Lifetime of the arrow of time inherent in chaotic eigenstates
: case of coupled kicked rotors

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A linear oscillator very weakly coupled with the object quantum system is proposed as a detector measuring the lifetime of irreversibility exhibited by the system, and classically chaotic coupled kicked rotors are examined as ideal examples. The lifetime increases drastically in close correlation with the enhancement of entanglement entropy (EE) between the kicked rotors. In the transition regime to the full entanglement, the EE of individual eigenstates fluctuates anomalously, and the lifetime also fluctuates in correlation with the EE. In the fully entangled regime the fluctuation disappears, but the lifetime is not yet unique but increases in proportion to the number of superposed eigenstates and is proportional to the square of Hilbert space dimension in the full superposition.

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The classical chaotic system can be the simplest origin of irreversibility and the arrow of time [1]. Its quantum counterpart can also be the minimal unit exhibiting quantum irreversibility. Indeed, normal diffusion [2, 3], energy dissipation [4], energy spreading [5] and many apparently irreversible phenomena can be realized in classically chaotic quantum systems with a small number of degrees of freedom. Appearance of irreversibility in closed quantum system is also a severe problem limiting the performance of quantum computation algorithm [6]. If the quantum system is also a severe problem limiting the performance of quantum computation algorithm [6]. If the quantum system prevents the system from the complete manifestation of the irreversibility [2, 3]. However, the multidimensional unbounded quantum chaos systems can recover classical chaotic nature [7] and recover a complete irreversible behavior via the Anderson-transition-like mechanism under appropriate condition [8, 9].

Irreversibility in quantum system has been explored by the time-reversal characteristics [10,12]. In particular, it has been extensively investigated by many investigators in the context of fidelity [13,14]. The time scale on which quantum chaos can show the exponential sensitivity is quite short and is up to the Ehrenfest time proportional to \( \log(N_{\text{dim}}) \) at the most [15], but the time scale on which the irreversibility is maintained is much longer and is said to be as long as the Heisenberg time, which is proportional to \( N_{\text{dim}} \) [14].

It seems to be very difficult to give a general definition for the lifetime in which the quantum irreversibility is sustained, but it would be a very important parameter characterizing quantitatively the irreversibility, i.e., the time of arrow self-organized in closed quantum systems. The main purpose of the present article is to propose a general method for observing the irreversibility inherent in quantum systems and its associated quantum states and to measure its lifetime, taking typical quantum chaos systems as examples. An important result is that the lifetime measured by the proposed method is in general much longer than the conventional Heisenberg time.

We have the fidelity as a powerful tool extracting the irreversible characteristics of quantum system [14]. However, the fidelity is not convenient for the purpose of observing irreversible characteristics with adequate numerical accuracy over an extremely long time scale without disturbing the examined system.

We introduce a linear oscillator which converts the quantum motion of the object system to a Brownian motion in the infinitely extended linear oscillator’s action space. Let the Hamiltonian of the examined quantum system “S” be \( \hat{H}(\hat{p}, \hat{q}, t) \) (\( \hat{q} \) and \( \hat{p} \) are coordinate and momentum vector operator, respectively) and it is very weakly coupled with the linear oscillator “L”, which is represented by angle-action canonical pair operators \( \hat{\theta} \) and \( \hat{J} = -i\hbar d/d\hat{\theta} \) with the Hamiltonian \( \omega \hat{J} \) of the frequency \( \omega \). The total Hamiltonian reads

\[
\hat{H}_{\text{tot}}(t) = \hat{H}(\hat{p}, \hat{q}, t) + \eta \hat{v}(\hat{p}, \hat{q}) \hat{g}(\hat{\theta}) + \omega \hat{J},
\]

(1)

where \( \hat{v} \) is a Hermitian operator, and \( \hat{g}(\hat{\theta}) \) is \( 2\pi \)-periodic function with null average [10]. By monitoring the fictitious diffusion in the \( J \)-space we can measure the lifetime of irreversibility without disturbing the object by reducing \( \eta \) as small as possible in accordance with the time of observation.

\( \hat{J} \) has the eigenvalue \( J = j\hbar \ (j \in \mathbb{Z}) \) for the eigenstate \( \langle \theta | J \rangle \propto e^{-iJ\hat{\theta}/\hbar} \) because of the \( 2\pi \)-periodicity in \( \theta \)-space. We further impose the action periodic boundary condition identifying \( J = L\hbar \) with \( J = -L\hbar \), which quantize \( \theta \)
as \( \theta_k = \frac{2\pi k}{2L} \) \((k \in \mathbb{Z})\), where \(-L < k \leq L\).

We are interested in the lifetime of irreversibility realized by classically chaotic quantum systems defined in a bounded phase space. To be concrete, we hereafter confine ourselves to the particular case that the system \( \hat{H}(\hat{p}, \hat{q}, t) \) consists of the coupled kicked rotors (CKR) which exhibit ideal chaos. More general properties of the model will be presented in a forthcoming publication [18].

We consider here the coupled kicked rotors (CKR) \( \hat{H}(\hat{p}, \hat{q}, t) = \sum_{i=1}^{N} \left[ \frac{\hat{p}_{i}^{2}}{2} + V(\hat{q}_{i})\Delta(t) \right] + cV_{12}(\hat{q}_{1}, \hat{q}_{2})\Delta(t) \), as a sample system \( S \), where \( \Delta(t) = \sum_{j}(\hat{n}(t - iT) \), because the lifetime of irreversibility will vary with the development of entanglement between the two constituent chaotic rotors controlled by the coupling strength \( \epsilon \). [17]

KR is observed at the integer multiple of the fundamental period \( T \) as \( t = \tau T + 0 \) to \( t + \tau \) is described by the unitary operator \( \hat{U} = e^{-i\left[\hat{p}\hat{q}/2h + iV(\hat{q}) + cV_{12}(\hat{q}_{1}, \hat{q}_{2})\right]/h} \). Each KR, say KR1 and KR2, is defined in the bounded phase space \( [0, 2\pi] \times [0, 2\pi] \) and \( \hbar = 2\pi/N_{i} \), where \( N_{1} = N_{2} \) is a positive integer, and the dimension of the Hilbert space for the CKR is \( N = N_{1}N_{2} = N_{1}^{2} \), which is an important parameter. We take mainly the Arnold’s cat map \( V(\hat{q}) = -K\hat{q}^{2}/2 \) (and also the standard map \( V(\hat{q}) = K\cos\hat{q} \), if necessary) with the interaction \( V_{12}(\hat{q}_{1}, \hat{q}_{2}) = \cos(\hat{q}_{1} - \hat{q}_{2}) \). Hereafter, we couple \( L \) with \( S \) via only the first KR (KR1) as \( \hat{v}(\hat{p}, \hat{q}) = \hat{v}(\hat{p}_{1}) = \sin(\hat{p}_{1}) \). so that the entanglement between the KRs may sensitively be reflected in the dynamics of \( L \) and \( S \) as \( |J = 0\rangle \) and \(|\Psi_{0}\rangle \), respectively. We reduce the coupling \( \eta \) weak enough to eliminate the backaction from \( L \) to \( S \), then the mean square displacement (MSD) of action of \( L \) reads

\[
\langle \hat{J}^{2}(\tau) \rangle = \sum_{s=0}^{\tau-1} D(s), \quad D(\tau) = D_{0} \sum_{s=-\tau}^{\tau} C_{\tau}(s) \cos(\omega Ts)(2)
\]

where \( \langle \ldots \rangle \) means the expectation value with the initial condition \(|\Psi_{0}\rangle \otimes |J = 0\rangle \), and \( D_{0} = 2\eta^{2} \sin^{2}(\omega T/2)/\omega^{2} \). Note that \( \langle \hat{J}(\tau) \rangle = 0 \) thanks to the initial state \(|J = 0\rangle \). \( C_{\tau}(s) = \langle (\Psi_{0}|\hat{v}_{\tau}\hat{v}_{\tau-s}|\Psi_{0}) + \text{c.c.} \rangle /2 \), where \( \hat{v}_{\tau} = U^{-\tau}\hat{U}\tau \) and \( g(\theta) = \cos(\theta - \omega T/2) \), is the autocorrelation function satisfying \( C_{\tau}(-s) = C_{\tau}(s) \) labelled by \( \tau \in \mathbb{Z} \).

For classical KR having the ideally chaotic property such as K- and C-systems, the correlation function decays exponentially, and \( D(t) \) coincides with the classical diffusion constant \( D_{cl} \). However, in finitely bounded quantum KR, \( D(t) \) tentatively approaches to \( D_{cl} \) but finally it goes down to 0 on average, which means \( \langle \hat{J}^{2}\rangle \) saturates at a finite value, say \( J_{0}^{2} \) as \( \tau \to \infty \).

We define the lifetime \( \tau_{L} \) of the irreversibility that is represented as the stationary diffusion of \( L \) as follows: decide the diffusion exponent \( \alpha(\tau) \) defined for \( s \) in an appropriate interval \([\tau - \Delta\tau/2, \tau + \Delta\tau/2]\) around \( \tau \) such that \( \langle \hat{J}^{2}(s) \rangle \propto s^{\alpha(\tau)} \), we define \( \tau_{L} \) as the first step that deviate appreciably from 1 such that \(|\alpha(\tau_{L}) - 1| > \epsilon \), where we choose \( \epsilon = 0.5 \) typically.

The lifetime thus defined in general varies sensitively with the choice of \(|\Psi_{0}\rangle \). In order to eliminate such accidental fluctuations, we add small term such as \( \xi_{1n}\cos(\hat{q}_{1} - q_{1R}) \) (\( \xi_{1n} \) \( \sim \) \( O(\hbar) \)) to the potential \( V(\hat{q}) \), and take the average of \( \tau_{L} \) with respect to the ensemble of the potential parameter \( q_{1R} \) (\( i = 1, 2 \)). We refer it hereafter as the average lifetime \( \langle \tau_{L}\rangle \).

Figure 1(a) shows \( \langle \hat{J}^{2}\rangle \) vs \( \tau \) at various \( \epsilon \) for the coupled Arnold’s cats. They all saturate at a finite level, but there seems to be a critical value \( \epsilon_{c} \sim \hbar^{2} \) beyond which the saturation level is drastically enhanced. We are interested in how this enhancement is related to the development of entanglement between two KRs. To measure the entanglement degree we introduce the von-Neumann entropy as the entanglement entropy (EE). Let the quasi-eigenstate of \( \hat{U} \) be \(|m\rangle \), that is, \( \hat{U}|m\rangle = e^{-i\gamma_{m}^{(1)}}|m\rangle \), where \( \gamma_{m} \) is the eigenangle, then the EE is given by \( S_{m}^{(1)} = -\text{Tr}[^{1} \rho_{m}^{(1)} \log^{1} \rho_{m}^{(1)}] \) for the reduced density operator \( \rho_{m}^{(1)} = \text{Tr}_{2}|\langle m| \rangle|m\rangle \) traced over the second cat.

As shown in Fig 1(b), the average entropy \( S = \frac{\sum_{m=1}^{N} S_{m}^{(1)}}{N} \) increases around a threshold value \( \epsilon_{c} \) from 0 to a saturated value. This fact means the two systems are entangled beyond \( \epsilon = \epsilon_{c} \) to form a fully entangled two-degrees of freedom system [17]. A quite important fact is that the average lifetime increases in agreement with the increase of EE, which means that the lifetime of irreversibility grows in a strong correlation with the entanglement on average.

Unlike the unbounded CKR, the above transition is not accompanied by critical phenomena [9], but the entropy of individual eigenstate shows an anomalous fluctuation in the transition regime, as shown in Fig 2(a). Widely distributed EE means a rich variety of the degree of entanglement in eigenstates, which leads us to expect the

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**FIG. 1:** (a) \( \langle \hat{J}^{2} \rangle \) vs \( \tau \) for various values of \( \epsilon \), where \( N = 32 \times 32 \) and \( K = 10 \). (b) The EE \( S \) vs \( \epsilon \) (the right vertical axis) and \( \langle \tau_{L}\rangle \) vs \( \epsilon \) (the left vertical axis) for \( N = 16 \times 16 \), \( 32 \times 32 \), and \( 64 \times 64 \), respectively.
lifetime $\tau_L$ of each eigenstate also fluctuates in a correlation with EE. However, the accidental fluctuation of $\tau_L$ is so large that it is very difficult to observe a direct correlation between $S_m^{(1)}$ and $\tau_L$.

Instead, we can show that the lifetime and EE is correlated indirectly. Since $\hat{\rho}(\psi_1)$ is free of KR2, the connection between eigenstates represented by $\langle m|\hat{\rho}|n\rangle$ is enhanced only if KR1 and KR2 become entangled. The transition entropy $S_m^T$ representing the variety of transition by the perturbation $\hat{\nu}$ can be defined as $S_m^T = -\sum_n t_{mn} \log t_{mn}$, where $t_{mn} = |\langle m|\hat{\nu}|n\rangle|^2 / \sum_{n'} |\langle m|\hat{\nu}|n'\rangle|^2$. It would have a strong correlation with the EE. In Fig. 2(a) we the plot $(S_m^{(1)}, S_m^T)$ for every eigenstate $|m\rangle$, where the wide spread of $S_m^{(1)}$ in the transition region of $\epsilon$ indicate the anomalous fluctuation stated above. It is evident that $S_m^{(1)}$ has a strong correlation with $S_m^T$.

On the other hand, the entropy $S_m^T$ measures the number $B_m$ of the bonds connecting $|m\rangle$ with other states $|n\rangle$ by the relation $S_m^T \sim \log(B_m)$ or $B_m \sim e^{S_m^T}$. Here we suppose that the Heisenberg time is the time scale representing the lifetime, then it is proportional to the number of the relevant quantum states contributing to the irreversible behavior, which is nothing more than $B_m$ in the present case. We are thus lead to examine the correlation between lifetime and transition entropy $S_m^T \sim \log B_m$. Sorting $|m\rangle$ in the order of $S_m^T$ and superposing the same number of $|m\rangle$ with $S_m^{(1)}$ around a given $S^T$, we construct $|\Psi_0\rangle$ and measure the lifetime $\tau_L$ of the state. We further take its ensemble average and show $\langle \tau_L \rangle$ as a function of $S^T$ in Fig. 2(b). Evidently $\log(\langle \tau_L \rangle)$ increases linearly with $S^T$.

Figure 2(a) and (b) tell $\langle \tau_L \rangle$ is correlated with entanglement entropy $S_m^{(1)}$ via the transition entropy. This fact implies a quite interesting fact that at the birth of the irreversibility the degree of entanglement fluctuates anomalously, which is reflected in the broad variety of lifetime of irreversibility realized by each eigenstate.

Beyond the threshold value ($\epsilon >> \epsilon_c$), EE saturates, and $B_m$ becomes the largest value $N$, and the lifetime $\tau_L$ becomes the longest one. Here the classicalization of dynamics of CKR is completed, which drives the classical diffusion of $L$ for $\tau \leq \tau_L$. However, a nontrivial problem arises here. See Fig. 3 the time evolution pattern of MSD markedly changes with the number $M$ of superposing eigenstates taken as the initial state $|\Psi_0\rangle = \sum_{m=1}^M |C_m|^2 |m\rangle$ with the typical weight $|C_m|^2 \sim O(1/M)$. The lifetime at which the diffusion terminates increases remarkably with $M$ although the variation is not systematic. To make sure the above observation we show in Fig. 3(b) how the average lifetime $\langle \tau_L \rangle$ varies with $M$.

The lifetime $\langle \tau_L \rangle$ is proportional to $N$ if $M = 1$, and it increases in proportion to $M$, which means $\langle \tau_L \rangle \propto N^2$ in the fully mixed limit $M = N$. Such a behavior is observed irrespective of the natural frequency $\omega$ as long as $\omega \neq 0$.

The above results are hardly expected, and we hereafter consider the reason closely. To this end we evaluate the saturation level $J_\infty^2$ of MSD by averaging Eq. (4) over $\tau$. A straightforward calculation yields

$$J_\infty^2 D_0 = \sum_{m=1}^M |C_m|^2 \sum_{n=1}^N |\langle m|\hat{\nu}|n\rangle|^2 \left( 1 \sin^2 \left( \frac{\delta_{m-n}}{2} \right) + \frac{1}{\sin^2 \left( \frac{\delta_{m+n}}{2} \right)} \right).$$

where $\delta_{m-n} = \gamma_m - \gamma_n \pm \omega$. Here the interference terms between different $ms$ are neglected (the diagonal approximation) because their contribution is negligibly small in the regime we are concerned.

First we consider the very particular case of $\omega = 0$ in order to compare with general case discussed later. As is well-known in the study of fidelity, if $\hat{\nu}$ has diagonal components, it leads to a ballistic diffusion of $L$ beyond the
Heisenberg time [14]. Therefore, we consider the case of null diagonal component. Then the summation over $n$ in the RHS of Eq. $3$ is dominated by the component with minimal $\sin^2(\delta_{mn}/2) \sim |\gamma_m - \gamma_n|^2/4$. It is nothing more than the nearest neighbouring eigenangle distance following the Wigner distribution which have a definite peak around $2\pi/N$. Thus the magnitude of each term is dominated by $\sin(\delta_{mn}/2) \sim (\pi/N)^3$ irrespective of $m$, while using by the expression for the autocorrelation function at $s = 0$ it follows that $C(r(0)) = \sum_m |C_m|^2 \sum_n (|m\langle \hat{\nu}|n\rangle|^2$ (the subscript $\sigma$ of $C$ can now be omitted) the representative value $|\langle |\hat{\nu}^2\rangle|_n$ of the matrix element $|\langle m\hat{\nu}|n\rangle|^2$ is evaluated as $\langle |\hat{\nu}^2\rangle | = C(r(0))/N$, because the full mixing in this regime makes the magnitude of matrix elements $|\langle m\hat{\nu}|n\rangle|^2$ almost equal. As a result we evaluate the saturation level as $J_\infty^2 \sim D_0 C(0) N/2\pi^2$. Lifetime is the time at which the classical diffusion $D_{cl}\tau$ terminates at the saturation level, namely $D_{cl}\tau_L = J_\infty^2$. It leads to $\tau_L = D_{cl}C(0)\pi/N$ independent of $M$, which is confirmed numerically [18]. It again coincides with the Heisenberg time.

However, in the general case of $\omega \neq 0$ the lifetime is enhanced much more than the case of $\omega = 0$. If $\omega \neq 0$ the statistical distribution of $|\delta_{mn}^x| = |\gamma_m - \gamma_n \pm \omega T|$ does not suffer from any restriction like level repulsion in the vicinity of $|\delta_{mn}^x| \sim 0$. Hence, unlike the case of $\omega = 0$, $|\delta_{mn}^x|$ can be arbitrarily small, which will make the term $1/\sin^2(\delta_{mn}/2)$ larger. As $M$ increases the chance of encountering smaller $|\delta_{mn}^x|$ increases. In the summation over $n$ in the RHS of Eq. $3$ the term with the smallest $\delta_{mn}^x$ is most dominant. We take here a hypothesis that $|\delta_{mn}^x| (n = 1, 2...N)$ are uniformly distributed independent stochastic variables in the range $[0, 2\pi]$. This is a rather bold hypothesis that the distribution of chaotic eigenangle is under no restriction except for the repulsion of the nearest neighbouring levels, but this is not correct in a strict sense. It is now easy to show that the probability of the minimal $\{\delta_{mn}^x\}_n$ takes a value $x_m$ is given by $p(x_m) = N/2\pi e^{-x_m^2/2\pi}$. Next, we have to take the sum over $n$. At this second stage it is quite plausible to suppose that $x_m$ is now statistically independent variable, then it is shown that the probability of min$\{x_1, ..., x_M\}$ takes a value $x$ is $P(x) = \frac{M N}{2\pi} e^{-\frac{M x^2}{2\pi}}$, and the average of the minimal $|\delta_{mn}^x|/2$ is $\pi/MN$. Then the most dominant term of $1/\sin^2(\delta_{mn}/2)$ in the RHS is $(MN/\pi^2)$. Since $|\langle m\hat{\nu}|n\rangle|^2 \sim |\langle \hat{\nu}^2\rangle | = C(r(0))/N$ and $|C_m|^2 \sim 1/M$ we may evaluate $J_\infty^2 = D_0 C(0) M N/2\pi^2$. The $\tau_L$ is decided by $D_{cl}\tau_L = J_\infty^2$ to yield

$$\tau_L = \frac{D_0 C(0) MN}{2\pi^2 D_{cl}} \quad (4)$$

This is our final result and it agrees well with the numerical results except for a numerical factor $\sim 3$ as depicted in Fig.4(b). We have confirmed all the results discussed above are valid also for the coupled quantum standard maps. We can expect that it is the general features of ideally chaotic quantum maps.

In conclusion, we proposed a method to observe the irreversibility potential in quantum system and the associated quantum states. Applying it to coupled KRs, the relation of the organization of irreversibility to the development of the entanglement between the constituent systems has been elucidated: the lifetime of irreversibility largely fluctuates in correlation with the anomalous fluctuation of entanglement at the birth of irreversibility, and it further grows up to proportional to square of Hilbert space dimension in the full entanglement regime. We finally comment that the linear oscillator $L$ can be replaced by a two level system without any essential modifications which opens the possibility of experimental implementation on the optical lattice setting [13].

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