Approximate analytical solutions of the generalized Woods-Saxon potentials including the spin-orbit coupling term and spin symmetry

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Abstract

We study the approximate analytical solutions of the Dirac equation for the generalized Woods-Saxon potential with the pseudo-centrifugal term. In the framework of the spin and pseudospin symmetry concept, the approximately analytical bound state energy eigenvalues and the corresponding upper- and lower-spinor components of the two Dirac particles are obtained, in closed form, by means of the Nikiforov-Uvarov method which is based on solving the second-order linear differential equation by reducing it to a generalized equation of hypergeometric type. The special cases $\kappa = \pm 1$ ($l = \tilde{l} = 0$, s-wave) and the non-relativistic limit can be reached easily and directly for the generalized and standard Woods-Saxon potentials. Also, the non-relativistic results are compared with the other works.

Keywords: Dirac equation, spin symmetry, pseudospin symmetry, Woods-Saxon potential; Nikiforov-Uvarov method.

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I. INTRODUCTION

It is well known that the exact solutions play an important role in quantum mechanics since they contain all the necessary information regarding the quantum model under study. The exact solutions of the Schrödinger equation are only possible for a hydrogen atom and for a harmonic oscillator in three dimensions [1-3]. However, when a particle is in a strong potential field, the relativistic effect must be considered, which gives the correction for non-relativistic quantum mechanics [4]. Taking the relativistic effects into account, a particle in a potential field should be described with the Klein-Gordon (KG) and Dirac equations. In recent years, there has been an increased interest in finding exact solutions to Schrödinger, KG, Dirac and Salpeter equations for various potential schemes [4–20]. The problems that can be exactly solved for the KG and/or Dirac equations are seldom except a few examples, such as hydrogen atom and electrons in a uniform magnetic field. Recently some authors solved such relativistic equations for some potentials. Ikhdair [6] obtained the bound-state solution of the $D$-dimensional KG equation for the vector and scalar general Hulthén-type potentials with any arbitrary $l$-state using the Nikiforov-Uvarov (NU) method. Moreover, Eğrifes and Sever [7] investigated the bound state solutions of the $1D$ Dirac equation with $\mathcal{PT}$-symmetric real and complex forms of generalized Hulthén potential. Yi et al. [10] obtained the energy equations in the KG theory with equally mixed vector and scalar Rosen-Morse-type potentials. We have solved the spinless $1D$ Salpeter equation analytically for its exact bound state spectra and wave functions with real and complex forms of the $\mathcal{PT}$-symmetric generalized Hulthén potential [11]. We obtained a quasi-exact analytic bound-state solutions within the framework of the position-dependent effective mass KG equation for scalar and vector Hulthén potentials in any arbitrary $D$-dimension and with any orbital quantum number $l$ using the NU method combined with a new approximation scheme for the centrifugal potential term [20].

The Woods-Saxon (WS) potential and it’s various modifications have played an essential role in microscopic physics in the determination of the energy level spacing, particle number dependence of energy quantities and universal properties electron distributions in atoms, nuclei and atomic clusters since it can be used to describe the interaction of neutron with one heavy-ion nucleus and also for optical potential model [21,22]. Although the non-relativistic Schrödinger equation with this potential has been solved for $s$-states [21] and the single-
particle motion in atomic nuclei has been explained quite well, the relativistic effects for a
particle under the action of this potential are more important, especially for a strong-coupling
system. Berkdemir et al [22] obtained the bound-state solution of the 1D Schrödinger
equation for the generalized WS potential by means of the NU method. We investigated the
bound-state solutions of the 1D KG equation with real and complex forms of the generalized
WS potential [12]. Kennedy [23] obtained the scattering and bound-state solutions of the
one-dimensional Dirac equation for the WS potential. Guo and Sheng [24] solved exactly the
s(\tilde{s})-wave Dirac equation (l = 0, i.e., \kappa = -1 for spin and \kappa = 1 for pseudospin symmetry) for
a single particle with spin and pseudospin symmetry moving in a central WS potential. They
obtained the energy spectra of the bound-states and the corresponding wave functions for the
two-component spinor in terms of the hypergeometric functions. Alhaidari has developed
a new two-component approach in the case of three-dimensional Dirac equation for the
spherically symmetric potential and solved a class of shape-invariant Morse, Rosen-Morse,
Eckart, Pöschl-Teller and Scarf, potentials and given their relativistic bound-state spectra
and spinor wave functions [25-27]. Guo et al followed Alhaidari’s approach and discussed
the Dirac equation with WS and Hulthén potentials for spherical system and given their
bound-state spectra and the spinor wave functions for s-states [28,29].

For the more realistic nuclear system where the nucleons are described in the relativistic
mean field with the attractive \( S(\vec{r}) \) and repulsive vector \( V(\vec{r}) \) potential, although some
numerical techniques have been developed to Dirac equation, any analytic solution has not
still been obtained for the Dirac-WS problem. In the special cases of spin symmetry \( \Delta =
V - S = A = \text{constant} \) and pseudospin symmetry \( \Sigma = V + S = A = \text{constant} \), Ginocchio
et al solved triaxial, axial and spherical harmonic oscillators for the case \( \Delta = 0 \) and applied
it to the study of antinucleons embedded in nuclei [30]. Lisboa et al studied the generalized
relativistic harmonic oscillator for spin 1/2 particles and obtained the analytic solutions for
bound states of the corresponding Dirac equations by setting \( A = 0 \) [31]. Very recently,
Ikhdair [32] studied the exact solution of the spatially dependent Dirac equation with the
Rosen-Morse potential for arbitrary spin-orbit quantum number \( \kappa \). Under the conditions
of the spin symmetry \( S \sim V \) and pseudospin symmetry \( S \sim -V \), the bound state energy
eigenvalues and corresponding upper- and lower-spinor wave functions are investigated in
the framework of the NU method. Furthermore, We have solved the constant mass KG
equation for the Eckart potential [32] and the spatially-dependent mass KG equation for the
Coulomb-like potential [33] and obtained the bound-state solutions of the energy eigenvalues and wavefunctions.

In this work, we will solve the (3+1) dimensional Dirac equation for a particle trapped in the spherically symmetric generalized WS potential under the conditions of exact spin and pseudospin symmetry combined with approximation for the centrifugal and pseudocentrifugal terms, and give the two-component spinor wavefunctions and the energy spectra for any arbitrary spin-orbit $\kappa$ bound states. The NU method [34] is used in the calculations. For the 1D case, upon changing the values of the potential parameters from real to pure imaginary, we obtain Hamiltonians that may or may not be $\mathcal{PT}$-symmetric. In addition, the spinor wavefunctions and the energy spectra of $s$-wave ($\kappa = \pm 1$) bound states and the non-relativistic limit are also discussed for the generalized and for a special case of the standard WS potential.

The paper is structured as follows: In sect. 2, we outline the NU method and derive a parametric generalization version. Section 3 is devoted for the exact analytic bound state energy eigenvalues and two lower- and upper-spinor components wave functions of the Dirac equation with generalized WS quantum system obtained by means of the NU method. The spin symmetry and pseudo-spin symmetry solutions are investigated using the NU method. In sect. 4, we study the cases $\kappa = \pm 1$ ($l = \tilde{l} = 0$, $s$-wave) and the non-relativistic limit and compare with other wave equations and models. Finally, the relevant conclusions are given in sect. 5.

II. THE NIKIFOROV-UVAROV METHOD

The NU method has been used to solve the Schrödinger [19], KG [20,33] and Dirac [28,32] wave equations for central and non-central potentials. Let us briefly outline the basic concepts of the method [34]. This method was proposed to solve the second-order linear differential equation of the hypergeometric-type:

$$\sigma^2(z)u''(z) + \sigma(z)\tilde{\tau}(z)u'(z) + \tilde{\sigma}(z)u(z) = 0,$$  \hspace{1cm} (1)

where the prime denotes the differentiation with respect to $z$, $\sigma(z)$ and $\tilde{\sigma}(z)$ are analytic polynomials, at most of second-degree, and $\tilde{\tau}(s)$ is of a first-degree polynomial. Let us discuss
the exact particular solution of Eq. (1) by choosing

\[ u(z) = y_n(z)\phi_n(z), \quad (2) \]

resulting in a hypergeometric type equation of the form:

\[ \sigma(z)y''_n(z) + \tau(z)y'_n(z) + \lambda y_n(z) = 0. \quad (3) \]

The first part \( y_n(z) \) is the hypergeometric-type function whose polynomial solutions are given by the Rodrigues relation

\[ y_n(z) = A_n \frac{d^n}{dz^n} [\sigma^n(z)\rho(z)]. \quad (4) \]

where \( A_n \) is a normalization factor and \( \rho(z) \) is the weight function satisfying the condition

\[ [\sigma(z)\rho(z)]' = \tau(z)\rho(z), \quad (5) \]

with

\[ \tau(z) = \tilde{\tau}(z) + 2\pi(z), \tau'(z) < 0. \quad (6) \]

Since \( \rho(z) > 0 \) and \( \sigma(z) > 0 \), the derivative of \( \tau(z) \) has to be negative for bound states \([32-34]\) which is the main essential condition for any choice of particular solution. The other part of the wave function is defined as a logarithmic derivative

\[ \frac{\phi'(z)}{\phi(z)} = \frac{\pi(z)}{\sigma(z)}, \quad (7) \]

where

\[ \pi(z) = \frac{1}{2} [\sigma'(z) - \tilde{\tau}(z)] \pm \sqrt{\frac{1}{4} [\sigma'(z) - \tilde{\tau}(z)]^2 - \tilde{\sigma}(z) + k\sigma(z)}, \quad (8) \]

with

\[ k = \lambda - \pi'(z). \quad (9) \]

The determination of \( k \) is the key point in the calculation of \( \pi(z) \), for which the discriminant of the square root in the last equation is set to zero. This results in the polynomial \( \pi(z) \) which is dependent on the transformation function \( z(r) \). Also, the parameter \( \lambda \) defined in Eq. (9) takes the form

\[ \lambda = \lambda_n = -n\tau'(z) - \frac{1}{2} n(n-1)\sigma''(z), \quad n = 0, 1, 2, \ldots. \quad (10) \]
At the end, the energy equation and consequently its eigenvalues can be obtained by comparing Eqs. (9) and (10).

Let us now construct a parametric generalization of the NU method valid for any central and non-central exponential-type potential. Comparing the following generalized hypergeometric equation

\[ z (1 - c_3 z)^2 u''(z) + [z (1 - c_3 z) (c_1 - c_2 z)] u'(z) + \left( -B_1 z^2 + B_2 z - B_3 \right) u(z) = 0, \tag{11} \]

with Eq. (1), we obtain

\[ \tilde{\tau}(z) = c_1 - c_2 z, \quad \sigma(z) = z (1 - c_3 z), \quad \tilde{\sigma}(z) = -B_1 z^2 + B_2 z - B_3, \tag{12} \]

where the parameters \( c_j \) and \( B_j \) \((j = 1, 2, 3)\) are to be determined during the solution procedure. Thus, by following the method, we may obtain all the analytic polynomials and their relevant constants necessary for the solution of a radial wave equation. These analytic expressions are cited in Appendix A.

### III. SOLUTIONS OF THE DIRAC-GENERALIZED WS PROBLEM

The Dirac equation for fermionic massive spin-1/2 particles moving in an attractive scalar potential \( S(r) \) and a repulsive vector potential \( V(r) \) can be written as \[35\]

\[ [\alpha \cdot \mathbf{p} + \beta \left( mc^2 + S(r) \right) + V(r) - E] \psi(\mathbf{r}) = 0, \quad \psi(\mathbf{r}) = \psi(r, \theta, \phi), \tag{13} \]

where \( E \) is the relativistic energy of the system, \( \mathbf{p} = -i\hbar \nabla \) is the momentum operator, and \( \alpha \) and \( \beta \) are \( 4 \times 4 \) Dirac matrices

\[ \alpha = \begin{pmatrix} 0 & \sigma \\ \sigma & 0 \end{pmatrix}, \quad \beta = \begin{pmatrix} \mathbf{I} & 0 \\ 0 & -\mathbf{I} \end{pmatrix}, \quad \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{14} \]

where \( \mathbf{I} \) denotes the \( 2 \times 2 \) identity matrix and \( \sigma \) are three-vector Pauli spin matrices. For a spherical symmetrical nuclei, the total angular momentum operator of the nuclei \( \mathbf{J} \) and spin-orbit matrix operator \( \mathbf{K} = -\beta (\sigma \cdot \mathbf{L} + \mathbf{I}) \) commute with the Dirac Hamiltonian, where \( \mathbf{L} \) is the orbital angular momentum operator. The spinor wavefunctions can be classified according to the radial quantum number \( n \) and the spin-orbit quantum number \( \kappa \) and can be written using the Pauli-Dirac representation:

\[ \psi_{nk} = \begin{pmatrix} f_{nk} \\ g_{nk} \end{pmatrix} = \frac{1}{r} \begin{pmatrix} F_{nk}(r) Y_{jm\kappa}^l(\tilde{\tau}) \\ iG_{nk}(r) Y_{jm(-\kappa)}^l(\tilde{\tau}) \end{pmatrix}, \tag{15} \]
where $F_{nk}(r)$ and $G_{nk}(r)$ are the radial wave functions of the upper- and lower-spinor components, respectively, $Y_{jm\kappa}^l(\hat{r})$ and $Y_{jm(-\kappa)}^l(\hat{r})$ denote the spin spherical harmonic functions coupled to the total angular momentum $j$ and its projection $m$ on the $z$ axis and $l(l + 1) = \kappa(\kappa + 1)$ and $\tilde{l}(\tilde{l} + 1) = \kappa(\kappa - 1)$. The quantum number $\kappa$ is related to the quantum numbers for spin symmetry $l$ and pseudospin symmetry $\tilde{l}$ as

$$\kappa = \begin{cases} 
-(l + 1) = -(j + \frac{1}{2}), & (s_{1/2}, p_{3/2}, \text{etc.}), \quad j = l + \frac{1}{2}, \quad \text{aligned spin } (\kappa < 0), \\
+l = +(j + \frac{1}{2}), & (p_{1/2}, d_{3/2}, \text{etc.}), \quad j = l - \frac{1}{2}, \quad \text{unaligned spin } (\kappa > 0).
\end{cases} \quad (16)$$

and the quasi-degenerate doublet structure can be expressed in terms of a pseudospin angular momentum $\tilde{s} = 1/2$ and pseudo-orbital angular momentum $\tilde{l}$ which is defined as

$$\kappa = \begin{cases} 
-\tilde{l} = -(j + \frac{1}{2}), & (s_{1/2}, p_{3/2}, \text{etc.}), \quad j = \tilde{l} - 1/2, \quad \text{aligned spin } (\kappa < 0), \\
+\tilde{l} + 1 = +(j + \frac{1}{2}), & (p_{1/2}, d_{3/2}, \text{etc.}), \quad j = \tilde{l} + 1/2, \quad \text{unaligned spin } (\kappa > 0).
\end{cases} \quad (17)$$

where $\kappa = \pm 1, \pm 2, \cdots$. For example, $(3s_{1/2}, 2d_{3/2})$ and $(3p_{1/2}, 2p_{3/2})$ can be considered as pseudospin doublets.

Substituting Eq. (15) into Eq. (13), we obtain two radial coupled Dirac equations for the spinor components

$$\left( \frac{d}{dr} + \frac{\kappa}{r} \right) F_{nk}(r) = \left[ mc^2 + E_{nk} - \Delta(r) \right] G_{nk}(r), \quad (18a)$$

$$\left( \frac{d}{dr} - \frac{\kappa}{r} \right) G_{nk}(r) = \left[ mc^2 - E_{nk} + \Sigma(r) \right] F_{nk}(r), \quad (18b)$$

where $\Delta(r) = V(r) - S(r)$ and $\Sigma(r) = V(r) + S(r)$ are the difference and sum potentials, respectively. By eliminating $G_{nk}(r)$ in Eq. (18a) and $F_{nk}(r)$ in Eq. (18b), one is able to obtain two second-order differential equations for the upper- and lower-spinor components as follows:

$$\left\{ \frac{d^2}{dr^2} - \frac{\kappa(\kappa + 1)}{r^2} - \frac{1}{\hbar^2 c^2} \left[ U_-(r)U_+(r) - \frac{d\Delta(r)}{dr} \left( \frac{d}{dr} + \frac{\kappa}{r} \right) \right] mc^2 + E_{nk} - \Delta(r) \right\} F_{nk}(r) = 0, \quad F_{nk}(0) = 0, \quad (19a)$$

$$\left\{ \frac{d^2}{dr^2} + \frac{\kappa(\kappa - 1)}{r^2} - \frac{1}{\hbar^2 c^2} \left[ U_-(r)U_+(r) + \frac{d\Sigma(r)}{dr} \left( \frac{d}{dr} - \frac{\kappa}{r} \right) \right] mc^2 - E_{nk} + \Sigma(r) \right\} G_{nk}(r) = 0, \quad G_{nk}(0) = 0, \quad (19b)$$

where $U_-(r) = mc^2 + E_{nk} - \Delta(r)$ and $U_+(r) = mc^2 - E_{nk} + \Sigma(r)$, are the difference and the sum functions, respectively. From the above equations, the energy eigenvalues depend
on the quantum numbers \( n \) and \( \kappa \), and also the pseudo-orbital angular quantum number \( \tilde{l} \) according to \( \kappa(\kappa - 1) = \tilde{l}(\tilde{l} + 1) \), which implies that \( j = \tilde{l} \pm 1/2 \) are degenerate for \( \tilde{l} \neq 0 \).

The above non-linear radial wave equations having very complicated solutions are required to satisfy the necessary boundary conditions \( F_{n\kappa}(0) = G_{n\kappa}(0) \) and \( F_{n\kappa}(\infty) = G_{n\kappa}(\infty) \) for bound state solutions.

In this context, we take the sum potential in the form of an attractive generalized WS potential, i.e., \( \Sigma(r) = V_{GW\!S}(r) \) [36]. The interaction among nuclei is commonly described by using a potential which consists of the Coulomb and the nuclear potentials. It is usually taken in the form of WS potential. Here we take the following Hermitian real-valued generalized WS potential which is specified by the shape (deformation) parameter, \( q \), [13,22,37]

\[
V_{GW\!S}(x) = V_q(x) = -V_0 \frac{e^{-ax}}{1 + q e^{-ax}}, \quad R_0/a \equiv \alpha, \quad (r - R_0)/R_0 \equiv x, \quad R_0 \gg a, \quad q > 0 \quad (20)
\]

where \( r \in (0, \infty) \) or \( x \in (-1, \infty) \) refers to the center-of-mass distance between the projectile and the target nuclei. The relevant parameters of the inter-nuclear potential are given as follows: \( R_0 = r_0 A_0^{1/3} \) is to define the confinement barrier position value of the corresponding spherical nucleus or the range of the potential well, \( A_0 \) is the atomic mass number of target nucleus, \( r_0 \) is the radius parameter, the parameter \( V_0 \) is the potential depth, \( a \) is the surface thickness and has to control it’s slope, which is usually adjusted to the experimental values of ionization energies. Note further, \( q \) is a real shape (deformation) parameter, the strength of the exponential part other than unity, set to determine the shape of potential and is arbitrarily taken to be a real constant within the potential. In addition, it should be noted that the spatial coordinates in the potential are not deformed and thus the potential still remains spherical.

It is worth noting that under radial coordinate transformation, \( r \to r + \Delta \), then the generalized WS potential in Eq. (20) changes into the standard WS potential (when \( q \) is taken equal to 1 in the calculation) but with the displacement parameter \( \Delta \) satisfies the expression \( \exp(\Delta/a) = q \) and with a field strength \( V'_0 = V_0 \exp(-\Delta/a) \) [13,22]. The sense of generalization or deformation of the potential becomes clear. For completeness, it could be stated that if \( \Delta \) is positive (corresponding to \( q > 1 \)) then one may need to impose the condition on the choice of \( \Delta \), that is, \( |\Delta| \ll R_0 \).

Obviously, for some specific \( q \) values this potential reduces to the well-known types, such as for \( q = 0 \) to the exponential potential and for \( q = -1 \) and \( a = \delta^{-1} \) to the generalized
Hulthén potential (cf. [11,13,22] and the references therein). Obviously, the solutions in Ref. [22] are at best valid for $R_0 = 0$, in which the potential can be expanded in terms of hyperbolic functions [19,22]. The standard WS potential turns to become the well-known Rosen-Morse potential shifted by the term $-V_0/2$ (cf. Ref. [22] and the references therein), that is, $V_{SRM}(r) = -V_1 \sec h^2(\alpha x) + V_2 \tanh(\alpha x) - V_3$, where $V_1 = C/4$ and $V_2 = V_3 = V_0/2$ [32].

A. Spin symmetric solution

In the case of exact spin symmetry $S(r) \sim V(r)$ ($d\Delta(r)/dr = 0$, i.e., $\Delta(r) = A =$ constant), Eq. (19a) can be approximately written as

$$
\left\{ \frac{d^2}{dr^2} - \frac{\kappa (\kappa + 1)}{r^2} - \frac{1}{\hbar^2 c^2} \left[ m(r)c^2 + E_{n\kappa} - A \right] \left[ m(r)c^2 - E_{n\kappa} + \Sigma(r) \right] \right\} F_{n\kappa}(r) = 0, \quad (21)
$$

where $\kappa (\kappa + 1) = l (l + 1)$, $\kappa = l$ for $\kappa < 0$ and $\kappa = -(l + 1)$ for $\kappa > 0$. The spin symmetric energy eigenvalues depend on $n$ and $\kappa$, i.e., $E_{n\kappa} = E(n,\kappa (\kappa + 1))$. For $l \neq 0$, the states with $j = l \pm 1/2$ are degenerate. This is the exact spin symmetry. Taking the $\Sigma(r) = 2V(r) \rightarrow V_{GW}(r)$ as mentioned in Ref. [36] enables one to reduce the resulting relativistic solutions into their non-relativistic limit under appropriate transformations. We are set out to obtain bound state solutions (relativistic energy spectrum and upper- and lower spinor wavefunctions) of a spin-zero particle for a four parameter $\{V_0, q, a, R_0\}$ generalized WS potential by means of the NU method. Moreover, if $\kappa$ is not too large, the case of the vibrations of small amplitude about the minimum, we can then use the approximate expansion of the centrifugal potential near the minimum point $r = R_0$ ($x = 0$) as [38]

$$
\frac{\kappa (\kappa + 1)}{r^2} = \frac{\kappa (\kappa + 1)}{R_0^2} (1 + x)^{-2}
$$

$$
\approx \frac{\kappa (\kappa + 1)}{R_0^2} \left\{ D_0 + D_1 \frac{1 + \exp(-\alpha x)}{1 + q \exp(-\alpha x)} + D_2 \left[ \frac{-\exp(-\alpha x)}{1 + q \exp(-\alpha x)} \right]^2 \right\}, \quad (22)
$$

where

$$
D_0 = 1 - \left[ \frac{1 + \exp(-\alpha R_0)}{\alpha R_0} \right]^2 \left[ \frac{4\alpha R_0}{1 + \exp(-\alpha R_0)} - 3 - \alpha R_0 \right], \quad (23a)
$$

$$
D_1 = 2 \left[ \exp(\alpha R_0) + 1 \right] \left\{ 3 \left[ \frac{1 + \exp(-\alpha R_0)}{\alpha R_0} \right] - (3 + \alpha R_0) \left[ \frac{1 + \exp(-\alpha R_0)}{\alpha R_0} \right] \right\}, \quad (23b)
$$

$$
D_2 = \left[ \exp(\alpha R_0) + 1 \right]^2 \left[ \frac{1 + \exp(-\alpha R_0)}{\alpha R_0} \right]^2 \left[ 3 + \alpha R_0 - \frac{2\alpha R_0}{1 + \exp(-\alpha R_0)} \right], \quad (23c)
$$
and higher order terms are neglected. It is worth noting that for \( \kappa \neq 0 \) case, we have to use an approximation for the centrifugal term similar to the non-relativistic cases which is valid only for \( q = 1 \) value [6,38]. However, for \( s \)-waves, we remark that the problem can be solved exactly and the solution is valid for any deformation parameter \( q \).

We define the following new dimensionless parameter, \( z(x) = -e^{-\alpha x} \in [-e^{R_0/a}, 0] \), which maintains the finiteness of the transformed wave functions on the boundary conditions. Thus, using Eqs. (22) and (23), we can reduce Eq. (21) to generalized equation of the hypergeometric type for the upper-spinor component \( F_{n\kappa}(r) \),

\[
\left[ \frac{d^2}{dz^2} + \frac{(1 - qz)}{z(1 - qz)} \frac{d}{dz} + \frac{1}{[z(1 - qz)]^2} (-\beta_1 z^2 + \beta_2 z - \varepsilon_{n\kappa}^2) \right] F_{n\kappa}(z) = 0, \quad F_{n\kappa}(0) = F_{n\kappa}(\infty) = 0,
\]

where \( F_{n\kappa}(r) = F_{n\kappa}(z) \) and we introduce the definitions

\[
\varepsilon_{n\kappa} = \sqrt{\frac{a^2}{h^2c^2} [m^2c^4 - E_{n\kappa}^2 - A (mc^2 - E_{n\kappa})] + \frac{\omega a^2}{r_0^2} D_0 > 0}, \quad (25a)
\]

\[
\beta_1 = q^2 \varepsilon_{n\kappa}^2 - \frac{qa^2 V_0}{h^2c^2} (mc^2 + E_{n\kappa} - A) - \frac{\omega a^2}{r_0^2} (q D_1 - D_2), \quad (25b)
\]

\[
\beta_2 = 2q \varepsilon_{n\kappa}^2 - \frac{a^2 V_0}{h^2c^2} (mc^2 + E_{n\kappa} - A) - \frac{\omega a^2}{r_0^2} D_1, \quad (25c)
\]

with \( \omega = \kappa (\kappa + 1) \) where \( \kappa = \pm 1, \pm 2, \cdots \), for bound states (i.e., real \( \varepsilon_{n\kappa} \)). Before we can proceed, it is necessary to compare the last equation with Eq. (1) to obtain the following polynomials:

\[
\tilde{\tau}(z) = 1 - qz, \quad \sigma(z) = z(1 - qz), \quad \overline{\sigma}(z) = -\beta_1 z^2 + \beta_2 z - \varepsilon_{n\kappa}^2. \quad (26)
\]

We follow Appendix A to calculate the specific values of the parametric constants and then display them in Table 1 for the present potential model. Also, with the aid of Table 1, the key polynomials given in Appendix A take the following particular analytic forms:

\[
\pi(z) = \varepsilon_{n\kappa} - \frac{q}{2} (1 + 2 \varepsilon_{n\kappa} + \xi) z, \quad (27)
\]

\[
k = \beta_2 - q (2 \varepsilon_{n\kappa} + \xi) \varepsilon_{n\kappa}, \quad (28)
\]

\[
\tau(z) = 1 + 2\varepsilon_{n\kappa} - q (2 + 2 \varepsilon_{n\kappa} + \xi) z, \quad (29)
\]
where $\tau'(z) = -q(2 + 2\varepsilon_{\kappa \kappa} + \xi) < 0$ with $\xi = \sqrt{1 + \frac{4\omega^2}{q^2 R_0^2}} D_2$. We insert the values of the constants given in Table 1 into the energy equation cited in Appendix A and then obtain

$$
\varepsilon_{\kappa \kappa} = -\left(\frac{a}{q}\right)^2 \left[ \frac{q V_0}{\hbar c^2} (mc^2 + E_{\kappa \kappa} - A) + \frac{\omega}{q R_0} (q D_1 - D_2) \right] \left(1 + 2n + \sqrt{1 + \frac{4\omega^2}{q^2 R_0^2} D_2}\right) + \left(\frac{q}{2a}\right)^2 \left(1 + 2n + \sqrt{1 + \frac{4\omega^2}{q^2 R_0^2} D_2}\right) ,
$$

(30)

where $\kappa = \pm 1, \pm 2, \cdots$. Hence, the above equation gives explicitly the energy equation with exact spin symmetry for arbitrary spin-orbit coupling quantum $\kappa$ of the Dirac equation as follows

$$
[m^2 c^4 - E_{\kappa \kappa}^2 - A (mc^2 - E_{\kappa \kappa})] = -\frac{\hbar^2 c^2 \omega}{R_0^2} D_0
$$

$$
+ \left(\frac{a \hbar c}{q}\right)^2 \left[ \frac{V_0}{\hbar c^2} (mc^2 + E_{\kappa \kappa} - A) + \frac{\omega}{q R_0} (q D_1 - D_2) \right] \left(1 + 2n + \sqrt{1 + \frac{4\omega^2}{q^2 R_0^2} D_2}\right) + \frac{q}{4a^2} \left(1 + 2n + \sqrt{1 + \frac{4\omega^2}{q^2 R_0^2} D_2}\right) ,
$$

(31)

The energy level $E_{\kappa \kappa}$ is determined by energy equation (31), which is a rather complicated transcendental equation. Now, let us consider a few special cases of much concern. (i) If we choose $q = 1$, the potential (20) turns to the shifted WS potential:

$$
V(x) = -V_0 + \frac{V_0}{1 + e^{-\alpha x}},
$$

(32)

and then it’s energy spectra yield

$$(mc^2 - E_{\kappa \kappa}) (mc^2 + E_{\kappa \kappa} - A) = \frac{\hbar^2 c^2 \omega}{R_0^2} D_0
$$

$$
+ \hbar^2 c^2 a^2 \left[ \frac{V_0}{\hbar c^2} (mc^2 + E_{\kappa \kappa} - A) + \frac{\omega}{q R_0} (D_1 - D_2) \right] \left(1 + 2n + \sqrt{1 + \frac{4\omega^2}{q^2 R_0^2} D_2}\right) + \frac{1}{4a^2} \left(1 + 2n + \sqrt{1 + \frac{4\omega^2}{q^2 R_0^2} D_2}\right) ,
$$

(33)

(ii) If we choose $q = -1$, the potential (20) turns to the standard shifted Hulthén potential:

$$
V(x) = V_0 - \frac{V_0}{1 - e^{-\alpha x}},
$$

(34)

and then the resulting energy eigenvalues become

$$(mc^2 - E_{\kappa \kappa}) (mc^2 + E_{\kappa \kappa} - A) = -\frac{\hbar^2 c^2 \omega}{R_0^2} D_0
$$

$$
+ \hbar^2 c^2 a^2 \left[ \frac{V_0}{\hbar c^2} (mc^2 + E_{\kappa \kappa} - A) + \frac{\omega}{q R_0} (D_1 + D_2) \right] \left(1 + 2n + \sqrt{1 + \frac{4\omega^2}{q^2 R_0^2} D_2}\right) - \frac{1}{4a^2} \left(1 + 2n + \sqrt{1 + \frac{4\omega^2}{q^2 R_0^2} D_2}\right) ,
$$

(35)
(iii) If we choose \( q \to 0 \), the potential (20) turns to the exponential potential:

\[
V(x) = -V_0 e^{-\alpha x},
\]

(36)

the eigenvalues expression (31) does not give an explicit form, i.e., the NU method is not applicable to the exponential potential (36). Note that for this potential there is no explicit form of the energy expression of bound states for Schrödinger [9], KG [12] and also Dirac [8] equations.

In addition, for the \( s \)-wave (\( \kappa = -1 \)) and \( V(r) = S(r) \) (i.e., \( A = 0 \)), we obtain

\[
m^2c^4 - E^2_{n(-1)} = \frac{h^2c^2a^2}{q^2} \left[ \frac{V_0}{2\hbar^2c^2} \left( \frac{mc^2 + E_{n(-1)}}{n + 1} \right) + \frac{q(n + 1)}{2a^2} \right]^2, \quad n = 0, 1, 2, \ldots,
\]

(37)

and it can be seen easily that while the field strength \( V_0 \to 0 \), the energy states yield:

\[
E^\pm_{n(-1)} = \pm \frac{1}{2a} \sqrt{4a^2m^2c^4 - h^2c^2 (n + 1)^2}, \quad n = 0, 1, 2, \ldots,
\]

(38)

for particles and anti-particles. Note that in the above equation there exist bound states for the ground and excited states (\( n = 0, 1 \)) which are \( E_0 = \pm \sqrt{3mc^2}/2 \) and \( E_1 = 0 \), respectively, for positive \( q \) values and \( a = \lambda_c \), where \( \lambda_c = \hbar/mc \) denotes the Compton wavelength of the Dirac particle. Otherwise, there are no bound states for \( n \geq 2 \) states.

On the other hand, for the same value of \( \alpha \) and negative \( q \) values when \( V_0 \to 0 \), all energy eigenvalues go to zero. If the value of \( q \) is increasing, all positive bound states go to zero, from (38), asymptotically.

An inspection of the energy expression given by Eq. (37), for any given \( \alpha \), shows that we deal with a family of generalized WS potentials. The sign of \( V_0 \) does not affect the bound states. The spectrum consists of complex eigenvalues depending on \( q \). As we shall see the role played by the range parameter \( \alpha \) is very crucial in this regard. Of course, it is clear that by imposing appropriate changes in the parameters \( \{\alpha, V_0, q\} \), the energy spectrum in Eq. (37) for any modified parameter can be also calculated by resolving Dirac equation for every parameter change.

The upper-spinor wave functions for \( F_{n\kappa}(r) \) will be presented. In order to establish \( F_{n\kappa}(r) \), use will be made of Appendix A and Table 1. Hence, the first part of wave functions reads:

\[
\phi_n(z) = z^{\varepsilon_{n\kappa}} (1 - qz)^{\frac{1}{2}(1 + \xi)}, \quad \varepsilon_{n\kappa} > 0, \quad \xi > -1.
\]

(39)
In addition, to find the function, \( y_n(z) \), which is the polynomial solution of hypergeometric-type equation, we firstly calculate the weight function:

\[
\rho(z) = z^{2\varepsilon_{nk}} (1 - qz)^\xi. \tag{40}
\]

and thus the second part of wave functions (7) can be obtained as

\[
y_n(z) = D_n z^{-2\varepsilon_{nk}} (1 - qz)^{-\xi} \frac{d^n}{dz^n} \left[ z^{n+2\varepsilon_{nk}} (1 - z)^{n+\xi} \right], \quad \xi > -1, \tag{41}
\]

where \( D_n \) is a normalization constant. In the limit \( q \to 1 \), the polynomial solutions of \( y_n(z) \) are expressed in terms of Jacobi Polynomials, which is one of the classical orthogonal polynomials, with weight function given by Eq. (40) for \( z \in [0, 1] \), giving \( y_n(z) \approx P_n^{(2\varepsilon_{nk}, \xi)}(1 - 2z), \quad 2\varepsilon_{nk}, \quad \xi > -1 \) [39]. Thus the associated uppercomponent \( F_{nk}(z) \) for arbitrary the spin-orbit coupling quantum number \( \kappa \) can be obtained by substituting Eqs. (39) and (41) into Eq. (2) as

\[
F_{nk}(z) = N_{nk} z^{\varepsilon_{nk}} (1 - qz)^{\frac{1}{2} (1 + \xi)} P_n^{(2\varepsilon_{nk}, \xi)}(1 - 2qz)
= N_{nk} z^{\varepsilon_{nk}} (1 - qz)^{\frac{1}{2} (1 + \xi)} 2F_1 (-n, 1 + 2\varepsilon_{nk} + \xi + n; 1 + 2\varepsilon_{nk}; qz), \tag{42}
\]

where \( z(r) = -e^{-(r-R_0)/a} \) and \( N_{nk} \) are normalization constants calculated in Appendix B.

Before presenting the corresponding lower-component \( G_{nk}(r) \), let us recall a recurrence relation of hypergeometric function, which is used to solve Eq. (18a) and present the corresponding lower component \( G_{nk}(r) \),

\[
\frac{d}{dz} \left[ 2F_1 (a; b; c; z) \right] = \left( \frac{ab}{c} \right) 2F_1 (a + 1; b + 1; c + 1; z), \tag{43}
\]

with which the corresponding lower component \( G_{nk}(r) \) can be given by solving Eq. (18a) as follows

\[
G_{nk}(r) = \frac{1}{mc^2 + E_{nk} - A} \left[ \frac{dF_{nk}(r)}{dr} + \frac{\kappa}{r} F_{nk}(r) \right]
= \frac{N_{nk} \left( -e^{-(r-R_0)/a} \right)^{\varepsilon_{nk}} (1 + q e^{-(r-R_0)/a})^{\frac{1}{2} (1 + \xi)}}{mc^2 + E_{nk} - A} \left[ \varepsilon_{nk} a - \frac{q (1 + \xi) e^{-(r-R_0)/a}}{2a (1 + q e^{-(r-R_0)/a})} + \frac{\kappa}{r} \right]
\times 2F_1 (-n, 1 + 2\varepsilon_{nk} + \xi + n; 1 + 2\varepsilon_{nk} + 1; -qe^{-(r-R_0)/a})
+ N_{nk} \left[ \frac{qn (1 + 2\varepsilon_{nk} + \xi + n) \left( -e^{-(r-R_0)/a} \right)^{\varepsilon_{nk} + 1} (1 + q e^{-(r-R_0)/a})^{\frac{1}{2} (1 + \xi)}}{a (2\varepsilon_{nk} + 1) (mc^2 + E_{nk} - A)} \right]
\times 2F_1 (-n + 1; n + \xi + 2 (1 + \varepsilon_{nk}); 2 (1 + \varepsilon_{nk}); -qe^{-(r-R_0)/a}). \tag{44}
\]
Here, it should be noted that the hypergeometric series
\[ _2F_1\left(-n, 1 + 2\varepsilon_{nk}; \xi + n; 2\varepsilon_{nk} + 1; -\frac{qe^{-(r-R_0)/a}}{a}\right) \]
does not terminate for \( n = 0 \) and thus does not diverge for all values of real parameters \( \xi \) and \( \varepsilon_{nk} \).

For \( A > mc^2 + E_{nk} \) and \( E_{nk} < mc^2 \) or \( A < mc^2 + E_{nk} \) and \( E_{nk} > mc^2 \), we note that parameters given in Eq. (25a) turn to be imaginary, i.e., \( \varepsilon_{nk}^2 < 0 \) in the s-state (\( \kappa = -1 \)). As a result, the condition of existing bound states are \( \varepsilon_{nk} > 0 \) and \( \xi > 0 \), that is to say, in the case of \( A > mc^2 + E_{nk} \) and \( E_{nk} < mc^2 \), bound-states do not exist for some quantum number \( \kappa \) such as the s-state (\( \kappa = -1 \)). Of course, if these conditions are satisfied for existing bound-states, the energy equation and wave functions are the same as these given in Eq. (31) and Eqs. (42)-(44).

### B. Pseudospin symmetric solution

Under the condition of the pseudospin symmetry \( S(r) \sim -V(r) \) (i.e., \( d\Sigma(r)/dr = 0 \), or \( \Sigma(r) = A' = \text{constant} \), Eq. (19b) can be exactly written as

\[
\left\{ \frac{d^2}{dr^2} - \frac{\tilde{\omega}}{r^2} - \frac{1}{h^2c^2} \left[ mc^2 + E_{nk} - \Delta(r) \right] \left[ mc^2 - E_{nk} + A' \right] \right\} G_{nk}(r) = 0,
\]

where \( \tilde{\omega} = \kappa(\kappa - 1) = \tilde{l}(\tilde{l} + 1) \), the energy eigenvalues \( E_{nk} \) depend only on \( n \) and \( \tilde{l} \), i.e., \( E_{nk} = E(n, \tilde{l}(\tilde{l} + 1)) \). Taking the \( \Delta(r) = V_G S(r) \) allows us to reduce our results to the non-relativistic limit. For \( \tilde{l} \neq 0 \), the states with \( j = \tilde{l} \pm 1/2 \) are degenerate. This is the exact pseudospin symmetry. We follow the procedures in the previous subsection to obtain Dirac equation satisfying \( G_{nk}(r) \),

\[
\left[ \frac{d^2}{dz^2} + \frac{(1-qz)}{z(1-qz)} \frac{d}{dz} + \frac{1}{[z(1-qz)]^2} \left( -\tilde{\beta}_1 z^2 + \tilde{\beta}_2 z - \varepsilon_{nk}^2 \right) \right] G_{nk}(z) = 0, \quad G_{nk}(0) = G_{nk}(\infty) = 0,
\]

where \( G_{nk}(r) = G_{nk}(z) \) and we have used the definitions

\[
\varepsilon_{nk} = \sqrt{\frac{a^2}{h^2c^2} \left[ m^2c^4 - E_{nk}^2 + A' mc^2 + E_{nk} \right] + \frac{\tilde{\omega}a^2}{R_0^2} D_0 > 0, \quad \kappa = \pm1, \pm2, \cdots, (47a)
\]

\[
\tilde{\beta}_1 = q^2 \varepsilon_{nk}^2 + \frac{qa^2V_0}{h^2c^2} \left( mc^2 - E_{nk} + A' \right) - \frac{\tilde{\omega}a^2}{R_0^2} (qD_1 - D_2), (47b)
\]

\[
\tilde{\beta}_2 = 2qa^2 \varepsilon_{nk}^2 + \frac{a^2V_0}{h^2c^2} \left( mc^2 - E_{nk} + A' \right) - \frac{\tilde{\omega}a^2}{R_0^2} D_1. (47c)
\]
To avoid repetition in the solution of Eq. (46), a first inspection for the relationship between the present set of parameters \((\tilde{\varepsilon}_{n\kappa}, \tilde{\beta}_1, \tilde{\beta}_2)\) and the previous set \((\varepsilon_{n\kappa}, \beta_1, \beta_2)\) tells us that the negative energy solution for pseudospin symmetry, where \(S(r) \sim -V(r)\), can be obtained directly from those of the positive energy solution above for spin symmetry using the following parameter mapping [39-41]:

\[
F_{n\kappa}(r) \leftrightarrow G_{n\kappa}(r), \quad V(r) \to -V(r) \quad \text{or} \quad V_0 \to -V_0, \quad E_{n\kappa} \to -E_{n\kappa} \quad \text{and} \quad A \to -A'.
\]

(48)

Following the previous results with the above transformations, we finally arrive at the energy equation. The relativistic transcendental energy equation is

\[
\left[ m^2 c^4 - E_{n\kappa}^2 + A' \left( mc^2 + E_{n\kappa} \right) \right] = -\frac{\hbar^2 c^2 \tilde{\omega}}{R_0^2} D_0
\]

\[+ \left( \frac{a \hbar c}{q} \right)^2 \left[ \frac{-V_0}{\hbar^2 \tilde{\omega}^2} \left( mc^2 - E_{n\kappa} + A' \right) + \frac{\tilde{\omega}}{q^2 \tilde{n}_0} (qD_1 - D_2) \right] + \frac{q}{4a^2} \left( 1 + 2n + \sqrt{1 + \frac{4\tilde{\omega}^2}{q^2 R_0^2} D_2} \right)^2 \],

(49)

and the lower-spinor wave functions:

\[
G_{n\kappa}(z) = \tilde{N}_{n\kappa} z^{\tilde{\varepsilon}_{n\kappa}} (1 - qz)^{\frac{1}{2} (1 + \tilde{\xi})} P_n^{(2\tilde{\varepsilon}_{n\kappa}, \tilde{\xi})} (1 - 2qz)
\]

\[
= \tilde{N}_{n\kappa} z^{\tilde{\varepsilon}_{n\kappa}} (1 - qz)^{\frac{1}{2} (1 + \tilde{\xi})} \binom{2n}{2n} \binom{n + 2\tilde{\varepsilon}_{n\kappa} + \tilde{\xi} + 2n + 2\tilde{\varepsilon}_{n\kappa} + 1; qz}
\]

(50)

with \(\tilde{\xi} = \sqrt{1 + \frac{4\tilde{\omega}^2}{q^2 R_0^2} D_2}\) and \(\tilde{\varepsilon}_{n\kappa}\) is defined in Eq. (47a).

IV. DISCUSSIONS

Now, let us study three special cases. We first study the \(s(\tilde{s})\)-states \((l = \tilde{l} = 0, \text{ i.e.,} \kappa = \mp 1)\). In this case, we have the spin-orbit coupling term \(\kappa(\kappa + 1)/r^2 = 0\), and also the corresponding approximation to it in Eq. (22). The corresponding energy equation reduces to the \(s\)-states \((\kappa = -1)\), i.e.,

\[
\left[ m^2 c^4 - E_{n(-1)}^2 - A \left( mc^2 - E_{n(-1)} \right) \right] = \frac{\hbar^2 c^2}{4} \left[ \frac{aV_0 \left( mc^2 + E_{n(-1)} - A \right)}{q \hbar^2 c^2 (1 + n)} + \frac{1}{a} \left( 1 + n \right) \right]^2,
\]

(51)

and the upper-spinor component of the wave functions:

\[
F_n(z) = N_n z^{\varepsilon_n} (1 - qz) P_n^{(2\varepsilon_n, 1)} (1 - 2qz)
\]
The nonrelativistic limit with the mapping vector WS potential as follows

\[ E_n = \frac{\hbar^2 a^2}{8\mu} \left( \frac{2\mu}{\hbar^2 q (1 + n)} + \frac{(1 + n)}{a^2} \right)^2, \]

where \( N_n \) is calculated in Appendix B. As mentioned above, in the s-wave \((\kappa = -1)\) the condition of existing bound-states is for \( A < mc^2 + E_{n(-1)} \) and \( E_{n(-1)} < mc^2 \). Furthermore, in the nonrelativistic limit with the mapping \( mc^2 - E_{n(-1)} \to -E_{n0} \) and \( \frac{1}{\hbar^2 c^2} (mc^2 + E_{n(-1)}) \to \frac{2\mu}{\hbar^2} \), then we have

Second, we study the special case \( \Delta(r) = A = 0. \) If so we have \( V(r) = S(r) = 1/2 \Sigma(r) \) and it turns to the KG solution. Obviously, in this case the energy equation given in Eq. (31) reduces to the energy equation of arbitrary \( \kappa \) state Dirac equation for equal scalar and vector WS potential as follows

\[ m^2 c^4 - E_{n\kappa}^2 = -\frac{\hbar^2 c^2 \omega}{R_0^2} D_0 \]

\[ + \left( \frac{a \hbar c}{q} \right)^2 \left[ \frac{V_0}{\hbar^2 c^2} (mc^2 + E_{n\kappa}) + \frac{\omega}{q R_0} (q D_1 - D_2) \right] + \frac{q}{4a^2} \left( 1 + 2n + \sqrt{1 + \frac{4\omega a^2}{q^2 R_0^2} D_2} \right)^2, \]

and the upper component of the wave functions

\[ F_{n\kappa}(z) = N_{n\kappa} z^{\varepsilon_{nl}} (1 - qz)^{1/2} (1 + \varepsilon_0) \)

\[ = N_{n\kappa} z^{\varepsilon_{nl}} (1 - qz)^{1/2} (1 + \varepsilon_0) \]

\[ = N_{n\kappa} z^{\varepsilon_{nl}} (1 - qz)^{1/2} (1 + \varepsilon_0) \] \( -n, 2(1 + \varepsilon_n) + n; 1 + 2\varepsilon_n; qz \),

\[ \varepsilon_{n\kappa} = \sqrt{\frac{a^2}{\hbar^2 c^2} (m^2 c^4 - E_{n\kappa}^2) + \frac{\omega a^2}{R_0^2} D_0}; \]

\[ \xi_0 = \sqrt{1 + \frac{4\kappa(\kappa + 1)a^2}{q^2 R_0^2} D_2} \]

where \( E_{n\kappa}^2 \leq m^2 c^4 + \frac{\omega a^2}{R_0^2} D_0 \) is the essential condition for existing bound-states.

Third, the non-relativistic energy state limit for arbitrary \( l \) state are

\[ E_{nl} = \frac{\hbar^2 l(l + 1)}{2\mu R_0^2} D_0 \]

\[ -\frac{\hbar^2 a^2}{2\mu} \left[ \frac{2\mu V_0}{\hbar^2 q} + \frac{l(l+1)}{q^2 R_0^2} (q D_1 - D_2) \right] + \frac{1}{4a^2} \left( 1 + 2n + \sqrt{1 + \frac{4l(l+1)a^2}{q^2 R_0^2} D_2} \right)^2. \]
The above result is identical to Eq. (23) in Ref. [22] where Berkdemir et al. used the usual approximation to the centrifugal term in the potential expression (10) (cf. [42] and Eq. (2) in J. Math. Chem. 42, 461 (2007)).

It is worthy to note that in the calculations of Ref. [22], $R_0$ was neglected. Accordingly, the solutions of the energy spectra Eq. (23) of the original paper [22] are at best valid for $R_0 = 0$ in which case the standard WS potential Eq. (10) in the original paper reduces to the shifted Rosen-Morse (RM) potential (cf. Eq. (3) in Phys. Rev. C 74, 039902(E) (2006)). The additional potential besides the standard WS potential considered by Berkdemir et al. [22] provides the flexibility to construct the surface structure of the related nucleus [42]. Thus, the non-relativistic solutions obtained in [22] are only reasonable for the hyperbolic [43] exponential (RM) potential [32], not WS potential. This clear when we rewrite Eq. (20) in the following form

$$V_{cWS}(r) = -V_0' \frac{e^{-ar}}{1 + qe^{-ar}}, \quad \alpha = \frac{1}{a}, \quad V_0' = V_0 e^{R_0/a}, \quad q = e^{R_0/a}, \quad r \in (0, \infty), \quad (59)$$

when $R_0 = 0$, it implies that $q = 1$ and then the above potential reduces to the standard WS-type potential. In addition, the authors of Ref. [22] approximated the centrifugal potential term $l(l+1)\approx l(l+1)\alpha^2 \frac{e^{-ar}}{1-e^{-ar}}$ [13,43], where $C = l(l+1)\alpha^2$ [20]. However, in the present work, Eq. (58) contains the width of the potential $R_0$. Also, an expansion for the centrifugal potential term has been performed around the point $r \approx R_0$ ($x = 0$) [38], and without loss of generality we put $x \equiv (r - R_0)/R_0$ at the end of our calculations.

The empirical values found by Perey et al. are given as $R_0 = 1.285 \, fm$ and $a = 0.65 \, fm$ [44]. In addition, the following WS potential strength parameter is $V_0 \approx 40.5 + 0.13A_0 \, MeV$ in the non-relativistic limit. Here, $A_0$ is the atomic mass number of target nucleus and is defined through $R_0 = r_0A_0^{1/3}$. On the other hand, the associated upper-spinor component of the wave functions is

$$F_{nl}(z) = N_n z^{\varepsilon_{nl}} (1 - qz)^{\frac{1}{2}(1+\xi_1)} P_{n}^{(2\varepsilon_{nl};\xi_1)}(1 - 2qz)$$

$$= N_n z^{\varepsilon_{nl}} (1 - qz)^{\frac{1}{2}(1+\xi_1)} _2F_1 \left(-n, 1 + 2\varepsilon_{nl} + \xi_1 + n; 2\varepsilon_{nl} + 1; qz\right), \quad (60)$$

where

$$\varepsilon_{nl} = a \sqrt{-\frac{2\mu}{\hbar^2} E_{nl} + \frac{l(l+1)}{R_0^2} D_0} > 0, \quad \xi_1 = \sqrt{1 + \frac{4l(l+1)a^2}{q^2R_0^2} D_2}, \quad l = 0, 1, 2, \cdots, \quad (61)$$
where \( E_{nl} < \frac{\mu R^2}{2\hbar^2} D_0 \) is the essential condition for existing bound-states. As a numerical example, we impose appropriate values for the parameters in Eq. (51) to calculate the bound state energies of the spin symmetry generalized WS potential for special case \( \kappa = -1 \) and using \( \hbar = c = 1 \). The results obtained by using the following parameters \( V_0 = 2.2 \), \( m = 15 \), \( A = -5 \) and \( a = 1.425 \) are given in Table 2. The condition of existing bound states is the \( A < 0 \). When the \( A \geq 0 \), there are no bound states in the limit of exact spin symmetry. For \( q = \pm 1 \), there is only one attractive bound state \( E_{0,-1} \approx -(A + m) \).

V. CONCLUSIONS

We have discussed the approximate bound state solutions of the Dirac equation for the generalized WS potential with any arbitrary spin-orbit \( \kappa \) state under the conditions of the spin (pseudospin) symmetry \( V - S = A \) \((V + S = A)\) by means of the NU method combined with the approximation for the centrifugal term. By setting \( V + S \) \((V - S)\) to the spherically symmetric WS potential, we have derived the solutions of the Dirac equation for the relativistic energy eigenvalues and associated two-component spinor wave functions for arbitrary spin-orbit \( \kappa \) state that provides an approximate solution to the spin- and pseudo-spin symmetric Dirac equations. The most stringent interesting result is that the present spin and pseudo-spin symmetric can be easily reduced to the KG solution once \( S(r) = V(r) \) and \( S(r) = -V(r) \) \((i.e., A = 0)\), respectively. The non-relativistic limits of our solution are obtained by imposing appropriate transformations. The resulting solutions of the wave functions are being expressed in terms of the Jacobi polynomials. If we choose the spin-orbit quantum number \( \kappa = -1 \) \((\kappa = 1)\) for spin \((\text{pseudospin})\) symmetry, the problem reduces to the exact \( s(\tilde{s})\)-wave Dirac solution. The generalized Hulthén potential bound state solutions are simply derived when letting \( q \to -q \). We have also discussed the relation between the non-relativistic and relativistic solutions and the possibility of existing the bound states. It should be noted that the numerical calculation for energy levels involved in Eq. (51) is terribly sensitive to the choice of those parameters. In Table 2, we choose \( E_{n,-1}^1 \) as the physical solution for the transcendental equation (51).

At the end, the solutions that constitute the main results regarding the energy equations (31) and (49) for the spin and pseudospin symmetry, respectively, may have some interesting applications in many areas in physics. For example, the work is helpful to understand
spectroscopy with high field physics and useful to understand the nuclear properties like nuclear scattering systems [45]. In addition, the present results play an essential role in microscopic physics, since it can be used to describe the interaction of a nucleon with a heavy nucleus [21,22]. In the non-relativistic limits, the energy eigenvalues, Eq. (58), is physical and is in good agreement with the results obtained previously by other methods and works [22].

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APPENDIX A: PARAMETRIC GENERALIZATION VERSION OF THE NU METHOD

We complement the theoretical formulation of the NU method in presenting the essential polynomials, energy equation and wave functions together with their relevant constants as follows [46].

(i) The key polynomials:

\[ \pi(z) = c_4 + c_5 z - \left[ (\sqrt{c_9} + c_3 \sqrt{c_8}) z - \sqrt{c_8} \right], \]
\[ k = -(c_7 + 2c_3c_8) - 2 \sqrt{c_8c_9}. \]
\[ \tau(z) = 1 - (c_2 - 2c_5) z - 2 \left[ (\sqrt{c_9} + c_3 \sqrt{c_8}) z - \sqrt{c_8} \right], \]
\[ \tau'(z) = -2c_3 - 2 (\sqrt{c_9} + c_3 \sqrt{c_8}) < 0, \]

(ii) The energy equation:

\[ (c_2 - c_3) n + c_3 n^2 - (2n + 1) c_5 + (2n + 1) (\sqrt{c_9} + c_3 \sqrt{c_8}) + c_7 + 2c_3c_8 + 2 \sqrt{c_8c_9} = 0. \]

(iii) The wave functions:

\[ \rho(z) = z^{c_{10}} (1 - c_3 z)^{c_{11}}, \]
\[ \phi(z) = z^{c_{12}} (1 - c_3 z)^{c_{13}}, \quad c_{12} > 0, \quad c_{13} > 0, \]
\[ y_n(z) = P_n^{(c_{10}, c_{11})}(1 - 2c_3z), \quad c_{10} > -1, \quad c_{11} > -1, \quad (A8) \]

\[ u(z) = N_n z^{c_{12}} (1 - c_3z) c_{13} P_n^{(c_{10}, c_{11})}(1 - 2c_3z), \quad (A9) \]

where \( P_n^{(\mu, \nu)}(x) \), \( \mu > -1, \nu > -1 \) and \( x \in [-1, 1] \) are the Jacobi polynomials with

\[ P_n^{(\alpha, \beta)}(1 - 2s) = \frac{(\alpha + 1)_n}{n!} {}_2F_1 (-n, 1 + \alpha + \beta + n; \alpha + 1; s), \quad (A10) \]

and \( N_n \) is a normalization constant. Also, the above wave functions can be expressed in terms of the hypergeometric function as

\[ u_{nn}(z) = N_{nn} z^{c_{12}} (1 - c_3z) c_{13} {}_2F_1 (-n, 1 + c_{10} + c_{11} + n; c_{10} + 1; c_3z), \quad (A11) \]

where \( c_{12} > 0, c_{13} > 0 \) and \( z \in [0, 1/c_3] \).

(iv) The relevant constants:

\[ c_4 = \frac{1}{2} (1 - c_1), \quad c_5 = \frac{1}{2} (c_2 - 2c_3), \quad c_6 = c_5^2 + B_1, \]

\[ c_7 = 2c_4c_5 - B_2, \quad c_8 = c_4^2 + B_3, \quad c_9 = c_3 (c_7 + c_3c_5) + c_6, \]

\[ c_{10} = c_1 + 2c_4 + 2\sqrt{c_8} - 1 > -1, \quad c_{11} = 1 - c_1 - 2c_4 + \frac{2}{c_3} \sqrt{c_9} > -1, \]

\[ c_{12} = c_4 + \sqrt{c_8} > 0, \quad c_{13} = -c_4 + \frac{1}{c_3} (\sqrt{c_9} - c_5) > 0. \quad (A12) \]

### APPENDIX B: NORMALIZATION OF THE RADIAL WAVE FUNCTION

In order to find the normalization constants \( N_{nn} \), we start by writing the normalization condition:

\[ a N_{nn}^2 \int_0^1 z^{2\xi_{nn} - 1} (1 - z)^{\xi + 1} \left[ P_n^{(2\xi_{nn}, \xi)}(1 - 2z) \right]^2 dz = 1. \quad (B1) \]

where \( q = 1 \). Unfortunately, there is no formula available to calculate this key integration. Nevertheless, we can find the explicit normalization constant \( N_{nn} \). For this purpose, it is not difficult to obtain the results of the above integral by using the following formulas [46,47]

\[ \int_0^1 (1 - s)^{\mu - 1} s^{\nu - 1} {}_2F_1 (\alpha, \beta; \gamma; as) dz = \frac{\Gamma(\mu)\Gamma(\nu)}{\Gamma(\mu + \nu)} {}_3F_2 (\nu, \alpha, \beta; \mu + \nu, \gamma; a), \quad \mu > -1, \quad \nu > -1, \quad (B2) \]
and \( \binom{2}{1}(a, b; c; z) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(b)} \sum_{p=0}^{\infty} \frac{\Gamma(a+p)\Gamma(b+p)}{\Gamma(c+p)} \frac{z^p}{p!} \). Following Ref. [46], we calculate the normalization constants:

\[
\mathcal{N}_{n\kappa} = \left[ \frac{a\Gamma(2\varepsilon_{n\kappa} + 1)\Gamma(\xi + 2)}{\Gamma(n)} \sum_{m=0}^{\infty} \frac{(-1)^m (1 + n + 2\varepsilon_{n\kappa} + \xi)_m \Gamma(n + m)}{m! (m + 2\varepsilon_{n\kappa})! \Gamma(m + 2\varepsilon_{n\kappa} + \xi + 2)} C_{n\kappa} \right]^{-1/2}
\]  

\( (B3) \)

where

\[
C_{n\kappa} = \binom{3}{2} (2\varepsilon_{n\kappa} + m, -n, n + 1 + \xi + 2\varepsilon_{n\kappa}; m + 2\varepsilon_{n\kappa} + \xi + 2; 1 + 2\varepsilon_{n\kappa}; 1), \quad (B4)
\]

Furthermore, the normalization constants for the s-wave can be also found as

\[
\mathcal{N}_{n\kappa} = \left[ \frac{a\Gamma(2\varepsilon_{n\kappa} + 1)\Gamma(\xi + 2)}{\Gamma(n)} \sum_{m=0}^{\infty} \frac{(-1)^m (1 + n + 2\varepsilon_{n\kappa} + \xi)_m \Gamma(n + m)}{m! (m + 2\varepsilon_{n\kappa})! \Gamma(m + 2\varepsilon_{n\kappa} + \xi + 2)} C_{n\kappa} \right]^{-1/2}
\]  

\( (B5) \)

where

\[
C_{n} = \binom{3}{2} (2\varepsilon_{n} + m, -n, n + 2\varepsilon_{n} + 2; m + 2\varepsilon_{n} + 3; 1 + 2\varepsilon_{n}; 1), \quad (B6)
\]

and \( \varepsilon_{n} \) is given in Eq. (53).
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TABLE I: The specific values for the parametric constants necessary for the present potential.

| Constant | Value | Constant | Value |
|----------|-------|----------|-------|
| $c_1$    | 1     | $c_2$    | $q$   |
| $c_3$    | $q$   | $c_4$    | 0     |
| $c_5$    | $-\frac{q}{2}$ | $c_6$ | $\frac{1}{4} (q^2 + 4\beta_1)$ |
| $c_7$    | $-\beta_2$ | $c_8$ | $\varepsilon_{n\kappa}$ |
| $c_9$    | $(\frac{q}{2})^2 \xi^2$ | $c_{10}$ | $2\varepsilon_{n\kappa}$ |
| $c_{11}$ | $\xi$   | $c_{12}$ | $\varepsilon_{n\kappa}$ |
| $c_{13}$ | $\frac{1}{2} (1 + \xi) B_1$ | $B_1$ | $\beta_1$ |
| $B_2$    | $\beta_2$ | $B_3$ | $\varepsilon_{n\kappa}^2$ |

TABLE II: The bound state energy levels $E_{n\kappa}$ for the special case $\kappa = -1$.

| $q = 1$ | $q = 2$ | $q = -1$ | $q = -2$ |
|---------|---------|----------|----------|
| $n$     | $E_{n,-1}^1$ | $E_{n,-1}^2$ | $E_{n,-1}^1$ | $E_{n,-1}^2$ | $E_{n,-1}^1$ | $E_{n,-1}^2$ | $E_{n,-1}^1$ | $E_{n,-1}^2$ |
| 0       | -10.197602 | -19.996367 | 1.337420 | -19.996426 | -9.561001 | -19.996589 | 2.018956 | -19.996536 |
| 1       | 0.985745 | -19.985463 | 9.849615 | -19.985698 | 2.349482 | -19.986350 | 10.803623 | -19.986141 |
| 2       | 6.597152 | -19.967274 | 12.216763 | -19.967804 | 8.327343 | -19.969273 | 13.247481 | -19.968802 |
| 3       | 9.328912 | -19.941777 | 13.119033 | -19.942722 | 11.239603 | -19.945338 | 14.180142 | -19.944500 |
| 4       | 10.775321 | -19.908939 | 13.534261 | -19.910418 | 12.784029 | -19.914518 | 14.610668 | -19.913204 |
| 5       | 11.602814 | -19.868715 | 13.742899 | -19.870854 | 13.670315 | -19.876777 | 14.828469 | -19.874879 |
| 6       | 12.102341 | -19.821054 | 13.847458 | -19.823977 | 14.208309 | -19.832071 | 14.939340 | -19.829478 |
| 7       | 12.412538 | -19.765892 | 13.892224 | -19.769728 | 14.545655 | -19.780348 | 14.988985 | -19.776946 |
| 8       | 12.605105 | -19.703154 | 13.898750 | -19.708035 | 14.758727 | -19.721546 | 14.999654 | -19.717219 |
| 9       | 12.719786 | -19.632753 | 13.878495 | -19.638818 | 14.889869 | -19.655595 | 14.983185 | -19.650223 |
| 10      | 12.779911 | -19.554593 | 13.837978 | -19.561982 | 14.963949 | -19.582415 | 14.946314 | -19.575874 |