n} \ln(2n(1 - p_n) + 1) - n\varepsilon^2/9\) which is equivalent to

\[
n(1/9 - 2p_n)\varepsilon^2 \leq \frac{4\tilde{C}}{1 - p_n} \ln(2n(1 - p_n) + 1) \\
\]

and also \(\varepsilon \leq \sqrt{\frac{4\tilde{C} \ln(2n(1 - p_n) + 1)}{n(1/9 - 2p_n)\varepsilon^2}} := \varepsilon_n\).
Thus if $\varepsilon \leq \varepsilon_n$, it follows that $\min(B(n, p_n, \varepsilon), V(n, p_n, \varepsilon)) = \exp(-2np_n\varepsilon^2)$, thus if $\delta = \exp(-2np_n\varepsilon^2)$ we deduce that $\varepsilon = \sqrt{\frac{\ln(1/\delta)}{2np_n}}$. If $\varepsilon > \varepsilon_n$, $\min(B(n, p_n, \varepsilon), V(n, p_n, \varepsilon)) = (2n(1-p_n)+1)^{\frac{4V_n}{1-p_n}} \exp(-n\varepsilon^2/9)$. Thus if $\delta = (2n(1-p_n)+1)^{\frac{4V_n}{1-p_n}} \exp(-n\varepsilon^2/9)$, we then deduce that $\varepsilon = 3\sqrt{\frac{4V_n\ln(2n(1-p_n)+1)}{(1-p_n)(1/9-2p_n)}} = (2n(1-p_n)+1)^{\frac{4p_nV_n}{n(1-p_n)\ln(2n(1-p_n)+1)}}$. Denote $\delta_n = \exp(-2np_n\varepsilon_n^2) = \exp(-\frac{4p_nV_n\ln(2n(1-p_n)+1)}{(1-p_n)(1/9-2p_n)}) = (2n(1-p_n)+1)^{\frac{4p_nV_n}{n(1-p_n)\ln(2n(1-p_n)+1)}}$.

In conclusion, if $\delta \geq \delta_n$, we have:

$$f(n, p_n, \delta) = \sqrt{\frac{\ln(1/\delta)}{2np_n}}$$

and if $\delta < \delta_n$,

$$f(n, p_n, \delta) = 3\sqrt{\frac{4V_n\ln(2n(1-p_n)+1)}{n(1-p_n)\ln(1/\delta)}} \leq 6\sqrt{\frac{V_n\ln(2n+1) + \ln(1/\delta)}{n(1-p_n)}}.$$

□

In summary, the probability of the deviation between the out-of-bag cross-validation estimate and the generalization error is bounded by the minimum of a Hoeffding-type bound and a Vapnik-Chernovenkis-type bounds, and thus it is smaller than 1 even for small learning sets. Finally, we also give a simple rule on how to subbag the predictor. However, in the case of classification, we show that subagging strong learners can give a strong learner. It would be more interesting to answer the following question: can we obtain a similar result with the subagging of weak learners?
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6 Appendices

We will use the definition of strong difference bounded introduced by [KUT02] and a corollary of his main theorem inspired by [McD89].

**Definition 35 (Kutin [KUT02])** Let $\Omega_1, \ldots, \Omega_n$ be probability spaces. Let $\Omega = \prod_{k=1}^n \Omega_k$ and let $X$ a random variable on $\Omega$. We say that $X$ is strongly difference bounded by $(b,c,\delta)$ if the following holds: there is a "bad" subset $B \subset \Omega$, where $\delta = P(B)$. If $\omega, \omega' \in \Omega$ differ only in $k$-th coordinate, and $\omega \notin B$, then

$$|X(\omega) - X(\omega')| \leq c$$

Furthermore, for any $\omega, \omega' \in \Omega$,

$$|X(\omega) - X(\omega')| \leq b$$

We will need the following theorem. It says in substance that a strongly difference bounded function of independent variables is closed to its expectation with high probability.

**Theorem 36 (Kutin [KUT02])** Let $\Omega_1, \ldots, \Omega_n$ be probability spaces. Let $\Omega = \prod_{k=1}^n \Omega_k$ and let $X$ a random variable on $\Omega$, which is strongly difference bounded by $(b,c,\delta)$. Assume $b \geq c \geq 0$ and $\alpha > 0$. Let $\mu = E(X)$. Then, for any $\tau > 0$,

$$\Pr(X - \mu \geq \tau) \leq 2 \exp(-\frac{\tau^2}{8n(c + b\alpha)^2}) + \frac{n\delta}{\alpha}$$

We will use the definition of weak difference bounded introduced by [KUT02] and a corollary of his main theorem.

**Definition 37 (Kutin)** Let $\Omega_1, \ldots, \Omega_n$ be probability spaces. Let $\Omega = \prod_{k=1}^n \Omega_k$ and let $X$ a random variable on $\Omega$. We say that $X$ is weakly difference bounded by $(b,c,\delta)$ if the following holds: for any $k$,

$$\forall^k(\omega, v) \in \Omega \times \Omega_k, \, P(|X(\omega) - X(\omega')|) \leq c$$

where $\omega'_k = v$ and $\omega'_i = \omega_i$ for $i \neq k$, and the notation $\forall^k \omega, \Phi(\omega)$ means "$\Phi(\omega)$ holds for all but but a $\delta$ fraction of $\Omega$".

$$|X(\omega) - X(\omega')| \leq c$$

Furthermore, for any $\omega, \omega' \in \Omega$, differing only one coordinate:

$$|X(\omega) - X(\omega')| \leq b$$

We will need the following theorem. It says in substance that a weakly difference bounded function of independent variables is closed to its expectation with probability.

**Theorem 38 (Kutin)** Let $\Omega_1, \ldots, \Omega_n$ be probability spaces. Let $\Omega = \prod_{k=1}^n \Omega_k$ and let $X$ a random variable on $\Omega$, which is weakly difference bounded by $(b,c,\delta)$. Assume $b \geq c \geq 0$ and $\alpha > 0$. Let $\mu = E(X)$. Then, for any $\epsilon > 0$

$$\Pr(|X - \mu| \geq \epsilon) \leq 2 \exp(-\frac{\epsilon^2}{10nc^2(1 + \frac{2\epsilon}{15nc})^2}) + \frac{2nb\delta^{1/2}}{c} \exp(\frac{\epsilon b}{4nc^2}) + 2n\delta^{1/2}$$

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