Yang-Mills Instantons and Dyons on Group Manifolds

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Abstract

We consider Euclidean SU(N) Yang-Mills theory on the space $G \times \mathbb{R}$, where $G$ is a compact semisimple Lie group, and introduce first-order BPS-type equations which imply the full Yang-Mills equations. For gauge fields invariant under the adjoint $G$-action these BPS equations reduce to first-order matrix equations, to which we give instanton solutions. In the case of $G = SU(2) \cong S^3$, our matrix equations are recast as Nahm equations, and a further algebraic reduction to the Toda chain equations is presented and solved for the SU(3) example. Finally, we change the metric on $G \times \mathbb{R}$ to Minkowski and construct finite-energy dyon-type Yang-Mills solutions. The special case of $G = SU(2) \times SU(2)$ may be used in heterotic flux compactifications.
1 Introduction and summary

Instantons [1] play an important role in modern gauge field theories [2, 3]. They are nonperturbative BPS configurations in four Euclidean dimensions solving the first-order self-duality equations and forming a subset of solutions to the full Yang-Mills equations. On $\mathbb{R}^4$ the construction of instantons was described systematically in the framework of the twistor approach [4] and by the ADHM method [5]. They were also generalized to noncommutative $\mathbb{R}^4$ (see e.g. [6] and references therein) which resolved the problem of ‘zero size’ instantons.

On the other hand, in the celebrated AdS/CFT correspondence [7], $\mathcal{N}=4$ supersymmetric Yang-Mills theory appears on $S^3 \times \mathbb{R}$, which is the boundary of AdS$_5$. Hence, it is of interest to study instanton configurations on $S^3 \times \mathbb{R}$ with Euclidean signature and on its further compactification $S^3 \times S^1$ (thermal time circle) [8]. For the group SU(2) embedded into SU($N$), their construction was discussed e.g. in [9, 10]. Furthermore, for $N>2$, [10] reduced the self-duality equations on $S^3 \times \mathbb{R}$ to Toda-like equations and gave some explicit solutions.

In space-time dimensions higher than four, BPS configurations can still be found as solutions to first-order equations, known as generalized self-duality equations [11]-[13]. These appear in superstring compactifications as conditions for the survival of at least one supersymmetry [14]. Various solutions to these first-order equations were found e.g. in [11, 13, 15, 16], and their noncommutative generalizations have been considered e.g. in [17, 18], alongside with their brane interpretation.

In this paper, we revisit the first-order Yang-Mills BPS-type equations on the Euclidean group manifold $G \times \mathbb{R}$, where $G$ is a compact semisimple Lie group. For Ad$_G$-invariant gauge potentials this system reduces to a particular matrix mechanics admitting finite-action (instanton) solutions. For the case of $G = SU(2) \cong S^3$, which was also studied in [10], we reformulate the matrix equations as Nahm equations and reduce these further to the Toda chain equations, whose explicit solutions we present for the SU(3) example. Generically, the action for the Toda chain solutions turns out to be infinite, hence these configurations describe an instanton gas or liquid (cf. [19]).

Considering Minkowski signature on the space $G \times \mathbb{R}$, we also construct non-BPS dyon solutions of the Yang-Mills equations on this space, generalizing those on $S^3 \times \mathbb{R}$ found in [20]. Their energy is proportional to the volume of the group $G$, hence is finite for compact Lie groups.

As an application of our results one may consider the special case of $G = SU(2) \times SU(2)$, which appears in the compactification of heterotic strings on AdS$_3 \times S^3 \times S^3$ with an SU(3) structure (nonvanishing $H$-field) on the internal six-manifold (see e.g. [21] and references therein). It would be interesting to generalize our solutions to such a general setting.

2 Instantons in Yang-Mills theory on $G \times \mathbb{R}$

Let $\mathcal{G}$ be a Lie algebra of a compact semisimple Lie group $G$. We assume that the structure constants $f_{abc}$ of $G$ are normalized so that $f_{acd}f_{bcd} = 2\delta_{ab}$ for $a, b, \ldots = 1, \ldots, n-1$ and $n = \dim \mathcal{G} + 1$. Consider the vector space $\mathbb{R}^n = \mathcal{G} \oplus \mathbb{R}$ with the Euclidean metric $(\delta_{\mu\nu}) = (\delta_{ab}, \delta_{nn})$ with $\mu, \nu, \ldots = 1, \ldots, n$.

It is obvious that $\mathcal{G} \oplus \mathbb{R}$ can be regarded as a Lie algebra with the commutation relations

$$[I_a, I_b] = f_{abc} I_c \quad \text{and} \quad [I_a, I_n] = 0,$$

(2.1)

where the $I_a$ are generators of $G$ and $I_n$ generates the translations along $\mathbb{R}$. So with a coordinate $\tau$ on $\mathbb{R}$ we multiply $(g_1, \tau_1) \cdot (g_2, \tau_2) = (g_1g_2, \tau_1+\tau_2)$ on $G \times \mathbb{R}$.

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On $G \times \mathbb{R}$ we consider left-invariant one-forms $\{e^a, e^n\}$ defined by the equations
\[ g^{-1}dg = -\frac{1}{R} e^a I_a \quad \text{and} \quad -d\tau = e^n , \quad (2.2) \]
where $g \in G$ and $R$ is a dimensional parameter characterizing the ‘size’ of $G$. These one-forms satisfy the Maurer-Cartan equations
\[ de^a - \frac{1}{2R} f_{abc} e^b \wedge e^c = 0 \quad \text{and} \quad de^n = 0 \quad (2.3) \]
which easily follow from the fact that $g^{-1}dg$ is the canonical flat connection on $G$.

Let us now specialize to $\text{Ad}_G$-invariant $su(N)$-valued gauge potentials $A$, which in the ‘temporal gauge’ $A_\tau = 0$ take the form
\[ A(g, \tau) = X_a(\tau) e^a(g) \quad \text{with} \quad X_a(\tau) \in su(N) . \quad (2.4) \]
The corresponding gauge field reads
\[ F \equiv dA + A \wedge A = \dot{X}_a e^a \wedge e^n + \frac{1}{2} \left( \frac{1}{R} f_{abc} X_c + [X_a, X_b] \right) e^a \wedge e^b , \quad (2.5) \]
where $\dot{X}_a := dX_a/d\tau$. From (2.5) we extract the components $F_{\mu\nu}$ of $F$,
\[ F_{ab} = \frac{1}{R} f_{abc} X_c + [X_a, X_b] \quad \text{and} \quad F_{cn} = \dot{X}_c , \quad (2.6) \]
in the nonholonomic basis $\{e^\mu\} = \{e^a, e^n\}$ of one-forms on $G \times \mathbb{R}$. Note that the metric on the group manifold $G \times \mathbb{R}$ has the form
\[ ds^2 = \delta_{\mu\nu} e^\mu e^\nu = \delta_{ab} e^a e^b + (e^n)^2 =: R^2 d\Omega^2_{n-1} + d\tau^2 . \quad (2.7) \]

On $\mathbb{R}^n = G \oplus \mathbb{R}$ we introduce the following completely antisymmetric four-index tensor $T_{\mu\nu\lambda\sigma}$:
\[ T_{abcd} = 0 \quad \text{and} \quad T_{abcn} = f_{abc} . \quad (2.8) \]
Consider now the first order (BPS) equations
\[ \frac{1}{2} T_{\mu\nu\lambda\sigma} F_{\lambda\sigma} = F_{\mu\nu} , \quad \text{i.e.} \quad F_{ab} = f_{abc} F_{cn} , \quad (2.9) \]
written in the nonholonomic basis $\{e^\mu\}$ on $G \times \mathbb{R}$ defined by (2.2). These BPS equations generalize the self-duality equations [1] on $\mathbb{R}^4$. The group $G$ can be embedded into the rotation group $\text{SO}(n-1)$ and acts on $G$ via the adjoint representation $\text{Ad}_G$, leaving invariant the tensor $T_{\mu\nu\lambda\sigma}$ and therefore the equations (2.9). From (2.5) it follows that (2.9) is equivalent to
\[ f_{abc} \dot{X}_c = \frac{1}{R} f_{abc} X_c + [X_a, X_b] . \quad (2.10) \]
Note that the full Yang-Mills equations on $G \times \mathbb{R}$ after substitution of (2.4)-(2.6) reduce to the matrix equations
\[ \dot{X}_a = \frac{1}{R^2} X_a + \frac{3}{2R} f_{abc} [X_b, X_c] + \frac{1}{2} f_{abc} f_{cde} [X_b, [X_d, X_e]] , \quad (2.11) \]
and each solution of (2.10) satisfies (2.11) as can be seen after differentiation (2.10) and multiplication by the structure constants.
To find a solution of (2.10) we embed $G$ into $su(N)$ with the ansatz (cf. [10, 20])

$$X_a(\tau) = -\left(\phi(\tau) + \frac{1}{2R}\right) I_a,$$  \hspace{1cm} (2.12)

where $\phi(\tau) \in \mathbb{R}$ and the $I_a$ generate an $N$-dimensional representation of $G$. Substituting (2.12) into (2.10), we arrive at

$$\dot{\phi} = \frac{1}{4R} - \phi^2$$  \hspace{1cm} (2.13)

whose solution is the kink

$$\phi = \frac{1}{2R} \tanh\left(\frac{\phi}{2R}\right),$$  \hspace{1cm} (2.14)

where one can shift $\tau \mapsto \tau - \tau_0$ due to translational invariance.

Inserting (2.14) into (2.4) and (2.5), we obtain the corresponding solution of the Yang-Mills equations on $G \times \mathbb{R}$,

$$A = -\frac{1}{2R} \left(\tanh\left(\frac{\phi}{2R}\right) + 1\right) e^a I_a$$  \hspace{1cm} and  \hspace{1cm} $F = \frac{1}{4R \cosh^2\left(\frac{\phi}{2R}\right)} \left(e^a \wedge e^b + \frac{1}{2} f_{bc}^a e^b \wedge e^c\right) I_a,$$  \hspace{1cm} (2.15)

which is an instanton since

$$A \to 0 \quad \text{for} \quad \tau \to -\infty \quad \text{and} \quad A \to -\frac{1}{R} e^a I_a = g^{-1} dg \quad \text{for} \quad \tau \to +\infty,$$  \hspace{1cm} (2.16)

where $g : G \to SU(N)$ is a degree one map. Furthermore, the action functional for this solution is

$$S = -\int_{G \times \mathbb{R}} \text{tr}(F \wedge *F) = 2c (n-1) \text{Vol}(G) \int_{-\infty}^{\infty} d\tau \dot{\phi}^2 = \frac{1}{3} c (n-1) \text{Vol}(G) R^{-3} < \infty,$$  \hspace{1cm} (2.17)

where $c$ is the Dynkin index of the $N$-dimensional $G$-representation generated by $I_a$.

### 3 Yang-Mills theory on $S^3 \times \mathbb{R}$ and Toda chain equations

In the special case of $A \in su(2)$ on $G = SU(2)$ with size $R$, i.e. instantons on $S^3_{2R} \times \mathbb{R}$, from (2.17) with $c=\frac{1}{2}$, $n-1=3$ and $\text{Vol}(G)=2\pi^2 (2R)^3$ we obtain $S=8\pi^2$, which is exactly the action of the one-instanton BPST solution. For this case, the matrix equations (2.10) read

$$\dot{X}_a = \frac{1}{R} X_a + \frac{1}{2} \varepsilon_{abc} [X_b, X_c] \quad \text{with} \quad a, b, \ldots = 1, 2, 3.$$  \hspace{1cm} (3.1)

To further simplify we introduce a new variable $r \in (0, \infty)$,

$$r = R \exp\left(\frac{\tau}{R}\right) \quad \Leftrightarrow \quad \tau = R \log\left(\frac{r}{R}\right)$$  \hspace{1cm} (3.2)

so that

$$ds^2 = \delta_{ab} e^a e^b + dr^2 = d\tau^2 + R^2 d\Omega_3^2 = \frac{R^2}{r^2} \left(dr^2 + r^2 d\Omega_3^2\right).$$  \hspace{1cm} (3.3)

At the same time, we redefine our matrix functions (cf. [22]) as

$$X_a(\tau) \mapsto Y_a(r) := \exp\left(-\frac{r}{R}\right) X_a(\tau(r)),$$  \hspace{1cm} (3.4)

which transforms (3.1) to the well-known Nahm equations [23]

$$\frac{d}{dr} Y_a = \frac{1}{2} \varepsilon_{abc} [Y_b, Y_c].$$  \hspace{1cm} (3.5)
Nahm’s equations admit a Lax representation, i.e. they can be written as

$$\frac{d}{dr} L(\zeta) = [L(\zeta), M(\zeta)],$$

with

$$L(\zeta) = (1+\zeta^2) Y_1 + i(1-\zeta^2) Y_2 - 2i \zeta Y_3 \quad \text{and} \quad M(\zeta) = \zeta(Y_1-iY_2) - iY_3,$$

where \(\zeta \in \mathbb{C}P^1\) is an extra ‘spectral’ parameter. Therefore, to \((3.5)\) one can apply the integrable systems’ machinery for constructing solutions, conserved charges (see e.g. \([22]\)) etc. In fact, a general solution of \((3.5)\) can be given in terms of theta functions.

Here, we are interested in a special class of solutions which arise by reducing \((3.6)\) to (periodic) Toda lattice equations (see e.g. \([13,24]\)) and to Toda chain equations, in particular. Namely, let us consider the Chevalley basis \(\{H_\alpha, E_\alpha, E_{-\alpha}\}\) for the Lie algebra \(\text{su}(N)\),

$$[H_\alpha, H_\beta] = 0, \quad [E_\alpha, E_{-\beta}] = \delta_{\alpha\beta} H_\beta, \quad [H_\alpha, E_\beta] = K_{\alpha\beta} E_\beta, \quad [H_\alpha, E_{-\beta}] = -K_{\alpha\beta} E_{-\beta},$$

where \((K_{\alpha\beta})\) is the Cartan matrix and \(\alpha,\beta,\ldots = 1,\ldots,N-1\). We then put

$$Y_1 = \frac{1}{2} \sum_{\alpha=1}^{N-1} f_\alpha(r) (E_\alpha - E_{-\alpha}), \quad Y_2 = \frac{i}{2} \sum_{\alpha=1}^{N-1} f_\alpha(r) (E_\alpha + E_{-\alpha}), \quad Y_3 = \frac{i}{2} \sum_{\alpha=1}^{N-1} h_\alpha(r) H_\alpha,$$

introducing real-valued functions \(f_\alpha\) and \(h_\alpha\). In addition, we relate the latter via ‘potentials’ \(\phi_\alpha\) as

$$f_\alpha = \exp\left(\frac{i}{2} \phi_\alpha\right) \quad \text{and} \quad h_\alpha = \sum_{\beta=1}^{N-1} K^{-1}_{\alpha\beta} \frac{d}{dr} \phi_\beta,$$

where \(K^{-1}\) is the inverse Cartan matrix. With this, the Nahm equations \((3.5)\) as well as their Lax representation \((3.6)\) turn into the Toda chain equations

$$\frac{d^2}{dr^2} \phi_\alpha = \sum_{\beta=1}^{N-1} K_{\alpha\beta} \exp(\phi_\beta).$$

The general solution to \((3.11)\) is known (see e.g. \([25]\)). As an example we display it for \(N=3\):

$$f_1 = A_1 \exp(A_1 r + B_1) \frac{\sqrt{\phi}}{\sqrt[4]{\psi}}, \quad f_2 = A_2 \exp(A_2 r + B_2) \frac{\sqrt{\phi}}{\sqrt[4]{\psi}},$$

$$h_1 = \frac{4A_1 + 2A_2}{3} + \frac{A_1 \exp(2A_1 r + 2B_1)}{\sqrt[4]{\psi}} \left(1 - \frac{A_1}{A_1 + A_2} \exp(2A_2 r + 2B_2)\right),$$

$$h_2 = \frac{2A_1 + 4A_2}{3} + \frac{A_2 \exp(2A_2 r + 2B_2)}{\sqrt[4]{\psi}} \left(1 - \frac{A_2}{A_1 + A_2} \exp(2A_1 r + 2B_1)\right),$$

$$\Psi = 1 - \exp(2A_1 r + 2B_1) + \frac{A_1^2}{(A_1 + A_2)^2} \exp(2A_1 r + 2B_1) \exp(2A_2 r + 2B_2),$$

$$\Phi = 1 - \exp(2A_2 r + 2B_2) + \frac{A_2^2}{(A_1 + A_2)^2} \exp(2A_1 r + 2B_1) \exp(2A_2 r + 2B_2),$$

$$\psi = 1 - \exp(2A_1 r + 2B_1) + \frac{A_1^2}{(A_1 + A_2)^2} \exp(2A_1 r + 2B_1) \exp(2A_2 r + 2B_2),$$

$$\psi = 1 - \exp(2A_2 r + 2B_2) + \frac{A_2^2}{(A_1 + A_2)^2} \exp(2A_1 r + 2B_1) \exp(2A_2 r + 2B_2),$$

$$\phi = 1 - \exp(2A_1 r + 2B_1) + \frac{A_1^2}{(A_1 + A_2)^2} \exp(2A_1 r + 2B_1) \exp(2A_2 r + 2B_2),$$

$$\phi = 1 - \exp(2A_2 r + 2B_2) + \frac{A_2^2}{(A_1 + A_2)^2} \exp(2A_1 r + 2B_1) \exp(2A_2 r + 2B_2).$$
where $A_1, A_2, B_1, B_2$ are arbitrary constants. Substituting (3.2), (3.10) and (3.12) into (2.4) and (2.5), we obtain a smooth solution of the self-dual Yang-Mills equations in four dimensions. However, the field strength $\mathcal{F}$ does not vanish for $\tau$ or $r \to \infty$. Therefore, the action and topological charge for such solutions are infinite, i.e. they describe an instanton gas or instanton liquid (see e.g. [19]). Only for limiting values of the parameters $A_1, A_2, B_1, B_2$, the action functional for the solution (3.12) becomes finite – in this case it reduces to the one-instanton solution discussed earlier.

4 Dyons in Yang-Mills theory on $G \times \mathbb{R}$

Finally let us change the signature of $G \times \mathbb{R}$ from Euclidean to Minkowski by choosing on $\mathbb{R}$ a coordinate

$$x^0 = t = -i\tau$$

so that $\mathcal{E} = dt$.

For $A$ we copy the ansatz (2.4) but now

$$\mathcal{F} = \frac{d}{dt} X_a e^0 \wedge e^a + \frac{1}{2} \left( \frac{1}{R} f_{abc} X_c + [X_a, X_b] \right) e^a \wedge e^b .$$

After substituting (2.4) and (4.2) into the Yang-Mills equations on $G \times \mathbb{R}$, we obtain the matrix equations

$$\frac{d^2}{dt^2} X_a + \frac{1}{R^2} X_a + \frac{3}{2R^2} f_{abc} [X_b, X_c] + \frac{1}{2} f_{abcde} [X_b, [X_d, X_e]] = 0 ,$$

which also follow from (2.11) by $\tau \mapsto it$. Note that for the Minkowski signature we cannot write a first-order form for these equations inside $su(N)$.

Still, the previous ansatz (2.12) reduces the equations (4.3) to

$$\frac{d^2}{dt^2} \phi - \frac{1}{2R^2} \phi + 2\phi^3 = 0 ,$$

which is solved by

$$\phi = \left[ \sqrt{2} R \cosh \left( \frac{t}{\sqrt{2} R} \right) \right]^{-1} .$$

For the case of $G = SU(2)$, such solutions were discussed in [20].

By inserting (4.5) into (2.4) and (4.2), we arrive at a dyon configuration

$$\mathcal{A} = -\frac{1}{2R} \left( 1 + \frac{\sqrt{2}}{\cosh \left( \frac{t}{\sqrt{2} R} \right)} \right) e^a I_a ,$$

$$\mathcal{F} = \frac{1}{2R^2} \sinh \left( \frac{t}{\sqrt{2} R} \right) e^0 \wedge e^a + \frac{1}{8R^2} \left( \frac{2}{\cosh^2 \left( \frac{t}{\sqrt{2} R} \right)} - 1 \right) f_{abc} e^b \wedge e^c I_a ,$$

from which we extract the components

$$\mathcal{F}_{0a} = \frac{1}{2R^2} \sinh \left( \frac{t}{\sqrt{2} R} \right) I_a$$

and

$$\mathcal{F}_{ab} = \frac{1}{4R^2} \frac{2 - \cosh^2 \left( \frac{t}{\sqrt{2} R} \right)}{\cosh^2 \left( \frac{t}{\sqrt{2} R} \right)} f_{abc} I_c .$$

Hence, the energy density becomes

$$\mathcal{E} = -\tr \left( 2\mathcal{F}_{0a} \mathcal{F}_{0a} + \mathcal{F}_{ab} \mathcal{F}_{ab} \right) = \frac{1}{8} c (n-1) R^{-4} ,$$

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where $c$ is the Dynkin index of the embedding representation. Integrating over the compact Lie group $G$ we finally obtain the energy

$$E = \frac{1}{8} c (n-1) \text{Vol}(G) R^{-4} < \infty .$$

(4.9)

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