Gradient-limited surfaces.

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A simple scenario of the formation of geological landscapes is suggested and the respective lattice model is derived. Numerical analysis shows that the arising non-Gaussian surfaces are characterized by the scale-dependent Hurst exponent, which varies from 0.7 to 1, in agreement with experimental data.

PACS numbers: PACS numbers: 05.40.-a, 91.10.Jf, 68.35.Bs, 64.60.Ak

Rough interfaces around us, such as Earth’s surface, cloud perimeters, fracture surfaces, etc., are common objects, the properties and formation of which have been studied for several decades. In some cases, significant advances in theoretical understanding have been achieved. In particular, this applies to the surface growth processes, the analysis of which has led to a wide variety of kinetic roughening models, cf. [1, 2]. However, many processes leading to rough surfaces, eg. the formation of fractures and Earth’s landscapes are less understood.

The formation of the Earth’s surface is a complex process, affected by various phenomena, such as seismic and tectonic activity, erosion, sedimentation, etc. These phenomena, in their turn, can be of diverse nature. Thus, erosion can be caused by meandering rivers, oceanic and atmospheric influence, by the motion of ice, avalanches, and so on. Furthermore, the physical properties of the ground vary in a very wide range. Incorporating all this diversity into a concise embraceable mathematical model is a hopeless task. However, the scale-invariant properties of the geologic landscapes appear to be surprisingly universal: in a reasonable approximation, they are typically self-affine, with the Hurst exponent ranging between $H ≈ 0.7$ and $0.9$ [1, 2]. Therefore, it is natural to expect that there is a simple, universal and robust mechanism leading to such surfaces. In what follows we show that such a mechanism can be provided by the competition of erosion and tectonic activity: the model of gradient-limited surfaces incorporates these two effects in their simplest form and leads to realistic landscapes.

Most models of geological landscapes are based on the evolution of river networks [1, 11, 12]. The evolution of rivers plays undoubtedly an important role in the formation of landscapes, but is not able to increase the height of the mountains and will not lead to self-affine surfaces. The only attempt of constructing a robust self-affine model of Earth’s surface has been made by Mandelbrot [1, 2]. He modelled roughening due to tectonic activity, and his method can be outlined as follows. Inside a polygon (Earth’s surface), a random point is coined. Through that point, a line of random direction is drawn. This “fault line” divides the polygon into two parts, one of which is elevated (with respect to the other) by a unit height. The procedure is repeated $N → \infty$ times. The Brownian growth of height differences is eliminated by normalizing the surface height to $\sqrt{N}$. This results in a self-affine surface with $H = 0.5$. In order to address the discrepancy between the model and empirical values $H ≈ 0.7 – 0.9$, the model has been generalized by replacing the Heaviside profile of the “fault” by a profile with singularity (so that the height change is given by $\Delta h = x^{\alpha} \text{sign } x$, where $-0.5 < \alpha < 0.5$ and the x-axis is perpendicular to the “fault line”).

The tectonic activity and formation of faults, as captured by the Mandelbrot’s model, plays certainly an important role in the evolution of the Earth’s surface. Meanwhile, the singular shape of the fault profile is artificial, with no physical motivation. Besides, there are no physical processes which would normalize the surface height to the number of faults. Instead of that, the basic effect reducing the height differences is erosion. As mentioned above, erosion itself is a very complex phenomenon which, in particular, has been addressed by the models of the evolution of river networks [1, 11, 12]. The excessively detailed erosion models, however, are not suited for revealing the most generic aspects of landscape roughening. Therefore, we opt for the simplest possible approach and assume that effectively, erosion imposes an upper limit to the modulus of the gradient of the surface height. More specifically, we assume that, as soon as a slope becomes steeper than a threshold value, the excess of the height drop is spread over the neighboring regions. Such a smoothing of too steep slopes can be accomplished, for instance, by avalanches.

To begin with, let us define the model of gradient-limited surfaces for a continuous medium. A random point $P$ and direction $\tau$ define a “fault line” inside a polygon of diameter $L \gg 1$. This line divides the polygon into two parts, one of which (leftmost, with respect to the direction $\tau$) is elevated by a unit height. The height drop is spread over the nearest neighboring regions in such a way that the modulus of the local gradient remains everywhere below a threshold value, i.e. $|\nabla \psi| < 1$. If the fault line goes through a region of a saturated slope, the avalanches can affect large areas. Then, the actual elevation (the change of surface slope) will take place far from the fault, at the edges of the saturated slope. These edges will be referred to as the elevation lines. The procedure is repeated $N → \infty$ times. Note that unlike the Mandelbrot’s model, this model has a lower cut-off scale (the ratio of the height drop at a single fault line
and threshold gradient).

The model can also be formulated on a lattice. A natural basis for the lattice formulation is given by the six-vertex (restricted solid-on-solid) model [3, 13]. On the square lattice of the six vertex model, all the edges are marked with arrows so that each site has equal number of incoming and outgoing arrows, see Fig. 1. The arrows define an incompressible flow, the streamfunction of which is our surface. Thus, each arrow represents a unit jump in the surface height. The mean slope is defined by the mean density of counter-directed arrows, which is limited by the step of the grid; hence, the slope steepness is constrained automatically.

Consider an oriented chain of arrows dividing the lattice into two parts (see Fig. 1). Swapping the direction of all the arrows of the chain is legitimate, because for all the affected sites, the number of incoming arrows is conserved. It corresponds to the lowering of one part of the surface with respect to the other part by two units. Therefore, directed chains of arrows can play the role of elevation lines. Gradient-limited surfaces are obtained as follows. A random site of the lattice $P$ and a random direction $\tau$ are coined, they define an aim line, the site of the “fault”. That part of the surface, which is leftwards to the aim line, is to be elevated. The elevation is accomplished along such a directed chain of arrows, which follows as closely as possible the aim line. Similarly to the continuous case, if the aim line goes through a region of uni-directional arrows (saturated slope), the elevation line is forced to go around those regions. There are different technical options, how to minimize the distance between the elevation line and aim line. In particular, the distance can be optimized locally and globally. However, the scaling properties of the resulting surfaces are insensitive with respect to the particular choice. For our main series of simulations, we used a local algorithm. The elevation line was traced step-by-step, starting from the origin $P$. At each step, there are two possibilities to continue the line, because there are two outgoing arrows from each site (however, if the site has been already visited, one of the outgoing arrows is occupied, and there is only one possibility left). That option is to be selected, which leads closer to the aim line. The (signed) departure from the aim line is easily tracked as $s = \Delta x \sin \alpha - \Delta y \cos \alpha$, where $\Delta x$ and $\Delta y$ are displacements along $x$ and $y$ axes, and $\alpha$ — the angle between the $x$-axis and the aim line. The elevation line is terminated as soon as it reaches the boundary of the polygon. The gradient-limited surfaces are obtained at the long-time limit, when the number of elevations exceeds the relaxation time (which scales as the number of sites in the polygon, see below), and the initial shape of the surface becomes irrelevant.

A simulation result is presented in Fig. 2. Observe the shape of the elevation line: in the region of a saturated slope, it has to depart far from the straight aim line. These regions are responsible for the long-range correlation of the surface height increments. Qualitatively, the saturated slopes are caused by accumulated excess of the “faults” of a certain direction. If there were no avalanches, that excess would fluctuate as the square root of the number of “faults”, tending to infinity. Therefore, the presence of large saturated slopes should not be surprising. As it will be shown below, in one-dimensional (1D) case, the accumulation phenomenon gives rise to $H = 1$ (i.e. typically, a saturated slope occupies the whole polygon). In 2D geometry, the accumulation effect is weaker and leads to a more interesting scaling behavior.

The one-dimensional (1D) version of the model is most...
conveniently formulated as a spin exchange problem, and admits analytical solution. This analytic approach helps us to understand the features of the 2D-model. Suppose there is a sequence of spins, \( \varphi_i = \pm 1 \). It is convenient to consider infinite periodic sequence, \( \varphi_{i+N} = \varphi_i \), where \( i \in Z \) and \( N \) is the period. The spins \( \varphi_i \) can be interpreted as the increments of a self-affine curve \( \psi_j = \sum_{i=0}^j \varphi_i \). A random point \( k \) and a random spin increment \( \nu = \pm 2 \) are coined. The increment is to be added to the \( k \)-th spin, or, if it is not possible (resulting in \( \varphi_k = \pm 3 \)), to the nearest suitable spin (i.e. to \( \varphi_l = -\nu \pm 1 \) with minimal value of \( |l-k| \)). If there is no suitable spin at all, the next pair of \( k \) and \( \nu \) are coined. The procedure is repeated ad infinitum. Let us denote the relative number of positive spins by \( \xi \). Then, the height drop of the above defined self-affine curve \( \psi_j \) at distance \( N \) is \( N(2\xi - 1) \). The quantity \( \xi \) performs Brownian fluctuations, because at each time step, it is randomly incremented by \( \pm N^{-1} \). At the limit \( N \rightarrow \infty \), the probability density function \( n(\xi, t) \) evolves according to the diffusion equation, \( n_t = D n_{\xi\xi} \) with \( D = (2N^2)^{-1} \) and no flux at the boundaries, \( n_\xi(0, t) = n_\xi(1, t) = 0 \). The stationary solution \( n(\xi, t) = 1 \) allows us to calculate the delta variance \( \langle (\psi_{i+N} - \psi_i)^2 \rangle = N^2 \int_0^1 (2\xi - 1)^2 d\xi = N^2/3 \), which corresponds to \( H = 1 \). The relaxation time of the spin exchange problem can be found as the diffusion time, \( \tau \approx N^2 \) (time is measured in the number of spin exchanges). The relaxation time of the 2D gradient-limited surfaces can be estimated in the same way, because the height difference between left and right edges of the polygon performs also nearly-Brownian fluctuations.

For two-dimensional geometry, the simulations indicate that the gradient-limited surfaces are not strictly speaking self-affine. However, the data collapse of the simulation results can be achieved, when we introduce the scale-dependent differential Hurst exponent, defined as

\[
h(\lambda) = \frac{1}{2} d \log \left( \frac{a_L^2}{d \log L} \right), \quad \lambda = \log L / \log L_{\text{max}}.
\]

Here \( a_L \) is the height of the surface at the distance \( L \) from the center of the polygon of size \( L_{\text{max}} \). The angular braces denote averaging over different realizations of the surface. In Fig. 3, the differential Hurst exponent is approximated by \( h(\lambda, L_{\text{max}}) = \log \left( \frac{\langle a^2 \rangle}{\langle a^2_{i+1} \rangle} \right) \frac{\log(L_i/L_{i+1})}{\log(L_i/L_{i+1})}^{-1}, \) where \( i \) and \( i+1 \) are neighboring data-points and \( \lambda = \log(L_i/L_{i+1})/\log L_{\text{max}}^2 \). At the limit \( L_{\text{max}} \rightarrow \infty \), the curves converge to the asymptotic function \( h(\lambda) \). Note that at the extreme right-hand-side of the plot, the convergence of the \( h(\lambda, L_{\text{max}}) \)-curves is not as good as elsewhere; this is explained by finite-size effects and by the fact that for \( \lambda \approx 1 \), the finite differences fail providing an acceptable approximation for the derivative in Eq. (1). The rapid fall-off of the curves at the limit \( \lambda \rightarrow 1 \) has the same origin, and therefore, the values \( h \lesssim 0.65 \) are not reliable.

For large scales (which are most interesting in the context of the Earth’s surface) with \( \lambda \gtrsim 0.9 \), a better approach is to extrapolate the asymptotic curve, see dashed line in Fig. 2. The conclusion \( h \approx 0.7-0.9 \) for \( 0.5 < \lambda < 1 \) is in a good agreement with the values recorded for geological landscapes, cf. [6, 14, 15]. Intriguingly, the roughness exponents of the fracture surfaces vary in the same range, cf. [12, 13].

Finally, the scaling law of the overall (edge-to-edge) height drop of the gradient-limited surfaces is given by the integral Hurst exponent \( H = \int_0^1 d\lambda: \langle a(L_{\text{max}})^2 \rangle \propto L_{\text{max}}^{2H} \); according to the simulations, \( H = 0.91 \pm 0.01 \). This exponent equals to the area under the asymptotic
We have observed that the differential Hurst exponent is scale-dependent. Such a generalized scale-invariance when critical exponents depend on scale, is not unique. For instance, similar behavior has been observed for certain forest fire models [16]. In our case, the Hurst exponent increasing towards small scales is caused by the presence of large areas of saturated slope. Indeed, consider a random pair of points. If the distance L between them is small, the points are likely to reside inside a single region of saturated slope. Hence, their average height difference scales almost as L implying h ≈ 1. On the other hand, larger saturated slopes are more rare than the smaller ones. Therefore, for a more distant pair of points, falling inside a single saturated slope is a rare event, and the conclusion h ≈ 1 is no more valid.

For Gaussian self-affine surfaces, all the scaling exponents of statistical topography are functions of the Hurst exponent H. However, the gradient-limited surfaces are not Gaussian, as evidenced by the presence of large saturated slopes. Therefore, the differential Hurst exponent h(λ) alone does not provide a complete description of the surface. First we consider the differential fractal dimension of the contour loops (“coastlines”), 

\[ d = d \log (l_L)/d \log L. \]

Here l is the length of a contour loop, and L its diameter. The numerical results are given in Fig. 4. For Gaussian surfaces, the fractal dimension of contour loops \( D(H) \approx 1.5 - 0.5H \) [7, 17]. Dotted line in Fig. 4 is the asymptotic \( (L_{\text{max}} \to \infty) \) dependence \( d(\lambda) \), and dashed line is the curve, calculated on the basis of the functions \( h(\lambda) \) (from Fig. 3), and \( D(H) \). Evidently, \( D(h(\lambda)) \neq d(\lambda) \).

Even more pronounced mismatch between the Gaussian surfaces and the gradient-limited surfaces is observed for size-distribution of the contour loops. Let \( p(L) \) denote the probability that a randomly chosen point belongs to such a contour loop, the diameter of which is larger than L, but smaller than 2L. Then, for Gaussian surfaces we would expect that \( p(L) = L^k \), where \( k = D(H) - (2 - H) < 0 \) [8, 17]. For gradient-limited surfaces, the curves \( k(\lambda, L_{\text{max}}) \) converge again to the asymptotic dependence \( k(\lambda) \). This convergence with \( k(\lambda) > 0 \) is clearly observed up to the scale \( \lambda \approx 0.85 \); above of that scale, convergence is very slow due to finite-size effects. The inequality \( k > 0 \) means that, as compared with the Gaussian surfaces, there is significantly lesser number of small contour loops, which is quite a natural observation. Indeed, the saturated slopes can be embraced by large contour loops, but leave almost no room for small ones.

In conclusion, the new model of gradient-limited surfaces leads to non-Gaussian surfaces which are characterized by the scale-dependent differential Hurst exponent. The latter varies from \( h \approx 1 \) for small scales, up to \( h \approx 0.7 \) for large scales. This is in a reasonable agreement with the experimentally observed roughness of real geological landscapes. The relevance of the model to fracture surfaces deserves further analysis.

The support of ESF grant No 5036 is acknowledged.

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[1] B. B. Mandelbrot, Proc. Natl. Acad. Sci. U.S.A. 72, 3825 (1975).
[2] B. B. Mandelbrot, The Fractal Geometry of Nature (Freeman, New York, 1982).
[3] P. Meakin, P. Ramanlal, L. M. Sander, and R. C. Ball, Phys. Rev. A 34, 5091 (1986).
[4] S. V. Buldyrev et al, Phys. Rev. A 45, R8313 (1992).
[5] Science 26, 185 (1982).
[6] E. Bouchaud, G. Lapasset, and J. Planèes, Phys. Rev. B 48, 2917 (1993).
[7] P. Meakin, Fractals, Scaling and Growth Far From Equilibrium, Cambridge University Press, Cambridge, England (1998).
[8] J. Kondev, C. L. Henley and D. G. Salinas, Phys. Rev. E 61, 164 (2000).
[9] A. Giacometti, A. Maritan, and J. R. Banavar, Phys. Rev. Lett. 75, 577 (1995).
[10] K. Sinclair and R. C. Ball, Phys. Rev. Lett. 76, 3360 (1996).
[11] J. Banavar et al, Phys. Rev. Lett. 78, 4522 (1997).
[12] K. K. Chan and D. H. Rothman, Phys. Rev. E 63, 055102(R) (2001).
[13] F. Rys, Helv. Phys. Acta 36, 537 (1963).
[14] M. Sahimi, Phys. Rep. 306, 213 (1998).
[15] K. J. Mályov, A. Hansen, E. L. Hinrichsen, and S. Roux, Phys. Rev. Lett. 68, 213 (1992).
[16] K. Chen and P. Bak, Phys. Rev. E 62, 1613 (2000).
[17] J. Kalda, Phys. Rev. E 64, 020101(R) (2001).
[18] J. Kalda, In *Transport, Chaos and Plasma Physics*, edited by S. Benkadda, F. Doveil, and Y. Elskens (World Scientific, Singapore, 1993), p. 272.