MV-algebras as Sheaves of ℓ Groups on Fuzzy Topological Spaces

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Abstract

We introduce the concept of fuzzy sheaf as a natural generalisation of a sheaf over a topological space in the context of fuzzy topologies. Then we prove a representation for a class of MV-algebras in which the representing object is an MV-sheaf of lattice-ordered Abelian groups, namely, a fuzzy sheaf in which the base (fuzzy) topological space is an MV-topological space and the stalks are Abelian ℓ-groups.

1 Introduction

MV-topological spaces are fuzzy topological spaces in which Łukasiewicz t-norm and t-conorm play the role of strong intersection and union of fuzzy sets. They were introduced by the second author [19] with the aim of extending Stone duality to semisimple MV-algebras. Many basic notions and results of general topology have been successfully extended to MV-topologies in [19] and [4], and the results obtained so far indicate that MV-topological spaces constitute a pretty well-behaved fuzzy generalization of classical topological spaces.

In this paper, we extend the concept of sheaf to fuzzy topological spaces with particular emphasis to the class of MV-topologies; then we represent a class of MV-algebras as MV-sheaves of lattice-ordered Abelian groups. More precisely, we show that every MV-algebra of that class is isomorphic to the algebra of global sections of an MV-sheaf of ℓ-groups. Our representation is strongly connected to Filipoiu and Georgescu’s sheaf representation for MV-algebras [11]. Indeed, from a strictly algebraic viewpoint, we use essentially the same tool, that is, the fact that any MV-algebra $A$ is subdirectly embeddable in the product of a family of local MV-algebras — the quotients of $A$ w.r.t. its minimal prime ideals.

However, our representation differs from the one in [11] in the way the “information is encoded”. In Filipoiu and Georgescu’s representation, each MV-algebra is obtained as an algebra of global sections of a (classical) sheaf over the maximal spectrum of the

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algebra and whose stalks are local MV-algebras. So, grossly speaking, we can say that each element of the algebra is represented as an open set of maximal ideals, carrying just the Boolean information, with an element of a local MV-algebra attached to each of its points, the latter encoding the “non-idempotent” part. In our representation, the base space is the maximal MV-spectrum (see [19]) and is in charge of encoding the whole semisimple skeleton of the given algebra, while the stalks only carry the non-semisimple (or infinitesimal) information of the elements of the algebra. Therefore, using the same description, each element of the algebra is a fuzzy open set along with ℓ-group elements attached to its (fuzzy) points; the fuzzy points of the open set form the semisimple part and the group elements represent exclusively the infinitesimal one.

Our representation, however, has a limit. It is not possible for any MV-algebra, and therefore we must restrict to the class of MV-algebras for which the quotients over minimal prime ideals have retractive radical [6]. As we shall see, the local algebras with retractive radical are particularly well-behaved, as they allow us to use some results on lexicographic MV-algebra from [5].

Throughout the paper, unless otherwise specified, we refer the reader to [2] for any definition about MV-algebras not explicitly reported here.

2 Preliminaries.

In this preliminary section we shall recall basic notions and results on MV-topological spaces, from [19], and lexicographic MV-algebras, from [5], which are strictly necessary for the comprehensibility of the present work.

MV-Topological Spaces

Both crisp and fuzzy subsets of a given set will be identified with their membership functions and usually denoted by lower case latin or greek letters. In particular, for any set $X$, we shall use also $1$ and $0$ for denoting, respectively, $X$ and $\emptyset$. In some cases, we shall use capital letters in order to emphasize that the subset we are dealing with is crisp.

We recall that an MV-topological space is basically a special fuzzy topological space in the sense of C. L. Chang [1]. Moreover, most of the definitions and results of the present section are simple adaptations of the corresponding ones of the aforementioned work to the present context or directly derivable from the same work or from the results presented in the papers [12–15, 17, 18, 20, 21].

Definition 2.1. Let $X$ be a set, $A$ the MV-algebra $[0, 1]^X$ and $\tau \subseteq A$. We say that $(X, \tau)$ is an MV-topological space (or MV-space) if $\tau$ is a subuniverse both of the quantale $([0, 1]^X, \lor, \oplus)$ and of the semiring $([0, 1]^X, \land, \otimes, 1)$. More explicitly, $(X, \tau)$ is an MV-topological space if

1. $0, 1 \in \tau$,
2. for any family $\{o_i\}_{i \in I}$ of elements of $\tau$, $\lor_{i \in I} o_i \in \tau$,

and, for all $o_1, o_2 \in \tau$,
(iii) \( a_1 \odot a_2 \in \tau \),
(iv) \( a_1 \oplus a_2 \in \tau \),
(v) \( a_1 \wedge a_2 \in \tau \).

\( \tau \) is also called an \( \text{MV-topology} \) on \( X \) and the elements of \( \tau \) are the \( \text{open MV-subsets} \) of \( X \). The set \( \tau^* = \{ o^* | o \in \tau \} \) is easily seen to be a subquantale of \( ([0, 1]^X, \wedge, \odot) \) (where \( \wedge \) has to be considered as the join w.r.t. to the dual order \( \geq \)) and a subsemiring of \( ([0, 1]^X, \lor, \oplus, 0) \), i.e., it verifies the following properties:

1. \( 0, 1 \in \tau^* \),
2. for any family \( \{ c_i \} \) of elements of \( \tau^* \), \( \bigwedge_i c_i \in \tau^* \),
3. for all \( c_1, c_2 \in \tau^* \), \( c_1 \odot c_2, c_1 \oplus c_2, c_1 \lor c_2 \in \tau^* \).

The elements of \( \tau^* \) are called the \( \text{closed MV-subsets} \) of \( X \).

Let \( X \) and \( Y \) be sets. Any function \( f : X \to Y \) naturally defines a map

\[
f^\odot^\ast : [0, 1]^Y \to [0, 1]^X \quad \alpha \mapsto \alpha \circ f.
\] (1)

Obviously \( f^\odot^\ast(0) = 0 \); moreover, if \( \alpha, \beta \in [0, 1]^Y \), for all \( x \in X \) we have \( f^\odot^\ast(\alpha \oplus \beta)(x) = (\alpha \oplus \beta)(f(x)) = \alpha(f(x)) \oplus \beta(f(x)) = f^\odot^\ast(\alpha)(x) \oplus f^\odot^\ast(\beta)(x) \) and, analogously, \( f^\odot^\ast(\alpha^*) = f^\odot^\ast(\alpha)^* \). Then \( f^\odot^\ast \) is an MV-algebra homomorphism and we shall call it the \( \text{MV-preimage} \) of \( f \). The reason of such a name is essentially the fact that \( f^\odot^\ast \) can be seen as the preimage, via \( f \), of the fuzzy subsets of \( Y \). From a categorical viewpoint, once denoted by \( \mathcal{Set}, \mathcal{Boole} \) and \( \mathcal{M}^\mathcal{V} \) the categories of sets, Boolean algebras, and MV-algebras respectively (with the obvious morphisms), there exist two contravariant functors \( \mathcal{P} : \mathcal{Set} \to \mathcal{Boole}^\circ \) and \( \mathcal{F} : \mathcal{Set} \to \mathcal{M}^\mathcal{V}^\circ \) sending each map \( f : X \to Y \), respectively, to the Boolean algebra homomorphism \( f^- : \mathcal{P}(Y) \to \mathcal{P}(X) \) and to the MV-homomorphism \( f^\odot^\ast : [0, 1]^Y \to [0, 1]^X \).

**Definition 2.2.** [1] Let \( (X, \tau_X) \) and \( (Y, \tau_Y) \) be two MV-topological spaces. A map \( f : X \to Y \) is said to be

- **continuous** if \( f^\odot^\ast[\tau_Y] \subseteq \tau_X \),
- **open** if \( f^- (o) \in \tau_Y \) for all \( o \in \tau_X \),
- **closed** if \( f^- (c) \in \tau_Y \) for all \( c \in \tau_X^c \),
- an **MV-homeomorphism** if it is bijective and both \( f \) and \( f^{-1} \) are continuous.

We can use the same words of the classical case because, as it is trivial to verify, if a map between two classical topological spaces is continuous, open, or closed in the sense of the definition above, then it has the same property in the classical sense.
Definition 2.3. [23] As in classical topology, we say that, given an MV-topological space \((X, \tau)\), a subset \(B\) of \([0, 1]^X\) is called a base for \(\tau\) if \(B \subseteq \tau\) and every open set of \((X, \tau)\) is a join of elements of \(B\).

Lemma 2.4. [19] Let \((X, \tau_X)\) and \((Y, \tau_Y)\) be two MV-topological spaces and let \(B\) be a base for \(\tau_Y\). A map \(f : X \to Y\) is continuous if and only if \(f^{-1}[B] \subseteq \tau_X\).

A covering of \(X\) is any subset \(\Gamma\) of \([0, 1]^X\) such that \(\bigvee \Gamma = 1\), while an additive covering (⊕-covering, for short) is a finite family \(\{\alpha_i\}_{i=1}^n\) of elements of \([0, 1]^X\), \(n < \omega\), such that \(\alpha_1 \oplus \cdots \oplus \alpha_n = 1\). It is worthwhile remarking that we used the expression “finite family” in order to include the possibility for such a family to have repetitions. In other words, an additive covering is a finite subset \(\{\alpha_1, \ldots, \alpha_k\}\) of \([0, 1]^X\), along with natural numbers \(n_1, \ldots, n_k\), such that \(n_1\alpha_1 \oplus \cdots \oplus n_k\alpha_k = 1\).

Definition 2.5. An MV-topological space \((X, \tau)\) is said to be compact if any open covering of \(X\) contains an additive covering; it is called strongly compact if any open covering contains a finite covering.1

Definition 2.6. Let \((X, \tau)\) be an MV-topological space. \(X\) is called a Hausdorff (or separated) space if, for all \(x \neq y \in X\), there exist \(o_x, o_y \in \tau\) such that

(i) \(o_x(x) = o_y(y) = 1\),

(ii) \(o_x \land o_y = 0\).

Lexicographic MV-algebras

We recall that a partially-ordered Abelian group is an Abelian group \((G, +, -, 0)\) endowed with a partial order relation \(\leq\) which is compatible with the sum. The positive cone \(G_+\) of \(G\) is the set \(\{x \in G \mid 0 \leq x\}\), while the negative cone \(G_-\) is \((-G_+)\), i.e., the set of all the elements of \(G\) which are \(\leq 0\). When the order relation is total, \(G\) is called a totally-ordered Abelian group (o-group for short), and if the order of \(G\) is a lattice order the group is called a lattice-ordered Abelian group (ℓ-group henceforth). An element \(u \in G\) is a strong (order) unit if \(u \geq 0\) and, for every \(x \in G\) there is a natural number \(n\) such that \(x \leq nu\). An ℓ-group (respectively: an o-group) \(G\) with a strong unit \(u\) is called a unital ℓ-group (ℓu-group) (resp.: unital o-group, ou-group) and is usually denoted by \((G, u)\).

An ideal \(I\) of an MV-algebra \(A\) is called lexicographic if the following hold:

(LMV1) \(I \neq \{0\}\),

(LMV2) \(I\) is strict

(LMV3) \(I\) is retractive,

(LMV4) \(I\) is prime,

(LMV5) \(\rho \leq x \leq \rho^*, \text{ for any } \rho \in I \text{ and any } x \in A \setminus \langle I \rangle\).

1What we call strong compactness here is called simply compactness in the theory of lattice-valued fuzzy topologies [1].
The set of all lexicographic ideals of $A$ is denoted by $\text{Lex Id}(A)$.

**Definition 2.7.** An MV-algebra $A$ is called lexicographic if $\text{Lex Id}(A) \neq \emptyset$.

**Theorem 2.8.** [5, Theorem 4.1] The following are equivalent:

1. $A$ is a lexicographic MV-algebra,
2. there exists an ou-group $(H, u)$ and a non-trivial $\ell$-group $G$ such that
   
   $$A \cong \Gamma(H \times_{\text{lex}} G, (u, 0)).$$

This representation theorem says that the class of lexicographic MV-algebras is the largest class of MV-algebras which can be represented, via Mundici’s functor $\Gamma$ (see [2, Section 2.1] or [16]), as lexicographic products of ou-groups and non-trivial $\ell$-groups, with strong unit of the form $(u, 0)$. We recall here a sketch of the proof in order to provide the reader with some technical tools that will be used later.

**Proof.** (Sketch)

$\Rightarrow$) It follows from [5, Proposition 3.1].

$\Leftarrow$) Let $A$ be a lexicographic MV-algebra and $I$ a lexicographic ideal of $A$. We have the following:

- Let $\delta_I$ be the retraction of the canonical projection $\pi_I : A \to A/I$.
- Let $S_I = \delta_I(A/I)$ the MV-subalgebra of $A$ which is isomorphic to $A/I$.
- For any $a \in A$, set $s_a = \delta_I(\pi_I(a))$ as the unique element of $S_I$ such that $[s_a]_I = [a]_I$.
- Set $e_a = a \odot s_a$ and $\tau_a = a^* \odot s_a$.
- There exist an isomorphism of MV-algebras $\zeta_I : S_I \to \Gamma(H, u)$ and an isomorphism of lattice-ordered monoids $\eta_I : I \to G_+.$
- Let $(H, u) \cong \Gamma^{-1}(A/I)$ and $G \cong \Delta^{-1}(\langle I \rangle)$

The function $f_I : A \to \Gamma(H \times_{\text{lex}} G, (u, 0))$ defined by

$$f_I(a) = (\zeta_I(s_a), \eta_I(e_a) - \eta_I(\tau_a)),$$

for any $a \in A$

is an isomorphism of MV-algebras.

$\square$

**Corollary 2.9.** If $A$ is a lexicographic MV-algebra the following are equivalent:

1. $\text{Rad} A \in \text{Lex Id}(A)$,
2. there exists an $\ell u$-subgroup $(R', 1)$ of $(\mathbb{R}, 1)$ and a non-trivial $\ell$-group $G$ such that

   $$A \cong \Gamma(R' \times_{\text{lex}} G, (1, 0)).$$
Moreover, if the above equivalent conditions are satisfied the \( \ell u \)-subgroup \( (\mathbb{R}', 1) \) of \( (\mathbb{R}, 1) \) and the \( \ell \)-group \( G \) are uniquely determined, up to isomorphisms.

In [5], the authors also showed the following inclusions which give an interesting classification of some classes of MV-algebras:

\[
\text{Perfect} \subset \text{Local with retractive radical} \subset \text{Lexicographic} \subset \text{Local}.
\]

3 The Spectral Topology

In this section we recall a natural topology on the set \( \text{Spec} A \) of prime ideals of an MV-algebra \( A \). We also display some properties of \( \text{Spec} A \), and of its subspaces \( \text{Max} A \) and \( \text{Min} A \). For more about this topology, the reader may refer to [8].

For any ideal \( I \) of \( A \) we define

\[
r(I) := \{ P \in \text{Spec} A : I \not\subseteq P \}
\]

If we define \( \tau := \{ r(I) : I \in \text{Id}(A) \} \), we have that \( (\text{Spec} A, \tau) \) is a topological space. Indeed,

(i) \( r(\{0\}) = \emptyset \),

(ii) \( r(A) = \text{Spec} A \),

(iii) \( r(I \land J) = r(I) \cap r(J) \) for all \( I, J \in \text{Id}(A) \),

(iv) \( r(\bigvee \{ I_\lambda : \lambda \in \Lambda \}) = \bigcup \{ r(I_\lambda) : \lambda \in \Lambda \} \) for any \( \{ I_\lambda : \lambda \in \Lambda \} \subseteq \text{Id}(A) \).

In the sequel \( \tau \) or \( O(\text{Spec} A) \) will be referred to as the spectral topology or the Zariski topology. Other properties of the sets \( r(I) \) are the following:

- \( I \subseteq J \) iff \( r(I) \subseteq r(J) \) for any \( I, J \in \text{Id}(A) \),

- if \( X \subseteq A \) then \( \{ P \in \text{Spec} A : X \not\subseteq P \} = r((X)) \).

For any \( a \in A \) we define

\[
r(a) := \{ P \in \text{Spec} A : a \not\in P \}
\]

and we have the following properties:

Lemma 3.1. [8]

(i) \( r(a) = r((a)) \) for any \( a \in A \),

(ii) \( r(0) = \emptyset \),

(iii) \( r(1) = \text{Spec} A \),

(iv) \( r(a \lor b) = r(a \oplus b) = r(a) \cup r(b) \) for any \( a, b \in A \),

(v) \( r(a \land b) = r(a) \cap r(b) \) for any \( a, b \in A \),
(vi) $r(I) = \bigcup \{ r(a) : a \in I \}$ for any $I \in \text{Id}(A)$.

By properties (i) and (vi) in the last lemma we have that $\{ r(a) : a \in A \}$ is a basis for the topology $\tau$. It is well-known also that the compact open subsets of $\text{Spec} A$ are exactly the sets of the form $r(a)$ for some $a \in A$. In particular, $\text{Spec} A$ is compact because $r(1) = \text{Spec} A$ (see [8]).

For each $a \in A$, the set

$$ H(a) := \{ P \in \text{Spec} A : a \in P \} $$

is an open set of $\text{Spec} A$ [10, Lemma 3.6].

Since $\text{Max} A, \text{Min} A \subseteq \text{Spec} A$ we can endow $\text{Max} A$ and $\text{Min} A$ with the topology induced by the spectral topology $\tau$ on $\text{Spec} A$. This means that the open sets of $\text{Max} A$ are

$$ R(I) = r(I) \cap \text{Max} A = \{ M \in \text{Max} A : I \not\subseteq M \} $$

So, for any $a \in A$ and $I \in \text{Id} A$

$$ R(a) = r(a) \cap \text{Max} A = \{ M \in \text{Max} A : a \not\in M \} \text{ and } R(I) = \bigcup \{ R(a) : a \in I \} $$

Hence the family $\{ R(a) : a \in A \}$ is a basis for the induced topology on $\text{Max} A$. The set of opens in $\text{Max} A$ will be denoted by $\mathcal{O}(\text{Max} A)$.

By [8, Theorem 3.6.10], we have that for any MV-algebra $A$ the maximal ideal space, $\text{Max} A$, is a compact Hausdorff topological space with respect to the topology induced by the spectral topology on $\text{Spec} A$.

Analogously, we have that the open sets of $\text{Min} A$ are

$$ d(I) = r(I) \cap \text{Min} A = \{ M \in \text{Min} A : I \not\subseteq M \} $$

We conclude the section recalling that also the coZariski topology on $\text{Spec} A$ has been considered in the literature (see, for instance, Dubuc and Poveda [9]). Such a topology has the family $\{ W_a : a \in A \}$ where $W_a = \{ P \in \text{Spec} A : a \in P \}$ as a basis. In particular, $W_0 = \text{Spec} A$, $W_1 = \emptyset$ and $W_a \cap W_b = W_{a \oplus b}$.

4 MV-presheaves

Let $(X, \tau)$ be an MV-topological space. The poset of open fuzzy subsets $\tau \subseteq [0, 1]^X$, with the fuzzy inclusion $\leq$, can be viewed as a category in the usual manner, namely, $\tau$ is the object class and, for all $\alpha, \beta \in \tau$, there is exactly one morphism $\alpha \rightarrow \beta$ if $\alpha \leq \beta$, there are none otherwise.

**Definition 4.1.** Let $(X, \tau)$ be an MV-topological space and let $C$ be a category (of algebras). An MV-presheaf of $\text{Obj}(C)$ on $X$ is a contravariant functor $F : \tau \rightarrow C$, that is:

(i) for each fuzzy open set $\alpha$ in $\tau$, $F(\alpha)$ is an object of $C$, called the set of sections of $F$ over $\alpha$;
We define the constant MV-presheaf $A$.

**Example 4.3.** Let $\tau$ be a fuzzy open set in $\tau_X$.

**Definition 4.2.** Let $F$ and $G$ be MV-presheaves of $\text{Obj}(C)$ over $(X, \tau_X)$. A morphism of MV-presheaves from $F$ to $G$ is a natural transformation $f : F \Rightarrow G$, that is, a family $\{f(\alpha) : F(\alpha) \to G(\alpha)\}_{\alpha \in \tau}$ such that, whenever $\beta \leq \alpha$ are open fuzzy sets in $\tau$, the diagram

\[ \begin{array}{ccc}
F(\alpha) & \xrightarrow{f(\alpha)} & G(\alpha) \\
\rho_{\beta} & \downarrow & \rho_{\beta}^\prime \\
F(\beta) & \xrightarrow{f(\beta)} & G(\beta)
\end{array} \]

commutes.

**Example 4.4.** Let $(X, \tau_X)$ and $(Y, \tau_Y)$ be MV-topological spaces. Let us consider $C^Y : \tau_X \to \text{Set}$ defined by

\[ C^Y(\alpha) = \{ f : \text{supp}(\alpha) \to Y \mid f \text{ is continuous} \}, \]

with $\rho_{\beta}^\alpha : C^Y(\alpha) \to C^Y(\beta)$ such that $\rho_{\beta}^\alpha(f) = f|_{\text{supp}(\beta)}$ for $\beta \leq \alpha$ in $\tau_X$. $C^Y$ is an MV-presheaf of sets over $X$. Note that $\text{supp}(\beta) \subseteq \text{supp}(\alpha)$ if $\beta \leq \alpha$.

## 5 MV-sheaves

**Definition 5.1.** An MV-presheaf of sets over the MV-topological space $(X, \tau_X)$ satisfying the following two conditions is called an MV-sheaf of $\text{Obj}(C)$.

1. If $\alpha$ is a fuzzy open set of $X$ and the family $\{\alpha_i\}_{i \in I} \subseteq [0, 1]^X$ is an open covering of $\alpha$, i.e., $\alpha = \bigvee_{i \in I} \alpha_i$, and $s, s' \in F(\alpha)$ are two sections of $F$ such that for all $i \in I$

   \[ \rho_{\alpha_i}^\alpha(s) = \rho_{\alpha_i}^\alpha(s') \]

   then $s = s'$.

2. If $\alpha$ is a fuzzy open set of $X$ and the family $\{\alpha_i\}_{i \in I} \subseteq [0, 1]^X$ is an open covering of $\alpha$; and if there is a family $\{s_i\}_{i \in I}$ of sections of $F$ with $s_i \in F(\alpha_i)$ for all $i \in I$, such that for all $i, j \in I$

   \[ \rho_{\alpha_i,\alpha_j}^{\alpha_i}(s_i) = \rho_{\alpha_i,\alpha_j}^{\alpha_j}(s_j) \]

   then there is $s \in F(\alpha)$ such that for all $i \in I$

   \[ \rho_{\alpha_i}^\alpha(s) = s_i. \]
In other words, if the system \((s_i)_{i \in I}\) is given on a covering and is consistent on all of the overlaps, then it comes from a section over all of the \(\alpha\)'s.

**Definition 5.2.** If \(F, G\) are MV-sheaves of \(\text{Obj}(G)\) and \(f : F \Rightarrow G\) is an MV-presheaf morphism, we also call \(f\) a morphism of MV-sheaves.

**Example 5.3.** The MV-presheaf \(C^Y\), described in the Example 4.4, is an MV-sheaf. Let \(\alpha\) be a fuzzy open set of \(X\) and let \(\{\alpha_i\}_{i \in I} \subseteq [0, 1]^X\) be an open covering of \(\alpha\), i.e., \(\alpha = \bigvee_{i \in I} \alpha_i\).

1. let \(f, f' \in C^Y(\alpha)\) be two sections of \(C^Y\) such that for all \(i \in I\),

\[
\rho^\alpha_i(f) = \rho^\alpha_i(f')
\]

that is,

\[
f_{\{\text{supp}(\alpha_i)\}} = f'_{\{\text{supp}(\alpha_i)\}}
\]

where \(f, f' : \text{supp}(\alpha) \to Y\).

Note that \(\bigcup_{i \in I} \text{supp}(\alpha_i) = \text{supp}(\alpha)\) because \(\alpha = \bigvee_{i \in I} \alpha_i\). Let us see that \(f = f'\).

If \(x \in \text{supp}(\alpha)\), then there exists \(i \in I\) such that \(x \in \text{supp}(\alpha_i)\), so

\[
f(x) = f_{\{\text{supp}(\alpha_i)\}}(x) = f'_{\{\text{supp}(\alpha_i)\}}(x) = f'(x)
\]

then \(f = f'\).  

2. For the second condition, suppose that there is a family \(\{f_i\}_{i \in I}\) of sections of \(C^Y\) with \(f_i \in C^Y(\alpha_i)\) for all \(i \in I\), such that for all \(i, j \in I\)

\[
\rho^\alpha_{i/\alpha_j}(f_i) = \rho^\alpha_{i/\alpha_j}(f_j)
\]

We define \(f := \bigcup_{i \in I} f_i : \text{supp}(\alpha) \to Y\) by \(f(x) = f_i(x)\) if \(x \in \text{supp}(\alpha_i) = \text{dom}(f_i)\). We know that \(\text{supp}(\alpha) = \bigcup_{i \in I} \text{supp}(\alpha_i)\) then \(f\) is well defined because for all \(i, j \in I\), \(x \in \text{supp}(\alpha_i) \cap \text{supp}(\alpha_j)\) iff \(x \in \text{supp}(\alpha_i \land \alpha_j)\), and by hypothesis

\[
f_{\{\text{supp}(\alpha_i \land \alpha_j)\}}(x) = f_{\{\text{supp}(\alpha_i \land \alpha_j)\}}(x)
\]

where \(f_i : \text{supp}(\alpha_i) \to Y\) and \(f_j : \text{supp}(\alpha_j) \to Y\). It is clear that \(f_{\{\text{supp}(\alpha_i)\}} = f_i\) for each \(i \in I\).

Now, let us prove that \(f\) is continuous.

Let \(\gamma \in \tau_Y\), and let us prove that \(\gamma \circ f \in \tau_{\text{supp}(\alpha)}\). For each \(i \in I\), \(\gamma \circ f_i \in \tau_{\text{supp}(\alpha_i)}\), i.e., \(\gamma \circ f_i = \beta \land \text{supp}(\alpha_i)\) with \(\beta \in \tau_X\). As \(\text{supp}(\alpha_i) = \text{supp}(\alpha_i) \land \text{supp}(\alpha)\), then \(\gamma \circ f_i = \beta \land \text{supp}(\alpha_i) \land \text{supp}(\alpha)\). Thus, for each \(i \in I\), \(\gamma \circ f_i \in \tau_{\text{supp}(\alpha_i)}\) because \(\beta \land \text{supp}(\alpha_i) \in \tau_X\). Therefore, \(\gamma \circ f = \bigvee_{i \in I} (\gamma \circ f_i) \in \tau_{\text{supp}(\alpha)}\).

**Definition 5.4.** A directed set \(I\) is a set with a pre-order \(\leq\) which satisfies the following:

(a) for all \(i, j \in I\), there exists \(k \in I\) such that \(i \leq k\) and \(j \leq k\).
A **direct system** of sets indexed by a directed set \( I \) is a family \( \{\alpha_i\}_{i \in I} \) of sets together with maps \( \rho_{ij} : \alpha_i \rightarrow \alpha_j \), for each \( i \leq j \in I \), satisfying

(b) For all \( i \in I \), \( \rho_{ii} = \text{id}_{\alpha_i} \);

(c) For all \( i, j, k \in I \), \( i \leq j \leq k \) implies \( \rho_{ik} = \rho_{jk} \circ \rho_{ij} \).

Let \( F \) be an MV-presheaf of \( \text{Obj}(C) \) over an MV-topological space \((X, \tau)\) and fix \( x \in X \). Then \( \{F(\alpha) : x \in \text{supp}(\alpha)\} \), forms a direct system with maps \( \rho^\alpha_{\beta} : F(\alpha) \rightarrow F(\beta) \), whenever \( \beta \leq \alpha \), and \( x \in \text{supp}(\beta) \subseteq \text{supp}(\alpha) \). We have the following definition:

**Definition 5.5.** The **MV-stalk** \( F_x \) of \( F \) at \( x \) is

\[
\lim_{x \in \text{supp}(\alpha)} F(\alpha),
\]

which comes equipped with maps \( F(\alpha) \rightarrow F_x \) such that \( s \mapsto s_x \) whenever \( x \in \text{supp}(\alpha) \) for \( x \in \tau \). The members of \( F_x \) are also called **germs** (of sections of \( F \)).

**Definition 5.6.** Let \((X, \tau_X)\) be an MV-topological space. An **MV-sheaf space** over \( X \) is a triple \((E, p, X)\) where \((E, \tau_E)\) is an MV-topological space and \( p : E \rightarrow X \) is a **local MV-homeomorphism**, that is, \( p \) is continuous and, for all \( x \in E \), there exists an open fuzzy set \( \alpha \in \tau_E \) such that \( \alpha(x) > 0 \) and an open fuzzy set \( \beta \in \tau_X \) such that \( p|_{\text{supp}(\alpha)} : \text{supp}(\alpha) \rightarrow \text{supp}(\beta) \) is an MV-homeomorphism.

A morphism of MV-sheaf spaces over \( X \), \( f : (E, p, X) \rightarrow (E', p', X) \), is a continuous map \( f : E \rightarrow E' \) such that \( p = p' \circ f \).

We can construct an MV-sheaf of sets from an MV-sheaf space and reciprocally, we can construct an MV-sheaf space from an MV-sheaf. These constructions follow the canonical rules of sheaf theory on topological spaces (see [3, 22]).

### 6 MV-sheaf Representation I: the representable class and the MV-sheaf

In this section, we will present the main tools of our MV-sheaf representation, including the class of algebras that are isomorphically representable in our framework. As a first step, we shall recall some necessary results about the Maximal MV-Spectrum and the minimal prime ideals of an MV-algebra.

Let \((\text{Max}A, \tau_A)\) be the Maximal MV-Spectrum of \( A \), as defined in [19]. Let us see some of its properties and its relation with the topological space \( \text{Max}A \) with the Zariski topology. The basic opens of \( \text{Max}A \) denoted by \( R(a) \) with \( a \in A \), were defined in Section 3.

**Proposition 6.1.** Let \( A \) be an MV-algebra and \((\text{Max}A, \tau_A)\) be the associated MV-topological space, according [19]. For each basic fuzzy open \( b \in \tau_A \), we have that \( R(b) = \text{supp}(b) \). So, each Zariski basic open \( R(b) \) on \( \text{Max}A \) is a fuzzy open of \( \tau_A \), i.e., \( R(b) \in \tau_A \) for each \( a \in A \).

**Proof.** In fact, for each \( M \in \text{Max}A \), \( \hat{b}(M) = \frac{b}{M} = 1 \) if and only if \( b \in M \). That is, \( M \in \text{supp}b \) iff \( \hat{b}(M) = \frac{b}{M} > 0 \) iff \( b \notin M \) iff \( M \in R(b) \).

\( \square \)
Let $A$ be an MV-algebra and $P$ a prime ideal of $A$. We set

$$O_P = \bigcap \{ Q \in \text{Spec} \, A : Q \subseteq P \}. \quad (4)$$

We note that $O_P$ is an ideal of $A$ and $O_P \subseteq P$. In the following, we display some properties of these ideals which will be useful in this work. The interested reader may refer to [10] for more information about such ideals.

**Proposition 6.2.** [10] For each $P \in \text{Spec} \, A$, $O_P = \bigcup \{ a^+ : a \notin P \},$ where $a^+ = \{ b \in A : a \land b = 0 \}$.

**Proposition 6.3.** [10] For each $P \in \text{Spec} \, A$, the ideal $O_P$ is primary.

**Proposition 6.4.** For each $a \in A$, the set $H(a) = \{ M \in \text{Max} \, A : a \in O_M \}$ is an element of $\tau_A$.

**Proof.** We will prove that $H(a)$ is the support of a fuzzy open of $\tau_A$. If $M \in H(a)$ then $a \in O_M$, so by Proposition 6.2 there exists $b_M \notin M$ such that $a \land b_M = 0$. That is, $M \in R(b_M) = \text{supp}(b_M)$. Let us see that

$$H(a) = \text{supp}(\bigvee_{M \in H(a)} \widehat{b_M}).$$

In fact, if $N \in H(a)$ then there exists $b_N$ such that $b_N \notin N$ and $a \land b_N = 0$, then $\widehat{b_N}(N) > 0$, and therefore $(\bigvee_{M \in H(a)} \widehat{b_M})(N) = \bigvee_{M \in H(a)} \widehat{b_M}(N) > 0$, i.e., $N \in \text{supp}(\bigvee_{M \in H(a)} \widehat{b_M})$.

For the other inclusion, if $(\bigvee_{M \in H(a)} \widehat{b_M})(N) > 0$ then there exists $\widehat{b_M}$ with $M \in H(a)$ such that $\widehat{b_M}(N) > 0$, i.e., $b_M \notin N$ and $a \land b_M = 0$, then $a \in O_N$ and therefore $N \in H(a)$. \hfill \Box

In the following we show a representation of a class of MV-algebras through an MV-sheaf. In this part, we use some theory and results of lexicographic MV-algebras, which can be consulted in [5]. We also use strongly the Filipoiu and Georgescu sheaf representation given in [11].

In order to represent a class of MV-algebras through of an MV-space, let us consider the following functors.

1. Let $(X, \tau)$ be an MV-topological space and $(X, B(\tau))$ its corresponding skeleton topological space defined in [19], where $B(\tau) = \tau \cap [0, 1]^X$. As usual, we consider the posets $\tau$ and $B(\tau)$ with their natural order as categories, that is, the objects are the elements of $\tau$ and $B(\tau)$ respectively, and the morphisms are given by $\alpha \leq \beta$ in $\tau$ and $U \subseteq V$ in $B(\tau)$, respectively. The following map obviously define a covariant functor:

$$\text{Sk} : \tau \quad \rightarrow \quad B(\tau)$$

$$\quad \alpha \quad \mapsto \quad \text{supp}(\alpha)$$

For $\alpha \leq \beta$, we have the unique morphism $\alpha \xrightarrow{f} \beta$ in $\tau$, and its corresponding morphism $\text{supp}(\alpha) \xrightarrow{\text{Sk}(f)} \text{supp}(\beta)$ in $B(\tau)$ is also uniquely determined, because $\alpha \leq \beta$ implies $\text{supp}(\alpha) \subseteq \text{supp}(\beta)$. 

2. According Filipoiu and Georgescu’s representation [11], each MV-algebra $A$ can be represented as the MV-algebra of global sections of a sheaf whose stalks are local MV-algebras and the base space is the space of maximal ideals of $A$ with the Zariski topology, $O(\text{Max } A)$. The associated sheaf in that representation is the following contravariant functor:

$$\mathcal{F} : O(\text{Max } A) \to \text{MV}$$

$$U \mapsto A/O_U$$

where $O_U = \bigcap_{M \in U} O_M$, and the unique morphism between two open sets (if it exists) is sent to the natural projection between the corresponding quotient algebras.

3. We recall the category $\ell G^\text{Ab}$ whose objects are Abelian $\ell$-groups and whose morphisms are $\ell$-group homomorphisms. The following mapping defines a functor from the category of MV-algebras to the category $\ell G^\text{Ab}$:

$$G : \text{MV} \to \ell G^\text{Ab}$$

$$A \mapsto G(\text{Rad}(A))$$

where $G(\text{Rad}(A))$ is the Abelian $\ell$-group generated by the ordered cancellative monoid $(\text{Rad}(A), \oplus, 0)$. Actually, $G(\text{Rad}(A)) = D(A)$ where $D$ is the inverse of the functor $\Delta : \ell G^\text{Ab} \to \text{MV}^\text{perf}$ between Abelian $\ell$-groups and perfect MV-algebras presented in [7] (note that the group $D(A)$ can be constructed for any MV-algebra $A$, not necessarily perfect). The action on morphisms of the functor $G$ is exactly the same as for $D$.

Now, for each $\alpha \in \tau A$, by Proposition 6.1 we have that $\text{supp}(\alpha) \in O(\text{Max } A)$. So, set $A_\alpha := \mathcal{F}(\text{supp}(\alpha))$ for each $\alpha \in \tau A$. We obtain the following MV-presheaf:

$$\mathcal{H} : \tau A \to \ell G^\text{Ab}$$

$$\alpha \mapsto G(\text{Rad}(A_\alpha))$$

Note that in the construction performed by Filipoiu and Georgescu, the stalks are the local algebras $A/O_M$, then for each $M \in \text{Max } A$,

$$\lim_{M \in \text{supp}(\alpha)} \mathcal{H}(\text{supp}(\alpha)) = A/O_M$$

Such a limit can be extended to the presheaf $\mathcal{H}$ on the category $\ell G^\text{Ab}$, thus obtaining the following two limits:

$$\lim_{M \in \text{supp}(\alpha)} \text{Rad}(\mathcal{H}(\text{supp}(\alpha))) = \text{Rad}(A/O_M)$$

and

$$\lim_{M \in \text{supp}(\alpha)} \mathcal{H}(\alpha) = G(\text{Rad}(A/O_M)).$$

Since $\text{Rad}(A/O_M) = M/O_M$ for each $M \in \text{Max } A$, we have an MV-sheaf on $\ell G^\text{Ab}$ where the stalks are the $\ell$-groups $G(M/O_M)$. 
Now, let $A$ be an MV-algebra. Let $M$ be a maximal ideal of $A$. By Proposition 6.3, we have that $A/O_M$ is a local MV-algebra. Note that $A/M \cong (A/O_M)/(M/O_M)$ and $(M/O_M, \oplus, 0)$ is a lattice ordered cancellative monoid. We can construct, in the usual manner, the lattice ordered group $G(M/O_M)$ generated by $(M/O_M, \oplus, 0)$.

For every $M \in \text{Max } A$, let us suppose that $A/O_M$ has retractive radical. Then $A/O_M$ is a local MV-algebra with retractive radical and therefore it is lexicographic, because the class of local MV-algebras with retractive radical is strictly included in the class of lexicographic MV-algebras [5].

According to Representation Theorem for lexicographic MV-algebras (Theorem 2.8), we have that

$$A/O_M \cong \Gamma(H \times_{\text{lex}} G_M, (u, 0))$$

where

$$\Gamma(H, u) \cong \Gamma^{-1}(A/M)$$

and $G_M \cong \Delta^{-1}(\langle M/O_M \rangle)$. We can construct, in the usual manner, the lattice ordered group $G(M/O_M)$ generated by $(M/O_M, \oplus, 0)$.

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Now, since $A/O_M$ is an MV-subalgebra of the standard MV-algebra $[0, 1]$, for each $M \in \text{Max } A$, we have:

$$A/O_M \cong \Gamma(A/M \times_{\text{lex}} G(M/O_M), (1, 0)).$$

So, in accordance with the proof of Theorem 2.8, and with an abuse of notation, we can see each element $a/O_M$ of $A/O_M$ in the following way:

$$\tilde{a} : \text{Max } A \rightarrow H_A : M \mapsto (g_{aM}, M)$$

where $g_{aM} = (a/O_M \ominus a/M) - (a/M \ominus a/O_M) \in G(M/O_M)$. In fact,

$$G(M/O_M) = \langle g_{aM} : a \in A, M \in \text{Max } A \rangle.$$

7 MV-sheaf Representation II: the representing algebra and the isomorphism

As a consequence of what we discussed in the previous section, we have an MV-sheaf space $(H_A, \pi, \text{Max } A)$ whose total MV-space, $H_A$, will be the disjoint union of stalks $\tilde{S}_M = G(M/O_M)$ and $\pi$ will be the trivial projection:

$$H_A = \{ (g_{aM}, M) : a \in A, M \in \text{Max } A \},$$

and

$$\pi : H_A \rightarrow \text{Max } A$$

$$\{ (g_{aM}, M) \} \mapsto M.$$

Now for each $a \in A$ we define:

$$\tilde{a} : \text{Max } A \rightarrow H_A$$

$$M \rightarrow (g_{aM}, M)$$

It is clear that $(\pi \circ \tilde{a})(M) = \pi(g_{aM}, M) = M$ for all $M \in \text{Max } A$. 

As usual in sheaf representations, we shall use \( \{\tilde{a}^{-1}(\tilde{b})\}_{a, b \in A} \) as a subbase for an MV-topology on \( H_A \), where

\[
\tilde{a}^{-1}(\tilde{b})(g_{cM}, M) = \bigvee_{\tilde{a}(N) = (g_{cM}, M)} \tilde{b}(N) = \begin{cases} \tilde{b}(M) & \text{if } g_{aM} = g_{cM} \\ 0 & \text{otherwise} \end{cases}.
\]

Let us see that \( \alpha_{a,b} := \{M \in \text{Max}A : g_{aM} = g_{bM}\} \) is an element of \( \mathcal{O}(\text{Max}A) \). If \( g_{aM} = g_{bM} \), we have the following cases:

- If \( \frac{a}{aM} = \frac{b}{bM} \) then \( \left( \frac{a}{aM}, g_{aM} \right) = \left( \frac{b}{bM}, g_{bM} \right) \). Hence \( \frac{a}{aM} = \frac{b}{bM} \) and, therefore,
  \[ \alpha_{a,b} = H(d(a, b)) \in \mathcal{O}(\text{Max}A). \]

- If \( \frac{a}{aM} \neq \frac{b}{bM} \) then necessarily \( \frac{a}{aM} \neq \frac{b}{bM} \). That is \( \frac{a}{aM} < \frac{b}{bM} \) or \( \frac{b}{bM} > \frac{a}{aM} \). Since \( A/\text{OM} \) has a lexicographic order, this implies that \( \frac{a}{aM} < \frac{b}{bM} \) or \( \frac{b}{bM} > \frac{a}{aM} \). Therefore there exists \( c \in A \) such that \( \frac{a}{aM} = \frac{b+c}{bM} \) or \( \frac{b}{bM} = \frac{a+c}{aM} \). Hence
  \[
  \alpha_{a,b} = \bigcup_{c \in \text{A}} (M \in \text{Max}A : \frac{a}{aM} = \frac{b+c}{bM}) \cup \bigcup_{c \in \text{A}} (M \in \text{Max}A : \frac{b}{bM} = \frac{a+c}{aM}) = \\
  = \bigcup_{c \in \text{A}} H(d(a, b \oplus c)) \cup \bigcup_{c \in \text{A}} H(d(a \oplus c, b)) \in \mathcal{O}(\text{Max}A)
  \]

As a consequence, each \( \alpha_{a,b} \) is an element of \( \tau_A \), and this guarantees that \( (H_A, \pi, \text{Max}A) \) is indeed an MV-sheaf space.

The MV-sheaf defined above is an MV-sheaf of lattice-ordered Abelian groups. We want to obtain a representation of the MV-algebra \( A \) through this MV-sheaf.

First, let us consider for each \( a \in A \), the function \( \tilde{a} \) restricting the codomain \( H_A \) to its image \( \text{Im}(\tilde{a}) = \{ (g_{aM}, M) : M \in \text{Max}A \} \). Actually, the new \( \tilde{a} \) acts exactly like the previous one on the elements of the domain, so we shall use the same notation for them. Then, we have the bijective maps:

\[
\begin{array}{ccc}
\tilde{a} : & \text{Max}A & \rightarrow & \text{Im}(\tilde{a}) \\
& M & \mapsto & (g_{aM}, M)
\end{array}
\]

and for each basic open set \( \tilde{a} \) in \( \text{Max}A \) we have the open fuzzy set \( \tilde{a}^{-1}(\tilde{a}) \) in \( H_A \) satisfying

\[ \tilde{a}^{-1}(\tilde{a})(g_{aM}, M) = \frac{a}{aM} \text{ for each } (g_{aM}, M) \in \text{Im}(\tilde{a}). \]

Now, let us consider the inverse of the graphic of \( \tilde{a}^{-1}(\tilde{a}) \) given by

\[ a := G^{-1}(\tilde{a}^{-1}(\tilde{a})) = \left\{ \left( \frac{a}{M}, g_{aM} \right) \right\}_{M \in \text{Max}A} \]

**Definition 7.1.** Let \( \mathcal{A} = \{a : a \in A\} \). We define the structure \( (\mathcal{A}, \oplus^*, \circ) \) with the operations and the constant defined as follows:

- for each \( a, b \in \mathcal{A} \),
  - (i) \( a \circ := G^{-1}(\tilde{a}^{-1}(\tilde{a}))(a) \)
  - (ii) \( a \oplus b := G^{-1}(\tilde{a}^{-1}(\tilde{a}))(a \oplus b) \)
(iii) \( a^* := G^{-1}(\tilde{a}^*(\tilde{a}^*)) \).

**Theorem 7.2.** \((\mathfrak{A}, \oplus, *, o)\) is an MV-algebra.

**Proof.** Let us see that \( \mathfrak{A} \) satisfies the equations defining MV-algebras, as listed in [2, Definition 1.1.1].

\([MV1]\)

\[
(a \oplus b) \oplus c = \left\{ (a \oplus b) \oplus c \right\}_{M \in \text{Max}_A}_{M \in \text{Max}_A} = a \oplus (b \oplus c)
\]

\([MV2]\)

\[
a \oplus b = \left\{ a \oplus \left( b \oplus c \right) \right\}_{M \in \text{Max}_A} = b \oplus a
\]

\([MV3]\)

\[
a \oplus 0 = \left\{ a \oplus 0 \right\}_{M \in \text{Max}_A} = a
\]

\([MV4]\)

\[
(a^*)^* = \left\{ \left( a^* \right)^* \right\}_{M \in \text{Max}_A} = a
\]

\([MV5]\)

\[
a \oplus 0^* = \left\{ a \oplus 0^* \right\}_{M \in \text{Max}_A} = 0^*
\]
Theorem 7.3. The MV-algebras $A$ and $\mathfrak{A}$ are isomorphic.

Proof. The natural map $\Psi : a \in A \mapsto a \in \mathfrak{A}$ preserves the operations $\oplus, \ast$, and the constant $0$. Indeed, by Definition 7.1, for each $a, b \in A$, $\Psi(a \oplus b) = G^{-1}(\tilde{a} \oplus \tilde{b}, \tilde{a} \oplus \tilde{b}) = a \oplus b = \Psi(a) \oplus \Psi(b)$, and $\Psi(a^*) = a^* = (\Psi(a))^*$. Analogously, we have that $\Psi(0) = 0$.

It is clear that $\Psi$ is a surjection, and let us prove that $\Psi$ is injective. Let $a, b \in A$, and suppose that $\Psi(a) = \Psi(b)$, that is,

\[
\left\{ \left( \frac{a}{M}, g_{aM} \right) \right\}_{M \in \text{Max} A} = \left\{ \left( \frac{b}{M}, g_{bM} \right) \right\}_{M \in \text{Max} A},
\]

then for each $M \in \text{Max} A$,

\[
\left( \frac{a}{M}, g_{aM} \right) = \left( \frac{b}{M}, g_{bM} \right),
\]

that is, $\frac{a}{aM} = \frac{b}{bM}$ for every $M \in \text{Max} A$, then $d(a, b) \in O_M$ for every $M \in \text{Max} A$. So $a = b$ because $\bigcap \{O_M : M \in \text{Max} A\} = 0$. □

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