A VALUATION-THEORETIC APPROACH TO THE DEEP JACOBIAN CONJECTURE

SUSUMU ODA

Professor Emeritus, Kochi University
334 Hiradani, Taki-cho, Taki-gun, Mie 519-2178, JAPAN
E-mail : ssmoda@ma.mctv.ne.jp

(Dedicated to the Author’s Father, Tadashi ODA, who passed away on August 25, 2010, and to Mother Kikue ODA, who is 102 years old.)

Abstract. Our goal is to settle the following result:

Theorem: Let $R$ be a Krull domain and let $\Delta_1$ and $\Delta_2$ be subsets of $Ht_1(R)$ such that $\Delta_1 \cup \Delta_2 = Ht_1(R)$ and $\Delta_1 \cap \Delta_2 = \emptyset$. Put $R_i := \bigcap_{Q \in \Delta_i} R_Q$ (i = 1, 2), subintersections of $R$. Assume that $\Delta_2$ is a finite set and that $R \rightarrow R_1$ is flat. If $R_1$ is factorial and $R^\times = (R_1)^\times$, then $\Delta_2 = \emptyset$ and $R = R_1$.

As an application, this gives us the following:

The Deep Jacobian Conjecture (DJC): Let $\varphi : S \rightarrow T$ be an unramified homomorphism of Noetherian normal domains with $T^\times = \varphi(S^\times)$. Assume that $T$ is factorial and that $S$ is algebraically simply connected. Then $\varphi$ is an isomorphism.

Besides, as a corollary, the following faded problem is settled:

The Jacobian Conjecture (JCn): If $f_1, \ldots, f_n$ are elements in a polynomial ring $k[X_1, \ldots, X_n]$ over a field $k$ of characteristic 0 such that $\det(\partial f_i / \partial X_j)$ is a nonzero constant, then $k[f_1, \ldots, f_n] = k[X_1, \ldots, X_n]$.

For the consistency of our discussion, we observe on Example: “An open embedding of $\mathbb{C}^2$ in a 2-dimensional $\mathbb{C}$-affine variety” sited in [10,(10.3) in p.305].

1. Introduction

Let $k$ be an algebraically closed field, let $\mathbb{A}^n_k = \text{Spec}^n(k[X_1, \ldots, X_n])$ be an affine space of dimension $n$ over $k$ and let $f : \mathbb{A}^n_k \rightarrow \mathbb{A}^n_k$ be a morphism of affine spaces over $k$ of dimension $n$. Note here that for a ring $R$, $\text{Spec}(R)$ (resp. $\text{Spec}^n(R)$) denotes the prime spectrum of $R$ (or merely the set of prime ideals of $R$) (resp. the

2010 Mathematics Subject Classification. Primary: 14R15; Secondary: 13B25

Key words and phrases. The Jacobian Conjecture, the Deep Jacobian Conjecture, unramified, étale, simply connected and polynomial rings

1
maximal spectrum (or merely the set of the maximal ideals of \(R\)). Then \(f\) is given by

\[ \mathbb{A}^n_k \ni (x_1, \ldots, x_n) \mapsto (f_1(x_1, \ldots, x_n), \ldots, f_n(x_1, \ldots, x_n)) \in \mathbb{A}^n_k, \]

where \(f_i(X_1, \ldots, X_n) \in k[X_1, \ldots, X_n].\) If \(f\) has an inverse morphism, then the Jacobian \(J(f) := \det(\partial f_i/\partial X_j)\) is a nonzero constant. This follows from the easy chain rule of differentiations without specifying the characteristic of \(k\). The Jacobian Conjecture asserts the converse.

If \(k\) is of characteristic \(p > 0\) and \(f(X) = X + X^p\), then \(df/dX = f'(X) = 1\) but \(X\) cannot be expressed as a polynomial in \(f\). It follows that the inclusion \(k[X + X^p] \hookrightarrow k[X]\) is finite and étale but \(f : k[X] \rightarrow k[X]\) is not an isomorphism. This implies that \(k[X]\) is not simply connected (i.e., \(\text{Spec}(k[X]) = \mathbb{A}^1_k\) is not simply connected, see §2. Definition 5.2) when \(\text{char}(k) = p > 0\). Thus we must assume that the characteristic of \(k\) is 0.

The algebraic form of The Jacobian Conjecture \((JC_n)\) (or the Jacobian Problem \((JC_n)\)) is the following:

**The algebraic form \((JC_n)\).** If \(f_1, \ldots, f_n\) are elements in a polynomial ring \(k[X_1, \ldots, X_n]\) over a field \(k\) of characteristic 0 such that \(\det(\partial f_i/\partial X_j)\) is a nonzero constant, then \(k[f_1, \ldots, f_n] = k[X_1, \ldots, X_n]\).

Note that when considering \((JC_n)\), we may assume that \(k = \mathbb{C}\) by “Lefschetz-principle” (See [10,(1.1.12)]).

The Jacobian Conjecture \((JC_n)\) has been settled affirmatively under a few special assumptions below (See [6]). Let \(k\) denote a field of characteristic 0. We may assume that \(k\) is algebraically closed. Indeed, we can consider it in the case \(k = \mathbb{C}\), the field of complex numbers. So we can use all of the notion of Complex Analytic Geometry. But in this paper, we go forward with the algebraic arguments.

For example, under each of the following assumptions, the Jacobian Conjecture \((JC_n)\) has been settled affirmatively (Note that we may assume that \(k\) is algebraically closed and of characteristic 0):

**Case(1)** \(f : \mathbb{A}^n_k = \text{Spec}(k[X_1, \ldots, X_n]) \rightarrow \text{Spec}(k[f_1, \ldots, f_n]) = \mathbb{A}^n_k\) is injective;

**Case(2)** \(k[X_1, \ldots, X_n] = k[f_1, \ldots, f_n]\);

**Case(3)** \(k[X_1, \ldots, X_n]\) is a Galois extension of \(k[f_1, \ldots, f_n]\);

**Case(4)** \(\deg f_i \leq 2\) for all \(i\);

**Case(5)** \(k[X_1, \ldots, X_n]\) is integral over \(k[f_1, \ldots, f_n]\).

A fundamental reference for The Jacobian Conjecture \((JC_n)\) is [6] which includes the above Cases.

See also the reference [6] for a brief history of the developments and the state of the art again since it was first formulated and partially proved by Keller in 1939 ([13]), together with a discussion on several false proofs that have actually appeared in print, not to speak of so many other claims of prospective proofs being announced.
but proofs not seeing the light of the day. The Jacobian Conjecture ($JC_n$), due to the simplicity of its statement, has already fainted the reputation of leading to solution with ease, especially because an answer appears to be almost at hand, but nothing has been insight even for $n = 2$.

The conjecture obviously attracts the attention of one and all. It is no exaggeration to say that almost every makes an attempt at its solution, especially finding techniques from a lot of branches of mathematics such as algebra (Commutative Ring Theory), algebraic geometry/topology, analysis (real/complex) and so on, having been in whatever progress (big or small) that is made so far (cf. E. Formanek, Bass’ Work on The Jacobian Conjecture, Contemporary Mathematics 243 (1999), 37-45).

For more recent arguments about The Jacobian Conjecture, we can refer to [W] and [K-M].

Throughout this paper, unless otherwise specified, we use the following notations:

**Basic Notations**
- All fields, rings and algebras are assumed to be commutative with unity.
- For a ring $R$,
  - A *factorial* domain $R$ is also called a unique factorization domain,
  - $R^\times$ denotes the set of units of $R$,
  - $\text{nil}(R)$ denotes the nilradical of $R$, i.e., the set of the nilpotent elements of $R$,
  - $K(R)$ denotes the total quotient ring (or the total ring of fractions) of $R$, that is, letting $S$ denote the set of all non-zero-divisors in $R$, $K(R) := S^{-1}R$,
  - When $R$ is an integral domain, for $x \in R \setminus \{0\}$ \( R_x := \{ r/x^n \mid r \in R, n \in \mathbb{Z}_{\geq 0} \} \subseteq K(R) $,
  - $\text{Ht}_1(R)$ denotes the set of all prime ideals of height one in $R$,
  - $\text{Spec}(R)$ denotes the *affine scheme* defined by $R$ (or merely the set of all prime ideals of $R$),
  - Let $A \to B$ be a ring-homomorphism and $p \in \text{Spec}(A)$. Then $B_p$ means $B \otimes_A A_p$.
- Let $k$ be a field.
  - A (separated) scheme over a field $k$ is called *$k$-scheme*. A $k$-scheme locally of finite type over $k$ is called a (algebraic) *variety* over $k$ or a (algebraic) $k$-variety if it is integral (i.e., irreducible and reduced).
  - A $k$-variety $V$ is called a *$k$-affine variety* or an *affine variety* over $k$ if it is isomorphic to an affine scheme $\text{Spec}(R)$ for some $k$-affine domain $R$ (i.e., $R$ is a finitely generated domain over $k$).
  - An integral, closed $k$-subvariety of codimension one in a $k$-variety $V$ is called a *hypersurface* of $V$.
  - A closed $k$-subscheme (possibly reducible or not reduced) of pure codimension one in a $k$-variety $V$ is called an (effective) *divisor* of $V$, and thus an irreducible and reduced divisor (i.e., a prime divisor) is the same as a hypersurface in our terminology.
Our Main Objective is to settle the Deep version as follows:

**The Deep Jacobian Conjecture (DJC).** Let $\varphi : S \to T$ be an unramified homomorphism of Noetherian domains with $T^\times = \varphi(S^\times)$. Assume that $T$ is factorial and that $S$ is a simply connected normal domain. Then $\varphi$ is an isomorphism.

In Section 3, to begin with a theorem, Theorem 3.5 about a Krull domain and its flat subintersection with the same units group is discussed and in the next place Conjecture (DJC) is settled as a main result, Theorem 3.10, and consequently the Jacobian Conjecture ($JC_n$) ($\forall n \in \mathbb{N}$) is resolved.

For the consistency of our discussion, we assert that the examples appeared in the papers ([12], [2] and [20]) published by the certain excellent mathematicians, which would be against our original target Conjecture (DJC), are imperfect or incomplete counter-examples. We discuss them in detail in another paper (See S. Oda: Some Comments around The Examples against The Generalized/Deep Jacobian Conjecture, ArXive:0706.1138 v99 [math.AC] – Nov 2022).

By the way, the Jacobian Conjecture ($JC_n$) is a problem concerning a polynomial ring over a field $k$ (characteristic 0), so that investigating the structure of automorphisms $\text{Aut}_k(k[X_1, \ldots, X_n])$ seems to be substantial. Any member of $\text{Aut}_k(k[X_1, X_2])$ is known to be tame, but for $n \geq 3$ there exists a wild automorphism of $k[X_1, \ldots, X_n]$ (which was conjectured by M. Nagata with an explicit example and was settled by Shestakov and Umirbaev (2003)).

In such a sense, to attain a positive solution of ($JC_n$) by an abstract argument like this paper may be far from its significance.

Our general references for unexplained technical terms of Commutative Algebra are [14] and [15].

Remark that we often say in this paper that
a ring $A$ is "simply connected" if $\text{Spec}(A)$ is simply connected, and
a ring homomorphism $f : A \to B$ is "unramified, étale, an open immersion, a closed immersion, · · · · ·" when "so" is its morphism $\alpha f : \text{Spec}(B) \to \text{Spec}(A)$, respectively.

2. **Some Comments about a Krull Domain**

To confirm the known facts about Krull domains, we give some explanations about Krull domains for our usage. Our fundamental source is [11].

Let $R$ be an integral domain and $K = K(R)$ its quotient field. We say that an $R$-submodule $I$ in $K$ is a fractional ideal of $R$ if $I \neq 0$ and there exists a non-zero element $\alpha \in R$ such that $\alpha I \subseteq R$. In the word of Fossum [11], a fractional ideal of $R$ is the same as an $R$-lattice in $K$ (See [11,p.12]).

It is easy to see that for fractional ideals $I$ and $J$ of $R$, $IJ, I \cap J, I + J$ and $R :_{K(R)} I := \{x \in K \mid xI \subseteq R\}$ are fractional ideal of $R$. 

If a fractional ideal $I$ of $R$ satisfies $I = R : K : (R : K I)$, we say that $I$ is a \textit{divisorial fractional ideal} of $R$. Note that $I$ is an $R$-submodule of $R : K(R) : (R : K(R) I)$ by \([11,p.10]\). Let $I^* := R : K I$ and $I^{**} := (I^*)^* = R : K : (R : K I)$, which are divisorial fractional ideals of $R$ (\([11,(2.4)]\)). It is clear that $R$ itself is a divisorial fractional ideal of $R$. We call $I^{**}$ the \textit{divisorialization} of $I$, which is the minimal divisorial fractional ideal of $R$ containing $I$.

We say that $I$ is an \textit{invertible} fractional ideal of $R$ if there exists a fractional ideal $J$ of $R$ such that $IJ = R$, that is, $I \otimes_R J \cong_R IJ = R$ (and hence $J = I^*$ in this case) (cf.\([14,p.80]\)), and moreover if in addition $IJ$ is a principal fractional ideal of $R$ then both $I$ and $J$ are invertible ideals of $R$. It is easy to see that an invertible fractional ideal $R$ is finitely generated flat over $R$, that is, a finitely generated projective $R$-module. Every invertible fractional ideal of $R$ is divisorial, and if $I$ and $J$ are invertible fractional ideals of $R$ then so is $IJ$.

Let $A$ be a Krull domain\(^1\). Note that a Krull domain is completely integrally closed (\([11,(3.6)]\)). For $P \in \text{Ht}_1(A)$, $v_P(\ )$ denotes the (additive or exponential) \textit{valuation} on $K(A)$ associated to the principal valuation ring $A_P$.

\textbf{Remark 2.1.} According to \([11,\text{§}(5.2)+(5.4)+(5.5)(b)]\), we see the following equivalences:

\begin{align*}
IA_P &= JA_P \ (\forall P \in \text{Ht}_1(A)) \iff \bigcap_{P \in \text{Ht}_1(A)} IA_P = \bigcap_{P \in \text{Ht}_1(A)} JA_P \iff I = J \quad (+).
\end{align*}

\textbf{Remark 2.2.} In \([14,p.29]\) (See \[EGA,IV,\text{§}13\] or Bourbaki:Commutative Algebra Chap.1-7, Springer-Verlag (1989)), we see the following definition:

If $P$ is a prime ideal of a Krull domain $A$ and $n \in \mathbb{Z}_{\geq 0}$ then the \textit{symbolic $n$-th power} of is the ideal $P^{(n)}$ defined by $P^{(n)} := P^n A_P \cap A$, which is a $P$-primary ideal of $A$. Moreover, an integral ideal of $A$ is divisorial if and only if it can be expressed as an intersection of a finite number of height one primary ideals of $A$ (See \([14,\text{Ex}(12.4)]\)). In particular, each $P \in \text{Ht}_1(A)$ is divisorial. This definition is adopted in many texts of algebra. But $n$ must be non-negative.

On the other hand, in Fossum\([11,p.26]\), we see the following definition:

If $P$ is in $\text{Ht}_1(A)$ and $n \in \mathbb{Z}$ then the \textit{symbolic $n$-th power} of $P$ is a divisorial ideal $P^{(n)} := \{x \in K(A) \mid v_P(x) \geq n\}$.

\(^1\)Let $A$ be an integral domain which is contained in a field $K$. The integral domain $A$ is said to be \textit{Krull domain} provided there is a family $\{V_i\}_{i \in I}$ of principal valuation rings (i.e., discrete rank one valuation rings), with $V_i \subseteq K$, such that

(i) $A = \bigcap_{i \in I} V_i$

(ii) Given $0 \neq f \in A$, there is at most a finite number of $i$ in $I$ such that $f$ is not a unit in $V_i$.

Such a family as $\{V_i\}_{i \in I}$ (or $\{p_i \cap A \mid p_i$ is a non-zero prime ideal of $V_i (i \in I)\}$) is called a \textit{defining family} of a Krull domain $A$. The property (ii) is called the \textit{finite character} of $\{V_i\}_{i \in I}$ (or $A$). Note that $\{A_P \mid P \in \text{Ht}_1(A)\}$ (or $\text{Ht}_1(A)$) is indeed a defining family of a Krull domain $A$ (cf.\([11,(1.9)]\)), where $A_P$ is called an \textit{essential} valuation over-ring of $A$ (\(\forall P \in \text{Ht}_1(A)\)). The defining family $\{A_P \mid P \in \text{Ht}_1(A)\}$ is the minimal one.
However, for even $n > 0$ and $P \in \text{Ht}_1(A)$, it is clear that $P^nA_P = (PA_P)^n = \{x \in K(A) \mid v_P(x) \geq n\}$, where in general, $P^nA_P$ is not a fractional ideal of $A$ because $P^nA_P$ can not be contained in any finitely generated $A$-submodule of $K$. Thus $\{x \in K(A) \mid v_P(x) \geq n\}$ is not a fractional ideal of $A$ regrettable (See [11,p.6]). So we should modify the definition in Fossum[11,p.26] above.

As a modification of Definition[11,p.26] or a generalization of Definition[14,p.29], in our paper, we adopt the following Definition:

**Definition 2.3.** Let $A$ be a Krull domain. If $P$ is in $\text{Ht}_1(A)$ and $n \in \mathbb{Z}$ then the *symbolic n-th power* $P^{(n)}$ of $P$ is defined as follows:

$$P^{(n)} = \begin{cases} A :_{K(A)} (A :_{K(A)} P^n) & (n \geq 0) \\ A :_{K(A)} P^{-n} & (n < 0). \end{cases}$$

**Remark 2.4.** In this definition, we see that $P^{(n)} (P \in \text{Ht}_1(A), n \in \mathbb{Z})$ is a divisorial fractional ideal of $A$ (See [11,(2.6)]).

We see easily from Definition 2.3 that for $P \in \text{Ht}_1(A)$ and $n \in \mathbb{Z}$ and for each $Q \in \text{Ht}_1(A)$,

$$P^{(n)}A_Q = \begin{cases} (PA_P)^n & \text{(if } Q = P) \\ A_Q & \text{(if } Q \neq P) \end{cases} \quad (+)$$

because for $Q \neq P$, $(PA_P)^nA_Q = K(A)$ by [11,(5.1)].

**NOTE:** A simple verification shows that we can use the results in [11] freely except the definition of the symbolic power [11,p.26] and Corollary [11,(5.7)] in Fossum[11].

Considering the localization at each member in $\text{Ht}_1(A)$ and [11,(5.5)(b)+(5.2)(c)] together with (+), (+), Definition 2.3 can be expressed as follows:

**Proposition 2.5.** Let $A$ be a Krull domain and $P \in \text{Ht}_1(A)$. Let $n \in \mathbb{Z}$. Then

$$P^{(n)} = \bigcap_{Q \in \text{Ht}_1(A)} (PA_Q)^n = (PA_P)^n \cap \bigcap_{Q \in \text{Ht}_1(A) \setminus \{P\}} A_Q,$$

that is, $P^{(n)} = \{x \in K(A) \mid v_P(x) \geq n \text{ and } v_Q(x) \geq 0 \text{ for } \forall Q \in \text{Ht}_1(A) \setminus \{P\}\}.$

**Corollary 2.6.** Let $A$ be a Krull domain and $P_1 \neq P_2$ in $\text{Ht}_1(A)$. For $n_1, n_2 \in \mathbb{Z}$, put $I := \{x \in K(A) \mid v_{P_1}(x) \geq n_1, v_{P_2}(x) \geq n_2 \text{ and } v_Q(x) \geq 0 \text{ for } \forall Q \in \text{Ht}_1(A) \setminus \{P_1, P_2\}\}$. Then $I = \bigcap_{Q \in \text{Ht}_1(A)} I_Q = I_{P_1} \cap I_{P_2} \cap (\bigcap_{Q \in \text{Ht}_1(A) \setminus \{P_1, P_2\}} A_Q)$, which is a divisorial fractional ideal of $A$. In other words, for distinct $P_1, P_1 \in \text{Ht}_1(A)$, $(P_1A_{P_1})^{n_1} \cap (P_2A_{P_2})^{n_2} \cap (\bigcap_{Q \in \text{Ht}_1(A) \setminus \{P_1, P_2\}} A_Q)$ is a divisorial fractional ideal of $A$ for any $n_1, n_2 \in \mathbb{Z}$.

**Proof.** We see easily that $I$ is a fractional ideal of $A$ and that the two equalities hold. So $I$ is divisorial by [11,(5.5)(b)].
Remark 2.7. We see easily the following:

Let $A$ be a Krull domain, let $P \neq Q$ in $\text{Ht}_1(A)$ and let $m, n \in \mathbb{Z}_{\geq 0}$. Then (i) $P^{(n)} = (PA_P)^n \cap A$, which is $P$-primary (See [14,(4.2)]), (ii) $P^{(n)} \cap P^{(m)} = P^{(\text{Max}(n,m))}$ by Proposition 2.5, (iii) $P^{(1)} = P$, $P^{(0)} = A$ and $A : K(A) P^{(n)} P^{(-n)} = A$.

Therefore Definition 2.3 is the generalization of the one seen in [14,p.29]. However $P^{(n)} P^{(-n)} \neq A$ for $n \neq 0$ if $P$ is not an invertible fractional ideal of $A$ (even though $A : K(A) (P^{(n)} P^{(-n)}) = A : K(A) (P^{(n)} (A : K(A)) P^{(n)}) = A$ (See [11,pp.12-13]), noting that either $n > 0$ or $-n > 0$).

In general, from $(++)$ and Proposition 2.5 we have

Lemma 2.8. Let $A$ be a Krull domain and let $P$ be in $\text{Ht}_1(A)$. Then for $m, n \in \mathbb{Z}$, $P^{(n+m)} = A : K(A) (A : K(A) P^{(n)} P^{(m)})$.

If in addition $P$ is invertible then $P^{(n+m)} = P^{(n)} P^{(m)}$, which is also invertible.

Proof. We see the following by Definition 2.3: for each $Q \in \text{Ht}_1(A)$, if $Q \neq P$ then $P^{(n+m)} A_Q = A_Q = (A_Q : K(A) (A_Q : K(A) P^{(n)} A_Q P^{(m)} A_Q) = (A : K(A) (A : K(A) P^{(n)} P^{(m)})) A_Q$, and if $Q = P$ then $P^{(n+m)} A_P = (PA_P)^{n+m} = A_P : K(A) (A_P : K(A) P^{(n)} P^{(m)} A_P).$ Thus $P^{(n+m)} = A : K(A) (A : K(A) P^{(n)} P^{(m)})$ by $(++)$.

If $P$ is invertible, so is $P^{(n)}$ for all $n \in \mathbb{Z}$, and $P^{(n)} P^{(m)}$ is also invertible for all $m \in \mathbb{Z}$. So the second statement holds indeed by Lemma 2.5.

Notations: For a non-zero fractional ideal $I$ of a Krull domain $A$, we use the following notations:

- $v_P(I) := \inf\{v_P(a) \mid a \in I\}$, (which is non-zero for finitely many members $P \in \text{Ht}_1(A)$ according to the finite character property of a defining family of a Krull domain),
- $\text{Supp}^*(I) := \{P \in \text{Ht}_1(A) \mid v_P(I) \neq 0\}$,
- for fractional ideals $I_1, \ldots, I_n$ of $A$, $\prod_{i=1}^n I_i$ means a product $I_1 \cdots I_n$ (not a direct product), which is also a fractional ideal of $A$.

Lemma 2.9. For a divisorial fractional ideal $I$ of a Krull domain, $\text{Supp}^*(I)$ is a finite subset of $\text{Ht}_1(A)$.

Proof. First, we see that a divisorial (integral) ideal is contained in only finitely many prime ideals in $\text{Ht}_1(A)$ by [11,(3.6)+(3.12)(e)]. Since $I$ is a divisorial fractional ideal of $A$, it is easy to see that: for $P \in \text{Ht}_1(A)$, $I_P \neq A_P \Leftrightarrow (A : K(A)) I_P \neq A_P \Leftrightarrow (A : K(A) (A : K(A)) I_P \neq A_P = I_P$. Thus

$\text{Supp}^*(I) = \text{Supp}^*(A : K(A) I)$ (*).

Put $\Delta(+) := \{P \in \text{Ht}_1(A) \mid v_P(I) > 0\}$ and $\Delta(-) := \{P \in \text{Ht}_1(A) \mid v_P(I) < 0\}$. Then $\text{Supp}(I) = \Delta(+) \cup \Delta(-)$. Let $I^{(+)} := \bigcap_{P \in \Delta(+)} I_P \cap A$ and let $I^{(-)} := \bigcap_{P \in \Delta(-)} (A : K(A) I_P) \cap A$. Then both $I^{(+)}$ and $I^{(-)}$ are divisorial (integral) ideals.
of \(A\) by \([14, \text{Ex}(12.4)]\). Thus \(\Delta(+) = \text{Supp}^*(I^{(+)})\) and \(\Delta(-) = \text{Supp}^*(I^{(-)})\) and they are finite subsets of \(\text{Ht}_1(A)\). Therefore \(\text{Supp}^*(I)\) is a finite subset of \(\text{Ht}_1(A)\) by \((*)\).

**Remark 2.10** ([11, p.12]). We see the following:

For a fractional ideal \(I\) of a Krull domain \(A\),

(i) \(A : K(A) \langle A : K(A) \rangle I = \bigcap_{x \in A \cap K(A)} xA\) \([11, \text{p.12}]\).

(ii) For a fractional ideal \(I\) of a Krull domain \(A\), \(I\) is divisorial \(\iff I = \bigcap_{P \in \text{Ht}_1(A)} IP\)
\(\iff\) every regular \(A\)-sequence of length 2 is a regular \(I\)-sequence \([11,(5.2)(c)\) and \((5.5)(f)]\).

Viewing \([11,(5.3) + (5.5)(b)]\) with Proposition 2.5, we then see:

**Lemma 2.11.** Let \(A\) be a Krull domain, let \(P_1, P_2 \in \text{Ht}_1(A)\) with \(P_1 \neq P_2\) and let \(n_1, n_2 \in \mathbb{Z}\). Then

(i) If \(n_1, n_2 \neq 0\), then \(\text{Supp}^*(P_i^{(n_i)}) = \{P_i\} (i = 1, 2)\) and \(\text{Supp}^*(P_1^{(n_1)}P_2^{(n_2)}) = \{P_1, P_2\},\)

(ii) \(\bigcap_{Q \in \text{Ht}_1(A) \setminus \{P\}} A_Q \bigcap_{Q \in \text{Ht}_1(A) \setminus \{P_1, P_2\}} A_Q = \bigcap_{Q \in \text{Ht}_1(A) \setminus \{P_1, P_2\}} A_Q,\)

(iii) \(P_1^{(n_1)}P_2^{(n_2)} = (P_1A_{P_1})^{n_1} \cap (P_2A_{P_2})^{n_2} \cap (\bigcap_{Q \in \text{Ht}_1(A) \setminus \{P_1, P_2\}} A_Q),\) which is a divisorial fractional ideal of \(A\).

**Proof.**

(i) is obvious by \((++)\).

(ii) See \([11, (5.1)]\).

(iii) We may assume that \(n_1 \neq 0\) and \(n_2 \neq 0\) because in the case \(n_1 = n_2 = 0\) (iii) holds obviously.

Note that \(A_Q A_P = K(A)\) and \(P A_Q = A_Q\) if \(Q \neq P\) and that \((\bigcap_{Q \in \text{Ht}_1(A) \setminus \{P\}} A_Q) A_P = K(A)\) by (i) and (ii) above.

(1) From Proposition 2.5, \((i)\) and \((ii)\) above, we see:

\[
P_1^{(n_1)}P_2^{(n_2)} = \left( (P_1A_{P_1})^{n_1} \bigcap_{Q \in \text{Ht}_1(A) \setminus \{P\}} A_Q \right) \left( (P_2A_{P_2})^{n_2} \bigcap_{Q \in \text{Ht}_1(A) \setminus \{P_2\}} A_Q \right)
\]

\[
\subseteq (P_1A_{P_1})^{n_1} (P_2A_{P_2})^{n_2} \cap (P_1A_{P_1})^{n_1} \cap (P_2A_{P_2})^{n_2} \cap \left( \bigcap_{Q \in \text{Ht}_1(A) \setminus \{P_1, P_2\}} A_Q \right)
\]

\[
= (P_1A_{P_1})^{n_1} \cap (P_2A_{P_2})^{n_2} \cap \left( \bigcap_{Q \in \text{Ht}_1(A) \setminus \{P_1, P_2\}} A_Q \right)
\]

\[
\therefore (P_1A_{P_1})^{n_1} (P_2A_{P_2})^{n_2} = K(A)\) and \((i)\) above.

(2) It is easy to see that \(P_i^{(n_i)}\) is a divisorial \(A\)-fractional ideal \((i = 1, 2)\). By \((i)\) and Remark 2.10 \((i)\),

\[
P_i^{(n_i)} = \bigcap_{x_i \in A} x_i A = \bigcap_{Q \in \text{Ht}_1(A)} \bigcap_{x_i \in A} (x_i A_Q),
\]
and for each \(i = 1, 2\), \(\text{Supp}^*(x_iA) = \text{Supp}^*(\bigcap_{P_i^{(n_i)}} \subseteq x_iA) = \text{Supp}^*(\{P_i\}) = \{P_i\}\) by (i) above and Remark 2.10(i). It follows that \(x_1, x_2A_Q = x_1A_Q\) (if \(Q = P_1\)), \(x_2A_Q\) (if \(Q = P_2\)) or \(A_Q\) (if \(Q \in \text{Ht}_1(A) \setminus \{P_1, P_2\}\)) respectively. By use of these properties, we then have

\[
P_{1}^{(n_1)}P_{2}^{(n_2)} = \bigcap_{P_1^{(n_1)} \subseteq x_1A \, (x_1 \in K(A))} \bigcap_{P_2^{(n_2)} \subseteq x_2A \, (x_2 \in K(A))} x_1A \bigcap_{Q \in \text{Ht}_1(A)} (\bigcap_{P_1^{(n_1)} \subseteq x_1A \, (x_1 \in K(A))} \bigcap_{P_2^{(n_2)} \subseteq x_2A \, (x_2 \in K(A))} x_1A_Q) \bigcap_{Q' \in \text{Ht}_1(A)} (\bigcap_{P_2^{(n_2)} \subseteq x_2A \, (x_2 \in K(A))} x_2A_{Q'})
\]

(\(\because\) the properties about \(x_1, x_2A_Q\) explained above)

\[
= \bigcap_{Q \in \text{Ht}_1(A)} (\bigcap_{P_1^{(n_1)} \subseteq x_1A \, (x_1 \in K(A))} x_1A_Q) \bigcap_{Q' \in \text{Ht}_1(A)} (\bigcap_{P_2^{(n_2)} \subseteq x_2A \, (x_2 \in K(A))} x_2A_{Q'})
\]

(\(\because\) \(P_1 \neq P_2\) and \(x_1A_Qx_2A_{Q'} = K(A)\) if \(Q \neq Q'\))

\[
= \bigcap_{Q \in \text{Ht}_1(A)} (\bigcap_{P_1^{(n_1)} \subseteq x_1A \, (x_1 \in K(A))} x_1A_Q) \bigcap_{Q' \in \text{Ht}_1(A)} (\bigcap_{P_2^{(n_2)} \subseteq x_2A \, (x_2 \in K(A))} x_2A_{Q'})
\]

(\(\because\) \(\bigcap_{P_2^{(n_2)} \subseteq x_2A \, (x_2 \in K(A))} x_2A_{Q'}\) is a principal fractional ideal of \(A_Q\))

\[
= \bigcap_{Q \in \text{Ht}_1(A)} (\bigcap_{P_1^{(n_1)} \subseteq x_1A \, (x_1 \in K(A))} x_1A_Q) \bigcap_{Q' \in \text{Ht}_1(A)} (\bigcap_{P_2^{(n_2)} \subseteq x_2A \, (x_2 \in K(A))} x_2A_{Q'})
\]

(\(\because\) \(\bigcap_{P_2^{(n_2)} \subseteq x_2A \, (x_2 \in K(A))} x_2A_{Q'}\) is a principal fractional ideal of \(A_Q\))

\[
= \bigcap_{Q \in \text{Ht}_1(A)} (\bigcap_{P_1^{(n_1)} \subseteq x_1A \, (x_1 \in K(A))} x_1A_Q) \bigcap_{Q' \in \text{Ht}_1(A)} (\bigcap_{P_2^{(n_2)} \subseteq x_2A \, (x_2 \in K(A))} x_2A_{Q'})
\]

(by \([11, (5.2)(c)])

\[
\subseteq A :K(A) (A :K(A) P_1^{(n_1)} P_2^{(n_2)}) \quad \text{(by Remark 2.10(i)).}
\]

Therefore \(P_1^{(n_1)}P_2^{(n_2)} = A :K(A) (A :K(A) (P_1^{(n_1)} P_2^{(n_2)}))\) and \(P_1^{(n_1)}P_2^{(n_2)} = (P_1A_{P_1} P_2^{(n_2)})^{n_1} \cap (P_2A_{P_2} P_2^{(n_2)})^{n_2} \cap (\bigcap_{Q \in \text{Ht}_1(A)} \{P_1, P_2\} A_Q)\) by (1).

\(\Box\)

Now enlarging the argument like the preceding Lemma 2.11 (with the escalation of “\(\{P_1, P_2\}\)” into “a finite subset of \(\text{Ht}_1(A)\)”), we have
Proposition 2.12. Let $A$ be a Krull domain. Then for a finite subset $\Delta$ of $\text{Ht}_1(A)$ and $n_P \in \mathbb{Z}$ ($P \in \Delta$),
\[
\prod_{P \in \Delta} P^{(n_P)} \quad \text{(each } P \text{ ranges in } \Delta \text{ just once)}
\]
\[
= \prod_{P \in \Delta} \left( (PA_P)^{n_P} \cap \left( \bigcap_{Q \in \text{Ht}_1(A) \setminus \{P\}} A_Q \right) \right)
\]
\[
= \left( \bigcap_{P \in \Delta} (PA_P)^{n_P} \right) \cap \left( \bigcap_{Q \in \text{Ht}_1(A) \setminus \Delta} A_Q \right),
\]
which is a divisorial fractional ideal of $A$.

Consequently, by Lemma 2.9 we have

Corollary 2.13. For a divisorial fractional ideal $I$ of a Krull domain $A$,
\[
I = \left( \bigcap_{P \in \text{Supp}^*(I)} (PA_P)^{v_P(I)} \right) \cap \left( \bigcap_{Q \in \text{Ht}_1(A) \setminus \text{Supp}^*(I)} A_Q \right)
\]
\[
= \prod_{P \in \text{Supp}^*(I)} \left( (PA_P)^{v_P(I)} \cap \left( \bigcap_{Q \in \text{Ht}_1(A) \setminus \{P\}} A_Q \right) \right)
\]
\[
= \prod_{P \in \text{Supp}^*(I)} P^{(v_P(I))}.
\]

From Corollary 2.13 we have the following result:

Theorem 2.14. Let $A$ be a Krull domain and let $I$ be a divisorial fractional ideal of $A$. Then
\[
IA_P = (PA_P)^{v_P(I)} = P^{(v_P(I))} A_P \quad (\forall P \in \text{Ht}_1(A)),
\]
and
\[
I = \left( \prod_{P \in \text{Supp}^*(I)} P^{(v_P(I))} \right)^{**} = \prod_{P \in \text{Supp}^*(I)} P^{(v_P(I))}
\]
\[
= \left( \prod_{P \in \text{Ht}_1(A)} P^{(v_P(I))} \right)^{**} = \prod_{P \in \text{Ht}_1(A)} P^{(v_P(I))},
\]
where $\prod$ denotes a (substantially finite) product of fractional ideals of $A$ in $K(A)$, noting here that $v_P(I) = 0$ for almost all $P \in \text{Ht}_1(A)$, that is, $\text{Supp}^*(I)$ is finite, and each $P$ ranges in $\text{Supp}^*(I)$ (resp. $\text{Ht}_1(A)$) just once. In other words, any divisorial fractional ideal is expressed uniquely (up to permutation) as a finite product of the symbolic powers of some distinct members in $\text{Ht}_1(A)$.

Corollary 2.15. Let $A$ be a Krull domain and let $I$ and $J$ be divisorial fractional ideals of $A$. If $\text{Supp}^*(I) \cap \text{Supp}^*(J) = \emptyset$, then $IJ$ is also a divisorial fractional ideal of $A$ and $\text{Supp}^*(IJ) = \text{Supp}^*(I) \cup \text{Supp}^*(J)$.

We recall the following example:
Example 2.16 ([15,(8.H)]). Let $k$ be field and let $B := k[x, y]$ be a polynomial ring in indeterminates $x$ and $y$. Put $A := k[x, xy, y^2, y^3] \subseteq B$ and $P := yB \cap A$. Then $P^2 = (x^2y^2, x^3y^4, y^5)$. Since $y = xy/x \in AP$, we have $B = k[x, y] \subseteq AP$ and $AP = B_{yB}$. Let $P^{(2)} := y^2B_{yB}/A = y^2B/A = (y^2, y^3)$, which is characterized as a (unique) $P$-primary component of $P^2$, and is called the 2nd-symbolic power of $P$ in [15]. It is clear that $P^{(2)} \neq P^2$. An irredundant primary decomposition of $P^2$ is given by $P^2 = (y^2, y^3) \cap (x^2, xy^3, y^4, y^5) = P^{(2)} \cap (x^2, xy^3, y^4, y^5)$, where $\text{ht}((x^2, xy^3, y^4, y^5)) = 2$. If $A$ were a Krull domain, $P^{(2)}$ could be the same as the one in our definition, but $A$ is not a normal domain ($A$ is not $(2,3)$-closed and hence not semi-normal).

Some arguments after [11]: Let $D(A)$ be the collection of non-zero divisorial fractional ideals of $A$, and define $\odot : D(A) \times D(A) \rightarrow D(A)$ by $(I, J) \mapsto A : K(A) (A : K(A) IJ)$, that is, $I \odot J = A : K(A) (A : K(A) IJ)$. (Note that $A : K(A) (A : K(A) IJ)$ if $\text{Supp}(I) \cap \text{Supp}(J) = \emptyset$.) With this operation $\odot$, $D(A)$ becomes an abelian group by [11,(3.4)] and [11,(3.6)] ($A$ is completely integrally closed), where the identity element in $D(A)$ is $A$ and the inverse of $I \in D(A)$ is $A : K(A) I$ because $A : K(A) (A : K(A) I(A : K(A) I)) = A$ ([11,p.13]), and moreover $D(A)$ is free on the primes in $Ht_1(A)$ ([11,(3.14)]). Then $I = A : K(A) (A : K(A) I) = A : K(A) (A : K(A) \prod_{P \in Ht_1(A)} P^{(n_P)}) = A : K(A) (A : K(A) \prod_{P \in \text{Supp}(I)} P^{(n_P)})$, where $n_P := v_P(I)$ (Theorem 2.14).

Considering the isomorphism $\text{div} : D(A) \rightarrow \text{Div}(A)$ from the (multiplicative) group of divisorial fractional ideals to the (additive) free group on the set $Ht_1(A)$ ($I \mapsto \sum_{P \in Ht_1(A)} v_P(I)P$) (cf.[11,p.27]) and the isomorphism $\text{div} : P(A) \rightarrow \text{Prin}(A)$ from the subgroup $P(A)$ of principal fractional ideals to its image $\text{Prin}(A)$ in $\text{Dvi}(A)$, the definitions of the divisor groups $D(A)$ and $\text{Div}(A)$, and the divisor class group $\text{Cl}(A) := D(A)/P(A)$ and $\text{Dvi}(A)/\text{Prin}(A)$ can be identified as abelian groups by these definitions, respectively. (For details, see [11,pp.12-29]). These are based on the facts that the symbolic $n$-th power $P^{(n)}$ of $P \in Ht_1(A)$ is a divisorial ideal and that any divisorial fractional ideal of $A$ is expressed as a finite product of some symbolic powers of prime ideals in $Ht_1(A)$.

The rest of this section will be devoted to preparation of Theorem 3.5 in the next section.

Remark 2.17 (cf.[11] or [14,§10-§12]). Let $R$ be a Krull domain and let $\Delta$ be a subset of $Ht_1(R)$, let $R_\Delta = \bigcap_{P \in \Delta} R_P$, a subintersection of $R$ and let $P \in Ht_1(R)$. Note first that any $P \in Ht_1(R)$ (resp. similar for $R_\Delta$) is a maximal divisorial prime ideal of $R$ (resp. $R_\Delta$) (cf.[11,(3.6)]+(3.12)] or [14,Ex(12.4)]) and consequently a defining family $Ht_1(R)$ (resp. $Ht_1(R_\Delta)$) of a Krull domain $R$ (resp. $R_\Delta$) is minimal among defining families of $R$ (resp. $R_\Delta$) (cf.[14,(12.3)], [11,(1.9)]) and that any DVRs $R_P$, $P \in Ht_1(R)$ (resp. $(R_\Delta)_Q$, $Q \in Ht_1(R_\Delta)$) are independent DVRs in $K(R)$ (resp. in $K(R_\Delta) = K(R)$), that is, $(R_P)_{P'} = K(R)$ for $P \neq P'$ (resp. similar
for $R_{\Delta}$ in $\text{Ht}_1(R)$ (resp. in $\text{Ht}_1(R_{\Delta})$) (cf. [11,(5.1)]), which gives indeed ‘The Approximation Theorem for Krull domains’ (Lemma 2.18).

Assume that $R \hookrightarrow R_{\Delta}$ is flat.

We can see in [11] that the following statements (i) $\sim$ (vi) hold:

(i) $R_{\Delta}$ is a Krull domain with $K(R_{\Delta}) = K(R)$ (cf. [11,(1.5)]), and $P' \cap R \in \text{Ht}_1(R)$ for any $P' \in \text{Ht}_1(R_{\Delta})$ (Lemma 2.19 below).

(ii) Any $P \in \text{Ht}_1(R)$ is a divisorial prime ideal of $R$. So $PR_{\Delta}$ is divisorial by [11,p.31] (or [11,(3.5)] and Lemma 2.19 below) according to the flatness of $R \hookrightarrow R_{\Delta}$. So we have by [11,(5.5)] (or Remark 2.18 below),

$$PR_{\Delta} = \begin{cases} PR_P \cap R_{\Delta} \neq R_{\Delta} & (P \in \Delta) \\ R_{\Delta} & (P \in \text{Ht}_1(R) \setminus \Delta) \end{cases}$$

Hence $PR_{\Delta}$ ($P \in \Delta$) is in $\text{Ht}_1(R_{\Delta})$.

(iii) $i : R \hookrightarrow R_{\Delta}$ (i.e., “$i : \text{Spec}(R_{\Delta}) \to \text{Spec}(R)$”) induces a bijection $\text{Ht}_1(R_{\Delta}) \to \Delta$ (cf. [11,(3.15)]), and $R_{\Delta} = \bigcap_{P' \in \text{Ht}_1(R_{\Delta})} (R_{\Delta})_{P'} = \bigcap_{P' \in \text{Ht}_1(R_{\Delta})} (R_{P' \cap R})$ by (i) and (ii). (Note that the first statement is not necessarily required that $R \hookrightarrow R_1$ is flat in this part.)

(iv) $R = R_{\Delta} \iff \text{Ht}_1(R) = \Delta$. (In fact, $\text{Ht}_1(R) \supseteq \Delta \iff R = \bigcap_{P' \in \text{Ht}_1(R)} R_{P'} \subseteq \bigcap_{P \in \Delta} R_P = \bigcap_{P' \in \text{Ht}_1(R_{\Delta})} (R_{\Delta})_{P'} = R_{\Delta}$ ($\subseteq K(R)$) by the minimality of their respective defining families. Note that it is not necessarily required that $R \hookrightarrow R_1$ is flat in this part.)

(v) For $P \in \Delta$ and $n \in \mathbb{Z}$, $P^{(n)}R_{\Delta} = (PR_{\Delta})^{(n)}$, where $P^{(n)}$ denotes the symbolic $n$-th power of $P$ (Definition 2.20).

(vi) $R^X = R \setminus \bigcup_{P \in \text{Ht}_1(R)} P$ and $(R_{\Delta})^X = R_{\Delta} \setminus \bigcup_{P \in \Delta} PR_{\Delta}$ ($\subseteq PR_{\Delta} R_{\Delta} \cap R_{\Delta} = PR_{\Delta}$, a prime ideal of $R_{\Delta}$ ($\forall P \in \Delta$)).

Remark 2.18. Let $A$ be a Krull domain. Then a fractional ideal $I$ of $A$ is divisorial if and only if $I = \bigcap_{P \in \text{Ht}_1(A)} IA_P$ (See [11,(5.5)]). In addition, let $B$ be a Krull domain containing $A$ such that $A \hookrightarrow B$ is flat. If $I$ is a divisorial fractional ideal of $A$, then $IB$ is also a divisorial fractional ideal of $B$ (See [11,p.31]). Note here that there exists an example of a non-flat subintersection of a Noetherian Krull domain (See [11,p.32]).

Lemma 2.19 ([11,(6.5)]). Let $A$ be an integral domain whose quotient field is $K$. Let $B$ be a ring between $A$ and $K$. Then $B$ is flat over $A$ if and only if $A_{M \cap A} = B_M$ for every maximal ideal $M$ of $B$.

Corollary 2.20. Let $R$ be an integral domain, let $\Delta \subseteq \Delta'$ be subsets of $\text{Ht}_1(R)$ and put $R_{\Delta'} := \bigcap_{Q \in \Delta'} R_Q$ and $R_{\Delta} := \bigcap_{Q \in \Delta} R_Q$. If $R \hookrightarrow R_{\Delta}$ is flat then $R_{\Delta'} \hookrightarrow R_{\Delta}$ is also flat.

Proof. It is easy to see that $R_{\Delta'} \subseteq R_{\Delta}$ and $K(R_{\Delta'}) = K(R_{\Delta}) = K(R)$. Since $R \hookrightarrow R_{\Delta}$ is flat, $R_{M \cap R} = (R_{\Delta'})_{M \cap R_{\Delta'}} = (R_{\Delta})_M$ for every maximal ideal $M$ of $R_{\Delta}$ by Lemma 2.19. So using Lemma 2.19 again, $R_{\Delta'} \hookrightarrow R_{\Delta}$ is also flat. \[\square\]
3. The Main Results, Conjectures(DJC) and (JCn)

In this section, we discuss Conjecture(DJC). To make sure, we begin with the following definitions.

**Definition 3.1** (Unramified, Étale). Let \( f : A \to B \) be a ring-homomorphism of finite type of Noetherian rings. Let \( P \in \text{Spec}(B) \) and put \( P \cap A := f^{-1}(P) \), a prime ideal of \( A \). The homomorphism \( f \) is called unramified at \( P \in \text{Spec}(B) \) if \( P B_P = (P \cap A)B_P \) and \( k(P) := B_P/PB_P \) is a finite separable field-extension of \( k(P \cap A) := A_P/(P \cap A)P \cap A \). If \( f \) is not unramified at \( P \), we say \( f \) is ramified at \( P \). The set \( R_f := \{ P \in \text{Spec}(B) \mid \text{\( a\) } f \text{ is ramified at } P \in \text{Spec}(B) \} \) is called the the ramification locus of \( f \), which is a closed subset of \( \text{Spec}(B) \). The homomorphism \( f \) is called étale at \( P \) if \( f \) is unramified and flat at \( P \). The homomorphism \( f \) is called unramified (resp. étale) if \( f \) is unramified (resp. étale) at every \( P \in \text{Spec}(B) \). The morphism \( \psi : \text{Spec}(B) \to \text{Spec}(A) \) is called unramified (resp. étale) if \( \psi \) is unramified (resp. étale).

**Definition 3.2** ((Scheme-theoretically or Algebraically) Simply Connected). A Noetherian ring \( R \) is called (algebraically or scheme-theoretically) simply connected if the following condition holds: Provided any 'connected' ring \( A \) (i.e., \( \text{Spec}(A) \) is connected) with a finite étale ring-homomorphism \( \varphi : R \to A \), \( \varphi \) is an isomorphism.

**Remark 3.3.** Let \( K \) be a field. It is known that there exists the algebraic closure \( \overline{K} \) of \( K \) (which is determined uniquely up to \( K \)-isomorphisms). Let \( K_{\text{sep}} \) denote the separable algebraic closure of \( K \) (i.e., the set consisting of all separable elements in \( \overline{K} \) over \( K \)). Note that \( \overline{K} \) and \( K_{\text{sep}} \) are fields. (See [23] for details.) Let \( K \to L \) is a finite algebraic extension field. We know that \( K \to L \) is étale \( \iff \) \( L \) is a finite separable \( K \)-algebra \( \iff \) \( L \) is a finite algebraic separable extension field of \( K \). So \( K \) is simply connected if and only if \( K = K_{\text{sep}} \) by Definition 3.2 and hence if \( K \) is algebraically closed, then \( K \) is simply connected. In particular, \( Q \) is not simply connected because \( Q \nsubseteq Q_{\text{sep}} = \overline{Q} \). But \( \mathbb{C} \) is simply connected because \( \mathbb{C} \) is algebraically closed.

---

\(^1\)In general, let \( X \) and \( Y \) be of locally Noetherian schemes and let \( \psi : Y \to X \) be a morphism locally of finite type. If for \( y \in Y \), \( \psi^*_y : O_{Y, \psi(y)} \to O_{Y, y} \) is unramified at \( y \), then \( \psi \) is called unramified at \( y \in Y \). The set \( R_{\psi} := \{ y \in Y \mid \psi^*_y \text{ is ramified} \} \subseteq Y \) is called the ramification locus of \( \psi \) and \( \psi(R_{\psi}) \subseteq X \) is called the branch locus of \( \psi \). Note that the ramification locus \( R_{\psi} \) defined here is often called the branch locus of \( \psi \) instead of \( \psi(R_{\psi}) \) in some texts (indeed, see e.g. [4] etc.).

\(^2\)In general, let \( X \) and \( Y \) be of locally Noetherian schemes and let \( \psi : Y \to X \) be a morphism locally of finite type. If \( \psi \) is finite and surjective, then \( \psi \) (or \( Y \)) is called a ramified cover of \( X \) (cf.[4,VI(3.8)]). If \( \psi \) covers \( X \), \( \psi \) is called an étale cover of \( X \). If every connected étale cover of \( X \) is isomorphic to \( X \), \( X \) is said to be (scheme-theoretically or algebraically) simply connected. Remark that if \( X \) is an algebraic variety over \( \mathbb{C} \) then \( \text{"X is a (geometrically) simply connected in the usual \( \mathbb{C} \)-topology \( \Rightarrow X \) is (algebraically) simply connected "} \), but in general the converse \( \text{" \( \Rightarrow \) "} \) does not hold. In this paper, 'simply connected' means (geometrically) simply connected'.

\(^3\)Let \( k \) be a field and \( A \) a \( k \)-algebra. We say that \( A \) is separable over \( k \) (or \( A \) is a separable \( k \)-algebra) if for every extension field \( k' \) of \( k \), the ring \( A \otimes_k k' \) is reduced (See [14,p.198].)
Remark 3.4. Let $k$ be an algebraically closed field and put $k[X] := k[X_1, \ldots, X_n]$, a polynomial ring over $k$.

(i) If $\text{char}(k) = 0$, then the polynomial ring $k[X]$ ($n \geq 1$) is simply connected (See [23]).

(ii) If $\text{char}(k) = p > 0$, then the polynomial ring $k[X]$ ($n \geq 1$) is not simply connected. (Indeed, for $n = 1$, $k[X_1 + X_1^p] \hookrightarrow k[X_1]$ is a finite étale morphism, but is not an isomorphism as mentioned before.)

(iii) An algebraically closed field $k$ is simply connected (See Remark 3.3). However we see that for a simply connected Noetherian domain $A$, a polynomial ring $A[X]$ is not necessarily simply connected (See the case of $\text{char}(A) = p > 1$).

Moreover any finite field $\mathbb{F}_q$, where $q = p^n$ for a prime $p \in \mathbb{N}$, is not simply connected because it is a perfect field.

Now we start on showing our main result.

The following theorem is a core result which leads us to a positive solution to Conjecture (DJC).

Theorem 3.5. Let $R$ be a Krull domain domain and let $\Delta_1$ and $\Delta_2$ be subsets of $\text{Ht}_1(R)$ such that $\Delta_1 \cup \Delta_2 = \text{Ht}_1(R)$ and $\Delta_1 \cap \Delta_2 = \emptyset$. Put $R_i := \bigcap_{Q \in \Delta_i} R_Q$ ($i = 1, 2$), subintersections of $R$. Assume that $\Delta_2$ is a finite set and that $R \hookrightarrow R_1$ is flat. Assume moreover that $R_1$ is factorial and that $R^\times = (R_1)^\times$. Then $R = R_1$.

(Note: If every $P \in \Delta_2$ is a principal (prime) ideal of $R$ and $R_1$ is factorial, it follows from Nagata’s Theorem [11, (7.1)] that $R$ is factorial, and our conclusion follows from the assumption $R^\times = (R_1)^\times$.)

Proof. We have only to show the following statements hold:

1. Every $Q_i \cap R$ ($Q_i \in \text{Ht}_1(R_1)$) is a principal ideal of $R$.
2. More strongly, $\Delta_2 = \emptyset$ and $R = R_1$.

Recall first that $R = R_1 \cap R_2$ and let $\Delta_2 = \{Q_1', \ldots, Q_r'\}$.

We will show that (1) and (2) hold under the assumption $\# \Delta_2 \leq 1$. Our conclusion for the finite set $\Delta_2$ is given by induction on $r = \# \Delta_2$. [Indeed, put $R'' := \bigcap_{Q \in \Delta''} R_Q$ (a Krull domain by Remark 2.11(i)), where $\Delta'' := \Delta_1 \cup \{Q_r'\}$. Then we see that $R \subseteq R'' \subseteq R_1$ with $R'' \hookrightarrow R_1$ being flat by Corollary 2.20 and that $R^\times = (R'')^\times = (R_1)^\times$. So we can consider $R''$ instead of $R$.]

If $\# \Delta_2 = 0$, then Theorem holds trivially.

Thus from now on we assume that $\Delta_2 = \{Q_\infty\}$ and $R_2 = R_{Q_\infty}$, and show that $Q_\infty$ cannot appear in $\text{Ht}_1(R)$ after all.

Then

$$R = R_1 \cap R_2 = \left( \bigcap_{Q \in \Delta_1} R_Q \right) \cap R_{Q_\infty}.$$ 

Note that $R_2 = R_{Q_\infty}$ is a DVR (factorial domain), and that $R \hookrightarrow R_{Q_\infty} = R_2$ is flat. Then we have the canonical bijection $\Delta_i \rightarrow \text{Ht}_1(R_i)$ ($\Delta_i \ni Q \mapsto QR_i \in$...
Ht₁(R₁) (cf. Remark 2.17(iii)). So for Q ∈ Ht₁(R), QR₁ is either a prime ideal of height one (if Q ∈ Δ₁) or R₁ itself (if Q ∈ Δ₂) for each i = 1, 2 (cf. Remark 2.17(ii)).

Let \( v_Q(\cdot) \) be the (additive) valuation on \( K(R) \) associated to the principal valuation ring \( R_Q(\subseteq K(R)) \) for Q ∈ Ht₁(R). Note here that for each Q ∈ Δ₁ (i = 1, 2), \( R_Q = (R₁)QR₁ \), and \( v_Q(\cdot) = v_{QR₁}(\cdot) \) by Remark 2.17(i),(ii),(iii).

**Proof of (1)**: Put \( P_s = Q_s \cap R \). Since \( Q_s \in Ht₁(R₁) \), \( P_s \in Δ₁ \) by Remark 2.17(i).

Apply Lemma 2.8 to \( R \) (or \( Ht₁(R₁) = Δ₁ \cup Δ₂ \) with \( Δ₂ \) (a finite set)). Then there exists \( t ∈ K(R) \) such that

\[ \forall j, v_{P_s}(t) = 1, v_{Q∞}(t) = 0 \text{ and } v_{Q_s^∞}(t) ≥ 0 \text{ otherwise } (\ast). \]

It is easy to see that \( t ∈ R \) and \( tR_{Q∞} = R_{Q∞} \) (i.e., \( t ∈ R \setminus Q∞ \)) by (\ast). Since \( R₁ \) is factorial, we have

\[ t = t₁^{n₁} \cdots t_s^{n_s} \quad (\exists s, n_j ≥ 1 \ (1 ≤ j ≤ s)) \quad (\ast\ast) \]

with some prime elements \( t_j \) (1 ≤ j ≤ s) in \( R₁ \), where each prime element \( t_j \) in \( R₁ \) is determined up to modulo \( (R₁)^s = R^s \).

Moreover \( v_Q(t) = n₁v_Q(t₁) + \cdots + n_nv_Q(t_n) \) for every \( Q ∈ Ht₁(R) \), where \( n_j = v_{Q_jR₁}(t) = v_Q(t) > 0 \) (1 ≤ j ≤ s) and \( v_{Q∞}(t) = 0 \). In particular,

\[ \sum_{i=1}^{s} n_iv_{Q∞}(t_i) = v_{Q∞}(t) = 0 \quad (\ast\ast\ast). \]

Put \( Q_j := t_j¹R₁ \cap R \) (1 ≤ j ≤ s). Then \( Q_jR₁ = t_j¹R₁ \) and \( Q_jR₁ \cap R = t_j¹R₁ \cap R = Q_j \) by Remark 2.17(i),(ii).

Let \( Δ' := \{ Q ∈ Ht₁(R) \mid v_Q(t) > 0 \} \). Then \( Δ' \) is a finite subset of \( Δ₁ \) by (\ast).

Considering that \( t_j¹ \) is a prime factor of \( t \) in \( R₁ \) and that \( (R₁)^s = R^s \), we have a one-to-one correspondences: for each \( j \) (1 ≤ j ≤ s),

\[ t_j¹R^s = t_j¹(R₁)^s \rightarrow t_j¹R₁ = Q_jR₁ \rightarrow t_j¹R₁ \cap R = Q_jR₁ \cap R = Q_j \quad (\#). \]

Note that for each \( j \), the value \( v_Q(t_j¹) \) (\( Q ∈ Ht₁(R) \)) remains unaffected by the choice of \( t_j¹ \) in (\ast\ast\ast).

Considering an irredundant primary decomposition of \( tR \) (\( tR \) is a Krull domain), we have \( tR = Q₁^{(n₁)} \cap \cdots \cap Q_s^{(n_s)} \cap Q_{∞}^{(m')} \) for some \( m' ∈ Z ≥ 0 \). Then \( m' = v_{Q∞}(t) = 0 \), and hence \( tR = Q₁^{(n₁)} \cap \cdots \cap Q_s^{(n_s)} \). Thus

\[ Δ' = \{ Q₁, \ldots, Q_s \} \]

by considering the one-to-one correspondences (\#) above. So \( Δ' \) consists of the prime divisors of \( tR \), and \( Q∞ ∉ Δ' \). Therefore considering Corollary 2.7(ii), we have

\[ tR_{Q∞} = R_{Q∞} \quad \text{and} \quad tR = Q₁^{(n₁)} \cap \cdots \cap Q_s^{(n_s)} = Q₁^{(n₁)} \cdots Q_s^{(n_s)} \quad (n_j = v_{Q_j}(t) > 0). \]
Thus we have \( \sum_{i=1}^{s} n_i [Q_i] = 0 \) in \( \text{Cl}(R) \) \((n_i > 0 \ (1 \leq i \leq s))\) since \( \sum_{i=1}^{s} n_i v_{Q_i}(t'_i) = v_{Q_\infty}(t) = 0 \) as mentioned above. We see Nagata’s Theorem[11,(7.1)] : the divisor class group \( \text{Cl}(R) = \mathbb{Z} \cdot [Q_\infty] \).

(1-1) Now we shall show that \( Q_j = t'_j R \) for every \( j \) \((1 \leq j \leq s)\).

Since \( t'_j R \) is a divisorial fractional ideal of \( R \), we have \( t'_j R = \prod_{Q \in \text{Ht}_1(R)} Q^{(v_{Q_\infty}(t'_j))} \) by Theorem 2.14 where \( v_Q(t'_j) = 0 \) for almost all \( Q \in \text{Ht}_1(R) \). Note that \( \text{Ht}_1(R) = \Delta_1 \cup \{ Q_\infty \} \) and that \( R \hookrightarrow R_1 \) is flat with \( R_{Q_j} = (R_1)_{Q_j R_1} \) and that \( t'_j R_1 = Q_j R_1 = R_1 \) \((1 \leq j \leq s)\), \( Q_\infty R_1 = R_1 \) and \( Q_j \neq Q_i \) \((\forall i \neq j)\). Thus noting that \( t'_j \) does not belong to \( Q_i R_1 \) \((i \neq j)\), we have \( Q_j R_1 = t'_j R_1 = Q_j^{(v_{Q_\infty}(t'_j))} R_1 = (Q_j R_1)^{(v_{Q_\infty}(t'_j))}, \) and consequently \( v_{Q_j}(t'_j) = 1 \).

Hence we have

\[
 t'_j R = Q_j^{(v_{Q_\infty}(t'_j))} = Q_j Q_\infty^{(v_{Q_\infty}(t'_j))},
\]

which also means that both \( Q_j \) and \( Q_\infty^{(v_{Q_\infty}(t'_j))} \) are invertible fractional ideals of \( R \). Thus by Lemma 2.8 we have

\[
 t'_j Q_\infty^{-(v_{Q_\infty}(t'_j))} = Q_j (\subseteq R) \quad (\star).
\]

Then we have \( t'_j Q_\infty^{-(v_{Q_\infty}(t'_j))} R_{Q_\infty} = Q_j R_{Q_\infty} = R_{Q_\infty} \). Hence there exists \( x_j \in Q_\infty^{-(v_{Q_\infty}(t'_j))} \) such that \( v_{Q_\infty}(t'_j x_j) = 0 \). Then

\[
 t'_j x_j R_{Q_\infty} = R_{Q_\infty} \quad \text{and hence} \quad x_j R_{Q_\infty} = (Q_\infty R_{Q_\infty})^{-v_{Q_\infty}(t'_j)}. \]

So \( x_j R \subseteq Q_\infty^{-(v_{Q_\infty}(t'_j))} = (Q_\infty R_{Q_\infty})^{-v_{Q_\infty}(t'_j)} \cap (\bigcap_{Q \in \text{Ht}_1(R) \setminus \{ Q_\infty \}} R_Q) \) \((\text{cf. Proposition 2.8})\), which yields

\[
 t'_j x_j R \leq t'_j x_j R_{Q_\infty} \cap t'_j (\bigcap_{Q \in \text{Ht}_1(R) \setminus \{ Q_\infty \}} R_Q) = t'_j x_j R_{Q_\infty} \cap (\bigcap_{Q \in \text{Ht}_1(R) \setminus \{ Q_\infty \}} t'_j R_Q) = t'_j R_{Q_\infty} \cap (\bigcap_{Q \in \text{Ht}_1(R) \setminus \{ Q_j \}} R_Q) = t'_j R
\]

\((\because R_Q = (R_1)_{Q R_1} \ (\forall Q \in \text{Ht}_1(R) \setminus \{ Q_\infty \}) \) and \( t'_j R \) is divisorial \((\text{See \S 2 for detail})\). Thus \( t'_j x_j R \subseteq t'_j R \). Whence \( x_j \in R \) \((1 \leq \forall j \leq s)\).

It follows that \( v_{Q_\infty}(t'_j) \leq 0 \) because \( v_{Q_\infty}(t'_j x_j) = 0 \) and \( v_{Q_\infty}(x_j) \geq 0 \). Since \( \sum_{j=1}^{s} n_j v_{Q_\infty}(t'_j) = 0 \) by \((***)\) together with \( n_j > 0 \) \((1 \leq \forall j \leq s)\), we have \( v_{Q_\infty}(t'_j) = 0 \) \((1 \leq \forall j \leq s)\).
Therefore according to (●), we have \( Q_j = t'_j Q^\infty \) (1 \( \leq \) \( \forall \) \( j \leq s \)), that is,

\[
Q_j = t'_j R \quad (1 \leq \forall j \leq s).
\]

\textbf{(1-2)} Now since \( P_1 \in \Delta_1 \) by (⋆) and \( 1 = v_{P_1}(t) = v_{P_1 R_1}(t) = \sum v_{P_1 R_1_i}(t'_j) \) for some prime elements \( P_1 \), there exists \( i \) such that \( v_{P_1}(t'_i) = v_{P_1 R_1}(t'_i) = 1 \) with \( n_i = 1 \) and \( v_{P_1}(t'_i) = 0 \) for \( \forall j \neq i \), say \( i = 1 \), and then \( P_1 = Q_1 \in \Delta_1 \).

Therefore from (1-1), \( P_1 = Q_1 \cap R \) is a principal ideal \( t'_1 R \) of \( R \).

(Though it may be redundant, we can add something natural to the last argument: using \( t'_1 \in (R Q_\infty) \times (1 \leq \forall j \leq s) \) above, it follows that \( P_1 = P_1 (R_1 \cap R Q_\infty) \subseteq P_1 R_1 \cap P_1 R Q_\infty = t'_1 R_1 \cap t'_1 R Q_\infty = t'_1 R_1 \cap t'_1 R Q_\infty = t'_1 R_1 \cap R Q_\infty = t'_1 R \cap R R Q_\infty = t'_1 R \subseteq R \). Since \( P_1 \in \text{Ht}_1 \), we have \( P_1 = t'_1 R \).)

\textbf{Proof of (2) :} \textbf{Suppose} that \( \Delta_2 \neq \emptyset \) and so that \( \Delta_2 = \{ Q_\infty \} \) (See the first paragraph of \textit{Proof}).

We divide the proof of (2) into the following two cases.

\textbf{(2-1)} Consider the case that \( Q_\infty \not\subseteq \bigcup_{P \in \Delta_1} P \). Take \( t' \in Q_\infty \) such that \( t' \not\in \bigcup_{P \in \Delta_1} P = \bigcup_{P \in \Delta_1} PR_1 \cap R \). Then \( t' \in R \) is a unit in \( R_1 \). Therefore \( t' \in (R_1)^\times = R^\times \), \textit{a contradiction}.

\textbf{(2-2)} Consider the case that \( Q_\infty \subseteq \bigcup_{P \in \Delta_1} P \) (with \( Q_\infty \not\subseteq P \) \( \forall P \in \Delta_1 \)) (\( \therefore Q_\infty \not\subseteq \Delta_1 \)). Take \( t' \in Q_\infty \) such that \( t' R Q_\infty = Q_\infty R Q_\infty \). Since \( t' \in \bigcup_{P \in \Delta_1} P \subseteq \bigcup_{P \in \Delta_1} PR_1 \) by Remark 2.17(iii), we have \( t' \in Q_\infty \subseteq \bigcup_{P \in \Delta_1} PR_1 \) (which is contained in the set of the non-units in \( R_1 \)), whence \( t' \) is a non-unit in \( R_1 \), that is, \( t' \not\in (R_1)^\times = R^\times \). Thus we have an irredundant primary decomposition \( t' R = P_1^{(m_1)} \cap \cdots \cap P_s^{(m_s)} \cap Q_\infty \) for some \( s \geq 1 \), \( m_i \geq 1 \) and \( P_i \in \Delta_1 (1 \leq i \leq s) \) with \( P_i \not= P_j \) (\( i \neq j \)). It is clear that \( P_i \not= Q_\infty \) (\( i = 1, \ldots, s' \)). Hence noting that \( R \not\rightarrow R_1 \) is a flat subintersection of \( R \) and \( R_1 \) is factorial, we have \( t' R_1 = P_1^{(m_1)} R_1 \cap \cdots \cap P_s^{(m_s)} R_1 \cap Q_\infty R_1 = P_1^{(m_1)} R_1 \cap \cdots \cap P_s^{(m_s)} R_1 = a_1^{m_1} \cdots a_s^{m_s} R_1 \) for some prime elements \( a_1 \) in \( R_1 \) with \( a_1 R_1 = P_1 R_1 \) because \( Q_\infty R_1 = R_1 \) and \( P_i R_1 \not= R_1 \) (\( 1 \leq \forall i \leq s' \)) (cf.\textit{Remark} 2.17(iii) and (iv)). Thus we may assume that \( t' = a_1^{m_1} \cdots a_s^{m_s} \) in \( R_1 \).

Since each \( P_i \in \Delta_1 \) is a principal ideal \( a_i R_1 \) of \( R \) generated by a prime element \( a_i \in R \) by the preceding argument \( (1) \), we have \( a_i R_1 = P_i R_1 = a_i R_1 \) and hence both \( a_i / a_i' \) and \( a'_i / a_i \) belong to \( R_1 \). Thus \( a'_i / a_i \in (R_1)^\times = R^\times \). So we can assume that \( a_i \in P_i \subseteq R \) and \( a_i R = P_i \) for all \( i \). Since \( t' = a_1^{m_1} \cdots a_s^{m_s} \in Q_\infty \), \( a_i \in Q_\infty \) (\( \exists i \)) and \( P_i = a_i R \subseteq Q_\infty \) Thus \( P_i = Q_\infty \in \Delta_1 \cap \Delta_2 = \emptyset \), \textit{a contradiction}.

Therefore in any case, we conclude that \( \Delta_2 = \emptyset \) and \( R = R_1 \). \hfill \Box

Here we emphasize the result in \textit{Theorem} 3.6 as follows.

\textbf{Corollary 3.6.} \textbf{Let} \( R \) \textbf{be a Krull domain domain and let} \( \Gamma \) \textbf{be a finite subset of} \textit{Ht}_1 \textit{(R)}. \textbf{Assume} \textbf{that a subintersection} \( R_0 := \bigcap_{P \in \textit{Ht}_1 \textit{(R)} \setminus \Gamma} R_0 \) \textbf{is factorial, that} \( (R_0)^\times = R^\times \) \textbf{and that} \( R \not\rightarrow R_0 \) \textbf{is flat}. \textbf{Then} \( \Gamma = \emptyset \) \textbf{and} \( R = R_0 \).
Proof. Putting $\Delta_1 := \text{Ht}_1(R) \setminus \Gamma$ and $\Delta_2 := \Gamma$, we can apply Theorem 3.5 to this case, and we have our conclusion. \hfill \Box

Now we know the following lemma:

**Lemma 3.7 ([11, (6.6)])**. A flat extension of a Krull domain within its quotient field is a subintersection.

The following proposition gives us a chance of a fundamental approach to the Deep Jacobian Conjecture (DJC).

**Proposition 3.8.** Let $i : C \hookrightarrow B$ be Noetherian normal domains such that $i^* : \text{Spec}(B) \to \text{Spec}(C)$ is an open immersion. If $B$ is factorial and $C^\times = B^\times$, then $C = B$.

**Proof.** Note first that $C$ is a Krull domain because a Noetherian normal domain is completely integrally closed (See [11, (3.13)]). Since $i : C \hookrightarrow B$ is flat and $K(C) = K(B)$, $B$ is a subintersection $C_\Gamma = \bigcap_{P \in \text{Ht}_1(C)} P C$ with a finite subset $\Gamma$ of $\text{Ht}_1(C)$ by Lemma 3.7. Therefore it follows from Corollary 3.6 that $\Gamma = \emptyset$ and $C = B$. \hfill \Box

**Corollary 3.9.** Let $i : A \hookrightarrow B$ be a quasi-finite homomorphism of Noetherian normal domains such that $K(B)$ is finite separable algebraic over $K(A)$. If $B$ is factorial and $i$ induces an isomorphism $A^\times \to B^\times$ of groups, then $i : A \hookrightarrow B$ is finite.

**Proof.** Let $C$ be the integral closure of $A$ in $K(B)$. Then $A \hookrightarrow C$ is finite and $C \hookrightarrow B$ is an open immersion by Lemma A.7 and Lemma A.11. So $C$ is a Noetherian normal domain and $C$ is a subintersection of $B$ by Lemma 3.7. Since $A^\times = C^\times = B^\times$, we have $C = B$ by Proposition 3.8. Therefore we conclude that $i : A \hookrightarrow C = B$ is finite. \hfill \Box

Here is our main result as follows.

**Theorem 3.10 (The Deep Jacobian Conjecture (DJC)).** Let $\varphi : S \to T$ be an unramified homomorphism of Noetherian normal domains with $T^\times = \varphi(S^\times)$. Assume that $T$ is factorial and that $S$ is an (algebraically) simply connected domain. Then $\varphi$ is an isomorphism.

---

1Let $f : A \to B$ be a ring-homomorphism of finite type. Then $f$ is said to be quasi-finite at $P \in \text{Spec}(B)$ if $B_P/(P \cap A)B_P$ is a finite dimensional vector space over the field $k(P \cap A) := A_{P \cap A}/(P \cap A)A_{P \cap A}$. We say that $f$ is quasi-finite over $A$ if $f$ is quasi-finite over $A$ at every point in $\text{Spec}(B)$. Equivalently, for every $p \in \text{Spec}(A)$, the fiber ring $B \otimes_A k(p)$ is finite over $k(p)$, where $k(p) := A_p/p A_p$. Note that $\text{Spec}(B \otimes_A k(p))$ is the fiber over $p$, where $a f : \text{Spec}(B) \to \text{Spec}(A)$ (cf. [14, p.47 and p.116], [21, pp.40-41]). For a fiber $a f^{-1}(p)$ with $p \in \text{Spec}(A)$, its fiber ring $B \otimes_A k(p)$ has possibly a non-trivial nilpotent element. In general, let $X$ and $Y$ be schemes and $\varphi : X \to Y$ a morphism locally of finite type. Then $\varphi$ is said to be quasi-finite if, for each point $x \in X$, $O_x/m_{\varphi(x)}O_x$ is a finite dimensional vector space over the field $k(\varphi(x))$. In other words, the fiber $X_y := X \times_Y \text{Spec}(k(y))$ is finite over $\text{Spec}(k(y))$ ($\forall y \in Y$). In particular, a finite morphism and an unramified morphism are quasi-finite (cf. [21], [4.VI(2.1)]).
Proof. Note first that \( \varphi : S \to T \) is an étale (and hence flat) homomorphism by Lemmas \( \text{A.12} \) and \( \text{A.10} \) and that \( \varphi \) is injective by Lemmas \( \text{A.5} \) and \( \text{A.3} \). We can assume that \( \varphi : S \to T \) is the inclusion \( S \hookrightarrow T \). Let \( C \) be the integral closure of \( S \) in \( K(T) \). Then \( S \hookrightarrow C \) is finite and \( C \) is a Noetherian normal domain by Lemma \( \text{A.7} \) since \( K(T) \) is a finite separable (algebraic) extension of \( K(S) \) and \( C \hookrightarrow T \) is an open immersion by Lemma \( \text{A.11} \) with \( S^\times = C^\times = T^\times \). Thus we have \( C = T \) by Corollary \( 3.9 \). So \( S \hookrightarrow C = T \) is étale and finite, and hence \( S = T \) because \( S \) is (algebraically) simply connected. \( \square \)

**Corollary 3.11.** Let \( k \) be a field of characteristic 0 and let \( \psi : V \to W \) be an unramified morphism of simply connected \( k \)-affine varieties whose affine rings \( K[V] \) and \( K[W] \). If \( K[W] \) is normal and \( K[V] \) is factorial, then \( \psi \) is an isomorphism.

**Proof.** We may assume that \( k \) is an algebraically closed field. By the simple connectivity, we have \( K[V]^\times = K[W]^\times = k^\times \) by Proposition \( \text{A.3} \). So our conclusion follows from Theorem \( \text{A.10} \). \( \square \)

On account of Remark \( \text{A.1} \) Corollary \( 3.11 \) resolves The Jacobian Conjecture (\( JC_n \)) as follows:

**Corollary 3.12 (The Jacobian Conjecture (\( JC_n \))).** If \( f_1, \ldots, f_n \) are elements in a polynomial ring \( k[X_1, \ldots, X_n] \) over a field \( k \) of characteristic 0 such that \( \det(\partial f_i/\partial X_j) \) is a nonzero constant, then \( k[f_1, \ldots, f_n] = k[X_1, \ldots, X_n] \).

**Example 3.13 (Remark).** In Theorem \( \text{B.10} \) the assumption \( T^\times \cap S = S^\times \) seems to be sufficient. However, the following Example implies that it is not the case. It seems that we must really require at least such strong assumptions that \( T \) is simply connected or that \( S^\times = T^\times \) as a certain mathematician pointed out.

Let \( S := \mathbb{C}[x^3 - 3x] \), and let \( T := \mathbb{C}[x, 1/(x^2 - 1)] \). Then obviously \( \text{Spec}(T) \to \text{Spec}(S) \) is surjective and \( T^\times \cap S = S^\times = \mathbb{C}^\times \), but \( T \) is not simply connected and \( T^\times \not\supset S^\times \). Since \( S = \mathbb{C}[x^3 - 3x] \to \mathbb{C}[x] := C \) is finite, indeed \( C[x] \) is the integral closure of \( S \) in \( K(\mathbb{C}[x]) \). Note here that \( S, C \) and \( T \) are factorial but that \( T^\times \neq C^\times = \mathbb{C}^\times \). Since \( \frac{\partial(x^3 - 3x)}{\partial x} = 3(x - 1)(x + 1) \), \( T \) is unramified (indeed, étale) over \( S \) (by Lemma \( \text{A.10} \)).

Precisely, put \( y = x^3 - 3x \). Then \( S = \mathbb{C}[y], \quad C = \mathbb{C}[x] \) and \( T = \mathbb{C}[x, 1/(x - 1), 1/(x + 1)] \). It is easy to see that \( y - 2 = (x + 1)^2(x - 2) \) and \( y + 2 = (x - 1)^2(x + 2) \) in \( C = \mathbb{C}[x] \). So \((x + 1)C \cap S = (y - 2)S \) and \((x - 1)C \cap S = (y + 2)S \). Since \( T = C_{x^2 - 1} = \mathbb{C}[x]_{x^2 - 1}, (x - 2)T = (y - 2)T \) and \((x + 2)T = (y + 2)T \), that is, \( y + 2, y - 2 \not\in T^\times \). It is easy to see \((y - b)T \neq T \) for any \( b \in \mathbb{C} \), which means that \( S \hookrightarrow T \) is faithfully flat and \( T^\times \cap S = S^\times = \mathbb{C}^\times \).

**Remark 3.14.** We see the following result of K. Adjamagbo (cf. [10], (4.4.2)) and [3]): Let \( k \) be an algebraically closed field of characteristic 0. Let \( f : V \to W \) be an injective morphism between irreducible \( k \)-affine varieties of the same dimension. If \( K[W] \), the coordinate ring of \( W \), is factorial then there is equivalence between
(i) \( f \) is an isomorphism and (ii) \( f^*: K[W] \to K[V] \) induces an isomorphism \( K[W]^\times \to K[V]^\times \).

This is indeed interesting and is somewhat a generalization of Case(1) in Introduction. But “the factoriality of \( K[W] \)” seems to be a too strong assumption (for the Deep Jacobian Conjecture(DJC)). In our paper, we dealt with the case that \( K[V] \) is factorial instead of \( K[W] \) (see Theorem 3.5).

Finally we show the following, which is probably interesting from another point of view though it is only a corollary to Proposition 3.8.

**Theorem 3.15.** Let \( k \) be a field of characteristic 0. Let \( X \) be a \( k \)-affine (irreducible) variety of dimension \( n \). Then \( X \) contains a \( k \)-affine open subvariety \( U \) which is isomorphic to a \( k \)-affine space \( \mathbb{A}^n_k \) if and only if \( X = U \cong \mathbb{A}^n_k \).

In other words, a \( k \)-affine variety \( X \) contains a \( k \)-affine space as an open \( k \)-subvariety if and only if \( X \) is a \( k \)-affine space.

(Note here that we say that a \( k \)-variety is a \( k \)-affine space if it is isomorphic to \( \mathbb{A}^n_k \) for some \( n \)).

**Proof.** We may assume that \( k = \mathbb{C} \). We have only to show ” only if ”.

If \( X \) contains an open \( \mathbb{C} \)-affine space \( U \), then \( \pi_1(U) \to \pi_1(X) \) is surjective (cf. [12]). The affine space \( \mathbb{A}^n_k \) is simply connected by Proposition A.2.

(1) Suppose that \( X \) is normal. In this case, \( X \) is also simply connected and hence \( K[U]^\times = \mathbb{C}^\times = K[X]^\times \) by Proposition A.3 and our conclusion follows from Proposition 3.8 immediately.

(2) Let \( X \) be a \( k \)-affine variety. Let \( \tilde{X} \) and \( \tilde{U} \) be the normalizations of \( X \) and \( U \) with \( K(\tilde{X}) = K(X) \), respectively. Then \( \tilde{U} = U \) and \( K(\tilde{X})^\times = K(U)^\times = \mathbb{C}^\times = K[U]^\times = K[X]^\times \). Since \( \tilde{X} \) is a normal \( k \)-affine (irreducible) variety, if \( X \) contains a \( k \)-affine open subvariety \( U \) which is isomorphic to a \( k \)-affine space \( \mathbb{A}^n_k \) then \( \tilde{X} \) contains a \( k \)-affine open subvariety \( U = \tilde{U} \) which is isomorphic to a \( k \)-affine space \( \mathbb{A}^n_k \). Thus \( \tilde{X} = \tilde{U} = U \cong \mathbb{A}^n_k \) by (1), and hence \( \tilde{X} = \tilde{U} = U \subseteq X \), which is a finite open immersion. So \( X \) is normal by Lemma A.12. Thus our conclusion follows from (1), that is, \( X = U \).

Concerning Theorem 3.15 we will discuss some comments in Section 4.

**Remark 3.16.** (1) Let \( R \) be a normal \( \mathbb{C} \)-affine domain. If \( R^\times \neq \mathbb{C}^\times \) then \( \text{Spec}''(R) \) is not simply connected. In particular, if \( R^\times \neq \mathbb{C}^\times \), then \( \text{Spec}''(R) \) does not contain a simply connected open \( \mathbb{C} \)-affine subvariety.

(2) Let \( V \) be a normal \( \mathbb{C} \)-affine variety.

(2-i) If \( V \) has a non-constant invertible regular function on \( V \), then \( V \) is not simply connected. We know that the affine-space \( \mathbb{A}^n_{\mathbb{C}} \) is simply connected by Proposition A.2. So if \( V \) is a simply connected, then by Proposition A.3 every invertible regular functions on \( V \) is constant.

(2-ii) If there exists a simply connected open \( \mathbb{C} \)-variety \( U \) of \( V \), then the canonical morphism \( U \to V \) induces a surjective homomorphism \( \pi_1(U) \to \pi_1(V) \) of
multiplicative groups (cf. [12]) and hence $1 = \pi_1(U) = \pi_1(V)$, that is, $V$ is simply connected. So $K[U]^\times = K[V]^\times = \mathbb{C}^\times$ by Proposition A.3.

--- SUPPLEMENT ---

We would like to consider the following Question (SC) which could yield a simpler solution to The Jacobian Conjecture if it has a positive answer.

This is regarded as a purely topological approach to $(JC_n)$. The problem is whether the following Question is true or not.

**Question (SC)**: Let $X$ be a normal $\mathbb{C}$-affine variety and let $F$ be a hypersurface in $X$. If $X$ is simply connected and $X \setminus F$ is a $\mathbb{C}$-affine subvariety, then is $X \setminus F$ not simply connected?

**NOTE 1**: Let $\mathbb{P}^n_\mathbb{C}$ denote the projective space and let $F$ be a hypersurface in $\mathbb{P}^n_\mathbb{C}$. Then $\mathbb{P}^n_\mathbb{C} \setminus F$ is simply connected if $F$ is a hyperplane, and is not simply connected if $F$ is a hypersurface (possibly reducible) except a hyperplane (Corollary A.15).

**NOTE 2**: Let $U \hookrightarrow V$ be an open immersion of normal $\mathbb{C}$-affine varieties. Then $V \setminus U$ is a hypersurface (possibly reducible). If $U$ is simply connected, then $V$ is also simply connected. Thus $K[V]^\times = K[U]^\times = \mathbb{C}^\times$.

(Note that the simple-connectivity of $U$ gives that of $V$. Indeed, it is known that $\pi_1(U, p_0) \to \pi_1(V, p_0)$ ($p_0 \in U$) is surjective by forgetting unnecessary loops around the hypersurface (possibly reducible) $V \setminus U$.)

If Question (SC) has a positive answer, then we have $U = V$. So in this case, the following Problem has a positive solution:

**Problem (SC-GJC)**. Let $\varphi : X \to Y$ be an unramified morphism of normal $\mathbb{C}$-affine varieties. If both $X$ and $Y$ are simply connected, then $\varphi$ is an isomorphism.

Since $Y$ is normal, $\varphi$ is étale by Lemma A.10. So a proof is obtained immediately by use of Zariski’s Main Theorem (Lemma A.11) once Question (SC) is answered positively.

4. About A Certain Example Concerning Theorem 3.15

We observe some comments about “Example” in [10, (10.3) in p.305] (See below) which could be possibly a counter-example to Theorem 3.15 and Theorem 3.5. So we discuss it for a while. The argument here is independent of Sections 2 and 3.

Though we should be going without saying, we prepare some notations for our purpose.
Notations: For a $C$-affine domain $R$ and $I$ its ideal, $\text{Spec}^m(R)$ denotes the maximal-spectrum of $R$, and $V^m(I)$ denotes $V(I) \cap \text{Spec}^m(R) = \{ M \in \text{Spec}^m(R) \mid I \subseteq M \}$. It is known that if $C[z_1, \ldots, z_n]$ is a polynomial ring, then the correspondence $\text{Spec}^m(C[z_1, \ldots, z_n]) \ni M = (z_1 - a_1, \ldots, z_n - a_n) \leftrightarrow C[a_1, \ldots, a_n] \in C^n$ induces the isomorphism $\text{Spec}^m(C[z_1, \ldots, z_n]) \cong C^n$ as $C$-varieties, and for an ideal $J$ of $C[z_1, \ldots, z_n]$, $V^m(J) = V^m(\sqrt{J})$ corresponds to a (closed) algebraic set $\{(a_1, \ldots, a_n) \in C^n \mid g(a_1, \ldots, a_n) = 0 (\forall g \in J)\}$ of $C^n$ (Hilbert’s Nullstellensatz [10,(A.5.2)]), which can be identified.

Let $R$ be an integral domain with quotient field $K$ and let $I$ be an ideal of $R$. The set $S(I; R) := \{ f \in K \mid fI^n \subseteq R \ (\exists n \in \mathbb{Z}_{\geq 0}) \}$, which is an integral domain containing $R$. For any integer $n \geq 0$, set $I^{-n} := \{ f \in K \mid fI^n \subseteq R \}$. Then $S(I; R) = \bigcup_n I^{-n}$. We call $S(I; R)$ an $I$-transformation of $R$, and abbreviate $S(I; R)$ to $S$ when there is no confusion. We say that $S(I; R)$ is finite if $S(I; R) = R/I^{-n}$ for some $n$. (See [16, Ch. V]).

Lemma 4.1 ([16, Theorem 3’, Ch. V]). Let $k$ be a field. Let $X$ be a $k$-affine variety defined by a $k$-affine domain $R$ and let $V$ be a closed set defined by an ideal $I$ of $R$. Then the open subset $X \setminus V$ of $X$ is $k$-affine if and only if $1 \in IS$, where $S$ is the $I$-transform of $R$. In this case, $V$ is pure of codimension 1 and $S$ is finite, that is, the $k$-affine domain of $X \setminus V$.

Note here that $\mathbb{P}^1_k$ is decomposed into $A^1_k \cup A^1_k$ with the certain glueing of two affine lines $A^1_k$ and that the projective line $\mathbb{P}^1_k$ is not $k$-affine indeed.

The following is a corollary to Theorem 3.15. We will prove it by another direct argument (without Theorem 3.15).

Proposition 4.2. Let $X$ be a $C$-affine variety with $\dim(X) = n$ and let $U_1, U_2$ be open $C$-subvarieties of $X$ such that $X = U_1 \cup U_2$ and both $U_1$ and $U_2$ are isomorphic to $A^n_C$. Then $X = U_1 = U_2$.

Proof. Let $K[X]$ denote the coordinate ring of $X$, which is $C$-affine domain. Then $X \setminus U_j = V^m(I_j)$ for an ideal $I_j$ of $K[X]$ ($j = 1, 2$). By Lemma 4.1 every prime divisor of $I_1$ and $I_2$ is of height 1. Since $X = U_1 \cup U_2 = (X \setminus V^m(I_1)) \cup (X \setminus V^m(I_2)) = X \setminus V^m(I_1 + I_2)$, which implies $I_1 + I_2 = K[X]$, that is, $V^m(I_1) \cap V^m(I_2) = \emptyset$. Thus $V^m(I_1) \subseteq U_2 = C^n$ and $V^m(I_2) \subseteq U_1 = C^n$, where $\approx$ means homeomorphic in the usual $C$-topology. Therefore $U_1 \setminus V^m(I_2)$ has a non-contractible loop around the hypersurface $V^m(I_2)$ ([9, Prop(4.1.4)]), and hence $X \setminus V^m(I_2) = (U_1 \setminus V^m(I_2)) \cup V^m(I_1)$ has a non-contractible loop around $V^m(I_2)$ (cf. [12, Lemma 4.4]).

\footnote{Let $f : X \to Y$ be a map between topological spaces. $f$ is called continuous if $f^{-1}(U)$ is open in $X$ for every open subset $U$ of $Y$, and $f$ is called a homeomorphism if $f$ is a bijective continuous map and is an open map (i.e., $f$ maps every open subset of $X$ to an open subset of $Y$), that is, there exists a continuous map $g : Y \to X$ such that $f \circ g = id_Y$ and $g \circ f = id_X$. Two topological spaces are called homeomorphic if there exists a homeomorphism between them.}
in §1). However, \( X \setminus V^m(I_2) = U_2 \approx \mathbb{C}^n \) is simply connected, which is absurd if \( V^m(I_2) \neq 0 \). So \( U_1 = X \). By symmetric argument, we have \( U_2 = X \). □

By the way, we find an Example sited in [10,(10.3) in p.305] as follows:

**Example** (An open embedding of \( \mathbb{C}^2 \) in a 2-dimensional \( \mathbb{C} \)-affine variety).

Let \( X := \{(s,t,u) \in \mathbb{C}^3 \mid su - t^2 + t = 0\} \), a closed algebraic set in \( \mathbb{A}^3_{\mathbb{C}} = \mathbb{C}^3 \) and let \( F: \mathbb{C}^2 \to X \) be a polynomial map given by

\[
F(x,y) := (y, xy, x^2y - x).
\]

Then \( F \) is an open embedding (open immersion) of \( \mathbb{C}^2 \) in (into) \( X \).

More precisely

\[
(F) \colon \mathbb{C}^2 \cong \{(s,t,u) \in X \mid s \neq 0\} \cup \{(s,t,u) \in X \mid t - 1 \neq 0\}.
\]

**Note.** A word ‘embedding’ is defined in [10,(5.3.1)] and a word ‘open immersion’ is defined in [10,p.285], but we can not find a word ‘open immersion’. In **Example**, the parts ( ) are the author’s interpretations.

This example fascinates us unbelievably, however it could be possibly a counter-example to Proposition 3.15 and Theorem 5.5.

To make sure, we will check Example above in more detail.

Suppose that **Example** above is valid.

It is easy to see that \( X \) is a non-singular \( \mathbb{C} \)-affine variety by the Jacobian criterion. So its ring \( \mathbb{C}[s, \bar{t}, \bar{u}] \) of regular functions on \( X \) is a regular domain and hence is a locally factorial domain, where \( \bar{s}, \bar{t}, \bar{u} \) is the images of \( s, t, u \) by the canonical homomorphism \( \mathbb{C}[s,t,u] \to \mathbb{C}[s,t,u]/(su - t^2 + t) = \mathbb{C}[\bar{s}, \bar{t}, \bar{u}] \). (Thus any \( P \in H^1_{\mathbb{C}}(\mathbb{C}[\bar{s}, \bar{t}, \bar{u}]) \) is an invertible ideal of \( \mathbb{C}[\bar{s}, \bar{t}, \bar{u}] \).

The morphism \( F \) induces \( F^* : \mathbb{C}[\bar{s}, \bar{t}, \bar{u}] \to \mathbb{C}[x,y] \) by \( F^*(\bar{s}) = x, F^*(\bar{t}) = xy \) and \( F^*(\bar{u}) = x^2y - x \). Then \( F^* : \mathbb{C}[s, \bar{t}, \bar{u}] \to \mathbb{C}[x,y, x^2y - x] \to \mathbb{C}[x,y] \).

We see that \( \{(s,t,u) \in X \mid s \neq 0\} \) and \( \{(s,t,u) \in X \mid t \neq 1\} \) are open \( \mathbb{C} \)-affine subvarieties of \( X \) and hence that \( \{(s,t,u) \in X \mid s \neq 0\} \cup \{(s,t,u) \in X \mid t \neq 1\} \) is an open subset of \( X \). Since we supposed that **Example** is valid, \( F : \text{Spec}^m(\mathbb{C}[x,y]) = \mathbb{C}^2 \cong \{(s,t,u) \in X \mid s \neq 0\} \cup \{(s,t,u) \in X \mid t - 1 \neq 0\} \).

Note that \( \{(s,t,u) \in X \mid s \neq 0\} = \text{Spec}^m(\mathbb{C}[s,t,u]_{s}) \cap X = \text{Spec}^m(\mathbb{C}[\bar{s}, \bar{t}, \bar{u}]_s) \) and \( \{(s,t,u) \in X \mid t \neq 1\} = \text{Spec}^m(\mathbb{C}[s,t,u]_{t-1}) \cap X = \text{Spec}^m(\mathbb{C}[\bar{s}, \bar{t}, \bar{u}]_{t-1}) \).

So we see the following:

\[
\{(s,t,u) \in X \mid s \neq 0\} \cup \{(s,t,u) \in X \mid t \neq 1\} = \text{Spec}^m(\mathbb{C}[\bar{s}, \bar{t}, \bar{u}]_s) \cup \text{Spec}^m(\mathbb{C}[\bar{s}, \bar{t}, \bar{u}]_{t-1})
\]

\[
= \text{Spec}^m(\mathbb{C}[\bar{s}, \bar{t}, \bar{u}]_s) \setminus (V^m(\bar{s}) \cap V^m(\bar{t} - 1))
\]

\[
= \text{Spec}^m(\mathbb{C}[\bar{s}, \bar{t}, \bar{u}]_s) \setminus V^m(\bar{s}, \bar{t} - 1),
\]

which is an open subvariety of \( \text{Spec}^m(\mathbb{C}[\bar{s}, \bar{t}, \bar{u}]) = X \).
The ideal \((\bar{s}, \bar{t} - 1)C[\bar{s}, \bar{t}, \bar{u}]\) is a prime ideal of height 1 and \(V^m(\bar{s}, \bar{t} - 1)\) is isomorphic to a line \(\mathbb{A}^1_C\), a contractible hypersurface of \(X\) in the usual \(C\)-topology.

[Indeed, \(C[\bar{s}, \bar{t}, \bar{u}]/(\bar{s}, \bar{t} - 1) = \left(\mathbb{C}[s, t, u]/((su - t^2 + t))\right)/(\left(s, t - 1)/(su - t^2 + t)\right) \cong C[s, t, u]/(s, t - 1) \cong C[u]\) and \(V^m(\bar{s}, \bar{t} - 1)\) is of codimension 1 in \(X\).]

Thus the prime ideal \((\bar{s}, \bar{t} - 1)\) is in \(Ht_1(C[\bar{s}, \bar{t}, \bar{u}])\) and hence is a divisorial ideal of the regular domain \(C[\bar{s}, \bar{t}, \bar{u}]\). So it is an invertible ideal. Similarly, the ideal \((\bar{s}, \bar{t})C[\bar{s}, \bar{t}, \bar{u}]\) is a prime ideal of height 1 and \(V^m(\bar{s}, \bar{t})\) is isomorphic to a line \(\mathbb{A}^1_C\), a contractible hypersurface of \(X\) in the usual \(C\)-topology.

It is easy to see that \(F\) is a non-surjective open immersion because
\[ F(\text{Spec}^m(C[x, y])) \not\ni (0, 1, c) \in X \ (\forall c \in C). \]

We may identify \(\bar{s}, \bar{t}, \bar{u}\) with \(F^*(\bar{s}), F^*(\bar{t}), F^*(\bar{u}) \in C[x, y]\), respectively and \(F^* : C[\bar{s}, \bar{t}, \bar{u}] \to C[x, y]\). So \(\bar{s} = y, \bar{t} = xy, \bar{u} = x^2y - x\).

Incidentally, a \(C\)-automorphism \(\sigma\) of \(C[s, t, u]\) defined by \(\sigma(s) = s, \sigma(t) = 1 - t\), \(\sigma(u) = u\) induces a \(C\)-automorphism of \(C[\bar{s}, \bar{t}, \bar{u}]\), where we use the same \(\sigma\).

[Indeed, \(\sigma((su - t(t - 1)) = su - (1 - t)((1 - t) - 1)) = su - (1 - t)(-t) = su - t(t - 1).\]

Then \(^a\sigma \in \text{Aut}(\text{Spec}^m(C[\bar{s}, \bar{t}, \bar{u}]))\) with \(^a\sigma^2 = id_X\). Besides, \(\sigma\) can be seen an automorphism of the quotient field \(C(\bar{s}, \bar{t}, \bar{u}) = C(x, y)\).

We see
\[ \text{Spec}^m(C[x, y]) \xrightarrow{F} X \setminus V^m(\bar{s}, \bar{t} - 1) \cong X \setminus ^a\sigma(V^m(\bar{s}, \bar{t} - 1)) = X \setminus V^m(\bar{s}, \bar{t}) \]
and
\[ ^a\sigma(V^m(\bar{s}, \bar{t} - 1)) = V^m(\bar{s}, \bar{t}). \]

Since \((\bar{s}, \bar{t})C[\bar{s}, \bar{t}, \bar{u}] + (\bar{s}, \bar{t} - 1)C[\bar{s}, \bar{t}, \bar{u}] = C[\bar{s}, \bar{t}, \bar{u}]\), it follows that \(V^m(\bar{s}, \bar{t}) \cap V^m(\bar{s}, \bar{t} - 1) = \emptyset\) and \((X \setminus V^m(\bar{s}, \bar{t})) \cup (X \setminus V^m(\bar{s}, \bar{t} - 1)) = X \setminus (V^m(\bar{s}, \bar{t} - 1) \cap V^m(\bar{s}, \bar{t})) = X\).

Thus
\[ X = F(\text{Spec}^m(C[x, y])) \cup ^a\sigma F(\text{Spec}^m(C[x, y])), \]
where \(\text{Spec}^m(C[x, y]) \cong \mathbb{C}^2 \cong ^a\sigma F(\text{Spec}^m(C[x, y]))\).

Therefore \(X \neq F(\text{Spec}^m(C[x, y]))\) by Proposition 4.2. However \(X \neq F(\text{Spec}^m(C[x, y]))\) as was seen before. This is a contradiction.

5. An Extension of The Jacobian Conjecture (JC\(_n\))

In this section we enlarge a coefficient ring of a polynomial ring and consider the Jacobian Conjecture about it. This is seen in [6, I(1.1)] by use of the observation on the formal inverse [6, III]. We can also see it in [10, (1.1.14)]. Our proof is simpler than that of [6] even though considering only the case of integral domains.

**Theorem 5.1.** Let \(A\) be an integral domain whose quotient field \(K(A)\) is of characteristic 0. Let \(f_1, \ldots, f_n\) be elements of a polynomial ring \(A[X_1, \ldots, X_n]\) such that
\[ f_i = X_i + \text{higher degree terms} \quad (1 \leq i \leq n) \quad (*) \]
If \( K(A)[X_1, \ldots, X_n] = K(A)[f_1, \ldots, f_n] \), then \( A[X_1, \ldots, X_n] = A[f_1, \ldots, f_n] \).

Proof. It suffices to prove \( X_1, \ldots, X_n \in A[f_1, \ldots, f_n] \).

We introduce a linear order in the set \( \{ k := (k_1, \ldots, k_n) \mid k_r \in \mathbb{Z}_{\geq 0} \ (1 \leq r \leq n) \} \) of lattice points in \( \mathbb{R}_{\geq 0}^n \) (where \( \mathbb{R} \) denotes the field of real numbers) in the following way:

\[ k = (k_1, \ldots, k_n) > j = (j_1, \ldots, j_n) \text{ if } k_r > j_r \text{ for the first index } r \text{ with } k_r \neq j_r. \]

(This order is so-called the lexicographic order in \( \mathbb{Z}_{\geq 0}^n \).)

**Claim.** Let \( F(s) := \sum_{j=0}^{s} c_j f_1^{j_1} \cdots f_n^{j_n} \in A[X_1, \ldots, X_n] \) with \( c_j \in K(A) \). Then \( c_j \in A \ (0 \leq j \leq s) \).

(Proof.) If \( s = 0(= (0, \ldots, 0)) \), then \( F(0) = c_0 \in A \).

Suppose that for \( k(< s) \), \( c_j \in A \ (0 < j \leq k) \). Then \( F(k) \in A[X_1, \ldots, X_n] \) by \((*)\), and \( F(s) - F(k) = G := \sum_{j>k} c_j f_1^{j_1} \cdots f_n^{j_n} \in A[X_1, \ldots, X_n] \). Let \( k' = (k'_1, \ldots, k'_n) \) be the next member of \( k \) with \( k = (k_1, \ldots, k_n) < (k'_1, \ldots, k'_n) = k' \) with \( c_{k'} \neq 0 \).

We must show \( c_{k'} \in A \). Note that \( F(s) = F(k) + G \) with \( G \in A[X_1, \ldots, X_n] \). Developing \( F(s) := \sum_{j=0}^{s} c_j f_1^{j_1} \cdots f_n^{j_n} \in A[X_1, \ldots, X_n] \) with respect to \( X_1, \ldots, X_n \), though the monomial \( X_1^{k_1} \cdots X_n^{k_n} \) with some coefficient in \( A \) maybe appears in \( F(k) \), it appears in only one place of \( G \) with a coefficient \( c_{k'} \) by the assumption \((*)\). Hence the coefficient of the monomial \( X_1^{k_1} \cdots X_n^{k_n} \) in \( F(s) \) is a form \( b + c_{k'} \) with \( b \in A \) because \( F(k) \in A[X_1, \ldots, X_n] \). Since \( F(s) \in A[X_1, \ldots, X_n] \), we have \( b + c_{k'} \in A \) and hence \( c_{k'} \in A \). Therefore we have proved our Claim by induction.

Next, considering \( K(A)[X_1, \ldots, X_n] = K(A)[f_1, \ldots, f_n] \), we have

\[ X_1 = \sum c_j f_1^{j_1} \cdots f_n^{j_n} \]

with \( c_j \in A \) by Claim above. Consequently, \( X_1 \) is in \( A[f_1, \ldots, f_n] \). Similarly \( X_2, \ldots, X_n \) are in \( A[f_1, \ldots, f_n] \) and the assertion is proved completely. Therefore \( A[f_1, \ldots, f_n] = A[X_1, \ldots, X_n] \).  

The Jacobian Conjecture for \( n \)-variables can be Deep as follows.

**Corollary 5.2 (The Extended Jacobian Conjecture).** Let \( A \) be an integral domain whose quotient field \( K(A) \) is of characteristic 0. Let \( f_1, \ldots, f_n \) be elements of a polynomial ring \( A[X_1, \ldots, X_n] \) such that the Jacobian \( \det(\partial f_i/\partial X_j) \) is in \( A^\times \). Then \( A[X_1, \ldots, X_n] = A[f_1, \ldots, f_n] \).

Proof. We see that \( K(A)[X_1, \ldots, X_n] = K(A)[f_1, \ldots, f_n] \) by Corollary 3.12. It suffices to prove \( X_1, \ldots, X_n \in A[f_1, \ldots, f_n] \). We may assume that \( f_i \ (1 \leq i \leq n) \) have no constant term. Since \( f_i \in A[f_1, \ldots, f_n] \),

\[ f_i = a_{i1} X_1 + \cdots + a_{in} X_n + \text{(higher degree terms)} \]

with \( a_{ij} \in A \), where \( (a_{ij}) = (\partial f_i/\partial X_j)(0, \ldots, 0) \). The assumption implies that the determinant of the matrix \((a_{ij})\) is a unit in \( A \).

Let \[ Y_i = a_{i1} X_1 + \cdots + a_{in} X_n \quad (1 \leq i \leq n). \]
Then \( A[X_1, \ldots, X_n] = A[Y_1, \ldots, Y_n] \) and \( f_i = Y_i + \) (higher degree terms). So to prove the assertion, we can assume that without loss of generality

\[
f_i = X_i + \text{(higher degree terms)} \quad (1 \leq i \leq n)
\]

(\ast).

Therefore by Theorem 5.1 we have \( A[f_1, \ldots, f_n] = A[X_1, \ldots, X_n] \).

\[\square\]

Example 5.3. Let \( \varphi : \mathbb{A}^n \to \mathbb{A}^n \) be a morphism of affine spaces over \( \mathbb{Z} \), the ring of integers. If the Jacobian \( J(\varphi) \) is equal to either \( \pm 1 \), then \( \varphi \) is an isomorphism.

Appendix A. A Collection of Tools Required in This Paper

Recall the following well-known results, which are required in this paper. We write down them for convenience.

Remark A.1 (cf.\([10, (1.1.31)]\)). Let \( k \) be an algebraically closed field of characteristic 0 and let \( k[X_1, \ldots, X_n] \) denote a polynomial ring and let \( f_1, \ldots, f_n \in k[X_1, \ldots, X_n] \). If the Jacobian \( \det(\partial f_i/\partial X_j) \in k^\times = k \setminus \{0\} \), then \( k[X_1, \ldots, X_n] \) is étale over the subring \( k[f_1, \ldots, f_n] \). Consequently \( f_1, \ldots, f_n \) are algebraically independent over \( k \). Moreover, \( \text{Spec}(k[X_1, \ldots, X_n]) \to \text{Spec}(k[f_1, \ldots, f_n]) \) is surjective, which means that \( k[f_1, \ldots, f_n] \hookrightarrow k[X_1, \ldots, X_n] \) is faithfully flat.

In fact, put \( T = k[X_1, \ldots, X_n] \) and \( S = k[f_1, \ldots, f_n] \subseteq T \). We have an exact sequence by [15,(26.H)]:

\[
\Omega_k(S) \otimes_S T \xrightarrow{\omega} \Omega_k(T) \to \Omega_S(T) \to 0,
\]

where

\[
v(df_i \otimes 1) = \sum_{j=1}^{n} \frac{\partial f_i}{\partial X_j} dX_j \quad (1 \leq i \leq n).
\]

So \( \det(\partial f_i/\partial X_j) \in k^\times \) implies that \( v \) is an isomorphism. Thus \( \Omega_S(T) = 0 \) and hence \( T \) is unramified over \( S \) by [4,VI,(3.3)]. So \( T \) is étale over \( S \) by Lemma A.10 below. Thus \( df_1, \ldots, df_n \in \Omega_k(S) \) compounds a free basis of \( T \otimes_S \Omega_k(S) = \Omega_k(T) \), which means \( K(T) \) is algebraic over \( K(S) \) and that \( f_1, \ldots, f_n \) are algebraically independent over \( k \).

The following proposition is related to the ‘simple-connectivity’ of affine spaces \( \mathbb{A}^n_k \) \((n \in \mathbb{Z}_{\geq 0}) \) over a field \( k \) of characteristic 0. Its (algebraic) proof is given without the use of the geometric fundamental group \( \pi_1(\ ) \) after embedding \( k \) into \( \mathbb{C} \) (the Lefschetz Principle).

Proposition A.2 ([23]). Let \( k \) be an algebraically closed field of characteristic 0. Then a polynomial ring \( k[Y_1, \ldots, Y_n] \) over \( k \) is (algebraically) simply connected.

Proposition A.3 ([2,Theorem 3]). Any invertible regular function on a normal, (algebraically) simply connected \( \mathbb{C} \)-variety is constant.
Lemma A.4 ([15,(6.D)]). Let \( \varphi : A \to B \) be a homomorphism of rings. Then \( \varphi \) is an \( A \)-algebra of finite type. If \( B \) is flat over \( A \), then the canonical morphism \( \varphi : \text{Spec}(B) \to \text{Spec}(A) \) is dominating (or dominant) (i.e., \( \varphi(\text{Spec}(B)) \) is dense in \( \text{Spec}(A) \)) if and only if \( \varphi \) has a kernel \( \subseteq \text{nil}(A) := \sqrt{(0)_A} \). If, in particular, \( A \) is reduced, then \( \varphi \) is dominant \( \iff \varphi(\text{Spec}(B)) \) is dense in \( \text{Spec}(A) \) \( \iff \varphi \) is injective.

Lemma A.5 ([14,(9.5)], [15,(6.I)]). Let \( A \) be a Noetherian ring and let \( B \) be an \( A \)-algebra of finite type. If \( B \) is flat over \( A \), then the canonical morphism \( \varphi : \text{Spec}(B) \to \text{Spec}(A) \) is an open map. (In particular, if \( A \) is reduced (e.g., normal) in addition, then \( A \to B \) is injective.)

The following is well-known, but we write it down here for convenience.

Lemma A.6 ([14]). Let \( k \) be a field, let \( R \) be a \( k \)-affine domain and let \( L \) be a finite algebraic field-extension of \( K(R) \). Then the integral closure \( R_L \) of \( R \) in \( L \) is finite over \( R \).

Moreover the above lemma can be generalized as follows.

Lemma A.7 ([15,(31.B)]). Let \( A \) be a Noetherian normal domain with quotient field \( K \), let \( L \) be a finite separable algebraic extension field of \( K \) and let \( A_L \) denote the integral closure of \( A \) in \( L \). Then \( A_L \) is finite over \( A \).

Lemma A.8 (The Approximation Theorem for Krull Domains [11,(5.8)]). Let \( A \) be a Krull domain. Let \( n(P) \) be a given integer for each \( P \in \text{Ht}_1(A) \) such that \( n(P) = 0 \) for almost all \( P \). For any preassigned set \( P_1, \ldots, P_r \) there exists \( x \in K(R)^\times \) such that \( v_P(x) = n(P_i) \) with \( v_P(x) \geq 0 \) otherwise, where \( v_P( \cdot ) \) denotes the (additive) valuation associated to the DVR \( A_P \).

For a Noetherian ring \( R \), the definitions of its normality (resp. its regularity) is seen in [15,p.116], that is, \( R \) is a normal ring (resp. a regular ring) if \( R_p \) is a normal domain (resp. a regular local ring) for every \( p \in \text{Spec}(R) \).

Lemma A.9 ([14,(23.8)], [15,(17.I)])(Serre’s Criterion on normality). Let \( A \) be a Noetherian ring. Consider the following conditions:

\[
(R_1) : A_p \text{ is regular for all } p \in \text{Spec}(A) \text{ with } \text{ht}(p) \leq 1 ; \\
(S_2) : \text{depth}(A_p) \geq \min(\text{ht}(p), 2) \text{ for all } p \in \text{Spec}(A).
\]

Then \( A \) is a normal ring if and only if \( A \) satisfies \( (R_1) \) and \( (S_2) \). (Note that \( (S_2) \) is equivalent to the condition that any prime divisor of \( fA \) for any non-zerodivisor \( f \) of \( A \) is not an embedded prime.)

Lemma A.10 ([SGA, (Exposé I, Cor.9.11)]). Let \( S \) be a Noetherian normal domain, let \( R \) be an integral domain and let \( \varphi : S \to R \) be a ring-homomorphism of finite type. If \( \varphi \) is unramified, then \( \varphi \) is étale.

Lemma A.11 ([21,p.42](Zariski’s Main Theorem)). Let \( A \) be a ring and let \( B \) be an \( A \)-algebra of finite type which is quasi-finite over \( A \). Let \( \overline{A} \) be the integral closure of \( A \) in \( B \). Then the canonical morphism \( \text{Spec}(B) \to \text{Spec}(\overline{A}) \) is an open immersion.
Lemma A.12 (cf.\cite[23.9]{14}). Let \((A,m)\) and \((B,n)\) be Noetherian local rings and \(A \to B\) a local homomorphism. Suppose that \(B\) is flat over \(A\). Then

(i) if \(B\) is normal (or reduced), then so is \(A\),
(ii) if both \(A\) and the fiber rings of \(A \to B\) are normal (or reduced), then so is \(B\).

Lemma A.13 (\cite[Prop(4.1.3)]{9}). Let \(V_i \ (1 \leq i \leq k)\) be different hypersurfaces of \(\mathbb{P}^n_C\) which have \(\deg(V_i) = d_i\). Let \(V := \bigcup_{i=1}^k V_i\). Then

\[
\pi_1(\mathbb{P}^n_C \setminus V)/[\pi_1(\mathbb{P}^n_C \setminus V), \pi_1(\mathbb{P}^n_C \setminus V)] = H_1(\mathbb{P}^n_C \setminus V) = \mathbb{Z}^{k-1} \oplus (\mathbb{Z}/(d_1, \ldots, d_k)\mathbb{Z}),
\]

where \((d_1, \ldots, d_k)\) denotes the greatest common divisor and \([, , ]\) denotes a commutator group.

Corollary A.14 (\cite[Prop(4.1.4)]{9}). If \(X \subseteq C^n\) is a hypersurface (not necessarily irreducible) with \(k\) irreducible components, then

\[
\pi_1(C^n \setminus X) \to \mathbb{Z}^k
\]
is surjective.

Corollary A.15. Let \(V_i \ (1 \leq i \leq k)\) be different hypersurfaces of \(\mathbb{P}^n_C\) which have \(\deg(V_i) = d_i\). Let \(V := \bigcup_{i=1}^k V_i\). Then \(\mathbb{P}^n_C \setminus V\) is simply connected \iff \(V\) is a hyperplane in \(\mathbb{P}^n_C\) \iff \(\mathbb{P}^n_C \setminus V \sim A^n_C\).

Proof. By Lemma A.13, \(\mathbb{P}^n_C \setminus V\) is simply connected if and only if \(k = 1\) and \(d_1 = \deg(V) = 1\) if and only if \(V\) is a hyperplane in \(\mathbb{P}^n_C\) if and only if \(\mathbb{P}^n_C \setminus V \sim A^n_C\).

Acknowledgment: The author would like to be grateful to Moeko ODA for walking with him for a long time, to his son Shuhei ODA and to young grandsons Naoki ODA and Takuma KUSHIDA, and finally to Unyo ODA, the beloved dog of his family, for having always cheered him up (who passed away on July 1, 2014 in Kochi City, JAPAN).

References

[1] S.Abhyankar, Expansion technique in algebraic geometry, Tata Institute of Fundamental Research, Springer-Verlag (1977).
[2] K.Adjamagbo, The complete solution to Bass Deep Jacobian Conjecture, (arXive:1210.5281 v1[math AG] 18 Oct 2012). (Preprint)
[3] K.Adjamagbo, Sur les morphismes injectifs et les isomorphismes des varieties algébriques affines, Comm. in Alg., 24(3) (1996), 1117-1123.
[4] A.Altmann and S.Kleiman, Introduction to Grothendieck Duality Theory, Lecture Notes in Math, 146, Springer-Verlag (1970).
[5] M.F.Atiyah and L.G.MacDonald, Introduction to Commutative Algebra, Addison-Wesley, London (1969).
[6] H.Bass, E.H.Connel and D.Wright, The Jacobian conjecture : Reduction of degree and formal expansion of the inverse, Bull. A.M.S., 7(2) (1982),287-330.
[7] H.Bass, Differential structure of étale extensions of polynomial algebra, Proceedings of a Microprogram Held, June 15-July 2, 1987, M.Hoster, C.Hunecke, J.D.Sally, ed., Springer-Verlag, New York, 1989.
AN APPROACH TO THE Deep JACOBIAN CONJECTURE

[8] L.A.Campbell, A condition for a polynomial map to be invertible, Math. Ann., 205 (1973), 243-248.

[9] A.Dimca, Singularities and Topology of Hypersurfaces, Universitext, Springer-Verlag, New York (1992).

[10] A.van den Essen, Polynomial Automorphisms and The Jacobian Conjecture, Birkhäuser Verlag (2000).

[11] R.Fossum, The Divisor Class Group of a Krull Domain, Springer, Berlin Heidelberg New York, 1973.

[12] V.S.Kulikov, Generalized and local Jacobian Problems, Russian Acad. Sci. Izv. Math, Vol 41 (1993), No.2, 351-365. (There is an error in translation.)

[13] O.H.Keller, Gamze Cremona-Transformationen, Monatsheft fur Math. und Phys., Vol. 47 (1939), 299-306.

[14] H.Matsumura, Commutative Ring Theory, Cambridge Univ. Press (1980).

[15] H.Matsumura, Commutative Algebra, Benjamin New York (1970).

[16] M.Nagata, Lectures on the fourteenth problem of Hilbert, Tata Inst. of Fund. Res. Lect. on Math., No.31, Tata Institute of Fundamental Research, Bombay (1965).

[17] M.Nagata, Field Theory, Marcel Dekker, New York (1977).

[18] M.Nagata, Local Rings, Interscience (1962).

[19] S.Oda and K.Yoshida, A short proof of the Jacobian conjecture in the case of degree ≤ 2, C.R. Math. Rep. Acad. Sci. Canada, Vol.V (1983), 159-162.

[20] F.Oort, Units in number fields and function fields, Exposition Math., 17 no.2 (1999), 97-115.

[21] M.Raynaud, Anneaux Locaux Henséiens, Lecture Notes in Math. 169, Springer-Verlag (1970).

[22] S.Wang, A Jacobian criterion for separability, J. Algebra 65 (1980), 453-494.

[23] D.Wright, On the Jacobian conjecture, Illinois J. Math., 25 (1981), 423-440.

[Ha] R.Hartshorne, Algebraic Geometry, Graduate Texts in Math. 52, Springer-Verlag, New York-Heidelberg (1977).

[EGA] A.Grothendieck and J.Dieudonné, Eléments de Géométrie Algébrique. I.HES Publ. Math. No.4, 1960. IV., No.20, 1964. IV., No.24, 1965. IV., No.28, 1966. IV., No.32, 1967.

[SGA 1] A.Grothendieck et al., Revêtements étale et Groupe Fondamental, Lec. Note in Math. 224, Springer-Verlag, Heidelberg (1971).

[SGA] A.Grothendieck, Séminaire de Géométrie Algébrique,1960-1961,I.H.E.S.,(1960)Fascicule 1.

[W] D.Wright, The Jacobian Conjecture :ideal membership questions and recent advances, Contemporary Math.,AMS 369 (2005), 261-276.

[K-M] T.Kambayashi and M.Miyanishi, On two recent views of the Jacobian Conjecture, Contemporary Math.,AMS 369 (2005), 113-138.

11 Again I saw that under the sun
the race is nor to the swift,
nor the battle to the strong,
nor bread to the wise,
nor riches to the intelligent,
nor favour to those with knowledge,
but time and chance happen to them all.

12 For man does not know his time.

——— ECCLESIASTES 9 (ESV)

14 For He Himself knows our frame;
He is mindful that we are (made of) but dust.

15 As for man, his days are like grass;
As a flower of the field, so he flourishes.

16 When the wind has passed over it,
it is no more, and its place acknowledges it no longer.

——— PSALM 103 (NASB)