Contractions with necessarily unbounded matrices

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We prove that for each dimension not less than five there exists a contraction between solvable Lie algebras that can be realized only with matrices whose Euclidean norms necessarily approach infinity at the limit value of contraction parameter. Therefore, dimension five is the lowest dimension of Lie algebras between which contractions of the above kind exist.

1 Introduction

The study of ways for implementing contractions between Lie algebras plays an important role in the theory of contractions from its very outset. The general notion of limiting processes between Lie algebra structures was first introduced by Segal [16], who was inspired by examples of physical theories being a limit case of others. Contracting Lie algebras and their representations became physicists’ operating tool after the papers by Inönü and Wigner [8, 9]. They intended to consider (linear) contractions whose matrices are linear in the contraction parameter but in fact they merely studied contractions that can be realized, in properly chosen bases of initial and target algebras, by diagonal matrices with only zero and first powers of the contraction parameter on their diagonals. These particular linear contractions are called Inönü–Wigner contractions, or briefly IW-contractions. General linear contractions were more comprehensively analyzed by Saletan [15]; hence they are sometimes called Saletan contractions. Therein a rigorous definition of contraction in terms of the right action of the general linear group on Lie brackets was presented, becoming conventional in physical literature. Another generalization of IW-contractions, where the diagonal elements are allowed to be real powers of the contraction parameter, was suggested in [5]. To realize these p-contractions, called also Doebner–Melsheimer contractions or, more often, generalized IW-contractions [7], it in fact suffices to use only integer powers of the contraction parameter [13].

In the course of exploring possibilities for realizing contractions naturally arises a problem on existence of contraction matrices that have well-defined (finite) limits at the limit value of the contraction parameter [17]. The analysis of the results on contractions of real and complex Lie algebras up to dimension four [4, 12, 14] shows that all of these contractions can be realized by such matrices. Is the same true for Lie algebras of higher dimension? The first study of this problem was carried out in [17] for the contraction between two specially chosen five-dimensional Lie algebras.

Consider the n-dimensional ($n \geq 5$) solvable real Lie algebras $\mathfrak{a}$ and $\mathfrak{a}_0$ that are defined by the following nonzero commutation relations:

$\mathfrak{a}:$ $[e_1, e_3] = e_3, \quad [e_2, e_4] = e_4, \quad [e_1, e_2] = e_5,$

$\mathfrak{a}_0: \quad [e_1, e_3] = e_3, \quad [e_2, e_4] = e_4.$

Using Mubarakzyanov’s classification of low-dimensional Lie algebras [11], these algebras can be denoted by $A_{5,38} \oplus (n - 5)A_1$ and $A_{2,1} \oplus A_{2,1} \oplus (n - 4)A_1$. Note that each five-dimensional solvable Lie algebra with one-dimensional center and three-dimensional nilradical is isomorphic to either $A_{5,38}$ or $A_{2,1} \oplus A_{2,1} \oplus A_1$, and three is the minimal dimension of nilradical for five-dimensional solvable Lie algebras. It is obvious that the contraction $\mathfrak{a} \rightarrow \mathfrak{a}_0$ is realized by the
diagonal matrix $U = \text{diag}(1, 1, 1, 1, \varepsilon^{-1}, 1, \ldots, 1)$, whose fifth diagonal entry goes to infinity as $\varepsilon \to +0$. The same is true for the contraction $\tilde{a} \to \tilde{a}_0$ between the complexifications $\tilde{a}$ and $\tilde{a}_0$ of $a$ and $a_0$. It was shown in [17] that for $n = 5$ any realization of the contraction $a \to a_0$ as a generalized Inönü–Wigner contraction necessarily involves a negative power of the contraction parameter, and hence some entries of the corresponding contraction matrix approach infinity at zero. The purpose of the present paper is to prove the following stronger and more general assertion:

**Theorem 1.** The Euclidean norm of any contraction matrix that realizes the contraction of the algebra $a$ to the algebra $a_0$ approaches infinity at the limit point. The same is true for the complex counterpart of this contraction.

In other words, for any dimension $n \geq 5$ Theorem 1 constructively gives a positive answer to the question whether there exist contractions between $n$-dimensional Lie algebras that can be realized only by unbounded matrices, and dimension five is the lowest dimension for which contractions of the above kind exist.

We additionally show that, up to automorphisms of the algebra $a$, the Euclidean norm of the tuple formed by the $(5, 5)$th, $\ldots$, $(5, n)$th entries of any contraction matrix in the chosen bases of the algebras $a$ and $a_0$ approaches infinity at the limit point of the contraction parameter. In particular, in the case $n = 5$ it is the $(5, 5)$th entry of a contraction matrix whose absolute value goes to infinity.

Both Theorem 1 and the last claim are directly extended to the complex case.

## 2 Auxiliary results

Given a finite-dimensional vector space $V$ over the field $\mathbb{F} = \mathbb{R}$ or $\mathbb{F} = \mathbb{C}$, by $\mathcal{L}_n = \mathcal{L}_n(\mathbb{F})$ we denote the set of all possible Lie brackets on $V$, where $n = \dim V < \infty$. Each element $\mu$ of $\mathcal{L}_n$ corresponds to a Lie algebra with the underlying space $V$, $\mathfrak{g} = (V, \mu)$. Fixing a basis $\{e_1, \ldots, e_n\}$ of $V$ leads to a bijection between $\mathcal{L}_n$ and

$$\mathcal{C}_n = \{(c_{ij}^k) \in \mathbb{F}^{n^2} \mid c_{ij}^k + c_{ji}^k = 0, c_{ij}^k c_{kl}^r + c_{ki}^j c_{lj}^r + c_{jk}^i c_{il}^r = 0\}.$$  

The structure constant tensor $(c_{ij}^k) \in \mathcal{C}_n$ associated with a Lie bracket $\mu \in \mathcal{L}_n$ is given by the formula $\mu(e_i, e_j) = c_{ij}^k e_k$. Here and in what follows, the indices $i$, $j$, $k$, $i'$, $j'$ and $k'$ run from 1 to $n$ and the summation convention over repeated indices is assumed. The right action of the group $\text{GL}(V)$ on $\mathcal{L}_n$, which is conventional for the physical literature, is defined as

$$(U \cdot \mu)(x, y) = U^{-1}(\mu(x, U y)) \quad \forall U \in \text{GL}(V), \forall \mu \in \mathcal{L}_n, \forall x, y \in V.$$  

**Definition 1.** Given a Lie bracket $\mu \in \mathcal{L}_n$ and a continuous matrix function $U : (0, 1] \to \text{GL}(V)$, we construct the parameterized family of Lie brackets $\mu_\varepsilon = U_\varepsilon \cdot \mu$, $\varepsilon \in (0, 1]$. Each Lie algebra $\mathfrak{g}_\varepsilon = (V, \mu_\varepsilon)$ is isomorphic to $\mathfrak{g} = (V, \mu)$. If the limit

$$\lim_{\varepsilon \to +0} \mu_\varepsilon(x, y) = \lim_{\varepsilon \to +0} U_\varepsilon^{-1}(\mu_\varepsilon(x, U_\varepsilon y)) =: \mu_0(x, y)$$

exists for any $x, y \in V$, then $\mu_0$ is a well-defined Lie bracket. The Lie algebra $\mathfrak{g}_0 = (V, \mu_0)$ is called a one-parametric continuous contraction (or simply a contraction) of the Lie algebra $\mathfrak{g}$. We call a limiting process that provides $\mathfrak{g}_0$ from $\mathfrak{g}$ with a matrix function a realization of the contraction $\mathfrak{g} \to \mathfrak{g}_0$.

The notion of contraction is extended to the case an arbitrary algebraically closed field in terms of orbit closures in the variety of Lie brackets [1, 2, 3, 6, 10].
If a basis \( \{e_1, \ldots, e_n\} \) of \( V \) is fixed, then the operator \( U_\varepsilon \) can be identified with its matrix \( U_\varepsilon \in \text{GL}_n(\mathbb{R}) \), which is denoted by the same symbol, and Definition 1 can be reformulated in terms of structure constants. Let \( C = (c^k_{ij}) \) be the tensor of structure constants of the algebra \( g \) in the basis chosen. Then the tensor \( C_\varepsilon = (\varepsilon_{\varepsilon,ij}) \) of structure constants of the algebra \( g_\varepsilon \) in this basis is the result of the action by the matrix \( U_\varepsilon \) on the tensor \( C \), \( C_\varepsilon = C \circ U_\varepsilon \). In term of components this means that
\[
\varepsilon^k_{\varepsilon,ij} = (U_\varepsilon)^j_i \varepsilon^j_k (U_\varepsilon^{-1})^k_i c^l_{ij}.
\]
Then Definition 1 is equivalent to that the limit
\[
\lim_{\varepsilon \to +0} \varepsilon^k_{\varepsilon,ij} =: \varepsilon^k_{0,ij}
\]
even for all values of \( i, j \) and \( k \) and, therefore, \( \varepsilon^k_{0,ij} \) are components of the well-defined structure constant tensor \( C_0 \) of the Lie algebra \( g_0 \). The parameter \( \varepsilon \) and the matrix-function \( U_\varepsilon \) are called a contraction parameter and a contraction matrix, respectively.

Sequential contractions are defined analogously to continuous contractions using matrix sequences, \( \{U_p, p \in \mathbb{N}\} \subset \text{GL}(V) \), instead of continuous matrix functions. For each Lie bracket \( \mu \) from the sequence \( \{\mu_p = U_p \cdot \mu, p \in \mathbb{N}\} \), the Lie algebra \( g_p = (V, \mu_p) \) is isomorphic to \( g = (V, \mu) \). If the limit
\[
\lim_{p \to \infty} \mu_p(x, y) = \lim_{p \to \infty} U_p^{-1} \mu(U_p x, U_p y) =: \mu_0(x, y)
\]
even for any \( x, y \in V \), then \( \mu_0 \) is a well-defined Lie bracket on \( V \). The Lie algebra \( g_0 = (V, \mu_0) \) is called a sequential contraction of the Lie algebra \( g \). Within the basis-dependent approach, each algebra \( g_p \) is associated with the structure constant tensor \( C_p = C \circ U_p \) with the components
\[
\varepsilon^k_{p,ij} = (U_p)^j_i \varepsilon^j_k (U_p^{-1})^k_i c^l_{ij}.
\]
The existence of the above limit of \( \{\mu_p\} \) is equivalent to the existence of the limit
\[
\lim_{p \to \infty} \varepsilon^k_{p,ij} =: \varepsilon^k_{0,ij}
\]
even for all values of \( i, j \) and \( k \), where \( \varepsilon^k_{0,ij} \) are components of the structure constant tensor \( C_0 \) of the Lie algebra \( g_0 \).

Any continuous contraction from \( g \) to \( g_0 \) gives an infinite family of matrix sequences resulting in sequential contractions from \( g \) to \( g_0 \). More precisely, if \( U_\varepsilon \) is the matrix of the continuous contraction and the sequence \( \{\varepsilon_p, p \in \mathbb{N}\} \) satisfies the conditions \( \varepsilon_p \in (0, 1], \varepsilon_p \to +0, p \to \infty \), then matrix sequence \( \{U_{\varepsilon_p}, p \in \mathbb{N}\} \) generates a sequential contraction from \( g \) to \( g_0 \).

Definitions of special types of contractions, statements on properties and their proofs in the case of sequential contractions can be easily obtained via reformulation of those for the case of continuous contractions. It is enough to replace continuous parametrization by discrete one.

The following useful assertion is obvious.

**Lemma 1.** If the matrix \( U_\varepsilon \) of a contraction \( g \to g_0 \) can be represented in the form \( U_\varepsilon = \hat{U}_\varepsilon \hat{U}_\varepsilon \), where \( \hat{U} \) and \( \hat{U} \) are continuous functions from \((0, 1] \) to \( \text{GL}_n(\mathbb{R}) \) and the function \( \hat{U} \) has a limit \( \hat{U}_0 \in \text{GL}_n(\mathbb{R}) \) at \( \varepsilon \to +0 \), then \( \hat{U}_\varepsilon \hat{U}_0 \) also is a matrix of the contraction \( g \to g_0 \).

The same is true for sequential contractions. We will need a more particular lemma, which is related to the \( LQ \) matrix decomposition and is in fact a computational counterpart of Proposition 1.7 from 2 for the real and complex cases.

**Lemma 2.** A Lie algebra \( g \) is sequentially contracted to a Lie algebra \( g_0 \) if and only if in the fixed basis \( \{e_1, \ldots, e_n\} \) of the underlying space \( V \) there exists the sequence \( \{L_p, p \in \mathbb{N}\} \) of nondegenerate lower triangular \( n \times n \) matrices and an orthogonal (resp. unitary) \( n \times n \) matrix \( Q \) in the real (resp. complex) case such that \( C \circ L_p \to C_0 \circ Q \) as \( p \to \infty \).
Proof. Using the sequential realization of contractions, we prove the lemma only for the real case since the complex case is considered in a similar way with replacing orthogonal matrices by unitary ones. Let \( \{U_p, p \in \mathbb{N}\} \) be a sequence of matrices that realize the contraction \( g \to g_0 \), i.e., \( C \circ U_p \to C_0, p \to \infty \). For each \( p \), we decompose the matrix \( U_p \) into triangular and orthogonal multipliers, \( U_p = L_p Q_p \), where \( L_p \) is a lower triangular matrix and \( Q_p \) is an orthogonal matrix. As the set of \( n \times n \) orthogonal matrices is compact in the Euclidean topology, the sequence \( \{Q_p, p \in \mathbb{N}\} \) contains a convergent subsequence. Any subsequence of a matrix sequence realizes the same sequential contraction as the whole sequence. Therefore, without loss of generality, we can assume that the sequence \( \{Q_p\} \) itself is convergent. Its limit \( Q_0 \) is also an orthogonal matrix. Since \( C \circ U_p \to C_0, Q_p^T \to Q_0^T \) and the matrices \( Q_p \) are orthogonal,

\[
C \circ L_p = C \circ L_p Q_p Q_p^T = C \circ U_p Q_p^T \to C_0 \circ Q_0^T
\]
as \( p \to \infty \).

We denote \( Q_0^T \) by \( Q \), completing the proof of the lemma. \( \square \)

Remark 1. The sequence of triangular matrices \( \{L_p, p \in \mathbb{N}\} \) and the orthogonal matrix \( Q \) are defined in Lemma 2 up to the transformation

\[
\hat{L}_p = M_p L_p D_p, \quad \hat{Q} = K Q D_0,
\]

where \( K \) is the matrix of an orthogonal automorphism of \( g_0 \), \( D_0 \) is a diagonal orthogonal (resp. unitary) matrix in the real (resp. complex) case, \( M_p \) for each \( p \in \mathbb{N} \) is the matrix of an automorphism of \( g \), and the sequence of the triangular matrices \( \{D_p, p \in \mathbb{N}\} \) approaches the matrix \( D_0 \).

3 Proof

We prove Theorem 1 in the real case. For the complex case, orthogonal matrices should be replaced by unitary ones, and the other differences are indicated explicitly.

First we consider an arbitrary sequential realization of the contraction \( a \to a_0 \) with a matrix sequence \( \{U_p, p \in \mathbb{N}\} \). If we suppose that the Euclidean norm of \( U_p \) does not approach infinity, then the sequence \( \{U_p\} \) contains a bounded subsequence \( \{U_{p_s}, s \in \mathbb{N}\} \). Following the proof of Lemma 2, we factorize each matrix \( U_{p_s} \) into its lower triangular and orthogonal parts, choose a subsequence of elements of \( \{U_{p_s}\} \) with convergent orthogonal parts and apply the algebraic limit theorem. As a result, we construct a bounded sequence of lower triangular matrices and an orthogonal matrix \( Q \), which satisfy Lemma 2 for \( g = a \) and \( g_0 = a_0 \). At the same time, as we will see below, the sequence of Euclidean norms of such triangular matrices necessarily approaches infinity. The contradiction obtained means that the Euclidean norm of \( U_p \) approaches infinity.

Suppose that there exists a continuous realization of the contraction \( a \to a_0 \) with a continuous function \( U : (0, 1] \to \text{GL}(V) \) for which the Euclidean norm of its values \( U_{\varepsilon} \) does not go to infinity as \( \varepsilon \to +0 \). Then we can choose a sequence \( \{\varepsilon_p, p \in \mathbb{N}\} \subset (0, 1] \) such that its limit equals zero and the matrix sequence \( \{U_{\varepsilon_p}, p \in \mathbb{N}\} \) is bounded. As the last sequence realizes a sequential contraction \( a \to a_0 \), this immediately leads to a contradiction.

Given the above, it suffices to prove that for any sequence \( \{L_p = (l_{p,j}), p \in \mathbb{N}\} \) of lower triangular matrices (and orthogonal matrix \( Q = (q^T_{k}) \)) satisfying Lemma 2 for \( g = a \) and \( g_0 = a_0 \) the corresponding sequence of Euclidean norms goes to infinity.

Let us look into the constraints on the matrix \( Q \). We denote the structure constant tensors of the algebras \( a \) and \( a_0 \) in the chosen basis \( \{e_1, \ldots, e_n\} \) of the underlying vector space by \( C = (c^i_{jk}) \) and \( C_0 = (c^i_{0,ij}) \) respectively. Then \( C_p = C \circ L_p \) and \( \tilde{C}_0 = C_0 \circ Q \) are the structure constant tensors of the algebras \( a_p \) and \( a_0 \) that are isomorphic to the algebras \( a \) and \( a_0 \) with respect to the operators \( L_p \) and \( Q \). By the construction, \( \lim_{p \to \infty} \tilde{c}_{k_{p,i,j}}^{i} = c_{0,i,j}^{k} \). Since for any \( i, j, k \)
and \( j^* = 5, \ldots, n \) we have that \( c_{ij^*}^k = c_{ij} = c_{ij}^0 = 0 \) and, for any \( p, \), \( l_{p,i}^j = 0 \) if \( i < j \), then \( c_{p,ij^*}^k = c_{p,ij}^1 = c_{p,ij}^2 = 0 \) holds true for any \( i, j, k \) and \( p \). Hence the same is true for elements of \( \hat{C}_0 \). \( c_{0,ij}^1 = c_{0,ij}^2 = 0 \). At the same time, the corresponding components of \( C_0 \) also vanish by the definition of \( a_0 \). Geometrically, this means that \( Q(e_5, \ldots, e_n) = \langle e_5, \ldots, e_n \rangle \) and \( Q(e_3, e_4) \subset \langle e_3, \ldots, e_n \rangle \). As the matrix \( Q \) is orthogonal, then it is a block diagonal matrix of the form

\[
Q = \begin{pmatrix} q_1^1 & q_1^2 & q_1^3 \\ q_1^4 & q_1^5 & q_1^6 \\ q_1^7 & q_1^8 & q_1^9 \end{pmatrix} \oplus \begin{pmatrix} q_2^1 & q_2^2 & q_2^3 \\ q_2^4 & q_2^5 & q_2^6 \\ q_2^7 & q_2^8 & q_2^9 \end{pmatrix} \oplus \begin{pmatrix} q_3^1 & q_3^2 & q_3^3 \\ q_3^4 & q_3^5 & q_3^6 \\ q_3^7 & q_3^8 & q_3^9 \end{pmatrix} \oplus \begin{pmatrix} q_4^1 & q_4^2 & q_4^3 \\ q_4^4 & q_4^5 & q_4^6 \\ q_4^7 & q_4^8 & q_4^9 \end{pmatrix}, \quad \text{where } i^*, j^* = 5, \ldots, n. \tag{1}
\]

There are three more values of the triplet \((i, j, k)\), namely \((1, 4, 3), (2, 4, 3) \) and \((2, 3, 3)\), for which the structure constants \( c_{ij}^k, c_{p,ij}^k \) (for all values of \( p \)) and hence \( c_{ij}^k \) vanish. In other words, we obtain the equations

\[
c_{14}^3 = q_2^1 q_3^3 q_1^4 + q_2^3 q_1^3 q_1^4 = 0, \quad (q_2^1 q_3^3 q_1^4 + q_2^3 q_1^3 q_1^4 = 0),
\]

\[
c_{24}^3 = q_2^1 q_3^3 q_1^4 + q_2^3 q_1^3 q_1^4 = 0, \quad (q_2^1 q_3^3 q_1^4 + q_2^3 q_1^3 q_1^4 = 0),
\]

\[
c_{23}^3 = q_2^1 (q_3^3)^2 + q_2^3 (q_3^4)^2 = 0, \quad (q_2^1 q_3^3 q_1^4 + q_2^3 q_1^3 q_1^4 = 0).
\]

In the brackets we present the corresponding equations for the complex case, and the bar denotes the complex conjugation. Because of \( q_1^3 q_2^3 - q_2^3 q_1^3 \neq 0 \), the first two equations imply that \( q_3^3 q_1^4 = q_3^4 q_1^3 = 0 \). Combining the orthogonality of \( Q \) with the above equations gives the following two possibilities:

1. \( q_3^3 = q_3^4 = 0 \). Then \( q_3^3 q_1^4 \neq 0, q_1^3 = q_2^3 = 0 \) and \( q_1^3 q_1^3 \neq 0 \).

2. \( q_3^3 q_1^4 \neq 0 \). Then \( q_3^3 = q_3^4 = 0, q_1^3 = q_2^3 = 0 \) and \( q_1^3 q_1^3 \neq 0 \).

The corresponding forms of the matrix \( Q \) are

\[
Q = \begin{pmatrix} 0 & q_1^1 & q_1^2 & q_1^3 \\ q_1^4 & 0 & q_1^5 & q_1^6 \\ q_1^7 & q_1^8 & 0 & q_1^9 \end{pmatrix} \oplus \begin{pmatrix} q_2^1 & q_2^2 & q_2^3 \\ q_2^4 & 0 & q_2^5 & q_2^6 \\ q_2^7 & q_2^8 & 0 & q_2^9 \end{pmatrix} \oplus \begin{pmatrix} q_3^1 & q_3^2 & q_3^3 \\ q_3^4 & 0 & q_3^5 & q_3^6 \\ q_3^7 & q_3^8 & 0 & q_3^9 \end{pmatrix} \oplus \begin{pmatrix} q_4^1 & q_4^2 & q_4^3 \\ q_4^4 & 0 & q_4^5 & q_4^6 \\ q_4^7 & q_4^8 & 0 & q_4^9 \end{pmatrix}.
\]

Recall that the matrix \( Q \) is defined up to the multiplication by the matrix of an orthogonal automorphism of \( a_0 \) from the left and by an orthogonal diagonal matrix from the right, cf. Remark [1]. The change of the basis \( (\hat{e}_1, \hat{e}_2, \hat{e}_3, \hat{e}_4, \hat{e}_5, \ldots, \hat{e}_n) = (e_2, e_1, e_4, e_3, e_5, \ldots, e_n) \), which is an orthogonal automorphism of the algebra \( a_0 \), reduces the first case to the second one. In the second case the matrix \( Q \) can be made diagonal by the orthogonal automorphism

\[
(\hat{e}_1, \hat{e}_2, \hat{e}_3, \hat{e}_4) = (e_1, e_2, e_3, e_4), \quad \hat{e}_j^* = e_j q_j^*.
\]

of the algebra \( a_0 \). Therefore, it suffices to consider only the case of \( Q \) being the identity matrix., i.e., \( \hat{C}_0 = C_0 \).

For values of the triplet \((i, j, k)\) with \( i, j, k = 1, \ldots, 5 \), that have not been used yet, we represent the conditions \( \lim_{n \to \infty} c_{p,ij}^k = c_{0,ij}^k \) in the form

\[
c_{p,ij}^k := \tilde{l}_{p,i}^j l_{p,j}^i c_{ij}^k = e_{0,ij}^k + c_{p,ij}^k,
\]

where \( \tilde{L}_p = (\tilde{l}_{p,j}^i = L_p^{-1} \) denotes the inverse of the matrix \( L_p \) and \( \lim_{p \to \infty} c_{p,ij}^k = 0 \). The algebra \( a \) is the sum of the ideal spanned by the first five basis elements of this algebra and the abelian ideal spanned by the other basis elements. The matrix \( L_p \) is lower triangular. This is why the expressions for the structure constants \( c_{p,ij}^k \) with \( i, j, k = 1, \ldots, 5 \) do not involve entries \( l_{p,j}^i \) of the matrix \( L_p \) where \( i > 5 \) or \( j > 5 \). As a result, we derive a system of equations
on \( l_{p,j} \) and \( o_{p,i,j} \) with \( i, j, k = 1, \ldots, 5 \) (in what follows we generally omit the subscript \( p \) for concise presentation),

\[
\begin{align*}
    l_1^1 &= 1 + o_{13}^3, \quad l_2^2 = 1 + o_{24}^2, \quad l_1^4 = o_{14}^4, \quad l_1^5 = o_{12}^5, \quad l_2^5 = o_{23}^5, \quad -l_2^4 = o_{24}^4, \\
    -l_1^4 + l_2^4 + l_1^5 - l_2^5 &= o_{12}^5, \quad -l_1^4 + l_2^4 - l_1^5 + l_2^5 = o_{13}^5, \\
    -l_1^5 + l_2^5 &= o_{14}^5, \quad -l_1^4 + l_2^4 - l_1^5 + l_2^5 = o_{13}^5, \\
    l_1^4 l_2^5 - l_1^4 l_2^5 - l_2^4 l_5^4 + l_1^4 l_2^5 - l_1^5 l_2^4 &= o_{12}^5.
\end{align*}
\]

We solve the equations in the first two rows with respect to \( l_2^5, l_1^3, l_2^3, l_1^4 \) and \( l_2^5 \) and substitute the obtained expressions into the last equation, which gives

\[
\frac{l_1^4 l_2^5}{l_2^5} = o_{12}^5 - \frac{o_{24}^2}{l_2^5} o_{12}^4 - \left( o_{13}^5 + \frac{l_2^4}{l_2^5} o_{23}^5 \right) \frac{o_{12}^5}{l_1^4}.
\]

The last equality obviously implies that \( l_{p,i} l_{p,j} / l_{p,5} \to 0 \), i.e., \( |l_{p,5}| \to \infty \) as \( p \to \infty \). Therefore, the sequence of Euclidean norms of the matrices \( L_p, p \in \mathbb{N} \), also goes to infinity. Note that the equations in the third row of the system do not lead to additional constraints for entries of \( L_p \), and the sixth and eighth equations imply that \( l_{p,4}^5 / l_{p,5}^5 \to 0 \) and \( l_{p,3}^5 / l_{p,5}^5 \to 0 \) as \( p \to \infty \).

Now we additionally show that, up to automorphisms of the algebra \( a \), the Euclidean norm of the tuple formed by \( (5, 5) \)th, \( (5, n) \)th entries of any contraction matrix in the chosen bases of the algebras \( a \) and \( a_0 \) goes to infinity at the limit point of the contraction parameter.

Given a sequential contraction \( a \to a_0 \) with a matrix sequence \( \{U_p, p \in \mathbb{N}\} \), we again factorize each matrix \( U_p \) into its lower triangular and orthogonal parts \( L_p, Q_p, U_p = L_p Q_p \). As the limit of any convergent subsequence of \( \{Q_p, p \in \mathbb{N}\} \) has the form (1), for each such subsequence and hence for the entire sequence \( \{Q_p, p \in \mathbb{N}\} \) we have that \( q_{p,j}^5 \to 0 \) as \( p \to \infty \) if \( i = 1, \ldots, 4 \) and \( j = 5, \ldots, n \) or if \( i = 5, \ldots, n \) and \( j = 1, \ldots, 4 \). For the corresponding subsequences of \( \{L_p, p \in \mathbb{N}\} \) the limits \( l_{p,5}^5 \to \infty, l_{p,4}^5 / l_{p,5}^5 \to 0 \) and \( l_{p,3}^5 / l_{p,5}^5 \to 0 \) as \( p \to \infty \) hold true. Hence, the same limits hold true for the whole sequence \( \{L_p, p \in \mathbb{N}\} \) (otherwise, we obtain a contradiction).

Using Remark 1 for each \( p \) we multiply the matrix \( L_p \) from the left by the matrix

\[
M_p = E - \frac{1}{l_{p,1}^5} \left( l_{p,1}^5 - l_{p,1}^4 l_{p,2}^2 \right) E_1^5 - \frac{l_{p,2}^5}{l_{p,2}} E_2^5,
\]

which is associated with an automorphism of \( a \). Here \( E \) denotes the \( n \times n \) identity matrix and \( E_j^i \) denotes the \( n \times n \) matrix with the unit entry on the cross of the \( i \)-th row and the \( j \)-th column and zero otherwise. The entries \( l_{p,1}^5 \) and \( l_{p,2}^5 \) of the matrix \( L_p \) are equal to zero. Then for the \( (5, j) \)th entries of the matrix \( \tilde{U}_p = \tilde{L}_p Q_p = M_p U_p \) with \( j \geq 5 \) we have

\[
\begin{align*}
    \lim_{p \to \infty} \sum_{j=5}^{n} \left( \tilde{U}_p^j \right)^2 &= \lim_{p \to \infty} \sum_{j=5}^{n} \left( l_{p,3}^5 q_{p,j}^3 + l_{p,4}^5 q_{p,j}^4 + l_{p,5}^5 q_{p,j}^5 \right)^2 \\
    &= \lim_{p \to \infty} \left( l_{p,3}^5 \right)^2 \sum_{j=5}^{n} \left( l_{p,3}^5 q_{p,j}^3 + l_{p,4}^5 q_{p,j}^4 + q_{p,j}^5 \right)^2 = \lim_{p \to \infty} \left( l_{p,3}^5 \right)^2 \sum_{j=5}^{n} \left( q_{p,j}^5 \right)^2 \\
    &= \lim_{p \to \infty} \left( l_{p,3}^5 \right)^2 = \infty.
\end{align*}
\]

We additionally use the facts that \( \sum_{j=1}^{n} q_{p,j}^5 q_{p,j}^5 = 1 \) and \( q_{p,j}^5 \to 0 \) as \( p \to \infty \) if \( j < 5 \).
The proof for the case of continuous contractions is similar. The only additional feature is continuity with respect to the contraction parameter $\varepsilon$. The Gram–Schmidt process applied to the contraction matrix $U_\varepsilon$ leads to a factorization in which both the lower triangular and orthogonal parts $L_\varepsilon$ and $Q_\varepsilon$ are continuous matrix-functions of $\varepsilon$. Then the corresponding automorphism $M_\varepsilon$ of $\mathfrak{a}$ that annihilates the $(5,1)$th and $(5,2)$th entries of $L_\varepsilon$ is also continuous with respect to $\varepsilon$, which implies the continuity of $\tilde{U}_\varepsilon = M_\varepsilon U_\varepsilon$.

4 Conclusion

We have constructed a single example of the solvable Lie algebras $\mathfrak{a}$ and $\mathfrak{a}_0$ for each dimension greater than four such that the contraction $\mathfrak{a} \to \mathfrak{a}_0$ cannot be realized by a bounded matrix-function. Moreover, we have showed that, up to automorphisms of the algebras $\mathfrak{a}$ and $\mathfrak{a}_0$, the Euclidean norm of the tuple formed by $(5,5)$th, $\ldots$, $(5,n)$th entries of any contraction matrix in the chosen bases of the algebras $\mathfrak{a}$ and $\mathfrak{a}_0$ necessarily approaches infinity at the limit point of the contraction parameter.

The proof of Theorem 1 involves several techniques. The first step in managing the contraction matrix is to factorize it into lower triangular and orthogonal parts and then apply Lemma 2 in order to move the orthogonal part from under the limit to the contracted structure constants. Due to the special structure of the considered Lie algebras it is possible to prove that the orthogonal part is an automorphism matrix of the contracted algebra $\mathfrak{a}_0$ and hence can be set to the identity matrix, which is neglected. For each fixed $(i, j, k)$ we consider the difference between the corresponding transformed and contracted structure constants as a new unknown value, which should approach zero. This reduces the limit relations between the structure constants to the system of algebraic equations in entries of the lower triangular part and new vanishing values. For the completion of the proof, it suffices to find out that the obtained algebraic equations for $i, j, k = 1, \ldots, 5$ involve only entries of the lower triangular part and new vanishing values with indices that run in the same range. The algebraization of the limit relations between the structure constants and considering a subsystem of algebraic equations that does not depend on the dimension $n$ allow for the verification of all computations using a computer program.

It is not understandable yet what properties lead to the above phenomenon, which does not appear in lower dimensions. We can only note that in the case $n = 5$ the contraction $\mathfrak{a} \to \mathfrak{a}_0$ is direct, i.e. there is no intermediate algebra $\tilde{\mathfrak{a}}_0$ such that $\mathfrak{a} \to \tilde{\mathfrak{a}}_0$ and $\tilde{\mathfrak{a}}_0 \to \mathfrak{a}_0$ are well-defined proper contractions. This follows from the fact that the derivation algebras of $\mathfrak{a}$ and $\mathfrak{a}_0$ are of dimensions six and seven, respectively, and any contraction leads to the increase of the dimension of the derivation algebra.

Since this is the first example in the literature, it is not clear how common are contractions with necessarily unbounded contraction matrices. At the same time, we have no reason to assume the above phenomenon unique, and we could guess that the number of such contractions grows when dimension of Lie algebras increases.

We can pose one more problem related to the subject considered. Given a generalized IW-contraction that necessarily involves negative powers of the contraction parameter, does there exist a realization of this contraction with a bounded matrix-function of another kind?

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