THE FACTORIZATION AND SIMULATION FOR FUNDAMENTAL SOLUTION OF CAUCHY PROBLEM

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Abstract. In this paper, we demonstrate the simulation of fundamental solution for the parabolic equation by the relationship with Ito diffusion. The factorization and Monte Carlo methods of the fundamental solution are considered. With the fact that the fundamental solution can be written as a product of the transition function and the expectation of a bridge path integral, we give an novel and efficient algorithm to simulate the fundamental solution by importance sampling method, especially for dealing with the multi-dimensional case.

Key words. diffusion, fundamental solution, Feynman-Kac formula, bridge path integral, Girsanov transformation, importance sampling

AMS subject classifications. 65M75, 65M80

1. Introduction. Ever since 1950s, the Cauchy problem has attracted much attention in front scientific research. It asks for the solution of a partial differential equation with initial value conditions, which has wide applications on many different fields. In financial field, a typical illustration of Cauchy problem is Black-Scholes-Merton model, that tells a story about pricing theory of options, proposed by Fisher Black, Myron Scholes and Robert Merton. They derived the famous Black-Scholes differential equation for European-style options. It was Robert Merton who gave a more generalized pricing formula. In 1997, the importance of the model was recognized when Robert Merton and Myron Scholes were awarded the Nobel prize for economics. In physics, a special case of Cauchy problem is Fokker-Planck equation, which describes transition density function for the velocity of a particle or other observables with time evolution. In modern probability theory, this type equation is connected with stochastic process via Kolmogorov backward and forward equations. The foundation of the solution was represented by the central result in the modern theory of stochastic processes, Feynman-Kac formula.

However, the simulation for the statistical representation of Feynman-Kac formula needs many painstaking efforts, especially the annoying frequent change of the initial condition. Then many mathematicians and physicists pay attention to fundamental solution, which plays an central role in the research of partial differential equation (PDE). In Feller (1952), the author discussed the backward and forward equation by its adjointness, duality and boundary properties of fundamental solution. In Friedman (1975), the author discussed the existence, uniqueness and the boundary of the fundamental solution. Gronwall’s inequality, Harnack’s inequality and the maximum principle are commonly used and effective technology. In the current paper, we follow these assumptions to guarantee these basic properties of fundamental solution and focus the factorization and simulation of the fundamental solution for Cauchy problem via an importance sampling with the help of Girsanov transformation. By this method, to get the solution we only need to calculate the integral which is much more easier than other methods. Further more, we must point out that the fundamental

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solution can be treated as adjusted transitional function by the relationship with the transition probability density function of diffusion.

This rest of this paper is organized as follows. In section 2, we recall some facts about PDE and stochastic differential equation (SDE), especially the factorization of fundamental solution. In section 3, we give the algorithm with the help of importance sampling method, one dimensional case and multivariate case. We will dedicate to the variance and efficiency of our estimate in section 4. In section 5, we give some examples, figures and tables to show the fundamental solution.

2. Factorization. Suppose $d$-dimensional Ito diffusion $X_t$ satisfies the following SDE,

\begin{equation}
\frac{dX_t}{dt} = \mu(X_t)dt + \sigma(X_t)dB_t \quad X_0 = x
\end{equation}

where $B_t$ is $m$-dimensional diffusion, $\mu : \mathbb{R}^d \to \mathbb{R}^d$ is a vector drift function, $\sigma : \mathbb{R}^d \to \mathbb{R}^d \times \mathbb{R}^m$ the diffusion matrix. For any $f \in C^2_0$ and suitable $\lambda(x)$, let $L$ be the infinitesimal generator of $X_t$ with the form

\[ Lu = \sum_{i=1}^{d} \frac{\partial u}{\partial x_i} \mu_i(x) + \frac{1}{2} \sum_{i,j} (\sigma \sigma^T)_{ij} \frac{\partial^2 u}{\partial x_i \partial x_j} \]

Consider the following parabolic equation,

\[ \left\{ \begin{array}{l}
u_t = Lu - \lambda(x)u \\
u(0, x) = f(x) \end{array} \right. \]

there exists a functional $q(t, x, y)$ such that the solution $u(t, x)$ have the representation

\[ u(t, x) = \int q(t, x, y)f(y)dy \]

On the other hand, by Feynman-Kac formula, the solution $u(t, x)$ can be written as

\begin{equation}
\begin{aligned}
u(t, x) &= \mathbb{E}_x \left[ \exp \left( -\int_0^t \lambda(X_s)ds \right) f(X_t) \right] \\
\end{aligned}
\end{equation}

We denote the transition probability density of $X_t$ as $p(t, x; s, y)$, since the diffusion is time homogeneous, we consider $p(0, x; t, y)$ and write it as $p(t, x, y) := p(0, x; t, y)$. The relationship between $q(t, x, y)$ and $p(t, x, y)$ is given by following theorem,

**Theorem 2.1.** Consider the diffusion $X_t$ satisfies SDE (2.1) with killing/renewal rate $\lambda(x)$, then the fundamental solution $q(t, x, y)$ of the partial differential equation

\[ \frac{\partial u}{\partial t} = Lu - \lambda(x)u; \quad t > 0, x \in \mathbb{R}^d \]

can be presented as follows,

\begin{equation}
\begin{aligned}
u(t, x, y) &= p(t, x, y) \cdot w(t, x, y) \\
\end{aligned}
\end{equation}

where $p(t, x, y)$ is transitional probability density of $X_t$, and

\begin{equation}
w(t, x, y) = \mathbb{E}_x^y \left[ \exp \left( -\int_0^t \lambda(X_s)ds \right) \right]
\end{equation}
Proof. The existence and uniqueness of \( q(t, x, y) \) as the fundamental solution for this PDE can be found in Friedman (1975), and we have

\[
q(t, x, y) = E_x \left[ \delta(X_t - y) \exp \left( - \int_0^t \lambda(X_s) ds \right) \right]
\]

\[
= E \left\{ E_x \left[ \delta(X_t - y) \exp \left( - \int_0^t \lambda(X_s) ds \right) \right] | X_t \right\}
\]

\[
= E \left\{ \delta(X_t - y) E_x \left[ \exp \left( - \int_0^t \lambda(X_s) ds \right) | X_t \right] \right\}
\]

\[
= \int \delta(z - y) E_x \left[ \exp \left( - \int_0^t \lambda(X_s) ds \right) \right] p(t, x, z) dz
\]

\[
= p(t, x, y) \cdot E_x \left[ \exp \left( - \int_0^t \lambda(X_s) ds \right) \right]
\]

In fact, this relation between \( p(t, x, y) \) and \( q(t, x, y) \) was proposed firstly by Kac (1949), but sadly there is no further investigation especially the Monte Carlo simulation. Crucially speaking, with this factorization of the fundamental solution, \( q(t, x, y) \) can be treated as the weighted average of \( p(t, x, y) \) by the weight \( w(t, x, y) \) which is the bridge path integral.

3. Algorithm. Suppose that \( p(t, x, y) \) can be easily calculated (for popular diffusions, it is not difficult), then the simulation of the fundamental solution \( q(t, x, y) \) is equivalent to that of \( p(t, x, y) \). When \( \lambda(x) \) is taken as constant, the result is trivial, and \( q(t, x, y) = e^{-\lambda t} p(t, x, y) \). But for more general cases, the analytic representation of \( q(t, x, y) \) is difficult to obtain. In fact there are many methods can be applied here. One way is to get the simulation of the diffusion \( X_t \) conditional on \( X_t = y \), denoted by \( \tilde{X}_t \), also known as diffusion bridge. In the past decades, simulation for diffusion bridge has been investigated by statisticians. It was previously believed impossible to draw the diffusion bridges by means of simple schemes, because of the low efficiency of the rejection sampler. In Lyons and Zheng (1990), the authors showed that the distribution of \( \tilde{X}_t \) satisfies

\[
d\tilde{X}_t = \tilde{\mu}(\tilde{X}_t) dt + \sigma(\tilde{X}_t) dB_t, \quad \tilde{X}_0 = x
\]

where

\[
\tilde{\mu}(x) = \mu(x) + [\sigma(x) \sigma^T(x)] \nabla_x (\log p(t, x, y))
\]

and \( p(t, x, y) \) is the transition function of \( X_t \). Of course we can use the SDE (3.1) to impute the trajectories of \( \tilde{X}_t \) and calculate the functional \( w(t, x, y) \), however, in many cases the diffusion bridge \( \tilde{X}_t \) is not stable (since the \( \log p(t, x, y) \)) and difficult to draw since the Euler scheme is complicated and not exact enough. Lin, Chen and Mykland (2010) proposed a sequential Monte Carlo method for simulating diffusion bridges with a resampling scheme guided by the empirical distribution of backward paths. Bladt and Sorensen (2012) demonstrates an accurate and effective algorithm with an analogous spirit. The rejection sampler recommended by Bladt and Sorensen (2012) has a quite acceptable rejection probability because they use backward paths as surrogates for the end points. A comprehensive exposition of these problems can be found in Papaspiliopoulos and Roberts (2012).
In fact, it is not necessary to simulate the diffusion bridges since \( w(t, x, y) \) is an expectation form on the exponential function of these bridges. We use the importance sampling method to describe \( w(t, x, y) \), which is discussed in DiCesare and McLeish (2006). Girsanov transformation must be introduced as the base of our importance sampling scheme.

**Lemma 3.1. (Girsanov transformation).** Let \( X_t \) be a Ito diffusion satisfying

\[
dX_t = \alpha(X_t)dt + dB_t
\]

assume that the drift coefficient \( \alpha \) satisfies Novikov's condition:

\[
E \left[ \exp \left( \frac{1}{2} \int_0^t \alpha^2(B_s)ds \right) \right] < \infty
\]

It is then true that

\[
d\mathbb{P}_X \frac{d\mathbb{P}_Y}{d\mathbb{W}} = \exp \left( \int_0^t \alpha(B_s)dB_s - \frac{1}{2} \int_0^t \alpha^2(B_s)ds \right) := G(B)
\]

where \( \mathbb{P}_X \) is the probability measure induced by \( X \), \( \mathbb{W} \) is the Wiener measure.

If we want to estimate the path integral w.r.t \( \mathbb{P}_X \) on \( \{\Omega, \mathcal{F}\} \), but \( \mathbb{P}_X \) is difficult to sample, with Girsanov theorem, we can find the proportion \( G(B) \) and Brownian motion or Brownian bridge is easy to draw. Therefore, an importance sampling method can be employed here.

Our scheme is as follows, since \( w(t, x, y) \) relies on three functions \( \mu(x) \), \( \sigma(x) \), \( \lambda(x) \), we will deal with \( \sigma(x) \) by Lamperti transformation first, then to carry out \( \mu(x) \) we use Girsanov transformation and compute the weight for importance sampling, at last we will calculate the integral for the term \( \lambda(x) \).

**3.1. One dimensional diffusion.** Consider one dimensional diffusion \( X_t \) satisfying SDE (2.1). Suppose this integral of Lamperti transformation:

\[
s(x) = \int_x^\infty \frac{1}{\sigma(z)}dz
\]

is well-defined and the inverse function \( s^{-1} \) exists, where \( \varepsilon \) is an arbitrary point from the state space of \( X_t \), 0 for example. Define diffusion \( Y_t \) as follows,

\[
Y_t = \int_0^{X_t} \frac{1}{\sigma(z)}dz
\]

By Ito’s Lemma, diffusion \( Y_t \) satisfies the differential equation

\[
dY_t = \alpha(Y_t)dt + dB_t
\]

where

\[
\alpha(Y_t) = \frac{\mu(s^{-1}(Y_t))}{\sigma(s^{-1}(Y_t))} - \frac{1}{2} \sigma'(s^{-1}(Y_t))
\]

Therefore the simulation for diffusion bridge \( X_t \) conditional on two points \( X_0 = x, X_t = y \), is equivalent to the simulation for \( Y_t \) conditional on \( Y_0 = s(x), Y_t = s(y) \).

Let \( \mathbb{P}_Y \) be the probability measure induced by \( Y_t \), by Girsanov theorem,

\[
d\mathbb{P}_Y \frac{d\mathbb{P}_Y}{d\mathbb{W}} = \exp \left( \int_0^t \alpha(B_s)dB_s - \frac{1}{2} \int_0^t \alpha^2(B_s)ds \right) := G(B)
\]
where \( W \) is the Wiener measure.

Suppose \( \alpha \) is everywhere differentiable, let \( A(x) = \int_0^x \alpha(z)dz \), using Ito’s formula \( G(B) \) can be simplified to

\[
G(B) = \exp \left( A(B_t) - A(B_0) - \frac{1}{2} \int_0^t (\alpha^2(B_s) + \alpha'(B_s))ds \right)
\]

Since \( w(t, x, y) \) is bridge path integral, we consider the Radon-Nikodym derivative for the law induced by Brownian bridge and Wiener measure first.

**Proposition 3.2.** Let \( Z \) be a Brownian bridge, and \( \mathbb{P}_y^Z \) be the probability measure for the \( Z \) conditioned on the terminal value \( y \). If the support of \( h \) is the real line, then

\[
\frac{d\mathbb{P}_y^Y}{dW} = \frac{h(B_t)}{(1/\sqrt{2\pi t}) \exp(-B_t^2/(2t))}
\]

**Proof.** see the proof in the appendix.

The following equation can be obtained now,

\[
\frac{d\mathbb{P}_y^Y}{d\mathbb{P}_y^Z} \propto \exp \left( -\int_0^t \frac{1}{2} (\alpha^2(Z_s) + \alpha'(Z_s))ds \right)
\]

if we choose some appropriate function \( h \), we have

\[
\frac{d\mathbb{P}_y^Y}{d\mathbb{P}_y^Z} \propto \exp \left( -\int_0^t \frac{1}{2} (\alpha^2(Z_s) + \alpha'(Z_s))ds \right)
\]

Let \( \mathbb{P}_y^Y \) denote the measure with respect to the conditional process \( Y \) on terminal value \( y \), analogously with the analysis above, the following proposition can be obtained about the Radon-Nikodym derivative for the measure \( \mathbb{P}_y^Y \) and \( \mathbb{P}_y^Z \).

**Proposition 3.3.** Suppose \( Y_t \) satisfies the following equation,

\[
dY_t = \alpha(Y_t)dt + dB_t \quad Y_0 = y_0
\]

and \( Z \) is a Brownian bridge conditional on \( y \), then

(3.6) \[
\frac{d\mathbb{P}_y^Y}{d\mathbb{P}_y^Z} \propto \exp \left( -\int_0^t \frac{1}{2} (\alpha^2(Z_s) + \alpha'(Z_s))ds \right)
\]

This proposition describes the Radon-Nikodym derivative between the law of the conditional diffusion with diffusion coefficient 1 and the Brownian bridge, then the importance sampling algorithm can be summarized as follows,

1. Generate \( n \) Brownian bridge paths \( Z^{(i)}_t \) with \( Z_0 = Y_0 = s(x) \) and \( Z_t = Y_t = s(y) \), for \( i = 1 \) to \( n \).
2. Calculate \( v^{(i)} = \exp \left( -\int_0^t \frac{1}{2} (\alpha^2(Z^{(i)}_s) + \alpha'(Z^{(i)}_s))ds \right) \) as weight function for every path.
3. Estimate of \( w(t, x, y) \) using the weighted average

(3.7) \[
\hat{w}(t, x, y) = \frac{\sum_{i=1}^n v^{(i)} \exp \left( -\int_0^t \lambda^{-1}(Z^{(i)}_s)ds \right)}{\sum_{i=1}^n v^{(i)}}
\]
3.2. Multivariate dimensional diffusion. We wish a similar method for multivariate diffusions as one dimensional case. Suppose that a $d$-dimensional Ito diffusion $X_t$ with the same form as SDE 2.1. Transformations similar with Lamperti discussed by Ait-Sahalia, (2002), (2004), (2007), (2008) which was named reducible can be applied here.

**Definition 3.4.** (Reducibility) The $d$-dimensional diffusion $X_t$ is said to be reducible if and only if there exists an invertible function $\gamma(x)$, infinitely differentiable in $X$ on the state space, such that $Y_t = \gamma(X_t)$ satisfies

$$dY_t = \mu_Y(Y_t)dt + dB_t$$

To deal with the diffusion matrix $\sigma$ in $d$-dimensional case, we need the Radon-Nikodym derivative of $Y$ with respect to that of Wiener measure similar with Proposition 3.3. The following lemma given by Dacunha-Castelle and Florens-Zmirou (1986) will be used,

**Lemma 3.5.** Consider a $d$-dimensional diffusion satisfying

$$dY_t = \mu(Y_t)dt + dB_t$$

Under certain restrictions, the Radon-Nikodym derivative of the distribution of $Y_t$ with respect to Wiener measure is given by

$$\exp \left[ G(B_t) - G(x) - \sum_{k=1}^{d} \int_0^t h_k(B_s)ds \right]$$

where $G$ is potential function, $\nabla G = \mu$, $h_k = \frac{1}{2} \left( \mu_k^2 + \frac{\partial \mu_k}{\partial x_k} \right)$.

**Proof.** The proof is given in the appendix. We consider a reducible diffusion $X_t$ with $\sigma$ nonsingular, transformation $\gamma$ introduced by Ait-Sahalia (2002) must satisfy the equation

$$\nabla \gamma = \sigma^{-1}(x)$$

where $\nabla \gamma_{ij} = \frac{\partial \gamma_j(x)}{\partial x_i}$. Let $Y_t = \gamma(X_t)$, with the help of Ito’s lemma, we have

$$dY_t = \alpha(Y_t)dy + dB_t$$

where the drift functions $\alpha(y) = (\alpha_k(y))_{k=1}^{d}$ are given as follows

$$\alpha_k(y) = \nabla \gamma_k(\gamma^{-1}(y))\mu(\gamma^{-1}(y)) + \frac{1}{2} \sum_i \sum_j \left( \frac{\partial^2}{\partial y_i \partial y_j} \gamma_k(\gamma^{-1}(y)) \xi_{ij}(\gamma^{-1}(y)) \right)$$

where $\xi(x) = \sigma(x)\sigma(x)^T$.

The importance sampling algorithm to approximate $w(t,x,y)$ for $d$ dimensional case can be summarized as follows

1. Calculate the transformation $\gamma$

   $$\nabla \gamma = \sigma^{-1}(x)$$

   the drift function $\alpha(y)$ with

   $$\alpha_k(y) = \nabla \gamma_k(\gamma^{-1}(y))\mu(\gamma^{-1}(y)) + \frac{1}{2} \sum_i \sum_j \left( \frac{\partial^2}{\partial y_i \partial y_j} \gamma_k(\gamma^{-1}(y)) \xi_{ij}(\gamma^{-1}(y)) \right)$$
where \( \xi(x) = \sigma(x)\sigma(x)^T \). And the functions

\[
h_k(y) = \frac{1}{2} \left( \alpha_k^2(y) + \frac{\partial \alpha_k(y)}{\partial y_k} \right) \quad k = 1, \ldots, d
\]

2. Generate \( n \) multivariate Brownian bridges \( Z_t^{(i)} \) with \( Z_0 = \gamma(x) \), \( Z_t = \gamma(y) \), for \( i = 1 \) to \( n \). Calculate

\[
v^{(i)} = \exp \left[ -\sum_{k=1}^d \int_0^t h_k(Z_s^{(i)}) ds \right]
\]

3. Estimate of \( w(t, x, y) \) using the weighted average

\[
\hat{w}(t, x, y) = \frac{\sum_{i=1}^n v^{(i)} \exp \left( -\int_0^t \lambda^{-1}(Z_s^{(i)}) ds \right)}{\sum_{i=1}^n v^{(i)}} \quad (3.8)
\]

4. Variance analysis and efficiency. Suppose the density of \( X \) is \( \pi(x) \) difficult to sample, importance sampling attempts to estimate \( w = E[f(X)] \), by the following approximation

\[
\tilde{w} = \frac{1}{n} \sum_{i=1}^n f(X_i) \pi(X_i) / p(X_i)
\]

where \( X_i \) are generated from an easy-to-sample distribution \( p(x) \) which is close to the shape \( f(x)\pi(x) \). With the delta method based on Kong (1992) to develop the approximation above, we have

\[
\hat{w} = \frac{\sum_{i=1}^n v(X_i) f(X_i)}{\sum_{i=1}^n v(X_i)} = \frac{\bar{U}}{\bar{V}} \quad (4.1)
\]

where \( U = v(X)f(X) \), \( V = v(X) \). The estimate \( \tilde{w} \) is applied in our scheme to impute \( \hat{w}(t, x, y) \), since we only get the ratio between measures deduced by Brownian bridge and conditional diffusion in Proposition 2. Using standard delta method, we get the variance of \( \hat{w} \),

\[
Var_p(\hat{w}) \approx \frac{1}{n}(w^2Var_p(V) + Var_p(U) - 2wCov_p(U, V))
\]

In Liu (1996), the author gave the following approximation to the variance of \( \hat{w} \),

\[
Var_p(\hat{w}) \approx \frac{1}{n}Var_\pi(f(X))[1 + Var_p(v(X))]
\]

and the efficiency of \( \hat{w} \) with respect to other estimate \( \hat{w} = \sum_{i=1}^n f(Y_i)/n \) where \( Y_i \sim \pi \) is

\[
\frac{Var_\pi(f(Y))}{Var_p(f(X)v(X))} \approx \frac{1}{1 + Var_p(v(X))}
\]
Therefore, \( \text{Var}_s(v(X)) \) plays a most important role to both variance of \( \dot{w} \) and efficiency. In the algorithm for one dimensional diffusion, \( X \) is taken as Brownian bridge starting at \( s(x) \) and ending at \( s(y) \) and following the symbol in the previous section, denote it by \( Z \),

\[
v(Z) = \exp \left( -\int_0^t \frac{1}{2} (\alpha^2(Z_s) + \alpha'(Z_s)) ds \right)
\]

5. Examples.

5.1. One-dimensional example. Consider time-dependent Schrödinger equation with harmonic oscillator,

\[
i\hbar \frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2m} \frac{\partial^2 \psi}{\partial x^2} + \frac{m\omega^2}{2} x^2 \psi
\]

where \( \omega \) is the classical oscillator speed. The exact expression of the fundamental solution for the PDE (5.1) is

\[
\varphi(t, x, y) = \left[ \frac{m\omega}{2\pi \hbar \sin(\omega t)} \right]^{1/2} \exp \left( -\frac{i m\omega}{2\hbar \sin(\omega t)} \left[ (x^2 + y^2) \cos(\omega t) - 2xy \right] \right)
\]

With the help of \( \sin(y/i) = \sinh(y)/i \) and \( \cos(y/i) = \cosh(y) \), we can get the analytic continuation for the fundamental solution for the parabolic equation with harmonic oscillator of the following form

\[
\frac{\partial u}{\partial t} = \frac{\hbar^2}{2m} \frac{\partial^2 u}{\partial x^2} - \frac{m\omega^2 x^2}{2\hbar} u
\]

the explicit expression of \( q(t, x, y) \) is

\[
q(t, x, y) = \sqrt{\frac{m\omega}{2\pi \hbar \sinh(\omega t)}} \exp \left( -\frac{m\omega}{2\hbar \sinh(\omega t)} \left[ (x^2 + y^2) \cosh(\omega t) - 2xy \right] \right)
\]

Let the parameter \( \hbar, m \) and \( \omega \) be taken as 1, we have the following figure. The standard deviation of \( \hat{q}(t, x, y) \) w.r.t different terminal value \( y \) is shown in the next table, time lap \( t \) is 1, the initial position is \( x = 0 \) and the number of paths is 10000.

| \( y \) | -3 | -2 | -1 | 0 | 1 | 2 | 3 |
|---|---|---|---|---|---|---|---|
| \( q \) | 9.994e-4 | 0.02663 | 0.3680 | 0.1909 | 0.02663 | 9.994e-4 |
| \( \hat{q} \) | 9.996e-4 | 0.02659 | 0.3681 | 0.1911 | 0.02659 | 1.003e-3 |
| error | 1.210e-7 | -4.006e-5 | -3.688e-4 | 1.424e-4 | 2.658e-4 | 5.926e-5 | 3.431e-6 |
| r.e | 1.211e-4 | 1.504e-3 | 1.932e-3 | 3.870e-4 | 1.387e-3 | 2.223e-3 | 3.354e-3 |
| \( \hat{\sigma}(q) \) | 4.155e-6 | 7.305e-5 | 2.860e-4 | 2.499e-4 | 2.830e-4 | 7.267e-5 | 4.171e-6 |

5.2. Multi-dimensional example. It is well known that the 2-dim O-U process satisfies

\[
dX_t = -AX_t dt + \sigma dB_t, \quad X_0 = x \in \mathbb{R}^2
\]
The fundamental solution for harmonic oscillator

The rate of log(sigma) with respect to log(N)

The rate of log(sigma) with respect to log(N)

Fig. 5.1. The first picture is the fit to the exact solution for simple harmonic oscillator with our algorithm, the time lap is 1, the initial position is $x = 0$. The second picture is the rate of standard deviation with respect to the sample size, the time lap is 1, the initial and terminal position are $x = 0$ and $y = 1$. The third picture is the standard deviation for different $y$. The fourth one is the relative error.

where $A$ and $\sigma$ are both real $2 \times 2$ matrices. We can solve this multivariate SDE by Ito’s lemma,

$$X_t = e^{-At}x + \int_0^t e^{(t-s)A} \sigma dB_s$$

From the above formula, it is clear that the distribution of $X_t$ with $X_0 = x$ is the Gaussian measure $N(e^{-At}x, \Sigma_t)$ with mean $e^{-tA}x$ and covariance operator $\Sigma_t$

$$\Sigma_t = e^{-At} \left( \int_0^t e^{As} \sigma \sigma^T e^{As} ds \right) e^{-AT}, \quad t \geq 0$$

Take $A$ and $\sigma$ as follows for convenience,

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \quad \sigma = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$
the SDEs can be written as
\[
\begin{align*}
\frac{dX_t^{(1)}}{dt} &= -X_t^{(2)} dt + dB_t^{(1)} \\
\frac{dX_t^{(2)}}{dt} &= -X_t^{(1)} dt + dB_t^{(2)}
\end{align*}
\]

The transition functions for \(x = (x_1, x_2)^T, y = (y_1, y_2)^T\)

\[
p(t, x, y) = \frac{1}{2\pi\sqrt{\det \Sigma}} \exp \left( - \frac{(y - e^{-At}x)^T \Sigma^{-1} (y - e^{-At}x)}{2} \right)
\]

where \(\Sigma = \int_0^t e^{2(s-t)A} ds\). Then we take \(\lambda(x_1, x_2) = x_1 + x_2\), \(p(t, x, y)\) and \(q(t, y, x)\) are shown in Figure.

**Fig. 5.2.** The symmetry 2-dim OU process and its dual drifting in opposite directions when \(\lambda(x_1, x_2) = x_1 + x_2\). In the figure, the time lap is 0.25, the initial/terminal position are \(x = (0, 0)^T\). It should be noted that \(\lambda > 0, p(t, x, y) > q(t, x, y)\) while \(\lambda < 0, p(t, x, y) < q(t, x, y)\). The surface in this figure just shows this, on the region \(x_1 + x_2 > 0\), \(p\) is on the top but on \(x_1 + x_2 < 0\), \(q\) is on the top.

**Appendix A. Proof.**

**Proof of Proposition 1.** Choose any \(F \in \sigma(\{B_s, 0 < s < t\})\). Let

\[
f := \frac{h(B_t)\sqrt{2\pi t}}{\exp(-B_t^2/(2t))}
\]

Let \(P^y_F\) denote the measure with respect to the conditional process \(Y\) on terminal value \(y\), our goal is \(E_{P^y_F}[I_F] = E_W[I_F f]\). Since \(f\) is \(\sigma(B_t)\)-measurable and the definition of Brownian bridge, we have

\[
P^y_F[F|\sigma(B_t)] = W[F|\sigma(B_t)] =: g(B_t) \quad W \text{ - a.s.}
\]

for some Borel-measurable function \(g : R \to R\). Then since \(B_t\) is normal distributed with mean 0 and variance \(t\),

\[
E_W[I_F f] = E_W[E_W[I_F f|\sigma(B_t)]] = E_W[fW[fW|\sigma(B_t)]]
\]
\[
= \int_R \frac{h(z) \sqrt{2\pi t}}{\exp(-z^2/(2t))} g(z)dz \\
= \int_R h(z) g(z)dz 
\]

hence it is clear that
\[
E_{\mathcal{P}_Y}[I_F] = E_{\mathcal{P}_Y}[E_{\mathcal{P}_Y}[I_F|\sigma(B_t)]] = \int_R h(z) g(z)dz 
\]

so we conclude \( E_{\mathcal{P}_Y}[I_F] = E_{\mathcal{W}}[I_F f] \). The proof is completed.

**Proof of Lemma 1.** Denote \( \mathcal{P}_{Z,0}^{0,x} \) as the measure induced by a diffusion \( Z \) when \( Z_0 = x \). By Girsanov’s theorem, the density \( \rho_{0,x}^0 \) of the measure \( \mathcal{P}_{Z,0}^{0,x} \) with respect to the Wiener measure \( \mathcal{P}_B^{0,x} \):
\[
\rho_{0,x}^0 = \frac{d\mathcal{P}_{Z,0}^{0,x}}{d\mathcal{P}_B^{0,x}} = \exp\left( \int_0^t \mu(B_s)dB_s - \frac{1}{2} \int_0^t \| \mu(B_s) \|^2 ds \right) \\
= \exp\left( \sum_{k=1}^d \int_0^t \mu_k(B_s)dB_s^k - \frac{1}{2} \sum_{k=1}^d \int_0^t \mu_k(B_s)^2ds \right) 
\]

To calculate the integration \( \int_0^t \mu_k(B_s)dB_s^k \), by Ito’s lemma we have
\[
\begin{align*}
G(B_t) &= G(x) + \sum_{k=1}^d \int_0^t \frac{\partial G}{\partial x_k}(B_s)dB_s^k + \frac{1}{2} \sum_{k,j=1}^d \int_0^t \frac{\partial^2 G}{\partial x_k \partial x_j}(B_s)d[B_s^k,B_s^j] \\
&= G(x) + \sum_{k=1}^d \int_0^t \frac{\partial G}{\partial x_k}(B_s)dB_s^k + \frac{1}{2} \sum_{k=1}^d \int_0^t \frac{\partial^2 G}{\partial x_k^2}(B_s)ds \\
&= G(x) + \sum_{k=1}^d \int_0^t \mu_k(B_s)dB_s^k + \frac{1}{2} \sum_{k=1}^d \int_0^t \frac{\partial \mu}{\partial x_k}(B_s)ds 
\end{align*}
\]

Thus
\[
\sum_{k=1}^d \int_0^t \mu_k(B_s)dB_s^k = G(B_t) - G(x) - \frac{1}{2} \sum_{k=1}^d \int_0^t \frac{\partial \mu}{\partial x_k}(B_s)ds 
\]

substitute this into \( \rho_{0,x}^0 \),
\[
\begin{align*}
\rho_{0,x}^0 &= \exp \left[ G(B_t) - G(x) - \frac{1}{2} \sum_{k=1}^d \int_0^t \frac{\partial \mu}{\partial x_k}(B_s)ds - \frac{1}{2} \sum_{k=1}^d \int_0^t \mu_k(B_s)^2ds \right] \\
&= \exp \left[ G(B_t) - G(x) - \frac{1}{2} \sum_{k=1}^d \int_0^t h_k(B_s)ds \right] 
\end{align*}
\]

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