NEW TWO-DIMENSIONAL QUANTUM MODELS PARTIALLY SOLVABLE
BY SUPERSYMMETRICAL APPROACH

M.V. Ioffe\textsuperscript{a}, P.A. Valinevich\textsuperscript{b}

Department of Theoretical Physics, Sankt-Petersburg State University,
198504 Sankt-Petersburg, Russia

New solutions for second-order intertwining relations in two-dimensional SUSY QM are found via the repeated use of the first order supersymmetrical transformations with intermediate constant unitary rotation. Potentials obtained by this method - two-dimensional generalized Pöschl-Teller potentials - appear to be shape-invariant. The recently proposed method of $SUSY-$separation of variables is implemented to obtain a part of their spectra, including the ground state. Explicit expressions for energy eigenvalues and corresponding normalizable eigenfunctions are given in an analytic form. Intertwining relations of higher orders are discussed.

PACS numbers: 03.65.-w, 03.65.Fd, 11.30.Pb

1. Introduction

The importance of each new exactly solvable model in one-dimensional (1D) Quantum Mechanics is well known, especially because the list of such models is quite small. The elegant modern approach used for the study and classification of these "elite" models was provided by Supersymmetrical Quantum Mechanics (SUSY QM) \cite{1,2}, which is in essence an alternative formulation of the famous Factorization Method \cite{3} in one-dimensional Quantum Mechanics. Furthermore, the introduction in the framework of SUSY QM of a new notion - the shape invariance \cite{4,2} - gave a novel, algebraic, tool to deal with such kind of models. There are different ways of going beyond the scope of the standard Witten’s SUSY QM in order to enlarge the class of involved models. The Higher Order SUSY QM (HSUSY QM), or, equivalently, Polynomial and $N-$fold SUSY QM \cite{5,6} as well as constructions for multidimensional coordinate spaces \cite{7,8} are among the most promising ones.

\textsuperscript{a}E-mail: m.ioffe@pobox.spbu.ru
\textsuperscript{b}E-mail: pasha@PV7784.spb.edu
From the very beginning, after 1D SUSY QM was formulated by Witten [1], the question of finding the opportunity to generalize it for higher dimensions of space attracted considerable attention. A direct $d-$dimensional generalization was built in [8] by means of methods originating from SUSY Quantum Field Theory. In this approach the Superhamiltonian (of block-diagonal form) includes both scalar and matrix components and can be used to analyse different physical problems with matrix potentials [9].

In the particular case of two-dimensional space an alternative SUSY QM approach was proposed, which directly generalizes the HSUSY QM ideas, namely, the use of the SUSY intertwining relations with the second order supercharges. This method avoids an appearance of matrix potentials and provides the intertwining of two scalar Schrödinger Hamiltonians. A large class of such intertwined Hamiltonians was found in [10] - [13].

In the framework of the latter approach two new methods for the study of the spectra and the (normalizable) eigenfunctions of the two-dimensional quantum models were proposed recently in [14], [15]: SUSY—separation of variables and the two-dimensional shape invariance (see also the review-like paper [16]). The combination of both of them was explored to investigate a specific model - a generalized 2D Morse potential with three free parameters - which is not amenable to the conventional separation of variables. As a result, this model turned out to be partially solvable, i.e. only a part of the variety of its normalizable wave functions and corresponding values of energies were found analytically. Thus the transfer from one-dimensional to two-dimensional shape invariance was accompanied by the loss of the complete solvability with only the partial one remaining. It is worth mentioning here that each of the 2D Hamiltonians involved in the second order intertwining relation is integrable: the symmetry operator of the fourth order in derivatives was constructed explicitly in terms of supercharges [10], [12].

In this paper both approaches of two-dimensional SUSY QM - the direct two-dimensional generalization [7], [8] and the second order construction [10]-[13] - will be used to build and to investigate some new models, to which no standard separation of variables can be applied. Again SUSY—separation of variables turns out to be applicable to the model, providing a set of normalizable wave functions. This model, which is shown to be partially solvable, will be called a 2D-generalized Pöschl-Teller potential.

As for the method of the two-dimensional shape invariance [14] - [16], the situation is more delicate. Though the considered model possesses the property of shape invariance, the
corresponding solutions of the Schrödinger equation turned out to be unnormalizable.

The paper is organized as follows. In Section 2 the known methods of 2D SUSY QM will be described briefly in order to simplify the comprehension of the new results. A new technique of searching for solutions of the two-dimensional second order intertwining relations will be presented in Section 3, and, in particular, two-dimensional generalizations of Pöschl-Teller potentials will be constructed. In Section 4 SUSY—separation of variables will be used to find a part of the spectrum of this model and analytical expressions for its wave functions, including the ground state. The peculiarities of shape invariance are also investigated. In Section 5 an additional structure with two different superpartners for the same Hamiltonian is presented, new intertwining relations of fourth and sixth orders in derivatives are constructed (the last ones are shape-invariant).

2. Basics of 2D SUSY QM

2.1. 2D representation of SUSY algebra

The SUSY algebra of quantum mechanics is given by the following (anti)commutation relations [1]:

\[ \{ \hat{Q}^+, \hat{Q}^- \} = \hat{H}; \quad \{ \hat{Q}^+, \hat{Q}^+ \} = \{ \hat{Q}^-, \hat{Q}^- \} = 0; \quad [\hat{Q}^\pm, \hat{H}] = 0. \]  

(1)

In the case of two dimensions it can be realized [7], [8] by the following 4×4 matrix operators:

\[
\hat{H} = \begin{pmatrix}
H^{(0)}(\vec{x}) & 0 & 0 \\
0 & H_{ik}^{(1)}(\vec{x}) & 0 \\
0 & 0 & H^{(2)}(\vec{x})
\end{pmatrix}; \quad i, k = 1, 2; \quad \hat{Q}^+ = (\hat{Q}^-)^\dagger =
\begin{pmatrix}
0 & 0 & 0 & 0 \\
q_1^- & 0 & 0 & 0 \\
q_2^- & 0 & 0 & 0 \\
0 & p_1^+ & p_2^+ & 0
\end{pmatrix},
\]  

(2)

where two scalar Hamiltonians \( H^{(0)}, H^{(2)} \) and one 2×2 matrix Hamiltonian \( H_{ik}^{(1)} \) of Schrödinger type can be expressed in a quasifactorized form (compare to the factorized form in one-dimensional case [1], [2]):

\[
H^{(0)} = q_i^+ q_i^- = -\partial_i^2 + V^{(0)}(\vec{x}) = -\partial_i^2 + \left( \partial_i \chi(\vec{x}) \right)^2 - \partial_i^2 \chi(\vec{x}); \quad \partial_i^2 \equiv \partial_t^2 + \partial_x^2;
\]

\[
H^{(2)} = p_i^+ p_i^- = -\partial_i^2 + V^{(2)}(\vec{x}) = -\partial_i^2 + \left( \partial_i \chi(\vec{x}) \right)^2 + \partial_i^2 \chi(\vec{x});
\]

\[
H_{ik}^{(1)} = q_i^- q_k^+ + p_i^- p_k^+ = -\delta_{ik} \partial_i^2 + \delta_{ik} \left( (\partial_i \chi(\vec{x}))^2 - \partial_i^2 \chi(\vec{x}) \right) + 2 \partial_i \partial_k \chi(\vec{x}),
\]  

(3)
with components of supercharges of first order in derivatives:

\[ q_i^\pm \equiv \mp \partial_i + \partial_i \chi(\vec{x}); \quad p_i^\pm \equiv \epsilon_{ik} q_k^\mp, \]  

(4)

where \( \partial_i \equiv \partial/\partial x_i \) and summation over repeated indices is implied. Anticommutators in (1) produce the following intertwining relations for the component Hamiltonians \( H^{(0)}, H^{(1)}, H^{(2)} \) of the Superhamiltonian (2):

\[ H^{(0)} q_i^+ = q_k^+ H^{(1)}_{ki}; \quad H^{(1)}_{ik} q_k^- = q_i^- H^{(0)}; \quad H^{(1)}_{ik} p_k^- = p_i^- H^{(2)}; \quad H^{(2)} p_i^+ = p_k^+ H^{(1)}_{ki}. \]  

(5)

They connect the spectrum of the matrix hamiltonian with spectra of two scalar ones. In general, \( H^{(0)} \) and \( H^{(2)} \) are not isospectral since \( q_k^+ p_k^- \equiv 0 \) due to (4).

2.2. Second-order supercharges in 2D SUSY QM

Two-dimensional SUSY QM models without any matrix Hamiltonians were constructed [10], [11], [16] by means of second order supercharges

\[ Q^+ = (Q^-)^\dagger = g_{ik}(\vec{x}) \partial_i \partial_k + C_i(\vec{x}) \partial_i + B(\vec{x}) \]  

(6)

where \( g_{ik}, C_i, B \) are arbitrary real functions. Some particular solutions for two scalar Hamiltonians \( H^{(0,1)} \) which satisfy the intertwining relations

\[ H^{(1)}(\vec{x}) Q^+ = Q^+ H^{(0)}(\vec{x}); \quad H^{(0)}(\vec{x}) Q^- = Q^- H^{(1)}(\vec{x}), \]  

(7)

were found. They both possess the symmetry operators \( R^{(1,2)} \) of fourth order in derivatives [10], [12]:

\[ [R^{(i)}, H^{(i)}] = 0; \quad i = 0, 1; \quad R^{(0)} = Q^- Q^+; \quad R^{(1)} = Q^+ Q^-; \]

which are not, in general, polynomials of \( H^{(i)} \).

In terms of unknown functions \( g_{ik}, C_i, B, V^{(0)}, V^{(1)} \) Eq. (7) has the form [10] of seven nonlinear partial differential equations, and its general solution is not known. To obtain particular solutions different ansätze for ”metrics” \( g_{ik} \) were used.

Only the choice of Laplacian (elliptic) metrics \( g_{ik}(\vec{x}) = diag(1,1) \) leads to Hamiltonians amenable to \( R \)–separation [17] of variables. All other choices of metrics give nontrivial results. The case of Lorentz (hyperbolic) metrics \( g_{ik} = diag(1,-1) \) was investigated in
papers [10] - [13]. In particular, the intertwining relations (7) were reduced to the pair of differential equations:

\[ \partial_1 \partial_2 F = 0; \quad \partial_- (C_- F) = -\partial_+ (C_+ F). \]  

(8)

where \( x_\pm = (x_1 \pm x_2)/\sqrt{2} \) and \( C_{1,2} \) were proven to satisfy \( C_\pm \equiv C_1 \mp C_2 \equiv C_\pm (\sqrt{2} x_\pm) \). Then potentials \( V^{(0),(1)} \) and the supercharges \( Q^+ \) are expressed in terms of functions \( C_\pm (\sqrt{2} x_\pm) \) and \( F(\vec{x}) \) which obviously can be written as \( F = F_1(2x_1) + F_2(2x_2) \) according to (8):

\[ V^{(0),(1)} = \pm \frac{1}{2} \left( C'_+(\sqrt{2}x_+) + C'_-(\sqrt{2}x_-) \right) + \frac{1}{8} \left( C^2_+(\sqrt{2}x_+) + C^2_-(\sqrt{2}x_-) \right) + \frac{1}{4} \left( F_2(2x_2) - F_1(2x_1) \right); \]  

\[ Q^+ = (\partial^2 - \partial^2_0) + C_1 \partial_1 + C_2 \partial_2 + B; \]  

\[ B = \frac{1}{4} \left( C_+(\sqrt{2}x_+)C_-(\sqrt{2}x_-) + F_1(2x_1) + F_2(2x_2) \right), \]  

(9)

(10)

(11)

where the prime denotes the derivative of function with respect to its argument. A list of particular solutions of (8) was obtained in [11] - [13]. In the next Section we will obtain new solutions for the case of hyperbolic metrics.

3. New solutions for the Lorentz (hyperbolic) metrics

3.1. Intertwining of second order with reducible supercharges

Let us consider two Superhamiltonians \( \hat{H} \) and \( \hat{\tilde{H}} \) of 2D SUSY QM:

\[ \hat{H} = \begin{pmatrix} H^{(0)}(\vec{x}) & 0 & 0 \\ 0 & H^{(1)}_{ik}(\vec{x}) & 0 \\ 0 & 0 & H^{(2)}(\vec{x}) \end{pmatrix}; \quad \hat{\tilde{H}} = \begin{pmatrix} H^{(0)}(\vec{x}) & 0 & 0 \\ 0 & \tilde{H}^{(1)}_{ik}(\vec{x}) & 0 \\ 0 & 0 & \tilde{H}^{(2)}(\vec{x}) \end{pmatrix}, \]  

(12)

with superpotentials \( \chi(\vec{x}) \) and \( \tilde{\chi}(\vec{x}) \), correspondingly.

In addition, let \( H^{(1)}_{ik} \) and \( \tilde{H}^{(1)}_{ik} \) be linked by an unitary \( 2 \times 2 \) matrix transformation \( U \):

\[ U_{ik} \tilde{H}^{(1)}_{kl} = \hat{H}^{(1)}_{im} U_{ml}; \]  

(13)

\[ U = \alpha_0 \sigma_0 + i \alpha \vec{\sigma}; \quad \alpha_0^2 + \alpha^2 = 1; \quad \alpha_0, \alpha_i \in \mathbb{R}, \]  

(14)

\( ^c \)We use here the definition of \( x_\pm \) slightly different from the analogous one in [11] - [13].
where $\sigma_i$ are the Pauli matrices and $\sigma_0$ is the unit matrix.

Then (due to (3)) the scalar Hamiltonians $H^{(0)}$ and $\tilde{H}^{(0)}$ can be included in the chain:

$$H^{(0)} \leftrightarrow H^{(1)}_{ik} \leftrightarrow U_{lm} \leftrightarrow \tilde{H}^{(1)}_{ik} \leftrightarrow \tilde{H}^{(0)}_{ik},$$

leading to the intertwining relations between a pair of scalar Hamiltonians:

$$H^{(0)} Q^- = Q^- \tilde{H}^{(0)}, \quad Q^+ H^{(0)} = \tilde{H}^{(0)} Q^+$$

with second order operators

$$Q^- = (Q^+)^\dagger = q^+_i U_{ik} \tilde{q}^-_k.$$  \hspace{1cm} (17)

This intertwining operator is constructed from two first order ones with intermediate matrix transformation $U_{ik}$. Precisely this matrix provides that such supercharges $Q^\pm$ are nontrivial and, contrary to Subsection 2.1., can be naturally described as reducible (compare with the case of one-dimensional reducibility introduced in [5]).

In contrast to the approach of [14] (see Subsection 2.2.), the first Hamiltonian in the chain (15) is quasifactorized according to Eq. (3). Therefore the solution of the corresponding Schrödinger equation with zero energy can be written as $\Psi^{(0)}_0 \sim \exp(-\chi)$. Due to expression (17), $\exp(-\chi)$ is a zero mode of supercharge $Q^+$ as well. In general, until the specific form of $\chi(\vec{x})$ is chosen, the normalizability of this solution is not guaranteed. But in the concrete model [31] analyzed below in Subsection 3.3. the zero energy solution is normalizable due to asymptotic properties of $\chi(\vec{x})$ for corresponding ranges [16] of parameters.

Target Hamiltonians $H^{(0)}$ and $\tilde{H}^{(0)}$ are expressed in terms of two unknown functions $\chi$ and $\tilde{\chi}$ (see [3]). To determine these functions one should substitute (3), (14) and (17) into (16). After some manipulations one obtains the system of equations for $\chi_\pm = (\chi \pm \tilde{\chi})/2$:

$$\begin{align*}
\alpha_3 \Box \chi_- + 2\alpha_1 \partial_1 \partial_2 \chi_- &= 0; & \alpha_1 \Box \chi_+ - 2\alpha_3 \partial_1 \partial_2 \chi_+ &= 0; \\
\alpha_2 \Box \chi_+ + 2\alpha_0 \partial_1 \partial_2 \chi_- &= 0; & \alpha_0 \Box \chi_- + 2\alpha_2 \partial_1 \partial_2 \chi_+ &= 0; \\
(\partial_k \chi_-)(\partial_k \chi_+) &= 0, \hspace{1cm} (20)
\end{align*}$$

where $\Box \equiv \partial^2_1 - \partial^2_2$. Eq. (20) (which is equivalent to $(\partial_k \chi)^2 = (\partial_k \tilde{\chi})^2$) can be used to simplify expressions (3) for the Hamiltonians $H^{(0)}$ and $\tilde{H}^{(0)}$:

$$H^{(0)}, \tilde{H}^{(0)} = -\partial^2_1 + \left( (\partial_1 \chi_+)^2 - \partial^2_1 \chi_+ \right) + \left( (\partial_1 \chi_-)^2 \mp \partial^2_1 \chi_- \right).$$

(21)

Linear partial differential equations (18)-(19) can be easily solved, but the solution of the nonlinear Eq. (20) is a nontrivial problem.
3.2. The particular solutions of the intertwining relations

It can be shown that in the case when all coefficients \( \alpha_i \) and \( \alpha_0 \) in (14) do not vanish, potentials \( V^{(0)} \) and \( \tilde{V}^{(0)} \) are 4-th order polynomials on \( x_{1,2} \) with some additional constraints for their coefficients. In the present paper we will consider potentials beyond this rather narrow class, restricting ourselves to the particular case \( \alpha_0 = \alpha_1 = \alpha_2 = 0; \ \alpha_3 \neq 0 \), i.e. \( U = \sigma_3 \). Then the metrics of supercharges \( Q^\pm \) is Lorentz, i.e. they belong to the class discussed in Subsection 2.2.

For this case equations (18) - (19) read:

\[
\Box \chi_- = 0; \quad \partial_1 \partial_2 \chi_+ = 0.
\]

Their solution is

\[
\chi_- = \mu_+(x_+) + \mu_-(x_-), \\
\chi_+ = \mu_1(x_1) + \mu_2(x_2),
\]

with \( \mu_1, \mu_2, \mu_\pm \) being arbitrary functions. The last equation (20) takes the form

\[
\mu'_1(x_1) [\mu'_+(x_+) + \mu'_-(x_-)] + \mu'_2(x_2) [\mu'_+(x_+) - \mu'_-(x_-)] = 0.
\]

By substitutions \( \phi \equiv \mu' \), it becomes purely functional (without derivatives) equation:

\[
\phi_1(x_1) [\phi_+(x_+) + \phi_-(x_-)] = -\phi_2(x_2) [\phi_+(x_+) - \phi_-(x_-)]. \tag{22}
\]

The general solution of (22) is given in the Appendix\(^d\). Some particular cases will be discussed in Subsection 3.3.

The Hamiltonians (21) and intertwining operators (17) can be expressed in terms of \( \phi \):

\[
V^{(0)}, \tilde{V}^{(0)} = \left( \phi_1^2(x_1) - \phi_1'(x_1) \right) + \left( \phi_2^2(x_2) - \phi_2'(x_2) \right) + \left( \phi_+^2(x_+) + \phi_-'(x_+) \right) + \left( \phi_-^2(x_-) + \phi_-'(x_-) \right),
\]

\[
Q^\pm = \partial_1^2 - \partial_2^2 \pm \sqrt{2} \left( \phi_+(x_+) + \phi_-(x_-) \right) \partial_1 \mp \sqrt{2} \left( \phi_+(x_+) - \phi_-(x_-) \right) \partial_2 - \left( \phi_1^2(x_1) - \phi_1'(x_1) \right) + \left( \phi_2^2(x_2) - \phi_2'(x_2) \right) + 2 \phi_+(x_+ \phi_-(x_-). \tag{23}
\]

By rearrangement of terms Eq. (22) can be rewritten as:

\[
\phi_+(x_+) [\phi_1(x_1) + \phi_2(x_2)] = -\phi_-(x_-) [\phi_1(x_1) - \phi_2(x_2)]. \tag{24}
\]

\(^d\)It was derived by D.N. Nishnianidze (private communication).
i.e. in the form similar to the initial Eq. (22). This means that (22) possesses the symmetry property (which will be called $S_1$ symmetry in the subsequent text): if \( \{\phi_1(x_1), \phi_2(x_2), \phi_+(x_+), \phi_-(x_-)\} \) is a solution, then \( \{\phi_+(x_1), \phi_-(x_2), \phi_1(x_+), \phi_2(x_-)\} \) is also a solution. Let us mention one more discrete symmetry of (20), $S_2$ symmetry: \( \{\phi_1(x_1), \phi_2(x_2), \phi_+(x_+), \phi_-(x_-)\} \rightarrow \{\phi_1(x_1), -\phi_2(x_2), \phi_1^{-1}(x_+), \phi_2^{-1}(x_-)\} \). The $S_1$-symmetry produces another supersymmetrical model:

\[
\mathcal{V}^{(0)}, \mathcal{\tilde{V}}^{(0)} = \left( \phi^2_1(x_+) \mp \phi'_1(x_+) \right) + \left( \phi^2_2(x_-) \mp \phi'_2(x_-) \right) + \left( \phi^2_+(x_1) - \phi'_+(x_1) \right) + \left( \phi^2_-(x_2) - \phi'_-(x_2) \right),
\]

\[
\mathcal{Q}^\pm = \partial_1^2 - \partial_2^2 \pm \sqrt{2} \left( \phi_1(x_+) + \phi_2(x_-) \right) \partial_1 \mp \sqrt{2} \left( \phi_1(x_+) - \phi_2(x_-) \right) \partial_2 - \left( \phi^2_+(x_1) - \phi'_+(x_1) \right) + \left( \phi^2_-(x_2) - \phi'_-(x_2) \right) + 2\phi_1(x_+)\phi_2(x_-).
\]

Below both forms (23) and (25) will be explored.

To compare the new notations of this Section with those of [11] - [13] (see Subsection 2.2.) one can use the following relations:

\[
C_\pm(\sqrt{2}x_\pm) = 2\sqrt{2}\phi_\pm(x_\pm);
\]

\[
F_{1,2}(2x_{1,2}) = \mp 4 \left( \phi^2_{1,2}(x_{1,2}) - \phi'_{1,2}(x_{1,2}) \right).
\]

### 3.3. Nonperiodical solutions for potentials $V^{(0)}$, $\mathcal{\tilde{V}}^{(0)}$.

From Eq. (A8) one can conclude that for an arbitrary choice of the parameters $a, b, c$ the functions $\phi_{1,2}$ are expressed in terms of elliptic (Jacobi or Weierstrass) functions [18] (volume 3). In this paper we restrict ourselves by considering the limiting cases, for which the potentials are not periodical (the models with periodicity properties in $x_{1,2}$ will be studied elsewhere).

The integral in the r.h.s. in (A8) is an elementary function only if either some of coefficients $a, b, c$ are zero or the quadratic polynomial is a full square. There are two families of solutions of (22), with members interconnected by symmetries $S_1$ and $S_2$ (and their combinations):

\[
(i) \quad \phi_1(x) = \phi_2(x) = A/x; \quad \phi_+ = \phi_- = B/x \quad (A, B = \text{const}).
\]

The multiparticle potentials of this type were found in [19] to be quasi-exactly solvable [20]. All other members of this family allow the separation of variables.

\[
(ii) \quad \phi_1(x) = \phi_2(x) = M \left( \delta_+ e^{\alpha x} + \delta_- e^{-\alpha x} \right);
\]
\[ \phi_+(x) = -L \frac{\delta_+ e^{\alpha x/\sqrt{2}} - \delta_- e^{-\alpha x/\sqrt{2}}}{\delta_+ e^{\alpha x/\sqrt{2}} + \delta_- e^{-\alpha x/\sqrt{2}}}; \quad \phi_-(x) = L \coth \left( \alpha x/\sqrt{2} \right). \quad (27) \]

For the particular case \( \delta_- = 0 \) one has:

\[ V^{(0)}, \tilde{V}^{(0)} = (B^2 e^{-2\alpha x_1} + B\alpha e^{-\alpha x_1}) + (B^2 e^{-2\alpha x_2} + B\alpha e^{-\alpha x_2}) + 4A^2 + A(2A \mp \alpha) \left[ \sinh \left( \frac{\alpha}{2}(x_1 - x_2) \right) \right]^2 \quad (28) \]

with two new constants \( A, B \) instead of \( M, L, \delta_+ \). This potentials (up to translations in \( x_{1,2} \)) were analyzed in [14] and were found to be shape-invariant.

Another particular case \( \delta_+ = -\delta_- \) for (27) after using symmetries and redefinition of parameters gives:

\[ \phi_1 = -\phi_2 = \frac{A}{\sinh \sqrt{2} \alpha x} \]
\[ \phi_+ = \phi_- = B \tanh \alpha x, \quad (29) \]

The corresponding potentials and intertwining operators for this model due to (28) are:

\[ V^{(0)}, \tilde{V}^{(0)} = \left( B^2 - \frac{B(B \pm \alpha)}{\cosh^2 \left( \frac{\alpha}{\sqrt{2}}(x_1 + x_2) \right)} \right) + \left( B^2 - \frac{B(B \pm \alpha)}{\cosh^2 \left( \frac{\alpha}{\sqrt{2}}(x_1 - x_2) \right)} \right) + \]
\[ + A \left[ \frac{A - \sqrt{2}\alpha \cosh \left( \sqrt{2} \alpha x_1 \right)}{\sinh^2 \left( \sqrt{2} \alpha x_1 \right)} + \frac{A + \sqrt{2}\alpha \cosh \left( \sqrt{2} \alpha x_2 \right)}{\sinh^2 \left( \sqrt{2} \alpha x_2 \right)} \right], \quad (30) \]

\[ Q^\pm = \partial_1^2 - \partial_2^2 \pm \sqrt{2}B \left[ \tanh \left( \frac{\alpha}{\sqrt{2}}(x_1 + x_2) \right) + \tanh \left( \frac{\alpha}{\sqrt{2}}(x_1 - x_2) \right) \right] \partial_1 \mp \]
\[ \mp \sqrt{2}B \left[ \tanh \left( \frac{\alpha}{\sqrt{2}}(x_1 + x_2) \right) - \tanh \left( \frac{\alpha}{\sqrt{2}}(x_1 - x_2) \right) \right] \partial_2 - \]
\[ -A \left[ \frac{A - \sqrt{2}\alpha \cosh \left( \sqrt{2} \alpha x_1 \right)}{\sinh^2 \left( \sqrt{2} \alpha x_1 \right)} - \frac{A + \sqrt{2}\alpha \cosh \left( \sqrt{2} \alpha x_2 \right)}{\sinh^2 \left( \sqrt{2} \alpha x_2 \right)} \right] + \]
\[ + 2B^2 \tanh \left( \frac{\alpha}{\sqrt{2}}(x_1 + x_2) \right) \tanh \left( \frac{\alpha}{\sqrt{2}}(x_1 - x_2) \right). \]

Solution (29), obtained by the discrete symmetry \( S_1 \), is:

\[ V^{(0)}, \tilde{V}^{(0)} = \left( B^2 - \frac{B(B + \alpha)}{\cosh^2 \left( \alpha x_1 \right)} \right) + \left( B^2 - \frac{B(B + \alpha)}{\cosh^2 \left( \alpha x_2 \right)} \right) + \]
\[ + A \left[ \frac{A \mp \sqrt{2}\alpha \cosh \left( \alpha(x_1 + x_2) \right)}{\sinh^2 \left( \alpha(x_1 + x_2) \right)} + \frac{A \pm \sqrt{2}\alpha \cosh \left( \alpha(x_1 - x_2) \right)}{\sinh^2 \left( \alpha(x_1 - x_2) \right)} \right] \quad (31) \]

\[ Q^\pm = \partial_1^2 - \partial_2^2 \pm \sqrt{2}A \left[ \frac{1}{\sinh \left( \alpha(x_1 + x_2) \right)} + \frac{1}{\sinh \left( \alpha(x_1 - x_2) \right)} \right] \partial_1 \mp \]
\[\pm \sqrt{2}A \left[ \frac{1}{\sinh(\alpha(x_1 + x_2))} - \frac{1}{\sinh(\alpha(x_1 - x_2))} \right] \partial_2 - \left[ B^2 - \frac{B(B + \alpha)}{\cosh^2(\alpha x_1)} \right] + \left[ B^2 - \frac{B(B + \alpha)}{\cosh^2(\alpha x_2)} \right] + \frac{2A^2}{\sinh(\alpha(x_1 + x_2))\sinh(\alpha(x_1 - x_2))} \]

Both potentials (30) and (31) can be treated as superpositions of two one-dimensional Pöschl-Teller terms plus a singular term (so we will refer to them as 2D-generalized Pöschl-Teller potentials). Each of them possesses a term which prevents application of the conventional method of separation of variables to determine their eigenfunctions and eigenvalues. Meanwhile, a part of the spectrum and corresponding eigenfunctions will be found by the method of SUSY – separation of variables (see \[14\], \[16\]) in the next Section.\footnote{Other members of the same family (S2- and S2S1-symmetric to (29)) can be treated analogously.}

4. Partial solvability of 2D-generalized Pöschl-Teller potentials

4.1. SUSY – separation of variables

As far as Hamiltonians with potentials (31) are intertwined by operators \( Q^\pm \) with Lorentz metrics, we shall briefly remind the reader of the general method for searching for eigenvalues and eigenfunctions proposed in \[14\]. From intertwining relations \( Q^+\mathcal{H}^{(0)} = \tilde{\mathcal{H}}^{(0)}Q^+ \) (where \( \mathcal{H}^{(0)} = -\partial_1^2 + V^{(0)} \) and \( \tilde{\mathcal{H}}^{(0)} = -\partial_1^2 + \tilde{V}^{(0)} \)) one obtains that the subspace of zero modes\footnote{Here we suppose that \((N+1)\) normalizable zero modes \( \Omega_n(\vec{x}) \) are known, and \( \vec{\Omega}(\vec{x}) \) is a column vector with components \( \Omega_n(\vec{x}), n = 0, 1, \ldots N \).} of the supercharge \( Q^+ \):

\[ Q^+\vec{\Omega}(\vec{x}) = 0, \] (32)

is closed under the action of \( \mathcal{H}^{(0)} \):

\[ \mathcal{H}^{(0)}\vec{\Omega}(\vec{x}) = \hat{C}\vec{\Omega}(\vec{x}) \] (33)

with some constant matrix \( \hat{C} \).

To determine the eigenvalues \( E_k \) and eigenfunctions \( \Psi_k(\vec{x}) \) of \( \mathcal{H}^{(0)} \) one needs (see more details in \[14\]) a matrix \( \hat{B} \), which satisfies the matrix equation \( \hat{B}\hat{C} = \hat{\Lambda}\hat{B} \) with an unknown
yet diagonal matrix $\hat{\Lambda} = \text{diag}(\lambda_0, \lambda_1, \ldots, \lambda_N)$. Then the matrix $\hat{B}$ transforms zero-modes $\Omega_n$’s into wave functions $\Psi_n$’s.

Operator $Q^+$ belongs to type (10). For this type of supercharges the problem (32) permits the conventional separation of variables in $Q^+$ by means of “gauge” transformation, which separates variables in the supercharge:

$$q^+ = e^{-\kappa(\vec{x})} Q^+ e^{\kappa(\vec{x})} = \partial_1^2 - \partial_2^2 + \frac{1}{4}(F_1(2x_1) + F_2(2x_2)), \quad (34)$$

$$h(\vec{x}) \equiv e^{-\kappa(\vec{x})} \mathcal{H}^{(0)}(\vec{x}) e^{\kappa(\vec{x})} = -\partial_1^2 - \partial_2^2 + C_1(\vec{x}) \partial_1 - C_2(\vec{x}) \partial_2 - \frac{1}{4}F_1(2x_1) + \frac{1}{4}F_2(2x_2). \quad (35)$$

$$\kappa(\vec{x}) \equiv -\frac{\sqrt{2}}{4} \left[ \int C_+(\sqrt{2}x_+) \, dx_+ + \int C_-(\sqrt{2}x_-) \, dx_- \right].$$

Then the zero modes $\omega_n(\vec{x})$ of $q^+$ can be written as products $\omega_n(\vec{x}) = \eta_n(x_1) \rho_n(x_2)$, where $\rho_n$ and $\eta_n$ are eigenfunctions of the one-dimensional Schrödinger equations with ”potentials” $(\mp \frac{1}{4}F_{1,2}(2x_{1,2}))$, correspondingly (see (31)), and common eigenvalues (constants of separation) $\epsilon_n$.

It is obvious that $h\vec{\omega} = \hat{C}\vec{\omega}$ with the same matrix $\hat{C}$ as in (33). The simplest way to find $\hat{C}$ is to calculate the r.h.s. of (35), which can be rewritten as:

$$h\omega_n = [2\epsilon_n + C_1(\vec{x})\partial_1 - C_2(\vec{x})\partial_2] \omega_n. \quad (36)$$

As a result, after construction of the matrix $\hat{B}$ one will obtain part of the spectrum $E_k$ and corresponding wave functions $\Psi_k(\vec{x})$.

### 4.1.1. Calculation of $\hat{C}$

This general method, proposed in [14] and used there successfully to investigate the 2D Morse potential, can be applied to the pair $V^{(0)}$, $\tilde{V}^{(0)}$ as well. In this case both one-dimensional equations for multipliers $\eta_n(x_1)$ and $\rho_n(x_2)$ have the same ”potentials” - one dimensional Pöschl-Teller potentials - being exactly-solvable:

$$(-\partial_1^2 + B^2 - B(B + \alpha) \cosh^{-2}(\alpha x_1)) \eta_n(x_1) = \epsilon_n \eta_n(x_1) \quad (37)$$

and a similar equation for $\rho_n(x_2)$. By the change of variable $\xi \equiv \tanh \alpha x_1$ Eq.(37) can be reduced to the generalized Legendre equation [21] :

$$\frac{d}{d\xi} \left[ (1 - \xi^2) \frac{d\eta}{d\xi} \right] + \left[ s(s + 1) - (s^2 - \frac{\epsilon}{\alpha^2}) \frac{1}{1 - \xi^2} \right] \eta = 0,$$
where \( s = B/\alpha \). To have the finite solution at \( \xi = -1 \) the condition \( \sqrt{(B^2 - \epsilon)/\alpha^2} - s = -n; \) \( n \in \mathbb{N} \) must be satisfied. It gives a discrete set of values for the separation constant \( \epsilon \):

\[
\epsilon_n = \alpha^2 n(2s - n)
\]

for \( n < s \). Corresponding functions (up to normalization factors) are \( \eta_n = P_{s-n}^s(\xi) \), where \( P_{\nu}^\mu(x) \) are the (generalized) Legendre functions. Thus, one achieves the expression for \( \omega_n \):

\[
\omega_n = P_{s-n}^s(\xi_1)P_{s-n}^s(\xi_2),
\]

with \( \xi_i = \tanh \alpha x_i, \ i = 1, 2 \).

The next step in calculating the eigenfunctions is evaluating the r.h.s. of (36):

\[
h\omega_n(\vec{x}) = 2\alpha^2 n(2s - n)\omega_n - 2\sqrt{2}A\alpha(2s - n)(n + 1)\frac{(1 - \xi_1^2)(1 - \xi_2^2)(1 - s)^{1/2}}{(\xi_1^2 - \xi_2^2)}\Pi(n, s; \xi_1, \xi_2), \quad (38)
\]

where me made the shorthand notation

\[
\Pi(n, s; \xi_1, \xi_2) = \xi_2(\xi_1^2 - 1)^{1/2}P_{s-n}^s(\xi_1)P_{s-n}^s(\xi_2) - \xi_1(\xi_1^2 - 1)^{1/2}P_{s-n}^s(\xi_1)P_{s-n}^s(\xi_2).
\]

Our goal is to represent the r.h.s. of (36) as a linear combination of \( \omega_k \)'s. For this purpose we use the recurrent formula for Legendre functions (see [18], v.1, p.161, eq. (1))

\[
(z^2 - 1)^{1/2}P_{\nu+2}^\mu(z) + 2(\mu + 1)zP_{\nu+1}^\mu(z) = (z^2 - 1)^{1/2}(\nu - \mu)(\nu + \mu + 1)P_{\nu}^\mu.
\]

Applying it twice to \( \Pi(n, s; \xi_1, \xi_2) \) one obtains the following recurrent formula:

\[
\Pi(n, s; \xi_1, \xi_2) = \frac{1}{(n + 1)n(2s - n)(2s - n + 1)}\left[ 2(s - n + 1)(\xi_1^2 - \xi_2^2)(1 - \xi_1^2)^{1/2}(1 - \xi_2^2)^{1/2}\omega_{n-1} + \Pi(n-2, s; \xi_1, \xi_2) \right]. \quad (39)
\]

To stop this procedure at \( n = 0 \), one has to consider \( s \in \mathbb{N} \). In this case the Legendre functions turns into the associate Legendre polynomials, for which \( P_{m}^n(z) \equiv 0 \) for \( m > n \). So, applying (39) several times,

\[
\Pi(n, s; \xi_1, \xi_2) = \frac{\xi_1^2 - \xi_2^2}{(1 - \xi_1^2)^{1/2}(1 - \xi_2^2)^{1/2}}\sum_{k=0}^{n} a_{nk}\omega_k
\]

with constants \( a_{nk} \). The matrix elements \( c_{nk} \) of matrix \( \hat{C} \) are:

\[
c_{nk} = 2\alpha^2 n(2s - n)\delta_{nk} - 2\sqrt{2}A\alpha(2s - n)(n + 1)a_{nk}; \quad (40)
\]

\[
a_{nk} = \begin{cases} 0, & k \geq n; \\ 0, & k = n - 2m - 2; m = 0, 1, 2, \ldots \\ 2(s - k)\frac{(k-1)!((2s-n-1))!}{(n+1)!(2s-k)!}, & k = n - 2m - 1; m = 0, 1, 2, \ldots 
\end{cases} \quad (41)
\]
4.1.2. Calculation of eigenfunctions

Matrix $\hat{C}$ for the model (31) appeared to be triangular, and hence its eigenvalues coincide with the diagonal elements:

$$E_k = c_{kk} = 2\alpha^2 k(2s - k).$$

(42)

This formula gives $E_0 = 0$, demonstrating that the zero energy solution of $\mathcal{H}^{(0)}$ is a zero mode of $Q^+$ as well: $\Psi_0 \sim \exp(-\chi(\vec{x}))$.

In order to avoid zeros on the diagonal of $\hat{C}$ one can shift Hamiltonians by a constant $\gamma$. This transformation does not destroy the intertwining relations (16) and changes $\hat{C}$ as follows:

$$c_{ik} \rightarrow c_{ik} + \gamma \delta_{ik}.$$ 

This new $\hat{C}$ can be diagonalized by the method, presented in [14]. Namely, the formal solution for the matrix elements of $\hat{B}$ reads:

$$b_{m,p} = b_{m,N-m} \left[ \sum_{l=1}^{N-p-1} (\tau^{(m)})^l \right]_{N-m,p}$$

(43)

where $(N + 1)$ matrices $\tau^{(m)}$ are defined by

$$\tau_{n,k}^{(m)} = \frac{c_{n,k}}{c_{N-m,N-m} - c_{k,k}},$$

and label $(m)$ has values $m = 0, 1, ..., N$. In Eq.(43) the repeated index $N - m$ is not summed over, and (to avoid misunderstanding) $\tau^{(m)}_l$ means the $l$th power of matrix $\tau^{(m)}$.

Thus one obtains the recipe for the construction of eigenfunctions for $\mathcal{H}^{(0)}$ in (31):

$$\Psi_{N-n}(\vec{x}) = \sum_{k=0}^{N} b_{n,k} \Omega_k(\vec{x}).$$

(44)

Formula (43) gives us the opportunity to express an element $b_{m,p}$ by means of the $\tau^{(m)}$ matrices and an arbitrary element $b_{m,N-m}$ on the crossed diagonal. This last element can be fixed by the normalization condition for $\Psi_{N-m}$. The reason for the ”inverted” numeration of $\Psi$ in (44) is to make $\Psi_k$ dependent only on $\Omega_l; l = 0, 1, ..., k$. In particular, $\Psi_0 \sim \Omega_0$.

So, applying the method of SUSY—separation of variables to $\mathcal{H}^{(0)}$, $\tilde{\mathcal{H}}^{(0)}$ one obtains a set of eigenvalues and eigenfunctions for $\mathcal{H}^{(0)}$.

Keeping in mind that $s = B/\alpha > 0$, we can restrict ourselves with $B > 0, \alpha > 0$. The conditions of normalizability of $\Omega_n$ (and therefore, of $\Psi_n$ ) for all $\omega_n$ can be derived from the explicit expressions:

$$\Omega_n = \omega_n \exp \kappa = \left( \frac{\cosh(\alpha(x_1 + x_2)) - 1}{\sinh(\alpha(x_1 + x_2))(\cosh(\alpha(x_1 - x_2)) - 1)} \right)^{A/(\sqrt{2}\alpha)} P_s^{-n}(\xi_1) P_s^{-n}(\xi_2).$$

(45)
The constraint is: \(-\frac{1}{\sqrt{2}} < \frac{A}{\alpha} < \frac{1}{\sqrt{2}}\). The condition \(A > 0\) keeps the strength of attractive singularities of both superpartners \(V(0), \tilde{V}(0)\) at \(x_\pm \to 0\) not exceeding the standard bound \(-1/(4x_\pm^2)\). The resulting range of parameters for both \(\Omega_n\) and \(\tilde{\Omega}_n\) is:

\[
\alpha > 0; \quad B > 0; \quad \frac{B}{\alpha} \in \mathbb{N}; \quad 0 < A < \frac{\alpha}{\sqrt{2}}. \tag{46}
\]

At first it seems that the energy eigenvalues (42), which were built above from the analysis of the zero-modes of \(Q^+\), should be absent in the spectrum of its superpartner \(\tilde{H}(0)\), since the corresponding eigenfunctions are annihilated by \(Q^+\). However, the whole procedure of SUSY—separation of variables can also be implemented for the spectral problem for \(\tilde{H}(0)\) by suitably replacing (32) with \(Q^-\tilde{\Omega} = 0\). Since \(Q^-\) and \(Q^+\) differ only by sign in front of the first derivatives (see (25)), one should use the ”gauge” transformation with \(\exp (-\kappa(\vec{x}))\). In this case one will obtain the same equations (37), as for the problem (32). Then the zero-modes of \(Q^-\) can be written as:

\[
\tilde{\Omega}_n(\vec{x}) = \exp (-\kappa(\vec{x}))\omega_n(\vec{x}) = \exp (-2\kappa(\vec{x}))\Omega_n(\vec{x}),
\]

and the corresponding matrix \(\tilde{C}\) is again triangular. To be more precise, it is the same as (40)-(41) up to the sign of the last term in (40). Therefore, its eigenvalues, i.e. values of energy for \(\tilde{H}(0)\), coincide with (42). One can check that the eigenfunctions are normalizable in the same range of parameters (46). Thus the obtained part of the spectra of superpartners \(\mathcal{H}(0)\) and \(\tilde{\mathcal{H}}(0)\) totally coincide. In a certain sense this result is similar to one of the variants of the second order intertwining in 1D HSUSY QM [5]: the equal number of bosonic and fermionic zero modes does not signal the spontaneous breaking of the supersymmetry.

4.2. The method of shape-invariance

Shape-invariance [4], [2], [14] is an additional property of intertwined superpartner Hamiltonians which gives the opportunity to determine their spectra algebraically. Namely, if both Hamiltonians depend on some extra parameter (or set of parameters) \(a\), this property reads:

\[
\tilde{H}(a_0) = H(a_1) + \mathcal{R}(a_0), \tag{47}
\]

where \(a_1 = f(a_0)\) is another value of parameter, and \(\mathcal{R}(a_0)\) does not depend on \(\vec{x}\), i.e. \(\tilde{H}\) has the same (up to an additive constant) shape as \(H\), but with another set of parameters.
Let us assume that we know some eigenfunction \( \Psi^{(0)} \) of \( H \) and the corresponding eigenvalue \( E^{(0)} \) in some range of the parameter \( a \). Starting from

\[
H(a_1)\Psi^{(0)}(a_1) = E^{(0)}(a_1)\Psi^{(0)}(a_1)
\]

and employing (47), one obtains:

\[
\tilde{H}(a_0)\Psi^{(0)}(a_1) = (E^{(0)}(a_1) + R(a_0))\Psi^{(0)}(a_1).
\]

Using intertwining relations (16) in (49),

\[
H(a_0)\left[ Q^-(a_0)\Psi^{(0)}(a_1) \right] = (E^{(0)}(a_1) + R(a_0)) \left[ Q^-(a_0)\Psi^{(0)}(a_1) \right].
\]

This means that \( H(a_0) \) has the eigenvalue \( E^{(1)}(a_0) = E^{(0)}(a_1) + R(a_0) \) with the wave function \( \Psi^{(1)}(a_0) = Q^-(a_0)\Psi^{(0)}(a_1) \) (its normalizability is not guaranteed). Starting from (48) with parameter \( a_2 = f(f(a_0)) \) and repeating the described procedure twice one can find:

\[
H(a_0)\left[ Q^-(a_0)Q^-(a_1)\Psi^{(0)}(a_2) \right] = (E^{(0)}(a_2) + R(a_1) + R(a_0)) \left[ Q^-(a_0)Q^-(a_1)\Psi^{(0)}(a_2) \right],
\]

which gives one more point \( (\Psi^{(2)}(a_0),E^{(2)}(a_0)) \) in the spectrum of \( H(a_0) \). The general formulas are

\[
\Psi^{(n)}(a_0) = Q^-(a_0)Q^-(a_1)...Q^-(a_{n-1})\Psi^{(0)}(a_n),
\]

\[
E^{(n)}(a_0) = E^{(0)}(a_n) + \sum_{k=0}^{n-1} R(a_k).
\]

So, we have constructed a "shape-invariance chain" of eigenfunctions starting from one given. The natural idea is to combine this method with SUSY—separation of variables (if the Hamiltonians possess shape-invariance): having \((N+1)\) eigenfunction from SUSY—separation, we use each of them to start the described above shape-invariance chain. This procedure was implemented for the generalized 2D Morse potential (28) in [14].

For the 2D-generalized Pöschl-Teller potential the situation becomes more complicated. Indeed, the Hamiltonians \( H^{(0)}, \tilde{H}^{(0)} \) with potentials (30) are shape-invariant:

\[
\tilde{H}^{(0)}(\vec{x};B,\alpha) = H^{(0)}(\vec{x};B - \alpha, \alpha) + 2\left(B^2 - (B - \alpha)^2\right),
\]

where, in the notations introduced above, \( a_0 \equiv B; a_1 = f(a_0) \equiv B - \alpha \). But, contrary to the model [14], the method of SUSY—separation of variables does not work here, since the zero modes \( \omega_n \) are unnormalizable for all values of the parameter \( A \).
This obstacle can be overcome by exploring the relation between systems (30) and (31):

\[ H^{(0)}(x_1, x_2) = \mathcal{H}^{(0)}(x_+, x_-), \]

where in the r.h.s. arguments \(x_1, x_2\) are substituted by \(x_+, x_-\). Because a part of the spectrum (and the eigenfunctions) of \(\mathcal{H}^{(0)}\) was found by the method of \(SUSY\)—separation of variables (Subsection 4.1.), one can use the relation (55) to obtain the corresponding part of the spectrum (and the eigenfunctions) for \(H^{(0)}\). Then one can use these data to start shape invariance chains for the system \(H^{(0)}, \tilde{H}^{(0)}\) according to (52) with operators \(Q^-\). For example, starting from the first zero mode one obtains:

\[ \Psi^{(n)}(x_1, x_2; B) = Q^- (x_1, x_2; B) \ldots Q^- (x_1, x_2; B - (n - 1)\alpha) \Omega_0(x_+, x_-; B - n\alpha). \]

The general formula for the spectrum can be obtained from (53), where the \(E^{(0)}\)'s for each chain are taken from (12):

\[ E_{mn} = 2\alpha^2 [m(2s - 2n - m) + n(2s - n)] = 2\alpha^2 (m + n)[2s - (m + n)], \]

where \(0 < m < s\) corresponds to the number of the chain (number of eigenfunction constructed by \(SUSY\)–separation), and \(0 < n < s\) in order to keep positive all of \(R(a_k), k = 0, \ldots, (N - 1)\), since the ground state energies \(E^{(0)}(a_k) = 0\). Comparing (57) with (12) one will find that these points of the spectrum coincide exactly \((k \equiv m + n)\). But at closer examination all ”wave functions” of the form (56) with \(n \geq 1\) are unnormalizable due to the singular behaviour of the supercharges (31) at \(x_1, x_2 \to 0\). Thus the seeming \((k + 1)\)–fold degeneracy of \(k\)–th energy level in (57) is spurious since only one of the solutions of Schrödinger equation (namely, the linear combination of zero modes \(\Omega_n\)) is normalizable. Therefore, in contrast to the method of \(SUSY\)–separation of variables, the method of 2D shape invariance is powerless to give normalizable shape invariance chains of wave functions for the 2D Pöschl-Teller potential.

### 5. Two-dimensional intertwining relations of more than second order

In this Section we will imply equivalence of \(H^{(0)}\) and \(\mathcal{H}^{(0)}\) up to a change of variables for the new construction. Due to (55), the intertwining relation \(\mathcal{H}^{(0)} Q^- = Q^- \tilde{H}^{(0)}\) can be rewritten
as:

\[ H^{(0)} \tilde{Q}^- = \tilde{Q}^- \tilde{H}^{(0)}, \]

where \( \tilde{Q}^\pm(x_1, x_2) = Q^\pm(x_+, x_-) \) and \( \tilde{H}^{(0)}(x_1, x_2) = \tilde{H}^{(0)}(x_+, x_-) \). Comparing it with intertwining relations for the pair \( (H^{(0)}, \tilde{H}^{(0)}) \), one can conclude, that \( H^{(0)} \) has two different superpartners:

\[ \tilde{H}^{(0)} \xleftarrow{\tilde{Q}^\pm} H^{(0)} \xrightarrow{Q^\pm} \tilde{H}^{(0)}, \tag{58} \]

intertwined by different supercharges. Therefore, the Hamiltonians \( \tilde{H}^{(0)} \) and \( \tilde{H}^{(0)} \) can be considered as superpartners intertwined by the fourth order operators \( \tilde{Q}^+Q^- \). This pair does not obey the shape-invariance property.

Because, the Hamiltonian \( \tilde{H}^{(0)} \) is shape-invariant, one can develop the construction (57):

\[ \tilde{H}^{(0)}(a_0) \xleftarrow{\tilde{Q}^\pm(a_0)} H^{(0)}(a_0) \xrightarrow{Q^\pm(a_0)} \tilde{H}^{(0)}(a_0) = H^{(0)}(a_1) + R(a_0) \xleftarrow{\tilde{Q}^\pm(a_1)} \tilde{H}^{(0)}(a_1) + R(a_0), \tag{59} \]

where \( a_0 = B, \quad a_1 = B - \alpha \) (see Subsection 4.2.). The outermost operators in (59) are intertwined by sixth order supercharges according to:

\[ \tilde{H}^{(0)}(a_0) \left[ \tilde{Q}^+(a_0)Q^-(a_0)\tilde{Q}^-(a_1) \right] = \left[ \tilde{Q}^+(a_0)Q^-(a_0)\tilde{Q}^-(a_1) \right] \left[ \tilde{H}^{(0)}(a_1) + R(a_0) \right], \tag{60} \]

and contrary to the previous, fourth order, case, obey the shape-invariance property. Thus one can continue the construction of the spectrum of \( \tilde{H}^{(0)} \).

**Appendix. The general solution of the functional equation**

Applying operator \( (\partial_1^2 - \partial_2^2) \) to both sides of (22), one has:

\[
2 \left( \frac{\tilde{\phi}'_1(x_1)}{\phi_1(x_1)} \partial_1 - \frac{\tilde{\phi}'_2(x_2)}{\phi_2(x_2)} \partial_2 \right) (\phi_-(x_-) - \phi_+(x_+)) = \left( \frac{\phi''_2(x_2)}{\phi_2(x_2)} - \frac{\phi''_1(x_1)}{\phi_1(x_1)} \right) (\phi_-(x_-) - \phi_+(x_+)),
\]

(A1)

where the notation \( \tilde{\phi}_1(x_1) = 1/\phi_1(x_1) \) was used. The general solution of (A1) is:

\[
\phi_-(x_-) - \phi_+(x_+) = (\tilde{\phi}'_1(x_1)\phi'_2(x_2))^{-1/2} \Lambda \left( \int \frac{\phi'_1(x_1)}{\phi'_1(x_1)} dx_1 + \int \frac{\phi'_2(x_2)}{\phi'_2(x_2)} dx_2 \right)
= \phi_1(x_1)(\phi'_1(x_1)\phi'_2(x_2))^{-1/2} \Lambda \left( \int \frac{\phi_1(x_1)}{\phi'_1(x_1)} dx_1 - \int \frac{\phi_2(x_2)}{\phi'_2(x_2)} dx_2 \right), \tag{A2}
\]
where \( \Lambda \) is an arbitrary function. The corresponding expression for \( (\phi_-(x_-) + \phi_+(x_+)) \) can be obtained using initial Eq. (22):

\[
\phi_-(x_-) + \phi_+(x_+) = \phi_2(x_2)/(\phi'_1(x_1)\phi'_2(x_2))^{-1/2}\Lambda \left( \int \frac{\phi_1(x_1)}{\phi'_1(x_1)} dx_1 - \int \frac{\phi_2(x_2)}{\phi'_2(x_2)} dx_2 \right). 
\] (A3)

Expressions for \( \phi_\pm \) in terms of \( \Lambda \) should depend on proper argument, i.e. \( \partial_\pm \phi_\pm = 0 \), leading to additional constraints for the function \( \Lambda \):

\[
\frac{1}{2} \left( \frac{\phi''_2(x_2)}{\phi_2(x_2)} + \frac{\phi''_1(x_1)}{\phi_1(x_1)} \right) \Lambda = \left( \frac{1}{\phi'_1(x_1)} - \frac{1}{\phi'_2(x_2)} \right) \Lambda', 
\] (A4)

\[
\left( \phi'_1(x_1) + \phi'_2(x_2) - \frac{\phi_1(x_1)\phi''_1(x_1)}{2\phi'_1(x_1)} - \frac{\phi_2(x_2)\phi''_2(x_2)}{2\phi'_2(x_2)} \right) \Lambda = \left( \frac{\phi''_2(x_2)}{\phi_2(x_2)} - \frac{\phi''_1(x_1)}{\phi'_1(x_1)} \right) \Lambda'. 
\] (A5)

Its trivial solution \( \Lambda \equiv 0 \) gives \( \phi_+ = \phi_- = 0 \), \( \phi_{1,2} \) - arbitrary, and the potentials \( (23) \) and \( (24) \) are amenable to separation of variables.

Otherwise one can exclude \( \Lambda \) from (A4)-(A5):

\[
\frac{\phi''_1(x_1)\phi''_2(x_2)}{\phi_1(x_1)} - \frac{\phi''_2(x_2)\phi''_1(x_1)}{\phi_2(x_2)} = 2\phi''_2(x_2) - 2\phi''_1(x_1) + \phi_1(x_1)\phi''_1(x_1) - \phi_2(x_2)\phi''_2(x_2).
\] (A6)

Though there is no separation of variables in (A6), it will appear after applying the operator \( \partial_1\partial_2 \), so that:

\[
\frac{(\phi''_1/\phi_1)'}{(\phi''_2/\phi_2)'} = 2a = \text{const.} 
\] (A7)

Integrating, multiplying by \( \phi'_{1,2} \), integrating again and taking into account (A6), one obtains the general solution of (22) in the form:

\[
\phi^2_{1,2} = a\phi^4_{1,2} + b\phi^2_{1,2} + c; \quad x = \pm \int \frac{d\phi_1}{\sqrt{a\phi^4_1 + b\phi^2_1 + c}}; \quad b, c = \text{const.} 
\] (A8)

**Acknowledgements**

Authors are indebted to A.A. Andrianov for useful discussions, and especially to D.N. Nishnianidze for careful reading of manuscript and for derivation of the general solution of Eq. (22). P.V. is grateful to International Centre of Fundamental Physics in Moscow and Non-profit Foundation "Dynasty" for financial support. This work was partially supported by the grant No.05-01-01090 of the Russian Foundation for Basic Research.
References

[1] Witten E. 1981 Nucl. Phys. B188 513.

[2] Junker G., Supersymmetric Methods in Quantum and Statistical Physics, Springer, Berlin, 1996;
Cooper F., Khare A., Sukhatme U., Phys.Rep. 251 (1995) 268;
Bagchi B.K., Supersymmetry in Quantum and Classical Mechanics, Chapman and Hall/CRC, Boca Raton, 2001.

[3] Infeld L. and Hull T.E. 1951 Rev. Mod. Phys. 23 21.

[4] Gendenstein L.E. 1983 JETP Lett. 38 356.

[5] Andrianov A.A., Cannata F., Dedonder J.-P. and Ioffe M.V. 1995 Int.J.Mod.Phys. A10 2683.

[6] Andrianov A.A., Ioffe M.V. and Spiridonov V.P. 1993 Phys.Lett. A174 273;
Bagrov V.G. and Samsonov B.F. 1995 Theor.Math.Phys. 104 1051;
Samsonov B.F. 1996 Mod.Phys.Lett. A11 1563;
Fernandez D.J. 1997 Int.J.Mod.Phys. A12 171;
Klishevich S. and Plyushchay M. 1999 Mod. Phys. Lett. A14 2739;
Plyushchay M. 2000 Int. J. Mod. Phys. A15; 3679
Fernandez D.J., Negro J. and Nieto L.M. 2000 Phys.Lett. A275 338;
Aoyama H., Sato M. and Tanaka T. 2001 Phys.Lett. B503 423;
Aoyama H., Sato M. and Tanaka T. 2001 Nucl.Phys. B619 105;
Aoyama H., Nakayama N., Sato M. and Tanaka T. 2001 Phys.Lett. B519 260;
Klishevich S. and Plyushchay M. 2001 Nucl.Phys. B606[PM] 583;
Sasaki R. and Takasaki K. 2001 J.Phys.A:Math.Gen. A34 9533;
Andrianov A.A. and Sokolov A.V. 2003 Nucl.Phys. B660 25;
Andrianov A.A. and Cannata F. 2004 J.Phys.A:Math.Gen. A37 10297;
Ioffe M.V. and Nishnianidze D.N. 2004 Phys.Lett.A A327 425.

[7] Andrianov A.A., Borisov N.V. and Ioffe M.V. 1984 JETP Lett. 39 93;
Andrianov A.A., Borisov N.V. and Ioffe M.V. 1984 Phys. Lett. 105A 19.
[8] Andrianov A.A., Borisov N.V., Ioffe M.V. and Eides M.I. 1985 Phys.Lett. 109A 143.

[9] Andrianov A.A., Borisov N.V., Ioffe M.V. 1986 Phys.Lett. B181 141;
    Andrianov A.A., Borisov N.V., Ioffe M.V. 1988 Theor.Math.Phys. 72 748 [Translated from Teor.Mat.Fiz. 1987 72 97];
    Andrianov A.A. and Ioffe M.V. 1988 Phys. Lett. B205 507;
    Cannata F. and Ioffe M.V. 2001 J. Phys. A: Math. Gen. 34 1129;
    Ioffe M.V. and Neelov A.I. 2003 J.Phys.A:Math.Gen. A36 2493.

[10] Andrianov A.A., Ioffe M.V. and Nishnianidze D.N. 1995 Phys.Lett., A201 103.

[11] Andrianov A.A., Ioffe M.V. and Nishnianidze D.N. 1995 Theor.Math.Phys. 104 1129.

[12] Andrianov A.A., Ioffe M.V. and Nishnianidze D.N. 1996 solv-int/9605007; Published in: 1995 Zapiski Nauch. Seminarov POMI RAN ed.L.Faddeev et.al. 224 68 (In Russian);
    Translation in: Problems in QFT and Statistical Physics ed.L.D.Faddeev et.al. 13.

[13] Andrianov A.A., Ioffe M.V. and Nishnianidze D.N. 1999 J.Phys.A:Math.Gen. A32 4641.

[14] Cannata F., Ioffe M.V. and Nishnianidze D.N. 2002 J. Phys. A: Math. Gen 35 1389.

[15] Cannata F., Ioffe M.V. and Nishnianidze D.N. 2003 Phys.Lett. A310 344.

[16] Ioffe M.V. 2004 J.Phys.A:Math.Gen. 37 10363.

[17] Miller W.,Jr. 1977 Symmetry and Separation of Variables (Addison-Wesley Publishing Company, London).

[18] Bateman H. and Erdelyi E. 1953-55 Higher transcendental functions. v.1 - 3. (New-York: McGraw-Hill).

[19] Tanaka T. 2004 Ann.Phys.(NY) 309 239.

[20] Turbiner A.V. 1988 Commun. Math. Phys. 118 467;
    Ushveridze A.G. 1989 Sov. J. Part. Nucl. 20 504.

[21] Landau L. and Lifshitz E. 1965 Quantum Mechanics (Non-relativistic Theory) 2nd Edition (London: Pergamon).