SUSPENSION AND LEVITATION IN NONLINEAR THEORIES

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I investigate stable equilibria of bodies in potential fields satisfying a generalized Poisson equation
\[ \mathbf{\nabla} \cdot [\mu(\mathbf{\nabla} \varphi)/a_0] \mathbf{\nabla} \varphi] = \rho. \]
This describes diverse systems such as nonlinear dielectrics, certain flow problems, magnets, and superconductors in nonlinear magnetic media; equilibria of forced soap films; and equilibria in certain nonlinear field theories such as Born-Infeld electromagnetism. Earnshaw's theorem, totally barring stable equilibria in the linear case, breaks down. While it is still impossible to suspend a test, point charge or dipole, one can suspend point bodies of finite charge, or extended test-charge bodies. I examine circumstances under which this can be done, using limits and special cases. I also consider the analogue of magnetic trapping of neutral (dipolar) particles.

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I. INTRODUCTION

The possibility to suspend bodies, or to levitate them against the pull of gravity, by applying various long-range fields to them, is of obvious interest and importance. This has to be done by fields whose sources are outside the body to be suspended. The fields usually considered (electrostatic, magnetostatic, nonrelativistic gravitational) are derived from scalar potentials that satisfy the Laplace equation outside sources. There is a sweeping statement, under the name of Earnshaw's theorem \[1\], of the impossibility to suspend, statically, in stable equilibrium, a body carrying a distribution of charges (in rigid distributions) in any setting of the fields, when there is no overlap between suspended charges and sources. This has to do with the fact that a solution of the Laplace equation can attain true extrema only on the boundary of the domain of solution—the maximum (extremum) principle, see e.g. \[2\].

Many of the Earnshaw-type results, beyond the basic statement of the impossibility to suspend a point charge, rest heavily on the linearity of the Laplace operator. The Earnshaw statement may thus be expected to break down in non-linear systems. Here I revisit the question of stable equilibria of bodies subject to fields described by a nonlinear generalization of the Poisson equation: A source distribution \( \rho(\mathbf{r}) \), in \( D \)-dimensional Euclidean space, produces a potential \( \varphi \) through
\[ N\varphi = \mathbf{\nabla} \cdot [\mu(\mathbf{\nabla} \varphi)/a_0] \mathbf{\nabla} \varphi = \alpha_0 N \rho, \] 
where 
\[ \alpha_0 = \frac{2\pi}{2D}\Gamma(D/2)\]

\( D \)-dimensional solid angle (introduced for convenience), \( \mu(x) = dF(y)/dy \) is positive except perhaps that \( \mu(0) \) may vanish. I normalize to \( F(0) = 0 \). The nonlinear operator \( N \) generalizes the Laplacian, and can also be written as \( A_{ij} \partial_i \partial_j \varphi \), with \( A_{ij} = \delta_{ij} + \mu \varphi, \varphi, |\nabla \varphi|^2 \), where \( \hat{\mu} \equiv \mu x^i / \mu \). Points where \( \mathbf{\nabla} \varphi = 0 \) need special treatment in theories where \( \mu(0) = 0 \), but this does not modify any of the results, and is ignored for brevity. (Summation over repeated indices is implied throughout.) Only the case where eq.\[3\] is elliptic is considered, so \( A_{ij} \) is positive definite, tantamount to \( \hat{\mu} > -1 \). For \( G > 0 \), like, point sources attract each other, and opposite sources repel (as in gravity), and \textit{vice versa} for \( G < 0 \) (as in electostatics) \[3\].

In \[3\] I derive general results pertaining to forces on bodies in such theories. Here I concentrate on the existence of, and criteria for, stable equilibria of bodies. Solving the general problem requires numerical means, but much can be learnt from closed-form solutions that can be found in certain limits and special cases.

Equation\[3\] describes a variety of physical problems. Some examples are: (a) Nonlinear dielectric, and diamagnetic, media; \( \mu \) is then the dielectric, or diamagnetic coefficient, which depends on the field strength (\( G < 0 \)). (b) Subsonic-potential-flow problems of non-viscous fluids with an equation of state of the form \( \rho = \rho(\varphi) \) (subsonicity is equivalent to ellipticity). The form of \( \mu(x) \) depends on the equation of state. For instance, when \( \rho \propto \varphi^2, \mu(x) \propto [1-(\gamma-1)x^2]^{1/(\gamma-1)} \) for \( \gamma \geq 1 \) (the limit \( \gamma \to 1 \) exists: \( \exp(-x^2) \)) \( G > 0 \). Our results then pertain to equilibria of sources, sinks, and obstacles in the flow. (c) Equation\[3\] was used in \[3\] as an effective-action approximation to Abelianized QCD, with \( \mu(x) \propto \ln x \); it is elliptic for \( x < e^{-1} \), or \( x > 1 \). (d) Nonlinear (vacuum) electrostatics as formulated e.g. in the

\[ \sqrt{x} \]
Born-Infeld nonlinear electromagnetism, which also appears in effective Lagrangians resulting from string theory (see review and references in [3]). In the electrostatic case $\mu(x) \propto (1 - x^2)^{-1/2}$, and $G < 0$. (e) A formulation of an alternative nonrelativistic gravity to replace the dark-matter hypothesis in galactic systems [4]. Here $\mu(x) \approx x$ for $x \ll 1$, and $\mu \approx 1$ for $x \gg 1$ ($G > 0$). (f) Problems of nonlinear electric-current flows in systems with field-dependent conductivity (nonlinear current-voltage relation), and nonlinear diffusion problems; $\mu(x)$ is the transport coefficient. Here, a force on a body signifies the gradient of the entropy-production rate with position of the body, so stable equilibria are configurations of extremal entropy generation rates. (g) Area (volume) minimization problems: If $\varphi(r)$ is understood as the height of a $D$-dimensional surface above position $r$ on a $D$-dimensional hyperplane $H$, then eq. (4) describes the problem of the minimization of the volume of the surface. The sources could represent a prescribed vertical-force density. Forces on sources as will appear below correspond to lateral forces (parallel to $H$). In this problem $\mu(x) \propto (1 + x^2)^{-1/2}$, and $G > 0$.

II. THE GENERAL PROBLEM

I start with the analogue of Earnshaw’s original question: can an equilibrium configuration $\rho(r)$ exist subject to the $\varphi$ field alone? This would require that $\rho \nabla \varphi$ vanish everywhere, and can probably be precluded, as I show for a wide class of theories. In [3] I derive an expression for the virial integral $V = \int \rho \cdot \nabla \varphi \, d^D r = a_0^2/2\alpha_\nu G \int F(D - 2\hat{F}) \, d^D r$ [with $F(y) \equiv y F'(y)/F(y)$], that holds for theories in which $\hat{F}(0) < D/2$. (In this case the potential vanishes asymptotically for a bounded charge; for $\hat{F}(0) = D/2$ it diverges logarithmically.) We learn from this that in problems with $\hat{F}(y) < D/2$ we have $V \neq 0$, and $\rho \nabla \varphi$ cannot vanish everywhere (actually true even for $\hat{F} = D/2$ where the expression for $V$ has an extra term). This applies e.g., to the flow, and volume-minimization, problem (where $\hat{\varphi} \leq 1$), and to the modified dynamics (where $1 \leq \hat{\varphi} \leq D/2$). Such complete-equilibrium configurations seem, anyway, to be of academic interest only. The bodies making up the systems we want to keep in equilibrium are themselves not held together, internally, in static equilibrium by the same forces: atoms and stars consist of moving constituents, held, intrinsically by forces other than electricity, and gravity, respectively.

Consider then equilibria of a charged body $B$, given by a rigid charge distribution $\rho^\alpha(r)$, in the presence of some fixed distribution $\hat{\rho}(r)$ of “holding” sources, with no overlap: $\hat{\rho}\rho^\alpha = 0$. (Body and sources might be each held rigid by forces other than the $\varphi$ field.) The force on $B$ is

$$F = - \int \rho^\alpha \nabla \varphi \, d^D r, \quad \text{ (3)}$$

with $\varphi$ determined from eq. (1) with $\rho(r) = \rho^\alpha(r) + \hat{\rho}(r)$. $F$ is also the gradient of the energy with respect to rigid translations of $B$ by $\delta a$: $\delta E = -\delta a \cdot F$. I also want to consider the possibility of several types of charge $\rho_\alpha$ coupled to fields $\varphi_\alpha$ through equations of type (1), possibly with different forms of $\mu(x)$ (e.g. levitation in some $\varphi$ field against gravity). Under $\rho \to -\rho$, $\varphi \to -\varphi$, and $F \to -F$; under $G \to -G$, $\varphi$ and all forces change sign.

Under what conditions (choice of $\mu(x)$, boundary conditions, $\hat{\rho}$, and $\rho^\alpha$) can $B$ be suspended stably (with $E$ at a minimum)? I am mainly concerned with translational stability, so define $E(R)$ as the translational energy as a function of the position of some reference point $R$ in $B$, keeping the orientation fixed. $F = -\nabla_R E(R)$. A stable equilibrium where, $R = 0$, requires that the Hessian of $E$, $E_{ij} = -\partial^2 E/\partial R_i \partial R_j$, be positive definite. (More generally, that the first non-vanishing derivative-must be of even order–is positive definite: $E_{a_{11}...a_{2n}} \geq 0$ for any non-zero $a$.)

In the linear case we can separate $\varphi = \hat{\varphi} + \varphi^\alpha$, with $\hat{\varphi}$, and $\varphi^\alpha$ coming, respectively from the ”holding” sources, and from $B$ alone. The contribution of $\varphi^\alpha$ to the force in eq. (3) vanishes, because a body does not exert a force on itself, and we then have (for the many-field case)

$$E(R) = \sum_\alpha \int \rho^\alpha(x) \hat{\varphi}_\alpha(R + x) \, dv \quad \text{ (4)}$$

($x$ is the position inside the body). Clearly, when all $\hat{\varphi}_\alpha$ satisfy the Laplace equation, so does $E(R)$, and $E$ does not have a true extremum, leading to the general Earnshaw statement.

In the nonlinear case eq. (4) remains valid only when $R$ is a test body, whose contribution to the sources can be neglected when calculating the potential. For a point, test body, in a single field we have $E(R) = q\hat{\varphi}$, and it is still not possible to have a stable equilibrium because solutions $\hat{\varphi}$ of eq. (1) still satisfy an extremum principle. However, any departure from this restricted case might permit stable suspension. (i) Unlike a test point charge, a point body of finite charge is not denied a stable equilibrium. The extremum principle for $\hat{\varphi}$ does not exclude this, as $\hat{\varphi}$ is no
more the effective potential (energy) of the body. (ii) Even test charges arranged rigidly in an extended body can sometimes be suspended in a $\varphi$ field. (iii) A point, test particle that carries test charges of different types can be suspended in a combination of fields $\varphi_\alpha$ satisfying eq.(1). Now, $E = \sum_\alpha q_\alpha \varphi_\alpha$ is not subject to an extremum principle even though the separate $\varphi_\alpha$s are.

### III. EQUILIBRIA OF POINT CHARGES

Take the case of a point charge $q$ "held" by a distribution $\hat{\rho}$, when the only boundary condition dictated is $\varphi \to 0$ at infinity. The problem can be solved in closed form in the limit where $q$ is very large (relative to $\hat{\rho}$). The total force on the whole system vanishes, so the force on $q$ is equal and opposite that on $\hat{\rho}$. In the above limit, $\hat{\rho}$ may be treated as a distribution of test charges, so the force on it is $-\int d^3r \hat{\rho}(r)\nabla\varphi$, where $\varphi$ is produced by $B$ alone. This is gotten straightforwardly by applying Gauss theorem to eq.(1) for a point charge. Putting all this together we obtain for the force $F_q(R)$ on $q$, at position $R$:

\[ F_q(R) = s(qG) a_0 \int d^3r \hat{\rho}(r) S \left( \frac{|qG|}{a_0 |r - R|^b} \right) \frac{r - R}{|r - R|}. \]  (5)

Here, $s(x) = \text{sign}(x)$, and $S(y)$ is the inverse of $\nu(x) = x\mu(x)$ ($\nu$ is increasing, by the ellipticity condition). The energy $E$ ($F_q(R) = -\nabla_R E$) is thus linear in $\hat{\rho}$:

\[ E(R) = \int d^3r \hat{\rho}(r) G_q(|r - R|). \]  (6)

From eq.(1), the effective Green’s function, $G_q$, is integrated from $|x| \neq 0$ in light of the non-overlap assumption:

\[ \nabla_x G_q(x) = s(qG) a_0 S(z) \frac{x}{|x|}. \]  (7)

with $z = |qG|/a_0 |x|^{d-1}$. The Hessian of $G_q$ is

\[ \mathcal{G}_{q,ij} = s(qG) a_0 S(z) \frac{\partial^2}{|x|} \mathcal{B}_{ij}, \quad \mathcal{B}_{ij} = \delta_{ij} - \frac{D + \hat{\mu}}{1 + \hat{\mu}} e_i e_j, \]  (8)

with $e_i = x_i/|x|$. $\mathcal{B}$ has $D - 1$ eigenvalues equal 1, and one that equals $-(D - 1)/(1 + \hat{\mu}) < 0$. So,

\[ \Delta G_q(x) = s(qG) (D - 1) a_0 S(z) |x|^{d-1} \frac{\hat{\mu}(z)}{1 + \hat{\mu}(z)} \]  (9)

($\hat{\mu}(z) = z\mu'/\mu$). Also,

\[ E_{ij}(R) = \int d^3r \hat{\rho}(r) \mathcal{G}_{q,ij}(|r - R|). \]  (10)

Since $S$ depends nontrivially on its argument, the dependence of $F_q$, $G_q$, and $E$ on $q$ can be nontrivial. Among other things, the location of equilibria may depend on $q$. Take $\hat{\rho}$ to have a center of symmetry, which must then be an equilibrium point; when is it stable? If $\hat{\rho}$ has cubic symmetry the question hinges on the sign $\Delta E$ at the center (taken at the origin), because from the symmetry

\[ E_{ij}(0) = D^{-1} \Delta E(0) \delta_{ij}. \]  (11)

If $\hat{\mu}$ and $\hat{\rho}$ have fixed signs, then from eqs.(1) $\Delta E$ has the fixed sign $s = \text{sign}(q\mu G\hat{\rho})$ at all $R$ for which there is no overlap. If $s$ is positive, $\Delta E > 0$, and there might be a true minimum (while $\Delta E < 0$ precludes a stable equilibrium). In particular, in the cubic-symmetric case the center is then per force a stable equilibrium. By the same token, the charge distribution $\hat{\rho}$ can be suspended (stably against translations) in the field of $q$, exemplifying statement (ii) below eq.(1). Thus, e.g., in the flow, Born-Infeld, and volume-minimization problems, where $\mu G < 0$, $\hat{\rho}$ has to have a sign opposite that of $q$ to be able to suspend $q$. If the cubic symmetry of $\hat{\rho}$ is slightly disturbed there remains an equilibrium point (from continuity), which then may depend on $q$ if the center of symmetry is lost.

When $\hat{\rho}$ has lower-than-cubic symmetry, the center might still be a stable equilibrium: if $G_{q,ij}$ does not have a definite sign, $E_{ij}$ might be positive definite upon integration by eq.(1). For example, in $D > 2$, if $\hat{\rho}$ is any planar
(two-dimensional) distribution with a four-fold symmetry, angular integration of $B$ gives a matrix whose eigenvalues are all 1, but two that equal $(\mu - D + 2)/2(1 + \mu)$. We get a stable minimum at the center if $\mu(z) > D - 2$, for all $z$ contributing to the radial integration. This is straightforwardly generalized to other symmetries. Clearly, because of the linearity in $\hat{\rho}$ stability of a composite configuration can be assessed by adding up $E_{ij}$ of the components. So, any combination of configurations that each gives a stable equilibrium also does so (e.g. a combination of any number of concentric, cubic-symmetric $\hat{\rho}(r)$).

What happens when $q$ is decreased away from the large-$q$ limit? In the general case, equilibrium might conceivably be lost, as $q$ becomes comparable with the the "holding" charges, when one of the eigenvalues of $E_{ij}$ changes sign. For cubic-symmetric $\hat{\rho}$, however, where eq. (11) is valid for all $q$, the answer depends on the sign of $\Delta E(0)$. In the test-particle limit, $q \to 0$, we have $E \to q\hat{\phi}$. Since the Hessian of $\hat{\phi}$ cannot have a definite sign at the center, we must have $d_q = q^{-1}\Delta E(0) \to \Delta \hat{\phi}(0) \to 0$ in the limit. It is likely that stability is lost in the test-charge limit with $d_q$ remaining positive all the way down to $q = 0$. Then, the center remains a stable equilibrium for all finite $q$. If $d_q$ changes sign and approach 0 from below, then the same configuration has a stable equilibrium in a theory with $G \to -G$. (The possibility that $d_q = 0$ for a finite stretch of $q$ may be rejected from analyticity. In the linear case $d_q = 0$ for all $q$.)

The above treatment (for large $q/\hat{\rho}$) is applicable whenever we have the solution for the field $\varphi^0$ of $B$ alone. For instance, all the above expressions still hold if $B$ is any spherically symmetric charge (not overlapping with $\hat{\rho}$), with $q$ its total charge. Exact solutions can also be found for any body with 1-D symmetry (plane, cylindrical, etc.). Another example: in [8] I derive the exact field for a pair of opposite point charges in the theory with its total charge. Exact solutions can also be found for any body with 1-D symmetry (plane, cylindrical, etc.). Another example: in [8] I derive the exact field for a pair of opposite point charges in the theory with its total charge. Exact solutions can also be found for any body with 1-D symmetry (plane, cylindrical, etc.).

**IV. SOME IMPOSSIBILITIES**

Even if stable equilibria are possible in the nonlinear case, some elementary feats are still impossible (a) The impossibility to suspend a point, test charge in a single field stems from the fact that $E = q\hat{\phi}$, and so $A_{ij}E_{ij} = 0$ away from sources. As $A$ is positive definite, $E_{ij}$ cannot be positive definite as stability requires. (I assume here and below that not all the $E_{ij}$ vanish at the equilibrium point. More generally, it can be shown that the first, relevant (even) derivative that does not vanish cannot be positive definite.) (b) It is impossible to balance stably a test point charge by two $\varphi$ fields in a region where one of them has a constant gradient $\nabla \varphi' = a$ (e.g. levitate the charge in a $\varphi$ field against earth-surface gravity). Here $E = q(a \cdot r + \varphi)$, and also satisfies $A_{ij}(\nabla \varphi)E_{ij} = 0$ away from sources, and so $E$ cannot have a minimum. (c) Even if extended, test bodies can, in general be suspended, it is not possible to suspend a test dipole in a $\varphi$ field. The force on a dipole $p = -(p \cdot \nabla)\nabla \varphi$. The translational energy is then $E = p \cdot \nabla \varphi$, and $A_{ij}(\nabla \varphi)E_{ij} = (p \cdot \nabla)(A_{ij} \varphi_{ij}) - \varphi_{ij} \partial A_{ij}/\partial \varphi_{ij} (p \cdot \nabla)\varphi_{ij}$. The first term vanishes everywhere away from sources; the second term vanishes at equilibrium, because $(p \cdot \nabla)\nabla \varphi$ is the force. Again, $E_{ij}$ cannot be positive definite at an equilibrium–no stable minimum, not even against translations. (d) Consider now a small test body that has both a charge $q$ and a dipole $p$ in equilibrium at a point where $\nabla \varphi \neq 0$. Here $E = q\varphi + p \cdot \nabla \varphi$. As in case (c) we have $A_{ij}E_{ij} = -\varphi_{ij} \partial A_{ij}/\partial \varphi_{ij} (p \cdot \nabla)\varphi_{ij}$, but now the vanishing of the force implies $(p \cdot \nabla)\varphi_{ij} = -q\varphi_{ij}$, so $A_{ij}E_{ij} = q\varphi_{ij} \partial A_{ij}/\partial \varphi_{ij}$. Take, specifically, the orientation for which the moment on the dipole vanishes (also a requirement of equilibrium): $p = -p\nabla \varphi / |\nabla \varphi| (p > 0)$ calculated at the equilibrium point, and then consider stability against pure translations. For this value of $p$ one finds that at the equilibrium point $A_{ij}E_{ij} = p^{-1}q^2 |\nabla \varphi|^2 \partial \mu / \partial x$. (putting $a_0 = 1$.) A stable equilibrium might then be possible only where $\mu$ is increasing. This is precluded, e.g., in the flow, and the volume-minimization, problem, and in the modified dynamics, where $\mu$ is always decreasing, but not in Born-Infeld electrostatics. As we have only checked stability to translations, increasing $\mu$ does not guarantee full stability.

**V. SUSPENSION OF ALIGNED DIPOLES**

Neutral, dipolar bodies can be levitated stably, even in the linear case, provided the (constant-magnitude) dipole is forced to remain aligned with $\nabla \varphi$. For example, under conditions of adiabaticity, precession of a spin (aligned with the dipole) about $\nabla \varphi$ insures alignment. This fact is used, e.g., in the construction of magnetic traps for atoms and neutrons [8], and for macroscopic, magnetized tops [10]. Take then $p = -pe$, where $e = \nabla \varphi / |\nabla \varphi|$ (restricted to equilibrium at points where $\nabla \varphi \neq 0$), and consider levitation of the body against a constant-force field.
The total force is $\mathbf{f} + p\nabla|\nabla\varphi|$, the energy (for translation plus realignment) is $E(\mathbf{R}) = -\mathbf{f} \cdot \mathbf{R} - p|\nabla\varphi|$, and $E_{ij} = -p|\nabla\varphi|^{-1}(\varphi_{,i,j} - \varphi_{,j,i} + |\nabla\varphi|\varphi_{,z,z,j})$, with $z$ the direction of $\mathbf{e}$ (and is not summed over). As before, we require, as a necessary condition for stability, $A_{ij}E_{ij} > 0$ at the equilibrium, where we can put $\varphi_{,z,i} = -p^{-1}\mathbf{f}_i$.

After some algebra we get

$$A_{ij}E_{ij} = -p^{-1}|\nabla\varphi|^{-1}[p^2\varphi_{,i,k}\varphi_{,i,k} - |\mathbf{f}|^2 + \hat{\mu}(f_z^2 - |\mathbf{f}|^2 - \hat{\mu}'|\nabla\varphi|f_z^2)].$$

(12)

In the linear case ($\hat{\mu} = 0$) the expression in parentheses is positive (unless all the $\varphi_{,i,j} = 0$.) and we restore $p < 0$ as the necessary condition. In the case of suspension in a pure $\varphi$ field ($\mathbf{f} = 0$) it remains so in the nonlinear case. When $\mathbf{f} \neq 0$, stability depends also on $\hat{\mu}$ and $\hat{\mu}'$. Take, e.g., the case where $\mathbf{f}$ is along a symmetry axis of the field so $f_z^2 = |\mathbf{f}|^2$. Then, $A_{ij}E_{ij} = -p^{-1}|\nabla\varphi|^{-1}[p^2\varphi_{,i,k}\varphi_{,i,k} - |\mathbf{f}|^2(1 + \hat{\mu}'|\nabla\varphi|)].$ With large enough $\hat{\mu}'$ it may be possible to get stable suspension even with $p > 0$. For example, when can such a dipole be levitated in the $\varphi$ field of a point charge against a constant $\mathbf{f}$? I find that this can be done if and only if one takes $p > 0$ (dipole aligned with $-\nabla\varphi$), and $\hat{\mu}$ satisfies at the equilibrium position $\hat{\mu}\frac{d\mathbf{f}}{dt} > 1 + (1 + \hat{\mu})/(D - 1)$. Of the problems listed above, the condition on $\hat{\mu}$ can be met only in the Born-Infeld theory; in the others $\hat{\mu}\hat{\mu}' < 0$.

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