Parametric instability of linear oscillators with colored time-dependent noise

F.M.Izrailev 1, V. Dossetti-Romero 1, A.A.Krokhin1-2, and L.Tessieri 3
1 Instituto de Física, Universidad Autónoma de Puebla, Apdo. Postal J-48, Puebla, Pue. 72570, México
2 Center for Nonlinear Science, University of North Texas, P.O. Box 311427, Denton, Texas 76203-1427, USA
3 Instituto de Física y Matemáticas, Universidad Michoacana de San Nicolás de Hidalgo, Ciudad Universitaria, 58060 Morelia, Mich. México

February 20, 2022

The goal of this paper is to discuss the link between the quantum phenomenon of Anderson localization on the one hand, and the parametric instability of classical linear oscillators with stochastic frequency on the other. We show that these two problems are closely related to each other. On the base of analytical and numerical results we predict under which conditions colored parametric noise suppresses the instability of linear oscillators.

I. INTRODUCTION

Let us consider the one-dimensional (1D) model described by the Schrödinger equation

$$\frac{\hbar^2}{2m} \psi''(x) + U(x)\psi(x) = E\psi(x). \quad (1)$$

Here the $\psi$–function is a stationary solution for a particle of energy $E$ moving in a random potential $U(x)$. To simplify the form of the analytical expressions, in what follows we use energy units such that $\hbar^2/2m = 1$, and we set the zero of the energy scale so that the mean value of the disordered potential is zero, $\langle U(x) \rangle = 0$. Here the angular brackets $\langle ... \rangle$ denote the average over the disorder (i.e., over different realizations of $U(x)$). We restrict our considerations to the case of weak disorder, defined by the condition $\varepsilon^2 = \langle U^2(x) \rangle \ll 1$.

In the analysis of the model (1) one of the main questions is about global structure of the eigenstates $\psi(x)$ in an infinite configuration space, $-\infty < x < +\infty$. Of a particular interest is the problem of whether the eigenstates are localized or extended for $x \to \pm \infty$. As was shown in Ref. [1], in 1D models any amount of disorder (even an infinitesimal one) results in the localization of all eigenstates (with the exception of a zero-measure set) provided that the potential $U(x)$ is completely random. This means that the amplitude of every eigenstate decays exponentially towards infinity, therefore, far away from the localization center $x_0$, one can write,

$$|\psi(x)| \sim \exp(-|x-x_0|/l(E)).$$

Here $l(E)$ is the so-called localization length that characterizes, in average, the decrease of the amplitude of the eigenstate corresponding to the energy $E$. Analytical expression of $l(E)$ that is relatively easy to obtain for a weak disorder, for which it is known that the localization length is inversely proportional to the square of the disorder strength, $l \sim 1/\varepsilon^2$, see below.

Taking into account that the energy of a free electron is $E = k^2$, the equation (1) can be written in the form of wave equation,

$$\psi''(x) + k^2\psi(x) = U(x)\psi(x) \quad (2)$$

that describes wave propagation in different classical systems. One example is the propagation of electromagnetic waves in single-mode waveguides with a rough surface [2]. In this application the potential $U(x)$ is determined by the horizontal profile $\xi(x) = \varepsilon \varphi(x/R_c)$, where $R_c$ is the correlation length of the profile, $\varepsilon \ll d$ is the amplitude of the profile with $d$ being the transverse size of the waveguide. Note that in this case the parameter $k$ in Eq.(2) has the meaning of the longitudinal wave number $k = \sqrt{(\omega/c)^2 - (\pi/d)^2}$ where $\omega$ is the frequency of the wave.

II. DISCRETE MODELS

When the potential is constituted by a succession of delta scatterers, the model (2) takes the specific form

$$\psi''(x) + k^2\psi(x) = \sum_{n=-\infty}^{n=\infty} U_n \psi(x_n) \delta(x - x_n). \quad (3)$$
Here $U_n$ is the amplitude of the $n$th delta-scatterer situated at $x = x_n$. In experiments, potentials of this kind are quite easy to construct; in particular, one can obtain a realization of delta scatterers by inserting an array of screws with predetermined lengths and appropriate positions in a single-mode waveguide \[3\]. Typically, one considers two limit cases. The first one occurs when all amplitudes $U_n$ are random variables, while the scatterers are periodically spaced, i.e., $x_n = an$. In this case one can speak of amplitude disorder. The second case is represented by the opposite situation in which the amplitudes of the scatterers are constant, $U_n = U_0$ while the positions $x_n$ are randomly distributed around their mean values, i.e., $x_n = an + \eta_n$ with $\langle \eta_n \rangle = 0$ and $\langle \eta_n^2 \rangle \ll a^2$. Clearly, in the latter case the mean value of the potential is not zero; however one can handle this case within the framework of zero-mean potentials by making use of the special transformation to new variables, see details in Ref \[4\]. This second limit case can be referred to as positional disorder.

Due to the delta-like form of a random potential, the model \(3\) can be considered as a discrete one. In fact, its analysis can be reduced to the study of an equivalent classical two-dimensional map which can be obtained by integrating Eq. \(3\) between two successive kicks of the scattering potential \[4\],

\[
\begin{align*}
 p_{n+1} &= (p_n + A_n q_n) \cos \mu_n - q_n \sin \mu_n, \\
 q_{n+1} &= (p_n + A_n q_n) \sin \mu_n + q_n \cos \mu_n.
\end{align*}
\]

(4)

Here $q_n$ and $p_n$ are conjugate coordinates and momenta defined by the identities

$$q_n = \psi_n \text{and} \ p_n = (\psi_n \cos \mu_{n-1} - \psi_{n-1}) / \sin \mu_{n-1}$$

where $\psi_n$ is the value of the $\psi$-function at the position $x = x_n$. The parameter $\mu_n$ is the phase shift of the $\psi$-function between two scatters,

$$\mu_n = k(x_{n+1} - x_n)$$

(5)

and the amplitude $A_n$ of the $n$th kick is given by the value of the potential at the position $x_n$,

$$A_n = U_n / k.$$  

(6)

Free rotation in \(4\) between two successive kicks corresponds to free propagation between scatterers, and each kick is due to the scattering from a $\delta$ spike of the potential.

In the case of amplitude disorder, the phase shift between two successive scatterers is the same, $\mu_n = \mu = ka$, and the model \(3\) is known as the Kronig-Penney model. In this case the two-dimensional map \(4\) is equivalent to the following relation between $\psi_{n+1}, \psi_{n-1}$ and $\psi_n$,

$$\psi_{n+1} + \psi_{n-1} = \left( 2 \cos \mu + \frac{U_n}{k} \sin \mu \right) \psi_n.$$  

(7)

One can see that the relation \(7\) has the same form as discrete Schrödinger equation for the standard 1D Anderson tight-binding model,

$$\psi_{n+1} + \psi_{n-1} = (E + \epsilon_n) \psi_n,$$  

(8)

and describes electrons on a discrete lattice with the site energies $\epsilon_n$. Therefore, many of the results for the Kronig-Penney model can be obtained by a formal comparison with the Anderson model, as discussed below.

### III. THE HAMILTONIAN MAP APPROACH

One of the tools to find the localization length for discrete disordered models, either analytically or numerically, is based on the transfer matrix method. In this approach the localization length can be expressed as the inverse of the Lyapunov exponent $\lambda$ which characterizes the growth of the eigenstates $\psi(x)$ of the stationary Schrödinger equation for increasing $x$. An alternative approach can be obtained by interpreting the stationary Schrödinger equation as the equation of motion of a classical particle (in this scheme the space coordinate $x$ of the disordered model is to be seen as the time coordinate for its dynamical counterpart). In particular, in the case of discrete disordered models, this approach leads to the study of classical maps.

It is instructive to illustrate this approach by discussing its application to the simplest case of the Anderson model \(8\). Comparing Eq. \(8\) with Eq. \(4\), one can obtain that there is an exact correspondence between them by letting $\mu_n = \mu$ and
\[ E = 2 \cos \mu; \quad A_n = -\epsilon_n / \sin \mu. \] (9)

It is clear that for weak disorder the energy spectrum of the Anderson model (8) is close to the unperturbed one which is defined by the condition \(|E| \leq 2\); this legitimates the first equality in Eq. (9).

To analyze the dynamics of the two-dimensional map (4), it is convenient to introduce the action-angle variables \((r_n, \theta_n)\) according to the standard transformation, \(q = r \sin \theta, p = r \cos \theta\). As a result, the map gets the following form,

\[
\begin{align*}
\sin \theta_{n+1} &= D_n^{-1} (\sin(\theta_n - \mu) - A_n \sin \theta_n \sin \mu) \\
\cos \theta_{n+1} &= D_n^{-1} (\cos(\theta_n - \mu) + A_n \sin \theta_n \cos \mu), 
\end{align*}
\] (10)

where

\[ D_n = \frac{r_{n+1}}{r_n} = \sqrt{1 + A_n \sin(2\theta_n) + A_n^2 \sin^2 \theta_n}. \]

Note that the following results for the localization length do not depend on the sign of \(\mu\). It is important that the equation for the angle \(\theta_n\) can be written in the form of the one-dimensional map,

\[ \cot(\theta_{n+1} + \mu) = \cot \theta_n + A_n. \] (11)

This fact simplifies the analysis of the distribution of \(\theta_n\). The localization length \(l\) is defined as the inverse Lyapunov exponent, and the latter is determined by the standard relation [5]

\[ l^{-1} = \lambda = \lim_{N \to \infty} \left( \frac{1}{N} \sum_{n=0}^{N-1} \ln \left| \frac{q_{n+1}}{q_n} \right| \right) = \left\langle \ln \left| \frac{q_{n+1}}{q_n} \right| \right\rangle. \]

Here the overbar stays for time average and the brackets for the average over different disorder realizations. The expression for \(l^{-1}\) can be splitted in two terms

\[ l^{-1} = \left\langle \ln \left( \frac{r_{n+1}}{r_n} \right) \right\rangle + \left\langle \ln \left| \frac{\sin \theta_{n+1}}{\sin \theta_n} \right| \right\rangle. \] (12)

The second term on the r.h.s. is negligible because it is the average of a bounded quantity. It becomes important only when the first term is also small, i.e. at the band edge \(|E| \approx 2\) or \(\mu \approx 0\). Thus, apart from this specific case, the localization length can be evaluated from the map (10) using only the dependence of the radius \(r_n\) on discrete time \(t_n = n\). It is important to note that the ratio \(r_{n+1}/r_n\) depends only on the angle \(\theta_n\) and not on the radius \(r_n\); as a consequence, one can compute the localization length just by averaging the first term in (12) over the invariant measure \(\rho(\theta)\) associated with the 1D angular map (11).

In a direct analytical evaluation of (12) one can write,

\[ l^{-1} = \int P(\epsilon) \int_0^{2\pi} \ln(D(\epsilon, \theta)) \rho(\theta) d\theta d\epsilon, \] (13)

where \(P(\epsilon)\) is the density of the distribution of \(\epsilon_n\), and \(\rho(\theta)\) is the invariant measure for the angle variable. We use here the fact that \(\rho(\theta)\) does not depend on the specific sequence \(\epsilon_n\), but can depend on the moments of \(P(\epsilon)\), particularly on its second moment \(\sigma^2\) (see discussion in [6]). As one can see, in order to evaluate the expression (13), first one has to determine the invariant measure \(\rho(\theta)\).

In the case of weak disorder and not close to the band edges we have \(|A_n| \ll 1\) and one can use the standard perturbation theory. This allows one to cast Eq. (13) in the form

\[ l^{-1} = \frac{1}{2 \sin^2 \mu} \int \epsilon^2 P(\epsilon) d\epsilon \int_0^{2\pi} \rho(\theta) \left( \frac{1}{4} - \frac{1}{2} \cos(2\theta) + \frac{1}{4} \cos(4\theta) \right) d\theta. \] (14)

This expression is valid for all energies within the band, but fails at the band edges, where one must include the contribution of the second term of Eq. (12) in the computation of the inverse localization length (see Ref. [6]). One can also see from Eq. (11) that in the limit of weak disorder the map for \(\theta_n\) has the approximate form

\[ \theta_{n+1} = \theta_n - \mu - A_n \sin^2 \theta_n + A_n^2 \sin^3 \theta_n \cos \theta_n \quad \{\text{mod } 2\pi\}. \] (15)
Therefore, in the first approximation the invariant measure for $\theta_n$ is flat and this makes possible an explicit evaluation of the expression (14). Thus, one easily obtains

$$l^{-1} = \frac{\sigma^2}{8 \sin^2 \mu} = \frac{\sigma^2}{8 \sqrt{1 - \frac{\mu^2}{4}}}$$

where $\sigma^2 = \langle \xi_n^2 \rangle$ is the variance of the disorder. It is interesting to note that the expression (16) is not correct at the band center, i.e., for $E = 0$ (see discussion and references in Ref [6]). The reason is that for this energy the standard perturbation theory fails and one must use specific methods to obtain the correct expression of $l^{-1}$. As was found, the anomaly at the band center originates from the fact that for $E = 0$ the density $\rho(\theta)$ is not flat, instead, it has a slight modulation with $\theta$. This additional $\theta$-dependence of the invariant measure is due to the special circumstance that the case $E = 0$ corresponds to $\mu = \pi/2$ so that the map (15) becomes approximatively periodic of period four. As a consequence, $\rho(\theta)$ has a weak modulation of period $\pi/2$ and, therefore, the fourth harmonic $\cos(4\theta)$ in the expression (14) also gives a contribution.

Due to the analogy between the Anderson model (8) (A-model) and the Kronig-Penney model (3) (KP-model) with $\mu_n = \mu$, one can derive from the result (16) the expression for the localization length of the KP-model

$$l^{-1}(E) = \frac{\varepsilon_0^2 \sin^2(ka)}{8k^2 \sin^2 \gamma}.$$ 

Here the phase $\gamma$ ($0 \leq \gamma \leq \pi$) is given by the equation,

$$2\cos(ka) + \frac{\varepsilon}{k} \sin(ka) = 2\cos \gamma.$$ 

This equation is the well-known dispersion relation for the periodic Kronig-Penney model; the parameter $\gamma$ plays the role of the Bloch number.

**IV. ANDERSON LOCALIZATION AND PARAMETRIC INSTABILITY**

It is easy to see that the Schrödinger equation (3) for the quantum 1D disordered model can be interpreted as the dynamical equation of a linear classical oscillator with a parametric perturbation constituted by a succession of delta-kicks at times $t_n = x_n$. In particular, the map (4) corresponding to the Kronig-Penney model ($\mu_n = \mu$) can be obtained by integrating the dynamical equations between two successive kicks for a stochastic oscillator with Hamiltonian of the form

$$H = \omega \left( \frac{q^2}{2} + \frac{p^2}{2} \right) + \frac{q^2}{2} \left( \sum_{n=-\infty}^{\infty} A_n \delta(t - nT) \right).$$

Therefore, $q_n$ and $p_n$ in Eq. (4) stand for the position and momentum of the oscillator immediately before the $n$th kick of amplitude $A_n$, occurring at the time $t = nT$. Correspondingly, the phase shift between two successive kicks is given by $\mu = \omega T$ where $\omega$ is the unperturbed frequency of the oscillator and $T$ is the period between the kicks.

In this description the exponential localization of the eigenstates of Eq. (3) corresponds to a parametric instability of the stochastic oscillator (17). The instability manifests itself as an exponential divergence of initially nearby orbits (orbit instability) and, correspondingly, as an exponential growth of the average energy of the parametric oscillator (energy instability). The Lyapunov exponent $\lambda$, which gives the inverse localization length in the solid-state the divergence of classical trajectories (or, the rate of the energy growth).

In the previous section we have considered the case of a weak *uncorrelated* disorder which is characterized by its variance $\sigma^2$ only. In application to classical oscillators this corresponds to a white noise perturbation. In the following, we consider the general case of colored noise and show that noise correlations can lead to a quite unexpected phenomenon. To discuss the effects of correlated noise in parametric oscillators, we apply the approach of Ref. [7] to the continuous model described by the Hamiltonian,

$$H = \omega \left( \frac{q^2}{2} + \frac{p^2}{2} \right) + \frac{q^2}{2} \xi(t)$$

where $\xi(t)$ is a continuous and stationary noise. This model is slightly different from the one defined by Eq. (17) because the noise $\xi(t)$ is a continuous function of time rather than a succession of $\delta$-kicks. We assume that the noise $\xi(t)$ has zero average and that its binary correlator is a known function,
Here and below, in contrast to previous sections the symbol \( \langle \ldots \rangle \) will refer to the time average, \( \langle f(t) \rangle = \lim_{T_0 \to \infty} \frac{1}{T_0} \int_0^{T_0} f(t) dt \), which is assumed to coincide with the ensemble average for the process \( \xi(t) \).

We define the Lyapunov exponent as follows,

\[
\lambda = \lim_{T_0 \to \infty} \lim_{\delta \to 0} \frac{1}{T_0} \int_0^{T_0} \ln \frac{q(t + \delta)}{q(t)} dt.
\]

As in the previous section, we introduce polar coordinates via the standard relations \( q = r \sin \theta, p = r \cos \theta \). This allows us to represent Eq. (20) in the form

\[
\lambda = \lim_{T_0 \to \infty} \frac{1}{T_0} \int_0^{T_0} \frac{\dot{\theta}}{r} dt.
\]

To proceed further, we consider the equations for the random oscillator in polar coordinates

\[
\dot{\theta} = \omega + \xi(t) \sin^2 \theta,
\]

\[
\dot{r} = -\frac{1}{2} r \xi(t) \sin 2\theta.
\]

Using the last equation, the expression for the Lyapunov exponent can be finally written in the form

\[
\lambda = \lim_{T_0 \to \infty} \frac{1}{2T_0} \int_0^{T_0} \xi(t) \sin (2\theta(t)) dt = \frac{1}{2} \langle \xi(t) \sin (2\theta(t)) \rangle.
\]

Therefore, the problem of computing the Lyapunov exponent (20) is reduced to that of calculating the noise-angle correlator that appears in Eq. (21). This was done in Ref. [7] by extending the procedure, originally introduced in Ref. [8] for discrete models, to the continuum case. As a result, the expression for the Lyapunov exponent takes the simple but non-trivial form,

\[
\lambda = \frac{1}{8} \int_{-\infty}^{+\infty} \langle \xi(t)\xi(t + \tau) \rangle \cos(2\omega \tau) d\tau.
\]

One can see that the Lyapunov exponent for the stochastic oscillator (18) is proportional to the Fourier transform \( \tilde{\chi}(2\omega) \) of the correlation function at twice the frequency of the unperturbed oscillator.

A similar result can be obtained for the parametric oscillator (17) with discrete noise. In this case the inverse localization length can be written as [8]

\[
\lambda = \frac{\langle A_n^2 \rangle}{8T_0} \varphi(\omega T).
\]

Here the function \( \varphi(\omega T) \) is the Fourier transform,

\[
\varphi(\omega T) = 1 + 2 \sum_{k=1}^{+\infty} \zeta(k) \cos(2\omega Tk)
\]

of the binary correlator

\[
\zeta(k) = \frac{\langle A_{n+k} A_n \rangle}{\langle A_n^2 \rangle}
\]

of the colored noise. Therefore, the final expression is given by the product of two factors, namely, the Lyapunov exponent for the white noise case and the function \( \varphi(\omega T) \), which describes the effect of the noise correlations (the color). In the case of white noise we have \( \varphi(\omega T) = 1 \).

\[5\]
V. SUPPRESSION OF THE PARAMETRIC INSTABILITY

Expressions (22) and (23) of the Lyapunov exponent for stochastic oscillators with weak frequency noise give a remarkable result: within the limits of the second-order approximation the rate of parametric instability depends only on the binary correlator of the noise. In application to solid state models this fact has suggested a way to construct random potentials with specific spatial correlations that result in “windows of transparency” in the energy spectrum. Indeed, if the Lyapunov exponent vanishes within some range of the energy (or, the wave number \( k \)), then the corresponding eigenstates are extended in that energy interval. When one considers finite samples, this means that the transmission coefficient has to be one in the energy windows where the Lyapunov exponent vanishes.

The possibility of engineering random potentials in order to obtain Lyapunov exponents with predefined energy dependence can be deduced from the expression (22) for continuous model, or from Eq. (23) for the discrete one. Both expressions show that if the Lyapunov exponent is known, the two-point correlator of the corresponding noise can be computed with an inverse Fourier transform. Since a stochastic process is not completely determined by its two-point correlator, one can conclude that there is actually an infinite set of noises which give rise to the same Lyapunov exponent because they have an identical binary correlator.

As one can see, in order to have suppression of the parametric instability in classical oscillators with colored noise, one needs to have \( \lambda(\omega) = 0 \) in some range of \( \omega \). Although at first sight the construction of a random potential \( A_n \) or \( \xi(t) \) with a given binary correlator seems a difficult task, a rather simple method to solve this problem was presented in Ref. [8] for discrete models. This method was subsequently extended to oscillators with continuous noise. Here we describe how the method works for both classes of oscillators.

As an example, we consider the Lyapunov exponent \( \lambda(\omega) = \begin{cases} 1 & \text{if } |\omega| < 1/2 \\ 0 & \text{otherwise} \end{cases} \), whose frequency dependence implies that the random oscillator undergoes a sharp transition for \( |\omega| = 1/2 \), passing from an energetically stable condition to an unstable one. Following the described procedure, it is easy to see that the Lyapunov exponent (27) is generated by a noise of the form

\[
\xi(t) = \frac{\sqrt{8}}{\pi} \int_{-\infty}^{+\infty} \frac{\sin(s)}{s} \eta(s + t) \, ds,
\]

with \( \eta(t) \) being any random process with the statistical properties (26). Working along these lines, it is easy to see that one can also construct the frequency noises such that the parametric oscillator is stable for all values of \( \omega \) except those contained in a narrow frequency window.
We now turn our attention to discrete models of the form (17). To show how the expression that is equivalent to Eq. (25) can be worked out for this class of systems, we refer to the case of the Anderson model (8) with correlated disorder. Since this model can be put into one-to-one correspondence with the kicked oscillator (17), it is perfectly legitimate to analyse each of the two models in terms of the other; this approach has also the advantage of enhancing the physical understanding of the problem because it allows one to interpret the parametric instability of a stochastic oscillator in terms of localization of electronic states for the Anderson model.

When we transpose the result (23) to the case of the Anderson model (8), we obtain that the expression for the localization length has the form,

\[ l^{-1} = \frac{\langle \epsilon_n^2 \rangle}{\sin^2 \mu} \varphi(\mu), \]  

(28)

where

\[ \varphi(\mu) = 1 + 2 \sum_{k=1}^{\infty} \zeta(k) \cos(2\mu k). \]

Here \( \zeta(k) \) is the binary correlator (24) which can be written in terms of the site energies \( \epsilon_n \) of the Anderson model as

\[ \zeta(k) = \langle \epsilon_n \epsilon_{n+k} \rangle / \langle \epsilon_n^2 \rangle. \]

If the Lyapunov exponent (28) (therefore, the function \( \varphi(\mu) \)) is known, the binary correlators (24) can be derived with an inverse Fourier transform,

\[ \zeta(k) = \frac{2}{\pi} \int_{0}^{\pi/2} \varphi(\mu) \cos(2\mu k) \, d\mu. \]  

(29)

As for the continuous model, the sequence of site energies \( \epsilon_n \) with the correlator of the specific form (29) can then be constructed with the convolution product,

\[ \epsilon_n = \sqrt{\langle \epsilon_n^2 \rangle} \sum_{k=-\infty}^{\infty} \beta_k Z_{n+k}, \]  

(30)

where

\[ \beta_k = \frac{2}{\pi} \int_{0}^{\pi/2} \sqrt{\varphi(\mu)} \cos(2\mu k) \, d\mu \]

and \( Z_n \) are random numbers with the zero mean and unitary variance. It is easy to check that the correlators of the site potential (30) coincide with the Fourier coefficients (29).

As an illustration of the method, we construct the random potential which results in the following function \( \varphi(\mu) \) for the Lyapunov exponent of the discrete Anderson model,

\[ \varphi(\mu) = \begin{cases} 
C_0^2 & \text{if } \mu_1 < \mu < \mu_2,
0 & \text{if } 0 < \mu < \mu_1 \text{ or } \mu_2 < \mu < \pi/2.
\end{cases} \]

Here, \( C_0^2 = \pi/2(\mu_2 - \mu_1) \) is the normalization constant that results from the condition \( \zeta_0 = 1 \). The corresponding localization length exhibits two sharp mobility edges at the values \( E_1 = 2 \cos \mu_1 \) and \( E_2 = 2 \cos \mu_2 \). Specifically, in the energy window \( E_1 < E < E_2 \) the eigenstates are strongly localized, while they are extended outside of this window. The binary correlators \( \zeta(k) \) for a random potential resulting in such a situation, are given by

\[ \zeta(k) = \frac{C_k^2}{\pi k} [\sin(2k\mu_2) - \sin(2k\mu_1)]. \]

As a result, the expression for the inverse localization length reads

\[ \lambda = l^{-1} = \frac{\pi \sigma^2}{16 \sin^2 \mu} \left[ \arccos\left( \frac{E_2}{2} \right) - \arccos\left( \frac{E_1}{2} \right) \right], \]

where \( \sigma^2 = \langle \epsilon_n^2 \rangle \). If the energy window \( \Delta E = E_1 - E_2 \) is narrow, one can write,
One can see that the narrower the window \( \Delta_E \), the sharper the transition which occurs at the mobility edges. This effect can be easily observed numerically, and it may have interesting applications for parametric oscillators. Indeed, small localization lengths correspond to large values of the Lyapunov exponent. Therefore, for values of the frequency of the kicked oscillator (17) which correspond to energy values within the localization window in the related Anderson model, the instability of the oscillator is very strong and one can speak of a kind of “parametric stochastic resonance”.

VI. DISCUSSION

In the previous section we have shown how a proper choice of colored noise (random potentials in the solid state models) can suppress the parametric instability of a stochastic oscillator in a prescribed frequency range. One should note, however, that the theoretical analysis has been focused on the case of weak noise and that almost all analytical results have been obtained using perturbative approach. Therefore, the conclusion that the Lyapunov exponent can vanish within some frequency region is valid only within the framework of second-order perturbation theory. Going beyond the second-order approximation, it is possible to estimate the correction to the present results for the inverse localization length and to show that the correction is represented by a term of order \( O(\sigma^4) \) (with \( \sigma^2 = \langle c_n^2 \rangle \)). It is not clear whether one can make this fourth-order correction vanish with an appropriate choice of the statistical properties of the noise [9]; from a practical point of view, however, in the case of weak noise there is a well-defined separation between the time scale \( t \sim 1/\sigma^2 \) over which the suppression of instability holds, and the much longer time scale \( t \sim 1/\sigma^4 \) over which the effects of fourth-order corrections become relevant. When the second-order results for the inverse localization length are applied to wave-guides or solid state models, fourth-order effects can also be generally avoided by a proper choice of size of an experimental device [3].

The transfer matrix method in the Hamiltonian form described above is also very useful for finite times. In application to solid state models this question refers to transport properties through finite samples of size \( L \). As is known [5,10], all transport properties can be directly related to the classical trajectories of the Hamiltonian map (4). Specifically, by studying general properties of these trajectories, one can find statistical properties of the transmission coefficient or the resistance. The transmission coefficient through a \( L \)-site sample can be expressed in terms of dynamical variables of the classical map (4) as

\[
T_L = \frac{4}{2 + r_1^2 + r_2^2}
\]

where \( r_1 \) and \( r_2 \) represent the radii at the \( L \)th step of the map trajectories starting from the phase-space points \( P_1 = (x_0 = 1, p_0 = 0) \) and \( P_2 = (x_0 = 0, p_0 = 1) \), respectively. As for the resistance \( R_L \), it is defined as the inverse of the transmission coefficient \( R_L = T_L^{-1} \). The key feature of these formulae is that they express the transport properties of a disordered sample in terms of the radii of map trajectories in the phase space. On the other hand, the square radius \( r^2 \) of a map trajectory is a quadratic function of the coordinate and momentum of the corresponding kicked oscillator, \( r^2 = p^2 + q^2 \), and is therefore proportional to the energy of the latter. This fact makes possible to relate transport properties of quantum models with the time dependence of the energy of classical parametric oscillators.

It is possible to obtain quite easily the moments of the energy \( r^2 \) of the parametric oscillator described by the Hamiltonian (17), see details in Refs. [7,11]. In particular, one can obtain that in the asymptotic limit (i.e., for times \( t \gg \lambda^{-1} \)) the mean value of the energy grows exponentially as

\[
\langle r^2(t) \rangle = r^2(0) \exp(4\lambda t)
\]

where \( \lambda \) is the Lyapunov exponent (23). This formula shows that the exponential rate of the energy growth for the parametric oscillator is four times the Lyapunov exponent, i.e., the rate of exponential separation of nearby orbits [7].

Another important question concerns the fluctuations of \( r^2(t) \) for fixed times \( t \) depending on different realizations of the noise. Using the results of Ref. [7,11], one should distinguish between two different situations. The first one corresponds to small times when the value of \( r(t) \) is close to the initial value \( r(0) \). In solid state models this case is known as the ballistic transport for which the localization length \( l = \lambda^{-1} \) is much larger than the size \( L = t \) of the sample, \( \lambda t \ll 1 \). Another limit case corresponds to large times, \( \lambda t \gg 1 \), or to the strongly localized regime in quantum models. One of the most interesting effects is that in this case the fluctuations of the energy of the classical oscillator (resistance in quantum models) are huge and the quantity \( R = r^2 \) is not self-averaging. To deal with a well-behaved (that is, self-averaging) statistical property, one has to consider the logarithm of the oscillator energy, which has a Gaussian distribution for large times. It turns out that the energy \( r^2 \) has log-normal distribution:
\[ P(r^2, t) = \frac{1}{\sqrt{8\pi \lambda t}} \exp \left[ -\frac{(\ln r^2 - 2\Lambda)^2}{8\lambda t} \right]. \]

This distribution implies that the energy of parametric oscillator, or the resistance \( R = r^2 \) of disordered samples, satisfy the relations

\[ \langle \ln R \rangle = 2\Lambda; \langle \ln^2 R \rangle = 4\Lambda + 4\Lambda^2, \]

\[ \text{Var} (\ln R) = \langle \ln^2 R \rangle - \langle \ln R \rangle^2 = 2\langle \ln R \rangle. \]

where \( \Lambda = \lambda t. \)

In conclusion, we have discussed the analogy between properties of quantum 1D models with random potentials and classical linear oscillators governed by parametric noise. We have shown that many results known for quantum models can be mapped unto corresponding properties of classical oscillators. One of the important questions is about the time-dependence of the energy of stochastic oscillators with frequency perturbed by a white noise. Another, even more exciting problem, is the behavior of the oscillators when the frequency noise has long-range correlations. It was shown that in the case of weak noise all statistical properties of the classical trajectories depend on the binary correlator of the noise only. This fact opens the door to the construction of colored noises with specific long-range correlations which result in a sharp change in the dynamical behavior of the parametric oscillator at some threshold value of the unperturbed frequency. Specifically, the characteristic instability of parametric oscillators can be suppressed in a certain frequency range (with a brisk transition), thanks to long-range temporal correlations of the noise. These results may find different applications in the field of classical systems with colored noise.

ACKNOWLEDGMENTS

The authors are very thankful to N.M.Makarov for fruitful discussions and valuable comments.

[1] K. Ishii, Suppl. Progr. Theor. Phys. 53, p. 77, 1973.
[2] F. M. Izrailev and N. M. Makarov, Optics Lett. 26, p. 1604, 2001.
[3] U. Kuhl, F. M. Izrailev, A. A. Krokhin, and H.-J. Stöckmann, Appl. Phys. Lett. 77, p. 633, 2000; A. A. Krokhin, F. M. Izrailev, U. Kuhl, H.-J. Stöckmann, and S. Ulloa, Physica E 13, p. 695, 2002.
[4] F. M. Izrailev, A. A. Krokhin, and S. E. Ulloa, Phys. Rev. E, 63, p. 041102, 2001.
[5] I. M. Lifshitz, S. Gredeskul, and L. Pastur, Introduction to the Theory of Disordered Systems, Wiley, New York, 1988.
[6] F. M. Izrailev, S. Ruffo and L. Tessieri, J. Phys. A: Math. Gen. 31, p. 5263, 1998.
[7] L. Tessieri and F. M. Izrailev, Phys. Rev. E 64, p. 66120, 2001.
[8] F. M. Izrailev and A. A. Krokhin, Phys. Rev. Lett. 82, p. 4062, 1999.
[9] L. Tessieri, J. Phys. A: Math. Gen. 35, p. 9585, 2002.
[10] T. Kottos, G. P. Tsironis and F. M. Izrailev, J. Phys.: Condens. Matter 9, p. 1777, 1997.
[11] V. Dossetti, F. M. Izrailev, and A. A. Krokhin, Phys. Lett. A 320, p. 276, 2004.