We review classical BPS monopoles, their moduli spaces, twistor descriptions and dynamics. Particular emphasis is placed upon symmetric monopoles, where recent progress has been made. Some remarks on the role of monopoles in S-duality and Seiberg-Witten theory are also made.

1. Introduction

Monopoles are topological soliton solutions in three space dimensions, which arise in Yang-Mills-Higgs gauge theories where the non-abelian gauge group \( G \) is spontaneously broken by the Higgs field to a residual symmetry group \( H \). The Higgs field at infinity defines a map from \( S^2 \) to the coset space of vacua \( G/H \), so if \( \pi_2(G/H) \) is non-trivial then all solutions have a topological characterization. The simplest case is to take \( G = SU(2) \) broken to \( H = U(1) \) by a Higgs field in the adjoint representation. We shall concentrate upon this case and indicate how the methods and results extend to higher rank gauge groups in Section 5.

The Lagrangian density, with the usual symmetry breaking potential is

\[
\mathcal{L} = \frac{1}{8} \text{tr}(F_{\mu\nu}F^{\mu\nu}) - \frac{1}{4} \text{tr}(D_{\mu}\Phi D^{\mu}\Phi) - \frac{1}{8} \lambda(|\Phi|^2 - 1)^2 \tag{1.1}
\]

where \(|\Phi|^2 = -\frac{1}{2} \text{tr}\Phi^2\) is the square of the length of the Higgs field. In the \( SU(2) \) case we have that \( \pi_2(SU(2)/U(1)) = \mathbb{Z} \), so that each configuration has an associated topological integer, or winding number, \( k \), which may be expressed as

\[
k = \frac{1}{8\pi} \int \text{tr}(B_i D_i \Phi) \, d^3x. \tag{1.2}
\]

Here \( B_i \) denotes the magnetic part of the gauge field, \( B_i = \frac{1}{2} \epsilon_{ijk} F_{jk} \), and indices run over the spatial values 1, 2, 3. The integer \( k \) is the degree of the map given by the Higgs field at infinity

\[
\hat{\Phi} : S^2 \mapsto S^2 \tag{1.3}
\]

*Email P.M.Sutcliffe@ukc.ac.uk
and is known as the monopole number, or charge, since it determines the magnetic charge of the solution as follows. The $U(1)$ residual symmetry group is identified as the group of electromagnetism, and the only component of the non-abelian gauge field which survives at infinity is the one in the direction of $\hat{\Phi}$, the asymptotic Higgs field. This allows us to define abelian magnetic and electric fields

$$b_i = \frac{1}{2} \text{tr}(B_i \hat{\Phi}), \quad e_i = \frac{1}{2} \text{tr}(E_i \hat{\Phi}) \quad (1.4)$$

and it can be shown, by using Stokes’ theorem, that the associated magnetic charge is exactly $4\pi k$.

In order to make progress with the classical solutions of the second order field equations which follow from (1.1) it is helpful to consider the BPS limit, in which the Higgs potential is removed by setting $\lambda = 0$. We shall consider only this case from now on. Then, as Bogomolny pointed out, by completing a square in the energy integral

$$E = \frac{1}{4} \int -\text{tr}(D_i \Phi D_i \Phi + B_i B_i + E_i E_i + D_i \Phi D_i \Phi) \, d^3x \quad (1.5)$$

a bound on $E$, in terms of the monopole number $k$, can be obtained

$$E \geq 4\pi |k|. \quad (1.6)$$

Moreover, all solutions of the second order equations which attain this bound are static solutions that solve the first order Bogomolny equation

$$D_i \Phi = -B_i \quad (1.7)$$

(or the one obtained from above by a change of sign for $k < 0$).

For solutions of the Bogomolny equation (1.7) the energy may be expressed in the convenient form

$$E = \frac{1}{2} \int \partial_i \partial_i \|\Phi\|^2 \, d^3x \quad (1.8)$$

allowing the energy density to be computed from knowledge of the Higgs field alone.

The charge one solution of equation (1.7) has a spherically symmetric form and was first written down by Prasad & Sommerfield. It is given by

$$\Phi = ix_j \sigma_j \frac{r \cosh r - \sinh r}{r^2 \sinh r}, \quad A_i = -i \epsilon_{ijk} \sigma_j x_k \frac{\sinh r - r}{r^2 \sinh r}. \quad (1.9)$$

This monopole is positioned at the origin, but clearly three parameters can be introduced into the solution by a translation which places the monopole at an arbitrary location in $\mathbb{R}^3$. A further parameter can also be introduced into this solution, but its appearance is more subtle. Consider gauge transformations of the form $g = \exp(\chi \Phi)$, which is a zero mode since the potential energy is independent of the constant $\chi$. However, if one returns to the second order field equations then it turns out that a transformation of this form, but where $\chi$ is now linearly time
dependent, also gives a solution, and now the monopole becomes a dyon, with
electric charge proportional to the rate of change of $\chi$. Thus it is useful to include the
phase $\chi$ as one of the moduli in the solutions of the Bogomolny equation (1.7). Hence
the moduli space, $\mathcal{M}_1$, of charge one solutions is 4-dimensional. Mathematically,
to correctly define this 4-dimensional moduli space of gauge inequivalent solutions
one must carefully state the type of gauge transformations which are allowed, and
consider framed monopoles and transformations.

In the BPS limit of a massless Higgs there is now both a long-range magnetic
repulsion and a long-range scalar attraction between the fields of two well-separated
and equally charged monopoles. By studying the field equations Manton was able
to show that these long-range forces exactly cancel, thus providing for the possibility
of static multi-monopoles ie. solutions of (1.7) with $k > 1$.

Multi-monopole solutions indeed exist and are the topic of discussion in Section
2. Briefly, it is now known that the moduli space of charge $k$ solutions, $\mathcal{M}_k$, is a
$4k$-dimensional manifold. Roughly speaking, if the monopoles are all well-
separated then the $4k$ parameters represent three position coordinates and a phase
for each of the $k$ monopoles. However, for points in $\mathcal{M}_k$ which describe monopoles
that are close together, we shall see that the situation is more interesting. The
explicit construction of any multi-monopole solution turned out to be a difficult
task, which required the application of powerful twistor methods, as we now review.

2. Twistor Methods

As a first attempt to construct multi-monopole solutions one might try to generalize
the spherically symmetric $k=1$ solution (1.7). However, Bogomolny showed
that this is in fact the unique spherically symmetric solution, so there are no spherically
symmetric monopoles for $k > 1$. This makes the task of explicitly constructing any
multi-monopole solution more difficult, and indeed it would probably be impossible
if it were not for the fact that the Bogomolny equation (1.7) has the special property
of being integrable, as we explain in the following.

Ward’s twistor transform for self-dual Yang-Mills gauge fields relates
instanton solutions in $\mathbb{R}^4$ to certain holomorphic vector bundles over the standard
complex 3-dimensional twistor space $\mathbb{C}P^3$. By constructing such bundles the self-
dual gauge fields can be extracted and this can be done explicitly in a number of cases.

Now to be able to apply this technique to monopoles we need to find the
connection between self-dual gauge fields in $\mathbb{R}^4$ and monopoles, which are objects
in $\mathbb{R}^3$.

The observation of Manton is that if one considers the self-dual equation

$$ F_{\mu\nu} = \frac{1}{2} \epsilon_{\mu\nu\alpha\beta} F_{\alpha\beta} \quad (2.1) $$

for the gauge field in four dimensions (so greek indices take the values 1, 2, 3, 4) and
dimensionally reduces, by setting all functions independent of the $x_4$ coordinate, then (2.1) reduces to the Bogomolny equation (1.7) after the identification $A_4 = \Phi$. 
Thus monopoles may be thought of as some particular kinds of self-dual gauge fields, though they are not instantons, since they are required to have infinite action in order to have the required $x_4$ dependence. For the particular case of the charge one monopole the solution can be obtained using the form of the Corrigan-Fairlie-t’Hooft-Wilczek ansatz which gives a subset of self-dual gauge fields, though no multi-monopoles can be obtained in this way.

Knowing the form of the $k = 1$ monopole solution, one can construct the associated vector bundle to which it corresponds and then hope to generalize this to obtain the vector bundle of a $k = 2$ solution and hence a charge two monopole. Ward’s original description of this procedure was in terms of bundles over $\mathbb{C}P^3$, which have a particular special form to obtain the required $x_4$ independence of the gauge fields. However, as later described by Hitchin, the dimensional reduction can be made at the twistor level too, to obtain a direct correspondence between monopoles and bundles over the mini-twistor space $\mathcal{T}\mathbb{T}$, which is a 2-dimensional complex manifold isomorphic to the holomorphic tangent bundle to the Riemann sphere $\mathbb{T}\mathbb{P}^1$. It is helpful in connecting with other approaches if we adopt this reduced description.

To put coordinates on $\mathcal{T}\mathbb{T}$ let $\zeta$ be the standard inhomogeneous coordinate on the base space and $\eta$ the complex fibre coordinate. For the twistor transform these twistor coordinates are related to the space coordinates $x_1, x_2, x_3$ via the relation

$$\eta = \frac{(x_1 + ix_2)}{2} - x_3\zeta - \frac{(x_1 - ix_2)}{2}\zeta^2. \quad (2.2)$$

Monopoles correspond to certain rank two vector bundles over $\mathcal{T}\mathbb{T}$, which may be characterized by a $2 \times 2$ patching matrix which relates the local trivializations over the two patches $U_1 = \{ \zeta : |\zeta| \leq 1 \}$ and $U_2 = \{ \zeta : |\zeta| \geq 1 \}$. For charge $k$ monopoles the patching matrix, $F$, may be taken to have the Atiyah-Ward form

$$F = \begin{pmatrix} \zeta^k & \Gamma \\ 0 & \zeta^{-k} \end{pmatrix}. \quad (2.3)$$

To extract the gauge fields from the bundle over twistor space requires the patching matrix to be ‘split’ as $F = H_2H_1^{-1}$ on the overlap $U_1 \cap U_2$, where $H_1$ and $H_2$ are regular and holomorphic in the patches $U_1$ and $U_2$ respectively.

For a patching matrix of the Atiyah-Ward form this ‘splitting’ can be done by a contour integral. From the Taylor-Laurent coefficients

$$\Delta_p = \frac{1}{2\pi i} \oint_{|\zeta| = 1} \Gamma\zeta^{p-1} \, d\zeta \quad (2.4)$$

the gauge fields can be computed. For example there is the elegant formula

$$\|\Phi\|^2 = 1 - \partial_i\partial_i \log D \quad (2.5)$$

where $D$ is the determinant of the $k \times k$ banded matrix with entries

$$D_{pq} = \Delta_{p+q-k-1}, \quad 1 \leq p, q \leq k. \quad (2.6)$$
For charge $k$ monopoles the function $\Gamma$ in the Atiyah-Ward ansatz (2.3) has the form

$$\Gamma = \frac{\zeta^k}{S}(e^{-(-x_1 + ix_2)\zeta - x_3} + (-1)^k e^{-(-x_1 - ix_2)\zeta^{-1} + x_3})$$

(2.7)

where $S$ is a polynomial in $\eta$ of degree $k$, with coefficients which are polynomials in $\zeta$.

The $k = 1$ solution (1.3) is obtained by taking $S = \eta$. In this case the contour integral (2.4) gives

$$\Delta_0 = \frac{2\sinh r}{r}$$

(2.8)

and since $k = 1$ then $D = \Delta_0$ and (2.3) gives

$$\|\Phi\|^2 = 1 - \partial_i \partial^i \log \frac{2\sinh r}{r} = \frac{(rcos hr - sinh r)^2}{(rsinh r)^2}$$

(2.9)

which clearly agrees with (1.9). Similarly, by taking

$$S = \eta - \frac{(a_1 + ia_2)}{2} + a_3 \zeta + \frac{(a_1 - ia_2)}{2} \zeta^2$$

(2.10)

a monopole with position $x = (a_1, a_2, a_3)$ is obtained.

Ward was able to generalize the $k = 1$ solution to present a 2-monopole solution which is given by taking

$$S = \eta^2 + \frac{\pi^2}{4} \zeta^2.$$  

(2.11)

As stated earlier, it was known that such a solution could not be spherically symmetric and in fact Ward’s 2-monopole solution has an axial symmetry, so that a surface of constant energy density is a torus. The traditional definition of the position of a monopole is taken to be where the Higgs field is zero. This toroidal monopole has a double zero at the origin and no others, so this configuration may be thought of as two monopoles both of which are located at the origin.

For the general 2-monopole solution there is one important parameter, related to the separation of the monopoles, with the other seven parameters being accounted for by the position of the centre of mass, an overall phase and spatial $SO(3)$ rotations of the whole configuration. This one-parameter family of solutions was also constructed by Ward and corresponds to the function

$$S = \eta^2 - \frac{K^2}{4}(m + 2(m - 2)\zeta^2 + m\zeta^4)$$

(2.12)

where $m \in [0, 1)$ and $K$ is the complete elliptic integral of the first kind with parameter $m$. If $m = 0$ then $K = \pi/2$ and (2.12) reduces to (2.11), representing coincident monopoles. In the limit as $m \to 1$ then $K \to \infty$ and (2.12) becomes asymptotic to the product

$$S = (\eta + \frac{K}{2}(1 - \zeta^2))(\eta - \frac{K}{2}(1 - \zeta^2))$$

(2.13)
which, by comparison with (2.10), can be seen to describe two well-separated monopoles with positions \((\pm K, 0, 0)\).

It should be noted that at around the same time that Ward produced his 2-monopole solutions using twistor methods, a more traditional integrable systems approach was taken by Forgács, Horváth & Palla\(^2\) and the same results obtained. This method makes use of the fact that the Bogomolny equation (1.7) (in a suitable formulation) can be written as the compatibility condition of an overdetermined linear system. The linear system can be solved in terms of projectors and the corresponding gauge fields extracted.

An axially symmetric monopole exists for all \(k > 1\), with the functions \(S\) that generalize the \(k = 2\) example (2.11) being found by Prasad & Rossi\(^2\) to be

\[
S = \prod_{l=0}^{g}\{(\eta^2 + (l + \frac{1}{2})^2\pi^2\zeta^2}\} \quad \text{for} \quad k = 2g + 2 \tag{2.14}
\]

\[
S = \eta \prod_{l=1}^{g}\{(\eta^2 + l^2\pi^2\zeta^2}\} \quad \text{for} \quad k = 2g + 1 \tag{2.15}
\]

A main difficulty with this direct twistor approach is finding the polynomials \(S\) so that a non-singular solution of the Bogomolny equation (1.7) is obtained. By explicit computation Ward was able to show that the axially symmetric 2-monopole solution corresponding to (2.11) is smooth, and this can then be used to show that, at least for solutions corresponding to (2.12) which are close to this one, then they too are smooth. A general discussion of non-singularity was given by Hitchin\(^1\) and we shall return to this shortly.

An analysis of the degrees of freedom in the function \(S\) together with the constraints of reality and non-singularity, allowed Corrigan & Goddard\(^8\) to deduce that a charge \(k\) monopole solution has \(4k\) degrees of freedom, as we have mentioned earlier. However, for \(k > 2\) very few explicit examples of functions \(S\) are known, although some have recently been obtained by considering particularly symmetric cases (see later). Note that even if the function \(S\) is known it is still a difficult task to perform the contour integrals (2.4) and extract the gauge fields explicitly, with the tractable cases only likely to be those in which \(S\) factors into a product containing no greater than quadratic polynomials in \(\eta\).

We have seen that a monopole is determined by a vector bundle over \(\mathcal{T}\) and that this in turn is determined by a polynomial \(S\). In fact it turns out that this polynomial occurs in many of the different twistor descriptions of monopoles and is a central object. A more direct relation between this polynomial and the monopole was introduced by Hitchin\(^1\) and we shall now review this topic of spectral curves.

In Hitchin’s approach \(\mathcal{T}\) is identified with the space of directed lines in \(\mathbb{R}^3\). The base space coordinate \(\zeta\) defines a direction in \(\mathbb{R}^3\) (via the usual Riemann sphere description of \(S^2\)) and the fibre \(\eta\) is a complex coordinate in a plane orthogonal to this line. Given a line in \(\mathbb{R}^3\), determined by a point in \(\mathcal{T}\), one then considers the linear differential equation

\[
(D_u - i\Phi)v = 0 \tag{2.16}
\]
for the complex doublet \( v \), where \( u \) is the coordinate along the line and \( D_u \) denotes the covariant derivative in the direction of the line. This equation has two independent solutions and a basis \((v_0, v_1)\) can be chosen such that

\[
\lim_{u \to \infty} v_0(u) u^{-k/2} e^{u} = e_0, \\
\lim_{u \to \infty} v_1(u) u^{k/2} e^{-u} = e_1
\]

where \( e_0, e_1 \) are constant in some asymptotically flat gauge. Thus \( v_0 \) is bounded and \( v_1 \) is unbounded as \( u \to \infty \). Clearly a similar description exists in terms of a basis \((v_0', v_1')\) of bounded and unbounded solutions in the opposite direction as \( u \to -\infty \).

Now the line along which we consider Hitchin’s equation (2.16) is called a spectral line if the solution is decaying in both directions \( u \to \pm \infty \). The set of all spectral lines defines a curve of genus \( (k-1)^2 \) in \( \mathbb{T} \) called the spectral curve, which for a charge \( k \) monopole has the form

\[
S = \eta^k + \eta^{k-1} a_1(\zeta) + \ldots + \eta^r a_{k-r}(\zeta) + \ldots + \eta a_{k-1}(\zeta) + a_k(\zeta) = 0
\]

where, for \( 1 \leq r \leq k \), \( a_r(\zeta) \) is a polynomial in \( \zeta \) of maximum degree \( 2r \). However, general curves of this form will only correspond to \( k \)-monopoles if they satisfy the reality condition

\[
a_r(\zeta) = (-1)^r \zeta^{2r} a_r(\zeta) (1/\zeta)
\]

and some difficult non-singularity conditions.

This function \( S \), whose zero set gives the spectral curve, is precisely the function which occurred earlier in the determination of the Atiyah-Ward patching matrix. Moreover, Hitchin was able to prove that all monopole solutions can be constructed from the Atiyah-Ward class of bundles using Ward’s method. By an analysis of the spectral curve singularity constraints Hurubise was able to derive the spectral curve (2.12) of the general 2-monopole solution.

Although the spectral curve has now appeared in two approaches the main problem still lies in satisfying the difficult non-singularity conditions which a general curve must satisfy in order to be a spectral curve and thus correspond to a monopole. A third approach was introduced by Nahm and this has the great advantage that non-singularity is manifest, although there is a price to be paid for this, which is that the transform requires the solution of a matrix nonlinear ordinary differential equation.

The Atiyah-Drinfeld-Hitchin-Manin (ADHM) construction is a formulation of the twistor transform for instantons on \( \mathbb{R}^4 \). It allows their construction in terms of linear algebra in a vector space whose dimension is related to the instanton number. Since monopoles correspond to infinite action instantons, then an adaptation of the ADHM construction involving an infinite dimensional vector space, which can be represented by functions of an auxiliary variable, might be possible. Nahm was able to formulate such an adaptation, which is now known as the Atiyah-Drinfeld-Hitchin-Manin-Nahm (ADHMN) construction, or the Nahm transform.
The ADHMN construction is an equivalence between $k$-monopoles and Nahm data $(T_1, T_2, T_3)$, which are three $k \times k$ matrices which depend on a real parameter $s \in [0, 2]$ and satisfy the following:

(i) Nahm’s equation

$$\frac{dT_i}{ds} = \frac{1}{2} \epsilon_{ijk}[T_j, T_k],$$

(ii) $T_i(s)$ is regular for $s \in (0, 2)$ and has simple poles at $s = 0$ and $s = 2$,

(iii) the matrix residues of $(T_1, T_2, T_3)$ at each pole form the irreducible $k$-dimensional representation of SU(2),

(iv) $T_i(s) = -T_i^\dagger(s)$,

(v) $T_i(s) = T_i^\dagger(2-s)$.

Finding the Nahm data effectively solves the nonlinear part of the monopole construction but in order to calculate the fields themselves the linear part of the ADHMN construction must also be implemented. Given Nahm data $(T_1, T_2, T_3)$ for a $k$-monopole we must solve the ordinary differential equation

$$(\mathbb{1}_{2k} + \mathbb{1}_k \otimes \frac{x_j \sigma_j}{2} + iT_j \otimes \sigma_j) \mathbf{v} = 0$$

for the complex $2k$-vector $\mathbf{v}(s)$, where $\mathbb{1}_k$ denotes the $k \times k$ identity matrix and $x = (x_1, x_2, x_3)$ is the point in space at which the monopole fields are to be calculated. Introducing the inner product

$$\langle \mathbf{v}_1, \mathbf{v}_2 \rangle = \int_0^2 \mathbf{v}_1^\dagger \mathbf{v}_2 \, ds$$

then the solutions of (2.21) which are required are those which are normalizable with respect to (2.22). It can be shown that the space of normalizable solutions to (2.21) has (complex) dimension two. If $\hat{\mathbf{v}}_1, \hat{\mathbf{v}}_2$ is an orthonormal basis for this space then the Higgs field $\Phi$ and gauge potential $A_i$ are given by

$$\Phi = i \begin{bmatrix} (s - 1)\hat{\mathbf{v}}_1 & (s - 1)\hat{\mathbf{v}}_1 \cr (s - 1)\hat{\mathbf{v}}_2 & (s - 1)\hat{\mathbf{v}}_2 \end{bmatrix}, \quad A_i = \begin{bmatrix} \langle \hat{\mathbf{v}}_1, \partial_i \hat{\mathbf{v}}_1 \rangle & \langle \hat{\mathbf{v}}_1, \partial_i \hat{\mathbf{v}}_2 \rangle \cr \langle \hat{\mathbf{v}}_2, \partial_i \hat{\mathbf{v}}_1 \rangle & \langle \hat{\mathbf{v}}_2, \partial_i \hat{\mathbf{v}}_2 \rangle \end{bmatrix}.$$  

Nahm’s equation (2.20) has the Lax formulation

$$\frac{d\Lambda}{ds} = [\Lambda, \Lambda_+]$$

where

$$\Lambda = (T_1 + iT_2) - 2iT_3\zeta + (T_1 - iT_2)\zeta^2, \quad \Lambda_+ = -iT_3 + (T_1 - iT_2)\zeta.$$  

(2.24)
Hence the spectrum of $\Lambda$ is $s$-independent, giving the associated algebraic curve

$$S = \det(\eta + \Lambda) = 0$$

(2.26)

whose coefficients are the constants of motion associated with the dynamical system (2.20). It can be shown \cite{19} that this curve is again the same spectral curve that we have met twice already.

The power of the ADHMN construction may be demonstrated by considering the case $k = 1$. In this case each Nahm matrix is just a purely imaginary function of $s$, so the solution of Nahm’s equation (2.20) is simply that each of these functions must be constant. Thus the required Nahm data is simply $T_i = -ia_i/2$, where the three real constants $a_i$ determine the position of the monopole to be $x = (a_1, a_2, a_3)$. To construct a monopole at the origin the Nahm data is thus $T_i = 0$ and since this monopole is spherically symmetric we can simplify the presentation by restricting to the $x_3$-axis by setting $x = (0, 0, r)$.

Writing $v = (w_1, w_2)^t$ then the $2 \times 2$ system (2.21) becomes the pair of decoupled equations

$$\frac{dw_1}{ds} + \frac{r}{2}w_1 = 0, \quad \frac{dw_2}{ds} - \frac{r}{2}w_2 = 0$$

(2.27)

which are elementary to solve as

$$w_1 = c_1e^{-rs/2}, \quad w_2 = c_2e^{rs/2}$$

(2.28)

where $c_1$ and $c_2$ are arbitrary constants.

An orthonormal basis $\hat{v}_1, \hat{v}_2$, with respect to the inner product (2.22), is obtained by the choice

$$c_1^2 = 0, \quad c_2^2 = r/(e^{2r} - 1) \quad \text{for } \hat{v}_1$$

$$c_2^2 = 0, \quad c_1^2 = r/(1 - e^{-2r}) \quad \text{for } \hat{v}_2.$$  

(2.29)

(2.30)

Note that a different choice of orthonormal basis corresponds to a different choice of gauge for the monopole fields. In the gauge we have chosen $\Phi = i\varphi\sigma_3$ where

$$\varphi = \langle (s - 1)\hat{v}_1, \hat{v}_1 \rangle = \frac{r}{(e^{2r} - 1)} \int_0^2 (s - 1)e^{rs} \, ds = \frac{r\cosh r - \sinh r}{r\sinh r}$$

(2.31)

which reproduces the expression (1.9).

After a suitable orientation the Nahm data of a 2-monopole has the form

$$T_1 = \frac{f_1}{2} \left( \begin{array}{cc} 0 & i \\ i & 0 \end{array} \right), \quad T_2 = \frac{f_2}{2} \left( \begin{array}{cc} 0 & 1 \\ -1 & 0 \end{array} \right), \quad T_3 = \frac{f_3}{2} \left( \begin{array}{cc} -i & 0 \\ 0 & i \end{array} \right)$$

(2.32)

with corresponding spectral curve

$$\eta^2 + \frac{1}{4}((f_1^2 - f_2^2) + (2f_1^2 + 2f_2^2 - 4f_3^2)\zeta^2 + (f_1^2 - f_2^2)\zeta^4) = 0.$$  

(2.33)
With this ansatz Nahm’s equation reduces to the Euler top equation

$$\frac{df_1}{ds} = f_2 f_3 \tag{2.34}$$

and cyclic permutations. The solution satisfying the appropriate Nahm data boundary conditions is

$$f_1 = -\frac{K \text{dn}(Ks)}{\text{sn}(Ks)}, \quad f_2 = -\frac{K}{\text{sn}(Ks)}, \quad f_3 = -\frac{K \text{cn}(Ks)}{\text{sn}(Ks)} \tag{2.35}$$

where \(\text{sn}(u), \text{cn}(u), \text{dn}(u)\) denote the Jacobi elliptic functions with argument \(u\) and parameter \(m\), and, as earlier, \(K\) is the complete elliptic integral of the first kind with parameter \(m\). Substituting these expressions for \(f_i\) into the curve (2.33) and using the standard identities \(\text{sn}^2(u) + \text{cn}^2(u) = 1\) and \(m \text{sn}^2(u) + \text{dn}^2(u) = 1\) we once again obtain the 2-monopole spectral curve (2.12).

The fact that the spectral curve of a \(k\)-monopole has genus \((k - 1)^2\) means that Nahm’s equation, whose flow is linearized on the Jacobian of this curve, can be solved in terms of theta functions defined on a Riemann surface of genus \((k - 1)^2\). For \(k = 2\) then the curve is elliptic, which explains why the general solution of Nahm’s equation can be obtained in terms of elliptic functions and why the parameters in the spectral curve are obtained in terms of elliptic integrals. However, for \(k > 2\) it appears a very difficult task to attempt to express the general solution of Nahm’s equation in terms of theta functions and then try and impose the required boundary conditions necessary to produce Nahm data. This is the underlying mathematical obstruction which has prevented the general 3-monopole solution from being constructed, or even its spectral curve.

Nonetheless, there are simplifying special cases where extra symmetry of the monopole means that progress can be made. In such symmetric cases, where the monopole has some rotational symmetry given by a group \(G \subset SO(3)\), then it is not the genus of the spectral curve \(S\) which is the important quantity, but rather the genus \(\tilde{g}\) of the quotient curve \(\tilde{S} = S/G\). If \(\tilde{g} < 2\) then the situation is greatly simplified and there is a hope of some form of construction (either the monopole fields, Nahm data or spectral curve) in terms of, at worst, elliptic functions and integrals. The axially symmetric \(k\)-monopoles for \(k > 2\), given by (2.14) and (2.15), are such examples and we shall see some other recent examples in Section 3, where \(G\) is a Platonic symmetry group.

By making use of the Nahm transform, Donaldson was able to prove that the monopole moduli space \(\mathcal{M}_k\) is diffeomorphic to the space of degree \(k\) based rational maps \(R : \mathbb{C} \mathbb{P}^1 \rightarrow \mathbb{C} \mathbb{P}^1\). Explicitly, Donaldson showed how every \(k\)-monopole gives rise to a unique rational map

$$R(z) = \frac{p(z)}{q(z)} \tag{2.36}$$

where \(q(z)\) is a monic polynomial of degree \(k\) in the complex variable \(z\) and \(p(z)\) is a polynomial of degree less than \(k\), with no factors in common with \(q(z)\).
To understand this diffeomorphism better it is useful to follow the analysis of Hurtubise. We have already noted that a study of Hitchin’s equation (2.16) along a line shows that there are two independent solutions with a basis \((v_0, v_1)\) consisting of solutions which are respectively bounded and unbounded as \(u\), the coordinate along the line, tends to infinity. In addition we introduced the basis \((v'_0, v'_1)\) of bounded and unbounded solutions in the opposite direction \(u \to -\infty\). Now fix a direction in \(\mathbb{R}^3\), which gives the decomposition

\[
\mathbb{R}^3 \cong \mathbb{C} \times \mathbb{R}.
\]  

(2.37)

For convenience, we choose this direction to be that of the positive \(x_3\)-axis and denote by \(z\) the complex coordinate on the \(x_1x_2\)-plane. Thus the coordinate \(u\) in the above analysis of Hitchin’s equation (2.16) is now \(x_3\). The approach of Hurtubise is to consider the scattering along all such lines and write

\[
v'_0 = a(z)v_0 + b(z)v_1,
\]

(2.38)

\[
v_0 = a'(z)v'_0 + b(z)v'_1.
\]

(2.39)

The rational map is then given by

\[
R(z) = \frac{a(z)}{b(z)}.
\]

(2.40)

Furthermore, since the spectral curve \(S(\eta, \zeta)\) of a monopole corresponds to the bounded solutions of (2.16) then

\[
b(z) = S(z, 0).
\]

(2.41)

Finally, it can be shown that the full scattering data is given by

\[
\begin{bmatrix}
a & b \\
-b' & -a'
\end{bmatrix}
\begin{pmatrix}
v_0 \\
v_1
\end{pmatrix} =
\begin{pmatrix}
v'_0 \\
v'_1
\end{pmatrix}
\]

(2.42)

where

\[
aa' = 1 + b'b.
\]

(2.43)

The advantage of rational maps is that monopoles are easily described in this approach, since one simply writes down any rational map. The disadvantage is that the rational map tells us very little about the monopole. In particular, since the construction of the rational map requires the choice of a direction in \(\mathbb{R}^3\) it is not possible to study the full symmetries of a monopole from its rational map. However, the following isometries are known. Let \(\lambda \in U(1)\) and \(\nu \in \mathbb{C}\) define a rotation and translation respectively in the plane \(\mathbb{C}\). Let \(x \in \mathbb{R}\) define a translation perpendicular to the plane and let \(\mu \in U(1)\) be a constant gauge transformation. Under the composition of these transformation a rational map \(R(z)\) transforms as

\[
R(z) \mapsto \mu^2 e^{2x} \lambda^{-2k} R(\lambda^{-1}(z - \nu)).
\]

(2.44)
Furthermore, under the reflection $x_3 \mapsto -x_3$, $R(z) = p(z)/q(z)$ transforms as

$$\frac{p(z)}{q(z)} \mapsto \frac{I(p)(z)}{q(z)}$$  \hspace{1cm} (2.45)$$

where $I(p)(z)$ is the unique polynomial of degree less than $k$ such that $(I(p)p)(z) = 1 \mod q(z)$.

Some information regarding the monopole configuration can be determined from the rational map in special cases, corresponding to well-separated monopoles. Bielawski has proved that for a rational map $p(z)/q(z)$ with well-separated poles $\beta_1, \ldots, \beta_k$ the corresponding monopole is approximately composed of unit charge monopoles located at the points $(x_1, x_2, x_3)$, where $x_1 + ix_2 = \beta_i$ and $x_3 = \frac{1}{2} \log |p(\beta_i)|$. This approximation applies only when the values of the numerator at the poles is small compared to the distance between the poles.

For the complimentary case of monopoles strung out in well-separated clusters along (or nearly along) the $x_3$-axis, the large $z$ expansion of the rational map $R(z)$ is

$$R(z) \sim e^{2x+ix} \frac{z^L}{z^{2L+M}} + \ldots$$  \hspace{1cm} (2.46)$$

where $L$ is the charge of the topmost cluster with $x$ its elevation above the plane and $M$ is the charge of the next highest cluster with elevation $y$.

On a technical point, it is often convenient to restrict to what are known as strongly centred monopoles, which roughly means that the total phase is unity and the centre of mass is at the origin. More precisely, in terms of the rational map a monopole is strongly centred if and only if the roots of $q$ sum to zero and the product of $p$ evaluated at each of the roots of $q$ is equal to one.

Although this rational map description of monopoles is very useful, particularly as a convenient description of the monopole moduli space, it does suffer from the drawback of requiring a choice of direction in $\mathbb{R}^3$, which is not helpful in several contexts. Atiyah has suggested that a second rational map description might exist, in which the full rotational symmetries around the origin of $\mathbb{R}^3$ are not broken, and indeed this has recently been confirmed by Jarvis. To construct the Jarvis rational map one considers Hitchin’s equation (2.16) along each radial line from the origin to infinity. The coordinate $u$ is now identified with the radius $r$ and as we have already seen there is just one solution, with basis $v_0$, which is bounded as $r \to \infty$. Writing $v_0(r) = (w_1(r), w_2(r))$ then define $J$ to be the ratio of these components at the origin ie. $J = w_1(0)/w_2(0)$. Now take $z$ to determine a particular radial line, by giving its direction as a point on the Riemann sphere. Then, in fact, $J$ is a holomorphic function of $z$, since by virtue of the Bogomolny equation (1.7) the covariant derivative in the angular direction, $D_\bar{z}$, commutes with the operator $D_r - i\Phi$ appearing in Hitchin’s equation (2.16). It can be verified that $J$ is a rational function of degree $k$ so that we have a new rational map $J : \mathbb{C}P^1 \to \mathbb{C}P^1$. Note that the Jarvis rational map is unbased, since a gauge transformation replaces $J$ by an $SU(2)$ Möbius transformation determined by the gauge transformation evaluated
at the origin. Thus the correspondence is between a monopole and an equivalence class of Jarvis maps, where two maps are equivalent if they can be mapped into each other by a reorientation of the target Riemann sphere. In Section 3 we shall see some explicit examples of Jarvis maps corresponding to symmetric monopoles.

3. Platonic Monopoles

In the last Section we commented on how the construction of monopoles can be simplified by restricting to special symmetric cases. In this Section we discuss in detail some particular examples of this, where the symmetry group $G$ is Platonic ie. tetrahedral, octahedral or icosahedral.

The first issue to confront is how to impose a given symmetry in the monopole construction and this is most easily discussed in terms of the spectral curve approach. Recall that in terms of the coordinates $(\eta, \zeta)$ on $\mathbb{T}$ the lines through the origin are parameterized by $\zeta$ with $\eta = 0$. An $SO(3)$ rotation in $\mathbb{R}^3$ acts on the Riemann sphere coordinate via an $SU(2)$ Möbius transformation and since $\eta$ is the coordinate in the tangent space to this Riemann sphere then it transforms via the derivative of this Möbius transformation. For example, rotation through an angle $\phi$ around the $x_3$-axis is given by

$$R_\phi : (\eta, \zeta) \mapsto (\eta', \zeta') = (e^{i\phi} \eta, e^{i\phi} \zeta).$$

A monopole is symmetric under a symmetry group $G$ if its transformed spectral curve $S(\eta', \zeta') = 0$ is the same as its original curve $S(\eta, \zeta) = 0$, for all Möbius transformations corresponding to the elements of $G$. For example, it is now clear that the spectral curves (2.14) and (2.15) describe monopoles which are axially symmetric, since they are invariant under the transformation (3.1) for all $\phi$, due to the fact that these curves are homogeneous in the twistor coordinates.

It is thus straightforward to write down, for example, that the candidate axially symmetric 2-monopole spectral curve has the form

$$\eta^2 + a\zeta^2 = 0$$

where $a$ is real due to the reality constraint (2.19). However, symmetry alone gives no information regarding the possible values (if any) of $a$. This is determined by the non-singularity condition which can be analyzed directly or, more easily, using the ADHMN formulation as detailed in the previous Section. Either way, the result that $a = \pi^2/4$ is obtained, which also proves the non-existence of a spherically symmetric 2-monopole, which would require $a = 0$, to be invariant under all $SU(2)$ Möbius transformations.

It is certainly not obvious what symmetry to attempt to impose in order to simplify the monopole construction, but the proposal by Hitchin, Manton & Murray to consider Platonic symmetries turns out to be a very fruitful one. Its motivation lies with some numerical work of Braaten, Townsend and Carson on another kind of topological soliton, the Skyrmion, where it was found that Skyrmions of charge three and four appear to have tetrahedral and cubic symmetry respectively.
Let us first consider the tetrahedral case, $G = T_d$. One way to proceed would be as described above, that is, to write down the generators of $T_d$, compute the associated $SU(2)$ Möbius transformations and hence derive the form of the invariant polynomials which are allowed in the candidate spectral curve. However, there is a short cut available, since the result which one would obtain by this method is precisely the computation of the invariant tetrahedral Klein polynomial.\cite{ref35} The Klein polynomials associated with a Platonic solid are the polynomials obtained by taking either the vertices, faces or edges of the solid, projecting these onto the unit 2-sphere and computing the monic polynomial whose complex roots are exactly these points, when thought of as points on the Riemann sphere. These Klein polynomials are all listed in ref.\cite{ref35} and of relevance to the tetrahedral case is

$$T_e = \zeta (\zeta^4 - 1)$$

which is the edge polynomial of a (suitably oriented) tetrahedron. Note that this polynomial should really be thought of projectively as a degree six polynomial with one root at infinity. The other two tetrahedral Klein polynomials

$$T_{v,f} = \zeta^4 \pm 2\sqrt{3}i\zeta^2 + 1$$

corresponding to vertex and face points, are not appropriate in this case since they are not strictly invariant, as they pick up a multiplying factor under some of the tetrahedral transformations.

From the restriction that the coefficients $a_r(\zeta)$ in the general curve (2.18) must be a polynomial of maximum degree $2r$, then it is clear that in order to be able to accommodate the tetrahedral term (3.3) requires $k \geq 3$. Hence the smallest charge candidate tetrahedral spectral curve is the charge three curve

$$\eta^3 + ic_3 T_e = 0$$

where $c_3$ is a real constant to satisfy reality. The determination of the possible (if any) values of $c_3$ that are allowed by non-singularity is a more difficult problem, to which we shall return shortly, but the result is that\cite{ref35}

$$c_3 = \pm \frac{\Gamma(1/6)^3\Gamma(1/3)^3}{48\sqrt{3}\pi^3/2}$$

where the $\pm$ corresponds to the tetrahedron and its dual. This proves the existence and uniqueness, up to translations and rotations, of a tetrahedrally symmetric 3-monopole. We shall study this monopole and other symmetric examples in more detail at the end of this Section.

Turning now to the octahedral group $G = O_h$, the relevant Klein polynomial is the face polynomial of the octahedron

$$O_f = \zeta^8 + 14\zeta^4 + 1.$$
Since this is a degree eight polynomial then the charge is required to satisfy \( k \geq 4 \), with the smallest charge case having the form

\[
\eta^4 + c_4 \mathcal{O}_I = 0
\]  

(3.8)

for \( c_4 \) real. Again a non-singularity investigation reveals the unique value

\[
c_4 = \frac{3\Gamma(1/4)^8}{1024\pi^2}.
\]  

(3.9)

There is also a unique octahedrally symmetric 5-monopole given by

\[
\eta^5 + 4c_4 \eta \mathcal{O}_I = 0.
\]  

(3.10)

In the icosahedral case, \( G = I_h \), all the Klein polynomials are strictly invariant, thus the smallest degree polynomial to use is the icosahedron vertex polynomial

\[
I_v = \zeta(\zeta^{10} + 11\zeta^5 - 1).
\]  

(3.11)

This implies that \( k \geq 6 \) so the smallest charge candidate is

\[
\eta^6 + c_6 I_v = 0.
\]  

(3.12)

However, it is rather surprising to find that there are no allowed values of \( c_6 \) compatible with non-singularity and hence no icosahedrally symmetric monopoles exist with \( k \leq 6 \).

A unique icosahedrally symmetric 7-monopole does exist with

\[
\eta^7 + c_7 \eta I_v = 0, \quad \text{where } c_7 = \frac{\Gamma(1/6)^6\Gamma(1/3)^6}{64\pi^3}.
\]  

(3.13)

For the case of the axially symmetric 2-monopole we have seen that the simplest method of determining the value taken by the constant in the spectral curve is through the computation of the Nahm data, from which the spectral curve can be read off using (2.26). By a simple application of the Riemann-Hurwitz formula it can be shown that in all the Platonic examples given above the quotient curve \( \tilde{S} = S/G \) is elliptic. Thus the corresponding reduction of Nahm’s equation is solvable in terms of elliptic functions and this provides the easiest method of computing the spectral curve coefficients.

With this aim in mind we need to formulate an algorithm for constructing Nahm data invariant under any discrete symmetry group \( G \subset SO(3) \). This is explained in refs. and we review it below.

The Nahm matrices are traceless, so they transform under the rotation group as

\[
\mathfrak{g} \otimes sl(k) \cong \mathfrak{g} \otimes (2k - 1 \oplus 2k - 3 \oplus \ldots \oplus 3) \\
\cong (2k + 1 \oplus 2k - 1 \oplus 2k - 3 \oplus \ldots \oplus 3) \oplus \ldots \\
\ldots \oplus (2r + 1 \oplus 2r - 1 \oplus 2r - 3 \oplus \ldots \oplus 3) \oplus \ldots \oplus (3 \oplus 3 \oplus 1)
\]  

(3.14)

\( ^a \)In refs. there is a factor of 16 error.
where \( \mathbb{CP}^1 \) denotes the unique irreducible \( r \)-dimensional representation of \( su(2) \) and the subscripts \( u, m \) and \( l \) (which stand for upper, middle and lower) are a convenient notation allowing us to distinguish between \( (2r + 1) \)-dimensional representations occurring as

\[
3 \otimes 2r - 1 \cong 2r + 1_u \oplus 2r - 1_m \oplus 2r - 3_l,
\]

\[
3 \otimes 2r + 1 \cong 2r + 3_u \oplus 2r + 1_m \oplus 2r - 1_l
\]

and

\[
3 \otimes 2r + 3 \cong 2r + 5_u \oplus 2r + 3_m \oplus 2r + 1_l.
\]

We can then use invariant homogeneous polynomials over \( \mathbb{CP}^1 \), that is we use the homogeneous coordinates \( \zeta_1/\zeta_0 = \xi \), to construct \( G \)-invariant Nahm triplets.

The vector space of degree \( 2r \) homogeneous polynomials \( a_{2r}\zeta^r + a_{2r-1}\zeta^{r-1} + \ldots + a_0\zeta^0 \) is the carrier space for \( 2r + 1 \) under the identification

\[
X = \zeta_1 \frac{\partial}{\partial \zeta_0}, \quad Y = \zeta_0 \frac{\partial}{\partial \zeta_1}, \quad H = -\zeta_0 \frac{\partial}{\partial \zeta_0} + \zeta_1 \frac{\partial}{\partial \zeta_1}
\]  

(3.15)

where \( X, Y \) and \( H \) are the basis of \( su(2) \) satisfying

\[
[X, Y] = H, \quad [H, X] = 2X, \quad [H, Y] = -2Y.
\]  

(3.16)

If \( p(\zeta_0, \zeta_1) \) is a \( G \)-invariant homogeneous polynomial we can construct an \( L^2 \) charge \( k \) Nahm triplet by the following scheme.

(i) The inclusion

\[
2r + 1 \rightarrow 3 \otimes 2r - 1 \cong 2r + 1_u \oplus 2r - 1_m \oplus 2r - 3_l
\]  

(3.17)

is given on polynomials by

\[
p(\zeta_0, \zeta_1) \mapsto \xi_1^2 \otimes p_{11}(\zeta_0, \zeta_1) + 2\xi_0\xi_1 \otimes p_{10}(\zeta_0, \zeta_1) + \xi_0^2 \otimes p_{00}(\zeta_0, \zeta_1)
\]  

(3.18)

where we have used the notation

\[
p_{ab}(\zeta_0, \zeta_1) = \frac{\partial^2 p}{\partial \zeta_a \partial \zeta_b}(\zeta_0, \zeta_1).
\]  

(3.19)

(ii) The polynomial expression \( \xi_1^2 \otimes p_{11}(\zeta_0, \zeta_1) + 2\xi_0\xi_1 \otimes p_{10}(\zeta_0, \zeta_1) + \xi_0^2 \otimes p_{00}(\zeta_0, \zeta_1) \) is rewritten in the form

\[
\xi_1^2 \otimes q_{11}(\zeta_0 \frac{\partial}{\partial \zeta_1})\xi_1^{2r} + (\xi_0 \frac{\partial}{\partial \zeta_1})\xi_1^2 \otimes q_{10}(\zeta_0 \frac{\partial}{\partial \zeta_1})\xi_1^{2r} + \frac{1}{2}(\xi_0 \frac{\partial}{\partial \zeta_1})^2 \xi_1^2 \otimes q_{00}(\zeta_0 \frac{\partial}{\partial \zeta_1})\xi_1^{2r}.
\]  

(3.20)

(iii) This then defines a triplet of \( k \times k \) matrices. Given a \( k \times k \) representation of \( X, Y \) and \( H \) above, the invariant Nahm triplet is given by

\[
(S'_1, S'_2, S'_3) = (q_{11}(adY)X^r, q_{10}(adY)X^r, q_{00}(adY)X^r),
\]  

(3.21)
where \( \text{ad}Y \) denotes the adjoint action of \( Y \) and is given on a general matrix \( M \) by \( \text{ad}YM = [M, Y] \).

(iv) The Nahm isospace basis is transformed. This transformation is given by

\[
(S_1, S_2, S_3) = \left( \frac{1}{2} S'_1 + S'_3, \frac{i}{2} S'_1 + iS'_3, -iS'_2 \right).
\]

Relative to this basis the \( \text{SO}(3) \)-invariant Nahm triplet corresponding to the \( 1 \) representation in (3.14) is given by \( (\rho_1, \rho_2, \rho_3) \) where

\[
\rho_1 = X - Y, \quad \rho_2 = i(X + Y), \quad \rho_3 = iH.
\]

It is also necessary to construct invariant Nahm triplets lying in the \( 2r + 1 \) representations. To do this, we first construct the corresponding \( 2r + 1 \) triplet. We then write this triplet in the canonical form

\[
|c_0 + c_1(\text{ad}Y \otimes 1 + 1 \otimes \text{ad}Y) + \ldots + c_i(\text{ad}Y \otimes 1 + 1 \otimes \text{ad}Y)^i + \ldots + c_{2r}(\text{ad}Y \otimes 1 + 1 \otimes \text{ad}Y)^{2r}| X \otimes X^r
\]

and map this isomorphically into \( 2r + 1 \) by mapping the highest weight vector \( X \otimes X^r \) to the highest weight vector

\[
X \otimes \text{ad}YX^{r+1} - \frac{1}{r+1}\text{ad}YX \otimes X^{r+1}.
\]

As an example, applying this scheme to determine charge seven Nahm data with icosahedral symmetry, we take the icosahedron vertex polynomial \( I_v \) (3.11) in homogeneous form

\[
I_v = \zeta_1^{11}\zeta_0 + 11\zeta_0^6c_6 - \zeta_1^{11}.
\]

This leads to an icosahedrally invariant Nahm triplet

\[
T_i(s) = x(s)\rho_i + z(s)Z_i
\]

where

\[
Z_1 = \begin{bmatrix}
0 & 5\sqrt{6} & 0 & 0 & 7\sqrt{6} \sqrt{10} & 0 & 0 \\
-5\sqrt{6} & 0 & -9\sqrt{10} & 0 & 0 & 0 & 0 \\
0 & 9\sqrt{10} & 0 & 5\sqrt{12} & 0 & 0 & -7\sqrt{6} \sqrt{10} \\
0 & 0 & -5\sqrt{12} & 0 & 5\sqrt{12} & 0 & 0 \\
-7\sqrt{6} \sqrt{10} & 0 & 0 & -5\sqrt{12} & 0 & -9\sqrt{10} & 0 \\
0 & 0 & 0 & 0 & 9\sqrt{10} & 0 & 5\sqrt{6} \\
0 & 0 & 7\sqrt{6} \sqrt{10} & 0 & 0 & -5\sqrt{6} & 0
\end{bmatrix}
\]

\[
Z_2 = i
\begin{bmatrix}
0 & 5\sqrt{6} & 0 & 0 & -7\sqrt{6} \sqrt{10} & 0 & 0 \\
5\sqrt{6} & 0 & -9\sqrt{10} & 0 & 0 & 0 & 0 \\
0 & -9\sqrt{10} & 0 & 5\sqrt{12} & 0 & 0 & 7\sqrt{6} \sqrt{10} \\
0 & 0 & 5\sqrt{12} & 0 & 5\sqrt{12} & 0 & 0 \\
-7\sqrt{6} \sqrt{10} & 0 & 0 & 5\sqrt{12} & 0 & -9\sqrt{10} & 0 \\
0 & 0 & 0 & 0 & -9\sqrt{10} & 0 & 5\sqrt{6} \\
0 & 0 & 7\sqrt{6} \sqrt{10} & 0 & 0 & 5\sqrt{6} & 0
\end{bmatrix}
\]
and the basis has been chosen in which

$$Z_3 = i \begin{bmatrix} -12 & 0 & 0 & 0 & -14\sqrt{6} & 0 & 0 \\ 0 & 48 & 0 & 0 & 0 & -14\sqrt{6} \\ 0 & 0 & -60 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -14\sqrt{6} & 0 & 0 & 0 & -48 & 0 \\ 0 & -14\sqrt{6} & 0 & 0 & 0 & 12 \end{bmatrix} \quad (3.28)$$

Substitution of the Nahm data (3.27) into Nahm’s equation (2.20) produces the reduced equations

$$\frac{dx}{ds} = 2x^2 - 750z^2 \quad (3.31)$$
$$\frac{dz}{ds} = -10xz + 90z^2$$

with corresponding spectral curve

$$\eta^7 + c_7\eta\eta_v = 0 \quad (3.32)$$

where

$$c_7 = 552960(14xz - 175z^2)(x + 5z)^4 \quad (3.33)$$

is a constant.

These equations have the solution

$$x(s) = \frac{2\kappa}{7} \left[ -3\sqrt{\wp(2\kappa s)} + \frac{\wp'(2\kappa s)}{4\wp(2\kappa s)} \right] \quad (3.34)$$
$$z(s) = -\frac{\kappa}{35} \left[ \sqrt{\wp(2\kappa s)} + \frac{\wp'(2\kappa s)}{2\wp(2\kappa s)} \right] \quad (3.35)$$

where \(\wp(t)\) is the Weierstrass function satisfying

$$\wp^2 = 4(\wp^3 - 1) \quad (3.36)$$

and to satisfy the Nahm data boundary conditions \(\kappa\) is the real half-period of this elliptic function ie.

$$\kappa = \frac{\Gamma(1/6)\Gamma(1/3)}{8\sqrt{3}\pi} \quad (3.37)$$
Then
\[ c_7 = 110592\kappa^6 = \frac{\Gamma(1/6)^6\Gamma(1/3)^6}{64\pi^3} \]  
(3.38)
and we obtain the icosahedrally symmetric 7-monopole spectral curve (3.13).

In a similar way the Nahm data and spectral curves of the other Platonic monopoles given earlier can be computed. Once the Nahm data is known it would be nice to complete the linear part of the ADHMN construction and derive the monopole gauge fields explicitly. However, the form of the Nahm data in these examples is sufficiently complicated that the explicit solution of equation (2.21) appears a difficult task. Fortunately, a numerical solution of this equation and a subsequent numerical implementation of the linear part of the ADHMN construction can be achieved, to display surfaces of constant energy density.

In Fig.1 we display surfaces of constant energy density (not to scale) for the four Platonic monopoles discussed in this Section.

We see that each of the monopoles resembles a Platonic solid (tetrahedron, cube, octahedron and dodecahedron for \( k = 3, 4, 5, 7 \) respectively) with the energy density taking its maximum value on the vertices of this solid.

It is interesting to ask where the zeros of the Higgs field occur. We know from the general discussion in Section 1 that the number of Higgs zeros, when counted with multiplicity, is \( k \). Consider then, for example, the tetrahedral 3-monopole. It is clear that the only way to arrange three points with tetrahedral symmetry is to put all three points at the origin. Thus if the tetrahedral monopole has three zeros of the Higgs field then they must all be at the origin, as in the case of the axisymmetric 3-monopole. This would be a little intriguing, but the true situation is even more interesting. In fact, by examining the Higgs field from the numerical ADHMN construction, it transpires that there are five zeros of the Higgs field. There are four positive zeros (i.e. each corresponding to a local winding of +1) on the vertices of a regular tetrahedron and an anti-zero (i.e. corresponding to a local winding of −1) at the origin. Here a local winding at a point is defined as the winding number of the normalized Higgs field, \( \Phi/|\Phi| \), on a small 2-sphere centred at this point. This integer winding number counts the number of zeros of the Higgs field, counted with multiplicity, inside this 2-sphere and thus, by definition, the sum of these local windings around all Higgs zeros must equal \( k \). Therefore the tetrahedral 3-monopole is a solution in which the Higgs field has both positive multiplicity and negative multiplicity zeros but nonetheless saturates the Bogomolny energy bound. This is a recently discovered new phenomenon for monopoles which is still not fully understood.

One possible approach to investigating this interesting phenomenon may be to examine analogous cases for monopoles in hyperbolic spaces. Just as monopoles in \( \mathbb{R}^3 \) may be interpreted as self-dual gauge fields in \( \mathbb{R}^4 \) with a translation invariance, then hyperbolic monopoles may be identified as self-dual gauge fields which are invariant under a circle action. To be topologically correct one should really consider the self-dual gauge fields to be defined in the compactification of \( \mathbb{R}^4 \) to \( S^4 \), then
Fig. 1. Energy density surfaces for a) Tetrahedral 3-monopole; b) Cubic 4-monopole; c) Octahedral 5-monopole; d) Dodecahedral 7-monopole.
a circle action leaves invariant an $S^2$. A subclass of self-dual gauge fields can be obtained from the Jackiw-Nohl-Rebbi ansatz which, to determine an $n$-instanton, requires a choice of $n + 1$ points in $\mathbb{R}^4$ together with $n + 1$ weights. By choosing equal weights and placing the points suitably in the invariant $S^2$, we could construct symmetric hyperbolic monopoles. For example, taking $n = 3$ and placing the four points on the vertices of a regular tetrahedron in the invariant $S^2$ a tetrahedrally invariant hyperbolic monopole can be computed. The advantage of working with hyperbolic monopoles is that the Jackiw-Nohl-Rebbi instanton has a simple and explicit form, so that it should be possible to derive an explicit expression for the hyperbolic monopole fields. With such an explicit expression it may be possible to understand the monopole fields better and hopefully learn something about the Euclidean case.

An examination of the other Platonic monopoles reveals that in some cases these extra anti-zeros are present and in others they are not. In Section 4, when we investigate monopole dynamics, we shall discuss some aspects of these extra Higgs zeros.

After the existence of these Platonic monopoles was proved in the above way, using the Nahm transform, the new rational map correspondence of Jarvis, which we discussed in Section 2, was proved. This allows a much easier study of the existence of monopoles with rotational symmetries, since there are no differential equations which need to be solved. Recall that the Donaldson rational map is of no use in tackling this problem, since it requires the choice of a direction in $\mathbb{R}^3$, thereby breaking the rotational symmetry of the problem.

A Jarvis map, $J : \mathbb{CP}^1 \to \mathbb{CP}^1$, is symmetric under a subgroup $G \subset SO(3)$ if there is a set of Möbius transformation pairs \{$g, D_g$\} with $g \in G$ acting on the domain and $D_g$ acting on the target, such that

$$J(g(z)) = D_g J(z). \tag{3.39}$$

where the transformations \{$D_g$\} form a 2-dimensional representation of $G$.

The simplest case is the spherically symmetric 1-monopole, given by $J = z$. The axially symmetric $k$-monopole, with symmetry around the $x_3$-axis, has the Jarvis map $J = z^k$, for $k \geq 2$.

As a more complicated example, let us construct the Jarvis map of a tetrahedrally symmetric 3-monopole and hence prove its existence.

To begin with, we can impose $180^\circ$ rotation symmetry about all three Cartesian axes by requiring the symmetries under $z \mapsto -z$ and $z \mapsto 1/z$ as

$$J(-z) = -J(z) \quad \text{and} \quad J(1/z) = 1/J(z). \tag{3.40}$$

When applied to general degree three rational maps this can be used to restrict to a one-parameter family of the form

$$J(z) = \frac{cz^2 - 1}{z(z^2 - c)} \tag{3.41}$$
with $c$ complex. The easiest way to find the value of the constant $c$ for tetrahedral symmetry is to examine the branch points of the rational map, which must be invariant under the tetrahedral group. These are given by the zeros of the numerator of the derivative

$$
\frac{dJ}{dz} = \frac{-c(z^4 + (c - 3/c)z^2 + 1)}{z^2(z^2 - c)^2}.
$$

(3.42)

By comparison of the numerator with the vertex and face Klein polynomials of the tetrahedron $T_{v,f}$, given by (3.4), it is clear that tetrahedral symmetry results only if $c = \pm i\sqrt{3}$. It can be checked that only for these two values is the remaining 120° rotation symmetry of the tetrahedron attained as

$$
J\left(\frac{iz + 1}{-iz + 1}\right) = \frac{iJ(z) + 1}{-iJ(z) + 1}.
$$

(3.43)

In a similar manner the Jarvis maps of the other Platonic monopoles can be constructed. These maps, together with many other symmetric maps, can be found in ref. In particular, a monopole corresponding to the remaining Platonic solid, the icosahedron, is shown to exist for $k = 11$, as conjectured in ref.

The simplicity of the construction of Jarvis maps for Platonic monopoles, as compared to the computation of their Nahm data, is clearly evident. However, if more detailed information regarding the monopole is sought, such as the distribution of energy density or zeros of the Higgs field, then the ADHMN construction is still the most efficient approach. In principle it is possible to construct the monopole fields from the Jarvis map, but this requires the solution of a nonlinear partial differential equation that is just as difficult to solve as the original Bogomolny equation (1.7). It is thus not as effective as the ADHMN construction, which reduces the problem to the solution of ordinary differential equations only. Nonetheless, for cases in which the Nahm data is not known the construction of monopoles from the Jarvis map is more appealing than a direct solution of the Bogomolny equation, even when both need to be implemented numerically, since it allows an elegant specification of exactly which monopole is to be constructed.

4. Dynamics and Moduli Space Metrics

So far we have discussed only static monopoles, which are solutions of the Bogomolny equation (1.7). As we have seen this equation is integrable, which for the present case we shall take to mean that it has a twistor correspondence. Unfortunately the full time-dependent field equations which follow from the Lagrangian (1.1) are not integrable and so we do not expect to be able to solve this equation explicitly, or apply a twistor transform, to investigate monopole dynamics. However, progress can be made on the study of slowly moving monopoles by applying Manton’s moduli space approximation (sometimes also called the geodesic approximation).

The moduli space approximation, which can also be applied to the dynamics of other kinds of topological solitons, was first proposed for the study of
monopoles\(^4\). Here one assumes that the \(k\)-monopole configuration at any fixed time may be well-approximated by a static \(k\)-monopole solution. The only time dependence allowed is therefore in the dynamics of the \(4k\)-dimensional \(k\)-monopole moduli space \(\mathcal{M}_k\). A Lagrangian on \(\mathcal{M}_k\) is inherited from the field theory Lagrangian \(^1\), but since all elements of \(\mathcal{M}_k\) have the same potential energy the kinetic part of the action completely determines the dynamics. It defines a metric on \(\mathcal{M}_k\) and the dynamics is given by geodesic motion on \(\mathcal{M}_k\) with respect to this metric.

Intuitively one should think of the potential energy landscape in the charge \(k\) configuration space as having a flat valley given by \(\mathcal{M}_k\), and the low energy monopole dynamics takes place in, or at least close to, this valley. A rigorous mathematical analysis of the validity of the moduli space approximation has been performed by Stuart\(^4\).

In the case of a single monopole the moduli space approximation is rather trivial. \(\mathcal{M}_1 = \mathbb{R}^3 \times S^1\) is flat, with constant motion in the \(\mathbb{R}^3\) giving the momentum of the monopole and constant angular speed in the \(S^1\) determining the electric charge of the monopole, as mentioned in Section 1.

To study multi-monopole dynamics, as well as for other reasons which we shall see later, it is therefore of interest to find the metric on \(\mathcal{M}_k\). By a formal application of the hyperkähler quotient construction\(^5\), it can be shown\(^6\) that \(\mathcal{M}_k\) is a hyperkähler manifold, which means that there are three covariantly constant complex structures satisfying the quaternionic algebra. The rotational symmetry of the system means that there is also an \(SO(3)\) action, which permutes the complex structures. Physically, the motion of the centre of mass of the system and the total phase decouples from the relative motion, which mathematically means that there is an isometric splitting

\[
\tilde{\mathcal{M}}_k = \mathbb{R}^3 \times S^1 \times \mathcal{M}_k^0
\]  \(4.1\)

where \(\tilde{\mathcal{M}}_k\) is a \(k\)-fold covering of \(\mathcal{M}_k\) and \(\mathcal{M}_k^0\) is the \(4(k-1)\)-dimensional moduli space of strongly centred \(k\)-monopoles. The fact that a \(k\)-fold covering occurs is because the \(k\) monopoles are indistinguishable.

The non-trivial structure of the moduli space of 2-monopoles is therefore contained in the totally geodesic 4-dimensional submanifold \(\mathcal{M}_2^0\). Using the fact that this is a hyperkähler manifold with an \(SO(3)\) symmetry, Atiyah & Hitchin\(^5\) were able to reduce the computation of its metric to the solution of a single ordinary differential equation, which can be done explicitly in terms of elliptic integrals. They were thus able to explicitly determine the metric on \(\mathcal{M}_2^0\), which is now known as the Atiyah-Hitchin manifold. The easiest way to present the Atiyah-Hitchin metric is in terms of the left-invariant 1-forms

\[
\sigma_1 = -\sin \psi d\theta + \cos \psi \sin \theta d\phi
\]  \(4.2\)
\[
\sigma_2 = \cos \psi d\theta + \sin \psi \sin \theta d\phi
\]  \(4.3\)
\[
\sigma_3 = d\psi + \cos \theta d\phi
\]  \(4.4\)
where \( \theta, \phi, \psi \) are the usual Euler angles. It is given by
\[
ds^2 = \frac{b^2}{K^2} dK^2 + a^2 \sigma_1^2 + b^2 \sigma_2^2 + c^2 \sigma_3^2
\] (4.5)
\[
a^2 = 2K(K - E)(E - m'K)/E
\] (4.6)
\[
b^2 = 2EK(K - E)/(E - m'K)
\] (4.7)
\[
c^2 = 2EK(E - m'K)/(K - E)
\] (4.8)

where \( K \) and \( E \) denote the complete elliptic integrals of the first and second kind with parameter \( m \) and \( m' = 1 - m \) is the complimentary parameter.

The coordinate \( m \) determines the separation of the two monopoles in the same way as appears in the 2-monopole spectral curve (2.12). That is, \( m = 0 \) represents the axially symmetric 2-monopole (which is referred to as the bolt in the metric context) and \( m \to 1 \) represents the monopoles moving off to infinity.

There is an interesting totally geodesic 2-dimensional submanifold of the Atiyah-Hitchin manifold which can be obtained by the imposition of a reflection symmetry. This 2-dimensional submanifold is a surface of revolution, which metrically is a rounded cone. An interesting geodesic is one which passes directly over the cone, i.e., a generator of the surface of revolution. In the moduli space approximation this geodesic describes the head-on collision of two monopoles, which pass instantaneously through the axially symmetric 2-monopole and emerge at 90° to the initial direction of approach. This famous right-angle scattering of monopoles has now been observed in many other systems and appears to be a general feature of multi-dimensional topological solitons.

Another interesting geodesic in the Atiyah-Hitchin manifold was discovered by Bates & Montgomery. This is a closed geodesic and thus, within the moduli space approximation, describes a bound state of two orbiting monopoles.

The metric on \( M_0^k \) for \( k > 2 \) is unknown and its difficulty of computation is related to the earlier comment that there is an associated algebraic curve of genus \((k - 1)^2\), which is therefore no-longer simply elliptic. However, some recent progress has been made regarding the computation of the metric on certain totally geodesic submanifolds of \( M_0^k \). In order to describe these results we now turn to a consideration of the moduli space metric in terms of the Nahm transform.

The discussion of the ADHMN construction in Section 2 was presented in terms of Nahm data consisting of three Nahm matrices \((T_1, T_2, T_3)\), but in order to discuss the metric we must, following Donaldson, introduce a fourth Nahm matrix \( T_0 \). Then we have that charge \( k \) monopoles are equivalent to Nahm data \((T_0, T_1, T_2, T_3)\), which satisfy the full Nahm equations
\[
\frac{dT_i}{ds} + [T_0, T_i] = \frac{1}{2} \epsilon_{ijk} [T_j, T_k] \quad i = 1, 2, 3.
\] (4.9)

The Nahm data conditions remain the same as before, but are supplemented by the requirement that \( T_0 \) is regular for \( s \in [0, 2] \).
Let $H$ be the group of analytic $su(k)$-valued functions $h(s)$, for $s \in [0, 2]$, which are the identity at $s = 0$ and $s = 2$ and satisfy $h^t(2 - s) = h^{-1}(s)$. Then gauge transformations $h \in H$ act on Nahm data as

$$T_0 \mapsto hT_0h^{-1} - \frac{dh}{ds}h^{-1}, \quad T_i \mapsto hT_ih^{-1} \quad i = 1, 2, 3. \quad (4.10)$$

Note that the gauge $T_0 = 0$ may always be chosen, which is why this fourth Nahm matrix is usually not introduced. However, when discussing the metric on Nahm data we need to consider the action of the gauge group and so this extra Nahm matrix needs to be kept.

To find the metric on the space of Nahm data a basis for the tangent space needs to be found, by solving the linearized form of Nahm’s equation (4.9). Let $(V_0, V_1, V_2, V_3)$ be a tangent vector corresponding to the point with Nahm data $(T_0, T_1, T_2, T_3)$. It is a solution of the equations

$$\frac{dV_i}{ds} + [V_0, T_i] + [T_0, V_i] = \epsilon_{ijk}[T_j, V_k] \quad i = 1, 2, 3 \quad (4.11)$$

and

$$\frac{dV_0}{ds} + \sum_{i=0}^3 [T_i, V_i] = 0 \quad (4.12)$$

where $V_i, \ i = 0, 1, 2, 3$, is an analytic $su(k)$-valued function of $s \in [0, 2]$. If $W_i$ is a second tangent vector then the metric component corresponding to these two tangent vectors is defined as

$$< V_i, W_i > = - \int_0^2 \sum_{i=0}^3 tr(V_i W_i) \ ds. \quad (4.13)$$

Nakajima has proved that the transformation between the monopole moduli space metric and the metric on Nahm data is an isometry. Therefore it is in principle possible to compute the metric on the monopole moduli space if the corresponding Nahm data is known. For example, by using the known 2-monopole Nahm data, (2.32) with (2.35), the tangent vectors can be found explicitly in terms of elliptic functions and the resulting integrals in (4.13) performed in terms of complete elliptic integrals to recover the Atiyah-Hitchin metric (4.5).

Given the above discussion it is therefore of interest to look for families of symmetric monopoles corresponding to totally geodesic submanifolds of $M^0_3$, where the Nahm data can be found explicitly in terms of elliptic functions. We shall refer to such submanifolds as elliptic. It is then possible that the metric on elliptic submanifolds can be computed exactly, in terms of elliptic integrals.

It can be shown that there is a 4-dimensional submanifold of strongly centred 3-monopoles which are symmetric under the inversion $x \mapsto -x$. Since the fixed point set of a group action is always totally geodesic, then this is a 4-dimensional totally geodesic submanifold of $M^0_3$. The corresponding Nahm data is a generalization of
the 2-monopole Nahm data (2.32) where the same functions (2.35) occur but the matrices which form the spin $\frac{1}{2}$ representation of $su(2)$ are replaced by those of the spin 1 representation. The upshot of this is that the only modification which arises in the computation of the metric from the Nahm data is the overall multiplication by a constant. Thus, after checking that rotations act in the same way as before, this proves that this submanifold is a totally geodesic Atiyah-Hitchin submanifold of $M_3^0$.

Physically, the three monopoles are collinear, with the third monopole fixed at the origin and the other two behaving in an Atiyah-Hitchin manner, as if they do not see this third one. Simultaneously with this discovery Bielawski found the same result using the hyperkähler quotient construction and also showed that it generalizes to a totally geodesic Atiyah-Hitchin submanifold of $M_k^0$, for all $k > 2$, corresponding to a string of $k$ equally spaced collinear monopoles. The Nahm data in each case is again obtained by replacing the spin $\frac{1}{2}$ representation of $su(2)$ by the spin $(k - 1)/2$ representation, but the submanifold can not be obtained as the fixed point set of a group action. However, the fact that it is totally geodesic can be deduced from the knowledge that it is a hyperkähler submanifold of a hyperkähler manifold. Note that once the manifold is shown to have an Atiyah-Hitchin submanifold then it is guaranteed to possess a closed geodesic describing a bound state of $k$ monopoles, since it is merely the inherited Bates & Montgomery geodesic in $M_3^0$.

A geodesic in $M_3^0$ can be obtained by the imposition of tetrahedral symmetry on 4-monopoles. This 1-dimensional submanifold is elliptic, allowing the computation of the associated Nahm data, which can be used to study the scattering of 4-monopoles and also the calculation of the 1-dimensional metric. This family obviously includes the cubic 4-monopole and thus via the moduli space approximation it describes the scattering of four monopoles which pass through the cubic configuration. In Fig.2 we display surfaces of constant energy density at increasing times throughout this process. It can be seen that four monopoles approach from infinity on the vertices of a contracting regular tetrahedron, coalesce to form a configuration with instantaneous cubic symmetry and emerge on the vertices of an expanding tetrahedron dual to the incoming one. Note that in the above example it is not necessary to know the metric in order to determine the trajectories of the monopoles. This is because an application of a symmetry resulted in a totally geodesic 1-dimensional submanifold, which by definition is a geodesic. Thus the metric only determines the rate at which motion takes place along the geodesic and is not needed to determine the form of the motion.

This observation is useful and suggests the scheme of searching for appropriate symmetry groups $G$, for various charges $k$, so that the moduli space of $G$-symmetric $k$-monopoles is 1-dimensional, thus furnishing a geodesic.

Searching for symmetric monopoles using the ADHMN construction is not a simple task, since in each case it relies upon a careful study of solutions to Nahm’s equation. Furthermore, unless the submanifold in question is elliptic, it is unlikely that this will be a tractable problem. However, using rational maps it is a much
Fig. 2. Energy density surfaces for tetrahedral 4-monopole scattering.

Recall that Donaldson rational maps require a choice of direction in $\mathbb{R}^3$, so only symmetries which preserve this direction can be studied using these maps. Nonetheless, it turns out that this still provides a substantial collection of interesting geodesics.

The obvious symmetry to impose which fixes a direction is cyclic symmetry and the corresponding rational maps were investigated by Hitchin, Manton & Murray. Requiring invariance of a $k$-monopole under cyclic $C_k$ symmetry and an additional reflection symmetry, leads to a number of geodesics, $\Sigma^l_k$, in the $k$-monopole moduli space $M_k$. Essentially there are $(2k+3+(-1)^k)/4$ different types of these geodesics, corresponding to $l = 0, 1, \ldots k/2$ if $k$ is even and $l = 0, 1, \ldots (k-1)/2$ if $k$ is odd. Physically, for $l \neq 0$, the associated monopole scatterings are distinguished by having the out state (or in state by time reversal) consisting of two clusters of monopoles with charges $k - l$ and $l$. This explains why we do not allow $l > k/2$, since this is basically the same scattering event as one of the geodesics with $l < k/2$.

If $l = 0$ then the monopoles remain in a plane and scatter instantaneously through the axisymmetric $k$-monopole and emerge with a $\pi/k$ rotation. In this case if $k = 2$ then this is just the Atiyah-Hitchin right-angle scattering that we have already discussed. In fact the case $k = 2$ is special, in that the geodesics $\Sigma_2^0$ and $\Sigma_2^1$ are isomorphic, so that there is only this one type of scattering. For all $k$ with $l = 0$ this kind of $\pi/k$ scattering is essentially a two-dimensional process and has been extensively studied in planar systems.

For $l \neq 0$ we see that the scatterings are more exotic, since the clustering of
monopoles changes during the scattering process. It can be shown\textsuperscript{[2]} that for these cases Nahm’s equation reduces to the $A_{k-1}^{(1)}$ Toda chain and the quotient spectral curve has genus $k-1$. Again this curve is not elliptic (except for the case $k = 2$ when the Toda chain is equivalent to the static sine-Gordon equation with the appropriate solution being the kink) but perhaps relating the ADHMN construction to other well-studied integrable systems may prove fruitful. In fact other aspects of the ADHMN construction can also be connected with more traditional integrable systems, such as Lamé equations\textsuperscript{[60, 61]}, and it would be worthwhile exploring these relationships further.

Even though the Nahm data for these cyclic monopoles has not been found explicitly, it is possible to find an approximation to it and hence display the scattering events\textsuperscript{[37, 39]} Fig. 3 shows one such example for the geodesic $\Sigma_3^1$. It shows three individual monopoles which lie on the vertices of a contracting equilateral triangle in the plane, which merge to instantaneously form the tetrahedral 3-monomopole and finally split to form a charge two torus and a single monopole moving apart along an axis orthogonal to the plane of the incoming monopoles.

Another class of symmetries which it is worth investigating using Donaldson maps are the twisted cyclic symmetries obtained by the composition of a rotation with a reflection in a plane orthogonal to the axis of rotation. Geodesics can be obtained in this way\textsuperscript{[37] which describe monopoles scattering through all the Platonic configurations discussed earlier. In the simplest case of 3-monomopoles with a twisted 90° rotation symmetry the corresponding geodesic is elliptic and the Nahm data has been computed exactly. Explicitly the one-parameter family of spectral curves is given by

\[ y^3 - 6(a^2 + 4\epsilon)^{1/3} \kappa^2 \eta \zeta^2 + 2i \kappa^3 a(\zeta^5 - \zeta) = 0, \tag{4.14} \]

\[(\epsilon = \pm 1), \text{ where } \kappa \text{ is the real half-period of the elliptic curve} \]

\[ y^2 = 4x^3 - 3(a^2 + 4\epsilon)^{2/3}x + 4\epsilon. \tag{4.15} \]

This includes the tetrahedral 3-monomopole and its dual ($a = \pm 2, \epsilon = -1$), the axisymmetric 3-monomopole ($a = 0, \epsilon = -1$) with symmetry axis $x_3$, and asymptotically ($a \to 0, \epsilon = 1$) describes three collinear monopoles which lie along the $x_3$-axis.

As mentioned earlier, some of the above included configurations have spurious anti-zeros of the Higgs field, while some do not, such as the well-separated limit. Thus there must be special ‘splitting points’ at which anti-zeros appear, or disappear. In this example it seems that these ‘splitting points’ occur at $a = \pm \sqrt{8}, \epsilon = -1$ and $a = 0, \epsilon = -1$, which are all the points where the discriminant of the elliptic curve (4.15) vanishes, so that the curve is rational. The motion of the zeros and anti-zeros throughout this scattering is discussed in detail in ref.\textsuperscript{[37]}

The recently introduced Jarvis maps, which we reviewed in Section 2, are much better for identifying geodesics obtainable by symmetry considerations. Some examples are given in ref.\textsuperscript{[41]} such as an interesting scattering process involving seven monopoles which pass through two dodecahedra and a cube.
Fig. 3. Energy density surfaces for cyclic 3-monopole scattering.
Although the full metric on $\mathcal{M}_k^0$ is unknown for $k > 2$ it is known asymptotically on regions which correspond to all the $k$ monopoles being well-separated\cite{63}. First consider the Atiyah-Hitchin metric\cite{13} for large monopole separations. The asymptotic form can be found by using the standard expansions for elliptic integrals as $m \to 1$. Writing $\rho = 2K$, which asymptotically is the separation of the two monopoles, it is given by

$$ds^2 = \left(1 - \frac{2}{\rho}\right)(d\rho^2 + \rho^2 d\theta^2 + \rho^2 \sin^2 \theta d\phi^2) + 4\left(1 - \frac{2}{\rho}\right)^{-1} (d\psi + \cos \theta d\phi)^2. \quad (4.16)$$

This is the Taub-NUT metric with a negative mass parameter and is also a hyperkähler metric. It has a singularity at $\rho = 2$ but this is not a value for which the approximation is valid, since it assumes that $\rho \gg 1$. Note that (4.16) also has a $U(1)$ symmetry which the Atiyah-Hitchin metric lacks and corresponds physically to the conservation of the relative electric charge. Asymptotically the metric is correct up to terms which are exponentially suppressed.

Manton\cite{64} pointed out that this asymptotic metric could be derived from a point particle approximation, by treating each monopole as a source of electric, magnetic and scalar charge. A similar calculation for the general charge $k$ case was performed by Gibbons & Manton\cite{63} and results in the following. For $k$ monopoles located at $\{\rho_i\}$ with phases $\{\theta_i\}$ the asymptotic metric is

$$ds^2 = g_{ij} d\rho_i \cdot d\rho_j + g_{ij}^{-1} (d\theta_i + W_{ik} \cdot d\rho_k) (d\theta_j + W_{jl} \cdot d\rho_l) \quad (4.17)$$

where $\cdot$ denotes the usual scalar product on $\mathbb{R}^3$ vectors, repeated indices are summed over and

$$g_{jj} = 1 - \sum_{i \neq j} \frac{1}{\rho_{ij}} \quad \text{(no sum over } j) \quad (4.18)$$

$$g_{ij} = \frac{1}{\rho_{ij}} \quad (i \neq j)$$

$$W_{jj} = - \sum_{i \neq j} w_{ij} \quad \text{(no sum over } j)$$

$$W_{ij} = w_{ij} \quad (i \neq j),$$

$\rho_{ij} = \rho_i - \rho_j$ and $\rho_{ij} = |\rho_{ij}|$. The approximation is valid for $\rho_{ij} \gg 1$. The $w_{ij}$ are Dirac potentials and are defined by

$$\text{curl } w_{ij} = \text{grad } \frac{1}{\rho_{ij}} \quad (4.19)$$

where the curl and grad operators are taken with respect to the $i$th position coordinate $\rho_i$.

It can be checked, for example, that the metric on the moduli space of tetrahedrally symmetric charge four monopoles, which is known exactly in terms of complete elliptic integrals\cite{55}, agrees with this formula asymptotically.

5. Higher Rank Gauge Groups
So far we have dealt only with the case of \( SU(2) \) monopoles. The kind of analysis we have reviewed can, of course, be extended to more general gauge groups, where things usually become more complicated. In this Section we sketch how the ideas and results are modified for \( SU(N) \) gauge groups and discuss some special situations in which the problem simplifies.

Recall from Section 1 that in a gauge theory where the non-abelian gauge group \( G \) is spontaneously broken by the Higgs field to a residual symmetry group \( H \), then the monopoles have a topological classification determined by the elements of \( \pi_2(G/H) \).

For \( G = SU(N) \) then the boundary conditions at spatial infinity are that \( \Phi \) takes values in the gauge orbit of the matrix

\[
M = i \text{diag} (\mu_1, \mu_2, \ldots, \mu_N). \tag{5.1}
\]

By convention it is assumed that \( \mu_1 \leq \mu_2 \leq \ldots \leq \mu_N \) and since \( \Phi \) is traceless then \( \mu_1 + \mu_2 + \ldots + \mu_N = 0 \). This \( M \) is the vacuum expectation value for \( \Phi \) and the residual symmetry group \( H \) is the symmetry group of \( M \) under gauge transformations. Thus, for example, if all the \( \mu_p \) are distinct then the residual symmetry group is the maximal torus \( U(1)^{N-1} \) and this is known as maximal symmetry breaking. In this case

\[
\pi_2 \left( \frac{SU(N)}{U(1)^{N-1}} \right) = \pi_1(U(1)^{N-1}) = \mathbb{Z}^{N-1} \tag{5.2}
\]

so the monopoles are associated with \( N - 1 \) integers.

In contrast, the minimal symmetry breaking case is that in which all but one of the \( \mu_p \) are identical, so the residual symmetry group is \( U(N-1) \). Since

\[
\pi_2 \left( \frac{SU(N)}{U(N-1)} \right) = \mathbb{Z} \tag{5.3}
\]

there is only one topological integer characterization of a monopole. Nonetheless, a given solution has \( N - 1 \) integers associated with it, which arise in the following way.

A careful analysis of the boundary conditions\(^{3,4,5}\) indicates that there is a choice of gauge such that the Higgs field for large \( r \), in a given direction, is given by

\[
\Phi(r) = i \text{diag} (\mu_1, \mu_2, \ldots, \mu_N) - \frac{i}{r} \text{diag} (k_1, k_2, \ldots, k_N) + O(r^{-2}). \tag{5.4}
\]

In the maximal symmetry breaking case the topological charges are given by

\[
m_p = \sum_{q=1}^{p} k_q. \tag{5.5}
\]

In the case of minimal symmetry breaking only the first of these numbers, \( m_1 \), is a topological charge. Nonetheless, the remaining \( m_p \) constitute an integer characterization of a solution, which is gauge invariant up to reordering of the integers \( k_p \).
The \( m_p \) are known as magnetic weights, with the matrix \( \text{diag}(k_1, k_2, \ldots, k_N) \) often called the charge matrix and \( \text{diag}(\mu_1, \mu_2, \ldots, \mu_N) \) the mass matrix.

There are some obvious ways of embedding \( su(2) \) in \( su(N) \), for example,

\[
\begin{pmatrix}
\alpha & \beta \\
-\bar{\beta} & -\alpha
\end{pmatrix} \mapsto
\begin{pmatrix}
\ddots & \alpha & \ldots & \beta \\
& \ddots & \ddots & \ddots \\
& & \ddots & \ddots \\
& & & -\bar{\beta} & \ldots & -\alpha \\
\end{pmatrix}.
\tag{5.6}
\]

Important \( SU(N) \) monopoles can be produced by embedding the \( SU(2) \) charge one monopole fields, which are known \( su(2) \)-valued fields, in \( su(N) \). Some care must be taken in producing these embedded monopoles to ensure that the asymptotic behaviour is correct. The \( SU(2) \) monopole may need to be scaled and it may be necessary to add a constant diagonal field beyond the plain embedding described by (5.6); details may be found in refs. 66, 67. Obviously there is an embedding of the form (5.6) for each choice of two columns in the target matrix. The embedded 1-monopoles have a single \( k_p = 1 \) and another \( k_{p'} = -1 \), the rest are zero. The choice of columns for the embedding dictates the values for \( p, p' \), so there are \( N-1 \) different types of fundamental monopole with \( m_p = 1 \) and the rest zero, corresponding to the choice \( p' = p + 1 \).

Recall that in the case of minimal symmetry breaking the choice of order of the \( k_p \) is a gauge choice. In fact, in the case of minimal symmetry breaking, the embedded 1-monopole is unique up to position and gauge transformation. Solutions with \( k_1 = k \) have \( k \) times the energy of this basic solution and so it is reasonable to call these \( k \)-monopoles. There are of course different types of such \( k \)-monopoles corresponding to different magnetic weights.

For other intermediate cases of symmetry breaking the residual symmetry group is \( H = U(1)^r \times K \), where \( K \) is a rank \( N - r - 1 \) semi-simple Lie group, the exact form of which depends on how the entries in the mass matrix coincide with each other. Such monopoles have \( r \) topological charges.

The twistor methods of Section 2 can be formulated for the case of general gauge groups. Ward\textsuperscript{[24]} has constructed some explicit \( SU(3) \) monopoles via the splitting of appropriate patching matrices over \( \Pi \). The spectral curve approach for maximal symmetry breaking has been formulated by Hurtubise & Murray\textsuperscript{[68]} and consists of a specification of rank(\( G \)) algebraic curves in \( \Pi \), satisfying reality and non-singularity conditions. For higher rank gauge groups the Donaldson rational map correspondence has been extended by Murray\textsuperscript{[69]} to maps into flag manifolds and a similar extension exists for the new rational maps of Jarvis\textsuperscript{[33]}. The ADHMN construction for general \( G \) is outlined in the original work of Nahm\textsuperscript{[26]} and is discussed further in ref.\textsuperscript{[68]}. Briefly, for \( G = SU(N) \) the Nahm data are triplets of anti-hermitian matrix functions \( (T_1, T_2, T_3) \) of \( s \) over the intervals \( (\mu_p, \mu_{p+1}) \). The size of the matrices depends on the corresponding values of \( m_p \);
the matrices \((T_1, T_2, T_3)\) are \(m_p \times m_p\) matrices in the interval \((\mu_p, \mu_{p+1})\). They are required to be non-singular inside each interval and to satisfy Nahm’s equation \((2.20)\) but there are complicated boundary conditions at the ends of each of the intervals. These boundary conditions are designed so that the linear equation \((2.21)\) has the correct number of solutions required to yield the right type of monopole fields.

The simplest case is maximal symmetry breaking in an \(SU(3)\) theory. There are then two types of monopole, so the charge is a 2-component vector \((m_1, m_2)\). The simplest multi-monopole is therefore of charge \((1, 1)\), and its Nahm data was studied by Connell\(^7\). Since there is only one of each type of monopole then the Nahm data is 1-dimensional over each of the two intervals, so Nahm’s equations are trivially satisfied by taking the Nahm data to be constants over each of the two intervals. These two triplets of constants determine the positions of the two monopoles and the matching condition at the common boundary of the two intervals determines the relative phase.

The moduli space of these monopoles is 8-dimensional but, as in the \(SU(2)\) case, there is an isometric splitting to factor out the position of the centre of mass and the overall phase. The relative moduli space, \(\mathcal{M}_{(1,1)}^0\), is thus 4-dimensional. By computing the metric on Nahm data and using a uniqueness argument, Connell\(^6\) was able to show that the metric on \(\mathcal{M}_{(1,1)}^0\) is the Taub-NUT metric with a positive mass parameter. This result was rediscovered some years later\(^8\). Recall that the asymptotic Atiyah-Hitchin metric is also a Taub-NUT metric, but with a negative mass parameter, so the asymptotic metric has a singularity outside its region of validity. This difference in sign results from the fact that in the \(SU(3)\) case the two monopoles are electrically charged with respect to different \(U(1)\) factors in the residual symmetry group. There is thus a conservation of the individual electric charge of each monopole, providing a \(U(1)\) symmetry in the metric which is absent in the Atiyah-Hitchin metric, since charge exchange occurs between \(SU(2)\) monopoles. This results in a simplified dynamics of charge \((1, 1)\) monopoles, which bounce back off each other in a head-on collision in comparison with the right-angle scattering of two \(SU(2)\) monopoles.

Similar simplifications can be expected in all cases where there is at most a single monopole of each type. Thus the \(4(N-2)\)-dimensional relative moduli space \(\tilde{\mathcal{M}}_N^0\) of charge \((1, 1, \ldots, 1)\) monopoles in an \(SU(N)\) theory should be tractable. Indeed, Lee, Weinberg & Yi\(^9\) have computed the asymptotic metric, which is a generalization of the Taub-NUT case, and conjectured that it is the exact metric. This is supported by a computation of the metric on the space of Nahm data by Murray\(^10\), which obtains the same result. Note that this last calculation is not quite a proof, since although it is believed that the transformation between the monopole moduli space metric and the metric on Nahm data is an isometry for all gauge groups and symmetry breaking, it has only been proved for \(SU(2)\) monopoles\(^11\) and special cases for minimally broken \(SU(N)\). More recently, these and other monopole metrics have also been obtained by Gibbons & Rychenkova\(^12\) using the hyperkähler
quotient construction.

There is a method which can be used to give a local construction of hyperkähler metrics known as the generalized Legendre transform. This can be used, for example, to give yet another derivation of the Atiyah-Hitchin manifold. Using this method Chalmers was able to rederive the Lee-Weinberg-Yi metric and prove that it is the correct metric throughout the moduli space.

In order to examine if there are any other special choices of gauge group, symmetry breaking and monopole charges for which there may be a simplification we need to review a few more details of the Nahm data boundary conditions.

For ease of notation we shall only describe the case where \( m_{p-1} > m_p \), since this will be the one of interest in what follows. Define the function

\[
k(s) = \sum_{p=1}^{N} k_p \theta(s - \mu_p)
\]

where \( \theta(s) \) is the usual step function. In the interval \( (\mu_p, \mu_{p+1}) \) then \( k(s) = m_p \), so it is a rectilinear skyline whose shape depends on the charge matrix of the corresponding monopole.
If \( k(s) \) near \( \mu_p \) looks like

![Diagram of a staircase with steps down of unit height](image)

then as \( s \) approaches \( \mu_p \) from below it is required that

\[
T_i(s) = \begin{pmatrix}
\frac{1}{2} R_i + O(1) & O(z(\lvert k_p \rvert - 1)/2) \\
O(z(\lvert k_p \rvert - 1)/2) & T'_i + O(z)
\end{pmatrix}
\]

where \( z = s - \mu_p \) and where

\[
T_i(s) = T'_i + O(z)
\]

as \( s \) approaches \( \mu_p \) from above. It follows from Nahm’s equation (2.20) that the \( \lvert k_p \rvert \times \lvert k_p \rvert \) residue matrices \((R_1, R_2, R_3)\) in (8.7) form a representation of \( su(2) \). The boundary conditions require that this representation is the unique irreducible \( \lvert k_p \rvert \)-dimensional representation of \( su(2) \).

In summary, at the boundary between two intervals, if the Nahm matrices are \( m_{p-1} \times m_{p-1} \) on the left and \( m_p \times m_p \) on the right an \( m_p \times m_p \) block continues through the boundary and there is an \( (m_{p-1} - m_p) \times (m_{p-1} - m_p) \) block simple pole whose residues form an irreducible representation of \( su(2) \).

These conditions now suggest a simplifying case, since if \( k_p = -1 \) for all \( p > 1 \) then \( k(s) \) is a staircase with each step down of unit height. We shall refer to this situation as the countdown case since the magnetic weights are given by \((N-1, N-2, ..., 2, 1)\). Thus, since all the 1-dimensional representations of \( su(2) \) are trivial, the Nahm data has only one pole, which is at \( s = \mu_1 \). Taking the limiting case of minimal symmetry breaking, by setting \( \mu_1 = -(N-1) \) and \( \mu_2 = ... = \mu_N = 1 \), we find that the Nahm data is defined on a single interval \([-N+1, 1]\) with the only pole occurring at the left-hand end of the interval. This is very similar to the Nahm data for \( SU(2) \) monopoles, except that the pole at the right-hand end of the interval is lost. This allows a construction of Nahm data for charge \( N-1 \) monopoles in a minimally broken \( SU(N) \) theory in terms of rescaled Nahm data for \( SU(2) \) monopoles, where the rescaling moves the second pole in the Nahm data outside the interval. With this in mind it is convenient to shift \( s \) so that the Nahm data is defined over the interval \([0, N]\).
As an illustration of the simplification that occurs in the countdown case we present the Nahm data for an $SU(N)$ charge $N - 1$ spherically symmetric monopole. It is given by $T_i = -\rho_i / 2s$ where $\rho_1, \rho_2, \rho_3$ form the standard spin $(N - 2)/2$ representation of $su(2)$. The associated spectral curves are simply $\eta^{N-1} = 0$. Spherically symmetric monopoles were first studied by Bais & Wilkinson, Leznov & Saveliev and Ganoulis, Goddard & Olive all using a radial ansatz in the Bogomolny equation.

The simplest countdown example to consider further is the case of charge two $SU(3)$ monopoles with minimal symmetry breaking. For $k = 2$ there are two distinct types of monopoles corresponding to magnetic weights $(2, 0)$ and $(2, 1)$. (The cases $(2, 2)$ and $(2, 0)$ are equivalent by a reordering of $k_2$ and $k_3$). For weights $(2, 0)$ the monopoles are all embeddings of $su(2)$ 2-monopoles and this case is not interesting as an example of $su(3) 2$-monopoles. For weights $(2, 1)$ this is a countdown case and was first studied by Dancer. Given the comments above it is fairly clear that the appropriate Nahm data has the same form as the $SU(2) 2$-monopole Nahm data (2.32) where the functions $f_1, f_2, f_3$ are almost the same as in the $SU(2)$ case (2.33) except that the complete elliptic integral $K$, whose value was required to place the second pole at $s = 2$, is now replaced by a parameter $D$, whose range is such that no second pole occurs in the interval $i e$. $D < 2K/3$. Explicitly the Nahm data is

$$T_1 = -D \frac{dn(Ds)}{2 sn(Ds)} \sigma_1, \quad T_2 = -D \frac{D}{2 sn(Ds)} \sigma_2, \quad T_3 = D \frac{cn(Ds)}{2 sn(Ds)} \sigma_3 \quad (5.10)$$

The moduli space of such monopoles is 12-dimensional, so after centering we are left with an 8-dimensional relative moduli space $M^8_0$. There is an isometric $SO(3) \times SU(2)/\mathbb{Z}_2$ action on $M^8_0$. The $SU(2)/\mathbb{Z}_2$ action is a gauge transformation on the four Nahm matrices, which is the identity at $s = 0$, while the $SO(3)$ action both rotates the three Nahm matrices as a vector and gauge transforms all four Nahm matrices. Taking the quotient of $M^8_0$ by the $SU(2)/\mathbb{Z}_2$ action gives a 5-dimensional manifold $M^5_0$ which has an $SO(3)$ action, since the $SU(2)/\mathbb{Z}_2$ and $SO(3)$ actions on $M^8_0$ commute. The Nahm data for $M^5_0$ is precisely the orbit under $SO(3)$ of the 2-parameter family of Nahm data (5.10). Using this Nahm data, Dancer computed an explicit expression for the metric on $M^5_0$ and an implicit form for the metric on the whole of $M^8_0$. A more explicit form for the metric on $M^8_0$, in terms of invariant 1-forms corresponding to the two group actions, together with a study of the corresponding asymptotic monopole fields has been given by Irwin.

A totally geodesic 2-dimensional submanifold, $Y$, of $M^8_0$ is obtained by imposition of a $\mathbb{Z}_2 \times \mathbb{Z}_2$ symmetry, representing monopoles which are symmetric under reflection in all three Cartesian axes. In fact $Y$ consists of six copies of the space $M^5_0/SO(3)$. This submanifold was introduced by Dancer & Leese and the geodesics and corresponding monopole dynamics investigated. There are two interesting new phenomena which occur. The first is that there can be double scatterings, where the two monopoles scatter at right-angles in two orthogonal planes. The second is that there are unusual geodesics which describe monopole dynamics where
two monopoles approach from infinity but stick together, with the motion taking the configuration asymptotically towards an embedded $SU(2)$ field, which is on the boundary of the $SU(3)$ monopole moduli space and metrically at infinity. This kind of behaviour is still not completely understood but the interpretation is that there is a non-abelian cloud, whose radius is related to the parameter $D$ in the Nahm data (5.10). It is the motion of this cloud which carries off the kinetic energy when the monopoles stick. Lee, Weinberg & Yi interpret this cloud as the limit of a charge $(2,1)$ monopole in a maximally broken theory, in which the mass of the $(2,1)$ monopole is taken to zero, thereby losing its identity and becoming the cloud.

For the case of charge $(2,1)$ monopoles in the maximally broken $SU(3)$ theory, Chalmers has conjectured an implicit form for the metric. This uses the generalized Legendre transform technique and modifies the same construction of the Atiyah-Hitchin metric.

Nahm data for other $SU(N)$ countdown examples can be obtained by a modification of $SU(2)$ Nahm data. For example, Platonic $SU(N)$ monopoles can be studied from the $SU(2)$ Nahm data discussed in Section 3. Again exotic phenomena are found such as double scatterings and pathological geodesics where the monopoles never separate.

6. S-Duality and Seiberg-Witten Theory

There has been a recent revival of interest in BPS monopoles due to their central role in S-duality and Seiberg-Witten theory. It is beyond the scope of this review to discuss these topics in any detail but a few remarks regarding the application of results in monopole theory will be made.

Montonen-Olive duality is a conjectured weak to strong coupling $\mathbb{Z}_2$ symmetry which exchanges electric and magnetic charge and thus interchanges the fundamental particles and monopoles. It was soon realised that the best chance of this duality existing occurs in an $\mathcal{N} = 4$ supersymmetric Yang-Mills theory. Much later it was observed, following work on string theory and lattice models, that this $\mathbb{Z}_2$ symmetry should be extended to a conjectured $SL(2,\mathbb{Z})$ symmetry, which is now known as S-duality. The real explosion in this subject took place with the work of Sen who showed that a consequence of S-duality is the existence of certain monopole-fermion bound states, which are described by self-dual normalizable harmonic forms on the classical centred monopole moduli space $\mathcal{M}_k^0$. There is thus a testable prediction of S-duality and moreover Sen was able to explicitly present the appropriate self-dual 2-form on the Atiyah-Hitchin manifold thus confirming the prediction in the simplest case of a 2-monopole bound state. As we have seen the metrics on the higher charge ($k > 2$) monopole moduli spaces $\mathcal{M}_k^0$ are not known, so there is no hope of a similar explicit construction of the predicted Sen forms. However, the existence of these forms can be answered by a study of the appropriate cohomology of $\mathcal{M}_k^0$ and the predictions of S-duality have essentially been confirmed in this way by Segal & Selby, making use of Donaldson’s rational map description of the monopole moduli space. An alternative confirmation of the existence of the
Sen forms has been given by Porrati\textsuperscript{93}.

As we have seen in the last Section there are simplifying special cases for higher rank gauge groups where the metric is known explicitly. For $G = SU(N)$ with maximal symmetry breaking the metric on the moduli space of charge $(1,1,...,1)$ monopoles is known. The appropriate Sen forms for the simplest case of $N = 3$ were presented by Lee, Weinberg & Yi\textsuperscript{71} and also Gauntlett & Lowe\textsuperscript{72} and for the case of general $N$ by Gibbons\textsuperscript{94}.

For classical monopoles all configurations of charge $k$ have equal energy, so there are no special configurations from this point of view. However, in the quantum case the Sen form determines a probability density on the classical monopole moduli space which is peaked over certain sets of classical monopole solutions. In the work of Segal & Selby\textsuperscript{92} certain low-dimensional cycles in $\mathcal{M}_0^k$ are also important. We have seen in Section 3 that there are special configurations of monopoles, such as those with Platonic symmetry or extra zeros of the Higgs field and it would be interesting if these were the ones of interest in the above context. By constructing a Morse function on $\mathcal{M}_0^k$ certain distinguished cycles can be found which include these special configurations\textsuperscript{41} but more work in this direction is required.

Turning now to the case of $N = 2$ supersymmetric theories the celebrated work of Seiberg & Witten\textsuperscript{95} allows the computation of the nonperturbative low energy effective action by application of a version of duality, but this time which describes how the theory varies as a function of the expectation value of the Higgs field. The important object is an algebraic curve of genus $N - 1$ for gauge group $SU(N)$\textsuperscript{95,96,97}. A special differential exists such that the spectrum of BPS states is given by integration of this differential over the $2(N - 1)$ one-cycles of the curve. There has been much work, which began with that of Gorski et al\textsuperscript{98}, to connect these curves to those occurring in certain integrable systems. However, there is also a connection with monopoles, as follows, but it may be only a mathematical coincidence. The observation\textsuperscript{59} is that the Seiberg-Witten curves for gauge group $SU(N)$ have the same form as the quotient spectral curves of $SU(2)$ cyclically symmetric charge $N$ monopoles. At the time it seemed very strange that the size of the gauge group was connected to the charge of the monopole but, as we shall discuss briefly below, exactly the same phenomenon was later found for gauge theories in three dimensions and a physical explanation discovered in terms of string theory. It was checked by Chalmers & Hanany\textsuperscript{99} that extending this correspondence to the gauge group $SO(2N)$ by considering the quotient spectral curves of dihedral monopoles also works and it seems likely that it extends to the exceptional groups by considering Platonic monopoles. However, from the point of view of the monopole construction it is not so clear how to incorporate the exceptional algebras. In the case of the affine algebra $A^{(1)}_{k-1}$ there is a simple ansatz\textsuperscript{59} to determine the form of the Nahm data for $SU(2)$ cyclic $k$-monopoles in terms of the generators of the algebra. The problem with the exceptional algebras is that if one attempts to use the same ansatz then the construction will work but the charge of the monopoles will be ridiculously high. It seems plausible that the charges can be reduced to reasonable values by
considering principle $su(2)$ subalgebras but still these charges are higher than the minimal values known to allow Platonic configurations. For these reasons the connection between the Platonic monopoles discussed in Section 3 and the exceptional algebras is still not understood.

Finally, let us discuss the case of the Coulomb branch of $\mathcal{N} = 4$ supersymmetric theories in three dimensions. These are the latest developments in Seiberg-Witten theory but appear to be those in which a direct and interesting connection with results in monopole theory can be made.

The $\mathcal{N} = 4$ supersymmetric theory in three dimensions can be obtained from a dimensional reduction of the $\mathcal{N} = 1$ supersymmetric theory in six dimensions. The theory therefore contains three Higgs fields and, since the Higgs potential is the trace of the commutators of these fields, the vacuum configurations of Higgs fields are where they all lie in the Cartan subalgebra of the gauge group $G$. Consider the case $G = SU(N)$, then the moduli space of the Higgs vacuum is $3(N - 1)$-dimensional. In addition, for maximal symmetry breaking, there are $N - 1$ photons which in three dimensions are dual to $N - 1$ scalars. So in all, the classical moduli space of vacua is the $4(N - 1)$-dimensional space of these massless scalars. The fact that there is $\mathcal{N} = 4$ supersymmetry means that the low energy effective action can be understood in terms of the metric on this manifold, which is hyperkähler. Classically the metric is flat but there are both perturbative and instanton corrections to this. Note that since the theory is in three dimensions then the classical instantons are actually the BPS monopoles themselves.

Using ideas of duality Seiberg & Witten computed the quantum metric in the $SU(2)$ case and found it to be the Atiyah-Hitchin metric i.e. the classical metric on the moduli space $M_0^2$ of $SU(2)$ centred 2-monopoles. The generalization of the quantum metric to gauge group $SU(N)$ was conjectured by Chalmers & Hanany to be the metric on $M_N^0$ i.e. the classical metric on the moduli space of centred $SU(2)$ monopoles of charge $N$. An explanation of this intriguing identification in terms of string theory has been given by Hanany & Witten and consists of considering certain configurations of fivebranes and threebranes in type IIB superstring theory in ten dimensions.

The obvious question for monopole theorists is how these conjectured appearances of multi-monopole moduli spaces can be verified. For the quantum theory in three dimensions the perturbative loop corrections can be computed as can the non-perturbative instanton corrections, at least for low instanton numbers. Indeed for the $SU(2)$ case Dorey et al have computed the perturbative and one-instanton corrections and shown that they agree with the asymptotic form of the Atiyah-Hitchin metric plus the leading order exponential correction. This is enough to verify the $SU(2)$ result. However, recall from the discussion in Section 4 that the multi-monopole moduli space metric is not known on $M_k^0$ for $k > 2$. Thus even if the computations of the quantum metric are performed there is no known result for comparison.

The ball is now in the court of the monopole theorist to provide some results
to which instanton calculations can be compared. Excluding the Atiyah-Hitchin case (and the known Atiyah-Hitchin submanifolds), there is only one case in which the exact monopole metric is known even on any geodesic submanifold of \( \mathcal{M}_k \). This is the metric on the moduli space of tetrahedrally symmetric charge four monopoles. The correspondence between the scalar fields in the quantum gauge theory and the moduli of the monopoles is that the vacuum expectation values of the three Higgs fields gives the relative positions of the monopoles and the scalars dual to the photons give the relative phases. Thus by computation of the perturbative and instanton contributions in the \( SU(4) \) quantum theory where the vacuum expectation values correspond to the vertices of a tetrahedron, there is a known result against which the answer can be checked. If required, the metric on other totally geodesic submanifolds discussed in Section 4, in which the Nahm data is known in terms of elliptic functions, could also be computed.

Seiberg & Witten also discuss the quantum field theory when massive hypermultiplets are included. If a single hypermultiplet is included, whose mass vector is \( q \in \mathbb{R}^3 \), they find that the appropriate 4-dimensional hyperkähler metric is a one-parameter deformation, \( \mathcal{M}(|q|) \), of the Atiyah-Hitchin metric. This metric was discovered by Dancer as the hyperkähler quotient of the manifold \( \mathcal{M}_0 \) (which is the centred moduli space of charge two \( SU(3) \) monopoles as discussed in the last Section) by a \( U(1) \) subgroup of the \( SU(2) \) action. The vector \( q \) is the level set of the moment map and the manifold is a deformation of the Atiyah-Hitchin manifold in the sense that \( \mathcal{M}(0) \) is the double cover of \( \mathcal{M}_0 \).

Recently, Houghton has rederived the manifold \( \mathcal{M}(|q|) \) as another monopole moduli space. It is the moduli space of charge \((2, 1)\) monopoles in a maximally broken \( SU(3) \) theory in which the mass of the \((2, 1)\) monopole is taken to infinity. This infinite mass limit fixes the position of the \((2, 1)\) monopole which is determined by the constant vector \( q \). An advantage of Houghton’s description is that the asymptotic metric can be computed using a point particle approximation. However, it should be possible to go further and compute the exponential corrections to this asymptotic metric using the Nahm data. This would be a useful result since a comparison with, at least, the one-instanton correction in the quantum theory should be possible.

The process of taking massless and infinitely massive limits of monopole moduli spaces has been investigated within the framework of the hyperkähler quotient construction by Gibbons & Rychenkova and other metrics of relevance to quantum gauge theories in three dimensions obtained.

Acknowledgements

Over a few years I have benefited from useful interactions with several monopole experts. In particular I would like to thank Ed Corrigan, Nigel Hitchin, Conor Houghton, Werner Nahm, Richard Ward and especially Nick Manton. I am also grateful to Nick Dorey and Ami Hanany for discussions of their work on duality.
References

1. E.B. Bogomolny, Sov. J. Nucl. Phys. 24, 449 (1976).
2. M.K. Prasad and C.M. Sommerfield, Phys. Rev. Lett. 35, 760 (1975).
3. R.S. Ward, Commun. Math. Phys. 79, 317 (1981).
4. B. Julia and A. Zee, Phys. Rev. D 11, 2727 (1975).
5. M.F. Atiyah and N.J. Hitchin, ‘The geometry and dynamics of magnetic monopoles’, Princeton University Press, 1988.
6. N.S. Manton, Nucl. Phys. B 126, 525 (1977).
7. E.J. Weinberg, Phys. Rev. D 20, 936 (1979).
8. E. Corrigan and P. Goddard, Commun. Math. Phys. 80, 575 (1981).
9. S.K. Donaldson, Commun. Math. Phys. 96, 387 (1984).
10. A. Jaffe and C. Taubes, ‘Vortices and monopoles’, Boston, Birkhäuser, 1980.
11. R.S. Ward, Phys. Lett. A 61, 81 (1977).
12. R.S. Ward and R.O. Wells, ‘Twistor Geometry and Field Theory’, Camnbridge University Press, 1990.
13. M.F. Atiyah and R.S. Ward, Commun. Math. Phys. 55, 111 (1977).
14. N.S. Manton, Nucl. Phys. B 135, 319 (1978).
15. E. Corrigan and D.B. Fairlie, Phys. Lett. 67, 69 (1977).
16. G. t’Hooft, unpublished.
17. F. Wilczek, in ‘Quark confinement and field theory’, ed. D. Stump and D. Wein- garten, John Wiley, New York, 1977.
18. N.J. Hitchin, Commun. Math. Phys. 83, 579 (1982).
19. N.J. Hitchin, Commun. Math. Phys. 89, 145 (1983).
20. M.K. Prasad, Physica D 1, 167 (1980).
21. R.S. Ward, Phys. Lett. B 102, 136 (1981).
22. P. Forgács, Z. Harváth and L. Palla, Phys. Lett. B 99, 232 (1981); 102, 131 (1981); Ann. of Phys. 136, 371 (1981); Nucl. Phys. B 192, 141 (1981); 229, 77 (1983).
23. M.K. Prasad and P. Rossi, Phys. Rev. Lett. 46, 806 (1981).
24. M.K. Prasad, Commun. Math. Phys. 80, 137 (1981).
25. J. Hurtubise, Commun. Math. Phys. 92, 195 (1983).
26. W. Nahm, ‘The construction of all self-dual multimonopoles by the ADHM method’, in Monopoles in quantum field theory, eds. N.S. Craigie, P. Goddard and W. Nahm, World Scientific, 1982.
27. M.F. Atiyah, N.J. Hitchin, V.G. Drinfeld and Yu.I. Manin, Phys. Lett. A 65, 185 (1978).
28. S.A. Brown, H. Panagopoulos and M.K. Prasad, Phys. Rev. D 26, 854 (1982); 28, 380 (1983).
29. J. Hurtubise, Commun. Math. Phys. 100, 463 (1985).
30. N.J. Hitchin, N.S. Manton and M.K. Murray, Nonlinearity 8, 661 (1995).
31. R. Bielawski, Ann. Glob. Anal. Geom. 14, 123 (1995).
32. C.J. Houghton and P.M. Sutcliffe, Nonlinearity 9, 385 (1996).
33. S. Jarvis, ‘A rational map for Euclidean monopoles via radial scattering’, Oxford preprint (1996).
34. E. Braaten, S. Townsend and L. Carson, Phys. Lett. B 235, 147 (1990).
35. F. Klein, ‘Lectures on the icosahedron’, London, Kegan Paul, 1913.
36. C.J. Houghton and P.M. Sutcliffe, Commun. Math. Phys. 180, 343 (1996).
37. C.J. Houghton and P.M. Sutcliffe, Nucl. Phys. B 464, 59 (1996).
38. M.F. Atiyah, ‘Magnetic monopoles in hyperbolic spaces’, in M.F. Atiyah Collected Works, vol. 5, 579.
39. R. Jackiw, C. Nohl and C. Rebbi, Phys. Rev. D 15, 1642 (1977).
40. P.M. Sutcliffe, Phys. Lett. B 376, 103 (1996).
41. C.J. Houghton, N.S. Manton and P.M. Sutcliffe, ‘Rational maps, monopoles and Skyrmions’, hep-th/9705151.
42. N.S. Manton, Phys. Lett. B 110, 54 (1982).
43. T.M. Samols, Commun. Math. Phys. 145, 149 (1992).
44. P.J. Ruback, Nucl. Phys. B 296, 669 (1988).
45. I.A.B. Strachan, J. Math. Phys. 33, 102 (1992).
46. R.S. Ward, Phys. Lett. B 158, 424 (1985).
47. R.A. Leese, Nucl. Phys. B 344, 33 (1990).
48. D. Stuart, Commun. Math. Phys. 166, 149 (1994).
49. N.J. Hitchin, A. Karlhede, U. Lindström and M. Roček, Commun. Math. Phys. 108, 535 (1987).
50. R.A. Leese, M. Peyrard and W.J. Zakrzewski, Nonlinearity 3, 387 (1990).
51. P.M. Sutcliffe, Nonlinearity 4, 1109 (1991).
52. L. Bates and R. Montgomery, Commun. Math. Phys. 118, 635 (1988).
53. C.J. Houghton and P.M. Sutcliffe, Nonlinearity 9, 1609 (1996).
54. R. Bielawski, Nonlinearity 9, 1463 (1996).
55. H.W. Braden and P.M. Sutcliffe, Phys. Lett. B 391, 366 (1997).
56. H. Nakajima, ‘Monopoles and Nahm’s equations ’, in Sanda 1990, Proceedings, Einstein metrics and Yang-Mills connections.
57. P.M. Sutcliffe, Phys. Lett. B 357, 335 (1995).
58. A. Kudryavtsev, B. Piette and W.J. Zakrzewski, Phys. Lett. A 180, 119 (1993).
59. P.M. Sutcliffe, Phys. Lett. B 381, 129 (1996).
60. R.S. Ward, J. Phys. A 20, 2679 (1987).
61. P.M. Sutcliffe, J. Phys. A 29, 5187 (1996).
62. P.M. Sutcliffe, ‘Cyclic Monopoles’, hep-th/9610030, to appear in Nucl. Phys. B.
63. G.W. Gibbons and N.S. Manton, Phys. Lett. B 356, 32 (1995).
64. N.S. Manton, Phys. Lett. B 154, 397 (1985); (E) B 157, 475 (1985).
65. P. Goddard, J. Nuyts and D. Olive, Nucl. Phys. B 125, 1, (1977).
66. E.J. Weinberg, Nucl. Phys. B 167, 500 (1980).
67. R.S. Ward, Commun. Math. Phys. 86, 437 (1982).
68. J. Hurtubise and M.K. Murray, Commun. Math. Phys. 122, 35 (1989).
69. M.K. Murray, Commun. Math. Phys. 125, 661 (1989).
70. S. Connell, ‘The dynamics of the SU(3) (1, 1) magnetic monopole’. PhD Thesis. The Flinders University of South Australia, 1991.
71. K. Lee, E.J. Weinberg and P. Yi, Phys. Lett. B 376, 97 (1996).
72. J.P. Gauntlett and D.A. Lowe, Nucl. Phys. B 472, 194 (1996).
73. K. Lee, E.J. Weinberg and P. Yi, Phys. Rev. D 54, 1633 (1996).
74. M.K. Murray, ‘A note on the (1, 1, ..., 1) monopole metric’, hep-th/9605054.
75. M. Takahasi, PhD Thesis, University of Tokyo.
76. G.W. Gibbons and P. Rychenkova, ‘Hyperkähler quotient construction of BPS monopole moduli spaces’, hep-th/9608083.
77. I.T. Ivanov and M. Roček, Commun. Math. Phys. 182, 291 (1996).
78. G. Chalmers, ‘Multi-monopole moduli spaces for SU(N) gauge groups’, hep-th/9605182.
79. F.A. Bais and D. Wilkinson, Phys. Rev. D 19, 2410 (1979).
80. A.N. Leznov and M.V. Saveliev, Lett. Math. Phys. 3, 489 (1979); Commun. Math. Phys. 74, 111 (1980).
81. N. Ganoulis, P. Goddard and D. Olive, Nucl. Phys. B 205 [FS5] 601 (1982).
82. A.S. Dancer, Commun. Math. Phys. 158, 545-568 (1993).
83. A.S. Dancer, Nonlinearity 5, 1355 (1992).
84. P. Irwin, ‘SU(3) monopoles and their fields’, hep-th/9704155.
85. A.S. Dancer and R.A. Leese, Proc. R. Soc. 440, 421 (1993).
86. A.S. Dancer and R.A. Leese, Phys. Lett. B 390, 252 (1997).
87. K. Lee, E.J. Weinberg and P. Yi, Phys. Rev. D 54, 6351 (1996).
88. C.J. Houghton and P.M. Sutcliffe, ‘SU(N) monopoles and Platonic symmetry’, preprint UKC/IMS/96-70.
89. C. Montonen and D. Olive, Phys. Lett. B 72, 117 (1977).
90. H. Osborn, Phys. Lett. B 83, 321 (1979).
91. A. Sen, Phys. Lett. B 329, 217 (1994).
92. G. Segal and A. Selby, Commun. Math. Phys. 177, 775 (1996).
93. M. Porrati, Phys. Lett. B 377, 67 (1996).
94. G.W. Gibbons, Phys. Lett. B 382, 53 (1996).
95. N. Seiberg and E. Witten, Nucl. Phys. B 426, 19 (1994).
96. A. Klemm, W. Lerche, S. Yankielowicz and S. Theisen, Phys. Lett. B344, 169 (1995).
97. P. Argyres and A. Faraggi, Phys. Rev. Lett. 73, 3931 (1995).
98. A. Gorskii, I. Krichever, A. Marshakov, A. Mironov and A. Morozov, Phys. Lett. B355, 466 (1995).
99. G. Chalmers and A. Hanany, Nucl. Phys. B 489, 223 (1997).
100. N. Seiberg and E. Witten, ‘Gauge dynamics and compactification to three dimensions’, hep-th/9607163.
101. A. Hanany and E. Witten, Nucl. Phys. B 492, 152 (1997).
102. N. Dorey, V.V. Khoze, M.P. Mattis, D. Tong and S. Vandoren, ‘Instantons, three-dimensional gauge theory, and the Atiyah-Hitchin manifold’, hep-th/9703228.
103. A.S. Dancer, Quart. J. Math. 45, 463 (1994).
104. C.J. Houghton, ‘New hyperkahler manifolds by fixing monopoles’, hep-th/9702161.