Bose-Einstein condensation of relativistic Scalar Field Dark Matter

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Standard thermodynamical results of ideal Bose gases are used to study the possible formation of a cosmological Bose-Einstein condensate in Scalar Field Dark Matter models; the main hypothesis is that the boson particles were in thermal equilibrium in the early Universe. It is then shown that the only relevant case needs the presence of both particles and anti-particles, and that it corresponds to models in which the bosonic particle is very light. Contrary to common wisdom, the condensate should be a relativistic phenomenon. Some cosmological implications are discussed in turn.

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I. INTRODUCTION

Scalar Field Dark Matter (SFDM) models, in which the dark matter (DM) particle is a spin-0 boson, are becoming a serious alternative to the Cold DM paradigm that invokes the existence of weakly interacting massive particles (WIMP’s)\(^6\).

The WIMP hypothesis has been widely studied in many proposed extensions of the Standard Model of Particle Physics, and there are high expectations for its detection in the near future. Support for the WIMP hypothesis comes from the comparison between high-resolution N-body simulations and the large scale structure we observe in the Universe. The resulting DM distribution seems to be in good agreement with observations, from clusters of galaxies down to the scales of the galaxies themselves.

Good agreement means that there are fine details for which the WIMP scenario does not have a definite answer. Those details refer to the internal DM density profile in galaxies, the low number of galaxy satellites, etc., see for instance. It is not clear if the observed discrepancies are due to corrections to the WIMP hypothesis that have not been taken into account yet, but the possibility exists that the DM particle is something rather different from a WIMP.

It is here where SFDM models enter into play. The usual approximation in cosmological studies so far is to describe SFDM particles by means of a scalar field \(\phi\) minimally coupled to gravity that is endowed with a scalar field potential \(V(\phi)\). The dynamics of any SFDM model depends upon the particular form of the scalar potential, but it is well known that a massive potential as simple as \(V(\phi) = m^2 \phi^2/2\), where \(m\) is the mass parameter of the SFDM particles, is enough to successfully describe DM in a cosmological setting up to linear order in the formation of large scale structure. The key point here is that the scalar field oscillations around the minimum of the aforementioned quadratic potential make the SFDM particles behaves as cold DM candidates.

More difficult tasks are those of non-linear structure formation, though some progress in that direction has been recently reported. Other studies show that the scalar field \(\phi\) can form gravitationally stable structures and that some of them may explain the particular features observed in the rotation curves of galaxies. Nonetheless, the full formation of cosmological structure still is an open question in SFDM models.

The mass \(m\) is a free parameter that has to be constrained by cosmological observations. The values considered in the literature cover an ample range of masses, from very heavy values of the order of \(10^{13}\) GeV to values as light as \(10^{-13}\) eV. It is usually agreed that very heavy SFDM particles would be indistinguishable from WIMP’s in many respects, and the reason is that cosmological bosonic features show up only if the Compton length of the scalar field \(\lambda_C \equiv m^{-1}\) is of the order of galactic dimensions. This is usually the case in galactic studies, in which very light values of the scalar field mass are ubiquitous.

On the other hand, the existence of SFDM particles, because of its bosonic nature, opens the possibility for the formation of a cosmologically relevant Bose-Einstein condensate (BEC). Actually, the scalar field \(\phi\) mentioned before is supposed to represent the wave function that describes the evolution of a cosmological BEC, see for instance. And, the scalar potential \(V(\phi)\) encloses, in general, all the possible self-interaction terms that may be present among the SFDM particles. Therefore, the scalar field description of DM and the formation of a BEC are closely related, but their relation is yet to be fully understood.

In this paper, I study the conditions for the formation of a cosmological SFDM BEC using known results of the thermodynamics of ideal Bose gases. I shall show that the formation of a BEC is not a generic process in the history of the Universe, but one that only appears for a relativistic SFDM gas composed of very light massive particles.

To set up the general framework, I describe first the general assumptions that will be present throughout the calculations below. First of all, the SFDM particles are supposed to have been in thermal equilibrium with other matter components in the early universe, at a time when they were relativistic. That means the temperature \(T_i\) of the early thermal bath was such that \(m \ll T_i \simeq 10^9\) GeV,
as this is the usual initial value assumed for the temperature at the beginning of the so-called Hot Big Bang.

Moreover, the SFDM particles will also decouple from the thermal bath while still relativistic. That is, the temperature at decoupling \( T_D \) is much larger than the mass of the particles, \( m \ll T_D < T_i \). Because we expect the Universe to expand adiabatically, the temperature of the SFDM gas will always be very similar to the temperature of the Universe itself.

Another assumption is that SFDM particles may or may not become non-relativistic at some point in the evolution of the Universe. This will allow us to study a ample range of masses for the SFDM particles, including mass values that are well below the present temperature of the Universe, \( m < T_0 \sim 10^{-4} \text{ eV} \).

We shall find that, for all practical purposes, the number density of SFDM particles at any given time suffices to decide about the formation of a BEC. Other thermodynamical quantities will be needed at some point, but only to clarify some features about the expansion of the Universe. All calculations below are given in units for which \( h = c = k_B = 1 \).

A brief description of the paper is as follows. In Sec. III I revise the formation of a BEC using standard formulæ for Bose ideal gases; the same is done in Sec. III but with formulæ that include the coexistence of particles and anti-particles. Finally, Sec. IV is devoted to conclusions.

II. STANDARD THERMAL HISTORY OF SFDM

I start with standard textbook results\[19\] to calculate the relic abundance of relativistic SFDM particles that become non-relativistic at late times; the latter is a necessary condition for them to be cold DM particles. The number density \( n_\phi \) of an ideal Bose-Einstein gas of SFDM particles is\[20\]

\[
n_\phi(T, \mu) = \frac{1}{2\pi^2} \int_0^\infty \frac{k^2 dk}{\exp[\beta(E_k - \mu(T))] - 1},
\]

where \( E_k^2 = k^2 + m^2 \), \( \beta = 1/T \) is the usual Boltzmann factor, \( T \) is the temperature of the thermal bath, and \( \mu(T) \) is the chemical potential. Relativistic means that \( m \ll T \); in addition, we must impose the condition \( |\mu| < m \) in order to keep \( n_\phi \) positive definite\[21\].

After the decoupling of the SFDM particles, the form of the distribution function remains that of a relativistic (massless) component, and the temperature of the SFDM particles, that I will denote by \( T_\phi \) henceforth, redshifts strictly as \( T_\phi \propto a^{-1} \), where \( a \) is the scale factor of the universe. For the rest of the species which continue to be in thermal equilibrium, the temperature falls a bit slower since entropy conservation implies that \( T \propto g_S \frac{1}{3}(T)a^{-1} \), where \( g_S(T) \) represents the entropy degrees of freedom at temperature \( T \). The number density \[1\] then acquires the usual form of a relativistic bosonic component

\[
n_\phi = \frac{\zeta(3)}{\pi^2} T_\phi^3,
\]

where \( \zeta(x) \) is the Riemann zeta function of \( x \).

It is useful to define the SFDM mass per photon \( \xi_\phi = \rho_\phi/n_\phi \[22\] \) where \( \rho_\phi \) denotes the energy density of the SFDM. While in the relativistic regime, \( m \ll T_\phi \), \( \xi_\phi \) evolves as

\[
\xi_\phi = \frac{\pi^4}{60\zeta(3)} \frac{T_\phi^4}{T^3}.
\]

Both temperatures \( T_\phi \) and \( T \) evolve approximately at the same rate, and then we conclude that \( \xi_\phi \sim a^{-1} \).

Later on in the history of the Universe, the SFDM particles become non-relativistic once \( T_\phi \approx m \). Their energy is \( E_k \approx m \), and then the energy density is simply given by \( \rho_\phi \approx m n_\phi \). Therefore, the SFDM mass per photon is now given by

\[
\xi_\phi = \frac{m}{2} \frac{T_\phi^3}{T^3} = \frac{m g_S(T_D)}{2 g_S(T)}.
\]

The last equality in the above equation arises from the fact that the SFDM particles were relativistic at the time they decoupled from the thermal bath.

If we take the present measured value \( \xi_{\phi,0} = 3.3 \times 10^{-28} m_p \[22\] \), Eq. (4) imposes tight constraints on the mass value of any SFDM particle that decoupled at a time when it was relativistic. Taking standard values such as \( g_S(T_D) \approx 10^2 \) and \( g_S(T_0) \approx 3.6 \[19\] \), where the subscript ‘0’ denotes present values, we find that the mass of the SFDM particles should be \( m \approx 1 \text{ eV} \). This particle became non-relativistic at a redshift of \( z \approx m/T_0 \approx 10^4 \), just on time to become a cold DM candidate.

The simple calculation above, which parallels those on generic hot DM relics\[19\], readily shows that present data would prefer a SFDM candidate as a massive as the neutrino, and that it would be quite difficult to accommodate either heavier or lighter SFDM particles that were in thermal equilibrium in the early universe, unless the entropy degrees of freedom at the time of decoupling \( g_S(T_D) \) take some unexpected and extreme values. But this seems to be very unlikely.

It is now time to turn our attention to the non-standard textbook case in which part of the SFDM particle budget is in the form of a BEC. As we said before, because the SFDM were relativistic at decoupling, the evolution of \( n_\phi \) is given by Eq. (2) at all later times.

We then have to take a look at the relativistic conditions for the formation of a BEC\[21\]; in this case, the critical temperature \( T_c \) and the critical number density \( n_{\phi, c} \) are given, respectively, by

\[
T_c = \left[ \frac{\pi^2}{\zeta(3)} n_\phi \right]^{1/3}, \quad n_{\phi, c} = \frac{\zeta(3)}{\pi^2} T_\phi^3.
\]
Hence, we can say that the formation of a BEC proceeds for temperatures $T_\phi < T_c$, or for number densities such that $n_{\phi,c} < n_\phi$.

In other words, Eq. (4) gives, for a given temperature, the maximum number density that can be accounted by the excited states; then, $n_{\phi,c} < n_\phi$ means that some particles must necessarily reside in the (condensed) ground state.

A quick comparison of Eqs. (5) with Eq. (2) shows that the excited states are just able to accommodate all SFDM particles, and then no BEC is necessary.

The properties of the charge density (6) have been widely explored in the literature. For our purposes, it suffices to know that at high temperatures, $m \ll T$, the charge density in excited states is

$$ q_{\phi,c} = \frac{\mu(T_\phi)}{3} T_\phi^2. $$

The above formula is true only for the case $\mu(T_\phi) \leq m$, as for larger values of the chemical potential the charge density is not positive definite. We conclude that the maximum charge density allowed by the excited states at a given relativistic temperature is $q_\phi(T_\phi) = m T_\phi^2 / 3$.

We have then given the needed description to define a critical temperature $T_c$ and a critical charge density $q_{\phi,c}$ for the formation of a BEC in an ultra-relativistic Bose gas of SFDM particles. The new formulae that replace Eqs. (3) are

$$ T_c = \left( \frac{3 q_\phi}{m} \right)^{1/2}, \quad q_{\phi,c} = \frac{m}{3} T_\phi^2. $$

On the other hand, cosmological expansion is adiabatic, and then we expect the usual scaling of the SFDM charge density $q_\phi = \eta_\phi T_\phi^3$ after decoupling, where $\eta_\phi$ is an appropriate constant we determine below. The reason is that the entropy density of an ultra-relativistic ideal Bose gas is

$$ s_\phi(T_\phi) = \frac{4 \pi^2}{45} T_\phi^3. $$

If the evolution of SFDM particles proceeds separately at constant entropy, $S_\phi = s_\phi a^3 = \text{const}$, we find the usual relationship $a T_\phi = \text{const}$.

## III. THERMAL HISTORY OF SFDM REVISITED

I now show that the formation of a relativistic BEC is possible if we take into account the coexistence of particles and anti-particles in a relativistic ideal Bose gas[16, 21] (see also[14]). To begin with, the equation that replaces (1) is

$$ q_{\phi,c} < q_\phi \quad \text{for the formation of a BEC} $$

The condition $q_{\phi,c} < q_\phi$ for the formation of a BEC translates into

$$ T_\phi > T_{\phi,c} = \frac{m}{3 \eta_\phi}. $$

In the same way, if we solve the first of Eqs. (8) for the critical temperature, we find

$$ T_c(T_\phi) = \left( \frac{T_c}{T_{\phi,c}} \right)^{1/2} T_\phi. $$

As long as Eq. (10) is satisfied, the above equation confirms the standard wisdom that $T \ll T_c$ is necessary for the formation of a BEC.\(^1\)

As shown in[17], the measurements on the present contribution of the DM can help to determine the $\eta_\phi$-constant that appears in Eq. (10). We expect the present charge density in SFDM particles to be of the order of $q_{\phi,0} \simeq n_{b,0}$, where $n_{b,0} \simeq 10^{-10}$ is the present baryon charge density. Finally, under the assumption that the SFDM particles were once part of the thermal bath in the early universe, a good estimation is

$$ \eta_\phi \simeq 10^{-10} \xi(3) \frac{g_S(T_D)}{\pi^2} \frac{g_S(T_0)}{g_S(T_D)} \simeq 10^{-10}. $$

Once the BEC is formed in the early Universe at high temperatures, its corresponding charge density $q_{BEC}$ is

$$ q_{BEC}(T_\phi) = n_\phi(T_\phi) - \frac{m}{3} T_\phi^2 = \eta_\phi T_\phi^3 \left( 1 - \frac{T_{\phi,c}}{T_\phi} \right). $$

\(^1\) It is said in Refs.[14, 17] that the critical temperature is $T_c = m / \eta_\phi$. This statement is confusing, as the very definition of $T_c$ is Eq. (3) for a given value of the charge density $q_\phi$. Thus, my interpretation is that Eq. (10) determines the minimum temperature $T_{\phi,c}$ of the SFDM particles at which a cosmological BEC can form.
The BEC charge density in the above equation is equivalent to other results obtained before\cite{21}; to show the equivalence one only needs to take the critical temperature $T_c$ defined in Eq. \eqref{11} and substitute it in Eq. \eqref{13}.

\section*{A. Non-relativistic SFDM}

Even if I have thoroughly supposed the existence of relativistic SFDM particles in the early universe, I include here, for completeness, a short description of the case in which SFDM particles are non-relativistic at decoupling.

It is well known that the density of particles for a decoupled non-relativistic species, for which $m \gg T_D$, is given by\cite{19}

$$n_\phi = e^{-(m-\mu_D)/T_D} \left( \frac{m T_\phi}{2 \pi} \right)^{3/2}, \quad \text{(14)}$$

where we notice an exponential suppression in which $\mu_D$ and $T_D$ are the values of the chemical potential and of the temperature at the time of decoupling, respectively. It must be noticed that the non-relativistic expression above is the same for the two relativistic cases discussed previously\cite{21}.

The non-relativistic conditions for the formation of a BEC are\cite{20}

$$T_c = \frac{2 \pi}{m} \left[ \frac{n_\phi}{\zeta(3/2)} \right]^{2/3}, \quad n_{\phi,c} = \zeta(3/2) \left( \frac{m T_\phi}{2 \pi} \right)^{3/2}. \quad \text{(15)}$$

Because of the constraint $|\mu| \leq m$, a comparison of Eqs. \eqref{14} and \eqref{15} leads us to the conclusion that a BEC does not appear for a non-relativistic species; this is but the known result that we cannot change between the uncondensed and condensed phases in an adiabatic process\cite{21}.

\section*{IV. CONCLUSIONS}

I have described above the main criteria that can lead to the formation of a cosmological BEC within the SFDM hypothesis. The different formulae presented must give a correct description of the SFDM thermodynamics as long as there are no sudden critical changes in the evolution of the Universe that may require a separate study\cite{19}. At the same order of approximation, the effects of the expansion of the Universe are taken into account in the adiabatic behavior of the thermodynamical quantities.

First of all, it is remarkable that the assumption of thermal equilibrium of the SFDM particles imposes tight constraints in the formation of a cosmological BEC. In general, the condensed phase is unreachable because of the adiabatic expansion of the universe.

The existence of a BEC is possible if the presence of anti-particles is explicitly taken into account. The discussion led us to Eq. \eqref{10}, a simple formula that gives the threshold temperature $T_{\phi,c}$ for the formation of a BEC in terms of basic physical quantities, the mass $m$ of the SFDM particles and their net charge contribution, see $\eta_\phi$ in Eq. \eqref{12}, to the total matter contents of the universe.

On the other hand, Eq. \eqref{10} points out that for massive enough SFDM particles, the cosmological BEC disappears at late times; the reason lies in the fact that the SFDM gas is diluted by the expansion of the Universe. Actually, we can easily estimate the mass values for which a SFDM BEC exists up to the present time. By taking $T_{\phi,c} = T_0$ in Eq. \eqref{10}, we find that a BEC always exists for $m < 10^{-14}$ eV. It is exactly this latter case in which a cosmological scalar field $\phi$ can be considered the right description of SFDM.

For particles with $m > 10^{-14}$ eV, the SFDM gas is still relativistic at the temperature at which the BEC disappears, $m \ll T_{\phi,c}$, see Eq. \eqref{10}. After that, the total charge density is entirely accounted by the excited states, $q_\phi = \eta_\phi T_\phi^3 = \mu(T_\phi) T_\phi / 3$, from which we find that the functional form of the chemical potential is the expected adiabatic one, $\mu(T_\phi) = 3 \eta_\phi T_\phi$\cite{10,21}.

If $m > 1eV$, the SFDM gas should also become non-relativistic before the present time. But because its number density continues to be that of an ultra-relativistic species, no BEC will be further form at low temperatures. A similar result arises for very large masses so that the SFDM particles decoupled while non-relativistic; the formation of a BEC is not allowed by the adiabatic expansion of the universe\cite{2}.

It is interesting that the existence of a BEC in a cosmological setting requires both high temperatures and relativistic particles. This is contrary to usual expectations according to which the formation of a BEC is considered a low-temperature and non-relativistic phenomenon (see for instance\cite{12,23}).

I should mention here that there is also the possibility that the SFDM particles were produced through a non-thermal process in the early universe (like the typical cases of the inflaton and axion fields\cite{19}), so that the appearance of a condensed phase could not be prevented. Such cases, however, are beyond the purposes of this paper.

The overall conclusion of this work is then that the formation of a relativistic BEC is the only possibility for SFDM models if the scalar particles were in thermal equilibrium in the early Universe.

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