Sáez-Ballester and Einstein-massless-scalar systems are one and the same theory!

Israel Quiros\textsuperscript{1,}\textsuperscript{a} and Francisco Antonio Horta-Rangel\textsuperscript{1,}\textsuperscript{b}

\textsuperscript{1}Dpto. Ingeniería Civil, División de Ingeniería, Universidad de Guanajuato, Gto., México.

Here we discuss a topic that comes up more often than expected: A same theory or theoretical model arises in two different presentations which are assumed to be actually different theories so that these are independently developed. Sometimes this leads to an unwanted doubling of the results. In this paper we illustrate this issue with the example of two apparently different gravitational theories: (i) the (minimally coupled) Einstein-massless-scalar system and (ii) the Sáez-Ballester theory. We demonstrate that the latter is not a scalar-tensor theory of gravity, as widely acknowledged. Moreover, Sáez-Ballester theory is identified with the Einstein-massless-scalar theory. As illustrations of this identification we show that several known solutions of Sáez-Ballester theory are also solutions of the Einstein-massless-scalar system and vice versa. Cosmological arguments are also considered. In particular, a dynamical systems-based demonstration of the dynamical equivalence between these theories is given. This paper is written in a rather pedagogical way so that it is suitable for teachers, undergraduate and postgraduate students from physics and mathematics disciplines.

I. INTRODUCTION

Scalar fields have played a major role in the gravitational theories as well as in the standard model of particles (SMP). Within the framework of the metric theories of gravitation \cite{1}, the so called scalar-tensor theories of gravity \cite{2,3} have received much attention as viable alternatives to general relativity, in the search for solutions to outstanding problems of the latter theory \cite{6,7}: dark matter (DM) and dark energy (DE) problems, among others. The close connection of scalar-tensor theories (STT) and the $f(R)$ modified theories has been investigated as well \cite{8}.

Scalar-tensor theories of gravity are distinguished by the property that, in addition to the graviton, the scalar field is also a carrier of the gravitational interactions. In general one have to differentiate its use as an additional – perhaps exotic – matter field in general relativity, from its use as one of the carriers of the gravitational interactions of matter itself. In this regard, for instance, theories of the kind,\textsuperscript{1}

\[
S_{GR} = \frac{1}{2} \int d^4x \sqrt{-g} \left[ R - (\partial \varphi)^2 \right], \tag{1}
\]

where $\varphi$ is a canonical massless scalar field and we have adopted that $(\partial \varphi)^2 \equiv g^{\mu\nu} \partial_\mu \varphi \partial_\nu \varphi$, are not scalar-tensor theories. In this case what we have is GR plus a matter field whose Lagrangian $\mathcal{L}_\varphi = - (\partial \varphi)^2/2$. As a matter of fact the equations of motion (EOM) derived from the action (\ref{eq:1}) are the Einstein’s equations with a massless scalar field as matter source,

\[
G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = T^{(\varphi)}_{\mu\nu},
\]

\[
\nabla^\lambda T^{(\varphi)}_{\lambda\mu} = 0 \Rightarrow \nabla^2 \varphi = 0, \tag{2}
\]

where $\nabla^2 \equiv g^{\mu\nu} \nabla_\mu \nabla_\nu$ is the d’Alembert operator and the stress-energy tensor of the scalar field is given by,

\[
T^{(\varphi)}_{\mu\nu} = - \frac{2\delta \left( \sqrt{-g} \mathcal{L}_\varphi \right)}{\sqrt{-g} g^{\mu\nu}} \partial_\mu \varphi \partial_\nu \varphi - \frac{1}{2} g_{\mu\nu} (\partial \varphi)^2. \tag{3}
\]

This is also known as Einstein-massless-scalar (EMS) system\textsuperscript{2} which has been studied in detail \cite{9,31}.

The EMS system is not a scalar-tensor theory. In contrast, theories given by the following action,

\[
S_{STT} = \int d^4x \sqrt{-g} \left[ \phi R - \frac{\omega(\phi)}{\phi} (\partial \phi)^2 \right], \tag{4}
\]

where $\omega(\phi)$ is the coupling function, are indeed scalar-tensor theories. The main difference of action (\ref{eq:4}) with (\ref{eq:1}) is in the non minimal coupling between the scalar field and the curvature, through the term $\phi R$. This coupling entails that the metric and the scalar field both propagate the gravitational interactions (this is why these are called as scalar-tensor theories of gravity in the first place.) The resulting effective gravitational coupling $16\pi G_{\text{eff}}(\phi) = \phi^{-1}$ is a point dependent quantity. In consequence the scalar field determines the strength of the gravitational interactions point by point. For the gravitational constant measured in Cavendish type experiments one gets \cite{2,33},

\[
8\pi G_{\text{cav}} = \frac{1}{\phi_0} \left[ 3 + 2\omega(\phi_0) \right], \tag{5}
\]

\textsuperscript{1}In this paper we use units where the speed of light and the reduced Planck constant $c = \hbar = 1$, while the Newton’s constant $G_N = (8\pi)^{-1}$.

\textsuperscript{a}Electronic address: iquiros@fisica.ugto.mx

\textsuperscript{b}Electronic address: anthort@ugto.mx

\textsuperscript{2}Here we shall use, interchangeably, the whole name “minimally coupled EMS” system and the abbreviated one “EMS” system.
where $\phi_0 = \phi(t_0)$ is the scalar field evaluated at present cosmic time. Notice that only in the limit $\omega(\phi) \to \infty$, under the normalization where $\phi_0 = 1$, the measured gravitational constant coincides with the Newton’s constant: $G_{\text{cm}} \to G_N = (8\pi)^{-1}$.

Having in mind these facts, one can easily identify a STT, i.e., one can differentiate these theories from theories where the scalar field is non-gravitational and acts only as a matter source in Einstein’s equations (2). But not always STT-s are correctly classified. This is the case of the so called Sáez-Ballester theory (SBT) [33, 34]. In [32] it was demonstrated that SBT is not a scalar-tensor theory but it is just the minimally coupled EMS theory. In spite of this demonstration, several recent works on the subject have been published where this demonstration is ignored and SBT is considered as an STT [33, 34]. For this reason we feel that further discussion on SBT is required in order to make clear that this is not a STT but it is just the EMS system!

Aim of the present paper will be to set the above discussion on solid mathematical basis and in as much pedagogical way as we can, so that young and future scientists working in gravitational physics, would be able to find equivalences between seemingly different gravitational theories, so as to evade any doubling of the results. For this purpose in Sec. II we shall identify SBT with the EMS theory. Then, in Sec. III we shall discuss on several known solutions of Sáez-Ballester theory and we shall show that these are solutions of minimally coupled EMS system and viceversa. In Sec. IV we shall discuss on cosmological models that are based in SBT/EMS theories and we shall show that the SBT-based cosmological models can not explain neither the DM nor the DE problems, as incorrectly claimed [33, 34]. In Sec. V, we shall show that these are solutions of minimally coupled EMS system and viceversa. In Sec. VI we shall discuss on cosmological models that are based in SBT/EMS theories and we shall show why the SBT-based cosmological models can not explain neither the DM nor the DE problems, as incorrectly claimed [33, 34]. In Sec. VI where we include a brief study of the asymptotic dynamics of these cosmological models through the use of the dynamical systems tools, we show that SBT and EMS asymptotic cosmological dynamics are the same. Discussion of the results and brief conclusions are given in Sec. VI. In this paper we use the following signature of the metric: $(+, +, +, +)$.

3. The confusion often comes from the indiscriminate use of the conformal transformation of the metric, $g_{\mu\nu} \to \Omega^2 g_{\mu\nu}, g^{\mu\nu} \to \Omega^{-2} g^{\mu\nu},$

where the positive function $\Omega$ is called as conformal factor, and its geometrical/physical interpretation. For a detailed discussion of this controversial issue see section 6 of Ref. [33].

4. We sympathize with the complains in [33] on the lack of efforts on finding equivalences between seemingly different theories of modified gravity. According to the authors of this bibliographic reference, the lack of efforts in the mentioned direction makes the landscape of related theories larger than what it really is, and makes its classification confusing and misleading.

II. IDENTIFICATION OF SÁEZ-BALLESTER THEORY AND EINSTEIN-MASSLESS-SCALAR SYSTEM

Sáez-Ballester theory is given by the following action [33, 34]:

$$S_{SBT} = \int d^4x \sqrt{-g} \left[ R - \omega(\phi)(\partial\phi)^2 \right],$$

where $\phi$ is the SBT scalar field while $\omega$ and $n$ are free constant parameters. The SBT EOM that can be derived from the above action read,

$$G_{\mu\nu} = \omega(\phi) \left[ \partial_\mu \phi \partial_\nu \phi - \frac{1}{2} g_{\mu\nu} (\partial\phi)^2 \right],$$

$$2\omega(\phi) \nabla^2 \phi + n(\phi)^n (\partial\phi)^2 = 0.$$  

According to the authors of the original paper [33], the coupling of the scalar field to the metric would lead to more important departures from GR than the $G_N$-varying theories (strictly speaking the STT-s.)

That SBT is just GR with a minimally coupled massless scalar field as a source of Einstein’s equations – in simpler words; minimally coupled EMS system – has been demonstrated in [3] and also in [32]. Although the demonstration is straightforward, here we include it again since it has been ignored in several papers that have appeared after publication of [3, 32] (see, for instance, Refs. [33, 44] to quote a few of them.)

Let us perform the following innocuous redefinition of the SBT scalar field,

$$\varphi = \frac{2\sqrt{\omega}}{n + 2} \phi^{n+2}.$$  

After this redefinition, the action (10) is transformed into the action (11), which corresponds to minimally coupled EMS system. In the same way, under the redefinition [33] the SBT EOM (7) transforms into the EMS EOM (2).

In the bibliography one also encounters works that are based in the so called “generalized SBT,” where the scalar field’s kinetic term in the action (6) is replaced by the more general term [40]:

$$S_{SBT}^{gen} = \int d^4x \sqrt{-g} \left[ R - F(\phi)(\partial\phi)^2 \right],$$

where $F(\phi)$ is an arbitrary function. We should notice that in this case the replacement $\varphi = \int \sqrt{F(\phi)} d\phi$ transforms (11) into the EMS action (1). This suffices to show that SBT must be identified with EMS theory, contrary to expectation in [33]. Based on the latter identification, below we shall look for solutions of the EMS theory on the basis of existing solutions of SBT and viceversa.
Those who are familiar with the k-essence theories\textsuperscript{47–52} might think that (10) belongs in this class of gravitational theories. The action of k-essence is given by (for simplicity of writing we use the following notation $X \equiv - (\partial \phi)^2 / 2$),

$$S_K = \int d^4 x \sqrt{-g} \left[ \frac{1}{2} R + K(\phi, X) \right],$$

where $K(\phi, X)$ is a function of the scalar field and of its kinetic energy density. In the bibliography it is mostly used the following decomposition: $K(\phi, X) = K_1(\phi)K_2(X)$. Although k-essence is not a scalar-tensor theory since the scalar field does not modify neither the gravitational coupling nor the measured value of the gravitational constant, it may have cosmological implications differing from those of Einstein-massless-scalar theory, since perturbations of the k-essence field propagate at a sound speed squared $c_s^2$ different from the one obtained in the EMS system ($c_s^2 = 1$)\textsuperscript{52}.

As discussed in\textsuperscript{18}, the linear case $K_2(X) = aX + b$ in\textsuperscript{10}, where $a$ and $b$ are free constants, corresponds to GR with a minimally coupled self-interacting scalar field. Only for non-linear functions $K_2(X)$ can we speak of a k-essence field. The SBT action\textsuperscript{13} corresponds, precisely, to the linear case $K_2(X) = X$ ($K_1(\phi) = \omega \phi^\omega$) so that it is GR with a minimally coupled (massless) scalar field, also known as EMS system.

Before we end up this section we want to mention a famous case where ignorance of such a simple scalar field redefinition as in Eq. (8), led to unphysical (incorrect) bounds on the parameters of the theory, which led, in turn, to erroneous physical estimates. In\textsuperscript{53}, in order to explain the present stage of accelerated expansion of the Universe, a self-interacting Einstein-scalar model was proposed where the kinetic energy density of the scalar field entered with the wrong sign. It was dubbed phantom DE. The obvious problem with this model is the instability due to the negative sing of the energy density, which can be avoided only if the instability time scale is greater than the age of the Universe. In\textsuperscript{53} arguments were given in support of this possibility. One of the analyzed possibilities yielding appropriate constraints was based on an interaction (effective) Lagrangian $\mathcal{L}_{\text{eff}} \propto \phi (\partial \phi)^2$. As shown in\textsuperscript{54}, this interaction is an artifact of a noncanonical formulation. It is removed if in $\mathcal{L}_{\text{eff}}$ make the innocuous replacement $\phi \to 4\phi^{3/2} / 9 \Rightarrow \phi (\partial \phi)^2 \to (\partial \phi)^2$.

### III. LOCAL SOLUTIONS

Local spherically symmetric solutions of the EMS system have been found\textsuperscript{10,11,17,18}. All of these solutions are really the same but expressed in terms of different coordinates. This has been demonstrated in\textsuperscript{23} for the solutions found in Refs.\textsuperscript{11,17} and in\textsuperscript{20} for the solutions found in Refs.\textsuperscript{10,11}.

In\textsuperscript{18}, in particular, the static, spherically symmetric solution to (2) is found to be,

$$ds^2 = - \left( 1 - \frac{2\eta}{r} \right)^{\frac{\alpha}{\beta}} dt^2 + \left( 1 - \frac{2\eta}{r} \right)^{-\frac{\alpha}{\beta}} dr^2 + \left( 1 - \frac{2\eta}{r} \right)^{1 - \frac{\alpha}{\beta}} r^2 d\Omega^2,$$

$$\varphi(r) = \frac{\sigma}{\sqrt{2\eta}} \ln \left( 1 - \frac{2\eta}{r} \right),$$

where we use spherical coordinates, $(t, r, \theta, \varphi)$, $d\Omega^2 = d\theta^2 + \sin^2 \theta d\varphi^2$, $\eta = \sqrt{m^2 + \sigma^2}$ (m and $\sigma$ are free constants) and

$$R = \left( 1 - \frac{2\eta}{r} \right)^{\frac{\alpha m}{\beta \sigma}} r,$$

is the standard radial coordinate. In this case the event horizon at $r = 2\eta$ shrinks to a point, thus preventing the formation of a black hole\textsuperscript{18}.

If we substitute the static, spherically symmetric metric,

$$ds^2 = - e^{\gamma(r)} dt^2 + e^{-\gamma(r)} dr^2 + e^{\beta(r)} r^2 d\Omega^2,$$

into the SBT EOM (7), one gets the same solution for the line element than in (11), while the SBT scalar field is given by,

$$\phi(r) = \left[ \frac{(n + 2)\alpha}{2 \sqrt{2 n \eta}} \right]^{\frac{1}{\beta + 2}} \ln \left( 1 - \frac{2\eta}{r} \right).$$

This can be found as well by directly substituting (8) into (11). Notice that the new free parameters $\omega$ and $n$ play no role in the solution for the line-element. Hence, the physical (also geometrical) results are just the same as in\textsuperscript{18}.

#### A. Non-static spherically symmetric solutions

The non-static, spherically symmetric solution of the EMS system (2) was investigated in\textsuperscript{21}. The solution is given by,

$$ds^2 = (at + b) \left[ - f^2(r) dt^2 + f^{-2}(r) dr^2 \right] + R^2 d\Omega^2,$$

$$\varphi(t, r) = \pm 2\sqrt{\pi} \ln \left[ d (at + b)^{\sqrt{\pi}} \left( 1 - 2c / r \right)^{\sqrt{\pi}} \right].$$

where $a$, $b$, $c$ and $d$ are free constants, $\alpha = \pm \sqrt{3}/2$, and
\[ f^2(r) = \left(1 - \frac{2c}{r}\right)\alpha, \]
\[ R^2 = R^2(t, r) = (at + b) \left(1 - \frac{2c}{r}\right)^{1-\alpha} r^2. \]

Although this solution does not shed light on the scalar field collapse problem in asymptotically flat space (the solution is not asymptotically flat), it provides an example of spacetimes with evolving apparent horizons [21]. We can perform the redefinition [8] to find the corresponding solution of the SBT system [7], but this is a futile exercise since, as we have demonstrated, the SBT is one and the same as the EMS system.

### B. Other local solutions

There are found in the bibliography wormhole solutions of EMS as well [55]. In the latter bibliographic reference Bronnikov-type wormhole is investigated. This wormhole solution is possible thanks to a small departure from the Ellis wormhole. The wormhole solution is possible thanks to a small departure from the Ellis wormhole. The wormhole solution is given by [55],

\[ ds^2 = -h(r)dt^2 + h^{-1}(r)dr^2 + R^2(r)d\Omega^2, \]
\[ \varphi(r) = \frac{\sqrt{2q}}{\sqrt{q^2 - M^2}} \ln f(r), \]

where

\[ h(r) = f^{-\frac{2M}{\sqrt{q^2 - M^2}}}(r), \]
\[ R^2(r) = (r^2 + q^2 - M^2) f^{-\frac{2M}{\sqrt{q^2 - M^2}}}(r), \]
\[ f(r) = \exp \left[ \arctan \left( \frac{r}{\sqrt{q^2 - M^2}} \right) \right]. \]

In the above equations \( q \) and \( M \) are integration constants. When \( M = 0 \), the above solution corresponds to the Ellis wormhole. The wormhole [15] connects two asymptotic Minkowski spacetimes with different values of the speed of light, so that the wormhole connects two different worlds.

Through using the redefinition [8], one can bring the above wormhole solution of the EMS system into the corresponding wormhole solution of the SBT theory. As a matter of fact we can do that with any solution of the EMS theory and also one can bring back any solution of S´aez-Ballester theory into the corresponding solution of the EMS theory. Hence, with the help of the innocuous scalar field redefinition [8] one can construct a “dictionary” of solutions of either SBT or EMS. This, however, will be a futile exercise since, as already shown, both are one and the same theory.

There are many other known solutions of the SBT theory, for instance Bianchi type solutions [56–60], as well as of the EMS theory, such as Petrov type [61] and rotating solutions [62], etc. So that one may “translate” these solutions to the EMS system and to the SBT theory, respectively, without difficulty.

### IV. FRW COSMOLOGY

One of the main physical implications of SBT was to (seemingly) take account of the missing-matter problem [33], presently known as the DM problem. Today we know that S´aez-Ballester theory can not explain neither the DM nor the DE problems [63 [65].

In order to show why this theory can not explain these problems, let us write the EOM in terms of the Friedmann-Robertson-Walker (FRW) metric with flat spatial sections,

\[ ds^2 = -dt^2 + a^2(t)\delta_{ij}dx^idx^j, \]

where \( a(t) \) is the dimensionless scale factor and \( t \) is the cosmic time. In place of the SBT EOM (7) we shall write the simpler and completely equivalent EMS EOM (2),

\[ 3H^2 = \frac{1}{2} \dot{\varphi}^2, \]
\[ 2\dot{H} = -\ddot{\varphi}, \]
\[ \ddot{\varphi} + 3H\dot{\varphi} = 0, \]

where \( H \equiv \dot{a}/a \) is the Hubble parameter, the dot accounts for derivative with respect to the cosmic time and only two of the above equations are independent of each other. Straightforward integration of equation (19) yields,\(^5\)

\[ \dot{\varphi} = \sqrt{2k^2 \rho_K^2} \Rightarrow \rho_K = \frac{\dot{\varphi}^2}{2} = \frac{k^4}{a^6}, \]

where \( k^2 \) is an integration constant. Hence, the kinetic energy density of the scalar field \( \rho_K \propto a^{-6} \) dies off much faster than the radiation \( \rho_r \propto a^{-4} \) and, obviously, much faster than CDM energy density \( \rho_m \propto a^{-3} \). Hence, \( \varphi \) may have played a role at early times through replacing the matter bigbang by a stronger stiff-matter dominated bigbang, but not at late time (this includes our present stage of the cosmic expansion.)

\(^5\) Due to absence of a potential (self-interacting) term, the scalar field \( \varphi \) behaves as stiff matter fluid.
Let us, for completeness, to present a general cosmological solution of EMS/SBT system in the presence of a matter component of the cosmic fluid characterized by energy density $\rho_m$ and barotropic pressure $p_m = (\gamma - 1)\rho_m$, where $\gamma$ is the barotropic index of the fluid. The latter cosmological parameter is related with the equation of state (EOS) parameter $w$ of the fluid: $\gamma = w + 1$. In this case the EMS EOM read,

$$3H^2 = \rho_m + \frac{1}{2} \dot{\varphi}^2, \quad (21)$$

$$2\dot{H} = -\gamma \rho_m - \dot{\varphi}^2, \quad (22)$$

$$\ddot{\varphi} + 3H \dot{\varphi} = 0, \quad (23)$$

$$\dot{\rho}_m + 3H(\rho_m + p_m) = 0, \quad (24)$$

where the matter fluid continuity equation $\rho_m$ has been included. Integration of this last equations leads to

$$\int a^3 dv, \quad (26)$$

If one replaces the cosmic time in this equation by the new variable $v$,

$$t = \int a^3 dv, \quad (26)$$

one can integrate $a(v)$ in quadratures to obtain that,

$$a(v) = a_0 \sinh \frac{\sqrt{3}}{\sqrt{\gamma - 2}} \left[ \frac{3(\gamma - 2)k^2}{2\sqrt{3}} (v - v_0) \right], \quad (27)$$

where $v_0$ is an integration constant and,

$$a_0 \equiv \left[ \frac{M^4}{k^4} \right]^{\frac{1}{\gamma - 2}}.$$

The scale factor in $a(v)$ can be written in terms of the cosmic time. Actually, by substituting $a(v)$ from (27) back into (26) and performing the integration, one gets $t = t(v)$. Then one finds the inverse $v = v(t)$ and substitutes in (27). The latter is the general solution of (25).

**V. DYNAMICAL SYSTEMS-BASED DEMONSTRATION**

In spite of the apparent simplicity of the EOMs $\rho_m$–$p_m$ (same for the system (36)–(38) below) as with any system of non-linear second-order differential equations, it is a very difficult task to find exact solutions. Even when an analytic solution can be found it will not be unique but just one in a large set of them. In this case the dynamical systems tools are very useful. These very simple tools give us the possibility to correlate such important concepts like past and future attractors (also saddle equilibrium points) in the phase space, with generic solutions of the cosmological EOMs without the need to analytically solve them.\(^6\)

The basic idea of the application of the dynamical systems in the cosmological framework is to map the original set of differential equations which constitute the cosmological EOMs onto an equivalent phase space spanned by appropriate phase variables. Then the original cosmological EOMs are traded by an autonomous system of ordinary differential equations (ASODE) on the phase space variables. What matters are the critical points (also stationary or equilibrium points) of the obtained ASODE and the study of their stability properties. These (isolated or spiral) critical points can be either source (past attractors) or sink (future attractors) or saddle equilibrium points. One then goes back to the original equations and finds which generic solutions of the cosmological EOMs these critical points in the phase space correspond to. These solutions will describe the asymptotic dynamics of the given cosmological model.

The idea is to demonstrate that the SBT and the EMS system share the same phase space properties, so that their asymptotic dynamics is the same. This will be a much more solid demonstration of the dynamical equivalence of both theories than just looking for particular classes of solutions, since the critical points in the phase space correspond to those classes of solutions which, in a sense, are “preferred” by the cosmological EOMs, i. e., those which decide the past and future (also the intermediate) asymptotic behavior.

**A. Asymptotic dynamics of the EMS system**

Let us start by looking for adequate phase space variables in the case of the EMS system, which is described by the EOMs (21)–(24). In this case it is well-known that an adequate bounded variable could be

$$x \equiv \frac{\dot{\varphi}}{\sqrt{6H}}, \quad -1 \leq x \leq 1,$$

where, since we are focused on expanding Universe exclusively, then we shall consider nonegative $H \geq 0$. This means that negative $x$-s amount to decaying scalar field $\dot{\varphi} < 0$, while positive $x$-s entail growing $\dot{\varphi}$-s: $\dot{\varphi} > 0$.

In terms of the $x$ variable the cosmological EOMs can be written in the following way. The Friedmann equation (21) amounts to the following constraint:

\(^6\) For a compact pedagogical introduction to the application of the dynamical system tools in cosmology, specifically in scalar field models, see Ref. [67].
\[ \Omega_m = 1 - x^2, \quad \text{(29)} \]

where \( \Omega_m \equiv \rho_m/3H^2 \) is the normalized (dimensionless) matter energy density. The cosmological EOM (22) is written as,

\[ 2 \frac{\dot{H}}{H^2} = -3\gamma \Omega_m - 6x^2, \]

which, if take into account the Friedmann constraint (29), can be written either as,

\[ 2 \frac{\dot{H}}{H^2} = -3\gamma - 3(2 - \gamma)x^2, \quad \text{(30)} \]

or as,

\[ 2 \frac{\dot{H}}{H^2} = -6 + 3(2 - \gamma)\Omega_m. \quad \text{(31)} \]

Meanwhile, the scalar field’s EOM (23) is written as,

\[ \frac{\dot{\varphi}}{H^2} = -3\sqrt{6}x, \quad \text{(32)} \]

and the continuity equation (24) can be written in the following way:

\[ \frac{\dot{\rho}_m}{3H^3} = -3\gamma \Omega_m, \]

or, equivalently

\[ \dot{\Omega}_m = -H\Omega_m \left( 3\gamma + 2\frac{\dot{H}}{H^2} \right). \quad \text{(33)} \]

It is clear that, thanks to the Friedmann constraint (29), only one phase space variable is required: either \( x \) or \( \Omega_m \). Let us choose first the variable \( x \). Hence a phase line is enough to represent the asymptotic cosmological dynamics. The autonomous ordinary differential equation (ODE) reads,

\[ \dot{x} = -\frac{3}{2} (2 - \gamma) Hx(1 - x^2), \]

or

\[ x' = -\frac{3}{2} (2 - \gamma) x(1 - x^2), \quad \text{(34)} \]

where the prime denotes derivative with respect to the time variable \( \tau = \int H dt \). If instead of the variable \( x \) one wants to work with the normalized energy density, one should consider equations (33) and (31). We get that,

\[ \Omega_m = 3(2 - \gamma)\Omega_m \left( 1 - \Omega_m \right). \quad \text{(35)} \]

One of these equations is enough to uncover the asymptotic dynamics of the EMS system. These are not independent of each other since through the Friedmann constraint (29) one can transform one equation into the other one.

The critical points of the dynamical system (35) are:  
(i) the scalar field’s kinetic energy dominated solution (also known as stiff-matter solution), \( \Omega_m = 0 \Rightarrow x^2 = 1 \Rightarrow 3H^2 = \varphi^2/2 \), and (ii) the matter-dominated solution \( \Omega_m = 1 \Rightarrow 3H^2 = \rho_m \). In order to check the linear stability of these critical points one expands the dynamical system around each one of the critical points and considers only the linear approximation. For the stiff-matter solution \( \Omega_m = 0 \) one substitutes \( \Omega_m \rightarrow 0 \pm \delta \) in (35) (\( \delta \) is a small perturbation) and keeps only terms \( \propto \delta \): \( \delta' = 3(2 - \gamma)\delta \). Straightforward integration of this equation yields \( \delta(\tau) = \delta_0 \exp[3(2 - \gamma)\tau] \) (\( \delta_0 \) is an integration constant and for standard matter \( \gamma \leq 2 \)), so that the scalar field dominated solution is unstable: the linear perturbation \( \delta \) around \( \Omega_m = 0 \) grows up in \( \tau \)-time. This means that the stiff-matter solution is the past attractor in the phase line. Notice that, since \( \Omega_m = 0 \Rightarrow x^2 = 1 \Rightarrow x = \pm 1 \), then in the \( x \)-phase line there are two stiff-matter past attractors: the one for growing values of the scalar field \( (x = -1) \) and the other one for decreasing values of \( \varphi \) \((x = 1) \). In a similar way, for the matter-dominated solution \( \Omega_m = 1 \), in Eq. (35) one substitutes \( \Omega_m \rightarrow 1 \pm \delta \) and considers the linear approximation. We obtain that \( \delta' = -3(2 - \gamma)\delta \), whose integration leads to: \( \delta(\tau) = \delta_0 \exp[-3(2 - \gamma)\tau] \). This means that the perturbation decreases with \( \tau \)-time so that this point is stable. I. e., the matter-dominated solution is the future attractor in the phase line.

### B. Dynamical equivalence of EMS system and SBT

In order to demonstrate that SBT and EMS are dynamically equivalent models, let us write the FRW EOM for the SBT theory:

\[ 3H^2 = \rho_m + \frac{\omega}{2}\phi^n\dot{\phi}^2, \quad \text{(36)} \]

\[ 2\dot{H} = -\gamma\rho_m - \omega\phi^n\dot{\phi}^2, \quad \text{(37)} \]

\[ \ddot{\phi} + 3H\dot{\phi} + \frac{n}{2}\phi = 0, \quad \text{(38)} \]

where the continuity equation (24) is the same in both EMS and SBT models. Following the same procedure as above, let us write a variable of the phase space:

\[ u \equiv \frac{\ddot{\phi}}{\sqrt{6}H} = \frac{\phi'}{\sqrt{6}}, \quad \text{(39)} \]
The Friedmann equation \(\text{[36]}\) can be written in the following way:

\[
\Omega_m = 1 - \omega \phi^n u^2. \tag{40}
\]

From this equation it seems that another phase space variable is required. However, instead of trying to find such a variable adequate to this problem, we go a different way. Let us write the SBT EOMs in terms of the variable \(u\). We get,

\[
\frac{\dot{\Omega}}{H^2} = -\frac{3}{2} \Omega_m - 3 \omega \phi^n u^2 = -3 + \frac{3(2 - \gamma)}{2} \Omega_m \tag{41}
\]

or, if take into account \(\text{[41]}\),

\[
u = -\sqrt{\frac{3}{2} \phi} u^2 - \frac{3(2 - \gamma)}{2} \Omega_m u, \tag{43}\]

where it is evident that this is not a closed equation (another variable is required). Next let us take the \(\tau\)-derivative of Eq. \(\text{[40]}\):

\[
\dot{\Omega}_m = -2 \omega \phi^n u \left(\sqrt{\frac{3}{2} \phi} u^2 + u\right), \tag{44}\]

and substitute \(u\) from \(\text{[43]}\) into \(\text{[44]}\), we obtain

\[
\dot{\Omega}_m = 3(2 - \gamma) \omega \phi^n u^2 \Omega_m. \tag{45}\]

Finally, if in this last equation take into account Eq. \(\text{[40]}\): \(\omega \phi^n u^2 = 1 - \Omega_m\), then we obtain the same dynamical system \(\text{[35]}\) of the EMS system. This means that both share the same asymptotic dynamics, which is what we wanted to demonstrate.

VI. DISCUSSION AND CONCLUSION

Although we have already mentioned that the Sáez-Ballester theory is not a scalar-tensor theory of gravity, let us further discuss on this subject. First we need to answer the following question: what is a STT of gravity? As suggested by its name, in a scalar-tensor theory of gravity both the metric and the scalar field are propagators of the gravitational interactions. This is reflected, in particular, in the measured value of the Newton’s constant \(G_N\). In the introduction we have shown this in the case of Brans-Dicke (BD) type scalar-tensor theories of gravity. Below we go a step further and we shall show what a STT is in a most generalized case.

Among the most general scalar-tensor theories of gravity are those which are included in the Horndeski classification \(\text{[4, 32, 69–71]}\). The Horndeski class is given by the following action,

\[
S_H = \int d^4x \sqrt{-g} \left[ G_4R + K - G_3(\nabla^2 \phi) \right. \]

\[
\left. + G_5 G_{\mu\nu} \nabla^\mu \nabla^\nu \phi \right], \tag{46}\]

where \(G_{\mu\nu} = R_{\mu\nu} - g_{\mu\nu} R/2\) is the Einstein’s tensor, \(K = K(\phi, X)\) and \(G_3 = G_3(\phi, X)\) are functions of the scalar field and of its kinetic energy density, while, for simplicity, here we assume that \(G_4 = G_4(\phi)\) and \(G_5 = G_5(\phi)\) can be functions of the scalar field exclusively. Although the Horndeski class \(\text{[46]}\) includes self-interacting Einstein-scalar system (recall that in our units system \(8\pi G_N = 1\)),

\[
K = X - V(\phi), \quad G_3 = G_5 = 0, \quad G_4 = \frac{1}{2}, \tag{47}\]

and the k-essence theories,

\[
K = f(\phi) g(X), \quad G_3 = G_5 = 0, \quad G_4 = \frac{1}{2}, \tag{48}\]

which are not STTs, scalar-tensor theories beyond BD-type are also included. For instance \(\text{[4, 32]}\):

- BD theory.

\[
K = \frac{\omega_{BD}}{\phi} X - V(\phi), \quad G_3 = G_5 = 0, \quad G_4 = \phi, \tag{49}\]

where \(\omega_{BD}\) is the BD coupling constant.

- Cubic galileon in the Einstein frame.

\[
G_3 = 2\alpha X, \quad G_4 = \frac{1}{2}, \quad G_5 = 0, \tag{50}\]

where \(\sigma\) is the cubic self-coupling.

- Kinetic coupling to the Einstein’s tensor.

\[
K = X - V, \quad G_3 = 0, \quad G_4 = \frac{1}{2}, \quad G_5 = -\frac{\alpha}{2} \phi, \tag{51}\]

where \(\alpha\) is the coupling constant.
In order to demonstrate that the above are STTs let us write the expression for the measured (Cavendish-type) gravitational constant. According to [22] for those Horndeski theories where the PPN formalism can be applied we get that,

$$8\pi G_{\text{cav}} = \frac{1}{2G_4} \left[ \frac{3G^2_{4,\phi} + G_4K_{,X} + G^2_{4,\phi} e^{-M'r}}{3G^2_{4,\phi} + G_4K_{,X}} \right],$$  \hspace{1cm} (49)$$

where $Y_{,X} \equiv dY/dX$, $Z_{,\phi \phi} \equiv d^2Z/d\phi^2$, etc., and

$$M = \sqrt{-\frac{2G_4K_{,\phi \phi}}{K_{,X} + 3G^2_{4,\phi}}}.$$  

Above we have taken into account that for the cubic galileon a Vainshtein-like screening takes place [23] so that the PPN formalism can not be applied in this case. For that reason we have set $G_3 = 0$. For kinetic coupling theory (49) is not valid either, although in this case the coupling of the derivative of the scalar field to the curvature through $G_{\mu \nu}$ already suggests that it is a STT. For further explanation why the cubic galileon and the kinetic coupling theory are actually scalar-tensor theories of gravity we recommend the discussion in [4].

From equation (49) it is evident that for constant $G_4 = 1/2$, the measured gravitational constant coincides with the Newton’s constant $G_{\text{cav}} = (8\pi)^{-1}$, as in GR. This result holds true for any $K = K(\phi, X)$, so that k-essence may be identified with general relativity with an exotic scalar field as source of the Einstein’s equations. This includes, of course, the EMS system.

We expect that the present discussion will suffice to amend a frequent and long standing misconception. As a matter of fact such a thing like Sáez-Ballester theory does not exist because it is just EMS theory. SBT was proposed in a 1986 paper [33] while, as long as we know, EMS system was introduced as early as in 1957 year [8]. By the time when [33] was published dozen of papers on EMS already existed in scientific bibliography. What is worse is to incorrectly classify the SBT as a scalar-tensor theory. We hope this misunderstanding would be fixed as well.

Acknowledgments. We thank María José Guzmán and Andronikos Paliathanasis for encouraging comments and Kazuhiro Bamba and Vasilis Oikonomou for pointing to us bibliographic references. We also acknowledge FORDECYT-PRONACES-CONACYT for support of the present research under grant CF-MG-2558591.

[1] C.M. Will, Living Rev. Rel. 17 (2014) 4 [e-Print: 1403.7377].
[2] Y. Fujii, K. Maeda, The Scalar-Tensor Theory of Gravitation (Cambridge University Press, 2003).
[3] V. Faraoni, Cosmology in scalar tensor gravity (Kluwer Academic Publishers, 2004); Fundam. Theor. Phys. 139 (2004).
[4] I. Quiros, Int. J. Mod. Phys. D 28 (2019) 1930012 [e-Print: 1901.08690].
[5] T. Clifton, P.G. Ferreira, A. Padilla, C. Skordis, Phys. Rept. 513 (2012) 1 [e-Print: 1106.2476].
[6] K. Bamba, S. Capozziello, S. Nojiri, S.D. Odintsov, Astrophys. Space Sci. 342 (2012) 155 [e-Print: 1205.3421].
[7] S. Nojiri, S.D. Odintsov, V.K. Oikonomou, Phys. Rept. 692 (2017) 1 [e-Print: 1705.11098].
[8] T.P. Sotiriou, V. Faraoni, Rev. Mod. Phys. 82 (2010) 451 [e-Print: 0805.1726].
[9] O. Bergmann, R. Leipnik, Phys. Rev. 107 (1957) 1157.
[10] H.A. Buchdahl, Phys. Rev. 115 (1959) 1325.
[11] A.I. Janis, E.T. Newman, J. Winicour, Phys. Rev. Lett. 20 (1968) 878.
[12] R. Penney, Phys. Rev. 174 (1968) 1578.
[13] A.I. Janis, D.C. Robinson, J. Winicour, Phys. Rev. 186 (1969) 1729.
[14] R. Penney, Phys. Rev. 182 (1969) 1383.
[15] S. Deser, J. Higbie, Annals Phys. 58 (1970) 56.
[16] J.D. Bekenstein, Annals Phys. 82 (1974) 535.
[17] M. Wyman, Phys. Rev. D 24 (1981) 839.
[18] A.G. Agnese, M. La Camera, Phys. Rev. D 31 (1985) 1280.
[19] S. Abe, Phys. Rev. D 38 (1988) 1053.
[20] B.C. Xanthopoulos, T. Zannias, Phys. Rev. D 40 (1989) 2564.
[21] V. Husain, E.A. Martinez, D. Nunez, Phys. Rev. D 50 (1994) 3783.
[22] K.S. Virbhadra, S. Jhingan, P.S. Joshi, Int. J. Mod. Phys. D 6 (1997) 357.
[23] K.S. Virbhadra, Int. J. Mod. Phys. A 12 (1997) 4831.
[24] K.S. Virbhadra, D. Narasimha, S.M. Chitre, Astron. Astrophys. 337 (1998) 1 [e-Print: astro-ph/9801174].
[25] M. Gremm, Phys. Rev. D 62 (2000) 044017 [e-Print: hep-th/0002040].
[26] A. Bhadra, K.K. Nandi, Mod. Phys. Lett. A 16 (2001) 2079.
[27] C. Vuiile, Gen. Rel. Grav. 39 (2007) 621.
[28] J.B. Formiga, Phys. Rev. D 83 (2011) 087502.
[29] J. Tafel, Gen. Rel. Grav. 46 (2014) 1645 [e-Print: 1112.2687].
[30] M. Cadoni, E. Franzin, Phys. Rev. D 91 (2015) 104011 [e-Print: 1503.04734].
[31] M. Cadoni, E. Franzin, M. Serra, JHEP 01 (2016) 125 [e-Print: 1511.03986].
[32] I. Quiros, R. De Arcia, R. García-Salcedo, T. González, F.A. Horta-Rangel, Int. J. Mod. Phys. D 29 (2020) 2050047 [e-Print: 1905.08177].
[33] D. Sáez, V.J. Ballester, Phys. Lett. A 113 (1986) 467.
[34] D. Sáez, Phys. Rev. D 35 (1987) 2027.
[35] J. Daimary, R.R. Baruah, Front. Astron. Space Sci. 9 (2022) 878653.
[36] S.M.M. Rasouli, M. Sakellariadou, P. Vargas Moniz, e-Print: 2203.00766.
