Mirror symmetry, D-brane superpotentials and Ooguri–Vafa invariants of Calabi–Yau manifolds

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Abstract: The D-brane superpotential is very important in the low energy effective theory. As the generating function of all disk instantons from the worldsheet point of view, it plays a crucial role in deriving some important properties of the compact Calabi–Yau manifolds. By using the generalized GKZ hypergeometric system, we will calculate the D-brane superpotentials of two non-Fermat type compact Calabi–Yau hypersurfaces in toric varieties, respectively. Then according to the mirror symmetry, we obtain the A-model superpotentials and the Ooguri–Vafa invariants for the mirror Calabi–Yau manifolds.

Key words: D-brane, superpotential, mirror symmetry, Ooguri–Vafa invariant

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1 Introduction

The theory of topological strings, which is derived from the two dimensional \((N,\hat{N})=(2,2)\) superconformal field theory, has developed considerably over the past few years and has had a deep influence on mathematics. The D-brane superpotential, the generating function of correlation functions, is a particularly vital physical quantity which is a section of special holomorphic line bundles of the moduli space from the mathematical perspective. Through the superpotential we can derive a series of important properties for the CY manifolds, such as Yukawa couplings, Ooguri–Vafa invariants and so on. Therefore the calculation of the superpotential is very meaningful.

Some important properties of the moduli spaces for various Calabi–Yau manifolds [1–3] have been well studied via the mirror symmetry which was first mentioned in the local operator algebra of the \(N = 2\) string theory [4]. It is well known that mirror symmetry connects two different moduli spaces which are respectively parameterized by Kahler geometry deformation and complex geometry deformation in A- and B-models. In the A-model there exist contributions from the instantons while there are none in the B-model. So calculating the superpotential directly in the A-model is rather difficult. In fact only in several special cases do we know the corresponding brane configuration on mirror A-model side for a given brane configuration in the compact CY manifold [5] derived from the GKZ system in the B-model. In the GKZ system the superpotential is related to the period integral, and the Hodge theoretic approach [6] provides a useful insight into studying the period integrals for CY manifolds which satisfies the Picard–Fuchs differential equation, which is closely related to the GKZ system.

Recently, for compact CY manifolds, there have been some great developments in calculating the quantum corrected domain wall tensions on the CY threefolds via open–closed mirror symmetry [7, 8]. The properties of some compact Calabi–Yau manifolds have been studied in Refs. [9–16]. In this note, we compute the D-brane superpotentials for two non-Fermat CY threefolds in detail via mirror maps and GKZ hypergeometric system.

The structure of this paper is as follows. In Section 2 we describe the generalized GKZ hypergeometric system. The solution of the GKZ hypergeometric system is just the integral period. We also outline the approach to construct the corresponding polyhedron \(\Delta\) and its mirror polyhedron \(\Delta^*\) for the Calabi–Yau manifold. Then we review how to calculate the superpotential. In Section 3 we analyze two non-Fermat type compact CY manifolds in toric varieties, respectively, and compute their superpotentials as well as some disk invariants with the method referred to previously. The last section is the conclusion.
2 Toric geometry, relative period integrals and the GKZ system

We divide this section into two parts to review some related background.

2.1 Superpotential on the D-brane

In the presence of some background fluxes and space-filling D-branes, type II string theory compactification on the Calabi–Yau manifold gives rise to the \( N = 1 \) low energy effective theories [13], whose effective superpotential is captured by the relative period integral of the holomorphic three-form \( \Omega(z) \) over the relative cycles with boundaries wrapped by D-branes [17]. As is listed in Refs. [18–22], the above integral is derived from the action of a holomorphic Chern–Simons theory on the brane which wraps the holomorphic curves.

For the D-brane wrapping internal cycles of Calabi–Yau manifold \( X \), the corresponding effective superpotential is [19]

\[
\mathcal{W}_{\text{brane}} = \int_X \Omega \wedge \text{Tr} \left[ A \wedge \partial A + \frac{2}{3} A \wedge A \wedge A \right],
\]

where if there are \( N \) branes, \( A \) is a holomorphic \( U(N) \) gauge connection on \( X \) and \( \Omega \) is the holomorphic three-form on \( X \). For a type IIB string, the effective superpotential is a linear combination of the relative period integrals [21, 23, 25].

\[
\mathcal{W}_{\text{brane}}(\varphi, \xi) = \hat{N}_a \hat{H}^a(\varphi, \xi),
\]

where \( \hat{N}_a \) stands for the homology class, which is wrapped with the D-brane. \( \hat{H}^a(\varphi, \xi) \) represents the period integral

\[
\hat{H}^a(\varphi, \xi) = \int_{\gamma^a(\xi)} \Omega(\varphi).
\]

Here, \( \xi \) and \( \varphi \) stand for the open- and closed-string moduli respectively. The internal background fluxes \( H = H^{RR} + H^{NS} \) lead to an effective superpotential [13–15, 26–29] which is defined by

\[
\mathcal{W}_{\text{flux}} = \int_X \Omega \wedge H = \int_X \Omega \wedge (H^{RR} + \tau H^{NS}),
\]

where \( \Omega \) denotes the holomorphic three-form on the Calabi–Yau manifold, and \( \tau \) denote the complex couplings for the type II string of the B-model. In this note, we only consider the RR flux, so the induced superpotential becomes

\[
\mathcal{W}_{\text{flux}} = \int_X \Omega \wedge H^{RR} = \sum_a N_a \Pi^a(\varphi).
\]

Therefore the combined superpotential generated by the D-brane and flux is [30, 31]

\[
\mathcal{W}(\varphi, \xi) = \mathcal{W}_{\text{brane}}(\varphi, \xi) + \mathcal{W}_{\text{flux}}(\varphi) = \sum_a N_a \Pi^a(\varphi, \xi).
\]

Here the coefficient \( N_a \) denotes both the D-brane topological charge and the RR flux quantum data and \( \Pi_a(\varphi, \xi) \) denotes the integral of the three-form \( \Omega(\varphi) \) over the three-chains in the relative integrer homology group, which is defined by

\[
\Pi_a(\varphi, \xi) = \int_{\gamma^a(\xi)} \Omega(\varphi), \quad \Pi^a(\xi) \in H_a(Y, S, Z).
\]

The relative period integral referred to previously, \( \hat{H}^a(\varphi, \xi) \), is equal to the domain wall tension \( T(\varphi, \xi) \) [5, 20, 21, 30, 31]. \( T(\varphi, \xi) \) is defined as

\[
T(\varphi, \xi) = \mathcal{W}(C^+_{\varphi, \xi}) - \mathcal{W}(C^-_{\varphi, \xi}).
\]

At its critical point \( \xi = z \), \( T(\varphi, \xi) \) is identical to the on-shell domain wall tension \( T = \mathcal{W}(C^+_{\varphi}) - \mathcal{W}(C^-_{\varphi}) \). As shown in Refs. [3, 8, 32, 33], at the critical points, the domain wall tensions are considered as a normal function from which the Abel–Jacobi invariants can be derived.

For the D-brane in the A-model, the superpotential which is expressed in terms of the flat closed/open coordinates can be calculated as the generating function of the correlation functions [21, 30, 34–36]. It is defined by

\[
\mathcal{W}(t, \hat{t}) = \sum_{\hat{\kappa}, \hat{\eta}} G_{\hat{\kappa}, \hat{\eta}} q^{\hat{n}_{\hat{\kappa}, \hat{\eta}}} = \sum_{\hat{\kappa}, \hat{\eta}} \sum_{d} n_{\hat{\kappa}, \hat{\eta}}^d \frac{q^{\hat{n}_{\hat{\kappa}, \hat{\eta}}}}{k^d}. \quad (9)
\]

Here, \( q = e^{2 \pi i t}, \hat{q} = e^{2 \pi i \hat{t}} \) and \( n_{\hat{\kappa}, \hat{\eta}} \) is the Ooguri–Vafa invariant. Mirror symmetry, which indicates that the two superpotentials for D-branes in the A- and B-models, are related to each other by the mirror map, gives us a method to compute the Ooguri–Vafa invariant which is closely related to the open Gromov–Witten invariant \( G_{\hat{\kappa}, \hat{\eta}} \). The superpotentials and the Ooguri–Vafa invariants are as follows [7, 37]:

\[
\frac{W^{(z)}(z(q))}{\omega_0(z(q))} = \frac{1}{(2\pi i)^2} \sum_{\hat{\kappa}, \hat{\eta}, d_1, d_2} \sum_{\hat{\kappa}, \hat{\eta}, d_1, d_2} n_{\hat{\kappa}, \hat{\eta}}^{d_1} \frac{q^{d_1} \hat{q}^{d_2} / k^d}{\omega_0(z)}. \quad (10)
\]

where \( q_a = e^{\lambda a} \) (\( a = 1, 2 \)) and

\[
\omega_0(z) = \sum_{n} \epsilon(n) z^n = \sum_{n} \prod_{a} F^a(\sum_{i} I_{ij}^{(n)} z^{i+1}) z^n. \quad (11)
\]

Here the mirror map \( t_a \) is defined as \( t_a = \frac{\partial \omega_0}{\partial \omega_0} \).

2.2 Toric geometry and the GKZ System

The generalized hypergeometric system was first introduced in Ref. [38], and soon developed quickly in mirror symmetry [6, 39–42]. Let us define a mirror pair of hypersurfaces \( (X, X^*) \) in two toric ambient spaces \( (V, V^*) \), respectively. The toric varieties \( (V, V^*) \) are related to the fans \( (\Sigma(\Delta), \Sigma(\Delta^*)) \) induced by the two dual poly-
The torus invariant algebraic coordinates \( \{ x_i \} \). The defining polynomial for the hypersurface is defined as:

\[
P = \sum_{i=0}^{p-1} a_i \prod_{k=1}^{4} x_i^{v_{i,k}}. \tag{12}
\]

Or we can write the above equation in another way

\[
P = \sum_{i=0}^{p-1} a_i \prod_{v_j \in \Delta} x_i^{(v_j, \lambda_j^{(i)})+1}. \tag{13}
\]

Here \( a_i \) are complex parameter and \( x_k \) are inhomogeneous coordinates on the open torus, \( x_i \) are the homogeneous coordinates.

The general integral period is expressed as

\[
\Pi(a_i) = \frac{1}{(2\pi i)^4} \int_{|x_i|=1} \prod_{k=1}^{4} \frac{dx_k}{x_k}. \tag{14}
\]

It is shown in Refs. [39, 40] that the period can be annihilated by a GKZ hypergeometric differential system

\[
\mathcal{D}_l \Pi(a_i) = 0 \quad (l \in L), \quad \mathcal{Z}_l \Pi(a_i) = 0 \quad (i=0,1,\ldots,p), \tag{15}
\]

The operators \( \mathcal{D}_l \) and \( \mathcal{Z}_l \) are expressed as

\[
\mathcal{D}_l = \prod_{i_k>0} \left( \frac{\partial}{\partial a_i} \right)^{l_i} \prod_{j_k<0} \left( \frac{\partial}{\partial \theta_j} \right)^{-l_j}, \quad (l \in L), \tag{16}
\]

\[
\mathcal{Z}_j = \sum_{i=0}^{p} \bar{v}_{i,j} \theta_i - \beta_j, \quad (j=0,1,\ldots,n). \tag{17}
\]

The torus invariant algebraic coordinates \( z_a \) in the large complex structure limit is [3]

\[
z_a = (-1)^{i_a} \prod_j \bar{a}_j^{i_j}, \tag{18}
\]

where \( l^a \) is the set of basic vectors which denote the generators of the Mori cone. Then, by \( \theta_a = z_a \partial z_a \), (2.16) changes into

\[
\mathcal{D}_l \equiv \prod_{k=1}^{l_0} \left( \theta_a - k \right) \prod_{k>0} \left( \theta_a - k \right) - \left( -1 \right)^{l_0} z_a \prod_{k=1}^{l_0} \left( \theta_a - k \right) \prod_{k<0} \left( -1 \right)^{l_0} z_a \prod_{k=0}^{l_0} \left( -\theta_a - k \right), \tag{19}
\]

where \( l \) is the linear combination of \( l^a \). One can refer to [32, 43–45] for more details. The result for the GKZ system is described as

\[
\mathcal{B}_{\{l^a\}}(z_a; \rho_a) = \sum_{n_1,\ldots,n_N \in \mathbb{Z}_0^N} \Gamma \left( \frac{1}{2} \sum_{a} l^a(n_a + \rho_a) \right) \prod_{a} \frac{1}{\prod_{i} x_i^{n_a + \rho_a}} \times \prod_{a} z_a^{(n_a + \rho_a)}. \tag{20}
\]

3 Study of two compact non-Fermat type Calabi–Yau manifolds

In this section we will calculate the superpotentials and disk invariants for two compact CY in the weighted projective space, with the method described in Section 2.

3.1 Calabi–Yau hypersurface \( X_{1}(1, 1, 1, 1, 3) \)

\( X_{1}(1, 1, 1, 1, 3) \) is a hypersurface in the weighted projective space \( P^4(1, 1, 1, 1, 3) \). Let \( \tilde{X} \) denote this hypersurface; then, its mirror manifold \( X^* \) is denoted by \( X^* = \tilde{X}/H \), where \( \tilde{X} \) represents the CY 3-fold \( X_{14}(1, 2, 2, 2, 7) \) and \( H \) is defined by \( (h_i^j) = \frac{1}{7}(1, 0, 6, 0, 0), \frac{1}{7}(1, 0, 0, 0, 0) \). So, \( X_{1}(1, 1, 1, 1, 3) \) is isomorphic to \( X_{14}(1, 2, 2, 2, 7) \), which had been checked in Ref. [46]. The hypersurface \( X_{1}(1, 1, 1, 1, 3) \) is defined as the zero locus of the polynomial \( P \).

\[
P = x_1^2 + x_2^2 + x_3^2 + x_4^2. \tag{21}
\]

The weighted projective space \( P^4(1, 1, 1, 1, 3) \) is a toric variety, and the vertices of its corresponding polyhedron \( \Delta \) are as follows:

\[
v_1 = (-1, -1, -1, -1), \quad v_2 = (-1, -1, -1, -1), \quad v_3 = (-1, -1, 1, 0), \quad v_4 = (6, -1, -1, -1), \quad v_5 = (-1, 1, -1, -1), \quad v_6 = (-1, 3, -1, -1), \quad v_7 = (0, -1, 3, -1), \quad v_8 = (-1, 1, -1, 1), \quad v_9 = (0, 3, -1, -1).
\]

The vertices of the corresponding dual polyhedron \( \Delta^* \) are

\[
v^*_1 = (-1, -1, -1, -1, -3), \quad v^*_2 = (1, 0, 0, 0), \quad v^*_3 = (0, 1, 0, 0), \quad v^*_4 = (0, 0, 1, 0), \quad v^*_5 = (0, 0, 0, 1), \quad v^*_6 = (0, 0, 0, -1).
\]

There exists only one integral point denoted as \( v^*_0 = (0, 0, 0, 0) \) in \( \Delta^* \). For \( \Delta^* \), the charge vector of the Mori cone is

\[
l^{(1)} = (-2, 0, 0, 0, 1, 1, 0, -3). \tag{22}
\]

Consider the divisor

\[
Q(D) = x_3^2 + x_4^3, \tag{22}
\]

at the critical point \( z_3 = 1 \). Let

\[
u_1 = \frac{z_3}{z_3} (1-z_3)^2, \quad u_2 = z_2, \tag{23}
\]

then according to (2.19) the GKZ system of the two-parameters family become

\[
\mathcal{D}_1 = \tilde{\theta}_1 \tilde{\theta}_2 - 2\tilde{\theta}_1 \tilde{\theta}_2 - (2\tilde{\theta}_1 \tilde{\theta}_2 + 2\tilde{\theta}_1 + \tilde{\theta}_2 - 1) z_1, \tag{24}
\]

\[
\mathcal{D}_2 = \tilde{\theta}_1 \tilde{\theta}_2 (2\tilde{\theta}_1 + \tilde{\theta}_2 - 1) - 4\tilde{\theta}_1 (2\tilde{\theta}_1 + \tilde{\theta}_2 - 1) z_1
\]

\[
- 7 \prod_{i=1}^{3} (2\tilde{\theta}_1 + \tilde{\theta}_2 - 1). \tag{25}
\]
The solution to this GKZ system is written as
\[ \Pi_1(u_1,u_2) = \frac{c}{2} B(t) \left( u_1,u_2,0,\frac{1}{2} \right), \]
\[ \Pi_2(u_1,u_2) = \frac{c}{2} B(t) \left( u_1,u_2,1,\frac{1}{2} \right). \]
(26)
(27)
At the critical point \( z_3 = 1 \), the on-shell superpotential satisfies \( W_C = W_C^* \) according to the \( Z_2 \) symmetry. So the on-shell superpotential is described as
\[ W^\pm(z_1,z_2) = \frac{1}{2\pi i} \int_{\partial D} \Pi(z_1,z_2,\xi) \frac{dx}{\xi} |_{\xi=1}. \]
(28)
For this model the on-shell superpotentials are expressed as
\[ W^+_1 = \frac{c}{8} \sum_{n_1,n_2>0} \Gamma \left( 2n_1+n_2+\frac{3}{2} \right) z_1^{n_1} z_2^{n_2+\frac{3}{2}}, \]
\[ W^+_2 = \frac{c}{8} \sum_{n_1,n_2>0} \Gamma \left( \frac{1}{n_1} + \frac{1}{n_2} + \frac{3}{2} \right) \Gamma \left( n_1-3n_2, \frac{1}{2} \right) \Gamma \left( n_2+3, \frac{3}{2} \right), \]
(29)
(30)
The flat coordinates in the A- and B-models are connected via the mirror map \( t_a = \frac{d_a}{\omega_0} \). The mirror map is as follows:
\[ t_1 = \log(z_1)+2z_2+3z_1^2+\frac{20}{3} z_1^3+68 z_1^2 z_2-10 z_1 z_2 \]
\[ +2 z_2-15 z_2^2+66 z_2 z_2^2+\frac{560}{3} z_2^3+o(z^3), \]
\[ t_2 = \log(z_2)+2z_2+45 z_2^2+65 z_2 z_2^2-198 z_2 z_2^2+30 z_2 z_2^2 \]
\[ +9 z_1-204 z_2 z_2^2+\frac{43}{2} z_2^3+62 z_2^3+o(z^3), \]
(31)
(32)
and the corresponding inverse mirror map is
\[ z_1 = q_1-2q_1^2+3q_1^3-2q_1 q_2+5q_1 q_2^2+3q_1 q_2^3+o(q^4), \]
\[ z_2 = q_2+6q_2^2+9q_2^3-9q_1 q_2-12q_1 q_2^2+37 q_2^3+o(q^4). \]
(33)
(34)
According to (2.10) we can derive the Ooguri–Vafa invariants from the on-shell superpotentials. The results are listed in the following Tables (Table 1, Table 2).

### 3.2 Calabi–Yau hypersurface \( X_0(1, 1, 2, 2, 3) \)

\( X_0(1, 1, 2, 2, 3) \) is a hypersurface defined in the weighted projective space \( P^4(1, 1, 2, 2, 3) \). Let \( X \) denote the corresponding Calabi–Yau three-fold; then, its mirror manifold \( X^* \) is denoted by \( X^*=X/H \), where \( X \) represents the CY 3-fold \( X_{12}(1, 1, 3, 4) \) and \( H \) is defined by \( h_1 = \frac{1}{9} (1, 8, 0, 0, 0), \frac{1}{4} (1, 0, 3, 0, 0) \), So, \( X_0(1, 1, 2, 2, 3) \) is isomorphic to \( X_{12}(1, 1, 3, 4) \). The polyhedron \( \Delta \) for this model has the vertices
\[ v_1=(-1, -1, -1, -1, 2), \]
\[ v_2=(-1, -1, -1, -1, 1), \]
\[ v_3=(-1, -1, -2, -1, 0), \]
\[ v_4=(-1, -1, -1, -1, 0), \]
\[ v_5=(-1, -1, -3, -1, 1), \]
\[ v_6=(0, 1, -1, -1), \]
\[ v_7=(0, 3, -1, -1), \]
then the dual polyhedron \( \Delta^* \) has vertices
\[ v_1^*=(1, 0, 0, 0, 0, 0), \]
\[ v_2^*=(0, 0, 0, 0, 1), \]
\[ v_3^*=(0, 0, 1, 0, 0, 0), \]
\[ v_4^*=(0, 0, 0, 1, 0, 0), \]
\[ v_5^*=(0, 0, 1, -1, -1), \]
\[ v_6^*=(0, -1, -1, -1, 1), \]
The polyhedron \( \Delta \) for this model has the vertices
\[ v_1=(1, 0, 0, 0, 0, 0), \]
\[ v_2=(0, 0, 0, 0, 1), \]
\[ v_3=(0, 0, 1, 0, 0, 0), \]
\[ v_4=(0, 0, 0, 1, 0, 0), \]
\[ v_5=(0, 0, 1, -1, -1), \]
\[ v_6=(0, -1, -1, -1, 1), \]
\[ v_7=(0, 3, -1, -1), \]

### Table 1. \( n_{d_1,d_2}^{(1,++)} \)

| \( d_1 \) \( \times \) \( d_2 \) | 1 | 3 | 5 | 7 | 9 | 11 |
|---|---|---|---|---|---|---|
| 0 | 0 | -2 | 2 | -10 | 84 | -858 | 13820 |
| 1 | 28 | -28 | 252 | -2828 | 36400 | -729130 |
| 2 | 70 | 112 | -2702 | 42910 | -714410 | 17644120 |
| 3 | 0 | 252 | 16716 | -391580 | 8645280 | -262033434 |
| 4 | 0 | 6832 | -83286 | 2543170 | -741112654 | 2741674588 |
| 5 | 0 | 84364 | 315880 | -13445796 | 496350736 | -22527070394 |

### Table 2. \( n_{d_1,d_2}^{(2,++)} \)

| \( d_1 \) \( \times \) \( d_2 \) | 1 | 3 | 5 | 7 | 9 | 11 |
|---|---|---|---|---|---|---|
| 1 | 0 | 0 | 0 | 0 | 0 | 0 |
| 2 | 0 | 0 | 0 | 0 | 0 | 0 |
| 3 | -70 | 0 | 0 | 0 | 0 | 0 |
| 5 | -28 | 56 | -140 | 896 | -8088 | -5747938 |
| 7 | 2 | -792 | 3164 | -27608 | 315050 | 108056554 |
| 9 | 0 | -38962 | -32424 | 391104 | -5784408 | -1230725514 |
| 11 | 0 | -84364 | 358288 | -3814860 | 6819744 | 10291361734 |
The solution to this GKZ system is written as

\[ W(z) = \frac{1}{2\pi i} \int_{\Gamma} \frac{\Gamma}{\Gamma(n + 1)\Gamma(n + 3/2)} \frac{1}{z_1^{n_1}z_2^{n_2}}, \]

expressed as

\[ W_i^\pm = \pm \sum_{n_1, n_2 \geq 0} \frac{\Gamma(3n_1 + n_2 + 5/2)}{\Gamma(n_1 + n_2 + 3/2)} \times \frac{1}{\Gamma(n_1 + 3/2)} \frac{1}{z_1^{n_1}z_2^{n_2}}. \]

The mirror map is

\[ t_1 = \log(z_1) + 30z_2 - 2\log(1 + e^{1/2z_1}) - 10z_1^2 \]
\[ + 927z_1^2 - 288z_1^2 - 105 - 105 z_1^2 + o(z^4), \]

\[ t_2 = \log(z_2) + 2z_2 - 3z_2 + 20/3 - 74z_1z_2 - 168z_1^2 \]
\[ - 192z_1^3 - 8853z_1^2z_2^2 + 35/2 z_1^2 + o(z^4), \]

and the corresponding inverse mirror map is

\[ z_1 = q_1 + 3q_1q_2 + 3q_1q_3 + q_1^2q_2 + 30q_1^2q_3 - 594q_1^2q_2 + o(q^4), \]
\[ z_2 = q_2 - 2q_2 - 3q_1^3 - 4q_1^3 - 74q_1^3 + 390q_1^3 + o(q^4). \]

Table 3. \( n_{d_1, d_2}^{(1,+)} \)

| \( d_1 \) | \( d_2 \) | 0 | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|---|---|
| 1 | 2 | 18 | 2 | 2 | 2 | 2 | 2 |
| 3 | 0 | 0 | 1584 | -710 | -626 | -616 | -616 |
| 5 | 0 | 0 | 38018 | 208244 | 49382 | 49382 | 49382 |
| 7 | 0 | 0 | 0 | 0 | 1745190 | 111219514 | 111219514 |
| 9 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 11 | 0 | 0 | 0 | 0 | 0 | 0 | -94519326 |

Table 4. \( n_{d_1, d_2}^{(2,+)} \)

| \( d_1 \) | \( d_2 \) | 1 | 3 | 5 | 7 | 9 | 11 |
|---|---|---|---|---|---|---|---|
| 1 | 16 | 0 | -2 | -2 | -2 | -2 | 46 |
| 3 | 0 | -80 | 2208 | 608 | 584 | 26342 | 26342 |
| 5 | 0 | 0 | 720 | 158874 | -4120 | 3039542 | 3039542 |
| 7 | 0 | 0 | 0 | -8848 | 15004928 | 94908448 | 94908448 |
| 9 | 0 | 0 | 0 | 0 | 126608 | 3473058320 | 3473058320 |
| 11 | 0 | 0 | 0 | 0 | 0 | 0 | -434510208 |
4 Conclusion

The D-brane superpotential plays a crucial role in both physics and mathematics. From the physical point of view, it determines the vacuum of the low energy $N=1$ effective theory. From the A-model worldsheet viewpoint, it is the generating function of the Ooguri–Vafa invariants of the Calabi–Yau manifold and the submanifold which is wrapped by the D-branes in the A-model. These Ooguri–Vafa invariants are closely related to the number of BPS states. From the mirror geometric viewpoint, it is the integral period which is the solution to the generalized GKZ system. It is very hard to calculate directly the D-brane superpotential for compact Calabi–Yau manifolds in the A-model because these superpotentials are essentially non-perturbative, and are impossible to obtain in the perturbative or localization ways which are important methods to compute the D-brane superpotential in non-compact Calabi–Yau manifolds in the A-model. An effective approach to obtain the D-brane superpotential is using the blown-up geometry of target space along the submanifold wrapped by the D-branes [9, 47]. The alternative approach to compute the superpotential of the D-brane in compact Calabi–Yau manifolds in the A-model is via the algebraic geometric method and mirror symmetry.

In this paper, we extend the generalized GKZ system in a Fermat Calabi–Yau three-folds to the compact non-Fermat Calabi–Yau three-folds which have been less studied so far in contrast to the Fermat type Calabi–Yau three-folds. We first constructed the generalized GKZ system for the compact non-Fermat type Calabi–Yau manifolds, then worked out the corresponding D-brane superpotential in the mirror B-model by the algebraic geometric method. The superpotential in the A-model was obtained according to mirror symmetry. Finally the Ooguri–Vafa invariants were extracted from the A-model superpotential.

These superpotentials have potential phenomenological applications. Furthermore, according to the type II string/M-theory/F-theory duality, in the weak decoupling limit $g_s \rightarrow 0$, these superpotentials of Type II strings give the Gukov–Vafa–Witten superpotentials $W_{GVW}$ of F-theory compactified on the dual fourfold. On the other hand, since there is not yet a systematic mathematical method to compute them, it is difficult to get from other approaches, those Ooguri–Vafa invariants predicted in this paper. Those Ooguri–Vafa invariants provide some concrete data which could potentially be checked by an independent mathematical calculation.

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