The geodesic Vlasov equation and its integrable moment closures

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Abstract

Various integrable geodesic flows on Lie groups are shown to arise by taking moments of a geodesic Vlasov equation on the group of canonical transformations. This was already known for both the one- and two-component Camassa-Holm systems [GiHoTr2005, GiHoTr2007]. The present paper extends our earlier work to recover another integrable system of ODE’s that was recently introduced by Bloch and Iserles [BlIs2006]. Solutions of the Bloch-Iserles system are found to arise from the Klimontovich solution of the geodesic Vlasov equation. These solutions are shown to form one of the legs of a dual pair of momentum maps. The Lie-Poisson structures for the dynamics of truncated moment hierarchies are also presented in this context.

1 Introduction

Kinetic equations govern the evolution of probability distributions in the phase space of many-particle systems in non-equilibrium statistical mechanics. For example, the phase-space probability distribution of a many-particle system whose correlations are negligible is governed by the collisionless Boltzmann equation, also known as Vlasov equation [Vl1961]. This equation encodes evolution of the Vlasov single-particle probability distribution \( f(q,p,t) \) as conservation along phase space trajectories, written as

\[
\frac{df}{dt} = 0 \quad \text{along} \quad \dot{q} = \frac{\partial h}{\partial p}, \quad \dot{p} = -\frac{\partial h}{\partial q},
\]  

(1)

where \( h \) is the single-particle Hamiltonian, often expressed as the sum of kinetic and potential energies

\[
h = \frac{|p|^2}{2} + V(q)
\]

in physical applications. Applying the canonical Poisson bracket \( \{ \cdot, \cdot \} \) in the phase space variables \((q,p)\) expresses the Vlasov equation in its familiar form

\[
\frac{\partial f}{\partial t} + \{ f, h \} = 0.
\]  

(2)

The Hamiltonian structure of this system is well known [MaWe1981]. Namely, the Vlasov equation possesses a Lie-Poisson bracket defined on the Lie algebra of canonical transformations, such that the Liouville theorem for preservation of the volume on phase space entirely characterizes Vlasov dynamics.

The moment method, widely used in probability theory, provides approximate descriptions of the Vlasov solutions. Moments are functionals of the distribution function \( f \) obtained by projections onto the space of phase space polynomials (symmetric tensors). Since the Vlasov distribution depends on both position \( q \) and momentum \( p \), one may define two different types of moments. These
are the \textit{kinetic moments} and the \textit{statistical moments}. Kinetic moments are given by projection of the Vlasov distribution $f(q, p)$ onto the $n$-th power of the single-particle momentum, $p$. In contrast, statistical moments are integrals of $f(q, p)$ against the $n$-th power of the full phase space vector, $z = (q, p)$. The remarkable property of these two hierarchies of moment projections is that they each define equivariant momentum maps \cite{HoLySc1990, ScWe1994, GiHoTr2008}. Consequently, the resulting moment dynamics is again Lie-Poisson. Moment equations possess interesting closures, which are given by the particular Lie algebra structure determining their Lie-Poisson bracket. For example, kinetic moments of the Vlasov equation at zero-th and first order yield the familiar closure known as \textit{ideal fluid dynamics}.

Remarkably, these kinetic moment equations are associated to a family of integrable dynamical systems, whose most famous example is probably the Benney system for shallow water dynamics \cite{Be1973, Gi1981}. This convergence of different areas of mathematical physics also occurs for several other integrable equations. For example, as shown in \cite{GiHoTr2005}, a specific form of the first-order kinetic moment equation yields the Camassa-Holm equation \cite{CaHo1993}. Extending the system to include the zero-th order moment yields another integrable system; the two-component Camassa-Holm system \cite{GiHoTr2007}. Interestingly enough, these Camassa-Holm systems are geodesic flows on different Lie groups, arising as moment closures of the same kinetic equation, called here the \textbf{geodesic Vlasov equation} or EPCan. The latter acronym refers to the Euler-Poincaré (EP) equation on the group of canonical transformations $\text{Can}$ acting on phase space $T^*Q$. A special case of EPCan for canonical transformations whose generating functions are \textit{linear} in the canonical momentum has recently appeared in the theory of \textit{metamorphoses} in imaging science \cite{HoTrYo2007}.

This paper reviews the theory of geodesic equations on the statistical moments and shows how such equations possess an additional interesting closure, which is related to the space of purely quadratic Hamiltonian functions. We find that such a closure yields a particular case of yet another integrable system, recently discovered by Bloch and Iserles \cite{BlIsMaRa2005, BlIs2006}. Moreover, extending to inhomogeneous quadratic Hamiltonians yields complete equivalence between moment equations and the Bloch-Iserles (BI) system.

\textbf{Plan} The rest of this section adds a few more remarks about the Lie-Poisson bracket for the Vlasov equation. Section 2 is devoted to the Hamiltonian structure of the Vlasov moments and their truncations. Section 3 formulates the geodesic Vlasov equation, presents its dual pair and illustrates the geometric footing of kinetic theory. Section 4 shows how both of the Camassa-Holm systems are obtained as geodesic equations on kinetic moments. The last section derives the BI system from the statistical moment equations and presents the corresponding Klimontovich solutions.

1\textsuperscript{1}The group of canonical transformations $\text{Can}$ is also known as the Hamiltonian diffeomorphisms $\text{Diff}_{\text{Ham}}$. 
1.1 The Vlasov kinetic equation

The Vlasov equation is a Lie-Poisson Hamiltonian system on the group of canonical transformations of the phase space $T^*Q$ for a configuration manifold $Q$ [MaWe1981]. The dynamics of Lie-Poisson systems takes place on the dual $g^*$ of the Lie algebra $g$ of the symmetry group $G$. In this case $G = \text{Can}(T^*Q)$ and $g = \mathfrak{X}_{\text{can}}(T^*Q)$. That is, the Lie algebra is the infinite-dimensional space of Hamiltonian vector fields. Given the Lie algebra isomorphism between Hamiltonian vector fields and phase-space functions ($\mathfrak{X}_{\text{can}} \simeq \mathcal{F}$), the dynamical variable is a phase-space distribution $f(q,p)$, i.e., a density on phase space ($f \in \mathcal{F}^* \simeq \text{Den}$). Upon using the definition of canonical Poisson bracket $\{\cdot, \cdot\}$, the Vlasov Lie-Poisson structure is found to be

$$\{F, H\}[f] = \iint f(q,p) \left\{ \frac{\delta F}{\delta f}, \frac{\delta H}{\delta f} \right\} d^Kq \, d^Kp$$

where $K = \dim(Q)$. The Vlasov equation (2) is recovered upon choosing $F = f$ and $h = \delta H/\delta f$.

In many physical applications, the Vlasov Hamiltonian is the sum of kinetic and potential energy. For example, electrostatic or gravitational interactions are governed in the absence of collisions by the Poisson-Vlasov system whose Hamiltonian is given by

$$H[f] = \iint f(q,p) \left( \frac{1}{2} |p|^2 + \Delta^{-1} \int f(q,p') \, d^K p' \right) d^Kq \, d^Kp,$$

where $\Delta^{-1}$ denotes convolution with the Green’s function of the Laplace operator.

2 Hamiltonian structure of Vlasov moments

The moment method is a popular approach in kinetic systems theory. This approach is justified geometrically because taking moments of the Vlasov distribution is a momentum map [HoLySc1990, ScWe1994, GiHoTr2008]. This momentum map arises via the dual of a Lie algebra homomorphism arising from the well-known isomorphism between symmetric tensors and polynomials. The main point is that this momentum map endows the space of symmetric tensors with a Lie bracket, thereby generating a well defined Lie algebra. In what follows, we shall analyze the cases of kinetic and statistical moments separately and then discuss their similarities.
2.1 Kinetic moments and the Schouten concomitant

Kinetic moments are constructed from the following fiber integral [QiTa2004]

\[ A_n(q, t) := \int_{T^*Q} (p \cdot dq)^n f(q, p, t) \, d^K q \wedge d^K p \]

\[ = \sum_{i_1, \ldots, i_n=1}^{K} \int_{T^*Q} p_{i_1} \cdots p_{i_n} \, dq^{i_1} \otimes \cdots \otimes dq^{i_n} f(q, p, t) \, d^K q \wedge d^K p \]

\[ = \sum_{i_1, \ldots, i_n=1}^{K} (A_n(q, t))_{i_1 \ldots i_n} \, dq^{i_1} \otimes \cdots \otimes dq^{i_n} \otimes d^K q, \quad (5) \]

where \( p \cdot dq \) denotes the canonical one form (canonical momentum) and \( d^K q \) is the volume element on the configuration space \( Q \). This construction projects the Vlasov distribution onto the space of symmetric tensors. In particular, kinetic moments are defined as symmetric covariant tensor fields carrying the volume element. That is, they are symmetric covariant tensor densities.

The moments are functionals of the Vlasov density \( f \). Hence, their variational derivative may be computed by applying the chain rule as

\[ \frac{\delta F}{\delta f} = \sum_{n=0}^{\infty} \frac{\delta F}{\delta A_n} \, \frac{\delta A_n}{\delta f} := \sum_{n=0}^{\infty} \sum_{i_1, \ldots, i_n=1}^{K} \frac{\delta (A_n)_{i_1 \ldots i_n}}{\delta f} \, \frac{\delta F}{\delta (A_n)_{i_1 \ldots i_n}} \]

\[ = \sum_{n=0}^{\infty} \sum_{i_1, \ldots, i_n=1}^{K} p_{i_1} \cdots p_{i_n} \frac{\delta F}{\delta (A_n)_{i_1 \ldots i_n}} \]

\[ = \sum_{n=0}^{\infty} \frac{\delta F}{\delta A_n} \wedge p^n, \]

which explicitly defines the contraction operation \( \wedge \). This chain rule formula expresses the Lie algebra homomorphism (isomorphism) from symmetric tensors to polynomials, whose dual is the momentum map associated to the moments [GiHoTr2008]. Inserting the chain rule formula into the Vlasov bracket yields the Lie-Poisson bracket for moments,

\[ \{ F, G \} [A] = - \sum_{n,m=0}^{\infty} \int_{A_{m+n-1}(q)} \left[ \frac{\delta F}{\delta A_n}, \frac{\delta G}{\delta A_m} \right] \, d^3 q \]

\[ \quad \text{in which the bracket} \left[ \frac{\delta F}{\delta A_n}, \frac{\delta G}{\delta A_m} \right] = \mathcal{S} \left( n \left( \frac{\delta F}{\delta A_n} \cdot \nabla \right) \otimes \frac{\delta G}{\delta A_m} - m \left( \frac{\delta G}{\delta A_m} \cdot \nabla \right) \otimes \frac{\delta F}{\delta A_n} \right) \quad (7) \]

is inherited from the canonical Poisson bracket. Here the notation \( A \cdot B \) for one-index contraction between covariant and contravariant tensors is written as

\[ a^i \rightarrow a_i, \quad b_i \rightarrow b^i. \]
\[(A \cdot B)^{hl\ldots}_{ij\ldots k} = A_{ij\ldots k}B^{hl\ldots} \text{ and analogously for } B \cdot A = (B \cdot A)^{km\ldots}_{ij\ldots l} = B^{km\ldots l}A_{ij\ldots l}\.\]

This bracket is well known in differential geometry as an invariant differential operator of first order \[\text{Ni1955}\]. In fact, this operation is a Lie bracket, which is known as the Schouten concomitant or symmetric Schouten bracket. See, e.g., \[\text{GiHoTr2008}\] for more discussions and references.

**Remark 2.1 (History of Lie-Poisson structure for kinetic moments)

In one dimension, the moment Lie-Poisson structure \[\text{[4]}\] is the Kupershmidt-Manin bracket \[\text{[KuMa1978]}\] which was found in the context of the integrable Benney system in shallow water theory. Lebedev was the first to establish its relation with the Lie algebra of Hamiltonian vector fields in \[\text{[Le1979]}\] and Gibbons recognized later \[\text{[Gi1981]}\] its direct relation to the Vlasov flow. In higher dimensions, Kupershmidt introduced a multi-index notation \[\text{[Ku1987]}\], corresponding to the occupation number representation of the symmetric Schouten bracket. This observation suggested the quantum-like framework for kinetic moments in \[\text{[GiHoTr2008]}\], where the moment space is described in terms of a bosonic Fock space.

The moment algebra comprises symmetric contravariant tensor fields and these may be characterized as the Fock space represented by a direct sum of symmetric powers of vector fields given by

\[
\mathfrak{g} := \bigoplus_{n=0}^{\infty} \left( \bigwedge_{i=0}^{n} X(Q) \right) \quad \text{with} \quad \bigwedge_{i=0}^{n} X := \mathcal{S} \left( \bigotimes_{i=0}^{n} X \right) =: \mathfrak{g}_n. \quad (8)
\]

This is reminiscent of the universal enveloping algebra of the diffeomorphism group \(\text{Diff}(Q)\), which is the enveloping algebra \(\mathcal{U}(\mathfrak{X})\) of vector fields \(\mathfrak{X}(Q)\) on the configuration space \(Q\). It is a standard result that the graded structure of an enveloping algebra possesses a Poisson bracket structure \[\text{[DaSWe1999]}\].

**Remark 2.2 (Kinetic moments and Poisson-Lie groups)

Interestingly enough, the Schouten concomitant identifies the kinetic moment algebra with the Lie algebra of symbols of differential operators. This identification is quite suggestive, since differential symbols are known to be a subalgebra of the Lie algebra of pseudo-differential symbols \[\text{[KhZa1995]}\]. While the differential symbols (the moment algebra) are supposed to have no underlying Lie group structure, the group \(\Psi D\) of pseudo-differential operators is a well-defined Poisson-Lie group \[\text{[KhZa1995]}\]. This suggests that the characterization of coadjoint orbits for moment dynamics requires the complete Poisson-Lie group structure of pseudo-differential operators. A similar direction involving vector fields was followed by Ovsienko and Roger \[\text{[OvRo1999]}\]. Also, the appearance of the Wick-ordered product from quantum theory in this Poisson-Lie group context \[\text{[KhZa1995]}\] \[\text{[OvRo1999]}\] implies a further relation to moment dynamics, whose quantum-like creation and annihilation operators were presented in \[\text{[GiHoTr2008]}\].
The moment algebra carries a graded structure \((\mathfrak{g} = \bigoplus_i \mathfrak{g}_i)\) with filtration
\[
[\mathfrak{g}_n, \mathfrak{g}_m] \subseteq \mathfrak{g}_{n+m-1}
\] (9)
thereby recovering the space of vector fields \(\mathfrak{g}_1 = \mathcal{X}\) as a particular subalgebra. The largest subalgebra is however \(\mathfrak{g}_0 \oplus \mathfrak{g}_1 \simeq \mathcal{X} \otimes \mathcal{F}\), i.e. the semidirect product of vector fields with scalar functions. This space appears in the description of ideal compressible fluids, where the \(\mathcal{X}\)-variable is the fluid velocity and the \(\mathcal{F}\)-variable is associated to the fluid density. Geodesic motion on this space has also recently appeared in the metamorphosis process in imaging science [HoTrYo2007].

The moment equations may be written in Lie-Poisson form as
\[
\frac{\partial A_m}{\partial t} = - \sum_{m,n=0}^{\infty} \text{ad}^*_{\alpha_{k,n}} A_{n+m-1}
\] (10)
where \(\text{ad}^*\) is the Lie algebraic coadjoint operation defined using the pairing
\[
\sum_{k,n=0}^{\infty} \langle \text{ad}^*_{\alpha_{k,n}} A_k, \alpha_{k-n+1} \rangle := \sum_{k,n=0}^{\infty} \langle A_k, [\beta_n, \alpha_{k-n+1}] \rangle.
\]
The explicit expression for \(\text{ad}^*\) is given in [GiHoTr2008] in any number of dimensions. Here we present the one-dimensional case which will be needed in the following sections. The Schouten concomitant in equation (7) assumes a particularly simple form in 1D
\[
[\alpha_m, \beta_n] = m \alpha_n \partial_q \beta_n - n \beta_n \partial_q \alpha_m
\] (11)
where \(\alpha\) is the spatial variable. Simple use of integration by parts yields the following covariant tensor density of rank \(k-n+1\):
\[
\text{ad}^*_{\alpha_{k,n}} A_k = (k+1) \ A_k \partial_q \beta_n + n \beta_n \partial_q A_k.
\] (12)
This operation was introduced by Kirillov [Ki1982], who first envisioned the possibility of a Lie-Poisson bracket on the symmetric Schouten algebra. Familiar versions of this operator with \(n = 1\) or \(k = n\) arise in the theory of ideal fluid dynamics, soliton dynamics and image matching, while very little is known for other values on \(n, k\). Some features of this intriguing open question are investigated further below, in dealing with truncations of moment hierarchies.

### 2.2 Statistical moments and their Lie-Poisson structure

As we have seen, the fiber integral defining the kinetic moment hierarchy in \([\ref{5}]\) requires a kinetic equation on a cotangent bundle. In contrast, the notion of statistical moments is given on a symplectic vector space. Upon denoting \(z = (q, p)\), the definition of the \(n\)-th statistical moment is given by
\[
X^n(t) := \int z^n f(z, t) \ d^N z
\] (13)
where the upper index $n$ in the integrand denotes tensor power ($z^n = \otimes^n z$), while for the time-dependent tensors $X^n(t)$ it denotes the tensor rank. This definition places the statistical moments and kinetic moments into the same mathematical framework. The first observation is that statistical moments are symmetric contravariant tensors on phase space, which is now a symplectic vector space $V$ of even dimension $N = 2K$ (eventually $V = \mathbb{R}^N$) with elements $z = z^i e_i \in V$.

The moment Poisson bracket for statistical moments may be obtained by following exactly the same steps as in the previous discussion for kinetic moments. That is, one inserts the chain rule formula

$$\frac{\delta F}{\delta f} = \sum_{n=0}^{\infty} \left( \frac{\partial F}{\partial X^n} \right)_{i_1...i_n} \left( \frac{\delta X^n}{\delta f} \right)^{i_1...i_n}$$

and definition of $\mathcal{J}$ into the Vlasov structure, which may then be written as

$$\{F,G\}[f] = \int f(z) \left\{ \frac{\delta F}{\delta f}, \frac{\delta G}{\delta f} \right\} \, d^N z = \int f(z) \left[ \mathcal{J} \left( \frac{\partial}{\partial z} \frac{\delta F}{\delta f} \otimes \frac{\partial}{\partial z} \frac{\delta G}{\delta f} \right) \right] \, d^N z$$

where $\mathcal{J}$ is a non-degenerate two form. That is, $\mathcal{J}$ is a $N \times N$ antisymmetric matrix of maximal rank. Since equation (14) implies

$$\frac{\partial}{\partial z} \frac{\delta F}{\delta f} = \sum_n \frac{\partial F}{\partial X^n} \mathcal{J} z^{n-1},$$

it follows that the moment Poisson structure is

$$\{F,G\}(X) = \sum_{n,m=0}^{\infty} X^{n+m-2} \left[ \frac{\partial F}{\partial X^n}, \frac{\partial G}{\partial X^m} \right]$$

where

$$\left[ \frac{\partial F}{\partial X^n}, \frac{\partial G}{\partial X^m} \right] := n \mathcal{S} \left( \frac{\partial F}{\partial X^n} \cdot \mathcal{J} \cdot \frac{\partial G}{\partial X^m} \right)$$

is the moment Lie bracket, in which again $\mathcal{S}$ operates to take the symmetric part of its argument. Recall that $\mathcal{J}$ is considered as a contravariant antisymmetric matrix, i.e. it possesses upper indexes $\mathcal{J}^{ij} = -\mathcal{J}^{ji}$.

Thus, again, the isomorphism between symmetric tensors and polynomials produces the momentum map associated with the moments [ScWe1994]. In turn, this means that the Lie-Poisson bracket for statistical moments is inherited from the Vlasov Lie-Poisson structure. In contrast to the Schouten concomitant (7) for kinetic moments, the Lie bracket for statistical moments in (17) still
involves the symplectic matrix $J$ (Poisson tensor). Thus, the dynamics of the statistical moments depends explicitly on the original symplectic structure. This allows, for example, the direct construction of moment invariants (Casimirs) as presented in [HoLySc1990]. Also, the moment algebra $\mathcal{M}$ involves symmetric tensors that are covariant, rather than contravariant as happens for the Schouten concomitant.

**Remark 2.3 (Statistical moments and accelerator beam optics)**

Statistical moments are important, for example, in the study of beam dynamics in particle accelerators. In this framework, they are defined as [Ch1983]

$$M_n^b(t) := \int \int p^n q^n f(q, p, t) \, dp \, dq$$

where $e_k$ is a basis element of the configuration vector space $Q$, while $e^k$ is its dual (so that $V = Q \times Q^*$). In 1D, the beam emittance

$$\epsilon := (M_0^2 \mathcal{M}_2 - (\mathcal{M}_1)^2)^{1/2},$$

known as the Courant-Snyder invariant [CoSn1958], was recognized as a moment invariant (Casimir). This observation led to the study of more general moment invariants [Dr1990, HoLySc1990]. A geometric investigation of statistical moments was carried out in [ScWe1994], where the moment algebra was related to the Heisenberg algebra on phase space. For particle accelerator design, moments are often used in computational efforts to account for space charge effects and other beam-related problems.

As for the kinematic moments, one characterizes the Lie algebra of statistical moments by using the grading,

$$\mathfrak{g} := \bigoplus_{n=0}^{\infty} \left( \bigvee_{i=0}^{n} V^i \right) \cong \bigoplus_{i=0}^{\infty} \mathfrak{g}_i \quad (18)$$

with the filtration

$$[\mathfrak{g}_n, \mathfrak{g}_m] \subseteq \mathfrak{g}_{n+m-2} \quad (19)$$

which shows how symmetric matrices $\mathfrak{g}_2 = V^* \vee V^* = \text{Sym}^*(N)$ form a particular subalgebra (here we denote by $\text{Sym}^*(N)$ covariant symmetric matrices). The largest subalgebra is given by $\mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_2 = \mathbb{R} \oplus V^* \oplus \text{Sym}^*(N)$ and it will play a central role in the remainder of this paper.

**Remark 2.4 (Occupation number representation)**

Analogously to the Lie algebra of kinetic moments, the statistical moments also carry a bosonic Fock space structure, where appropriate occupation numbers may be defined through the introduction of the multi-index notation $z^n := (z^1)^{n_1} \cdots (z^N)^{n^N}$. Indeed, taking moments by $X^n(t) = \int z^n f(z, t) \, d^N z$ yields the occupation number representation for statistical Vlasov moments.
The moment equations are written in terms of the Lie algebra coadjoint operator $\text{ad}^*$ as

$$
\frac{dX_m}{dt} = -\sum_{n=0}^{\infty} \text{ad}^*_n X^{n+m-2}
$$

$$
= - m \sum_{n=0}^{\infty} n S \left( \left( \frac{\partial H}{\partial X^n} \right) \mathcal{J} X^{m+n-2} \right)
$$

so that

$$
\left( \dot{X}^m \right)_{i_1 \ldots i_m} = - m \sum_{n=0}^{\infty} n S \left( \left( \frac{\partial H}{\partial X^n} \right) \mathcal{J}^{k i_m} \left( X^{m+n-2} \right)_{i_1 \ldots i_{m-1} j_1 \ldots j_{n-1}} \right)
$$

in which repeated tensorial indexes are summed. Having established the notation for the various operations among moments, we turn next to their application in geodesic Vlasov flows.

### 2.3 Truncation of moment hierarchies

The truncation of Vlasov moment hierarchies is a typical problem in kinetic theory and, for statistical moments, this question was addressed by Channell [Ch1995] by using Levi’s decomposition theorem. Channell’s result may be summarized by saying that, for a moment Hamiltonian not depending on the first-order moment ($\partial H / \partial X^1 = 0$), the moment hierarchy can be truncated at any order, thereby yielding a truncated Lie-Poisson system. In order to see how this works in practice, let us write the truncated equations at the order $K$ for the hierarchy of statistical moments $\dot{X}^n = \text{ad}^*_{n m} X^{m+n-2}$, where $h_m = \partial H / \partial X^m$. We have

$$
\begin{align*}
\dot{X}^1 &= \text{ad}^*_{h_2} X^1 + \text{ad}^*_{h_3} X^2 + \cdots + \text{ad}^*_{h_K} X^{K-1} \\
\dot{X}^2 &= \text{ad}^*_{h_2} X^2 + \text{ad}^*_{h_3} X^3 + \cdots + \text{ad}^*_{h_K} X^K \\
\dot{X}^3 &= \text{ad}^*_{h_2} X^3 + \cdots + \text{ad}^*_{h_{K-1}} X^K \\
&\vdots \\
\dot{X}^{K-1} &= \text{ad}^*_{h_2} X^{K-1} + \text{ad}^*_{h_3} X^K \\
\dot{X}^K &= \text{ad}^*_{h_2} X^K
\end{align*}
$$

and we recognize that the equation for $X^1$ is decoupled, so we restrict to the equations for $X^2, \ldots, X^K$. At this point, one verifies that the Lie Poisson bracket for the truncated moment system is given by

$$
\{F, G\}(X) = \sum_{n=2}^{K} \sum_{m=2}^{K-n+2} \left< X^{n+m-2}, \left[ \frac{\partial F}{\partial X^n}, \frac{\partial G}{\partial X^m} \right] \right>
$$

(22)
with the same notation as in (17). We recognize that the truncated structure is completely determined by the Lie algebra filtration (19) and does not depend on the particular expression of the Lie bracket, which was not used in deriving the truncated system above.

Following similar arguments, one can write the truncated system for kinetic moments as

\[
\begin{align*}
\partial_t A_0 &= -\text{ad}_{h_{1}}^{*} A_0 - \text{ad}_{h_{2}}^{*} A_1 - \cdots - \text{ad}_{h_{K}}^{*} A_{K-1} \\
\partial_t A_1 &= -\text{ad}_{h_{1}}^{*} A_1 - \text{ad}_{h_{2}}^{*} A_2 - \cdots - \text{ad}_{h_{K}}^{*} A_{K} \\
\partial_t A_2 &= -\text{ad}_{h_{1}}^{*} A_2 - \cdots - \text{ad}_{h_{K-1}}^{*} A_{K} \\
\vdots \\
\partial_t A_{K-1} &= -\text{ad}_{h_{1}}^{*} A_{K-1} - \text{ad}_{h_{2}}^{*} A_{K} \\
\partial_t A_{K} &= -\text{ad}_{h_{1}}^{*} A_{K}
\end{align*}
\]

(23)

where \( h_{m} = \frac{\delta H}{\delta A_{m}} \) and we have assumed \( h_{0} \equiv 0 \). The zero-th moment equation decouples and one is left with the remaining equations for \( A_1, \ldots, A_K \).

These equations possess the following bracket structure:

\[
\{F, G\}(A) = -\sum_{n=1}^{K} \sum_{m=1}^{K-n+1} \left\langle A_{n+m-1}, \left[ \frac{\delta F}{\delta A_{n}}, \frac{\delta G}{\delta A_{m}} \right] \right\rangle
\]

(24)

with the same notation as in (17). One recognizes again that the truncated structure is uniquely determined by the Lie algebra filtration (9). This fact suggests that a similar approach would also apply to the BBGKY moments of the Liouville equation (reduced probability distributions), whose corresponding Lie algebra is known to possess a similar filtration [MaMoWe84]. However, we leave the investigation of the BBGKY moment hierarchy for another time.

3 The geodesic Vlasov equation

We have seen that moment hierarchies are equivalent descriptions of the Vlasov equation, which allow for geometric closures of the kinetic system (e.g. the ideal fluid closure for kinetic moments). The Vlasov equation is a Lie-Poisson equation on the Lie algebra of the group \( \text{Can}(T^{*}Q) \) of canonical transformations, and this property is reflected in the Lie-Poisson structure of moment dynamics. It is well known that physical systems with interesting geometric behavior are often geodesic flows on Lie groups with respect to a metric provided by the system’s kinetic energy. The most familiar example is probably rigid body motion, which is governed by geodesic motion on \( \text{SO}(3) \). Likewise, Euler’s equations for ideal fluids may be interpreted as geodesic motion on the volume-preserving diffeomorphisms \( \text{Diff}_{\text{vol}}(\mathbb{R}^{3}) \) of the 3D flow domain \( \mathbb{R}^{3} \) [Ar1966]. Another interesting example of geodesic motion is provided by the EPDiff equation [HoMaRa1998], which governs geodesic motion on the full diffeomorphism group \( \text{Diff}(\mathbb{R}^{n}) \). In
many cases, geodesic flows on Lie groups turn out to be completely integrable Hamiltonian systems. For example, in a one-dimensional flow domain $\mathbb{R}$, EPDiff recovers the Camassa-Holm equation for shallow water waves \cite{CaHo1993}.

The problem of geodesic flow on the symplectic group (symplecto-hydrodynamics) was introduced by Arnold and Khesin in \cite{ArKe1998}. The present paper pursues this idea by considering geodesic flow on the canonical transformations within the context of Vlasov dynamics. In particular, we develop a **geodesic Vlasov equation** called EPCan (Euler-Poincaré equation on the canonical transformations) as an extension of previous work in \cite{GiHoTr2005}.

**Remark 3.1 (Symplectomorphisms vs canonical transformations)**

The name EPCan refers to the Euler-Poincaré equation for geodesic motion on the subgroup of the symplectic transformations arising from Hamiltonian vector fields. The subgroup Can (which could equally well be called DiffCan) may be identified as the group of smooth invertible canonical transformations with smooth inverses. These transformations coincide with symplectic transformations in simple domains such as $\mathbb{R}^{2K}$. In those simpler cases, the geodesic Vlasov equation is also known as EPSymp \cite{GiHoTr2005, GiHoTr2007}.

As mentioned in the introduction, the idea to investigate EPCan was motivated by the observation that geodesic equations for kinetic moments were found to include the Camassa-Holm equation \cite{GiHoTr2005}. In fact, geodesic moment equations arise from EPCan whenever the norm may be expanded as a Taylor series. Later, it was recognized \cite{GiHoTr2007} that the geodesic moment equations also recover the two-component Camassa-Holm equation \cite{ChLiZh2005}. The latter is a geodesic flow on a semidirect-product Lie group \cite{Kuzmin2007}. In order to explain these issues, we shall introduce the EPCan equation and show how it specializes to each integrable case.

Given a symplectic manifold $\mathcal{P}$ of even dimension $N = 2K$, the EPCan Vlasov Hamiltonian is defined by

$$H[f] = \frac{1}{2} \iint f(z) G(z, z') f(z') \, d^Nz \, d^Nz' = \frac{1}{2} \|f\|_G^2$$

where $z \in \mathcal{P}$ and the kernel $G$ is chosen so that it defines an appropriate norm on $\text{Den} \mathcal{P}$. When dealing with moments, we shall restrict to the special cases $\mathcal{P} = T^*Q$, with $Q$ a general configuration manifold, and $\mathcal{P} = V$, with $V$ a symplectic vector space. The geodesic Vlasov equation (aka EPCan) is written simply as

$$\frac{\partial f}{\partial t} = -\left\{f, \frac{\delta H}{\delta f}\right\} = -\left\{f, G * f\right\}$$

which coincides with Euler’s vorticity equation in 2D when $G = (-\Delta)^{-1}$.

### 3.1 Euler-Poincaré equations on Hamiltonian vector fields

In order to understand how the geodesic Vlasov equation arises from an Euler-Poincaré approach, one starts with an invariant Lagrangian defined on the tangent space of the canonical transformations $\mathcal{L} : TCan \to \mathbb{R}$, which is purely
quadratic. By the invariance property, we can write the associated variational principle on the Lie algebra of Hamiltonian vector fields as follows

$$\delta \int_{t_0}^{t_1} L[X_h] \, dt = 0$$

(27)

where $X_h = J \nabla h$ and

$$L[X_h] := \frac{1}{2} \langle \hat{Q} X_h, X_h \rangle = \frac{1}{2} \langle \hat{Q} J \nabla h, J \nabla h \rangle = \frac{1}{2} \langle \text{div}(J \hat{Q} J \nabla h), h \rangle =: L[h],$$

(28)

in which $\hat{Q}$ is taken to be a positive-definite symmetric operator so that $L[X_h]$ defines a nondegenerate norm. The Legendre transform

$$f = \frac{\delta L}{\delta h} = \text{div}(J \hat{Q} J \nabla h) \Rightarrow h = \left(\text{div} J \hat{Q} J \nabla \right)^{-1} f$$

(29)

yields the EPCan Hamiltonian in the Vlasov form

$$H[f] = \frac{1}{2} \langle f, \hat{O}^{-1} f \rangle$$

(30)

with

$$\hat{O} := \text{div} J \hat{Q} J \nabla .$$

(31)

This formula specifies the relation between the geodesic Vlasov equation and the geodesic motion on the Hamiltonian vector fields. An interesting case occurs when $\hat{Q}$ is the ‘flat’ operation $\hat{Q} X_h = X^k_h$ which takes contravariant vectors to covariant vectors, so that

$$\text{div} J (J \nabla h)^k = - \Delta h.$$  

(32)

Then the operator $\hat{O}$ reduces to minus the Laplacian

$$\hat{O} = - \Delta$$

(33)

and in two dimensions one obtains the Euler Hamiltonian for vorticity dynamics,

$$H[\omega] = 1/2 \langle \omega, (-\Delta)^{-1} \omega \rangle ,$$

with $\omega = f$. In the more general case when $\hat{Q}$ is a purely differential operator, one finds that $\hat{Q}$ and $J$ commute and thus $\hat{O} = - \text{div} \hat{Q} \nabla$. Also if $\hat{Q}$ commutes with the divergence, then, one has $\hat{O} = - \hat{Q} \Delta$. However in general, $\hat{Q}$ is a matrix differential operator that does not commute with $J$. 

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3.2 The ideal fluid dual pair for the Vlasov equation

Both the Poisson-Vlasov system (4) in plasma physics and the geodesic Vlasov equation (25) possess the single-particle solution (Klimontovich solution) expressed as a delta function in phase space,

\[ f(z, t) = \sum_{a=1}^{N} w_a \delta(z - \zeta_a(t)), \quad (34) \]

where the index \( a \) is summed over \( a = 1, \ldots, N \) and the number \( w_a \) is a constant weight associated to each particle. This apparently trivial solution is of fundamental importance in physics and leads to the Klimontovich theory of kinetic equations [Kl1982]. In the remainder of this paper, we shall see how such a solution emerges within the analysis of integrable moment closures of EPCan.

In particular, an interesting situation occurs when one allows for more general solutions of the form

\[ f(z, t) = \sum_{a=1}^{N} \int w_a(s) \delta(z - \zeta_a(s, t)) \, d^k s \quad (35) \]

where \( s \) is a coordinate on the immersed submanifold \( S_a \subset \mathbb{R}^{2K} \) with \( k = \dim(S) \) (e.g. \( k = 1 \) for a curve, or \( k = 2 \) for a surface). For simplicity of notation, we have already suppressed the implied subscript \( a \) in the arclength \( s \) for each \( w_a \) and \( \zeta_a \). For further simplicity and without loss of generality, we take \( N = 1 \) and so suppress the index \( a \) in what follows. This is equivalent to treating an isolated singular solution. As one might expect, this is only a notational simplification; not a real restriction.

The \( \zeta \)'s in (35) belong to the space of embeddings \( \text{Emb}(S, \mathbb{R}^{2K}) \). That is, \( \zeta : S \hookrightarrow \mathbb{R}^{2K} \). Remarkably, these solutions define a momentum map\[ J_{\text{Sing}} : \text{Emb}(S, \mathbb{R}^{2K}) \rightarrow X^*_\text{can}(\mathbb{R}^{2K}) \quad (36) \]

where \( X^*_\text{can}(\mathbb{R}^{2K}) \simeq \text{Den}(\mathbb{R}^{2K}) \). This momentum map is produced by the left action of canonical transformations on \( P \) by composition of functions; that is,

\[ \eta : \zeta = \eta \circ \zeta. \]

Importantly, the same kind of momentum map arises in the motion of ideal fluids (e.g., for point vortices in 2D). See [MaWe1983], where these momentum maps are shown to possess a dual pair structure [We1983]. Namely, if one considers \( S \) as a manifold with volume form

\[ \omega_{\text{vol}} = w(s) \, d^k s, \]

3Here \( X_{\text{can}} \) denotes the Lie algebra of Hamiltonian vector fields, which should not be confused with the larger Lie algebra corresponding to the tangent space at the identity of the symplectomorphism group \( \text{Symp} \).
then the right action of Diff_{vol} on Emb(S, \mathbb{R}^{2K})
\[ \zeta \cdot \eta = \zeta \circ \eta , \]
yields another momentum map
\[ J_S : \text{Emb}(S, \mathbb{R}^{2K}) \rightarrow \mathfrak{x}_{\text{vol}}^*(S). \] (37)

In more generality, if (S, w) is a volume manifold and (P, \omega) is a symplectic manifold, then the right action momentum map is (cf. [MaWe1983])
\[ J_S : \zeta \mapsto \zeta^* \omega : \text{Emb}(S, P) \rightarrow \mathfrak{x}_{\text{vol}}^*(S). \] (38)

To summarize, we have the following dual pair structure

\[
\begin{array}{c}
\text{Emb}(S, P) \\
\downarrow J_{\text{Sing}} \\
\mathfrak{x}_{\text{can}}^*(P) \\
\downarrow J_S \\
\mathfrak{x}_{\text{vol}}^*(S)
\end{array}
\]

which is formally equivalent to the dual pair structure for ideal fluids [MaWe1983]. Moreover, the left leg yields a solution of the Vlasov equation regardless the number of dimensions and this makes the above dual pair a natural object in kinetic theory.

In order to write explicit formulas, we specialize to the case $P = \mathbb{R}^{2K}$. Upon denoting $\zeta(s) = (Q(s), P(s))$, one defines the following Poisson structure on $\text{Emb}(S, P)$
\[
\{ F, G \}_\text{Emb} = \sum_{i=1}^{K} \int w(s) \left( \frac{\delta F}{\delta Q^i} \frac{\delta G}{\delta P_i} - \frac{\delta G}{\delta Q^i} \frac{\delta F}{\delta P_i} \right) d^k s 
\]
where we see that the factor $1/w(s)$ is needed for functionals of the form $G(\zeta) = \int \omega_{\text{vol}} g(\zeta) = \int w(s) g(\zeta(s)) d^k s$, whose functional derivative $\delta G/\delta \zeta = w(s) dg/d\zeta$ takes values in $\text{Den}(S)$.

Remark 3.2 (Comparison with point vortices)
The factor $1/w(s)$ is reminiscent of the vortex strength factors in the Poisson bracket for point vortices [MaWe1983]. Indeed, the bracket above appears as the higher dimensional version of the vortex bracket, so that the one dimensional vortex strengths are replaced by appropriate densities (the weights $w(s)$) on the embedded space $S$.

Finally one checks that, for any Hamiltonian function $h \in \mathcal{F}(T^*\mathbb{R}^{2K})$,
\[
\{ F, \langle J_{\text{Sing}}, h \rangle \}_\text{Emb} = X_h[F] 
\]
(40)
where \( X_h[F] \) is the infinitesimal generator of the action of canonical transformations \( \text{Can}(\mathbb{R}^{2k}) \) on \( \text{Emb}(S, \mathbb{R}^{2k}) \). Thus, \( J_{\text{Sing}} \) satisfies the classical definition of a momentum map.

The singular solution momentum map \( J_{\text{Sing}} \) produces the collective Vlasov Hamiltonian \( H \circ J_{\text{Sing}} \). In particular, substituting the singular solution momentum map \( J_{\text{Sing}} \) into the EPCan Hamiltonian \( \mathcal{H} \) yields the collective Hamiltonian

\[
H_N = \frac{1}{2} \sum_{a,b=1}^{N} \int w_a(s) w_b(s') \mathcal{G}(Q_a(s), P_a(s), Q_b(s'), P_b(s')) \, dk s \, dk' \quad (41)
\]

thereby producing the following collective equations of motion

\[
\frac{\partial Q_a(s,t)}{\partial t} = \frac{\delta H_N}{\delta \dot{P}_a} = w_a(s) \sum_{b=1}^{N} \int w_b(s') \frac{\partial}{\partial P_a} \mathcal{G}(Q_a(s), P_a(s), Q_b(s'), P_b(s')) \, dk' \quad (42)
\]

\[
\frac{\partial P_a(s,t)}{\partial t} = -w_a(s) \sum_{b=1}^{N} \int w_b(s') \frac{\partial}{\partial Q_a} \mathcal{G}(Q_a(s), P_a(s), Q_b(s'), P_b(s')) \, dk' \quad (43)
\]

**Remark 3.3 (Possible divergent terms in collective motion)**

The collective dynamics of singular solutions deserves some care, depending on the form of the Vlasov Hamiltonian. The existence of such solutions does not guarantee the existence of a well defined collective Vlasov Hamiltonian \( H \circ J_{\text{Sing}} \). This is because the singular solution momentum map \( J_{\text{Sing}} \) may produce divergent terms in the collective Hamiltonian \( H_N \). For example, this is the case of the Vlasov Poisson system \( \mathcal{H} \), where the divergence is generated by potential terms such as \( 1/2 \sum w_a w_b \lvert Q_a - Q_b \rvert^{-1} \), when \( a = b \). The same situation occurs for point vortex solutions of the planar Euler’s vorticity equation; these solutions correspond to the 2D phase space Hamiltonian \( H_N = 1/2 \sum w_a w_b \log \lvert (Q_a - Q_b, P_a - P_b) \rvert \). On the other hand, these problems are absent, for instance, in the Vlasov-Helmholtz system (see \[\text{GiHoTr2008}\] and references therein), since the potential terms are given by \( 1/2 \sum w_a w_b e^{\lvert Q_a - Q_b \rvert} \).

As for the right-action momentum map, the expression

\[
J_S(\zeta) = \zeta^* \omega = \zeta^* (dq \wedge dp) = \zeta^* dq \wedge \zeta^* dp = d(\zeta^* q) \wedge d(\zeta^* p) \quad \quad (42)
\]

yields the following simple expression

\[
J_S(Q, P) = dq(s) \wedge dp(s) = \sum_{n,m=1}^{k} \frac{\partial Q}{\partial s^n} \frac{\partial P}{\partial s^m} ds^n \wedge ds^m \quad \quad (43)
\]

The conservation law \( dJ_S/dt = 0 \) is recovered by Noether’s theorem, due to the \( \text{Diff}(S) \)-invariance of the collective Vlasov Hamiltonian \( H_N = H \circ J_{\text{Sing}} \) in \( [11] \).

When \( S \) is a Lagrangian submanifold (this requires \( \text{dim}(S) = 1/2 \text{dim}(P) \)), the momentum map \( J_S \) restricts to \( J_S(\zeta) = 0 \). Likewise, the case \( \text{dim}(S) = 0 \)}
recovers the usual Klimontovich solution (34) of particle motion used in kinetic theory. This fact, together with the geometric results on moment hierarchies of kinetic equations, illustrates the geometric basis of kinetic theory, in analogy to Arnold’s formulation of the ideal fluid [Ar1966].

3.3 Klimontovich solution and the Lagrange-to-Euler map

This section discusses the two limiting cases of the singular solution momentum map $J_{\text{Sing}}$, that is $\dim(S) = 0$ and $S = P$. As mentioned above, the first case yields the Klimontovich solution (34), which is then a momentum map

$$J_{\text{Sing}} : \times_{a=1}^{N} P \to \mathcal{X}_{\text{can}}^{*}(P).$$

In this case, the solution identifies the particle trajectories, subject to initial conditions $\zeta_{a}(0) = z_{0}^{a}$, so that the particles are transported in the phase space $P$ by canonical transformations as $\{\zeta_{a}(t)\} = \psi_{t} \circ \{z_{0}^{a}\}$, where $\psi_{t} \in \text{Can}(\times_{a} P)$ is generated by the collective Hamiltonian $H \circ J_{\text{Sing}}$. In two phase-space dimensions, the Klimontovich solution is the usual point vortex solution of the Euler’s vorticity equation.

Another suggestive case of the above treatment is given by $S = P$, so we may denote $s = z^{0}$. Then, one has $\zeta_{a}(\cdot, t) = \eta^{(a)}_{t} \in \text{Can}(P)$ and the momentum map $J_{\text{Sing}} : \times_{a} \text{Can}(P) \to \mathcal{X}_{\text{can}}^{*}(P)$ is written as

$$f(z, t) = \sum_{a} \int w_{a}(z^{0}) \delta(z - \eta^{(a)}_{t} \cdot z^{0}) \, d^{2}K_{z^{0}}. \quad (44)$$

This expression coincides with the well known Lagrange-to-Euler map for fluids, whose importance is well established in continuum dynamics. The Lagrange-to-Euler map is equivalent to the characteristic form of the Vlasov equation (1). Notice that both the Klimontovich and the Lagrange-to-Euler maps are produced by the same Lie group $\text{Can}(P)$ acting on $\times_{a} P$ and $\times_{a} \text{Can}(P)$ respectively, with the same left action by composition of functions, that is $\eta \cdot \{\zeta_{a}\} = \{\eta \circ \zeta_{a}\}$ in the first case and $\eta \cdot \{\eta^{(a)}\} = \{\eta \circ \eta^{(a)}\}$ in the second. On the other hand, for the Klimontovich case, the collective dynamics generated by the Hamiltonian $H \circ J_{\text{Sing}} : \times_{a} P \to \mathbb{R}$ produces the canonical transformations $\psi \in \text{Can}(\times_{a} P) \neq \times_{a} \text{Can}(P)$. This point is of fundamental importance because the Lie group $\text{Can}(\times_{a} P)$ is the symmetry group of the Liouville equation [MaMoWe84] and the difference between $\text{Can}(\times_{a} P)$ and $\times_{a} \text{Can}(P)$ is related to the particle correlations, which are neglected in the second situation. (The latter is the Vlasov mean field approximation.)

The fact that these two fundamental maps each arise from the left leg of a dual pair of momentum maps again illuminates the geometric footing of kinetic theory. The above arguments also provide mathematical support for the wide success of Klimontovich method in kinetic equations [Kl1982].

3.4 Geometric kinetic theory

The presence of dual pairs in kinetic theory illuminates the Liouville and Vlasov equations in the light of their Lie symmetry properties. (Something similar
happens for Euler’s vorticity equation.) Namely, the presence of momentum maps is not accidental in kinetic approaches. Indeed, a reasonable summary of the results in [MaMoWe84, HoLySc1990, GiHoTr2008] could be made by saying that the process

\[
\begin{array}{ccc}
\text{Liouville equation} & \rightarrow & \text{Vlasov equation} \\
\uparrow \text{ideal fluid} & & \downarrow \text{beam optics}
\end{array}
\]

is given by a composition of momentum maps. In other words, taking the moments (BBGKY, kinetic or statistical) of a Lie-Poisson kinetic equation is always a momentum map [MaMoWe84, HoLySc1990, GiHoTr2008]. Moreover, the closures adopted to obtain Vlasov from BBGKY, fluid theory from kinetic moments and beam optics from statistical moments are also momentum maps arising from particular subgroups of the symmetry group of the starting system. More explicitly, passing from Liouville to Vlasov requires the subgroup \( \text{Can}(\mathcal{P}) \subset \text{Can}(\mathfrak{x},\mathcal{P}) \). Likewise, passing from Vlasov to fluid requires the fiber preserving subgroup \( \text{Can}_e(T^*Q) \subset \text{Can}(T^*Q) \). Finally, passing from Vlasov to beam optics requires the subgroup \( \text{Sp}(2K,\mathbb{R}) \subset \text{Can}(\mathbb{R}^{2K}) \). We can summarize the situation in the following statement

\[ \text{All these moment approximations in kinetic theory are momentum maps.} \]

That the BBGKY distributions are momentum maps is a remarkable fact. One may ask whether the Klimontovich averages in plasma theory also share this property. In this case, the autocorrelations considered in the latter approach would again be naturally included in the geometry of the theory. We leave this promising question open, as a direction for future research.

4 Moment closures of EP\( \text{Can} \): integrable cases

As explained in [GiHoTr2005, GiHoTr2007], the geodesic Vlasov equation may be represented in terms of the moments. Indeed, upon supposing that the metric \( \mathcal{G} \) in (25) and (26) is sufficiently smooth, may can expand \( \mathcal{G}(z,z') \) in a Taylor series.

4.1 Integrable closures of kinetic moments

In this section, we present the kinetic moment hierarchy for EP\( \text{Can} \). Upon denoting \( z = (q,p) \), one may expand \( \mathcal{G} \) in a Taylor series, as follows,

\[
\mathcal{G}(z,z') = \sum_{n,m=0}^{\infty} p^n \otimes p'^m \mathcal{G}_{nm}(q,q')
\]

\[
= \sum_n \sum_m \sum_{i_1,...,i_n,j_1,...,j_m} \left( p^n \right)_{i_1,...,i_n} \left( p'^m \right)_{j_1,...,j_m} \left( G_{nm}(q,q') \right)_{i_1,...,i_n,j_1,...,j_m}
\]
where $G_{nm}$ is now a contravariant tensor field of rank $n + m$. Inserting this Taylor expansion in the EPCan Hamiltonian yields (with the notation above)

$$H = \frac{1}{2} \sum_{n,m=0}^{\infty} \int A_n(q) \otimes A_m(q') \, J G_{nm}(q,q') \, d^K q \, d^K q' = \frac{1}{2} \|\{A_n\}\|_G \quad (46)$$

Thus, upon denoting

$$G_{nm} * A_m := \int G_{nm}(q,q') \, J A_m(q') \, d^K q' \quad (47)$$

the geodesic moment equations become

$$\frac{\partial A_n}{\partial t} = - \sum_{m,k=0}^{\infty} \text{ad}^*_G G_{mk} * A_k A_{n+m-1} \quad (48)$$

where $\text{ad}^*$ is the coadjoint Lie-Schouten operator. The singular solutions of the Vlasov moment hierarchy may be expressed in the following form:

$$A_n(q,t) = \int w(s) \, P^n(s,t) \, \delta(q - Q(s,t)) \, d^k s \quad (49)$$

which is a momentum map $J : \text{Emb}(S,T^*Q) \to g^*$, where $g$ is the kinetic moment algebra.

As we have seen in the preceding discussions, the moment algebra possesses the important subalgebra $g_1 = X(Q)$ of vector fields on the configuration manifold. In terms of canonical transformations, this corresponds to Hamiltonian generating functions that are linear in the momentum coordinate, i.e. point transformations. These are cotangent lifts $T^*\text{Diff}(Q)$ of diffeomorphisms on the configuration manifold $Q$ [HoMa2004, GiHoTr2007]. Remarkably, when the moment hierarchy of EPCan is closed such that $G_{11} := G_1$ is the only non-vanishing term of $G_{nm}$, we obtain the Hamiltonian on the one-form density $A_1 := m(q) \cdot dq \otimes d^K q \in X^*$

$$H = \frac{1}{2} \int m(q) \cdot G_1(q,q') \, m(q') \, d^K q \, d^K q' \quad (50)$$

By using the property of the Schouten bracket $[\beta_1, \alpha_n] = J \beta_1 \alpha_n$ (where $J$ denotes Lie derivative), one finds the EPDiff equation,

$$\frac{\partial m}{\partial t} + J G_1 * m = 0 . \quad (51)$$

EPDiff is the Euler-Poincaré equation on the diffeomorphisms [HoMa2004]. This equation has the important property of exhibiting emergent singular $\delta$-like solutions from any confined smooth initial configuration. In 1D, the particular case $G_1 = (1 - \alpha_2 \partial^2)^{-1}$ yields the integrable Camassa-Holm equation, which is well known in the community of integrable systems.

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Remarkably, if we also allow for $G_0 =: \neq G_0 \neq 0$, we obtain the Hamiltonian

$$H[A_0, A_1] = \frac{1}{2} \int \int A_1(q) \cdot G_1(q, q') A_1(q') \, d^Kq \, d^Kq'$$

$$+ \frac{1}{2} \int \int A_0(q) \cdot G_0(q, q') A_0(q') \, d^Kq \, d^Kq'$$

which yields a geodesic flow on the semidirect-product Lie group Diff$\circ F$, introduced in [HoTrYo2007] in the context of image matching, discussed further in [HoTr2009] and shown in numerical simulations to exhibit emergent singularities in both of its variables [HoOnTr2009]. Interestingly enough, the EP(Diff$\circ F$) equations (with notation $\beta_0 \diamond A_0 = - \text{ad}_{\beta_0}^* A_0$)

$$A_{0,t} + \mathcal{L}_{G_1 \ast A_1} A_0 = 0,$$

$$A_{1,t} + \mathcal{L}_{G_1 \ast A_1} A_1 = A_0 \diamond (G_0 \ast A_0).$$

are a geodesic flow on the extended point transformations, i.e., compositions of cotangent lifts and fiber translations [MaWeRaScSp1983].

**Remark 4.1 (Geodesic motion on fiber-preserving transformations)**

The semidirect-product Lie group Diff$\circ F$ is identified with the compositions of cotangent lifts with fiber translations on the phase space $T^*Q$ with coordinates $(q, p)$. This identification is of fundamental importance in plasma physics [MaWeRaScSp1983]. It also yields the interpretation of EP(Diff$\circ F$) as a geodesic motion on the Lie group Can$\pi(T^*Q)$ of fiber-preserving canonical transformations on the cotangent bundle $T^*Q$. In fact, any transformation by Can$\pi(T^*Q)$ can be realized as the composition of a fiber translation and a cotangent lift (or vice versa) [BaWe1997]. On the other hand, such a transformation is always a canonical transformation characterized by a generating function that is linear (and inhomogeneous) in the canonical momentum. Therefore, since Diff$(Q) \circ F(Q) \simeq \text{Can}_\pi(T^*Q)$ then \text{EP(Diff$\circ F$)} \simeq \text{EPCan}_\pi.

In 1D, the special case $G_0 = \delta$-function yields yet another integrable system,

$$\lambda_t = - (u\lambda)_q$$

$$u_t - u_{qqq} = -3uu_q + 2u_q u_{qq} + uu_{qqq} - \lambda_q$$

known as two-component Camassa-Holm equation [ChLiZh2005] (here $u = (1 - \partial^2)^{-1} A_1$ and $\lambda = A_0$). This system first appeared in [OlRo1996]. Its singular solutions were studied in [Colv2008] who pointed out their relation between this system and shallow water equations. Upon slightly modifying the Hamiltonian by $G_0 = (1 - \partial^2)^{-1}$, one has the spontaneous emergence of the Klimontovich solution [49] in $(A_0, A_1)$, as shown in [HoOnTr2009]. The two-component Camassa-Holm system and its Klimontovich solutions have also found applications in the metamorphosis approach to image matching, e.g., for magnetic resonance images [HoTrYo2007].
4.2 Geodesic flow on statistical moments

Section 5 shows that representing the geodesic Vlasov equation in terms of statistical moments in a special case recovers the well-known Bloch-Iserles integrable system [BlIsMaRa2005, BlIs2006]. If $G$ in (25) and (26) is assumed to be analytic in both position and momentum, its Taylor expansion may be written as:

$$G(z, z') = \sum_{n,m=0}^{\infty} \Gamma_{nm} z^n \otimes z'^m$$  \hspace{1cm} (52)

where $\Gamma_{nm}$ is a covariant tensor of rank $n + m$. Then, the EPCan Hamiltonian (25) is written in terms of the moments $X^n(t) = \int z^n f(z, t) \, d^Nz$ as

$$H = \frac{1}{2} \sum_{n,m=0}^{\infty} \Gamma_{nm} X^n \otimes X^m$$  \hspace{1cm} (53)

so that the moment equations for $(n, m) \in \mathbb{Z}_+$ turn out to be

$$\dot{X}^m = - m \sum_{n=0}^{\infty} n S \left( \left( \Gamma_{nk} X^k \right) \cdot \underline{J} \right) \cdot \underline{J} X^{m+n-2}$$  \hspace{1cm} (54)

where, as before, the symbol $\cdot$ denotes contraction between upper and lower indices of the various tensors. In explicit index notation this reads as

$$(\dot{X}^m)^{i_1 \cdots i_m} = - m \sum_{n=0}^{\infty} n S \left( \Gamma_{j_1 \cdots j_n i_1 \cdots i_k} (X^k)^{i_1 \cdots i_k} \underline{J} \right) X^{m+n-2}^{i_1 \cdots i_{m-1} j_1 \cdots j_{n-1}}.$$  \hspace{1cm} (55)

An example of this equation is discussed in Section 5 for the case $m = n = 2$.

4.3 Klimontovich dynamics for statistical moments

As in the case of kinetic moments, the single-particle Klimontovich solution

$$f(z, t) = \sum_a w_a \delta(z - \zeta_a(t))$$  \hspace{1cm} (55)

of the geodesic Vlasov equation offers an interesting opportunity for presenting solutions of the moment hierarchy. These solutions may be represented as

$$X^n(t) = \sum_a w_a \zeta_a^n(t)$$  \hspace{1cm} (56)

where $\zeta_a^n := \zeta_a \otimes \cdots \otimes \zeta_a$ (n times) and $\zeta_a$ satisfies Hamilton’s canonical equations with Hamiltonian

$$H_N = \frac{1}{2} \sum_{a,b} \sum_{n,m} w_a w_b \Gamma_{nm} \underline{J} \cdot \underline{J} \zeta_a^n \otimes \zeta_b^m.$$  \hspace{1cm} (57)
This means that \( \zeta_a \) satisfies
\[
\dot{\zeta}_a = J \nabla \zeta_a H_N = J \sum_{nm} n w_a \sum_b w_b \Gamma_{nm} \mathcal{J} \zeta_a^{n-1} \otimes \zeta_b^m,
\tag{58}
\]
where the moment order \( m \) or \( n \) ranges from zero to infinity, and \( a, b \) range over the number of particles \( a, b = 1, \ldots, N \). Each \( \zeta_a \) undergoes Hamiltonian dynamics because the Vlasov single-particle solution is an equivariant momentum map. Equivariance of the corresponding momentum map for dynamics of point vortices in an Euler fluid was proven in \cite{MaWe1983}. The same argument holds for the singular solutions of kinetic moment equations, including the EPDiff equation \cite{HoMa2004}.

**Remark 4.2 (Truncation of geodesic moment hierarchies)**

*It is important to notice that higher-order moment truncations do not possess the Klimontovich solution. The latter exists only for genuine moment closures, such as the fluid closure for the kinetic moments (consisting of \( A_0, A_1 \)), or the 2\textsuperscript{nd}-order closure for statistical moments used in linear beam optics (consisting of \( X^1, X^2 \)).*

5 **Bloch-Iserles system as a moment equation**

The case \( (n, m) = (2, 2) \) of the moment equation \cite{51} yields an important moment subalgebra, given by **homogeneous quadratic polynomials on phase space**, i.e. quadratic forms on \( V \). In this case, formula \cite{17} implies the following Lie-Poisson moment bracket
\[
\{F, G\}(X^2) = 4 \left\langle X^2, S \left( \frac{\partial F}{\partial X^2} \mathcal{J} \frac{\partial G}{\partial X^2} \right) \right\rangle
\tag{59}
\]
where the moment \( X^2 \) is now a \( N \times N \) symmetric matrix. Because of the antisymmetry of \( \mathcal{J} \), the bracket above may be rewritten as
\[
\{F, G\}(X^2) = \left\langle X^2, \left[ \frac{\partial F}{\partial X^2}, \frac{\partial G}{\partial X^2} \right]_{2\mathcal{J}} \right\rangle
\tag{60}
\]
where
\[
\left[ \frac{\partial F}{\partial X^2}, \frac{\partial G}{\partial X^2} \right]_{2\mathcal{J}} := \frac{\partial F}{\partial X^2} 2\mathcal{J} \frac{\partial G}{\partial X^2} - \frac{\partial G}{\partial X^2} 2\mathcal{J} \frac{\partial F}{\partial X^2}
\tag{61}
\]
which is the Lie bracket for the integrable Bloch-Iserles (BI) system of equations introduced in \cite{BlIsMaRa2005, BlIs2006}. In that case, an antisymmetric matrix \( \mathcal{N} \) of any dimension defines the following Lie bracket on the space of symmetric matrices of the same dimension. In formulas one has
\[
[X, Y]_{\mathcal{N}} := XYN - YNX
\tag{62}
\]
where \( X \) and \( Y \) are symmetric matrices. The Bloch-Iserles system

\[
\dot{X} = [(X)^2, N] \tag{63}
\]

is Lie-Poisson on this Lie algebra, with Hamiltonian \( H_{BI} = \frac{1}{4} \text{Tr}(\dot{X}X) \). Here the notation \((X)^2\) denotes standard matrix multiplication of \( X \) by itself, in order to distinguish from second-order moments \( X^2 \). In addition, \( \dot{X} \) denotes the transpose of the matrix \( X \), so that \( \dot{X} = X \) when \( X \) is symmetric. One concludes the following.

**Theorem 5.1** In the even-dimensional case, the integrable Bloch-Iserles system is the Vlasov moment equation \( \text{[20]} \) obtained from the quadratic Hamiltonian

\[
H_{BI} = \frac{1}{2} \langle X^2, X^2 \rangle \tag{64}
\]

associated with the antisymmetric matrix \( N := 2J \).

Let us now look at the moment bracket for functions of \((X^0, X^1, X^2) \in g^*_0 \oplus g^*_1 \oplus g^*_2 \simeq \mathbb{R} \oplus V \oplus (V \vee V) \) \( \text{[65]} \) where \( \vee \) is again the symmetric tensor product defined in \( \text{[8]} \) and \( X^0 = \text{const} \) is naturally taken as the probability normalization. The moment bracket \( \text{[17]} \) becomes

\[
\{F, G\}(X) = X^0 \frac{\partial F}{\partial X^1} \cdot J \frac{\partial G}{\partial X^1} + X^1 \cdot \left( \frac{\partial F}{\partial X^2} 2J \frac{\partial G}{\partial X^1} - \frac{\partial G}{\partial X^2} 2J \frac{\partial F}{\partial X^1} \right)
\]

\[
+ \left( X^2, \left[ \frac{\partial F}{\partial X^2}, \frac{\partial G}{\partial X^2} \right]_{2J} \right) \tag{66}
\]

which is given by the direct sum of the canonical Poisson bracket on \( V \) in first term, plus the semidirect-product Lie-Poisson bracket on \( g^*_0 \oplus g^*_1 \simeq \text{Sym}(V) \) in the second and third terms. Thus, the specialization of the moment bracket \( \text{[17]} \) to \( \text{[66]} \) in this case defines a Lie-Poisson bracket on \( (\text{Sym}(V)) \oplus \mathbb{R} \).

We now turn our attention to the odd-dimensional BI system. In this system, one has a degenerate antisymmetric matrix \( \bar{N} \) of odd dimension \( n \) and rank \( K \).

Upon defining \( N \) as the non-degenerate minor of maximal dimension \( (2K) \), the degenerate matrix \( \bar{N} \)

\[
\bar{N} = \begin{bmatrix} \bar{N} & 0 \\ 0 & 0 \end{bmatrix}
\]

produces the Lie bracket \( \text{[62]} \) associated to the BI equation \( \text{[63]} \).

The odd-dimensional BI system is known to be a geodesic flow on the space \((\text{Sym}(2K) \oplus M_{2K \times d}) \oplus \text{Sym}(d)\) endowed with the Lie bracket (cf. equation (2.14) in \text{[BlIsMaRa2005]})

\[
[(S, A, B), (S', A', B')] :=
(S \bar{N} S' - S' \bar{N} S, S \bar{N} A' - S' \bar{N} A, \ iA \bar{N} A' - iA' \bar{N} A) \tag{67}
\]
where one denotes $d = n - 2K$, for any $S, S' \in \text{Sym}(2K)$, $A, A' \in M_{2K \times d}$, and $B, B' \in \text{Sym}(d)$.

We will show that the bracket (17) for the Vlasov moment system is Lie-Poisson on the dual to the Lie algebra $(\text{Sym}(2K) \otimes V) \oplus \mathbb{R}$. That is, we choose $d = 1$, which is the case of interest here. Proposition 2.5 in [BlIsMaRa2005] shows that the geodesic equations on $\text{Sym}(2K+1)$ are equivalent to the geodesic equations on $(\text{Sym}(2K) \otimes V) \oplus \mathbb{R}$, where $V$ is a $2K$-dimensional symplectic space carrying a non-degenerate symplectic structure $\tilde{N}/2$. This leads us to the identification of the Bloch-Iserles system with geodesic moment dynamics.

**Theorem 5.2** When $d = 1$, the odd-dimensional BI system is a Vlasov moment equation of the form (24) on the statistical moments. This equation is generated by the quadratic Hamiltonian

$$H(X) = \frac{1}{2} \langle X^2, X^2 \rangle + \frac{1}{2} X^1 \cdot X^1$$

yielding a geodesic moment flow. The corresponding BI Hamiltonian $H_{BI}(X) = 1/2 \text{Tr}(\{XX\})$ is written in terms of the symmetric $(2K+1)$-dimensional matrices of the form

$$X = \begin{bmatrix} X^2 & X^1 \\ X^1 & 4X^0 \end{bmatrix}$$

These matrices are endowed with the Bloch-Iserles Lie bracket (62), where the (degenerate) antisymmetric matrix $\bar{N}$ takes the form

$$\bar{N} = \begin{bmatrix} 2J & 0 \\ 0 & 0 \end{bmatrix}$$

and $J$ is the canonical symplectic matrix.

**Proof.** Upon denoting

$$\bar{N} = 2J, \quad S = \frac{\partial F}{\partial X^2}, \quad S' = \frac{\partial G}{\partial X^2}, \quad A = \frac{\partial F}{\partial X^1}, \quad A' = \frac{\partial G}{\partial X^1}, \quad (69)$$

one sees that the only difference between the Lie bracket in (67) and the bracket in (66) resides in a constant factor in the first term of (66)

$$\left( \frac{\partial F}{\partial X^1} \right) \cdot J \frac{\partial G}{\partial X^1} = \frac{1}{4} \left[ \left( \frac{\partial F}{\partial X^1} \right) \cdot 2J \frac{\partial G}{\partial X^1} - \left( \frac{\partial G}{\partial X^1} \right) \cdot 2J \frac{\partial F}{\partial X^1} \right]$$

where the square bracket in the right hand side is identical to the last component of (67). This difference however can be easily overcome. Indeed, one can always re-define the Lie bracket on $(\text{Sym}(2K) \otimes V) \oplus \mathbb{R}$ as

$$[(S, A, B), (S', A', B')] := \left( S\bar{N}S' - S'\bar{N}S, S\bar{N}A' - S'\bar{N}A, \frac{1}{4} \left( 'A \bar{N}A' - 'A' \bar{N}A \right) \right)$$

24
and verify that the map

\[ \Psi : (\text{Sym}(2K) \otimes V) \oplus \mathbb{R} \to \text{Sym}(2K + 1) \]

\[(S, A, B) \mapsto \begin{bmatrix} S & A \\ tA & 4B \end{bmatrix} \]

is a Lie algebra isomorphism for \( \text{Sym}(2K + 1) \), which is endowed with the BI Lie bracket \([X, Y]_N = XNY - YNX\). The isomorphism property is a direct verification identical to Proposition 2.5 in [BlIsMaRa2005]. In particular, let \((S, A, B), (S', A', B') \in (\text{Sym}(2K) \otimes V) \oplus \mathbb{R}\) and compute directly that

\[ \Psi ((S, A, B), (S', A', B')) = \Psi (S \bar{NI} S' - S' \bar{NI} S, S \bar{NI} A' - S' \bar{NI} A, 1/4 (tA \bar{NI} A' - tA' \bar{NI} A)) \]

\[ = \begin{bmatrix} S & A \\ tA & 4B \end{bmatrix} \begin{bmatrix} \bar{N} & 0 & S' & A' \\ 0 & \bar{N} & tA' & 4B' \end{bmatrix} - \begin{bmatrix} S' & A' \\ tA' & 4B' \end{bmatrix} \begin{bmatrix} \bar{N} & 0 & S & A \\ 0 & \bar{N} & tA & 4B \end{bmatrix} \]

\[ = [\Psi (S, A, B), \Psi (S', A', B')]_N \]

as required. \( \blacksquare \)

Thus, we conclude that the BI system and the geodesic moment equations are equivalent.

5.1 Klimontovich solutions of the Bloch-Iserles system

The Klimontovich map for the geodesic Vlasov equation provides simple solutions of the Bloch-Iserles system in any dimension. For example, in even dimensions, the dynamics of the BI solution

\[ X(t) = \sum_a w_a \zeta^2_a(t) \]  

(71)

is given by the system

\[ \dot{\zeta}_a = w_a \sum_b w_b \bar{N} \zeta_b^2 \zeta_a \]  

(72)

where \( \bar{N} = 2J \) and \( \zeta_b^2 \zeta_a \) is a covector such that \( (\zeta_b^2 \zeta_a)_i = (\zeta_b^2)(\zeta_a)^j \). In explicit index notation, one has

\[ (\zeta_a)^i = 2 w_a \sum_b w_b \bar{N}^i j (\zeta_b^2)_{jk} (\zeta_a)^k. \]

This system is a Hamiltonian system with the homogeneous quartic Hamiltonian

\[ H_N = \frac{1}{2} \sum_{a,b=1}^{N} w_a w_b \text{Tr} (tA \bar{NI} A') \zeta_b^2 \]  

(73)
Notice that, by writing the equation for $\zeta_a$ as

$$\dot{\zeta}_a = w_a \sum_{b \neq a} w_b \mathbb{N} \zeta_b^2 \zeta_a + w_a^2 \||\zeta_a\||^2 \mathbb{N} \zeta_a$$  \hspace{1cm} (74)$$

we can specialize the above to the simple case when $w_a = 1$ for a fixed $a$ and $w_b = 0 \forall b \neq a$, so that

$$X(t) = \zeta^2(t) \with \dot{\zeta} = \||\zeta||^2 \mathbb{N} \zeta$$  \hspace{1cm} (75)$$

This case leads however to trivially linear dynamics, since the norm $||\zeta||$ is evidently conserved. This does not happen for different norms in the quadratic moment Hamiltonian, such as $H = 1/2 (\Gamma_{22} \mathcal{J} X^2 \otimes X^2)$.

The above arguments also provide solutions to the Bloch-Iserles system in $2K + 1$ dimensions. Indeed, the particle solution of EPCan (55) becomes a solution of the BI system in the following form

$$X(t) = \sum_a w_a \begin{bmatrix} \zeta_a^2(t) \\ \zeta_a(t) \\ 4 \end{bmatrix} \in \text{Sym}(2K + 1).$$  \hspace{1cm} (76)$$

Here $\zeta_a$ undergoes Hamiltonian dynamics with

$$H_N = \frac{1}{2} \sum_{a,b} w_a w_b \left( \zeta_a \cdot \zeta_b + \text{Tr} \left( \left( i \zeta_a^2 \right) \zeta_b^2 \right) \right).$$  \hspace{1cm} (77)$$

whose collective EPCan equations are

$$\dot{\zeta}_a = w_a \sum_b w_b \mathbb{N} \zeta_b^2 \zeta_a + \frac{1}{2} w_a \sum_b w_b \mathbb{N} \zeta_b.$$

(78)

As we shall see, these solutions are also momentum maps in both the even and odd-dimensional cases, since the Klimontovich solution (55) is a momentum map. In particular, upon fixing $N = 1$, these are solution momentum maps $V \to \text{Sym}(n)$ ($\zeta \mapsto X$), where $V$ is a symplectic space and $X \in \text{Sym}(n)$ is the Bloch-Iserles dynamical variable. This construction arises from a special case of the momentum map in (35), with $\dim(S) = 0$ (Klimontovich case), $P = V$ (symplectic vector space) and where the Lie algebra $\mathfrak{X}_{can}(V)$ is restricted to the Lie subalgebra of linear Hamiltonian vector fields. In the more general case when $\dim(S) \geq 1$ one has also the conserved quantity in (43). That is, the operation of taking moments preserves the dual pair structure of the Vlasov equation.

**Theorem 5.3** Upon fixing $N = 1$, the solution (76) of the odd-dimensional Bloch-Iserles system is a momentum map

$$J_{2K+1} : \left( V, w\mathcal{J} \right) \to \text{Sym}(2K + 1)$$

where $w\mathcal{J}$ is the symplectic form on the vector space $V$. Moreover, the solution (77) in the even-dimensional case is also a momentum map

$$J_{2K} : \left( V, w\mathcal{J} \right) \to \text{Sym}(2K).$$
Proof. In what follows we shall use the isomorphisms

\[ \text{Sym}(2K + 1) \simeq (\text{Sym}(2K) \otimes V) \oplus \mathbb{R} \]

and

\[ \text{Sym}(2K) \simeq \mathfrak{sp}^*(2K, \mathbb{R}), \]

where \( \mathfrak{sp}(2K, \mathbb{R}) \) denotes the Lie algebra of Hamiltonian matrices. We prove the first statement, which comprises the second as a particular case. Let \( V \) be endowed with the Poisson structure

\[ \{ F, G \} = \frac{1}{w} \left( \frac{\partial F}{\partial \zeta} \right) \cdot J \frac{\partial G}{\partial \zeta} \]

then, the definition of momentum map can be verified by inserting \( G = \langle J, \beta \rangle \), with \( J = w(\zeta^1, \zeta, 1) \in (\text{Sym}(2K) \otimes V) \oplus \mathbb{R} \) and \( \beta = (\beta_2, \beta_1, \beta_0) \) its dual. Then, we have

\[ \{ F, (J, \beta) \} = \frac{1}{w} \left( \frac{\partial F}{\partial \zeta} \right) \cdot J \cdot \beta_1 + 2J \beta_2 \zeta \frac{\partial F}{\partial \zeta} \]

which identifies the infinitesimal action of linear (inhomogeneous) Hamiltonian vector fields on the phase space functions \( F \in \mathcal{F}(V) \).

Restricting to even \( 2K \) dimensions requires setting \( \beta_0 = 0 = \beta_1 \), thereby producing the action of (homogeneous) Hamiltonian vector fields, i.e. Hamiltonian matrices in \( \mathfrak{sp}(2K, \mathbb{R}) \).

The integrability properties of these solutions will be discussed elsewhere.

6 Conclusions and outlook

After reviewing the geometric basis of Vlasov moment dynamics, this paper showed how moment closures of the geodesic Vlasov equation produce interesting known integrable systems, which include CH, CH2 and Bloch-Iserles (BI) equations in both odd and even dimensions. While the CH and CH2 cases were already known to arise in 1D [GiHoTr2005, GiHoTr2007], the higher dimensional moment bracket showed that these moment closures also recover the EPDiff equation and extend CH2 to higher dimensions. The paper also recovered the BI system from (26) by a finite-dimensional moment closure, corresponding to inhomogeneous quadratic phase-space functions. Thus, a kinetic theory approach led to special solutions of BI.

The moment closures preserve the two equivariant momentum maps in the Vlasov dual pair. Preservation of this structure guaranteed that the resulting closed moment systems discussed here were still Poisson. This preservation also enabled reduction to finite-dimensional systems by using the Klimontovich particle solutions from plasma theory. For example, the peakon solutions of the CH equation arose from a Klimontovich approach in [GiHoTr2005]. Singular solutions also arose upon allowing extra smoothing in the CH2 moment Hamiltonian [GiHoTr2007, HoOnTr2009]. In addition, this paper showed in Section 5.1 that the same approach also produced solutions of the finite-dimensional
BI system. However, Klimontovich solutions are not admitted by arbitrary approximations. They are prevented, for example, when moment hierarchies are simply truncated at a certain weight. That is, moment closures preserve the Vlasov dual pair, while moment truncations do not, even though they may be shown to still be Lie-Poisson. Open questions concern both the construction of a Lax pair for the Klimontovich dynamics of the BI system and potential integrability properties of the truncated equations (e.g. the $(A_1, A_2)$ truncation for kinetic moments).

The geometric setting showed how the left momentum map in the Vlasov dual pair recovers both the Klimontovich solution and the Lagrange-to-Euler map. That this geometry applies also to the Liouville equation illuminates the geometric footing of kinetic theory. Indeed, this paper explained how all the standard moment approximations in kinetic theory are momentum maps preserving the same dual pair. This construction would certainly be destroyed by introducing the collision integral, whose celebrated Boltzmann version implies irreversibility and produces a preferred direction of time via the $H$-theorem. This irreversibility prevents the Klimontovich solutions, which are solutions of a time-reversal invariant system.

The kinetic-theory interpretation of geodesic Vlasov moment dynamics also provided insight into the physical description of the integrable cases. For example, the CH2 case was interpreted in this light as a charged fluid in the context of the one-component CH equation, thereby extending the CH model to include space-charge effects. On the other hand, CH2 has also been related to shallow water dynamics by applying a series of approximations to the Green-Naghdi equations [Coly2008]. Remarkably, a modified version of CH2 dynamics has also been applied in image matching [HoTrYo2007]. (In image matching, the Hamiltonian is the norm in which one applies optimal control.) Emergent peakon solutions were found to result from applying $H^1$ smoothing to the Hamiltonian in [HoOnTr2009].

The paper also identified several other potentially interesting open problems. One of these is the problem of making physical applications of the Kirillov ad*-action (12) for the Lie-Poisson bracket on the symmetric Schouten algebra for arbitrary values of $(n, k)$. Another is to determine whether the Klimontovich average in plasma kinetic theory is a momentum map. One may also ask how the family of symplectically conserved quantities corresponding to statistical-moment versions of the Poincaré invariants found in [HoLySc1990] may fit into the theory of kinetic moments.

The kinetic approach used here may also provide physical interpretations of use in applying the BI system. For example, particle beams in linear accelerator lattices are described in terms of symplectic transfer matrices. (The same holds for linear ray optics.) In formulas, one has the relation $z(t) = M(t)z(0)$, where $M(t)$ is a one parameter subgroup of $\text{Sp}(6, \mathbb{R})$ determining the beam evolution $z(t) \subset \mathbb{R}^6$. In this sense, geodesics in $\text{Sp}(6, \mathbb{R})$ would correspond to optimal transfer maps for particle or optical beams. It is interesting that similar approaches have recently emerged in quantum computation, where the Hamiltonian of the system is constrained to an optimal trajectory (i.e., a geodesic) by a
cost function from optimal control theory \[\text{BrElHo08}] \text{NiDoGuDo08}\]. These additional open problems bode well for the potential success in future applications of using the geometric approach to Vlasov dynamics discussed here. The application of ideas from optimal control to the Vlasov moments may be especially fruitful.

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