An equivalent representation of the Jacobi field of a Lévy process

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Abstract

In [8], the Jacobi field of a Lévy process was derived. This field consists of commuting self-adjoint operators acting in an extended (interacting) Fock space. However, these operators have a quite complicated structure. In this note, using ideas from [1, 17], we obtain a unitary equivalent representation of the Jacobi field of a Lévy process. In this representation, the operators act in a usual symmetric Fock space and have a much simpler structure.

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1 Lévy process and its Jacobi field

The notion of a Jacobi field in the Fock space first appeared in the works by Berezansky and Koshmanenko [3, 7], devoted to the axiomatic quantum field theory, and then was further developed by Brüning (see e.g. [10]). These works, however, did not contain any relations with probability measures. A detailed study of a general commutative Jacobi field in the Fock space and a corresponding spectral measure was carried out in a series of works by Berezansky, see e.g. [3, 5] and the references therein.

In [8] (see also [10, 18]), the Jacobi field of a Lévy process on a general manifold was studied. Let us shortly recall these results.

Let X be a complete, connected, oriented $C^\infty$ (non-compact) Riemannian manifold and let $\mathcal{B}(X)$ be the Borel $\sigma$-algebra on X. Let $\sigma$ be a Radon measure on $(X, \mathcal{B}(X))$ that is non-atomic and non-degenerate (i.e., $\sigma(O) > 0$ for any open set $O \subset X$). As a typical example of measure $\sigma$, one can take the volume measure on X.
We denote by $D$ the space $C_0^\infty(X)$ of all infinitely differentiable, real-valued functions on $X$ with compact support. It is known that $D$ can be endowed with a topology of a nuclear space. Thus, we can consider the standard nuclear triple

$$D \subset L^2(X, \sigma) \subset D',$$

where $D'$ is the dual space of $D$ with respect to the zero space $L^2(X, \sigma)$. (Here and below, all the linear spaces we deal with are real.) The dual pairing between $\omega \in D'$ and $\varphi \in D$ will be denoted by $\langle \omega, \varphi \rangle$. We denote the cylinder $\sigma$-algebra on $D'$ by $C(D')$.

Let $\nu$ be a measure on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ whose support contains an infinite number of points and assume $\nu(\{0\}) = 0$. Let

$$\tilde{\nu}(ds) := s^2 \nu(ds).$$

We further assume that $\tilde{\nu}$ is a finite measure on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$, and moreover, there exists an $\varepsilon > 0$ such that

$$\int_\mathbb{R} \exp\left(\varepsilon |s|\right) \tilde{\nu}(ds) < \infty. \quad (1)$$

We now define a centered Lévy process on $X$ (without Gaussian part) as a generalized process on $D'$ whose law is the probability measure $\mu$ on $(D', C(D'))$ given by its Fourier transform

$$\int_{D'} e^{i \langle \omega, \varphi \rangle} \mu(d\omega) = \exp\left(\int_{\mathbb{R} \times X} (e^{is\varphi(x)} - 1 - is\varphi(x)) \nu(ds) \sigma(dx)\right), \quad \varphi \in D. \quad (2)$$

Thus, $\nu$ is the Lévy measure of the Lévy process $\mu$. Without loss of generality, we can suppose that $\tilde{\nu}$ is a probability measure on $\mathbb{R}$. (Indeed, if this is not the case, define $\nu' := c^{-1} \nu$ and $\sigma' := c \sigma$, where $c := \tilde{\nu}(\mathbb{R})$.)

It follows from (1) that the measure $\tilde{\nu}$ has all moments finite, and furthermore, the set of all polynomials is dense in $L^2(\mathbb{R}, \tilde{\nu})$. Therefore, by virtue of (2), there exists a unique (infinite) Jacobi matrix

$$J = \begin{pmatrix}
a_0 & b_1 & 0 & 0 & 0 & \cdots \\
b_1 & a_1 & b_2 & 0 & 0 & \cdots \\
0 & b_2 & a_2 & b_3 & 0 & \cdots \\
0 & 0 & b_3 & a_3 & b_4 & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{pmatrix}, \quad a_n \in \mathbb{R}, \ b_n > 0,$n}

whose spectral measure is $\tilde{\nu}$.

Next, we denote by $\mathcal{P}(D')$ the set of continuous polynomials on $D'$, i.e., functions on $D'$ of the form $F(\omega) = \sum_{i=0}^n \langle \omega^{\otimes i}, f_i \rangle$, $\omega^{\otimes 0} := 1$, $f_i \in D^{\otimes i}$, $i = 0, \ldots, n$, $n \in \mathbb{Z}_+$. 

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Here, \( \hat{\otimes} \) stands for symmetric tensor product. The greatest number \( i \) for which \( f^{(i)} \neq 0 \) is called the power of a polynomial. We denote by \( \mathcal{P}_n(D') \) the set of continuous polynomials of power \( \leq n \).

By (11), (12), and [13, Sect. 11], \( \mathcal{P}(D') \) is a dense subset of \( L^2(D', \mu) \). Let \( \mathcal{P}^\sim_n(D') \) denote the closure of \( \mathcal{P}_n(D') \) in \( L^2(D', \mu) \), let \( \mathcal{P}_n(D') \), \( n \in \mathbb{N} \), denote the orthogonal difference \( \mathcal{P}_n^\sim(D') \ominus \mathcal{P}^\sim_{n-1}(D') \), and let \( \mathcal{P}_0(D') := \mathcal{P}^\sim_0(D') \). Then, we evidently have:

\[
L^2(D', \mu) = \bigoplus_{n=0}^{\infty} \mathcal{P}_n(D').
\] (3)

The set of all projections \( \langle \cdot \otimes f_n, \rangle \) of continuous monomials \( \langle \cdot \otimes f_n, \rangle, f_n \in \mathcal{D}^\otimes \), onto \( \mathcal{P}_n(D') \) is dense in \( \mathcal{P}_n(D') \). For each \( n \in \mathbb{N} \), we define a Hilbert space \( \mathfrak{F}_n \) as the closure of the set \( \mathcal{D}^\otimes \) in the norm generated by the scalar product

\[
(f_n, g_n)_{\mathfrak{F}_n} := \frac{1}{n!} \int_{D'} \langle \omega^\otimes, f_n \rangle : \langle \omega^\otimes, g_n \rangle: \mu(d\omega), \quad f_n, g_n \in \mathcal{D}^\otimes.\] (4)

Denote

\[
\mathfrak{F} := \bigoplus_{n=0}^{\infty} \mathfrak{F}_n n!,
\] (5)

where \( \mathfrak{F}_0 := \mathbb{R} \). By (3)–(5), we get the unitary operator

\[
\mathcal{U} : \mathfrak{F} \rightarrow L^2(D', \mu)
\]

that is defined through \( \mathcal{U} f_n := \langle \cdot \otimes f_n, \rangle \), \( f_n \in \mathcal{D}^\otimes \), \( n \in \mathbb{Z}_+ \), and then extended by linearity and continuity to the whole space \( \mathfrak{F} \).

An explicit formula for the scalar product \( \langle \cdot, \cdot \rangle_{\mathfrak{F}_n} \) looks as follows. We denote by \( \mathbb{Z}^+ \) the set of all sequences \( \alpha \) of the form

\[
\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n, 0, 0, \ldots), \quad \alpha_i \in \mathbb{Z}_+, \ n \in \mathbb{N}.
\]

Let \( |\alpha| := \sum_{i=1}^{\infty} \alpha_i \). For each \( \alpha \in \mathbb{Z}^+, 1\alpha_1 + 2\alpha_2 + \cdots = n, \ n \in \mathbb{N} \), and for any function \( f_n : X^n \rightarrow \mathbb{R} \) we define a function \( D_\alpha f_n : X^{|\alpha|} \rightarrow \mathbb{R} \) by setting

\[
(D_\alpha f_n)(x_1, \ldots, x_{|\alpha|}) := f(x_1, \ldots, x_{\alpha_1}, x_{\alpha_1+1}, x_{\alpha_1+2}, \ldots, x_{\alpha_1+\alpha_2}, x_{\alpha_1+\alpha_2+1}, \ldots, x_{\alpha_1+\alpha_2+1}, x_{\alpha_1+\alpha_2+2}, \ldots) \quad (2 \text{ times})
\]

\[
\ldots \quad (3 \text{ times})
\]

We have (cf. [15]):
Theorem 1 For any $f^{(n)}, g^{(n)} \in \mathcal{D}^{\hat{\otimes} n}$, we have:

$$
(f^{(n)}, g^{(n)})_{\mathfrak{F}_n} = \sum_{\alpha \in \mathbb{Z}_{+ \infty}^+ : 1\alpha_1 + 2\alpha_2 + \cdots = n} K_\alpha \int_{X^{\mid \alpha \mid}} (D_\alpha f_n)(x_1, \ldots, x_{\mid \alpha \mid}) \\
\times (D_\alpha g_n)(x_1, \ldots, x_{\mid \alpha \mid}) \sigma^{\mid \alpha \mid}(dx_1, \ldots, dx_{\mid \alpha \mid}),
$$

where

$$
K_\alpha = \frac{(1\alpha_1 + 2\alpha_2 + \cdots)!}{\alpha_1! \alpha_2! \cdots} \prod_{k \geq 2} \left( \frac{\prod_{i=1}^{k-1} b_i}{k!} \right)^{2\alpha_k}.
$$

(6)

Next, we find the elements which belong to the space $\mathfrak{F}_n$ after the completion of $\mathcal{D}^{\hat{\otimes} n}$. To this end, we define, for each $\alpha \in \mathbb{Z}_{+ \infty}^+$, the Hilbert space

$$
L^2_{\alpha}(X^{\mid \alpha \mid}, \sigma^{\mid \alpha \mid}) := L^2(X, \sigma)^{\hat{\otimes} \alpha_1} \otimes L^2(X, \sigma)^{\hat{\otimes} \alpha_2} \otimes \cdots.
$$

Define a mapping

$$
U^{(n)} : \mathcal{D}^{\hat{\otimes} n} \rightarrow \bigoplus_{\alpha \in \mathbb{Z}_{+ \infty}^+ : 1\alpha_1 + 2\alpha_2 + \cdots = n} L^2_{\alpha}(X^{\mid \alpha \mid}, \sigma^{\mid \alpha \mid}) K_\alpha
$$

by setting, for each $f^{(n)} \in \mathcal{D}^{\hat{\otimes} n}$, the $L^2_{\alpha}(X^{\mid \alpha \mid}, \sigma^{\mid \alpha \mid}) K_\alpha$-coordinate of $U^{(n)} f^{(n)}$ to be $D_\alpha f^{(n)}$. By virtue of Theorem 1, $U^{(n)}$ may be extended by continuity to an isometric mapping of $\mathfrak{F}_n$ into

$$
\bigoplus_{\alpha \in \mathbb{Z}_{+ \infty}^+ : 1\alpha_1 + 2\alpha_2 + \cdots = n} L^2_{\alpha}(X^{\mid \alpha \mid}, \sigma^{\mid \alpha \mid}) K_\alpha.
$$

Furthermore, we have (cf. [9, 16]):

Theorem 2 The mapping

$$
U^{(n)} : \mathfrak{F}_n \rightarrow \bigoplus_{\alpha \in \mathbb{Z}_{+ \infty}^+ : 1\alpha_1 + 2\alpha_2 + \cdots = n} L^2_{\alpha}(X^{\mid \alpha \mid}, \sigma^{\mid \alpha \mid}) K_\alpha
$$

is a unitary operator.

By virtue of Theorem 2 and (5), we can identify $\mathfrak{F}_n$ with the space

$$
\bigoplus_{\alpha \in \mathbb{Z}_{+ \infty}^+ : 1\alpha_1 + 2\alpha_2 + \cdots = n} L^2_{\alpha}(X^{\mid \alpha \mid}, \sigma^{\mid \alpha \mid}) K_\alpha
$$

and the space $\mathfrak{F}$ with

$$
\bigoplus_{\alpha \in \mathbb{Z}_{+ \infty}^+} L^2_{\alpha}(X^{\mid \alpha \mid}, \sigma^{\mid \alpha \mid}) K_\alpha (1\alpha_1 + 2\alpha_2 + \cdots)!.
$$
For a vector \( f \in \mathcal{F} \), we will denote its \( \alpha \)-coordinate by \( f_\alpha \).

Note that, for for \( \alpha = (n, 0, 0, \ldots) \), we have

\[
L^2_\alpha(X^{\lvert \alpha \rvert}, \sigma^\lvert \alpha \rvert) = L^2(X, \sigma) \hat{\otimes} \mathcal{F}_n, \quad K_\alpha = 1, \quad (1\alpha_1 + 2\alpha_2 + \cdots)! = n!.
\]

Hence, the space \( \mathcal{F} \) contains the symmetric Fock space

\[
\mathcal{F}(L^2(X, \sigma)) = \bigoplus_{n=0}^{\infty} L^2(X, \sigma) \hat{\otimes} \mathcal{F}_n
\]

as a proper subspace. Therefore, we call \( \mathcal{F} \) an extended Fock space. We also note that the space \( \mathcal{F} \) satisfies the axioms of an interacting Fock space, see [11].

In the space \( L^2(D', \mu) \), we consider, for each \( \varphi \in D \), the operator \( M(\varphi) \) of multiplication by the function \( \langle \cdot, \varphi \rangle \). Let \( J(\varphi) := UM(\varphi)U^{-1} \). Denote by \( \mathcal{F}_{\text{fin}}(D) \) the set of all vectors of the form \( (f_0, f_1, \ldots, f_n, 0, 0, \ldots) \), \( f_i \in D \hat{\otimes} i \), \( i = 0, \ldots, n \), \( n \in \mathbb{Z}_+ \). Evidently, \( \mathcal{F}_{\text{fin}}(D) \) is a dense subset of \( \mathcal{F} \). We have the following theorem, see [3].

**Theorem 3** For any \( \varphi \in D \), we have:

\[
\mathcal{F}_{\text{fin}}(D) \subset \text{Dom}(J(\varphi)), \quad J(\varphi) \upharpoonright \mathcal{F}_{\text{fin}}(D) = J^+(\varphi) + J^0(\varphi) + J^-(\varphi).
\]

Here, \( J^+(\varphi) \) is the usual creation operator:

\[
J^+(\varphi)f_n = \varphi \hat{\otimes} f_n, \quad f_n \in D \hat{\otimes} n, \quad n \in \mathbb{Z}_+.
\]

Next, for each \( f^{(n)} \in D \hat{\otimes} n \), \( J^0(\varphi)f^{(n)} \in \mathcal{F}_n \) and

\[
(J^0(\xi)f^{(n)})_\alpha(x_1, \ldots, x_{\lvert \alpha \rvert}) = \sum_{k=1}^{\infty} \alpha_k a_{k-1} S_{\alpha}(\xi(x_{\alpha_1 + \cdots + \alpha_k})(D_{\alpha f^{(n)}})(x_1, \ldots, x_{\lvert \alpha \rvert}))
\]

\( \sigma^{\otimes \lvert \alpha \rvert} \)-a.e., \( \alpha \in \mathbb{Z}^\infty_+, \quad 1\alpha_1 + 2\alpha_2 + \cdots = n \),

\[
J^-(\xi)f^{(n)} = 0 \text{ if } n = 0, \quad J^-(\xi)f^{(n)} \in \mathcal{F}_{n-1} \text{ if } n \in \mathbb{N} \text{ and }
\]

\[
(J^-(\xi)f^{(n)})_\alpha(x_1, \ldots, x_{\lvert \alpha \rvert}) = nS_{\alpha} \left( \int_X \xi(x)(D_{\alpha_1 1 f^{(n)}})(x_1, \ldots, x_{\lvert \alpha \rvert}) \sigma(dx) \right)
\]

\( \sigma^{\otimes \lvert \alpha \rvert} \)-a.e., \( \alpha \in \mathbb{Z}^\infty_+, \quad 1\alpha_1 + 2\alpha_2 + \cdots = n - 1 \).
In formulas (9) and (10), we denoted by $S_\alpha$ the orthogonal projection of $L^2(X^{[\alpha]}, \sigma^{\otimes [\alpha]})$ onto $L^2_\alpha(X^{[\alpha]}, \sigma^{\otimes [\alpha]})$, 

$$\alpha \pm 1_n := (\alpha_1, \ldots, \alpha_{n-1}, \alpha_n \pm 1, \alpha_{n+1}, \ldots), \quad \alpha \in \mathbb{Z}_{+0}, \ n \in \mathbb{N}.$$ 

Finally, each operator $J(\varphi), \varphi \in \mathcal{D}$, is essentially self-adjoint on $\mathcal{F}_{\text{fin}}(\mathcal{D})$.

By (7), the operator $J(\varphi) \upharpoonright \mathcal{F}_{\text{fin}}(\mathcal{D})$ is a sum of creation, neutral, and annihilation operators, and hence $J(\varphi) \upharpoonright \mathcal{F}_{\text{fin}}(\mathcal{D})$ has a Jacobi operator’s structure. The family of operators $(J(\varphi))_{\varphi \in \mathcal{D}}$ is called the Jacobi field corresponding to the Lévy process $\mu$.

2 An equivalent representation

As shown in [4, 13, 15, 16], in some cases, the formulas describing the operators $J^0(\varphi)$ and $J^-(\varphi)$ can be significantly simplified. However, in the case of a general Lévy process this is not possible, see [8]. We will now present a unitarily equivalent description of the Jacobi field $(J(\varphi))_{\varphi \in \mathcal{D}}$, which will have a simpler form.

Let us consider the Hilbert space $\ell_2$ spanned by the orthonormal basis $(e_n)_{n=0}^\infty$ with 

$$e_n = (0, \ldots, 0, \underbrace{1}_{(n+1)\text{-st place}}, 0, 0 \ldots).$$

Consider the tensor product $\ell_2 \otimes L^2(X, \sigma)$, and let 

$$\mathcal{F}(\ell_2 \otimes L^2(X, \sigma)) = \bigoplus_{n=0}^\infty (\ell_2 \otimes L^2(X, \sigma))^{\otimes n} n!$$

be the (usual) symmetric Fock space over $\ell_2 \otimes L^2(X, \sigma)$.

Denote by $\ell_{2,0}$ the dense subset of $\ell_2$ consisting of all finite vectors, i.e.,

$$\ell_{2,0} := \{(f^{(n)})_{n=0}^\infty : \exists N \in \mathbb{Z}_+ \text{ such that } f^{(n)} = 0 \text{ for all } n \geq N\}.$$ 

The Jacobi matrix $J$ determines a linear symmetric operator in $\ell_2$ with domain $\ell_{2,0}$ by the following formula:

$$Je_n = b_{n+1}e_{n+1} + a_ne_n + b_ne_{n-1}, \quad n \in \mathbb{Z}_+, \ e_{-1} := 0.$$ 

Denote by $J^+, J^0, J^-$ the corresponding creation, neutral, and annihilation operators in $\ell_{2,0}$, so that $J = J^+ + J^0 + J^-.$

Denote by $\Phi$ the linear subspace of $\mathcal{F}(\ell_2 \otimes L^2(X, \sigma))$ that is the linear span of the vacuum vector $(1, 0, 0, \ldots)$ and vectors of the form $(\xi \otimes \varphi)^{\otimes n}$, where $\xi \in \ell_{2,0}$, $\varphi \in \mathcal{D}$, and $n \in \mathbb{N}$. The set $\Phi$ is evidently a dense subset of $\mathcal{F}(\ell_2 \otimes L^2(X, \sigma))$. 

6
Now, for each $\varphi, \psi \in \mathcal{D}$ and $\xi \in \ell_{2,0}$, we set

\[
A^+(\varphi)(\xi \otimes \psi)^{\otimes n} := (e_0 \otimes \varphi) \hat{\otimes} (\xi \otimes \psi)^{\otimes n} + n((J^+ \xi) \otimes (\varphi \psi)) \hat{\otimes} (\xi \otimes \varphi)^{\otimes (n-1)},
\]

\[
A^0(\varphi)(\xi \otimes \psi)^{\otimes n} := n((J^0 \xi) \otimes (\varphi \psi)) \hat{\otimes} (\xi \otimes \varphi)^{\otimes (n-1)},
\]

\[
A^-(\varphi)(\xi \otimes \psi)^{\otimes n} := n(\xi, e_0)(\varphi, \psi)(\xi \otimes \varphi)^{\otimes (n-1)} + n((J^- \xi) \otimes (\varphi \psi)) \hat{\otimes} (\xi \otimes \varphi)^{\otimes (n-1)},
\]

and then extend these operators by linearity to the whole $\Phi$. Thus,

\[
A^+(\varphi) = a^+(e_0 \otimes \varphi) + a^0(J^+ \otimes \varphi),
\]

\[
A^0(\varphi) = a^0(J^0 \otimes \varphi),
\]

\[
A^-(\varphi) = a^-(e_0 \otimes \varphi) + a^0(J^- \otimes \varphi),
\]

where $a^+(\cdot), a^0(\cdot), a^-(\cdot)$ are the usual creation, neutral, and annihilation operators in $\mathcal{F}^0(\ell_2 \otimes L^2(X, \sigma))$. (In fact, under, e.g., $a^0(J^+ \otimes \varphi)$ we understand the differential second quantization of the operator $J^+ \otimes \varphi$ in $\ell_2 \otimes L^2(X, \sigma)$, which, in turn, is the tensor product of the operator $J^+$ in $\ell_2$ defined on $\ell_{2,0}$ and the operator of multiplication by $\varphi$ in $L^2(X, \sigma)$ defined on $\mathcal{D}$.) Note also that

\[
A(\varphi) := A^+(\varphi) + A^0(\varphi) + A^-(\varphi)
\]

\[
= a^+(e_0 \otimes \varphi) + a^0((J^+ + J^0 + J^-) \otimes \varphi) + a^-(e_0 \otimes \varphi)
\]

\[
= a^+(e_0 \otimes \varphi) + a^0(J \otimes \varphi) + a^-(e_0 \otimes \varphi).
\]

In the following theorem, for a linear operator $A$ in a Hilbert space $H$, we denote by $\overline{A}$ its closure (if it exists).

**Theorem 4** There exists a unitary operator

\[
I : \mathfrak{g} \to \mathcal{F}(\ell_2 \otimes L^2(X, \sigma))
\]

for which the following assertions hold. Let $J^+(\varphi), J^0(\varphi)$, and $J^-(\varphi), \varphi \in \mathcal{D}$, be linear operators in $\mathfrak{g}$ with domain $\mathcal{F}_{\text{fin}}(\mathcal{D})$ as in Theorem 3. Then, for all $\varphi \in \mathcal{D}$,

\[
IJ^+(\varphi)I^{-1} = A^+(\varphi),
\]

\[
IJ^0(\varphi)I^{-1} = A^0(\varphi),
\]

\[
IJ^-(\varphi)I^{-1} = A^-(\varphi),
\]

and

\[
IJ(\varphi)I^{-1} = A(\varphi).
\]
Remark 1 Note, however, that the image of $F_n$ under $I$ does not coincide with the subspace $(\ell_2 \otimes L^2(X,\sigma))^\otimes n$ of the Fock space $\mathcal{F}(\ell_2 \otimes L^2(X,\sigma))$.

The proof of Theorem 4 is a straightforward generalization of the proof of Theorem 1 in [17], so we only outline it.

First, we recall the classical unitary isomorphism between the usual Fock space over $L^2(\mathbb{R} \times X, \nu \otimes \sigma)$ and $L^2(D',\mu)$:

$$U_1 : \mathcal{F}(L^2(\mathbb{R} \times X, \nu \otimes \sigma)) \rightarrow L^2(D',\mu).$$

This isomorphism was proved by Itô [12] and extended in [16] to a general Lévy process on $X$. Note also that

$$L^2(\mathbb{R} \times X, \nu \otimes \sigma) = L^2(\mathbb{R}, \nu) \otimes L^2(X,\sigma).$$

Next, we have the unitary operator

$$U_2 : L^2(\mathbb{R},\tilde{\nu}) \rightarrow L^2(\mathbb{R},\nu)$$

defined by

$$(U_2f)(s) := \frac{1}{s} f(s).$$

Let

$$U_3 : \ell_2 \rightarrow L^2(\mathbb{R},\tilde{\nu})$$

be the Fourier transform in generalized joint eigenvectors of the Jacobi matrix $J$, see [2]. The $U_3$ can be characterized as a unique unitary operator which maps the vector $(1,0,0,\ldots)$ into the function identically equal to 1, and which maps the closure $\overline{J}$ of the symmetric operator $J$ in $\ell_2$ into the multiplication operator by the variable $s$.

Let

$$U_4 : \ell_2 \otimes L^2(X,\sigma) \rightarrow L^2(\mathbb{R} \times X, \nu \otimes \sigma)$$

be given by

$$U_4 := (U_2 U_3) \otimes \text{id},$$

where id denotes the identity operator. Using $U_4$, we naturally construct the unitary operator

$$U_5 : \mathcal{F}(\ell_2 \otimes L^2(X,\sigma)) \rightarrow \mathcal{F}(L^2(\mathbb{R} \times X, \nu \otimes \sigma)).$$

We now define a unitary operator

$$I := U U_1^{-1} U_5^{-1} : \mathcal{F} \rightarrow \mathcal{F}(\ell_2 \otimes L^2(X,\sigma)).$$

Then, the assertions of Theorem 4 about the unitary operator $I$ follow from Theorem 3, the construction of the unitary operator $I$ (see in particular Theorem 3.1 in [16], [12]), and a limiting procedure.
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