n-body dynamics of stabilized vector solitons.

Gaspar D. Montesinos
Departamento de Matemáticas, Escuela Técnica Superior de Ingenieros Industriales,
Universidad de Castilla-La Mancha, 13071 Ciudad Real, Spain

María I. Rodas-Verde
Área de Óptica, Facultade de Ciencias de Ourense,
Universidade de Vigo, As Lagoas s/n, Ourense, ES-32005 Spain.

Víctor M. Pérez-García
Departamento de Matemáticas, Escuela Técnica Superior de Ingenieros Industriales,
Universidad de Castilla-La Mancha, 13071 Ciudad Real, Spain.

Humberto Michinel
Área de Óptica, Facultade de Ciencias de Ourense,
Universidade de Vigo, As Lagoas s/n, Ourense, ES-32005 Spain.
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In this work we study the interactions between stabilized Townes solitons. By means of effective Lagrangian methods, we have found that the interactions between these solitons are governed by central forces, in a first approximation. In our numerical simulations we describe different types of orbits, deflections, trapping and soliton splitting. Splitting phenomena are also described by finite-dimensional reduced models. All these effects could be used for potential applications of stabilized solitons.

The cubic Nonlinear Schrödinger Equation (NLSE) is among the most important physical models in the field of nonlinear waves. Besides its fundamental value as a first order nonlinear wave equation, it is an integrable model in the one dimensional case [1] and represents many different physical systems: from laser wavepackets propagating in nonlinear materials to matter waves in Bose-Einstein Condensates (BEC), gravitational models for quantum mechanics, plasma physics or wave propagation in geological systems, between others [2, 3, 4]. In this paper we consider the vector version of the NLSE and study the dynamics of some particular solutions which are stabilized by means of a periodic modulation of the nonlinearity.

I. INTRODUCTION

One of the types of NLS equations arising frequently in the applications is the two-dimensional cubic Nonlinear Schrödinger equation, which is of the form

$$\frac{\partial u}{\partial t} = \left[ \frac{1}{2} \Delta + g(t)|u|^2 \right] u$$

$$u(r, 0) = u_0(r) \in H^1(\mathbb{R}^2)$$

(1a)

(1b)

where $u(r, t) : \mathbb{R}^2 \times \mathbb{R}^+ \rightarrow \mathbb{C}$ is the complex wave amplitude, $\Delta = \partial^2/\partial x^2 + \partial^2/\partial y^2$ and $g(t)$ is a real function (the nonlinear coefficient) so that if $g < 0$ the nonlinearity is attractive whereas for $g > 0$ the nonlinearity is repulsive.

When $g$ is a real constant Eq. (1a) is the cubic NLSE, which is one of the most important models of mathematical physics.

It is well known that for $g < 0$ if $N = \int_{\mathbb{R}^2} |u_0|^2$, is above a threshold value $N_c$, solutions of Eq. (1a) can self-focus and become singular in a finite time. This phenomenon is called wave collapse or blowup of the wave amplitude. More precisely, there is never blowup when $N < N_c$ but for any $\epsilon > 0$, there exist solutions with $N = N_c + \epsilon$ for which blowup takes place [5, 6].

Eq. (1a) admits stationary solutions of the form $u(r, t) = e^{i\mu t} \Phi_\mu(r)$, where $\Phi_\mu(r)$ verifies

$$\Delta \Phi_\mu - 2\mu \Phi_\mu - 2g|\Phi_\mu|^2 \Phi_\mu = 0.$$  (2)

As it is precisely stated in [4], when $g$ is negative, for each positive $\mu$ there exists only one solution of Eq. (2) which is real, positive and radially symmetric and for which $\int |\Phi_\mu|^2 \, dr$ has the minimum value between all of the possible solutions of Eq. (2). Moreover, the positivity of $\mu$ ensures that this solution decays exponentially at infinity. This solution is called the ground state or Townes soliton. We will denote it as $R_\mu(r)$ which satisfies

$$\Delta R_\mu - 2\mu R_\mu - 2gR_\mu^3 = 0$$

$$\lim_{r \to \infty} R_\mu(r) = 0, \quad R_\mu(0) = 0.$$  (3)

(4)

From the theory of nonlinear Schrödinger equations it is known that the Townes soliton has exactly the critical norm for blowup $N_c$, therefore, it separates in some sense the region of collapsing and expanding solutions. Moreover, the Townes soliton is unstable, i.e. small perturbations of this solution lead to either expansion of the initial
After an expansion and contraction regime the solution \( \partial \Delta = \) tons,\( \triangledown \) is an arbitrary number of components. In Optics, for spatial solitons \( t \) plays the role of the propagation coordinate and \( \Delta \) is of the form 
\[
\Delta = \Delta_1 \Delta_2 + \Delta_3 \Delta_4 \mid a_{j1} \mid^2 + \ldots + a_{jn} \mid u_n \mid^2 \]
(5)

where \( j = 1, \ldots, n \), \( u_j \) are the complex amplitudes \( \Delta = \partial^2/\partial x^2 + \partial^2/\partial y^2 \), \( a_{jk} \in \mathbb{R} \) are the nonlinear coupling coefficients and \( g(t) \) is a periodic function accounting for the modulation of the nonlinearity.

Eqs. (5) are the natural extension of the Manakov system \( \Delta \) to two transverse dimensions and an arbitrary number of components. In Optics, for spatial solitons, \( t \) plays the role of the propagation coordinate and \( u_j \) are \( n \) mutually incoherent laser wavepackets. One-dimensional Manakov-type models have been extensively studied in nonlinear Optics, mainly due to the potential applications of Manakov solitons in the design of all-optical computing devices. \( \Delta \). In BEC these equations (with an additional trapping term) describe the dynamics of multicomponent two-dimensional condensates, \( u_j \) being the wavefunctions for each of the atomic species involved \( \Delta \). Some features of this model have been described in Ref. \( \Delta \). In particular it is clear that if \( u_j(r, 0) = R_{jk}(\|r - r_j\|) \), where \( R_{jk} \) is a Stabilized Townes Soliton (STS), and the centers of the distributions are much more separated than the width of the Townes solitons, then we may have states with STS on each component. Because of the Galilean invariance we can also construct solutions propagating with uniform velocity along straight trajectories \( r_j(t) \). Again, if the trajectories are well separated it is reasonable to expect that (as it happens in the case of generic time-independent nonlinearities \( \Delta \) the STS will propagate without interactions. The purpose of this paper is to provide a first systematic exploration of collisions of STS. A few results were already reported in Ref. \( \Delta \). One of the main contributions of that paper was to realize that for a given set of parameters \( a_{jk} \) it is possible to use STS to build explicit solutions of Eqs. (5). These solutions are constructed by taking
\[
u_j = a_j R_{jk}(r), j = 1, \ldots, n
\]
(6)
for any set of coefficients \( a_j \) satisfying
\[
a_{j1} \alpha_1^2 + \ldots + a_{jn} \alpha_n^2 = 1, j = 1, \ldots, n.
\]
(7)
These solutions of Eqs. (5) are called Stabilized Vector Solitons (SVS) because they correspond to solutions with appreciable overlapping of the different components \( u_j \). In Ref. \( \Delta \) the stability of these structures as well as the way they arise from collisions between stabilized solitons was studied.

In this paper we present many more examples of collisions between STS and the associated phenomenology.

III. EFFECTIVE-PARTICLE MODEL FOR COLLISIONS OF STABILIZED TOWNES SOLITONS

A. Motivation

Before describing the direct numerical simulations of Eqs. (5) in detail and the many different phenomenologies observed we first present an effective-particle model for collisions of STS. This model will give us a few hints on the expected dynamics of the system. The idea, as in many other problems in Physics, is to assume that during the collisions stabilized solitons behave as particles in the sense that can be described qualitatively by a bell-type ansatz with a few free parameters.

This type of assumptions allows to reduce the dynamics to a finite number degrees of freedom and receives many different names depending on the field of application: method of collective coordinates, averaged Lagrangian description, time-dependent variational method, effective-particle method, etc.
In our case we take as trial Gaussian functions, which is one of the standard choices

\[ u_j = A_j \exp \left[ -\frac{(x-x_j)^2}{2w_{jx}^2} - \frac{(y-y_j)^2}{2w_{jy}^2} + i \left( v_{jx}x + v_{jy}y + \beta_j x^2 + \beta_j y^2 \right) \right]. \] (8)

The \( t \)-dependent parameters have the following meaning: \( A_j \) is the wavefunction amplitude; \( x_j, y_j \) are the coordinates of the centroid; \( w_{jx}, w_{jy} \) are the widths along the \( x \) and \( y \) axis; \( v_{jx}, v_{jy} \) the initial velocities and \( \beta_j, \beta_j \) are phase factors which are required to obtain reliable results [22]. Although Gaussians do not have the same asymptotic decay as STS, our choice simplifies the calculations while leading to the same qualitative relations for the parameters and, as we will see below, the resulting equations provide an elegant and simple picture for the dynamics of the centroids of the STS.

### B. Equations for the effective-particle parameters

Eqs. (5) can be derived by means of a variational formalism from a Lagrangian density which can be written as a sum of the Lagrangians for the linear operators plus the nonlinear interaction in the following simple form:

\[ \mathcal{L} = \frac{1}{2} \sum_{j,k=1}^{N} (\mathcal{L}_j + \mathcal{L}_{jk}) , \] (9a)

\[ \mathcal{L}_j = i \left( u_j \dot{u}_j^* - u_j^* \dot{u}_j \right) + |\nabla u_j|^2 , \] (9b)

\[ \mathcal{L}_{jk} = g(t) \alpha_{jk} |u_j|^2 |u_k|^2 . \] (9c)

The standard calculations of the method [22] consist of minimizing the trial functions from Eqs. (5) over the Lagrangian density given by Eqs. (9). The final result is a set of second order ordinary differential equations for the evolution of the effective-particle parameters. In the case of the centroids we obtain:

\[ \ddot{x}_j = - \sum_{k=1}^{N} \frac{\partial I_{jk}}{\partial x_j} , \] (10a)

\[ \ddot{y}_j = - \sum_{k=1}^{N} \frac{\partial I_{jk}}{\partial y_j} . \] (10b)

With \( j \neq k \) these equations are in the form of Newton’s second law with \( I_{jk} \) playing the role of a potential ruling the interaction between pairs of solitons according to the expression:

\[ I_{jk} = \frac{a_{jk} N_k g(t)}{\pi} \frac{e^{-\frac{(x-x_j)^2}{2w_{jx}^2} + \frac{(y-y_j)^2}{2w_{jy}^2}}}{\sqrt{w_{jx}^2 + w_{ky}^2}} \times \frac{e^{-\frac{(x-x_k)^2}{2w_{jx}^2} + \frac{(y-y_k)^2}{2w_{ky}^2}}}{\sqrt{w_{jx}^2 + w_{ky}^2}} , \] (11)

being \( N_k = \int |u_k|^2 dx dy \) the square norm of the \( k \)-th wavepacket. The corresponding widths along \( x \) and \( y \) are determined by the following equations

\[ \ddot{x}_j = \frac{1}{w_{jx}^2} \sum_{k=1}^{N} \frac{\partial I_{jk}}{\partial w_{jx}} , \] (12a)

\[ \ddot{y}_j = \frac{1}{w_{jy}^2} \sum_{k=1}^{N} \frac{\partial I_{jk}}{\partial w_{jy}} . \] (12b)

Finally, some complementary relationships (first integrals of motion) for the velocities and phase coefficients are also obtained

\[ v_{jx} = \dot{x}_j - 2x_j \beta_j , \] (13a)

\[ v_{jy} = \dot{y}_j - 2y_j \beta_j , \] (13b)

\[ \beta_j = \frac{w_{jx}}{2w_{jy}} , \] (13c)

\[ \beta_j = \frac{w_{jy}}{2w_{jx}} . \] (13d)

From Eqs. (13) it is evident that the quantities \( v_{jx}, v_{jy} \) play the role of the initial velocities of the distribution centroids.

### C. Fast modulation approximation

In this paper we take the modulation \( g(t) = g_0 + g_1 \cos \omega t \) although the same qualitative results are obtained with other choices for \( g \). To get STS the period of \( g(t) \) must be very short [7, 12, 13] and the mean value \( \langle g \rangle = \frac{g_0 + g_1}{2} \) must satisfy the relation \( \langle g \rangle < -N_c \), where \( N_c \) is the critical square norm for blowup. The oscillations of the wavepacket widths, induced by \( g(t) \), are very fast compared with the dynamics of the centroids ruled by the potential from Eq. (11) with an effective range of the order of the sizes of the wavepackets. Therefore, as a first approximation, the evolution of \( x_j \) and \( y_j \) can be decoupled from the oscillations of \( \dot{x}_j \) and \( \dot{y}_j \). With these assumptions the interaction between different wavepackets will be determined by a constant nonlinearity with \( g = \langle g \rangle = g_0 \).

On the other hand, the parameters of the modulation can be chosen to minimize the amplitude of the wavepacket oscillations [13]. An example can be seen in Fig. 11 where the results of direct numerical integrations of Eqs. (5) in the one-component case \( n = 1 \) are shown. The peak amplitude and width of the solution present a small variation when suitable parameters are taken. Thus, as a first approximation we can consider wavepackets of constant circular section \( w_{jx}(t) = w_{jy}(t) = w_j \). With all these considerations the potential between pairs of solitons is given by

\[ P^0_{jk} = \frac{a_{jk} N_k g_0}{\pi w^2} e^{-r_{jk}^2/w^2} , \] (14)

where \( w^2 = w_j^2 + w_k^2 \) and \( r_{jk} \) is the distance between centroids.
Therefore, the interaction between two stabilized solitons is governed, in a first approximation, by an attractive force which only depends on the distance \( r_{jk} \) between the centroids. That is, the motion of the centroids will be similar to planetary mechanics, with a “gravitational” potential \( J_{jk}^0 \) decaying as a Gaussian instead as \( 1/r \). In fact, several common properties of central forces like conservation of angular momentum and the reduction of the 2-body problem to a single body motion in the center of mass frame are straightforwardly derived. Specifically the reduced particle moves under the action of an effective potential given by

\[
V = \frac{a_{jk}N_k g_0}{\pi w^2} e^{-r^2/w^2} + \frac{J^2}{2r^2}
\]

where \( g_0 < 0 \) and \( J \) is the conserved angular momentum.

Using elementary techniques it is easy to obtain a necessary condition for the existence of circular orbits, namely:

\[
CA^2 = e^\Delta
\]

where \( C = 2a_{jk}N_k g_0/\pi J^2 \) is a constant of motion and \( \Delta = r^2/w^2 \). If \( C < e^2/4 \) Eq. (16) has no solutions. If \( C = e^2/4 \) there is only one (\( \Delta_0 = 2 \)) and for \( C > e^2/4 \) there are two solutions (\( \Delta_1 < \Delta_0 \) and \( \Delta_2 > \Delta_0 \)). This means that depending on the initial velocity of the wavepackets \( v_0 \) two closed orbits can exit. The most internal one (\( r_u < \sqrt{2w} \)) is stable whereas the external one (\( r_u > \sqrt{2w} \)) is unstable. If \( v_0 \) is increased up to a threshold value \( v_f \) (escape velocity) there exists only one unstable closed orbit (\( r_0 = \sqrt{2w} \)). Finally if \( v_0 > v_f \) closed orbits are not possible because the wavepackets move too fast to be captured.

Taking this simple picture in mind, we will perform in the next section a numerical exploration of the dynamics of stabilized solitons. Our main interest will be to check the existence of orbits and other dynamical processes.

### IV. NUMERICAL SIMULATIONS OF STABILIZED SOLITONS COLLISIONS

In this section we study the collisions of STS in the framework of Eqs. (15). We take initial data of the form

\[
u_j(r, 0) = R_{0.5}((|r - r_j|) e^{iw_j} r,
\]

and study their evolution by means of a pseudospectral Fourier scheme with time evolution of split-step type [13]. The scheme incorporates absorbing boundary conditions to get rid of the small amounts of radiation which are generated both by the STS and by the collisional processes.

Now, we proceed to describe the different behaviors observed.

#### A. Fast collisions

As it was shown in Ref. [13] when “fast collisions” of STS take place the solitons emerge with only slight variations of the amplitudes and widths. These behaviors can be accounted for by the effective-particle model proposed in Sec. [13]. To see this we focus on the collisions of two stabilized solitons initially placed at \( r_1 = (x_1, y_1), r_2 = (x_2, y_2) \) with initial velocities \( v_1 = (v_{1x}, v_{1y}), v_2 = (v_{2x}, v_{2y}) \). Following the results of Sec. [13] and assuming trial Gaussian functions of equal size \( (w_{1x} = w_{2x} = w_x, w_{1y} = w_{2y} = w_y) \) we obtain the equation for the effective-particle parameters

\[
\begin{align}
\dot{x}_1 &= -\frac{Ng(t)}{2\pi w_x^2 w_y} e^{-\left(\frac{r_x^2}{2w_x^2} + \frac{r_y^2}{2w_y^2}\right)} \ell_x, \\
\dot{x}_2 &= -\dot{x}_1, \\
\dot{y}_1 &= -\frac{Ng(t)}{2\pi w_x^2 w_y} e^{-\left(\frac{r_x^2}{2w_x^2} + \frac{r_y^2}{2w_y^2}\right)} \ell_y, \\
\dot{y}_2 &= -\dot{y}_1,
\end{align}
\]

\[
\begin{align}
\dot{w}_x &= \frac{1}{w_x^3} + \frac{Ng(t)}{2\pi w_x^2 w_y} \left[1 + e^{-\left(\frac{r_x^2}{2w_x^2} + \frac{r_y^2}{2w_y^2}\right)} \left(1 - \frac{r_x^2}{w_x^2}\right)\right], \\
\dot{w}_y &= \frac{1}{w_y^3} + \frac{Ng(t)}{2\pi w_x^2 w_y} \left[1 + e^{-\left(\frac{r_x^2}{2w_x^2} + \frac{r_y^2}{2w_y^2}\right)} \left(1 - \frac{r_y^2}{w_y^2}\right)\right],
\end{align}
\]

where \( \ell_x = x_2 - x_1 \) and \( \ell_y = y_2 - y_1 \). Moreover we have the complementary relations [13] and the conservation law \( N(t) = \pi |A|^2 \omega_x \omega_y = \pi |A(0)|^2 \omega_x(0) \omega_y(0) \).
The different terms in Eqs. 18 account for the phenomenology observed in “fast collisions”. For example, they contain an asymmetric interaction (notice the differences between Eqs. 18e and 18f) due to the fact that they contain an asymmetric interaction (notice the differences the effective-particle model reproduces the observed dynamics and the qualitative behavior of the system.

B. Collapsing orbits

We have studied the evolution of two stabilized solitons which initially are placed at \( r_1 = (-2, 0) \) and \( r_2 = (2, 0) \) with initial velocities \( v_{1y} = -0.06 \) and \( v_{2y} = 0.06 \) along the \( y \)-axis. In this situation we have observed that the solitons move inwards on a spiraling orbit. Therefore, the distance between the centroids of the two wavepackets decreases monotonically and the solitons join at the origin periodically.

This behavior is shown in Fig. 3 where we plot the evolution of \( |u_1|^2 \) and \( |u_2|^2 \) for different times. It can also be observed that the solitons interact continuously and that a partial splitting takes place, leading to the formation of a pair of SVS [14].

The effective-particle method cannot take into account the readjustment phenomenon between stabilized solitons observed in Fig. 3. Therefore, this simplified description is not valid any more and can be used only as a first approximation to the problem. A more sophisticated model will be presented in Sec. V.

C. Expanding orbits

For initial velocities larger than in the previous case it is possible to overcome the collapsing character of the orbit. If the initial velocity is above a threshold value (which is analogous to a escape velocity) the stabilized solitons follow outward trajectories. In Fig. 4 and Fig. 5 we show the results obtained for two stabilized solitons initially placed at \( r_1 = (-2, 0) \) and \( r_2 = (2, 0) \) with input velocities \( v_{1y} = -0.08 \) and \( v_{2y} = 0.08 \) along the \( y \)-axis.
FIG. 4: [Color online] Pseudocolor plots of $|u_1|^2$ for different times corresponding to the evolution of component $u_1$ from simulation of Eqs. (5) with $g(t) = -2\pi + 8\pi \cos(40t)$, $r_1 = (-2, 0)$ and $v_{1y} = -0.08$. The last picture corresponds to the superposition of several snapshots from $t = 0$ to $t = 200$.

Again we observe the formation of SVS. Since what it is plotted is the amplitude of only one component ($|u_1|^2$ or $|u_2|^2$) the existence of two main spots in many subplots of Fig. 4 and Fig. 5 show, once more, that the interaction between stabilized solitons not only affects the trajectories of their centers, but also induces a readjustment of the distributions and the dynamical formation of SVS. We will try to describe this phenomenon in the next section.

D. Wavepacket splitting with deflection

In this subsection we consider another interesting case: one stabilized soliton initially at rest and another one approaching to it. The situation is shown in Fig. 6 and Fig. 7 where we plot respectively the evolution of $u_1$ and $u_2$. The first wavepacket starts at $r_1 = (0, 0)$ with zero initial velocity and the second one is initially placed at $r_2 = (3, -3)$ with initial velocity $v_{2y} = 0.3$ along the $y$-axis. It can be appreciated how the interaction leads to the formation of SVS by means of the same splitting mechanism observed in previous figures. Pictures show that both wavepackets split simultaneously and part of $u_1$ is dragged by one half of $u_2$ forming a vector soliton. The remaining vector soliton is deflected.

FIG. 5: [Color online] Same as Fig. 4 for the evolution of component $u_2$ with $r_2 = (2, 0)$ and $v_{2y} = 0.08$.

E. Three interacting solitons

We have also studied the three-body problem which corresponds to three solitons in the corners of an equilateral triangle of side $d$. Therefore, by taking the origin of coordinates in the barycenter the solitons are placed at $r_1 = (0, d/\sqrt{3})$, $r_2 = (-d/2, -d/2\sqrt{3})$, $r_3 = (d/2, -d/2\sqrt{3})$. The initial velocities are $v_{1x} = v \sin(\pi/2)$, $v_{1y} = 0$, $v_{2x} = v \sin(\pi/2 + 2\pi/3)$, $v_{2y} = v \cos(\pi/2 + 2\pi/3)$, $v_{3x} = -v \sin(\pi/2 + 4\pi/3)$ and $v_{3y} = v \cos(\pi/2 + 4\pi/3)$ being $v$ the input velocity. The numerical results for $d = 4$ and $v = 0.12$ are shown in Fig. 8 where we see the evolution of one of the wavepackets. The initial velocities are such that the attraction between solitons is not enough to keep them bounded and the wavepackets move outwards. As in the previous cases the distributions split during evolution, the initial distributions readjust due to the interaction with the other two components and three two-part vector solitons are generated.

V. THEORETICAL ANALYSIS OF THE WAVEPACKET SPLITTING

A. Motivation and ansatz

From the previous section it is clear that the effective-particle model used in Sec. III cannot explain the wavepacket splitting observed in the most of our numerical simulations. Therefore, in this section we develop another approach to capture the main characteristics of
FIG. 6: [Color online] Pseudocolor plots of $|u_1|^2$ for different times corresponding to the evolution of component $u_1$ from simulation of Eqs. (5) with $g(t) = -2\pi + 8\pi \cos(40t)$, $r_1 = (0,0)$ and $v_1 = 0$. The last pseudocolor plot corresponds to the superposition of several snapshots from $t = 0$ to $t = 50$. It is also shown as a surface plot in the bottom-right picture.

such splitting. We will use again an effective Lagrangian technique but now we consider a two-mode model described by the following ansatzs

$$u_1 = A_{11} \exp \left[ -\frac{(x - \ell)^2}{2w_{1x}^2} - \frac{y^2}{2w_{1y}^2} \right] + A_{12} \exp \left[ -\frac{(x + \ell)^2}{2w_{2x}^2} - \frac{y^2}{2w_{2y}^2} \right],$$  \hspace{0.5cm} (19a)

$$u_2 = A_{21} \exp \left[ -\frac{(x - \ell)^2}{2w_{1x}^2} - \frac{y^2}{2w_{1y}^2} \right] + A_{22} \exp \left[ -\frac{(x + \ell)^2}{2w_{2x}^2} - \frac{y^2}{2w_{2y}^2} \right].$$  \hspace{0.5cm} (19b)

This choice now allows readjustment of mass between two parts related to each component of the vector system, i.e. $u_1$ is now composed of two parts which can be used to emulate the splitting mechanism and the formation of SVS.

However, the full analysis of Eqs. (19) by effective Lagrangian methods is cumbersome and to achieve some results we will make several simplifications which can be justified by the observation of the simulations. In the first place, when the initial stabilized solitons $u_1$ and $u_2$ are equal, the splitting mechanism is a symmetric process meaning that the growth rate of one part of $u_1$ is the same that the growth rate of the corresponding part of $u_2$. Therefore we can consider $A_{11} = A_{22} = \alpha$.

FIG. 7: [Color online] Same as Fig. 6 for the evolution of component $u_2$ with $r_2 = (3,-3)$ and $v_2 = 0.3$.

FIG. 8: [Color online] Surface plots of $|u_3|^2$ for different times corresponding to the evolution of component $u_3$ from simulation of Eqs. (5) with $r_1 = (0,d/\sqrt{3})$, $r_2 = (-d/2,-d/2\sqrt{3})$, $r_3 = (d/2,-d/2\sqrt{3})$ and initial velocities $v_{1x} = -v \sin(\pi/2)$, $v_{1y} = 0$, $v_{2x} = -v \sin(\pi/2 + 2\pi/3)$, $v_{2y} = v \cos(\pi/2 + 2\pi/3)$, $v_{3x} = -v \sin(\pi/2 + 4\pi/3)$, $v_{3y} = v \cos(\pi/2 + 4\pi/3)$ being $d = 4$, $v = 0.12$ and $g(t) = -2\pi + 8\pi \cos(40t)$. 
$A_{12} = A_{21} = \beta$, $w_{1x} = \tilde{w}_{2x}$, $w_{1y} = \tilde{w}_{2y}$, $\tilde{w}_{1x} = w_{2x}$, $\tilde{w}_{1y} = w_{2y}$. Secondly, we will assume that the sizes of every part are very similar and remain close to their mean values except for the fast oscillations. So we consider that $w_{1x} = w_{2x} = \tilde{w}_{1x} = \tilde{w}_{2x} = w_{y}$ and $w_{1y} = w_{2y} = \tilde{w}_{1y} = \tilde{w}_{2y} = w_{y}$ being $w_{x}, w_{y}$ constants. Finally, to get some insight on the problem we consider the case where the distance between centroids $\Delta t$ is approximately constant which can happen for instance during a fast switching process. Thus, the only free parameters are $\alpha$ and $\beta$.

### B. Model equations

The Lagrangian $L = \int L dx dy$ obtained from the density Lagrangian $\mathcal{L}$ given by Eqs. (19) when the ansätze are as in Eqs. (19) is

$$L = \pi w_x w_y \left\{ \left( \alpha \alpha^{*} - \alpha^{*} \alpha + \beta \beta^{*} - \beta^{*} \beta \right) + \right.$$  
$$\left. + i \left( \alpha \beta^{*} - \alpha^{*} \beta + \beta \alpha^{*} - \beta^{*} \alpha \right) e^{-\frac{\beta^2}{w_{x}^2}} \right.$$  
$$\left. + \left( |\alpha|^2 + |\beta|^2 \right) \left( \frac{1}{2 w_x^2} + \frac{1}{2 w_y^2} \right) \right.$$  
$$\left. + \left( \alpha \beta^{*} + \beta \alpha^{*} \right) \left( \frac{w_x^2 - 2 \beta^2}{2 w_x^2} + \frac{1}{2 w_y^2} \right) e^{-\frac{\beta^2}{w_{x}^2}} \right.$$  
$$\left. + \frac{g(z)}{2} \left[ \left( |\alpha|^4 + |\beta|^4 \right) \left[ 1 + e^{-\frac{2 \beta^2}{w_{x}^2}} \right] \right. \right.$$  
$$\left. \left. + 6 |\alpha|^2 |\beta|^2 + 2 (|\alpha|^2 (|\beta|^2 + 2 (|\alpha|^2 |\beta|^2) e^{-\frac{\beta^2}{w_{x}^2}} \right.$$  
$$\left. \left. + 2 |\alpha|^2 |\beta|^2 + 4 (|\alpha|^2 + |\beta|^2) (\alpha \beta^{*} + \beta \alpha^{*}) e^{-\frac{\beta^2}{w_{x}^2}} \right) \right] \right) \right) \right) \tag{20}$$

where we have taken $a_{11} = a_{12} = a_{21} = a_{22} = 1$.

The standard calculations yield to a system of ordinary differential equations for $\alpha$ and $\beta$,

$$\begin{pmatrix} 1 \ e^{-\frac{\beta^2}{w_{x}^2}} \ 1 \end{pmatrix} \begin{pmatrix} \alpha \ 
\beta \end{pmatrix} = -i \left( \frac{1}{2} \begin{pmatrix} \Lambda_1 & \Lambda_2 \\
\Lambda_2 & \Lambda_1 \end{pmatrix} \begin{pmatrix} \alpha \\
\beta \end{pmatrix} \right), \tag{21}$$

where

$$\Lambda_1(t) = \frac{1}{2 w_x^2} + \frac{1}{2 w_y^2} + g(t) \left[ \frac{2 N(t)}{\pi w_x w_y} e^{-\frac{\beta^2}{w_{x}^2}} + \right.$$  
$$\left. + \left( |\alpha|^2 + |\beta|^2 \right) (1 + e^{-\frac{2 \beta^2}{w_{x}^2}} - 2 e^{-\frac{\beta^2}{w_{x}^2}}) \right), \tag{22a}$$

$$\Lambda_2(t) = \left\{ \frac{w_x^2 - 2 \beta^2}{2 w_x^2} + \frac{1}{2 w_y^2} + g(t) \left[ \frac{2 N(t)}{\pi w_x w_y} + \right.$$  
$$\left. + \left( |\alpha|^2 + |\beta|^2 \right) (2 e^{-\frac{\beta^2}{w_{x}^2}} - 2) \right] \right\} e^{-\frac{\beta^2}{w_{x}^2}}, \tag{22b}$$

and the function $N(t)$ is the square norm of the ansätze

$$N(t) = ||u_t||^2 = \pi w_x w_y |\alpha|^2 + |\beta|^2 + (\alpha \beta^{*} + \beta \alpha^{*}) e^{-\frac{\beta^2}{w_{x}^2}}.$$

System (21) can be written as

$$\begin{pmatrix} \dot{\alpha} \\
\dot{\beta} \end{pmatrix} = i \begin{pmatrix} \nu_1 & \nu_2 \\
\nu_2 & \nu_1 \end{pmatrix} \begin{pmatrix} \alpha \\
\beta \end{pmatrix}, \tag{23}$$

where

$$\nu_1(t) = \frac{\Lambda_2(t) e^{-\frac{\beta^2}{w_{x}^2}} - \Lambda_1(t)}{2 (1 - e^{-\frac{2 \beta^2}{w_{x}^2}})} \tag{24a}$$

and

$$\nu_2(t) = \frac{\Lambda_1(t) e^{-\frac{\beta^2}{w_{x}^2}} - \Lambda_2(t)}{2 (1 - e^{-\frac{2 \beta^2}{w_{x}^2}})} \tag{24b}$$

Using system (23) and its complex conjugate and taking into account that $\nu_1, \nu_2 \in \mathbb{R}$ can be immediately proved that

$$\frac{d}{dt} (|\alpha|^2 + |\beta|^2) = 0, \tag{25a}$$

$$\frac{d}{dt} (\alpha \beta^{*} + \alpha^{*} \beta) = 0, \tag{25b}$$

and therefore three invariants exist

$$|\alpha|^2 + |\beta|^2 = q_0, \tag{26a}$$

$$\alpha \beta^{*} + \alpha^{*} \beta = q_1, \tag{26b}$$

$$N(t) = N_0, \tag{26c}$$

being $q_1, q_2, N_0$ constants.

System (23) can be solved numerically to find the evolution of the amplitudes $\alpha$ and $\beta$. Nevertheless we can solve it analytically by considering that $\nu_1$ and $\nu_2$ are constants since the only dependence on $t$ is given by the nonlinear coefficient $g(t)$ and, as we discussed in Sec. III C, the dynamics is determined by an averaged nonlinearity $g = g_0$. By writing $\alpha = \alpha_R + i \alpha_I, \beta = \beta_R + i \beta_I$ with $\alpha_R, \alpha_I, \beta_R, \beta_I \in \mathbb{R}$ the solutions of system (23) are obtained by using basic techniques from the theory of ordinary differential equations and the result is

$$\begin{pmatrix} \alpha_R \\
\beta_R \end{pmatrix} = \frac{1}{2 M} \begin{pmatrix} \sin(\nu_1 - \nu_2) t \\
\cos(\nu_1 - \nu_2) t \end{pmatrix} \cos(\nu_1 + \nu_2) t, \tag{27}$$

where

$$M = \begin{pmatrix} -M_1 & M_2 & -M_3 & M_4 \\
M_1 & -M_2 & -M_3 & M_4 \\
-M_2 & M_1 & M_4 & M_3 \\
-M_3 & M_2 & M_1 & M_3 \end{pmatrix}, \tag{28}$$

and

$$M_1 = \alpha_I(0) - \beta_I(0), \tag{29a}$$

$$M_2 = \alpha_R(0) - \beta_R(0), \tag{29b}$$

$$M_3 = \alpha_I(0) + \beta_I(0), \tag{29c}$$

$$M_4 = \alpha_R(0) + \beta_R(0), \tag{29d}$$

being $\alpha_R(0), \alpha_I(0), \beta_R(0), \beta_I(0)$ the initial conditions.
FIG. 9: Comparison of the evolution of the peak amplitude of the wavepacket \( u_1 \) corresponding to the simulation of Fig. 4 and Fig. 5 obtained by direct numerical simulation of Eqs. 31, with the periodic oscillations from Eqs. 31 obtained by the wavepacket splitting model.

C. Validation

Taking as initial conditions \( \alpha_R(0) = \alpha_0, \alpha_I(0) = \beta_R(0) = \beta_I(0) = 0 \) from Eq. 27 we obtain

\[
\begin{align*}
\alpha_R &= \alpha_0 \cos \nu t \cos \nu_2 t, \quad (30a) \\
\beta_R &= -\alpha_0 \sin \nu t \sin \nu_2 t, \quad (30b) \\
\alpha_I &= \alpha_0 \sin \nu t \cos \nu_2 t, \quad (30c) \\
\beta_I &= \alpha_0 \cos \nu t \sin \nu_2 t, \quad (30d)
\end{align*}
\]

and the amplitude evolutions are straightforwardly derived

\[
|\alpha| = |\alpha_0| \cos \nu_2 t, \quad (31a) \\
|\beta| = |\alpha_0| \sin \nu_2 t. \quad (31b)
\]

Eqs. 31 mean that the splitting mechanism is oscillatory and periodic as can be seen in Figs. 4 and 5. In Fig. 4 we compare the evolution of the peak amplitude of the wavepacket \( u_1 \) corresponding to the simulation of Fig. 4 and Fig. 5 (for which \( \ell = 2 \), obtained by direct numerical simulation of Eqs. 31 with Eqs. 31). The value of \( \nu_2 \) is calculated according to Eq. 24b, with a slight correction in the value of \( \ell \) which is taking as \( \ell = 1.88 \). This correction is due to the fact that numerical simulations of Fig. 4 and Fig. 5 are made with Townes solitons initial data whereas the analysis of the wavepacket splitting is made under the assumption of Gaussian initial data. Therefore, since Townes solitons decay at infinity as \( \exp(-\nu r) \) and Gaussians do it as \( \exp(-\nu r^2) \), for a fixed separation \( 2\ell \), the overlapping between Townes solitons is greater than between Gaussians. For this reason to compare both situations it is necessary to take a smaller value of \( \ell \) in the case of Gaussian data. We see that there is a very good agreement between both curves up to around \( t = 70 \). After that numerical simulations show that solitons repel each other and the theoretical analysis is not valid any more, because we supposed constant separation between solitons. Therefore, we can conclude that our two-mode model is valid to predict the readjustment of mass between the wavepackets provided that the distance between parts remains nearly constant. As in the one dimensional case, the final repulsion of the wavepackets can be explained by taking into account that the relative phases between coherent solitons tend to separate them. In this case, the solution of Eqs. 30 gives a phase difference which takes the values \( \pm \pi/2 \).

Finally, we must notice that the formation of vector solitons is the dominant tendency of the system in the slow collision regime. Therefore, the effective-particle model must be combined with the analysis of the partial splitting in order to get a more accurate picture of the dynamics. In fact this kind of solitons exhibit a teleportation behavior as they suddenly vanish to appear in another place. This effect is very fast and can have potential applications in optical information processing.

VI. CONCLUSIONS

In this paper we have presented a detailed study of interactions between wavepackets which are stabilized against collapse by means of a modulation of the nonlinear term, i.e. stabilized solitons.

We have studied their dynamics by direct numerical simulations of the model equations. In this study, we have found that the system presents collapsing and expanding orbits depending on the initial configuration as well as other phenomena as deflection of one wavepacket due to the attraction and split in several parts in the corresponding \( n \)-body interactions.

We have also developed a theoretical explanation of the wavepacket splitting based on effective Lagrangian methods and corroborated that effective-particle models allow to obtain conclusions for this system only in very limited situations.

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