On the Striated Regularity for the 2D Anisotropic Boussinesq System

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Abstract
In this paper, we investigate the global existence and uniqueness of strong solutions to the 2D anisotropic Boussinesq system for rough initial data with striated regularity. We prove the global well-posedness of the Boussinesq system with anisotropic thermal diffusion with initial vorticity being discontinuous across some smooth interface. In the case of an anisotropic horizontal viscosity, we study the propagation of the striated regularity for the smooth temperature patches problem. The proofs rely on the idea of Chemin to solve the 2-D vortex patch problem for ideal fluids, namely the striated regularity can help to bound the gradient of the velocity.

Keywords Anisotropic Boussinesq equations · Littlewood–Paley theory · Striated regularity

Mathematics Subject Classification 35Q30 · 76D03 · 76D05

1 Introduction
The Boussinesq system is a classical model in geophysical fluid dynamics which describes the large-scale atmospheric and oceanic flows and also plays an important role in the study of Rayleigh–Bénard convection (see Pedlosky 2013 for example). This system describes the phenomenon of convection in an incompressible viscous...
fluid, under the effect of the upward buoyancy force induced by the temperature. In the present paper, we investigate the 2D anisotropic Boussinesq equations with horizontal temperature diffusion or horizontal velocity dissipation. These are derivative models from the classical Boussinesq system for geophysical fluid where the vertical dimension of the domain is very small compared with the horizontal dimension of the domain. In this case, after rescaling the domain, the dissipation is not isotropic and we have to deal with the anisotropic problem. More precisely, we study the following system which is the Euler equations coupled with a transport-diffusion temperature equation with diffusion only in the horizontal direction,

\[
\begin{aligned}
\partial_t u + u \cdot \nabla u &= -\nabla p + \theta e_2, \\
\partial_t \theta + u \cdot \nabla \theta - \kappa \partial_1^2 \theta &= 0, \\
\nabla \cdot u &= 0, \\
u(0, x) &= u_0(x), \theta(0, x) = \theta_0(x),
\end{aligned}
\]  

(1.1)

and a system where the Navier–Stokes equations with no vertical viscosity couples with a transport temperature equation,

\[
\begin{aligned}
\partial_t u + u \cdot \nabla u - \nu \partial_1^2 u &= -\nabla p + \theta e_2, \\
\partial_t \theta + u \cdot \nabla \theta &= 0, \\
\nabla \cdot u &= 0, \\
u(0, x) &= u_0(x), \theta(0, x) = \theta_0(x).
\end{aligned}
\]  

(1.2)

Here, \( u = (u^1(x, t), u^2(x, t)) \) denotes the velocity field, \( p = p(x, t) \) is a scalar function denoting the pressure, \( \theta = \theta(x, t) \) is a scalar representing the temperature in the content of thermal convection and the density in the modeling of geophysical fluids. \( e_2 = (0, 1) \) is the vertical unit vector field, and the forcing term \( \theta e_2 \) on behalf of the buoyancy force due to the gravity field. The parameters \( \kappa \) and \( \nu \) denote the molecular diffusion and the viscosity respectively. These anisotropic systems are important, modeling dynamics of geophysical flows (see, e.g., Chemin et al. 2000, 2006; Iftimie 2002; Paicu 2005).

The general 2D anisotropic Boussinesq equations can be read as,

\[
\begin{aligned}
\partial_t u + u \cdot \nabla u - \nu_1 \partial_1^2 u - \nu_2 \partial_2^2 u &= -\nabla p + \theta e_2, \\
\partial_t \theta + u \cdot \nabla \theta - \kappa_1 \partial_1^2 \theta - \kappa_2 \partial_2^2 \theta &= 0, \\
\nabla \cdot u &= 0, \\
u(0, x) &= u_0(x), \theta(0, x) = \theta_0(x),
\end{aligned}
\]  

(1.3)

where \( \nu_1, \nu_2, \kappa_1 \) and \( \kappa_2 \) are real parameters. Systems (1.1) and (1.2) are two special cases for (1.3). When \( \nu_1 = \nu_2 > 0, \kappa_1 = \kappa_2 > 0 \), the global well-posedness theory for (1.3) has been established in Cannon and DiBenedetto (1980) and Guo (1989). On the contrary, when these four parameters are zero, whether (1.3) has a unique global solution is a challenging problem and still unsolved. This system has
many similarities with the classical 3D incompressible Euler equations such as the vortex-stretching mechanism (which will be explained later). So it has both physical motivation and mathematical significance to investigate the intermediate cases (only partial dissipation) and some improvements have been made in the past few years.

The global regularity for the case when $\nu_1 = \nu_2 > 0$ and $\kappa_1 = \kappa_2 = 0$ was proven by Chae (2006) and by Hou and Li (2004) with smooth initial data. Later, Abidi and Hmidi studied this system in the Besov space in Abidi and Hmidi (2007). The global weak solution with finite energy has been constructed in Hmidi and Keraani (2007) and has been proved to be unique later in Danchin and Paicu (2008). For the case $\nu_1 = \nu_2 = 0$ and $\kappa_1 = \kappa_2 > 0$, Chae (2006) also studied the global regularity for smooth data. This result was improved by Hmidi and Keraani (to appear), Danchin and Paicu (2009) for rough initial data. The global well-posedness for (1.1) and (1.2) was considered by Danchin and Paicu (2011), and they established the global existence and uniqueness theory. Then, the global well-posedness for the anisotropic Boussinesq equations with vertical dissipation, namely (1.3) with only $\nu_2, \kappa_2 > 0$, was studied by Cao and Wu (2013). Later, Adhikaria et. al. investigated other mixed dissipation cases Adhikari et al. (2016). Other interesting recent results of the 2D anisotropic Boussinesq equations and other related systems can be found in Larios et al. (2013), Lai et al. (2011), Li and Titi (2016), Li et al. (2015), Adhikari et al. (2010), Adhikari et al. (2011), Jiu and Liu (2016) and Xu and Zhu (2018).

Our main goal in this paper is to study the vortex patch problem for the 2D Boussinesq system with anisotropic viscosity and to study the propagation of a smooth front of temperature for the 2D Boussinesq system with anisotropic thermal diffusion. Also, for both systems we improve the global well-posedness results obtained in Danchin and Paicu (2011).

We denote by the quantity $\omega \triangleq \partial_1 u^2 - \partial_2 u^1$, the vorticity of the flow which measures how fast the fluid rotates. This quantity is widely utilized in the literature we have mentioned above. Taking curl operator to the first equation of (1.1), we obtain the corresponding vorticity equation,

$$\partial_t \omega + u \cdot \nabla \omega = \partial_1 \theta. \quad (1.4)$$

Similarly, the vorticity form of system (1.2),

$$\partial_t \omega + u \cdot \nabla \omega - \partial_1^2 \omega = \partial_1 \theta. \quad (1.5)$$

The forcing term $\partial_1 \theta$ is making this system become more complex than the 2D Euler system. Considering to the formal analogy between the 2D Boussinesq system and the 3D axisymmetric swirling flows, we can refer to the forcing term $\partial_1 \theta$ as a “vortex-stretching” term (see Majda and Bertozzi 2002).

Another part of our paper is devoted to study the vortex (temperature) patch problem. Before we describe this problem, we need first to introduce some notations. Let us denote by $\psi(\cdot, t)$ the flow associated with the vector field $u$, that is
\[ \frac{d}{dt} \psi(x, t) = u(\psi(t, x), t), \]
\[ \psi(0, x) = x. \]  

(1.6)

The classical vortex patch problem is associated to the 2D Euler equations: if the initial vorticity is given by the characteristic function supported in some connected bounded domain, whether the regularity of the boundary can be preserved through the evolution of the flow \( \psi \)? It has been proved by Chemin that the regularity of the boundary is preserved for all the time in some Hölder class (see Chemin 1993, 1998 for details). Other results about the vortex (temperature) patch problems corresponding to the Euler equations, homogeneous (inhomogeneous) Navier–Stokes equations and other fluid models can be found in Bertozzi and Constantin (1993), Danchin (1997a), Danchin (1997b), Danchin et al. (2018), Danchin and Mucha (2018), Danchin and Zhang (2017a), Danchin and Zhang (2017b), Danchin and Mucha (2018), Danchin and García-Juárez (2018), Hassainia and Hmidi (2015), Hmidi (2005), Liao and Zhang (2016), Liao and Zhang (2019), Paicu and Zhang (2019) and the references therein.

We introduce now the notion of striated regularity which generalizes the classical vortex patch problem. This more general geometric structure means that the vorticity is more regular along with some special directions, given by a non-degenerate family of vector fields. In order to understand the striated regularity clearly, we need first to introduce some notations and definitions which will be used to describe the boundary regularity. Let \( X_0 \) be a vector field defined on \( D_0 \) (a connected bounded domain), \( X \) is the evolution of \( X_0 \) along with the flow \( \psi \) defined as follows,  
\[ X(x, t) \triangleq \partial X_0 \psi(\psi^{-1}(x, t), t), \]  

(1.7)

where \( \partial X_0 f \triangleq X_0 \cdot \nabla f \) denoting the standard directional derivative.

Taking the time derivative of (1.7), we see that \( X \) satisfies the following transport equation,  
\[ \begin{cases} 
\partial_t X + u \cdot \nabla X = \partial X u, \\
X(0, x) = X_0(x). 
\end{cases} \]  

(1.8)

It is not hard to check that \( \partial X \) satisfies,  
\[ [\partial X, D_t] = 0, \]  

(1.9)

where \([A, B] \triangleq AB - BA\) represents the standard commutator, and \(D_t \triangleq \partial_t + u \cdot \nabla\) denotes the material derivative.

We need also the following two definitions, which can be found in Bahouri et al. (2011) and Chemin (1998).

**Definition 1.1** Let \( 0 < s < 1 \) and \( \Omega \) be a bounded domain in \( \mathbb{R}^2 \). We say that \( \Omega \) is of class \( C^{1+s} \) if there exists a compactly supported function \( f \in C^{1+s}(\mathbb{R}^2) \) and a neighborhood \( V \) of \( \partial \Omega \) such that  
\[ \partial \Omega = f^{-1}(\{0\}) \cap V \quad \text{and} \quad \nabla f(x) \neq 0 \quad \forall \ x \in V. \]
Definition 1.2 A family \((X_\lambda)_{\lambda \in \Lambda}\) of vector fields over \(\mathbb{R}^2\) is said to be non-degenerate whenever
\[
I(X) \triangleq \inf_{x \in \mathbb{R}^d} \sup_{\lambda \in \Lambda} |X_\lambda(x)| > 0.
\]

Let \(r \in (0, 1)\) and \((X_\lambda)_{\lambda \in \Lambda}\) be a non-degenerate family of \(C^r\) vector fields over \(\mathbb{R}^2\). A bounded function \(f\) is said to be in the function space \(C^r_X\) if it satisfies
\[
\|f\|_{C^r_X} \triangleq \sup_{\lambda \in \Lambda} \left( \|f\|_{L^\infty} \|X_\lambda\|_{C^r} + \|\nabla \cdot (X_\lambda f)\|_{C^{r-1}} \right) < \infty.
\]

Next, we present the main results for our paper. Since the concrete values of the constants \(\kappa\) in system (1.1) and \(v\) in (1.2) play no role in our discussion, for this reason, we shall assume \(\kappa = v = 1\) throughout this paper.

The main result pertaining to system (1.1) can be stated as follows.

**Theorem 1.1** Assume \(u_0 \in L^2\) be a divergence-free vector field and the corresponding vorticity \(\omega_0 \triangleq \partial_1 u_0^2 - \partial_2 u_0^1 \in L^\infty\). Let \((\omega_0, \theta_0) \in H^s \times H^{1+s}\) with \(0 < s < 1\). Then, system (1.1) has a unique global solution \((u, \theta)\) satisfying:
\[
\begin{align*}
  u &\in L^\infty([0, T]; H^{1+s}), \quad \omega \in L^\infty([0, T]; L^\infty), \quad \theta \in L^\infty([0, T]; H^{1+s}), \\
  \partial_1 \theta &\in L^2([0, T]; H^{1+s}),
\end{align*}
\]
for any \(T > 0\).

Furthermore, for any non-degenerate vector field \(X_0 \in C^s\) such that \(\partial X_0 \omega_0 \in L^p\) (for some \(2 < p < \infty\)), there exists a unique global solution \(X \in L^\infty([0, T]; C^s)\) to Eq. (1.8) and we have
\[
\partial_X \omega \in L^\infty([0, T]; L^p), \quad \nabla u \in L^1([0, T]; L^\infty).
\]

As a direct application, this theorem can be used to deal with the so called “vortex patch” problem as follows.
For
\[
\omega_0(x) = \chi_{D_0}(x) \triangleq \begin{cases} 
1 & x \in D_0, \\
0 & x \notin D_0,
\end{cases}
\] (1.10)
where \(D_0\) is a connected bounded domain, \(\chi_{D_0}\) is the standard characteristic function of \(D_0\). Let \(\omega(x, t) = \omega^1(x, t) + \omega^2(x, t)\) where \(\omega^1\) is the solution of the system
\[
\begin{align*}
  \partial_t \omega^1 + u \cdot \nabla \omega^1 &= 0, \\
  \omega^1(x, 0) &= \omega_0(x),
\end{align*}
\] (1.11)
and \(\omega^2\) is the solution of the system
\[
\begin{align*}
  \partial_t \omega^2 + u \cdot \nabla \omega^2 &= \partial_1 \theta, \\
  \omega^2(x, 0) &= 0.
\end{align*}
\] (1.12)
Then, the main result can be stated as follows.

**Corollary 1.1** Assume \( \omega_0 \) defined as in (1.10) and \( D_0 \) be a connected bounded domain with its boundary \( \partial D_0 \) in Hölder class \( C^{1+s} \) (\( 0 < s < 1 \)). Then, system (1.1) has a unique global solution that satisfies the properties shown in Theorem 1.1. Moreover, the solution of systems (1.11) and (1.12) satisfies:

\[
\omega^1 = \chi_{D_t}, \quad \partial_X \omega^2 \in L^\infty([0, T]; C^{s-1}),
\]

with \( D_t \triangleq \psi(D_0, t) \) and the boundary of the domain remains in the class \( C^{1+s} \).

Then, we present our main result pertaining to system (1.2).

**Theorem 1.2** Assume \( u_0 \in L^2 \) be a divergence-free vector field and the corresponding vorticity \( \omega_0 \triangleq \partial_1 u_0^2 - \partial_2 u_0^1 \in \sqrt{\mathcal{L}} \). Let \( (\omega_0, \theta_0) \in H^s \times H^\beta \) with \( \frac{1}{2} < s < \beta \). Then, system (1.2) has a unique global solution \( (u, \theta) \) which satisfies:

\[
u \in L^\infty([0, T]; H^{1+s}), \quad \partial_1 u \in L^2([0, T]; H^{1+s}), \quad \nabla u \in L^1([0, T]; L^\infty),
\]

\[
\theta \in L^\infty([0, T]; H^s).
\]

Furthermore, for any vector field \( X_0 \in H^s \), there exists a unique global solution \( X \in L^\infty([0, T]; H^s) \) to Eq. (1.8). Moreover, \( X \in L^\infty([0, T]; H^s') \) for \( s' > 1 \) if provided \( \omega_0 \in \dot{W}^{1,p} \cap H^s \), \( \theta_0 \in \dot{W}^{1,p} \cap H^s \) with some \( 2 < p < \infty \) and \( X_0 \in H^{s'} \).

**Remark 1** Compared with the result of the paper of Danchin and the first author Danchin and Paicu (2011) where the velocity was only Log-Lipschitz, here we obtain that the velocity \( u \) is Lipschitz.

**Remark 2** In the critical case \( s = 1/2 \), we can prove the global well-posedness and obtain the Lipschitz norm of the velocity with \( \omega_0 \in B_{2,1}^{1/2} \) and \( \theta_0 \in H^\beta \), \( 1/2 < \beta \). We can even obtain the control of the Lipschitz norm of the velocity with initial vorticity \( \omega_0 \) in anisotropic Besov space \( B^{0,1/2} \) through a similar idea. Here, \( B^{0,1/2} \) is the space given by the norm

\[
\| f \|_{B^{0,1/2}} = \sum_{q \in \mathbb{Z}} 2^q \| \Delta_q f \|_{L^2} \quad \text{and} \quad \Delta_q^v = \mathcal{F}^{-1}(\varphi(2^{-q}\xi_2)\hat{f}(\xi))
\]

is the dyadic bloc in the vertical Fourier variable and the definition of \( \varphi(\xi) \) will be given in the next section. Because the proof is more complicated, we left it in the “Appendix”.

The above result can be used to solve the smooth “temperature patch” problem. Defining

\[
\theta_0^\varepsilon(x) = \chi_{D_0} \ast \eta_\varepsilon(x) = \begin{cases} 1 & x \in D_\varepsilon^-, \\ 0 & x \in D_\varepsilon^+ \end{cases},
\]

(1.13)
where $\chi_{D_0}$ is the characteristic function of the domain $D_0$. $\eta_\varepsilon$ is the standard mollified function. $D^-_\varepsilon$ and $D^+_\varepsilon$ are two domains defined by

$$D^-_\varepsilon \triangleq \{ x \in D_0 : \text{dist}(x, \partial D_0) > \varepsilon \},$$
$$D^+_\varepsilon \triangleq \{ x \in \mathbb{R}^2 : \text{dist}(x, D_0) > \varepsilon \}. $$

We denote by $d(t)$ the distance between $\psi(D^-_\varepsilon, t)$ and $\psi(D^+_\varepsilon, t)$, which are the evolution by the flow at time $t > 0$ of the initial domains $D^-_\varepsilon$ and $D^+_\varepsilon$, with $d(0) = 2\varepsilon$. Then, the following result holds true.

**Corollary 1.2** Let $\frac{1}{2} < s < 1$, assume the initial data $\theta_0 = \theta_0^\varepsilon$ defined as in (1.13) with $\partial D_0 \in H^{1+s}$, $\omega_0 \in L^\infty \cap H^s$. Then, there exists a unique solution $(u, \theta)$ to system (1.2) satisfying the properties listed in Theorem 1.2. Furthermore, $\theta(x, t)$ satisfies the same form as $\theta_0$ that

$$\theta(x, t) = \begin{cases} 
1 & x \in \psi(D^-_\varepsilon, t), \\
0 & x \in \psi(D^+_\varepsilon, t), 
\end{cases}$$

and the distance $d(t)$ satisfies,

$$2\varepsilon e^{-\int_0^t \|\nabla u(\tau)\|_{L^\infty} d\tau} \leq d(t) \leq 2\varepsilon e^{\int_0^t \|\nabla u(\tau)\|_{L^\infty} d\tau}.$$  \hfill (1.14)

Moreover, the flow $\psi(\cdot, t) \in H^{1+s}$ and the boundary $\partial D^-_\varepsilon$, $\partial D^+_\varepsilon \in H^{1+s}$ for all $t \geq 0$.

**Remark** We can propagate higher-order regularity of the boundary for the temperature patch if we improve the regularity condition of the initial data.

The rest of this paper is divided into three sections and an “Appendix.” In Sect. 2, we provide some definitions and lemmas which will be used in the next sections. Section 3 is devoted to the study of system (1.1) which is divided into three subsections. The first one gives some regularity estimates, the second subsection shows the estimate for striated regularity, and the last subsection gives the proof of Corollary 1.1. Section 4 deals with system (1.2) which is divided into five subsections. Section 4.1 obtains the estimate for the Lipschitz norm of the velocity, and Sect. 4.2 gives the estimate of $X$. In Sects. 4.3 and 4.4, we investigate the higher-order estimates for $(\omega, \theta)$ and $X$. Then in Sect. 4.5, we deal with the temperature patch problem. Finally, “Appendix A” provides the technical proof for some lemmas presented in the second section.

### 2 Preparations

In this section, we will give some definitions and lemmas which will be used in the next several sections. First, we give some notations. Throughout this paper, $C$ stands for some real positive constant which may vary from line to line. We denote by $C(t)$ a generic continuous function depending on time and on various norms on $(u_0, \theta_0)$,
the initial data which arbitrarily but fixed. \( \{b_q\} \) stands for a generic sequence in \( \ell^1 \) which may be different in each occurrence. Here, we have denoted by \( \ell^p \) the space of summable sequences with the norm \( \|\{b_q\}_q\|_{\ell^p} = (\sum_q |b_q|^p)^{1/p} < +\infty \). And \( |D| \triangleq (-\Delta)^{1/2} \) denotes the Zygmund operator which is defined through the Fourier transform that

\[
|D| f = |\xi| \hat{f},
\]

where

\[
\hat{f} \triangleq \mathcal{F}(f) = \frac{1}{(2\pi)^2} \int_{\mathbb{R}^2} e^{-ix \cdot \xi} f(x) \, dx.
\]

Similarly, we can define

\[
|D|^s f = |\xi|^s \hat{f}, \quad |\partial_1|^s f = |\xi_1|^s \hat{f}.
\]

Next, we present the classical Littlewood–Paley theory in \( \mathbb{R}^d \) which plays an important role in the proof of our results. Let \( \chi \) be a smooth function supported on the ball \( B \triangleq \{\xi \in \mathbb{R}^d : |\xi| \leq \frac{4}{3}\} \) and \( \varphi \) be a smooth function supported on the ring \( C \triangleq \{\xi \in \mathbb{R}^d : \frac{3}{4} \leq |\xi| \leq \frac{8}{3}\} \) such that

\[
\chi(\xi) + \sum_{q \geq 0} \varphi(2^{-q} \xi) = 1, \quad \text{for all} \quad \xi \in \mathbb{R}^d,
\]

\[
\sum_{q \in \mathbb{Z}} \varphi(2^{-q} \xi) = 1, \quad \text{for all} \quad \xi \in \mathbb{R}^d \setminus \{0\}.
\]

Then for every \( u \in \mathcal{S}' \) (tempered distributions), we define the non-homogeneous Littlewood–Paley operators as follows,

\[
\Delta_q u = 0 \quad \text{for} \quad q \leq -2, \quad \Delta_{-1} u = \chi(D) u = \mathcal{F}^{-1}(\chi(\xi) \hat{u}(\xi)),
\]

\[
\Delta_q u = \varphi(2^{-q} D) u = \mathcal{F}^{-1}(\varphi(2^{-q} \xi) \hat{u}(\xi)), \quad \forall \, q \geq 0, \quad S_q u = \sum_{j=-1}^{q-1} \Delta_j u.
\]

Next, we state the definition of non-homogeneous Besov spaces through the dyadic decomposition.

**Definition 2.1** For \( s \in \mathbb{R} \) and \( 1 \leq p, r \leq \infty \), the non-homogeneous Besov space \( B^s_{p,r} \) is defined by

\[
B^s_{p,r} = \{ f \in \mathcal{S}'; \| f \|_{B^s_{p,r}} < \infty \},
\]

where

\[
\| f \|_{B^s_{p,r}} = \begin{cases} 
\sum_{q \geq -1} (2^{qs} \| \Delta_q f \|_{L^p})^{1/r} & \text{for} \, r < \infty, \\
\sup_{q \geq -1} 2^{qs} \| \Delta_q f \|_{L^p} & \text{for} \, r = \infty.
\end{cases}
\]
We point out that when \( p = r = 2 \), for all \( s \in \mathbb{R} \), we have \( B^s_{2,2}(\mathbb{R}^d) = H^s(\mathbb{R}^d) \), and when \( p = r = \infty \), \( B^s_{\infty,\infty}(\mathbb{R}^d) = C^s(\mathbb{R}^d) \).

**Lemma 2.1**  
(Bernstein inequality Bahouri et al. 2011; Chemin 1998) Let \( k \in \mathbb{N} \cup \{0\} \), \( 1 \leq a \leq b \leq \infty \). Assume that 
\[
\text{supp} \hat{f} \subset \{ \xi \in \mathbb{R}^d : |\xi| \leq 2^q C \},
\]
for some integer \( q \), then there exists a constant \( C_1 \) such that 
\[
\| \nabla^\alpha f \|_{L^b} \leq C_1 2^{q(k+d(\frac{1}{2}-\frac{1}{b}))} \| f \|_{L^a}, \quad k = |\alpha|.
\]
If \( f \) satisfies 
\[
\text{supp} \hat{f} \subset \{ \xi \in \mathbb{R}^d : |\xi| = 2^q C \},
\]
for some integer \( q \), then 
\[
C_2 2^{q \alpha} \| f \|_{L^b} \leq \| \nabla^\alpha f \|_{L^b} \leq C_3 2^{q(k+d(\frac{1}{2}-\frac{1}{b}))} \| f \|_{L^a}, \quad k = |\alpha|,
\]
where \( C_2 \) and \( C_3 \) are constants depending on \( \alpha \), \( a \) and \( b \) only.

Notice that if \( u \) is a divergence-free vector field in \( \mathbb{R}^2 \), then it can be recovered from the corresponding vorticity \( \omega \) utilizing the following Biot–Savart law
\[
u = \nabla \perp \Delta^{-1} \omega. \tag{2.3}
\]
Combining the classical Calderón–Zygmund estimate and (2.3) can lead to the following lemma Chemin (1998).

**Lemma 2.2**  
For any smooth divergence-free vector field \( u \) with its vorticity \( \omega \in L^p \) and \( p \in (1, \infty) \), there exists a constant \( C \) such that 
\[
\| \nabla u \|_{L^p} \leq C \frac{p^2}{p-1} \| \omega \|_{L^p}. \tag{2.4}
\]

The next lemma shows the Hölder estimate for the transport equation, which is useful in the estimate of the striated regularity. The proof can be found in Chemin (1998).

**Lemma 2.3**  
Let \( v \) be a smooth divergence-free vector field, \( r \in (-1, 1) \). Consider two functions \( f \in L^\infty_{\text{loc}}(\mathbb{R}; C^r) \) and \( g \in L^1_{\text{loc}}(\mathbb{R}; C^r) \), such that \( f \) satisfies the transport equation 
\[
\partial_t f + u \cdot \nabla f = g.
\]
Then, we have
\[
\| f(t) \|_{C^r} \leq C \| f(0) \|_{C^r} e^{C \int_0^t \| \nabla u(\tau) \|_{L^\infty} \, d\tau} + C \int_0^t \| g(\tau) \|_{C^r} e^{C \int_0^{\tau} \| \nabla u(s) \|_{L^\infty} \, ds} \, d\tau,
\]
and the constant \( C \) depends only on \( r \).

The following logarithmic inequality plays an important role in the process of bounding the Lipschitz norm of the velocity in system (1.1). The proof of this lemma can be found in Bahouri et al. (2011) and Chemin (1998).

**Lemma 2.4** Let \( r \in (0, 1) \) and \( (X_\lambda)_{\lambda \in \Lambda} \) be a non-degenerate family of \( C^r \) vector fields over \( \mathbb{R}^2 \). Let \( u \) be a divergence-free vector field over \( \mathbb{R}^2 \) with vorticity \( \omega \in C^r X \). Assume, in addition that \( u \in L^q \) for some \( q \in [1, +\infty] \) or that \( \nabla u \in L^p \) for some finite \( p \). Then, there exists a constant \( C \) depending on \( p \) and \( r \) such that
\[
\| \nabla u \|_{L^\infty} \leq C \left( \min(\| u \|_{L^q}, \| \omega \|_{L^p}) + \| \omega \|_{L^\infty} \log \left( e + \frac{\| \omega \|_{C^r X}}{\| \omega \|_{L^\infty}} \right) \right).
\]

Then, we give the definition of the space \( \sqrt{L} \) and \( LL^{\frac{1}{2}} \).

**Definition 2.2** The space \( \sqrt{L} \) stands for the space of functions \( f \) in \( \bigcap_{2 \leq p < \infty} L^p \) such that
\[
\| f \|_{\sqrt{L}} \triangleq \sup_{p \geq 2} \frac{\| f \|_{L^p}}{\sqrt{p - 1}} < \infty.
\]
And the space \( LL^{\frac{1}{2}} \) denotes by
\[
LL^{\frac{1}{2}} \triangleq \left\{ f \in S' : \| f \|_{LL^{\frac{1}{2}}} \triangleq \sup_{j \geq 0} \frac{\| S_j f \|_{L^\infty}}{\sqrt{j + 1}} < \infty \right\},
\]
where \( S_j f \) denotes the “low frequencies” part given by \( S_j f = \mathcal{F}^{-1}(\chi(2^{-j} \xi) \hat{f}(\xi)) \).

**Remark** It is not hard to check that \( \sqrt{L} \hookrightarrow LL^{\frac{1}{2}} \).

The following lemma plays a significant role in the estimate of the convection term. The proof of this lemma shall be shown in the “Appendix.”

**Lemma 2.5** Assume \( u \) is a smooth divergence-free vector field with \( u \in L^2 \), \( \nabla u \in L^\infty \), \( f \in H^s \) with \( s \in (0, 1) \), then we have
\[
- \int_{\mathbb{R}^2} \Delta_q (u \cdot \nabla f) \Delta_q f \, dx \leq C b_q 2^{-2q s} \| \nabla u \|_{L^\infty} \| f \|_{H^s}^2,
\]
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with $b_q \in \ell^1$. Moreover, if $\omega, \partial_1 \omega \in L^2$, $\partial_1 f \in H^s$, then we have
\[
- \int_{\mathbb{R}^2} \Delta_q (u \cdot \nabla f) \Delta_q f \, dx \leq C b_q 2^{-2qs} (\|u\|_{L^2} + \|\omega\|_{L^2} + \|\partial_1 \omega\|_{L^2}) \\
\times (\|f\|_{H^s}^2 + \|f\|_{H^s} \|\partial_1 f\|_{H^s} + \|f\|_{H^s}^3 \|\partial_1 f\|_{H^s}^2),
\] (2.7)
where $\omega$ is the corresponding vorticity of $u$.

Then, we give a lemma which shows the classical losing regularity estimate for the transport equation, and the result can be found in Bahouri et al. (2011) and Danchin and Paicu (2011). For the sake of completeness, we will give the proof of this lemma in the “Appendix.”

**Lemma 2.6 (Losing regularity estimate for transport equation)** Let $\rho$ satisfy the transport equation
\[
\begin{cases}
\partial_t \rho + u \cdot \nabla \rho = f, \\
\rho(0, x) = \rho_0(x),
\end{cases}
\] (2.8)
where $\rho_0 \in B^s_{2,r}$, $f \in L^1([0, T]; B^s_{2,r})$ with $r \in [1, \infty]$. If $u \in L^2$ is a divergence-free vector field and for some $V(t) \in L^1([0, T])$, $u$ satisfies
\[
\sup_{N \geq 0} \frac{\|\nabla S_N u(t)\|_{L^\infty}}{\sqrt{1 + N}} \leq V(t).
\]
Then for all $s > 0$, $\varepsilon \in (0, s)$ and $t \in [0, T]$, we have the following estimate,
\[
\|\rho(t)\|_{B^s_{2,r}} \leq C \left(\|\rho_0\|_{B^s_{2,r}} + \int_0^T \|f(\tau)\|_{B^s_{2,r}} \, d\tau\right) e^{\frac{C}{\varepsilon} \left(\int_0^T V(\tau) \, d\tau\right)^2},
\]
where $C$ is a constant independent of $T$ and $\varepsilon$.

The last lemma of this section gives the classical Kato–Ponce type inequality, which can be found in Kato (1990), Kenig et al. (1991) and Kato and Ponce (1988).

**Lemma 2.7** Assume $s > 0$ and $p \in (1, +\infty)$. Let $f$ satisfies $f \in L^{p_1}$, $\nabla f \in L^{p_1}$, $|D|^s f \in L^{p_2}$, $g$ satisfies $|D|^s g \in L^{p_2}$, $|D|^s g \in L^{p_2}$, $g \in L^{p_2}$, then we have
\[
\|[D]^s f g\|_{L^p} \leq C \left(\|\nabla f\|_{L^{p_1}} \|[D]^s g\|_{L^{p_2}} + \|[D]^s f\|_{L^{p_3}} \|g\|_{L^{p_4}}\right),
\] (2.9)
\[
\|[D]^s (fg)\|_{L^p} \leq C \left(\|f\|_{L^{p_1}} \|[D]^s g\|_{L^{p_2}} + \|[D]^s f\|_{L^{p_3}} \|g\|_{L^{p_4}}\right),
\] (2.10)
where $p_2, p_3 \in (1, +\infty)$ satisfies
\[
\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p_3} + \frac{1}{p_4}.
\]
3 The Case of Horizontal Diffusivity

This section is devoted to study the first model (1.1). In the first subsection, we will give some regularity estimates for \((\omega, \theta)\). Then, we will show the Hölder estimate of \(X\) and prove Corollary 1.1.

3.1 A Priori Estimates for \(\omega\) and \(\theta\)

Before we give the regularity estimate for \((\omega, \theta)\), we need first to give the definition of strong solutions for system (1.1).

**Definition 3.1** Let \((u_0, \theta_0) \in H^\sigma, \sigma > 0\) and \(T > 0\). We say that \((u, \theta)\) is strong solution to (1.1) on the interval \([0, T]\) if \(u \in L^\infty([0, T]; H^\sigma)\) and \(\theta \in L^\infty([0, T]; H^\sigma)\) with \(\partial_1 \theta \in L^2([0, T]; H^\sigma)\) and verify system (1.1) in the sense of distributions on the interval \([0, T]\).

We recall the following existence and uniqueness result in Danchin and Paicu (2011) about system (1.1).

**Theorem 3.1** Let \(1 < s < \frac{3}{2}\) and \(\theta_0 \in H^1\) such that \(|\partial_1|^s \theta_0 \in L^2\). Let \(u_0 \in H^1\) be a divergence-free vector field and the corresponding vorticity \(\omega_0\) in \(L^\infty\). Then, system (1.1) with initial data \((\theta_0, u_0)\) admits a global unique solution \((\theta, u)\) in \(C_w(\mathbb{R}_+; H^1)\) such that

\[
\theta \in L^\infty(\mathbb{R}_+; H^1), \quad \partial_1 \theta \in L^2(\mathbb{R}_+; H^1 \cap L^\infty), \quad \omega \in L^\infty_{loc}(\mathbb{R}_+; L^\infty),
\]

\[
|\partial_1|^s \theta \in L^\infty(\mathbb{R}_+; L^2), \quad |\partial_1|^{1+s} \theta \in L^2_{loc}(\mathbb{R}_+; L^2).
\]

**Remark 3.1** In the following of the paper, we only present formally a priori estimates on the strong solutions of the system. In order to construct the global solution, we can begin by generating an approximate sequence of solutions by adding an artificial viscosity \(-\epsilon \Delta(u, \theta)\) on system (1.1) and by regularizing the initial data. For this fully parabolic system with smooth initial data, we have a unique global solution by the classical result on the Boussinesq system. We shall prove here below, a priori estimates that we obtained here below, which are uniform in the parameter \(\epsilon > 0\). By the classical Aubin–Lions compactness theorem, we show that the sequence of approximate solutions has a subsequence converging to a limit in appropriate function spaces. Passing to the limit in the weak formulation of the regularized Boussinesq system when \(\epsilon\) is converging to zero, we obtain that the limit is a strong solution for system (1.1). Because this step is classical, we will not give all the details about the construction of the solution, but we will let them for the reader. Also, we will not mention the proof of the uniqueness because is a consequence of the result from Danchin and Paicu (2011).

Then, we give a proposition which shows the regularity estimate of \((\omega, \theta)\).
Proposition 3.1 Let $0 < s < 1$, assume the initial data $\omega_0 \in L^2 \cap L^\infty \cap H^s$ and $\theta_0 \in H^{1+s}$. Then, the following estimate holds true,

$$
\|\omega(t)\|_{H^s}^2 + \|\theta(t)\|_{H^{1+s}}^2 + \int_0^t \|\partial_1 \theta(\tau)\|_{H^{1+s}}^2 \, d\tau \leq C(t) e^{\int_0^t \|\nabla u(\tau)\|_{L^\infty} \, d\tau}. \quad (3.1)
$$

Proof We first estimate $\omega$. Applying $\Delta_q$ to (1.4), we get

$$
\partial_t \Delta_q \omega + \Delta_q (u \cdot \nabla \omega) = \partial_1 \Delta_q \theta. \quad (3.2)
$$

Taking $L^2$ inner product with $\Delta_q \omega$, one can deduce

$$
\frac{1}{2} \frac{d}{dt} \|\Delta_q \omega(t)\|_{L^2}^2 = -\int_{\mathbb{R}^2} \Delta_q (u \cdot \nabla \omega) \Delta_q \omega \, dx + \int_{\mathbb{R}^2} \partial_1 \Delta_q \theta \Delta_q \omega \, dx \triangleq N_1 + N_2. \quad (3.3)
$$

For $N_1$, making use of Lemma 2.5,

$$
N_1 \leq C b_q 2^{-2q^s} \|\nabla u\|_{L^\infty} \|\omega\|_{H^s}^2. \quad (3.4)
$$

Then, we estimate $N_2$, by Hölder inequality and Young’s inequality,

$$
N_2 \leq C \|\partial_1 \Delta_q \theta\|_{L^2} \|\Delta_q \omega\|_{L^2} \leq C b_q 2^{-2q^s} \|\partial_1 \theta\|_{H^1} \|\omega\|_{H^s} \leq C b_q 2^{-2q^s} (\|\theta\|_{H^{1+s}}^2 + \|\omega\|_{H^s}^2). \quad (3.5)
$$

Inserting estimate (3.4) and (3.5) into (3.3), then multiplying both side by $2^{2q^s}$ and summing up over $q \geq -1$, it follows that

$$
\frac{1}{2} \frac{d}{dt} \|\omega(t)\|_{H^s}^2 \leq C (1 + \|\nabla u\|_{L^\infty}) \times (\|\omega\|_{H^s}^2 + \|\theta\|_{H^{1+s}}^2). \quad (3.6)
$$

Then, we estimate $\theta$. Applying $\Delta_q$ to the second equation of (1.1), we have

$$
\partial_t \Delta_q \theta + \Delta_q (u \cdot \nabla \theta) - \partial_1^2 \Delta_q \theta = 0. \quad (3.7)
$$
Multiplying (3.7) by $\Delta_q \theta$ and integrating over $\mathbb{R}^2$ with respect to $x$, after integration by parts, one can deduce

$$\frac{1}{2} \frac{d}{dt} \| \Delta_q \theta(t) \|_{L^2}^2 + \| \partial_1 \Delta_q \theta \|_{L^2}^2 = -\int_{\mathbb{R}^2} \Delta_q (u \cdot \nabla \theta) \Delta_q \theta \, dx$$

$$= -\sum_{|k-q| \leq 2} \int_{\mathbb{R}^2} \Delta_q (S_{k-1} u \cdot \nabla \Delta_k \theta) \Delta_q \theta \, dx$$

$$- \sum_{|k-q| \leq 2} \int_{\mathbb{R}^2} \Delta_q (\Delta_k u \cdot \nabla S_{k-1} \theta) \Delta_q \theta \, dx$$

$$- \sum_{k \geq q-1} \sum_{|l| \leq 1} \int_{\mathbb{R}^2} \Delta_q (\Delta_k u \cdot \nabla \Delta_{k+l} \theta) \Delta_q \theta \, dx$$

$$\triangleq \Theta_1 + \Theta_2 + \Theta_3. \quad (3.8)$$

For $\Theta_1$, along the same method as in the proof of Lemma 2.5 which showed in the “Appendix,” we can obtain

$$\Theta_1 \leq C b_q 2^{-2q(1+s)} \| \nabla u \|_{L^\infty} \| \theta \|_{H^{1+s}}^2. \quad (3.9)$$

For $\Theta_2$, we can write it explicitly,

$$\Theta_2 = -\sum_{|k-q| \leq 2} \int_{\mathbb{R}^2} \Delta_q (\Delta_k u \cdot \nabla S_{k-1} \theta) \Delta_q \theta \, dx$$

$$= -\sum_{|k-q| \leq 2} \int_{\mathbb{R}^2} \Delta_q (\Delta_k u^1 \partial_1 S_{k-1} \theta) \Delta_q \theta \, dx$$

$$- \sum_{|k-q| \leq 2} \int_{\mathbb{R}^2} \Delta_q (\Delta_k u^2 \partial_2 S_{k-1} \theta) \Delta_q \theta \, dx$$

$$\triangleq \Theta_{21} + \Theta_{22}. \quad (3.10)$$

We will use overall that $\Delta_q$ is bounded operator in any $L^p$ so we can get rid of it in the following. Making use of Hölder inequality, $\Theta_{21}$ can be bounded by

$$\Theta_{21} \leq C \sum_{|k-q| \leq 2} \| \Delta_k u^1 \|_{L^2} \| \partial_1 \theta \|_{L^\infty} \| \Delta_q \theta \|_{L^2}$$

$$\leq C \sum_{|k-q| \leq 2} \| \Delta_k \partial_2 \Delta^{-1} \omega \|_{L^2} \| \partial_1 \theta \|_{L^\infty} \| \Delta_q \theta \|_{L^2}$$

$$\leq C \sum_{|k-q| \leq 2} 2^{-k} 2^{-sk} 2^{sk} \| \Delta_k \omega \|_{L^2} \| \partial_1 \theta \|_{L^\infty} 2^{-(1+s)k} 2^{(1+s)k} \| \Delta_q \theta \|_{L^2}$$

$$\leq C 2^{-2(1+s)+} b_q \| \partial_1 \theta \|_{L^\infty} \| \omega \|_{H^s} \| \theta \|_{H^{1+s}}.$$
where we have used the Biot–Savart law (2.3).

Also making use of (2.3), combining with integration by parts, we can write \( \Theta_{22} \) as

\[
\Theta_{22} = - \sum_{|k-q| \leq 2} \int_{\mathbb{R}^2} \Delta_q (\Delta_k u^2 \partial_2 S_{k-1} \theta) \Delta_q \theta \, dx
\]

\[
= - \sum_{|k-q| \leq 2} \int_{\mathbb{R}^2} \Delta_q (\Delta_k \partial_1 \Delta^{-1} \omega \partial_2 S_{k-1} \theta) \Delta_q \theta \, dx
\]

\[
= \sum_{|k-q| \leq 2} \int_{\mathbb{R}^2} \Delta_q (\Delta_k \Delta^{-1} \omega \partial_2 S_{k-1} \theta) \partial_1 \Delta_q \theta \, dx
\]

\[
+ \sum_{|k-q| \leq 2} \int_{\mathbb{R}^2} \Delta_q (\Delta_k \Delta^{-1} \omega \partial_1 \partial_2 S_{k-1} \theta) \Delta_q \theta \, dx
\]

\[
\triangleq \Theta_{221} + \Theta_{222}.
\]

For \( \Theta_{221} \), by Hölder inequality and Bernstein inequality in Lemma 2.1,

\[
\Theta_{221} \leq C \sum_{|k-q| \leq 2} \| \Delta_k \Delta^{-1} \omega \|_{L^\infty} \| \partial_2 S_{k-1} \theta \|_{L^2} \| \partial_1 \Delta_q \theta \|_{L^2}
\]

\[
\leq C \sum_{|k-q| \leq 2} 2^{-2k} \| \Delta_k \omega \|_{L^\infty} \| \theta \|_{H^1} 2^{-(1+s)q} 2^{(1+s)q} \| \partial_1 \Delta_q \theta \|_{L^2}
\]

\[
\leq C \sum_{|k-q| \leq 2} 2^{-2k} 2^{k} \| \Delta_k \omega \|_{L^2} \| \theta \|_{H^1} 2^{-(1+s)q} 2^{(1+s)q} \| \partial_1 \Delta_q \theta \|_{L^2}
\]

\[
\leq C 2^{-2(1+s)d} b_q \| \theta \|_{H^1} \| \omega \|_{H^s} \| \partial_1 \theta \|_{H^{1+s}}.
\]

Next, we bound \( \Theta_{222} \) by Hölder inequality and Bernstein inequality,

\[
\Theta_{222} \leq C \sum_{|k-q| \leq 2} \| \Delta_k \Delta^{-1} \omega \|_{L^2} \| \partial_1 \partial_2 S_{k-1} \theta \|_{L^\infty} \| \Delta_q \theta \|_{L^2}
\]

\[
\leq C \sum_{|k-q| \leq 2} 2^{-2k} \| \Delta_k \omega \|_{L^2} \left( \sum_{k' \leq k-2} \| \partial_1 \partial_2 \Delta_{k'} \theta \|_{L^\infty} \right) \| \Delta_q \theta \|_{L^2}
\]

\[
\leq C \sum_{|k-q| \leq 2} 2^{-2k} \| \Delta_k \omega \|_{L^2} 2^{(1-s)q} \left( \sum_{k' \leq k-2} 2^{(k'-k)(1-s)} 2^{sk'} \| \partial_1 \partial_2 \Delta_{k'} \theta \|_{L^2} \right) \| \Delta_q \theta \|_{L^2}
\]

\[
\leq C 2^{-2(1+s)d} b_q \| \omega \|_{L^2} \| \theta \|_{H^s} \| \partial_1 \theta \|_{H^{1+s}},
\]

where we have used the discrete Young’s inequality in the last step.

Then inserting the estimates of \( \Theta_{21}, \Theta_{221} \) and \( \Theta_{222} \) into (3.10), one can obtain

\[
\Theta_2 \leq C 2^{-2(1+s)d} b_q \| \partial_1 \theta \|_{L^\infty} \| \omega \|_{H^s} \| \theta \|_{H^{1+s}}
\]

\[
+ C 2^{-2(1+s)d} b_q (\| \omega \|_{H^s} + \| \theta \|_{H^{1+s}}) \| \partial_1 \theta \|_{H^{1+s}}.
\]

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Finally, we estimate $\Theta_3$, by Hölder inequality and Bernstein inequality,

$$
\Theta_3 = - \sum_{k \geq q-1} \sum_{|k-l| \leq 1} \int_{\mathbb{R}^2} \Delta_q \nabla \cdot (\Delta_k u \Delta_l \theta) \Delta_q \theta \, dx
\leq C \sum_{k \geq q-1} 2^q \|\Delta_k u\|_{L^\infty} \|\Delta_k \theta\|_{L^2} \|\Delta_q \theta\|_{L^2}
\leq C \sum_{k \geq q-1, k \geq 0} 2^{q-k} \|\Delta_k \nabla u\|_{L^\infty} \|\Delta_k \theta\|_{L^2} \|\Delta_q \theta\|_{L^2} + C \|\Delta_{-1} u\|_{L^\infty} \|\Delta_{-1} \theta\|_{L^2}^2
\leq C 2^{-2(1+s)q} b_q (1 + \|\nabla u\|_{L^\infty}) \|\theta\|_{H^{1+s}}^2.
\tag{3.12}
$$

Inserting estimates (3.9), (3.11) and (3.12) into (3.8), and making use of Young’s inequality, we obtain

$$
\frac{1}{2} \frac{d}{dt} \|\Delta_q \theta(t)\|_{L^2}^2 + \|\partial_1 \Delta_q \theta\|_{L^2}^2 \leq C 2^{-2(1+s)q} (1 + \|\nabla u\|_{L^\infty} + \|\partial_1 \theta\|_{L^\infty}) \times \left( \|\omega\|_{H^s}^2 + \|\theta\|_{H^{1+s}}^2 \right) + \frac{\varepsilon}{b_q} b_q 2^{-2(1+s)q} \|\partial_1 \theta\|_{H^{1+s}}^2.
$$

Multiplying both sides by $2^{(1+s)q}$ and summing up from $-1$ to $\infty$ with respect to $q$, choosing $\varepsilon = \frac{1}{2}$, one can deduce

$$
\frac{d}{dt} \|\theta(t)\|_{H^{1+s}}^2 + \|\partial_1 \theta\|_{H^{1+s}}^2 \leq C (1 + \|\nabla u\|_{L^\infty} + \|\partial_1 \theta\|_{L^\infty}) \times \left( \|\omega\|_{H^s}^2 + \|\theta\|_{H^{1+s}}^2 \right).
\tag{3.13}
$$

Combining (3.6) with (3.13) and by Grönwall’s Lemma, because $\partial_1 \theta \in L^2_t (L^\infty_x)$ (see Theorem 3.1), it follows that

$$
\|\omega(t)\|_{H^s}^2 + \|\theta(t)\|_{H^{1+s}}^2 + \int_0^t \|\partial_1 \theta(\tau)\|_{H^{1+s}}^2 \, d\tau \leq C(t) e^{\int_0^t \|\nabla u(\tau)\|_{L^\infty} \, d\tau},
$$

which completes the proof of this proposition. \(\square\)

### 3.2 A Priori Estimates for the Striated Regularity

In this subsection, we will give the estimates of tangential derivatives of $\omega$ and the regularity estimates of $X$. The first lemma gives $L^p$ ($p \in [1, \infty]$) estimate of $X$.

**Lemma 3.1** Let $r \in [1, \infty]$, $X_0 \in L^r$ and $(\omega_0, \theta_0)$ satisfies the assumptions in Lemma 4.1. Then, the solution $X$ of equation (1.8) satisfies

$$
\|X_0\|_{L^r} e^{-\int_0^t \|\nabla u(\tau)\|_{L^\infty} \, d\tau} \leq \|X(t)\|_{L^r} \leq \|X_0\|_{L^r} e^{\int_0^t \|\nabla u(\tau)\|_{L^\infty} \, d\tau}. \tag{3.14}
$$
Proof Multiplying both sides of Eq. (1.8) by $|X|^{r-2}X$ ($1 < r < \infty$) and integrating over $\mathbb{R}^2$ with respect to $x$, we can obtain

$$
\frac{1}{r} \frac{d}{dt} \|X(t)\|_{L^r}^r \leq C \|\nabla u\|_{L^\infty} \|X\|_{L^r}^r,
$$

(3.15)

which implies the right-hand side inequality of (3.14). Using the time reversibility of this equation and the same $L^r$ estimate, we can obtain the first inequality of (3.14). Then taking $r \to \infty$, we can deduce the result for the case $r = \infty$, which completes the proof of this lemma. □

Applying $\partial X$ to the vorticity equation, according to (1.9), we get

$$
\partial_t \partial X \omega + u \cdot \nabla \partial X \omega = \partial X(\partial_t \theta) = X \cdot \nabla \partial_t \theta.
$$

(3.16)

The next lemma deals with the $L^p$ estimate of $\partial X \omega$.

Lemma 3.2 Let $\partial X_0 \omega_0 \in L^p$ ($2 \leq p < \infty$), and $(\omega, \theta)$ satisfies the assumptions in Proposition 3.1, then we have

$$
\|\partial X \omega(t)\|_{L^p} \leq \|\partial X_0 \omega_0\|_{L^p} + C(t) e^{2 \int_0^t \|\nabla u(\tau)\|_{L^\infty} d\tau}.
$$

Proof Multiplying Eq. (3.16) by $|\partial X \omega|^{p-2} \partial X \omega$ ($2 \leq p < \infty$), and integrating over $\mathbb{R}^2$ with respect to $x$, because $u$ satisfies the divergence-free condition, by Hölder inequality,

$$
\frac{1}{p} \frac{d}{dt} \|\partial X \omega(t)\|_{L^p}^p \leq \|X\|_{L^\infty} \|\partial_1 \nabla \theta\|_{L^p} \|\partial X \omega\|_{L^p}^{p-1}.
$$

Because of the embedding $H^s \hookrightarrow L^p$ with $\frac{2}{p} = 1 - s$, we obtain

$$
\frac{d}{dt} \|\partial X \omega(t)\|_{L^p} \leq \|X\|_{L^\infty} \|\partial_1 \nabla \theta\|_{H^s}.
$$

Then integrating in time and combining with the results of Lemma 3.1 and Proposition 3.1, we have

$$
\|\partial X \omega(t)\|_{L^p} \leq \|\partial X_0 \omega_0\|_{L^p} + \int_0^t \|X(\tau)\|_{L^\infty} \|\partial_1 \nabla \theta(\tau)\|_{H^s} d\tau

\leq \|\partial X_0 \omega_0\|_{L^p} + \|X\|_{L^\infty} \int_0^t \|\partial_1 \nabla \theta(\tau)\|_{H^s} d\tau

\leq \|\partial X_0 \omega_0\|_{L^p} + C(t) e^{2 \int_0^t \|\nabla u(\tau)\|_{L^\infty} d\tau},
$$

which completes the proof of this lemma. □
Then, we give the Hölder estimate of \(X\). The next proposition obtains the bound of Lipschitz norm of the velocity \(u\) and the \(C^s\) norm of \(X\) simultaneously.

**Proposition 3.2** Let \(0 < s < 1\), assume \(X_0 \in C^s\), \(\partial X_0 \omega_0 \in L^p\) and \((\omega_0, \theta_0)\) satisfies the assumptions in Proposition 3.1, then we have the velocity \(u\) satisfies

\[
\nabla u \in L^1([0, t]; L^\infty).
\]

Moreover,

\[
X \in L^\infty([0, t]; C^s), \quad \omega \in L^\infty([0, t]; H^s), \quad \partial X \omega \in L^\infty([0, t]; L^p),
\]

\[
\theta \in L^\infty([0, t]; H^{1+s}), \quad \partial_1 \theta \in L^2([0, t]; H^{1+s}).
\]

**Proof** Firstly, we compute the Hölder estimate of \(X\). Applying Lemma 2.3 to (1.8), we obtain

\[
\|X(t)\|_{C^s} \leq C \|X_0\|_{C^s} e^{\tilde{C} \int_0^t \|\nabla u(\tau)\|_{L^\infty} \, d\tau} + C \int_0^t \|\partial X u(\tau)\|_{C^s} e^{\tilde{C} \int_0^\tau \|\nabla u(s)\|_{L^\infty} \, ds} \, d\tau,
\]

where we can choose \(\tilde{C} > 2\). In order to estimate Hölder norm of \(\partial X u\), we need the following estimate which proof can be found in Chemin (1998) and Bahouri et al. (2011),

\[
\|\partial X u\|_{C^s} \leq C (\|\nabla u\|_{L^\infty} \|X\|_{C^s} + \|\partial X \omega\|_{C^{s-1}}).
\]

By Sobolev embedding \(L^p \hookrightarrow C^{s-1} (1 - s = \frac{2}{p})\) and Lemma 3.2, it follows that

\[
\|\partial X \omega\|_{C^{s-1}} \leq C \|\partial X \omega\|_{L^p} \leq C (\|\partial X \omega_0\|_{L^p} + C(t) e^{2 \int_0^t \|\nabla u(\tau)\|_{L^\infty} \, d\tau}.
\]

Inserting (3.20) and (3.21) into (3.19), one can deduce that

\[
\|X(t)\|_{C^s} \leq C e^{\tilde{C} \int_0^t \|\nabla u(\tau)\|_{L^\infty} \, dr} \left(\|X_0\|_{C^s} + \int_0^t (C(\tau) \right)
\]

\[
\left. + \|\nabla u(\tau)\|_{L^\infty} \|X(\tau)\|_{C^s} e^{\tilde{C} \int_0^\tau \|\nabla u(s)\|_{L^\infty} \, ds} \, d\tau) \right).
\]

Denoting

\[
F(t) \triangleq \|X(t)\|_{C^s} e^{\tilde{C} \int_0^t \|\nabla u(\tau)\|_{L^\infty} \, d\tau}.
\]

Then according to the above estimates, we obtain

\[
F(t) \leq C F(0) + \int_0^t C(\tau)(\|\nabla u(\tau)\|_{L^\infty} + 1)(F(\tau) + 1) \, d\tau.
\]

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By Grönwall’s Lemma,
\[ F(t) \leq C(F(0) + 1)e^{\int_0^t C(\tau)\|\nabla u(\tau)\|_{L^\infty} + 1)\,d\tau}. \]

According to the definition of \( F(t) \), we obtain the Hölder estimate of \( X \) that,
\[ \|X(t)\|_{C^s} \leq C(t)e^{\int_0^t \|\nabla u(\tau)\|_{L^\infty} \,d\tau}. \]
\[ (3.22) \]

Recalling the logarithmic inequality in Lemma 2.4 that
\[ \|\nabla u\|_{L^\infty} \leq C\left( \|\omega\|_{L^2} + \|\omega\|_{L^\infty} \log \left( e + \frac{\|\omega\|_{C^s_X}}{\|\omega\|_{L^\infty}} \right) \right). \]
\[ (3.23) \]
where \( \|\omega\|_{C^s_X} \) is defined in Definition 1.2.

Because \( \|\omega\|_{L^2 \cap L^\infty} \) is bounded, inserting estimates (3.21), (3.22) into (3.23), we obtain
\[ \|\nabla u\|_{L^\infty} \leq C\left( 1 + \log \left( e + C(t)e^{\int_0^t \|\nabla u(\tau)\|_{L^\infty} \,d\tau} \right) \right) \]
\[ \leq C\left( 1 + \int_0^t C(t)(1 + \|\nabla u(\tau)\|_{L^\infty}) \,d\tau \right). \]

Then by Grönwall’s Lemma,
\[ \|\nabla u(t)\|_{L^\infty} \leq C(t), \quad \forall t > 0. \]
\[ (3.24) \]

Combining estimates (3.22) and (3.24), we can obtain the desired Hölder norm of \( X \),

Then inserting estimate (3.22) into Proposition 3.1 and Lemma 3.2, we can complete the proof of this proposition. \( \square \)

### 3.3 The Vortex Patch Problem

In this subsection, we devote to prove Corollary 1.1, which solving the vortex patch problem. Because
\[ \omega_0 = \chi_{D_0}(x) \triangleq \begin{cases} 1 & x \in D_0, \\ 0 & x \notin D_0, \end{cases} \]

where \( D_0 \) is a connected bounded domain with \( \partial D_0 \in C^{1+s} \) for \( 0 < s < 1 \). Then according to Definition 1.1, there exists a real function \( f_0 \in C^{1+s} \) and a neighborhood \( V_0 \) such that \( \partial D_0 = V_0 \cap f_0^{-1}(0) \) and \( \nabla f_0 \neq 0 \) on \( V_0 \). Noticing that at time \( t \), the boundary \( \partial D_t = \psi(D_0, t) \) is the level set of the function \( f(\cdot, t) = f_0(\psi^{-1}(\cdot, t)) \), where \( \psi \) is the flow map associated with the velocity \( u \) defined in (1.6) and \( f \) being transported by \( \psi \) as:
\[
\begin{aligned}
\frac{\partial f}{\partial t} + u \cdot \nabla f &= 0, \\
f(x, 0) &= f_0(x).
\end{aligned}
\] (3.25)

Setting the vector field \(X \triangleq \nabla \bot f\) with initial data \(X_0 \triangleq \nabla \bot f_0\), it is not hard to verify that \(X\) satisfies (1.7) and the corresponding system (1.8). Then, we can parametrize \(\partial D_0\) as

\[
\gamma_0 : S^1 \rightarrow \partial D_0, \quad \sigma \mapsto \gamma_0(\sigma),
\]

with

\[
\begin{aligned}
\partial \sigma \gamma_0 &= X_0(\gamma_0(\sigma)), \quad \forall \sigma \in S^1, \\
\gamma_0(0) &= x_0 \in \partial D_0.
\end{aligned}
\] (3.26)

In order to conclude the proof of Corollary 1.1, we observe that a parametrization for \(\partial D_t\) is given by \(\gamma_t(\sigma) \≜ \psi(\gamma_0(\sigma), t)\) and by differentiating with respect to the parameter \(\sigma\), we get

\[
\begin{aligned}
\partial \sigma \gamma_t(\sigma) &= X(\gamma_t(\sigma)), \quad \forall \sigma \in S^1, \\
\gamma_t(0) &= \psi(x_0, t) \in \partial D_t.
\end{aligned}
\] (3.27)

According to Theorem 1.1, \(X \in L^\infty([0, T]; C^s)\), thus \(\gamma_t \in C^{1+s}(S^1)\) for all \(t \geq 0\). This completes the proof of Corollary 1.1.

4 The Case of Horizontal Viscosity

In this section, we focus on system (1.2). We start by recalling the definition of strong solutions for (1.2).

**Definition 4.1** Let \((u_0, \theta_0) \in H^\sigma, \sigma > 0\) and \(T > 0\). We say that \((u, \theta)\) is strong solution to (1.1) on the interval \([0, T]\) if \(u \in L^\infty([0, T]; H^\sigma)\) with \(\partial_t u \in L^2([0, T]; H^\sigma)\) and \(\theta \in L^\infty([0, T]; H^\sigma)\) verify system (1.1) in the sense of distributions on the interval \([0, T]\).

Before we begin to prove the result in Theorem 1.2, we need to review the following existence and uniqueness result for system (1.2) which can be found in Danchin and Paicu (2011).

**Theorem 4.1** Let \(s \in (\frac{1}{2}, 1]\). For all functions \(\theta_0 \in H^s \cap L^\infty\) and divergence-free vector field \(u_0 \in H^1\) with vorticity \(\omega_0 \in \sqrt{L}\). System (1.2) with data \((u_0, \theta_0)\) admits a unique global solution \((u, \theta)\) such that \(\theta \in C_w(\mathbb{R}_+; L^\infty) \cap C(\mathbb{R}_+; H^{s-\varepsilon})\) for all \(\varepsilon > 0\) and

\[
\begin{aligned}
u &\in C_w(\mathbb{R}_+; H^1), \quad \omega \in L^\infty_{\text{loc}}(\mathbb{R}_+; \sqrt{L}) \quad \text{and} \quad \nabla u \in L^\infty_{\text{loc}}(\mathbb{R}_+; \sqrt{L}).
\end{aligned}
\] (4.1)

In the following, we obtain formal a priori estimates on the strong solutions of the system; the construction of the solution is obtained as usual, by regularizing the initial data and by adding an artificial viscosity \(-\varepsilon \Delta(u, \theta)\) on system (1.2). By using
classical results for 2D Boussinesq system with positive viscosity and positive thermal diffusion, we can construct a sequence of smooth solutions for the regularized system, and using classical compactness theorem we can pass to the limit and obtain a strong solution for system (1.2). Also, we do not give the proof of the uniqueness as it is a consequence of the result from Danchin and Paicu (2011). We let these details for the reader.

In the rest of this section, we will first show that the solution \( u \) of system (1.2) actually can be in \( L^1([0, t]; L^\infty) \) in the first subsection. Then, we estimate the striated regularity in the second subsection. In Sects. 4.3 and 4.4, we exam the higher-order regularity estimates of \( (\omega, \theta) \) and the vector field \( X \). The proof of Corollary 1.2 will be given in the last subsection.

### 4.1 A Priori Estimates for the Lipschitz Norm of the Velocity Field

In this subsection, we will give the estimates for the Lipschitz norm of the velocity field and \( H^s \left( \frac{1}{2} < s < 1 \right) \) norm of \( (\omega, \theta) \). Those estimates will be based on the results of Theorem 4.1.

First we give the estimate for \( \| \nabla u \|_{L^\infty} \), which plays an important role in the estimate for striated regularity in the next subsections. The main results can be stated as following.

**Lemma 4.1** Assume \( \omega_0 \in H^s \) and \( \theta_0 \in H^\beta \) with \( \beta > s > \frac{1}{2} \), then the solution \( (\omega, \theta) \) satisfies

\[
\| \omega \|_{L^\infty_t(H^s)}^2 + \| \partial_1 \omega \|_{L^2_t(H^s)}^2 \leq C(t), \quad \| \theta \|_{L^\infty_t(H^s)}^2 \leq C,
\]

moreover,

\[
\| \nabla u \|_{L^2_t(L^\infty)} \leq C(t).
\]

**Proof** Because of Theorem 4.1, we already know \( \nabla u \in \sqrt{L} \). Then according to the definition of space \( \sqrt{L} \) and Lemma 2.6, we have

\[
\| \theta \|_{L^\infty([0, t]; H^s)} \leq C. \quad (4.2)
\]

Then, we give the estimate of \( \omega \). Applying \( \Delta_q \) to the vorticity Eq. (1.5) and taking \( L^2 \) inner product with \( \Delta_q \omega \), one can obtain

\[
\frac{1}{2} \frac{d}{dt} \| \Delta_q \omega(t) \|_{L^2}^2 + \| \partial_1 \Delta_q \omega \|_{L^2}^2 = \int_{\mathbb{R}^2} \partial_1 \Delta_q \theta \Delta_q \omega \, dx - \int_{\mathbb{R}^2} \Delta_q (u \cdot \nabla \omega) \Delta_q \omega \, dx. \quad (4.3)
\]

After integration by parts, according to Hölder inequality and Young’s inequality,

\[
\frac{1}{2} \frac{d}{dt} \| \Delta_q \omega(t) \|_{L^2}^2 + \frac{1}{2} \| \partial_1 \Delta_q \omega \|_{L^2}^2 \leq C \| \Delta_q \theta \|_{L^2}^2 - \int_{\mathbb{R}^2} \Delta_q (u \cdot \nabla \omega) \Delta_q \omega \, dx. \quad (4.4)
\]
For the last term in (4.4), by Lemma 2.5 and Young’s inequality,

\[- \int_{\mathbb{R}^2} \Delta_q (u \cdot \nabla) \Delta_q \omega \, dx \leq C b_q 2^{-2qs} \left( \|u\|_{L^2} + \|\omega\|_{L^2} + \|\partial_1 \omega\|_{L^2} \right) \times \left( \|\omega\|^2_{H^s} + \|\omega\|_{H^{s+\frac{1}{2}}} \|\partial_1 \omega\|_{H^{s+\frac{1}{2}}} + \|\omega\|_{H^{s+\frac{1}{2}}} \|\partial_1 \omega\|_{H^{s+\frac{1}{2}}} \right) \leq C b_q 2^{-2qs} \|\omega\|^2_{H^s} + \frac{1}{4b_q} b_q 2^{-2qs} \|\partial_1 \omega\|^2_{H^s}. \]

(4.5)

According to bound (4.2),

\[\|\Delta_q \theta\|^2_{L^2} \leq C b_q 2^{-2qs} \|\theta\|^2_{H^s}, \tag{4.6}\]

with \(b_q = \frac{2^{2q} \|\Delta q \theta\|^2_{L^2}}{\|\theta\|^2_{H^s}} \in \ell^1\).

Inserting (4.5), (4.6) into (4.4) and taking summation of \(q\), after calculation we obtain

\[\frac{d}{dt} \|\omega(t)\|^2_{H^s} + \|\partial_1 \omega\|^2_{H^s} \leq C (1 + \|\omega\|^2_{H^s}).\]

Then by Grönwall’s Lemma, we get

\[\|\omega(t)\|^2_{H^s} + \int_0^t \|\partial_1 \omega(\tau)\|^2_{H^s} d\tau \leq C(t).\]

According to trace theory, one can deduce

\[\|f(x_1, x_2)\|_{L_\infty^x(\mathcal{H}^{\alpha-\frac{1}{2}}_{\mathbb{R}^{1+2}})} \leq C \|f(x_1, x_2)\|_{\mathcal{H}^\alpha}, \quad \text{for } \alpha > \frac{1}{2}.\]

Thus by Sobolev embedding,

\[\int_0^t \|\omega(t)\|^2_{L_\infty} \, dt \leq \int_0^t \|\omega(t)\|^2_{L_\infty(\mathcal{H}^{s+\frac{1}{2}}_{\mathbb{R}^{1+2}})} \, dt \leq \int_0^t \|\partial_1 \omega(t)\|_{L_\infty(\mathcal{H}^{s-\frac{1}{2}}_{\mathbb{R}^{1+2}})} \|\omega(t)\|_{L_\infty(\mathcal{H}^{s-\frac{1}{2}}_{\mathbb{R}^{1+2}})} \, dt \leq \int_0^t \|\partial_1 \omega(t)\|_{H^s} \|\omega(t)\|_{H^s} \, dt \leq \left( \int_0^t \|\partial_1 \omega(t)\|^2_{H^s} \, dt \right)^{\frac{1}{2}} \left( \int_0^t \|\omega(t)\|^2_{H^s} \, dt \right)^{\frac{1}{2}} \leq C(t).\]
Noticing that $\partial_1 \omega = \Delta u^2$ and $\partial_1 u^1 + \partial_2 u^2 = 0$, we have
\[ \int_0^t \| \partial_i u^j (\tau) \|_{H^{s+1}}^2 \, d\tau \leq C(t), \quad \text{for } i, j = 1, 2, \ (i, j) \neq (2, 1). \]

Then by Sobolev embedding,
\[ \int_0^t \| \partial_i u^j (\tau) \|_{L^\infty}^2 \, d\tau \leq C(t), \quad \text{for } i, j = 1, 2, \ (i, j) \neq (2, 1). \]

As for $(i, j) = (2, 1)$, according to the definition of vorticity $\omega$,
\[ \partial_2 u^1 = \partial_1 u^2 - \omega, \]
from this,
\[ \int_0^t \| \partial_2 u^1 (\tau) \|_{L^\infty}^2 \, d\tau \leq \int_0^t \| \partial_1 u^2 (\tau) \|_{L^\infty}^2 \, d\tau + \int_0^t \| \omega (\tau) \|_{L^\infty}^2 \, d\tau \leq C(t). \]

Thus, we obtain $\| \nabla u \|_{L^\infty_t(L^\infty)}$ is bounded, which completes the proof of this lemma. □

4.2 A Priori Estimates for Striated Regularity

In this section, we will give some estimates about the vector field $X$. Along the same method of Lemma 3.14 and combining with Lemma 4.1, one can deduce for any $r \in [1, \infty]$,
\[ \| X(t) \|_{L^r} \leq C \| X_0 \|_{L^r} e^{\int_0^t \| \nabla u (\tau) \|_{L^\infty} \, d\tau} \leq C(t). \] (4.7)

The next lemma shows the $H^s$ ($\frac{1}{2} < s < 1$) estimate for $X$.

**Lemma 4.2** Let $s > \frac{1}{2}$, $X_0 \in H^s$ and $(\omega_0, \theta_0) \in H^s \times H^\beta$ with $\beta > s$. Then, the solution $X$ of (1.8) satisfies
\[ X \in L^\infty([0, t]; H^s), \]
for any $t > 0$.

**Proof** Applying operator $\Delta_q$ to (1.8),
\[ \partial_t \Delta_q X + \Delta_q (u \cdot \nabla X) = \Delta_q (u \cdot \nabla u). \] (4.8)

Taking the $L^2$ inner product of the above equality with $\Delta_q X$, we get
\[ \frac{1}{2} \frac{d}{dt} \| \Delta_q X(t) \|_{L^2}^2 = - \int_{\mathbb{R}^2} \Delta_q (u \cdot \nabla X) \cdot \Delta_q X \, d\tau + \int_{\mathbb{R}^2} \Delta_q (u \cdot \nabla u) \cdot \Delta_q X \, d\tau. \] (4.9)
For the first term of the right-hand side in (4.9). By Lemma 2.5, we have

$$-\int \Delta_q(u \cdot \nabla X) \Delta_q X \, dx \leq C b_q 2^{-2qs} \| \nabla u \|_{L^\infty} \| X \|_{H^s}^2. \quad (4.10)$$

Then, we estimate the last term of (4.9), by Hölder inequality,

$$\int \Delta_q \partial X u \cdot \Delta_q X \, d\tau \leq \| \Delta_q \partial X u \|_{L^2} \| \Delta_q X \|_{L^2} \leq C b_q 2^{-2qs} \| \partial X u \|_{H^s} \| X \|_{H^s}.$$

For $H^s$ norm of $\partial X u$, we can bound it by

$$\| \partial X u \|_{H^s} = \| X \cdot \nabla u \|_{H^s} \leq (\| X \|_{L^\infty} \| \nabla u \|_{H^s} + \| X \|_{H^s} \| \nabla u \|_{L^\infty}).$$

By Lemmas 4.1 and 3.1, we see that

$$\| X \|_{L^\infty} \leq C(t), \quad \| \nabla u \|_{H^s} \leq C \| \omega \|_{H^s} \leq C(t).$$

Thus, we obtain

$$\int_0^t \Delta_q \partial X u \cdot \Delta_q X \, d\tau \leq C(t) b_q 2^{-2qs} (\| X \|_{H^s} + \| \nabla u \|_{L^\infty} \| X \|_{H^s}^2). \quad (4.11)$$

Inserting estimates (4.10) and (4.11) into (4.9) then multiplying both sides by $2^{2qs}$ and taking summation over $q \geq -1$, it follows that

$$\frac{1}{2} \frac{d}{dt} \| X(t) \|_{H^s}^2 \leq C(t) (\| X \|_{H^s} + \| \nabla u \|_{L^\infty} \| X \|_{H^s}^2). \quad (4.12)$$

Then by Grönwall’s Lemma and combining with Lemma 4.1, we conclude that

$$\| X \|_{H^s} \leq C(t),$$

which completes the proof of this lemma.

\[ Q.E.D. \]

### 4.3 A Priori Estimates for $\omega$ and $\theta$

In this subsection, we will give some regularity estimates for $(\omega, \theta)$ based on the Lipschitz norm $\| \nabla u \|_{L^1_t(L^\infty)}$. The following lemma gives the $H^1$ estimate of $(\omega, \theta)$.

**Lemma 4.3** Assume $\omega_0 \in H^1$ and $\theta_0 \in H^1$, then the solution $(\omega, \theta)$ satisfies

$$\| \nabla \omega \|_{L^\infty_t(L^2)}^2 + \| \nabla \theta \|_{L^\infty_t(L^2)}^2 + \| \partial_1 \nabla \omega \|_{L^2_t(L^2)}^2 \leq C(t). \quad (4.13)$$
Proof Applying $\partial_k$ ($k = 1, 2$) to the vorticity equation (1.5), we can obtain $\partial_k \omega$ satisfies
\begin{equation}
\partial_t \partial_k \omega + u \cdot \nabla \partial_k \omega + \partial_k u \cdot \nabla \omega - \partial_k^2 \partial_k \omega = \partial_1 \partial_k \theta. \tag{4.14}
\end{equation}

Multiplying $\partial_k \omega$ to (4.14) and integrating over $\mathbb{R}^2$ with respect to $x$, we have
\begin{equation}
\frac{1}{2} \frac{d}{dt} \|\partial_k \omega(t)\|_{L^2}^2 + \|\partial_1 \partial_k \omega\|_{L^2}^2 = \int_{\mathbb{R}^2} \partial_1 \partial_k \theta \partial_k \omega \, dx - \int_{\mathbb{R}^2} \partial_k u \cdot \nabla \omega \partial_k \omega \, dx \triangleq N_1 + N_2. \tag{4.15}
\end{equation}

After integration by parts and using Hölder inequality and Young’s inequality, one can deduce
\begin{equation}
N_1 \leq \frac{1}{2} \|\partial_1 \nabla \omega\|_{L^2}^2 + \frac{1}{2} \|\nabla \theta\|_{L^2}^2. \tag{4.16}
\end{equation}

For $N_2$, by Hölder inequality,
\begin{equation}
N_2 \leq \|\nabla u\|_{L^\infty} \|\nabla \omega\|_{L^2}^2. \tag{4.17}
\end{equation}

Applying $\partial_k$ ($k = 1, 2$) to the temperature equation of (1.2), we deduce $\partial_k \theta$ satisfies
\begin{equation}
\partial_t \partial_k \theta + u \cdot \nabla \partial_k \theta + \partial_k u \cdot \nabla \theta = 0. \tag{4.18}
\end{equation}

Similarly, we can prove
\begin{equation}
\frac{1}{2} \frac{d}{dt} \|\partial_k \theta(t)\|_{L^2}^2 \leq C \|\nabla u\|_{L^\infty} \|\nabla \theta\|_{L^2}^2. \tag{4.19}
\end{equation}

Inserting (4.16) and (4.17) into (4.15) and combining with (4.19), we can deduce
\begin{equation}
\frac{d}{dt} (\|\nabla \omega(t)\|_{L^2}^2 + \|\nabla \theta(t)\|_{L^2}^2) + \|\partial_1 \nabla \omega(t)\|_{L^2}^2 \leq C \|\nabla u\|_{L^\infty} (\|\nabla \omega\|_{L^2}^2 + \|\nabla \theta\|_{L^2}^2). \tag{4.20}
\end{equation}

Then by virtue of the Grönwall’s Lemma and Lemma 4.1,
\begin{equation}
\|\nabla \omega\|_{L_t^\infty(L^2)}^2 + \|\nabla \theta\|_{L_t^\infty(L^2)}^2 + \|\partial_1 \nabla \omega\|_{L_t^\infty(L^2)}^2 \leq C(t),
\end{equation}

which completes the proof of this lemma. \qed

The following lemma establishes the $L^p$ estimate of $(\nabla \omega, \nabla \theta)$.

Lemma 4.4 Assume $\omega_0 \in H^s, \theta_0 \in H^\beta$ with $\beta > s > \frac{1}{2}$. $\nabla \omega_0, \nabla \theta_0 \in L^p$ ($2 < p < \infty$), then the solution $(\omega, \theta)$ satisfies
\begin{equation}
\|\nabla \omega\|_{L_t^\infty(L^p)}^2 + \|\nabla \theta\|_{L_t^\infty(L^p)}^2 \leq C(t). \tag{4.20}
\end{equation}
Proof Multiplying by \(|\partial_k \omega|^{p-2} \partial_k \omega\) to (4.14) and integrating over \(\mathbb{R}^2\) with respect to \(x\), by Hölder inequality and Young’s inequality,

\[
\frac{1}{p} \frac{d}{dt} \|\partial_k \omega(t)\|_{L^p}^p + (p - 1) \int_{\mathbb{R}^2} |\partial_1 \partial_k \omega|^2 |\partial_k \omega|^{p-2} \, dx
\]

\[
= \int_{\mathbb{R}^2} \partial_1 \partial_k \theta |\partial_k \omega|^{p-2} \partial_k \omega \, dx - \int_{\mathbb{R}^2} \partial_k u \cdot \nabla \omega |\partial_k \omega|^{p-2} \partial_k \omega \, dx
\]

\[
= -(p - 1) \int_{\mathbb{R}^2} \partial_1 \partial_k \theta |\partial_k \omega|^{p-2} \partial_k \omega \, dx - \int_{\mathbb{R}^2} \partial_k u \cdot \nabla \omega |\partial_k \omega|^{p-2} \partial_k \omega \, dx
\]

\[
\leq \frac{p-1}{2} \int_{\mathbb{R}^2} |\partial_1 \partial_k \omega|^2 |\partial_k \omega|^{p-2} \, dx + C \int_{\mathbb{R}^2} |\partial_k \theta|^2 |\partial_k \omega|^{p-2} \, dx
\]

\[
\leq \frac{p-1}{2} \int_{\mathbb{R}^2} |\partial_1 \partial_k \omega|^2 |\partial_k \omega|^{p-2} \, dx + C \|\nabla\theta\|_{L^p}^2 \|\nabla\omega\|_{L^p}^{p-2} + \|\nabla u\|_{L^\infty} \|\nabla\omega\|_{L^p}.
\]

Thus, we obtain

\[
\frac{d}{dt} \|\nabla \omega(t)\|_{L^p}^2 \leq C \|\nabla \theta\|_{L^p}^2 + \|\nabla u\|_{L^\infty} \|\nabla\omega\|_{L^p}^2.
\] (4.21)

Similarly, we can prove

\[
\frac{d}{dt} \|\nabla \theta(t)\|_{L^p}^2 \leq C \|\nabla u\|_{L^\infty} \|\nabla\theta\|_{L^p}^2.
\] (4.22)

Combining (4.21) with (4.22), we can deduce

\[
\frac{d}{dt} (\|\nabla \omega(t)\|_{L^p}^2 + \|\nabla \theta(t)\|_{L^p}^2) \leq C (1 + \|\nabla u\|_{L^\infty}) (\|\nabla \omega\|_{L^p}^2 + \|\nabla \theta\|_{L^p}^2).
\]

Then by virtue of the Grönwall’s Lemma and Lemma 4.1,

\[
\|\nabla \omega\|_{L^p}^2 + \|\nabla \theta\|_{L^p}^2 \leq C(t),
\]

which completes the proof of this lemma. \(\square\)

Next, we discuss the higher-order regularity estimate for \((\omega, \theta)\). Applying \(|D|^s (s > 0)\) to the vorticity equation (1.5) and temperature equation of (1.2), we can get \((|D|^s \omega, |D|^s \theta)\) satisfies the following system,

\[
\begin{aligned}
\partial_t |D|^s \omega + u \cdot \nabla |D|^s \omega - \partial_2 |D|^s \omega = \partial_1 |D|^s \theta - [|D|^s, u \cdot \nabla] \omega,
\partial_t |D|^s \theta + u \cdot \nabla |D|^s \theta = -[|D|^s, u \cdot \nabla] \theta.
\end{aligned}
\] (4.23)

The follow lemma gives the \(H^s (s > 1)\) estimate of \((\omega, \theta)\).
Lemma 4.5 Assume \( \omega_0 \in \dot{W}^{1,p} \cap H^s \) and \( \theta_0 \in \dot{W}^{1,p} \cap H^s \) \( (2 < p < \infty, s > 1) \), then the solution \((\omega, \theta)\) satisfies

\[
\|D|^s \omega\|_{L_t^\infty(L^2)}^2 + \|D|^s \theta\|_{L_t^\infty(L^2)}^2 + \|\partial_1 |D|^s \omega\|_{L_t^2(L^2)}^2 \leq C(t). \tag{4.24}
\]

**Proof** Taking \( L^2 \) inner product with \((|D|^s \omega, |D|^s \theta)\) and adding them up, we have

\[
\frac{1}{2} \frac{d}{dt} (\|D|^s \omega(t)\|_{L^2}^2 + \|D|^s \theta(t)\|_{L^2}^2) + \|\partial_1 |D|^s \omega\|_{L^2}^2 \\
= \int_{\mathbb{R}^2} \partial_1 |D|^s \theta |D|^s \omega \ dx - \int_{\mathbb{R}^2} [\|D|^s, u \cdot \nabla] \omega |D|^s \omega \ dx \\
- \int_{\mathbb{R}^2} [\|D|^s, u \cdot \nabla] \theta |D|^s \theta \ dx
\]

\( \triangleq K_1 + K_2 + K_3. \)

For \( K_1 \), after integration by parts and Young’s inequality,

\[
K_1 \leq \frac{1}{2} \|D|^s \theta\|_{L^2}^2 + \frac{1}{2} \|\partial_1 |D|^s \omega\|_{L^2}^2. \tag{4.26}
\]

For \( K_2 \), by virtue of the H"older inequality and inequality (2.9) in Lemma 2.7,

\[
K_2 \leq \|\|D|^s, u \cdot \nabla\|_{L^2} \|D|^s \omega\|_{L^2} \\
\leq C(\|\nabla u\|_{L^\infty} \|\|D|^{s-1} \nabla \|_{L^2} + \|D|^s u\|_{L^p} \|\nabla \omega\|_{L^{p'}}) \|D|^s \omega\|_{L^2},
\]

with \( \frac{1}{p} + \frac{1}{p'} = \frac{1}{2}, p \in (2, \infty) \). By interpolation

\[
\|f\|_{L^p} \leq C \|f\|_{L^2}^{\frac{2}{p}} \|\nabla f\|_{L^{2^{*}}}^{\frac{1}{p'}},
\]

this clearly forces

\[
\|D|^s u\|_{L^p} \leq C \|D|^s u\|_{L^2}^{\frac{2}{p}} \|\nabla D|^s u\|_{L^2}^{\frac{1}{2^{*}}} \leq C(\|u\|_{L^2} + \|D|^s \omega\|_{L^2}).
\]

Thus, we have

\[
K_2 \leq C(\|\nabla u\|_{L^\infty} + \|\nabla \omega\|_{L^{p'}}) \times (\|D|^s \omega\|_{L^2}^2 + 1). \tag{4.27}
\]

In the same way,

\[
K_3 \leq C(\|\nabla u\|_{L^\infty} + \|\nabla \theta\|_{L^{p'}}) \times (\|D|^s \omega\|_{L^2}^2 + \|D|^s \theta\|_{L^2}^2 + 1). \tag{4.28}
\]

Inserting (4.26), (4.27) and (4.28) into (4.25), making use of Lemma 4.4 and Grönewall’s Lemma, we can deduce

\[
\|D|^s \omega\|_{L_t^\infty(L^2)}^2 + \|D|^s \theta\|_{L_t^\infty(L^2)}^2 + \|\partial_1 |D|^s \omega\|_{L_t^2(L^2)}^2 \leq C(t),
\]
which completes the proof of this lemma. □

4.4 A Priori Estimates for the Higher-Order Striated Regularity

In this subsection, we will give the higher-order estimates of the vector field $X$. The first lemma asserts the $H^1$ estimate of $X$.

**Lemma 4.6** Let $\omega_0 \in H^1$, $\theta_0 \in H^1$ and $X_0 \in H^1$, then we have

$$\| \nabla X \|_{L^\infty_t(L^2)}^2 \leq C(t). \quad (4.29)$$

**Proof** Applying $\partial_k (k = 1, 2)$ to the first equation of (1.8), we can obtain $\partial_k X$ satisfies

$$\partial_t \partial_k X + u \cdot \nabla \partial_k X + \partial_k u \cdot \nabla X = \partial_k X u. \quad (4.30)$$

Multiplying $\partial_k X$ to (4.30) and integrating over $\mathbb{R}^2$ with respect to $x$, we have

$$\frac{1}{2} \frac{d}{dt} \| \partial_k X(t) \|_{L^2}^2 = \int_{\mathbb{R}^2} \partial_k \partial_k u \cdot \partial_k X \, dx - \int_{\mathbb{R}^2} \partial_k u \cdot \nabla \cdot \partial_k X \, dx$$

$$= \int_{\mathbb{R}^2} \partial_k X \cdot \nabla u \cdot \partial_k X \, dx + \int_{\mathbb{R}^2} X \cdot \nabla \partial_k u \cdot \partial_k X \, dx$$

$$- \int_{\mathbb{R}^2} \partial_k u \cdot \nabla X \cdot \partial_k X \, dx \quad (4.31)$$

By Hölder inequality, $B_1$ can be bounded by

$$B_1 = \int_{\mathbb{R}^2} \partial_k X \cdot \nabla u \cdot \partial_k X \, dx \leq \| \nabla u \|_{L^\infty} \| \nabla X \|_{L^2}^2. \quad (4.32)$$

Similarly,

$$B_3 = - \int_{\mathbb{R}^2} \partial_k u \cdot \nabla X \cdot \partial_k X \, dx \leq \| \nabla u \|_{L^\infty} \| \nabla X \|_{L^2}^2. \quad (4.33)$$

Then by virtue of anisotropic Hölder inequality,

$$B_2 = \int_{\mathbb{R}^2} X \cdot \nabla \partial_k u \cdot \partial_k X \, dx$$

$$\leq C \| X \|_{L^\infty_t(L^2)} \| \partial_k \nabla u \|_{L^2_t(L^2)} \| \partial_k X \|_{L^2(\mathbb{R}^2)} \quad (4.34)$$

$$\leq C \| X \|_{L^2}^{\frac{1}{2}} \| \partial_2 X \|_{L^2}^{\frac{1}{2}} \| \nabla \omega \|_{L^2}^{\frac{1}{2}} \| \partial_1 \nabla \omega \|_{L^2}^{\frac{1}{2}} \| \partial_k X \|_{L^2}$$

$$\leq C (\| \nabla \omega \|_{L^2} + \| \partial_1 \nabla \omega \|_{L^2}) \times (\| X \|_{L^2}^{\frac{1}{2}} + \| \nabla X \|_{L^2}^{\frac{1}{2}}).$$
After substituting (4.32), (4.34) and (4.33) into (4.31), we find that
\[
\frac{d}{dt} \| \nabla X(t) \|_{L^2}^2 \leq C(\| \nabla u \|_{L^\infty} + \| \nabla \omega \|_{L^2} + \| \partial_1 \nabla \omega \|_{L^2}) \times (\| X \|_{L^2}^2 + \| \nabla X \|_{L^2}^2).
\]
Combining with estimates (4.7) and (4.35), using Gronwall’s Lemma and by Lemmas 4.1 and 4.15, we can deduce
\[
\| \partial_1 X u \|_{L^\infty_t(L^2)}^2 + \| \partial_1 \partial_1 X u \|_{L^2_t(L^2)}^2 + \| \nabla X \|_{L^\infty_t(L^2)}^2 \leq C(t),
\]
which completes the proof of this lemma. \(\square\)

The next lemma shows the \(H^s\) (\(s > 1\)) estimate for \(X\).

**Lemma 4.7** Assume \(\omega_0 \in \dot{W}^{1,p} \cap H^s, \theta_0 \in \dot{W}^{1,p} \cap H^s, X_0 \in H^s\) (\(2 < p < \infty, s > 1\)), then we have
\[
\| |D|^s X \|_{L^\infty_t(L^2)}^2 \leq C(t).
\]

**Proof** Applying \(|D|^s\) to the first equation of (1.8), making use of the definition of commutator, we can obtain \(|D|^s X\) satisfies the following equation
\[
\partial_t |D|^s X + u \cdot \nabla |D|^s X = -[|D|^s, u \cdot \nabla] X + |D|^s (X \cdot \nabla u).
\]
Taking \(L^2\) inner product with \(|D|^s X\),
\[
\frac{1}{2} \frac{d}{dt} (\| |D|^s X(t) \|_{L^2}^2) = -\int_{\mathbb{R}^2} [|D|^s, u \cdot \nabla] X \cdot |D|^s X \, dx + \int_{\mathbb{R}^2} |D|^s (X \cdot \nabla u) \cdot |D|^s X \, dx \triangleq M_1 + M_2.
\]
For \(M_1\), by Hölder inequality and inequality (2.9) in Lemma 2.7,
\[
M_1 \leq \| |D|^s, u \cdot \nabla \|_{L^2} \| |D|^s X \|_{L^2} \leq C(\| \nabla u \|_{L^\infty} \| |D|^{s-1} \nabla X \|_{L^2} + \| |D|^s u \|_{L^p} \| \nabla X \|_{L^{p'}}) \| |D|^s X \|_{L^2},
\]
with \(\frac{1}{p} + \frac{1}{p'} = \frac{1}{2}, p \in (2, \infty).\) Choosing \(p\) such that
\[
\| \nabla X \|_{L^{p'}} \leq C \| |D|^s X \|_{L^2},
\]
and noticing that by interpolation
\[
\| |D|^s u \|_{L^p} \leq C \| |D|^s u \|_{L^2}^{\frac{2}{p}} \| |D|^{s+1} u \|_{L^2}^{1-\frac{2}{p}} \leq C(\| u \|_{L^2} + \| |D|^s \omega \|_{L^2}),
\]
one can deduce,
\[
M_1 \leq C(\| \nabla u \|_{L^\infty} + \| |D|^s \omega \|_{L^2} + 1) \| |D|^s X \|_{L^2}^2.
\]
Next we estimate $M_2$, making use of the Hölder inequality and inequality (2.10) in Lemma 2.7,

$$M_2 \leq \| |D|^{s}(X \cdot \nabla u)\|_{L^2} \| |D|^{s}X\|_{L^2}$$

$$\leq C \left( \| X\|_{L^{\infty}} \| |D|^{s} \nabla u\|_{L^2} + \| |D|^{s}X\|_{L^2} \| \nabla u\|_{L^{\infty}} \right) \| |D|^{s}X\|_{L^2}.$$  

By Sobolev embedding,

$$\| X\|_{L^{\infty}} \leq C \| |D|^{s}X\|_{L^2}, \quad \text{for } s > 1.$$  

Thus, we have

$$M_2 \leq C(\| |D|^{s}\omega\|_{L^2} + \| \nabla u\|_{L^{\infty}}) \| |D|^{s}X\|_{L^2}^2. \quad (4.40)$$

Inserting (4.39) and (4.40) into (4.38), using Grönwall’s Lemma, we can deduce

$$\| |D|^{s}X\|_{L^\infty(L^2)}^2 \leq C(t),$$

which completes the proof of this lemma. \qed

4.5 The Temperature Patch Problem

This subsection is devoted to the proof of Corollary 1.2. Because most of the proof is the same as Corollary 1.1, we just need to verify inequality (1.14). Choosing two points arbitrarily so that $x_1 \in D^{-}_\varepsilon$, $x_2 \in D^{+}_\varepsilon$, consider the difference

$$\frac{|x_1 - x_2|}{\| \nabla \psi^-\|_{L^{\infty}}} \leq |\psi(x_1, t) - \psi(x_2, t)| \leq \| \nabla \psi\|_{L^{\infty}} |x_1 - x_2|, \quad \text{for any } t > 0. \quad (4.41)$$

Noticing that from (1.6), we have

$$\| \nabla \psi^-\|_{L^{\infty}} \leq e^{\int_0^t \| \nabla u(\tau)\|_{L^{\infty}} \, d\tau}. \quad (4.42)$$

Then inserting estimate (4.42) into (4.41) and taking infimum of $x_1, x_2$, we can obtain

$$2\varepsilon e^{\int_0^t \| \nabla u(\tau)\|_{L^{\infty}} \, d\tau} \leq |d(t)| \leq 2\varepsilon e^{\int_0^t \| \nabla u(\tau)\|_{L^{\infty}} \, d\tau}.$$  

which is the desired bound (1.14).

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A. Appendix

The first two parts of this appendix are to give the proof of Lemmas 2.5 and 2.8. Then, we show the proof of the results given in Remark 2 in the last of this appendix.

Proof of Lemma 2.5 The proof of (2.6) can be found in Danchin and Paicu (2011) which used the standard Bony’s decomposition (see Chemin 1998; Bahouri et al. 2011). Here we focus on proving (2.7) using the anisotropic idea. Firstly, we divide the first term of (2.7) into two terms,

$$ - \int_{\mathbb{R}^2} \Delta_q (u \cdot \nabla f) \Delta_q f \, dx = - \int_{\mathbb{R}^2} \Delta_q (u_1 \partial_1 f) \Delta_q f \, dx - \int_{\mathbb{R}^2} \Delta_q (u_2 \partial_2 f) \Delta_q f \, dx $$

$$ \triangleq P + Q. $$

For $P$, by Bony’s decomposition, we can divide it into the following three terms,

$$ - \int_{\mathbb{R}^2} \Delta_q (u_1 \partial_1 f) \Delta_q f \, dx $$

$$ = - \sum_{|k-q| \leq 2} \int_{\mathbb{R}^2} \Delta_q (S_{k-1} u_1 \Delta_k \partial_1 f) \Delta_q f \, dx $$

$$ - \sum_{|k-q| \leq 2} \int_{\mathbb{R}^2} \Delta_q (\Delta_k u_1 S_{k-1} \partial_1 f) \Delta_q f \, dx \quad (A.1) $$

$$ - \sum_{k \geq q-1} \sum_{|k-l| \leq 1} \int_{\mathbb{R}^2} \Delta_q (\Delta_k u_1 \Delta_l \partial_1 f) \Delta_q f \, dx $$

$$ \triangleq P_1 + P_2 + P_3. $$

For $P_1$, we can rewrite it as

$$ P_1 = - \sum_{|q-k| \leq 2} \int_{\mathbb{R}^2} \Delta_q (S_{k-1} u_1 \partial_1 \Delta_k f) \Delta_q f \, dx $$

$$ = - \sum_{|q-k| \leq 2} \int_{\mathbb{R}^2} [\Delta_q, S_{k-1} u_1 \partial_1] \Delta_k f \Delta_q f \, dx $$

$$ - \sum_{|q-k| \leq 2} \int_{\mathbb{R}^2} S_{k-1} u_1 \partial_1 \Delta_q \Delta_k f \Delta_q f \, dx $$
where we have used the fact \( \sum |q-k| \leq 2, \partial_1 \Delta_q \Delta_k f = \partial_1 \Delta_q f \). For \( P_{11} \), by Hölder inequality,

\[
|P_{11}| \leq \sum_{|q-k| \leq 2} \left| \int_{\mathbb{R}^2} [\Delta_q, S_{k-1} u^1 \partial_1] \Delta_k f \Delta_k f \, dx \right| \\
\leq C \sum_{|q-k| \leq 2} \| [\Delta_q, S_{k-1} u^1 \partial_1] \Delta_k f \|_{L^2} \| \Delta_q f \|_{L^2}.
\]

According to the definition of \( \Delta_q \),

\[
[\Delta_q, S_{k-1} u^1 \partial_1] \Delta_k f = \int_{\mathbb{R}^2} \phi_q(x-y)(S_{k-1} u^1(y) \partial_1 \Delta_k f(y)) \, dy \\
- S_{k-1} u^1(x) \int_{\mathbb{R}^d} \phi_q(x-y) \partial_1 \Delta_k f(y) \, dy \\
= \int_{\mathbb{R}^2} \phi_q(x-y)(S_{k-1} u^1(y) - S_{k-1} u^1(x)) \partial_1 \Delta_k f(y) \, dy \\
= \int_{\mathbb{R}^2} \phi_q(x-y) \int_0^1 (y-x) \cdot \nabla S_{k-1} u^1(sy + (1-s)x) \, ds \partial_1 \Delta_k f(y) \, dy \\
= \int_{\mathbb{R}^2} \int_0^1 \phi_q(z) z \cdot \nabla S_{k-1} u^1(x-sz) \partial_1 \Delta_k f(x-z) \, ds \, dz,
\]

where \( \phi_j(x) \triangleq 2^jd \mathcal{F}^{-1}(\varphi)(2^j x) \). Thus, we have by Hölder inequality and Bernstein inequality,

\[
\| [\Delta_q, S_{k-1} u^1 \partial_1] \Delta_k f \|_{L^2} \\
= \left\| \int_{\mathbb{R}^2} \int_0^1 \phi_q(z) z \cdot \nabla S_{k-1} u^1(x-sz) \partial_1 \Delta_k f(x-z) \, ds \, dz \right\|_{L^2} \\
\leq C \int_{\mathbb{R}^2} \| \phi_q(z) \|_{L^1} \| z \| \| \nabla S_{k-1} u^1(x-sz) \|_{L^\infty} \| \partial_1 \Delta_k f(x-z) \|_{L^2} \\
\leq C \int_{\mathbb{R}^2} \| \phi_q(z) \|_{L^1} \| z \| \| \nabla S_{k-1} u^1 \|_{L^\infty} \| \partial_1 \Delta_k f \|_{L^2} \\
\leq C 2^{-q} 2^k \| \nabla S_{k-1} u^1 \|_{L^2} \| \partial_1 \Delta_k f \|_{L^2} \\
\leq C 2^{k-q} \| \omega \|_{L^2} \| \partial_1 \Delta_k f \|_{L^2}.
\]
Then, we obtain

\[
|P_{11}| \leq C \sum_{|q-k| \leq 2} \| [\Delta_q, S_{k-1} u^1 \partial_1] \Delta_k f \|_{L^2} \| \Delta_q f \|_{L^2} \\
\leq C \sum_{|q-k| \leq 2} 2^{k-q} \| \omega \|_{L^2} \| \partial_1 \Delta_k f \|_{L^2} \| \Delta_q f \|_{L^2} \\
\leq C b q 2^{-2qs} \| \omega \|_{L^2} \| f \|_{H^s} \| \partial_1 f \|_{H^s}.
\]

For \( P_{12} \), by Hölder inequality and Bernstein inequality,

\[
|P_{12}| = \sum_{|q-k| \leq 2} \left| \int_{\mathbb{R}^2} ((S_{k-1} u^1 - S_q u^1) \partial_1 \Delta_q \Delta_k f) \Delta_q f \, dx \right| \\
\leq C \sum_{|q-k| \leq 2} \| (S_{k-1} u^1 - S_q u^1) \partial_1 \Delta_q \Delta_k f \|_{L^1} \| \Delta_q f \|_{L^\infty} \\
\leq C \sum_{|q-k| \leq 2} \| \Delta_k u^1 \|_{L^2} \| \Delta_q \Delta_k \partial_1 f \|_{L^2} 2^q \| \Delta_q f \|_{L^2}.
\]

For the case \( k = -1 \), by Bernstein inequality,

\[
|P_{12}| \leq C \| \Delta_{-1} u^1 \|_{L^2} 2^{-1} \| \Delta_q \Delta_{-1} f \|_{L^2} 2^{-1} \| \Delta_q f \|_{L^2} \\
\leq C b q 2^{-2qs} \| u^1 \|_{L^2} \| f \|_{H^s}^2.
\]

For the case \( k \geq 0 \), by Bernstein inequality,

\[
|P_{12}| \leq C \sum_{|q-k| \leq 2} 2^{-k} \| \nabla \Delta_k u^1 \|_{L^2} \| \Delta_q \Delta_k \partial_1 f \|_{L^2} 2^q \| \Delta_q f \|_{L^2} \\
\leq C \sum_{|q-k| \leq 2} 2^{-k} \| \omega \|_{L^2} 2^q \| \Delta_q \partial_1 f \|_{L^2} \| \Delta_q f \|_{L^2} \\
\leq C b q 2^{-2qs} \| \omega \|_{L^2} \| f \|_{H^s} \| \partial_1 f \|_{H^s}.
\]

From the above, it follows that

\[
|P_1| \leq C b q 2^{-2qs} (\| u \|_{L^2} + \| \omega \|_{L^2}) (\| f \|_{H^s}^2 + \| f \|_{H^s} \| \partial_1 f \|_{H^s}). \tag{A.2}
\]

For \( P_2 \), we can bound it by Hölder inequality that

\[
|P_2| \leq C \sum_{|q-k| \leq 2} \| \Delta_k u^1 \|_{L^2} \| \partial_1 S_{k-1} f \|_{L^\infty} \| \Delta_q f \|_{L^2}.
\]
Applying Bernstein inequality, similar as \( P_{12} \),

\[
|P_2| \leq C \sum_{|q-k| \leq 2} \| \Delta_k u^1 \|_{L^2} \sum_{m \leq k-2} 2^m \| \Delta_m \partial_1 f \|_{L^2} \| \Delta_q f \|_{L^2} \\
\leq C \sum_{|q-k| \leq 2} 2^q \| \Delta_k u^1 \|_{L^2} \sum_{m \leq q-2} 2^{m-q} \| \Delta_m \partial_1 f \|_{L^2} \| \Delta_q f \|_{L^2} \\
\leq C 2^{-q^s} \sum_{|q-k| \leq 2} 2^{q-k} 2^k \| \Delta_k u^1 \|_{L^2} \sum_{m \leq q-2} 2^{(m-q)(1-s)} 2^{m} \| \Delta_m \partial_1 f \|_{L^2} \| \Delta_q f \|_{L^2} \\
\leq C b_q 2^{-2q^s} (\| u^1 \|_{L^2} + \| \omega \|_{L^2}) \| f \|_{H^s} \| \partial_1 f \|_{H^s},
\]

where we have used discrete Young’s inequality in the last step.

Next we estimate \( P_3 \). By Hölder inequality and Bernstein inequality,

\[
|P_3| \leq \left| \sum_{k \geq q-1} \sum_{|k-l| \leq 1} \int_{\mathbb{R}^2} \Delta_q (\Delta_k u^1 \partial_1 \Delta_q f) \Delta_q f \ dx \right| \\
\leq C \sum_{k \geq q-1} \sum_{|k-l| \leq 1} \| \Delta_q (\Delta_k u^1 \partial_1 f) \|_{L^1} \| \Delta_q f \|_{L^\infty} \\
\leq C 2^q \sum_{k \geq q-1} 2^{-k} 2^k \| \Delta_k u^1 \|_{L^2} \| \Delta_k \partial_1 f \|_{L^2} \| \Delta_q f \|_{L^2} \\
\leq C 2^{-q^s} \sum_{k \geq q-1} 2^{(q-k)(1+s)} 2^k \| \Delta_k \partial_1 f \|_{L^2} (\| u^1 \|_{L^2} + \| \omega \|_{L^2}) \| \Delta_q f \|_{L^2} \\
\leq C b_q 2^{-2q^s} (\| u^1 \|_{L^2} + \| \omega \|_{L^2}) \| f \|_{H^s} \| \partial_1 f \|_{H^s},
\]

where discrete Young’s inequality has been used in the last two lines.

For \( Q \), we can also divide it into three parts,

\[
- \int_{\mathbb{R}^2} \Delta_q (u^2 \partial_2 f) \Delta_q f \ dx = Q_1 + Q_2 + Q_3,
\]

with

\[
Q_1 = - \sum_{|k-q| \leq 2} \int_{\mathbb{R}^2} \Delta_q (S_{k-1} u^2 \Delta_k \partial_2 f) \Delta_q f \ dx,
\]

\[
Q_2 = - \sum_{|k-q| \leq 2} \int_{\mathbb{R}^2} \Delta_q (\Delta_k u^2 S_{k-1} \partial_2 f) \Delta_q f \ dx
\]

and

\[
Q_3 = - \sum_{k \geq q-1} \sum_{|k-l| \leq 1} \int_{\mathbb{R}^2} \Delta_q (\Delta_k u^2 \Delta_l \partial_2 f) \Delta_q f \ dx.
\]
Similar as $P_1$, we can rewrite $Q_1$ as

$$
Q_1 = - \sum_{|q-k| \leq 2} \int_{\mathbb{R}^2} [\Delta_q, S_{k-1} u^2 \partial_2] \Delta_k f \Delta_q f \, dx
- \sum_{|q-k| \leq 2} \int_{\mathbb{R}^2} (S_{k-1} u^2 - S_q u^2) \partial_2 \Delta_q \Delta_k f \Delta_q f \, dx
- \int_{\mathbb{R}^2} S_q u^2 \partial_2 \Delta_q f \Delta_q f \, dx
\triangleq Q_{11} + Q_{12} + Q_{13}.
$$

Here, we should notice that $P_{13} + Q_{13} = 0$ because of the divergence-free condition of $u$, so we do not need to estimate these two terms.

For $Q_{11}$, by Hölder inequality,

$$
|Q_{11}| \leq \sum_{|q-k| \leq 2} \left| \int_{\mathbb{R}^2} [\Delta_q, S_{k-1} u^2 \partial_2] \Delta_k f \Delta_q f \, dx \right|
\leq C \sum_{|q-k| \leq 2} \| [\Delta_q, S_{k-1} u^2 \partial_2] \Delta_k f \|_{L^2} \| \Delta_q f \|_{L^2}.
$$

According to the definition of $\Delta_q$ and similar as $P_{11}$,

$$
[\Delta_q, S_{k-1} u^2 \partial_2] \Delta_k f = \int_{\mathbb{R}^2} \int_0^1 \phi_q(z) z \cdot \nabla S_{k-1} u^2 (x - s z) \partial_2 \Delta_k f (x - z) \, ds \, dz.
$$

Making use of the anisotropic Hölder inequality and Bernstein inequality,

$$
\| [\Delta_q, S_{k-1} u^2 \partial_2] \Delta_k f \|_{L^2}
= \left\| \int_{\mathbb{R}^2} \int_0^1 \phi_q(z) z \cdot \nabla S_{k-1} u^2 (x - s z) \partial_2 \Delta_k f (x - z) \, ds \, dz \right\|_{L^2}
\leq C \int_{\mathbb{R}^2} \| \phi_q(z) \|_{L^2} \, dz \| \nabla S_{k-1} u^2 (x - s z) \|_{L^\infty_{x_2}(L^\infty_{x_1})} \| \partial_2 \Delta_k f (x - z) \|_{L^2_{x_2}(L^\infty_{x_1})}
\leq C 2^{-q} \| \nabla S_{k-1} u^2 \|_{L^2} \| \partial_2 \nabla S_{k-1} u^2 \|_{L^2} \| \partial_2 \Delta_k f \|_{L^2} \| \partial_1 \partial_2 \Delta_k f \|_{L^2} \frac{1}{2}.
$$

Noticing that by Biot–Savart law $u^2 = \partial_1 \Delta^{-1} \omega$, and combining with the boundedness of Riesz transform in $L^2$,

$$
\| [\Delta_q, S_{k-1} u^2 \partial_2] \Delta_k f \|_{L^2} \leq C 2^{-q} \| \omega \|_{L^2} \| \partial_2 \nabla \partial_1 \Delta^{-1} \omega \|_{L^2} \| \Delta_k f \|_{L^2} \| \partial_1 \Delta_k f \|_{L^2} \frac{1}{2}
\leq C 2^{-q} \| \omega \|_{L^2} \| \partial_1 \omega \|_{L^2} \| \Delta_k f \|_{L^2} \| \partial_1 \Delta_k f \|_{L^2} \frac{1}{2}.
$$

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Then, $Q_{11}$ is bounded by

$$|Q_{11}| \leq C \sum_{|q-k| \leq 2} \|\Delta_q S_{k-1}u^2 \partial_2 \Delta_k f \|_{L^2} \|\Delta_q f \|_{L^2}$$

$$\leq C \sum_{|q-k| \leq 2} 2^{k-q} \|\omega\|_{L^\infty}^{1/2} \|\partial_1 \omega\|_{L^\infty}^{1/2} \|\Delta_k f \|_{L^2} \|\Delta_q f \|_{L^2}$$

$$\leq C b_q 2^{-2q} \|\omega\|_{L^2}^{1/2} \|\partial_1 \omega\|_{L^2}^{1/2} \|f\|_{H^s} \|\partial_1 f\|_{H^s}.$$

For $Q_{12}$, by the anisotropic Hölder inequality and interpolation inequality,

$$|Q_{12}| = \sum_{|q-k| \leq 2} \left| \int_{\mathbb{R}^2} \left( (S_{k-1}u^2 - S_q u^2) \partial_2 \Delta_q \Delta_k f \right) \Delta_q f \, dx \right|$$

$$\leq C \sum_{|q-k| \leq 2} \|\Delta_q \Delta_k f \|_{L^2} \|\Delta_q f \|_{L^2}$$

$$\leq C \sum_{|q-k| \leq 2} \|\Delta_k u^2 \|_{L^2_{x_1} (L^\infty_{x_2})} \|\Delta_q \Delta_k \partial_2 f \|_{L^2_{x_1} (L^\infty_{x_2})} \|\Delta_q f \|_{L^2}$$

$$\leq C \sum_{|q-k| \leq 2} \|\Delta_k u^2 \|_{L^2} \|\Delta_q \Delta_k \partial_2 u^2 \|_{L^2} \|\Delta_k \Delta_q \partial_2 f \|_{L^2} \|\Delta_k \partial_1 \partial_2 f \|_{L^2} \|\Delta_q f \|_{L^2}$$

For the case $k = -1$, by Bernstein inequality,

$$|Q_{12}| \leq C \|\Delta_{-1} u^2 \|_{L^2} \|\Delta_q \Delta_{-1} f \|_{L^2} \|\Delta_q f \|_{L^2}$$

$$\leq C b_q 2^{-2q} \|u^2\|_{L^2} \|f\|_{H^s}.$$

For the case $k \geq 0$, by Bernstein inequality and the relation $u^2 = \partial_1 \Delta^{-1} \omega$,

$$|Q_{12}| \leq C \sum_{|q-k| \leq 2} 2^{q-k} \|\nabla \Delta_k u^2 \|_{L^2}^{1/2} \|\nabla \Delta_k \partial_1 \Delta^{-1} \omega \|_{L^2}^{1/2} \|\Delta_q \partial_1 f \|_{L^2}^{1/2} \|\Delta_q f \|_{L^2}^{1/2}$$

$$\leq C \sum_{|q-k| \leq 2} 2^{q-k} \|\omega\|_{L^2}^{1/2} \|\partial_1 \omega\|_{L^2}^{1/2} \|\Delta_q \partial_1 f \|_{L^2}^{1/2} \|\Delta_q f \|_{L^2}^{1/2}$$

$$\leq C b_q 2^{-2q} \|\omega\|_{L^2} \|\partial_1 \omega\|_{L^2} \|f\|_{H^s} \|\partial_1 f\|_{H^s}.$$

Then, it follows that

$$|Q_1| \leq C b_q 2^{-2q} \left( \|u\|_{L^2} + \|\omega\|_{L^2} \|\partial_1 \omega\|_{L^2} \right) \left( \|f\|_{H^s} + \|f\|_{H^s} \|\partial_1 f\|_{H^s} \right). \quad (A.6)$$
Similar as $Q_{12}$, applying anisotropic Hölder inequality and Bernstein inequality, $Q_2$ can be bounded by

$$|Q_2| \leq C \sum_{|q-k| \leq 2} \| \Delta_k u^2 \|_{L^\infty_x(L^2)} \| \partial_2 S_{k-1} f \|_{L^2_x(L^\infty)} \| \Delta_q f \|_{L^2}$$

$$\leq C \sum_{|q-k| \leq 2} \| \Delta_k u^2 \|_{L^2} \| \Delta_k \partial_2 u^2 \|_{L^2} \| \partial_2 S_{k-1} f \|_{L^2} \| \partial_1 \partial_2 S_{k-1} f \|_{L^2} \| \Delta_q f \|_{L^2}$$

$$\leq C b_q 2^{-2q} \| u \|_{L^2} \| f \|_{H^s}^2 + C \| \partial_1 \|_{L^2} \| \partial_1 \omega \|_{L^2} \left( \sum_{m \leq q-2} 2^{m-q} \| \Delta_m f \|_{L^2} \right)^{\frac{1}{2}}$$

$$\times \left( \sum_{n \leq q-2} 2^n \| \Delta_n \partial_1 f \|_{L^2} \right)^{\frac{1}{2}} \| \Delta_q f \|_{L^2}$$

$$\leq C b_q 2^{-2q} \left( \| u \|_{L^2} + \| \partial_1 \|_{L^2} \right) (\| f \|_{H^s}^2 + \| f \|_{H^s}^2 \| \partial_1 f \|_{H^s}^2).$$

(A.7)

Finally, we estimate $Q_3$. By Hölder inequality and Bernstein inequality,

$$|Q_3| \leq \left| \sum_{k \geq q-1} \sum_{|k-l| \leq 1} \int \Delta_q (\Delta_k u^2 \partial_2 \Delta_q f) \Delta_q f \, dx \right|$$

$$\leq C \sum_{k \geq q-1} \sum_{|k-l| \leq 1} \| \Delta_q (\Delta_k u^2 \Delta_l f) \|_{L^1} \| \Delta_q f \|_{L^\infty}$$

$$\leq C 2^q \sum_{k \geq q-1} \| \Delta_k u^2 \|_{L^2} \| \Delta_k \partial_2 f \|_{L^2} \| \Delta_q f \|_{L^2}$$

(A.8)

$$\leq C 2^q \sum_{k \geq q-1} (\| u \|_{L^2} + \| \partial_1 \|_{L^2}) 2^{-2k} 2^k \| \Delta_k f \|_{L^2} \| \Delta_q f \|_{L^2}$$

$$\leq C b_q 2^{-2q} \left( \| u \|_{L^2} + \| \partial_1 \|_{L^2} \right) \| f \|_{H^s}^2.$$

Taking all these estimates into account, we can obtain

$$- \int \Delta_q (u \cdot \nabla f) \Delta_q f \, dx \leq C b_q 2^{-2q} \left( \| u \|_{L^2} + \| \partial_1 \|_{L^2} \right)$$

$$\times \left( \| f \|_{H^s}^2 + \| f \|_{H^s}^2 \| \partial_1 f \|_{H^s}^2 + \| f \|_{H^s}^2 \| \partial_1 f \|_{H^s}^2 \right),$$

which completes the proof of this lemma. \hfill \Box

**Lemma A.1** (Losing regularity estimate for transport equation) Let $\rho$ satisfy the transport equation

\[
\begin{align*}
\partial_t \rho + u \cdot \nabla \rho &= f, \\
\rho(0, x) &= \rho_0(x),
\end{align*}
\]  

(A.9)
where $\rho_0 \in B^2_{2,r}$, $f \in L^1([0, T]; B^2_{2,r})$ with $r \in [1, \infty]$. Here, $u \in L^2$ is a divergence-free vector field and for some $V(t) \in L^1([0, T])$, $v$ satisfies

$$\sup_{N \geq 0} \frac{\|\nabla S_{Nu}(t)\|_{L^\infty}}{\sqrt{1 + N}} \leq V(t).$$

Then for all $s > 0$, $\varepsilon \in (0, s)$ and $t \in [0, T]$, we have the following estimate,

$$\|\rho(t)\|_{B^2_{2,r} - \varepsilon} \leq C\left(\|\rho_0\|_{B^2_{2,r}} + \int_0^T \|f(\tau)\|_{B^2_{2,r}} \, d\tau\right)e^{C(\int_0^T V(\tau) \, d\tau)^2},$$

where $C$ is a constant independent of $T$ and $\varepsilon$.

**Proof** The case $r = \infty$ has been shown in Danchin and Paicu (2011), here we just discuss $1 \leq r < \infty$. Applying $\Delta_q$ to (2.8), we obtain

$$\partial_t \Delta_q \rho + \Delta_q (u \cdot \nabla \rho) = \Delta_q f. \quad (A.10)$$

Taking $L^2$ inner product with $\Delta_q \rho$,

$$\frac{1}{2} \frac{d}{dt} \|\Delta_q \rho\|_{L^2}^2 = -\int_{\mathbb{R}^2} \Delta_q (u \cdot \nabla \rho) \Delta_q \rho \, dx + \int_{\mathbb{R}^2} \Delta_q f \Delta_q \rho \, dx \triangleq I + II. \quad (A.11)$$

For $II$, by Hölder inequality,

$$II = \int_{\mathbb{R}^2} \Delta_q f \Delta_q \rho \leq \|\Delta_q f\|_{L^2} \|\Delta_q \rho\|_{L^2}. \quad (A.12)$$

For $I$, along a similar argument as Lemma 2.5, we can divide it as

$$I = -\int_{\mathbb{R}^2} \Delta_q (u \cdot \nabla \rho) \Delta_q \rho \, dx$$

$$= -\sum_{|k-q| \leq 2} \int_{\mathbb{R}^2} \Delta_q (S_{k-1} u \cdot \Delta_k \nabla \rho) \Delta_q \rho \, dx$$

$$-\sum_{|k-q| \leq 2} \int_{\mathbb{R}^2} \Delta_q (\Delta_k u \cdot \nabla S_{k-1} \rho) \Delta_q \rho \, dx$$

$$-\sum_{k \geq 1} \sum_{|k-l| \leq 1} \int_{\mathbb{R}^2} \Delta_q (\Delta_k u \cdot \nabla \Delta_l \rho) \Delta_q \rho \, dx$$

$$\triangleq L_1 + L_2 + L_3.$$
For $L_1$, we can rewrite it as

\[
L_1 = - \sum_{|q-k| \leq 2} \int_{\mathbb{R}^2} [\Delta_q, S_{k-1}u \cdot \nabla] \Delta_k \rho \Delta_q \rho \, dx
\]

\[
- \sum_{|q-k| \leq 2} \int_{\mathbb{R}^2} (S_{k-1}u - S_q u) \cdot \nabla \Delta_q \Delta_k \rho \Delta_q \rho \, dx
\]

\[
- \int_{\mathbb{R}^2} S_q u \cdot \nabla \Delta_q \rho \Delta_q f \, dx
\]

\[\triangleq L_{11} + L_{12} + L_{13},\]

According to divergence-free condition of $u$, it is not difficult to find that $L_{13} = 0$. For $L_{11}$, by Hölder inequality,

\[
|L_{11}| \leq \sum_{|q-k| \leq 2} \left| \int_{\mathbb{R}^2} [\Delta_q, S_{k-1}u \cdot \nabla] \Delta_k \rho \Delta_k \rho \, dx \right|
\]

\[
\leq C \sum_{|q-k| \leq 2} \| [\Delta_q, S_{k-1}u \cdot \nabla] \Delta_k \rho \|_{L^2} \| \Delta_q \rho \|_{L^2}.
\]

According to the definition of $\Delta_q$,

\[
[\Delta_q, S_{k-1}u \cdot \nabla] \Delta_k \rho = \int_{\mathbb{R}^2} \phi_q(x-y)(S_{k-1}u(y) \cdot \nabla \Delta_k \rho(y)) \, dy
\]

\[
- S_{k-1}u(x) \cdot \int_{\mathbb{R}^d} \phi_q(x-y) \nabla \Delta_k \rho(y) \, dy
\]

\[
= \int_{\mathbb{R}^2} \phi_q(x-y)(S_{k-1}u(y) - S_{k-1}u(x)) \cdot \nabla \Delta_k \rho(y) \, dy
\]

\[
= \int_{\mathbb{R}^2} \phi_q(x-y) \int_0^1 (y-x) \cdot \nabla S_{k-1}u(sy + (1-s)x) \, ds \cdot \nabla \Delta_k \rho(y) \, dy
\]

\[
= \int_{\mathbb{R}^2} \int_0^1 \phi_q(z) z \cdot \nabla S_{k-1}u(x-sz) \cdot \nabla \Delta_k \rho(x-z) \, dx \, dz.
\]

Thus, we have

\[
\| [\Delta_q, S_{k-1}u \cdot \nabla] \Delta_k \rho \|_{L^2}
\]

\[
= \left\| \int_{\mathbb{R}^2} \int_0^1 \phi_q(z) z \cdot \nabla S_{k-1}u(x-sz) \cdot \nabla \Delta_k \rho(x-z) \, dx \, dz \right\|_{L^2}
\]

\[
\leq C \int_{\mathbb{R}^2} |\phi_q(z)| |z| \, dz \| \nabla S_{k-1}u(x-sz) \|_{L^\infty} \| \nabla \Delta_k \rho(x-z) \|_{L^2}
\]

\[
\leq C 2^{-q} \int_{\mathbb{R}^2} |\phi_q(z)| |z| \, dz \| \nabla S_{k-1}u \|_{L^\infty} \| \nabla \Delta_k \rho \|_{L^2}
\]

\[
\leq C 2^{-q} \| \nabla S_{k-1}u \|_{L^\infty} 2^k \| \Delta_k \rho \|_{L^2}.
\]
Then, we obtain

\[ |L_{11}| \leq C \sum_{|q-k| \leq 2} \| [\Delta_q, S_{k-1}u \cdot \nabla] \Delta_k \rho \|_{L^2} \| \Delta_q \rho \|_{L^2} \]

\[ \leq C \sum_{|q-k| \leq 2} 2^{k-q} \| \nabla S_{k-1}u \|_{L^\infty} \| \Delta_q \rho \|_{L^2}^2 \]

\[ \leq C \sqrt{q} V(t) \| \Delta_q \rho \|_{L^2}^2 \]

\[ \leq C d_q 2^{-\sigma q} \sqrt{q} V(t) \| \rho \|_{B_{2,r}^q} \| \Delta_q \rho \|_{L^2}, \]

where \( d_q \in \ell^r \).

For \( L_{12} \), by Hölder inequality,

\[ |L_{12}| = \sum_{|q-k| \leq 2} \left| \int_{\mathbb{R}^2} ((S_{k-1}u - S_q u) \cdot \nabla \Delta_q \Delta_k \rho) \Delta_q \rho \, dx \right| \]

\[ \leq C \sum_{|q-k| \leq 2} 2^{q-k} \| \nabla \Delta_k u \|_{L^\infty} \| \Delta_q \rho \|_{L^2}^2 + \| u \|_{L^2} \| \Delta_q \rho \|_{L^2}^2 \]

\[ \leq C (\sqrt{q} + 2 V(t) + \| u \|_{L^2}) \| \Delta_q \rho \|_{L^2}^2 \]

\[ \leq C d_q 2^{-\sigma q} (\sqrt{q} + 2 V(t) + \| u \|_{L^2}) \| \rho \|_{B_{2,r}^q} \| \Delta_q \rho \|_{L^2}. \]

For \( L_2 \), we can bound it by Hölder inequality that

\[ |L_2| \leq C \sum_{|q-k| \leq 2} \| \Delta_k u \|_{L^\infty} \| \nabla S_{k-1} \rho \|_{L^2} \| \Delta_q \rho \|_{L^2}. \]

According to Bernstein inequality,

\[ |L_2| \leq C \sum_{|q-k| \leq 2} \| \Delta_k u \|_{L^\infty} \sum_{m \leq q-2} 2^m \| \Delta_m \rho \|_{L^2} \| \Delta_q \rho \|_{L^2} \]

\[ \leq C \sum_{|q-k| \leq 2} 2^q \| \Delta_k u \|_{L^\infty} \sum_{m \leq q-2} 2^{m-q} \| \Delta_m \rho \|_{L^2} \| \Delta_q \rho \|_{L^2} \]

\[ \leq C \sum_{|q-k| \leq 2} 2^q \| \Delta_k u \|_{L^\infty} \sum_{m \leq q-2} 2^{m-q} \| \Delta_m \rho \|_{L^2} \| \Delta_q \rho \|_{L^2} \]

\[ \leq C (\sqrt{q} + 2 V(t) + \| u \|_{L^2}) \sum_{m \leq q-2} 2^{m-q} \| \Delta_m \rho \|_{L^2} \| \Delta_q \rho \|_{L^2} \]

\[ \leq C d_q 2^{-\sigma q} (\sqrt{q} + 2 V(t) + \| u \|_{L^2}) \| \rho \|_{B_{2,r}^q} \| \Delta_q \rho \|_{L^2}. \]
Then, we bound $L_3$. By Hölder inequality and Bernstein inequality,

$$|L_3| \leq \left| \sum_{k \geq q-1}\sum_{|k-l| \leq 1} \int_{\mathbb{R}^2} \Delta_q (\Delta_k u \cdot \nabla \Delta_l \rho) \| \Delta_q \rho \|_{L^2} \, dx \right|$$

$$\leq C \sum_{k \geq q-1}\sum_{|k-l| \leq 1} \| \Delta_q \nabla \cdot (\Delta_k u \Delta_l \rho) \|_{L^2} \| \Delta_q \rho \|_{L^2}$$

$$\leq C2^q \sum_{k \geq q-1} \| \Delta_k u \|_{L^\infty} \| \Delta_l \rho \|_{L^2} \| \Delta_q \rho \|_{L^2}$$

$$\leq C(\sqrt{q} + 2V(t) + \|u\|_{L^2}) \sum_{k \geq q-1} 2^{q-k} \| \Delta_k \rho \|_{L^2} \| \Delta_q \rho \|_{L^2}$$

$$\leq Cd_q 2^{-\alpha q} (\sqrt{q} + 2V(t) + \|u\|_{L^2}) \| \rho \|_{B^{\alpha}_{2,r}} \| \Delta_q \rho \|_{L^2}.$$

Thus, we obtain $I$ can be bounded by

$$I \leq Cd_q 2^{-\alpha q} (\sqrt{q} + 2V(t) + 1) \| \rho \|_{B^{\alpha}_{2,r}} \| \Delta_q \rho \|_{L^2}. \quad (A.13)$$

Inserting (A.12) and (A.13) into (A.11), one can obtain

$$\frac{d}{dt} \| \Delta_q \rho(t) \|_{L^2} \leq \| \Delta_q f \|_{L^2} + Cd_q 2^{-\alpha q} (\sqrt{q} + 2V(t) + 1) \| \rho \|_{B^{\alpha}_{2,r}}. \quad (A.14)$$

Denoting $s_t \triangleq s - \eta \int_0^t V(\tau) \, d\tau$ for $t \in [0, T]$ with $\eta = \varepsilon (\int_0^T V(\tau) \, d\tau)^{-1}$. Choosing $\sigma = s_t$ and integrating (A.14) from 0 to $t$ with respect to time variable and then multiplying by $2^{st q}$,

$$2^{st q} \| \Delta_q \rho(t) \|_{L^2} \leq dq \| \rho_0 \|_{B^{\alpha}_{2,r}} + dq \int_0^t \| f(\tau) \|_{B^{\alpha}_{2,1}} \, d\tau$$

$$+ Cd_q \int_0^t 2^{-\eta \int_s^t V(s) \, ds} q (\sqrt{q} + 2V(\tau) + 1) \| \rho \|_{B^{\alpha}_{2,r}} \, d\tau. \quad (A.15)$$

Choosing $q_0 > 0$ is the smallest integer such that

$$\frac{4C^2 \| dq \|_{L^{\infty}}^2}{(\log 2)^2 \eta^2} \leq q_0 + 2.$$

Then for $q \geq q_0$, we have

$$C \int_0^t 2^{-\eta \int_s^t V(s) \, ds} q (\sqrt{q} + 2V(\tau) \, d\tau \leq \frac{1}{2\| b_q \|_{L^{\infty}}}. \quad (A.16)$$
Inserting these results into \((A.15)\) and taking \(\ell^r\) norm of \(q\), on can deduce

\[
\|\rho(t)\|_{B^r_{2,r}} \leq C\|\rho_0\|_{B^r_{2,r}} + C\int_0^t \|f(\tau)\|_{B^r_{2,r}} \, d\tau
\]

\[
+ C\left(\sum_{q \geq q_0} \left(d_q \int_0^t 2\left(-\eta_j f^r \sqrt{q + 2V(\tau)}\|\rho\|_{B^r_{2,r}} \, d\tau\right)\right) \right)^{1 \over 2}
\]

\[
+ C\left(\sum_{1 \leq q < q_0} \left(d_q \int_0^t 2\left(-\eta_j f^r \sqrt{q + 2V(\tau)}\|\rho\|_{B^r_{2,r}} \, d\tau\right)\right) \right)^{1 \over 2}
\]

\[
\leq C\|\rho_0\|_{B^r_{2,r}} + C\int_0^t \|f(\tau)\|_{B^r_{2,r}} \, d\tau
\]

\[
+ \frac{1}{2} \sup_{t \in [0,T]} \|\rho\|_{B^r_{2,r}} + C\sqrt{q_0 + 1} \int_0^t V(\tau)\|\rho\|_{B^r_{2,r}} \, d\tau.
\]

\((A.17)\)

Taking supremum of time \(t\) from 0 to \(T\) and applying the Grönwall’s Lemma, it follows that

\[
\sup_{t \in [0,T]} \|\rho(t)\|_{B^r_{2,r}} \leq C\left(\|\rho_0\|_{B^r_{2,r}} + \int_0^T \|f(\tau)\|_{B^r_{2,r}} \, d\tau\right) e^{\sqrt{q_0 + 1} \int_0^T V(\tau) \, d\tau}.
\]

According to the definition of \(q_0\), finally we conclude that

\[
\sup_{t \in [0,T]} \|\rho(t)\|_{B^r_{2,r}} \leq C\left(\|\rho_0\|_{B^r_{2,r}} + \int_0^T \|f(\tau)\|_{B^r_{2,r}} \, d\tau\right) e^{C \left(\int_0^T V(\tau) \, d\tau\right)^2},
\]

which entails the desired inequality given that \(s \geq s_t \geq s - \varepsilon\) for all \(t \in [0, T]\).

\[\square\]

The next proposition gives the estimate for Lipschitz norm of the velocity with anisotropic initial data \(\omega_0 \in B^{0, \frac{1}{2}}\) for system \((1.2)\), which shows the proof of Remark 2.

**Proposition A.1** Assume \(u_0\) is a divergence-free vector in \(H^1\), \(\omega_0 \in \sqrt{L} \cap B^{0, \frac{1}{2}}\) and \(\theta_0 \in L^\infty \cap H^s\) with \(s \in (\frac{1}{2}, 1]\), then the solution \(u\) of Theorem 4.1 satisfies \(\nabla u \in L^2_{loc}(\mathbb{R}_+; L^\infty)\).

**Proof** Applying \(\Delta_q^u\) to \((1.5)\), and taking \(L^2\) inner product with \(\Delta_q^u\omega\), we have

\[
\frac{1}{2} \frac{d}{dt} \|\Delta_q^u \omega(t)\|_{L^2}^2 + \|\partial_1 \Delta_q^u \omega\|_{L^2}^2 = -\int_{\mathbb{R}^2} \Delta_q^u \theta \partial_1 \Delta_q^u \omega \, dx - \int_{\mathbb{R}^2} \Delta_q^u (u \cdot \nabla \omega) \Delta_q^u \omega \, dx.
\]

\((A.18)\)
For the first term in the right-hand side of (A.18), by Hölder inequality, Young’s inequality and the definition of the space $B^{0,\frac{1}{2}}$, we obtain

$$- \int_{\mathbb{R}^2} \Delta_q^v \theta \partial_1 \Delta_q^v \omega \, dx \leq \| \Delta_q^v \theta \|_{L^2} \| \partial_1 \Delta_q^v \omega \|_{L^2}$$

$$\leq \frac{1}{2} \| \Delta_q^v \theta \|_{L^2}^2 + \frac{1}{2} \| \partial_1 \Delta_q^v \omega \|_{L^2}^2$$

$$\leq \frac{1}{2} 2^{-q} \| \Delta_q^v \theta \|_{L^2}^2 + \frac{1}{2} \| \partial_1 \Delta_q^v \omega \|_{L^2}^2$$

$$\leq C 2^{-q} a_q \| \theta \|_{B^{0,\frac{1}{2}}}^2 + \frac{1}{2} \| \partial_1 \Delta_q^v \omega \|_{L^2}^2. \quad (A.19)$$

with $a_q = \frac{(2^q/2 \| \Delta_q^v \theta \|_{L^2})^2}{\| \theta \|_{B^{0,\frac{1}{2}}}^2} \in \ell^{\frac{1}{2}}$. For the last term of (A.18), we divide it as,

$$- \int_{\mathbb{R}^2} \Delta_q^v (u \cdot \nabla \omega) \Delta_q^v \omega \, dx = - \int_{\mathbb{R}^2} \Delta_q^v (u^1 \partial_1 \omega) \Delta_q^v \omega \, dx - \int_{\mathbb{R}^2} \Delta_q^v (u^2 \partial_2 \omega) \Delta_q^v \omega \, dx \triangleq Y_1 + Y_2.$$

For $Y_1$, by Bony’s decomposition, we can divide it into the following three terms,

$$- \int_{\mathbb{R}^2} \Delta_q^v (u^1 \partial_1 \omega) \Delta_q^v \omega \, dx$$

$$= - \sum_{|k-q| \leq 2} \int_{\mathbb{R}^2} \Delta_q^v (S_{k-1}^v u^1 \Delta_k^v \partial_1 \omega) \Delta_q^v \omega \, dx$$

$$- \sum_{|k-q| \leq 2} \int_{\mathbb{R}^2} \Delta_q^v (S_{k-1}^v u^1 \Delta_k^v \partial_1 \omega) \Delta_q^v \omega \, dx$$

$$- \sum_{k \geq q-1} \sum_{|k-l| \leq 1} \int_{\mathbb{R}^2} \Delta_q^v (S_{k-1}^v u^1 \Delta_l^v \partial_1 \omega) \Delta_q^v \omega \, dx \triangleq Y_{11} + Y_{12} + Y_{13}. \quad (A.20)$$

For $Y_{11}$, we can rewrite it as

$$Y_{11} = - \sum_{|q-k| \leq 2} \int_{\mathbb{R}^2} \Delta_q^v (S_{k-1}^v u^1 \partial_1 \Delta_k^v \omega) \Delta_q^v \omega \, dx$$

$$= - \sum_{|q-k| \leq 2} \int_{\mathbb{R}^2} [\Delta_q^v, S_{k-1}^v u^1 \partial_1] \Delta_k^v \omega \Delta_q^v \omega \, dx$$
\[
- \sum_{|q-k| \leq 2} \int_{\mathbb{R}^2} (S_{k-1}^u u^1 - S_q^u u^1) \partial_1 \Delta_q^v \Delta_k^v \omega \Delta_q^v \omega \, dx \\
- \int_{\mathbb{R}^2} S_q^u u^1 \partial_1 \Delta_q^v \omega \Delta_q^v \omega \, dx \\
\triangleq Y_{111} + Y_{112} + Y_{113},
\]

where we have used the fact \(\sum_{|q-k| \leq 2} \partial_1 \Delta_q^v \Delta_k^v \omega = \partial_1 \Delta_q^v \omega\). For \(Y_{111}\), the commutator can be written as,

\[
\left[ \Delta_q^v, S_{k-1}^u u^1 \partial_1 \right] \Delta_k^v \omega = \int_{\mathbb{R}} \phi_q(x_1, x_2 - y) (S_{k-1}^u u^1(x_1, y) \partial_1 \Delta_k^v \omega(x_1, y)) \, dy \\
- S_{k-1}^u u^1(x_1, x_2) \int_{\mathbb{R}} \phi_q(x_1, x_2 - y) \partial_1 \Delta_k^v \omega(x_1, y) \, dy \\
= \int_{\mathbb{R}} \phi_q(x_1, x_2 - y) (S_{k-1}^u u^1(x_1, y) - S_{k-1}^u u^1(x_1, x_2)) \partial_1 \Delta_k^v \omega(x_1, y) \, dy \\
= \int_{\mathbb{R}} \phi_q(x_1, x_2 - y) \int_0^1 (y - s x_2) \partial_2 S_{k-1}^u u^1(s y + (1 - s)x_2) \, ds \partial_1 \Delta_k^v \omega(x_1, y) \, dy.
\]

where \(\phi_j(x) \triangleq 2^j \mathcal{F}^{-1}(\varphi)_x(2^j x)\). By anisotropic Hölder inequality and Bernstein inequality, we can bound \(Y_{111}\) by

\[
|Y_{111}| \leq C \sum_{|q-k| \leq 2} \| \left[ \Delta_q^v, S_{k-1}^u u^1 \partial_1 \right] \Delta_k^v \omega \|_{L^2} \| \Delta_q^v \omega \|_{L^2} \\
\leq C \sum_{|q-k| \leq 2} 2^{-q} \| \partial_2 S_{k-1}^u u^1 \|_{L^2_{x_2} L^\infty_{x_1}} \| \partial_1 \Delta_k^v \omega \|_{L^\infty_{x_2} L^2_{x_1}} \| \Delta_q^v \omega \|_{L^2} \\
\leq C \sum_{|q-k| \leq 2} 2^{q - \frac{k}{2}} \| \partial_2 S_{k-1}^u u^1 \| \| \partial_1 \Delta_k^v \omega \|_{L^2} \| \Delta_q^v \omega \|_{L^2} \\
\leq C \sum_{|q-k| \leq 2} \frac{2^{k-q}}{2^{q/2}} \| \omega \|_{L^2} \| \partial_1 \Delta_k^v \omega \|_{L^2} \| \Delta_q^v \omega \|_{L^2} \\
\leq C \sum_{|q-k| \leq 2} 2^{-q} a_q \| \omega \|_{L^2} \| \partial_1 \Delta_k^v \omega \|_{B^0_{1/2}} \| \Delta_q^v \omega \|_{B^0_{1/2}}.
\]

For \(Y_{112}\), by anisotropic Hölder inequality and Bernstein inequality,

\[
Y_{112} = - \sum_{|q-k| \leq 2} \int_{\mathbb{R}^2} ((S_{k-1}^u u^1 - S_q^u u^1) \partial_1 \Delta_q^v \Delta_k^v \omega) \Delta_q^v \omega \, dx \\
\leq C \sum_{|q-k| \leq 2} \| \Delta_k^v u^1 \|_{L^2_{x_2} L^\infty_{x_1}} \| \partial_1 \Delta_k^v \Delta_q^v \omega \|_{L^2} \| \Delta_q^v \omega \|_{L^\infty_{x_2} L^2_{x_1}}.
\]
Next we estimate $Y_{12}$, using anisotropic Hölder inequality and Bernstein inequality,

$$Y_{12} = - \sum_{|k-q| \leq 2} \int \Delta^v q (\Delta^v_k u^1 S^v_{k-1} \partial_1 \omega) \Delta^v q \omega \ dx$$

$$\leq C \sum_{|k-q| \leq 2} \| \Delta^v q u^1 \|_{L^\infty \to L^1_2} \| S^v_{k-1} \partial_1 \omega \|_{L^2} \| \Delta^v q \omega \|_{L^2} L^\infty$$

$$\leq C \sum_{|q-k| \leq 2} 2^{q-k} \| \partial^2_2 \Delta^v_k u^1 \|_{L^2} \| \partial^2_2 \omega \|_{L^2} 2^{-q} \| \Delta^v q \partial_1 \omega \|_{L^2}$$

$$\leq C \sum_{|q-k| \leq 2} 2^{q-k} \| \partial^2_2 \Delta^v_k u^1 \|_{L^2} \| \partial^2_2 \omega \|_{L^2} 2^{-q} \| \Delta^v q \partial_1 \omega \|_{L^2}$$

$$+ C \| \Delta^v_0 u^1 \|_{L^2} \| \Delta^v_0 \partial_1 \omega \|_{L^2} \| \partial^2_2 \omega \|_{L^2} \| \Delta^v q \partial_1 \omega \|_{L^2}$$

$$\leq C 2^{-q} a_q \left( \| \partial^2_2 \omega \|_{L^2} \| \omega \|_{B^0 \frac{1}{2}} \| \partial^2_2 \omega \|_{B^0 \frac{1}{2}} + \| u \|_{L^2} \| \partial^2_2 \omega \|_{L^2} \| \partial^2_2 \omega \|_{B^0 \frac{1}{2}} \right).$$

In the same manner, $Y_{13}$ can be handled by,

$$Y_{13} = - \sum_{k \geq q-1} \sum_{|k-l| \leq 1} \int \Delta^v q (\Delta^v_k u^1 \Delta^v_l \partial_1 \omega) \Delta^v q \omega \ dx$$

$$\leq C \sum_{k \geq q-1} \| \Delta^v_k u^1 \|_{L^\infty \to L^1_2} \| \Delta^v_0 \partial_1 \omega \|_{L^2} \| \Delta^v q \omega \|_{L^2} L^\infty$$

$$\leq C \sum_{k \geq q-1} \| \Delta^v_k u^1 \|_{L^\infty \to L^1_2} \| \Delta^v_k \partial_1 \omega \|_{L^2} \| \Delta^v q \omega \|_{L^2} 2^{-q} \| \Delta^v q \omega \|_{L^2}$$

(A.21)

$$\leq C \sum_{|q-k| \leq 2} 2^{q-k} \| \partial^2_2 \Delta^v_k u^1 \|_{L^2} \| \partial^2_2 \partial^2_2 \omega \|_{L^2} 2^{-q} \| \Delta^v q \partial_1 \omega \|_{L^2}$$

$$+ C \| \Delta^v_0 u^1 \|_{L^2} \| \Delta^v_0 \partial_1 \omega \|_{L^2} \| \Delta^v q \partial_1 \omega \|_{L^2} \| \Delta^v q \omega \|_{L^2}$$

$$\leq C (\| u \|_{L^2} + \| \omega \|_{L^2}) 2^{-q} a_q \| \omega \|_{B^0 \frac{1}{2}} \| \partial^2_2 \omega \|_{B^0 \frac{1}{2}}.$$
Then, we estimate $Y_2$. Similar as $Y_1$, we can also divide it into the following three terms by the Bony’s decomposition,

$$
Y_2 = - \int_{\mathbb{R}^2} \Delta_q^v (u^2 \partial_2 \omega) \Delta_q^v \omega \, dx
$$

$$
= - \sum_{|k-q| \leq 2} \int_{\mathbb{R}^2} \Delta_q^v (S_{k-1}^v u^2 \Delta_k^v \partial_2 \omega) \Delta_q^v \omega \, dx
$$

$$
- \sum_{|k-q| \leq 2} \int_{\mathbb{R}^2} \Delta_q^v (\Delta_k^v u^2 S_{k-1}^v \partial_2 \omega) \Delta_q^v \omega \, dx
$$

$$
- \sum_{k \geq q-1} \sum_{|k-l| \leq 1} \int_{\mathbb{R}^2} \Delta_q^v (\Delta_k^v u^2 \Delta_l^v \partial_2 \omega) \Delta_q^v \omega \, dx
$$

$$
\triangleq Y_{21} + Y_{22} + Y_{23}.
$$

For $Y_{21}$, we can write it as,

$$
Y_{21} = - \sum_{|q-k| \leq 2} \int_{\mathbb{R}^2} \Delta_q^v (S_{k-1}^v u^2 \partial_2 \Delta_k^v \omega) \Delta_q^v \omega \, dx
$$

$$
= - \sum_{|q-k| \leq 2} \int_{\mathbb{R}^2} [\Delta_q^v, S_{k-1}^v u^2 \partial_2] \Delta_k^v \omega \Delta_q^v \omega \, dx
$$

$$
- \sum_{|q-k| \leq 2} \int_{\mathbb{R}^2} \partial_2 (S_{k-1}^v u^2 - S_{q}^v u^2) \Delta_q^v \Delta_k^v \omega \Delta_q^v \omega \, dx
$$

$$
- \int_{\mathbb{R}^2} S_q^v u^2 \partial_2 \Delta_q^v \omega \Delta_q^v \omega \, dx
$$

$$
\triangleq Y_{211} + Y_{212} + Y_{213},
$$

Here, we should notice that $Y_{113} + Y_{213} = 0$ because of the divergence-free condition of $u$, so we do not need to estimate these two terms.

For $Y_{211}$, similar as $Y_{111}$, the commutator can be written as,

$$
[\Delta_q^v, S_{k-1}^v u^2 \partial_2] \Delta_k^v \omega = \int_{\mathbb{R}} \phi_q(x_1, x_2-y) \int_0^1 (y-x_2) \partial_2 S_{k-1}^v u^2 (sy + (1-s)x_2) \, ds \, \partial_2 \Delta_k^v \omega (x_1, y) \, dy.
$$

Thus by anisotropic Hölder inequality and Biot–Savart law $u^2 = \partial_1 \Delta^{-1} \omega$,

$$
|Y_{211}| \leq C \sum_{|q-k| \leq 2} 2^{-q} \| \partial_2 S_{k-1}^v u^2 \|_{L^\infty_x L^2_y} \| \Delta_k^v \partial_2 \omega \|_{L^2_x L^\infty_y} \| \Delta_q^v \omega \|_{L^2}
$$

$$
\leq C \sum_{|q-k| \leq 2} 2^{k-q} \| \partial_2 S_{k-1}^v u^2 \|_{L^2_x} \| \partial_2 \partial_2 S_{k-1}^v u^2 \|_{L^2_x} \| \Delta_k^v \partial_2 \omega \|_{L^2_x} \| \Delta_q^v \omega \|_{L^2} \| \Delta_q^v \partial_1 \omega \|_{L^2_x} \| \Delta_k^v \partial_1 \omega \|_{L^2_x} \| \Delta_q^v \omega \|_{L^2}
$$

$$
\leq C 2^{-q} a_q (\| \omega \|_{L^2} + \| \partial_1 \omega \|_{L^2}) \| \omega \|_{B^{\frac{3}{2}}_{\infty, 2}} \| \partial_1 \omega \|_{B^{\frac{3}{2}}_{\infty, 2}}.
$$

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Similarly, \( Y_{212} \) can be bounded by

\[
|Y_{212}| \leq C2^{-q}a_q (\|u\|_{L^2} + \|\omega\|_{L^2} + \|\partial_1\omega\|_{L^2}) \|\omega\|_{B^{0,\frac{1}{2}}}^3 \|\partial_1\omega\|_{B^{0,\frac{1}{2}}}^{\frac{1}{2}}.
\]

Next we estimate \( Y_{22} \), by the anisotropic Hölder inequality and discrete Young’s inequality,

\[
Y_{22} = - \sum_{|k-q| \leq 2} \int_{\mathbb{R}^2} \Delta_q^v (\Delta_k^v u^2 S_{k-1}^v \partial_2 \omega) \Delta_q^v \omega \, dx
\]

\[
\leq C \sum_{|q-k| \leq 2} \|\Delta_k^v u^2\|_{L^\infty_{x_1}L^1_{x_2}} \|S_{k-1}^v \partial_2 \omega\|_{L^\infty_{x_1}L^2_{x_2}} \|\Delta_q^v \omega\|_{L^2}
\]

\[
\leq C \sum_{|q-k| \leq 2} \|\Delta_k^v u^2\|_{L^\infty} \|\partial_2 \Delta_k^v u^2\|_{L^2} \|S_{k-1}^v \partial_2 \omega\|_{L^\infty} \|\partial_1 S_{k-1}^v \partial_2 \omega\|_{L^2} \|\Delta_q^v \omega\|_{L^2}
\]

\[
\leq C \sum_{|q-k| \leq 2} 2^q \|\Delta_k^v u^2\|_{L^2} \|\partial_2 \Delta_k^v u^2\|_{L^2} \left( \sum_{m \leq q-2} 2^{\frac{m-q}{2}} 2^\frac{m}{2} \|\Delta_m^v \omega\|_{L^2} \right)^\frac{1}{2}
\]

\times \left( \sum_{n \leq q-2} 2^\frac{n-q}{2} 2^\frac{n}{2} \|\Delta_n^v \partial_1 \omega\|_{L^2} \right)^\frac{1}{2} \|\Delta_q^v \omega\|_{L^2}
\]

\[
\leq C2^{-q}a_q (\|u\|_{L^2} + \|\omega\|_{L^2} + \|\partial_1\omega\|_{L^2}) \|\omega\|_{B^{0,\frac{1}{2}}}^3 \|\partial_1\omega\|_{B^{0,\frac{1}{2}}}^{\frac{1}{2}}.
\]

The estimate of \( Y_{23} \) is the same as \( Y_{13} \); thus, we can obtain

\[
|Y_{23}| \leq C2^{-q}a_q (\|u\|_{L^2} + \|\omega\|_{L^2} + \|\partial_1\omega\|_{L^2}) \|\omega\|_{B^{0,\frac{1}{2}}}^2 \|\partial_1\omega\|_{B^{0,\frac{1}{2}}}^{\frac{1}{2}}.
\]

Summing up all these estimates above, noticing that \( u \) and \( \omega \) are bounded in \( L^2 \), we deduce from (A.18) that

\[
\frac{1}{2} \frac{d}{dt} \|\Delta_q^v \omega(t)\|_{L^2}^2 + \frac{1}{2} \|\partial_1 \Delta_q^v \omega\|_{L^2}^2 \leq C2^{-q}a_q \|\theta\|_{B^{0,\frac{1}{2}}}^2 + C2^{-q}a_q (1 + \|\partial_1\omega\|_{L^2})
\]

\[
\times \left( \|\omega\|_{B^{0,\frac{1}{2}}} \|\partial_1\omega\|_{B^{0,\frac{1}{2}}} + \|\omega\|_{B^{0,\frac{1}{2}}}^\frac{3}{2} \|\partial_1\omega\|_{B^{0,\frac{1}{2}}}^{\frac{1}{2}}
\]

\[
+ \|\omega\|_{B^{0,\frac{1}{2}}}^2 \right).
\]
Then integrating from 0 to \( t \) with respect to time variable, taking square root on both sides and summing up of \( q \), by Minkowski inequality we obtain

\[
\| \omega(t) \|_{B^{0, \frac{1}{2}}} + \left( \int_0^t \| \partial_1 \omega(\tau) \|_{B^{0, \frac{1}{2}}} \, d\tau \right)^{\frac{1}{2}} \\
\leq \| \omega_0 \|_{B^{0, \frac{1}{2}}} + C \left( \int_0^t \| \theta \|_{B^{0, \frac{1}{2}}}^2 \, d\tau \right)^{\frac{1}{2}} + C \left( \int_0^t (1 + \| \partial_1 \omega \|_{L^2}) \right)^{\frac{1}{2}} \times \left( \| \omega \|_{B^{0, \frac{1}{2}}} \| \partial_1 \omega \|_{B^{0, \frac{1}{2}}} + \| \omega \|_{B^{0, \frac{1}{2}}} \| \partial_1 \omega \|_{B^{0, \frac{1}{2}}} + \| \omega \|_{B^{0, \frac{1}{2}}}^2 \, d\tau \right)^{\frac{1}{2}} \\
\leq \| \omega_0 \|_{B^{0, \frac{1}{2}}} + C(t)\| \theta \|_{L^\infty(B^{0, \frac{1}{2}})} + C \left( \int_0^t (1 + \| \partial_1 \omega \|_{L^2})^2 \| \omega \|_{B^{0, \frac{1}{2}}}^2 \, d\tau \right)^{\frac{1}{2}} + \frac{1}{2} \left( \int_0^t \| \partial_1 \omega \|_{B^{0, \frac{1}{2}}}^2 \, d\tau \right)^{\frac{1}{2}}. 
\]

According to Lemma 2.6, we have \( \theta \) is bounded in \( B^{\frac{1}{2}, 1}_2 \), then by the Besov embedding \( B^{\frac{1}{2}, 1}_2 \hookrightarrow B^{0, \frac{1}{2}}_0 \), we get

\[
\| \omega(t) \|_{B^{0, \frac{1}{2}}}^2 \leq C(t) + C \int_0^t (1 + \| \partial_1 \omega \|_{L^2})^2 \| \omega \|_{B^{0, \frac{1}{2}}} \, d\tau.
\]

Then, using Grönwall’s Lemma, We deduce

\[
\| \omega(t) \|_{B^{0, \frac{1}{2}}}^2 + \int_0^t \| \partial_1 \omega(\tau) \|_{B^{0, \frac{1}{2}}}^2 \, d\tau \leq C(t). \quad (A.23)
\]

According to Biot–Savart law (2.3), divergence-free condition of the velocity \( u \) and Besov embedding, it follows that

\[
\int_0^t \| \partial_1 u^1, \partial_1 u^2, \partial_2 u^2 \|_{L^\infty} \, d\tau \leq C(t).
\]

Also, using inequality \( \| f \|_{L^\infty_{2,1}} \leq C \| f \|_{L^2_{2,1}} \| \partial_1 f \|_{L^2_{2,1}} \) and estimate (A.23), we have

\[
\int_0^t \| \omega \|_{L^\infty} \, d\tau \leq C \int_0^t \| \omega \|_{B^{0, \frac{1}{2}}} \| \partial_1 \omega \|_{B^{0, \frac{1}{2}}} \, d\tau \leq C(t).
\]

From the above, it follows that

\[
\int_0^t \| \partial_2 u^1 \|_{L^\infty} \, d\tau \leq \int_0^t \| \omega \|_{L^\infty} \, d\tau + \int_0^t \| \partial_1 u^2 \|_{L^\infty} \, d\tau \leq C(t).
\]
Finally, we obtain the estimate
\[
\int_0^t \| \nabla u \|_{L^\infty}^2 \, d\tau \leq C(t),
\]
which completes the proof of this proposition. \qed

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