On the Vertex-Connectivity of an Uncertain Random Graph

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ABSTRACT In many practical problems, randomness and uncertainty simultaneously appear in one complex system or network. When graph theory is applied to these problems, these complex systems or networks are usually represented by uncertain random graphs, in which some edges exist with degrees in probability measure, and some other edges exist with degrees in uncertain measure. In this paper, we focus on the connectivity of uncertain random graphs with respect to vertices. We propose the concepts of $k$-vertex-connectivity index and the connectivity of an uncertain random graph. The former is the chance measure that an uncertain random graph is $k$-vertex-connected, and the latter is an uncertain random variable, which characterizes the connectivity of the uncertain random graph with respect to vertices. We discuss some properties of these concepts. Methods and formulas are also presented for calculating the $k$-vertex-connectivity index, and the distribution and expected value of the connectivity of an uncertain random graph.

INDEX TERMS Chance theory, graph theory, uncertain random graph, uncertainty theory, vertex-connectivity.

I. INTRODUCTION

The study of graph theory could be traced back to the year of 1736, when Euler published the first paper on graph theory for solving the Seven Bridges Problem of Königsberg. In his paper, Euler used vertices to represent the areas of land, and used edges to represent the bridges. This is the basic model of graphs of graph theory, and is still widely used today. After two centuries, D. König published his book Theory of Finite and Infinite Graphs in 1936, which is the first book on graph theory. (König’s book was republished in 1990 [18].) Since then, this subject experienced explosive growth, due in large measure to its role as an essential structure underpinning modern applied mathematics. In recent years, graph theory has been widely applied to many fields, especially to all kinds of networks, such as social network [3], web network [4] and biological protein network [29]. Therefore, the research on structural properties and parameters of graphs became more and more important, and drew interest of researchers from many different fields.

In Euler’s model of graphs, vertices and edges are deterministic. However, indeterminate factors frequently appear in networks and systems of practical problems today. So when graph theory is applied to these networks and systems, it is inappropriate to use deterministic graphs to represent them. It is reasonable to assume that some edges exist with some belief degrees.

In order to deal with indeterminacy phenomena, probability theory was developed by Kolmogorov [17] in 1933 for modeling frequencies. When there was sufficient data to generate probability distribution functions for random variables, Erdős and Rényi suggested that whether the edge between two vertices exists or not, could be represented by a random variable. Under this assumption, the model of random graphs was proposed by them [6]. The interested readers could read the book Random Graphs [1] written by Bollobás. However, when there were no samples or observation data, it was impossible to determine the distributions of random variables. In 2007, uncertainty theory was founded by Liu [20] for modeling belief degrees. Then uncertainty theory was applied to evaluating the belief degree based on the experience and knowledge of experts [21]. In 2013, the model of uncertain graphs was proposed by Gao and Gao [10]. In an uncertain graph, the existence of the edge between two vertices, could be represented by an uncertain variable. Some structural parameters of an uncertain graph were soon discussed, such
as Euler index [32], matching index [33], connectivity index [10], cycle index [7], regularity index [8] and tree index [9]. In addition, some important problems of graph theory were also discussed in uncertain graphs, such as the shortest path problem [37] and the algorithm for edge connectivity [12]. For more information, the interested readers may refer to [11], [19], [27], [28], [13], [34].

In recent years, systems and networks became more and more complex. In many cases, randomness and uncertainty simultaneously appear in one complex system or network, because some indeterminate factors had sufficient samples and data while some other indeterminate factors did not. In order to represent these systems and networks, uncertain random graph was proposed via chance theory by Liu [22] in 2014. In an uncertain random graph, all edges are independent, and some edges exist with degrees in probability measure while other edges exist with degrees in uncertain measure. In 2014, Liu [22] discussed the connectivity index of an uncertain random graph. The Euler index of an uncertain random graph was discussed by Zhang et al. [35] in 2017. They also discussed the matching index of an uncertain random graph in 2018 [36]. In 2018, Chen et al. [5] discussed the cycle index of an uncertain random graph. For more knowledge about chance theory and uncertain random graphs, the readers may refer to [15], [16], [23]–[25], [26], [30], [31], [38].

In this paper, we will discuss the connectivity properties of an uncertain random graph with respect to vertices. We will propose the concepts of $k$-vertex-connectivity index and the connectivity of an uncertain random graph. The former is the chance measure that an uncertain random graph is $k$-connected, and the latter is an uncertain random variable, which characterizes the connectivity of this uncertain random graph with respect to vertices. The properties of them will be discussed. Methods and algorithms will be presented for determining the $k$-vertex-connectivity index, and the distribution and expected value of the connectivity of an uncertain random graph.

The remainder of the paper is organized as follows. In Section 2, we will introduce some necessary definitions and notations of graph theory, and give a brief introduction of chance theory and uncertain random graphs. In Section 3, we will propose the concept of $k$-vertex-connectivity index, discuss its properties, and give a formula for calculating the index. In Section 4, we will propose the concept of connectivity, and present how to determine its distribution and expected value. In Section 5, algorithms will be presented, and an example will be given to illustrate the ideas. The last section will conclude this paper with a brief summary.

II. PRELIMINARIES

In this section, we first introduce some definitions of graph theory. Then we introduce some preliminary knowledge about chance theory and uncertain random graphs.

A. NOTATIONS OF GRAPH THEORY

Graphs in this paper are finite simple graphs, which have no multi-edges and loops. Terms and notations not defined here are referred to [2].

A graph $G$ is an ordered pair $(V, E)$ consisting of a set $V$ of vertices and a set $E$ of edges. A graph is of order $n$ if it has $n$ vertices. Without loss of generality, in the rest of this paper, we assume that graphs are always of order $n$, and $V = \{1, 2, \ldots, n\}$.

Two vertices $i$ and $j$ are called adjacent if there is an edge $e$ between them. They are called the endpoints of edge $e$. And we say $e$ is incident with vertices $i$ and $j$. We also use $(i, j)$ to represent edge $e$. For a vertex $i$, the degree of $i$, denoted by $d(i)$, is the number of edges incident with this vertex. The minimum degree of a graph $G$ is denoted by $\delta(G)$. For a vertex set $U \subseteq V$, we use $G - U$ to denote the graph $(V \setminus U, E')$, where $E' = \{(i, j) \in E | i, j \in V \setminus U\}$. For an edge set $E_0 \subseteq E$, we use $G - E_0$ to denote the graph $(V, E \setminus E_0)$. If all the vertices of $G$ are pairwise adjacent, then $G$ is called a complete graph, which is denoted by $K_n$.

A walk is a sequence $i_1e_1i_2e_2i_3\cdots i_ke_ki_{k+1}$ such that $(i_j, i_{j+1}) = e_j$, for $j = 1, 2, \ldots, k$. If the vertices of a walk are disjoint, then the walk is called a path. A graph is called connected if for every pair of distinct vertices, there is a path linking them. For a connected graph $G$ and a positive integer $k$, $G$ is called $k$-connected or $k$-vertex-connected, if $G - U$ is connected for every set $U \subseteq V$ of fewer than $k$ vertices. The connectivity of $G$, denoted by $\kappa(G)$, is the greatest integer $k$ such that $G$ is $k$-connected. For a connected graph $G$ and a positive integer $k$, $G$ is called $k$-edge-connected, if $G - E_0$ is connected for every set $E_0 \subseteq E$ of fewer than $k$ edges. The edge-connectivity of $G$, denoted by $\kappa'(G)$, is the greatest integer $k$ such that $G$ is $k$-edge-connected. If $G$ is not a complete graph, then $\kappa(G) \leq n - 2$. The reason is that there are two non-adjacent vertices in $G$, and deleting all other $n-2$ vertices leaves a disconnected graph. In particular, $\kappa(G) = \kappa'(G) = n - 1$ if and only if $G = K_n$, and $\kappa(G) = \kappa'(G) = 0$ if and only if $G$ is disconnected.

Proposition 1: [2] Let $G$ be a graph. Then

$$\kappa(G) \leq \kappa'(G) \leq \delta(G).$$

As deleting a vertex would deleting all edges incident to it, $\kappa$ is smaller than $\kappa'$ for most graphs. For example, let $H = (V, E)$ be the graph, where $V = \{1, 2, 3, 4, 5\}$ and $E = \{(1, 2), (2, 3), (1, 3), (3, 4), (4, 5), (3, 5)\}$. Then $\kappa(H) = 1$, while $H - \{3\}$ is disconnected. However, we have to delete at least two edges of $H$ to obtain a disconnected graph. So $\kappa'(H) = 2$.

Let $i$ and $j$ be two distinct non-adjacent vertices of $G$. We say that a vertex set $U \subseteq V \setminus \{i, j\}$ separates $i$ and $j$ if there are no paths linking $i$ and $j$ in $G - U$. The vertex set $U$ is also called a vertex cut separating $i$ and $j$. The minimum size of a vertex cut separating $i$ and $j$ is denoted by $c(i, j)$. Note that $c(i, j)$ could be computed by running the Max-Flow Min-Cut Algorithm on an auxiliary digraph of order $2n - 2$ with unit capacities ( [2] Section 9.1). By the definition of
connectivity of $G$,
\[
\kappa(G) = \min \{c(i,j) : i, j \in V, i \neq j, (i,j) \notin E\}. \quad (1)
\]
As Max-Flow Min-Cut Algorithm is a polynomial-time algorithm, $\kappa(G)$ could be computed in polynomial time.

The adjacency matrix of $G$, denoted by $A$, is an $n \times n$ matrix
\[
\begin{pmatrix}
a_{11} & a_{12} & \cdots & a_{1n} \\
a_{21} & a_{22} & \cdots & a_{2n} \\
\vdots & \vdots & \ddots & \vdots \\
a_{n1} & a_{n2} & \cdots & a_{nn}
\end{pmatrix},
\]
where
\[
a_{ij} = \begin{cases} 1, & \text{if } (i,j) \in E \\ 0, & \text{otherwise}. \end{cases}
\]
As $G$ is a simple graph, $A$ is a symmetric matrix with $a_{ii} = 0$, for $i = 1, 2, \cdots, n$. The following is a well-known result.

**Proposition 2**: Let $G$ be a graph with adjacency matrix $A$. Then $G$ is connected if and only if $I + A + A^2 + \cdots + A^{n-1} > 0$, where $I$ is the identity matrix.

For a set $U \subset V$, we use $A_U$ to denote the matrix obtained from $A$ by deleting the $i$th rows and the $i$th columns, $i \in U$. Note that $A_U$ is the adjacency matrix of $G - U$. For any matrix $X$, define a function
\[
f(X) = \begin{cases} 1, & \text{if } I + X + X^2 + \cdots + X^{n-1} > 0 \\ 0, & \text{otherwise}. \end{cases}
\]
Then by the definition of $\kappa(G)$ and Proposition 2, we have the following theorem.

**Theorem 1**: Let $G = (V, E)$ be a graph with adjacency matrix $A$. Then
\[
\kappa(G) = \min_{U \subset V, f(A_U) = 0} |U| \quad (2)
\]

**Proof**: By Proposition 2, for a vertex set $U \subset V$, $G - U$ is disconnected if and only if $f(A_U) = 0$. Assume that $\min_{U \subset V, f(A_U) = 0} |U| = k$. Choose a vertex set $U_0$ such that $|U_0| = k$ and $f(A_{U_0}) = 0$. Then $G$ is not $(k+1)$-connected, as $G - U_0$ is disconnected. However, for any vertex set $V_0$ with $|V_0| < k$, by the choice of $U_0$, $f(A_{V_0}) = 1$, and $G - V_0$ is connected. Thus $G$ is $k$-connected. Therefore, by the definition of connectivity,
\[
\kappa(G) = k = \min_{U \subset V, f(A_U) = 0} |U|.
\]

**B. CHANCE THEORY**

Let $(\Gamma, \mathcal{L}, \mathcal{M})$ and $(\Omega, \mathcal{A}, \operatorname{Pr})$ be an uncertainty space and a probability space, respectively. The product $(\Gamma, \mathcal{L}, \mathcal{M}) \times (\Omega, \mathcal{A}, \operatorname{Pr})$ is called a chance space. The product $\sigma$-algebra $\mathcal{L} \times \mathcal{A}$ is the smallest $\sigma$-algebra containing all measurable rectangles of the form $\Lambda \times A$, where $\Lambda \in \mathcal{L}$ and $A \in \mathcal{A}$. For each $\Theta \in \mathcal{L} \times \mathcal{A}$, $\Theta$ is called an event of the chance space. The chance measure of event $\Theta$ was defined by Liu [24] as
\[
\operatorname{Ch}(\Theta) = \int_0^1 \operatorname{Pr}(\omega \in \Theta | M(\gamma \in \Gamma | (\gamma, \omega) \in \Theta) \geq x) dx.
\]

An uncertain random variable is a function $\xi$ from a chance space $(\Gamma, \mathcal{L}, \mathcal{M}) \times (\Omega, \mathcal{A}, \operatorname{Pr})$ to the set of real numbers such that $(\xi \in B)$ is an event in $\mathcal{L} \times \mathcal{A}$ for any Borel set $B$.

**Theorem 2**: [24] The chance measure is self-dual. That is, for any Borel set $B$ of real numbers, we have
\[
\operatorname{Ch}(\xi \in B) + \operatorname{Ch}(\xi \in B^c) = 1
\]

For an uncertain random variable $\xi$, its chance distribution function is defined by Liu [24] as
\[
\Phi(x) = \operatorname{Ch}(\xi \leq x) \quad (3)
\]
for any real number $x$. The expected value of $\xi$, denoted by $E[\xi]$, could be calculated by Theorem 3.

**Theorem 3**: [24] Let $\xi$ be an uncertain random variable with chance distribution $\Phi$. If the expected value of $\xi$ exists, then
\[
E[\xi] = \int_{-\infty}^{+\infty} (1 - \Phi(x)) dx - \int_{-\infty}^{0} \Phi(x) dx \quad (4)
\]
A random variable is called a Boolean random variable if it takes values 0 or 1. Similarly, an uncertain variable or an uncertain random variable is called a Boolean uncertain variable or a Boolean uncertain random variable, respectively, if it takes values 0 or 1. A function is called a Boolean function if the range of the function is $\{0, 1\}$. The following theorem is frequently used for determining parameters of uncertain random graphs.

**Theorem 4**: [23] Assume that $\eta_1, \eta_2, \cdots, \eta_m$ are independent Boolean random variables, i.e.
\[
\eta_i = \begin{cases} 1 & \text{with probability measure } a_i \\ 0 & \text{with probability measure } 1 - a_i \end{cases}
\]
for $i = 1, 2, \cdots, m$, and the variables $\tau_1, \tau_2, \cdots, \tau_n$ are independent Boolean uncertain variables, i.e.
\[
\tau_j = \begin{cases} 1 & \text{with uncertain measure } b_j \\ 0 & \text{with uncertain measure } 1 - a_j \end{cases}
\]
for $j = 1, 2, \cdots, n$. If $f$ is a Boolean function, then $\xi = f(\eta_1, \eta_2, \cdots, \eta_m, \tau_1, \tau_2, \cdots, \tau_n)$ is a Boolean uncertain random variable such that
\[
\operatorname{Ch}(\xi = 1) = \sum_{(x_1, \ldots, x_m) \in \{0, 1\}^m} \left( \prod_{i=1}^m \mu_i(x_i) \right) f^*(x_1, \ldots, x_m),
\]
where
\[
f^*(x_1, \ldots, x_m)
\]
\[
= \begin{cases} \sup_{f(x_1, \ldots, x_m, y_1, \ldots, y_n) = 1} \min_{y \leq 0} v_f(y), & \text{if } \sup_{f(x_1, \ldots, x_m, y_1, \ldots, y_n) = 1} \min_{y \leq 0} v_f(y) < 0.5 \\ 1 - \sup_{f(x_1, \ldots, x_m, y_1, \ldots, y_n) = 1} \min_{y \leq 0} v_f(y), & \text{if } \sup_{f(x_1, \ldots, x_m, y_1, \ldots, y_n) = 1} \min_{y \leq 0} v_f(y) \geq 0.5, \end{cases}
\]
This paper are simple graphs, $G$ adjacency matrix. Then the quartette collection of random edges, and $A$ exist, $i$ and $j$ are uncertain edges, $R = \{(i, j) | 1 \leq i < j \leq n \}$ and (i, j) are random edges), with $U \cup R = \{(i, j) | 1 \leq i < j \leq n \}$. Deterministic edges are regarded as special uncertain ones.

The uncertain random adjacency matrix is an $n \times n$ matrix

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix},$$

where $a_{ij}$ represent the truth values in uncertain measure or probability measure that the edges between vertices $i$ and $j$ exist, $i, j = 1, 2, \cdots, n$, respectively. As graphs considered in this paper are simple graphs, $A$ is a symmetric matrix, and $a_{ii} = 0$ for $i = 1, 2, \cdots, n$.

**Definition 1:** (Liu [22]) Assume $\mathcal{V}$ is the collection of vertices, $U$ is the collection of uncertain edges, $R$ is the collection of random edges, and $A$ is the uncertain random adjacency matrix. Then the quartette $\mathcal{G} = (\mathcal{V}, U, R, A)$ is said to be an uncertain random graph.

For an uncertain random graph $\mathcal{G} = (\mathcal{V}, U, R, A)$, write

$$X = \begin{pmatrix} x_{11} & x_{12} & \cdots & x_{1n} \\ x_{21} & x_{22} & \cdots & x_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ x_{n1} & x_{n2} & \cdots & x_{nn} \end{pmatrix}$$

and

$$\mathbb{X} = \begin{cases} x_{ij} = 0 \text{ or } 1, & \text{if } (i, j) \in R \\ x_{ij} = 0, & \text{if } (i, j) \notin U \\ x_{ij} = x_{ji}, & i, j = 1, 2, \cdots, n \\ x_{ij} = 0, & i = 1, 2, \cdots, n \end{cases}.$$  

(5)

For any $X \in \mathbb{X}$, the extension class of $X$ is a set of $n \times n$ matrices, which is defined by

$$X^* = \begin{cases} y_{ij} = x_{ij}, & \text{if } (i, j) \in R \\ y_{ij} = 0 \text{ or } 1, & \text{if } (i, j) \notin U \\ y_{ij} = y_{ji}, & i, j = 1, 2, \cdots, n \\ y_{ii} = 0, & i = 1, 2, \cdots, n \end{cases}.$$  

(6)

As there are $n(n-1)$ edges in $G$, there are $2^{n(n-1)}$ possible realizations of edges. Each realization of edges could be represented by a simple graph, which is called a realization graph. Note that a simple graph could be fully characterized by its adjacency matrix. So there is a bijection $F$ from the set of all realization graphs to the set $\{Y | Y \in \mathbb{X}, X \in X\}$, such that for any realization graph $H$, $F(H) = Y$ if $Y$ is the adjacency matrix of $H$. And the chance measure that the realization graph $H$ appears, is

$$\left( \prod_{(i,j) \in R} v_{ij}(Y) \right) \left( \min_{(i,j) \in U} v_{ij}(Y) \right),$$

where

$$v_{ij}(Y) = \begin{cases} a_{ij}, & \text{if } y_{ij} = 1 \\ 1 - a_{ij}, & \text{if } y_{ij} = 0. \end{cases}$$

**Example 1:** Let $\mathcal{G} = (\mathcal{V}, U, R, A)$ be an uncertain random graph (shown in Fig. 1), where $\mathcal{V} = \{1, 2, 3, 4\}$, $R = \{(1, 3), (2, 4)\}$, $U = \{(1, 2), (2, 3), (3, 4)\}$, and

$$A = \begin{pmatrix} 0 & 0.7 & 0.6 & 0 \\ 0.7 & 0 & 0.1 & 0.3 \\ 0.6 & 0.1 & 0 & 0.4 \\ 0 & 0.3 & 0.4 & 0 \end{pmatrix}.$$  

As $\mathcal{G}$ has 5 edges, it has $2^5$ realizations graphs, two of which are shown in Fig. 1. The chance measure of the event that $H_1$ appears is

$$(1 - 0.6) \times 0.3 \times \min\{0.7, 1 - 0.1, 0.4\},$$

which equals to 0.048. Similarly, $H_2$ appears with chance measure $0.6 \times 0.3 \times \min\{0.7, 0.1, 1 - 0.4\} = 0.018$.

An uncertain random graph $\mathcal{G} = (\mathcal{V}, U, R, A)$ becomes a random graph (Erdős-Rényi [6], Gilbert [14]) if $U = \emptyset$. This is the frequently used random graph model $\mathcal{G}(n, (p_{ij}))$ [1]. Then

$$\mathbb{X} = \begin{cases} x_{ij} = 0 \text{ or } 1, & i, j = 1, 2, \cdots, n \\ x_{ij} = x_{ji}, & i, j = 1, 2, \cdots, n \\ x_{ij} = 0, & i = 1, 2, \cdots, n \end{cases}.$$  

(7)

For any $X \in \mathbb{X}$, $X$ is the adjacency matrix of a realization graph, which appears with probability

$$\prod_{1 \leq i < j \leq n} v_{ij}(X).$$

An uncertain random graph $\mathcal{G} = (\mathcal{V}, U, R, A)$ becomes an uncertain graph (Gao-Gao [10]) if $R = \emptyset$. Then let $\mathbb{X}$ be the set of matrices satisfying (7). For any $X \in \mathbb{X}$, $X$ is the adjacency matrix of a realization graph, which appears with uncertain measure

$$\min_{1 \leq i < j \leq n} v_{ij}(X).$$
III. k-VERTEX-CONNECTIVITY INDEX OF AN UNCERTAIN RANDOM GRAPH

An uncertain random graph is connected for some realizations and is disconnected for other realizations. The connectivity index of an uncertain random graph $G$, denoted by $\rho(G)$ [22], is the chance measure of the event that $G$ is connected. However, in the applications of graph theory, graphs are always required to have better connectivity properties than just being connected. In this section, we investigate how likely an uncertain random graph is $k$-connected.

Definition 2: Let $G = (V, U, R, A)$ be an uncertain random graph. The $k$-vertex-connectivity index of $G$, denoted by $\eta_k(G)$, is the chance measure of the event that $G$ is $k$-connected.

As being connected is actually being 1-connected, $\rho(G) = \eta_1(G)$. Therefore, $\eta_k(G)$ is a generalization of $\rho(G)$. As a $(k+1)$-connected graph is also a $k$-connected graph, the event that an uncertain random graph $G$ is $(k+1)$-connected, is contained in the event that $G$ is $k$-connected. Thus, $\eta_{k+1}(G) \leq \eta_k(G)$. Also note that, a graph of order $n$ is at most $(n-1)$-connected. Therefore, we have the following proposition.

Proposition 3: Let $G = (V, U, R, A)$ be an uncertain random graph. Then

$$1 \geq \rho(G) = \eta_1(G) \geq \eta_2(G) \geq \cdots \geq \eta_{n-1}(G) \geq 0,$$

and

$$\eta_k(G) = 0, \text{ for } k \geq n.$$

Write $N = \{1, 2, \ldots, n\}$. Let $X$ be an $n \times n$ symmetric matrix. For a set $U \subseteq N$, recall that $X_U$ denotes the matrix obtained from $X$ by deleting the $i$th rows and the $i$th columns, $i \in U$. We define the following function

$$F(X) = \min_{U \subseteq N, f(\alpha_i) = 0} |U|,$$

where

$$f(X) = \begin{cases} 1, & \text{if } I + X + X^2 + \cdots + X^{n-1} > 0 \\ 0, & \text{otherwise} \end{cases}$$

By Theorem 1, if $X$ is the adjacency matrix of a graph, then $F(X)$ is its connectivity. Then the $k$-vertex-connectivity index of an uncertain random graph could be determined by the following theorem.

Theorem 5: Let $G = (V, U, R, A)$ be an uncertain random graph. Then for $1 \leq k \leq n-1$,

$$\eta_k(G) = \sum_{X \in \mathcal{X}} \left( \prod_{(i,j) \in R} v_{ij}(X) \right) \varphi_k^*(X)$$

where

$$v_{ij}(X) = \begin{cases} \alpha_{ij}, & \text{if } x_{ij} = 1 \\ 1 - \alpha_{ij}, & \text{if } x_{ij} = 0 \end{cases}, \quad (i, j) \in R,$$

and

$$\varphi_k(Y) = \begin{cases} 1, & \text{if } F(Y) \geq k \\ 0, & \text{otherwise}, \end{cases}$$

$F$ is the function defined by (8), $\mathcal{X}$ is the class of matrices satisfying (5), and $X^*$ is the extension class of $X$ satisfying (6).

Proof: The existence of edge $(i, j)$ could be represented by a Boolean variable $\beta_{ij}$. If $(i, j) \in R$, then $\beta_{ij}$ is a Boolean random variable. If $(i, j) \in U$, then $\beta_{ij}$ is a Boolean uncertain variable. As all edges are independent, all the random variables and uncertain variables are independent.

For any $X \in \mathcal{X}$ and for any $Y \in X^*$, $Y$ is the adjacency matrix of a realization graph. By Theorem 1, the realization graph is $k$-connected if and only if $\varphi_k(Y) = 1$. Thus, $\varphi_k$ is a Boolean function for determining whether a realization graph is $k$-edge-connected or not. Therefore, by Theorem 4, this theorem is proved.

When an uncertain random graph becomes a random graph, we have the following corollary.

Corollary 1: Let $G = (V, A)$ be a random graph. Then for $1 \leq k \leq n-1$,

$$\eta_k(G) = \sum_{X \in \mathcal{X}, \varphi_k(X) = 1} \left( \prod_{1 \leq i < j \leq n} v_{ij}(X) \right)$$

where

$$\varphi_k(X) = \begin{cases} 1, & \text{if } F(X) \geq k \\ 0, & \text{otherwise}. \end{cases}$$
\[ v_{ij}(X) = \begin{cases} \alpha_{ij}, & \text{if } x_{ij} = 1 \\ 1 - \alpha_{ij}, & \text{if } x_{ij} = 0, \end{cases} \]

\[ F \] is the function defined by (8), and \( X \) is the class of matrices satisfying (7).

When an uncertain random graph becomes an uncertain graph, we have the following corollary.

**Corollary 2:** Let \( G = (V, A) \) be an uncertain graph. Then for \( 1 \leq k \leq n - 1, \)

\[ \eta_k(G) = \begin{cases} \sup_{x \in X, \phi_k(x) = 1} \min_{1 \leq i \leq n} v_{ij}(X), & \text{if } x_{ij} < 0.5 \\ 1 - \sup_{x \in X, \phi_k(x) = 0} \min_{1 \leq i \leq n} v_{ij}(X), & \text{if } x_{ij} \geq 0.5, \end{cases} \]

where

\[ \varphi_k(X) = \begin{cases} 1, & \text{if } F(X) \geq k \\ 0, & \text{otherwise}, \end{cases} \]

\[ v_{ij}(X) = \begin{cases} \alpha_{ij}, & \text{if } x_{ij} = 1 \\ 1 - \alpha_{ij}, & \text{if } x_{ij} = 0, \end{cases} \]

\( F \) is the function defined by (8), and \( X \) is the class of matrices satisfying (7).

### IV. CONNECTIVITY OF AN UNCERTAIN RANDOM GRAPH AND ITS DISTRIBUTION

The connectivity of a graph \( G \), denoted by \( \kappa(G) \), is the greatest integer \( k \) such that \( G \) is \( k \)-connected. For an uncertain random graph, the connectivity differs for different realizations. So the connectivity of an uncertain random graph is an uncertain random variable.

**Definition 3:** Let \( G = (V, \mathcal{U}, \mathcal{R}, A) \) be an uncertain random graph. The connectivity of \( G \), denoted by \( \kappa(G) \), is a discrete uncertain random variable, taking values \( 0, 1, 2, \ldots, n - 1 \).

Note the fact that a graph with connectivity \( k \) is \( k \)-connected, but the converse is not true. Thus, compared with the event that \( G \) is \( k \)-connected, the event that \( \kappa(G) = k \) appears with a relatively smaller measure. Therefore, for \( 1 \leq k \leq n - 1, \)

\[ \eta_k(G) \geq \text{Ch}[\kappa(G) = k]. \]

The chance distribution of the connectivity of an uncertain random graph could be determined by the following theorem.

**Theorem 6:** Let \( G = (V, \mathcal{U}, \mathcal{R}, A) \) be an uncertain random graph. Then

\[ \text{Ch}[\kappa(G) = 0] = 1 - \eta_1(G), \]

and for \( 1 \leq k \leq n - 1, \)

\[ \text{Ch}[\kappa(G) = k] = \sum_{X \in X, \phi_k(X) = 1} \left( \prod_{1 \leq i < j \leq n} v_{ij}(X) \right) \phi_k(X) \]

where

\[ v_{ij}(X) = \begin{cases} \alpha_{ij}, & \text{if } x_{ij} = 1 \\ 1 - \alpha_{ij}, & \text{if } x_{ij} = 0 \end{cases} \]

\( F \) is the function defined by (8), and \( X \) is the class of matrices satisfying (7).

When an uncertain random graph becomes an uncertain graph, the connectivity becomes a discrete uncertain variable, taking values \( 0, 1, \ldots, n - 1 \).

**Corollary 4:** Let \( G = (V, A) \) be an uncertain graph. Then

\[ \mathcal{M}[\kappa(G) = 0] = 1 - \eta_1(G), \]
and for each \( k (1 \leq k \leq n - 1) \),
\[
M \{ \kappa (G) = k \} = \begin{cases} 
\sup_{x \in X, \phi_k(x) = 1} \min_{1 \leq i < j \leq n} \nu_{ij}(X), & \text{if } \min_{1 \leq i < j \leq n} \nu_{ij}(X) < 0.5 \\
1 - \sup_{x \in X, \phi_k(x) = 0} \min_{1 \leq i < j \leq n} \nu_{ij}(X), & \text{if } \min_{1 \leq i < j \leq n} \nu_{ij}(X) \geq 0.5
\end{cases}
\]
where
\[
\phi_k(X) = \begin{cases} 
1, & \text{if } F(X) = k \\
0, & \text{otherwise},
\end{cases}
\quad \nu_{ij}(X) = \begin{cases} 
\alpha_{ij}, & \text{if } x_{ij} = 1 \\
1 - \alpha_{ij}, & \text{if } x_{ij} = 0,
\end{cases}
\]

\( F \) is the function defined by (8), and \( X \) is the class of matrices satisfying (7).

Next, we will determine the chance distribution function and expected value of the connectivity of an uncertain random graph.

For an uncertain random graph \( G = (V, U, R, A) \), let \( \Phi(x) \) be the chance distribution function of \( \kappa (G) \). Then by Theorem 2,
\[
\Phi(x) = \text{Ch}[\kappa (G) \leq x] = 1 - \text{Ch}[\kappa (G) > x] = 1 - \text{Ch}[\kappa (G) \geq |x| + 1] = 1 - \text{Ch}[G \text{ is } (|x| + 1) \text{- connected}] = 1 - \eta_{|x|+1}(G).
\]

Therefore,
\[
\Phi(x) = \begin{cases} 
0, & \text{if } x < 0 \\
1 - \eta_{|x|+1}(G), & \text{if } k \leq x < k + 1, \\
1, & \text{if } x \geq n - 1.
\end{cases}
\] (9)

By (9) and (4), the expected value of \( \kappa (G) \) is
\[
E[\kappa (G)] = \int_0^{+\infty} (1 - \Phi(x)) dx - \int_0^{-\infty} \Phi(x) dx = \sum_{k=0}^{n-2} (1 - (1 - \eta_{k+1}(G))) = \sum_{k=0}^{n-2} \eta_{k+1}(G) = \sum_{k=1}^{n-1} \eta_k(G).
\] (10)

**Remark:** In a probability space \( (\Omega, A, Pr) \), the probability measure satisfies additivity axiom. That is,
\[
\text{Pr}\left( \bigcup_{k=1}^{\infty} A_k \right) = \sum_{k=1}^{\infty} \text{Pr}(A_k),
\]
if \( A_1, A_2, \ldots, A_k, \ldots \) are mutually disjoint events. However, the chance measure and the uncertain measure satisfy subadditivity axiom [20], [24] in a chance space and an uncertainty space, respectively. That is,
\[
\text{Ch}\left( \bigcup_{k=1}^{\infty} A_k \right) \leq \sum_{k=1}^{\infty} \text{Ch}(A_k),
\]
if \( A_1, A_2, \ldots, A_k, \ldots \) are mutually disjoint events in an uncertain random space. Therefore,
\[
\Phi(x) = \text{Ch}[\kappa (G) \leq x] = \text{Ch}[\kappa (G) \leq |x|] \leq \sum_{k=0}^{x} \text{Ch}(\kappa(G) = k).
\] (11)

**Algorithm 1** Algorithm for Calculating the \( k \)-Vertex-Connectivity Index of an Uncertain Graph

Step 1. Generate the set of matrices \( X \) satisfying (7). Set \( X_0 = X_1 = \emptyset \).

Step 2. Choose \( X = (x_{ij})_{n \times n} \in X \) and let \( H_X \) be the graph, whose adjacency matrix is \( X \). For every pair of vertices \( i \) and \( j \) satisfying \( i < j \) and \( x_{ij} = 0 \), determine \( c(i, j) \) in \( H_X \) by Max-Flow Min-Cut Algorithm. Then \( \kappa (H_X) = \min[c(i, j) : i, j \in V, i < j, x_{ij} = 0] \).

Step 3. If \( \kappa (H_X) \geq k \), then reset \( X_1 = X_1 \cup \{ X \} \); otherwise, reset \( X_0 = X_0 \cup \{ X \} \).

Step 4. Reset \( X = X \setminus \{ X \} \). If \( X \neq \emptyset \), then go to Step 2; otherwise, stop, and the \( k \)-connectivity index is
\[
\eta_k(G) = \begin{cases} 
\sup_{x \in X_1} \min_{1 \leq i < j \leq n} \nu_{ij}(X), & \text{if } \min_{1 \leq i < j \leq n} \nu_{ij}(X) < 0.5 \\
1 - \sup_{x \in X_0} \min_{1 \leq i < j \leq n} \nu_{ij}(X), & \text{if } \min_{1 \leq i < j \leq n} \nu_{ij}(X) \geq 0.5.
\end{cases}
\]

**V. ALGORITHMS AND EXAMPLES**

Recall that in a deterministic graph \( G \), for every pair of nonadjacent vertices \( i \) and \( j \), the minimum size of a vertex cut separating \( i \) and \( j \) is denoted by \( c(i, j) \), which could be computed by Max-Flow Min-Cut Algorithm. By (1), \( \kappa (G) \) could be computed by running the Max-Flow Min-Cut Algorithm for every pair of nonadjacent vertices. As Max-Flow Min-Cut Algorithm is a polynomial-time algorithm, \( \kappa (G) \) could be computed in polynomial time.

We first give algorithms for calculating the \( k \)-vertex-connectivity index and the distribution of connectivity of an uncertain graph. Let \( G = (V, A) \) be an uncertain graph.

Let \( G = (V, U, R, A) \) be an uncertain random graph. Let \( \bar{X} \) be the class of matrices satisfying (5) of \( G \). For each \( X \in \bar{X} \), let \( A_X = (a_{ij})_{n \times n} \) be the matrix such that
\[
a_{ij} = \begin{cases} 
x_{ij}, & \text{if } (i, j) \in U \\
a_{ij}, & \text{if } (i, j) \in R \\
0, & \text{if } i = j.
\end{cases}
\]
Algorithm 2 Algorithm for Calculating the Distribution of Connectivity of an Uncertain Graph
Step 1. Calculate $\eta_1(G)$ by Algorithm V, for $k = 1, 2, \cdots, n - 1$. Then the distribution function and expected value of $\kappa(G)$ could be calculated by (9) and (10), respectively.
Step 2. $M(\kappa(G) = 0) = 1 - \eta_1(G)$. Generate the set of matrices $X$ satisfying (5). Set $k = 1$.
Step 3. Set $X_{k0} = X_{k1} = 0$.
Step 4. Choose $X = (x_{ij})h_{X^n} \in X_k$, and let $H_X$ be the graph, whose adjacency matrix is $X$. For every pair of vertices $i$ and $j$ satisfying $i < j$ and $x_{ij} = 0$, determine $c(i, j)$ in $H_X$ by Max-Flow Min-Cut Algorithm. Then $\kappa(H_X) = \min\{c(i, j) : i, j \in V, i < j, x_{ij} = 0\}$.
Step 5. If $\kappa(H_X) = k$, then reset $X_{k1} = X_{k1} \cup \{X\}$; otherwise, reset $X_{k0} = X_{k0} \cup \{X\}$.
Step 6. Reset $X_k = X_k \setminus \{X\}$. If $X_k \neq \emptyset$, then go to Step 4; otherwise,

$$M(\kappa(G) = k) = \left\{ \begin{array}{ll}
\sup_{X \in X_{k11}} \min_{1 \leq i, j \leq n} v_j(X), \\
\text{if } \sup_{X \in X_{k11}} \min_{1 \leq i, j \leq n} v_j(X) < 0.5 \\
1 - \sup_{X \in X_{k01}} \min_{1 \leq i, j \leq n} v_j(X), \\
\text{if } \sup_{X \in X_{k01}} \min_{1 \leq i, j \leq n} v_j(X) \geq 0.5.
\end{array} \right.$$

Step 7. If $k \leq n - 2$, then set $k = k + 1$, and go to Step 3. Otherwise, stop.

Algorithm 3 Algorithm for Calculating the $k$-Vertex-Connectivity Index of an Uncertain Random Graph
Step 1. Generate the set of matrices $X$ satisfying (5).
Step 2. For each $X \in X$, generate the uncertain graph $G_X$, and determine $\eta_k(G_X)$ by Algorithm V.
Step 3. Calculate $\eta_k(G)$ by equation (12).

As deterministic edges could be viewed as special uncertain edges, $G_X = (V, A_X)$ is an uncertain graph. By Theorem 5 and Corollary 2,

$$\eta_k(G) = \sum_{X \in X} \left( \prod_{(i, j) \in R} v_j(X) \right) \phi_k^1(X) = \sum_{X \in X} \left( \prod_{(i, j) \in R} v_j(X) \right) \eta_k(G_X).$$

So $\eta_k(G)$ could be calculated by the following algorithm.

Similarly, by Theorem 6 and Corollary 4, for each $k (1 \leq k \leq n - 1)$,

$$\text{Ch}(\kappa(G) = k) = \sum_{X \in X} \left( \prod_{(i, j) \in R} v_j(X) \right) \phi_k^0(X) = \sum_{X \in X} \left( \prod_{(i, j) \in R} v_j(X) \right) M(\kappa(G_X) = k).$$

Therefore, the distribution of $\kappa(G)$ could be calculated by the following algorithm.

Example 2: Let $G = (V, U, R, A)$ be an uncertain random graph (shown in Fig. 2), where $V = \{1, 2, 3, 4\}, U = \{(1, 2), (2, 3), (3, 4), (4, 1)\}, R = \{(1, 3), (2, 4)\}$ and

$$A = \begin{pmatrix}
0 & 0.6 & 0.2 & 0.9 \\
0.6 & 0 & 0.7 & 0.4 \\
0.2 & 0.7 & 0 & 0.3 \\
0.9 & 0.4 & 0.3 & 0
\end{pmatrix}.$$

As $G$ has 2 random edges, there are $2^2$ matrices in $X$, which are listed as follows:

$$X_1 = \begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix},
X_2 = \begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix},
X_3 = \begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix},
X_4 = \begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0
\end{pmatrix}.$$

Therefore, there are four uncertain graphs $G_{X_1} = (V, A_{X_1}), G_{X_2} = (V, A_{X_2}), G_{X_3} = (V, A_{X_3})$ and $G_{X_4} = (V, A_{X_4})$ (shown in Fig. 2), where

$$A_{X_1} = \begin{pmatrix}
0 & 0.6 & 0 & 0.9 \\
0.6 & 0 & 0.7 & 0 \\
0 & 0.7 & 0 & 0.3 \\
0.9 & 0 & 0.3 & 0
\end{pmatrix},
A_{X_2} = \begin{pmatrix}
0 & 0.6 & 1 & 0.9 \\
0.6 & 0.7 & 0 & 0.3 \\
0.9 & 0 & 0.3 & 0 \\
0 & 0.6 & 0 & 0.9
\end{pmatrix},
A_{X_3} = \begin{pmatrix}
0 & 0.6 & 0 & 0.9 \\
0.6 & 0 & 0.7 & 1 \\
0 & 0.7 & 0 & 0.3 \\
0.9 & 1 & 0.3 & 0
\end{pmatrix},
A_{X_4} = \begin{pmatrix}
0 & 0.6 & 1 & 0.9 \\
0.6 & 0.7 & 1 & 0 \\
1 & 0.7 & 0 & 0.3 \\
0.9 & 1 & 0.3 & 0
\end{pmatrix}.$
For $1 \leq i \leq 4$, the $k$-connectivity index of $G_{X_i}$ could be calculated by Algorithm V. The results are shown in Table 1. It is easy to see that $\eta_1(G_{X_i}) \geq \eta_2(G_{X_i}) \geq \eta_3(G_{X_i})$. The distribution of connectivity of $G_{X_i}$ could be calculated by Algorithm V. The results are shown in Table 2. Note that $M[\kappa(G_{X_i}) = 0] = 1 - \eta_1(G_{X_i})$ and $M[\kappa(G_{X_i}) = k] \leq \rho_k(G_{X_i})$, for $i \in \{1, 2, 3, 4\}$ and $k \in \{1, 2, 3\}$, respectively.

The $k$-vertex-connectivity index of $G$ could be calculated by Algorithm V. By equation (12) and values of Table 1,

$$\eta_1(G) = \sum_{X_i \in X} \left( \prod_{(i,j) \in R} v_{ij}(X_i) \right) \rho_1(G_{X_i}) = 0.8 \times 0.6 \times 0.6 + 0.2 \times 0.6 \times 0.7 + 0.8 \times 0.4 \times 0.7 + 0.2 \times 0.4 \times 0.9 = 0.62.$$

Similarly, we have $\eta_2(G) = 0.332$ and $\eta_3(G) = 0.024$ (Table 3). It is easy to see that $\eta_1(G) \geq \eta_2(G) \geq \eta_3(G)$.

The distribution of connectivity of $G$ could be calculated by Algorithm V. Note that

$$\text{Ch}[\kappa(G) = 0] = 1 - \eta_1(G) = 1 - 0.668 = 0.332.$$

By equation (13) and values of Table 2,

$$\text{Ch}[\kappa(G) = 1] = \sum_{X_i \in X} \left( \prod_{(i,j) \in R} v_{ij}(X_i) \right) M[\kappa(G_{X_i}) = 1] = 0.8 \times 0.6 \times 0.6 + 0.2 \times 0.6 \times 0.7 + 0.8 \times 0.4 \times 0.7 + 0.2 \times 0.4 \times 0.3 = 0.62.$$
Similarly, we have $\text{Ch}(\kappa(G)) = 0.332$ and $\text{Ch}(\kappa(G)) = 0.024$ (Table 3). Note that $\text{Ch}(\kappa(G)) = k \leq \eta_k(G)$, $k = 1, 2, 3$.

By (9), the distribution function of $\kappa(G)$ is

$$\Phi(x) = \begin{cases} 0, & \text{if } x < 0 \\ 0.332, & \text{if } 0 \leq x < 1 \\ 0.668, & \text{if } 1 \leq x < 2 \\ 0.976, & \text{if } 2 \leq x < 3 \\ 1, & \text{if } 3 \leq x, \end{cases}$$

whence $E[\kappa(G)] = \sum_{k=1}^{3} \eta_k(G) = 0.668 + 0.332 + 0.024 = 1.024$.

VI. CONCLUSIONS

In this paper, we discussed the connectivity properties of an uncertain random graph with respect to vertices. We proposed the concept of $k$-vertex-connectivity index and gave a formula for determining the index. We also proposed the connectivity of an uncertain random graph and determined its distribution and expected value. The properties of both concepts were discussed. In addition, related algorithms were presented and examples were also presented to illustrate the ideas and methods.

Further research will focus on the following aspects. First, we may improve the efficiency of some related algorithms. Second, some important optimization problems, such as the minimum spanning tree problem and the shortest path problem, could be researched in the frame of uncertain random graphs. Finally, similar models could be introduced, such as uncertain fuzzy graph.

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