On an analogue of Schwarz’s reflection principle

V.V. Napalkov (Jr.)

December 12, 2013

Abstract

We consider the Bergman space on the complex plane. We prove an analogue of Schwarz’s reflection principle for unbounded quasidisks.

Keywords: Schwarz’s reflection principle, Bergman space, quasiconformal reflection, orthosimilar system.

Adress: Valerii V. Napalkov, Institute of Mathematics with Computer Center of the Ufa Science Center of the Russian Academy of Sciences 112, Chernyshevsky str., Ufa, Russia, 450008.

E-mail: vnap@matem.anrb.ru, vnap@mail.ru.

Let $G$ be an unbounded simple connected Jordan domain in the complex plane and $\partial G$ its boundary. Assume that $\infty \in \partial G$. The curve $\partial G$ divides the complex plane into two domains $G$ and $\mathbb{C} \setminus \overline{G}$. Bergman space $B_2(G)$ consists of all holomorphic functions $f(z), z \in G$ such that

$$\|f\|_{B_2(G)}^2 \overset{def}{=} \int_G |f(z)|^2 \, dv(z) < \infty,$$

where $dv(z)$ is the Lebesgue area measure on $G$. Let us show that the system $\{ \frac{1}{(z-\xi)^2} \}_{\xi \in \mathbb{C} \setminus \overline{G}}$ belongs to $B_2(G)$ as functions of the variable $z$. Suppose that $\xi_0 \in \mathbb{C} \setminus \overline{G}$, $d = \frac{\text{dist}(\xi_0, \partial G)}{2}$ and $R_d$ is a disk with center at the point $\xi_0$ and
radius \( d \). We have

\[
\int_{G} \left| \frac{1}{(z - \xi_0)^2} \right|^2 dv(z) \leq \int_{C \setminus R_d} \left| \frac{1}{(z - \xi_0)^2} \right|^2 dv(z) = \\
= \int_{d}^{\infty} \int_{0}^{2\pi} \frac{1}{r^4} \cdot r \, dr \, d\varphi = \frac{4\pi}{d^2} < \infty.
\]

(1)

Hence, the function \( \frac{1}{(z - \xi_0)^2} \), \( z \in G \) belongs to \( B_2(G) \) and the system \( \left\{ \frac{1}{(z - \xi)^2} \right\}_{\xi \in \partial G} \) belongs to \( B_2(G) \) as functions of the variable \( z \).

Let us show that the system \( \left\{ \frac{1}{(z - \xi)^2} \right\}_{\xi \in \partial G} \) is complete in the space \( B_2(G) \).

By the Banach Theorem we must prove that the condition

\[
\left( \frac{1}{(z - \xi)^2}, g(z) \right)_{B_2(G)} = 0, \quad \forall \xi \in \partial G, \; g \in B_2(G),
\]

implies \( g \equiv 0 \).

Without loss of generality it can be assumed that \( 0 \in \partial G \). The conformal mapping \( w = \varphi(z) = 1/z \) takes domain \( G \) (\( \partial G \)) to bounded domain \( G_{\varphi} \) (\( \partial G_{\varphi} \)). The mapping \( \varphi \) generates an isometry \( T_{\varphi} \) (see, e.g., [1])

\[
f \in B_2(G), \; f(z) \xrightarrow{T_{\varphi}} f_{\varphi}(w) = f(\varphi^{-1}(w)) \cdot \varphi^{-1}(w) \in B_2(G_{\varphi}), \quad (f_{\varphi}, g_{\varphi})_{B_2(G_{\varphi})} = (f, g)_{B_2(G)}.
\]

The operator \( T_{\varphi} \) takes the system of function \( \left\{ \frac{1}{(z - \xi)^2} \right\}_{\xi \in \partial G} \in B_2(G) \) to system

\[
\left\{ \frac{1}{w^2} \cdot \left( \frac{1}{(1/w - 1/\eta)^2} \right) \right\}_{\eta \in \partial G_{\varphi}} \in B_2(G_{\varphi}).
\]

We have

\[
0 = \left( \frac{1}{(z - \xi)^2}, g(z) \right)_{B_2(G)} = \left( \frac{1}{w^2} \cdot \left( \frac{1}{(1/w - 1/\eta)^2} \right), g_{\varphi}(w) \right)_{B_2(G_{\varphi})} = \\
= \eta^2 \cdot \left( \frac{1}{w_\eta^2} \cdot \left( \frac{1}{(1/w - 1/\eta)^2} \right), g_{\varphi}(w) \right)_{B_2(G_{\varphi})} = \\
= \eta^2 \cdot \left( \frac{1}{(w - \eta)^2}, g_{\varphi}(w) \right)_{B_2(G_{\varphi})}, \quad \forall \eta \in \partial G_{\varphi}. \quad (2)
\]

It follows from the paper [2] that \( \left( \frac{1}{(w - \eta)^2}, g_{\varphi}(w) \right)_{B_2(G_{\varphi})}, \eta \in \partial G_{\varphi} \) is holomorphic function. Using \( (2) \), we get

\[
\left( \frac{1}{(w - \eta)^2}, g_{\varphi}(w) \right)_{B_2(G_{\varphi})} = 0, \forall \eta \in \partial G_{\varphi}. \quad (3)
\]
The system of functions \( \{ \frac{1}{(w-\eta)^2} \}_{\eta \in \mathbb{C} \setminus \mathcal{G}_\varphi} \) is complete in the space \( B_2(G) \) (see, e.g. [2]). By (3), it follows that \( g_\varphi(w) \equiv 0, \; w \in \mathbb{C} \setminus \mathcal{G}_\varphi \). Hence,
\[
g(z) \equiv 0, \; z \in \mathbb{C} \setminus \mathcal{G}.
\]

Also, the system \( \{ \frac{1}{(z-\xi)^2} \}_{\xi \in \mathbb{C} \setminus \mathcal{G}} \) is complete in the space \( B_2(G) \). Let us associate every linear continuous functional \( f^* \) on \( B_2(G) \) generates by the function \( f \in B_2(G) \), to the function
\[
\tilde{f}(\xi) \overset{def}{=} \left( \frac{1}{(z-\xi)^2}, f(z) \right)_{B_2(G)} = \int_{\mathbb{C} \setminus \mathcal{G}} f(z) \cdot \frac{1}{(z-\xi)^2}, dv(z) \quad \xi \in \mathbb{C} \setminus \mathcal{G}.
\]

**Definition 1** The function \( \tilde{f} \) is called Hilbert transform of the functional generated by \( f \in B_2(G) \).

Since the system of functions \( \{ \frac{1}{(z-\xi)^2} \}_{\xi \in \mathbb{C} \setminus \mathcal{G}} \) is complete in the space \( B_2(G) \), we see that the mapping \( f^* \to f \) is injective. The family of functions \( \tilde{f} \) forms a space
\[
\{ \tilde{f} : \tilde{f}(\xi) = \left( \frac{1}{(z-\xi)^2}, f(z) \right)_{B_2(G)} = \tilde{B}_2(G) \},
\]
where the induced structure of the Hilbert space is considered, i.e.
\[
(\tilde{f}, \tilde{g})_{\tilde{B}_2(G)} \overset{def}{=} (g, f)_{B_2(G)}, \; f, g \in B_2(G).
\]

**Definition 2** It is said that a bounded simply connected domain \( G \subset \mathbb{C} \) is quasidisk if there exists a constant \( C > 0 \) such that for any \( z_1, z_2 \in \partial G \)
\[
diam l(z_1, z_2) \leq C |z_1 - z_2|,
\]
where \( l(z_1, z_2) \) is the part of \( \partial G \) between \( z_1 \) and \( z_2 \) which has the smaller diameter.

The following theorem is valid (see [2])

**Theorem 1** 1. For any \( g \in B_2(G) \) the function \( \tilde{g} \) lies in \( B_2(\mathbb{C} \setminus \mathcal{G}) \) and
\[
\|\tilde{g}\|_{B_2(\mathbb{C} \setminus \mathcal{G})} \leq \|g\|_{B_2(G)}.
\]
2. Suppose the domain \( G \) is a quasidisk. Then the Hilbert transform operator acting from \( B_2(G) \) to \( B_2(\mathbb{C} \setminus \overline{G}) \) is surjective operator: for any function \( h \in B_2(\mathbb{C} \setminus \overline{G}) \) there exist a unique function \( g \in B_2(G) \) such that \( \tilde{g} = h \) and

\[
\|g\|_{B_2(G)} \leq c \|\tilde{g}\|_{B_2(\mathbb{C} \setminus \overline{G})},
\]

were \( 0 < c \leq 1 \) is a some constant.

3. If domain \( G \) is not a quasidisk, then the Hilbert transform operator acting from \( B_2(G) \) to \( B_2(\mathbb{C} \setminus \overline{G}) \) is not surjective operator. The image of the Hilbert transform operator is dense in \( B_2(\mathbb{C} \setminus \overline{G}) \).

Let the domain \( G \) be an unbounded quasidisk and \( \infty \in \partial G \).

From Ahlfors’s theorem (see [3], p. 48) it follows that there exist a quasiconformal reflection \( \rho(z) \) such that

1. The map \( \rho(z) \) is homeomorphism of the extended complex plane.

2. There exist constants \( C_1, C_2 > 0 \) such that

\[
C_1|z_1 - z_2| \leq |\rho(z_1) - \rho(z_2)| \leq C_2|z_1 - z_2|, \quad \forall z_1, z_2 \in \mathbb{C}.
\]

3. \( \rho(z) = z, \quad z \in \partial G \).

The map \( \rho \) is called Ahlfors’s quasiconformal reflection.

Take a point \( z_0 \in G \). The functional \( \delta_{z_0} \) is linear continuos functional on \( B_2(G) \) (see [1]), so \( B_2(G) \) is reproducing kernel Hilbert space([4]). By \( K_{B_2(G)}(z, \xi), z, \xi \in G \) we denote the reproducing kernel of the space \( B_2(G) \).([4],[5])

In this paper we obtain

**Theorem 2** Let \( G \) be an unbounded quasidisk, \( \infty \in \partial G \). Then there exist a linear continuous one-to-one operator \( \mathcal{B} \) in the space \( B_2(G) \) such that

\[
\mathcal{B} K_{B_2(G)}(z, \rho(\xi)) = \frac{1}{(z - \xi)^2}, \quad \forall \xi \in \mathbb{C} \setminus \overline{G}.
\]
1 Auxiliary information

Definition 3 ([6]) Let $H$ be the Hilbert space over the field $\mathbb{R}$ or $\mathbb{C}$, and $\Omega$ is a space with a countably additive measure $\mu$ (see [7], p.95–101). The system of elements $\{e_\omega\}_{\omega \in \Omega}$ is called an orthosimilar system (similar to orthogonal) with respect to the measure $\mu$ in $H$, if any element $y \in H$ can be represented in the form

$$y = \int_\Omega (y,e_\omega)_H e_\omega \, d\mu(\omega).$$

Here the integral is interpreted as a proper or improper Lebesgue integral of a function with the values in $H$. In the latter case there is an exhaustion $\{\Omega_k\}_{k=1}^\infty$ of the space $\Omega$ possibly depending on $y$ (it is called suitable for $y$), that the function $(y,e_\omega)_H \cdot e_\omega$ is Lebesgue integrable on $\Omega_k$ and

$$y = \int_\Omega (y,e_\omega)_H e_\omega \, d\mu(\omega) = \lim_{k \to \infty} (L) \int_{\Omega_k} (y,e_\omega)_H e_\omega \, d\mu(\omega).$$

Note that all $\Omega_k$ are measurable by $\mu$, $\Omega_k \subset \Omega_{k+1}$ for $k \in \mathbb{N}$ and $\bigcup_{k=1}^\infty \Omega_k = \Omega$.

Examples:

1. Any orthonormal basis $\{e_k\}_{k=1}^\infty \subset H$ in an arbitrary Hilbert space $H$ is an orthosimilar system; any element $y \in H$ can be represented in the form

$$y = \sum_{k=1}^\infty (y,e_k)e_k.$$

Here one can take a set $\mathbb{N}$ as $\Omega$, and as of the measure $\mu$ one can take the counting measure, i.e. a measure of the set from $\mathbb{N}$ is the amount of different natural numbers belonging to the set.

2. Let $H$ be the Hilbert space, $H_1$ is a subspace of $H$, and $P$ is the operator of orthogonal projection of elements from $H$ onto $H_1$. Let $\{e_k\}_{k=1}^\infty \subset H$ be an orthogonal basis in $H$. Then, the system of elements $\{P(e_k)\}_{k=1}^\infty \subset H_1$ is an orthosimilar system in $H_1$. (see [6], Theorem 9). Note, that if $\{e_k\}_{k=1}^\infty$ is an orthogonal basis in $H$, then the system $\{P(e_k)\}_{k=1}^\infty$, in general, is not an orthogonal basis in $H_1$.

3. Let $H = L_2(\mathbb{R})$. The function $\psi \in L_2(\mathbb{R})$, $\|\psi\|_{L_2(\mathbb{R})} = 1$. A system of Morlet wavelets $\psi_{a,b}(x) = \frac{1}{\sqrt{|a|}} \psi \left( \frac{t-b}{a} \right)$, $a \in \mathbb{R}\backslash\{0\}, b \in \mathbb{R}$ is an
orthosimilar system in the space \( L_2(\mathbb{R}) \); any function \( L_2(\mathbb{R}) \) can be represented in the form

\[
f(x) = \int_{\mathbb{R}\setminus\{0\}} \int_{\mathbb{R}} (f(\tau), \psi_{a,b}(\tau))L_2(\mathbb{R}) \psi_{a,b}(x) \frac{dbda}{C_\psi|a|^2}.
\]

where \( C_\psi > 0 \) is a constant. The set \( (\mathbb{R}\setminus\{0\}) \times \mathbb{R} \) with the measure \( \frac{dbda}{C_\psi|a|^2} \) is taken as the space \( \Omega \) here. (see [8], [6]).

**Definition 4 ([6])** An orthosimilar system is said to be nonnegative if the measure \( \mu \) is nonnegative.

**Theorem 3 (see [9])** Let \( H \) be a reproducing kernel Hilbert space of functions on the domain \( G \subset \mathbb{C} \). The norm in the space \( H \) has an integral form

\[
\|f\|_H = \sqrt{\int_G |f(\xi)|^2 \, d\nu(\xi)}
\]

in the space \( H \) if and only if the system of functions \( \{K_H(\xi,t)\}_{t \in G} \) is a nonnegative orthosimilar system with respect to the measure \( \nu \) in the space \( H \).

**Lemma 1** Let a domain \( G \subset \mathbb{C} \) be an unbounded quasidisk, \( \infty \in \partial G \). Then there exist a linear continuous self-adjoint operator \( \mathcal{R} \), which is an automorphism of the Hilbert space \( B_2(G) \), such that the system

\[
\{ \mathcal{R}K_{B_2(G)}(z, \rho(\xi)) \}_{\xi \in \mathbb{C}\setminus\overline{G}}
\]

is an orthosimilar system with respect to the Lebesgue area measure in the space \( B_2(G) \), i.e. any function \( f \in B_2(G) \) can be represented in the form

\[
f(z) = \int_{\mathbb{C}\setminus\overline{G}} (f(\tau), R_zK_{B_2(G)}(\tau, \rho(\xi)))_{B_2(G)} \cdot R_zK_{B_2(G)}(z, \rho(\xi)) \, d\nu(\xi), \ z \in G.
\]

**Proof.** The norm of \( B_2(G) \) has the form:

\[
\|f\|_{B_2(G)} = \sqrt{\int_G |f(z)|^2 \, dv(z)}.
\]
After change of variable $z = \rho(\xi)$:

$$
\int_G |f(z)|^2 \, dv(z) = \int_{C \setminus G} |f(\rho(\xi))|^2 \, dv(\rho(\xi)).
$$

From relation [4] it follows that

$$
C^2_1 \int_{C \setminus G} |f(\rho(\xi))|^2 \, dv(\rho(\xi)) \leq \int_{C \setminus G} |f(\rho(\xi))|^2 \, dv(\rho(\xi)) \leq C^2_2 \int_{C \setminus G} |f(\rho(\xi))|^2 \, dv(\rho(\xi)). \quad (6)
$$

Define

$$
(f, g)_{B_2(G)} \overset{\text{def}}{=} \int_{C \setminus G} f(\rho(\xi)) \cdot \overline{g(\rho(\xi))} \, dv(\xi); \|f\|_1 \overset{\text{def}}{=} \sqrt{(f, f)_1}.
$$

Using relation (6), we get

$$
C_1 \|f\|_1 \leq \|f\|_{B_2(G)} \leq C_2 \|f\|_1, \forall f \in B_2(G).
$$

The norms $\| \cdot \|_{B_2(G)}, \| \cdot \|_1$ are equivalent.

From Lemma 1, [9] it follows that there exist a linear continuous self-adjoint operator $T$, which is the automorphism of the Hilbert space $B_2(G)$, such that

$$
(f, g)_{B_2(G)} = (Tf, g)_{B_2(G)}, f, g \in B_2(G).
$$

We get

$$
(f, g)_{B_2(G)} = (Tf, g)_{B_2(G)} = \int_{C \setminus G} Tf(\rho(\xi)) \cdot \overline{g(\rho(\xi))} \, dv(\xi) =
$$

$$
= \int_{C \setminus G} (Tf(\tau), K_{B_2(G)}(\tau, \rho(\xi)))_{B_2(G)} \cdot (g(\tau), K_{B_2(G)}(\tau, \rho(\xi)))_{B_2(G)} \, dv(\xi). \quad (7)
$$

Consider a fixed point $z \in G$. We take the function $g(\tau) = K_{B_2(G)}(\tau, z)$. 7
Hence,
\[
f(z) = (f(\tau), K_{B_2(G)}(\tau, z))_{B_2(G)} = (Tf(\tau), K_{B_2(G)}(\tau, z))_1 =
\int_{\mathbb{C} \setminus G} (Tf(\tau), K_{B_2(G)}(\tau, \rho(\xi)))_{B_2(G)} \times
(K_{B_2(G)}(\tau, z), K_{B_2(G)}(\tau, \rho(\xi)))_{B_2(G)} d\nu(\xi) =
\int_{\mathbb{C} \setminus G} (Tf(\tau), K_{B_2(G)}(\tau, \rho(\xi)))_{B_2(G)} \cdot K_{B_2(G)}(z, \rho(\xi)) d\nu(\xi) =
\int_{\mathbb{C} \setminus G} (Tf(\tau), K_{B_2(G)}(\tau, \rho(\xi)))_{B_2(G)} \cdot K_{B_2(G)}(z, \rho(\xi)) d\nu(\xi). \quad (8)
\]

Any function \( f \in B_2(G) \) can be represented in the form
\[
f(z) = \int_{\mathbb{C} \setminus G} (Tf(\tau), K_{B_2(G)}(\tau, \rho(\xi)))_{B_2(G)} \cdot K_{B_2(G)}(z, \rho(\xi)) d\nu(\xi).
\]

Operator \( T \) is a self-adjoint operator, so it has a unique positive square root
\[
\mathcal{R} : B_2(G) \to B_2(G)
\]
(see, e.g., [10], pp.264, 265) such that \( T = \mathcal{R} \circ \mathcal{R} \). The operator \( \mathcal{R} \) is an automorphism of the space \( B_2(G) \) as well. Therefore,
\[
f(z) = \int_{\mathbb{C} \setminus G} (\mathcal{R} \circ \mathcal{R} f(\tau), K_{B_2(G)}(\tau, \rho(\xi)))_{B_2(G)} \cdot K_{B_2(G)}(z, \rho(\xi)) d\nu(\xi) =
\int_{\mathbb{C} \setminus G} (\mathcal{R} f(\tau), \mathcal{R} K_{B_2(G)}(\tau, \rho(\xi)))_{B_2(G)} \cdot K_{B_2(G)}(\tau, \rho(\xi)) d\nu(\xi). \quad (9)
\]

Since \( \mathcal{R} \) is one-to-one operator in the space \( B_2(G) \), it follows from (9) that
\[
\mathcal{R}^{-1} f(z) = \int_{\mathbb{C} \setminus G} (f(\tau), R_\tau K_{B_2(G)}(\tau, \rho(\xi)))_{B_2(G)} \cdot K_{B_2(G)}(z, \rho(\xi)) d\nu(\xi),
\]
where operator \( \mathcal{R}^{-1} \) is the inverse operator to the operator \( \mathcal{R} \). Using the theorem from ([7], 113), one can demonstrate that
\[
f(z) = \int_{\mathbb{C} \setminus G} (f(\tau), R_\tau K_{B_2(G)}(\tau, \rho(\xi)))_{B_2(G)} \cdot R_z K_{B_2(G)}(z, \rho(\xi)) d\nu(\xi), z \in G.
\]

(10)
Thus, the system
\[ \{ R K_{B_2(G)}(z, \rho(\xi)) \}_{\xi \in \mathbb{C} \setminus G} \]
is an orthosimilar system with respect to the Lebesgue area measure in the space \( B_2(G) \), i.e. any function \( f \in B_2(G) \) can be represented in the form
\[
 f(z) = \int_{\mathbb{C} \setminus G} (f(\tau), R_{z} K_{B_2(G)}(\tau, \rho(\xi)))_{B_2(G)} \cdot R_{z} K_{B_2(G)}(z, \rho(\xi)) \, dv(\xi), \quad z \in G.
\]
Lemma 1 is proved.

Lemma 2 Let a domain \( G \) be a quasisdisk. Then there exist a linear continuous self-adjoint operator \( S \), which is an automorphism of the Hilbert space \( B_2(G) \), such that the system
\[ \{ S_{z^{-1}(z-\xi)^2} \}_{\xi \in \mathbb{C} \setminus G} \]
is an orthosimilar system with respect to the Lebesgue area measure in the space \( B_2(G) \), i.e. any function \( f \in B_2(G) \) can be represented in the form
\[
 f(z) = \int_{\mathbb{C} \setminus G} (f(\tau), S_{z^{-1}(z-\xi)^2})_{B_2(G)} \cdot S_{z^{-1}(z-\xi)^2} \, dv(\xi), \quad z \in G. \tag{11}
\]
The reproducing kernel of the space \( B_2(G) \) has the form
\[
 K_{B_2(G)}(z, \eta) = \int_{\mathbb{C} \setminus G} S_{z^{-1}(z-\xi)^2} \cdot \overline{S_{\eta^{-1}(\eta-\xi)^2}} \, dv(\xi), \quad z, \eta \in G. \tag{12}
\]

Proof. Since \( G \) is a quasisdisk, it follows from Theorem 1 that the space \( \tilde{B}_2(G) \) has an equivalent norm
\[
 \| \tilde{g} \|_{1} \overset{\text{def}}{=} \sqrt{\int_{\mathbb{C} \setminus G} |\tilde{g}(\xi)|^2 \, dv(\xi)}.
\]
There exist a constant \( c > 0 \) such that
\[
 c \| \tilde{g} \|_{\tilde{B}_2(\mathbb{C} \setminus \mathbb{G})} \leq \| \tilde{g} \|_1 \leq \| \tilde{g} \|_{\tilde{B}_2(\mathbb{C} \setminus \mathbb{G})}, \forall \tilde{g} \in \tilde{B}_2(\mathbb{C} \setminus \mathbb{G}).
\]
We use the following theorem from [9].
Theorem 4 ([9]) In order to introduce into the space $\tilde{B}_2(G,\mu)$ a norm

$$\|\tilde{f}\|_{\nu} = \sqrt{\int_{\mathbb{C}\setminus G} |\tilde{f}(\xi)|^2 d\nu(\xi)}$$

($\nu$ is a nonnegative Borel measure on $\mathbb{C}\setminus G$), which is equivalent to the original one, it is necessary and sufficient that there exist a linear continuous operator $S$, realizing an automorphism of the Banach space $B_2(G,\mu)$, such that the system $\{S\left(\frac{1}{(z-\xi)^2}\right)\}_{\xi \in \mathbb{C}\setminus G}$ is an orthosimilar system with respect to the measure $\nu$ in the space $B_2(G,\mu)$, i.e. any element $f \in B_2(G,\mu)$ can be represented in the form

$$f(z) = \int_{\mathbb{C}\setminus G} (f(\tau), S_{\tau} \left(\frac{1}{(\tau-\xi)^2}\right)_{B_2(G,\mu)}) S_{z} \left(\frac{1}{(z-\xi)^2}\right) d\nu(\xi), \quad z \in G.$$

Let the measure $\mu$ be the Lebesgue area measure on $G$. Let the measure $\nu$ be the Lebesgue area measure on $\mathbb{C}\setminus G$ as measure $\nu$. By theorem 4 it follows that the system $\{S_{z} \left(\frac{1}{(z-\xi)^2}\right)\}_{\xi \in \mathbb{C}\setminus G}$ is an orthosimilar system with respect to the Lebesgue area measure on $\mathbb{C}\setminus G$ in the space $B_2(G)$, i.e. any element $f \in B_2(G)$ can be represented in the form

$$f(z) = \int_{\mathbb{C}\setminus G} (f(\tau), S_{\tau} \left(\frac{1}{(\tau-\xi)^2}\right)_{B_2(G)}) S_{z} \left(\frac{1}{(z-\xi)^2}\right) d\nu(\xi), \quad z \in G.$$  

Fixing a point $\eta \in G$, we take the function $f(z) = K_{B_2(G)}(z,\eta)$. We obtain

$$K_{B_2(G)}(z,\eta) = \int_{\mathbb{C}\setminus G} S_{z} \left(\frac{1}{(z-\xi)^2}\right) \cdot S_{\eta} \left(\frac{1}{(\eta-\xi)^2}\right) d\nu(\xi), \quad z,\eta \in G.$$

Lemma 2 is proved.

Let us define a linear manifold of functions $\mathcal{L}$ as a set of functions $\varphi \in B_2(G)$ such that there is a finite set of points $\{\xi_k\}_{k=1}^n \in \mathbb{C}\setminus G$, and a set of complex numbers $\{c_k\}_{k=1}^n \in \mathbb{C}$, and the function $\varphi$ has the form

$$\varphi(z) = \sum_{k=1}^n c_k R_z K_{B_2(G)}(z,\rho(\xi_k)), \quad z \in G.$$  

Thus, $\mathcal{L}$ is a linear span of the system of functions

$$\{R_z K_{B_2(G)}(z,\rho(\xi))\}_{\xi \in \mathbb{C}\setminus G} \subset B_2(G).$$
Lemma 3 Let \( \{\xi_k\}_{k=1}^n \) be a set of \( n \) distinct points belonging to the domain \( \mathbb{C}\setminus \overline{\mathbb{G}} \), and \( \varphi(z) \) be a function of the form

\[
\varphi(z) = \sum_{k=1}^{n} c_k R_z K_{B_2(G)}(z, \rho(\xi_k)), \quad z \in G,
\]

where \( c_k, k = 1, \ldots, n \) are some constants. Then, the condition \( \varphi(z) \equiv 0 \) implies that \( c_k = 0, k = 1, \ldots, n \).

Let us define a linear manifold of functions \( \mathcal{M} \) as a set of functions \( \psi \in B_2(G) \) such that there is a finite set of points \( \{\xi_k\}_{k=1}^n \in \mathbb{C}\setminus \overline{\mathbb{G}} \), and a set of complex numbers \( \{c_k\}_{k=1}^n \in \mathbb{C} \), and the function \( \psi \) has the form

\[
\psi(z) = \sum_{k=1}^{n} c_k S_z \frac{1}{(z-\xi_k)^2}, \quad z \in G.
\]

Therefore, \( \mathcal{M} \) is a linear span of the system of functions

\[
\left\{ S_z \frac{1}{(z-\xi_k)^2} \right\}_{\xi \in \mathbb{C}\setminus \overline{\mathbb{G}}} \subset B_2(G).
\]

Lemma 4 Let \( \{\xi_k\}_{k=1}^n \) be a set of \( n \) distinct points belonging to the domain \( \mathbb{C}\setminus \overline{\mathbb{G}} \), and \( \psi(z) \) be a function of the form

\[
\psi(z) = \sum_{k=1}^{n} c_k S_z \frac{1}{(z-\xi_k)^2}, \quad z \in G,
\]

where \( c_k, k = 1, \ldots, n \) are some constants. Then, the condition \( \psi(z) \equiv 0 \) implies that \( c_k = 0, k = 1, \ldots, n \).

2 Proof of Theorem 2.

Let \( G \) be an unbounded quasidisk, \( \infty \in \partial G \). By Lemma 1, it follows that the system

\[
\{ R K_{B_2(G)}(z, \rho(\xi)) \}_{\xi \in \mathbb{C}\setminus \overline{\mathbb{G}}}
\]

is an orthosimilar system with respect to the Lebesgue area measure in the space \( B_2(G) \), i.e. any function \( f \in B_2(G) \) can be represented in the form

\[
f(z) = \int_{\mathbb{C}\setminus \overline{\mathbb{G}}} (f(\tau), R_z K_{B_2(G)}(\tau, \rho(\xi)))_{B_2(G)} \cdot R_z K_{B_2(G)}(z, \rho(\xi)) \, dv(\xi), \quad z \in G.
\]
From Lemma 2 it follows that the system
\[
\left\{ S_{z_1 \cdot \frac{1}{(z-\xi)^2}} \right\}_{\xi \in \mathbb{C} \setminus G}
\]
is an orthosimilar system with respect to the Lebesgue area measure in the
space $B_2(G)$, i.e. any function $f \in B_2(G)$ can be represented in the form
\[
f(z) = \int_{\mathbb{C} \setminus G} (f(\tau), S_{\tau \cdot \frac{1}{(\tau-\xi)^2}})_{B_2(G)} \cdot S_{z \cdot \frac{1}{(z-\xi)^2}} dv(\xi), \ z \in G.
\]
Define the operator $A : \mathcal{L} \to \mathcal{M}$ by the rule
\[
\varphi(z) = \sum_{k=1}^{n} c_k R_z K_{B_2(G)}(z, \rho(\xi_k)) \to A\varphi(z) = \sum_{k=1}^{n} c_k S_{z \cdot \frac{1}{(z-\xi_k)^2}}, \ z \in G.
\]
By Lemma 1, Lemma 2 it follows that operator $A$ is well defined. Thus, operator $A$ is an one-to-one operator acting from $\mathcal{L}$ to $\mathcal{M}$. Define the norm on $\mathcal{M}$:
\[
\|A\varphi\|_{\mathcal{M}} \overset{\text{def}}{=} \|\varphi\|_{B_2(G)}, \ \varphi \in \mathcal{L}.
\]
Let $H_M$ be a complement of $\mathcal{M}$ with respect to the norm $\| \cdot \|_{\mathcal{M}}$. $H_M$ is a Hilbert space. Since $\mathcal{L}$ is a dense set in $B_2(G)$ (see [4]), we see that the operator $A$ has a continuous extension to Hilbert space $B_2(G)$.
\[
\|Af\|_{H_M} = \|f\|_{B_2(G)}, \ f \in B_2(G).
\]
Indeed, any function $f \in B_2(G)$ can be approximated by the sequence of elements $\{\varphi_k\}_{k \geq 0}$ from $\mathcal{L}$. The image $\{A\varphi_k\}_{k \geq 0}$ is a fundamental sequence in $H_M$. Then it follows that there exist an element $Af$ in $H_M$ such that
\[
Af = \lim_{k \to \infty} A\varphi_k.
\]
It is clear that the element $Af$ is well defined. Therefore, the operator
\[
A : B_2(G) \to H_M
\]
is a linear continuous bijective operator acting from $B_2(G)$ to $H_M$. This operator is well defined.

**Lemma 5** The space $H_M$ is reproducing kernel Hilbert space. Reproducing kernel of the space $H_M$ has the form:
\[
K_{H_M}(z, \eta) = \int_{\mathbb{C} \setminus G} S_{z \cdot \frac{1}{(z-\xi)^2}} \cdot S_{\eta \cdot \frac{1}{(\eta-\xi)^2}} dv(\xi), \ z, \eta \in G.
\]
Proof. By Lemma 1 it follows that any function $f \in B_2(G)$ can be represented in the form:

$$f(z) = \int_{C \setminus G} (f(\tau), R_\tau K_{B_2(G)}(\tau, \rho(\xi)))_{B_2(G)} \cdot R_z K_{B_2(G)}(z, \rho(\xi)) \, dv(\xi), \quad z \in G.$$ 

We have

$$A \circ R_z K_{B_2(G)}(z, \rho(\xi)) = S_z \frac{1}{(z - \xi)^2}, \quad \xi \in C \setminus G.$$ 

Taking into account theorem (13), we obtain

$$Af(z) = \int_{C \setminus G} (f(\tau), R_\tau K_{B_2(G)}(\tau, \rho(\xi)))_{B_2(G)} \cdot A \circ R_z K_{B_2(G)}(z, \rho(\xi)) \, dv(\xi) =$$

$$= \int_{C \setminus G} (Af(\tau), S_\tau \frac{1}{(\tau - \xi)^2})_{HM} \cdot S_z \frac{1}{(z - \xi)^2}, \quad z \in G.$$ 

Thus, the system

$$\{ S_z \frac{1}{(z - \xi)^2} \}_{\xi \in C \setminus G}$$

is an orthosimilar system with respect to the Lebesgue area measure in $B_2(G)$. We use the following theorem (see [6], Theorem 1).

**Theorem 5 ([6], An analogue of the Parseval identity)** Let $\{e_\omega\}_{\omega \in \Omega} \subset H$ be a nonnegative orthosimilar system with respect to the measure $\mu$ in $H$. Then, for any element $y \in H$ one has

$$\|y\|^2_H = \int_{\Omega} |(y, e_\omega)|^2 \, d\mu(\omega)$$

and for any two elements $x, y \in H$ one has

$$(x, y)_H = \int_{\Omega} (x, e_\omega) \cdot (y, e_\omega) \, d\mu(\omega).$$

This implies that,

$$\|Af\|^2_{HM} = \int_{C \setminus G} |(Af(\tau), S_\tau \frac{1}{(\tau - \xi)^2})_{HM}|^2 \, dv(\xi).$$
Consider the point $z_0 \in \mathbb{C} \setminus G$. It follows from Schwarz’s inequality that

$$|A_z f(z_0)| \leq \sqrt{\int_{\mathbb{C} \setminus G} |(Af(\tau), S_{\tau} \frac{1}{(\tau - \xi)^2})|_{H_M}^2 \, dv(\xi) \times \sqrt{\int_{\mathbb{C} \setminus G} |S_z \frac{1}{(z_0 - \xi)^2}|^2 \, dv(\xi)}} = \|Af\|_{H_M} \cdot \sqrt{\int_{\mathbb{C} \setminus G} |S_z \frac{1}{(z_0 - \xi)^2}|^2 \, dv(\xi)}. \quad (14)$$

Thus, $H_M$ is a reproducing kernel Hilbert space.

Using (13), we get

$$K_{H_M}(z, \eta) = \int_{\mathbb{C} \setminus G} S_z \frac{1}{(z_0 - \xi)^2} \cdot \overline{S_{\eta} \frac{1}{(\eta - \xi)^2}} \, dv(\xi), \, z, \eta \in G. \quad (15)$$

Lemma 5 is proved.

By Lemma 2

$$K_{B_2}(G)(z, \eta) = \int_{\mathbb{C} \setminus G} S_z \frac{1}{(z_0 - \xi)^2} \cdot \overline{S_{\eta} \frac{1}{(\eta - \xi)^2}} \, dv(\xi), \, z, \eta \in G.$$  

Using (15), we get

$$K_{H_M}(z, \eta) \equiv K_{B_2}(G)(z, \eta).$$

It follows from Moore–Aronszajn’s theorem (see [4], p. 243) that the space $H_M$ coincides with the space $B_2(G)$. Thus, the operator $A$ is a linear continuous one-to-one operator $B$ in the space $B_2(G)$ such that

$$A \circ R K_{B_2(G)}(z, \rho(\xi)) = S_{\frac{1}{(z - \xi)^2}}, \, \forall \xi \in \mathbb{C} \setminus G.$$

Denote $B \overset{\text{def}}{=} S \circ A \circ R$. We have

$$B K_{B_2(G)}(z, \rho(\xi)) = \frac{1}{(z - \xi)^2}, \, \forall \xi \in \mathbb{C} \setminus G.$$  

Theorem 2 is proved.

We prove the following analogue of Schwarz’s reflection principle.
Corollary 1 Let $G$ be an unbounded quasidisk, $\infty \in \partial G$, and $\rho$ be Ahlfors’s quasiconformal reflection. If $f \in B_2(G)$ then $\overline{f(\rho(\xi))} \in B_2(\mathbb{C}\setminus \overline{G})$. For any function $g \in B_2(\mathbb{C}\setminus G)$ there exists a unique $f \in B_2(G)$ such that
\[ g(\xi) = \overline{f(\rho(\xi))}, \quad \xi \in \mathbb{C}\setminus \overline{G}, \]
\[ C_1 \|\tilde{f}\|_{B_2(\mathbb{C}\setminus \overline{G})} \leq \|g\|_{B_2(\mathbb{C}\setminus \overline{G})} \leq C_2 \|\tilde{f}\|_{B_2(\mathbb{C}\setminus \overline{G})}, \quad C_1, C_2 > 0. \tag{16}\]

Proof. It follows from theorem [2] that there exists a linear continuous one-to-one operator $\mathcal{B}$ in the space $B_2(D, \mu)$ such that
\[ \mathcal{B}zK_{B_2(G)}(z, \rho(\xi)) = 1/(z-\xi)^2, \quad \xi \in \mathbb{C}\setminus \overline{G} \]
and
\[ \mathcal{B}_z^{-1}1/(z-\xi)^2 = K_{B_2(G)}(z, \rho(\xi)), \quad \xi \in \mathbb{C}\setminus \overline{G}, \]
where operator $\mathcal{B}^{-1}$ is the inverse operator to the operator $\mathcal{B}$. If $f \in B_2(G)$ then
\[ \overline{f(\rho(\xi))} = (K_{B_2(G)}(z, \rho(\xi)), f(z))_{B_2(G)} = (\mathcal{B}_z^{-1}1/(z-\xi)^2, f(z))_{B_2(G)} = \]
\[ = (1/(z-\xi)^2, \mathcal{B}_z^{-1}f(z))_{B_2(G)} = \mathcal{B}^{-1}f(\xi), \quad \xi \in \mathbb{C}\setminus \overline{G}. \tag{17}\]
Since $\mathcal{B}^{-1}f$ belongs to $B_2(\mathbb{C}\setminus \overline{G})$, we see that $g(\xi) \overset{\text{def}}{=} \overline{f(\rho(\xi))}$ belongs to $B_2(\mathbb{C}\setminus \overline{G})$. By theorem [1] it follows that
\[ \|g\|_{B_2(\mathbb{C}\setminus \overline{G})} = \|\mathcal{B}^{-1}f\|_{B_2(\mathbb{C}\setminus \overline{G})} \preceq \|\mathcal{B}^{-1}f\|_{B_2(G)}. \tag{18}\]
Symbol $\preceq$ means that norms are equivalent. Since
\[ \|\mathcal{B}^{-1}f\|_{B_2(G)} \preceq \|f\|_{B_2(G)}, \quad f \in B_2(G) \tag{19}\]
it follows from Theorem [1] that
\[ \|f\|_{B_2(G)} \preceq \|\tilde{f}\|_{B_2(\mathbb{C}\setminus \overline{G})}, \quad f \in B_2(G). \tag{20}\]
The relations (18), (19), (20) implies (16). Corollary [1] is proved.
References

[1] D. Gaier, Lectures on complex approximation, Boston ; Basel ; Stuttgart: Birkhauser, 1985. 208 p.

[2] V. V. Napalkov Jr., R. S. Yulmukhametov, On the Hilbert Transform in Bergman Space// Mathematical notes, July 2001, Volume 70, Issue 1-2, pp. 61-70.

[3] L.V. Ahlfors, Lectures on quasiconformal mappings, University Lecture series, AMS, 2006, 2nd. ed., 164 p.

[4] N. Aronszajn, Theory of reproducing kernels//Transactions of the AMS V. 68. 3. P. 337–404.

[5] H. Hedenmalm, B. Korenblum, K. Zhu, Theory of Bergman spaces, Springer-Verlag, New York, Inc. 2000. 289 p.

[6] T.P. Lukashenko, Properties of expansion systems similar to orthogonal ones// Izvestiya: Mathematics, 1998, 62(5): 1035-1054

[7] N. Dunford, J. Schwartz, Linear operators. Part I: General theory, New York, Interscience Publishers, 1958. 858 pp.

[8] A. Grossmann, J. Morlet, Decomposition of Hardy functions into square integrable wavelets of constant shape// SIAM J. Math. Anal. 1984. V. 15. P. 723-736.

[9] V.V. Napalkov Jr., On orthosimilar systems in a space of analytical functions and the problem of describing the dual space// Ufa mathematical journal. Volume 3. 1, 2011, pp. 30-41.

[10] F. Riesz, B. Szokefalvy-Nagy, Functional analysis, London, Translated from Second French edition, 1956, 478 pp.