Faraday instability in a two-component Bose–Einstein condensate

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Received 10 March 2008
Accepted for publication 21 August 2008
Published 2 October 2008
Online at stacks.iop.org/PhysScr/78/045009

Abstract
Motivated by recent experiments on Faraday waves in Bose–Einstein condensates (BECs), we investigate the dynamics of a two-component cigar-shaped BEC subjected to periodic modulation of the strength of the transverse confinement. It is shown that two coupled Mathieu equations govern the dynamics of the system. We found that the two-component BEC in a phase mixed state is relatively more unstable towards pattern formation than the phase segregated state.

PACS numbers: 03.75.Kk, 03.75.Mn, 05.45.-a, 47.54.-r

(Some figures in this article are in colour only in the electronic version.)

1. Introduction
Faraday waves are generated when the free surface of a fluid layer is subjected to a periodic vertical acceleration [1]. When the acceleration exceeds a threshold value, surface waves appear, oscillating at half the forcing frequency. Recently, Faraday waves, which come under the category of spontaneous pattern formation, a very general phenomenon studied in different fields of nonlinear science, were observed in a cigar-shaped Bose–Einstein condensate (BEC) by periodically modulating the radial trap frequency [2]. The radial modulation leads to a periodic modulation of the density of the cloud in time, which in turn leads to a periodic change in the nonlinear interactions. This leads to the parametric excitation of longitudinal sound-like waves (Faraday waves) in the direction of weak confinement. It has been shown theoretically that for a BEC, Faraday instability can be generated either by modulating the scattering length by Feshbach resonance [3] or by modulating the trap frequency in the tight confinement direction [4]. In both cases, the dynamics are governed by a Mathieu equation that is typical of parametrically driven systems. Floquet analysis reveals that a series of resonances exists, consisting of a main resonance at half the driving frequency and higher resonance tongues at integer multiples of half the driving frequency. A theoretical analysis of Faraday instability based on a Mathieu-type analysis of the nonpolynomial Schrödinger equation was given recently [5]. From the perspective of phonon number occupation, the Faraday-type modulation has been analysed theoretically [6]. In general, the nonlinear spatiotemporal dynamics of BECs is attracting increasing interest in recent years, with a major focus on controlling structures like solitons by temporal modulation of the atomic scattering length [7]. The evolution of a BEC in a time-dependent trap has been addressed earlier by some authors [8].

The aim of the present paper is to analyse Faraday instability in a two-component cigar-shaped BEC. The two components could be, for example, $^{87}$Rb atoms in two different hyperfine states in different external trapping potentials. A periodic modulation of the radial trap frequencies would give rise to Faraday instability in both the components. The dynamics of this system is now governed by two coupled Mathieu equations. If $\Omega_1$ and $\Omega_2$ are the natural frequencies of the two components, then the parametric resonances will occur not only at $2\Omega_1/n$ and $2\Omega_2/n$ (where $n$ is an integer) but also at the combination frequencies ($|\pm\Omega_1 \pm \Omega_2|/n$). The behaviour of this system would then be similar to spontaneous pattern formation in miscible and immiscible binary classical liquids [9].

An important question is the stability of such Faraday waves in binary mixtures because depending on the atom–atom interaction strengths $g_{ii}$ (intraspecies interactions) and $g_{ij}, i \neq j$ (interspecies interaction), binary mixtures can be phase separated ($g_{11} < g_{22}$) or phase mixed ($g_{11} > g_{22}$) [10]. The nonlinear interaction can be conveniently manipulated by Feshbach resonances [11]. The purpose of the present paper is to analyse the influence of the interactions on the stability of the parametric excitations that are generated theoretically.
Later qualitatively, we will discuss the role of damping and, by doing so, we do not lose the essential physics. In the presence of the trapping potentials, we can define the internal spin states, e.g., \( g \), \( N \) is the intraspecies scattering length. The \( N \) is the number of particles of the \( i \)th condensate atoms is specified by \( g_{ii, \kappa} = (4 \pi \hbar^2 a_{ii, \kappa})/m \) and that between 1 and 2 by \( g_{12} = g_{21} = \frac{(4 \pi \hbar^2 a_{12})}{m} \). Here \( a_{ii} \) is the intraspecies s-wave scattering length and \( a_{12} = a_{21} \) is the interspecies scattering length. \( N_i \) is the number of particles of the \( i \)th species. Here \( a_{ii, \kappa} \) is the trapping potential of the \( i \)th component. The wavefunctions are normalized according to \( \int |\psi_i|^2 \, d^3r = 1 \) in the absence of damping. Here we are ignoring damping and, by doing so, we do not lose the essential physics. Later qualitatively, we will discuss the role of damping in the stability chart and include it while studying the spatio-temporal dynamics of the BEC.

When the condition \( g_{11, \kappa} \ll g_{12}^2 \) is satisfied, the condensates are phase segregated. In the opposite limit, i.e. \( g_{11} g_{22} > g_{12}^2 \), the condensates are in a mixed state. Our study covers the one-dimension (1D) case of a cigar-shaped BEC extended along the \( z \)-direction, \( V_i(r, t) = \left( m/2 \right) \left[ \Omega_i(t)^2 \left( x^2 + y^2 \right) + \Omega_{z, i}^2 z^2 \right] \), where \( \Omega_{z, i} \ll \Omega_i \) is assumed and \( \Omega_{z, i} \) and \( \Omega_i \) are the frequencies of the trap for the \( i \)th component along the weak and tight confinement directions, respectively. We assume that \( \Omega_i \) is subjected to periodic modulation: \( \Omega_i(t) = \Omega_i [1 + \alpha_i \cos \Omega_i t] \), \( \alpha_i \ll 1 \). Here \( \Omega_i \) is the trap modulation frequency of the \( i \)th component. The radial modulation leads to a periodic change of the density of the cloud in time, which is equivalent to a change in the nonlinear interactions and speed of sound. In this paper we will always work either in the deep phase mixed regime or in the deep phase segregated regime. At the boundary separating the phase mixed regime and phase separated regime, the dynamics could be very complex as the trap modulations can continuously switch the system from one regime to the other. In terms of scaled variables, \( \tau = \Omega_2 t, \bar{R} \equiv (X, Y, Z) = (x, y, z)/\Omega_2, \Omega_1 = \hbar/\Omega_2^2 \) and \( \bar{\xi}_i = \sqrt{2} \tilde{\Omega}_N \tilde{\Omega}_2 \psi_i \), equations (1) and (2) take the following dimensionless form:

\[
\frac{\partial \bar{\xi}_1}{\partial \tau} = \left\{ -\frac{\hbar^2}{2m} \nabla_i^2 + V_1(\bar{r}, \tau) + g_{i, 1} N_1 |\psi_1|^2 + g_{12, 2} |\psi_2|^2 - \mu_i \right\} \bar{\psi}_1,
\]

\[
\frac{\partial \bar{\psi}_2}{\partial \tau} = \left\{ -\frac{\hbar^2}{2m} \nabla_i^2 + V_2(\bar{r}, \tau) + g_{22, 2} |\psi_2|^2 + g_{21, 1} |\psi_1|^2 - \mu_2 \right\} \bar{\psi}_2.
\]

As in [4], we reduce the BEC 3D dynamics to an effective 1D description using a multiple scale analysis [12]. We write the dimensionless wavefunction as

\[
\bar{\xi}_i = \sqrt{\omega_i} \exp \left( -\frac{\alpha_i}{\bar{\Omega}_2} (X^2 + Y^2) \right) \phi_i(Z, \tau), \quad i = 1, 2.
\]

For a flat trap in a weakly confined space \( \omega_{w, i} = 0 \), consequently equations (3) and (4) become

\[
2 \bar{t} \frac{\partial \phi_1}{\partial \tau} = \frac{-\delta^2}{\delta Z^2} + g_1 \bar{\omega}_1 |\phi_1|^2 + \omega_R |\phi_2|^2 - \mu_1^i \phi_1, \quad i = 1, 2,
\]

\[
2 \bar{t} \frac{\partial \phi_2}{\partial \tau} = \frac{-\delta^2}{\delta Z^2} + g_2 \bar{\omega}_2 |\phi_2|^2 + \omega_R |\phi_1|^2 - \mu_2^i \phi_2,
\]

where \( \omega_R = (2 \bar{\omega}_1 \bar{\omega}_3 / (\bar{\omega}_1 + \bar{\omega}_3)) \) and \( \mu_i^i = 2(\bar{\mu}_i - \bar{\omega}_i) \). In order to proceed analytically, we study equations (6) and (7) for the simple case of \( \alpha_1 = \alpha_2 = \alpha, \omega_1 = \omega_2 = \omega \) and \( \omega_{w, 1} = \omega_{w, 2} = \omega_w \). In this case, \( \omega_R = \bar{\omega}_1 (1 + \alpha \cos \omega \tau) \). For a flat potential and in the absence of any perturbation (\( \alpha = 0 \), \( \phi_i = \phi_i^0 \) is a constant independent of \( Z \). Equations (6) and (7) yield

\[
|\phi_1^0|^2 = \frac{\beta_2 - \beta_1 \bar{g}_1}{\bar{g}_1 (1 - \bar{g}_1)} = \delta_1,
\]

\[
|\phi_2^0|^2 = \frac{\beta_1 - \beta_2 \bar{g}_1}{\bar{g}_1 (1 - \bar{g}_1)} = \delta_2,
\]

where \( \beta_i = (2 \bar{\mu}_i - \bar{\omega}_i), \quad i = 1, 2 \). In the presence of the modulations, equations (6) and (7) admit the following homogeneous states:

\[
\phi_1^0 = \sqrt{\delta_1} \exp \left( -\frac{i \alpha}{2 \omega_{w}} \sin \omega \tau \right),
\]

\[
\phi_2^0 = \sqrt{\delta_2} \exp \left( -\frac{i \alpha}{2 \omega_{w}} \sin \omega \tau \right).
\]
This yields $\mu_1 = \mu_2 = 1$, $|\phi_1^0|^2 = \left| \frac{1 - q_{11}}{a_{11}(1 - q_{11})} \right|$ and $|\phi_2^0|^2 = \left| \frac{1 - q_{12}}{a_{12}(1 - q_{12})} \right|$.

Now we wish to know whether the spatially homogeneous external driving fields are able to induce a spontaneous spatial symmetry breaking of this homogeneous state of the coupled BECs. For that, we perform a linear stability analysis of equations (6) and (7) by adding a small perturbation in the form

$$\phi_i = \bar{\phi}_i^0 \left(1 + W_i \cos \tilde{k}_n Z\right), \quad i = 1, 2,$$

where $W_i$ and $\tilde{k}_n$ (where $i = 1, 2$) are the complex valued amplitude and the wave vector of the perturbation of the $i$th component, respectively. Substituting equation (12) into (6) and (7) and linearizing with respect to $W_i$ leads to the following coupled Mathieu equations for $u_i = Re W_i$

$$\frac{\partial^2 u_1}{\partial \tau^2} + (\Omega_1^2 + \alpha Q_{11} \cos \omega \tau) u_1 + Q_{12} (1 + \alpha \cos \omega \tau) u_2 = 0,$$

$$\frac{\partial^2 u_2}{\partial \tau^2} + (\Omega_2^2 + \alpha Q_{22} \cos \omega \tau) u_2 + Q_{21} (1 + \alpha \cos \omega \tau) u_1 = 0,$$

(13)

(14)

where $\Omega_i(\tilde{k}_n) = (\tilde{k}_n/2)\sqrt{\tilde{k}_n^2 + 2q_{ii}}$ is the dispersion relation of the perturbation of the $i$th condensate in the absence of driving. Also $q_{11} = \frac{g_{11}(1 - g_{11})}{(1 - g_{11}) \tilde{k}_n}$, $q_{22} = \frac{g_{22}(1 - g_{22})}{(1 - g_{22}) \tilde{k}_n}$, $q_{12} = q_{11}/g_{11}$, $q_{21} = q_{22}/g_{22}$, $Q_{11} = k_n^2 q_{11}/2$, $Q_{22} = k_n^2 q_{22}/2$, $Q_{12} = k_n^2 q_{12}/2$ and $Q_{21} = k_n^2 q_{21}/2$.

As known from the general theory of equations of the type (13) and (14), instability may occur for $\omega$ near twice the natural frequencies and their subharmonics, $2\Omega_i/n$, (where $i = 1, 2$, and $n$ is an integer) and also close to the
so-called combination frequencies and their subharmonics, \( |±Ω₁±Ω₂|/n \). The boundary curves for the instability regions must therefore emerge from these frequencies. The wavenumbers \( k_{s,n} \) corresponding to the resonance tongues near the natural frequencies and their subharmonics are

\[
k_{s,n} = \sqrt{-q_{ii} + q_{ii}^2 + 4(\alpha \omega/2)^2}.
\]

In order to find the stability diagrams for the two coupled Mathieu equations, we employ the method indicated in [13]. The method involves infinite determinants which are truncated and then the eigenvalues are found using Mathematica, together with a condition on the characteristic exponent for the solutions on the boundary curves. The method is briefly discussed in the appendix. The result for the phase separated case \((g₁ = 0.5, g₂ = 0.75)\) and the phase mixed case \((g₁ = 1.25, g₂ = 1.5)\) is shown in figures 1 and 2 respectively. The shaded regions are the regions of instability. Clearly, one notices that the two sets of resonance tongues, namely \( Ω₁, Ω₂ \) and \((Ω₁ + Ω₂)/2\) and \( 2Ω₁, 2Ω₂ \) and \( Ω₁ + Ω₂ \), for the phase separated case are more widely spaced with larger stability region in between the tongues than for the phase mixed case. The stable regions between the resonance tongues for the phase separated mixture extend up to \( α = 0.2 \), while those for the phase mixed case extend only up to \( α = 0.075 \). The deeper one goes into the phase mixed state, and the stable regions between the resonance tongues further decrease. If the detuning of the driving frequency is of the same order for two adjacent modes, then these modes can enter in competition. The competition between nearly degenerate modes has been shown to lead to chaotic behaviour in fluids [14]. Since the resonance tongues are more closely spaced in the phase mixed state, the mode competition will be stronger for this case. In the absence of damping, all the modes are excited at \( α = 0 \). A finite damping sets nonzero threshold values for the parametric instability and removes the degeneracy of threshold values for higher resonances tongues.

Now, we study the time and space evolution of the deviation from the initial density \( f_i = \frac{\left|\phi_i\right|^2 - \left|\phi_{i0}\right|^2}{\left|\phi_{i0}\right|^2} \) by including phenomenologically a damping term proportional to \( γ(du_i/dτ) \) in equations (13) and (14). Here \( γ \) is the damping coefficient. The coupled Mathieu equations (13) and (14) are solved using Mathematica. Figures 3 and 4 illustrate the time and space evolution of \( f_1 \) (density deviation of the first condensate) and \( f_2 \) (density deviation of the second condensate) in the phase mixed state, respectively. The parameters are \( ω = 0.47 \times 10^{-3}, α = 0.1, γ = 0.1, g₁ = 1.25 \) and \( g₂ = 1.5 \). The periodicity of Faraday patterns generated is always more for the condensate with lower \( q_{ii} \). We found that this observation is also true for the phase separated case. The influence of interactions on the periodicity of pattern formation is clear from the expression \( k_{s,n} = \sqrt{-q_{ii} + q_{ii}^2 + 4(α \omega/2)^2} \). Tuning the interactions, controls the \( k_{s,n} \) and hence the periodicity. Also noticed is the fact that the onset of pattern formation is not immediate but takes a certain time due to damping. For long enough times in the presence of a continuous modulation, the excitations will grow and eventually the condensates will be destroyed. This contrasts with the everlasting periodic revivals of the spatial modulations in the one-component 1D case [A.1]. The spontaneous pattern formation for combination frequencies is much more complex and simple analytical formulations cannot be done and would require a full numerical simulation of the coupled equations (1) and (2). Time evolution of the amplitudes \( u_i \), where \( i = 1 \) and 2, is shown in figure 5 for the phase mixed (plots (a) and (b)) and the phase separated (plots (c) and (d)) cases. The time \( τ = ατ/2 \). The parameters chosen are \( ω = 1.414 \times 10^{-5}, γ = 0.05 \). The values of \( g₁ \) and \( g₂ \) are the same as before. For plots (a) and (c), the strength of the perturbation is \( α = 0.15 \), and for plots (b) and (d), \( α = 0.4 \). For a given \( ω \) for the phase mixed case, the perturbation amplitude \( (u_i) \) which is damped for \( α = 0.15 \) grows in time.
on increasing the perturbation strength ($\alpha = 0.4$). The time evolution of $u_1$ and $u_2$ for the phase mixed case is also found to evolve in phase. On the other hand, for the phase separated case, the amplitudes $u_1$ and $u_2$ evolve out of phase. On increasing $\alpha$ from 0.15 to 0.4, no substantial change is seen in the growth rate of $u_1$ and $u_2$ for the phase separated case. The influence of damping on the phase mixed case is shown in figure 6. Clearly, on increasing the damping coefficient, the amplitudes $u_i$ are damped instead of growing. In the phase separated case, increasing the damping only reduces the growth rate (data not shown).

3. Conclusion

In conclusion, we have demonstrated that the dynamics of Faraday instability in a two-component elongated cigar-shaped BEC is described by two coupled Mathieu equations. The dynamics depend on the two-body intra- and interspecies interactions. In contrast with the single-component case, we found that parametric resonances can occur not only at integral multiples of half the natural frequencies of the two components but also at integral multiples of half the combination frequencies. The stability chart in the $[\omega, \alpha]$ plane reveals that the pattern formation in the phase mixed case is more sensitive towards variation in the strength of perturbation ($\alpha$) than the phase separated mixture. This was confirmed by the spatiotemporal dynamics of the BEC where we found that the amplitude of the perturbation in the phase mixed case changes substantially with the perturbation and damping parameter as compared with the phase segregated case.

Appendix

The stability problem for the $n$ coupled Mathieu equations

$$
\frac{d^2 \mathbf{y}}{dt^2} + (A^* + 2gQ^* \cos \omega t) \mathbf{y} = 0
$$

is studied by first substituting $\omega t = 2\pi$. Here $A^*$ and $Q^*$ are $n \times n$ matrices. This reduces the above equation into standard form

$$
\omega^2 \frac{d^2 \mathbf{y}}{dt^2} + (A + 2gQ \cos 2\tau) \mathbf{y} = 0
$$

with $A = 4A^*$ and $Q = 4Q^*$.

According to the Floquet theory, the solutions of equation (A.2) can be written as

$$
y = e^{\phi} \phi(t),
$$

where $\phi(t)$ is $\pi$ periodic and $h = \alpha + i\beta$ is a complex quantity. We must have $Re(h)$ greater than zero in order to have $y$ unstable, and since we have no damping, $Re(h)$ is equal to zero in the stable domains. This fact is used to determine the boundary curves. We now expand $\phi(t)$ in its Fourier series

$$
\phi(t) = \sum_{j=1}^{\infty} b_j v_j,
$$

where $v_j = 1$ for $j = 1$, $v_j = \sin j\tau/2$ for $j = 2, 4, \ldots$ and $v_j = \cos (j - 1)\tau/2$ for $j = 3, 5, \ldots$.

Combining equations (A.2)–(A.4), cutting after $N$ terms and dividing by $e^{\phi t}$ gives

$$
\sum_{j=1}^{N} \left( \omega^2 \frac{d^2 v_j}{dt^2} + 2h\omega^2 \frac{dv_j}{dt} \right) + 2h \omega \sum_{j=1}^{N} (A + 2gQ \cos 2\tau) v_j b_j = 0,
$$

where $I$ is the identity matrix. If we then denote the $4th$ component of $b_j$ by $b_{kj}$ and write equation (A.5) in its component form, we will get $n$ equations in the $n \times N$ unknowns $b_{kj}$. We then argue that in each of the $n$ equations, the coefficients of each $v_j$ must be equal to zero in order to make the equations zero for all $\tau$. Thus we have $n \times N$ equations for the $n \times N$ unknowns. In order for $b$ to have a solution different from zero, the determinant of equation (A.5) must be zero. The boundary curves can be determined by setting $\beta$ to zero.

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Figure 6. Influence of damping on the phase mixed case. The damping parameter is increased from $\gamma = 0.05$ (figure 5(b)) to $\gamma = 0.07$. The amplitudes now are damped compared with that in figure 5(b).