Solution Of Tensor Complementarity Problem Using Homotopy Function

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Abstract

The paper aims to propose a suitable method in finding the solution of tensor complementarity problem. The tensor complementarity problem is a subclass of nonlinear complementarity problems for which the involved function is defined by a tensor. We propose a new homotopy function with smooth and bounded homotopy path to obtain solution of the tensor complementarity problem under some conditions. A homotopy continuation method is developed based on the proposed homotopy function. Several numerical examples are provided to show the effectiveness of the proposed homotopy continuation method.

Keywords: Tensor complementarity problem, homotopy function, homotopy path, bounded smooth curve, semipositive tensor, linear complementarity problem.

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1 Introduction

During last several decades, researchers are of keen interest to study the complementarity theory finding solution of linear complementarity problem. The linear programming problem, linear fractional programming problem, convex quadratic programming problem, bimatrix game problem and quadratic multiobjective programming problem are some of the optimization problems which can be modelled to be linear complementarity problem. For details see [4], [22], [28] and [29] and references cited therein. Several matrix classes are introduced for the purpose of study of theoretical properties, applications and solution methods of complementarity problem.
For details see [12], [13], [19], [18], [27], [26], [3] and [24] and references cited therein. The linear complementary problem has a key role to obtain the value vector and optimal stationary strategies for discounted and undiscounted zero-sum stochastic games. For details see [21], [31] and [23]. The complementarity problem establishes a vital connection with multiobjective programming problem for its weighted problem and the solution point [20]. The complementarity problems are considered with respect to principal pivot transforms and pivotal method to its solution point of view. See [2], [30], [5] and [25]. The linear complementarity problem arising from a free boundary problem can be reformulated as a fixed-point equation. Zhang [40] presented a modified modulus-based multigrid method to solve this fixed-point equation. Recently there has been renewed interest in finding the solution of tensor complementarity problem. The tensor complementarity problem is considered to be a subclass of nonlinear complementarity problems with special class of polynomials using tensor. The function involved in the tensor complementarity problem is more complex than that of the function involving matrix. It is of interest to develop numerical methods to solve tensor complementarity problem. The tensor complementarity problem has several applications. For details see [6], [39], [9], [38], [34], [35], [17], [16], [11], [10].

The purpose of the study is to develop a suitable homotopy function to find the solution of tensor complementarity problem. The basic idea of homotopy method is to construct a homotopy continuation path from the auxiliary mapping \( g \) to the object mapping \( f \). Suppose the given problem is to find a root of the non-linear equation \( f(x) = 0 \) and suppose \( g(x) = 0 \) is an auxiliary equation with \( g(x_0) = 0 \). Then the homotopy function \( H : \mathbb{R}^{n+1} \rightarrow \mathbb{R}^n \) can be defined as \( H(x, \mu) = (1 - \mu)f(x) + \mu g(x) \), \( 0 \leq \mu \leq 1 \). Then we consider the homotopy equation \( H(x, \mu) = 0 \), where \((x_0, 1)\) is a known solution of the homotopy equation. Our aim is to find the solution of the equation \( f(x) = 0 \) from the known solution of \( g(x) = 0 \) by solving the homotopy equation \( H(x, \mu) = 0 \) varying the values of \( \mu \) from 1 to 0. Kojima et al showed that under some conditions nonlinear complementarity problem can be solvable by homotopy continuation method. For details see [14], [15], [36]. Han [9] introduced a homotopy continuation method for solving tensor complementarity problems. Han [8], Yan et al. [37] solve multilinear systems with strong completely positive tensors by homotopy method.

The paper is organized as follows. Section 2 presents some basic notations and results. In section 3, we propose a new homotopy function to find the solution of tensor complementarity problem. We construct a smooth and bounded homotopy path obtaining the solution of the tensor complementarity problem under some conditions as the homotopy parameter \( \mu \) tends to 0. We prove necessary and sufficient conditions to obtain the solution of tensor complementarity problem from the solution of the homotopy equation. We also find the sign of the positive tangent direction of the homotopy path. We use a modified interior-point bounded homotopy path algorithm for solving the linear complementarity problem in section 4. Finally, in section 4, some numerical results are given to illustrate the effectiveness of the homotopy function.
2 Preliminaries

We denote \([n] = \{1, 2, \cdots, n\}\). We denote the \(n\) dimensional real space by \(\mathbb{R}^n\) where \(\mathbb{R}_n^+\) and \(\mathbb{R}_n^{++}\) denote the nonnegative and positive orthant of \(\mathbb{R}^n\). Any vector \(x \in \mathbb{R}^n\) is a column vector and \(x^t\) denotes the row transpose of \(x\). \(e\) denotes the vector of all 1. An \(m\)th order \(n\)th dimensional tensor \(\mathcal{A} = (a_{i_1i_2\cdots i_m})\) is a multi-array of entries \(a_{i_1i_2\cdots i_m}\), where \(i_j \in [n]\) for \(j \in [m]\). The set of all \(m\)th order, \(n\)th dimensional tensors is denoted by \(T_{m,n}\). Here we consider vectors, matrices and tensors with real entries. For \(\mathcal{A} \in T_{m,n}\) and \(x \in \mathbb{R}^n\), \(\mathcal{A}x^{m-2} \in \mathbb{R}^{n\times n}\) is a matrix defined by \((\mathcal{A}x^{m-2})_{ij} = \sum_{i_3, \cdots, i_m=1}^n a_{i_1i_2i_3\cdots i_m}x_{i_3}x_{i_4}\cdots x_{i_m}\) for all \(i, j \in [n]\). \(\mathcal{A}x^{m-1} \in \mathbb{R}^n\) is a vector defined by \((\mathcal{A}x^{m-1})_i = \sum_{i_3, \cdots, i_m=1}^n a_{i_1i_2i_3\cdots i_m}x_{i_3}x_{i_4}\cdots x_{i_m}\) for all \(i \in [n]\) and \(\mathcal{A}x^m \in \mathbb{R}\) is a scalar defined by \(x^T \mathcal{A}x^{m-1} = \mathcal{A}x^m = \sum_{i_1, i_2, i_3, \cdots, i_m=1}^n a_{i_1i_2i_3\cdots i_m}x_{i_1}x_{i_2}\cdots x_{i_m}\).

Now we consider the tensor \(\mathcal{A} \in T_{m,n}\) and \(q \in \mathbb{R}^n\) then the problem is to find \(x\) such that
\[
x \geq 0, \quad \omega = \mathcal{A}x^{m-1} + q \geq 0, \quad \text{and} \quad x^T(\mathcal{A}x^{m-1} + q) = 0 \tag{2.1}\]
is called a tensor complementarity problem, denoted by the TCP\((q, \mathcal{A})\).

We define the partially symmetrized tensor of a tensor \(\mathcal{A} = (a_{i_1i_2\cdots i_m})\) with respect to the indices \(i_2, i_3, \cdots, i_m\) is defined by \(\hat{\mathcal{A}} = \hat{\mathcal{A}}_{i_1i_2\cdots i_m} = \frac{1}{(m-1)!} \sum_{\pi} a_{i_1 \pi(i_2i_3\cdots i_m)}\), where the sum is over all the permutations \(\pi(i_2i_3\cdots i_m)\). For details see \([5]\). The partial derivative matrix of \(\mathcal{A}x^{m-1}\) with respect to \(x\) is \(D_x \mathcal{A}x^{m-1} = (m-1)\hat{\mathcal{A}}x^{m-2}\). Note that \(\mathcal{A}x^{m-1} = \hat{\mathcal{A}}x^{m-1} \forall x \in \mathbb{R}^n\).

Now we state some results which will be required in the next section.

**Lemma 2.1:** (Generalizations of Sard’s Theorem\([1]\)) Let \(U \subset \mathbb{R}^n\) be an open set and \(f : \mathbb{R}^n \to \mathbb{R}^p\) be smooth. We say \(y \in \mathbb{R}^p\) is a regular value for \(f\) if \(\text{Range}Df(x) = \mathbb{R}^p \forall x \in f^{-1}(y)\), where \(Df(x)\) denotes the \(n\times p\) matrix of partial derivatives of \(f(x)\).

**Lemma 2.2:** (Parameterized Sard Theorem \([33]\)) Let \(V \subset \mathbb{R}^n, U \subset \mathbb{R}^m\) be open sets, and let \(\phi : V \times U \to \mathbb{R}^k\) be a \(C^\alpha\) mapping, where \(\alpha > \max\{0, m-k\}\). If \(0 \in \mathbb{R}^k\) is a regular value of \(\phi\), then for almost all \(a \in V, 0\) is a regular value of \(\phi_a = \phi(a, .)\).

**Lemma 2.3:** (The inverse image theorem \([33]\)) Let \(\phi : U \subset \mathbb{R}^n \to \mathbb{R}^p\) be \(C^\alpha\) mapping, where \(\alpha > \max\{0, n-p\}\). Then \(\phi^{-1}(0)\) consists of some \((n-p)\)-dimensional \(C^\alpha\) manifolds.

**Lemma 2.4:** (Classification theorem of one-dimensional smooth manifold \([11]\)) One-dimensional smooth manifold is diffeomorphic to a unit circle or a unit interval.

3 Main Results

In 2019 Han \([9]\) proposed the following homotopy function to solve the TCP\((\mathcal{A}, q)\) where \(\mathcal{A}\) is a strong strictly semipositive tensor. We begin by the followings:
We have that 0 is a regular value of\(\text{triangular matrix with nonzero diagonals}\). Now \(\det(\text{of } H)\) of a nonempty set. It is clear that the set (0 \(\text{of } H\)) is bounded. So there exists a sequence of \(\text{starting from } (v^{(0)}, 1)\).

\[H(v, v^{(0)}, \mu) = \begin{bmatrix} (1 - \mu)(w - z_1 + (m - 1)(\hat{A} x^{m-2})^T(x - z_2)) + \mu(x - x^{(0)}) \\
Z_1 x - \mu Z_1^{(0)} x^{(0)} \\
Z_2 w - \mu Z_2^{(0)} w^{(0)} + (1 - \mu)X w \\
w - (1 - \mu)(Ax^{m-1} + q) - \mu w^{(0)} \end{bmatrix} = 0\] (3.1)

where \(\hat{A}\) is partially symmetrize tensor, \(e = [1, 1, \ldots, 1]^T\); \(X = \text{diag}(x)\); \(Z_1 = \text{diag}(z_1)\); \(Z_2 = \text{diag}(z_2)\); \(Z_1^{(0)} = \text{diag}(z_1^{(0)})\); \(Z_2^{(0)} = \text{diag}(z_2^{(0)})\); \(v = (x, w, z_1, z_2) \in \mathcal{F}_1\); \(v^{(0)} = (x^{(0)}, w^{(0)}, z_1^{(0)}, z_2^{(0)}) \in \mathcal{F}_1\); \(\mu \in (0, 1]\).

We prove that the proposed homotopy function contains a smooth and bounded path.

**Theorem 3.1:** For almost all initial points \(v^{(0)} \in \mathcal{F}_1\), 0 is a regular value of the homotopy function \(H : \mathbb{R}^n \times (0, 1] \to \mathbb{R}^n\) and the zero point set \(H_{v^{(0)}}^{-1}(0) = \{(v, \mu) \in \mathcal{F}_1 \times (0, 1] : H(v, v^{(0)}, \mu) \equiv 0\}\) contains a smooth curve \(\Gamma_v^{(0)}\) starting from \((v^{(0)}, 1)\).

**Proof.** The jacobian matrix of the above homotopy function \(H(v, v^{(0)}, \mu)\) is denoted by \(DH(v, v^{(0)}, \mu)\) and we have \(DH(v, v^{(0)}, \mu) = \left[ \frac{\partial H(v,v^{(0)},\mu)}{\partial v} \frac{\partial H(v,v^{(0)},\mu)}{\partial v^{(0)}} \frac{\partial H(v,v^{(0)},\mu)}{\partial \mu} \right] \). For all \(v^{(0)} \in \mathcal{F}_1\) and \(\mu \in (0, 1]\), we have

\[
\frac{\partial H(v,v^{(0)},\mu)}{\partial v^{(0)}} = \begin{bmatrix}
-\mu I & 0 & 0 \\
-\mu Z_1^{(0)} & 0 & -\mu X^{(0)} \\
0 & -\mu Z_2^{(0)} & 0 \\
0 & 0 & -\mu I \\
\end{bmatrix} = L_1;
\]

where \(W^{(0)} = \text{diag}(w^{(0)})\), \(X^{(0)} = \text{diag}(x^{(0)})\), \(Z_1^{(0)} = \text{diag}(z_1^{(0)})\), \(Z_2^{(0)} = \text{diag}(z_2^{(0)})\).

Note that after some elementary row operations the block matrix \(L_1\) becomes a lower triangular matrix with nonzero diagonals. Now \(\det(\frac{\partial H}{\partial v^{(0)}}) = \mu^n \prod_{i=1}^n x_i^{(0)} w_i^{(0)} \neq 0\) for \(\mu \in (0, 1]\). Thus \(DH(v, v^{(0)}, \mu)\) is of full row rank. Therefore, 0 is a regular value of \(H(v, v^{(0)}, \mu)\). By Lemmas 2.2 and 2.3 for almost all \(v^{(0)} \in \mathcal{F}_1\), 0 is a regular value of \(H(v, v^{(0)}, \mu)\) and \(H_{v^{(0)}}^{-1}(0)\) consists of some smooth curves and \(H(v^{(0)}, v^{(0)}, 1) = 0\). Hence there must be a smooth curve \(\Gamma_v^{(0)}\) starting from \((v^{(0)}, 1)\). \(\square\)

**Theorem 3.2:** Let \(\mathcal{F}_1\) be a nonempty set. Assume that there exists a sequence of points \(\{m_i\} \subset \Gamma_v^{(0)} \subset \mathcal{F}_1 \times (0, 1]\), where \(m_i = (x^k, w^k, z_1^i, z_2^k, \mu^k)\) such that \(\|x^k\| < \infty\) as \(k \to \infty\) and \(\|z_2^k\| < \infty\) as \(k \to \infty\). For a given \(v^{(0)} \in \mathcal{F}_1\), if 0 is a regular value of \(H(v, v^{(0)}, \mu)\), then \(\Gamma_v^{(0)}\) is a bounded curve in \(\mathcal{F}_1 \times (0, 1]\).

**Proof.** We have that 0 is a regular value of \(H(v, v^{(0)}, \mu)\) by theorem 3.1 and \(\mathcal{F}_1\) be a nonempty set. It is clear that the set \((0, 1]\) is bounded. So there exists a sequence of
Let \( z \in \mathbb{R}^k \) be an unbounded curve. Then there exists a sequence of points \( z_k \in \Gamma_1 \) such that \( \lim_{k \to \infty} z_k = z_0 \). Note that the boundedness of the sequence \( \{z_k\} \) guarantees the boundedness of the sequence \( \{w_k\} \). By contradiction we assume that \( \Gamma_1(0,1) \) is an unbounded curve. Then there exists a sequence of points \( \{m_k\} \), where \( m_k = (v_k, \mu_k) \in \Gamma_1(0) \) such that \( \|m_k\|_1 \to \infty \). Assume that \( \|z_k\|_1 \to \infty \) as \( k \to \infty \). Since \( \Gamma_1(0,1) \subset H^{-1}(0,1) \), we have

\[
(1 - \mu^k)[w_k - z_k^k] + (m - 1)(\mathcal{A}(x)^{m-2})^T(x_k^k - z_2^k) + \mu^k(x_k - x^{(0)}) = 0 \quad (3.2)
\]

\[
Z_1^k x_k - \mu^k Z_1(0)^k x^{(0)} = 0 \quad (3.3)
\]

\[
Z_2^k w_k - \mu^k Z_2(0)^k w^{(0)} + (1 - \mu^k)X w_k = 0 \quad (3.4)
\]

\[
w_k - (1 - \mu^k)(\mathcal{A}(x)^m_q + q - \mu w_k w^{(0)} = 0 \quad (3.5)
\]

where \( Z_1^k = \text{diag}(z^k_1) \), \( X = \text{diag}(x_k) \), \( W_k = \text{diag}(w_k) \) and \( Z_2^k = \text{diag}(z^k_2) \).

Let \( \bar{\mu} \in [0,1] \), \( \|z_k^k\| = \infty \) and \( \|z_k^k\| < \infty \) as \( k \to \infty \). Then \( \exists i \in \{1, 2, \ldots, n\} \) such that \( z_k^i \to \infty \) as \( k \to \infty \). Let \( I_1 = \{i \in \{1, 2, \ldots, n\} : \lim_{k \to \infty} z_k^i = \infty \} \). For \( \bar{\mu} \in [0,1] \) and \( i \in I_1 \), we obtain from equation (3.2)

\[
(1 - \mu^k)[w_k^i - z_k^{i_1} + (m - 1)(\mathcal{A}(x)^m_q)^T(x_k^i - z_2^i)] + \mu^k(x_k^i - x^{(0)}) = 0
\]

\[
\Rightarrow (1 - \mu^k)z_k^{i_1} = (1 - \mu^k)[w_k^i + (m - 1)(\mathcal{A}(x)^m_q)^T(x_k^i - z_2^i)] + \mu^k(x_k^i - x^{(0)})
\]

\[
\Rightarrow z_k^{i_1} = [w_k^i + (m - 1)(\mathcal{A}(x)^m_q)^T(x_k^i - z_2^i)] + \mu^k(x_k^i - x^{(0)})
\]

As \( k \to \infty \) right hand side is bounded, but left hand side is unbounded. It contradicts that \( \|z_k^i\| = \infty \).

When \( \bar{\mu} = 1 \), from equation (3.3) we obtain, \( x_k^i = \frac{\mu^k z_k^{i_1}}{z_k^{i_1}} \) for \( i \in I_1 \). As \( k \to \infty \), \( x_k^i \to 0 \).

Again from equation (3.2) we obtain \( x_k^{(0)} = \frac{(1 - \mu^k)[w_k^i + (m - 1)(\mathcal{A}(x)^m_q)^T(x_k^i - z_2^i)] + x_k^i}{\mu^k} \) for \( i \in I_1 \). As \( k \to \infty \), we have \( x_k^{(0)} = - \lim_{k \to \infty} \frac{(1 - \mu^k)z_k^{i_1}}{\mu^k} \leq 0 \). It contradicts that \( \|z_k^{i_1}\| = \infty \). So \( \Gamma_1(0) \) is a bounded curve in \( \mathcal{F}_1 \times (0,1) \).

Therefore the boundedness of the sequences \( \{x_k\} \) and \( \{z_k^2\} \) guarantee the boundedness of the sequence \( \{z_k^1\} \), i.e. the boundedness of the sequence \( \{m_k\} \). Now we prove the boundedness of the sequences \( \{x_k\} \) and \( \{z_k^2\} \).

**Theorem 3.3**: For a given \( v^{(0)} \in \mathcal{F}_1 \), if 0 is a regular value of \( H(v, v^{(0)}, \mu) \), then \( \Gamma_v^{(0)} \) is a bounded curve in \( \mathcal{F}_1 \times (0,1) \).

**Proof.** Suppose the solution set \( \Gamma_v^{(0)} \) is unbounded for \( \mu \in [0,1] \). Then there exists a sequence of points \( \{m_k\} \subset \Gamma_v^{(0)} \subset \mathcal{F}_1 \times (0,1) \), where \( m_k = (v_k, \mu_k) = (x_k^i, w_k^i, z_k^1, z_k^2, \mu_k) \) such that \( \lim_{k \to \infty} \mu_k = \bar{\mu} \in [0,1] \). There also exist \( (\xi, \zeta, \eta, \sigma) \in R^m_v \) such that \( e^T \xi = 1 \).

Now we consider the following two cases.

**Case 1**: \( \|z_k^1\| < \infty \) as \( k \to \infty \). Since the solution set \( \Gamma_v^{(0)} \) is unbounded, we consider the following two subcases.
Subcase (i) $\lim_{k \to \infty} e^T x^k = \infty$:

Let $\lim_{k \to \infty} \frac{x^k}{x} = \xi \geq 0$, $\lim_{k \to \infty} \frac{w^k}{w} = \zeta \geq 0$ and $\lim_{k \to \infty} \frac{z^k}{z} = \eta \geq 0$. So it is clear that $e^T \xi = 1$. Multiplying $(x^k)^T$ in both sides of (3.2) and taking limit $k \to \infty$ from equation (3.2), we write

$$(1 - \mu)(x^k)^T w^k - (x^k)^T z^k + (m-1)(x^k)^T (\hat{A}(x^k)^{m-2}) (x^k - z^k) + \mu (x^k)^T (x^k - x(0)) = 0.$$  

(3.6)

Dividing the equations 3.6, 3.3 and 3.4 by $(x^k)^T$ and taking limit $k \to \infty$ and dividing by $(e^T x^k)^2$ and taking limit $k \to \infty$ from equation (3.5) we write

$$(1 - \bar{\mu})[\xi^T \zeta - \xi^T \eta + (m-1)(\hat{A} \xi^m)^T \xi] + \bar{\mu} \xi^T \xi = 0 \quad (3.7)$$

$$\xi_i \eta_i = 0 \forall i$$  

(3.8)

$$(1 - \bar{\mu}) \xi_i \zeta_i = 0 \forall i.$$  

(3.9)

$$\zeta - (1 - \bar{\mu}) \hat{A} \xi^m = 0.$$  

(3.10)

From equation (3.10) $\zeta = (1 - \bar{\mu}) \hat{A} \xi^m$. Now multiplying $\xi^T$ in both sides we obtain

$$\xi^T \zeta = (1 - \bar{\mu}) \xi^T \hat{A} \xi^m = (1 - \bar{\mu}) \xi^T \hat{A} \xi^m = (\hat{A} \xi^m)^T \xi = \frac{1}{(1 - \bar{\mu})} \xi^T \xi.$$  

Hence from the equation (3.7), we obtain

$$(1 - \bar{\mu})(m-1)(\hat{A} \xi^m)^T \xi + \bar{\mu} \xi^T \xi = 0 \implies \xi^T \xi = -\frac{\bar{\mu}}{(m-1)} (\hat{A} \xi^m)^T \xi \leq 0.$$

for $\bar{\mu} \in [0, 1]$. Specifically for $\bar{\mu} \in (0, 1)$, $\xi^T \xi < 0$, contradicts that $\xi, \zeta \geq 0$. Hence the solution set $\Gamma_v^{(0)}$ is bounded for $\bar{\mu} \in (0, 1)$. Note that for $\bar{\mu} = 0, \xi^T \xi = 0$ and for $\bar{\mu} = 1, \xi = 0$. This contradicts that $e^T \xi = 1$. Therefore the solution set $\Gamma_v^{(0)}$ is bounded for $\bar{\mu} \in [0, 1]$.

Subcase (ii) $\lim_{k \to \infty} (1 - \mu^k) e^T x^k = \infty$:

Let $\lim_{k \to \infty} \frac{(1 - \mu^k) x^k}{(1 - \mu^k) x^k} = \zeta' \geq 0$. Then $e^T \zeta' = 1$. Let $\lim_{k \to \infty} \frac{w^k}{(1 - \mu^k) x^k} = \zeta' \geq 0$, $\lim_{k \to \infty} \frac{w^k}{(1 - \mu^k) x^k} = \eta' \geq 0$. Then multiplying the equations 3.6, 3.3 with $(1 - \mu^k)$ and dividing by $((1 - \mu^k) e^T x^k)^2$ and dividing the equations 3.5 and 3.3 by $(1 - \mu^k) e^T x^k$ and $((1 - \mu^k) e^T x^k)^2$ respectively and taking limit $k \to \infty$, we write

$$(1 - \bar{\mu})[(\zeta')^T \zeta' - (\zeta')^T \eta'] + (m-1)(\hat{A} \xi^m)^T \zeta' + \frac{\bar{\mu}}{1 - \bar{\mu}}(\xi')^T \zeta' = 0.$$  

(3.11)

$$\zeta_i' \eta_i' = 0 \forall i$$  

(3.12)

$$\zeta_i' \zeta_i' = 0 \forall i.$$  

(3.13)

$$\zeta' - \hat{A} \xi^m = 0.$$  

(3.14)

Using equations (3.12) 3.13 and 3.14 in the equation 3.11, we obtain

$$\zeta^T \xi' = -\frac{\bar{\mu}}{m-1(\bar{\mu})} (\zeta')^T \xi' \leq 0.$$  

Specifically for $\bar{\mu} \in (0, 1)$, $\zeta^T \xi < 0$, contradicts that $\xi, \zeta \geq 0$. Hence the solution set $\Gamma_v^{(0)}$ is bounded for $\bar{\mu} \in (0, 1)$. Note that for $\bar{\mu} = 0, \zeta^T \xi = 0 \implies \xi = 0$ and for $\bar{\mu} = 1, \xi = 0$. This contradicts that $e^T \xi = 1$. Therefore the solution set $\Gamma_v^{(0)}$ is bounded for $\bar{\mu} \in [0, 1]$. 

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Case 2: \( \lim_{{k \to \infty}} e^T x_k = \infty \). Since the solution set of \( \Gamma_v^{(0)} \) is unbounded we consider following two subcases.

Subcase (i) \( \lim_{{k \to \infty}} e^T x^k = \infty \):
Let \( \lim_{{k \to \infty}} \frac{x^k}{{e^T x^k}} = \xi \geq 0 \), \( \lim_{{k \to \infty}} \frac{w^k}{{e^T x^k}} = \zeta \geq 0 \), \( \lim_{{k \to \infty}} \frac{\mu^k}{{e^T x^k}} = \eta \geq 0 \), \( \lim_{{k \to \infty}} \frac{\sigma^k}{{e^T x^k}} = \sigma \geq 0 \). It is clear that \( e^T \xi = 1 \). Then dividing by \( e^T x^k \) and taking limit \( k \to \infty \) from equation 3.5 and dividing by \((e^T x^k)^2\) and taking limit \( k \to \infty \) from equations 3.6, 3.3 and 3.4 we write

\[
(1 - \bar{\mu})[(\xi)^T \zeta - (\xi)^T \eta + (m - 1)(\hat{A}\xi^{m-1})^T (\xi - \sigma)] + \bar{\mu}(\xi)^T \xi = 0 \quad (3.15)
\]

\[
\xi_i \eta_i = 0 \quad \forall \, i \quad (3.16)
\]

\[
\sigma_i \zeta_i + (1 - \bar{\mu}) \xi_i \zeta_i = 0 \quad \forall \, i \quad (3.17)
\]

\[
\zeta - (1 - \bar{\mu}) A \xi^{m-1} = 0 \quad (3.18)
\]

Using the equations 3.16, 3.17 and 3.18 from equation 3.15, we obtain

\[
(m - 1)\zeta^T (\xi - \sigma) - \zeta^T \sigma = -\bar{\mu}(\xi)^T \xi \implies (m - 1)\zeta^T \xi + (m - 1)(1 - \bar{\mu})\zeta^T \xi + ((1 - \bar{\mu}))\zeta^T \xi = -\bar{\mu}(\xi)^T \xi \implies \zeta^T \xi = \frac{-\bar{\mu}}{m - 1 + m(1 - \bar{\mu})} (\xi)^T \xi \leq 0.
\]

Specifically for \( \bar{\mu} \in (0, 1) \), \( \zeta^T \xi < 0 \), contradicts that \( \xi, \zeta \geq 0 \). Hence the solution set \( \Gamma_v^{(0)} \) is bounded for \( \bar{\mu} \in (0, 1) \). Note that for \( \bar{\mu} = 0, \zeta^T \xi = 0 \implies \xi = 0 \) and for \( \bar{\mu} = 1, \xi = 0 \). This contradicts that \( e^T \xi = 1 \). Therefore the solution set \( \Gamma_v^{(0)} \) is bounded for \( \bar{\mu} \in [0, 1] \).

Subcase (ii) \( \lim_{{k \to \infty}} (1 - \mu^k)e^T x^k = \infty \):
Let \( \lim_{{k \to \infty}} \frac{(1 - \mu^k)x^k}{(1 - \mu^k)e^T x^k} = \xi' \geq 0 \), \( \lim_{{k \to \infty}} \frac{\mu^k}{(1 - \mu^k)e^T x^k} = \eta' \geq 0 \), \( \lim_{{k \to \infty}} \frac{\sigma^k}{(1 - \mu^k)e^T x^k} = \sigma' \geq 0 \). Then multiplying the equations 3.6, 3.3 with \((1 - \mu^k)\) and dividing by \((1 - \mu^k)e^T x^k)^2\) and dividing the equations 3.5 and 3.4 by \((1 - \mu^k)e^T x^k\) and \((1 - \mu^k)e^T x^k)^2\) respectively and taking limit \( k \to \infty \), we obtain

\[
(1 - \bar{\mu})[(\xi')^T \zeta' - (\xi')^T \eta'] + (m - 1)(\hat{A}\xi^{m-1})^T (\xi' - \sigma') + \frac{\bar{\mu}}{1 - \bar{\mu}} (\xi')^T \xi' = 0 \quad (3.19)
\]

\[
\xi'_i \eta'_i = 0 \quad \forall \, i \quad (3.20)
\]

\[
\sigma'_i \zeta'_i + \xi'_i \zeta'_i = 0 \quad \forall \, i \quad (3.21)
\]

\[
\zeta' - \hat{A}\xi^{m-1} = 0 \quad (3.22)
\]

Using the equations 3.20, 3.21 and 3.22 from equation 3.19, we obtain

\[
(m - 1)((\xi')^T (\xi' - \sigma') + (1 - \bar{\mu})\xi'^T \zeta') = -\frac{\bar{\mu}}{1 - \bar{\mu}} (\xi')^T \xi' \implies (m - 1)((\xi')^T \xi' + (\zeta')^T \xi') + \frac{\bar{\mu}}{1 - \bar{\mu}} (\xi')^T \xi' \implies \zeta'^T \xi' = \frac{-\bar{\mu}}{2(m - 1)(1 - \mu^2)} (\xi')^T \xi' \leq 0,
\]
contradicts that \( \xi, \zeta \geq 0 \). Hence the solution set \( \Gamma_v^{(0)} \) is bounded for \( \bar{\mu} \in (0, 1) \). Note that for \( \bar{\mu} = 0, \zeta^T \xi = 0 \implies \xi = 0 \) and for \( \bar{\mu} = 1, \xi = 0 \). This contradicts that \( e^T \xi = 1 \). Therefore the solution set \( \Gamma_v^{(0)} \) is bounded for \( \bar{\mu} \in [0, 1] \).

Hence considering all the cases it is proved that the solution set \( \Gamma_v^{(0)} \) of the homotopy function 3.1 \( H(v, v^{(0)}, \mu) = 0 \) is bounded for \( \bar{\mu} \in [0, 1] \). For an initial point
\(v(0) \in \mathcal{F}_1\) we obtain a smooth bounded homotopy path which leads to the solution of homotopy function 3.1 as the parameter \(\mu \to 0\).

**THEOREM 3.4**: For \(v(0) = (x(0), w(0), z_1(0), z_2(0)) \in \mathcal{F}_1\), the homotopy continuation method finds a bounded smooth curve \(\Gamma_v(0) \subset \mathcal{F}_1 \times (0, 1]\) which starts from \((v(0), 1)\) and approaches the hyperplane at \(\mu = 0\). As \(\mu \to 0\), the limit set \(l \times \{0\} \subset \mathcal{F}_1 \times \{0\}\) of \(\Gamma_v(0)\) is nonempty and every point in \(l\) is a solution of the following system:

\[
\begin{align*}
  w - z_1 + (m - 1)(\hat{A}x^{m-2})^T(x - z_2) &= 0 \\
  Z_1x &= 0 \\
  Z_2w + Xw &= 0 \\
  w - (Ax^{m-1} + q) &= 0
\end{align*}
\]

(3.23)

**Proof.** Note that in view of lemma 2.4, \(\Gamma_v(0)\) is diffeomorphic to a unit circle or a unit interval \((0, 1]\). Since \(\frac{\partial H(v, w, 0)}{\partial v}\) is nonsingular, \(\Gamma_v(0)\) is diffeomorphic to a unit interval \((0, 1]\). \(\Gamma_v(0)\) is a bounded smooth curve by the theorem 3.1 and 3.3. Let \((\bar{v}, \bar{\mu})\) be a limit point of \(\Gamma_v(0)\). Now we consider the following four cases:

(i) \((\bar{v}, \bar{\mu}) \in \mathcal{F}_1 \times \{1\}\) : As the homotopy function \(H(v, 1) = 0\) has only one solution \(v(0) \in \mathcal{F}_1\), this case is impossible.

(ii) \((\bar{v}, \bar{\mu}) \in \partial \mathcal{F}_1 \times \{1\}\) : There exists a subsequence of \((v^k, \mu^k) \in \Gamma_v(0)\) such that \(x^k_i \to 0\) or \(w^k_i \to 0\) for \(i \subseteq \{1, 2, \cdots, n\}\). From the second and third equalities of the homotopy function 3.1 we have \(z_1^k \to \infty\) or \(z_2^k \to \infty\). Hence it contradicts the boundedness of the homotopy path by the theorem 3.3.

(iii) \((\bar{v}, \bar{\mu}) \in \partial \mathcal{F}_1 \times (0, 1]\) : Also impossible followed by the case (ii).

(iv) \((\bar{v}, \bar{\mu}) \in \mathcal{F}_1 \times \{0\}\) : The only possible case.

Hence \(\bar{v} = (\bar{x}, \bar{w}, \bar{z}_1, \bar{z}_2)\) is a solution of the system 3.23

\[
\begin{align*}
  w - z_1 + (m - 1)(\hat{A}x^{m-2})^T(x - z_2) &= 0 \\
  Z_1x &= 0 \\
  Z_2w + Xw &= 0 \\
  w - (Ax^{m-1} + q) &= 0
\end{align*}
\]

(3.23)

\(\blacksquare\)

**REMARK 3.1:** So, from the homotopy function 3.1 as \(\mu \to 0\) we get \(\bar{w} - \bar{z}_1 + (m - 1)(\hat{A}x^{m-2})^T(\bar{x} - \bar{z}_2) = 0, \bar{w} = \hat{A}x^{m-1} + q\) and \(\bar{z}_1, \bar{z}_2, \bar{w}_i = 0, \bar{x}_i \bar{w}_i = 0 \forall i \in \{1, 2, \cdots, n\}\). Hence \(\mu \to 0\) we get the solution of the homotopy function as well as TCP(\(A, q\)).

**REMARK 3.2:** We find the homotopy path \(\Gamma_v(0) \subset \mathcal{F}(m_1) \times (0, 1]\) from the initial point \((v(0), 1)\) until \(\mu \to 0\) and find the solution of the given complementarity problem 2.1 under some assumptions. Let \(s\) denote the arc length of \(\Gamma_v(0)\). We can parameterize the homotopy path \(\Gamma_v(0)\) for \(s\) in the following form

\[
H(v(s), \mu(s)) = 0, \ v(0) = v(0), \ \mu(0) = 1.
\]

(3.24)
Now differentiating \(3.23\) with respect to \(s\) we obtain the following system of ordinary differential equations with given initial values [7]

\[
H'(v(s), \mu(s)) \begin{bmatrix} \frac{dv}{ds} \\ \frac{d\mu}{ds} \end{bmatrix} = 0, \quad \|\begin{bmatrix} \frac{dv}{ds} \\ \frac{d\mu}{ds} \end{bmatrix}\| = 1, \quad v(0) = v(0), \quad \mu(0) = 1, \quad \frac{d\mu}{ds}(0) < 0, \quad (3.25)
\]

and the \(v\)-component of \((v(s), \mu(s))\) gives the solution of the complementarity problem for \(\mu(s) = 0\).

Now we use the modified homotopy continuation method to trace the homotopy path \(\Gamma_v^{(0)}\) numerically to find the solution of TCP\((q, A)\).

**Algorithm 1 Modified Homotopy Continuation Method**

**Step 0:** Initialize \((v^{(0)}, \mu_0)\) and a natural number \(m \in (0, 50)\). Set \(l_0 \in (0, 1)\). Choose \(\epsilon_2 \gg \epsilon_3 \gg \epsilon_1 > 0\) which are very small positive values.

**Step 1:** \(\tau^{(0)} = \xi^{(0)} = (\frac{1}{n_0}) \begin{bmatrix} s^{(0)} \\ -1 \end{bmatrix} \) for \(i = 0\), where \(n_0 = \|\begin{bmatrix} s^{(0)} \\ -1 \end{bmatrix}\| \) and \(s^{(0)} = (\frac{\partial H}{\partial v}(v^{(0)}, \mu_0))^{-1} (\frac{\partial H}{\partial \mu}(v^{(0)}, \mu_0))\).

For \(i > 0\), \(s^{(i)} = (\frac{\partial H}{\partial v}(v^{(i)}, \mu_i))^{-1} (\frac{\partial H}{\partial \mu}(v^{(i)}, \mu_i))\), \(n_i = \|\begin{bmatrix} s^{(i)} \\ -1 \end{bmatrix}\|\), \(\xi^{(i)} = (\frac{1}{n_i}) \begin{bmatrix} s^{(i)} \\ -1 \end{bmatrix}\).

If \(\det(\frac{\partial H}{\partial v}(v^{(i)}, \mu_i)) > 0\), \(\tau^{(i)} = \xi^{(i)}\) else \(\tau^{(i)} = -\xi^{(i)}, i \geq 1\).

Set \(l = 0\).

**Step 2:** (Predictor and corrector point calculation) \((\hat{v}^{(i)}, \hat{\mu}_i) = (v^{(i)}, \mu_i) + a \tau^{(i)}\), where \(a = l^{i-1}_0\). Compute \((\hat{v}^{(i)}, \hat{\mu}_i) = H'_{v^{(0)}}(\hat{v}^{(i)}, \hat{\mu}_i) + H(\hat{v}^{(i)}, \hat{\mu}_i)\) and \((\hat{v}^{(i)}, \hat{\mu}_i) = (v^{(i)}, \mu_i) - (\hat{v}^{(i)}, \hat{\mu}_i)\). Now compute \((\hat{v} \hat{v}^{(i)}), \hat{\mu} \hat{\mu}_i) = (H'_{v^{(0)}}(\hat{v}^{(i)}, \hat{\mu}_i) + H'_{v^{(0)}}(\hat{v}^{(i)}, \hat{\mu}_i)) + H(\hat{v}^{(i)}, \hat{\mu}_i)\) and \((\hat{v} \hat{v}^{(i)}), \hat{\mu} \hat{\mu}_i) = (v^{(i)}, \mu_i) - 2(\hat{v} \hat{v}^{(i)}, \hat{\mu} \hat{\mu}_i)\).

Compute \((v^{(i+1)}, \mu_{i+1}) = (\hat{v} \hat{v}^{(i)}, \hat{\mu} \hat{\mu}_i) - H'_{v^{(0)}}(\hat{v}^{(i)}, \hat{\mu}_i) + H(\hat{v}^{(i)}, \hat{\mu}_i)\).

Continue the method from the computation of \((\hat{v}^{(i)}, \hat{\mu}_i)\) to the computation of \((v^{(i+1)}, \mu_{i+1})\) for \(m\) times. In each step after repeating the computation for \(m\) times, can obtain the value for next iteration \((v^{(i+1)}, \mu_{i+1})\).

If \(0 < \|\mu_{i+1} - \mu_i\| < 1\), go to step 3. Otherwise if \(m' = \min(a, \|v^{(i+1)} - \mu_{i+1}\|) > a_0\), update \(l\) by \(l + 1\), and recompute \((\hat{v}_i, \hat{\mu}_i)\) else go to step 3.

**Step 3:** Determine the norm \(r = \|H(v^{(i+1)}, \mu_{i+1})\|\). If \(r \leq 1\) and \(v^{(i+1)} > 0\) go to step 5, otherwise if \(a > \epsilon_3\), update \(l\) by \(l + 1\) and go to step 2 else go to step 4.

**Step 4:** If \(|\mu_{i+1} - \mu_i| < \epsilon_2\), then if \(|\mu_{i+1}| < \epsilon_2\), then stop with the solution \((v^{(i+1)}, \mu_{i+1})\), else terminate (unable to find solution) else \(i = i + 1\) and go to step 1.

**Step 5:** If \(|\mu_{i+1}| \leq \epsilon_1\), then stop with solution \((v^{(i+1)}, \mu_{i+1})\), else \(i = i + 1\) and go to step 1.

Note that in step 2, \(H'_{v^{(0)}}(v, \mu) = H'_{v^{(0)}}(v, \mu)^T (H'_{v^{(0)}}(v, \mu) H'_{v^{(0)}}(v, \mu)^T)^{-1}\) is the Moore-Penrose inverse of \(H'_{v^{(0)}}(v, \mu)\).

Now we obtain the sign of the positive tangent direction of the homotopy path.
THEOREM 3.5: If the homotopy curve $\Gamma_v^{(0)}$ is smooth, then the positive predictor direction $\tau^{(0)}$ at the initial point $v^{(0)}$ satisfies $\text{sign}(\det \frac{\partial H}{\partial \theta}(v, v^{(0)}, 1)_{\tau^{(0)}} < 0$.

Proof. From equation 3.1 we have

$$H(v, v^{(0)}, \mu) = \begin{bmatrix} (1 - \mu)(w - z_1 + (m - 1)(\hat{A}x^m - 2)^T(x - z_2)) + \mu(x - x^{(0)}) \\ Z_1x - \mu Z_1^{(0)}x^{(0)} \\ Z_2w - \mu Z_2^{(0)}w^{(0)} + (1 - \mu)Xw \\ w - (1 - \mu)(Ax^{m-1} + q) - \mu w^{(0)} \end{bmatrix} = 0.$$  

Now at the initial point $(v^{(0)}, 1)$ the partial derivative of the homotopy function 3.1 is given by, $\frac{\partial H}{\partial \theta}(v, \mu) = [L_5 \quad L_6]$, where

$$L_5 = \begin{bmatrix} I & 0 & 0 & 0 \\ Z_1^{(0)} & 0 & X^{(0)} & 0 \\ 0 & Z_2^{(0)} & 0 & W^{(0)} \\ 0 & I & 0 & 0 \end{bmatrix},$$

$$L_6 = \begin{bmatrix} A_1 \\ B_1 \\ C_1 \\ D_1 \end{bmatrix},$$

$$X^{(0)} = \text{diag}(x^{(0)}), W^{(0)} = \text{diag}(w^{(0)}), Z_1^{(0)} = \text{diag}(z_1^{(0)}), Z_2^{(0)} = \text{diag}(z_2^{(0)}),$$

$$A_1 = (x - x^{(0)}) - [w - z_1 + (m - 1)(\hat{A}x^m - 2)^T(x - z_2)], B_1 = -Z_1^{(0)}x^{(0)}, C_1 = -Z_2^{(0)}w^{(0)} - X^{(0)}w^{(0)}, D_1 = A x^{m-1} - w^{(0)}.$$  

Let positive predictor direction be $\tau^{(0)} = \begin{bmatrix} t \\ -1 \end{bmatrix} = \begin{bmatrix} (\mathbb{R}_1^{(0)})^{-1} R_2^{(0)} \\ -1 \end{bmatrix}$,  

where $\mathbb{R}_1^{(0)} = \begin{bmatrix} I & 0 & 0 & 0 \\ Z_1^{(0)} & 0 & X^{(0)} & 0 \\ 0 & Z_2^{(0)} & 0 & W^{(0)} \\ 0 & I & 0 & 0 \end{bmatrix}$

and $\mathbb{R}_2^{(0)} = \begin{bmatrix} A_1 \\ B_1 \\ C_1 \\ D_1 \end{bmatrix}$, where $A_1 = (x - x^{(0)}) - [w - z_1 + (m - 1)(\hat{A}x^m - 2)^T(x - z_2)], B_1 = -Z_1^{(0)}x^{(0)}, C_1 = -Z_2^{(0)}w^{(0)} - X^{(0)}w^{(0)}, D_1 = A x^{m-1} - w^{(0)}.$

Here $\det(\mathbb{R}_1^{(0)}) = \prod_{i=1}^n x_i^{(0)} w_i^{(0)} > 0$.

Therefore, $\det \frac{\partial H}{\partial \theta}(v, v^{(0)}, 1)_{\tau^{(0)}} = \det \begin{bmatrix} \mathbb{R}_1^{(0)} & \mathbb{R}_2^{(0)} \\ 0 & -1 - (\mathbb{R}_2^{(0)})^T(\mathbb{R}_1^{(0)})^{-1} (\mathbb{R}_1^{(0)})^{-1} R_2^{(0)} \end{bmatrix}$

$= \det(\mathbb{R}_1^{(0)}) \det(-1 - (\mathbb{R}_2^{(0)})^T(\mathbb{R}_1^{(0)})^{-1} (\mathbb{R}_1^{(0)})^{-1} R_2^{(0)})$

$\begin{bmatrix} \mathbb{R}_1^{(0)} & \mathbb{R}_2^{(0)} \\ 0 & -1 - (\mathbb{R}_2^{(0)})^T(\mathbb{R}_1^{(0)})^{-1} (\mathbb{R}_1^{(0)})^{-1} R_2^{(0)} \end{bmatrix}$

$= \det(\mathbb{R}_1^{(0)}) \det(1 + (\mathbb{R}_2^{(0)})^T(\mathbb{R}_1^{(0)})^{-1} (\mathbb{R}_1^{(0)})^{-1} R_2^{(0)})$
\[ \prod_{i=1}^{n} x_i^{(0)} w_i^{(0)} \det (1 + (\mathbb{R}_2^{(0)})^T (\mathbb{R}_1^{(0)})^{-1} (\mathbb{R}_1^{(0)})^{-1} R_2^{(0)}) < 0. \]

\section{Numerical Examples}

\textbf{Example 4.1:} Let consider a column sufficient tensor \( A \in T_{4,2} \) such that \( a_{1112} = -2, a_{2111} = 1, a_{222} = 1 \). Other entries are zero. \( q = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \). The initial point is \( v^{(0)} = (1, 1, 1, 1, 1, 1, 1, 1) \). After 10 iterations the solution of the tensor complementarity problem is \( z = \begin{bmatrix} .7937 \\ .7937 \end{bmatrix} \) and \( w = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \).

\textbf{Example 4.2:} Let consider a column competent tensor \( A \in T_{3,2} \) such that \( a_{11} = 1, a_{121} = 1, a_{12} = 1, a_{21} = 1, a_22 = 1, a_{22} = 1 \). Other entries are zero. \( q = \begin{bmatrix} -2 \\ 1 \end{bmatrix} \). The initial point is \( v^{(0)} = (1, 1, 1, 1, 1, 1, 1, 1) \). After 51 iterations the solution of the tensor complementarity problem is \( z = \begin{bmatrix} 1697.278 \\ 0 \end{bmatrix} \) and \( w = \begin{bmatrix} 0 \\ 3 \end{bmatrix} \).

\textbf{Example 4.3:} Let consider a tensor \( A \in T_{4,2} \) such that \( a_{111} = 1, a_{112} = -1, a_{211} = 1 \). All other entries are zero. \( q = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \). It is column adequate tensor. The initial point is \( v^{(0)} = (1, 1, 1, 1, 1, 1, 1, 1) \). After 51 iterations the solution reaches to \( z = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \) and \( w = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \).

\textbf{Example 4.4:} Let consider a \( P_0 \)-tensor \( A \in T_{4,2} \) such that \( a_{111} = 2, a_{112} = 1, a_{212} = 4, a_{222} = 2 \). All other entries are zero. \( q = \begin{bmatrix} -1 \\ -1 \end{bmatrix} \). The initial point is \( v^{(0)} = (1, 1, 1, 1, 1, 1, 1, 1) \). After 11 iterations the solution of the tensor complementarity problem is \( z = \begin{bmatrix} 0.717516 \\ 0.50706 \end{bmatrix} \) and \( w = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \).

\textbf{Example 4.5:} Let consider a strictly semi positive tensor \( A \in T_{3,2} \) such that \( a_{111} = 1, a_{112} = 2, a_{121} = 1, a_{12} = 1, a_{21} = -1, a_{22} = -1 \). All other entries are zero. But this tensor is not strong strictly semi positive. \( q = \begin{bmatrix} -1.5 \\ 1 \end{bmatrix} \). The initial point is \( v^{(0)} = (1, 1, 1, 1, 1, 1, 1, 1, 1, 1) \). After 16 iterations the solution is \( z = \begin{bmatrix} 0.901703 \\ 0.3230419 \end{bmatrix} \) and \( w = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \).

\textbf{Example 4.6:} Let consider a semi positive tensor \( A \in T_{3,2} \) such that \( a_{111} = 1, a_{112} = -3, a_{122} = 1, a_{222} = 1, a_{211} = 1, a_{121} = -2 \). All other entries are zero. But this tensor is neither strictly semi positive nor strong strictly semipositive. \( q = \begin{bmatrix} -2 \\ -1 \end{bmatrix} \). The
initial point is $v^{(0)} = (1, 1, 1, 1, 1, 1, 1, 1, 1)$. After 19 iterations the solution of the tensor complementarity problem reaches to $z = \begin{bmatrix} 1.414214 \\ 0 \end{bmatrix}$ and $w = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

5 Conclusion

In this study we introduce a new homotopy function to process the tensor complementarity problem under some conditions. We show that the proposed function is smooth and the homotopy path is bounded. We prove necessary and sufficient conditions to obtain the solution of tensor complementarity problem from the solution of the homotopy equation. In this connection we obtain the sign of the positive tangent direction of the homotopy path. We construct a homotopy continuation method and show that the method reaches to solution. Finally, several numerical illustrations are presented to show the effectiveness of the proposed approach.

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