Topography of real toric surfaces

Sam Payne

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Abstract

We determine the homeomorphism type of the set of real points of a smooth projective toric surface.¹

The purpose of this note is to prove the following topological classification of real toric surfaces.

Theorem [De1, Proposition 4.1.2.] Let \( X = X(\Delta) \) be a smooth projective toric surface. If \( X \) is isomorphic to an even Hirzebruch surface \( F_{2a} \), for some integer \( a \geq 0 \), then \( X(\mathbb{R}) \) is homeomorphic to \( S^1 \times S^1 \). Otherwise, \( X(\mathbb{R}) \) is homeomorphic to a connect sum of \#\( \Delta(1) \) copies of \( \mathbb{RP}^2 \).

Here, \( \Delta(1) \) denotes the set of 1-dimensional cones in \( \Delta \).

We begin with a few preliminaries on the topology of real projective toric varieties in general, roughly following [GKZ, Chapter 11, Section 5]. Let \( X = X(\Delta) \) be a projective toric variety, and let \( P \subset M_{\mathbb{R}} \) be the polytope corresponding to an ample toric divisor on \( X \). For each group homomorphism \( \epsilon : M \to \{\pm 1\} \), define

\[
T_\epsilon := \{ t \in T(\mathbb{R}) \subset X(\mathbb{R}) : \text{sgn}(\chi^u(t)) = \epsilon(u) \text{ for all } u \in M \}.
\]

Let \( X_\epsilon \subset X(\mathbb{R}) \) be the closure of \( T_\epsilon \) in the real analytic topology. Let \( \mu : X(\mathbb{R}) \to P \)

be the moment map, and \( \mu_\epsilon \) the restriction of \( \mu \) to \( X_\epsilon \). We claim that \( \mu_\epsilon \) is a homeomorphism. The case \( \epsilon_0(M) = +1 \) is proved in [Ful, Section 4.2]. For general \( \epsilon \), the semigroup homomorphism \( \epsilon : M \to \mathbb{R} \) corresponds to a point \( t_\epsilon \in T(\mathbb{R}) \). Translation by \( t_\epsilon \) takes \( X_{\epsilon_0} \) to \( X_\epsilon \) and commutes with the moment map, so the claim follows.

Now \( X(\mathbb{R}) = \bigcup \epsilon X_\epsilon \). The following proposition describes \( X_\epsilon \cap X_{\epsilon'} \) and the induced construction of \( X(\mathbb{R}) \) by gluing \( 2^{\dim X} \) copies \( P_\epsilon \) of \( P \), indexed by the group homomorphisms \( \epsilon : M \to \{\pm 1\} \). For a face \( F \subset P \), let \( F_\epsilon \) denote the

¹When this note was prepared and submitted to the arXiv, the author was not aware that these results (and much more) had already appeared in C. Delaunay’s work on real toric varieties [De1, De2]. We hope that this note may serve as an expository introduction to some of the ideas and techniques in Delaunay’s work.
corresponding face of $P$, and let $M_F$ be the subgroup of $M$ parallel to $F$, i.e. if $\tau \in \Delta$ is the cone corresponding to $F$, then $M_F = \tau^\perp \cap M$.

**Proposition 1** [Dei, Proposition 4.1.1.] Let $X = X_F$ be a projective toric variety, and let $\epsilon, \epsilon' : M \rightarrow \{\pm 1\}$ be group homomorphisms. Then $X_\epsilon \cap X_{\epsilon'}$ is the union of the preimages under $\mu_e$ of the faces $F \subset P$ such that $\epsilon|_{M_F} = \epsilon'|_{M_F}$. In particular, $X(\mathbb{R})$ is homeomorphic to the space constructed by gluing the $P_e$ along faces as follows: $F_e$ and $F_{e'}$ are identified if and only if $\epsilon|_{M_F} = \epsilon'|_{M_F}$.

**Proof:** Let $F$ be a face of $P$, and let $F^\circ$ be the relative interior of $F$. Let $x \in \mu_e^{-1}(F^\circ)$. It will suffice to show that $x \in X_\epsilon$ if and only if $\epsilon|_{M_F} = \epsilon'|_{M_F}$. The rational functions $\{\chi_u : u \in M_F\}$ are regular on $\mu_e^{-1}(F^\circ)$ and separate points. For $u \in M_F$, the absolute value of $\chi_u(x)$ is determined by $\mu(x)$ and the sign of $\chi_u(x)$ is $\epsilon(u)$. Hence $x \in X_\epsilon$ if and only if $\epsilon'(u) = \epsilon(u)$ for all $u \in M_F$.  

Proposition 1 is a correction of Theorem 5.4 from [GKZ, Chapter 11], which says that $F_\epsilon$ and $F_{\epsilon'}$ are identified if and only if $\epsilon$ and $\epsilon'$ agree on the intersection of $M$ with the affine span of $F$. If $\epsilon$ and $\epsilon'$ agree on $M \cap \text{Aff}(F)$, then they also agree on $M_F$, since every point in $M_F$ is a difference of points in $M \cap \text{Aff}(F)$, but the converse is false in general.

The gluing construction does not depend on the choice of ample divisor. Indeed, $P$ can be replaced by the unit ball in $N_a$ with a cell structure on the boundary sphere dual to that induced by intersection with the cones of $\Delta$. If $F$ is the cell corresponding to a cone $\tau \in \Delta$, then $F_\epsilon$ and $F_{\epsilon'}$ are identified if and only if $\epsilon$ and $\epsilon'$ agree on $\tau^\perp \cap M$.

**Example** Suppose $X \cong \mathbb{F}_a$ is a Hirzebruch surface, where $a$ is a nonnegative integer. Choose coordinates on $N$ so that the rays of $\Delta$ are generated by $e_1, e_2, -e_2$, and $-e_1 + ae_2$. To visualize the construction of $X(\mathbb{R})$ by the gluing recipe in Proposition 1, draw four copies $P_e$ of the polytope $P$ corresponding to some ample divisor on $X$, and label the interior of $P_e$ with an ordered pair of plus or minus signs for $\epsilon(e_1^\perp), \epsilon(e_2^\perp)$. Similarly, label each edge $F_e$ of $P_e$ with a plus or minus sign for $\epsilon(u_F)$, where $u_F$ is a primitive generator of the 1-dimensional lattice $M_F$. The resulting four labeled copies of $P$ are as follows:

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+ | - , +  { + a even
- | -          - a odd
+ | + , +

- | - , -  { - a even
-          + a odd
- | + , -
                  +
```

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The upper left copy of $P$, where $\epsilon(e_1^*) = -1$ and $\epsilon(e_2^*) = +1$. The right (diagonal) edge of $P$ is marked with $+$ if $a$ is even and with $-$ if $a$ is odd because it is parallel to the subgroup generated by $ae_1^* + e_2^*$, and
\[ \epsilon(ae_1^* + e_2^*) = \epsilon(e_1^*)^a \cdot \epsilon(e_2^*) = (-1)^a. \]

With this notation, $F_\epsilon$ and $F_\epsilon'$ are identified if and only if they are marked with the same sign. Gluing the top, bottom, and left edges produces a cylinder whose ends are formed by the right edges. The way the ends of the cylinder are glued depends on the parity of $a$. If $a$ is even the result is a torus; if $a$ is odd, the result is a Klein bottle.

Let $\Delta(i)$ be the set of $i$-dimensional cones of $\Delta$.

**Proposition 2** The Euler characteristic of $X(\mathbb{R})$ is given by
\[ \chi(X(\mathbb{R})) = \sum_{i=0}^{\dim X} (-2)^{\dim X - i} \cdot \#\Delta(i). \]

Proof: Proposition 1 gives a cell decomposition of $X(\mathbb{R})$ with $2^{\dim X - i} \cdot \#\Delta(i)$ cells of dimension $\dim X - i$. $\square$

For the remainder of this note, we assume that $X$ is a smooth projective toric surface.

**Lemma** Let $C$ be a $T$-invariant curve in $X$, and let $U$ be a tubular neighborhood of $C(\mathbb{R})$ in $X(\mathbb{R})$. If $C^2$ is even then $U$ is homeomorphic to a cylinder; if $C^2$ is odd, then $U$ is homeomorphic to a Möbius band.

Proof: Say $C = V(\rho)$ and $C^2 = a$. We can choose coordinates on $N$ so that $\rho$ is generated by $e_2$ and the adjacent rays in $\Delta$ are generated by $e_1$ and $-e_1 - ae_2$. Then $\mu(C)$ is the bottom edge of $P$ (the edge whose inward normal is generated by $e_2$) and a tubular neighborhood of $C(\mathbb{R})$ may be constructed by gluing neighborhoods of the bottom edges of the $P_\epsilon$ according to the recipe given in Proposition 1. With notation as in the Example, the pieces to be glued are drawn in the figure below.
Gluing the bottom and left edges yields a strip homeomorphic to $[0, 1] \times (0, 1)$ whose ends are formed by the right edges. The gluing of the ends of the strip depends on the parity of $a$. If $a$ is even the result is a cylinder; if $a$ is odd the result is a Möbius band.

Proof of Theorem: The homeomorphism type of a surface is determined by its Euler characteristic and orientability. By Proposition 2 \[ \chi(X(\mathbb{R})) = 4 - \#\Delta(1). \] It remains to show that $X(\mathbb{R})$ is orientable if and only if $X$ is isomorphic to an even Hirzebruch surface $\mathbb{F}_a$.

If $X$ is not a minimal surface, then it contains a $-1$-curve $C$, which must be $T$-invariant (otherwise $C$ would move and hence have nonnegative self-intersection). By the Lemma, a tubular neighborhood of $C(\mathbb{R})$ is homeomorphic to a Möbius band, so $X(\mathbb{R})$ is nonorientable. Therefore, if $X(\mathbb{R})$ is orientable, then $X$ must be minimal. The minimal rational surfaces are $\mathbb{P}^2$ and the Hirzebruch surfaces. Of course, $\mathbb{P}^2(\mathbb{R})$ is homeomorphic to $\mathbb{R}\mathbb{P}^2$, which is nonorientable. As seen in the Example, $\mathbb{F}_a(\mathbb{R})$ is orientable if and only if $a$ is even. \qed

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