Testing Improved Actions

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We discuss testing improved actions in the context of finite volume gauge theories, where both results for the continuum and the Wilson lattice action are known analytically for volumes up to 0.7 fermi across. A new improved action is introduced, obtained by adding a $2 \times 2$ plaquette to the Lüscher-Weisz Symanzik action, for which the gauge field propagator greatly simplifies. We call this the square Symanzik action. We present the tree-level parameters of this improved action and the value of its Lambda parameter. We also give some Monte Carlo results and discuss some of the issues related to violations of unitarity at the scale of the lattice cutoff due to next-to-nearest coupling in the time direction.

1 Introduction

Lattice gauge theory has proven to be a viable tool for non-perturbative studies in field theories and non-Abelian gauge theories in particular. In perturbation theory it can be proven to have a continuum limit, and all results indicate that non-perturbatively the same will hold. Despite the tremendous increase of computer power since the first Monte Carlo calculations, it remains a technical challenge to make the lattice spacing $a$ small and the physical volume large enough. In particular problems of critical slowing down at small values of the coupling, implying that the algorithm is not efficient in probing the relevant portions of field space, makes a straightforward approach of decreasing the lattice spacing (while keeping the physical volume large enough) a costly procedure. Instead one can use the increased computer power of today to study alternative lattice actions, which are chosen to remove as much as possible the scaling violations introduced by a finite lattice cutoff. The computational overhead in using a more complicated lattice action is usually not more than a few to ten times the cost for the Wilson action.

1.1 Symanzik improvement

The notion of improved actions was introduced more than a decade ago and in particular the study of Symanzik for scalar theories and Lüscher and Weisz.

\textsuperscript{a}Talk presented by the last author at the second workshop “Continuous advances in QCD”, Univ. of Minnesota, March 28-31, 1996.
for non-Abelian gauge theories have been influential. The last few years have seen a surge in applying these ideas in actual simulations. When considering a lattice action, here restricting ourselves to pure gauge theories, the dynamical variables live on the (directed) links of the lattice and are elements of the gauge group $U_\mu(x)$. The connection to a continuum configuration is provided in terms of parallel transport of the vector potential $A_\mu(x)$ along the link $x \rightarrow x + \hat{\mu}$ ($\hat{\mu}$ the directional vectors of the hypercubic lattice)

$$U_\mu(x) = P\exp\left(\int_0^a A_\mu(x + s\hat{\mu})\,ds\right).$$  \hspace{1cm} (1)

This allows us to express any of the lattice actions in powers of the lattice spacing. Terms that are of order $a^4$ (in four dimensions), being the volume of a unit cell on the lattice, correspond to the naive continuum limit. Higher order terms of which we will quote the powers of $a$ relative to this volume factor, correspond to irrelevant operators. For example, for the Wilson action one finds the result

$$S = \sum_P \text{Tr}(1 - U(P)) = \sum_{x,\mu,\nu} \text{Tr}\left(1 - j^2\right)$$

$$= \sum_{x,\mu,\nu} \text{Tr}(1 - U_\mu(x)U_\nu(x + \hat{\mu})U_\mu^\dagger(x + \hat{\nu})U_\nu^\dagger(x))$$

$$= \sum_{x,\mu,\nu} -\text{Tr}\left[\frac{a^4}{2}F_{\mu\nu}(x) - \frac{a^6}{24}((D_\mu F_{\mu\nu}(x))^2 + (D_\nu F_{\mu\nu}(x))^2) + \frac{a^8}{24}(F_{\mu\nu}(x)$$

$$+ \frac{1}{30}((D_\mu^2 F_{\mu\nu}(x))^2 + (D_\nu^2 F_{\mu\nu}(x))^2) + \frac{1}{3}D_\mu^2 F_{\mu\nu}(x)D_\nu^2 F_{\mu\nu}(x)$$

$$- \frac{1}{4}(D_\mu D_\nu F_{\mu\nu}(x))^2]\right] + \mathcal{O}(a^{10}),$$  \hspace{1cm} (2)

Note that apart from the corrections to the action density, sums need to be converted to integrals (which for smooth fields would only give exponential corrections, but for rough fields is part of the renormalization procedure) and the measure for integration over the fields needs to be converted from (compact) link variables to vector potentials. The main advantage of the lattice formulation is of course its intrinsic gauge invariance. In the continuum, even in a finite volume, the integral over the irrelevant gauge degrees of variables is hard to define unambiguously. We note that modifications in the integration measure can, in principle, be absorbed in the action.

To compensate for the irrelevant higher dimensional operators in the action one can follow two routes. One is using Wilson’s renormalization group, introducing blocking transformations which for finite couplings and lattice spacings
define a renormalized trajectory along which the physical quantities remain constant (giving a perfect lattice action). Its limit for zero coupling is the classically perfect lattice action. In a sense this can be seen, at least in principle, as a tree-level improved action to all orders in the lattice spacing, but through the blocking transformations is supposed to also correct for the other errors we mentioned above. It may be expected, as demonstrated to one-loop order in the O(3) model, that power-like cut-off effects in loop corrections are absent too. In this should lie the strength of classically perfect lattice actions, despite the need for truncations in actual numerical calculations. A large redundancy in the way one parametrizes these actions is reflected in the freedom of choosing the blocking transformations. One will try to keep the action as local as possible.

The large redundancy is also clear in the Symanzik improvement scheme, which is based on perturbative computations of physical quantities. At tree-level there are many ways of choosing the action in terms of the gauge invariant Wilson loops that extend further than a single plaquette, such that the order $a^2$ correction has a vanishing coefficient. The class of actions we will consider is defined by \((\langle \rangle)\) implies averaging over orientations)

\[
S(\{c_i\}) = \sum_x \text{Tr}\left\{ c_0 \left\langle 1- \right\rangle + 2c_1 \left\langle 1- \right\rangle + \frac{4}{3} c_2 \left\langle 1- \right\rangle \right\} \\
+ 4c_3 \left\langle 1- \right\rangle + c_4 \left\langle 1- \right\rangle \right\} \\
= -\frac{a^4}{2} (c_0 + 8c_1 + 8c_2 + 16c_3 + 16c_4) \sum_{x,\mu,\nu} \text{Tr}(F_{\mu\nu}^2(x)) \\
+ \frac{a^6}{12} (c_0 + 20c_1 - 4c_2 + 4c_3 + 64c_4) \sum_{x,\mu,\nu} \text{Tr}(D_\mu F_{\mu\nu}(x))^2 \\
+ a^6 \left( \frac{c_2}{3} + c_3 \right) \sum_{x,\mu,\nu,\lambda} \text{Tr}(D_\mu F_{\mu\lambda}(x)D_\nu F_{\nu\lambda}(x)) \\
+ a^6 \frac{c_2}{3} \sum_{x,\mu,\nu,\lambda} \text{Tr}((D_\mu F_{\nu\lambda})^2) + \mathcal{O}(a^8) .
\]

We have added a $2 \times 2$ plaquette to the parametrization of the lattice action employed in ref. 8. (Note that sometimes in the literature conventions are used where $c_2$ and $c_3$ are interchanged, e.g. refs. 10,14). We will motivate further on the advantage of adding this new plaquette. Using this action in perturbative calculations, one finds that the coefficients $c_i$ need to be corrected at higher
order in $g_0^2$. This is achieved by computing a physical quantity to the required order and impose the vanishing of the $O(a^2)$ term to the relevant order in the coupling. This is called on-shell improvement and should be independent of the set of quantities required to remove the $O(a^2)$ terms. Note that some redundancies appear due the invariance under field redefinitions. As was shown by Lüscher and Weisz \cite{15} this allows one to choose $c_3 \equiv 0$.

1.2 Tadpole improvement

The most urgent question would be how small the coupling constant (and hence the lattice spacing) has to be chosen to be able to truncate the perturbative construction of these improved actions. Relevant for questions related to truncating a perturbative series is which definition of the coupling constant to take. Formally, lattice perturbation theory is defined in the bare coupling constant. Usually, however, it helps greatly to convert the coupling constant to one that is defined at the scale of the process for which one is computing the perturbative corrections. This works well in the computation of physical quantities. However, one may doubt if this is valid for the computations of the improved action, as the corrections are by definition not physical. Still, a coupling at the scale of the cutoff would presumably perform better than the bare coupling. It has become customary to choose for this Parisi’s mean field coupling \cite{16}, extracted from the plaquette expectation value,

$$g_P^2 \equiv -a(c_i, N) \log(u_0), \quad u_0 = \frac{1}{N} \text{ReTr} U(P) > 1/4$$

where $a$ is a constant, defined such that $g_P = g_0 + O(g_0^3)$. It is clear that this is just one possible choice. Nevertheless in terms of this coupling, deviations from asymptotic scaling seem to be much smaller than in terms of the bare lattice coupling. One usually relates this coupling to one that is obtained by a resummation of tadpole diagrams, although this is hard to make precise \cite{17}. Similarly one may attempt to apply “mean field” type corrections to the improvement coefficients that take into account the apparent large renormalization factors for the links, implied by the large difference between $g_P$ and $g_0$ at moderate and large couplings. This has been known as tadpole improvement and is observed to work very well, in particular when considering the rotational invariance of the heavy quark potential \cite{18}. It should be emphasized that tadpole improvement can be seen as a rearrangement of perturbation theory. When only tree-level improvement coefficients are known the coefficients $c_i$ are divided by a factor $u_0$ for each extra link (compared to the single plaquette),

$$c_1 \to c_1/u_0^2, \quad c_2 \to c_2/u_0^2, \quad c_3 \to c_3/u_0^2, \quad c_4 \to c_4/u_0^4$$

(5)
For the only case so far where the one-loop improvement corrections to $c_i$ are known, one adjusts the coefficients according to the expansion of $g_P$ in the bare coupling $g_0$. The implicit assumption of tadpole improvement is that corrections of $O(g_P^4)$ can be neglected. Although in many cases tadpole improvement is shown to reduce the size of the perturbative corrections and to improve numerical results for very coarse lattices indeed, it remains a rather ad hoc procedure. As long as one needs to show the validity by comparing with the carefully extrapolated Wilson action results, one should shed doubt on the usefulness of this procedure in particular when one starts to “fudge” with alternate definitions of $u_0$ (for example extracted from more extended Wilson loops). What would be required is a well motivated definition of (a class of) improved lattice actions where similar to the Wilson action careful scaling studies are performed, that allow one to extrapolate results independently to the continuum and infinite volume. Working only at the coarsest lattices will probably prove to be insufficient to obtain reliable results. In particular systematic studies that use finite-size techniques as developed by the alpha collaboration for determining the running coupling constant and renormalization factors seem mandatory for a careful study of the systematic errors involved.

1.3 The setting

Motivated by all these problems we set out to test improved actions in the context of finite volume spectroscopy. The clear advantage is that for volumes below 0.7 fm, results for the low-lying spectrum are known more or less analytically, both for the continuum and for the Wilson action. In such a case we can be precise in saying how much improvement is achieved. We can not reach the coarsest lattices employed so far. Our volume has to be at least three lattice spacings across, which means that below 0.7 fm we can only reliably reach lattice spacings of the order of 0.2 fm. The data presented here will be for much smaller lattice spacing, where nevertheless scaling violations in mass ratios are particularly big.

In order to also study the results of the improved actions analytically we introduced a new improved lattice action for which the gauge field propagator simplifies greatly. The explicit form of the propagator also will make clear that there are violations of unitarity (poles in the propagator with negative residue) at the scale of the cutoff. In the continuum limit these are harmless, in the same way as Pauli-Villars regulator fields are harmless below the scale of the cutoff. It does, however, cause difficulties in extracting a Hamiltonian.
2 The new improved action

To study perturbation theory on the lattice we write \( U_{\mu}(x) = \exp(q_{\mu}(x)) \) and expand the lattice action to second order in \( q \). At tree level we have, as for the Lüscher-Weisz Symanzik action, \( c_2 = c_3 = 0 \), cf. Eq. 3. To remove the remaining \( O(a^2) \) irrelevant operator we have the three constants \( c_0, c_1 \) and \( c_4 \) at our disposal. One is eliminated by requiring the appropriate continuum limit without the need of rescaling the coupling constant. For the LW Symanzik action, corresponding to \( c_4 = 0 \), one finds \( c_0 = 5/3 \) and \( c_1 = -1/12 \). The disadvantage of this choice in perturbation theory becomes clear when writing the quadratic part of the action

\[
S_2 = \sum_{x,\mu,\nu} \frac{-1}{4} \text{Tr}[c_0 (\partial_{\mu} q_{\nu}(x) - \partial_{\nu} q_{\mu}(x))^2] + 2c_1 (2 + \partial_{\nu}) (\partial_{\mu} q_{\nu}(x) - \partial_{\nu} q_{\mu}(x))^2] ,
\]

where the lattice derivative \( \partial_{\mu} \) is the difference operator

\[
\partial_{\mu} \varphi(x) \equiv \varphi(x + \hat{\mu}) - \varphi(x) .
\]

There is no “covariant” gauge condition that will make the gauge field propagator diagonal in the space-time indices. In analytic calculations this makes perturbative computations cumbersome, in particular in the presence of a background field, but of course not impossible. Nevertheless, if we choose

\[
c_4 \cdot c_0 = c_1^2 ,
\]

a condition invariant under the tadpole modification in Eq. 5, we can rearrange Eq. 6 by completing squares, from which we read off a simple gauge condition

\[
S_2 = -\sum_{x,\mu,\nu} c_0 \text{Tr}[\partial_{\mu} q_{\nu}(x) (1 + z(2 + \partial_{\mu})(2 + \partial_{\nu})) (1 + z(2 + \partial_{\nu})(2 + \partial_{\nu})) \partial_{\mu} q_{\nu}(x)] + \sum_x \text{Tr} F^2_{gf}(x) , \quad F_{gf}(x) \equiv \sqrt{c_0} \sum_{\mu} \partial_{\mu} \left( 1 + z(2 + \partial_{\mu})(2 + \partial_{\mu}) \right) q_{\mu}(x) ,
\]

where \( z \equiv c_1/c_0 \). At tree level we find \( c_0 = 16/9 \), \( c_1 = -1/9 \), \( c_2 = c_3 = 0 \) and \( c_4 = 1/144 \), whereas for \( u_0 \neq 1 \) we have \( z = -1/16u_0^2 \) and \( c_0 = 1/(1 + 4z)^2 \). We propose to call this new action the square Symanzik action, or square action for short. We find the following propagators

\[
\text{Ghost :} \quad P(k) = \frac{1}{\sqrt{c_0} \sum_{\lambda} (4 \sin^2(k_{\lambda}/2) + 4z \sin^2 k_{\lambda})} ,
\]

\[
\text{Vector :} \quad P_{\mu\nu}(k) = \frac{P(k) \delta_{\mu\nu}}{\sqrt{c_0 (1 + 4z \cos^2(k_{\mu}/2))}} .
\]
This simple form of the propagator allows us to demonstrate a general feature of improved actions, which have couplings in the time directions that are not nearest neighbor. We introduce the standard lattice propagators

\[
P_s \equiv \frac{1}{4 \sin^2(k_0/2) + \omega_s^2}, \quad \omega_s^2 \equiv -\frac{(1 + 4z)}{2z} \left(1 \pm \sqrt{1 + \frac{4z\omega^2}{(1 + 4z)^2}}\right),
\]

\[
\omega_0^2 \equiv -\frac{1 + 4z}{z}, \quad \omega^2 \equiv 4 \sum_{i=1}^{3} \sin^2(k_i/2) (1 + 4z \cos^2(k_i/2)),
\]

which have a single pole in the \(k_0\) Brillouin zone with the standard residue as would have been obtained from the Wilson action (except that \(\omega \equiv \omega(z = 0)\) is replaced by \(\omega_s\)). This allows us to write

\[
P = Z(P_0 - P_+), \quad P_{jj} = \frac{(1 + 4z)}{(1 + 4z \cos^2(k_j/2))} P_j,
\]

\[
P_{00} = (1 + 4z) \left(\frac{\omega^2 Z}{\omega_s^2} P_0 - \frac{\omega^2 Z}{\omega_s^2} P_+ + \frac{\omega^2}{\omega_s^2} P_0\right),
\]

where the \(Z\) factor is given by \(Z \equiv 1/\sqrt{1 + 4z\omega^2/(1 + 4z)^2}\), showing that there are unphysical propagating negative (and positive) norm states. This is a direct consequence of the fact that the transfer matrix, as defined in ref. 23, is not hermitian. The unphysical masses involved are at the scale of the cutoff (in lattice units \(O(1)\)). For \(k = 0\) and \(u_0 = 1\) one finds \(\omega_- = 0\) and \(\omega_+ = \omega_0 = \sqrt{12}\). Although the unphysical states are interacting in a complicated way, and do not just appear in closed loops (thereby making it somewhat misleading to call them ghosts) they can be shown not to lead to imaginary eigenvalues and violations of unitarity in the low-lying spectrum.

In particular the low-lying spectrum for gauge theories in an intermediate volume is obtained by integrating out the non-zero momentum modes in a loop expansion and solving the resulting effective theory using a Rayleigh-Ritz analysis which includes imposing proper boundary conditions in field space implied by the gauge invariance. The effect of the spurious poles is integrated out at the same level of accuracy as the high momentum modes. Here we will only use our square Symanzik action to calculate the effective potential for an abelian constant background field, which greatly simplifies due to the diagonal structure of the gauge field propagator. Also it will be used to compute the Lambda parameter.
3 The effective potential

The effective potential for a constant abelian background field $C_\mu$ is particularly simple to calculate since the background field will give rise to a shifted momentum, $k_\mu \rightarrow k_\mu + sC_\mu/N_\mu$, where $N_\mu$ is the size of the lattice in each of the four directions. For the modes $q$ that commute with the background field one has $s = 0$, whereas for the two charged components with respect to the abelian direction defined by the background field one takes $s = \pm 1$. To one-loop order the effective potential is simply obtained by summing over the logarithm of the eigenvalues of the quadratic fluctuation operator, properly corrected for by the ghost contributions. We take $C_0 = 0$ and $N_i = N$, such that

$$V_1(C) = \frac{1}{N_0} \sum_{k,s} \left\{ \frac{1}{2} \sum_\mu \log \lambda_\mu(k + sC/N) - \log \lambda_{gh}(k + sC/N) \right\}$$  \hspace{1cm} (13)

The eigenvalues $\lambda(k)$ are directly read off from the explicit expressions for the propagators in Eq. 10.

$$\lambda_{gh}(k) = \sqrt{c_0} \sum_\nu 4 \sin^2(k_\nu/2)(1 + 4z\cos^2(k_\nu/2)),$$

$$\lambda_\mu(k) = \sqrt{c_0}(1 + 4z\cos^2(k_\mu/2))\lambda_{gh}(k).$$  \hspace{1cm} (14)

This formula holds equally well for the Wilson action (where $z \equiv 0$). As follows from the factorization of the propagator we can write $\lambda_{gh}$ as the product of two eigenvalues. They will come in complex conjugate pairs when the momentum gets too close to the edge of the Brillouin zone. Nevertheless splitting $\log \lambda_{gh}$ in these separate contributions, that depend on $k_0$ as in the background field calculation for the Wilson action, allows one to perform the sum over $k_0$ exactly. In particular for $N_0 \rightarrow \infty$ one finds the following explicit result

$$V_1(C) = N \sum_{\vec{n} \in \mathbb{Z}^3} \left\{ 4 \text{asinh} \left( 2u_0 \sqrt{1 + 4z + \frac{\omega^2}{2} + \omega \sqrt{1 + \frac{\omega^2}{4}}} \right) + \sum_i \log (\Omega_i) \right\},$$  \hspace{1cm} (15)

up to an irrelevant overall constant. Here $\Omega_i \equiv 1 + 4z\cos^2[(2\pi n_i + C_i)/(2N)]$ and $\omega \equiv \omega(\vec{k} = (\pi \vec{n} + \vec{C})/N)$ as defined in Eq. 11. For $z \rightarrow 0$ this gives the well known result obtained for the Wilson action, whereas $N \rightarrow \infty$ recovers the result for the continuum theory. In figure 1 we compare the results of $V_1(C)$ for the square Symanzik action (lower two curves for $N = 3$ and $N = 4$), with those for the Wilson action (upper three curves for $N = 3, 4$ and 6). The full curve gives the result of $V_1(C)$ for the continuum (we have chosen...
The effective potential for a constant Abelian background field $A_1 = \frac{1}{2} i C \sigma_3 / N$. The full line represents the continuum result. The lower two dashed curves are for the square Symanzik action with $N = 3$ and 4 ($N = 6$ is indistinguishable from the continuum curve). The upper three dotted curves are for the Wilson action with $N = 3, 4$ and 6.

$\vec{C} = (C, 0, 0)$. For the new improved action the result for $N = 6$ can already not be distinguished from that in the continuum at the scale of this figure. One might even fear that choosing $u_0 \neq 1$ will make the agreement worse. However, the effective potential is not a spectral quantity and deviations of $V_1(C)$ from the continuum can in principle be compensated by other corrections in the effective Hamiltonian for the zero-momentum modes.

4 The Lambda parameter

One can now follow the same strategy as in calculating the zero-momentum effective Hamiltonian for the Wilson action. The difficulty lies in converting a non-hermitian transfer matrix, defined by the euclidean path integral, to a Hamiltonian. We postpone this to a future publication, where it will be shown that by a suitable field redefinition one can map the effective action with next-to-nearest neighbor couplings in the time direction to one with nearest neighbor coupling. After this transformation (whose Jacobian will give rise to corrections of odd powers in the lattice spacing), one can convert the path integral through a hermitian transfer matrix to a Hamiltonian. This conversion will introduce corrections that are of even powers in the lattice spacing. At first, finding scaling violations that are of odd powers in the lattice spacing may seem rather puzzling, but it turns out they are required to exactly cancel similar scaling violations that appear in some coefficients of the effective action.
However, to determine the Lambda parameter for the square Symanzik action, it is sufficient to determine the effective action and study the finite difference in the renormalization as compared to (say) the Wilson action in the limit of zero lattice spacing. Although in the renormalized action the tree level kinetic and potential terms, \( \frac{1}{2} (\partial_t c_i)^2 \) and \( \frac{1}{2} (F_{ij}^a)^2 \), will have coefficients that differ by finite renormalizations (due to the breaking of Lorentz invariance), the difference of these finite renormalizations between the different regularizations is expected to be the same for both terms. It should be noted that in principle one can perform a rescaling of the fields, but this would upset the periodicity (a consequence of gauge invariance) of the effective theory along the abelian constant potential. Indeed in background field calculations no field renormalizations appear.

Consequently two numerically independent determinations of the Lambda parameter can be extracted from this background field calculation, as for the Wilson action. In addition one can follow the alternative determination of the Lambda parameter through the heavy quark potential. All three determinations gave to a high degree of accuracy the same result for SU(2). The result for SU(3) was obtained with the latter method only.

\[
\frac{\Lambda_{S^2}}{\Lambda_W} = \begin{cases} 
4.0919901(1) & \text{for SU(2)}, \\
5.2089503(1) & \text{for SU(3)}. 
\end{cases}
\]  

(16)

The index \( S^2 \) stands for the square Symanzik and \( W \) for the Wilson action. Details of both methods to compute these ratios will be presented elsewhere.

5 Monte Carlo data

In figure 2 we present the SU(2) Monte Carlo results for mass ratios in a small volume. Full details of the analysis will be presented elsewhere. We chose the volume such that the lattice artefacts in the mass ratios, the difference between the full curves (continuum) and the dotted curves (Wilson action for a 4\(^3\) \times \infty \) lattice), were maximal. For the Wilson action, at \( \beta = 4/g_0^2 = 3 \) this corresponds to a lattice spacing of approximately 0.018 fm. Another reason to pick these parameters was to compare with earlier high precision data for the Wilson action by Michael (crosses in the figure), so as to make sure no errors were made in the implementation and in the measurements of the masses. We also improved here somewhat on the statistical errors. The data corresponding to the LW Symanzik action is represented by the triangles and for our new square Symanzik action by the squares. In both cases we used tree-level improvement only. The improvement is significant. For the LW Symanzik action the data is within two sigma of the continuum values. The results seem
to indicate that the square Symanzik action is somewhat less effective, although the difference is not significant. A comparison at coarser lattices will be more interesting. The lattice results for the new action can be shown to confirm the value of the Lambda parameter computed from perturbation theory. The value of $\beta$ for the Monte Carlo analysis with the square Symanzik action was chosen before the the perturbative calculations were completed.

The mass ratios shown are all with respect to the scalar mass in the singlet representation $A_1^+$ of the cubic group. On the left we plot the square root of the finite volume “string tension” for one, two and three units of electric flux, whereas on the right we consider the tensor doublet $E^+$ and triplet $T^+_2$, which become degenerate in large volumes when rotational invariance is restored. The finite volume “string tension” is simply defined by dividing the energy of
the electric flux states (also called torelon masses\cite{24}) by the length of the box.
The electric flux is defined with the help of twisted boundary conditions\cite{25}.
The Wilson loop operator that closes due to the periodic boundary conditions
can be seen to create electric flux. Two (or three) units of electric flux are
obtained from loops that wind around two (or three) directions of the torus.

6 Outlook

Our contribution has been modest as we have not probed very coarse lattices.
The Monte Carlo data presented here were for a lattice spacing of approximately 0.02 fm. New Monte Carlo data will be presented elsewhere, studying
lattice spacings of approximately 0.12 fm. We would also like to stress that we
used mass ratios to test improvement. It has been well know that the Parisi
mean field coupling constant improves the approach to asymptotic scaling quite
well. We see this as a separate issue, quite (but not completely) unrelated to
the issue of scaling. The reason is that the scale in lattice calculations is usually set by one of the masses (in pure gauge theories by the string tension)
anyhow.

From the practical point of view we presented an alternative action, motivated by the requirement to simplify the perturbative calculations, rather than
to minimize the lattice artefacts. Nevertheless, comparisons will give a clue on
how big the higher order corrections are. Of course at tree level it is trivial
to write down many types of improved actions, so we have set out to compute
the one-loop improvement coefficients too, to bring our new square Symanzik
improved action to the same footing as the L"uscher-Weisz Symanzik improved
action. These results will be presented elsewhere. This allows one to test the
"universality" of tadpole improvement. Again we wish to stress that the most
interesting tests are those for the scaling of mass ratios.

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References

1. K. Wilson, Phys. Rev. D 10, 2445 (1974).
2. T. Reisz, *Nucl. Phys.* B 318, 417 (1989); *Comm. Math. Phys.* 116, 81 (1988).
3. M. Lüscher, in *Fields, strings and critical phenomena*, Les Houches 1988, p. 451-528.
4. M. Creutz, *Quarks, gluons and lattices*, (Cambridge Univ. Press, 1983).
5. I. Montvay and G. Münster, *Quantum fields on a lattice*, (Cambridge Univ. Press, 1994).
6. M. Creutz, *Phys. Rev. Lett.* 43, 553, 890 (1979).
7. K. Symanzik, *Nucl. Phys.* B 226, 187, 205 (1983).
8. M. Lüscher and P. Weisz, *Phys. Lett.* B 158, 250 (1985).
9. G.P. Lepage, *Nucl. Phys.* B (Proc.Suppl) 47, 3 (1996).
10. M. García Pérez, e.a., *Nucl. Phys.* B 413, 535 (1994).
11. K. Wilson, in *Recent developments in gauge theories*, eds. G. ’t Hooft et. al., Plenum Press, New York, 1980, p.363.
12. P. Hasenfratz and F. Niedermayer, *Nucl. Phys.* B 414, 785 (1994); T. DeGrand, e.a., *Nucl. Phys.* B 454, 587, 615 (1995).
13. F. Farchioni, e.a., *Nucl. Phys.* B 454, 638 (1995).
14. P. Weisz, *Nucl. Phys.* B 212, 1 (1983); P. Weisz and R. Wohler, *Nucl. Phys.* B 236, 397 (1984).
15. M. Lüscher and P. Weisz, *Comm. Math. Phys.* 97, 59 (1985).
16. G. Parisi, in *High Energy Physics-1980*, eds. L. Durand and L.G. Pondrom (American Institute of Physics, New York, 1981).
17. V. Periwal, *Phys. Rev. D* 53, 2605 (1996).
18. G.P. Lepage and P.B. Mackenzie, *Phys. Rev. D* 48, 2250 (1993).
19. M. Alford, e.a., *Phys. Lett.* B 361, 87 (1995); C. Morningstar and M. Peardon, *Nucl. Phys.* B (Proc.Suppl.) 47, 258 (1996); hep-lat/9606008.
20. G. de Divitiis, e.a., *Nucl. Phys.* B 437, 447 (1995).
21. K. Jansen, e.a., *Phys. Lett.* B 372, 275 (1996).
22. P. van Baal, *Phys. Lett.* B 224, 397 (1989); *Nucl. Phys.* B 351, 183 (1991).
23. M. Lüscher and P. Weisz, *Nucl. Phys.* B 240[FS12], 349 (1984).
24. M. Lüscher and P. Weisz, *Nucl. Phys.* B 266, 309 (1986).
25. P. van Baal, *Hamiltonian from improved actions*, presented at Lattice’96, St.Louis, June 4-8, 1996, to appear in the proceedings.
26. C. Michael, *Nucl. Phys.* B 329, 225 (1990).
27. G. ’t Hooft, *Nucl. Phys.* B 153, 141 (1979).