Conservative invariant finite-difference schemes for the modified shallow water equations in Lagrangian coordinates

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Abstract

The one-dimensional modified shallow water equations in Lagrangian coordinates are considered. It is shown the relationship between symmetries and conservation laws in Lagrangian coordinates, in mass Lagrangian variables, and Eulerian coordinates. For equations in Lagrangian coordinates an invariant finite-difference scheme is constructed for all cases for which conservation laws exist in the differential model. Such schemes possess the difference analogues of the conservation laws of mass, momentum, energy, the law of center of mass motion for horizontal, inclined and parabolic bottom topographies. Invariant conservative difference scheme is tested numerically in comparison with naive approximation invariant scheme.

Keywords: shallow water, Lagrangian coordinates, Lie point symmetries, conservation law, Noether’s theorem, numerical scheme, direct method

1. Introduction

Mathematical modeling of physical phenomena is one of the main streams in continuum mechanics. Such phenomena as hydraulic currents, coastal currents, currents in rivers and lakes, currents in water intakes, technical troughs and trays, tsunami simulation, propagation breakthrough of waves and tidal pine forests in rivers, the spread of heavy gases and impurities in the atmospheres of planets, atmospheric movements scales used in weather forecasting require mathematical consideration.

Motion of ideal fluid flow under the force of gravity can be modeled by means of the Euler equations. However, the full Euler equations, even under the assumption of incompressibility, barotropy and absence of rotation, are still rather complicated for describing waves on a surface. One of these difficulties is that the free surface is a part of the solution. This difficulty has motivated scientists to derive simpler equations. For this reason development of approximate models and their analysis by analytical and numerical methods is an actual problem.

The need to reduce the original equations to simpler equations led to the construction of asymptotic expansion models with respect to a small parameter determined by the ratio of

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the depth of the fluid to the characteristic linear size. One class of such equations is the class of shallow water equations. There are many approaches for deriving shallow water models, a review of which can be found in \[1, 2\].

The shallow water equations describe the motion of incompressible fluid in the gravitational field if the depth of the liquid layer is small enough. They are widely used in the description of processes in the atmosphere, water basins, modeling of tidal oscillations, tsunami waves and gravitational waves (see the classical papers such as \[3, 4\] and detailed description in, for example, \[5, 6\]).

For solving real-world problems it is also necessary to consider additional impacts determined by specific flow conditions. This leads to the appearance of additional terms in the system of shallow water equations. In particular, shallow water flows on plane surface in the presence of the weak vertical inhomogeneities in the initial conditions contain additional terms appearing as the result of depth averaging of the nonlinear terms in the initial fluid equations \[7, 8\]. One of the models including this effect in consideration is proposed in \[9, 10\]:

\[
\begin{align*}
\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho u) &= 0, \\
\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + g \left(1 + \frac{g_1}{\rho}\right) \frac{\partial \rho}{\partial x} - g H'(x) &= 0,
\end{align*}
\]

where \(t\) is time, \(x\) is the Eulerian space coordinate, \(u\) is the velocity, \(\rho\) is the depth of the layer of fluid, the differentiable function \(H(x)\) describes the bottom topography, \(g\) is the gravitational acceleration, and \(g_1/\rho\) is the additional term. The latter term describes an advective transport of impulse as a result of the dependence of horizontal shallow water flows on vertical coordinate \(9\). It is assumed that \(g_1 \neq 0\) in this paper, otherwise equations \(1\) become standard one-dimensional shallow water equations. We use the letter \(\rho\) for the depth of the fluid instead of usually used \(h\) to differ it from the standard notation for the finite-difference mesh spacing.

It is wellknown that symmetry of mathematical model is intrinsic property inherited from physical phenomena. One of the tools for studying symmetries is Lie group analysis \[11, 12\] which is a basic method for constructing exact solutions of ODEs and partial differential equations. Even in case of the one-dimensional shallow water equations for the flat bottom one meets certain difficulties to obtain nontrivial exact solutions. Applications of Lie groups to differential equations is the subject of many books and review articles \[11–16\].

The group properties of the shallow water equations were studied in numerous papers (see \[17–20\]). Group classification and first integrals of these equations can be found in \[21, 22\]. It was shown (see, e. g., \[22, 23\]) that the shallow water equations in Lagrangian coordinates can be obtained as Euler–Lagrange equations of Lagrangian functions of a special kind. Some exact solutions also can be found in \[5, 24, 25\].

More recently applications of Lie groups have been extended to difference equations \[26–42\].

The applications have certain peculiarities related with nonlocal character of difference operators and geometrical structure of difference mesh \[39\]. The results were obtained both for the ordinary and partial difference equations and systems (e. g., \[28, 29, 39, 43–45\]). The finite-difference analogues of Lagrangian \[30, 31, 46\] and the Hamiltonian \[46, 47\] formalism were developed. Moreover, it was shown that in case of an absence of the Lagrangian and Hamiltonian there is a more general approach based on the Lagrange operators identity and adjoint equations method \[48\], the difference analogue of which was developed in \[42, 49\]. In all three approaches the starting point is the symmetry of the differential equations, preserved in
difference equations and meshes. In this article we follow Lagrangian approach and the Noether theorem. We also discuss so called direct method \cite{48, 50, 51} for constructing conservation laws.

The invariant finite-difference schemes are of particular interest as far it should preserve symmetries and since the geometric properties of the original equations, in particular it have symmetry reductions on subgroups and exact invariant solutions. In this paper we mostly concentrate on invariant schemes in the Lagrange coordinate system.

The present paper is devoted to the construction of invariant conservative difference schemes for the modified shallow water equations \cite{1} in Lagrangian coordinates and mass Lagrangian coordinates. The base of the construction is the invariant scheme already developed for standard shallow water model \cite{52}.

The paper is organized as follows. In Section 2, the modified shallow water equations in Lagrange coordinates are given. It is shown that they can be obtained as the Euler–Lagrange equations for a certain Lagrangian. Symmetries and conservation laws for the modified shallow water equations for various bottom topographies are given in Lagrangian, mass Lagrangian and Eulerian coordinates in Section 3. In Section 4, the equations in Lagrangian coordinates are discretized. The constructed finite-difference schemes for various bottom topographies are invariant and possess finite-difference analogues of the differential conservation laws. In addition, a scheme defined on a reduced finite-difference stencil is constructed in mass Lagrangian coordinates. Approaches to the numerical comparison of various schemes in Lagrangian coordinates are also discussed in the section. In Section 5, the numerical implementation of the constructed schemes is carried out. The obtained schemes and numerical results are discussed in Conclusion.

2. One-dimensional modified shallow water equations in Lagrangian coordinates

Modelling physical phenomena in continuum mechanics is considered in two distinct ways. The typical approach uses Eulerian coordinates, where flow quantities at each instant of time during motion are described at fixed points. Alternatively, the Lagrangian description is used, where the particles are identified by the positions which they occupy at some initial time.

Following the classical gas dynamics equations, introduce Lagrangian variables \((\xi, t)\), relating the Eulerian variables \((x, t)\) and Lagrangian variables by the equation

\[ x = \tilde{\varphi}(t, \xi), \]

where the function \(\tilde{\varphi}(t, \xi)\) satisfies the Cauchy problem

\[ \tilde{\varphi}_t = u(t, \tilde{\varphi}), \quad \tilde{\varphi}(t_0, \xi) = \xi. \]

Subindex of a function \(f\) denotes a corresponding partial derivative.

Let the dependent variables in Lagrangian variables are denoted as \(\tilde{\rho}(t, \xi)\) and \(\tilde{u}(t, \xi)\). Their relations with counterparts are given by the formulas

\[ \tilde{\rho}(t, \xi) = \rho(t, \tilde{\varphi}(t, \xi)), \quad \tilde{u}(t, \xi) = u(t, \tilde{\varphi}(t, \xi)). \]

Differentiating the latter with respect to \(\xi\) and \(t\), one finds

\[ \rho_x = \tilde{\rho}_\xi/\tilde{\varphi}_\xi, \quad \rho_t + u\rho_x = \tilde{\rho}_t, \quad u_x = \tilde{u}_\xi/\tilde{\varphi}_\xi, \quad u_t + uu_x = \tilde{u}_t. \]
Noticing that $\tilde{u}_\xi = \tilde{\varphi}_t$, the continuity equation becomes

$$(\tilde{\rho}\tilde{\varphi})_t = 0.$$  

Hence,

$$\tilde{\rho}(t, \xi)\tilde{\varphi}_t(t, \xi) = \tilde{\rho}_0(\xi),$$

where $\tilde{\rho}_0(\xi) = \tilde{\rho}(\xi, 0)$. Introducing the change $\xi = \alpha(s)$, where

$$\alpha'(s)\tilde{\rho}_0(\alpha(s)) = 1,$$

one derives the independent variables $(s, t)$, which are called in the classical gas dynamics by the mass Lagrangian coordinates. In the mass Lagrangian coordinates one obtains that

$$\tilde{\rho}(t, s) = \frac{1}{\varphi_s(t, s)},$$

where

$$\tilde{\rho}(t, s) = \tilde{\rho}(t, \alpha(s)), \quad \tilde{u}(t, s) = \tilde{u}(t, \alpha(s)), \quad \tilde{\varphi}(t, s) = \tilde{\varphi}(t, \alpha(s)).$$

Further the sign bar ‘$\bar{}$’ is omitted.

In the mass Lagrangian coordinates, the second equation of system [1] becomes

$$\varphi_{tt} - g\varphi_{ss} - gg_1\varphi_{ss} - gH'(\varphi) = 0. \quad (2)$$

One of the advantages of using the mass Lagrangian coordinates is the existence of a Lagrangian whose Euler-Lagrange equations is equivalent to equation (2). In Lagrangian mechanics the problem of determining whether a given system of differential equations can arise as the Euler-Lagrange equations for some Lagrangian function is called the Helmholtz problem.

For finding a Lagrangian for which equation (2) is the Euler-Lagrange equations one has to solve the following problem. Let $L(t, s, \varphi, \varphi_s, \varphi_t)$ be a corresponding Lagrangian. Then, substituting $L$ into the equation

$$\frac{\delta L}{\delta \varphi} = 0, \quad (3)$$

excluding the derivative $\varphi_{tt}$ found from equation (2), and splitting them with respect to the parametric derivatives $\varphi_{ts}$, $\varphi_{ss}$, one obtains an overdetermined system of equations for the function $L$. Here $\frac{\delta}{\delta \varphi}$ is the variational derivative:

$$\frac{\delta F}{\delta \varphi} = \frac{\partial F}{\partial \varphi} - D_t^L \left( \frac{\partial F}{\partial \varphi_t} \right) - D_s \left( \frac{\partial F}{\partial \varphi_s} \right),$$

where $F$ is an arbitrary function, $D^L_t$, $D_s$ are total derivatives with respect to $t$ and $s$, respectively. It supposed that

$$\Delta = \frac{\partial^2 L}{\partial \varphi_t^2} \neq 0. \quad (4)$$

Any such solution of this overdetermined system of equations satisfying condition (4) gives a sought Lagrangian.
The calculations give that
\[ L = k \left( \frac{\varphi^2}{2} + gg_1 \ln(\varphi_s) - g\varphi_s^{-1} + gH(\varphi) \right) + G, \]
where \( G = \varphi_t L_1 + \varphi_s L_2 + L_3, \) and the functions \( L_i(t, s, \varphi), \ (i = 1, 2, 3) \) satisfy the condition
\[ L_{1t} + L_{2s} = L_{1\varphi} + L_{2\varphi}. \]
Noticing that \( \frac{\delta G}{\delta \varphi} = 0, \) and because of the condition (4), one derives that the seeking Lagrangian is can be chosen as
\[ L = \frac{\varphi^2}{2} + gg_1 \ln(\varphi_s) - g\varphi_s^{-1} + gH(\varphi). \] (5)

By means of the transformation
\[ t = \tilde{t}/\sqrt{2}, \quad s = \tilde{s}/g, \quad g_1 = 2\gamma_1/g, \quad H = 2(\tilde{H} - \gamma_1 \ln g)/g, \]
the Lagrangian (5) and equation (2) are brought to the forms\(^1\)
\[ L = \frac{\varphi^2}{2} + \gamma_1 \ln(\varphi_s) - \frac{1}{2\varphi_s} + H(\varphi), \] (6)
\[ \varphi_{tt} - \frac{\varphi_{ss}}{\varphi_s^2} - \gamma_1 \frac{\varphi_{ss}}{\varphi_s^2} - H'(\varphi) = 0. \] (7)

3. Symmetries and conservation laws of the modified shallow water equations

3.1. Symmetries and conservation laws in Lagrangian coordinates

In [22] the group classification of the one-dimensional Euler–Lagrange equations of continuum mechanics was carried out by the authors. In particular, it was shown that the modified shallow water equation (7) is the Euler–Lagrange equation for the Lagrangian
\[ L = \frac{\varphi^2}{2} - \frac{1}{2\varphi_s} + \gamma_1 \ln \varphi_s + H(\varphi). \] (8)

The group classification states that in case \( H'(\varphi) \) is arbitrary, equation (7) admits two generators
\[ X_1 = \frac{\partial}{\partial t}, \quad X_2 = \frac{\partial}{\partial s}. \] (9)
In case \( H'(\varphi) = 0, \) there is the following extension of the admitted Lie algebra
\[ X_3 = \frac{\partial}{\partial \varphi}, \quad X_4 = t \frac{\partial}{\partial \varphi}, \quad X_5 = t \frac{\partial}{\partial t} + s \frac{\partial}{\partial s} + \varphi \frac{\partial}{\partial \varphi}. \] (10)

It is shown below that the case of an inclined bottom \( H'(\varphi) = \text{const} \) is reduced to the case of a horizontal bottom by a point transformation. Therefore, we do not consider this case here.

\(^{1}\)Here and further the symbol “\( \tilde{\} \) is also omitted.
In case of a parabolic bottom $H'(\varphi) = \varphi$, the extension of the admitted Lie algebra is

$$X_3^+ = e^t \frac{\partial}{\partial \varphi}, \quad X_4^+ = e^{-t} \frac{\partial}{\partial \varphi}. \quad (11)$$

In case $H'(\varphi) = -\varphi$, one has

$$X_3^- = \sin t \frac{\partial}{\partial \varphi}, \quad X_4^- = \cos t \frac{\partial}{\partial \varphi}. \quad (12)$$

In case $H' = 1/\varphi$, the extension only consists of the generator

$$X_5 = t \frac{\partial}{\partial t} + s \frac{\partial}{\partial s} + \varphi \frac{\partial}{\partial \varphi}. \quad (13)$$

Usually the Lagrangian admits some subset of generators of the Lie algebra admitted by the corresponding Euler–Lagrange equations. Such generators are called variational symmetries. The presence of variational symmetries greatly simplifies the search for conservation laws for the Euler–Lagrange equations. This can be done using Noether’s theorem \[13, 54\] which establishes a connection between variational symmetries and conservation laws.

Recall that (local) conservation laws of the system

$$F^i(t, s, \varphi, \varphi_t, \varphi_s, \varphi_{tt}, \varphi_{ts}, \varphi_{ss}) = 0, \quad i = 1, 2, ..., n, \quad (14)$$

can be represented in the form

$$(T^t)_t + (T^s)_s = \Lambda_j F^j = 0, \quad (15)$$

where $T^t = T^t(t, s, \varphi, \varphi_t, \varphi_s)$ is a conserved density, $T^s = T^s(t, s, \varphi, \varphi_t, \varphi_s)$ is a conserved flux, and $\Lambda_j = \Lambda_j(t, s, \varphi, \varphi_t, \varphi_s)$, $j = 1, ..., n$ are so called conservation law multipliers \[48\]. Further on $\Lambda_{ij}$ denote the multipliers corresponding to the conservation law obtained using Noether’s theorem for the generator $X_i$. For brevity, in case $n = 1$, the subscript $j$ is omitted.

There is a one-to-one correspondence between the equivalence classes of conservation laws and conservation law multipliers \[12\], and it is often easier to find the multipliers first. Conservation law multipliers for a given system (14) can be found by means of direct method \[48, 50\]. The method consists of applying the Euler operator

$$E_\varphi \equiv \frac{\partial}{\partial \varphi} - D_t \frac{\partial}{\partial \varphi_t} + \cdots + (-1)^s D_{i_1} \cdots D_{i_s} \frac{\partial}{\partial \varphi_{i_1 \cdots i_s}} + \cdots \quad (16)$$

to the expression $\Lambda_j F^j$ for unknown multipliers. (Here $D_i$ denotes the total differentiation operator with respect to $i$-th independent variable.) Since the Euler operator identically vanishes divergence expressions, to find conservation law multipliers one has to solve the equation

$$E_\varphi(\Lambda_j F^j) \equiv 0. \quad (17)$$

In the present section, Noether’s theorem is used. It turned out that the finite-difference analogue of the direct method is effective when constructing conservative finite-difference schemes. This is discussed in the related sections below.
By means of Noether’s theorem one finds the following conservation laws, and the corresponding conservation law multipliers of equation (7) by the direct method.

For an arbitrary differentiable function \( H(x) \) and the generators \( X_1 \) and \( X_2 \) one obtains the conservation law of energy

\[
\Lambda_1 = \varphi_t, \quad \left( \frac{\varphi_s^2}{2} + \frac{1}{2\varphi_s} - \gamma_1 \ln \varphi_s - H \right)_t + \left( \varphi_t \left( \frac{\gamma_1}{\varphi_s} + \frac{1}{2\varphi_s^2} \right) \right)_s = 0, \quad \text{(18)}
\]

and the conservation law of momentum

\[
\Lambda_2 = \varphi_s, \quad (\varphi_t \varphi_s)_t + \left( \frac{1}{\varphi_s} - \frac{1}{2} \varphi_t^2 - \gamma_1 \ln \varphi_s - H \right)_s = 0. \quad \text{(19)}
\]

Conservation laws obtained for \( H' = 0 \) and the generators \( X_3 \) and \( X_4 \) are the alternative form of the momentum conservation law

\[
\Lambda_3 = 1, \quad (\varphi_t)_t + \left( \frac{1}{2\varphi_s^2} + \frac{\gamma_1}{\varphi_s} \right)_s = 0, \quad \text{(20)}
\]

and the center of mass law

\[
\Lambda_4 = t, \quad (t\varphi_t - \varphi)_t + \left( \frac{t}{2\varphi_s^2} + \frac{\gamma_1 t}{\varphi_s} \right)_s = 0. \quad \text{(21)}
\]

The conservation laws corresponding to \( H' = \varphi \) and the generators \( X_3^+ \) and \( X_4^+ \) are

\[
\Lambda_3^+ = e^t, \quad \left( e^t (\varphi - \varphi_t) \right)_t - \left( e^t \left( \frac{1}{2\varphi_s^2} + \frac{\gamma_1}{\varphi_s} \right) \right)_s = 0, \quad \text{(22)}
\]

\[
\Lambda_4^+ = e^{-t}, \quad \left( e^{-t} (\varphi + \varphi_t) \right)_t + \left( e^{-t} \left( \frac{1}{2\varphi_s^2} + \frac{\gamma_1}{\varphi_s} \right) \right)_s = 0. \quad \text{(23)}
\]

Similarly, for \( H' = -\varphi \) and the generators \( X_3^- \) and \( X_4^- \) one obtains

\[
\Lambda_3^- = \sin t, \quad \left( \varphi \cos t - \varphi_t \sin t \right)_t - \left( \sin t \left( \frac{1}{2\varphi_s^2} + \frac{\gamma_1}{\varphi_s} \right) \right)_s = 0, \quad \text{(24)}
\]

\[
\Lambda_4^- = \cos t, \quad \left( \varphi \sin t + \varphi_t \cos t \right)_t + \left( \cos t \left( \frac{1}{2\varphi_s^2} + \frac{\gamma_1}{\varphi_s} \right) \right)_s = 0. \quad \text{(25)}
\]

To the best of authors’ knowledge, conservation laws (22)–(25) have no clear physical interpretation.

Finally, in case \( H' = 1/\varphi \), the generator \( X_5 \) does not satisfy the Noether theorem.

One can find a more detailed discussion of the symmetries and conservation laws of the modified shallow water equations with different bottom topographies in [22].

Remark 1. In case of an inclined bottom \( H(z) = C_1 z + C_2 \), where \( z = z(t', s') \) the modified shallow water equations in Lagrangian coordinates are reduced to the equations for a horizontal bottom \( H = \text{const} \) by means of the transformation

\[
z = \varphi + \frac{C_1}{2} t'^2, \quad t' = t, \quad s' = s. \quad \text{(26)}
\]

The same transformation was used in [22] for the one-dimensional shallow water equations in Lagrangian coordinates.
3.2. Conservation laws in mass Lagrangian coordinates

It is often possible to represent equations originally given in Eulerian or Lagrangian coordinates in a simpler form using mass Lagrangian coordinates. Equations represented in mass coordinates are also often more suitable for solving problems numerically [55, 56]. Moreover, the transition from Lagrangian coordinates to mass Lagrangian coordinates is especially simple. The mass Lagrangian coordinates are introduced with the differential form

\[ ds = \rho dx - u dt. \]  

Taking into account \( x = \varphi(t, s) \), from the latter one derives

\[ \varphi_t = u, \quad \varphi_s = \frac{1}{\rho}. \]  

Equations \( \varphi_{ts} = \varphi_{st} \) and (7) are brought to the one-dimensional modified shallow water equations in mass Lagrangian coordinates, namely

\[ \left( \frac{1}{\rho} \right)_t - u_s = 0, \]  
\[ u_t + \rho u_s + \gamma_1 \rho_s - H' = 0, \]

where \( H = H(x) \).

**Remark 2.** In discretization process we will use another representation of the system introducing a new variable [52, 53, 57]

\[ p = \rho^2. \]

Then, one can rewrite the second equation of (29) as

\[ u_t + \frac{1}{2} p_s + \gamma_1 p_s - H' = 0, \quad \text{or} \quad u_t + \frac{1}{2} \left( 1 + \frac{\gamma_1}{\rho} \right) p_s - H' = 0. \]  

Below we will use such representation to reduce a mesh stencil for finite-difference schemes.

Consider the conservation laws possessed by system (29). The first equation of (29) is the conservation law of mass. By means of (28) the conservation laws of energy (18) and momentum (19) in mass Lagrangian coordinates are brought to

\[ \left( \frac{u^2}{2} + \frac{\rho}{2} + \gamma_1 \ln \rho - H \right)_t + \left( u \left( \frac{\rho^2}{2} + \gamma_1 \rho \right) \right)_s = 0, \]  
\[ \left( \frac{u}{\rho} \right)_t + \left( \rho - \frac{u^2}{2} + \gamma_1 \ln \rho - H \right)_s = 0. \]

The conservation laws (20) and (21) become

\[ u_t + \left( \frac{\rho^2}{2} + \gamma_1 \rho \right)_s = 0, \]  
\[ (tu - x)_t + \left( \frac{t\rho^2}{2} + \gamma_1 t \rho \right)_s = 0. \]
The conservation laws (22) and (23) become
\[
(e^t(x-u))_t - \left( e^t \left( \frac{\rho^2}{2} + \gamma_1 \rho \right) \right)_s = 0, \tag{36}
\]
\[
(e^{-t}(x+u))_t + \left( e^{-t} \left( \frac{\rho^2}{2} + \gamma_1 \rho \right) \right)_s = 0, \tag{37}
\]
and the conservation laws (24) and (25) are
\[
(x \cos t - u \sin t)_t - \left( \sin t \left( \frac{\rho^2}{2} + \gamma_1 \rho \right) \right)_s = 0, \tag{38}
\]
\[
(x \sin t + u \cos t)_t + \left( \cos t \left( \frac{\rho^2}{2} + \gamma_1 \rho \right) \right)_s = 0. \tag{39}
\]

### 3.3. Conservation laws in Eulerian coordinates

Denote by \(D^E_t\) and \(D^L_t\) Eulerian and Lagrangian total differentiation operators with respect to \(t\). Total differentiation operators with respect to \(s\) and \(x\) are denoted by \(D_s\) and \(D_x\). By means of (27), the total differentiations in Eulerian and Lagrangian coordinates are related as follows
\[
D^L_t = D^E_t + u D_x, \quad D_s = \frac{1}{\rho} D_x. \tag{40}
\]
The conserved quantities \((T^t, T^s)\) of conservation laws in Lagrangian coordinates are related to the corresponding quantities \((\rho^T^t, \rho^T^x)\) in Eulerian coordinates as
\[
\rho^T^t = \rho T^t, \quad \rho^T^x = \rho u T^t + T^s. \tag{41}
\]
Formulas (41) allow one to obtain the following Eulerian counterparts of the conservation laws (32)–(39)
\[
D^E_t \left( \rho \left( \frac{u^2 + \rho^2}{2} + \gamma_1 \ln \rho - H(x) \right) \right) + D_x \left( \rho u \left( \frac{u^2}{2} + \rho + \gamma_1 (1 + \ln \rho) - H(x) \right) \right) = 0, \tag{42}
\]
\[
D^E_t (u) + D_x \left( \frac{u^2}{2} + \rho + \gamma_1 \ln \rho - H(x) \right) = 0. \tag{43}
\]
In case \(H = \text{const}\),
\[
D^E_t (\rho u) + D_x \left( \rho u^2 + \frac{\rho^2}{2} + \gamma_1 \rho \right) = 0. \tag{44}
\]
\[
D^E_t (\rho (tu - x)) + D_x \left( \rho (tu - x) + \frac{t \rho^2}{2} + t \gamma_1 \rho \right) = 0. \tag{45}
\]
In case \(H = \frac{x^2}{2}\),
\[
D^E_t (e^t \rho (x - u)) - D_x \left( e^t \rho \left( \frac{\rho}{2} + u^2 - xu + \gamma_1 \right) \right) = 0. \tag{46}
\]
\[
D^E_t (e^{-t} \rho (x + u)) + D_x \left( e^{-t} \rho \left( \frac{\rho}{2} + u^2 + xu + \gamma_1 \right) \right) = 0. \tag{47}
\]
In case \(H = -\frac{x^2}{2}\),
\[
D^E_t \left( \rho (x \cos t - u \sin t) \right) + D_x \left( \rho \left( xu \cos t - \left( \frac{u^2 + \rho^2}{2} + \gamma_1 \right) \sin t \right) \right) = 0. \tag{48}
\]
\[
D^E_t \left( \rho (x \sin t + u \cos t) \right) + D_x \left( \rho \left( xu \sin t + \left( \frac{u^2 + \rho^2}{2} + \gamma_1 \right) \cos t \right) \right) = 0. \tag{49}
\]
4. Discretization of the one-dimensional modified shallow water equations in Lagrangian coordinates

In the present section conservative invariant finite-difference schemes are constructed for the one-dimensional modified shallow water equations in Lagrangian and mass Lagrangian coordinates. The schemes early constructed [52, 58] for the shallow water equations are taken as starting point.

Schemes in Lagrangian coordinates are further considered on a 9-point stencil

\[
(x^n, x^m, x^{n+1}, x^{m+1}, x^{n+1, m+1}, x^{n-1, m+1}, x^{n-1, m-1}, x^{n+1, m-1}) \equiv (x, x_-, \hat{x}, \hat{x}_-, \hat{x}_+, \hat{x}_+),
\]

where, in accordance with the notation of [52], we denote \( x = x^n_m = \varphi \). The indices \( n \) and \( m \) are changed along time and space axes \( t \) and \( s \) correspondingly. The time and space steps are defined as

\[
\tau_n = \hat{\tau} = t_{n+1} - t_n = \hat{t} - t, \quad \tau_{n-1} = \hat{\tau} = t_n - t_{n-1} = t - \hat{t},
\]

\[
h_m = \hat{h} = s_{m+1} - s_m = s_+ - s, \quad h_{m-1} = \hat{h} = s_m - s_{m-1} = s - s_-. \tag{51}
\]

We seek for finite-difference approximations for equation (7) defined on uniform orthogonal meshes as depicted in Figure 1.

Consider the following scheme on a uniform orthogonal mesh

\[
\Phi(x, x_-, x_+, \hat{x}, \hat{x}_-, \hat{x}_+, \hat{x}_-) = 0, \tag{52a}
\]

\[
h_+ = h_- = h, \quad \hat{\tau} = \tau, \quad (\hat{\tau}, \hat{h}) = 0. \tag{52b}
\]

The group generator

\[
X = \xi_t \frac{\partial}{\partial t} + \xi_s \frac{\partial}{\partial s} + \eta \frac{\partial}{\partial x} \tag{53}
\]

is prolonged to the finite-difference space as follows [28, 39]

\[
\tilde{X} = \sum_{k,l=-\infty}^{\infty} S^k \ S^l(X), \tag{54}
\]

where the finite-difference shifts along the time and space axes are

\[
S_{\pm \tau}(f(t_n, s_m, x^n_m)) = f(t_{n\pm1}, s_m, x^{n\pm1}_m),
\]

\[
S_{\pm s}(f(t_n, s_m, x^n_m)) = f(t_n, s_{m\pm1}, x^{n\pm1}_m).
\]
The criterion of invariance of system (52a), (52b) is formulated as follows \[39\]
\[
\tilde{X}\Phi\bigg|_{(52a), (52b)} = 0, \quad \tilde{X}\big(\hat{\tau} - \check{\tau}\big)\bigg|_{(52a), (52b)} = 0, \quad \tilde{X}(h_+ - h_-)\bigg|_{(52a), (52b)} = 0, \quad \tilde{X}(\hat{\tau}, \check{h})\bigg|_{(52a), (52b)} = 0.
\]

Scheme (52a), (52b) is called \textit{invariant} if conditions (55) hold.

Notice that symmetries (9)–(12) transform variables \(t\) and \(s\) independently of solution. Therefore, the criterion of mesh invariance can be considered separately.

To preserve uniformness and orthogonality of the mesh it is needed \[28, 39\]
\[
D_{+s} + D_{-s} (\xi_t) = 0, \quad D_{+\tau} - D_{-\tau} (\xi_t) = 0, \quad D_{\pm s} (\xi_t) = -D_{\pm s} (\xi_s),
\]

where \(D_{\pm \tau}\) and \(D_{\pm s}\) are finite-difference differentiation operators
\[
D_{\pm \tau} = \frac{S - 1}{t_{n+1} - t_n}, \quad D_{-\tau} = \frac{1 - S}{t_n - t_{n-1}}, \quad D_{+s} = \frac{S - 1}{s_{m+1} - s_m}, \quad D_{-s} = \frac{1 - S}{s_m - s_{m-1}}.
\]

One can verify that symmetries (9)–(12) satisfy conditions (56) and (56). This means that we can construct symmetry-preserving schemes for the modified shallow water equations in Lagrangian coordinates on an invariant uniform orthogonal mesh.

We are seeking for conservative schemes, i.e. schemes possessing finite-difference conservation laws. A finite-difference conservation law of scheme (52a) is a divergent expression of the form
\[
D_{-\tau} (T_t) + D_{-s} (T_s) = 0
\]
that vanishes on solutions of (52a), (52b). The quantities \(T_t\) and \(T_s\) are called density and flux. Notice that expressions of the form (58) approximate \textit{local} differential conservation laws, so they may not hold for discontinuous solutions such as shock waves.

4.1. \textit{Conservative schemes for the modified shallow water equations in Lagrangian coordinates}

Consider the invariant finite-difference scheme on a uniform orthogonal mesh for the one-dimensional shallow water equations constructed by the authors in [52].

\[
x_{ti} + D_{-s} \left( \frac{1}{2x_s x_s} \right) - \frac{D(H^h + \hat{H}^h)}{x_t + \check{x}_t} = 0, \quad h_+ = h_- = h, \quad \hat{\tau} = \check{\tau} = \tau,
\]

where \(H^h = H^h(x)\) is some approximation for the function \(H(x)\), \(h\) and \(\tau\) are constant.

The scheme possesses the conservation law of energy
\[
D_{-\tau} \left( \frac{x_i^2}{2} + \frac{1}{4x_s} + \frac{1}{4\check{x}_s} - \frac{H^h + \hat{H}^h}{2} \right) + D_{-s} \left( \frac{x_i^+ + \check{x}_i^+}{4\check{x}_s} \right) = 0
\]
with the corresponding conservation law multiplier

$$\Lambda_1^h = \frac{x_t + \check{x}_t}{2}. \quad (61)$$

The conservation law of mass

$$D(\check{x}_s) - D(x_t^+) = 0 \quad (62)$$

holds automatically on a uniform orthogonal mesh.

In case $H^h = \text{const}$, scheme [59] also possesses the conservation law of momentum

$$\Lambda_3^h = 1, \quad D \left( x_t \right) + D \left( \frac{1}{2\check{x}_s\check{x}_s} \right) = 0 \quad (63)$$

and center of mass law

$$\Lambda_4^h = t, \quad D \left( t x_t - x \right) + D \left( \frac{t}{2\check{x}_s\check{x}_s} \right) = 0. \quad (64)$$

Notice that there are additional finite-difference conservation laws exist for some specific bottom topographies [52, 53]. We will consider them in the sequel of the paper.

In [57] an invariant conservative finite-difference scheme for the one-dimensional Green–Naghdi equations in Lagrangian coordinates was constructed by extending scheme (59). A nonlinear invariant higher-order finite-difference term was added to the first equation of the scheme. That allowed the authors to construct the scheme possessing finite-difference analogues of all the differential local conservation laws of the one-dimensional Green–Naghdi equations. Scheme (59) was also used as a basis to construct invariant conservative schemes for the two-dimensional shallow water equations in [59]. Since the modified shallow water equations extend the shallow water equations, it seems natural to extend scheme (59) in such a way that it approximates the modified shallow water equations. Thus, here we again refer to the idea of extending the scheme previously constructed for a simpler model.

The peculiarity of such an extension is related to the fact that the conservation law of energy (18) for the modified shallow water equations includes a logarithmic term. In contrast to the differential case, in finite differences one cannot pass from logarithmic expressions to rational expressions by means of differentiation or integration. This imposes restrictions on possible approximations for a conservative difference scheme.

Suppose that there is a finite-difference scheme that can be represented in the form of some rational function on 9-point stencil (50). Then, it is natural to extend scheme (59) for equation (7) as follows

$$F_0 = x_{tt} + D \left( \frac{1}{2\check{x}_s\check{x}_s} \right) + D \left( \frac{\gamma_1}{\alpha_1 x_s + \alpha_2 \check{x}_s + (1 - \alpha_1 - \alpha_2)\check{x}_s} \right) - \frac{D(H^h + \check{H}^h)}{x_t + \check{x}_t} = 0, \quad (65)$$

where $h_+ = h_- = h, \quad \check{\tau} = \check{\tau} = \tau,$

$$h = h, \quad \check{\tau} = \check{\tau} = \tau,$$

where $\alpha_1$ and $\alpha_2$ are some constant coefficients.
Scheme (59) possesses the conservation law of energy (60) with the conservation law multiplier (61). If the extended scheme possesses an energy conservation law with multiplier (61), then

\[ \Lambda^h_1 F_0 = \left( x_t + \hat{x}_t \right) \cdot \left( x_{tt} + D \left( \frac{1}{2\hat{x}_s \hat{x}_s} + \frac{\gamma_1}{\alpha_1 x_s + \alpha_2 \hat{x}_s + (1 - \alpha_1 - \alpha_2) \hat{x}_s} \right) - \frac{D(H^h + \hat{H}^h)}{x_t + \hat{x}_t} \right) \]

is a finite-difference divergent expression at least for some particular values of \( \alpha_1 \) and \( \alpha_2 \). To find out if this is really the case, one uses the finite-difference analogue of the direct method [52, 60]. The direct method requires to consider the equation

\[ \mathcal{E}_x (\Lambda^h_1 F_0) |_{h_\tau = h, \tau = 0} = 0, \]

where \( \mathcal{E}_x \) is the Euler operator on a uniform orthogonal mesh at point \( x \)

\[ \mathcal{E}_x = \frac{\partial}{\partial x} - \sum_{k=-\infty}^{\infty} \sum_{l=-\infty}^{\infty} S^k S^l \left[ \frac{D}{\tau} \left( \frac{\partial}{\partial x} \right) \left( \rho^k \rho^l \right) \right] + D \left[ \frac{\partial}{\partial x} \left( \rho^k \rho^l \right) \right] \]

taking into account the commutation relation (62).

But, as one can verify by direct computation, there are no values \( \alpha_1, \alpha_2 \) satisfying equation (67). Thus, approximation (66) does not possess a conservation law of energy under the chosen constraints. Despite of the fact that the considered case is quite simple, it can be verified that more complex rational approximations for equation (7) as well as for the integrating factor \( \Lambda_1 = x_t \) still do not bring one closer to the goal. The direct method is only effective enough when considering polynomial or rational expressions. In the most general case, it is problematic to obtain the form of a scheme or a conservation law multiplier with its help. The problem requires some additional assumptions or an ansatz to be involved.

The following considerations give a better result.

One can notice that a scheme possessing a conservation law of energy has to include some logarithmic terms. To demonstrate this, consider the following terms of (18)

\[ \left( \ln \varphi_s \right)_t - \left( \frac{\varphi_t}{\varphi_s} \right)_s = \frac{\varphi_{st} - \varphi_{ts}}{\varphi_s} \frac{\rho}{\varphi_s} \left( \frac{1}{\varphi_s} \right)_s. \]

The first and the second terms in the right hand side cancel each other out. (This is necessary in order to write (18) as the product of (7) and the multiplier \( \varphi_t \).) On the contrary, in the finite-difference case this is not the true as one can see from the following

\[ D \left( \ln x_s \right) - D \left( \frac{x_t}{x_s} \right) = \frac{1}{\tau} \ln \frac{x_s}{\hat{x}_s} - \frac{1}{x_s} - x_t \cdot \frac{\rho}{x_s} \left( \frac{1}{x_s} \right)_s. \]

The first and the second terms of the latter expression are obviously not canceled. This suggests that the term under \( D \) in the left hand side of (70) should also include a logarithmic expression. Then, one can notice that (69) can be rewritten in the following equivalent form

\[ \left( \ln \varphi_s \right)_t - \left( \frac{\varphi_t}{\varphi_s} (\ln \varphi_s)_s \right)_s. \]
The latter can be used as an ansatz for construction of conservative schemes. For example, one can construct the following finite-difference analogue of (71)

\[
D \left( \frac{\ln x_s}{x_{st}} \right) - D \left( x_t \frac{D(\ln x_s)}{x_{st}} \right) = \frac{1}{\tau} \ln \frac{x_s}{\tau x_{st}} - \frac{\gamma_1}{x_{s} - \hat{x}_s} \ln \hat{x}_s - x_t D \left( \frac{1}{\tau x_{st}} \ln \frac{x_s}{\hat{x}_s} \right) = -x_t D \left( \frac{1}{\tau x_{st}} \ln \frac{x_s}{\hat{x}_s} \right)
\]

which is an approximation of a desired form. The resulting expression (72) corresponds to the conservation law multiplier \( x_t \). For the multiplier (61) some minor changes required, namely one should consider the expression

\[
D \left( \frac{\ln(x_s\hat{x}_s)}{x_{st}} \right) - D \left( \frac{x_t + \hat{x}_t}{\tau(x_s + \hat{x}_s)} \right) = -\Lambda_1^n D \left( \frac{2}{\tau(x_s + \hat{x}_s)} \ln \frac{x_s}{\hat{x}_s} \right) = \Lambda_1^n D \left( \frac{2}{\hat{x}_s - \hat{x}_s} \ln \frac{x_s}{\hat{x}_s} \right).
\]

instead of (72). The latter can be obtained by means of algebraic transformations or, more systematically, with the help of the direct method.

Based on the above, one extends scheme (59) to the following invariant conservative scheme for the one-dimensional modified shallow water equations (7)

\[
x_{ti} + \frac{D(\ln x_s)}{x_{st}} + D \left( \frac{\gamma_1}{\hat{x}_s - \hat{x}_s} \ln \hat{x}_s \right) - \frac{D(H^h + \hat{H}^h)}{x_t + \hat{x}_t} = 0,
\]

\[
h_+ = h_- = h, \quad \hat{\tau} = \tau = \tau.
\]

This scheme is defined on 9-point stencil (50). It approximates (7) up to \( O(\tau^2 + h^2) \).

Scheme (74) is an invariant one. In case \( H^h = H^h(x) \), it admits the generators \( X_1 \) and \( X_2 \). In case \( H^h = \text{const} \), it also admits the generators \( X_3 \), \( X_4 \) and \( X_5 \).

Scheme (74) possesses conservation law of mass (62) and the conservation law of energy (with the conservation law multiplier \( \Lambda_1^n = \frac{1}{2}(x_t + \hat{x}_t) \))

\[
D \left( \frac{x_t^2}{2} + \frac{1}{4x_s} + \frac{1}{4\hat{x}_s} - \frac{\gamma_1}{2} \ln(x_s\hat{x}_s) - \frac{H^h + \hat{H}^h}{2} \right) + D \left( \frac{\gamma_1}{2} \ln \frac{x_s}{\hat{x}_s} \right) = 0.
\]

Notice that there is no finite-difference analogue of the conservation law (19). As alternative, one can construct schemes that conserve momentum but do not possess the energy conservation law. See discussion in [52]. For a horizontal bottom topography one can also consider the conservation law of momentum (76).

4.1.1. Case of a horizontal bottom

In case \( H^h = \text{const} \), there are two additional conservation laws, momentum and center-of-mass law, with the conservation law multipliers 1 and \( t \).

\[
D \left( x_t \right) + D \left( \frac{1}{2\hat{x}_s\hat{x}_s} \right) + \frac{\gamma_1}{\hat{x}_s - \hat{x}_s} \ln \frac{\hat{x}_s}{\hat{x}_s} = 0,
\]

\[
D \left( tx_t - x \right) + D \left( \frac{t}{2\hat{x}_s\hat{x}_s} \right) + \frac{\gamma_1 t}{\hat{x}_s - \hat{x}_s} \ln \frac{\hat{x}_s}{\hat{x}_s} = 0.
\]
4.1.2. Case of an inclined bottom

In case \( H(z) = C_1 z + C_2 \) (i.e., \( H' = C_1 \)), where \( z = z(t', s') \), the corresponding scheme

\[
z_{tt'} + D_{-s'} \left( \frac{1}{2} \hat{z}_{s'} \hat{z}_{s'} \right) + D_{-s'} \left( \frac{\gamma_1}{\hat{z}_{s'} - \hat{z}_{s'}} \ln \frac{\hat{z}_{s'}}{\hat{z}_{s'}} \right) - C_1 = 0,
\]

\[
s'_{+} - s'_{-} = s'_{-} = h',
\]

\[
\hat{t}' - t' = t' - \hat{t}' = \tau'
\]

can be transformed into the form (74) by means of the following finite-difference analogue of (26)

\[
z = x + \frac{C_1}{2} \hat{t}', \quad t' = t, \quad s' = s.
\]

Thus, in Lagrangian coordinates the case of an inclined bottom is reduced to the case of a horizontal bottom (see also [52]).

4.1.3. Case of parabolic bottoms

In the paper [53] the authors have constructed the schemes for the parabolic bottoms \( H(x) = \pm \frac{x^2}{2} \). By analogy to the previous case, they are extended as follows.

In case \( H(x) = \frac{x^2}{2} \), the approximation

\[
H^h = \cosh \tau - \frac{1}{\tau^2} x \hat{x} = \frac{x^2}{2} + O(\tau)
\]

for the function \( H(x) \) was found with the help of the direct method. It was found that the latter approximation is necessary for the scheme to possess the conservation laws with the conservation law multipliers \( e^{\pm t} \). (See [53] for details.) Thus, the extended scheme for the modified shallow water equations is

\[
x_{tt} + D_{-s} \left( \frac{1}{2x_s \hat{x}_s} + \frac{\gamma_1}{\hat{x}_s - \hat{x}_s} \ln \frac{\hat{x}_s}{\hat{x}_s} \right) - \frac{2(\cosh \tau - 1)}{\tau^2} x = 0,
\]

\[
h_+ = h_- = h, \quad \hat{\tau} = \hat{\tau} = \tau.
\]

The conservation law of energy for the latter scheme is obtained by formula (75)

\[
\Lambda^h_1 = \frac{x_t + \hat{x}_t}{2}, \quad D_{-\tau} \left( \frac{x_t^2}{2} + \frac{1}{4x_s} + \frac{1}{4\hat{x}_s} - \frac{\gamma_1}{2} \ln(x_s \hat{x}_s) - \frac{2(\cosh \tau - 1)}{\tau^2} x \hat{x} \right)
\]

\[
+ D_{-s} \left( \frac{x_t^2}{2} + \frac{1}{2x_s \hat{x}_s} + \frac{\gamma_1}{\hat{x}_s - \hat{x}_s} \ln \frac{\hat{x}_s}{\hat{x}_s} \right) = 0.
\]

The additional finite-difference conservation laws and the multipliers which correspond to (22) and (23) are the following

\[
(A^+_3)^h = e^t, \quad D_{-\tau} \left( \frac{e^i - e^{-t}}{\tau} - e^t x_t \right) - D_{-s} \left( e^t \left( \frac{1}{2x_s \hat{x}_s} + \frac{\gamma_1}{\hat{x}_s - \hat{x}_s} \ln \frac{\hat{x}_s}{\hat{x}_s} \right) \right) = 0,
\]

\[
(A^+_4)^h = e^{-t}, \quad D_{-\tau} \left( \frac{e^{-t} - e^{-t}}{\tau} - e^{-t} x_t \right) + D_{-s} \left( e^{-t} \left( \frac{1}{2x_s \hat{x}_s} + \frac{\gamma_1}{\hat{x}_s - \hat{x}_s} \ln \frac{\hat{x}_s}{\hat{x}_s} \right) \right) = 0.
\]
In case $H = -\frac{x^2}{2}$, the scheme is extended the similar way, and one derives

$$x_{ti} + D \left( \frac{1}{2\hat{x}_s\hat{x}_s} + \frac{\gamma_1}{\hat{x}_s - \hat{x}_s} \ln \frac{\hat{x}_s}{\hat{x}_s} \right) = \frac{2(\cos \tau - 1)}{\tau^2} x = 0,$$

$$h_+ = h_- = h, \quad \hat{\tau} = \tilde{\tau} = \tau. \quad (85)$$

The conservation law of energy and two additional conservation laws which correspond to (24) and (25) are

$$\Lambda^h_1 = \frac{x_t + \hat{x}_t}{2}, \quad D \left( \frac{x_t^2}{2} + \frac{1}{4\hat{x}_s} - \frac{1}{4\hat{x}_s} - \gamma_1 \ln(x_s\hat{x}_s) - \frac{2(\cos \tau - 1)}{\tau^2} x\hat{x} \right) + D \left( \frac{x_t^+ + \hat{x}_t^+}{2} + \frac{\gamma_1}{2\hat{x}_s\hat{x}_s} + \frac{\gamma_1}{\hat{x}_s - \hat{x}_s} \ln \frac{\hat{x}_s}{\hat{x}_s} \right) = 0, \quad (86)$$

$$(\Lambda^3_3)^h = \cos t, \quad D \left( x_t \cos t - x\frac{\cos \hat{t} - \cos t}{\tau} \right) + D \left( \cos t \left( \frac{1}{2\hat{x}_s\hat{x}_s} + \frac{\gamma_1}{\hat{x}_s - \hat{x}_s} \ln \frac{\hat{x}_s}{\hat{x}_s} \right) \right) = 0. \quad (87)$$

$$(\Lambda^3_3)^h = \sin t, \quad D \left( x_t \sin t - x\frac{\sin \hat{t} - \sin t}{\tau} \right) + D \left( \sin t \left( \frac{1}{2\hat{x}_s\hat{x}_s} + \frac{\gamma_1}{\hat{x}_s - \hat{x}_s} \ln \frac{\hat{x}_s}{\hat{x}_s} \right) \right) = 0. \quad (88)$$

4.1.4. **Comparison with a naive approximation**

In Section 5 devoted to the numerical implementation of schemes, the analysis of energy preservation by various schemes will be performed. The present section provides some preliminary ideas.

Consider the following naive scheme in case of the horizontal bottom $H = 0$

$$x_{ti} + D \left( \frac{1}{2\hat{x}_s\hat{x}_s} + \frac{\gamma_1}{\hat{x}_s} \right) = 0,$$

$$h_+ = h_- = h, \quad \hat{\tau} = \tilde{\tau} = \tau. \quad (89)$$

This scheme is invariant and approximates the modified shallow water equations up to $O(\tau^2 + h^2)$.

Scheme (89) is constructed in obvious way by extending scheme (59) with the rational term

$$D(-\frac{\gamma_1}{\hat{x}_s}).$$

As it was shown in the beginning of Section 4.1.1 such a scheme cannot possess a conservation law of energy corresponding to the conservation law multiplier (61). One can verify the similar way that more general multipliers of polynomial or even rational form do not lead to a conservation law of energy. Thus, either scheme (89) does not possess a conservation law of energy, or it possesses a conservation law of energy of a very complicated form.

In order to compare how the energy is preserved by schemes (74) and (89), we will use two approaches.
1. Assuming that scheme (89) does not possess an energy conservation law in principle, we construct a quite reasonable approximation of it, based on the known conservation laws (60) and (75) for schemes (59) and (74). Based on (60), we consider the following approximation for the conservation law of energy

\[
\frac{x_t + 6_t}{2} \left( x_{tt} + D \left( \frac{1}{2} x_{s} x_{s} + \frac{\gamma_1}{x_s} \right) \right) = D \left( \frac{x_t^2}{2} + \frac{1}{4x_s} + \frac{1}{4x_s} \right) + D \left( \frac{x_t^+ + x_t^-}{4x_s x_s} \right) + \gamma_1 \frac{x_t + 6_t}{2} - \gamma_1 \frac{1}{x_s} . \tag{90}
\]

The term at \( \gamma_1 \) is not a divergent expression. In order for expression (90) to approximate conservation law (18), a logarithmic term should be added to the density of the conservation law. It cannot arise from rational expressions, and therefore we introduce it artificially. There are various ways to do this, and we introduce it in the same form as in conservation law (75) for scheme (74). To do this, we add and subtract new divergent terms to (90). Then, rearranging the terms, we derive

\[
D \left( \frac{x_t^2}{2} + \frac{1}{4x_s} + \frac{1}{4x_s} - \frac{\gamma_1}{2} \ln x_s x_s \right) + D \left( \frac{x_t^+ + x_t^-}{4x_s x_s} + \frac{\gamma_1 x_t^+ + x_t^-}{2x_s} \right) - \gamma_1 \left\{ \frac{x_t + 6_t x_s}{2} x_s x_s + D \left( \frac{x_t^+ + x_t^-}{2x_s} \right) - \frac{1}{2} \frac{D}{\tau} (\ln x_s x_s) \right\} . \tag{91}
\]

The terms in the first two brackets form a reasonable approximation for the conservation law (18). Indeed, with infinite mesh refinement, they vanish on the solutions of the scheme. The expression in the curly braces tends to zero in continuous limit, so that one can consider it as the energy preservation error \( \delta \varepsilon \). The Taylor series expansion gives the estimation

\[
\delta \varepsilon = \left( \frac{1}{2} \varphi \right)_t + \frac{(\ln \varphi_s)_t}{3} \gamma_1^2 + O(h^2). \tag{92}
\]

Notice that representation (91) of (90) is not unique. We have chosen an appropriate approximation based on (60) analyzing schemes (89) and (74). It can be shown that other approximations give similar results.

2. Another approach is to measure the total energy evolution in time without regard to a particular scheme. In Lagrangian coordinates, we consider the following sum

\[
\mathcal{H}(n) = \frac{h}{2} \sum_{(i)} \left[ \left( \frac{x_i^{n+1} - x_i^n}{\tau} \right)^2 + \frac{1}{x_{i+1} - x_i} - 2\gamma_1 \ln \frac{x_i^{n+1} - x_i^n}{h} \right] \tag{93}
\]

whose value gives the total energy in the computational domain. Its value should tend to constant in the continuous limit \[46\]. \( \mathcal{H}(n) \) corresponds to the total energy for the modified shallow water equations

\[
\mathcal{H}(t, s) = \int \left( \frac{\varphi_t^2}{2} + \frac{1}{2} \varphi_s - \gamma_1 \ln \varphi_s \right) d\varphi \tag{94}
\]

at time \( t = n\tau \). Recall that the function under integral is the sum of kinetic and potential energy. To estimate the change in total energy over time, we will consider the relative error

\[
e_R(n) = \frac{|\mathcal{H}(n) - \mathcal{H}(0)|}{|\mathcal{H}(0)|} \tag{95}
\]
for the schemes under comparison.

The advantage of the second approach is that we do not make any assumptions about the existence of conservation laws for the schemes. On the other hand, the first approach allows one to obtain a reasonable estimate of energy preservation for a fixed moment of time under certain assumptions about the form of the finite-difference conservation laws.

4.2. Conservative schemes in mass Lagrangian coordinates

A straightforward transition to mass Lagrangian coordinates using simple approximations of (28) leads to three-layer schemes. Now we demonstrate that by choosing an appropriate approximation for (28) and (30) one can rewrite the scheme in mass Lagrangian coordinates on two time layers. Following [52], we choose the approximation

\[
\dot{x}_s + x_s = \frac{2}{\dot{\rho}}, \quad x_t = u
\]  

(96)

for equation (28), and the implicit approximation

\[
\frac{1}{\sqrt{\rho}} + \frac{1}{\sqrt{\rho}} = \frac{2}{\dot{\rho}}
\]

(97)

for (30). Notice that by virtue of (96) the latter can be considered equivalent to \( p = 1/x_s^2 \). It is also important to note that the change of variables does not affect the independent variables \( t \) and \( s \), so the uniform orthogonal mesh remains invariant.

Then, scheme (74) can be expressed in mass Lagrangian coordinates on two time layers as\(^2\)

\[
\begin{align*}
D_{-\tau} \left( \frac{1}{\rho} \right) - D_{-s} \left( \frac{u^+ + \tilde{u}^+}{2} \right) &= 0, \\
D_{-\tau} (u) + D_{-s} (Q) - \frac{\tau}{u + \tilde{u}} &= 0, \\
x_t = u, \quad \dot{x}_s + x_s = \frac{2}{\dot{\rho}}, \quad \frac{1}{\sqrt{\rho}} + \frac{1}{\sqrt{\rho}} = \frac{2}{\dot{\rho}}, \\
\hat{\tau} = \tau = \tau, \quad h_+ = h_- = h, \quad (\hat{\tau}, \hat{h}) = 0,
\end{align*}
\]

(98)

where

\[
Q = \frac{1}{2} \left[ \frac{4}{\rho \dot{\rho}} - \frac{2}{\sqrt{\rho}} \left( \frac{1}{\rho} + \frac{1}{\dot{\rho}} \right) + \left( \frac{1}{\rho} \right)^{-1} - \frac{\gamma_1 \rho \dot{\rho}}{\rho - \dot{\rho}} \ln \left( \sqrt{\rho} \left( \frac{2}{\rho} - \frac{1}{\sqrt{\rho}} \right) \right) \right] = \frac{\rho^2}{2} + \gamma_1 \rho + O(\tau).
\]

Thus, scheme (98) is defined on two time layers (see Figure 2) by including additional equations (96) and (97) into the system. Notice that for the first time such an approach was proposed in [61] for constructing schemes possessing an extended set of conservation laws.

\(^2\)To derive the first equation of the scheme one gets the sum of (62) and the shifted one, i.e.,

\[
D_{-\tau} (\dot{x}_s + x_s) - D_{-s} (x_t^+ + \tilde{x}_t^+) = 0.
\]
By means of (96) and (97) one obtains the following finite-difference conservation laws for scheme (98).

The conservation law of mass is the first equation of scheme (98), i.e.,

\[ D_{-\tau} \left( \frac{1}{\rho} \right) - D_{-s} \left( \frac{u^+ + \bar{u}^+}{2} \right) = 0. \] (100)

For an arbitrary \( H(x) \), the conservation law of energy (32) is brought to

\[ D_{-\tau} \left( \frac{u^2}{2} + \frac{1}{2} \rho - 2\sqrt{\rho} \right) - \gamma_1 \ln \left( \frac{2}{\rho} \frac{1}{\sqrt{\rho}} \right) - \frac{H^h + \bar{H}^h}{2} + D_{-s} \left( \frac{u^+ + \bar{u}^+}{2} Q \right) = 0. \] (101)

The conservation laws of energy for specific bottom topographies are obtained from (101) in quiet straightforward way so they are not presented here.

The conservation law of momentum (76) and the center of mass law (77) for the case \( H(x) = \text{const} \) in mass Lagrangian coordinates are

\[ D_{-\tau} (u) + D_{-s}(Q) = 0, \] (102)

\[ D_{-\tau} (tu - x) + D_{-s}(tQ) = 0, \] (103)

The additional conservation laws (83) and (84) for a parabolic bottom profile \( H(x) = \frac{x^2}{2} \) become

\[ D_{-\tau} \left( x \frac{e^{i} - e^{-i}}{\tau} - e^{i}u \right) - D_{-s}(Qe^{i}) = 0, \] (104)

\[ D_{-\tau} \left( x \frac{e^{-i} - e^{i}}{\tau} - e^{-i}u \right) + D_{-s}(Qe^{-i}) = 0. \] (105)

Similarly, the additional conservation laws (87) and (88) for \( H(x) = -\frac{x^2}{2} \) in mass coordinates are

\[ D_{-\tau} \left( u \cos t - x \frac{\cos \hat{t} - \cos t}{\tau} \right) + D_{-s}(Q \cos t) = 0. \] (106)

\[ D_{-\tau} \left( u \sin t - x \frac{\sin \hat{t} - \sin t}{\tau} \right) + D_{-s}(Q \sin t) = 0. \] (107)
5. Numerical implementation of the constructed schemes

In the present section the numerical implementation of conservative scheme \([74]\) and “naive” scheme \([89]\) are considered on two test problems. The first problem is the dam-break problem over a parabolic bottom. For the one-dimensional shallow water equations in Lagrangian coordinates it was considered in \([53]\) where it was numerically implemented for scheme \([59]\). The second problem is the collapse of a fluid column above an inclined bottom which was considered in \([62]\) for the modified shallow water equations in Eulerian coordinates.

As explained in Section 2, for a given initial height \(\rho_0(\xi)\) of the fluid over the bottom \(H\), one can derive the function \(\alpha(s)\) by solving the Cauchy problem

\[
\rho_0(\alpha(s)) = \frac{1}{\alpha'(s)}, \quad \alpha(0) = 0. \tag{108}
\]

Solving the equation \(x = \varphi(t, \alpha(s))\) with respect to \(s = A(t, x)\), and using the identity

\[
x - \varphi(t, \alpha(A(t, x))) = 0,
\]

one obtains that

\[
A_x(t, x) = \rho(\alpha(A(t, x)), t).
\]

Hence, the initial distribution \(A(t_0, x)\) becomes

\[
A(t_0, x) = \int_0^x \rho_0(\xi)d\xi. \tag{109}
\]

For more detailed discussion on calculations in Lagrangian coordinates see \([53, 57]\).

To implement schemes \([74]\) and \([89]\) numerically, we represent them in the following form

\[
\dot{x} - 2x + \ddot{x} + \frac{h^2\tau^2}{2} \left( \ddot{x} - \ddot{x}_- \right) (\ddot{x} - \ddot{x}_-) - \left( \dot{x}_+ - \dot{x}_- \right) \left( \dot{x}_+ - \dot{x}_- \right) + \tau^2 D(\gamma H^h) + \tau^2 (H^h)'(x) = 0, \tag{110}
\]

where \((H^h)'\) is a chosen approximation for the derivative \(H'(x)\), and \(\Gamma^h\) is an approximation for the term \(1/\varphi_s\).

Then, we linearize equation \([110]\) with respect to the solution on the upper time layer \([53]\)

\[
\frac{h^2\tau^2}{\Delta_1} (x_m^{n-1} - x_m^{n-1}) x_m^{(j+1)} - \left( 1 + \frac{h^2\tau^2}{\Delta_1} (x_m^{n+1} - x_m^{n-1}) \right) x_m^{(j+1)} = 2x_m^n - x_m^{n-1} - \tau^2 D(\gamma_1 H^h) - \tau^2 (H^h)'(x_m), \tag{111}
\]

where the indices \((j)\) denote the number of iteration,

\[
\Delta_1 = 2 \left( x_m^{(j)} - x_m^{(j-1)} \right) (x_m^{(j)} - x_m^{(j-1)}) (x_m^{n-1} - x_m^{n-1}) (x_m^{n-1} - x_m^{n-1}),
\]

\[
n = 2, 3, \ldots, \quad m = 2, 3, \ldots, \left\lfloor L/h \right\rfloor - 1.
\]

The advantage of representation \([111]\) is that on each iteration it can be solved with the help of tridiagonal matrix algorithm (one can find description and the stability conditions of this well-known algorithm, e. g., in \([63]\)).
For scheme (89), \( \Gamma^h = \frac{1}{x_s} = \frac{h}{x_{m+1} - x_m} \), and for scheme (74) it is
\[
\Gamma^h = \frac{1}{\hat{x}_s - \bar{x}_s} \ln \frac{\hat{x}_s}{\bar{x}_s} = \frac{h}{x_{m+1} - x_{m+1} - x_{m+1} + x_{m+1}} \ln \frac{x_{m+1} - x_{m+1}}{x_{m+1} - x_{m+1}}. \tag{112}
\]

If the value of \( x_s \) remains unchanged or almost does not change between the time layers \( n - 1 \) and \( n + 1 \), the numerical calculation of expression \( \Gamma^h \) causes practical difficulties. Linearizing this expression and representing it in the form of an iterative process only complicates the situation, so here we consider \( \Gamma^h \) without regard to the iterative process. Notice that if
\[
|\hat{x}_s/\bar{x}_s| = 1 + \epsilon, \quad |\epsilon| \ll 1,
\]
then the following expansion can be considered instead of \( \Gamma^h \)
\[
\frac{1}{\hat{x}_s - \bar{x}_s} \ln \frac{\hat{x}_s}{\bar{x}_s} = \ln(1 + (\hat{x}_s/\bar{x}_s - 1)) \sim \frac{1}{\hat{x}_s} \sum_{\kappa=0}^{\infty} \frac{1}{\kappa + 1} \left(1 - \hat{x}_s/\bar{x}_s\right)^\kappa. \tag{113}
\]

In the regions of small change in \( x_s \), the latter expansion is used, and the first eight terms of the expansion are taken. Outside such regions, calculations can still be performed using equation \( \Gamma^h \).

5.0.1. Dam break over a parabolic bottom

The dam-break problem is considered over a parabolic bottom
\[
H(x) = d_1 \left[\left(\frac{2}{L}\right)^2 \left(x - \frac{L}{2}\right)^2 - 1\right] \tag{114}
\]
where \( L \) is the length of the river segment and \( d_1 \) is the height of the parabolic bottom at the point \( x = L/2 \). According to [53], the following approximation for \( H'(x) \) is chosen
\[
(H')^h = \frac{2(\cosh(\sqrt{\beta} \tau) - 1)}{\tau^2} \left(x - \frac{L}{2}\right), \tag{115}
\]
where \( \beta = 8d_1/L^2 \).

In order to provide smoother initial data, the initial free surface profile is described by the function
\[
\eta(\xi) = \eta_L - \frac{\eta_L - \eta_R}{1 + \exp(\sigma(\xi - L/2))}, \tag{116}
\]
where \( \sigma = 20 \) is the curve steepness coefficient, and the constants \( \eta_L = 2 \) and \( \eta_R = 0.5 \) are given in Figure 3. We also put \( d_1 = 10 \) and \( L = 100 \). By means of (109) one states that the total mass of the fluid has value \( s \sim 791.7 \). Here and further we choose \( h = 0.1 \) and \( \tau = 0.01 \).
To investigate the qualitative effect of the value of $\gamma_1$ on the solution, calculations have been performed at the moment $t = 0.2$ for different values of $\gamma_1$. Figure 4 shows that an increase in $\gamma_1$ leads to an approximately linear increase in the fluid velocity. Based on Figure 4, further on we put $\gamma_1 = 10$.

The solutions of the problem for $t_1 = 0.2$ and $t_2 = 1$ are given in Figure 5. Here and further no artificial viscosity is used since this allows better control of the energy preservation on solutions. The profiles of solutions obtained by schemes (89) and (74) on the given scale practically do not differ, therefore, solutions obtained by scheme (74) are presented throughout the text. In contrast to solution profiles, the quality of energy conservation on solutions varies considerably for the schemes. In Figure 6, the energy preservation on solutions is given using the conservation law (82) and the estimation (91). Scheme (74) conserves energy much better than “naive” scheme (89).

The estimates of the total energy conservation by formula (95) for the two schemes practically do not differ as it shown in Figure 7. The total energy conservation pattern will be
significantly different for the next problem.

![Figure 5: Solutions for $\gamma_1 = 10$, $t_1 = 0.2$ (dash line) and $t_2 = 1$ (solid line).](image)

![Figure 6: Energy conservation law errors comparison for the conservative scheme (dot line) and for “naive” scheme (solid line) at time $t_1 = 0.2$ and $t_2 = 1$.](image)

![Figure 7: Total energy relative error for the schemes on time interval $0 \leq t \leq 1$.](image)

5.0.2. **Collapse of a fluid column above an inclined bottom**

A column of liquid above an inclined bottom, which collapses due to gravity, is considered. By means of transformation (79), the problem is reduced to the case of a horizontal bottom. The initial data is depicted in Figure 8 by dot line. The initial profile is smoothed in the same way as in the previous problem. It is described by the function

$$
\eta(\xi) = \eta_L - \frac{\eta_L - \eta_R}{1 + \exp(\sigma(\xi - L/2 + dL))} + \frac{\eta_L - \eta_R}{1 + \exp(\sigma(\xi - L/2 - dL))},
$$

(117)

where $dL = 2$, and the remaining parameters have the same meaning and values as in the previous section.
In Figure 8 the solid line shows the solution at time $t = 2$. The dashed line shows the solution profile calculated using artificial viscosity. The solution for an inclined bottom is given in Figure 9. This solution is very similar to the solution obtained in Eulerian coordinates in [62]. In [62], the case is also considered when the liquid rises at a certain velocity $u_0 < 0$ up the inclined plane. In contrast to Eulerian coordinates, in Lagrangian coordinates the profiles of solutions for this case do not differ from the case with zero velocity, since the mass distribution remains the same. In these cases, only the trajectories of the particles differ as shown in Figure 10.
Symmetries and conservation laws of the one-dimensional modified shallow water equations in Lagrangian coordinates for various bottom topographies are considered. These results are based on group classification \[22\]. Variational formulation of equations in Lagrangian coordinates allows one to obtain their conservation laws by means of the Noether theorem. The corresponding conservation laws in mass Lagrangian coordinates and Eulerian coordinates derived from conservation laws in Lagrangian coordinates are also given.

On the basis of invariant finite-difference schemes for the shallow water equations recently obtained in \[52, 53\], invariant schemes for the modified shallow water equations for various bottom topographies are constructed. These schemes possess finite-difference analogues of the local conservation laws of mass and energy for arbitrary shape of bottom, as well as additional...
conservation laws that appear for special cases of bottom topography. All the schemes are constructed on uniform orthogonal meshes which are invariant with respect to all symmetries inherited from the differential model.

To construct conservative schemes on such meshes, it is often convenient to use the finite-difference analogue of the direct method. The direct method is well suited for schemes that can be written in terms of rational expressions. This approach successfully used to construct conservative schemes for the standard shallow water equations, but for the schemes for the modified shallow water equations it doesn’t work by straightforward application. Indeed, in the case of the modified shallow water equations the conservation laws of energy and momentum include logarithmic terms which significantly complicates the problem. For the problem under consideration, the authors managed to find a special ansatz which solves the problem of logarithmic term. This example shows the importance of such methods as the finite-difference analogue of the Noether theorem \[30, 31, 36\] and the Lagrange identity and adjoint equation method \[42, 49\].

The numerical implementation of the constructed finite-difference schemes is carried out for the examples of a dam break over a parabolic bottom and a collapsing liquid column over an inclined bottom. All calculations are performed in Lagrangian coordinates. The constructed schemes are compared with a naive invariant scheme constructed without invariant Lagrangian consideration. It is shown that the specially constructed conservative schemes preserve energy much better than the naive approximation. This emphasizes the importance of the criteria of invariance and conservativeness in the construction of schemes.

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