Tangent metric spaces to starlike sets on the plane

Oleksiy Dovgoshey:

Institute of Applied Mathematics and Mechanics of NASU, R. Luxemburg str. 74, Donetsk 83114, Ukraine
aleksdov@mail.ru

Fahreddin Abdullayev and Mehmet Küçükaslan:

Mersin University Faculty of Literature and Science, Department of Mathematics, 33342 Mersin, Turkey
fabdul@mersin.edu.tr and mkucukaslan@mersin.edu.tr

Abstract

Let $A \subseteq \mathbb{C}$ be a starlike set with a center $a$. We prove that every tangent space to $A$ at the point $a$ is isometric to the smallest closed cone, with the vertex $a$, which includes $A$. A partial converse to this result is obtained. The tangent space to convex sets is also discussed.

Key words: Metric spaces; Tangent spaces to metric spaces; Tangent space to convex sets; Tangent space to starlike sets.

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1 Introduction and main results

Analysis on metric spaces with no a priory smooth structure has been rapidly developed recently. This development is closely related to some generalizations of the differentiability. Important examples of such generalizations and even an axiomatics of so-called “pseudo-gradients” can be found
in \([1,8,5,10,12,17]\) and respectively in \([2]\). In almost all above-mentioned books and papers the generalized differentiations involve an induced linear structure that makes possible to use the classical differentiations in the linear normed spaces. A new intrinsic approach to the introduction of the “smooth” structure by means of the construction of “tangent spaces” for general metric spaces was proposed by O. Martio and by the first author of the present paper in \([8]\).

In the present paper we prove that for every starlike set \(A \subseteq \mathbb{C}\) with a center \(a\) all tangent spaces to \(A\) at the point \(a\) are isometric to the smallest closed cone which includes \(A\) and has the vertex \(a\). A partial converse to this result is also obtained. Important particular cases \(A = \mathbb{R}\), \(A = \mathbb{R}^+\) and \(A = \mathbb{C}\) are considered in details. The results of the paper were partly published in the preprint from \([7]\).

For convenience we recall the main notions from \([6–8]\).

Let \((X,d)\) be a metric space and let \(a\) be a point of \(X\). Fix a sequence \(\tilde{r}\) of positive real numbers \(r_n\) which tend to zero. In what follows this sequence \(\tilde{r}\) is called a normalizing sequence. Let us denote by \(\tilde{X}\) the set of all sequences of points from \(X\).

**Definition 1.1.** Two sequences \(\tilde{x}, \tilde{y} \in \tilde{X}, \tilde{x} = \{x_n\}_{n \in \mathbb{N}}\) and \(\tilde{y} = \{y_n\}_{n \in \mathbb{N}}\), are mutually stable (with respect to a normalizing sequence \(\tilde{r} = \{r_n\}_{n \in \mathbb{N}}\)) if there is a finite limit

\[
\lim_{n \to \infty} \frac{d(x_n, y_n)}{r_n} = \tilde{d}(\tilde{x}, \tilde{y}) = \tilde{d}(\tilde{x}, \tilde{y}).
\]

We shall say that a family \(\tilde{F} \subseteq \tilde{X}\) is self-stable (w.r.t. \(\tilde{r}\)) if every two \(\tilde{x}, \tilde{y} \in \tilde{F}\) are mutually stable. A family \(\tilde{F} \subseteq \tilde{X}\) is maximal self-stable if \(\tilde{F}\) is self-stable and for an arbitrary \(\tilde{z} \in \tilde{X}\) either \(\tilde{z} \in \tilde{F}\) or there is \(\tilde{x} \in \tilde{F}\) such that \(\tilde{x}\) and \(\tilde{z}\) are not mutually stable.

A standard application of Zorn’s Lemma leads to the following

**Proposition 1.2.** Let \((X,d)\) be a metric space and let \(a \in X\). Then for every normalizing sequence \(\tilde{r} = \{r_n\}_{n \in \mathbb{N}}\) there exists a maximal self-stable family \(\tilde{X}_a = \tilde{X}_{a,\tilde{r}}\) such that \(\tilde{a} := \{a, a, \ldots\} \in \tilde{X}_a\).

Note that the condition \(\tilde{a} \in \tilde{X}_a\) implies the equality

\[
\lim_{n \to \infty} d(x_n, a) = 0
\]

for every \(\tilde{x} = \{x_n\}_{n \in \mathbb{N}} \in \tilde{X}_a\).
Consider a function $\tilde{d} : \tilde{X}_a \times \tilde{X}_a \rightarrow \mathbb{R}$ where $\tilde{d}(\tilde{x}, \tilde{y}) = \tilde{d}_p(\tilde{x}, \tilde{y})$ is defined by (1.3). Obviously, $\tilde{d}$ is symmetric and nonnegative. Moreover, the triangle inequality for the original metric $d$ implies

$$\tilde{d}(\tilde{x}, \tilde{y}) \leq \tilde{d}(\tilde{x}, \tilde{z}) + \tilde{d}(\tilde{z}, \tilde{y})$$

for all $\tilde{x}, \tilde{y}, \tilde{z} \in \tilde{X}_a$. Hence $(\tilde{X}_a, \tilde{d})$ is a pseudometric space.

**Definition 1.3.** The pretangent space to the space $X$ at the point a w.r.t. $\tilde{r}$ is the metric identification of the pseudometric space $(\tilde{X}_{a, \tilde{r}}, \tilde{d})$.

Since the notion of pretangent space is basic for the present paper, we remind this metric identification construction.

Define a relation $\sim$ on $\tilde{X}_a$ by $\tilde{x} \sim \tilde{y}$ if and only if $d(\tilde{x}, \tilde{y}) = 0$. Then $\sim$ is an equivalence relation. Let us denote by $\Omega_a = \Omega_{a, \tilde{r}} = \Omega_{X, \tilde{r}}$ the set of equivalence classes in $\tilde{X}_a$ under the equivalence relation $\sim$. It follows from general properties of pseudometric spaces, see, for example, [13, Chapter 4, Th. 15], that if $\rho$ is defined on $\Omega_a$ by

$$\rho(\alpha, \beta) := \tilde{d}(\tilde{x}, \tilde{y})$$

(1.2)

where $\tilde{x} \in \alpha$ and $\tilde{y} \in \beta$, then $\rho$ is the well-defined metric on $\Omega_a$. The metric identification of $(\tilde{X}_a, \tilde{d})$ is, by definition, the metric space $(\Omega_a, \rho)$.

Remark that $\Omega_{a, \tilde{r}} \neq \emptyset$ because the constant sequence $\tilde{a}$ belongs to $\tilde{X}_{a, \tilde{r}}$, see Proposition 1.2.

Let $\{n_k\}_{k \in \mathbb{N}}$ be an infinite, strictly increasing sequence of natural numbers. Let us denote by $\tilde{r}'$ a subsequence $\{r_{n_k}\}_{k \in \mathbb{N}}$ of the normalizing sequence $\tilde{r} = \{r_n\}_{n \in \mathbb{N}}$ and let $\tilde{x}' := \{x_{nk}\}_{k \in \mathbb{N}}$ for every $\tilde{x} = \{x_n\}_{n \in \mathbb{N}} \in \tilde{X}$.

It is clear that if $\tilde{x}$ and $\tilde{y}$ are mutually stable w.r.t. $\tilde{r}$, then $\tilde{x}'$ and $\tilde{y}'$ are mutually stable w.r.t. $\tilde{r}'$ and

$$\tilde{d}_p(\tilde{x}, \tilde{y}) = \tilde{d}_p(\tilde{x}', \tilde{y}')$$

(1.3)

If $\tilde{X}_{a, \tilde{r}}$ is a maximal self-stable (w.r.t. $\tilde{r}$) family, then, by Zorn’s Lemma, there exists a maximal self-stable (w.r.t. $\tilde{r}'$) family $\tilde{X}_{a, \tilde{r}'}$ such that

$$\{\tilde{x}' : \tilde{x} \in \tilde{X}_{a, \tilde{r}}\} \subseteq \tilde{X}_{a, \tilde{r}'}.$$

(1.4)

Denote by $\text{in}_{\tilde{r}'}$ the mapping from $\tilde{X}_{a, \tilde{r}}$ to $\tilde{X}_{a, \tilde{r}'}$ with $\text{in}_{\tilde{r}'}(\tilde{x}) = \tilde{x}'$ for all $\tilde{x} \in \tilde{X}_{a, \tilde{r}}$. If follows from (1.3) that after the metric identifications $\text{in}_{\tilde{r}'}$ passes to an isometric embedding $\text{em}' : \Omega_{a, \tilde{r}} \rightarrow \Omega_{a, \tilde{r}'}$ under which the diagram

\[
\begin{array}{ccc}
\tilde{X}_{a, \tilde{r}} & \xrightarrow{\text{in}_{\tilde{r}'}} & \tilde{X}_{a, \tilde{r}'} \\
\downarrow p & & \downarrow p' \\
\Omega_{a, \tilde{r}} & \xrightarrow{\text{em}'} & \Omega_{a, \tilde{r}'}
\end{array}
\]

(1.5)
is commutative. Here \( p \) and \( p' \) are metric identification mappings, \( p(\tilde{x}) := \{ \tilde{y} \in \tilde{X}_{a,\tilde{r}} : \tilde{d}_{\tilde{r}}(\tilde{x}, \tilde{y}) = 0 \} \), \( p'(\tilde{x}) := \{ \tilde{y} \in \tilde{X}_{a,\tilde{r}'} : \tilde{d}_{\tilde{r}'}(\tilde{x}, \tilde{y}) = 0 \} \).

Let \( X \) and \( Y \) be two metric spaces. Recall that a map \( f : X \to Y \) is called an isometry if \( f \) is distance-preserving and onto.

**Definition 1.4.** A pretangent \( \Omega_{a,\tilde{r}} \) is tangent if \( em' : \Omega_{a,\tilde{r}} \to \Omega_{a,\tilde{r}'} \) is an isometry for every \( \tilde{r}' \).

Note that the property to be tangent does not depend on the choice of \( \tilde{X}_{a,\tilde{r}} \) in (1.4), see Proposition 2.1 in the present paper.

Let \( X \) be a metric space with a marked point \( a, \tilde{r} \) a normalizing sequence, \( \tilde{X}_{a,\tilde{r}} \) a maximal self-stable family and \( \Omega_{a,\tilde{r}} \) the corresponding pretangent space.

**Definition 1.5.** The pretangent space \( \Omega_{a,\tilde{r}} \) lies in a tangent space if there is a maximal self-stable family \( \tilde{X}_{a,\tilde{r}'} \) such that (1.4) holds and if \( \Omega_{a,\tilde{r}'} \), the metric identification of \( \tilde{X}_{a,\tilde{r}'} \), is tangent.

Let \((X,d)\) be a metric space with a marked point \( a \), let \( Y \) and \( Z \) be subspaces of \( X \) such that \( a \in Y \cap Z \) and let \( \tilde{r} = \{r_n\}_{n \in \mathbb{N}} \) be a normalizing sequence.

**Definition 1.6.** The subspaces \( Y \) and \( Z \) are tangent equivalent at the point \( a \) w.r.t. \( \tilde{r} \) if for every \( \tilde{y}_1 = \{y_n^{(1)}\}_{n \in \mathbb{N}} \in \tilde{Y} \) and for every \( \tilde{z}_1 = \{z_n^{(1)}\}_{n \in \mathbb{N}} \in \tilde{Z} \) with finite limits

\[
\tilde{d}_{\tilde{r}}(\tilde{a}, \tilde{y}_1) = \lim_{n \to \infty} \frac{d(y_n^{(1)}, a)}{r_n} \quad \text{and} \quad \tilde{d}_{\tilde{r}}(\tilde{a}, \tilde{z}_1) = \lim_{n \to \infty} \frac{d(z_n^{(1)}, a)}{r_n}
\]

there exist \( \tilde{y}_2 = \{y_n^{(2)}\}_{n \in \mathbb{N}} \in \tilde{Y} \) and \( \tilde{z}_2 = \{z_n^{(2)}\}_{n \in \mathbb{N}} \in \tilde{Z} \) such that

\[
\lim_{n \to \infty} \frac{d(y_n^{(1)}, z_n^{(2)})}{r_n} = \lim_{n \to \infty} \frac{d(y_n^{(2)}, z_n^{(1)})}{r_n} = 0.
\]

We shall say that \( Y \) and \( Z \) are strongly tangent equivalent at \( a \) if \( Y \) and \( Z \) are tangent equivalent at \( a \) for all normalizing sequences \( \tilde{r} \).

Let \( A \) be a set in a linear topological space \( X \) over \( \mathbb{R} \). The set \( A \) is termed starlike with a center \( a \) if

\[
[a, b] = \{ x \in X : x = a + t(b - a), \quad t \in [0, 1] \} \subseteq A
\]

for all \( b \in A \). Moreover, \( A \) is a cone with the vertex \( a \) if the ray

\[
l_a(b) := \{ x \in X : x = a + t(b - a), t \in \mathbb{R}^+ \}
\]

(1.6)
lies in $A$ for every $b \in A$. For nonvoid sets $X \subseteq \mathbb{C}$ and $a \in X$ define $\text{Con}_a X$ ($\text{Conv}_a X$) as the intersection of all closed (closed convex) cones $A \supseteq X$ with the common vertex $a$.

Now we are ready to formulate the first result of the paper.

**Theorem 1.7.** Let $X \subseteq \mathbb{C}$ be a set with a marked point $a$. If $X$ is starlike with the center $a$, then for each tangent space $\Omega^X_{a, \tilde{r}}$ there is an isometry $\psi : \Omega^X_{a, \tilde{r}} \rightarrow \text{Con}_a (X)$, $\psi(\alpha) = a$, where $\alpha = p(\tilde{a})$, see (1.5), and, moreover, every pretangent space $\Omega^X_{a, \tilde{r}}$ lies in some tangent space $\Omega^X_{a, \tilde{r}'}$.

This theorem can be rewritten in a slightly more general form.

**Theorem 1.8.** Let $X \subseteq \mathbb{C}$ be a set with a marked point $a$. Suppose that $X$ is strongly tangent equivalent (at the point $a$) to a starlike set with the center $a$. Then all pretangent spaces to $X$ at the point $a$ lie in tangent spaces and there is a closed cone $B \subseteq \mathbb{C}$ with a vertex $b$ such that for every tangent space $\Omega^X_{a, \tilde{r}}$ there exists an isometry $\psi : \Omega^X_{a, \tilde{r}} \rightarrow B$, $\psi(\alpha) = b$, where $\alpha = p(\tilde{a})$, see (1.5).

Theorem 1.7 admits a partial converse.

Let $l = l_a(b)$ be a ray with a vertex $a$, let $X \subseteq \mathbb{C}$, $a \in X$ and let $\beta > 0$. Consider the two-sided angular sector

$$\Gamma(a, l, \beta) := \{z \in \mathbb{C} : \text{dist}(z, l) \leq \beta |z - a|\}$$

(1.7)

where, as usual,

$$\text{dist}(z, l) = \inf_{w \in l} |z - w|.$$

Write

$$R(X, l, \beta) := \{|z - a| : z \in X \cap \Gamma(a, l, \beta)\},$$

(1.8)

i.e., a positive number $t$ belongs to $R(X, l, \beta)$ if and only if the sphere $S(a, t) = \{z \in X : |z - a| = t\}$ with the center $a$ and the radius $t$ and the sector $\Gamma(a, l, \beta)$ have a nonvoid intersection. In what follows we will use a porosity of the set $R(X, l, \beta)$, so recall a definition.

**Definition 1.9.** Let $A \subseteq \mathbb{R}$ and let $x \in A$. The right-side porosity of $A$ at the point $x$ is the quantity

$$p(A) := \limsup_{h \to 0} \frac{l(x, h, A)}{h}$$

(1.9)

where $l(x, h, A)$ is the length of the longest interval in $[x, x + h] \setminus A$. 

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Theorem 1.10. Let $X \subseteq \mathbb{C}$ be a set with a marked point $a$. Suppose that all pretangent spaces to $X$ at the point $a$ lie in tangent spaces and there is a closed cone $B \subseteq \mathbb{C}$ with a vertex $b$ such that for every tangent space $\Omega_{a,t}^X$ there exists an isometry $\psi : \Omega_{a,t}^X \to B$, $\psi(\alpha) = b$, where $\alpha = p(\tilde{a})$, see (1.5).

Then for every ray $l$ with the vertex $a$ we have either

$$\lim_{\beta \to 0} p(R(X,l,\beta)) = 0 \quad \text{or} \quad \lim_{\beta \to 0} p(R(X,l,\beta)) = 1.$$  \hspace{1cm} (1.10)

where $p(R(X,l,\beta))$ is the right-side porosity of $R(X,l,\beta)$ at the point $0$.

Since every convex set $X$ is starlike, Theorem 1.7 implies the following

Corollary 1.11. Let $Y$ be a convex subset of $\mathbb{C}$ with a marked point $a$ and let $\tilde{r}$ be a normalizing sequence. The following statements hold for every pretangent space $\Omega_{a,\tilde{r}}^Y$.

(i) If the space $\Omega_{a,\tilde{r}}^Y$ is tangent, then $\Omega_{a,\tilde{r}}^Y$ and $\text{Conv}_a(Y)$ are isometric.

(ii) If $\Omega_{a,\tilde{r}}^Y$ is pretangent, then $\Omega_{a,\tilde{r}}^Y$ lies in some tangent $\Omega_{a,\tilde{r}'}^Y$.

For $X = \mathbb{R}$, $X = \mathbb{R}^+ = [0,\infty]$ or $X = \mathbb{C}$ all pretangent spaces $\Omega_{a,\tilde{r}}^X$ are tangent, see Section 3 of the present paper, but for an arbitrary convex $X \subseteq \mathbb{C}$ pretangent spaces can cease to be tangent.

Recall that convex set $X$ is termed a convex body if $\text{Int}X \neq \emptyset$.

Proposition 1.12. Let $X$ be a convex body in the plane and let $a \in \partial X$. Then for every normalizing sequence $\tilde{r}$ there is a maximal self-stable family $\tilde{X}_{a,\tilde{r}}$ such that the corresponding space $\Omega_{a,\tilde{r}}$ is not tangent.

2 Auxiliary results

In this section we collect some results related to pretangent and tangent spaces of general metric spaces.

Proposition 2.1. Let $X$ be a metric space with a marked point $a$, $\tilde{r}$ a normalizing sequence and $\tilde{X}_{a,\tilde{r}}$ a maximal self-stable family with the corresponding pretangent space $\Omega_{a,\tilde{r}}$. The following statements are equivalent.

(i) $\Omega_{a,\tilde{r}}$ is tangent.

(ii) For every subsequence $\tilde{r}'$ of the sequence $\tilde{r}$ the family $\{\tilde{x}' : \tilde{x} \in \tilde{X}_{a,\tilde{r}}\}$ is maximal self-stable w.r.t. $\tilde{r}'$.

(iii) A function $\text{em}': \Omega_{a,\tilde{r}} \to \Omega_{a,\tilde{r}'}$ is surjective for every $\tilde{r}'$.

(iv) A function $\text{in}': \tilde{X}_{a,\tilde{r}} \to \tilde{X}_{a,\tilde{r}'}$ is surjective for every $\tilde{r}'$. 


For the proof see [6, Proposition 1.2] or [8, Proposition 1.5].

Let \( \tilde{F} \subseteq \tilde{X} \). For a normalizing sequence \( \tilde{r} \) we define a family \( [\tilde{F}]_Y = [\tilde{F}]_{Y,\tilde{r}} \) by the rule

\[
(\tilde{y} \in [\tilde{F}]_Y) \iff ((\tilde{y} \in \tilde{Y}) \& (\exists \tilde{x} \in \tilde{F} : \tilde{d}_{\tilde{r}}(\tilde{x}, \tilde{y}) = 0)).
\]  

(2.1)

**Proposition 2.2** ([6]). Let \( Y \) and \( Z \) be subspaces of a metric space \( X \) and let \( \tilde{r} \) be a normalizing sequence. Suppose that \( Y \) and \( Z \) are tangent equivalent (w.r.t. \( \tilde{r} \)) at a point \( a \in Y \cap Z \). Then following statements hold for every maximal self-stable (in \( \tilde{Z} \)) family \( \tilde{Z}_{a,\tilde{r}} \).

(i) The family \( [\tilde{Z}_{a,\tilde{r}}]_Y \) is maximal self-stable (in \( \tilde{Y} \)) and we have the equalities

\[
[[\tilde{Z}_{a,\tilde{r}}]_Y]_Z = \tilde{Z}_{a,\tilde{r}} = [\tilde{Z}_{a,\tilde{r}}]_Z.
\]  

(2.2)

(ii) If \( \Omega^Z_{a,\tilde{r}} \) and \( \Omega^Y_{a,\tilde{r}} \) are metric identifications of \( \tilde{Z}_{a,\tilde{r}} \) and, respectively, of \( \tilde{Y}_{a,\tilde{r}} := [\tilde{Z}_{a,\tilde{r}}]_Y \), then the mapping

\[
\Omega^Z_{a,\tilde{r}} \ni \alpha \longrightarrow [\alpha]_Y \in \Omega^Y_{a,\tilde{r}}
\]

is an isometry. Furthermore, if \( \Omega^Z_{a,\tilde{r}} \) is tangent, then \( \Omega^Y_{a,\tilde{r}} \) also is tangent.

The following lemma is a partial generalization of Proposition 2.2 (i).

**Lemma 2.3.** Let \( Z \) and \( Y \) be subspaces of a metric space \( (X, d) \), \( a \in X \cap Y \), \( \tilde{r} \) a normalizing sequence, \( \tilde{Z}_{a,\tilde{r}} \) and \( \tilde{Y}_{a,\tilde{r}} \) maximal self-stable families such that

\[
\tilde{Y}_{a,\tilde{r}} = [\tilde{Z}_{a,\tilde{r}}]_Y_{Y,\tilde{r}}.
\]  

(2.4)

Suppose \( Y \) and \( Z \) are strongly tangent equivalent at the point \( a \). Then the equality

\[
\{\tilde{z} : \tilde{z} \in \tilde{Z}_{a,\tilde{r}}\} = \{[\tilde{y} : y \in \tilde{Y}_{a,\tilde{r}}]\}_Z_{Z,\tilde{r}'}
\]  

(2.5)

holds for every subsequence \( \tilde{r}' \) of the sequence \( \tilde{r} \).

**Proof.** Let \( \tilde{r}' = \{r_{n_k}\}_{k \in \mathbb{N}} \) be a subsequence of \( \tilde{r} = \{r_n\}_{n \in \mathbb{N}} \). We first note that (2.4) and (2.2) imply the equality

\[
\tilde{Z}_{a,\tilde{r}} = [\tilde{Y}_{a,\tilde{r}}]_Z_{Z,\tilde{r}}.
\]

Consequently, if \( \tilde{z}' = \{z_{n_k}\}_{k \in \mathbb{N}} \) belongs to the set in the left-hand side of (2.5), then there is \( \tilde{y} = \{y_n\} \in \tilde{Y}_{a,\tilde{r}} \) such that

\[
\lim_{n \to \infty} \frac{d(y_n, z_n)}{r_n} = 0.
\]
Hence

$$\lim_{k \to \infty} \frac{d(y_{n_k}, z_{n_k})}{r_{n_k}} = 0.$$  \hfill (2.6)

The last equality means that $\tilde{z}'$ belongs to the set in the right-hand side of (2.5). Conversely, if $\tilde{z}' = \{z_n\}_{n \in \mathbb{N}} \in \{\{\tilde{y}': \tilde{y} \in \tilde{Y}_{a, \tilde{r}}\}\}_{Z, \tilde{r}'}$, then (2.6) holds with some $\tilde{y} = \{y_n\} \in \tilde{Y}_{a, \tilde{r}}$. Hence, by (2.4), there is $\tilde{z}_1 = \{z_n^{(1)}\}_{n \in \mathbb{N}}$ such that

$$\lim_{n \to \infty} \frac{d(y_n, z_n^{(1)})}{r_n} = 0.$$  \hfill (2.7)

Let us define $\tilde{z}_2 = \{z_n^{(2)}\}_{n \in \mathbb{N}} \in \tilde{Z}$ by the rule

$$z_n^{(2)} := \begin{cases} z_{n_k} & \text{if } n = n_k \text{ for some } k \\ z_n & \text{otherwise.} \end{cases} \hfill (2.8)$$

Limit relations (2.6) and (2.7) imply that $d_{\tilde{r}}(\tilde{z}_1, \tilde{z}_2) = 0$. Moreover, by the second equality in (2.2), we have $\tilde{z}_2 \in Z_{a, \tilde{r}}$. Consequently, by (2.8), we have $\tilde{z}' = \tilde{z}_2' \in \{\tilde{z}' : \tilde{z} \in \tilde{Z}_{a, \tilde{r}}\}$.

\medskip

**Proposition 2.4.** Let $Y$ and $Z$ be subspaces of a metric space $X$ and let $a$ be a point in $Y \cap Z$. Suppose that $Y$ and $Z$ are strongly tangent equivalent at the point $a$ and that each pretangent space $\Omega^Y_{a, \tilde{r}}$ lies in some tangent space $\Omega^Z_{a, \tilde{r}'}$. Then each pretangent space $\Omega^Z_{a, \tilde{r}}$ lies in some tangent space $\Omega^Z_{a, \tilde{r}'}$.

**Proof.** Let $\Omega^Z_{a, \tilde{r}}$ be a pretangent space to $Z$ at the point $a$ and let $\tilde{Z}_{a, \tilde{r}}$ be the corresponding maximal (in $\tilde{Z}$), self-stable family. Write

$$\tilde{Y}_{a, \tilde{r}} := [\tilde{Z}_{a, \tilde{r}}]_{Y, \tilde{r}}.$$  

Then, by Proposition 2.2 (i), $\tilde{Y}_{a, \tilde{r}}$ is maximal (in $\tilde{Y}$), self-stable family and, by the supposition, there are a subsequence $\tilde{r}'$ of $\tilde{r}$ and a maximal self-stable family $\tilde{Y}_{a, \tilde{r}'}$ such that

$$\{\tilde{y}' : \tilde{y} \in \tilde{Y}_{a, \tilde{r}}\} \subseteq \tilde{Y}_{a, \tilde{r}'}$$  \hfill (2.9)
and $\Omega^Y_{a,\tilde{r}}$, the metric identification of $\tilde{Y}_{a,\tilde{r}}$, is tangent. For this $\tilde{r}'$ consider the family $\{\tilde{z}': \tilde{z} \in \tilde{Z}_{a,\tilde{r}}\}$. By Lemma 2.3 we have equality (2.5). Write $\tilde{Z}_{a,\tilde{r}} := [\tilde{Y}_{a,\tilde{r}}]_{Z,\tilde{r}}$. It follows from (2.9) and (2.5) that $\{\tilde{z}': \tilde{z} \in Z_{a,\tilde{r}}\} \subseteq \tilde{Z}_{a,\tilde{r}}$ and, moreover, Proposition 2.2 implies that $\Omega^Z_{a,\tilde{r}}$, the metric identification of $\tilde{Z}_{a,\tilde{r}}$, is tangent.

Let $Y$ be a subspace of a metric space $(X,d)$. For $a \in Y$ and $t > 0$ we denote by

$$S_t^Y = S^Y(a,t) := \{y \in Y : d(a,y) = t\}$$

the sphere (in the subspace $Y$) with the center $a$ and the radius $t$. Similarly for $a \in Z \subseteq X$ and $t > 0$ define

$$S_t^Z = S^Z(a,t) := \{z \in Z : d(a,z) = t\}.$$ 

Write

$$\varepsilon_a(t, Z, Y) := \sup_{z \in S_t^Z} \inf_{y \in Y} d(z,y) \quad \text{and} \quad \varepsilon_a(t) := \varepsilon_a(t, Z, Y) \vee \varepsilon_a(t, Y, Z).$$

**Proposition 2.5** ([6,8]). Let $Y$ and $Z$ be subspaces of a metric space $(X,d)$ and let $a \in Y \cap Z$. Then $Y$ and $Z$ are strongly tangent equivalent at the point $a$ if and only if the equality

$$\lim_{t \to 0} \frac{\varepsilon_a(t)}{t} = 0$$

holds.

**Corollary 2.6.** Let $Y$ be a dense subset of a metric space $X$. Then $X$ and $Y$ are strongly tangent equivalent at every point $a \in Y$.

**Lemma 2.7.** Let $(X,d)$ be a metric space with a marked point $a$, $\tilde{r} = \{r_n\}_{n \in \mathbb{N}}$ a normalizing sequence and let $\tilde{X}_{a,\tilde{r}}$ be a maximal self-stable family. Then for every $\varepsilon > 0$ and every $\tilde{x} = \{x_n\} \in \tilde{X}_{a,\tilde{r}}$ with $d(\tilde{x}, \tilde{a}) > 0$ there is $n_0 \in \mathbb{N}$ such that the double inequality

$$(1 - \varepsilon)d(\tilde{x}, \tilde{a}) < \frac{d(a,x_n)}{r_n} < (1 + \varepsilon)d(\tilde{x}, \tilde{a})$$

holds for all natural numbers $n \geq n_0$.

A simple proof is omitted here.
Lemma 2.8. Let \((X,d)\) be a metric space with a marked point \(a\), \(\tilde{r} = \{r_n\}_{n \in \mathbb{N}}\) a normalizing sequence and \(\tilde{X}_{a,\tilde{r}}\) a maximal self-stable family and \(\tilde{f} = \{f_n\}_{n \in \mathbb{N}}\) a sequence of isometries \(f_n : X \to X\) with \(f_n(a) = a\) for all \(n \in \mathbb{N}\). Then the family
\[
\tilde{f}(\tilde{X}_{a,\tilde{r}}) := \left\{\{f_n(x_n)\}_{n \in \mathbb{N}} : \{x_n\}_{n \in \mathbb{N}} \in \tilde{X}_{a,\tilde{r}}\right\}
\tag{2.12}
\]
is a maximal self-stable family and, in addition, the pseudometric spaces \((\tilde{X}_{a,\tilde{r}},\tilde{d}), (\tilde{f}(\tilde{X}_{a,\tilde{r}}),\tilde{d})\) are isometric. Moreover, \(\Omega_{a,\tilde{r}}\) and \(\Omega_{a,\tilde{r}}^{\tilde{f}}\), metric identifications of \(\tilde{X}_{a,\tilde{r}}\) and, respectively, of \(\tilde{f}(\tilde{X}_{a,\tilde{r}})\), are simultaneously tangent or not.

Proof. Since
\[
\tilde{d}(\tilde{x},\tilde{y}) = \lim_{n \to \infty} \frac{d(x_n,y_n)}{r_n} = \lim_{n \to \infty} \frac{d(f_n(x_n),f_n(y_n))}{r_n},
\]
every two \(\{f_n(x_n)\}_{n \in \mathbb{N}}\) and \(\{f_n(y_n)\}_{n \in \mathbb{N}}\) are mutually stable if \(\tilde{x} = \{x_n\}_{n \in \mathbb{N}}\) and \(\tilde{y} = \{y_n\}_{n \in \mathbb{N}}\) are mutually stable and the mapping
\[
\tilde{X}_{a,\tilde{r}} \ni \tilde{x} = \{x_n\}_{n \in \mathbb{N}} \mapsto \{f_n(x_n)\}_{n \in \mathbb{N}} := \tilde{f}(\tilde{x}) \in \tilde{f}(\tilde{X}_{a,\tilde{r}})
\]
is an isometry. It is clear that \(\tilde{f}((a,a,...,a,...)) \in \tilde{f}(\tilde{X}_{a,\tilde{r}})\). Hence it suffices to show that \(\tilde{f}(\tilde{X}_{a,\tilde{r}})\) is maximal self-stable. Suppose there is \(\tilde{z} = \{z_n\}_{n \in \mathbb{N}} \in \tilde{X}\) such that \(\tilde{z} \notin \tilde{f}(\tilde{X}_{a,\tilde{r}})\) but \(\tilde{z}\) and \(\tilde{x}\) are mutually stable for all \(\tilde{x} \in \tilde{f}(\tilde{X}_{a,\tilde{r}})\). It is easy to see that \(\tilde{f}^{-1}(\tilde{z}) := \{f_n^{-1}(z_n)\}_{n \in \mathbb{N}} \notin \tilde{X}_{a,\tilde{r}}\)
where \(f_n^{-1}\) is the inverse isometry of the isometry \(f_n\) and that \(\tilde{x}\) and \(\tilde{f}^{-1}(\tilde{z})\) are mutually stable for each \(\tilde{x} \in \tilde{X}_{a,\tilde{r}}\). Hence \(\tilde{X}_{a,\tilde{r}}\) is not a maximal self-stable family, contrary to the condition of the lemma.

Suppose that \(\Omega_{a,\tilde{r}}\) is tangent. Let \(\tilde{r}' = \{r_{nk}\}_{n \in \mathbb{N}}\) be a subsequence of \(\tilde{r}\). Then, by Proposition 2.1, the family \(\{x_{nk}\}_{k \in \mathbb{N}}: \{x_n\}_{n \in \mathbb{N}} \in \tilde{X}_{a,\tilde{r}}\) is maximal self-stable. The first part of the proof implies that the family \(\{f_n(x_{nk})\}_{k \in \mathbb{N}}: \{x_n\}_{n \in \mathbb{N}} \in \tilde{X}_{a,\tilde{r}}\) is also maximal self-stable. Consequently, by Proposition 2.1 \(\Omega_{a,\tilde{r}}^{\tilde{f}'}\) is tangent. \(\blacksquare\)
3 Tangent spaces to some model metric spaces

In this section we describe tangent spaces to \( \mathbb{R}^+, \mathbb{R} \) and \( \mathbb{C} \).

Example 3.1. Let \( X = \mathbb{R} \) or \( X = \mathbb{R}^+ = [0, \infty] \) and let \( d(x, y) = \| x - y \| \). We claim that each pretangent space \( \Omega_{0,\tilde{r}}^X \) (to \( X \) at the point 0) is tangent and isometric to \((X, d)\) for all normalizing sequences \( \tilde{r} \).

Consider the more difficult case \( X = \mathbb{R} \).

Proposition 3.2. Let \( \tilde{X}_{0,\tilde{r}} \) be maximal self-stable family and let \( \tilde{b} = \{ b_n \}_{n \in \mathbb{N}} \) be an element of \( \tilde{X}_{0,\tilde{r}} \) such that
\[ \tilde{d}_{\tilde{r}}(\tilde{0}, \tilde{b}) = \lim_{n \to \infty} \frac{|b_n|}{r_n} \neq 0. \]

The following statements are true.

(i) For every \( \tilde{y} = \{ y_n \}_{n \in \mathbb{N}} \in \tilde{X}_{0,\tilde{r}} \) there is a finite limit \( \lim_{n \to \infty} \frac{y_n}{b_n} \) and, conversely, if \( \tilde{y} \in \tilde{X} \) and this limit is finite, then \( \tilde{y} \in \tilde{X}_{0,\tilde{r}} \).

(ii) For every two \( \tilde{x} = \{ x_n \}_{n \in \mathbb{N}} \) and \( \tilde{y} = \{ y_n \}_{n \in \mathbb{N}} \) from \( \tilde{X}_{0,\tilde{r}} \) the equality
\[ \tilde{d}_{\tilde{r}}(\tilde{x}, \tilde{y}) = 0 \]
holds if and only if
\[ \lim_{n \to \infty} \frac{x_n}{b_n} = \lim_{n \to \infty} \frac{y_n}{b_n}. \]

(iii) The pretangent space \( \Omega_{0,\tilde{r}}^X \) which corresponds to \( \tilde{X}_{0,\tilde{r}} \) is isometric to \((\mathbb{R}, |.|,.)\) and tangent.

Proof. (i) If \( \tilde{y} = \{ y_n \}_{n \in \mathbb{N}} \in \tilde{X}_{0,\tilde{r}} \), then there are finite limits
\[ \tilde{d}(\tilde{0}, \tilde{y}) = \lim_{n \to \infty} \frac{|y_n|}{r_n} \] and \( \tilde{d}(\tilde{b}, \tilde{y}) = \lim_{n \to \infty} \frac{|y_n - b_n|}{r_n} \).

For the case where \( \tilde{d}(\tilde{0}, \tilde{y}) = 0 \) we obtain
\[ 0 = \frac{\tilde{d}(\tilde{0}, \tilde{y})}{\tilde{d}(\tilde{0}, \tilde{b})} = \lim_{n \to \infty} \frac{|y_n|}{|b_n|} = \lim_{n \to \infty} \frac{y_n}{b_n} \]
because \( \tilde{d}(\tilde{0}, \tilde{b}) \neq 0 \). Suppose \( \tilde{d}(\tilde{0}, \tilde{y}) \neq 0 \), then we have
\[ 0 < \lim_{n \to \infty} \frac{|y_n|}{|b_n|} = \frac{\tilde{d}(\tilde{0}, \tilde{y})}{\tilde{d}(\tilde{0}, \tilde{b})} < \infty. \] (3.1)
Write for every \( t \in \mathbb{R} \)
\[
t = |t| \text{sgn}(t)
\]
where, as usual,
\[
\text{sgn}(t) = \begin{cases} 
1 & \text{if } t > 0 \\
0 & \text{if } t = 0 \\
-1 & \text{if } t < 0.
\end{cases}
\]
Then, it follows from (3.1), the limit \( \lim_{n \to \infty} y_n b_n \) exists if and only if there is the limit \( \lim_{n \to \infty} \text{sgn}(y_n) \text{sgn}(b_n) \). If the last limit does not exist, then there are two infinite sequences \( \tilde{n} = \{n_k\}_{k \in \mathbb{N}} \) and \( \tilde{m} = \{m_k\}_{k \in \mathbb{N}} \) of natural numbers such that
\[
\text{sgn}(y_{n_k}) = \text{sgn}(b_{n_k}) \quad \text{and} \quad \text{sgn}(y_{m_k}) = \text{sgn}(b_{m_k}).
\]
for all \( k \in \mathbb{N} \). Consequently, we obtain
\[
\tilde{d}(\tilde{y}, \tilde{b}) = \lim_{k \to \infty} \frac{|y_{n_k} - b_{n_k}|}{r_{n_k}} = \lim_{k \to \infty} \frac{|y_{n_k}| - |b_{n_k}|}{r_{n_k}} = |\tilde{d}(0, \tilde{y}) - \tilde{d}(0, \tilde{b})|
\]
and similarly we have
\[
\tilde{d}(\tilde{y}, \tilde{b}) = \lim_{k \to \infty} \frac{|y_{m_k} - b_{m_k}|}{r_{m_k}} = \tilde{d}(0, \tilde{y}) + \tilde{d}(0, \tilde{b}).
\]
Thus we have the equality
\[
\tilde{d}(0, \tilde{y}) + \tilde{d}(0, \tilde{b}) = |\tilde{d}(0, \tilde{y}) - \tilde{d}(0, \tilde{b})|
\]
which implies that
\[
\tilde{d}(0, \tilde{y}) \land \tilde{d}(0, \tilde{b}) = 0.
\]
It is shown that for every \( \tilde{y} \in \tilde{X}_{0, \tilde{r}} \) there is a finite limit \( \lim_{n \to \infty} \frac{y_n}{b_n} \). Conversely, let \( \tilde{y} \in \tilde{X} \) and
\[
\lim_{n \to \infty} \frac{y_n}{b_n} = c \in \mathbb{R}. \tag{3.2}
\]
We must show that for every \( \tilde{x} \in \tilde{X}_{0, \tilde{r}} \) there is a finite limit \( \lim_{k \to \infty} \frac{|y_n - b_n|}{r_n} \), i.e. \( \tilde{x} \) and \( \tilde{y} \) are mutually stable w.r.t. \( \tilde{r} \). Since \( \tilde{x} \in \tilde{X}_{0, \tilde{r}} \), we have a finite limit
\[
\lim_{n \to \infty} \frac{x_n}{b_n} = k \in \mathbb{R}. \tag{3.3}
\]
Hence
\[
\lim_{n \to \infty} \frac{|y_n - x_n|}{r_n} = \lim_{n \to \infty} \frac{|b_n|}{r_n} \left| \frac{x_n}{b_n} - \frac{y_n}{b_n} \right| = \tilde{d}(0, \tilde{b}) |c - k| \tag{3.4}
\]
where constants $c, k$ are defined by (3.2) and, respectively, by (3.3).

(ii) Statement (ii) follows from (3.4).

(iii) Statement (i) implies that the sequence $\tilde{r}^* = \{r^*_n\}_{n \in \mathbb{N}}$ with

$$r^*_n = r_n \text{sgn}(b_n), \quad n \in \mathbb{N},$$

belongs to $\tilde{X}_{0,\tilde{r}}$. If we take $\tilde{r}^*$ instead $\tilde{b}$ in (3.2) and (3.3), then we obtain the mapping $f : \tilde{X}_{0,\tilde{r}} \to \mathbb{R}$ where

$$f(\tilde{x}) = \lim_{n \to \infty} \frac{x_n}{r^*_n}, \quad \tilde{x} = \{x_n\}_{n \in \mathbb{N}} \in \tilde{X}_{0,\tilde{r}}.$$

It is easy to see that there is a unique mapping $\psi : \Omega_{0,\tilde{r}} \to \mathbb{R}$ such that the diagram

$$\begin{array}{ccc}
\tilde{X}_{0,\tilde{r}} & \xrightarrow{p} & \Omega_{0,\tilde{r}} \\
\downarrow f & & \downarrow \psi \\
\mathbb{R} & & \\
\end{array}$$

(3.5)

is commutative, where $p$ is the metric identification mapping, see (1.5). Relations (3.2)–(3.4) imply that $\psi$ is an isometry. It remains to prove that $\Omega_{a,\tilde{r}}$ is tangent. Let $\tilde{n} = \{n_k\}_{k \in \mathbb{N}}$ be a strictly increasing, infinite sequence of natural numbers and let $\tilde{r}^\prime = \{r_n\}_{k \in \mathbb{N}}$ be the corresponding subsequence of the normalizing sequence $\tilde{r}$. If $\tilde{X}_{0,\tilde{r}^\prime}$ is a maximal self-stable family such that

$$\tilde{X}_{0,\tilde{r}^\prime} \supseteq \{x' : \tilde{x} \in \tilde{X}_{0,\tilde{r}}\},$$

then, by Statement (i), for every $\tilde{x} = \{x_k\}_{k \in \mathbb{N}} \in \tilde{X}_{0,\tilde{r}}$ there is a finite limit

$$\lim_{k \to \infty} \frac{x_k}{r_{nk} \text{sgn}(b_{nk})} := p.$$

Define $\tilde{y} = \{y_n\}_{n \in \mathbb{N}} \in \tilde{X}$ by the rule

$$y_n := \begin{cases} 
  x_k & \text{if there is } n_k \text{ such that } n_k = n, \\
  r_n \text{sgn}(b_n) & \text{otherwise}.
\end{cases}$$

A simple calculation shows that

$$\lim_{n \to \infty} \frac{y_n}{r_n \text{sgn}(b_n)} = p.$$

Hence, by Statement (i), $\tilde{y}$ belongs to $\tilde{X}_{0,\tilde{r}}$. Using Proposition 2.1 we see that $\Omega_{0,\tilde{r}}$ is tangent.

\[ \square \]
Example 3.3. Let $X = \mathbb{C}$ be the set of all complex numbers with the usual metric $|.,.|$ and the marked point 0 and let $\tilde{r} = \{r_n\}_{n \in \mathbb{N}}$ be a normalizing sequence.

Proposition 3.4. Let $\tilde{X}_{0,\tilde{r}}$ be a maximal self-stable family with the corresponding pretangent space $\Omega_{0,\tilde{r}}$. Then $\Omega_{0,\tilde{r}}$ is tangent and isometric to $\mathbb{C}$.

The proof is divided into four lemmas.

Lemma 3.5. Let $\tilde{x} = \{x_n\}_{n \in \mathbb{N}}$ and $\tilde{y} = \{y_n\}_{n \in \mathbb{N}}$ be elements of $\tilde{X}_{0,\tilde{r}}$ such that

$$2 \max \left\{ \tilde{d}(0, \tilde{x}), \tilde{d}(0, \tilde{y}), \tilde{d}(\tilde{x}, \tilde{y}) \right\} < \tilde{d}(0, \tilde{x}) + \tilde{d}(0, \tilde{y}) + \tilde{d}(\tilde{x}, \tilde{y}).$$

(3.6)

Then following statements are equivalent for every $\tilde{z} = \{z_n\}_{n \in \mathbb{N}} \in \tilde{X}$:

(a) $\tilde{z}$ belongs to $\tilde{X}_{0,\tilde{r}}$;

(b) There are finite limits $\tilde{d}(\tilde{0}, \tilde{z}) = \lim_{n \to \infty} |z_n| r_n$, $\tilde{d}(\tilde{x}, \tilde{z}) = \lim_{n \to \infty} |x_n - z_n| r_n$ and $\tilde{d}(\tilde{y}, \tilde{z}) = \lim_{n \to \infty} |y_n - z_n| r_n$.

(3.7)

Proof. The implication (a)$\implies$(b) is trivial. Suppose that (b) holds with $\tilde{z} = \tilde{z}_1 = \{z_n^{(1)}\}_{n \in \mathbb{N}}$ and with $\tilde{z} = \tilde{z}_2 = \{z_n^{(2)}\}_{n \in \mathbb{N}}$. We must prove that there is a finite limit

$$\tilde{d}(\tilde{z}_1, \tilde{z}_2) = \lim_{n \to \infty} \frac{|z_n^{(1)} - z_n^{(2)}|}{r_n}. \quad (3.8)$$

Write

$$x_n := |x_n| e^{i\beta_n}, \quad y_n := |y_n| e^{i\theta_n}, \quad z_n^{(1)} := |z_n^{(1)}| e^{i\gamma_n^{(1)}}, \quad z_n^{(2)} := |z_n^{(2)}| e^{i\gamma_n^{(2)}}. \quad (3.9)$$

Since $\tilde{x}, \tilde{y} \in \tilde{X}_{0,\tilde{r}}$ and the first relation in (3.7) holds, there are finite limits

$$R_x := \lim_{n \to \infty} \frac{|x_n|}{r_n}, \quad R_y := \lim_{n \to \infty} \frac{|y_n|}{r_n}, \quad R_{1,z} := \lim_{n \to \infty} \frac{|z_n^{(1)}|}{r_n}, \quad R_{2,z} := \lim_{n \to \infty} \frac{|z_n^{(2)}|}{r_n}. \quad (3.10)$$

Consequently we have the limit relations

$$\tilde{d}(\tilde{x}, \tilde{y}) = \lim_{n \to \infty} \left| R_x e^{i\beta_n} - R_y e^{i\theta_n} \right| = \lim_{n \to \infty} \left| R_x - R_y e^{i(\theta_n - \beta_n)} \right|, \quad (3.11)$$

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\[ \tilde{d}(\tilde{z}_1, \tilde{y}) = \lim_{n \to \infty} \left| R_{1,z} e^{i(\gamma_n^{(1)} - \beta_n)} - R_y e^{i(\theta_n - \beta_n)} \right|, \]
\[ \tilde{d}(\tilde{z}_2, \tilde{y}) = \lim_{n \to \infty} \left| R_{2,z} e^{i(\gamma_n^{(2)} - \beta_n)} - R_y e^{i(\theta_n - \beta_n)} \right|, \]
\[ \tilde{d}(\tilde{x}, \tilde{z}_1) = \lim_{n \to \infty} \left| R_{1,z} e^{i(\gamma_n^{(1)} - \beta_n)} - R_x \right|, \tilde{d}(\tilde{x}, \tilde{z}_2) = \lim_{n \to \infty} \left| R_{2,z} e^{i(\gamma_n^{(2)} - \beta_n)} - R_x \right| \]
\[ \text{(3.10)} \]

and must prove the existence of
\[ \tilde{d}(\tilde{z}_1, \tilde{z}_2) = \lim_{n \to \infty} \left| R_{2,z} e^{i(\gamma_n^{(2)} - \beta_n)} - R_{1,z} e^{i(\gamma_n^{(1)} - \beta_n)} \right|. \]
\[ \text{(3.11)} \]

It is clear from (3.10), (3.11) that, without loss of generality, it is sufficient to take \( \beta_n = 0 \) for all \( n \in \mathbb{N} \).

Moreover, (3.11) evidently holds if \( R_{1,z} \cdot R_{2,z} = 0 \). Hence we may also put
\[ R_{1,z} \neq 0 \neq R_{2,z}. \]
\[ \text{(3.12)} \]

Note that (3.6) implies
\[ R_x \neq 0 \neq R_y. \]
\[ \text{(3.13)} \]

Since
\[ \left| R_x - R_y e^{i\theta_n} \right|^2 = R_x^2 + R_y^2 - 2R_x R_y \cos(\theta_n), \]
the first relation in (3.10) and (3.13) imply that there exists \( \lim_{n \to \infty} \cos(\theta_n) \) and, in addition, if follows from (3.6) that
\[ \lim_{n \to \infty} \cos(\theta_n) \neq \pm 1, \]
\[ \text{(3.14)} \]

see Remark 3.6 below. Similarly using (3.12), (3.13) and last two relations from (3.10) we see that there are limits
\[ \lim_{n \to \infty} \cos(\gamma_n^{(1)}) \text{ and } \lim_{n \to \infty} \cos(\gamma_n^{(2)}). \]
\[ \text{(3.15)} \]

The remaining relations from (3.10) imply the existence of
\[ \lim_{n \to \infty} \cos(\gamma_n^{(1)} - \theta_n) \text{ and } \lim_{n \to \infty} \cos(\gamma_n^{(2)} - \theta_n). \]
\[ \text{(3.16)} \]

Since there are limits (3.15) and
\[ \left| R_{2,z} e^{i\gamma_n^{(2)}} - R_{1,z} e^{i\gamma_n^{(1)}} \right|^2 = R_{2,z}^2 + R_{1,z}^2 - 2R_{1,z} R_{2,z} \cos(\gamma_n^{(1)} - \gamma_n^{(2)}) \]
and
\[ \cos(\gamma_n^{(1)} - \gamma_n^{(2)}) = \cos \gamma_n^{(1)} \cos \gamma_n^{(2)} + \sin \gamma_n^{(1)} \sin \gamma_n^{(2)}, \]
limit (3.11) exists if and only if there is the limit
\[ \lim_{n \to \infty} \sin \gamma_n^{(1)} \sin \gamma_n^{(2)}. \tag{3.17} \]

Using (3.16) and (3.14) we obtain
\[ \lim_{n \to \infty} \sin \gamma_n^{(1)} \sin \gamma_n^{(2)} = \lim_{n \to \infty} \frac{(\sin \gamma_n^{(1)} \sin \theta_n)(\sin \gamma_n^{(2)} \sin \theta_n)}{1 - \cos^2 \theta_n} \]
and
\[ \lim_{n \to \infty} \sin \gamma_n^{(j)} \sin \theta_n = \lim_{n \to \infty} \cos(\gamma_n^{(j)} - \theta_n) - \lim_{n \to \infty} \cos \theta_n \cos \gamma_n^{(j)} \]
for \( j = 1, 2 \). It implies the existence of (3.17).

**Remark 3.6.** Menger’s notion of betweenness is well known for the metric spaces, see, for example, [15, p. 55]. For points \( x, y, z \) belonging to a pseudometric space \((Y, d)\), we may say that \( x \) lies between \( y \) and \( z \) if
\[ d(x, z) \cdot d(y, z) \neq 0 \text{ and } d(y, z) = d(y, x) + d(x, z). \]
Suppose in Lemma 3.5 we have \( \tilde{d}(\tilde{x}_0, \tilde{x}) \cdot \tilde{d}(\tilde{x}_0, \tilde{y}) \neq 0 \), then inequality (3.6) does not hold if and only if some point from the set \( \{\tilde{x}_0, \tilde{x}, \tilde{y}\} \) lies between two other points of this set (in the pseudometric space \((\tilde{X}, \tilde{d})\)).

**Remark 3.7.** For the future it is useful to note that if \( \beta_n = 0 \) and \( \theta_n \in [0, \pi] \) for all \( n \in \mathbb{N} \), then the sequences \( \{ |y_n| e^{i \theta_n} \}_{n \in \mathbb{N}} \), \( \{ |z_n^{(1)}| e^{i \gamma_n^{(1)}} \}_{n \in \mathbb{N}} \), and \( \{ |z_n^{(2)}| e^{i \gamma_n^{(2)}} \}_{n \in \mathbb{N}} \), see, (3.9), are convergent. Indeed, the function
\[ [0, \pi] \ni t \mapsto \cos t \in [-1, 1] \]
is a homeomorphism. Hence \( \{ \theta_n \}_{n \in \mathbb{N}} \) is convergent because there is \( \lim_{n \to \infty} \cos \theta_n \).
It implies the convergence of \( \{ |y_n| e^{i \theta_n} \}_{n \in \mathbb{N}} \). Moreover, it follows from (3.14) that
\[ \lim_{n \to \infty} \sin(\theta_n) \neq 0. \]
This relation and the existence of limits (3.16), (3.15) imply the convergence of the sequences \( \{ \sin \gamma_n^{(1)} \}_{n \in \mathbb{N}} \) and \( \{ \sin \gamma_n^{(2)} \}_{n \in \mathbb{N}} \). Consequently \( \{ |z_n^{(1)}| e^{i \gamma_n^{(1)}} \}_{n \in \mathbb{N}} \) and \( \{ |z_n^{(2)}| e^{i \gamma_n^{(2)}} \}_{n \in \mathbb{N}} \) are also convergent.
Lemma 3.8. Let $X = \mathbb{C}$ be the set of all complex numbers with the usual metric $|.|$ and with the marked point $0$ and let $\tilde{r} = \{r_n\}_{n \in \mathbb{N}}$ be a normalizing sequence. Let $\tilde{X}_0, \tilde{r}$ be a maximal self-stable family for which (3.6) holds with some $\tilde{x}, \tilde{y} \in \tilde{X}_0, \tilde{r}$. Then there is a maximal self-stable family $\tilde{X}^*_0, \tilde{r} \subseteq \tilde{X}$ such that:

(i) $\tilde{X}^*_0, \tilde{r}$ and $\tilde{X}_0, \tilde{r}$ are isometric;

(ii) There are $\tilde{x}^* = \{x^*_n\}_{n \in \mathbb{N}}$ and $\tilde{y}^* = \{y^*_n\}_{n \in \mathbb{N}}$ in $\tilde{X}^*_0, \tilde{r}$ for which the inequality

$$2 \max \left\{ \tilde{d}(\tilde{0}, \tilde{x}^*), \tilde{d}(\tilde{0}, \tilde{y}^*), \tilde{d}(\tilde{x}^*, \tilde{y}^*) \right\} < \tilde{d}(\tilde{0}, \tilde{x}^*) + \tilde{d}(\tilde{0}, \tilde{y}^*) + \tilde{d}(\tilde{x}^*, \tilde{y}^*) \quad (3.18)$$

holds and

$$x^*_n = |x^*_n|, \quad y^*_n = |y^*_n| e^{i\theta_n} \quad \text{and} \quad \theta_n \in [0, \pi]$$

for all $n \in \mathbb{N}$.

Proof. Let $\tilde{x} = \{|x_n| e^{i\beta_n}\}_{n \in \mathbb{N}}$ and $\tilde{y} = \{|y_n| e^{i\beta_n}\}_{n \in \mathbb{N}}$ be elements of $\tilde{X}_0, \tilde{r}$ for which (3.6) holds. Consider the sequence $\tilde{f} = \{f_n\}_{n \in \mathbb{N}}$ of the isometries

$$f_n : \mathbb{C} \to \mathbb{C}, \quad f_n(z) = e^{-\beta_n} z.$$ 

Then we have

$$\tilde{f}(\tilde{x}) = \{|x_n|\}_{n \in \mathbb{N}} \quad \text{and} \quad \tilde{f}(\tilde{y}) = \{|y_n| e^{i(\theta_n - \beta_n)}\}_{n \in \mathbb{N}}.$$ 

We may assume that

$$-\pi < \theta_n - \beta_n \leq \pi$$

for all $n \in \mathbb{N}$. Define a new sequence $\tilde{g}$ of isometries $g_n$ by the rule

$$g_n(z) := \begin{cases} z & \text{if} \quad 0 \leq \theta_n - \beta_n \leq \pi, \\ \overline{z} & \text{if} \quad -\pi < \theta_n - \beta_n < 0. \end{cases}$$

Using Lemma 2.8 we see that the family

$$\tilde{X}^*_0, \tilde{r} := \tilde{g}(\tilde{f}(\tilde{X}_0, \tilde{r}))$$

satisfies all desirable conditions with

$$\tilde{x}^* := \{g_n(f_n(x_n))\}_{n \in \mathbb{N}} \quad \text{and} \quad \tilde{y}^* := \{g_n(f_n(y_n))\}_{n \in \mathbb{N}}.$$ 

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Lemma 3.9. Let $X = \mathbb{C}$ be the set of all complex numbers with the usual metric $|.|$ and $\tilde{r} = \{r_n\}_{n \in \mathbb{N}}$ a normalizing sequence and $\tilde{X}_{0,\tilde{r}}$ a maximal self-stable family for which the conditions of Lemma 3.5 are satisfied. If $\Omega_{0,\tilde{r}}$ is a pretangent spaces which corresponds $\tilde{X}_{0,\tilde{r}}$, then $\Omega_{0,\tilde{r}}$ is tangent and isometric to $\mathbb{C}$.

Proof. Let $\tilde{x}$ and $\tilde{y}$ be two elements of $\tilde{X}_{0,\tilde{r}}$ for which (3.6) holds. By Lemma 3.8 there exists a maximal self-stable family $\tilde{X}^*_{0,\tilde{r}} \subseteq \tilde{X}$ which is isometric to $\tilde{X}_{0,\tilde{r}}$ and contains $\tilde{x}^* = \{x^*_n\}_{n \in \mathbb{N}}$ and $\tilde{y}^* = \{y^*_n\}_{n \in \mathbb{N}}$ such that (3.18) holds and

$$x^*_n = |x^*_n|, \quad y^*_n = |y^*_n| e^{i\theta_n}, \quad \theta_n \in [0, \pi]$$

for all $n \in \mathbb{N}$. Relations (3.19) imply that for every $\tilde{z}^* = \{z^*_n\}_{n \in \mathbb{N}} \in \tilde{X}^*_{0,\tilde{r}}$ the sequence

$$\frac{\tilde{z}^*}{\tilde{r}} := \left\{ \begin{array}{c} z^*_n \\ r_n \end{array} \right\}_{n \in \mathbb{N}}$$

is convergent, see Remark 3.7. Write

$$z^* := \lim_{n \to \infty} \frac{z^*_n}{r_n}$$

for every $\tilde{z}^* = \{z^*_n\}_{n \in \mathbb{N}} \in \tilde{X}^*_{0,\tilde{r}}$. In particular, we have

$$x^* := \lim_{n \to \infty} \frac{x^*_n}{r_n} \quad \text{and} \quad y^* := \lim_{n \to \infty} \frac{y^*_n}{r_n}.$$  \hspace{1cm} (3.20)

We claim that the function

$$\tilde{X}^*_{0,\tilde{r}} \ni \tilde{z}^* \mapsto z^* \in \mathbb{C}$$

is distance-preserving and onto. (It immediately implies that $\Omega^*_{0,\tilde{r}}$, the metric identification of $\tilde{X}^*_{0,\tilde{r}}$, and $\mathbb{C}$ are isometric, so $\Omega_{0,\tilde{r}}$ also is isometric to $\mathbb{C}$.) Indeed, if $\tilde{w}^* = \{w_n\}_{n \in \mathbb{N}} \in \tilde{X}^*_{0,\tilde{r}}$, then

$$d(\tilde{w}^*, \tilde{z}^*) = \lim_{n \to \infty} \frac{|w^*_n - z^*_n|}{r_n} = \lim_{n \to \infty} \frac{|w^*_n - z^*_n|}{r_n} = |w^* - z^*|.$$

Consequently it is sufficient to show that for every $p \in \mathbb{C}$ there is $\tilde{p}^* \in \tilde{X}^*_{0,\tilde{r}}$ such that $p^* = p$. Write

$$\tilde{p}^* = \{r_n p\}_{n \in \mathbb{N}}.$$
It is clear that 
\[ p^* = \lim_{n \to \infty} \frac{r_n p}{r_n} = p, \]
thus it is enough to prove that \( \tilde{p}^* \in \tilde{X}_{0,\tilde{r}}^* \). It follows from (3.20) that
\[ \lim_{n \to \infty} \frac{|r_n x^* - x_n^*|}{r_n} = \lim_{n \to \infty} \frac{|r_n y^* - y_n^*|}{r_n} = 0. \]
Hence
\[ \tilde{d}(\tilde{p}^*, \tilde{x}^*) = \lim_{n \to \infty} \frac{|r_n p - x_n^*|}{r_n} = \lim_{n \to \infty} \frac{|r_n p - r_n x^*|}{r_n} = |p - x^*| \]
and similarly we have \( \tilde{d}(\tilde{p}^*, \tilde{y}^*) = |p - \tilde{y}^*| \). Therefore, by Lemma 3.5, \( \tilde{p}^* \in \tilde{X}_{0,\tilde{r}}^* \).

It remains to show that \( \Omega_{0,\tilde{r}}^* \) is tangent, because, by Lemma 2.8, in this case \( \Omega_{0,\tilde{r}} \) is also tangent. Let \( \tilde{r}' = \{ r_{n_k} \}_{k \in \mathbb{N}} \) be a subsequence of \( \tilde{r} \) and let \( \tilde{X}_{0,\tilde{r}'}^* \) be maximal self-stable family such that
\[ \tilde{X}_{0,\tilde{r}'}^* \supseteq \left\{ \{ x_{n_k}^* \}_{k \in \mathbb{N}} : \{ x_n^* \}_{n \in \mathbb{N}} \in \tilde{X}_{0,\tilde{r}}^* \right\}. \]
Then \( \tilde{X}_{0,\tilde{r}'}^* \) satisfies all conditions of the lemma which is being proved. Hence, similarly (3.21), we can define a function
\[ \tilde{X}_{0,\tilde{r}'}^* \ni \tilde{z}^* \mapsto f'(\tilde{z}^*) = z^* = \lim_{n \to \infty} \frac{x_k^*}{r_{n_k}} \]
for \( \tilde{z}^* = \{ z_k \}_{k \in \mathbb{N}} \in \tilde{X}_{0,\tilde{r}'}^* \). Let \( \Omega_{0,\tilde{r}'}^* \) be the metric identification of \( \tilde{X}_{0,\tilde{r}'}^* \) and let \( i_{s'} : \mathbb{C} \to \Omega_{0,\tilde{r}'}^* \) be isometries such that the diagrams
\[ \begin{array}{ccc}
\tilde{X}_{0,\tilde{r}}^* & \xrightarrow{f} & \mathbb{C} \\
\downarrow p & & \uparrow is \\
\Omega_{0,\tilde{r}}^* & \xrightarrow{i_s} & \Omega_{0,\tilde{r}}^* 
\end{array} \quad \text{ and } \quad \begin{array}{ccc}
\tilde{X}_{0,\tilde{r}'}^* & \xrightarrow{f'} & \mathbb{C} \\
\downarrow p' & & \uparrow is' \\
\Omega_{0,\tilde{r}'}^* & \xrightarrow{i_{s'}} & \Omega_{0,\tilde{r}'}^* 
\end{array} \] (3.22)
are commutative. Similarly (1.5) we can define an isometric embedding \( e_{m'} : \Omega_{0,\tilde{r}}^* \to \Omega_{0,\tilde{r}'}^* \) such that the diagram
\[ \begin{array}{ccc}
\tilde{X}_{0,\tilde{r}}^* & \xrightarrow{\text{in}_{m'}} & \tilde{X}_{0,\tilde{r}'}^* \\
\downarrow p & & \downarrow p' \\
\Omega_{0,\tilde{r}}^* & \xrightarrow{e_{m'}} & \Omega_{0,\tilde{r}'}^* 
\end{array} \] (3.23)
is commutative. We claim that the following diagram

Also is commutative. To prove the commutativity of (3.24) it is sufficient to show that

\[
em'(is(z)) = is'(z)
\]  

(3.25)

for every \(z \in \mathbb{C}\). Let \(z\) be a point of \(\mathbb{C}\). Since \(f\) is a surjection, there is \(\tilde{x} \in \tilde{X}_{0,\tilde{r}}\) such that \(z = f(\tilde{x})\). Hence, using the commutativity of diagrams (3.22)–(3.23) and the equality \(f = f' \circ \text{in}_{\tilde{r}{\prime}}\), we obtain

\[
em'(is(z)) = em'(is(f(\tilde{x}))) = em'(p(\tilde{x})) = p'(\text{in}_{\tilde{r}{\prime}}(\tilde{x})) = is'(f'(\text{in}_{\tilde{r}{\prime}}(\tilde{x}))) = is'(f(\tilde{x})) = is'(z).
\]

Consequently (3.25) holds. The diagram

also is commutative, because (3.24) is commutative. Since \(f\) and \(is'\) are surjections, \(em'\) is surjective. Hence, by Proposition 2.1, \(\Omega_{0,\tilde{r}}\) is tangent.

The following lemma shows that if \(X = \mathbb{C}\), then every maximal self-stable \(\tilde{X}_{0,\tilde{r}}\) satisfies the conditions of Lemma 3.5. It is a final part of the proof of Proposition 3.4.

**Lemma 3.10.** Let \(X = \mathbb{C}\) be the set of all complex numbers with the usual metric \(|.,.|\), let be \(\tilde{r} = \{r_n\}_{n \in \mathbb{N}}\) a normalizing sequence and let \(\tilde{X}_{0,\tilde{r}}\) be a maximal self-stable family. Then there are \(\tilde{x}, \tilde{y} \in \tilde{X}_{0,\tilde{r}}\) such that (3.6) holds, i.e.,

\[
2 \max \left\{d(0, \tilde{x}), d(0, \tilde{y}), d(\tilde{x}, \tilde{y}) \right\} < d(0, \tilde{x}) + d(0, \tilde{y}) + d(\tilde{x}, \tilde{y})
\]
Proof. Suppose that the equality
\[ 2 \max \left\{ d(0, \tilde{x}), d(0, \tilde{y}), \tilde{d}(\tilde{x}, \tilde{y}) \right\} = \tilde{d}(0, \tilde{x}) + \tilde{d}(0, \tilde{y}) + \tilde{d}(\tilde{x}, \tilde{y}) \] (3.26)
holds for all \( \tilde{x}, \tilde{y} \in \tilde{X}_{0,\tilde{r}} \).

Consider first the simplest case where \( \tilde{d}(0, \tilde{x}) = 0 \) for all \( \tilde{x} \in \tilde{X}_{0,\tilde{r}} \). The last equality implies
\[ \tilde{d}(\tilde{r}, \tilde{x}) = \lim_{n \to \infty} \frac{\tilde{d}(x_n, r_n)}{r_n} = \lim_{n \to \infty} \left| \frac{x_n}{r_n} - 1 \right| = 1 \]
for all \( \tilde{x} = \{x_n\}_{n \in \mathbb{N}} \in \tilde{X}_{0,\tilde{r}} \). Consequently \( \tilde{X}_{0,\tilde{r}} \cup \{\tilde{r}\} \) is a self-stable family, so \( \tilde{X}_{0,\tilde{r}} \) is not maximal self-stable, contrary to the conditions.

Hence there is \( \tilde{x} = \{x_n\}_{n \in \mathbb{N}} \in \tilde{X}_{0,\tilde{r}} \) such that
\[ \tilde{d}(0, \tilde{x}) = c_0 > 0. \] (3.27)
Without loss of generality we may suppose that
\[ \tilde{x} = \tilde{r}. \] (3.28)
Indeed, passing, if necessary, to an isometric \( \tilde{X}_{0,\tilde{r}}^{*} \), see Lemma 2.8, we may put \( x_n = |x_n| \) for all \( n \in \mathbb{N} \). Next, since the family
\[ c\tilde{X}_{0,\tilde{r}} = \left\{ \{cz_n\}_{n \in \mathbb{N}} : \{z_n\}_{n \in \mathbb{N}} \in \tilde{X}_{0,\tilde{r}} \right\} \]
is maximal self-stable if \( 0 \neq c \in \mathbb{C} \), we can take \( \frac{1}{c_0} \tilde{X}_{0,\tilde{r}} \) instead of \( \tilde{X}_{0,\tilde{r}} \).

Moreover, since
\[ \frac{1}{c_0} \tilde{x} := \left\{ \frac{1}{c_0} x_n \right\}_{n \in \mathbb{N}} \in \frac{1}{c_0} \tilde{X}_{0,\tilde{r}} \text{ and } \lim_{n \to \infty} \frac{1}{c_0} x_n = 1, \]
we see that \( \tilde{d}(\frac{1}{c_0} \tilde{x}, \tilde{r}) = 0 \) and, consequently,
\[ \tilde{r} \in \frac{1}{c_0} \tilde{X}_{0,\tilde{r}}. \]
The equalities (3.26) and (3.28) imply that
\[ 2 \max \left\{ 1, \tilde{d}(0, \tilde{y}), \tilde{d}(\tilde{r}, \tilde{y}) \right\} = 1 + \tilde{d}(0, \tilde{y}) + \tilde{d}(\tilde{r}, \tilde{y}) \] (3.29)
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for all \( \tilde{y} \in \tilde{X}_{0,\tilde{r}} \). There exist only the following three possibilities under which (3.29) holds:

\[
\tilde{d}(\tilde{r}, \tilde{y}) = \tilde{d}(\tilde{y}, \tilde{0}) + 1, \quad 1 = \tilde{d}(\tilde{0}, \tilde{y}) + \tilde{d}(\tilde{y}, \tilde{r}) \text{ and } \tilde{d}(\tilde{0}, \tilde{y}) = 1 + \tilde{d}(\tilde{r}, \tilde{y}).
\]

(3.30)

Write for \( \tilde{y} \in \tilde{X}_{0,\tilde{r}} \)

\[
t = t(\tilde{y}) := \begin{cases} -\tilde{d}(\tilde{y}, \tilde{0}) & \text{if } \tilde{d}(\tilde{r}, \tilde{y}) = \tilde{d}(\tilde{y}, \tilde{0}) + 1, \\ \tilde{d}(\tilde{0}, \tilde{y}) & \text{otherwise.} \end{cases}
\]

(3.31)

We claim that the limit relation

\[
\lim_{n \to \infty} \frac{|t_{n} - y_{n}|}{r_{n}} = 0
\]

(3.32)

holds for every \( \tilde{y} = \{ y_{n} \}_{n \in \mathbb{N}} \in \tilde{X}_{0,\tilde{r}} \). Indeed, fix \( \tilde{y} \in \tilde{X}_{0,\tilde{r}} \), and suppose that the first equality from (3.30) holds. (Note that (3.32) is trivial if \( \tilde{d}(\tilde{0}, \tilde{y}) = 0 \) or \( \tilde{d}(\tilde{r}, \tilde{y}) = 0 \).) Let us denote by \( y^{*} \) an arbitrary limit point of the sequence \( \{ y_{n} / r_{n} \}_{n \in \mathbb{N}} \). To prove (3.32) it is sufficient to show that

\[
y^{*} = t.
\]

(3.33)

The first equality in (3.30) and definition (3.31) imply that

\[
|y^{*}| = \tilde{d}(\tilde{y}, \tilde{0}) = |t| \quad \text{and} \quad |y^{*} - 1| = \lim_{n \to \infty} \frac{|y_{n} - r_{n}|}{r_{n}} = \tilde{d}(\tilde{r}, \tilde{y}) = 1 + \tilde{d}(\tilde{y}, \tilde{0}).
\]

Hence \( y^{*} \) belongs to the intersection of the circumferences

\[
\left\{ z \in \mathbb{C} : |z| = \tilde{d}(\tilde{0}, \tilde{y}) \right\} \quad \text{and} \quad \left\{ z \in \mathbb{C} : |z - 1| = 1 + \tilde{d}(\tilde{y}, \tilde{0}) \right\}.
\]

Since this intersection has the unique element \( t \), see Fig. 1, we obtain (3.33). Similarly we have

\[
y^{*} \in \left\{ z \in \mathbb{C} : |z| = \tilde{d}(\tilde{0}, \tilde{y}) \right\} \cap \left\{ z \in \mathbb{C} : |z - 1| = 1 - \tilde{d}(\tilde{0}, \tilde{y}) \right\} = \{ t \}
\]

if \( 1 = \tilde{d}(\tilde{0}, \tilde{y}) + \tilde{d}(\tilde{y}, \tilde{r}) \) and

\[
y^{*} \in \left\{ z \in \mathbb{C} : |z| = \tilde{d}(\tilde{0}, \tilde{y}) \right\} \cap \left\{ z \in \mathbb{C} : |z - 1| = \tilde{d}(\tilde{0}, \tilde{y}) - 1 \right\} = \{ t \}
\]

for the case \( \tilde{d}(\tilde{0}, \tilde{y}) = 1 + \tilde{d}(\tilde{y}, \tilde{z}) \), i.e., \( y^{*} = t \) for all possible cases.
Figure 1. Points $\tilde{x}, \tilde{y}$ and $\tilde{0}$ are situated on the “real axis”.

To complete the proof let us consider the family

$$\tilde{X}_{0,\tilde{r}} := \{\{cr_n\}_{n\in\mathbb{N}} : c \in \mathbb{C}\}.$$  

Since (3.32) holds for all $\tilde{y} \in \tilde{X}_{0,\tilde{r}}$, it is easy to prove that $[\tilde{X}'_{0,\tilde{r}}]_X \supseteq \tilde{X}_{0,\tilde{r}}$ where the operation $[\cdot]_X$ was defined in (2.1). Note that $[\tilde{X}'_{0,\tilde{r}}]_X$ is self-stable and that

$$\tilde{d}(i\tilde{r}, t\tilde{r}) = \lim_{n \to \infty} \frac{|i\tilde{r}_n - t\tilde{r}_n|}{\tilde{r}_n} = \sqrt{1 + t^2} \neq 0$$

for every $t \in \mathbb{R}$. Hence $i\tilde{r} \notin \tilde{X}_{0,\tilde{r}}$, contrary to the supposition about the maximality of $\tilde{X}_{0,\tilde{r}}$. \hfill \qed

4 Tangent spaces to starlike sets

Examples 3.1 and 3.3 are some particular cases of starlike sets on the plane. The next our goal is to prove Theorem 1.7 which describes tangent spaces to arbitrary starlike subsets of $\mathbb{C}$.

For convenience we repeat this theorem here.
**Theorem 4.1.** Let $Y \subseteq \mathbb{C}$ be a starlike set with a center $a$ and let $\tilde{r}$ be a normalizing sequence. Then all pretangent spaces to $Y$ at the point $a$ lie in tangent spaces and for each tangent space $\Omega^Y_{a, \tilde{r}}$ there is an isometry

$$\psi : \Omega^Y_{a, \tilde{r}} \to \text{Con}_a(Y) \quad \text{with} \quad \psi(\alpha) = a.$$  

Before proving the theorem we consider its particular cases. If $Y$ is an one-point set, then all rays (1.6) are also one-point and so $\text{Con}_a(Y) = \{a\}$. Moreover, it is easy to see that, in this case, all pretangent spaces are tangent and one-point. Hence the theorem is valid if $Y = \{a\}$. For the case when each three point of $Y$ are collinear, desirable isometry (4.1) was, in fact, constructed in the proof of Proposition 3.2, see diagram (3.5). If $\text{Int}(Y) \neq \emptyset$ and $a \in \text{Int}(Y)$, then the theorem follows from Proposition 3.4. Consequently it is enough examine the case where $a \in \partial Y$ and $Y$ contains at least three noncollinear points.

**Lemma 4.2.** Let $X \subseteq \mathbb{C}$ be a closed cone with a vertex $0$. Let $a = 0$ be a marked point of $X$, $\tilde{r} = \{r_n\}_{n \in \mathbb{N}}$ a normalizing sequence and $\bar{X}_{0, \tilde{r}}$ a maximal self-stable family. Suppose $X$ contains at least three noncollinear points and there exist $\bar{x} = \{x_n\}_{n \in \mathbb{N}}$ and $\bar{y} = \{y_n\}_{n \in \mathbb{N}}$ from $\bar{X}_{0, \tilde{r}}$ such that

$$2 \max\{\tilde{d}(\tilde{0}, \bar{x}), \tilde{d}(\tilde{0}, \bar{y}), \tilde{d}(\bar{x}, \bar{y})\} < \tilde{d}(\tilde{0}, \bar{x}) + \tilde{d}(\tilde{0}, \bar{y}) + \tilde{d}(\bar{x}, \bar{y})$$

(4.2)

holds and sequences

$$\bar{x} = \left\{\frac{x_n}{r_n}\right\}_{n \in \mathbb{N}}, \quad \bar{y} = \left\{\frac{y_n}{r_n}\right\}_{n \in \mathbb{N}}$$

(4.3)

are convergent. Then the following statements hold.

(i) If $\bar{z} = \{z_n\}_{n \in \mathbb{N}} \in \bar{X}_{0, \tilde{r}}$, then there exists a limit

$$z^* = \lim_{n \to \infty} \frac{z_n}{r_n}$$

(4.4)

and

$$z^* \in X.$$  

(4.5)

(ii) Conversely, if $\bar{z} \in \bar{X}$ and if relations (4.4), (4.5) hold, then $\bar{z} \in \bar{X}_{0, \tilde{r}}$.

(iii) The mapping $\bar{X}_{0, \tilde{r}} \xrightarrow{f} X$, $f(\bar{z}) = z^*$, is distance-preserving and onto.

**Proof.** Suppose that

$$\tilde{z} = \{|z_n|e^{i\gamma_n}\}_{n \in \mathbb{N}} \in \bar{X}_{0, \tilde{r}}.$$
Since $\tilde{x}$ and $\tilde{z}$ are mutually stable, there is a limit

$$\lim_{n \to \infty} \left| \frac{z_n - x_n}{r_n} \right|^2 = R_x^2 + R_z^2 - 2R_xR_z \lim_{n \to \infty} \cos(\gamma_n - \beta) \quad (4.6)$$

where

$$R_x = \lim_{n \to \infty} \frac{|x_n|}{r_n}, \quad R_z = \lim_{n \to \infty} \frac{|z_n|}{r_n}, \quad R_x e^{i\beta} = \lim_{n \to \infty} \frac{x_n}{r_n}.$$ 

Similarly we have

$$\lim_{n \to \infty} \left| \frac{z_n - y_n}{r_n} \right|^2 = R_y^2 + R_z^2 - 2R_yR_z \lim_{n \to \infty} \cos(\gamma_n - \theta) \quad (4.7)$$

where

$$R_y = \lim_{n \to \infty} \frac{|y_n|}{r_n} \quad \text{and} \quad R_y e^{i\theta} = \lim_{n \to \infty} \frac{y_n}{r_n}.$$ 

Consider the system

$$\begin{align*}
\cos \theta \cos \gamma_n + \sin \theta \sin \gamma_n &= \cos(\gamma_n - \theta) \\
\cos \beta \cos \gamma_n + \sin \beta \sin \gamma_n &= \cos(\gamma_n - \beta).
\end{align*} \quad (4.8)$$

The inequality (4.2) implies that

$$\begin{vmatrix}
\cos \theta & \sin \theta \\
\cos \beta & \sin \beta
\end{vmatrix} = \sin(\beta - \theta) \neq 0.$$

Hence, by Cramer’s rule, we obtain from (4.8)

$$\begin{align*}
\cos \gamma_n &= \frac{\cos(\gamma_n - \theta) \sin \beta - \cos(\gamma_n - \beta) \sin \theta}{\sin(\beta - \theta)}, \\
\sin \gamma_n &= \frac{\cos(\gamma_n - \beta) \cos \theta - \cos(\gamma_n - \theta) \cos \beta}{\sin(\beta - \theta)}.
\end{align*}$$

Consequently, the existence of limits (4.6) and (4.7) implies the existence of limit (4.4). Note also that the elements of the sequence $\tilde{z}$ are some points of $X$. Hence we have (4.5) because $X$ contains all its limit points as a closed set.

Now suppose that $\tilde{z} \in \tilde{X}$ and relations (4.4), (4.5) hold. Since $\tilde{X}_{0, \tilde{r}}$ is maximal self-stable, to prove $\tilde{z} \in \tilde{X}_{0, \tilde{r}}$ it is sufficient to show that that $\tilde{z}$ and $\tilde{w} = \{w_n\}_{n \in \mathbb{N}}$ are mutually stable for each $\tilde{w} \in \tilde{X}_{0, \tilde{r}}$. Let $\tilde{w}$ be an arbitrary element of $\tilde{X}_{0, \tilde{r}}$. Statement (i) implies that there is $w^* \in X$ such that

$$\lim_{n \to \infty} \left| \frac{w_n - r_n w^*}{r_n} \right| = 0.$$
Hence, by (4.4),
\[ d(\tilde{w}, \tilde{z}) = \lim_{n \to \infty} \frac{|z_n - w_n|}{r_n} = |z^* - w^*|, \] (4.9)
i.e., \( \tilde{z} \) and \( \tilde{w} \) are mutually stable.

To prove Statement (iii) note that (4.9) means that the function
\[ \tilde{X}_0, \tilde{r} \ni \tilde{z} \mapsto z^* \in X \]
is distance-preserving. Moreover, Statement (ii) implies that for every \( z^* \in X \) we have \( \{z^*r_n\}_{n \in \mathbb{N}} \in \tilde{X}_0, \tilde{r} \), i.e., \( f \) is onto. \( \square \)

A modification of the proof of Lemma 3.10 gives the following.

**Lemma 4.3.** Let \( X \) be a set from Lemma 4.2. Then for every maximal self-stable \( \tilde{X}_0, \tilde{r} \) there are \( \tilde{r}' \) and \( \tilde{X}_0, \tilde{r}' \) such that
\[ \tilde{X}_0, \tilde{r} \supseteq \{ \tilde{x}' : \tilde{x} \in \tilde{X}_0, \tilde{r} \} \]
and (4.2) holds for some \( \tilde{x}, \tilde{y} \in \tilde{X}_0, \tilde{r}' \).

The following proposition is a model case of Theorem 4.1.

**Proposition 4.4.** Let \( X \) be a set from Lemma 4.2. Then the conclusion of Theorem 4.1 is valid for every \( \Omega_{0, \tilde{r}} \).

**Proof.** Let \( \tilde{r} \) be a normalizing sequence and let \( \tilde{X}_0, \tilde{r} \) be a maximal self-stable family with a corresponding pretangent space \( \Omega_{0, \tilde{r}} \). By Lemma 4.3 we may suppose, passing, if necessary, to a subsequence of \( \tilde{r} \), that there exist \( \tilde{x} = \{x_n\}_{n \in \mathbb{N}} \) and \( \tilde{y} = \{y_n\}_{n \in \mathbb{N}} \) in \( \tilde{X}_0, \tilde{r} \) such that (4.2) holds. Since the sequences
\[ \left\{ \frac{x_n}{r_n} \right\}_{n \in \mathbb{N}}, \left\{ \frac{y_n}{r_n} \right\}_{n \in \mathbb{N}} \]
are bounded, there is a subsequences \( \tilde{r}' = \{r_{n_k}\}_{k \in \mathbb{N}} \) of sequence \( \tilde{r} \) such that
\[ \left\{ \frac{x_{n_k}}{r_{n_k}} \right\}_{k \in \mathbb{N}}, \left\{ \frac{y_{n_k}}{r_{n_k}} \right\}_{k \in \mathbb{N}} \]
are convergent. Let \( \tilde{X}_0, \tilde{r}' \) be a maximal self-stable family such that
\[ \tilde{X}_0, \tilde{r}' \supseteq \{ \tilde{x}' : \tilde{x} \in \tilde{X}_0, \tilde{r} \} \]. (4.10)
Write $Ω_{0,F'}$ for the metric identifications of $X_{0,F'}$. We claim that $Ω_{0,F'}$ is tangent. Indeed, since all suppositions of Lemma 4.2 are valid, a limit

$$z^* = \lim_{k \to \infty} z_{n_k}$$

exists for every $\tilde{z} = \{ z_k \}_{k \in \mathbb{N}} \in \tilde{X}_{0,F'}$ and the mapping

$$\tilde{X}_{0,F'} \ni \tilde{z} \mapsto f' \colon z^* \in X$$

(4.11)

is distance-preserving and onto. Hence there is an isometry $is' : X \rightarrow Ω_{0,F'}$ such that the diagram

$$\begin{array}{ccc}
\tilde{X}_{0,F'} & \xrightarrow{f'} & X \\
p' \downarrow & & \downarrow is' \\
Ω_{0,F'} & & 
\end{array}$$

(4.12)

is commutative. In particular we have $is'(0) = p'(\tilde{0})$. Similarly, for every infinite subsequence $\tilde{r}''$ of $\tilde{r}'$ and for every maximal self-stable

$$\tilde{X}_{0,F''} \ni \tilde{x} \mapsto \tilde{x}' \in \tilde{X}_{0,F}$$

there are a distance-preserving surjection $f'' : \tilde{X}_{0,F''} \rightarrow X$ and an isometry $is'' : X \rightarrow Ω_{0,F''}$ with the commutative diagram

$$\begin{array}{ccc}
\tilde{X}_{0,F} & \xrightarrow{f''} & X \\
p'' \downarrow & \downarrow is'' \\
Ω_{0,F''} & & 
\end{array}$$

and with $is''(0) = p''(\tilde{0})$, where $p''$ is the metric identification mapping of the pseudometric space $\tilde{X}_{0,F''}$. As in the case of diagram (3.23) we can find an isometric embedding $em'' : Ω_{0,F''} \rightarrow Ω_{0,F''}$ such that $em'' \circ p'' = p'' \circ in''$ where $in''(\tilde{x}') = (\tilde{x}')' \in \tilde{X}_{0,F''}$ for $\tilde{x}' \in \tilde{X}_{0,F'}$. Repeating the proof of the
commutativity of (3.24) we see that the diagram

\[
\begin{array}{ccc}
\tilde{X}_{0,\tilde{r}} & \xrightarrow{in''} & \tilde{X}_{0,\tilde{r}''} \\
p' \downarrow & f' & f'' \\
X & \quad & \tilde{r}' \\
\Omega_{0,\tilde{r}} & \xrightarrow{em''} & \Omega_{0,\tilde{r}''}
\end{array}
\]  

is also commutative. Hence \(em''\) is surjective because \(f'\) and \(is''\) are surjections. Consequently, by Proposition 2.1, \(\Omega_{0,\tilde{r}}\) is tangent and, as was shown above, for the isometry \(is' : X \to \Omega_{0,\tilde{r}}\) we have \(is'(0) = \alpha\) where \(\alpha = p'(\tilde{0})\).

Thus, by definition, \(\Omega_{0,\tilde{r}}\) lies in the tangent space \(\Omega_{0,\tilde{r}'}\).

Suppose now that \(\Omega_{0,\tilde{r}}\), the metric identification of \(\tilde{X}_{0,\tilde{r}}\), is tangent. To prove the existence of an isometry

\[
\psi : \Omega_{0,\tilde{r}} \to X \quad \text{with} \quad \psi(\alpha) = 0,
\]

consider, as in the first part of the present proof, a maximal self-stable family \(\tilde{X}_{0,\tilde{r}}\) such that inclusion (4.10) holds and diagram (4.12) is commutative for function (4.11). Since \(is'\) is an isometry, the commutativity of (4.12) implies \(f' = (is')^{-1} \circ p'\) where \((is')^{-1}\) is the inverse function of \(is'\). Then combining the last equality with (4.13) we obtain the commutative diagram

\[
\begin{array}{ccc}
\tilde{X}_{0,\tilde{r}} & \xrightarrow{in''} & \tilde{X}_{0,\tilde{r}''} \\
p \downarrow & f' & \quad \qquad \quad \tilde{r}' \\
\Omega_{0,\tilde{r}} & \xrightarrow{em'} & \Omega_{0,\tilde{r}'}
\end{array}
\]

Write \(\psi = (is')^{-1} \circ em'\). For tangent spaces the mapping \(em'\) is an isometry, so \(\psi\) is an isometry as a superposition of two isometries. Note that the commutativity of diagram (4.14) implies the equality \(\psi(\alpha) = 0\) for \(\alpha = p(\tilde{0})\).

**Lemma 4.5.** Let \(Y \subseteq \mathbb{C}\) be a starlike set with a center \(a\) and let \(X := \text{Conv}_a(Y)\). Then \(X\) and \(Y\) are strongly tangent equivalent at the point \(a\).

**Proof.** If \(Y = \{a\}\) then \(X = \{a\}\) and this lemma is trivial. Consequently, we may suppose that \(a\) is a limit point of \(Y\).
Let us denote by $\hat{Y}$ the smallest (but not necessarily closed) cone with the vertex $a$ and such that $\hat{Y} \supseteq Y$. Then we have the equality
\[
\text{Cl}(\hat{Y}) = X \tag{4.15}
\]
where $\text{Cl}(\hat{Y})$ is the closure of $\hat{Y}$ in $\mathbb{C}$. Indeed, it is easy to prove that $\text{Cl}(\hat{Y})$ is a closed cone. Consequently, the inclusion $\text{Cl}(\hat{Y}) \supseteq \text{Con}_a(Y) = X$ holds. On the other hand $\text{Con}_a(Y)$ is a cone. Hence $\text{Con}_a(Y) \supseteq \hat{Y}$. It implies
\[
X = \text{Cl}(\text{Con}_a(Y)) \supseteq \text{Cl}(\hat{Y})
\]
and (4.15) follows.

Write for $t > 0$
\[
S^Y_t := \{ y \in Y : |a - y| = t \} \quad \text{and} \quad S^X_t := \{ x \in X : |a - x| = t \},
\]
i.e., $S^Y_t$ and $S^X_t$ are the spheres in $Y$ and, respectively, in $X$ with the center $a$ and radius $t$.

Since $Y \subseteq X$ it is sufficient, by Proposition 2.5, to prove that
\[
\lim_{t \to 0} \frac{\varepsilon_a(t, X, Y)}{t} = 0 \tag{4.16}
\]
where
\[
\varepsilon_a(t, X, Y) = \sup_{x \in S^X_t} \inf_{y \in Y} |x - y|.
\]
If equality (4.16) does not hold then there exists $\delta_0 > 0$ such that
\[
\limsup_{t \to 0} \frac{\varepsilon_a(t, X, Y)}{t} = \delta_0. \tag{4.17}
\]
Equality (4.16) implies that
\[
\text{Cl}(S^Y_1) = S^X_1 \tag{4.18}
\]
where $S^Y_1 = \{ y \in \hat{Y} : |y - a| = 1 \}$. Since $S^X_1$ is a compact subset of $\mathbb{C}$, it follows from (4.18) that there is a finite $\frac{\delta_0}{2}$-net $\{y_1, \ldots, y_n\} \subseteq S^Y_1$ for the set $S^X_1$. The starlikeness of $Y$ implies that there is $\gamma > 0$ such that the implication
\[
(|z - a| < \gamma) \Rightarrow (z \in Y)
\]
is true for every point $z \in \bigcap_{i=1}^n l_a(y_i)$ where $l_a(y_i)$ are rays starting from $a$ and passing through $y_i$, see (1.6). Hence for all $t \in ]0, \gamma[$ we have the inequality
\[
\frac{\varepsilon_a(t, X, Y)}{t} < \frac{\delta_0}{2},
\]
contrary to (4.17). \(\square\)
Proof of Theorem 4.1. Without loss of generality we may put \( a = 0 \). Let \( \tilde{Y}_{a, \tilde{a}} \) be a maximal self-stable family and let \( \Omega^X_{a, \tilde{a}} \) be the corresponding pretangent space. Write \( X = \text{Con}_a(Y) \). Then by Lemma 4.5 the sets \( X \) and \( Y \) are strongly tangent equivalent at the point \( a \). It follows from Proposition 4.4 that every pretangent space \( \Omega^X_{a, \tilde{a}} \) lies in some tangent \( \Omega^X_{a, \tilde{a}}' \). Consequently, using Proposition 2.4, we have that \( \Omega^X_{a, \tilde{a}} \) lies in some tangent space \( \Omega^X_{a, \tilde{a}}' \).

Suppose now that \( \Omega^X_{a, \tilde{a}} \) is tangent. Write \( \tilde{X} := [\tilde{Y}_{a, \tilde{a}}]_X \). Then Statement (ii) of Proposition 2.2 implies that \( \Omega^X_{a, \tilde{a}}, \) the metric identification of \( \tilde{X}_{a, \tilde{a}} \), also is tangent. Hence, by Proposition 4.4 there is an isometry

\[
\psi^X : \Omega^X_{a, \tilde{a}} \to X, \quad \psi^X(\alpha) = a,
\]

with

\[
\tilde{X}_{a, \tilde{a}} \ni \tilde{a} = (a, \ldots, a, \ldots) \overset{p}{\to} \alpha \in \Omega^X_{a, \tilde{a}}.
\]

where \( p \) is the projection of \( \tilde{X}_{a, \tilde{a}} \) on \( \Omega^X_{a, \tilde{a}} \). Statement (ii) of Proposition 2.2 implies that the mapping

\[
\Omega^Y_{a, \tilde{a}} \ni \alpha \overset{\nu}{\to} [\alpha]_X \in \Omega^X_{a, \tilde{a}}
\]

is an isometry. It is easy to see that the mapping \( \Omega^Y_{a, \tilde{a}} \overset{\nu}{\to} \Omega^X_{a, \tilde{a}} \overset{\psi^X}{\to} X \) is an isometry with the desirable properties. \( \square \)

In the next proof we use the notation from the proof of Lemma 4.5.

Proof of Corollary 1.11. It follows from the first part of Theorem 1.7 that we must only to prove the equality

\[
\text{Con}_a(Y) = \text{Conv}_a(Y)
\]

(4.19)

for convex sets \( Y \subseteq C \) and \( a \in Y \). To prove this, note that the cone \( \hat{Y} \) is convex for the convex \( Y \), see, for example, [16, Chapter I, §2, Corollary 2.6.3] and that \( \text{Cl}(\hat{Y}) \) also is convex [16, Chapter II, §6, Theorem 6.1]. Moreover, as has been stated in the proof of Lemma 4.5 the closure of every cone is a cone. Hence

\[
\text{Cl}(\hat{Y}) \supseteq \text{Conv}_a(Y).
\]

This inclusion and equality (4.15) imply the inclusion \( \text{Con}_a(Y) \supseteq \text{Conv}_a(Y) \). Since the reverse inclusion \( \text{Conv}_a(Y) \supseteq \text{Con}_a(Y) \) is trivial, we obtain (4.19). \( \square \)
Proof of Theorem 1.8. Let $Z$ be a starlike set with the center $a$ such that $Z$ and $X$ are strongly tangent equivalent at the point $a$. To prove the theorem under consideration we can repeat the proof of Theorem 4.1 using $Z$ instead of $\text{Con}_a(Y)$, $X$ instead of $Y$ and Theorem 4.1 in place of Proposition 4.4. 

Lemma 4.6. Let $X \subseteq \mathbb{C}$ be a set with a marked point $a$ and let $l$ be a ray with the vertex $a$. Then we have the equality

$$
\lim_{\beta \to 0} p(R(X, l, \beta)) = \lim_{\beta \to 0} p(R(\text{Cl}(X), l, \beta)).
$$

(4.20)

Proof. To prove (4.20) note that

$$
p(R(X, l, \beta)) \geq p(R(\text{Cl}(X), l, \beta))
$$

(4.21)

because $X \subseteq \text{Cl}(X)$. On the other hand, Definition 1.9 implies the equality

$$
p(A) = p(\text{Cl}(A))
$$

for every $A \subseteq \mathbb{R}$. Consequently,

$$
p(R(X, l, \beta)) = p(R(\text{Cl}(X, l, \beta))).
$$

(4.22)

Applying the well-known criterion, see [9, Proposition 2.1.15], we see that a distance function

$$
\mathbb{C} \ni x \mapsto |x - a| \in \mathbb{R}
$$

is closed. The characteristic property of closed maps, [14, Chapter 1, §13, XIV, formula (7)] implies

$$
\text{Cl}(R(X, l, \beta)) = \{|z - a| : z \in \text{Cl}(X \cap \Gamma(a, l, \beta))\}.
$$

(4.23)

Moreover, it is easy to see that

$$
\text{Cl}(X \cap \Gamma(a, l, \beta)) \supseteq \text{Cl}(X) \cap \Gamma(a, l, \gamma)
$$

(4.24)

for every $\gamma < \beta$. Relations (4.22)–(4.24) imply the inequality

$$
p(R(X, l, \beta)) \leq p(R(\text{Cl}(X), l, \gamma))
$$

for every $\gamma < \beta$. For example we have

$$
p(R(X, l, \beta)) \leq p(R(\text{Cl}(X), l, \frac{\beta}{1 + \beta})).
$$

Letting $\beta \to 0$ in the last inequality and in (4.21) we obtain (4.20). 

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Lemma 4.7. Let $X \subseteq \mathbb{C}$ be a set with a marked point $a$, $l$ a ray with the vertex $a$, $\tilde{r} = \{r_n\}_{n \in \mathbb{N}}$ a normalizing sequence, $\beta_0$ a positive constant and let $\tilde{x}, \tilde{y}$ belong to $X$, $\tilde{x} = \{x_n\}_{n \in \mathbb{N}}$, $\tilde{y} = \{y_n\}_{n \in \mathbb{N}}$. Suppose the following conditions are satisfied:

(i) The family $\{\tilde{x}, \tilde{y}, \tilde{a}\}$ is self-stable w.r.t. $\tilde{r}$;

(ii) The point $\tilde{x}$ lies between $\tilde{a}$ and $\tilde{y}$, i.e.,
\[
\tilde{d}_r(\tilde{a}, \tilde{y}) = \tilde{d}_r(\tilde{a}, \tilde{x}) + \tilde{d}_r(\tilde{x}, \tilde{y}) \quad \text{and} \quad \tilde{d}_r(\tilde{a}, \tilde{x}) \tilde{d}_r(\tilde{x}, \tilde{y}) \neq 0;
\]  (4.25)

(iii) For every $\beta > 0$ there is $n_0 \in \mathbb{N}$ such that $x_n \in \Gamma(a, l, \beta)$ for all $n \geq n_0$.

Then there is $m_0 \in \mathbb{N}$ such that $y_m \in \Gamma(a, l, \beta_0)$ for all $m \geq m_0$.

Proof. If the conclusion of the lemma does not hold, then there is a strictly increasing sequence $\{n_k\}_{k \in \mathbb{N}}$ of natural numbers such that
\[
y_{n_k} \in \mathbb{C} \setminus \Gamma(a, l, \beta_0)
\]  (4.26)
for all $n_k$. Condition (i) implies that $\{\frac{y_{n_k}}{r_{n_k}}\}_{k \in \mathbb{N}}$ is a bounded sequence. Hence, passing if necessary to a subsequence, we may suppose that $\tilde{y}'$ is convergent. Condition (iii) and the existence of $d(\tilde{a}, \tilde{x})$ imply that $\{\frac{x_{n_k}}{r_{n_k}}\}_{k \in \mathbb{N}}$ also is convergent and
\[
x^* := \lim_{k \to \infty} \frac{x_{n_k}}{r_{n_k}} \in l.
\]  (4.27)
Write
\[
y^* := \lim_{k \to \infty} \frac{y_{n_k}}{r_{n_k}}.
\]
It follows from (4.26) that
\[
y^* \in \mathbb{C} \setminus \text{Int}(\Gamma(a, l, \beta_0))
\]  (4.28)
where $\text{Int}(\Gamma(a, l, \beta_0))$ is the interior of the sector $\Gamma(a, l, \beta)$. Using (4.27) and (4.28) it is easy to show that
\[
|a - y^*| < |a - x^*| + |x^* - y^*|,
\]
see Fig. 2. The last inequality contradicts (4.25) because $\tilde{d}_r(\tilde{a}, \tilde{y}) = |a - x^*|$, $\tilde{d}_r(\tilde{a}, \tilde{y}) = |a - y^*|$ and $\tilde{d}_r(\tilde{x}, \tilde{y}) = |x^* - y^*|$. □
**Proof of Theorem 1.10.** Firstly, note that without loss of generality we may assume $X$ to be closed. Indeed, by Corollary 2.6, the sets $X$ and $Cl(X)$ are strongly tangent equivalent for every $a \in X$, so using Statement $(ii)$ of Proposition 2.2 we see that for $X$ and for $Cl(X)$ the supposition of Theorem 1.10 is true (or false) simultaneously. Analogously, using Lemma 4.6 we can replace $X$ by $Cl(X)$ in the conclusion of Theorem 1.10.

Suppose there is a ray $l = l_a(b)$ such that

$$1 > \lim_{\beta \to 0} p(R(X, l, \beta)) := \gamma_0 > 0.$$  \hspace{1cm} (4.29)

Since function $p(R(X, l, \beta))$ is decreasing in $\beta$, Definition 1.9 implies that for every $k \in ]0, 1[$ there is $\beta_0 > 0$ with

$$\gamma_0 \geq p_0 := p(R(X, l, \beta_0)) = \limsup_{h \to 0} \frac{l(0, h, R(X, l, \beta_0))}{h} > k\gamma_0 \quad (4.30)$$

where $l(0, h, R(X, l, \beta_0))$ is the length of the longest interval in $[0, h] \setminus R(X, l, \beta_0)$. Consequently, there is a decreasing sequence $\{h_n\}_{n \in \mathbb{N}}$, $h_n > 0$, with $\lim_{n \to \infty} h_n = 0$ such that

$$\lim_{n \to \infty} \frac{l(0, h_n, R(X, l, \beta_0))}{h_n} = p_0. \quad (4.31)$$

Write $]r_n, t_n[$ for the longest open interval in $[0, h_n] \setminus R(X, l, \beta_0)$. If $r_{m_0} = 0$ for some $m_0 \in \mathbb{N}$, then (4.29) does not hold. Thus we may suppose $r_m > 0$.

---

**Figure 2.** The point $x^*$ cannot lie between $a$ and $y^*$. 
for all $m \in \mathbb{N}$. Let $\{\beta_n\}_{n \in \mathbb{N}}$ be a decreasing sequence of positive numbers with $\lim_{n \to \infty} \beta_n = 0$. Write

$$\tau_n := \sup \{d(x,a) : x \in B(a,r_n) \cap X \cap \Gamma(a,l,\beta_n)\}$$

(4.32)

where $B(a,r_n) = \{x \in X : |x-a| \leq r_n\}$. The previous reasoning gives the strict inequality

$$\tau_n > 0 \quad (4.33)$$

for all $n \in \mathbb{N}$. Since $X$ is closed and $B(a,r_n)$ is compact, there is a sequence $\tilde{x} = \{x_n\}_{n \in \mathbb{N}}$ such that

$$|x_n - a| = \tau_n \quad (4.34)$$

and

$$x_n \in B(a,\tau_n) \cap X \cap \Gamma(a,l,\beta_n) \quad (4.35)$$

for all $n \in \mathbb{N}$. Let us obtain now some estimations for $\lim_{m \to \infty} r_m \tau_m$ and for $\lim_{m \to \infty} t_m \tau_m$. Using (4.31) and the definition of intervals $[r_m - t_m]$ and of the porosity of $R(X,l,\beta_0)$ we see that

$$p_0 \geq \limsup_{m \to \infty} \frac{|r_m - t_m|}{t_m} \geq \liminf_{m \to \infty} \frac{|r_m - t_m|}{h_m} = p_0,$$

that is

$$p_0 = \lim_{m \to \infty} \frac{|r_m - t_m|}{t_m} \quad (4.36)$$

and so

$$\lim_{m \to \infty} \frac{r_m}{t_m} = 1 - p_0. \quad (4.37)$$

Moreover, since the inclusion

$$R(X,l,\beta_n) \supseteq R(X,l,\beta_{n+1})$$

holds for all $n \in \mathbb{N}$, we obtain the inequality

$$\limsup_{m \to \infty} \frac{|r_m - t_m|}{t_m} \leq p(R(X,l,\beta_n)) \quad (4.38)$$

for all $n$. Letting $n \to \infty$ we have

$$\limsup_{m \to \infty} \frac{|r_m - t_m|}{t_m} \leq \gamma_0.$$

Passing to a subsequence we may suppose that there exists a limit

$$\lim_{m \to \infty} \frac{|r_m - t_m|}{t_m} \leq \gamma_0. \quad (4.39)$$
It is clear that $|r_m - t_m| \leq |\tau_m - t_m|$ for all $m \in \mathbb{N}$. Hence, by (4.30), (4.36) and by (4.39),

$$k\gamma_0 < \lim_{m \to \infty} \frac{|r_m - t_m|}{t_m} \leq \lim_{m \to \infty} \frac{|\tau_m - t_m|}{t_m} \leq \gamma_0.$$  \hfill (4.40)

By the construction we have $\tau_m \leq r_m \leq t_m$. Thus (4.40) implies

$$k\gamma_0 < 1 - \lim_{m \to \infty} \frac{\tau_m}{t_m} \leq \gamma_0$$  \hfill (4.41)

or, in an equivalent form,

$$\frac{1}{1 - k\gamma_0} < \lim_{m \to \infty} \frac{t_m}{\tau_m} \leq \frac{1}{1 - \gamma_0}.$$  \hfill (4.42)

Similarly we can rewrite (4.41) as

$$k\gamma_0 < \lim_{m \to \infty} \frac{t_m - r_m}{t_m} + \lim_{m \to \infty} \frac{r_m - \tau_m}{t_m} \leq \gamma_0.$$  \hfill (4.43)

From this, using (4.36), (4.37), we obtain

$$k\gamma_0 < p_0 + \lim_{m \to \infty} \frac{r_m - \tau_m}{1 - p_0} \leq \gamma_0$$

and, after simple calculations,

$$\frac{1 - p_0}{1 - k\gamma_0} < \lim_{m \to \infty} \frac{r_m}{\tau_m} \leq \frac{1 - p_0}{1 - \gamma_0}.$$  \hfill (4.43)

Note that the inequality

$$\frac{1 - p_0}{1 - \gamma_0} < \frac{1}{1 - k\gamma_0}$$  \hfill (4.44)

holds if

$$1 > k > \frac{1}{2}(\gamma_0 + 1).$$  \hfill (4.45)

Indeed, inequality (4.44) is equivalent to

$$p_0 + k\gamma_0 - kp_0\gamma_0 > \gamma_0.$$ \hfill (4.46)

By (4.30) we have $p_0 > k\gamma_0$. Consequently, to prove (4.46) it is sufficient to show

$$2k\gamma_0 - kp_0\gamma_0 > \gamma_0$$

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that is equivalent to $2k > kp_0 + 1$ because $\gamma_0 > 0$. But $kp_0 < p_0 \leq \gamma_0$, see (4.30), so (4.45) implies (4.44).

Let us take the sequence $\tilde{\tau} = \{\tau_n\}_{n \in \mathbb{N}}$ as a normalizing sequence. Let $\tilde{X}_{a, \tilde{\tau}}$ be a maximal self-stable family such that $\tilde{x} \in \tilde{X}_{a, \tilde{\tau}}$ and let $\Omega^X_{a, \tilde{\tau}}$ be the corresponding pretangent spaces. The supposition of the theorem which is being proved, implies that there is a subsequence $\{n_k\}_{k \in \mathbb{N}}$ of natural numbers such that $\Omega^X_{a, \tilde{\tau}}$ lies in tangent space $\Omega^X_{a, \tilde{\tau}'}$, $\tilde{\tau}' = \{\tau_{n_k}\}_{k \in \mathbb{N}}$. Replacing $n$ by $n_k$ in (4.34), (4.35) we may assume that $\Omega^X_{a, \tilde{\tau}'} = \Omega^X_{a, r}$, i.e. $\Omega^X_{a, \tilde{\tau}'}$ is tangent. Write $\mu := p(\tilde{x})$ where $p$ is the metric identification mapping of $\tilde{X}_{a, \tilde{\tau}}$ and where $\tilde{x} = \{x_n\}_{n \in \mathbb{N}}$ was defined by (4.34), (4.35). It follows from (4.44) and from the supposition of the theorem that there is $\tilde{y} = \{y_n\}_{n \in \mathbb{N}} \in \tilde{X}_{a, \tilde{\tau}}$ with

$$1 = \tilde{d}_\tilde{\tau}(\tilde{x}, \tilde{a}) \leq \frac{1 - p_0}{1 - \gamma_0} < \frac{1}{1 - k\gamma_0} \quad (4.47)$$

and such that $\mu$ lies between $\alpha := p(\tilde{a})$ and $\xi := p(\tilde{y})$. Since all conditions of Lemma 4.7 are satisfied by the triple $\tilde{x}, \tilde{y}, \tilde{a}$, there is $n_0 \in \mathbb{N}$ such that

$$y_n \in \Gamma(a, l, \beta_0) \quad (4.48)$$

for all $n \geq n_0$. Lemma 2.7 and relation (4.47) imply that there is $\varepsilon_0 > 0$ such that the double inequality

$$(1 + \varepsilon_0)\tau_n \frac{1 - p_0}{1 - \gamma_0} < d(a, y_n) < \frac{\tau_n}{1 - k\gamma_0} \quad (4.49)$$

holds for all sufficiently large $n$. Moreover, it follows from (4.42), (4.43) that there is $N(\varepsilon) \in \mathbb{N}$ such that

$$\frac{\tau_n}{1 - k\gamma_0} < t_n < (1 + \varepsilon_0)\frac{\tau_n}{1 - \gamma_0} \quad (4.50)$$

and

$$\frac{\tau_n}{1 - k\gamma_0} < r_n < (1 + \varepsilon_0)\frac{\tau_n(1 - p_0)}{1 - \gamma_0} \quad (4.51)$$

for all $n \geq N(\varepsilon)$. The left inequality in (4.49) and the right one in (4.51) give $r_n < d(a, y_n)$. Similarly, from the right inequality in (4.49) and from the left one in (4.50) we obtain $d(a, y_n) < t_n$. Thus we have

$$d(a, y_n) \in ]r_n, t_n[ \quad (4.52)$$

for sufficiently large $n$. In addition (4.48) implies

$$d(a, y_n) \in R(X, l, \beta_0). \quad (4.53)$$

Relations (4.52), (4.53) contradict the definition of intervals $]r_n, t_n[$. Hence double inequality (4.29) does not hold and the theorem follows. \qed
Example 4.8. Let \( X = \{re^{i\varphi} \in \mathbb{C} : r \in \mathbb{R}^+ \) and \( \varphi \in [\theta_1, \theta_2] \}, \) \( 0 < \theta_1 - \theta_2 \leq \pi \) be the closed convex cone, \( a = 0 \) a marked point of \( X \) and \( \tilde{r} = \{r_n \}_n \in \mathbb{N} \) a normalizing sequence. Write for all \( n \in \mathbb{N} \)

\[
  z_n := \begin{cases} 
  r_ne^{i\theta_1} & \text{if } n \text{ is odd} \\
  r_ne^{i\theta_3} & \text{if } n \text{ is even}
  \end{cases}
\]  

(4.54)

where a number \( \theta_3 \) belongs to \( [\theta_1, \theta_2] \).

Let \( \tilde{X}_{0, \tilde{r}} \ni \tilde{z} = \{z_n \}_n \in \mathbb{N} \) be a maximal self-stable family with the corresponding pretangent space \( \Omega_{0, \tilde{r}} \). We claim that \( \Omega_{0, \tilde{r}} \) is not tangent.

Indeed, suppose that \( \Omega_{0, \tilde{r}} \) is tangent. Write \( \tilde{r}^{(1)} = \{r_{2n+1} \}_n \in \mathbb{N} \) and \( \tilde{r}^{(2)} = \{r_{2n} \}_n \in \mathbb{N} \). By Statement \((ii)\) of Proposition 2.1 the families

\[
  \tilde{X}_{0, \tilde{r}}^{(1)} := \{\{x_{2n+1} \}_n : \{x_n \}_n \in \tilde{X}_{0, \tilde{r}}\}
\]

and

\[
  \tilde{X}_{0, \tilde{r}}^{(2)} := \{\{x_{2n} \}_n : \{x_n \}_n \in \tilde{X}_{0, \tilde{r}}\}
\]

are also maximal self-stable and corresponding spaces \( \Omega_{0, \tilde{r}}^{(1)}, \Omega_{0, \tilde{r}}^{(2)} \) are tangent. Passing, if necessary, to subsequences we may suppose that for every \( \tilde{x} = \{x_n \}_n \in \tilde{X}_{0, \tilde{r}} \) there are limits

\[
  f^{(1)}(\tilde{x}) := \lim_{n \to \infty} \frac{x_{2n+1}}{r_{2n+1}} \quad \text{and} \quad f^{(2)}(\tilde{x}) := \lim_{n \to \infty} \frac{x_{2n}}{r_{2n}}
\]  

(4.55)

Let \( \lfloor \tilde{X}_{0, \tilde{r}} \rfloor \) be a system of distinct representatives of the factor space \( \Omega_{0, \tilde{r}} \), i.e., for every \( \tilde{x} \in \tilde{X}_{0, \tilde{r}} \) there is a unique \( \tilde{y} \in \lfloor \tilde{X}_{0, \tilde{r}} \rfloor \) such that \( d(\tilde{x}, \tilde{y}) = 0 \). Then, by Statement \((iii)\) of Lemma 4.2 the mappings

\[
  \lfloor \tilde{X}_{0, \tilde{r}} \rfloor \ni \tilde{x} \mapsto f^{(1)}(\tilde{x}) \in X \quad \text{and} \quad \lfloor \tilde{X}_{0, \tilde{r}} \rfloor \ni \tilde{x} \mapsto f^{(2)}(\tilde{x}) \in X
\]

are isometries, where \( f^{(1)}(\tilde{x}) \) and \( f^{(2)}(\tilde{x}) \) are defined by (4.55). Therefore, we have

\[
  f^{(1)}(\tilde{z}) = e^{i\theta_1} \quad \text{and} \quad f^{(2)}(\tilde{z}) = e^{i\theta_3}.
\]

Hence, there is an isometry \( \psi : X \to X \) such that

\[
  \psi(e^{i\theta_3}) = e^{i\theta_1}.
\]  

(4.56)

Note that \( e^{i\theta_3} \in \text{Int}X \) and \( e^{i\theta_1} \in \partial X \). Consequently, equality (4.56) contradicts the Brouwer Theorem on the invariance of the open sets.

This example and Statement \((ii)\) of Proposition 2.2 imply Proposition 1.12.

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