Quantum Shift Register

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We consider a quantum circuit in which shift and rotation operations on qubits are performed by swap gates and controlled swap gates. These operations can be useful for quantum computers performing elementary arithmetic operations such as multiplication and a bit-wise comparison of qubits.

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During the last decade there has been a growing interest in the theory of quantum computation [1]. Due to the quantum parallelism, quantum computers have the potential to outperform the classical computers. The shift operation in classical computers is useful for a bitwise manipulation of data, pseudo-random number generation [2], and elementary arithmetic operation such as multiplication and division. So it has been used as one of the basic operations performed in the CPUs of classical computers [3].

In this paper, we investigate the function of the quantum shift register made of swap gates. The quantum shift register means a quantum circuit which can shift every data qubit to the nearest qubit in a specific direction. We further study its applications to arithmetic calculation and bit-wise operations on two quantum registers. As is well known, the swap gate consists of three CNOT gates [4] and can be realized by NMR [5], which is useful for reordering of qubits such as in the quantum Fourier transform [6].

Let us consider the quantum shift register consisted of swap gates instead of the flip-flops. In Fig.2, a quantum circuit which can perform both shift-left and rotation-left on a n-qubit data is presented.

The input(Q) of each flip-flop is connected to the output (D) of the next bit in lower flip-flop in this classical shift register. The lowest bit is usually set to 0. Due to the irreversibility of the classical shift operation, the information originally stored in the highest bit(b_n)will be lost after the shift. Each flip-flop is a feedback circuit which is constituted of classical logic gates such as XOR gates. Unfortunately, since the feedback required for the flip-flop is not a unitary operation, it is impossible to make a quantum shift register by straightforward conversion of the classical logic gates in the flip-flop into corresponding quantum logic gates.

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It can be easily shown that sequential exchanges of qubits is equivalent to the shift operation. Since a unitary evolution is reversible, the information contained in the quantum shift register does not disappear contrary to the case of the classical shift register [6]. Hence, to preserve the information, we need at least k extra qubits.
forms the state of the register as $|a\rangle = |a_1 a_2 ... a_k\rangle$ and the data qubits as $|b\rangle = |b_1 b_2 ... b_n\rangle$, then the initial state of the shift register (omitting the control qubit) is $|a\rangle \otimes |b\rangle \equiv |a_1 a_2 ... a_k; b_1 b_2 ... b_n\rangle$. In this quantum circuit, the types of the operation activated is governed by the control qubit. To activate the shift operation, the control qubit $c$ is set to $|0\rangle$, and to activate the rotation operation, $c$ is set to $|1\rangle$.

The ancilla qubits can be initially set to $|0\rangle$ for convenience. In this case, each swap operation transforms the state of the register as $|a_1 a_2 ... a_k; b_1 b_2 ... b_n\rangle \rightarrow |a_2 a_1 ... a_k; b_1 b_2 ... b_n\rangle \rightarrow |a_2 a_k b_n; b_1 b_2 b_3 a_1 b_n ... b_n-1\rangle \rightarrow |a_2 a_k b_n; a_1 b_1 b_2 b_3 b_n ... b_n-1\rangle$. Since the control qubit $c$ is set to $|0\rangle$, the last swap between $a_k$ and $b_1$ is inhibited. After one shift (i.e., after $n + k - 1$ swaps), the ancilla qubits becomes $|a_2 ... a_k b_n\rangle$. Note that, during the swaps and the shift, the ancilla qubits remain disentangled from the data qubits.

Second, let us consider the rotation left operation. In this case, the control bit is set to $|1\rangle$ to allow the swap between the last ancilla qubit and the first data qubit after the shift-left operation on the data qubits. Then we obtain $|a_1 a_2 ... a_k; b_1 b_2 ... b_n\rangle \rightarrow \ldots |a_2 ... a_k b_n; a_1 b_1 b_2 b_3 b_n ... b_n-1\rangle \rightarrow |a_2 ... a_k a_1; b_1 b_2 b_3 b_n ... b_n-1\rangle$, which is equal to output obtained by operating left-rotation both on $n$-data qubits and on $k$-ancilla qubits.

It is straightforward to generalize the shift register to a shift-right and a rotation-right register by simply reversing the order of the swaps.

Here, we present some applications of the shift register. We use a conventional abbreviation which omits explicit representation of the ancilla qubits. With the quantum shift register a quantum computer can easily perform multiplication $\mathcal{A}$. Let us consider the multiplication of $n$ numbers ($a_i$) in a state $A = (n)^{-1/2} \sum_{i=0}^{n-1} |a_i\rangle$ by a classical binary number $1100_{(2)}$. This is achieved by performing addition of two outputs obtained by operating shift-left two times on $A$ and three times on $A$ provided that the shift register has enough size to accommodate the results. The reason is as follows. Generally, if a multiplier $l$ is denoted by $l = l_{k-1} \cdots l_0_{(2)} = \sum_{p=0}^{k-1} 2^p l_p$ in a binary form, then multiplication of $a_i$ by $l$ can be written as

$$a_i \times l = \sum_{p=0}^{k-1} a_i 2^p l_p.$$

Here the multiplication of $a_i$ by $2^p$ can be performed by operating shift-left $p$-times on $n$ numbers ($a_i$, $i = 0 \cdots n - 1$). Therefore, summation of all those terms obtained by performing shift-left operation $p$-times on $a_i$ when $l_p = 1$ results in the multiplication of $a_i$ and $l$.

However, due to the No-Cloning theorem we cannot make a copy of $A$ necessary for the generation of $2^2 A$ and $2^3 A$ [8]. This difficulty can be overcome by preparing a work state $B = |0\rangle$, defining a quantum adder $ADD(A, B) = (A, A + B)$ [8], and adding $B$ to the output obtained by performing appropriate shift operation on $A$ as shown in Fig. 3.

By using the quantum adder we can make $B$ state contain the desired final results. In this respect, $B$ plays the role of an 'accumulator' in a classical CPU. Following illustration offers a clear explanation. For the initial state $A = (|0\rangle_A + |1\rangle_A) / \sqrt{2}$, $B = |0\rangle_B$ and the same multiplier $l = 1100_{(2)}$, one can get entangled states ($|0000\rangle_A |0000\rangle_B + |1000\rangle_A |1100\rangle_B) / \sqrt{2}$ (omitting ancilla qubits) after three shifts and two additions on the initial $A$ and $B$, respectively. It is easy to verify that the number of operations required for multiplication of $A = (2^n)^{-1/2} \sum_{m=0}^{2^n-1} |m\rangle$ by a $k$ digits positive binary number using the quantum shift register is $O(k)$ (each shift is composed of $n + k - 1$ swaps. See Fig. 2). On the other hand, classical computers require $O(k)$ shifts and additions for every $2^n$ numbers ($m$), separately. Thus the total number of operations required for this multiplication in a classical computers is $O(2^n k)$. Hence, a quantum computer will outperform a classical computer in multiplications, if the quantum shift register can be utilized. Of course, with the quantum computer, we can obtain superposition of all results of multiplication but only one result is available after measurement, contrary to the classical computers. However, the multiplication circuit can be used as a part of larger quantum algorithm which does not require measurement just after the multiplications, just as the modular multiplication is a part of the Shor’s factoring algorithm [8]. So embedded in a larger quantum algorithm, the multiplication circuit using the quantum shift register can show exponential speed up compared to the classical multiplication circuit.
Now, it is straightforward to extend our arguments to the case where the $k$ digits binary multipliers themselves are $m$ numbers ($c_j, j = 1, 2, ..., m$) contained in a superposition of states $C = (m)^{-1/2} \sum_{j=1}^{m} |c_j\rangle$. (See Fig.4)

![Quantum algorithm performing multiplication of two states A and C. Here RSH is the quantum right shift register and cADD is a controlled quantum adder. The lines with a slash denote qubit wires. The lowest qubit of C is shown separately for clarity.](image)

Let us consider the operations depicted in Fig. 4. To begin with, performing a conditional quantum addition of $A = (n)^{-1/2} \sum_{i=0}^{n-1} |a_i\rangle$ and $B = |0\rangle$ only when the lowest qubit of $C$ is $|1\rangle$ leads to $B = (nm)^{-1/2} \sum_{j=1}^{m} \sum_{i=0}^{n-1} |a_i c_j^0\rangle$. Here $c_j$ is denoted in a binary form $c_j^{k-1} \cdots c_j^0 (2)$. Then one can find that repetitions of above conditional quantum addition of $A$ and $B$ only when the lowest qubit of $C$ is $|1\rangle$ after operating shift-left operation on $A$ and shift-right on $C$ is equivalent to performing the quantum parallel multiplications of all numbers $a_i$ and $c_j$ simultaneously and storing the result in $B$.

Fig.4 also shows another possible application of the shift register. It can be used to select out a specific qubit in the register by shifting the qubit, for example, to the lowest qubit as the right-shift register does.

As is shown in Fig.1, the shift register in our model requires many swaps. Hence, it will be interesting to consider the quantum shift register performing the shift operation simultaneously on each qubits to speed up the calculation using, for example, flying qubits.

In summary, we have constructed the quantum shift and rotation register utilizing swap gates and a controlled swap gate with the help of the ancilla qubits. The quantum circuit can be used for fast numerical operations such as multiplication and qubit selections.

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