COMPATIBLE FILTERS FOR ISOMORPHISM TESTING

JOSHUA MAGLIONE

Abstract. We provide the necessary framework to use filters in computational settings, in particular for finitely generated nilpotent groups. The main motivation for this is to take advantage of the associated graded Lie ring to make computations for isomorphism testing easier. We prove that if $G$ is a nilpotent polycyclic group, then for every filter on $G$, there exists a filter refinement such that the associated Lie ring maps onto $G$. Under additional hypotheses, this is a bijection that induces a bijection between graded bases of the associated Lie ring to the set of polycyclic generating sequences of $G$. This, for example, enables us to define functions between groups from isomorphisms between their associated Lie rings.

1. Introduction

There are several progressively more general methods to associate a Lie ring to a nilpotent group, see [H2, K2]. While the applications vary, a common theme is to make group-theoretic problems easier by employing linear algebra in the context of the Lie ring. In particular, this helps in the study of isomorphism and automorphism problems for groups [ELGO, H1, K1, M1, M3]. Amongst the most general approaches of associating a Lie algebra $L(G)$ to a group $G$ are described in [W] generalizing approaches from Magnus [M4, M5] and Lazard [L] when $L(G)$ is like a linearization of $G$. For example, we give criteria for $|L(G)| = |G|$ and produce a bijection between the set of bases of $L(G)$ and the set of polycyclic generating sets of $G$. These properties are immediately satisfied by the previous correspondences to Lie rings. We also give examples of where these claims fail for general examples.

We close with implications to isomorphism testing.

Definition. A filter is a function $\phi : M \to 2^G$ ($2^G$ denotes the power set of $G$) from a commutative pre-ordered monoid $M = \langle M, +, 0, \preceq \rangle$ into the normal subgroups of $G$ satisfying, for all $s, t \in M$,

$$[\phi_s, \phi_t] \leq \phi_{s+t} \quad s \preceq t \implies \phi_s \geq \phi_t.$$ 

The boundary filter is $\partial \phi_s = \langle \phi_s+t \mid t \neq 0 \rangle$, for $s \in M$, and it follows that

$$L(\phi) = \bigoplus_{s \neq 0} \phi_s/\partial \phi_s$$

is an $M$-graded Lie algebra of $\mathbb{Z}[\phi_0/\partial \phi_0]$-modules under the following product on homogeneous components, $[\partial \phi_s x, \partial \phi_t y] = \partial \phi_{s+t} x^{-1} y^{-1} x y$. It is possible that the cardinalities of $G$ and $L(\phi)$ are different; for examples see Section 2.6. However, we show that this situation can be remedied.

Date: May 11, 2018.
This research was supported in part by NSF grant DMS 1620454.
Theorem A. Suppose every subgroup of the nilpotent group $G$ is finitely generated. If $\phi : M \to 2^G$ is a filter, then there exists a filter $\theta : M' \to 2^G$ such that $\text{im}(\phi) \subseteq \text{im}(\theta)$ and a surjection from $L(\theta)$ to $\partial \theta_0$.

Next, we provide a means to relate the bases of $L(\phi)$ to nice generating sets of $G$. This allows for calculations in $L(\phi)$ to be translated to $G$ and vice-verse. For example, it gives coordinates for which we can relate automorphisms of $L(\phi)$ to ones of $G$. To do this, we need to choose generating sets for $G$ that are compatible with filters: ones that interact nicely with the complete lattice generated by $\text{im}(\phi)$.

Definition 1.1. A generating set $X \subseteq G$ is filtered by $\phi$ if

(i) for all $s \in M$, $\langle \phi_s \cap X \rangle = \phi_s$ and

(ii) $H \mapsto H \cap X$ is a lattice embedding from the complete lattice generated by $\text{im}(\phi)$ into the subset lattice of $X$.

We say that $X$ is faithfully filtered by $\phi$ if $X$ is filtered by $\phi$ and if for each $x \in X$, there exists a unique $s \in M$ such that $x \in \phi_s - \partial \phi_s$.

Theorem B. If $X$ is faithfully filtered by $\phi$, then there exists a bijection between $L(\phi)$ and $\partial \phi_0$ that induces a bijection between the set of bases of $L(\phi)$, respecting the graded direct sum decomposition, and the set of polycyclic generating sequences of $\partial \phi_0$ that are filtered by $\phi$.

When the underlying monoid is cyclic, e.g. $M = \mathbb{N}$ and $\phi$ is the lower central series like in the case of the Lazard correspondence, every polycyclic generating set of $G$ is faithfully filtered and corresponds to a basis for $L(\phi)$. Even when the monoid $M$ is totally ordered, it is simple to construct faithfully filtered generating sets for $G$, see [M1]. In these examples, the above definitions are expected to always be satisfied, but in the case where $\leq$ is a partial order, many issues arise, and we illustrate a few in Section 2.6.

One of the driving motivations for this work comes from the isomorphism problem of groups\(^1\). Algorithms to decide isomorphism or compute automorphism groups of finite nilpotent groups rely on induction and the ability to construct characteristic subgroups (i.e. subgroups fixed by every automorphism), see [CH, ELGO, O]. The algorithms for finite nilpotent groups boil down to just finite $p$-groups, groups of order $p^n$ for some prime $p$. For a fixed order $|G| = p^n$, the most challenging groups to compute $\text{Aut}(G)$ are the nonabelian $p$-groups with the fewest known characteristic subgroups: $p$-groups where the only known characteristic subgroups are contained in the set $\{G, G', 1\}$.

Because of Theorem B, we can leverage the structure of the Lie algebra $L(\phi)$ to constrain the possible automorphisms of $G$. For example, if each $\phi_s$ is characteristic in $G$, then each automorphism of $G$ induces an automorphism of $L(\phi)$. While the reverse direction may not always hold: it could be starting point to construct automorphisms of $G$, especially if $\text{Aut}(L(\phi))$ can be done more efficiently than inductively constructing $\text{Aut}(G)$. One example is a large family of graded algebras $L$ in [BOW] for which $\text{Aut}(L)$ can be constructed in time polynomial in $|L|$.

Filters (in particular, their associated $M$-graded Lie algebras) have proved to be a significant resource for efficiently constructing characteristic subgroups, see [M1, M2, W]. The main benefit is that the inclusion of one new subgroup in a filter can

\(^1\)Filters naturally apply to rings and algebras as well, but we focus on groups here.
drastically change the filter and the associated Lie algebra. This opens the door to efficient recursive methods as \( \dim L(\phi) = \log_p |G| \). To contrast, other methods to construct characteristic subgroups typically partition subgroups based on certain isomorphism invariants like, for example, properties of centralizers and isomorphism types of certain quotients. While different properties can further refine these partitions, these operations are not recursive and often require searching through sets of size polynomial in \(|G|\). Frequently, constructing just one characteristic subgroup not in \( \{G, G', 1\} \) leads to more characteristic subgroups with filters.

1.1. **Overview.** Section 2 details preliminary definitions and theorems needed for the rest of the paper. We discuss topics concerning lattices, polycyclic groups, and filters. We also include examples of filters, some of which illustrate justification for future definitions. In Sections 3 and 4 we define properties necessary for filters to construct an algorithm for constructing group automorphisms from Lie algebra automorphisms. This involves defining when a generating set is filtered by \( \phi : M \to 2^G \) and studying the structure this condition imposes on the filter. In Section 4, we also tackle the issue of inertia. We give alternate characterizations of inert subgroups that get used throughout the paper, and we provide a process to remove inert subgroups from a given filter. Thus, proving Theorem A.

Even with all the work from Sections 3 and 4, we still cannot construct automorphisms of \( G \) from automorphisms of \( L(\phi) \)—let alone bijections of \( G \). In Section 5, we define what it means for a generating set to be faithfully filtered by \( \phi \), and we prove that such a generating set provides \( \phi \) with structure. Moreover, the existence of such a generating set, one faithfully filtered by \( \phi \), implies that \( L(\phi) \) and \( \partial \phi_0 \) are in bijection, provided every subgroup of \( G \) is finitely generated, which proves Theorem B. We close with some examples and further questions.

## 2. Preliminaries

We give a brief overview of definitions and theorems that will be used throughout.

### 2.1. Notation and assumptions.** We use notation from [R] for groups. We let \( 2^X \) denote the set of subsets of \( X \). Furthermore, \( \mathbb{N} \) will denote the set of nonnegative integers.

Throughout, \( G \) is a group. For \( x, y \in G \), set \( [x, y] = x^{-1}y^{-1}xy \). For subsets \( X, Y \subseteq G \), set \( [X, Y] = \langle [x, y] : x \in X, y \in Y \rangle \), and for \( X, Y, Z \subseteq G \), set \( [X, Y, Z] = [[X, Y], Z] \) and \( [X] = X \). Let \( \gamma_1 = G \) and for \( i \geq 1 \), set \( \gamma_{i+1} = [\gamma_i, G] \). A nilpotent group is class \( c \) if \( \gamma_c > \gamma_{c+1} = 1 \).

A commutative monoid \( \langle M, +, 0 \rangle \) is pre-ordered by a pre-order \( \preceq \) if \( s \preceq t \) and \( s' \preceq t' \) imply that \( s + s' \preceq t + t' \). Throughout, we will use \( + \) for the (commutative) monoid operation, 0 for the additive identity in \( M \), and \( \preceq \) for the partial order on \( M \). For \( s, t \in M \), we let \( s \parallel t \) denote the case when \( s \) and \( t \) are incomparable under \( \preceq \). That is, \( s \npreceq t \) and \( t \npreceq s \). We assume that 0 is the minimal element of \( M \). That is, for all \( s \in M \), \( 0 \preceq s \). Thus, our monoids are conical. A commutative monoid \( M \) is conical if \( s + t = 0 \) implies \( s = t = 0 \), for all \( s, t \in M \).

**Lemma 2.1.** Suppose \( M \) is a commutative, pre-ordered monoid. If 0 is the minimal element of \( M \), then \( M \) is conical.

**Proof.** Suppose \( s + t = 0 \). Since \( 0 \preceq s \), it follows that \( t = 0 + t \preceq s + t = 0 \). \( \square \)
2.2. Commutative monoids. We first define some standard finite commutative monoids. It is possible to describe all cyclic monoids up to isomorphism. Let \( r \in \mathbb{N} \), \( s \in \mathbb{Z}_+ \) (this is the index and period, respectively). Define a congruence \( \sim \) on \( \mathbb{N} \) where \( i, j \in \mathbb{N} \),

\[
\begin{align*}
    i \sim j & \iff \begin{cases} 
    i = j & \text{if } i, j < r \\
    i \equiv j \pmod{s} & \text{if } i, j \geq r.
    \end{cases}
\end{align*}
\]

Define \( C_{r,s} = \mathbb{N}/\sim \), and note that \( |C_{r,s}| = r + s \).

**Proposition 2.2** ([G, Proposition 5.8]). If \( M \) is a cyclic monoid, then either \( M \cong \mathbb{N} \) or there exists \( r, s \in \mathbb{N} \) such that \( M \cong C_{r,s} \).

2.3. Partially-ordered sets and lattices. We pull from [B] for notation and definitions of partially-ordered sets and lattices. We will summarize a few definitions and theorems that will be used later.

A partial order \( \preceq \) on a set \( M \) is reflexive, anti-symmetric, and transitive. For \( s, t \in M \) we let \( s \parallel t \) denote \( s \not\preceq t \) and \( t \not\preceq s \). Given partially-ordered sets \((S, \leq)\) and \((T, \preceq)\), a map \( f : S \to T \) is isotone if for all \( x, y \in S \),

\[
x \leq y \implies f(x) \preceq f(y).
\]

An order isomorphism \( f : S \to T \) is a isotone bijection whose inverse is also isotone.

We define two important partial orders on a direct product of pre-ordered monoids: the lexicographical (abbreviated lex) and direct product ordering. Suppose \((M, \leq)\) and \((N, \preceq)\) are two pre-ordered (commutative) monoids. The lexic order \( \leq_\ell \) of \( M \times N \), denoted \( \leq_\ell \), is defined as follows. For all \((m, n), (m', n') \in M \times N \),

\[
(m, n) \leq_\ell (m', n') \iff (m < m') \lor (m = m' \land n \preceq n').
\]

The direct product order \( \leq_d \) of \( M \times N \), denoted \( \leq_d \), is define so that

\[
(m, n) \leq_d (m', n') \iff (m \leq m') \land (n \preceq n').
\]

These orders will be used in examples throughout.

There is another important ordering on monoids: the algebraic order denoted \( \preceq_+ \) where for \( s, t \in M \),

\[
s \preceq_+ t \iff \exists u \in M, s + u = t.
\]

The only cyclic monoids where \( \preceq_+ \) is a pre-order are \( \mathbb{N} \) and \( C_{r,1} \). The reason is akin to why finite fields cannot be (totally) ordered. Of course every monoid has a partial order: let 0 be the minimal element and every pair of nonzero elements are incomparable, but this is not helpful for us.

Lattices play an important role in our study of filters. All of our lattices are sub-lattices of either the power set of a pre-ordered monoid or the normal subgroups of a group. Both of these lattices are well-studied, so we do not state the general definitions. Therefore, \( \cap \) and \( \cup \) are understood to be either set or subgroup intersection and union (join).

**Definition 2.3.** A partially ordered set \( L \) is a lattice if for all \( X, Y \in L \) both \( X \cap Y \in L \) and \( X \cup Y \in L \).

**Definition 2.4.** A partially ordered set \( L \) is a complete lattice if for all \( X \subseteq L \) both \( \bigcap_{x \in X} x \in L \) and \( \bigcup_{x \in X} x \in L \).
For lattices $L$ and $M$, $f : L \to M$ is a lattice homomorphism if for all $X, Y \in L$
\[
f(X \cap Y) = f(X) \cap f(Y) \quad \text{and} \quad f(X \cup Y) = f(X) \cup f(Y).
\]
Similarly define a complete lattice homomorphism over arbitrary intersections and unions. (Complete) lattices $L$ and $M$ are isomorphic if there exists a bijective (complete) lattice homomorphism.

2.4. Polycyclic groups. Much of what we focus on are polycyclic groups. We state some definitions and theorems about polycyclic groups from [S2, Chapter 9].

Definition 2.5. A group is polycyclic if it has a finite subnormal series whose factors are all cyclic. That is, if there exists subgroups $G_i$ such that
\[
G = G_1 \supseteq G_2 \supseteq \cdots \supseteq G_n \supseteq G_{n+1} = 1,
\]
where $G_i/G_{i+1}$ is cyclic.

The series in (1) is called a polycyclic series. Moreover, for each $i \geq 1$, there exists $a_i \in G_i$ such that $\langle G_{i+1} a_i \rangle = G_i/G_{i+1}$.

Definition 2.6. The sequence $A = (a_1, \ldots, a_n)$ is a polycyclic generating sequence (pcgs) if the following is a polycyclic series
\[
G = \langle a_1 \rangle > \langle a_2 \rangle > \cdots > \langle a_n \rangle > 1.
\]

Note that the order matters for a pcgs $A$, so we will call a set $\{a_1, \ldots, a_n\}$ a polycyclic generating set if, under some relabeling, it is a pcgs. We will not need to construct the composition series from a pcgs, so we will use pcgs to mean a polycyclic generating set.

Proposition 2.7 ([S2, Chapter 9, Proposition 3.9]). A group is polycyclic if, and only if, it is solvable and all subgroups are finitely generated.

For each $i$ where $G_i/G_{i+1}$ is finite, let $m_i = [G_i : G_{i+1}]$. Define the set $E_i = \{0, \ldots, m_i - 1\}$, and if $G_i/G_{i+1}$ is infinite, let $E_i = \mathbb{Z}$.

Proposition 2.8 ([S2, Chapter 9, p.395]). If $A = (a_1, \ldots, a_n)$ is a pcgs for $G$, then for every $g \in G$ there exists unique $e_i \in E_i$ such that
\[
g = a_1^{e_1} \cdots a_n^{e_n}.
\]

2.5. Filters. One of the main uses of filters is to have a process for refining by including known characteristic subgroups. When “inserting” a new subgroup into a filter, we define a new function containing the image of the original filter together with the new subgroup. Often, more subgroups are generated so that this new function is a filter. We describe this process below, and later in Section 4.1, we apply similar constructions to prove Theorem A.

Definition 2.9. A function $\pi : X \to 2^G$ is a prefilter if it satisfies the following conditions.
\[
(a) \ 0 \in X \subseteq M \text{ and } \langle X \rangle = M;
(b) \ \text{if } x \in X \text{ and } y \in M \text{ with } y \leq x, \text{ then } y \in X;
(c) \ \text{for all } x \in X, \pi_x \leq G;
(d) \ \text{for all } x, y \in X, \ x \leq y \text{ implies } \pi_x \geq \pi_y.
\]
For $s \in \langle X \rangle$, a partition of $s$ with respect to $X$ is a sequence $(s_1, \ldots, s_k)$ where each $s_i \in X$ and $s = \sum_{i=1}^{k} s_i$. Let $\mathcal{P}_X(s)$ denote the set of partitions of $s \in \langle X \rangle$ with respect to $X$, and if $P = (s_1, \ldots, s_k) \in \mathcal{P}_X(s)$, then set

$$[\pi_P] = [\pi_{s_1}, \ldots, \pi_{s_k}].$$

For a function $\pi : X \to 2^G$, define a new function $\pi : \langle X \rangle \to 2^G$ where

$$\pi_s = \prod_{P \in \mathcal{P}_X(s)} [\pi_P].$$

Because each $\pi_x \subseteq G$, the subgroups $[\pi_P]$ are permutable and the order of the product in (2) does not matter.

**Theorem 2.10** ([W, Theorem 3.3]). If $\pi$ is a prefilter, then $\pi$ is a filter.

See [M1] for efficient algorithms and examples of this process on totally-ordered monoids.

### 2.6. Examples of filters

The definition of a filter is not very restrictive, and in this section, we give some possibly surprising examples of what constitutes a filter. We also allude to some important properties of filters for the coming sections.

We show that the property

$$(\forall s, t \in M) \quad [\phi_s, \phi_t] \leq \phi_{s+t}$$

does not need to be an equality. This is not the case for, say, the lower central series as $\gamma_{s+1} := [\gamma_s, \gamma_1]$. Furthermore, it should come as no surprise that a nilpotent group of finite class can have a filter containing a chain of infinite length.

**Example 2.11.** Suppose

$$G = \left\{ \begin{bmatrix} 1 & a & b \\ 1 & a & 1 \\ 1 & 1 & 1 \end{bmatrix} \bigg| a, b \in \mathbb{Z}_p[x] \right\}.$$ 

Let $M = \mathbb{N}^2$ with the lexicographical ordering, and define a filter $\phi : M \to 2^G$ where $\phi_0 = G$, $\phi_{(1,0)} = \gamma_2$,

$$\phi_{(0,s)} = \left\{ \begin{bmatrix} 1 & u & v \\ 1 & u & 1 \\ 1 & 1 & 1 \end{bmatrix} \bigg| u \in (x^s), \ v \in \mathbb{Z}_p[x] \right\},$$

$$\phi_{(1,s)} = \left\{ \begin{bmatrix} 1 & 0 & v \\ 1 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix} \bigg| v \in (x^s) \right\},$$

and $\phi_t = 1$ otherwise. The condition $[\phi_s, \phi_t] \leq \phi_{s+t}$ is always satisfied, and provided $\phi_s \neq 1 \neq \phi_t$, it is always a strict containment. Even though $G$ is class 2, the filter $\phi$ has infinite length and its associated Lie algebra is abelian. \qed

In Section 2.1, we asserted that 0 is the minimal element of $M$. In the following example, we provide some justification for this assumption. If, for example, that no element is a minimal element, then we cannot properly define an associated Lie ring. Indeed, although not explicitly stated in [W], it is assumed that 0 is the minimal element.
Example 2.12. Suppose $M = \mathbb{N}$, with the partial order $s \preceq t$ if, and only if, $s = t$. In other words, for distinct $s, t$, then $s \parallel t$, and therefore 0 is not the minimal element of $M$. Let $G = \mathbb{Z}$, and define the filter $\phi : M \to 2^G$ such that

$$\phi_s = \begin{cases} s\mathbb{Z} & \text{if } s \text{ is prime}, \\ 0 & \text{otherwise}. \end{cases}$$

Note that $\phi$ is a filter since $G$ is abelian and all distinct $s, t \in M$ are incomparable. If $p$ is a prime, then $\phi_p \not\supseteq \partial\phi_p$. If $s \in M$ is not prime, then $\phi_s = 0$, and because there exists a prime larger than $s$, $\phi_s \prec \partial\phi_s$. Therefore, the quotient, $\phi_s/\partial\phi_s$, has no meaning as $\partial\phi_s$ is not necessarily contained in $\phi_s$.\hspace{1cm} \Box

We need not limit ourselves to solvable groups. The next example is a filter of an almost quasi-simple group. The nonabelian simple composition factor makes no contribution to the associated Lie ring. These subgroups—subgroups not “seen” by the Lie ring—are studied in detail in Section 4.

Example 2.13. Let $G = \text{GL}(2, 7)$ and $M = \mathbb{N}^2$ with the direct product ordering. Define a filter $\phi : M \to 2^G$ where

$$\phi_s = \begin{cases} \text{GL}(2, 7) & \text{if } s \in \{0, e_1, e_2\}, \\ \text{SL}(2, 7) & \text{otherwise}. \end{cases}$$

Therefore, $L(\phi) = \mathbb{Z}_6 \oplus \mathbb{Z}_6$. Observe that for $\phi$ to be a filter, the trivial subgroup cannot be contained in the image of $\phi$; otherwise, both filter properties imply that the derived subgroup of $\text{SL}(2, 7)$ be trivial.\hspace{1cm} \Box

3. Filters and Lattices

Our main objective in this section is to develop a generating set that interacts nicely with filters. A common theme for using groups effectively in computational settings is to have a structured generating set for the group. Some examples include bases and strong generating sets for permutation groups [S1, Chapter 4], (special) polycyclic generating sequences for solvable groups [CELG, EW], [S2, Chapter 9], and power-commutator presentations for $p$-groups [HN, NO]. These generating sets are all based on a series in the group, so, influenced by these generating sets, we define an appropriate generating set in the context of filters.

The key motivation is that we want a compatible generating set for filters over partially-ordered monoids for isomorphism testing. If we want to construct group isomorphisms from associated Lie algebras $L(\phi)$, then we need to define functions on the original groups from homomorphisms on the Lie algebras. The lack of a total order and the full generality of allowing for any commutative, pre-ordered monoid makes this task challenging.

For now, we say a generating set $\mathcal{X} \subseteq G$ is filtered by $\phi : M \to 2^G$ if it contains a generating set for each $\phi_s$ and is compatible with the induced complete lattice of $\text{im}(\phi)$, see Definition 3.4 below. In this section we prove the following theorem.

Theorem 3.1. If $\mathcal{X} \subseteq G$ is filtered by $\phi : M \to 2^G$, then

(i) the complete lattice induced by $\text{im}(\phi)$ is distributive, and
(ii) $\mathcal{X}$ is filtered by $\partial\phi : M \to 2^G$.

Since the lattice of normal subgroups is not distributive, Theorem 3.1 shows that not all filters have such a generating set. This structure question is addressed later.
in Section 5. We start with arbitrary groups $G$ and begin with a natural condition on generating sets, akin to strong generating sets.

**Definition 3.2.** A generating set $\mathcal{X} \subseteq G$ is weakly-filtered by $\phi : M \to 2^G$ if for all $s \in M$, $\langle \phi_s \cap \mathcal{X} \rangle = \phi_s$.

The property of a generating set $\mathcal{X}$ being weakly-filtered can be rephrased in the context of partially-ordered sets. Suppose $\mathcal{X} \subseteq G$ is weakly-filtered by $\phi$. Define functions on partially-ordered sets $2^G$ and $2^\mathcal{X}$; namely, $\cap : 2^G \to 2^G$ where $H \mapsto H \cap \mathcal{X}$ and $\langle \cdot \rangle : 2^\mathcal{X} \to 2^G$ where $Y \mapsto \langle Y \rangle$. These functions are isotone because $H, K \in 2^G$ with $H \subseteq K$ implies $H \cap \mathcal{X} \subseteq K \cap \mathcal{X}$, and if $Y, Z \in 2^\mathcal{X}$ with $Y \subseteq Z$, then $\langle Y \rangle \leq \langle Z \rangle$. This proves the following lemma.

**Lemma 3.3.** If $\mathcal{X} \subseteq G$ is weakly-filtered by $\phi$, then the restriction of $\cap \mathcal{X}$ on $\text{im}(\phi)$ is an (order) isomorphism with inverse $\langle \cdot \rangle : \text{im}(\phi) \cap \mathcal{X} \to 2^G$.

We need a stronger definition for our purposes. The set $\text{im}(\phi)$ is, in general, not a lattice, so let $\text{Lat}(\phi)$ denote the complete intersection and join closure of $\text{im}(\phi)$. That is, any family of meets and joins are contained in $\text{Lat}(\phi)$. Let $\text{Lat}(\phi) \cap \mathcal{X}$ denote the image of $\text{Lat}(\phi)$ in $2^\mathcal{X}$ under $\cap \mathcal{X}$. The surjection $\cap \mathcal{X} : \text{Lat}(\phi) \to \text{Lat}(\phi) \cap \mathcal{X}$ is isotone; however, if $H, K \in \text{Lat}(\phi)$ and $H \cap \mathcal{X} \subseteq K \cap \mathcal{X}$, then $H$ need not be a subgroup of $K$. Even as partially-ordered sets $\text{Lat}(\phi)$ need not be isomorphic to $\text{Lat}(\phi) \cap \mathcal{X}$. The strength of the following definition comes when $\cap \mathcal{X}$ and $\langle \cdot \rangle$ are complete lattice homomorphisms.

**Definition 3.4.** A generating set $\mathcal{X} \subseteq G$ is filtered by $\phi$ if it is weakly-filtered and for all $S \subseteq M$,

$$\bigcap_{s \in S} \phi_s = \left(\bigcap_{s \in S} (\phi_s \cap \mathcal{X})\right) \quad \text{and} \quad \left(\bigcap_{s \in S} \phi_s \cap \mathcal{X} \right) = \bigcup_{s \in S} (\phi_s \cap \mathcal{X}).$$

This definition is not just a weakly-filtered analogue for the complete lattice $\text{Lat}(\phi)$. If $\mathcal{X}$ satisfies the property that for all $H \in \text{Lat}(\phi)$, $\langle H \cap \mathcal{X} \rangle = H$, then $\mathcal{X}$ may still not be filtered by $\phi$. Some subtleties of this definition are revealed in Example 3.5. It is worth pointing out that the generating set that is filtered by $\phi$ in Example 3.5 induces a basis for associated Lie ring $L(\phi)$.

**Example 3.5.** Let $G = \mathbb{Z}_{60}$, and $M = (\mathbb{N}^2, \leq_d)$, ordered by the direct product ordering. Define a filter $\phi : M \to 2^G$ where $\phi_0 = G$,

$$\phi_s = \begin{cases} (2) & \text{if } s = e_1, \\ (3) & \text{if } s = e_2, \\ (10) & \text{if } s = 2e_1, \\ (15) & \text{if } s = 2e_2, \\ \end{cases}$$

and $\phi_s = 0$ otherwise. Set $\mathcal{X} = \{2, 3, 10, 15\}$, and observe that $\mathcal{X}$ is weakly-filtered by $\phi$. In Figure 1, we plot the Hasse diagram of $\phi$ and the lattice of $\text{Lat}(\phi)$. Set $H = \langle 6 \rangle$ and $K = \langle 30 \rangle$. Then $H \cap \mathcal{X} = \emptyset = K \cap \mathcal{X}$, but $\langle 6 \rangle = H \not\subseteq K = \langle 30 \rangle$. Thus, $\cap \mathcal{X} : \text{Lat}(\phi) \to \text{Lat}(\phi) \cap \mathcal{X}$ is not isotone and, hence, is not an isomorphism. By Proposition 3.6, $\mathcal{X}$ is not filtered by $\phi$.

If, instead, we set $\mathcal{X} = \{2, 3, 5, 6, 10, 15, 30\}$, then for all $H \in \text{Lat}(\phi)$, $\langle H \cap \mathcal{X} \rangle = H$. However, $\mathcal{X}$ is not filtered by $\phi$: for example,

$$(\phi_{2e_1} \phi_{2e_2}) \cap \mathcal{X} = \langle 5 \rangle \cap \mathcal{X} = \{5, 10, 15, 30\}$$
and

\[(\phi_{2e_1} \cap \mathcal{X}) \cup (\phi_{2e_2} \cap \mathcal{X}) = (\langle 10 \rangle \cap \mathcal{X}) \cup (\langle 15 \rangle \cap \mathcal{X}) = \{10, 15, 30\}.
\]

If \(\mathcal{X} = \{6, 10, 15, 30\}\), then \(\mathcal{X}\) is filtered by \(\phi\).

\[\begin{array}{c}
\text{(a) The Hasse diagram of im(\phi).} \\
\end{array}\]

\[\begin{array}{c}
\text{(b) The lattice Lat(\phi).}
\end{array}\]

\textbf{Figure 1.} Hasse diagrams related to \(\phi : M \to 2^G\) from Example 3.5.

If \(\mathcal{X}\) is filtered by \(\phi\), then \(\cap \mathcal{X}\) is not just a lattice homomorphism but an isomorphism, and therefore, the lattice \(\text{Lat}(\phi)\) inherits properties of the subset lattice \(\text{Lat}(\phi) \cap \mathcal{X}\). The next proposition proves Theorem 3.1 (i).

\textbf{Proposition 3.6.} The set \(\mathcal{X} \subseteq G\) is filtered by \(\phi\) if, and only if, \(\cap \mathcal{X} : \text{Lat}(\phi) \to \text{Lat}(\phi) \cap \mathcal{X}\) and \(\langle \cdot \rangle : \text{Lat}(\phi) \cap \mathcal{X} \to \text{Lat}(\phi)\) are complete lattice isomorphisms. In such a case, \(\text{Lat}(\phi)\) is a distributive lattice.

\textbf{Proof.} Suppose \(\mathcal{X}\) is filtered by \(\phi\) and \(S \subseteq M\). Since meet is associative and \(\mathcal{X}\) is filtered,

\[\left(\bigcap_{s \in S} \phi_s\right) \cap \mathcal{X} = \bigcap_{s \in S} (\phi_s \cap \mathcal{X}) \quad \text{and} \quad \left(\bigcap_{s \in S} \phi_s\right) \cap \mathcal{X} = \bigcup_{s \in S} (\phi_s \cap \mathcal{X}).\]

Hence \(\cap \mathcal{X} : \text{Lat}(\phi) \to \text{Lat}(\phi) \cap \mathcal{X}\) is a lattice homomorphism. Since \(\mathcal{X}\) is filtered by \(\phi\) it is also weakly-filtered. Therefore,

\[\left(\bigcup_{s \in S} (\phi_s \cap \mathcal{X})\right) = \prod_{s \in S} (\phi_s \cap \mathcal{X}) = \prod_{s \in S} \phi_s \quad \text{and} \quad \left(\bigcap_{s \in S} (\phi_s \cap \mathcal{X})\right) = \bigcap_{s \in S} \phi_s.\]

Therefore, \(\langle \cdot \rangle : \text{Lat}(\phi) \cap \mathcal{X} \to \text{Lat}(\phi)\) is a lattice homomorphism. Both homomorphisms \(\cap \mathcal{X}\) and \(\langle \cdot \rangle\) are isotone. Since \(\mathcal{X}\) is weakly-filtered, \(\langle \cdot \rangle\) is the inverse of \(\cap \mathcal{X}\), and hence, \(\text{Lat}(\phi) \cong \text{Lat}(\phi) \cap \mathcal{X}\).

Conversely, suppose \(\cap \mathcal{X}\) and \(\langle \cdot \rangle\) are complete lattice isomorphisms. It follows then that \(\mathcal{X}\) is weakly-filtered by \(\phi\). Let \(S \subseteq M\), so \(\bigcap_{s \in S} \phi_s \in \text{Lat}(\phi)\). Since \(\langle \cdot \rangle\) is a complete lattice homomorphism,

\[\bigcap_{s \in S} \phi_s = \left(\bigcap_{s \in S} \phi_s \cap \mathcal{X}\right) = \left(\bigcap_{s \in S} (\phi_s \cap \mathcal{X})\right).\]
Furthermore, since $\cap X$ is a complete lattice homomorphism,

$$\left( \prod_{s \in S} \phi_s \right) \cap X = \bigcup_{s \in S} (\phi_s \cap X).$$

Now we can prove the second part of Theorem 3.1 by employing Proposition 3.6. The key to the next proof is to use the fact that $\cap X$ and $\langle \cdot \rangle$ are complete lattice homomorphisms when $X$ is filtered by $\phi$.

**Proof of Theorem 3.1 (ii).** Suppose $X$ is filtered by $\phi : M \to 2^G$; we will prove that $X$ is filtered by $\partial \phi$. First we show that for all $S \subseteq M$,

$$\bigcap_{s \in S} \partial \phi_s = \left\langle \bigcap_{s \in S} \left( \partial \phi_s \cap X \right) \right\rangle.$$

By Proposition 3.6, $\cap X$ and $\langle \cdot \rangle$ are complete lattice homomorphisms, so

$$\left\langle \bigcap_{s \in S} \left( \partial \phi_s \cap X \right) \right\rangle = \left\langle \bigcap_{s \in S} \bigcup_{t \in M-0} \left( \phi_{s+t} \cap X \right) \right\rangle = \bigcap_{s \in S} \left( \bigcup_{t \in M-0} \phi_{s+t} \cap X \right) = \bigcap_{s \in S} \left( \bigcap_{t \in M-0} \phi_{s+t} \right).$$

For the second part, we show that

$$\left( \prod_{s \in S} \partial \phi_s \right) \cap X = \bigcup_{s \in S} (\partial \phi_s \cap X).$$

Again, we use the fact that $\cap X$ is a complete lattice homomorphism:

$$\left( \prod_{s \in S} \partial \phi_s \right) \cap X = \left( \prod_{s \in S} \prod_{t \in M-0} \phi_{s+t} \right) \cap X = \bigcup_{s \in S} (\partial \phi_s \cap X).$$

Therefore, $X$ is filtered by $\partial \phi$. □

**3.1. The descending chain condition.** Our goal is to develop the framework so that group automorphisms can be constructed via Noetherian induction from $\partial \phi_0$ down to the bottom of the lattice. We will assume that every chain in $\text{im}(\phi)$ has finite length.

**Definition 3.7.** A filter $\phi : M \to 2^G$ satisfies the **descending chain condition (DCC)** if there does not exist a strictly decreasing infinite chain of subgroups in $\text{im}(\phi)$.

We prove that if a filter satisfies DCC, then it has a unique minimal subgroup owing to the fact that 0 is the unique minimal element of $M$.

**Lemma 3.8.** Let $\phi : M \to 2^G$ be a filter satisfying DCC. If $H \in \text{im}(\phi)$ is the minimal subgroup of some maximal descending series in $\text{im}(\phi)$, then $H = \bigcap_{s \in M} \phi_s$.

**Proof.** There exists $s \in M$ such that $\phi_s = H$. Let $t \in M$. Since $t \geq 0$, it follows that $\phi_{s+t} \leq \phi_s \cap \phi_t$. By minimality of $H$, $\phi_{s+t} = H$; otherwise, $H$ is not the minimal subgroup of a maximal descending chain. Hence, $H \leq \phi_t$, and the statement follows. □

If $H \in \text{im}(\phi)$ is the minimal subgroup and $H \neq 1$, then we will instead consider the filter $\mu : M \to 2^G/H$, where $\mu_s = \phi_s/H$. Note that for all $s \in M$, $L_s(\mu) \cong L_s(\phi)$, even further as $M$-graded Lie rings, $L(\mu) \cong L(\phi)$. Therefore, we assume $1 \in \text{im}(\phi)$.
This can be achieved superficially as well by altering the monoid. If \( \phi : M \to 2^G \), with \( 1 \notin \text{im}(\phi) \), then define a new filter \( \tilde{\phi} : M \cup \{\infty\} \to 2^G \), where
\[
\tilde{\phi}_s = \begin{cases} \phi_s & \text{if } s \in M, \\ 1 & \text{if } s = \infty. \end{cases}
\]

The addition in \( M \cup \{\infty\} \) is standard: if \( s \in M \cup \{\infty\} \), then \( s + \infty = \infty \). Of course, if no minimal subgroup \( H \in \text{im}(\phi) \) exists, then this implies that there exists an infinite descending chain of subgroups in \( \text{im}(\phi) \).

This construction—artificially including 1 in \( \text{im}(\phi) \) in this way—illustrates a potential problem with filters and their associated Lie rings. Observe that with the above monoid, \( M \cup \{\infty\} \), if \( s, t \in M \), then \( s + t \in M \). In order for \( s + t = \infty \), either \( s = \infty \) or \( t = \infty \), and if \( H = \bigcap_{s \in M} \phi_s \neq 1 \), then \( H \) has the property that if \( \phi_s = H \), then \( \partial\phi_s = H = \phi_s \). Therefore, \( H \) makes no contribution to \( L(\phi) \), and \( L(\mu) \cong L(\phi) \) as rings. This brings us to the topic of inert subgroups.

4. Inert Subgroups of Filters

We start with an explicit example of a property of filters we want to avoid. We construct a filter \( \phi \) such that \( \text{im}(\phi) \) contains the lower central series, but the associated Lie ring \( L(\phi) \) is trivial.

**Example 4.1.** Let \( G \) be the Heisenberg group over a field \( K \), so
\[
G = \left\{ \begin{bmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{bmatrix} \mid a, b, c \in K \right\}.
\]

Let \( M = (\mathbb{N}^2, \preceq_{\ell}) \), ordered by the lex ordering, and set \( Y = \{ s \in M \mid s \prec_{\ell} (2, 0) \} \). Define a prefilter \( \pi : Y \to 2^G \) where \( \text{im}(\pi) = \{G\} \), and set \( \phi = \pi \). Thus,
\[
\phi_s = \begin{cases} G & s \prec_{\ell} (2, 0), \\ Z(G) & (2, 0) \preceq_{\ell} s \prec_{\ell} (3, 0), \\ 1 & (3, 0) \preceq_{\ell} s. \end{cases}
\]

We claim that \( \phi \) and \( \partial\phi \) have generating sets that are filtered, but \( L(\partial\phi) = 0 \). To see this, consider \( \partial\phi_s \geq \phi_{s+(0,1)} = \phi_s \), which holds for all \( s \in M \). Therefore, as functions \( \partial\phi = \phi \). Let
\[
\mathcal{X} = \left\{ \begin{bmatrix} 1 & a & 0 \\ 0 & 1 & b \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & c \\ 1 & 0 & b \\ 0 & 0 & 1 \end{bmatrix} \mid a, b, c \in K \right\}.
\]

Thus, \( \mathcal{X} \) is filtered by \( \phi \) and, by Theorem 3.1, by \( \partial\phi \) as well.

**Remark 4.2.** Observe that \( \mathbb{N}^2 \) with the lex-order is isomorphic, as pre-ordered monoids, to the set of ordinals \( \{ \alpha \mid \alpha < \omega^2 \} \), where \( (a, b) \mapsto \omega \cdot a + b \). A problem with Example 4.1 is that for every group \( H \in \text{im}(\phi) \), the set \( \{ s \in \mathbb{N}^2 \mid \phi_s = H \} \) contains no maximal element. For filters over totally-ordered monoids, this is remedied in \([M1, \text{Section 3.2}]\).

In Example 4.1, the filter \( \phi : M \to 2^G \) has the property that \( L(\phi) = 0 \). This is an extreme example, but this illustrates a property we want to avoid: essentially, a subgroup \( H \in \text{im}(\phi) \) is inert if it makes no contribution in \( L(\phi) \). We give a more precise definition below in Definition 4.3. The example further shows that inertia is
inert subgroups of $\phi$ of $G$. We do this by applying induction to replace the obstructions in $B$ with an appropriate selection. It follows that $\phi_s$ is inert when there is no appropriate choice of replacement. The next proposition determines a method of Noetherian induction on $\text{im}(\phi)$, which will be used constantly throughout.

The following subset of $M$ plays an important role in characterizing inertia. Throughout, fix the subset $\mathcal{I} = \{ t \in M \mid \partial \phi_t \neq \phi_t \}$, so as abelian groups $L(\phi) \cong \bigoplus_{s \in \mathcal{I}^{-0}} \phi_s / \partial \phi_s$.  

**Proposition 4.6.** Suppose $\phi : M \to 2^G$ is a filter satisfying DCC. The following are equivalent.

1. For all $s \in M$, there exists $I_s \subseteq \mathcal{I}$ such that $\partial \phi_s = \langle \phi_t \mid t \in I_s \rangle$.
2. For all $s \in M$, $\phi_s \in \mathcal{B}$. In particular, if every subgroup of $G$ is finitely generated, then there exists $n \in \mathbb{N}$ such that $\mathcal{B}_n = \text{im}(\phi)$.

**Proof.** (i) $\Rightarrow$ (ii): By the assumption, for every $s \in M$, there exists $I_s \subseteq \mathcal{I}$ such that $\partial \phi_s = \langle \phi_t \mid t \in I_s \rangle$. If $t_1 \in I_s$, then $\phi_s \geq \partial \phi_s \geq \phi_t > \partial \phi_t$. 

By (i), there exists $I_{t_1} \subseteq I$ such that $\partial \phi_{t_1} = \langle \phi_t \mid t \in I_{t_1} \rangle$. If $t_2 \in I_{t_1}$, then continue this series

$$\phi_s \geq \partial \phi_s \geq \phi_{t_1} > \partial \phi_{t_1} \geq \phi_{t_2} > \partial \phi_{t_2}.$$  

Continue this indefinitely, so we have the following descending series in $G$

$$(3) \quad \phi_{t_1} > \phi_{t_2} > \phi_{t_3} > \cdots .$$

Since $\phi$ satisfies DCC, the series in (3) must stabilize, say, at $\phi_{t_1}$. By (i), it follows that $\phi_{t_1} = 1$, and every possible choice in the above series (3) ends in the same way, with possibly different values for $r$. By induction for every $t \in I_s$, $\phi_t \in \mathcal{B}$, so $\phi_s \in \mathcal{B}$.

(ii) $\implies$ (i): Conversely, suppose for all $s \in M$, $\phi_s \in \mathcal{B}$. If there exists $s \in M$ such that $\phi_s$ does not satisfy (i), then, because $\phi$ satisfies DCC, there exists a minimal $\phi_s$ does that satisfy (i). Since $\phi_s \in \mathcal{B}$, it follows that there exist $B \in \bigcup_{s \geq s} \mathcal{B}$ such that $\partial \phi_s = \langle H \mid H \in B \rangle$. Because $\phi_s$ does not satisfy (i), there exists $\phi_t \in B$ such that $t \notin I$. There exists $m \geq 0$ such that $\phi_t \in \mathcal{B}_m$.

We will invoke the minimality of $\phi_s$ to show that $\phi_s$ actually satisfies (i); however, currently $\phi_s \geq \phi_t$, so we first rule out the equality. If $\phi_s = \phi_t$, then there exists $B' \subseteq \mathcal{B}_{m-1}$ such that $\partial \phi_t = \langle H \mid H \in B' \rangle$. Since $t \notin I$, it follows that $\partial \phi_t = \phi_t = \phi_s = \partial \phi_s$. Hence, $\phi_s \in \mathcal{B}_{m-1}$. Continuing this argument yields $\phi_s = 1$, which satisfies (i). Therefore, $\phi_s \neq \phi_t$.

Now, for all $\phi_t \in B$, $\phi_s > \phi_t$. By minimality of $\phi_s$, it follows that for all $\phi_t \in B$, there exists $I_t \subseteq I$ such that $\partial \phi_t = \langle \phi_u \mid u \in I_t \rangle$. If $\phi_t \in B$ and $t \notin I$, replace $\phi_t$ with the set $\{ \phi_u \mid u \in I_t \}$ as $\partial \phi_t = \phi_t$. Hence, $\phi_s$ satisfies (i). 

If $H$ is inert and $I = \{ s \in M : \phi_s = H \}$, then by Proposition 4.6, for all $s \in I$, $\phi_s = \partial \phi_s$. This is only a necessary condition and is not sufficient, c.f. Example 3.5: the group $G \in \text{im}(\phi)$ is always equal to its boundary but is not inert. Note that all of the subgroups in the image of the filter $\partial \phi$ in Example 4.1 are inert, so $\text{L}(\partial \phi) = 0$.

The following example will be revisited later, and seems like a more typical example of how inert subgroups can arise: when refining a filter with too many subgroups.

**Example 4.7.** Let

$$G = \left\{ \begin{array}{ccc} I_2 & a & 0 \\ & 0 & a \\ I_2 & b & c \\ & & 1 \end{array} \right\}. a, b, c, x, y \in \mathbb{F}_p \right\}.$$  

For $k \in \mathbb{F}_p$, let $X(k) = I_5 + kE_{15}$ and $Y(k) = I_5 + kE_{25}$. For each $k \in \{ 0, 1, \ldots, p-1 \}$ define $H_k = \langle X(1)Y(k) \rangle$. Each $H_1$ is normal since it is central.

Let $\gamma : \mathbb{N} \to 2^G$ be the lower central series of $G$, with $\gamma_0 = \gamma_1 = G$. For $M = (\mathbb{N}^{p+1}, \leq_d)$, ordered by the direct product ordering, and define a filter $\phi : M \to 2^G$ where $\phi_0 = G$ and for $t > 0$,

$$\phi_s = \begin{cases} \gamma_t & \text{if } s = te_1, \\ H_{k-2} & \text{if } s = te_k (k \geq 2), \\ 1 & \text{otherwise.} \end{cases}$$
The boundary filter is then $\partial \phi_0 = G$ and for $t > 0$,

$$\partial \phi_s = \begin{cases} 
\gamma_{t+1} & s = te_1, \\
H_{k-2} & s = te_k (k \geq 2), \\
1 & \text{otherwise}.
\end{cases}$$

Therefore, $L(\phi) \cong L(\gamma)$ as $\mathbb{F}_p$-vector spaces. Moreover, $L(\phi)$ and $L(\gamma)$ have the same Hilbert series.

In section 4.1 we remove the inertness from the subgroups. The process produces a new filter $\theta : M \rightarrow 2^G$ where $\theta_0 = G$ and for $t > 0$,

$$\theta_s = \begin{cases} 
\gamma_t & s = te_1, \\
H_{k-2} & s = e_k (k \geq 2), \\
1 & \text{otherwise}.
\end{cases}$$

The boundary filter then is then $\partial \theta_0 = G$ and for $t > 0$,

$$\theta_s = \begin{cases} 
\gamma_{t+1} & s = te_1, \\
1 & \text{otherwise}.
\end{cases}$$

It follows that $|L(\theta)| = p^{p+5} > p^5 = |G|$. \qed

**Remark 4.8.** All $H_i \vartriangleleft G$. We could have easily described an example of $G$ where all $H_i$ are characteristic in $G$. So even in the context of refining filters to make computing $\text{Aut}(\partial \phi_0)$ potentially simpler by computing $\text{Aut}(L(\phi))$ first, via $L(\phi)$, this issue of inertia must be addressed.

### 4.1. Refreshing filters.

In this section we show that for every filter $\phi$, there is a way to fix the inertness. Our method requires that $G$ be nilpotent as any filter of $\text{GL}(2, p)$, for example, will have inert subgroups, see Example 2.13. To do this, we localize to the indices of a particular inert subgroup and redefine the filter on these indices. There is a two-step process to accomplish this: first, apply the generation formula from Theorem 2.10, then a closure operation to force the order-reversing property.

Our immediate goal is just to fix the inertness of one subgroup. Suppose $\phi : M \rightarrow 2^G$ is a filter and $H \in \text{im}(\phi)$ is inert. Throughout this section, we fix the following notation. Let $I = \{ s \in M \mid \phi_s = H \}$, and let $J \subset I$ be defined such that

1. $J$ contains all the minimal elements of $I$, and
2. $(M - I) \cup J$ generates $M$.

Define a set of restricted partitions of $M$ as follows: if $s \in M$, then

$$\mathcal{R}(s) = \{(r_1, \ldots, r_k) \mid k \in \mathbb{N}, r_1 + \cdots + r_k = s, \ r_i \in (M - I) \cup J \}.$$  

For each $s \in M$, define

$$\nu_s = \prod_{r \in \mathcal{R}(s)} \lfloor \phi_r \rfloor,$$

where $\lfloor \phi_r \rfloor = [\phi_{r_1}, \ldots, \phi_{r_k}]$. Because $\phi$ is a filter $\nu_s \leq \phi_s$, for all $s \in M$. Observe that if $s \in (M - I) \cup J$, then $(s) \in \mathcal{R}(s)$, so $\nu_s \geq \phi_s$. Therefore, when $s \in (M - I) \cup J$, $\nu_s = \phi_s$.

In general, $\nu_s$ is not order-reversing, so define a function $\hat{\phi} : M \rightarrow 2^G$ such that

$$\hat{\phi}_s = \prod_{s \leq t} \nu_t = \prod_{s \leq t} \left( \prod_{r \in \mathcal{R}(s)} \lfloor \phi_r \rfloor \right).$$
We will prove that \( \hat{\phi} \) is a filter where \( \text{im}(\phi) \subseteq \text{im}(\hat{\phi}) \) and \( H \) is not inert.

The next lemma follows the spirit of [W, Lemma 3.4] by applying the Three Subgroups Lemma, c.f. [R, 5.1.10, p. 126].

**Lemma 4.9.** If \( s, t \in M \), then \( [\nu_s, \nu_t] \leq \nu_{s+t} \).

**Proof.** First, we consider the case when \( s, t \in M - I \). If \( s + t \notin I \), then the statement follows as \( \phi \) is a filter, so if \( s + t \in I \), then \((s, t) \in R(s + t)\). Therefore, \([\nu_s, \nu_t] = [\phi_s, \phi_t] \leq \nu_{s+t}\).

Suppose now that \( s, t \in I \). If \( s + t \notin I \), then \([\nu_s, \nu_t] \leq [\phi_s, \phi_t] = \nu_{s+t} \) as \( \phi \) is a filter and \( \nu_u \leq \phi_u \) for all \( u \in I \). Now consider the case when \( s + t \in I \). Suppose \( s \in R(s) \). If \( s = (s) \), then \( s \in J \), and therefore, \( \nu_s = \phi_s \). Hence, for all \( t \in R(t) \), \((s, t) = (s, t_1, \ldots, t_\ell) \in R(s + t) \), and

\[
[\phi_s, [\phi_t]] = [\phi_t, \phi_s] \leq \nu_{s+t}.
\]

Therefore, in the case where \((s) \in R(s)\),

\[
[\nu_s, \nu_t] = [\nu_t, \phi_s] \leq \nu_{s+t}.
\]

Now we proceed by induction on the size of the partition \( s = (s_1, \ldots, s_k) \in R(s) \). Let \( s' = (s_1, \ldots, s_{k-1}) \), and let \( A = [\phi_{s'}], B = \phi_{s_k}, \) and \( C = [\phi_t] \). Then

\[
[[\phi_s], \phi_t)] = [A, B, C].
\]

Since \((s, t) \in R(s + t)\), all permutations of \((s, t)\) are also contained in \( R(s + t) \). Hence, \((t_1, \ldots, t_\ell, s_k, s_1, \ldots, s_{k-1}) \in R(s + t) \). If \( t' = (t_1, \ldots, t_\ell, s_k) \), then by induction

\[
[B, C, A] = [C, B, A] = [\phi_{t'}, \phi_{s'}] \leq \nu_{s+t}.
\]

Although \(-s_k\) may not be contained \( M \), we let \( s - s_k \) denote \( s_1 + \cdots + s_{k-1} \). Again, by induction

\[
[C, A, B] \leq [\nu_{s-s_k+t}, \phi_{s_k}] \leq \nu_{s+t}.
\]

By the Three Subgroups Lemma,

\[
[[\phi_s], \phi_t)] = [A, B, C] \leq [B, C, A][C, A, B] \leq \nu_{s+t}.
\]

Therefore, in this case, \([\nu_s, \nu_t] \leq \nu_{s+t}\).

For the final case, suppose \( s \in I \) and \( t \in M - I \). This is similar to the base case above. If \( s \in R(s) \), then \((s, t) \in R(s + t)\), so

\[
[[\phi_s], \phi_t)] = [\phi_s, \phi_t] \leq \nu_{s+t}.
\]

Since \( \nu_t = \phi_t \), it follows that \([\nu_s, \nu_t] \leq \nu_{s+t} \). Therefore, the statement of the lemma follows. \( \square \)

Problems can arise due to the structure of the monoid: particularly, when a subset \( S \subseteq M - 0 \) forms a group under +. An element \( s \in M \) is **cancellative** if for all \( t, u \in M \), \( s + t = s + u \) implies \( t = u \). For the next theorem we need a weaker version of cancellative.

**Definition 4.10.** An element \( s \in M \) is **semi-cancellative** if \( s + t = 0 \) implies \( t = 0 \). Otherwise, \( s \) is a sink.

**Definition 4.11.** A filter is **progressive** if \( \phi_s \neq 1 \) implies that \( s \) is semi-cancellative.
Example 4.12. Consider the cyclic monoid $C_{3,5}$ for $r, s \in \mathbb{Z}^+$, see Section 2 for definitions. We identify the elements of $M = C_{3,5}$ by the smallest integer in the equivalence class, so as sets $M = \{0, \ldots, 7\}$. Because the index $(r = 3)$ is positive, $M$ is not a group, and so $M$ contains semi-cancellative elements and sinks. The elements $\{3, \ldots, 7\}$ are sinks and $\{0, 1, 2\}$ are semi-cancellative.

Define an ordering $\preceq$ on $C_{3,5}$ as follows. If $s \in \{0, 1, 2\}$ and $t \in C_{5,3}$ then $s \preceq t$ if and only if $s \leq t$. Let $G$ be a class 2 nilpotent group, and define a filter $\gamma : C_{3,5} \to 2^G$ given by the lower central series in $G$ with $\gamma_0 = G$. Let $\{H_3, H_4, \ldots, H_7\}$ be a collection of subgroups of $G$. Define a filter $\phi : C_{3,5} \to 2^G$ where

$$\phi_s = \begin{cases} \gamma_s & s \in \{0, 1, 2\}, \\ H_s & s \in \{3, \ldots, 7\}. \end{cases}$$

If $\phi \neq \gamma$, then $\gamma$ is a progressive filter while $\phi$ is not.

Observe that if $s \in M$ is cancellative, then $s$ is semi-cancellative, and a filter is progressive if for all sinks $s \in M$, $\phi_s = 1$. Now we are ready to prove that we can refresh inert subgroups and construct filters with more vigor.

Theorem 4.13. Suppose $\phi : M \to 2^G$ is an progressive filter satisfying DCC and that $G$ is nilpotent. If $H \in \text{im}(\phi)$ is a minimal inert subgroup, then there exists a filter satisfying DCC where $\text{im}(\phi) \subseteq \text{im}(\hat{\phi})$ and $H$ is not inert.

**Proof.** Let $s, t \in M$. By Lemma 4.9,

$$[\hat{\phi}_s, \hat{\phi}_t] = \prod_{s \preceq u \preceq t} \prod_{u \preceq v} [\nu_u, \nu_v] \leq \prod_{s \preceq u \preceq t} \prod_{u \preceq v} \nu_{u+v} \leq \hat{\phi}_{s+t}.$$ 

If $s \preceq t$, then

$$\hat{\phi}_s = \prod_{s \preceq u} \nu_u \geq \prod_{s \preceq u} \nu_u = \hat{\phi}_t.$$ 

Therefore, $\hat{\phi}$ is a filter.

For each $s \in (M - I) \cup J$, $\nu_s = \phi_s$ since $\phi$ is a filter and $(s) \in R(s)$. Moreover, $\hat{\phi}_s = \nu_s = \phi_s$. Therefore, $\text{im}(\phi) \subseteq \text{im}(\hat{\phi})$.

Now we show that $H$ is not inert in $\hat{\phi}$. Let $s \in J$ be a maximal element. By definition, $\hat{\phi}_s = H$. Let $t \in M - 0$; we will show that $\hat{\phi}_{s+t}$ is not inert and therefore, $H$ is not inert. Since all semi-cancellative elements in $M$ evaluate to $1 \in \text{im}(\phi)$, it follows that $s \neq s+t$. If $s+t \notin I$, then $\hat{\phi}_{s+t} = \phi_{s+t} \neq H$ by definition. Furthermore, $H = \phi_s > \phi_{s+t}$, so by minimality of $H$, $\hat{\phi}_{s+t}$ is not inert. Suppose, on the other hand, $s+t \in I$. Because $G$ is nilpotent and $H$ is minimal, $\phi_{s+t}$ is not inert. Therefore, if $\hat{\phi}_{s+t} = \partial\hat{\phi}_{s+t}$, then by Proposition 4.6, there exists $I_{s+t} \subseteq I = \{u \in M \mid \partial\hat{\phi}_u \neq \hat{\phi}_u\}$ such that $\partial\hat{\phi}_{s+t} = \langle \hat{\phi}_u \mid u \in I_{s+t} \rangle$, and if $\hat{\phi}_{s+t} \neq \partial\hat{\phi}_{s+t}$, then $s+t \in I$. Hence, there exists $I_s \in I$ such that $\partial\hat{\phi}_s = \langle \hat{\phi}_u \mid u \in I_s \rangle$. By Proposition 4.6, $H$ is not inert. 

We can remove all inertness from a progressive filter from the bottom up by iterating Theorem 4.13.

**Corollary 4.14.** If $\phi : M \to 2^G$ is a progressive filter satisfying DCC and $G$ is nilpotent, then there exists a filter $\hat{\phi} : M \to 2^G$ with no inert subgroups such that $\text{im}(\phi) \subseteq \text{im}(\hat{\phi})$. 

4.2. **Proof of Theorem 4.4.** By Corollary 4.14, if the monoid structure is nice enough, then we can fix the inertness of the filter. On the other hand if \( \phi \) is not progressive, we can still fix the filter but over a different monoid: we move to the free commutative monoid \( \mathbb{N}^d \), which eliminates sinks. Care is needed when constructing a partial order that is compatible with the partial order on \( M \). We let \( \prec \) denote the strict partial order: when \( s \preceq t \) and \( s \neq t \).

**Proof of Theorem 4.4.** Since \( M \) is finitely generated, there exists \( d \in \mathbb{Z} \) and a congruence \( \sim \) of \( \mathbb{N}^d \) such that \( \mathbb{N}^d / \sim \cong M \). Let \( \mu : \mathbb{N}^d \to M \) be the induced surjection. Let \( \preceq_+ \) be the algebraic partial order on \( \mathbb{N}^d \). Define a ordering \( \preceq' \) on \( \mathbb{N}^d \) as follows. For \( s, t \in \mathbb{N}^d \),

\[
 s \preceq' t \iff (\mu(s) \prec \mu(t)) \lor (\mu(s) = \mu(t) \land s \preceq_+ t).
\]

Since \( \mu \) is a monoid homomorphism, \( \preceq \) a partial ordering of \( M \), and \( \preceq_+ \) a partial order of \( \mathbb{N}^d \), it follows that \( \preceq' \) is a partial order for \( \mathbb{N}^d \).

Set \( M' = (\mathbb{N}^d, \preceq') \), and define a function \( \bar{\phi} : M' \to 2^G \) such that \( \bar{\phi}_s = \phi_{\mu(s)} \).

Since \( \mu \) is a monoid homomorphism respecting the partial orders, it follows that \( \bar{\phi} \) is a filter. Moreover, by construction, \( \text{im}(\phi) = \text{im}\left(\bar{\phi}\right) \). Since every element of \( M' \) is semi-cancellative, it follows that \( \bar{\phi} \) is progressive. Now apply Corollary 4.14 to \( \bar{\phi} \) for the desired result. \( \square \)

4.3. **Finitely generated groups.** Throughout, we assume that \( G \) is finitely generated, and we make the following assumptions on filters \( \phi : M \to 2^G \):

- (a) \( \phi \) contains no inert subgroups,
- (b) \( \phi \) has DCC, and
- (c) \( 1 \in \text{im}(\phi) \).

Under these assumptions, then, we prove that \( L(\phi) \) maps onto \( \partial \phi_0 \), a basic requirement if we are to construct automorphisms from automorphisms of \( L(\phi) \). These assumptions force the composition factors of \( \partial \phi_0 \) to be contained in the composition factors of \( L(\phi) \). Since \( L(\phi) \) is an abelian group and \( L_0 = 0 \), it follows that \( \partial \phi_0 \) must be solvable.

**Lemma 4.15.** If \( \phi : M \to 2^G \) is a filter, then \( \partial \phi_0 \) is solvable.

**Proof.** Assume via induction that every \( \phi_s \in \mathcal{B}_{n-1} \) is solvable. If \( \phi_s \in \mathcal{B}_n \), then by definition there exists \( B \subseteq \mathcal{B}_{n-1} \) such that \( \partial \phi_s = \langle H \mid H \in B \rangle \). By induction, \( \partial \phi_s \) is a product of solvable normal subgroups, \( \partial \phi_s \) is solvable. Since \( \phi_s \) is an abelian-by-solvable group, \( \phi_s \) is solvable. Since \( \partial \phi_0 \) is a product of solvable normal subgroups, the lemma follows. \( \square \)

If we remove the assumption that \( 1 \in \text{im}(\phi) \), then \( \partial \phi_0 / \bigcap_{s \in M} \phi_s \) is solvable.

**Definition 4.16.** A subset \( \mathcal{Y} \subseteq L \) is a graded basis if

- (i) for all \( y \in \mathcal{Y} \), there exists \( s \in M \) such that \( y \in L_s \) and
- (ii) for all \( s \in M \), the subset \( \mathcal{Y} \cap L_s \) is a basis for \( L_s \).

**Lemma 4.17.** Suppose \( \phi : M \to 2^G \) is a filter and \( \mathcal{Y} \) a graded basis for \( L(\phi) \). If \( \mathcal{X} \) is a pre-image of \( \mathcal{Y} \) in \( G \), then for all \( s \in M - 0 \), \( \langle \phi_s \cap \mathcal{X} \rangle = \phi_s \), and \( \mathcal{X} \) is weakly-filtered by \( \partial \phi \).
Proof. Suppose \( \phi_s \in \mathcal{B}_n - \mathcal{B}_{n-1} \) for \( n \geq 1 \). If \( \partial \phi_s = \phi_s \), then by induction \( \langle \phi_s \cap \mathcal{X} \rangle = \phi_s \), so assume \( \phi_s \neq \partial \phi_s \). Since \( \mathcal{Y} \) is a graded basis of \( L(\phi) \), there exists a unique subset of \( \mathcal{Y} \) that is a basis for \( L_s(\phi) \), where \( s \neq 0 \). Let \( \mathcal{X}_s \) be a preimage of this unique subset generating \( L_s(\phi) \). Therefore, \( \langle \phi_s \cap \mathcal{X} \rangle = \langle \mathcal{X}_s \cup (\partial \phi_s \cap \mathcal{X}) \rangle \).

By induction,

\[
\langle \partial \phi_s \cap \mathcal{X} \rangle = \langle \{ \phi_u \mid u \in B \} \cap \mathcal{X} \rangle \geq \langle \{ \phi_u \mid u \in B \} \rangle = \langle \{ \phi_u \mid u \in B \} \rangle = \partial \phi_s.
\]

Therefore, for all \( \phi_s \in \mathcal{B} \), where \( s \neq 0 \), \( \langle \phi_s \cap \mathcal{X} \rangle = \phi_s \). Since \( \phi \) has no inert subgroups, the lemma follows. \( \square \)

From the proof of Lemma 4.17, if \( \phi_0 = \partial \phi_0 \), then \( \mathcal{X} \) is weakly-filtered by \( \phi \).

The above two lemmas basically prove that \( \mathcal{X} \) contains a polycyclic generating set, provided \( \partial \phi_0 \) is polycyclic.

**Proposition 4.18.** Suppose \( \phi : M \to 2^G \) is a filter where every subgroup of \( \partial \phi_0 \) is finitely generated. If \( \mathcal{Y} \) is a graded basis for \( L(\phi) \), then a preimage \( \mathcal{X} \) contains a pcgs for \( \partial \phi_0 \).

Proof. By Lemma 4.15, \( \partial \phi_0 \) is solvable, and since every subgroup of \( \partial \phi_0 \) is finitely generated, by Proposition 2.7, \( \partial \phi_0 \) is polycyclic.

By Proposition 4.6, for all \( s \in M \), there exists \( I_s \subseteq \mathcal{I} = \{ t \in M \mid \phi_t \neq \partial \phi_t \} \) such that \( \partial \phi_s = \langle \phi_t \mid t \in I_s \rangle \). Let \( B_s = \langle \partial \phi_t \mid t \in I_s \rangle \). From the filter properties it follows that \( \partial \phi_s / B_s \) is abelian. By Lemma 4.17,

\[
\langle B_s \cap \mathcal{X} \rangle \geq \langle \partial \phi_t \cap \mathcal{X} \mid t \in I_s \rangle = B_s.
\]

Define \( \mathcal{X}_s = \{ x \in \mathcal{X} \mid x \in \partial \phi_s - B_s \} \), so \( \langle \mathcal{X}_s \rangle B_s = \partial \phi_s \). Since every \( x \in \mathcal{X} \) comes from a graded basis \( \mathcal{Y} \), there exists a subset of \( \mathcal{X} \) that is a pcgs of \( \partial \phi_s / B_s \). Since \( B_s = \langle \partial \phi_t \mid t \in I_s \rangle \), for each \( t \in I_s \), there exists \( I_t \subseteq \mathcal{I} \) such that \( \partial \phi_t = \langle \phi_u \mid u \in I_t \rangle \). Thus, by induction there exists a pcgs in \( \mathcal{X} \) for \( B_s \) and, hence, for \( \partial \phi_s \). \( \square \)

### 4.4. Proof of Theorem A

By Proposition 4.18, there is little work left to do to prove Theorem A. We define a map \( \pi : L(\phi) \to \partial \phi_0 \), since the image of a basis contains a pcgs of \( \partial \phi_0 \), the map is surjective.

**Proof of Theorem 4.5.** Let \( \mathcal{Y} \) be a graded basis of \( L(\phi) \). Assign some total order to \( \mathcal{Y} \) so that \( \mathcal{Y} \) is an ordered basis for \( L(\phi) \). For each \( x \in L(\phi) \) and \( y \in \mathcal{Y} \), there exists unique \( k_y \) such that

\[
x = \sum_{y \in \mathcal{Y}} k_y y,
\]

where the sum runs through \( \mathcal{Y} \) in order. For each \( y \in \mathcal{Y} \), let \( x_y \in \mathcal{X} \) be the corresponding preimage of \( y \). Define a function \( \pi : L(\phi) \to G \) such that

\[
(7) \quad x = \sum_{y \in \mathcal{Y}} k_y y \to \prod_{y \in \mathcal{Y}} x_y^{k_y},
\]

where the product runs through \( \mathcal{Y} \) in ascending order. By Proposition 4.18, \( \{x_y \mid y \in \mathcal{Y}\} \) contains a pcgs of \( \partial \phi_0 \), so \( \pi \) is surjective. \( \square \)
5. Faithful filters

In this section, we impose one more property on our filters so that the sets \( L(\phi) \) and \( \partial \phi_0 \) are in bijection. Recall the subjection \( \pi : L(\phi) \to G \) from Theorem 4.5, c.f. equation (7). The main issue for \( \pi : L(\phi) \to G \) not being injective comes down to the fact that \( (\phi_a - \partial \phi_a) \cap (\phi_t - \partial \phi_t) \) might be nonempty. So there exists \( x \in L_s(\phi) \) and \( y \in L_t(\phi) \) that get mapped to the same image in \( G \). This is problematic if we want to extract group automorphisms from Lie automorphisms. If there is such a collision, where \( g = \pi(x) = \pi(y) \) but \( x \neq y \), then constructing an automorphism of \( G \) from \( \delta \) requires a choice of where \( g \) gets mapped. We address this issue with the following definitions.

It is convenient to assume that \( \phi_0 = \partial \phi_0 \). The proceeding theorems still apply without this equality. However, as \( \phi_0 / \partial \phi_0 \) is not a homogeneous component of \( L(\phi) \) (as it may not even be solvable), pre-images of graded bases of \( L(\phi) \) cannot, in general, be filtered by \( \phi \). We are more concerned with \( \partial \phi_0 \) than we are with \( \phi_0 \), so we assume that \( \phi_0 = \partial \phi_0 \).

**Definition 5.1.** A filter \( \phi : M \to 2^G \) is full if a preimage of a graded basis of \( L(\phi) \) is filtered by \( \phi \).

In general, \( \phi \) is full if a preimage of a graded basis of \( L(\phi) \) induces a generating set filtered by \( \phi \): adjoin a generating set for \( \phi_0 / \partial \phi_0 \) to a preimage of the graded basis.

**Definition 5.2.** A generating set \( \mathcal{X} \subseteq G \) is faithful if for each \( x \in \mathcal{X} \), there exists a unique \( s \in M \) such that \( x \in \phi_s - \partial \phi_s \). If such a generating set \( \mathcal{X} \) is also filtered, then \( \mathcal{X} \) is faithfully filtered by \( \phi \), and in this case, we say that \( \phi \) is a faithful filter.

We summarize the properties we developed in the following definition.

**Definition 5.3.** A filter \( \phi : M \to 2^G \) is fully faithful if it faithful, has no inert subgroups, satisfies DCC, and \( \phi_0 = \partial \phi_0 \).

We prove the following theorems in this section.

**Theorem 5.4.** Assume \( \phi : M \to 2^G \) is a fully faithful filter. If \( \mathcal{X} \subseteq G \) is filtered by \( \phi \), then

(i) \( \phi \) is full and

(ii) every pre-image of every graded basis of \( L(\phi) \) is filtered by \( \phi \).

**Theorem 5.5** (Theorem B). Suppose \( \phi : M \to 2^G \) is a fully faithful filter, and assume that every subgroup of \( G \) is finitely generated. If \( \mathcal{X} \) is faithfully filtered by \( \phi \), then there exists a bijection from \( L(\phi) \) to \( \partial \phi_0 \) that induces a bijection between the set of graded bases of \( L(\phi) \) and the set of preimages of \( \partial \phi_0 \) that are filtered by \( \phi \).

The next lemma is fundamental to the proofs for the above theorems. In essence, if \( \mathcal{X} \) is faithfully filtered by \( \phi \), then the structure of \( \phi \) is constrained so that every element \( x \) contained in \( \phi_s \cap \phi_t \) must also be contained in \( \partial \phi_s \cap \partial \phi_t \). From the faithful property, \( x \) must be contained in either \( \partial \phi_s \) or \( \partial \phi_t \). Say \( x \) is contained in, say, \( \phi_s \cap \partial \phi_t \), but since \( \phi \) has no inert subgroups, \( \partial \phi_t \) is generated by subgroups \( \phi_u \) strictly contained in \( \phi_t \). Since \( \mathcal{X} \) is faithfully filtered, \( x \) must be contained in each \( \partial \phi_u \). Continue this argument and eventually we reach the trivial subgroup as \( \phi \) satisfies DCC. Recall that \( \| \) denotes when two elements of a partially-ordered set are incomparable.
Lemma 5.6. Suppose \( \phi : M \to 2^G \) is a fully faithful filter. If \( S \subseteq M \) where every distinct pair \( s, t \in S \), \( \phi_s \parallel \phi_t \), then
\[
\bigcap_{s \in S} \phi_s = \bigcap_{s \in S} \partial \phi_s.
\]

Proof. We suppose that \( \bigcap_{s \in S} \phi_s \neq 1 \). Since \( X \) is filtered, there exists
\[
x \in \left( \bigcap_{s \in S} \phi_s \right) \cap X = \bigcap_{s \in S} (\phi_s \cap X).
\]
Since \( X \) is faithful, \( x \in \bigcup_{s \in S} \partial \phi_s \subseteq \prod_{s \in S} \partial \phi_s \). Without loss of generality, suppose \( x \notin \partial \phi_s \) and \( x \in \prod_{t \in S - s} \partial \phi_t \). Since \( \phi \) contains no inert subgroups, by Proposition 4.6, for all \( u \in M \) there exists \( I_u \subseteq \mathcal{I} = \{ v \in M \mid \partial \phi_v \neq \phi_c \} \) such that
\[
\partial \phi_u = \langle \phi_v \mid v \in I_u \rangle.
\]
In particular, for all \( t \in S - s \), there exists \( I_t \subseteq \mathcal{I} \) such that \( \partial \phi_t = \langle \phi_v \mid v \in I_t \rangle \). By Proposition 3.6,
\[
\partial \phi_t \cap X = \langle \phi_v \mid v \in I_t \rangle \cap X = \bigcup_{v \in I_t} (\phi_v \cap X).
\]
Since \( x \in \left( \prod_{t \in S - s} \partial \phi_t \right) \cap X \), for each \( t \in S - s \), there exists \( u \in I_t \), such that \( x \in \phi_u \). There exists \( I_u \subseteq \mathcal{I} \) such that \( \partial \phi_u = \langle \phi_v \mid v \in I_u \rangle \) and for all \( v \in I_u \), \( \phi_u > \phi_v \). Since \( X \) is faithful and since \( x \in \phi_u - \partial \phi_u \), it follows that \( x \in \partial \phi_u \).

Therefore, by the same reasoning as before, there exists \( v \in I_u \) such that \( x \in \phi_v \). Continue this ad infinitum.

By Proposition 4.6, this stops at \( \mathcal{B}_0 = \{1\} \). This implies that \( x = 1 \), so \( x \in \partial \phi_s \), a contradiction. Therefore, if \( x \in \bigcap_{s \in S} \phi_s \), then \( x \in \bigcap_{s \in S} \partial \phi_s \). Since \( X \) is filtered,
\[
\bigcap_{s \in S} \phi_s = \left( \bigcap_{s \in S} \phi_s \right) \cap X \leq \bigcap_{s \in S} \partial \phi_s.
\]
Since \( \phi_s \geq \partial \phi_s \), the other containment follows. \( \square \)

From Lemma 5.6, we are led to the following definition concerning filters— independent of generating sets.

Definition 5.7. A filter is faithful if for all \( S \subseteq M \), where every distinct pair \( s, t \in S \), \( \phi_s \parallel \phi_t \), implies that \( \bigcap_{s \in S} \phi_s = \bigcap_{s \in S} \partial \phi_s \).

Note then that if \( \phi \) is a faithful filter and \( X \) is filtered by \( \phi \), then \( X \) is faithfully filtered by \( \phi \). From the above lemma, faithful filters are highly structured filters. We show that faithful implies full, provided there exists \( X \) that is filtered by \( \phi \). The basic argument is that the image of \( X \) in \( L(\phi) \) will contain a graded basis \( X \) of \( L(\phi) \), and because \( X \) is filtered, a pre-image of \( X \) will be filtered as well.

Lemma 5.8. Suppose \( \phi : M \to 2^G \) is a faithful filter with no inert subgroups, satisfying DCC. If \( X \subseteq G \) is filtered by \( \phi \), then \( \phi \) is full.

Proof. We show that \( X \) induces a graded basis of \( L(\phi) \). Since \( X \) is faithful, there exists a function \( \omega : X \to M \) such that if \( x \in X \), then \( x \in \phi_{\omega(x)} - \partial \phi_{\omega(x)} \). Let \( \mathcal{Z} = \{ \partial \phi_{\omega(x)} x \mid x \in X \} \). Since \( X \) is filtered by \( \phi \), \( X \) is filtered by \( \partial \phi \) by Theorem 3.1. Therefore there exists \( X_s \subseteq X \) such that
\[
\langle \phi_s \cap X \rangle = \langle X_s \cup (\partial \phi_s \cap X) \rangle = \phi_s.
\]
Furthermore, the image of $\mathcal{X}_s$ in $L(\phi)$ spans $L_s(\phi)$. Since $\mathcal{X}$ is faithful, this holds for all $s \in M - 0$. Therefore, $Z$ spans $L(\phi)$.

Let $\mathcal{Y} \subseteq Z$ be a basis for $L(\phi)$, and let $W \subseteq X$ correspond to $\mathcal{Y}$. Since $W$ is a preimage of a basis $\mathcal{Y}$, by Lemma 4.17, $W$ is weakly-filtered by $\phi$.

Let $S \subseteq M$, and by definition, $(\prod_{s \in S} \phi_s) \cap W \supseteq \bigcup_{s \in S}(\phi_s \cap W)$. Hence, $(\prod_{s \in S} \phi_s) \cap W = \emptyset$ if, and only if, $\prod_{s \in S} \phi_s = 1$. Let $w \in (\prod_{s \in S} \phi_s) \cap W$. Since

$$w \in \left( \prod_{s \in S} \phi_s \right) \cap X = \bigcup_{s \in S}(\phi_s \cap X),$$

it follows that there exists $s \in S$ such that $w \in \phi_s \cap X$. Therefore,

$$(8) \quad \left( \prod_{s \in S} \phi_s \right) \cap W = \bigcup_{s \in S}(\phi_s \cap W).$$

Finally, we show that for $S \subseteq M$,

$$(9) \quad \bigcap_{s \in S} \phi_s = \left( \bigcap_{s \in S} (\phi_s \cap W) \right),$$

by induction up the sequence of $\mathfrak{B}_0 \subseteq \mathfrak{B}_1 \subseteq \cdots$. As $W$ is weakly-filtered by $\phi$, we assume without loss of generality that for each distinct pair $s, t \in S$, $\phi_s \parallel \phi_t$.

By Lemma 5.6, $\bigcap_{s \in S} \phi_s = \bigcap_{s \in S} \partial \phi_s$. Assume by induction that for all $S \subseteq M$ with $\phi_s \in \mathfrak{B}_n$, the equation in (9) is satisfied, and $S \subseteq M$ such that for all $s \in S$, $\phi_s \in \mathfrak{B}_{n+1}$. For each $s \in S$, there exists $B_s \subseteq \mathfrak{B}_n$ such that $\partial \phi_s = (H \mid H \in B_s)$.

Since $B_s \subseteq \mathfrak{B}_n$,

$$\left\langle \bigcap_{s \in S} \phi_s \cap W \right\rangle = \left\langle \bigcap_{s \in S} \partial \phi_s \cap W \right\rangle \quad \text{(Lemma 5.6)}$$

$$= \left\langle \bigcap_{s \in S} \left( \prod_{H \in B_s} H \right) \cap W \right\rangle \quad \text{(inert free)}$$

$$= \left\langle \bigcap_{s \in S} \bigcup_{H \in B_s} (H \cap W) \right\rangle \quad \text{(equation (8))}$$

$$= \bigcap_{s \in S} \bigcup_{H \in B_s} (H \cap W) \quad \text{(induction and Proposition 3.6)}$$

$$= \bigcap_{s \in S} \partial \phi_s. \quad \text{(weakly-filtered)}$$

Equation (8) follows from the fact that if $\phi_s \in \mathfrak{B}$, then there exists $B \subseteq \bigcup_{i \geq 0} \mathfrak{B}_i$ such that $\partial \phi_s = (H \mid H \in B)$. Since $\phi$ has no inert subgroups every $S \subseteq M$ satisfies the property that for all $s \in S$, $\phi_s \in \mathfrak{B}$. Hence, $W$ is filtered by $\phi$.

**5.1. Proof of Theorem 5.4.** Now we are ready to prove that if $\mathcal{X}$ is faithfully filtered by $\phi : M \to 2^G$, then every graded basis of $L(\phi)$ induces a faithfully filtered generating set of $G$. This can be turned into an algorithm to decide if there exists a generating set $\mathcal{X}$ that is filtered by the faithful filter $\phi$.

The following proof uses Noetherian induction, going up the sequence

$$\{1\} = \mathfrak{B}_0 \subseteq \mathfrak{B}_1 \subseteq \cdots.$$
The idea is to assume that a pre-image $\mathcal{X}$ of an arbitrary graded basis $\mathcal{Y}$ of $L(\phi)$ is filtered by $\phi$ up to some $\mathfrak{B}_n$. This is certainly true for $\mathfrak{B}_0$. Then for every group $\phi_s \in \mathfrak{B}_{n+1}$, there exists $B \in \mathfrak{B}_n$ such that $\partial\phi_s = (H \mid H \in B)$. Thus, $\partial\phi_s$ is handled by induction, and all that is left are quotients $\phi_s/\partial\phi_s = L_s(\phi)$.

**Proof of Theorem 5.4.** For (i), apply Lemma 5.8. For (ii), let $\mathcal{Y}$ be the graded basis whose pre-image $\mathcal{X}$ is filtered by $\phi$ (using Lemma 5.8), and suppose $Z$ is some other graded basis of $L(\phi)$. By Lemma 4.17, a pre-image $\mathcal{W}$ of $Z$ is weakly-filtered by $\phi$.

Suppose that for all $B \subseteq \mathfrak{B}_n$,

$$
\bigcap_{H \in B} H = \left( \bigcap_{H \in B} (H \cap \mathcal{W}) \right) \quad \text{and} \quad \left( \prod_{H \in B} H \right) \cap \mathcal{W} = \bigcup_{H \in B} (H \cap \mathcal{W}).
$$

In some sense, $\mathcal{W}$ is filtered by $\phi$ up to $\mathfrak{B}_n$. Let $B \subseteq \mathfrak{B}_{n+1}$, and set $\partial B = \{\partial\phi_u \mid \phi_u \in B\} \subseteq \mathfrak{B}_n$. First we show that

$$
\left( \prod_{H \in B} H \right) \cap \mathcal{W} \subseteq \bigcup_{H \in B} (H \cap \mathcal{W})
$$

as the reverse containment is already satisfied. Since

$$
\left( \prod_{H \in B} H \right) \cap \mathcal{W} = \left( \left( \prod_{H \in B} H - \prod_{K \in \partial B} K \right) \cap \mathcal{W} \right) \cup \left( \left( \prod_{K \in \partial B} K \right) \cap \mathcal{W} \right),
$$

by induction, the inequality in (10) follows if

$$
\left( \prod_{H \in B} H - \prod_{K \in \partial B} K \right) \cap \mathcal{W} \subseteq \bigcup_{H \in B} (H \cap \mathcal{W}).
$$

Suppose there exists $w \in \left( \prod_{H \in B} H - \prod_{K \in \partial B} K \right) \cap \mathcal{W}$. Let $\overline{w}$ denote the corresponding basis vector in $Z$. Since $Z$ is a graded basis, there exists a unique $s \in M$ such that $\overline{w} \in L_s(\phi)$. Therefore, $w \in \phi_s - \partial\phi_s$ and $\phi_s \in B - \partial B$. Hence, $w \in \bigcup_{H \in B} (H \cap \mathcal{W})$, and the inequality in (10) follows.

Now we prove the other equation holds to show that $\mathcal{W}$ is filtered. There exists a subset $C \subseteq B \subseteq \mathfrak{B}_{n+1}$ such that for every distinct pair $H, K \in C$, $H \parallel K$ and

$$
\bigcap_{H \in B} H = \bigcap_{H \in C} H.
$$

By Lemma 5.6, if $\partial C = \{\partial\phi_u \mid \phi_u \in C\}$, then by induction

$$
\bigcap_{H \in B} H = \bigcap_{H \in C} H = \bigcap_{H \in \partial C} H = \left( \bigcap_{H \in \partial C} H \cap \mathcal{W} \right) = \left( \bigcap_{H \in B} H \cap \mathcal{W} \right).
$$

Therefore, $\mathcal{W}$ is filtered on every $\mathfrak{B}_n$, and so $\mathcal{W}$ is filtered by $\phi$. \qed

The following example illustrates one instance where a filter cannot have an associated $\mathcal{X}$ that is faithful. This problem comes up naturally and is dealt with in a more comprehensive example later on (Example 6.1). It is not known if a method exists in general to address the issue in Example 5.9, see Question 3 in Section 7.
Example 5.9. Let $G$ be the Heisenberg group over the finite field $K$. Let $\gamma : \mathbb{N} \to 2^G$ be the lower central series, so

$$G = \gamma_0 = \gamma_1 = \left\{ \begin{bmatrix} 1 & * & * \\ 1 & 1 & * \\ 1 & 1 & 1 \end{bmatrix} \right\}, \quad \gamma_2 = Z(G) = \left\{ \begin{bmatrix} 1 & 0 & * \\ 1 & 0 & 0 \end{bmatrix} \right\},$$

and $\gamma_i = 1$ for $i \geq 3$. If

$$\mathcal{X} = \left\{ \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \right\},$$

then $\mathcal{X}$ is faithfully filtered by $\gamma$. Moreover, $\gamma$ is fully faithful.

Let $Y = \{(0, 0), (1, 0), (0, 1)\} \subseteq \mathbb{N}^2$, and define $\pi : Y \to 2^G$ to be the constant function where $\text{im}(\pi) = \{G\}$. Then the closure, $\phi = \pi : \mathbb{N}^2 \to 2^G$, is realized as

$$\phi_{(i, j)} = \gamma_{i+j}.$$  

Since $\text{im}(\phi) = \text{im}(\gamma)$ and since $\mathcal{X}$ is filtered by $\gamma$, $\mathcal{X}$ is also filtered by $\phi$. However, $\mathcal{X}$ is not faithful:

$$G = \phi_{(1,0)} = \phi_{(0,1)}, \quad Z(G) = \phi_{(2,0)} = \phi_{(1,1)} = \phi_{(0,2)},$$

$$1 = \phi_{(3,0)} = \phi_{(2,1)} = \phi_{(1,2)} = \phi_{(0,3)} = \ldots.$$  

Since $G$ is finite, $|L(\phi)| = |G/\gamma_2|^2|\gamma_2|^3$.

5.2. Proof of Theorem B. To prove Theorem B, we first apply Theorem A to produce a filter $\phi : M \to 2^G$ with no inert subgroups. This also yields a surjection from $L(\phi)$ to $\partial\phi_0$. We use the fact that elements of a polycyclic group have a unique normal word with respect to a pcgs—this gives us injectivity.

Proof of Theorem 5.5. Let $\mathcal{Y}$ be a graded basis for $L(\phi)$. By Theorem 5.4, if $\mathcal{X}$ is a pre-image of $\mathcal{Y}$, then $\mathcal{X}$ is filtered by $\phi$. From the proof of Theorem 4.5, the map $\pi : L(\phi) \to \partial\phi_0$ given by

$$x = \sum_{y \in \mathcal{Y}} k_y y \mapsto \prod_{y \in \mathcal{Y}} x_y^{k_y}$$

is a surjection.

By Proposition 4.18, $\mathcal{X}$ contains a pcgs of $\partial\phi_0$. Suppose for some $x \in \mathcal{X}$, the set $\mathcal{X} - x$ still contains a pcgs for $\partial\phi_0$, say $\{x_1, \ldots, x_n\} \subseteq \mathcal{X} - x$ is a pcgs. Then there exists some unique normal word for $x$:

$$x = x_1^{e_1} \cdots x_n^{e_n},$$

for integers $e_i$. This implies that there exists $S \subseteq M$ such that

$$x = x_1^{e_1} \cdots x_n^{e_n} \in \prod_{t \in S} \phi_t.$$

Because $\mathcal{X}$ is faithfully filtered by $\phi$, there exists a unique $s \in M$ such that $x \in \phi_s - \partial\phi_s$. By construction, the image of $\mathcal{Y}$ in $L(\phi)$ is a basis vector in $\mathcal{Y}$, but by the uniqueness in equation (11), there exists a linear combination in vectors $\mathcal{Y}$, a contradiction. Therefore, $\mathcal{X} - x$ cannot contain a pcgs of $\partial\phi_0$. Hence, $\mathcal{X}$ is a pcgs for $\partial\phi_0$. Because every element in $\partial\phi_0$ is expressed by a unique normal word in $\mathcal{X}$, it follows that $\pi$ is injective. 

$\square$
The crux of Theorem B is not the bijection between $\partial_0\phi$ and $L(\phi)$, though that is necessary for our purposes. The main point is actually the induced bijection between graded bases of $L(\phi)$ and pegs of $\partial_0\phi$ filtered by $\phi$. This allows us to get a well-defined bijection on $\partial_0\phi$ from a linear transformation on $L(\phi)$ as graded bases $L(\phi)$ induce pegs of $G$.

6. Some examples

Example 6.1. Let $G$ be the group of $d \times d$ upper unitriangular matrices over the ring $K$. The terms of the lower central series can be easily visualized

$$G = \begin{bmatrix} 1 & * & * & * \\ 1 & * & * \\ 1 & * \\ 1 \end{bmatrix}, \quad \gamma_2 = \begin{bmatrix} 1 & 0 & * \\ 1 & 0 \end{bmatrix},$$

$$\gamma_3 = \begin{bmatrix} 1 & 0 & 0 & * \\ 1 & 0 & 0 \\ 1 & 0 \end{bmatrix}, \quad \gamma_4 = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}.$$

Here $*$ denotes that every element of $K$ can be an entry. We define three more characteristic subgroups

$$H = \begin{bmatrix} 1 & * & * & * \\ 1 & 0 & * \\ 1 & 0 \\ 1 \end{bmatrix}, \quad K = \begin{bmatrix} 1 & 0 & * & * \\ 1 & 0 & * \\ 1 \end{bmatrix}, \quad L = \begin{bmatrix} 1 & 0 & 0 & * \\ 1 & 0 \end{bmatrix}.$$

Note that $H$ has class 3 and $K$ has class 2.

Let $M = C_{4,1} \times C_{3,1}$, and set $e_1 = (1, 0)$ and $e_2 = (0, 1)$. We let $M$ be ordered by the direct product ordering. Define a function $\pi : \{0, e_1, e_2\} \to 2^G$, where $\pi_0 = G$, $\pi_{e_1} = H$ and $\pi_{e_2} = K$. Set $\phi = \pi : M \to 2^G$; the image of $\phi$ is plotted in Figure 2a. Notice there exists no generating set faithfully filtered by $\phi$ because $\phi(2, 2) = \phi(3, 1) \neq \partial_0(2, 2) = \partial_0(3, 1)$. This can be fixed by altering $\phi$ slightly; see Figure 2b. That is, define $\lambda : M \to 2^G$ where for all $s \in M - \{(2, 2), (3, 2)\}$, $\lambda_s = \phi_s$, $\lambda_{(2, 2)} = \gamma_3$, and $\lambda_{(3, 2)} = \gamma_4$. Suppose $E_{ij}$ is a $5 \times 5$ matrix over $K$ with 1 in the $(i,j)$ entry and 0 elsewhere. If $X = \{I_5 + E_{ij} \mid 1 \leq i < j \leq 5\}$, then $X$ is strongly-filtered by $\lambda$.

Example 6.2. We consider a group examined in [ELGO, Section 12.1] and [M1, Section 5]. For a fixed prime $p$, we define a $p$-group $G$ by a power-commutator presentation, where all trivial commutators are omitted

$$G = \langle g_1, \ldots, g_{13} \mid [g_{10}, g_6] = g_{11}, [g_{10}, g_7] = g_{12}, [g_2, g_1] = [g_4, g_3] = [g_6, g_5] = [g_8, g_7] = [g_{10}, g_9] = g_{13}, \text{exponent } p \rangle.$$

In [M1], we defined a filter on $\mathbb{N}^2$, with a total ordering; here, we define the same filter, except over $M = C_{3,1} \times C_{5,1}$, totally-ordered by the lexicographical ordering. Denote this filter by $\phi$. 

Observe from the presentation that $G$ has class 2 and $\gamma_2 = \langle g_{11}, g_{12}, g_{13} \rangle$. The following subgroups are characteristic

$$J_1 = \langle g_1, \ldots, g_9, \gamma_2 \rangle,$$
$$J_2 = \langle g_1, \ldots, g_5, g_8, g_9, \gamma_2 \rangle,$$
$$J_3 = \langle g_5, g_8, g_9, \gamma_2 \rangle,$$
$$J_4 = \langle g_9, \gamma_2 \rangle,$$
$$H = \langle g_{13} \rangle.$$  

The details of this are given in [M1]. The image of $\phi$ produces the following characteristic series

$$G > J_1 > J_2 > J_3 > J_4 > \gamma_2 > H > 1.$$  

Using techniques developed in [BW], the tensor $\circ : G/\gamma_2 \times G/\gamma_2 \rightarrow \gamma_2$ yields more characteristic subgroups. In fact, as $\ast$-algebras, the adjoint ring has the following Taft decomposition

$$\text{Adj}(\circ) \cong J \times (\mathbf{X}(2, p) \oplus \mathbf{S}(4, p)),$$

where the simple $\ast$-algebras $\mathbf{X}(n, p)$ and $\mathbf{S}(n, p)$ are defined in [BW]. The simple $\ast$-algebras $\mathbf{X}(2, p)$ and $\mathbf{S}(4, p)$ determine new characteristic subgroups:

$$E = \langle g_5, \ldots, g_{10}, \gamma_2 \rangle,$$
$$S = \langle g_1, \ldots, g_4, \gamma_2 \rangle.$$  

Let $M' = M \times \mathbb{N} \times \mathbb{N}$, where $M'$ is ordered by the direct product ordering. Set $T = \{ (m, 0, 0) \mid m \in M \} \cup \{ e_2, e_3 \}$, and define a function $\pi : T \rightarrow 2^G$, where $\pi_{(m,0,0)} = \phi_m$, $\pi_{e_2} = E$, and $\pi_{e_3} = S$. Let $\lambda = \pi$. If $\mathcal{X} = \{ g_1, \ldots, g_{13} \}$, then $\mathcal{X}$ is filtered by $\lambda$. We cannot easily plot the refinement of $\lambda$ as we did in Figures 2a and 2b, so we display the lattice of characteristic subgroups in Figure 3.

7. Closing remarks and questions

There are many directions to go from the work here. One major direction is to develop efficient algorithms for constructing filters with the various properties from Sections 3–5. It seems unlikely that there exists an efficient algorithm to produce a faithful filter from a given arbitrary filter. If there was, then computing the
intersection of normal subgroups would be Turing reducible to such an algorithm. We explicitly state a few questions in computational directions.

**Question 1.** Is there an polynomial-time algorithm that returns a filter with no inert subgroups, given a filter \( \phi : M \to 2^G \) for a nilpotent group \( G \)?

In order to address Question 1, it seems as though a polynomial-time algorithm for closures of prefilters is required. On the other hand, a polynomial-time algorithm for closures is certainly essential for efficiently refining filters, so it is of interest on its own.

**Question 2.** Is there an algorithm that, given a prefilter \( \pi \), returns \( \pi \) in polynomial time?

An answer to Question 2 has applications to computing automorphism groups, but as we have seen the definition of a filter is not very restrictive. In [M1], we give an affirmative answer in the case when the monoid is totally-ordered. Currently it seems that all prefilters come from refining a filter. Of course, we should not limit ourselves only to this case, but presumably Question 2 becomes easier when there exists a (faithfully) filtered generating set \( X \subseteq G \) for the filter we are refining. Does there exist such an algorithm for prefilters in this case?

It seems like there is an efficient algorithm that decides if, for a given filter \( \phi : M \to 2^G \), there exists \( X \subseteq G \) that is faithfully filtered by \( \phi \). By Theorem 5.4, it seems sufficient to test if a pre-image of a graded basis is faithfully filtered. However, it is not known if there is an efficient algorithm that returns a filter \( \phi' \) and a set \( X \) faithfully filtered by \( \phi' \), given a filter \( \phi \), even in favorable conditions.

**Definition 7.1.** A filter \( \rho : M' \to 2^G \) refines a filter \( \phi : M \to 2^G \) if \( \text{im}(\phi) \subseteq \text{im}(\rho) \) and for all \( s \in M' \), there exists \( t \in M \) such that \( \partial \phi_t \leq \rho_s \leq \phi_t \).

For the next question, suppose \( \phi : M \to 2^G \) is a filter where \( L_s \) is elementary abelian for all \( s \neq 0 \), and in addition, there exists \( X_\phi \subseteq G \) that is faithfully filter by \( \phi \). If \( G \) is a \( p \)-group, the lower exponent-\( p \) series is one example \( \eta : \mathbb{N} \to 2^G \).
Question 3. If $\rho : M' \to 2G$ refines $\phi$, then does there exist a polynomial-time algorithm that returns $X_\rho$ that is faithfully filtered by $\rho$?

Asserting that $\rho$ refines $\phi$ means that intersections between $\rho_s$ and $\rho_t$ can be computed in polynomial time. Thus, when $\rho$ is faithful, it seems that such an $X_\rho$ can be efficiently computed. It may be the case that $\rho$ is not faithful to begin with, and in this case can an optimal compromise be obtained? Presently, such a compromise is not well-defined and may never be. Regardless, if $\rho$ is not faithful there seems to be two methods to fix this issue: (1) refine $\rho$ by including appropriate intersections and (2) remove obstructions. Are there general ways to address these two procedures?

Acknowledgements

The author is indebted to J. B. Wilson for encouraging this research and providing endless feedback along the way. Thanks also to P. A. Brooksbank, A. Hulpke, and T. Penttila for many helpful discussions.

References

[B] T. S. Blyth, Lattices and ordered algebraic structures, Universitext, Springer-Verlag London, Ltd., London, 2005. MR2126425

[BOW] Peter A. Brooksbank, E.A. O’Brien, and James B. Wilson, Testing isomorphism of graded algebras, preprint. arXiv:1708.08873.

[BW] Peter A. Brooksbank and James B. Wilson, Computing isometry groups of Hermitian maps, Trans. Amer. Math. Soc. 364 (2012), no. 4, 1975–1996. MR2869196

[CELG] John J. Cannon, Bettina Eick, and Charles R. Leedham-Green, Special polycyclic generating sequences for finite soluble groups, J. Symbolic Comput. 38 (2004), no. 5, 1445–1460. MR2168723

[CH] J. J. Cannon and D. F. Holt, Automorphism group computation and isomorphism testing in finite groups, J. Symbolic Comput. 35 (2003), no. 3, 241–267.

[ELGO] Bettina Eick, C. R. Leedham-Green, and E. A. O’Brien, Constructing automorphism groups of $p$-groups, Comm. Algebra 30 (2002), no. 5, 2271–2295. MR1904637

[EW] Bettina Eick and Charles R. B. Wright, Computing subgroups by exhibition in finite soluble groups, J. Symbolic Comput. 33 (2002), no. 2, 129–143. MR1879377

[G] P. A. Grillet, Commutative semigroups, Advances in Mathematics (Dordrecht), vol. 2, Kluwer Academic Publishers, Dordrecht, 2001. MR2017849

[HN] George Havas and M. F. Newman, Application of computers to questions like those of Burnside, Burnside groups (Proc. Workshop, Univ. Bielefeld, Bielefeld, 1977), Lecture Notes in Math., vol. 806, Springer, Berlin, 1980, pp. 211–230.

[H1] Graham Higman, Groups and rings having automorphisms without non-trivial fixed elements, J. London Math. Soc. 32 (1957), 321–334. MR0089204

[H2] , Lie ring methods in the theory of finite nilpotent groups, Proc. Internat. Congress Math. 1958, Cambridge Univ. Press, New York, 1960, pp. 307–312. MR0116050

[K1] E. I. Khukhro, Lie rings and groups that admit an almost regular automorphism of prime order, Mat. Sb. 181 (1990), no. 9, 1207–1219 (Russian); English transl., Math. USSR-Sb. 71 (1992), no. 1, 51–63. MR1085151

[K2] , $p$-automorphisms of finite $p$-groups, London Mathematical Society Lecture Note Series, vol. 246, Cambridge University Press, Cambridge, 1998. MR1615819

[L] Michel Lazard, Sur les groupes nilpotents et les anneaux de Lie, Ann. Sci. Ecole Norm. Sup. (3) 71 (1954), 101–190 (French). MR0088496 (19,529b)

[M1] Joshua Maglione, Efficient characteristic refinements for finite groups. part 2, J. Symbolic Comput. 80 (2017), no. part 2, 511–520. MR3574524

[M2] , Longer nilpotent series for classical unipotent subgroups, J. Group Theory 18 (2015), no. 4, 569–585. MR3365818

[M3] , Most small $p$-groups have an automorphism of order 2, Arch. Math. (Basel) 108 (2017), no. 3, 225–232. MR3614700
[M4] Wilhelm Magnus, *A connection between the Baker-Hausdorff formula and a problem of Burnside*, Ann. of Math. (2) 52 (1950), 111–126. MR0038964

[M5] , *Über Gruppen und zugeordnete Liesche Ringe*, J. Reine Angew. Math. 182 (1940), 142–149 (German). MR0003411

[NO] M. F. Newman and E. A. O'Brien, *Application of computers to questions like those of Burnside. II*, Internat. J. Algebra Comput. 6 (1996), no. 5, 593–605. MR1419133

[O] E. A. O'Brien, *Isomorphism testing for p-groups*, J. Symbolic Comput. 17 (1994), no. 2, 131, 133–147.

[R] Derek J. S. Robinson, *A course in the theory of groups*, 2nd ed., Graduate Texts in Mathematics, vol. 80, Springer-Verlag, New York, 1996. MR1357169

[S1] Ákos Seress, *Permutation group algorithms*, Cambridge Tracts in Mathematics, vol. 152, Cambridge University Press, Cambridge, 2003. MR1970241

[S2] Charles C. Sims, *Computation with finitely presented groups*, Encyclopedia of Mathematics and its Applications, vol. 48, Cambridge University Press, Cambridge, 1994. MR1267733

[W] James B. Wilson, *More characteristic subgroups, Lie rings, and isomorphism tests for p-groups*, J. Group Theory 16 (2013), no. 6, 875–897. MR3198722

Department of Mathematical Sciences, Kent State University, Kent, OH 44242, USA
E-mail address: jmaglion@math.kent.edu