ZERO VISCOSITY-RESISTIVITY LIMIT FOR THE 3D INCOMPRESSIBLE MAGNETOHYDRODYNAMIC EQUATIONS IN GEVREY CLASS

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(Communicated by Chongchun Zeng)

Abstract. We study the zero viscosity-resistivity limit for the 3D incompressible magnetohydrodynamic (MHD) equations in a periodic domain in the framework of Gevrey class. We first prove that there exists an interval of time, independent of the viscosity coefficient and the resistivity coefficient, for the solutions to the viscous incompressible MHD equations. Then, based on these uniform estimates, we show that the solutions of the viscous incompressible MHD equations converge to that of the ideal incompressible MHD equations as the viscosity and resistivity coefficients go to zero. Moreover, the convergence rate is also given.

1. Introduction. We consider the 3D viscous incompressible magnetohydrodynamic (MHD) equations:

\[
\begin{align*}
\boldsymbol{u}^{\nu,\mu} & + \nu \nabla \boldsymbol{u}^{\nu,\mu} \cdot \nabla \boldsymbol{u}^{\nu,\mu} - \nabla \nabla \cdot \nabla p^{\nu,\mu} - \nabla \nabla - \nu \Delta \boldsymbol{u}^{\nu,\mu} = 0, \\
\boldsymbol{H}^{\nu,\mu} & + \nabla \nabla - \nabla \nabla - \mu \Delta \boldsymbol{H}^{\nu,\mu} = 0, \\
\text{div} \boldsymbol{u}^{\nu,\mu} & = \text{div} \boldsymbol{H}^{\nu,\mu} = 0
\end{align*}
\]

in \( \mathbb{T}^3 = (-\pi, \pi)^3 \), a periodic domain of \( \mathbb{R}^3 \), together with the following initial condition

\[
(\boldsymbol{u}^{\nu,\mu}, \boldsymbol{H}^{\nu,\mu})|_{t=0} = (\boldsymbol{u}_0(x), \boldsymbol{H}_0(x)), \quad x \in \mathbb{T}^3.
\]

The unknowns are \( \boldsymbol{u}^{\nu,\mu} = (u_1^{\nu,\mu}, u_2^{\nu,\mu}, u_3^{\nu,\mu}) \), \( \boldsymbol{H}^{\nu,\mu} = (H_1^{\nu,\mu}, H_2^{\nu,\mu}, H_3^{\nu,\mu}) \) and \( p^{\nu,\mu} \), denoting the velocity field, the magnetic field and the pressure, respectively. The parameters \( \nu > 0 \) and \( \mu > 0 \) denote the viscosity coefficient and the magnetic resistivity coefficient, respectively.

The system (1) describes the macroscopic behavior of electrically conducting incompressible fluids in a given magnetic field. Due to the significance of the physical background, the incompressible MHD equations have been studied by many physicists and mathematicians on various topics, for example, see [6, 11, 18, 15] and the references cited therein. One of the most important problems in magnetohydrodynamics is to understand the inviscid limit in a domain with boundary. The outcomes of the inviscid limit and convergence rate are helpful to understand the turbulent

2010 Mathematics Subject Classification. Primary: 35Q30, 35Q35; Secondary: 76D03, 76D09.

Key words and phrases. Incompressible MHD equations, Gevrey class, zero viscosity-resistivity limit, convergence rate.

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phenomena governed by the viscous MHD equations. Thus, it is very interesting to consider the vanishing viscosity-resistivity limit of the viscous incompressible MHD equations.

When we take $H^{\nu,\mu} = 0$ in the system (1), it is reduced into the classical incompressible Navier-Stokes equations and many results are available on the inviscid limit to it in $\mathbb{R}^n$ or torus, for instance, see [7, 8, 9, 22, 34] and the references cited therein. Specially, Cheng, Li, and Xu [7] studied the vanishing viscosity limit of the incompressible Navier-Stokes equations in Gevrey class in a torus and got the convergence rate. However, in the presence of a physical boundary, according to the classical Prandtl boundary layers theory [37, 38], the inviscid limit of the incompressible Navier-Stokes equations in a domain with non-slip boundary condition is still an outstanding open problem up to now and only a few results for some specific cases available. Sammartino and Caflisch first investigated the Prandtl theory and the vanishing viscosity limit problem in the analytic setting in [40, 41]. In 2014, Maekawa [33] studied the same problems for 2D case under the assumption that the initial vorticity of outer Euler flows should vanish in a neighborhood of boundary. Additional, the results in [40, 41] were generalized to Gevrey class in [16], see also [44]. Xin and Zhang [48] proved the global existence of weak solutions to Prandtl equations for the favorable pressure. Alexandre et al. [1] and Masmoudi and Wong [35] independently verified the local well-posedness for the Prandtl equations in Sobolev space by the direct energy method. Gerard-Varet and Dormy [17] showed that the ill-posedness in Sobolev space for the linearized Prandtl equation around non-monotonic shear flows. Let us also mention some conditional convergence results [10, 23, 24, 25, 42, 43], which were first considered by Kato in [23]. He proved that the vanishing viscosity limit is equivalent to having sufficient control of the gradient of the velocity in a boundary layer of width proportional to the viscosity. Recently, Kelliher [24] showed that the gradient of the velocity can be replaced by the vorticity in Kato’s condition. For more mathematical results on the Prandtl boundary layers theory, we can see [1, 12, 17, 21, 29, 30, 35, 48] and the references cited therein. It should point out that although the rigorous verification of the Prandtl boundary layers theory was achieved for some special cases, the Prandtl boundary layer equations are ill-posed in Sobolev space for many cases. Therefore, the justification for the Prandtl boundary layers theory in general setting is still open.

In addition, we note that when we consider the incompressible Navier-Stokes equations with Navier type boundary conditions, compared with the non-slip case, the problem becomes easier. The uniform $H^3$ bound and a uniform existence time interval with respect to the viscosity coefficient were obtained by Xiao and Xin [46] for the flat boundary. Subsequently, the conclusions in [46] were extended to $W^{k,p}$ with $k \geq 3$ and $p \geq 2$ in [4]. The main reason is that the boundary integrals vanish on the flat boundary, see also [3, 5]. Masmoudi and Rousset [36] considered the vanishing viscosity limit for the incompressible Navier-Stokes equations with Navier boundary conditions in anisotropic conormal Sobolev spaces. Moreover, based on the results in [36], Gie and Kelliher [20] proved the convergence rates in different spaces by constructing an explicit corrector.

When considering the zero viscosity-resistivity limit to the incompressible MHD equations, the similar situations happen. Compared with the incompressible Navier-Stokes equations, there are only a few results on the well-posedness theory of the MHD boundary layer equations and the inviscid limit problem of MHD equations.
When the viscous incompressible MHD equations are supplemented with Navier boundary conditions, the authors in [28, 47] studied the inviscid limit problem in anisotropic conormal Sobolev spaces and classical Sobolev spaces, respectively. The inviscid limit problem of the viscous incompressible MHD equations with non-slip boundary condition for velocity is very challenging due to the appearance of non-trivial boundary layer. Recently, Liu, Xie, and Yang [31] studied the well-posedness theory of the MHD boundary layer equations in weighted Sobolev spaces in 2D case. In addition, based on the well-posedness theory of the MHD boundary layer equations, they [32] also studied the inviscid limit. Wang and Xin [45] investigated the zero viscosity-resistivity limit to the incompressible MHD equations by constructing the boundary layer directly.

In this paper, motivated by [7] on the incompressible Navier-Stokes equations, we study the zero viscosity-resistivity limit of the viscous incompressible MHD equations in the torus $\mathbb{T}^3$ in Gevrey class. Due to the strong coupling between $u^{\nu, \mu}$ and $H^{\nu, \mu}$, we need to estimate some new nonlinear terms. In order to overcome the difficulty in handling the nonlinear terms, we shall establish a more general result (see Lemma 2.2 below) than that in Lemma 2.3 of [7]. Besides, we also need to construct more complicated energy estimates to obtain the desired convergence results. The details will be presented in the proceeding arguments.

Formally, let $(\nu, \mu) = (0, 0)$ in the system (1), it is reduced into the ideal incompressible MHD equations

$$
\begin{align*}
\dot{u}_t^0 + u^0 \cdot \nabla u^0 - H^0 \cdot \nabla H^0 + \nabla p^0 &= 0, \\
\dot{H}_t^0 + u^0 \cdot \nabla H^0 - H^0 \cdot \nabla u^0 &= 0, \\
\text{div} u^0 &= \text{div} H^0 = 0,
\end{align*}
$$

with the same initial data as to (1), i.e.

$$(u^0, H^0)|_{t=0} = (u_0(x), H_0(x)), \quad x \in \mathbb{T}^3. \quad (4)$$

Before giving a rigorous justification for the above formal procedure and stating our main results, we first introduce the functions spaces used throughout this paper. Let $L^2(\mathbb{T}^3)$ be the vector function space

$$L^2(\mathbb{T}^3) = \left\{ u = \sum_{k \in \mathbb{Z}^3} \hat{u}_k e^{i k \cdot x} \right\}$$

where $\hat{u}_k$ is the $k$-th order Fourier coefficient of $u$ and $i = \sqrt{-1}$. The condition $k \cdot \hat{u}_k = 0$ implies that $\nabla \cdot u = 0$ in the weak sense, so it is the standard $L^2$ space with the divergence-free condition. Let $\mathcal{H}^r(\mathbb{T}^3)$ be the vector periodic Sobolev space: for $r \geq 1$,

$$\mathcal{H}^r(\mathbb{T}^3) = \left\{ u = \sum_{k \in \mathbb{Z}^3} \hat{u}_k e^{i k \cdot x} \right\}$$

where the condition $k \cdot \hat{u}_k = 0$ also means $\nabla \cdot u = 0$, so it is the standard Sobolev space $H^r$ with the divergence-free condition. Denote $(\cdot, \cdot)$ the $L^2$ inner product of two vector functions. Let us define the fractional differential operator $\Lambda = (-\Delta)^{1/2}$ and the exponential operator $e^{r \Lambda^{1/2}}$ as follows,

$$
\Lambda u = \sum_{j \in \mathbb{Z}^3} |j| \hat{u}_j e^{ij \cdot x}, \quad e^{r \Lambda^{1/2}} u = \sum_{j \in \mathbb{Z}^3} e^{r |j|^{1/2}} \hat{u}_j e^{ij \cdot x}.
$$

(5)
The vector Gevrey space \( G^s_{r,\tau} \) \((s \geq 1, \tau > 0, r \in \mathbb{R})\) is
\[
G^s_{r,\tau}(\mathbb{T}^3) = \left\{ u \in H^r(\mathbb{T}^3) \mid \|u\|^2_{G^s_{r,\tau}} = (2\pi)^3 \sum_{k \in \mathbb{Z}^3} |k|^{2r}e^{2\tau|k|^2} |\hat{u}_k|^2 < \infty \right\}. \tag{6}
\]

We point out that the Gevrey space is an intermediate space between the spaces of smooth functions and analytic functions. The name is given in honor of M. Gevrey, who gave the first motivating example, refer to [19], in which the regularity estimates of the heat kernel were deduced. Thanks to [39], we know that when \( s = 1 \), this space is the same as the space of analytic functions, and for \( s > 1 \), there are non-trivial Gevrey functions having compact support, which is different from analytic functions. Foias and Temam [13] first used the Fourier space method, namely the Gevrey class regularity, to prove the analyticity of the solutions for the incompressible Navier-Stokes equations. Later, the techniques developed in [13] were improved and used to other problems, see [2, 26, 27] and the references cited therein.

Now our main results can be stated as follows.

**Theorem 1.1.** Let \( r > \max \left\{ \frac{5}{2} + \frac{7}{s}, \frac{7}{2} + \frac{1}{2s} \right\}, \tau_0 > 0, \) and \( s \geq 1 \). Consider the initial data \((u_0, H_0) \in G^s_{r,\tau_0}(\mathbb{T}^3)\). Then there exist a time \( T > 0 \), a constant \( \widetilde{C}_0 > 0 \), and a decreasing function \( \tau(t) > 0 \), independent of \( \mu, \nu \in (0,1] \) such that, for any fixed \( \nu, \mu \), there exists a unique solution \((u^\nu,\mu,H^\nu,\mu,p^\nu,\mu)\) of the problem (1)-(2) on \([0,T]\) satisfying
\[
\|u^\nu,\mu,H^\nu,\mu\|_{L^\infty(0,T;G^s_{r,\tau(t)}(\mathbb{T}^3))} + \|p^\nu,\mu\|_{L^\infty(0,T;G^s_{r+1,\tau(t)}(\mathbb{T}^3))} \leq \widetilde{C}_0. \tag{7}
\]
As for the problem (3)-(4), there also exists a unique solution \((u^0, H^0, p^0)\) on \([0,T]\) satisfying
\[
\|u^0, H^0\|_{L^\infty(0,T;G^s_{r,\tau(t)}(\mathbb{T}^3))} + \|p^0\|_{L^\infty(0,T;G^s_{r+1,\tau(t)}(\mathbb{T}^3))} \leq \widetilde{C}_0. \tag{8}
\]
Moreover, we have the following convergence rate: for any \( 0 < t \leq T \),
\[
\|u^\nu,\mu(t,\cdot) - u^0(t,\cdot)\|^2_{G^s_{r-1,\tau(t)}} + \|H^\nu,\mu(t,\cdot) - H^0(t,\cdot)\|^2_{G^s_{r-1,\tau(t)}} \leq \widetilde{C}_1(\nu + \mu), \tag{9}
\]
\[
\|p^\nu,\mu(t,\cdot) - p^0(t,\cdot)\|^2_{G^s_{r,\tau(t)}} \leq \widetilde{C}_3(\nu + \mu), \tag{10}
\]
where \( \widetilde{C}_1 \) is a constant depending on \( r, s, u_0, H_0 \) and \( T \).

**Remark 1.1.** The uniform Gevrey radius \( \tau(t) \) of the solution is
\[
\tau(t) = \frac{1}{1_\tau e^{C_2 t} + C_3(e^{C_2 t} - 1)}, \tag{11}
\]
where \( C_2 \) and \( C_3 \) are two positive constants depending on \( r, s, u_0, H_0 \) and \( T \).

The rest of this paper is organized as follows. In Section 2, we list a local existence result of the \( H^r \) solution to the problem (1)-(2), and some elementary inequalities which will be used later. Next, we derive a priori estimates in Gevrey space, and use the a priori estimates to prove the existence of the solution in Gevrey space in Section 3. In Section 4, we show the convergence rate of the zero viscosity-resistivity limit of the problem (1)-(2) in Gevrey space. Finally, in the Appendix, we give the proof of the existence of the \( H^r \) solution.
2. Preliminaries. We first give a result on the local existence of the $H^r$ solution to the viscous incompressible MHD equations.

**Theorem 2.1.** Let $(u_0, H_0) \in H^r(\mathbb{T}^3)$ for $r > 5/2$. Then there exists a time $T^* > 0$ depending on $(u_0, H_0)$ but not on $\nu, \mu$, such that, for any fixed $\nu, \mu \in (0, 1]$, the problem (1)-(2) has a unique solution $(u^{\nu, \mu}, H^{\nu, \mu})$ on $[0, T^*]$ that satisfies the following estimate

$$
\|u^{\nu, \mu}(t, \cdot)\|_{H^r} + \|H^{\nu, \mu}(t, \cdot)\|_{H^r} \leq \frac{2(\|u_0\|_{H^r}^2 + \|H_0\|_{H^r}^2)^{1/2}}{1 - Ct(\|u_0\|_{H^r}^2 + \|H_0\|_{H^r}^2)^{1/2}}, \quad t \in [0, T^*],
$$

where $C > 0$ is a constant depending only on $r$.

The proof of Theorem 2.1 will be given in the Appendix. As for the ideal MHD equations, by taking the similar arguments to that in Theorem 2.1, we can also obtain the local existence of the $H^r$ solution in $\mathbb{T}^3$. Next, we give two important inequalities, which will be used repeatedly throughout this paper. The first one is that, for any $j, k \in \mathbb{Z}^3 \setminus \{0\}$,

$$
|k - j| \leq 2|j||k|.
$$

The proof is a simple application of triangle inequality. So we omit the details here. The second one is the following lemma.

**Lemma 2.1 ([7]).** Given two real numbers $\xi, \eta \geq 1$ and $s \geq 1$, then the following inequality holds

$$
|\xi^s - \eta^s| \leq C\frac{|\xi - \eta|}{|\xi|^{1 - \frac{s}{2}} + |\eta|^{1 - \frac{s}{2}}},
$$

where $C > 0$ is a constant depending only on $s$.

Finally, with the help of (13) and Lemma 2.1, we have the following estimate which will be used to estimate the nonlinear terms.

**Lemma 2.2.** Let $r > 0$, $s \geq 1$ and $\tau > 0$ be given constants and $r > \max \{\frac{5}{2} + \frac{\tau}{s}, \frac{7}{2}, \frac{3}{2} + \frac{\tau}{s}\}$. Then, for any $f, g, h \in \mathcal{G}_r^s(\mathbb{T}^3)$, we have

$$
\begin{align*}
&\left|\left(\Lambda^r e^{\Lambda^s} (f \cdot \nabla g), \Lambda^r e^{\Lambda^s} \frac{1}{r} h\right) - \left(f \cdot \nabla \Lambda^r e^{\Lambda^s} \frac{1}{r} g, \Lambda^r e^{\Lambda^s} \frac{1}{r} h\right)\right| \\
&\leq C\|f\|_{H^r}\|g\|_{H^r}\|h\|_{\mathcal{G}_r^s} + C(1 + \tau^2)\|f\|_{H^r}\|g\|_{\mathcal{G}_r^s} + C\tau\|g\|_{H^r}\|h\|_{\mathcal{G}_r^s} \\
&\quad + C\tau\|g\|_{H^r}\|h\|_{\mathcal{G}_r^s} + C\tau\|g\|_{H^r}\|h\|_{\mathcal{G}_r^s} + C\tau\|g\|_{H^r}\|h\|_{\mathcal{G}_r^s},
\end{align*}
$$

where the constant $C$ depends only on $r$ and $s$.

**Proof.** It follows from the definition of the vector function space $\mathcal{G}_r^s(\mathbb{T}^3)$ that

$$
f = \sum_{j \in \mathbb{Z}^3} \hat{f}_je^{ij \cdot x}, \quad g = \sum_{k \in \mathbb{Z}^3} \hat{g}_ke^{ik \cdot x}, \quad h = \sum_{l \in \mathbb{Z}^3} \hat{h}_le^{il \cdot x},
$$

and $\hat{f}_0 = \hat{g}_0 = \hat{h}_0 = 0$. In view of the Fourier series convolution property, we have

$$
f \cdot \nabla g = i \sum_{k \in \mathbb{Z}^3} \sum_{j \in \mathbb{Z}^3} \left(\hat{f}_j \cdot (k - j)\right) \hat{g}_{k-j} e^{ik \cdot x}.
$$

(16)
Applying $\Lambda^r e^{r \Lambda^k}$ to $f \cdot \nabla g$, we obtain that
\[
\Lambda^r e^{r \Lambda^k} (f \cdot \nabla g) = i \sum_{k \in \mathbb{Z}^3} \sum_{j \in \mathbb{Z}^3} \left( \hat{f}_j \cdot (k - j) \right) \hat{g}_{k-j} |k| e^{r |k|^2} e^{i k \cdot x}.
\] (17)

In addition, $\Lambda^r e^{r \Lambda^k} h = \sum_{l \in \mathbb{Z}^3} |l| e^{r |l|^2} \hat{h} e^{i l \cdot x}$. Now we take the $L^2$ inner product of $\Lambda^r e^{r \Lambda^k} (f \cdot \nabla g)$ with $\Lambda^r e^{r \Lambda^k} h$ over $\mathbb{T}^3$. The orthogonality of the exponentials in $L^2$ yields
\[
\langle \Lambda^r e^{r \Lambda^k} (f \cdot \nabla g), \Lambda^r e^{r \Lambda^k} h \rangle = i(2\pi)^3 \sum_{k \in \mathbb{Z}^3} \sum_{j \in \mathbb{Z}^3} \left( \hat{f}_j \cdot (k - j) \right) \hat{g}_{k-j} \hat{h}_{j-k} |k| e^{2r |k|^2}.\] (18)

Similarly, we also have
\[
\langle f \cdot \nabla (\Lambda^r e^{r \Lambda^k} g), \Lambda^r e^{r \Lambda^k} h \rangle = i(2\pi)^3 \sum_{k \in \mathbb{Z}^3} \sum_{j \in \mathbb{Z}^3} \left( \hat{f}_j \cdot (k - j) \right) \hat{g}_{k-j} \hat{h}_{j-k} |k| e^{r |k|^2}.\] (19)

Then, (18) minus (19) yields
\[
\langle \Lambda^r e^{r \Lambda^k} (f \cdot \nabla g), \Lambda^r e^{r \Lambda^k} h \rangle - \langle f \cdot \nabla (\Lambda^r e^{r \Lambda^k} g), \Lambda^r e^{r \Lambda^k} h \rangle = \mathcal{I}_1 + \mathcal{I}_2,
\] (20)

where
\[
\mathcal{I}_1 = i(2\pi)^3 \sum_{k \in \mathbb{Z}^3} \sum_{j \in \mathbb{Z}^3} (|k|^r - |k - j|^r) e^{r |k-j|^2} \left( \hat{f}_j \cdot (k - j) \right) \hat{g}_{k-j} \hat{h}_{j-k} |k| e^{r |k|^2},
\]
\[
\mathcal{I}_2 = i(2\pi)^3 \sum_{k \in \mathbb{Z}^3} \sum_{j \in \mathbb{Z}^3} (|k|^r e^{r |k|^2} - e^{r |k-j|^2}) \left( \hat{f}_j \cdot (k - j) \right) \hat{g}_{k-j} \hat{h}_{j-k} |k| e^{r |k|^2}.
\]

Before beginning to control $\mathcal{I}_1$ and $\mathcal{I}_2$, we recall the following mean value theorem, for any $\xi, \eta \in \mathbb{R}^+$, there exist two constants $0 \leq \theta, \theta' \leq 1$ such that
\[
|\xi^r - \eta^r| = r(\eta - \xi) \left( (\theta \xi + (1 - \theta)\eta)^{r-1} - \eta^{r-1} \right) + r(\xi - \eta)\eta^{r-2}.
\] (21)

which implies that
\[
|\xi^r - \eta^r| \leq C|\xi|^2 ((\xi^r - 2) + |\eta^r - 2|) + C|\xi||\eta^r - 1|,
\] (22)

where the constant $C$ depends only on $r$. On the other hand, the inequality $e^\xi \leq e + \xi^2 e^\xi$ holds for all $\xi \in \mathbb{R}$, which yields
\[
e^{r |k-j|^2} \leq e + r^2 |k-j|^2 e^{r |k-j|^2}.
\] (23)

Then it follows from (22) and (23) that
\[
|\mathcal{I}_1| \leq C \sum_{k \in \mathbb{Z}^3} \sum_{j \in \mathbb{Z}^3} (|\xi^r| + |\xi|^2 |k-j|^r) \hat{f}_j \hat{g}_{k-j} |k| e^{r |k|^2} (e + r^2 |k-j|^2 e^{r |k-j|^2})
\]
\[
\times |\hat{h}_{k-j}||k| e^{r |k|^2} + C \sum_{k \in \mathbb{Z}^3} \sum_{j \in \mathbb{Z}^3} |\xi^r| \hat{f}_j \hat{g}_{k-j} |k| e^{r |k|^2} |\hat{h}_{k-j}||k| e^{r |k|^2}.
\]
In view of the discrete Hölder’s inequality and Minkowski’s inequality, we can obtain the estimates of $I_{1i}(i = 1, \ldots, 5)$. For the first term $I_{11}$, we have

$$I_{11} = C \sum_{k \in \mathbb{Z}^3} \sum_{j \in \mathbb{Z}^3} |j| \hat{f}_j |k - j||\hat{g}_{k-j}||k| \hat{r}^e \hat{k} \frac{1}{2} |\hat{h}_{-k}|,$$

where

$$I_{11} = Ce \sum_{k \in \mathbb{Z}^3} \sum_{j \in \mathbb{Z}^3} |j|^r \hat{f}_j |k - j||\hat{g}_{k-j}||k|^\frac{1}{2} |\hat{h}_{-k}|,$$

$$I_{12} = Ce \sum_{k \in \mathbb{Z}^3} \sum_{j \in \mathbb{Z}^3} |j|^2 \hat{f}_j |k - j||\hat{g}_{k-j}||k| \hat{r}^e \hat{k} \frac{1}{2} |\hat{h}_{-k}|,$$

$$I_{13} = C \tau^2 \sum_{k \in \mathbb{Z}^3} \sum_{j \in \mathbb{Z}^3} |j||\hat{f}_j |k - j|^{|-1|} \hat{r}^e \hat{k} \frac{1}{2} |\hat{g}_{k-j}||k| \hat{r}^e \hat{k} \frac{1}{2} |\hat{h}_{-k}|,$$

$$I_{14} = C \tau^2 \sum_{k \in \mathbb{Z}^3} \sum_{j \in \mathbb{Z}^3} |j|^r \hat{f}_j |k - j|^{|-1+2|} \hat{r}^e \hat{k} \frac{1}{2} |\hat{g}_{k-j}||k| \hat{r}^e \hat{k} \frac{1}{2} |\hat{h}_{-k}|,$$

$$I_{15} = C \sum_{k \in \mathbb{Z}^3} \sum_{j \in \mathbb{Z}^3} |j| \hat{f}_j |k - j| \hat{r}^e \hat{k} \frac{1}{2} |\hat{g}_{k-j}||h_{-k}||k| \hat{r}^e \hat{k} \frac{1}{2}.$$
where $C$ is a constant depending on $r$, $s$, and $e$, and the assumption $r > \frac{5}{2} + \frac{2}{s}$ has been used to estimate $I_{13}$. As for $I_{14}$, using the facts $s \geq 1$ and $r > \frac{5}{2} + \frac{1}{2s}$, and the inequality $|k - j|^{\frac{1}{r}} \leq C|k|^{\frac{1}{r}}|j|^{\frac{1}{r}}$, we obtain that

$$I_{14} = C\tau^2 \sum_{k \in \mathbb{Z}^3} \sum_{j \in \mathbb{Z}^3} |j|^2 |\hat{f}_j| |k - j|^{r-1+\frac{s}{r}} e^{-\tau|k-j|^\frac{1}{r}} \hat{g}_{k-j} |k|^{r} e^{\tau|k|^\frac{1}{r}} |\hat{h}_{-k}|$$

$$\leq C\tau^2 \sum_{k \in \mathbb{Z}^3} \sum_{j \in \mathbb{Z}^3} |j|^{2+\frac{s}{r}} |\hat{f}_j| |k - j|^{r-1+\frac{s}{r}} e^{-\tau|k-j|^\frac{1}{r}} \hat{g}_{k-j} |k|^{r} \frac{1}{r} e^{\tau|k|^\frac{1}{r}} |\hat{h}_{-k}|$$

$$\leq C\tau^2 \|f\|_{H^r} \|g\|_{\dot{G}^{s, r}_{-1+\frac{s}{r}, r}} \|h\|_{\dot{G}^{s, r}_{r+\frac{s}{r}, r}}.$$  

(29)

Hence, substituting (25)-(29) into (24), we obtain that

$$|Z_1| \leq C\|f\|_{H^r} \|g\|_{H^{-\frac{1}{2}}(\mathbb{R})} + C\tau^2 \|f\|_{H^r} \|g\|_{\dot{G}^{s, r}_{-1+\frac{s}{r}, r}} \|h\|_{\dot{G}^{s, r}_{r+\frac{s}{r}, r}} + C(1 + \tau^2) \|f\|_{H^r} \|g\|_{\dot{G}^{s, r}_{-1+\frac{s}{r}, r}} \|h\|_{\dot{G}^{s, r}_{r+\frac{s}{r}, r}}.$$  

(30)

For the term $I_2$, we first have the following observations:

$$|e^{\xi} - 1| \leq |\xi| |e^{\xi}|$$

holds for any $\xi \in \mathbb{R}$. Then

$$|e^{\tau(|k|^\frac{1}{r} - |k-j|^\frac{1}{r})} - 1| \leq \tau \left| |k|^\frac{1}{r} - |k-j|^\frac{1}{r} \right| |e^{\tau|k-j|^\frac{1}{r}} |.$$  

(31)

Due to $s \geq 1$, we have

$$\left| |k|^\frac{1}{r} - |k-j|^\frac{1}{r} \right| \leq |j|^\frac{1}{r},$$  

(32)

which, together with (31), imply that

$$|e^{\tau(|k|^\frac{1}{r} - |k-j|^\frac{1}{r})} - 1| \leq \tau \left| |k|^\frac{1}{r} - |k-j|^\frac{1}{r} \right| |e^{\tau|j|^\frac{1}{r}} |.$$  

(33)

Additional, in view of Lemma 2.1, we have

$$\left| |k|^\frac{1}{r} - |k-j|^\frac{1}{r} \right| \leq C|\tau| \frac{1}{|k|^{1-\frac{1}{r}} + |k-j|^{1-\frac{1}{r}}}.$$  

(34)

Therefore, it follows from (33) and (34) that

$$|Z_2| \leq C \sum_{k \in \mathbb{Z}^3} \sum_{j \in \mathbb{Z}^3} |k|^{r} e^{-\tau|k-j|^\frac{1}{r}} |e^{\tau(|k|^\frac{1}{r} - |k-j|^\frac{1}{r})} - 1| |k-j| e^{\tau|j|^\frac{1}{r}} |\hat{f}_j| |\hat{g}_{k-j}||k|^{r} e^{\tau|k|^\frac{1}{r}} |\hat{h}_{-k}|$$

$$\leq C \sum_{k \in \mathbb{Z}^3} \sum_{j \in \mathbb{Z}^3} |k|^{r-\frac{1}{r}} e^{-\tau|k-j|^\frac{1}{r}} \tau \left| |k|^\frac{1}{r} - |k-j|^\frac{1}{r} \right| |e^{\tau|j|^\frac{1}{r}} |k-j| |\hat{f}_j|$$

$$\times |\hat{g}_{k-j}| |\hat{h}_{-k}| |k|^{r} e^{\tau|k|^\frac{1}{r}}$$

$$\leq C\tau \sum_{k \in \mathbb{Z}^3} \sum_{j \in \mathbb{Z}^3} (|j|^{r-\frac{1}{r}} + |k-j|^{r-\frac{1}{r}}) e^{-\tau|k-j|^\frac{1}{r}} \frac{|j||k-j|}{|k|^{1-\frac{1}{r}} + |k-j|^{1-\frac{1}{r}}}$$

$$\times e^{\tau|j|^\frac{1}{r}} |\hat{f}_j| |\hat{g}_{k-j}||\hat{h}_{-k}| |k|^{r} e^{\tau|k|^\frac{1}{r}}$$

$$\leq I_{21} + I_{22},$$

(35)

where

$$I_{21} = C\tau \sum_{k \in \mathbb{Z}^3} \sum_{j \in \mathbb{Z}^3} |j|^{r-\frac{1}{r}} e^{\tau|j|^\frac{1}{r}} |\hat{f}_j| |k-j| e^{\tau|k-j|^\frac{1}{r}}$$

$$\times (1 + \tau|k-j|^\frac{1}{r}) e^{-\tau|k-j|^\frac{1}{r}}$$

and

$$I_{22} = C\tau \sum_{k \in \mathbb{Z}^3} \sum_{j \in \mathbb{Z}^3} |j|^{r-\frac{1}{r}} e^{\tau|j|^\frac{1}{r}} |\hat{f}_j| |k-j| e^{\tau|k-j|^\frac{1}{r}}.$$
Here we have used the inequalities
\[|k|^{1+\frac{1}{2}} + |k-j|^{1+\frac{1}{2}} \geq |j|^{1+\frac{1}{2}}, \quad |k|^{1+\frac{1}{2}} + |k-j|^{1+\frac{1}{2}} \geq |k-j|^{1+\frac{1}{2}}\]
and \(e^\xi \leq 1 + \xi e^\xi\) for \(\xi \in \mathbb{R}^+\). Using the discrete Hölder’s inequality and Minkowski's inequality, we conclude that
\[
\begin{align*}
|I_{21}| &\leq C\tau \left( \|g\|_{H^r} \|f\|_{\mathcal{G}^r} + C\tau^2 \|g\|_{\mathcal{G}^r} \|f\|_{H^r} \right) + C\tau^2 \|g\|_{\mathcal{G}^r} \|f\|_{H^r} \|h\|_{\mathcal{G}^r}, \\
|I_{22}| &\leq C\tau \left( \|f\|_{H^r} \|g\|_{\mathcal{G}^r} + C\tau^2 \|g\|_{\mathcal{G}^r} \|f\|_{H^r} \right) + C\tau^2 \|g\|_{\mathcal{G}^r} \|f\|_{H^r} \|h\|_{\mathcal{G}^r}.
\end{align*}
\]
Thus, we derive from (35)-(37) that
\[
|I_2| \leq C\tau \left( \|g\|_{H^r} \|f\|_{\mathcal{G}^r} + \|f\|_{H^r} \|g\|_{\mathcal{G}^r} \right) + C\tau^2 \|g\|_{\mathcal{G}^r} \|f\|_{H^r} \|h\|_{\mathcal{G}^r}.
\]
Finally, inserting (30) and (38) into (20), we obtain (15). This completes the proof of Lemma 2.2. \(\square\)

3. Uniform regularity of the solutions. In this section, we prove the existence of Gevrey class solutions \((u^{+\mu}, H^{+\mu})\) to the viscous incompressible MHD equations. And the existence of Gevrey class solution \((u^0, H^0)\) to the ideal MHD equations can be verified similarly. The methods of our proof are based on Galerkin approximation. To this end, we introduce the following equivalent problem for (1)-(2):
\[
\begin{cases}
\left. u_t^{+\mu} + \mathbb{P}(u^{+\mu} \cdot \nabla u^{+\mu}) - \mathbb{P}(H^{+\mu} \cdot \nabla H^{+\mu}) + \nu u^{+\mu} = 0, \right. \\
\left. H_t^{+\mu} + u^{+\mu} \cdot \nabla H^{+\mu} - H^{+\mu} \cdot \nabla u^{+\mu} + \mu AH^{+\mu} = 0, \right. \\
\left. (u^{+\mu}, H^{+\mu})|_{t=0} = (u_0, H_0), \right.
\end{cases}
\]
where \(A = -\mathbb{P}\Delta\) is the well-known Stokes operator and \(\mathbb{P}\) is Leray projector. Similarly, we have the following equivalent problem for (3)-(4):
\[
\begin{cases}
\left. u_t^0 + \mathbb{P}(u^0 \cdot \nabla u^0) - \mathbb{P}(H^0 \cdot \nabla H^0) = 0, \right. \\
\left. H_t^0 + u^0 \cdot \nabla H^0 - H^0 \cdot \nabla u^0 = 0, \right. \\
\left. (u^0, H^0)|_{t=0} = (u_0, H_0). \right.
\end{cases}
\]
We first recall some properties of the Stokes operator \(A\).

**Proposition 3.1** ([9]). The Stokes operator \(A\) is symmetric and self-adjoint, moreover, the inverse of the Stokes operator, \(A^{-1}\), is a compact operator in \(L^2\). The Hilbert theorem implies that there exists a sequence of positive numbers \(\lambda_j\) and an orthonormal basis \(\omega_j\) of \(L^2\), which satisfies
\[A\omega_j = \lambda_j \omega_j, \quad 0 < \lambda_1 < \ldots < \lambda_j \leq \lambda_{j+1} \leq \ldots, \quad \lim_{j \to \infty} \lambda_j = \infty.\]
Particularly, in the case of $\mathbb{T}^3$, the eigenvector functions and the eigenvalues have the following definite expression

$$\omega_{k,j}(x) = \left(e_j - \frac{k_j k_j}{|k|^2}\right)e^{ik \cdot x}, \quad \lambda_{k,j} = |k|^2,$$

(41)

where $k = (k_1, k_2, k_3) \in \mathbb{Z}^3$, $k \neq 0$, $j = 1, 2, 3$ and $\{e_j\}_{j=1,2,3}$ are the canonical basis in $\mathbb{R}^3$. Hence, we find that each $\omega_j$ is not only in $\mathcal{L}^2$, but also in $\mathcal{G}^s_{r,\tau}$ for any $r > 0$. Now we will show that there exists a solution to (39) with the initial data $(u_0, H_0) \in \mathcal{G}^s_{r,\tau_0}$, where $s \geq 1$, $r > \max\{\frac{3}{2} + \frac{3}{2}, \frac{2}{7} + \frac{1}{2}\}$, and $\tau(t) > 0$ is a differentiable decreasing function of $t$. For this purpose, we first derive the following a priori estimate which is the crucial step in the proof of Theorem 1.1. For notational convenience, we shall drop the superscripts $\nu$ and $\mu$ throughout this section.

**Proposition 3.2.** Let $s \geq 1$, $r > \max\{\frac{3}{2} + \frac{3}{2}, \frac{2}{7} + \frac{1}{2}\}$, $(u_0, H_0) \in \mathcal{G}^s_{r,\tau_0}$ and $\tau(t) > 0$ be a differentiable decreasing function of $t$ defined on $[0, T]$ with $\tau(0) = \tau_0$, where $0 < T < T^*$ and $T^*$ is the lifespan time of the $H^r$ solution to (39) obtained in Theorem 2.1. Let $(u, H) \in L^\infty(0, T; \mathcal{G}^s_{r,\tau(t)}(\mathbb{T}^3)) \cap L^2(0, T; \mathcal{G}^s_{r+1,\tau(t)}(\mathbb{T}^3))$ be the solution to (39). Then, for any $0 < t \leq T$,

$$\|u(t, .)\|^2_{\mathcal{G}^s_{r,\tau(t)}} + \|H(t, .)\|^2_{\mathcal{G}^s_{r,\tau(t)}} \leq \bar{C}_T^1,$$

(42)

$$\int_0^t \left(\nu\|u(s, .)\|^2_{\mathcal{G}^s_{r+1,\tau(s)}} + \mu\|H(s, .)\|^2_{\mathcal{G}^s_{r+1,\tau(s)}}\right) ds \leq \bar{C}_T^2.$$  

(43)

Under the same assumptions as above, let $(u^0, H^0) \in L^\infty(0, T; \mathcal{G}^s_{r,\tau(t)}(\mathbb{T}^3))$ be the solution to (40), then we have

$$\|u^0(t, .)\|^2_{\mathcal{G}^s_{r,\tau(t)}} + \|H^0(t, .)\|^2_{\mathcal{G}^s_{r,\tau(t)}} \leq \bar{C}_T^1, \forall t \in (0, T].$$

(44)

Moreover, the uniform radius $\tau(t)$ is given by

$$\tau(t) = \frac{1}{\tau_0 \bar{C}_2 + \bar{C}_3(e^{\bar{C}_2 t} - 1)},$$

(45)

where $\bar{C}_2$, $\bar{C}_3$, $\bar{C}_1^1$ and $\bar{C}_2^2$ are positive constants depending on $u_0$, $H_0$, $r$, $s$ and $T$.

**Proof.** Applying $\Lambda^r e^{\tau \Lambda^\frac{1}{2}}$ to both sides of (39), and taking the $L^2$ inner product of both sides with $\Lambda^r e^{\tau \Lambda^\frac{1}{2}} u$, we have

$$\frac{1}{2} \frac{d}{dt}\|u\|^2_{\mathcal{G}^s_{r,\tau}} + \nu\|u\|^2_{\mathcal{G}^s_{r+1,\tau}} = \tau'(t)\|u\|^2_{\mathcal{G}^s_{r,\tau}} + \left(\Lambda^r e^{\tau \Lambda^\frac{1}{2}} (H \cdot \nabla H), \Lambda^r e^{\tau \Lambda^\frac{1}{2}} u\right)$$

$$- \left(\Lambda^r e^{\tau \Lambda^\frac{1}{2}} (u \cdot \nabla u), \Lambda^r e^{\tau \Lambda^\frac{1}{2}} u\right),$$

(46)

where we have used the facts that $\mathbb{P}$ commutes with $\Lambda^r e^{\tau \Lambda^\frac{1}{2}}$ and $\mathbb{P}$ is symmetric. Arguing analogously to (46), we can also obtain that

$$\frac{1}{2} \frac{d}{dt}\|H\|^2_{\mathcal{G}^s_{r,\tau}} + \mu\|H\|^2_{\mathcal{G}^s_{r+1,\tau}} = \tau'(t)\|H\|^2_{\mathcal{G}^s_{r,\tau}} + \left(\Lambda^r e^{\tau \Lambda^\frac{1}{2}} (H \cdot \nabla u), \Lambda^r e^{\tau \Lambda^\frac{1}{2}} H\right)$$

$$- \left(\Lambda^r e^{\tau \Lambda^\frac{1}{2}} (u \cdot \nabla u), \Lambda^r e^{\tau \Lambda^\frac{1}{2}} H\right).$$

(47)
Now we come to estimate the right-hand side terms of (48). First, in view of the incompressible condition of $H$, we infer that

\[
|I_3 + I_4| = \left| \left( \Lambda^r e^{r\Lambda^{\frac{1}{2}}} (H \cdot \nabla H), \Lambda^r e^{r\Lambda^{\frac{1}{2}}} u \right) + \left( \Lambda^r e^{r\Lambda^{\frac{1}{2}}} (H \cdot \nabla u), \Lambda^r e^{r\Lambda^{\frac{1}{2}}} H \right) \right|
\]

\[
- (H \cdot \nabla \Lambda^r e^{r\Lambda^{\frac{1}{2}}} u, \Lambda^r e^{r\Lambda^{\frac{1}{2}}} H) - (H \cdot \nabla \Lambda^r e^{r\Lambda^{\frac{1}{2}}} H, \Lambda^r e^{r\Lambda^{\frac{1}{2}}} u) \leq |(\Lambda^r e^{r\Lambda^{\frac{1}{2}}} (H \cdot \nabla H), \Lambda^r e^{r\Lambda^{\frac{1}{2}}} u) - (H \cdot \nabla \Lambda^r e^{r\Lambda^{\frac{1}{2}}} H, \Lambda^r e^{r\Lambda^{\frac{1}{2}}} u) |
\]

\[
+ |(\Lambda^r e^{r\Lambda^{\frac{1}{2}}} (H \cdot \nabla u), \Lambda^r e^{r\Lambda^{\frac{1}{2}}} H) - (H \cdot \nabla \Lambda^r e^{r\Lambda^{\frac{1}{2}}} u, \Lambda^r e^{r\Lambda^{\frac{1}{2}}} H) |.
\]

It follows from Lemma 2.2 that

\[
|I_3 + I_4| \leq C \|H\|_{H^2} \|u\|_{\tilde{H}^r} + C \|u\|_{H^r} \|H\|_{H^r} \|H\|_{\tilde{H}^r}
\]

\[
+ C(1 + \tau^2) \|H\|_{H^r} \|H\|_{\tilde{H}^r} \|u\|_{\tilde{H}^r} + \|u\|_{H^r} \|H\|_{\tilde{H}^r}
\]

\[
+ C\tau^2 \left( \|H\|_{H^r} \|H\|_{\tilde{H}^r} \|u\|_{\tilde{H}^r} + \|u\|_{H^r} \|H\|_{\tilde{H}^r} \right) + C\tau^2 \left( \|H\|_{H^r} \|H\|_{\tilde{H}^r} \|u\|_{\tilde{H}^r} + \|u\|_{H^r} \|H\|_{\tilde{H}^r} \right)
\]

\[
+ C\tau^2 \left( \|H\|_{H^r} \|H\|_{\tilde{H}^r} \|u\|_{\tilde{H}^r} + \|u\|_{H^r} \|H\|_{\tilde{H}^r} \right) \|H\|_{\tilde{H}^r}.
\]

Next, using Lemma 2.2 straightforwardly, we have

\[
|I_5| \leq C \|u\|_{H^r} \|H\|_{H^r} \|H\|_{\tilde{H}^r} + C(1 + \tau^2) \|u\|_{H^r} \|u\|_{\tilde{H}^r} + C\tau \|u\|_{H^r} \|u\|_{H^r} \|H\|_{\tilde{H}^r}
\]

\[
+ C\tau^2 \left( \|u\|_{H^r} \|u\|_{\tilde{H}^r} + \|u\|_{H^r} \|H\|_{\tilde{H}^r} \right) \|H\|_{\tilde{H}^r}.
\]

Finally, due to the incompressible condition of $u$, we find that

\[
|I_6| = - \left( \Lambda^r e^{r\Lambda^{\frac{1}{2}}} (H \cdot \nabla H), \Lambda^r e^{r\Lambda^{\frac{1}{2}}} H \right) + (u \cdot \nabla \Lambda^r e^{r\Lambda^{\frac{1}{2}}} H, \Lambda^r e^{r\Lambda^{\frac{1}{2}}} H).
\]

By virtue of Lemma 2.2, we have

\[
|I_6| \leq C \|u\|_{H^r} \|H\|_{H^r} \|H\|_{\tilde{H}^r} \|H\|_{\tilde{H}^r}
\]

\[
+ C(1 + \tau^2) \|u\|_{H^r} \|H\|_{\tilde{H}^r} \|H\|_{\tilde{H}^r}
\]

\[
+ C\tau^2 \left( \|u\|_{H^r} \|H\|_{\tilde{H}^r} \|u\|_{\tilde{H}^r} + \|H\|_{\tilde{H}^r} \|u\|_{\tilde{H}^r} \|H\|_{\tilde{H}^r} \right) \|H\|_{\tilde{H}^r}.
\]
Noting that \( \|u\|_{\mathcal{G}^{r,\tau}_{r}} \leq \|u\|_{\mathcal{G}^{r,\tau}_{r+1},r} \|u\|_{\mathcal{G}^{r,\tau}_{r+1},r} \leq \|u\|_{\mathcal{G}^{r,\tau}_{r+1},r} \), \( \|H\|_{\mathcal{G}^{r,\tau}_{r+1},r} \leq \|H\|_{\mathcal{G}^{r,\tau}_{r+1},r} \) and \( \|H\|_{\mathcal{G}^{r,\tau}_{r+1},r} \leq \|H\|_{\mathcal{G}^{r,\tau}_{r+1},r} \) for \( s \geq 1 \). Hence, substituting (49)-(52) into (48), we obtain that
\[
\frac{1}{2} \frac{d}{dt} (\|u\|_{\mathcal{G}^{r,\tau}_{r},r}^2 + \|H\|_{\mathcal{G}^{r,\tau}_{r},r}^2) + (\nu \|u\|_{\mathcal{G}^{r+1,\tau}_{r+1},r}^2 + \mu \|H\|_{\mathcal{G}^{r+1,\tau}_{r+1},r}^2) \\
\leq C(\|u\|_{\mathcal{H}^{r},r}^2 + \|H\|_{\mathcal{H}^{r},r}^2)(\|u\|_{\mathcal{G}^{r,\tau}_{r},r} + \|H\|_{\mathcal{G}^{r,\tau}_{r},r}) \\
+ C(\|u\|_{\mathcal{H}^{r},r} + \|H\|_{\mathcal{H}^{r},r})(\|u\|_{\mathcal{G}^{r,\tau}_{r},r} + \|H\|_{\mathcal{G}^{r,\tau}_{r},r}) \\
+ (\tau' + C\tau(\|u\|_{\mathcal{H}^{r},r} + \|H\|_{\mathcal{H}^{r},r}) + C\tau^2(\|u\|_{\mathcal{H}^{r},r} + \|H\|_{\mathcal{H}^{r},r} + \|u\|_{\mathcal{G}^{r,\tau}_{r},r} + \|H\|_{\mathcal{G}^{r,\tau}_{r},r})) \\
(\|u\|_{\mathcal{G}^{r,\tau}_{r+1},r}^2 + \|H\|_{\mathcal{G}^{r,\tau}_{r+1},r}^2).
\] (53)

Now if the radius of Gevrey class \( \tau(t) \) is smooth and decreasing fast enough such that the following inequality holds,
\[
\tau' + C\tau(\|u\|_{\mathcal{H}^{r},r} + \|H\|_{\mathcal{H}^{r},r}) + C\tau^2(\|u\|_{\mathcal{H}^{r},r} + \|H\|_{\mathcal{H}^{r},r} + \|u\|_{\mathcal{G}^{r,\tau}_{r},r} + \|H\|_{\mathcal{G}^{r,\tau}_{r},r}) \leq 0.
\] (54)

Then we obtain that
\[
\frac{1}{2} \frac{d}{dt} (\|u\|_{\mathcal{G}^{r,\tau}_{r},r}^2 + \|H\|_{\mathcal{G}^{r,\tau}_{r},r}^2) + (\nu \|u\|_{\mathcal{G}^{r+1,\tau}_{r+1},r}^2 + \mu \|H\|_{\mathcal{G}^{r+1,\tau}_{r+1},r}^2) \\
\leq C(\|u\|_{\mathcal{H}^{r},r}^2 + \|H\|_{\mathcal{H}^{r},r}^2)(\|u\|_{\mathcal{G}^{r,\tau}_{r},r} + \|H\|_{\mathcal{G}^{r,\tau}_{r},r}) \\
+ C(\|u\|_{\mathcal{H}^{r},r} + \|H\|_{\mathcal{H}^{r},r})(\|u\|_{\mathcal{G}^{r,\tau}_{r},r} + \|H\|_{\mathcal{G}^{r,\tau}_{r},r}).
\] (55)

Furthermore, since \( \mu, \nu > 0 \) and by Cauchy-Schwarz inequality, we infer that
\[
\frac{1}{2} \frac{d}{dt} (\|u\|_{\mathcal{G}^{r,\tau}_{r},r}^2 + \|H\|_{\mathcal{G}^{r,\tau}_{r},r}^2) \leq C(\|u\|_{\mathcal{H}^{r},r} + \|H\|_{\mathcal{H}^{r},r})(\|u\|_{\mathcal{G}^{r,\tau}_{r},r} + \|H\|_{\mathcal{G}^{r,\tau}_{r},r}) \\
+ C(\|u\|_{\mathcal{H}^{r},r} + \|H\|_{\mathcal{H}^{r},r})^3.
\] (56)

An application of Gronwall’s inequality to (56) then implies that, for \( 0 < t < T \),
\[
\|u\|_{\mathcal{G}^{r,\tau}_{r},r}^2 + \|H\|_{\mathcal{G}^{r,\tau}_{r},r}^2 \leq \hat{g}(t) \left( \|u_0\|_{\mathcal{G}^{r,\tau}_{r_0},r_0}^2 + \|H_0\|_{\mathcal{G}^{r,\tau}_{r_0},r_0}^2 + \right. \\
\left. + \int_0^t C(\|u\|_{\mathcal{H}^{r},r} + \|H\|_{\mathcal{H}^{r},r})^3 \, ds \right) \triangleq A(t),
\] (57)

where
\[
\hat{g}(t) = \exp \left\{ \int_0^t C(\|u\|_{\mathcal{H}^{r},r} + \|H\|_{\mathcal{H}^{r},r}) \, ds \right\}.
\]

Moreover, it follows from (12) that the \( \mathcal{H}^r \) solution satisfies the following inequality: for \( 0 < t < T \),
\[
\|u\|_{\mathcal{H}^{r},r} + \|H\|_{\mathcal{H}^{r},r} \leq \frac{2(\|u_0\|_{\mathcal{H}^{r},r}^2 + \|H_0\|_{\mathcal{H}^{r},r}^2) \frac{1}{2}}{1 - C\tau(\|u_0\|_{\mathcal{H}^{r},r}^2 + \|H_0\|_{\mathcal{H}^{r},r}^2) \frac{1}{2}} \\
\leq \frac{2(\|u_0\|_{\mathcal{H}^{r},r}^2 + \|H_0\|_{\mathcal{H}^{r},r}^2) \frac{1}{2}}{1 - C\tau(\|u_0\|_{\mathcal{H}^{r},r}^2 + \|H_0\|_{\mathcal{H}^{r},r}^2) \frac{1}{2}} \triangleq C_T,
\] (58)

which, together with (57), yields
\[
\|u\|_{\mathcal{G}^{r,\tau}_{r},r}^2 + \|H\|_{\mathcal{G}^{r,\tau}_{r},r}^2 \leq A(t) \leq e^{CC_T \tau}(\|u_0\|_{\mathcal{G}^{r,\tau}_{r_0},r_0}^2 + \|H_0\|_{\mathcal{G}^{r,\tau}_{r_0},r_0}^2 + CC_T \tau t)
\]
We start with a sequence of approximate functions (\textit{Proof}.

Similarly, there exists a unique solution (Theorem 3.1. There exists a unique solution

\[ \text{existence of the solutions } (u, H) \text{, the Galerkin approximation, following the arguments in [9], to establish the local solution } (u, H) \text{, and proceeding exactly as above, we can obtain similarly a priori estimates for the solution } (u^0, H^0) \text{ to (40). Hence the proof of Proposition 3.2 is completed.} \]

Next, with the help of the a priori estimates obtained in Proposition 3.2, we use the Galerkin approximation, following the arguments in [9], to establish the local existence of the solutions \((u, H)\) and \((u^0, H^0)\) in Gevrey class \(\mathcal{G}_{r,T}^s\).

**Theorem 3.1.** There exists a unique solution \((u, H)\) to the problem (39) such that

\[ (u, H) \in L^\infty(0, T; \mathcal{G}_{r,T}^s). \]

Similarly, there exists a unique solution \((u^0, H^0)\) to the problem (40) such that

\[ (u^0, H^0) \in L^\infty(0, T; \mathcal{G}_{r,T}^s). \]

**Proof.** We start with a sequence of approximate functions \((u^{(m)}, H^{(m)})\):

\[ u^{(m)} = \sum_{j=1}^{m} u_{j,m} \omega_j, \quad H^{(m)} = \sum_{j=1}^{m} H_{j,m} \omega_j, \]

where \(\{\omega_j\}_{j=1}^\infty\) are the orthonormal basis given in Proposition 3.1, and \(u_{j,m}\) and \(H_{j,m}\) \((j = 1, 2, ..., m)\) solve the following differential equations:

\[
\begin{aligned}
\frac{d}{dt} u_{j,m} + \nu \lambda_j u_{j,m} + \sum_{k,j=1}^{m} b(\omega_k, \omega_l, \omega_j) (u_{k,m} u_{l,m} - H_{k,m} H_{l,m}) &= 0, \\
\frac{d}{dt} H_{j,m} + \mu \lambda_j H_{j,m} + \sum_{k,j=1}^{m} b(\omega_k, \omega_l, \omega_j) (u_{k,m} H_{l,m} - H_{k,m} u_{l,m}) &= 0,
\end{aligned}
\]

(65)

with \(b(\omega_k, \omega_l, \omega_j) = \int_{\Omega} \omega_k \cdot \nabla \omega_l \cdot \omega_j \, dx\). In view of the standard nonlinear ordinary differential equation theory, the problem (65) is locally well-posed, say on \([0, T_m]\). Now, we need to show that \(T_m\) can be extended to \(T\). Taking the inner products
Moreover, for any \( t \) we obtain that
\[
\sum_{j,k,l=1}^{m} b(\omega_k, \omega_l, \omega_j) u_{j,m} u_{k,m} u_{l,m} = \sum_{j,k,l=1}^{m} b(\omega_k, \omega_l, \omega_j) u_{k,m} H_{l,m} H_{j,m} = 0,
\]
\[
\sum_{j,k,l=1}^{m} b(\omega_k, \omega_l, \omega_j) u_{j,m} H_{k,m} H_{l,m} + \sum_{j,k,l=1}^{m} b(\omega_k, \omega_l, \omega_j) H_{k,m} u_{l,m} H_{j,m} = 0,
\]
we obtain that
\[
\frac{1}{2} \frac{d}{dt} \sum_{j=1}^{m} \left( u_{j,m}^2 + H_{j,m}^2 \right) + \sum_{j=1}^{m} \lambda_j (\nu u_{j,m}^2 + \mu H_{j,m}^2) = 0,
\]
(66)
which yields that, for \( 0 < t \leq T_m \),
\[
\| u^{(m)}(t, \cdot) \|_{L^2}^2 + \| H^{(m)}(t, \cdot) \|_{L^2}^2 = \sum_{j=1}^{m} \left( u_{j,m}(t)^2 + H_{j,m}(t)^2 \right) 
\leq \sum_{j=1}^{m} \left( u_{j,m}(0)^2 + H_{j,m}(0)^2 \right) \leq \| u_0 \|_{L^2}^2 + \| H_0 \|_{L^2}^2.
\]
(67)
Therefore, for every \( T_m \), it can be extended to \( T \), and
\[
(u^{(m)}, H^{(m)}) \text{ is bounded in } L^\infty(0, T; L^2), \text{ uniformly for } m.
\]
Moreover, for any \( t \in [0, T) \), \((u^{(m)}, H^{(m)})\) solves the following system
\[
\begin{cases}
\frac{d}{dt} u^{(m)} + \chi_m \mathbb{P}(u^{(m)} \cdot \nabla u^{(m)}) - \chi_m \mathbb{P}(H^{(m)} \cdot \nabla H^{(m)}) + \nu A u^{(m)} = 0, \\
\frac{d}{dt} H^{(m)} + \chi_m (u^{(m)} \cdot \nabla H^{(m)}) - \chi_m (H^{(m)} \cdot \nabla u^{(m)}) + \mu AH^{(m)} = 0,
\end{cases}
\]
(68)
where \( \chi_m \) denotes the orthogonal projector from \( L^2 \) into the space spanned by \{\omega_j\}_{j=1}^{m}. Then, we turn to obtain the uniform Gevrey class norm bound for \((u^{(m)}, H^{(m)})\). To this end, we first apply \( \Lambda e^{\tau \Lambda^\frac{1}{2}} \) to (68)\(_1\) and (68)\(_2\), take the inner product with \( \Lambda e^{\tau \Lambda^\frac{1}{2}} u^{(m)} \) and \( \Lambda e^{\tau \Lambda^\frac{1}{2}} H^{(m)} \), respectively, and then use the properties that the operators \( \chi_m \) and \( \mathbb{P} \) commute with \( \Lambda e^{\tau \Lambda^\frac{1}{2}} \) to obtain that
\[
\frac{1}{2} \frac{d}{dt} (\| u^{(m)} \|_{\mathcal{F}^r}^2 + \| H^{(m)} \|_{\mathcal{F}^r}^2) + (\nu \| u^{(m)} \|_{\mathcal{F}^r+1}^2 + \nu \| H^{(m)} \|_{\mathcal{F}^r+1}^2)
\]
\[
= \tau'(t) (\| u^{(m)} \|_{\mathcal{F}^r+1}^2 + \| H^{(m)} \|_{\mathcal{F}^r+1}^2) 
+ (\Lambda e^{\tau \Lambda^\frac{1}{2}} (H^{(m)} \cdot \nabla H^{(m)}), \Lambda e^{\tau \Lambda^\frac{1}{2}} u^{(m)})
+ (\Lambda e^{\tau \Lambda^\frac{1}{2}} (H^{(m)} \cdot \nabla u^{(m)}), \Lambda e^{\tau \Lambda^\frac{1}{2}} H^{(m)})
- (\Lambda e^{\tau \Lambda^\frac{1}{2}} (u^{(m)} \cdot \nabla u^{(m)}), \Lambda e^{\tau \Lambda^\frac{1}{2}} u^{(m)})
- (\Lambda e^{\tau \Lambda^\frac{1}{2}} (u^{(m)} \cdot \nabla H^{(m)}), \Lambda e^{\tau \Lambda^\frac{1}{2}} H^{(m)}).
\]
(69)
Arguing analogously to Proposition 3.2, we have
\[ \|u^{(m)}(t, \cdot)\|_{L^2_{\tau, \tau}} + \|H^{(m)}(t, \cdot)\|_{L^2_{\tau, \tau}} \leq C_T^1, \quad 0 < t < T^*, \quad \forall \ m. \tag{70} \]

Thus
\[ (u^{(m)}, H^{(m)}) \text{ is bounded in } L^\infty(0, T; \mathcal{G}^r_{\tau, \tau}), \quad \text{uniformly for } m, \tag{71} \]
\[ (\nu^{\frac{1}{2}} u^{(m)}, \mu^{\frac{1}{2}} H^{(m)}) \text{ is bounded in } L^2(0, T; \mathcal{G}^s_{r+1, \tau}), \quad \text{uniformly for } m. \tag{72} \]

In order to pass to the limit in the nonlinear terms by using compactness arguments, we need to give the temporal derivative estimate of \((u^{(m)}, H^{(m)})\). From (68), we have
\[ \left\| \left( \frac{d}{dt} u^{(m)}(t), \frac{d}{dt} H^{(m)}(t) \right) \right\|_{L^2} \leq \left\| \chi_m \nabla u^{(m)}(t) \right\|_{L^2} + \left\| \chi_m \nabla H^{(m)}(t) \right\|_{L^2} \]
\[ + \left\| \chi_m (u^{(m)} \cdot \nabla H^{(m)}) (t) \right\|_{L^2} + \left\| \chi_m (H^{(m)} \cdot \nabla u^{(m)}) (t) \right\|_{L^2} \]
\[ + \left\| \nu \Delta u^{(m)} (t) \right\|_{L^2} + \left\| \mu \Delta H^{(m)} (t) \right\|_{L^2} \]
\[ \leq C \left( \|u^{(m)}\|_{H^r}^2 + \|H^{(m)}\|_{H^r}^2 + \|u^{(m)}\|_{H^r} \|H^{(m)}\|_{H^r} \right) \]
\[ + \nu \|u^{(m)}\|_{H^r} + \mu \|H^{(m)}\|_{H^r}. \tag{73} \]

Then, it follows from Theorem 2.1 and (73) that
\[ \left( \frac{d}{dt} u^{(m)}, \frac{d}{dt} H^{(m)} \right) \text{ is bounded in } L^\infty(0, T; \mathcal{L}^2), \quad \text{uniformly for } m. \tag{74} \]

Based on Aubin-Lions-Simon theorem in [14] and the fact that \(H^r\) is compactly embedded in \(\mathcal{L}^2\), we prove that there exists a subsequence \(\{(u^{(m)}_n, H^{(m)}_n)\}_{n=1}^\infty \subset \{(u^{(m)}, H^{(m)})\}_{m=1}^\infty\) and \((u, H) \in L^\infty(0, T; H^r) \cap C(0, T; \mathcal{L}^2)\) such that
\[ (u^{(m)}_n, H^{(m)}_n) \to (u, H) \text{ weak-* in } L^\infty(0, T; H^r), \tag{75} \]
\[ (u^{(m)}_n, H^{(m)}_n) \to (u, H) \text{ strongly in } C(0, T; \mathcal{L}^2). \tag{76} \]

Choosing \(v \in \mathcal{L}^2\) arbitrarily, taking the inner product of (68)\_1 and (68)\_2 with \(v\) and integrating in time, we obtain that
\[ \begin{cases} 
(u^{(m)}(t), v) - (u^{(m)}(0), v) + \int_0^t (u^{(m)}(s) \cdot \nabla u^{(m)}(s), \chi_m v) \, ds \\
= \int_0^t (H^{(m)}(s) \cdot \nabla H^{(m)}(s), \chi_m v) \, ds + \nu \int_0^t (\Delta u^{(m)}(s), v) \, ds, \\
(H^{(m)}(t), v) - (H^{(m)}(0), v) + \int_0^t (u^{(m)}(s) \cdot \nabla H^{(m)}(s), \chi_m v) \, ds \\
= \int_0^t (H^{(m)}(s) \cdot \nabla u^{(m)}(s), \chi_m v) \, ds + \mu \int_0^t (\Delta H^{(m)}(s), v) \, ds. 
\end{cases} \tag{77} \]
According to (75) and (76) and passing to the limit in (77), we have
\[
\begin{aligned}
(u(t), v) - (u(0), v) + \int_0^t (u(s) \cdot \nabla u(s), v) \, ds \\
= \int_0^t \left( H(s) \cdot \nabla H(s), v \right) \, ds + \nu \int_0^t (\Delta u(s), v) \, ds,
\end{aligned}
\]
\begin{equation}
(78)
\end{equation}
\[
\begin{aligned}
(H(t), v) - (H(0), v) + \int_0^t (u(s) \cdot \nabla H(s), v) \, ds \\
= \int_0^t \left( H(s) \cdot \nabla u(s), v \right) \, ds + \mu \int_0^t (\Delta H, v) \, ds.
\end{aligned}
\]
So \((u, H)\) is a weak solution to (39). Furthermore, due to the lower semicontinuity of norms, \((u, H)\) is the \(H^r\) solution to (39). Now, we pay attention to the regularity of \((u, H)\) in Gevrey class. The arguments are similar to that in [27]. We first give a uniform bound for \((\frac{d}{dt} u^{(m)}), \frac{d}{dt} H^{(m)})\) in Gevrey class. To this end, we recall an inequality that \(\|f\|_{G^r} \leq C_r \|f\|_{G^{r-1}}\|g\|_{G^r}\) for \(s \geq 1\), whose proof is similar to Lemma 1 of [13]. It follows from (68) that
\[
\begin{aligned}
\|\frac{d}{dt} u^{(m)}\|_{G^{r-1}} &\leq \|\chi_{m}\|_{\mathcal{P}(u^{(m)} \cdot \nabla u^{(m)})}\|_{\mathcal{P}(H^{(m)} \cdot \nabla H^{(m)})}\|_{G^{r-1}} + \|\nu A u^{(m)}\|_{G^{r-1}} \\
&\leq \|u^{(m)}\|_{G^{r-1}} \|u^{(m)}\|_{G^{r}} + \|H^{(m)}\|_{G^{r-1}} \|H^{(m)}\|_{G^{r}} \\
&\quad + \nu \|u^{(m)}\|_{G^{r+1}}.
\end{aligned}
\]
\begin{equation}
(79)
\end{equation}
Thus, by virtue of (71) and (72), we can verify that \(\frac{d}{dt} u^{(m)}\) is uniformly bounded in \(L^2(0, T; G^{r-1})\) with respect to \(t\). Arguing analogously to (79), we also have that \(\frac{d}{dt} H^{(m)}\) is uniformly bounded in \(L^2(0, T; G^{r-1})\) with respect to \(t\). Let \(0 < \epsilon \ll 1\). Thanks to the compact embedding \(G^{r-\epsilon, \tau} \hookrightarrow G^{r-\epsilon, \tau}\) and the above bounds, we can prove by the Aubin-Lions-Simon Theorem that there exists a subsequence of \(\{(u^{(m_k)}(r, \tau), (H^{(m_k)})_k\} \) for all \(k = 1, \ldots, \), which converges to an element \((v, B)\) in \(C(0, T; G^{r-\epsilon, \tau})\). As shown in (76), this sequence also converges to \((u, H)\) in \(C(0, T; L^2)\). By the uniqueness of limits, \((u, H) = (v, B)\) in \(C(0, T; G^{r-\epsilon, \tau})\). Furthermore, we also have
\[
\sup_{t \in [0, T]} \left( \|u\|^2_{G^{r-\epsilon, \tau}} + \|H\|^2_{G^{r-\epsilon, \tau}} \right) \leq \tilde{C}_T^1.
\]
\begin{equation}
(80)
\end{equation}
Since this bound holds uniformly for all \(0 < \epsilon \ll 1\), it also holds for \(\epsilon = 0\), which can be proved by using the Fourier series representation
\[
\|u\|^2_{G^{r-\epsilon, \tau}} + \|H\|^2_{G^{r-\epsilon, \tau}} = (2\pi)^3 \sum_{k \in \mathbb{Z}^3} |k|^{2(r-\epsilon)} e^{2\pi |k| \frac{1}{2} (|\tilde{u}_k|^2 + |\tilde{H}_k|^2)},
\]
\begin{equation}
(81)
\end{equation}
taking the limit as \(\epsilon \to 0\), and passing the limit inside the infinite sum by the Levi Monotone Convergence theorem for the counting measure. The uniqueness of the solution is clear since we work with functions with Lipschitz regularity. Consequently, we obtain the first conclusion in Theorem 3.1.

For the ideal MHD equations, according to the similar approaches described above, we can obtain the existence of solution in Gevrey class space and we omit the details here. Thus we complete the proof of Theorem 3.1.
It remains to show that \((u,H)\) is the solution of (1) and (2). In fact, based on Theorem 3.1, we have
\[
\mathbb{P}\left\{ \frac{du}{dt} - \nu \Delta u + u \cdot \nabla u - H \cdot \nabla H \right\} = 0.
\]
Thus there exists a scalar function \(p\) such that
\[
\frac{du}{dt} - \nu \Delta u + u \cdot \nabla u - H \cdot \nabla H + \nabla p = 0,
\]
where \(p\) is unique up to a constant, and \(p\) satisfies
\[
-\Delta p = \nabla \cdot (u \cdot \nabla u - H \cdot \nabla H)
\]
with periodic boundary condition. For the regularity of the pressure \(p\) in Gevrey class space, we have the following proposition.

**Proposition 3.3.** Let \(p\) satisfies (82), then the following estimate holds,
\[
\|p(t, \cdot)\|_{\mathcal{G}^{r + 1, \tau}(t)} \leq C \widetilde{C}^{1}_{T}, \quad 0 < t \leq T,
\]
where \(T\) and \(\widetilde{C}^{1}_{T}\) are defined in Proposition 3.2. And for the pressure \(p^{0}\) in (3), we also have
\[
\|p^{0}(t, \cdot)\|_{\mathcal{G}^{r + 1, \tau}(t)} \leq C \widetilde{C}^{1}_{T}, \quad 0 < t \leq T.
\]

**Proof.** To begin with, from the standard elliptic equations theory, the existence of the pressure \(p\) in \(H^{r+1}\) can be guaranteed. Now we focus on the regularity of \(p\) in Gevrey class space. Formally, applying \(\Lambda^{r} e^{\frac{m-1}{2}r\Lambda^{\frac{1}{2}}}\) to both sides of (82) where \(m\) is an arbitrary positive integer and taking \(L^{2}\) inner product of both sides with \(\Lambda^{r} e^{\frac{m-1}{2}r\Lambda^{\frac{1}{2}}} p\), we have that
\[
\|p\|_{\mathcal{G}^{r + 1, \tau}(t)} = \langle \nabla \cdot (u \cdot \nabla u), \Lambda^{2r} e^{\frac{m-1}{2}r\Lambda^{\frac{1}{2}}} p \rangle - \langle \nabla \cdot (H \cdot \nabla H), \Lambda^{2r} e^{\frac{m-1}{2}r\Lambda^{\frac{1}{2}}} p \rangle.
\]

The first term of the right-hand side of (85) can be bounded by
\[
\left| \langle \nabla \cdot (u \cdot \nabla u), \Lambda^{2r} e^{\frac{m-1}{2}r\Lambda^{\frac{1}{2}}} p \rangle \right|
\leq (2\pi)^{3} \sum_{k \in \mathbb{Z}^{3}} \sum_{j \in \mathbb{Z}^{3}} |k|^{r-1} (\hat{u}_{j} \cdot (k-j)) (k \cdot \hat{u}_{k-j}) \hat{p}_{-k} |k|^{r+1} e^{\frac{m-1}{2}r |k|^{\frac{1}{2}}}
\leq C \|p\|_{\mathcal{G}^{r + 1, \tau}(t)} \left( \sum_{k \in \mathbb{Z}^{3}} \left( \sum_{j \in \mathbb{Z}^{3}} |j|^{r-1} + |k-j|^{r-1} \right) |j||k-j| \right)^{\frac{1}{2}}
\times |\hat{u}_{j}| e^{r|j|^{\frac{1}{2}}} e^{r|k-j|^{\frac{1}{2}}} |\hat{u}_{k-j}|^{2}\frac{1}{2}
\leq C \|p\|_{\mathcal{G}^{r + 1, \tau}(t)} \|u\|_{\mathcal{G}^{r + 1, \tau}}^{2},
\]
(86)
where the constant $C$ is independent on $m$. For the second term of the right-hand side of (85), taking the same arguments as those in (86), we have

$$
\left| (\nabla \cdot (H \cdot \nabla H), \Lambda^{2r} e^{2m-1-r} X^{r_1} p) \right| \leq C \|p\|_{G^r_{r+1, \frac{2m-1}{2}}} \|H\|_{G^r_{r+1, \frac{2m-1}{2}}},
$$

(87)

Then, substituting (86) and (87) into (85), we obtain that

$$
\|p\|_{G^r_{r+1, \frac{2m-1}{2}}} \leq C \|p\|_{G^r_{r+1, \frac{2m-1}{2}}} \left( \|u\|_{G^r_{r+1, \frac{2m-1}{2}}} + \|H\|_{G^r_{r+1, \frac{2m-1}{2}}} \right).
$$

(88)

We first note that when $m = 1$, $\|p\|_{G^r_{r+1, 0}} \leq \|p\|_{H^{r+1}}$. So, by the induction on $m$, (88) and the fact $p \in G^s_{r+1, \frac{2m-1}{2}}$ for any positive integer $m$.

Now we turn to prove (83). Arguing analogously to (86), we can also obtain that

$$
\|p\|_{G^r_{r+1, \frac{2m-1}{2}}} \leq C \left( \|u\|_{G^r_{r+1, \frac{2m-1}{2}}} + \|H\|_{G^r_{r+1, \frac{2m-1}{2}}} \right),
$$

(89)

where $C$ is a constant independent on $m$. It follows from (59) that

$$
\|p\|_{G^r_{r+1, \frac{2m-1}{2}}} \leq C\tilde{C}_T, \quad t \in [0, T].
$$

(90)

Since the bound in (90) holds uniformly for all positive integer $m$, using the definition of $\| \cdot \|_{G^r_{r+1, \frac{2m-1}{2}}}$ in (6) and the Levi Monotone Convergence theorem for the counting measure, we have that

$$
\|p\|_{G^r_{r+1, \frac{2m-1}{2}}} \leq C\tilde{C}_T, \quad t \in [0, T].
$$

(91)

Therefore, (83) holds.

For the pressure $p^0(t, x)$ in the ideal MHD equations, we can first obtain the following elliptic equation

$$
-\Delta p^0 = \nabla \cdot (u^0 \cdot \nabla u^0 - H^0 \cdot \nabla H^0).
$$

(92)

Then by using the same arguments as above, we can get (84). \hfill \Box

4. Zero viscosity-resistivity limit. In this section, we will show the zero viscosity-resistivity limit of the viscous incompressible MHD equations in Gevrey class space. Moreover, we give the convergence rate with respect to $\mu$ and $\nu$.

**Theorem 4.1.** Let $(u^{\nu, \mu}, H^{\nu, \mu}, p^{\nu, \mu})$ and $(u^0, H^0, p^0)$ are the solutions obtained in the previous section. Then the following inequalities hold,

$$
\|u^{\nu, \mu}(t, \cdot) - u^0(t, \cdot)\|_{G^r_{r-1, \frac{2m-1}{2}}(\Omega)} + \|H^{\nu, \mu}(t, \cdot) - H^0(t, \cdot)\|_{G^r_{r-1, \frac{2m-1}{2}}(\Omega)} \leq \tilde{C}_1(\mu + \nu),
$$

(93)

$$
\|p^{\nu, \mu}(t, \cdot) - p^0(t, \cdot)\|_{G^r_{r-1, \frac{2m-1}{2}}(\Omega)} \leq \tilde{C}_1(\mu + \nu)
$$

(94)

for any $0 < t \leq T$, where $\tilde{C}_1$ is a constant depending on $r$, $s$, $u_0$, $H_0$ and $T$.

**Proof.** We denote by $w = u^{\nu, \mu} - u^0$, $B = H^{\nu, \mu} - H^0$ and $q = p^{\nu, \mu} - p^0$, and find that $(w, B, q)$ satisfies

$$
\begin{align*}
&w_t - \nu \Delta w^{\nu, \mu} + w \cdot \nabla w^{\nu, \mu} + u^0 \cdot \nabla w - B \cdot \nabla H^{\nu, \mu} - H^0 \cdot \nabla B + \nabla q = 0, \\
&B_t - \mu \Delta B^{\nu, \mu} + w \cdot \nabla B^{\nu, \mu} + u^0 \cdot \nabla B - B \cdot \nabla H^{\nu, \mu} - H^0 \cdot \nabla w = 0, \\
&\text{div } w = \text{div } B = 0, \\
&(w, B)|_{t=0} = 0.
\end{align*}
$$

(95)
Applying $\Lambda^{-1}e^{\tau\Lambda^s}$ to (95) and (95)$_2$, taking the inner product with $\Lambda^{-1}e^{\tau\Lambda^s}w$ and $\Lambda^{-1}e^{\tau\Lambda^s}B$, respectively, and using $\text{div} \ w = 0$, we obtain that

$$
\frac{1}{2} \frac{d}{dt} (\|w\|_{L^2_{\tau+1-\tau}^2}^2 + \|B\|_{L^2_{\tau+1-\tau}^2}^2) = \tau' (\|w\|_{L^2_{\tau+1-\tau}^2}^2 + \|B\|_{L^2_{\tau+1-\tau}^2}^2) + \sum_{i=1}^{10} T_i, \quad (96)
$$

where

$$
T_1 = \nu (\Lambda^{-1}e^{\tau\Lambda^s} \Delta u^{\nu,\mu}, \Lambda^{-1}e^{\tau\Lambda^s} w),
$$

$$
T_2 = \mu (\Lambda^{-1}e^{\tau\Lambda^s} \Delta H^{\nu,\mu}, \Lambda^{-1}e^{\tau\Lambda^s} B),
$$

$$
T_3 = -(\Lambda^{-1}e^{\tau\Lambda^s} (w \cdot \nabla u^{\nu,\mu}), \Lambda^{-1}e^{\tau\Lambda^s} w),
$$

$$
T_4 = -(\Lambda^{-1}e^{\tau\Lambda^s} (u^0 \cdot \nabla w), \Lambda^{-1}e^{\tau\Lambda^s} w),
$$

$$
T_5 = (\Lambda^{-1}e^{\tau\Lambda^s} (B \cdot \nabla H^{\nu,\mu}), \Lambda^{-1}e^{\tau\Lambda^s} w),
$$

$$
T_6 = (\Lambda^{-1}e^{\tau\Lambda^s} (H^0 \cdot \nabla B), \Lambda^{-1}e^{\tau\Lambda^s} w),
$$

$$
T_7 = -(\Lambda^{-1}e^{\tau\Lambda^s} (w \cdot \nabla H^{\nu,\mu}), \Lambda^{-1}e^{\tau\Lambda^s} B),
$$

$$
T_8 = -(\Lambda^{-1}e^{\tau\Lambda^s} (u^0 \cdot \nabla B), \Lambda^{-1}e^{\tau\Lambda^s} B),
$$

$$
T_9 = (\Lambda^{-1}e^{\tau\Lambda^s} (B \cdot \nabla u^{\nu,\mu}), \Lambda^{-1}e^{\tau\Lambda^s} B),
$$

$$
T_{10} = (\Lambda^{-1}e^{\tau\Lambda^s} (H^0 \cdot \nabla w), \Lambda^{-1}e^{\tau\Lambda^s} B).
$$

Now we need to control $T_i$ ($i = 1, 2, \ldots, 10$). For $T_1$ and $T_2$, using the discrete Hölder’s inequality directly, we obtain that

$$
|T_1| = \nu (2\pi)^3 \sum_{k \in \mathbb{Z}^3} |k|^{2\tau} e^{2\tau|k|^s} \left( \hat{u}_k^{\nu,\mu} \cdot \hat{w}_{-k} \right) \leq C \nu \|u^{\nu,\mu}\|_{L^2_{\tau+1-\tau}^2} \|w\|_{L^2_{\tau+1-\tau}^2}, \quad (97)
$$

$$
|T_2| = \mu (2\pi)^3 \sum_{k \in \mathbb{Z}^3} |k|^{2\tau} e^{2\tau|k|^s} \left( \hat{H}_k^{\nu,\mu} \cdot \hat{B}_{-k} \right) \leq C \mu \|H^{\nu,\mu}\|_{L^2_{\tau+1-\tau}^2} \|B\|_{L^2_{\tau+1-\tau}^2}, \quad (98)
$$

As for $T_3$, we first write it into the sum of their Fourier coefficients,

$$
T_3 = i(2\pi)^3 \sum_{k \in \mathbb{Z}^3} \sum_{j \in \mathbb{Z}^3} \left( \hat{w}_j \cdot (k - j) \right) \left( \hat{u}_{k-j}^{\nu,\mu} \cdot \hat{w}_{-k} \right) |k|^{2\tau} e^{2\tau|k|^s}. \quad (99)
$$

Since $r > \max \{ \frac{s}{2} + \frac{3}{2}, \frac{s}{2} + \frac{1}{2} \}$, there exists a constant $C$ such that

$$
|k|^r \leq C (|j|^r + |k - j|^r). \quad (100)
$$

In addition, $s \geq 1$ implies that

$$
e^{\tau|k|^s} \leq e^{\tau|j|^\frac{s}{2}} e^{\tau|k-j|^\frac{s}{2}}. \quad (101)
$$

Then, it follows from (100) and (101) that

$$
|T_3| \leq C \sum_{k \in \mathbb{Z}^3} \sum_{j \in \mathbb{Z}^3} \left( |j|^r + |k - j|^r \right) |\hat{w}_j| |k - j| |\hat{u}_{k-j}^{\nu,\mu}| e^{\tau|j|^\frac{s}{2}} e^{\tau|k-j|^\frac{s}{2}}
$$

$$
\times |\hat{w}_{-k}| |k|^r e^{\tau|k|^\frac{s}{2}}. \quad (102)
$$
By the discrete Hölder’s inequality and Minkowski’s inequality, we obtain that

$$|T_3| \leq C\|u^*, \|\|g_{\nu, \mu}||w\|^2_{G_{\nu, \mu}}.$$  \hspace{1cm} (103)

For $T_5$, $T_7$, and $T_9$, by taking the same arguments as those to $T_3$, we get

$$|T_5| \leq C\|H_{\nu, \mu}||g_{\nu, \mu}||w\|^2_{G_{\nu, \mu}},$$  \hspace{1cm} (104)

$$|T_7| \leq C\|H_{\nu, \mu}||g_{\nu, \mu}||w\|^2_{G_{\nu, \mu}},$$  \hspace{1cm} (105)

$$|T_9| \leq C\|u^*, \|\|B\|^2_{G_{\nu, \mu}}.$$  \hspace{1cm} (106)

As for $T_4$, we can rewrite it as

$$T_4 = (u^0 \cdot \nabla e^{-r} (\frac{1}{e^{r}} w, \lambda w) - (\lambda) e^{-r} \lambda^\frac{1}{2} w) = T_4^1 + T_4^2,$$  \hspace{1cm} (107)

where

$$T_4^1 = -i(2\pi)^3 \sum_{k \in \mathbb{Z}^3} \sum_{j \in \mathbb{Z}^3} (|k|^{-1} - |k - j|^{-1}) e^{\tau|k|} (\hat{u}_j^0 \cdot (k - j))$$

$$\times (\hat{w}_{k-j} \cdot \hat{w}_k)|k|^{-1} e^{\tau|k|},$$

$$T_4^2 = -i(2\pi)^3 \sum_{k \in \mathbb{Z}^3} \sum_{j \in \mathbb{Z}^3} |k - j|^{-1} e^{\tau|k|} (e^{|k|^\frac{1}{2}} - e^{-|k - j|^\frac{1}{2}}) (\hat{u}_j^0 \cdot (k - j))$$

$$\times (\hat{w}_{k-j} \cdot \hat{w}_k)|k|^{-1} e^{\tau|k|}.$$  

In view of the mean value theorem, there exists a constant $\theta \in [0, 1]$ such that

$$|k|^{-1} - |k - j|^{-1} = |(r - 1)(|k| - |k - j|) (\theta|k| + (1 - \theta)|k - j|)^{-1} \right| \leq C|j||(|k|^{-2} + |k - j|^{-2})^{-1}.$$  \hspace{1cm} (108)

Then it follows from (108) that

$$|T_4^1| \leq C \sum_{k \in \mathbb{Z}^3} \sum_{j \in \mathbb{Z}^3} |j|(|j|^2 - |k - j|^2) e^{\tau|j|} e^{\tau|k-j|} |u_j^0||k - j|$$

$$\times |\hat{w}_{k-j}||\hat{w}_k||k|^{-1} e^{\tau|k|}$$

$$\leq C\|u^0\|_{G_{\nu, \mu}} \|w\|_{G_{\nu, \mu}}.$$  \hspace{1cm} (109)

For $T_4^2$, by the inequality $|e^x - 1| \leq |x|e^{|x|}$ for $x \in \mathbb{R}$ and Lemma 2.1, we obtain that

$$|T_4^2| \leq C \sum_{k \in \mathbb{Z}^3} \sum_{j \in \mathbb{Z}^3} |k - j|^{-1} e^{\tau|k-j|} |e^{\tau(|k|^{\frac{1}{2}} - |k - j|^{\frac{1}{2}})} - 1| |\hat{u}_j^0||k - j|$$

$$\times |\hat{w}_{k-j}||\hat{w}_k||k|^{-1} e^{\tau|k|}$$

$$\leq C \sum_{k \in \mathbb{Z}^3} \sum_{j \in \mathbb{Z}^3} |k - j|^{-1} e^{\tau|k-j|} e^{\tau|j|} \frac{|j||k-j|}{|k|^{1-\frac{1}{2}} + |k - j|^{1-\frac{1}{2}}} |u_j^0|$$

$$\times |\hat{w}_{k-j}||\hat{w}_k||k|^{-1} e^{\tau|k|}.$$  

Using the inequality $|k - j| \leq 2|k||j|$ for $k, j \neq 0$, we have

$$\frac{|k-j|}{|k|^{1-\frac{1}{2}} + |k - j|^{1-\frac{1}{2}}} \leq C|k - j|^\frac{1}{2} |k|^{\frac{1}{2}} |j|^{\frac{1}{2}},$$  \hspace{1cm} (110)
where $C$ is a constant depending on $s$. Then it follows from (110) that

$$|T_4| \leq C \tau \sum_{k \in \mathbb{Z}^3} \sum_{j \in \mathbb{Z}^3} |k - j|^{r-1+\frac{1}{2}} e^{\tau |k-j|^{\frac{1}{2}}} \left| j \right|^{1+\frac{1}{2}} |\hat{u}_j|$$

$$\times |\hat{w}_{k-j}| |\hat{w}_{-k}| |k|^{r-1+\frac{1}{2}} e^{\tau |k|^{\frac{1}{2}}}$$

$$\leq C \tau \sum_{k \in \mathbb{Z}^3} \sum_{j \in \mathbb{Z}^3} |k - j|^{r-1+\frac{1}{2}} e^{\tau |k-j|^{\frac{1}{2}}} (1 + \tau |j|^{\frac{1}{2}} e^{\tau |j|^{\frac{1}{2}}}) |j|^{1+\frac{1}{2}} |\hat{u}_j|$$

$$\times |\hat{w}_{k-j}| |\hat{w}_{-k}| |k|^{r-1+\frac{1}{2}} e^{\tau |k|^{\frac{1}{2}}}$$

$$\leq C \tau \|u_0\|_{H^r} \|w\|_{\tilde{G}^{r-\frac{1}{2},r}_w}^2 + C \tau^2 \|u_0\|_{G^{r-\frac{1}{2},r}} \|w\|_{\tilde{G}^{r-\frac{1}{2},r}_w}^2 .$$

(111)

Here we have used the inequality $e^{\xi} \leq 1 + \xi e^{\xi}$ for any $\xi \in \mathbb{R}^+$ with respect to $e^{\tau |j|^{\frac{1}{2}}}$, the discrete Hölder’s inequality and Mintkovski’s inequality. Substituting (109) and (111) into (107), we obtain that

$$|T_4| \leq C \|u_0\|_{G^{r-\frac{1}{2},r}} \|w\|_{\tilde{G}^{r-\frac{1}{2},r}_w}^2 + C \tau \|u_0\|_{H^r} \|w\|_{\tilde{G}^{r-\frac{1}{2},r}_w}^2$$

$$+ C \tau^2 \|u_0\|_{G^{r-\frac{1}{2},r}} \|w\|_{\tilde{G}^{r-\frac{1}{2},r}_w}^2 .$$

(112)

As for $T_8$, taking the same arguments as $T_4$, we obtain that

$$|T_8| \leq C \|u_0\|_{G^{r-\frac{1}{2},r}} \|B\|_{\tilde{G}^{r-\frac{1}{2},r}_B}^2 + C \tau \|u_0\|_{H^r} \|B\|_{\tilde{G}^{r-\frac{1}{2},r}_B}^2$$

$$+ C \tau^2 \|u_0\|_{G^{r-\frac{1}{2},r}} \|B\|_{\tilde{G}^{r-\frac{1}{2},r}_B}^2 .$$

(113)

It remains to estimate $T_6$ and $T_{10}$. We observe that

$$T_6 + T_{10}$$

$$= (\Lambda^{-1} e^{\tau \Lambda^\frac{1}{2}} (H^0 \cdot \nabla B), \Lambda^{-1} e^{\tau \Lambda^\frac{1}{2}} w) - (H^0 \cdot \nabla \Lambda^{-1} e^{\tau \Lambda^\frac{1}{2}} B, \Lambda^{-1} e^{\tau \Lambda^\frac{1}{2}} w)$$

$$+ (\Lambda^{-1} e^{\tau \Lambda^\frac{1}{2}} (H^0 \cdot \nabla w), \Lambda^{-1} e^{\tau \Lambda^\frac{1}{2}} B) - (H^0 \cdot \nabla \Lambda^{-1} e^{\tau \Lambda^\frac{1}{2}} w, \Lambda^{-1} e^{\tau \Lambda^\frac{1}{2}} B)$$

By applying the same approaches as $T_4$, we can obtain that

$$|T_6 + T_{10}| \leq C \|H^0\|_{G^{r-\frac{1}{2},r}} \|w\|_{G^{r-\frac{1}{2},r}} \|B\|_{G^{r-\frac{1}{2},r}}$$

$$+ C \tau \|H^0\|_{H^r} \|w\|_{G^{r-\frac{1}{2},r}} \|B\|_{G^{r-\frac{1}{2},r}}$$

$$+ C \tau^2 \|H^0\|_{G^{r-\frac{1}{2},r}} \|w\|_{G^{r-\frac{1}{2},r}} \|B\|_{G^{r-\frac{1}{2},r}} .$$

(114)

Substituting (97), (98), (103)-(106), (112), (113) and (114) into (96), we obtain that

$$\frac{1}{2} \frac{d}{dt} \left( \|w\|_{\tilde{G}^{r-\frac{1}{2},r}_w}^2 + \|B\|_{\tilde{G}^{r-\frac{1}{2},r}_B}^2 \right)$$

$$\leq \left( \tau' + C \tau \left( \|u_0\|_{H^r} + \|H^0\|_{H^r} \right) + C \tau^2 \left( \|u_0\|_{G^{r-\frac{1}{2},r}} + \|H^0\|_{G^{r-\frac{1}{2},r}} \right) \right) \left( \|w\|_{\tilde{G}^{r-\frac{1}{2},r}_w}^2 + \|B\|_{\tilde{G}^{r-\frac{1}{2},r}_B}^2 \right)$$

$$+ C \nu \|u^{\nu \mu}\|_{G^{r-\frac{1}{2},r}} \|w\|_{G^{r-\frac{1}{2},r}} + C \mu \|H^{\nu \rho\mu}\|_{G^{r-\frac{1}{2},r}} \|B\|_{G^{r-\frac{1}{2},r}}$$

$$+ C \left( \|u_0\|_{G^{r-\frac{1}{2},r}} + \|H^0\|_{G^{r-\frac{1}{2},r}} + \|u^{\nu \mu}\|_{G^{r-\frac{1}{2},r}} + \|H^{\nu \rho\mu}\|_{G^{r-\frac{1}{2},r}} \right)$$

$$\times \left( \|w\|_{\tilde{G}^{r-\frac{1}{2},r}_w}^2 + \|B\|_{\tilde{G}^{r-\frac{1}{2},r}_B}^2 \right) .$$

(115)
By the choice of $\tau$ in (60), noting that (57), (58) and (59) also hold for $(u^0, H^0)$, and choosing the appropriate constant $C$, we have

$$\tau' + C\tau'(\|u^0\|_{H^r} + \|H^0\|_{H^r}) + C\tau^2(\|u^0\|_{\bar{H}^{r+1}} + \|H^0\|_{\bar{H}^{r+1}}) \leq 0.$$ (116)

Then we can obtain from (59) and (115) that

$$\frac{1}{2} \frac{d}{dt} (\|u\|^2_{\bar{H}^{r+1}} + \|B\|^2_{\bar{H}^{r+1}}) \leq C(\nu^2 \|u^{\nu,\mu}\|^2_{\bar{H}^{r+1}} + \mu^2 \|H^{\nu,\mu}\|^2_{\bar{H}^{r+1}}) + (4C(C_1^1)^2 + 1) (\|u\|^2_{\bar{H}^{r+1}} + \|B\|^2_{\bar{H}^{r+1}}).$$ (117)

By Gronwall’s inequality and (95), we obtain that, for $0 < t \leq T$,

$$\|u(t, \cdot)\|^2_{\bar{H}^{r+1}} + \|B(t, \cdot)\|^2_{\bar{H}^{r+1}} \leq e^{8C(C_1^1)^2} T + 2t \int_0^t 2C(\nu^2 \|u^{\nu,\mu}\|^2_{\bar{H}^{r+1}} + \mu^2 \|H^{\nu,\mu}\|^2_{\bar{H}^{r+1}}) \, ds.$$ (118)

Recalling from (63), we have, for $0 < t \leq T$,

$$\int_0^t (\nu \|u^{\nu,\mu}\|^2_{\bar{H}^{r+1}} + \mu \|H^{\nu,\mu}\|^2_{\bar{H}^{r+1}}) \, ds \leq C_2 T.$$ (119)

Thus, we obtain that

$$\|u(t, \cdot)\|^2_{\bar{H}^{r+1}} + \|B(t, \cdot)\|^2_{\bar{H}^{r+1}} \leq 4Ce^{8C(C_1^1)^2} T + 2T C_2 \nu + \nu \nu.$$(120)

This proves (93) by arranging the constant.

Now we turn to estimate the convergence rate of the pressure. To this end, applying the operator $\text{div}$ to the both sides of (95), we obtain the following elliptic equation

$$-\Delta q = \nabla \cdot (w \cdot \nabla u^{\nu,\mu} + u^0 \cdot \nabla w - B \cdot \nabla H^{\nu,\mu} - H^0 \cdot \nabla B).$$ (121)

Applying $\Lambda^{r-1} e^{\tau \Lambda^{1/2}}$ to the both sides of (121) and taking the inner product with $\Lambda^{r-1} e^{\tau \Lambda^{1/2}} q$, we obtain that

$$\|q\|^2_{\bar{H}^{r+1}} = - (2\pi)^3 \sum_{k \in \mathbb{Z}^3} \sum_{j \in \mathbb{Z}^3} (\hat{u}_j \cdot (k - j))(k \cdot \hat{u}_{k-j}^\nu \mu) |k|^{2(r-1)} e^{2\tau |k|^{1/2}} \hat{q}_{k}$$

$$- (2\pi)^3 \sum_{k \in \mathbb{Z}^3} \sum_{j \in \mathbb{Z}^3} (\hat{u}_j^0 \cdot (k - j))(k \cdot \hat{u}_{k-j}) |k|^{2(r-1)} e^{2\tau |k|^{1/2}} \hat{q}_{k}$$

$$+ (2\pi)^3 \sum_{k \in \mathbb{Z}^3} \sum_{j \in \mathbb{Z}^3} (\hat{B}_j \cdot (k - j))(k \cdot \hat{H}_{k-j}^\nu \mu) |k|^{2(r-1)} e^{2\tau |k|^{1/2}} \hat{q}_{k}$$

$$+ (2\pi)^3 \sum_{k \in \mathbb{Z}^3} \sum_{j \in \mathbb{Z}^3} (\hat{H}_j^0 \cdot (k - j))(k \cdot \hat{B}_{k-j}) |k|^{2(r-1)} e^{2\tau |k|^{1/2}} \hat{q}_{k}. $$ (122)

For simplicity, we only estimate the first term on the right-hand side of (122) and the other terms can be estimated in the same way. Using the equality $k \cdot \hat{u}_{k-j}^\nu \mu = j \cdot \hat{u}_{k-j}^\nu \mu$, we have

$$\left| \sum_{k \in \mathbb{Z}^3} \sum_{j \in \mathbb{Z}^3} (\hat{u}_j \cdot (k - j))(k \cdot \hat{u}_{k-j}^\nu \mu) |k|^{2(r-1)} e^{2\tau |k|^{1/2}} \hat{q}_{k} \right|$$

$$\leq C \sum_{k \in \mathbb{Z}^3} \sum_{j \in \mathbb{Z}^3} (|j|^{r-2} + |k - j|^{r-2}) e^{\tau |k-j|^{1/2}} |k - j|^{1/2} |k - j|.$$
\[ \times |\hat{w}_j| |\hat{u}_{k,\mu}^r| |k|^r e^{c|k|^\frac{1}{2}} |\hat{q}_{-k}| \]
\[ \leq C ||w||_{\mathcal{G}_{r-1,r}} ||u^r_{\nu,\mu}||_{\mathcal{G}_{r,r}} ||q||_{\mathcal{G}_{r,r}}. \]
(123)

Taking the same arguments as (123) to the last three terms on the right hand-side of (122), we can obtain that
\[ \left| \sum_{k \in \mathbb{Z}^3} \sum_{j \in \mathbb{Z}^3} (\hat{u}_j^r \cdot (k - j))(k \cdot \hat{w}_{k-j}) |k|^{2(r-1)} e^{2\tau|k|^\frac{1}{2}} |\hat{q}_{-k}| \right| \]
\[ \leq C ||w||_{\mathcal{G}_{r-1,r}} ||u^0||_{\mathcal{G}_{r,r}} ||q||_{\mathcal{G}_{r,r}}, \]
(124)
\[ \left| \sum_{k \in \mathbb{Z}^3} \sum_{j \in \mathbb{Z}^3} (\hat{B}_j \cdot (k - j))(k \cdot \hat{H}_{k-j}^{\nu,\mu}) |k|^{2(r-1)} e^{2\tau|k|^\frac{1}{2}} |\hat{q}_{-k}| \right| \]
\[ \leq C ||B||_{\mathcal{G}_{r-1,r}} ||H^{\nu,\mu}_{\nu,\mu}||_{\mathcal{G}_{r,r}} ||q||_{\mathcal{G}_{r,r}}, \]
(125)
\[ \left| \sum_{k \in \mathbb{Z}^3} \sum_{j \in \mathbb{Z}^3} (\hat{H}_j \cdot (k - j))(k \cdot \hat{B}_{k-j}) |k|^{2(r-1)} e^{2\tau|k|^\frac{1}{2}} |\hat{q}_{-k}| \right| \]
\[ \leq C ||B||_{\mathcal{G}_{r-1,r}} ||H^0||_{\mathcal{G}_{r,r}} ||q||_{\mathcal{G}_{r,r}}. \]
(126)

Then substituting (123)-(126) into (122), we infer
\[ ||q||_{\mathcal{G}_{r,r}} \leq C(C_1\Rightarrow \Phi) \left( ||w||_{\mathcal{G}_{r-1,r}} + ||B||_{\mathcal{G}_{r-1,r}} \right). \]
(127)

Then it follows from (120) that
\[ ||q||_{\mathcal{G}_{r,r}} \leq C(C_1\Rightarrow \Phi) \left( e^{4C(\Phi)\frac{1}{2}T} + \mu \frac{1}{2} + \nu \frac{1}{2} \right). \]
(128)

This proves (94) by arranging the constants. Thus we complete the proof of Theorem 1.1.

Appendix. In this section, we give the proof of Theorem 2.1. The key step is to derive an a priori estimate. To this end, let \((u^{\nu,\mu}, H^{\nu,\mu})\) be a \(H^r\) solution of (1) and (2). In addition, we assume that the solution \((u^{\nu,\mu}, H^{\nu,\mu})\) possesses proper regularity, so that the procedure of formal calculations make sense.

Proof. Since the inequality
\[ C_1(1 + |\xi|)^r \leq (1 + |\xi|^2)^\frac{r}{2} \leq C_2(1 + |\xi|)^r \]
holds for any \(\xi \in \mathbb{Z}^3\), where \(C_1\) and \(C_2\) are constants depending only on \(r\), we can regard \(((2\pi)^3)^r \sum_{k \in \mathbb{Z}^3} (1 + |k|)^{2r} |\hat{u}_k|^2\)^{1/2} as the \(H^r\) norm of the function \(u\). For the convenience of the following calculations, we use this definition of the \(H^r\) norm. Applying \((1 + \Lambda)^r\) to both sides of \((1)_{1,1}\), and taking the \(L^2\) inner product of both sides with \((1 + \Lambda)^r u^{\nu,\mu}\), we obtain that
\[ \frac{1}{2} \frac{d}{dt} ||u^{\nu,\mu}||^2_{H^r} + \nu ||\nabla u^{\nu,\mu}||^2_{H^r} = \left( (1 + \Lambda)^r (H^{\nu,\mu} \cdot \nabla H^{\nu,\mu}), (1 + \Lambda)^r u^{\nu,\mu} \right) \]
\[ - \left( (1 + \Lambda)^r (u^{\nu,\mu} \cdot \nabla u^{\nu,\mu}), (1 + \Lambda)^r u^{\nu,\mu} \right). \]
(130)

Similar to the derivation of (130), we also have
\[ \frac{1}{2} \frac{d}{dt} ||H^{\nu,\mu}||^2_{H^r} + \mu ||\nabla H^{\nu,\mu}||^2_{H^r} = \left( (1 + \Lambda)^r (H^{\nu,\mu} \cdot \nabla H^{\nu,\mu}), (1 + \Lambda)^r H^{\nu,\mu} \right) \]
\[ - \left( (1 + \Lambda)^r (u^{\nu,\mu} \cdot \nabla H^{\nu,\mu}), (1 + \Lambda)^r H^{\nu,\mu} \right). \]
(131)
Adding (130) and (131) up, we have

$$\frac{1}{2} \frac{d}{dt} (\|u^{r,\mu}\|^2_{H^r} + \|H^{r,\mu}\|^2_{H^r}) + \nu \|\nabla u^{r,\mu}\|^2_{H^r} + \mu \|\nabla H^{r,\mu}\|^2_{H^r} = \sum_{i=1}^{3} \mathcal{J}_i,$$  \hspace{1cm} (132)

where

$$\mathcal{J}_1 = - ((1 + \Lambda)^r (u^{r,\mu} \cdot \nabla u^{r,\mu}), H^{r,\mu}),$$

$$\mathcal{J}_2 = - (1 + \Lambda)^r (H^{r,\mu} \cdot \nabla H^{r,\mu}),$$

$$\mathcal{J}_3 = (1 + \Lambda)^r (u^{r,\mu} \cdot \nabla H^{r,\mu}),$$

and

$$\mathcal{J}_4 = (1 + \Lambda)^r (H^{r,\mu} \cdot \nabla u^{r,\mu}).$$

Now, we proceed to control the terms $\mathcal{J}_1$, $\mathcal{J}_2$ and $\mathcal{J}_3$. Utilizing the incompressible condition of $u^{r,\mu}$, we obtain that

$$|\mathcal{J}_1| = \left| (1 + \Lambda)^r (u^{r,\mu} \cdot \nabla u^{r,\mu}) \right|$$

$$\leq C \sum_{k \in \mathbb{Z}^d} \sum_{j \in \mathbb{Z}^d} |(1 + |k|)^r - (1 + |k - j|)^r| \|\hat{u}_{j}^{r,\mu}\||k - j|$$

$$
\leq C \sum_{k \in \mathbb{Z}^d} \sum_{j \in \mathbb{Z}^d} |j|(|j|^{-1} + (1 + |k - j|)^{-1}) \|\hat{u}_{j}^{r,\mu}\||k - j| \|\hat{u}_{j}^{r,\mu}\|(1 + |k|)^r$$

$$\leq C \|u^{r,\mu}\|_{H^r}^3,$$  \hspace{1cm} (133)

where we have used (21), the discrete Hölder’s inequality, Minkowski’s inequality and the fact $r > \frac{5}{2}$. As for $\mathcal{J}_2$ and $\mathcal{J}_3$, arguing analogously to $\mathcal{J}_1$, we have

$$|\mathcal{J}_2| = \left| (1 + \Lambda)^r (u^{r,\mu} \cdot \nabla H^{r,\mu}) \right|$$

$$\leq C \|u^{r,\mu}\|_{H^r} \|H^{r,\mu}\|^2_{H^r},$$  \hspace{1cm} (134)

$$|\mathcal{J}_3| \leq \left| (1 + \Lambda)^r (H^{r,\mu} \cdot \nabla u^{r,\mu}) \right|$$

$$\leq C \|u^{r,\mu}\|_{H^r} \|H^{r,\mu}\|^2_{H^r}.$$  \hspace{1cm} (135)

Inserting (133)-(135) into (131), we conclude that

$$\frac{1}{2} \frac{d}{dt} (\|u^{r,\mu}\|^2_{H^r} + \|H^{r,\mu}\|^2_{H^r}) \leq C \|u^{r,\mu}\|_{H^r} \left( \|u^{r,\mu}\|^2_{H^r} + \|H^{r,\mu}\|^2_{H^r} \right).$$  \hspace{1cm} (136)

Then, we obtain that

$$\|u^{r,\mu}\|_{H^r} + \|H^{r,\mu}\|_{H^r} \leq \frac{2 \left( \|u_0\|^2_{H^r} + \|H_0\|^2_{H^r} \right)^{\frac{1}{2}}}{1 - C t \left( \|u_0\|^2_{H^r} + \|H_0\|^2_{H^r} \right)^{\frac{1}{2}}},$$  \hspace{1cm} (137)

where the constant $C$ depends only on $r$. Similar to the derivation of Theorem 3.1, combining the a priori estimate (137) with Galerkin approximation, we can obtain the conclusions in Theorem 2.1. \hfill $\square$
Acknowledgments. The authors are very grateful to the anonymous referees for their constructive comments and helpful suggestions, which improved the earlier version of this paper. Li is supported in part by NSFC (Grant No. 11671193), the Fundamental Research Funds for the Central Universities (Grant No. 020314380014), and PAPD. Zhang is supported by the China Scholarship Council.

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Received May 2017; revised October 2017.

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