We study cosmological consequences of the dark spinor model when torsion is included. Only some components of the torsion are allowed to be non-vanishing in homogeneous and isotropic cosmology, but there exist freedoms in the choice of these components which is consistent with the evolution equations. We exploit this and discuss several cases which can result in interesting cosmological consequences. Especially, we show that there exist exact cosmological solutions in which the Universe began its acceleration only recently and this solution is an attractor. This corresponds to a specific form of the torsion with a mild fine-tuning which can address the coincidence problem.

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I. INTRODUCTION

To search for the theoretical foundation of the current accelerating Universe [1] has become one of the most fundamental problems in modern cosmology. An unknown source of energy which is repulsive in nature referred to as the dark energy is one of the most promising candidates and various proposals for its origin have been put forward [2]. But so far, relatively little attention has been paid to the possibility of fermionic sources in cosmology even though it has been known for some time that spinor fields could play some important roles in the evolution of the Universe [3–6]. For example, in Ref. [4] it is shown that a spinor field can accommodate any desired behavior of its energy density if an appropriate self-interaction of the spinor field is introduced and it is suggested that it can account for the dark energy and allows for realistic cyclic non-singular solutions. In Ref. [5] it is investigated whether fermionic sources could be responsible for the accelerating periods during the evolution of a universe after a matter field would guide for the decelerating period. In Ref. [6] the authors have shown that it is possible to simulate perfect fluid and dark energy by means of a nonlinear spinor field.

Recently, the so-called ELKO spinor, which is a new spin-half field defined as the eigenspinor of the charge conjugation operator with mass dimension one, was introduced in Ref. [7]. They have many special properties, one being that their dominant coupling is via the gravitational field and therefore are naturally dark. It has been proposed to be a candidate of dark matter [7] and dark energy [8] and many cosmological aspects of the ELKO spinor model have been investigated [9]. In addition, dark spinor was shown to be a potential candidate to probe non-trivial topologies in spacetime [10] and to be incorporated in a natural extension of the Standard Model by representing a mass dimension-transmuting operator between Dirac and ELKO spinor [11].

In Ref. [12], however, it was pointed out that the construction of ELKO spinors itself implicitly violates Lorentz invariance. The authors have proposed a non-local but Lorentz invariant version of the dark spinors. Also it is pointed out that some crucial errors of neglecting the spin connection part of the fermionic sector in the computations of stress energy tensors were made in the previous literatures on ELKO models. The correct expression for the ELKO spinor field was applied to dark energy and the result showed the existence of the de Sitter acceleration of the Universe. Another interesting consequences of dark spinor in cosmological aspects is that the equation of the state $\omega$ for the dark spinor can be negative and can cross the phantom-divide even without any phantom-like matter carrying negative kinetic energy [13]. It converges to a parameter value within the bound produced by using WMAP data [14] as was discussed in Ref. [8].
In spite of these nice features, further dynamical analysis of the ELKO model following this work using the correct form of the stress energy momentum tensor revealed that scaling attractor solutions do not exist in this model [15], and it has the difficulty of addressing the coincidence problem. This aspect of the dark spinor model draws our attention and motivates to extend to the case in which torsion [16] is included in the theory. First, recall that when the cosmological principle is extended to the general relativity theory with torsion, only the time component of the trace and totally anti-symmetric components of the torsion tensor can be nonzero [17]. For ordinary Dirac spinor, it is well-known that it interacts only with the totally anti-symmetric components of the torsion tensor and the spin connection part does not contribute to the total energy density [18]. On the other hand, the dark spinor interacts with all components of the torsion and we investigate the cosmological consequence of the torsion in the dark spinor model. In particular, we find the time component of the non-vanishing trace vector could provide a novel feature of explaining the coincidence problem. That is, with a mild fine-tuning of the parameters, we show that there exist a cosmological solution in which the Universe began its acceleration only recently and that the solution is an attractor.

The paper is organized as follows: In Sec. II, we give a brief summary of the spinor field in curved background with torsion in order to set the notation. In Sec. III, the action of the dark spinor is given and the equations of motions are calculated. In Sec. IV, we discuss various cosmological solutions focusing on the solution which can address the coincidence problem and stability analysis of this solution is performed. Sec. VI contains conclusion and discussions.

II. SPINOR IN CURVED BACKGROUND

In this section, we give a brief summary of the spinor field in curved background which is essential for our analysis. More detailed descriptions can be found in Ref. [19]. The gamma matrices are given by

$$\gamma^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \gamma^i = \begin{pmatrix} 0 & \sigma_i \\ -\sigma_i & 0 \end{pmatrix}$$

with \{γ^a, γ^b\} = −2η^{ab}, \ η^{ab} = \text{diag.}(−1, 1, 1, 1)

We introduce the metric and tetrad via

$$g_{\mu
u} = e^a_{\mu} e_b^\nu \eta_{ab}.$$  \hspace{1cm} (1)

Imposing the metricity \(\nabla_\mu g_{\alpha\beta} = 0\), the connection is given by

$$\Gamma^\rho_{\mu\nu} = \{ \rho_{\mu\nu} \} - K^\rho_{\mu\nu},$$ \hspace{1cm} (2)

where \{\} is the Christoffel symbol and the contortion \(K^\rho_{\mu\nu}\) is given by the torsion tensor \(S^\rho_{\mu\nu} = \Gamma^\rho_{[\mu\nu]}\) via [16].

$$K^\rho_{\mu\nu} = -(S^\rho_{\mu\nu} + S^\rho_{\nu\mu} + S^\rho_{\nu\mu}).$$ \hspace{1cm} (3)

The covariant derivative of the spinor and its dual are given by

$$\nabla_\mu \psi \equiv \partial_\mu \psi - \Gamma_\mu \psi$$ \hspace{1cm} (4)

and

$$\nabla_\mu \bar{\psi} \equiv \partial_\mu \bar{\psi} + \bar{\psi} \Gamma_\mu,$$ \hspace{1cm} (5)

where \(\Gamma_\mu\) is the connection on the spinor.

We introduce the spin connection by imposing the metricity for the tetrad

$$0 = \nabla_\mu e^a_{\nu} = \partial_\mu e^a_{\nu} - \Gamma^\rho_{\mu\nu} e^a_{\rho} + \omega^a_{\mu b} e^b_{\nu},$$ \hspace{1cm} (6)

where the spin connection satisfy \(\omega^a_{\mu b} = -\omega^a_{\mu b}\) and

$$\omega^a_{\mu b} = e^a_{\nu} (\partial_\mu e^b_{\nu} + \Gamma^\nu_{\mu\rho} e^b_{\rho}).$$ \hspace{1cm} (7)

We also impose the covariant constancy of the gamma matrices;

$$0 = \nabla_\mu \gamma^0 = \partial_\mu \gamma^0 + \Gamma^\rho_{\mu\nu} \gamma^\nu - [\Gamma_\mu, \gamma^\rho].$$ \hspace{1cm} (8)
From the eqs (7) and (9), we obtain
\[ \Gamma_\mu = \frac{1}{4} \omega^{ab}_\mu \gamma_a \gamma_b + cI. \] (10)

We consider only the case when the constant \( c \) is equal to zero.

We can decompose the contortion tensor (4) into a traceless part and trace [20]:
\[ K_{\mu \nu} = \tilde{K}_{\mu \nu} - \frac{2}{3} (\delta_\mu^\rho S_\nu - g_{\mu \nu} S^\rho), \] (11)

where \( \tilde{K}_{\mu \nu} \) is the traceless part, \( \tilde{K}_\mu^\mu = 0 \) and \( S_\nu \) is the trace of the torsion tensor, \( S_\nu = S_{\nu \mu}^\mu \). Making use of (11), we can write curvature scalar as follows:
\[ \bar{R} = R - 4\nabla_\mu S_\mu - \frac{8}{3} S_\mu S^\mu - \tilde{K}_{\nu \rho \alpha} \tilde{K}^{\alpha \nu \rho}. \] (12)

In the right hand side of the above equation, scalar curvature and the covariant derivative are calculated with respect to the Christoffel symbol.

### III. TORSION DARK SPINOR MODEL

Let us consider the dark spinor action of the form
\[ S = \int d^4x \sqrt{-g} \left[ \bar{R} + \frac{1}{2} g^{\mu \nu} \nabla_\mu \bar{\psi} \nabla_\nu \psi - V(\bar{\psi} \psi) \right] + S_m, \]
where \( \bar{R} \) is the scalar curvature including the torsion piece and \( V \) is the potential for the dark spinor (ELKO) field \( \psi \) and its dual \( \bar{\psi} \). \( S_m \) is the matter field action.

To discuss the Friedman cosmology in the flat Robertson-Walker space-time, consider a metric of the form
\[ ds^2 = -N^2 dt^2 + a(t)^2 dx^i dx^i, \] (13)
where \( a(t) \) is the scale factor of our three dimensional universe. We assume
\[ \psi(x^\mu) = \phi(t) \xi \] (14)
with a homogeneous real scalar field \( \phi(t) \) and a constant spinor \( \xi \) such that \( \bar{\xi} \xi = 1 \) but \( \nabla_\mu \xi \neq 0 \). It is to be pointed out that the spinor nature of the dark spinor is remnantal in the real scalar field \( \phi(t) \) defined in (14) and the ensuing evolution equations contain the contributions from the spin connection which are absent in the case of a genuine scalar field. It was shown that these new contributions supply fundamentally different aspects in the dark spinor cosmology [12].

For isotropy and homogeneity of Universe, one assumes the following non-vanishing components of the torsion [17]
\[ S_{10} = S_{20} = S_{30} = h(t)/2, \quad S_{jk} = S_{ijk} = f(t)/3\epsilon_{ijk}. \] (15)

Eqs. (13) and (15) give the following connection components:
\[ \Gamma^0_{00} = \frac{\dot{N}}{N}, \quad \Gamma^0_{ij} = \frac{ab}{N^2} \delta_{ij}, \quad \Gamma^i_{j0} = \frac{b}{a} \delta_{ij}, \quad \Gamma^i_{0j} = \frac{\dot{a}}{a} \delta_{ij}, \quad \Gamma^i_{jk} = f \epsilon_{ijk}, \] (16)

with \( b = \dot{a} + a h \). Using the above components, we obtain the scalar curvature
\[ \bar{R} = 6 \left[ \frac{1}{aN} \frac{d}{dt} \left( \frac{b}{N} \right) + \left( \frac{b}{aN} \right)^2 - \left( \frac{f}{a} \right)^2 \right]. \] (17)

Note that the above result [17] agrees with Eq. (12) up to a total derivative term. From the metric (13), the tetrad is given by
\[ e^a_\mu = (N, a, a, a), \quad e^\mu_a = (N, a, a, a). \] (18)
Using (16), the only non-vanishing components of the spin connection (8) are given by

\[ \omega_{x^i}^j = \omega^{\mu}_{x^i} = \frac{b}{N} \delta_{ij}, \quad \omega_{x^i}^j = f \epsilon_{ijk}. \]  

(19)

Then, we have from eq. (10)

\[ \Gamma_0 = 0, \quad \Gamma_{x^i}^j = \frac{b}{2N} \gamma_0 \gamma_i + \frac{f}{4} \epsilon_{ijk} \gamma_k. \]  

(20)

Putting everything together, we have (\( \kappa = 1 \))

\[ S = \int dt \left[ \frac{1}{N} \left( -3a \dot{a}^2 + 3a^3 \dot{h}^2 + \frac{3}{2} a^3 \dot{\phi}^2 + \frac{3}{8} \dot{a}^2 \dot{\phi}^2 \right) - N \left( 3a f^2 + \frac{3}{8} \dot{a}^2 \phi^2 + a^3 V \right) \right] + S_m \]  

(21)

The equation related with the energy density coming from the variation with respect to \( N \) is given by

\[ 3H^2 = \frac{1}{2} \dot{\phi}^2 + V + \frac{3}{8} H^2 \dot{\phi}^2 + \frac{3}{4} H \dot{h} \phi^2 + \frac{3}{2} \left( 1 + \frac{1}{8} \dot{\phi}^2 \right) h \dot{h} + 3 \left( 1 + \frac{1}{8} \dot{\phi}^2 \right) \frac{f^2}{a^2} + \rho_m. \]  

(22)

and the one related with the pressure coming from the variation with respect to \( a \) is

\[ -2 \dot{H} - 3H^2 = \frac{1}{2} \dot{\phi}^2 - V - \frac{3}{8} H^2 \dot{\phi}^2 - \frac{1}{4} \frac{d}{dt} \left[ (H + h) \phi^2 \right] + 3 \left( 1 + \frac{1}{8} \dot{\phi}^2 \right) h \dot{h} + 3 \left( 1 + \frac{1}{8} \dot{\phi}^2 \right) \frac{f^2}{a^2} + p_m. \]  

(23)

The scalar field equation is given by

\[ 0 = \ddot{\phi} + 3H \dot{\phi} + V'(\phi) - \frac{3}{4} \left( (H + h)^2 - \frac{f^2}{a^2} \right) \phi. \]  

(24)

Note that the above equations (22), (24), (23) are different from the equations one would obtain by directly using \( \bar{G}_{\mu\nu} = T^m_{\mu\nu} \), where \( \bar{G}_{\mu\nu} \) is the Einstein tensor calculated with the connections including the torsion as in Eq. (16). \( S_m \). The discrepancy comes from the fact that when torsion is included, the variation \( \delta R_{\mu\nu} \) of the Ricci tensor in calculating the equations of motion from the scalar curvature is not a total derivative and cannot be neglected as in the case without torsion. Therefore, \( \bar{G}_{\mu\nu} = T^m_{\mu\nu} \) is not justified in the case when torsion is included. Differentiation Eq. (22) with respect to time and using Eqs. (24) and (23), we obtain the following consistency equation:

\[ 0 = 3h \left[ 2 \dot{h} + 6Hh + \frac{1}{4} (H + h) \phi \dot{\phi} + \frac{3}{4} H (H + h) \phi^2 \right] + 6 \frac{f^2}{a^2} \left( 1 + \frac{1}{8} \dot{\phi}^2 \right) + \dot{\rho}_m + 3H (\rho_m + p_m), \]  

(25)

which serves as a dynamical equations for the torsion.

IV. COSMOLOGICAL SOLUTIONS

In this section, we discuss diverse cosmological solutions depending on the torsion.

A. Torsion as a dark matter candidate

We first analyze the pure torsion case where the dark spinor field is absent, \( \phi \equiv 0 \). Let us also assume that \( \rho_m = p_m = 0 \). If we define \( \rho_h = 3h^2 \) and \( \rho_f = f^2/a^2 \), we have from Eqs. (22) and (23)

\[ 3H^2 = \rho_h + \rho_f, \quad -2H - 3H^2 = \rho_h - \frac{1}{3} \rho_f. \]  

(26)

The second equation says that \( p_h = \rho_h \), whereas \( p_f = -\rho_f/3 \). Eq. (25) gives the continuity equation for the torsion energy density:

\[ \dot{\rho}_h + 6H \rho_h + \dot{\rho}_f + 2H \rho_f = 0. \]  

(27)
Let us suppose that \( \rho_h \) and \( \rho_f \) are separately conserved. Then, \( \rho_h \propto 1/a^6 \) corresponding to the equation of state \( \omega_h = 1 \) and \( f = f_0 \) is a constant corresponding to \( \omega_f = -1/3 \) as expected. However, if we assume \( \rho_f = r \rho_h \) with a constant \( r > 0 \) and \( \rho_t = \rho_h + \rho_f = (1 + r) \rho_h \), and the pressure \( p_t = \rho_h - \rho_f/3 = (1 - r/3) \rho_h \), Eq. (27) becomes

\[
\dot{\rho}_t + 3AH \rho_t = 0, \quad A = \frac{6 + 2r}{3 + 3r}.
\]

Therefore, \( \rho_t \) can describe an ideal fluid with the range of the equation of state given by \( -1/3 \leq \omega_t = 3 - r/3 + 3r \leq 1 \).

When \( r = 3 \), it describes the cold dark matter. It is to be commented that the relation \( \rho_f = r \rho_h \) is very ad hoc, nevertheless nothing forces separate conservation as far as continuity equation (27) is concerned.

### B. Torsion and cosmological constant

Next, we consider the cases \( f = 0 \). Let us define a quantity

\[
X = 2h + \frac{1}{4} (H + h) \phi^2.
\]

We assume that there is no interaction between the matter and spinor dark energy with torsion, and the matter and the dark spinor are separately conserved.

\[
\dot{\rho}_m + 3H(\rho_m + p_m) = 0.
\]

Then from Eq. (25), we have

\[
3h \left( \dot{X} + 3HX \right) = 0.
\]

We consider \( h \neq 0 \) and then, Eq. (31) can be immediately integrated to give

\[
X = \frac{c_1}{a^3}.
\]

Eq. (23) by using (22) becomes

\[
-2 \left( \dot{H} + h \right) = \dot{\phi}^2 + 3(H + h)X + \rho_m + p_m.
\]

Also, Eqs. (22) and (24) can be rewritten as

\[
3H^2 = \frac{1}{2} \dot{\phi}^2 + V + \frac{6}{\phi^2} (X - 2h)^2 + 3h^2 + \rho_m.
\]

and

\[
0 = \ddot{\phi} + 3H \dot{\phi} + V'(\phi) - 12 \frac{(X - 2h)^2}{\phi^3}, \quad \left( \tau \equiv \frac{d}{d\phi} \right)
\]

We have four equations (31), (32), (33), and (34). These equations are not independent, and we have three equations to be solved including Eq. (33).

The torsion component \( h \) which was introduced for homogeneity and isotropy via (15) is an external input as long as it satisfies the evolution equations. We exploit this arbitrariness to assume an existence of the functional dependence of \( h(t) \equiv h(\phi) \) and discuss cases where some analytic properties of the above equations can be investigated.

Let us discuss the case where \( \rho_m = p_m = 0 \). Under the further simplifying assumption of \( c_1 = 0 \), some exact cosmological solutions can be obtained in a couple of cases which describe the accelerating Universe. First, \( X = 0 \) and we have

\[
H = -h \left( 1 + \frac{8}{\phi^2} \right).
\]

We are interested in the case of \( h < 0 \). From Eq. (32), we have

\[
\dot{\phi} = \frac{16}{\phi^3} (h \phi - 2h),
\]
which yields from Eq. (33) the potential of the form
\[ V = \frac{24 h^2}{\phi^4} \left(8 + \phi^2\right) - \frac{128}{\phi^6} (h,\phi - 2h)^2. \] (37)

One can check that Eq. (34) is satisfied with Eqs. (35), (36) and (37).

Let us discuss a couple of cases where exact cosmological solution can be obtained. The first case is when the potential is of the form
\[ V(\phi) = \frac{1}{2} m^2 \phi^2 + V_0. \] (38)

It corresponds to the choice of \( h \propto \phi^2 \). Comparing with the potential (37), we find they coincide if we choose \( h = \pm m \phi^2 / \sqrt{48} \). Moreover, the cosmological constant \( V_0 \) is given by
\[ V_0 = 4m^2. \] (39)

One can check that Eq. (34) is also satisfied. From Eq. (36), we have \( \dot{\phi} = 0 \), and \( \phi = \phi_0 \). Choosing \( h = -m \phi^2 / \sqrt{48} \), the Hubble parameter is given by
\[ H = \frac{m}{\sqrt{48}} (8 + \phi_0^2), \] (40)

for the de Sitter accelerating solution. A couple of comments are in order: First, \( \phi \) need not be the minimum of the potential (38) for its stable equilibrium point and can stay away from zero. This can be seen dynamically from Eq. (34) as follows: The restoring force coming from \( V' \) is exactly canceled by the repulsive force coming from the torsion if \( h \propto \phi^2 \). Therefore, the scalar field undergoes a damped motion without any force acting; and this solution corresponds to the solution in which if it takes initial value \( \phi_0 \), it stays there forever. Also, we note the difference with an ordinary quintessence model with mass term and cosmological constant. In this case, the \( \phi = 0 \) is dynamically preferred, and the acceleration can be driven by the cosmological constant. However, the crucial difference is that there is a priori no relation between the cosmological constant and the mass of the scalar field, whereas in the dark spinor model with torsion, there is a relation given by eq. (39). This implies that the smallness of the cosmological constant is directly interwoven with the ultra light scalar field.

C. Torsion and coincidence problem

Another case of interest is when \( h \) is given by
\[ h = -|c| \exp(-\lambda \phi / 2) \phi^2 \ (\lambda > 0). \] (41)

In this case, we have potential of the form
\[ V = V_* e^{-\lambda \phi} \left[ \phi^2 + c_* \right], \ V_* = 24c^2, \ c_* = 8 - \frac{3}{2} \lambda^2. \] (42)

Eq. (36) gives
\[ \dot{\phi} = \lambda_* e^{-\lambda \phi / 2} \ (\lambda_* = 8 |c| \lambda), \] (43)

which can be integrated to yield
\[ \phi(t) = \frac{2}{\lambda} \ln t + A. \] (44)

We choose the integration constant \( A \) of the above equation to be zero which fixes a relation \( 4 |c| \lambda^2 = 1 \). The Hubble parameter \( H \) by using Eq. (35) and its first time derivative are given by
\[ H = \frac{2}{\lambda^2 t} \left[ 1 + \frac{(\ln t)^2}{2 \lambda^2} \right], \ \dot{H} = -\frac{2}{\lambda^2 t^2} \left[ 1 - \frac{\ln t}{\lambda^2} + \frac{(\ln t)^2}{2 \lambda^2} \right]. \] (45)
To calculate the scale factor, let us define $H_\phi = d\ln a/d\phi$. Then, Eq. (35) can be readily integrated to give
\begin{equation}
a(\phi) = a_e e^{\frac{1}{2}(\phi+\phi^3)} = a_e e^{\frac{1}{2}(\ln t+\frac{(\ln t)^3}{6\lambda^2})},
\end{equation}
(46)

To study the existence of the accelerating solution, we first check
\begin{equation}
\frac{\ddot{a}}{a} = \dot{H} + H^2 = \frac{1}{t^2} \left[ -\frac{2}{\lambda^2} \left( 1 - \frac{2}{\lambda^2} \right) + 2 \ln t - \left( 1 - \frac{4}{\lambda^2} \right) \frac{(\ln t)^2}{\lambda^4} + \frac{(\ln t)^4}{\lambda^8} \right].
\end{equation}
(47)
We see that at early times when $\lambda^{3/2} < \ln t < \lambda^2 \ (\lambda > 1)$, Eq. (47) describes a decelerating universe. This is the epoch when the first term in $\dot{H}$ is dominant. When $\ln t > \lambda^2$ the $(\ln t)^4$ term begins to become dominant gradually and at later times, $H^2$ term becomes dominant. The acceleration would begin around $\ln t_{acc} = \lambda^2$. If we adjust the parameter $\lambda$ to be $\sim 12$ we get the late time acceleration occurring around $t_{acc} \sim 10^{18} t_{pl}$, which roughly corresponds to the current age of the Universe. This could explain the coincidence problem. The dependence of $t_{acc}$ on the numerical value of $\lambda$ is depicted in Fig. 1. Comparing with the $\Lambda - CDM$ which requires a fine-tuning of the order of $\sim 10^{-120}$ for the cosmological constant, it is of $O(1)$ and poses no fine-tuning issue.

![FIG. 1: The relation between $t_{acc}$ and the parameter $\lambda$ in the Planck unit. The current acceleration can be achieved with $\lambda \approx 12$.](image)

We perform a stability analysis of the solution (44) and (46) with $A = 0$. Variation of the Eq. (34) with $X = 0$ gives
\begin{equation}
0 = \delta \ddot{\phi} + 3H \delta \dot{\phi} + 3\dot{\phi} \delta H + V''(\phi) \delta \phi + 144 \frac{h^2}{\phi^2} \delta \phi - 9e^{\frac{1}{6}} \frac{h}{\phi^3} \delta \phi.
\end{equation}
(48)
$\delta H$ can be eliminated through Eq. (35) via
\begin{equation}
\delta H = \left[ -h'(1 + \frac{8}{\phi^2}) + 16 \frac{h}{\phi^3} \right] \delta \phi.
\end{equation}
(49)
Using Eqs. (41), (42), and (44), we find that Eq. (48) reduces to
\begin{equation}
\delta \ddot{\phi} + \frac{3}{\lambda^4} \left( 2\lambda^2 + (\ln t)^2 \right) \delta \dot{\phi} + \frac{3}{\lambda^4} \left( f(\lambda) + (\ln t)^2 \right) \delta \phi = 0,
\end{equation}
(50)
with $f(\lambda)$ being
\begin{equation}
f(\lambda) = -\frac{3}{4} \lambda^4 + 2\lambda^2 - \frac{1}{3}.
\end{equation}
(51)
A numerical analysis of Eq. (50) shows the attractor behavior of the solution, as is shown in Fig. 2.
FIG. 2: Attractor behavior: The initial condition is given at $N_i \equiv \ln t_i \sim 130$ which corresponds approximately to the decoupling time. The red curve is with $\phi(N_i) = 0.005, \dot{\phi}(N_i) = 0.01$, blue with $\phi(N_i) = -0.001, \dot{\phi}(N_i) = 0.01$, green with $\phi(N_i) = -0.005, \dot{\phi}(N_i) = -0.01$, and pink: $\phi(N_i) = 0.001, \dot{\phi}(N_i) = -0.01$.

A dynamical estimate how the solution becomes an attractor can be given. To check the stability in the asymptotic region where $\ln t >> \lambda^2$, $\phi >> 2\lambda$, we first note that from Eq. (42)

$$V'' \delta \phi \sim 24e^2 e^{-\lambda \phi^2} = \frac{6}{\lambda^4 t^2} (\ln t)^2 \delta \phi,$$

and among the last three terms of Eq. (48), $V''(\phi)$ term dominates. This is a positive quantity and provides a restoring force in the asymptotic region. Also, from Eq. (49), we have

$$3 \delta H \dot{\phi} \sim -\frac{\lambda}{2} H \delta \phi = -\frac{3}{\lambda^4 t^2} (\ln t)^2 \delta \phi.$$

This term acts as a source of a repulsive force, but their magnitude is smaller then the attractive force of Eq. (52) Over all, in the asymptotic region, the perturbation equation becomes

$$\delta \ddot{\phi} + \frac{3}{\lambda^4 t} \delta \dot{\phi} + \frac{3}{\lambda^4 t^2} (\ln t)^2 \delta \phi = 0.$$  

It can be easily checked that the perturbation decays as

$$\delta \phi \sim \frac{(\ln t)^\alpha}{t},$$

and the solution is an attractor in the asymptotic region.

V. CONCLUSION

We have examined the possible cosmological role that the torsion could play in the dark spinor model. Freedom exists in the choice of the torsion, which is consistent with the continuity equation. This has been exploited for the
choice of diverse forms of the torsion, which resulted in the various cosmological consequences. In the pure torsion case without dark spinor, torsion could play the role of the dark matter. When the torsion is proportional to $\phi^2$, we have an accelerating de Sitter Universe. In particular, we found an exact cosmological solution in which the Universe began its acceleration only recently and it is an attractor. This corresponds to a specific form of the torsion with a mild fine-tuning and it can address the coincidence problem.

In this work, the torsion was assumed to be a kind of external matter field whose dynamics is dictated by the continuity equation. Even though this does not confront with any theoretical inconsistency, their choices were restrictive to certain forms and could be regarded as rather ad hoc. In the more fundamental approach like the Einstein-Cartan-Sciama-Kibble theory of gravity \cite{skordis} where the torsion tensor is regarded as a dynamical variable from the beginning, the equations of motion coming from the variation with respect to the torsion generates a spin-spin interaction equation. In the Dirac case, this interaction is significant only at extremely high densities and it was shown that such an interaction averts the unphysical big-bang singularity, replacing it with a cusp-like bounce at a finite minimum scale factor \cite{amendola}. This approach could be extended to the ELKO case \cite{ribas}. In Ref. \cite{oh}, the most general ELKO matter fields including the dynamical torsion was constructed, whose cosmology leads to anisotropic universes. To the knowledge of the authors, the relation between the two approaches has not been explored, and it seems that at present only the former approach could provide the novel cosmological feature of addressing the coincidence problem for the dark spinor.

One interesting feature of the potential \cite{amendola} is that in contrast with the pure exponential potential in the quintessence model where the parameter $\lambda$ is restricted to be within a certain range to produce the attractor solution, the solution with the torsion is an attractor for each value of $\lambda > 0$. Moreover, the term $e^{-\lambda \phi/2} g^2$ is reminiscent of the Albrecht-Skordis potential \cite{bskordis} which appears in the low energy limit of $M$ theory. The presence of the torsion is a key ingredient and possible implications of this aspect deserve further investigations.

We conclude with the following remarks. As shown in Eq. \eqref{eq:potential} of Sec. IV B, there is a relation between the cosmological constant and the mass of the cosmological particle $\phi$ in our dark spinor model with torsion, which might represent the main component of our present universe and is about $10^{-3} g$, as in the massive vector model \cite{ahluwalia}. When the analysis is extended to the case including a conformal coupling of dark spinor fields to gravity, it will provide with a richer variety of cosmological consequences than the case without torsion \cite{hoff-da-silva}. Last but not least, possible implications of the experimental limits on torsion \cite{hoff-da-silva} in the dark spinor case need to be explored.

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