Revising Partially Ordered Beliefs

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Abstract

This paper deals with the revision of partially ordered beliefs. It proposes a semantic representation of epistemic states by partial orders on interpretations and a syntactic representation by partially ordered belief bases. Two revision operations, the revision stemming from the history of observations and the probabilistic revision, defined when the epistemic state is represented by a total pre-order, are generalized, at a semantic level, to the case of a partial pre-order on interpretations, and at a syntactic level, to the case of a partially ordered belief base. The equivalence between the two representations is shown for the two revision operations.

1 Introduction

Most of the time, an intelligent agent faces incomplete, uncertain or inaccurate information. The arrival of a new item of information, more reliable or more certain leads the agent to refine (specify) his beliefs, to revise them. Belief revision is a well known problem in Artificial intelligence [1], [2], [3], in this context, an epistemic state encodes a set of beliefs about the real world (based on the available information). An epistemic state is generally interpreted as a plausibility ordering between possible states of the world, or as a preference relation between information sources from which an agent can derive his beliefs.

On a semantic level, epistemic states have been represented by a total pre-order on interpretations of the underlying logical language [4]. This total pre-order models the agent’s preferences between several situations. This pre-order has been encoded according to several ways, ordinals [5], [6], possibilities [7], polynomials [15].

However, the agent has not always a total pre-order between situations at his disposal, but is only able to define a partial pre-order between situations. The arrival of successive items of information can help him to refine this partial pre-order in order to converge to a total pre-order between situations.

In other respects, total pre-orders are not suitable to model the case where decision making is impossible or not arbitrary. Suppose the agent has to make a decision on the cultivation of a new plot according to three rules given by experts. A first rule, $R_1$ specifies that if some agronomical conditions hold (warm climate, deep soil, acidity, etc...), called condition 1, then the cultivation of tobacco is feasible. The second rule $R_2$ specifies that if the agronomical conditions (condition 1) hold and the zone is mildewed (mildew parasite could ruin the plantation) called condition 2, then the cultivation of tobacco is not feasible. The third rule $R_3$ specifies that according to the regulation of production of tobacco, if the area of the plot is not greater than the authorized area, called condition 3, then the cultivation of tobacco is feasible.

The question is to define a pre-order on the three rules in order to make a decision on the cultivation of tobacco. Since $R_2$ is more specific than $R_1$, it is natural to prefer $R_2$ over $R_1$ (namely $R_2 < R_1$ holds). However, condition 3 is not related with condition 1 nor with condition 2, then it seems reasonable to consider $R_3$ incomparable with $R_1$ and $R_2$. On contrast, if we want to impose a total pre-order on rules, we have to set $R_3$ relatively to $R_2$, therefore there are two intuitive choices, either $R_1 \leq R_3$ or $R_3 \leq R_2$. In the first case, the following total pre-order holds: $R_2 < R_1 \leq R_3$ this means that if condition 1, condition 2 and condition 3 are both satisfied, according to the total pre-order we make the decision that the cultivation of tobacco is not feasible. In the second case, the following total pre-order holds: $R_3 \leq R_2 < R_1$ this means that if condition 1, condition 2 and condition 3 are satisfied,
to the total pre-order we make the decision that the cultivation of tobacco is feasible. These two total pre-orders lead to contradictory decisions, and there is no reason to choose the first one or the second one, the choice can only be arbitrary. Since we think that arbitrary choices have to be excluded, we think that a better solution is to consider \( R3 \) and \( R2 \) as incomparable (and hence all total pre-orders are considered), and thus to define a partial pre-order between rules. In such cases, an epistemic state has to be represented by a partial pre-order on interpretations and revision operations of partial pre-orders by formulas have to be defined. Partial pre-orders on interpretations have been used to represent update operations \[12\], this paper does not address updates but revisions.

In the present paper, we propose a generalization of two revision operators to the case of partial pre-orders. Section 2 presents the problematics of the representation of epistemic states by a partial pre-order and focuses on the difficulties of this generalization. Section 3 presents the generalization of the semantic revision operations to partial pre-orders and the generalization of the syntactic counterparts of these operations to partially ordered belief bases. These generalizations are rather direct. On contrast the generalization of the mapping from the syntactic level to the semantic level is more problematical as shown in Section 4, because the definition of a partial pre-order between formulas leads to two possible partial pre-orders between subsets of formulas. We choose one of them, however the results presented hold for the other one. We finally present the syntactic computation of the belief set corresponding to an epistemic state in Section 5 before a concluding discussion in Section 6.

## 2 Presentation of the problem

### 2.1 Basic definitions of partial pre-orders

In this paper we use propositional calculus, denoted by \( \mathcal{L}_PC \), as knowledge representation language with usual connectives \( \neg, \land, \lor, \to, \equiv \) (logical equivalence). The lower case letters \( a, b, c, \ldots \), are used to denote propositional variables, lower case Greeks letters \( \phi, \psi, \ldots \), are used to denote formulas, upper case letters \( A, B, C \), are used to denote sets of formulas, and upper case Greeks letters \( \Psi, \Phi \ldots \), are used to denote epistemic states. We denote by \( W \) the set of interpretations of \( \mathcal{L}_PC \) and by \( \text{Mod}(\psi) \) the set of models of a formula \( \psi \), that is \( \text{Mod}(\psi) = \{ \omega \in W, \omega \models \psi \} \) where \( \models \) denotes the inference relation used for drawing conclusions.

A partial pre-order, denoted by \( \preceq \) on a set \( A \) is a reflexive and transitive binary relation. Let \( x \) and \( y \) be two members of \( A \), the equality is defined by \( x = y \) iff \( x \preceq y \) and \( y \preceq x \). The corresponding strict partial pre-order, denoted by \( \prec \), is such that, \( x \prec y \) iff \( x \preceq y \) holds but \( x \not\preceq y \) does not hold. We denote by \( \sim \) the incomparability relation \( x \sim y \) iff \( x \preceq y \) does not hold nor \( y \preceq x \).

\( \preceq \) can equivalently be defined from \( = \) and \( \prec \): \( a \preceq b \) iff \( a \prec b \) or \( a = b \).

Given \( \preceq \) on a set \( A \), the minimal elements of \( A \) are defined by: \( \text{min}(A, \preceq) = \{ x : x \in A, \forall y \in A, y \sim x \} \)

For the purpose of this paper, we need to define a partial pre-order on subsets of elements of a set \( A \). According to Halpern’s works \[10\], \[11\] (see also Cayrol et al. \[4\], Dubois et al. \[6\] and Lafage et al. \[14\]) there are two meaningful ways to compare subsets of \( A \). We denote these two partial pre-orders on \( 2^A \) by \( \preceq_{A,w} \) and by \( \preceq_{A,s} \).

Let \( X \) and \( Y \) be two subsets of \( A \). The sets \( X \) and \( Y \) are considered equals if for each preferred element in \( X \) there exists a so preferred element in \( Y \) and conversely, more formally:

**Definition 1** Let \( X \) and \( Y \) be two subsets of \( A \) and \( \preceq_A \) a partial pre-order on \( A \), \( X = Y \) iff:

\[
\forall x \in \text{min}(X, \preceq_A), \exists y \in \text{min}(Y, \preceq_A) \text{ such that } x = y \quad \text{and} \quad \forall y \in \text{min}(Y, \preceq_A), \exists x \in \text{min}(X, \preceq_A) \text{ such that } x = y.
\]

The first way to define a partial pre-order on subsets \( X \) and \( Y \) of \( A \) is to consider that \( X \) is preferred to \( Y \) if for each element of \( Y \) there exists at least one element in \( X \), which is preferred to it, more formally (we assume that \( X \) and \( Y \) are not both empty):

**Definition 2** \( X \) is weakly preferred to \( Y \), denoted by \( X \prec_{A,w} Y \), iff \( \forall y \in \text{min}(Y, \preceq_A), \exists x \in \text{min}(X, \preceq_A) \) such that \( x \prec_{A} y \).

The second way to define a partial pre-order on subsets \( X \) and \( Y \) of \( A \) is to consider that \( X \) is preferred to \( Y \) if there exists at least one element in \( X \) which is preferred to all elements in \( Y \), more formally:

**Definition 3** Let \( \preceq_A \) be a partial pre-order on \( A \) and \( X,Y \subseteq A \). \( X \) is strongly preferred to \( Y \), denoted by \( X \prec_{A,s} Y \), iff \( \exists x \in \text{min}(X, \preceq_A) \) such that \( \forall y \in \text{min}(Y, \preceq_A), x \prec_{A} y \).

It can be shown that the definition of \( \prec_{A,s} \) implies the definition of \( \prec_{A,w} \), namely, if \( X \prec_{A,s} Y \) then \( X \prec_{A,w} Y \). The converse does not hold.
Example 1 Let $A = \{x_1, x_2, y_1, y_2\}$ and $\leq_A$ be a partial pre-order on $A$ such that $x_1 \preceq_A y_1$ and $x_2 \preceq_A y_2$. Let $X$ and $Y$ be two subsets of $A$, $X = \{x_1, x_2\}$ and $Y = \{y_1, y_2\}$, we have $X \not\preceq_{A,w} Y$, indeed $x_1$ is preferred to $y_1$ and $x_2$ is preferred to $y_2$. However, $X \not\preceq_{A,s} Y$ does not hold, there is no element in $X$ which is preferred to all elements of $Y$.

In the case where $\leq_A$ is a total pre-order, the definition of $\preceq_{A,s}$ is equivalent to the definition of $\preceq_{A,w}$, more formally, $X \preceq_{A,s} Y$ iff $X \preceq_{A,w} Y$.

For lack of space, we will only focus on the weakly preference definition. But all provided results are valid for the strong preference.

2.2 Semantic representation of epistemic states

Let $\Psi$ be an epistemic state, $\Psi$ is first represented by a partial pre-order on interpretations, denoted by $\preceq_{\Psi}$. The interpretation $\omega$ is preferred (or more plausible than) to $\omega'$, denoted $\omega \preceq_{\Psi} \omega'$. $\omega \sim_{\Psi} \omega'$ denotes that the agent has no preference between $\omega$ and $\omega'$. The belief set corresponding to $\Psi$, denoted by $\text{Bel}^\omega(\Psi)$, modeling the agent’s current beliefs is such that $\text{Mod}(\text{Bel}^\omega(\Psi)) = \min(W, \preceq_{\Psi})$. We illustrate this representation with the example informally described in the introduction.

Example 2 The agent has to make a decision on the cultivation of a new plot according to rules given by experts. A first rule, $R_1$ specifies that if some agronomical conditions hold (warm climate, deep soil, acidity, etc...), called condition 1, then the cultivation of tobacco is feasible. The second rule $R_2$ specifies that if the agronomical conditions (condition 1) hold and the zone is mildewed (mildew parasite could ruin the plantation) called condition 2, then the cultivation of tobacco is not feasible. The third rule $R_3$ specifies that according to the regulation of the production of tobacco, if the area of the plot is not greater than the authorized area, called condition 3, then the cultivation of tobacco is feasible. More formally, the three rules can be represented as follows: $R_1$: $b \rightarrow a$, $R_2$: $b \land c \rightarrow \neg a$, $R_3$: $d \rightarrow a$, where $a$ encodes the cultivation of tobacco is feasible, $b$ encodes condition 1, $c$ encodes condition 2 and $d$ encodes condition 3.

There are four propositional variables $a$, $b$, $c$ and $d$. The sixteen interpretations are $\omega_0 = \{\neg a, \neg b, \neg c, \neg d\}$, $\omega_1 = \{\neg a, \neg b, \neg c, d\}$, $\omega_2 = \{\neg a, \neg b, c, \neg d\}$, $\cdots$, $\omega_{15} = \{a, b, c, d\}$. Let $\Psi$ be an epistemic state which corresponds to the belief set $\text{Bel}^\omega(\Psi) = (b \rightarrow a) \land (b \land c \rightarrow \neg a) \land (d \rightarrow a)$, we represent the epistemic state by a partial pre-order, denoted $\preceq_{\Psi}$ as follows:

Since $R_2$ is more specific than $R_1$, and since $R_2$ and $R_3$ are incomparable, then:

- the interpretations satisfying all constraints are preferred to all other interpretations,
- the interpretations which falsify $R_2$ are preferred to the interpretations falsifying $R_1$,
- the interpretations which falsify $R_2$ and the ones which falsify $R_3$ are incomparable.

The partial pre-order $\leq_{\Psi}$ is represented by Figure 1 (an arrow $x \rightarrow y$ means $x < y$, the transitivity and the reflexivity are not represented for sake of clarity).

![Figure 1: Representation of initial epistemic state $\Psi$](image-url)

2.3 Syntactic representation of epistemic states

An epistemic state $\Psi$ is syntactically represented by a partially ordered belief base, denoted by $\preceq_{\Sigma}$, where $\Sigma$ is a set of propositional formulas, and $\preceq_{\Sigma}$ is a partial pre-order on $\Sigma$. Let $\phi$ and $\phi' \in \Sigma$, $\phi \preceq_{\Sigma} \phi'$ means that $\phi$ is preferred (more important than) to $\phi'$ and $\phi \sim_{\Sigma} \phi'$ denotes that the agent has no preference between $\phi$ and $\phi'$. We illustrate this representation with the example informally described in the introduction.

Example 3 Let $\Psi$ be the epistemic state, where $\Sigma = \{b \rightarrow a, b \land c \rightarrow \neg a, d \rightarrow a\}$, we represent the epistemic state by a partial pre-order on $\Sigma$, denoted by $\preceq_{\Sigma}$, as follows: Since $b \land c \rightarrow \neg a$ is more specific than $b \rightarrow a$, and $d \rightarrow a$, we represent the partial pre-order on formulas holds:

$b \land c \rightarrow \neg a \preceq_{\Sigma} b \rightarrow a$ and $b \land c \rightarrow \neg a \sim_{\Sigma} d \rightarrow a$ and $b \rightarrow a \sim_{\Sigma} d \rightarrow a$.

The generalization of the representation and revision of an epistemic state to a partial pre-order leads to the following diagram:

\[
\begin{array}{c}
\preceq_{\Sigma} \rightarrow \preceq_{\Psi} \\
\downarrow \quad \downarrow \\
\preceq_{\Sigma} \circ \mu \rightarrow \preceq_{\Psi} \circ \mu \\
\downarrow \quad \downarrow \\
\text{Bel}^\Psi(\Psi \circ \mu) \equiv \text{Bel}^\Psi(\Psi \circ \mu)
\end{array}
\]
In the special case of a total pre-order, the diagram has been shown valid and the equivalence between the syntactic approach and the semantic approach has been proved \(\text{2}\). The question is what does remain true when this diagram is extended to the representation by a partial pre-order. We show in Section 3 that we get the mappings \(\preceq_{\Sigma} \rightarrow \preceq_{\Sigma \circ^{SY} \mu}, \preceq_{\Psi} \rightarrow \preceq_{\Psi \circ^{SY} \mu}\) and \(\preceq_{\Psi \circ^{SY} \mu} \rightarrow \text{Bel}^{se}(\Psi \circ^{SY} \mu)\) rather directly. On contrast, the mappings \(\preceq_{\Sigma} \rightarrow \preceq_{\Psi}\) and \(\preceq_{\Sigma \circ^{SY} \mu} \rightarrow \text{Bel}^{SY}(\Psi \circ^{SY} \mu)\) are less straightforward.

We now present the revision extended to a partial pre-order.

## 3 Semantic and syntactic revision of partial pre-orders

### 3.1 Extension of the revision stemming from the history of observations

We extend the revision operation, defined in \(\text{4} \quad \text{5}\), to the case of partial pre-orders. The underlying intuition stems from the fact that the agent remembers all his previous observations. However these observations are not at the same level, according to whether there are more plausible or not in the next epistemic state. The general philosophy is that an old assertion persists until it becomes contradictory with a more recent one. The revision operation uses the history of the sequence of previous observations to perform revision.

#### 3.1.1 Semantic extension

When an epistemic state, \(\Psi\), is represented by a propositional formula \(\mu\) leads to a revised epistemic state \(\Psi \circ^{SE} \mu\), represented by a partial pre-order on interpretations. This new epistemic state is such that the relative ordering between models of \(\mu\) is preserved, the relative ordering between counter-models of \(\mu\) is preserved, and the models of \(\mu\) are preferred to its counter-models. More formally:

**Definition 4** Let \(\Psi\) be an epistemic state and \(\mu\) be a propositional formula, the revised epistemic state \(\Psi \circ^{SE} \mu\) corresponds to the following partial pre-order:

- If \(\omega, \omega' \in \text{Mod}(\mu)\) then \(\omega \preceq_{\Psi \circ^{SE} \mu} \omega' \) iff \(\omega \preceq_{\Psi} \omega'\),
- If \(\omega \in \text{Mod}(\mu)\) and \(\omega' \notin \text{Mod}(\mu)\) then \(\omega \preceq_{\Psi \circ^{SE} \mu} \omega'\).

According to this definition it is easy to check that \(\text{Mod}(\text{Bel}^{SE}(\Psi \circ^{SE} \mu)) = \min(\text{Mod}(\mu), \preceq_{\Psi})\).

**Example 4** We come back to example \(\text{3}\), where the initial epistemic state \(\Psi\) is represented by the Figure \(\text{4}\).

The corresponding belief set \(\text{Bel}^{se}(\Psi)\) is such that:

\[
\text{Mod}(\text{Bel}^{se}(\Psi)) = \{\omega_0, \omega_2, \omega_3, \omega_9, \omega_10, \omega_11, \omega_12, \omega_13\}.
\]

Suppose we learn that condition 3 holds, namely we revise \(\Psi\) by the propositional formula \(\mu = d\). According to the definition, \(\text{the revised epistemic state } \Psi \circ^{se} \mu\) is represented by the partial pre-order on interpretations \(\preceq_{\Psi \circ^{SE} \mu}\) graphically represented on Figure \(\text{4}\).

**Figure 2:** Representation of \(\Psi \circ^{SE} \mu\)

\[
\text{Mod}(\neg \mu) = \omega_1 \quad \text{Mod}(\mu) = \omega_6
\]

\[
\omega_0 = \omega_2 = \omega_8 = \omega_10 = \omega_12
\]

\[
\omega_1 = \omega_3, \omega_5 = \omega_7 \quad \omega_9 = \omega_11 = \omega_13
\]

The belief set corresponding to \(\Psi \circ^{SE} \mu\) is such that:

\[
\text{Mod}(\text{Bel}^{se}(\Psi \circ^{SE} \mu)) = \{\omega_9, \omega_{11}, \omega_{13}\}
\]

It can be checked that:

\[
\text{Mod}(\text{Bel}^{se}(\Psi \circ^{SE} \mu)) = \min(\text{Mod}(\mu), \preceq_{\Psi}).
\]

#### 3.1.2 Syntactic extension

We now present the syntactic extension of this revision operation to partial pre-orders. Let \(\Psi\) be an epistemic state, \(\Psi\) is syntactically represented by a partially ordered belief base, denoted by \(\preceq_{\Sigma}\), where \(\Sigma\) is a set of propositional formulas, and \(\preceq_{\Sigma}\) is a partial pre-order on \(\Sigma\). The revision of \(\preceq_{\Sigma}\) by a propositional formula \(\mu\) leads to a partially ordered belief base denoted by \(\preceq_{\Sigma \circ^{SY} \mu}\) as follows:

Let us denote by \(U\) the set of the disjunctions between \(\mu\) and the formulas of \(\Sigma\), more formally, \(U = \{\phi \vee \mu, \phi \in \Sigma\}\), such that \(\phi \in \Sigma\) and \(\phi \neq \top\).

**Definition 5** Let \(\Psi\) be an epistemic state, represented by a partially ordered belief base \(\preceq_{\Sigma}\), the revision of \(\Psi\) by \(\mu\) leads to a revised epistemic state \(\Psi \circ^{SY} \mu\) represented by a partially ordered belief base where \(\Sigma \circ^{SY} \mu = \Sigma \cup U \cup \{\mu\}\) and \(\Sigma \circ^{SY} \mu\) is such that:
Let \( \Psi \) be an epistemic state and \( \mu \) be a propositional formula, the revised epistemic state \( \Psi \circ^{se}_\pi \mu \) corresponds to the following partial pre-order:

- if \( \omega, \omega' \in \text{Mod}(\mu) \) then \( \omega \preceq_{\Psi \circ^{se}_\pi \mu} \omega' \iff \omega \preceq \Psi, \omega' \).
- if \( \omega, \omega' \notin \text{Mod}(\mu) \) then \( \omega =_{\Psi \circ^{se}_\pi \mu} \omega' \).
- if \( \omega \in \text{Mod}(\mu) \) and \( \omega' \notin \text{Mod}(\mu) \) then \( \omega <_{\Psi \circ^{se}_\pi \mu} \omega' \).

According to this definition it is easy to check that \( \text{Mod}((\text{Bel}^{se}(\Psi \circ^{se}_\pi \mu))) = \min(\text{Mod}(\mu), \leq_{\Psi}) \).

**Example 6** Let us consider again example \([3]\). We revise \( \Psi \) by the propositional formula \( \mu = d \). According to the definition \([4]\), the revised epistemic state \( \Psi \circ^{se}_\pi \mu \) is represented by the following partial pre-order on interpretations \( \leq_{\Psi \circ^{se}_\pi \mu} \), graphically represented on Figure \([4]\).

The belief set corresponding to \( \Psi \circ^{se}_\pi \mu \) is such that:

\[ \text{Mod}(\neg \mu) = \{ \omega_0 = \omega_2 = \omega_4 = \omega_6 = \omega_8 = \omega_{10} = \omega_{12} = \omega_{14} \} \]

\[ \text{Mod}(\mu) = \{ \omega_1 = \omega_3 = \omega_5 = \omega_7 \} \]

\[ \text{Mod}(\neg \mu) \cup \{ \omega_0 \} \]

**3.2 Extension of possibilistic revision**

We now present the extension of the possibilistic revision to partial pre-orders. In a possibility theory framework \([1]\) an epistemic state \( \Psi \) is represented by a possibility distribution \( \pi \). Each interpretation is assigned with a real number belonging to the interval \([0,1] \). The value 1 means that the interpretation is totally possible, whereas the value 0 means that the interpretation is totally impossible. A possibility distribution induces a total pre-order \( \leq_{\pi} \) on interpretations in the following way: \( \omega \leq_{\pi} \omega' \iff \pi(\omega) \geq \pi(\omega') \). For more details on possibility theory and the revision of possibility distributions, see \([7,8]\).

**3.2.1 Semantic extension**

Let \( \Psi \) be an epistemic state represented by a partial pre-order \( \leq_{\Psi} \). The possibilistic revision of \( \Psi \) by a propositional formula \( \mu \) leads to a revised epistemic state \( \Psi \circ^{se}_\pi \mu \), represented by a partial pre-order on interpretations, denoted by \( \leq_{\Psi \circ^{se}_\pi \mu} \) which considers that all the counter-models of the new item of information \( \mu \) as impossible and preserves the relative ordering between the models of \( \mu \). More formally:

**Definition 6** Let \( \Psi \) be an epistemic state and \( \mu \) be a propositional formula, the revised epistemic state \( \Psi \circ^{se}_\pi \mu \) corresponds to the following partial pre-order:

- if \( \omega, \omega' \in \text{Mod}(\mu) \) then \( \omega \leq_{\Psi \circ^{se}_\pi \mu} \omega' \iff \omega \leq_{\Psi} \omega' \).
- if \( \omega, \omega' \notin \text{Mod}(\mu) \) then \( \omega =_{\Psi \circ^{se}_\pi \mu} \omega' \).
- if \( \omega \in \text{Mod}(\mu) \) and \( \omega' \notin \text{Mod}(\mu) \) then \( \omega <_{\Psi \circ^{se}_\pi \mu} \omega' \).

We now present the syntactic extension of the possibilistic revision operation to partial pre-orders. Let \( \Psi \) be an epistemic state, syntactically represented by a partially ordered belief base. The revision of \( \leq_{\Sigma} \) by a propositional formula \( \mu \) leads to a partially ordered belief base denoted by \( \leq_{\Sigma \circ^{se}_\pi^\mu} \) as follows:

**Definition 7** The revision of \( \Psi \) by \( \mu \) leads to a revised epistemic state \( \Psi \circ^{se}_\pi \mu \) represented by a partially ordered belief base \( \Sigma \circ^{se}_\pi^\mu = \Sigma \cup \{ \mu \} \) where \( \leq_{\Sigma \circ^{se}_\pi^\mu} \) is such that:

- if \( \phi \in \Sigma \) then \( \mu <_{\Sigma \circ^{se}_\pi^\mu} \phi \).
- if \( \phi, \phi' \in \Sigma \) then \( \phi \leq_{\Sigma} \phi' \iff \phi <_{\Sigma \circ^{se}_\pi^\mu} \phi' \).

**Example 7** We come back to example \([3]\), where \( \Sigma = \{ b \rightarrow a, b \land c \rightarrow \neg a, d \rightarrow a \} \), and \( \leq_{\Sigma} \) only contains one constraint: \( b \land c \rightarrow \neg a <_{\Sigma} b \rightarrow a \).
Let us revise $\Psi$ by the propositional formula $\mu = d$. According to the definition [3], the revised epistemic state $\Psi \circ_\mu^y \mu$ is represented by the following partial pre-order on $\Sigma \circ_\mu^y \mu$ denoted by $\leq_{\Sigma \circ_\mu^y \mu}$ and graphically represented by Figure 5.

![Figure 5: Representation of $\Psi \circ_\mu^y \mu$](image)

Note that revising a partial pre-order, with $\circ_\mu^y \mu$ and $\circ_\mu^{sc}$, carry away some incomparabilities. Hence, after a certain number of successive revisions the resulting partial pre-order on interpretations converges to a total pre-order on interpretations, more formally:

**Proposition 1** Let $\preceq_{\Psi}$ be a partial pre-order on interpretations, there exists a sequence of formulas $(\mu_1, \mu_2, \ldots, \mu_n)$ such that the resulting partial pre-order after successive revisions

- $((\preceq_{\Psi} \circ_\mu^{sc} \mu_1) \circ_\mu^{sc} \mu_2) \circ_\mu^{sc} \ldots \circ_\mu^{sc} \mu_n)$ is a total pre-order, and
- $((\preceq_{\Psi} \circ_\mu^{sy} \mu_1) \circ_\mu^{sy} \mu_2) \circ_\mu^{sy} \ldots \circ_\mu^{sy} \mu_n)$ is a total pre-order.

The interest of such a result stems from the fact that starting from total ignorance about a topic, successive revisions lead to a partial pre-order on interpretations, and we now know how to perform these revisions. Moreover, after a certain number of revision the partial pre-order converges to a total pre-order that can be revised according to the results previously obtained in [2, 3].

4 From syntax to semantics

We now present the mapping from a partially ordered belief base $\preceq_{\Sigma}$ to a partial pre-order on interpretation $\preceq_{\Psi}$.

**Definition 8** Let $\Sigma$ be a partially ordered belief base and $\omega$ be an interpretation. We denote by $[\omega, \Sigma]$ the set of preferred formulas of $\Sigma$ falsified by $\omega$. We define a partial pre-order on interpretations as follow:

$$\omega \preceq_{\Psi,w} \omega' \text{ iff } [\omega', \Sigma] \preceq_{\Sigma,w} [\omega, \Sigma].$$

Where $\preceq_{\Sigma,w}$ is given by definition [4].

**Example 8** We come back to example [1], where $\Sigma = \{b \rightarrow a, b \land c \rightarrow \neg a, d \rightarrow a\}$, and $\preceq_{\Sigma}$ is defined as $b \land c \rightarrow \neg a, \preceq_{\Sigma} b \rightarrow a$.

The sets of preferred formulas of $\Sigma$ falsified by the interpretations are the following:

- $[\omega_0, \Sigma] = [\omega_2, \Sigma] = [\omega_8, \Sigma] = [\omega_9, \Sigma] = \emptyset$,
- $[\omega_{10}, \Sigma] = [\omega_{11}, \Sigma] = [\omega_{12}, \Sigma] = [\omega_{13}, \Sigma] = \emptyset$,
- $[\omega_{1}, \Sigma] = [\omega_{3}, \Sigma] = \{d \rightarrow a\}$,
- $[\omega_{4}, \Sigma] = [\omega_{6}, \Sigma] = \{b \rightarrow a\}$,
- $[\omega_{5}, \Sigma] = [\omega_{7}, \Sigma] = \{b \rightarrow a, d \rightarrow a\}$,
- $[\omega_{14}, \Sigma] = [\omega_{15}, \Sigma] = \{b \land c \rightarrow \neg a\}$.

According to definition of $\preceq_{\Sigma,w}$, it can be easily checked that the computation of $\preceq_{\Sigma,w}$ leads to the same partial pre-order than the one used in the semantic representation of $\Psi$ in example [1], namely $\preceq_{\Sigma,w} = \preceq_{\Psi}$.

We are now able to establish the equivalence between the syntactic representation of epistemic states by means of partially ordered belief bases and the semantic representation of epistemic states by means partial pre-orders on interpretations.

**Theorem 1** Let $\Psi$ be an epistemic state represented, on one hand by a partially ordered belief base $\preceq_{\Sigma}$ and on the other hand by a partial pre-orders on interpretations $\preceq_{\Psi}$. Let $\circ_\mu^{sy}$ and $\circ_\mu^{sc}$ be the syntactic and semantic revision operators stemming from the history of the observations, and $\circ_\mu^{sy}$ and $\circ_\mu^{sc}$ be the syntactic and semantic possibilistic revision operators. Let we be the mapping from a partially ordered belief base $\preceq_{\Sigma}$ to a partial pre-order on interpretation $\preceq_{\Psi}$. The following result holds:

- $\text{we}(\preceq_{\Sigma} \circ_\mu^{sy} \mu) = \text{we}(\preceq_{\Sigma}) \circ_\mu^{sc} \mu$,
- $\text{we}(\preceq_{\Sigma} \circ_\mu^{sy} \mu) = \text{we}(\preceq_{\Sigma}) \circ_\mu^{sc} \mu$.

We illustrate this theorem by the following example.

**Example 9** Let $\Psi$ be the epistemic state of example [1], where $\Sigma = \{b \rightarrow a, b \land c \rightarrow \neg a, d \rightarrow a\}$, and $\preceq_{\Sigma}$ is such that $b \land c \rightarrow \neg a, \preceq_{\Sigma} b \rightarrow a$.

According to example [1], the revised epistemic state $\Psi \circ_\mu^{sy} \mu$ is represented by the partial pre-order on $\Sigma \circ_\mu^{sy} \mu$ given by figure [1].

The sets of preferred formulas of $\Sigma \circ_\mu^{sy} \mu = \{b \rightarrow a, b \land c \rightarrow \neg a, d \rightarrow a\}$ falsified by the interpretations are the following:
According to the definition of $\preceq_{c,w}$, we obtain the partial pre-order graphically represented by Figure 4.

Figure 6: Representation of $\preceq_{c,w}$

\[
C_0 = C_2 \quad \quad C_3 \quad \quad C_4 \quad \quad C_7
\]

The syntactic definition of $\text{Bel}^{sy}(\Psi)$ is:

\[
\text{Bel}^{sy}(\Psi) = \bigvee_{C \in \text{Cons}_{sy}(\Sigma)} C.
\]

This means that the syntactic inference can be defined as: $\phi$ is inferred from $\preceq_{sy} \Psi$ if $\forall C \in \text{Cons}_{sy}(\Sigma)$; $C \cup \overline{\phi}$ is consistent.

Theorem 2 Let $\Psi$ be the epistemic state, let $\phi^{sy}$ and $\phi^{se}$ be the syntactic and semantic revision operators stemming from the history of observations, and $\phi^{se}$ be the syntactic and semantic possibilistic revision operators. The following result holds:

- $\text{Bel}^{sy}(\Psi \phi^{sy} \mu) \equiv \text{Bel}^{se}(\Psi \phi^{se} \mu)$,
- $\text{Bel}^{sy}(\Psi \phi^{sy} \mu) \equiv \text{Bel}^{se}(\Psi \phi^{se} \mu)$.

We illustrate this theorem with an example:

Example 11 Let $\Psi$ be the epistemic state defined in example 3 where the revision by $\mu = d$ leads, to the belief set $\text{Bel}^{se}(\Psi \phi^{se} \mu)$ such that $\text{Mod}(\text{Bel}^{se}(\Psi \phi^{se} \mu)) = \{\omega_9, \omega_{11}, \omega_{13}\}$.

On the syntactic level, we can check that, since $\Sigma \cup \{\mu\} \cup U$ is consistent, that the preferred elements for $\preceq_{c,w}$, $\text{Cons}_{sy}(\Sigma \phi^{sy}) \mu$ is simply the subset composed of all elements of $\Sigma \phi^{sy} \mu$. Namely we have $\text{Cons}_{sy}(\Sigma \phi^{sy}) \mu = \{\{b \to a, b \cup c \to \neg a, d \to a, (b \to a) \vee s, (b \cup c \to \neg a) \vee d, (d \to a) \vee d\} \}$ and hence $\text{Mod}(\text{Cons}_{sy}) = \{\omega_9, \omega_{11}, \omega_{13}\}$, as it is excepted.

6 Concluding discussion

Since in certain situations an agent faces incomplete information and has to deal with partially ordered information, this paper proposed a semantic representation of an epistemic state by a partial pre-order on interpretations as well as a syntactic representation by a
partially ordered belief base. The extension to partial pre-orders of two revision strategies already defined for total pre-orders are presented, and the equivalence between the representations is shown. We showed that after a certain number of successive revisions the partial pre-order converges to a total pre-order.

In a future work, we have to investigate the properties of these revision operators and the presented approach could be generalized to the revision of a partial pre-order by a partial pre-order, generalizing the approach proposed in [3]. Moreover, in order to provide reversibility the encoding by polynomials of partial pre-orders on interpretations and partially ordered belief base could be investigated.

Another future work is to develop algorithms for computing the belief set at the syntactical case, and to apply them in geographical information systems where available information is often partially ordered.

7 Acknowledgments

This work was supported by European Community with the REVIGIS project IST-1999-14189. http://www.cmi.uni-mrs.fr/REVIGIS

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