Non-linear resonance in the accretion disk of a millisecond pulsar

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ABSTRACT

Twin quasi-periodic millisecond modulations of the X-ray flux (kHz QPOs) have recently been reported from an accreting 2.5 ms X-ray pulsar. We identify modes of disk oscillations whose frequencies are in agreement with the observed ones when the rotating neutron star is modeled with realistic equations of state. The frequency difference of the twin QPOs, equal to about one half of the neutron-star spin rate, clearly indicates that resonant oscillations of the accretion disk have been observed. Similar non-linear resonances may also be spontaneously excited in the accretion disk. The two QPO frequencies in the pulsar system are close to a 5:7 ratio and this suggests a link with the QPOs in black hole systems, where frequency ratios of 2:3 and 3:5 have been reported.

Subject headings: Stars: neutron – X-rays: general

1. Introduction

A millisecond pulsar that accretes matter from a binary stellar companion provides a unique probe of the physics of accretion flow a few Schwarzschild radii away from the neutron star. In contrast with the case of the slowly rotating steady X-ray pulsars, the magnetic field of the (X-ray transient) millisecond pulsar is too weak to prevent the accreting gas from orbiting the neutron star very close to its surface. Hence, in such a pulsar oscillations in the inner parts of the disk may be observed at their characteristic frequency of several hundred Hertz. The pulsar is expected to disturb the accretion disk at its spin frequency. That it does, is demonstrated by the discovery in the 2.5 ms accreting pulsar of a frequency difference between the two QPOs equal to one-half the pulsar spin rate.
We point out that the two QPO frequencies themselves may be commensurable with the spin frequency and with each other. The two simultaneously observed QPO frequencies are close to 500 Hz and 700 Hz, i.e., they may be in a 5:7 ratio. Twin QPOs have been proposed to be a manifestation of non-linear resonance in relativistic accretion disks (Kluźniak and Abramowicz 2001), and the resulting suggestion that twin QPO frequencies should be in the ratio of small integers has subsequently been confirmed for a number of black hole transients (Abramowicz and Kluźniak 2001; Remillard et al. 2002; Kluźniak and Abramowicz 2002; McClintock and Remillard 2003) and for the bright steady source Sco X-1 (Abramowicz et al. 2003a).

The two QPOs discovered in the 2.5 ms accreting pulsar (Wijnands et al. 2003) are similar to QPOs previously detected in many unpulsed X-ray sources, in which an accretion disk is thought to feed matter and angular momentum to the neutron star or black hole (van der Klis et al. 2000; Lewin et al. 1995). Such QPOs in black hole systems were interpreted as oscillations of the accretion disk (Wagoner 1999; Kato 2001), but a direct proof of this was lacking, and for neutron star systems other models had been proposed (van der Klis et al. 2000). SAX J1808.4-3658 is the first kHz QPO source in which the rotational period of the neutron star has been unambiguously determined, and the first in which the frequency difference of the twin QPOs is known to be a subharmonic of the stellar rotation rate. For the first time, we have direct evidence that kHz QPOs are caused by disk oscillations.

2. Non-linear resonance and epicyclic frequencies

The presence of a sub-harmonic frequency is a clear signature of non-linear resonance (Nayfeh and Mook 1979). Theodor von Karman observed in 1940 that high-frequency vibrations of an airplane engine excite lower-frequency resonances in the airframe (von Karman 1940; Nayfeh and Mook 1979). A tragic example is known when a subharmonic resonance was excited in an airplane wing, which in turn excited a subharmonic resonance in the rudder, at 1/2 the wing eigenfrequency (Lefschetz 1956; Nayfeh and Mook 1979). In the present case, the 2.5 ms pulsar SAX J1808.4-3658 plays the role of the engine, while the accretion disk (whose response is observed as the quasi-periodic oscillations) is the airframe with its own set of known eigenfrequencies. The fact that the QPO frequency difference coincides with one half the known pulsar spin frequency,

\[(694 \pm 4) \text{ Hz} - (499 \pm 4) \text{ Hz} = (195 \pm 6) \text{ Hz} \approx \frac{1}{2}(401 \text{ Hz}),\]

follows directly from the nature of the system. The structure of the accretion disk is determined by the non-linear equations of hydrodynamics, and the effective gravitational potential (in which the accreting gas moves) is not harmonic, so a non-linear response is expected. Below, we discuss in detail how this comes about for an accretion disk in the space-time of a rotating neutron-star.

The motion of a test particle in nearly circular orbits close to the equatorial plane can be decomposed into three components, circular planar motion at the orbital frequency \(\Omega \equiv 2 \pi \nu_{\text{orb}},\)
harmonic radial motion at the radial epicyclic frequency $\kappa \equiv 2\pi \nu_r$, and harmonic vertical motion at the meridional epicyclic frequency $\zeta \equiv 2\pi \nu_{\text{vert}}$. In Newtonian theory of spherically symmetric gravitating bodies the three frequencies coincide, but in Einstein’s gravity $\kappa < \Omega$. (The relativistic precession of the perihelion of Mercury occurs at the rate $\Omega - \kappa$.) The same three frequencies are important in a discussion of fluid motion about a gravitating body in general relativity (Kato 1998). The accretion disk is a body of hot gas that is supported against infall primarily by rotation and is nearly in hydrostatic equilibrium (Lewin et al. 1995; Kato 1998). Theories of accretion (Shakura and Sunyaev 1973; Jaroszynski et al. 1980) admit solutions in which the disk thickness is much smaller than its radial extent, but also solutions in which the disk has the geometry of a torus. Like other extended bodies in equilibrium, the disk is capable of motion in a variety of modes. In the linear regime, small oscillations of geometrically thin accretion disks, as well as waves in this body of fluid, have been extensively studied in the Kerr metric (Wagoner 1999; Kato 2001). The radial vibrations of two-dimensional models of geometrically thick disks (accretion torii) have been investigated numerically in the Schwarzschild metric (Rezzolla et al. 2003). Qualitatively, these black-hole metrics are similar to the metrics of neutron stars.

We have studied numerically the response to a transient external perturbation of an ideal-gas torus that is initially in equilibrium rotation about a (Schwarzschild) black hole or neutron star. We find that the torus performs harmonic oscillations both in the radial and vertical directions, even if the impulse imparted to it at the beginning of the computation is purely radial. Further, the vertical oscillations have variable amplitude. Both these effects speak of a non-linearity in the system. The vertical and radial displacements of the center of the torus (defined as the point of maximum pressure in a meridional cross-section) are shown as a function of time in Fig. 1. The radial displacement has been rescaled to fit the graph—in the simulation described here, the amplitude of radial motion is actually larger than the amplitude of vertical motions.

For the numerical simulation (Fig. 1), we use a Newtonian 2-d smooth particle hydrodynamics (Monaghan 1992, SPH) code (Lee and Ramirez-Ruiz 2002) simulating ideal gas with adiabatic index 4/3 in a pseudo-potential (Kluźniak and Lee 2002) $\psi_{KL}(r) = [1 - \exp(r_{ms}/r)]GM/r_{ms}$, which reproduces the Schwarzschild ratios of the orbital and epicyclic frequencies, $\zeta/\Omega = 1$, $(\kappa/\zeta)^2 = 1 - r_{ms}/r$. Here, $r_{ms} = 6M(G/c^2)$ is the radius of the marginally stable orbit. A torus in equilibrium, with $r_0 = 12.25M(G/c^2)$, was perturbed at time $t = 0$ by imparting to it a radial velocity field with magnitude proportional to $\sqrt{r_{ms}/r}$. The frequencies of the resulting motions are found to be inversely proportional to the central mass $M$, whose value was adjusted to match the observed QPO frequencies.

The two frequencies observed in this numerical simulation coincide, within errors, with the known epicyclic frequencies, $\kappa$ and $\zeta$, at the equilibrium radial position, $r_0$, of the locus of maximum pressure, $r_c$. We conclude that a slender torus responds to a radial external perturbation with oscillations occurring at frequencies equal to the two epicyclic frequencies, $\kappa(r_0)$ and $\zeta(r_0)$, at that radius where the pressure of the torus in equilibrium is highest. An important implication is that these frequencies can vary if the torus varies with time as it accumulates less or more of the matter.
flowing through it.

For the radial perturbation applied in the simulation, the amplitude of vertical motion is smaller than that of the radial motion. Presumably the ratio of the amplitudes depends on the perturbation applied. Quantifying this requires further study. A tilted magnetic dipole may perturb the disk quite strongly in the vertical direction. In the remainder of this section we assume that the perturbation is such that only one of the two motions is typically excited with a large amplitude, and that the other would not be as easily discernible in an X-ray observation, unless its initially much smaller amplitude were amplified by a resonance.

A pulsar is a rotating neutron star with a strong magnetic dipole not aligned with the rotation axis. As a result, in the 2.5 ms accreting pulsar (Wijnands and van der Klis 1998) the accretion disk suffers a periodic disturbance at the spin frequency $\nu_1 = 401$ Hz. Observations indicate the presence of a QPO varying in frequency between 280 and 750 Hz (Wijnands et al. 2003). We interpret this as one of the two epicyclic frequencies, $\nu_0$, and expect that a second QPO will be present at the second epicyclic frequency, when a resonance occurs at frequency $\nu_{\text{res}}$ between the corresponding disk oscillation and the disturbing forces.

Twin QPOs have been observed in this source on only one occasion, and it is not clear to us whether the single variable frequency $\nu_0$ present in the remaining observations corresponds to the upper or the lower QPO, i.e., to $\nu_{\text{vert}}$ or to $\nu_r$ (but see below). A simple and general reason why the difference is equal to one half the spin frequency, $\nu_{\text{res}} - \nu_0 = \nu_1/2$, and not the spin frequency, $\nu_1$, can be given if $\nu_0 = \nu_r$. Then $\nu_{\text{res}} = \nu_{\text{vert}}$ at a particular radius, and the resonance would occur in an oscillator without quadratic anharmonicity terms. The restoring force in the vertical motion of the torus is obtained by taking the $z$ derivative of the effective potential expanded about the equatorial plane. Because the potential is symmetric under reflection in this plane, this expansion has only even powers of the cylindrical co-ordinate $z$, $V(z) = V_0 + (\zeta z)^2/2 + \beta z^4/4 + \ldots$. The vertical motion of the torus can then be described as that of an oscillator with (angular) eigen-frequency $\zeta$, and a cubic anharmonicity in leading order, $\beta z^3$.

Consider resonance in an accretion disk which can be described as a damped oscillator with a cubic anharmonicity. In view of the very strong disturbance by the pulsar, the resonance may be driven by a combination of the pulsar spin frequency $\nu_1$ and another, much weaker, disturbance at frequency $\nu_2$, which we take to be a harmonic (i.e., an integer multiple) of the QPO frequency $\nu_0$. In the present case the frequencies of the two harmonic driving forces satisfy $\nu_2 > \nu_1$. The oscillator may be in resonance with both of these forces at the same time (Nayfeh and Mook 1979, simultaneous resonance) in one of three cases:

a) $\nu_{\text{res}} \approx 2\nu_1 \pm \nu_2$, or
b) $\nu_{\text{res}} \approx 2\nu_2 \pm \nu_1$, or
c) $\nu_{\text{res}} \approx (\nu_2 \pm \nu_1)/2$.

The single occasion when two QPOs were detected in the 2.5 ms pulsar corresponds to case c), with
\[ \nu_2 = 2\nu_0, \text{ i.e., } \nu_{\text{res}} \approx \nu_0 \pm \nu_1/2. \] Since for neutron stars in general relativity \( \nu_{\text{vert}} > \nu_r \), the upper sign corresponds to the case \( \nu_{\text{res}} = \nu_{\text{vert}}, \nu_0 = \nu_r \), the lower sign to the case \( \nu_{\text{res}} = \nu_r, \nu_0 = \nu_{\text{vert}} \).

Indeed, the two observed QPO frequencies, are in this relation to the spin frequency of 401 Hz, \( \nu_{\text{upper}} - \nu_{\text{lower}} = 694 \text{ Hz} - 499 \text{ Hz} \approx (401 \text{ Hz})/2 = \nu_1/2 \).

We note that the two QPOs cannot be separated by the spin frequency in this scheme. A separation between the two QPO frequencies of twice the spin frequency, \( \nu_{\text{res}} - \nu_0 = 802 \text{ Hz}, \text{ case a with } \nu_0 = \nu_2 \) is in principle also possible in this scheme if resonance occurs at the very inner edge of the disk, at frequencies of about 650 Hz and 1450 Hz, but it does not seem likely that the bulk of the torus could reside so close to the neutron star.

We have numerically computed realistic models of rotating neutron stars and their exterior metrics. Fig. 2 presents a typical result consistent with observations of the two QPOs in the 2.5 ms pulsar. The vertical and radial epicyclic frequencies for a 1.22 solar mass neutron star, modeled with equation of state FPS, and rotating at 401 Hz are illustrated as a function of the radius. The case c) resonance discussed above occurs when pressure in the torus has a maximum at radius \( r_0 = 11.27M(G/c^2) = 20.6 \text{ km} \). The stellar mass and resonance radius are only indicative of the true values for this pulsar, as the frequencies for a flattened torus of large radial extent may differ from \( \kappa(r_0) \) and \( \zeta(r_0) \) by several per cent (Rezzolla et al. 2003).

Wijnands et al. (2003) give phenomenological arguments in favor of the frequency being the upper QPO. In this case, in addition to the resonance described below, a resonance at the combination frequency \( \nu_r = \nu_{\text{vert}} - \nu_1 \) could also occur in principle if the radial oscillator with its quadratic anharmonicity were perturbed simultaneously at the spin frequency and the frequency of the vertical oscillation. This would happen at \( r_0 = 16 \text{ km} \) for the same model of the neutron star, when the upper QPO frequency has the (as yet unobserved value) 1015 Hz and the lower frequency is 615 Hz (dashed vertical line in Fig. 2).

To compute the frequencies in Figs. 2 and 3, we constructed numerical models of rotating neutron stars and their exterior metrics using the relativistic code of Stergioulas and Friedman ( 1995), which solves Einstein’s field equations for arbitrarily large rotation rate in an integral form. Details of the numerical method and extensive accuracy tests can be found in (Stergioulas 2003). The results presented here are not very sensitive to the choice of equation of state (EOS) of neutron-star matter.

3. Parametric resonance

So far we have been discussing a resonance directly caused by an external disturbance. But when the frequencies \( \kappa(r) \) and \( \zeta(r) \) happen to be in particular ratios, large-amplitude motions of the torus may be excited spontaneously. This mechanism, and specifically parametric resonance between the two modes of oscillation, has been suggested (Kluźniak and Abramowicz 2002) as the origin of the high-frequency QPOs in black hole systems, where the frequency ratios 5:3 and 3:2
have been noted (Abramowicz and Kluźniak 2001; Remillard et al. 2002; Kluźniak and Abramowicz 2002; McClintock and Remillard 2003), as well as in the candidate neutron-star system Sco X-1 (Abramowicz et al. 2003a,b). The presence of subharmonics in some of the black hole systems has also been discussed (Remillard et al. 2002; Kluźniak and Abramowicz 2003). On the other hand, in several X-ray bursters the frequency difference between the twin QPOs has been reported to be close to the suspected spin frequency (or one half of it) of the neutron star (van der Klis et al. 2000). It is interesting to note that in addition to being separated by one-half the pulsar spin frequency, the two QPO frequencies in SAX J1808.4-3658 are approximately in a 7:5 = 1.4 ratio. The system may thus be the missing link between the QPOs in black hole systems and X-ray bursters. It is known that if the frequency difference of two modes is accidentally close to that of an external disturbance, a resonance involving the three frequencies may occur (Kato 1974). In neutron star systems this could favor the appearance of strong QPOs with a frequency difference selected by the perturbing neutron-star rotation, while in black hole disks only the spontaneous resonance would occur.

Parametric resonance between two oscillations is possible in accretion disks because the coupling of modes is non-linear in hydrodynamics. For thin disks, this can lead to excitation of two modes whose frequency ratio is closer to 1.4 than to 1.5. In the Schwarzschild metric, the damping or excitation of \( g \)-modes with azimuthal dependence \( \exp(i m \phi) \) has been considered by Kato (Kato 2003), who found that an \( m = 0 \) mode and an \( m = 1 \) mode will undergo resonant amplification when interacting with a non-rotating one-armed \((m = 1)\) stationary warp, assumed to be present in the accretion disk. We assume that the same modes are unstable also in a gravitational field that is not spherically symmetric, and compute their frequencies for the metric of rotating neutron stars. The sum of the two frequencies turns out to be \( \nu_{\text{orb}} \), and if they are identified with the two observed QPO frequencies this constrains the mass of the 2.5 ms pulsar to be less than 1.5 solar mass. The two mode frequencies, \( \nu_{\text{lower}} \) and \( \nu_{\text{upper}} \), as well as \( \nu_{\text{orb}} \), \( \nu_\tau \), and \( \nu_{\text{orb}} - \nu_\tau \) are shown in Fig. 3 for a 1.49 solar mass neutron star, spinning at a frequency of 406 Hz. The resonance occurs in the region where \( \nu_{\text{lower}} < \nu_\tau \), with the largest modulation of X-ray flux expected when the two frequencies are evaluated at the edge of the resonance zone, i.e., when they are \( \nu_{\text{lower}} \approx \nu_\tau \) and \( \nu_{\text{upper}} \approx \nu_{\text{orb}} - \nu_\tau \).

The two unstable modes in Fig. 3 were taken, following (Kato 2003), to be \( m = 1, n = 1 \), and \( m = 0, n = 1 \), \( g \)-modes excited by parametric resonance, which is mediated by a warp assumed to be present in the thin accretion disk. For a non-rotating warp their frequencies (Kato 2003) would be \( 2\nu_{\text{orb}} - \sqrt{2}\nu_{\text{vert}} \) and \( \sqrt{2}\nu_{\text{vert}} - \nu_{\text{orb}} \), respectively, with their ratio tending to \( \sqrt{2} = 1.41 \) in the limit of non-rotating stars, or in the limit of large radial distance from the star, \( r \to \infty \). The warp can be identified with a \( c \)-mode of disk oscillations and is expected to rotate at the Lense-Thirring angular frequency \( 2\pi(\nu_{\text{orb}} - \nu_{\text{vert}}) \), evaluated at a certain characteristic radius, \( r_w \) (Silbergleit et al. 2001). The two curves plotted in Fig. 3 are \( \nu_{\text{lower}}(r) = (\sqrt{2} - 1)\nu_{\text{vert}}(r) \) and \( \nu_{\text{upper}}(r) = \nu_{\text{orb}} + (1 - \sqrt{2})\nu_{\text{vert}}(r) \). The actual mode frequencies coincide with these at \( r = r_w \), and at arbitrary radius differ from these by \( [\nu_{\text{orb}}(r_w) - \nu_{\text{vert}}(r_w)] - [\nu_{\text{orb}}(r) - \nu_{\text{vert}}(r)] \), i.e., by a few
percent. For the present purposes, we take the quantity \( r_w \) to be an unknown parameter.

### 4. Conclusions

In conclusion, we note again that it is not yet known whether the accretion disk resembles a torus or, to the contrary, is geometrically thin. The 2.5 ms pulsar is the first source in which the frequencies of the twin QPOs are at once clearly related to the neutron-star spin frequency and are in a 1.4 ratio. With one observation of twin frequencies in only one source with a known rotation rate (but an unknown mass), it is impossible to determine which of the two pairs of oscillation modes discussed here is responsible for the non-linear resonance. The ambiguity will be removed if twin QPOs are observed in one or more of the four other accreting pulsars (Markwardt and Swank 2003) known, where the spin periods are different.

It is worth noting that the fact that the frequency difference between the two observed kHz QPOs is equal to one half the spin frequency of the neutron star is the first clear indication that the commonly observed millisecond modulations of the X-ray flux in low mass X-ray binaries are caused by oscillatory motions of the accretion disk around the neutron star (or black hole in other systems).

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Fig. 1.— The vertical and radial motions of an accretion torus in response to an impulsive radial perturbation at time $t = 0$. The vertical displacement of the torus (dashed line) about the equatorial plane and the radial displacement (continuous line) about the equilibrium radius, $r_0$, of the circle of maximum pressure are plotted as a function of time. In this pseudo-Schwarzschild simulation, $r_0 = 12.25M(G/c^2)$. It is clear that the torus performs two harmonic motions. We find that their frequencies coincide with the meridional and radial epicyclic frequencies, $\nu_{\text{vert}}$ and $\nu_r$, here 700 Hz and 500 Hz, respectively. The ratio of the frequencies is a function of $r_0$, their value a function of the mass $M$ of the neutron star. See the text for details.
Fig. 2.— The radial and vertical epicyclic frequencies for a neutron star rotating at a rate of 401 Hz. The two thick dots indicate where the two epicyclic frequencies are separated by 1/2 the spin frequency (within grid errors). A torus whose circle of maximum pressure has radius \( r_0 = 11.27M \) (vertical line) would oscillate at frequencies 496 Hz and 694 Hz. The vertical dashed line indicates the radius \( (r = 8.72M) \) where the epicyclic frequencies are separated by the spin frequency. Also shown is ten times the Lense-Thirring frequency. The neutron star was modeled with equation of state FPS and has 1.22 solar mass. All frequencies are plotted as a function of the radius, given in units of \( M(G/c^2) = 1.22 \times 1.5 \) km.
Fig. 3.— The frequencies (thick curves) of two $g$-modes of oscillation excited by parametric resonance in a thin accretion disk. The two thick dots at $r = 6.69M$ mark the boundary of the instability region—the resonant interaction with a rotating warp occurs to the right of the dots, where the lower frequency is less than the radial epicyclic frequency (dashed curve). The largest modulation of X-rays is expected close to the indicated boundary of the resonance region, i.e., at about 508 Hz and 737 Hz for the model computed (1.49 solar mass neutron star rotating at 406 Hz, EOS AU). For a star of lower mass, $\nu_{\text{lower}}$ and $\nu_{\text{upper}}$ would have the values 500 and 700 Hz at a larger value of $r/M$. The computed frequencies of the two modes are uncertain by several percent because of the unknown rotation rate of the warp, which coincides at a poorly constrained radius with the Lense-Thirring frequency (also shown, in ten-fold magnification).