A FAMILY OF MAPS WITH MANY SMALL FIBERS

HANNAH ALPERT AND LARRY GUTH

Abstract. The waist inequality states that for a continuous map from $S^n$ to $\mathbb{R}^q$, not all fibers can have small $(n-q)$-dimensional volume. We construct maps for which most fibers have small $(n-q)$-dimensional volume and all fibers have bounded $(n-q)$-dimensional volume.

Let $n, q \in \mathbb{N}$ with $n > q \geq 1$, and let $f : S^n \to \mathbb{R}^q$ be a continuous map. Let $\hat{p} : \mathbb{R}^{n+1} \to \mathbb{R}^q$ be a surjective linear map, and let $p = \hat{p}|_{S^n}$. The waist inequality states that the largest fiber of $f$ is at least as large as the largest fiber of $p$:

$$\sup_{y \in \mathbb{R}^q} \text{Vol}_{n-q} f^{-1}(y) \geq \sup_{y \in \mathbb{R}^q} \text{Vol}_{n-q} p^{-1}(y).$$

See [1], [3], [4], and [6] for proofs of the waist inequality, or [5] for a survey. In the case $q = 1$, the waist inequality is a consequence of the isoperimetric inequality on $S^n$. The isoperimetric inequality can also be used to prove that the portion of $S^n$ covered by small fibers of $f$ is not very big; that is, for all $\varepsilon$, we have

$$\text{Vol}_n \{ y : \text{Vol}_{n-q} f^{-1}(y) < \varepsilon \} \leq \text{Vol}_n \{ y : \text{Vol}_{n-q} p^{-1}(y) < \varepsilon \}.$$

The theorem presented in this paper describes how the same statement does not hold in the case $q > 1$. We have also included an appendix with a more precise statement of the waist inequality and the isoperimetric inequality.

Theorem 1. For every $n, q \in \mathbb{N}$ with $n > q \geq 1$, and for every $\varepsilon > 0$, there is a continuous map $f : S^n \to \mathbb{R}^q$ such that all but $\varepsilon$ of the $n$-dimensional volume of $S^n$ is covered by fibers that have $(n-q)$-dimensional volume at most $\varepsilon$. Moreover, we may require that every fiber of $f$ has $(n-q)$-dimensional volume bounded by $C_{n,q}$, a constant not depending on $\varepsilon$.

In what follows, $I^n = [0,1]^n$ denotes the $n$-dimensional unit cube, and $\partial I^n$ denotes its boundary. A tree refers to the topological space corresponding to a graph-theoretic tree: topologically, a tree is a finite 1-dimensional simplicial complex that is contractible.

The bulk of the construction comes from the following lemma, in which we construct a preliminary “tree map” $t_{n,r,\delta}$ from $I^n$ to a tree. Later, to construct $f$ we will change the domain from $I^n$ to $S^n$ by gluing several tree maps together, and we will change the range from the tree to $\mathbb{R}^q$ by composing with a map from a thickened tree to $\mathbb{R}^q$. In the tree map $t_{n,r,\delta}$, the parameter $r$ corresponds to the depth of the tree. As $r$ increases, the typical fiber of the map becomes smaller. The parameter $\delta$ corresponds to the total volume of the larger fibers.

Lemma 1. For every $n, r \in \mathbb{N}$, there is a rooted tree $T_{n,r}$ such that for every $\delta > 0$ there is a continuous map $t_{n,r,\delta} : I^n \to T_{n,r}$ with the following properties:

2010 Mathematics Subject Classification. 53C23.
Figure 1. Every fiber of $t_{2,2,\delta}$ has length at most 6, and most fibers have length at most 1.

(1) Every fiber of $t_{n,r,\delta}$ is either a single point, the boundary of an $n$-dimensional cube of side length at most 1, or the $(n-1)$-skeleton of a $2 \times 2 \times \cdots \times 2$ array of $n$-dimensional cubes each of side length at most $\frac{1}{\delta}$.

(2) All but $\delta$ of the volume of $I^n$ is covered by fibers of $t_{n,r,\delta}$ that are boundaries of $n$-dimensional cubes of side length at most $2^{-r}$.

(3) $t_{n,r,\delta}(\partial I^n)$ is a single point, the root of $T_{n,r}$.

(4) Each vertex has at most $2^n$ daughter vertices.

Proof. We construct the tree and tree map recursively in $r$. For $r = 0$, the tree $T_{n,0}$ is a single edge which we may identify with the interval $[0, \frac{1}{2}]$, with 0 being the root. For any $\delta$, we set $t_{n,0,\delta}(x) = \text{dist}(x, \partial I^n)$ for all $x \in I^n$.

Now let $r > 0$. To construct $T_{n,r}$, we take the disjoint union of one copy of $[0,1]$ and $2^n$ copies of $T_{n,r-1}$, and identify the root of every copy of $T_{n,r-1}$ with $1 \in [0,1]$. The root of $T_{n,r}$ is $0 \in [0,1]$. We define $t_{n,r,\delta}$ piecewise as follows. For some small choice of $\delta_1 > 0$, we define $t_{n,r,\delta}$ on the closed $\delta_1$-neighborhood of $\partial I^n$ to $[0,1] \subset T_{n,r}$ by

$$t_{n,r,\delta}(x) = \frac{1}{\delta_1} \text{dist}(x, \partial I^n).$$
Then, translating the coordinate hyperplanes to pass through the center of \( I^n \) we divide the remainder of the cube into a \( 2 \times 2 \times \cdots \times 2 \) array of cubes \( Q_1, \ldots, Q_{2^n} \) each of side length slightly less than \( \frac{1}{2} \). For each \( j = 1, \ldots, 2^n \), let \( \lambda_j : Q_j \to I^n \) be the map that scales \( Q_j \) up to unit size, and let \( i_j : T_{n,r-1} \to T_{n,r} \) be the inclusion of the \( j \)th copy of \( T_{n,r-1} \) into \( T_{n,r} \). Then for some small choice of \( \delta_2 > 0 \), we put
\[
t_{n,r,\delta}(Q_j) = i_j \circ t_{n,r-1,\delta_2} \circ \lambda_j.
\]

Properties 1, 3, and 4 are easily satisfied by the construction. To ensure property 2, we need to choose \( \delta_1 \) and \( \delta_2 \). The volume of \( I^n \) that is covered by large fibers—fibers not equal to the boundary of a cube of side length at most \( 2^{-r} \)—is at most \( \delta_1 \cdot 2n + 2^n \cdot \delta_2 \cdot 2^{-n} \), because the area of \( \partial I^n \) is \( 2n \) and because the portion of each \( Q_j \) that is covered by large fibers has volume at most \( \delta_2 \cdot \text{Vol}(Q_j) < \delta_2 \cdot 2^{-n} \). Thus we may choose \( \delta_1 = \frac{1}{2n} \) and \( \delta_2 = \frac{1}{2} \).

**Proof of Theorem 1.** We may replace \( S^n \) by \( \partial I^{n+1} \) by composing with the (bi-Lipschitz) homeomorphism \( \psi : S^n \to \partial I^{n+1} \) given by lining up the centers of \( S^n \) and \( \partial I^{n+1} \) in \( \mathbb{R}^{n+1} \) and projecting radially. We start by constructing a tree \( T \) and a tree map \( t : \partial I^{n+1} \to T \). For some large choice of \( r \), let \( T \) be the tree obtained by identifying the roots of \( 2(n+1) \) copies of \( T_{n,r} \), one for each \( n \)-dimensional face of \( \partial I^{n+1} \). For some small choice of \( \delta \), define \( t \) on each \( n \)-dimensional face of \( \partial I^{n+1} \) to be the composition of \( t_{n,r,\delta} \) with the inclusion of the corresponding \( T_{n,r} \) into \( T \).

The fibers of \( t \) have dimension \( n-1 \). In order to cut the fibers down to dimension \( n-q \), we next construct a projection map \( p : \partial I^{n+1} \to \mathbb{R}^{q-1} \) such that the fibers of \( p \) intersect the fibers of \( t \) transversely. The fibers of \( p \) have codimension \( 2 \) in \( \mathbb{R}^{n+1} \) and are aligned with the standard coordinates, so we achieve transversality by using other linear coordinates to construct \( p \). We choose \( q-1 \) linearly independent vectors \( v_1, \ldots, v_{q-1} \in \mathbb{R}^{n+1} \) such that for every two standard basis vectors \( e_i, e_j \in \mathbb{R}^{n+1} \) the spaces \( \text{span}\{e_i, e_j\}^\perp \) and \( \text{span}\{v_1, \ldots, v_{q-1}\}^\perp \) intersect transversely; equivalently, the set \( e_i, e_j, v_1, \ldots, v_{q-1} \) is linearly independent. For \( k = 1, \ldots, q-1 \), define the \( k \)-th component of \( p \) to be the dot product of the input with \( v_k \). Then the fibers of \( t \times p : \partial I^{n+1} \times T \to \mathbb{R}^{q-1} \) are codimension \( q-1 \) transverse linear cross-sections of the \((n-1)\)-dimensional fibers of \( t \), and have \((n-q)\)-dimensional volume bounded by some constant depending on \( n \) and \( q \).

There exists \( M \) large enough that \( p(\partial I^{n+1}) \) is contained in the \((q-1)\)-dimensional ball \( B(M) \) of radius \( M \). We define a map \( \phi : T \times B(M) \to \mathbb{R}^q \) such that the number of points in each fiber of \( \phi \) is at most the maximum degree of \( T \), which is \( 2^n + 1 \). Then we define \( f = \phi \circ (t \times p) \). The fibers of \( f \), like the fibers of \( t \times p \), have \((n-q)\)-dimensional volume bounded by a constant \( C_{n,q} \).

The map \( \phi \) is constructed as follows. Let \( \phi|_{T \times \{0\}} \) be an embedding of \( T \) into \( \mathbb{R}^q \) in which the edges map to straight line segments and each daughter vertex has \( x_1 \)-coordinate greater than that of its parent. Let \( d \) be the minimum distance between disjoint edges of \( \phi(T \times \{0\}) \). Then for every \( p \in T \) and \( x \in B(M) \), we set
\[
\phi(p, x) = \phi(p, 0) + \frac{d}{4} \left( 0, \frac{x}{M} \right),
\]
where \( (0, \frac{x}{M}) \) denotes the point in \( \mathbb{R}^q \) constructed by adding onto \( \frac{x}{M} \in \mathbb{R}^{q-1} \) a first coordinate of 0. If \( \phi(p, x) = \phi(p', x') \), then \( \phi(p, 0) \) and \( \phi(p', 0) \) are at most \( \frac{d}{2} \) apart, so \( p \) and \( p' \) lie on two incident edges of \( T \); also, \( \phi(p, 0) \) and \( \phi(p', 0) \) have the same
$x_1$-coordinate, so these two edges are between two daughters and a common parent, rather than a daughter, a parent, and a grandparent.

To finish the proof, we show that $\delta$ and $r$ may be chosen such that all but $\varepsilon$ of the $n$-dimensional volume of $\partial I^{n+1}$ is covered by fibers with $(n-q)$-dimensional volume at most $\varepsilon$. The maximum number of daughter vertices of any vertex of $T$ is $2^n$, and most of $\partial I^{n+1}$ is covered by fibers of $f$ that are unions of at most $2^n$ codimension $q-1$ transverse linear cross-sections of boundaries of $n$-dimensional cubes of side length at most $2^{-1}$. We choose $r$ large enough that every codimension $q-1$ transverse linear cross-section of $2^{-r}\partial I^n$ has $(n-q)$-dimensional volume at most $\frac{\varepsilon}{2(n+1)}$. The volume of the portion of $\partial I^{n+1}$ covered by larger fibers is at most $2(n+1)\cdot\delta$, so we choose $\delta < \frac{\varepsilon}{2(n+1)}$. \hfill \Box

**Appendix: The waist inequality and the isoperimetric inequality**

In order to be precise about the waist inequality, we need a notion of $(n-q)$-dimensional volume of arbitrary closed subsets in $S^n$. Gromov’s version of the waist inequality is stated in terms of the Lebesgue measures $\text{Vol}_n$ of the $\varepsilon$-neighborhoods $f^{-1}(y)_\varepsilon$ of the fibers $f^{-1}(y)$ of a continuous map $f$.

**Theorem 2** (Waist inequality, [4]). Let $f: S^n \to \mathbb{R}^q$ be a continuous map. Then there exists a point $y \in \mathbb{R}^q$ such that for all $\varepsilon > 0$, we have

$$\text{Vol}_n(f^{-1}(y)_\varepsilon) \geq \text{Vol}_n(S^n_{\varepsilon} - q),$$

where $S^n_{\varepsilon} - q \subset S^n$ denotes an equatorial $(n-q)$-sphere.

The paper [6] gives a detailed exposition of the proof of the waist inequality and fills in some small gaps in the original argument. For convenience we introduce a notation for comparing the $\varepsilon$-neighborhoods of two sets: given $E, F \subseteq S^n$, we say that $E$ is **larger in neighborhood** than $F$, denoted $E \geq_{nbd} F$, if for all $\varepsilon > 0$ we have

$$\text{Vol}_n(E_\varepsilon) \geq \text{Vol}_n(F_\varepsilon).$$

Then the waist inequality states that for some $y \in \mathbb{R}^q$ we have $f^{-1}(y) \geq_{nbd} S^n_{\varepsilon} - q$.

In the case $q = 1$, we would like to say that the waist inequality is a consequence of the isoperimetric inequality. The classical isoperimetric inequality applies only to regions with smooth boundary, so we need the following version, which is stated and proved in [4] and attributed to [7]:

**Theorem 3** (Isoperimetric inequality). Let $A \subseteq S^n$ be a closed set and $B \subseteq S^n$ be a closed ball with $\text{Vol}_n(B) = \text{Vol}_n(A)$. Then we have

$$A \geq_{nbd} B.$$

In the introduction we claimed that in the case $q = 1$, the isoperimetric inequality could be used to prove, in addition to the waist inequality, another statement about the volume of $S^n$ covered by small fibers. Here we formulate the statement more precisely and prove it. The proof implies the waist inequality for $q = 1$.

**Theorem 4.** Let $f: S^n \to \mathbb{R}$ be a continuous map, and $p: S^n \to \mathbb{R}$ be the restriction to $S^n$ of a surjective linear map $\tilde{p}: \mathbb{R}^{n+1} \to \mathbb{R}$. Then for all $y \in p(S^n)$, we have

$$\text{Vol}_n\{x \in S^n : f^{-1}(f(x)) \geq_{nbd} p^{-1}(y)\} \geq \text{Vol}_n\{x \in S^n : \tilde{p}^{-1}(p(x)) \geq_{nbd} \tilde{p}^{-1}(y)\}.$$
Lemma 2. Let \( X, Y \subset S^n \) be closed sets with \( X \cup Y = S^n \). Let \( B^X, B^Y \subset S^n \) be closed balls such that their two centers are antipodal in \( S^n \) and \( \text{Vol}_n(B^X) = \text{Vol}_n(X) \) and \( \text{Vol}_n(B^Y) = \text{Vol}_n(Y) \). Then we have
\[
X \cap Y \supseteq \text{nbd} B^X \cap B^Y.
\]

Proof. First we claim that \((X \cap Y)_\varepsilon \) is the disjoint union of \( X_\varepsilon \setminus X \), \( Y_\varepsilon \setminus Y \), and \( X \cap Y \). It is clear that \((X \cap Y)_\varepsilon \) is the disjoint union of its intersections with \( S^n \setminus X \), \( S^n \setminus Y \), and \( X \cap Y \). Thus it suffices to show that
\[
(X \cap Y)_\varepsilon \cap (S^n \setminus X) = X_\varepsilon \setminus X.
\]
Because \((X \cap Y)_\varepsilon \subseteq X_\varepsilon \), we immediately have
\[
(X \cap Y)_\varepsilon \cap (S^n \setminus X) \subseteq X_\varepsilon \setminus X.
\]
For the reverse inclusion, let \( y \in X_\varepsilon \setminus X \), and let \( \gamma : [0, 1] \to S^n \) be a curve of length at most \( \varepsilon \) with \( \gamma(0) = y \) and \( \gamma(1) = x \in X \). Let \( t \in [0, 1] \) be the greatest value with \( \gamma(t) \in Y \). Then \( \gamma(t) \in X \cap Y \), so \( y \in (X \cap Y)_\varepsilon \).

Thus, applying the isoperimetric inequality and additivity of measure, we have
\[
\text{Vol}_n((X \cap Y)_\varepsilon) = \text{Vol}_n(X_\varepsilon) - \text{Vol}_n(X) + \text{Vol}_n(Y_\varepsilon) - \text{Vol}_n(Y) + \text{Vol}_n(X \cap Y) \geq \text{Vol}_n(B^X_\varepsilon) - \text{Vol}_n(B^X) + \text{Vol}_n(B^Y_\varepsilon) - \text{Vol}_n(B^Y) + \text{Vol}_n(B^X \cap B^Y) = \text{Vol}_n((B^X \cap B^Y)_\varepsilon).
\]

\( \square \)

Proof of Theorem 4. Without loss of generality we assume \( p(S^n) = [0, 1] \) and \( y \leq \frac{1}{2} \). Then on the right-hand side of the desired inequality we have
\[
\{ x \in S^n : p^{-1}(p(x)) \supseteq \text{nbd} p^{-1}(y) \} = p^{-1}[y, 1 - y].
\]
Define \( \alpha, \beta \in \mathbb{R} \) as
\[
\alpha = \sup \{ t \in \mathbb{R} : \text{Vol}_n f^{-1}(-\infty, t) \leq \text{Vol}_n p^{-1}[0, y] \},
\]
\[
\beta = \inf \{ t \in \mathbb{R} : \text{Vol}_n f^{-1}(t, \infty) \leq \text{Vol}_n p^{-1}[y, 1] \}.
\]
For each \( t \in [\alpha, \beta] \), apply the lemma with \( X = f^{-1}(-\infty, t) \) and \( Y = f^{-1}[t, \infty) \) to get \( f^{-1}(t) \supseteq \text{nbd} p^{-1}[y_1, y_2] \) for some \( y_1, y_2 \in [y, 1 - y] \). In particular, we have
\[
f^{-1}(t) \supseteq \text{nbd} p^{-1}(y_1) \supseteq \text{nbd} p^{-1}(y).
\]
Thus, we have
\[
f^{-1}[\alpha, \beta] \subseteq \{ x \in S^n : f^{-1}(f(x)) \supseteq \text{nbd} p^{-1}(y) \}.
\]
Because \( \text{Vol}_n f^{-1}(-\infty, \alpha) \leq \text{Vol}_n p^{-1}[0, y] \) and \( \text{Vol}_n f^{-1}(\beta, \infty) \leq \text{Vol}_n p^{-1}[y, 1] \) we have
\[
\text{Vol}_n f^{-1}[\alpha, \beta] \geq \text{Vol}_n p^{-1}[y, 1 - y].
\]
\( \square \)
References

[1] F.J. Almgren, The theory of varifolds — a variational calculus in the large for the $k$-dimensional area integrated, Mimeographed notes, 1965.

[2] T. Figiel, J. Lindenstrauss, and V. D. Milman, The dimension of almost spherical sections of convex bodies, Acta Math. 139 (1977), no. 1-2, 53–94. MR 0445274 (56 #3618)

[3] M. Gromov, Filling Riemannian manifolds, J. Differential Geom. 18 (1983), no. 1, 1–147. MR 697984 (85h:53029)

[4] ______, Isoperimetry of waists and concentration of maps, Geom. Funct. Anal. 13 (2003), no. 1, 178–215. MR 1978494 (2004m:53073)

[5] L. Guth, The waist inequality in Gromov’s work, The Abel Prize 2008–2012 (H. Holden and R. Piene, eds.), Springer, 2014, pp. 181–195.

[6] Y. Memarian, On Gromov’s waist of the sphere theorem, J. Topol. Anal. 3 (2011), no. 1, 7–36. MR 2784762 (2012g:53066)

[7] E. Schmidt, Die Brunn-Minkowskische Ungleichung und ihr Spiegelbild sowie die isoperimetrische Eigenschaft der Kugel in der euklidischen und nichteuklidischen Geometrie. I, Math. Nachr. 1 (1948), 81–157. MR 0028600 (10,471d)