On a Shape Optimization Problem for Tree Branches

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Abstract

This paper is concerned with a shape optimization problem, where the functional to be maximized describes the total sunlight collected by a distribution of tree leaves, minus the cost for transporting water and nutrient from the base of trunk to all the leaves. In the case of 2 space dimensions, the solution is proved to be unique, and explicitly determined.

Keywords: shape optimization, sunlight functional, branched transport.

MSC: 49Q10, 49Q20.

1 Introduction

In the recent papers [7, 9] two functionals were introduced, measuring the amount of light collected by the leaves, and the amount of water and nutrients collected by the roots of a tree. In connection with a ramified transportation cost [1, 14, 18], these lead to various optimization problems for tree shapes.

Quite often, optimal solutions to problems involving a ramified transportation cost exhibit a fractal structure [2, 3, 4, 12, 15, 16, 17]. In the present note we analyze in more detail the optimization problem for tree branches proposed in [7], in the 2-dimensional case. In this simple setting, the unique solution can be explicitly determined. Instead of being fractal, its shape reminds of a solar panel.

The present analysis was partially motivated by the goal of understanding phototropism, i.e., the tendency of plant stems to bend toward the source of light. Our results indicate that this
behavior cannot be explained purely in terms of maximizing the amount of light collected by the leaves (Fig. 1). Apparently, other factors must have played a role in the evolution of this trait, such as the competition among different plants. See [6] for some results in this direction.

The remainder of this paper is organized as follows. In Section 2 we review the two functionals defining the shape optimization problem, and state the main results. Proofs are then worked out in Sections 3 to 5.

![Figure 1: A stem $\gamma_1$ perpendicular to the sun rays is optimally shaped to collect the most light. For the stem $\gamma_2$ bending toward the light source, the upper leaves put the lower ones in shade.](image)

2 Statement of the main results

We begin by reviewing the two functionals considered in [7, 9].

2.1 A sunlight functional

Let $\mu$ be a positive, bounded Radon measure on $\mathbb{R}^d_+ = \{(x_1, x_2, \ldots, x_d); \ x_d \geq 0\}$. Thinking of $\mu$ as the density of leaves on a tree, we seek a functional $S(\mu)$ describing the total amount of sunlight absorbed by the leaves. Fix a unit vector $n \in S^{d-1} = \{x \in \mathbb{R}^d; \ |x| = 1\}$, and assume that all light rays come parallel to $n$. Call $E_n^\perp$ the $(d-1)$-dimensional subspace perpendicular to $n$ and let $\pi_n : \mathbb{R}^d \mapsto E_n^\perp$ be the perpendicular projection. Each point $x \in \mathbb{R}^d$ can thus be expressed uniquely as

$$x = y + sn$$

(2.1)

with $y \in E_n^\perp$ and $s \in \mathbb{R}$.

On the perpendicular subspace $E_n^\perp$ consider the projected measure $\mu^n$, defined by setting

$$\mu^n(A) = \mu\left(\{x \in \mathbb{R}^d; \ \pi_n(x) \in A\}\right).$$

(2.2)

Call $\Phi^n$ the density of the absolutely continuous part of $\mu^n$ w.r.t. the $(d-1)$-dimensional Lebesgue measure on $E_n^\perp$. 

2
**Definition 2.1** The total amount of sunlight from the direction $n$ captured by a measure $\mu$ on $\mathbb{R}^d$ is defined as
\[
S^n(\mu) = \int_{E^n_\mu} \left(1 - \exp\{-\Phi^n(y)\}\right) dy.
\] (2.3)

More generally, given an integrable function $\eta \in L^1(S^{d-1})$, the total sunshine absorbed by $\mu$ from all directions is defined as
\[
S^n(\mu) = \int_{S^{d-1}} \left(\int_{E^n_\mu} \left(1 - \exp\{-\Phi^n(y)\}\right) dy\right) \eta(n) \, dn.
\] (2.4)

In the formula (2.4), $\eta(n)$ accounts for the intensity of light coming from the direction $n$.

**Remark 2.2** According to the above definition, the amount of sunlight $S^n(\mu)$ captured by the measure $\mu$ only depends on its projection $\mu^n$ on the subspace perpendicular to $n$. In particular, if a second measure $\tilde{\mu}$ is obtained from $\mu$ by shifting some of the mass in a direction parallel to $n$, then $S(\tilde{\mu}) = S(\mu)$.

### 2.2 Optimal irrigation patterns

Consider a positive Radon measure $\mu$ on $\mathbb{R}^d$ with total mass $M = \mu(\mathbb{R}^d)$, and let $\Theta = [0, M]$. We think of $\xi \in \Theta$ as a Lagrangian variable, labeling a water particle.

**Definition 2.3** A measurable map
\[
\chi : \Theta \times \mathbb{R}_+ \mapsto \mathbb{R}^d
\] (2.5)

is called an **admissible irrigation plan** if

(i) For every $\xi \in \Theta$, the map $t \mapsto \chi(\xi, t)$ is Lipschitz continuous. More precisely, for each $\xi$ there exists a stopping time $T(\xi)$ such that, calling
\[
\dot{\chi}(\xi, t) = \frac{\partial}{\partial t} \chi(\xi, t)
\]
the partial derivative w.r.t. time, one has
\[
|\dot{\chi}(\xi, t)| = \begin{cases} 
1 & \text{for a.e. } t \in [0, T(\xi)], \\
0 & \text{for } t > T(\xi).
\end{cases}
\] (2.6)

(ii) At time $t = 0$ all particles are at the origin: $\chi(\xi, 0) = 0$ for all $\xi \in \Theta$.

(iii) The push-forward of the Lebesgue measure on $[0, M]$ through the map $\xi \mapsto \chi(\xi, T(\xi))$ coincides with the measure $\mu$. In other words, for every open set $A \subset \mathbb{R}^d$ there holds
\[
\mu(A) = \operatorname{meas}\left\{\xi \in \Theta; \ \chi(\xi, T(\xi)) \in A\right\}.
\] (2.7)
One may think of \( \chi(\xi, t) \) as the position of the water particle \( \xi \) at time \( t \).

To define the corresponding transportation cost, we first compute how many particles travel through a point \( x \in \mathbb{R}^d \). This is described by

\[
| x |_\chi \doteq \text{meas}\left( \{ \xi \in \Theta \mid \chi(\xi, t) = x \text{ for some } t \geq 0 \} \right).
\] (2.8)

We think of \( | x |_\chi \) as the total flux going through the point \( x \). Following [13, 14], we consider

**Definition 2.4 (irrigation cost).** For a given \( \alpha \in [0, 1] \), the total cost of the irrigation plan \( \chi \) is

\[
\mathcal{E}^\alpha(\chi) \doteq \int_\Theta \left( \int_0^{T(\xi)} | \chi(\xi, t)|^{\alpha-1} dt \right) d\xi.
\] (2.9)

The \( \alpha \)-irrigation cost of a measure \( \mu \) is defined as

\[
\mathcal{T}^\alpha(\mu) \doteq \inf_{\chi} \mathcal{E}^\alpha(\chi),
\] (2.10)

where the infimum is taken over all admissible irrigation plans for the measure \( \mu \).

**Remark 2.5** Sometimes it is convenient to consider more general irrigation plans where, in place of (2.6), for a.e. \( t \in [0, T(\xi)] \) the speed satisfies \( |\dot{\chi}(\xi, t)| \leq 1 \). In this case, the cost (2.9) is replaced by

\[
\mathcal{E}^\alpha(\chi) \doteq \int_\Theta \left( \int_0^{T(\xi)} | \chi(\xi, t)|^{\alpha-1} |\dot{\chi}(\xi, t)| dt \right) d\xi.
\] (2.11)

Of course, one can always re-parameterize each trajectory \( t \mapsto \chi(\xi, t) \) by arc-length, so that (2.6) holds. This does not affect the cost (2.11).

**Remark 2.6** In the case \( \alpha = 1 \), the expression (2.9) reduces to

\[
\mathcal{E}^1(\chi) \doteq \int_\Theta \left( \int_{\mathbb{R}_+} |\dot{\chi}(\xi, t)| dt \right) d\xi = \int_\Theta \text{total length of the path } \chi(\xi, \cdot) \, d\xi.
\]

Of course, this length is minimal if every path \( \chi(\cdot, \xi) \) is a straight line, joining the origin with \( \chi(\xi, T(\xi)) \). Hence

\[
\mathcal{T}(\mu) \doteq \inf_{\chi} \mathcal{E}(\chi) = \int_\Theta |\chi(\xi, T(\xi))| \, d\xi = \int |x| d\mu.
\]

On the other hand, when \( \alpha < 1 \), moving along a path which is traveled by few other particles comes at a high cost. Indeed, in this case the factor \( |\chi(\xi, t)|^{\alpha-1} \) becomes large. To reduce the total cost, is thus convenient that many particles travel along the same path.

For the basic theory of ramified transport we refer to the monograph [1]. For future use, we recall that optimal irrigation plans satisfy

**Single Path Property:** If \( \chi(\xi, \tau) = \chi(\xi', \tau') \) for some \( \xi, \xi' \in \Theta \) and \( 0 < \tau \leq \tau' \), then

\[
\chi(\xi, t) = \chi(\xi', t) \text{ for all } t \in [0, \tau].
\] (2.12)
2.3 The general optimization problem for branches.

Combining the two functionals (2.4) and (2.10), one can formulate an optimization problem for the shape of branches:

\[
\text{(OPB) Given a light intensity function } \eta \in \mathcal{L}^1(S^{d-1}) \text{ and two constants } c > 0, \alpha \in [0, 1], \text{ find a positive measure } \mu \text{ supported on } R^d_\ast \text{ that maximizes the payoff}
\]

\[
S^\theta(\mu) - c I^\alpha(\mu).
\tag{2.13}
\]

2.4 Optimal branches in dimension \(d = 2\).

We consider here the optimization problem for branches, in the planar case \(d = 2\). We assume that the sunlight comes from a single direction \(\mathbf{n} = (\cos \theta_0, \sin \theta_0)\), so that the sunlight functional takes the form (2.3). Moreover, as irrigation cost we take (2.10), for some fixed \(\alpha \in [0, 1]\). For a given constant \(c > 0\), this leads to the problem

\[
\text{maximize: } S^\theta(\mu) - c I^\alpha(\mu),
\tag{2.14}
\]

over all positive measures \(\mu\) supported on the half space \(R^2_+ \doteq \{x = (x_1, x_2); \ x_2 \geq 0\}\). To fix the ideas, we shall assume that \(0 < \theta_0 < \pi/2\). Our main goal is to prove that for this problem the “solar panel” configuration shown in Fig. 2 is optimal, namely:

**Theorem 2.7** Assume that \(0 < \theta_0 \leq \pi/2\) and \(1/2 \leq \alpha \leq 1\). Then the optimization problem (2.14) has a unique solution. The optimal measure is supported along two rays, namely

\[
\text{Supp}(\mu) \subset \{(r \cos \theta, r \sin \theta); \ r \geq 0, \ \text{either } \theta = 0 \text{ or } \theta = \theta_0 + \pi/2\} \doteq \Gamma_0 \cup \Gamma_1.
\tag{2.15}
\]

When \(0 < \alpha < 1/2\), the same conclusion holds provided that the angle \(\theta_0\) satisfies

\[
\cos \left(\frac{\pi}{2} - \theta_0\right) \geq 1 - 2^{2^2 - 1}.
\tag{2.16}
\]

![Figure 2: When the light rays impinge from a fixed direction \(\mathbf{n}\), the optimal distribution of leaves is supported on the two rays \(\Gamma_0\) and \(\Gamma_1\).](image)

In the case \(\alpha = 1\) the result is straightforward. Indeed, for any measure \(\mu\) we can consider its projection \(\tilde{\mu}\) on \(\Gamma_0 \cup \Gamma_1\), obtained by shifting the mass in the direction parallel to the vector
n. In other words, for \( x \in \mathbb{R}^2 \) call \( \phi^n(x) \) the unique point in \( \Gamma_0 \cup \Gamma_1 \) such that \( \phi^n(x) - x \) is parallel to \( n \). Then let \( \tilde{\mu} \) be the push-forward of the measure \( \mu \) w.r.t. \( \phi^n \). Since this projection satisfies \( |\phi^n(x)| \leq |x| \) for every \( x \in \mathbb{R}^2_+ \), the transportation cost decreases. On the other hand, by Remark 2.2 the sunlight captured remains the same. We conclude that

\[
S^n(\tilde{\mu}) - I^1(\tilde{\mu}) \geq S^n(\mu) - I^1(\mu),
\]

with strict inequality if \( \mu \) is not supported on \( \Gamma_0 \cup \Gamma_1 \).

In the case \( 0 < \alpha < 1 \), the result is not so obvious. Indeed, we do not expect that the conclusion holds if the hypothesis (2.16) is removed. A proof of Theorem 2.7 will be worked out in Sections 3 and 4.

Having proved that the optimal measure \( \mu \) is supported on the two rays \( \Gamma_0 \cup \Gamma_1 \), the density of \( \mu \) w.r.t. one-dimensional measure can then be determined using the necessary conditions derived in [6]. Indeed, the density \( u_1 \) of \( \mu \) along the ray \( \Gamma_1 \) provides a solution to the scalar optimization problem

\[
\text{maximize: } J_1(u) = \int_0^{+\infty} \left( 1 - e^{-u(s)} \right) ds - c \int_0^{+\infty} \left( \int_s^{+\infty} u(r) dr \right)^\alpha ds, \tag{2.17}
\]

among all non-negative functions \( u : \mathbb{R}_+ \to \mathbb{R}_+ \). Here \( s \) is the arc-length variable along \( \Gamma_1 \). Similarly, the density \( u_0 \) of \( \mu \) along the ray \( \Gamma_0 \) provides a solution to the problem

\[
\text{maximize: } J_0(u) = \int_0^{+\infty} \sin \theta_0 \left( 1 - e^{-u(s)/\sin \theta_0} \right) ds - c \int_0^{+\infty} \left( \int_s^{+\infty} u(r) dr \right)^\alpha ds. \tag{2.18}
\]

We write (2.17) in the form

\[
\text{maximize: } J_1(u) = \int_0^{+\infty} \left( 1 - e^{-u(s)} \right) ds - c z^{\alpha} ds, \tag{2.19}
\]

subject to

\[
\dot{z} = -u, \quad z(+\infty) = 0. \tag{2.20}
\]

The necessary conditions for optimality (see for example [8, 11]) now yield

\[
u(s) = \arg\max_{\omega \geq 0} \{ -e^{-\omega} - \omega q(s) \} = -\ln q(s), \tag{2.21}\]

where the dual variable \( q \) satisfies

\[
\dot{q} = c\alpha z^{\alpha - 1}, \quad q(0) = 0. \tag{2.22}
\]

Notice that, by (2.21), \( u > 0 \) only if \( q < 1 \). Combining (2.20) with (2.22) one obtains an ODE for the function \( q \mapsto z(q) \), with \( q \in [0, 1] \). Namely

\[
\frac{dz(q)}{dq} = \frac{z^{1-\alpha} \ln q}{c\alpha}, \quad z(1) = 0. \tag{2.23}
\]

This equation admits the explicit solution

\[
z(q) = c^{-1/\alpha} \left[ 1 + q \ln q - q \right]^{1/\alpha}. \tag{2.24}
\]
Inserting (2.24) in (2.22), we obtain an implicit equation for \( q(s) \):

\[
s = \frac{1}{\alpha c^{1/\alpha}} \int_0^{q(s)} [1 + t \ln t - t]^{1-\alpha} dt.
\] (2.25)

In turn, the density \( u(s) \) of the optimal measure \( \mu \) along \( \Gamma_1 \), as a function of the arc-length \( s \), is recovered from (2.21). Notice that this measure is supported only on an initial interval \([0, \ell_1]\), determined by

\[
\ell_1 = \frac{1}{\alpha c^{1/\alpha}} \int_0^1 [1 + s \ln s - s]^{1-\alpha} ds.
\]

The density of the optimal measure along the ray \( \Gamma_0 \) is computed in an entirely similar way. In this case, the equations (2.21) and (2.25) are replaced respectively by

\[
u(s) = - (\sin \theta_0) \ln q(s),
\]

\[
s = \frac{(\sin \theta_0)^{1-\alpha}}{\alpha c^{1/\alpha}} \int_0^{q(s)} [1 + t \ln t - t]^{1-\alpha} dt.
\]

Again, the condition \( u(s) > 0 \) implies \( q(s) < 1 \). Along \( \Gamma_0 \), the optimal measure \( \mu \) is supported on an initial interval \([0, \ell_0]\), where

\[
\ell_0 = \frac{(\sin \theta_0)^{1-\alpha}}{\alpha c^{1/\alpha}} \int_0^1 [1 + s \ln s - s]^{1-\alpha} ds.
\]

### 2.5 The case \( \alpha = 0 \).

In the analysis of the optimization problem (OPB), the case \( \alpha = 0 \) stands apart. Indeed, the general theorem on the existence of an optimal shape proved in [7] does not cover this case.

When \( \alpha = 0 \), a measure \( \mu \) is irrigable only if it is concentrated on a set of dimension \( \leq 1 \). When this happens, in any dimension \( d \geq 3 \) we have \( S^\eta(\mu) = 0 \) and the optimization problem is trivial. The only case of interest occurs in dimension \( d = 2 \). In the following, \( \langle \cdot, \cdot \rangle \) denotes the inner product in \( \mathbb{R}^2 \).

**Theorem 2.8** Let \( \alpha = 0 \), \( d = 2 \). Let \( \eta \in L^1(S^1) \) and define

\[
K \doteq \max_{|w| = 1} \int_{S^1} \left| \langle w, n \rangle \right| \eta(n) \, dn.
\] (2.26)

(i) If \( K > c \), then the optimization problem (OPB) has no solution, because the supremum of all possible payoffs is \( +\infty \).

(ii) If \( K \leq c \), then the maximum payoff is zero, which is trivially achieved by the zero measure.

A proof will be given in Section 5.
3 Properties of optimal branch configurations

In this section we consider the optimization problem (2.14) in dimension \( d = 2 \). As a step toward the proof of Theorem 2.7, some properties of optimal branch configurations will be derived.

By the result in [7] we know that an optimal measure \( \mu \) exists and has bounded support, contained in \( \mathbb{R}^2_+ \). Call \( M = \mu(\mathbb{R}^2_+) \) the total mass of \( \mu \) and let \( \chi : [0,M] \times \mathbb{R}_+ \rightarrow \mathbb{R}^2_+ \) be an optimal irrigation plan for \( \mu \).

![Diagram](image)

Figure 3: According to the definition (3.3), the set \( \chi^-(x) \) is a curve joining the origin to the point \( x \). The set \( \chi^+(x) \) is a subtree, containing all paths that start from \( x \).

Next, consider the set of all branches, namely

\[
\mathcal{B} = \{ x \in \mathbb{R}^2_+ ; |x|_x > 0 \}. \tag{3.1}
\]

By the single path property, we can introduce a partial ordering among points in \( \mathcal{B} \). Namely, for any \( x, y \in \mathcal{B} \) we say that \( x \preceq y \) if for any \( \xi \in [0, M] \) we have the implication

\[
\chi(t, \xi) = y \implies \chi(t', \xi) = x \text{ for some } t' \in [0, t]. \tag{3.2}
\]

This means that all particles that reach the point \( y \) pass through \( x \) before getting to \( y \).

For a given \( x \in \mathcal{B} \) the subsets of points \( y \in \mathcal{B} \) that precede or follow \( x \) are defined as

\[
\chi^-(x) = \{ y \in \mathcal{B} ; \ y \preceq x \}, \quad \chi^+(x) = \{ y \in \mathcal{B} ; \ x \preceq y \}, \tag{3.3}
\]

respectively (see Fig. 3).

We begin by deriving some properties of the sets \( \chi^+(x) \). Introducing the unit vectors \( \mathbf{e}_1 = (1,0), \mathbf{e}_2 = (0,1) \), we denote by \( \mathbf{Re}_1 \) the set of points on the \( x_1 \)-axis. As before, \( \mathbf{n} = (\cos \theta_0, \sin \theta_0) \) denotes the unit vector in the direction of the sunlight. Throughout the following, the closure of a set \( A \) is denoted by \( \overline{A} \), while \( \langle \cdot, \cdot \rangle \) denotes an inner product.

**Lemma 3.1** Let the measure \( \mu \) provide an optimal solution to the problem (2.14), and let \( \chi \) be an optimal irrigation plan for \( \mu \). Then, for every \( x \in \mathcal{B} \), one has

\[
\chi^+(x) \subset \Gamma_x \triangleq \{ y \in \mathbb{R}^2_+ ; \ \langle \mathbf{n}, y \rangle \in [a_x, b_x] \}, \tag{3.4}
\]

where \( a_x = \langle \mathbf{n}, x \rangle \), while \( b_x \) is defined as follows.
Figure 4: If the set $\chi^+(x)$ is not contained in the slab $\Gamma_x$ (the shaded region), by taking the perpendicular projections $\pi^\#$ and $\pi^\♭$ we obtain another irrigation plan with strictly lower cost, which irrigates a new measure $\tilde{\mu}$ gathering exactly the same amount of sunlight. Notice that here $P$ is the point in the closed set $\overline{\chi^+(x)} \cap \mathbb{R}e_1$ which has the largest inner product with $n$.

- If $\overline{\chi^+(x)} \cap \mathbb{R}e_1 = \emptyset$, then $b_x = a_x = \langle n, x \rangle$.
- If $\overline{\chi^+(x)} \cap \mathbb{R}e_1 \neq \emptyset$, then
  \[ b_x = \max \{ a_x, b'_x \}, \quad b'_x = \sup \left\{ \langle n, z \rangle ; \ z \in \overline{\chi^+(x)} \cap \mathbb{R}e_1 \right\}. \]

**Proof.** The right hand side of (3.4) is illustrated in Fig. 4. To prove the lemma, consider the set of all particles that pass through $x$, namely
\[ \Theta_x = \{ \xi \in [0, M] ; \ \chi(\tau, \xi) = x \text{ for some } \tau \geq 0 \}. \]

1. We first show that, by the optimality of the solution,
\[ \langle n, \chi(\xi, t) \rangle \geq a_x \text{ for all } \xi \in \Theta_x, \ t \geq \tau. \tag{3.5} \]
Indeed, consider the perpendicular projection on the half plane
\[ \pi^\#: \mathbb{R}^2 \mapsto S^2 = \{ y \in \mathbb{R}^2 ; \ \langle n, y \rangle \geq a_x \}. \]
Define the projected irrigation plan
\[ \chi^\#(t, \xi) \doteq \begin{cases} \pi^\# \circ \chi(t, \xi) & \text{if } \xi \in \Theta_x, \ t \geq \tau, \\ \chi(t, \xi) & \text{otherwise.} \end{cases} \]
Then the new measure $\mu^\#$ irrigated by $\chi^\#$ is still supported on $\mathbb{R}^2_+$ and has exactly the same projection on $E^+_n$ as $\mu$. Hence it gathers the same amount of sunlight. However, if the two irrigation plans do not coincide a.e., then the cost of $\chi^\#$ is strictly smaller than the cost of $\chi$, contradicting the optimality assumption.

2. Next, we show that
\[ \langle n, \chi(\xi, t) \rangle \leq b_x \text{ for all } \xi \in \Theta_x, t \geq \tau. \tag{3.6} \]
Indeed, call

\[ b'' = \sup \left\{ (n, z); z \in \chi^+(x) \right\}. \]

If \( b'' \leq b_x \), we are done. In the opposite case, by a continuity and compactness argument we can find \( \delta > 0 \) such that the following holds. Introducing the perpendicular projection on the half plane

\[ \pi^b : \mathbb{R}^2 \rightarrow S^b \doteq \{ y \in \mathbb{R}^2; \langle n, y \rangle \leq b'' - \delta \}, \]

one has

\[ \{ \pi^b(y); y \in \chi^+(x) \} \subseteq \mathbb{R}^2_+. \tag{3.7} \]

Similarly as before, define the projected irrigation plan

\[ \chi^b(t, \xi) \doteq \begin{cases} 
\pi^b \circ \chi(t, \xi) & \text{if } \xi \in \Theta_x, \ t \geq \tau, \\
\chi(t, \xi) & \text{otherwise.}
\end{cases} \]

Then the new measure \( \mu^b \) irrigated by \( \chi^b \) is supported on \( \mathbb{R}^2_+ \cap S^b \) and has exactly the same projection on \( E^+_n \) as \( \mu \). Hence it gathers the same amount of sunlight. However, if the two irrigation plans do not coincide a.e., then the cost of \( \chi^b \) is strictly smaller than the cost of \( \chi \), contradicting the optimality assumption. This completes the proof of the Lemma. \( \square \)

![Figure 5: After a rotation of coordinates, the sunlight comes from the vertical direction. Here the blue lines correspond to the set \( B^* \) in (3.8).](image)

Based on the previous lemma, we now consider the set

\[ B^* \doteq \{ x \in B; \ \chi^b(x) \cap \mathbb{R}e_1 \neq \emptyset \}. \tag{3.8} \]

It will be convenient to rotate coordinates by an angle of \( \pi/2 - \theta_0 \), and choose new coordinates \((z_1, z_2)\) oriented as in Fig. 5. In these new coordinates, the direction of sunlight becomes vertical, while the positive \( x_1 \)-axis corresponds to the line

\[ S \doteq \{ (z_1, z_2); z_1 \geq 0, \ z_2 = -\lambda z_1 \}, \quad \text{where} \ \lambda = \tan \theta_0. \tag{3.9} \]

Calling \((z_1(\xi, t), z_2(\xi, t))\) the corresponding coordinates of the point \( \chi(\xi, t) \), from Lemma 3.1 we immediately obtain
Corollary 3.2 Let \( \chi \) be an optimal irrigation plan for a solution to (2.14). Then

(i) For every \( \xi \in [0, M] \), the map \( t \mapsto z_1(\xi, t) \) is non-decreasing.

(ii) If \( \mathbf{z} = (\bar{z}_1, \bar{z}_2) \notin \mathcal{B}^* \), then \( \chi^+(\mathbf{z}) \) is contained in a horizontal line. Namely,

\[
\chi^+(\mathbf{z}) \subset \{(\bar{z}_1, s) ; \ s \in \mathbb{R}\}.
\]

(3.10)

To make further progress, we define

\[
z_1^{\text{max}} = \sup \{z_1 ; (z_1, z_2) \in \mathcal{B}^*\}.
\]

Moreover, on the interval \([0, z_1^{\text{max}}]\) we consider the function

\[
\varphi(z_1) = \sup \{s ; (z_1, s) \in \mathcal{B}^*\}.
\]

(3.11)

Figure 6: The construction used in the proof of Lemma 3.3.

Lemma 3.3 For every \( z_1 \in [0, z_1^{\text{max}}[ \), the supremum \( \varphi(z_1) \) is attained as a maximum.

Proof. 1. Assume that, on the contrary, for some \( \bar{z}_1 \) the supremum is not a maximum. In this case, as shown in Fig. 6, there exist a sequence of points \( P_n \to P \) with \( P_n = (\bar{z}_1, s_n) \), \( P = (\bar{z}_1, \bar{z}_2) \), \( s_n \uparrow \bar{z}_2 \). Here \( P_n \in \mathcal{B}^* \) for every \( n \geq 1 \) but \( P \notin \mathcal{B}^* \).

2. Choose two values \( a, b \) such that

\[-\lambda \bar{z}_1 < b < a < \varphi(\bar{z}_1).\]

By construction, for every \( n \geq 1 \) the set \( \overline{\chi^+(P_n)} \) intersects \( S \). Therefore we can find points

\[
P_n \prec A_n \prec B_n
\]

all in \( \mathcal{B}^* \), with

\[
A_n = (t_n, a), \quad B_n = (t_n', b), \quad \bar{z}_1 \leq t_n \leq t_n' \leq z_1^{\text{max}}.
\]

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3. Since the branches $\chi^+(A_n)$ are all disjoint, we have
\[ \sum_{n \geq 1} |A_n|_{\chi} \leq M = \mu(\mathbb{R}^2). \]
We can thus find $N$ large enough so that
\[ \varepsilon_N = |A_N|_{\chi} < (a - b)^{1 - \alpha}. \tag{3.12} \]
Consider the modified transport plan $\tilde{\chi}$, obtained from $\chi$ by removing all particles that go through the point $B_N$. More precisely, $\tilde{\chi}$ is the restriction of $\chi$ to the domain
\[ \tilde{\Theta} = \Theta \setminus \{ \xi; \; \chi(\xi, \tau) = B_N \; \text{for some} \; \tau \geq 0 \}. \]
Let $\tilde{\mu}$ be the measure irrigated by $\tilde{\chi}$.
Since $\tilde{\mu} \leq \mu$, the total amount of sunlight gathered by the measure $\tilde{\mu}$ satisfies
\[ S(\mu) - S(\tilde{\mu}) \leq (\mu - \tilde{\mu})(\mathbb{R}^2). \tag{3.13} \]
We now estimate the reduction in the transportation cost, achieved by replacing $\mu$ with $\tilde{\mu}$. Since all water particles reaching $B_N$ must pass through $A_N$, they must cover a distance $\geq |B_N - A_N| \geq a - b$ traveling along a path whose maximum flux is $\leq \varepsilon_N$. The difference in the transportation costs can thus be estimated by
\[ T^\alpha(\mu) - T^\alpha(\tilde{\mu}) \geq (a - b) \cdot \varepsilon_N^{\alpha - 1} \cdot (\mu - \tilde{\mu})(\mathbb{R}^2). \tag{3.14} \]
If (3.12) holds, combining (3.13)-(3.14) we obtain
\[ S(\mu) - T^\alpha(\mu) < S(\tilde{\mu}) - T^\alpha(\tilde{\mu}). \]
Hence the measure $\mu$ is not optimal. This contradiction proves the lemma. \hfill \Box

By the previous result, the graph of $\varphi$ is contained in one single maximal trajectory of the transport plan $\chi$. As in Figure 7, we let $s \mapsto \gamma(s)$ be the arc-length parameterization of this curve, which provides the left boundary of the set $B^*$.

Along the curve $\gamma$, we now consider the set of points $C_j = (z_{1,j}, z_{2,j})$ where some horizontal branch bifurcates on the left. A property of such points is given below.

**Lemma 3.4** In the above setting, for every $j$, one has
\[ \varphi(s) < z_{2,j} \quad \text{for all} \; s < z_{1,j}. \tag{3.15} \]

**Proof.** If (3.15) fails, there exists another point $C_j^* = (z_{1,j}^*, z_{2,j})$ along the curve $\gamma$, with $z_{1,j}^* < z_{1,j}$. We can now replace the measure $\mu$ by another measure $\tilde{\mu}$ obtained as follows. All the mass lying on the horizontal half-line $\{(z_{1,j}, s); \; s \geq z_{2,j}\}$ is shifted downward on the half-line $\{(z_{1,j}^*, s); \; s \geq z_{2,j}\}$. Since the functional $S^n$ is invariant under vertical shifts, we have $S^n(\tilde{\mu}) = S^n(\mu)$. However, the transportation cost is strictly smaller: $T^\alpha(\tilde{\mu}) < T^\alpha(\mu)$. This contradicts the optimality of $\mu$. \hfill \Box
Next, as shown in Fig. 7, we consider a point $P^* = (p_1^*, p_2^*) \in \gamma$ where the component $z_2$ achieves its maximum, namely

$$p_2^* = \max\{z_2; (z_1, z_2) \in \gamma\} \geq 0.$$  \hfill (3.16)

Notice that such a maximum exists because $\gamma$ is a continuous curve, starting at the origin. If this maximum is attained at more than one point, we choose the one with smallest $z_1$-coordinate, so that

$$p_1^* = \min\{z_1; (z_1, p_2^*) \in \gamma\}.$$  \hfill (3.17)

Moreover, call $q_2^* = \inf\{z_2; (z_1, z_2) \in \text{Supp}(\mu)\}$, and let $Q^* = (q_1^*, q_2^*) \in S$ be the point on the ray $S$ whose second coordinate is $p_2^*$. We observe that, by the optimality of the solution, all paths of the irrigation plan $\chi$ must lie within the convex set

$$\Sigma^* \doteq \{(z_1, z_2); \ z_1 \in [0, q_1^*], \ z_2 \geq q_2^*\}.$$

Otherwise, calling $\pi^* : \mathbb{R}^2 \mapsto \Sigma^*$ the perpendicular projection on the convex set $\Sigma^*$, the composed plan

$$\chi^*(\xi, t) \doteq \pi^*(\chi(\xi, t))$$

would satisfy

$$\mathcal{S}^\alpha(\chi^*) = \mathcal{S}^\alpha(\chi), \quad \mathcal{E}^\alpha(\chi^*) < \mathcal{E}^\alpha(\chi),$$

contradicting the optimality assumption.

By a projection argument we now show that, in an optimal solution, all the particle paths remain below the segment $\gamma^*$ with endpoints $P^*$ and $Q^*$.

**Lemma 3.5** In the above setting, let

$$\gamma^* = \{(z_1, z_2); \ z_1 = a + bz_2, \ z_2 \in [q_2^*, p_2^*]\}$$
be the segment with endpoints $P^*, Q^*$. If
\[
(\xi, t) \mapsto \chi(\xi, t) = (z_1(\xi, t), z_2(\xi, t))
\]
is an optimal irrigation plan for the problem (2.14), then we have the implication
\[
z_2(\xi, t) \in [q_2^*, p_2^*] \implies z_1(\xi, t) \leq a + b z_2(\xi, t).
\]

**Proof.** 1. It suffices to show that the maximal curve $\gamma$ lies below $\gamma^*$. If this is not the case, consider the set of particles which go through the point $P^*$ and then move to the right of $P^*$, namely
\[
\Omega^* = \left\{\xi \in [0, M] ; \chi(\xi, t^*) = P^* \text{ for some } t^* \geq 0, \ z_2(\xi, t) < p_2^* \text{ for } t > t^* \right\}.
\]

2. Consider the convex region below $\gamma^*$, defined by
\[
\Sigma = \left\{(z_1, z_2) ; \ 0 \leq z_1 \leq a + b z_2, \ z_2 \in [q_2^*, p_2^*] \right\}.
\]
Let $\pi : \mathbb{R}^2 \mapsto \Sigma$ be the perpendicular projection. Then the irrigation plan
\[
\chi^\dagger(\xi, t) = \begin{cases} 
\pi(\chi(\xi, t)) & \text{if } \xi \in \Omega^*, \ t > t^*, \\
\chi(\xi, t) & \text{otherwise},
\end{cases}
\]
has total cost strictly smaller than $\chi$. Indeed, for all $x, \xi, t$ we have
\[
|\pi(x)|_{\chi^\dagger} \geq |x|_{\chi}, \quad |\dot{\chi}^\dagger(\xi, t)| \leq |\dot{\chi}(\xi, t)|.
\]
Notice that, in (3.22), equality can hold for a.e. $\xi, t$ only in the case where $\chi = \chi^\dagger$.

3. We now observe that the perpendicular projection on $\Sigma$ can decrease the $z_2$-component. As a consequence, the measures $\mu$ and $\mu^\dagger$ irrigated by $\chi$ and $\chi^\dagger$ may have a different projections on the $z_2$ axis. If this happens, we may have $S^\alpha(\mu) \neq S^\alpha(\mu^\dagger)$.

To address this issue, we observe that all particles $\xi \in \Omega^*$ satisfy $\chi^\dagger(\xi, t^*) = \chi(\xi, t^*) = P^*$. In terms of the $z_1, z_2$ coordinates, this implies
\[
z^\dagger_1(\xi, t^*) = z_2(\xi, t^*) = p_2^*, \quad z^\dagger_2(\xi, T(\xi)) \leq z_2(\xi, T(\xi)) < p_2^*.
\]
By continuity, for each $\xi \in \Omega^*$ we can find a stopping time $\tau(\xi) \in [t^*, T(\xi)]$ such that
\[
z^\dagger_2(\xi, \tau(\xi)) = z_2(\xi, T(\xi)).
\]
Call $\tilde{\chi}$ the truncated irrigation plan, such that
\[
\tilde{\chi}(\xi, t) = \begin{cases} 
\chi^\dagger(\xi, t) & \text{if } \xi \in \Omega^*, \ t \leq \tau(\xi), \\
\chi(\xi, \tau(\xi)) & \text{if } \xi \in \Omega^*, \ t \geq \tau(\xi), \\
\chi(\xi, t) & \text{if } \xi \notin \Omega^*.
\end{cases}
\]
By construction, the measures $\mu$ and $\tilde{\mu}$ irrigated by $\chi$ and $\tilde{\chi}$ have exactly the same projections on the $z_2$ axis. Hence $S^\alpha(\tilde{\mu}) = S^\alpha(\mu)$. On the other hand, the corresponding costs satisfy
\[
E^\alpha(\tilde{\chi}) \leq E^\alpha(\chi^\dagger) < E^\alpha(\chi).
\]
This contradicts optimality, thus proving the lemma.
\[\square\]
4 Proof of Theorem 2.7

In this section we give a proof of Theorem 2.7. As shown in Fig. 7, let \( P^* = (p_1^*, p_2^*) \) be the point defined at (3.16) We consider two cases:

(i) \( P^* = 0 \in \mathbb{R}^2 \),

(ii) \( P^* \neq 0 \).

Assume that case (i) occurs. Then, by Lemma 3.4, the only branch that can bifurcate to the left of \( \gamma \) must lie on the \( z_2 \)-axis. Moreover, by Lemma 3.5, the path \( \gamma \) cannot lie above the segment with endpoints \( P^*, Q^* \). Therefore, the restriction of the measure \( \mu \) to the half space \( \{ z_2 \leq 0 \} \) is supported on the line \( S \). Combining these two facts we achieve the conclusion of the theorem.

The remainder of the proof will be devoted to showing that the case (ii) cannot occur, because it would contradict the optimality of the solution.

To illustrate the heart of the matter, we first consider the elementary configuration shown in Fig. 8, left, where all trajectories are straight lines. We call \( \kappa \) the flux along the segment \( P^*Q \) and \( \sigma \) the flux along the horizontal line bifurcating to the left of \( P^* \). As in Fig. 8, right, we then replace the segments \( PP^* \) and \( P^*Q \) by a single segment with endpoints \( P, Q \). To fix the ideas, the lengths of these two segments will be denoted by

\[
\ell_a = |P - P^*|, \quad \ell_b = |Q - P^*|.
\]

The angles between these segments and a horizontal line will be denoted by \( \theta_a, \theta_b \), respectively. Our main assumption is

\[
0 \leq \theta_a \leq \frac{\pi}{2}, \quad 0 \leq \theta_b < \frac{\pi}{2} - \theta_0.
\]

![Diagram](image_url)

Figure 8: The basic case: in a neighborhood of \( P^* \) the trajectories are straight lines. To show that the configuration on the left is not optimal, we replace the portion of the trajectory between \( P \) and \( Q \) with a single segment.
Having performed this modification, the previous transportation cost along \( PP' \) and \( P'Q \)
\[(\kappa + \sigma)^\alpha \ell_a + \kappa^\alpha \ell_b \]
is replaced by
\[
\kappa^\alpha \sqrt{\ell_a^2 + \ell_b^2 - 2\ell_a \ell_b \cos(\theta_a + \theta_b)} + \sigma^\alpha \ell_a \cos \theta_a.
\] (4.3)
Notice that the last term in (4.3) accounts for the fact that an amount \( \sigma \) of particles need to cover a longer horizontal distance, reaching \( P \) instead of \( P' \).

The difference in the cost is thus expressed by the function
\[
f(\ell_a, \ell_b) = (\kappa + \sigma)^\alpha \ell_a - \sigma^\alpha \ell_a \cos \theta_a + \kappa^\alpha \left[ \ell_b - \sqrt{\ell_a^2 + \ell_b^2 - 2\ell_a \ell_b \cos(\theta_a + \theta_b)} \right].
\]
Notice that this function is positively homogeneous of degree 1 w.r.t. the variables \( \ell_a, \ell_b \). We observe that, by choosing the angle \( \theta_c \) between the segment \( PQ \) and a horizontal line to be just slightly larger than \( \theta_b \), we can render the ratio \( \ell_a/\ell_b \) as small as we like. Taking advantage of this fact, we set
\[
\ell_a = \varepsilon \ell, \quad \ell_b = \ell
\]
for some \( \varepsilon > 0 \) small. By the homogeneity of \( f \) it follows
\[
f(\varepsilon \ell, \ell) = \ell \left[ \varepsilon(\kappa + \sigma)^\alpha - \varepsilon \sigma^\alpha \cos \theta_a + \kappa^\alpha \left( 1 - \sqrt{1 + \varepsilon^2 - 2\varepsilon \cos(\theta_a + \theta_b)} \right) \right].
\]
This yields
\[
\frac{d}{d\varepsilon} f(\varepsilon \ell, \ell) = \varepsilon(\kappa + \sigma)^\alpha - \varepsilon \sigma^\alpha \cos \theta_a + \kappa^\alpha \left( 1 - \sqrt{1 + \varepsilon^2 - 2\varepsilon \cos(\theta_a + \theta_b)} \right)
\]
\[
= \varepsilon \left[ (\kappa + \sigma)^\alpha - \sigma^\alpha \cos \theta_a + \kappa^\alpha \cos(\theta_a + \theta_b) + O(1) \cdot \varepsilon \right].
\] (4.4)
Setting
\[
\lambda = \frac{\sigma}{\kappa + \sigma}
\]
we now study the function
\[
F(\lambda, \theta_a, \theta_b) \doteq 1 - \lambda^\alpha \cos \theta_a + (1 - \lambda)^\alpha \cos(\theta_a + \theta_b),
\] (4.5)
and find under which conditions on \( \theta_b \) this function \( F \) it remains positive for all \( \lambda \in [0, 1] \), \( \theta_a \in [0, \pi/2] \).

**Lemma 4.1**

(i) For \( \alpha \geq 1/2 \) and any \( \theta_a, \theta_b \in [0, \pi/2] \), we always have \( F(\lambda, \theta_a, \theta_b) \geq 0 \).

(ii) When \( 0 < \alpha < 1/2 \) we have \( F(\lambda, \theta_a, \theta_b) \geq 0 \) for every \( \theta_a, \theta_b \in [0, \pi/2] \) provided that \( \theta_b \) satisfies the additional bound
\[
\cos \theta_b \geq 1 - 2^{2\alpha - 1}.
\] (4.6)

**Proof.** The function \( F \) in (4.5) can be written in terms of an inner product:
\[
F(\lambda, \theta) = 1 - \cos \theta_a [\lambda^\alpha - (1 - \lambda)^\alpha \cos \theta_b] - \sin \theta_a (1 - \lambda)^\alpha \sin \theta_b
\]
\[
= 1 - \langle (\cos \theta_a, \sin \theta_a), (\lambda^\alpha - (1 - \lambda)^\alpha \cos \theta_b, (1 - \lambda)^\alpha \sin \theta_b) \rangle.
\] (4.7)
To prove that $F \geq 0$ it thus suffices to show that the second vector on the right hand side of (4.7) has length less than or equal to one. Namely

$$
\lambda^{2\alpha} + (1 - \lambda)^{2\alpha} - 2\lambda^\alpha (1 - \lambda)^\alpha \cos \theta_b \leq 1.
$$

This inequality holds provided that

$$
\cos \theta_b \geq \frac{\lambda^{2\alpha} + (1 - \lambda)^{2\alpha} - 1}{2\lambda^\alpha (1 - \lambda)^\alpha}.
$$

(4.8)

In the case where $\alpha \geq 1/2$ we have

$$
\lambda^{2\alpha} + (1 - \lambda)^{2\alpha} \leq 1 \quad \text{for all } \lambda \in [0, 1],
$$

hence (4.8) holds.

To study the case where $\alpha < 1/2$, consider the function

$$
g(\lambda) = \frac{\lambda^{2\alpha} + (1 - \lambda)^{2\alpha} - 1}{2\lambda^\alpha (1 - \lambda)^\alpha} = 1 + \frac{(\lambda^\alpha - (1 - \lambda)^\alpha)^2 - 1}{2\lambda^\alpha (1 - \lambda)^\alpha}.
$$

We observe that, for $0 \leq \alpha \leq \frac{1}{2}$, one has

$$
0 \leq g(\lambda) \leq g\left(\frac{1}{2}\right) = 1 - 2^{2\alpha - 1},
$$

(4.9)

while

$$
\lim_{\lambda \to 0^+} g(\lambda) = \lim_{\lambda \to 1} g(\lambda) = 0.
$$

From (4.9) it now follows that the condition (4.6) guarantees that (4.8) holds, hence $F \geq 0$, as required.

\[\square\]

Figure 9: A more general configuration, compared with the one in Fig. 8.

We now consider the more general configuration shown in Fig. 9. Water is transported along the path $\gamma$ up to the point $P^*$. Then the flux is split into a finite number of straight paths.
One goes horizontally to the left, with flux $\sigma \geq 0$. The other pipes go to the right, with fluxes $\kappa_1, \ldots, \kappa_n > 0$, at angles

$$0 \leq \theta_n < \cdots < \theta_2 < \theta_1 < \frac{\pi}{2} - \theta_0. \quad (4.10)$$

We compare this configuration with a modified irrigation plan, where a “bypass” is inserted along a segment $\tilde{\gamma}$ with endpoints $P, P_1$, at an angle $\beta$ satisfying

$$\theta_1 < \beta < \frac{\pi}{2} - \theta_0. \quad (4.11)$$

In this case, water particles travel along $\gamma$ until they reach $P$. Then, an amount $\sigma$ of particles bifurcates to the left. All the remaining particles are transported along the segment $\tilde{\gamma}$, until they reach the points $P_n, \ldots, P_1$ along the old pipes. The next lemma estimates the saving in the irrigation cost achieved by inserting the “bypass” along the segment $PP_1$.

**Lemma 4.2** As in Theorem 2.7, assume that either $1/2 \leq \alpha \leq 1$, or else (2.16) holds. In the above setting, one has

$$[\text{old cost}] - [\text{new cost}] \geq |P_1 - P^*| \cdot \delta(\theta_1, \kappa), \quad (4.12)$$

where $\delta(\theta_1, \kappa)$ is a continuous function, strictly positive for $0 \leq \theta_1 < \frac{\pi}{2} - \theta_0$ and $\kappa = \kappa_1 + \cdots + \kappa_n > 0$.

**Proof.** 1. As in the previous lemmas, we call $\theta_a$ the angle between the segment $PP^*$ and a horizontal line. The difference between the old cost and the new cost can be expressed as

$$|P - P^*| \left(\sigma + \sum_{j=1}^{n} \kappa_j \right)^{\alpha} + \sum_{j=1}^{n} \kappa_j^\alpha |P^* - P_j| - \sigma^\alpha \cos \theta_a |P - P^*| - \sum_{j=1}^{n} \left(\sum_{i=1}^{j} \kappa_i \right)^{\alpha} |P_{j+1} - P_j|, \quad (4.13)$$

where, for notational convenience, we set $P_{n+1} = P$. According to (4.13) we can write

$$[\text{old cost}] - [\text{new cost}] = A + S_n, \quad (4.14)$$

where

$$A = |P - P^*| \left[\left(\sigma + \sum_{j=1}^{n} \kappa_j \right)^{\alpha} - \sigma^\alpha \cos \theta_a \right] + \left(\sum_{j=1}^{n} \kappa_j \right)^{\alpha} \left(|P^* - P_1| - |P - P_1|\right) \quad (4.15)$$

$$S_n = \sum_{j=1}^{n} \kappa_j^\alpha |P^* - P_j| - \left(\sum_{j=1}^{n} \kappa_j \right)^{\alpha} \left(|P^* - P_1| - |P_{n+1} - P_1|\right) - \sum_{j=1}^{n} \left(\sum_{i=1}^{j} \kappa_i \right)^{\alpha} |P_{j+1} - P_j| \quad (4.16)$$

2. Notice that the quantity $A$ in (4.15) would describe the difference in the costs if all the mass $\kappa = \kappa_1 + \cdots + \kappa_n$ were flowing through the point $P_1$. Using Lemma 4.1, we can thus choose $P = P_1$ close enough to $P^*$ such that this difference is strictly positive. More precisely, for a fixed $\kappa > 0$, we claim that one can achieve the lower bound

$$A \geq |P - P^*| \left[(\sigma + \kappa)^{\alpha} - \sigma^\alpha \cos \theta_a + \kappa^\alpha \cos(\theta_a + \theta_1) - \frac{\kappa^\alpha |P - P^*|}{2 |P_1 - P^*|} \right] \quad (4.17)$$

$$\geq |P_1 - P^*| \cdot \delta(\theta_1, \kappa) > 0.$$
Indeed, the last two terms within the square brackets in (4.17) are derived from

\[ |P^* - P_1| - |P - P_1| = |P^* - P_1| \left[ 1 - \sqrt{1 - 2 \left| \frac{P - P^*}{P^* - P_1} \right| \cos(\theta_a + \theta_1) + \left| \frac{P - P^*}{P^* - P_1} \right|^2} \right]. \]

Moreover, since we have the strict inequalities

\[ \cos \theta_1 < \frac{\lambda^{2\alpha} + (1 - \lambda)^{2\alpha} - 1}{2\lambda^\alpha (1 - \lambda)^\alpha}. \]

Given \( \kappa > 0 \) and \( P_1 \), we can then choose \( P \) close enough to \( P^* \) so that

- the terms within the square brackets in (4.17) is strictly positive,
- the ratio \( |P - P^*|/|P_1 - P^*| \) is small but uniformly positive, as long as \( \theta_1 \) remains bounded away from \( \frac{\pi}{2} \) or from \( \frac{\pi}{2} - \theta_0 \) respectively, in the two cases considered in (4.18).

This proves our claim (4.17).

3. To complete the proof of the lemma, it remains to prove that \( S_n \geq 0 \). This will be proved by induction on \( n \). Starting from (4.16) and using the inequalities

\[ |P_n - P_1| \leq |P^* - P_1|, \quad \left( \sum_{i=1}^{n} \kappa_i \right)^\alpha \leq \kappa_n^\alpha + \left( \sum_{i=1}^{n-1} \kappa_i \right)^\alpha, \]

we obtain

\[ S_n = \sum_{j=1}^{n} \kappa_j^\alpha |P^* - P_j| - \left( \sum_{j=1}^{n} \kappa_j \right)^\alpha \left( |P^* - P_1| - |P_n - P_1| \right) - \sum_{j=1}^{n-1} \left( \sum_{i=1}^{j} \kappa_i \right)^\alpha |P_{j+1} - P_j| \geq S_{n-1} + \kappa_n^\alpha \left( |P^* - P_n| - |P_n - P_1| \right) \geq 0. \]

Repeating this same argument, by induction we obtain

\[ S_n \geq S_{n-1} \geq \cdots \geq S_1. \]

Observing that

\[ S_1 = \kappa_1^\alpha |P^* - P_1| - \kappa_1^\alpha \left( |P^* - P_1| - |P_2 - P_1| \right) - \kappa_1^\alpha |P_2 - P_1| = 0, \]

we complete the proof of the lemma.
We now consider the most general situation, shown in Fig. 10. Differently from the setting of Lemma 4.2, various scenarios must be considered.

- In addition to the horizontal path bifurcating to the left of $P^*$ with flux $\sigma$, there can be countably many additional horizontal branches bifurcating to the left of $\gamma$, below $P^*$. We shall denote by $\sigma_n$, $n \geq 1$, the fluxes through these branches, at the bifurcation points.

- There can be countably many distinct branches bifurcating to the right of $P^*$, say with fluxes $\kappa^*_j$, $j \geq 1$.

- Furthermore, there can be countably many additional branches bifurcating to the right of $\gamma$, at points close to $P^*$. We shall denote by $\kappa'_i$, $i \geq 1$, the fluxes through these branches, at the bifurcation points.

- Finally, the measure $\mu$ could concentrate a positive mass along the arc $PP^*$.

We observe that, by optimality, all particle trajectories to the right of $\gamma$ move in the right-upward direction. Namely, setting $\chi(\xi,t) = (z_1(\xi,t), z_2(\xi,t))$, for these paths we have

$$\dot{z}_1(\xi,t) \geq 0, \quad \dot{z}_2(\xi,t) \leq 0.$$ 

We now construct a “bypass”, choosing a segment $PQ$ with endpoints both lying on the curve $\gamma$, making an angle $\beta$ with the horizontal direction such that

$$\beta^* < \beta < \frac{\pi}{2} - \theta_0.$$ 

Here $\beta^*$ denotes the angle between the segment $P^*Q^*$ and a horizontal line.

Given $\varepsilon > 0$, we can choose $N \geq 1$ large enough so that, among the branches bifurcating from $P^*$, one has

$$\sum_{j > N} \kappa^*_j < \varepsilon.$$ 

Moreover, by choosing $Q$ sufficiently close to $P^*$, the following can be achieved:

(i) The total flux along the horizontal branches bifurcating to the left of $\gamma$ below $P^*$ satisfies

$$\sum_{n \geq 1} \sigma_n < \varepsilon.$$ 

(ii) The total flux along the branches bifurcating to the right of $\gamma$ between $P$ and $P^*$, and between $P^*$ and $Q$ satisfies

$$\sum_{i \geq 1} \kappa'_i < \varepsilon.$$ 

(iii) For each $j = 1, \ldots, N$, there exists a path $\gamma_j$ connecting $P^*$ with a point $P_j$ on the segment $PQ$, along which the flux remains $\geq \kappa_j \geq \kappa^*_j - (\varepsilon/N)$. Here we denote by $\kappa_j$ the flux reaching $P_j$.

In other words, even if the $j$-th branch through $P^*$ further bifurcates, most of the particles along this branch cross the segment $PQ$ at the same point $P_j$. 

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Figure 10: In the fully general situation, we have additional branches bifurcating to the left of $\gamma$ between $P$ and $P^*$, and to the right of $\gamma$ at any point between $P$ and $Q$. In addition, there can be an additional absolutely continuous source along the arc $PP^*$.

(iv) The total mass of $\mu$ along $\gamma$, between $P$ and $P^*$ is $< \varepsilon$.

We estimate the difference in the new cost produced by these additional branches. Call $P = (p_1, p_2), Q = (q_1, q_2)$.

- The additional mass on the left branches, together with the mass of $\mu$ present between $P$ and $P^*$ now travels along a horizontal line through $P$. By (i) and (iv) this mass is $< 2\varepsilon$. Hence:

$$[\text{additional cost}] \leq (2\varepsilon)^{1-\alpha} (z_2^* - z_2). \tag{4.25}$$

- The additional mass bifurcating to the right of $\gamma$, not crossing the segment $PQ$ at one of the finitely many points $P_1, \ldots, P_N$ is $< \varepsilon$. The additional cost in transporting this mass from $P$ to some point between $P$ and $Q$ satisfies

$$[\text{additional cost}] < \kappa_0^{\alpha-1} \cdot 3\varepsilon|P - Q|. \tag{4.26}$$

We now use Lemma 4.2. Combining (4.12) with (4.25)-(4.26) we obtain

$$[\text{old cost}] - [\text{new cost}] \geq |P_1 - P^*| \cdot \delta(\theta_1, \kappa) - (2\varepsilon)^{1-\alpha} |P - P^*| - \kappa_0^{\alpha-1} \cdot 3\varepsilon|P - Q|. \tag{4.27}$$

By choosing $\varepsilon > 0$ small enough, the right hand side of (4.27) is strictly positive. Hence the configuration with $P^* \neq 0$ is not optimal. This completes the proof of Theorem 2.7.

5 The case $d = 2, \alpha = 0$

We give here a proof of Theorem 2.8.

1. Assume that there exists a unit vector $w^* \in \mathbb{R}^2$ such that

$$K = \int_{n \in S^1} |\langle w^*, n \rangle| \eta(n) dn > c.$$
Let $v = (\cos \beta, \sin \beta)$ be a unit vector perpendicular to $w^*$, with $\beta \in [0, \pi]$. Let $\mu$ be the measure supported on the segment $\{rv; \ r \in [0, \ell]\}$, with constant density $\lambda$ w.r.t. 1-dimensional Lebesgue measure.

Then the payoff achieved by $\mu$ is estimated by

$$S^\eta(\mu) - cI^0(\mu) = \ell \cdot \int_{S^1} \left( 1 - \exp \left\{ - \frac{\lambda}{|\langle w^*, n \rangle|} \right\} \right) |\langle w^*, n \rangle| \eta(n) \, dn - c \ell$$

$$\geq \ell \cdot (1 - e^{-\lambda}) \int_{S^1} |\langle w^*, n \rangle| \eta(n) \, dn - c \ell$$

$$= \left[(1 - e^{-\lambda}) K - c\right] \ell.$$ \hspace{1cm} (5.1)

By choosing $\lambda > 0$ large enough, the first factor on the right hand side of (5.1) is strictly positive. Hence, by increasing the length $\ell$, we can render the payoff arbitrarily large.

2. Next, assume that $K \leq c$. Consider any Lipschitz curve $s \mapsto \gamma(s)$, parameterized by arc-length $s \in [0, \ell]$. Then, for any measure $\mu$ supported on $\gamma$, the total amount of sunlight from the direction $n$ captured by $\mu$ satisfies the estimate

$$S^n(\mu) \leq \int_0^\ell |\langle \dot{\gamma}(s)^\perp, n \rangle| \, ds.$$ 

Indeed, it is bounded by the length of the projection of $\gamma$ on the line $E_n^\perp$ perpendicular to $n$. Integrating over the various sunlight directions, one obtains

$$S^n(\mu) \leq \int_0^\ell \int_{S^1} |\langle \dot{\gamma}(s)^\perp, n \rangle| \eta(n) \, dn \, ds \leq K \ell.$$ 

More generally, $\mu = \sum_i \mu_i$ can be the sum of countably many measures supported on Lipschitz curves $\gamma_i$. In this case, since the sunlight functional is sub-additive, one has

$$S^n(\mu) \leq \sum_i S^n(\mu_i) \leq \sum_i K \ell_i.$$ 

Hence

$$S^n(\mu) - cI^0(\mu) \leq \sum_i K \ell_i - c \sum_i \ell_i \leq 0.$$ 

This concludes the proof of case (ii) in Theorem 2.8. \hspace{1cm} $\Box$

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