A CARTAN TYPE IDENTITY FOR ISOPARAMETRIC HYPERSURFACES IN SYMMETRIC SPACES

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Abstract. In this paper, we obtain a Cartan type identity for curvature-adapted isoparametric hypersurfaces in symmetric spaces of compact type or non-compact type. This identity is a generalization of Cartan-D’Atri’s identity for curvature-adapted (=amenable) isoparametric hypersurfaces in rank one symmetric spaces. Furthermore, by using the Cartan type identity, we show that certain kind of curvature-adapted isoparametric hypersurfaces in a symmetric space of non-compact type are principal orbits of Hermann actions.

1. Introduction. An isoparametric hypersurface in a (general) Riemannian manifold is a connected hypersurface whose sufficiently close parallel hypersurfaces are of constant mean curvature (see [12] for example). In this paper, we assume that all isoparametric hypersurfaces are complete. It is known that all isoparametric hypersurfaces in a symmetric space of compact type are equifocal in the sense of [37] and that, conversely all equifocal hypersurfaces are isoparametric (see [12]). Also, it is known that all isoparametric hypersurfaces in a symmetric space of non-compact type are complex equifocal in the sense of [18] and that, conversely, all curvature-adapted complex equifocal hypersurfaces are isoparametric (see [19, Theorem 15]), where the curvature-adaptedness implies that, for a unit normal vector $v$, the (normal) Jacobi operator $R(\cdot, v)v$ preserves the tangent space invariantly and commutes with the shape operator $A$ for $v$, where $R$ is the curvature tensor of the ambient space. It is known that principal orbits of a Hermann action (i.e., the action of a symmetric subgroup of $G$) of cohomogeneity one on a symmetric space $G/K$ of compact type are curvature-adapted and equifocal (see ([11])). Hence they are isoparametric hypersurfaces. On the other hand, we [20, 23] showed that the principal orbits of a Hermann action (i.e., the action of a (not necessarily compact) symmetric subgroup of $G$) of cohomogeneity one on a symmetric space $G/K$ of non-compact type are curvature-adapted and complex equifocal, and they have no focal point of non-Euclidean type on the ideal boundary of $G/K$. Hence they are isoparametric hypersurfaces.
For an isoparametric hypersurface $M$ in a real space form $N$ of constant curvature $c$, it is known that the following Cartan’s identity holds:

\[
\sum_{\lambda \in \text{Spec}A \setminus \{\lambda_0\}} c + \frac{\lambda \lambda_0}{\lambda - \lambda_0} \times m_\lambda = 0
\]

for any $\lambda_0 \in \text{Spec}A$, where $A$ is the shape operator of $M$ and $\text{Spec}A$ is the spectrum of $A$, $m_\lambda$ is the multiplicity of $\lambda$. Here we note that all hypersurfaces in a real space form are curvature-adapted. In general cases, this identity is shown in algebraic method. Also, it is shown in geometrical method in the following three cases:

(i) $c = 0$, $\lambda_0 \neq 0$,
(ii) $c > 0$, $\lambda_0$ : any eigenvalue of $A$,
(iii) $c < 0$, $|\lambda_0| > \sqrt{-c}$.

In detail, it is shown by showing the minimality of the focal submanifold for $\lambda_0$ and using this fact.

Let $H \curvearrowright G/K$ be a cohomogeneity one action of a compact group $H (\subset G)$ on a rank one symmetric space $G/K$ and $M$ a principal orbit of this action. Since the $H$-action is of cohomogeneity one, it is hyperpolar. Hence $M$ is an equifocal (hence isoparametric) hypersurface (see [13]). In 1979, D’Atri [8] obtained a Cartan type identity for $M$ in the case where $M$ is amenable (i.e., curvature-adapted). On the other hand, in 1989–1991, Berndt [1, 2] obtained a Cartan type identity (in algebraic method) for curvature-adapted hypersurfaces with constant principal curvature in rank one symmetric spaces other than spheres and hyperbolic spaces. Here we note that, for a curvature-adapted hypersurface in a rank one symmetric space of non-compact type, it has constant principal curvature if and only if it is isoparametric.

In this paper, we obtain the Cartan type identities for curvature-adapted isoparametric hypersurfaces in symmetric spaces and, furthermore, by using the Cartan type identity, we prove that certain kind of curvature-adapted isoparametric hypersurfaces in a symmetric space of non-compact type are principal orbits of Hermann actions. Let $M$ be a hypersurface in a symmetric space $N = G/K$ of compact type or non-compact type and $v$ a unit normal vector field of $M$. Set $R(v_x) := R(\cdot, v_x, v_x)|_{T_xM}$, where $R$ is the curvature tensor of $N$. For each $r \in \mathbb{R}$, we define a function $\tau_r$ over $[0, \infty)$ by

\[
\tau_r(s) := \begin{cases} 
\frac{\sqrt{s}}{\tan (r \sqrt{s})} & (s > 0) \\
\frac{1}{r} & (s = 0)
\end{cases}
\]

Also, for each $r \in \mathbb{C}$, we define a complex-valued function $\hat{\tau}_r$ over $(-\infty, 0]$ by

\[
\hat{\tau}_r(s) := \begin{cases} 
\frac{i \sqrt{-s}}{\tan (br \sqrt{-s})} & (s < 0) \\
\frac{1}{r} & (s = 0)
\end{cases}
\]
where $i$ is the imaginary unit. First we prove the following Cartan type identity for a curvature-adapted isoparametric hypersurface in a simply connected symmetric space of compact type.

**Theorem A.** Let $M$ be a curvature-adapted isoparametric hypersurface in a simply connected symmetric space $N := G/K$ of compact type. For each focal radius $r_0$ of $M$, we have

$$\sum_{(\lambda, \mu) \in S_{r_0}^x} \frac{\mu + \lambda \tau_{r_0}(\mu)}{\lambda - \tau_{r_0}(\mu)} \cdot m_{\lambda, \mu} = 0,$$

where $S_{r_0}^x := \{ (\lambda, \mu) \in \text{Spec} A_x \times \text{Spec} R(v_x) : \text{Ker}(A_x - \lambda I) \cap \text{Ker}(R(v_x) - \mu I) \neq \{0\}, \lambda \neq \tau_{r_0}(\mu) \}$ and $m_{\lambda, \mu} := \dim(\text{Ker}(A_x - \lambda I) \cap \text{Ker}(R(v_x) - \mu I))$.

**Remark 1.1.**
(i) If $\text{Ker}(A_x - \lambda_0 I) \cap \text{Ker}(R(v_x) - \mu_0 I) = \{0\}$, we have $\tau_{r_0}(\mu_0) = \lambda_0$.
(ii) If $G/K$ is a sphere of constant curvature $c$, then $\text{Spec} R(v_x) = \{c\}$ and $\hat{\tau}_{r_0}(c)$ is equal to the principal curvature corresponding to $r_0$. Hence the identity (1.2) coincides with (1.1).
(iii) In the case where $G/K$ is a rank one symmetric space of compact type, the identity (1.2) coincides with the identity obtained by D’Atri [8] (see [8, Theorems 3.7 and 3.9]).
(iv) In the case where $G/K$ is a rank one symmetric space of compact type other than spheres, the identity (1.2) is different from the identity obtained by Berndt [1, 2].

Next, in this paper, we prove the following Cartan type identity for a curvature-adapted isoparametric $C^\omega$-hypersurface in a symmetric space of non-compact type, where $C^\omega$ means the real analyticity.

**Theorem B.** Let $M$ be a curvature-adapted isoparametric $C^\omega$-hypersurface in a symmetric space $N := G/K$ of non-compact type. Assume that $M$ has no focal point of non-Euclidean type on the ideal boundary $N(\infty)$ of $N$. Then $M$ admits a complex focal radius and, for each complex focal radius $r_0$ of $M$, we have

$$\sum_{(\lambda, \mu) \in S_{r_0}^x} \frac{\mu + \lambda \hat{\tau}_{r_0}(\mu)}{\lambda - \hat{\tau}_{r_0}(\mu)} \cdot m_{\lambda, \mu} = 0,$$

where $S_{r_0}^x := \{ (\lambda, \mu) \in \text{Spec} A_x \times \text{Spec} R(v_x) : \text{Ker}(A_x - \lambda I) \cap \text{Ker}(R(v_x) - \mu I) \neq \{0\}, \lambda \neq \hat{\tau}_{r_0}(\mu) \}$ and $m_{\lambda, \mu} := \dim(\text{Ker}(A_x - \lambda I) \cap \text{Ker}(R(v_x) - \mu I))$.

**Remark 1.2.**
(i) The notion of a complex focal radius was introduced in [18]. This quantity indicates the position of a focal point of the complexification $M^C \subset G^C/K^C$ of a submanifold $M$ in a symmetric space $G/K$ of non-compact type (see [19]).
(ii) If $\text{Ker}(A_x - \lambda_0 I) \cap \text{Ker}(R(v_x) - \mu_0 I) = \{0\}$, we have $\hat{\tau}_{r_0}(\mu_0) = \lambda_0$.
(iii) If $G/K$ is a hyperbolic space of constant curvature $c$, then $\text{Spec} R(v_x) = \{c\}$ and $\hat{\tau}_{r_0}(c)$ is equal to the principal curvature corresponding to $r_0$. Hence the identity (1.3) coincides with (1.1).
(iv) In the case where $G/K$ is a rank one symmetric space of non-compact type and $r_0$ is a real focal radius, the identity (1.3) coincides with the identity obtained by D’Atri [8] (see [8, Theorems 3.7 and 3.9]).

(v) In the case where $G/K$ is a rank one symmetric space of non-compact type other than hyperbolic spaces, the identity (1.3) is different from the identity obtained by J. Berndt [1, 2].

(vi) For a curvature-adapted and isoparametric hypersurface $M$ in $G/K$, the following conditions (a)–(c) are equivalent:

(a) $M$ has no focal point of non-Euclidean type on $N(\infty)$,

(b) $M$ is proper complex equifocal in the sense of [20],

(c) $\text{Ker}(A_x \pm \sqrt{-\mu I}) \cap \text{Ker}(R(v_x) - \mu I) = \{0\}$ holds for each $\mu \in \text{Spec} R(v_x) \setminus \{0\}$.

(vii) Principal orbits of a Hermann type action of cohomogeneity one on $G/K$ are curvature-adapted isoparametric $C^\omega$-hypersurface having no focal point of non-Euclidean type on $N(\infty)$ (see [20, Theorem B] and the above (iii)).

The proof of Theorem B is performed by showing the minimality of the focal submanifold $F := \{\exp^+((\text{Re} r_0)v_x + (\text{Im} r_0)Jv_x) ; x \in M^C\}$ of the complexification $M^C$ of $M$ (see Figure 1), where $\exp^+$ is the normal exponential map of the submanifold $M^C$ in $G^C/K^C$, $J$ is the complex structure of $G^C/K^C$ and $v$ is a unit normal vector field of $M$ (in $G/K$). Here we note that $\exp^+((\text{Re} r_0)v_x + (\text{Im} r_0)Jv_x)$ is equal to the point $\gamma_x^C(r_0)$ of the complexified geodesic $\gamma_x^C$ in $G^C/K^C$. In the case where $G/K$ is of rank greater than one and $M$ is not homogeneous, the proof of the minimality of $F$ is performed by showing the minimality of the lift $\tilde{F} := (\pi \circ \phi)^{-1}(F)$ of $F$ to the path space $H^0([0, 1], g^C)$, where $\phi$ is the parallel transport map for $G^C$ (which is an anti-Kaehlerian submersion of $H^0([0, 1], g^C)$ onto $G^C$) and $\pi$ is the natural projection of $G^C$ onto $G^C/K^C$ (which also is an anti-Kaehlerian submersion). Here we note that the minimality of $F$ is trivial in the case where $M$ is homogeneous. By using Theorem B, we prove the following fact for the number of distinct principal curvatures

![Figure 1](image_url)
of a curvature-adapted isoparametric $C^\infty$-hypersurfaces in a symmetric space of non-compact type.

By using Theorem B, we prove the following main result.

**Theorem C.** Let $M$ be a curvature-adapted isoparametric $C^\infty$-hypersurface in a symmetric space $N$ of non-compact type. Assume that $M$ has no focal point of non-Euclidean type on $N(\infty)$. Then $M$ is a principal orbit of a Hermann action.

**Remark 1.3.** In this theorem, are indispensable both the condition of the curvature-adaptedness and the condition for the non-existenceness of non-Euclidean type focal point on the ideal boundary. In fact, we have the following examples. Let $G/K$ be an irreducible symmetric space of non-compact type such that the (restricted) root system of $G/K$ is non-reduced. Let $g = \mathfrak{h} + \mathfrak{p}$ ($g = \text{Lie } G$, $\mathfrak{h} = \text{Lie } K$) be the Cartan decomposition associated with a symmetric pair $(G, K)$ and $\mathfrak{a}$ a maximal abelian subspace of $\mathfrak{p}$. Also, let $\Delta_+$ be the positive root system of $G/K$ with respect to $\mathfrak{a}$ and $\Pi$ the simple root system of $\Delta_+$, where we fix a lexicographic ordering of the dual space $\mathfrak{a}^*$ of $\mathfrak{a}$. Set $n := \sum_{\lambda \in \Delta_+} g_\lambda$ and $N := \exp n$, where $g_\lambda$ is the root space for $\lambda$ and $\exp$ is the exponential map of $G$. If $G/K$ is of rank one, then any orbit of the $N$-action on $G/K$ is a full irreducible curvature-adapted isoparametric $C^\infty$-hypersurface but it has a focal point of non-Euclidean type on $N(\infty)$ (see [25]). On the other hand, it is a principal orbit of no Hermann action. Thus, in this theorem, is indispensable the condition for the non-existenceness of a focal point of non-Euclidean type on the ideal boundary. Let $H_\lambda$ be the element of $\mathfrak{a}$ defined by $(H_\lambda, \bullet) = \lambda(\bullet)$. Assume that the (restricted) root system of $G/K$ is of type $(BC_n)$. Take an element $\lambda$ of $\Pi$ such that $2\lambda$ belongs to $\Delta_+$, and one-dimensional subspaces $l$ of $\mathbb{R}H_\lambda + \mathfrak{g}_e$. Set $S := \exp(\mathfrak{a} + n) \ominus l$, where $\exp$ is the exponential map of $G$ and $(\mathfrak{a} + n) \ominus l$ is the orthogonal complement of $l$ in $\mathfrak{a} + n$. Then $S$ is a subgroup of $AN := \exp(\mathfrak{a} + n)$ and any orbit of the $S$-action on $G/K$ is a full irreducible isoparametric $C^\infty$-hypersurface but it is not curvature-adapted (see [25]). Furthermore, we can find an orbit having no focal point of non-Euclidean type on $N(\infty)$ among orbits of the $S$-action. On the other hand, it is a principal orbit of no Hermann action. Thus the condition of the curvature-adaptedness is indispensable in this theorem.

In Section 2, we recall basic notions. In Section 3, we prove Theorem A. In Section 4, we define the mean curvature of a proper anti-Kaehlerian Fredholm submanifold and prepare a lemma to prove Theorem B. In Section 5, we prove Theorems B and C.

### 2. Basic notions

In this section, we recall basic notions which are used in the proof of Theorems A and B. First we recall the notion of an equifocal hypersurface in a symmetric space. Let $M$ be a complete (oriented embedded) hypersurface in a symmetric space $N = G/K$ and fix a global unit normal vector field $v$ of $M$. Let $\gamma_v$ be the normal geodesic of $M$ with $\gamma_v(0) = x$, where $x \in M$ and $\gamma_v'(0)$ is the velocity vector of $\gamma_v$, at $0$. If $\gamma_v(s_0)$ is a focal point of $M$ along $\gamma_v$, then $s_0$ is called a focal radius of $M$ at $x$. Denote by $\mathcal{FR}_{M,x}$ the set of all focal radii of $M$ at $x$. If $M$ is compact and if $\mathcal{FR}_{M,x}$ is independent of the choice
of $x$, then it is called an \textit{equivocal hypersurface}. This notion is the hypersurface version of an equivocal submanifold defined in \cite{37}.

Next we recall the notion of a complex equivocal hypersurface in a symmetric space of non-compact type. Let $M$ be a complete (oriented embedded) hypersurface in a symmetric space $N = G/K$ of non-compact type and fix a global unit normal vector field $v$ of $M$. Let $g$ be the Lie algebra of $G$ and $\theta$ be the Cartan involution of $G$ with $\text{Fix} \theta = K$, where $\text{Fix} \theta$ is the fixed point group of $\theta$. Denote by the same symbol $\theta$ the involution of $g$ induced from $\theta$. Set $p := \ker(\theta + \text{id})$. The subspace $p$ is identified with the tangent space $T_x N$ of $N$ at $eK$, where $e$ is the identity element of $G$. Let $M$ be a complete (oriented embedded) hypersurface in $N$. Fix a global unit normal vector field $v$ of $M$. Denote by $A$ the shape operator of $M$ (for $v$). Take $X \in T_x M$ ($x = gK$). The $M$-Jacobi field $Y$ along $\gamma_x$ with $Y(0) = X$ (hence $Y'(0) = -A_x X$) is given by

$$Y(s) = (P_{\gamma_x|[0,s]} \circ (D_{svx}^{\text{co}} - sD_{sxv}^{\text{ii}} \circ A_x))(X),$$

where $P_{\gamma_x|[0,s]}$ is the parallel translation along $\gamma_x|[0,s]$, $D_{svx}^{\text{co}}$ (resp. $D_{sxv}^{\text{ii}}$) is given by

$$D_{svx}^{\text{co}} = g_s \circ \cos(\text{lad}(s g_s^{-1} v_x)) \circ g_s^{-1}$$

(resp. $D_{sxv}^{\text{ii}} = g_s \circ \sin(\text{lad}(s g_s^{-1} v_x)) \circ g_s^{-1}$).

Here $\text{lad}$ is the adjoint representation of the Lie algebra $g$ of $G$. All focal radii of $M$ at $x$ are caught as real numbers $s_0$ with $\ker(D_{s_0 v_x}^{\text{co}} - s_0 D_{s_0 v_x}^{\text{ii}} \circ A_x) \neq \{0\}$. So, we \cite{18} defined the notion of a complex focal radius of $M$ at $x$ as a complex number $z_0$ with $\ker(D_{z_0 v_x}^{\text{co}} - z_0 D_{z_0 v_x}^{\text{ii}} \circ A_x) \neq \{0\}$, where $D_{z_0 v_x}^{\text{co}}$ (resp. $D_{z_0 v_x}^{\text{ii}}$) is a $\mathbb{C}$-linear transformation of $(T_x N)^{\mathbb{C}}$ defined by

$$D_{z_0 v_x}^{\text{co}} = g_z \circ \cos(\text{lad}^{\mathbb{C}}(z_0 g_z^{-1} v_x)) \circ (g_z^{-1})^{-1}$$

(resp. $D_{z_0 v_x}^{\text{ii}} = g_z \circ \frac{\sin(\text{lad}^{\mathbb{C}}(z_0 g_z^{-1} v_x))}{\text{lad}^{\mathbb{C}}(z_0 g_z^{-1} v_x)} \circ (g_z^{-1})^{-1}$),

where $g_z$ (resp. $\text{lad}^{\mathbb{C}}$) is the complexification of $g_s$ (resp. $\text{lad}$). Also, we call $\ker(D_{z_0 v_x}^{\text{co}} - z_0 D_{z_0 v_x}^{\text{ii}} \circ A_x)$ the focal space of the complex focal radius $z_0$ and its complex dimension the \textit{multiplicity} of the complex focal radius $z_0$. In \cite{19}, it was shown that, in the case where $M$ is of class $C^0$, complex focal radii of $M$ at $x$ indicate the positions of focal points of the extrinsic complexification $M^{\mathbb{C}} (\hookrightarrow G^{\mathbb{C}}/K^{\mathbb{C}})$ of $M$ along the complexified geodesic $\gamma_x^G$, where $G^{\mathbb{C}}/K^{\mathbb{C}}$ is the anti-Kaehlerian symmetric space associated with $G/K$. See \cite{19} (also \cite{26}) about the detail of the definition of the extrinsic complexification. Denote by $\mathcal{C} FR_x$ the set of all complex focal radii of $M$ at $x$. If $\mathcal{C} FR_x$ is independent of the choice of $x$, then $M$ is called a \textit{complex equivocal hypersurface}. Here we note that we should call such a hypersurface an equi-complex focal hypersurface but, for simplicity, we call it a complex equivocal hypersurface. This notion is the hypersurface version of a complex equivocal submanifold defined in \cite{18}.
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Next we recall the notion of an anti-Kaehlerian equivocal hypersurface in an anti-Kaehlerian symmetric space. Let \( J \) be a parallel complex structure on an even dimensional pseudo-Riemannian manifold \((M, \langle \cdot, \cdot \rangle)\) of half index. If \( \langle JX, JY \rangle = -\langle X, Y \rangle \) holds for every \( X, Y \in TM \), then \((M, \langle \cdot, \cdot \rangle, J)\) is called an anti-Kaehlerian manifold. Let \( N = G/K \) be a symmetric space of non-compact type and \( G^C/K^C \) the anti-Kaehlerian symmetric space associated with \( G/K \). See [19] about the anti-Kaehlerian structure of \( G^C/K^C \). Let \( f \) be an isometric immersion of an anti-Kaehlerian manifold \((M, \langle \cdot, \cdot \rangle, J)\) into \( G^C/K^C \). If \( \tilde{J} \circ f_\ast = f_\ast \circ J \), then \( M \) is called an anti-Kaehlerian submanifold immersed by \( f \). Let \( A \) be the shape tensor of \( M \). We have \( A_{\tilde{J}v}X = A_v(JX) = J(A_vX) \), where \( X \in TM \) and \( v \in T^\perp M \). If \( A_vX = aX + bJX \) \((a, b \in \mathbb{R})\), then \( X \) is called a \( J \)-eigenvector for \( a + bi \). Let \( \{e_i\}_{i=1}^n \) be an orthonormal system of \( T_vM \) such that \( \{e_i\}_{i=1}^n \cup \{je_i\}_{i=1}^n \) is an orthonormal base of \( T_vM \). We call such an orthonormal system \( \{e_i\}_{i=1}^n \) a \( J \)-orthonormal base of \( T_vM \). If there exists a \( J \)-orthonormal base consisting of \( J \)-eigenvectors of \( A_v \), then we say that \( A_v \) is diagonalizable with respect to a \( J \)-orthonormal base. Then we set \( \operatorname{Tr}_J A_v := \sum_{i=1}^n \lambda_i \) as \( A_v e_i = (\text{Re} \lambda_i)e_i + (\text{Im} \lambda_i)J e_i \) \((i = 1, \ldots, n)\). We call this quantity the \( J \)-trace of \( A_v \).

If, for each unit normal vector \( v \in M \), the shape operator \( A_v \) is diagonalizable with respect to a \( J \)-orthonormal tangent base, if the normal Jacobi operator \( R(v) \) preserves the tangent space \( T_vM \), then \( A_v \) and \( R(v) \) commute, then we call \( M \) a curvature-adapted anti-Kaehlerian submanifold, where \( R \) is the curvature tensor of \( G^C/K^C \). Assume that \( M \) is an anti-Kaehlerian hypersurface \( (i.e., \text{codim} M = 2) \) and that it is orientable. Denote by \( \exp^+ \) the normal exponential map of \( M \). Fix a global parallel orthonormal normal base \( \{v, Jv\} \) of \( M \). \( \exp^+(av_v + bJv_v) \) is a focal point of \( (M, x) \), then we call the complex number \( a + bi \) a complex focal radius along the geodesic \( \gamma_v \). Assume that the number \( (which may be 0 and \infty) \) of distinct complex focal radii along the geodesic \( \gamma_v \) is independent of the choice of \( x \in M \). Furthermore assume that the number is not equal to 0. Let \( \{r_{i,x}; i = 1, 2, \ldots\} \) be the set of all complex focal radii along \( \gamma_v \), where \( |r_{i,x}| < |r_{i+1,x}| \) or \( |r_{i,x}| = |r_{i+1,x}| \) and \( \text{Re} r_{i,x} > \text{Re} r_{i+1,x} \) or \( |r_{i,x}| = |r_{i+1,x}| \) and \( \text{Re} r_{i,x} = \text{Re} r_{i+1,x} \) and \( \text{Im} r_{i,x} + \text{Im} r_{i+1,x} = 0 \). Let \( r_i \) \((i = 1, 2, \ldots)\) be complex-valued functions on \( M \) defined by assigning \( r_{i,x} \) to each \( x \in M \). We call this function \( r_i \) the \( i \)-th complex focal radius function for \( v \). If the number of distinct complex focal radii along \( \gamma_v \) is independent of the choice of \( x \in M \), complex focal radius functions for \( v \) are constant on \( M \) and they have constant multiplicity, then \( M \) is called an anti-Kaehlerian equivocal hypersurface. We \((19)\) showed the following fact.

**Fact 3.** Let \( M \) be a complete embedded \( C^\infty \)-hypersurface in \( G/K \). Then \( M \) is complex equivocal if and only if \( M^C \) is anti-Kaehler equivocal.

Next we recall the notion of an anti-Kaehlerian isoparametric hypersurface in an infinite dimensional anti-Kaehlerian space. Let \( f \) be an isometric immersion of an anti-Kaehlerian Hilbert manifold \((M, \langle \cdot, \cdot \rangle, J)\) into an infinite dimensional anti-Kaehlerian space \((V, \langle \cdot, \cdot \rangle, \tilde{J})\).

See [19, Section 5] about the definitions of an anti-Kaehlerian Hilbert manifold and an infinite dimensional anti-Kaehlerian space. If \( \tilde{J} \circ f_\ast = f_\ast \circ J \) holds, then we call \( M \) an
anti-Kaehlerian Hilbert submanifold in $(V, \langle , \rangle, \tilde{J})$ immersed by $f$. If $M$ is of finite codimension and there exists an orthogonal time-space decomposition $V = V_{\pm} \oplus V_+$ such that $\tilde{J}V_{\pm} = V_{\mp}$, $(V, \langle , \rangle)_{V_{\pm}}$ is a Hilbert space, the distance topology associated with $(\langle , \rangle)_{V_{\pm}}$ coincides with the original topology of $V$ and, for each $v \in T^\perp M$, the shape operator $A_v$ is a compact operator with respect to $f^* (\langle , \rangle)_{V_{\pm}}$, then we call $M$ an anti-Kaehlerian Fredholm submanifold (rather than anti-Kaehlerian Fredholm Hilbert submanifold). Let $(M, \langle , \rangle, J)$ be an orientable anti-Kaehlerian Fredholm hypersurface in an anti-Kaehlerian space $(V, \langle , \rangle, \tilde{J})$ and $A$ be the shape tensor of $(M, \langle , \rangle, J)$. Fix a global unit normal vector field $v$ of $M$. If there exists $X(\neq 0) \in T_x M$ with $A_v X = aX + bJX$, then we call the complex number $a + bi$ a $J$-eigenvalue of $A_v$ (or a complex principal curvature of $M$ at $x$) and call $X$ a $J$-eigenvector of $A_v$ for $a + bi$. Here we note that this relation is rewritten as $A_v^C X^{(1,0)} = (a + bi)X^{(1,0)}$, where $X^{(1,0)} := \frac{1}{2}(X - iJX)$. Also, we call the space of all $J$-eigenvectors of $A_v$ for $a + b\sqrt{-1}$ a $J$-eigenspace of $A_v$ for $a + bi$. We call the set of all $J$-eigenvalues of $A_v$, the $J$-spectrum of $A_v$, and denote it by $\text{Spec}_J A_v$. $\text{Spec}_J A_v \setminus \{0\}$ is described as follows: 
\[
\text{Spec}_J A_v \setminus \{0\} = \{ \lambda_i ; i = 1, 2, \ldots \}
\]
\[
\left( |\lambda_i| > |\lambda_{i+1}| \quad \text{or} \quad "|\lambda_i| = |\lambda_{i+1}| \quad \text{and} \quad \text{Re} \lambda_i > \text{Re} \lambda_{i+1}" \right).
\]
Also, the $J$-eigenspace for each $J$-eigenvalue of $A_v$, other than 0 is of finite dimension. We call the $J$-eigenvalue $\lambda_i$ the $i$-th complex principal curvature of $M$ at $x$. Assume that the number (which may be $\infty$) of distinct complex principal curvatures of $M$ is constant over $M$. Then we can define functions $\tilde{\lambda}_i (i = 1, 2, \ldots)$ on $M$ by assigning the $i$-th complex principal curvature of $M$ at $x$ to each $x \in M$. We call this function $\tilde{\lambda}_i$ the $i$-th complex principal curvature function of $M$. If the number of distinct complex principal curvatures of $M$ is constant over $M$, each complex principal curvature function is constant over $M$ and it has constant multiplicity, then we call $M$ an anti-Kaehlerian isoparametric hypersurface. Let $\{e_1\}_{i=1}^\infty$ be an orthonormal system of $(T_x M, \langle , \rangle)$, and $\{e_1\}_{i=1}^\infty \cup \{J e_1\}_{i=1}^\infty$ is an orthonormal base of $T_x M$, then we call $\{e_1\}_{i=1}^\infty$ a $J$-orthonormal base. If there exists a $J$-orthonormal base consisting of $J$-eigenvectors of $A_v$, then $A_v$ is said to be diagonalized with respect to the $J$-orthonormal base. If $M$ is anti-Kaehlerian isoparametric and, for each $x \in M$, the shape operator $A_v$ is diagonalized with respect to a $J$-orthonormal base, then we call $M$ a proper anti-Kaehlerian isoparametric hypersurface.

In [18], we defined the notion of the parallel transport map for a semi-simple Lie group $G$ as a pseudo-Riemannian submersion of a pseudo-Hilbert space $H^0([0, 1], g)$ onto $G$. See [18] in detail. Also, in [19], we defined the notion of the parallel transport map for the complexification $G^C$ of a semi-simple Lie group $G$ as an anti-Kaehlerian submersion of an infinite dimensional anti-Kaehlerian space $H^0([0, 1], g^C)$ onto $G^C$. See [19] in detail. Let $G/K$ be a symmetric space of non-compact type and $\phi : H^0([0, 1], g^C) \to G^C$ the parallel transport map for $G^C$ and $\pi : G^C \to G^C/K^C$ the natural projection. We [19] showed the following fact.
FACT 4. Let $M$ be a complete anti-Kaehlerian hypersurface in an anti-Kaehlerian symmetric space $G^C/K^C$. Then $M$ is anti-Kaehlerian equifocal if and only if each component of $(\pi \circ \phi)^{-1}(M)$ is anti-Kaehlerian isoparametric.

Next we recall the notion of a focal point of non-Euclidean type on the ideal boundary $N(\infty)$ of a hypersurface $M$ in a Hadamard manifold $N$ which was introduced in [23] for a submanifold of general codimension. Assume that $M$ is orientable. Let $v$ be a unit normal vector field of $M$ and $\gamma_{v_x} : [0, \infty) \to N$ the normal geodesic of $M$ of direction $v_x$. If there exists an $M$-Jacobi field $Y$ along $\gamma_{v_x}$ satisfying $\lim_{t \to \infty} ||Y(t)||/t = 0$, then we call $\gamma_{v_x}(\infty) (\in N(\infty))$ a focal point of $M$ on the ideal boundary $N(\infty)$ along $\gamma_{v_x}$, where $\gamma_{v_x}(\infty)$ is the asymptotic class of $\gamma_{v_x}$. Also, if there exists an $M$-Jacobi field $Y$ along $\gamma_{v_x}$ satisfying $\lim_{t \to \infty} ||Y(t)||/t = 0$ and $\text{Sec}(v_x, Y(0)) \neq 0$, then we call $\gamma_{v_x}(\infty)$ a focal point of non-Euclidean type of $M$ on $N(\infty)$ along $\gamma_{v_x}$, where $\text{Sec}(v_x, Y(0))$ is the sectional curvature for the 2-plane spanned by $v_x$ and $Y(0)$. If, for any point $x$ of $M$, $\gamma_{v_x}(\infty)$ and $\gamma_{-v_x}(\infty)$ are not a focal point of non-Euclidean type of $M$ on $N(\infty)$, then we say that $M$ has no focal point of non-Euclidean type on the ideal boundary $N(\infty)$. According to [19, Theorem 1] and [23, Theorem A], we have the following fact.

FACT 5. Let $M$ be a curvature-adapted and isoparametric $C^\omega$-hypersurface in a symmetric space $N := G/K$ of non-compact type. Then the following conditions (i) and (ii) are equivalent:

(i) $M$ has no focal point of non-Euclidean type on the ideal boundary $N(\infty)$.

(ii) Each component of $(\pi \circ \phi)^{-1}(M^C)$ is proper anti-Kaehlerian isoparametric.

3. Proof of Theorem A. In this section, we shall prove Theorem A. Let $M$ be a curvature-adapted isoparametric hypersurface in a simply connected symmetric space $G/K$ of compact type, $v$ a unit normal vector field of $M$ and $C(\subset T^*_x M)$ the Coxeter domain (i.e., the fundamental domain (containing 0) of the Coxeter group of $M$ at $x$). The boundary $\partial C$ of $C$ consists of two points and it is described as $\partial C = \{r_1 v_x, r_2 v_x\} (r_2 < 0 < r_1)$. We may assume that $|r_1| \leq |r_2|$ by replacing $v$ with $-v$ if necessary. Note that the set $\mathcal{F}R_M$ of all focal radii of $M$ is equal to $\{k r_1 + (1-k)r_2 : k \in \mathbb{Z}\}$. Set $F_i := \{\gamma_{v_x}(r_i) : x \in M\}$ ($i = 1, 2$), which are all of focal submanifolds of $M$. The hypersurface $M$ is the $r_i$-tube over $F_i$ ($i = 1, 2$). Let $\pi$ be the natural projection of $G$ onto $G/K$ and $\phi$ the parallel transport map for $G$. Let $\tilde{M}$ be a component of $(\pi \circ \phi)^{-1}(M)$, which is an isoparametric hypersurface in $H^0([0, 1], g)$. The set $\mathcal{P}C_{\tilde{M}}$ of all principal curvatures other than zero of $\tilde{M}$ is equal to $\{\frac{1}{k r_1 + (1-k)r_2} : k \in \mathbb{Z}\}$. Set $\lambda_{2k-1} := \frac{1}{k r_1 + (1-k)r_2} (k = 1, 2, \ldots)$ and $\lambda_{2k} := -\frac{1}{(k-1)r_1 + kr_2} (k = 1, 2, \ldots)$. Then we have $|\lambda_{i+1}| < |\lambda_i|$ or $\lambda_i = -\lambda_{i+1} > 0$ for any $i \in \mathbb{N}$. Denote by $m_i$ the multiplicity of $\lambda_i$. Denote by $A (resp. \tilde{A})$ the shape operator of $M$ for $v$ (resp. $\tilde{M}$ for $v^\perp$), where $v^\perp$ is the horizontal lift of $v$ to $\tilde{M}$ with respect to $\pi \circ \phi$. Fix $r_0 \in \mathcal{F}R_M$. The focal map $f_{r_0} : M \to G/K$ is defined by $f_{r_0}(x) := \gamma_{v_x}(r_0) (x \in M)$. Let $F := f_{r_0}(M)$, which is either $F_1$ or $F_2$. Denote by $A^F$ the shape tensor of $F$ and $\psi_t$ the geodesic flow of $G/K$. 
PROOF OF THEOREM A. Define a set $S_x$ by

$$S_x := \{(\lambda, \mu) \in \text{Spec} A_x \times \text{Spec} R(v_x); \text{Ker}(A_x - \lambda I) \cap \text{Ker}(R(v_x) - \mu I) \neq \{0\}\}.$$ \hfill (1.1)

Since $M$ is curvature adapted, we have

$$T_x M = \bigoplus_{(\lambda,\mu) \in S_x} \left(\text{Ker}(A_x - \lambda I) \cap \text{Ker}(R(v_x) - \mu I)\right).$$ \hfill (1.2)

Define a distribution $D$ on $M$ by

$$D_x := \bigoplus_{(\lambda,\mu) \in S_x} \left(\text{Ker}(A_x - \lambda I) \cap \text{Ker}(R(v_x) - \mu I)\right).$$ \hfill (1.3)

Let $X \in \text{Ker}(A_x - \lambda I) \cap \text{Ker}(R(v_x) - \mu I)$ and $Y$ be the Jacobi field along $\gamma_{vr_0}$ with $Y(0) = X$ and $Y'(0) = -A_{r_0,v_x} X$. This Jacobi field $Y$ is described as

$$Y(s) = \left(\cos(s r_0 \sqrt{\mu}) - \frac{\lambda \sin(s r_0 \sqrt{\mu})}{\sqrt{\mu}}\right) P_{\gamma_{r_0}[0,s]}(X).$$ \hfill (2.1)

Since $Y(1) = f_{r_0} X$, we have

$$f_{r_0} X = \left(\cos(r_0 \sqrt{\mu}) - \frac{\lambda \sin(r_0 \sqrt{\mu})}{\sqrt{\mu}}\right) P_{\gamma_{r_0}}(X),$$ \hfill (2.2)

which is not equal to 0 because $(\lambda,\mu) \in S_{r_0}$. From this relation, we have $T f_{r_0}(X) F = P_{\gamma_{r_0}}(D)$. On the other hand, we have

$$\nabla_X Y_{\gamma_{r_0}} = \frac{1}{r_0} Y'(1) = -\left(\sqrt{\mu} \sin(r_0 \sqrt{\mu}) + \lambda \cos(r_0 \sqrt{\mu})\right) P_{\gamma_{r_0}}(X).$$ \hfill (2.3)

From (2.1) and (2.2), we have

$$A^F_{\gamma_{r_0}} f_{r_0} X = \frac{\mu + \lambda t_{r_0}(\mu)}{\lambda - t_{r_0}(\mu)} f_{r_0} X.$$
Hence we can derive the following relation:

\[
\text{Tr} A^F_{\phi_0(v_x)} = - \sum_{(\lambda, \mu) \in S^1_0} \frac{\mu + \lambda \tau_0(\mu)}{\lambda - \tau_0(\mu)} \times m_{\lambda, \mu},
\]

where \(S^1_0\) and \(m_{\lambda, \mu}\) are as in the statement of Theorem A. On the other hand, it is not difficult to show the existence of a transnormal function on \(G/K\) having \(M\) and \(F\) as a regular level and a singular level, respectively. Hence, according to [28, Theorem 1.3], \(F\) is austere and hence minimal. Therefore, we obtain the desired identity from (3.3). \(\square\)

4. The mean curvature of a proper anti-Kaehlerian Fredholm submanifold. In this section, we define the notion of a proper anti-Kaehlerian Fredholm submanifold and its mean curvature vector. Let \(M\) be an anti-Kaehlerian Fredholm submanifold in an infinite dimensional anti-Kaehlerian space \(V\) and \(A\) be the shape tensor of \(M\). Denote by the same symbol \(J\) the complex structures of \(M\) and \(V\). If \(A_v\) is diagonalized with respect to a \(J\)-orthonormal base for each unit normal vector \(v\) of \(M\), then we call \(M\) a proper anti-Kaehlerian Fredholm submanifold. Assume that \(M\) is such a submanifold. Let \(v\) be a unit normal vector of \(M\). If the series \(\sum_{i=1}^{\infty} m_i \lambda_i\) exists, then we call it the \(J\)-trace of \(A_v\) and denote it by \(\text{Tr}_J A_v\), where \(\{\lambda_i : i = 1, 2, \ldots\} = \text{Spec}_J A_v \setminus \{0\}\) (\(\lambda_i\)'s are ordered as stated in Section 2) and \(m_i = \frac{\dim \ker(A_v - \lambda_i I)}{\lambda_i I} (i = 1, 2, \ldots)\), where \(\lambda_i I\) means \((\text{Re} \lambda_i) I + (\text{Im} \lambda_i) J\). Note that, if \(\sharp(\text{Spec}_J A_v)\) is finite, then we promise \(\lambda_i = 0\) and \(m_i = 0\) \((i > \sharp(\text{Spec}_J A_v \setminus \{0\}))\), where \(\sharp(\cdot)\) is the cardinal number of \((\cdot)\). Define a normal vector field \(H\) of \(M\) by \((H_x, v) = \text{Tr}_J A_v (x \in M, v \in T^\perp_x M)\). We call \(H\) the mean curvature vector of \(M\).

Let \(G/K\) be a symmetric space of non-compact type and \(\phi : H^0([0, 1], g^C) \to G^C\) be the parallel transport map for the complexification \(G^C\) of \(G\) and \(\pi\) be the natural projection of \(G^C\) onto the anti-Kaehlerian symmetric space \(G^C/K^C\). We have the following fact, which will be used in the proof of Theorem B in the next section.

**Lemma 4.1.** Let \(M\) be a curvature-adapted anti-Kaehlerian submanifold in \(G^C/K^C\) and \(A\) (resp. \(\tilde{A}\)) be the shape tensor of \(M\) (resp. \((\pi \circ \phi)^{-1}(M))\). Assume that, for each unit normal vector \(v\) of \(M\) and each \(J\)-eigenvalue \(\mu\) of \(R(v)\), \(\ker(A_v - \sqrt{-1} \mu I) \cap \ker(R(v) - \mu I) = \{0\}\) holds. Then the following statements (i) and (ii) hold:

(i) \((\pi \circ \phi)^{-1}(M)\) is a proper anti-Kaehlerian Fredholm submanifold.

(ii) For each unit normal vector \(v\) of \(M\), \(\text{Tr}_J \tilde{A}_v = \text{Tr}_J A_v\) holds, where \(v^L\) is the horizontal lift of \(v\) to \((\pi \circ \phi)^{-1}(M)\) and \(\text{Tr}_J A_v\) is the \(J\)-trace of \(A_v\).

**Proof.** We can show the statement (i) in terms of [19, Lemmas 9, 12 and 13]. By imitating the proof of [18, Theorem C], we can show the statement (ii), where we also use the above lemmas in [19]. \(\square\)

5. Proofs of Theorems B and C. In this section, we first prove Theorem B. Let \(M\) be a curvature-adapted isoparametric \(C^\infty\)-hypersurface in a symmetric space \(G/K\) of non-compact type. Assume that \(M\) admits no focal point of non-Euclidean type on the ideal boundary of \(G/K\). Denote by \(A\) the shape tensor of \(M\) and \(R\) the curvature tensor of \(G/K\).
Let \( v \) be a unit normal vector field of \( M \), which is uniquely extended to a unit normal vector field of the extrinsic complexification \( M^C(\subset \mathbb{C}^C/K^C) \) of \( M \). Since \( M \) is a curvature-adapted isoparametric hypersurface admitting no focal point of non-Euclidean type on the ideal boundary \( N(\infty) \), it admits a complex focal radius. Let \( r_0 \) be one of complex focal radii of \( M \). The focal map \( f_{r_0} : M^C \to G^C/K^C \) for \( r_0 \) is defined by \( f_{r_0}(x) := \exp^\pm(r_0v_x)(:= y_0^C(r_0)) \) \((x \in M^C)\), where \( r_0v_x \) means \((\text{Re}r_0)v_x + (\text{Im}r_0)Jv_x \) \((J : \text{the complex structure of } G^C/K^C)\). Let \( F := f_{r_0}(M^C) \), which is an anti-Kaehlerian submanifold in \( G^C/K^C \) (see Figure 1). Without loss of generality, we may assume \( o := eK \in M \). Denote by \( \tilde{A} \) and \( A^F \) the shape tensor of \( M^C \) and \( F \), respectively. Let \( \psi_t \) be the geodesic flow of \( G^C/K^C \). Then we have the following fact.

**Lemma 5.1.** For any \( x \in M (\subset M^C) \), the following relation holds:

\[
\text{Tr}_F A^F_{\psi_0}\left(\frac{\gamma_0}{r_0}\right) = \frac{-r_0}{|r_0|} \sum_{(\lambda, \mu) \in S^0_c} \frac{\mu + \lambda \tilde{r}_0(\mu)}{\lambda - \tilde{r}_0(\mu)} \times m_{\lambda, \mu},
\]

where \( S^0_c \) and \( m_{\lambda, \mu} \) are as in the statement of Theorem B.

**Proof.** Let \( S_c := \{(\lambda, \mu) \in \text{Spec}A_{v_x} \times \text{Spec}R(v_x) : \text{Ker}(A_{v_x} - \lambda I) \cap \text{Ker}(R(v_x) - \mu I) \neq \{0\}\} \). Since \( M \) is curvature adapted, we have \( T_{x_k}M = \bigoplus_{(\lambda, \mu) \in S^0_c}(\text{Ker}(A_x - \lambda I) \cap \text{Ker}(R(v_x) - \mu I)) \) and \( D_x^+ \) the orthogonal complement of \( D_x \) in \( T_xM \). The tangent space \( T_x(M^C) \) is identified with the complexification \( T_xM^C \). Under this identification, the shape operator \( A_{v_x} \) is identified with the complexification \( A^C \) of \( A_x \). Let \( X \in \text{Ker}(A_x - \lambda I)^C \cap \text{Ker}(R(v_x) - \mu I)^C \) \((\lambda, \mu) \in S^0_c\) and \( Y \) be the Jacobi field along \( y_{v_0}y_{v_x} \) with \( Y(0) = X \) and \( Y'(0) = -\lambda \tilde{r}_0v_xX = -\lambda ((\text{Re}r_0)X + (\text{Im}r_0)JX) \), where \( y_{v_0}y_{v_x} \) is the geodesic in \( G^C/K^C \) with \( y_{v_0}v_0(0) = r_0v_x = (\text{Re}r_0)v_x + (\text{Im}r_0)Jv_x \). This Jacobi field \( Y \) is described as

\[
Y(s) = \left( \cos(is\sqrt{-\mu}) - \frac{\lambda \sin(i\sqrt{-\mu})}{\sqrt{-\mu}} \right) P_{y_{v_0}v_0}(X).
\]

Since \( Y(1) = f_{r_0}*X \), we have

\[
f_{r_0}*X = \left( \cos(is\sqrt{-\mu}) - \frac{\lambda \sin(i\sqrt{-\mu})}{\sqrt{-\mu}} \right) P_{y_{v_0}v_0}(X)
\]

which is not equal to 0 because \((\lambda, \mu) \in S^0_c\). This relation implies that \( T_{f_{r_0}(x)}F = P_{y_0}v_0(D^C_x) \).

On the other hand, we have

\[
\tilde{\nabla}_{f_{r_0}*X} \psi_{y_0}\left(\frac{r_0}{|r_0|}v_x \right) = \frac{1}{|r_0|}Y'(1)
\]

\[
= -\frac{1}{|r_0|} \left( i\sqrt{-\mu} \sin(i\sqrt{-\mu}) + \lambda \cos(i\sqrt{-\mu}) \right) P_{y_0}v_0(X).
\]

From (5.1) and (5.2), we have

\[
A^F_{\psi_0}\left(\frac{r_0}{|r_0|}v_x \right) f_{r_0}*X = \frac{-\frac{\mu}{|r_0|}}{\lambda - \tilde{r}_0(\mu)} \left( \mu + \lambda \tilde{r}_0(\mu) \right) f_{r_0}*X.
\]
The desired relation follows from this relation.

\[ \kappa(\lambda, \mu) := \frac{\kappa(\lambda + \mu, \nu)}{\lambda - \nu} \quad (\lambda, \mu \in \mathbb{R}). \]

**Lemma 5.2.** Let \((\lambda_1, \mu_1) \in S_0^\ell\). Then we have

1. \((\exp_{G^C} r_0 v_x)_x^{-1} \psi_{[\rho]} \left( \frac{r_0}{n_0} v_x \right) = \frac{r_0}{n_0} v_x\)
2. \((\exp_{G^C} r_0 v_x)_x^{-1} \left( \ker(A_{v_0}) \cap \ker(R(v_x) - \mu I) \right) = \bigoplus_{(\lambda, \mu) \in S_0^\ell} \left( \ker(A_{v_0} - \lambda I) \cap \ker(R(v_x) - \mu I) \right)^C\)

where \(S_0^\ell \cap \ker(R(v_x) - \mu I) \neq \emptyset\).

**Proof.** The relation of (i) is trivial. Let \((\lambda, \mu) \in S_0^\ell \cap \ker(R(v_x) - \mu I)\). The restriction \(f_{r_0, v_x}^{G^C} \cap \ker(R(v_x) - \mu I)^C\) of \(f_{r_0, v_x}\) is equal to \(P_{\rho_0, v_x} \cap \ker(R(v_x) - \mu I)^C\) up to constant multiple by (5.1). Also, we have \(P_{\gamma_0, v_x} = (\exp_{G^C} r_0 v_x)_x\). These facts together with (5.3) deduce

\[ (\exp_{G^C} r_0 v_x)_x \left( \ker(A_{v_0} - \lambda I) \cap \ker(R(v_x) - \mu I) \right) = f_{r_0, v_x} \left( \ker(A_{v_0} - \lambda I) \cap \ker(R(v_x) - \mu I) \right)^C \subset \ker(A_{v_0}) \cap \ker(A_{v_0} - \mu I) \right)^C \]

From this fact, the relation of (ii) follows. Now we shall show the statement (iii). Let \(r_0 = a_0 + b_0 \sqrt{-1} \in \mathbb{C}\). Suppose that \(\kappa(\lambda_1, \mu_1) = \pm \sqrt{-1}\). By squaring both sides of this relation, we have

\[ (\hat{r}_0 (\mu_1)^2 + \mu_1) (\hat{\nu}_0^2 + \mu_1) = 0. \]

Hence we have \(\lambda_1 = \pm \sqrt{-1}\). Thus the statement (iii) is shown.

Denote by \(\hat{R}\) the curvature tensor of \(G^C / K^C\). By using these lemmas, we prove Theorem B. According to Lemma 5.1, we have only to show \(\text{Tr}_A F = 0 \quad (x \in M)\). In the case where \(M\) is homogeneous, we can show this relation by imitating the process of the proof of [15, Corollary 1.1].

**Simple proof of Theorem B in Rank One Case.** We have only to show \(\text{Tr}_A F = 0\). Assume that \(G / K\) is of rank one. Define a complex linear function \(\Phi : T_{r_0(v_x)}^+ F \to \mathbb{C}\) by \(\Phi(w) = \text{Tr}_A F (w \in T_{r_0(v_x)}^+ F)\). Since \(M\) is curvature-adapted, we have \(T_A M = \bigoplus_{(\lambda, \mu) \in S_0^\ell} \left( \ker(A_{v_0} - \lambda I) \cap \ker(R(v_x) - \mu I) \right)\). Set

\[ S_0^\ell := \{ (\lambda, \mu) \in \text{Spec} \hat{A}_{v_0} \times \text{Spec} \hat{R}(v_x); \ker(A_{v_0} - \lambda I) \cap \ker(R(v_x) - \mu I) \neq \{0\} \}\]

where \(\lambda \neq \hat{r}_0 (\mu_1)\).
(\gamma \in M^C). Define a distribution \( \hat{D} \) on \( M^C \) by
\[
\hat{D}_\gamma := \bigoplus_{(\lambda,\mu) \in \hat{S}^r_0} (\text{Ker}(\hat{A}_{\gamma} - \lambda I) \cap \text{Ker}(\hat{R}(v_\gamma) - \mu I)) \quad (\gamma \in M^C)
\]
and \( \hat{D}^\perp \) the orthogonal complementary distribution of \( \hat{D} \) in \( T(M^C) \). Also, define a distribution \( \hat{D} \) on \( M \) by \( D_x := \bigoplus_{(\lambda,\mu) \in \hat{S}^r_0} (\text{Ker}(A_x - \lambda I) \cap \text{Ker}(R(v_x) - \mu I)) \) (\( x \in M \)) and \( D^\perp \) the orthogonal complementary distribution of \( D \) in \( TM \). Under the identification of \( T_x(M^C) \) with \( (T_xM)_1 \), \( \hat{D}_x \) is identified with the complexification \( (D_x)_1 \) of \( D_x \). The focal map \( f_{r_0} \) is a submersive of \( \hat{S}^r \) onto \( F \) and the fibres of \( f_{r_0} \) are integral manifolds of \( \hat{D}^\perp \). Let \( L \) be the integral manifold of \( \hat{D}^\perp \) through \( x \) and set \( L_{\mathbb{R}} := L \cap M \). It is shown that \( L \) is the extrinsic complexification of \( L_{\mathbb{R}} \). Set \( Q := \{ \psi_{r_0}(\frac{r_0}{|r_0|}v_x) ; x \in L \} \) and \( Q_{\mathbb{R}} := \{ \psi_{r_0}(\frac{r_0}{|r_0|}v_x) ; x \in L_{\mathbb{R}} \} \). It is shown that \( Q \) is the extrinsic complexification of \( Q_{\mathbb{R}} \) and that \( Q \) is a complex hypersurface without geodesic point in \( T^L_{f_{r_0}(x)}F \), that is, it is not contained in any complex affine hyperplane of \( T^L_{f_{r_0}(x)}F \). According to Lemma 5.1, we have
\[
\Phi(\psi_{r_0}(\frac{r_0}{|r_0|}v_x)) = -\frac{r_0}{|r_0|} \sum_{(\lambda,\mu) \in \hat{S}^r_0} \frac{\mu + \lambda \tilde{r}_0(\mu)}{\lambda - \tilde{r}_0(\mu)} \times m_{\lambda,\mu}.
\]
Let \( (\tilde{\lambda}, \tilde{\mu}) \) be a pair of continuous functions on \( L_{\mathbb{R}} \) such that \( (\tilde{\lambda}(y), \tilde{\mu}(y)) \in \hat{S}^r_y \) for any \( y \in L \). Since \( G/K \) is of rank one, \( \tilde{\mu} \) is constant on \( L_{\mathbb{R}} \). The complex focal radius having \( \text{Ker}(A_x - \tilde{\lambda}(y) I) \cap \text{Ker}(R(v_x) - \tilde{\mu}(y) I) \) as a part of the focal space is the complex number \( z_0 \) satisfying \( \text{Ker}(D_0^{C^0} - z_0 D_0^{C^0} \circ A_0^C) \cap \text{Ker}(A_x - \tilde{\lambda}(y) I) \cap \text{Ker}(R(v_x) - \tilde{\mu}(y) I) \neq \{0\} \), that is, it is equal to \( (1/\sqrt{\tilde{\mu}(y)}) \arctan(\sqrt{\frac{\tilde{\lambda}(y)}{\tilde{\mu}(y)}}) \), which is independent of the choice of \( y \in L_{\mathbb{R}} \) by the isoparametricity (hence complex equifocality) of \( M \). Hence \( \tilde{\lambda} \) is constant on \( L_{\mathbb{R}} \). Therefore \( \Phi \) is constant along \( Q_{\mathbb{R}} \). Since \( \Phi \) is of class \( C^0 \) and \( Q_{\mathbb{R}} \) is a half-dimensional totally real submanifold in \( Q \), \( \Phi \) is constant along \( Q \). Furthermore, this fact together with the linearity of \( \Phi \) imply \( \Phi \equiv 0 \). In particular, we have \( \text{Tr} A^F_{\psi_{r_0}(v_x)} = 0 \). \( \square \)

**Proof of Theorem B (General Case).** According to Lemma 5.1, we have only to show \( \text{Tr} A^F_{\psi_{r_0}(\frac{r_0}{|r_0|}v_x)} = 0 \) (\( x_0 \in M \)). We shall show this relation by investigating the focal submanifold of \( (\pi \circ \phi)^{-1}(M^C) \) corresponding to \( r_0 \), where \( \phi : H^0([0, 1], g^C) \to G^C \) is the parallel transport map for \( G^C \) and \( \pi \) is the natural projection of \( G^C \) onto \( G^C/K^C \). Let \( M^C \) be the complete extension of \( (\pi \circ \phi)^{-1}(M^C) \). Let \( v^L \) be the horizontal lift of \( v \) to \( \tilde{M}^C \). Since \( \pi \circ \phi \) is an anti-Kaehlerian submersion, the complex focal radii of \( M^C \) (hence \( M \)) are those of \( \tilde{M}^C \). Let \( r_0 \) be a complex focal radius of \( M \) (hence \( \tilde{M}^C \)). The focal map \( \tilde{f}_{r_0} \) for \( r_0 \) is defined by \( \tilde{f}_{r_0}(x) = x + r_0 v^L_x \) (\( x \in \tilde{M}^C \)). Set \( \tilde{F} := \tilde{f}_{r_0}(\tilde{M}^C) \). Denote by \( \tilde{A} \) (resp. \( A^F \)) the shape tensor of \( \tilde{M}^C \) (resp. \( \tilde{F} \)). Let \( \text{Spec} \tilde{A}_{\tilde{x}_0} \setminus \{0\} = \{ \lambda_i ; i = 1, 2, \ldots \} \) ("\( |\lambda_i| > |\lambda_{i+1}| \) or "\( |\lambda_i| = |\lambda_{i+1}| \) & \( \text{Re}\lambda_i > \text{Re}\lambda_{i+1} \) or "\( |\lambda_i| = |\lambda_{i+1}| \) & \( \text{Re}\lambda_i = \text{Re}\lambda_{i+1} \) & \( \text{Im}\lambda_i = -\text{Im}\lambda_{i+1} > 0 \)"). The set of all complex focal radii of \( M^C \) (hence \( M \)) is equal to \( \{1/\lambda_i; i = 1, 2, \ldots \} \). We have \( r_0 = 1/\lambda_{i_0} \) for some \( i_0 \). Define a distribution \( \tilde{D}_i \) (\( i = 0, 1, 2, \ldots \)) on \( \tilde{M}^C \) by
\( (\tilde{D}_0)_u := \text{Ker}\tilde{\Lambda}_{\tilde{g}}^{\perp} \) and \( (\tilde{D}_i)_u := \text{Ker}(\tilde{\Lambda}_{\tilde{g}}^{\perp} - \lambda_i I) \) \((i = 1, 2, \ldots)\), where \( u \in \tilde{M}^c \). Since \( M \) is a curvature-adapted isoparametric submanifold admitting no focal point of non-Euclidean type on \( N(\infty) \), \( \tilde{M}^c \) is proper anti-Kaehlerian isoparametric by Fact 5. Therefore, we have \( T\tilde{M}^c = \tilde{D}_0 \oplus (\bigoplus_{i} \tilde{D}_i) \) and \( \text{Spec}_j\tilde{\Lambda}_{\tilde{g}}^{\perp} \) is independent of the choice of \( u \in \tilde{M}^c \). Take \( u_0 \in \tilde{M}^c \) with \((\pi \circ \phi)(u_0) = x_0\). Let \( X_i \in (\tilde{D}_i)_{u_0} \) \((i \neq i_0)\) and \( X_0 \in (\tilde{D}_0)_{u_0} \). Then we have \( \tilde{f}_{r_0}X_i = (1 - r_0\lambda_i)X_i \) and \( \tilde{f}_{r_0}X_0 = X_0 \). Hence we have \( T\tilde{f}_{r_0}(u_0)\tilde{F} = (\tilde{D}_0)_{u_0} \oplus (\bigoplus_{i \neq i_0} (\tilde{D}_i)_{u_0}) \) and \( \text{Ker}(\tilde{f}_{r_0})_{u_0} = (\tilde{D}_0)_{u_0} \), which implies that \( \tilde{D}_0 \) is integrable. On the other hand, we have \( \tilde{A}_{\tilde{F}(0)}(\frac{m_i}{\lambda_i}v_{u_0})\tilde{f}_{r_0}X_i = (\lambda_i r_0)/|r_0|X_i \) and \( \tilde{A}_{\tilde{F}(0)}(\frac{m_i}{\lambda_i}v_{u_0})\tilde{f}_{r_0}X_0 = 0 \), where \( \tilde{\psi} \) is the geodesic flow of \( H^0([0, 1], \mathfrak{g}^C) \). Therefore, we obtain \( \tilde{A}_{\tilde{F}(0)}(\frac{m_i}{\lambda_i}v_{u_0})\tilde{f}_{r_0}X_i = \frac{\lambda_i |r_0|}{\lambda_i - \lambda_i} \tilde{f}_{r_0}X_i \). Hence we have

\[
\text{Tr}_J \tilde{A}_{\tilde{F}(0)}(\frac{m_i}{\lambda_i}v_{u_0}) = \sum_{i \neq 0} \frac{\lambda_i |r_0|}{\lambda_i - \lambda_i} \times m_i, \quad \text{where} \quad m_i := \frac{4}{\dim \tilde{D}_i}.
\]

According to Theorem 2 of [19], each leaf of \( \tilde{D}_0 \) is a complex sphere. Let \( L \) be the leaf of \( \tilde{D}_0 \) through \( u_0 \) and \( u_0^* \) be the anti-podal point of \( u_0 \) in the complex sphere \( L \). Similarly we can show

\[
\text{Tr}_J \tilde{A}_{\tilde{F}(0)}(\frac{m_i}{\lambda_i}v_{u_0}) = \sum_{i \neq 0} \frac{\lambda_i |r_0|}{\lambda_i - \lambda_i} \times m_i. \quad \text{Thus we have} \quad \text{Tr}_J \tilde{A}_{\tilde{F}(0)}(\frac{m_i}{\lambda_i}v_{u_0}) = \text{Tr}_J \tilde{A}_{\tilde{F}(0)}(\frac{m_i}{\lambda_i}v_{u_0}^*).
\]

On the other hand, it follows from \( \tilde{\psi}_{[r_0]}(\frac{m_i}{\lambda_i}v_{u_0}^*) = -\tilde{\psi}_{[r_0]}(\frac{m_i}{\lambda_i}v_{u_0}) \) that \( \text{Tr}_J \tilde{A}_{\tilde{F}(0)}(\frac{m_i}{\lambda_i}v_{u_0}) = -\text{Tr}_J \tilde{A}_{\tilde{F}(0)}(\frac{m_i}{\lambda_i}v_{u_0}^*). \)

Hence we obtain

\[
(5.4) \quad \text{Tr}_J \tilde{A}_{\tilde{F}(0)}(\frac{m_i}{\lambda_i}v_{u_0}) = 0.
\]

It follows from (i) and (ii) of Lemma 5.2 that \( F := f_{r_0}(M^C) \) is a curvature adapted anti-Kaehlerian submanifold. Also, it follows from (iv) of Remark 1.2, (5.3), (i) and (iii) of Lemma 5.2 that, for each unit normal vector \( w \) of \( F \) and each \( \mu \in \text{Spec}_jR(w) \setminus \{0\} \), Ker\((A^F_w \pm \sqrt{-\mu}I) \cap \text{Ker}(R(w) - \mu I) = \{0\} \) holds. Therefore, it follows from Lemma 4.1 that \( \tilde{F} \) is a proper anti-Kaehlerian Fredholm submanifold and, for each unit normal vector \( w \) of \( F \), we have \( \text{Tr}_J A^F_w = \text{Tr}_J A^F_w \). It is clear that \( \tilde{\psi}_{[r_0]}(\frac{m_i}{\lambda_i}v_{u_0}) \) is the horizontal lift of \( \psi_{[r_0]}(\frac{m_i}{\lambda_i}v_{u_0}) \) to \( \tilde{f}_{r_0}(u_0) \). Hence we have

\[
(5.5) \quad \text{Tr}_J \tilde{A}_{\tilde{F}(0)}(\frac{m_i}{\lambda_i}v_{u_0}) = \text{Tr}_J \tilde{A}_{\tilde{F}(0)}(\frac{m_i}{\lambda_i}v_{u_0}^*).
\]

From (5.4) and (5.5), we have \( \text{Tr}_J \tilde{A}_{\tilde{F}(0)}(\frac{m_i}{\lambda_i}v_{u_0}) = 0 \). This completes the proof. \( \square \)

Now we prepare the following lemma to prove Theorem C.

**Lemma 5.3.** Let \( M \) be a curvature-adapted isoparametric \( C^\infty \)-hypersurface in a symmetric space \( N := G/K \) of non-compact type. Assume that \( M \) has no focal point of non-Euclidean type on \( N(\infty) \). Then, for any complex focal radius \( r \) of \( M \), we have

\[
\text{Spec} \left( A_{r|\text{Ker}R(v_i)} \right) \subset \left\{ \frac{1}{\text{Re } r}, 0 \right\}
\]
and
\[
\text{Spec} \left( A_x |_{\text{Ker}(R(v_x) - \mu I)} \right) \subset \left\{ \frac{\sqrt{-\mu}}{\tanh(\sqrt{-\mu}r)}, \sqrt{-\mu} \tanh(\sqrt{-\mu}r) \right\}
\]
for \( \mu \in \text{Spec} R(v_x) \setminus \{0\} \), where \( x \) is an arbitrary point of \( M \).

PROOF. For simplicity, we set \( D_\mu := \text{Ker}(R(v_x) - \mu \text{id}) \) for each \( \mu \in \text{Spec} R(v_x) \). Let \( r_0 \) be the complex focal radius of \( M \) with \( \text{Re} r_0 = \max \text{Re} r \), where \( r \) runs over the set of all complex focal radii of \( M \). Let \( (\lambda, \mu) \in S_{r_0}^* \setminus \{(0, 0)\} \) and \( r \) a complex focal radius including \( \text{Ker}(A_\mu - \lambda I) \cap D_\mu \) as the focal space, that is, \( \lambda = \hat{\tau}(\mu) \) (see (ii) of Remark 1.2).

Set \( c_{\lambda, \mu} := -\frac{\mu + \sqrt{\lambda}}{\lambda - r_0(\mu)} \). We shall show \( \text{Re} c_{\lambda, \mu} \leq 0 \). The argument divides into the following three cases:

(i) \( \mu = 0 \), (ii) \( 0 < \sqrt{-\mu} < |\lambda| \), (iii) \( |\lambda| < \sqrt{-\mu} \).

First we consider the case (i). Then we have \( c_{\lambda, \mu} = \frac{\lambda}{1 - \lambda r_0} \). Also, we can show \( \lambda = 1/r \).

Hence we have
\[
(5.6) \quad c_{\lambda, \mu} = \frac{1}{r - r_0}.
\]

Furthermore, we have \( \text{Re} c_{\lambda, \mu} \leq 0 \) from the choice of \( r_0 \). Next we consider the case (ii). Since \( \lambda = \hat{\tau}(\mu) \) and \( \lambda \) is a real number with \( |\lambda| > \sqrt{-\mu} \), we can show \( \lambda = \hat{\tau}(\mu) = \frac{\sqrt{-\mu}}{\tanh(\sqrt{-\mu}r)} \) and \( r \equiv \text{Re} r \mod (\pi i)/\sqrt{-\mu} \). Hence we have \( c_{\lambda, \mu} = \hat{\tau}(r_0 - \text{Re} r)(\mu) \), where we note that \( \text{Re} r \not\equiv r_0 \mod (\pi i)/\sqrt{-\mu} \) because \( (\lambda, \mu) \in S_{r_0}^* \). Therefore, we obtain
\[
(5.7) \quad \text{Re} c_{\lambda, \mu} = \frac{\sqrt{-\mu}}{\text{tanh}(\sqrt{-\mu}r_0)} \text{tanh}(\sqrt{-\mu}(\text{Re} r - r_0)) \frac{1}{\text{tan}^2(\sqrt{-\mu}r_0) + \text{tan}^2(\sqrt{-\mu}r_0)} \leq 0.
\]

because \( \text{Re} r \leq r_0 \). Next we consider the case (iii). Since \( \lambda = \hat{\tau}(\mu) \) and \( \lambda \) is a real number with \( |\lambda| < \sqrt{-\mu} \), we can show \( \lambda = \hat{\tau}(r_0 - \text{Re} r)(\mu) \) and \( r \equiv \text{Re} r \mod \left(\frac{\pi}{\sqrt{-\mu}}\right) \). Hence we have \( c_{\lambda, \mu} = \hat{\tau}(r_0 - \text{Re} r + \frac{\pi}{\sqrt{-\mu}})(\mu) \). Therefore, we obtain
\[
(5.8) \quad \text{Re} c_{\lambda, \mu} = \frac{\sqrt{-\mu}}{\text{tanh}(\sqrt{-\mu}(\text{Re} r - r_0))} \text{tanh}(\sqrt{-\mu}(r_0 - \text{Re} r))(\mu) \frac{1}{\text{tan}^2(\sqrt{-\mu}(r_0 - \text{Re} r)) + \text{tan}^2(\sqrt{-\mu}(r_0 - \text{Re} r))} \leq 0.
\]

Thus \( \text{Re} c_{\lambda, \mu} \leq 0 \) is shown in general. Hence, from the identity in Theorem B, \( \text{Re} c_{\lambda, \mu} = 0 \) \((\lambda, \mu) \in S_{r_0}^* \) follows, where we note that \( c_{0, 0} = 0 \). In case of (i), it follows from (5.6) that \( \text{Re} \left( \frac{1}{r_0} \right) = 0 \). Hence we have \( \text{Re} r \equiv \text{Re} r_0(< \infty) \) or \( r = \infty \). If \( \text{Re} r = \text{Re} r_0(< \infty) \), then we have \( \lambda = 1/r = 1/\text{Re} r_0 = \hat{\tau}(r_0)(0) \) (which does not happen if \( r_0 \) is real because \( (\lambda, 0) \in S_{r_0}^* \)). Also, if \( r = \infty \), then we have \( \lambda = 0 \). Thus we have
\[
(5.9) \quad \text{Spec}(A_x |_{D_0}) \subset \left\{ \frac{1}{\text{Re} r_0}, 0 \right\}.
\]

In case of (ii), it follows from (5.7) that \( \text{Re} r = \text{Re} r_0 \). Hence we have \( \lambda = \hat{\tau}(r_0)(\mu) \) (which does not happen if \( r_0 \equiv \text{Re} r_0 \mod (\pi i)/\sqrt{-\mu} \) because \( (\lambda, \mu) \in S_{r_0}^* \)). In case of (iii), it
follows from (5.8) that \( \text{Re} r = \text{Re} r_0 \). Hence we have \( \lambda = \tilde{r}_{(\text{Re} r_0 + \frac{i}{2\sqrt{-\mu}})}(\mu) \) (which does not happen if \( r_0 \equiv \text{Re} r_0 + \frac{i}{2\sqrt{-\mu}} \pmod{\pi i}/\sqrt{-\mu} \)) because \( \lambda, \mu \in S^1_{r_0} \). Hence we have

\[
\text{Spec}(A_x \mid D_{\mu}) \subset \left\{ \frac{\sqrt{\mu}}{\tanh(\sqrt{-\mu}R_0)}, \frac{\sqrt{-\mu}}{\tanh(\sqrt{-\mu}R_0)} \right\}.
\]

This completes the proof. \( \square \)

Next we prove Theorem C in terms of this Lemma and its proof.

**Proof of Theorem C.** According to the proof of Lemma 5.3, the real parts of complex focal radii of \( M \) coincide with one another. Denote by \( s_0 \) this real part. Then, according to Lemma 5.3, we have

\[
\text{Spec}(A_x \mid D_{\mu}) \subset \left\{ \frac{1}{s_0} \right\}
\]

and

\[
\text{Spec}(A_x \mid D_{\mu}) \subset \left\{ \frac{\sqrt{\mu}}{\tanh(\sqrt{-\mu}S_0)}, \frac{\sqrt{-\mu}}{\tanh(\sqrt{-\mu}S_0)} \right\} \quad (\mu \in \text{Spec}(R(v_x) \setminus \{0\})).
\]

Set \( D_0^V := \text{Ker} \left( A_x \mid D_{\mu} - \frac{1}{s_0} \text{id} \right) \), \( D_0^H := \text{Ker} A_x \mid D_{\mu} \),

\[
D_0^V := \text{Ker} \left( A_x \mid D_{\mu} - \frac{\sqrt{\mu}}{\tanh(\sqrt{-\mu}S_0)} \text{id} \right)
\]

and

\[
D_0^H := \text{Ker} \left( A_x \mid D_{\mu} - \frac{\sqrt{\mu}}{\tanh(\sqrt{-\mu}S_0)} \text{id} \right).
\]

According to (ii) of Remark 1.2, if \( D_0^V \oplus \left( \bigoplus_{\mu \in \text{Spec}(R(v_x) \setminus \{0\})} D_0^V \right) \neq \{0\} \), then \( s_0 \) is a (real) focal radius of \( M \) whose focal space is equal to \( D_0^V \oplus \left( \bigoplus_{\mu \in \text{Spec}(R(v_x) \setminus \{0\})} D_0^V \right) \neq \{0\} \). Let \( \eta_{sv} \) \((s \in \mathbb{R})\) be the end-point map for \( sv \). Set \( M_{sv} := \eta_{sv}(M) \). Set \( F := M_{sv} \). If \( s_0 \) is a (real) focal radius of \( M \), then \( F \) is the only focal submanifold of \( M \), and if \( s_0 \) is not a (real) focal radius of \( M \), then \( F \) is a parallel submanifold of \( M \). Without loss of generality, we may assume that \( \epsilon K \in F \). Define a unit normal vector field \( v^s \) of \( M_{sv} \) \((0 \leq s < s_0)\) by \( v^s_{\eta_{sv}(x)} = \gamma^s_{sv}(x) \) \((x \in M)\). Denote by \( A^t \) \((0 \leq s < s_0)\) the shape operator of \( M_{sv} \) \((\text{for} v^s)\) and \( A^f \) the shape tensor of \( F \). Set \( (D_0^V)^t := (\eta_{sv})_*(D_0^V) \) \((0 \leq s < s_0)\) and \( (D_0^H)^t := (\eta_{sv})_*(D_0^H) \) \((0 \leq s < s_0, \mu \in \text{Spec}(R(v_x) \setminus \{0\})\). Also, set \( (D_0^V)^f := (\eta_{sv})_*(D_0^V) \) \((s \in \mathbb{R})\) and \( (D_0^H)^f := (\eta_{sv})_*(D_0^H) \) \((s \in \mathbb{R}, \mu \in \text{Spec}(R(v_x) \setminus \{0\})\). Easily we have

\[
T_{\eta_{sv}(x)} F = (D_0^V)^t_{\eta_{sv}(x)} \bigoplus \left( \bigoplus_{\mu \in \text{Spec}(R(v_x) \setminus \{0\})} (D_0^H)^f_{\eta_{sv}(x)} \right).
\]

Also, we can show

\[
A^f_{\eta_{sv}(x)}|_{(D_0^H)^f_{\eta_{sv}(x)}} = 0 \quad (0 \leq s < s_0)
\]

and

\[
A^f_{\eta_{sv}(x)}|_{(D_0^V)^f_{\eta_{sv}(x)}} = \mu \tanh(\sqrt{-\mu}(s_0 - s)) \text{id} \quad (0 \leq s < s_0).
\]
FIGURE 3.

\[ D_x^H := (D_0^H)_x \oplus \left( \bigoplus_{\beta \in \Delta_+ \cap \mathbb{R}_x} (D_\beta^H)_x \right) \]

\[ (D_H^s)_{\psi}(s) := \left( \bigoplus_{\beta \in \Delta_+ \cap \mathbb{R}_x} (D_\beta^H)^s_{\psi}(s) \right) \]

Hence we have

\[ A^F_{\psi_{t_0}(v_x)} | (D_0^H)_{\psi_{t_0}(x) = 0} = 0 \]

and

\[ A^F_{\psi_{t_0}(v_x)} | (D_0^H)_{\psi_{t_0}(x)} = 0 \]

where \( \psi \) is the geodesic flow of \( G/K \). From these relations and (5.11), we obtain \( A^F_{\psi_{t_0}(v_x)} = 0 \). Since this relation holds for any \( x \in M \), \( F \) is totally geodesic. Denote by \( \exp^\perp \) the normal exponential map for \( F \). Since the real parts of complex focal radii of \( M \) coincide with one another, the normal umbrella \( \exp^\perp(T^\perp F) \) do not intersect with one another. From this fact, an involutive diffeomorphism \( \tau : G/K \to G/K \) having \( F \) as the fixed point set is well-defined by \( \tau(\exp^\perp(w)) := \exp^\perp(-w) \) \((w \in T^\perp F)\). For each \( s \in \mathbb{R} \setminus \{s_0\} \), the restriction \( \tau|_{M_s} \) of \( \tau \) to \( M_s \) coincides with the end-point map \( \eta_{2(s_0 - s)^v} \) for \( 2(s_0 - s)^v \). Since \( F \) is totally geodesic, we see that \( \eta_{2(s_0 - s)^v} \) (hence \( \tau|_{M_s} \)) is an isometry of \( M_s \). From this fact, it follows that \( \tau \) is an isometry of \( G/K \). Hence \( F \) is reflective. Furthermore, by imitating the proof of [16, Proposition 1.12], we can show that \( F \) is an orbit of a Hermann action on \( G/K \) as follows. Take \( \text{Exp} \) \( Z_0 \in F \), where \( \text{Exp} \) is the exponential map of \( G/K \) at \( o \). Set \( m := \text{Ad}(\text{exp}(-Z_0))(\text{exp}(-Z_0))^{-1}(T_{\text{Exp} Z_0} F) \), where \( \text{Ad} \) is the adjoint operator of \( G \). Define a subalgebra \( \mathfrak{t}' \) of \( \mathfrak{g} \) by \( \mathfrak{t}' := \{ X \in \mathfrak{t} ; \text{ad}(X)m = m \} \) and set \( \mathfrak{h} := \mathfrak{t}' + m \), which is a subalgebra of \( \mathfrak{g} \). Set \( H := I(\text{exp} Z_0)(\text{exp}(h)) \), where \( I(\text{exp} Z_0) \) is the inner automorphism of \( G \) by
exp Z_0. Easily we can show that T_{\exp Z_0}(H \exp Z_0) = T_{\exp Z_0} F and hence H \exp Z_0 = F. Define an involution \hat{\tau} of G by \hat{\tau}(g) := \tau \circ g \circ \tau^{-1} (g \in G). It is easy to show that (Fix \hat{\tau})_0 \subset H \subset Fix \hat{\tau}. Thus H \lhd G/K is a Hermann action. Let H^C be the complexification of H and M^C(\subset G^C/K^C) be the complete complexification of M. See [22] about the definition of the complete complexification of M. Since both H^C \cdot o and M^C are anti-Kaehler equifocal submanifolds having F^C as a focal submanifold, they are equal to one of the partial tubes over F^C stated in Section 5 in [22]. Thus they coincides with each other. Furthermore, from this fact, we can derive H \cdot o = M. This completes the proof.

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