ON THE $T$-LEAVES OF SOME POISSON STRUCTURES RELATED TO PRODUCTS OF FLAG VARIETIES

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ABSTRACT. For a connected abelian Lie group $T$ acting on a Poisson manifold $(Y, \pi)$ by Poisson isomorphisms, the $T$-leaves of $\pi$ in $Y$ are, by definition, the orbits of the symplectic leaves of $\pi$ under $T$, and the leaf stabilizer of a $T$-leaf is the subspace of the Lie algebra of $T$ that is everywhere tangent to all the symplectic leaves in the $T$-leaf. In this paper, we first develop a general theory on $T$-leaves and leaf stabilizers for a class of Poisson structures defined by Lie bialgebra actions and quasitriangular r-matrices. We then apply the general theory to four series of holomorphic Poisson structures on products of flag varieties and related spaces of a complex semi-simple Lie group $G$. We describe their $T$-leaf decompositions, where $T$ is a maximal torus of $G$, in terms of (open) extended Richardson varieties and extended double Bruhat cells associated to conjugacy classes of $G$, and we compute their leaf stabilizers and the dimension of the symplectic leaves in each $T$-leaf.

1. INTRODUCTION AND STATEMENTS OF RESULTS

1.1. Introduction. A holomorphic Poisson structure on a complex manifold $Y$ is a holomorphic bi-vector field $\pi$ on $Y$ such that the bracket $\{\phi, \psi\} = \pi(d\phi \wedge d\psi)$ on the sheaf of holomorphic functions satisfies the Jacobi identity. Holomorphic Poisson structures form an important class of Poisson structures, and they have recently also been studied in the context of generalized complex geometry and deformation theory (see [22, 28] and references therein).

A triple $(Y, \pi, \lambda)$, where $(Y, \pi)$ is a complex Poisson manifold and $\lambda$ a holomorphic action on $Y$ by a connected abelian complex Lie group $T$ preserving $\pi$, is called a complex $T$-Poisson manifold. A complex $T$-Poisson manifold $(Y, \pi, \lambda)$ gives rise to a decomposition of $Y$ into the $T$-orbits of symplectic leaves of $\pi$, also called $T$-leaves, which are of the form $\bigcup_{t \in T} t\Sigma$, where $\Sigma$ is a symplectic leaf of $\pi$ in $Y$ (see §2.2 for the precise definition). While a complex manifold can not support non-symplectic holomorphic Poisson structures with finitely many symplectic leaves, as the degeneracy locus of such a Poisson structure, being a non-empty divisor, can not be the union of finitely many symplectic leaves, it is easy to construct examples of $T$-Poisson manifolds with finitely many $T$-leaves: for a complex torus $T$, any smooth toric $T$-variety with the zero Poisson structure is such an example.

If $L$ is a $T$-leaf in a $T$-Poisson manifold $(Y, \pi, \lambda)$, the subspace $t_L$ of the Lie algebra $t$ of $T$ which is tangent to every symplectic leaf in $L$ under the action $\lambda$ is called the leaf stabilizer of $(\lambda$ in) $L$ (see §2.2). Every $T$-leaf $L$ in $(Y, \pi, \lambda)$ admits a nowhere vanishing anti-canonical section, called the Poisson $T$-Pfaffian, constructed as an exterior product of the Poisson bi-vector field $\pi$ and vector fields on $Y$ coming from any complement of $t_L$ in $t$ (see Remark 2.7), and the co-rank of $\pi$ in $L$ is equal to the codimension of $t_L$ in $t$. For a given $T$-Poisson manifold $(Y, \pi, \lambda)$, it is a natural and interesting problem to determine the $T$-leaf decomposition of $\pi$ in $Y$ and the leaf stabilizers for the $T$-leaves.

Let $G$ be a connected complex semi-simple Lie group with Lie algebra $\mathfrak{g}$, and fix a pair $(B, B_-)$ of opposite Borel subgroups of $G$ and a symmetric non-degenerate invariant bilinear form $\langle , \rangle_\mathfrak{g}$ on $\mathfrak{g}$. The choice of $(B, B_-, \langle , \rangle_\mathfrak{g})$ gives rise to a standard multiplicative holomorphic Poisson structure $\pi_{st}$ on $G$, and the pair $(G, \pi_{st})$ is known as a standard complex semi-simple Poisson Lie group (see §6.1 for detail). The Poisson structure $\pi_{st}$ is invariant under the action by the maximal torus $T = B \cap B_-$.
by left translation. It is well-known \[23\,25\] that the $T$-leaves of $\pi_{st}$ in $G$ are the double Bruhat cells $G^{u,v} = BuB \cap B_- v B_-$, where $u, v \in W$, the Weyl group of $(G,T)$. Double Bruhat cells have been studied intensively and have served as motivating examples of the theories of total positivity and cluster algebras (see \[3\,18\,21\] and references therein).

The Poisson structure $\pi_{st}$ on $G$ projects to a well-defined Poisson structure, denoted by $\pi_{G/B}$, on the flag variety $G/B$ of $G$. The $T$-leaves of $\pi_{G/B}$ in $G/B$ have been shown in \[20\] to be precisely the open Richardson varieties, i.e. non-empty intersections $(BuB/B) \cap (B_- wB/B)$, where $u, w \in W$ and $w \leq u$ in the Bruhat order on $W$. Open Richardson varieties and their Zariski closures in $G/B$, called Richardson varieties, have also been studied intensively from the points of view of geometric representation theory, combinatorics, and cluster algebras (see, for example, \[4\,5\,29\,31\,32\,33\]).

There are many other natural examples of $T$-Poisson manifolds associated to the Poisson Lie group $(G,\pi_{st})$, including the generalized flag varieties $G/P$ \[20\], where $P$ is a parabolic subgroup of $G$, twisted conjugacy classes in $G$ \[16\,35\,36\], symmetric spaces of $G$ \[15\], the wonderful compactification of $G$ when $G$ is of adjoint type \[15\], and the variety of Lagrangian subalgebras \[15\,39\]. In these examples, the $T$-leaves, and leaf stabilizers in some cases, have been determined by somewhat ad-hoc methods (but see \[52\] for the method of weak splittings in the study of $T$-leaves and symplectic leaves for a class of Poisson structures including $\pi_{G/B}$ on $G/B$).

In this paper, we describe the $T$-leaves and the leaf stabilizers for four series of $T$-Poisson manifolds associated to a standard complex semi-simple Poisson Lie group $(G,\pi_{st})$, respectively denoted as

\[(1.1) \quad (F_n, \pi_n), \quad (\tilde{F}_n, \tilde{\pi}_n), \quad (\bar{F}_n, \bar{\pi}_n), \quad (\bar{\bar{F}}_n, \bar{\bar{\pi}}_n), \quad n \geq 1.\]

When $n = 1$, we have

\[(F_1, \pi_1) = (G/B, \pi_{G/B}), \quad (\tilde{F}_1, \tilde{\pi}_1) = (G, \pi_{st}), \quad (\bar{F}_1, \bar{\pi}_1) = (G \times G, \Pi_{st}),\]

where $(G \times G, \Pi_{st})$ is the Drinfeld double Poisson Lie group of $(G,\pi_{st})$ (see \[6\,1\]), and $\Pi_{(G \times G)/(B \times B_-)}$ is the projection of $\Pi_{st}$ to $(G \times G)/(B \times B_-)$. For $n \geq 1$, both $F_n$ and $\tilde{F}_n$ are quotient manifolds of $G^n$, and the Poisson structures $\pi_n$ and $\tilde{\pi}_n$ are projections of the $n$-fold product Poisson structure $\pi_{st}^n$ on $G^n$. Similarly, $\bar{F}_n$ and $\bar{\bar{F}}_n$ are quotient manifolds of $(G \times G)^n$, with $\Pi_n$ and $\bar{\Pi}_n$ projections of the product Poisson structure $\Pi_{st}^n$ on $(G \times G)^n$. Precise definitions of the Poisson manifolds in \(1.1\) are given in \[1.2\].

This paper, a sequel to \[35\], is the second of a series of papers devoted to a detailed study of the four series of Poisson manifolds in \(1.1\). In \[38\], we have identified the Poisson structures in \(1.1\) as mixed product Poisson structures defined by quasitriangular $r$-matrices. In the present paper, we develop a general theory on $T$-leaves and $T$-leaf stabilizers for a class of Poisson structures defined by quasitriangular $r$-matrices and apply the theory to the Poisson manifolds in \(1.1\). The general theory also provides a unified approach to many other Poisson structures such as those mentioned earlier from \[15\,16\,20\,35\,36\,39\] (see \[6\,4\]).

Our descriptions of $T$-leaves for the Poisson manifolds in \(1.1\) naturally lead to what we call extended Bruhat cells, extended Richardson varieties, and extended double Bruhat cells associated to conjugacy classes (see \[16\,20\,35\,36\,39\]). In \[13\], the third in the series of papers on the Poisson manifolds in \(1.1\), we express explicitly in the so-called Bott-Samelson coordinates the Poisson structures $\pi_n$ on extended Bruhat cells in terms of the root strings and the structure constants of the Lie algebra $\mathfrak{g}$ of $G$. In particular, we show in \[13\] that each extended Bruhat cell of dimension $m$ gives rise to a polynomial
Poisson algebra $\mathbb{C}[z_1, \ldots, z_m]$ which is a symmetric nilpotent semi-quadratic Poisson-Ore extension of $\mathbb{C}$ in the sense of [21 Definition 4]. Moreover, when the bilinear form $(\cdot, \cdot)_g$ on $\mathfrak{g}$ that comes into the definition of $\pi_{st}$ is suitably chosen, the Poisson bracket $\{z_i, z_j\}$ between any two coordinate functions is in fact a polynomial with integer coefficients. In separate papers, we will further study extended Bruhat cells and extended double Bruhat cells in the context of symplectic groupoids.

Let $\mathbb{T}$ be an algebraic torus. The $\mathbb{T}$-leaves in a $\mathbb{T}$-Poisson manifold $(Y, \pi, \lambda)$ are the semi-classical analogs of $\mathbb{T}$-prime ideals of a quantum algebra $A$ with rational $\mathbb{T}$-actions by automorphisms [6], and the $\mathbb{T}$-leaf decomposition of $(Y, \pi, \lambda)$ is the semi-classical analog of the Goodearl-Letzler partition of the spectrum $\text{Spec}(A)$ of $A$ into tori indexed by $\mathbb{T}$-invariant prime ideals [19]. In particular, if a $\mathbb{T}$-invariant prime ideal $I$ in $A$ corresponds to a $\mathbb{T}$-leaf $L$ of $(Y, \pi, \lambda)$, the torus in the Goodearl-Letzler partition of $\text{Spec}(A)$ indexed by $I$ should correspond to the quotient torus $\mathbb{T}/\mathbb{T}_L$, where $\mathbb{T}_L$ is the sub-torus of $\mathbb{T}$ preserving the symplectic leaves in $L$. In the case of the Bruhat cell $BuB/B \subset G/B$ with the Poisson structure $\pi_1 = \pi_{G/B}$, where $u \in W$, one has the quantum algebra $U^\pi$ constructed by De Concini, Kac, and Procesi [9] as a quantization of the algebra of regular functions on $BuB/B$ (see [19]), and the explicit correspondence between the Goodearl-Letzler partition of $\text{Spec}(U^\pi)$ and the $\mathbb{T}$-leaves of $\pi_{G/B}$ in $BuB/B$, namely the open Richardson varieties $(BuB/B) \cap (BwB/B)$, $w \leq u$, have been studied in detail in [31, 49, 50, 49]. Similar studies for $(\tilde{F}_1, \tilde{\pi}_1) = (G, \pi_{st})$ can be found in [24, 26]. It would thus be very interesting to study the quantizations of the four series of Poisson manifolds in (1.1) (or of their Poisson submanifolds) and establish explicit correspondences between the Goodearl-Letzler partitions of the spectra of the quantizations and the $\mathbb{T}$-leaf decompositions and the leaf stabilizers described in the current paper.

As the Poisson manifolds treated in this paper are special classes of Poisson homogeneous spaces, the general theory established in the paper can also be regarded a further development of Drinfeld’s theory on Poisson homogeneous spaces [12, 39]. In particular, a generalization of Drinfeld’s Lagrangian subalgebras associated to points in a Poisson homogeneous space [12] plays an important role in our general theory (see Lemma 3.16 and and formulas (4.8) and (4.10) for detail).

We now give an outline and the main results of the paper.

1.2. Holomorphic Poisson structures related to flag varieties. If $G$ is a group and $n \geq 1$ an integer, let the product group $G^n$ act on itself from the right by

$$(g_1, g_2, \ldots, g_n) \cdot (h_1, h_2, \ldots, h_n) = (g_1 h_1, g_2 h_2, \ldots, g_n h_n), \quad g_j, h_j \in G.$$  

For subgroups $Q_1, \ldots, Q_n$ of $G$ and subsets $S_1, \ldots, S_n$ of $G$ such that $S_1$ is right $Q_1$-invariant and $S_j$ is left $Q_{j-1}$ and right $Q_j$-invariant for $j = 2, \ldots, n$, let $S_1 \times Q_1 \cdots \times Q_{n-1} S_n$ denote the quotient $g_1 \times \cdots \times g_n$ by the action of $Q_1 \times \cdots \times Q_n$ as a subgroup of $G^n$.

If $(G, \pi_G)$ is a Poisson Lie group and if $Q_1, \ldots, Q_n$ are closed Poisson Lie subgroups of $(G, \pi_G)$, then the product Poisson structure $\pi^n_G$ on $G^n$ projects to a well-defined Poisson structure on the quotient space $G \times Q_1 \cdots \times Q_{n-1} G/Q_n$ (see [38, §7]). Throughout the paper, if $G \times Q_1 \cdots \times Q_{n-1} G/Q_n$ is denoted by $Z_n$, we will denote by $\pi_{Z_n}$ the projection of $\pi^n_G$ to $Z_n$ and also refer to $\pi_{Z_n}$ as a quotient Poisson structure. Denote the image $(g_1, \ldots, g_n) \in G^n$ in $Z_n$ by $[g_1, \ldots, g_n]_{Z_n}$, and define

$$\mu_{Z_n} : Z_n \rightarrow G/Q_n, \quad [g_1, g_2, \ldots, g_n]_{Z_n} \mapsto g_1 g_2 \cdots g_n Q_n \in G/Q_n.$$  

Then the map $\mu_{Z_n} : (Z_n, \pi_{Z_n}) \rightarrow (G/Q_n, \pi_{G/Q_n})$ is Poisson, and the action

$$(g, g_1, g_2, \ldots, g_n) \cdot [g_1, g_2, \ldots, g_n]_{Z_n} \mapsto [gg_1, g_2, \ldots, g_n]_{Z_n}, \quad g, g_j \in G,$$
is a Poisson action of the Poisson Lie group \((G, \pi_G)\) on the Poisson manifold \((Z_n, \pi_{Z_n})\). This class of quotient Poisson structures was introduced in [35].

Let \((G, \pi_{st})\) be a standard complex semi-simple Poisson Lie group, determined by the choice of a pair \((B, B_-)\) of opposite Borel subgroups of \(G\) and a symmetric non-degenerate invariant bilinear form \(\langle \cdot, \cdot \rangle_\mathfrak{g}\) on \(\mathfrak{g}\). Let \((G \times G, \Pi_4)\) be its Drinfeld double ([35], (1.6)). Both \(B\) and \(B_-\) are Poisson Lie subgroups of \((G, \pi_{st})\), while \(B \times B_-\) is a Poisson Lie subgroup of \((G \times G, \Pi_{st})\). For an integer \(n \geq 1\), let

\[
F_n = G \times B \times \cdots \times B G / B, \quad \tilde{F}_n = (G \times G) \times \cdots \times (G \times G) / (B \times B_-),
\]

and let \(\pi_n\) and \(\tilde{\pi}_n\) be the projections of \(\pi_{st}\) from \(G^n\) to \(F_n\) and \(\tilde{F}_n\) respectively, and let \(\Pi_n\) and \(\tilde{\Pi}_n\) be the projections of \(\Pi_{st}\) from \((G \times G)^n\) to \(F_n\) and \(\tilde{F}_n\) respectively. The maximal torus \(T = B \cap B_-\) of \(G\) acts on \((F_n, \pi_n), (\tilde{F}_n, \pi_n), (\tilde{F}_n, \tilde{\pi}_n), (\tilde{F}_n, \tilde{\Pi}_n)\) by Poisson diffeomorphisms via

\[
t \cdot [g_1, g_2, \ldots, g_n]_{\pi_n} = [tg_1, g_2, \ldots, g_n]_{\pi_n},
\]

where \(t \in T\) and \(g_j \in G\) for \(1 \leq j \leq n\). Let \(\mathfrak{h}\) be the Lie algebra of \(T\). For \(Z_n \in \{F_n, \tilde{F}_n, \tilde{F}_n, \tilde{\Pi}_n\}\), let \(\lambda_{z_n} : \mathfrak{h} \to \mathfrak{h}^1(Z_n)\) be the Lie algebra action of \(\mathfrak{h}\) on \(Z_n\) generated by the action of \(T\) on \(Z_n\), and for \(z \in Z_n\), define the leaf stabilizer of \(\lambda_{z_n}\) at \(z\) to be

\[
t_z = \{x \in \mathfrak{h} : \lambda_{z_n}(x)(z) \in T_z \Sigma_z\},
\]

where \(\Sigma_z\) is the symplectic leaf of \(\pi_{Z_n}\) in \(Z_n\) through \(z\).

Let \(W = N_G(T) / T\) be the Weyl group of \((G, T)\), where \(N_G(T)\) is the normalizer of \(T\) in \(G\). Let \(\Delta\) be the Bruhat order on \(W\), and recall the monoidal product \(*\) on \(W\) defined by

\[
BuB'vB = B(u * v)B, \quad u, v \in W,
\]

where for a subset \(X\) of \(G\), \(\overline{X}\) denotes the Zariski closure of \(X\) in \(G\). For \(u = (u_1, \ldots, u_n) \in W^n\), let \(l(u) = l(u_1) + \cdots + l(u_n)\), where \(l : W \to \mathbb{N}\) is the length function on \(W\), and let

\[
BuB = (Bu_1B) \times B \cdots \times B (Bu_nB) \subset \tilde{F}_n.
\]

For another sequence \(v = (v_1, \ldots, v_n) \in W^n\), let

\[
(B \times B_-)(u, v)(B \times B_-) = (Bu_1B \times B_- v_1B_-) \times \cdots \times (Bu_nB \times B_- v_nB_-) \subset \tilde{F}_n.
\]

The images of \(BuB \subset \tilde{F}_n\) and \((B \times B_-)(u, v)(B \times B_-) \subset \tilde{F}_n\) under the projections

\[
F_n \to F_n, \quad [g_1, g_2, \ldots, g_n]_{\tilde{\pi}_n} \to [g_1, g_2, \ldots, g_n]_{\pi_n}, \quad g_j \in G,
\]

will be respectively denoted by \(BuB / F_n\) and \((B \times B_-)(u, v)(B \times B_-) / (B \times B_-) \subset F_n\).

We now state our results on the \(T\)-leaves and leaf stabilizers for each one of the four series in (1.1).

The following Theorem 1.1 on \((F_n, \pi_n), n \geq 1\), will be proved in ([6], 2) for \(n = 1\), Parts 1) and 2) of Theorem 1.1 have been proved in [20, Theorem 0.4] and [51, Theorem 3.1].

**Theorem 1.1.** For \(u = (u_1, \ldots, u_n) \in W^n\) and \(w \in W\), let

\[
R^u_w = (Bu_B / B) \cap \mu_{\pi_n}^{-1}(B_- w B / B) \subset F_n \quad \text{and} \quad \mathfrak{h}_w = \{x + u_1 \cdots u_n w^{-1}(x) : x \in \mathfrak{h}\} \subset \mathfrak{h}.
\]
1) \( R_w^u \neq \emptyset \) if and only if \( w \leq u_1 \cdots u_n \), and in this case, \( R_w^u \) is a connected smooth submanifold of \( F_n \) with \( \dim R_w^u = l(u) - l(w) \).

2) The decomposition of \( F_n \) into \( T \)-leaves of the Poisson structure \( \pi_n \) is

\[
F_n = \bigsqcup_{u \in W^n, w \in W} R_w^u,
\]

and all the symplectic leaves of \( \pi_n \) in \( R_w^u \) have dimensions equal to

\[
l(u) - l(w) - \dim \ker(1 + u_1u_2 \cdots u_nw^{-1});
\]

3) The leaf stabilizer of \( \lambda_{F,n} \) at \( z \in R_w^u \) is \( t_z = h_w^u \).

The following Theorem 1.3 on \( (\tilde{F}_n, \tilde{\pi}_n) \), \( n \geq 1 \), will be proved in \( \S 6.3 \). For \( n = 1 \), Theorem 1.2 has been proved in \( [36] \).

**Theorem 1.2.** For \( u = (u_1, \ldots, u_n), v = (v_1, \ldots, v_n) \in W^n \), and \( w \in W \), let

\[
G(w) = G_{\text{diag}}(w, e)(B \times B_-)/(B \times B_-) \subset (G \times G)/(B \times B_-),
\]

\[
P_w^{u,v} = ((B \times B_-)(u, v)(B \times B_-)/(B \times B_-)) \cap \mu_{\tilde{n}}^{-1}(G(w)) \subset \mathcal{F}_n,
\]

\[
h_w^{u,v} = \{ x + u_1 \cdots u_nw^{-1}(v_1 \cdots v_n)^{-1}(x) : x \in h \} \subset h,
\]

where \( G_{\text{diag}} = \{(g, g) : g \in G\} \).

1) \( P_w^{u,v} \neq \emptyset \) if and only if \( w \leq (v_1 \cdots v_n)^{-1} u_1 \cdots u_n \), and in this case, \( P_w^{u,v} \) is a connected smooth submanifold of \( \mathcal{F}_n \) with \( \dim P_w^{u,v} = l(u) + l(v) - l(w) \);

2) The decomposition of \( \mathcal{F}_n \) into \( T \)-leaves of the Poisson structure \( \pi_n \) is

\[
\mathcal{F}_n = \bigsqcup_{u,v \in W^n, w \in W} P_w^{u,v},
\]

and all the symplectic leaves of \( \pi_n \) in \( P_w^{u,v} \) have dimensions equal to

\[
l(u) + l(v) - l(w) - \dim \ker(1 + u_1 \cdots u_nw^{-1}(v_1 \cdots v_n)^{-1});
\]

3) The leaf stabilizer of \( \lambda_{\mathcal{F},n} \) at \( z \in P_w^{u,v} \) is \( t_z = h_w^{u,v} \).

The following Theorem 1.3 on \( (\tilde{F}_n, \tilde{\pi}_n) \), \( n \geq 1 \), will be proved in \( \S 6.3 \). For \( n = 1 \), Parts 1) and 2) of Theorem 1.3 have been proved in \( [30] \).

**Theorem 1.3.** 1) For any \( u = (u_1, \ldots, u_n) \in W \) and \( v \in W \), the intersection \( (BuB) \cap \mu_{\tilde{n}}^{-1}(B_- vB_-) \) in \( \tilde{F}_n \) is a non-empty smooth connected submanifold of \( \tilde{F}_n \) of dimension \( l(u) + l(v) + \dim T \);

2) The decomposition of \( \tilde{F}_n \) into \( T \)-leaves of the Poisson structure \( \tilde{\pi}_n \) is

\[
\tilde{F}_n = \bigsqcup_{u \in W^n, v \in W} (BuB) \cap \mu_{\tilde{n}}^{-1}(B_- vB_-),
\]

and all the symplectic leaves of \( \tilde{\pi}_n \) in \( (BuB) \cap \mu_{\tilde{n}}^{-1}(B_- vB_-) \) have dimensions equal to

\[
l(u) + l(v) + \dim \text{Im}(1 - u_1 \cdots u_nv^{-1});
\]

3) The leaf stabilizer of \( \lambda_{\mathcal{F},n} \) at \( z \in (BuB) \cap \mu_{\tilde{n}}^{-1}(B_- vB_-) \) is \( t_z = \{ x - u_1 \cdots u_nv^{-1}(x) : x \in t \} \).

Let \( C \) be the set of all conjugacy classes in \( G \). The following Theorem 1.4 on \( (\tilde{F}_n, \tilde{\Pi}_n) \), \( n \geq 1 \), will be proved in \( \S 6.3 \). For \( n = 1 \), Theorem 1.4 has been proved in \( [30] \).
Theorem 1.4. For $C \in \mathcal{C}$, let $\Omega_C = G_{\text{diag}} \cdot (C \times \{e\}) \cdot G_{\text{diag}} = \{(g, h) \in G \times G : gh^{-1} \in C\}$, and for $u = (u_1, \ldots, u_n), \; v = (v_1, \ldots, v_n) \in W^n$, let

$$R_{C,u,v}^n = ((B \times \mathcal{B}_u)(u,v)(B \times \mathcal{B}_v)) \cap \mu_{\mathcal{Z}}^{-1}(\Omega_C) \subset \overline{\mathcal{F}}_n,$$

$$h_{u,v} = \{x - u_1 \cdots u_n(v_1 \cdots v_n)^{-1} : x \in h\} \subset h.$$

1) For any $u, v \in W^n$ and $C \in \mathcal{C}$, $R_{C,u,v}^n$ is a connected smooth submanifold of $\overline{\mathcal{F}}_n$ of dimension $l(u) + l(v) + \dim C + \dim T$;

2) The decomposition of $\overline{\mathcal{F}}_n$ into $T$-leaves of the Poisson structure $\overline{\Pi}_n$ is

$$\overline{\mathcal{F}}_n = \bigsqcup_{u,v \in W^n, C \in \mathcal{C}} R_{C,u,v}^n,$$

and all symplectic leaves of $\overline{\Pi}_n$ in $R_{C,u,v}^n$ have dimensions equal to

$$l(u) + l(v) + \dim C + \dim \text{Im}(1 - u_1 \cdots u_n(v_1 \cdots v_n)^{-1});$$

3) The leaf stabilizer of $\lambda_{C,u,v}$ at $z \in R_{C,u,v}^n$ is $t_z = h_{u,v}$.

1.3. Extended Bruhat cells, Bott-Samelson varieties, and extended Richardson varieties.

Let $n \geq 1$ and consider the disjoint union

$$F_n = \bigsqcup_{u \in W^n} BuB/B.$$

For $u = (u_1, \ldots, u_n) \in W^n$, we will call

$$BuB/B = (Bu_1B \times_B \cdots \times_B Bu_nB)/B \subset F_n$$

an extended Bruhat cell. By Theorem 1.1, extended Bruhat cells in $F_n$ are Poisson submanifolds with respect to the Poisson structure $\pi_n$. If $u = (s_1, \ldots, s_n)$, where each $s_j \in W$ is a simple reflection, we say that the extended Bruhat cell $BuB/B$ in $F_n$ is of Bott-Samelson type.

Every extended Bruhat cell $BuB/B \subset F_n$ with the Poisson structure $\pi_n$ is Poisson isomorphic to an extended Bruhat cell of Bott-Samelson type in $F_{l(u)}$ with the Poisson structure $\pi_{l(u)}$. Indeed, consider first the case of $n = 1$: if $u \in W$ and $u = s_1 \cdots s_{l(u)}$ is a reduced decomposition of $u$, the Poisson morphism $\mu_{F_{l(u)}} : (F_{l(u)}, \pi_{l(u)}) \to (G/B, \pi_1)$ then restricts to a Poisson isomorphism

$$(Bu(u)B/B, \pi_{l(u)}) \longrightarrow (BuB/B, \pi_1),$$

where $s(u) = (s_1, \ldots, s_{l(u)})$. For any arbitrary $u = (u_1, \ldots, u_n) \in W^n$, the choice of a reduced word $s(u_j) \in W^{l(u_j)}$ for each $u_j$ gives rise to a Poisson isomorphism from $(BuB/B, \pi_n)$ to the Bott-Samelson type extended Bruhat cell $B(s(u_1), \ldots, s(u_n))B/B$ in $F_{l(u)}$.

On the other hand, recall that for any sequence $(s_1, \ldots, s_n)$ of simple reflections in $W^n$, the Bott-Samelson variety $Z(s_1, \ldots, s_n)$ can be defined as the Zariski closure of $B(s_1, \ldots, s_n)B/B$ in $F_n$ and thus carries the Poisson structure $\pi_n$. Bott-Samelson varieties, with the Poisson structures thus defined, are therefore the building blocks for the Poisson manifolds $(F_n, \pi_n)$, $n \geq 1$, and are important even for the study of $(F_1, \pi_1)$. In [13], a sequel to the current paper, the Poisson structure $\pi_n$ on any Bott-Samelson variety $Z(s_1, \ldots, s_n)$ is explicitly computed in each of the $2^n$ natural affine coordinate charts. In particular, it is shown in [13] that in each of these affine coordinate charts, $\pi_n$ gives rise to a polynomial Poisson algebra $\mathbb{C}[z_1, \ldots, z_n]$ which is a Poisson-Ore extension of $\mathbb{C}$ compatible with the $T$-action. When the bilinear form $\langle \cdot, \cdot \rangle_{\mathfrak{g}}$ is such that $\frac{1}{2}\langle \alpha, \alpha \rangle_{\mathfrak{g}} \in \mathbb{Z}$ for every root of $\mathfrak{g}$, it is shown in [13] that the Poisson structure on $\mathbb{C}[z_1, \ldots, z_n]$ corresponding to each of the $2^n$ affine coordinate charts has the property that the Poisson brackets $\{z_i, z_k\}$ between the coordinate functions are polynomials with coefficients in $\mathbb{Z}$, thus giving rise to a Poisson-Ore extension of any field $k$ of arbitrary characteristic.
For \( u = (u_1, \ldots, u_n) \in W^n \) and \( w \in W \) such that \( w \leq u_1 \ast \cdots \ast u_n \), it is natural to refer to

\[
F^u_w = (BuB/B) \cap \mu^{-1}_n(B_wB/B) \subset F_n
\]
as an (open) extended Richardson variety and its closure \( \overline{F^u_w} \) in \( F_n \) an extended Richardson variety in \( F_n \). By Theorem \[1.1\] extended Richardson varieties in \( F_n \) are precisely closures of \( T \)-leaves of the Poisson structure \( \pi_n \) in \( F_n \). It would be very interesting to extend the work of Lenagan and Yakimov in \[33\] on cluster structures on Richardson varieties in \( F_1 = G/B \) to extended Richardson varieties.

Analogous to taking Zariski closures in \( F_n \) of extended Bruhat cells of Bott-Samelson type, one can consider the Zariski closures in \( \mathbb{F}_n \) of \((B \times B_\ast)(u, v)(B \times B_\ast)/(B \times B_\ast)\), where \( u, v \in W^n \) are sequences of simple reflections. Such closures are examples of double Bott-Samelson varieties introduced in \[43\], and carry the Poisson structure \( \Pi_n \). Some combinatorial aspects of double Bott-Samelson varieties and calculations of the Poisson structure \( \Pi_n \) in special coordinate charts have been given in \[42, 43\].

### 1.4. Extended double Bruhat cells associated to conjugacy classes

Analogous to the Poisson manifold \((\widetilde{F}_n = G \times_{B_\ast} \cdots \times_{B_\ast} G, \tilde{\pi}_n)\), \( n \geq 1 \), one also has the quotient manifold \( \overline{\widetilde{F}_n} = G \times_{B_\ast} \cdots \times_{B_\ast} G \)
of \( G^n \) with the well-defined Poisson structure \( \tilde{\pi}_n \), the projection of the Poisson structure \( \pi_n^n \) from \( G^n \) to \( \overline{\widetilde{F}_n} \). Consider, on the other hand, the diffeomorphism \( S_{\tilde{\pi}_n} : \tilde{\Pi}_n \to \overline{\widetilde{F}_n} \times \overline{\widetilde{F}_n} \) given by

\[
[g_1, k_1, \ldots, g_n, k_n]_{\tilde{\pi}_n} \mapsto ([g_1, \ldots, g_n]_{\tilde{\pi}_n}, [k_1, \ldots, k_n]_{\tilde{\pi}_n}),
\]
and set \( \tilde{\pi}_n = S_{\tilde{\pi}_n}(\tilde{\Pi}_n) \). By a result in \[33\] §8, \( \tilde{\pi}_n \) is a two-fold mixed product Poisson structure on the product manifold \( \overline{\widetilde{F}_n} \times \overline{\widetilde{F}_n} \), i.e., it is the sum of the product Poisson structure \( \tilde{\pi}_n \times \tilde{\pi}_n \) and a certain mixed term defined by the action of \( B \) on \( \overline{\widetilde{F}_n} \) and \( B_\ast \) on \( \overline{\widetilde{F}_n} \) by left translation. For \( (u, v) = (u_1, \ldots, u_n, v_1, \ldots, v_n) \in W^{2n} \) and any conjugacy class \( C \) in \( G \), define

\[
G^u_v^C = \{([g_1, \ldots, g_n]_{\tilde{\pi}_n}, [k_1, \ldots, k_n]_{\tilde{\pi}_n}) \in (BuB) \times (B_vB) : g_1g_2 \cdots g_n(k_1k_2 \cdots k_n)^{-1} \in C\}
\]
closed in \( \overline{\widetilde{F}_n} \times \overline{\widetilde{F}_n} \),

where \( (B \ast vB\ast) = (B \ast v_1B) \ast \cdots \ast (B \ast v_nB) \subset \overline{\widetilde{F}_n} \). By Theorem \[1.3\] each \( G^u_v^C \) is a \( T \)-leaf of \( \tilde{\pi}_n \) in \( \overline{\widetilde{F}_n} \) for the diagonal action of \( T \). We call \( G^u_v^C \) an extended double Bruhat cell associated to the conjugacy class \( C \). The case of \( n = 1 \) has been considered in \[36\] §7.3, where \( G^u_v^C \subset C \), for \( u, v \in W \) and a conjugacy class \( C \) in \( G \), is called a double Bruhat cell associated to the conjugacy class \( C \). Note that for \( u, v \in W^n \) and \( C = \{e\} \), we have

\[
G^u_v := G^u_v^{\{e\}} = \{([g_1, \ldots, g_n]_{\tilde{\pi}_n}, [k_1, \ldots, k_n]_{\tilde{\pi}_n}) \in (BuB) \times (B_vB) : g_1g_2 \cdots g_n = k_1k_2 \cdots k_n\},
\]
a direct generalization of double Bruhat cells in \( G \). In a separate paper, we will study extended Bruhat cells and extended double Bruhat cells via Poisson groupoids and symplectic groupoids.

### 1.5. General theory

Let \( r \in \mathfrak{g} \otimes \mathfrak{g} \) be a quasitriangular \( r \)-matrix on a Lie algebra \( \mathfrak{g} \) (definition recalled in \[43\]), and let \( \lambda : \mathfrak{g} \to \mathcal{Y}(Y) \) be a Lie algebra action of \( \mathfrak{g} \) on a manifold \( Y \). A simple observation made in \[33\] (see also \[34\] §2.1) is that if the 2-tensor field \( \lambda(r) \) on \( Y \) is skew-symmetric, then it is Poisson, and in this case we say that the Poisson structure \( -\lambda(r) \) (or \( \lambda(r) \)) on \( Y \) is defined by the Lie algebra action \( \lambda \) and the quasitriangular \( r \)-matrix \( r \). We also refer to Poisson structures obtained this way simply as Poisson structures defined by quasitriangular \( r \)-matrices.

Consider again the quotient manifold \( Z_n = G \times_{Q_1} \cdots \times_{Q_n} G/Q_n \) with the quotient Poisson structure \( \pi_{z_n} \) for an arbitrary Poisson Lie group \((G, \pi_G)\) and closed Poisson Lie subgroups \( Q_1, \ldots, Q_n \) described...
in $\mathbf{[1.2]}$ Note that the manifold $Z_n$ is diffeomorphic to the product manifold $G/Q_1 \times \cdots \times G/Q_n$ via the diffeomorphism $J_{z_n}: Z_n \to G/Q_1 \times \cdots \times G/Q_n$ given by

\[
J_{z_n}([g_1, g_2, \ldots, g_n]) = (g_1Q_1, g_1g_2Q_2, \ldots, g_1g_2 \cdots g_nQ_n), \quad g_1, \ldots, g_n \in G.
\]

A key fact established in $\mathbf{[35]} \mathbf{[57]}$ is that the Poisson structure $J_{z_n}(\pi_{z_n})$ on $G/Q_1 \times \cdots \times G/Q_n$ is defined by a quasitriangular $r$-matrix (see $\mathbf{[5.3]}$).

After a review on the notion of $T$-leaves in $\mathbf{[2]}$ and on Poisson structures defined by quasitriangular $r$-matrices in $\mathbf{[3]}$ we devote $\mathbf{[1]} - \mathbf{[3]}$ of the paper to a general theory on $T$-leaves and $T$-leaf stabilizers for a class of $T$-invariant Poisson structures defined by quasitriangular $r$-matrices. Theorem $\mathbf{1.4}$ is proved in $\mathbf{[6]}$ as immediate examples of the general theory.

To give an outline of the general theory, let $G$ be a connected Lie group with Lie algebra $\mathfrak{g}$ and $r \in \mathfrak{g} \otimes \mathfrak{g}$ a factorizable quasitriangular $r$-matrix on $\mathfrak{g}$. The symmetric part of $r$ determines a symmetric non-degenerate invariant bilinear form $\langle , \rangle_{\mathfrak{g}}$ on $\mathfrak{g}$ (see $\mathbf{[3.2]}$). If $Y$ is a manifold with a Lie group action $\lambda$ by $G$, we say that the quadruple $(G, r, Y, \lambda)$ is strongly admissible if the stabilizer subgroup $Q_y$ of $G$ at every $y \in Y$ is connected and its Lie algebra $\mathfrak{q}_y$ satisfies

\[
[q_y, q_y] \subset q_y^\perp \subset q_y,
\]

where $v^\perp = \{x \in \mathfrak{g} : \langle x, v \rangle_{\mathfrak{g}} = 0\}$ for a subspace $v$ of $\mathfrak{g}$. If $(G, r, Y, \lambda)$ is strongly admissible, the 2-tensor field $\lambda(r)$ on $Y$ is necessarily skew-symmetric (see $\mathbf{[3.3]}$), so it is a Poisson structure on $Y$. Associated to the factorizable quasitriangular $r$-matrix $r$ one also has two distinguished Lie subalgebras $\mathfrak{f}_+ = \text{Im}(r^+_r)$ and $\mathfrak{f}_- = \text{Im}(r^-_r)$ of $\mathfrak{g}$, where if $r = \sum_i x_i \otimes x'_i \in \mathfrak{g} \otimes \mathfrak{g}$, then $r^+_r, r^-_r : \mathfrak{g} \to \mathfrak{g}$ are respectively given by

\[
r^+_r(x) = \sum_i \langle x, x'_i \rangle_{\mathfrak{g}} x'_i \quad \text{and} \quad r^-_r(x) = -\sum_i \langle x, x'_i \rangle_{\mathfrak{g}} x_i, \quad x \in \mathfrak{g}.
\]

A pair $(M_+, M_-)$ of connected Lie subgroups of $G$ is said to be $r$-admissible if their respective Lie algebras $\mathfrak{m}_+$ and $\mathfrak{m}_-$ satisfy

\[
\mathfrak{f}_+ \subset \mathfrak{m}_+, \quad \mathfrak{f}_- \subset \mathfrak{m}_-, \quad [\mathfrak{m}_+, \mathfrak{m}_+] \subset \mathfrak{m}_+^\perp, \quad [\mathfrak{m}_-, \mathfrak{m}_-] \subset \mathfrak{m}_-^\perp.
\]

In $\mathbf{[1]}$ we consider a six-tuple $(G, r, Y, \lambda, M_+, M_-)$, where $(G, r, Y, \lambda)$ is a strongly admissible quadruple, and $(M_+, M_-)$ is a pair of $r$-admissible Lie subgroups of $G$. Let $\mathcal{T}$ be the connected component of $M_+ \cap M_-$ containing the identity element. Then $\mathcal{T}$ is necessarily abelian and acts on $(Y, \lambda(r))$ through $\lambda$ by Poisson isomorphisms. On the other hand, one has the disjoint union

\[
Y = \bigcup_{\mathcal{O}_+, \mathcal{O}_-} \mathcal{O}_+ \cap \mathcal{O}_-,
\]

where $\mathcal{O}_+$ and $\mathcal{O}_-$ are respectively $M_+$-orbits and $M_-$-orbits in $Y$. The conditions $\mathfrak{f}_+ \subset \mathfrak{m}_+$ and $\mathfrak{f}_- \subset \mathfrak{m}_-$ imply that each non-empty intersection $\mathcal{O}_+ \cap \mathcal{O}_-$ in $\mathbf{[1.12]}$ is a $T$-invariant Poisson submanifold of the $T$-Poisson manifold $(Y, \lambda(r), \lambda)$. We say that the six-tuple $(G, r, Y, \lambda, M_+, M_-)$ is admissible if the $T$-leaves of the Poisson structure $\lambda(r)$ in $Y$ are precisely the connected components of all the non-empty intersections $\mathcal{O}_+ \cap \mathcal{O}_-$ in $\mathbf{[1.12]}$.

For each pair $(\mathcal{O}_+, \mathcal{O}_-)$ of $M_+$- and $M_-$-orbits contained in the same $G$-orbit in $Y$, we introduce an integer $\delta_{\mathcal{O}_+, \mathcal{O}_-}$, given in $\mathbf{[1.8]}$, and a subspace $\mathfrak{t}_{\mathcal{O}_+, \mathcal{O}_-}$ of the Lie algebra $\mathfrak{t} = \mathfrak{m}_+ \cap \mathfrak{m}_-$ of $\mathcal{T}$, given in $\mathbf{[1.10]}$. Both $\delta_{\mathcal{O}_+, \mathcal{O}_-}$ and $\mathfrak{t}_{\mathcal{O}_+, \mathcal{O}_-}$ are defined using arbitrary points $y_+ \in \mathcal{O}_+$ and $y_- \in \mathcal{O}_-$ but are independent of the choices. The main results of $\mathbf{[1]}$ are summarized in the following Theorem $\mathbf{1.5}$.

**Theorem 1.5.** Suppose that $\delta_{\mathcal{O}_+, \mathcal{O}_-} = 0$ for every pair $(\mathcal{O}_+, \mathcal{O}_-)$ of $M_+$- and $M_-$-orbits. Then the six-tuple $(G, r, Y, \lambda, M_+, M_-)$ is admissible; the leaf stabilizer of each $T$-leaf in $\mathcal{O}_+ \cap \mathcal{O}_-$ is $\mathfrak{t}_{\mathcal{O}_+, \mathcal{O}_-}$, and the co-rank of the Poisson structure $\lambda(r)$ in $\mathcal{O}_+ \cap \mathcal{O}_-$ is equal to the co-dimension of $\mathfrak{t}_{\mathcal{O}_+, \mathcal{O}_-}$ in $\mathfrak{t}$.
The second part of the general theory, presented in [5] is an application of the “test” described in Theorem 1.5. Assume that \( Q \) is a closed and connected Lie subgroup of \( G \) whose Lie algebra satisfies \( \mathfrak{f}_+ \subset \mathfrak{q} \) and \([\mathfrak{q}, \mathfrak{q}] \subset \mathfrak{q}^\perp\). Equip \( G \) with the Poisson structure \( \pi_G = r^L - r^R \), where \( r^L \) (resp. \( r^R \)) is the left (resp. right) invariant tensor field on \( G \) with value \( r \) at the identity element of \( G \). Then \([38, \S 7]\) \( Q \) is a Poisson Lie subgroup of the Poisson Lie group \((G, \pi_G)\), and one thus has the quotient Poisson structure \( \pi_{Z_n} \) on the quotient manifold \( Z_n = G \times_Q \cdots \times_Q G/Q \), as explained in \([12,2]\). Let \((M_+, M_-)\) be again an \( r \)-admissible pair of Lie subgroups of \( G \) with respective Lie algebras \( \mathfrak{m}_+ \) and \( \mathfrak{m}_- \). The connected component \( T \) of \( M_+ \cap M_- \) containing the identity element then acts on \((Z_n, \pi_{Z_n})\) by Poisson isomorphisms via \([13]\). We study the \( T \)-leaves of \( \pi_{Z_n} \) in \( Z_n \) via the \( T \)-leaves of the Poisson structure \( J_{Z_n}(\pi_{Z_n}) \) on the \( n \)-fold product manifold \((G/Q)^n\), where \( J_{Z_n} : Z_n \to (G/Q)^n \) is the diffeomorphism given in \((1.11)\), and \( T \) acts on \((G/Q)^n\) diagonally.

More precisely, let \( \lambda \) be the Lie algebra action of the direct product Lie algebra \( \mathfrak{g}^n \) on \((G/Q)^n\) induced from the left action of \( G^n \) on \((G/Q)^n\) by left translation. By a result in \([38, \S 8]\) (see \([5.3]\) for more detail), the Poisson structure \( J_{Z_n}(\pi_{Z_n}) \) on \((G/Q)^n\) is defined by the Lie algebra action \( \lambda \) of \( \mathfrak{g}^n \) and a certain quasitriangular \( r \)-matrix \( r^{(n)} \) on \( \mathfrak{g}^n \). One can thus study the \( T \)-leaves of \( J_{Z_n}(\pi_{Z_n}) \) in \((G/Q)^n\) via the six-tuple

\[
(\mathfrak{g}^n, \ r^{(n)}, \ (G/Q)^n, \ \lambda, \ M_+^{(n)}, \ M_-^{(n)})
\]

where \((M_+^{(n)}, M_-^{(n)})\) is an \( r^{(n)} \)-admissible pair of Lie subgroups of \( G^n \) determined by \((M_+, M_-)\). Applying Theorem 1.5 we obtain in Proposition 5.3 sufficient conditions for the six-tuple in \((1.13)\) to be admissible. Translating to \((Z_n, \pi_{Z_n})\) using the Poisson diffeomorphism

\[
J_{Z_n} : (Z_n, \pi_{Z_n}) \to ((G/Q)^n, J_{Z_n}(\pi_{Z_n})),
\]

we obtain in Theorem 6.8 a description of \( T \)-leaves and leaf stabilizers for \((Z_n, \pi_{Z_n})\) under some sufficient conditions on the triple \((M_+, M_-, Q)\). Theorem 1.1 and Theorem 1.2 are then proved in \([6,2]\) as special cases of Theorem 6.8. Theorem 1.3 and Theorem 1.4 are similarly proved in \([6,3]\) as special cases of Theorem 5.10 as an analog of Theorem 5.8.

### 1.6. Notation

Throughout the paper, the pairing between a finite dimensional vector space \( V \) (over \( \mathbb{R} \) or \( \mathbb{C} \)) and its dual space \( V^* \) will always be denoted by \( \langle , \rangle \). The annihilator of a vector subspace \( U \) of \( V \) is, by definition, the subspace of \( V^* \) given by \( U^0 = \{ \xi \in V^* : \langle \xi, U \rangle = 0 \} \). For each integer \( k \geq 1 \), \( \wedge^k V \) is identified with the subspace of skew-symmetric elements in \( V^\otimes k \), and for \( v_1, \ldots, v_k \in V \),

\[
(1.14) \quad v_1 \wedge v_2 \cdots \wedge v_k = \sum_{\lambda \in S_k} \text{sign}(\lambda) v_{\lambda(1)} \otimes v_{\lambda(2)} \cdots \otimes v_{\lambda(k)} \in \wedge^k V \subset V^\otimes k.
\]

For an element \( r = \sum_i u_i \otimes v_i \in V \otimes V \), define \( r^{(2)} = \sum_i v_i \otimes u_i \in V \otimes V \) and

\[
(1.15) \quad r^\# : V^* \to V, \quad r^\#(\xi) = \sum_i \langle \xi, u_i \rangle v_i, \quad \xi \in V^*.
\]

Then \((r^\#)^* = (r^{(2)})^# : V^* \to V\).

If \( \mathfrak{g} \) is a Lie algebra over \( \mathbb{R} \) (resp. \( \mathbb{C} \)), by a left Lie algebra action of \( \mathfrak{g} \) on a manifold (resp. complex manifold) \( Y \) we mean a Lie algebra anti-homomorphism \( \lambda : \mathfrak{g} \to \mathcal{V}^1(Y) \), where \( \mathcal{V}^1(Y) \) is the space of smooth (resp. holomorphic) vector fields on \( Y \). For \( k \geq 1 \) and \( X = \sum x_{i_1} \otimes \cdots \otimes x_{i_k} \in \mathfrak{g}^\otimes k \), let \( \lambda(X) \) be the \( k \)-tensor field on \( Y \) given by

\[
(1.16) \quad \lambda(X) = \sum \lambda(x_{i_1}) \otimes \cdots \otimes \lambda(x_{i_k}).
\]
When $G$ is a connected Lie group with Lie algebra $g$ and $\lambda : G \times Y \to Y$, $(g, y) \mapsto gy$, is a left Lie group action, we use $\lambda$ to also denote the induced left action of $g$ on $Y$, i.e.,

\begin{equation}
\lambda : g \to V^1(Y), \quad \lambda(x)(y) = \frac{d}{dt}|_{t=0} \exp(tx)y, \quad x \in g, \ y \in Y.
\end{equation}

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2. Poisson actions and $T$-leaves

2.1. Regular and full Poisson actions by Lie bialgebras. Recall that a Lie bialgebra over the field $k = \mathbb{C}$ or $\mathbb{R}$ is a pair $(g, \delta)$, where $g$ is a Lie algebra over $k$ and $\delta : g \to \wedge^2 g$, called the co-bracket, is a linear map satisfying the co-Jacobi identity and the cocycle condition

\begin{equation}
\delta[x, y] = \text{ad}_x(\delta(y)) - \text{ad}_y(\delta(x)), \quad x, y \in g.
\end{equation}

Given a Lie bialgebra $(g, \delta)$, the dual map of $\delta$ is then a Lie bracket on the dual space $g^*$ of $g$.

A left Poisson action of the Lie bialgebra $(g, \delta)$ on a Poisson manifold $(Y, \pi)$ is a left Lie algebra action $\lambda : g \to V^1(Y)$ such that

\[ L_{\lambda(x)} \pi = \lambda(\delta(x)), \quad x \in g, \]

where for $x \in g$, $L_{\lambda(x)} \pi$ is the Lie derivative of $\pi$ in the direction of the vector field $\lambda(x)$. Here when $g$ is a Lie bialgebra over $\mathbb{C}$, we understand $(Y, \pi)$ to be a complex Poisson manifold, i.e., $Y$ is a complex manifold and $\pi$ a holomorphic Poisson structure on $Y$.

Lie bialgebras and Poisson actions of Lie bialgebras are the infinitesimal counterparts of Poisson Lie groups and Poisson actions of Poisson Lie groups, and we refer to \cite{8, 10, 11, 14, 46, 47} for the basics of the theory. In this paper we follow the notation and sign conventions in the review section §2 of \cite{38}.

Let $\lambda : g \to V^1(Y)$ be a left Poisson action of a Lie bialgebra $(g, \delta)$ on a Poisson manifold $(Y, \pi)$. For $y \in Y$, set $\pi_y = \pi(y) \in \wedge^2 T_y Y$, and

\[ \lambda_y : g \to T_y Y : \quad \lambda_y(x) = \lambda(x)(y), \quad y \in Y, \ x \in g. \]

Recall that $\text{Im}(\pi_y^\#)$, where $\pi_y^\# : T^*_y Y \to T_y Y$ is defined using \eqref{eq:1.15}, is the tangent space to the symplectic leaf of $\pi$ through $y$, and the rank of $\pi$ at $y$ is, by definition, $\dim(\text{Im}(\pi_y^\#))$. Set

\begin{equation}
\Phi_y = \lambda_y(g) + \text{Im}(\pi_y^\#) \subset T_y Y.
\end{equation}

Definition 2.1. The Poisson action $\lambda$ of $(g, \delta)$ on $(Y, \pi)$ is said to be regular (resp. full) if $\dim \Phi_y$ is independent of $y \in Y$ (resp. $\Phi_y = T_y Y$ for all $y \in Y$).

Given a left Poisson action $\lambda$ of a Lie algebra $(g, \delta)$ on a Poisson manifold $(Y, \pi)$, it is shown in \cite{37} that the vector bundle $(Y \times g) \oplus T^* Y$, the direct product of the trivial vector bundle over $Y$ with fiber $g$ and the cotangent bundle $T^* Y$ of $Y$, has the structure of a Lie algebroid over $Y$ with $-\lambda + \pi^\#$ as the anchor map. We denote this Lie algebroid by

\begin{equation}
A_\lambda = (Y \times g) \bowtie T^* Y
\end{equation}

and refer to \cite{37} for details. Thus the Poisson action $\lambda$ is regular (resp. full) if and only if the Lie algebroid $A_\lambda$ is regular (resp. transitive), i.e., the anchor map $-\lambda + \pi^\#$ of $A_\lambda$ has constant rank (resp. everywhere surjective).

For a Poisson action $\lambda$ of a Lie bialgebra $(g, \delta)$ on $(Y, \pi)$, and for $y \in Y$, we also define

\begin{equation}
g_y = \{ x \in g : \lambda_y(x) \in \text{Im}(\pi_y^\#) \} \subset g.
\end{equation}
The linear map $\lambda_y : g \to T_y Y$ then induces a well-defined vector space isomorphism
\[(2.5) \quad g/\mathfrak{g}_y \to \Phi_y/\text{Im}(\pi^\#_y) : \quad x + \mathfrak{g}_y \mapsto \lambda_y(x) + \text{Im}(\pi^\#_y), \quad x \in g.\]

2.2. $T$-leaves and leaf stabilizers. In (2.2) let $(Y, \pi)$ be a real (resp. complex) Poisson manifold and $T$ a connected real (resp. complex) abelian Lie group acting on $Y$ by Poisson isomorphisms. Denote the action by $\lambda : T \times Y \to Y$ and the Lie algebra of $T$ by $\mathfrak{t}$. Then $\lambda$ is a Poisson action of the Lie bialgebra $(\mathfrak{t}, 0)$ on $(Y, \pi)$, where $0$ denotes the zero map $t \to \wedge^2 \mathfrak{t}$. We will refer to the triple $(Y, \pi, \lambda)$ as a $T$-Poisson manifold. Let $A_\lambda = (Y \times t) \cong T^* Y$ be the Lie algebroid of over $Y$ given in (2.3).

Recall (see, for example, [17, 40]) that the orbits of the Lie algebroid $A_\lambda$ are the integral submanifolds of the distribution on $Y$ (not necessarily of constant rank) defined as the image of the anchor map of $A_\lambda$, i.e., of the distribution $\bigcup_{y \in Y} (\lambda_y(t) + \text{Im}(\pi^\#_y))$ on $Y$.

Definition 2.2. Orbits of the Lie algebroid $A_\lambda$ are called the $T$-orbits of symplectic leaves, or simply the $T$-leaves, of $\pi$ in $Y$.

The following Lemma 2.3 justifies the terminology in Definition 2.2.

Lemma 2.3. Let $L$ be any $T$-leaf of $\pi$ in $Y$ and let $\Sigma$ be any symplectic leaf of $\pi$ such that $L \cap \Sigma \neq \emptyset$. Then $\Sigma \subset L$ and the action map $\lambda : T \times \Sigma \to L$ is a surjective submersion.

Proof. As $\bigcup_{y \in Y} \text{Im}(\pi^\#_y)$ is a sub-distribution of $\bigcup_{y \in Y} (\lambda_y(t) + \text{Im}(\pi^\#_y))$, one has $\Sigma \subset L$. Let $t \in T$ and $y \in \Sigma$. The differential of $\lambda$ at $(t, y)$ is the linear map
\[\lambda_\ast : T_{(t, y)}(T \times \Sigma) \to T_t L, \quad (r_t(x), v_y) \mapsto \lambda_t y(x) + (\lambda_t)_\ast v_y, \quad x \in t, v_y \in T_y \Sigma,\]
where for $x \in t$, $r_t(x) \in T_t \mathfrak{t}$ is the right translate of $x$ by $t$, and $(\lambda_t) : T_y Y \to T_y Y$ is the differential at $y$ of the map $\lambda_t : Y \to Y, y_1 \mapsto ty_1, y_1 \in Y$. Since $\lambda_t$ preserves $\pi$, $(\lambda_t)_\ast (T_y \Sigma) = \text{Im}(\pi^\#_y)$. Thus the action map $\lambda$ is a submersion, and $T\Sigma := \bigcup_{t \in T} t\Sigma$ is open in $L$. If $\Sigma'$ is an other symplectic leaf of $\pi$ contained in $L$, then either $T\Sigma = T\Sigma'$ or $T\Sigma \cap T\Sigma' = \emptyset$, and since $L$ is connected, one must have $T\Sigma = T\Sigma'$. Thus $\lambda$ is surjective.

Q.E.D.

Definition 2.4. For $y \in Y$, the subspace
\[t_y = \{x \in t : \lambda_y(x) \in \text{Im}(\pi^\#_y)\}\]
of $t$ is called the $T$-leaf stabilizer (or simply the leaf stabilizer) of $\lambda$ at $y$.

The following Lemma 2.5 is due to A. Knutson through private communication.

Lemma 2.5. If $y$ and $y'$ are in the same $T$-leaf of $(Y, \pi)$, then $t_y = t_{y'}$.

Proof. Let $L$ be any $T$-leaf of $\pi$ in $Y$. Let $2r$ be the rank of $\pi$ in $L$, and let $\pi^r$ be the $r$th exterior product of $\pi$ with itself. For $x \in t$, let $V_x = \lambda(x) \wedge \pi^r$, a $(2r + 1)$-vector field on $L$. As $T$ is abelian and acts on $L$ by Poisson isomorphisms, $V_x$ is $T$-invariant. For any local (smooth or holomorphic) function $f$ on $L$, if $X_f = \pi^\#(df)$ is the Hamiltonian vector field of $f$, then
\[L_{X_f} V_x = (L_{X_f} \lambda(x)) \wedge \pi^r = X_{\lambda(x)(f)} \wedge \pi^r = 0.\]
Thus $V_x$ is also invariant under local Hamiltonian flows. It follows that the vanishing locus $Z(V_x)$ of $V_x$ is open in $L$. As $Z(V_x)$ is also closed and as $L$ is connected, $V_x$ vanishes everywhere on $L$ if it vanishes.
at one point. But for any \( y \in L \), \( V_x(y) = 0 \) if and only if \( x \in t_y \). It follows that the subspace \( t_y \) of \( t \) is independent of \( y \in L \).

Q.E.D.

**Definition 2.6.** For a \( \mathbb{T} \)-leaf \( L \) in \( Y \), the subspace \( t_L := t_y \) of \( t \), where \( y \in L \) is arbitrary, will be called the *leaf stabilizer* of \( \lambda \) in \( L \).

**Remark 2.7.** (1) Note that by the vector space isomorphism in (2.5), the co-rank of \( \pi \) in \( L \) is the same as the co-dimension of \( t_L \) in \( t \);

(2) Let \( L \subset Y \) be a \( \mathbb{T} \)-leaf in \( Y \) with \( \dim L = l \), and let \( 2r \) be the rank of \( \pi \) in \( L \). Let \( t_L' \subset t \) be any vector space complement of \( t_L \) in \( t \). Then \( \dim t_L' = l - 2r \). For any \( \xi \in \wedge^{l-2r} t_L' \), \( \xi \neq 0 \), the \( \ell \)-vector field (2.6)

\[
\eta_L = \lambda(\xi) \wedge \pi \wedge \ldots \wedge \pi_{r}
\]
on \( Y \) then restricts to a nowhere vanishing anti-canonical section of \( L \). It is also clear that up to non-zero scalar multiples, the restriction of \( \eta_L \) to \( L \) is independent of the choices of the complement \( t_L' \) of \( t_L \) in \( t \) and of the non-zero element \( \xi \in \wedge^{l-2r} t_L' \). The restriction to \( L \) of \( \eta_L \) in (2.6) is called a Poisson \( \mathbb{T} \)-Pfaffian of the Poisson structure \( \pi \) on \( L \), a term suggested by M. Yakimov and A. Knutson.

3. Poisson structures defined by quasitriangular \( r \)-matrices

3.1. Quasitriangular \( r \)-matrices. We recall some basic facts on quasitriangular \( r \)-matrices and Lie bialgebras. Our references are [8, 10, 11, 14, 38].

Let \( g \) be a Lie algebra. Recall that the Classical Yang-Baxter operator for \( g \) is the map \( \text{CYB} : g \otimes g \rightarrow g \otimes g \otimes g \) given by

\[
\text{CYB}(r) = \sum_{i,j} ([x_i, x_j] \otimes y_i \otimes y_j + x_i \otimes [y_i, x_j] \otimes y_j + x_i \otimes x_j \otimes [y_i, y_j]), \quad \text{if } r = \sum x_i \otimes y_i.
\]

A *quasitriangular \( r \)-matrix* on \( g \) is an element \( r \in g \otimes g \) such that \( r + r^{21} \in (S^2 g)^g \) and that \( \text{CYB}(r) = 0 \), where \( (S^2 g)^g \) is the space of \( g \)-invariant elements in \( S^2 g \) with respect to the adjoint action. A quasitriangular \( r \)-matrix \( r \) on \( g \) is said to be *factorizable* if \( r + r^{21} \in (S^2 g)^g \) is non-degenerate.

Let \( r \in g \otimes g \) be a quasitriangular \( r \)-matrix on \( g \). Define

\[
(3.1) \quad \delta_r : g \rightarrow g \otimes g, \quad \delta_r(x) = \text{ad}_r(x), \quad x \in g.
\]

As \( r + r^{21} \in (S^2 g)^g \), \( \delta_r \) takes values in \( \wedge^2 g \), and \( (g, \delta_r) \) is a Lie bialgebra. Define \( r_{\pm} : g^* \rightarrow g \) by

\[
(3.2) \quad r_{+} = r^\# , \quad r_{-} = -(r^{21})^\# = -r_{+}^*.
\]

(see [11, 14]). The Lie bracket on \( g^* \), defined as the dual map of \( \delta_r \), is then given by

\[
(3.3) \quad [\xi, \eta] = \text{ad}_r^*(\xi) \eta - \text{ad}_r^*(\eta) \xi = \text{ad}_{r_{-}(-\xi)}^*(\eta) - \text{ad}^*_{r_{+}}(\eta) \xi, \quad \xi, \eta \in g^*,
\]

where for \( x \in g \) and \( \zeta \in g^* \), the element \( \text{ad}_r^* \zeta \in g^* \) is given by \( \langle \text{ad}_r^* \zeta, y \rangle = \langle \zeta, [y, x] \rangle \) for \( y \in g \). It is well-known ([14] Lecture 4, [14]) that both \( r_{+} \) and \( r_{-} \) are Lie algebra homomorphisms. Set

\[
(3.4) \quad f_+ = \text{Im}(r_{+}), \quad f_- = \text{Im}(r_{-}).
\]

Then both \( f_+ \) and \( f_- \) are Lie subalgebras of \( g \), and \( \delta_{r_+} f_{\pm} \subset \wedge^2 f_{\pm} \). In other words, both \( f_+ \) and \( f_- \) are sub-Lie bialgebras of the Lie bialgebra \( (g, \delta_r) \). A different proof of the following Lemma 3.1 can be found in [25, §7.1].

**Lemma 3.1.** Any Lie subalgebra \( m \) of \( g \) containing \( f_+ \) or \( f_- \) is a sub-Lie bialgebra of \( (g, \delta_r) \).
Proof. Assume that $m \supset f_+$. One needs to show that $m^0$, the annihilator of $m$ in $g^*$, is a Lie ideal with respect to the Lie bracket on $g^*$ given in (3.3). Let $\xi \in m^0$ and $\eta \in g^*$. Since $m \supset f_+$, one has $m^0 \subset f^0_+ = \ker r_-$, and it follows from (3.3) that $[\xi, \eta] = -\text{ad}^*_{r_-(\eta)} \xi$. As $r_+(\eta) \in f_+ \subset m$, one has $\text{ad}^*_{r_+(\eta)} \xi \in m^0$. Thus $m^0$ is a Lie ideal of $g^*$. The case when $m \supset f_-$ is proved similarly.

Q.E.D.

3.2. Factorizable quasitriangular r-matrices. Recall that a quadratic Lie algebra is a pair $(g, \langle \cdot, \cdot \rangle_g)$, where $g$ is a Lie algebra and $\langle \cdot, \cdot \rangle_g$ is a symmetric non-degenerate invariant bilinear form on $g$. Given a quadratic Lie algebra $(g, \langle \cdot, \cdot \rangle_g)$, for any vector subspace $v$ of $g$, let

\[(3.5) \quad v^\perp = \{ x \in g : \langle x, v \rangle_g = 0 \}.
\]

By a Lagrangian subalgebra of $(g, \langle \cdot, \cdot \rangle_g)$ we mean a Lie subalgebra $l$ of $g$ that is also Lagrangian with respect to $\langle \cdot, \cdot \rangle_g$, i.e., $l^\perp = l$. By a Lagrangian splitting of $(g, \langle \cdot, \cdot \rangle_g)$ we mean a decomposition $g = u + u'$ where both $u$ and $u'$ are Lagrangian subalgebras of $g$. The notion of quadratic Lie algebras with Lagrangian splittings is then equivalent to that of Manin triples [13, Lecture 4].

Given a Lie bialgebra $(u, \delta_u)$, recall that the Drinfeld double Lie algebra of $(u, \delta_u)$ is the quadratic Lie algebra $(g, \langle \cdot, \cdot \rangle_g)$, where $g = u \oplus u^*$ as a vector space, $\langle \cdot, \cdot \rangle_g$ is the symmetric bilinear form on $g$ given by

\[ \langle x + \xi, y + \eta \rangle_g = \langle x, \eta \rangle + \langle \xi, y \rangle, \quad x, y \in u, \xi, \eta \in u^*, \]

and the Lie bracket on $g$ is the unique one with respect to which the bilinear form $\langle \cdot, \cdot \rangle_g$ is invariant and both $u \cong u \oplus 0$ and $u^* \cong 0 \oplus u^*$ are Lie subalgebras. The decomposition $g = u + u^*$ is thus a Lagrangian splitting of $(g, \langle \cdot, \cdot \rangle_g)$. One also refers to $(g, \langle \cdot, \cdot \rangle_g, u, u^*)$ as the Manin triple of the Lie bialgebra $(u, \delta_u)$. Conversely, a Lagrangian splitting $g = u + u'$ of a quadratic Lie algebra $(g, \langle \cdot, \cdot \rangle_g)$ gives rise to a Lie bialgebra $(u, \delta_u)$, where $\delta_u : u \to \wedge^2 u$ is the map dual to the Lie bracket on $u'$, the latter being identified with $u^*$ via the pairing between $u$ and $u'$ defined by $\langle \cdot, \cdot \rangle_g$, and the Manin triple $(g, \langle \cdot, \cdot \rangle_g, u, u^*)$ is isomorphic to the Manin triple of $(u, \delta_u)$.

Assume now that $r$ is a factorizable quasitriangular $r$-matrix on a Lie algebra $g$, i.e., the linear map

\[ r_+ - r_- = (r + r^{21})' : g^* \to g \]

is invertible. The symmetric bilinear form $\langle \cdot, \cdot \rangle_g$ on $g$ given by

\[(3.6) \quad \langle x_1, x_2 \rangle_g = \langle (r_+ - r_-)^{-1} x_1, x_2 \rangle = \langle x_1, (r_+ - r_-)^{-1} x_2 \rangle, \quad x_1, x_2 \in g.
\]

is then non-degenerate and invariant, making $(g, \langle \cdot, \cdot \rangle_g)$ into a quadratic Lie algebra. We will refer to $\langle \cdot, \cdot \rangle_g$ as the symmetric bilinear form on $g$ associated to $r$. Set

\[ (3.7) \quad r^b_\pm = r_\pm \circ (r_+ - r_-)^{-1} : g \to g.
\]

One thus has $r^b_+ - r^b_- = \text{Id}_g$, and

\[ \langle r^b_+(x_1), x_2 \rangle_g + \langle x_1, r^b_-(x_2) \rangle_g = 0, \quad x_1, x_2 \in g.
\]

It also follows from the definitions that $f_\pm = \text{Im}(r^b_\pm)$ and that

\[ \text{ker}(r_+) = f^0_-, \quad \text{ker}(r_-) = f^0_+, \quad \text{ker}(r^b_+) = f^1_+, \quad \text{ker}(r^b_-) = f^1_-.
\]

In particular, if $x \in f^1_-$, then $x = r^b_+(x) - r^b_-(x) = r^b_+(x) \in f_+$, so $f^1_+ \subset f_+$. Similarly, $f^1_- \subset f_-$. Still assuming that $r$ is factorizable, consider now the direct product Lie algebra $g \oplus g$ and the bilinear form $\langle \cdot, \cdot \rangle_{g \oplus g}$ on $g \oplus g$ given by

\[ \langle (x_1, x_2), (x'_1, x'_2) \rangle_{g \oplus g} = \langle x_1, x'_1 \rangle_g - \langle x_2, x'_2 \rangle_g, \quad x_1, x_2, x'_1, x'_2 \in g.
\]
One then has the Lagrangian splitting
\begin{equation}
\label{eq:lagrangian-splitting}
g \oplus g = g_{\text{diag}} + \mathfrak{l}_r,
\end{equation}
of the quadratic Lie algebra \((g \oplus g, \langle \cdot, \cdot \rangle_{g \oplus g})\), where \(g_{\text{diag}} = \{(x, x) : x \in g\} \subset g \oplus g\), and
\begin{equation}
\label{eq:lie-algebra-splitting}
\mathfrak{l}_r = \{(r_+(\xi), r_-(\xi)) : \xi \in g^*\} = \{(r_+^x(x), r_-^x(x)) : x \in g\} \subset g \oplus g.
\end{equation}

One checks that \(\delta_r : g \to \wedge^2 g\) coincides with the co-bracket on \(g \cong g_{\text{diag}}\) induced by the Lagrangian splitting in \(\text{(3.10)}\). The assignment \(g \otimes g \ni r \mapsto \mathfrak{l}_r \subset g \oplus g\) gives a one to one correspondence between the set of all factorizable quasitriangular \(r\)-matrices on \(g\) that have \(\langle \cdot, \cdot \rangle_{g}\) as the associated symmetric bilinear form and the set of Lagrangian subalgebras \(I\) of \((g \oplus g, \langle \cdot, \cdot \rangle_{g \oplus g})\) such that \(g \oplus g = g_{\text{diag}} + I\).

**Example 3.2.** Let \((g, \langle \cdot, \cdot \rangle_{g})\) be any quadratic Lie algebra and let \(g = u + u'\) be a Lagrangian splitting of \((g, \langle \cdot, \cdot \rangle_{g})\). The quadratic Lie algebra \((g \oplus g, \langle \cdot, \cdot \rangle_{g \oplus g})\) (see \(\text{(3.9)}\)) has the Lagrangian splitting \(g \oplus g = g_{\text{diag}} + I\), where \(I = \{(\xi, x) : \xi \in u', x \in u\} \subset g \oplus g\). The factorizable quasitriangular \(r\)-matrix \(r_{(u,u')}\) on \(g\) such that \(\mathfrak{l}_{r_{(u,u')}} = I\) is given by
\begin{equation}
\label{eq:factorizable-r-matrix}
r_{(u,u')} = \sum_{i=1}^{m} x_i \otimes \xi_i \in g \otimes g,
\end{equation}
where \(\{x_i\}_{i=1}^{m}\) is any basis of \(u\) and \(\{\xi_i\}_{i=1}^{m}\) the basis of \(u'\) such that \(\langle x_i, \xi_j \rangle_{g} = \delta_{ij}\) for \(1 \leq i, j \leq m\). We will call \(r_{(u,u')}\) the \(r\)-matrix on \(g\) defined by the Lagrangian splitting \(g = u + u'\) of \((g, \langle \cdot, \cdot \rangle_{g})\). It is easy to see that the Lie subalgebras \(f_-\) and \(f_+\) of \(g\) associated to \(r_{(u,u')}\) (see definitions in \(\text{(3.3)}\)) are respectively given by \(f_- = u\) and \(f_+ = u'\). In particular, \(f_- \cap f_+ = 0\). Conversely, let \(r\) be a factorizable quasitriangular \(r\)-matrix on \(g\) with \(\langle \cdot, \cdot \rangle_{g}\) as the associated symmetric form. If \(f_- \cap f_+ = 0\), then \(g = f_- + f_+\) is a Lagrangian splitting and \(r = r_{(f_-f_+)}\).

**Example 3.3.** Let \(g\) be a simple complex Lie algebra, let \(\langle \cdot, \cdot \rangle_{g}\) be any non-zero scalar multiple of the Killing form of \(g\), and let \(\langle \cdot, \cdot \rangle_{g \oplus g}\) be the bilinear form on \(g \oplus g\) given in \(\text{(3.9)}\). In \(\text{[2]}\), Belavin and Drinfeld classified all Lagrangian splittings of \((g \oplus g, \langle \cdot, \cdot \rangle_{g \oplus g})\) of the form \(g \oplus g = g_{\text{diag}} + I\) and wrote down explicitly the corresponding factorizable quasitriangular \(r\)-matrices on \(g\). Details on the so-called standard \(r\)-matrix \(r_{\text{st}}\) on \(g\) will be recalled in \(\text{(6.1)}\).

### 3.3. Poisson structures defined by quasitriangular \(r\)-matrices.

Let \(Y\) be a manifold with a left Lie algebra action \(\lambda : g \to \mathcal{V}(Y)\). For \(r \in g \otimes g\), let \(\lambda(r)\) be the tensor field on \(Y\) given by
\begin{equation}
\label{eq:lambda-r}
\lambda(r) = \sum_{i} \lambda(x_i) \otimes \lambda(y_i), \quad \text{if} \quad r = \sum_{i} x_i \otimes y_i.
\end{equation}

The special case of the following Lemma \(\text{(5.4)}\) when \(r\) is the \(r\)-matrix on a quadratic Lie algebra defined by a Lagrangian splitting (see Example \(\text{3.2}\)) is proved in \(\text{[39]}\) Theorem 2.3).

**Lemma 3.4.** \(\text{[38]}\) Let \(r\) be a quasitriangular \(r\)-matrix on \(g\). If \(\lambda(r)\) is skew-symmetric, i.e., if \(\lambda(r)\) is a bivector field on \(Y\), then it is Poisson, and \(\lambda\) is a left Poisson action of the Lie bialgebra \((g, \delta_r)\) on the Poisson manifold \((Y, \{-\lambda(r)\})\).

Note that \(\lambda(r)\) is skew-symmetric if and only if \(\lambda(r + r^{21}) = 0\), or, equivalently,
\begin{equation}
\label{eq:poisson-condition}
(r_+ - r_-)(q_y^0) \subset q_y, \quad y \in Y,
\end{equation}
where for \(y \in Y\), \(q_y\) is the stabilizer of \(\lambda\) at \(y\), i.e., \(q_y = \ker(\lambda_y) \subset g\), and \(q_y^0\) the annihilator of \(q_y\) in \(g^*\). In particular, \(\text{(3.14)}\) is independent on the skew-symmetric part of \(r\). Note also that if \(r \in g \otimes g\) is
factorizable defining the symmetric bilinear form \(\langle , \rangle_g\) on \(g\), then (3.14) is equivalent to \(q_y \subset g\) being coisotropic with respect to \(\langle , \rangle_g\) for each \(y \in Y\), i.e.,

\[
(3.15) \quad q_y^+ \subset q_y, \quad \forall y \in Y.
\]

**Definition 3.5.** By an admissible quadruple we mean a quadruple \((g, r, Y, \lambda)\), where \(g\) is a Lie algebra, \(r \in g \otimes g\) is a quasitriangular \(r\)-matrix on \(g\), \(Y\) is a manifold, and \(\lambda\) is a left Lie algebra action of \(g\) on \(Y\) such that \(\lambda(r)\) is skew-symmetric, i.e., (3.14) holds for every \(y \in Y\). Given an admissible quadruple \((g, r, Y, \lambda)\), we refer to \(-\lambda(r)\) (and sometimes \(\lambda(r)\)) as the Poisson structure on \(Y\) defined by \((r, \lambda)\).

Let \(G\) be a connected Lie group with Lie algebra \(g\), and let \(r \in g \otimes g\) be a quasitriangular \(r\)-matrix on \(g\). Let \(r^L\) (resp. \(r^R\)) be the left (resp. right) invariant tensor field on \(G\) with value \(r\) at the identity element of \(G\). Then the bivector field on \(G\) given by

\[
\pi_G = r^L - r^R
\]

is Poisson, making \((G, \pi_G)\) into a Poisson Lie group [13]. If \(\lambda : G \times Y \to Y\) is a left Lie group action of \(G\) on a manifold \(Y\) such that \(\lambda(r)\) is skew-symmetric, where \(\lambda : g \to \mathcal{V}^1(Y)\) also denotes the induced left Lie algebra action of \(g\) on \(Y\) (see (1.16)), we also call \((G, r, \lambda, \pi_G)\) an admissible triple. In this case, \(\lambda\) is a left Poisson action of the Poisson Lie group \((G, \pi_G)\) on \((Y, -\lambda(r))\). When the action \(\lambda\) is transitive, we also say that \((G, r, Y, \lambda)\) is a homogeneous admissible quadruple.

**Remark 3.6.** It is clear from the definition that a quadruple \((G, r, Y, \lambda)\) is admissible if and only if \((G, r, \mathcal{O}, \lambda)\) is admissible for every \(G\)-orbit \(\mathcal{O}\) in \(Y\). In studying admissible quadruples, we may therefore restrict ourselves to homogeneous ones. Let \(Q\) be any closed subgroup of \(G\) with Lie algebra \(q\), and let \(\lambda_{G/Q}\) be the left action of \(G\) on \(G/Q\) by left translation. Then the quadruple \((G, r, G/Q, \lambda_{G/Q})\) is admissible if and only if \((r_+ - r_-)(q^0) \subset q\), or equivalently, \(q^+ \subset q\) when \(r\) is factorizable. As a special case, assume that the Lie algebra \(q\) of \(Q \subset G\) satisfies \(q \supset f_+\). Then

\[
(r_+ - r_-)(q^0) \subset (r_+ - r_-)(f_+^0) \subset r_+(f_+^0) \subset f_+ \subset q,
\]

so \((G, r, G/Q, \lambda_{G/Q})\) is admissible. On the other hand, by Lemma 3.1, \(Q\) is a Poisson Lie subgroup of \((G, \pi_G)\), so \(\pi_G\) projects to a well-defined Poisson structure, denoted by \(\pi_{G/Q}\), on \(G/Q\). It is shown in [38 \S 7] that \(\pi_{G/Q} = -\lambda_{G/Q}\).

Let \((G, r, Y, \lambda)\) be an admissible quadruple and consider the Poisson structure \(\pi = -\lambda(r)\) on \(Y\). By the definition of \(\pi\) and by (3.14), one has

\[
(3.16) \quad \pi^\#_g(\alpha_y) = \lambda_y (r_+ (\lambda^\#_g(\alpha_y))) = \lambda_y (r_- (\lambda^\#_g(\alpha_y))), \quad \alpha_y \in T^*_yY.
\]

It follows that for any Lie subalgebra \(m\) of \(g\) containing \(f_+\) or \(f_-\), and for any \(y \in Y\), one has

\[
(3.17) \quad \text{Im}(\pi^\#_m) \subset \lambda_y(m) \subset T_yY.
\]

The following Lemma 3.7 follows immediately from (3.17) and Lemma 3.4

**Lemma 3.7.** Let \((G, r, Y, \lambda)\) be an admissible quadruple, and let \(M\) be a connected Lie subgroup of \(G\) such that the Lie algebra \(m\) of \(M\) contains \(f_+\) or \(f_-\). Then every orbit \(\mathcal{O}_M\) of \(M\) in \(Y\) is a Poisson submanifold of \(Y\) with respect to the Poisson structure \(\pi = -\lambda(r)\), and \(\lambda\) restricts to a left Poisson action of the Poisson Lie group \((M, \pi_{|\mathcal{O}_M})\) on \((\mathcal{O}_M, \pi_{|\mathcal{O}_M})\).

In particular, by taking \(M = G\) in Lemma 3.7, every \(G\)-orbit in \(Y\), equipped with the Poisson structure \(\pi\), is a Poisson homogeneous space of the Poisson Lie group \((G, \pi_G)\).
Let $\mathcal{O}$ be any $G$-orbit in $Y$. For $y \in \mathcal{O}$, let $(\text{Im}(\pi_y^#))^0 \subset T_y^*\mathcal{O}$ be the co-normal space in $\mathcal{O}$ at $y$ of the symplectic leaf of $\pi$ through $y$. Let $[\lambda_y] : g/\mathfrak{q}_y \rightarrow T_y\mathcal{O}$ be the vector space isomorphism induced by $\lambda_y : g \rightarrow T_yY$. Identify $(g/\mathfrak{q}_y)^*$ with $\mathfrak{q}_y^0 \subset g^*$. Then one has the vector space isomorphism $[\lambda^*_y] : T^*_y\mathcal{O} \rightarrow \mathfrak{q}_y^0$.

**Lemma 3.8.** For any $y \in \mathcal{O}$, one has

$$[\lambda^*_y]((\text{Im}(\pi_y^#))^0) = \mathfrak{q}_y^0 \cap r_+^{-1}(\mathfrak{q}_y).$$

Consequently, $\dim \mathcal{O} - \dim(\text{Im}(\pi_y^#)) = \dim(\mathfrak{q}_y^0 \cap r_+^{-1}(\mathfrak{q}_y)) = \dim(\mathfrak{q}_y^0 \cap r_+^{-1}(\mathfrak{q}_y))$.

**Proof.** By (3.16), one has $\text{Im}(\pi_y^#) = \lambda_y(r_+(\mathfrak{q}_y^0)) = [\lambda^*_y](\langle \mathfrak{q}_y + r_+(\mathfrak{q}_y^0) \rangle/\mathfrak{q}_y)$. Thus

$$[\lambda^*_y]((\text{Im}(\pi_y^#))^0) = \mathfrak{q}_y^0 \cap (r_+(\mathfrak{q}_y^0))^0 = \mathfrak{q}_y^0 \cap r_+^{-1}(\mathfrak{q}_y) = \mathfrak{q}_y^0 \cap r_+^{-1}(\mathfrak{q}_y).$$

Q.E.D.

We recall a result of Drinfeld [12] on Poisson homogeneous spaces: suppose that $(\mathcal{O}, \pi, \lambda)$ is a Poisson homogeneous space of a Poisson Lie group with Lie bialgebra $(g, \delta_g)$. Then the Lie algebroid $(\mathcal{O} \times g) \rightrightarrows T^*\mathcal{O}$ over $\mathcal{O}$ in (2.3) is transitive, so the kernel of its anchor map $-\lambda + \pi^#$ is a bundle of Lie algebras over $\mathcal{O}$. Consider the map

$$\Psi : (\mathcal{O} \times g) \rightrightarrows \mathfrak{d}, \ (x, \alpha_y) \mapsto x + \lambda^*_y(\alpha_y), \quad x \in g, \ y \in \mathcal{O}, \ \alpha_y \in T^*_y\mathcal{O},$$

where $(\mathfrak{d}, (\cdot, \cdot)_\delta)$ is the Drinfeld double Lie algebra of $(g, \delta_g)$. For $y \in \mathcal{O}$, let

$$l_y = \Psi(\ker(-\lambda_y + \pi^#_y)) = \{x + \xi : \ x \in g, \ \xi \in \mathfrak{q}_y^0, \ \lambda_y(x) = \pi^#_y([\lambda^*_y]^{-1}(\xi)) \} \subset \mathfrak{d}.$$ 

By [37], for $y \in Y$, $\Psi|_{\ker(-\lambda_y + \pi^#_y)} : \ker(-\lambda_y + \pi^#_y) \rightarrow \mathfrak{d}$ is injective and $l_y$ is a Lagrangian subalgebra of $(\mathfrak{d}, (\cdot, \cdot)_\delta)$, called the **Drinfeld Lagrangian subalgebra** at $y$ associated to the Poisson structure $\pi$.

The following Lemma 3.9 follows directly from the definition of $l_y$ and is basic to Drinfeld’s theory on Poisson homogeneous spaces [12].

**Lemma 3.9.** (i). The conormal subspace $(\text{Im}(\pi_y^#))^0 \subset T^*_y\mathcal{O}$ can be identified with $g^* \cap l_y$ under the vector space isomorphism $[\lambda^*_y] : T^*_y\mathcal{O} \rightarrow \mathfrak{q}_y^0$.

(ii). For $x \in g$, $\lambda_y(x) \in \text{Im}(\pi_y^#)$ if and only if $x \in g \cap (g^* + l_y) = \text{pr}_g(l_y)$, where $\text{pr}_g : \mathfrak{d} \rightarrow g$ is the projection with respect to the decomposition $\mathfrak{d} = g + g^*$.

In the context of Lemma 3.9 one checks from the definition that

$$(3.18) \quad l_y = \{x + \xi \in \mathfrak{d} : \ x \in g, \ \xi \in \mathfrak{q}_y^0, \ x + r_+(\xi) \in \mathfrak{q}_y\}.$$

Thus $g^* \cap l_y = \mathfrak{q}_y^0 \cap r_+^{-1}(\mathfrak{q}_y)$. This gives another proof of Lemma 3.8 using Drinfeld’s general theory in [12]. We will return to the Drinfeld Lagrangian subalgebras in 3.4.

3.4. **Strongly admissible quadruples.**

**Definition 3.10.** By a **strongly admissible quadruple** we mean a quadruple $(G, r, Y, \lambda)$, where $G$ is a connected Lie group, $r$ is a factorizable quasi-triangular $r$-matrix on the Lie algebra $g$ of $G$, $Y$ is a manifold, and $\lambda$ is a left action of $G$ on $Y$, such that the stabilizer subgroup $Q_y$ of $G$ at every $y \in Y$ is connected and its Lie algebra $\mathfrak{q}_y$ satisfies

$$(3.19) \quad [\mathfrak{q}_y, \mathfrak{q}_y] \subset \mathfrak{q}_y^+ \subset \mathfrak{q}.$$
Remark 3.11. A strongly admissible quadruple is thus an admissible quadruple \((G, r, Y, \lambda)\) with the additional requirements that \(r\) be factorizable, the stabilizer subgroup \(Q_y\) of \(G\) at each \(y \in Y\) be connected and its Lie algebra satisfy \([q_y, q_y] \subset q_y^\perp\).

Example 3.12. Let \(G\) be a connected Lie group with Lie algebra \(\mathfrak{g}\) and \(r\) a factorizable quasitrivially \(r\)-matrix on \(\mathfrak{g}\). Homogeneous strongly admissible quadruples are then of the form \((G, r, G/Q, \lambda)\), where \(Q\) is a closed and connected Lie subgroup of \(G\) whose Lie algebra \(\mathfrak{q}\) satisfies

\[ [\mathfrak{q}, \mathfrak{q}] \subset \mathfrak{q}^\perp \subset \mathfrak{q}, \tag{3.20} \]

and \(\lambda_{G/Q}\) is the left action of \(G\) on \(G/Q\) by left translation. In the special case when \(r = r(u, u')\) is the \(r\)-matrix on \(\mathfrak{g}\) defined by a Lagrangian splitting \(\mathfrak{g} = u + u'\) of \((\mathfrak{g}, \langle \cdot, \cdot \rangle_{\mathfrak{g}}\), Condition \((3.20)\) was first introduced in \([39]\) and some properties of the Poisson structure \(\lambda_{G/Q}(r)\) on \(G/Q\) were also studied in \([39]\). Note also that \((3.20)\) holds automatically if \(\mathfrak{q}\) is Lagrangian with respect to \(\langle \cdot, \cdot \rangle_{\mathfrak{g}}\).

Example 3.13. Continuing with Example \(3.3\), let \(G\) be a connected complex simple Lie group with Lie algebra \(\mathfrak{g}\), and let \(\langle \cdot, \cdot \rangle_{\mathfrak{g}}\) be a fixed non-zero scalar multiple of the Killing form of \(\mathfrak{g}\). Recall from Example \(3.3\) that Lagrangian splittings of \((\mathfrak{g} \oplus \mathfrak{g}, \langle \cdot, \cdot \rangle_{\mathfrak{g} \oplus \mathfrak{g}})\) of the form \(\mathfrak{g} \oplus \mathfrak{g} = \mathfrak{g}_{\text{diag}} + I\) have been classified by Belavin-Drinfeld in \([2]\). Let \(r_{BD} \in \mathfrak{g} \otimes \mathfrak{g}\) be the factorizable quasitrivially \(r\)-matrix on \(\mathfrak{g}\) corresponding to such an \(I \subset \mathfrak{g} \oplus \mathfrak{g}\). Let \(P\) be any parabolic subgroup of \(G\). As the Lie algebra \(\mathfrak{p}\) of \(P\) is coisotropic with respect to \(\langle \cdot, \cdot \rangle_{\mathfrak{g}}\), the quadruple \((G, r_{BD}, G/P, \lambda_{G/P})\) is admissible. On the other hand, the Lie algebra \(\mathfrak{p}\) of \(P\) satisfies \([\mathfrak{p}, \mathfrak{p}] \subset \mathfrak{p}^\perp\) if and only if \(\mathfrak{p}\) is Borel. Thus \((G, r_{BD}, G/P, \lambda_{G/P})\) is strongly admissible if and only if \(\mathfrak{p}\) is a Borel subgroup of \(G\).

We now prove some preliminary properties of strongly admissible quadrupules.

Lemma 3.14. Assume that \((G, r, Y, \lambda)\) is strongly admissible. Then for any \(y_1, y_2 \in Y\) in the same \(G\)-orbit in \(Y\), the linear isomorphism

\[ I_{y_2, y_1} : \mathfrak{q}_{y_1}/\mathfrak{q}_{y_1}^\perp \to \mathfrak{q}_{y_2}/\mathfrak{q}_{y_2}^\perp, \quad x + \mathfrak{q}_{y_1}^\perp \to \text{Ad}_y(x) + \mathfrak{q}_{y_2}^\perp, \quad x \in \mathfrak{q}_{y_1}, \tag{3.21} \]

is independent of the choice of \(g \in G\) such that \(gy_1 = y_2\).

Proof. For any \(y \in Y\), as the stabilizer subgroup \(Q_y\) of \(G\) at \(y \in Y\) is connected, condition \((3.19)\) implies that the action of \(Q_y\) on \(\mathfrak{q}_y/\mathfrak{q}_y^\perp\) induced by the adjoint action of \(Q_y\) on \(\mathfrak{q}_y\) is trivial. Consequently, the map \(I_{y_2, y_1}\) is independent of the choice of \(g \in G\) such that \(gy_1 = y_2\).

Q.E.D.

It is also clear that if \(y_1, y_2, y_3\) are in the same \(G\)-orbit, then

\[ I_{y_2, y_1}^{-1} = I_{y_1, y_2} : \mathfrak{q}_{y_2}/\mathfrak{q}_{y_2}^\perp \to \mathfrak{q}_{y_1}/\mathfrak{q}_{y_1}^\perp, \tag{3.22} \]

\[ I_{y_3, y_2} \circ I_{y_2, y_1} = I_{y_3, y_1} : \mathfrak{q}_{y_1}/\mathfrak{q}_{y_1}^\perp \to \mathfrak{q}_{y_3}/\mathfrak{q}_{y_3}^\perp. \tag{3.23} \]

Definition 3.15. Let \((G, r, Y, \lambda)\) be a strongly admissible quadruple. For \(y_1, y_2 \in Y\) in the same \(G\)-orbit, define

\[ I_{y_1, y_2} = \{(x_1, x_2) \in \mathfrak{q}_{y_1} \oplus \mathfrak{q}_{y_2} : I_{y_2, y_1}(x_1 + q_{y_1}^\perp) = x_2 + q_{y_2}^\perp\} \subset \mathfrak{g} \oplus \mathfrak{g}. \tag{3.24} \]

Recall from \((3.2)\) that, as \(r\) is factorizable, the Drinfeld double Lie algebra of the Lie bialgebra \((\mathfrak{g}, \delta_r)\) can be identified with the quadratic Lie algebra \((\mathfrak{g} \oplus \mathfrak{g}, \langle \cdot, \cdot \rangle_{\mathfrak{g} \oplus \mathfrak{g}})\), where \(\langle \cdot, \cdot \rangle_{\mathfrak{g} \oplus \mathfrak{g}}\) is defined in \((3.9)\).
Lemma 3.16. Let \((G, r, Y, \lambda)\) be a strongly admissible quadruple. For any \(y_1, y_2 \in Y\) in the same \(G\)-orbit, \(\mathfrak{l}_{y_1, y_2}\) is a Lagrangian subalgebra of \((\mathfrak{g} \oplus \mathfrak{g}, \{\cdot, \cdot\}_\mathfrak{g} \mathfrak{g})\). Moreover, for any \(y \in Y\),
\begin{equation}
\mathfrak{l}_y = \{(x_1, x_2) \in \mathfrak{q}_y \oplus \mathfrak{q}_y : x_1 - x_2 \in \mathfrak{q}_y^+\}
\end{equation}
is the Drinfeld Lagrangian subalgebra \(\mathfrak{l}_y\) at \(y\) associated to the Poisson structure \(\pi = -\lambda(r)\) on the \(G\)-orbit \(Gy \subset Y\).

Proof. Let \(y \in Y\) be arbitrary and let \(\mathcal{O} = Gy \subset Y\). By (3.18), the Drinfeld Lagrangian subalgebra \(\mathfrak{l}_y\) at \(y\) associated to the Poisson structure \(\pi = -\lambda(r)\) on \(\mathcal{O}\) is given by
\[\mathfrak{l}_y = \{(x, x) + (r_+^y(x'), r_-^y(x')) : x \in \mathfrak{g}, x' \in \mathfrak{q}_y^+\},\]
from which it follows that \(\mathfrak{l}_y = \mathfrak{l}_{y,y}\). For any \(y_1, y_2 \in Gy\), let \(y_1 = g_1y\) and \(y_2 = g_2y\), where \(g_1, g_2 \in G\). It follows from the definitions that
\begin{equation}
\mathfrak{l}_{y_1, y_2} = (\text{Ad}_{g_1} \oplus \text{Ad}_{g_2})(\mathfrak{l}_{y,y}).
\end{equation}
Thus \(\mathfrak{l}_{y_1, y_2}\) is a Lagrangian subalgebra of \((\mathfrak{g} \oplus \mathfrak{g}, \{\cdot, \cdot\}_\mathfrak{g} \mathfrak{g})\).

Q.E.D.

Remark 3.17. By Lemma 3.9 and Lemma 3.16 for a strongly admissible quadruple \((G, r, Y, \lambda)\), the rank of \(\pi\) at \(y \in Y\) is equal to \(\dim(Gy) - \dim(\mathfrak{l}_y \cap \mathfrak{l}_{y,y})\).

4. Orbit intersections for strongly admissible quadruples

4.1. The set-up. Assume that \((G, r, Y, \lambda)\) is an admissible quadruple, in which \(r\) is factorizable. Let \(M_+\) and \(M_-\) be connected Lie subgroups of \(G\) whose respective Lie algebras \(\mathfrak{m}_+\) and \(\mathfrak{m}_-\) satisfy
\begin{equation}
f_+ \subset \mathfrak{m}_+, \quad f_- \subset \mathfrak{m}_-.
\end{equation}
Let \(T\) be the connected component of \(M_+ \cap M_-\) containing the identity element. The Lie algebra \(\mathfrak{t}\) of \(T\) is then given by \(\mathfrak{t} = \mathfrak{m}_+ \cap \mathfrak{m}_-\). By Lemma 3.11 \(T\) is a Poisson Lie subgroup of \((G, \pi_\mathcal{O})\), where \(\mathcal{O} = r^L - r^R\), and \((\mathfrak{t}, \delta_{\mathcal{O}}|_{\mathfrak{t}})\) is a sub-Lie bialgebra of the Lie bialgebra \((\mathfrak{g}, \delta_{\mathcal{O}})\). Consider the decomposition
\[Y = \bigcup_{\mathcal{O}_+, \mathcal{O}_-} \mathcal{O}_+ \cap \mathcal{O}_-\]
where \(\mathcal{O}_+\) and \(\mathcal{O}_-\) are respectively \(M_+\)-orbits and \(M_-\)-orbits in \(Y\). As \(r\) is factorizable, one has \(f_+ + f_- = \mathfrak{g}\), and thus \(\mathfrak{m}_+ + \mathfrak{m}_- = \mathfrak{g}\). Consequently, each non-empty intersection \(\mathcal{O}_+ \cap \mathcal{O}_-\) is a smooth submanifold of \(Y\), and by Lemma 3.7 also a Poisson submanifold with respect to the Poisson structure \(\pi = -\lambda(r)\) on \(Y\). Denote the restriction of \(\pi\) to \(\mathcal{O}_+ \cap \mathcal{O}_-\) also by \(\pi\). As \(\mathcal{O}_+ \cap \mathcal{O}_-\) is \(T\)-invariant, the Poisson action \(\lambda\) of \((G, \pi_\mathcal{O})\) on \((Y, \pi)\) restricts to a Poisson action on \((\mathcal{O}_+ \cap \mathcal{O}_-, \pi)\) by the Poisson Lie group \((T, \pi_\mathcal{O}|_T)\) and the Lie bialgebra \((\mathfrak{t}, \delta_{\mathcal{O}}|_{\mathfrak{t}})\).

Definition 4.1. The six-tuple \((G, r, Y, \lambda, M_+, M_-)\) is said to be admissible if the quadruple \((G, r, Y, \lambda)\) is strongly admissible and if \(T\) is abelian, acting on \((Y, \lambda(r))\) by Poisson isomorphisms, and the \(T\)-leaves of \(\lambda(r)\) in \(Y\) are precisely all the connected components of non-empty intersections \(\mathcal{O}_+ \cap \mathcal{O}_-\) where \(\mathcal{O}_+\) is an \(M_+\)-orbit and \(\mathcal{O}_-\) an \(M_-\)-orbit in \(Y\).

Remark 4.2. If \(G\) is an affine algebraic group over \(\mathbb{C}\) and if \(M_+\) and \(M_-\) are algebraic subgroups of \(G\) such that \(M_+ \cap M_-\) is connected, by [15, Corollary 1.5], every non-empty intersection of an \(M_+\)-orbit and an \(M_-\)-orbit in \(Y\) is irreducible and thus connected.
In (4.1) we develop a test for a six-tuple \((G, r, Y, \lambda, M_+, M_-)\) to be admissible and we compute the \(T\)-leaf stabilizers. The test will be used in (4.2) to identify a class of admissible six-tuples.

More precisely, for a strongly admissible quadruple \((G, r, Y, \lambda)\), and a pair \((M_+, M_-)\) of connected Lie subgroups of \(G\) satisfying (4.1), we show that (4.2) that the Lie bialgebra action \(\lambda|_t\) of \((t,\delta|_t)\) on any non-empty intersection \(O_+ \cap O_-\) of \(M_+\) and \(M_-\)-orbits is regular, and we give a formula for the integer \(\delta_{O_+, O_-}\) that measures how far the action \(\lambda|_t\) of \((t,\delta|_t)\) on \(O_+ \cap O_-\) is from being full (see Definition 2.1). When \(\delta|_t = 0\), so \(T\) acts on \((Y, \pi)\) by Poisson isomorphisms, we show that the Poisson structure \(\pi\) is also regular on each non-empty intersection \(O_+ \cap O_-\). In (4.3) under a further assumption on the pair \((M_+, M_-)\), called \(r\)-admissible, which implies that \(T\) is abelian and \(\delta|_t = 0\), we compute the leaf stabilizer for each \(T\)-leaf in \(Y\). Theorem 4.5 a summary of the main results in (4.1) is proved in (4.3).

**Remark 4.3.** The assumptions that \(f_+ \subset m_+\) and \(f_- \subset m_-\) are equivalent to

\[
(4.2) \quad t_{\text{diag}} + l_r = m_+ \oplus m_-,
\]

where \(l_r\) is the Lagrangian subalgebra of \(g \oplus g\) given in (3.11), and \(t_{\text{diag}} = \{(x, x) : x \in t\}\). Indeed, clearly \(f_+ \subset m_+\) and \(f_- \subset m_-\) if and only if \(l_r \subset m_+ \oplus m_-\), which, by the direct sum decomposition \(g \oplus g = g_{\text{diag}} \oplus l_r\), is equivalent to \(m_+ \oplus m_- = g_{\text{diag}} \cap (m_+ \oplus m_-) + l_r = t_{\text{diag}} + l_r\).

**4.2. The action of \((t,\delta|_t)\) on \((O_+ \cap O_-, \pi)\) is regular.** Assume first that \((G, r, Y, \lambda)\) is admissible, in which \(r\) is factorizable, and let \((M_+, M_-)\) be a pair of connected Lie subgroups of \(G\) satisfying (4.1). Let \(\pi = -\lambda(r)\) on \(Y\). For \(y \in Y\), define

\[
(4.3) \quad \delta_y = \dim((M_+ y) \cap (M_- y)) - \dim(\lambda_y(t) + \text{Im}(\pi_y^\#)).
\]

Then for any pair \((O_+ \cap O_-)\) of \(M_+\) and \(M_-\)-orbits in \(Y\) such that \(O_+ \cap O_- \neq \emptyset\), the Poisson action \(\lambda|_t\) of the Lie bialgebra \((t,\delta|_t)\) on \((O_+ \cap O_-, \pi)\) is regular (see Definition 2.1) if and only of the integer-valued function \(y \mapsto \delta_y\) on \(O_+ \cap O_-\) is a constant function. For \(y \in Y\), let \(p_y : q_y \rightarrow q_y/q_y^\perp\) be the natural projection, and define two subspaces \(a_y^+\) of \(q_y/q_y^\perp\) by

\[
(4.4) \quad a_y^+ = p_y((m_+ \cap q_y)) = \frac{m_+^\perp \cap q_y}{m_+^\perp \cap q_y}, \quad a_y^- = p_y((m_- \cap q_y)) = \frac{m_-^\perp \cap q_y}{m_-^\perp \cap q_y}.
\]

**Lemma 4.4.** One has \(\delta_y = \dim(a_y^+ \cap a_y^-)\) for every \(y \in Y\).

**Proof.** Let \(O = G Y \subset Y\) and and let \(\lambda_y : g/q_y \cong T_y O\) be the vector space isomorphism induced by the linear map \(\lambda_y : g \rightarrow T_y Y\). Let \(O_+ = M_+ y \subset O\) and \(O_- = M_- y \subset O\). Then

\[
T_y(O_+ \cap O_-) = \lambda_y(m_+ \cap \lambda_y(m_-) = \lambda_y(m_+ \cap (m_+ + q_y)) = [\lambda_y]((q_y + m_+ \cap (m_+ + q_y))/q_y),
\]

\[
\lambda_y(t) + \text{Im}(\pi_y^\#) = [\lambda_y]((q_y + m_+ \cap m_- + r_+(q_y^\#))/q_y).
\]

On the other hand, it follows from \((r_+ - r_-)(q_y^\#) = q_y^\perp\) that

\[
(4.5) \quad m_+ \cap m_- + r_+(q_y^\#) = m_+ \cap (m_- + q_y^\perp).
\]

Indeed, as \(f_+ \subset m_+\), one has \(m_+ \cap m_- + r_+(q_y^\#) \subset m_+\), and as \((r_+ - r_-)(q_y^\#) = q_y^\perp\), one has

\[
m_+ \cap m_- + r_+(q_y^\#) \subset m_+ + (r_+ - r_-)(q_y^\#) \subset m_- + q_y^\perp,
\]

so \(m_+ \cap m_- + r_+(q_y^\#) \subset m_+ \cap (m_- + q_y^\perp)\). Conversely, let \(x \in m_+ \cap (m_- + q_y^\#)\) and write \(x = x_- + (r_+ - r_-)(\xi)\) for \(x_- \in m_-\) and \(\xi \in q_y^\perp\). Then \(x_- - r_- (\xi) = x - r_+(\xi) \in m_+ \cap m_-\) and \(x = x_- - r_-(\xi) + r_+(\xi) \in m_+ \cap m_- + r_+(q_y^\#)\). Thus \(m_+ \cap (m_- + q_y^\perp) \subset m_+ \cap m_- + r_+(q_y^\#)\), and this proves (4.5). Consequently, under the vector space isomorphism \(\lambda_y : g/q_y \cong T_y O\), one has

\[
T_y(O_+ \cap O_-) \cong (q_y + m_+ \cap (m_- + q_y))/q_y, \quad \lambda_y(t) + \text{Im}(\pi_y^\#) \cong (q_y + m_+ \cap (m_- + q_y))/q_y.
\]
Using \( q_y^+ \subset q_y \), one has
\[
(q_y + m_+ \cap (m_- + q_y^+)) \downarrow = q_y^+ \cap (m^+_+ + m_-^- \cap q_y) = q_y^+ \cap (m^+_+ \cap q_y + m_-^- \cap q_y),
\]
\[
(q_y + m_+ \cap (m_- + q_y)) \downarrow = q_y^+ \cap (m^+_+ + m_-^- \cap q_y^+) = q_y^+ \cap (m^+_+ \cap q_y^+ + m_-^- \cap q_y^+).
\]

It follows that
\[
\delta_y = \dim \left( \frac{q_y^+ \cap (m^+_+ \cap q_y + m_-^- \cap q_y)}{q_y^+ \cap (m^+_+ \cap q_y^+ + m_-^- \cap q_y^+)} \right).
\]

Note that \( m^+_+ \cap m_-^- = (m_+ + m_-^-)^0 = q^0 = 0 \). Writing an element \( x \in q_y^+ \cap (m^+_+ \cap q_y + m_-^- \cap q_y) \) uniquely as \( x = x_+ + x_- \), where \( x_+ \in m^+_+ \cap q_y \) and \( x_- \in m^-_- \cap q_y \), the map
\[
q_y^+ \cap (m^+_+ \cap q_y + m_-^- \cap q_y) \to q_y^+ / q_y^+, \ x \mapsto x_+ + q_y^+ = -x_- + q_y^+
\]
induces a well-defined vector space isomorphism
\[
\frac{q_y^+ \cap (m^+_+ \cap q_y + m_-^- \cap q_y)}{q_y^+ \cap (m^+_+ \cap q_y^+ + m_-^- \cap q_y^+)} \to a_y^+ \cap a_y^-.
\]
It follows that \( \delta_y = \dim (a_y^+ \cap a_y^-) \).

Q.E.D.

For the remainder of \( \{4.12\} \) assume that \((G, r, Y, \lambda)\) is strongly admissible, and let \((M_+, M_-)\) be a pair of connected Lie subgroups of \( G \) satisfying \((4.1)\).

**Lemma 4.5.** For any \( M_+ \)-orbit \( O_+ \) and any \( M_- \)-orbit \( O_- \) in \( Y \), one has
1) \( I_{y_2} (a^+_{y_1}) = a^+_{y_2} \) for all \( y_1, y_2 \in O_+ \);
2) \( I_{y_2} (a^-_{y_1}) = a^-_{y_2} \) for all \( y_1, y_2 \in O_- \).

**Proof.** We only prove 1), the proof of 2) being similar. Assume thus \( y_2 = m_+ y_1 \), where \( m_+ \in M_+ \). As \( \text{Ad}_{m_+} m^+_+ = m^+_+ \), one has
\[
I_{y_2} (a^+_{y_1}) = \frac{\text{Ad}_{m_+} (m^+_+ \cap q_{y_1})}{\text{Ad}_{m_+} (m^+_+ \cap q_{y_2})} = \frac{m^+_+ \cap q_{y_2}}{m^+_+ \cap q_{y_2}} = a^+_{y_2}.
\]
Q.E.D.

Let \((O_+, O_-)\) be any pair of \( M_+ \)- and \( M_- \)-orbits contained in the same \( G \)-orbit \( O \) in \( Y \), possibly \( O_+ \cap O_- = \emptyset \). Let \( y_0 \in O \), \( y_+ \in O_+ \) and \( y_- \in O_- \) be arbitrary, and let \( g_+, g_- \in G \) be such that \( y_+ = g_+ y_0 \) and \( y_- = g_- y_0 \). Define
\[
\delta_{O_+, O_-} = \dim \left( (I_{y_0} (a^+_{y_+})) \cap (I_{y_0} (a^-_{y_-})) \right)
\]
\[
= \dim \left( \left( \left( \frac{\text{Ad}_{g_{y_0}} (m^+_+)}{\text{Ad}_{g_{y_0}} (m^+_+)} \right) \cap q_{y_0} \right) \cap \left( \left( \frac{\text{Ad}_{g_{y_0}} m^+_+}{\text{Ad}_{g_{y_0}} m^+_+} \right) \cap q_{y_0} \right) \right),
\]
where the intersection on the right hand side of \((4.6)\) is in the vector space \( q^{y_0} / q^{y_0}_{y_0} \).

**Proposition 4.6.** For any pair \((O_+, O_-)\) of \( M_+ \)- and \( M_- \)-orbits contained in the same \( G \)-orbit in \( Y \), the integer \( \delta_{O_+, O_-} \) in \((4.6)\) is independent of the choices of \( y_0 \in O \), \( y_+ \in O_+ \) and \( y_- \in O_- \). Moreover, when \( O_+ \cap O_- \neq \emptyset \), one has \( \delta_{O_+, O_-} = \delta_y \) for any \( y \in O_+ \cap O_- \).
Proof. For \( y_0 \in O, y_+ \in O_+, and y_- \in O_- \), let
\[
(4.7) \quad a_{O_+,y_0} = I_{y_0,y_+}(a_{y_+}^+) \subset q_{y_0}/q_{y_0}, \quad a_{O_-,y_0} = I_{y_0,y_-}(a_{y_-}^-) \subset q_{y_0}/q_{y_0}.
\]
By Lemma 3.14 and 3.23, \( a_{O_+,y_0} \) and \( a_{O_-,y_0} \) are independent of the choices of \( y_+ \in O_+ \) and \( y_- \in O_- \), and the integer \( \delta_{O_+,O_-} = \dim(a_{O_+,y_0} \cap a_{O_-,y_0}) \) is independent of the choice of \( y_0 \in O \).

Assume that \( O_+ \cap O_- \neq \emptyset \), and let \( y \in O_+ \cap O_- \). Then
\[
I_{y_0,y}(a_{y_+}^+) = I_{y_0,y_+}(I_{y_+,y}(a_{y_+}^+)) = I_{y_0,y_+}(a_{y_+}^+) = a_{O_+,y_0},
\]
\[
I_{y_0,y}(a_{y_-}^-) = I_{y_0,y_-}(I_{y_-,y}(a_{y_-}^-)) = I_{y_0,y_-}(a_{y_-}^-) = a_{O_-,y_0}.
\]
It follows that \( \delta_{O_+,O_-} = \dim(a_{O_+,y_0} \cap a_{O_-,y_0}) = \dim(a_{y_+}^+ \cap a_{y_-}^-) = \delta_y \).

Q.E.D.

Corollary 4.7. Assume that \((G, r, Y, \lambda)\) is strongly admissible and let \( (M_+, M_-) \) be a pair of connected Lie subalgebras of \( G \) satisfying \((4.1)\). Then for every pair \((O_+, O_-)\) of \( M_+ \) - and \( M_- \) -orbits in \( Y \) such that \( O_+ \cap O_- \neq \emptyset \), the Poisson action \( \lambda_{|t} \) of the Lie bialgebra \((t, \delta_{|t})\) on \((O_+ \cap O_-, -\lambda(r))\) is regular.

Remark 4.8. For a pair \((O_+, O_-)\) of \( M_+ \) and \( M_- \) -orbits in \( Y \) such that \( O_+ \cap O_- \neq \emptyset \), the Poisson action \( \lambda_{|t} \) of the Lie bialgebra \((t, \delta_{|t})\) on \((O_+ \cap O_-, -\lambda(r))\) is full if and only if \( \delta_{O_+,O_-} = 0 \). Thus, the integer \( \delta_{O_+,O_-} \) measures how far the Poisson action \( \lambda_{|t} \) of \((t, \delta_{|t})\) on \((O_+ \cap O_-, -\lambda(r))\) is from being full. In examples, one can compute \( \delta_{O_+,O_-} \) using \((4.6)\) by choosing \( y_0 \in O \) and \( y_\pm \in O_\pm \) that are convenient for the computation. This is the case for the Poisson structures to be considered in \([\mathfrak{m}] \odot \mathfrak{m}\).

Corollary 4.9. Assume that \((G, r, Y, \lambda)\) is strongly admissible and let \((M_+, M_-) \) be a pair of connected Lie subalgebras of \( G \) satisfying \((4.1)\). If \( q_y \subset \mathfrak{g} \) is Lagrangian for every \( y \in Y \), then for every pair \((O_+, O_-)\) of \( M_+ \) - and \( M_- \) -orbits in \( Y \) such that \( O_+ \cap O_- \neq \emptyset \), the Poisson action \( \lambda_{|t} \) of the Lie bialgebra \((t, \delta_{|t})\) on \((O_+ \cap O_-, -\lambda(r))\) is full.

Proof. As \( q_{y_0} = q_{y_0}^0 \) for any \( y_0 \in Y \), it follows trivially from Proposition 4.6 that \( \delta_{O_+,O_-} = 0 \).

Q.E.D.

Still assuming that \((G, r, Y, \lambda)\) is strongly admissible and that \((M_+, M_-)\) satisfies \((4.1)\), we now give another formula for the integer \( \delta_{O_+,O_-} \). Let \((O_+, O_-)\) be any pair of \( M_+ \) - and \( M_- \) -orbits in the same \( G \) -orbit in \( Y \), and let \( y_+ \in O_+ \) and \( y_- \in O_- \). Recall the Lagrangian Lie subalgebra \( l_{y_+,y_-} \) of \((\mathfrak{g} \oplus \mathfrak{g}, \langle \cdot, \cdot \rangle_{|\mathfrak{g} \oplus \mathfrak{g}})\) given by
\[
l_{y_+,y_-} = \{(x_+, x_-) : x_+ \in q_{y_+}, x_- \in q_{y_-}, I_{y_-y_+}(x_+ + q_{y_+}^+) = x_- + q_{y_-}^+\}.
\]
Let \( p_+ : l_{y_+,y_-} \rightarrow q_{y_+}/q_{y_+}^+ \) be the given by
\[
p_+(x_+, x_-) = x_+ + q_{y_+}^+ = I_{y_+,y_-}(x_- + q_{y_-}^+), \quad (x_+, x_-) \in l_{y_+,y_-}.
\]

Lemma 4.10. Let \((O_+, O_-)\) be any pair of \( M_+ \) - and \( M_- \) -orbits in \( Y \) contained in the same \( G \) -orbit, and let \( y_+ \in O_+ \) an \( y_- \in O_- \) be arbitrary. Then
\[
(4.8) \quad \delta_{O_+,O_-} = \dim \left(p_+ \left((m_+^+ \oplus m_-^+) \cap l_{y_+,y_-}\right)\right).
\]

Proof. Let \( y_0 \) be any point in the unique \( G \) -orbit \( O \) containing both \( O_+ \) and \( O_- \). Then the vector space isomorphism \( I_{y_0,y_+} : q_{y_+}/q_{y_+}^+ \rightarrow q_{y_0}/q_{y_0}^+ \) induces an isomorphism
\[
p_+ \left((m_+^+ \oplus m_-^+) \cap l_{y_+,y_-}\right) \rightarrow I_{y_0,y_+}(a_{y_+}^+) \cap I_{y_0,y_-}(a_{y_-}^-).
\]
Consequently, \( \delta_{O_+,O_-} = \dim(I_{y_0,y_+}(a_{y_+}^+) \cap I_{y_0,y_-}(a_{y_-}^-)) \) is also given by \((4.8)\).
We now turn to the ranks of the Poisson structure $\pi = -\lambda(r)$ in $Y$. Recall that $\pi$ is said to be regular on a Poisson submanifold $X$ of $(Y, \pi)$ if it has constant rank on $X$.

Recall our assumptions that $f_+ \subset m_+$ and $f_- \subset m_-$ in (4.11), which are equivalent to $l_r \subset m_+ \oplus m_-$, which is in turn equivalent to $t_{\text{diag}} + l_r = m_+ \oplus m_-$, where $t = m_+ \cap m_-$ and $l_r$ is the Lagrangian subalgebra of $g \oplus g$ given by (3.11) (see Remark 3.13). Let $n_{\oplus g}(l_r) \subset g \oplus g$ be the normalizer of $l_r$ in $g \oplus g$. Recall the co-bracket $\delta : g \to \wedge^2 g$ on $g$ defined in (3.11).

**Lemma 4.11.** Under the assumption that $l_r \subset m_+ \oplus m_-$, the following are equivalent:

(4.9) \[ m_+ \oplus m_- \subset n_{\oplus g}(l_r), \]

(4.10) \[ [m_+, f_+] \subset f_+^1, \quad [m_-, f_-] \subset f_-^1, \]

(4.11) \[ \delta_r(x) = 0, \quad \forall x \in \mathfrak{t}. \]

**Proof.** It is clear that (4.9) is equivalent to

$$ m_+ \oplus 0 \subset n_{\oplus g}(l_r) \quad \text{and} \quad 0 \oplus m_- \subset n_{\oplus g}(l_r). $$

By definition, $m_+ \oplus 0 \subset n_{\oplus g}(l_r)$ if and only if for any $x_+ \in m_+$ and $x \in g$, there exists $x' \in g$ such that $[(x_+, 0), (r_{\mathfrak{t}}^e(x), r_{\mathfrak{t}}^e(x'))] = (r_{\mathfrak{t}}^e(x), r_{\mathfrak{t}}^e(x'))$, which is equivalent to $[x_+, r_{\mathfrak{t}}^e(x)] = r_{\mathfrak{t}}^e(x')$ and $r_{\mathfrak{t}}^e(x') = 0$, which, in turn, are equivalent to $[x_+, r_{\mathfrak{t}}^e(x)] = x'$ and $x' \in f_+^1$. Thus $m_+ \oplus 0 \subset n_{\oplus g}(l_r)$ if and only if $[m_+, f_+] \subset f_+^1$. Similarly, $m_- \oplus 0 \subset n_{\oplus g}(l_r)$ if and only if $[m_-, f_-] \subset f_-^1$. This shows that (4.9) is equivalent to (4.10).

As $t_{\text{diag}} + l_r = m_+ \oplus m_-$, it is also clear that (4.10) is equivalent to $t_{\text{diag}} \subset n_{\oplus g}(l_r)$. Using the identification of the quadratic Lie algebra $(g \oplus g, \langle , \rangle_{g \oplus g})$ with the Drinfeld double of the Lie bialgebra $(g, \delta_r)$, one sees that $t_{\text{diag}} \subset n_{\oplus g}(l_r)$ if and only if $\text{ad}^*_x x = 0$ for all $x \in \mathfrak{g}^*$ and $x \in \mathfrak{t}$, where $\text{ad}^*_{\xi}$ is the co-adjoint action of $\xi \in \mathfrak{g}^*$ on $g$, and $\mathfrak{g}^*$ has the Lie bracket defined by the dual map of $\delta_r$. It follows that $t_{\text{diag}} \subset n_{\oplus g}(l_r)$ if and only if $\delta_r(x) = 0$ for all $x \in \mathfrak{t}$. Thus (4.9) is equivalent to (4.11).

**Q.E.D.**

**Proposition 4.12.** Let $(G, r, Y, \lambda)$ be strongly admissible and assume furthermore that

(4.12) \[ l_r \subset m_+ \oplus m_- \subset n_{\oplus g}(l_r). \]

Then $T$ acts on $(Y, \lambda(r))$ by Poisson isomorphisms, and the Poisson structure $\lambda(r)$ is regular on every non-empty intersection $O_+ \cap O_- \cap \mathcal{O}_+$ of an $M_+$-orbit $\mathcal{O}_+$ and an $M_-$-orbit $\mathcal{O}_-$ in $Y$.

**Proof.** Fix a pair $(O_+, O_-)$ of $M_+$- and $M_-$-orbits in $Y$ such that $O_+ \cap O_- \neq \emptyset$, and let $\mathcal{O}$ be the unique $G$-orbit in $Y$ containing $O_+ \cap O_-$. Let $y_1, y_2 \in O_+ \cap O_-$, and let $m_+ \in M_+$ and $m_- \in M_-$ be such that $y_2 = m_+ y_1 = m_- y_1$. By Remark 3.17 and (3.20) and the fact that $(\text{Ad}_{m_+} \oplus \text{Ad}_{m_-})(l_r) = l_r$,  

$$ \dim \mathcal{O} - \dim(\text{Im}(\pi_{y_2}^m)) = \dim(l_r \cap l_{y_2, y_2}) = \dim(\mathfrak{t}_r \cap (\text{Ad}_{m_+} \oplus \text{Ad}_{m_-})(l_{y_1, y_1})) = \dim(l_r \cap l_{y_1, y_1}) = \dim \mathcal{O} - \dim(\text{Im}(\pi_{y_1}^m)). $$

This proves Proposition 4.12

**Q.E.D.**

**Example 4.13.** When $r = r_{(u, u')}$, the $r$-matrix on a quadratic Lie algebra $(g, \langle , \rangle_g)$ defined by a Lagrangian splitting $g = u + u'$ (see Example 3.2), one can take $m_- = u$ and $m_+ = u'$, so that $l_r = m_+ \oplus m_-$. Proposition 4.12 in this case was proved in [39, Theorem 2.7]. In the case of $m_- = n_g(u)$...
and $m_+ = n(g(u'),$ the normalizer subalgebras in $g$ of $u$ and $u'$ respectively, one has $m_+ \oplus m_- = n_{g \oplus g}(l_r)$, and Proposition 4.12 in this case was proved in [39 Proposition 2.13].

4.3. T-leaf stabilizers. Let again $G$ be a connected Lie group with Lie algebra $g$, and let $r$ be a factorizable quasitriangular $r$-matrix on $g$. Recall the symmetric bilinear form $(, )_g$ on $g$ associated to $r$ and the two Lie subalgebras $f_+$ and $f_-$ of $g$.

Definition 4.14. A pair $(M_+, M_-)$ of Lie subgroups of $G$ is said to be $r$-admissible if they are connected and their respective Lie algebras $m_+$ and $m_-$ satisfy

$$f_+ \subset m_+, \quad f_- \subset m_-, \quad [m_+, m_+] \subset m_+^\perp, \quad [m_-, m_-] \subset m_-^\perp.$$ (4.13)

Let $(M_+, M_-)$ be a pair of $r$-admissible Lie subgroups of $G$ with respective Lie algebras $m_+$ and $m_-$. Recall that $T$ is the connected component of $M_+ \cap M_-$ containing the identity element and $t = m_+ \cap m_-$ is the Lie algebra of $T$. Since $m_+^\perp \cap m_-^\perp = (m_+ + m_-)^\perp = g^\perp = 0$, (4.13) implies that $t = m_+ \cap m_-$ is abelian, and thus $T$ is also abelian. By Lemma 4.11, (4.12) implies (4.13). If $(G, r, Y, \lambda)$ is a strongly admissible quadruple, then Proposition 4.12 applies and $T$ acts on $(Y, \lambda(r))$ by Poisson isomorphisms. Thus $(Y, \pi = -\lambda(r), \lambda)$ is a $T$-Poisson manifold. Recall from Definition 2.4 that for each $y \in Y$, one has the leaf stabilizer

$$t_y = \{ x \in t : \lambda_y(x) \in \text{Im}(\pi_y^\#) \} \subset t.$$ By Lemma 2.3, $t_{y_1} = t_{y_2}$ for any $y_1, y_2$ on the same $T$-leaf of $\pi$ in $Y$. To compute $t_y$ for $y \in Y$, consider the projection

$$p_t : m_+ \oplus m_- = t_{\text{diag}} + l_r \rightarrow t, \quad (x, x) + (r_+^\#(x'), r_-^\#(x')) \mapsto x, \quad x \in t, x' \in g.$$ (4.14)

Note that $p_t$ is the restriction to $m_+ \oplus m_-$ of the projection

$$p_g : g \oplus g = g_{\text{diag}} + l_r \rightarrow g, \quad (x, x) + (r_+^\#(x'), r_-^\#(x')) \mapsto x, \quad x \in g, x' \in g.$$ Using the fact that $r_+ - r_- = \text{Id}_g$, one sees that $p_g : g \oplus g \rightarrow g$ is also given by

$$p_g(x_1, x_2) = -r_+^\#(x_1) + r_-^\#(x_2), \quad x_1, x_2 \in g.$$ (4.15)

Let $(O_+, O_-)$ be any pair of $M_+$- and $M_-$-orbits contained in the same $G$-orbit in $Y$, possibly $O_+ \cap O_- = \emptyset$. Let $y_+ \in O_+$ and $y_- \in O_-$ be arbitrary, and define

$$t_{O_+, O_-} = p_t((m_+ \oplus m_-) \cap l_{y_+, y_-}),$$ (4.16)

where $l_{y_+, y_-}$ is the Lagrangian subalgebra of $(g \oplus g, (, )_{g \oplus g})$ defined in [3, 24].

Proposition 4.15. Assume that $(G, r, Y, \lambda)$ is strongly admissible and that $(M_+, M_-)$ is an $r$-admissible pair of Lie subgroups of $G$. Then for any pair $(O_+, O_-)$ of $M_+$- and $M_-$-orbits contained in the same $G$-orbit in $Y$, $t_{O_+, O_-} \subset t$ is independent of the choices of $y_+ \in O_+$ and $y_- \in O_-$. Moreover, when $O_+ \cap O_- \neq \emptyset$, one has $t_{O_+, O_-} = t_y$ for any $y \in O_+ \cap O_-$. 

Proof. Let $y'_+ = m_+ y_+ \in O_+$ and $y'_- = m_- y_- \in O_-$, where $(m_+, m_-) \in M_+ \times M_-$. Then

$$(m_+ \oplus m_-) \cap l_{y'_+, y'_-} = (m_+ \oplus m_-) \cap \text{Ad}(m_+ \oplus m_-)(y_+, y_-) = \text{Ad}(m_+ \oplus m_-) (m_+ \oplus m_-) \cap l_{y_+, y_-}).$$

Let $x \in t$. Then $x \in p_t((m_+ \oplus m_-) \cap l_{y'_+, y'_-})$ if and only if $\text{Ad}^{-1}(m_+, m_-)(x, x) \in l_r + l_{y_+, y_-}$. Writing

$$\text{Ad}^{-1}(m_+, m_-)(x, x) = (x, x) + \left( \text{Ad}_{m_+}(x) - x, \text{Ad}_{m_-}(x) - x \right)$$

and noting that (4.12) implies (4.13) implies

$$\text{Ad}_{m_+}(x) - x, \text{Ad}_{m_-}(x) - x \in m_+^\perp \oplus m_-^\perp \subset l_r,$$ one has

$$\text{Ad}_{m_+}(x) - x, x \in l_r + l_{y_+, y_-} \text{ iff } (x, x) \in l_r + l_{y_+, y_-}.$$
Lemma 4.17. Let \((y_+, y_-)\) be homogeneous admissible six-tuples. Thus \(t_{O_+, O_-}\) given in (4.16) is independent of the choice of \(y_+ \in O_+\) and \(y_- \in O_-\). Assume now that \(O_+ \cap O_- \neq \emptyset\) and let \(y_0 \in O_+ \cap O_-\). Let \(x \in t\). By (ii) of Lemma 3.9 \(x \in t_y\) if and only if \((x, x) \in t_{\text{diag}} \cap (t_y + l_{y,y})\), which is equivalent to \(x \in p_t((m_+ + m_-) \cap l_{y,y}) \in t_{O_+, O_-}\). Thus \(t_y = t_{O_+, O_-}\).

Q.E.D.

4.4. Proof of Theorem 1.5. Theorem 1.5 which is a summary of the main results of [1] now follows directly from Proposition 4.6 Proposition 4.15 and Remark 2.7.

For the convenience of the reader, we repeat the statements of Theorem 1.5 here: Let \((G, r, Y, \lambda)\) be a strongly admissible quadruple and let \((M_+, M_-)\) be an \(r\)-admissible pair of Lie subgroups of \(G\) with respective Lie algebras \(m_+\) and \(m_-\). For each pair \((O_+, O_-)\) of \(M_+\) and \(M_-\)-orbits contained in the same \(G\)-orbit in \(Y\), let \(\delta_{O_+, O_-}\) be the integer defined in (4.8) and let the subspace \(t_{O_+, O_-}\) of \(t = m_+ \cap m_-\) be defined in (4.16). Let \(T\) be the connected component of \(M_+ \cap M_-\) containing the identity element. If \(\delta_{O_+, O_-} = 0\) for every pair \((O_+, O_-)\) of \(M_+\) and \(M_-\)-orbits contained in the same \(G\)-orbit in \(Y\), then the six-tuple \((G, r, Y, \lambda, M_+, M_-)\) is admissible, i.e., the \(T\)-leaves of \(\lambda(r)\) in \(Y\) are precisely the connected components of the non-empty intersections \(O_+ \cap O_-\) of \(M_+\) and \(M_-\)-orbits in \(Y\); Moreover, the leaf stabilizer of each \(T\)-leaf in \(O_+ \cap O_-\) is \(t_{O_+, O_-}\), and the co-rank of the Poisson structure \(\pi = -\lambda(r)\) in \(O_+ \cap O_-\) is equal to the co-dimension of \(t_{O_+, O_-}\) in \(t\).

4.5. Homogeneous admissible six-tuples. Consider a homogeneous strongly admissible quadruple \((G, r, G/Q, \lambda_{G/Q})\) (see Example 3.12), where \(Q\) is a closed and connected Lie subgroup of \(G\) whose Lie algebra \(q\) satisfies

\[ [q, q] \subset q^\perp \subset q.\]

Here recall that \(q^\perp\) is defined in (3.16) using the symmetric non-degenerate bilinear form \(\langle \cdot, \cdot \rangle_g\) on \(g\) associated to \(r\). We first state a consequence of the assumption that \(q^\perp \subset q\).

Lemma-Notation 4.16. For any subspace \(c\) of \(q\) such that \(q = c + q^\perp\) is a direct sum decomposition, the restriction of \(\langle \cdot, \cdot \rangle_g\) on \(c\) is non-degenerate, and \(g = c + c^\perp\) is a direct sum decomposition. Let \(p_c : g \to c\) be the projection with respect to the decomposition \(g = c + c^\perp\). Then \(p_c \circ \text{Ad}_g = \text{Ad}_c \circ p_c\) : \(g \to c\), where \(g\) is any element in the normalizer \(N_G(c)\) of \(c\) in \(G\);

Proof. If \(x \in c \cap c^\perp\), then \(x \in c \cap q^\perp\), so \(x = 0\). The second statement follows from the fact that \(\text{Ad}_g c^\perp = c^\perp\) for every \(g \in N_G(c)\).

Q.E.D.

Lemma 4.17. Let \(q = c + q^\perp\) be a direct sum decomposition. If \(y_+, y_- \in G/Q\) are of the form \(y_+ = g_+ Q\) and \(y_- = g_- Q\), where \(g_+, g_- \in N_G(c)\), then the Lagrangian subalgebra \(I_{y_+, y_-}\) of \((g \oplus g, \langle \cdot, \cdot \rangle_g \oplus \langle \cdot, \cdot \rangle_g)\) defined in (3.24) is given by

\[ I_{y_+, y_-} = \{(x_+, x_-) \in \text{Ad}_{g_+} q \oplus \text{Ad}_{g_-} q : p_c(x_+) = \text{Ad}_{g_+} c \oplus \text{Ad}_{g_-} c; p_c(x_-) = \text{Ad}_{g_+} c \oplus \text{Ad}_{g_-} c\}. \]

Proof. Lemma 4.17 follows directly from the definition of \(I_{y_+, y_-}\) and the decompositions

\[ q_{y_+} = \text{Ad}_{g_+} q = c + \text{Ad}_{g_-} q^\perp \quad \text{and} \quad q_{y_-} = \text{Ad}_{g_-} q = c + \text{Ad}_{g_-} q^\perp. \]

Q.E.D.
Let again \((M_+, M_-)\) be a pair of \(r\)-admissible Lie subgroups of \(G\) with respective Lie algebras \(m_+\) and \(m_-\). If every \(M_+\)-orbit \(O_+\) and every \(M_-\)-orbit \(O_-\) in \(G/Q\) contain elements of the form \(gQ\) with \(g \in N_G(e)\), one can use (4.17) to compute the integers \(\delta_{O_+, O_-}\) and the subspaces \(t_{O_+, O_-}\) of \(t\), as in the following Proposition 4.18.

**Proposition 4.18.** Suppose that \(q = \mathfrak{e} + q^+\) is a direct decomposition, and assume that all \((M_+, Q)\) and \((M_-, Q)\)-double cosets in \(G\) contain elements in \(N_G(q)\). Let \((O_+, O_-)\) be any pair of \(M_+\) and \(M_-\)-orbits in \(G/Q\), and choose any \(g_+, g_- \in N_G(q)\) such that \(g_+Q \in O_+\) and \(g_-Q \in O_-\). Then

\[
\delta_{O_+, O_-} = \dim \left( p_t \left( m_+^+ \cap \text{Ad}_{g_+} q \right) \cap \text{Ad}_{g_+ g_-}^{-1} \left( p_t \left( m_-^- \cap \text{Ad}_{g_-} q \right) \right) \right),
\]

\[
t_{O_+, O_-} = p_t(V_{g_+, g_-}),
\]

where \(V_{g_+, g_-} = \left\{ (x_+, x_-) \in (m_+ \cap \text{Ad}_{g_+} q) \oplus (m_- \cap \text{Ad}_{g_-} q) : p_t(x_+) = \text{Ad}_{g_+ g_-}^{-1} p_t(x_-) \right\}\). In particular, the six-tuple \((G, r, G/Q, \lambda_{G/Q}, M_+, M_-)\) is admissible if

\[
p_t \left( m_+^+ \cap \text{Ad} q \right) \cap \text{Ad}_h \left( p_t \left( m_-^- \cap \text{Ad} q \right) \right) = 0, \quad \forall g, h, k \in N_G(q).
\]

**Proof.** Proposition 4.18 follows directly from (4.18) for \(\delta_{O_+, O_-}\) and (4.16) for \(t_{O_+, O_-}\).

Q.E.D.

**Remark 4.19.** Note that if \(q\) is Lagrangian with respect to \(\langle , \rangle_q\), by taking \(e = 0\), the six-tuple \((G, r, G/Q, \lambda_{G/Q}, M_+, M_-)\) is automatically admissible, and the leaf stabilizer \(t_{O_+, O_-}\) is given by

\[
t_{O_+, O_-} = p_t \left( (m_+^+ \cap \text{Ad}_{g_+} q) \oplus (m_-^- \cap \text{Ad}_{g_-} q) \right)
\]

for any \(g_+Q \in O_+\) and \(g_-Q \in O_-\).

**Example 4.20.** Let \((G, \pi_G)\) be a connected Poisson Lie group, where \(\pi_G = r^L - r^R\) for a factorizable quasitriangular \(r\)-matrix \(r\) on the Lie algebra \(g\) of \(G\). Equip the direct product Lie algebra \(g \oplus g\) with the direct product factorizable quasitriangular \(r\)-matrix \((r, -r)\), and let \(\lambda\) be the left action of \(G \times G\) on \(G\) given by

\[
(g_1, g_2) \cdot g = g_1 g g_2^{-1}, \quad g_1, g_2, g \in G.
\]

As the stabilizer subgroup of \(\lambda\) at \(g \in G\) is \(\left\{ (g_1, g_1^{-1} g) : g_1 \in G \right\}\), which is connected and its Lie algebra Lagrangian with respect to the the symmetric bilinear form on \(g \oplus g\) associated to \((r, -r)\), the quadruple \((G \times G, (r, -r), G, \lambda)\) is strongly admissible. It is easy to see that \(-\lambda(r, -r) = \pi_G\). If \((M_+, M_-)\) is a pair of \(r\)-admissible Lie subgroups of \(G\), then \((M_+ \times M_+, M_- \times M_-)\) is a pair of \((r, -r)\)-admissible Lie subgroups of \(G \times G\). By Remark 4.19, the six-tuple \((G \times G, (r, -r), G, \lambda, M_+ \times M_+, M_- \times M_-)\) is admissible, and hence the \((T, T)\)-leaves of \((G, \pi_G)\) are precisely the connected components of non-empty intersections of \((M_+, M_+)-double\ cosets and \((M_-, M_-)-double\ cosets in \(G\). As a special case, if \(r\) is defined by a Lagrangian splitting \(g = u + u'\) by taking \(M_+ = U'\) and \(M_- = U\), where \(U\) and \(U\) are the connected Lie subgroups of \(G\) with Lie algebras \(u\) and \(u'\) respectively, the symplectic leaves of \((G, \pi_G)\) are precisely the connected components of \((U, U')\)-double cosets and \((U', U')\)-double cosets in \(G\) (see, for example, [46]).

5. Mixed product Poisson structures associated to admissible quadruples

5.1. The construction. Let \(r\) be a quasitriangular \(r\)-matrix on a Lie algebra \(g\), and let \(n \geq 1\) be any integer. Writing \(r = \sum x_i \otimes y_i \in g \otimes g\), we introduced in [38] a quasitriangular \(r\)-matrix \(r^{(n)}\) on the
direct product Lie algebra $g^n = g \oplus \cdots \oplus g$ $(n$-copies), which is given by

\begin{equation}
(5.1) \quad r^{(n)} = \sum_{1 \leq j \leq n, \ j \text{ is odd}} (r)_j + \sum_{1 \leq j \leq n, \ j \text{ is even}} (-r^{2j})_j - \sum_{1 \leq j < k \leq n} \sum_i (y_i)_j \wedge (x_i)_k,
\end{equation}

where for $X \in \mathfrak{g}^\otimes l = l = 1, 2$, and $1 \leq j \leq n$, $(X)_j \in (\mathfrak{g}^n)^\otimes l$ is the image of $X$ under the embedding of $\mathfrak{g}$ into $\mathfrak{g}^n$ as the $j$'th summand. Since the symmetric part of $r^{(n)}$ is $(s, -s, s, -s, \ldots)$, where $s$ is the symmetric part of $r$, if $r$ is factorizable, so is $r^{(n)}$.

Assume that $(G, r, Y, \lambda_i)$ is an admissible quadruple for each $1 \leq i \leq n$, and consider the product manifold $Y = Y_1 \times \cdots \times Y_n$ and the direct product action $\lambda$ of $\mathfrak{g}^n$ on $Y$, i.e.,

\[ \lambda : \mathfrak{g}^n \to \mathcal{V}^1(Y), \quad \lambda(x_1, x_2, \ldots, x_n) = (\lambda_1(x_1), \lambda_2(x_2), \ldots, \lambda_n(x_n)), \quad x_j \in \mathfrak{g}. \]

Then the quadruple $(G^n, r^{(n)}, Y, \lambda)$ is admissible. Moreover, when each $(G, r, Y, \lambda_i)$ is strongly admissible, so is $(G^n, r^{(n)}, Y, \lambda)$. Being the sum of the direct product Poisson structure $-\lambda(r^{(n)})$ on $Y$ and some “mixed terms”, the Poisson structure $-\lambda(r^{(n)})$ on $Y$ is an example of a mixed product Poisson structure on the product manifold $Y$ (see [38]).

In [35] we apply the general theory in [34] to the admissible quadruples $(G^n, r^{(n)}, Y, \lambda)$. We first establish a property of the quasitriangular $r$-matrix $r^{(n)}$ on $\mathfrak{g}^n$. For notational simplicity, set

\begin{equation}
(5.2) \quad \tilde{r} = r^{(n)} \in \mathfrak{g}^n \otimes \mathfrak{g}^n, \quad \tilde{f}_{\pm} = \text{Im}(\tilde{r}_{\pm}) \subset \mathfrak{g}^n.
\end{equation}

Recall that when $r$ is factorizable, one has the Lie subalgebra $L_r$ of $\mathfrak{g} \oplus \mathfrak{g}$ given by

\[ L_r = \{(r_+ (\xi), r_- (\xi)) : \xi \in \mathfrak{g}^*\} = \{(r^+_\xi(x), r^-_\xi(x)) : x \in \mathfrak{g}\} \subset \mathfrak{g} \oplus \mathfrak{g}, \]

and the Lie subalgebras $f_{\pm} = \text{Im}(r^\pm_\xi)$ of $\mathfrak{g}$. Let $\tau$ be the automorphism of $\mathfrak{g}^n$ defined by

\[ \tau(x_1, x_2, x_3, \ldots, x_{n-1}, x_n) = (x_1, x_3, \ldots, x_{n-1}, x_n, x_2), \quad x_j \in \mathfrak{g}. \]

**Lemma 5.1.** Assume that $r$ is factorizable. Then

\begin{align*}
\tilde{f}_+ &= \tau(L_r \oplus \mathfrak{g}_{\text{diag}} \oplus \cdots \oplus \mathfrak{g}_{\text{diag}}), & \tilde{f}_- &= \mathfrak{g}_{\text{diag}} \oplus \cdots \oplus \mathfrak{g}_{\text{diag}}, & \text{if } n = 2m \text{ is even}, \\
\tilde{f}_+ &= \tilde{f}_+ \oplus \mathfrak{g}_{\text{diag}} \oplus \cdots \oplus \mathfrak{g}_{\text{diag}}, & \tilde{f}_- &= \mathfrak{g}_{\text{diag}} \oplus \cdots \oplus \mathfrak{g}_{\text{diag}} \oplus \tilde{f}_-, & \text{if } n = 2m + 1 \text{ is odd}.
\end{align*}

**Proof.** Assume first that $n = 2m$ is even, and, for notational simplicity, set

\[ \tilde{f}_+ = \tau(L_r \oplus \mathfrak{g}_{\text{diag}} \oplus \cdots \oplus \mathfrak{g}_{\text{diag}}), \quad \tilde{f}_- = \mathfrak{g}_{\text{diag}} \oplus \cdots \oplus \mathfrak{g}_{\text{diag}}. \]

Let $\tilde{\xi} = (\xi_1, \xi_2, \ldots, \xi_n) \in (\mathfrak{g}^*)^n \cong (\mathfrak{g}^n)^*$, and write

\begin{equation}
(5.3) \quad \tilde{r}_+ (\tilde{\xi}) = (x_1, x_2, \ldots, x_n), \quad \tilde{r}_- (\tilde{\xi}) = (y_1, y_2, \ldots, y_n).
\end{equation}

By the definition of $\tilde{r}$, one has, for $1 \leq j \leq n$,

\begin{align*}
x_j &= \begin{cases} r_- (\xi_1 + \cdots + \xi_{j-1}) + r_+ (\xi_j + \xi_{j+1} + \cdots + \xi_n), & \text{if } j \text{ odd,} \\
r_- (\xi_1 + \cdots + \xi_{j-1} + \xi_j) + r_+ (\xi_{j+1} + \cdots + \xi_n), & \text{if } j \text{ even,}
\end{cases} \\
y_j &= \begin{cases} r_- (\xi_1 + \cdots + \xi_{j-1} + \xi_j) + r_+ (\xi_{j+1} + \cdots + \xi_n), & \text{if } j \text{ odd,} \\
r_- (\xi_1 + \cdots + \xi_{j-1} + \xi_j) + r_+ (\xi_{j+1} + \cdots + \xi_n), & \text{if } j \text{ even.}
\end{cases}
\end{align*}

As $x_1 = r_+ (\xi_1 + \cdots + \xi_n)$, $x_n = r_- (\xi_1 + \cdots + \xi_n)$, and for $1 \leq k \leq m - 1$,

\[ x_{2k} = x_{2k+1} = r_- (\xi_1 + \cdots + \xi_{2k}) + r_+ (\xi_{2k+1} + \cdots + \xi_n), \]

\[ = r_+ (\xi_1 + \cdots + \xi_n) + (r_- - r_+)(\xi_1 + \cdots + \xi_{2k}), \]
one has \( \tilde{r}_+ (\xi) \in \tilde{T}_+ \). Moreover, since \( r_- \rightarrow r_+ : \mathfrak{g}^* \to \mathfrak{g} \) is invertible, \( \tilde{r}_+ (\xi) = 0 \) if and only if \( \xi_{2k-1} + \xi_{2k} = 0 \) for every \( 1 \leq k \leq m \). Thus \( \dim \ker \tilde{r}_+ = m (\dim \mathfrak{g}) \). It follows that \( \dim (\text{Im}(\tilde{r}_+)) = \dim (\tilde{T}_+) \) and hence \( \tilde{T}_+ = \tilde{T}_+ \). Similarly, since
\[
y_{2k-1} = y_{2k} = r_- (\xi_1 + \cdots + \xi_{2k-1}) + r_+ (\xi_{2k} + \cdots + \xi_n) = (r_- - r_+) (\xi_1 + \cdots + \xi_{2k-1}) + r_+ (\xi_1 + \cdots + \xi_n)
\]
for every \( 1 \leq k \leq m \), \( \tilde{r}_- (\xi) \in \tilde{T}_- \). Moreover, \( \tilde{r}_- (\xi) = 0 \) if and only if \( y_j = 0 \) and \( y_j - y_{j+1} = 0 \) for every \( 1 \leq j \leq n - 1 \), which is equivalent to \( r_- (\xi_1) + r_+ (\xi_n) = 0 \) and \( \xi_{2k} + \xi_{2k+1} = 0 \) for every \( 1 \leq k \leq m - 1 \). Since the map \((\mathfrak{g}^*)^2 \to \mathfrak{g} \), \( (\xi, \eta) \mapsto r_- (\xi) + r_+ (\eta) \) is surjective, one sees that \( \dim (\text{Im}(\tilde{r}_-)) = \dim (\tilde{T}_-) \) and hence \( \tilde{T}_- = \tilde{T}_- \).

Q.E.D.

Assume again that \( r \) is factorizable, and let \((M_+, M_-)\) be an \( r \)-admissible pair of Lie subgroups of \( G \) with respective Lie algebras \( \mathfrak{m}_+ \) and \( \mathfrak{m}_- \). Let \( M_+^{(n)}, M_-^{(n)} \subset G^n \) be given by
\[
M_+^{(n)} = M_+ \times G_{\text{diag}} \times \cdots \times G_{\text{diag}} \times M_-, \quad M_-^{(n)} = G_{\text{diag}} \times \cdots \times G_{\text{diag}} \times M_-,
\]
if \( n = 2m \) is even, and
\[
M_+^{(n)} = M_+ \times G_{\text{diag}} \times \cdots \times G_{\text{diag}} \times M_-, \quad M_-^{(n)} = G_{\text{diag}} \times \cdots \times G_{\text{diag}} \times M_-,
\]
if \( n = 2m + 1 \) is odd. Then their Lie algebras \( \mathfrak{m}_+^{(n)} \) and \( \mathfrak{m}_-^{(n)} \) are respectively given by
\[
\mathfrak{m}_+^{(n)} = \mathfrak{m}_+ \oplus \mathfrak{g}_{\text{diag}} \oplus \cdots \oplus \mathfrak{g}_{\text{diag}} \oplus \mathfrak{m}_-,
\]
if \( n = 2m \) is even, and
\[
\mathfrak{m}_+^{(n)} = \mathfrak{m}_+ \oplus \mathfrak{g}_{\text{diag}} \oplus \cdots \oplus \mathfrak{g}_{\text{diag}} \oplus \mathfrak{m}_-,
\]
if \( n = 2m + 1 \) is odd.

It is clear from Lemma 5.1 that the pair \((M_+^{(n)}, M_-^{(n)})\) of subgroups of \( G^n \) is \( r^{(n)} \)-admissible. Let again \( T \) be the connected component of \( M_+ \cap M_- \) containing the identity element of \( G \). Then \( T^{(n)} := \{ (g, \ldots, g) : g \in T \} \) is the connected component of \( M_+^{(n)} \cap M_-^{(n)} \) containing the identity element of \( G^n \).

5.2. A homogeneous case. As in \([4, 5]\) consider a six-tuple \((G, r, G/Q, \lambda_{G/Q}, M_+, M_-)\), where \( G \) is a connected Lie group with Lie algebra \( \mathfrak{g} \), \( r \) a factorizable quasitriangular \( r \)-matrix on \( \mathfrak{g} \), \( Q \) a closed and connected Lie subgroup of \( G \) whose Lie algebra \( \mathfrak{q} \) satisfies
\[
\mathfrak{q}, \mathfrak{q} \subset \mathfrak{q}^\perp \subset \mathfrak{q},
\]
and \((M_+, M_-)\) a pair of \( r \)-admissible Lie subgroups of \( G \) with respective Lie algebras \( \mathfrak{m}_+ \) and \( \mathfrak{m}_- \). In \([5, 2]\) we consider, for each integer \( n \geq 1 \), the six-tuple
\[
(G^n, r^{(n)}, (G/Q)^n, \lambda, M_+^{(n)}, M_-^{(n)}),
\]
where \( \lambda \) is the direct product of the action \( \lambda_{G/Q} \) of \( G \) on each factor of \((G/Q)^n\). Identify
\[
T \xrightarrow{\sim} T^{(n)}, \quad t \mapsto (t, t, \ldots, t), \quad t \in T.
\]
Then \( T \) acts on \((G/Q)^n\) diagonally. As the case of \( n = 1 \) is covered in Proposition 4.11, we assume that \( n \geq 2 \).
Assumption 5.2. There exists a direct sum decomposition \( q = c + q^\perp \) such that
1) every \((K, Q)\)-double cosets in \(G\), where \(K \in \{M_+, M_-, Q\}, \) contains elements in \(N_G(c)\);
2) \( p_c(\mathfrak{m}_+^\perp \cap \text{Ad}_g q) \cap \text{Ad}_h p_c(\mathfrak{m}_-^\perp \cap \text{Ad}_h q) = 0 \) for all \(g, h, k \in N_G(c)\),
where recall that \( p_c : g \mapsto c \) is the projection with respect to the decomposition \( g = c + c^\perp \).

Remark 5.3. Note that by taking \( c = 0 \), Assumption 5.2 holds automatically if \( q = q^\perp \).

Notation 5.4. For \( g = (g_1, \ldots, g_n) \in G^n \), let \( g = (g_1 Q, g_2 Q, \ldots, g_n Q) \in (G/Q)^n \). Let \( e \in G \) be the identity element \( G \). Let \((\tilde{O}_+, \tilde{O}_-)\) be any pair of \( M_+^{(n)} \) and \( M_-^{(n)} \)-orbits in \((G/Q)^n \). Assumption 5.2 implies that there exist \( g, h \in (N_G(c))^n \) of the form
\[
\begin{align*}
g &= \begin{cases} (g_1, e, g_3, e, g_5, \ldots, e, g_{2m-1}, g_{2m}), & \text{if } n = 2m \text{ is even} \\ (g_1, e, g_3, e, g_5, \ldots, e, g_{2m-1}, e, g_{2m+1}), & \text{if } n = 2m + 1 \text{ is even} \
\end{cases}, \\
h &= \begin{cases} (e, h_2, e, h_4, \ldots, e, h_{2m-2}, e, h_{2m}), & \text{if } n = 2m \text{ is even} \\ (e, h_2, h_4, \ldots, e, h_{2m-2}, e, h_{2m+1}), & \text{if } n = 2m + 1 \text{ is odd} \end{cases},
\end{align*}
\]
such that \( g \in \tilde{O}_+ \) and \( h \in \tilde{O}_- \). With \( g, h \in (N_G(c))^n \) so chosen, let
\[
\begin{align*}
g \triangleright h &= \begin{cases} (g_1, h_2, g_3, h_4, \ldots, g_{2m-1}, h_{2m}), & \text{if } n = 2m \text{ is even} \\ (g_1, h_2, g_3, h_4, \ldots, g_{2m-1}, h_{2m}) & \text{if } n = 2m + 1 \text{ is odd} \end{cases}, \\
\end{align*}
\]
and let \((g \triangleright h)_{n+1} = g_n\) if \( n \) is even and \((g \triangleright h)_{n+1} = h_n\) if \( n \) is odd. Let \( c = (c_1, c_2, \ldots, c_{n+1}) \), where \((c_1, c_2, \ldots, c_{n+1}) = g \triangleright h \) and \( c_{n+1} = (g \triangleright h)_{n+1} \), and let
\[
V_c = \left\{ (x_+, x_-) \in (\mathfrak{m}_+ \cap \text{Ad}_c q) \oplus (\mathfrak{m}_- \cap \text{Ad}_{c_{n+1}} q) : p_c(x_+) = \text{Ad}_{c_1 c_2 \cdots c_{n+1}} (p_c(x_-)) \right\}.
\]
Recall again that \( t = m_+ \cap m_- \), and one has the projection \( p_t : m_+ \oplus m_- \to t \cong t_{\text{diag}} \) with respect to the decomposition \( m_+ \oplus m_- = t_{\text{diag}} + t_r \).

Proposition 5.5. Under Assumption 5.5,
\(1\) the six-tuple \((G^n, \tau^{(n)}, (G/Q)^n, \lambda, M_+^{(n)}, M_-^{(n)})\) is admissible for every \( n \geq 2 \);
\(2\) for any pair \((\tilde{O}_+, \tilde{O}_-)\) of \( M_+^{(n)} \) and \( M_-^{(n)} \)-orbits in \((G/Q)^n \), the leaf stabilizer of every \( T \)-leaf in \( \tilde{O}_+ \cap \tilde{O}_- \) is given by \( t_{\tilde{O}_+, \tilde{O}_-} = p_t(V_c) \subset t \) with \( V_c \subset m_+ \oplus m_- \) given in (5.9).

Proof. Let \((\tilde{O}_+, \tilde{O}_-)\) be an arbitrary pair of \( M_+^{(n)} \) and \( M_-^{(n)} \)-orbits in \((G/Q)^n \), and let \( g, h \in (N_G(c))^n \) be as in (5.6) and (5.7) such that \( y_+ = g \in \tilde{O}_+ \) and \( y_- = h \in \tilde{O}_- \). We use the description in Lemma 4.11 of the Lagrangian subalgebra \( t_{y_+, y_-} \) of \( g^n \oplus g^n \) to show that \( \delta_{\tilde{O}_+, \tilde{O}_-} = 0 \) and to compute the subspace \( t_{\tilde{O}_+, \tilde{O}_-} \) of \( t = m_+ \cap m_- \). For notational simplicity, set \( \tilde{M}_\pm := M_\pm^{(n)} \) and let \( \tilde{m}_\pm \) be their respective Lie algebras.

Assume first that \( n = 2m \) is even. By Lemma 4.17 \((\tilde{m}_+ \oplus \tilde{m}_-^\perp) \cap t_{y_+, y_-}\) consists of elements \( a \in g^n \oplus g^n \) of the form
\[
a = (x_1 + x_1', x_2 + x_2', \ldots, x_n + x_n', z_1 + z_1', z_2 + z_2', \ldots, z_n + z_n'),
\]
where, by writing \( g = (g_1, g_2, \ldots, g_n) \) and \( h = (h_1, h_2, \ldots, h_n) \),
\[
x_j, z_j \in c, \quad x_j' \in \text{Ad}_{g_j} q^1, \quad z_j' \in \text{Ad}_{g_j} q^1, \quad x_j = \text{Ad}_{g_j h_j^{-1}} z_j, \quad j = 1, \ldots, n,
\]
x_1 + x_1' \in \mathfrak{m}_+^\perp \cap \text{Ad}_g q, \quad x_{2m} + x_{2m}' \in \mathfrak{m}_-^\perp \cap \text{Ad}_h q,
\]
and
\[
x_{2j} = x_{2j+1}, \quad x_{2j}' = x_{2j+1}', \quad j = 1, \ldots, m - 1,
\]
\[
z_{2j-1} = z_{2j}, \quad z_{2j-1}' = z_{2j}', \quad j = 1, \ldots, m.
\]
It follows that

\[ x_1 = \text{Ad}_{g_1, g_2, h_1, \ldots, g_{2m-1}, h_{2m}^{-1}} (x_{2m}) \in p_c (m^+ \cap \text{Ad}_{c_1}) \cap \text{Ad}_{c_2} \cdots c_{n+1}^{-1} p_c (m^+ \cap \text{Ad}_{c_{n+1}}). \]

By 2) of Assumption 5.2, \( x_1 = 0 \), and it follows that \( x_j = 0 \) for every \( j = 1, \ldots, n \). By (4.3), \( \delta_{\theta, \phi} = 0 \). The case of \( n = 2m + 1 \) is odd is proved similarly. This proves (1).

To prove (2), note that by Lemma \( 5.1 \)

\[ t_{\theta, \phi} = p_t \left( (p (m^+ \oplus \tilde{m}^+)) \cap t_+ \right), \]

where \( p : g^n \oplus g^n \to g \oplus g \) is given by

\[ p (a_1, a_2, \ldots, a_n, b_1, b_2, \ldots, b_n) = \begin{cases} (a_1, a_n), & \text{if } n \text{ is even}, \\ (a_1, b_n), & \text{if } n \text{ is odd}. \end{cases} \]

Replacing \( \tilde{m}^+ \oplus \tilde{m}^- \) in the proof of (1) by \( m^+ \oplus m^- \), one sees that

\[ p \left( (m^+ \oplus m^-) \cap t_+ \right) \subset V_c. \]

Assume again that \( n = 2m \) is even and let \( (x_+, x_-) \in V_c \). Then there is a unique element \( a \in (m^+ \oplus m^-) \cap t_+ \) of the form \( (5.10) \) with \( x_1 + x'_1 = x_+, x_n + x'_n = x_-, x'_j = 0 \) for \( j = 2, \ldots, n - 1 \), and \( z'_j = 0 \) for all \( j = 1, \ldots, n \). Moreover, \( p(a) = (x_+, x_-) \). This shows that \( p \left( (m^+ \oplus m^-) \cap t_+ \right) = V_c \) when \( n \) is even, and by Proposition \( 4.15 \) one has \( t_{\theta, \phi} = p_t (V_c) \subset t \). The case when \( n \) is odd is proved similarly. This proves (2).

Q.E.D.

5.3. Main examples. Consider again a homogeneous strongly admissible quadruple \((G, r, G/Q, \lambda_{G/Q})\), but where we assume that the Lie algebra \( q \) of \( Q \) satisfies

\[ (5.11) \quad f_+ \subset q \quad \text{and} \quad [q, q] \subset q^+. \]

Since \( q^+ \subset f^- \subset f_+ \subset q \), the quadruple \((G, r, G/Q, \lambda_{G/Q})\) is indeed strongly admissible. Moreover, by Remark 3.6, \( Q \) is a Poisson Lie subgroup of the Poisson Lie group \((G, \pi_G)\), where \( \pi_G = r^L - r^R \), and the Poisson structure \(-\lambda_{G/Q}(r)\) on \( G/Q \) coincides with the projection \( \pi_{G/Q} \) of \( \pi_G \) to \( G/Q \).

Let \( n \geq 1 \) be an integer, and let \( G^n \) acts on itself from the right by \((1.2)\). Then we have the two quotient manifolds

\[ Y_n = G \times_Q \cdots \times_Q G/Q \quad \text{and} \quad X_n = G \times_Q \cdots \times_Q G \]

of \( G^n \), each with the quotient Poisson structure, respectively denoted by \( \pi_{Y_n} \) and \( \pi_{X_n} \), which, by definition, are the projections of the direct product Poisson structure \( \pi_G^n \) on \( G^n \). Let \((M_+, M_-)\) be a pair of \( r \)-admissible Lie subgroups of \( G \), and let again \( T \) be the connected component of \( M_+ \cap M_- \) containing the identity element. Then \( T \) acts on \((Y_n, \pi_{Y_n})\) and \( T^2 = T \times T \) acts on \((X_n, \pi_{X_n})\) by Poisson isomorphisms via

\[ t \cdot [g_1, g_2, \ldots, g_n]_{Y_n} = [tg_1, g_2, \ldots, g_n]_{Y_n}, \quad t \in T, \quad g_j \in G, \]

\[ (t_1, t_2) \cdot [g_1, g_2, \ldots, g_n]_{X_n} = [t_1 g_1, t_2 g_2, \ldots, g_n t_2^{-1}]_{X_n}, \quad t_1, t_2 \in T, \quad g_j \in G. \]

In this section, we study the \( T \)-leaves of \((Y_n, \pi_{Y_n})\) and the \( T^2 \)-leaves of \((X_n, \pi_{X_n})\). As the case for \((Y_1, \pi_{Y_1})\) is covered in Proposition 4.15 and the case of \((X_1 = G, \pi_{X_1} = \pi_G)\) is covered in Example 4.20 we will assume that \( n \geq 2 \).

We first look at the case of \((Y_n, \pi_{Y_n})\). Consider the diffeomorphism

\[ (5.12) \quad J_{Y_n} : Y_n \longrightarrow (G/Q)^n, \quad [g_1, g_2, \ldots, g_n]_{Y_n} \longrightarrow (g_1 Q, g_1 g_2 Q, \ldots, g_1 g_2 \cdots g_n Q). \]
Let $\lambda : g^n \to V^1((G/Q)^n)$ be again the direct product of the action $\lambda_{G/Q}$ on each factor. We have the following crucial Proposition 5.6 from [38] §8.

**Proposition 5.6.** As Poisson structures on $(G/Q)^n$, one has $J_{Y_n}(\pi_{Y_n}) = -\lambda(r^{(n)})$.

We can thus apply Proposition 5.5 to the Poisson structure $\lambda(r^{(n)})$ on $(G/Q)^n$ and use $J_{Y_n}^{-1} : (G/Q)^n \to Y_n$, $(k_1Q, k_2Q, \ldots, k_nQ) \mapsto [k_1, k_1^{-1}k_2, k_2^{-1}k_3, \ldots, k_n^{-1}k_1]_{Y_n}$, to translate the results to the Poisson structure $\pi_{Y_n}$ on $Y_n$. For $a = (a_1, a_2, \ldots, a_n) \in G^n$, let

$$C(a) = (M_+a_1Q) \times_Q (Qa_2Q) \times_Q \cdots \times_Q (Qa_nQ)/Q \subset Y_n.$$ 

Let $\mu_{Y_n} : Y_n \to G/Q$ be given by $\mu_{Y_n}([a_1, a_2, \ldots, a_n]_{Y_n}) = a_1a_2\cdots a_nQ$.

**Lemma 5.7.** For any $g, h \in G^n$ of the form (5.6) and (5.7), one has

$$J_{Y_n}^{-1}\left(\left[M_+^{(n)}(g) \cap \left[M_-^{(n)}(h)\right]\right]\right) = C(g \bowtie h) \cap \mu_{Y_n}^{-1}(M_-g_nQ/Q).$$

**Proof.** Write $g = (g_1, g_2, \ldots, g_n)$ and $h = (h_1, h_2, \ldots, h_n)$, and consider first the case when $n = 2m$ is even. Let $k = (k_1, k_2, \ldots, k_n) \in G^n$. Then

1) $k \in M_+^{(n)}g$ if and only if $k_1 \in M_+g_1Q$ and $k_2^{-1}k_{2i+1} \in Qg_2^{-1}g_{2i+1}Q$ for $i = 1, \ldots, m-1$, and $k_{2m} \in M_-g_{2m}Q$, and

2) $k \in M_-^{(n)}g$ if and only if $k_{2i-1}^{-1}k_{2j} \in Qg_{2i-1}^{-1}g_{2j}Q$ for $j = 1, \ldots, m$.

It follows that $k \in \left(M_+^{(n)}g \cap \left[M_-^{(n)}h\right]\right)$ if and only if $J_{Y_n}(k) \in C(g \bowtie h) \cap \mu_{Y_n}^{-1}(M_-g_nQ/Q)$. The case when $n = 2m + 1$ is odd is proved similarly.

Q.E.D.

**Theorem 5.8.** Under Assumption 5.2 the following holds for every $n \geq 2$:

(a) Every non-empty intersection

$$(5.13) \quad Y_n(c) = ((M_+c_1Q) \times_Q (Qc_2Q) \times_Q \cdots \times_Q (Qc_nQ)/Q) \cap \mu_{Y_n}^{-1}(M_-c_{n+1}Q/Q) \subset Y_n,$$

where $c = (c_1, \ldots, c_{n+1}) \in (N_G(c))^{n+1}$, is transversal, and their connected components are precisely all the $T$-leaves of $\pi_{Y_n}$ in $Y_n$;

(b) The leaf stabilizer for every $T$-leaf in $Y_n(c)$ in (5.13) is $p_t(V_c)$, where

$$V_c = \{ (x_+, x_-) \in (m_+ \cap Ad_{c_1}q) \oplus (m_- \cap Ad_{c_{n+1}}q) : p_t(x_+) = Ad_{c_2\cdots c_n}c_{n+1}^{-1}(p_t(x_-)) \}.$$ 

Moreover, if $G$ is an affine algebraic group over $\mathbb{C}$ and $M_+, M_-$ and $Q$ are algebraic subgroups such that $M_+ \cap M_-$ is connected, then every non-empty intersection $Y_n(c)$ in (5.13) is irreducible and is thus a single $T$-leaf of $\pi_{Y_n}$ in $Y_n$.

**Proof.** Parts (a) and (b) follow immediately from Proposition 5.5 and Lemma 5.7 by relabeling $g \bowtie h$ by $(c_1, \ldots, c_n)$ and $(g \bowtie h)_{n+1}$ by $c_{n+1}$. The last part of Theorem 5.8 follows from Remark 4.2

Q.E.D.

We now turn to the case of $(X_n, \pi_{X_n})$, where $n \geq 2$. Let

$$\mu_{X_n} : X_n \to G, \quad [g_1, g_2, \ldots, g_n]_{X_n} \mapsto g_1g_2\cdots g_n, \quad g_j \in G.$$

**Assumption 5.9.** There exists a direct sum decomposition $g = c + q^\perp$ such that

1) every $(K, Q)$-double cosets in $G$, where $K \in \{M_+, Q\}$, contains elements in $N_G(c)$;

2) $p_t \left( m_+ \cap Ad_{c}q \right) \cap Ad_{c}p_k \left( m_- \cap Ad_{c}q \right) = 0$ for all $g, h, k \in N_G(c)$,

where again $p_t : g \to c$ is the projection with respect to the decomposition $g = c + c^\perp$. 

Theorem 5.10. Under Assumption 5.9, the following holds for every $n \geq 2$:

(a) Every non-empty intersection

$$X_n(c) = ((M+c_1Q) \times_Q (Q+c_2Q) \times_Q \cdots \times_Q (Q+c_nM)) \cap \mu_n^{-1}(M_{-c_{n+1}}M_-) \subset X_n,$$

where $c = (c_1, \ldots, c_{n+1})$ with $(c_1, \ldots, c_n) \in (N_G(e))^n$ and $c_{n+1} \in G$, is transversal, and their connected components are precisely all the $T^2$-leaves of $\pi_{X_n}$ in $X_n$:

(b) The leaf stabilizer for every $T^2$-leaf in $X_n(c)$ in (5.14) is $(p_t \oplus r_t)(V_c)$, where

$$V_c = \left\{ \left( x_+, x_-, z_+, \text{Ad}_{-1}(x_-) \right) : x_+ \in m_+ \cap \text{Ad}_{c_1}q, \ x_- \in m_- \cap \text{Ad}_{c_{n+1}}m_-, \ z_+ \in m_+ \cap \text{Ad}_{c_1}q, \ p_t(x_+) = \text{Ad}_{c_1 \cdots c_n}(p_t(z_+)) \right\}.$$

Moreover, if $G$ is an affine algebraic group over $\mathbb{C}$ and $M_+M_-$ and $Q$ are algebraic subgroups such that $M_+M_-$ is connected, then every non-empty intersection $X_n(c)$ in (5.14) is irreducible and is thus a single $T^2$-leaf of $\pi_{X_n}$ in $X_n$.

Proof. Consider the diffeomorphism $J_{X_n} : X_n \to (G/Q)^{n-1} \times G$ given by

$$J_{X_n}([g_1, g_2, \ldots, g_n]_{X_n}) = (g_1Q, g_1g_2Q, \ldots, g_1g_2 \cdots g_{n-1}Q, g_1g_2 \cdots g_n), \quad g_j \in G.$$ We will again study $\pi_{X_n}$ by studying the Poisson structure $J_{X_n}(\pi_{X_n})$ on $(G/Q)^{n-1} \times G$. Let $\lambda$ be the left action of $G^{n+1}$ on the product manifold $(G/Q)^{n-1} \times G$ by

$$(g_1, \ldots, g_n, g_{n+1}) \cdot (h_1Q, \ldots, h_{n-1}Q, h_n) = (g_1h_1Q, \ldots, g_{n-1}h_{n-1}Q, g_nh_n, g_n^{-1}),$$

where $g_j, h_k \in G$, and define the direct sum quasitriangular $r$-matrix $r^{(n+1)}$ on $g^{n+1}$ by

$$(r^{(n+1)}) = \begin{cases} (r^{(n)}, 0) + (0, -r), & \text{if } n = 2m + 1 \text{ is odd}, \\ (r^{(n)}, 0) + (0, r^{21}), & \text{if } n = 2m \text{ is even}. \end{cases}$$

Then the homogeneous quadruple $\left( (G^{n+1}, r^{(n+1)}), (G/Q)^{n-1} \times G, \lambda \right)$ is strongly admissible. It is proved in [35, §8] that, as Poisson structures on $(G/Q)^{n-1} \times G$,

$$J_{X_n}(\pi_{X_n}) = -\lambda \left( r^{(n+1)} \right).$$

Define the Lie subgroups $M_+^{(n+1)} \subset G^{n+1}$ and $M_-^{(n+1)} \subset G^{n+1}$ respectively by

$$M_+^{(n+1)} = M_+^{(n)} \times M_- \quad \text{and} \quad M_-^{(n+1)} = M_-^{(n)} \times M_+ \quad \text{if } n = 2m \text{ is even,}$$

$$M_+^{(n+1)} = M_+^{(n)} \times M_+ \quad \text{and} \quad M_-^{(n+1)} = M_-^{(n)} \times M_- \quad \text{if } n = 2m + 1 \text{ is odd.}$$

Then the pair $\left( M_+^{(n+1)}, M_-^{(n+1)} \right)$ is $r^{(n+1)}$-admissible, and one thus has the six-tuple

$$\left( (G^{n+1}, r^{(n+1)}), (G/Q)^{n-1} \times G, \lambda, M_+^{(n+1)}, M_-^{(n+1)} \right).$$

Same as in Theorem 5.5, we will apply Theorem 5.9 to the six-tuple in (5.17) and use the diffeomorphism $J_{X_n} : X_n \to (G/Q)^{n-1} \times G$ to prove Theorem 5.10. Note that by identifying

$$T^2 \xrightarrow{\sim} M_-^{(n+1)} \cap M_-^{(n+1)}, \quad (t_1, t_2) \mapsto (t_1, t_1, \ldots, t_1, t_2), \quad t_1, t_2 \in T$$

the diffeomorphism $J_{X_n} : X_n \to (G/Q)^{n-1} \times G$ is $T^2$-equivariant. For notational simplicity, set again $\tilde{M}_\pm = M_+^{(n+1)}$ and let $m_\pm$ be their respective Lie algebras.

Assume first that $n = 2m$ is even with $m \geq 1$. Let $(\tilde{O}_+, \tilde{O}_-)$ be an arbitrary pair of $\tilde{M}_+$ and $\tilde{M}_-$-orbits in $(G/Q)^{n-1} \times G$. By 1) of Assumption 5.9 there exist

$$g = (g_1, e, g_3, e, g_5, \ldots, e, g_{2m-1}, g_{2m}, e) \in (N_G(e))^{n-1} \times G \times G,$$

$$h = (e, h_2, e, h_4, e, \ldots, h_{2m-2}, e, h_{2m}, e) \in (N_G(e))^{n+1},$$
such that $y_+ := \mathfrak{g} \in \bar{\mathcal{O}}_+$ and $y_- := \mathfrak{h} \in \bar{\mathcal{O}}_-$, where $\mathfrak{g} = (a_1 Q, \ldots, a_n Q, a_0 a_n^{-1}) \in (G/Q)^{n-1} \times G$ for $a = (a_1, \ldots, a_{n+1}) \in G^{n+1}$. Let

$$c = (g_1, h_2, g_3, h_4, \ldots, g_{2m-1}, h_{2m}, g_{2m}) \in (N_G(c))^n \times G.$$

By Lemma 4.17, $(\mathfrak{m}^+ \oplus \mathfrak{m}^\perp) \cap t_{y_+, y_-}$ consists of elements $a \in \mathfrak{g}^{n+1} \oplus \mathfrak{g}^{n+1}$ of the form

$$a = (x_1 + x_1', \ldots, x_{n-1} + x_{n-1}', \ Ad_{g_n}(x_n), x_n, z_1 + z_1', \ldots, z_{n-1} + z_{n-1}', \ Ad_{h_n}(z_n), z_n),$$

where, by writing $g = (g_1, g_2, \ldots, g_{n+1})$ and $h = (h_1, h_2, \ldots, h_{n+1})$,

$$x_j, \ z_j \in \mathfrak{c}, \quad x_j' \in \text{Ad}_{g_j} \mathfrak{q}^\perp, \quad z_j' \in \text{Ad}_{h_j} \mathfrak{q}^\perp, \quad x_j = \text{Ad}_{g_1} h_{j-1}^{-1} z_j, \quad j = 1, \ldots, n-1,$$

$$x_1 + x_1' \in \mathfrak{m}^+ \cap \text{Ad}_{g_1} \mathfrak{q}, \quad x_n \in \mathfrak{m}^+ \cap \text{Ad}_{g_n} \mathfrak{m}^\perp, \quad z_1 \in \mathfrak{m}^+, \quad z_n \in \mathfrak{m}^\perp, \quad z_{n-1} + z_{n-1}' = \text{Ad}_{h_n}(z_n),$$

$$x_{2j} = x_{2j+1}, \quad x'_{2j} = x'_{2j+1}, \quad z_{2j-1} = z_{2j}, \quad z'_{2j-1} = z'_{2j}, \quad j = 1, \ldots, m-1.$$

It follows that

$$x_1 = \text{Ad}_{g_1} h_{2g_2} h_{3g_3} \cdots g_{2m-1} h_{2m} (z_{2m}) \in p_c(\mathfrak{m}^+ \cap \text{Ad}_{\mathfrak{c}} \mathfrak{q}) \cap \text{Ad}_{\mathfrak{c}1 \mathfrak{c}2} \cdots \mathfrak{c}_n p_c(\mathfrak{m}^+ \cap \text{Ad}_{\mathfrak{c}n-1} \mathfrak{q}).$$

By 2) of Assumption 5.9, $x_1 = 0$, and it follows that $x_j = 0$ for every $j = 1, \ldots, n-1$. By (4.8), $\delta_{\mathfrak{c}+, \mathfrak{c}} = 0$. Let $p_{\mathfrak{c} \oplus \mathfrak{t}}$ be the projection from $\mathfrak{m}^+ \oplus \mathfrak{m}^\perp$ to $\mathfrak{m}^+ \cap \mathfrak{m}^\perp \cong \mathfrak{t} \oplus \mathfrak{t}$. An argument similar to that in the proof of Theorem 5.8 also shows that

$$p_{\mathfrak{c} \oplus \mathfrak{t}}((\mathfrak{m}^+ \oplus \mathfrak{m}^\perp) \cap t_{y_+, y_-}) = (p_1 \oplus p_t)(V_0).$$

Moreover, an argument similar to that in the proof of Lemma 6.7 shows that $J_{\mathfrak{c}}^{-1}((\bar{\mathcal{O}}_+ \cap \bar{\mathcal{O}}_-)) = X_n(c)$. Theorem 6.10 is now a consequence of Theorem 1.7. The case of $n = 2m + 1$ is odd is proved similarly.

Q.E.D.

6. APPLICATIONS TO POISSON STRUCTURES RELATED TO FLAG VARIETIES

After reviewing the standard complex semi-simple Lie groups, we prove Theorem 1.1- Theorem 1.4 stated in [11,2] on the four series of Poisson manifolds given in [11,1].

6.1. Standard complex semi-simple Poisson Lie groups. Let $\mathfrak{g}$ be a complex semi-simple Lie algebra, and let $\langle \cdot, \cdot \rangle_{\mathfrak{g}}$ be a fixed symmetric non-degenerate invariant bilinear form on $\mathfrak{g}$. Fix also a choice $(\mathfrak{b}, \mathfrak{b}_-)$ of opposite Borel subalgebras of $\mathfrak{g}$ and let $\mathfrak{h} = \mathfrak{b} \cap \mathfrak{b}_-$, a Cartan subalgebra of $\mathfrak{g}$. Let $\Delta$ and $\Delta_+ \subset \Delta$ be respectively the set of roots for the pairs $(\mathfrak{g}, \mathfrak{h})$ and $(\mathfrak{b}, \mathfrak{h})$, and let $\mathfrak{g} = \mathfrak{h} + \sum_{\alpha \in \Delta} \mathfrak{g}_\alpha + \sum_{\alpha \in \Delta_+} \mathfrak{g}_{-\alpha}$ be the corresponding root decomposition. Let $n_+ = \sum_{\alpha \in \Delta_+} \mathfrak{g}_\alpha$ and $n_- = \sum_{\alpha \in \Delta_+} \mathfrak{g}_{-\alpha}$. Equip again the direct product Lie algebra $\mathfrak{g} \oplus \mathfrak{g}$ with the bilinear form $\langle \cdot, \cdot \rangle_{\mathfrak{g} \oplus \mathfrak{g}}$ as in (3.9). Then

$$L_{st} = \{ (x_+ + x_0, -x_0 + x_-) : \ x_+ \in n_+, \ x_0 \in \mathfrak{h} \}$$

is a Lagrangian subalgebra of $(\mathfrak{g} \oplus \mathfrak{g}, \langle \cdot, \cdot \rangle_{\mathfrak{g} \oplus \mathfrak{g}})$ such that $\mathfrak{g} \oplus \mathfrak{g} = \mathfrak{g}_{\text{diag}} + L_{st}$. The decomposition $\mathfrak{g} \oplus \mathfrak{g} = \mathfrak{g}_{\text{diag}} + L_{st}$ is called a standard Lagrangian splitting of $(\mathfrak{g} \oplus \mathfrak{g}, \langle \cdot, \cdot \rangle_{\mathfrak{g} \oplus \mathfrak{g}})$ and the element $r_{st} \in \mathfrak{g} \oplus \mathfrak{g}$ such that $L_{st} = L_{st}$ a standard (factorizable quasitriangular) $r$-matrix on $\mathfrak{g}$. Let $(\delta_{ij})_{i,j=1}^{r}$ be a basis of $\mathfrak{h}$ such that $\langle h_i, h_j \rangle_{\mathfrak{g}} = \delta_{ij}$ for $1 \leq i, j \leq r$, and let $E_\alpha \in \mathfrak{g}_\alpha$ and $E_{-\alpha} \in \mathfrak{g}_{-\alpha}$ be root vectors for $\alpha \in \Delta_+$ such that $\langle E_\alpha, E_{-\alpha} \rangle_{\mathfrak{g}} = 1$. Then $r_{st}$ is explicitly given by

$$(6.1) \quad r_{st} = \frac{1}{2} \sum_{i=1}^{r} h_i \otimes h_i + \sum_{\alpha \in \Delta_+} E_{-\alpha} \otimes E_\alpha.$$
It is clear from the definition in \([3.3]\) that the Lie subalgebras \(\mathfrak{f}_-\) and \(\mathfrak{f}_+\) of \(\mathfrak{g}\) associated to \(\pi_{\text{st}}\) are respectively given by \(\mathfrak{f}_- = \mathfrak{b}_-\) and \(\mathfrak{f}_+ = \mathfrak{b}_+.\) It is also clear that \(\pi_{\text{st}}\) is factorizable and that the symmetric bilinear form on \(\mathfrak{g}\) associated to \(\pi_{\text{st}}\) (see \([3.2]\)) is precisely \((\cdot, \cdot)_g\).

By \((5.1)\), one also has the factorizable quasi-triangular \(r\)-matrix \(r^{(2)}_{\text{st}}\) on \(\mathfrak{g} \oplus \mathfrak{g}\). Explicitly,

\[
(6.2) \quad r^{(2)}_{\text{st}} = \frac{1}{2} \sum_i \left( (h_i, 0) \otimes (h_i, 0) - (0, h_i) \otimes (0, h_i) - (h_i, 0) \wedge (0, h_i) \right) + \sum_{\alpha \in \Delta^+} \left( (E_{\alpha} \otimes E_{\alpha}, 0) - (0, E_{\alpha} \otimes E_{\alpha}) - (E_{\alpha}, 0) \wedge (0, E_{\alpha}) \right)
\]

\[
= \frac{1}{2} \sum_i (h_i, h_i) \otimes (h_i, -h_i) + \sum_{\alpha \in \Delta^+} \left( (E_{\alpha}, E_{\alpha}) \otimes (0, -E_{\alpha}) + (E_{\alpha}, -E_{\alpha}) \otimes (E_{\alpha}, 0) \right).
\]

One checks directly (see also \([38, \S 6]\)) that \(r^{(2)}_{\text{st}}\) coincides with the \(r\)-matrix on \(\mathfrak{g} \oplus \mathfrak{g}\) defined by the Lagrangian splitting \(\mathfrak{g} \oplus \mathfrak{g} = \mathfrak{g}_{\text{diag}} \oplus \mathfrak{h}\) (see the definition in Example \([3.2]\)). The Lie subalgebras \(\mathfrak{f}_+\) and \(\mathfrak{f}_-\) of \(\mathfrak{g} \oplus \mathfrak{g}\) associated to \(r^{(2)}_{\text{st}}\) are then \(\mathfrak{h}\) and \(\mathfrak{g}_{\text{diag}}\), respectively.

Let \(G\) be any connected complex Lie group with Lie algebra \(\mathfrak{g}\). The Poisson structure

\[
\pi_{\text{st}} := r^L_{\text{st}} - r^R_{\text{st}}
\]

on \(G\) is called a standard multiplicative holomorphic Poisson structure on \(G\), and the pair \((G, \pi_{\text{st}})\) a standard complex semi-simple Poisson Lie group. One also has the Drinfeld double Poisson Lie group \((G \times G, \Pi_{\text{st}})\) of the Poisson Lie group \((G, \pi_{\text{st}})\), where

\[
\Pi_{\text{st}} = \left( r^{(2)}_{\text{st}} \right)^L - \left( r^{(2)}_{\text{st}} \right)^R.
\]

Let \(B\) and \(B_-\) be the Borel subgroups of \(G\) with Lie algebras \(\mathfrak{b}\) and \(\mathfrak{b}_-\) respectively, and let \(T = B \cap B_-\), a maximal torus of \(G\). Let \(Q = B\) for the Poisson Lie group \((G, \pi_{\text{st}})\) and \(Q = B \times B_-\) for the Poisson Lie group \((G \times G, \Pi_{\text{st}})\). Then Condition \((5.1)\) is satisfied, and one arrives at the four quotient Poisson manifolds \((F_n, \pi_n)\), \((F_n, \Pi_n)\), \((\bar{F}_n, \bar{\pi}_n)\) and \((\bar{F}_n, \bar{\Pi}_n)\), as in \((1.1)\) in \(\S 1.2\).

For \((F_n, \pi_n)\) and \((\bar{F}_n, \bar{\pi}_n)\), take \(M_+ = B\) and \(M_- = B_-\). Then \(M_+ \cap M_- = T\). The projection \(p_1 : m_+ \oplus m_- \to t = \mathfrak{h}\) in \((1.1)\) is now given by

\[
(6.3) \quad p_1 : B \oplus B_- \to \mathfrak{h}, \quad (x_0 + x_+, y_0 + y_-) \mapsto x_0 + y_0, \quad x_0, y_0 \in \mathfrak{h}, x_+ \in \mathfrak{n}, y_- \in \mathfrak{n}_-.
\]

For \((F_n, \Pi_n)\) and \((\bar{F}_n, \bar{\Pi}_n)\), take \(M_+ = B \times B_-\) and \(M_- = G_{\text{diag}} = \{(g, g) : g \in G\}\). Then \(M_+ \cap M_- = T_{\text{diag}} = \{ (t, t) : t \in T \}\). The projection \(p_1 : m_+ \oplus m_- \to t = \mathfrak{h}_{\text{diag}} \cong \mathfrak{h}\) in \((1.1)\) in this case given by

\[
(6.4) \quad p_1 : (B \oplus B_-) \oplus \mathfrak{g}_{\text{diag}} \to \mathfrak{h}, \quad ((x_0 + x_+, y_0 + y_-), (x, x)) \mapsto x_0 + y_0,
\]

where \(x_0, y_0 \in \mathfrak{h}, x_+ \in \mathfrak{n}, y_- \in \mathfrak{n}_-,\) and \(x \in \mathfrak{g}\.\)

We make some further preparation for the proofs of Theorem \([1.1]\) - Theorem \([1.4]\).

**Lemma 6.1.** For \(u_1, \ldots, u_n, v_1, \ldots, v_n, w \in W\), the following are equivalent:

1) \((B_{u_1}B_{u_2} \cdots B_{u_n}) \cap (B_{v_1}B_{v_2} \cdots B_{v_n}B_- w B) \neq \emptyset;\)

2) \(w \leq (v_1 \ast \cdots \ast v_n)^{-1} \ast u_1 \ast \cdots \ast u_n.\)

**Proof.** For \(n = 1\), the equivalent between 1) and 2) is proved in \([38, \text{Proposition 4.1}]\). Assume that \(n \geq 2\) and write \(u = u_1 \ast \cdots \ast u_n\) and \(v = v_1 \ast \cdots \ast v_n\). Suppose that 1) holds. As

\[
B_{u_1}B \cdots B_{u_n}B = \bigsqcup_{x \in V} BxB \quad \text{and} \quad B_{-v_1}B_{-v_2} \cdots B_{-v_n}B_- = \bigsqcup_{y \in V} B_{-y}B_-,
\]
where \( \mathcal{U} \subset \{ x \in W : x \leq u \} \) and \( \mathcal{V} \subset \{ y \in W : y \leq v \} \), there exist \( x \leq u \) and \( y \leq v \) such that \((B_x B) \cap (B_y B - w B) \neq \emptyset\), and hence \( w \leq y^{-1} \times x \leq v^{-1} \times u \). Conversely, if \( w \leq v^{-1} \times u \), then \((B_u B) \cap (B_v B - w B) \neq \emptyset\). As \( B_u B \subset B_{u_1} B \cdots B_{u_n} B \) and \( B_v B - \subset B_{v_1} B \cdots B_{v_n} B - \), 1) holds.

Q.E.D.

**Lemma 6.2.** For any \( u, v \in W \) and any conjugacy class \( C \) in \( G \), \((BuBB_v B -) \cap C \neq \emptyset\).

**Proof.** As \( B_ - \cap (Bu B) \neq \emptyset \), one has \( B_ - \cap (BuBB_v B -) \neq \emptyset \), which implies that \( BB_ - \subset BuBB_v B - \). Similarly, \( BB_ - \subset BB_v B - \). Thus \( B \subset BB_ - \subset BuBB_v B - \). As \( B \cap C \neq \emptyset \), one has \((BuBB_v B -) \cap C \neq \emptyset\).

Q.E.D.

For \( n \geq 1 \), consider \( P_n : (G \times G)^n \rightarrow G^n \) given by \( P_n(g_1, k_1, \ldots, g_n, k_n) = (g_1, \ldots, g_n) \), and the induced projections, both denoted as \([P_n]\), from \( F_n \) to \( F_n \) and from \( \tilde{F}_n \) to \( \tilde{F}_n \), i.e.,

\[
P_n[g_1, k_1, \ldots, g_n, k_n]|_{\tilde{F}_n} = [g_1, \ldots, g_n]|_{\tilde{F}_n}, \quad P_n(g_1, k_1, \ldots, g_n, k_n)|_{\tilde{F}_n} = [g_1, \ldots, g_n]|_{\tilde{F}_n}.
\]

As \( P_1 : (G \times G, \Pi_{st}) \rightarrow (G, \pi_{st}) \) is Poisson (this follows, for example, from the fact that \( P_1(r_{st}) = r \)), one knows that \([P_n] : (F_n, \Pi_{st}) \rightarrow (F_n, \pi_{st}) \) and \([P_n] : (\tilde{F}_n, \Pi_n) \rightarrow (\tilde{F}_n, \tilde{\pi}_n) \) are Poisson. This observation will be used to show that Theorem 1.1 is a special case of Theorem 1.2 (see 6.3) and that Theorem 1.3 is a special case of Theorem 1.4 (see 6.3).

6.2. **Proofs of Theorem 1.1 and Theorem 1.2.** We first prove Theorem 1.2 by applying Theorem 5.8 to the Poisson Lie group \((G \times G, \Pi_{st})\) and by taking \( Q = M_+ = B \times B_+ \) and \( M_- = G_{\text{diag}} \). As \( M_+ \cap M_- = \text{T}_{\text{diag}} := \{(t, t) : t \in \mathbb{T} \} \), the intersection \( R_{w, v}^{u, v} \), whenever non-empty, is smooth and connected and has dimension \( l(u) + l(v) - l(w) \). The fact that \( R_{w, v}^{u, v} \neq \emptyset \) if and only if \( w \leq (v_1 \cdots v_n)^{-1} u_1 \cdots u_n \) follows directly from Lemma 6.1 (it is also proved in [22 Proposition 3.32] using distinguished double subexpressions). This proves 1) of Theorem 1.2. Letting \( c = (u_1, v_1, \ldots, u_n, v_n) \) and \( c_{n+1} = (w, v) \) in (b) of Theorem 5.8, one proves the first part of 2) of Theorem 1.2. Computing explicitly the subspace \( V \subset M_+ \cap M_- \) as described in Theorem 6.8 and using 6.4 for \( P_1 : M_+ \cap M_- \rightarrow \mathbb{T} \), one sees that the leaf stabilizer of \( \lambda_{n, v} \) in \( R_{w, v}^{u, v} \) is precisely \( h_{w, v} \subset h \). This proves 3) of Theorem 1.2. The second part of 2) follows from 3) by Theorem 6.8. This finishes the proof of Theorem 1.2.

Theorem 1.3 is proved either similarly, by applying Theorem 5.8 to the Poisson Lie group \((G, \pi_{st})\) and taking \( Q = M_+ = B \) and \( M_- = B_- \), or by the following observation: let

\[
\tilde{F}_n := (G \times B_-) \times (B \times B_-) \cdots (B \times B_-) / (B \times B_-) \subset F_n.
\]

By Lemma 6.3, \( G \times B_- \) is a Poisson Lie subgroup of \((G \times G, \Pi_{st})\). Thus \( \tilde{F}_n \) is a Poisson submanifold of \((F_n, \Pi_n)\). It is then clear that the Poisson morphism \([P_n] : (F_n, \Pi_n) \rightarrow (F_n, \pi_{st})\) in 6.5 restricts to a Poisson isomorphism from \((\tilde{F}_n, \Pi_n)\) to \((F_n, \pi_{st})\). Theorem 1.1 now follows by applying Theorem 1.2 to the Poisson submanifold \((\tilde{F}_n, \Pi_n)\) of \((F_n, \Pi_n)\).

6.3. **Proofs of Theorem 1.3 and Theorem 1.4.** We first prove Theorem 1.4 by applying Theorem 5.10 to the Poisson Lie group \((G \times G, \Pi_{st})\) and taking \( Q = M_+ = B \times B_+ \) and \( M_- = G_{\text{diag}} \). As \( M_+ \cap M_- = \text{T}_{\text{diag}} \), the intersection \( R_{C, v}^{u, v} \), whenever non-empty, is smooth and connected, and

\[
\dim R_{C, v}^{u, v} = \dim((B \times B_-)(u, v)(B \times B_-)) + \dim \mu_{v}^{-1}(\Omega_C) - \dim \tilde{F}_n = l(u) + l(v) + \dim C + \dim T.
\]

Let \( u = (u_1, \ldots, u_n), v = (v_1, \ldots, v_n) \in W^n \), and \( C \subset C \) be arbitrary. By definition,

\[
\mu_{v}^{-1}(\Omega_C) = \{[g_1, k_2, \ldots, g_n, k_n]|_{\tilde{F}_n} : (g_1 g_2 \cdots g_n)(k_1 k_2 \cdots k_n)^{-1} \in C\}.
\]
By Lemma 6.2, \((Bu_1Bu_2 \cdots Bu_nBBu_{n-1}^{-1} \cdots B_2B_1^{-1}B_1^{-1}B \cap C \neq \emptyset, \text{ so } R^{u,v}_g \neq \emptyset.\) This proves 1) of Theorem 5.10. Letting \(c = (u_1, v_1, \ldots, u_n, v_n)\) and \(c_{n+1} = (c, e)\) in (b) of Theorem 5.10 where \(c \in C\) is arbitrary, one sees that the \(R^{u,v}_g\)'s are precisely the \(T^2\)-leaves of \(\widehat{\Pi}_n\) in \(\widehat{F}_n\), where \(T^2\)-acts on \(\widehat{F}_n\) by
\[(t_1, t_2) \cdot [g_1, k_1, \ldots, g_n, k_n]_{\pi} = [tg_1, tk_1, g_2, k_2, \ldots, g_{n-1}, k_{n-1}, g_n, k_n, t_2^{-1}, k_n, t_2^{-1}]_{\pi}.
\]
Computing explicitly the subspace \(V_c \subset m_+ \oplus m_-\) as described in Theorem 5.10 and using (6.4) for \(p_t : m_+ \oplus m_- \to t = \mathfrak{h}\), one sees that the leaf stabilizer for the \(T^2\)-action on \(R^{u,v}_g\) is given by
\[(h \oplus 0)^{u,v} := \{(u(x) + v(y), x, y) : x, y \in \mathfrak{h}\},
\]where \(u = u_1 * \cdots * u_n\) and \(v = v_1 * \cdots * v_n\). Note that \((\mathfrak{h} \oplus 0) + (\mathfrak{h} \oplus \mathfrak{h})^{u,v} = \mathfrak{h} \oplus \mathfrak{h}\), the action \(\lambda_{\pi_n}\) of \(T \cong T \times \{e\} \subset T \times T\) on \(\widehat{F}_n\) given in (1.9) is also full, so each \(R^{u,v}_g\) is a single \(T\)-leaf for the action \(\lambda_{\pi_n}\).

Moreover, the leaf-stabilizer of \(\lambda_{\pi_n}\) is \(R^{u,v}_g\) is
\[(\mathfrak{h} \oplus 0) \cap ((\mathfrak{h} \oplus \mathfrak{h})^{u,v}) = \{u(x) - v(x) : x \in \mathfrak{h}\} = \mathfrak{h}^{u,v}.
\]
This finishes the proof of Theorem 5.14.

Theorem 5.13 is proved either similarly, by applying Theorem 5.10 to the Poisson Lie group \((G, \pi_{st})\) and taking \(Q = M_+ = B\) and \(M_- = B_-\), or by the following observation: let
\[F_n^{\vee} := ((G \times B_-) \times_{(B \times B_-)} \cdots \times_{(B \times B_-)} (G \times B_-) \times_{(B \times B_-)} (G \times G)) \cap \mu_n^{-1}(G_{\text{diag}}) \subset \widehat{F}_n.
\]By Theorem 5.14, \(F_n^{\vee}\) is a Poisson submanifold of \((\widehat{F}_n, \widehat{\Pi}_n)\). It is clear that the Poisson morphism \([P_n] : (\widehat{F}_n, \widehat{\Pi}_n) \to (\widehat{F}_n, \widehat{\Pi}_n)\) in (6.5) restricts to a Poisson isomorphism from \((F_n^{\vee}, \widehat{\Pi}_n)\) to \((\widehat{F}_n, \widehat{\Pi}_n)\).

Theorem 5.13 now follows by applying Theorem 5.14 to the Poisson submanifold \((F_n^{\vee}, \widehat{\Pi}_n)\) of \((\widehat{F}_n, \widehat{\Pi}_n)\).

6.4. Other examples. The results in [4] and [5] can be used to give a unified approach to many other examples, old or new, of \(T\)-Poisson manifolds related to real or complex semi-simple Lie groups. We give two such example here, leaving other examples to be treated elsewhere.

Example 6.3. Let \((G, \pi_{st})\) be again a standard complex semi-simple Lie group as in [6.3] and let \(\theta \in \text{Aut}(G)\) be such that \(\theta(T) = T\) and \(\theta(B) = B\), and denote by the same letter the induced automorphism on \(\mathfrak{g}\). Let \(\lambda\) be the left action of \(G \times G\) on \(G\) given by
\[(g_1, g_2) \cdot g = g_1g\theta(g_2)^{-1}, \quad g_1, g_2, g \in G.
\]Orbits of \(G_{\text{diag}} \subset G \times G\) on \(G\) under \(\lambda\) are called \(\theta\)-twisted conjugacy classes of \(G\). Let \(r = r_{st}^{(2)}\) be the \(r\)-matrix on \(\mathfrak{g} \oplus \mathfrak{g}\) defined by the Lagrangian splitting \(\mathfrak{g} \oplus \mathfrak{g} = \mathfrak{g}_{\text{diag}} + t_{st}\) (see [6.2]). As the stabilizer subalgebras for \(\lambda\) are Lagrangian with respect to \((\cdot, \cdot)_{\mathfrak{g} \oplus \mathfrak{g}}\), the quadruple \((G \times G, r, G, \lambda)\) is strongly admissible. The Poisson structure \(\pi_\theta = -\lambda(r)\) is studied in [50].

Recall that the Lie subalgebras \(f_- \oplus f_+\) of \(\mathfrak{g} \oplus \mathfrak{g}\) associated to \(r\) is \(f_- = \mathfrak{g}_{\text{diag}}\) and \(f_+ = t_{st}\). Let \(M_+ = G_{\text{diag}}\) and let \(M_- = B \times B_-\), so that \(M_- \cap M_+ = T_{\text{diag}} = \{(t, t) : t \in T\}\) which we identify with \(T\). Note that \(M_-\)-orbits in \(G\) are precisely the \(\theta\)-twisted conjugacy classes in \(G\), and each \(M_+\)-orbit is of the form \(BwB_-\) for a unique \(w \in W\). It is shown in [7, §2.4] that for a \(\theta\)-twisted conjugacy class \(C\), there is a unique element \(m_C \in W\) such that for \(w \in W\), \(C \cap (BwB) \neq \emptyset\) if and only if \(w \leq m_C\). Again as the stabilizer subalgebras of \(\lambda\) are Lagrangian with respect to \((\cdot, \cdot)_{\mathfrak{g} \oplus \mathfrak{g}}\), it follows trivially from Proposition 4.6 that \(\delta_{\mathfrak{c}_-C_-} = 0\) for every \(\mathfrak{c}_- = C\) and \(\mathfrak{c}_+ = BwB_-\). Thus the \(T\)-leaf decomposition of \((G, \pi_\theta)\) is given by \(G = \bigsqcup_{C \cap (BwB_-) \neq \emptyset} C \cap (BwB_-),\) where \(C\) is a \(\theta\)-twisted conjugacy class in \(G\) and \(w \in W\) such that \(w \leq m_C\). Denote by \(t_{C, w}\) the leaf stabilizer of \(C \cap (BwB_-)\) in \(\mathfrak{h}\). Pick any \(g \in C\) and let \(y_- = g\) and \(y_+ = \dot{w}\). Then
\[l_{y_-, y_+} = q_{y_+} \oplus q_{y_-} = \text{Ad}_{(\dot{w}, e)}\mathfrak{g}_\theta \oplus \text{Ad}_{(g, e)}\mathfrak{g}_\theta \subset (\mathfrak{g} \oplus \mathfrak{g}) \oplus (\mathfrak{g} \oplus \mathfrak{g}),
\]
where $\mathfrak{g}_0 = \{ (\theta(x), x) : x \in \mathfrak{g} \}$. It follows from [4, 16] that $t_{C,w} = \Im(w\theta + 1)$, a result obtained in [36].

**Example 6.4.** Let $G$ be a connected complex semi-simple Lie group with Lie algebra $\mathfrak{g}$, and let $\Im(\langle \cdot, \cdot \rangle_{\mathfrak{g}}$ be the imaginary part of Killing form $\langle \cdot, \cdot \rangle_{\mathfrak{g}}$ of $G$. Let $G = KAK$ be an Iwasawa decomposition of $G$, and let $\mathfrak{k}, \mathfrak{a},$ and $\mathfrak{n}$ be respectively the Lie algebras of $K$, $A$, and $N$. Regarding $(\mathfrak{g}, \Im(\langle \cdot, \cdot \rangle_{\mathfrak{g}})$ as a real quadratic Lie algebra, one has the Lagrangian splitting $\mathfrak{g} = \mathfrak{t} + (\mathfrak{a} + \mathfrak{n})$ of $(\mathfrak{g}, \Im(\langle \cdot, \cdot \rangle_{\mathfrak{g}})$. Let $\pi_C$ be the real analytic Poisson structure on $G$ given by $\pi_C = (r_0)_L - (r_0)_R$, where $r_0 \in \mathfrak{g} \otimes \mathfrak{g}$ is the quasitriangular $r$-matrix on $\mathfrak{g}$ defined by the Lagrangian splitting $\mathfrak{g} = \mathfrak{t} + (\mathfrak{a} + \mathfrak{n})$. Then $K$ is a Poisson Lie subgroup of $(G, \pi_C)$, and for each integer $n \geq 1$, one has the quotient space $\mathcal{P}_n = G \times_K \cdots \times_K G/K$ with the quotient Poisson structure $\pi_{\mathcal{P}_n}$ on $\mathcal{P}_n$. One can regard $\mathcal{P}_n$ as the space of $n$-gons in the Riemannian symmetric space $G/K$ (see [11, 27]). Taking $M_+ = K$ and $M_- = AN$ so that $M_+ \cap M_- = \{ e \}$, it follows from Theorem 5.8 and the decomposition $G = KAK$ that the symplectic leaves of $\pi_{\mathcal{P}_n}$ in $\mathcal{P}_n$ are of the forms $(K\alpha_1 K) \times_K \cdots \times_K (K\alpha_n K)/K$ with $w = (a_1, \ldots, a_n) \in A^n$, the space of $n$-gons with fixed side lengths $a$.

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