DEFORMATION PRINCIPLE AND ANDRÉ MOTIVES OF PROJECTIVE HYPERKÄHLER MANIFOLDS

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Abstract. Let $X_1$ and $X_2$ be deformation equivalent projective hyperkähler manifolds. We prove that the André motive of $X_1$ is abelian if and only if the André motive of $X_2$ is abelian. Applying this to manifolds of $K3^{[n]}$, generalized Kummer and OG6 deformation types, we deduce that their André motives are abelian. As a consequence, we prove that all Hodge classes in arbitrary degree on such manifolds are absolute. We discuss applications to the Mumford-Tate conjecture, showing in particular that it holds for even degree cohomology of such manifolds.

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1. Introduction

In this paper, we study André motives of projective hyperkähler manifolds. By a hyperkähler manifold we mean a compact simply connected Kähler manifold $X$ such that $H^0(X, \Omega^2_X)$ is spanned by a symplectic form. We generalize the results of [3] and [30], showing that for most of the known deformation types of hyperkähler manifolds their André motives are abelian.

1.1. André motives and hyperkähler manifolds. André motives were introduced in [2] as a refinement of Deligne’s motives [11]. They form a semi-simple Tannakian category, whose construction we briefly recall in section 2. The motives of abelian varieties generate a full Tannakian subcategory whose objects are called abelian motives. The theory of André motives has found applications to the study of various arithmetic and Hodge-theoretic questions about algebraic varieties. Recall the theorem of Deligne [11] stating that any Hodge cohomology class on an abelian variety is absolute Hodge. More generally, it was shown in [2] that for a projective manifold $X$ whose André motive is abelian, all Hodge classes on $X$ are absolute. This shows that one part of the Hodge conjecture holds for varieties with abelian motives. Another application is related to the Mumford-Tate conjecture, which predicts a relation between the Mumford-Tate groups and the Galois group action on the cohomology of a projective variety. We review the absolute Hodge classes and the Mumford-Tate conjecture in more detail in sections 1.2 and 1.3 below. We recommend [26] for a general overview of the recent developments in this area. We remark that the Mumford-Tate conjecture for all known deformation classes of hyperkähler manifolds was independently proven in [13], the proof relying on Theorem 1.1 below.

The main new tool used in this paper is the generalized Kuga-Satake construction for hyperkähler manifolds, which was introduced in [22]. For any projective hyperkähler manifold $X$, this construction gives an embedding of the cohomology groups of $X$ into the cohomology of an abelian variety, which respects the Hodge structures. Therefore, our main goal is to prove that the Kuga-Satake embedding lifts to the category of André motives. To do this, we need to show that the cohomology class defining the embedding is motivated in the sense of [2], see also section 2.

Our approach is based on the deformation principle for motivated cohomology classes [2, Theorem 0.5]. More precisely, assume that

$$\pi: \mathcal{X} \to B$$

is a smooth projective morphism, $B$ a connected quasi-projective variety and the fibres of $\pi$ are hyperkähler manifolds. Assume that for some point $b_0 \in B$ the André motive of $\mathcal{X}_{b_0} = \pi^{-1}(b_0)$ is abelian. This implies that the Kuga-Satake embedding for $\mathcal{X}_{b_0}$ is motivated, and the deformation principle [2, Theorem 0.5] implies that it is motivated for any fibre $\mathcal{X}_{b_1}$, $b_1 \in B$. Therefore, it suffices to prove that there is one hyperkähler manifold in each deformation class whose motive is abelian. This is usually possible to do using some explicit geometric construction. For example, in the case of K3 surfaces one can assume that $\mathcal{X}_{b_0}$ is a Kummer surface. Other deformation types of hyperkähler manifolds are discussed in section 4.

The approach outlined above has one subtlety. Namely, assume that $X_1$ and $X_2$ are deformation equivalent projective hyperkähler manifolds. In the moduli space of all hyperkähler manifolds the projective ones are parametrized by a countable collection of divisors. We show in section 6 that it is possible to realize $X_1$ and $X_2$ as fibres of a smooth analytic family as in (1.1), but it is a priori not clear if one can make the family algebraic. To resolve this issue, we prove in section 5 a generalization of the deformation principle, which applies to the case when all fibres of $\pi$ in (1.1) are projective, but the morphism $\pi$ is projective only over a dense Zariski-open subset of the base.
Before stating our main result we recall that two compact hyperkähler manifolds $X_1$ and $X_2$ are called deformation equivalent if they can be realized as two fibres of a smooth family (1.1) where $B$ is a connected complex analytic space and all fibres of $\pi$ are compact hyperkähler manifolds. So $X_1$ and $X_2$ are deformations of each other in the complex analytic sense, and in general no polarization is preserved along the deformation. Our main result is the following statement.

**Theorem 1.1.** Let $X_1$ and $X_2$ be deformation equivalent projective hyperkähler manifolds. The André motive of $X_1$ is abelian if and only if the André motive of $X_2$ is abelian.

The proof is given in section 6.3.

We apply this theorem to several known deformation types of hyperkähler manifolds. We leave out only the OG10 type, which has recently been treated by Floccari, Fu and Zhang in [13].

**Corollary 1.2.** Let $X$ be a projective hyperkähler manifold of $K3^{[n]}$, generalized Kummer, or OG6 deformation type. Then the André motive of $X$ is abelian.

**Proof.** By Theorem 1.1, it suffices to find one manifold with abelian motive in each deformation class. For the Hilbert schemes of points on K3 surfaces and generalized Kummer varieties, the motives are abelian by [7], [8] and [37]. For OG6 deformation type, one can find a manifold with abelian motive using the construction from [25]. We recall all these constructions in section 4. □

The above Corollary recovers the results of [2], [3] and [30], where the case of K3 surfaces and, more generally, $K3^{[n]}$-type varieties has been considered. Unlike [30], we do not use the results of Markman on the structure of the cohomology ring, which are specific for $K3^{[n]}$-type varieties. The approach using the Kuga-Satake construction is more general, and therefore allows us to treat other deformation types.

Let us also mention that a substantial amount of recent research has been devoted to the study of Chow motives of hyperkähler manifolds and related questions, see e.g. [14] and references therein. Proving that Chow motives of hyperkähler manifolds are abelian seems to be a more difficult problem, and the methods of the present paper are not sufficiently strong to deal with it.

Let us next review the applications of our results to the absolute Hodge classes and the Mumford-Tate conjecture.

1.2. **Absolute Hodge classes.** Let $X$ be a non-singular projective variety over $\mathbb{C}$. Denote by $X^{an}$ the corresponding complex manifold. Recall that the de Rham cohomology $H^\bullet_{dR}(X)$ is the hypercohomology of the algebraic de Rham complex $\Omega^\bullet_{X/\mathbb{C}}$. For every $k$, the $\mathbb{C}$-vector space $H^k_{dR}(X)$ is endowed with a decreasing Hodge filtration $F^\bullet$. On the other hand, the singular cohomology $H^k(X^{an},\mathbb{C})$ is endowed with the $\mathbb{Q}$-structure given by the subspace $H^k(X^{an},\mathbb{Q})$. Comparison results between the algebraic and the analytic cohomology of coherent sheaves and the quasi-isomorphism $\Omega^\bullet_{X^{an}} \simeq \mathbb{C}$ induce natural isomorphisms of the cohomology groups $H^k_{dR}(X) \simeq H^k(X^{an},\mathbb{C})$.

Recall that an element $\alpha \in F^pH^2_{dR}(X)$ is called a Hodge class, if its image in $H^{2p}(X^{an},\mathbb{C})$ under the isomorphism described above is contained in the subspace $(2\pi i)^pH^{2p}(X^{an},\mathbb{Q})$. The Hodge conjecture implies that the property of $\alpha$ being Hodge should be stable under automorphisms of the field of complex numbers. More precisely, let $\sigma \in \text{Aut}(\mathbb{C}/\mathbb{Q})$ be an automorphism and $X_{\sigma}$ be the variety obtained by base change via $\sigma$. We have a chain of isomorphisms of $\mathbb{Q}$-vector spaces:

$$H^k_{dR}(X) \simeq H^k_{dR}(X) \otimes_{\mathbb{C},\sigma} \mathbb{C} \simeq H^k_{dR}(X_{\sigma}) \simeq H^k(X^{an}_{\sigma},\mathbb{C}). \tag{1.2}$$

A cohomology class $\alpha \in F^pH^2_{dR}(X)$ is called absolute Hodge, if its image under the isomorphism (1.2) lies in $(2\pi i)^pH^p(X^{an}_{\sigma},\mathbb{Q})$ for any $\sigma \in \text{Aut}(\mathbb{C}/\mathbb{Q})$. If $\alpha$ is an algebraic class, i.e. it is contained it the
Q-subspace spanned by the classed of algebraic subvarieties, then it is absolute Hodge. According to the Hodge conjecture, every Hodge class should be algebraic, therefore absolute Hodge.

According to Deligne [11], any Hodge class on an abelian variety is absolute. Using the results of [2] and Theorem 1.1, we deduce the following statement.

**Corollary 1.3.** Let $X$ be a projective hyperkähler manifold of $K3^{[n]}$, generalized Kummer, or OG6 deformation type. Then all Hodge classes on $X$ are absolute.

**Proof.** By Corollary 1.2, the André motive of $X$ is abelian, and we can apply [2, section 6]. □

1.3. **The Mumford-Tate conjecture.** The purpose of this section is to explain how the results of Floccari [12] combined with Theorem 1.1 lead to the proof of some cases of the Mumford-Tate conjecture for hyperkähler manifolds. A similar proof was independently found in [13].

Assume that $X$ is a non-singular projective variety defined over a subfield $k \subset \mathbb{C}$ finitely generated over $\mathbb{Q}$. Recall the comparison isomorphism between the $\ell$-adic and singular cohomologies of $X$:

\[(1.3)\quad H^k_{\ell}(X, \mathbb{Q}_\ell) \simeq H^k(X^\text{an}, \mathbb{Q}) \otimes \mathbb{Q}_\ell.\]

Let us briefly denote by $H^k_\ell$ this $\mathbb{Q}_\ell$-vector space.

The left-hand side of (1.3) is a representation of the Galois group $\text{Gal}(\bar{k}/k)$. Denote by $G^k_\ell$ the Zariski closure of the image of $\text{Gal}(k/k)$ in $\text{GL}(H^k_\ell)$. Let $G^k_\ell,0$ be the connected component of the identity in $G^k_\ell$. The right-hand side of (1.3) is naturally a representation of the Mumford-Tate group $\text{MT}^k(X) \otimes \mathbb{Q}_\ell$. The Mumford-Tate conjecture predicts that these two subgroups of $\text{GL}(H^k_\ell)$ are equal:

\[(1.4)\quad G^k_\ell,0 = \text{MT}^k(X) \otimes \mathbb{Q}_\ell.\]

This conjecture has been a subject of an active research. For an overview of the recent developments, see [26]. There is a number of recent works on the Mumford-Tate conjecture that use methods similar to ours. In [3], the Mumford-Tate conjecture in degree 2 was proven for hyperkähler manifolds. In [27] this result was generalized to a wider class of varieties with $h^{2,0} = 1$; the proof relies on the Kuga-Satake construction. In [6], it was shown that for varieties with abelian André motive the validity of the Mumford-Tate conjecture does not depend on $\ell$. In [12], the Mumford-Tate conjecture in arbitrary degree for $K3^{[n]}$-type varieties was proven; the method of the proof relies on the results of Markman about the structure of the cohomology algebra, similarly to [30].

Let us remark, that at present the Mumford-Tate conjecture is not known even for general abelian varieties. On the other hand, we know from [3] that it holds in degree 2 for any hyperkähler manifold. Using Theorem 1.1 and the results of [12], we can deduce the Mumford-Tate conjecture in all even degrees for the same type of varieties as in Corollary 1.2.

One special type of abelian varieties for which the Mumford-Tate conjecture is known are the varieties of CM type. In this case, one can use the results of [34], see also [26, Theorem 3.3.2 and Corollary 4.3.15]. We deduce analogous results for hyperkähler manifolds. We will say that a projective hyperkähler manifold $X$ is of CM type, if the Mumford-Tate group of $H^2(X, \mathbb{Q})$ is abelian. In this case the Mumford-Tate groups of $H^k(X, \mathbb{Q})$ are abelian for all $k$. This follows from the fact that the Hodge structures on all cohomology groups of $X$ are induced by the natural Lie algebra action, as we recall in section 3.

We summarize our discussion in the following statement. The idea of its proof is entirely due to Floccari, and we follow his arguments from [12]. The only ingredient that was missing in [12] is the deformation principle, Theorem 1.1. In the case of $K3^{[n]}$-type it was obtained in [12] independently.
Corollary 1.4. Let $X$ be a projective hyperkähler manifold of $K3^{[n]}$, generalized Kummer, or OG6 deformation type. Then the Mumford-Tate conjecture holds for the cohomology of $X$ in all even degrees. If $X$ is moreover of CM type, then the Mumford-Tate conjecture holds for the cohomology in all degrees.

Proof. Let $H^+_\ell = \oplus_\ell H^2_{\ell}(X^{an},\mathbb{Q}_\ell)$ and $H^2_\ell = H^2(X^{an},\mathbb{Q}_\ell)$. Denote by $G^{+,\circ}_\ell \subset \text{GL}(H^+_\ell)$ and $G^{2,\circ}_\ell \subset \text{GL}(H^2_\ell)$ the connected algebraic groups obtained from the Galois representations as explained above. Let $\text{MT}^+ \subset \text{GL}(H^+_\ell)$ and $\text{MT}^2 \subset \text{GL}(H^2_\ell)$ be the Mumford-Tate groups. Since the André motive of $X$ is abelian, by [26, formula (3.3)] we have an inclusion $G^{+,\circ}_\ell \subset \text{MT}^+$. By the work of André [3], the analogous inclusion in degree two is an isomorphism $G^{2,\circ}_\ell \simeq \text{MT}^2$. We have the following diagram of groups

\[
\begin{array}{ccc}
G^{+,\circ}_\ell & \longrightarrow & \text{MT}^+ \\
\downarrow & & \downarrow \\
G^{2,\circ}_\ell & \simeq & \text{MT}^2
\end{array}
\]

where the vertical arrows are induced by the projection $H^+_\ell \to H^2_\ell$. The arrow on the right is an isomorphism because the Hodge structures on the cohomology of $X$ are induced by the action of the orthogonal Lie algebra $\mathfrak{g}_{\text{tran}}$ (see section 3 and [23, (1.7)]).

It follows from the diagram above that the upper arrow is also surjective, which proves the first part of the corollary. The second part follows from [26, Theorem 3.3.2], see also [26, Corollary 4.3.15]. \qed

1.4. Organization of the paper. In section 2, we discuss the notions of a motivated cohomology class and André motive. We recall all necessary definitions and constructions from [2]. In section 3, we recall basic results about the cohomology of hyperkähler manifolds and the Kuga-Satake construction from [22]. The main results of this section are Proposition 3.2 and Corollary 3.3, which generalize the results of [22] to the relative setting. In section 4, we recall some known results about the Hilbert schemes of points, generalized Kummer varieties and O’Grady’s 6-dimensional varieties. We explain that in each of these deformation types one can find a variety with abelian motive. In section 5, we prove a generalization of the deformation principle for motivated cohomology classes, Proposition 5.1. In section 6, we discuss the construction of families of hyperkähler manifolds and in 6.3 prove the main result, Theorem 1.1.

2. Motivated cohomology classes and André motives

In this section, we briefly recall some results of [2]. Let $(\text{Var}/\mathbb{C})$ be the category of non-singular complex projective varieties and their morphisms. For a variety $X \in (\text{Var}/\mathbb{C})$, we will denote by $H^k(X)$ the singular cohomology group $H^k(X^{an},\mathbb{C})$. Let $(\text{GrAlg}/\mathbb{Q})$ be the the category of finite-dimensional graded $\mathbb{Q}$-algebras.

2.1. Motivated cohomology classes. Assume that $X \in (\text{Var}/\mathbb{C})$ and let $\mathcal{L}$ be an ample line bundle on $X$. Denote by $h \in H^2(X)$ the first Chern class of $\mathcal{L}$. The Lefschetz operator $L_h \in \text{End}(H^\bullet(X))$ is defined as the cup product with $h$, and it induces isomorphisms $L_h^k : H^{n-k}(X) \to H^{n+k}(X)$ for every $k = 0,\ldots,n$, where $n = \dim_{\mathbb{C}}(X)$. The subspace of primitive elements $H^k_{\text{pr}}(X) \subset H^k(X)$, $k = 0,\ldots,n$, is by definition the kernel of $L_h^{n-k+1}$.

We will denote by $*_h \in \text{End}(H^\bullet(X))$ the Lefschetz involution associated with $h$. Recall that for $x \in H^k_{\text{pr}}(X)$ and $i = 0,\ldots,n-k$ we have $*_h(L_h^{i}x) = L_h^{n-k-i}x$. This uniquely determines $*_h$, since $H^\bullet(X)$ is spanned by the elements of the form $L^{i}x$ with $x$ primitive.
For $X,Y \in (\text{Var}/\mathbb{C})$ and two ample line bundles $\mathcal{L}_1 \in \text{Pic}(X)$, $\mathcal{L}_2 \in \text{Pic}(Y)$, let $p_X$, $p_Y$ denote the two projections from $X \times Y$, and let $h = c_1(p_X^*\mathcal{L}_1 \otimes p_Y^*\mathcal{L}_2) \in H^2(X \times Y)$. For any two classes of algebraic cycles $\alpha, \beta \in H_{m0}(X \times Y)$, consider the class

\begin{equation}
(2.1) \quad p_{X*}(\alpha \cup *_h \beta).
\end{equation}

Let $H^*_M(X)$ be the $\mathbb{Q}$-subspace of $H^*(X)$ spanned by the classes (2.1) for all $Y$, $\mathcal{L}_1$, $\mathcal{L}_2$, $\alpha$, $\beta$ as above. Elements of $H^*_M(X)$ will be called motivated cohomology classes (or motivated cycles, in the terminology of [2]). Let us list a few properties of the motivated classes.

1. $H^*_M(X)$ is a graded $\mathbb{Q}$-subalgebra of $H^*(X)$;
2. For $f : X \to Y$, we have $f^*H^*_M(X) \subset H^*_M(Y)$, hence we have a functor $H^*_M : (\text{Var}/\mathbb{C})^{op} \to (\text{GrAlg}/\mathbb{Q})$;
3. For $f$ as above, we have $f_*H^*_M(X) \subset H^*_M(\dim(X)Y)$;
4. All classes in $H^*_M(X)$ are absolute Hodge;
5. The Künneth components of the diagonal are contained in $H^*_M(X \times X)$ for any $X \in (\text{Var}/\mathbb{C})$.

Let us also recall the following deformation principle:

**Theorem 2.1** ([2, Theorem 0.5]). Let $\pi : X \to B$ be a smooth projective morphism. Assume that the base $B$ is a connected quasi-projective variety. Let $\nu \in \Gamma(B, R^{2p}\pi_!\mathbb{C})$. Assume that for some $b_0 \in B$ the element $\nu_{b_0} \in H^{2p}(X_{b_0})$ is a motivated cohomology class. Then $\nu_{b_1}$ is a motivated class on $X_{b_1}$ for any $b_1 \in B$.

We will generalize the deformation principle in section 5, relaxing the condition of projectivity for the morphism $\pi$, see Proposition 5.1.

### 2.2. André motives

Define a $\mathbb{Q}$-linear category $(\text{Mot}_A)$ whose objects are triples $(X,p,n)$, where $X \in (\text{Var}/\mathbb{C})$, $p \in \text{Cor}^0_M(X,X)$, $p \circ p = p$, and $n \in \mathbb{Z}$. The space of morphisms from $(X,p,n)$ to $(Y,q,m)$ is by definition $q \circ \text{Cor}^m_{M-n}(X,Y) \circ p \subset \text{Cor}^m_{M-n}(X,Y)$. The tensor product on $(\text{Mot}_A)$ is defined using the Cartesian product of varieties. It is shown in [2], that $(\text{Mot}_A)$ is a semi-simple graded neutral Tannakian category. We will denote the André motives by boldface letters: for $X \in (\text{Var}/\mathbb{C})$ define

\[ \text{M}(X)(n) = (X, [\Delta_X], n) \in (\text{Mot}_A), \]

where $\Delta_X$ denotes the diagonal in $X \times X$. More generally, since the Künneth components of the diagonal are motivated, we can define the motives representing the cohomology groups of $X$:

\[ H^k(X)(n) = (X, \delta_k, n) \in (\text{Mot}_A), \]

where $\delta_k$ is the $k$-th Künneth component of $[\Delta_X]$. Therefore, we have the direct sum decomposition

\[ \text{M}(X)(n) = \oplus_k H^k(X)(n). \]

The subcategory of abelian motives $(\text{Mot}^{ab})$ is the minimal full Tannakian subcategory of $(\text{Mot}_A)$ containing $\text{M}(A)(n)$ for all abelian varieties $A$ and $n \in \mathbb{Z}$. It is shown in [2] that for any $X \in (\text{Var}/\mathbb{C})$, if $\text{M}(X) \in (\text{Mot}^{ab}_A)$, then all Hodge classes on $X$ are motivated, in particular absolute Hodge.
We will use the well-known fact that the class of varieties with abelian motives is stable under two basic operations. Namely, let \( X, Y \in (\text{Var}/\mathbb{C}) \) and let \( f : X \to Y \) be a surjective generically finite morphism. Then \( f^* : \text{Mot}(Y) \to \text{Mot}(X) \) is injective, and \( \text{Mot}(X) \in (\text{Mot}_k^\text{ab}) \) implies \( \text{Mot}(Y) \in (\text{Mot}_k^\text{ab}) \). The second basic operation is the blow-up: if \( i : X \to Y \) is a closed immersion and \( \text{Mot}(X), \text{Mot}(Y) \in (\text{Mot}_k^\text{ab}) \), then \( \text{Mot}(\text{Bl}_X Y) \in (\text{Mot}_k^\text{ab}) \). As an example, consider a complex abelian surface \( A \) and denote by \( A[2] \) its 2-torsion points. Then \( S = (\text{Bl}_{A[2]} A)/\pm 1 \) is the corresponding Kummer K3 surface. We see that \( \text{Mot}(S) \in (\text{Mot}_k^\text{ab}) \).

3. Hyperkähler manifolds and the Kuga-Satake embedding

In this section, we recall basic properties of hyperkähler manifolds. We refer to [33], [18], [4] for more details. We also recall the results of [22] and extend them to the relative setting.

3.1. Hyperkähler manifolds. In this paper, a hyperkähler manifold is a compact simply connected Kähler manifold \( X \) such that \( H^0(X, \Omega_X^2) \) is spanned by a symplectic form. The dimension of any symplectic manifold is even; we let \( \dim_{\mathbb{C}}(X) = 2n \).

Let \( V_\mathbb{Z} = H^2(X, \mathbb{Z}) \) and \( V = V_\mathbb{Z} \otimes \mathbb{Q} \). Let \( q \in S^2 V^* \) denote the Beauville-Bogomolov-Fujiki (BBF) form. Recall that this form has the following property: there exists a constant \( c_X \in \mathbb{Q} \) such that for all \( h \in H^2(X, \mathbb{Q}) \) the equality \( q(h) = c_X h^{2n} \) holds. We can assume that \( q \) is integral and primitive on \( V_\mathbb{Z} \), and \( q(h) > 0 \) for a Kähler class \( h \). The signature of \( q \) is \( (3, b_2(X) - 3) \). Recall also that \( V \) carries a Hodge structure of K3 type.

There exists a natural action of the orthogonal Lie algebra on the total cohomology of \( X \). Let us briefly recall how to define this action. An element \( h \in V \) has Lefschetz property, if the cup product with \( h^k \) induces an isomorphism \( H^{2n-k}(X, \mathbb{Q}) \cong H^{2n+k}(X, \mathbb{Q}) \) for all \( k = 0, \ldots, 2n \). In this case one can consider the corresponding Lefschetz \( \mathfrak{sl}_2 \)-triple. Let \( \mathfrak{g}_{\text{tot}} \subset \text{End}(H^*(X, \mathbb{Q})) \) be the Lie subalgebra generated by all such \( \mathfrak{sl}_2 \)-triples.

Let \( \tilde{V} = \langle e_0 \rangle \oplus V \oplus \langle e_4 \rangle \) be the graded \( \mathbb{Q} \)-vector space with \( e_i \) of degree \( i \), and \( V \) in degree 2. Let \( \tilde{q} \in S^2 \tilde{V}^* \) be the quadratic form such that: \( \tilde{q}|_V = q \), \( e_0 \) and \( e_4 \) are isotropic and orthogonal to \( V \) and span a hyperbolic plane. It was shown in [35] and [23] that there exists an isomorphism of graded Lie algebras \( \mathfrak{g}_{\text{tot}} \cong \mathfrak{so} (\tilde{V}, \tilde{q}) \).

One can show that Hodge structures on the cohomology groups of \( X \) are induced by the action of \( \mathfrak{g}_{\text{tot}} \). More precisely, let \( W \) be the Weil operator that induces the Hodge structure on \( V \), i.e. it acts on \( V_{p,q} \) as multiplication by \( i(p-q) \). Then \( W \in \mathfrak{so}(V, q) \subset \mathfrak{so}(\tilde{V}, \tilde{q}) \) and \( W \) induces the Hodge structures on \( H^k(X, \mathbb{Q}) \) for all \( k \) (see [35]).

3.2. The Kuga-Satake construction. To the K3 type Hodge structure \( V \) we can associate a Hodge structure of abelian type, which is called the Kuga-Satake Hodge structure. Let us briefly recall the construction. Let \( H = \text{Cl}(V, q) \) be the Clifford algebra. There exists a natural embedding \( V \to H \). Define \( H^{0,-1} \) to be the right ideal \( V^{2,0} \cdot H_C \) (see [31, Lemma 3.3]), and \( H^{-1,0} = \overline{H^{0,-1}} \). This defines a rational Hodge structure on \( H \).

Note that \( H \) is canonically an \( \mathfrak{so}(V, q) \)-module, and the Hodge structure on it is induced by the action of the Weil operator \( W \) (see e.g. [31]). Since the hyperkähler manifold \( X \) is projective, one can show that \( H \) is polarizable. Moreover, the polarization can be chosen \( \text{Spin}(V, q) \)-invariant (see e.g. [22] or [15]).

Let \( d = \frac{1}{4}\dim_{\mathbb{Q}}(H) \). The following theorem was proven in [22].
Theorem 3.1. There exists a structure of graded $\mathfrak{so}(\tilde{V}, \tilde{q})$-module on $\Lambda^*H^*$ that extends the $\mathfrak{so}(V, q)$-module structure. For some $m > 0$ there exists an embedding of $\mathfrak{so}(\tilde{V}, \tilde{q})$-modules

$$H^{*\cdot+2n}(X, \mathbb{Q}) \hookrightarrow \Lambda^{*\cdot+2d}(H^{*\oplus m}).$$

This induces embeddings of Hodge structures

$$\nu_i: H^{i\cdot+2n}(X, \mathbb{Q}(n)) \hookrightarrow \Lambda^{i\cdot+2d}(H^{*\oplus m})(d),$$

where $i = -2n, \ldots, 2n$.

3.3. The Kuga-Satake construction in families. Let us consider a smooth projective morphism $\varphi: X \to B$ whose fibres are hyperkähler manifolds. Let us fix a base point $b_0 \in B$ and denote by $X_{b_0}$ the fibre $X_{b_0}$. We would like to apply Theorem 3.1 to the family $\varphi$ and obtain an embedding of the corresponding variations of Hodge structures. In order to do so, we need to construct a family of Kuga-Satake abelian varieties such that the embeddings $\nu_i$ from Theorem 3.1 are $\pi_1(B, b_0)$-equivariant. We will explain below, that it is possible to do this after we pass to a finite étale covering of $B$. The base $B$ can be an arbitrary complex analytic space.

Denote by $\text{Aut}^P(X) \subset \text{GL}(H^\bullet(X, \mathbb{Q}))$ the subgroup of algebra automorphisms that fix the Pontryagin classes of $X$. We will denote by $\text{Aut}^P(X)_{\mathbb{Z}}$ the arithmetic subgroup of $\text{Aut}^P(X)_{\mathbb{Q}}$ that consists of all elements preserving the integral cohomology lattice.

Recall that $\mathfrak{so}(V, q)$ acts on $H^\bullet(X, \mathbb{Q})$ by derivations, and this action induces a homomorphism of algebraic groups

$$\alpha: \text{Spin}(V, q) \to \text{Aut}^P(X),$$

see section 3.1 in [32] and references therein. Denote by $\text{Aut}^+(X)$ the image of $\alpha$. Let $\text{MC}(X) = \text{Diff}(X)/\text{Diff}^\circ(X)$ be the mapping class group of $X$. Here $\text{Diff}(X)$ is the group of diffeomorphisms of $X$, and $\text{Diff}^\circ(X)$ is the subgroup of diffeomorphisms isotopic to the identity. The mapping class group acts on the cohomology of $X$ fixing Pontryagin classes, hence a group homomorphism

$$\beta: \text{MC}(X) \to \text{Aut}^P(X)_{\mathbb{Z}}.$$
The construction of $\text{MC}'$ is summarized in the following commutative diagram:

\[
\begin{array}{ccc}
\text{MC}' & \longrightarrow & \text{MC}'' \\
\downarrow & & \downarrow \beta \\
\Gamma \cap \Gamma' & \longrightarrow & \Gamma \\
\downarrow & & \downarrow \\
\Gamma' & \longrightarrow & \text{O}(V, q)_{\mathbb{Q}}
\end{array}
\]

Here $\Gamma \cap \Gamma'$ is of finite index in $\Gamma$, because both $\Gamma$ and $\Gamma'$ are arithmetic subgroups of $\text{O}(V, q)_{\mathbb{Q}}$. By the definition of $\Gamma$, it has finite index in $\text{Aut}^0(X)_{\mathbb{Z}}$. We then let $\text{MC}' = \beta^{-1}(\Gamma \cap \Gamma')$, and define $\beta'$ to be the composition of the two maps in the left column of the diagram and the embedding $\Gamma' \subset \text{Spin}(V, q)_{\mathbb{Q}}$.

**Corollary 3.3.** Let $\varphi: X \to B$ be a smooth family of hyperkähler manifolds with $B$ connected, $b_0 \in B$ a base point, $X$ the fibre of $\varphi$ over $b_0$, and $2n = \dim_{\mathbb{C}}(X)$. Let $V = H^2(X, \mathbb{Q})$ and denote by $q$ the BBF form on $V$. Then the following statements hold:

1. There exists a finite étale covering $B' \to B$ such that the action of $\pi_1(B', b_0)$ on $H^*(X, \mathbb{Q})$ factors through a homomorphism $\rho: \pi_1(B', b_0) \to \text{Spin}(V, q)_{\mathbb{Q}}$;

2. There exists a smooth family of compact complex tori $\psi: A \to B'$ and an embedding of the variations of Hodge structures

\[\tilde{\nu}_i: R^{i+2n}\varphi'_i,\mathbb{Q}(n) \hookrightarrow R^{i+2d}\psi_i,\mathbb{Q}(d),\]

where $i = -2n, \ldots, 2n$, $2d = \dim_{\mathbb{C}}(A_{b_0})$ and $\varphi': X' = X \times_B B' \to B'$;

3. Assume that there exists a monodromy-invariant cohomology class $h \in H^2(X, \mathbb{Q})^{\pi_1(B, b_0)}$ such that $q(h) > 0$. Then one can find $B'$ as above such that $\psi$ is a projective family of abelian varieties.

**Proof.** (1) The monodromy action on the cohomology is induced by a homomorphism $\rho: \pi_1(B, b_0) \to \text{MC}(X)$. Using the subgroup $\text{MC}' \subset \text{MC}(X)$ from Proposition 3.2, we get a finite index subgroup $\rho^{-1}(\text{MC}') \subset \pi_1(B, b_0)$ and the corresponding covering $B'$. It satisfies the required properties by the definition of $\text{MC}'$.

(2) Denote by $\tilde{B}$ the universal covering of $B$. Let $\mathcal{H}$ be the variation of the Kuga-Satake Hodge structures over $\tilde{B}$ that can be fibrewise described as in section 3.2. The fact that the fibrewise construction indeed defines a variation of Hodge structures was proved e.g. in [31, section 3.2, in particular Proposition 3.6]. By Proposition 3.2, the image of $\pi_1(B, b_0)$ under $\rho$ is contained in an arithmetic subgroup of $\text{Spin}(V, q)_{\mathbb{Q}}$. This implies that the action of the fundamental group of $B'$ preserves some lattice $\Lambda \subset \mathcal{H}$. Taking the quotient of $\mathcal{H}/\Lambda$ by the action of $\pi_1(B', b_0')$, we get a family of complex tori $\psi: A \to B'$. The embeddings of Hodge structures $\nu_i$ from Theorem 3.1 are monodromy-equivariant. This implies that they induce the embeddings $\tilde{\nu}_i$ of the corresponding variations of Hodge structures over $B'$, see [32, Proposition 3.7].

(3) Under our assumptions, the Kuga-Satake Hodge structures admit a $\text{Spin}(V, q)$-invariant polarization, see e.g. [22]. We can therefore fix a polarization type for the Kuga-Satake abelian varieties. Choosing appropriate finite étale cover $B' \to B$, we obtain a map from $B'$ to an arithmetic quotient of the Siegel half-space $\Gamma\backslash \mathbb{H}_{2d}$. For a suitable choice of $\Gamma$, there exists a projective universal family of abelian varieties over $\Gamma\backslash \mathbb{H}_{2d}$, see e.g. [29, Theorem 8.11]. We can then construct the family $\psi$ as the pull-back of the universal family. \qed
4. The manifolds of $K3^{[n]}$, Generalized Kummer and OG6 deformation types

In this section, we recall the results of [7], [8], [37] and [25]. They provide the necessary geometric input for the proof of Corollaries 1.2, 1.3 and 1.4, showing that in each of the $K3^{[n]}$, generalized Kummer and OG6 deformation types there exists at least one variety with abelian motive.

4.1. Hilbert schemes of points on K3 surfaces and generalized Kummer varieties. Let $S$ be a non-singular complex projective surface. We will denote by $S^{[n]}$ the Hilbert scheme of length $n$ subschemes of $S$ and by $S^{(n)}$ the $n$-th symmetric power of $S$. Recall that $S^{[n]}$ is non-singular and there exists a Hilbert-Chow morphism $\chi: S^{[n]} \to S^{(n)}$.

In [7], the natural stratification of $S^{(n)}$ was used to describe the motive of $S^{[n]}$. We briefly recall the construction. Let $\nu = (\nu_1, \ldots, \nu_n)$ denote a partition of $n$, so that $n = \nu_1 + 2\nu_2 + \ldots + n\nu_n$ and $\nu_i \geq 0$. Let $l(\nu) = \sum \nu_i$ and denote by $S^{(\nu)}$ the product $\prod S^{(\nu_i)}$. Recall that the points of $S^{(\nu_i)}$ are the 0-cycles of length $\nu_i$ in $S$. Consider the morphism $S^{(\nu)} \to S^{(n)}$ that sends a collection of cycles $(x_1, \ldots, x_n)$ to the cycle $x_1 + 2x_2 + \ldots + nx_n$. Let $Z_\nu$ denote the product $S^{(\nu)} \times_{S^{(n)}} S^{[n]}$ with the reduced scheme structure. We get a commutative square:

\[
\begin{array}{ccc}
Z_\nu & \xrightarrow{q_\nu} & S^{[n]} \\
\downarrow p_\nu & & \downarrow \chi \\
S^{(\nu)} & \xrightarrow{\nu} & S^{(n)}
\end{array}
\]

By construction, $\dim(S^{(\nu)}) = 2l(\nu)$ and one can show that $\dim(Z_\nu) = n + l(\nu)$. Note that the symmetric powers of $S$ have quotient singularities. Hence all the natural operations on Chow groups with rational coefficients are well-defined. Consider the morphisms $q_\nu \circ p_\nu^*: CH^k_Q(S^{[n]}) \to CH^k_Q(S^{(n)})$. It is shown in [7, Theorem 5.4.1] that the sum of these morphisms gives an isomorphism of Chow groups

\[
CH^*_Q(S^{[n]}) \cong \oplus_\nu CH^*_{Q}(-n+l(\nu))(S^{(\nu)}),
\]

where the direct sum is taken over all partitions of $n$. One deduces from this an isomorphism of motives

\[
M(S^{[n]}) \cong \oplus_\nu M(S^{(\nu)})(n - l(\nu)),
\]

which actually holds on the level of Chow motives with rational coefficients, cf. [7, Theorem 6.2.1]. The motive of $S^{(\nu)}$ is by definition a sub motive of $\otimes M(S^{n})$. The latter is abelian if the motive of $S$ is abelian.

It is shown in [2, Theorem 7.1] that for any complex projective K3 surface $S$ the motive of $S$ is abelian, hence $M(S^{[n]}) \in (\text{Mot}_A^{ab})$. Analogous arguments show that the motive of a generalized Kummer variety is abelian. Namely, let $A$ be a complex abelian surface. Consider the Albanese morphism $a: A^{[n+1]} \to A$ which sends an $(n+1)$-tuple of points on $A$ into their sum. The fibre $K^nA = a^{-1}(0)$ is called the generalized Kummer variety.

To describe the motive of $K^nA$ one uses the construction described above, replacing the symmetric power $A^{(n)}$ by the fibre of the Albanese morphism $A^{(\nu)} \to A$. The fibres of the morphisms $A^{(\nu_i)} \to A$ are finite quotients of abelian varieties. Therefore, repeating the arguments of [7] or using more general results of [8], we find that $M(K^nA) \in (\text{Mot}_A^{ab})$. This was also shown in [37, Theorem 1.1] using similar methods.

4.2. Manifolds of OG6 deformation type. The manifolds of OG6 deformation type were discovered by O’Grady, see [28]. These 6-dimensional manifolds have originally been constructed as desingularizations of certain moduli spaces of sheaves on abelian surfaces. To produce one manifold of this deformation type
with abelian motive, one can use a construction from [25]. In [25, section 6] one finds a diagram of the form

$$X \xleftarrow{f} Y_2 \xrightarrow{g_2} Y_1 \xrightarrow{g_1} Y.$$ 

In this diagram: $X$ is an OG6-type manifold; $Y$ is a K3$^3$-type manifold; $f$ is a surjective generically finite morphism; $g_1$ and $g_2$ are blow-ups with centres $Z_1$ and $Z_2$, where $Z_1$ is the disjoint union of 256 projective spaces and $Z_2$ is isomorphic to $\text{Bl}_{(A \times A')/\mathbb{Z}}(A \times A')/\pm 1$ for some abelian surface $A$ (see the proof of [25, Proposition 6.1]). The motives of $Y$, $Z_1$ and $Z_2$ are abelian, hence the motive of $Y_2$ is also abelian. By projection formula, the motive of $X$ embeds into the motive of $Y_2$, hence it is also abelian.

5. **Deformation principle**

5.1. **The setting.** In this section, we will assume that

$$\pi: X \to B$$

is a smooth proper morphism with connected fibres between complex analytic spaces, the base $B$ is a connected quasi-projective variety and for any $b \in B$ the fibre $X_b$ is projective. Moreover, we will assume that there exists a line bundle $L \in \text{Pic}(X)$ and a dense Zariski-open subset $U \subset B$ such that $L|_{X_b}$ is ample for any $b \in U$. Let $X'_U = \pi^{-1}(U)$. We will assume that $X'_U$ is a quasi-projective variety, and that the bundle $L|_{X'_U}$ and the morphism $\pi|_{X'_U}$ are algebraic.

Let us emphasize that we do not assume the total space $X$ to be an algebraic variety, and for the families that we consider below (see Proposition 6.3) it will typically not be algebraic.

5.2. **The statement and preliminary constructions.** The following proposition is a version of the deformation principle for motivated cohomology classes, see section 2 and [2, Theorem 0.5]. We remark that similar results about specialization of motivated cycles were obtained in [5, Corollary 4.7]. We can not apply those results in our setting, because we do not assume the total space $X$ of our family to be an algebraic variety.

**Proposition 5.1.** In the above setting 5.1, let $b_0, b_1 \in B$ be two points and consider a section $\xi \in H^0(B, R^{2p}\pi_*\mathbb{C})$. Let $\xi_{b_i} \in H^{2p}(X_{b_i}, \mathbb{C})$, $i = 0, 1$ be the values of $\xi$ at $b_i$. Then $\xi_{b_0}$ is motivated if and only if $\xi_{b_1}$ is motivated.

The proof is given below in 5.3. It uses the same idea as in [2], but in our case the variety $X$ is not algebraic, so we start by explaining some preliminary constructions.

First note that since $B$ is quasi-projective, we can connect any two points in it by a chain of integral curves. Since $U$ is dense in $B$, we can also make sure that each of those curves intersects $U$. Choosing intermediate points between $b_0$ and $b_1$ at the intersections of the curves in the chain, we reduce to the case when $B$ is an integral quasi-projective curve. We may pull back the family $X$ to the normalization of this curve, and then assume that $B$ is smooth.

The case when both $b_0$ and $b_1$ lie in $U$ reduces to [2, Theorem 0.5], since in this case we can shrink $B$ and assume that $L$ is $\pi$-ample. If both $b_0$ and $b_1$ lie in $B \setminus U$, we can choose an intermediate point $b_2 \in U$, and hence we reduce to the case when one of the points from the statement of Proposition 5.1 lies in $U$ and the other in $B \setminus U$. To fix the notation, we will assume that $b_0 \in U$, $b_1 \in B \setminus U$. After shrinking $B$, we may moreover assume that $U = B \setminus \{b_1\}$. We will denote by $X'_1 = \pi^{-1}(b_1)$ the two fibres.

The curve $B$ is quasi-projective, hence there exists a projective curve $\bar{B}$ that contains $B$ as a Zariski-open subset. Let us denote the boundary $\bar{B} \setminus B = \{z_1, \ldots, z_n\}$, and let $\bar{U} = \bar{B} \setminus \{b_1\}$. Our next goal is
Lemma 5.2. There exists a compact complex manifold $\tilde{X}$, a flat morphism $\tilde{\pi}: \tilde{X} \to \tilde{B}$, a line bundle $\tilde{L} \in \text{Pic}(\tilde{X})$ and an open embedding $j: X \to \tilde{X}$ that satisfy the following conditions:

1. $\tilde{\pi}|_{j(X)} = \pi$ and $\tilde{\pi}^{-1}(z_i)$;
2. The open subset $\tilde{X}_U = \tilde{\pi}^{-1}(U) \subset \tilde{X}$ is a quasi-projective variety;
3. The restriction of $\tilde{\pi}$ to $\tilde{X}_U$ is an algebraic morphism onto $\tilde{U}$ and $\tilde{L}|_{\tilde{X}_U}$ is $\tilde{\pi}$-ample algebraic bundle.

Proof. Recall that $\pi_U: X_U \to U$ is a smooth morphism of quasi-projective varieties with relatively ample line bundle $L_U = L|_{X_U}$. For a big enough integer $k$, consider the vector bundle $E = \pi_U^*(L_U^k)$ over $U$. We can find such $k$ that the canonical morphism $\pi_U^*(E) \to L_U$ induces a closed embedding $X_U \hookrightarrow \mathbb{P}(E)$. Note that $E$ is an algebraic vector bundle over the quasi-projective curve $U$, so we can extend it to a vector bundle $\tilde{E}$ over $\tilde{U}$. Denote by $\tilde{Z}$ the closure of $X_U$ in $\mathbb{P}(E)$. Then $\tilde{Z}$ is a quasi-projective variety fibred over $\tilde{U}$. It may be singular, the singularities lying over the points $z_i$. By a theorem of Hironaka, there exists a resolution of singularities $\tilde{\rho}: \tilde{Z} \to Z$. This means that $\tilde{Z}$ is a non-singular quasi-projective variety and the morphism $\tilde{\rho}$ is a composition of blow-ups with centres lying over the singular locus of $Z$, see e.g. [16, Definition 7.1 and Theorem 7.5]. In our case the blow-ups occur in the fibres over the points $z_i$, and $\tilde{\rho}$ is an isomorphism over $X_U$. Hence $\tilde{Z}$ contains $X_U$ as an open subset, and the morphism $\tilde{Z} \to \tilde{U}$ agrees with $\pi_U$ on this subset. We get the manifold $\tilde{X}$ by gluing $\tilde{Z}$ with $X'$ along their common open subset $X_U$. Then $X'$ is embedded into $\tilde{X}$ by construction, and we get a morphism $\tilde{\pi}: \tilde{X} \to \tilde{B}$ that has the claimed property (1). The morphism $\tilde{\pi}$ is flat, because it is equidimensional and $\tilde{X}, \tilde{U}$ are smooth manifolds (see [16, page 114]).

The open subset $\tilde{X}_U$ is by construction identified with $\tilde{Z}$, hence it is quasi-projective and (2) is satisfied. The first part of (3) is also satisfied because $\tilde{Z}$ is a blow-up of the algebraic variety $Z$.

Next we prove the existence of the line bundle $\tilde{L}$. Consider the line bundle $\mathcal{L}' = \mathcal{O}_{\mathbb{P}(E)/\tilde{U}}(1)|_Z$. The bundle $\mathcal{L}'$ is relatively ample over $\tilde{U}$ and $\mathcal{L}'|_{X_U} \simeq \mathcal{L}_U^k$. The blow-up morphism $\rho$ is projective, and if we denote by $E_j$ the exceptional divisors of $\rho$, the bundle $\mathcal{L}' = \rho^*(\mathcal{L}(\sum a_j E_j))$ is relatively ample over $\tilde{U}$ for suitably chosen integers $a_j$. We have $\mathcal{L}'|_{X_U} \simeq \mathcal{L}_U^k$ and $\mathcal{L}'|_{X_U} \simeq \mathcal{L}_U^k$, and we define $\tilde{L}$ by gluing $\mathcal{L}'$ and $\mathcal{L}_U^k$ over the open subset $X_U$. The second part of (3) is then satisfied because $\tilde{L}|_{\tilde{X}_U} \simeq \mathcal{L}'$ is algebraic and relatively ample over $\tilde{U}$.

In what follows we will implicitly identify $X$ with its image in $\tilde{X}$ under the embedding $j$ from the lemma above.

Lemma 5.3. The manifold $\tilde{X}$ from Lemma 5.2 is Moishezon. There exists a projective manifold $\hat{X}$ and a birational morphism $r: \hat{X} \to \tilde{X}$ that is an isomorphism over $\hat{X} \setminus X_1$ and such that $r^{-1}(X_1)$ is a simple normal crossing divisor.

Proof. We use the notation from the statement of Lemma 5.2. We refer to [16, chapter VII, §6] for the discussion of Moishezon manifolds. Our proof is standard: we use the line bundle $\tilde{L}$ (or rather its power) to produce a birational map from $\tilde{X}$ to a projective variety.

Let $\tilde{\pi}_U = \tilde{\pi}|_{\tilde{X}_U}$ and $E'' = \tilde{\pi}_U^*(\tilde{L}|_{\tilde{X}_U})$. After possibly replacing the bundle $\tilde{L}$ by its power, we may assume that the canonical morphism $\tilde{\pi}_U^*(E) \to \tilde{L}|_{\tilde{X}_U}$ defines a closed embedding $\tilde{X}_U \hookrightarrow \mathbb{P}(E)$. Now consider the vector bundle $E' = \tilde{\pi}_*(\tilde{L})$ over $\tilde{B}$ and note that $E'|_{\tilde{U}} \simeq E$. Since we do not assume that
\( L|_{X_1} \) is ample (or even globally generated) the morphism \( \tilde{\pi}^*(E') \to \tilde{\ell} \) only defines a meromorphic map \( \varphi: \tilde{X} \to \mathbb{P}(E') \) whose restriction to \( \tilde{X}_0 \) coincides with the embedding \( \tilde{X}_0 \to \mathbb{P}(E) \). The indeterminacy locus of \( \varphi \) is contained in the fibre \( X_1 \). We can resolve the indeterminacy of \( \varphi \) by a sequence of blow-ups \( r': \tilde{X} \to \tilde{X} \), so that \( \varphi \) lifts to a morphism \( \tilde{\varphi}: \tilde{X} \to \mathbb{P}(E') \). The latter morphism is proper, hence its image is an analytic subvariety that we denote by \( \mathcal{Y} \). Since \( \mathcal{Y} \) is a subvariety of the projective manifold \( \mathbb{P}(E') \), it is also projective. This shows that \( \tilde{X} \) is Moishezon.

Next we show that there is a birational morphism \( r''': \tilde{\mathcal{Y}} \to \tilde{\mathcal{X}} \) for some projective manifold \( \tilde{\mathcal{Y}} \). We consider the birational map \( \tilde{\varphi}^{-1}: \tilde{\mathcal{Y}} \to \tilde{\mathcal{X}} \). We resolve the indeterminacy of the latter map and the singularities of \( \mathcal{Y} \) by a sequence of blow-ups \( s: \tilde{\mathcal{Y}} \to \mathcal{Y} \). Then \( \tilde{\mathcal{Y}} \) is a projective manifold and there is a birational morphism \( r'': \tilde{\mathcal{Y}} \to \tilde{\mathcal{X}} \). Note that the blow-ups occur only in the fibre over the point \( b_1 \in B \), so all the constructed birational morphisms are isomorphisms over \( \tilde{X}_0 \). The preimage of \( b_1 \) in \( \tilde{\mathcal{Y}} \) is a divisor. By blowing up \( \tilde{\mathcal{Y}} \) further, we make sure that this divisor has simple normal crossings. We define \( \tilde{\mathcal{X}} \) to be the result of this final sequence of blow-ups and \( r \) to be the induced map to \( \tilde{\mathcal{X}} \). By construction, \( \tilde{\mathcal{X}} \) has the claimed properties.

Consider the morphism \( r: \tilde{\mathcal{X}} \to \tilde{\mathcal{X}} \) constructed in Lemma 5.3. Then \( D = (\tilde{\pi} \circ r)^{-1}(b_1) \) is a simple normal crossing divisor by the lemma, and \( r(D) = X_1 \). Let us denote by \( \tilde{\mathcal{X}}_1 \) the irreducible component of \( D \) that dominates \( X_1 \). It is a non-singular projective variety. The restriction of \( r \) to \( \tilde{\mathcal{X}}_1 \) is a birational morphism \( r_1: \tilde{\mathcal{X}}_1 \to X_1 \). We denote by \( \alpha_1: \tilde{X}_1 \to \tilde{\mathcal{X}}_1 \) the closed immersions. Since \( r \) is an isomorphism over \( \tilde{\mathcal{X}} \setminus X_1 \), the immersion \( \alpha_0 \) lifts to \( \alpha_0: X_0 \to \tilde{\mathcal{X}}_1 \). We summarize our constructions in a diagram:

\[
\begin{array}{ccc}
X_0 & \xrightarrow{\alpha_0} & \tilde{\mathcal{X}}_1 \\
\downarrow & & \downarrow \alpha_1 \\
X_0 & \xrightarrow{\alpha_0} & \tilde{\mathcal{X}} \\
& \downarrow r & \downarrow r_1 \\
& \tilde{\mathcal{X}} & \tilde{\mathcal{X}}_1 \\
\end{array}
\]

(5.1)

The projective manifold \( \tilde{\mathcal{X}} \) can be used as the 0-th term of a simplicial resolution for \( \tilde{\mathcal{X}} \), which is an augmented simplicial variety \( \mathcal{S}_* \to \tilde{\mathcal{X}} \) with \( \mathcal{S}_1 \) smooth projective and \( \mathcal{S}_0 = \tilde{\mathcal{X}} \). Such resolution can be constructed as in [10, 6.2.5]. Note that both \( \tilde{\mathcal{X}} \) and \( \tilde{\mathcal{X}} \) are smooth manifolds of the same dimension, hence by projection formula \([20, \text{IX.7.3}]\) the pull-back on cohomology \( r^* \) is injective. Thus we have the following exact sequence (cf. \([10, \text{Proposition 8.2.5}]\)):

\[
0 \to H^{2p}(\tilde{\mathcal{X}}, \mathbb{C}) \xrightarrow{r^*} H^{2p}(\tilde{\mathcal{X}}, \mathbb{C}) \xrightarrow{\delta_0^* - \delta_1^*} H^{2p}(\mathcal{S}_1, \mathbb{C}),
\]

where \( \delta_0, \delta_1: \mathcal{S}_1 \to \mathcal{S}_0 = \tilde{\mathcal{X}} \) are the face maps.

**Definition 5.4.** Define the motive \( H^{2p}(\tilde{\mathcal{X}}) \) to be the kernel of the morphism

\[
\delta_0^* - \delta_1^*: H^{2p}(\tilde{\mathcal{X}}) \to H^{2p}(\mathcal{S}_1).
\]

Let \( G_i = \pi_1(B, b_i) \). There exist canonical isomorphisms \( H^0(B, R^{2p}\pi_* \mathbb{C}) \cong H^{2p}(X_i, \mathbb{C})^{G_i} \).

**Lemma 5.5.** We have the following equalities:

\[
\ker(\alpha_0^*: H^{2p}(\tilde{\mathcal{X}}, \mathbb{C}) \to H^{2p}(X_0, \mathbb{C})) = \ker(\alpha_1^*: H^{2p}(\tilde{\mathcal{X}}, \mathbb{C}) \to H^{2p}(X_1, \mathbb{C})), \tag{5.2}
\]

and

\[
\im(\alpha_1^*: H^{2p}(\tilde{\mathcal{X}}, \mathbb{C}) \to H^{2p}(X_1, \mathbb{C})) = H^{2p}(X_i, \mathbb{C})^{G_i}, \quad i = 0, 1. \tag{5.3}
\]
Proof. The kernels in (5.2) can be identified with the kernel of the composition

\[ H^{2p}(\bar{X}, \mathbb{C}) \rightarrow H^{2p}(X, \mathbb{C}) \rightarrow H^0(B, R^{2p}\pi_*\mathbb{C}), \]

where the last morphism comes from the Leray spectral sequence. This proves (5.2).

To prove (5.3), it is sufficient to check that \( H^{2p}(\bar{X}, \mathbb{C}) \rightarrow H^{2p}(X_0, \mathbb{C})^{G_0} \) is surjective, because this would imply surjectivity of the composition (5.4). Since \( X_0 \subset X_U \), we have the composition

\[ H^{2p}(\bar{X}, \mathbb{C}) \xrightarrow{\beta} H^{2p}(X_U, \mathbb{C}) \xrightarrow{\gamma} H^{2p}(X_0, \mathbb{C})^{G_0}. \]

The line bundle \( L \) defines a polarization on the fibres over \( U \), hence the Leray spectral sequence degenerates at \( E_2 \). Since the action of \( \pi_1(U, b_0) \) on the cohomology of \( X_0 \) factors through \( G_0 \), it follows that the morphism \( \gamma \) is surjective. The cohomology of \( X_U \) carries a mixed Hodge structure, and since the Hodge structure on \( H^{2p}(X_0, \mathbb{C}) \) is pure, we deduce that the restriction of \( \gamma \) to \( W_{2p}H^{2p}(X_U, \mathbb{C}) \) is still surjective. On the other hand, \( \bar{X} \) is a smooth Moishezon compactification of \( X_U \), and [9, Corollaire (3.2.17)] shows that \( W_{2p}H^{2p}(X_U, \mathbb{C}) \) is the image of \( \beta \). This completes the proof of (5.3).

\[ \square \]

Definition 5.6. Define the following motives:

\[ H^{2p}(X_0)^{G_0} = \mathrm{im}(\hat{\alpha}_0 : H^{2p}(\bar{X}) \rightarrow H^{2p}(X_0)), \]

\[ H^{2p}(X_1)^{G_1} = \mathrm{im}(r_{1*} \circ \hat{\alpha}_1 : H^{2p}(\bar{X}) \rightarrow H^{2p}(X_1)). \]

Note that \( \hat{\alpha}_0 \circ r^* = \alpha_0^* \) and by projection formula \( r_{1*} \circ \hat{\alpha}_1^* \circ r^* = r_{1*} \circ r_{1*}^* \circ \alpha_1^* = \alpha_1^* \), see the diagram (5.1). Therefore, it follows from (5.3) that \( H^{2p}(X_i)^{G_i} \) are the motives representing \( H^{2p}(X_i, \mathbb{C})^{G_i} \).

5.3. Proof of Proposition 5.1. Consider the motives \( H^{2p}(X_i)^{G_i} \), introduced in Definition 5.6. The two quotient maps \( H^{2p}(\bar{X}) \rightarrow H^{2p}(X_i)^{G_i} \) have the same kernel by (5.2). Hence the quotients \( H^{2p}(X_i)^{G_i} \) are canonically isomorphic, and the corresponding isomorphism in cohomology is induced by \( H^{2p}(X_i, \mathbb{C})^{G_i} \cong H^{2p}(X, \mathbb{C})^{G_0} \). We conclude that the isomorphism \( H^{2p}(X_0, \mathbb{C})^{G_0} \cong H^{2p}(X_1, \mathbb{C})^{G_1} \) lifts to the category of André motives. It follows that for any section \( \xi \in H^0(B, R^{2p}\pi_*\mathbb{C}) \) the cohomology class \( \xi_{b_0} \in H^{2p}(X_0, \mathbb{C}) \) is motivated if and only if \( \xi_{b_1} \in H^{2p}(X_1, \mathbb{C}) \) is motivated. This finishes the proof.

6. Motives of hyperkähler manifolds

6.1. Moduli theory of compact hyperkähler manifolds. In what follows, we consider hyperkähler manifolds of a fixed deformation type. We start by recalling some necessary facts about moduli spaces of such manifolds.

First recall that two compact hyperkähler manifolds \( X_1 \) and \( X_2 \) are called deformation equivalent if there exists a smooth analytic family \( \pi : \mathcal{X} \rightarrow B \) over a connected complex analytic base \( B \) such that all fibres are compact hyperkähler manifolds and there exist two fibres isomorphic to \( X_1 \) and \( X_2 \). In particular, the underlying topological manifolds are diffeomorphic, and one can show that properly normalized Beauville-Bogomolov-Fujiki forms on \( X_1 \) and \( X_2 \) are equal, see e.g. [33, section 2.2]. Hence we can fix a lattice \( \Lambda \) representing the second integral cohomology with the BBF form for our deformation equivalence class.

We will formulate the moduli theory in the language of marked hyperkähler manifolds, following [19]. Recall that a marked hyperkähler manifold is a pair \((X, \varphi)\), where \( X \) is a hyperkähler manifold of the fixed deformation type and \( \varphi : H^2(X, \mathbb{Z}) \xrightarrow{\sim} \Lambda \) is an isomorphism of lattices which is called a marking. If \( \sigma \in H^0(X, \Omega^2_X) \) is the symplectic form, then \( q(\sigma) = 0 \) and \( q(\sigma, \sigma) > 0 \), where \( q \) is the BBF form. We denote by \( \mathcal{D} \subset \mathbb{P}(\Lambda \otimes \mathbb{C}) \) the period domain defined as \( \mathcal{D} = \{ x \in \mathbb{P}(\Lambda \otimes \mathbb{C}) \mid q(x) = 0, q(x, \bar{x}) > 0 \} \), which
is an open subset of the quadric in $\mathbb{P}(\Lambda \otimes \mathbb{C})$ defined by the BBF form. We define the period of a marked hyperkähler manifold $(X, \varphi)$ to be the point $\rho(X, \varphi) = [\varphi(\sigma)] \in \mathcal{D}$.

We call two marked hyperkähler manifolds $(X_1, \varphi_1)$ and $(X_2, \varphi_2)$ isomorphic if there exists a biholomorphic isomorphism $f: X_1 \cong X_2$ such that $\varphi_2 = \varphi_1 \circ f^*$. In this case clearly $\rho(X_1, \varphi_1) = \rho(X_2, \varphi_2)$, since $f^*$ is an isomorphism of Hodge structures. We let $\mathfrak{M}$ be the set of isomorphism classes of marked hyperkähler manifolds of the fixed deformation type. Then $\rho$ is a well-defined map from $\mathfrak{M}$ to $\mathcal{D}$ called the period map. It is known (see e.g. [19, Proposition 4.3]) that one can endow $\mathfrak{M}$ with the structure of a complex manifold in such a way that $\rho$ becomes holomorphic. We briefly recall how to define the topology and complex analytic charts on $\mathfrak{M}$. Let $(X, \varphi) \in \mathfrak{M}$ and consider the universal deformation $\pi: \mathcal{X} \to \Delta^k$ of the complex manifold $X$. Here $\Delta$ is the unit disc, $k = \dim H^1(X, T_X)$, and the fibre of $\pi$ over zero is isomorphic to $X$. The base of the universal deformation is smooth by the well known result of Bogomolov-Tian-Todorov. The total space $\mathcal{X}$ is diffeomorphic to the product $X \times \Delta$, and we can canonically identify $H^2(X, \mathbb{Z})$ with $H^2(X_0, \mathbb{Z}) = H^2(X, \mathbb{Z})$, where $t \in \Delta^k$ and $X_t = \pi^{-1}(t)$. Composing with $\varphi$, we get a marking $\varphi_t$ on $X_t$ for every $t \in \Delta^k$, hence a map $\mu: \Delta^k \to \mathfrak{M}$. By the local Torelli theorem (see [4]) the differential at $0 \in \Delta^k$ of the composition $\rho \circ \mu$ is an isomorphism, in particular $\rho \circ \mu$ maps some smaller polydisc $U \subset \Delta^k$ biholomorphically onto an open subset of $\mathcal{D}$. This implies that $\mu|_U$ is an embedding. We endow $\mathfrak{M}$ with the finest topology for which all such embeddings are continuous. We use the images of polydiscs under the described embeddings as a system of complex analytic charts on $\mathfrak{M}$.

The constructed complex manifold $\mathfrak{M}$ is usually non-Hausdorff. Recall that two points $x, y \in \mathfrak{M}$ are called inseparable if for any non-empty open neighbourhoods $U_x$ of $x$ and $U_y$ of $y$ we have $U_x \cap U_y \neq \emptyset$. The inseparability is not an equivalence relation for points in arbitrary topological spaces, but it is shown in [19, section 4.3] that for $\mathfrak{M}$ it actually is. So we can define the Hausdorff reduction $\overline{\mathfrak{M}}$ to be the set of equivalence classes of inseparable points in $\mathfrak{M}$. One checks that $\overline{\mathfrak{M}}$ is a Hausdorff complex manifold, and that the period map $\rho$ factors through $\overline{\rho}: \overline{\mathfrak{M}} \to \mathcal{D}$.

Let $\mathcal{M}$ be a connected component of $\mathfrak{M}$ and let $\overline{\mathcal{M}}$ be its Hausdorff reduction. One of the central results of the moduli theory of compact hyperkähler manifolds can be formulated as follows, see [36, Theorem 4.29], [19, Corollary 5.9]:

**Theorem 6.1 (The global Torelli theorem).** The period map $\overline{\rho}: \overline{\mathcal{M}} \to \mathcal{D}$ is a biholomorphic isomorphism.

In what follows, we will always fix a connected component $\mathcal{M}$ of the moduli space as above. Let $(X_1, \varphi_1) \in \mathcal{M}$ and $X_2$ be a hyperkähler manifold deformation equivalent to $X_1$. Then $X_1$ and $X_2$ are two fibres of a smooth family $\pi: \mathcal{X} \to \mathcal{B}$ over a connected base. Pulling back the family $\mathcal{X}$ to the universal covering $\tilde{\mathcal{X}}$ of the base, we get a family $\tilde{\pi}: \tilde{\mathcal{X}} \to \tilde{\mathcal{B}}$. The space $\tilde{\mathcal{B}}$ being simply connected, the local system $\mathbb{R}^2 \tilde{\pi}_* \mathbb{Z}$ is trivial and by parallel transport we identify its fibres with $H^2(X_1, \mathbb{Z})$. Composing with $\varphi_1$, we get markings on all fibres of the map $\tilde{\pi}$, in particular on $X_2$. We denote the latter marking by $\varphi_2$. Thus $\tilde{\mathcal{X}}$ becomes a family of marked hyperkähler manifolds, and we get a holomorphic map from $\tilde{\mathcal{B}}$ to $\mathfrak{M}$. Since $\tilde{\mathcal{B}}$ is connected, the image of this map is contained in $\mathcal{M}$. In particular we observe that $(X_2, \varphi_2) \in \mathcal{M}$. Hence it is possible to find a marking for $X_2$, so that the corresponding marked manifold defines a point in the same connected component $\mathcal{M}$. We will use this observation later.

Given a non-zero element $h \in \Lambda$, we denote by $\Lambda_h \subset \Lambda$ the orthogonal complement to $h$ and let $\mathcal{D}_h = \mathcal{D} \cap \mathbb{P}(\Lambda_h \otimes \mathbb{C})$, $\mathcal{M}_h = \rho^{-1}(\mathcal{D}_h)$. Recall the projectivity criterion for a hyperkähler manifold $X$: by [18, Theorem 3.11] $X$ is projective if and only if there exists a class $a \in H^2(X, \mathbb{Z}) \cap H^{1,1}(X)$ such that $q(a) > 0$. Equivalently, $X$ is projective if and only if there exists a line bundle $L \in \text{Pic}(X)$ with
Let $q(c_1(L)) > 0$ (although it is not always true that this line bundle is ample). It follows that for $h \in \Lambda$ with $q(h) > 0$ all hyperkähler manifolds parametrized by $\mathcal{M}_h$ are projective.

Finally, let us recall some necessary facts about inseparable points in $\mathcal{M}$. Consider a point $(X, \varphi) \in \mathcal{M}$. If another point $(X', \varphi') \in \mathcal{M}$ is inseparable from $(X, \varphi)$, it is known from [18, Theorem 4.3] that $X$ and $X'$ are bimeromorphic. It is possible to distinguish between different bimeromorphic models of $X$ using their Kähler cones. Namely, we consider the open set $\{ x \in H^{1,1}(X, \mathbb{R}) \mid q(x) > 0 \}$ which has two connected components, because the signature of $q|_{H^{1,1}(X, \mathbb{R})}$ is $(1, h^{1,1}(X) - 1)$. We define the positive cone $C_X$ to be the connected component of this set that contains the Kähler cone $K_X$, the latter being the set of all cohomology classes represented by Kähler forms on $X$. If $f: X \dashrightarrow X'$ is a bimeromorphic map, then $f^*: H^2(X', \mathbb{R}) \to H^2(X, \mathbb{R})$ is well-defined, see [18, Lemma 2.6]. We define the birational Kähler cone $B\mathcal{K}_X \subset \mathcal{C}_X$ to be the union of all preimages $f^*(\mathcal{K}_{X'})$ for all bimeromorphic models $X'$ of $X$. These preimages are the connected component of $B\mathcal{K}_X$, and $\mathcal{K}_X$ is one of them.

Below we will need a way to make sure that $\mathcal{M}$ contains no points inseparable from $(X, \varphi)$. To do this one has to show that $\mathcal{K}_X = \mathcal{C}_X$, see [24, Theorem 2.2(4)], because then $B\mathcal{K}_X = \mathcal{K}_X$ and $X$ has only one bimeromorphic model. To understand when $\mathcal{K}_X = \mathcal{C}_X$, one can use the description of the boundary of $\mathcal{K}_X$ given in [1, Theorem 1.19]. It is shown in loc. cit. that when $\mathcal{K}_X$ is strictly contained in $\mathcal{C}_X$, then its boundary contains a face that is cut out by a hyperplane orthogonal to a certain cohomology class $z \in H^2(X, \mathbb{Z}) \cap H^{1,1}(X)$ called an MBM class, see [1, Definition 1.13]. Such a class must have negative BBF square because of the signature of $q$ mentioned above. Hence to make sure that $\mathcal{K}_X = \mathcal{C}_X$ it suffices to show that there exist no negative classes in $H^2(X, \mathbb{Z}) \cap H^{1,1}(X)$. This is the case e.g. when $\text{Pic}(X) = 0$ or when $\text{Pic}(X)$ is spanned by a line bundle with first Chern class of positive BBF square, cf. [24, Theorem 2.2(5)]. In particular, if $h \in \Lambda$, $q(h) > 0$ and $\mathcal{D}_h$ is the divisor in the period domain introduced above, then for a very general point $p \in \mathcal{D}_h$ the Picard group of a marked manifold with period $p$ has rank one. The generator of the Picard group of such a manifold has positive BBF square, hence the condition discussed above is satisfied. We conclude that all points of $\mathcal{M}_h$ lying over very general points of $\mathcal{D}_h$ are Hausdorff. A more detailed discussion of inseparable points can be found in [24, section 5.3], see in particular [24, Theorem 5.16].

6.2. Constructing smooth families of hyperkähler manifolds. We will need the following lemma about filling in families of hyperkähler manifolds over the punctured disc. We denote the unit disc by $\Delta$ and the punctured disc by $\Delta^*$. If $\pi': X' \to \Delta^*$ is a family of hyperkähler manifolds, its marking is an isomorphism $\varphi': \mathbb{R}^2 \pi'_!\mathbb{Z} \cong \underline{\Lambda}$, where $\underline{\Lambda}$ is the constant sheaf with fibre $\Lambda$. In particular, the monodromy action on $H^2$ is trivial for a marked family. The marking induces a period map $\gamma': \Delta^* \to \mathcal{D}$. We will say that some condition is satisfied for a very general point $t \in \Delta$, if there exists a countable subset $Z \subset \Delta$ such that the condition is satisfied for any $t \in \Delta \setminus Z$.

**Lemma 6.2.** Let $\pi: X \to \Delta$ be a flat projective morphism and $\pi': X' \to \Delta^*$ its restriction to $\Delta^*$. Assume that $\pi'$ is a smooth family of marked hyperkähler manifolds, and the period map $\gamma': \Delta^* \to \mathcal{D}$ extends to a morphism $\gamma: \Delta \to \mathcal{D}$. Assume that a very general fibre of $\pi$ has Picard rank one. Let $(X, \varphi)$ be a marked hyperkähler manifold such that $\rho(X, \varphi) = \gamma(0)$. Then there exists a finite ramified cover $\alpha: \Delta \to \Delta$, and a smooth family of hyperkähler manifolds $\bar{\pi}: \bar{X} \to \Delta$ such that $\bar{X}_0 \simeq X$ and $\alpha^*\bar{X}|_{\Delta^*} \simeq \bar{X}|_{\Delta^*}$, after possibly shrinking $\Delta$. Any line bundle $L' \in \text{Pic}(\bar{X}|_{\Delta^*})$ can be extended to a line bundle $L \in \text{Pic}(\bar{X})$.

**Proof.** The statement is essentially equivalent to [21, Theorem 0.8]. Following the argument from [21, section 3], we pull back the universal family of $X$ via $\gamma$ and obtain a smooth family of hyperkähler manifolds
\[ \xi: \mathcal{Y} \to \Delta \] with central fibre \( X \). There exists a finite covering \( \alpha: \Delta \to \Delta \), and a cycle \( Z \subset \alpha^* \mathcal{X} \times \Delta \alpha^* \mathcal{Y} \) that induces a birational isomorphism between \( \alpha^* \mathcal{X} \) and \( \alpha^* \mathcal{Y} \) over \( \Delta \). We define \( \widetilde{X} = \alpha^* \mathcal{Y} \).

The local system \( R^2 \xi_* \mathcal{Z} \) is trivial and by parallel transport we identify its fibres with \( H^2(\mathcal{Y}_0, \mathbb{Z}) \cong H^2(X, \mathbb{Z}) \). Using the marking \( \phi \) we then obtain an isomorphism \( R^2 \xi_* \mathcal{Z} \cong \Lambda \), i.e. \( \mathcal{Y} \) becomes a family of marked hyperkähler manifolds. For \( t \in \Delta^* \) let us denote by \( (\mathcal{X}_t, \phi_t) \) and \( (\mathcal{Y}_t, \phi'_t) \) the fibres of the two families \( \mathcal{X} \) and \( \mathcal{Y} \) with the induced markings. Thus we obtain two maps from \( \Delta^* \) to \( \mathcal{M} \) whose compositions with the period map \( \rho \) are equal, by construction. Recall that we denote by \( \mathcal{M} \) one connected component of the moduli space of marked hyperkähler manifolds, and that by the global Torelli Theorem 6.1 \( \rho \) induces an isomorphism between the Hausdorff reduction of \( \mathcal{M} \) and \( \mathcal{D} \). It follows that for \( t \in \Delta^* \) the marked manifolds \( (\mathcal{X}_t, \phi_t) \) and \( (\mathcal{Y}_t, \phi'_t) \) represent either the same point in \( \mathcal{M} \) or a pair of inseparable points. By our assumption, for a very general \( t \in \Delta^* \) the Picard group of \( \mathcal{X}_t \) is generated by an ample line bundle, hence the Kähler cone of \( \mathcal{X}_t \) coincides with a connected component of the positive cone. This implies that \( \mathcal{M} \) contains no inseparable points over \( \gamma(t) \), see the discussion at the end of section 6.1 or [24, section 5.3 and Theorem 5.16]. Hence the cycle \( Z_s \) is the graph of an isomorphism between \( \mathcal{X}_{\alpha(s)} \) and \( \mathcal{Y}_{\alpha(s)} \) for a very general \( s \in \Delta \). The subset of \( s \in \Delta \) for which \( Z_s \) defines an isomorphism of the fibres is Zariski-open. This implies that the families \( \alpha^* \mathcal{X}|_{\Delta^*} \) and \( \mathcal{X}|_{\Delta^*} \) are isomorphic, after possibly shrinking \( \Delta \).

To prove the last claim of the lemma, note that we have the following isomorphism

\[ \text{Pic}(\widetilde{X}) \cong \ker(H^0(\Delta, R^2 \bar{\pi}_s \mathcal{Z}) \to H^0(\Delta, R^2 \bar{\pi}_s \mathcal{O}_X)). \]

This isomorphism follows from the exponential exact sequence using the fact that \( \Delta \) is a Stein manifold and the fibres of \( \bar{\pi} \) are simply connected. The local system \( (R^2 \bar{\pi}_s \mathcal{Z})|_{\Delta^*} \) is trivial, and its section that defines \( \mathcal{L}' \) extends to a section of \( R^2 \bar{\pi}_s \mathcal{Z} \). This extension still lies in the kernel on the right hand side of the above formula, because the sheaf \( R^2 \bar{\pi}_s \mathcal{O}_{\widetilde{X}} \) is locally free. Thus we get a line bundle \( \mathcal{L} \in \text{Pic}(\widetilde{X}) \) that extends \( \mathcal{L}' \).

As we recalled above, given a hyperkähler manifold \( X_1 \), we can choose a marking \( \varphi_1: H^2(X_1, \mathbb{Z}) \cong \Lambda \) such that \( (X_1, \varphi_1) \in \mathcal{M} \). Assume that \( X_1 \) is projective with a very ample line bundle \( L \), and let \( h = \varphi_1(c_1(L)) \). Then \( (X_1, \varphi_1) \in \mathcal{M}_h \).

**Proposition 6.3.** In the above setting, assume that we have another marked hyperkähler manifold \( (X_2, \varphi_2) \in \mathcal{M}_h \). Then there exists a connected quasi-projective curve \( C \), a smooth analytic family of hyperkähler manifolds \( \pi: \mathcal{X} \to C \) and two points \( x_1, x_2 \in C \) such that \( X_i \cong \pi^{-1}(x_i) \) and \( \pi \) is a projective morphism of algebraic varieties over \( C \setminus \{x_2\} \). There exists a line bundle \( \mathcal{L} \in \text{Pic}(\mathcal{X}) \) that is algebraic and \( \pi \)-ample over \( C \setminus \{x_2\} \).

**Proof.** We embed \( X_1 \) into \( \mathbb{P}^N = \mathbb{P}H^0(X_1, L) \) and denote by \( P \) its Hilbert polynomial. Let \( \mathcal{H} \) be the Hilbert scheme \( \text{Hilb}^P(\mathbb{P}^N) \) with the reduced scheme structure and let \( \psi: \widetilde{X} \to \mathcal{H} \) be the restriction to \( \mathcal{H} \) of the universal family. Let \( U \subset \mathcal{H} \) be the open subset over which the morphism \( \psi \) is smooth. If \( U \) is disconnected, we replace it by the connected component containing \( [X_1] \). Let \( \mu: \tilde{U} \to U \) be the universal covering, and let \( \psi': \widetilde{X}' \to \tilde{U} \) be the pull back of the family \( \widetilde{X} \) to \( \tilde{U} \). The local system \( R^2 \psi'_* \mathcal{Z} \) is trivial and by parallel transport its fibres can be identified with \( H^2(X_1, \mathbb{Z}) \). Using the marking \( \varphi_1 \) we identify the fibres of the latter local system with \( \Lambda \), and the relative Hodge to de Rham spectral sequence induces an embedding of vector bundles \( \rho: \psi'_*(\Omega^2_{\widetilde{X}'/\tilde{U}}) \to \Lambda \otimes \mathcal{O}_{\tilde{U}} \). The family \( \widetilde{X}' \) is polarized, the class of the polarization being \( h \in \Lambda \), hence \( \rho \) factors via \( \Lambda_h \otimes \mathcal{O}_{\tilde{U}} \). So \( \psi'_*(\Omega^2_{\widetilde{X}'/\tilde{U}}) \) is a rank one subbundle of \( \Lambda_h \otimes \mathcal{O}_{\tilde{U}} \), and this gives us a period map from \( \tilde{U} \) to \( \mathcal{D}_h \). We can find a torsion-free arithmetic subgroup \( \Gamma \subset \text{O}(\Lambda_h, q) \).
and a finite covering $U' \to U$ such that the period map descends to a morphism $\mu: U' \to D_h/\Gamma$ of quasi-projective varieties. Note that the morphism $\mu$ is dominant (see e.g. the proof of \cite[Lemma 4.5]{32}).

Let $p_1$ and $p_2$ be the images of $(X_1, \varphi_1)$ and $(X_2, \varphi_2)$ in $D_h/\Gamma$. By construction, $p_1 \in \text{im}(\mu)$. We can find a smooth quasi-projective curve $C_1 \subset D_h/\Gamma$ such that $p_1, p_2 \in C_1$. For a very general point of $D_h$, the corresponding hyperkähler manifold has Picard group of rank one (generated by $h$), and we can assume that the same is true for a very general point of $C_1$. Since $\mu$ is dominant, there exists a curve $C_2 \subset U'$ that maps dominantly to $C_1$. Taking the normalization of $C_1$ in the function field of $C_2$, we get a curve $C_3$ and a finite morphism $\nu: C_3 \to C_1$. By construction, there exists a rational map from $C_3$ to $\mathcal{H}$. Since $\mathcal{H}$ is a projective variety, this map extends to a morphism $\xi: C_3 \to \mathcal{H}$. We obtain a projective family $X' = C_3 \times_\mathcal{H} X$ over $C_3$.

\begin{equation}
\begin{array}{ccc}
  C_3 & \xrightarrow{\nu} & C_1 \\
  \downarrow & & \downarrow \\
  \mathcal{H} & \xleftarrow{U'} & D_h/\Gamma
\end{array}
\end{equation}

Let $q_1, q_2 \in C_3$ be two points with $\nu(q_1) = p_1$. Note that the fibre of $X'$ over $q_1$ is isomorphic to $X_1$ by construction. The fibre over $q_2$ might be non-smooth or not isomorphic to $X_2$. We use Lemma 6.2 to modify the family $X'$ over a disk around $q_2$, and produce a new family with fibre $X_2$. Let $\Delta \subset C_3$ and $\Delta' \subset C_1$ be two small disks around $q_2$ and $p_2$ such that $\nu(\Delta) \subset \Delta'$ and the covering map $D_h \to D_h/\Gamma$ splits over $\Delta'$. Let $\Delta'' \subset D_h$ map isomorphically onto $\Delta'$ under the covering map. We obtain a morphism $\gamma: \Delta \to \Delta'' \subset D_h$. Let $\Delta^* = \Delta \setminus \{q_2\}$ and note that $\gamma|_{\Delta^*}$ is the period map for $\mathcal{X}'|_{\Delta^*}$ for some choice of the marking. We can now apply Lemma 6.2. After passing to some finite ramified cover $\alpha: C_4 \to C_3$, we obtain a smooth family $\mathcal{Y} \to \alpha^{-1}(\Delta)$ with central fibre $X_2$ such that its restriction to $\alpha^{-1}(\Delta^*)$ is isomorphic to the restriction of $X'' = C_4 \times_{C_3} X'$. We modify $X''$ over $\alpha^{-1}(\Delta)$ by gluing in $\mathcal{Y}$, and obtain a new family $\mathcal{X} \to C_4$ that contains both $X_1$ and $X_2$ as fibres. We restrict to an open subset $C \subset C_4$ over which this family is smooth. Since the family $X''$ is projective, there exists a relatively ample line bundle $\mathcal{L}'' \in \text{Pic}(X'')$. Using the last claim in Lemma 6.2, after gluing in $\mathcal{Y}$ we get a line bundle $\mathcal{L} \in \text{Pic}(X)$ whose restriction to all fibres except possibly $X_2$ is ample. This completes the proof.

\textbf{Lemma 6.4.} Assume that $\pi: \mathcal{X} \to C$ is a smooth family of hyperkähler manifolds such that: $C$ is a smooth quasi-projective curve; all fibres of $\pi$ are projective; $\pi$ is a projective morphism of algebraic varieties over a dense Zariski-open subset $U \subset C$; there exists a line bundle $\mathcal{L} \in \text{Pic}(\mathcal{X})$ that is algebraic and $\pi$-ample over $U$. Assume that the André motive of the fibre $\mathcal{X}_{b_0}$ is abelian for some $b_0 \in C$. Then for any $b_1 \in C$ the André motive of $\mathcal{X}_{b_1}$ is abelian.

\textbf{Proof.} Using part (1) of Corollary 3.3 and possibly replacing $C$ by a finite cover, we may assume that $\pi_1(C, b_0)$ acts on $H^*(\mathcal{X}_{b_0}, \mathbb{Q})$ via a homomorphism $\rho: \pi_1(C, b_0) \to \text{Spin}(V, q)_{\mathbb{Q}}$, where $V = H^2(\mathcal{X}_{b_0}, \mathbb{Q})$ and $q$ is the Beauville-Bogomolov-Fujiki form. Since the morphism $\pi$ is projective over an open subset of $C$, there exists a line bundle defining the polarization. The first Chern class of this line bundle gives a monodromy-invariant element $h \in V$ such that $q(h) > 0$. Hence we can use parts (2) and (3) of Corollary 3.3 to obtain a projective family $\psi: \mathcal{A} \to C$ of Kuga-Satake abelian varieties.

Consider the product $\mathcal{Y} = \mathcal{X} \times_C \mathcal{A}$ and denote by $\xi: \mathcal{Y} \to C$ the induced morphism. The embeddings $\tilde{\nu}_i$ from Corollary 3.3 can be viewed as global sections of the local system $\mathbb{R}^{2n+2d}\xi_i\mathbb{Q}(n + d)$. We need to prove that the corresponding cohomology class $\tilde{\nu}_{i, b_0} \in H^{2n+2d}(\mathcal{X}_{b_0} \times \mathcal{A}_{b_1}, \mathbb{Q}(n + d))$ is motivated. By Proposition 5.1, it is enough to prove that $\tilde{\nu}_{i, b_0}$ is motivated. But the fibre $\mathcal{Y}_{b_0} \simeq \mathcal{X}_{b_0} \times \mathcal{A}_{b_0}$ has abelian André motive, hence any cohomology class on it is motivated by \cite[section 6]{2}. \hfill \hbox{\ }
6.3. Proof of theorem 1.1. We use the notation introduced in section 6.1. We choose two markings \( \varphi_i: H^2(X_i, \mathbb{Z}) \to \Lambda \), so that \((X_i, \varphi_i) \in \mathcal{M} \). We choose very ample line bundles on \( X_i \) and denote by \( h_i \in \Lambda \) their classes. We use Proposition 6.3 to connect \( X_1 \) and \( X_2 \) by several smooth families of hyperkähler manifolds and then apply Lemma 6.4 to these families. We connect \( X_1 \) to \( X_2 \) in several steps, depending on the relative position of the divisors \( \mathcal{D}_{h_1} \) and \( \mathcal{D}_{h_2} \) inside \( \mathcal{D} \).

Case 1. Assume that \( \rho(X_1, \varphi_1) \in \mathcal{D}_{h_1} \cap \mathcal{D}_{h_2} \) or \( \rho(X_2, \varphi_2) \in \mathcal{D}_{h_1} \cap \mathcal{D}_{h_2} \). In this case we can apply Proposition 6.3 to construct a family connecting \( X_1 \) and \( X_2 \).

Case 2. Assume that \( \mbox{sign}(q|_{(h_1, h_2)}) = (1, 1) \). This condition implies that \( \mathcal{D}_{h_1} \cap \mathcal{D}_{h_2} \neq \emptyset \). By the surjectivity of the period map \( \rho: \mathcal{M} \to \mathcal{D} \) (see [19, Theorem 5.5]), we can pick \((X_3, \varphi_3) \in \mathcal{M} \) such that \( \rho(X_3, \varphi_3) \in \mathcal{D}_{h_1} \cap \mathcal{D}_{h_2} \) and reduce to Case 1 above.

Case 3. Assume the \( q|_{(h_1, h_2)} \) is positive definite. In this case \( \mathcal{D}_{h_1} \cap \mathcal{D}_{h_2} = \emptyset \), but we will find \( h_3 \in \Lambda \) such that \( q(h_3) > 0 \) and \( \mathcal{D}_{h_1} \cap \mathcal{D}_{h_3} \neq \emptyset, \mathcal{D}_{h_2} \cap \mathcal{D}_{h_3} \neq \emptyset \), reducing to Case 2. Consider the set \( \mathcal{V} = \{ v \in \Lambda \otimes \mathbb{R} \mid q(v) > 0, \mbox{sign}(q|_{(h_1, v)}) = \mbox{sign}(q|_{(h_2, v)}) = (1, 1) \} \).

This set is an open cone in \( \Lambda \otimes \mathbb{R} \), and it suffices to prove that \( \mathcal{V} \neq \emptyset \). Choose three vectors \( e_1, e_2, e_3 \in \Lambda \otimes \mathbb{R} \) such that: \( q(e_1) = q(e_2) = 1, q(e_3) = -1 \); \( e_i \) are pairwise orthogonal; \( h_1 = ae_1, h_2 = be_1 + cc_2 \) for some \( a, b, c \in \mathbb{R} \). If \( b = 0 \), then \( v = e_1 + e_2 + de_3 \in \mathcal{V} \) for \( 1 < d^2 < 2 \). If \( b \neq 0 \), then \( v = be_1 + ce_2 + de_3 \in \mathcal{V} \) for \( c^2 < d^2 < b^2 + c^2 \).

Case 4. Assume the \( q|_{(h_1, h_2)} \) is degenerate. Then we will find \( h_3 \in \Lambda \) such that \( q(h_3) > 0 \) and the restrictions \( q|_{(h_1, h_3)}, q|_{(h_2, h_3)} \) are non-degenerate, reducing to the previous cases. Consider the set \( \mathcal{V} = \{ v \in \Lambda \otimes \mathbb{R} \mid q(v) > 0, q|_{(h_1, v)} \) and \( q|_{(h_2, v)} \) non-degenerate \}.

As above, \( \mathcal{V} \) is an open cone and we need to check that \( \mathcal{V} \neq \emptyset \). The condition that \( q|_{(h_1, v)} \) is degenerate is given by the vanishing of the determinant of the Gram matrix, hence it defines a hypersurface in \( \Lambda \otimes \mathbb{R} \). Since the set \( \{ v \in \Lambda \otimes \mathbb{R} \mid q(v) > 0 \} \) is clearly open and non-empty, \( \mathcal{V} \) is also non-empty. This completes Case 4, and the proof of the theorem.

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