Explicit formulas for the zeta functions $\zeta_\alpha(s)$ corresponding to bosonic ($\alpha = 2$) and to fermionic ($\alpha = 3$) quantum fields living on a noncommutative, partially toroidal spacetime are derived. Formulas for the most general case of the zeta function associated to a quadratic+linear+constant form (in $Z$) are obtained. They provide the analytical continuation of the zeta functions in question to the whole complex $s$–plane, in terms of series of Bessel functions (of fast, exponential convergence), thus being extended Chowla-Selberg formulas. As well known, this is the most convenient expression that can be found for the analytical continuation of a zeta function, in particular, the residua of the poles and their finite parts are explicitly given there. An important novelty is the fact that simple poles show up at $s = 0$, as well as in other places (simple or double, depending on the number of compactified, noncompactified, and noncommutative dimensions of the spacetime), where they had never appeared before. This poses a challenge to the zeta-function regularization procedure.

Abstract

Explicit formulas for the zeta functions $\zeta_\alpha(s)$ corresponding to bosonic ($\alpha = 2$) and to fermionic ($\alpha = 3$) quantum fields living on a noncommutative, partially toroidal spacetime are derived. Formulas for the most general case of the zeta function associated to a quadratic+linear+constant form (in $Z$) are obtained. They provide the analytical continuation of the zeta functions in question to the whole complex $s$–plane, in terms of series of Bessel functions (of fast, exponential convergence), thus being extended Chowla-Selberg formulas. As well known, this is the most convenient expression that can be found for the analytical continuation of a zeta function, in particular, the residua of the poles and their finite parts are explicitly given there. An important novelty is the fact that simple poles show up at $s = 0$, as well as in other places (simple or double, depending on the number of compactified, noncompactified, and noncommutative dimensions of the spacetime), where they had never appeared before. This poses a challenge to the zeta-function regularization procedure.

1This paper is dedicated to Aleix E. T., on the unique and promising occasion of his eighteenth birthday.

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1 Introduction

For its application in practice, the zeta function regularization method relies on the existence of quite simple formulas that give the analytical continuation of the zeta function, $\zeta(s)$, from the region of the complex plane extending to the right of the abscissa of convergence, $\text{Re } s > s_0$, to the rest of the complex plane [1, 2, 3]. These are not only the reflection formula of the corresponding zeta function in each case, but also some other, very fundamental expressions, as the Jacobi theta function identity, Poisson’s and Plana’s resummation formulas, and the Chowla-Selberg formula. However, some of these powerful expressions are often restricted to specific zeta functions, and their explicit derivation is usually quite involved. For instance, until very recently, the Chowla-Selberg (CS) formula was only known to exist for the homogeneous, two-dimensional Epstein zeta function. Ultimate extensions of it to more general zeta functions in any number of dimensions will be given in Sect. 2 of the present paper.

A fundamental property shared by all zeta functions is the existence of a reflection formula. For the Riemann zeta function it is: $\Gamma(s/2)\zeta(s) = \pi^{s-1/2}\Gamma(1 - s/2)\zeta(1 - s)$. For a generic zeta function, $Z(s)$, we may write it as: $Z(\omega - s) = F(\omega, s)Z(s)$. It allows for its analytic continuation in a very easy way —what is, in simple cases, the whole story of the zeta function regularization procedure. But the analytically continued expression thus obtained is just another series, which has again a slow convergence behavior, of power series type [4] (actually the same that the original series had, on its convergence domain). Some years ago, S. Chowla and A. Selberg found a formula, for the Epstein zeta function in the two-dimensional case [5], that yields \textit{exponentially quick convergence everywhere}, not just in the reflected domain. They were very proud of it. In Ref. [6], a first attempt was done to try to extend this expression to inhomogeneous zeta functions (very important for physical applications, see [7]), but remaining always in \textit{two} dimensions, for this was commonly believed to be an insurmountable restriction of the original formula (see, for instance, Ref. [8]). More recently, extensions to an \textit{arbitrary} number of dimensions [9, 10], both for the homogeneous (quadratic form) and non-homogeneous (quadratic plus affine form) cases were constructed. However, some of the new formulas (remarkably the ones corresponding to the zero-mass case, e.g., the original CS framework!) are \textit{not explicit}, since they involve solving a rather non-trivial recurrence. (By the way, these explains why the CS formula had not been extended to higher-dimensional Epstein zeta functions before.)

In Sect. 2 we shall finish this program, by providing for the first time explicit, Chowla-
Selberg-like extended formulas for all possible cases involving forms of the very general type: quadratic+linear+constant. This will complete the construction initiated in Refs. [9, 10].

In Sect. 3 we will move to specific applications of these formulas in noncommutative field theory. In particular, we will obtain the explicit analytic continuation of the zeta functions corresponding to scalar and vector fields defined on a quite general, partially noncommutative toroidal manifold. Their pole structure will be discussed in detail. The existence of simple poles at $s = 0$ comes as a novelty in the zeta function regularization method in this case, confirming a result obtained in Ref. [11]. In other places, up to double poles will be shown to appear. The corresponding residua and finite parts at the poles are immediately obtained from these expressions.

2 Extended Chowla-Selberg formulas, associated with arbitrary forms of quadratic+linear+constant type

Let $A$ a positive-definite elliptic ΨDO of positive order $m \in \mathbb{R}$, acting on the space of smooth sections of $E$, an $n$-dimensional vector bundle over $M$, a closed $n$-dimensional manifold. The zeta function $\zeta_A$ is defined as

$$\zeta_A(s) = \text{tr} A^{-s} = \sum_j \lambda_j^{-s}, \quad \text{Re } s > \frac{n}{m} \equiv s_0, \quad (1)$$

where $s_0 = \dim M/\text{ord} A$ is called the abscissa of convergence of $\zeta_A(s)$. Under these conditions, it can be proven that $\zeta_A(s)$ has a meromorphic continuation to the whole complex plane $\mathbb{C}$ (regular at $s = 0$), provided that the principal symbol of $A$ (that is $a_m(x, \xi)$) admits a spectral cut: $L_\theta = \{\lambda \in \mathbb{C}; \text{Arg } \lambda = \theta, \theta_1 < \theta < \theta_2\}$, $\text{Spec } A \cap L_\theta = \emptyset$ (Agmon-Nirenberg condition). The definition of $\zeta_A(s)$ depends on the position of the cut $L_\theta$. The only possible singularities of $\zeta_A(s)$ are simple poles at $s_k = (n - k)/m$, $k = 0, 1, 2, \ldots, n-1, n+1, \ldots$. M. Kontsevich and S. Vishik have managed to extend this definition to the case when $m \in \mathbb{C}$ (no spectral cut exists) [12].

Consider now the following zeta function (Re $s > p/2$):

$$\zeta_{A, \vec{c}, q}(s) = \sum_{\vec{n} \in \mathbb{Z}^p} \left[ \frac{1}{2} (\vec{n} + \vec{c})^T A (\vec{n} + \vec{c}) + q \right]^{-s} \equiv \sum_{\vec{n} \in \mathbb{Z}^p} \left[ Q (\vec{n} + \vec{c}) + q \right]^{-s}. \quad (2)$$

The prime on a summation sign means that the point $\vec{n} = \vec{0}$ is to be excluded from the sum. As we shall see, this is irrelevant when $q$ or some component of $\vec{c}$ is non-zero but, on the
contrary, it becomes an inescapable condition in the case when \( c_1 = \cdots = c_p = q = 0 \). Note that, alternatively, we can view the expression inside the square brackets of the zeta function as a sum of a quadratic, a linear, and a constant form, namely, \( Q(\vec{n} + \vec{c}) + q = Q(\vec{n}) + L(\vec{n}) + \vec{q} \).

Our aim is to obtain a formula that gives (the analytic continuation of) this multidimensional zeta function in terms of an exponentially convergent series, and which is valid in the whole complex plane, exhibiting the singularities (poles) of the meromorphic continuation—with the corresponding residua—explicitly. The only condition on the matrix \( A \) is that it correspond to a (non-negative) quadratic form, which we call \( Q \). The vector \( \vec{c} \) is arbitrary, while \( q \) will be (for the moment) a positive constant. As we shall see, the solution to this problem will depend very much (its explicit form) on the fact that \( q \) and/or \( \vec{c} \) are zero or not. According to this, we will have to distinguish different cases, leading to unrelated final formulas, all to be viewed as different non-trivial extensions of the CS formula (they will be named ECS formulas, and will carry additional tags, for the different cases).

Use of the Poisson resummation formula in Eq. (2) yields [9, 10]

\[
\zeta_{A,\vec{c},q}(s) = \left(2\pi\right)^{p/2}q^{p/2-s} \frac{\Gamma(s-p/2)}{\Gamma(s)} + \frac{2^{s/2+p/4+2\pi^2q^{-s/2+p/4}}}{\sqrt{\det A} \Gamma(s)} \times \sum_{\vec{m} \in \mathbb{Z}^{p}_{1/2}} ' \cos(2\pi \vec{m} \cdot \vec{c}) \left(\vec{m}^T A^{-1} \vec{m}\right)^{s/2-p/4} K_{p/2-s} \left(2\pi \sqrt{2q} \vec{m}^T A^{-1} \vec{m}\right),
\]

where \( K_{\nu} \) is the modified Bessel function of the second kind and the subindex \( 1/2 \) in \( \mathbb{Z}^{p}_{1/2} \) means that in this sum, only half of the vectors \( \vec{m} \in \mathbb{Z}^{p} \) enter. That is, if we take an \( \vec{m} \in \mathbb{Z}^{p} \) we must then exclude \( -\vec{m} \) (as a simple criterion one can, for instance, select those vectors in \( \mathbb{Z}^{p}\backslash\{\vec{0}\} \) whose first non-zero component is positive). Eq. (3) fulfills all the requirements of a CS formula. But it is very different from the original one, constituting a non-trivial extension to the case of a quadratic-linear-constant form, in any number of dimensions, with the constant term being non-zero. We shall denote this formula, Eq. (3), by the acronym ECS1.

It is notorious how the only pole of this inhomogeneous Epstein zeta function appears, explicitly, at \( s = p/2 \), where it belongs. Its residue is given by:

\[
\text{Res}_{s=p/2} \zeta_{A,\vec{c},q}(s) = \frac{(2\pi)^{p/2}}{\Gamma(p/2)} \left(\det A\right)^{-1/2}.
\]
2.1 Limit \( q \to 0 \)

After some work, one can obtain the limit of expression (3) as \( q \to 0 \) (for simplicity we also set \( \vec{c} = \vec{0} \))

\[
\zeta_{A,\vec{0},0}(s) = 2^{1+s}a^{-s}\zeta(2s) + \sqrt{\frac{\pi}{a}} \Gamma(s-1/2) \zeta_{\Delta_{p-1},\vec{0},0}(s-1/2) + \frac{4\pi^s}{a^{s/2+1/4} \Gamma(s)} \times \sum_{\vec{n}_2 \in \mathbb{Z}^{p-1}} \sum_{n_1=1}^{\infty} \cos \left( \frac{\pi n_1}{a} \vec{b}^T \vec{n}_2 \right) n_1^{s-1/2} \left( \vec{n}_2^T \Delta_{p-1} \vec{n}_2 \right)^{1/4-s/2} K_{s-1/2} \left( \frac{2\pi n_1}{\sqrt{a}} \sqrt{n_2^T \Delta_{p-1} n_2} \right).
\]

(5)

In Eqs. (3) and (5), \( A \) is a \( p \times p \) symmetric matrix \( A = (a_{ij})_{i,j=1,2,...,p} = A^T \), \( A_{p-1} \) the \((p-1) \times (p-1)\) reduced submatrix \( A_{p-1} = (a_{ij})_{i,j=2,...,p} \), \( a \) the first component, \( a = a_1 \), \( \vec{b} \) the \( p-1 \) vector \( \vec{b} = (a_{21},...,a_{p1})^T = (a_{12},...,a_{1p})^T \), and \( \Delta_{p-1} \) is the following \((p-1) \times (p-1)\) matrix: \( \Delta_{p-1} = A_{p-1} - \frac{1}{4d} \vec{b} \otimes \vec{b} \). More precisely, what one actually obtains by taking the limit is the reflected formula, as one would get after using the Epstein zeta function reflection \( \Gamma(s)Z(s;A) = \pi^{2s-p/2} (\det A)^{-1/2} \Gamma(p/2-s)Z(p/2-s; A^{-1}) \), being \( Z(s;A) \) the Epstein zeta function \([13] \). Finally, it can be written as (5). (It is a rather non-trivial exercise to perform this calculation.) Note that Eq. (3) has all the properties demanded from a CS formula, but it is actually not explicit. It is in fact a recurrence, rather lengthy to solve as it stands. In fact, it can be viewed as the straightforward extension of the original CS formula to higher dimensions. It was the top result of previous work on this subject, for the case \( q = c_1 = \cdots = c_p = 0 \) \([9,10] \).

Using a different strategy, this recurrence will be now solved explicitly, in a much more simple way. Indeed, let us proceed in a complementary way, namely, by doing the inversion provided by the Poisson resummation formula (or the Jacobi identity), with respect to \( p-1 \) of the indices (say, \( j = 2,3,...,p \)). This leaves us with three sums, corresponding to positive, zero, and negative values of the remaining index (\( n_1 \), in this case). The zero value of \( n_1 \) (in correspondence with the rest of the \( n_i \)'s not being all zero) classifies the number of different situations (according to the values of the \( c_i \)'s an \( q \) being all zero or not) into just two cases. (As is immediate, from start all \( c_i \)'s can be taken to be between 0 and 1: \( 0 \leq c_i < 1, i = 1,2,...,p \).) (i) The first case is, thus, when at least one of the \( c_i \)'s or \( q \geq 0 \) is not zero. Since the case \( q \neq 0 \) has been solved already, we will mean by this case now that, say \( c_1 \neq 0 \). (ii) The second case is when all \( q = c_1 = \cdots = c_p = 0 \).

2.2 Case with \( q = 0 \) but \( c_1 \neq 0 \)
2.2.1 General (non-diagonal) subcase

By doing the inversion provided by the Poisson resummation formula (or the Jacobi identity), with respect to \( p - 1 \) of the indices (here, \( j = 2, 3, \ldots, p \)), we readily obtain:

\[
\zeta_{A_p, c, 0}(s) = \frac{2^n}{\Gamma(s)} \left( \det A_{p-1} \right)^{-1/2} \left\{ \pi^{(p-1)/2} \left( a_{11} - \bar{a}_{p-1}^T A_{p-1}^{-1} \bar{a}_{p-1} \right)^{(p-1)/2-s} \Gamma (s - (p - 1)/2) \times \left[ \zeta_H(2s - p + 1, c_1) + \zeta_H(2s - p + 1, 1 - c_1) \right] + 4\pi^s \left( a_{11} - \bar{a}_{p-1}^T A_{p-1}^{-1} \bar{a}_{p-1} \right)^{(p-1)/4-s/2} \times \sum_{n_1 \in \mathbb{Z}} \sum_{\bar{m} \in \mathbb{Z}^{(p-1)}} \left( 2\pi n_1 + c_1 \right) \left( 2\pi \bar{m}^T \bar{c}_{p-1} + A_{p-1}^{-1} \bar{a}_{p-1} (n_1 + c_1) \right) \right\} \right|_{n_1 + c_1}^{(p-1)/2-s} \left( \bar{m}^T A_{p-1}^{-1} \bar{m} \right)^{s/2-(p-1)/4} \times K_{(p-1)/2-s} \left( 2\pi |n_1 + c_1| \sqrt{\left( a_{11} - \bar{a}_{p-1}^T A_{p-1}^{-1} \bar{a}_{p-1} \right) \bar{m}^T A_{p-1}^{-1} \bar{m}} \right) - \left( \frac{1}{2} \bar{c}^T A \bar{c} \right)^{-s}. \tag{6}
\]

Here, and in what follows, \( A_{p-1} \) is (as before) the submatrix of \( A_p \) composed of the last \( p - 1 \) rows and columns. Moreover, \( a_{11} \) is the first diagonal component of \( A_p \), while \( \bar{a}_{p-1} = (a_{12}, \ldots, a_{1p})^T = (a_{21}, \ldots, a_{p1})^T \), and \( \bar{m} = (n_2, \ldots, n_p)^T \). Note that this is an explicit formula, that the only pole at \( s = p/2 \) appears also explicitly, and that the second term of the rhs is a series of exponentially fast convergence. It has, therefore (as Eq. (3)), all the properties required to qualify as a CS formula. We shall name this expression ECS2.

2.2.2 Diagonal subcase

In this very common, particular case the preceding expression reduces to the more simple form:

\[
\zeta_{A_p, c, 0}(s) = \frac{2^n}{\Gamma(s)} \left( \det A_{p-1} \right)^{-1/2} \left\{ \pi^{(p-1)/2} a_{11}^{(p-1)/2-s} \Gamma (s - (p - 1)/2) \times \left[ \zeta_H(2s - p + 1, c_1) + \zeta_H(2s - p + 1, 1 - c_1) \right] + 4\pi^s a_{11}^{(p-1)/4-s/2} \sum_{n_1 \in \mathbb{Z}} \sum_{\bar{m} \in \mathbb{Z}^{(p-1)}} \left( 2\pi n_1 + c_1 \right) \left( 2\pi \bar{m}^T \bar{c}_{p-1} \right) \left| n_1 + c_1 \right|^{(p-1)/2-s} \left( \bar{m}^T A_{p-1}^{-1} \bar{m} \right)^{s/2-(p-1)/4} \times K_{(p-1)/2-s} \left( 2\pi |n_1 + c_1| \sqrt{a_{11} \bar{m}^T A_{p-1}^{-1} \bar{m}} \right) - \left( \frac{1}{2} \bar{c}^T A \bar{c} \right)^{-s}. \tag{7}
\]

We shall call this formula ECS2d.

2.3 Case with \( c_1 = \cdots = c_p = q = 0 \)

2.3.1 General (non-diagonal) subcase

As remarked in [8, 10], we had not been able to obtain here yet a closed formula, but just a (rather non-trivial) recurrence, Eq. (6), relating the \( p \)-dimensional case with the
(p - 1)-dimensional one. After a second look, we have now realized that we can actually still proceed as if we had in fact c₁ = 1 ≠ 0, both for positive and for negative values of n₁. A sum, though, remains with n₁ = 0 —and the rest of the nᵢ’s not all being zero— what yields, once more, the same zeta function of the beginning, but corresponding to p - 1 indices. All in all:

\[
\zeta_{A_p, \tilde{a}, 0}(s) = \zeta_{A_{p-1}, \tilde{a}, 0}(s) + \frac{2^{1+s}}{\Gamma(s)} \left( \det A_{p-1} \right)^{-1/2} \left\{ \pi^{(p-1)/2} \left( a_{11} - \tilde{a}_{p-1}^T A_{p-1}^{-1} \tilde{a}_{p-1} \right)^{(p-1)/2-s} \times \Gamma \left( s - \frac{(p-1)}{2} \right) \zeta_R(2s - p + 1) \right\} + 4\pi^s \left( a_{11} - \tilde{a}_{p-1}^T A_{p-1}^{-1} \tilde{a}_{p-1} \right)^{(p-1)/4-s/2} \sum_{n=1}^{\infty} \sum_{\tilde{m} \in \mathbb{Z}_{p-1}^2} \cos \left( 2\pi \tilde{m}^T A_{p-1}^{-1} \tilde{a}_{p-1} n \right) n^{(p-1)/2-s} \times \left( \tilde{m}^T A_{p-1}^{-1} \tilde{m} \right)^{s/2-(p-1)/4} K_{(p-1)/2-s} \left[ 2\pi n \sqrt{\left( a_{11} - \tilde{a}_{p-1}^T A_{p-1}^{-1} \tilde{a}_{p-1} \right) \tilde{m}^T A_{p-1}^{-1} \tilde{m} } \right].
\]

This is also a recurrent expression, an alternative to [3], obtained with the help of a different strategy.

Remarkably enough, it is easy to resolve this recurrence explicitly, and indeed to obtain a closed formula for this case (we shall write the dimensions of the submatrices of A as subindices). The result being

\[
\zeta_{A_p}(s) = \zeta_{A_{p-j}, \tilde{a}_{p-j}, 0}(s) = \frac{2^{1+s}}{\Gamma(s)} \left( \det A_{p-j} \right)^{-1/2} \left\{ \pi^{(p-j)/2} \left( a_{jj} - \tilde{a}_{p-j}^T A_{p-j}^{-1} \tilde{a}_{p-j} \right)^{(p-j)/2-s} \times \Gamma \left( s - \frac{(p-j)}{2} \right) \zeta_R(2s - p + j) \right\} + 4\pi^s \left( a_{jj} - \tilde{a}_{p-j}^T A_{p-j}^{-1} \tilde{a}_{p-j} \right)^{(p-j)/4-s/2} \sum_{n=1}^{\infty} \sum_{\tilde{m} \in \mathbb{Z}_{p-j}^2} \cos \left( 2\pi \tilde{m}^T A_{p-j}^{-1} \tilde{a}_{p-j} n \right) n^{(p-j)/2-s} \times \left( \tilde{m}^T A_{p-j}^{-1} \tilde{m} \right)^{s/2-(p-j)/4} K_{(p-j)/2-s} \left[ 2\pi n \sqrt{\left( a_{jj} - \tilde{a}_{p-j}^T A_{p-j}^{-1} \tilde{a}_{p-j} \right) \tilde{m}^T A_{p-j}^{-1} \tilde{m} } \right].
\]

With a similar notation as above, here Aᵢ₋ₗ is the submatrix of Aᵢ composed of the last p - j rows and columns. Moreover, aᵢᵢ is the j-th diagonal component of Aᵢ, while \( \tilde{a}_{p-j} = (a_{jj+1}, \ldots, a_{jp})^T = (a_{j+1}, \ldots, a_{pj})^T \), and \( \tilde{m}_{p-j} = (n_{j+1}, \ldots, n_p)^T \). Again, this is an extension of the Chowla-Selberg formula to the case in question. It exhibits all the same good properties. Physically, it corresponds to the homogeneous, massless case. It is to be viewed, in fact, as the genuine multidimensional extension of the Chowla-Selberg formula. We shall call it ECS3.
2.3.2 Diagonal subcase

Let us particularize once more to the diagonal case, with $\vec{c} = \vec{0}$, which is quite important in practice and gives rise to more simple expressions. For the recurrence, we have

$$\zeta_{A_p}(s) = \zeta_{A_{p-1}}(s) + \frac{2^{1+s}}{\Gamma(s)} (\det A_{p-1})^{-1/2} [\pi^{(p-1)/2}a_1^{(p-1)/2-s} \Gamma (s - (p - 1)/2) \zeta_R(2s - p + 1)]$$

$$+ 4\pi^s a_1^{(p-1)/4 - s/2} \sum_{n=1}^{\infty} \sum_{\vec{m} \in \mathbb{Z}^p} t^{(p-1)/2-s} \left( \vec{m}^T A_{p-1}^{-1} \vec{m} \right)^{s/2 - (p-1)/4} K_{(p-1)/2-s} \left( 2\pi n \sqrt{a_1 \vec{m}^T A_{p-1}^{-1} \vec{m}} \right).$$

As above, we can solve this finite recurrence and obtain the following simple and explicit formula for this case:

$$\zeta_{A_p}(s) = \frac{2^{1+s}}{\Gamma(s)} \sum_{j=0}^{p-1} (\det A_j)^{-1/2} [\pi^{j/2}a_j^{2-s} \Gamma (s - j/2) \zeta_R(2s - j)]$$

$$+ 4\pi^s a_j^{j/4 - s/2} \sum_{n=0}^{\infty} \sum_{\vec{m} \in \mathbb{Z}^j} t^{j/2-s} \left( \vec{m}^T A_j^{-1} \vec{m} \right)^{s/2 - j/4} K_{j/2-s} \left( 2\pi n \sqrt{a_j \vec{m}^T A_j^{-1} \vec{m}} \right),$$

with $A_p = \text{diag}(a_1, \ldots, a_p)$, $A_j = \text{diag}(a_{p-j+1}, \ldots, a_p)$, $\vec{m}_j = (n_{p-j+1}, \ldots, n_p)^T$, and $\zeta_R$ the Riemann zeta function. Note again the fact that this and Eq. (10) are explicit expression for the multidimensional, generalized Chowla-Selberg formula and, in this way, they go beyond any result obtained previously. We name this formula ECS3d.

It is immediate to see that the term for $j = 0$ in the sum yields the last term, $\zeta_{A_1}(s)$, of the recurrence, that is:

$$\zeta_{A_1}(s) = \sum_{n_p = -\infty}^{+\infty} \left( \frac{a_p}{2} n_p^2 \right)^{-s} = 2^{1+s} a_p^{-s}\zeta_R(2s).$$

It exhibits a pole, at $s = 1/2$ which is spurious—it is actually not a pole of the whole function (since it cancels, in fact, with another one coming from the next term, with further cancellations of this kind going on, each term with the next). Concerning the pole structure of the resulting zeta function, as given by Eq. (11), it is not difficult to see that only the pole at $s = p/2$ is actually there (as it should). It is in the last term, $j = p - 1$, of the sum, and it has the correct residue, namely

$$\text{Res } \zeta_{A_p}(s) \big|_{s=p/2} = \frac{(2\pi)^{p/2}}{\Gamma(p/2)} (\det A_p)^{-1/2}.$$  

The rest of the seem-to-be poles at $s = (p - j)/2$ are not such: they compensate among themselves, one term of the sum with the next, adding pairwise to zero.
Summing up, this formula, Eq. (11), provides a convenient analytic continuation of the zeta function to the whole complex plane, with its only simple pole showing up explicitly. Aside from this, the finite part of the first sum in the expression is quite easy to obtain, and the remainder—an awfully looking multiple series—is in fact an extremely-fast convergent sum, yielding a small contribution, as happens in the CS formula. In fact, since it corresponds to the case \( q = 0 \), this expression should be viewed as the extension of the original Chowla-Selberg formula—for the zeta function associated with an homogeneous quadratic form in two dimensions—to an arbitrary number, \( p \), of dimensions. The rest of the formulas above provide also extensions of the original CS expression.

The general case of a quadratic+linear+constant form has been here thus completed. As we clearly see, the main formulas corresponding to the three different subcases, namely ECS1 Eq. (3), ECS2 Eq. (6), and ECS3 Eq. (9), are in fact quite distinct and one cannot directly go from one to another by adjusting some parameters.

For the sake of completeness, we must mention the following. Notice that all cases considered here correspond to having a non-identically-zero quadratic form \( Q \). For \( Q \) identically zero, that is, the linear+constant (or affine) case, the formulas for the analytic continuation are again quite different from the ones above. The corresponding zeta function is called Barnes’ zeta function. This case has been throughly studied in Ref. [10].

### 3 Spectral zeta function for both scalar and vector fields on a spacetime with a noncommutative toroidal part

We shall now consider the physical example of a quantum system consisting of scalars and vector fields on a \( D \)—dimensional noncommutative manifold, \( M \), of the form \( \mathbb{R}^{1,d} \otimes T_p^\theta \) (thus \( D = d + p + 1 \)). \( T_p^\theta \) is a \( p \)—dimensional noncommutative torus, its coordinates satisfying the usual relation: \( [x_j, x_k] = i\theta \sigma_{jk} \). Here \( \sigma_{jk} \) is a real nonsingular, antisymmetric matrix of \( \pm 1 \) entries, and \( \theta \) is the noncommutative parameter.

This physical system has attracted much interest recently, in connection with \( M \)—theory and with string theory [14, 13, 16, 17, 18, 19, 20], and also because of the fact that those are perfectly consistent theories by themselves, which could lead to brand new physical situations. It has been shown, in particular, that noncommutative gauge theories describe
the low energy excitations of open strings on $D-$branes in a background Neveu-Schwarz two-form field [14, 15, 16].

This interesting system provides us with a quite non-trivial case where the formulas derived above are indeed useful. For one, the zeta functions corresponding to bosonic and fermionic operators in this system are of a different kind, never considered before. And, moreover, they can be most conveniently written in terms of the zeta functions in the previous section. What is also nice is the fact that a unified treatment (with just one zeta function) can be given for both cases, the nature of the field appearing there as a simple parameter, together with those corresponding to the numbers of compactified, noncompactified, and noncommutative dimensions of the spacetime.

### 3.1 Poles of the zeta function

The spectral zeta function for the corresponding (pseudo)differential operator can be written in the form [11]

$$
\zeta_\alpha(s) = \frac{V}{(4\pi)^{(d+1)/2}} \frac{\Gamma(s-(d+1)/2)}{\Gamma(s)} \sum_{\vec{n} \in \mathbb{Z}^p} \left[ Q(\vec{n}) \left( \frac{d+1}{2} \right)^2 \left[ 1 + \Lambda \theta^{2-2\alpha} Q(\vec{n})^{-\alpha} \right] \left( \frac{d+1}{2} - s \right) \right],
$$

where $V = \text{Vol}(\mathbb{R}^{d+1})$, the volume of the non-compact part, and $Q(\vec{n}) = \sum_{j=1}^p a_j n_j^2$, a diagonal quadratic form, being the compactification radii $R_j = a_j^{-1/2}$. Moreover, the value of the parameter $\alpha = 2$ for scalar fields and $\alpha = 3$ for vectors, distinguishes between the two different fields. In the particular case when we set all the compactification radii equal to $R$, we obtain:

$$
\zeta_\alpha(s) = \frac{V}{(4\pi)^{(d+1)/2}} \frac{\Gamma(s-(d+1)/2)}{\Gamma(s) R^{d+1-2s}} \sum_{\vec{n} \in \mathbb{Z}^p} \left[ I(\vec{n}) \left( \frac{d+1}{2} - s \right) \left[ 1 + \Lambda \theta^{2-2\alpha} R^{2\alpha} I(\vec{n})^{-\alpha} \right] \left( \frac{d+1}{2} - s \right) \right],
$$

being now the quadratic form: $I(\vec{n}) = \sum_{j=1}^p n_j^2$.

After some calculations, this zeta function can be written in terms of the Epstein zeta function of the previous section, with the result:

$$
\zeta_\alpha(s) = \frac{V}{(4\pi)^{(d+1)/2}} \sum_{l=0}^{\infty} \frac{\Gamma(s+l-(d+1)/2)}{l! \Gamma(s)} (-\Lambda \theta^{2-2\alpha})^l \zeta_{Q_i,0}(s+\alpha l -(d+1)/2),
$$

which reduces, in the particular case of equal radii, to

$$
\zeta_\alpha(s) = \frac{V}{(4\pi)^{(d+1)/2} R^{d+1-2s}} \sum_{l=0}^{\infty} \frac{\Gamma(s+l-(d+1)/2)}{l! \Gamma(s)} (-\Lambda \theta^{2-2\alpha})^l \zeta_E(s+\alpha l -(d+1)/2),
$$

10
where we use here the notation $\zeta_E(s) \equiv \zeta_{l,\tilde{\alpha},0}(s)$, e.g., the Epstein zeta function for the standard quadratic form.

The pole structure of the resulting zeta function deserves a careful analysis. It differs, in fact, very much from all cases that were known in the literature till now. This is not difficult to understand, from the fact that the pole of the Epstein zeta function at $s = p/2 - \alpha k + (d + 1)/2 = D/2 - \alpha k$, when combined with the poles of the gamma functions, yields a very rich pattern of singularities for $\zeta_\alpha(s)$, on taking into account the different possible values of the parameters involved. The pole structure is straightforwardly found from the explicit expressions for the zeta functions in Sect. 2.

Having already given the formula (16) above —that contains everything needed to perform such calculation of pole position, residua and finite part— for its importance for the calculation of the determinant and the one-loop effective action from the zeta function, we will here start by specifying what happens at $s = 0$. Remarkably enough, a pole appears in many cases (depending on the values of the different parameters). This will also serve as an illustration of what one has to expect for other values of $s$. The general case will be left for the following subsection.

It is convenient to classify the different possible subcases according to the values of $d$ and $D = d + p + 1$. We obtain, at $s = 0$, the pole structure given in Table 1.

| $d = 2k$ | if $D \neq \frac{2\alpha}{\alpha}$ | $\Rightarrow \zeta_\alpha(0) = 0$, |
| | if $D = \frac{2\alpha}{\alpha}$ | $\Rightarrow \zeta_\alpha(0) = \text{finite}$. |

For $d = 2k - 1$

| if $D \neq \frac{2\alpha}{\alpha}$ | $\left\{ \begin{array}{l} \text{finite, for } l \leq k \\ 0, \text{ for } l > k \end{array} \right\} \Rightarrow \zeta_\alpha(0) = \text{finite},$ |
| if $D = 2\alpha l$ | $\left\{ \begin{array}{l} \text{pole, for } l \leq k \\ \text{finite, for } l > k \end{array} \right\} \Rightarrow \zeta_\alpha(0) = \text{pole}$. |

Table 1: Pole structure of the zeta function $\zeta_\alpha(s)$, at $s = 0$, according to the different possible values of $d$ and $D$ ($\frac{2\alpha}{\alpha}$ means multiple of $2\alpha$.)

Here $l$ is the summation index in Eq. (16). The appearance of a pole of the zeta function $\zeta_\alpha(s)$, for both values of $\alpha$, at $s = 0$ is, let us repeat, an absolute novelty, bound to have
important physical consequences for the regularization process. It is necessary to observe, that this fact is *not* in contradiction with the well known theorems on the pole structure of a (elliptic) differential operator \[21\]. The situation that appears in the noncommutative case is completely different. (i) To begin with, we do not have any longer a standard differential operator, but a strictly *pseudodifferential* one, from the beginning. (ii) Moreover, the new spectrum is *not* perturbatively connected (for \(\theta \to 0\)) with the corresponding one for the commutative case.

### 3.2 Explicit analytic continuation of \(\zeta_\alpha(s)\), \(\alpha = 2, 3\), in the complex \(s\)–plane

Substituting the corresponding formula, from the preceding section, for the Epstein zeta functions in Eq. (14), we obtain the following explicit analytic continuation of \(\zeta_\alpha(s)\) (\(\alpha = 2, 3\), for bosonic and fermionic fields, to the \textit{whole} complex \(s\)–plane:

\[
\zeta_\alpha(s) = \frac{2^{s-d} V}{(2\pi)^{(d+1)/2}} \Gamma(s) \sum_{l=0}^{\infty} \frac{\Gamma(s + l - (d + 1)/2)}{l! \Gamma(s + \alpha l - (d + 1)/2)} (-2^\alpha \Lambda \theta^{2-2\alpha}) ^{p-1} \sum_{j=0}^{(d+1)/2} (\det A_j)^{-1/2} \\
\times \left[ \pi^{j/2} a_{p-j}^{-s-\alpha l+(d+j+1)/2} \Gamma(s + \alpha l - (d + j + 1)/2) \zeta_R(2s + 2\alpha l - d - j - 1) \right. \\
+ 4 \pi^{s+\alpha l-(d+1)/2} a_{p-j}^{-(s+\alpha l)/2-(d+j+1)/2} \sum_{n=1}^{\infty} \sum_{\vec{m}_j \in \mathbb{Z}^n} n^{(d+j+1)/2-s-\alpha l} \\
\left. \times \left( \vec{m}_j A^{-1} \vec{m}_j \right)^{(s+\alpha l)/2-(d+j+1)/4} K_{(d+j+1)/2-s-\alpha l} \right] 
\]

As discussed in the previous subsection in detail, the non-spurious poles of this zeta function are to be found in the terms corresponding to \(j = p - 1\). With the knowledge we have gained from the analytical continuation of the Epstein zeta functions in Sect. 2, the final analysis can be here completed at once. Note that the situation here corresponds to the case of Subsect. 2.3.2, namely, the diagonal case with \(c_1 = \cdots = c_p = q = 0\).

To be remarked again is that, what we have in the end, by using our method, is an exponentially fast convergent series of Bessel functions together with a first, finite part, where a pole (simple or double, as we shall see) may show up, for specific values of the dimensions of the different parts of the manifold, depending also on the nature (scalar vs. vectorial) of the fields (the value of \(\alpha\), see Table 1 and Eq. (18)).

To summarize the discussion at the end of Sect. 2, the pole structure of Eq. (18) is in fact best seen from Eq. (19) (for \(s = 0\) it has been analyzed in Table 1 already). For a
fixed value of the summation index $l$, the contribution to the only pole of the zeta function $\zeta_E(s + \alpha l - (d + 1)/2)$, at $s = D/2 - \alpha l$, comes from the last term of the $j-$sum only, namely from $j = p - 1$. It is easy to check that it yields the corresponding residuum (13). This corresponds to the second sum in Eq. (18). Combined now with the poles of the gamma functions, and taking into account the first series in $l$, this yields the following expression for the residua of the zeta function $\zeta_\alpha(s)$ at the poles $s = D/2 - \alpha l$, $l = 0, 1, 2, \ldots$

\[
\text{Res } \zeta_\alpha(s) \big|_{s=D/2-\alpha l} = \frac{2^{p/2-d} \pi^{(p-d-1)/2} V}{\Gamma(p/2)} (\det A_p)^{-1/2} \frac{(-\Lambda \theta^{2-2\alpha})^l}{l!} \frac{\Gamma(p/2 + (1 - \alpha)l)}{\Gamma(D/2 - \alpha l)},
\]

(19)

Actually, depending on $D$ and $p$ being even or odd, completely different situations arise, for different values of $l$: from the disappearance of the pole, giving rise to a finite contribution, to the appearance of a simple or a double pole. We shall distinguish four different situations and, to simplify the notation, we will denote by $U$ the whole factor in the expression (19) for the residuum, that multiplies the last fraction of two gamma functions (in short, $\text{Res } \zeta_\alpha = U \Gamma_1/\Gamma_2$).

1. For $D - 2\alpha l = -2h$, $h = 0, -1, -2, \ldots$

(a) for $p/2 + (1 - \alpha)l \neq 0, -1, -2, \ldots \implies$ finite, 

$\text{Res } \zeta_\alpha = -h! \ U \ \Gamma(p/2 + (1 - \alpha)l);$ 

(b) for $p = 2(\alpha - 1)l - 2k$, $k = 0, -1, -2, \ldots \implies$ pole, 

$\text{Res } \zeta_\alpha = (h!/k!) \ U.$

2. For $D - 2\alpha l \neq -2h$, $h = 0, -1, -2, \ldots$

(a) for $p/2 + (1 - \alpha)l \neq 0, -1, -2, \ldots \implies$ pole, 

$\text{Res } \zeta_\alpha = U \ \Gamma(p/2 + (1 - \alpha)l)/\Gamma(D/2 + \alpha l);$ 

(b) for $p = 2(\alpha - 1)l - 2k$, $k = 0, -1, -2, \ldots \implies$ double pole, 

$\text{Res } \zeta_\alpha = (-1/k!) \ U /\Gamma(D/2 + \alpha l).$

Note that we here just quote the generic situation that occurs for $l$ large enough in each case. For instance, if $p = 2$ a double pole appears for $l = 1, 2, \ldots$. For $p = 4$, a double pole appears for $l = 1, 2, \ldots$, if $\alpha = 3$, but only for $l = 2, 3, \ldots$, if $\alpha = 2$. For $p = 6$, a double pole appears for $l = 2, 3, \ldots$, if $\alpha = 3$, but only for $l = 3, 4, \ldots$, if $\alpha = 2$, and so on. The case with both $D$ and $p$ even (what implies $d$ odd) is the most involved one. For $p = 2$ and $D = 4$, for
instance, there is a transition from a pole for \( l = 0 \) corresponding to the zeta function factor, to a pole for \( l = 1 \) and higher, corresponding to the gamma function in the numerator (the compensation of the pole of the zeta function factor with the one coming from the gamma function in the denominator prevents the formation of a double pole). In any case, the explicit analytic continuation of \( \zeta_\alpha(s) \) given by Eq. [18] contains all the information one needs for calculating the poles and corresponding residua in a straightforward way.

The pole structure can be summarized as in Table 2.

| p \ D | even | odd |
|------|------|-----|
| odd  | (1a) pole / finite \((l \geq l_1)\) | (2a) pole / pole |
| even | (1b) double pole / pole \((l \geq l_1, l_2)\) | (2b) pole / double pole \((l \geq l_2)\) |

Table 2: General pole structure of the zeta function \( \zeta_\alpha(s) \), according to the different possible values of \( D \) and \( p \) being odd or even. In italics, the type of behavior corresponding to lower values of \( l \) is quoted, while the behavior shown in roman characters corresponds to larger values of \( l \).

An application of these formulas to the calculation of the one-loop partition function corresponding to quantum fields at finite temperature, on a noncommutative flat spacetime, will be given elsewhere [22].

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