The inducibility of small oriented graphs

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Abstract: We use Razborov’s flag algebra method [7] to show an asymptotic upper bound for the maximal induced density \( i(\vec{P}_3) \) of the orgraph \( \vec{P}_3 \) in an arbitrary orgraph. A conjecture of Thomassé states that \( i(\vec{P}_3) = \frac{2}{5} \). The hitherto best known upper bound \( \frac{12}{25} \) was given by Bondy. We can show that \( \frac{12}{25} \leq i(\vec{P}_3) \). Further, we consider such a maximal density for some other small orgraphs. With easy arguments one can see that \( i(\vec{C}_3) = \frac{1}{4} \), \( i(\vec{K}_2 \cup \vec{E}_1) = \frac{3}{4} \) and \( \frac{21}{22} \leq i(\vec{C}_4) \). We show that \( \frac{21}{22} \leq i(\vec{C}_4) \). Furthermore we show that the extremal orgraphs of \( \vec{P}_3 \) and \( \vec{C}_4 \) are the same. Furthermore we show that \( \frac{6 - 4\sqrt{2}}{8} \leq i(\vec{K}_{1,2}) \leq \frac{6}{8} \).

1 Introduction

The whole paper deals with oriented graphs (we will call them orgraphs), thus graphs with directed edges, no loops, no bidirected edges and no multiple edges. For an orgraph \( \Gamma = (V, E) \) we write \( V_\Gamma \) for its set of vertices and \( |\Gamma| \) for its number of vertices as well as \( E_\Gamma \) for its set edges and \( |\Gamma| \) for its number of edges. The most famous and surely most studied problem on orgraphs is the Cacetta-Häggvist conjecture [3], which was made in 1978. In this paper we want to turn our attention to another problem in this area. For an orgraph \( \Gamma \) let \( \text{max}(\Gamma; n) \) denote the maximal number of sets \( T \subseteq V_\Gamma \) with \( |T| = |\Gamma| \) such that \( T \) induces a subgraph which is isomorphic to \( \Gamma \) in an \( n \)-vertex orgraph. Remark, that we don’t count possible symmetries of \( \Gamma \). Now the inducibility of an orgraph \( \Gamma \) is defined as

\[
i(\Gamma) := \limsup_{n \to \infty} \frac{\text{max}(\Gamma; n)}{|\Gamma|},
\]

the asymptotic value of the maximal density of \( \Gamma \) in any orgraph. Thus, in an arbitrary orgraph the maximal density of orgraphs \( \Gamma \) is \( i(\Gamma) + o(1) \).

There are several papers (see [1], [2], [4], [5] and [6]), where the inducibility on undirected simple graphs is investigated. There are some small orgraphs, whose inducibility is not known yet. On these we want to focus now.

For two orgraphs \( \Gamma_1 = (V_1, E_1) \) and \( \Gamma_2 = (V_2, E_2) \) the lexicographic graph product \( \circ \) is defined as the following orgraph.

\[
\Gamma_1 \circ \Gamma_2 := (V_1 \times V_2, \{(d_1, d_2)(d'_1, d'_2) \mid d_1d'_1 \in E_1 \text{ or } (d_1 = d'_1 \text{ and } d_2d'_2 \in E_2)\})
\]

Furthermore, we define

\[
\Gamma^{\circ n} := \underbrace{\Gamma \circ \Gamma \circ \ldots \circ \Gamma}_{n \text{-times}}.
\]
Thus, the lexicographic product $\Gamma_1 \circ \Gamma_2$ is a copy of the orgraph $\Gamma_1$, where each vertex of $\Gamma_1$ is replaced by a copy of $\Gamma_2$. Figure 1 defines some orgraphs, which we will need in the following.

![Figure 1: The definition of the orgraphs $\vec{P}_3$, $\vec{C}_3$, $\vec{C}_4$, $\vec{K}_2 \cup \vec{E}_1$, $\vec{K}_{1,2}$ and $\vec{K}_{2,1}$.](image)

**Observation 1.**

\[ i(\vec{P}_3) \geq \frac{2}{5} \text{ and } i(\vec{C}_4) \geq \frac{2}{21}. \]

**Proof.** We have a look on the limit orgraph $\lim_{n \to \infty} \left( \vec{C}_4 \right)^{\text{on}}$. Let $x$ be the density of $\vec{P}_3$ and $y$ the density of $\vec{C}_4$ in $\lim_{n \to \infty} \left( \vec{C}_4 \right)^{\text{on}}$. Then

\[
\begin{align*}
x &= 1 \cdot \frac{3}{4} \cdot \frac{2}{4} + 1 \cdot \frac{1}{4} \cdot \frac{1}{4} \cdot x, \\
y &= 1 \cdot \frac{3}{4} \cdot \frac{2}{4} \cdot \frac{1}{4} + 1 \cdot \frac{1}{4} \cdot \frac{1}{4} \cdot \frac{1}{4} \cdot y.
\end{align*}
\]

In both equations the first summand is the density of $\vec{P}_3$ (resp. $\vec{C}_4$), where each vertex of $\vec{P}_3$ (resp. $\vec{C}_4$) is from a different part. The second summand is the density that each vertex is from the same part. Thus, $x = \frac{2}{5}$ and $y = \frac{2}{21}$. \qed

In [9] a conjecture of Thomassé can be found which claims that $i(\vec{P}_3) = \frac{2}{5}$. We conjecture that the extremal graphs are the same for $\vec{P}_3$ and $\vec{C}_4$. Maybe, the reason for this could be that every induced subgraph of $\vec{C}_4$ on 3 vertices is isomorphic to $\vec{P}_3$. The best known upper bound for $i(\vec{P}_3)$ states that $i(\vec{P}_3) \leq \frac{12}{25}$ and was given by Bondy. This upper bound can be found in [9] too. In section 3 we will prove that $i(\vec{P}_3) \leq 0.4446$, $i(\vec{C}_4) \leq 0.1104$ and $6 - 4\sqrt{2} \leq i(\vec{K}_{1,2}) = i(\vec{K}_{2,1}) \leq 0.4644$. To do this we need some parts of Razborov’s flag algebra method [7]. In section 2 we will roughly explain everything we need from it and show as an example for the application of the described method that $i(\vec{P}_3) \leq \frac{1}{7}$, $i(\vec{C}_4) = \frac{1}{4}$ and $i(\vec{K}_2 \cup \vec{E}_1) = \frac{4}{7}$.

Let $[k] := \{1, 2, \ldots, k\}$. We write vectors underlined, e.g. $\underline{v} = (u(1), u(2), u(3))$ is a vector with three coordinates. A collection $V_1, \ldots, V_t$ of finite sets is a sunflower with center $C$ if $V_i \cap V_j = C$ for every two distinct $i, j \in [t]$. 

2
2 Flag Algebras

With his theory of flag algebras, Razborov developed a very strong tool for solving some classes of problems in extremal graph theory. For our proof we will just need a small part of his method, which can be thought of as an application of the Cauchy-Schwarz inequality in the theory of orgraphs. For a detailed study of flag algebras we refer the reader to Razborov’s original paper [7]. In this section we will just define the most important ingredients for our calculation. Furthermore, we will give a short introduction into flag algebras.

Let $O$ be the family of all unlabeled orgraphs considered up to isomorphism. By $O_\ell$ we denote the set of all $\Gamma \in O$ with order $\ell$. A type $\sigma$ of order $k$ is a labeled orgraph of order $k$. Thus, each vertex of a type can be uniquely identified by its label. Usually, we use the elements of $[k]$ as labels.

One denotes by $0$ the unique type of order $0$. Likewise one denotes by $1$ the unique type of order $1$.

If $\sigma$ is a type of order $k$, we define a $\sigma$-flag as a pair $F = (\Gamma, \theta)$, where $\Gamma \in O$ with $|\Gamma| \geq k$ and $\theta : [k] \to V_\Gamma$ is an injective function such that the labeled vertices define an induced embedding of $\sigma$ into $\Gamma$. An isomorphism between two $\sigma$-flags $(\Gamma, \theta)$ and $(\Gamma', \theta')$ is an isomorphism $\phi$ between $\Gamma$ and $\Gamma'$ where $\phi(\theta(i)) = \theta'(i)$. We write $F^\sigma$ for the set of all $\sigma$-flags up to isomorphism. Again, we define $F^\sigma_\ell \subseteq F^\sigma$ as the set of all $\sigma$-flags of order $\ell$. For example, $F^0_\ell = O_\ell$. If $\sigma$ is a type of order $k$, then $F^\sigma_k$ consists only of $(\sigma, id)$. One denotes this element simply by $1_\sigma$.

Follow the notation of [7], we write \textbf{math bold face} for denoting random objects.

**Definition 1.** (from [7])

Fix a type $\sigma$ of order $k$, assume that integers $\ell, \ell_1, \ldots, \ell_t \geq k$ are such that

$$\ell_1 + \ldots + \ell_t - k(t - 1) \leq \ell,$$

and $F = (M, \theta) \in F^\sigma_\ell$, $F_1 \in F^\sigma_{\ell_1}, \ldots, F_t \in F^\sigma_{\ell_t}$ are $\sigma$-flags. We define the (key) quantity $p(F_1, \ldots, F_t; F) \in [0, 1]$ as follows. Choose in $V(M)$ uniformly at random a sunflower $(V_1, \ldots, V_t)$ with center $im(\theta)$ and $\forall i \ |V_i| = \ell_i$. We let $p(F_1, \ldots, F_t; F)$ denote the probability of the event \"$\forall i \in [t] \ |F|_{V_i}$ is isomorphic to $F_i$.\" When $t = 1$, we use the notation $p(F_1; F)$ instead of $p(F_1; F)$.

In the following we can identify a $\sigma$-flag $F$ by the probability $p(F, \hat{F})$, where $\hat{F}$ is an arbitrary large enough $\sigma$-flag. Thus, for example if we write

$$1,$$

we can think of it to be the normalized number of outneighbours of a fixed vertex (called \"1\") in an arbitrary large enough orgraph. Or if we write

$$,$
we can think of it to be the density of oriented triangles in an arbitrary large enough graph. Remark that these examples are not formal definitions. It should just allow an easier understanding of the following definitions.

Now, we build formal finite linear combinations of $\sigma$-flags. We denote the space which contains these linear combinations by $R_F^\sigma$. Roughly speaking, if we think of the $F$-density in a graph of sufficiently large order for a flag $F \in F^\sigma_{\ell}$, it seems sensible to call the subspace $K^\sigma$ which is generated by all elements of the form

$$F_1 - \sum_{F \in F^\sigma_{\ell'}} p(F_1, F) F,$$

where $F_1 \in F^\sigma_{\ell_1}$ with $\ell_1 \leq \ell$, the subspace of "identically zero flag parameters". We want to illustrate this by an example. It can be seen by an easy double-counting argument that

$$F_1 = \sum_{F \in F^\sigma_{\ell_1}} p(F_1, F) F. \quad (1)$$

For example, the edge-density in an arbitrary large enough graph can be expressed as a linear combination of induced subgraph-densities of graphs of order 3 in this graph. Thus,

$$= + + + + + .$$

Now it is natural to define $A^\sigma := R_F^\sigma / K^\sigma$ as the flag algebra of the type $\sigma$. This means, we factor $R_F^\sigma$ by the subspace $K^\sigma$. In Lemma 2.4 of [7] Razborov shows that $A^\sigma$ is naturally endowed with the structure of a commutative associative algebra. He defines a bilinear mapping for flags in the following way. Let $\sigma$ be a type of order $k$. For two $\sigma$-flags $F_1 \in F^\sigma_{\ell_1}$, $F_2 \in F^\sigma_{\ell_2}$ and $\ell \geq \ell_1 + \ell_2 - k$ we define

$$F_1 \cdot F_2 := \sum_{F \in F^\sigma_{\ell}} p(F_1, F_2; F) F.$$

Remark that this definition is not well defined on $R_F^\sigma$, but on $A^\sigma$ it is. The disadvantage of this definition is that this product is just asymptotically the same as the product one would expect, if we interpret the $\sigma$-flags in the above way, because

$$p(F_1, F_2; F) = p(F_1, F)p(F_2, F) + o(1).$$

That is why flagalgebraic proofs using this product operation are only asymptotically true.

Additionally, we want to remark in a bit crude words, that the function $F \rightarrow p(F, \hat{F})$ for very large $\hat{F}$ asymptotically corresponds to an algebra homomorphism $\phi \in \text{Hom}(A^\sigma, \mathbb{R})$. Razborov now considers the set

$$\text{Hom}^+(A^\sigma, \mathbb{R}) := \{ \phi \in \text{Hom}(A^\sigma, \mathbb{R}) | \forall F \in F^\sigma \phi(F) \geq 0 \}.$$
and shows in Corollary 3.4 of [7] that \( \text{Hom}^+ (\mathcal{A}^\sigma, \mathbb{R}) \) captures all asymptotically true relations in extremal combinatorics. Thus, we have seen the basic idea of flag algebras. It is useful to define for \( f, g \in \mathcal{A}^0 \) that \( f \geq g \) if \( \forall \phi \in \text{Hom}^+ (\mathcal{A}^\sigma, \mathbb{R}) (\phi(f) \geq \phi(g)) \). This is a partial preorder on \( \mathcal{A}^0 \). Now we want to turn our attention to an application of the Cauchy-Schwarz inequality in flag algebras. We define the averaging operator \( [\cdot]_\sigma : \mathcal{A}^\sigma \to \mathcal{A}^0 \) as follows. For a type \( \sigma \) of order \( k \) and \( F = (\Gamma, \theta) \in \mathcal{F}^\sigma \), let \( q_\sigma(F) \) be the probability that a uniformly at random chosen injective mapping \( \theta : [k] \to V_\Gamma \) defines an induced embedding of \( \sigma \) in \( \Gamma \) and the resulting \( \sigma \)-flag \( (\Gamma, \theta) \) is isomorphic to \( F \). Now, we define

\[
[F]_\sigma := q_\sigma(F) \cdot \Gamma
\]

partially on \( \mathcal{F}^\sigma \). In section 2.2 in [7], Razborov proves that this operator can be extended linearly to \( \mathcal{A}^\sigma \) and he explains why it corresponds to averaging.

**Theorem 1. Cauchy-Schwarz inequality** (from [7], Theorem 3.14)

Let \( f, g \in \mathcal{F}^\sigma \), then

\[
[f^2]_\sigma \cdot [g^2]_\sigma \geq [fg]_\sigma^2.
\]

In particular \( (g = 1_\sigma) \),

\[
[f^2]_\sigma \cdot 1_\sigma \geq [f]_\sigma^2,
\]

which in turn implies

\[
[f^2]_\sigma \geq 0.
\]

It is an easy consequence of Theorem 1 that \( \|v^T A v\|_\sigma \geq 0 \), if \( v \) is a vector of \( n \) \( \sigma \)-flags and \( A \in \mathbb{R}^{n \times n} \) is symmetric positive semidefinite.

**Example 1.** As an example, we want to show with the described method of flag algebras that \( i(\overline{P}_3) \leq \frac{4}{3} \) and that \( i(\overline{C}_3) = \frac{1}{2} \). For this purpose we look at the following equalities.

\[
1 = \bullet + \bullet + \overline{V} + \overline{V} + \overline{V} + \overline{V} + \overline{V} + \overline{V} \]

\[
\left[ \begin{pmatrix} \bullet \\ 1 \bullet \end{pmatrix} \right]^2_1 = \left[ \begin{pmatrix} \bullet \\ 1 \bullet \end{pmatrix} + \overline{V} \right]_1 = \bullet + 1 \frac{1}{3}
\]

\[
\frac{3}{2} \left[ \begin{pmatrix} \bullet \\ 1 \bullet \end{pmatrix} \right]^2_1 = \frac{3}{2} \left[ \begin{pmatrix} \overline{V} + \overline{V} \end{pmatrix} \right]_1 = \frac{1}{2} \overline{V} + \frac{1}{2} \overline{V}
\]

\[
\frac{3}{2} \left[ \begin{pmatrix} \bullet \\ 1 \bullet \end{pmatrix} \right]^2_1 = \frac{3}{2} \left[ \begin{pmatrix} \overline{V} + \overline{V} \end{pmatrix} \right]_1 = \frac{1}{2} \overline{V} + \frac{1}{2} \overline{V}
\]

Now these equations tell us everything we need. For an easier notation we denote the flag \( \uparrow \) by \( \rho \). Notice that each evaluation of \( \rho \) with an orgraph homomorphism \( \phi \in \text{Hom}^+ (\mathcal{A}^\sigma, \mathbb{R}) \)
belong to the edge-density of the orgraph "corresponding" to \( \phi \).

Now, we have

\[
\begin{align*}
\begin{array}{c}
\begin{tikzpicture}
\node (v1) at (0,0) [circle,fill] {};
\node (v2) at (1,0) [circle,fill] {};
\end{tikzpicture}
\end{array}
+ \\
\begin{array}{c}
\begin{tikzpicture}
\node (v1) at (0,0) [circle,fill] {};
\node (v2) at (1,0) [circle,fill] {};
\end{tikzpicture}
\end{array}
\leq 1 - \left[ \left( \begin{array}{c}
1
\end{array} \right)^2 \right]_1 - \frac{3}{2} \left[ \left( \begin{array}{c}
1
\end{array} \right)^2 \right]_1 - \frac{3}{2} \left[ \left( \begin{array}{c}
1
\end{array} \right)^2 \right]_1 \\
\leq 1 - (1 - \rho)^2 - \frac{3}{2} (\rho^2) - \frac{3}{2} (\rho^2) = -\frac{7}{4}\rho^2 + 2\rho
\end{align*}
\]

The righthandside depends just on the edge-density \( \rho \), which can be minimized by taking \( \rho = \frac{4}{7} \). Thus, we have

\[
\begin{array}{c}
\begin{tikzpicture}
\node (v1) at (0,0) [circle,fill] {};
\node (v2) at (1,0) [circle,fill] {};
\end{tikzpicture}
\end{array}
+ \\
\begin{array}{c}
\begin{tikzpicture}
\node (v1) at (0,0) [circle,fill] {};
\node (v2) at (1,0) [circle,fill] {};
\end{tikzpicture}
\end{array}
\leq \frac{4}{7} \Rightarrow \\
\begin{array}{c}
\begin{tikzpicture}
\node (v1) at (0,0) [circle,fill] {};
\node (v2) at (1,0) [circle,fill] {};
\end{tikzpicture}
\end{array}
\leq \frac{4}{7} \Rightarrow i(\vec{P}_3) \leq \frac{4}{7}.
\]

Additionally, we know that

\[
\begin{array}{c}
\begin{tikzpicture}
\node (v1) at (0,0) [circle,fill] {};
\node (v2) at (1,0) [circle,fill] {};
\end{tikzpicture}
\end{array}
\leq -\frac{7}{4}\rho^2 + 2\rho.
\]

We can assume, that there are extremal orgraphs for \( i(\vec{C}_3) \), where the edge-density \( \rho = 1 \), because in every extremal orgraph we can fill the missing edges in an arbitrary way by new edges without decreasing the number of \( \vec{C}_3 \) in this extremal orgraph. Thus, we get

\[
\begin{array}{c}
\begin{tikzpicture}
\node (v1) at (0,0) [circle,fill] {};
\node (v2) at (1,0) [circle,fill] {};
\end{tikzpicture}
\end{array}
\leq \frac{1}{4} \Rightarrow i(\vec{C}_3) \leq \frac{1}{4}.
\]

On the other hand, if we have a look at \( \lim_{n \to \infty} (\vec{C}_3)^{\text{on}} \), it is easy to see that \( i(\vec{C}_3) \geq \frac{1}{4} \).

**Example 2.** In our second example we show that \( i(\vec{K}_2 \cup \vec{E}_1) = \frac{3}{4} \). Let \( T_n \) be an arbitrary complete orgraph on \( n \) vertices, thus a tournament. By a look at the limit graph \( \lim_{n \to \infty} T_n \cup T_n \) its easy to see that \( i(\vec{K}_2 \cup \vec{E}_1) \geq \frac{3}{4} \). We define a vector of 1-flags as

\[
g := \left( \begin{array}{c}
\cdot \\
1 \\
\cdot \\
1 \\
\cdot \\
1
\end{array} \right)^T.
\]

Now, we get with the help of a positive semidefinite matrix

\[
\frac{3}{4} - \\
\begin{array}{c}
\begin{tikzpicture}
\node (v1) at (0,0) [circle,fill] {};
\node (v2) at (1,0) [circle,fill] {};
\end{tikzpicture}
\end{array}
\geq \frac{3}{4} \left[ \left( \begin{array}{cc}
1 & -1 \\
-1 & 1
\end{array} \right) \left( \begin{array}{c}
1 \\
-1 \\
1
\end{array} \right) \right]_1 \geq 0
\]

\[
\Rightarrow i(\vec{K}_2 \cup \vec{E}_1) \leq \frac{3}{4}.
\]
3 Main Results

A lot of calculations in our proofs deal with the 582 elements of $O_5 = \mathcal{F}_0^3$ and the 15 elements of $\mathcal{F}_3^1$ which are defined in figure 2. Let $\vec{f}_i^1$ be the vector with $\vec{f}_i^1(i) := F_i^1$ for $i \in \{0, 1, \ldots, 14\}$.

**Theorem 2.**

\[ i(\vec{P}_3) \leq 0.4446 \]
\[ i(\vec{C}_4) \leq 0.1104 \]

**Proof.** We define the symmetric matrices $A, B \in \mathbb{R}^{15 \times 15}$ by

\[
A := \begin{pmatrix}
739 & 1153 & -62 & -420 & -120 & -31 & -321 \\
1153 & 7013 & 2254 & 2355 & -18 & 4192 & -772 \\
-62 & 2254 & 3147 & 2726 & 1998 & 925 & -142 \\
-420 & 2355 & 2726 & 6798 & -488 & 545 & 804 \\
-120 & -18 & 1998 & -488 & 4573 & -2496 & -529 \\
-31 & 4192 & 925 & 545 & -2496 & 12610 & 777 \\
-321 & -772 & -142 & 804 & -529 & 777 & 1578 \\
-131 & 51 & 420 & 2622 & -1719 & -1327 & -495 \\
-91 & 1992 & 865 & 3770 & -2159 & 3317 & 564 \\
1153 & 1953 & -630 & -3913 & -18 & 4192 & -772 \\
-62 & -636 & -469 & -2970 & 1998 & 925 & -142 \\
-420 & -3913 & -2970 & 5390 & -488 & 545 & 804 \\
-91 & -798 & -1853 & -3172 & -2159 & 3317 & 564 \\
-131 & -2009 & -2958 & -2781 & -1719 & -1327 & -495 \\
663 & 1276 & -2042 & -527 & -4884 & 3548 & -491
\end{pmatrix}
\]
Thus, the smallest eigenvalue of $A$ and $B$ are symmetric positive semidefinite. Now, the proof is completed by the

\[ B := \frac{6}{10^4} \]

\[
\begin{pmatrix}
-131 & -91 & 1153 & -62 & -420 & -91 & -131 & 663 \\
51 & 1992 & 1953 & -636 & -3913 & -798 & -2009 & 1276 \\
420 & 865 & -636 & -469 & -2970 & -1853 & -2958 & -2042 \\
2622 & 3770 & -3913 & -2970 & -5390 & -3172 & -2781 & -527 \\
-1719 & -2159 & -18 & 1998 & -488 & -2159 & -1719 & -4884 \\
-1327 & 3317 & 4192 & 925 & 545 & 3317 & -1327 & 3548 \\
-495 & 564 & -772 & -142 & 804 & 564 & -495 & -491 \\
4221 & 1590 & -2009 & -2958 & -2781 & -1067 & -992 & 2025 \\
1590 & 4666 & -798 & -1853 & -3172 & -1329 & -1067 & 2603 \\
-2009 & -798 & 7013 & 2254 & 2355 & 1992 & 51 & 1276 \\
-2958 & -1853 & 2254 & 3147 & 2726 & 865 & 420 & -2042 \\
-2781 & -3172 & 2355 & 2726 & 6798 & 3770 & 2622 & -527 \\
-1067 & -1329 & 1992 & 865 & 3770 & 4666 & 1590 & 2603 \\
-992 & -1067 & 51 & 420 & 2622 & 1590 & 4221 & 2025 \\
2025 & 2603 & 1276 & -2042 & -527 & 2603 & 2025 & 8134 \\
\end{pmatrix}
\]

The smallest eigenvalue of $A$ is $\approx 0.00004$ and the smallest eigenvalue of $B$ is $\approx 0.00005$. Thus, $A$ and $B$ are symmetric positive semidefinite. Now, the proof is completed by the
following inequalities.

\[
0.446 - \frac{\sqrt{2}}{2} \geq \left[ (f_1^1)^T A f_1^1 \right]_1 \geq 0 \Rightarrow i(\tilde{P}_3) \leq 0.446
\]

\[
0.1104 - \frac{\sqrt{2}}{2} \geq \left[ (f_1^1)^T B f_1^1 \right]_1 \geq 0 \Rightarrow i(\tilde{C}_4) \leq 0.1104
\]

\[\square\]

**Theorem 3.**

\[6 - 4\sqrt{2} \leq i(\tilde{K}_{1,2}) = i(\tilde{K}_{2,1}) \leq 0.4644\]

**Proof.** At first, we observe that \(i(\tilde{K}_{1,2}) = i(\tilde{K}_{2,1})\), because each extremal orgraph of \(i(\tilde{K}_{1,2})\) can be transformed to an extremal orgraph of \(i(\tilde{K}_{2,1})\) by changing the directions of each edge. Thus, in the following we consider only \(\tilde{K}_{1,2}\).

The lower bound we will get from a recursive construction, which is a generalisation of the lexicographic product constructions, we had found for the other orgraphs before. Let \(s := \frac{\sqrt{2} - 1}{2}\) and \(\tilde{G}\) be an orgraph of infinite order. We separate \(V_{\tilde{G}}\) into three parts \(S_1, S_2\) and \(S_3\), such that \(S_1\) are \(s\) parts of all vertices of \(\tilde{G}\), \(S_2\) are \(\frac{1-s}{2}\) parts and \(S_3\) the remaining \(\frac{1-s}{2}\) parts. Now \(\tilde{G}\) has an edge from every vertex in \(S_1\) to every vertex in \(S_2 \cup S_3\). Furthermore there is no edge between a vertex from \(S_2\) and a vertex from \(S_3\). Finally, for each \(i \in [3]\) the vertices of \(S_i\) contain a copy of \(\tilde{G}\). See figure 3 for an illustration of this definition. Now we can get the non-edge-density \(\tilde{\rho}\) of \(\tilde{G}\) by
\[ \bar{\rho} = \frac{(1 - s)^2}{2} \] between \( S_2 \) and \( S_3 \) in each part.

Thus, by rearranging we have

\[ \bar{\rho} = 1 - s. \]

Now we are able to compute the density \( d \) of \( \overrightarrow{K}_{1,2} \)'s in \( \overrightarrow{G} \). We get

\[ d = 6s \left( \frac{1 - s}{2} \right)^2 + 6s \left( \frac{1 - s}{2} \right)^2 \bar{\rho} + \left( s^3 + 2 \left( \frac{1 - s}{2} \right)^3 \right) d, \]

where the first summand is for the \( \overrightarrow{K}_{1,2} \)'s, where each two vertices are from different parts. The second summand is for the \( \overrightarrow{K}_{1,2} \)'s, where one vertex is from \( S_1 \) and the other two vertices are both from \( S_2 \) or \( S_3 \). Finally, the last summand is for the \( \overrightarrow{K}_{1,2} \)'s in each of the three parts. Hence, again by rearranging we get

\[ d = \frac{4(1 - s)s}{(1 + s)(3s + 1)} = 6 - 4\sqrt{2} \Rightarrow i(\overrightarrow{K}_{1,2}) \geq 6 - 4\sqrt{2}. \]

The upper bound we obtain in the same way like in theorem 2. We define \( C \in \mathbb{R}^{15 \times 15} \) by

\[
\begin{pmatrix}
15476 & 18421 & 4912 & -8427 & -4151 & -17228 & -19517 \\
18421 & 124190 & 15060 & -26258 & 86563 & 12365 & -30370 \\
4912 & 15060 & 14103 & -24269 & 29162 & -8930 & -14418 \\
-8427 & -26258 & -24269 & 41867 & -50792 & 14927 & 24879 \\
-4151 & 86563 & 29162 & -50792 & 234128 & 50298 & -1715 \\
-17228 & 12365 & -8930 & 14927 & 50298 & 109464 & 19869 \\
-19517 & -30370 & -14418 & 24879 & -1715 & 19869 & 58166 \\
-2703 & 32824 & -28619 & 48783 & -9280 & 65336 & 11943 \\
-1825 & -23505 & -30259 & 52000 & -79330 & 12541 & 5415 \\
6567 & 74168 & 22951 & -39339 & 142094 & 35394 & -10797 \\
-8635 & 29076 & 23975 & -41342 & 97496 & 12362 & 18337 \\
-39099 & -16836 & 3066 & -6625 & 90402 & 62263 & 45590 \\
-18355 & -34480 & -15446 & 26604 & -32752 & 16811 & 23634 \\
-14216 & 27335 & -20578 & 34825 & 47003 & 88561 & 30468 \\
5928 & 29429 & -36650 & 62808 & -59524 & 46894 & 561
\end{pmatrix}
\]
\[ -2703 -1825 6567 -8635 -39099 -18355 -14216 5928 \\
32824 -23505 74168 -20078 -16836 -34480 27335 29429 \\
-28619 -30259 22951 23975 3666 -15446 -20578 36650 \\
48783 52000 -39939 -41342 10271 37361 39105 70076 \\
-9280 -79330 113660 142094 97496 90402 -32752 47003 \\
65336 12541 35394 12362 62263 16811 88561 46894 \\
11943 5415 -10797 10837 45590 23634 30468 561 \\
191285 71426 3893 -52488 41342 -6625 26604 34825 62808 \\
-52488 -72396 64490 80839 -17545 -9052 -89649 \\
-10271 -37361 39105 70076 155515 24670 44067 -59957 \\
20412 28315 -34596 17545 24670 32563 20238 23056 \\
113660 38299 34033 -9052 44067 20238 158708 103921 \\
167521 100215 -25666 -89649 -59957 23056 103921 232223 \]

Again, the smallest eigenvalue of \( C \) is \( \approx 0.000007 \). Hence, \( C \) is symmetric positive semidefinite. Now the proof of the upper bound is completed by the following inequality.

\[ 0.4644 - \sqrt{\left( f_1^T C f_1 \right)} \geq 0 \Rightarrow i(K_{1,2}) \leq 0.4644 \]

\[ \square \]

### 3.1 Some remarks

Most parts of the proofs of the upper bounds in the theorems 2 and 3 were done by a computer. At first we decided to work in \( O_5 \). Thus, if we take products of two 1-flags on 3 vertices, then our calculus works in \( O_5 \). After that we used a computer program to calculate the equation which the semidefinite matrix \( A \) have to fulfill such that we can prove the associated upper bound. Finally, the determination of the matrices was simply done by a sufficiently close rational approximation to the outcome of a numerical semidefinite-program-solver. The decision to took the type 1 came from computer experiments. Surprisingly, we don’t get better results, if we take larger types, because in another paper \[8\] we got better results with types of higher order. For example we tried the calculation with 9 types on 3 vertices and flags of appropriate order such that our calculation works in \( O_5 \), but we didn’t get better upper bounds in this case.

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