Magic State Distillation with the Ternary Golay Code

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Abstract

The ternary Golay code – one of the first and most beautiful classical error-correcting codes discovered – naturally gives rise to an 11-qutrit quantum error correcting code. We apply this code to magic state distillation, a leading approach to fault-tolerant quantum computing. We find that the 11-qutrit Golay code can distill the “most magic” qutrit state – an eigenstate of the qutrit Fourier transform known as the strange state – with cubic error-suppression and a remarkably high threshold. It also distills the “second-most magic” qutrit state, the Norell state, with quadratic error-suppression and an equally high threshold to depolarizing noise.
1 Introduction

The classical Golay codes [1, 2] are amongst the first and most beautiful ways discovered to protect classical information. Two Golay codes exist – the 23-bit binary Golay code and the 11-trit ternary Golay code. These codes are unique, in that they are the only linear perfect classical error correcting codes other than the Hamming codes. While they were discovered through a computer search, (and independently by a Finnish football enthusiast, apparently via trial and error), their discovery led to profound advancements in the theory of coding as well as the mathematical theory of finite groups. [3]

Can the Golay codes provide us better ways to protect quantum information from noise? Via the CSS construction, the Golay codes can be used to construct \([23, 1, 7]_2\) and \([11, 1, 5]_3\) quantum error correcting codes. Applications of the 23-qubit Golay code to fault-tolerant quantum computing exist [4, 5], but the 11-qutrit Golay code has apparently never been discussed. Here, we observe that the 11-qutrit Golay code is remarkably well-suited for a promising approach to fault-tolerant quantum computing known as magic state distillation [6, 7].

Magic state distillation [6–9] is a leading approach to fault tolerant quantum computing. In the past few years, magic state distillation for qudits of (typically odd prime) dimensions other than two has attracted some interest [10–14], and notably has been used to identify contextuality as an essential resource for universal quantum computation [15]. However, for the most part, qudit fault-tolerant quantum computing [16] appears relatively unexplored, although attractive experimental realizations of qutrits do exist, e.g., [17–19].

In the magic state model, a fault-tolerant quantum computer has the ability to measure and initialize states without error in the computational basis, and act without error on these states with a discrete subgroup of the full set of unitary operators known as the Clifford group [20, 16]. A quantum computer with only these capabilities is classically simulable [21–24] and therefore not sufficient for universal quantum computation. In addition, the computer is able to prepare ancilla qudits in certain non-stabilizer states, called magic states; but these states are produced with limited fidelity. To approximate a universal quantum computer within this model, we require arbitrarily pure magic states, which can be used to implement non-Clifford gates via state-injection. Using many low-fidelity magic states, it is sometimes possible to distill a small number of high-fidelity magic states via protocols involving only Clifford unitaries and stabilizer measurements. This process is only successful if the noise level of the low-fidelity input qudits is below a particular threshold associated with the particular distillation protocol employed. An open problem is to design a distillation protocol with as high a threshold as possible.

What constitutes a magic state for a qutrit? In entanglement theory, any state that is not a product state is defined to be entangled. By analogy, any (pure) state that is not a stabilizer state is defined to be magic [25]. One can then ask, which qutrit state is most magic? To answer this question, a natural measure to use is the regularized entropy of magic, which is
defined as the relative entropy between a large supply of qutrits in the candidate magic state and the nearest multi-qutrit stabilizer state. Unfortunately, the regularized entropy of magic is not feasible to compute. To place rigorous bounds on magic, two useful surrogate measures exist: the *mana*, [25] which is essentially a measure of the sum of negative entries in the discrete Wigner function [26–30] of the candidate magic state; and the *thauma* [31] which is the minimum relative entropy between the candidate magic state and a subnormalized state with positive Wigner function.

Two qutrit magic states were identified in [25] that maximize the mana – the strange state, an eigenstate of the qutrit Fourier transform (which was first discovered in [32]), and the so-called Norell state, which is the eigenstate of another single qutrit Clifford operator \( N \) defined below. It was recently shown that the strange state has larger thauma [31] than the Norell state, hence earning it the accolade of the “most magic” qutrit state. This accolade is conceptually satisfying because the strange state also maximally violates the contextuality inequality of [15], also the qutrit state for which distillation could be most robust to depolarizing noise. As we show in [33], the qutrit strange state is also the most symmetric of all qudit magic states, and has no natural analogue in higher odd-prime dimensions.

Distillation of the strange states is an exciting problem for both practical and theoretical reasons. The strange state is furthest from the Wigner polytope [23], and therefore has potential to be distilled with the greatest threshold to noise of any qutrit state, as first observed in [32]. Moreover, constructing a magic state distillation routine that distills strange states, with a threshold meeting the theoretical upper bound set by negativity of the Wigner function, would be tantamount to a proof that contextuality is sufficient for universal quantum computation. [15]

Previous works on qutrit and qudit magic state distillation [11,12,34] have mostly focused on distilling a class of equatorial magic states, which possess several useful properties [35], although they have non-maximal mana. In addition, eigenstates of the qutrit Fourier transform other than the strange state were distilled via the 5-qutrit code in [10], and [13] presented a distillation routine for the qutrit Norell state; although one should note that the protocols of [10], and [13] have only a linear reduction in noise rate. Prior to this work, no magic state distillation routine with the strange state as a stable endpoint was known.

Here, we show that an \([11,1,5]_3\) code obtained from the ternary Golay code distills the strange state, with a threshold to depolarizing noise that exceeds the best known threshold of any qutrit magic state distillation routine. This code also distills Norell states with a nearly-equal threshold to depolarizing noise.

2 The strange state and the Norell state

Eigenstates of Clifford operators that are not stabilizer states are natural candidates for attractive endpoints of distillation routines. A complete enumeration of qutrit Clifford eigenstates appears in [33].
The strange state,
\[ |S\rangle = \frac{1}{\sqrt{2}} (|1\rangle - |2\rangle), \] (2.1)
and the Norell state,
\[ |N\rangle = \frac{1}{\sqrt{2}} (|1\rangle + |2\rangle), \] (2.2)
are both eigenstates of the single-qutrit Clifford operator \( N \), defined as,
\[ N = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & \omega^2 \\ 0 & \omega^2 & 0 \end{pmatrix}. \] (2.3)

The strange state is also an eigenvector of the qutrit Fourier transform
\[ H = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 1 & 1 \\ 1 & \omega & \omega^2 \\ 1 & \omega^2 & \omega \end{pmatrix} \] (2.4)
with eigenvalue \( i \). The symmetries of these and other qutrit Clifford eigenstates are discussed in detail in [33]. Both states can be used to implement a non-Clifford gate via state injection, as we review in Appendix A, which closely follows [10].

In the magic state model, we will begin with a supply of noisy \(|S\rangle\) and \(|N\rangle\) states, that lie somewhere near \(|S\rangle\) or \(|N\rangle\) in the 8-dimensional space of single-qutrit density matrices. Via random application of Clifford unitaries, a process known as twirling, one can restrict the density matrices of noisy input qutrits to a more manageable form.

For the Norell state, we can randomly apply the unitary \( N \), to restrict our noisy states to lie in the two-dimensional plane spanned by convex combinations of \(|0\rangle\), \(|N\rangle\) and \(|S\rangle\).

The strange state \(|S\rangle\) is the unique simultaneous eigenstate of two Clifford unitaries \( H \) and \( N \). These two unitaries generate a subgroup of the Clifford group isomorphic to \( SL(2,\mathbb{Z}_3) \) [28, 36, 33]. By random application of any element of this finite group, any noisy input state can be brought into the form,
\[ \rho(\delta) = (1 - \delta) |S\rangle \langle S| + \delta \frac{1}{3}. \] (2.5)
The parameter \( \delta \) has the physical interpretation as the depolarizing noise rate. After twirling, our \( n \) noisy input qutrits are in the state \( \rho_{in}^\otimes n = \rho(\delta_{in})^\otimes n \).

A magic state distillation routine consists of projecting the \( n \) noisy qutrits onto the codespace of an \([n,1]\) stabilizer code, and returns the decoded logical qutrit as output. Assuming the stabilizer code employed has suitable symmetries, the output qutrit will be in a state of the same form \( \rho_{out} = \rho(\delta_{out}) \), thus giving rise to a function \( \delta_{out}(\delta_{in}) \) that characterizes its performance, much like the qubit case.

The existence of a twirling protocol that converts all noise to depolarizing noise is a unique feature of the \(|S\rangle\) state, that arises because of its exceptional symmetry properties under Clifford transformations. [33] This property is not shared by any other qutrit magic state, nor, is it expected to hold for any other qudit magic state, for any odd prime \( d > 3 \).
3 The 11-qutrit Golay code

Consider any ternary maximal self-orthogonal code [37] of odd length \( n \), with generator matrix \( \mathbf{M}_c \). We construct a quantum error correcting code from two copies of \( \mathbf{M}_c \), following the CSS construction [38,39], with the following symplectic matrix:

\[
\mathbf{M}_q = \begin{pmatrix} \mathbf{M}_c & 0 \\ 0 & \mathbf{M}_c \end{pmatrix}.
\] (3.1)

A maximal self-orthogonal classical ternary code of odd length \( n \) has dimension \( k = (n - 1)/2 \), so the quantum code generated by the CSS construction encodes 1 qutrit. Such a code also commutes both \( H^\otimes n \) and \( N^\otimes n \), and, more generally with any Clifford operator that is a symplectic rotation. Let us choose for \( \mathbf{M}_c \) the generator matrix for the ternary Golay code,

\[
\mathbf{M}_{cG} = \begin{pmatrix} 2 & 1 & 1 & 2 & 2 & 0 & 1 & 0 & 0 & 0 \\ 2 & 1 & 2 & 1 & 0 & 2 & 0 & 1 & 0 & 0 \\ 2 & 2 & 1 & 0 & 1 & 2 & 0 & 0 & 1 & 0 \\ 2 & 2 & 0 & 1 & 2 & 1 & 0 & 0 & 0 & 1 \\ 2 & 0 & 2 & 2 & 1 & 1 & 0 & 0 & 0 & 1 \end{pmatrix}.
\] (3.2)

The Golay code is of length \( 11 \equiv 2 \mod 3 \). \( X^\otimes n \) and \( Z^\otimes n \) cannot be stabilizers of this code since the classical ternary vector \((1,...,1)\) is not self-orthogonal. However, both \( X^\otimes n \) and \( Z^\otimes n \) can serve as logical Pauli operators for the code. (We denote logical operators with an overbar.) In particular, we make the choice

\[
\bar{X} = (X)^\otimes n, \quad \bar{Z} = (Z)^\otimes n.
\] (3.3)

Other choices are related to this by a Clifford transformation on the decoded qutrit.

With the above choice, the logical \( H \) and \( N \) operators are given by,

\[
\bar{H} = (H^\dagger)^\otimes n, \quad \bar{N} = (N^\dagger)^\otimes n.
\] (3.4)

To distill \(|S\rangle\) states, we require that \(|S\rangle^\otimes n \) decode to \(|\tilde{S}\rangle\) after projection onto the codespace. Since \( \bar{H} |\tilde{S}\rangle = i^{11} |\tilde{S}\rangle \), this is indeed the case for the ternary Golay code.

Before proceeding, we note that we could perform this construction for any self-orthogonal maximal ternary code of length \( n = 12m - 1 \), with the property that the ternary vector \((1 \, 1 \, \ldots \, 1)\) is orthogonal to all its generators. There are three such codes of length 11, given in [37]. Using such a code for distillation of \(|S\rangle\) states, we generically expect the noise rate of the distilled qutrit to depend linearly on the noise rate of the input qutrits, for the following reason. The eigenstates of the qutrit Hadamard operator are \(|H_+\rangle, |H_-\rangle \) and \(|S\rangle\), with eigenvalues 1, \(-1 \) and \( i \), respectively. The noisy input state can be rewritten as,

\[
\rho(\epsilon) = (1 - \epsilon) |S\rangle \langle S| + \epsilon/2 \left( \frac{|H_+\rangle \langle H_+| + |H_-\rangle \langle H_-|}{\sqrt{2}} \right),
\] (3.5)
where $\epsilon = 2\delta/3$. The term in $\rho(\epsilon)^{\otimes n}$ proportional to $\epsilon(1-\epsilon)^{n-1}$ will contain terms of the form $(|S\rangle\langle S|)^{\otimes n-1}|H\rangle\langle H|$. Each such term is an eigenvector of $\bar{H}$ with eigenvalue $\pm 1$. Unless this term is orthogonal to the codespace, it will decode to $|H\rangle$, resulting in a term linear in $\epsilon$ in the output density matrix. Similar comments apply for distillation of the $|N\rangle$ state and other qutrit magic states.

## 4 Distilling the strange state

A convenient way to calculate $\rho_{\text{out}}$ from $\rho_{\text{in}}$ is to use the exact simulation algorithm of [14]. This algorithm is essentially a geometric reformulation of magic state distillation in the language of discrete phase space, which gives an expression for the Wigner function of the distilled qutrit, $W_{\text{out}}(z,x)$, in terms of the Wigner function of the noisy input qutrits, $W_{\text{in}}(z,x)$. Given any $n$-qutrit stabilizer code described by symplectic matrix $M$, and logical $\bar{X}$ and $\bar{Z}$ operators given by symplectic vectors $(\vec{a}_x | \vec{b}_x)$ and $(\vec{a}_z | \vec{b}_z)$, [14] states that

$$W_{\text{out}}(z_L, x_L) = \frac{1}{P} \sum_{\vec{u} \in \mathbb{Z}^{n-1}_3} \prod_{i=1}^{n} W_{\text{in}}(z_i(\vec{u}, z_L, x_L), x_i(\vec{u}, z_L, x_L)), \quad (4.1)$$

where $P$ is a normalization constant chosen so

$$\sum_{z_L, x_L} W_{\text{out}}(z_L, x_L) = 1,$$

and also determines the success probability. The quantities $z_i$ and $x_i$ are the $i$th components of $\vec{z}$ and $\vec{x}$ given by,

$$
\begin{pmatrix}
\vec{z}(\vec{u}, z_L, x_L) \\
\vec{x}(\vec{u}, z_L, x_L)
\end{pmatrix} =
\begin{pmatrix}
\vec{M}^T & \vec{a}_x \\
\vec{b}_z & \vec{b}_x
\end{pmatrix}
\begin{pmatrix}
\vec{u} \\
-z_L \\
-x_L
\end{pmatrix}.
$$

We find $\delta_{\text{out}}(\delta)$ for the 11-qutrit Golay code is:

$$\delta_{\text{out}} = \frac{\delta^3}{2Q(\delta)} \frac{P(\delta)}{Q(\delta)}, \quad (4.3)$$

where

$$
\begin{align*}
P(\delta) &= 3021\delta^8 - 24816\delta^7 + 92180\delta^6 - 203280\delta^5 + 292710\delta^4 - 283536\delta^3 + 181764\delta^2 \\
&\quad - 71280\delta + 13365, \\
Q(\delta) &= 495\delta^{11} - 3960\delta^{10} + 13750\delta^9 - 25245\delta^8 + 18810\delta^7 - 23628\delta^6 - 86328\delta^5 \\
&\quad + 121770\delta^4 - 102465\delta^3 + 53460\delta^2 - 16038\delta + 2187.
\end{align*}
$$

This is plotted in Figure 1.

For small $\delta$,

$$\delta_{\text{out}} \approx \frac{55}{18}\delta^3. \quad (4.6)$$
As mentioned earlier, we generically expect a linear relation between $\delta_{\text{out}}$ and $\delta$, so this relation is fairly surprising. We hope to better understand the origin of this cubic rate of error-suppression in a future work.

The threshold for distillation is at

$$
\delta_* = \frac{3}{135} \left( 31 - 262^{3/3} \sqrt[3]{\frac{2}{405 \sqrt{109} - 2981}} + 2^{2/3} \sqrt[3]{405 \sqrt{109} - 2981} \right) \approx 0.38715
$$

This is slightly more than half of the theoretical upper bound for the threshold determined by the Wigner polytope [14, 15, 23], which is at $\delta_* = \frac{3}{4}$. This threshold is better than the best previously known threshold for any qutrit magic state distillation protocol. (The best previously known threshold to depolarizing noise was achieved by a distillation routine in [13] that had only linear error-suppression.)

5 Distilling the Norell state

The ternary Golay code can also be used to distill Norell states. While the Norell state is, strictly speaking, less magic than the strange state, if we restrict our operations to two-qutrit stabilizer measurements and Clifford unitaries, the Norell state is slightly more useful for state injection, as discussed in Appendix A.

The Norell state is an eigenstate of a single Clifford operator $N$. The best twirling protocol is to apply this operator a random number of times, after which all noisy input density matrices will lie in the triangle formed by convex mixtures of its three eigenvectors. The three eigenvectors of $N$ are $|0\rangle$, $|S\rangle$ and $|N\rangle$, so twirled states can be parameterized as,

$$
\rho(\epsilon_0, \epsilon_S) = (1 - \epsilon_0 - \epsilon_S) |N\rangle \langle N| + \epsilon_0 |0\rangle \langle 0| + \epsilon_S |S\rangle \langle S|.
$$

(5.1)
Figure 2: By randomly applying the Clifford operator \( N \), any state can be made to lie in the triangle spanned by convex combinations of \(|S\rangle\), \(|N\rangle\) and \(|0\rangle\). The purple region distills to the Norell state. The orange region distills to \(|S\rangle\) and the dark blue region distills to \(|0\rangle\). The teal and green regions distill to mixed states.

Our distillation routine takes 11 qutrits in the state \( \rho(\epsilon_0, \epsilon_S) \otimes 11 \) and outputs a single qutrit in the state \( \rho(\epsilon'_0, \epsilon'_S) \), and is thus characterized by the two functions \( \epsilon'_0(\epsilon_0, \epsilon_S) \), and \( \epsilon'_S(\epsilon_0, \epsilon_S) \). Complete expressions for these functions are presented in Appendix B. For small \( \epsilon_0 \) and \( \epsilon_S \), these come out to be:

\[
\epsilon'_0 = \epsilon_0^2 \left( \frac{55}{18} + \frac{55\epsilon_S}{9} + \frac{715\epsilon_S^2}{6} + O\left(\epsilon_S^3\right) \right) + O\left(\epsilon_0^3\right) \tag{5.2}
\]

\[
\epsilon'_S = \left( \frac{55\epsilon_S^3}{3} + O\left(\epsilon_S^4\right) \right) + \epsilon_0 \left( 55\epsilon_S^3 + O\left(\epsilon_S^4\right) \right) + \epsilon_0^2 \left( \frac{2915\epsilon_S^3}{54} + O\left(\epsilon_S^4\right) \right) + O\left(\epsilon_0^3\right) \tag{5.3}
\]

By iterating this procedure many times, we numerically determined the region of state space that distills to the Norell state. This is shown in Figure 2. To translate this two-dimensional region into a single number, let us assume only depolarizing noise \( (\epsilon_S = \epsilon_0 = \delta_N/3) \) on the input qutrits. We find the maximum depolarizing noise rate \( \delta_N \) for input states to eventually distill to \(|N\rangle\) is 0.38612. This approximately, but not exactly, equal to the threshold for \(|S\rangle\) state distillation. This threshold is substantially better than the threshold 0.32989 for Norell states using the distillation protocol of [13] which has only a linear error-supression.

The region of state space that distills to the strange state is also shown in Figure 2. We could have used this twirling scheme for distilling the strange state. However this does not offer any advantages over the simpler twirling scheme for strange states discussed earlier.

6 Discussion

The 11-qutrit Golay code distills strange states with a threshold to depolarizing noise of \( \delta_s = 0.38715 \). This is the highest threshold of any known qutrit magic state distillation
routine. Moreover, we emphasize that this threshold is a worst-case threshold that applies to all forms of noise, not just depolarizing noise, thanks to the twirling scheme presented above. The best threshold to depolarizing noise for a qubit magic state distillation routine is $\delta^* = 0.34535$, which arises for distillation of $|T\rangle$ states via the 5-qubit code [6]. So the 11-qutrit Golay code defines the first qutrit distillation protocol that also has a better threshold than any qubit distillation protocol, although it may not be meaningful to compare noise thresholds between qudits of different dimensionalities. Qudit codes for sufficiently large odd-prime dimension [12] do have higher thresholds to depolarizing noise, but, in these cases, the depolarizing noise threshold does not, on its own, completely characterize the distillable region of state space.

This noise threshold is only a little over half of the theoretical upper limit for the noise threshold $\delta^* = 3/4$, set by the necessity of contextuality (or positivity of the discrete Wigner function). Do other codes exist with better thresholds? We tried a similar construction with other self-orthogonal maximal ternary codes of length 11 and 13 [37, 40]; but the ternary Golay code is the only code we could find that is suitable for magic state distillation. At present, the 11-qutrit Golay code is the only code known to be able to distill the strange state.

A notable disadvantage of distillation via the ternary Golay code is that its probability of success is quite low, $\frac{1}{1728}$, for input qutrits with no error in either the strange state or the Norell state. The low success rate means that, in practice, approximately 19008 qutrits would be needed for a single successful round of distillation. This is offset slightly by the cubic error suppression, which implies that, starting with $n$ noisy copies of the strange state with depolarizing noise rate $\delta$, the noise rate of the distilled strange state scales with $n$ as

$$\delta_{out}(n, \delta) \approx \frac{1}{1.75} (1.75\delta)^{n^{0.112}}$$  \hspace{1cm} (6.1)

where $\xi = \frac{1}{\log_3 19008} \approx .112$ is the yield parameter. For comparison, with the 5-qubit code, we obtain a similar relation,

$$\delta_{out}^{5-\text{qubit}}(n, \delta) \approx \frac{1}{2.5} (2.5\delta)^{n^{0.204}}$$  \hspace{1cm} (6.2)

with yield parameter $\xi_{5-\text{qubit}} = \frac{1}{\log_2 30} \approx .204$.

Magic state distillation with the 23-qubit Golay code was discussed briefly in [8], where it was shown that it is not suitable for distilling $|H\rangle$ states. It is interesting to note that the 23-qubit Golay code is able to distill $|T\rangle$ states with a threshold that is just slightly less than that of the 5-qubit code. As we review in appendix C, the error-suppression for $|T\rangle$ state distillation using the 23-qubit code is quadratic, as one would expect for a generic code. On the other hand, the ternary Golay code is the best known code for distillation of strange and Norell magic states, and is able to distill the strange state with a somewhat miraculous cubic error-suppression, whose origin needs to be better understood.

One motivation for distilling strange states is to address whether contextuality can be shown to be a sufficient resource for universal quantum computation. [15] This requires us
to construct a distillation scheme that is tight to the boundary of the Wigner polytope, i.e., has a threshold to depolarizing noise of \( \frac{3}{4} \). While it can be shown that no magic state distillation routine based on a finite stabilizer code can achieve this threshold, \([14, 41]\), the possibility remains that a sequence of stabilizer codes exist which distill the strange state, whose threshold approaches \( \frac{3}{4} \). Of course, the ternary Golay code is an extremely special error-correcting code, and there is no reason to expect that one can generalize it to obtain such a sequence of codes. Nevertheless, demonstrating the existence of a single magic state distillation routine that distills the strange state, is an important first step for this program.

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Appendix A: State injection with the strange state

Let us show how the strange state can be used to implement a non-Clifford gate via state injection. We closely follow \([10]\).

We first review the standard approach to state injection. Let \( U \) be a unitary operator whose eigenbasis is a complete set of stabilizer states. By a Clifford transformation, such an operator can be brought into a form where it is diagonal in the computational basis,

\[
\begin{pmatrix}
1 & 0 & 0 \\
0 & e^{i\theta_1} & 0 \\
0 & 0 & e^{i\theta_2}
\end{pmatrix}.
\]  (6.3)

We denote such an operator as \( U_Z \). Such operators were referred to as “equatorial operators” in \([11]\).

Let us denote

\[
|+\rangle = \frac{1}{\sqrt{3}} \left( |0\rangle + |1\rangle + |2\rangle \right),
\]

as the eigenstate of \( X \) with eigenvalue 1. Define \( |U_Z\rangle = U_Z |+\rangle \), which can be written in the form,

\[
|U_Z(\theta_1, \theta_2)\rangle = |0\rangle + e^{i\theta_1} |1\rangle + e^{i\theta_2} |2\rangle.
\]  (6.4)

We refer to any state that can be brought into the above form via a Clifford unitary as an equatorial state following \([10, 11]\).

\( U_Z \) can be implemented by the following state-injection circuit using the state \( |U_Z\rangle \) as follows:
1. Let qudit 1 be in the state $|UZ\rangle$, and qudit 2 be in the state $|\psi\rangle$.

2. Apply a controlled-$X^2$ gate to the $|UZ\rangle|\psi\rangle$, with $|\psi\rangle$ as the target.

3. Measure $Z$ on qudit 2; if the outcome is $\omega^m$, apply $(U_ZX^mU_Z^\dagger)" to qudit 1. Qudit 1 is now in the state $U_Z|\psi\rangle$.

The above procedure works if $U_Z^{1/2}XU_Z$ is a Clifford operator, i.e., if $U_Z$ is in the third level of the Clifford hierarchy. \[42,43\]

If $U_Z$ is not in the third-level of the Clifford hierarchy, then it is not possible to apply the outcome-dependent correction at the end. In this case we end up with the state $U_ZX^{-m}|\psi\rangle$ with a random, but known value of $m$. In this case, it is convenient to modify the circuit by applying another controlled-$X$, with qudit 1 as target, to obtain $X^mU_ZX^{-m}|\psi\rangle$. If $m \neq 0$ we can repeat this state injection procedure in hopes of eventually reaching the state $U_Z|\psi\rangle$, or a state Clifford-equivalent to it. If the group $G$ generated by operators of the form $X^mU_ZX^{-m}$ is a finite group of relatively small (i.e., $O(1)$) order, this process is a random walk which reaches $|U_Z\rangle$ in $O(1)$ steps.

The magic states $|N\rangle$ and $|S\rangle$ are not equatorial states, but can be converted into equatorial states via a series of 2-to-1 stabilizer reductions. \[10\] showed how to convert the state $|N\rangle$ to an equatorial state via a 2-to-1 stabilizer reduction:

1. Start with two (very pure) qubits in the state $|N\rangle|N\rangle$.
2. Project onto the codespace of the $[2,1]$ code defined by the stabilizer $\omega X_1X_2$. This has a $1/4$ success probability.
3. Decode treating $X_2$ as the logical $X$ operator, and $Z_1^2Z_2$ as the logical $Z$ operator.

For input $|N\rangle$, the resulting state is $X^2|U_Z(\pi/3,2\pi/3)\rangle$, which is Clifford equivalent to $|U_Z(0,\pi)\rangle$. $U_Z(0,\pi)$ is a non-Clifford gate; while it is not an element of the third-level of the Clifford hierarchy, the group generated by $X^mU_Z(0,\pi)X^{-m}$ is finite, and can be used to implement a non-Clifford gate as discussed above.

There is no 2-to-1 stabilizer reduction which converts $|S\rangle$ to an equatorial state. However, we can convert two copies of an $|S\rangle$ state to a $|N\rangle$ state via the 2-to-1 stabilizer reduction with stabilizer $Z_1Z_2$, and decoding via logical operators $\bar{Z} = Z_2$ and $\bar{X} = X_1^2X_2$. This stabilizer reduction succeeds with probability $1/2$. This stabilizer reduction can also convert two copies of any state of the form $\gamma|1\rangle + \delta|2\rangle$, which is also a Clifford eigenstate \[33\], into the state $|N\rangle$, with success probability $2|\gamma\delta|^2$.

This scheme appears to be the optimal scheme using only 2-qutrit stabilizer projections and Clifford unitaries. It would be interesting to search for schemes involving $n$-qutrit stabilizer projections, for $n > 2$, that make better use of magic.
Appendix B: Distilled Norell states

The output qutrit is in the state $\rho(\epsilon'_0, \epsilon_S)$, where

$$
epsilon'_0(\epsilon_0, \epsilon_S) = \frac{1}{2} P_0(\epsilon_0, \epsilon_S) Q_N(\epsilon_0, \epsilon_S), \quad \epsilon_S(\epsilon_0, \epsilon_S) = 6 \epsilon_S P_S(\epsilon_0, \epsilon_S) Q_N(\epsilon_0, \epsilon_S),$$

where

$$P_0 = 55 - 495 \epsilon_0 + 1980 \epsilon_0^3 - 4092 \epsilon_0^5 + 3762 \epsilon_0^7 + 990 \epsilon_0^9 - 5940 \epsilon_0^{11} + 5940 \epsilon_0^{13} - 2673 \epsilon_0^{15} + 601 \epsilon_0^{17} - 495 \epsilon_0^{19} + 3960 \epsilon_0^{21} - 13860 \epsilon_0^{23} + 24552 \epsilon_0^{25} - 18810 \epsilon_0^{27} + 3960 \epsilon_0^{29} - 2673 \epsilon_0^{31} + 17820 \epsilon_0^{33} - 11880 \epsilon_0^{35} + 2673 \epsilon_0^{37} + 3960 \epsilon_0^{39} - 27720 \epsilon_0^{41} + 83160 \epsilon_0^{43} - 122760 \epsilon_0^{45} + 75240 \epsilon_0^{47} + 11880 \epsilon_0^{49} - 35640 \epsilon_0^{51} + 11880 \epsilon_0^{53} - 18480 \epsilon_0^{55} + 110880 \epsilon_0^{57} - 27720 \epsilon_0^{59} + 327360 \epsilon_0^{61} - 150480 \epsilon_0^{63} - 15840 \epsilon_0^{65} + 23760 \epsilon_0^{67} + 55440 \epsilon_0^{69} - 27720 \epsilon_0^{71} + 55440 \epsilon_0^{73} + 491040 \epsilon_0^{75} + 150480 \epsilon_0^{77} + 7920 \epsilon_0^{79} - 110880 \epsilon_0^{81} + 443520 \epsilon_0^{83} - 665280 \epsilon_0^{85} + 392832 \epsilon_0^{87} - 60192 \epsilon_0^{89} + 147840 \epsilon_0^{91} - 443520 \epsilon_0^{93} + 443520 \epsilon_0^{95} - 130944 \epsilon_0^{97} - 126720 \epsilon_0^{99} + 253440 \epsilon_0^{101} - 126720 \epsilon_0^{103} + 63360 \epsilon_0^{105} - 140800 \epsilon_0^{107} \quad (6.5)
$$

$$P_S = 220 \epsilon_0^{10} + 220 \epsilon_0^9 - 220 \epsilon_0^8 - 2585 \epsilon_0^7 - 3960 \epsilon_0^6 - 1980 \epsilon_0^5 - 1440 \epsilon_0^4 + 23320 \epsilon_0^3 - 220 \epsilon_0^2 - 2585 \epsilon_0^1 + 3960 \epsilon_0^0 = 220 \epsilon_0^{10} + 220 \epsilon_0^9 - 220 \epsilon_0^8 - 2585 \epsilon_0^7 - 3960 \epsilon_0^6 - 1980 \epsilon_0^5 - 1440 \epsilon_0^4 + 23320 \epsilon_0^3 - 220 \epsilon_0^2 - 2585 \epsilon_0^1 + 3960 \epsilon_0^0 \quad (6.6)
$$

$$Q_N = 220 \epsilon_0^{11} + 2475 \epsilon_0^9 - 1155 \epsilon_0^8 + 10890 \epsilon_0^7 - 9900 \epsilon_0^6 + 4290 \epsilon_0^5 + 21120 \epsilon_0^4 - 26730 \epsilon_0^3 + 8910 \epsilon_0^2 + 6930 \epsilon_0 + 3960 \epsilon_0^0 = 220 \epsilon_0^{11} + 2475 \epsilon_0^9 - 1155 \epsilon_0^8 + 10890 \epsilon_0^7 - 9900 \epsilon_0^6 + 4290 \epsilon_0^5 + 21120 \epsilon_0^4 - 26730 \epsilon_0^3 + 8910 \epsilon_0^2 + 6930 \epsilon_0 + 3960 \epsilon_0^0 \quad (6.8)
$$
Numerical basins computed in Figure 2 appear to be symmetric with respect to interchange of the $|N\rangle$ and $|S\rangle$ state. This is not quite the case, as the thresholds to depolarizing noise for $|S\rangle$ and $|N\rangle$ states are slightly different. Interchange of $|N\rangle$ and $|S\rangle$ corresponds to interchange of $1 - \epsilon_0 - \epsilon_S$ and $\epsilon_S$. The expressions above are not symmetric under this exchange.

**Appendix C: Distillation with the 23-qubit Golay code**

Distillation with the 23-qubit Golay code was first reported in [8]. There, it was found that 23-qubit Golay code is not suitable for distilling qubit $|H\rangle$ magic states, but it can distill $|T\rangle$ states. Here we present the results for $|T\rangle$ state distillation in some more detail.

The binary generator matrix for the classical Golay code is:

$$M^{(2)}_c = \begin{pmatrix}
0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
1 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
1 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 1 & 0 & 0 & 1 & 0 & 1 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 1 & 0 & 1 & 0 & 1 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 1 & 1 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 1 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\end{pmatrix} \quad (6.9)$$

The 23-qubit Golay code is defined as the code given by the symplectic matrix:

$$\begin{pmatrix}
M^{(2)}_c & 0 \\
0 & M^{(2)}_c \end{pmatrix} \quad (6.10)$$

The $|T\rangle$ magic state is defined as $|T\rangle \langle T| = \frac{1}{2}(1 + \frac{1}{\sqrt{3}}(X + Y + Z))$, and is an eigenstate of the Clifford operator $T$ defined in [6]. Noisy $|T\rangle$ states can be twirled to take the form:

$$\rho(\delta) = (1 - \delta) |T\rangle \langle T| + \delta \frac{1}{2}. \quad (6.11)$$

Distilling with respect to the Golay gives a relation $\delta_{\text{out}}(\delta)$ that takes the following form:

$$\delta_{\text{out}} = \frac{\delta^2}{\bar{Q}_T(\delta)} \approx \frac{253}{196} \delta^2 \quad (6.12)$$
Figure 3: The relation $\delta_{\text{out}}(\delta_{\text{in}})$ induced by a single round of distillation with the 23-qubit Golay code is shown by the solid red line. The dashed blue line is the line $\delta_{\text{out}} = \delta_{\text{in}}$, which is also shown for convenience, and the black line is the relation $\delta_{\text{out}}(\delta_{\text{in}})$ for the 5-qubit code. Both codes have quadratic reduction in noise, but the 5-qubit code has a better threshold.

where

\[
P_T(\delta) = 3895\delta^{21} - 117921\delta^{20} + 129713\delta^{19} - 6154225\delta^{18} + 1514205\delta^{17} + 142287453\delta^{16} - 869243991\delta^{15} + 2817045198\delta^{14} - 5579251128\delta^{13} + 5943010480\delta^{12} + 978697104\delta^{11} - 15862508256\delta^{10} + 30813957440\delta^9 - 35023976064\delta^8 + 26000789760\delta^7 - 11870031360\delta^6 + 1942262784\delta^5 + 1403652096\delta^4 - 1189658624\delta^3 + 435240960\delta^2 - 87048192\delta + 8290304
\]

and

\[
Q_T(\delta) = -28336\delta^{22} + 623392\delta^{21} - 5801796\delta^{20} + 28761040\delta^{19} - 70542472\delta^{18} - 19126800\delta^{17} + 798925677\delta^{16} - 3140863440\delta^{15} + 713803400\delta^{14} - 10619737744\delta^{13} + 10395332080\delta^{12} - 5839214976\delta^{11} + 931120960\delta^{10} - 346508800\delta^9 + 4146253056\delta^8 - 8139005952\delta^7 + 8906118144\delta^6 - 6659186688\delta^5 + 3627008000\delta^4 - 1450803200\delta^3 + 410370048\delta^2 - 73859072\delta + 6422528.
\]

This is plotted in Figure 3.

Note that error-suppression is quadratic, as expected for a generic code of length $n = 6m - 1$, that has $T^\otimes n$ as a transversal operator. The threshold is at $\delta^* = 0.32237$. This is slightly worse than the threshold of the 5-qubit code which is at 0.34535.

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