Abstract

In this paper, we give a criterion that two regular Cantor sets in higher dimensions have \( C^1 \)-stable intersection and provide a concrete example which satisfies the condition. This contrasts that no regular Cantors sets in the real line have \( C^1 \)-stable intersection. As an application of the criterion, we construct a hyperbolic basic set which exhibits \( C^2 \)-robust homoclinic tangency of the largest codimension for any higher dimensional manifold. This answers a question posed by Barrientos and A. Raibekas.

1 Introduction

In this paper, we give a criterion that two regular Cantor sets in higher dimension have \( C^1 \)-stable intersection and apply the criterion to a problem on persistence of degenerate homoclinic tangency.

Regular Cantor sets and their stable intersection A Cantor set is a topological space which is compact, totally disconnected, and without isolated points. Many examples of Cantor sets are given as the maximal invariant subset of a finite family of contracting maps on the Euclidean space. Such Cantor sets are called regular. We can define \( C^k \)-perturbation of a regular Cantor set by perturbation of the family of contracting maps that determines the Cantor set. We say that two regular Cantor sets have \( C^k \)-stable intersection if any \( C^k \)-perturbations of the Cantor sets have non-empty intersection (see Section 3 for the precise definition). Regular Cantor sets and their stable intersection have important applications to bifurcation problems. In [14] (see also [15, 17]), Newhouse defined a numerical invariant called thickness for Cantor sets in the real line and prove that two thick regular Cantor sets have \( C^2 \)-stable intersection. He applied this to show the persistence of homoclinic tangency and the abundance of diffeomorphisms with infinitely many attracting periodic orbits for \( C^2 \) surface diffeomorphisms. Palis and Viana [18] generalized Newhouse's
result to higher dimensional cases. With Newhouse’s thickness criterion, Kiriki and Soma [11] also proved the persistence of more degenerate tangency. In [7, 8], Buzzard studied stable intersection of regular Cantor sets generated by holomorphic maps of the complex line \( \mathbb{C} \) and proved results analogous to Newhouse’s one in holomorphic setting.

\( C^1 \)-stable intersection of Cantor sets As shown in [21], thickness is zero for \( C^1 \) generic regular Cantor sets in the real line. This means that the thickness criterion by Newhouse is useless for finding a pair having \( C^1 \)-stable intersection. Moreira proved the following negative result on \( C^1 \)-stable intersection of Cantor sets in the real line.

**Theorem 1.1** (Moreira [12]). There exists no pair of regular Cantor sets in the real line which have \( C^1 \)-stable intersection.

In this paper, we give a criterion that two regular Cantor sets in higher dimensional Euclidean space have \( C^1 \)-stable intersection and prove that a pair of Cantor sets having \( C^1 \)-stable intersection exists by applying the criterion.

**Theorem A.** For any positive integers \( l, m \) and positive real number \( \epsilon \), there exists a pair \((K_1, K_2)\) of regular Cantor sets in \( \mathbb{R}^{l+m} \) such that \( \dim_H(K_1) < l + \epsilon \), \( \dim_H(K_2) < m + \epsilon \), \( K_1 \) and \( K_2 \) have \( C^1 \)-stable intersection, where \( \dim_H(K_i) \) be the Hausdorff dimension of \( K_i \).

Let us remark on the inequality on Hausdorff dimension in the theorem. For Cantor sets \( K_1 \) and \( K_2 \) in the \( n \)-dimensional Euclidean space \( \mathbb{R}^n \), it is easy to see that the difference

\[
K_1 - K_2 = \{ x - y \mid x \in K_1, y \in K_2 \}
\]
satisfies \( \dim_H(K_1 - K_2) \leq \dim_H(K_1) + \dim_H(K_2) \). Hence, if \( \dim_H(K_1) + \dim_H(K_2) < n \) then \( K_1 - K_2 \) is nowhere dense subset of \( \mathbb{R}^n \). This means that \( K_1 \) and the translation \( K_2 + t = \{ y + t \mid y \in K_2 \} \) of \( K_2 \) do not intersect for dense \( t \in \mathbb{R}^n \). Therefore, any pair \((K_1, K_2)\) having stable intersection must satisfy the inequality \( \dim_H(K_1) + \dim_H(K_2) \geq n \). For Cantor sets in the real line with this inequality on Hausdorff dimension, Moreira and Yoccoz proved the abundance of pairs having \( C^2 \)-stable intersection.

**Theorem 1.2** (Moreira and Yoccoz [13]). There exists an open and dense subset \( U \) of the space of pairs \((K_1, K_2)\) of \( C^2 \) regular Cantor sets in the real line with \( \dim_H(K_1) + \dim_H(K_2) > 1 \) such that

\[
I_S(K_1, K_2) = \{ t \in \mathbb{R} \mid K_1 \text{ and } K_2 + t \text{ have } C^2 \text{-stable intersection} \}
\]
is a dense subset of \( K_1 - K_2 \) for any \((K_1, K_2)\) in \( U \).

It is natural to ask whether the analogy for higher dimension holds or not.

**Question 1.3.** Does there exist an open and dense subset \( U \) of the space of pairs \((K_1, K_2)\) of \( C^2 \) regular Cantor sets in \( \mathbb{R}^n \) with \( \dim_H(K_1) + \dim_H(K_2) > m \) such that \( I_S(K_1, K_2) \) is a dense subset of \( K_1 - K_2 \) for any pair \((K_1, K_2)\) in \( U \)?

\[1\] We refer [11] for recent progress in holomorphic case.
Robust homoclinic tangency of large codimensions  Like Newhouse’s thickness criterion in [14], our criterion for stable intersection in higher dimension has an application to bifurcation theory. Let $f$ be a $C^k$ diffeomorphism of a manifold $M$ and $\Lambda$ a topologically transitive hyperbolic set (see Section 4.1 for the precise definition). We say that $\Lambda$ exhibits homoclinic tangency if there exist $p_1, p_2 \in \Lambda$ such that the stable manifold $W^u(p_1)$ and the unstable manifold $W^s(p_2)$ intersect at a point $q$ where $T_q M \neq T_q W^u(p_1) + T_q W^s(p_2)$. The integer $c_T(q) = \dim(T_q W^u(p_1) \cap T_q W^s(p_2)) = \dim M - \dim(T_q W^u(p_1) + T_q W^s(p_2))$
is called the codimension of the tangency at $q$. The codimension $d$ satisfies that

$$1 \leq d \leq \min\{\dim T_q W^u(p_1), \dim T_q W^s(p_2)\} \leq \frac{1}{2} \dim M.$$

Remark that the tangent spaces $T_q W^u(p_1)$ and $T_q W^s(p_2)$ coincide when $\dim M$ is even and $2d = \dim M$.

Suppose that a hyperbolic invariant set $\Lambda$ of $f$ admits a continuation $(\Lambda(g))_{g \in \mathcal{U}}$ on a $C^k$-neighborhood $\mathcal{U}$ of $f$. We say that $\Lambda$ exhibits $C^k$-robust homoclinic tangency of codimension $d$ if $\Lambda(g)$ exhibits homoclinic tangency of codimension at least $d$ for any diffeomorphism $g$ sufficiently $C^k$ close to $f$. Study of such degenerate tangency was initiated by Díaz, Nogueira, and Pujals [9] for heterodimensional tangency.\footnote{Heterodimensional tangency is tangency of the unstable manifold and the stable manifold of hyperbolic basic sets $\Lambda_1$ and $\Lambda_2$ with different unstable indices.} Examples exhibiting $C^2$-robust heterodimensional tangency were given by Kiriki and Soma [11] for codimension one and by Barrientos and Raibekas [2] for larger codimensions. Barrientos and Perez [3] also gave examples of $C^1$-robust heterodimensional tangency. For homoclinic tangency, Barrientos and Raibekas [2] gave examples which exhibit $C^2$-robust homoclinic tangency of codimensions for $2 \leq d \leq \frac{1}{2} \dim M - 1$ for manifolds $M$ with $\dim M \geq 6$. The following question is natural to ask since the dimension of a manifold must be at least four for exhibiting homoclinic tangency of codimension at least two.

**Question 1.4** (Barrientos and Raibekas [2, p.4369], see also [3, Question 3]). Does there exist a diffeomorphism exhibiting $C^2$-robust homoclinic tangency of codimension 2 for a manifold $M$ with $\dim M = 4, 5$?

We apply our criterion (Theorem 2.13) to show the existence of diffeomorphisms exhibiting $C^2$-robust homoclinic tangency of the largest codimension.

**Theorem B.** For any manifold $M$ with $\dim M \geq 4$, there exists a $C^\infty$ diffeomorphism which exhibits $C^2$-robust homoclinic tangency of codimension $\lfloor \frac{1}{2} \dim M \rfloor$, where $\lfloor x \rfloor$ is the maximal integer not greater than $x$.

This gives an affirmative answer to Question 1.4.
Outline of Proofs  In Section 2 we give a criterion that two regular Cantor sets have $C^1$-stable intersection. The key observation is that a blender horseshoe, introduced by Bonatti and Díaz in [5] (see also [6] Section 6.2), behaves like a higher dimensional manifold. Our criterion, Theorem 2.13 can be phrased that two ‘mutually transverse’ blender horseshoes have $C^1$-stable intersection if they are sufficiently thin in the transverse directions. The first two subsections are devoted to prepare terminology which is needed to describe a blender. In Subsection 2.4 we define blenders and show the $C^1$-stable intersection of their invariant sets. Subsection 2.3 consists of some lemmas on the maximal invariant set in the union of rectangles, which we need in Section 4.

In Section 3, we prove Theorem A. It is done by applying Theorem 2.13 to concrete families of contracting maps on the Euclidean space.

In Section 4, we prove Theorem B. In [2], Barrientos and Raibekas constructed a blender horseshoe whose lift to the Grassmannian bundle also admits a blender horseshoe. The proof of Theorem B is done applying Theorem 2.13 to a pair of such blender horseshoes in the Grassmannian bundle. In the first two subsections, we prepare terminology on hyperbolic dynamics and lift of diffeomorphisms to the Grassmannian bundle. In Subsection 4.3 we review the blender horseshoe constructed by Barrientos and Raibekas in [2] in terms of our terminology prepared in Section 2. In Subsection 4.4 we give a diffeomorphism which realize ‘transverse intersection’ of the invariant sets of blenders in the Grassmannian. The proof of Theorem B finishes in Subsection 4.5 by applying Theorem 2.13 to diffeomorphisms given in previous subsections.

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2 The intersection of blenders

In this section, we give a criterion that two regular Cantor sets have $C^1$-stable intersection. Roughly speaking, the criterion is a transverse condition of two thin blenders. The first two subsections are devoted to prepare terminology which is needed to describe a blender. In Subsection 2.4 we define blenders and show the $C^1$-stable intersection of their invariant sets.

2.1 Splittings and the cone condition

For manifolds $M_1$ and $M_2$ and $k \geq 1$, let $C^k(M_1, M_2)$ be the set of $C^k$ maps from $M_1$ to $M_2$ with the compact-open $C^k$-topology. For a subset $S$ of $M_1$, we say that a map $f : S \to M_2$ is $C^r$ if it extends to a $C^r$ map from an open neighborhood of $S$ to $M_2$. For $m \geq 1$, we denote the $m$-dimensional Euclidean space by $\mathbb{R}^m$ and the box norm on $\mathbb{R}^m$ by $\| \cdot \|$, i.e., $\|(x_1, \ldots, x_m)\| = \max\{|x_1|, \ldots, |x_m|\}$

We identify the tangent space $T_x\mathbb{R}^m$ at each $x \in \mathbb{R}^m$ with $\mathbb{R}^m$ in a natural way.
Let $l, m$ be positive integers and $M$ an $(l+m)$-dimensional manifold. We say the pair $(P, Q)$ is an $(l,m)$-splitting of $M$ if $P : M \to \mathbb{R}^l$ are $Q : M \to \mathbb{R}^m$ $C^\infty$ maps and the map $(P, Q) : M \to \mathbb{R}^{l+m}$ given by $(P, Q)(x) = (P(x), Q(x))$ is a diffeomorphism onto an open subset of $\mathbb{R}^{l+m}$. Let $M$ be a manifold with an $(l,m)$-splitting $(P, Q)$. For $x \in M$ and a real number $\theta > 0$, we define the $\theta$-cone $C(x, \theta, P, Q)$ by

$$C(x, \theta, P, Q) = \{ v \in T_xM \mid \| DQ_xv \| \leq \theta\| DP_xv \| \}.$$ 

Let $M_1, M_2$ be manifolds with $(l,m)$-splitting $(P_1, Q_1), (P_2, Q_2)$ respectively, $U$ an open subset of $M_1$, and $f : U\to M_2$ a $C^1$ embedding. For $\theta > 0$ and a subset $S$ of $U$, we say that $f$ satisfies the $\theta$-cone condition on $S$ with respect to $(P_1, Q_1)$ and $(P_2, Q_2)$ if there exists $0 < \theta' < \theta$ such that

$$Df^{-1}(C(f(x), \theta, P_2, Q_2)) \subset C(x, \theta', P_1, Q_1),$$

$$Df(C(x, \theta, P_1)) \subset C(f(x), \theta', Q_2, P_2)$$

for any $x \in S$. When $M_1 = M_2$ and $(P_1, Q_1) = (P_2, Q_2)$, we say that $f$ satisfies the $\theta$-cone condition on $S$ with respect to $(P_1, Q_1)$. For $\theta, \lambda, \mu > 0$, we say that $f$ satisfies the $(\theta, \lambda, \mu)$-cone condition if it satisfies the $\theta$-cone condition on $S$ and there exist $\epsilon > 0$ such that

$$\| D(P_1 \circ f^{-1})v \| \geq (\lambda + \epsilon)\| DP_1v \|, \quad \| D(Q_2 \circ f)w \| \geq (\mu + \epsilon)\| DQ_1w \|$$

for any $x \in S$, $v \in C(f(x), \theta, P_2, Q_2)$, and $w \in C(x, \theta, Q_1, P_1)$.

**Remark 2.1.** If a $C^1$ embedding $f : U\to M_2$ satisfies the $(\theta, \lambda, \mu)$-cone condition on $S$ with respect to $(P_1, Q_1)$ and $(P_2, Q_2)$, then $f^{-1} : f(U)\to M_1$ satisfies the $(\theta, \mu, \lambda)$-cone condition on $f(S)$ with respect to $(Q_2, P_2)$ and $(Q_1, P_1)$.

**Remark 2.2.** If $f$ satisfies the $(\theta, \lambda, \mu)$-cone condition on a compact subset $K$ of $U$, then there exists a neighborhood $U_K$ of $K$ and $C^1$-neighborhood $U$ of $f$ such that any $g \in U$ satisfies the $(\theta, \lambda, \mu)$-cone condition on $U_K$.

**Remark 2.3.** Let $M$ be a manifold with an $(l,m)$-splitting $(P, Q)$ and $U$ an open subset of $M$. If $f : U\to M$ satisfies the $(\theta, 1, 1)$-cone condition with respect to $(P, Q)$ on $U$ for some $\theta > 0$, then there exist constants $\kappa > 1, 0 < \gamma < 1$, and a $Df$-invariant continuous splitting $E^s \oplus E^u$ of $TM$ on $\Lambda = \bigcap_{n\in\mathbb{Z}} f^n(U)$ which satisfies the following conditions for any $x \in \Lambda$:

- $\dim E^-(x) = l, \dim E^+(x) = m$,
- $E^-(x) \subset C(x, \theta, Q, P), E^+(x) \subset C(x, \theta, P, Q),$
- $\| Df^nv \| \leq \kappa^n\| v \|$ and $\| Df^{-n}w \| \leq \kappa\gamma^n\| w \|$ for $v \in E^-(x), w \in E^+(x)$ and $n \geq 0$.

A $Df$-invariant splitting with the third conditions is called the hyperbolic splitting on $\Lambda$ and we say that $\Lambda$ is a hyperbolic set.
2.2 Rectangles

Fix positive integers $l, m$ and an $(l+m)$-dimensional manifold $M$. For $C^1$-maps $G : M \to \mathbb{R}^l$, $H : M \to \mathbb{R}^m$ (the pair $(G, H)$ does not need to be an $(l, m)$-splitting of $M$), we say that a subset $R$ of $M$ is a $(G, H)$-rectangle if $(G, H)(R) = G(R) \times H(R)$ and the map $(G, H) : R \to G(R) \times H(R)$ is a diffeomorphism.

**Lemma 2.4.** Let $R \subset U$ be a compact $(G, H)$-rectangle. For any open neighborhood $U_R$ of $R$ and any compact subset $K_R$ of $\text{Int} R$, there exists a $C^1$-neighborhood $U$ of $(G, H)$ such that any $(G', H') \in U$ admits a compact $(G', H')$-rectangle $R'$ satisfying $G'(R') = G(R), H'(R') = H(R), K_R \subset \text{Int} R'$, and $R' \subset U_R$.

**Proof.** If a pair $(G', H')$ is sufficiently $C^1$-close to $(P, Q)$, then $(G', H')$ maps a neighborhood of $R$ to a neighborhood of $G(R) \times H(R)$ diffeomorphically. Then, $R' = (G', H')^{-1}(G(R) \times H(R))$ is a $(G', H')$-rectangle such that $G'(R') = G(R)$ and $H'(R') = H(R)$. If $(G', H')$ is sufficiently close to $(G, H)$, then the inclusions $K_R \subset \text{Int} R'$ and $R' \subset U_R$ hold.

Rectangles ’skewed’ by a diffeomorphism and their crossing are important to describe blenders and to investigate their properties. The setting is the following. Let $M_n$ be a manifold with an $(l, m)$-splitting $(P_n, Q_n)$ for $n = 1, \ldots, N+1$, $U_n$ an open subset of $M_n$, and $f_n : U_n \to M_{n+1}$ a $C^1$ embedding for $n = 1, \ldots, N$. Put $F_n = f_n \circ \cdots \circ f_1$ for $1 \leq n \leq N$ and let $F_0$ be the identity map on $M_1$. We give two lemmas for such sequence of maps.

**Lemma 2.5.** Let $(R_n)_{n=1}^N$ be a sequence such that $R_n$ is a compact $(P_n, Q_{n+1} \circ f_n)$-rectangle in $M_n$ and

$$P_{n+1}(f_n(R_n)) \subset \text{Int} P_{n+1}(R_{n+1}), \quad Q_{n+1}(R_{n+1}) \subset \text{Int} Q_{n+1}(f_n(R_n)),$$

for each $i = 1, \ldots, N$. Suppose that there exists a constant $0 < \theta \leq 1$ such that $f_n$ satisfies the $\theta$-cone condition with respect to $(P_n, Q_n)$ and $(P_{n+1}, Q_{n+1})$ on $R_n$ for any $n = 1, \ldots, N$. Then, $R_* = \bigcap_{n=1}^N F_n^{-1}(R_n)$ is a $(P_1, Q_{N+1} \circ F_N)$-rectangle such that

$$P_1(R_*) = P_1(R_1), \quad Q_{N+1}(F_N(R_*)) = Q_{N+1}(f_N(R_N)).$$
Proof. Proof is done by induction of $N$. The case $N = 1$ is trivial. Suppose that the lemma holds for the case $N - 1$. Then, $R'_{\ast} = \bigcap_{n=1}^{N-1} F_{n-1}^{-1}(R_n)$ is a $(P_1, Q_N \circ F^{N-1})$-rectangle such that $P_1(R'_\ast) = P_1(R_1)$ and $Q_N(F_{N-1}(R'_\ast)) = Q_N(f_{N-1}(R_{N-1}))$. Since each $f_n$ satisfies the $\theta$-cone condition on the compact subset $R_n$, there exists $0 < \theta' < \theta$ such that $f_n$ satisfies the $\theta'$-cone condition on $R_n$ for each $n = 1, \ldots, N$. Put $R_{\ast} = \bigcap_{n=1}^{N-1} F_{n-1}^{-1}(R_n)$. Then, $R_{\ast} = R'_\ast \cap F_{N-1}^{-1}(R_N)$ and $F_N$ satisfies the $\theta'$-cone condition on $R_{\ast}$.

Firstly, we show that the restriction of $(P_1, Q_{N+1} \circ F_N)$ to $R_{\ast}$ is an immersion. Take $x \in R_{\ast}$ and $v \in T_x M_1$ with $D(P_1, Q_{N+1} \circ F_N)(v) = 0$. Since $DP_1v = 0$, the vector $v$ is contained in $C(x, \theta, Q_1, P_1)$. The $\theta'$-cone condition on $R_{\ast}$ for $F_N$ implies $\|D(P_{N+1} \circ F_N)v\| \leq \theta' \|D(Q_{N+1} \circ F_N)v\|$. Since $D(Q_{N+1} \circ F_N)v = 0$, we have $DF_Nv = 0$. This implies that $v = 0$ since $F_N$ is an embedding. Hence, the map $(P_1, Q_{N+1} \circ F_N) : R_{\ast} \rightarrow P_1(R_1) \times (Q_{N+1} \circ f_N)(R_N)$ is an immersion.

Fix $s_0 \in P_1(R_1)$ and $t_0 \in (Q_{N+1} \circ f_N)(R_N)$ and we put

$$D^+ = \{ x \in R_1 \mid P_1(x) = s_0 \}, \quad D^- = \{ y \in R_N \mid (Q_{N+1} \circ f_N)(y) = t_0 \}.$$ 

We claim that $F_{N-1}(D^+)$ intersects with $D^-$ at a unique point. The claim implies that $(P_1, Q_{N+1} \circ F_N)$ is a bijection from $R_{\ast}$ to $P_1(R_1) \times (Q_{N+1} \circ f_N)(R_N)$. Since the map is an immersion on a neighborhood of $R_{\ast}$, it is a diffeomorphism.

Therefore, the claim implies the lemma for the case $N$ and the induction completes the proof of the lemma.

Let us show the claim. Since $R'_\ast$ is a $(P_1, Q_N \circ F_{N-1})$-rectangle, $F_{N-1}(R'_\ast)$ is a $(P_1 \circ F_{N-1}^{-1}, Q_N)$-rectangle. This implies that there exists a $C^1$ map $g^+_0 : Q_N(F_{N-1}(R'_\ast)) \rightarrow P_N(F_{N-1}(R'_\ast))$ such that $(P_N, Q_N)(F_{N-1}(D^+))$ is the graph of $g^+_0$ on $Q_N(F_{N-1}(R'_\ast))$, i.e.,

$$(P_N, Q_N)(F_{N-1}(D^+)) = \{ (g^+_0(t), t) \mid t \in Q_N(F_{N-1}(R'_\ast)) \}. \quad (1)$$

By the $\theta'$-cone condition for $F_{N-1} = f_{N-1} \circ \ldots \circ f_1$, we have $\|g^+_0\| \leq \theta' < \theta \leq 1$. Similarly, there exists a $C^1$ function $g^-_0 : P_N(R_N) \rightarrow Q_N(R_N)$ such that

$$(P_N, Q_N)(D^-) = \{ (s, g^-_0(s)) \mid s \in P_N(R_N) \} \quad (2)$$

and $\|g^-_0\| \leq \theta' < 1$. Since $P_N \circ F_{N-1}(R'_\ast) \subset P_N(R_N)$ and $Q_N(R_N) \subset Q_N \circ F_{N-1}(R'_\ast)$, we can define a map $G : P_N(R_N) \times (Q_N \circ F_{N-1})(R'_\ast) \rightarrow P_N(R_N) \times (Q_N \circ F_N)(R_N)$.
Lemma 2.6. Let \((Q_N \circ F_{N-1})(R'_s)\) by \(G(s,t) = (g_{n_0}(t), g_{n_0}^+(s))\). This map is a uniform contraction. Hence, it admits a unique fixed point \((s_*, t_*)\). By \(1\) and \(2\), \(x \in F_{N-1}(R'_s) \cap R_N\) is contained in \(F_{N-1}(D^+) \cap D^-\) if and only if \((P_N, Q_N)(x)\) is a fixed point of \(G\). Therefore, \(F_{N-1}(D^+)\) intersects with \(D^-\) at the unique point \((P_N, Q_N)^{-1}(s_*, t_*)\). This completes proof of the claim, and hence, of the lemma.

For \(m \geq 1\) and a subset \(S\) of \(\mathbb{R}^m\), we denote the diameter of \(S\) with respect to the Euclidean metric by \(\text{diam } S\). The following lemma gives a bound of the diameter of \(f^n(R_*)\) in the previous lemma.

**Lemma 2.6.** Let \(R\) be a compact subset of \(U\) where \(F_n = f_n \circ \cdots \circ f_1\) is well-defined for any \(n = 1, \ldots, N\). Suppose that \(R\) is a \((P_1, Q_{N+1} \circ F_N)\)-rectangle and there exist constants \(\theta, \lambda, \mu > 0\) such that \(f_n\) satisfies the \((\theta, \lambda, \mu)\)-cone condition on \(F_{n-1}(R)\) with respect to \((P_n, Q_n)\) and \((P_{n+1}, Q_{n+1})\) for each \(n = 1, \ldots, N\). Then,

\[
\text{diam } Q_{n+1}(F_n(R)) \leq \theta \lambda^{-n} \text{diam } P_1(R) + \mu^{-(N-n)} \text{diam } Q_{N+1}(F_N(R)),
\]

(3)

\[
\text{diam } P_{n+1}(F_n(R)) \leq \lambda^{-n} \text{diam } P_1(R) + \theta \mu^{-(N-n)} \text{diam } Q_{N+1}(F_N(R)),
\]

(4)

for any \(0 \leq n \leq N\).

**Proof.** Take \(x, x' \in R\). Since \(R\) is a \((P_1, Q_{N+1} \circ F_N)\)-rectangle, there exists \(x_* \in R\) such that \(P_1(x_*) = P_1(x)\) and \(Q_{N+1} \circ F_N(x_*) = Q_{N+1} \circ F_N(x')\). Put

\[
R^- = \{y \in R \mid Q_{N+1}(F_N(y)) = Q_{N+1}(F_N(x_*))\},
\]

\[
R^+ = \{y \in R \mid P_1(y) = P_1(x_*)\}.
\]

Then, \(\{x, x_*\} \subset R^-\) and \(\{x', x_*\} \subset R^+\). By the \((\theta, \lambda, \mu)\)-cone condition for \(f_1, \ldots, f_n\), the composition \(f_n' \circ \cdots \circ f_1\) satisfies the \((\theta, \lambda^{(n'-n)+1}, \mu^{(n'-n)+1})\)-cone condition for \(1 \leq n \leq n' \leq N\). For any \(y \in R^-,\) any vector \(v \in T_y R^-\) satisfies that \(D(Q_{N+1} \circ F_N)(v) = 0\), and hence, \(DF_N v\) is contained in \(C(F_N(y), \theta, P_{n+1}, Q_{N+1})\). By the cone condition, we have

\[
\|D(Q_{n+1} \circ F_N)v\| \leq \theta\|D(P_{n+1} \circ F_N)v\|, \quad \|DP_1 v\| \geq \lambda^{n}\|D(P_{n+1} \circ F_N)v\|.
\]

for any \(0 \leq n \leq N\). For any \(y' \in R^+\), any vector \(v' \in T_{y'} R^+\) satisfies that \(DP_1 v' = 0\), and hence, it is contained in \(C(y', \theta, Q_1, P_1)\). By the cone condition, we have

\[
\|D(Q_{n+1} \circ F_N)w\| \geq \mu^{N-n}\|D(Q_{n+1} \circ F_N)w\|
\]

for any \(0 \leq n \leq N\). These inequalities imply that

\[
\text{diam } Q_{n+1}(F_n(R^-)) \leq \theta \lambda^{-n} \text{diam } P_1(R^-),
\]

\[
\text{diam } Q_{n+1}(F_n(R^+)) \leq \mu^{-(N-n)} \text{diam } Q_{N+1}(F_N(R^+)).
\]
Since \( \{x, x'\} \subset R^- \) and \( \{x', x\} \subset R^+ \), we obtain that
\[
\|Q_{n+1} \circ F_n(x) - Q_{n+1} \circ F_n(x')\| \leq \text{diam } Q_{n+1}(F_n(R^-)) + \text{diam } Q_{n+1}(F_n(R^+)).
\]
Since \( x \) and \( x' \) are arbitrary points in \( R \), we obtain (3) by combining with the previous inequalities. The inequality (4) can be proved in the same way.

The case \( N = 1 \) of the above lemma implies

**Corollary 2.7.** Let \( M_1, M_2 \) be manifold with \((l, m)\)-splittings \((P_1, Q_1), (P_2, Q_2)\) respectively, \( U \) an open subset of \( M_1 \), \( f : U \to M_2 \) an \( C^1 \)-embedding, and \( R \) a \((P_1, Q_2 \circ f)\)-rectangle in \( U \). Suppose that \( f \) satisfies the \((\theta, \lambda, \mu)\)-cone condition on \( R \) with respect to \((P_1, Q_1)\) and \((P_2, Q_2)\). Then,
\[
\text{diam } Q_1(R) \leq \theta \text{diam } P_1(R) + \mu^{-1} \text{diam } Q_2(f(R)) \tag{5}
\]
\[
\text{diam } P_2(f(R)) \leq \lambda^{-1} \text{diam } P_1(R) + \theta \text{diam } Q_2(f(R)). \tag{6}
\]

**2.3 Invariant sets in the union of rectangles**

In this subsection, we show some properties of the maximal invariant set in the union of rectangles, which we need in Section 3.

Let \( M \) be a manifold with an \((l, m)\)-splitting \((P, Q)\), \( U \) an open subset of \( M \), \( f : U \to M \) a \( C^1 \) embedding, and \((R_i)_{i \in I} \) a family of mutually disjoint compact \((P, Q \circ f)\)-rectangles in \( U \) indexed by a finite set \( I \). Suppose that
\[
P(f(R_i)) \subset \text{Int } P(R_j), \quad Q(R_j) \subset \text{Int } Q(f(R_i))
\]
if \( f(R_i) \cap R_j \neq \emptyset \) for \( i, j \in I \). Put \( R = \bigcup_{i \in I} R_i \) and \( \Lambda = \bigcap_{n \in \mathbb{Z}} f^n(R) \).

**Lemma 2.8.** \( \Lambda = \bigcap_{n \in \mathbb{Z}} f^n(\text{Int } R) \). In particular, \( \Lambda \) is locally maximal.

**Proof.** It is sufficient to show that \( R_i \cap f(R_j) \cap f^{-1}(R_k) \subset \text{Int } R_i \) for any \( i, j, k \in I \). Without loss of generality, we may assume that \( R_i \cap f(R_j) \) and \( R_i \cap f^{-1}(R_k) \) are non-empty. By the assumption on the family \((R_i)_{i \in I}\), we have
\[
P(R_i \cap f(R_j)) \subset \text{Int } P(R_i),
\]
\[
(Q \circ f)(R_i \cap f^{-1}(R_k)) = Q(R_k \cap f(R_i)) \subset \text{Int } (Q \circ f)(R_i).
\]
Since \( R_i \) is a \((P, Q \circ f)\)-rectangle, \((P, Q \circ f)\) is a homeomorphism between \( \text{Int } R_i \) and \( \text{Int } P(R_i) \times \text{Int } (Q \circ f)(R_i) \). Hence, \( R_i \cap f(R_j) \cap f^{-1}(R_k) \subset \text{Int } R_i \).

We say that a homeomorphism \( h \) of a compact set \( X \) is topologically transitive if any pair \((U, V)\) of non-empty open subsets of \( X \) admits an integer \( n \) such that \( h^n(U) \cap V \neq \emptyset \).

**Lemma 2.9.** The restriction of \( f \) to \( \Lambda \) is topologically transitive if the map \( f \) satisfies the \((\theta, 1, 1)\)-cone condition on \( R \) for some \( 0 < \theta \leq 1 \) and for any given pair \((i, j)\) of elements of \( I \), there exists a sequence \((i_n)_{n=0}^N \) in \( I \) such that \( i_0 = i, i_N = j \), \( P(f(R_{i_0})) \subset P(R_{i_{n+1}}) \) and \( Q(R_{i_{n+1}}) \subset \text{Int } Q(f(R_{i_n})) \) for any \( n = 1, \ldots, N - 1 \).
Proof. For a sequence $s = (i_n)_{n \in \mathbb{Z}}$ of elements of $I$ and an integer $N \geq 0$, we put $R(s, N) = \bigcap_{|n| \leq N} f^{-n}(R_{i_n})$ and $R(s) = \bigcap_{N \geq 0} R(s, N) = \bigcap_{n \in \mathbb{Z}} f^{-n}(R_{i_n})$. If $f(R_{i_{n-1}}) \cap R_{i_n} \neq \emptyset$ for any $n \in \mathbb{Z}$, then $R(s, N)$ is a non-empty compact subset of $R(s, N - 1)$ for any $N \geq 1$ by Lemma 2.4, and hence, $R(s)$ is non-empty. Since $f$ satisfies the $(\theta, 1, 1)$-cone condition on $R$, it also satisfies the $(\theta, \nu, \nu)$-cone condition for some $\nu > 1$. By Lemma 2.6 we have

$$\max\{\text{diam } P(R(s, N)), \text{diam } Q(R(s, N))\} \leq (1 + \theta)\nu^{-N} \max\{\text{diam } P(R_i), \text{diam } Q(R_i) \mid i \in I\}.$$

Therefore, diam $R(s, N)$ converges to zero as $N$ goes to infinity.

Take $x^{-}, x^{+} \in \Lambda$ and their neighborhoods $U^{+}$ and $U^{-}$ in $M$. For $\sigma = \pm$, let $s^\sigma = (i_n^\sigma)_{n \in \mathbb{Z}}$ be the sequence in $I$ such that $f^n(x^\sigma) \in R_{i_n^\sigma}$ for $n \in \mathbb{Z}$. Choose $N \geq 1$ such that $R(s^\sigma, N) \subset U^\sigma$ for $\sigma = \pm$. By assumption, there exists a sequence $(j_n)_{n=0, \ldots, k}$ in $I$ such that $j_0 = i^-_N$, $j_k = i^+_N$, and $f(R_{j_{n-1}}) \cap R_{j_n} \neq \emptyset$ for any $n = 1, \ldots, k$. We define a sequence $s^{*} = (i_n^{*})_{n \in \mathbb{Z}}$ by

$$i^n_* = \begin{cases} i_{n+N}^- & \text{if } n \leq 0, \\ j_n & \text{if } 1 \leq n \leq k - 1, \\ i_{n-N-k}^+ & \text{if } n \geq k. \end{cases}$$

We can check that $R(s^{*})$ is a non-empty subset of $f^N(R(s^{-}, N)) \cap f^{-N}(R(s^{+}, N)) \cap \Lambda$. This implies that $f^{2N}(U^{-} \cap \Lambda) \cap (U^{+} \cap \Lambda) \neq \emptyset$. Therefore, the restriction of $f$ to $\Lambda$ is topologically transitive. \qed

Remark 2.10. Let $\Sigma_R$ be the set of sequences $s = (s(n))_{n \in \mathbb{Z}}$ such that $f(R_{s(n)}) \cap R_{s(n+1)} \neq \emptyset$ for any $n$. This set admits the shift map $\sigma_R$ given by $(\sigma_R(s))(n) = s(n + 1)$. If $f$ satisfies the $(\theta, 1, 1)$-cone condition on $R$, then $R(s)$ contains exactly one point by the proof of Lemma 2.9. The map $h : \Sigma_R \rightarrow \Lambda$ given by $R(s) = \{h(s)\}$ provides a topological conjugacy between $\sigma_R$ and $f$ (we will not use this fact in the this paper).

2.4 Blenders

Let $l, m$ be positive integers, $M$ an $(l + m)$-dimensional manifold with an $(l, m)$-splitting $(P, Q)$. We call a family $\{(f_i, R_i)\}_{i \in I}$ indexed by a finite set $I$ a blender with respect to $(P, Q)$ if there exist $\Delta > 0$ and a $(P, Q)$-rectangle $Z$ such that

1. $R_i$ is a compact subset of $M$ and $f_i$ is a $C^1$ embedding from an open neighborhood of $R_i$ to $M$;
2. $R_i$ is a $(P, Q \circ f_i)$-rectangle such that $P(R_i) = P(Z)$ and $Q(R_i) \subset \text{Int } Q(Z)$ for any $i \in I$;
3. for any compact subset $K$ of $Q(Z)$ with diam $K \leq \Delta$, there exists $i_K \in I$ such that $K \subset \text{Int } (Q \circ f_i)(R_{i_K})$.
We call \( \Delta \) the width and \( Z \) the blending region of \( B \).

**Example 2.11** (A contracting iterated function system in \( \mathbb{R}^2 \)). Let \( P, Q : \mathbb{R}^2 \to \mathbb{R} \) be the natural projections given by \( P(x, y) = x \) and \( Q(x, y) = y \). Then, \((P, Q)\) is a \((1,1)\)-splitting of \( \mathbb{R}^2 \). Define affine maps \( f_1, f_2 : \mathbb{R}^2 \to \mathbb{R}^2 \) by

\[
f_1(x, y) = \left( \frac{1}{8}x + \frac{1}{4}, \frac{9}{10}y \right), \quad f_2(x, y) = \left( \frac{1}{8} + \frac{3}{4}, \frac{9}{10}y + \frac{1}{10} \right).
\]

Put \( R_1 = R_2 = [0, 1] \times [1/6, 5/6] \), \( \Delta = 1/3 \), and \( Z = [0, 1] \times [3/19, 16/19] \). See Figure 3. Then, \( f_\tau \) is a contraction with \( f_\tau([0,1]^2) \subset \text{Int}[0,1]^2 \) for \( \tau = 1,2 \), there exists a unique compact subset \( K \) of \( \text{Int}[0,1]^2 \) such that \( K = f_1(K) \cup f_2(K) \). The set \( K \) is a Cantor set since \( f_1([0,1]^2) \cap f_2([0,1]^2) = \emptyset \).

**Example 2.12** (An affine blender horseshoe in \( \mathbb{R}^3 \)). Define a \((1,2)\)-splitting \((P, Q)\) of \( \mathbb{R}^3 \) by \( P(x, y, z) = x \) and \( Q(x, y, z) = (y, z) \). Let \( f_1, f_2 \) be affine maps in the previous example and \( F \) be a diffeomorphism of \( \mathbb{R}^3 \) satisfying that

\[
F(x, y, z) = \begin{cases} 
(f_1(x), 4z - \frac{1}{2}) & \text{if } (x, y, z) \in [0, 1]^2 \times [1/8, 3/8] \\
(f_2(x), 4z - \frac{1}{2}) & \text{if } (x, y, z) \in [0, 1]^2 \times [5/8, 7/8].
\end{cases}
\]

Put \( R'_1 = R_1 \times [1/8, 3/8] \) and \( R'_2 = R_2 \times [5/8, 7/8] \). Then, \( Q(F(R'_1)) = [3/20, 4/3] \times [0, 1] \) and \( Q(F(R'_2)) = [1/4, 17/20] \times [0, 1] \). Put \( \Delta = b - a - 2\epsilon \) and \( Z = [0, 1] \times [3/19, 16/19] \times [1/9, 8/9] \). Then, it is easy to check that \( \{(F, R'_1), (F, R'_2)\} \) is a blender with width \( \Delta \) and blending region \( Z \). See Figure 4. Remark that the restriction of \( F \) to \( \bigcap_{n \in \mathbb{Z}} F([0,1]^2 \times ([1/8, 3/8] \cup [5/8, 7/8])) \) is topologically conjugate to the shift map on \( \{0, 1\}^\mathbb{Z} \).

For a blender \( B = \{(f_i, R_i)\}_{i \in \mathbb{T}} \), let \( \Lambda^{-}(B) \) be a subset of \( M \) consisting of points \( x \) which admit sequences \((x_n)_{n \geq 1}\) and \((i_n)_{n \geq 1}\) such that \( x_n \in R_{i_n} \),

\[
\Lambda^{-}(B) = \{ x \in M : (x_n)_{n \geq 1}, (i_n)_{n \geq 1} \}. 
\]
Figure 4: A blender horseshoe

$x_n = f_{i_{n+1}}(x_{n+1})$ for any $n \geq 1$, and $x = f_{i_1}(x_1)$. Remark that if there exists an embedding $f : \bigcup_{i \in I} R_i \to M$ such that $f_i$ is the restriction of $f$ to $R_i$, then $\Lambda^-(\mathcal{B}) = \bigcap_{n \geq 1} f^n(\bigcup_{i \in I} R_i)$. The following criterion to intersection of the invariant sets of two blenders is a keystone to prove Theorems A and B.

**Theorem 2.13.** Let $l, m$ be positive integers, $M_1, M_2$ be $(l + m)$-dimensional manifolds with $(l, m)$-splitting $(P_1, Q_2)$, $(m, l)$-splitting $(P_2, Q_2)$ respectively. For each $\tau = 1, 2$, let $B_{\tau} = \{(f_{\tau,i}, R_{\tau,i})\}_{i \in I}$ a blender with respect to $(P_\tau, Q_\tau)$ whose width and blending region are $\Delta_\tau$ and $Z_\tau$. Suppose that there exist positive real numbers $\theta, \lambda_\tau, \mu_\tau (\tau = 1, 2, \sharp)$, a compact subset $R_\sharp$ of $M_1$, and a $C^1$ embedding $h_\sharp : R_\sharp \to M_2$ which satisfy the following properties:

1. $f_{\tau,i}$ satisfies the $(\theta, \lambda_\tau, \mu_\tau)$-cone condition on $R_{\tau,i}$ with respect to $(P_\tau, Q_\tau)$ for any $\tau = 1, 2$ and $i \in I_\tau$.
2. $h_\sharp$ satisfies the $(\theta, \lambda_\sharp, \mu_\sharp)$-cone condition on $R_\sharp$ with respect to $(P_1, Q_1)$ and $(P_2, Q_2)$.
3. $R_\sharp$ is a $(P_1, P_2 \circ h_\sharp)$-rectangle such that
   
   \begin{align*}
   P_1(R_\sharp) &= P_1(Z_1), & Q_1(R_\sharp) \subset \text{Int} \, Q_1(Z_1), & \text{diam } Q_1(R_\sharp) < \Delta_1, \\
   P_2(h_\sharp(R_\sharp)) &= P_2(Z_2), & Q_2(h_\sharp(R_\sharp)) \subset \text{Int} \, Q_2(Z_2), & \text{diam } Q_2(h_\sharp(R_\sharp)) < \Delta_2.
   \end{align*}

4. $\lambda_1 \mu_2 > 1$, $\mu_1 \lambda_2 > 1$, and
   \begin{align*}
   \theta \text{ diam } P_1(Z_1) + \mu_\sharp^{-1} \text{ diam } P_2(Z_2) &< \Delta_1, \\
   \theta \text{ diam } P_2(Z_2) + \lambda_\sharp^{-1} \text{ diam } P_1(Z_1) &< \Delta_2.
   \end{align*}

Then, $h_\sharp(\Lambda^-(B_1))$ intersects with $\Lambda^-(B_2)$.

**Proof.** We say that a pair $(h, R)$ satisfies the condition (\sharp) if
1. $h : U \to M_2$ is a $C^1$ embedding from an open subset $U$ of $M_1$ to $M_2$ which satisfies the $(\theta, \lambda, \mu)$-cone condition with respect to $(P_1, Q_1)$ and $(Q_2, P_2)$.

2. $R$ is a compact $(P_1, P_2 \circ h)$-rectangle in $U \cap Z_1 \cap h^{-1}(Z_2)$ such that

   $$P_1(R) = P_1(Z_1), \quad Q_1(R) \subset \text{Int} Q_1(Z_1), \quad \text{diam } Q_1(R) < \Delta_1,$$

   $$P_2(h(R)) = P_2(Z_2), \quad Q_2(h(R)) \subset \text{Int} Q_2(Z_2), \quad \text{diam } (Q_2 \circ h)(R) < \Delta_2.$$

It is sufficient to prove the following claim to show the theorem.

**Claim.** If $(h, R)$ is a pair satisfying the condition $(\sharp)$, then there exists $i \in I_1$ and $j \in I_2$ such that the pair $(h', R')$ with

$$h' = f_{2,j}^{-1} \circ h \circ f_{1,i}, \quad R' = R_{1,i} \cap f_{1,i}^{-1}(R) \cap (h')^{-1}(R_{2,j})$$

also satisfies the condition $(\sharp)$.

Once the claim is proved, we can inductively find a sequence $\{(h_n, R_n, i_n, j_n)\}_{n \geq 0}$ such that $h_0 = h$, $R_0 = R$, $h_{n+1} = f_{2,j_n}^{-1} \circ h_n \circ f_{1,i_n}$, $R_{n+1} = R_{1,i_n} \cap f_{1,i_n}^{-1}(R_n) \cap h_{n+1}^{-1}(R_{2,j_n})$, and the pair $(h_n, R_n)$ satisfies the condition $(\sharp)$ for any $n \geq 0$. Put $F_1^n = f_{1,i_0} \circ \cdots \circ f_{1,i_n}$ and $F_2^n = f_{2,j_0} \circ \cdots \circ f_{2,j_n}$. Then, $h_{n+1} = (F_2^n)^{-1} \circ h \circ F_1^n$ and

$$(h_n \circ F_1^n)(R_{n+1}) = \begin{cases} (h_1 \circ f_{1,i_0})(R_0) \cap h_2(R_{1,i_1}) \cap f_{2,j_1}(R_{2,j_1}) & (n = 0), \\ (h_n \circ F_1^n)(R_{i_n}) \cap (h_n \circ F_1^n)(R_{2,j_n}) & (n \geq 1). \end{cases}$$

This implies that the intersection $\bigcap_{n \geq 0} (h_n \circ F_1^n)(R_{n+1})$ of decreasing compact sets is a non-empty subset of $\bigcap_{n \geq 0} (h_n \circ F_1^n)(R_{1,i_n}) \cap F_2^n(R_{2,j_n})$. Since the latter intersection is contained in $h_n(B) \cap B_2$, the claim implies the theorem.

Let us start the proof of the claim. Take a pair $(h, R)$ which satisfies the condition $(\sharp)$. By the inequality $\text{diam } Q_1(R) < \Delta_1$ and the definition of a blender, there exists $i \in I_1$ such that $Q_1(R) \subset \text{Int}(Q_1 \circ f_{1,i})(R_{1,i})$. Put $R_* = R_{1,i} \cap f_{1,i}^{-1}(R)$. Then, $(h \circ f_{1,i})(R_*) \subset h(R) \subset Z_2$. In particular,

$$\text{diam } (Q_2 \circ h \circ f_{1,i})(R_*) \leq \text{diam } (Q_2 \circ h)(R) < \Delta_2.$$

Since $(P_1 \circ f_{1,i})(R_{1,i}) \subset \text{Int } P_1(Z_1) = \text{Int } P_1(R)$, Lemma 2.2 implies that the set $R_*$ is a $(P_1, P_2 \circ h \circ f_{1,i})$-rectangle such that $P_1(R_*) = P_1(R_{1,i}) = P_1(Z_1)$ and $(P_2 \circ h \circ f_{1,i})(R_*) = (P_2 \circ h)(R) = P_2(Z_2)$. By the inequality $\text{diam } (Q_2 \circ h \circ f_{1,i})(R_*) < \Delta_2$ and the definition of a blender, there exists $j \in I_2$ such that $(Q_2 \circ h \circ f_{1,i})(R_*) \subset \text{Int}(Q_2 \circ f_{2,j})(R_{2,j})$. Put

$$h' = f_{2,j}^{-1} \circ h \circ f_{1,i}, \quad R' = R_{1,i} \cap f_{1,i}^{-1}(R_{1,i}) \cap (h')^{-1}(R_{2,j}).$$

We will show that the pair $(h', R')$ satisfies the condition $(\sharp)$. It is easy to see that $R' \subset Z_1 \cap (h')^{-1}(Z_2)$. Notice that $f_{2,j}^{-1}$ satisfies the $(\theta, \mu_2, \lambda_2)$-cone condition.
with respect to $(Q_2, P_2)$, $f_{2,j}(R_{2,j})$ is $(Q_2, P_2 \circ f_{2,j}^{-1})$-rectangle, and $R' = R_\ast \cap (h')^{-1}(R_{2,j})$. Applying Lemma 2.5 to $R_\ast$ and $f_{2,j}(R_{2,j})$, we obtain that $R' = R_\ast \cap (h')^{-1}(R_{2,j})$ is a $(P_1, P_2 \circ h')$-rectangle such that $P_1(R') = P_1(R_\ast) = P_1(Z_1)$ and $(P_2 \circ h')(R') = P_2(R_{2,j}) = P_2(Z_2)$. By the cone conditions for $f_{1,i}$, $h$, and $(f_{2,j})^{-1}$, the map $h' = f_{2,j}^{-1} \circ h \circ f_{1,i}$ satisfies the $(\theta, \lambda_1 \lambda_2 \mu_2, \mu_1 \mu_2 \lambda_2)$-cone condition with respect to $(P_1, Q_1)$ and $(Q_2, P_2)$. Since $\lambda_1 \mu_2 > 1$ and $\mu_1 \lambda_2 > 1$, this implies the $(\theta, \lambda_2, \mu_2)$-cone condition for $h'$ on $R'$. Applying Lemma 2.6 for $R'$, we obtain

$$\text{diam } Q_1(R') \leq \theta \text{ diam } P_1(R') + \mu_2^{-1} \text{ diam } (P_2 \circ h')(R')$$

$$\leq \theta \text{ diam } P_1(Z_1) + \mu_2^{-1} \text{ diam } P_2(Z_2),$$

$$\text{diam } (Q_2 \circ h')(R') \leq \theta \text{ diam } (P_2 \circ h')(R') + \lambda_2^{-1} \text{ diam } P_1(R')$$

$$\leq \theta \text{ diam } P_2(Z_2) + \lambda_2^{-1} \text{ diam } P_1(Z_1).$$

By the assumption on the constants, we have $\text{diam } Q_1(R') < \Delta_1$ and $\text{diam } (Q_2 \circ h')(R') < \Delta_2$. Therefore, $(h', R')$ satisfies the condition (j). This completes the proof of the claim.

3 Stable intersection of Cantor sets

In this section, we prove Theorem A stated in the introduction.

**Theorem A.** For any positive integers $l, m$ and positive real number $\epsilon$, there exist $C^\infty$ regular Cantor sets $K_1, K_2$ in $\mathbb{R}^{l+m}$ which have $C^1$-stable intersection and satisfy that $\dim_H(K_1) < l + \epsilon$, $\dim_H(K_2) < m + \epsilon$, where $\dim_H(K_i)$ be the Hausdorff dimension of $K_i$. 

![Figure 5: Claim in Proof of Theorem 2.13](image)
Let us give precise definitions of regular Cantor sets and their intersection. We call a family of diffeomorphisms of $\mathbb{R}^m$ an iterated function system (IFS) on $\mathbb{R}^m$. An IFS is said to be finite if it is a finite family. For $0 < \alpha < 1$, we say that an IFS $\mathcal{F} = (f_i)_{i \in I}$ is $\alpha$-contracting if $\|Df_x v\| \leq \alpha \|v\|$ for any $x \in \mathbb{R}^m$ and $v \in T_x \mathbb{R}^m$. A contracting IFS is a IFS which is $\alpha$-contracting for some $0 < \alpha < 1$. Let $I^\mathbb{N}$ be the set of sequences valued in $I$. The discrete topology of $I$ induces the product topology on $I^\mathbb{N}$. It is known that the coding map $c : I^\mathbb{N} \to \mathbb{R}^m$ given by
\[
c((i_n)_{n \geq 1}) = \lim_{n \to \infty} f_{i_1} \circ f_{i_2} \circ \cdots \circ f_{i_n}(x)
\]
is a continuous map and does not depend on the choice of $x \in \mathbb{R}^m$. We call the image $\{c(s) \mid s \in I^\mathbb{N}\}$ the limit set of the IFS $\mathcal{F}$ and denote it by $K(\mathcal{F})$. The limit set $K(\mathcal{F})$ is the unique compact subset $K$ of $\mathbb{R}^m$ such that $K = \bigcup_{i \in I} f_i(K)$.

We say that an IFS $\mathcal{F} = (f_i)_{i \in I}$ is of Cantor type if it is a finite contracting IFS and there exists a non-empty compact subset $D$ of $\mathbb{R}^m$ such that $(f_i(D))_{i \in I}$ is a family of mutually disjoint subsets of Int $D$. The limit set of a finite IFS $\mathcal{B} = (f_i)_{i \in I}$ of Cantor type is a Cantor set. A Cantor set $K$ in $\mathbb{R}^m$ is called $C^k$-regular if there exists a finite $C^k$ IFS $\mathcal{F} = (f_i)_{i \in I}$ of Cantor type and a subshift $\Sigma \subset I^\mathbb{N}$ of finite type\footnote{A subset $\Sigma$ of $I^\mathbb{N}$ is a subshift of finite type if there exists a $\{0, 1\}$-valued matrix $(a_{ij})_{i,j \in I}$ such that $\Sigma = \{(i_n)_{n \geq 1} \in I^\mathbb{N} \mid a_{i_n i_{n+1}} = 1 \text{ for any } n \geq 1\}$.} such that $K = \{c(s) \mid s \in \Sigma\}$, where $c : I^\mathbb{N} \to K(\mathcal{F})$ is the coding map. In particular, the limit set of a finite $C^k$ IFS of Cantor type is a $C^k$-regular Cantor set.

Let $\text{IFS}_C^k(\mathbb{R}^m, I)$ be the set of $C^k$ IFSs $\mathcal{F} = (f_i)_{i \in I}$ of Cantor type on $\mathbb{R}^m$. This set admits a $C^k$-compact open topology as a family of $C^k$-diffeomorphism indexed by $I$. For $k \geq 1$, we say that two Cantor sets $K_1, K_2$ of Cantor sets in $\mathbb{R}^m$ have $C^k$-stable intersection if there exist finite IFSs $\mathcal{F}, \mathcal{G} \in \text{IFS}_C^k(\mathbb{R}^m, I)$ and their neighborhoods $\mathcal{U}_\mathcal{F}$ and $\mathcal{U}_\mathcal{G}$ respectively such that $K_1 = K(\mathcal{F})$, $K_2 = K(mc\mathcal{G})$, and $K(\mathcal{F})$ intersects with $K(\mathcal{G}')$ for any $\mathcal{F}' \in \mathcal{U}_\mathcal{F}$ and $\mathcal{G}' \in \mathcal{U}_\mathcal{G}$.

We apply Theorem 2.14 to prove Theorem A. Recall that $\|\cdot\|$ is the box norm on $\mathbb{R}^m$. For constants $\beta, \Delta > 0$ and a compact subset $D$ of $\mathbb{R}^m$, we say that a finite IFS $\mathcal{F} = (f_i)_{i \in I}$ is a covering IFS of type $(\beta, \Delta, D)$ if
\begin{enumerate}
  \item it is contracting IFS,
  \item $\|Df_x v\| \geq \beta \|v\|$ for any $x \in \mathbb{R}^m$ and $v \in T_x \mathbb{R}^m$, and
  \item for any compact subset $K$ of $D$ with $\text{diam} K \leq \Delta$ there exists $i \in I$ such that $K \subset f_i(\text{Int } D)$.
\end{enumerate}

For $m \geq 1$, $p \in \mathbb{R}^m$, and $r > 0$, let $B^m(p, r)$ be the $m$-dimensional closed $r$-ball \{$x \in \mathbb{R}^m \mid \|x - p\| \leq r$\}. When $p$ is the origin of $\mathbb{R}^m$, we write just $B^m(r)$ for it. Remark that $B^1(p, r) \times B^m(q, r) = B^{1+m}((p, q), r)$ for $p \in \mathbb{R}^1$, $q \in \mathbb{R}^m$, and $R > 0$ since $\|\cdot\|$ is the box norm on $\mathbb{R}^m$. The following proposition gives a sufficient condition to $C^1$-stable intersection for a pair of regular Cantor sets in $\mathbb{R}^{l+m}$. \footnote{A subset $\Sigma$ of $I^\mathbb{N}$ is a subshift of finite type if there exists a $\{0, 1\}$-valued matrix $(a_{ij})_{i,j \in I}$ such that $\Sigma = \{(i_n)_{n \geq 1} \in I^\mathbb{N} \mid a_{i_n i_{n+1}} = 1 \text{ for any } n \geq 1\}$.}
Proposition 3.1. Let $I, J$ be positive integers, $I, J$ finite sets, $(f^-_i)_{i \in I}$, $(g^+_j)_{j \in J}$ IFSs on $\mathbb{R}^1$, and $(f^+_i)_{i \in I}$, $(g^-_j)_{j \in J}$ IFSs on $\mathbb{R}^m$. Suppose that there exists constants $\alpha_\tau, \beta_\tau, \Delta_\tau$ for $\tau = 1, 2$ such that

1. $0 < \alpha_\tau < \beta_\tau < 1$, $0 < \Delta_\tau < 1$ for $\tau = 1, 2$,
2. $2\alpha_1\beta_2^{-1} < \Delta_2$, $2\alpha_2\beta_1^{-1} < \Delta_1$,
3. $(f^-_i)_{i \in I}$ is an $\alpha_1$-Cantor IFS such that $f^-_i(B^1(1)) \subset \text{Int} B^1(1)$,
4. $(g^-_j)_{j \in J}$ is an $\alpha_2$-Cantor IFS such that $g^-_j(B^m(1)) \subset \text{Int} B^m(1)$,
5. $(f^+_i)_{i \in I}$ is a covering IFS of type $(\beta_1, \Delta_1, B^m(1))$, and
6. $(g^+_j)_{j \in J}$ is a covering IFS of type $(\beta_2, \Delta_2, B^1(1))$,

and there exist $i_* \in I$ and $j_* \in J$ such that

$$f^-_{i_*}(B^1(1)) \subset \text{Int} g^+_{j_*}(B^1(1)),$$
$$g^-_{j_*}(B^m(1)) \subset \text{Int} f^+_{i_*}(B^m(1)).$$

Then, $(f^-_i \times f^+_i)_{i \in I}$ and $(g^+_j \times g^-_j)_{j \in J}$ are Cantor IFS's on $\mathbb{R}^{m+1}$ and their limit sets have $C^1$-stable intersection, where

$$(f^-_i \times f^+_i)(x, y) = (f^-_i(x), f^+_i(y)),$$
$$(g^+_j \times g^-_j)(x, y) = (g^+_j(x), g^-_j(y)).$$

Proof. Put $f_i = (f^-_i \times f^+_i)$ and $g_j = (g^+_j \times g^-_j)$. By the compactness of $B^m(1)$, there exists $\epsilon_1 > 0$ such that any compact subset $K$ of $B^m(1+\epsilon_1)$ with diam $K \leq \Delta_1$ satisfies $K \subset f^+_i(B^m(1))$ for some $i \in I$. Put $Z_1 = B^1(1) \times B^m(1+\epsilon_1)$ and $R = B^1(1+\epsilon_1)$. Define an $(l, m)$-splitting $(P, Q)$ of $\mathbb{R}^{l+m}$ by $P(x, y) = x$ and $Q(x, y) = y$. Then, $P(R) = P(Z_1)$ and $Q(R) \subset Q(\text{Int} Z_1)$. Since $(P, Q \circ f_i)(x, y) = (x, f^+_i(y))$, the compact set $R$ is a $(P, Q \circ f_i)$-rectangle. By the choice of $\epsilon_1$, the family $(f_i, R)_{i \in I}$ is a blender with respect to $(P, Q)$ with width

Figure 6: Blenders $\{(R, f_i)\}_{i \in I}$ and $\{(R, g_j)\}_{j \in J}$
$\Delta_1$ and blending region $Z_1$. Similarly, there exists $\epsilon_2 > 0$ such that $(g_j, R)_{j \in J}$ is a blender with respect to $(Q, P)$ with width $\Delta_2$ and blending region $Z_2 = B^l(1 + \epsilon_2) \times B^m(1)$.

Put $R_2 = B^l(1) \times (f^-_i)^{-1}(g^-_j, B^m(1))$ and define $h_2 : \mathbb{R}^{l+m} \to \mathbb{R}^{l+m}$ by $h_2 = g^-_j \circ f^-_i$. Since

$$h_2(x, y) = ((g^-_j)^{-1} \circ f^-_i(x), ((g^-_j)^{-1} \circ f^-_i(y))$$

for $x \in \mathbb{R}^l$ and $y \in \mathbb{R}^m$, we have

$$h_2(R_2) = (g^-_j)^{-1}(f^-_i(B^l(1))) \times B^m(1).$$

Recall that

$$f^-_i(B^l(1)) \subset \text{Int} g^+_j(B^l(1)), \quad g^-_j(B^m(1)) \subset \text{Int} f^+_i(B^m(1)).$$

These imply that

$$P(h_2(R_2)) \subset \text{Int} B^l(1) \subset \text{Int} P(Z_2), \quad Q(R_2) \subset \text{Int} B^m(1) \subset \text{Int} Q(Z_1).$$

Since $\|Df^-_i\| \leq \alpha_1$, $\|Dg^-_j\| \leq \alpha_2$, $\|D(f^-_i)^{-1}\| \leq \beta_1$, and $\|D(g^-_j)^{-1}\| \leq \beta_2$, we also have

$$\text{diam } Q(R_2) \leq \beta_1^{-1} \alpha_2 \text{diam } B^l(1) = 2\beta_1^{-1} \alpha_2 < \Delta_1$$

$$\text{diam } P(h_2(R_2)) \leq \beta_2^{-1} \alpha_1 \text{diam } B^m(1) = 2\beta_2^{-1} \alpha_1 < \Delta_2.$$ 

Take $\theta > 0$ such that $2\beta_1^{-1} \alpha_2 + 2\theta < \Delta_1$ and $2\beta_2^{-1} \alpha_1 + 2\theta < \Delta_2$. By the form of $f_i$, $g_j$, $h_2$, we can check that

1. $f_i$ satisfies the $(\theta, \alpha_1^{-1}, \beta_1^{-1})$-cone condition on $R$ with respect to $(P, Q)$,
2. $g_j$ satisfies the $(\theta, \alpha_2^{-1}, \beta_2^{-1})$-cone condition on $R$ with respect to $(Q, P)$,
3. $h_2$ satisfies the $(\theta, \alpha_1^{-1} \beta_2, \alpha_2^{-1} \beta_1)$-cone condition on $R_2$ with respect to $(P, Q)$

Therefore, $(f_i)_{i \in I}$, $(g_j)_{j \in J}$ and $h_2$ satisfy the assumption of Theorem 2.13 for $(P_1, Q_1) = (P, Q)$ and $(P_2, Q_2) = (Q, P)$. Hence, $K((f_i)_{i \in I}) = \Lambda^-((f_i, R)_{i \in I})$ intersects with $K ((g_j)_{j \in J}) = \Lambda^-((g_j, R)_{j \in J})$. The assumption of Theorem 2.13 is $C^1$-stable under perturbation of $(f_i)_{i \in I}$, $(g_j)_{j \in J}$ and $h_2$. Therefore, $K((f_i)_{i \in I})$ and $K((g_j)_{j \in J})$ have $C^1$-stable intersection.

The following proposition completes the proof of Theorem A.

**Proposition 3.2.** For any given integers $l, m \geq 1$ and $\delta > 0$, there exist finite sets $I$, $J$, IFSs $(f^+_i)_{i \in I}$, $(f^-_i)_{i \in I}$, $(g^+_j)_{j \in J}$, and $(g^-_j)_{j \in J}$ such that the assumptions of Proposition 3.7 hold for some constants $\alpha_\tau$, $\beta_\tau$, $\Delta_\tau$ ($\tau = 1, 2$) and the limit sets $K_1$ and $K_2$ of IFSs $(f^-_i \times f^+_i)_{i \in I}$ and $(g^-_j \times g^+_j)_{j \in J}$ on $\mathbb{R}^{m+l}$ satisfy that

$$\dim_H(K_1) < m \log(1 + 2\delta) \dim_H(K_2) < l \log(1 + 2\delta).$$
Proof. Fix positive integers \( l, m \) and a real number \( 0 < \delta < 1/8 \). Put \( w_* = (1, \ldots, 1) \in \mathbb{R}^l \). For each \( v = (v_1, \ldots, v_m) \in \{-1, 1\}^m \), we put \( p(v) = \sum_{n=1}^m v_n/4^n \) and define diffeomorphisms \( f_v^- \) of \( \mathbb{R}^l \) and \( f_v^+ \) of \( \mathbb{R}^m \) by

\[
f_v^-(x) = \frac{\delta}{4^m} x + p(v) w_*, \quad f_v^+(y) = \frac{1 + 2\delta}{2} y + \frac{1}{2} v.
\]

The family \( (f_v^-)_{v \in \{-1,1\}^m} \) is a \( \delta/4^m \)-contracting IFS. Since \( f_v^-(B^l(1)) = B^l(p(v) w_*, \delta/4^m) \), the family \( (f_v^-(B^l(1)))_{v \in \{-1,1\}^m} \) consists of mutually disjoint sets of \( \text{Int} \ B^l(1) \). In particular, the IFS \( (f_v^-)_{v \in \{-1,1\}^m} \) is of Cantor type. Since \( f_v^+(B^m(1)) = B^m((1/2)v, (1+2\delta)/2) \), if a compact subset \( K \) of \( B^m(1) \) satisfies that \( \dim K \leq \delta \) then \( K \subset \text{Int} f_v^+(B^m(1)) \) for some \( v \in \{-1,1\}^m \). Hence, \( (f_v^+)_{v \in \{-1,1\}^m} \) is a covering IFS of type \( ((1+2\delta)/2, \delta, B^m(1)) \). Similarly, for each \( w = (w_1, \ldots, w_l) \in \{-1,1\}^l \), we put \( q(w) = \sum_{n=1}^l w_n/4^n \), and define diffeomorphisms \( g_w^- \) of \( \mathbb{R}^m \) and \( g_w^+ \) of \( \mathbb{R}^l \) by

\[
g_w^-(x) = \frac{\delta}{4^m} x + q(w) v_*, \quad g_w^+(y) = \frac{1 + 2\delta}{2} y + \frac{1}{2} w,
\]

where \( v_* = (1, \ldots, 1) \in \mathbb{R}^m \). Then, \( (g_w^-)_{w \in \{-1,1\}^l} \) is a \( \delta/4^m \)-contracting IFS of Cantor type with \( g_w^- \) of type \( ((1+2\delta)/2, \delta, B^m(1)) \) is a covering IFS of type \( ((1+2\delta)/2, \delta, B^l(1)) \). Remark that

\[
f_v^-(B^l(1)) = B^l(p(v_* w_*), \delta) \subset \text{Int} B^l \left( \frac{1}{2} w_*, \frac{1 + 2\delta}{2} \right) = \text{Int} g_w^+(B^l(1)),
\]

\[
g_w^-(B^m(1)) = B^m(q(w_* v_*), \delta) \subset \text{Int} B^m \left( \frac{1}{2} v_*, \frac{1 + 2\delta}{2} \right) = \text{Int} f_v^+(B^m(1)).
\]

Therefore, the IFSs \( (f_v^+)_{v \in \{-1,1\}^m} \) and \( (g_w^+)_{w \in \{-1,1\}^l} \) satisfies the assumptions of Proposition 3.1 for \( \alpha_\tau = \delta/4^m, \beta_\tau = (1+2\delta)/2, \) and \( \Delta_\tau = \delta \) for \( \tau = 1, 2 \).

Put \( f_v = f_v^- \times f_v^+ \) and \( g_w = g_w^- \times g_w^+ \). Let \( K_1 \) and \( K_2 \) be the limit sets of the IFSs \( (f_v)_{v \in \{-1,1\}^m} \) and \( (g_w)_{w \in \{-1,1\}^l} \), respectively. For \( r > 0 \), let \( N(r) \) be the smallest number of \( r \)-balls to cover \( K_1 \). For \( n \geq 1 \) and \( v = (v(1), \ldots, v(n)) \in (\{-1,1\}^m)^n \), we put \( F_v = f_{v(1)} \circ \cdots \circ f_{v(n)} \) and \( F_v^+ = f_{v(1)}^+ \circ \cdots \circ f_{v(n)}^+ \). Then, \( K_1 \) is covered by \( (F_v(B^{l+m}(1)))_{v \in \{-1,1\}^m} \) and

\[
F_v(B^{l+m}(1)) = B^l \left( F_v^-(O), \frac{\delta}{4^m} \right) \times B^m \left( F_v^+(O), \frac{1 + 2\delta}{2} \right)^n.
\]

Hence, we have \( N(r) \leq 2^n m^n \) for \( ((1+2\delta)/2)^n < r < ((1+2\delta)/2)^{n+1} \). This implies that

\[
\dim_H(K_1) \leq \limsup_{r \to 0} \frac{\log N(r)}{-\log r} = m \log(1 + 2\delta).
\]

Similarly, we can show that \( \dim_H(K_2) \leq l \log(1 + 2\delta) \).
4 Robust homoclinic tangency of the largest codimension

In this section, we apply Theorem 2.13 to prove

Theorem B. For any manifold $M$ with $\dim M \geq 4$, there exists a $C^\infty$ diffeomorphism which exhibits $C^2$-robust homoclinic tangency of codimension $\left\lfloor \frac{1}{2} \dim M \right\rfloor$.

It is easy to obtain a diffeomorphism of $(2m+1)$-dimensional manifold which exhibits $C^2$-robust homoclinic tangency of codimension $m$ from $2m$-dimensional case by taking a product with a one-dimensional strong contraction. So, we will prove Theorem B for even-dimensional manifold.

4.1 Hyperbolic invariant sets and homoclinic tangency

In this subsection, we recall basic results of hyperbolic dynamics and homoclinic tangency. We refer [10] or [20] for details. Let $M$ be a smooth manifold and $f$ be a $C^r$ diffeomorphism of $M$. A subset $\Lambda$ is called an invariant set of $f$ if $f(\Lambda) = \Lambda$. We say that a compact invariant set $\Lambda$ of $f$ is hyperbolic if there exist a Riemannian metric on $M$, a constant $\lambda > 1$, and a continuous splitting $TM|_{\Lambda} = E^s \oplus E^u$ of the tangent bundle $TM$ on $\Lambda$ such that

$$\|Df v\| \leq \lambda^{-1}\|v\|,$$

$$\|Df w\| \geq \lambda\|w\|$$

for any $n \geq 0$, $x \in \Lambda$, $v \in E^s(x)$, and $w \in E^u(x)$, where $\| \cdot \|$ is the norm associated with the Riemannian metric. Let $\Lambda$ be a compact hyperbolic invariant set of $f$. For $x \in M$, we put

$$W^s(x) = \{y \in M \mid d(f^n(y), f^n(x)) \to 0 \ (n \to +\infty)\},$$

$$W^u(x) = \{y \in M \mid d(f^n(y), f^n(x)) \to 0 \ (n \to -\infty)\},$$

where $d$ is the distance induced from a Riemannian metric. By the stable manifold theorem, $W^s(x)$ and $W^u(x)$ are injectively immersed submanifolds of $M$ satisfying $T_xW^s(x) = E^s(x)$ and $T_xW^u(x) = E^u(x)$. They are called the stable and unstable manifolds of $f$ at $x$ respectively. We define the stable set $W^s(\Lambda)$ and the unstable set $W^u(\Lambda)$ by

$$W^s(\Lambda) = \{y \in M \mid \inf_{x \in \Lambda} d(f^n(y), x) \to 0 \ (n \to +\infty)\},$$

$$W^u(\Lambda) = \{y \in M \mid \inf_{x \in \Lambda} d(f^n(y), x) \to 0 \ (n \to -\infty)\}$$

It is known that $W^s(\Lambda) = \bigcup_{x \in \Lambda} W^s(x)$ and $W^u(\Lambda) = \bigcup_{x \in \Lambda} W^u(x)$. We call a point of $(W^s(\Lambda) \cap W^u(\Lambda)) \setminus \Lambda$ a homoclinic point of $\Lambda$. We say that $\Lambda$ exhibits homoclinic tangency if the intersection of $T_zW^s(x)$ and $T_zW^u(y)$ is not transverse for some $x, y \in \Lambda$ and $z \in (W^s(x) \cap W^u(y)) \setminus \Lambda$. The dimension of $T_zW^s(x) \cap T_zW^u(y)$ is called the codimension of tangency at $z$.

An $f$-invariant compact set $S$ is called locally maximal if there exists an open neighborhood $U$ of $S$ such that $S = \bigcap_{n \in \mathbb{Z}} f^n(U)$. Suppose that the hyperbolic
Proof. Take $\Lambda$ as Remark 2.3). Then, for any $N \geq 0$, the $f^n(U)$ of $f$ in the space of $C^1$ diffeomorphisms of $M$ with respect to the $C^1$-topology such that $\Lambda(g) = \bigcap_{n \in \mathbb{Z}} ^n g^n(U)$ is a hyperbolic compact invariant set of $g$ for any $g \in U_f$. The family $(\Lambda(g))_{g \in U_f}$ is called the continuation of $\Lambda$ on $U_f$. We say that $\Lambda$ exhibits $C^k$-robust homoclinic tangency if $\Lambda(g)$ exhibits homoclinic tangency for any diffeomorphism $g$ which is $C^k$-close to $f$.

4.2 Lifts of diffeomorphisms on splitted spaces

Fix integers $l, m \geq 1$. We denote the set of real $(l, m)$-matrices by $\text{Mat}(l, m)$. Let $M$ be an $(l + m)$-dimensional manifold. By $\text{Gr}_x(M, m)$, we denote the set of $m$-dimensional linear subspaces of the tangent space $T_x M$ at $x \in M$. The Grassmannian bundle $\text{Gr}(M, m) = \bigcup_{x \in M} \text{Gr}_x(M, m)$ is a $C^\infty$ fiber bundle over $M$ whose fiber $\text{Gr}_x(M, m)$ is of dimension $lm$. Let $\pi : \text{Gr}(M, m) \to M$ be the natural projection. We can choose local coordinates of $\text{Gr}(M, m)$ as follows: For $x \in M$ and $\xi \in \text{Gr}_x(M, m)$, take a smooth coordinate $(U, \varphi)$ of $M$ at $x$ such that $D\varphi(\xi) = \{0\} \oplus \mathbb{R}^m$. Put

$$V_\varphi = \{\eta \in \text{Gr}(M, m) \mid \pi(\eta) \in U, D\varphi(\eta) \text{ is transverse to } \mathbb{R}^l \oplus \{0\}\}.$$ 

We can define a map $\Pi_\varphi : V_\varphi \to \text{Mat}(l, m)$ by

$$D\varphi(\eta) = \{(\Pi_\varphi(\eta)v, v) \mid v \in \mathbb{R}^m\}.$$ 

Then, the map $(\pi, \Pi_\varphi) : \eta \mapsto (\pi(\eta), \Pi_\varphi(\eta))$ is a bijection from $V_\varphi$ to $\pi(U) \times \text{Mat}(l, m)$ and the family $\{(V_\varphi, \pi \times \Pi_\varphi)\}$ gives the smooth structure of $\text{Gr}(M, m)$.

For an open subset $U$ of $M$ and an $C^1$ embedding $f : U \to M$, we define the lift $\tilde{f} : \text{Gr}(U, m) \to \text{Gr}(M, m)$ of $f$ by $\tilde{f}(\xi) = \{Df v \mid v \in \xi\}$ for $\xi \in \text{Gr}(U, m)$. The lift $\tilde{f}$ is of class $C^{k-1}$ if $f$ is of class $C^k$.

Proposition 4.1. Let $M$ be a manifold with an $(l, m)$-splitting $(P, Q)$, $U$ an open subset of $M$, and $f : U \to M$ a $C^2$ embedding, and $\theta$ a positive constant. Put $\Lambda = \bigcap_{n \in \mathbb{Z}} f^n(U)$ and

$$\tilde{U} = \{\xi \in \text{Gr}_x(M, m) \mid x \in U, \xi \cap C(x, \theta, P, Q) = \{0\}\}.$$ 

Suppose that $\Lambda$ is a compact subset of $U$ and $f$ satisfies the $(\theta, 1, 1)$-cone condition with respect to $(P, Q)$ ($\Lambda$ is a hyperbolic invariant set of $f$ as Remark 2.3). Then, for any $N \geq 0$,

$$\bigcap_{n \geq N} \tilde{f}^n(\tilde{U}) \subset \{T_q W^u(p) \mid p \in \Lambda, q \in W^u(p, f)\}.$$ 

Proof. Take $\xi \in \bigcap_{n \geq N} \tilde{f}^n(\tilde{U})$ and put $q = \pi(\xi)$. Then, $q$ is contained in $\bigcap_{n \geq N} f^n(U)$. Since $\Lambda$ is the maximal compact hyperbolic invariant subset in $U$, we have

$$\bigcap_{n \geq N} f^n(U) \subset W^u(\Lambda, f) = \bigcup_{p \in \Lambda} W^u(p, f).$$
Hence, there exists $p \in \Lambda$ such that $q \in W^u(p, f)$. The proof will finish once we show that $\xi \subset T_q W^u(p, f)$ since the dimensions coincide.

Notice that any $w \in T_q W^u(p, f)$ satisfies that $\lim n \to \infty \|Df^{-n}w\| = 0$. There exists $0 < \theta' < \theta$ such that $Df^{-n}v \in C(f^{-n}(q), \theta', P, Q)$ and $\|Df^{-n}v\| \geq \|v\|$ for any $v \in C(q, \theta, P, Q)$ and $n \geq 1$. Since $\text{Ker } DQ_q \subset C(q, \theta, P, Q)$, we have a splitting $T_q M = \text{Ker } DQ_q \oplus T_q W^u(p, f)$. Take $v \in \text{Ker } DQ_q$ and $w \in T_q W^u(p, f)$ such that $v + w \in \xi$. To show that $\xi \subset T_q W^u(p, f)$, it is sufficient to see that $v = 0$. Suppose that $v \neq 0$. Then, $\|Df^{-n}v\| \geq \|v\|$, $Df^{-n}v \in C(f^{-n}, \theta', P, Q)$ for any $n \geq 1$, and $\|Df^{-n}w\|$ goes to zero as $n$ tends to infinity. These imply that $Df^{-n}(v + w) \in C(f^{-n}(q), \theta, P, Q)$, and hence, $\hat{f}^{-n}(\xi)$ intersects with $C(f^{-n}(q), \theta, P, Q)$ for any sufficiently large $n$. However, it contradicts that $\xi \in \bigcap_{n \geq N} \hat{f}^n(U)$. Therefore, $v = 0$. \hfill \Box

\section{Horseshoe whose lift is a blender}

We construct a horseshoe whose lift to the Grassmannian bundle contains a blender in the sense of Section 2.3. Our example is essentially same as the blender horseshoe given by Barrientos and Raibekas in [2].

Recall that $\text{Mat}(l, m)$ is the set of real $(l, m)$-matrices for $l, m \geq 1$. By $\|A\|$, we denote the operator norm of $A \in \text{Mat}(l, m)$ with respect to the box norms on $\mathbb{R}^l$ and $\mathbb{R}^m$ as a linear map from $\mathbb{R}^m$ to $\mathbb{R}^l$. For $A \in \text{Mat}(l, m)$ and $r > 0$, let $B(l, m)(A, r)$ be the $lm$-dimensional closed $r$-ball $\{A' \in \text{Mat}(l, m) \mid \|A' - A\| \leq r\}$ centered at $A$. We write $B(l, m)(r)$ when $A$ is the zero matrix $O$.

Fix a quadruple $m = (m_1, m_2, m_3, m_4)$ of positive numbers. Put $m_* = m_1 + m_2 + m_3 + m_4$, $m^- = n_1 + n_2$, and $m_+ = n_3 + n_4$. Define an $(m_-, m_+)$-splitting $(P, Q)$ of $\mathbb{R}^{m_*}$ by

$$P(x_1, x_2, x_3, x_4) = (x_1, x_2), \quad Q(x_1, x_2, x_3, x_4) = (x_3, x_4)$$

for $x_j \in \mathbb{R}^{m_j}$ $(j = 1, 2, 3, 4)$. Remark that $\text{Ker } DP = \{0\} \oplus \mathbb{R}^{m_+}$ and $\text{Ker } DQ = \mathbb{R}^{m^-} \oplus \{0\}$. Let $\tilde{M}$ be the subset of $\text{Gr}(\mathbb{R}^{m_*}, m_+)$ given by

$$\tilde{M} = \{\xi \in \text{Gr}(\mathbb{R}^{m_*}, m_+) \mid \xi \text{ is transverse to } \text{Ker } DQ = \mathbb{R}^{m_*} \text{-plus}\{0\}\}.\!

Let $\Pi : \tilde{M} \to \text{Mat}(m_-, m_+)$ be a map given by

$$D(P \times Q)(\xi) = \{(\Pi(\xi)w, w) \mid w \in \mathbb{R}^m\}.$$

In other words, $\Pi(\xi)$ is the unique element of $\text{Mat}(m_-, m_+)$ such that $DPv = \Pi(\xi)DQv$ for any $v \in \xi$. The map $(\pi, \Pi) : \xi \to (\pi(\xi), \Pi(\xi))$ is a diffeomorphism from $\tilde{M}$ to $\mathbb{R}^{m_*} \times \text{Mat}(m_-, m_+)$. We define $\Pi_{ij} : \tilde{M} \to \text{Mat}(m_i, m_j)$ for $i = 1, 2$ and $j = 3, 4$ by

$$\Pi(\xi) = \begin{pmatrix} \Pi_{13}(\xi) & \Pi_{14}(\xi) \\ \Pi_{23}(\xi) & \Pi_{24}(\xi) \end{pmatrix}.$$
For $i = 1, 2, 3, 4$, let $\pi_i : \hat{M} \to \mathbb{R}^{m_i}$ be the $i$-th component of the natural projection $\pi : \hat{M} \to \mathbb{R}^{m_*} = \mathbb{R}^{m_1} \times \mathbb{R}^{m_2} \times \mathbb{R}^{m_3} \times \mathbb{R}^{m_4}$. Put
\[
E^- = \mathbb{R}^{m_1} \times \text{Mat}(m_1, m_3) \times \text{Mat}(m_3, m_4) \times \text{Mat}(m_4, m_2),
\]
\[
E^+ = \mathbb{R}^{m_2} \times \mathbb{R}^{m_3} \times \mathbb{R}^{m_4} \times \text{Mat}(m_2, m_3),
\]
and define map $\mathcal{P} : \hat{M} \to E^-$ and $\mathcal{Q} : \hat{M} \to E^+$ by
\[
\mathcal{P} = (\pi_1, \Pi_{13}, \Pi_{14}, \Pi_{24}), \quad \mathcal{Q} = (\pi_2, \pi_3, \pi_4, \Pi_{23}).
\]
Then, the pair $(\mathcal{P}, \mathcal{Q})$ is an $(\hat{l}, \hat{m})$-splitting of $\hat{M}$ with $\hat{l} = m_1 + m_2m_3 + m_4 + m_2m_4$ and $\hat{m} = m_2 + m_3 + m_4 + m_2m_3$.

For a quadruple of integers $\mathbf{m} = (m_1, m_2, m_3, m_4)$, positive constants $\lambda, \mu, \Delta, \theta$, and a $(\mathcal{P}, \mathcal{Q})$-rectangle $Z$, we say that a $C^2$ diffeomorphism of $\mathbb{R}^{m_*}$ is a BR-blender horseshoe map\footnote{BR” means “Barrientos and Raibekas” because our proof of the existence of such a map is based on their construction in \cite{2}.} of type $(\mathbf{m}, \lambda, \mu, \Delta, \theta, Z)$ if there exist families $(\mathcal{R}_i)_{i \in I}$ and $(\mathcal{R}_i)_{i \in I}$ indexed by a finite set $I$ such that

1. $R_i$ is a compact $(P, Q \circ f)$-rectangle in $\mathbb{R}^{m_*}$ with
\[
P(R_i) = B^{m_1}(1), \quad (P \circ f)(R_i) \subset \text{Int } B^{m_1}(1) \setminus B^{m_1}(1/2),
\]
\[
Q(R_i) \subset \text{Int } B^{m_2}(1) \setminus B^{m_2}(1/2), \quad (Q \circ f)(R_i) = B^{m_2}(1)
\]
and $f$ satisfies the $(1, 1, 1)$-cone condition on $R_i$ for each $i \in I$,

2. $\mathcal{R}_i$ is a compact subset of $\hat{M}$ with $\pi(\mathcal{R}_i) \subset \text{Int } R_i$ and the lift $\hat{f}$ of $f$ satisfies the $(\theta, \lambda, \mu)$-cone condition on $\mathcal{R}_i$ for each $i \in I$, and

3. $\{\{\hat{f}|_{\mathcal{R}_i}, \mathcal{R}_i\}\}_{i \in I}$ is a blender with the width $\Delta$ and the blender region $Z$.

**Remark 4.2.** The set of $C^2$ BR-blender horseshoe maps of fixed type $(\mathbf{m}, \lambda, \mu, \Delta, Z)$ is $C^2$-open.

**Remark 4.3.** Let $(R_i)_{i \in I}$ and $B = (\mathcal{R}_i)_{i \in I}$ be the families of rectangles in the definition. Put
\[
R = \bigcup_{i \in I} R_i, \quad \Lambda = \bigcap_{n \in \mathbb{Z}} \hat{f}^n(R),
\]
\[
\mathcal{R} = \bigcup_{i \in I} \mathcal{R}_i, \quad \hat{\Lambda}^- = \bigcap_{n \geq 1} \hat{f}^n(\mathcal{R}).
\]

Then, by Lemma 2.38 and Proposition 4.11 we have $\Lambda = \bigcap_{n \in \mathbb{Z}} f^n(\text{Int } R)$ and
\[
\hat{\Lambda}^- \subset \{T_q W^u(p) \mid p \in \Lambda, q \in W^u(p, f)\}.
\]

Lemma 2.39 also implies that the restriction of $f$ on $\Lambda$ is topologically transitive.
Proposition 4.4. Let $\mathbf{m} = (m_1, m_2, m_3, m_4)$ be a quadruple of positive integers. For any constants $\lambda > 1 > \mu > 0$ with $\lambda \mu > 1$, there exists $\Delta > 0$, a $(P, Q)$-rectangle $Z$, and a $C^2$ diffeomorphism $f$ of $\mathbb{R}^m$ such that

- $\operatorname{diam} P(Z) \leq 2$ and $Q(\text{Int} Z)$ contains the origin of $E^+$;
- $f$ is a $C^2$ BR-blender horseshoe map of type $(\mathbf{m}, \lambda, \mu, \Delta, \theta, Z)$ for any $\theta > 0$.

The rest of this subsection is devoted to construct the diffeomorphism $f$ under suitable choices of $\Delta$ and $Z$. As mentioned above, our construction below is essentially same the one done by Barrientos and Raibekas in [2].

First, we fix positive constants $\alpha, r_2, r_3, r_4$, and $\Delta$ so that

\[
\mu^{1/2} < \alpha < 1, \quad 0 < r_2 < 1 - \alpha, \quad \alpha < r_3 < 1, \\
0 < 2r_4 < \alpha r_2, \quad 0 < 3\Delta < \alpha^2 r_4.
\]

By compactness of $B^{m_2}(r_2) \times B^{(m_2,m_3)}(r_4)$, there exist families $(p_i^2)_{i \in I}$ and $(C_i)_{i \in I}$ indexed by a finite set $I$ such that $p_i^2$ is a point in $\text{Int}(B^{m_2}(r_2))$, $C_i$ is a matrix in $\text{Int} B^{(m_2,m_3)}(r_4)$, and

\[
B^{m_2}(r_2) \times B^{(m_2,m_3)}(r_4) \subseteq \bigcup_{i \in I} \left( \text{Int} B^{m_2}(p_i^2, \Delta) \times \text{Int} B^{(m_2,m_3)}(C_i, \Delta) \right).
\]

For $\sigma = 1, 4$, take families $(p_i^\sigma)_{i \in I}$ of mutually distinct points in $\mathbb{R}^{m_\sigma}$ such that $1/2 < \|p_i^\sigma\| < 1$ for any $i \in I$. Then, there exists $1/2 < r_\sigma < 1$ and $\beta < \lambda^{-1}$ such that $(B^{m_\sigma}(p_i^\sigma, \beta))_{i \in I}$ is a family of mutually disjoint balls in $\text{Int} B^{m_\sigma}(r_\sigma) \setminus B^{m_\sigma}(1/2)$ for each $\sigma = 1, 4$. Put

\[
R_i^+ = B^{m_3}(\alpha) \times B^{m_4}(p_i^4, \beta), \quad R_i = B^{m_-}(1) \times R_i^+ \quad \text{for } i \in I.
\]

Let $f$ be a diffeomorphism of $\mathbb{R}^{m_\sigma}$ such that

\[
f \left( \begin{array}{c} x_1 \\ x_2 \\ x_3 \\ x_4 \end{array} \right) = \left( \begin{array}{c} \beta x_1 + p_i^1 \\ \alpha x_2 + C_i(\alpha^{-1} x_3) + p_i^2 \\ \alpha^{-1} x_3 \\ \beta^{-1} (x_4 - p_i^4) \end{array} \right)
\]

for any $(x_1, x_2, x_3, x_4) \in R_i$. We check that $f$ is a BR-blender horseshoe map of type $(\mathbf{m}, \lambda, \mu, \Delta, \theta, Z)$ for any $\theta > 0$. It is easy to see that $R_i$ of $\mathbb{R}^{m_\sigma}$ is a $(P, Q \circ f_0)$-rectangle with

\[
P(R_i) = B^{m_-}(1), \quad (P \circ f_0)(R_i) \subset \text{Int} B^{m_-}(1) \setminus B^{m_-}(1/2), \\
(Q \circ f)(R_i) = B^{m_+}(1), \quad Q(R_i) = R_i^+ \subset \text{Int} B^{m_+}(1) \setminus B^{m_+}(1/2).
\]

Hence, the former half of the first item in the definition of a BR-blender horseshoe map holds. Put $R = \bigcup_{i \in I} R_i$ and $\Lambda = \bigcap_{n \in \mathbb{Z}} f^n(R)$. The following lemma gives the latter half of the first item.
Lemma 4.5. The diffeomorphism $f$ satisfies the $(1,1,1)$-cone condition on $R$ with respect to the $(m_-, m_+)$-splitting $(P, Q)$. In particular, $\Lambda$ is a hyperbolic invariant set of $f$.

Proof. Take $x \in R_i$ and $v \in T_x \mathbb{R}^{m_*}$ and put $v' = Df v$. Then,
\[
\|DQv\| \leq \alpha \|DQv'\|,
\]
\[
\|DPv'\| \leq \alpha \|DPv\| + \|C_i(\alpha^{-1}\|DQv\|)\| \leq \alpha \|DPv\| + \alpha^{-1} r_* \|DQv\|.
\]
If $v \in C(x, 1, Q, P)$, then we have
\[
\|DPv'\| \leq (\alpha + \alpha^{-1} r_*) \|DQv\| \leq (\alpha^2 + r_*) \|DQv'\|
\]
If $v' \in C(f(x), 1, P, Q)$, then we have
\[
(1 - r_*) \|DQv\| \leq \alpha \|DPv'\| - r_* \|DQv\|
\]
\[
\leq \alpha \|DPv\| - r_* \|DQv\|
\]
\[
\leq \alpha^2 \|DPv\|,
\]
\[
(1 - r_*) \|DPv'\| \leq \alpha \|DPv\| + r_* (\alpha^{-1}\|DQv\| - \|DPv'\|)
\]
\[
\leq \alpha \|DPv\| + r_* (\|DQv'\| - \|DPv'\|)
\]
\[
\leq \alpha \|DPv\|.
\]
Since $\alpha > 1$ and $\alpha + r_* < 1$, the $f$ satisfies the $(1,1,1)$-cone condition on $R_i$. In particular, $f$ admits a hyperbolic splitting on $\Lambda = \bigcap_{n \in \mathbb{Z}} f^n(\bigcup_{i \in I} R_i)$.

Let $\hat{f} : \text{Gr}(\mathbb{R}^{m_*}, m_+) \to \text{Gr}(\mathbb{R}^{m_*}, m_+)$ be the lift of $f$. Put
\[
\hat{U} = \{ \xi \in \hat{M} \mid \pi(\xi) \in \text{Int } R, \|\Pi(\xi)\| < 1 \}.
\]
Since $f$ satisfies the 1-cone condition on $R$, we have $\hat{f}(\hat{U}) \subset \hat{M}$. For $i \in I$, we define maps $F_i^- : E^- \to E^-$ and $F_i^+ : E^+ \to E^+$ by
\[
F_i^- \begin{pmatrix} x_1 \\ A_{13} \\ A_{14} \\ A_{24} \end{pmatrix} = \begin{pmatrix} \beta x_1 + p_1' \\ \alpha \beta A_{13} \\ \beta^2 A_{14} \\ \alpha \beta A_{24} \end{pmatrix},
\]
\[
F_i^+ \begin{pmatrix} x_2 \\ x_3 \\ x_4 \\ A_{23} \end{pmatrix} = \begin{pmatrix} \alpha x_2 + C_i (\alpha^{-1} x_3) + p_2' \\ \alpha^{-1} x_3 \\ \beta^{-1} (x_4 - p_4') \\ \alpha^2 A_{23} + C_i \end{pmatrix}.
\]

Figure 7: A BR blender horseshoe map $f$
By direct computation, we can see that
\[ P \circ \hat{f}(\xi) = F_i^{-1} \circ P(\xi), \quad Q \circ \hat{f}(\xi) = F_i^+ \circ Q(\xi) \]
for any \( \xi \in \hat{U} \) with \( \pi(\xi) \in R_i \).

Let \( Z^- \), \( Z^+ \), \( Z \) be compact subsets of \( E^- \), \( E^+ \), and \( \hat{M} \) given by
\[
\begin{align*}
Z^- &= B^{m_1}(r_1) \times B^{(m_2, m_3)}(1/4) \times B^{(m_1, m_4)}(1/4) \times B^{(m_2, m_3)}(1/4), \\
Z^+ &= B^{m_2}(r_2) \times B^{m_3}(r_3) \times B^{m_4}(r_4) \times B^{(m_2, m_3)}(r_4), \\
Z &= \{ \xi \in \hat{M} \mid P(\xi) \in Z^-, Q(\xi) \in Z^+ \}.
\end{align*}
\]

For \( \gamma > 0 \) and \( i \in I \), we define compact subsets \( R_i^+(\gamma) \) of \( E^+ \) and \( R_i \) of \( \hat{M} \) by
\[
R_i^+(\gamma) = B^{m_2}(p_i^2, \gamma) \times B^{m_4}(C_i, \gamma), \\
R_i = \{ \xi \in \hat{M} \mid P(\xi) \in Z^-, F_i^+(Q(\xi)) \in R_i^+(3\Delta) \}.
\]

Since \( P \circ \hat{f} = F_i^{-1} \circ P \) and \( Q \circ \hat{f} = F_i^+ \circ Q \) on \( R_i \), the compact set \( R_i \) is a \((P, Q \circ \hat{f})\)-rectangle with
\[
\hat{f}(R_i) = \{ \xi \in \hat{M} \mid P(\xi) \in F_i^+(Z^-), Q(\xi) \in R_i^+(3\Delta) \}.
\]

The the inverse of \( F_i^+ \) can be written as
\[
(F_i^+)^{-1} = \begin{pmatrix}
y_2 \\
y_3 \\
y_4 \\
A_{23}
\end{pmatrix} = \begin{pmatrix}
\alpha^{-1}(y_2 - p_i^2 - C_i y_3) \\
\alpha y_3 \\
\beta y_4 + p_i^4 \\
\alpha^{-2}(A_{23} - C_i)
\end{pmatrix}.
\]

Since \( \|C_i\| \leq r_s \), we have
\[
(F_i^+)^{-1}(R_i^+(3\Delta)) \subset B^{m_2}(\alpha^{-1}(3\Delta + r_s)) \times B^{m_3}(\alpha) \\
\times B^{m_4}(p_i^4, \beta) \times B^{(m_2, m_3)}(3\alpha^{-2}\Delta).
\]

By the choice of constants, this implies that
\[
Q(R_i) = (F_i^+)^{-1}(R_i^+(3\Delta)) \subset \text{Int } Z^+, \\
\pi(R_i) = \pi(\{ \xi \in \hat{M} \mid P(\xi) \in Z^-, Q(\xi) \in (F_i^+)^{-1}(R_i^+(3\Delta)) \}) \\
\subset B^{m_1}(1) \times B^{m_3}(\alpha) \times B^{m_4}(p_i^4, \beta) = R_i.
\]

We also have
\[
F_i^-(Z^-) = B^{m_1}(p_i^1, \beta) \times B^{(m_1, m_3)}(\alpha, \beta/4) \times B^{(m_1, m_4)}(\beta^2/4) \times B^{(m_2, m_3)}(\alpha, \beta/4),
\]
and hence, \((P \circ \hat{f})(R_i) = F_i^{-}(Z^-) \subset \text{Int } Z^-\). \((P \circ \hat{f})(R_i) = F_i^{-}(Z^-) \subset \text{Int } Z^-\).

Therefore, the former half of the second item in the definition of the BR-blender
We also have for any $\xi \lambda \mu > 1$, the latter implies that since
\[ \text{Proof.} \]
\[
(\text{Lemma 4.6.}) \]
4.4 A map connecting blenders

4.4 A map connecting blenders

The following lemma shows that the latter half of the second item in the
definition of a BR-blender horseshoe map holds. Hence, it complites the proof
of the proposition.

**Lemma 4.6.** For any positive number $\theta > 0$, the map $\tilde{f} : \hat{U} \to \hat{M}$ satisfies the
$(\theta, \lambda, \mu)$-cone condition with respect to the splitting $(\mathcal{P}, \mathcal{Q})$ on $\mathcal{R}$

\[
\text{Proof. Since } \mu^{1/2} < \alpha < 1, r_2 < 1 - \alpha, \|C_i\| \leq r_*, \text{ and } \beta > \lambda, \text{ we have}
\]
\[
\|DF^\perp\| \leq \max\{\beta, \alpha \beta, \beta^2\} = \beta < \lambda^{-1}
\]
\[
\|D(F_i^\perp)^{-1}\| \leq \max\{\alpha^{-1}(1 + r_*), \alpha, \beta, \alpha^{-2}\} < \alpha^{-2} < \mu^{-1}.
\]

We also have
\[
\|D\mathcal{P}(D\tilde{f}(v))\| = \|Df^\perp(\mathcal{D}\mathcal{P}(v))\| \leq \lambda^{-1}\|D\mathcal{P}(v)\|
\]
\[
\|D\mathcal{Q}(D\tilde{f}(v))\| = \|Df^\perp(\mathcal{D}\mathcal{Q}(v))\| \geq \mu\|D\mathcal{Q}(v)\|
\]

for any $\xi \in \mathcal{R}_i$ and $v \in T_{\xi} \hat{M}$. This implies that $(\mathcal{Q} \circ \tilde{f})(\mathcal{R}_i) = \mathcal{R}_i^+(3\Delta)$. Since
$\lambda \mu > 1$, the latter implies that $\tilde{f}$ satisfies the $(\theta, \lambda, \mu)$-cone condition for any
$\theta > 0$.

\[ \square \]

4.4 A map connecting blenders

To apply Theorem 4.13, we need a map $h_\sharp$ which connects two blenders. In this
subsection, we construct the map $h_\sharp$ for BR-blender horseshoe maps on $\mathbb{R}^{2m}$
with $m = (1, m - 1, m - 1, 1)$ and $(m - 1, 1, m - 1, 1)$.

Fix $m \geq 2$. Let $P, Q : \mathbb{R}^{2m} \to \mathbb{R}^m$ be the projection to the first and second
components of the splitting $\mathbb{R}^{2m} = \mathbb{R}^m \times \mathbb{R}^m$ and put
\[
\hat{M} = \{ \xi \in \text{Gr}(\mathbb{R}^{2m}, m) \mid \xi \cap \text{Ker} \, DQ = \{0\} \}.
\]

In the same way as in Section 4.3, $\hat{M}$ admits a $(2m, m^2)$-splitting $(P_1, Q_1) : \hat{M} \to E_1^\perp \times E_1^+$ and an $(m^2, 2m)$-splitting $(P_2, Q_2) : \hat{M} \to E_2^\perp \times E_2^+$ associated with
\[ m_1 = (1, m - 1, m - 1, 1) \] and \[ m_2 = (m - 1, 1, m - 1, 1) \] respectively, where
\[
E_1^\perp = \mathbb{R} \times \text{Mat}(1, m - 1) \times \text{Mat}(1, 1) \times \text{Mat}(m - 1, 1),
\]
\[
E_1^+ = \mathbb{R}^{m-1} \times \mathbb{R}^{m-1} \times \mathbb{R} \times \text{Mat}(m - 1, m - 1),
\]
\[
E_2^\perp = \mathbb{R}^{m-1} \times \text{Mat}(m - 1, m - 1) \times \text{Mat}(m - 1, 1) \times \text{Mat}(1, 1),
\]
\[
E_2^+ = \mathbb{R} \times \mathbb{R}^{m-1} \times \mathbb{R} \times \text{Mat}(1, m - 1).
\]

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For \( \tau = 1, 2 \), let \( \mathcal{Y}_\tau \) be a compact \((\mathcal{P}_\tau, \mathcal{Q}_\tau)\)-rectangle. For constants \( \theta, \nu > 0 \), let \( \mathcal{V}(\theta, \nu, \mathcal{Y}_1, \mathcal{Y}_2) \) be the set of \( C^2 \) diffeomorphisms \( h \) of \( \mathbb{R}^{2m} \) such that there exists a compact subset \( \mathcal{R}_\tau \) of \( \hat{\mathcal{M}} \) which satisfies the following conditions:

1. \( \mathcal{R}_\tau \) is a \((\mathcal{P}_1, \mathcal{P}_2 \circ \hat{h})\)-rectangle such that
   
   \[
   \mathcal{P}_1(\mathcal{R}_\tau) = \mathcal{P}_1(\mathcal{Y}_1), \quad (\mathcal{P}_2 \circ \hat{h})(\mathcal{R}_\tau) = \mathcal{P}_2(\mathcal{Y}_2),
   \]
   
   \[
   \mathcal{Q}_1(\mathcal{R}_\tau) \subset \text{Int } \mathcal{Q}_1(\mathcal{Y}_1), \quad (\mathcal{Q}_2 \circ \hat{h})(\mathcal{R}_\tau) \subset \text{Int } \mathcal{Q}_2(\mathcal{Y}_2),
   \]
   
   where \( \hat{h} \) is the lift of \( h \) to \( \text{Gr}(\mathbb{R}^{2m}, m) \).

2. The lift \( \hat{h} \) of \( f \) satisfies the \((\theta, \nu, \nu)\)-cone condition on \( \mathcal{R}_\tau \) with respect to the \((m^2, 2m)\)-splittings \((\mathcal{P}_1, \mathcal{Q}_1)\) and \((\mathcal{P}_2, \mathcal{Q}_2)\).

**Remark 4.7.** The set \( \mathcal{V}(\theta, \mu, \mathcal{Y}_1, \mathcal{Y}_2) \) is \( C^2 \)-open.

**Remark 4.8.** \( \mathcal{R}_\tau \) is always contained in the \((\mathcal{P}_1, \mathcal{Q}_1)\)-rectangle \( \mathcal{Y}_1 \) and \( \hat{h}(\mathcal{R}_\tau) \) is contained in the \((\mathcal{P}_2, \mathcal{Q}_2)\)-rectangle \( \mathcal{Y}_2 \).

**Proposition 4.9.** For \( \tau = 1, 2 \), let \( \mathcal{Y}_\tau \) be a compact \((\mathcal{P}_\tau, \mathcal{Q}_\tau)\)-rectangle such that \( \text{Int } \mathcal{P}(\mathcal{Y}_\tau) \) and \( \text{Int } \mathcal{Q}(\mathcal{Y}_\tau) \) contain the origins of \( E^-_\tau \) and \( E^+_\tau \) respectively. Then, \( \mathcal{V}(\theta, \nu, \mathcal{Y}_1, \mathcal{Y}_2) \) is non-empty for any given \( \theta, \nu > 0 \).

The rest of this subsection is devoted to construction of a map in \( \mathcal{V}(\theta, \nu, \mathcal{Y}_1, \mathcal{Y}_2) \).

First, we define a map \( h_0 : \mathbb{R}^{2m} \to \mathbb{R}^{2m} \) by

\[
h_0 \begin{pmatrix} x_1 \\ \bar{x}_2 \\ \bar{x}_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} \bar{x}_2 + x_4 \cdot \bar{x}_3 \\ x_1 + x_4^2/2 \\ \bar{x}_3 \\ \bar{x}_4 \end{pmatrix},
\]

where \( x_1, x_4 \in \mathbb{R} \) and \( \bar{x}_2, \bar{x}_3 \in \mathbb{R}^{m-1} \). Then,

\[
(Dh_0)_{(x_1, \bar{x}_2, \bar{x}_3, x_4)} = \begin{pmatrix}
O & I_{m-1} & x_4 I_{m-1} & \bar{x}_3 \\
1 & 0 & O & x_4 \\
O & O & I_{m-1} & O \\
O & O & O & 1
\end{pmatrix},
\]

where \( I_{m-1} \) is the identity matrix of size \((m-1)\). Since \( Dh_0 \) preserves \( \text{Ker } D\mathcal{Q} = \mathbb{R}^m \oplus \{0\} \), the lift \( \hat{h}_0 \) of \( h_0 \) to \( \text{Gr}(\mathbb{R}^{2m}, m) \) preserves \( \hat{\mathcal{M}} \). Define \( H_0 : E_1^- \times E_1^+ \to E_2^- \times E_2^+ \) by

\[
H_0 = (\mathcal{P}_2, \mathcal{Q}_2) \circ \hat{h}_0 \circ (\mathcal{P}_1, \mathcal{Q}_1)^{-1}.
\]

Then, we can check that

\[
H_0 \begin{pmatrix} x_1 \\ A_{13} \\ a_{14} \\ \bar{a}_{24} \end{pmatrix}, \begin{pmatrix} \bar{x}_2 \\ \bar{x}_3 \\ \bar{x}_4 \\ A_{23} \end{pmatrix} = \begin{pmatrix} \bar{x}_2 + x_4 \cdot \bar{x}_3 \\ \bar{x}_3 + x_4 A_{13} \\ \bar{x}_4 \\ \bar{x}_3 + x_4 A_{13} \end{pmatrix}.
\]

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where we identify Mat($m - 1$) with $\mathbb{R}^{m-1}$ and Mat($1, 1$) with $\mathbb{R}$. Let $P^{\pm}_\tau : E^1_\tau \times E^\mp_\tau \to E^\pm_\tau$ be the projection to $E^\pm_\tau$ with respect to the splitting for $\tau = 1, 2$.

Then, $(P_1, P_2 \circ \hat{h}_0) = (P^{-}_1, P^{-}_2 \circ H_0) \circ (P_1, Q_1)$ and the map $(P^+_1, P^+_2 \circ H_0) : E^+_1 \times E^+_1 \to E^+_1 \times E^+_2$ has the inverse

$$\begin{pmatrix}
  x_1 \\ A_{13} \\ a_{14} \\ \vec{a}_{24}
\end{pmatrix}
\to
\begin{pmatrix}
  \vec{y}_1 \\ B_{13} \\ \vec{b}_{14} \\ b_{24}
\end{pmatrix} =
\begin{pmatrix}
  \vec{y}_1 - (b_{24} - a_{14})(\vec{b}_{14} - \vec{a}_{24}) \\ \vec{b}_{14} - \vec{a}_{24} \\ b_{24} - a_{14} \\ B_{13} - (b_{24} - a_{14}m_{1m-1})
\end{pmatrix}.
$$

This implies that $(P_1, P_2 \circ \hat{h}_0)$ is a diffeomorphism. In particular, for any compact subsets $K^- \subset E^1_-$ and $K^+ \subset E^+_2$, there exists a unique $(P_1, P_2 \circ \hat{h}_0)$-rectangle $R_K$ such that $P_1(R_K) = K^-$ and $(P_2 \circ \hat{h}_0)(R_K) = K^+$.

For $t > 0$, we define linear maps $L_{1,t} : \mathbb{R}^{2m} \to \mathbb{R}^{2m}$ and $L_{2,t} : \mathbb{R}^{2m} \to \mathbb{R}^{2m}$ by

$$
L_{1,t}
\begin{pmatrix}
  x_1 \\ \vec{x}_2 \\ \vec{x}_3 \\ x_4
\end{pmatrix}
= 
\begin{pmatrix}
  e^{-t}x_1 \\ e^{2t}\vec{x}_2 \\ e^{2t}\vec{x}_3 \\ e^{3t}x_4
\end{pmatrix},
L_{2,t}
\begin{pmatrix}
  \vec{y}_1 \\ y_2 \\ y_3 \\ y_4
\end{pmatrix}
= 
\begin{pmatrix}
  e^{-t}\vec{y}_1 \\ e^{2t}y_2 \\ e^{2t}y_3 \\ e^{3t}y_4
\end{pmatrix},
$$

where $x_1, x_4, y_2, y_4 \in \mathbb{R}$ and $\vec{x}_2, \vec{x}_3, \vec{y}_1, \vec{y}_3 \in \mathbb{R}^{m-1}$. Let $L_{j,t}$ be the lift of $L_{j,t}$ to $\text{Gr}(\mathbb{R}^{2m}, m)$. Since $DL_{1,t}$ preserves Ker $Q_0$, we have $L_{1,t}(\hat{M}) = \hat{M}$. Define linear maps $L_{1,t}^\pm : E^1_\tau \to E^\mp_1$ and $L_{1,t}^\pm : E^2_\tau \to E^\mp_2$ by

$$
L_{1,t}^-
\begin{pmatrix}
  x_1 \\ A_{13} \\ a_{14} \\ \vec{a}_{24}
\end{pmatrix}
= 
\begin{pmatrix}
  e^{-t}x_1 \\ e^{-2t}A_{13} \\ e^{-3t}a_{14} \\ e^{-4t}\vec{a}_{24}
\end{pmatrix},
L_{1,t}^+
\begin{pmatrix}
  \vec{x}_2 \\ \vec{x}_3 \\ x_4 \\ A_{23}
\end{pmatrix}
= 
\begin{pmatrix}
  e^{2t}\vec{x}_2 \\ e^{2t}\vec{x}_3 \\ e^{3t}x_4 \\ e^{4t}A_{23}
\end{pmatrix}.
$$

Then, we can check that

$$
L_{1,t}^- \circ P_1 = P_1 \circ L_{1,t}, \quad L_{1,t}^+ \circ Q_1 = Q_1 \circ L_{1,t}
$$

and

$$
\max\{\|L_{1,t}^-\|, \|L_{1,t}^+\|\} \leq e^{-t}
$$

for any $t \geq 0$. Similarly, there exist linear maps $L_{2,t}^- : E^2_\tau \to E^\mp_2$ and $L_{2,t}^+ : E^2_\tau \to E^\mp_2$ such that

$$
L_{2,t}^- \circ P_2 = P_2 \circ L_{2,t}, \quad L_{2,t}^+ \circ Q_2 = Q_2 \circ L_{2,t}
$$

and

$$
\max\{\|L_{2,t}^-\|, \|L_{2,t}^+\|\} \leq e^{-t}.
$$

For any $t > 0, \xi, \xi' \in \hat{M}, \theta > 0$, and $\tau = 1, 2$, we have

$$
DL_{\tau,t}(C(\xi, \theta, Q_\tau, P_\tau)) \subset C(L_{\tau,t}(\xi), e^{-2t}\theta, Q_\tau, P_\tau) \quad (8)
$$

$$
DL_{\tau,-t}(C(\xi', \theta, P_\tau, Q_\tau)) \subset C(L_{\tau,-t}(\xi'), e^{-2t}\theta, P_\tau, Q_\tau). \quad (9)
$$

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Since \((P_1, P_2 \circ \hat{h}_0) : \hat{M} \rightarrow E_1^- \times E_2^-\) is a diffeomorphism, there exists a \((P_1, P_2 \circ \hat{h}_0)\)-rectangle \(\mathcal{R}'(t)\) of \(\hat{M}\) such that

\[
P_1(\mathcal{R}'(t)) = (L_{1,t}^- \circ P_1)(Y_1), \quad (P_2 \circ \hat{h}_0)(\mathcal{R}'(t)) = (L_{2,t}^+ \circ P_2)(Y_2)
\]

for any \(t \geq 0\). We put \(h_t = L_{2,-t} \circ h_0 \circ L_{1,t}\) and \(\mathcal{R}(t) = L_{1,-t}(\mathcal{R}'(t))\). The following lemma implies that \(h_t\) satisfies the first item in the definition of \(V(\theta, \nu, Y_1, Y_2)\) for \(\mathcal{R}(t)\) for any sufficiently large \(t\).

**Lemma 4.10.** For any \(t \geq 0\), the set \(\mathcal{R}(t)\) is a \((P_1, P_2 \circ \hat{h}_t)\)-rectangle satisfying

\[
L_{1,t}(\mathcal{R}(t)) \subset \mathcal{R}(0), \quad P_1(\mathcal{R}(t)) = P_1(Y_1), \quad (P_2 \circ \hat{h}_t)(\mathcal{R}(t)) = P_2(Y_2).
\]

There exists \(T_0 > 0\) such that

\[
Q_1(\mathcal{R}(t)) \subset \text{Int } Q_1(Y_1), \quad (Q_2 \circ \hat{h}_t)(\mathcal{R}(t)) \subset \text{Int } Q_2(Y_2)
\]

for any \(t \geq T_0\).

**Proof.** By the equations \(L_{\tau,t}^- \circ P_\tau = P_\tau \circ L_{\tau,t}^-\) and \(L_{\tau,t}^+ \circ Q_\tau = Q_\tau \circ L_{\tau,t}^+\), we have

\[
(P_1, P_2 \circ \hat{h}_t) \circ L_{1,t}^- = (L_{1,t}^- \circ P_1, L_{2,-t}^- \circ P_2 \circ \hat{h}_0), \quad (10)
\]

\[
(Q_1, Q_2 \circ \hat{h}_t) \circ L_{1,t}^- = (L_{1,t}^- \circ Q_1, L_{2,-t}^- \circ Q_2 \circ \hat{h}_0). \quad (11)
\]

Since \(L_{1,t}\) is a diffeomorphism from \(\mathcal{R}(t)\) to a \((P_1, P_2 \circ \hat{h}_0)\)-rectangle \(\mathcal{R}'(t)\), the equation (10) implies that \(\mathcal{R}(t)\) is a \((P_1, P_2 \circ \hat{h}_t)\)-rectangle with

\[
P_1(\mathcal{R}(t)) = (L_{1,-t}^- \circ P_1)(\mathcal{R}'(t)) = P_1(Y_1),
\]

\[
(P_2 \circ \hat{h}_t)(\mathcal{R}(t)) = (L_{2,-t}^- \circ P_2 \circ \hat{h}_0)(\mathcal{R}'(t)) = P_2(Y_2).
\]

We also have

\[
Q_1(\mathcal{R}(t)) = (L_{1,-t}^- \circ Q_1)(\mathcal{R}'(t)), \quad (Q_2 \circ \hat{h}_t)(\mathcal{R}(t)) = (L_{2,-t}^- \circ Q_2 \circ \hat{h}_0)(\mathcal{R}'(t)).
\]

Since \(\max\{\|L_{\tau,t}^-\|, \|L_{\tau,-t}^+\|\} \leq e^{-t}\) and \(\text{Int } Q_\tau(Y_\tau)\) contains the origin of \(E_\tau^-\), there exists \(T_0 > 0\) such that

\[
(L_{1,t}^- \circ Q_1)(\mathcal{R}(t)) \subset \text{Int } Q_1(Y_1), \quad (L_{2,-t}^- \circ Q_2 \circ \hat{h}_0)(\mathcal{R}'(t)) \subset \text{Int } Q_2(Y_2)
\]

for any \(t \geq T_0\). Then, we have \(Q_1(\mathcal{R}_1(t)) \subset \text{Int } Q_1(Y_1)\) and \((Q_2 \circ \hat{h}_t)(\mathcal{R}_2(t)) \subset \text{Int } Q_2(Y_2)\).

The following lemma implies that \(h_t\) satisfies the second item in the definition of \(V(\theta, \nu, Y_1, Y_2)\). This completes the proof of Proposition [13].

**Lemma 4.11.** For any given constant \(\theta, \nu > 0\), there exists \(T_1 > 0\) such that \(h_t\) satisfies the \((\theta, \nu, \nu)\)-cone condition on \(\mathcal{R}(t)\) with respect to \((P_1, Q_1)\) and \((Q_2, P_2)\) for any \(t \geq T_1\).
Proof. Recall that \((P_1, P_2 \circ \widehat{h}_0)\) is a diffeomorphism from \(\widehat{M}\) to \(E_1^* \times E_2^*\). This implies that \(\widehat{Dh}_0(\text{Ker}(DP_1)_{\xi})\) is transverse to \(\text{Ker}(DP_2)_{\xi}^{-}\) for any \(\xi \in \widehat{M}\). By the compactness of \(\mathcal{R}'(0)\), there exist \(0 < \varepsilon < 1\) and \(\eta > 0\) such that
\[
\begin{align*}
\|D(\widehat{P}_2 \circ \widehat{h}_0)(\xi)\| &\geq \eta \|DQ_1(\xi)\|, \\
\|D(\widehat{P}_1 \circ \widehat{h}_0^{-1})(\xi)\| &\geq \eta \|DQ_2(\xi)\|,
\end{align*}
\]
for any \(\xi \in \mathcal{R}'(0), v \in C(\xi, \varepsilon, Q_1, P_1), \) and \(w \in C(\widehat{h}_0(\xi), \varepsilon, Q_2, P_2)\). Take \(0 < \theta' < \theta\) and \(T > 0\) such that \(e^{-2T \theta} \varepsilon < \varepsilon\) and \(e^{-2T \theta} \varepsilon < \varepsilon\). By the inclusions \(8, 9, 12,\) and \(13\), \(\widehat{h}_1\) satisfies the \(\theta\)-cone condition with respect to \((P_1, Q_1)\) and \((Q_2, P_2)\) on \(L_{t, t}^{-1}(\mathcal{R}'(0))\) for any \(t \geq T\). Since \(\max\{\|L_{t, t}^{-1}\|, 2\|L_{t, t}^{-1}\|\} \leq e^{-t}\) for any \(t = 1, 2\) and \(t \geq 0\), we have \(\mathcal{R}'(t) \subset \mathcal{R}'(0)\). The inequalities \(14\) also implies that \(\widehat{h}_1\) satisfies the \((\theta, e^{2T} \eta, e^{2T} \eta)\)-cone condition on \(L_{t, t}^{-1}(\mathcal{R}'(0))\) for any \(t \geq T\). Take \(T > 1\) so that \(e^{2T} \eta > \nu\). Then, \(\widehat{h}_1\) satisfies the \((\theta, \nu, \nu)\)-cone condition on \(\mathcal{R}(t) = L_{t, t}^{-1}(\mathcal{R}'(t)) \subset L_{t, t}^{-1}(\mathcal{R}'(0))\) for any \(t \geq T_1\). \(\square\)

4.5 Robust tangency

Fix \(m \geq 2\). Let \((P, Q)\) be an \((m, m)\)-splitting of \(\mathbb{R}^{2m}\), \(\widehat{M}\) a subspace of \(\text{Gr}(\mathbb{R}^{2m}, m), (P_1, Q_1) : \widehat{M} \to E_1^* \times E_2^*\) an \((m^2, 2m)\)-splitting and \((P_2, Q_2) : \widehat{M} \to E_1^* \times E_2^*\) an \((m^2, 2m)\)-splitting as in Section 4.4. Put \(m_1 = (1, m - 1, m - 1, 1), m_2 = (m - 1, 1, m - 1), \) and fix constants \(\lambda > 1 > \mu\) with \(\lambda \mu > 1\). By Proposition 4.4 we can find \(\Delta_\tau > 0\), a \((P_\tau, Q_\tau)\)-rectangle \(Z_\tau\) in \(\widehat{M}\), and a diffeomorphism \(f_\tau\) of \(\mathbb{R}^{2m}\) such that \(\text{diam} \ P_\tau(Z_\tau) \leq 2, \text{Int} \ Q_\tau(Z_\tau)\) contains the origin of \(E_1^*\), \(f_\tau\) is a \(C^2\) BR-blender horseshoe map of type \((m_\tau, \lambda, \mu, \Delta_\tau, \theta, Z_\tau)\) for any \(\theta > 0\) for each \(\tau = 1, 2\). Put \(\theta = \min\{\Delta_1, \Delta_2\}/6\). Let \(\mathcal{U}_\tau\) be the set of \(C^2\) BR-blender horseshoe maps \(f\) of type \((m_\tau, \lambda, \mu, \Delta_\tau, \theta, Z_\tau)\). Then, it is an open subset of the set of \(C^2\) diffeomorphisms of \(\mathbb{R}^{2m}\) which contains \(f_\tau\). In particular, \(f_\tau\) is non-empty. For any \(\tau = 1, 2\) and \(f \in \mathcal{U}_\tau\), let \((R_{\tau,i})_{i \in I}\) and \(\mathcal{B}_\tau(f) = (R_{\tau,i})_{i \in I}\) be the families in the definition of a BR-blender horseshoe map. Put \(R_\tau(f) = \bigcup_{i \in I} R_{\tau,i}, \Lambda_\tau(f) = \bigcap_{n \in \mathbb{Z}} f^n(R_\tau(f)), \) and \(\Lambda^-(\mathcal{B}_\tau(f)) = \bigcap_{n \geq 1} f^n(\bigcup_{i \in I} R_{\tau,i}).\)

**Lemma 4.12.** For any \(\tau = 1, 2\) and \(f \in \mathcal{U}_\tau\), \(\Lambda_\tau(f)\) is a hyperbolic invariant set of unstable index \(m\) such that \(\Lambda_\tau(f) = \bigcap_{n \in \mathbb{Z}} f^n(\text{Int} \ R_\tau(f))\) and
\[
\Lambda^-(\mathcal{B}_\tau(f)) \subset \{T Q W^n(p) \mid p \in \Lambda_\tau(f), q \in W^u(p, f)\}.
\]

**Proof.** By Lemmas 4.8 and 1.15 we have \(\Lambda_\tau(f) = \bigcap_{n \in \mathbb{Z}} f^n(\text{Int} \ R_\tau(f))\) and it is a hyperbolic invariant set of unstable index \(m\). Since \(\pi(I_{\tau,i})\) is contained in \(\text{Int} \ R_\tau(f)\), we obtain the inclusion in the lemma by Proposition 1.11 \(\square\)

Let \(B_\tau^+(r)\) be the closed \(r\)-ball in \(E_1^*\) centered at the origin. Take \(0 < r_\gamma < 1/9\) such that \(B_\tau^+(r_\gamma) \subset \text{Int} \ (Q_\tau(Z_\tau) \cap B_\tau^+(1/2))\) and \(\text{diam} \ B_\tau^+(r_\gamma) < \Delta_\tau\) for
\[ \tau = 1, 2. \] We put
\[ Y_\tau = \{ \xi \in \hat{M}_\tau \mid P_\tau(\xi) \in P_\tau(Z_\tau), Q_\tau(\xi) \in B^+_r(r_\tau) \}. \]

Recall that \( \theta = \min\{\Delta_1, \Delta_2\}/6. \) Since \( \text{diam } P_\tau(Z_\tau) \leq 2 \) for each \( \tau = 1, 2, \) there exists \( \nu > 1 \) such that
\[ \theta \text{ diam } P_1(Z_1) + \nu^{-1} \text{ diam } P_2(Z_2) < \Delta_1, \]
\[ \theta \text{ diam } P_2(Z_2) + \nu^{-1} \text{ diam } P_1(Z_1) < \Delta_2. \]

Let \( V = V(\theta, \nu, Y_1, Y_2) \) be the set defined in the previous subsection. It is \( C^2 \)-open and non-empty by Proposition [4.9]. For \( h \in V, \) let \( R_4(h) \) be the \((P_1, P_2 \circ \hat{h})\)-rectangle in the definition of \( V(\theta, \nu, Y_1, Y_2). \)

**Lemma 4.13.** Let \( f_1, f_2 \) be elements in \( U_1, U_2 \) respectively, \( h \) an element of \( V, \) and \( \hat{h} \) the lift of \( h \) to \( \text{Gr}(\mathbb{R}^{2m}, m). \) Then,
\[ \hat{h}(\Lambda^+(B_1(f_1))) \cap \Lambda^-(B_2(f_2)) \neq \emptyset. \]

**Proof.** Check that the lifts \( \hat{f}_1, \hat{f}_2, \hat{h} \) and blenders \( B_1(f_1), B_2(f_2) \) satisfy the assumption of Theorem [2.13] for \((2m, m^2)\)-splittings \((P_1, Q_1)\) and \((m^2, 2m)\)-splitting \((P_2, Q_2). \)

Let \( W_1 \) be the set of \( C^2 \) diffeomorphisms \( g \) of \( \mathbb{R}^{2m} \) which satisfy the \((1, 1, 1)\)-cone condition on \( B^{2m}(2) \) and admit a compact \((P_1, P_2 \circ g)\)-rectangle \( R_5(g) \) such that
\[ P_1(R_5(g)) = B^m(1), \quad Q_1(R_5(g)) \subset \text{Int}(B^{m-1}(1/8) \times B^1(1/3, 1/8)), \]
\[ (P_2 \circ g)(R_5(g)) = B^m(1), \quad (Q_2 \circ g)(R_5(g)) \subset \text{Int}(B^{m-1}(1/8) \times B^1(1/3, 1/8)). \]

Then, \( W_1 \) is open and it contains a diffeomorphism \( g \) given by
\[ g(x_1, x_2, x_3, x_4) = \left( 9 \left( x_4 - \frac{1}{3} \right), 9x_3, \frac{9}{9}x_2, \frac{9}{9}x_1 + \frac{1}{3} \right). \]

Hence, \( W_1 \) is non-empty. Similarly, let \( W_2 \) be the set of \( C^2 \) diffeomorphisms \( g \) of \( \mathbb{R}^{2m} \) which satisfy the \((1, 1, 1)\)-cone condition on \( B^{2m}(2) \) and admit a compact \((Q_1, Q_2 \circ g)\)-rectangle \( R_5(g) \) such that
\[ Q_1(R_5(g)) = B^m(1), \quad P_1(R_5(g)) \subset \text{Int}(B^1(1/3, 1/8) \times B^{m-1}(1/3, 1/8)), \]
\[ (Q_2 \circ g)(R_5(g)) = B^m(1), \quad (P_2 \circ g)(R_5(g)) \subset \text{Int}(B^{m-1}(1/3) \times B^1(1/3, 1/8)). \]

Then, \( W_2 \) is \( C^1 \)-open and it contains a diffeomorphism \( g \) given by
\[ g(x_1, x_2, x_3, x_4) = \left( \frac{1}{9}x_4 + \frac{1}{3}, \frac{1}{9}x_3, 9x_2, 9 \left( x_1 - \frac{1}{3} \right) \right). \]

Hence, \( W_2 \) is non-empty.
Now, we prove Theorem B for a $2m$-dimensional manifold $M$ with $m \geq 2$. Fix smooth coordinates $(U_1, \varphi_1)$ and $(U_2, \varphi_2)$ of $M$ such that $\varphi_\tau(U_\tau) = \mathbb{R}^{2m}$ and $U_1 \cap U_2 = \emptyset$. Let $O$ be the set of $C^2$ diffeomorphisms $F$ of $M$ such that

$$F(x) = \begin{cases} 
\varphi_1^{-1} \circ f_1 \circ \varphi_1(x) & \text{if } x \in \varphi_1^{-1}(R_1(f_1)) \\
\varphi_2^{-1} \circ f_2^{-1} \circ \varphi_2(x) & \text{if } x \in \varphi_2^{-1}(f_2(R_2(f_2))) \\
\varphi_2^{-1} \circ h \circ \varphi_1(x) & \text{if } x \in \varphi_2^{-1}(\pi(R_2(h))) \\
\varphi_1^{-1} \circ g_1 \circ \varphi_1(x) & \text{if } x \in \varphi_1^{-1}(R_0(g_1)) \\
\varphi_1^{-1} \circ g_2^{-1} \circ \varphi_2(x) & \text{if } x \in \varphi_2^{-1}(g_2(R_2(g_2))) 
\end{cases} \quad (15)$$

for some $f_\tau \in U_\tau$, $h \in V$, and $g_\tau \in W$. Recall that

$$Q_1(R_1(f_1)) \subset \text{Int } B^m(1) \setminus B^m(1/2),$$
$$Q_1(\pi(R_2(h))) \subset Q(\pi(V_1)) \subset B^m(1/9),$$
$$Q_1(R_0(g_1)) \subset B^m(1/18) \times B^1(1/3, 1/8),$$
$$P_2(g_2(R_2(g_2))) \subset B^{m-1}(1/8) \times B^1(1/3, 1/8),$$
$$P_2(f_2(R_2(f_2))) \subset \text{Int } B^m \setminus B^m(1/2).$$

Hence, the subsets $\varphi_1^{-1}(R_1(f_1)), \varphi_2^{-1}(f_2(R_2(f_2))), \varphi_1^{-1}(\pi(R_2(h))), \varphi_1^{-1}(R_0(g_1)),$ and $\varphi_2^{-1}(g_2(R_2(g_2)))$ of $M$ are mutually disjoint. Similarly, we can check that $\varphi_1^{-1}(f_1(R_1(f_1))), \varphi_2^{-1}(R_2(f_2)), \varphi_2^{-1}(h(\pi(R_2(h)))), \varphi_2^{-1}(g_1(R_0(g_1))),$ and $\varphi_1^{-1}(R_0(g_2))$ are mutually disjoint. This implies that there exists a diffeomorphism $F \in O$ satisfying (15) for any given $f_1, f_2, h, g_1$ and $g_2$. In particular, the set $O$ is non-empty. Since $U_\tau$, $V$, and $W_\tau$ are $C^2$-open, the set $O$ is also an open subset of the set of $C^2$ diffeomorphisms of $M$.

We finish the proof of Theorem B for the manifold $M$ by checking $\Lambda_\ast(F)$ exhibits homoclinic tangency of codimension $m$ for any diffeomorphism $F$ in the $C^2$-open set $O$. Take a diffeomorphism $F$ in $O$. Let $f_1, f_2, h, g_1$ and $g_2$ be maps which satisfy (15). Put

$$R_\ast(F) = \varphi_1^{-1}(R_1(f_1)) \cup R_0(g_1) \cup \varphi_2^{-1}(f_2(R_2(f_2))) \cup g_2(R_2(g_2)),$$
$$\Lambda_\ast(F) = \bigcap_{n \in \mathbb{Z}} F^n(R_\ast(F)).$$

Then, $\Lambda_\ast(F)$ contains $\varphi_1^{-1}(\Lambda_1(f_1)) \cup \varphi_2^{-1}(\Lambda_2(f_2))$. By the $(1, 1, 1)$-cone conditions on $R_\ast(F)$ and Lemma 2.9, $\Lambda_\ast(F)$ is a hyperbolic invariant set such that the restriction of $F$ to $\Lambda_\ast(F)$ is topologically transitive. By Lemmas 4.12 and 4.13 there exist $p_1 \in \varphi_1^{-1}(\Lambda_1(f_1)), p_2 \in \varphi_2^{-1}(\Lambda_2(f_2))$, and $q \in W^u(p_1, F) \cap W^s(p_2, F)$ such that $T_qW^u(p_1, F) = T_qW^u(p_2, F)$. Therefore, the hyperbolic set $\Lambda_\ast(F)$ exhibits homoclinic tangency of codimension $m$ for any $F \in O$.

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