THE EXPONENTIAL MAP OF THE COMPLEXIFICATION OF HAM
IN THE REAL-ANALYTIC CASE.

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Abstract. Let \((M, \omega, J)\) be a Kähler manifold and \(K\) its group of hamiltonian symplectomorphisms. The complexification of \(K\) introduced by Donadson is not a group, only a “formal Lie group”. However it still makes sense to talk about the exponential map in the complexification. In this note we show how to construct geometrically the exponential map (for small time), in case the initial data are real-analytic. The construction is motivated by, but does not use, semiclassical analysis.

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1. INTRODUCTION

A real Lie algebra can be easily complexified by tensoring it with the complex numbers over the reals, and extending the Lie bracket bilinearly. Complexifying a Lie group is a much more subtle problem, for which a solution does not always exist. All compact Lie groups admit a complexification, but the proof of that is not trivial (see for example [8], §106).

Let \((M, \omega, J)\) be a Kähler manifold, and let \(K\) denote the group of hamiltonian symplectomorphisms of \((M, \omega)\), with Lie algebra \(C^\infty(M, \mathbb{R})/\mathbb{R}\). \(K\) is known to be “morally” an infinite-dimensional analogue of a compact group, and one can wonder whether it has a complexification. (One can rigorously prove that \(K\) is a diffeological Lie group, [5].) From a physical point of view this would correspond to finding a sensible way to associate a dynamical system to a complex-valued hamiltonian, \(h : M \to \mathbb{C}\), in a manner that extends the notion of Hamilton flow in case \(h\) is real-valued.

This issue was raised and taken on by Donaldson in a series of papers beginning with [1], [2], in connection with a set of lovely problems in Kähler geometry, and has generated a lot of research. From this point of view, the interest in finding a complexification of \(Ham\) is based on the fact, discovered independently by Atiyah and Guillemin and Sternberg, that if a compact Lie group acts in a Hamiltonian fashion on a Kähler manifold then the action extends to the complexified group in an interesting way that can be understood.

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Donaldson pointed out that $\text{Ham}$ acts on the infinite-dimensional space of Kähler potentials of $(M,\omega_0)$, and the moment map is a Hermitian scalar curvature. The appeal of extending the finite-dimensional picture to this infinite-dimensional setting of Kähler metrics is that it was hoped to lead to a way of constructing extremal metrics. In this note we will not enter into the connections with Kähler geometry, other than to point out here that exponentiating purely-imaginary hamiltonians leads to geodesics in the space of Kähler potentials. We will explore this in an expanded version of this note.

Our focus is on the following issue: Donaldson has put forward a notion of “formal Lie group” to conceptualize his notion of a formal complexification of $\text{Ham}$. Briefly, a formal Lie group with Lie algebra $G$ is a “manifold” (the notion is of interest only in infinite dimensions), $G$, together with a trivialization of its tangent bundle of the form $T G \cong G \times G$, such that the map $G \ni h \mapsto h^\#: G \to G$ is a Lie algebra homomorphism (with respect to the commutator of vector fields). The vector fields $h^#$ should be thought of as “left- invariant”, though no group structure on $G$ exists. (Again this definition can be made completely rigorous in the language of diffeologies, [5].)

In the present case $G = C^\omega(M,\mathbb{C})$ and the exponential map in our title refers to the problem of constructing the flow of the fields $h^#$. Corollary 1.5 states that the family of diffeomorphisms $\{f_t\}$ there exponentiate the complex-valued hamiltonian $h$, in one of Donaldson’s models for the complexification of $G$ (see [2] §4).

Our construction is based on the following existence result:

**Proposition 1.1.** Let $(M,J,\omega)$ be a real analytic Kähler manifold of real dimension $2n$. There exists a holomorphic complex symplectic manifold $(X,I)$ of complex dimension $2n$ and an inclusion $\iota : M \to X$ such that $\iota^*\Omega = \omega$, and with the following additional structure:

1. An anti-holomorphic involution $\tau : X \to X$ whose fixed point set is the image of $\iota$ and such that $\tau^*\Omega = \overline{\Omega}$.
2. A holomorphic projection $\Pi : X \to M$, $\Pi \circ \iota = \text{Id}_M$, whose fibers are holomorphic lagrangian submanifolds.

The local existence is simple: We take $X$ to be a neighborhood of the diagonal in $M \times M$, with the complex structure $I = (J,-J)$. $\Omega$ is the holomorphic extension of $\omega$ and $\tau(z,w) = (w,z)$. Finally, the projection is simply $\Pi(z,w) = z$. However there exist natural complexifications that make our results below much more global in some cases.

For example, if $M$ is a generic coadjoint orbit of a compact simply connected Lie group $K$ (one through the interior of a Weyl chamber), then one can take for $X$ the orbit of the complexification, $G$, of $K$ through the same element. If we let $G = KAN$ be the Iwasawa decomposition of $G$, then the orbits of the action of $N$ on $X$ are the fibers of a holomorphic projection $X \to M$. (This picture extends to non-generic orbits.)

To describe our results we need some notation. Given a function $h : M \to \mathbb{C}$ whose real and imaginary parts are real analytic, there is a holomorphic extension $H : X \to \mathbb{C}$ perhaps only defined near $\iota(M)$, but we will not make a notational distinction between $X$ and such a neighborhood, as our results are local in time.

Let $h$ and $H$ be as above. The fibers of $\Pi$ are the leaves of a holomorphic foliation, $\mathcal{F}$, of $X$. Denote by $\Phi_h : X \to X$ the Hamilton flow of $\Re H$ (where $\Re H$ is the real part of $H$) with respect to the real part of $\Omega$. We denote by $\mathcal{F}_h$ the image of the foliation $\mathcal{F}$ under $\Phi_h$ (so that $\mathcal{F}_0 = \mathcal{F}$) We assume that there exists $E \subset \mathbb{R}$ an open interval containing the
origin such that \( \forall t \in E \) the leaves of \( F_t \) are the fibers of a projection \( \Pi_t : X \to M \). We will denote \( \mathcal{F}_t^x := \Pi_t^{-1}(x) \) the fiber of \( F_t \) over \( x \).

**Theorem 1.2.** Let \( E \subset \mathbb{R} \) be an open set as above. Let \( \phi_t : M \to M \) be defined by

\[
\tag{1.1} \phi_t := \Pi_t \circ \Phi_t \circ \iota.
\]

Then, \( \forall t \in E \):

1. There is a complex structure \( J_t : TM \to TM \) such that \( J_t \circ d\Pi_t = d\Pi_t \circ I \) (and \( J_0 = J \)).
2. \( \phi_t : (M,J) \to (M,J_t) \) is holomorphic (\( J_t \circ d\phi_t = d\phi_t \circ J \)).
3. The infinitesimal generator of \( \phi_t \) is

\[
\tag{1.2} \dot{\phi}_t \circ \phi_t^{-1} = \Xi_{\omega}^{\mathbb{R}h} + J_t(\Xi_{\mathbb{R}h}^{\mathbb{R}h})
\]

where \( \Xi_{\omega}^{\mathbb{R}h} \) denotes the Hamilton vector field of \( \mathbb{R}h \) with respect to \( \omega \), etc.

**Remark 1.3.** Conditions (2) and (3) (together with the initial condition \( \phi_0 = 1_M \)) characterize the family \( \{\phi_t\} \). It is in fact not hard to see that in local real-analytic coordinates \( u_j \) on \( M \), these conditions show that the family solves a system of first-order non-linear PDEs of the form

\[
\dot{U}_j(u,t) = F_j(U,U_{u_k})
\]

with \( U_j(u,0) = u_j \) and where the \( F_j \) are real-analytic. Therefore local existence and uniqueness follow from the Cauchy-Kowalewsky theorem.

**Remark 1.4.** Suppose \( G \) is a compact Lie group acting on \( M \) in a Hamiltonian fashion and preserving \( J \). Then the action extends to a holomorphic action to the complexification \( G_{\mathbb{C}} \) (c.f. [4], §4). The extended action is as follows: If \( a, b : C^\infty(M) \to \mathbb{R} \) are two components of the moment map of the \( G \) action, then the infinitesimal action corresponding to \( a + ib \) is the vector field \( \Xi^a + J(\Xi^b) \). The corresponding one-parameter group of diffeomorphisms, \( \varphi_t : M \to M \), satisfies (2) and (3) of Theorem 1.2 with \( J_t = J_0 \) for all \( t \). By the uniqueness part of the previous remark we must have \( \varphi_t = \phi_t \). In other words, our construction is an extension of the process of complexifying the action of a compact group of symmetries of \( (M,\omega,J) \).

We now explain the geometry behind the construction of \( \phi_t \) summarized by \([1.1]\). To find the image of \( x \in M \) under \( \phi_t \) one flows the leaf \( \mathcal{F}_0^x = \Pi_0^{-1}(x) \) of the foliation \( \mathcal{F} = \mathcal{F}_0 \) by \( \Phi_t \), and intersects the image leaf with \( M \). In other words, \([1.1]\) can be stated equivalently as:

\[
\tag{1.3} \{\phi_t(x)\} = \Phi_t(\Pi_0^{-1}(x)) \cap M.
\]

The definition of \( \phi \) is summarized in the following figure, where \( \mathcal{F}_t^y := \Pi_t^{-1}(y) \):
In this figure $M$ is represented by the horizontal segment. Under $\Phi_t$ the fibers of the foliation $\mathcal{F}_0$ are transformed into the fibers of $\mathcal{F}_t$, and $y = \phi_t(x)$.

As we will see the following is an easy consequence of the previous result:

**Corollary 1.5.** Let $f_t := \Pi_0 \circ \Phi_t \circ \iota : M \to M$ and $\omega_t$ the symplectic form defined by $\omega = f_t^* \omega_1$. Then

\[ f_t \circ f_t^{-1} = \Xi_{\mathcal{R}^h \circ \phi_t} + J_0 \left( \Xi_{\mathcal{R}^h \circ \phi_t} \right), \]

where $\Xi_{\mathcal{R}^h}$ denotes the Hamilton vector field of $\mathcal{R}^h$ with respect to $\omega_t$, etc.

It is not hard to check that $f_t = (\phi_t^{-1})^{-1}$, therefore in the (relatively rare) cases when $\phi_t$ is a one-parameter subgroup of diffeomorphisms one has $f_t = \phi_t$.

The present construction is motivated by semiclassical analysis. Ignoring (possibly catastrophic) domain issues, the notion of the exponential of a non-hermitian quantum hamiltonian is clear. (For example, if $M$ is compact and Planck’s constant is fixed, this amounts to exponentiating a matrix.) Therefore a very natural approach to exponentiating a non-hermitian classical hamiltonian is to first quantize it, exponentiate it on the quantum side, and then take the semiclassical limit. This approach has been developed by Rubinstein and Zelditch, [6, 7], and raises a number of interesting but difficult analytical questions. In our construction we bypass these analytic difficulties by considering the following geometric remnants of quantization: Each leaf of the foliation $\mathcal{F}_0$ corresponds to a quantum state (element of the projectivization of the quantum Hilbert space) represented by a coherent state centered at a point on the leaf, that is, an element in the Hilbert space that semiclassically concentrates at the intersection of the leaf with $M$. The fact that $\Pi$ is holomorphic says the coherent states are associated to the metric of $(M, J, \omega)$. On the quantum side the evolution of a coherent state remains a coherent state, whose lagrangian is simply the image of the one at time $t = 0$ by the complexified classical flow.

As explained above our maps $\{\phi_t\}$ simply follow the evolution of the real center.

Finally, we mention that the paper [3] by Graefe and Schubert contains a very clear and detailed account of the case when $M$ is equal to $\mathbb{R}^{2n} \cong \mathbb{C}^n$, $\hbar$ is a quadratic complex hamiltonian, and the lagrangian foliations are by complex-linear positive subspaces, corresponding to standard Gaussian coherent states with possibly complex centers. By explicit
calculations on both quantum and classical sides, they show that the evolution of a coherent state centered at $x$ is another coherent state whose center may be complex, but that represents the same quantum state as a suitable Gaussian coherent state centered at $\phi_t(x)$.

2. The proofs of Theorem 1.2 and Corollary 1.3

Let $\xi$ denote the infinitesimal generator of $\Phi_t$. The following is easy to check in a local trivialization of the foliation of $X$ by fibers of $\Pi_t$:

**Lemma 2.1.** Fix $t \in E$ and $x \in M$, and let $y = \phi_t(x)$ and $\xi$ denote the infinitesimal generator of the flow $\Phi$. Then $\dot{\phi}_t(x) \in T_yM$ is

$$\dot{\phi}_t(x) = d(\Pi_t)_y (\xi_y).$$

**Proof.** Introduce coordinates $(u, v)$ in a neighborhood $U \subset X$ centered at $y$ so that $U \cap M$ is defined by $v = 0$ and the projection $\Pi_t$ is just $\Pi_t(u, v) = u$. Note that, since $\Phi$ is a one-parameter local subgroup of diffeomorphisms, for $s$ small $\phi_{t+s}(x) = \Pi_{t+s} \circ \Phi_s(y)$.

For $s$ near zero denote the map $\Phi_s$ in coordinates by

$$\Phi_s(u, v) = (U(s, u, v), V(s, u, v))$$

(in a smaller neighborhood of $y$). For each $s$ the image of the $v$-axis under $\Phi_s$ locally parametrize the fiber $F_{y,t+s}$, namely

$$v \mapsto (U(s, 0, v), V(s, 0, v)).$$

Therefore we can write $\phi_{t+s}(x) = U(s, 0, v(s))$, where $v(s)$ is defined implicitly by $V(s, 0, v(s)) = 0$ and $v(0) = 0$. It follows that

$$\dot{\phi}_t(x) = \dot{U}(0, 0, 0) + \frac{\partial U}{\partial v}(0, 0, 0) \cdot \dot{v}(0).$$

However $\Phi_0$ is the identity, so that $U(0, u, v) = u$ and therefore $\frac{\partial U}{\partial v}(0, 0, 0) = 0$. \hfill $\square$

To proceed further we will need some notation. We regard $X$ as a real manifold of real dimension $4n$ with an integrable complex structure $I : TX \to TX$. Let us write

$$\Omega = \omega_1 + i\omega_2$$

for the real and imaginary parts of $\Omega$. Thus the $\omega_j$ are real symplectic forms on $X$ and $M$ is $\omega_1$-symplectic and $\omega_2$-lagrangian. Let us write

$$H = F + iG$$

for the real and imaginary parts of $H$. Recall that, by definition, $\xi$ is the Hamilton field of $F$ with respect to $\omega_1$.

**Lemma 2.2.** $\xi - iI(\xi)$ is the holomorphic vector field on $X$ associated to $\Omega$ and to $2H$. Therefore $\Phi_t$ is a holomorphic automorphism of $(X, \Omega)$.

**Proof.** We first note that, since $\Omega$ is of type $(2, 0)$,

$$\omega_1[I \xi] = -\omega_2[I \xi] \quad \text{and} \quad \omega_2[I \xi] = \omega_1[I \xi].$$

From this it follows easily that $\Omega[I(\xi - iI(\xi))] = 2dH$. For the final statement just use that $\Omega$ and $H$ are holomorphic. $\square$
For future reference we note the relations
\begin{equation}
\omega_1|\xi = d\Re H \quad \text{and} \quad \omega_1|I(\xi) = -d\Im H
\end{equation}
that follow from (2.3).

**Lemma 2.3.** Suppose that \( h := \iota^*H \) is real. Then \( \xi \) is tangent to \( M \), and its restriction to \( M \) is the Hamilton field of \( h \) with respect to \( \omega \).

**Proof.** If \( h \) is real then \( \tau^*H = H \) (by uniqueness of analytic continuation of \( h \)), so \( \Re H \) is \( \tau \)-invariant. Since \( \tau^*\Omega = \Omega \), \( \omega_1 \) is also \( \tau \)-invariant, and therefore \( \tau \) maps \( \xi \) to itself and so \( \xi \) has to be tangent to the fixed-point set of \( \tau \). For the second part just note that \( \omega = \iota^*\omega_1 \). \( \square \)

**Proof of Theorem 1.2.** We take one point at a time:

1. Since the fibers of \( \Pi_t \) are the leaves of a holomorphic foliation, there is a well-defined complex structure in the abstract normal bundle to the fibers. The inclusion \( \iota : M \hookrightarrow X \) realizes \( M \) as a cross-section to the foliation and identifies \( TM \) with the normal bundle to the foliation along \( M \). Therefore it inherits a complex structure that makes \( \Pi_t \) holomorphic.

2. This follows from the interpretation of the structures \( J_t \) as arising from the normal bundle structures together with the fact that \( \Phi_t \) is holomorphic, or can be checked directly as follows. Let \( v \in T_xM \), then \( I(v) = J_0(v) + w \), where \( w \in T_xF_0 \). Since \( d\Phi_t \) is holomorphic, one has
\[
Id\Phi_t(v) = d\Phi_t(J_0(v)) + d\Phi_t(w).
\]
But \( d\Phi_t(w) \in T\mathcal{F}_t \) since \( \Phi_t \) maps fibers of \( \mathcal{F}_0 \) to fibers of \( \mathcal{F}_t \). Therefore, the previous relation implies that
\[
d(\Pi_t)(d\Phi_t(J_0(v))) = d(\Pi_t)(Id\Phi_t(v)) = J_td(\Pi_t)(d\Phi_t(v)),
\]
which precisely says that \( \phi_t \) is holomorphic.

3. Omitting the subscript \( t \) for simplicity, by Lemma 2.1 we need to show that
\[
d\Pi_x(\xi) = \Xi_{\Re h} + \nabla\Im h
\]
where:

1. \( \Xi_{\Re h} \) is the Hamilton field of the real part of \( h \) with respect to \( \omega \), and
2. \( \nabla\Im h \) is the gradient of the imaginary part of \( h \) with respect to the metric \( (\omega, J) \).

By the previous lemma, if \( h \) is real, \( d\Pi(\xi_x) = \xi_x \) and there is nothing more to prove. Suppose now that \( h \) is purely imaginary. By the second relation in (2.4), \( I(\xi) \) is the Hamilton field of \( -G \) (see 2.2), and by the lemma 2.3 \( I(\xi) \) is tangent to \( X \) and its restriction to \( X \) is the Hamilton field of \( ih = \iota^*(-G + iF) \) with respect to \( \omega \). Therefore in this case we can write
\begin{equation}
-\Xi_{\Im h} = d\Pi(I(\xi)) = Jd\Pi(\xi),
\end{equation}
and it suffices to apply \( J \) to both sides to get the result. The general case follows by \( \mathbb{R} \)-linearity of the composition \( h \mapsto H \mapsto \xi \), where the first arrow is analytic continuation. This concludes the proof of Theorem 1.2.
Proof of Corollary 1.5. Recalling that \( f_t = (\phi_{-t})^{-1} \), so we wish to compute \( \dot{f}_t \circ \phi_{-t} \).

Differentiating with respect to time the identity \( f_t \circ \phi_{-t}(x) = x \), we get

\[
\dot{f}_t \circ \phi_{-t}(x) = d(f_t)(\phi_{-t}(x)) = d(f_t)\left[\Sigma^\omega_{\mathbb{R}h} + J_{-t}(\Sigma^\omega_{\mathbb{H}_h})\right]_{\phi_{-t}(x)}
\]

Now it is not hard to check that \( d(f_t)\left[\Sigma^\omega_{\mathbb{R}h}\right] = \Sigma^\omega_{\mathbb{R}h \circ f_t} \) and that \( df_t \circ J_{-t} = J_0 \circ df_t \) (using that \( \phi_{-t} : (M, J_0) \to (M, J_{-t}) \) is holomorphic). Therefore

\[
\dot{f}_t \circ \phi_{-t}(x) = \left[\Sigma^\omega_{\mathbb{R}h \circ f_t} + J_0(\Sigma^\omega_{\mathbb{H}_h \circ f_t})\right]_{f_t^{-1}(x)}.
\]

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