Simulating bulk viscosity in neutron stars. I. Formalism

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(Dated: May 22, 2023)

The faithful inclusion of the effects of bulk viscosity induced by the presence of chemical reactions is an important issue for simulations of core-collapse supernovae, binary neutron star mergers, and neutron star oscillations, where particle abundances are locally pushed out of chemical equilibrium by rarefaction and compression of the fluid elements. In this work, we discuss three different approaches that can be used to implement bulk viscosity in general relativistic hydrodynamic simulations of neutron stars: the exact multi-component reacting fluid, and two Müller-Israel-Stewart theories, namely the second order Hiscock-Lindblom model and its linear limit, the Maxwell-Cattaneo model. After discussing the theory behind the three approaches, we specialize their dynamics equations to spherical symmetry in the radial gauge-polar slicing (i.e., Schwarzschild) coordinates. We also discuss a particular choice for the equation of state of the fluid and the associated neutrino emission rates, which are used in a companion paper for the numerical comparison of the three frameworks, and we obtain the effective sound speed for the Hiscock-Lindblom theory in the non-linear regime.

I. INTRODUCTION

With the detection of gravitational and electromagnetic emission from a binary neutron star merger [1, 2], we are now in an era when information on hot and dense nuclear matter can be extracted from the observation of extreme astrophysical phenomena [3]. With more observations to become available in the near future, reliable numerical relativity simulations will be necessary to interpret the data. For this, it is fundamental that the theoretical models used in simulations can faithfully reproduce the physics of the system.

Until recently [e.g., 4–14], most simulations of neutron star mergers assumed that matter behaves as a perfect fluid. However, several dissipative processes can take place in the merger remnant, which is a hot, rotating, and massive neutron star. While on one hand there is a consensus on the importance of magnetohydrodynamic viscosity in this phase [5, 6, 8, 10, 12, 14], on the other hand the role of bulk viscosity is more controversial: orders of magnitude arguments show that weak interactions (i.e., a single species with no internal degrees of freedom) weakly relativistic ideal gases can display bulk viscosity if driven out of thermodynamic equilibrium by a volume change [15, 16]. For example, r-modes are unstable to gravitational wave emission, but this mechanism is suppressed by the shear and bulk viscosity at low and high temperatures, respectively (see, e.g., [17] for a review).

Bulk viscosity is present also in core-collapse supernovae, where huge amounts of neutrinos (on the order of a tenth of a solar mass [36]) are released during the collapse. As the chemical composition of matter and neutrinos is of the utmost importance for the supernova explosion, core-collapse supernova simulations routinely employ a multi-component fluid description, see e.g. O’Connor and Ott [18].

In this paper, we discuss three different approaches to bulk viscosity (the multi-component fluid [19, 38], Hiscock-Lindblom [39], and Maxwell-Cattaneo [40]) and specialize their dynamics equations to the specific case of spherically symmetric perturbations of neutron stars. In doing this, we extend the multi-component framework of Gavassino et al. [19] to include neutrino luminosity. The numerical framework, implemented in the new code hydro-bulk-1D [41], and the results of our simulations, are presented in the companion paper [42]. Our aim is not to make physical predictions regarding bulk viscosity in neutron stars but to check if, in view of the mathematical duality between bulk-viscous fluids and reacting mixtures [44], the multi-component reacting fluid can be a numerically convenient alternative to Müller-Israel-Stewart hydrodynamics for neutron star simulations.

The paper is organized as follows. In Sec. I we give a simple example of bulk viscosity and we outline different approaches to bulk viscosity in the literature. In Sec. II we discuss in more detail the three different approaches to bulk viscosity adopted in this paper. In Sec. III we discuss the relationship between these approaches and we extend the Müller-Israel-Stewart theories to account for the energy loss during the reactions. In Sec. IV we discuss the propagation speed of a signal in the hydrodynamic...
we describe our choice of the equation of state and particle reaction rates. In Sec. VII we specialize the equations to radial gauge, polar slicing coordinates in spherical symmetry (i.e., Schwarzschild). We draw our conclusions in Sec. VIII. In Appendix A we derive some results concerning the speed of sound in bulk viscous systems. Finally, in Appendix B we compute the characteristic velocities of the Hiscock-Lindblom theory in non-linear regime.

We adopt the signature $(-, +, +, +)$ and, unless otherwise specified, we set $c = G = k_B = 1$.

II. A BRIEF INTRODUCTION TO BULK VISCOSITY

In this section, we give a brief introduction to bulk viscosity, including a brief overview of the main approaches existing in the literature.

A. A simple example of reaction-induced bulk viscosity

To illustrate how chemical reactions give rise to bulk viscosity, we consider a thermally isolated system (see Fig. 1) composed by a mixture of two reacting particle species in the initial equilibrium state ‘A’. When the system is driven out of equilibrium by, say, moving a piston, the reactions between the two particle species are imbalanced, with the consequence that chemical transfusion tends to adjust the composition to a new equilibrium on a timescale $\tau_{\text{reac}}$, defined by the kinetics of the reactions. This system is a simplified model of what happens, locally, in a fluid element of a neutron star, neglecting heat conduction and neutrino emission (see e.g. Sec. 22.2 of Misner et al. [43]). How the system reacts depends on how $\tau_{\text{reac}}$ compares with the timescale $\tau_{\text{pert}}$ of the volume perturbation: (i) If the perturbation is much faster than the reaction, we are in the ‘frozen’ regime [23]; the composition of the mixture is frozen and the perturbation is reversible. Since the system is thermally isolated [24], there is no heat exchange and the entropy is constant. (ii) If the perturbation is much slower than the reaction, we are in the ‘quasi-stationary’ regime: the mixture ‘instantaneously’ (when compared to the perturbation timescale) adjusts itself, the system evolves on a sequence of equilibrium states, the transformation is reversible, and since there is no heat exchange, the entropy is constant. (iii) If the perturbation and the reaction timescales are of the same order of magnitude, we are in the ‘viscous’ regime: the transfusion reactions change the matter composition, but they do not have enough time to push the matter to a new equilibrium. This process is irreversible, and as a consequence the entropy of the system grows even if there is no heat exchange [45], which means that after a perturbation cycle the system is no more in its initial state.

B. Main approaches to bulk viscosity in the literature

There are different approaches to bulk viscosity, and, more generally, to dissipation in relativity. We briefly mention only those that are relevant to the present paper.

The so-called first-order models (in the hydrodynamic gradients) are relativistic versions of the Navier-Stokes equations, where the dissipative fluxes (e.g., viscous stresses and heat current) are determined only by first-order spatial gradients of the primary hydrodynamic variables of the perfect fluid. This approach, as originally formulated by Eckart [46] and Landau and Lifshitz [47], leads to superluminal signal propagation and nonphysical instabilities [48]. In the context of first-order gradient-expansion models, the causality and stability problems have been only recently solved by Bemfica et al. [49], by changing the definition of temperature, chemical potential, and fluid velocity (see Kovtun [50]). However, like any model based on a gradient expansion, this approach (called Bemfica-Disconzi-Noronha-Kovtun) is applicable only in the low-frequency limit.

Müller, Israel and Stewart [51, 52] followed a different approach. They adopted the prescription of extended irreversible thermodynamics [53, 54] of promoting the dissipative fluxes to new dynamic degrees of freedom. The resulting theory is by construction causal [53], and, as a consequence [55], stable [56, 57]. Moreover, since the expansion is performed near local equilibrium, and not for small gradients [2], Müller-Israel-Stewart theories are applicable in some cases [61] to accurately describe the dynamics of the slowest non-equilibrium degrees of freedom [58, 59]. This makes them applicable also when $\tau_{\text{reac}} \approx \tau_{\text{hydro}}$, where $\tau_{\text{hydro}}$ is the timescale of the hydrodynamic process under consideration. The same is not true for gradient-expansion theories, which are built on the assumption $\tau_{\text{reac}} \ll \tau_{\text{hydro}}$ [50]. Therefore, given that our goal is precisely to explore the regime $\tau_{\text{reac}} \approx \tau_{\text{hydro}}$, we cannot rely on the Bemfica-Disconzi-Noronha-Kovtun theory [43, 50] and we must follow the Müller-Israel-Stewart approach instead.

We consider two classes of Müller-Israel-Stewart theories: the Hiscock-Lindblom theory [39] and its linearization, the Maxwell-Cattaneo [40] theory [1], which are both

2 Müller-Israel-Stewart theories have been historically confused with the more complicated ‘second-order’ theories (in the hydrodynamic gradients), because of similarities in the field equations [52]. However, strictly speaking, this is a misconception, because Israel and Stewart [52] performed a first-order expansion in deviations from local equilibrium [52], and not a second order expansion in the gradients. Failure to recognize this subtlety leads to incorrect estimates of the relaxation timescale [52].

3 The linearization is performed with respect to the additional dy-
FIG. 1: A mixture of two reactive particle species in an isolated container is a minimal example of a multi-component system with dissipative response to compression and expansion (see e.g. Sec. 22.2 of Misner et al. [43]), which gives rise to bulk viscosity. The relative timescales of the perturbation $\tau_{\text{pert}}$ and the reaction $\tau_{\text{react}}$ define the evolution regimes as discussed in the text (see also [19, 23]). With the label ‘composition’ we refer to, for example, the fraction of one particle species.

based on a perturbative expansion near local thermodynamic equilibrium. Therefore, it is not surprising that Müller-Israel-Stewart theories can still develop non-causal and unstable solutions when the deviations from equilibrium become large [16, 39, 40, 56, 63]. Note that in all these frameworks, bulk viscosity does not necessarily arise from chemical transfusion [16], and indeed the Müller-Israel-Stewart and the Navier-Stokes models do not include any chemical fraction.

C. Mathematical duality between Müller-Israel-Stewart theories and multi-component reacting fluids

Recently, Gavassino et al. [19] proposed a framework for bulk-viscous fluids that builds on the formalism of Carter [38] and naturally allows for hyperbolic equations and causal solutions [57], which are the requirements needed for numerical applications and for thermodynamic stability [55, 64]. As shown in Gavassino et al. [19], when the quasi-equilibrium state of a fluid departing from full equilibrium can be parametrized by a number of scalar variables, it can be mapped into a chemically reacting mixture. In other words, these scalar variables can always be chosen in such a way that they play the role of fictitious chemical fractions or, better, reaction coordinates. This means that there is a mathematical duality between multi-component reacting fluids mixtures and Müller-Israel-Stewart theories. The fact that this mapping is possible for a generic (possibly complex) fluid is not immediately obvious, for two reasons. First, chemical reactions are not necessary in order to have bulk viscosity: in general, a non-reacting fluid may have a non-zero bulk viscosity coefficient. Second, a bulk-viscous fluid modeled within the Müller-Israel-Stewart framework has an energy-momentum tensor that departs from the multi-component fluid one, while the model with fictitious chemical components of Gavassino et al. [19] has an energy-momentum tensor that is formally the same as the one of a multi-component fluid [38].
In the special case in which only one scalar variable is sufficient to parametrize the displacement from equilibrium, a near-equilibrium expansion of the multi-component framework reproduces the Müller-Israel-Stewart hydrodynamics of a purely bulk viscous fluid \[10\]. Therefore, since Carter’s approach is not based on any near equilibrium expansion, in principle the multi-component framework extends the Müller-Israel-Stewart one far from equilibrium and to a generic number of reaction coordinates, possibly including superfluid and heat conduction \[38\], \[59\], see the scheme in Fig. I in Gavassino et al. \[50\].

III. THREE FRAMEWORKS FOR BULK VISCOSITY

In this section we introduce the three frameworks for bulk viscous fluids considered in this paper and numerically implemented in the companion paper \[11\], \[42\].

A. Multi-component reacting fluid

As discussed in Sec.I, a fluid composed by two or more species whose transference timescale is comparable with the hydrodynamic timescale exhibits reaction-induced bulk viscosity, and in this case it is natural to adopt the exact multi-component fluid approach of Carter \[38\]. This means that the effect of reaction-induced bulk viscosity will be automatically implemented just by keeping track of the change of chemical composition and by the consistent implementation of the second law of thermodynamics \[10\].

The composition of a fluid with out-of-equilibrium reactions is not determined by the other thermodynamic quantities, but has to be treated as a set of independent hydrodynamic degrees of freedom. In this case of incomplete equilibrium, the first law of thermodynamics is written in terms of an equation of state (EOS) that depends on the chemical fractions:

\[
d u(\rho, s, \{Y_i\}) = \frac{p}{\rho^2} d\rho + \frac{T}{m_n} ds - \sum_i \frac{\dot{A}^i}{m_n} dY_i, \tag{1}
\]

where \(\rho\) is the rest mass density, \(u\) is the internal specific (per unit mass) energy such that \(\epsilon = (1 + u)\rho\) is the total (including rest mass) energy density, \(p\) is the pressure, \(m_n\) the neutron mass, \(T\) the temperature, \(s\) the entropy per baryon, \(Y_i\) the number fraction (with respect to the total baryon number) of the independent particle species \(i\), and \(\dot{A}^i\) is the affinity of particle \(i\). The system reaches complete thermodynamic equilibrium when \(\dot{A}^i = 0\) for all the independent species \(i\) in the EOS \(u(\rho, s, \{Y_i\})\).

Since we are interested in neutron stars, we consider a multi-component general relativistic fluid whose species react and emit neutrinos. To simplify the discussion, we assume that neutrinos are emitted isotropically in the fluid frame and immediately leave the star\(^4\). The hydrodynamic equations for such a system are determined by the continuity equation for each species and by the energy and momentum conservation equations \[57\]:

\[
\nabla_\mu (\rho u^\mu) = 0, \tag{2}
\]

\[
\nabla_\mu (T^{\mu \nu} u^\nu) = -Q u^\nu, \tag{3}
\]

\[
\nabla_\mu (\rho Y_i u^\mu) = m_n Y_i R_i, \tag{4}
\]

where \(u^\nu\) is the four-velocity of the matter, \(Q = \sum_i Q_i\) is the total luminosity (i.e., the rate of energy loss of all reactions; \(Q_i\) is the luminosity of reaction \(j\)), \(R_i\) is the number reaction rate of particle species \(i\). Finally, \(T^{\mu \nu}\) is the stress-energy tensor of a multi-component fluid,

\[
T^{\mu \nu} = (\epsilon + p)u^\mu u^\nu + pg^{\mu \nu}, \tag{5}
\]

where \(g^{\mu \nu}\) is the metric. Although this multi-component system is dissipative (see, e.g., Sec. III of \[19\]), its stress-energy tensor \[50\] retains the form it would have in the absence of dissipation (namely, when \(R_i = 0, Q = 0\)).

B. Hiscock-Lindblom

Bulk viscosity can also be accounted for if we astrey the composition from the description of the matter. In the Hiscock-Lindblom theory of bulk viscosity, the hydrodynamic equations are (Eqs. (18) and (19) of Hiscock and Lindblom \[52\]):

\[
\nabla_\mu (\rho u^\mu) = 0, \tag{6}
\]

\[
\nabla_\mu (T^{\mu \nu} u^\nu) = 0, \tag{7}
\]

\[
\nabla_\mu (\rho s u^\mu) = \frac{m_n \Pi^2}{\zeta T^{\text{eq}}(\rho, \epsilon)}, \tag{8}
\]

where \(\Pi\) is the bulk viscous stress, \(\zeta\) the viscous coefficient, and \(s\) (the entropy per baryon of the out-of-equilibrium system) is now expanded to second order in \(\Pi\):

\[
s = s^{\text{eq}}(\rho, \epsilon) - \frac{\chi \Pi^2}{2n T^{\text{eq}}(\rho, \epsilon)}, \tag{9}
\]

\[
\tau = \zeta \chi, \tag{10}
\]

where \(n = \rho/m_n\) is the baryon number density, \(\chi > 0\) is the bulk viscous parameter, \(\tau\) is the bulk viscous timescale, and the ‘eq’ superscript refers to the quantities at equilibrium for given \(\epsilon\) and \(\rho\), for which \({Y_i} = {Y^{\text{eq}}_i}\).
The bulk viscous stress-energy tensor $T^{\mu\nu}$ can be obtained from the multi-component one in Eq. (5) with the decomposition $p = p^{eq}(\rho, \epsilon) + \Pi$:

$$T^{\mu\nu} = \left(\epsilon + \rho p^{eq}(\rho, \epsilon) + \Pi\right) u^\mu u^\nu + \rho \left(p^{eq}(\rho, \epsilon) + \Pi\right) g^{\mu\nu}. \quad (11)$$

Eq. (8) for the evolution of the entropy is equivalent to (cf. Eq. (21) of Hiscock and Lindblom [39]):

$$\Pi = -\zeta \left[ \nabla_\mu u^\mu + \chi u^\mu \nabla_\mu \Pi + \Pi \frac{T^{eq}}{2} \nabla_\mu \left(\frac{u^\mu}{T^{eq}}\right)\right], \quad (12)$$

which is in the so-called ‘telegraph-type’ form and is the one used in many implementations, after rewriting it in a flux-conservative form:

$$\nabla_\mu (\Pi u^\mu) = -\Pi \left(1 - \frac{1}{\chi} \right) \nabla_\mu u^\mu - \Pi \frac{\Pi}{2} u^\mu \nabla_\mu \left(\log \frac{\chi}{T^{eq}}\right). \quad (13)$$

C. Maxwell-Cattaneo

The linearization in the bulk stress $\Pi$ of the Hiscock-Lindblom theory is obtained by dropping the last term of Eq. (12):

$$\Pi = -\zeta \nabla_\mu u^\mu - \tau u^\mu \nabla_\mu \Pi. \quad (14)$$

Eq. (14) is known as Maxwell-Cattaneo equation, and can be put in a flux-conservative form together with the other hydrodynamic equations:

$$\nabla_\mu (\rho u^\mu) = 0, \quad (15)$$

$$\nabla_\mu (T^{\mu\nu}) = 0, \quad (16)$$

$$\nabla_\mu (\Pi u^\mu) = -\frac{\Pi}{\tau} \left(1 - \frac{1}{\chi} \right) \nabla_\mu u^\mu. \quad (17)$$

Note that, while Maxwell-Cattaneo is linear in the bulk stress $\Pi$, it is not linear in general since at first order $\nabla_\mu u^\mu \propto \Pi$.

Note also that these formulations of Hiscock-Lindblom and Maxwell-Cattaneo do not take into account that neutrinos are leaving the star carrying out energy. We address this issue in Sec. IV C.

IV. DUALITY BETWEEN MÜLLER-ISRAEL-STEWART AND THE MULTI-COMPONENT FLUID

In this section we show how to map the multi-component fluid model introduced in Sec. III A that is based on the approach of Carter [38] and does not explicitly contain the bulk stress $\Pi$, into the Müller-Israel-Stewart model (either Hiscock-Lindblom or Maxwell-Cattaneo), and vice versa, for a purely bulk-viscous fluid (i.e., in the absence of heat conduction, superfluidity and shear viscosity). Further details on the mathematical aspects of this mapping are given in Gavassino et al. [19], while the extension of the mapping to superfluid and heat conducting fluids is developed in Gavassino et al. [59].

A. Mapping the multi-component fluid into the Müller-Israel-Stewart model

Given a multi-component model with a non-equilibrium EOS and reaction rates $R_i$, it is possible to derive the equivalent bulk viscous parameters that appear in its dual Müller-Israel-Stewart theories, see Eqs. (84) and (27) of Gavassino et al. [19]:

$$\zeta = n^4 \varepsilon_{ab} \frac{\partial Y^c_{eq}}{\partial n_i} \frac{\partial Y^c_{eq}}{\partial n_j} \left(\frac{1}{\rho} \delta^C_{ij}\right), \quad (18)$$

$$\Xi_{ab} = \frac{\partial R_k \left(\{f_i, = 0\}_{Y_j}\right)}{\partial Y^k_{eq}} \left(\frac{\partial Y^k_{eq}}{\partial Y^k_{eq}}\right), \quad (19)$$

where the indices $a, b$, that label the fluid components are raised and lowered by matrix inversion $(\Xi_{ab} \Xi^{bc} = \delta^C_{c})$, and are manipulated according to Einstein’s sum convention. If the system can be described by a single out-of-equilibrium fraction, we can introduce a reaction timescale, see Eq. (113) of Gavassino et al. [19]:

$$\tau = -n \frac{\partial Y (h = 0)}{\partial \lambda_{ps}}, \quad (20)$$

where we dropped the species index because there is only one independent reaction coordinate $(\Xi \equiv \Xi_{11})$. Moreover, we can generalize the bulk-viscous timescale to more than one species out-of-equilibrium by requiring that the speed of sound of the multi-component fluid coincide with that of the Müller-Israel-Stewart theory (see Sec. V and Appendix A):

$$\tau = \frac{\varepsilon_{ab} \frac{\partial Y^c_{eq}}{\partial n_i} \frac{\partial Y^c_{eq}}{\partial n_j} \left(\frac{1}{\rho} \delta^C_{ij}\right)}{\frac{\partial Y^c_{eq}}{\partial n_i} \frac{\partial Y^c_{eq}}{\partial n_j} \left(\frac{1}{\rho} \delta^C_{ij}\right)}, \quad (21)$$

where we recall that $u = \epsilon/\rho - 1$ is the specific internal energy. For a single chemical fraction, Eq. (21) reduces to (20).

B. Mapping Müller-Israel-Stewart into the multi-component fluid

As anticipated in Sec II both thermodynamic and chemical processes contributing to bulk viscosity can be described in terms of a set of scalar variables (reaction coordinates) that parametrize the displacement of the substance from local thermodynamic equilibrium. This results in a multi-component fluid with comoving and reacting chemical species, that can be interpreted as a universal model for purely bulk-viscous (no shear viscosity
and no heat conduction) fluids. Therefore, given a bulk-viscous Müller-Israel-Stewart fluid, it is possible to find an equivalent multi-component description and vice versa.

In Sec. IV A we have seen that it is straightforward to map the multi-component model into the Müller-Israel-Stewart one. However, the inverse mapping is more subtle. In fact, the Müller-Israel-Stewart models are defined by the parameters \( \zeta (\rho, s) \) and \( \tau (\rho, s) \), but these are not enough to uniquely define a multi-component fluid. Therefore, for a given Müller-Israel-Stewart model it is possible to build a whole family of multi-component fluids, that are all equivalent in the limit of small perturbations around equilibrium. This is due to the fact that the Müller-Israel-Stewart framework is a perturbative one, based on an expansion near equilibrium. Therefore, while a multi-component fluid always gives a well-defined Müller-Israel-Stewart model, the inverse process is similar to try to recreate a function from its truncated Taylor series: the result cannot be unique. Moreover, the reaction coordinate (i.e., the fictitious particle abundance \( Y \)) must behave as a frozen variable in the high-frequency limit.

As an example, we will show in this section that the two-component fluid defined by:

\[
  u(\rho, s, Y) = u^q(\rho, s) + \frac{\zeta}{2m_n n} (Y - \log n)^2, \quad (22)
\]

\[
  R(\rho, s, Y) = -\frac{n}{\tau} (Y - \log n), \quad (23)
\]

is equivalent to the Maxwell-Cattaneo model (which in turn is equivalent at first order to the Hiscok-Lindblom theory). First, we obtain the affinity and the equilibrium particle fraction:

\[
  \mathcal{A}(\rho, s, Y) = -m_n \frac{\partial u}{\partial Y}_{\rho, s} - \frac{\zeta}{n} (Y - \log n), \quad (24)
\]

\[
  Y^e(\rho, s) = Y(\rho, s, \mathcal{A} = 0) = \log n. \quad (25)
\]

We then verify, using Eqs. (18)-(20), that the reacting multi-component model is equivalent to the Müller-Israel-Stewart (whose quantities are identified with the ‘MIS’ subscript) one:

\[
  \Xi = \frac{\partial R}{\partial \mathcal{A}} = \frac{n^2}{\zeta}, \quad (26)
\]

\[
  \zeta_{\text{MIS}} = \frac{n^4}{\Xi} \left( \frac{\partial Y^eq}{\partial n} \right)^2 = \zeta, \quad (27)
\]

\[
  \tau_{\text{MIS}} = -\frac{n}{\Xi} \frac{\partial Y}{\partial \mathcal{A}} = \tau. \quad (28)
\]

From Eq. (26) we have that:

\[
  \Pi = n \mathcal{A}, \quad (29)
\]

\[
  Y = -\frac{\tau}{\zeta} \Pi + \log n, \quad (30)
\]

from which we obtain \( \nabla_u = u^\mu \nabla_\mu \):

\[
  \nabla_u Y = -\nabla_u \left( \frac{\tau}{\zeta} \Pi \right) + \frac{\nabla a \nu}{n} \simeq -\frac{\tau}{\zeta} \nabla_u \Pi + \frac{\nabla a \nu}{n}, \quad (31)
\]

where in the last step we kept the first order contributions in the deviations from equilibrium. From Eqs. (2) and (41) we have that

\[
  nu^\mu \nabla_\mu Y = \nabla \mu (nY u^\mu) = \mathcal{R} = \Xi \mathcal{A} = \frac{n\Pi}{\zeta}, \quad (32)
\]

and inserting Eq. (31) into (32) and using \( u^\mu \nabla_\mu n = -n \nabla_\mu u^\mu \) [where we used again Eq. (2)] we obtain the Maxwell-Cattaneo equation (41).

C. A simple application of the mapping: extending Müller-Israel-Stewart to account for radiative losses

A first advantage of the mapping outlined in Sec. IV A is that it allows us to consistently include the effects of neutrino emission in the Müller-Israel-Stewart theories. In general, the energy and momentum conservation equation (4) has a contribution accounting for the energy and momentum lost by the fluid due to the reactions (e.g., neutrinos escaping from the star). However, the original Müller-Israel-Stewart theory does not have this contribution, cf. Eqs. (5), (7), and (10). Realizing how the Müller-Israel-Stewart theory can be derived from the multi-component reacting fluid, and in particular that at first order the bulk stress is given by (cf. Eq. (65) of Gavassino et al. [19]):

\[
  \Pi = \mathcal{A} \frac{\partial Y^eq}{\partial n} \bigg|_{\nu, s}, \quad (33)
\]

we can approximately account for the energy loss by expanding the luminosity \( Q \) around equilibrium:

\[
  Q_{\text{bv}}(\rho, s, \Pi) = Q^eq(\rho, s^eq) + \frac{\partial Q}{\partial \Pi} \Pi + O(\Pi^2), \quad (34)
\]

\[
  \frac{\partial Q}{\partial \Pi} = \frac{1}{n^2} \left( \frac{\partial Y^eq}{\partial n} \right)_{\nu, s}^{-1} \frac{\partial Q(\mathcal{A} = 0)}{\partial \mathcal{A}} \bigg|_{\nu, s}, \quad (35)
\]

where the subscript ‘bv’ means ‘bulk viscous’ and we dropped the species index because we consider for simplicity only one independent species. Note that in Eq. (34) we could replace \( s \) with \( s^eq \) because \( s - s^eq \) is of second order in \( \Pi \), cf. Eq. (9). Finally, Eqs. (7) and (10) become:

\[
  \nabla_\mu (T^{\mu \nu}) = -Q_{\text{bv}} u^\nu. \quad (36)
\]

V. PROPAGATION SPEED OF A SIGNAL AND SOUND SPEED OF THE MATTER

In a hydrodynamic code, the propagation speed of a signal is used to compute the timestep from the Levy-Friedrichs-Courant condition and to solve the Riemann
problem, and is equal to the maximal characteristic speed of the hydrodynamic equations. In a hydrodynamic system, the speed of sound enters in the definition of the propagation speed of a signal and is therefore needed to evolve the equations.

The definition of ‘speed of sound’ of a reacting fluid depends on the frequency of the sound wave. If the period of the wave is much shorter than the reaction timescale \( \tau_{\text{pert}} \ll \tau_{\text{react}} \), we are in the ‘ultraviolet’ limit, namely the composition is frozen and

\[
c^2_{s,uv}(\rho, s, \{Y\}_i) = \frac{\partial p}{\partial \epsilon}\bigg|_{\{Y\}_i}, \tag{37}
\]

The sound speed in Eq. (37) is used in hydrodynamic codes that evolve a multi-component fluid, because the timestep must be short enough to resolve the dynamics, which is influenced by the particle fractions.

On the other hand, if the wave period is much longer than the reaction timescale \( \tau_{\text{pert}} \gg \tau_{\text{react}} \), we are in the ‘infrared’ limit, that means that the fluid is always at chemical equilibrium and

\[
c^2_{s,ir}(\rho, s) = \frac{\partial p(\{A\}_i = 0)}{\partial \epsilon}\bigg|_{s,\{A\}_i}. \tag{38}
\]

In the limit in which the EOS contains only one independent species we recover the hydrodynamics of a perfect fluid: in this case, Eq. (38) coincides with Eq. (37), because the thermodynamic quantities do not depend on the particle fractions. In general, however, if the equilibrium state is stable against fluctuations, one can prove that (see Appendix A2):

\[
c^2_{s,eq} \geq c^2_{s,ir}. \tag{39}
\]

We can also define an infrared and an ultraviolet speed of sound for the Müller-Israel-Stewart theories. In the infrared limit, the gradients are small, hence the viscous stresses become negligible and the fluid behaves as a perfect fluid:

\[
c^2_{s,ir} = \frac{\partial p_{\text{eq}}}{\partial \epsilon}\bigg|_{s,\text{eq}}. \tag{40}
\]

In the ultraviolet limit, perturbations propagate along the characteristics of the full Müller-Israel-Stewart theory. In the Maxwell-Cattaneo case, the speed of sound is

\[
c^2_{s,uv} = \frac{\partial p_{\text{eq}}}{\partial \epsilon}\bigg|_{\rho} + \frac{1}{\epsilon + p_{\text{eq}} + \Pi} \left( \rho \frac{\partial p}{\partial \rho}\bigg|_{\epsilon} + \frac{1}{\chi} \right), \tag{41}
\]

while in the Hiscock-Lindblom case, the speed of sound is given by Eq. (B16). Close to equilibrium, we have in both cases that:

\[
c^2_{s,eq} = c^2_{s,ir} = \frac{1}{\epsilon + p_{\text{eq}} + \Pi} \chi, \tag{42}
\]

and we recover the thermodynamic inequality (39). We remark that stability and causality require that the last term of Eq. (12) does not diverge for vanishing rest mass density (Sec. III (b) of Hiscock and Lindblom [31]).

It is interesting to note that, since the ultraviolet speed of sound is the actual signal propagation speed of the field equations, causality only demands that \( c^2_{s,uv} < 1 \) and not that \( c^2_{s,ir} < 1 \). This implies that a viscous fluid may be consistent with the principle of causality even if the infrared speed of sound is superluminal. However, such a fluid would be thermodynamically unstable, and Eq. (39) would not hold. This result is a generalization of the Bludman-Ruderman theorem [68] to reacting mixtures.

VI. MICROPHYSICS

In order to close the hydrodynamic equations, it is necessary to introduce an EOS. Moreover, since bulk viscosity is due to reactions, we also need the reaction rates \( \mathcal{R}_i \) and \( Q_i \). The most accurate way to do this is by using the results of microphysics calculations in form of tables [67, 68]. However, the focus of this paper and its companion paper [12] is to compare different bulk viscous frameworks and not to obtain accurate astrophysical predictions. For this reason, we opted to use simple, analytic, but at the same time realistic EOS and reaction rates, that allow us to study bulk viscosity without the additional complications of table interpolation.

A. Equation of state

We consider a neutrinoless fluid of protons ‘p’, neutrons ‘n’, electrons ‘e’, and muons ‘µ’, which describe a mature (i.e., cold) neutron star, where neutrinos are not trapped. We assume that the fluid undergoes only direct beta reactions (direct Urca):

\[
\begin{align*}
\beta^{-}_e : & \quad n \rightarrow p + e^- + \bar{\nu}_e, \quad (43) \\
\beta^{+}_e : & \quad p + e^- \rightarrow n + \nu_e, \quad (44) \\
\beta^{-}_\mu : & \quad n \rightarrow p + \mu^- + \bar{\nu}_\mu, \quad (45) \\
\beta^{+}_\mu : & \quad p + \mu^- \rightarrow n + \nu_\mu. \quad (46)
\end{align*}
\]

Since baryons are conserved \( (dY_n = -dY_p) \), the system is charge-neutral \( (dY_e = dY_\mu + dY_\bar{\mu}) \), and neutrinos immediately leave the star \( (Y_\nu = 0) \), then there are only 2 independent chemical fractions, e.g. electrons and muons, and the first law of thermodynamics is:

\[
\begin{align*}
\frac{du}{T} &= \frac{p}{\rho} \frac{dp}{T} + \frac{T}{m_n} \frac{dY_e}{T} - \frac{\Delta Y_e}{m_n} \frac{dY_\mu}{T}, \quad (47) \\
\Delta Y_i &= Y_i - Y^0_i \frac{\rho}{\rho_n}. \quad (48)
\end{align*}
\]

Our equation of state is defined by:

\[
\begin{align*}
u &= k_\nu \rho + k_\nu \Delta Y^{-1} + k_e \Delta Y_e^2 + k_\mu \Delta Y_\mu^2, \quad (49) \\
\Delta Y_i &= Y_i - Y^0_i \frac{\rho}{\rho_n}. \quad (50)
\end{align*}
\]
where $u = u(\rho, s, Y_e, Y_\mu)$, $\Gamma_{th}$ is the thermal polytropic exponent, $k_0, k_{th}, k_i, k_\mu > 0$ are (positive) polytropic parameters, and $Y_{e0}, Y_{\mu0}$ are the equilibrium values of the particle fractions at saturation density. We note that this EOS is an extension of a $\Gamma = 2$ polytropic EOS (the thermal component was introduced in Camelio et al. [42]), and that we are assuming that the electron and muon polytropic exponents are $\Gamma_e = \Gamma_\mu = 1$. The reason of these choices is that in this way the pressure and the speed of sound are always positive\(^6\) if $Y_i \in [0, 1]$ and if
\[ k_0 > 2 \frac{k_e Y_{e0} + k_\mu Y_{\mu0}}{\rho_n}. \tag{51} \]

Our EOS has the advantage of being simple and analytic, but at the same time it reproduces the qualitative features of a more realistic EOS. In particular, the cold EOS at equilibrium is a $\Gamma = 2$ polytrope, which is a common choice for testing new codes, and the temperature goes to zero as the entropy goes to zero [47]. Moreover, the equilibrium fraction is analytical and given by:
\[ Y_i^{eq}(\rho) = Y_i^{eq} \frac{\rho}{\rho_n}. \tag{52} \]

Note that, in order to have bulk viscosity, $Y_i^{eq}$ must depend on $\rho$, otherwise $\zeta$ and $\Pi$ would vanish identically, cf. Eq. (13) and (33).

The other thermodynamic quantities can be derived from the EOS Eq. (49) and the first law of thermodynamics [Eq. (57)]:
\[ p = k_0 \rho^2 + (\Gamma_{th} - 1) k_{th} s^2 \rho^{\Gamma_{th}}, \tag{53} \]
\[ T = 2 m_n k_{th} s \rho^{\Gamma_{th} - 1}, \tag{54} \]
\[ \zeta = -2 m_n k_i \Delta Y_i, \tag{55} \]

Moreover, the ultraviolet and infrared sound speeds defined in Sec. V are:
\[ c_{s, uv}^2 = \frac{\partial p^{eq}}{\partial \rho} \bigg|_{s} - 2 \frac{\varphi}{\rho_n} \sum_i k_i Y_{i0} \left( 2 \Delta Y_i - Y_{i0} \frac{\varphi}{\rho_n} \right), \tag{56} \]
\[ c_{s, ir}^2 = \frac{\partial p^{eq}}{\partial \rho} \bigg|_{s} \div \frac{\partial c_s^{eq}}{\partial \rho} \bigg|_{s}, \tag{57} \]
\[ \frac{\partial c_s^{eq}}{\partial \rho} \bigg|_{s} = 2 k_0 \rho + \Gamma_{th} (\Gamma_{th} - 1) k_{th} s^2 \rho^{\Gamma_{th} - 1}, \tag{58} \]
\[ \frac{\partial c_s^{eq}}{\partial \rho} \bigg|_{s} = 1 + 2 k_0 \rho + \Gamma_{th} k_i s^2 \rho^{\Gamma_{th} - 1}. \tag{59} \]

Note that, as pointed out in Sec. V, $c_{s, uv}^{eq} \geq c_{s, ir}$.

\[ \footnote{Strictly speaking, $p > 0$ is not a fundamental thermodynamic requirement: negative pressures are physically possible in many interacting systems, also in the liquid state [70]. However, we assume positive pressure to avoid possible complications in the numerical implementation [43] presented in Camelio et al. [42].} \]

### B. Reaction rates

In this section we use cgs units, assume that the particle species are Fermi liquids, that the matter is strongly degenerate, that $A_i \ll \mu_i, \mu_\mu, \mu_e$ ($i = \{e, \mu\}$), approximate the nucleon effective masses to their bare ones, and neglect the effects of the medium on the weak interactions [71]. With these approximations, the number $R_i^-$ and energy $Q_i^-$ rates of the neutron decay reactions $\beta_i^-$ [i.e. direct Urca reactions, see Eqs. (43) and (45)] are (Eqs. (6)–(9) of Haensel [71]):
\[ R_i^- = 8.86 \times 10^{31} \frac{Y_i \rho}{\rho_n} \left( \frac{T}{10^9 \text{K}} \right)^5 G(x_i), \tag{60} \]
\[ G(x) = \int_0^{\infty} y^2 \frac{\pi^2}{1 + \exp(y - x)^2} dy, \tag{61} \]
\[ Q_i^- (x) = 1.22 \times 10^{25} \frac{Y_i \rho}{\rho_n} \left( \frac{T}{10^9 \text{K}} \right)^6 F(x_i), \tag{62} \]
\[ F(x) = \int_0^{\infty} y^3 \frac{\pi^2}{1 + \exp(y - x)^2} dy, \tag{63} \]
\[ x_i = \frac{k_i}{k_{th} T}. \tag{64} \]

In our case, the number $R_i^+$ and energy $Q_i^+$ rates of the lepton capture reactions $\beta_i^+$ [Eqs. (44) and (46)] and the total number $R_i$ and energy $Q_i$ rates of the direct beta reactions are [71]:
\[ R_i^+ (x_i) = R_i^- (-x_i), \tag{65} \]
\[ Q_i^+ (x_i) = Q_i^- (-x_i), \tag{66} \]
\[ R_i = R_i^- - R_i^+, \tag{67} \]
\[ Q_i = Q_i^- + Q_i^+. \tag{68} \]

From the definition of polylogarithm (Eq. (A.3.8.2) of Lewin [72]):
\[ L_i (z) = \frac{1}{\Gamma (n)} \int_0^{\infty} \frac{y^{n-1}}{\exp(y - z)} dy = -Li_n (\exp (-z)), \tag{69} \]

and assuming that $z \in \mathbb{R}^-$ and $n \in \mathbb{N}^+$, we have
\[ \int_0^\infty \frac{y^{n-1}}{1 + \exp(y - x)} dy = -Li_n (-\exp (x)) (n - 1)!, \tag{70} \]
where $x = \log (-z)$. Substituting Eq. (70) in $R_i$ and $Q_i$, and using the properties of polylogarithms (Eqs. (A.4.2)
Urca reaction rates for lepton species longer. Following the procedure we detailed above rather than with an iterative procedure, see Camelio after tedious but straightforward calculations, we obtain and (A.2.7.6) of Lewin [72]):

\[
\begin{align*}
\text{Li}_3(-e^x) - \text{Li}_3(-e^{-x}) &= -\frac{\pi^2 x}{6} - \frac{x^3}{6}, \\
\text{Li}_4(-e^x) + \text{Li}_4(-e^{-x}) &= -\frac{7\pi^4}{360} + \frac{\pi^2 x^2}{12} - \frac{x^4}{24}, \\
\text{Li}_5(-e^x) - \text{Li}_5(-e^{-x}) &= -\frac{7\pi^4}{360} + \frac{\pi^2 x^2}{12} - \frac{x^4}{120}, \\
\text{Li}_6(-e^x) + \text{Li}_6(-e^{-x}) &= -\frac{31\pi^4}{1512} + \frac{7\pi^4 x^2}{720} - \frac{\pi^2 x^4}{144} - \frac{x^6}{720}.
\end{align*}
\]

after tedious but straightforward calculations, we obtain

\[
\begin{align*}
\mathcal{R}_i &= \frac{8.86 \times 10^{31}}{\text{cm}^3 \text{s}} \left( \frac{Y_i \rho}{\rho_n} \right) \left( \frac{T}{10^9 \text{K}} \right)^5 \times \frac{17\pi^4 x_i + 10\pi^2 x_i^3 + x_i^5}{30}, \\
\mathcal{Q}_i &= \frac{1.22 \times 10^{25}}{\text{erg}^{-1} \text{cm}^3 \text{s}} \left( \frac{Y_i \rho}{\rho_n} \right) \left( \frac{T}{10^9 \text{K}} \right)^6 \times \frac{457\pi^6 + 51\pi^4 x_i^2 + 15\pi^2 x_i^4 + x_i^6}{60}.
\end{align*}
\]

In order to simplify the code (i.e., in order to implement the implicit time evolution with direct inversion rather than with an iterative procedure, see Camelio [71]), we linearize Eqs. (75) and (76) in \( \Delta Y_i \) around the equilibrium configuration:

\[
\begin{align*}
\mathcal{R}_i &\approx \frac{8.86 \times 10^{31}}{\text{cm}^3 \text{s}} \left( \frac{Y_i \rho}{\rho_n} \right) \left( \frac{T}{10^9 \text{K}} \right)^5 \frac{17\pi^4}{30} \frac{A_i}{k_B T_{\text{eq}}}, \\
\mathcal{Q}_i &\approx \frac{1.22 \times 10^{25}}{\text{erg}^{-1} \text{cm}^3 \text{s}} \left( \frac{Y_i \rho}{\rho_n} \right) \left( \frac{T}{10^9 \text{K}} \right)^6 \frac{457\pi^6}{1260} \left( 1 + \frac{\Delta Y_i}{3 Y_i} \right).
\end{align*}
\]

For modified beta reactions (modified Urca, see Sec. 2.2 of Haensel [71]) the calculations are similar but longer. Following the procedure we detailed above, we compute the number (\( \mathcal{R}_i^m \)) and energy (\( \mathcal{Q}_i^m \)) modified Urca reaction rates for lepton species \( i \), linearized in \( \Delta Y_i \) around the equilibrium configuration:

\[
\begin{align*}
\mathcal{R}_i^m &\approx \frac{5.91 \times 10^{23}}{\text{cm}^3 \text{s}} \left( \frac{Y_i \rho}{\rho_n} \right) \left( \frac{T}{10^9 \text{K}} \right)^7 \frac{367\pi^6}{63} \frac{A_i}{k_B T_{\text{eq}}}, \\
\mathcal{Q}_i^m &\approx \frac{8.15 \times 10^{16}}{\text{erg}^{-1} \text{cm}^3 \text{s}} \left( \frac{Y_i \rho}{\rho_n} \right) \left( \frac{T}{10^9 \text{K}} \right)^8 \frac{11513\pi^6}{2520} \left( 1 + \frac{\Delta Y_i}{3 Y_i} \right).
\end{align*}
\]

Only 3 particles plus one neutrino partake in a direct Urca reaction (Eqs. (13)–(14), see also Eqs. (1) and (2) of Haensel [71]), while 5 particles plus one neutrino partake in a modified Urca reaction (Eqs. (10) and (11) of Haensel [71]). As a consequence, in contrast to the direct Urca reactions [cf. Eqs. (77)–(78) with Eqs. (72)–(73)], (i) the prefactor of the modified Urca rates is smaller, (ii) the modified Urca rates depend on a larger power of the temperature, and (iii) the modified Urca reactions are not kinematically inhibited at lower density and temperature.

The aforementioned consequence (iii) is the main reason why modified Urca reactions should be kept into account when studying neutron stars. Moreover, theoretical estimates [73] and some numerical simulations (e.g., Most et al. [28]) show that in some regimes (i.e., in neutron star post-merger remnants) bulk viscosity from modified Urca reactions may provide significant damping, although simulations with more sophisticated neutrino transport and higher resolution do not find a significant imprint of out-of-equilibrium effects on the gravitational wave emission [31, 74].

We finally remark that the linearization of the reactions around equilibrium [Eqs. (77)–(80)] is not always correct in the cases considered in the companion paper [42]. Indeed, in our simulations \( \Lambda/k_B T \) can become greater than one, in particular for the electrons and in particular close to the surface. However, also considering the other approximations employed (i.e., spherical symmetry, a polytropic EOS, and only direct beta reactions), our aim is not to obtain a quantitative description of a neutron star, but rather to study the equivalence between different approaches to bulk viscosity in neutron stars.

VII. HYDRODYNAMIC EQUATIONS FOR BULK VISCOSITY

In this section, we derive the hydrodynamic equations for the bulk stress in the Müller-Israel-Stewart theories in radial gauge-polar slicing coordinates in spherical symmetry (i.e. Schwarzschild):

\[
dl^2 = -\alpha^2(r, t) dt^2 + X^2(r, t) dr^2 + r^2 d\Omega^2,
\]

where \( l \) is the proper time, \( t \) and \( r \) are respectively the time and radial coordinates, \( d\Omega \) the angular element, \( \alpha \) is the lapse and \( X \) is a metric function.

Using that the divergence of a vector \( A^\mu \) and of a scalar field \( \varphi \) are (Appendix A of O’Connor and Ott [57]):

\[
\sqrt{-g} \nabla_\mu A^\mu = \partial_\mu (\sqrt{-g} A^\mu),
\]

\[
\nabla_\mu \varphi = \partial_\mu \varphi,
\]

and that in Schwarzschild coordinates the determinant

\[\text{(77)}\] Note that, for our EOS, (i) \( A_i \) is also linear in \( \Delta Y_i \), and (ii) \( T = T_{\text{eq}} \) at first order in \( \Delta Y_i \) (cf. Eq. (64) of Gavassino et al. [14]).

\[\text{(78)}\] In addition, \( \text{Li}_8(-e^x) - \text{Li}_8(-e^{-x}) = -127\pi^8/604800 + \mathcal{O}(x^2) \).
of the metric $\sqrt{-g}$ and the fluid 4-velocity $u^\mu$ are:

$$\sqrt{-g} = r^2 \alpha X,$$

$$u^\mu = \left( \frac{W}{\alpha}, \frac{W v}{X}, 0, 0 \right),$$

$$W = (1 - v^2)^{-1/2},$$

where $v = X u^r / \alpha u^t$ is the physical velocity of the fluid and $W = \alpha u^t$ a Lorentz-like factor, the bulk viscous equations \[8\], \[13\], and \[17\] become, respectively:

$$\partial_t (XW \rho s) + \frac{1}{r^2} \partial_r (\alpha r^2 W r \rho sv) = \alpha X m_n \frac{\Pi^2}{\zeta T_{\text{eq}}},$$

$$\partial_t (XW \Pi) + \frac{1}{r^2} \partial_r (\alpha r^2 W r \Pi) = -\frac{\alpha \Pi}{\tau},$$

$$\partial_t (XW \tau) + \frac{1}{r^2} \partial_r (\alpha r^2 W r \tau) = -\frac{\Pi \alpha}{\tau},$$

Note that there are time derivatives in the source of Eqs. \([87]\) and \([89]\), which are a complication for the numerical implementation. However, Eq. \([87]\) has bigger problems, since the inversion from the entropy $s$ to the bulk stress $\Pi$ is degenerate [see Eq. \(\text{IV} B\)]. In practice, we numerically implemented only Eqs. \([88]\) and \([89]\) in the companion paper \[41\], since we did not manage to evolve Eq. \([87]\) in a stable manner.

VIII. CONCLUSIONS

With this paper we aim to clarify how different approaches to bulk viscosity can be implemented in practice. In order to do so, we cast the equations of the multi-component fluid and of two Müller-Israel-Stewart theories (Hiscock-Lindblom and Maxwell-Cattaneo) in conservative form in radial gauge, polar slicing coordinates and spherical symmetry.

The novel results presented in this paper are an example of inversion from the Müller-Israel-Stewart formulation to the multi-component fluid one (Sec. \(\text{IV} B\)), the extension of the Müller-Israel-Stewart theories to include the energy loss due to the reactions (Sec. \(\text{IV} C\)), the specialization of the Müller-Israel-Stewart equations to the radial gauge-polar slicing coordinates in spherical symmetry (Sec. \(\text{VII}\)), and the computation of the signal propagation speed for the Hiscock-Lindblom theory in the non-linear regime (Appendix \[B\]). We also propose a simple but physically motivated equation of state (Sec. \(\text{VI} A\)) and an analytical form for the reaction rates (Sec. \(\text{VII} B\)).

This paper establishes the theoretical foundations for a companion paper \[42\], in which we implement a one-dimensional general relativistic hydrodynamic code \[41\] to study bulk viscosity in neutron stars. As numerically confirmed in the companion paper \[42\], the Müller-Israel-Stewart and the multi-component fluid formulations are equivalent for small thermodynamic perturbations \[13\]. In our opinion, when bulk viscosity is due to particle reactions, it is more easy, convenient, and accurate to directly evolve the multi-component fluid rather than implementing bulk viscosity with a perturbative formulation such as the Müller-Israel-Stewart theories, which is often adopted for binary neutron star mergers. Note that the multi-component fluid is already the normal approach in core-collapse supernova simulations (e.g., O’Connor and Ott \[37\]), where knowing the exact composition of the matter is considered very important to correctly determine the evolution of the supernova and its observables, like the neutrino luminosity \[36\].

Moreover, even if bulk viscosity is not due to particle reactions, it is possible to reformulate the problem in terms of a multi-component fluid, whose additional fictitious particle species (also called reaction coordinate) effectively accounts for bulk viscosity (see Sec. \(\text{IV} B\)).

This paper and its companion \[42\] make many strong approximations: spherical symmetry, a polytropic EOS, the focus on direct Urca reactions over the modified ones, and the linearization of the reactions in $\Delta \mu / k_B T$. These approximations have been made not to clutter the numerical implementation with technical details that would obscure our main aim: the comparison between the multi-component and the bulk stress descriptions of viscosity in the context of neutron stars. As a consequence, our model should be considered as a playground to experiment with bulk viscosity and obtain qualitative results; more physics should be added in order to obtain quantitative physical predictions.

As a final remark, we point out that the correspondence between the multi-component fluid and bulk stress can be used also to describe superfluidity \[58\] and in contexts different from neutron stars, as in cosmology.
Acknowledgments

This work was supported by the Polish National Science Centre (NCN) grant number OPUS 2019/33/B/ST9/00942. SB acknowledges support by the EU H2020 under ERC Starting Grant, no. BinGrApsp-714626. LG is partially supported by a Vanderbilt’s Seeding Success Grant.

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Appendix A: Comparing the speeds of sound

1. Some useful thermodynamic relations

Here, we derive Eqs. (A1) and (A3), which will be used in the next subsections.

At equilibrium, the internal specific energy must be in a minimum $\mathcal{F}$, for constant $\rho$ and $s$. Hence, the symmetric matrix

$$ u^{ab} := \frac{\partial^2 u}{\partial Y_a \partial Y_b} \bigg|_{\rho, s, \{Y_i\}} = -\frac{1}{\rho_m} \frac{\partial h_a}{\partial Y_b} \bigg|_{\rho, s, \{Y_i\}} $$

(A1)

is non-negative definite close to equilibrium. If we differ-
entiate the equilibrium condition
\[
\mathcal{A}_a(p, s, \{Y_i\}_{i=1}^n) = 0 \quad \forall a,
\] (A2)
with respect to \(\rho\) we obtain (repeated species indices are summed with the Einstein convention):
\[
\frac{\partial \mathcal{A}_a}{\partial \rho} \bigg|_{s,\{Y_i\}_i} = m_a u^a \frac{\partial Y^{eq}_a}{\partial \rho} \bigg|_s,
\] (A3)
which implies, since \(u^a\) is non-negative definite,
\[
\frac{\partial Y^{eq}_a}{\partial \rho} \bigg|_{s,\{Y_i\}_i} + \frac{\partial \mathcal{A}_a}{\partial \rho} \bigg|_{s,\{Y_i\}_i} \geq 0.\tag{A4}
\]
The differentiation of the pressure with respect to \(Y_a\) at constant \(\epsilon\) is connected to that at constant \(\rho\) by this formula:
\[
\frac{\partial \rho}{\partial Y_a} \bigg|_{\epsilon,\{s,Y_i\}_i,\rho} = \frac{\partial \rho}{\partial Y_a} \bigg|_{\rho,\{s,Y_i\}_i,\epsilon} + m_a u^{ab} \frac{\partial Y^{eq}_a}{\partial \rho} \bigg|_s \frac{\partial Y^{eq}_b}{\partial \rho} \bigg|_s.\tag{A5}
\]
From the thermodynamic differential
\[
d\epsilon = \frac{\rho T}{m_n} ds + \frac{\epsilon + p}{\rho} d\rho - \frac{\rho \mathcal{A}_a}{m_a} dY_a
\] (A6)
we obtain
\[
\frac{\partial \rho}{\partial Y_a} \bigg|_{\epsilon,\{s,Y_i\}_i,\rho} = \frac{\rho^2 \mathcal{A}_a}{m_n (\epsilon + p)}\tag{A7}
\] and from Eq. (ii) we obtain the Maxwell relation
\[
\frac{\partial \rho}{\partial Y_a} \bigg|_{\rho,\{s,Y_i\}_i,\epsilon} = \frac{\rho^2 \mathcal{A}_a}{m_n \rho} \bigg|_{s,\{Y_i\}_i},\tag{A8}
\]
such that Eq. (A5) becomes
\[
\frac{\partial \rho}{\partial Y_a} \bigg|_{\epsilon,\{s,Y_i\}_i,\rho} = \frac{\rho^2 \mathcal{A}_a}{m_n (\epsilon + p)} \bigg|_{s,\{Y_i\}_i} - \frac{\partial \mathcal{A}_a}{\partial \rho} \bigg|_{s,\{Y_i\}_i}.\tag{A9}
\]

2. Ultraviolet vs infrared

The differentiation of the pressure with respect to \(\epsilon\) at constant \(\{A^i\}_i\) is connected to that at constant \(\{Y_i\}_i\) by this formula:
\[
\frac{\partial \rho}{\partial \epsilon} \bigg|_{s,\{A^i\}_i} = \frac{\partial \rho}{\partial \epsilon} \bigg|_{s,\{Y_i\}_i} + \frac{\partial \rho}{\partial Y_a} \bigg|_{\epsilon,\{s,Y_i\}_i,\rho} \frac{\partial Y_a}{\partial \epsilon} \bigg|_{s,\{A^i\}_i}.\tag{A10}
\]
which, using Eqs. (ii) and (10), becomes
\[
\frac{\partial \rho}{\partial \epsilon} \bigg|_{s,\{A^i\}_i} = \frac{\partial \rho}{\partial \epsilon} \bigg|_{s,\{Y_i\}_i} + \frac{\rho^2 \mathcal{A}_a}{m_n} \frac{\partial Y^{eq}_a}{\partial \epsilon} \bigg|_s \frac{\partial Y^{eq}_a}{\partial \rho} \bigg|_s \bigg|_{s,\{A^i\}_i}\tag{A11}
\]
If we evaluate this formula at equilibrium, it reduces to
\[
(c_{s,uv}^2)^2 = c_{s,ir}^2 + \frac{\rho^2 \mathcal{A}_a}{m_n (\epsilon + p)} \frac{\partial Y^{eq}_a}{\partial \rho} \bigg|_s \frac{\partial Y^{eq}_a}{\partial \rho} \bigg|_s \bigg|_{s,\{Y_i\}_i},\tag{A12}
\]
where the inequality follows from Eq. (A4).

3. Müller-Israel-Stewart vs multi-constituent fluid

At equilibrium, Eqs. (ii) and (11) are clearly the same, because \(s^{eq} = s\) and \(p^{eq} = p\). Hence, comparing Eqs. (22) and (11), and recalling Eq. (A4), we obtain that the equilibrium ultraviolet speed of sound of the multi-component fluid coincides with that of the bulk stress theory if
\[
\chi^{-1} = \rho^2 u^{ab} \frac{\partial Y^{eq}_a}{\partial \rho} \bigg|_s \frac{\partial Y^{eq}_a}{\partial \rho} \bigg|_s.\tag{A13}
\]
Combining this equation with Eqs. (10) and (11), we finally obtain Eq. (21).

Appendix B: Propagation speed of a signal in the Hiscock-Lindblom theory

In this section we compute the characteristic speeds of the Hiscock-Lindblom theory in the non-linear regime, following Bemfica et al. [66]. The hydrodynamic equations can be written as [66]:
\[
u^a \nabla u^a + \mathcal{A} \nabla \mu = 0,\tag{B1}
\]
\[
a_1 \Delta u^a \nabla \epsilon + \mathcal{A} u^a \nabla \mu + a_2 \Delta u^a \nabla \mu n + \Delta u^a \nabla \Pi = 0,\tag{B2}
\]
\[
n \nabla u^a + u^a \nabla n = 0,\tag{B3}
\]
\[
b_1 u^a \nabla \epsilon + \mathcal{A} u^a \nabla \mu + b_2 u^a \nabla \mu n + \tau n \nabla \Pi = 0,\tag{B4}
\]

where in the last equation we have generalized Eq. (4) of Bemfica et al. \[65\] to the Hiscock-Lindblom theory and:

\[ \mathcal{L} = \epsilon + p_{eq} + \Pi, \]  
\[ \Delta^{\mu\nu} = g^{\mu\nu} + u^\mu u^\nu, \]
\[ a_1 = \frac{\partial p_{eq}}{\partial \epsilon} |_n, \]
\[ a_2 = \frac{\partial p_{eq}}{\partial n} |_\epsilon, \]
\[ \zeta = \zeta + \frac{\tau \Pi}{2}, \]
\[ \Delta_{\mu\nu} = g_{\mu\nu} + u_\mu u_\nu, \]
\[ a_1 = \frac{\partial p_{eq}}{\partial \epsilon} |_n, \]
\[ a_2 = \frac{\partial p_{eq}}{\partial n} |_\epsilon, \]
\[ \zeta = \zeta + \frac{\tau \Pi}{2}, \]
\[ b_1 = \frac{\xi T_{eq}}{2} \frac{\partial \chi / T_{eq}}{\partial \epsilon} |_n, \]
\[ b_2 = \frac{\xi T_{eq}}{2} \frac{\partial \chi / T_{eq}}{\partial n} |_\epsilon. \]

In order to compute the characteristics, we need first to write Eqs. (B1)–(B4) in the form

\[ \mathcal{M}^\mu \nabla_\mu \psi + \mathcal{N} \psi = 0, \]

where \( \mathcal{M}^\mu \) and \( \mathcal{N} \) are matrices and

\[ \psi = (\epsilon, u^\mu, n, \Pi). \]

The matrix \( \mathcal{M}^\mu \) is

\[ \mathcal{M}^\mu = \begin{bmatrix} u^\mu & \delta^\mu_\nu & 0 & 0 \\ a_1 \Delta^{\mu\nu} & 0 & a_2 \Delta^{\mu\nu} & \Delta^{\mu\nu} \\ 0 & 0 & \zeta \delta^\mu_\nu & b_2 u^\mu \\ b_1 u^\mu & 0 & \tilde{\zeta} \delta^\mu_\nu & b_2 u^\mu \end{bmatrix}. \]

The characteristics of the system are determined by the \( \xi_\mu \) such that \( \det(\mathcal{M}^\mu \xi_\mu) = 0 \). We have that:

\[ \det(\mathcal{M}^\mu \xi_\mu) = (u^\mu \xi_\mu)^5 \mathcal{L}^4 \tau \left( (u^\mu \xi_\mu)^2 \right) \]
\[ - \left( a_1 - \frac{b_1}{\tau} + \frac{\zeta}{\tau \mathcal{L}} + \frac{n}{\mathcal{L}} \left( a_2 - \frac{b_2}{\tau} \right) \right) \Delta^{\mu\nu} \xi_\mu \xi_\nu, \]

and therefore the propagation speed of a signal (i.e., the effective sound speed) squared is:

\[ c_{s,uv}^2 = a_1 - \frac{b_1}{\tau} + \frac{\zeta}{\tau \mathcal{L}} + \frac{n}{\mathcal{L}} \left( a_2 - \frac{b_2}{\tau} \right). \]

Close to equilibrium we obtain, as for Maxwell-Cattaneo, Eq. (42).

---

9 In order to obtain Eq. (B15) one can directly compute the determinant of \( \mathcal{M}^\mu \xi_\mu \), noting that the second column/row of Eq. (B14) are actually 4 columns/rows, or one can subtract column 4 multiplied by \( b_1/\tau \) (resp. by \( b_2/\tau \)) to column 1 (resp. column 3) and use the result of Bemfica et al. \[65\] with \( \zeta \to \tilde{\zeta} \), \( a_1 \to a_1 - b_1/\tau \), and \( a_2 \to a_2 - b_2/\tau \).