Equilibria of flat and round galactic disks.

C. Pichon\textsuperscript{1,2} and D. Lynden-Bell\textsuperscript{1}

1: Institute of Astronomy, The observatories, Cambridge CB3OHA, UK
2: CITA, 60 St. George Street, Toronto, Ontario M5S 1A7, Canada.

ABSTRACT

A general method is presented for constructing distribution functions for flat systems whose surface density and Toomre’s Q number profile is given. The purpose of these functions is to provide plausible galactic models and assess their critical stability with respect to global non axi-symmetric modes. The derivation may be carried out for an azimuthal velocity distribution (or a given specific energy distribution) which may either be observed or chosen to match a specified temperature profile. Distribution functions describing stable models with realistic velocity distributions for power law disks, the Isochrone and the Kuzmin disks are provided. Specially simple inversion formulae are also given for finding distribution functions for flat systems whose surface densities are known.

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1. Introduction

Over the next decade, ample and accurate observational data on the detailed kinematics of nearby disk galaxies will become available. It will be of great interest to link these observations with theoretical models for the underlying dynamics. The problem of finding the distribution function for an axially symmetrical system may formally be solved by using Laplace transforms (Lynden-Bell 1960 [12]) or by using power series (Fricke 1952 [5], or more recently Vauterin 1995 [4]). Kalnajs (1976) [11], followed by Miyamoto (1974) [14], chose specific forms of distribution function because the flat problem has no unique solution. Distribution functions for the power-law disks were also recently derived by Evans (1993) [7] while Hunter & Qian (1993) [9] presented an inversion scheme which can be applied for thickened disks with two integrals. For flat disks, it is desirable to use the functional freedom left in $f$ in order to construct a distribution function which accounts for all the kinematics, either observed or desired (i.e. which accounts for the line profiles observed, or which are marginally stable to radial modes). Independent measurement of the observed radial and azimuthal velocity distribution functions could, for instance, be contrasted with predictions arising from the gravitational nature of the interaction. Indeed the laws of the motion and the associated conserved quantities together with the assumption that the system is stationary put strong constraints on the possible velocity distributions. This is formally expressed by the existence of an underlying distribution function which characterises the dynamics completely. The determination of realistic distribution functions which could account for observed line profiles is therefore an important project vis a vis the understanding of galactic structure. Producing theoretical models that accounts globally for the observed line profiles of a given galaxy provides a unique opportunity to inspect the current understanding of the dynamics of S0 galaxies. It should then be possible to study quantitatively all departures from the flat axisymmetric stellar models. Indeed, axisymmetric distribution functions are the building blocks of all sophisticated stability analyses, and a good phase space portrait of the unperturbed configuration is often needed in order to assess the stability of a given equilibrium state. Numerical N-body simulations also require sets of initial conditions which should reflect the nature of the equilibrium.

For a flat galaxy all the stellar orbits are confined to a plane and by Jeans’ theorem the steady state mass-weighted distribution function must be of the form $f = f(\varepsilon, h)$, where the specific energy, $\varepsilon$, and the specific angular momentum, $h$, are given by

$$\varepsilon = \frac{1}{2} (v_R^2 + v_\phi^2) - \psi \quad \text{and} \quad h = R v_\phi.$$  (1.1)
The surface density, $\Sigma(R)$, arises from this distribution of stellar orbits provided that the integral of $f$ over all bound velocities is $\Sigma(R)$, i.e.

$$
\Sigma(R) = \int \int f(\varepsilon, h) \, dv_R \, dv_\phi,
$$

where the integral is over the region $\delta(v_R^2 + v_\phi^2) < \psi$. The following inversion methods assume that $\Sigma(R)$ is given and its potential on the plane, $\psi(R)$ is known. This is achieved either by requiring self-consistency via Poisson’s equation

$$
\nabla^2 \psi = -4\pi \delta(z) \Sigma(R),
$$

or alternatively by assuming that the disk is embedded in a halo and choosing both $\Sigma$ and $\psi$ independently. At constant $v_\phi$ and $R$, $v_R \, dv_R = d\varepsilon$ and hence, using Eq. (1.1),

$$
dv_R = d\varepsilon \left[2(\varepsilon + \psi) - h^2 R^{-2}\right]^{-1/2}.
$$

Furthermore, at constant $R$, $dh = R \, dv_\phi$, and therefore Eq. (1.2) yields

$$
\Sigma(R) = \int_{-\sqrt{2R^2\psi}}^{+\sqrt{2R^2\psi}} \int_0^{(\varepsilon, h)} 2 \frac{f(\varepsilon, h) \, d\varepsilon \, dh}{\sqrt{2(\varepsilon + \psi) R^2 - h^2}}.
$$

The factor of 2 arises because $v_R$ takes both positive and negative values which are mapped into the same range of $\varepsilon$ which depends only on $v_R^2$. Similarly,

$$
\Sigma(R) v_R^2 = \int_{-\sqrt{2R^2\psi}}^{+\sqrt{2R^2\psi}} \int_0^{(\varepsilon, h)} 2 \frac{f(\varepsilon, h) \sqrt{2(\varepsilon + \psi) R^2 - h^2} \, d\varepsilon \, dh}{\sqrt{2R^2\psi}}.
$$

The basic inversion problem is concerned with finding the class of $f(\varepsilon, h)$ compatible with the constraints of Eqs. (1.5) and (1.6).

Two classes of inversion methods are constructed and implemented in this paper. These lead to classical distribution functions compatible with a given a given surface density (section 3), or a given surface density and a given pressure profile (section 2). The former technique relies on an Ansatz and yield direct and general methods for the construction of distribution functions for the purpose of theoretical modelling. The latter technique yields distribution functions that correspond to given line profiles (or rather, to the shape of the line profiles induced by the velocity distributions which will here loosely be called line profiles) which may either be postulated or chosen to match the observations. In fact, only a sub-sample of the observation is required to construct a fully self-consistent model which accounts, in turn, for all the observed kinematics. The corresponding redundancy in the data may be exploited (as sketched in section 4) to address the limitations of the description.

2. Distribution functions for given kinematics.

This global inversion method introduces an intermediate observable, $F_\phi(R, v_\phi)$, the distribution function for the number of stars which have azimuthal velocity $v_\phi$ at radius $R$ (or alternatively $G_\varepsilon(\varepsilon, R)$ the distribution function for the stars which have specific energy $\varepsilon$ at radius $R$). This technique may be applied to the reconstruction of distribution functions accounting for observed line profiles. In fact, given some symmetry assumptions about the shape of an observed galaxy, it is shown that only a subset of the available line profiles – say the azimuthal velocity distribution – is required to re-derive the complete kinematics. Therefore this method provides a general procedure for constructing self-consistent models for the complete dynamics of disk galaxies. These models predict radial velocity distributions which may, in turn, be compared with observations as discussed in section 4.

Alternatively, a ‘natural’ functional form for $F_\phi$ or $G_\varepsilon$ may be postulated and parameterised so that it is compatible with imposing the surface density, the average azimuth velocity and the azimuthal pressure profiles (or equivalently the Toomre number $Q$, as $Q$ follows from the equation of radial support). The
distribution function \( f(\varepsilon, h) \) of this disk follows in turn from \( F_\phi \) or \( G_\varepsilon \) via simple Abel transforms. This prescription is very general and especially useful when setting the initial conditions of numerical N-body stability analysis.

### 2.1. Inversion via line profiles

The number of stars which have azimuthal velocity \( v_\phi \) within \( dv_\phi \) at radius \( R \) reads

\[
F_\phi (R, v_\phi) = \int f(\varepsilon, h) \, dv_\phi = \sqrt{2} \int_0^\varepsilon \frac{f(\varepsilon, h)}{\sqrt{\varepsilon + Y}} \, d\varepsilon ,
\]

where the effective potential, \( Y \), is given by \( Y = \psi - \frac{h^2 R^{-2}}{2} = \psi - v_\phi^2/2 \). The line profile \( F_\phi (R, v_\phi) \) may be expressed in terms of \( (h, Y) \). Indeed the identity \( Y = \psi - \frac{h^2 R^{-2}}{2} \) may be solved for \( R \) which yields \( R(h, Y) \). The azimuthal velocity \( v_\phi \) becomes in turn a function of \( h, Y \) for each branch corresponding to the two roots of \( Y = \psi - \frac{h^2 R^{-2}}{2} \) (as illustrated in Fig. 2.1). Calling \( \tilde{F}_\phi (h, Y) = F_\phi (R, v_\phi) \) allows the inversion of Eq. (2.1) by an Abel transform:

\[
f(\varepsilon, h) = \frac{1}{\sqrt{2} \pi} \int_{-\varepsilon}^{\varepsilon} \frac{\partial \tilde{F}_\phi}{\partial Y} \frac{dY}{\sqrt{(-\varepsilon) - Y}} ,
\]

where the partial derivative is taken at constant \( h \). It was assumed here that the distribution \( F_\phi (R, v_\phi) \) vanishes at the escape velocity. Note that \( \tilde{F} \) yields both the symmetric and the antisymmetric parts of the distribution function. The r.h.s of Eq. (2.2) is advantageously re-expressed in terms of the variables \( (R, h) \) given that (for monotonic integration)

\[
\left( \frac{\partial \tilde{F}_\phi}{\partial Y} \right)_h dY = \left( \frac{\partial F_\phi}{\partial R} \right)_h \left( \frac{\partial R}{\partial Y} \right)_h dY = \left( \frac{\partial F_\phi}{\partial R} \right)_h \text{sign} \left[ \left( \frac{\partial R}{\partial Y} \right)_h \right] dR .
\]

This yields two contributions for Eq. (2.2)

\[
f(\varepsilon, h) = \frac{1}{\pi} \int_{R_a}^{R_p} \frac{(\partial F_\phi / \partial R)_h \, dR}{\sqrt{h^2 / R^2 - 2 \psi (R) - 2 \varepsilon}} - \frac{1}{\pi} \int_{R_a}^{\infty} \frac{(\partial F_\phi / \partial R)_h \, dR}{\sqrt{h^2 / R^2 - 2 \psi (R) - 2 \varepsilon}} ,
\]

where \( R_p(h, \varepsilon) \) and \( R_a(h, \varepsilon) \) are, respectively, the apogee and perigee of the star with invariants \( h \) and \( \varepsilon \), and \( R_a(h) \) is the inner radius of a star on a “parabolic” (zero energy) orbit with momentum \( h \). Note that the derivative in Eq. (2.4) is performed keeping \( h = R v_\phi \) constant. The contributions to the two integrals is illustrated on Fig. 2.1.

Equation (2.4) yields the unique distribution function compatible with an observed azimuthal velocity distribution. All macroscopic properties of the flow follow from \( f \). For instance the surface density reads

\[
\Sigma \equiv \int \left[ \int f_+ (\varepsilon, h) \, dv_\phi \right] \, dv_\phi = 2 \int_0^{\varepsilon} F_\phi (R, v_\phi) \, dv_\phi ,
\]

while the radial velocity distribution, \( F_R (R, v_R) \), obeys

\[
F_R (R, v_R) = \frac{1}{R (2 \psi - v_R^2)^{1/2}} \int_{-R (2 \psi - v_R^2)^{1/2}}^{R (2 \psi - v_R^2)^{1/2}} f \left( \frac{v_R^2}{2} + \frac{h^2}{2 R^2} - \psi (R) , h \right) \, dh .
\]

Both \( F_R \) and \( \Sigma \) may in turn be compared with the corresponding observed quantities.
Figure 2.1: the relationship between integration in terms of the effective potential, $Y$, and $R$ integration. The effective potential is drawn here as a function of $R$ for two values of $h$. The area between that curve and the lines $Y = -\varepsilon$ and $R = R_e$ is shaded, defining 3 regions from left to right. The middle region does not contribute to Eq. (2.4). The equation $Y = -\varepsilon$ has two roots corresponding to the perigee and the apogee of the star, while $Y = 0$ has two roots corresponding to infinity and $R_e$, the inner bound of an orbit with zero energy and angular momentum $h$. The sign of the slope of $Y(R)$ gives the sign of the contribution for each branch in Eq. (2.4).

2.2. Disks with given $Q$ profiles

From the point of view of stability analysis, $F_\phi$ may be chosen to match given constraints such as a specified potential and temperature profile of a disk. The azimuthal pressure, $p_\phi = \Sigma ((v_\phi^2) - \langle v_\phi \rangle^2)$, is then fixed via the equation of radial support and by the temperature of the disk defined by Toomre’s $Q$ number:

$$p_\phi = \Sigma V_c^2 + \frac{\partial (R p_R)}{\partial R}, \quad \text{and} \quad p_R = Q^2 \frac{\pi^2 \Sigma^3}{\kappa^2},$$

where the circular velocity, the epicyclic frequency and the surface density are given by

$$V_c^2 = -\left[ \frac{1}{R} \frac{\partial \psi}{\partial R} \right], \quad \kappa^2 = \frac{1}{R^3} \left[ \frac{\partial (R V_c)}{\partial R} \right]^2, \quad \text{and} \quad \Sigma = \frac{1}{2\pi} \left[ \frac{\partial \psi}{\partial z} \right]^2.$$

Note that here the azimuthal stress $p_\phi$ contains the mean streaming stress. It is assumed here that the field is self-consistent and obeys Eq. (1.3); alternatively $\Sigma$ may be given independently of $\psi$ and the method described here still applies. The surface density, $\Sigma$, and the azimuthal pressure, $\Sigma (v_\phi^2)$, may in turn be expressed in terms of $F_\phi (R, v_\phi)$ via

$$\Sigma \equiv \int \left[ \int f_+ (\varepsilon, h) \, dv_R \right] \, dv_\phi = 2 \int_0^{v_e} F_\phi (R, v_\phi) \, dv_\phi,$$
\[
\Sigma(\epsilon^2) = \int \left[ \int f_+ (\epsilon, h) \, dv_R \right] v_R^2 \, dv_\phi = 2 \int_{0}^{\infty} F_\phi (R, v_\phi) \, v_\phi^2 \, dv_\phi, \tag{2.9b}
\]

where the circular escape velocity, \(v_c\), is equal to \(\sqrt{2\psi}\). The function \(f_+\) stands for the component of the distribution function even in \(h\). Any function \(F_\phi\) satisfying these moment equations corresponds to a state of equilibrium stable against ring formation when \(Q > 1\). Realistic choices for \(F_\phi\) are presented in the next sections.

### 2.3. Alternative inversion method

Another intermediate observable, \(G_\epsilon (R, \epsilon)\), the distribution function for the number of stars which have specific energy \(\epsilon\), may also be parameterised to fix the surface density and the average energy density profiles. For external galaxies \(G_\epsilon\) is not directly observed. For our Galaxy, \(G_\epsilon\) is measurable indirectly – via the distribution of stars with given radial and azimuthal velocity given by spectroscopy and proper motions – but only yields the even component in \(h\) of the distribution function. However, the corresponding inversion method still provides a route for the construction of models with specified temperature profiles. The number of stars which have specific energy \(\epsilon\) within \(d\epsilon\) at radius \(R\) reads

\[
G_\epsilon (R, \epsilon) = \int \frac{f_+ (\epsilon, h)}{v_R} \, dv_R = \frac{2}{\sqrt{2}} \int_{0}^{X} \frac{g (\epsilon, h^2/2)}{\sqrt{X - h^2/2}} \, d \left( \frac{h^2}{2} \right), \tag{2.10}
\]

where \(X\) is defined by

\[
X = R^2 (\psi + \epsilon). \tag{2.11}
\]

Here the auxiliary function \(g\) is given by

\[
g (\epsilon, h^2/2) = \frac{f_+ (\epsilon, h)}{|h|}. \tag{2.12}
\]

The surface density, and mean energy density, \(\langle \epsilon \rangle\), may be expressed in terms of \(G_\epsilon (R, \epsilon)\) via

\[
\Sigma \equiv \int \int f_+ (\epsilon, h) \, dv_\phi \, dv_R = \int_{0}^{\infty} G_\epsilon (R, \epsilon) \, d\epsilon, \tag{2.13a}
\]

\[
\Sigma \langle \epsilon \rangle \equiv \int \int f_+ (\epsilon, h) \left[ \frac{v_R^2}{2} + \frac{v_\phi^2}{2} - \psi \right] \, dv_\phi \, dv_R = \int_{0}^{\infty} G_\epsilon (R, \epsilon) \, \epsilon \, d\epsilon. \tag{2.13b}
\]

Note that only the symmetric component of the distribution function follows from Eq. (2.12). The local mean energy density \(\langle \epsilon \rangle = \Sigma^{-1} (p_R/2 + p_\phi/2) - \psi(R)\) is fixed by the temperature of the disk defined by Toomre’s \(Q\) number:

\[
\langle \epsilon \rangle = \frac{1}{2} \left[ \Sigma V_c^2 + \frac{\partial (R p_R)}{\partial R} + p_R \right] - \psi(R) \quad \text{given} \quad p_R = Q^2 \frac{\pi^2 \Sigma^3}{\kappa^2}, \tag{2.14}
\]

where \(\kappa\) and \(V_c\) are given by Eq. (2.8). The function \(G(R, \epsilon)\) may be expressed in terms of \(\epsilon, X\) via Eq. (2.11) and \(R = R(\psi)\). Calling \(G\) \(G_\epsilon (R, X) \equiv G_\epsilon (R, \epsilon)\) leads to the inversion of Eq. (2.10) by an Abel transform:

\[
g (\epsilon, h^2/2) = \frac{1}{\sqrt{2} \pi} \int_{0}^{h^2/2} \frac{dX}{\sqrt{h^2/2 - X}}. \tag{2.15}
\]

\(f_+ (\epsilon, h)\) follows from \(g(\epsilon, h^2/2)\) via Eq. (2.12).

Any function \(G_\epsilon\) satisfying the moment constraint Eqs. (2.13) yields, via Eq. (2.15) and Eq. (2.14), a maximally rotating disk stable against ring formation.
2.4. Implementation: Gaussian line profiles

Gaussian velocity distributions are desirable both as building blocks to fit measured line profiles and as “realistic” choices for \( F_\phi \) in the construction scheme of disks parameterised by their temperature. The former point is discussed in section 4. The construction of Gaussian line profiles compatible with a given temperature requires a supplementary assumption for the mean azimuthal velocity of the flow, \( \langle v_\phi \rangle \), on which the Gaussian should be centred. In additions to the the two constraints, Eqs. (2.9), this puts a third constraint on \( F_\phi \), namely

\[
\Sigma \langle v_\phi \rangle = \int \left[ \int f_-(\varepsilon, h) dv_R \right] v_\phi dv_\phi = 2 \int_0^{v_e} F_\phi(R, v_\phi) v_\phi dv_\phi ,
\]

(2.16)

where \( f_- \) is the odd component in \( h \) of the distribution function. Suppose the following functional form for \( F_\phi \)

\[
F_\phi(R, v_\phi) = S(R) W_n(R, v_\phi) \exp \left( -\frac{(v_\phi - v(R))^2}{2\sigma^2(R)} \right),
\]

(2.17)

where the window function \( W_n \) is intended to damp \( F_\phi(R, v_\phi) \) near the escape velocity \( v_e \). A possible expression for \( W_n \) is

\[
W_n(R, v_\phi) = \begin{cases} 
\exp \left[ -\frac{v_\phi^2}{n^2(v_e^2(R) - v_\phi^2)} \right] & \text{if } |v_\phi| < v_e, \\
0 & \text{elsewhere}.
\end{cases}
\]

(2.18)

Self-consistency requires that the unknown functions \((v, \sigma, S)\) of Eq. (2.17) are solved in terms of \( \langle v_\phi \rangle \), \( \Sigma \), and \( p_\phi \) via Eqs. (2.9) and Eq. (2.16). For practical purposes, the temperature range explored in realistic disk models is such that \((v_e - v)^2/2\sigma^2\) is quite large at all radii; if \( n \) is also chosen so that at temperature \( \sigma \)

\[
n \gg \frac{v_e \langle v_\phi \rangle \sigma}{(v_e^2 - \langle v_\phi \rangle^2)(v_e + \langle v_\phi \rangle)},
\]

(2.19)

then the tail of the Gaussian function need not be taken explicitly into account. The line profile \( F \) then reads directly in terms of \( \langle v_\phi \rangle \), \( \Sigma \), and \( p_\phi \):

\[
F_\phi(R, v_\phi) = \frac{\Sigma(R)}{\sqrt{2\pi} \sigma_\phi} \exp \left( -\frac{(v_\phi - \langle v_\phi \rangle)^2}{2\sigma_\phi^2} \right),
\]

(2.20)

where \( \sigma_\phi \) is the the azimuthal velocity dispersion

\[
\sigma_\phi^2 = p_\phi/\Sigma - \langle v_\phi \rangle^2.
\]

(2.21)

The azimuthal pressure \( p_\phi \) follows from the equation of radial support and the temperature of the disk defined by Toomre’s \( Q \) number given by Eq. (2.7). The expression of the average azimuthal velocity, \( \langle v_\phi \rangle \), may be taken to be that which leads to an exact asymmetric drift equation:

\[
\Sigma \langle v_\phi \rangle^2 = p_\phi - p_R \left( \frac{\kappa^2 R^2}{4V_e^2} \right).
\]

(2.22)

Equations (2.20)-(2.22), together with Eqs. (2.7) provide a prescription for the Gaussian azimuthal line profile \( F_\phi \) which yield the distribution function \( f(\varepsilon, h) \) of a disk at given temperature profile via Eq. (2.4).
Figure 2.2: isocontours of the Gaussian distribution functions for the Kuzmin disk with temperature: $Q = 1$, (top left panel), $Q = 1.25$, (top right panel), $Q = 1.75$ (bottom left panel), $Q = 2$ (bottom right panel). The construction scheme is described by Eq. (2.4) and Eqs. (2.23). Each diagram represents the number of star with angular momentum, $h$, and relative energy, $\eta = \varepsilon / \varepsilon_h$, where $\varepsilon_h$ is the energy of the star on a circular orbit with momentum $h$. The parameter $\eta$ therefore measures the eccentricity of the orbits. A hotter component is apparent at larger momentum for the $Q = 2$ disk where the contours are more widely spaced.

2.5. Example: constant temperature Gaussian Kuzmin disks

For the Kuzmin disk, the prescription described in the above for the parameters of the line profile yields

$$\sigma_\phi = \frac{Q}{4(1 + R^2)^{3/4}} \quad \text{and} \quad \langle v_\phi \rangle = \frac{R \left( 256 - 60 Q^2 + 128 R^2 - 21 Q^2 R^2 + 16 R^4 \right)^{1/2}}{4(1 + R^2)^{3/4} (4 + R^2)},$$
where $Q$ is Toomre’s number. From Eq. (2.17), $F_\phi$ becomes as a function of $h$

$$F_\phi [h,R] = \sqrt{2/\pi 3} Q^{-1} \exp \left[ -8 \left( 1 + R^2 \right)^{3/2} \left( \frac{h}{R} \left( \frac{256 - 60 Q^2 + 128 R^2 - 21 Q^2 R^2 + 16 R^4}{4 (1 + R^2)^{3/4} (4 + R^2)} \right)^{1/2} \right)^2 \right].$$

Differentiating with respect to $R$ gives

$$\frac{\partial \log F_\phi}{\partial R} = (4 + R^2) \left[ \frac{8 h^2 (2 - R^2) Q^2 R^3 (1 + R^2)^{1/4}}{3 R} \frac{4 R P_b}{4 h F_b R} + \frac{Q^2 (1 + R^2)^{3/4} (4 + R^2)^3}{6 h F_b R} \right], \quad (2.23)$$

with

$$F_b = \frac{-1024 + 216 Q^2 - 768 R^2 + 106 Q^2 R^2 - 192 R^4 + 7 Q^2 R^4 - 16 R^6}{(256 - 60 Q^2 + 128 R^2 - 21 Q^2 R^2 + 16 R^4)^{1/2}}, \quad (2.24a)$$

and

$$P_b = -256 + 60 Q^2 - 192 R^2 + 27 Q^2 R^2 - 48 R^4 - 4 R^6. \quad (2.24b)$$

The equations (2.23) and (2.24) together with Eq. (2.9) fully characterise a complete family of distribution functions parameterised by their temperature via Toomre’s number. Fig. 2.2 illustrates this inversion procedure. The parametrized azimuthal velocity profile is given on Fig. 2.3, together with their derived radial velocity profile. In contrast, Fig. 2.4 corresponds to the same method applied to the Isochrone disk, $\psi = 1/(1 + \sqrt{1 + R^2})$. Note that the relatively broader core of the Isochrone disk does not require as strong a hot component to match the imposed temperature profile.

**Figure 2.3:** Azimuthal (left) and radial (right) velocity distributions for the Gaussian Kuzmin-Toomre disks parametrized by their “$Q$” temperature. The left panel was parameterised and the right deduced using Eq. (2.6) and Eq. (2.4). The curves from top to bottom correspond to radii going from $R = 0.5, 1, \cdots 2.0$. The plain curve corresponds to a $Q = 1$ disk, the dashed curve to a $Q = 0.5$ disk, while the dotted curves corresponds to a $Q = 1.5$ disk. Note that, as expected, the hotter disks have broader radial velocity distributions.
2.6. Example: Power law disks

The potential and surface density for the power law disk read

$$\psi = R^{-\beta} \quad \text{and} \quad \Sigma = \frac{S_\beta}{2\pi} R^{-\beta-1}, \quad (0 < \beta < 1),$$  

where $S_\beta$ is given by:

$$S_\beta = \beta \left( \frac{\Gamma(1/2 + \beta/2) \Gamma(1 - \beta/2)}{\Gamma(1/2 - \beta/2) \Gamma(1 + \beta/2)} \right).$$  

It is assumed here that distances are expressed in terms of $R_0$, the reference radius, and energies in terms of $\psi_0 = \psi(R_0)$. (in units of $G = 1$). The pressures follow from Eq. (2.25) and the equation of radial support.
where the circular escape velocity, \( v \), can be determined too. The surface density, \( \Sigma \), and the azimuthal pressure, \( \Sigma \phi \), which make the solution of the resulting integral equation remarkably simple. Explicit inversions are given when \( f_\epsilon(h, \psi) \) scales like \( CY^{n+1}h^b \), the inversion yields

\[
\frac{1}{\sqrt{2\pi}} \int_0^\infty \frac{\partial \tilde{F}_\phi}{\partial \psi} d\psi = \frac{(n+1)\Gamma(n+1)}{\sqrt{2\pi}\Gamma(n+3/2)} (\psi^{n+1/2} h^b),
\]

The surface density, \( \Sigma \), and the azimuthal pressure, \( \Sigma \phi \), read in terms of \( \tilde{F}_\phi(h, \psi) \) via

\[
\Sigma = 2 \int_0^{\psi_c} F_\phi \psi d\psi = \frac{C 2^{b/2+1/2} \Gamma(2+n) \Gamma(b/2+1/2)}{(n+1)^{-1} \sqrt{2\pi} \Gamma(b/2+n+5/2)} R^{-\beta(3/2+n+b/2)-b},
\]

\[
\Sigma \langle \phi \rangle = 2 \int_0^{\psi_c} F_\phi \psi d\phi = \frac{C 2^{b/2+5/2} \Gamma(2+n) \Gamma(b/2+3/2)}{(n+1)^{-1} \sqrt{2\pi} \Gamma(b/2+n+7/2)} R^{-\beta(5/2+n+b/2)-b-1},
\]

where the circular escape velocity, \( \psi_c \), is equal to \( \sqrt{2\psi} \). Identifying Eqs. (29) with Eqs. (25)-(27) yields the relations

\[
b = \frac{2\beta n}{2-\beta} - 1, \quad n = \beta - 2 + \frac{2(\beta - 2)\beta}{Q^2 S_\beta^2}, \quad C = \frac{S_\beta 2^{-\beta/2-1/2} \Gamma(n+\beta/2+5/2)}{2\pi \Gamma(2+n) \Gamma(\beta/2+1/2)},
\]

which gives the distribution function

\[
f_+ (\epsilon, h) = S_\beta (-\epsilon)^{n+1/2} [h] \left[ \frac{2^{-5/2} \pi^{-3/2} \Gamma(\alpha+1)}{\Gamma(\alpha+3/2) \Gamma(\alpha-\alpha-1)} \left( \frac{h^2}{2} \right)^{\beta n/(2-\beta)-1} \right]
\]

where \( \alpha = 1 + 2n/(2-\beta) \). Here, the effect of the parameter \( n \) on temperature becomes clear when inverting Eq. (30) for \( Q \):

\[
Q = \frac{2-\beta}{S_\beta} \left[ \frac{2\beta}{2-\beta+n} \right]^{3/2}.
\]

These results for the power-law disks were also derived by Evans who used the Ansatz II described in section 3.1 and illustrated in section 3.3.

3. Distribution functions for a given surface density

As mentioned in section 1, the inversion problem for flattened disks has no unique solutions. Here the freedom in choosing the distribution function is exploited to choose special forms of distribution function which make the solution of the resulting integral equation remarkably simple. Explicit inversions are given for distribution functions whose even parts are of the form \( f_\phi(\epsilon, h) = (-\epsilon)^{n+1/2} G_\alpha(h^2/2) \) where \( h \) is the specific angular momentum and \( \epsilon \) the specific energy of a given star. Inversions are also provided when \( f_\phi(\epsilon, h) = h^{2n} \psi_\phi(\epsilon) \). Examples are given and illustrated graphically. The part of \( \phi \) which is antisymmetric in \( h = Rv_\phi \) cannot be determined from the surface density alone unless it is assumed that all the stars rotate in the same sense about the galaxy, in which case \( f_- = \text{sign}(h) f_+ \). More generally, if the mean velocity \( \langle \psi \rangle \) is specified — say by the asymmetric drift prescription — a similar Ansatz for \( f_- \) allows its explicit determination too. The inversion formulae require expressing powers of \( R \) times the surface density \( \Sigma(R) \) as functions of either the potential \( \phi(R) \)  of the related function \( Z(R) = R^2 \psi \). These formulae can only be applied to those density distributions for which the potential in the plane of the matter is known (either self-consistently or not). The other characteristics of the disk follow from the knowledge of \( f \) and cannot be specified à-priori.
3.1. Derivation

- Even component of \( f \): first Ansatz

A first Ansatz is to look for solutions for \( f_+ \) which take the form

\[
I \quad f_+(\varepsilon, h) = (-\varepsilon)^{n+\frac{1}{2}} |h| G_n \left( \frac{h^2}{2} \right),
\]

where \( n \) is first assumed to be an integer. Here the \(|h|\) is taken out for convenience; its appearance in no way implies that \( f_+(\varepsilon, 0) = 0 \) because \( G_n(h^2/2) \) may behave as \(|h|^{-1}\) for small \( h \). The quantity \( n \) measures the level of anisotropy in the disk. Large \( n \) correspond to cold, centrifugally supported disks; small \( n \) correspond to isotropic, pressure supported disks. Inserting the Ansatz into Eq. (1.5) gives

\[
\Sigma(R) = \frac{2^n}{R} \int_0^{R^2 \psi} G_n \left( \frac{h^2}{2} \right) \left[ \int_0^Y \frac{(-\varepsilon)^{n+\frac{1}{2}} d(-\varepsilon)}{\sqrt{Y - (-\varepsilon)}} \right] d \left( \frac{h^2}{2} \right).
\]

The inner integral can be transformed to \( Y^{n+1} I_n \) where

\[
I_n = \int_0^1 \frac{x^{n+\frac{1}{2}} dx}{\sqrt{1-x}} = \frac{\sqrt{\pi} \Gamma(n + \frac{3}{2})}{2 \Gamma(n + 3)} = \frac{(2n + 1)(2n - 1)\ldots1}{2^n(n + 1)!} \pi,
\]

with \( x = (-\varepsilon/Y) \). Re-expressing \( Y^{n+1} \) in terms of the potential, writing \( H \) for \( h^2/2 \) and \( Z \) for \( R^2 \psi \), this expression becomes on multiplication by \( R^{2n+3} \)

\[
S_n \equiv R^{2n+3} \Sigma(R) = 2^n I_n \int_0^Z (Z - H)^{n+1} G_n(H) \, dH,
\]

hence defining \( S_n \). \( Z \equiv R^2 \psi \) is known, so \( R^{2n+3} \Sigma(R) \) can be re-expressed as a function \( S_n(Z) \). Differentiating Eq. (3.4) \( n + 2 \) times with respect to \( Z \), using Eq. (3.3) and substituting \( H \) for \( Z \) gives:

\[
G_n(H) = \frac{1}{\sqrt{2} [(n + \frac{1}{2})(n - \frac{1}{2})\ldots\frac{1}{2}]} \pi \frac{1}{d\!dH} (d^n S_n(H))^{n+2}.
\]

Eq. (3.5) specifies the part of the distribution function which is even in \( h \). Demanding no counter-rotating stars would imply, for instance, \( f = f_+ + f_- = 0 \) for \( h < 0 \). In that case \( f_- = -f_+ \) for \( h < 0 \) which leads to \( f_- = \text{sign}(h) f_+(\varepsilon, h) \). These solutions are called maximally rotating disks. The formal solution, Eq. (3.5), must be non-negative for all \( h \) so that Eq. (1.5) corresponds to a realizable distribution of stars. Within that constraint, the choice of \( n \) is, in principle, free. There are many distribution functions which give the same surface density distribution because there are two functional freedoms in \( f(\varepsilon, h) \) and only one function \( \Sigma(R) \) is given. Here it has been shown that even within the special functional form of Ansatz I there is potentially a solution for any chosen integer \( n \). When \( n \) is not an integer, \( n = n_0 - \alpha, \quad 0 < \alpha < 1 \), Eq. (3.2) may still be inverted using Abel transforms, though the final solution involves an integral which might require numerical evaluation

\[
G_n(H) = \sin(\pi\alpha) 2^{-3/2} \pi^{-1} I_n^{-1} \left[ \int_0^H \left( \frac{d\!dZ}{dZ} S_n \right)^{n+1} \frac{dZ}{(H - Z)^{1-\alpha}} + \left( \frac{d\!dZ}{dZ} S_n \right)_{Z=0} \frac{1}{H^{1-\alpha}} \right],
\]

where \( I_n \) is defined for non integer \( n \) by the first identity in Eq. (3.12). Alternatively, there are many ways in which it is possible to re-express \( \Sigma(R) \) in the general form \( \Sigma(R) = \sum_n R^{-2n-3} S_n^*(R^2 \psi) \). For any particular sum, a distribution function \( G_n^*(H) \) can be found which gives rise to that part of the density given by the \( n \)th term \( S_n^* \). Indeed the relationship between \( G_n^* \) and \( S_n^* \) is just that given by Eq. (3.5) between \( G_n \) and \( S_n \). Since the original integral equation is linear, it follows that each expression for \( \Sigma(R) \) yields a distribution
function in the form \( f_+ (\varepsilon, h) = \sum_n (-\varepsilon)^{n+\frac{1}{2}} |h| G_n^* (h^2/2) \). There is no requirement that each \( G_n^* \) should be positive provided that the sum \( f_+ \) is positive for all \( \varepsilon \) and \( h \). Thus the expression for the density and distribution functions gives a very considerable extension of the original Ansatz Eq. (3.1).

- **Odd Component of \( f \) and asymmetric drift**

  When the average azimuthal velocity field is known, a similar inversion procedure is available in order to specify the odd part in \( h \) of the distribution function. This field may be assumed to be given by the asymmetric drift equation which takes the form

  \[
  \Sigma \langle v_\phi \rangle^2 = p_\phi - p_R \left( \frac{\kappa^2 R^2}{4V_c^2} \right),
  \tag{3.7}
  \]

  where the azimuthal pressure, \( p_\phi \), and the radial pressure, \( p_R \), are derived from the even component of \( f_- \), and the epicyclic frequency and the circular velocity \( \kappa \) and \( V_c \) follow from \( \psi \). Formally, the only difference from the previous analysis is that in place of Eq. (1.2), the integral equation relating \( f_- \) and \( \langle v_\phi \rangle \) becomes

  \[
  R \Sigma \langle v_\phi \rangle = 4 \int_0^{\sqrt{2R^2\psi}} \frac{f_- (\varepsilon, h)}{\sqrt{2(\varepsilon + \psi) R^2 - h^2}} h d\varepsilon dh .
  \tag{3.8}
  \]

  Replacing Ansatz I by Ansatz I’:

  \[
  I' \quad f_- (\varepsilon, h) = (-\varepsilon)^{n+\frac{1}{2}} \tilde{G}_n \left( \frac{h^2}{2} \right) \text{sign}(h) ,
  \tag{3.9}
  \]

  yields a solution for \( \tilde{G}_n \) which is formally equivalent to Eq. (3.5), but with \( \tilde{S}_n \) defined by \( \tilde{S}_n(Z) = R^{2n+4} \Sigma \langle v_\phi \rangle \). The extension to solutions for \( f_- \) when \( \Sigma \langle v_\phi \rangle \) is a linear combination of \( R^{-2n-4} \tilde{S}_n(Z) \) is straightforward.

- **Alternative Ansatz**

  In place of Ansatz I, consider a different Ansatz:

  \[
  II \quad f_+(\varepsilon, h) = h^{2n} F_n (\varepsilon) ,
  \tag{3.10}
  \]

  where \( n \) is also first assumed to be an integer. Inserting Eq. (3.10) into Eq. (1.5), and reversing the order of the integrations, yields

  \[
  \Sigma (R) = 4 \int_0^0 F_n (\varepsilon) \left[ \int_0^{X^{1/2}} \frac{h^{2n} dh}{\sqrt{X - h^2}} \right] d\varepsilon ,
  \tag{3.11}
  \]

  where \( X \) was defined in Eq. (2.11). The inner integral is \( X^n J_n \), where

  \[
  J_n = \int_0^1 \frac{x^{2n} dx}{\sqrt{1 - x^2}} = \sqrt{\pi} \frac{\Gamma \left( \frac{1}{2} + n \right)}{2 \Gamma (1 + n)} = \left[ (n - \frac{1}{2}) (n - \frac{3}{2}) \cdots \frac{1}{2} \right] \pi / (2n!) ,
  \tag{3.12}
  \]

  with \( x = h/X^{\frac{1}{2}} \). Thus Eq. (3.11) becomes, using Eq. (2.11) for \( X^n \)

  \[
  \Sigma (R) = 2^{n+2} J_n R^{2n} \int_{-\psi}^0 (\varepsilon + \psi)^n F_n (\varepsilon) d\varepsilon .
  \tag{3.13}
  \]

  Re-expressing \( R^{-2n} \Sigma(R) \equiv S_n(\psi) \) and writing \( \varepsilon \) for \( -\psi \) yields

  \[
  F_n (\varepsilon) = \frac{2^{n-1}}{\left[ (n - \frac{1}{2}) (n - \frac{3}{2}) \cdots \frac{1}{2} \right] \pi} \left( \frac{d}{d(-\varepsilon)} \right)^n S_n (-\varepsilon) ,
  \tag{3.14}
  \]
Equation Eq. (3.14) was derived independently by Sawamura (1987) [17]. When $n$ is not an integer, the solution to Eq. (3.13) can still be found, though with a supplementary integration (with $n = n_0 - \alpha$, $n_0$ the closest upper integer), to give

$$F_n (\varepsilon) = \frac{2^{-n-2} \pi^{-1} J_n^{-1} \sin \pi \alpha}{((n_0 + 1 - \alpha) (n_0 - \alpha) ... (1 - \alpha))} \left[ \int_0^\varepsilon \frac{\psi}{\varepsilon} \frac{d S_n}{d \psi} \left( \frac{\psi - \varepsilon}{\varepsilon - \psi} \right)^{1-\alpha} \right] \left[ \int_0^\varepsilon \frac{1}{\varepsilon^{1-\alpha}} \frac{d S_n}{d \psi} \right]_{\psi=0}, \quad (3.15)$$

where $J_n$ is defined for non integer $n$ by the first identity in Eq. (3.12). The Ansatz in Eq. (3.10) may be also generalised when $\Sigma(R)$ is expressed as linear combinations of $R^{2n} S_n^\alpha (\psi)$.

- **Alternative Ansatz: odd component**
  
  The above procedure is also straightforward to implement in order to constrain the antisymmetric component of the distribution function. Indeed Ansatz II':

$$II' \quad f_-(\varepsilon, h) = h^{2n-1} \tilde{F}_n (\varepsilon), \quad (3.16)$$

may be chosen in place of Ansatz I'. Then Eq. (3.11) becomes

$$R \Sigma \langle \psi \rangle = 4 \int_{-\psi}^0 \tilde{F}_n (\varepsilon) \int_0^{X^{1/2}} \frac{h^{2n} dh}{\sqrt{X - h^2}} d\varepsilon. \quad (3.17)$$

Hence the solution for $F_n$ is formally identical to Eq. (3.14) but with $\tilde{S}_n$ substituted for $S_n$, where $\tilde{S}_n (\psi) = R^1 2n \Sigma \langle \psi \rangle$. Again, the integral equation is linear, so the prescription defined by Eq. (3.14) when $\Sigma \langle \psi \rangle$ can expressed as linear combinations of $R^{2n-1} \tilde{S}_n (\psi)$.

### 3.2. Disk properties

- **Radial velocity dispersion**

  The average radial velocity dispersion, $\sigma_R^2$, defined by Eq. (1.6) is an important quantity for the local stability of the disk. Given Eq. (3.1) and using Ansatz I, Eq. (1.6) may be rearranged as

$$\Sigma \sigma_R^2 = \frac{2 \tilde{F}_n}{R^{2n+5}} \int_0^Z (Z - H)^{n+2} G_n (H) dH. \quad (3.18)$$

The calculation of $\sigma_R^2 (R)$ is therefore straightforward once the function $G_n$ has been found. Note the similarity between Eq. (3.18) and Eq. (3.4). In fact this similarity provides a check for self-consistency since one should have

$$\frac{\partial}{\partial Z} \left( \sigma_R^2 \Sigma R^{2n+5} \right) = 2 (n + 2) I_n/I_n R^{2n+3} \Sigma = R^{2n+3} \Sigma.\quad (3.19)$$

Putting Eq. (3.10) into Eq. (1.6) yields, for Ansatz II:

$$\Sigma \sigma_R^2 = 2^{n+3} J_n R^{2n} \int_{-\psi}^0 (\varepsilon + \psi)^{n+1} F_n (\varepsilon) d\varepsilon. \quad (3.20)$$

The similarity between Eqs. (3.13) and (3.20) yields the identity:

$$\frac{\partial}{\partial \psi} (\Sigma \sigma_R^2 R^{-2n}) = 2 (n + 1) \Sigma R^{-2n} J_n / J_n = \Sigma R^{-2n}. \quad (3.21)$$

- **Azimuthal velocity dispersion**
The average azimuthal velocity dispersion, $\sigma_\phi$, is an observable constraint on models which satisfies

$$\Sigma \left( \sigma_\phi^2 + \langle v_\phi^2 \rangle \right) = \Sigma \langle v_\phi^2 \rangle = \frac{4}{R^3} \int_0^{\sqrt{R^2 \psi}} \int_0^\psi \int_{\frac{\psi}{\sqrt{R^2 \psi}}} f_+ (\varepsilon, h) \frac{h^2}{\sqrt{2 (\varepsilon + \psi) - \frac{h^2}{R^2}}} \, dh \, d\psi$$

(3.22)

where the mean azimuthal velocity $\langle v_\phi \rangle$ is given in terms of $f_-$ by linear combinations of Eqs. (3.8) and (3.17). Putting Eq. (3.1) into Eq. (3.22) and following the substitutions Eq. (3.1), Eq. (3.3) yields, for the first Ansatz,

$$R^{2n+5} \Sigma \left( \sigma_\phi^2 + \langle v_\phi^2 \rangle \right) = 2^{\frac{n}{2}} I_n \int_0^Z (Z - H)^{n+1} HG_n (H) \, dH.$$  

(3.23)

Putting Eq. (3.10) into Eq. (3.22) gives, for the second Ansatz:

$$\Sigma \langle v_\phi^2 \rangle = 2^{n+4} J_{n+1} R^{2n} \int_0^\psi (\varepsilon + \psi)^{n+1} F_n (\varepsilon) \, d\varepsilon.$$  

(3.24)

Note that $\sigma_R$, $\psi$, $\Sigma$ and $\langle v_\phi^2 \rangle$ are related via the equation of radial support Eq. (2.7). The properties of the disks constructed with Ansatz I & II are illustrated in Figs. 3.2 and Fig. 3.1 for the examples described below.

3.3. Examples

- **Isochrone disks**
  Distribution functions for the flat Isochrone are simplest with the assumptions of Ansatz II. In the equatorial plane, the Isochrone potential reads

$$\psi = GM/(b + r_b) \quad \text{where} \quad r_b^2 = R^2 + b^2.$$  

(3.25)

The corresponding surface density is

$$\Sigma = \frac{Mb}{2\pi R^3} \left\{ \ln \left[ (R + r_b)/b \right] - R/r_b \right\} = \frac{Mb}{2\pi R^3} L.$$  

(3.26)

But $R^2 \psi = Z = (r_b^2 - b^2)GM/(r_b + b) = GBM(s - 1)$, where $s$ is the dimensionless variable $r_b/b$. Therefore

$$\Sigma R^{2n+3} = \frac{Mb}{2\pi} R^{2n} L(s) = \frac{Mb^{2n+1}}{2\pi} (s^2 - 1)^n L(s),$$  

(3.27)

where $L(s) = \ln(\sqrt{s^2 - 1} + s) + \sqrt{s^2 - 1}/s$. Note that $L'(s)$ is quite simple: $L'(s) = \sqrt{s^2 - 1}/s^2$. With this notation,

$$\left( \frac{\partial}{\partial Z} \right)^{n+2} (R^{2n+3} \Sigma) = \frac{Mb^{2n+1}}{2\pi (GBM)^{n+2}} \left( \frac{\partial}{\partial s} \right)^{n+2} \left[ (s^2 - 1)^n L(s) \right],$$  

(3.28)

which implies, in turn, that

$$G_n \left( \frac{h}{2} \right) = \frac{M^{n-1} b^{n-1}}{2^{5/2} \pi G b (n + 1)!!} \left( \frac{\partial}{\partial s} \right)^{n+2} \left[ (s^2 - 1)^n L(s) \right]_{s=1+h^2/2}.$$  

(3.29)

Therefore, the final distribution function (or rather its symmetric part, $f_+$) reads

$$f_+ (\varepsilon, h) = \frac{\varepsilon / (GMb)^{n+1/2}}{4\pi^2 G b (n + 1/2)!!} \left( \frac{h^2}{2GMb} \right)^{1/2} \left( \frac{\partial}{\partial s} \right)^{n+2} \left[ (s^2 - 1)^n L(s) \right].$$  

(3.30)

with $s = 1 + h^2/(2GMb)$. Using these distributions functions, Toomre’s Q criterion for radial stability ($Q = \sigma_R / \kappa_{3.36G\Sigma}$) may be calculated via Eq. (3.18). Note that for the Isochrone potential

$$\sigma_R^2 (R) = \frac{2^{5/2} T_n}{R^{2n+5} \Sigma (R)} \int_0^Z (Z - H)^{n+2} \left[ \left( \frac{\partial}{\partial s} \right)^{n+2} L(s) (s^2 - 1)^n \right]_{s=H+1} dH.$$  

(3.31)
The “Toomre Q number”, $Q = \sigma_R \kappa / (3.36 \Sigma_0)$, against radius $R$ for the Isochrone model described by Eqs. (3.30) and (3.31). The parameter $n$ in Eq. (3.30) corresponds to a measure of the temperature of the disk.

- **Toomre-Kuzmin disks**

  Ansatz II is more appropriate when seeking distribution functions for the Toomre-Kuzmin disks. In the equatorial plane, the potential of the disk reads $\psi = -GM/r_b$, while the surface density is $\Sigma = (2\pi)^{-1}GMb/r_b^2$. Ansatz II yields the distribution (in units of $b$ and $GM/b$)

  $$f_n (\varepsilon, h) = \frac{2^{n-2} h^{2n}}{[n - \frac{1}{2}](n - \frac{3}{2})...\frac{1}{2}} \pi^2 \left[ s^{2n+3} \right]_{\varepsilon \rightarrow -\varepsilon}^{(n+1)} \left[ 1 - s^2 \right]^{n}.$$  

  (3.32)

  This example illustrates the drawbacks of Ansatz II. The factor $h^{2n}$ removes all zero angular momentum orbits. Self-consistency yields a solution with circular orbits near the angular momentum origin ($f$ scales like powers of $h^2/[1 - (-\varepsilon)^{2}]^{1/2}$). In position space, this implies that the inner core of the galaxy is made of circular orbits! Realistic distribution functions will therefore require superpositions of solutions of the family defined by Eq. (3.32), as illustrated by Fig. 3.2. It can be shown from Eqs. (3.32) and Eq. (3.24) that $\Sigma \sigma_R^2$ does not fall off fast enough to yield cold disks (it ought to scale like $\psi^{\beta}$ to lead to constant $Q$ profiles).

- **Power law disks**

  The simplicity of power law disks leads to solutions for the inversion problem which may be extended to models with a continuous free “temperature” parameter as described above. This temperature may be varied continuously between the two extremes corresponding to the isotropic and cold disk. The potential and surface density for the power law disk were given by Eqs. (2.25)–(2.26). The inversion method corresponding to Ansatz I may be carried out and, using (3.6), yields the distribution function found in Eq. (2.31). The velocity dispersions are recovered from Eqs. (3.18) and (3.22):

  $$\sigma_\phi^2 + \langle v_\phi \rangle^2 = \frac{n \beta}{2 - \beta + n} R^{-\beta}, \quad \sigma_R^2 = \frac{2 - \beta}{2 (2 - \beta + n)} R^{-\beta}.$$  

  (3.33)
Figure 3.2: radial (bottom lines) and azimuthal (top lines) pressure for a Kuzmin disk constructed by linear superposition of Eqs. (3.32). The weights are $(2/3, 1/12, 1/4 - i/12, i/24, i/24)$ where $i = 10, 13 \cdots 22$ for the $n = 0, 1 \cdots$ models. It was checked that the superposition has positive mass everywhere.

By construction, these dispersions satisfy the equation of radial equilibrium:

$$
\langle v_{\phi}^2 \rangle - R \frac{\partial \psi}{\partial R} = - \frac{\beta (2 - \beta)}{(2 - \beta + n)} R^{-\beta} = \frac{\partial (R \Sigma \sigma_{\phi}^2)}{\Sigma \partial R}.
$$

(3.34)

4. Observational implications

The inversion method described in section 2 may be applied on observational data which are now becoming available (e.g. Bender et al. (1994) \cite{bender_1994} and Fisher (1994) \cite{fisher_1994} have presented line of sight velocity distributions of elliptical and S0 galaxies obtained from spectra with (60 km/s) resolution and S/N ~ 50 ). The observations may be carried out as follows: consider a disk galaxy seen almost edge on. It should be chosen so that it looks approximately axisymmetric and flat. It may contain a fraction of gas in order to derive the mean potential $\psi$ from the observed H I velocity curve (Alternatively, the potential may be derived directly from the kinematics if the asymmetric drift assumption holds). Putting a slit on its major axis and cross correlating the derived absorption lines with template stellar spectra yields an estimate of the projected velocity distribution which should essentially correspond to the azimuthal line profile $F_\phi(R, v_\phi)$. An estimate of the radial line profile $F_R(R, v_R)$ arises when the slit is placed along the minor axis. But $F_R(R, v_R)$, is given by Eq. (2.6) while Eq. (2.4) gave $f$ expressed as a function of $F_\phi(v_\phi, R)$. Therefore, putting Eq. (2.4) into Eq. (2.6) provides a simple way of predicting the radial line profiles if the azimuthal line profiles are given. This procedure is illustrated on Fig. 2.3 for simulated data without noise. The surface density $\Sigma$ follows from both $F_R(v_R, R)$ and $F_\phi(v_\phi, R)$ and could also be compared with the photometry of that galaxy. The likely discrepancy between the predicted and the residual data may be used to assess the limitations of
the reconstruction scheme. Indeed, the above prescription for the inversion relies on a set of hypotheses for the nature of the flow (axial symmetry, thin disk approximation etc.) and the quality of measurements. The following features of an observed galaxy may constrain the scope of this analysis.

- **Thick disk, with random motions perpendicular to the plane of the galaxy.**

  The measured line profile yields a mean value corresponding to the integrated emission through the width of the disk; as the galaxy is not edge on, the result of the cross correlation of the spectra does not give $F_\phi$ and $F_R$ exactly, but rather gives the projection onto the line of sight of the full velocity distributions $F_{\phi,z}(v_\phi, v_z, R)$ and $F_{R,z}(v_R, v_z, R)$. However, it is consistent with the thin disk approximation to assume that the motion in the plane is decoupled from that perpendicular to the plane. The observed line profile, $F_v(R, v)$, then corresponds to the convolution of these functions with the component of the velocity distribution perpendicular to the plane of the galaxy, $F_z(R, v_z)$. Formally, for the line profile measured along the major axis, this reads

$$F_v(R, v) = \int_{-\infty}^{\infty} F_z[R, v \sin(i) + u \cos(i)] F_\phi[R, v \cos(i) - u \sin(i)] \, du$$

where $v$ and $u$ are the velocities along the line of sight, and perpendicular to the line of sight respectively. A similar expression for the line profile measured along the minor axis involving $F_R$ follows. Here, the angle $i$ measures the inclination of the plane of the galaxy with respect to the line of sight. At this level of approximation the velocity distribution normal to the plane is accurately described by a centered Gaussian distribution; its variance follows roughly from the equation of radial support and is used to deconvolve Eq. (4.1) (or practically to convolve the predicted line profile pairs and compare them to the data). Note that a rough estimate of the error induced is $\sigma_d^2/\sigma_z^2 \tan^2(i) \sim 1\%$ in our Galaxy when $i = 20$ degrees. Note that when looking at an ellipse that corresponds to a circle in the galaxy’s plane one would measure the dispersion $(\sigma_R^2 + \sigma_z^2) \cos^2(i)/2 + (\sigma_R^2 - \sigma_\phi^2) \cos(2\phi) \cos^2(i)/2 + \sigma_z^2 \sin^2(i)$. If it is assumed that the asymmetric drift hypothesis holds, – which can be checked by measuring $\langle v_\phi \rangle$ – then it should in principle be possible to measure indirectly $\sigma_z$.

- **Non-axisymmetric halo or disk, with dust and absorption.**

  The departure from axisymmetry reduces the number of invariants – angular momentum is not conserved, which invalidates this method. Some fraction of the population of stars may then follow chaotic orbits which cannot be characterised by a distribution function. Some real galaxies either present intrinsic non axisymmetric features such as bars, spiral or lopsided arms. Triaxial halos may induce warps within the disk, hence compromising the evaluation of the kinematics. Non-axisymmetrically distributed dust or molecular clouds may also affect measurement of the light from the stellar disk. On the positive side, non-axisymmetric features often appear as small perturbations which may only contribute weakly to the overall mass distribution. Under such circumstances, an estimate of the underlying distribution function for the axisymmetric component of the disk may be extracted from the data.

  Sets of initial conditions for an N-body simulation may also be extracted from the distribution function. The time evolution of the code would yield constraints on the stability of the observed galaxy. The above analysis could be carried out on an isolated S0 galaxy and on an Sb galaxy showing a grand design spiral pattern (or alternatively on an Sa presenting an apparent lopsided HI component to isolate plausible different instability modes). The relative properties of the two galaxies and their supposedly distinct fates when evolved forward in time should provide an unprecedented link between the theory of galactic disks and the detailed observation of the kinematics of these objects. The authors [17] have written a numerical procedure to implement linear stability analysis on arbitrary mass models and kinematics for flat and round galactic disks.

- **Data quality.**

  The method described in section 2 involves a deconvolution of the measured line profiles which yields the velocity distributions which, in turn, are related by Abel transforms. Both transformations are very sensitive to the unavoidable noise in the data. Sources of noise are poor seeing, photon noise, detector noise, thermal and mechanical drift of the spectrograph, poor telescope tracking, etc... It is therefore desirable to construct from Eqs. (2.4), (2.6) and (2.17) a set of pairs of Gaussian velocity distributions $(F_\phi, F_R)$ on which to project the observed quantities since this will help to assess the errors induced by this noise. Indeed, Eq. (2.4) is linear; therefore any superposition of Gaussian profiles corresponding to a good fit of the observed line profile
will lead to a distribution function expressed in terms of sums of solutions of Eq. (2.4). These Gaussians may in turn be identified with populations at different temperatures – say corresponding to a young component and an older population of stars. Formally this translates as

\[ F_\phi(R, v_\phi) = \sum_i S_i(R) \exp \left( -\frac{(v_\phi - v_i(R))^2}{2\sigma_i^2(R)} \right), \tag{4.2} \]

where the functions \( S_i(R), \sigma_i(R) \) and \( v_i(R) \) should be fitted to the observed azimuthal line profiles. Each component in Eq. (4.2) may then be identified with those in Eq. (2.20), yielding the temperature profile of that population. This approach should yield the best compromise between fitting the radial and the azimuthal profile while avoiding the deconvolution of noisy data. It also provides directly a least squares estimate of the errors. The observations described in this section may be achieved with today’s technology.

5. Conclusions

Simple general inversion formulae to construct distribution functions for flat systems whose surface density and Toomre’s Q number profiles were given and illustrated. The purpose of these functions is to provide plausible galactic models and assess their critical stability with respect to global non-axisymmetric perturbations. The inversion is carried out for a given azimuthal velocity distribution (or a given specific energy distribution) which may either be observed or chosen accordingly. When the azimuthal velocity distribution is measured from data, only a subset of the observationally available line profiles, namely the line profiles measured on the major axis, is required to re-derive the complete kinematics from which the line profile measured along the minor axis is predicted. This prediction may then be compared with the observed minor axis line profile. The Ansatz presented in section 3 yield direct and general methods for the construction of distribution functions compatible with a given surface density for the purpose of theoretical modelling. These distribution functions describe stable models with realistic velocity distributions for power law disks, and also the Isochrone and the Kuzmin disks. Note that the inversion technique described in section 2 can be implemented in the relativistic régime as discussed by Pichon & Lynden-Bell [16].

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