A “dilute” generalisation of the Birman–Wenzl–Murakami algebra is considered. It can be “Baxterised” to a solution of the Yang–Baxter algebra. The $D_{n+1}^{(2)}$ vertex models are examples of corresponding solvable lattice models and can be regarded as the dilute version of the $B_n^{(1)}$ vertex models.

1 Introduction

The theory of two-dimensional solvable lattice models is intimately connected with a list of algebraic structures with a wide range of applications in mathematics and physics [1]. Among those are e.g., the braid group [2] and the Temperley–Lieb [3] and Hecke algebras [4]. The braid and Temperley–Lieb or monoid [5] operators were combined into a single (so-called braid–monoid) algebra by Birman and Wenzl [6] and independently by Murakami [7], see also [8]. Besides being related closely to solvable lattice models, these algebras have another important property. They admit a simple diagrammatic interpretation in terms of transformations of strands or strings and are of importance in the theory of knot and link invariants (see e.g. [8]). Recently [9], a generalisation of braid–monoid algebras has been considered which amounts to considering strings of different “colours”. These are connected to recently constructed critical solvable lattice models [10, 11, 12, 13] which are related to (coloured) dense or dilute loop models.

A braid–monoid algebra (also called knit or tangle algebra) is defined as the algebra generated by $b_j, b_j^{-1}$ and $e_j$ ($1 \leq j \leq N - 1$, where $N$ corresponds to the number of strings in the diagrammatic interpretation mentioned above) subject to the following list of relations

\[ b_j b_j^{-1} = b_j^{-1} b_j = I \]
\[ b_j b_k = b_k b_j \quad \text{for} \quad |j - k| > 1 \]  
\[ b_j b_{j+1} b_j = b_{j+1} b_j b_{j+1} \]  

(1.1)
where \( \sqrt{Q} \) and \( \omega \) are central elements (hence numbers in any representation) and \( I \) denotes the identity. (1.1) are the braid relations and (1.2) the defining relations of the Temperley–Lieb algebra [3]. Usually, two equations are added to these relations, which are

\[
\begin{align*}
  f(b_j) &= 0, & g(b_j) &= e_j,
\end{align*}
\]

where \( f \) and \( g \) are some polynomials. The algebra originally investigated in Refs. [6, 7] (the Birman–Wenzl–Murakami algebra) for instance corresponds to the case where \( f \) is a cubic and \( g \) a quadratic polynomial in the braids. The reason why these equations are listed separately is that they do not have a diagrammatic representation and that one might consider these as properties of certain representations rather than as part of the defining relations of the algebra.

Solvable lattice models are commonly constructed as vertex or as face or IRF (interaction-round-a-face) [14] models whose Boltzmann weights satisfy the Yang–Baxter equations. This property can equivalently be stated in the form that they define a representation of the Yang–Baxter algebra [14, 15, 16] defined by

\[
\begin{align*}
  X_j(u) X_{j+1}(u + v) X_j(v) &= X_{j+1}(v) X_j(u + v) X_{j+1}(u), \\
  X_j(u) X_k(v) &= X_k(v) X_j(u) \quad \text{for } |j - k| > 1
\end{align*}
\]

where \( X_j(u) \) (\( u \) denotes the spectral parameter) are local operators whose matrix elements are the Boltzmann weights of the model, see e.g. [8] for details. For vertex models, these Yang–Baxter operators (also called “local face operators”) are particularly simple as they act on an \( N \)-fold tensor space (\( N \) being the number of vertices in one row), acting as the \( R \) matrix (to be precise, as \( R(u) = PR(u) \)) at slots \( j \) and \( j + 1 \) and as the identity elsewhere.

Every crossing-symmetric (see e.g. [8]) representation of the Yang–Baxter algebra yields a representation of the braid–monoid algebra by setting

\[
\begin{align*}
  e_j &= X_j(\lambda), & b_j^\pm &= \lim_{u \to \pm i \infty} \frac{X_j(u)}{\varrho(u)},
\end{align*}
\]

where \( \lambda \) is the crossing parameter and \( k \) and \( \varrho(u) \) are appropriately chosen normalisation factors. Conversely, one may be able to “Baxterise” [17] a representation of the braid–monoid algebra to a representation of the full Yang–Baxter algebra. This is especially useful if one can find a general expression for the Yang–Baxter operator in terms of the braids...
and monoids which can be shown to fulfill the Yang–Baxter algebra as a consequence of the algebraic relations alone (maybe apart from some additional assumptions, for instance about the polynomial reduction relations (1.4)). In this way, every appropriate representation of the braid–monoid algebra gives rise to a solvable lattice model.

This paper is organised as follows. First, we give a short summary of the Birman–Wenzl–Murakami case which corresponds to a braid–monoid algebra where the braids satisfy a cubic reduction relation. Representations of this algebra occur in the $B_n^{(1)}$, $C_n^{(1)}$, $D_n^{(1)}$, and $A_n^{(2)}$ series of vertex models [18, 19] and associated face models. This is of course well known [20, 21], but still there are a few surprising observations which arise. In Sec. 3, we introduce the dilute generalisation of the Birman–Wenzl–Murakami algebra and a graphical interpretation in terms of diagrams acting on strings of two kinds. This algebra is then “Baxterised” [17] to a solution of the Yang–Baxter algebra (1.5) in Sec. 4. Here, the corresponding examples of known models are the $D_{n+1}^{(2)}$ vertex models [19]. The associated representation of the dilute algebra can be regarded as a dilute version of the Birman–Wenzl–Murakami algebra related to the $B_n^{(1)}$ models which becomes more transparent by a suitable change of basis in the expression of the $R$ matrix in Ref. [19]. Finally, the results are summarised in Sec. 5.

## 2 Birman–Wenzl–Murakami Algebra

We assume that the braid satisfies the cubic

$$(b_j - \sigma^{-1} I) (b_j + \sigma I) (b_j - \sigma\tau^2 I) = 0 \quad (2.1)$$

where the third eigenvalue is the twist $\omega$ and hence

$$\omega = \sigma\tau^2 \quad \sqrt{Q} = 1 + \frac{\omega - \omega^{-1}}{\sigma - \sigma^{-1}} \quad (2.2)$$

$$e_j = I + \frac{(b_j - b_j^{-1})}{\sigma - \sigma^{-1}} \quad (2.3)$$

The braid–monoid algebra (1.1)–(1.3) with these additional relations is known as the Birman–Wenzl–Murakami (BWM) algebra. Then it is easy to show that the following ansatz satisfies the Yang–Baxter algebra (1.5)

$$X_j(u) = I + \zeta^{-1} \eta^{-1} (z - z^{-1}) (\tau^{-1} z b_j - \tau z^{-1} b_j^{-1}) \quad (2.4)$$

where $z = \exp(iu)$ ($u$ denotes the spectral parameter), $\zeta = (\sigma - \sigma^{-1})$, and $\eta = (\tau - \tau^{-1})$. It is crossing symmetric with a crossing parameter $\lambda$ given by $\tau = \exp(\lambda)$ (note that $X_j(\lambda) = e_j$), and satisfies the inversion relation

$$X_j(u) X_j(-u) = \varrho(u) \varrho(-u) I \quad (2.5)$$

where

$$\varrho(u) = \zeta^{-1} \eta^{-1} (\sigma z^{-1} - \sigma^{-1} z) (\tau z^{-1} - \tau^{-1} z) \quad (2.5)$$

3
Examples of solvable lattice models which can be expressed in this form \[20, 21\] are given by the $B_n^{(1)}$, $C_n^{(1)}$, $D_n^{(1)}$, and $A_n^{(2)}$ vertex models \[18, 19\] and related face models \[20\]. In the notation of Ref. \[19\] (where the $R$ matrices are parametrised by $x = z^2$ and a complex parameter $k$), the corresponding values of $\sigma$ and $\tau$ are

\[
(\sigma, \tau) = \begin{cases} 
(k, \xi^{1/2}) & \text{for } B_n^{(1)} \text{ and } D_n^{(1)} \\
(-k^{-1}, \xi^{1/2}) & \text{for } C_n^{(1)} \text{ and } A_n^{(2)}
\end{cases} \tag{2.6}
\]

where as in Ref. \[19\], $\xi = k^{2n-1}, k^{2n+2}, k^{2n-2}, -k^{n+1}$ for $B_n^{(1)}$, $C_n^{(1)}$, $D_n^{(1)}$, and $A_n^{(2)}$, respectively.

An interesting observation is in order. None of the above expressions in the Birman–Wenzl–Murakami algebra are altered by interchanging $\sigma \leftrightarrow -\sigma^{-1}$. This means that for a given representation there are in fact two Yang–Baxter operators (which \textit{a priori} are not the same since the values of $\tau$ are different), the second one given by Eq. (2.3) with

\[
\sigma \quad \mapsto \quad \sigma' = -\sigma^{-1}
\]

\[
\tau^2 \quad \mapsto \quad (\tau')^2 = -\sigma^2 \tau^2 . \tag{2.7}
\]

Having a closer look at the examples provided by the vertex models one realises that the pairs $(A_{2n}^{(2)}, B_{n}^{(1)})$ and $(A_{2n-1}^{(2)}, D_{n}^{(1)})$ are built on identical representations of the Birman–Wenzl–Murakami algebra. This implies that face models associated to $A_{2n}$ and $A_{2n-1}$ can be directly deduced from the corresponding $B_{n}^{(1)}$ and $D_{n}^{(1)}$ models, respectively \[22\]. Note that the $A_{2n-1}$ face models of Ref. \[22\] are actually built on $C_{n}^{(1)}$ and not on $D_{n}^{(1)}$ (in contrast to the vertex models of \[18, 19\]), compare the discussion in \[22\] at the end of sec. 1.

Of course, one cannot obtain the recently constructed dilute A–D–E models \[10, 11\] (which are related to the $A_2^{(2)}$ (Izergin-Korepin \[23\]) $R$ matrix) in this way since these have a different algebraic structure (corresponding to a different gauge of the $A_2^{(2)}$ $R$ matrix, see Ref. \[9\] for details) as already mentioned above.

Surprisingly, there is no obvious partner for the $C_{n}^{(1)}$ models. This either means that the second solution defines a new additional series of solvable vertex models (and corresponding face models) or that they are related to other models (for instance to fusion $R$ matrices of one of the other series). This question is left open here, it certainly demands further clarification.

### 3 Dilute Braid-Monoid Algebra

The idea of considering multi-colour generalisations of braid–monoid algebras originates in the investigation of recently constructed face models \[10, 11, 12\] which are related to (coloured) loop models. In Ref. \[3\] it was shown that these models could be conveniently described in terms of two-colour generalisations of the Temperley–Lieb algebra \[3\]. This has been the motivation to look for similar generalisations of the Birman–Wenzl–Murakami algebra and associated solvable lattice models.

To generate the $m$-colour algebra, we need “coloured” braid and monoid operators $b_{\pm}^{(a,b)}_j$, $e^{(a,b)}_j$ (where $a, b = 1, 2, \ldots, m$ denote the colours) as well as projectors $P^{(a)}_j$ which project
onto colour $a$ at position $j$. Also, $\sqrt{Q^{(a)}}$ and $\omega^{(a)}$ become the colour-dependent “Temperley–Lieb eigenvalue” and twist. Note that here we use superscripts “$+$” and “$-$” (of which the “$+$” is usually omitted) to distinguish coloured braids and “inverse” braids, see [9] for details. The full set of relations which defines the algebra (for the general $m$-colour case) can also be found in Ref. [9], they are straightforward generalisations of the one-colour relations (1.1)–(1.3). In complete analogy to the one-colour case, all the relations can be interpreted graphically where has to consider strings of two different colours (see [9]) which can never join.

Here, we are only interested in a “dilute (two-colour) braid–monoid algebra” by which we mean a two-colour case where one colour (we choose colour “2”) is trivial in the sense that

$$b^{(2,2)}_j = e^{(2,2)}_j = P^{(2)}_j P^{(2)}_{j+1} \quad (3.1)$$

which implies $\sqrt{Q^{(2)}} = 1$ and $\omega^{(2)} = 1$. Furthermore,

$$b^{-(a,b)}_j = b^{(a,b)}_j \quad (a \neq b). \quad (3.2)$$

This means that the only non-trivial operators acting on two sites $j$ and $j + 1$ are $b^{\pm{(1,1)}}_j$, $e^{(1,1)}_j$, $P^{(a)}_{j} P^{(b)}_{j+1} (a, b \in \{1, 2\})$, $b^{(a,\bar{a})}_j$, and $e^{(a,\bar{a})}_j (a \in \{1, 2\}, \bar{a} = 3 - a)$.

Thinking in terms of the graphical representation, this means that the second colour can also be interpreted as a vacancy of a string - but it is easier to draw pictures with two types of strings as one has to keep in mind where these vacancies are. Still, the special properties of the second colour lead to a somewhat simplified graphical representation than for the full two-colour algebra (Eq. (3.2) for instance means that one does not have to distinguish between two types of crossings of strings of different kind), see Ref. [24].

4 Baxterisation of Dilute BWM Algebra

We now consider a dilute braid–monoid algebra as introduced in the preceding section where the subalgebra generated by objects of colour “1” is of Birman–Wenzl–Murakami type. Changing our notation of Sec. 2 slightly, we assume the following cubic relation for the braids $b^{(1,1)}_j$

$$\left( b^{(1,1)}_j - \sigma^{-1} p^{(1,1)}_j \right) \left( b^{(1,1)}_j + \sigma p^{(1,1)}_j \right) \left( b^{(1,1)}_j - \tau^2 p^{(1,1)}_j \right) = 0 \quad (4.1)$$

where the third eigenvalue is again the twist $\omega^{(1)}$ which yields

$$\omega^{(1)} = \tau^2$$

$$\sqrt{Q^{(1)}} = 1 + \frac{\omega^{(1)} - (\omega^{(1)})^{-1}}{\sigma - \sigma^{-1}} \quad (4.2)$$

$$e^{(1,1)}_j = I + \frac{b^{(1,1)}_j - b^{-(1,1)}_j}{\sigma - \sigma^{-1}}$$
The above relations together with the defining relations of the algebra (see Sec. 3 and Ref. [9]) are sufficient to show that the following ansatz satisfies the Yang–Baxter algebra (1.3)

\[ X_j(u) = p_j^{(1,1)} \]

\[ + \zeta^{-1} \eta^{-1} (z - z^{-1}) (\tau^{-1} z b_j^{(1,1)} - \tau z^{-1} b_j^{(1,1)}) \]

\[ + \eta^{-1} (\tau z^{-1} - \tau^{-1} z) (p_j^{(1,2)} + p_j^{(2,1)}) \]

\[ - \varepsilon_1 \zeta^{-1} \eta^{-1} (z - z^{-1}) (\tau z^{-1} - \tau^{-1} z) (b_j^{(1,2)} + b_j^{(2,1)}) \]

\[ + \varepsilon_2 \eta^{-1} (z - z^{-1}) (e_j^{(1,2)} + e_j^{(2,1)}) \]

\[ + \left(1 - \zeta^{-1} \eta^{-1} (z - z^{-1}) (\tau z^{-1} - \tau^{-1} z)\right) p_j^{(2,2)} \quad (4.3) \]

where as in Sec. 2, \( z = \exp(iu) \), \( \zeta = (\sigma - \sigma^{-1}) \), \( \eta = (\tau - \tau^{-1}) \) and where \( \varepsilon_1^2 = \varepsilon_2^2 = 1 \) are two arbitrary signs. The appearance of this freedom is actually trivial as all relations of the dilute Birman–Wenzl–Murakami algebra are invariant under the transformations \( b_j^{(a,b)} \to (-1)^{a-b} b_j^{(a,b)} \) and \( e_j^{(a,b)} \to (-1)^{a-b} e_j^{(a,b)} \).

The expression (4.3) is manifestly crossing symmetric with crossing parameter \( \lambda \) defined by \( \tau = \exp(i\lambda) \). Note that in order to have the crossing transformations of the braid and monoid operators as suggested by the diagrammatic interpretation (see Ref. [9]) one should use \( \varepsilon_2 = 1 \) in Eq. (4.3). This stems from the fact that the mixed monoid operators \( e_j^{(a,b)} \) are crossing related to the mixed projectors \( p_j^{(a,b)} (a \neq b) \) which have a fixed sign due to the requirement that the sum of the projectors gives the identity. The inversion relation (2.4) is satisfied by (4.3) with

\[ \sigma(u) = \zeta^{-1} \eta^{-1} (\sigma z^{-1} - \sigma^{-1} z) (\tau z^{-1} - \tau^{-1} z) \quad (4.4) \]

which looks exactly the same as Eq. (2.5).

Comparing the above expression (4.3) with Eq. (2.3) one observes that not only the inversion relation but also the part which only involves colour “1” has exactly the same form as for the pure Birman–Wenzl–Murakami case. But in both cases one has to keep in mind that for a given representation, \( \tau \) (and hence \( \eta \)) has a different meaning in the two expressions (2.3) and (4.3), because the twist is given by \( \omega^{(1)} = \tau^2 \) here whereas \( \omega = \sigma \tau^2 \) in the discussion of Sec. 2. Obviously, the colour “1” part of Eq. (4.3) alone does not satisfy a Yang–Baxter equation.

Alternatively, Eq. (4.3) (with \( \varepsilon_1 = 1 \)) can be expressed in a more “symmetric” form which reads as follows

\[ X_j(u) = \eta^{-1} (\tau z^{-1} - \tau^{-1} z) I \]

\[ - \zeta^{-1} \eta^{-1} (z^{1/2} - z^{-1/2}) (\tau z^{-1} - \tau^{-1} z) (z^{1/2} B_j + z^{-1/2} B_j^{-1}) \]

\[ + \eta^{-1} (z^{1/2} - z^{-1/2}) (\tau z^{-1/2} + \tau^{-1} z^{1/2}) (e_j^{(1,1)} + e_j^{(2,2)}) \]

\[ + \varepsilon_2 \eta^{-1} (z - z^{-1}) (e_j^{(1,2)} + e_j^{(2,1)}) \quad (4.5) \]
Here, we used the same notation as in Eq. (4.3) and
\[
I = p_{j}^{(1,1)} + p_{j}^{(1,2)} + p_{j}^{(2,1)} + p_{j}^{(2,2)}
\]
\[
B_{j}^{\pm} = b_{j}^{\pm(1,1)} + b_{j}^{\pm(1,2)} + b_{j}^{\pm(2,1)} + b_{j}^{\pm(2,2)}
\]
We include this second form since it treats both colours on an equal footing and might be more suitable for possible generalisations.

As in Sec. 2, exchanging \(\sigma \leftrightarrow -\sigma^{-1}\) leaves all algebraic expressions invariant. But contrary to the former case this does not lead to a different solution as the value of \(\tau\) (defined by \(\omega^{(1)} = \tau^{2}\)) is not affected by this transformation and hence the Yang–Baxter operator is also unchanged.

The remainder of this section deals with the \(B_{n}^{(1)}\) and \(D_{n+1}^{(2)}\) vertex models. This follows a dual purpose: on the one hand we want to show that the \(D_{n+1}^{(2)}\) models provide examples for the algebraic structure defined above, on the other hand we will show that the representations corresponding to the \(D_{n+1}^{(2)}\) vertex models can easily be obtained from those related to the \(B_{n}^{(1)}\) vertex models. The reason why this is important is simply that the same procedure should work for face models also (at least in the trigonometric case).

Let us commence with the braid–monoid algebra representation related to the \(B_{n}^{(1)}\) vertex models. We define
\[
b_{j}^{\pm} = I \otimes I \otimes \ldots \otimes I \otimes b_{j}^{\pm} \otimes I \otimes \ldots \otimes I \otimes I
\]
\[
e_{j} = I \otimes I \otimes \ldots \otimes I \otimes e \otimes I \otimes \ldots \otimes I \otimes I
\]
where \(b_{j}^{\pm}\) and \(e\) act at positions \(j\) and \(j+1\). Using the notation of Ref. [19], the explicit form of the \(d^{2} \times d^{2}\) \((d = 2n + 1)\) matrices \(b_{j}^{\pm}\) and \(e\) reads
\[
b = \sum_{\alpha} k^{-1} (1 + (k-1) \delta_{\alpha,\alpha'}) E_{\alpha,\alpha} \otimes E_{\alpha,\alpha}
\]
\[+ \sum_{\alpha \neq \beta} (1 + (k-1) \delta_{\alpha,\beta'}) E_{\alpha,\beta} \otimes E_{\beta,\alpha}
\]
\[- (k-k^{-1}) \sum_{\alpha < \beta} E_{\alpha,\alpha} \otimes E_{\beta,\beta}
\]
\[+ (k-k^{-1}) \sum_{\alpha > \beta} k^{\delta_{\alpha,\beta}} E_{\alpha',\beta} \otimes E_{\alpha,\beta'}
\]
\[
b^{-1} = \sum_{\alpha} k (1 + (k^{-1} - 1) \delta_{\alpha,\alpha'}) E_{\alpha,\alpha} \otimes E_{\alpha,\alpha}
\]
\[+ \sum_{\alpha \neq \beta} (1 + (k^{-1} - 1) \delta_{\alpha,\beta'}) E_{\alpha,\beta} \otimes E_{\beta,\alpha}
\]
\[+ (k-k^{-1}) \sum_{\alpha > \beta} E_{\alpha,\alpha} \otimes E_{\beta,\beta}
\]
\[- (k-k^{-1}) \sum_{\alpha < \beta} k^{\delta_{\alpha,\beta}} E_{\alpha',\beta} \otimes E_{\alpha,\beta'}
\]
\[
e = k^{2n-1} \sum_{\alpha, \beta} k^{\tilde{\alpha} - \tilde{\beta}} E_{\alpha', \beta} \otimes E_{\alpha, \beta'}
\]  
(4.10)

Here, \(1 \leq \alpha, \beta \leq d\), \(\alpha' = d + 1 - \alpha\) \((d = 2n + 1)\),

\[
\tilde{\alpha} = \begin{cases} 
\alpha + \frac{1}{2} & 1 \leq \alpha \leq n \\
\alpha & \alpha = n + 1 \\
\alpha - \frac{1}{2} & n + 2 \leq \alpha \leq 2n + 1 
\end{cases}
\]  
(4.11)

and \(E_{\alpha, \beta}\) are the \(d \times d\) matrices with elements \((E_{\alpha, \beta})_{i,j} = \delta_{i,\alpha}\delta_{j,\beta}\).

These matrices fulfill the equations

\[
(b - k^{-1} I) (b + k I) (b - k^{2n} I) = 0
\]  
(4.12)

and

\[
e = I + \frac{b - b^{-1}}{k - k^{-1}}
\]  
(4.13)

and hence \(b_j^\pm\) and \(e_j\) \((1.7)\) form a representation of the Birman–Wenzl–Murakami algebra with

\[
\omega = k^{2n}
\]
\[
\sqrt{Q} = 1 + \frac{k^{2n} - k^{-2n}}{k - k^{-1}}
\]  
(4.14)

The corresponding Yang–Baxter operator \((2.3)\) with \(\sigma = k\) and \(\tau = k^{n-1/2}\) yields exactly the \(R\) matrix of Ref. \([19]\) (with \(x = z^2\)).

In order to obtain a representation of the dilute Birman–Wenzl–Murakami algebra, we add one extra state to the local spaces which is going to correspond to the second colour. The corresponding matrices which act on the tensor product of two spaces now have dimension \((d + 1)^2 \times (d + 1)^2\). The (two-site) projectors \(p_{(a,b)}\) are given by \(p_{(a,b)} = P^{(a)} \otimes P^{(b)}\) with

\[
P^{(1)} = \sum_{\alpha} E_{\alpha, \alpha}
\]
\[
P^{(2)} = E_{d+1, d+1}
\]  
(4.15)

where here and in what follows the summation variables are restricted to values \(1 \leq \alpha, \beta \leq d\) which correspond to the states of colour “1” and the now \((d + 1) \times (d + 1)\) matrices \(E_{\alpha, \beta}\) are defined as above. The representation matrices for the Birman–Wenzl–Murakami part (which is the part that involves colour “1” only) \(b_j^{(1,1)}\) and \(e_j^{(1,1)}\) are just given by the same expressions as the matrices \(b_j^\pm\) \((Eqs. (1.8) and (1.9))\) and \(e_j\) \((1.10)\), respectively, but they now are of course \((d + 1)^2 \times (d + 1)^2\) matrices as well. The mixed braids \(b_j^{(1,2)}\) and \(b_j^{(2,1)}\) are

\[
b_j^{(1,2)} = \sum_{\alpha} E_{d+1, \alpha} \otimes E_{\alpha, d+1}
\]
\[
b_j^{(2,1)} = \sum_{\alpha} E_{\alpha, d+1} \otimes E_{d+1, \alpha}
\]  
(4.16)

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and the mixed Temperley–Lieb operators have the following form
\[
e^{(1,2)} = -k^{n+1} \sum_{\alpha} k^{-\bar{\alpha}} E_{d_{1},\alpha} \otimes E_{d_{1},\alpha'}
\]
\[
e^{(2,1)} = -k^{-(n+1)} \sum_{\alpha} k^{\bar{\alpha}} E_{\alpha',d_{1}} \otimes E_{\alpha,d_{1}}
\]
with \(\alpha'\) and \(\bar{\alpha}\) defined as before.

The above equations define a representation of the dilute Birman–Wenzl–Murakami algebra with
\[
\omega^{(1)} = k^{2n}
\]
\[
\sqrt{Q^{(1)}} = 1 + \frac{k^{2n} - k^{-2n}}{k - k^{-1}}
\]
\[
\omega^{(2)} = \sqrt{Q^{(2)}} = 1
\]
\[
(4.17)
\]
Correspondingly, we obtain a representation of the Yang–Baxter algebra from Eqs. (4.3) or (4.5) with \(\sigma = k\) and \(\tau = k^n\) and hence a solvable vertex model with \(2n+2\) states.

As it turns out, these models are just the \(D_{n+1}^{(2)}\) vertex models which play a somewhat singular role in Ref. [19] as they are the only series of \(R\) matrices which do not commute at one place (i.e., in general \([\hat{R}(u), \hat{R}(v)] \neq 0\) which already implies that the Yang–Baxter operator cannot be written as a polynomial in a braid operator alone. The expression obtained here (choosing \(\epsilon_1 = \epsilon_2 = 1\) in Eqs. (4.3) or (4.5)) is related to \(\hat{R}(x) = PR(x)\) of Ref. [19] (with \(x = z, \xi = k^n = \tau\), \(P\) is the permutation map \(P : v \otimes w \mapsto w \otimes v\)) by an orthogonal transformation with the matrix \(S \otimes S\) where \(S = S_1 \cdot S_2\) with
\[
S_1 = \sum_{\alpha=1}^{n+1} E_{\alpha,\alpha} + \sum_{\alpha=n+2}^{d} E_{\alpha,\alpha+1} + E_{d+1,n+2}
\]
\[
S_2 = \sum_{\alpha=1}^{n} E_{\alpha,\alpha} + \sum_{\alpha=n+3}^{d+1} E_{\alpha,\alpha}
\]
\[
+ \frac{1}{\sqrt{2}} \left( E_{n+1,n+1} + E_{n+1,n+2} + E_{n+2,n+1} - E_{n+2,n+2} \right)
\]
\[
(4.19)
\]
i.e., the additional colour “2” state corresponds in Jimbo’s basis [19] to the asymmetric combination of states \(n+1\) and \(n+2\). In particular, the projectors onto the two colours are not diagonal in that basis.

5 Summary and Outlook

A “dilute Birman–Wenzl–Murakami algebra” has been defined as a generalisation of the well-known Birman–Wenzl–Murakami algebra [6, 7]. This was done following the general ideas of Ref. [9] on multi-colour braid–monoid algebras. Similar to the Birman–Wenzl–Murakami case, the dilute algebra can be Baxterised to a two-parameter solution of the Yang–Baxter
algebra. This means that every appropriate matrix representation of the dilute algebra defines a solvable lattice model.

As an example, the representation of the Birman–Wenzl–Murakami algebra which corresponds to the $B_n^{(1)}$ vertex models was considered explicitly and was enlarged to a representation of the dilute algebra. It turned out that the such obtained solvable vertex models are the $D_{n+1}^{(2)}$ vertex models, the $R$ matrix differing from that of Ref. [19] just by a simple similarity transformation.

There are a number of question raised by the results of this paper.

The first, of course, is about the nature of the “second” series of solutions related to the $C_n^{(1)}$ representations of the Birman–Wenzl–Murakami algebra (see the last paragraph of Sec. 2). It would come as a surprise to the author if these would indeed correspond to new solvable models. It has to be checked if they do not in fact correspond to fusion $R$ matrices.

Another question concerns face models (IRF models [14]) related to the $D_{n+1}^{(2)}$ vertex models. The result of Sec. 4 means that one can construct such models (at least with trigonometric weights) on the basis of the known $B_n^{(1)}$ models [25] in a similar way as the dilute A–D–E models [10, 11, 12, 9] are related to the “usual” A–D–E models. To do this one has to find a dilute extension of the corresponding representations of the braid–monoid algebra.

Eqs. (4.3) or (4.5) give a solution of the Yang–Baxter equation for any representation of the dilute Birman–Wenzl–Murakami algebra. But if one has a representation of the Birman–Wenzl–Murakami algebra itself it appears to be quite straightforward to generalise it to the dilute case as one sees in the example of the $D_{n+1}^{(2)}$ models in Sec. 4. These could be constructed starting from the BWM representation provided by the $B_n^{(1)}$ models. On the other hand, we had three such series in Sec. 2, the other two being related to the $C_n^{(1)}$ and $D_n^{(1)}$ models. It therefore seems plausible that these should also give rise to corresponding series of dilute models which on the first view do not seem to fit in the list of known solvable models. But even if it turns out that they are related to known models (for instance by a gauge transformation) these expressions might still be of use. It is plausible that as it happens in other cases (for instance for the $A_2^{(1)}$ models [22, 10, 11], see also [12, 24]) there exist several series of solvable face models which are related to the vertex model $R$ matrix in different gauges.

These subjects are currently been investigated.

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