Research Article

Certain Classes of Operators on Some Weighted Hyperbolic Function Spaces

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1. Introduction

Complex function spaces are one of the interesting core subjects in mathematical analysis. This subject has a lot of various generalizations with many joyful branches. The study of hyperbolic function spaces has been at the rigorous research activity. The presented paper deals with discussion of some hyperbolic function classes. 

The known open-unit disc that defined in the concerned complex plane $\mathbb{C}$ and its specific boundary are symbolized by $\Delta = \{z : |z| < 1\}$ and $\partial\Delta$, respectively. The specific space of all holomorphic functions in $\Delta$ is denoted by $H(\Delta)$. Also, assuming that $B(\Delta)$ defines the subset of $H(\Delta)$, which contains those functions $f \in H(\Delta)$ with $|f(z)| < 1$ for all concerned points $z \in \Delta$.

If $(X, D)$ is a concerned metric space, the specific open and closed concerned balls with center $a$ and radius $R > 0$ are denoted by

$$B(a, R) = \{u \in X : d(u, a) < R\},$$
$$\overline{B}(a, R) = \{u \in X : D(u, a) = R\},$$

respectively.

Assume that $f^\ast (z) = (|f^\prime (z)|/1 - |f(z)|^2)$ defines the specific hyperbolic derivative of $f \in B(\Delta)$.

Throughout this paper, we will suppose that $0 < \alpha < 1$ and $\omega: (0, 1) \rightarrow [0, \infty)$ to be a nondecreasing and continuous function with $\omega \neq 0$.

A concerned function $f \in B(\Delta)$ is said to belong to the specific hyperbolic $\alpha$-Bloch class $\mathcal{B}_{\omega, \alpha}^\ast$ when

$$\|f\|_{\mathcal{B}_{\omega, \alpha}^\ast} = \sup_{z \in \Delta} f^\ast (z) \left(1 - |z|^2\right)^\alpha \omega(1 - |z|) < \infty. \quad (2)$$

The little specific hyperbolic Bloch-type class $\mathcal{B}_{\omega, \alpha, 0}^\ast$ contains all $f \in \mathcal{B}_{\omega, \alpha}^\ast$, with

$$\lim_{|z| \rightarrow 1} f^\ast (z) \left(1 - |z|^2\right)^\alpha \omega(1 - |z|) = 0. \quad (3)$$

One of the major aims of this study is to give emerging treatments of some properties of a certain class of operators acting between hyperbolic Bloch functions using the concerned framework of hyperbolic spaces, which is based on the concerned technique of their specific functions. The multiplication operator between two different types of functions, one is analytic and the other is hyperbolic, is treated in this paper. Dealing with corresponding concerned spaces of weighted functions on the disk $\Delta$, essential properties of the new type of operators are discussed. There are some certain attempts to study hyperbolic function spaces (see [1–14] and others). Most research studies were on composition operators. In this paper,
by defining some new classes of operators, we will study hyperbolic logarithmic Bloch functions.

2. Boundedness and Compactness

Assuming \( f(z) \) and \( g(z) \) are two functions in \( \mathcal{H}(\Delta) \), the following operators are

\[
F(f(z)) = (zf(z))^* = \frac{zf'(z) + f(z)}{1 - |zf(z)|^2}
\]  

Further, the operator

\[
\mathcal{L}_g f(z) = f(z)g^*(z),
\]

is one of the concerned aims in this study.

**Remark 1.** The operators \( F(f(z)) \) and \( \mathcal{L}_g f(z) \) are introduced for the first time to study hyperbolic function classes in this paper.

An interesting approach of unified criteria for boundedness and compactness properties of new type operators acting between hyperbolic logarithmic Bloch spaces is investigated in the present section.

**Lemma 1.** Suppose that \( f \in \mathcal{H}(\Delta) \). Thus, \( f \in \mathcal{B}_{w,\alpha}^* \) if and only if \( F(f(z)) \in \mathcal{B}_{w,\alpha}^* \)

**Proof.** Clearly, we have that when the function \( g \in \mathcal{B}_{w,\alpha}^* \), we can have

\[
\sup_{z \in \Delta} g^*(z)F(\omega, z) \leq \lambda < \infty,
\]

where

\[
F(\omega, z) = \frac{(1 - |z|^2)^\alpha}{\omega(1 - |z|)}.
\]

Let \( f \in \mathcal{B}_{w,\alpha}^* \). Therefore,

\[
\|f\|_{\mathcal{B}_{w,\alpha}^*} = \sup_{z \in \Delta} f^*(z)F(\omega, z) < \infty.
\]

Hence,

\[
\|F(f(z))\|_{\mathcal{B}_{w,\alpha}^*} = \sup_{z \in \Delta} \frac{|zf'(z) + f(z)|}{1 - |zf(z)|^2} F(\omega, z)
\]

\[
\leq \sup_{z \in \Delta} \frac{|zf'(z)|}{1 - |zf(z)|^2} F(\omega, z) + \sup_{z \in \Delta} \frac{|f(z)|}{1 - |zf(z)|^2} F(\omega, z)
\]

\[
\leq \|f\|_{\mathcal{B}_{w,\alpha}^*} + \lambda < \infty.
\]

Thus, \( F(f(z)) \in \mathcal{B}_{w,\alpha}^* \).

For the other direction, when \( F(f(z)) \in \mathcal{B}_{w,\alpha}^* \), we infer that

\[
\|F(f(z))\|_{\mathcal{B}_{w,\alpha}^*} = \sup_{z \in \Delta} F(f(z))^*(z)L(\omega, z) < \infty.
\]

Since the concerned function \( f(z) = (F(f(z))/z) \) is a hyperbolic function on \( \Delta \), then we deduce

\[
\sup_{z \in U_{(1/4)}} f^*(z)F(\omega, z) \leq \lambda_1 < \infty,
\]

where \( U_{(1/4)} = \{ z; |z| < (1/4) \} \) and \( \lambda_1 > 0 \) (positive concerned constant). Then, regarding to the specific function \( F(f(z)) \in \mathcal{B}_{w,\alpha}^* \), we clearly obtain

\[
\|f\|_{\mathcal{B}_{w,\alpha}^*} = \sup_{z \in \Delta} f^*(z)F(\omega, z),
\]

\[
\leq \lambda_1 + \sup_{z \in \Delta} F(f(z))^*(z)L(\omega, z) + 16 \sup_{z \in \Delta} F(f(z))F(\omega, z),
\]

\[
\leq \lambda_1 + 4|F(f)|_{\mathcal{B}_{w,\alpha}^*} + 16 \lambda < \infty,
\]

where \( \lambda > 0 \) and \( \lambda_1 > 0 \). Then, \( f \in \mathcal{B}_{w,\alpha}^* \).

**Lemma 2.** Suppose that the function \( \omega \) maps from \((0, 1]\) to \([0, \infty)\). Thus, we have the following inclusion:

\[
\mathcal{B}_{w,\alpha, \ln}^* \subset \mathcal{B}_{w,\alpha, \ln, 0}^*. 
\]

**Proof.** Assume that \( f \in \mathcal{B}_{w,\alpha, \ln}^* \) so

\[
\|f\|_{\mathcal{B}_{w,\alpha, \ln}^*} = \sup_{z \in \Delta} f^*(z)F(\omega, z) < \infty.
\]

Thus, we infer that

\[
\lim_{|z| \to 1^-} f^*(z)F(\omega, z) = \lim_{|z| \to 1^-} f^*(z)L(\omega, z) \left( \frac{\ln(1/1 - |z|^2)}{\ln(1/1 - |z|^2)} \right)
\]

\[
\leq \|f\|_{\mathcal{B}_{w,\alpha, \ln}^*} \lim_{|z| \to 1^-} \left( \log \frac{1}{1 - |z|^2} \right)^{-1} = 0.
\]

Thus, \( f \in \mathcal{B}_{w,\alpha, \ln, 0}^* \).

**Theorem 1.** Let \( g \in \mathcal{H}(\Delta) \). Then, we have equivalence between the following:

(a) The boundedness of the operator \( \mathcal{L}_g : \mathcal{B}_{w}^* \to \mathcal{B}_{w,\alpha}^* \) holds bounded

(b) The boundedness of the operator \( \mathcal{L}_g : \mathcal{B}_{w,0}^* \to \mathcal{B}_{w,\alpha,0}^* \) holds bounded

(c) The boundedness of the operator \( \mathcal{L}_g : \mathcal{B}_{w,0}^* \to \mathcal{B}_{w,\alpha}^* \) holds bounded

(d) The concerned function \( g \in \mathcal{B}_{w,\alpha}^* \) holds bounded.
Proof. Because of
\[ B^*_w \subset B^*_w, \]
\[ B^*_w,0 \subset B^*_w,0. \]
Hence, clearly, we can deduce that \((a) \Rightarrow (c)\) and \((b) \Rightarrow (c)\). Therefore, we now aim to deduce that \((d) \Rightarrow (a)\), \((d) \Rightarrow (b)\), and \((c) \Rightarrow (d)\).

Using Lemma 1 and in view of the operator,
\[ \mathcal{L}_g = F(f(z))g, \]
Assume the concerned function \(g \in B^*_w, \ln\). Hence,
\[ \|g\|_{B^*_w,\ln} = \sup_{z \in \Delta} g^*(z)F(\omega, z)\ln \frac{1}{1 - |z|^2} < \infty. \]
For \(f \in B^*_w,\ln\) we have
\[ |f(z)| \leq |f(0)| + \|f\|_{B^*_w,\ln} \ln \frac{1}{1 - |z|^2}. \]
From
\[ \mathcal{L}_g f(z) = f(z)g^*(z), \]
we obtain that
\[ \|\mathcal{L}_g f\|_{B^*_w,\ln} = \sup_{z \in \Delta} |f(z)|g^*(z)F(\omega, z), \]
\[ \leq |f(0)| \sup_{z \in \Delta} g^*(z)F(\omega, z) \]
\[ + \|f\|_{B^*_w,\ln} \sup_{z \in \Delta} g^*(z)F(\omega, z)\ln \frac{1}{1 - |z|^2}, \]
\[ \leq |f(0)| \times \|g\|_{B^*_w,\ln} + \|f\|_{B^*_w,\ln} \|g\|_{B^*_w,\ln}. \]
Applying Lemma 2 and considering that \(g \in B^*_w, \ln\), we obtain that
\[ \|g\|_{B^*_w,\ln} < \infty. \]
Hence,
\[ \|\mathcal{L}_g f\|_{B^*_w,\ln} < \infty. \]
The closed-graph theorem yields the boundedness of the concerned operator \(\mathcal{L}_g : B^*_w \rightarrow B^*_w,\ln\) so \((d) \Rightarrow (a)\).

Now, assume that \(f \in B^*_w,0\). Therefore, for every \(\epsilon > 0\), we can find \(R \in (0, 1)\), for which \(R < |z| < 1\) as well as
\[ |f(z)| \leq \epsilon \ln \frac{1}{1 - |z|^2}. \]
Then, we can infer that
\[ (\mathcal{L}_g f)^*(z)F(\omega, z) = |f(z)|g^*(z)F(\omega, z), \]
\[ \leq \epsilon g^*(z)F(\omega, z) \]
\[ \leq \epsilon \|g\|_{B^*_w,\ln}, \]
where \(\mathcal{L}_g f \in B^*_w,0\), and we deduce \((d) \Rightarrow (b)\).

Let us assume that \(\mathcal{L}_g : B^*_w,0 \rightarrow B^*_w,0\) is bounded. Also, let us choose arbitrary \(w_0 \in \Delta(0)\) and suppose that \(f(\omega) = \ln(1/1 - \omega)\). It is not hard to see that \(f \in B^*_w,0\). Then,
\[ \inf \|\mathcal{L}_g f\|_{B^*_w,\ln} = \sup_{z \in \Delta} \|f(z)|g^*(z)F(\omega, z), \]
\[ \leq \sup_{z \in \Delta} g^*(z)F(\omega, z)\ln \frac{1}{1 - |z|^2}, \]
and put \(z = w_0\), we obtain that
\[ g^*(w_0) \frac{1 - |w_0|^2}{\omega(1 - |w_0|^2)} \ln \frac{1}{1 - |w_0|^2} \leq \|\mathcal{L}_g f\|_{B^*_w,\ln} < \infty. \]

Because \(w_0\) is arbitrary on \(\Delta(0)\), we have
\[ \|g\|_{B^*_w,\ln} \leq \|\mathcal{L}_g f\|_{B^*_w,\ln} < \infty. \]
Hence, \(g \in B^*_w,\log\). Thus, the assertion \((c) \Rightarrow (d)\) follows, and therefore, the proof is completely established.

As in [15], the following lemma can be proved. \(\Box\)

Lemma 3. Suppose that \(M\) is a closed concerned set in \(B^*_w,0\). Then, \(M\) is compact, and \(\Rightarrow M\) is bounded, and the following condition holds:
\[ \lim_{|z| \rightarrow 1} \sup_{f \in M} f^*(z)L(\omega, z) = 0. \]

Theorem 2. For the concerned function \(g \in \mathcal{H}(\Delta)\), we can find equivalence between the following:

(i) The compactness of the operator \(\mathcal{L}_g : B^*_w \rightarrow B^*_w,0\) holds

(ii) The compactness of the operator \(\mathcal{L}_g : B^*_w,0 \rightarrow B^*_w,0\) holds

(iii) The concerned function \(g \in B^*_w,\ln\)

Proof. Because \(B^*_w,0 \subset B^*_w,\ln\), then (I) \(\Rightarrow\) (II). Therefore, the specific proof of (III) \(\Rightarrow\) (I) and (II) \(\Rightarrow\) (III) are needed.

By using concerned Lemma 1 and considering the operator \(\mathcal{L}_g\) for \(g \in B^*_w, \ln\) using Lemma 2, we infer that \(g \in B^*_w,0\). Hence, \[ \lim_{|z| \rightarrow 1} g^*(z)F(\omega, z)\ln \frac{1}{1 - |z|^2} = 0, \]
\[ \lim_{|z| \rightarrow 1} g^*(z)F(\omega, z) = 0. \]
Therefore, by (19), for every \(f \in B^*_w\), we have that
\[
\lim_{[z] \to 1^-} \left( \mathcal{L}_g f_n \right)(z) F(\omega, z) = \lim_{[z] \to 1^-} |f(z)| g^*(z) F(\omega, z)
\]
\[
\leq |f(0)| \lim_{[z] \to 1^-} g^*(z) F(\omega, z) + \|f\|_{\mathcal{B}_w}^* \sup_{z \in \Delta} g^*(z) F(\omega, z) \ln \frac{1}{1-|z|^2} = 0.
\]
(31)

Thus, \( \mathcal{L}_g : \mathcal{B}_w^* \to \mathcal{B}_{w,0}^* \) is actually bounded. To prove that the concerned operator \( \mathcal{L}_g \) is actually compact, assume that \( \{f_n\} \subset \mathcal{B}_w^* \) is defined such that \( \|f_n\|_{\mathcal{B}_w} \leq 1 \). We have to clear that \( \{\mathcal{L}_g f_n\} \) has a concerned subsequence which is converging on \( \mathcal{B}_{w,0}^* \). Using (19), we can find a concerned subsequence of the sequence \( \{f_n\} \) which converges uniformly on concerned compact subsets on the disc \( \Delta \) to the concerned function \( f \). For this purpose, we suppose that the concerned sequence \( \{f_n\} \) converges also to the concerned function \( f \). For any fixed \( z \in \Delta \), we infer that
\[
f^*(z) F(\omega, z) = \lim_{n \to \infty} f_n^*(z) F(\omega, z) \leq 1.
\]
(32)

Then, \( f \in \mathcal{B}_{w,0}^* \) and \( \|f\|_{\mathcal{B}_{w,0}^*} \leq 1 \). Thus, \( \mathcal{L}_g f \in \mathcal{B}_{w,0}^* \), and it is enough to be clear that
\[
\lim_{n \to \infty} \|\mathcal{L}_g f_n - \mathcal{L}_g f\|_{\mathcal{B}_{w,0}^*} = 0.
\]
(33)

Because \( g \in \mathcal{B}_{w,0}^* \), we deduce for every \( \varepsilon > 0 \) and we can find an \( R \in (0, 1) \), for which
\[
\sup_{z \in \Delta \setminus \Lambda_R} |f_n(z) - f(z)| g^*(z) F(\omega, z) \leq |f_n(0) - f(0)|
\]
\[
\sup_{z \in \Delta \setminus \Lambda_R} g^*(z) F(\omega, z)
\]
\[
+ \left( \|f_n(z)\|_{\mathcal{B}_{w,0}^*} + \|f(z)\|_{\mathcal{B}_{w,0}^*} \right) \sup_{z \in \Delta \setminus \Lambda_R} g^*(z) F(\omega, z) \ln \frac{1}{1-|z|^2} < \varepsilon, \quad n \geq 1.
\]
(34)

Because the sequence \( \{f_n\} \) converges to the function \( f \) uniformly on each compact subset on \( \Delta \), we deduce that there exists an \( N > 0 \), for every \( n > N \) and every \( z \in \Delta, f_n(z) - f(z) < \varepsilon \). Then, for \( n > N \), we deduce that
\[
\sup_{z \in \Delta} |f_n(z) - f(z)| g^*(z) F(\omega, z)
\]
\[
\leq \varepsilon \sup_{z \in \Delta} g^*(z) F(\omega, z)
\]
\[
\leq \varepsilon \|g\|_{\mathcal{B}_{w,0}^*}.
\]
(35)

From (34) and (35), we infer that
\[
\lim_{n \to \infty} \sup_{z \in \Delta} |f_n(z) - f(z)| g^*(z) F(\omega, z) = 0.
\]
(36)

Then,
\[
\lim_{n \to \infty} \|\mathcal{L}_g f_n - \mathcal{L}_g f\|_{\mathcal{B}_{w,0}^*} = 0.
\]
(37)

Hence, \( \mathcal{L}_g : \mathcal{B}_w^* \to \mathcal{B}_{w,0}^* \) is actually compact. Then, the assertion (III) (I) is completely proved.

Next, we are clear that (II) (III) and assume that \( \mathcal{L}_g : \mathcal{B}_{w,0}^* \to \mathcal{B}_{w,0}^* \) is compact. Suppose that \( g \in \mathcal{B}_{w,0}^* \). We can find a specific sequence of points \( \{z_n\} \) in \( \Delta, \|z_n\| \to 1 \), for which
\[
g^*(z_n) F(\omega, z_n) \to k > 0,
\]
(38)

where \( k \) is a positive constant and
\[
F^*(\omega, z_n) = \frac{(1-|z_n|)^a}{\omega(1-|z_n|)}
\]
(39)

Now, assume that \( f_n(z) = \ln (1/1 - z_n) \). Then, \( f_n \in \mathcal{B}_{w,0}^* \) with \( \|f_n\|_{\mathcal{B}_{w,0}^*} \leq 1 \). Because \( \mathcal{L}_g \) is actually compact, then \( \{\mathcal{L}_g f_n\} \) is also a compact concerned subset of \( \mathcal{B}_{w,0}^* \). In view of Lemma 3, we obtain that
\[
\lim_{n \to \infty} \sup_{z \in \Delta \setminus \Lambda} |f_n(z) - f(\omega, z)| = 0.
\]
(40)

This is equivalent to
\[
\lim_{[z] \to 1^-} \sup_{\|f\|_{\mathcal{B}_{w,0}^*} \leq 1} |f_n(z)| g^*(z) F(\omega, z) = 0.
\]
(41)

Putting \( \varepsilon = (k/2) \), then we can find \( \rho > 0 \) such that
\[
\sup_{\|f\|_{\mathcal{B}_{w,0}^*} \leq 1} |\sup_{\|f\|_{\mathcal{B}_{w,0}^*} \leq 1} F^*(\omega, z_n)| \to k > 0,
\]
(42)

Assume that \( N > 0 \), so for \( n > N, |z_n| > \rho \), we deduce that
\[
g^*(z_n) |f_n(z_n)| F^*(\omega, z_n) \leq \sup_{\|f\|_{\mathcal{B}_{w,0}^*} \leq 1} \sup_{\|f\|_{\mathcal{B}_{w,0}^*} \leq 1} g^*(z_n) |f_n(z)| F^*(\omega, z_n) < \frac{k}{2}
\]
(43)

and this contradicts (38). Hence, the proof is established.

\section{3. Hyperbolic Bloch Classes and Usual Distance}

Connections between hyperbolic Bloch spaces and the metric spaces are investigated in the following emerging result.

The concept of the general hyperbolic derivative is defined in [2] by
\[
h_k^*(w) = \frac{|h^{(k)}(w)|}{1 - |h(w)|^{k+1}}, \quad k \in \mathbb{N}.
\]
(44)

Remark that, when \( k = 1 \), we get the known usual hyperbolic derivative as given above.

Now, we introduce the following concepts:

The logarithmic space \( \mathcal{B}_{k,ln}^* \) is defined by
\[
\mathcal{B}_{k,ln}^* = \{h_k^* \in B(\Delta): \|h_k^*\|_{\mathcal{B}_{k,ln}^*} = \sup_{z \in \Delta} L(\omega, z, \ln)h_k^* < \infty,\}
\]
(45)
where

\[ L(\omega, z, \ln) = \frac{1 - |z|^2}{\omega (1 - |z|^2)} \left( \frac{1}{1 - |z|^2} \right). \]  (46)

**Theorem 3.** The considered space \( \mathcal{B}^*_{k, w, \ln} \) endowed by the usual metric (distance) \( D(\cdot; \mathcal{B}^*_{k, w, \ln}) \) is a complete metric space. Furthermore, \( \mathcal{B}^*_{k, w, \ln, 0} \) is a concerned specific closed subspace of the concerned space \( \mathcal{B}^*_{k, w, \ln} \).

**Proof.** Let \( h, h_1, \) and \( h_2 \in \mathcal{B}^*_{k, w, \ln} \); hence, we infer that

\[ D(h, h_1; \mathcal{B}^*_{k, w, \ln}) \geq 0, \]

\[ D(h, h, \mathcal{B}^*_{k, w, \ln}) = 0. \]  (47)

Therefore, we can deduce also that

\[ D(h, h_1; \mathcal{B}^*_{k, w, \ln}) = D(h, h_1; \mathcal{B}^*_{k, w, \ln}), \]

\[ D(h, h_2; \mathcal{B}^*_{k, w, \ln}) \leq D(h, h_1; \mathcal{B}^*_{k, w, \ln}) + D(h_1, h_2; \mathcal{B}^*_{k, w, \ln}). \]  (48)

Since, the classes are defined by an usual distance on the hyperbolic class \( \mathcal{B}^*_{k, w, \ln} \), \( D(h, h_1; \mathcal{B}^*_{k, w, \ln}) = 0 \), which results in \( h = h_1 \). Therefore, the distance \( D \) forms a metric space on the class \( \mathcal{B}^*_{k, w, \ln} \).

Let \( \{h_n\}_{n=0}^\infty \) be a concerned Cauchy sequence in the defined metric space \( \mathcal{B}^*_{k, w, \ln} \); therefore, for any \( \epsilon > 0 \) there exists an \( N = N(\epsilon) \), for which

\[ D(h_n, h_m; \mathcal{B}^*_{k, w, \ln}) < \epsilon, \]  (49)

for all \( m, n > N \) because \( \{f_n\} \subset (B(\Delta)) \); thus, the concerned family \( \{h_n\} \) is actually uniformly bounded in \( \Delta \). Therefore, we can obtain \( h \in B(\Delta) \) with the concerned subsequence \( \{h_{n_j}\}_{j=1}^\infty = 1 \) such that \( \{h_{n_j}\} \) converges uniformly on some compact concerned subsets of \( \Delta \) to the specific function \( h \), which implies that \( h_n \) converges on compact subsets in the uniform type to the function \( h \).

Now, let \( m > N \), using the uniform convergence, we infer that

\[ \left( \frac{h_n^{(k)}(z)}{1 - |h_n(z)|^{k+1}} - \frac{h_m^{(k)}(z)}{1 - |h_m(z)|^{k+1}} \right)L(\omega, z, \ln) \]

\[ = \lim_{n \to \infty} \left( \frac{h_n^{(k)}(z)}{1 - |h_n(z)|^{k+1}} - \frac{h_m^{(k)}(z)}{1 - |h_m(z)|^{k+1}} \right)L(\omega, z, \ln) \]

\[ \leq \lim_{n \to \infty} D(h_n, h_m; \mathcal{B}^*_{k, w, \ln}) \leq \epsilon, \]  (50)

for all \( z \in \Delta \), and hence, \( h \in \mathcal{B}^*_{k, w, \ln} \) as we need.

Moreover, the above inequality and the compactness of the usual \( \mathcal{B}^*_{k, w, \ln} \) space give us that \( \{h_n\}_{n=1}^\infty \) converges to \( h \) with respect to the distance \( D \). Because

\[ \lim_{n \to \infty} D(h_n, h_m; \mathcal{B}^*_{k, w, \ln}) \leq \epsilon, \]  (51)

the second assertion follows, and hence, the proof is established.

**Corollary 1.** The space \( \mathcal{B}^\ast \) endowed by the usual metric (distance) \( D(\cdot; \mathcal{B}^\ast) \) is also a complete metric space. Further, \( \mathcal{B}^\ast_0 \) is a concerned closed subspace of \( \mathcal{B}^\ast \).

**Proof.** The proof can be obtained by letting \( \alpha = 1 \), \( \omega (1 - |z|) \equiv 1 \), and \( \ln(1/1 - |z|^2) = 1 \) in Theorem 3.

**Remark 2.** Complex function spaces have an interesting and joyful extension in quaternion analysis (see [16–20] and others).

For this direction of study, can we use quaternion functions instead of hyperbolic functions to study the properties of the defined operators in this paper?

4. Conclusions

This paper presents a variety of techniques for defining and studying classes of operators on hyperbolic function spaces and features a lot of their properties, focusing on the new concepts and methods of clearing proofs. The paper begins with presenting basic concepts and notation to the studied hyperbolic spaces, followed by introducing the new definitions of the considered operators and detailed proofs of the new results. After giving the boundedness and compactness properties of the defined operators, it lastly presents a metric space characterization of the hyperbolic-type Bloch space. The results are excellently and carefully presented.

With the help of a certain nondecreasing and continuous function, some weighted hyperbolic classes are defined and researched in this study. Essential characterizations for the weighted Bloch space by new multiplication operators are introduced in the sense of hyperbolic settings. Completeness of metric spaces is also established on certain hyperbolic function spaces.

**Data Availability**

The present concerned study has no any type of applied or created data.

**Conflicts of Interest**

The authors declare that they have no conflicts of interests from any type regarding the publications of this paper.

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