Measurement of the space-time interval between two events using the retarded and advanced times of each event with respect to a time-like world-line

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ABSTRACT

Several recent studies have been devoted to investigating the limitations that ordinary quantum mechanics and/or quantum gravity might impose on the measurability of space-time observables. These analyses are often confined to the simplified context of two-dimensional flat space-time and rely on a simple procedure for the measurement of space-like distances based on the exchange of light signals. We present a generalization of this measurement procedure applicable to all three types of space-time intervals between two events in space-times of any number of dimensions. We also present some preliminary observations on an alternative measurement procedure that can be applied taking into account the gravitational field of the measuring apparatus, and briefly discuss quantum limitations of measurability in this context.
\section{Introduction}

The limitations on measurability encountered in ordinary (non-gravitational) applications of quantum mechanics concern conjoint measurement of pairs of noncommuting observables. In particular, any given observable can be measured with arbitrary accuracy at the cost of renouncing any attempt at measurement of a conjugate observable. However, a number of arguments have been offered in support of the possibility that gravitational observables, including both the chrono-geometric and inertio-gravitational structures, may be subject to more severe measurability limitations. In particular, there may be absolute limitations (i.e. not avoidable even by sacrificing all knowledge of their conjugates) on the measurability of certain of these observables. These results have been obtained within the framework of certain approaches to quantum gravity, most notably in string theory \cite{1}, and they are also supported by certain heuristic analyses \cite{2, 3, 4, 5, 6, 7, 8}, in which the gravitational degrees of freedom playing a role in the relevant measurement procedures are treated classically, while the non-gravitational degrees of freedom are treated using ordinary quantum field theory in flat or non-flat background space-times. In addition, there are good heuristic arguments (see, e.g., Ref. \cite{9}) essentially going back to Bronstein \cite{10}, that the very concept of a smooth space-time should break down at the Planck four-volume. An attempt to localize a massive object to within a volume approaching the Planck volume will give rise, according to the Heisenberg indeterminacy relations, to energy fluctuations that will produce a black hole lasting a Planck time, making any further space-time localization meaningless (see, e.g., Ref. \cite{11}, p. 117). These arguments suggest that space-time smoothness breaks down at the Planck scale; and the same conclusion is reached by exploring heuristically the possible consequences of quantization of the gravitation field (see, e.g., Ref. \cite{12}, p. 40).

The main interest of these results lies in the hope that they might provide key insights concerning the structure of a future theory encompassing both present-day quantum field theory and general relativity (“quantum gravity”). Of course, faith in the reliability of these hints depends not only on the rigour of the individual analyses, but also on the generality (or representative nature) of the measurement procedures analyzed in establishing the suggested limitations. We share the point of view emphasized by Heisenberg \cite{13} and Bohr and Rosenfeld \cite{14}, that the limits of \textit{definability} of a quantity within any formalism should coincide with the limits of \textit{measurability} of that quantity for all conceivable (ideal) measurement procedures. For well-established theories, this criterion can be tested. For example, in spite of a serious challenge \cite{15}, source-free quantum electrodynamics was shown to pass this test \cite{14}. In the case of quantum gravity, our situation is rather the opposite. In the absence of a fully accepted, rigorous theory, exploration of the limits of measurability of various quantities can serve as a tool to provide clues in the search for such a theory: If we are fairly certain of the results of our measurability analysis, the proposed theory must be fully consistent with these results.

This paper is concerned with the generalization of one of the most frequently-cited procedures in such discussions of limitations on quantum measurement of geometrical quantities, first discussed in detail by Wigner and Salecker \cite{16, 17}: the measurement of space-like intervals in two-dimensional space-time based on the exchange of light signals.

Analyses based on this measurement procedure have been used extensively (see, e.g., Refs. \cite{6, 7, 18, 19}) as the source of intuitive ideas about the quantum-gravity problem. In evaluating the reliability of such intuitions, one might be concerned by the fact that their starting point only considered space-like distances in two dimensions; this might appear to be too narrow a basis on which to establish
general conclusions. Another reason for possible concern is the way in which Wigner-Salecker handled the delicate issue of the relation between the microscopic aspects of quantum phenomena and the macroscopic devices used in recording measurements of them.

This paper provides a simple generalization of the Wigner-Salecker procedure to the case of a space-time interval of any type defined by two events; and, perhaps more significantly, in any number of space-time dimensions\(^1\). We also propose a strategy for development of an alternative measurement procedure that handles more carefully the relation between its microscopic and macroscopic aspects.

We could turn directly to the case of arbitrary space-time intervals in a space-time of an arbitrary number of dimensions. But we prefer to adopt a more pedagogical approach, proceeding inductively from the case of 2D (two-dimensional) space-time, to 3D and then to (4+n)D space-times, where \(n=0,1,2,\ldots\).

The paper is organized as follows. Section 2 provides a brief summary of the Wigner and Salecker measurement analysis for space-like intervals in 2D Minkowski space-time. Section 3 is still confined to the 2D case, but generalizes the result to space-time intervals of any type. Section 4 concerns 3D Minkowski space-time with emphasis on some new elements required for measurability analyses in more than two dimensions. The straightforward generalization to (4+n)D space-times is discussed briefly in Section 5. In Section 6 we consider some limitations of the Wigner-Salecker analysis that result from the nature of the clock they employ (microscopic versus macroscopic), and the way in which the microscopic aspects of the measurement process are eventually recorded by a macroscopic device. We argue that the Wigner-Salecker procedure can be improved by letting the macroscopic recording apparatus consist of a spherically symmetric shell, entirely surrounding the region under study. In Section 7 we observe that this arrangement still can be used when the gravitational field of the surrounding shell is taken into account, and briefly consider some implications for the quantum-gravitational measurement problem. Section 8 summarizes our findings and indicates some goals of our future work.

## 2 Summary of the SW analysis (2D, space-like interval)

Rather than following Wigner-Salecker’s original measurement procedure for 2D space-like intervals in detail, we present the relevant portion of their analysis in a language and notation that provide a better starting point for our generalization, emphasizing those aspects of their work that have been cited and used most in recent studies \([6, 7, 18]\).

Wigner and Salecker \([16, 17]\) determine the space-like interval between two events in terms of certain proper time intervals along a given time-like world line passing through one of the events: namely, the time intervals for massless probes (photons) to traverse the distance back and forth between the world line and the two events\(^3\). The crucial ingredient is the relation between these proper time intervals and the space-like interval, so we start by expressing a space-like interval between two events in 2D

\(^1\)For Wigner’s concerns with the limitation to two dimensions, see Ref. \([10]\), pages 261 and 262.

\(^2\)Note that, like Wigner and Salecker, we take for granted that the time interval between two time-like separated events can be directly measured by the proper time readings of a clock moving on the time-like geodesic between the two events. At the quantum level this assumption might call for further analysis.

\(^3\)Wigner and Salecker also provide \([16, 17]\) a critique of the use of rigid rods in the measurement of distances. For a defense of legitimacy of their use see Ref. \([21]\).
Figure 1: Proper-time measurements on the inertial time-like world-line PQ used to measure the length $d_e$ of the space-like interval between the event $e$ and the world line.

Minkowski space-time in terms of proper times on an inertial time-like world-line PQ (see Figure 1). The simplest case is when one of the two events lies on PQ and the interval between the two events is orthogonal to PQ. As we shall see in Section 3, the interval between any two events can be derived easily from this case.

Let $\tau_{e,A}$ be the (proper) time of the point on PQ such that a light signal sent from it reaches the event $e$, and $\tau_{e,R}$ the (proper) time a signal sent from $e$ will reach the world-line. (We shall sometimes refer to these as the “advanced” and “retarded” times of the event $e$ with respect to PQ.) Letting

$$\Delta \tau_e \equiv \tau_{e,R} - \tau_{e,A}$$

and calling $\vec{\Delta \tau}_e$ the corresponding (2D) space-time vector (in general, $\vec{V}$ will denote a space-time vector of magnitude $V$), we see that the interval (distance) $\vec{d}_e$ between the event $O_e$ at the midpoint of $\Delta \tau_e$ (corresponding to proper time $\tau_e = \tau_{e,A}/2 + \tau_{e,R}/2$) and the event $e$ can be found by “squaring”...
the vectorial equation\[^4\]
\[ \vec{d}_e + N^e_{e,R} = \Delta \tau_e / 2 \] (2)

where \( N^e_{e,R} \) is the retarded null vector connecting \( e \) and the event on the PQ world-line that corresponds to the proper time \( \tau_{e,R} \).

Using the indefinite Minkowski metric to “square (2)”, and observing that by construction \( \vec{d}_e \) is space-like while \( \Delta \tau_e \) is time-like and that the square of a null vector = 0, we get
\[ d_e = \Delta \tau_e / 2 = (\tau_{e,R} - \tau_{e,A}) / 2 \] (3)

Thus, one can determine \( d_e \), the space-like distance between the event \( e \) off the world-line PQ and the event \( O_e \) on PQ, by measuring the advanced and retarded times \( \tau_{e,R} \) and \( \tau_{e,A} \).

A comment is important for our generalization: In 3D Minkowski space, we only need the half-plane defined by PQ and \( e \) since one can generate three-dimensional Minkowski space by rotating this half-plane about the line PQ. Put another way, the spatial direction orthogonal to the world-line can be interpreted as a radial direction. In their 2D diagrams, Salecker and Wigner correctly picture the entire two-dimensional Minkowski plane, but, in calculating the distance between two events, they tacitly assume that both lie in the same half-plane, which need not always be the case. This observation is important because they assert \[^5\] that, confining themselves to two dimensions, they are able to avoid the problem of ascertaining the direction from which the light signals come when calculating the interval between two events. This is not quite correct, as can be seen by considering how one might distinguish between two events that have the same advanced and retarded times with respect to the world-line but lie in opposite half-planes. This is the remnant in two dimensions of the fact that, in the higher-dimensional cases, we cannot avoid the problem of ascertaining the angle between the directions of two light signals (rotating a half-plane through the angle \( \pi \) generates the other half-plane).

### 3 Arbitrary interval in 2D space-time

As we have seen, the Wigner-Salecker analysis is satisfactory, and even enlightening. However, it does have definite limitations, notably the restrictions to space-like intervals and 2D space-times. In this section, we remain in 2D Minkowski space-time but generalize the Wigner-Salecker analysis to arbitrary space-time intervals defined by two events that may both lie off the time-like world-line PQ.

Our measurement procedure is based on the calculation of the space-time interval defined by any two events off PQ in terms of proper time measurements on PQ Let \( e \) and \( u \) be the two events, \( d_{eu} \) the space-time interval between them, and \( d^e_{eu} \) the corresponding (2D) space-time vector. In the same way that \( O_e \) was defined with respect to \( e \) in Section 2, one can define an event \( O_u \) on the world-line with respect to \( u \), and a vector \( \vec{d}_u \) connecting \( O_u \) and \( u \). It is then easy to see from Figure 2 that
\[ \vec{d}_{eu} = \vec{D}_{eu} + \vec{d}_u - \vec{d}_e \] (4)

where \( \vec{D}_{eu} \) is the vector connecting the events \( O_e \) and \( O_u \) on the world-line PQ. Note that by construction \( \vec{D}_{eu} \) is time-like and \( \vec{d}_u \) and \( \vec{d}_e \) are space-like, while \( \vec{d}_{eu} \) can be any type of vector (e.g., time-like if \( \vec{d}_u = \vec{d}_e \) or space-like if \( \vec{D}_{eu} = 0 \)).

\[^4\] We use units with \( c = 1 \).

\[^5\] This also tacitly assumes the availability of a totally-reflecting mirror (i.e., with zero transmission coefficient).
Figure 2: The space-time interval $d_{eu}$ defined by the events $e$ and $u$ in terms of proper time measurements on the world-line PQ. $d_e$ and $d_u$ are the spatial intervals between the world-line PQ and $e$ and $u$, respectively.
Using simple generalizations of the notation and arguments used in Section 2, one easily finds that \( d_e = (\tau_{e,R} - \tau_{e,A})/2 \), \( d_u = (\tau_{u,R} - \tau_{u,A})/2 \), and \( D_{eu} = (\tau_{e,R} + \tau_{e,A} - \tau_{u,R} - \tau_{u,A})/2 \). It is then straightforward to show that (adopting Minkowski signature \(+,−,−,−\))

\[
\vec{d}_{eu} \cdot \vec{d}_{eu} = D_{eu}^2 - |d_u - d_e|^2 = \\
= \frac{1}{4}[(\tau_{e,R} + \tau_{e,A} - \tau_{u,R} - \tau_{u,A})^2 - (\tau_{u,R} - \tau_{u,A} - \tau_{e,R} + \tau_{e,A})^2] \\
= \tau_{e,R} \tau_{e,A} - \tau_{e,A} \tau_{u,R} - \tau_{e,R} \tau_{u,A} + \tau_{u,R} \tau_{u,A} ;
\]

which indeed expresses \( \vec{d}_{eu} \cdot \vec{d}_{eu} \) in terms of the (four) advanced and retarded times of the (two) events \( e \) and \( u \).

As mentioned in Section 1, we postpone to a later paper a detailed analysis of the implications of this result for measurements at the quantum level. It is however reassuring to note that our generalization to arbitrary intervals does not require any qualitatively new type of measurement. All required information can still be obtained by recording some (advanced and retarded) readings of a clock placed close to the reference world-line. It should therefore be straightforward to generalize to arbitrary intervals the arguments previously advanced \([6, 7, 18]\) in analyses of the implications at the quantum level of the measurement of space-like intervals originally considered by Wigner-Salecker.

4 Arbitrary interval in 3D space-time

As noted in Section 2, we can generate 3D Minkowski space-time from the 2D space-time of the previous two Sections by rotating the half-plane about the time-like world-line \( PQ \) through the angle \( 2\pi \). For any given event \( e \), one can still define the retarded and advanced null vectors \( N_{e,R} \) and \( N_{e,A} \) that, after the \( 2\pi \) rotation, lie respectively on the retarded and advanced null cones having their origin on \( PQ \) and including the point \( e \). These retarded and advanced null cones intersect in a circle that lies in the space-like plane bisecting the interval vector \( \Delta \tau_e \); the center of the circle is at the bisecting point \( O_e \) of the world-line and \( e \) of course lies on this circle.

Before discussing the most general space-time interval defined by any two events \( e \) and \( u \) off \( PQ \), let us first consider the special case, in which \( e \) and \( u \) are such that the points \( O_e \) and \( O_u \) coincide (see Figure 3), \( i.e. \) both \( e \) and \( u \) lie in the same space-like plane intersecting the world line \( PQ \) in a single point \( O_e = O_u \). In this case, the distance \( \vec{d}_{eu} \) is space-like and may be found from the space-like triangle with sides \( \vec{d}_{eu}, \vec{d}_e \) and \( \vec{d}_u \):

\[
d_{eu} = \vec{d}_u - \vec{d}_e .
\]

"Squaring" this (which amounts to rederiving the law of cosines), we get:

\[
\vec{d}_{eu} \cdot \vec{d}_{eu} = -|\vec{d}_u - \vec{d}_e|^2 = -|d_e|^2 - |d_u|^2 + 2|d_e||d_u| \cos(\phi) ,
\]

where \( d_e \) and \( d_u \) are given respectively by \( d_e = (\tau_{e,R} - \tau_{e,A})/2 \) and \( d_u = (\tau_{u,R} - \tau_{u,A})/2 \), and \( \phi \) is the angle (in the plane on which both lie) between \( \vec{d}_e \) and \( \vec{d}_u \); \( i.e. \), the angle between two radii of the circle, discussed above, defined by the intersection of retarded and advanced null cones.

It is not hard now to generalize formula \([7]\) to the case of arbitrary \( e \) and \( u \): In general \( O_e \) and \( O_u \) need not coincide (\( i.e. \), \( e \) and \( u \) do not generally lie in the same space-like plane). In this case, \( \vec{d}_{eu} \) can
Figure 3: The special case in which $e$ and $u$ are such that the points $O_e$ and $O_u$ coincide.
be a vector of any type, but of course $\vec{d}_e$ and $\vec{d}_u$ are (by construction) still space-like. As in Section 3, it is also convenient here to introduce the time-like vector $\vec{D}_{eu}$ connecting the points $O_e$ and $O_u$ on the world-line PQ, allowing us to generalize Eq. (4) and (6) for $\vec{d}_{eu}$:

$$\vec{d}_{eu} = D_{eu} + \vec{d}_u - \vec{d}_e.$$  

(8)

Squaring this, we obtain

$$\vec{d}_{eu} \cdot \vec{d}_{eu} = D_{eu}^2 - |\vec{d}_u - \vec{d}_e|^2 = D_{eu}^2 - |d_e|^2 - |d_u|^2 + 2|d_e||d_u| \cos(\phi),$$

(9)

where $\phi$ is still the angle$^6$ between $\vec{d}_e$ and $\vec{d}_u$.

Using (9), we can express $\vec{d}_{eu} \cdot \vec{d}_{eu}$ in terms of advanced and retarded times on the world-line PQ$^7$

$$\vec{d}_{eu} \cdot \vec{d}_{eu} = \frac{1}{4}[(\tau_{e,R} + \tau_{e,A} - \tau_{u,R} - \tau_{u,A})^2 - (\tau_{u,R} - \tau_{u,A})^2 - (\tau_{e,R} - \tau_{e,A})^2 - 2|\tau_{u,R} - \tau_{u,A}||\tau_{e,R} - \tau_{e,A}| \cos(\phi)]$$

$$- 2|\tau_{u,R} - \tau_{u,A}| |\tau_{e,R} - \tau_{e,A}| \cos(\phi)$$

$$= \tau_{e,R}\tau_{e,A} + \tau_{u,R}\tau_{u,A} - \cos^2(\phi)[\tau_{e,A}\tau_{u,R} + \tau_{e,R}\tau_{u,A}] - \sin^2(\phi)[\tau_{e,A}\tau_{u,A} + \tau_{e,R}\tau_{u,R}].$$

(10)

Anticipating the quantum measurability analysis, we observe that the most significant new element that has emerged in generalizing the Wigner-Salecker procedure to 3D space-times is the requirement that the angle $\phi$ also be measured. All other measurements still involve only (advanced and retarded) readings of a clock placed close to the reference world-line. Interestingly (if perhaps not surprisingly), the way to minimize the sensitivity of the interval measurement to a possible uncertainty in the angle $\phi$ is by means of an arrangement making $\phi \sim 0$ (or $\phi \sim \pi$), i.e. an arrangement in which the experimenter is on a world line such that $\vec{d}_e$ and $\vec{d}_u$ are parallel (or anti-parallel).

## 5 Generalization to 4D space-time

Having obtained the formulas for the 3D case, the generalization to 4D Minkowski space-time is now rather trivial. Instead of rotation through a circle, to get four-dimensional Minkowski space-time, we need merely rotate the 2D Minkowski half-plane through a two-sphere of spherical angle $4\pi$ around the line PQ to get four-dimensional Minkowski space-time. Instead of a circle, the advanced and retarded light cones with vertices on PQ and including the event $e$ now intersect in a two-sphere lying in a space-like hyperplane (i.e., a three-space) orthogonal to PQ and centered on the midpoint $O_e$ of the interval $\Delta \tau_e$. The geometrical constructions then proceed just as in the 3D case, but now, instead of circles, $e$ and $u$ lie on spheres centered on the world-line (instead of circles). One can again introduce the space-like vectors $\vec{d}_e$ and $\vec{d}_u$ and the time-like vector $\vec{D}_{cu}$ in complete analogy with the 2D and 3D cases. In case both events $e$ and $u$ lie in the same space-like hypersurface orthogonal to PQ, the two spheres are concentric; and in order to use Eq. (7) in 4D, we merely need to interpret $\phi$ as the angle between two radii of the sphere.

$^6$More precisely, taking into account that here $\vec{d}_u$ and $\vec{d}_e$ do not lie in the same plane, $\phi$ is the angle between $\vec{d}_u$ and the translation (which is actually a translation by $\vec{D}_{eu}$) of $\vec{d}_e$ to the plane, in which $\vec{d}_u$ lies.

$^7$We thank Andor Frenkel for pointing out to us the possibility of simplifying Eq. (10) to its final form.
Similarly, equations (9) and (10) are still valid, and, just as it is in the 3D case, the distance \( \sqrt{\vec{d}_{eu} \cdot \vec{d}_{eu}} \) is determined by the (four) advanced and retarded times of the (two) events with respect to the world-line and by the angle \( \phi \).

Should the need arise for generalization to higher dimensions, it is obvious how to proceed.

6 Beyond Wigner-Salecker: why introduce the microscopic clock?

The analysis carried out in the previous Sections eliminates several of the limitations of the original Wigner-Salecker analysis: rather than being restricted to the context of space-like intervals in 2D space-time, the measurement procedure can now be applied to arbitrary space-time intervals in a space-time of Minkowski signature and arbitrary dimension, using an arbitrary\(^8\) time-like world-line. So far we have only alluded to some other limitations of the Wigner-Salecker analysis relevant to the key issue of the transfer of information about the microscopic system under observation to the macroscopic devices used to ultimately record the measurement result. This information is needed to complete the process for which we must compute a probability. If the process is treated classically, this probability can be computed directly from the ensemble in phase space defined by the process. If the process is treated quantum mechanically, a probability amplitude for it must be computed\(^9\)

The first limitation of this type was recognized by Wigner and Salecker. Early in their discussion of the microscopic clock, they remark (p. 571 of Ref. \[17\]):

\[
\text{As is well known, and as was pointed out most clearly by von Neumann, the measurement is not completed until its result is recorded by some macroscopic object. If the macroscopic object were part of the clock, no microscopic clock could exist. The way out of this difficulty is to transmit the signal of the clock to a macroscopic recorder (which can be the "final observer") which is far away from the clock, considered from the point of view of the average motion of the latter. The transmitting signal will be considered part of the [microscopic] clock, not, however, the recording apparatus.}
\]

This is essentially the reason why they confined themselves to a world of one spatial dimension (pp. 571-572 of Ref. \[17\]):

\[
\text{If the transmitting signal is to be microscopic, that is, if it is to consist of only a few quanta (actually, our signals will be light quanta), it will reach the recording equipment with certainty only if it does not spread out in every direction. In order to guarantee this, we confine ourselves to a world which has, in addition to the time-like dimension, only one space-like dimension.}
\]

\(^8\) The original Wigner-Salecker analysis assumed that one of the two events defining the interval was on the reference world-line.

\(^9\) We use Feynman’s term “process” to describe what Bohr calls a “phenomenon”: The preparation by some external apparatus of a quantum system, which then undergoes some interaction(s), the result of which is then “registered” by another external apparatus. The aim of any quantum-mechanical formalism is the computation of a probability amplitude for any such process (see Ref. \[22\]). For a discussion of the relations between classical and quantum ensembles, see Ref. \[23\].
They do not discuss any further the nature of the distant macroscopic recording device.

Another, related limitation was not pointed out by Wigner and Salecker. They take as unproblematic the notion of inertial paths, and indeed parallel inertial paths, in space-time (see Fig. 3, p. 573, Ref. [17], for example); as well as the notion of an inertial frame of reference (see Fig. 1, p. 572, Ref. [17], for example). The relation between these two limitations is that the macroscopic recording device can also serve to fix the inertial frame of reference, or at least be rigidly related to the macroscopic system that serves this purpose.

The relation between the microscopic clock, including the light quanta that transmit the signal, and the macroscopic recording device serving to fix the frame of reference, is quite analogous to that described by Bohr when he considers two possible alternative one-slit diaphragm experiments. In one case, the diaphragm is rigidly attached to the apparatus defining the inertial reference frame; and in the other, it is suspended from the frame by springs (see pp. 697-698 of Ref. [24] and pp. 218-221, esp. Figs. 4, p. 219 and Fig. 5, p. 220, of Ref. [25]). Until the relation of the diaphragm to the macroscopic apparatus defining the inertial frame of reference is fixed, one cannot speak of a definite phenomenon or process. The rigidly-fixed diaphragm can only be used for position measurements; while the suspended diaphragm can also be used for momentum measurement (for a fuller discussion, see Refs. [26, 27]).

Applying Bohr’s point of view to the case of the (microscopic) clock, until the macroscopic recorder has been introduced, serving both to fix or define an inertial frame of reference and to register the outcome of the process, one cannot meaningfully discuss the motion of the clock with respect to an inertial frame. In the context of special relativity, no more need be added, since the points of space and the instants of time relative to the inertial frame (assuming the Poincaré-Einstein convention to define the global time of the inertial frame) can be individuated quite independently of the quantum physical processes under investigation. This is not the case in general relativity, the theory in which we are really interested. But before turning to that case, let us return to the problem (discussed in Section 2) that limited Wigner and Salecker to one spatial-dimension: the spreading of the light signal. We can get around that problem by imagining the macroscopic recording apparatus to entirely surround the region under study— for example, to consist of a spherical shell of matter, within which lies the clock, as well as all the events to be investigated with it. Then, a record of the places and the times on the shell, at which the signals from the clock are received, may be treated as final observer results in von Neumann’s terminology (see above) or just the results of the measurement defining the process or phenomenon under investigation in the Bohr-Feynman terminology. As we shall show, these data then can be used to ascertain the macroscopic clock times needed to define the space-time interval between two events.

7 Beyond Wigner-Salecker: why not introduce gravity?

So far, the discussion has entirely neglected gravity and hence has been confined to special relativity. However, the introduction of a massive spherical shell offers an obvious way to begin to remedy this defect. Inside a spherical “hole” (matter- and non-gravitational field-free region of space-time) within a

\[10\] This actually implements Wigner’s suggestion, at page 263 of Ref. [10]: “In our experiments we surround the microscopic objects with a very macroscopic framework and observe coincidences between the particles emanating from the microscopic system, and parts of the framework”
spherically-symmetric distribution of matter, both hole and shell having the same center of symmetry, the Minkowski metric is a solution to the field equations of general relativity. So it appears that we can, to a certain extent, have our cake and eat it: We can carry certain results of a special-relativistic analysis into a valid general-relativistic context.

But that is the case only to a certain extent. Due to the universality of gravity, the presence within the shell of a clock, light signals, and whatever physical processes are used to define the events, the interval between which we wish to measure, will destroy the spherical symmetry. We must therefore assume that the masses (or more properly the physical components of the stress-energy tensors) of all such entities are so small compared to the mass of the shell that, to a good first approximation, we may neglect their effect on the gravitational field inside the shell\textsuperscript{11}. To the next approximation, we might use the linearized Einstein equations to take account of modifications of the gravitational field inside the shell, but we postpone study of this question until a later paper\textsuperscript{12}.

Perhaps an even more important modification in the general-relativistic case is the loss of a field-independent definition of spatial points and instants of time in the empty region within the shell. We shall henceforth assume the shell to be static in the sense of general relativity, i.e., there exists a time-like, hypersurface-orthogonal Killing vector field of the metric within the shell, which Killing vector is also a symmetry of the shell’s stress-energy tensor. But although the metric inside the shell is flat (i.e., its Riemann tensor vanishes), in the case of general relativity there is no unique way to physically associate a Minkowski metric with the points of the interior. As a solution to the field equations, the hole argument applies just as well to the Minkowski metric as to any other empty-space solution (see, e.g., Refs. \textsuperscript{29, 30}). But the presence of non-gravitational physical processes, previously regarded as presenting a difficulty, because of their gravitational effects, now proves to be an advantage: They enable us to define physically the spatial points and temporal instants associated with physical events inside the shell by relating them to events on the shell. To see how this can be done, let us look first at the situation in two-dimensional space-time. Here the sphere reduces to two parallel material strips, representing the histories of the two sides of the “one-sphere.” In the inertial frame fixed by these strips, let the distance between the inner surfaces of the two sides of the “one-sphere” be 2R. Let an event $e$ in the interior be marked by the emission of two light rays, one travelling towards each of the two sides. Assuming that times on the two strips have been synchronized (even in general relativity, a unique global time exists in the case of static metrics), we easily see that the time and position of the event, $\tau_e$ and $R_e$ respectively, are given by (see Figure 4)

$$
\tau_e = \frac{\tau_e R_1 + \tau_e R_2}{2} - R, \quad R_e = \frac{\tau_e R_1 - \tau_e R_2}{2}.
$$

\textsuperscript{11}One feature of the general-relativistic version of the material shell model must be noted here, even though we postpone detailed consideration of its implications until a later paper. In order that such a static shell be possible in general relativity, we must ensure that the radius of the shell falls outside the Schwarzschild radius associated with the mass of the shell (see the Introduction). That is, we must assure that $2GM/c^2 R$ be less than unity. This means, of course, that if for any reason we are forced to increase the mass of the shell (for example, in order that we may treat objects inside the shell of mass of the order $m$ as test bodies to a better and better approximation, we shall be forced to make the ratio $m/M$ smaller and smaller), we must take care to increase the radius $R$ if we start to approach the Schwarzschild limit.

\textsuperscript{12}For a careful quantum measurability analysis of the linearized gravitational field, see Ref. \textsuperscript{28} (also see Ref. \textsuperscript{29}).
Figure 4: In two-dimensional space-time, a “spherical shell” reduces to two strips. We denote by $R$ the “radius” (the half of the distance between the strips) and by $R_e$ the position of the event $e$ with respect to the origin of the position axis, which coincides with the “center” of the spherical shell. In figure $W_O$ is the world-line of this origin of the position axis.
This procedure can be generalized to higher dimensions as follows. If we increase the number of dimensions by rotating one of the strips of matter (the other strip is not needed; cf. our discussion in Section 2 of the transition from two to three dimensions), then the strip of matter becomes a space-time “cylinder of matter”, i.e. the history in time of a circle (or more generally a spherical annulus for three spatial dimensions, etc). Again, we shall assume that clocks are distributed over the surface of this cylinder at known positions and synchronized, enabling the recording of the arrival time of any light signal from events inside the cylinder.

The light ray emitted from $e$ now becomes a retarded light cone (of the appropriate number of dimensions). That is, we must now postulate that the event $e$ includes the emission of a (retarded) light cone, which intersects the material cylinder in some curve. Let the times of arrival at the material cylinder of the light rays on this cone at the material cylinder be noted by synchronized clocks. Of course, the fixed places of these clocks are known.

We shall now show that our previous 2D analysis can be applied to this situation. First of all, note that, if the event $e$ does not lie on the central axis of the cylinder (the case in which it does is trivial: all the signals of the light cone will arrive at the same time), then the central axis and $e$ define a time-like two-plane. If we are able to pick out the signals of the light cone that lie in this two-plane, then we have reduced the problem to the two-dimensional one. But, as a moment’s thought shows, of all the times recording the arrival of the light-cone signals, the two that lie in this plane will be the earliest and the latest recorded, respectively. Taking the 3D case, for example, and looking at the circular spatial cross-section of the cylinder at any time, one sees that points on the circle not at the ends of the diameter through the position of $e$ are further away than the closer point of the diameter.

Thus one need merely determine the latest and the earliest times of arrival on the cylinder of signals from the event. Calling them $\tau_{eR_1}$ and $\tau_{eR_2}$, respectively, we can again apply Eq. (11) to find the time and position of $e$. Of course, Eq. (11) only gives us the magnitude of $R_e$, but the positions of the clocks recording the earliest and the latest times already fix the diameter along which $e$ lies.

There may be other and better ways of fixing the position and time of the event $e$, but at least we have demonstrated that one such procedure exists. It is also interesting to note that, since it uses the entire light cone, this method does not seem to require the measurement of an angle.

Having seen how to define the spatial point and temporal instant associated with an event inside the shell by relating them to events on the shell, we no longer need the Wigner-Salecker microscopic interior clock. As indicated above, to make a microscopic clock reading meaningful, we need to relate it to events on the shell, and we have shown how to do this directly for the events under study.

This might also have interesting implications for the problem of quantum limitations in the gravitational context, which is transformed into the problem of what limitations are imposed by the quantum of action on the ability to define the position and time of an event inside the shell in terms of measurements of positions and times by macroscopic clocks on the shell.

8 Summary and outlook

The analysis in Sections 3-5 generalizes the Wigner-Salecker procedure in such a way that it is now applicable to the measurement of arbitrary intervals in an arbitrary number of space-time dimensions. A later paper will take up a detailed analysis of quantum limits to their measurability, but some of the results presented here are already relevant to the ongoing debate on “Planck-scale uncertainty
principles”. Several studies [6, 7, 18] have proposed such uncertainty principles on the basis of the 2D Wigner-Salecker procedure for the measurement of space-like intervals. The key aspect of those studies is the role played by (advanced and retarded) proper time measurements. Our generalization to the measurement of arbitrary intervals renders straightforward a corresponding generalization of the analyses reported in Refs. [6, 7, 18] to the case of non-space-like intervals.

Extending this procedure to more than two space-time dimensions, the measurement of advanced and retarded times is still a key ingredient; but an angle measurement also may be needed. If so, this result suggests a clear path for generalization of these 2D analyses to the case of higher space-time dimensions. The search for possible quantum (-gravity) limitations on the accuracy of this angular measurement would then be a key issue for such a generalization [13].

In Section 6 we propose a modification of the Wigner-Salecker procedure using a “spherical shell setup”, and in Section 7 show how it may be extended to include gravitation, which should allow a more consistent and theoretically comprehensive derivation of measurability limits relevant to quantum-gravity research. This will of course require a detailed and, in large part, new quantum measurability analysis: It will probably be necessary to treat the time measurements on the shell in close analogy with what has been done for such measurements by a Wigner-Salecker microscopic clock, but some new elements will arise from the requirement that the system of clocks on the shell be macroscopic, having a fixed position and capable of recording the time of arrival of signals. While in classical general relativity there is no in-principle obstruction to rigid motions, in a quantum setting the shell of finite mass and the clocks included in it are subject to the Heisenberg indeterminacy principle and can be set in rigid motion (rather than forming a “rigid body”) only up to a certain accuracy, which must be established.

Another important point to be considered for future work concerns the problem of developing a measurement procedure for the physical components of the space-time curvature, and showing how the existence of the quantum of action leads to a “Planck-scale indeterminacy principle” for the curvature components, as already carried out in linear approximation in Ref. [28]. As argued in Ref. [16], this should be possible with a suitable modification of the Wigner-Salecker procedure, even though in its original form, and in our generalization, the procedure assumes the absence of curvature in the space-time region of interest for the measurement procedure.

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13Our classical angular measurement analysis can provide the basis for a corresponding quantum-mechanical analysis, but presumably, since the studies reported in Refs. [6, 7, 18] do not advocate exactly the same perspective on quantum-measurability analysis of the Wigner-Salecker procedure already in the 2D case, different authors might treat differently the angular analysis at the quantum level. In a future paper, we shall propose our own perspective.
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