1. Introduction

At the age of eighteen, Gauss established the constructibility of the 17-gon, a result that had eluded mathematicians for two millennia. At the heart of his argument was a keen study of certain sums of complex exponentials, known now as Gaussian periods. These sums play starring roles in applications both classical and modern, including Kummer’s development of arithmetic in $\mathbb{Z}[\zeta]$ and the optimized AKS primality test of H. W. Lenstra and C. Pomerance. In a poetic twist, this recent application of Gaussian periods realizes “Gauss’ dream” of an efficient algorithm for distinguishing prime numbers from composites.

We seek here to study Gaussian periods from a graphical perspective. It turns out that these classical objects, when viewed appropriately, exhibit a dazzling and eclectic host of visual qualities. Some images contain discretized versions of familiar shapes, while others resemble natural phenomena. Many can be colorized to isolate certain features; for details, see Section 3.

![Diagram](image_url)

Figure 1. Eye and jewel — Images of cyclic supercharacters correspond to sets of Gaussian periods. For notation and terminology, see Section 3.
2. Historical context

The problem of constructing a regular polygon with compass and straight-edge dates back to Euclid, who gave an explicit construction of the 15-gon in Book IV of his *Elements*. Descartes and others knew that with only these tools on hand, the motivated geometer could draw, in principle, any segment whose length could be written as a finite composition of additions, multiplications and square roots of rational numbers [18]. Gauss’ construction of the 17-gon relied on showing that

\[
16 \cos \left( \frac{2\pi}{17} \right) = -1 + \sqrt{17} + \sqrt{34 - 2\sqrt{17}} + 2\sqrt{17 + 3\sqrt{17}} - \sqrt{34 - 2\sqrt{17}} - 2\sqrt{34 + 2\sqrt{17}}
\]

was such a length. After reducing the constructibility of the \( n \)-gon to drawing the length \( \cos \left( \frac{2\pi}{n} \right) \), his result followed easily. So proud was Gauss of this discovery that he wrote about it throughout his career, purportedly requesting a 17-gon in place of his epitaph\(^1\). While this appeal went unhonored, sculptor Fritz Schaper did include a 17-pointed star at the base of a monument to Gauss in Brunswick, where the latter was born [29].

\[\text{(A) } n = 3 \cdot 5 \cdot 17 \cdot 29 \cdot 37, \omega = 184747, c = 3 \cdot 17\]
\[\text{(B) } n = 13 \cdot 127 \cdot 199, \omega = 6077, c = 13\]

**Figure 2.** Disco ball and loudspeaker — Images of *cyclic supercharacters* correspond to sets of Gaussian periods. For notation and terminology, see Section 3.

Gauss went on to demonstrate that a regular \( n \)-gon is constructible if Euler’s totient \( \varphi(n) \) is a power of 2. He stopped short of proving that these were the *only* cases of constructibility; this remained unsettled until J. Petersen completed a largely-neglected argument of P. Wantzel [31], nearly three quarters of a century later. Nonetheless, the chapter containing Gauss’ proof has persisted deservedly as perhaps the most well-known section of his *Disquisitiones Arithmeticae*.

\(^1\)H. Weber makes a footnote of this anecdote in [40, p. 362] but omits it curiously from later editions.
Without the language of abstract algebra, Gauss initiated the study of cyclotomy, literally “circle cutting,” from an algebraic point of view.

\[
\begin{align*}
(a) & \quad n = 13 \cdot 127 \cdot 199, \quad X = \omega = 9247, \quad c = 127 \\
(b) & \quad n = 3 \cdot 7 \cdot 211 \cdot 223, \quad \omega = 710216, \quad c = 211
\end{align*}
\]

**Figure 3.** Mite and moth — Images of cyclic supercharacters correspond to sets of Gaussian periods. For notation and terminology, see Section 3.

The main ingredient in Gauss’ argument is an exponential sum known as a Gaussian period. Denoting the cardinality of a set \( S \) by \(|S|\), if \( p \) is an odd prime number and \( \omega \) has order \( d \) in \((\mathbb{Z}/p\mathbb{Z})^\times\), then the \( d \)-nomial Gaussian periods modulo \( p \) are the complex numbers

\[
\sum_{j=0}^{d-1} e\left(\frac{\omega^j y}{p}\right),
\]

where \( y \) belongs to \( \mathbb{Z}/p\mathbb{Z} \) and \( e(\theta) \) denotes \( \exp(2\pi i \theta) \) for real \( \theta \). Following its appearance in Disquisitiones, Gauss’ cyclotomy drew the attention of several mathematicians who saw its potential use in their own work. In 1879, J. J. Sylvester wrote that “[c]yclotomy is to be regarded . . . as the natural and inherent centre and core of the arithmetic of the future” [37]. Exaggeration notwithstanding, Sylvester’s claim does reflect a certain prominence of cyclotomy in 19th-century number theory. In fact, two of Kummer’s most significant achievements depended critically on his study of Gaussian periods. To be more specific, Gauss’ work laid the foundation for Kummer’s proof of Fermat’s Last Theorem in the case of regular primes, and later for Kummer’s reciprocity law.

This success inspired Kummer to generalize Gaussian periods in [28] to the case of composite moduli. Essential to his work was a study of the polynomial \( x^d - 1 \) by his former student, Kronecker, who continued to receive encouragement from his boyhood mentor for the better part of both men’s careers [26]. Just as Gaussian periods for prime moduli had given rise to various families of difference sets [7], Kummer’s composite cyclotomy has been used to explain certain difference sets discovered by I. Singer arising in finite projective geometry [14]. Shortly after Kummer’s publication, L. Fuchs
presented a result in [22] concerning the vanishing of Kummer’s periods that has appeared in several applications by K. Mahler [32, 33]. A modern treatment of Fuchs’ result and a further generalization of Gaussian periods are given in [20].

For a positive integer \( n \) and positive divisor \( d \) of \( \varphi(n) \), Kummer “defined” a \( d \)-nomial period modulo \( n \) to be

\[
\sum_{j=0}^{d-1} e \left( \frac{\omega^j y}{n} \right),
\]

where \( \omega \) has order \( d \) in the unit group \((\mathbb{Z}/n\mathbb{Z})^\times\) and \( y \) ranges over \( \mathbb{Z}/n\mathbb{Z} \). Unlike the case of prime moduli, there is no guarantee that a generator \( \omega \) of order \( d \) will exist or be unique. A similar lack of specificity pervaded Kummer’s other definitions, including notably his introduction of “ideal prime factors” used to prove a weak form of prime factorization for cyclotomic integers. This should hardly be viewed as a deficiency, however. Instead, as H. M. Edwards suggests, these examples indicate “the mathematical culture . . . as Kummer saw it” [19].

Fortunately for our purposes, we can clear up the ambiguity in Kummer’s definition by characterizing the periods in terms of the generator \( \omega \) instead of the divisor \( d \) of \( n \). That is, for \( n \) as above and an element \( \omega \) of \((\mathbb{Z}/n\mathbb{Z})^\times\), we define the Gaussian periods generated by \( \omega \) modulo \( n \) to be the sums in (1), where \( d \) is the order of \( \omega \) and \( y \) ranges over \( \mathbb{Z}/n\mathbb{Z} \), as before. These periods are closely related to Gauss sums, another type of exponential sum [9].

**Figure 4.** Atoms — Images of cyclic supercharacters correspond to sets of Gaussian periods. For notation and terminology, see Section 3.

### 3. Cyclic supercharacters

The theory of supercharacters was introduced in an axiomatic fashion in 2008 by P. Diaconis and I.M. Isaacs [15], building upon seminal work of C. Andrè on the representation theory of
unipotent matrix groups \[3, 4\]. Among other things, supercharacter techniques have been used to study the Hopf algebra of symmetric functions of noncommuting variables \[2\], random walks on upper triangular matrices \[5\], and combinatorial properties of Schur rings \[16, 38, 39\].

Let \(G\) be a finite group with identity \(0\), \(\mathcal{K}\) a partition of \(G\), and \(\mathcal{X}\) a partition of the set \(\text{Irr}(G)\) of irreducible characters of \(G\). The ordered pair \((\mathcal{X}, \mathcal{K})\) is called a supercharacter theory for \(G\) if \(\{0\} \in \mathcal{K}\), \(|\mathcal{X}| = |\mathcal{K}|\), and for each \(X \in \mathcal{X}\), the function

\[
\sigma_X = \sum_{\chi \in X} \chi(0) \chi
\]

is constant on each \(K \in \mathcal{K}\). The functions \(\sigma_X : G \to \mathbb{C}\) are called supercharacters and the elements of \(\mathcal{K}\) are called superclasses.

The irreducible characters of \(G = \mathbb{Z}/n\mathbb{Z}\) are the functions \(\chi_x(y) = e(\frac{xy}{n})\) for \(x\) in \(\mathbb{Z}/n\mathbb{Z}\). For a subgroup \(\Gamma\) of \((\mathbb{Z}/n\mathbb{Z})^\times\), let \(\mathcal{K}\) denote the partition of \(\mathbb{Z}/n\mathbb{Z}\) arising from the action \(a \cdot x = ax\) of \(\Gamma\). The action \(a \cdot \chi_x = \chi_{a^{-1}x}\) of \(\Gamma\) on the irreducible characters of \(\mathbb{Z}/n\mathbb{Z}\) yields a compatible partition \(\mathcal{X}\) making \((\mathcal{X}, \mathcal{K})\) a supercharacter theory on \(\mathbb{Z}/n\mathbb{Z}\). The corresponding supercharacters are

\[
\sigma_X(y) = \sum_{x \in X} e\left(\frac{xy}{n}\right) .
\]

(2)

For a positive integer \(n\) and an orbit \(X\) of \(\mathbb{Z}/n\mathbb{Z}\) under the multiplication action of a cyclic subgroup \(\langle \omega \rangle\) of \((\mathbb{Z}/n\mathbb{Z})^\times\), we define the cyclic supercharacter \(\sigma_X : \mathbb{Z}/n\mathbb{Z} \to \mathbb{C}\) by \(2\). The values of these functions are Gaussian periods in the sense of Kummer \[17\]; for applications of supercharacter theory to other exponential sums see \[11, 12, 21\].

![Figure 5. Bird and spacecraft — images of cyclic supercharacters](image-url)
σ(ω) : \mathbb{Z}/n\mathbb{Z} \to \mathbb{C}, \text{ where } \omega \text{ belongs to } (\mathbb{Z}/n\mathbb{Z})^\times, \text{ and } \langle \omega \rangle 1 \text{ denotes the orbit of 1 under the action of the subgroup generated by } \omega. \text{ Conveniently, the image of any cyclic supercharacter is a scaled subset of the image of one having the form } \sigma(\omega) \text{ [17, Proposition 2.2]}, \text{ so a restriction of our attention to orbits of 1 is quite natural. Moreover, the reader will note that the image of } \sigma(\omega) \text{ is the set of Gaussian periods generated by } \omega \text{ modulo } n, \text{ bringing classical relevance to these figures.}

To colorize each image, we fix a proper divisor } c \text{ of } n \text{ and assign a color to each of the layers } \{\sigma(\omega_1)(y) \mid y \equiv j \pmod{c}\}, \text{ for } j = 0, 1, \ldots, c - 1. \text{ Different choices of } c \text{ result in different “layerings”; for many images, certain values of } c \text{ yield colorizations that separate distinct graphical components.}

Predictable layering occurs when the image of a cyclic supercharacter contains several rotated copies of a proper subset. We say that a subset of } \mathbb{C} \text{ has } k \text{-fold dihedral symmetry} \text{ if it is invariant under complex conjugation and rotation by } \frac{2\pi}{k} \text{ about the origin. For example, the image pictured in Figure 4(A) has } 11\text{-fold dihedral symmetry, while the symmetry in Figure 4(B)} \text{ is } 7\text{-fold. The image of a cyclic supercharacter } \sigma(\omega) : \mathbb{Z}/n\mathbb{Z} \to \mathbb{C} \text{ has } k\text{-fold dihedral symmetry, where } k = \gcd(n,\omega - 1) \text{ [17, Proposition 3.1]. In this situation, taking } c = k \text{ results in exactly } k \text{ layers that are rotated copies of one another.}

\begin{figure}[h]
\centering
\begin{subfigure}{0.45\textwidth}
\includegraphics[width=\textwidth]{figure1.png}
\caption{(A) } m = 251 \cdot 281, \omega_m = 54184 \end{subfigure}
\begin{subfigure}{0.45\textwidth}
\includegraphics[width=\textwidth]{figure2.png}
\caption{(B) } mn = 5 \cdot 251 \cdot 281, \omega = 54184, c = 5 \end{subfigure}
\caption{The image on the right is the product set of the image on the left and the image \{2, \frac{1}{2}(\pm\sqrt{5} - 1)\}, as in (3).}
\end{figure}

In addition to the behaviors above, certain cyclic supercharacters enjoy a multiplicative property [17, Theorem 2.1]. Specifically, if \gcd(m, n) = 1 \text{ and } \omega \mapsto (\omega_m, \omega_n) \text{ under the isomorphism } (\mathbb{Z}/mn\mathbb{Z})^\times \to (\mathbb{Z}/m\mathbb{Z})^\times \times (\mathbb{Z}/n\mathbb{Z})^\times \text{ afforded by the Chinese Remainder Theorem, where the multiplicative orders of } \omega_m \text{ and } \omega_n \text{ are coprime, then}

\[ \sigma(\omega)(\mathbb{Z}/mn\mathbb{Z}) = \{ wz \in \mathbb{C} : (w, z) \in \sigma(\omega_m)(\mathbb{Z}/m\mathbb{Z}) \times \sigma(\omega_n)(\mathbb{Z}/n\mathbb{Z}) \}. \] (3)
This can be used to explain the images of cyclic supercharacters featuring “nested” copies of a given shape. For examples of this phenomenon, see Figures 6 and 7.

4. Asymptotic behavior

In this section, we restrict our attention to cyclic supercharacters $\sigma_X : \mathbb{Z}/q\mathbb{Z} \to \mathbb{C}$, where $q$ is a power of an odd prime and $X = \langle \omega \rangle$ is an orbit of 1. The Gaussian periods attained as values of these supercharacters have been applied in various settings [6, 8, 25]. Plotting the functions $\sigma_X$ in this case reveals asymptotic patterns that have, until recently, gone unseen.

We recall several definitions. For each positive integer $n$, let $S_n \subset \mathbb{R}^m$ be a finite set and let

$$S'_n = \{ (s_1 - \lfloor s_1 \rfloor, \ldots, s_m - \lfloor s_m \rfloor) \in [0, 1)^m : (s_1, \ldots, s_m) \in S_n \}.$$
where \(|x|\) denotes the greatest integer function. We say that the sequence \((S_n)_{n=1}^\infty\) is uniformly distributed modulo 1 if
\[
\lim_{n \to \infty} \sup_B \left| \frac{|B \cap S_n'|}{|S_n'|} - \text{vol}(B) \right| = 0,
\]
where the supremum is taken over all boxes \(B = [a_1, b_1] \times \cdots \times [a_m, b_m] \subset [0,1)^m\) and \(\text{vol}(B)\) denotes the volume of \(B\). The following key fact is due to H. Weyl [41].

**Lemma 1** (Weyl’s criterion). A sequence \((S_n)_{n=1}^\infty\) of finite subsets of \(\mathbb{R}^m\) is uniformly distributed modulo 1 if and only if for each \(v\) in \(\mathbb{Z}^m\) we have
\[
\lim_{n \to \infty} \frac{1}{|S_n|} \sum_{u \in S_n} e(u \cdot v) = 0.
\]

We require another definition to proceed. The \(d\)th cyclotomic polynomial \(\Phi_d(x)\), given by
\[
\Phi_d(x) = \prod_{\substack{k=1,2,\ldots,d \\gcd(k,d)=1}} \left( x - e \left( \frac{k}{d} \right) \right),
\]
is the unique monic polynomial with integer coefficients that divides \(x^d - 1\) but not \(x^k - 1\) for any \(k = 1, 2, \ldots, d - 1\). It can be shown that \(\Phi_d(x)\) is of degree \(\varphi(d)\) and is the minimal polynomial in \(\mathbb{Q}[x]\) of any primitive \(d\)th root of unity. The first several cyclotomic polynomials are
\[
\Phi_1(t) = x - 1, \\
\Phi_2(t) = x + 1, \\
\Phi_3(t) = x^2 + x + 1, \\
\Phi_4(t) = x^2 + 1, \\
\Phi_5(t) = x^4 + x^3 + x^2 + x + 1.
\]
In these examples, the coefficients have absolute value at most 1. N. G. Chebotarëv asked in 1938 whether this phenomenon continued for all factors of \(x^d - 1\) and all values of \(d\) [24]. Three years later, V. Ivanov showed that while the pattern holds for \(d < 105\), one coefficient of \(\Phi_{105}(x)\) is \(-2\).

Unbeknownst to either mathematician, however, A. S. Bang had solved Chebotarëv’s challenge more than forty years earlier [10].

In what follows, we let \(\omega_q\) denote a primitive \(d\)th root of unity in \(\mathbb{Z}/q\mathbb{Z}\) and
\[
S_q = \left\{ \frac{\ell}{q} \left( 1, \omega_q, \omega_q^2, \ldots, \omega_q^{\varphi(d)-1} \right) \in [0,1)^{\varphi(d)} : \ell = 0, 1, \ldots, q - 1 \right\}.
\]

**Lemma 2** (Myerson [34]). The sequence \((S_q)_{q \equiv 1 \pmod{d}}\) is uniformly distributed modulo 1.

**Proof.** Let \(v = (v_0, \ldots, v_{\varphi(d)-1})\) be nonzero in \(\mathbb{Z}^{\varphi(d)}\), and let \(f \in \mathbb{Z}[x]\) be given by
\[
f(x) = v_0 + v_1 x + \cdots + v_{\varphi(d)-1} x^{\varphi(d)-1}.
\]
Writing \(r = \frac{q}{\gcd(q,f(\omega_q))}\), we notice that
\[
\sum_{u \in S_q} e(u \cdot v) = \sum_{\ell=0}^{q-1} e \left( \frac{\ell f(\omega_q)}{q} \right)
\]
Recall that \( \Phi_d \) is the minimal polynomial in \( \mathbb{Q}[x] \) of each of its roots. Since \( \deg f < \deg \Phi_d \), it follows that \( \gcd(f(x), \Phi_d(x)) \) in \( \mathbb{Q}[x] \) has degree 0, yielding \( a(x) \) and \( b(x) \) in \( \mathbb{Z}[x] \) such that

\[
a(x)f(x) + b(x)\Phi_d(x) = m
\]

for some \( m \in \mathbb{Z} \). This gives \( a(\omega_q)f(\omega_q) \equiv m \pmod{q} \), which implies \( q|m \) whenever \( q|f(\omega_q) \). Hence \( q|f(\omega_q) \) for only finitely many odd prime powers \( q \equiv 1 \pmod{d} \).

It follows from (5) that

\[
\lim_{q \to \infty} \frac{1}{|S_d|} \sum_{u \in S_q} e(u \cdot v).
\]

Appealing to Lemma [1] completes the proof. \( \square \)

The following theorem summarizes our current understanding of the asymptotic behavior of cyclic supercharacters. A special case supplies a geometric description of the set of \( |X| \)-nomial periods modulo \( p \) considered by Gauss, shedding new light on these classical objects. We let \( \mathbb{T} \) denote the unit circle in \( \mathbb{C} \).

**Theorem 1** (Duke–Garcia–Lutz). If \( \sigma_X: \mathbb{Z}/p^n\mathbb{Z} \to \mathbb{C} \) is a cyclic supercharacter, where \( p^n \) is a power of an odd prime, \( X \) is an orbit of 1, and \( |X| = d \) divides \( p - 1 \), then the image of \( \sigma_X \) is contained in the image of the Laurent polynomial function \( g_d: \mathbb{T}^{\varphi(d)} \to \mathbb{C} \) defined by

\[
g_d(z_1, z_2, \ldots, z_{\varphi(d)}) = \sum_{k=0}^{d-1} \prod_{j=0}^{\varphi(d)-1} z_{j+1}^{c_{jk}},
\]

where the \( c_{jk} \) are given by the relation

\[
x^k \equiv \sum_{j=0}^{\varphi(d)-1} c_{jk}x^j \pmod{\Phi_d(x)}.
\]

Moreover, for a fixed \( |X| = d \), as \( p^n \equiv 1 \pmod{d} \) tends to infinity, every nonempty open disk in the image of \( g \) eventually contains points in the image \( \sigma_X(\mathbb{Z}/p^n\mathbb{Z}) \).
Proof. Since the elements $1, \omega_q, \ldots, \omega_q^{(d)-1}$ form a $\mathbb{Z}$-basis for $\mathbb{Z}[\omega_q]$ \cite[p. 60]{35}, for $k = 0, 1, \ldots, d-1$ we can write

$$\omega_q^k \equiv \sum_{j=0}^{(d)-1} c_{jk} \omega_q^j \pmod{q},$$

where the integers $c_{jk}$ are given by (7). Letting $X = \langle \omega_q \rangle$, we see that

$$\sigma_X(y) = \sum_{x \in X} e \left( \frac{x y}{q} \right) = \sum_{k=0}^{d-1} e \left( \frac{\omega_q^k y}{q} \right) = \sum_{k=0}^{d-1} e \left( \sum_{j=0}^{(d)-1} c_{jk} \omega_q^j \frac{\omega(d) y}{q} \right) = \sum_{k=0}^{d-1} \prod_{j=0}^{(d)-1} e \left( \frac{\omega_q^j y}{q} \right)^{c_{jk}},$$

whence it follows that the image of $\sigma_X$ is contained in the image of the function $g_d : \mathbb{T}^{\phi(d)} \to \mathbb{C}$ defined in (6). The claim about open disks follows immediately from Lemma 2. \qed

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{fig8.png}
\caption{Cyclic supercharacters filling out regions bounded by hypocycloids, outlined in black (see Proposition 1).}
\end{figure}

Several remarks are in order. First, when the hypotheses of Theorem 1 are satisfied, we say that $\sigma_X$ fills out the image of $g_d$, as illustrated by Figure 8. The corresponding values of $d$ are given in captions. Second, since every divisor of $\phi(p^a) = p^a - 1$ is the cardinality of some orbit $X$ under the action of a cyclic subgroup of $(\mathbb{Z}/p^a\mathbb{Z})^\times$, the requirement that $|X|$ divides $p - 1$ might appear to be restrictive. However, it turns out that if $p$ divides $|X|$, then the image of $\sigma_X$ is equal to a scaled copy of the image of a supercharacter that satisfies the hypotheses of the theorem, except perhaps for a single point at the origin \cite[Proposition 2.4]{17}.

5. Examples

As a consequence of Theorem 1 the functions $g_d$ are instrumental in understanding the asymptotic behavior of cyclic supercharacters $\sigma_X : \mathbb{Z}/p^a\mathbb{Z} \to \mathbb{C}$. Fortunately, whenever the coefficients of $\Phi_d(x) = x^{\phi(p^a)}$ are relatively accessible, we can obtain a convenient formula for $g_d$. For example, it is not difficult to show that

$$\Phi_{2^a}(x) = x^{2^{a-1}} + 1,$$
for any $b$. With this, we can compute the integers $c_{jk}$ in (7) to see that

$$g_{2^b}(z_1, z_2, \ldots, z_{2^b-1}) = 2 \sum_{j=1}^{2^{b-1}} \Re(z_j),$$

where $\Re(z)$ denotes the real part of $z$. Hence the image of $g_{2^b}$ is the real interval $[-2^b, 2^b]$. Alternatively, if $r$ is an odd prime, then

$$\Phi_{2^r}(x) = x^{p-1} - x^{p-2} + \cdots - x + 1,$$

giving

$$g_{2^r}(z_1, z_2, \ldots, z_{2^r-1}) = 2\Re\left(\frac{z_1 z_3 \cdots z_{2^r-1}}{z_1 z_3 \cdots z_{2^r-1}}\right) + 2 \sum_{j=1}^{r-1} \Re(z_j).$$

It can be shown that $g_d$ is real valued whenever $d$ is even.

![Figure 9. Circles of radius 1 trace out hypocycloids as they roll within circles of radii (from left to right) 2, 3, and 4.](image)

A particularly novel and accessible behavior occurs when $d = r$ is an odd prime. The reader might recall that a *hypocycloid* is a planar curve obtained by tracing a fixed point on a circle of integral radius as it “rolls” within a larger circle of integral radius. Figure 9 illustrates this construction. We are interested in the hypocycloid that is centered at the origin and has $r$ cusps, one of which is at $r$. This curve is obtained by rolling a circle of radius 1 within a circle of radius $d$; it has the parametrization $\theta \mapsto (r-1)e(\theta) + e((1-r)\theta)$. Let $H_r$ denote the compact region bounded by this curve.

**Proposition 1.** If $r$ is an odd prime, then the image of $g_r$ is $H_r$.

**Proof.** Since

$$\Phi_r(x) = x^{r-1} + x^{r-2} + \cdots + x + 1,$$

we obtain the formula

$$g_r(z_1, z_2, \ldots, z_{r-1}) = z_1 + z_2 + \cdots + z_{r-1} + \frac{1}{z_1 z_2 \cdots z_{r-1}}.$$
The image of this map is seen to be the set of all traces of $SU(q)$, the group of $q \times q$ complex unitary matrices with determinant 1. This set is none other than $H_q$ [13, Theorem 3.2.3]. In particular, the image under $g_d$ of the diagonal $z_1 = z_2 = \cdots = z_{r-1}$ is the boundary of $H_q$. □

To expand on the previous example, suppose again that $r$ is an odd prime and $b$ is a positive integer. We have

$$\Phi_{r^b}(x) = \sum_{j=0}^{r-1} x^{j r^b - 1},$$

whence

$$g_{r^b}(z_1, z_2, \ldots, z_{r^b - 1}) = \sum_{j=1}^{r^b - 1} z_j + \sum_{j=1}^{r^b - 1} \prod_{\ell=0}^{r^b - 1} z_j^{r^b - 1}. \quad (8)$$

If $r = 3$ and $b = 2$, for instance, then the map is given by

$$g_9(z_1, z_2, z_3, z_4, z_5, z_6) = z_1 + z_4 + \frac{1}{z_1 z_4} + z_2 + z_5 + \frac{1}{z_2 z_5} + z_3 + z_6 + \frac{1}{z_3 z_6}.$$  

To discuss the image in this case, we require a definition. The Minkowski sum of two nonempty subsets $S$ and $T$ of $C$, denoted $S + T$, is the set

$$S + T = \{ s + t : s \in S \text{ and } t \in T \}.$$  

We define the Minkowski sum of an arbitrary finite collection by induction. As a consequence of Proposition 1, we discover that the image of $g_{9}$ is none other than the Minkowski sum $H_3 + H_3 + H_3$, as illustrated in Figure 10. A close look at (8) reveals a more general phenomenon.

**Corollary 1.** If $r^b$ is a power of an odd prime, then the image of $g_{r^b}$ is the Minkowski sum

$$\sum_{j=1}^{r^b - 1} H_r.$$  

The Shapley–Folkman–Starr Theorem [36, Corollary, p. 37], familiar to mathematical economists, gives an explicit upper bound on the distance between points in a Minkowski sum and its convex hull. In the context of Corollary 1 we obtain the bound

$$\min \{|w - z| : w \in \sum_{j=1}^{r^b - 1} H_r\} \leq 2\sqrt{2}r \sin \left(\frac{\pi}{r}\right),$$

for any $z$ in the filled $r$-gon with vertices at $r^b e^{i \frac{2\pi j}{r}}$ for $j = 1, 2, \ldots, r$. It follows easily that as $b \to \infty$, any point in the filled $r$-gon whose vertices are the $r$th roots of unity becomes arbitrarily close to points in the scaled Minkowski sum

$$\frac{1}{r^b - 1} \sum_{j=1}^{r^b - 1} H_r.$$  

That is, for large values of $b$, any supercharacter $\sigma_X : \mathbb{Z}/p^a \mathbb{Z} \to \mathbb{C}$ with $|X| = r^b$ fills out a region approximating a filled regular $r$-gon.
There is work to be done toward understanding the images of cyclic supercharacters. If we are to stay the course of inquiry set by Section 5, then a different approach is required; beyond the special cases discussed above, a general formula for the integers $c_{jk}$ in (7) is unobtainable, since there is no known simple closed-form expression for the coefficients of an arbitrary cyclotomic polynomial $\Phi_d(x)$.

There is, however, a remedy. To minimize headache, suppose that $d = rs$ is a product of distinct odd primes, and that $\omega_q \mapsto (\gamma_r, \gamma_s)$ under the isomorphism $(\mathbb{Z}/d\mathbb{Z})^\times \rightarrow (\mathbb{Z}/r\mathbb{Z})^\times \times (\mathbb{Z}/s\mathbb{Z})^\times$. Instead of wielding the elements $1, \omega_q, \ldots, \omega_q^{\varphi(d) - 1}$ as a $\mathbb{Z}$-basis for $\mathbb{Z}[\omega_q]$, we can use an analogous basis for $\mathbb{Z}[\gamma_r, \gamma_s]$. After some computation, we see that the image of the function $g_d$, formerly quite mysterious, is equal to the image of the function $h_d : \mathbb{T}^\varphi(d) \rightarrow \mathbb{C}$ given by

$$h((z_{ij})_{0 \leq i < r-1, 0 \leq j < s-1}) = \sum_{i=0}^{r-2} \sum_{j=0}^{s-2} z_{ij} + \sum_{i=0}^{r-2} \prod_{j=0}^{s-2} \frac{1}{z_{ij}} + \sum_{j=0}^{s-2} \prod_{i=0}^{r-2} \frac{1}{z_{ij}} + \prod_{i=0}^{r-2} \prod_{j=0}^{s-2} z_{ij}.$$  

This procedure, which amounts to a change of coordinates, can be used to obtain a closed formula for a Laurent polynomial map $h_d$ having the same image as $g_d$, for any integer $d$. This brings us one step closer to understanding the asymptotic behavior of the cyclic supercharacters in Section 4.

In practice, however, the functions $h_d$ are still difficult to analyze, despite being considerably easier to write down than the $g_d$. Even the simplest cases, described in (9), resist the accessible geometric description provided for the examples in Section 5.

While certain graphical features of cyclic supercharacters with composite moduli have been explained in [17], the mechanisms behind many of the striking patterns herein remain enigmatic. An important step toward deciphering the behavior of these supercharacters is to predict the layering
constant \( c \), discussed in Section 3, given only a modulus \( n \) and generator \( \omega \). As is apparent, these layerings betray an underlying geometric structure that allows us to decompose the images of \( \sigma_X \) into more manageable sets. We have been successful so far in finding appropriate values of \( c \) ad hoc; however, a general theory is necessary to formalize our intuition.

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