New algebraically solvable systems of two autonomous first-order ordinary differential equations with purely quadratic right-hand sides

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We identify many new solvable subcases of the general dynamical system characterized by two autonomous first-order ordinary differential equations with purely quadratic right-hand sides; the solvable character of these dynamical systems amounting to the possibility to obtain the solution of their initial value problem via algebraic operations. Equivalently—by considering the analytic continuation of these systems to complex time—the algebraically solvable character corresponds to the fact that their general solution is either singlevalued or features only a finite number of algebraic branch points as functions of complex time (the independent variable). Thus our results provide a major enlargement of the class of solvable systems beyond those with singlevalued general solution identified by Garnier about 60 years ago. An interesting property of several of these new dynamical systems is the elementary character of their general solution, identifiable as the roots of a polynomial with explicitly obtainable time-dependent coefficients. We also mention that, via a well-known time-dependent change of (dependent and independent) variables featuring the imaginary parameter $i\omega$ (with $\omega$ an arbitrary strictly positive real number), autonomous variants can be explicitly exhibited of each of the algebraically solvable models we identify: variants which all feature the remarkable property to be isochronous, i.e. their generic solution is periodic with a period that is a fixed integer multiple of the basic period $T = 2\pi/\omega$.

I. INTRODUCTION

Two general approaches can be distinguished in the investigation of dynamical systems, which in this paper are identified as autonomous systems of ordinary differential equations (ODEs) with, generally nonlinearly coupled, right-hand sides; of course beyond general theorems about the existence and uniqueness of solutions to such problems. One point of view—which might perhaps be symbolically characterized with the name of Painlevé—focusses on the identification of such systems which can, in some sense, be characterized as solvable: of course by restricting on a case-by-case manner the techniques to be used for their solution, or the mathematical properties to be satisfied by their general solution. A complementary point of view—which might perhaps be symbolically identified with the name of Poincaré—focusses instead on the identification of specific, interesting, features of the solution of given systems, such as the existence of equilibria and the behavior in their vicinity, asymptotic properties, the sensitivity of the dependence on initial data and related issues having to do with the notion of deterministic chaos.

Clearly these two points of view are complementary, while the approaches they entail are quite different: in the first (“Painlevé”) case the goal is to identify special dynamical systems—possibly within certain general classes—which feature some special property, in particular that to be in some sense solvable (see below); in the second (“Poincaré”) case the goal is to investigate special properties of some given dynamical system.

In this paper we adopt the first (“Painlevé”) point of view, and focus on what might be considered the prototypical class of nonlinear dynamical systems: we study the following system of ODEs:

\[
\begin{align*}
\dot{x}_1 &= c_{11}x_1^2 + c_{12}x_2^2 + c_{13}x_1x_2, \\
\dot{x}_2 &= c_{21}x_1^2 + c_{22}x_2^2 + c_{23}x_1x_2.
\end{align*}
\]

Notation 1.1. Hereafter $t$ (“time”) is the independent variable, superimposed dots denote $t$-differentiations, we generally use the last letters of the Latin alphabet (such as $x$, $y$, $z$; possibly equipped with indices) to denote time-dependent variables (generally without indicating explicitly their time-dependence: hence for instance $x_1 \equiv x_1(t)$, $x_2 \equiv dx_2(t)/dt$), and the first letters (such as $a$, $b$, $c$, $A$, $B$, $C$; possibly equipped with indices) of the Latin alphabet to denote time-independent quantities, such as parameters (as, say, $c_{12}$: other time-independent quantities are of course the initial values, such as, say, $x_1(0)$). Generally all quantities are complex numbers (and $i$ denotes the imaginary unit: $i^2 = -1$); except for the time $t$ which is generally the real independent variable (although consideration of the analytic continuation of the dependent variables to complex values of the independent variable $t$ shall also turn out to be quite useful in the following treatment: see below). $
\]

The main results of this paper are the following: we define a concept of algebraic solvability, see below, and identify a large number of systems of the form having this property.
We further show its importance, both because it allows us to display the systems’ solutions in a rather explicit manner, and also because it allows to associate to each algebraically solvable system of the form (1) a related system with added linear terms, see (2) below, with the property that all its solutions are periodic with the same period, except possibly for an exceptional set of orbits passing through a singularity.

We define an algebraically solvable system as having the property that their solutions feature at most finitely many algebraic branch point singularities, so that they are defined, for complex times, on a Riemann surface with a finite number of sheets. This property, which is central to the paper, extends the class studied by Garnier [3], who identified all systems of two homogeneous ODE’s having the property that their solutions, as functions of complex times, are singlevalued, in other words, that they have no branch point singularities. In the following, we identify a countable infinity of systems of quadratic ODE’s with the more general property of algebraic solvability, and show that their solutions can be displayed in an explicit manner.

Let us illustrate our meaning by showing an instance of such an algebraically solvable system: let us consider the special case of (1) given by

\[
\dot{x}_1 = x_1 x_2, \quad \dot{x}_2 = (3/4) x_1^2 + x_2^2. \tag{2a}
\]

It follows from our results (see below Section III), that it has the following general solution:

\[
x_1(t) = \frac{9C}{[3-w(t)^2]^2}, \quad \tag{3a}
\]

\[
x_2(t) = \frac{9Cw(t)}{[3-w(t)^2]^2}, \quad \tag{3b}
\]

\[
\frac{w(t)^3}{9} - w(t) + \frac{3C}{4}(t-t_0) = 0, \quad \tag{3c}
\]

with \(C\) and \(t_0\) given in terms of the initial data by the following formulas:

\[
C = \frac{\left[3x_1(0)^2 - x_2(0)^2\right]^2}{9x_1(0)^4}, \quad \tag{4a}
\]

\[
t_0 = \frac{4x_2(0) x_1(0)^2 - 9x_1(0)^2}{3 x_2(0)^2 - 3x_1(0)^2} \tag{4b}
\]

and also note (see (3a) and (3b)) that, for all values of \(t\),

\[
w(t) = x_2(t)/x_1(t). \quad (5)
\]

These formulas provide quite explicitly the solution of the initial value problem for the system (2), except for the relatively minor—but quite significant—problem of identifying which one of the three solutions \(w(t)\) of the cubic equation (3c) should be inserted in the right-hand side of the two formulas (3a) and (3b). The way to address this issue is described in detail in [6].

Let us emphasize that this example shows that the solution of the initial value problem for the system of nonlinearly-coupled ODEs (2) has been reduced to a purely algebraic problem: in this particular case, to the quite simple one of finding the roots of a cubic equation, a problem that can even be solved explicitly via the well-known Cardano formulas. This kind of reduction of the solution of the initial value problem for the dynamical system under consideration to algebraic operations is the characteristic feature of the subclass of the dynamical system (1) which are identified in this paper, which should therefore be characterized as algebraically solvable, although for brevity we often use the term solvable.

We further note that the simplicity of the solution of (2) described above, generally arises whenever, as in this example, the Riemann surface on which the ODE’s solution is defined, has finitely many sheets. This condition implies, of course, both that all branch points be algebraic and that there be only finitely many such branch points.

Let us briefly compare our approach to others. Apart from qualitative and geometric approaches, which are not readily applicable to systems of two complex ODE’s, an important technique for the study of the kind of polynomial ODE’s we address, has been the search for polynomial, and more generally speaking, algebraic, invariants, according to the approach initially pioneered by Darboux [1]. Searching for such invariants has been done for systems similar to ours, such as the Lotka—Volterra system. This is the special case of (1) in which \(c_{12} = c_{21} = 0\), but for which linear growth or decay terms are additionally taken into account, see for example [8]. The existence of such invariants can be related to the value of a set of explicitly computable quantities known as the Kowalevski exponents, see [6,8]. The cases for which the Kowalevski invariants are all integers have been determined in [9]. For these it can be shown that the corresponding solutions are single-valued in the plane, and that they therefore correspond to the class of solutions investigated by Garnier. The relation of our approach to the techniques involving the search for invariants is not clear to us. We assume our approach may be more general. Certainly it allows to address directly the problem of determining the time evolution explicitly, which does not follow immediately from the knowledge of an invariant quantity.

In order to proceed, it is convenient to introduce a canonical form of the system (1), reading as follows:

\[
\dot{x}_1 = x_1 x_2, \quad \tag{6a}
\]

\[
\dot{x}_2 = A(x_1^2 + x_2^2) + Bx_1 x_2, \quad \tag{6b}
\]

which features in the right-hand side of its ODE’s only the two parameters \(A\) and \(B\) (rather than the six parameters \(c_{nij}\), see [1]). This can be done (as explained in Section III) using the obvious possibility to perform a linear transformation (featuring four a priori arbitrary parameters) of the two dependent variables; a change which modifies only quite marginally the nature of the problem. It should be pointed out that there exist exceptional cases for which the reduction to (1) cannot be performed. These can be reduced to a number of simpler forms involving only one parameter, see Section IV.

Remark 1.1. Note that, since the systems (1) and (6) generally involve complex variables, they may also be viewed as a system of four real ODE’s, involving the real and imaginary parts of \(x_1\) and \(x_2\). It is therefore not immediately accessible to standard qualitative approaches valid for two-dimensional
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This canonical form allows us to discuss the classical investigation of the system (1), as made about 60 years ago by Garnier. The focus of those studies is indeed on the analyticity properties of the solution of the system (1) when considered as functions of the complex variable $t$, and it led to the identification of all the subsystems of (1) such that their general solution is a single-valued function of the complex variable $t$: either they are entire functions of $t$—i.e., free of singularities in the entire complex $t$-plane—or their only singularities are unramified. This restriction amounts to the exclusion of all the dynamical systems (1), beyond the class previously identified by Garnier's approach—as the dynamical systems (1) the general solution of which—as functions of the complex variable $t$—defines a Riemann surface with a finite number of sheets. This implies, of course, that it can only feature a finite number of algebraic branch points (i.e., singularities of type $(t-t_0)^p$ with $p$ restricted to be a rational number: as implied by the above discussion, the example (2) discussed above is such an example in which both singularities with $p = 1/3$ and $p = 1/2$ arise.

In the following we show—by discussing in detail how the solution of these dynamical systems can be obtained—that it is indeed justified to categorize these systems as solvable; and that this entails a very substantial enlargement of the solvable subclass of dynamical systems (1), beyond the class previously identified by Garnier.

These findings are important because obviously the class of nonlinear dynamical systems (1) is quite interesting, both due to its mathematical neatness and even more so because of its relevance in many applicative contexts.

There is an additional reason why the solvable subclass of systems we identify is particularly interesting. This is a consequence of the following observation (see for instance and references therein): let the system (1) be reexpressed in terms of a new independent variable $\tau$, as follows (note that this is the only place in the paper where we shall use an independent variable different from $t$):

$$
\begin{align*}
\frac{dx_1}{d\tau} &= c_{11}x_1^2 + c_{12}x_2^2 + c_{13}x_1x_2, \\
\frac{dx_1}{d\tau} &= c_{21}x_1^2 + c_{22}x_2^2 + c_{23}x_1x_2.
\end{align*}
$$

If one now performs the following simple change of dependent and independent variables,

$$
\begin{align*}
x_1(t) &= \exp(-i\omega t)x_1(\tau), \\
x_2(t) &= \exp(-i\omega t)x_2(\tau),
\end{align*}
$$

the autonomous system (7) gets transformed into the following, also autonomous, system:

$$
\begin{align*}
\dot{x}_1 &= i\omega x_1 + c_{11}\dot{x}_1^2 + c_{12}\dot{x}_2^2 + c_{13}\dot{x}_1\dot{x}_2, \\
\dot{x}_2 &= i\omega x_2 + c_{21}\dot{x}_1^2 + c_{22}\dot{x}_2^2 + c_{23}\dot{x}_1\dot{x}_2.
\end{align*}
$$

Here and hereafter $\omega$ is a strictly positive real parameter; note that its presence multiplied by the imaginary unit $i$ implies that the dependent variables $\dot{x}_1 \equiv \dot{x}_1(t)$ and $\dot{x}_2 \equiv \dot{x}_2(t)$ evolving according to this system of ODEs take necessarily complex values—so that this system might equivalently be considered a system of four real dependent variables; while here we mainly restrict attention to real values of the independent variable $t$ (“time”).

It is then obvious that the time evolution of this system, (9), corresponds in a quite straightforward manner to the evolution of the system (7) when the complex time $\tau$ of that system, see (8), goes round and round, counterclockwise, on the circle of radius $1/\omega$ centered at the point $i/\omega$ in the complex $\tau$-plane. This implies—rather obviously, in the context of the results described above (or see, if need be)—that the time evolution of the system (7)—if this system is obtained from one of the solvable systems identified in this paper, see below—features the highly remarkable property to be isochronous, namely its general solution is periodic with a period independent of the initial conditions and given by an integer multiple of the basic period $T = 2\pi/\omega$. Note that this need not hold for strictly all orbits, since the circle on which $\tau$ moves may hit a branch point singularity, after which the solution is not uniquely defined any more. In fact, at the boundary between two regions in which the solution has different periods $n_1T$ and $n_2T$, with $n_1$ and $n_2$ different integers, this must necessarily happen.

The existence of a common period for all orbits, given by $mT$ with $m$ a fixed strictly positive integer, rests essentially on the property that the Riemann surface describing the solution only has a finite number of sheets, so that the evolution of $\tau$ on the circle always eventually returns to the sheet from which it started. For more general systems, it may happen, for instance, that every orbit is periodic with a period $nT$, with $n \in \mathbb{N}$, but that $n$ can take arbitrarily high values, depending on the initial condition. In other cases, it may happen that asymptotically periodic orbits, as for instance described in and more general aperiodic orbits, are generated.

Let us end this introductory survey of the results reported in this paper—which has been mainly meant to illustrate what we mean by the statement that a certain dynamical system is solvable—with a final observation, introduced in order to eliminate a possible misunderstanding. It is sometimes stated that a system of ODEs is solvable by quadratures; this is indeed the case for the system (1), as is well-known, see for example and below. This actually means that the time $t$ can
be expressed via integrals involving the functions \( x_1(t) \) and \( x_2(t) \) in such a way that the determination of these variables as a function of time is achieved by inverting these relations: see below. This task, however, can be relatively simple or exceedingly complicated; so that in the first case this approach does indeed allow to obtain the solution of the dynamical system (1)—i.e., to determine for given initial values \( x_1(0) \) and \( x_2(0) \) the solutions \( x_1(t) \) and \( x_2(t) \) for all values of \( t > 0 \) or even for complex values of \( t \); while in the second case it would be hardly justified to assert that the solution of the dynamical system (1) can be obtained by quadratures. The distinction among these two alternative situations shall become more clear from the treatment given below. Our point of view is that only in the first case the solution of the dynamical system (1) has been achieved; indeed our approach will be to analyze the solution by quadratures and to identify those special cases in which the inversion reduces to an algebraic procedure.

As an additional result, we have shown that the case of (6) with \( B = 0 \) can be reduced to a particular case of the one-dimensional Newton equation (“acceleration equals force”), thus allowing some further results to be derived. In particular it is shown there that the very simple complex Newtonian equation
\[
\ddot{\zeta} = \zeta^k
\]
(10a)
is algebraically solvable for the following infinite sets of values of the parameter \( k \)
\[
k = -(2n+1)/(2n-1) \quad \text{and} \quad k = -(n-1)/n,
\]
(10b)
for \( n \in \mathbb{N} \). And in particular this implies that the following complex variant of the Newtonian equations (10),
\[
\ddot{\xi} = i \left( \frac{k+3}{k-1} \right) \omega \xi + \frac{2(k+1)}{(k-1)^2} \omega^2 \dot{\xi} + \dot{\xi}^k
\]
(11)
(with \( \omega \) an arbitrary strictly positive parameter, \( i \) the imaginary unit, and \( k \) given by one of the values of (10b)), is isochronous for real values of \( t \).

In Section IV we set up the problem in some detail and describe the results obtained by Garnier. In Section V we describe our general approach. In Section VI we show the reduction of the \( B = 0 \) case to a Newtonian equation discussed above. In Section VII we discuss tersely the properties of the generic solution of the systems described in this paper. Finally, in Section VII we present some conclusions and an outlook. The paper is completed by 2 terse Appendices detailing some aspects of the reduction of the system (1) to its canonical form, as treated in Section IV

II. THE PROBLEM AND ITS CANONICAL FORM

Clearly, for the purposes we are considering, it makes little difference whether we consider the system (1) for given values of the parameters \( c_{ij} \) or any other system obtained from it by a linear transformation with time-independent coefficients, such as
\[
x_1(t) = a_{11} x_1(t) + a_{12} x_2(t), \quad (12a)
x_2(t) = a_{21} x_1(t) + a_{22} x_2(t). \quad (12b)
\]
(Note that a rescaling of the time variable by a factor \( \lambda \) is equivalent to a rescaling of \( x_1 \) and \( x_2 \) by the same factor—since the system (1) is invariant under the transformation \( x_n(t) \Rightarrow \lambda x_n(\lambda t) \)). Viewing two systems that can be transformed into one another by such a linear transformation as equivalent, we search for a canonical form such that each equivalence class contains at most a finite number of systems in the canonical form. It would, of course, be ideal to have only one canonical representative in each class, but this does not occur for any of the canonical forms we have considered. Nevertheless, the identification of a canonical form has two advantages: first, it decreases the number of free parameters, and second it avoids having to check on a case-by-case basis whether two apparently different solvable cases are in fact equivalent in this sense. The canonical form given by (6) satisfies the above requirements. It features only the two parameters \( A \) and \( B \). It turns out that every generic system (1) can be transformed into the form (6), but that the corresponding transformation generically arises in six different ways, so that there exist six linear changes of dependent variables, see (12), transforming a system from the form (6) into another, different, such system. Moreover, changing the sign of \( x_1 \) clearly leaves \( A \) invariant but changes the sign of \( B \), see (6), so that these six canonical forms come in pairs featuring equal values of \( A \) and values of \( B \) of opposite sign. The general formulas connecting the different canonical forms are given in Appendix B.

On the other hand, there do exist specific systems which cannot be transformed into the canonical form (6). These can be put in the forms
\[
\dot{x}_1 = x_1 x_2, \quad (13a)
\dot{x}_2 = x_1^2 + B x_1 x_2, \quad (13b)
\]
or
\[
\dot{x}_1 = x_1 x_2, \quad (14a)
\dot{x}_2 = A x_1^2 + \sigma x_1 x_2, \quad (14b)
\]
with \( \sigma = 0 \) or 1. Note that systems (13, 14) only depend on one parameter, so they are in fact more special than the systems of type (6). Finally the form
\[
\dot{x}_1 = x_1 x_2, \quad (15a)
\dot{x}_2 = B x_1 x_2, \quad (15b)
\]
while being just the special case \( A = 0 \) of (6), deserves to be singled out because it corresponds to the only model in the Garnier subclass \( \mathbb{H} \) which features a free parameter (see below). The solution is elementary and given in (13), all the other cases in the Garnier subclass \( \mathbb{H} \) correspond to fixed values of both \( A \) and \( B \).

There also exist four uncoupled exceptional forms—which need not be considered hereafter, since their solution is quite
A. Statement of the results

We give below sufficient conditions on the values of $A$ and $B$ for the system (11) to be algebraically solvable, and we exhibit the corresponding solutions of their initial value problems:

Case 3.1:

\[
A = \frac{n + q - 1}{n + q}, \quad B = \pm\frac{n - q}{n + q} \sqrt{\frac{n + q - 1}{nq}},
\]

(17a, 17b)

where $n$ is an arbitrary strictly positive integer, $n > 0$, and $q$ is an arbitrary noninteger complex rational number.

Case 3.2:

\[
A = \frac{n + 1}{n}, \quad B = \pm\frac{2q}{n} \sqrt{\frac{n + 1}{q^2 - n^2}},
\]

(18a, 18b)

where $n$ is again an arbitrary strictly positive integer, $n > 0$, and $q$ a noninteger complex rational number.

Case 3.3:

\[
A = \frac{n + 1}{n}, \quad B = \pm\frac{2m}{n} \sqrt{\frac{n + 1}{m^2 - n^2}},
\]

(19a, 19b)

where $n$ is again an arbitrary strictly positive integer, $n > 0$, and $m$ is an arbitrary integer of parity different from that of $n$.

Case 3.4: this is the special case with

\[
A = -2, \quad B = 0,
\]

(20)

or equivalently $A = -1/5$ and $B = \pm 3\sqrt{6}/10$. Its solution is given in terms of the square root of a Weierstrass elliptic function; this case is discussed separately in Section IV.

Note that, additionally to Cases 3.1-3, all cases arising from them by the transformations described in Appendix B are also solvable, but as the corresponding expressions of the parameters $A$ and $B$ become somewhat cumbersome, we do not report them explicitly.

B. Proofs of the results

In the following, we discuss the details of the treatment of the system (11) by quadratures, and we identify a set of cases

| $A$ | $B$ | Garnier list | solution type |
|-----|-----|-------------|---------------|
| 0   | arbitrary | $s_{IV}$ | linearisable |
| $-1/2$ | 0 | $s_{V}$ | elliptic |
| $-1/3$ | $\pm \sqrt{3}/2$ | $s_{VI}$ | elliptic |
| $-1/2$ | $\pm i$ | $s_{VIII}$ | elliptic |

TABLE I. Values of $A$ and $B$ for which the system (11) has a single valued solution. The third column states the name in the article by Garnier [20] and the fourth gives the nature of the general solution. Note that the system $s_{IV}$ simply corresponds to the system (15) for $B = 0$, called $s_{II}$ by Garnier in [20], and the system (14) for $A = 1$ and $\sigma = 0$, denoted by Garnier as $S_{I}$. Four additional non-exceptional cases are listed in Table I. The first has a solution given in terms of powers and exponential functions, whereas the last three have a solution given in terms of elliptic functions. In other terms, all these five cases are either linearisable or reducible to an elliptic ODE. Let us re-emphasize that in all these cases except the one corresponding to (15), called $s_{IV}$ by Garnier in [20], these models do not feature any free parameters. It should, however, be emphasized that not all systems of type (11) having the property of being single valued for all values of $t$ can be reduced by linear transformations alone to the normal forms listed below.

The reduction as performed by Garnier occasionally requires nonlinear transformations such as birational transformations.

To summarize: if we perform appropriate nonlinear transformations, see (13), we reduce any system of the form (11) featuring only single valued solutions to one of the four specific cases listed in Table I or else to a model of the type (13) with $B = 0$. As an example of the usefulness of this reduction procedure, we may point out that, in Appendix B, 10 algebraically solvable cases of (11) were identified (as subcases of more general solvable systems). Following the approach of Appendix A all of these were found to reduce to one of Cases $s_{IV}$ and $s_{VI}$ shown in Table I.

In the following, we limit ourselves to considering systems of the canonical form (11), and we identify a countably infinite set of values of $A$ and $B$ such that these systems are algebraically solvable.
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in which the corresponding solution can be obtained by algebraic operations, implying that the analytic continuations of the general solution \( x_1(t) \) and \( x_2(t) \) of the dynamical system (9) has only a finite number of singularities of the type of algebraic branch points, or equivalently that—considered as functions of complex \( t \)—they live on Riemann surfaces featuring a finite number of sheets.

Define

\[
u(t) = x_1(t)/x_2(t).
\] (21)

From (9) one finds

\[
\dot{\nu} = -x_2u(Au^2 + Bu + A - 1)
= -Ax_2(u - u_+)/(u - u_-),
\] (22a)

where

\[
u_\pm = \frac{-B \pm \sqrt{4A^2 - 4A^2 + B^2}}{2A}.
\] (22b)

We now define \( R(u) \) (consistently with (21)):

\[
 x_2 = R(u), \quad x_1 = uR(u).
\] (23)

Substituting (23) into (6b) and using (22a) yields

\[
\dot{x}_2 = R'(u)\dot{\nu}
= -AR'(u)R(u)(u - u_+)/(u - u_-)
= x_2^2 \left[A(1 + u_+^2) + Bu\right]
= R(u)^2 \left[A(1 + u^2) + Bu\right]
\] (24)

(note that here and hereafter we use the standard mathematical notation according to which a prime appended to a function denotes a differentiation with respect to the argument of that function). From the second and fourth lines of these formulas we clearly get

\[
\frac{d}{du} \ln R(u) = -\frac{1 + u^2 + Bu/A}{u(u - u_+)/(u - u_-)}.
\] (25)

We now define \( v_0 \) and \( v_\pm \) through the relation

\[
u = \frac{v_0}{u - u_+ - (u - u_+)} = \frac{v_+ - v_-}{u - u_+ - (u - u_-)}, \tag{26a}
\]

\[
u = -1 + v_+ + v_-, \tag{26b}
\]

Here (26b) follows from (26a) by matching the asymptotic behavior as \( u \to \infty \) of the right-hand side and the left-hand side of (26a).

We thus obtain for \( R(u) \), see (25) and (26a),

\[
R(u) = Cu^{v_+ + v_- - 1}(u - u_+ - (u - u_-))^{-v_-}.
\] (27)

Here \( C \) is an integration constant depending on the initial condition, which enforces the relation

\[
 x_2(0) = R[u(0)].
\] (28a)

or equivalently

\[
x_2(0)^2 = Cx_1(0)^{v_+ + v_- - 1}[x_1(0) - u_+ x_2(0)]^{-v_+} 
\times [x_1(0) - u_- x_2(0)]^{-v_-}.
\] (28b)

The time-dependence of \( u \) is obtained by inserting the ansatz (23) into (22a):

\[
\dot{\nu} = -AR(u)(u - u_+)/(u - u_-)
= -ACu^{v_+ + v_- - 1}(u - u_+ - (u - u_-))^{-v_- + 1},
\] (29)

as stated in (22a). This integrates to

\[
AC\int = -\int_{u(0)}^{u(t)} du u^{v_+ - v_-}(u - u_+)u^{-v_- + 1} \times
(u - u_-)^{v_- - 1}.
\] (30a)

As stated in the Introduction, we aim to look for values of \( A \) and \( B \) such that the connection between \( u \) and \( t \) given by (30a) is algebraic, that is, that the right hand side of (30a) be a rational function of \( u^{1/m} \) for some \( m \in \mathbb{N} \).

Further evaluating the above integral we obtain:

\[
B_p(v_+, v_-) = \int_0^p dt t^{v_- - 1}(1 - t)^{v_+ - 1}
= \frac{\rho t^{v_+}}{v_+} \left[F(v_+, 1 - v_-; v_+ + 1; \rho) \right].
\] (31)

Here \( B_p(p, q) \) is the incomplete beta function (see for instance page 87 of [12]).

\[
F(a, b; c; x) \quad \text{is the hypergeometric function.} \]

It follows, from the fact that \( F(a, b; c; x) \) is a polynomial in \( x \) whenever either \( -a \) or \( -b \) is a non-negative integer (see page 57 of [12]), that the right-hand side of (30b) is an algebraic expression in \( u \) (see (30d)) whenever \( v_- \) is a strictly positive integer and \( v_+ \) is a (possibly complex) rational number.

Alternative cases in which the right-hand side of (30a) is also
an algebraic expression in $u$ (not necessarily a polynomial)

can be identified by taking advantage of the following three identities (see eqs. (4), (21), and (22) in Section 2.8 of [19]),

\[
F(a, b; c; z) = (1-z)^{-a},
\]

\[
(a)_m z^{-1} F(a + m, b; c; z) = \frac{d^m}{dz^m} \left[ z^{-a} F(a, b; c; z) \right],
\]

\[
(c - n)_n z^{-1} F(a, b; c - n; z) = \frac{d^n}{dz^n} \left[ z^{-a} F(a, b; c; z) \right].
\]

It is indeed easily seen that they imply that $F(a + m, b; c - n; z)$ is algebraic in $z$ provided $a$ is a (possibly complex) rational number and both $m$ and $n$ are arbitrary nonnegative integers. Note that, in accordance with our definition of algebraically solvable, we do not view irrational powers as algebraic. Thus—see the right-hand side of (31)—now we look for those cases in which

\[
v_+ = a + m, \quad 1 - v_- = b, \quad v_+ + 1 = b - n,
\]

with $m$ and $n$ arbitrary nonnegative integers. These relations imply that $v_+, v_-$ is a strictly negative integer or zero, indeed, as seen by subtracting eq. (33b) from eq. (33c),

\[
v_+ + v_- = -n.
\]

However, see (36c) below, the case in which $v_+, v_- = 0$ does not arise from any finite value of $A$, and we thus discard it. Moreover these relations require that $v_+$ is neither a negative integer nor zero, since in that case the corresponding integral in (30b) has a singularity of logarithmic type. Additionally, both $v_+$ and $v_-$ must be rational, for which it is enough to assume that, say, $v_+$ be rational, see (34).

Let us now express $A$ and $B$ in terms of $v_+$ and $v_-$. Clearly (see (26a) and (22b))

\[
v_+ = \frac{1}{A u_+ (u_+ - u_-)},
\]

\[
v_- = \frac{1}{A u_- (u_+ - u_-)}.
\]

From this follows

\[
v_+ + v_- = \frac{1}{1 - A},
\]

\[
A = \frac{v_+ + v_- - 1}{v_+ + v_-},
\]

\[
B^2 = \left( \frac{v_+ - v_-}{v_+ + v_-} \right)^2 \frac{v_+ + v_- - 1}{v_+ v_-}.
\]

Summarizing, we have at least two distinct cases in which the expression for $t$ (see (30)) is algebraic in $u$:

1. When one of the two $v$’s is a strictly positive integer and the other is a non-zero (possibly complex) rational number. This translates into the case described in (17), if we set $v_+ = n$ and $v_- = q$.

2. When $v_+ + v_-$ is a strictly negative integer, neither $v_+$ nor $v_-$ is a negative integer or zero, and both $v_+$ and $v_-$ are non-zero (possibly complex) rational numbers. This translates into the two possible cases (19) and (18) described in Subsection III A above, depending on whether $v_+ - v_-$ is or is not an integer. In the latter case, we set $v_+ - v_- = -n$ and $v_+ - v_- = q, and the conditions on $v_\pm$ not being a negative integer or zero are automatically fulfilled. If $v_+ - v_- = m$, we note that $m$ and $n$ must have opposite parity for the conditions on $v_\pm$ to be fulfilled, leading to the case shown in (19).

Let us give three examples, chosen to be typical of the three Cases 3.1-3.3. First, the two first belong to Case 3.1, the second to Case 3.2, and the third to Case 3.3. In general the solutions are given in terms of two integration constants, $C_1$ and $t_0$, which are determined by the initial conditions: $C$ is given always by (28b) whereas $t_0$ is determined by substituting $t$ by $0$ and $u$ by $u(0)$ in the relationship connecting $u$ and $t$. In general, it is given by:

\[
t_0 = -\frac{1}{AC} \left( 1 - \frac{u_+}{u_-} \right)^{v_+} \left( 1 - \frac{u_-}{u_+} \right)^{v_-} B \rho(0)(v_+, v_-),
\]

where $\rho(0)$ is the expression for $\rho$ defined in (50b), where $u$ is replaced by $u(0)$.

1) $v_- = 1$ and $v_+$ an arbitrary complex rational number. From (36) we find the relationship

\[
A = \frac{v_+ + v_- - 1}{v_+ + v_-},
\]

\[
B = \left( \frac{v_+ - v_-}{v_+ + v_-} \right)^2 \frac{v_+ + v_- - 1}{v_+ v_-}.
\]

These define an algebraically solvable system whenever $A$ is a complex rational number.

Somewhat atypically, these relations can be solved for all $A$, to yield explicit expressions for $u$ as a function of $t$, and

...
hence for \( x_1 \) and \( x_2 \) through (23). Thus:

\[
\begin{align*}
  u &= \frac{A - 1}{A} \left( [AC(t_0 - t)]^{(1-A)/A} - 1 \right) \quad (39a) \\
  x_1(t) &= uR(u) \quad (39b) \\
  x_2(t) &= R(u) \quad (39c)
\end{align*}
\]

The solution can therefore be said to extend naturally to irrational values of \( A \), though (38) is then, of course, no more an algebraically solvable model. Note that, to obtain the full solution of the initial value problem, it is enough to substitute (39a) into (39b) and (39c) using for \( R(u) \) the expression in (38b).

Note in passing that this result can also be obtained directly from (6): the equality,

\[
z = x_1 + x_2 = (1 + u)R(u) = \frac{1}{A(t_0 - t)}
\]

which directly follows from (38), also follows from (6) through the easily verified relation

\[
\dot{z} = Az^2.
\]

(6a), then yields for \( x_1 \)

\[
\dot{x}_1 = x_1 \left[ \frac{1}{A(t_0 - t)} - x_1 \right],
\]

which is a Bernoulli equation linearized by the transformation \( r = 1/x_1 \).

2) \( v_+ = (1 - A)^{-1} - 2 \) and \( v_- = 2 \), for arbitrary rational values of \( A \). From (36) we find that and \( B = (4A - 3)\sqrt{A/(4A - 2)} \). From (30) and (27) we obtain the following expression for the time dependence of \( u \) and the function \( R(u) \):

and thus finally the full solution is given by (23). Here note that the solution can, in fact, always be obtained in algebraic terms whenever \( A \) is rational. However, the complexity of the problem increases as the denominator of \( A \) grows, and the solution for irrational \( A \) cannot be obtained in elementary algebraic terms.

3) \( v_+ = -7/5 \) and \( v_- = 2/5 \). This corresponds to \( A = 2 \) and \( B = 9/\sqrt{7} \), see (36). We similarly obtain for the time dependence of \( u \) and the function \( R(u) \):

\[
2C(t - t_0) = \frac{\left[ 14u^2 + 9\sqrt{7}u + 7 \right]}{2^{2/5} (2u + \sqrt{7})}\left[ \frac{(9\sqrt{7}u + 7^{3/5})w - 7\sqrt{7}u - 7}{7u + \sqrt{7}} \right]^{2/5} w
\]

\[
w = \left( \frac{5}{\sqrt{7}u + 1} + 2 \right)^{2/5}
\]

\[
R(u) = C\cdot u + \sqrt{7}/2 \left( u^2 (u + 1/\sqrt{7}) \right)^{7/5}
\]

and again the full solution is given by (23).

4) \( v_+ = 1/2 \) and \( v_- = -5/2 \). This corresponds to \( A = 3/2 \) and \( B = 3\sqrt{3}/5 \), see (36). We similarly obtain for the time dependence of \( u \) and the function \( R(u) \):
Here again the full solution is given by (45).

In all cases the full evaluation of the solution is reduced to solving for the zeros of a polynomial, the coefficients of which depend polynomially on time. The solution thus defines a Riemann surface when \( t \) is taken as a complex variable. The common feature to all these solutions is, of course, that they all define Riemann surfaces with a finite number of sheets.

IV. A NEWTONIAN APPROACH TO THE CASE \( B = 0 \)

We now treat the specific cases of (46) with \( B = 0 \):

\[
\begin{align*}
\dot{x}_1 &= x_1 x_2, \quad (46a) \\
\dot{x}_2 &= A(x_1^2 + x_2^2), \quad (46b)
\end{align*}
\]

since it is possible to treat them in a different and more convenient way. First of all, several formulas of Section 3 simplify when \( B = 0 \); we show how (36c) specializes in this case: one then has

\[
\begin{align*}
\nu_+ &= \nu_-, \quad (47a) \\
A &= \frac{2v-1}{2v}, \quad (47b) \\
u_{\pm} &= \pm \sqrt{\frac{1-A}{A}} = \pm \frac{1}{\sqrt{2v-1}}. \quad (47c)
\end{align*}
\]

Finally (30) and (27) simplify to

\[
\begin{align*}
ACt &= \int_{u(0)}^{u^{-1}} dw \left( 1 + \frac{A-1}{A} w^2 \right)^{-v-1} \\
&= \int_{u(0)}^{u^{-1}} dw \left( 1 - \frac{w^2}{2v-1} \right)^{-v-1}, \quad (48a) \\
R(u) &= Cu^{2v-1} \left( u^2 + \frac{A-1}{A} \right)^{-v} \\
&= Cu^{2v-1} \left( u^2 - \frac{1}{2v-1} \right)^{-v}. \quad (48b)
\end{align*}
\]

The full list of values of \( A \) and \( v \) which lead to algebraic solutions of (46) due to the above arguments are

\[
\begin{align*}
A &= n/(n+1), \quad v = n, \quad n \in \mathbb{N}. \quad (49a) \\
A &= 2n/(2n-1), \quad v = -(2n-1)/2, \quad n \in \mathbb{N}. \quad (49b)
\end{align*}
\]

Remark 4.1. It is interesting to note that none of the values listed in Garnier [23] correspond to a strictly positive integer or negative half-integer value of \( v \). The solutions \( x_1(t), x_2(t) \) listed by Garnier are either elliptic functions, or correspond to degenerate systems of types (13) or (14). None correspond to the cases treated in this paper.

Remark 4.2. Note that via an obvious constant rescaling of the dependent variable \( x \) or of the independent variable \( t \) this ODE can be replaced by the ODE (10a).

This second-order ODE can clearly be integrated by quadratures (by multiplying it by \( \dot{x} \) and then integrating it over time). Carrying this out, one obtains expressions a bit simpler than, but essentially equivalent to, those obtained in Section III. It follows that, if \( A \) has the values described in (49a) and (49b), solving the Newtonian equation (51) with the corresponding values of \( k \) can be reduced to solving a polynomial equation. The corresponding values of \( k \) and \( v \) are

\[
\begin{align*}
k &= -(2n+1)/(2n-1), \quad v = n, \quad n \in \mathbb{N}, \quad (52a) \\
k &= -(n-1)/n, \quad v = -(n-1)/2, \quad n \in \mathbb{N}. \quad (52b)
\end{align*}
\]

see (45).

On the other hand, it follows from the work of Picard [21] that \( z(t) \) is meromorphic in \( t \) only for \( k \) a nonnegative integer less than 4, \( k = 0, 1, 2, 3 \). \( k = 1 \) does not correspond to any finite value of \( A \), whereas the other three values correspond to \( A = 2, -2, -1 \). We saw above (see Table II in Section III) that \( A = -1 \) is a case in which the solution is singlevalued, as discussed by Garnier [23], while the case \( A = 1 \) corresponding to \( v = -1/2 \) was discussed above, see (36b).

But the case \( A = -2 \) is new: specifically it corresponds to \( v = 1/6 \), which does not fall in the cases discussed in Section III nor does it belong to the Garnier list. This case corresponds to \( k = 2 \). The solution \( z(t) \) of (51) is an elliptic function, but for \( x_1(t) \) we have

\[
\begin{align*}
x_1(t) &= z(t)^{1/2}, \quad (53a) \\
z(t) &= -\exp(2\pi i/3) \left( \frac{3}{2} \right)^{1/3} \times \phi \left[ \left( -\frac{2}{3} \right)^{1/3} (t+C_1); 0, C_2 \right], \quad (53b)
\end{align*}
\]
New algebraically solvable systems... where $C_1$ and $C_2$ are integration constants, and where $\rho(x; g_2, g_3)$ denotes the Weierstrass function as a function of its invariants $g_2$ and $g_3$, $x_1(t)$ is thus not singlevalued, but can be obtained from the singlevalued function $z(t)$ by an algebraic operation, namely by taking the square root.

Remark 4.3. Note that the case $A = -1$ mentioned two paragraphs above is similar, but in this case the connection between $z(t)$ and $x_1(t)$ introduces no loss of analyticity, so that the result is in the class of singlevalued solutions, and it indeed appears in the list of Garnier.

Remark 4.4. The fact that the solutions of the simple Newtonian equation (51) for all the assignments (52) of $k$ are algebraic functions of $t$ does not seem to have been noticed earlier; note that this implies that all corresponding, appropriately modified, systems are isochronous (as detailed at the end of Section II, see (11)). On the other hand it seems likely that for all sufficiently large positive integer values of the exponent $k$ in (51), the solution of this simple Newtonian equation is not algebraic, leading—in the complex—to extremely complicated behavior (see (11) for a detailed treatment when $k$ is a strictly positive even number).]

V. QUALITATIVE PROPERTIES OF THE GENERIC SOLUTION

In the following we consider the qualitative properties of the systems treated in Section III when they start from generic initial conditions. Such initial conditions are, of course, complex. To be specific, we concentrate on the cases 3.1, 3.2 and 3.3 described in Subsection III A. We shall, in the following, always consider the solution’s behavior for real times. The first finding we report is the following: for generic initial conditions the solution remains bounded for all finite real times. In other words, the solution never blows up at a finite time. Moreover, this generic solution remains analytic for all finite times, i.e. it never hits a singularity. This is seen as follows: $u(t)$, or some algebraic function of it, is a zero of some $t$-dependent polynomial. A first possible way in which a singularity might arise, is if the coefficients of the $t$-dependent polynomial at some time take values such that the polynomial has a multiple zero. When this happens, the discriminant of the polynomial must vanish, which entails two real conditions. If the time is real, generically the curve in the space of polynomials will therefore not hit the set of polynomials with multiple zeros, since this set has real codimension two in the set of all polynomials. Another possibility is that at some point $u(t)$ takes a value for which the denominator of $R(u)$ vanishes, thus leading to the divergence of $x_2(t)$. There are only three such values, however, namely 0 and $u_{\pm}$, see (27). Again, going through one of these values corresponds to two real conditions and will therefore generically not happen.

What can we say concerning the behavior of $x_1(t)$ and $x_2(t)$ for large times? From (30a) we see that $u$ must approach one of the three values 0 or $u_{\pm}$ (going to infinity is not an option for $u$, as the integral in $u$ in (30a) converges as $u \to \infty$, since the integrand goes as $u^{-3}$). Under these circumstances, since $u$ is defined as the root of a polynomial equation, as $t$ diverges, it remains close to a fixed value; it will hence eventually—for $t$ sufficiently large—not move from one branch to another, and thus tend monotonically to a given value. For similar reasons, the function $R(u)$, see (27) will tend to a value which is either 0 or infinity, with a power-law that can be determined in each specific case.

Such smoothness properties are by no means obvious. For instance they fail in the case of arbitrary real $A$ and $B$ and real initial conditions: indeed in this case, it is well known, and also readily verified from the results shown in Section III that the solutions of (6) can diverge at finite time for an open set of initial conditions. The result similarly fails in the case $A = -2$ and $B = 0$, see (53b). The solutions then do not have the regular behavior at infinity described in the last paragraph. Indeed, for generic values of $C_1$ and $C_2$, the poles of the elliptic function there given, while they do not typically lie on the $x$ axis, in general come arbitrarily close to it, since they lie on a lattice. The function $x_1(t)$ thus becomes arbitrarily large infinitely often, but irregularly so, as $t \to \infty$, see Figure 1. In this sense the systems whose solutions are described in Section III are therefore remarkably simple in their regularity properties for finite $t$ as well as in their asymptotic behavior.

VI. CONCLUSIONS AND OUTLOOK

We have displayed a large variety of special cases of the system of ODE’s (6) with the property that their solution, in the complex $t$-plane, only takes a finite number of different values according to the path taken, or in other words, that they define a Riemann surface with only a finite number of sheets. These solutions have the additional remarkable feature of being rather simple: in all cases but one, they can be calculated from $t$ by purely algebraic operations. They are therefore significantly simpler than the only other explicit solutions of previously known, namely those listed by Garnier, which are elliptic functions. For the one exceptional case, (6) with
A = −2 and B = 0, it is sufficient to calculate an elliptic function of t and take its square root. This constitutes in itself a remarkably simple addition to the list presented by Garnier.

These systems are additional to those already listed by Garnier, which were obtained by looking for systems whose solution has no branch points at complex times, hence is a single-valued function of the complex variable t. Since there were only five such cases, it is seen that loosening the requirement of being single-valued by allowing the presence of a finite number of algebraic branch points significantly increases the number of examples, since we have found a doubly infinite set of such models (see (6) with (17), (18), (19)). Moreover we have noted that, via the change of dependent and independent variables (6), all these systems yield new autonomous nonlinearly-coupled dynamical systems featuring the remarkable property to be isochronous.

As an additional finding, we found that the special case B = 0 of (6) can be transformed to the Newtonian equation (10), so that our results of Section III can be extended to the Newtonian equation (10a) for those values of k stated in (10b). This finding, which implies that the very simple Newtonian equation (10) is algebraically solvable when the exponent k has one of the infinite series of values listed in (10b), and likewise that the Newtonian equation (11) is isochronous for the same values of k, seems to us quite remarkable.

Two interesting open problems are the following: first, we have only given sufficient conditions for the algebraic solvability of (1). It is clearly of interest to be able to give a complete list of systems with this property. On the other hand, it would also be important to know how our systems behave as far as the existence of invariants is concerned. Finally, let us point out that Sokolov and Wolf generalized Garnier’s results to the case of quadratic systems with non-commuting variables: it might be an interesting direction of research to search for a similar generalisation of our results. Another interesting question concerns the possibility of extending this approach to non-homogeneous systems, in particular such as involve both linear and quadratic terms; for preliminary work in this direction, see for example (16, 18).

AUTHORS’ CONTRIBUTIONS

All authors contributed equally to this work.

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AIP PUBLISHING DATA SHARING POLICY

Data sharing not applicable—no new data generated.

Appendix A: Canonical forms for the system (1)

We display here the reduction of an arbitrary system of the form (1) to either the form (6) or one of the exceptional forms (13), (14), and (15), via a linear transformation of the two dependent variables, see (12).

From u = x1/x2 (see (21)) we obtain

\[ \dot{u} = -x_2 P_3(u), \]  
\[ P_3(u) = c_{21} u^3 - (c_{11} - c_{23}) u^2 - (c_{13} - c_{22}) u - c_{12}. \]  

(A1a)

We now distinguish two main cases with several subcases.

Case A1. Let the polynomial P3(u) have (at least) one simple zero: the generic case is, of course, that all its three zeros are simple. In the presence of a double zero, the other zero is of course simple. In that case we take \( \beta \) to be such a zero. We then define (see (21))

\[ y_1 = x_1 - \beta x_2 = (u - \bar{u}) x_2, \quad y_2 = x_2. \]  

(A2)

Because \( \beta \) is a zero of P3(u), if u starts out at \( \beta \), it maintains that value always (see (A1a)). It follows that, if y1 starts out at zero, it always remains there. The equation for y1 is thus of the form

\[ \dot{y}_1 = y_1 (\alpha y_1 + \beta y_2). \]  

(A3)

Since \( \beta \) is a simple zero of P3(u), it follows that

\[ \beta \neq 0. \]  

(A4)

We can thus introduce \( z_1 = y_1 \) and \( z_2 = \alpha y_1 + \beta y_2 \) as 2 new independent variables. One then has

\[ \dot{z}_1 = z_1 z_2, \quad \dot{z}_2 = A_1 z_1^2 + A_2 y_1^2 + B y_1 y_2 \]  

(A5)

and, by appropriately rescaling \( y_1 \), we obtain (6) if neither A1 nor A2 vanish (up to the formal exchange of y1, y2 with x1, x2). Likewise, if either A1 or A2 vanish, we obtain forms (13) and (14) respectively; and if both A1 and A2 vanish, we obtain (15).

Case A2. The polynomial P3(u) is quadratic, i.e. \( c_{21} = 0 \), and it features a double zero, implying

\[ (c_{13} - c_{22})^2 - 4c_{23} c_{12} = 0. \]  

(A6)

In this case we first try to invert the roles of x1 and x2. If this leads to a generic polynomial of third degree, we are led back to the first case. If not, we have overall

\[ c_{21} = 0, \]  
\[ c_{12} = 0, \]  
\[ (c_{13} - c_{22})^2 - 4c_{23} c_{12} = 0, \]  
\[ (c_{23} - c_{11})^2 - 4c_{13} c_{21} = 0. \]  

(A7a)

(A7b)

(A7c)

(A7d)
New algebraically solvable systems...  

But the last two equations reduce to

\[ c_{13} = c_{22}, \quad (A8a) \]
\[ c_{23} = c_{11}, \quad (A8b) \]

which leads to \( P_5(u) = 0 \). This system can then be reduced to the uncoupled forms [16].

Appendix B: Equivalent canonical forms

Let a system in the canonical form (6) have the parameters \( A \) and \( B \). Clearly the system with the parameters \( -A \) and \( B \) is equivalent via a change of sign of \( x_1 \). On the other hand, it can be seen, using straightforward but tedious calculations, that the following four values are also equivalent:

\[ A(\sigma_1, \sigma_2) = \frac{1}{\Delta} \left[ 4A^2 - 2A - B^2 - \sigma_1 R \right], \quad (B1a) \]
\[ B(\sigma_1, \sigma_2) = \frac{\sigma_2}{B \Delta} \left[ B^2 \left( 1 - 4A(A - 1) + B^2 \right) + \sigma_1 \left( 1 - 4A^2 + B^2 \right) R \right], \quad (B1b) \]
\[ \Delta = 2A \left( (1 - 2A)^2 - B^2 \right), \quad (B1c) \]
\[ R = \sqrt{B^2(4A^2 - 4A^2 + B^2)}, \quad (B1d) \]

where \( \sigma_1 \) and \( \sigma_2 \) each takes the values +1 and −1. These four sets of values together with the two initial sets of values lead to 6 equivalent canonical forms.

The route to arrive at these results, see [B1], goes as follows: first the general substitution

\[ x_1 = a_{11}y_1 + a_{12}y_2, \quad (B2a) \]
\[ x_2 = a_{21}y_1 + a_{22}y_2 \quad (B2b) \]

is performed in (6) and the resulting equations for the new variables \( y_1(t) \) and \( y_2(t) \) are computed. These depend on \( a_{11} \), \( a_{12} \), \( a_{21} \) and \( a_{22} \). The conditions stating that these new equations are again in the canonical form (6) are then determined and solved using Mathematica and yield

\[ a_{11} = \sigma_2 \frac{2A - 4A^2 + B^2 - \sigma_1 R}{\Delta}, \quad (B3a) \]
\[ a_{12} = -\frac{B^2 + (2A - 1)\sigma_1 R}{B \Delta}, \quad (B3b) \]
\[ a_{21} = \sigma_2 \frac{-B^2 - (2A - 1)\sigma_1 R}{B \Delta}, \quad (B3c) \]
\[ a_{22} = \sigma_1 \sigma_2 \frac{(2A - 4A^2 + B^2) + \sigma_1 R}{\Delta}, \quad (B3d) \]

where again \( \sigma_1 \) and \( \sigma_2 \) each takes the values +1 and −1. Putting these values into the transformed equations yields the result stated above, see [B1].

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