A LARGE DEVIATION PRINCIPLE FOR NONLINEAR STOCHASTIC WAVE EQUATION DRIVEN BY ROUGH NOISE

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Abstract This paper is devoted to investigating Freidlin-Wentzell’s large deviation principle for one (spatial) dimensional nonlinear stochastic wave equation $\frac{\partial^2 u^\varepsilon(t,x)}{\partial t^2} = \frac{\partial^2 u^\varepsilon(t,x)}{\partial x^2} + \sqrt{\varepsilon} \sigma(t,x,u^\varepsilon(t,x)) \dot{W}(t,x)$, where $\dot{W}$ is white in time and fractional in space with Hurst parameter $H \in (\frac{1}{4}, \frac{1}{2})$. The variational framework and the modified weak convergence criterion proposed by Matoussi et al. [27] are adopted here.

Keywords Stochastic wave equation; fractional Brownian motion; large deviation principle; weak convergence approach.

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1. INTRODUCTION

In this paper, we consider the following one (spatial) dimensional nonlinear stochastic wave equation (SWE for short):

$$\left\{ \begin{array}{ll}
\frac{\partial^2 u^\varepsilon(t,x)}{\partial t^2} &= \frac{\partial^2 u^\varepsilon(t,x)}{\partial x^2} + \sqrt{\varepsilon} \sigma(t,x,u^\varepsilon(t,x)) \dot{W}(t,x), & t > 0, x \in \mathbb{R}, \\
u^\varepsilon(0,\cdot) &= u_0(x), \quad \frac{\partial u^\varepsilon(0,x)}{\partial t} = v_0(x),
\end{array} \right. \quad (1.1)$$

where $\varepsilon > 0$ and the noise $W = \{W(t,x), t \geq 0, x \in \mathbb{R}\}$ is a mean zero Gaussian random field defined on a complete probability space $(\Omega, \mathcal{F}, P)$. We assume that the noise $W$ is a standard Brownian motion in time and a fractional Brownian motion with Hurst parameter $H \in (\frac{1}{4}, \frac{1}{2})$ in the space and $\dot{W}(t,x) = \frac{\partial^2 W}{\partial t \partial x}(t,x)$. The covariance of the noise $W$ is given by

$$\mathbb{E}[W(\varphi)W(\psi)] = \int_{\mathbb{R}+ \times \mathbb{R}} \mathcal{F}\varphi(t,\xi)\overline{\mathcal{F}\psi(t,\xi)} \cdot \mu(d\xi)dt, \quad \varphi, \psi \in \mathcal{D}(\mathbb{R}_+ \times \mathbb{R}), \quad (1.2)$$

where $\mathcal{D}(\mathbb{R}_+ \times \mathbb{R})$ is the space of real-valued infinitely differentiable functions on $\mathbb{R}_+ \times \mathbb{R}$ with compact support, and $\mathcal{F}\varphi(t,\cdot)$ is the Fourier transform with respect to the spatial variable $x$ of the function $\varphi(t,x)$, which is defined as

$$\mathcal{F}\varphi(t,\cdot)(\xi) := \int_{\mathbb{R}} e^{-i\xi x} \varphi(t,x)dx, \xi \in \mathbb{R},$$

and the measure $\mu$ is given by

$$\mu(d\xi) = c_{1,H}|\xi|^{-2H}d\xi, \quad c_{1,H} = \frac{\Gamma(2H + 1)\sin(\pi H)}{2\pi}, \quad (1.3)$$

where $\Gamma$ is the Gamma function and $\frac{1}{4} < H < \frac{1}{2}$. When the noise is general Gaussian which is white in time and satisfies Dalang’s condition in [11] (namely the spatial parameter $H \geq \frac{1}{2}$), there are some results about the well-posedness and the properties of the solutions, see e.g., [12, 14, 20]. In the case of $\frac{1}{4} < H < \frac{1}{2}$, since the Fourier transform of $\mu$ in the space of tempered distributions on $\mathbb{R}$ is not a locally integrable function, the noise $W$ is not of the same form as

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the one considered in [11], and the stochastic integral with respect to $W$ was constructed using different methods. See [1, 18] for more details.

Recently, many authors studied the existence and uniqueness for the solutions of stochastic partial differential equations (SPDEs for short) driven by a Gaussian noise that is white in time and rough in space, see e.g., [18, 19, 21, 23, 33]. Balan et al. [1] considered the stochastic heat equation (SHE for short) and the SWE with the affine diffusion coefficient. Hu et al. [18] and Hu and Wang [21] investigated the SHE with the general nonlinear coefficient. Liu et al. [23] considered the SWE driven by the rough noise in space with the general nonlinear coefficient and Song et al. [33] considered the fractional SWE. Compared with the SHE, the case of the SWE is much more complicated because of the lack of the semigroup property of the wave kernel. To overcome this difficulty, Liu et al. [23] decomposed the wave kernel $G(t-s,x-y)$ to four complicated parts (see [23, Lemma 3.1] or lemma 6.1 below). We use the decomposition technique to establish some estimates in this paper.

The aim of this paper is to establish a large deviation principle (LDP) for the solution $u^\varepsilon$ of Eq. (1.1) as $\varepsilon \to 0$. There are several results of LDPs and moderate deviation principles for the SWE. For example, Ortiz-López and Sanz-Solé [30] considered LDPs for SWE in spatial dimension three, driven by a Gaussian noise, white in time and with a stationary spatial covariance. Martirosyan [28] considered LDPs for stationary measures of stochastic nonlinear wave equations with smooth white noise. Martirosyana and Nersesyan [29] established an LDP for occupation measures of the stochastic damped nonlinear wave equation. Brzeźniak et al. [2] investigated the LDP for solutions of the (1+1)-dimensional stochastic geometric wave equation in the case of vanishing noise. Cheng et al. [8] considered moderate deviations for a stochastic wave equation in dimension three.

An important approach of investigating the LDP is the well-known weak convergence method (see e.g., [3–7, 16]). For some relevant LDP results by using the weak convergence method, we refer to [24, 26, 32, 36, 37] and the references therein. There are mainly two difficulties to study the LDP for the solution of Eq. (1.1). The first one comes from the spatial rough noise with $H \in (\frac{1}{4}, \frac{1}{2})$. The second difficulty comes from the lack of the semigroup property of the Green function. In this paper, we adopt a new sufficient condition for the LDP (see Condition 3.5 below) which is proposed by Matoussi, Sabbagh and Zhang [27]. This approach has been proved to be successful in a wide range of SPDEs, see e.g., [15, 25, 34, 35].

The rest of the paper is organised as follows. The definition of the stochastic integral, the notion of the solution for the equation (1.1) and the functional spaces introduced in [23] are presented in Section 2. In Section 3, we recall a general criterion for large deviations based on the weak convergence and state our main result. Section 4 is devoted to showing the existence and uniqueness of a solution to the skeleton equation associated with the equation (1.1). The large deviation principle for equation (1.1) is proved in Section 5. Finally, some auxiliary results are presented in Appendix.

Some notations and mathematical conventions used in this work are as follows. We always use $C_\alpha$ to denote a constant dependent on the parameter $\alpha$, which may change from line to line. $A \lesssim B$ $(A \gtrsim B)$, resp.) means that $A \leq CB$ $(A \geq CB)$, resp.) for some positive universal constant $C$, and $A \simeq B$ if and only if $A \lesssim B$ and $A \gtrsim B$.

2. Preliminaries

This section is divided into two parts. In Section 2.1 we briefly recall some necessary concepts about the noise $W$ and the stochastic integral. In Section 2.2 we collect some preliminaries about the SWE.

2.1. Stochastic integral. We recall some results from [18, 21, 23]. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space and let $H \in (\frac{1}{4}, \frac{1}{2})$ be given and fixed. Our noise $\tilde{W}$ is a zero-mean Gaussian
family \( \{W(\varphi), \varphi \in \mathcal{D}(\mathbb{R}_+ \times \mathbb{R})\} \) with the covariance structure given by

\[
\mathbb{E} [W(\varphi)W(\psi)] = \int_{\mathbb{R}_+ \times \mathbb{R}} \mathcal{F}\varphi(t, \xi)\mathcal{F}\psi(t, \xi) \cdot \mu(d\xi) dt
\]

for any \( \varphi, \psi \in \mathcal{D}(\mathbb{R}_+ \times \mathbb{R}) \) with \( \mu \) given by (1.3). Notice that (2.1) defines a Hilbert scalar product on \( \mathcal{D}(\mathbb{R}_+ \times \mathbb{R}) \). Denote \( \mathcal{H} \) the Hilbert space obtained by completing \( \mathcal{D}(\mathbb{R}_+ \times \mathbb{R}) \) with respect to this scalar product, which can be expressed in terms of fractional derivatives as following.

**Proposition 2.1.** ([21, Proposition 2.1], [31, Theorem 3.1]) For any \( \varphi, \psi \in \mathcal{D}(\mathbb{R}_+ \times \mathbb{R}) \), the space \( \mathcal{H} \) is a Hilbert space equipped with the scalar product

\[
\langle \varphi, \psi \rangle_{\mathcal{H}} := c_{1,H} \int_{\mathbb{R}_+} \left( \int_{\mathbb{R}} \mathcal{F}\varphi(t, \xi)\mathcal{F}\psi(t, \xi) \cdot |\xi|^{-2H} d\xi \right) dt
\]

\[
= c_{2,H} \int_{\mathbb{R}_+} \left( \int_{\mathbb{R}^2} [\varphi(t, x + y) - \varphi(x)] \cdot [\psi(t, x + y) - \psi(x)] \cdot |y|^{2H-2} dxdy \right) dt,
\]

where \( c_{1,H} \) is defined by (1.3) and

\[
c_{2,H} := H^\frac{1}{2} \left( \frac{1}{2} - H \right) \frac{1}{2} \left[ \Gamma\left(H + \frac{1}{2}\right) \right]^{-1} \left( \int_0^\infty \left[ (1 + t)^{-1} - t^{H-\frac{1}{2}} \right]^2 dt + \frac{1}{2H} \right) \frac{1}{2}.
\]

One can check that \( \varphi(t, x) = 1_{[0,t] \times [0,x]}, t \in \mathbb{R}_+, x \in \mathbb{R}, \) is in \( \mathcal{H} \) (we set \( 1_{[0,t] \times [0,x]} = -1_{[0,t] \times [x,0]} \) if \( x \in \mathbb{R} \) is negative), where \( 1 \) denotes the indicator function. We denote \( W(t, x) = W(1_{[0,t] \times [0,x]}) \). For any \( t \geq 0, \mathcal{F}_t = \sigma(W(s, x), 0 \leq s \leq t, x \in \mathbb{R}) \) be the \( \sigma \)-algebra generated by \( W \).

The stochastic integral with respect to \( W \) is first defined for elementary integrands and then can be extended to general ones.

**Definition 2.2.** An elementary process \( g \) is a process given by

\[
g(t, x) = \sum_{i=1}^n \sum_{j=1}^m X_{i,j} 1_{[a_i, b_i]}(t) 1_{[h_j, l_j]}(x),
\]

where \( n \) and \( m \) are finite positive integers, \( 0 \leq a_1 < b_1 < \cdots < a_n < b_n < \infty, h_j < l_j \) and \( X_{i,j} \) are \( \mathcal{F}_t \)-measurable random variables for \( i = 1, \ldots, n, j = 1, \ldots, m \). The stochastic integral of an elementary process with respect to \( W \) is defined as

\[
\int_{\mathbb{R}_+} \int_{\mathbb{R}} g(t, x) W(dt, dx) = \sum_{i=1}^n \sum_{j=1}^m X_{i,j} W(1_{[a_i, b_i]} \otimes 1_{[h_j, l_j]})
\]

\[
= \sum_{i=1}^n \sum_{j=1}^m X_{i,j} \left[ W(b_i, l_j) - W(a_i, l_j) - W(b_i, h_j) + W(a_i, h_j) \right].
\]

Hu et al. [18, Proposition 2.3] extended the notion of integral with respect to \( W \) to a broad class of adapted processes in the following way.

**Proposition 2.3.** ([18, Proposition 2.3]) Let \( \Lambda_H \) be the space of predictable processes \( g \) defined on \( \mathbb{R}_+ \times \mathbb{R} \) such that almost surely \( g \in \mathcal{H} \) and \( \mathbb{E} [\|g\|_{\mathcal{H}}^2] < \infty \). Then, we have that:

(i). the space of the elementary processes defined by Definition 2.2 is dense in \( \Lambda_H \);

(ii). for any \( g \in \Lambda_H \), the stochastic integral \( \int_{\mathbb{R}_+} \int_{\mathbb{R}} g(t, x) W(dt, dx) \) is defined as the \( L^2(\Omega) \)-limit of stochastic integrals of elementary processes approximating \( g(t, x) \) in \( \Lambda_H \), and for this stochastic integral we have the following isometry equality

\[
\mathbb{E} \left[ \left( \int_{\mathbb{R}_+} \int_{\mathbb{R}} g(t, x) W(dt, dx) \right)^2 \right] = \mathbb{E} [\|g\|_{\mathcal{H}}^2].
\]
Denote by \( D(\mathbb{R}) \) the space of real-valued infinitely differential functions on \( \mathbb{R} \) with compact support. Let \( \mathcal{H} \) be the Hilbert space obtained by completing \( D(\mathbb{R}) \) with respect to the following scalar product:

\[
\langle \varphi, \psi \rangle_{\mathcal{H}} = c_{1,H} \int_{\mathbb{R}} \mathcal{F} \varphi(\xi) \overline{\mathcal{F} \psi(\xi)} : |\xi|^{1-2H} \, d\xi
\]

\[
= c_{2,H} \int_{\mathbb{R}^2} [\varphi(x+y) - \varphi(x)] \cdot [\psi(x+y) - \psi(x)] : |y|^{2H-2} \, dx \, dy, \quad \forall \varphi, \psi \in D(\mathbb{R}),
\]

where \( c_{1,H} \) and \( c_{2,H} \) are defined by (1.3) and (2.3), respectively. By Proposition 2.3, for any orthonormal basis \( \{e_k\}_{k \geq 1} \) of the Hilbert space \( \mathcal{H} \), the family of processes

\[
\left\{ B_t^k := \int_0^t \int_{\mathbb{R}} e_k(y) W(ds, dy) \right\}_{k \geq 1}
\]

is a sequence of independent standard Wiener processes and the process \( B_t := \sum_{k \geq 1} B_t^k e_k \) is a cylindrical Brownian motion on \( \mathcal{H} \). It is well-known that (see [10] or [13]) for any \( \mathcal{H} \)-valued predictable process \( g \in L^2(\Omega \times [0,T]; \mathcal{H}) \), we can define the stochastic integral with respect to the cylindrical Wiener process \( B \) as follows:

\[
\int_0^T g(s) dB_s := \sum_{k \geq 1} \int_0^T \langle g(s), e_k \rangle_{\mathcal{H}} dB_s^k.
\]

Note that the above series converges in \( L^2(\Omega, \mathcal{F}, \mathbb{P}) \) and the sum does not depend on the selected orthonormal basis. Moreover, each summand, in the above series, is a classical Itô integral with respect to a standard Brownian motion.

Let \( (B, \| \cdot \|_B) \) be a Banach space with the norm \( \| \cdot \|_B \). Let \( H \in \left( \frac{1}{4}, \frac{1}{2} \right) \) be a fixed number. For any function \( f : \mathbb{R} \to B \), denote

\[
\mathcal{N}^B_{\frac{1}{2}-H} f(x) := \left( \int_{\mathbb{R}} \| f(x + h) - f(x) \|_B^2 \cdot |h|^{2H-2} \, dh \right)^{\frac{1}{2}},
\]

if the above quantity is finite. When \( B = \mathbb{R} \), we abbreviate the notation \( \mathcal{N}^\mathbb{R}_{\frac{1}{2}-H} f \) as \( \mathcal{N}^{\frac{1}{2}-H}_f \). As in [18], when \( B = L^p(\Omega) \), we denote \( \mathcal{N}^B_{\frac{1}{2}-H} \) by \( \mathcal{N}^{\frac{1}{2}-H}_{H,p} \), that is,

\[
\mathcal{N}^{\frac{1}{2}-H}_{H,p} f(x) := \left( \int_{\mathbb{R}} \| f(x + h) - f(x) \|_{L^p(\Omega)}^2 \cdot |h|^{2H-2} \, dh \right)^{\frac{1}{2}}.
\]

The following Burkholder-Davis-Gundy’s inequality is well-known (see e.g., [18, 21, 23]).

**Proposition 2.4.** ([18, Proposition 3.2]) Let \( W \) be the Gaussian noise with the covariance (2.1), and let \( f \in \Lambda_H \) be a predictable random field. Then, we have that, for any \( p \geq 2 \),

\[
\left\| \int_0^t \int_{\mathbb{R}} f(s,y) W(ds, dy) \right\|_{L^p(\Omega)} \leq \sqrt{4pc_H} \left( \int_0^t \int_{\mathbb{R}} \left[ \mathcal{N}^{\frac{1}{2}-H}_{H,p} f(s,y) \right]^2 \, dy \, ds \right)^{\frac{1}{2}}, \tag{2.11}
\]

where \( c_H \) is a constant depending only on \( H \) and \( \mathcal{N}^{\frac{1}{2}-H}_{H,p} f(s,y) \) denotes the application of \( \mathcal{N}^{\frac{1}{2}-H}_{H,p} \) to the spatial variable \( y \).

### 2.2. Stochastic wave equation

Let \( C([0,T] \times \mathbb{R}) \) be the space of all continuous real-valued functions on \([0,T] \times \mathbb{R}\), equipped with the metric

\[
d_C(u, v) := \sum_{n=1}^{\infty} \frac{1}{2^n} \max_{0 \leq t \leq T, |x| \leq n} (|u(t,x) - v(t,x)| + 1).
\]
We introduce the solution space $Z^p(T)$, which is the space of all continuous functions $v : [0, T] \times \mathbb{R} \to L^p(\Omega)$ such that the following norm is finite:

$$
\|v\|_{Z^p(T)} := \sup_{t \in [0, T]} \|v(t, \cdot)\|_{L^p(\Omega \times \mathbb{R})} + \sup_{t \in [0, T]} \mathcal{N}^{\ast}_{\frac{1}{2} - H, p} v(t),
$$

(2.13)

where

$$
\|v(t, \cdot)\|_{L^p(\Omega \times \mathbb{R})} := \left( \int_{\mathbb{R}} \mathbb{E}[|v(t, x)|^p] \, dx \right)^{\frac{1}{p}}
$$

(2.14)

and

$$
\mathcal{N}^{\ast}_{\frac{1}{2} - H, p} v(t) := \left( \int_{\mathbb{R}} \|v(t, \cdot) - v(t, \cdot + h)\|_{L^p(\Omega \times \mathbb{R})}^2 \cdot |h|^{2H - 2} \, dh \right)^{\frac{1}{2}}.
$$

(2.15)

Denote by $Z^p(T)$ the space of all non-random functions in $Z^p(T)$, which is equipped with the following norm:

$$
\|v\|_{Z^p(T)} := \sup_{t \in [0, T]} \|v(t, \cdot)\|_{L^p(\mathbb{R})} + \sup_{t \in [0, T]} \mathcal{N}^{\ast}_{\frac{1}{2} - H, p} v(t),
$$

(2.16)

where

$$
\|v(t, \cdot)\|_{L^p(\mathbb{R})} := \left( \int_{\mathbb{R}} |v(t, x)|^p \, dx \right)^{\frac{1}{p}}
$$

(2.17)

and

$$
\mathcal{N}^{\ast}_{\frac{1}{2} - H, p} v(t) := \left( \int_{\mathbb{R}} \|v(t, \cdot) - v(t, \cdot + h)\|_{L^p(\mathbb{R})}^2 \cdot |h|^{2H - 2} \, dh \right)^{\frac{1}{2}}.
$$

(2.18)

Let $G(t, x) := \frac{1}{2} 1_{|x| < t}, \ t \in \mathbb{R}_+, \ x \in \mathbb{R}$, be Green’s function associated with Eq. (1.1). Notice that $G(t, x)$ does not satisfy the semigroup property.

Recall the following definition of the solution to Eq. (1.1) from [23].

**Definition 2.5.** ([23, Definition 2.4]) Let $\{u^\varepsilon(t, x)\}_{t \in [0, T], x \in \mathbb{R}}$ be a real-valued adapted stochastic field such that for all fixed $t \in [0, T]$ and $x \in \mathbb{R}$, the random field

$$
\{G(t - s, x - y)\sigma(s, y, u^\varepsilon(s, y)) \mathbf{1}_{[0, t]}(s), (s, y) \in \mathbb{R}_+ \times \mathbb{R}\}
$$

is integrable with respect to $W$. The stochastic process $u^\varepsilon$ is called a strong solution to (1.1), if for all $t \in [0, T]$ and $x \in \mathbb{R}$ we have almost surely

$$
u^\varepsilon (t, x) = I_0(t, x) + \sqrt{\varepsilon} \int_0^t \int_{\mathbb{R}} G(t - s, x - y)\sigma(s, y, u^\varepsilon(s, y)) W(ds, dy),
$$

(2.19)

where

$$
I_0(t, x) := \frac{\partial}{\partial t} G(t) * u_0(x) + G(t) * v_0(x)
$$

$$
= \frac{1}{2} [u_0(x + t) + u_0(x - t)] + \frac{1}{2} \int_{x-t}^{x+t} v_0(y) \, dy.
$$

(2.20)

The following Hypothesis guarantees that Eq. (1.1) admits a unique solution.

**Hypothesis 2.6.** We assume that the following two items hold:

(i) assume $\sigma(t, x, u) \in C^{0,1,1}([0, T] \times \mathbb{R}^2)$ (the space of all continuous functions $\sigma$, with continuous partial derivatives $\sigma_x'$, $\sigma_u'$ and $\sigma_x u'$), satisfying $\sigma(t, x, 0) = 0$ and there exists a constant $C > 0$ such that

$$
\sup_{t \in [0, T], x \in \mathbb{R}, u \in \mathbb{R}} |\sigma_u'(t, x, u)| \leq C;
$$

(2.21)
definitions and results of the large deviation theory. Let

\[
\sup_{t \in [0, T], x \in \mathbb{R}, u \in \mathbb{R}} |\sigma''_{vu}(t, x, u)| \leq C; \tag{2.22}
\]

\[
\sup_{t \in [0, T], x \in \mathbb{R}} |\sigma'_u(t, x, u_1) - \sigma'_u(t, x, u_2)| \leq C|u_1 - u_2|, \quad \forall u_1, u_2 \in \mathbb{R}; \tag{2.23}
\]

(ii). the initial condition \(u_0\) and \(v_0\) are \(\alpha\)-Hölder continuous with \(\alpha \in (0, 1]\). Furthermore, we assume that \(I_0(t, x)\) is in \(Z^p(T)\) for some \(p > \frac{2}{\alpha+1}\).

Remark 1. Notice that by (2.21), we get the Lipschitz continuity of \(\sigma\) in \(u\) (uniformly in \(t\) and \(x\)), i.e.,

\[
\sup_{t \in [0, T], x \in \mathbb{R}} |\sigma(t, x, u) - \sigma(t, x, v)| \leq C|u - v|, \quad \forall u, v \in \mathbb{R}; \tag{2.24}
\]

for some constant \(C > 0\), which together with the assumption \(\sigma(t, x, 0) = 0\) implies that

\[
\sup_{t \in [0, T], x \in \mathbb{R}} |\sigma(t, x, u)| \leq C|u|, \quad \forall u \in \mathbb{R}. \tag{2.25}
\]

The following theorem follows from [23].

**Theorem 2.7.** ([23, Theorem 2.6]) Assume that Hypothesis 2.6 holds. Then Eq. (1.1) admits a unique strong solution in \(\mathcal{C}([0, T] \times \mathbb{R})\) almost surely.

3. Freidlin-Wentzell large deviations and statement of the main result

3.1. A general criterion for the large deviation principle. Let us first recall some standard definitions and results of the large deviation theory. Let \(\{X^\varepsilon\}_{\varepsilon > 0}\) be a family of random variables defined on a probability space \((\Omega, \mathcal{F}, \mathbb{P})\) and taking values in a Polish space \(E\). Roughly speaking, the LDP concerns the exponential decay of the probability measures of certain kinds of extreme or tail events and the rate of such exponential decay is expressed by the rate function.

**Definition 3.1.** A function \(I : E \to [0, \infty]\) is called a rate function on \(E\), if for each \(M < \infty\) the level set \(\{y \in E : I(y) \leq M\}\) is a compact subset of \(E\).

**Definition 3.2.** (Large deviation principle) Let \(I\) be a rate function on \(E\). The sequence \(\{X^\varepsilon\}_{\varepsilon > 0}\) is said to satisfy a large deviation principle on \(E\) with the rate function \(I\), if the following two conditions hold:

(a). for each closed subset \(F\) of \(E\),

\[
\limsup_{\varepsilon \to 0} \varepsilon \log \mathbb{P}(X^\varepsilon \in F) \leq - \inf_{y \in F} I(y);
\]

(b). for each open subset \(G\) of \(E\),

\[
\liminf_{\varepsilon \to 0} \varepsilon \log \mathbb{P}(X^\varepsilon \in G) \geq - \inf_{y \in G} I(y).
\]

Due to the representation formula of (2.7), our noise \(W\) can be identified as a sequence of independent, standard, real valued Brownian motions. Set \(\mathcal{V} = \mathcal{C}([0, T]; \mathbb{R}^\infty)\). Let \(\{\Gamma^\varepsilon\}_{\varepsilon > 0}\) be a family of measurable maps from \(\mathcal{V}\) to \(E\). We recall a criterion for the LDP of the family \(X^\varepsilon = \Gamma^\varepsilon(W)\) as \(\varepsilon \to 0\).

Define the following space of stochastic processes:

\[
\mathcal{L}_2 := \left\{ \phi : \Omega \times [0, T] \to \mathcal{H} \text{ is predictable and } \int_0^T \|\phi(s)\|^2_{\mathcal{H}} ds < \infty, \ \mathbb{P}\text{-a.s.} \right\}. \tag{3.1}
\]
For each $N \geq 1$, let
\[
S^N = \left\{ g \in L^2([0, T]; \mathcal{H}) : \int_0^T \|g(s)\|_H^2 ds \leq N \right\},
\]
where $L^2([0, T]; \mathcal{H})$ is the space of square integrable $\mathcal{H}$-valued functions on $[0, T]$. Here and in the sequel of this paper, we will always refer to the weak topology on the set $S^N$. Set $S = \bigcup_{N \geq 1} S^N$, and
\[
\mathcal{U}^N = \{ g \in \mathcal{L}_2 : g(\omega) \in S^N, \ \mathbb{P}\text{-a.s.} \}.
\]

**Condition 3.3.** There exists a measurable mapping $\Gamma^0 : \mathcal{V} \to E$ such that the following two items hold:

(a). for every $N < +\infty$, the set $K_N = \{ \Gamma^0 \left( \int_0^\infty g(s) ds \right) : g \in S^N \}$ is a compact subset of $E$;
(b). for every $N < +\infty$ and $\{ g^\varepsilon \}_{\varepsilon > 0} \subset \mathcal{U}^N$ satisfying that $g^\varepsilon$ converges in distribution (as $S^N$-valued random elements) to $g$, $\Gamma^0 \left( W + \frac{1}{\sqrt{\varepsilon}} \int_0^\infty g^\varepsilon(s) ds \right)$ converges in distribution to $\Gamma^0 \left( \int_0^\infty g(s) ds \right)$.

Let $I : E \to [0, \infty]$ be defined by
\[
I(\phi) := \inf_{\{g \in S : \phi = \Gamma^0(\int_0^\infty g(s) ds)\}} \left\{ \frac{1}{2} \int_0^T \|g(s)\|^2_H ds \right\}, \ \phi \in E,
\]
where the infimum over an empty set is taken as $+\infty$.

The following result is due to Budhiraja et al. [6].

**Theorem 3.4.** ([6, Theorem 6]) For any $\varepsilon > 0$, let $X^\varepsilon = \Gamma^\varepsilon(W)$ and suppose that Condition 3.3 holds. Then, the laws of $\{X^\varepsilon\}_{\varepsilon > 0}$ satisfy an LDP with the rate function $I$ defined by (3.3).

We now formulate the following sufficient condition established in [27] for verifying the assumptions in Condition 3.3 for LDPs, which is convenient in some applications.

**Condition 3.5.** There exists a measurable mapping $\Gamma^0 : \mathcal{V} \to E$ such that the following two items hold:

(a). for every $N < +\infty$ and any family $\{g^n\}_{n \geq 1} \subset S^N$ that converges to some element $g$ in $S^N$ as $n \to \infty$, $\Gamma^0 \left( \int_0^\infty g^n(s) ds \right)$ converges to $\Gamma^0 \left( \int_0^\infty g(s) ds \right)$ in the space $E$;
(b). for every $N < +\infty$, $\{g^\varepsilon\}_{\varepsilon > 0} \subset \mathcal{U}^N$ and $\delta > 0$,
\[
\lim_{\varepsilon \to 0} \mathbb{P}(\rho(Y^\varepsilon, Z^\varepsilon) > \delta) = 0,
\]
where $Y^\varepsilon = \Gamma^\varepsilon \left( W + \frac{1}{\sqrt{\varepsilon}} \int_0^\infty g^\varepsilon(s) ds \right)$, $Z^\varepsilon = \Gamma^0 \left( \int_0^\infty g^\varepsilon(s) ds \right)$ and $\rho(\cdot, \cdot)$ stands for the metric in the space $E$.

3.2. **Statement of the main result.** In this paper, we are concerned with the following one (spatial) dimensional nonlinear SWE:
\[
\begin{cases}
\frac{\partial^2 u^\varepsilon(t, x)}{\partial t^2} = \frac{\partial^2 u^\varepsilon(t, x)}{\partial x^2} + \sqrt{\varepsilon} \sigma(t, x, u^\varepsilon(t, x)) \hat{W}(t, x), \quad t > 0, \ x \in \mathbb{R}, \\
u^\varepsilon(0, \cdot) = u_0(x), \quad \frac{\partial u^\varepsilon(0, x)}{\partial t} = v_0(x).
\end{cases}
\]

Under Hypothesis 2.6, by Theorem 2.7, there exists a unique solution $u^\varepsilon \in C([0, T] \times \mathbb{R})$ a.s.. Therefore, there exists a Borel-measurable function $\Gamma^\varepsilon : C([0, T] \times \mathbb{R}^\varepsilon) \to C([0, T] \times \mathbb{R})$ such that
\[
u^\varepsilon(\cdot) = \Gamma^\varepsilon(W(\cdot)),
\]
For every $g \in \mathbb{S}$, we consider the following deterministic integral equation (the skeleton equation)

$$u^g(t, x) = I_0(t, x) + \int_0^t \langle G(t - s, x - \cdot) \sigma(s, \cdot, u^g(s, \cdot)), g(s, \cdot) \rangle_{\mathcal{H}} ds, \quad t \geq 0, \ x \in \mathbb{R}.$$ 

The solution $u^g$, whose existence will be proved in the next section, defines a measurable mapping $\Gamma^0 : C([0, T]; \mathbb{R}^N) \to C([0, T] \times \mathbb{R})$ so that

$$u^g(\cdot) = \Gamma^0 \left( \int_0^\cdot g(s) ds \right). \quad (3.5)$$

Here is the main result of this paper.

**Theorem 3.6.** Assume that Hypothesis 2.6 holds. Then, the family $\{u^\varepsilon\}_{\varepsilon > 0}$ in Eq. (1.1) satisfies an LDP in the space $C([0, T] \times \mathbb{R})$ with the rate function $I$ given by

$$I(\phi) := \inf_{\{g \in \mathcal{S}; \phi = \Gamma^0(\int_0^\cdot g(s) ds)\}} \left\{ \frac{1}{2} \int_0^T \|g(s)\|_{\mathcal{H}}^2 ds \right\}. \quad (3.6)$$

**Proof.** According to Theorem 3.4, we only need to prove that Condition 3.5 is fulfilled. The verification of Condition 3.5 (a) will be given by Proposition 5.1. Condition 3.5 (b) will be established in Proposition 5.2. The proof is complete. $\square$

### 4. Skeleton equation

In this section, we study the well-posedness of the skeleton equation:

$$u^g(t, x) = I_0(t, x) + \int_0^t \langle G(t - s, x - \cdot) \sigma(s, \cdot, u^g(s, \cdot)), g(s, \cdot) \rangle_{\mathcal{H}} ds, \quad t \geq 0, \ x \in \mathbb{R}, \quad (4.1)$$

where $g \in \mathcal{S}$ and $I_0(t, x)$ is defined by (2.20).

For any $p \geq 2$, $H \in \left(\frac{1}{2}, 1\right)$, recall the space $Z^p(T)$ and its norm defined by (2.16). We have the following well-posedness result for the skeleton equation (4.1).

**Proposition 4.1.** Assume that Hypothesis 2.6 holds. Then, Eq. (4.1) admits a unique solution in $C([0, T] \times \mathbb{R})$. In addition, $\sup_{g \in \mathcal{S}} \|u^g\|_{Z^p(T)} < \infty$ for any $N \geq 1$ and $p \geq 2$.

Due to the complexity of the space $\mathcal{H}$, it is difficult to use a Picard iteration scheme to prove the existence of the solution of Eq. (4.1) directly. We use the approximation method by introducing a new Hilbert space $\mathcal{H}_\varepsilon$ as follows.

For every fixed $\varepsilon > 0$, let

$$f_\varepsilon(x) := F^{-1} \left( e^{-\varepsilon |\xi|^2} |\xi|^{1-2H} \right)(x) = \frac{1}{2\pi} \int_\mathbb{R} e^{i \xi x} e^{-\varepsilon |\xi|^2} |\xi|^{1-2H} d\xi. \quad (4.2)$$

For any $\varphi, \psi \in \mathcal{D}(\mathbb{R})$, we define

$$\langle \varphi, \psi \rangle_{\mathcal{H}_\varepsilon} := c_{1,H} \int_\mathbb{R} F\varphi(\xi) \overline{F\psi(\xi)} e^{-\varepsilon |\xi|^2} |\xi|^{1-2H} d\xi$$

$$= c_{1,H} \int_{\mathbb{R}^2} \varphi(x) \psi(y) f_\varepsilon(x - y) dxdy, \quad (4.3)$$

where $c_{1,H}$ is given by (1.3). Let $\mathcal{H}_\varepsilon$ be the Hilbert space obtained by completing $\mathcal{D}(\mathbb{R})$ with respect to the scalar product given by (4.3). Notice that for any $0 \leq \varepsilon_1 < \varepsilon_2$, we have that for any $\varphi \in \mathcal{H}_{\varepsilon_1}$,

$$\|\varphi\|_{\mathcal{H}_{\varepsilon_1}} \geq \|\varphi\|_{\mathcal{H}_{\varepsilon_2}}, \quad (4.4)$$

and for any $\varphi, \psi \in \mathcal{H}$, by the dominated convergence theorem,

$$\lim_{\varepsilon \to 0} \langle \varphi, \psi \rangle_{\mathcal{H}_\varepsilon} = \langle \varphi, \psi \rangle_{\mathcal{H}}.$$
For any $g \in \mathcal{S}$, let
\begin{equation}
  u^g_\varepsilon(t, x) = I_0(t, x) + \int_0^t \langle G(t - s, x - \cdot) \sigma(s, \cdot), u^0_\varepsilon(s, \cdot)\rangle_{\mathcal{H}_\varepsilon} ds.
\end{equation}

Since $|\xi|^{1-2H} e^{-\varepsilon|\xi|^2}$ is in $L^1(\mathbb{R})$, $|f_{\varepsilon}|$ is bounded. Thus, using the Picard iteration, the existence and uniqueness of the solution $u^g_{\varepsilon}$ to Eq. (4.5) can be proved in a similar (but easier) way to that for Eq. (3.24) in [23].

The following lemma asserts that the approximate solution $u^g_{\varepsilon}$ is uniformly bounded in the space $Z^p(T)$ with respect to $\varepsilon > 0$.

**Lemma 4.1.** Let $H \in (\frac{1}{4}, \frac{1}{2})$ and $g \in \mathcal{S}$. Assume that Hypothesis 2.6 holds. Then the approximate solution $u^g_{\varepsilon}$ satisfies that, for any $p \geq 2$,
\begin{equation}
  \sup_{g \in \mathcal{S}^N} \sup_{\varepsilon > 0} \|u^g_{\varepsilon}\|_{Z^p(T)} < \infty.
\end{equation}

**Proof.** This proof is inspired by the proof of [23, Lemma 3.4]. We define the Picard iteration sequence as follows. For $n = 0, 1, 2, \cdots$, let
\begin{equation}
  u^{g,n}_{\varepsilon}(t, x) = I_0(t, x) + \int_0^t \langle G(t - s, x - \cdot) \sigma(s, \cdot), u^{g,n}_{\varepsilon}(s, \cdot)\rangle_{\mathcal{H}_\varepsilon} ds,
\end{equation}
with $u^{g,0}_{\varepsilon}(t, x) = I_0(t, x)$. Assume that there exists some constant $N > 0$ such that $\int_0^T \|g(s, \cdot)\|_{\mathcal{H}_\varepsilon}^2 ds \leq N$. Then, by (4.4), we know that
\begin{equation}
  \int_0^T \|g(s, \cdot)\|_{\mathcal{H}_\varepsilon}^2 ds \leq N, \text{ for any } \varepsilon > 0.
\end{equation}

The rest of the proof is divided into four steps. In Step 1, we prove the convergence of $u^{g,n}_{\varepsilon}(t, \cdot)$ in $L^p(\mathbb{R})$ for any $t \in [0, T]$. In Steps 2 and 3, we will bound $\|u^{g,n}_{\varepsilon}(t, \cdot)\|_{L^p(\mathbb{R})}$ and $N^*_{1-H,p} u^{g,n}_{\varepsilon}(t)$ for each fixed $\varepsilon > 0$. Step 4 is devoted to proving that $u^{g}_{\varepsilon}$ is bounded in $(Z^p(T), \|\cdot\|_{Z^p(T)})$ uniformly over $\varepsilon > 0$.

**Step 1.** We will bound $\|u^{g,n}_{\varepsilon}(t, \cdot)\|_{L^p(\mathbb{R})}$ uniformly in $n$ and show that $u^{g}_{\varepsilon}(t, \cdot)$ is in $L^p(\mathbb{R})$ in this step. By Cauchy-Schwarz’s inequality, (2.24), (4.8), the boundedness of $f_{\varepsilon}$ and Jensen’s inequality with respect to $\frac{1}{t-s} G(t - s, x - y) dy$, we have that for any $t \in [0, T]$ and $x \in \mathbb{R}$,
\begin{align*}
  &\|u^{g,n+1}_{\varepsilon}(t, x) - u^{g,n}_{\varepsilon}(t, x)\|^2 \\
  &= \left\| \int_0^t \langle G(t - s, x - \cdot) \left[ \sigma(s, \cdot, u^{g,n}_{\varepsilon}(s, \cdot)) - \sigma(s, \cdot, u^{g,n-1}_{\varepsilon}(s, \cdot)) \right], g(s, \cdot) \rangle_{\mathcal{H}_\varepsilon} ds \right\|^2 \\
  &\leq c_{1,H} N \int_0^t \int_{\mathbb{R}^2} G(t - s, x - y) \left[ \sigma(s, y, u^{g,n}_{\varepsilon}(s, y)) - \sigma(s, y, u^{g,n-1}_{\varepsilon}(s, y)) \right] \sigma(s, z, u^{g,n}_{\varepsilon}(s, z)) - \sigma(s, z, u^{g,n-1}_{\varepsilon}(s, z)) ds \\
  &\quad \cdot G(t - s, x - z) \left[ \sigma(s, z, u^{g,n}_{\varepsilon}(s, z)) - \sigma(s, z, u^{g,n-1}_{\varepsilon}(s, z)) \right] f_{\varepsilon}(y - z) dy dz ds \\
  &\leq c_{1,H} TN \|f_{\varepsilon}\|_{\mathcal{H}_\varepsilon} \cdot \int_0^t \int_{\mathbb{R}^2} G(t - s, x - y) \left[ \sigma(s, y, u^{g,n}_{\varepsilon}(s, y)) - \sigma(s, y, u^{g,n-1}_{\varepsilon}(s, y)) \right]^2 dy dz ds \\
  &\leq c_{\varepsilon,T,N} \int_0^t \int_{\mathbb{R}^2} G(t - s, x - y) \left| u^{g,n}_{\varepsilon}(s, y) - u^{g,n-1}_{\varepsilon}(s, y) \right|^2 dy dz ds.
\end{align*}

By Jensen’s inequality with respect to $\frac{1}{t-s} G(t - s, x - y) dydz$ and a change of variable, we have that for any $p \geq 2$, $t \in [0, T]$,
\begin{align*}
  &\|u^{g,n+1}_{\varepsilon}(t, \cdot) - u^{g,n}_{\varepsilon}(t, \cdot)\|_{L^p(\mathbb{R})}^p \\
  &\leq c_{\varepsilon,T,N} \int_0^t \int_{\mathbb{R}^2} G(t - s, x - y) \left| u^{g,n}_{\varepsilon}(s, y) - u^{g,n-1}_{\varepsilon}(s, y) \right|^2 dy dz ds.
\end{align*}
\[
\begin{align*}
&\leq C_{\varepsilon, T, N, p, H} \int_{\mathbb{R}} \left[ \int_{0}^{t} \int_{\mathbb{R}} G(t - s, x - y) \left| u_{\varepsilon}^{g,n}(s, y) - u_{\varepsilon}^{g,n-1}(s, y) \right|^2 \, dy \, ds \right]^{\frac{p}{2}} \, dx \\
&\leq C_{\varepsilon, T, N, p, H} \int_{0}^{t} \int_{\mathbb{R}} G(t - s, x) \, dx \cdot \int_{\mathbb{R}} \left| u_{\varepsilon}^{g,n}(s, y) - u_{\varepsilon}^{g,n-1}(s, y) \right|^p \, dy \, ds \\
&\leq C_{\varepsilon, T, N, p, H} \int_{0}^{t} \left\| u_{\varepsilon}^{g,n}(s, \cdot) - u_{\varepsilon}^{g,n-1}(s, \cdot) \right\|^p_{L^p(\mathbb{R})} \, ds \\
&\leq C_{\varepsilon, T, N, p, H} \frac{T^n}{m!} \sup_{s \in [0, T]} \left\| u_{\varepsilon}^{g,1}(s, \cdot) - u_{\varepsilon}^{g,0}(s, \cdot) \right\|^p_{L^p(\mathbb{R})}. 
\end{align*}
\]

Thus, (4.9) implies that

\[
\sup_{n \geq 1} \sup_{t \in [0, T]} \left\| u_{\varepsilon}^{g,n}(t, \cdot) \right\|_{L^p(\mathbb{R})} < \infty, \quad \text{for each } \varepsilon > 0,
\]

and that \( \{ u_{\varepsilon}^{g,n}(t, \cdot) \}_{n \geq 1} \) is a Cauchy sequence in \( L^p(\mathbb{R}) \) for any \( t \in [0, T] \). Hence, for any fixed \( t \in [0, T] \), there exists \( u_{\varepsilon}^{g}(t, \cdot) \in L^p(\mathbb{R}) \) such that \( u_{\varepsilon}^{g,n}(t, \cdot) \) converges to \( u_{\varepsilon}^{g}(t, \cdot) \) in \( L^p(\mathbb{R}) \) when \( n \) goes to infinity.

**Step 2.** In this step, we will provide a uniform bound of \( \left\| u_{\varepsilon}^{g,n}(t, \cdot) \right\|_{L^p(\mathbb{R})} \) for any fixed \( \varepsilon > 0 \).

For the simplicity of writing, denote that

\[
D_{h} f(t, x) := f(t, x + h) - f(t, x)
\]

and

\[
\Box_{h} f(t, x) := f(t, x + h + l) - f(t, x + l) - f(t, x + h) + f(t, x),
\]

for any function \( f \).

By the Cauchy-Schwarz inequality, (2.6), (4.4) and (4.8), we have

\[
\begin{align*}
|u_{\varepsilon}^{g,n+1}(t, x)|^p &\leq |I_0(t, x)|^p + \left[ \int_{0}^{t} \left| G(t - s, x - \cdot) \sigma(s, \cdot, u_{\varepsilon}^{g,n}(s, \cdot)) \right| ds \right]^p \nonumber \\
&\leq |I_0(t, x)|^p + \left( \int_{0}^{t} \left\| G(t - s, x - \cdot) \sigma(s, \cdot, u_{\varepsilon}^{g,n}(s, \cdot)) \right\|_{H_x}^2 \, ds \right)^{\frac{p}{2}} \\
&\leq |I_0(t, x)|^p + \left( \int_{0}^{t} \left\| G(t - s, x - \cdot) \sigma(s, \cdot, u_{\varepsilon}^{g,n}(s, \cdot)) \right\|_{H_x}^2 \, ds \right)^{\frac{p}{2}} \\
&\leq |I_0(t, x)|^p + \left( \int_{0}^{t} \left\| G(t - s, x - y - z) \sigma(s, y + z, u_{\varepsilon}^{g,n}(s, y + z)) \right\|_{H_x}^2 \, ds \right)^{\frac{p}{2}} \\
&\leq |I_0(t, x)|^p + A_1(t, x) + A_2(t, x) + A_3(t, x),
\end{align*}
\]

where

\[
A_1(t, x) := \left( \int_{0}^{t} \int_{\mathbb{R}^2} \left| G(t - s, x - y - z) \right|^2 \cdot \left| \sigma(s, y + z, u_{\varepsilon}^{g,n}(s, y + z)) - \sigma(s, y, u_{\varepsilon}^{g,n}(s, y + z)) \right|^2 \right)^{\frac{p}{2}} ds dy dx;
\]

\[
A_2(t, x) := \left( \int_{0}^{t} \int_{\mathbb{R}^2} \left| G(t - s, x - y - z) \right|^2 \cdot \left| \sigma(s, y, u_{\varepsilon}^{g,n}(s, y + z)) - \sigma(s, y, u_{\varepsilon}^{g,n}(s, y)) \right|^2 \right)^{\frac{p}{2}} ds dy dx;
\]

and

\[
A_3(t, x) := \left( \int_{0}^{t} \int_{\mathbb{R}^2} \left| G(t - s, x - y - z) \right|^2 \cdot \left| \sigma(s, y, u_{\varepsilon}^{g,n}(s, y)) \right|^2 \right)^{\frac{p}{2}} ds dy dx.
\]
\[ A_3(t, x) := \left( \int_0^t \int_{\mathbb{R}^2} |\mathcal{D}_z G(t - s, x - y)|^2 \cdot |\sigma(s, y, u^g_n(s, y))|^2 \cdot |z|^{2H-2} \, dz \, dy \, ds \right)^{\frac{2}{p}}. \]

If \( |z| > 1 \), then we have that by (2.25),
\[
|\sigma(s, y + z, u^g_n(s, y)) - \sigma(s, y, u^g_n(s, y))|^2 \\
\leq |\sigma(s, y + z, u^g_n(s, y))|^2 + |\sigma(s, y, u^g_n(s, y))|^2
\]
(4.13)

If \( |z| \leq 1 \), due to \( \sigma(t, x, 0) = 0 \) and (2.22), we have
\[
|\sigma(s, y + z, u^g_n(s, y)) - \sigma(s, y, u^g_n(s, y))|^2 = \left| \int_0^{u^g_n} \left[ \sigma'(s, y + z, \xi) - \sigma'(s, y, \xi) \right] d\xi \right|^2
\]
(4.14)

Since \( H \in \left( \frac{1}{4}, \frac{1}{2} \right) \), by a change of variable, (4.13), (4.14) and Minkowski’s inequality, we have
\[
\left( \int_\mathbb{R} A_1(t, x) \, dx \right)^{\frac{2}{p}} \leq \left( \int_\mathbb{R} \left( \int_0^t \int_{\mathbb{R}^2} |G(t - s, y)|^2 \cdot |u^g_n(s, x)|^2 \, dy \, ds \right)^{\frac{2}{p}} \, dx \right)^{\frac{2}{p}}
\]
(4.15)

By a change of variable, (2.24) and Minkowski’s inequality, we have
\[
\left( \int_\mathbb{R} A_2(t, x) \, dx \right)^{\frac{2}{p}}
\]
(4.16)

By (2.25), a change of variable, Minkowski’s inequality and Lemma 6.2, we have
\[
\left( \int_\mathbb{R} A_3(t, x) \, dx \right)^{\frac{2}{p}} \leq \left( \int_0^t \int_{\mathbb{R}^2} |\mathcal{D}_z G(t - s, y)|^2 \cdot |u^g_n(s, x)|^2 \cdot |z|^{2H-2} \, dz \, dy \, ds \right)^{\frac{2}{p}}
\]
(4.17)
Thus, putting (4.12), (4.15), (4.16) and (4.17) together, we have
\[
\|u^{\alpha_{n+1}}_\varepsilon(t,\cdot)\|_{L^p(\mathbb{R})}^2 \lesssim \|I_0(t,\cdot)\|_{L^p(\mathbb{R})}^2 + \int_0^t \left( (t-s) + (t-s)^{2H} \right) \cdot \|u^{\alpha_n}_\varepsilon(s,\cdot)\|_{L^p(\mathbb{R})}^2 ds \\
+ \int_0^t (t-s) \left[ \mathcal{N}_{\frac{1}{2},H,p}^* u^{\alpha_n}_\varepsilon(s) \right]^2 ds.
\] (4.18)

**Step 3.** This step is devoted to estimating \( \mathcal{N}_{\frac{1}{2},H,p}^* u^{\alpha_n}_\varepsilon(t) \) for any fixed \( \varepsilon > 0 \). Using the Cauchy-Schwarz inequality, (4.4) and (4.8), we have
\[
\left| u^{\alpha_{n+1}}_\varepsilon(t,x+h) - u^{\alpha_{n+1}}_\varepsilon(t,x) \right|^p \lesssim |I_0(t,x+h) - I_0(t,x)|^p + \left( \int_0^t \|D_h G(t-s,x-\cdot)\sigma(s,\cdot,u^{\alpha_n}_\varepsilon(s,\cdot))\|_{H^p}^2 ds \right)^{\frac{p}{2}} \\
\lesssim |I_0(t,x+h) - I_0(t,x)|^p + \left( \int_0^t \int_{\mathbb{R}^2} |D_h G(t-s,x-y-z)| \sigma(s,y+z,u^{\alpha_n}_\varepsilon(s,y+z)) \\
- \mathcal{D}_h G(t-s,x-y) \sigma(s,y,u^{\alpha_n}_\varepsilon(s,y)) \right)^2 \cdot |z|^{2H-2} dydz ds \right)^{\frac{p}{2}} \\
\lesssim I_0(t,x,h) + I_1(t,x,h) + I_2(t,x,h) + I_3(t,x,h),
\]
where
\[
I_0(t,x,h) := |I_0(t,x+h) - I_0(t,x)|^p; \\
I_1(t,x,h) := \left( \int_0^t \int_{\mathbb{R}^2} |D_h G(t-s,x-y-z)|^2 \\
\cdot \left| \sigma(s,y+z,u^{\alpha_n}_\varepsilon(s,y+z)) - \sigma(s,y,u^{\alpha_n}_\varepsilon(s,y+z)) \right|^2 \cdot |z|^{2H-2} dydz ds \right)^{\frac{p}{2}}; \\
I_2(t,x,h) := \left( \int_0^t \int_{\mathbb{R}^2} |D_h G(t-s,x-y-z)|^2 \\
\cdot \left| \sigma(s,y,u^{\alpha_n}_\varepsilon(s,y+z)) - \sigma(s,y,u^{\alpha_n}_\varepsilon(s,y)) \right|^2 \cdot |z|^{2H-2} dydz ds \right)^{\frac{p}{2}}; \\
I_3(t,x,h) := \left( \int_0^t \int_{\mathbb{R}^2} \left| \mathcal{D}_h,z(t-s,x,y) \right|^2 \cdot \left| \sigma(s,y,u^{\alpha_n}_\varepsilon(s,y)) \right|^2 \cdot |z|^{2H-2} dydz ds \right)^{\frac{p}{2}}.
\]
Therefore, by (2.18), we have
\[
\left[ \mathcal{N}_{\frac{1}{2},H,p}^* u^{\alpha_{n+1}}_\varepsilon(t) \right]^2 \lesssim \sum_{j=0}^3 \left( \int_{\mathbb{R}} I_j(t,x,h) dx \right)^{\frac{2}{p}} \cdot |h|^{2H-2} dh.
\] (4.19)

By (4.13), (4.14), a change of variable, Minkowski’s inequality and Lemma 6.2, we have
\[
\int_{\mathbb{R}} \left[ \int_{\mathbb{R}} I_1(t,x,h) dx \right]^{\frac{2}{p}} \cdot |h|^{2H-2} dh \\
\lesssim \int_{\mathbb{R}} \left[ \int_{\mathbb{R}} \left( \int_0^t \int_{\mathbb{R}^2} |D_h G(t-s,y)|^2 \left| u^{\alpha_n}_\varepsilon(s,x) \right|^2 dydz ds \right)^{\frac{p}{2}} dx \right]^{\frac{2}{p}} \cdot |h|^{2H-2} dh \\
\lesssim \int_0^t \int_{\mathbb{R}^2} |D_h G(t-s,y)|^2 \cdot |h|^{2H-2} dhdy \left( \int_{\mathbb{R}} \left| u^{\alpha_n}_\varepsilon(s,x) \right|^p dx \right)^{\frac{2}{p}} ds \\
\lesssim \int_0^t (t-s)^{2H} \cdot \|u^{\alpha_n}_\varepsilon(s,\cdot)\|_{L^p(\mathbb{R})}^2 ds.
\] (4.20)
By (2.24), a change of variable, Minkowski’s inequality, Jensen’s inequality and Lemma 6.2, we have

\[
\int_{\mathbb{R}} \left( \int_{\mathbb{R}} I_2(t, x, h) dx \right)^{\frac{2}{p}} \cdot |h|^{2H-2} dh \\
\leq \int_{\mathbb{R}} \left( \int_{\mathbb{R}} \left( \int_{\mathbb{R}^2} |D_h G(t - s, y)|^2 \cdot |u^{q,n}_\varepsilon(s, x + z) - u^{q,n}_\varepsilon(s, x)|^2 \cdot |z|^{2H-2} dy dz ds \right)^{\frac{p}{2}} dx \right)^{\frac{2}{p}} \cdot |h|^{2H-2} dh \\
\leq \int_{0}^{t} \int_{\mathbb{R}^2} |D_h G(t - s, y)|^2 \cdot |h|^{2H-2} dh dy \cdot \int_{\mathbb{R}} \left( \int_{\mathbb{R}} |u^{q,n}_\varepsilon(s, x + z) - u^{q,n}_\varepsilon(s, x)|^p dx \right)^{\frac{2}{p}} \cdot |z|^{2H-2} dz ds \\
\leq \int_{0}^{t} (t - s)^{2H} \cdot \left[ N^{q}_{\frac{1}{2}-H, p} u^{q,n}_\varepsilon(s) \right]^{2} ds.
\]

(4.21)

By (2.25), a change of variable, Minkowski’s inequality and Lemma 6.2, we have

\[
\int_{\mathbb{R}} \left( \int_{\mathbb{R}} I_3(t, x, h) dx \right)^{\frac{2}{p}} \cdot |h|^{2H-2} dh \\
\leq \int_{\mathbb{R}} \left( \int_{\mathbb{R}} \left( \int_{\mathbb{R}^3} \Box_h z(t - s, y)|^2 \cdot |u^{q,n}_\varepsilon(s, x)|^2 \cdot |z|^{2H-2} dy dz ds \right)^{\frac{p}{2}} dx \right)^{\frac{2}{p}} \cdot |h|^{2H-2} dh \\
\leq \int_{0}^{t} \int_{\mathbb{R}^3} \Box_h z(t - s, y)|^2 \cdot \left( \int_{\mathbb{R}} |u^{q,n}_\varepsilon(s, x)|^p dx \right)^{\frac{2}{p}} \cdot |h|^{2H-2} \cdot |z|^{2H-2} dh dz dy ds \\
\leq \int_{0}^{t} (t - s)^{4H-1} \cdot \|u^{q,n}_\varepsilon(s, \cdot)\|_{L^p(\mathbb{R})}^2 ds.
\]

(4.22)

Thus, substitute (4.20), (4.21) and (4.22) back into (4.19), we have

\[
\left[ N^{q}_{\frac{1}{2}-H, p} u^{q,n+1}_\varepsilon(t) \right]^{2} \leq \left[ N^{q}_{\frac{1}{2}-H, p} I_0(t) \right]^{2} + \int_{0}^{t} ((t - s)^{2H} + (t - s)^{4H-1}) \cdot \|u^{q,n}_\varepsilon(s, \cdot)\|_{L^p(\mathbb{R})}^2 ds \\
+ \int_{0}^{t} (t - s)^{2H} \cdot \left[ N^{q}_{\frac{1}{2}-H, p} u^{q,n}_\varepsilon(s) \right]^{2} ds.
\]

(4.23)

**Step 4.** Define

\[
\Psi^{n}_\varepsilon(t) := \|u^{q,n}_\varepsilon(t, \cdot)\|_{L^p(\mathbb{R})}^2 + \left[ N^{q}_{\frac{1}{2}-H, p} u^{q,n}_\varepsilon(t) \right]^2.
\]

Combining (4.18) and (4.23), there exists a constant $C_{T, p, H, N}$ such that

\[
\Psi^{n+1}_\varepsilon(t) \leq C_{T, p, H, N} \left( \|I_0\|_{L^p(T)}^2 + \int_{0}^{t} (t - s)^{4H-1} \Psi^{n}_\varepsilon(s) ds \right).
\]

By the extension of Gronwall’s lemma [11, Lemma 15], we have

\[
\sup_{n \geq 1} \sup_{t \in [0, T]} \Psi^{n}_\varepsilon(t) \leq C,
\]

(4.24)

where $C$ is a constant independent of $\varepsilon$ and $q \in S^N$.

For any fixed $\varepsilon > 0$, since $u^{q,n}_\varepsilon(t, \cdot) \to u^q_\varepsilon(t, \cdot)$ in $L^p(\mathbb{R})$ as $n \to \infty$, we have that by Fatou’s lemma,

\[
\sup_{n \geq 0} \sup_{t \in [0, T]} \|u^{q}_\varepsilon(t, \cdot)\|_{L^p(\mathbb{R})} \leq \sup_{n \geq 0} \sup_{t \in [0, T]} \|u^{q,n}_\varepsilon(t, \cdot)\|_{L^p(\mathbb{R})} + \sup_{n \geq 0} \sup_{t \in [0, T]} N^{q}_{\frac{1}{2}-H, p} u^{q}_\varepsilon(t) < \infty.
\]

The proof is complete. □
According to Lemmas 6.3, 6.4 and 6.5, by the same argument as that in the proofs of [23, Proposition 3.4] and [23, Proposition 4.1], the following lemma can be proved and we omit the details here.

**Lemma 4.2.** Let $u^\varepsilon$ be the approximate mild solution defined by (4.5). Then we have the following results:

(i) if $p > \frac{2}{4H-1}$, then there exists a constant $C_{T,p,H,N} > 0$ such that

$$
\sup_{t \in [0,T], x \in \mathbb{R}} N^{1-H}_{\frac{1}{2}} u^\varepsilon(t, x) \leq C_{T,p,H,N} \| u^\varepsilon \|_{Z^p(T)}; \tag{4.25}
$$

(ii) if $p > \frac{1}{H}$ and $0 < \gamma < H - \frac{1}{p}$, then there exists a constant $C_{T,p,H,N,\gamma} > 0$ such that

$$
\sup_{t, t+h \in [0,T], x \in \mathbb{R}} |u^\varepsilon(t+h, x) - u^\varepsilon(t, x)| \leq C_{T,p,H,N,\gamma} |h|^\gamma \cdot \| u^\varepsilon \|_{Z^p(T)}; \tag{4.26}
$$

(iii) if $p > \frac{1}{H}$ and $0 < \gamma < H - \frac{1}{p}$, then there exists a constant $C_{T,p,H,N,\gamma} > 0$ such that

$$
\sup_{t \in [0,T], x, y \in \mathbb{R}} |u^\varepsilon(t, x) - u^\varepsilon(t, y)| \leq C_{T,p,H,N,\gamma} |x - y|^\gamma \cdot \| u^\varepsilon \|_{Z^p(T)}. \tag{4.27}
$$

**Proof of Proposition 4.1.** Existence. The uniform Hölder continuity of the type specified in Lemma 4.2 (ii) and (iii) is the key to show the existence of the solution to Eq. (4.1). This, together with the fact of $u^\varepsilon(0, 0) = u_0(0)$ yields that the family $\{u^\varepsilon\}_{\varepsilon > 0}$ is relatively compact in $(C([0,T] \times \mathbb{R}), d_C)$ by the Arzelà-Ascoli theorem. Hence, there is a subsequence $\varepsilon_n \downarrow 0$ such that $u^\varepsilon_n \to u^g$ in $(C([0,T] \times \mathbb{R}), d_C)$. By Lemma 4.1 and Lemma 6.6 (i), we have that, for any $p \geq 2$,

$$
\sup_{g \in S^N} \| u^g \|_{Z^p(T)} < \infty. \tag{4.28}
$$

By using the Cauchy-Schwarz inequality, (2.24), Lemma 6.6 (ii) and the dominated convergence theorem, we have that as $\varepsilon_n \downarrow 0$,

$$
u^\varepsilon_n(t, x) = I_0(t, x) + \int_0^t \langle G(t-s, x-\cdot) (\sigma(s, \cdot, u^\varepsilon_n(s, \cdot)) - \sigma(s, \cdot, u^g(s, \cdot))) , g(s, \cdot) \rangle_{H_{\varepsilon_n}} ds
$$

$$
+ \int_0^t \langle G(t-s, x-\cdot) \sigma(s, \cdot, u^g(s, \cdot)) , g(s, \cdot) \rangle_{H_{\varepsilon_n}} ds
$$

$$
\to I_0(t, x) + \int_0^t \langle G(t-s, x-\cdot)(x-\cdot) \sigma(s, \cdot, u^g(s, \cdot)) , g(s, \cdot) \rangle_{H} ds.
$$

The uniqueness of the limit of $\{u^\varepsilon_n\}_{n \geq 1}$ implies that $u^g$ satisfies Eq. (4.1).

**Uniqueness.** Let $u^g$ and $v^g$ be two solutions of Eq. (4.1) with the same initial value condition. Similarly to Lemma 4.2 (i), we can obtain that, for any $p > \frac{2}{4H-1}$,

$$
\sup_{t \in [0,T], x \in \mathbb{R}} N^{1-H}_{\frac{1}{2}} u^g(t, x) < \infty, \quad \sup_{t \in [0,T], x \in \mathbb{R}} N^{1-H}_{\frac{1}{2}} v^g(t, x) < \infty. \tag{4.29}
$$

Denote that

$$
S_1(t) = \int_\mathbb{R} |u^g(t, x) - v^g(t, x)|^2 dx
$$

and

$$
S_2(t) = \int_{\mathbb{R}^2} |u^g(t, x + h) - v^g(t, x + h) - u^g(t, x) + v^g(t, x)|^2 \cdot |h|^{2H-2} dh dx.
$$

According to (4.28) and the definition of the norm $\| \cdot \|_{Z^p(T)}$, we know that

$$
\sup_{t \in [0,T]} S_1(t) < \infty, \quad \sup_{t \in [0,T]} S_2(t) < \infty. \tag{4.30}
$$
Recall $D_h f(t, x)$ defined by (4.10) and denote that

$$\nabla(t, x, y) := \sigma(t, x, u^g(t, y)) - \sigma(t, x, v^g(t, y)).$$

Since $T_0^T \|g(s, \cdot)\|_H^2 ds \leq N$, by using the Cauchy-Schwarz inequality, (2.6) and a change of variable, we have

$$\int_\mathbb{R} |u^g(t, x) - v^g(t, x)|^2 dx$$

$$= \int_\mathbb{R} \left| \int_0^t \langle G(t - s, x - \cdot) \nabla(s, \cdot, \cdot), g(s, \cdot) \rangle_H ds \right|^2 dx$$

$$\leq \int_\mathbb{R} \int_0^t \|G(t - s, x - \cdot) \nabla(s, \cdot, \cdot)\|_{H^2}^2 ds dx$$

$$\leq \int_0^t \int_\mathbb{R}^3 |G(t - s, x - y)|^2 |\nabla(s, y, y + h) - \nabla(s, y, y)|^2 \cdot |h|^{2H-2} dhdxdyds$$

$$+ \int_0^t \int_\mathbb{R}^3 |D_{-h} G(t - s, x - y)|^2 |\nabla(s, y + h, y)|^2 \cdot |h|^{2H-2} dhdxdyds$$

$$+ \int_0^t \int_\mathbb{R}^3 |G(t - s, x - y)|^2 |\nabla(s, y + h, y) - \nabla(s, y, y)|^2 \cdot |h|^{2H-2} dhdxdyds$$

$$=: V_1(t) + V_2(t) + V_3(t).$$

According to (2.21) and (2.23), we have

$$|\nabla(s, y, y + h) - \nabla(s, y, y)|^2$$

$$= \left| \int_0^1 \left[ u^g(s, y + h) - v^g(s, y + h) \right] \sigma_\theta(s, y, \theta u^g(s, y + h) + (1 - \theta)v^g(s, y + h)) d\theta$$

$$- \int_0^1 \left[ u^g(s, y) - v^g(s, y) \right] \sigma_\theta(s, y, \theta u^g(s, y) + (1 - \theta)v^g(s, y)) d\theta \right|^2$$

$$\leq |u^g(s, y + h) - v^g(s, y + h) - u^g(s, y) + v^g(s, y)|^2$$

$$+ |u^g(s, y) - v^g(s, y)|^2 \cdot \left[ |u^g(s, y + h) - u^g(s, y)|^2 + |v^g(s, y + h) - v^g(s, y)|^2 \right].$$

This, together with (4.29), gives the following estimate:

$$V_1(t) \leq \int_0^t (t - s) \cdot [S_1(s) + S_2(s)] ds.$$

By Lemma 6.2 and (2.24), we have

$$V_2(t) \leq \int_0^t \int_\mathbb{R} \left( \int_\mathbb{R}^2 |D_h G(t - s, x - y)|^2 |h|^{2H-2} dhdx \right) \cdot |u^g(s, y) - v^g(s, y)|^2 dyds$$

$$\leq \int_0^t (t - s)^{2H} \cdot S_1(s) ds.$$

If $h > 1$, (2.21) implies that

$$|\nabla(s, y + h, y) - \nabla(s, y, y)|^2 = \left| \int_{u^g}^{v^g} \left( \sigma_\theta(s, y + h, \xi) - \sigma_\theta(s, y, \xi) \right) d\xi \right|^2$$

$$\leq |u^g(s, y) - v^g(s, y)|^2.$$
If \( h \leq 1 \), then we have that by (2.22)

\[
|\triangle(s, y + h, y) - \triangle(s, y, y)|^2 = \left| \int_{v^g}^{v^g} (\sigma^g(s, y + h, \xi) - \sigma^g(s, y, \xi)) \, d\xi \right|^2 \leq |v^g(s, y) - v^g(s, y)|^2 \cdot |h|^2.
\]

(4.36)

Thus, (4.35) and (4.36) yield that

\[
V_3(t) \leq \int_0^t \int_{\mathbb{R}^2} |G(t - s, x - y)|^2 |u^g(s, y) - v^g(s, y)|^2 \, dy \, dx \, ds
\]

\[
= \int_0^t (t - s) \cdot S_1(s) \, ds.
\]

(4.37)

Collecting the inequalities in (4.31), (4.33), (4.34) and (4.37) together, we arrive at

\[
S_1(t) \leq \int_0^t (t - s)^{2H} \cdot [S_1(s) + S_2(s)] \, ds.
\]

(4.38)

Applying the same procedure as in the proof of (4.38), we can show

\[
S_2(t) \leq \int_0^t (t - s)^{4H-1} \cdot [S_1(s) + S_2(s)] \, ds.
\]

(4.39)

Therefore, (4.38) and (4.39) imply that

\[
S_1(t) + S_2(t) \leq \int_0^t \left( (t - s)^{2H} + (t - s)^{4H-1} \right) \cdot [S_1(s) + S_2(s)] \, ds.
\]

(4.40)

Invoking the fact that \( S_1(t) \) and \( S_2(t) \) are uniformly bounded on \([0, T]\) by (4.30), by the Gronwall inequality, we have

\[
S_1(t) + S_2(t) = 0 \quad \text{for all} \quad t \in [0, T],
\]

which together with the continuities of \( u^g \) and \( v^g \) gives that \( u^g = v^g \).

The proof is complete.

5. Large deviation principle

Firstly, we show that Condition 3.5 (a) is satisfied. Recall that \( \Gamma^0 (\int_0 g(s) \, ds) = u^g \) for \( g \in S \), where \( u^g \) is the solution of Eq. (4.1).

**Proposition 5.1.** Assume that Hypothesis 2.6 holds. For every \( N < +\infty \), let \( \{g_n\}_{n \geq 1}, g \) be in \( S^N \) such that \( g_n \rightarrow g \) weakly as \( n \rightarrow \infty \). Then, as \( n \rightarrow \infty \),

\[
u^{g_n} \rightharpoonup u^g \quad \text{in} \quad C([0, T] \times \mathbb{R}),
\]

where \( u^{g_n}, u^g \) are the solutions of Eq. (4.1) associated with \( g_n \) and \( g \), respectively.

**Proof.** Recall Eq. (4.1) with \( g \) replaced by \( g_n \),

\[
u^{g_n}(t, x) = I_0(t, x) + \int_0^t \langle G(t - s, x - \cdot) \sigma(s, \cdot, \nu^{g_n}(s, \cdot)), g_n(s, \cdot) \rangle \, d\mathcal{H} \, ds, \quad t \geq 0, \quad x \in \mathbb{R}.
\]

(5.1)

Since \( \{u^{g_n}\}_{n \geq 1} \subset S^N \), by Proposition 4.1, we have that, for any \( p > 2 \),

\[
\sup_{n \geq 1} \|u^{g_n}\|_{L^p(\mathcal{H})} < \infty.
\]

(5.2)

Similarly to the Hölder continuity of \( u^\xi \) specified by (4.26) and (4.27), we can obtain the Hölder continuity of \( u^{g_n}(t, x) \) (uniformly in \( t \in [0, T] \) and \( x \in \mathbb{R} \)). Combining this with \( u^{g_n}(0, 0) = u_0(0) \), we know that \( \{u^{g_n}\}_{n \geq 1} \) is relatively compact on the space \( (C([0, T] \times \mathbb{R}), d_C) \) by the Arzelà-Ascoli theorem. Thus, there exists a subsequence of \( \{u^{g_n}\}_{n \geq 1} \) (still denoted by \( \{u^{g_n}\}_{n \geq 1} \) and
\[ u \in C([0, T] \times \mathbb{R}) \text{ such that } u^{g_n} \to u \text{ as } n \to \infty. \] By (5.2) and Lemma 6.6 (i), we have that, for any \( p > 2 \),

\[ \|u\|_{\mathcal{L}^p(T)} < \infty. \] (5.3)

We now prove that \( u = u^g \). Denote that

\[ D_1(t) = \int_{\mathbb{R}} |u^{g_n}(t, x) - u^g(t, x)|^2 \, dx \]

and

\[ D_2(t) = \int_{\mathbb{R}^2} |u^{g_n}(t, x + h) - u^g(t, x + h) - u^{g_n}(t, x) + u^g(t, x)|^2 \cdot |h|^{2H-2} \, dh \, dx. \]

According to (4.28), we know that \( \sup_{t \in [0, T]} D_1(t) < \infty \) and \( \sup_{t \in [0, T]} D_2(t) < \infty \). Denote \( \triangle_n(t, x, y) := \sigma(t, x, u^{g_n}(t, y)) - \sigma(t, x, u^g(t, y)) \). Recall \( D_n f(t, x) \) defined by (4.10). It follows from (4.1) and (5.1) that

\[ D_1(t) + D_2(t) \]

\[ \leq 2 \int_{\mathbb{R}} \left( |\int_0^t \langle G(t - s, x - \cdot) \triangle_n(s, \cdot), g_n(s, \cdot) \rangle_{\mathcal{H}} \, ds \|^2 \right) \, dx \]

\[ + 2 \int_{\mathbb{R}} \left( \int_0^t |\langle G(t - s, x - \cdot) \sigma(s, \cdot, u^g(s, \cdot)), g_n(s, \cdot) - g(s, \cdot) \rangle_{\mathcal{H}} \|^2 \, ds \right) \, dx \]

\[ + 2 \int_{\mathbb{R}} \left( \int_0^t |\langle D_n G(t - s, x - \cdot) \triangle_n(s, \cdot), g_n(s, \cdot) \rangle_{\mathcal{H}} \|^2 \cdot |h|^{2H-2} \, dh \, dx \right) \]

\[ + 2 \int_{\mathbb{R}} \left( \int_0^t |\langle D_n G(t - s, x - \cdot) \sigma(s, \cdot, u^g(s, \cdot)), g_n(s, \cdot) - g(s, \cdot) \rangle_{\mathcal{H}} \|^2 \cdot |h|^{2H-2} \, dh \, dx \right) \]

\[ =: (E_1(t) + E_2(t) + E_3(t) + E_4(t)). \]

Using the similar procedure as in the proof of (4.40), we can show that

\[ E_1(t) + E_3(t) \leq \int_0^t (t - s)^{4H-1} \cdot [D_1(s) + D_2(s)] \, ds. \] (5.5)

On the other hand, denote that

\[ F_n(t, x) := \left\| \int_0^t \langle G(t - s, x - \cdot) \sigma(s, \cdot, u^g(s, \cdot)), g_n(s, \cdot) - g(s, \cdot) \rangle_{\mathcal{H}} \, ds \right\|^2. \]

By Lemma 6.6 (ii), we know that, for almost all \( x \in \mathbb{R} \),

\[ \int_0^t \left\| G(t - s, x - \cdot) \sigma(s, \cdot, u^g(s, \cdot)) \right\|_{\mathcal{H}}^2 \, ds < \infty. \]

As \( g_n \) converges weakly to \( g \) when \( n \) tends to \( \infty \), we know that, for almost all \( x \in \mathbb{R} \),

\[ F_n(t, x) \to 0, \text{ as } n \to \infty. \] (5.6)

Since \( g_n, g \in S^N \), by using the Cauchy-Schwarz inequality, (2.6) and a change of variable, we have that, for any \( p > 2 \),

\[ \int_{\mathbb{R}} F_n(t, x)^{\frac{p}{2}} \, dx \]

\[ \leq \int_{\mathbb{R}} \left( \int_0^t \left( \int_0^t \left\| G(t - s, x - \cdot) \sigma(s, \cdot, u^g(s, \cdot)) \right\|_{\mathcal{H}}^2 \, ds \right)^{\frac{p}{2}} \, dx \right)^{\frac{2}{p}} \]
Putting (5.4), (5.5), (5.7) and (5.8) together, by the Gronwall lemma, we have

\[
\int_{\mathbb{R}} \left( \int_0^t \int_{\mathbb{R}^2} \left| G(t - s, x - y - h)\sigma(s, y + h, u^\vartheta(s, y + h)) \right| \cdot |h|^{2H - 2} dhdyds \right)^{\frac{\varepsilon}{2}} dx.
\]

By the similar technique as that in Step 2 in the proof of Lemma 4.1, we have

\[
\sup_{n \geq 1} \int_{\mathbb{R}} F_n(t, x)^{\frac{\varepsilon}{2}} dx \lesssim \|u^\vartheta\|^p_{Z^p(T)} < \infty.
\]

It follows that \( \{F_n(t, x)\}_{n \geq 1} \) is \( L^1 \)-uniformly integrable in \( (\mathbb{R}, dx) \). By (5.6) and the uniform integrability convergence theorem, we have

\[
\lim_{n \to \infty} E_2(t) = 0. \quad (5.7)
\]

Similarly, we can prove

\[
\lim_{n \to \infty} E_4(t) = 0. \quad (5.8)
\]

Putting (5.4), (5.5), (5.7) and (5.8) together, by the Gronwall lemma, we have

\[
\lim_{n \to \infty} [D_1(t) + D_2(t)] = 0, \text{ for all } t \in [0, T].
\]

In particular, \( u^\vartheta_n(t, \cdot) \to u^\vartheta(t, \cdot) \) as \( n \to \infty \) in the space \( L^2(\mathbb{R}) \) for all \( t \in [0, T] \). Since \( u^\vartheta_n \) also converges to \( u \) as \( n \to \infty \) in the space \( (C([0, T] \times \mathbb{R}), d_C) \), the uniqueness of the limit of \( u^\vartheta_n \) implies that \( u = u^\vartheta \). The proof is complete. \( \Box \)

We now verify the second part of Condition 3.5. For any \( \varepsilon \in (0, 1) \), recall the solution functional \( \Gamma^\varepsilon : C([0, T]; \mathbb{R}^p) \to C([0, T] \times \mathbb{R}) \) defined by

\[
\Gamma^\varepsilon(W(\cdot)) := u^\varepsilon, \quad (5.9)
\]

where \( u^\varepsilon \) stands for the solution of the equation of Eq. (1.1).

Let \( \{g^\varepsilon\}_{\varepsilon \in (0, 1)} \subset \mathcal{U}^N \) be a given family of stochastic processes. By the Girsanov theorem, we know that \( \tilde{u}^\varepsilon := \Gamma^\varepsilon \left( W(\cdot) + \frac{1}{\sqrt{\varepsilon}} \int_0^t G(t - s, x - y)\sigma(s, y, \tilde{u}^\varepsilon(s, y))W(ds, dy) \right) \) is the unique solution of

\[
\begin{align*}
\tilde{u}^\varepsilon(t, x) &= I_0(t, x) + \sqrt{\varepsilon} \int_0^t \int_{\mathbb{R}} G(t - s, x - y)\sigma(s, y, \tilde{u}^\varepsilon(s, y))W(ds, dy) \\
&\quad + \int_0^t \langle G(t - s, x - \cdot)\sigma(s, \cdot, \tilde{u}^\varepsilon(s, \cdot)), g^\varepsilon(s, \cdot) \rangle^\mathcal{H} ds.
\end{align*}
\]

Moreover, the following stochastic equation is held for \( \bar{u}^\varepsilon := \Gamma^0 \left( \int_0^t g^\varepsilon(s)ds \right) \), where \( \Gamma^0 \) is defined by (3.5):

\[
\bar{u}^\varepsilon(t, x) = I_0(t, x) + \int_0^t \langle G(t - s, x - \cdot)\sigma(s, \cdot, \bar{u}^\varepsilon(s, \cdot)), g^\varepsilon(s, \cdot) \rangle^\mathcal{H} ds. \quad (5.11)
\]

**Proposition 5.2.** Assume that Hypothesis 2.6 holds. Then, for any family \( \{g^\varepsilon\}_{\varepsilon \in (0, 1)} \subset \mathcal{U}^N \) \( (N < +\infty) \) and for any \( \delta > 0 \),

\[
\lim_{\varepsilon \to 0} \mathbb{P}(d_C(\bar{u}^\varepsilon, \bar{u}^\varepsilon) > \delta) = 0.
\]

Before proving Proposition 5.2, we give the following lemmas, which assert the finiteness of the \( \|\cdot\|_{Z^p(T)} \) norm and the Hölder continuity of \( \bar{u}^\varepsilon \) and \( \tilde{u}^\varepsilon \).

**Lemma 5.1.** Assume that Hypothesis 2.6 holds. Then, it holds that for any \( p > \frac{2}{1 - 2H} \),

\[
\sup_{\varepsilon \in (0, 1)} \|\tilde{u}^\varepsilon\|_{Z^p(T)} < \infty, \quad \sup_{\varepsilon \in (0, 1)} \|\bar{u}^\varepsilon\|_{Z^p(T)} < \infty. \quad (5.12)
\]
Proof. We provide the detailed proof for the result concerning $\tilde{u}^\varepsilon$, while the proof for that with respect to $u^n$ is similar but simpler and we omit it here. Following the idea developed by the proof of [23, Lemma 3.4], we approximate the noise $W$ by smoothing it with respect to the space variable. For each $\eta > 0$ and $\varphi \in \mathcal{F}$, we define

$$W_\eta(\varphi) = \int_0^t \int_\mathbb{R} [\rho_\eta * \varphi](s, y)W(ds, dy) = \int_0^t \int_\mathbb{R} \varphi(s, x)\rho_\eta(x - y)W(ds, dy)dx,$$

where $\rho_\eta(x) = \frac{1}{\sqrt{2\pi\eta}} \exp\left(-\frac{x^2}{2\eta}\right)$. Consider the equation

$$\tilde{u}_\eta^\varepsilon(t, x) = I_0(t, x) + \sqrt{\varepsilon} \int_0^t \int_\mathbb{R} G(t - s, x - y)\sigma(s, y, \tilde{u}^\varepsilon_\eta(s, y))W_\eta(ds, dy)$$

$$+ \int_0^t \langle\sigma(s, \cdot, \tilde{u}^\varepsilon_\eta(s, \cdot)), \tilde{\eta}(s, \cdot)\rangle_{\mathcal{H}_\eta} ds$$

$$=: I_0(t, x) + \sqrt{\varepsilon} \Phi^\varepsilon_{1, \eta}(t, x) + \Phi^\varepsilon_{2, \eta}(t, x).$$

We introduce the following Picard iteration:

$$\tilde{u}^\varepsilon_{n+1}(t, x) = I_0(t, x) + \sqrt{\varepsilon} \int_0^t \int_\mathbb{R} G(t - s, x - y)\sigma(s, y, \tilde{u}^\varepsilon_n(s, y))W_\eta(ds, dy)$$

$$+ \int_0^t \langle\sigma(s, \cdot, \tilde{u}^\varepsilon_n(s, \cdot)), \tilde{\eta}(s, \cdot)\rangle_{\mathcal{H}_\eta} ds$$

$$=: I_0(t, x) + \sqrt{\varepsilon} \Phi^\varepsilon_{1, \eta}(t, x) + \Phi^\varepsilon_{2, \eta}(t, x),$$

with $\tilde{u}^\varepsilon_0(t, x) = I_0(t, x)$. From the proof of [23, Lemma 3.5] and by the similar argument to that in Step 1 in the proof of Lemma 4.1, we know that, for any fixed $t \in [0, T]$, $\eta > 0$ and $\varepsilon \in (0, 1)$, $\tilde{u}^\varepsilon_{n+1}(t, \cdot)$ converges to $\tilde{u}^\varepsilon_\eta(t, \cdot)$ in $L^p(\Omega \times \mathbb{R})$ when $n$ goes to infinity.

A similar computation as that in Steps 2 and 3 in the proof of [23, Lemma 3.5] gives that

$$\|\Phi^\varepsilon_{1, \eta}(t, \cdot)\|^2_{L^p(\Omega \times \mathbb{R})} + \left[\mathcal{N}_{\frac{1}{2} - H, p}^* \Phi^\varepsilon_{1, \eta}(s)\right]^2$$

$$\lesssim \int_0^t \left((t - s) + (t - s)^{2H} + (t - s)^{4H-1}\right) \cdot \|\tilde{u}^\varepsilon_n(s, \cdot)\|^2_{L^p(\Omega \times \mathbb{R})} ds$$

$$+ \int_0^t \left((t - s) + (t - s)^{2H}\right) \cdot \left[\mathcal{N}_{\frac{1}{2} - H, p}^* \tilde{u}^\varepsilon_n(s)\right]^2 ds.$$ (5.15)

Furthermore, as in the proof of Lemma 4.1, we obtain that

$$\|\Phi^\varepsilon_{2, \eta}(t, \cdot)\|^2_{L^p(\Omega \times \mathbb{R})} + \left[\mathcal{N}_{\frac{1}{2} - H, p}^* \Phi^\varepsilon_{2, \eta}(t)\right]^2$$

$$\lesssim \int_0^t \left((t - s) + (t - s)^{2H} + (t - s)^{4H-1}\right) \cdot \|\tilde{u}^\varepsilon_n(s, \cdot)\|^2_{L^p(\Omega \times \mathbb{R})} ds$$

$$+ \int_0^t \left((t - s) + (t - s)^{2H}\right) \cdot \left[\mathcal{N}_{\frac{1}{2} - H, p}^* \tilde{u}^\varepsilon_n(s)\right]^2 ds.$$ (5.16)

For any $t \geq 0$, let

$$\tilde{\Psi}_\eta(t) := \|\tilde{u}^\varepsilon_n(t, \cdot)\|^2_{L^p(\Omega \times \mathbb{R})} + \left[\mathcal{N}_{\frac{1}{2} - H, p}^* \tilde{u}^\varepsilon_n(t)\right]^2.$$

Putting (5.14), (5.15) and (5.16) together, there exists a constant $C_{T, p, H, N} > 0$ such that

$$\tilde{\Psi}_\eta(t) \leq C_{T, p, H, N} \left(\|I_0\|^2_{Z_p(T)} + \int_0^t (t - s)^{4H-1} \cdot \tilde{\Psi}_\eta(s) ds\right).$$
Hence, by the extension of Gronwall’s lemma [11, Lemma 15], there exists a constant $C$ independent of $\eta$ and $\varepsilon$ such that

$$
\sup_{n \geq 1} \sup_{t \in [0, T]} \tilde{\Psi}^{\varepsilon,n}(t) \leq C.
$$

According to Fatou’s lemma, we have that for any $p > \frac{2}{4H-1}$, there exists a constant $C$ independent of $\varepsilon \in (0, 1)$ such that

$$
\sup_{\eta > 0} \| \tilde{u}_{\eta}^{\varepsilon} \|_{Z^p(T)} \leq C. \quad (5.17)
$$

By the same methods as that in the proof of both [23, Lemma 4.2 (ii), (iii)] and Lemma 4.2 (ii), (iii), we can get the uniform Hölder continuity of $\Phi_{i,p}^{\varepsilon}$, $i = 1, 2$. This, together with the fact $\tilde{u}^{\varepsilon}_{\eta}(0, 0) = u_0(0)$ implies that the laws of the family $\{\tilde{u}_{\eta}^{\varepsilon}\}_{\eta > 0}$ are tight in the space $(C([0, T] \times \mathbb{R}), d_C)$. Thus, $\tilde{u}_{\eta}^{\varepsilon} \rightarrow \tilde{u}^{\varepsilon}$ almost surely in the space $(C([0, T] \times \mathbb{R}), d_C)$ as $\eta \rightarrow 0$. Furthermore, we can obtain that $\tilde{u}^{\varepsilon}$ is the solution of Eq. (5.10) using the same method as that in the proofs of [21, Theorem 1.5] and Proposition 4.1. By (5.17) and [21, Lemma 4.6], we have that for any $p > \frac{2}{4H-1}$, $\sup_{\varepsilon \in (0,1)} \| \tilde{u}^{\varepsilon} \|_{Z^p(T)} < \infty$. The proof is complete. \hfill $\square$

For any $u \in Z^p(T)$ and $g \in U^N$, let

$$
Y(t, x) := \int_0^t \langle G(t-s, x, \cdot \rangle \sigma(s, \cdot) \cdot \sigma(s, \cdot) \rangle \eta \ ds. \quad (5.18)
$$

By Lemma 4.2 and Minkowski’s inequality, we have the following lemma.

**Lemma 5.2.** Assume that Hypothesis 2.6 holds. Then we have the following results:

(i). for any $p > \frac{2}{4H-1}$, there exists a constant $C_{T,p,H,N} > 0$ such that

$$
\mathbb{E} \left[ \sup_{t \in [0, T], x \in \mathbb{R}} \mathcal{N}_{\frac{1}{2}-H} \| Y(t, x) \|_p \right] \leq C_{T,p,H,N} \| u \|_{Z^p(T)}^p. \quad (5.19)
$$

(ii). if $p > \frac{1}{\gamma}$ and $0 < \gamma < H - \frac{1}{p}$, then there exists a positive constant $C_{T,p,H,N,\gamma}$ such that

$$
\mathbb{E} \left[ \sup_{t, t+h \in [0, T], x \in \mathbb{R}} \left| Y(t+h, x) - Y(t, x) \right| \right] \leq C_{T,p,H,N,\gamma} |h|^{\gamma} \cdot \| u \|_{Z^p(T)}^p. \quad (5.20)
$$

(iii). if $p > \frac{1}{\gamma}$ and $0 < \gamma < H - \frac{1}{p}$, then there exists a positive constant $C_{T,p,H,N,\gamma}$ such that

$$
\mathbb{E} \left[ \sup_{t \in [0, T], x, y \in \mathbb{R}} \left| Y(t, x) - Y(t, y) \right| \right] \leq C_{T,p,H,N,\gamma} \| x - y \|^{\gamma} \cdot \| u \|_{Z^p(T)}^p. \quad (5.21)
$$

Recall $\tilde{u}^{\varepsilon}$ and $\tilde{u}^{\varepsilon}$ defined by (5.10) and (5.11), respectively. For any $k \geq 1$ and $p > \frac{2}{4H-1}$, define the stopping time

$$
\tau_k := \inf \left\{ \tau \geq 0 : \sup_{0 \leq s \leq \tau, x \in \mathbb{R}} \mathcal{N}_{\frac{1}{2}-H} \tilde{u}^{\varepsilon}(s, x) \geq k, \text{ or } \sup_{0 \leq s \leq \tau, x \in \mathbb{R}} \mathcal{N}_{\frac{1}{2}-H} \tilde{u}^{\varepsilon}(s, x) \geq k \right\}. \quad (5.22)
$$

By [23, Lemma 4.2], Lemmas 5.1 and 5.2, we know that

$$
\sup_{\varepsilon \in (0,1)} \sup_{t \in [0, T], x \in \mathbb{R}} \mathcal{N}_{\frac{1}{2}-H} \tilde{u}^{\varepsilon}(t, x) \|_{L^p(\Omega)} < \infty, \quad \sup_{\varepsilon \in (0,1)} \sup_{t \in [0, T], x \in \mathbb{R}} \mathcal{N}_{\frac{1}{2}-H} \tilde{u}^{\varepsilon}(t, x) \|_{L^p(\Omega)} < \infty,
$$

which, together with Markov’s inequality, implies that

$$
\tau_k \uparrow \infty, \text{ a.s., as } k \rightarrow \infty. \quad (5.23)
$$
Proof. Let
\[
\Phi_1^\varepsilon(t, x) := \int_0^t \int_{\mathbb{R}} G(t - s, x - y) \sigma(s, y, \tilde{u}_k(s, y)) W(ds, dy)
\]
and
\[
\Phi_2^\varepsilon(t, x) := \int_0^t \langle G(t - s, x - \cdot) \Delta(s, \cdot, \cdot), g^\varepsilon(s, \cdot) \rangle_H ds.
\]
By using the same technique as in the proof of Lemmas 4.5 and 4.6 in [21], we have \(\|\Phi_1^\varepsilon\|_{L^p(\Omega \times \mathbb{R})} < \infty\) for any \(p > \frac{2}{4H - 1}\). By (2.24) and a similar computation as that in Step 2 in the proof of Lemma 4.1, we have
\[
\|\Phi_2^\varepsilon(t, \cdot)\|_{L^p(\Omega \times \mathbb{R})} \lesssim \int_0^t (t - s)^{2H} \cdot \|\tilde{u}_k^\varepsilon(s, \cdot) - \tilde{u}_k(s, \cdot)\|_{L^p(\Omega \times \mathbb{R})}^2 ds
\]
\[\quad + \int_0^t (t - s) \cdot \left[\mathcal{N}_{\frac{1}{2}}^{\varepsilon} (\tilde{u}_k^\varepsilon(s) - \tilde{u}_k(s))\right]^2 ds.
\]
Putting (5.28), (5.29) and (5.30) together, and by the Gronwall inequality, we have that, for any \(k \geq 1\) and \(p > \frac{2}{4H - 1}\),
\[
\lim_{\varepsilon \to 0} \|\tilde{u}_k^\varepsilon - \tilde{u}_k\|_{L^p(\Omega \times \mathbb{R})} = 0.
\]
It remains to prove (5.30). Since \(g^\varepsilon \in \mathcal{U}^N\), by the Cauchy-Schwarz inequality and (2.6), we have
\[
\mathbb{E} \left[\left|\int_0^t \langle \mathcal{D}_h G(t - s, x - \cdot) \Delta(s, \cdot, \cdot), g^\varepsilon(s, \cdot) \rangle_H ds\right|^2\right]
\]
\[\leq \mathbb{E} \left[\int_0^t \int_{\mathbb{R}^2} \left|\mathcal{D}_h G(t - s, x - y - z) \Delta(s, y + z, y + z) - \mathcal{D}_h G(t - s, x - y) \Delta(s, y, y)\right|^2 \cdot |z|^{2H - 2} dz dy ds\right]^2.
\]
where

\[ R_1^\varepsilon(t, x, h) := E \left[ \int_0^t \int_{\mathbb{R}^2} |D_h G(t - s, x - y - z)|^2 \right. \]
\[ \cdot \left| \Lambda(s, y + z, y + z) - \Lambda(s, y + z, y) \right|^2 \cdot |z|^{2H - 2} dz dy ds \] \]
\[ R_2^\varepsilon(t, x, h) := E \left[ \int_0^t \int_{\mathbb{R}^2} |\Box_{h, -z} G(t - s, x - y)|^2 \cdot |\Lambda(s, y + z, y)|^2 \cdot |z|^{2H - 2} dz dy ds \right] \]
\[ R_3^\varepsilon(t, x, h) := E \left[ \int_0^t \int_{\mathbb{R}^2} |D_h G(t - s, x - y)|^2 \cdot |\Lambda(s, y + z, y) - \Lambda(s, y, y)|^2 \cdot |z|^{2H - 2} dz dy ds \right]. \]

By a change of variable and (4.32), we have

\[ R_1^\varepsilon(t, x, h) \]
\[ \lesssim E \left[ \int_0^t \int_{\mathbb{R}^2} |D_h G(t - s, x - y)|^2 \cdot |z|^{2H - 2} dz \right. \]
\[ \cdot \left[ |\tilde{u}_k^\varepsilon(s, y + z) - \tilde{u}_k^\varepsilon(s, y + z) - \tilde{u}_k^\varepsilon(s, y)|^2 \right. \]
\[ \left. + |\tilde{v}_k^\varepsilon(s, y) - \tilde{v}_k^\varepsilon(s, y)|^2 \cdot \left( |\tilde{u}_k^\varepsilon(s, y + z) - \tilde{u}_k^\varepsilon(s, y)|^2 + |\tilde{u}_k^\varepsilon(s, y + z) - \tilde{u}_k^\varepsilon(s, y)|^2 \right) \right] dy ds \right] \frac{\tilde{p}}{2}. \]

Consequently, we have

\[ \int_{\mathbb{R}} \left( \int_{\mathbb{R}} R_1^\varepsilon(t, x, h) dx \right)^{\frac{\tilde{p}}{2}} \cdot |h|^{2H - 2} dh \]
\[ \lesssim \int_0^t \int_{\mathbb{R}^2} |D_h G(t - s, y)|^2 \cdot |h|^{2H - 2} dh dy ds \left[ k^2 \int_{\mathbb{R}} E \left[ |\tilde{u}_k^\varepsilon(s, x) - \tilde{u}_k^\varepsilon(s, x)|^p \right] dx \right]^{\frac{\tilde{p}}{2}} \]
\[ + \int_{\mathbb{R}} \left( \int_{\mathbb{R}} E \left[ |\tilde{u}_k^\varepsilon(s, x + z) - \tilde{u}_k^\varepsilon(s, x + z) - \tilde{u}_k^\varepsilon(s, x) + \tilde{u}_k^\varepsilon(s, x)|^p \right] dx \right)^{\frac{\tilde{p}}{2}} \cdot |z|^{2H - 2} dz ds \]
\[ \lesssim \int_0^t (t - s)^{2H} \cdot \left[ k^2 \cdot \|\tilde{u}_k^\varepsilon(s, \cdot) - \tilde{u}_k^\varepsilon(s, \cdot)\|_{L^p(\Omega \times \mathbb{R})}^2 + \left[ \mathcal{N}_{\frac{\tilde{p}}{2} - H, p}^\varepsilon (\tilde{u}_k^\varepsilon(s) - \tilde{u}_k^\varepsilon(s)) \right]^2 \right] ds, \]

where a change of variable, the Minkowski inequality and Lemma 6.2 have been used. Invoking the Lipschitz continuity of $\sigma$, a change of variable, Minkowski’s inequality and Lemma 6.2, we have

\[ \int_{\mathbb{R}} \left( \int_{\mathbb{R}} R_2^\varepsilon(t, x, h) dx \right)^{\frac{\tilde{p}}{2}} \cdot |h|^{2H - 2} dh \]
\[ \lesssim \int_0^t \int_{\mathbb{R}^2} |\Box_{h, -z} G(t - s, y)|^2 \cdot |z|^{2H - 2} \cdot |h|^{2H - 2} dz dy ds \left[ k^2 \int_{\mathbb{R}} E \left[ |\tilde{u}_k^\varepsilon(s, x) - \tilde{u}_k^\varepsilon(s, x)|^p \right] dx \right]^{\frac{\tilde{p}}{2}} \]
\[ \cdot |h|^{2H - 2} dh \]
\[ \lesssim \int_0^t (t - s)^{4H - 1} \cdot \|\tilde{u}_k^\varepsilon(s, \cdot) - \tilde{u}_k^\varepsilon(s, \cdot)\|_{L^p(\Omega \times \mathbb{R})}^2 ds. \]
It follows from the same analysis to (4.35) and (4.36) and Lemma 6.2 that

\[ \int_{\mathbb{R}} \left( \int_{\mathbb{R}} R^2_\varepsilon(t, x, h) \right)^{\frac{2}{p}} |h|^2H^{-2} dh \]

\[ \leq 0 \int_{0}^{\infty} (t - s)^{2H} \cdot |\tilde{u}_k^\varepsilon(s) - \tilde{u}_k^\varepsilon(s, \cdot)|_p^2 d\gamma \]

Putting (5.31)-(5.34) together, we have (5.30). The proof is complete. \(\square\)

We now prove Proposition 5.2.

**Proof of Proposition 5.2.** By [23, Lemma 4.2], Lemmas 5.1 and 5.2, we know that the probability measures on the space \((\mathcal{C}([0, T] \times \mathbb{R}), \mathcal{B}(\mathcal{C}([0, T] \times \mathbb{R})), dC)\) corresponding to the processes \(\{\tilde{u}^\varepsilon - \bar{u}^\varepsilon\}_{\varepsilon > 0}\) are tight. Thus, there is a subsequence \(\varepsilon_n \downarrow 0\) such that \(\tilde{u}^\varepsilon_n - \bar{u}^\varepsilon_n\) converges weakly to some stochastic process \(Z = \{Z(t, x), t \in [0, T], x \in \mathbb{R}\}\) in \((\mathcal{C}([0, T] \times \mathbb{R}), \mathcal{B}(\mathcal{C}([0, T] \times \mathbb{R})), dC)\).

On the other hand, for any \(k \geq 1, p > \frac{2}{2H-1}, \gamma > 0,\)

\[ \mathbb{P} \left( \sup_{0 \leq t \leq T} \int_{\mathbb{R}} |\tilde{u}^\varepsilon(t, x) - \bar{u}^\varepsilon(t, x)|^p dx > \gamma \right) \]

\[ \leq \mathbb{P} \left( \sup_{0 \leq t \leq T} \int_{\mathbb{R}} |\tilde{u}^\varepsilon(t, x) - \bar{u}^\varepsilon(t, x)|^p dx > \gamma, \tau_k > T \right) + \mathbb{P} (\tau_k \leq T) \]

\[ \leq \mathbb{P} \left( \sup_{0 \leq t \leq T} \int_{\mathbb{R}} |\tilde{u}_k^\varepsilon(t, x) - \bar{u}_k^\varepsilon(t, x)|^p dx > \gamma \right) + \mathbb{P} (\tau_k \leq T). \]

First letting \(\varepsilon \to 0\) and then letting \(k \to \infty,\) by Lemma 5.3 and (5.23), we have

\[ \sup_{0 \leq t \leq T} \int_{\mathbb{R}} |\tilde{u}^\varepsilon(t, x) - \bar{u}^\varepsilon(t, x)|^p dx \to 0 \]

in probability, as \(\varepsilon \to 0.\)

Thus, for any fixed \(t \in [0, T],\) the processes \(\{\tilde{u}^\varepsilon(t, x)\omega - \bar{u}^\varepsilon(t, x)\omega; (\omega, x) \in \Omega \times \mathbb{R}\}\) converges to 0 in probability in the product probability space \((\Omega \otimes \mathbb{R}, \mathbb{P} \otimes dx).\) This, together with the weak convergence of \(\{\tilde{u}^\varepsilon_n - \bar{u}^\varepsilon_n\}_{n \geq 1}\) and the uniqueness of the limit distribution of \(\tilde{u}^\varepsilon_n - \bar{u}^\varepsilon_n,\) implies that \(Z(t, x) \equiv 0, t \in [0, T], x \in \mathbb{R},\) almost surely. Thus, as \(\varepsilon \to 0, \tilde{u}^\varepsilon - \bar{u}^\varepsilon\) converges weakly to 0 in \((\mathcal{C}([0, T] \times \mathbb{R}), \mathcal{B}(\mathcal{C}([0, T] \times \mathbb{R})), dC)\). Equivalently, the sequence of real-valued random variables \(dC(\tilde{u}^\varepsilon, \bar{u}^\varepsilon)\) converges to 0 in distribution as \(\varepsilon\) goes to 0. It follows from [9, p. 98, Exercise 4] that

\[ dC(\tilde{u}^\varepsilon, \bar{u}^\varepsilon) \to 0 \]

in probability, as \(\varepsilon \to 0.\)

The proof is complete. \(\square\)

6. Appendix

In this section, we first give some lemmas related to the wave Green’s function \(G(t, x)\) from [23], which play a key role for the proofs in this paper. Then we present a lemma providing the Hölder continuity of \(I_0(t, x).\) Lastly, we give an auxiliary lemma used in the proof of Proposition 4.1.
Lemma 6.1. ([23, Lemma 3.1], [23, Remark 3.3]) The wave kernel $G(t, x) = \frac{1}{2} \mathbb{I}_{\{x \leq t\}}$ can be expressed as

$$G(t - s, x - y) = \int_{\mathbb{R}} C_\beta(t - r, x - z) S_\gamma(r - s, z - y) dz$$

$$+ \int_{\mathbb{R}} S_\alpha(t - r, x - z) C_1 - \alpha(r - s, z - y) dz$$

$$+ \int_{\mathbb{R}} S(t - r, x - z) E(r - s, z - y) dz$$

$$+ \int_{\mathbb{R}} E(t - r, x - z) S(r - s, z - y) dz,$$

where $\alpha, \beta \in (0, 1), S(t, x) = S_1(t, x) = G(t, x) = \frac{1}{2} \mathbb{I}_{\{x \leq t\}}$ and

$$\mathcal{E}(t, x) := \frac{1}{\pi} \frac{t}{(t^2 + x^2)},$$

$$\mathcal{S}_\alpha(t, x) := \frac{\Gamma(1 - \alpha)}{2\pi} \cos \left( \frac{\alpha\pi}{2} \right) \left[ (t + |x|)^{\alpha - 1} + \text{sgn}(t - |x|) t - |x|^{\alpha - 1} \right],$$

$$\mathcal{C}_1 - \alpha(t, x) := \frac{\Gamma(1 - \alpha)}{2\pi} \left[ (t + |x|)^{-\alpha} + |t - |x||^{-\alpha} \right]$$

$$- 2 \cos \left( \alpha \tan^{-1} \left( \frac{|x|}{t} \right) \right) \left[ t^2 + x^2 \right]^{-\alpha}.$$ (6.1)

Recall $\mathcal{D}_h f(t, x), \mathcal{K}_h f(t, x)$ defined by (4.10) and (4.11), respectively.

Lemma 6.2. ([23, (3.33), (3.34)]) For any $H \in \left( \frac{1}{4}, \frac{1}{2} \right)$ and any $t > 0$, there exists some constant $C_H$ such that

$$\int_{\mathbb{R}^2} |\mathcal{D}_h G(t, x)|^2 \cdot |h|^{2H - 2} dh dx \leq C_H t^{2H}$$

and

$$\int_{\mathbb{R}^3} |\mathcal{K}_h G(t, x)|^2 \cdot |y|^{2H - 2} |y|^{2H - 2} dy |h|^{2H - 2} dh dx \leq C_H t^{4H - 4}.$$ (6.3)

(6.4)

For convenience, we denote that

$$\mathcal{K}_1 = \mathcal{C}_\alpha, \quad \mathcal{K}_2 = \mathcal{S}_\alpha, \quad \mathcal{K}_3 = \mathcal{S}, \quad \mathcal{K}_4 = \mathcal{E}. \quad (6.5)$$

According to (6.1), we define $\tilde{K}_i$ to be the complements of $\mathcal{K}_i, i = 1, 2, 3, 4$, i.e.,

$$\tilde{K}_1 = \mathcal{S}_1 - \alpha, \quad \tilde{K}_2 = \mathcal{C}_1 - \alpha, \quad \tilde{K}_3 = \mathcal{E}, \quad \tilde{K}_4 = \mathcal{S}. \quad (6.6)$$

Let

$$\Phi^\theta_\varepsilon(t, x) := \int_0^t \langle G(t - s, x - \cdot) \sigma(s, \cdot, u^\theta_\varepsilon(s, \cdot)), g(s, \cdot) \rangle_{H_\varepsilon} ds.$$ (6.7)

where $\langle \cdot, \cdot \rangle_{H_\varepsilon}$ is defined by (4.3). Then, by a stochastic version of Fubini’s theorem and Lemma 6.1, we have that, for any $\theta \in (0, 1),$

$$\Phi^\theta_\varepsilon(t, x) \approx \sum_{i=1}^4 \frac{\sin(\theta \pi)}{\pi} \int_0^t \int_{\mathbb{R}} (t - r)^{-1} \tilde{K}_i(t - r, x - z) J^\theta_\varepsilon K_i(r, z) dz dr,$$ (6.8)

where

$$J^\theta_\varepsilon K_i(r, z) := \int_0^r (r - s)^{-\theta} \langle K_i(r - s, z - \cdot) \sigma(s, \cdot, u^\theta_\varepsilon(s, \cdot)), g(s, \cdot) \rangle_{H_\varepsilon} ds, \quad i = 1, 2, 3, 4. \quad (6.9)$$

We give the following two lemmas related to $J^\theta_\varepsilon K_i(r, z), i = 1, 2, 3, 4,$ for proving Lemma 4.2.
Lemma 6.4. Assume that $\sigma(t,x,y)$ satisfies Hypothesis 2.6. If $p > \frac{1}{4H}$, $1 - H < \alpha < 1 - \frac{1}{p}$ and $1 - \frac{2}{q} + \alpha < \theta < H + \alpha - \frac{1}{2}$, then there exists some constant $C$, independent of $r \in [0,T]$, such that

$$
\int_{\mathbb{R}} \left| J_{\theta}^{K_{r}}(r,z) \right|^p dz \leq C \| u_0^\alpha \|^p_{Z^p(T)}, \quad i = 1, 2, 3, 4. \tag{6.10}
$$

Proof. For notational simplicity, we assume $\sigma(t,x,u) = \sigma(u)$ without loss of generality because of Hypothesis 2.6. The proof is similar as that of [23, Lemma B.1] by replacing the stochastic integral by the deterministic integral. More precisely, before we use the argument in the proof of [23, Lemma B.1], we need to use (6.9) and the similar technique as that in (4.12) to have

$$
\int_{\mathbb{R}} \left| J_{\theta}^{K_{r}}(r,z) \right|^p dz \leq \int_{\mathbb{R}} \left[ I_1(r,z,h) + I_2(r,z,h) \right]^\frac{p}{2} dz, \quad i = 1, 2, 3, 4,
$$

where

$$
I_1(r,z,h) = \int_0^r \int_{\mathbb{R}^2} (r-s)^{-2\theta} |K_{i}(r-s, z-y-h)|^2 \cdot |D_h u_0^\alpha(s,y)|^2 \cdot \| h \|^2 H - 2 d y d h d s,
$$

$$
I_2(r,z,h) = \int_0^r \int_{\mathbb{R}^2} (r-s)^{-2\theta} |D_h K_{i}(r-s, z-y)|^2 \cdot |u_0^\alpha(s,y)|^2 \cdot \| h \|^2 H - 2 d y d h d s.
$$

The details are omitted here. The proof is complete. $\square$

The following lemma can be proved by the similar technique as that in the proofs of Lemma 6.3 and [23, Lemma B.2]. Here the proof is omitted.

Lemma 6.4. Assume that $\sigma(t,x,y)$ satisfies the hypothesis (H). If $p > \frac{1}{4H}$, $\frac{3}{2} - 2H < \alpha < 1 - \frac{1}{p}$ and $1 - \frac{2}{q} + \alpha < \theta < 2H + \alpha - 1$, then there exists a positive constant $C$, independent of $r \in [0,T]$, such that

$$
\int_{\mathbb{R}} \left[ \int_{\mathbb{R}} \left| J_{\theta}^{K_{r}}(r,z + h) - J_{\theta}^{K_{r}}(r,z) \right|^2 \cdot \| h \|^2 H - 2 d h \right]^\frac{p}{2} dz \leq C \| u_0^\alpha \|^p_{Z^p(T)}, \quad i = 1, 2, 3, 4. \tag{6.11}
$$

For the Hölder continuity of $I_0(t,x)$, we have the following lemma, which follows from the proof of Theorem 1.1 (1) in [17].

Lemma 6.5. Assume that $u_0$ and $v_0$ are $\alpha$-Hölder continuous with $\alpha \in (0,1]$. Then we have

$$
|I_0(t,x) - I_0(s,y)| \lesssim |t-s|^{\alpha} + |x-y|^{\alpha}, \quad t,s \in [0,T], x,y \in \mathbb{R}.
$$

We also need the following lemma to prove the existence of the solution to the skeleton equation (4.1), whose proof is similar as that of [22, Lemma 3.3] and is omitted here.

Lemma 6.6. Assume that $u_0 \in Z^p(T)$, $n \geq 1$, for some $p \geq 2$. If $u_n \to u$ in $(C([0,T] \times \mathbb{R}), d_C)$ as $n \to \infty$, then we have the following results:

(i). $u$ is also in $Z^p(T)$;

(ii). for any fixed $t \in [0,T]$,

$$
\int_0^t \int_{\mathbb{R}} \| G(t-s,x-\cdot) \sigma(s,\cdot,u(s,\cdot)) \|^2_{2t} ds dx < \infty. \tag{6.12}
$$

Hence, for almost all $x \in \mathbb{R}$,

$$
\int_0^t \| G(t-s,x-\cdot) \sigma(s,\cdot,u(s,\cdot)) \|^2_{2t} ds < \infty. \tag{6.13}
$$

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