Quantum Transformations

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Abstract

We show that the stationary quantum Hamilton–Jacobi equation of non–relativistic 1D systems, underlying Bohmian mechanics, takes the classical form with $\partial_q$ replaced by $\hat{\partial}_q$ where $d\hat{q} = dq / \sqrt{1 - \beta^2}$. The $\beta^2$ term essentially coincides with the quantum potential that, like $V - E$, turns out to be proportional to a curvature arising in projective geometry. In agreement with the recently formulated equivalence principle, these “quantum transformations” indicate that the classical and quantum potentials deform space geometry.
One of the main aspects of contemporary theoretical research concerns the quantization of gravity. Despite many efforts and results, such as those of superstring theory, the understanding of the problem is still incomplete. While Einstein’s general relativity, based on a simple principle, describes gravity in a purely geometrical framework, foundations of quantum mechanics rely on a set of axioms which apparently seem unrelated to any geometrical principle. It is then natural to think that the difficulties which arise in considering quantization of gravity merit a better understanding of the possible relationship between the foundations of general relativity and quantum mechanics.

Recently we proposed that quantum mechanics may in fact arise from an equivalence principle [1] [2]. While the original formulation considered the case of non–relativistic one–dimensional stationary systems with Hamiltonian of the form
\[
H = \frac{p^2}{2m} + V(q),
\]
which is also the case we consider in the present Letter, it will be shown in [3] [4] that the principle actually implies the higher dimensional time–dependent Schrödinger equation.

In this Letter we show that the Quantum Stationary Hamilton–Jacobi Equation (QSHJE), that we derived from the equivalence principle [1] [2], maps to the classical form under “quantum transformations” whose structure is strictly related to the quantum potential. This indicates that the classical and quantum potentials deform space geometry. We will also show that both the quantum potential and \( V - E \) are proportional to curvatures arising in projective geometry. These aspects, together with the investigation of \( p-q \) duality, related to the properties of the Legendre transformation, constitute the main results of the present Letter.

The solution \( S_0 \) of the QSHJE derived in [1] [2] is the quantum version of the Hamiltonian characteristic function (also called reduced action). In this respect the theory is consistently defined in terms of trajectories [5] [3] [3]. Although reminiscent of Bohmian mechanics [6] [7], the formulation we consider has some differences which will be further considered in [3]. In particular, as noticed also by Floyd [9], while in Bohm theory one identifies \( \psi = R e^{i\frac{\pi}{\hbar}S} \) with the Schrödinger wave function, one can see that in the 1D stationary case the natural identification is \( \psi = R(A e^{i\frac{\pi}{\hbar}S_0} + B e^{-i\frac{\pi}{\hbar}S_0}) \). While in Bohm theory the state described by a real wave function corresponds to \( S_0 = 0 \), this is never the case in the approach we consider. Furthermore, we note that the Schwarzian derivative \( \{ S_0, q \} \) is not defined for \( S_0 = \text{cnst.} \). As a consequence, while in Bohm theory the states described by a real wave function unavoidably have a vanishing conjugate momentum, this is never the case in the proposed formulation. While in Bohmian mechanics there is the issue of recovering the classical limit for states with real wave function, e.g. for the harmonic oscillator in which \( S_0 = 0 \) [7], this limit is rather natural in the formulation.
that we consider \[3\]. Another aspect is that there isn’t any wave guide in the proposed approach. Furthermore, a basic fact is that the conjugate momentum \( p = \partial_q S_0 \) is a real quantity even in classically forbidden regions. In \[3\] we will see that also the quantized energy spectra and their structure are a direct consequence of the equivalence principle.

Let us consider two 1D stationary non–relativistic systems with Hamilton’s characteristic functions \( S_0(q) \) and \( S_0^v(q^v) \). Setting

\[
S_0^v(q^v) = S_0(q),
\]

induces the “\( v \)–transformations”

\[
q \rightarrow q^v = v(q),
\]

where \( v = S_0^{-1} \circ S_0 \), with \( S_0^{-1} \) denoting the inverse of \( S_0^v \).

Recently, the following problem has been considered in \[1, 2\]

Given an arbitrary system with reduced action \( S_0(q) \), find the coordinate transformation \( q \rightarrow q^{v_0} = v_0(q) \), such that the new reduced action \( S_0^{v_0} \), defined by

\[
S_0^{v_0}(q^{v_0}) = S_0(q),
\]

corresponds to the free system with vanishing energy.

In the following we will use the notation \( q^0 = q^{v_0} \), \( S_0^0 = S_0^{v_0} \). We also set \( W(q) \equiv V(q) - E \), and denote the state \( W = 0 \) by

\[
W^0(q^0) \equiv 0.
\]

Observe that the structure of the states described by \( S_0^0 \) and \( S_0 \) determines the “trivializing coordinate” \( q^0 \) to be

\[
q \rightarrow q^0 = S_0^{-1} \circ S_0(q),
\]

Let us denote by \( \mathcal{H} \) the space of all possible \( W \)'s. Since the approach extends to arbitrary physical systems, the space \( \mathcal{H} \) is a rather general one and may include cases in which \( V(q) \) is a distribution. In particular, even if the possible potentials should be restricted to the ones physically realizable in nature, it is clear that the structure of this space cannot be defined a priori. Rather, for a given potential \( V(q) \), the possible values of \( E \) are determined by the properties of local homeomorphismicity of the \( v \)–maps which are natural to impose from the equivalence

\^1By \( W \) states we will mean for short the physical systems corresponding to a potential \( V \) and energy \( E \).
principle and that will be discussed later. This principle, suggested by the problem of finding the trivializing map (5), states that [1]

For each pair \( \mathcal{W}^a, \mathcal{W}^b \in \mathcal{H} \), there is a \( v \)-transformation such that

\[ \mathcal{W}^a(q) \longrightarrow \mathcal{W}^{a,v}(q^v) = \mathcal{W}^b(q^v). \] (6)

Note that this implies that there always exists the trivializing coordinate \( q^0 \) for which \( \mathcal{W} \longrightarrow \mathcal{W}^0 \).

Let us consider the properties that the \( v \)-maps should have in order that the equivalence principle be satisfied. First of all note that \( v \)-maps should be continuous: since both \( q \) and \( q^v \) take values continuously on \( \mathbb{R} \), it is clear that full equivalence between the two systems requires that the \( v \)-maps should be continuous. This is the general situation. However, depending on the structure of the potential, it may happen that the physical system is confined to an interval of the real line. This corresponds to a degenerate case. In particular, in studying the structure of the conjugate momentum \( p = \partial_q S_0 \), the case of the infinitely deep well is conveniently studied as a limiting procedure [5][3]. The equivalence principle is still satisfied with the trivializing map restricted to the finite interval delimited by the turning points [3].

Note that the equivalence principle implies that the transformation (2) should exist for any couple of physical systems. This provides the pseudogroup property (see below). In particular, one has to impose that the \( v \)-transformations be locally invertible. However, in discussing the properties of these maps, such as continuity, one should consider the extended real line \( \mathbb{R} = \mathbb{R} \cup \{ \infty \} \). Actually, since there are no reasons to restrict to global one-to-one self-maps of \( \mathbb{R} \), the issue of continuity of the \( v \)-maps forces us to consider \( \mathbb{R} \). This avoids considering the fictitious discontinuity arising at the points \( \pm \infty \), a property related to the structure of the real line and not to the intrinsic properties of the \( v \)-maps. Compactifying the real line allows us to select and discard the transformations which are intrinsically discontinuous. Therefore, the \( v \)-maps should be local homeomorphisms of \( \mathbb{R} \) into itself. In [3] we will see that this property also follows from the structure of the QSHJE.

To better understand the above aspect, it is useful to map the extended real line to the unit circle by means of a Cayley transformation and then consider the case of the trivializing map. While \( z = (q - i)/(-iq + 1) \) spans \( S^1 \) once, \( w = (q^0 - i)/(-iq^0 + 1) \) runs continuously around the unit circle. Since the Cayley transformation is a global univalent transformation, we have

This property implies the quantization of the energy spectra without making use of the axiomatic interpre-
that the $v$–maps induce local self–homeomorphisms of the unit circle. An interesting property of the $v$–maps is that associated to any physical state there is an integer number associated to the order of the covering of the trivializing map [2].

We note that since local homeomorphisms are closed under composition, it follows that local homeomorphicity of any $v$–map also follows from local homeomorphicity of the trivializing map. A similar aspect is called pseudogroup property. In this respect it is worth noting that this is the property of holomorphic functions which one uses for defining a complex analytic structure: this implies that the composition of two complex analytic local homeomorphisms is again a complex analytic local homeomorphism (see for example [3] and references therein).

In [1] it has been shown that the equivalence principle implies the quantum analogue of the Hamilton–Jacobi equation which in turns implies the Schrödinger equation. Subsequently, it has been shown in [2] that this is the unique possible solution. Let us shortly review the structure of the derivation in [1][2]. First of all one observes the basic fact that the equivalence principle cannot be consistently implemented in classical mechanics. This can be summarized in the following steps

2) Consider the Classical Stationary Hamilton–Jacobi Equation (CSHJE) $(\partial_q S_0^d)^2 = -2mW$.

Given another system with reduced action $S_0^{cl\,v}$, denote by $q^v$ the new space coordinate and set $q^v = v(q)$, with $v$ determined by $S_0^{cl\,v}(q^v) = S_0^{cl}(q)$, that is $v = S_0^{cl\,v}^{-1} \circ S_0^{cl}$.

2) compare the CSHJE for the system with reduced action $S_0^{cl\,v}$, that is $(\partial_q S_0^{cl\,v}(q^v))^2 = -2mW^v(q^v)$, with $(\partial_q S_0^{cl}(q))^2 = -2mW(q)$, and use $S_0^{cl\,v}(q^v) = S_0^{cl}(q)$ so that $W^v(q^v) = (\partial_q q^v)^{-2}W(q)$. Hence, consistency implies that in classical mechanics $W$ belongs to $Q$, the space of functions transforming as quadratic differentials under $v$–maps;

3) the fact that in classical mechanics one has $W \in Q$, implies that the state $W^0$ is a fixed point in $H$, i.e. under a coordinate transformation $W^0(q^0) \longrightarrow (\partial_q q^v)^{-2}W^0(q^0) = 0$.

It is therefore clear that in order to implement the equivalence principle the CSHJE should be modified. The most general form would be

$$
\frac{1}{2m} \left( \frac{\partial S_0(q)}{\partial q} \right)^2 + W(q) + Q(q) = 0.
$$

(7)
Since classical mechanics exists, it is clear that the above equation must reduce to the CSHJE in a suitable limit. That is in some limit we must have
\[ Q \rightarrow 0. \]  
(8)

Since the equivalence principle implies that \( W \notin Q \), it is clear that classical mechanics is the covariance breaking phase with \( Q \) having the role of covariantizing term.

The properties of \( W + Q \) under the \( v \)-transformations are determined by the transformed equation \( (\partial_{q^v} S_0^v(q^v))^2 / 2m + W^v(q^v) + Q^v(q^v) = 0 \), that by (1) and (7) yields
\[ W^v(q^v) + Q^v(q^v) = (\partial_q q^v)^{-2} (W(q) + Q(q)), \]  
(9)

that is
\[ (W + Q) \in Q. \]  
(10)

Let us recall how \( Q \) is determined by the equivalence principle [1] [2]. We have seen that if \( W \) transforms as a quadratic differential, then \( W_0 \) would be a fixed point in the \( H \) space. It follows that \( W \notin Q \) so that by (10) \( Q \notin Q \). Therefore
\[ W^v(q^v) = (\partial_q q^v)^{-2} W^a(q^a) + (q^a; q^v), \]  
(11)

and by (10)
\[ Q^v(q^v) = (\partial_q q^v)^{-2} Q^a(q^a) - (q^a; q^v). \]  
(12)

For \( W^a(q^a) = W^0(q^0) \) Eq.(11) gives
\[ W^v(q^v) = (q^0; q^v). \]  
(13)

This means that all the states correspond to the inhomogeneous part of the transformation of the state \( W_0 \) induced by some coordinate transformation.

Let \( a, b \) and \( c \) denote arbitrary \( v \)-transformations. Comparing
\[ W^b(q^b) = \left( \partial_q q^a \right)^2 W^a(q^a) + (q^a; q^b) = (q^0; q^b), \]  
(14)

with the same formula with \( q^a \) and \( q^b \) interchanged we have \( (q^b; q^a) = -(\partial_q q^b)^2 (q^a; q^b) \), in particular \( (q; q) = 0 \). More generally, comparing
\[ W^b(q^b) = \left( \partial_q q^c \right)^2 W^c(q^c) + (q^c; q^b) = \left( \partial_q q^c \right)^2 \left[ (\partial_q q^a)^2 W^a(q^a) + (q^a; q^c) \right] + (q^c; q^b) = \]
\[
(\partial_{\phi}q^a)^2 W^\alpha(q^a) + (\partial_{\phi}q^c)^2 (q^a; q^c) + (q^c; q^b),
\]
with (14), we obtain the basic relation [2]
\[
(q^a; q^c) = (\partial_{\phi}q^b)^2 (q^a; q^b) - (\partial_{\phi}q^b)^2 (q^c; q^b),
\]
which extends to higher dimensions [3][4]. This relation, which is a cocycle condition and directly follows from the equivalence principle, actually implies [2]
\[
(q^a; q^b) = -\frac{\beta^2}{4m} \{q^a, q^b\},
\]
where \(\beta\) is a dimensional constant and
\[
\{h(x), x\} = \frac{h'''(x)}{h'(x)} - \frac{3}{2} \left(\frac{h''(x)}{h'(x)}\right)^2 = (\ln h'(x))''' - \frac{1}{2}((\ln h'(x))')^2,
\]
develops the Schwarzian derivative. Since the inhomogeneous term in the transformation of \(W\) must disappear in the classical limit, we have by (17) that the classical phase corresponds to the \(\beta \longrightarrow 0\) limit. By (13) and (17) it follows that \(W\) itself is a Schwarzian derivative
\[
W'(q^v) = -\frac{\beta^2}{4m} \{q^0, q^v\},
\]
with \(q^0\) determined by the fact that the \(\beta \longrightarrow 0\) limit corresponds to the classical phase. One obtains [1][2]
\[
Q = \frac{\beta^2}{4m} \{S_0, q\},
\]
Eq.(7) and the identity
\[
(\partial_{\phi}S_0)^2 = \frac{\beta^2}{2} \{e^{\frac{2}{\beta}S_0}, q\} - \frac{\beta^2}{2} \{S_0, q\},
\]
imply that Eq.(20) is equivalent to
\[
W = -\frac{\beta^2}{4m} \{e^{\frac{2}{\beta}S_0}, q\}.
\]
By (7) and (20) it follows that the equation for \(S_0\) we were looking for is [1][2]
\[
\frac{1}{2m} \left(\frac{\partial S_0(q)}{\partial q}\right)^2 + W(q) + \frac{\beta^2}{4m} \{S_0, q\} = 0,
\]
\[3\]This identity admits a higher dimensional extension [3].
which is equivalent to (22). It follows that

$$e^{2\mathcal{S}_0} = \frac{A\psi^D + B\psi}{C\psi^D + D\psi},$$  \hspace{1cm} (24)

$AD - BC \neq 0$, with $\psi^D$ and $\psi$ linearly independent solutions of the stationary Schrödinger equation

$$\left(\frac{\beta^2}{2m} \frac{\partial^2}{\partial q^2} + V(q)\right)\psi = E\psi.$$  \hspace{1cm} (25)

Thus, for the “covariantizing parameter” we have

$$\beta = \hbar,$$  \hspace{1cm} (26)

where $\hbar = h/2\pi$ and $h$ is the Planck constant. We note that the QSHJE (23) has been already considered in literature [9][5][7].

In Ref.[1] the function $T_0(p)$, defined as the Legendre transformation of the reduced action, has been introduced

$$S_0(q) = pq - T_0(p).$$  \hspace{1cm} (27)

While $S_0(q)$ is the momentum generating function, its Legendre dual $T_0(p)$ is the coordinate generating function

$$p = \frac{\partial S_0}{\partial q}, \hspace{1cm} q = \frac{\partial T_0}{\partial p}.$$  \hspace{1cm} (28)

The second derivative of (27) with respect to $s = S_0(q)$ yields the “canonical equation”

$$\left(\partial_s^2 + U(s)\right)q\sqrt{p} = 0 = \left(\partial_s^2 + U(s)\right)\sqrt{p},$$  \hspace{1cm} (29)

with the “canonical potential” being

$$U(s) = \{q\sqrt{p}/\sqrt{p}, s\} = \{p, s\}.$$  \hspace{1cm} (30)

Observe that the choice of the coordinates $q$ and $q^v$, which of course does not imply any loss of generality as both $q$ and $q^v$ play the role of independent coordinate in their own system, allows us to look at the reduced action as a scalar function. In particular, since $S_0^v(q^v) = S_0(q)$, we see that the transformations (3) leave the Legendre transformation of $T_0$ (27) unchanged.

Consequently, from $\partial_q S_0^v(q^v) = (\partial_q q^v)^{-1} \partial_q S_0(q)$, we have $p \rightarrow p_v = (\partial_v q^v)^{-1} p$. However, while the Legendre transformation of $T_0$ is, by definition, invariant under $v$–transformations, this is not the case for the canonical potential $U$. Nevertheless, there is an important exception as under the $GL(2, \mathbb{C})$ transformations

$$q^v = (Aq + B)/(Cq + D) \hspace{1cm} \rightarrow \hspace{1cm} p_v = \rho^{-1}(Cq + D)^2 p,$$  \hspace{1cm} (31)
\[ \rho = AD - BC \neq 0, \] we have that the Möbius symmetry of the Schwarzian derivative implies

\[ U(s) = \{(Aq + B)/(Cq + D), s\}/2 = U(s). \tag{32} \]

Therefore we can speak of \( GL(2, \mathbb{C}) \)-symmetry of the canonical equation.

Involutivity of the Legendre transformation and the duality

\[ S_0 \leftrightarrow T_0, \quad q \leftrightarrow p, \]

imply another \( GL(2, \mathbb{C}) \)-symmetry, with the dual version of Eq. (29) being

\[ \left( \partial_t^2 + V(t) \right) p\sqrt{q} = 0 = \left( \partial_t^2 + V(t) \right) \sqrt{q}, \tag{33} \]

where

\[ V(t) = \{p\sqrt{q}/\sqrt{q}, t\}/2 = \{p, t\}/2, \tag{34} \]

with \( t = T_0(p) \). We note that for \( p = \gamma/q \) the solutions of (29) and (33) coincide. Therefore we have the self–dual states

\[ S_0 = \gamma \ln \gamma q, \quad T_0 = \gamma \ln \gamma p, \tag{35} \]

where the three constants satisfy

\[ \gamma_p \gamma_q \gamma = e. \tag{36} \]

It will be shown in [3] that this equation is connected to fundamental constants. Note that

\[ S_0 + T_0 = pq = \gamma, \quad U(s) = -1/4\gamma^2 = V(t). \tag{37} \]

We observe that the canonical equation (29) and its dual (33) correspond to two equivalent descriptions of physical systems that for the self–dual states overlap. Later we will consider another derivation of the self–dual states (35).

Remarkably, the QSHJE (23) can be also seen as modification by a “conformal factor” of the CSHJE. In particular, using the identity

\[ \{q, S_0\} = -\left( \partial_q S_0 \right)^{-2}\{S_0, q\}, \]

we have that the canonical potential determines the conformal rescaling [1][2]

\[ \frac{1}{2m} \left( \frac{\partial S_0}{\partial q} \right)^2 \left[ 1 - \hbar^2 U(S_0) \right] + V(q) - E = 0. \tag{38} \]
This shows the basic role of the purely quantum mechanical self–dual states \(^{(35)}\). Actually, observe that for the state \(\mathcal{W}(q^0)\), Eq.\(^{(38)}\) has the form
\[
\frac{1}{2m} \left( \frac{\partial \mathcal{S}_0^0}{\partial \tilde{q}^0} \right)^2 \left[ 1 - \hbar^2 \mathcal{U}(\mathcal{S}_0^0) \right] = 0. \tag{39}
\]
Setting
\[
\tilde{q}^0 = \frac{Aq^0 + B}{Cq^0 + D}, \tag{40}
\]
the solution of \(^{(38)}\) has the form
\[
\mathcal{S}_0^0 = \frac{\hbar}{2i} \ln \tilde{q}^0, \tag{41}
\]
that is
\[
1 - \hbar^2 \mathcal{U}(\frac{\hbar}{2i} \ln \tilde{q}^0) = 0. \tag{42}
\]
In the case in which \(\gamma_q = (A/D)^{\pm 1}, B = C = 0\), the solution \(^{(11)}\) corresponds to the two self–dual states defined by \(^{(35)}\) with
\[
\gamma = \gamma_{sd} \equiv \pm \hbar/2i. \tag{43}
\]

The solution \(\mathcal{S}_0^0 = \frac{\hbar}{2i} \ln q^0\) solves the problem of finding the trivializing coordinate for which \(\mathcal{W}(q) \rightarrow \mathcal{W}(q^0)\). Actually, by \(^{(5)}\) and \(^{(11)}\) we have
\[
\mathcal{S}_0^0(q^0) = \frac{\hbar}{2i} \ln \left( \frac{Aq^0 + B}{Cq^0 + D} \right) = \mathcal{S}_0(q), \tag{44}
\]
that is
\[
q \rightarrow q^0 = \frac{De^{\frac{2i}{\hbar} \mathcal{S}_0(q)} - B}{-Ce^{\frac{2i}{\hbar} \mathcal{S}_0(q)} + A}. \tag{45}
\]

We remark that related interesting issues have been recently considered in \(^{[10]}\).

Similarly to the case of general relativity in which the equivalence principle leads to the deformation of the geometry, also in quantum mechanics one should investigate whether the equivalence principle implies a space deformation. In this context, the structure of the QSHJE \(^{(38)}\) suggests considering an underlying geometrical structure. Actually, Eq.\(^{(38)}\) naturally leads to a coordinate transformation depending on the quantum potential. The key point is that \(^{(38)}\) can be written in the form
\[
\frac{1}{2m} \left( \frac{\partial \mathcal{S}_0}{\partial \tilde{q}} \right)^2 + V(q) - E = 0, \tag{46}
\]
where
\[
\left( \frac{\partial q}{\partial \tilde{q}} \right)^2 = \left[ 1 - \hbar^2 \mathcal{U}(\mathcal{S}_0) \right], \tag{47}
\]
or equivalently (we omit the solution with the minus sign)

\[
d\hat{q} = \frac{dq}{\sqrt{1 - \beta^2(q)}},
\]

with

\[
\beta^2(q) = \hbar^2 \mathcal{U}(S_0) = \frac{\hbar^2}{2} \{q, S_0\}.
\]

Integrating (47) yields

\[
\hat{q} = \int^q \frac{dx}{\sqrt{1 - \beta^2(x)}}.
\]

We observe that the nature of the coordinate transformation is purely quantum mechanical; in particular

\[
\lim_{\hbar \to 0} \hat{q} = q.
\]

Equation (50) indicates that in considering the differential structure one should take into account the effect of the quantum potential on the space geometry. In this context, the deformation of the CSHJE amounts to replacing the standard derivative with respect to the classical coordinate \( q \) with the derivative with respect to the deformed quantum coordinate \( \hat{q} \). In other words, the transition from the classical to the quantum regime amounts to a reconsideration of the underlying geometry which is modified by the quantum potential itself.

A property of the quantum transformation (48) is that it allows to put the QSHJE in the classical form. Namely, setting

\[
\hat{\mathcal{W}}(\hat{q}) = \mathcal{W}(q(\hat{q})),
\]

\[
\hat{S}_0(\hat{q}) = S_0(q(\hat{q})),
\]

it follows that Eq.(38), equivalent to Eq.(39), can be written in the form

\[
\frac{1}{2m} \left( \frac{\partial S_0(\hat{q})}{\partial \hat{q}} \right)^2 + \hat{\mathcal{W}}(\hat{q}) = 0.
\]

This can be seen as the opposite of the problem, considered by Schiller and Rosen [11], of determining the wave function representation for classical mechanics (see also [7]).

In the standard formulation of the quantum analogue of the Hamilton–Jacobi equation [12], one considers a couple of equations which arise by setting \( \psi = R e^{iS_0/\hbar} \), so that for the state \( \mathcal{W}^0(q^0) \) one chooses \( S_0^0 = \text{cnst} \) and \( R = Aq^0 + B \). Note that setting \( \psi = R e^{iS_0/\hbar} \) is suggested by
the interpretation of $|\psi|^2 = R^2$ as probability density. On the other hand, it is easy to see that any solution has the form

$$\psi = \frac{1}{\sqrt{S_0}} \left( A e^{-\frac{i}{\hbar} S_0} + B e^{\frac{i}{\hbar} S_0} \right).$$

(55)

However, while on the one hand it is not possible to define the Legendre transformation of a constant, so that the $S_0-T_0$ duality would be lost and $\{S_0^0, q^0\}$ cannot be defined, on the other hand, we have that the overlooked solution $S_0^0 = \frac{\hbar}{2i} \ln \bar{q}^0$ for the state $W^0$ still gives the same solution $\psi = A q^0 + B$ of the Schrödinger equation $-\frac{\hbar^2}{2m} \partial^2 \psi = 0$. In this context we observe that the non–linear relation between $S_0$ and the wave function, which can be also written in the form $S_0(q) = \frac{\hbar}{2i} \ln (A f^0 \psi^{-2} + B)/(C f^0 \psi^{-2} + D)$, is related to an incomplete equivalence between the Schrödinger equation and the QSHJE (23). Another interesting example of inequivalence between the Schrödinger equation and Eq.(23) is provided in [5][13] where it has been shown that for bound states the QSHJE (23) describes microstates not detected in the Schrödinger representation. This aspect will be considered in great detail in [3].

The fact that the QSHJE admits the classical representation (54) suggests that classically forbidden regions correspond to critical regions for the quantum coordinate. Actually, writing Eq.(50) in the equivalent form ($s = S_0(q)$)

$$\hat{q} = \int^q dx \frac{\partial_x S_0}{\sqrt{-2m W}} = \int^{S_0(q)} ds \frac{d_s}{\sqrt{-2m W}},$$

(56)

we see that the integrand is purely imaginary in the classically forbidden regions $W > 0$. Furthermore, since according to (33) the conformal factor for the state $W^0$ vanishes, it follows by (50) that the quantum coordinate for the free particle state with vanishing energy is divergent. To better understand the role of the state $W^0$ in (53) it is useful to first rederive the self–dual states (35) by another approach.

The $S_0-T_0$ duality implies that a given physical system may be described either by the $S_0$–picture or by the $T_0$–picture. On general grounds, it is clear that naturally selected states are the ones corresponding to the degenerate case in which the $S_0$ and $T_0$ pictures overlap. In order to find this common subspace we consider the interchange of the $S_0$ and $T_0$ pictures given by

$$q \rightarrow \tilde{q} = \alpha p, \quad p \rightarrow \tilde{p} = \beta q.$$

(57)

This implies that

$$\frac{\partial \tilde{T}_0}{\partial \tilde{p}} = \alpha \frac{\partial S_0}{\partial q}, \quad \frac{\partial \tilde{S}_0}{\partial \tilde{q}} = \beta \frac{\partial T_0}{\partial p}.$$

(58)
which is equivalent to
\[
\frac{\partial \tilde{T}_0}{\partial q} = \alpha \beta \frac{\partial S_0}{\partial q}, \quad \frac{\partial \tilde{S}_0}{\partial p} = \alpha \beta \frac{\partial T_0}{\partial p},
\]
that is
\[
\tilde{S}_0(\tilde{q}) = \alpha \beta T_0(p) + \text{cnst}, \quad \tilde{T}_0(\tilde{p}) = \alpha \beta S_0(q) + \text{cnst}.
\]
Furthermore, since we require that \((57)\) be of order two, we have up to an additive constant
\[
\tilde{S}_0 = S_0, \quad \tilde{T}_0 = T_0,
\]
so that
\[
(\alpha \beta)^2 = 1.
\]
We observe that \(\tilde{S}_0(\tilde{q})\) and \(\tilde{T}_0(\tilde{p})\) are basically the Legendre transformation of \(S_0(q)\) and \(T_0(p)\) respectively. The distinguished states are precisely those which are left invariant by \((57)\) and \((60)\), that is
\[
\tilde{S}_0(\tilde{q}) = S_0(q) + \text{cnst}.
\]
Let us now introduce the Legendre transformation of the Hamilton principal function \(S\)
\[
S = p \frac{\partial T}{\partial p} - T, \quad T = q \frac{\partial S}{\partial q} - S,
\]
\[
p = \frac{\partial S}{\partial q}, \quad q = \frac{\partial T}{\partial p},
\]
Observe that for stationary states
\[
S(q, t) = S_0(q) - Et, \quad T(p, t) = T_0(p) + Et.
\]
Let us consider the differentials
\[
dS = \frac{\partial S}{\partial q} dq + \frac{\partial S}{\partial t} dt = pdq + \frac{\partial S}{\partial t} dt,
\]
\[
dT = \frac{\partial T}{\partial p} dp + \frac{\partial T}{\partial t} dt = qdp + \frac{\partial T}{\partial t} dt,
\]
so that
\[
dS = d(pq - T) = pdq + qdp - qdp - \frac{\partial T}{\partial t} dt,
\]
that is
\[
\frac{\partial S}{\partial t} = -\frac{\partial T}{\partial t}.
\]
This equation connects the $S$ and $T$ pictures through the time evolution. By (60) (62) (63) and (66) we have that the distinguished states correspond to

$$S = \pm T + \text{cnst.}$$  \hfill (71)

As (71) should be stable under time evolution, the relation (70) fixes the sign ambiguity and sets

$$\alpha \beta = -1.$$  \hfill (72)

Therefore, the distinguished states correspond to

$$S = -T + \text{cnst.}$$  \hfill (73)

Since $S = pq - T$, we have

$$pq = \gamma,$$  \hfill (74)

where $\gamma$ is a constant. Therefore, the distinguished states are precisely the self–dual states (35).

We have seen that the self–dual states with $\gamma = \gamma_{sd} \equiv \pm \hbar / 2 i$ correspond to the state $W^0$. The fact that it corresponds to two of the distinguished states connecting the $S_0$ and $T_0$ pictures, indicates that $W^0$ corresponds to a critical point for the coordinate transformation. In this context the observed divergence for $\hat{q}$ corresponding to this state is not a surprise.

We note that due to the Möbius symmetry of the Schwarzian derivative, one has

$$W = -\frac{\hbar^2}{4m} \{ e^{\frac{2\pi}{\hbar}S_0}, q \} = -\frac{\hbar^2}{4m} \left\{ \frac{Ae^{\frac{2\pi}{\hbar}S_0} + B}{Ce^{\frac{2\pi}{\hbar}S_0} + D}, q \right\}. $$  \hfill (75)

This means that to find $S_0$ we need to fix three integration constants. Let us set

$$w = \frac{\psi^D}{\psi},$$  \hfill (76)

with $\psi^D$ and $\psi$ two linearly independent real solutions of the Schrödinger equation (note that these solutions always exist). Reality condition for $S_0$ restricts (24) to

$$e^{\frac{2\pi}{\hbar}S_0} = e^{i\alpha w + i\ell} \frac{w}{w - i\ell},$$  \hfill (77)

with $\alpha$ and $\ell$ real and complex integration constants respectively. Note that $\text{Re} \ell \neq 0$. The solution (77), derived in [2], is an equivalent form of the one previously derived in [1].

It follows by (21) that what is invariant is not $(\partial_q S_0)^2$ but rather $(\partial_q S_0)^2 + \hbar^2 \{ S_0, q \} / 2$. This shows that the “kinetic term” $\frac{1}{2m} (\partial_q S_0)^2$ and the quantum correction $Q = \frac{\hbar^2}{4m} \{ S_0, q \}$ mix under
a change of initial conditions. A property of this mixing is that this disappears in the classical regime only, where $Q \rightarrow 0$. We note that the role of the quantum correction $Q$ is somehow reminiscent of the relativistic rest energy, as it is an intrinsic property of the particle.

The above investigations, and the equivalence principle in particular, indicate that quantum mechanics is strictly connected to geometrical properties of space. It it then natural to investigate the existence of possible geometrical structures underlying the QSHJE. In order to do this we use a result obtained by Flanders [14] who showed that the Schwarzian derivative can be interpreted as an invariant (curvature) of an equivalence problem for curves in $\mathbb{P}^1$.

Let us introduce a frame for $\mathbb{P}^1$. This consists of a pair $x, y$ of points in affine space $\mathbb{A}^2$ such that $[x, y] = 1$, where $[x, y] = x_1y_2 - x_2y_1$ is the area function. Considering the moving frame $s \rightarrow \{x(s), y(s)\}$ and differentiating $[x, y] = 1$ yields the structure equations

$$ x' = ax + by, \quad y' = cx - ay, \quad (78) $$

where $a, b$, and $c$ depend on $s$. Given a map $\phi = \phi(s)$ from a domain to $\mathbb{P}^1$, one can choose a moving frame $x(s), y(s)$ in such way that $\phi(s)$ is represented by $x(s)$. Observe that this map can be seen as a curve in $\mathbb{P}^1$. Two mappings $\phi$ and $\psi$ are said to be equivalent if $\psi = \pi \circ \phi$ with $\pi$ a projective transformation on $\mathbb{P}^1$.

Flanders considered two extreme situations. Let $b(s) = 0$, for all $s$. In this case $\phi$ is constant. Actually, taking the derivative of $\lambda x$, for some $\lambda(s) \neq 0$, by (78) we have $(\lambda x)' = (\lambda' + a\lambda)x$. Choosing $\lambda \propto \exp[-\int_{s_0}^s dt a(t)] \neq 0$, we have $(\lambda x)' = 0$, so that $\lambda x$ is a constant representative of $\phi$.

The other case is for $b$ never vanishing. There are only two inequivalent situations. The first one is when $b$ is either complex or positive. It turns out that it is always possible to choose the following “natural moving frame” for $\phi$ [14]

$$ x' = y, \quad y' = -kx. \quad (79) $$

In the other case in which $b$ is real and negative, the natural moving frame for $\phi$ is

$$ x' = -y, \quad y' = kx. \quad (80) $$

A characterizing property of the natural moving frame is determined up to a sign and $k$ is an invariant. Thus, for example, suppose that for a given $\phi$ there is, besides (79), the natural moving frame $x'_1 = y_1, y'_1 = -k_1x_1$. Since both $x$ and $x_1$ are representatives of $\phi$, we have
\( \mathbf{x} = \lambda \mathbf{x}_1 \), so that \( \mathbf{y} = \mathbf{x}' = \lambda \mathbf{x}_1 + \lambda \mathbf{y}_1 \) and \( 1 = [\mathbf{x}, \mathbf{y}] = \lambda^2 \). Therefore, \( \mathbf{x}_1 = \pm \mathbf{x}, \mathbf{y}_1 = \pm \mathbf{y} \) and \( k_1 = k \) [14].

Let us now review the derivation of Flanders formula for \( k \). Consider \( s \rightarrow \mathbf{z}(s) \) to be an affine representative of \( \phi \) and let \( \mathbf{x}(s), \mathbf{y}(s) \) be a natural frame. Then \( \mathbf{z} = \lambda \mathbf{x} \) where \( \lambda(s) \) is never vanishing. Now note that, since \( \mathbf{z}' = \lambda' \mathbf{x} + \lambda \mathbf{y} \), we have that \( \lambda \) can be written in terms of the area function \( [\mathbf{z}, \mathbf{z}'] = \lambda^2 \). Computing the relevant area functions, one can check that that \( k \) has the following expression

\[
2k = \frac{[\mathbf{z}, \mathbf{z}'''] + 3[\mathbf{z}', \mathbf{z}'']}{[\mathbf{z}, \mathbf{z}']} - \frac{3}{2} \left( \frac{[\mathbf{z}, \mathbf{z}']}{[\mathbf{z}, \mathbf{z}']} \right)^2.
\]

(81)

Given a function \( \mathbf{z}(s) \), this can be seen as the non–homogeneous coordinate of a point in \( \mathbb{P}^1 \). Therefore, we can associate to \( \mathbf{z} \) the map \( \phi \) defined by \( s \rightarrow (1, \mathbf{z}(s)) = \mathbf{z}(s) \). In this case we have \( [\mathbf{z}, \mathbf{z}'] = z', [\mathbf{z}, \mathbf{z}'''] = z''', [\mathbf{z}', \mathbf{z}'''] = 0 \), and the curvature becomes \[14\]

\[
k = \frac{1}{2} \{z, s\}.
\]

(82)

Let us now consider a state \( \mathcal{W} \). We have

\[
\mathcal{W} = -\frac{\hbar^2}{4m} \{e^{\frac{2\pi i}{\hbar} \mathcal{S}_0}, q\} = -\frac{\hbar^2}{2m} k_{\mathcal{W}}.
\]

(83)

Similarly, for the quantum potential

\[
\mathcal{Q} = \frac{\hbar^2}{4m} \{\mathcal{S}_0, q\} = \frac{\hbar^2}{2m} k_{\mathcal{Q}},
\]

(84)

where \( k_{\mathcal{W}} \) is the curvature associated to the map

\[
q \rightarrow (1, e^{\frac{2\pi i}{\hbar} \mathcal{S}_0(q)}),
\]

(85)

while the curvature \( k_{\mathcal{Q}} \) is associated to the map

\[
q \rightarrow (1, \mathcal{S}_0(q)).
\]

(86)

The function defining the map (85) corresponds to the one defining the trivializing map (13).

The identification of \(-2m \mathcal{W}/\hbar^2 \) with the curvature \( k_{\mathcal{W}} \) allows us to write the Schrödinger equation in the geometrical form

\[
\left( \frac{\partial^2}{\partial q^2} + k_{\mathcal{W}} \right) \psi = 0.
\]

(87)

Furthermore, the identity (24) can be now seen as difference of curvatures

\[
(\partial_q \mathcal{S}_0)^2 = \hbar^2 k_{\mathcal{W}} - \hbar^2 k_{\mathcal{Q}},
\]

(88)
and the QSHJE (23) can be written in the form

\[
\frac{1}{2m} \left( \frac{\partial S_0(q)}{\partial q} \right)^2 + \mathcal{W}(q) + \frac{\hbar^2}{2m} k_q = 0.
\]  

(89)

Let us now consider the meaning of the natural moving frame in the framework of the QSHJE. First observe that the structure equations imply that

\[
x'' = -kx.
\]  

(90)

In the case of \( k = k_\mathcal{W} \), this equation is the Schrödinger equation, so that

\[
x = (\psi^D, \psi), \quad y = (\psi^{D'}, \psi'),
\]  

(91)

and the frame condition is nothing else but the statement that the Wronskian \( W \) of the Schrödinger equation is a constant

\[
[x, y] = \psi^D \psi' - \psi \psi'^D = W = 1.
\]  

(92)

Hence, the Schrödinger equation determines the natural moving frame associated to the curve in \( \mathbb{P}^1 \) given by the representative (85) with \(-2m\mathcal{W}/\hbar^2\) denoting the invariant associated to the map. In other words, the Schrödinger problem corresponds to finding the natural moving frame such that \(-2m\mathcal{W}/\hbar^2\) be the invariant curvature.

In the case \( k = k_Q \), Eq.(90) becomes

\[
\left( \frac{\hbar^2}{2m} \frac{\partial^2}{\partial q^2} + Q \right) \phi = 0,
\]  

(93)

so that if \( \phi^D \) and \( \phi \) are solutions of (23), then

\[
S_0 = \frac{A\phi^D + B\phi}{C\phi^D + D\phi}.
\]  

(94)

Note that the solutions of (94) are related to the solutions of the Schrödinger equation by

\[
\gamma_\psi \left( \psi^D / \psi \right) = e^{\frac{i}{\hbar} \gamma_\phi (\phi^D / \phi)},
\]  

(95)

where \( \gamma_\psi \) and \( \gamma_\phi \) denote two Möbius transformations.

Let us conclude this Letter with a few remarks. As we noticed above, with respect to the standard solution for the free quantum state with vanishing energy, \( S_0^0 = \text{cnst} \), which is the same as for classical mechanics, one has that what is vanishing is not \( \partial_{q^0} S_0^0 \) but \( 2(\partial_{q^0} S_0^0(q^0))^2 + \)
\( \hbar^2 \{ S_0^0, q^0 \} = 0 \), so that \( S_0^0 = \frac{\hbar}{2i} \ln q^0 \) and there is a non zero curvature term associated to the free particle with vanishing energy.

In the conventional approach to quantum mechanics the discretized spectra and its structure arise from the properties imposed on the wave function. For example, for the harmonic oscillator one requires that the wave function vanishes at infinity: a direct consequence of the axiomatic interpretation of the wave function as probability amplitude. An outcome of [3] is that the quantized energy spectra and their structure are a direct consequence of the equivalence principle. Therefore, two basic aspects of quantum mechanics, such as the tunnel effect and energy quantization (and its structure), strictly related to the wave function interpretation, arise in our approach as a consequence of the equivalence principle.

While this principle has been formulated for the 1D case, it actually implies the time dependent Schrödinger equation in \( D + 1 \) dimensions. The point is that this is the unique possibility if one requests that the deformed Hamilton–Jacobi equation reduces to the classical one in the \( \hbar \rightarrow 0 \) limit and reproduces \( D \) copies of the one–dimensional quantum Hamilton–Jacobi equation in the case in which \( V(q_1, \ldots, q_D, t) = \sum_{k=1}^{D} V_k(q_k) \). In doing this, one uses the higher dimensional generalizations of the cocycle condition (16) and of the identity (21) [3][4].

It is worth stressing that in the higher dimensional case the resulting quantum Hamilton–Jacobi equation reproduces the Bohmian one but with the some additional important conditions [3] such as the exclusion of the solution \( S_0 = \sum_{k=1}^{D} A_k q_k + B \). This aspect follows from a detailed analysis which includes the study of both the \( E \rightarrow 0 \) and \( \hbar \rightarrow 0 \) limits [3]. This fact is strictly related to the existence of the Schwarzian derivative and of the Legendre transformation of \( S_0 \), which in turn is related to the issue of \( p–q \) duality [3].

We observe that our investigation is related to the approach in [15], further developed by Carroll in [16], where it has been shown that the space coordinate is proportional to the Legendre transformation of the prepotential \( F \), defined by \( \psi^D = \partial_{\psi} F \), with respect to the square of the wave function.

Finally, we note that very recently related issues have been considered in [17].

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