ON MODULAR GROUP REPRESENTATIONS ASSOCIATED TO SO(p)\textsubscript{2}-TQFTS

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Abstract. In this paper, we prove that for any odd prime larger than 3, the modular group representation associated to the SO(p)\textsubscript{2}-TQFT can be defined over the ring of integers of a cyclotomic field. We will provide explicit integral bases. In the last section, we will relate these representations to the Weil representations over finite fields.

1. Introduction

Let \( p \) be an odd prime. According to [RT91], to each modular tensor category, we can associate a Reshetikhin-Turaev TQFT. This TQFT not only gives rise to quantum invariants of 3-manifolds, but also to a series of projective representations of the mapping class groups \( \text{MCG}(\Sigma_g) \) of closed oriented surfaces \( \Sigma_g \) of genus \( g \). In particular, in genus one, we get a projective representation of the modular group \( \text{MCG}(\Sigma_1) = \text{SL}(2, \mathbb{Z}) \).

A systematic way to construct modular categories is to consider the representation theory of quantum groups at roots of unity. The TQFT representations of mapping class groups arising from such modular categories are finite-dimensional and can be defined over a cyclotomic field \( \mathbb{Q}(\zeta) \) where \( \zeta \) is a root of unity. As a result, the corresponding quantum invariants are elements of \( \mathbb{Q}(\zeta) \).

The first integrality result was obtained by Murakami [Mur94, Mur95], who showed that the SU(2)- and SO(3)-invariants are algebraic integers when the order of \( \zeta \) is prime. More precisely, those invariants are elements of the ring of integers of \( \mathbb{Q}(\zeta) \), namely, \( \mathbb{Z}[\zeta] \). The result was reproved in [MR97], generalized to all classical Lie types in [MW98, TY99], then to all Lie types by Le [Le03]. These results helped us relate the quantum invariants to other invariants such as the Casson invariant [Mur94, Mur95] and the Ohtsuki series [Oht95, Oht96, Le03].

A natural question to ask then is whether one can define the whole TQFT over \( \mathbb{Z}[\zeta] \), or at least can one define the representations of the mapping class groups over \( \mathbb{Z}[\zeta] \). If that is the case, we can get more information about these representations. For example, in [GM07] studied the Frohman Kania-Bartoszynska ideal [FKB01] using the integral SO(3)-TQFT, and this method is supposed to be generalized to other integral TQFTs. Another possible application of integral TQFTs is that we can reduce these TQFTs by the natural reduction map \( \mathbb{Z}[\zeta] \to \mathbb{Z}/p\mathbb{Z} \) to get the so-called \( p \)-modular
TQFTs. $p$-modular TQFTs have rich connections to topological information of 3-manifolds, such as the Casson-Lescop invariant, the Milnor torsion, see for example, [Ker02]. We may also answer questions such as the finiteness of the images of these representations by the integrality results.

For the SO(3)-TQFT, Gilmer, Masbaum and van Wamelen first constructed integral bases for genus one and two [GMvW04]. Then Gilmer and Masbaum generalized the result to arbitrary genus in [GM07], hence completed the construction of the integral SO(3)-TQFT.

In this paper, we will focus on the integrality properties of the SO($p$)-TQFTs for $p \geq 5$, which comes from the representation theories of quantum groups associated to the Lie algebra $\mathfrak{so}(p)$ at certain roots of unity. These TQFTs emerge as important objects in the context of topological quantum computing [HNW14], and as interesting examples of classical Lie type quantum groups themselves. We will establish the integrality of them in genus one by proving

**Theorem 1.** Suppose $p \geq 5$ is an odd prime. Then the genus one mapping class group representation given by the SO($p$)-TQFT can be defined over $\mathcal{O}$, where

$$
\mathcal{O} = \begin{cases} 
\mathbb{Z}[\zeta_p, i] = \mathbb{Z}[\zeta_{4p}], & \text{if } p \equiv 3 \pmod{4} \\
\mathbb{Z}[\zeta_p], & \text{if } p \equiv 1 \pmod{4}.
\end{cases}
$$

In the proof, we give an explicit integral basis as the authors of [GMvW04] did for the SO(3)-TQFT. As a byproduct, we show that a part of the genus one mapping class group representation of SL(2, $\mathbb{Z}$) factors through a part of the Weil representation of SL(2, $\mathbb{Z}/p\mathbb{Z}$), then conclude as a corollary that the image of the SO($p$)-TQFT representation in genus one is finite. This confirms the theorem by [NS10] saying that the SL(2, $\mathbb{Z}$) TQFT representation given by any modular category is finite.

The paper is organized as follows. In [Section 2], we give a quick review of preliminaries on the TQFT representation and the SO($p$)-TQFT. In particular, we give explicit matrix presentations of the SL(2, $\mathbb{Z}$) representation under a fixed basis. In [Section 3], we use number theoretic tools to construct a new basis for the representation space and show that it is indeed integral. In [Section 4], we briefly recall the definition of the Weil representation of SL(2, $\mathbb{Z}/p\mathbb{Z}$). We prove in [Theorem 2] that a part of the genus one TQFT representation factors through a part of the Weil representation. As a result, we prove that the image of the SL(2, $\mathbb{Z}$) representation is finite.

**Notations and conventions.** In the discussion below, we will assume that $p \geq 5$ is an odd prime. Let $r = \frac{p-1}{2}$. Let $\zeta_n = e^{\frac{2\pi i}{n}}$ be an $n$-th root of unity. We will let $\mathcal{O}$ be as in [Theorem 1] and use $\mathcal{O}^\times$ to denote the group of units of $\mathcal{O}$. It is a well-known fact that $\sqrt{p} \in \mathcal{O}$, hence $\frac{1}{\sqrt{p}} \in \mathbb{Q}(\zeta_p)$ or $\mathbb{Q}(\zeta_{4p})$ depending on $p$. Let $*^t$ denote the transpose of $*$, and let $I_{dn}$ stands for the $n \times n$ identity matrix. We call a representation integral if
the matrix coefficients of the representation are in \( \mathcal{O} \) with respect to certain basis. Sometimes we will also call a matrix with entries in \( \mathcal{O} \) integral.

**Acknowledgments.** The author would like to thank his advisor Professor Thomas Kerler for his guidance and many stimulating discussions. The author would like to thank Professor James W. Cogdell for helping the author understanding the Weil representation. The author is also grateful to Professor Patrick Gilmer, Professor Eric Rowell and Professor Zhenghan Wang for helpful discussions and their advice.

## 2. Preliminaries

In this section, we briefly recall the definition of the modular category \( \text{SO}(p)_2 \) and the Reshetikhin-Turaev TQFT associated to it. For more details, the readers are referred to [RT91][HNW14].

The \( \text{SO}(p)_2 \) modular category is the unitary modular tensor category obtained from the representation theory of the quantum group \( U_q(\mathfrak{so}(p)) \), where \( q = e^{\pi i/2p} \). It has \( (r + 4) \) simple objects, which we will label as \( \text{Irr}(	ext{SO}(p)_2) = \{ 1, Z, Y_1, \ldots, Y_r, X, X' \} \). Here \( 1 \) is the tensor unit. The fusion rules can be completely determined by the following listed ones:

\[
\begin{align*}
Z \otimes Z & \cong 1, \\
Z \otimes X & \cong X', \\
Z \otimes Y_j & \cong Y_j, \quad \forall j = 1, \ldots, r, \\
X \otimes X & \cong 1 \oplus \bigoplus_{j=1}^r Y_j, \\
(2) \quad X \otimes X' & \cong Z \oplus \bigoplus_{j=1}^r Y_j, \\
X \otimes Y_j & \cong X \oplus X', \quad \forall j = 1, \ldots, r, \\
Y_j \otimes Y_j & \cong 1 \oplus Z \oplus Y_{\min\{2j, m-2j\}}, \quad \forall j = 1, \ldots, r, \\
Y_j \otimes Y_k & \cong Y_{|j-k|} \oplus Y_{\min\{j+k, m-j-k\}}, \forall 1 \leq j, k \leq r, \quad j \neq k.
\end{align*}
\]

In the following, we will let

\[
H = \text{span}_\mathbb{C}\{ 1, Z, Y_1, \ldots, Y_r, X, X' \},
\]

and we will view \( \text{Irr}(\text{SO}(p)_2) \) as the fixed basis of \( H \).

To any modular tensor category, we can construct a Reshetikhin-Turaev TQFT. The TQFT is, roughly speaking, a tensor functor from a suitably defined cobordism category to the category of finite-dimensional vector spaces. The cobordism category has closed oriented surfaces as objects and 3-manifolds bounding two such surfaces as morphisms. The tensor structure
on the cobordism category is the disjoint union, and the tensor structure on
the category of vector spaces is the tensor product over the ground field.

In particular, for any orientation preserving diffeomorphism $\Psi$ of a surface
$\Sigma_g$ of genus $g$, the image of the mapping cylinder of $\Psi$ under the TQFT
functor is a linear automorphism of the vector space associated to $\Sigma_g$. It is
unique up to scalar multiples and is invariant under isotopy. As a result, the
TQFT gives rise to projective representations of the mapping class groups
of surfaces of genus $g$ for each $g \in \mathbb{Z}_{>0}$.

As $\text{SO}(p)_2$ is a modular tensor category, we can of course consider the
TQFT associated to it. In the genus one case, we have $\text{MCG}(\Sigma_1) = \text{SL}(2, \mathbb{Z})$
and we give the representation explicitly in terms of the generators of $\text{SL}(2, \mathbb{Z})$
as follows.

Let $\sigma = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ and $\tau = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ be generators of $\text{SL}(2, \mathbb{Z})$. Let $A$ be the
$(r \times r)$-matrix with entries given by

$$A_{jk} = \frac{2}{\sqrt{p}} \cos \left( \frac{2\pi jk}{p} \right) = \frac{1}{\sqrt{p}} (\zeta^{jk} + \zeta^{-jk})$$

for all $1 \leq j, k \leq r$.

Let

$$a = \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix}$$

be an $(r \times 1)$-dimensional vector, and let

$$\psi = \zeta^r = e^{2\pi i \frac{r}{p}}.$$

Note that $\psi \notin \mathcal{O}$, but $\psi^2 \in \mathcal{O}$.

From now on, we will suppress the subscript of $\zeta_p$ by simply writing $\zeta$, while letting

$$\theta_j = \zeta^{rj^2} = e^{\frac{2\pi i}{p} j^2} = e^{\frac{2\pi i}{p} \frac{j(j-1)}{2}}, \quad \forall j \in \mathbb{Z}.$$

The projective representation of $\text{SL}(2, \mathbb{Z})$ derived from the $\text{SO}(p)_2$-TQFT is
given by (see, for example, [NR11]):

$$\rho_1 : \text{SL}(2, \mathbb{Z}) \to \text{PGL}(H),$$
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\[ \rho_1(\sigma) = \begin{bmatrix} \frac{1}{2\sqrt{p}} & \frac{1}{2\sqrt{p}} & \frac{1}{\sqrt{p}} \cdot a^t & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2\sqrt{p}} & \frac{1}{2\sqrt{p}} & \frac{1}{\sqrt{p}} \cdot a^t & -\frac{1}{2} & -\frac{1}{2} \\ \frac{1}{\sqrt{p}} \cdot a & \frac{1}{\sqrt{p}} \cdot a & A & 0_{r \times 1} & 0_{r \times 1} \\ \frac{1}{2} & -\frac{1}{2} & 0_{1 \times r} & \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} & 0_{1 \times r} & -\frac{1}{2} & \frac{1}{2} \end{bmatrix}, \]

and

\[ \rho_1(\tau) = \begin{bmatrix} 1 \\ 1 \\ \theta_1 \\ \vdots \\ \theta_r \\ \psi \\ -\psi \end{bmatrix}. \]

Here $b_{\mu \times \nu}$ in a matrix $M$ is understood as a block of $M$ whose dimension is $\mu \times \nu$ and whose entries are all $b$.

Now we are ready to proceed to the proof of Theorem 1.

3. Proof of Theorem 1

To prove the theorem, we will give an explicit change of basis matrix $W \in \text{GL}(H)$ so that $W^{-1}\rho_1(\sigma)W$ and $W^{-1}\rho_1(\tau)W$ have entries in $\mathcal{O}$. We will find $W$ in several steps. First, we decompose $H$ into a direct sum of two invariant subspaces in Lemma 1, and reduce the problem to Claim 1. We then investigate properties of the column vectors of the representation after the change of basis proposed in Claim 1. We will prove integrality of one of the column vector in Proposition 2. Finally, we will prove the integrality of the rest of the column vectors by proving Claim 2.
Let

\[ U = \begin{bmatrix} 1 & 0 & 0 & 1 & 0_{2 \times r} \\ -1 & 0 & 0 & 1 \\ 0 & 1 & \psi & 0 \\ 0 & 1 & -\psi & 0 \end{bmatrix}_{4 \times r}. \]

(11)

**Lemma 1.** After the change of basis by \( U \), \( H \) splits into a direct sum of two invariant subspaces \( H \cong H_1 \oplus H_2 \), with \( \rho_1 \mid_{H_1} \) integral under the new basis.

**Proof.** From the form of \( U \), it is easy to see that the new basis corresponding to \( U \) is \( B_U = \{1 - Z, X + X', \psi(X - X'), 1 + Z, Y_1, \ldots, Y_r\} \). To determine the matrix coefficients of \( \rho_1(\sigma) \) and \( \rho_1(\tau) \) after the change of basis, we simply have to calculate how the two linear operations act on the new basis vectors. Then we will write the resulting vectors as linear combinations of vectors in \( B_U \).

By (11), we have

\[
\begin{align*}
\rho_1(\sigma)(1) &= \frac{1}{2\sqrt{p}}1 + \frac{1}{2\sqrt{p}}Z + \frac{1}{\sqrt{p}}\sum_{k=1}^{r} Y_k + \frac{1}{2}X + \frac{1}{2}X', \\
\rho_1(\sigma)(Z) &= \frac{1}{2\sqrt{p}}1 + \frac{1}{2\sqrt{p}}Z + \frac{1}{\sqrt{p}}\sum_{k=1}^{r} Y_k - \frac{1}{2}X - \frac{1}{2}X', \\
\rho_1(\sigma)(Y_j) &= \frac{1}{\sqrt{p}}1 + \frac{1}{\sqrt{p}}Z + \sum_{k=1}^{r} A_{kj}Y_k, \quad \forall j = 1, \ldots, r, \\
\rho_1(\sigma)(X) &= \frac{1}{2}1 - \frac{1}{2}Z + \frac{1}{2}X - \frac{1}{2}X', \\
\rho_1(\sigma)(X') &= \frac{1}{2}1 - \frac{1}{2}Z - \frac{1}{2}X + \frac{1}{2}X'.
\end{align*}
\]
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So the action of $\rho_1(\sigma)$ on the new basis vectors (written as linear combinations of them) is given by

\begin{align*}
\rho_1(\sigma)(1 - Z) &= X + X', \\
\rho_1(\sigma)(X + X') &= 1 - Z, \\
\rho_1(\sigma)(\psi(X - X')) &= \psi(X - X'), \\
(13) \\
\rho_1(\sigma)(1 + Z) &= \frac{1}{\sqrt{p}}(1 + Z) + \frac{2}{\sqrt{p}} \sum_{k=1}^{r} Y_k, \\
\rho_1(\sigma)(Y_j) &= \frac{1}{\sqrt{p}}(1 + Z) + \sum_{k=1}^{r} A_{kj} Y_k.
\end{align*}

Therefore, the linear map $\rho_1(\sigma)$ has the following matrix presentation in the new basis $B_U$:

\begin{equation}
U^{-1} \rho_1(\sigma) U = 
\begin{bmatrix}
0 & 1 & 0 & 0 \\
1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0_{3 \times (r+1)} \\
0_{(r+1) \times 3} & \frac{1}{\sqrt{p}} & \frac{1}{\sqrt{p}} \cdot a^t & \frac{2}{\sqrt{p}} \cdot a \cdot A
\end{bmatrix}.
\end{equation}

By a similar argument, we have

\begin{align*}
\rho_1(\tau)(1 - Z) &= 1 - Z, \\
\rho_1(\tau)(X + X') &= \psi(X - X'), \\
(15) \\
\rho_1(\tau)(\psi(X - X')) &= \psi^2(X + X'), \\
\rho_1(\tau)(1 + Z) &= 1 + Z, \\
\rho_1(\tau)(Y_j) &= \theta_j Y_j.
\end{align*}
Therefore, under $\mathcal{B}_U$, $\rho_1(\tau)$ has matrix presentation

\[
U^{-1} \rho_1(\tau) U = \begin{bmatrix}
1 & 0 & 0 & 1 \\
0 & 0 & \psi^2 & \theta_1 \\
0 & 1 & 0 & \theta_2 \\
& & \ddots & \ddots \\
& & & \theta_r
\end{bmatrix}.
\]

The empty slots in the matrix are considered as 0-matrices of suitable size.

It is easy to see, either from the actions of $\rho_1(\sigma)$ and $\rho_1(\tau)$ or from the block form of their matrix presentations in the basis $\mathcal{B}_U$, that they preserve the subspaces $H_1 = \text{span}_\mathbb{C}\{1 - Z, X + X', \psi(X - X')\}$ and the subspace $H_2 = \text{span}_\mathbb{C}\{1 + Z, Y_1, \ldots, Y_r\}$ of $H$. So we have

\[
H \cong H_1 \oplus H_2.
\]

In addition, the matrix coefficients of $U^{-1} \rho_1(\sigma) U$ and $U^{-1} \rho_1(\tau) U$ restricted to $H_1$ are in $\mathcal{O}$. \qed

Given Lemma 1, we just have to find a change of basis for the $(r + 1)$-dimensional subspace $H_2$ so that $\rho_1$ restricted to $H_2$ is integral. For convenience, we introduce the following notations:

\[
S' = \rho_1|_{H_2}(\sigma) = \begin{bmatrix}
\frac{1}{\sqrt{p}} \\
\frac{1}{\sqrt{p}} \\
\frac{2}{\sqrt{p}} \\
\frac{2}{\sqrt{p}} \cdot a \\
\frac{1}{\sqrt{p}} \cdot a A
\end{bmatrix},
\]

and

\[
T' = \rho_1|_{H_2}(\tau) = \begin{bmatrix}
1 \\
\theta_1 \\
\theta_2 \\
\ddots \\
\theta_r
\end{bmatrix}.
\]

Instead of $S'$ and $T'$, we would prefer to work with their transposes. We define

\[
S := (S')^t = D^{-1} S' D = \begin{bmatrix}
\frac{1}{\sqrt{p}} & 2 \sqrt{p} \cdot a' \\
\frac{1}{\sqrt{p}} \cdot a \\
\frac{1}{\sqrt{p}} \cdot a A
\end{bmatrix}.
\]
and

\[(21) \quad T := (T')^t = D^{-1}T'D = T' = \begin{bmatrix} 1 & \theta_1 & \cdots & \theta_r \end{bmatrix}.\]

Here

\[(22) \quad D = \begin{bmatrix} \frac{1}{2} & 1 & \cdots & 1 \\ 1 & \theta_1 & \cdots & \theta_r \\ \vdots & \vdots & \ddots & \vdots \\ 1 & \theta_r & \cdots & \theta_r \end{bmatrix}.\]

Theorem 1 now follows from the claim below:

Claim 1. Let \( V \) be the following Vandermonde matrix

\[(23) \quad V = \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & \theta_1 & \theta_1^2 & \cdots & \theta_1^r \\ 1 & \theta_2 & \theta_2^2 & \cdots & \theta_2^r \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \theta_r & \theta_r^2 & \cdots & \theta_r^r \end{bmatrix}.\]

Then \( V^{-1}SV \) and \( V^{-1}TV \) are both integral.

Proof of Theorem 1. The change of basis matrix \((Id_3 \oplus DV)\) makes the block matrices \( U^{-1} \rho_1(\sigma)U \) and \( U^{-1} \rho_1(\tau)U \) integral. Hence \( W = U(Id_3 \oplus DV) \) is the desired change of basis matrix. \qed

To prove the claim, we need the following property of \( SV \). (Convention: in the following discussions, we will index the matrix entries from 0, and recall that by definition \( \theta_0 = 1 \).)

Proposition 1. The \((j, k)\)-th matrix coefficient of \( SV \) is given by

\[(24) \quad (SV)_{jk} = \begin{cases} \sqrt{p}, & \text{if } j = k = 0, \\ 0, & \text{if } 1 \leq j \leq r, \text{ and } k = 0, \\ \left(\frac{r^k}{p}\right)_j \cdot \iota(p) \cdot \theta_j^{-\frac{1}{2}}, & \text{if } k \geq 1. \end{cases}\]

Here \( \left(\frac{\ast}{\ast}\right)_j \) stands for the Jacobi symbol and

\[(25) \quad \iota(p) = \begin{cases} 1, & \text{if } p \equiv 1 \mod 4, \\ i, & \text{if } p \equiv 3 \mod 4. \end{cases}\]

Proof. Case 1. When \( j = k = 0 \), a direct computation shows that

\[(26) \quad (SV)_{00} = \frac{1}{\sqrt{p}} + \frac{2}{\sqrt{p}} \times r = \frac{1}{\sqrt{p}} \times p = \sqrt{p}.\]
Case 2. When $1 \leq j \leq r$, and $k = 0$, we have,

\[
(SV)_{j0} = \frac{1}{\sqrt{p}} + \sum_{l=1}^{r} A_{jl} \times 1
\]

\[
= \frac{1}{\sqrt{p}} + \sum_{l=1}^{r} \frac{1}{\sqrt{p}} (\zeta^{j} + \zeta^{-j})
\]

\[
= \frac{1}{\sqrt{p}} \sum_{l=0}^{2r} \zeta^{jl}
\]

\[
= 0.
\]

The third equality results from the fact that $-l \equiv p - l \pmod{p}$. The last equality stands on the fact that for any $1 \leq j \leq r$, $\zeta^j$ is an $p$-th root of unity, hence is a solution to the minimal polynomial $\Phi_p(x) = 1 + x + \cdots + x^{2r}$ (recall that by assumption $p$ is an odd prime).

Case 3. When $1 \leq k \leq r$, we have

\[
(SV)_{jk} = \frac{1}{\sqrt{p}} + \sum_{l=1}^{r} \frac{1}{\sqrt{p}} (\zeta^{jl} + \zeta^{-jl}) \times \theta^k_l
\]

\[
= \frac{1}{\sqrt{p}} + \frac{1}{\sqrt{p}} \sum_{l=1}^{r} (\zeta^{jl} + \zeta^{-jl}) \times \zeta^{rkl^2}
\]

\[
= \frac{1}{\sqrt{p}} \sum_{l=0}^{2r} \zeta^{jl+rkl^2}
\]

\[
= \frac{1}{\sqrt{p}} \sum_{l=0}^{2r} \zeta^{r(kl^2+rkl^1)}.
\]

Note that by assumption, $1 \leq k \leq r$, hence $\frac{j}{rk}$ is well-defined in the finite field $\mathbb{Z}/p\mathbb{Z}$. Letting $\gamma = \frac{j}{rk} \in \mathbb{Z}/p\mathbb{Z}$, we can continue our calculation as follows:

\[
\frac{1}{\sqrt{p}} \sum_{l=0}^{2r} \zeta^{rk(l^2+2\gamma l)} = \frac{1}{\sqrt{p}} \sum_{l=0}^{2r} \zeta^{rk(l+\gamma)^2-rk\gamma^2}
\]

\[
= \frac{1}{\sqrt{p}} \zeta^{-rk\gamma^2} \sum_{l=0}^{2r} \zeta^{rk(l+\gamma)^2}.
\]
Hence by the quadratic Gauss sum formula, we have

\[(SV)_{jk} = \frac{1}{\sqrt{p}} \times \zeta^{-rk^2} \times \left(\frac{rk}{p}\right)_J \times \varpi(p) \times \sqrt{p} \]
\[(30) \quad = \left(\frac{rk}{p}\right)_J \times \varpi(m) \times \zeta^{-rk^2}.\]

Note that

\[(31) \quad 4r^2 - 1 = (2r + 1)(2r - 1) = p(2r - 1) \equiv 0 \mod p.\]

Therefore,

\[(32) \quad k\gamma^2 = \frac{j^2}{4r^2k} \equiv \frac{j^2}{k} \mod p, \]

and consequently,

\[(33) \quad (SV)_{jk} = \left(\frac{rk}{p}\right)_J \times \varpi(p) \times \zeta^{(rj^2) \times (-\frac{1}{2})} = \left(\frac{rk}{p}\right)_J \times \varpi(p) \times \theta_j^{-\frac{j}{k}},\]

as desired. \(\square\)

To proceed further, let’s recall some basic facts in number theory.

**Lemma 2.** Let \(\epsilon = (-1)^r \times \zeta^{-\frac{r(r+1)}{2}} \in \mathcal{O}^\times.\) Then

\[(34) \quad p = 2r + 1 = \epsilon \prod_{k=1}^{r} (1 - \zeta^k)^2.\]

**Proof.** Recall that

\[(35) \quad \Phi_p(x) = 1 + x + \cdots + x^{2r} = \prod_{l=1}^{2r} (x - \zeta^l).\]
Putting $x = 1$, we have

$$p = 2r + 1 = \prod_{l=1}^{2r} (1 - \zeta^l)$$

$$= \prod_{k=1}^{r} (1 - \zeta^k) \times (1 - \zeta^{-k})$$

$$= \prod_{k=1}^{r} (1 - \zeta^k) \times \zeta^{-k} \times (\zeta^k - 1)$$

$$= \prod_{k=1}^{r} (-\zeta^{-k}) \times \prod_{k=1}^{r} (1 - \zeta^k)^2$$

$$= (-1)^r \times \zeta^{-\frac{r(r+1)}{2}} \times \prod_{k=1}^{r} (1 - \zeta^k)^2$$

$$= \epsilon \prod_{k=1}^{r} (1 - \zeta^k)^2.$$  \hfill \Box

Note that $\epsilon^2$ is also in $\mathcal{O}^\times$. Indeed, by definition, $(-1)^\frac{2}{p}$ is a power of $\psi^2$ and $-\frac{r(r+1)}{4}$ is well-defined in $\mathbb{Z}/p\mathbb{Z}$. Note also that for any integers $\alpha, \beta$ such that $\gcd(\alpha, p) = \gcd(\beta, p) = 1$, we have

$$\frac{1 - \zeta^\alpha}{1 - \zeta^\beta} \in \mathcal{O}^\times.$$  \hfill (37)

This is because in $\mathbb{Z}/p\mathbb{Z}$, we can write $\alpha$ as a multiple of $\beta$, so the quotient in (37) becomes a sum of elements in $\mathcal{O}$, so it is in $\mathcal{O}$. On the other hand, we can write $\beta$ as a multiple of $\alpha$, then the inverse of the quotient in (37) is also a sum of elements in $\mathcal{O}$, hence in $\mathcal{O}$.

Combining Lemma 2 and the above observation, we have

**Corollary 1.**

$$\sqrt{p} = \prod_{k=1}^{r} (1 - \theta_k) \times u,$$

and $u \in \mathcal{O}^\times$.

**Proof.** By Lemma 2, we know that

$$p = \epsilon \prod_{k=1}^{r} (1 - \zeta^k)^2.$$  \hfill (39)
Then
\[
\sqrt{p} = \epsilon^\frac{1}{2} \prod_{k=1}^{r} (1 - \zeta^k),
\]
where $\epsilon^\frac{1}{2} \in \mathcal{O}^\times$ as shown above. Also by the observation above, we have, for any $1 \leq k \leq 2r$,
\[
\eta_k = \frac{1 - \zeta^k}{1 - \theta_k} = \frac{1 - \zeta^k}{1 - \zeta^{rk^2}} \in \mathcal{O}^\times.
\]
Letting $u = \epsilon^\frac{1}{2} \times \prod_{k=1}^{r} \eta_k$, we have
\[
\sqrt{p} = \epsilon^\frac{1}{2} \prod_{k=1}^{r} (1 - \zeta^k) = \prod_{k=1}^{r} (1 - \theta_k) \times (\epsilon^\frac{1}{2} \times \prod_{k=1}^{r} \eta_k) = \prod_{k=1}^{r} (1 - \theta_k) \times u.
\]
Note that $u$, as product of elements in $\mathcal{O}^\times$, is in $\mathcal{O}^\times$.

**Proposition 2.** The 0-th column of $V^{-1}SV$ is a vector in $\mathcal{O}^{r+1}$.

**Proof.** By Proposition 1, for any $j$, we have
\[
(V^{-1}SV)_{j0} = \sum_{l=0}^{r} (V^{-1})_{jl} (SV)_{l0} = (V^{-1})_{j0} \times \sqrt{p}.
\]
To prove the proposition, we simply have to show that $(V^{-1})_{j0} \times \sqrt{p} \in \mathcal{O}$.

By definition, we have
\[
V \cdot (V^{-1}) = Id_{r+1}.
\]
In other words, for any $0 \leq k \leq r$,
\[
\sum_{j=0}^{r} \theta_k^j \times (V^{-1})_{j0} = \delta_{k,0},
\]
where $\delta_{*,*}$ is the Kronecker delta function. Consider the polynomial
\[
P_0(x) = \sum_{j=0}^{r} (V^{-1})_{j0} \times x^j.
\]
By (45), we have
\[
P_0(\theta_0) = 1, \quad P_0(\theta_k) = 0, \quad k = 1, \ldots, r.
\]
Therefore, by the Lagrangian interpolation formula, we have
\[
P_0(x) = \sum_{j=0}^{r} (V^{-1})_{j0} \times x^j = \prod_{n=1}^{r} \frac{x - \theta_n}{1 - \theta_n}.
\]
By comparing coefficients, we can write down explicit formulas for \((V^{-1})_{j0}\).
But what is more important here is that

\[(V^{-1})_{j0} \times \prod_{n=1}^{r} (1 - \theta_n) \in \mathcal{O},\]

since it is a coefficient of the integral polynomial \(\prod_{n=1}^{r} (x - \theta_n)\). On the other hand, by Corollary 1, we have \(\sqrt{p} = \prod_{n=1}^{r} (1 - \theta_n) \times u\) for some unit \(u \in \mathcal{O}^\times\), hence

\[(V^{-1})_{j0} \times \sqrt{p} = (V^{-1})_{j0} \times \prod_{n=1}^{r} (1 - \theta_n) \times u \in \mathcal{O}.\]

\[\square\]

By Proposition 2, we are left to show that the \(l\)-th column vector of \(SV\) for \(1 \leq l \leq r+1\) and all the column vectors of \(TV\) have the property that after multiplying them \(V^{-1}\) from left we get vectors in \(\mathcal{O}^{r+1}\).

In light of Proposition 1, we have the following observation:

**Lemma 3.** The \(l\)-th column vector of \(SV\) for \(1 \leq l \leq r+1\) and all the column vectors of \(TV\) are, up to a scalar multiplication by \(\pm i\) or \(\pm 1\), of the form \([1, \theta_1^j, \theta_2^j, \cdots, \theta_r^j]^t\) for some \(0 \leq j \leq 2r\).

**Proof.** This is a direct result of Proposition 1 and the definition of \(T\). \(\square\)

Hence we reduce our problem to the problem of showing

**Claim 2.** The vectors in Lemma 3, after being multiplied by \(V^{-1}\) from left, become vectors in \(\mathcal{O}^{r+1}\).

We will need the following lemma.

**Lemma 4.** Let \(f(x) = (x - x_0)(x - x_1)(x - x_2) \cdots (x - x_r) = x^{r+1} + a_1 x^r + \cdots + a_r x + a_{r+1}\). Then the matrix

\[
C = \begin{bmatrix}
0 & 1 & 0 & 0 & \cdots & 0 \\
0 & 0 & 1 & 0 & \cdots & 0 \\
0 & 0 & 0 & 1 & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & 0 & \cdots & 1 \\
-a_{r+1} & -a_r & -a_{r-1} & -a_{r-2} & \cdots & -a_1
\end{bmatrix}
\]

has \([1, x_i, x_i^2, \cdots, x_i^r]^t\) as eigenvectors corresponding to eigenvalues \(x_i\) for any \(0 \leq i \leq r\).
Proof. For any $0 \leq i \leq r$, we have

$$
C = \begin{bmatrix}
1 & x_i & x_i^2 & \cdots & x_i^r \\
x_i & x_i^2 & x_i^3 & \cdots & x_i^{r+1} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
x_i^r & x_i^{r+1} & x_i^{r+2} & \cdots & x_i^{2r}
\end{bmatrix}.
$$

Since $f(x_i) = x_i^{r+1} + a_1 x_i^r + \cdots + a_r x_i + a_{r+1} = 0$, we have $x_i^{r+1} = -a_{r+1} - a_r x_i - \cdots - a_1 x_i^r$. Hence

$$
C = \begin{bmatrix}
1 & x_i & x_i^2 & \cdots & x_i^r \\
x_i & x_i^2 & x_i^3 & \cdots & x_i^{r+1} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
x_i^r & x_i^{r+1} & x_i^{r+2} & \cdots & x_i^{2r}
\end{bmatrix} = \begin{bmatrix}
1 & x_i \\
x_i & x_i^2 \\
\vdots & \vdots \\
x_i^r & x_i^{r+1}
\end{bmatrix}.
$$

Corollary 2. $V^{-1}TV$ has entries in $\mathcal{O}$. Consequently, $V^{-1}T^jV$ has entries in $\mathcal{O}$ for every $0 \leq j \leq 2r$. In particular, their first columns are vectors in $\mathcal{O}^{r+1}$.

Proof. Let $x_k = \theta_k$ in Lemma 4. Then we have the corresponding polynomial $h(x) = (x - 1)(x - \theta_1) \cdots (x - \theta_r) = x^{r+1} + b_1 x^r + \cdots + b_{r+1}$ with its companion matrix

$$
C_h = \begin{bmatrix}
0 & 1 & 0 & 0 & \cdots & 0 \\
0 & 0 & 1 & 0 & \cdots & 0 \\
0 & 0 & 0 & 1 & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & 0 & \cdots & 1 \\
-b_{r+1} & -b_r & -b_{r-1} & -b_{r-2} & \cdots & -b_1
\end{bmatrix}.
$$

By Lemma 4, $V^t$ diagonalizes $C_h$:

$$
(V^t)^{-1}C_h(V^t) = T.
$$

Taking the transpose of both sides, we have

$$
V(C_h)^t(V^{-1}) = T.
$$

Note that $T$ is diagonal, then

$$
V^{-1}TV = (C_h)^t.
$$

As the $\mathcal{O}$ is a ring, $b_k \in \mathcal{O}$ for all $k$. Hence all entries in $C_h$ are in $\mathcal{O}$, so is its transpose therefore $V^{-1}TV$. The rest of the corollary follows immediately.

Observing that the first columns of $T^jV$ correspond exactly to the vectors in Claim 2, we can conclude that Claim 2 is true. As a result, Claim 1 as well as Theorem 1, is true.
4. Weil representation over finite fields

In Section 3 of [LW05], the genus one representation of $\text{SL}(2, \mathbb{Z})$ associated to the $\text{SO}(3)$-TQFT for a fixed odd prime $p \geq 5$ (in the sense of [BHMV95]) was considered, where the authors identified the representation with the odd part of the Weil representation of $\text{SL}(2, \mathbb{Z}/p\mathbb{Z})$. Here we will prove a result in some sense “dual” to that in [LW05]. Namely, for the fixed prime $p$, a factor of $\rho_1$ factors through the even part of the Weil representation of $\text{SL}(2, \mathbb{Z}/p\mathbb{Z})$.

To clarify the above paragraph, let us briefly recall the definition of the Weil representation over finite fields. The basic idea is to realize elements in $\text{SL}(2, \mathbb{Z}/p\mathbb{Z})$ as intertwining operators of the Heisenberg representation of the Heisenberg group, which will be defined below. There is a vast amount of research on the Weil representations, and we will only extract some essential ingredients of the representation of $\text{SL}(2, \mathbb{Z}/p\mathbb{Z})$ here. The interested readers are referred to [Ger77].

Fix an odd prime $p \geq 5$. We start by looking at a group called the Heisenberg group $\mathcal{H}_p$, defined by

$$\mathcal{H}_p = \left\{ \begin{bmatrix} 1 & y & z \\ 0 & 1 & x \\ 0 & 0 & 1 \end{bmatrix}, \ x, y, z \in \mathbb{Z}/p\mathbb{Z} \right\}.$$  

Here the group multiplication is the matrix multiplication. Considering the embedding

$$\mathbb{Z}/p\mathbb{Z} \rightarrow \mathcal{H}_p, \ z \mapsto \begin{bmatrix} 1 & 0 & z \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

we can view the group $\mathcal{H}_p$ as a central extension of $\mathbb{Z}/p\mathbb{Z}$. More precisely, we have a short exact sequence

$$0 \rightarrow \mathbb{Z}/p\mathbb{Z} \rightarrow \mathcal{H}_p \rightarrow (\mathbb{Z}/p\mathbb{Z})^2 \rightarrow 0.$$  

The quotient map is given by

$$\mathcal{H}_p \rightarrow (\mathbb{Z}/p\mathbb{Z})^2, \ \begin{bmatrix} y & z \\ 0 & 1 & x \\ 0 & 0 & 1 \end{bmatrix} \mapsto \begin{bmatrix} x \end{bmatrix}.$$

With a suitable choice of section to the quotient map above, it is not difficult to show that the defining action of $\text{SL}(2, \mathbb{Z}/p\mathbb{Z})$ on $(\mathbb{Z}/p\mathbb{Z})^2$,

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} ax + by \\ cx + dy \end{bmatrix}, \text{ where } \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \text{SL}(2, \mathbb{Z}/p\mathbb{Z}), \ \begin{bmatrix} x \\ y \end{bmatrix} \in (\mathbb{Z}/p\mathbb{Z})^2,$$

can be lifted to $\mathcal{H}_p$, and the lifted action is trivial on the center $Z(\mathcal{H}_p)$ of $\mathcal{H}_p$. 

Let $L^2(\mathbb{Z}/p\mathbb{Z})$ denote the space of complex-valued functions on $\mathbb{Z}/p\mathbb{Z}$. It is easily seen that $\dim(L^2(\mathbb{Z}/p\mathbb{Z})) = p$. Given any irreducible central character $\varphi : \mathbb{Z}/p\mathbb{Z} \to \mathbb{C}$, we can define a representation $\pi_\varphi : \mathcal{H}_p \to \text{GL}(L^2(\mathbb{Z}/p\mathbb{Z}))$ by

\[
(\pi_\varphi \left( \begin{bmatrix} 1 & y & z \\ 0 & 1 & x \\ 0 & 0 & 1 \end{bmatrix} \right) ) (f) = \varphi(-xa + z)f(a - y),
\]

for any $\left( \begin{bmatrix} 1 & y & z \\ 0 & 1 & x \\ 0 & 0 & 1 \end{bmatrix} \right) \in \mathcal{H}_p$ and $f \in L^2(\mathbb{Z}/p\mathbb{Z})$.

Since $\pi_\varphi$ is $p$-dimensional, by the representation theory of finite groups, it is either a direct sum of $p$ 1-dimensional representations or irreducible. However, in the first case, $\pi_\varphi|_{\mathcal{Z}(\mathcal{H}_p)}$ should be trivial, which contradicts to our assumption on $\varphi$.

By Theorem 3.1 of [Pra09], if two irreducible representations of $\mathcal{H}_p$ coincide on the center $\mathcal{Z}(\mathcal{H}_p)$, then they are equivalent. Now let $\varphi$ be any nontrivial irreducible central character. For any $\alpha \in \text{SL}(2, \mathbb{Z}/p\mathbb{Z})$, consider the representation $\pi_\varphi \circ \alpha$, a $p$-dimensional representation of $\mathcal{H}_p$ with the property

\[
(\pi_\varphi \circ \alpha)|_{\mathcal{Z}(\mathcal{H}_p)} = \varphi = \pi_\varphi|_{\mathcal{Z}(\mathcal{H}_p)}.
\]

By a similar argument as above, we know that $\pi_\varphi \circ \alpha$ is also irreducible. Hence $\pi_\varphi \circ \alpha$ is equivalent to $\pi_\varphi$, i.e., there is an intertwining operator (unique up to scalar by Schur’s lemma), denoted by $W_\varphi(\alpha) \in \text{GL}(L^2(\mathbb{Z}/p\mathbb{Z}))$ such that the diagram

\[
\begin{array}{ccc}
L^2(\mathbb{Z}/p\mathbb{Z}) & \xrightarrow{\pi_\varphi(h)} & L^2(\mathbb{Z}/p\mathbb{Z}) \\
W_\varphi(\alpha) \downarrow & & \downarrow W_\varphi(\alpha) \\
L^2(\mathbb{Z}/p\mathbb{Z}) & \xrightarrow{\pi_\varphi(\alpha(h))} & L^2(\mathbb{Z}/p\mathbb{Z})
\end{array}
\]

commutes for all $h \in \mathcal{H}_p$.

If we consider the class of $W_\varphi(\alpha)$ in the projective general linear group $\text{PGL}(L^2(\mathbb{Z}/p\mathbb{Z}))$ instead of $W_\varphi(\alpha) \in \text{GL}(L^2(\mathbb{Z}/p\mathbb{Z}))$, we can eliminate the scaling ambiguity and get a well-defined projective representation of $\text{SL}(2, \mathbb{Z}/p\mathbb{Z})$ (by abuse of notation this map is also denoted by $W_\varphi$):

\[
W_\varphi : \text{SL}(2, \mathbb{Z}/p\mathbb{Z}) \to \text{PGL}(L^2(\mathbb{Z}/p\mathbb{Z})).
\]

We call this projective representation the Weil representation of $\text{SL}(2, \mathbb{Z}/p\mathbb{Z})$ (with respect to $\varphi$).
Remark. We may omit the word “projective” when it does not cause confusions, and we will present an element in $\text{PGL}(\mathcal{L}(\mathbb{Z}/p\mathbb{Z}))$ by one of its representatives in $\text{GL}(\mathcal{L}(\mathbb{Z}/p\mathbb{Z}))$.

Again, let $\sigma = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ and $\tau = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ be the generators of $\text{SL}(2, \mathbb{Z})$. Their reductions mod $p$ generate $\text{SL}(2, \mathbb{Z}/p\mathbb{Z})$. By an abuse of notation, we will not distinguish $\sigma$ and $\tau$ from their reductions. For $j \in \mathbb{Z}/p\mathbb{Z}$, let $f_j : \mathbb{Z}/p\mathbb{Z} \to \mathbb{Z}/p\mathbb{Z}$ be the $j$-th Kronecker delta function defined by

\[ f_j(x) = \delta_{j,x}, \quad \forall x \in \mathbb{Z}/p\mathbb{Z}. \]

The set $\{ f_j | j \in \mathbb{Z}/p\mathbb{Z} \}$ is a basis of $\mathcal{L}(\mathbb{Z}/p\mathbb{Z})$, which we fix in the following.

Now, to describe the Weil representation with respect to a nontrivial character $\varphi$, it suffices to give the matrices of $W_\varphi(\sigma)$ and $W_\varphi(\tau)$ under the fixed basis defined above. It is not difficult to compute that

\[ W_\varphi(\sigma) = \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & \varphi(1) & \varphi(2) & \cdots & \varphi(p-1) \\ 1 & \varphi(2) & \varphi(4) & \cdots & \varphi(2(p-1)) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \varphi(p-1) & \varphi(2(p-1)) & \cdots & \varphi((p-1)^2) \end{bmatrix} \]

and that

\[ W_\varphi(\tau) = \begin{bmatrix} 1 & \varphi(-\frac{1^2}{2}) \\ \varphi(-\frac{2^2}{2}) & \varphi(-\frac{3^2}{2}) \\ \vdots & \vdots \\ \varphi(-\frac{(p-1)^2}{2}) \end{bmatrix}. \]

As before, $\frac{1}{2}$ is understood as the multiplicative reciprocal of 2 in $\mathbb{Z}/p\mathbb{Z}$.

Note that this representation is reducible. Indeed, it is easy to see that the $\mathbb{C}$-span of $\{ f_k + f_{p-k} | k = 0, 1, \cdots, r \}$ and $\{ f_k - f_{p-k} | k = 0, 1, \cdots, r \}$ are two invariant subspaces. If we denote the former vector space by $E^\text{even}$ and the latter by $E^\text{odd}$, then we have a decomposition of representation spaces $\mathcal{L}(\mathbb{Z}/p\mathbb{Z}) \cong E^\text{even} \oplus E^\text{odd}$.

We are mainly interested in the restriction of the Weil representation on the even subspace $E^\text{even}$. By (68) and (69), we have

\[ W^{\text{even}}_\varphi(\sigma) = W_\varphi|_{E^\text{even}}(\sigma) = \begin{bmatrix} 1 & a^r \\ 2 \cdot a & B \end{bmatrix}. \]

Here $B$ is an $r \times r$-matrix with entries given by
\[ B_{jk} = \varphi(jk) + \varphi(-jk), \quad \forall j, k = 1, \ldots, r. \]

In addition, we have

\[ W_{\varphi}^{\text{even}}(\tau) = W_{\varphi}|_{E^{\text{even}}} (\tau) = \begin{bmatrix}
1 \\
\varphi(-\frac{1^2}{2}) \\
\varphi(-\frac{2^2}{2}) \\
\ddots \\
\varphi(-\frac{r^2}{2})
\end{bmatrix}. \]

If we choose the special character \( \varphi : \mathbb{Z}/p\mathbb{Z} \to \mathbb{C} \) defined by

\[ \varphi(j) = \zeta^j, \]

we will have \( \sqrt{p}A = B \) and

\[ 2r \equiv -1 \mod p \Rightarrow r \equiv -\frac{1}{2} \mod p \Rightarrow \varphi(-\frac{j^2}{2}) = \zeta^{rj^2} = \theta_j. \]

Therefore,

\[ W_{\varphi}^{\text{even}}(\sigma) = \sqrt{p} \begin{bmatrix}
1 \\
\frac{1}{\sqrt{p}} \\
\frac{1}{\sqrt{p}} \cdot a^t \\
\frac{1}{\sqrt{p}} \cdot a \cdot A
\end{bmatrix}, \]

and

\[ W_{\varphi}^{\text{even}}(\tau) = \begin{bmatrix}
1 \\
\theta_1 \\
\ddots \\
\theta_r
\end{bmatrix}. \]

Recall from previous sections that \( H \cong H_1 \oplus H_2 \) and that \( H_2 \) is an \((r+1)\)-dimensional vector space. We can then identify \( H_2 \) and \( E^{\text{even}} \) via

\[ 1 + \mathbb{Z} \leftrightarrow 2f_0, \quad Y_j \leftrightarrow f_j + f_{-j}, \quad \forall j = 1, \ldots, r. \]

With all the ingredients ready, we now state the theorem of this section:

**Theorem 2.** Let \( \varphi \) be chosen as in Equation (73), then the restriction of the \( SO(p)_2 \)-TQFT representation of \( SL(2, \mathbb{Z}) \) to \( H_2, \rho_1|_{H_2} \), factors through \( W_{\varphi}^{\text{even}} \), the even part of the Weil representation of \( SL(2, \mathbb{Z}/p\mathbb{Z}) \) associated
to $\varphi$. In other words, we have the following commutative diagram:

\[
\begin{array}{ccc}
SL(2,\mathbb{Z}) & \xrightarrow{\rho_1|_H_2} & PGL(H_2) \\
\text{mod } p & & \downarrow \cong \\
SL(2,\mathbb{Z}/p\mathbb{Z}) & \xrightarrow{W_{\text{even}}^\rho} & PGL(E_{\text{even}})
\end{array}
\]  

(78)

Proof. By Equations (18), (19), (75) and (76), we know that $\rho_1|_H_2$ and $W_{\varphi}^{\text{even}}$ are only different by a scalar multiple, hence as projective representations, they are the same. \hfill \square

We immediately have the following corollaries:

**Corollary 3.** The image of $\rho_1|_H_2$ is finite.

From equations (16) and (14), it is easy to see that the image of $\rho_1|_{H_1}$ can be viewed as a subgroup of the permutation group of the finite set $\{\pm 1, \pm i\} \times \{1 - \mathbb{Z}, X \pm X'\}$, so $\rho_1|_{H_1}(SL(2,\mathbb{Z}))$ is also finite. Hence, together with the above corollary, we have:

**Corollary 4.** The image of $\rho_1$ is finite.

**Remark.** The above corollary is a special case of the famous finiteness result obtained by [NS10].

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