Rigid spheres in Riemannian spaces

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Abstract
We define a special family of topological two-spheres, which we call ‘rigid spheres’, and prove that there is a four-parameter family of rigid spheres in a generic Riemannian three-manifold (in case of the flat Euclidean three-space these four parameters are: three coordinates of the center and the radius of the sphere). The rigid spheres can be used as building blocks for various (‘spherical’, ‘bispherical’ etc) foliations of the Cauchy space. This way a supertranslation ambiguity may be avoided. Generalization to the full four-dimensional case is discussed. Our results generalize both the Huang foliations (cf [4]) and the foliations used by us (cf [8]) in the analysis of the two-body problem.

1. Introduction

In General Relativity Theory, the amount of gravitational energy (mass) contained in a portion \(V \subset \Sigma\) of a Cauchy three-surface \(\Sigma\) is assigned to its boundary \(S = \partial V\), rather than to the volume \(V\) itself (cf the notion of a ‘quasi-local’ mass introduced by Penrose, [10]). The above philosophy was also used in [11], where important quasi-local observables (like, e.g., momentum, angular momentum or center of mass) assigned to a generic two-dimensional (2D) surface (whose topology is that of \(S^2\)) have acquired a Hamiltonian interpretation as generators of the corresponding canonical transformations of the (appropriately defined) phase space of gravitational initial data. Recently, we were able to define energy contained in an asymptotically Schwarzschild–de Sitter spacetime (cf [12]), and again the quasi-local, Hamiltonian description of the field dynamics provided an adequate starting point for our analysis.
Typically, the $S^2$-spheres used for the quasi-local purposes come from specific spacetime foliations \( \{ t = \text{const.}; \ r = \text{const.} \} \), where a specific choice of coordinates \( t \) and \( r \) plays the role of a gauge. In literature, gauge conditions based on three-dimensional (3D)-elliptic problems have been mostly used (see e.g. ‘traceless-transversal’ condition advocated by York (see e.g. [14]) or a ‘p-harmonic gauge’ analyzed in [2]). Important results have been obtained by Huisken and Ilmanen (cf [5]) who used a parabolic gauge condition imposed for the radial coordinate \( r \). The same gauge was also used by one of us (Jezierski, see [6]) to prove stability of the Reissner–Nordström solution, together with a version of Penrose’s inequality.

For purposes of the quasi-local analysis, these approaches exhibit an obvious drawback consisting in the fact that we do not control intrinsic properties of the surfaces \( \{ r = \text{const.} \} \) constructed this way. This feature was partially removed by Huang in [4], where new 3D foliations were thoroughly analyzed. Their fibers \( \{ r = \text{const.} \} \) are selected by a 2D-elliptic condition: \( k = \text{const.} \), where \( k \) denotes the mean extrinsic curvature. In a generic Riemannian three-manifold \( \Sigma \), the above equation admits a one-parameter family of ‘spheres’. Physically, they are related to the ‘center of mass’ of the geometry (cf [4]). Unfortunately, the above condition is not stable with respect to small perturbations of the geometry. Indeed, in the (flat) Euclidean space \( E^3 \), this condition admits not ‘one-’ but a four-parameter family of solutions (parameterized e.g. by the radius \( R \) and the three coordinates of a center). Moreover, the exclusive use of the center of mass reference frame is often too restrictive for physical applications. In particular, it does not allow us to describe easily the momentum—i.e. the generator of space translations.

In the present paper we propose a new gauge condition, which is also 2D-elliptic but does not exhibit the above drawback. Indeed, in a generic Riemannian three-manifold our condition selects a four-parameter family of solutions, like in the Euclidean space \( E^3 \). Moreover, our condition is weaker than ‘\( k = \text{const.} \)’ (equivalent in the non-generic, Euclidean case, only). Topological two-spheres satisfying our condition will be called ‘rigid spheres’. They can be organized in topologically different ways: not necessarily standard ‘nested spheres foliations’, but also e.g. ‘bispherical foliations’ which already proved to be very useful in the analysis of the two-body problem\(^5\) (see [8]). We expect that various such arrangements, with rigid spheres used as building blocks, will provide useful gauge conditions in General Relativity Theory.

The present paper is a part of a bigger project, where we construct ‘spheres’ which are rigid not only with respect to 3D, but also with respect to four-dimensional (4D) deformations. More precisely, an eight-parameter family of similar ‘rigid spheres’ will be constructed in a generic 4D Lorentzian spacetime. In the present paper we limit ourselves to the 3D Riemannian case. It turns out, however, that our construction can be generalized to the entire pseudo-Riemannian spacetime \( M \), instead of the Riemannian Cauchy three-space \( \Sigma \subset M \). The idea of this extension is to mimic the case of the flat Minkowski space, where all possible round spheres, embedded in all possible flat subspaces \( \Sigma \) of \( M \), form an eight-parameter family. All of them can be obtained from a single one by the action of the product of the one-parameter group of dilations (changing the size of \( S \)) and the ten-parameter Poincaré group, quotiented by the three-parameter rotation group. The 4D version of our construction will take into account not only the external curvature of \( S \), but also its torsion (in section 2.5 we give a short outline of this construction, which will be presented in detail in a subsequent paper [1]). The rigid spheres obtained this way will form an eight-parameter family and will be used to construct useful coordinate systems not only on a given Cauchy surface \( \Sigma \), but also in the entire spacetime.

\(^5\) Initial data for the two black holes system can be easily obtained from the flat Euclidean geometry \( E^3 \) by two ‘punctures’. Such a space admits the ‘\( k = \text{const.} \)’ foliation only in the external region, far away from the two bodies. On the contrary, our ‘rigid spheres’ can be organized into a ‘bispherical system of coordinates’ which covers nicely the entire exterior of the two horizons.
The main advantage of such a construction consists in its rigidity at infinity. We very much hope to be able to eliminate supertranslations and to reduce the symmetry group of the ‘Scri’, otherwise infinite-dimensional, to the finite-dimensional one.

The construction which we propose in the present paper is based on the following idea. Given a surface \( S \) satisfying the rigid sphere condition, consider its infinitesimal deformations. They may be parameterized by sections of the normal bundle \( T^\perp S \). If we want our condition to admit a four-parameter family of solutions, like in the flat case, its linearization must admit a four-parameter family of deformations. This means that we are not allowed to constrain the complete information about the mean curvature \( k \): four real parameters describing \( k \) must be left free. In the flat case these four parameters which have to be left free are: the mean value (or the monopole part) of \( k \), which is responsible for the size of \( S \), and its dipole part (which vanishes exceptionally in flat case due to Gauss–Codazzi equations). The dipole part of the deformation is related to the group of translations. In fact, possible motions of a metric sphere are described by the group of Euclidean motions, quotiented by the subgroup of rotations which form the group of internal symmetries of every particular sphere \( S \).

To implement the above idea in a non-flat case, an intrinsic, geometric notion of a multipole expansion on an arbitrary Riemannian, topologically \( S^2 \)-surface is proposed in section 2. This construction is our main technical tool and we very much believe in its universal validity, going far beyond the purposes of the present paper. Section 2 is completed with the definition of a rigid sphere.

Section 3 contains formulation and the proof of theorem 3: a generic Riemannian three-space admits a four-parameter family of rigid spheres. Our proof is relatively simple, but is valid in the ‘weak field region’ only. This is sufficient for purposes of the quasi-local analysis of gravitational energy (in fact, the idea originates from our analysis of interaction between two black holes, cf [8]). Further development concerning strong fields will be given elsewhere.

Finally, discussion concerning less known (but necessary) technical results, like specific spectral properties of the Laplace operator on \( S^2 \) or the second variation of area, has been shifted to the appendix.

2. Equilibrated spherical coordinates. Multipole calculus on distorted spheres

2.1. Conformally spherical coordinates

Let \( S \) be a differential two-manifold, diffeomorphic to the two-sphere \( S^2 \subset \mathbb{R}^3 \) and equipped with a (sufficiently smooth) metric \( g \). Coordinates \( (\vartheta, \varphi) = (x^A, A = 1, 2) \), defined on a dense subset of \( S \setminus \ell \), where \( \ell \) is topologically a line interval, will be called conformally spherical coordinates if they have the same range of values as the standard spherical coordinates on \( S^2 \subset \mathbb{R}^3 \) and, moreover, if the corresponding metric tensor \( g_{AB} \) is conformally equivalent to the standard round metric on \( S^2 \), i.e. the following formula holds:

\[
g_{AB} = \psi \cdot \sigma_{AB},
\]

where \( \psi \) is a (sufficiently smooth) function on \( S \) and

\[
\sigma_{AB} = \begin{pmatrix} 1 & 0 \\ 0 & \sin^2 \varphi \end{pmatrix}.
\]
Example 1. A ‘proper’ conformal transformation, i.e. which is not a rotation: let \( n \in S \) and \( \tau > 0 \) be a positive number. Using appropriate rotation, choose conformally spherical coordinates \((\vartheta, \varphi)\) in such a way that \( n \) is a north pole, i.e. the coordinate \( \vartheta \) vanishes at \( n \). Define
\[
F_{n,\tau}(\vartheta, \varphi) = (\tilde{\vartheta}, \tilde{\varphi}),
\]
where
\[
\tilde{\vartheta} := 2 \arctan \left( \frac{\tau \cdot \tan \frac{\vartheta}{2}}{2} \right), \quad \tilde{\varphi} := \varphi,
\]
or, equivalently,
\[
\tan \frac{\tilde{\vartheta}}{2} = \tau \cdot \tan \frac{\vartheta}{2}.
\]

For the fixed point \( n \) these transformations form a one-parameter group\(^6\):
\[
F_{n,\tau} \circ F_{n,\sigma} = F_{n,\tau \sigma},
\]
generated by the vector field:
\[
\frac{d}{dt} \bigg|_{t=1} F_{n,\tau}(\vartheta, \varphi) = \frac{d}{dt} \bigg|_{t=1} \left[ 2 \arctan \left( \tau \cdot \tan \frac{\vartheta}{2} \right) \right] \frac{\partial}{\partial \vartheta} = \sin \vartheta \frac{\partial}{\partial \vartheta},
\]
which is the (minus) gradient of the function \( z = \cos \vartheta \).

In particular, \( F_{n,1} = I \) (the identity map) for every \( n \). Moreover, equation (5) implies the following identity:
\[
F^{-1}_{n,\tau} = F_{n,\tau^{-1}}.
\]

Using (4) and (5) we may easily derive the following formula:
\[
\frac{d\vartheta}{d\tilde{\vartheta}} = \frac{\tau}{1 + \tan^2 \frac{\vartheta}{2}} \frac{d\tilde{\vartheta}}{\tau^2 + \tan^2 \frac{\vartheta}{2}}.
\]

Similarly, we may prove:
\[
\sin \vartheta = \frac{\sin \tilde{\vartheta}}{\sin \tilde{\vartheta}} \sin \tilde{\vartheta} = 2 \frac{1}{\tau} \tan \frac{\tilde{\vartheta}}{2} \cdot \frac{1 + \tan^2 \frac{\vartheta}{2}}{2 \tan \frac{\vartheta}{2}} \sin \tilde{\vartheta} = \tau \frac{1 + \tan^2 \frac{\vartheta}{2}}{\tau^2 + \tan^2 \frac{\vartheta}{2}} \sin \tilde{\vartheta}.
\]

As a conclusion we obtain:
\[
(h^2 [ (d\tilde{\vartheta})^2 + \sin^2 \tilde{\vartheta} (d\varphi)^2 ]).
\]

where
\[
h = \tau \frac{1 + \tan^2 \frac{\vartheta}{2}}{\tau^2 + \tan^2 \frac{\vartheta}{2}},
\]
which proves the conformal character of the transformation. Indeed, we have
\[
g_{AB} dx^A dx^B = \psi [(d\tilde{\vartheta})^2 + \sin^2 \tilde{\vartheta} (d\varphi)^2] = \psi h^2 [ (d\tilde{\vartheta})^2 + \sin^2 \tilde{\vartheta} (d\varphi)^2].
\]

Hence, \( (\tilde{\vartheta}, \varphi) \) are conformally spherical coordinates if \( (\vartheta, \varphi) \) were.

\(^6\) In stereographic coordinates calculated with respect to the south pole this group is the dilation group: \( \zeta \to \tau \zeta \).
2.2. Barycenter of a conformally spherical system

Given a system of conformally spherical coordinates on $S$, consider the corresponding three functions:

\[ x := \sin \vartheta \cos \varphi, \quad (14) \]
\[ y := \sin \vartheta \sin \varphi, \quad (15) \]
\[ z := \cos \vartheta. \quad (16) \]

We have, therefore, a mapping $D : [0, \pi] \times [0, 2\pi] \mapsto \mathbb{R}^3$, given by:

\[ D(\vartheta, \varphi) = \begin{pmatrix} D^1(\vartheta, \varphi) \\ D^2(\vartheta, \varphi) \\ D^3(\vartheta, \varphi) \end{pmatrix} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}. \quad (17) \]

The following vector $X = \begin{pmatrix} \langle x \rangle \\ \langle y \rangle \\ \langle z \rangle \end{pmatrix} \in \mathbb{R}^3$, (18)

where by $\langle f \rangle$ we denote the average (mean value) of the function $f$ on $S$, i.e. the number

\[ \langle f \rangle := \frac{\int_S f \sqrt{\det g} \, d^2x}{\int_S \sqrt{\det g} \, d^2x}, \quad (19) \]

will be called a ‘barycenter’ of the system $(\vartheta, \varphi)$ on $S$. Of course, we have $\|X\| \leq 1$, because of the Hölder inequality:

\[ \|X\|^2 = \langle x \rangle^2 + \langle y \rangle^2 + \langle z \rangle^2 \leq \langle x^2 \rangle + \langle y^2 \rangle + \langle z^2 \rangle = 1. \]

**Example 2.** Consider the proper conformal transformation (4) and calculate the new barycenter $\widetilde{X} = \begin{pmatrix} \langle \tilde{x} \rangle \\ \langle \tilde{y} \rangle \\ \langle \tilde{z} \rangle \end{pmatrix} \in \mathbb{R}^3$, (20)

where

\[ \tilde{x} := \sin \tilde{\vartheta} \cos \varphi, \]
\[ \tilde{y} := \sin \tilde{\vartheta} \sin \varphi, \]
\[ \tilde{z} := \cos \tilde{\vartheta}. \]

The trigonometric identity:

\[ \cos \tilde{\vartheta} = \frac{1 - \tan^2 \frac{\vartheta}{2}}{1 + \tan^2 \frac{\vartheta}{2}}, \quad (21) \]

implies:

\[ \tan^2 \frac{\vartheta}{2} = \frac{1 - \cos \tilde{\vartheta}}{1 + \cos \tilde{\vartheta}} = \frac{1 - z}{1 + z}. \quad (22) \]

Hence, formula (5) implies:

\[ \frac{1 - \tilde{z}}{1 + \tilde{z}} = \tau^2 \frac{1 - z}{1 + z}, \quad (23) \]

or, equivalently,

\[ \tilde{z} = \frac{1 + z - \tau^2(1 - z)}{1 + \tau z + \tau^2(1 - z)}. \quad (24) \]

\[ \]
Moreover, formula (10) and its inverse:
\[
\sin \tilde{\vartheta} = \frac{1 + \tan^2 \frac{\vartheta}{2}}{1 + \tau^2 \tan^2 \frac{\vartheta}{2}} \sin \vartheta = \frac{1 + \frac{\tau}{1 + z + \tau^2 (1 - z)}}{1 + \frac{\tau}{1 + z + \tau^2 (1 - z)}} \sin \vartheta = \frac{2 \tau \sin \vartheta}{1 + z + \tau^2 (1 - z)},
\]
(25)
give
\[
\tilde{x} := \frac{2 \tau}{1 + z + \tau^2 (1 - z)} x,
\]
(26)
\[
\tilde{y} := \frac{2 \tau}{1 + z + \tau^2 (1 - z)} y.
\]
(27)

To calculate mean values of the functions (26), (27) and (24) we do not need to pass to new coordinates \((\tilde{\vartheta}, \tilde{\varphi})\), but we may use, as well, old coordinates \((\vartheta, \varphi)\). But we see that for \(\tau \to 0\) we have \(\tilde{x} \to 0, \tilde{y} \to 0, \tilde{z} \to 1\). The Lebesgue theorem implies, therefore, that for \(\tau \to 0\) we have
\[
\tilde{X} = \begin{pmatrix} \langle \tilde{x} \rangle \\ \langle \tilde{y} \rangle \\ \langle \tilde{z} \rangle \end{pmatrix} \to \begin{pmatrix} \langle 0 \rangle \\ \langle 0 \rangle \\ \langle 1 \rangle \end{pmatrix} = n.
\]
(28)

### 2.3. Equilibrated spherical coordinates

**Definition 1.** Conformally spherical coordinate system \((\vartheta, \varphi)\) is called equilibrated, if its barycenter vanishes: \(X = 0 \in \mathbb{R}^3\).

**Remark 2.** If there are two equilibrated spherical systems on \(S\) then they are related by a rotation.

**Theorem 1.** Each metric tensor on \(S\) admits a unique (up to rotations) equilibrated spherical system.

**Proof.** Given a metric tensor \(g\) on \(S\), choose first any system of conformally spherical coordinates \((\tilde{\vartheta}, \tilde{\varphi})\) on \(S\) and consider the corresponding identification of its points with the points of \(S^2 = \partial K(0, 1) \subset \mathbb{R}^3\). Consider now the mapping
\[
\mathbb{R}^3 \ni K(0, 1) \ni N \to \mathcal{F}(N) \in K(0, 1) \subset \mathbb{R}^3,
\]
(29)
given for \(N \neq 0\) by the following formula
\[
\mathcal{F}(N) := \tilde{X}_{n, \tau},
\]
(30)
where the latter is the barycenter of the coordinates \((\tilde{\vartheta}, \tilde{\varphi})\) obtained from \((\vartheta, \varphi)\) by the proper conformal transformation (3) with
\[
n := \frac{N}{\|N\|}
\]
(31)
and
\[
\tau := 1 - \|N\|.
\]
(32)
For \(N = 0\) formula (31) has no sense, but then (32) gives \(\tau = 1\) and, whence, equation (5) implies that the corresponding transformation (3) reduces to identity, no matter which vector \(n\) do we choose. Consequently, we define \(\mathcal{F}(0)\) as the barycenter of the original coordinates \((\vartheta, \varphi)\). Obviously, \(\mathcal{F}\) defined this way is continuous. Moreover, for \(\|N\| = 1\) we have \(\mathcal{F}(N) = N\) due to (32) and (28). This means that \(\mathcal{F}\) reduces to the identical mapping when restricted to the boundary \(S^2 = \partial K(0, 1) \subset K(0, 1)\). Consequently, there must be a
Theorem 2. Let $S$ be a differential two-manifold, diffeomorphic to the two-sphere $S^2 \subset \mathbb{R}^3$ and equipped with a metric $g$ of class $C^{(k,a)}$. For every pair $n, m \in S$, $n \neq m$, there is a unique equilibrated spherical system $(\vartheta, \phi)$ of coordinates on $S$, such that $\vartheta$ vanishes at $n$ and $\phi$ vanishes at $m$, and the metric components $g_{AB}$ are of the same class $C^{(k,a)}$. 

point $N_0$ which solves equation $\mathcal{F}(N_0) = 0$. This completes the existence proof. To prove the uniqueness, let us suppose that there is another solution: $\mathcal{F}(N_1) = 0$. Consider now the conformal transformation $F_{n,1} \circ F_{m,0}^{-1}$. Since the proper conformal transformations do not form any subgroup of the group of all conformal transformations, we cannot assume that it is again a proper transformation. But it may be decomposed into a product of rotations and a proper conformal transformation:

$$F_{n,1} \circ F_{m,0}^{-1} = O_1 \circ F_{m,\tau} \circ O_0,$$

where $O_1$ and $O_0$ are rotations. Denote by $(\vartheta_0, \varphi_0)$ the spherical coordinates obtained from $(\vartheta, \varphi)$ by the transformation $F_{n,1}$ and then rotation $O_0^{-1}$. Similarly, denote by $(\vartheta_1, \varphi_1)$ the ones obtained from $(\vartheta, \varphi)$ by $F_{m,1}$ and then by rotation $O_1^{-1}$. Because a rotation does not affect the spherical coordinates, both systems $(\vartheta_0, \varphi_0)$ and $(\vartheta_1, \varphi_1)$ are equilibrated. But the latter may be obtained from the former by a proper conformal transformation $F_{m,\tau}$. We shall prove that this is impossible unless $\tau = 1$ or, equivalently, transformation $F_{m,\tau}$ is trivial (identical).

For this purpose consider, for each value of $\tau$, the linear function $z_\tau$. Without any loss of generality we may assume that

$$m = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

(if this is not the case, it is sufficient to perform an appropriate rotation of coordinates). Formula (24) implies the following relation:

$$z_\tau = \frac{1 + z_0 - \tau^2(1 - z_0)}{1 + z_0 + \tau^2(1 - z_0)}.$$

Hence

$$\frac{d}{d\tau} z_\tau = \frac{-4 \tau (1 - z_0^2)}{[1 + z_0 + \tau^2(1 - z_0)]^2} \leq 0,$$

and it vanishes only at a single point $z_0 = 1$. Consequently, its mean value:

$$\frac{d}{d\tau} z_\tau$$

is strictly negative. This implies that starting from $\tau = 1$ (which corresponds to the identity mapping $F_{m,1}$) and moving towards the actual value $\tau < 1$, the $'z'$-component of the vector $\tilde{X}_{m,\tau}$ is strictly increasing. It vanishes at the beginning because $(\vartheta_0, \varphi_0)$ is equilibrated. Hence, it must be strictly positive at the end. This means that the final system $(\vartheta_1, \varphi_1)$ cannot be equilibrated unless $\tau = 1$ and, therefore, both systems coincide. \hfill \Box

Different equilibrated spherical systems of coordinates form, therefore, a 3D family. They can be parameterized by the position of a fixed point $n \in S$ (north pole) and the geographic longitude of a fixed point $m \in S$ (Greenwich). More precisely: given two points $n, m \in S$, $n \neq m$, there is a unique equilibrated spherical system $(\vartheta, \phi)$ of coordinates on $S$, such that $\vartheta$ vanishes at $n$ and $\phi$ vanishes at $m$.

Combining these observations with classical results (cf [9]), we obtain the following

**Theorem 2.** Let $S$ be a differential two-manifold, diffeomorphic to the two-sphere $S^2 \subset \mathbb{R}^3$ and equipped with a metric $g$ of class $C^{(k,a)}$. For every pair $n, m \in S$, $n \neq m$, there is a unique equilibrated spherical system $(\vartheta, \phi)$ of coordinates on $S$, such that $\vartheta$ vanishes at $n$ and $\phi$ vanishes at $m$, and the metric components $g_{AB}$ are of the same class $C^{(k,a)}$. 


Here, $C^{k,\alpha}(S)$ is a Hölder space $C^{k,\alpha}(S^2)$, defined for $1 \leq k \in \mathbb{N}$ and $0 < \alpha < 1$. The space consists of those functions on $S^2$ which have continuous derivatives up to order $k$ and such that the $k$th partial derivatives are Hölder continuous with exponent $\alpha$. This is a locally convex topological vector space.

The Hölder coefficient of a function $f$ is defined as follows:

$$|f|_{C^{\alpha}} = \sup_{x,y \in S^2, x \neq y} \frac{|f(x) - f(y)|}{|x - y|^{\alpha}}.$$ 

The function $f$ is said to be (uniformly) Hölder continuous with exponent $\alpha$ if $|f|_{C^{\alpha}}$ is finite. In this case the Hölder coefficient can be used as a seminorm.

The Hölder space $C^{k,\alpha}(S^2)$ is composed of those functions whose derivatives up to order $k$ are bounded and the derivatives of the order $k$ are Hölder continuous. It is a Banach space equipped with the norm

$$\|f\|_{C^{\alpha}} = \|f\|_{C^k} + \max_{|\beta| = k} |D^\beta f|_{C^{\alpha}},$$

where $\beta$ ranges over multi-indices and

$$\|f\|_{C^k} = \max_{|\beta| \leq k} |D^\beta f(x)|.$$

### 2.4. Rigid spheres in a Riemannian three-space

Given a manifold $S$ equipped with a metric tensor $g$, there is a 3D space of ‘linear functions’ uniquely defined on $S$ as linear combinations of functions (14)–(16), calculated in any equilibrated spherical system of coordinates $(\theta, \phi)$. We denote this space by $\mathcal{M}^3$. By $\mathcal{M}^4$ we denote the space spanned by $\mathcal{M}^3$ and the constant functions on $S$. Linear functions (14)–(16) on $S$ are eigenfunctions of the Laplace operator $\Delta_\sigma$, with the eigenvalue equal to $-2$, i.e.

$$\Delta_\sigma X^i = -2X^i,$$

where we denote $x = (x^1, x^2, x^3)$, $y = (y^1, y^2, y^3)$. Let us denote by $d\sigma := \sin \theta \, d\theta \, d\phi$ the measure associated with the metric $\sigma_{AB}$.

**Definition 2.** Let $f \in L^2(S, d\sigma)$. The projection of $f$ onto the subspace of constant functions:

$$P_m(f) := \frac{1}{4\pi} \int_S f \, d\sigma$$

will be called the monopole part of $f$, whereas the projection onto $\mathcal{M}^3 = \text{span}\{X^1, X^2, X^3\}$:

$$P_d(f) := \sum_{i=1}^3 \left( X^i \int_S X^i f \, d\sigma \right)$$

will be called the dipole part of $f$. In addition, we set

$$\mathcal{M}^4 := \text{span}\{1\} \oplus \mathcal{M}^3 = \text{span}\{1, X^1, X^2, X^3\},$$

and $P_{md}(f) := P_m(f) + P_d(f) \in \mathcal{M}^4$ denotes the mono-dipole part of $f$.

The above structure enables us to define the multipole decomposition of the functions defined on a topological sphere $S$ in terms of eigenspaces of the Laplace operator associated with the metric $\sigma_{AB}$. If $h$ is a function on $S$, then by $h^m := P_m(h)$ we denote its monopole (constant) part, by $h^d := P_d(h)$—the dipole part (projection to the eigenspace of the Laplacian with eigenvalue $-2$). By $h^w := (I - P_{md})(h) = h - h^m - h^d$ we denote the ‘wave’, or mono-dipole-free, part of $h$, $h^{d\omega} := (I - P_m)(h) = h - h^m = h^d + h^\omega$, and finally $h^{md} := P_{md}(h) = h^m + h^d$.

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7. By $\Delta_\sigma$ we denote the usual Laplace operator for the unit-sphere metric (2).
Remark 3. Mutually orthogonal projectors $P_{md}$ and $P_w := (I - P_{md})$ are, of course, continuous, when considered as operators in the Hilbert space $L^2(S, d\sigma)$. For our purposes we have to consider them as operators in the Banach space $C^{(k,\alpha)}$. Here, no ‘orthogonality’ is defined. Nevertheless, both operators are again continuous projectors. They define an isomorphism:

$$C^{(k,\alpha)} \cong C_{md}^{(k,\alpha)} \times C_w^{(k,\alpha)}.$$ 

where $C_{md}^{(k,\alpha)} = P_{md}(C^{(k,\alpha)}) \equiv \mathcal{M}^4$ and $C_w^{(k,\alpha)} = P_w(C^{(k,\alpha)})$. Hence, a function $f \in C^{(k,\alpha)}$ is uniquely characterized by its mono-dipole part $f_{md}$ and the remaining ‘wave’ part $f_w$, i.e. we have: $f = (f_{md}, f_w)$.

Definition 3. Let $\Sigma$ be a Riemannian three-manifold and let $S \subset \Sigma$ be a submanifold homeomorphic with $S^2 \subset \mathbb{R}^3$. We say that $S$ is a rigid sphere if its mean extrinsic curvature $k$ satisfies $k \in \mathcal{M}^4$, i.e. if the following equation holds:

$$k^w = 0.$$ 

(41)

2.5. The 4D spacetime case—an outline

Definition of a rigid sphere in a Lorentzian four-manifold is more complicated: to control ‘rigidity’ of a sphere, we must take into account more geometry. For this purpose we consider the extrinsic curvature vector of $S$: $k^a = k^a_{AB}g^{AB}$, where $k^a_{AB}$ denotes the external curvature tensor of $S$ (here, $a, b$ are indices corresponding to the subspace orthogonal to $S$ whereas $A, B$ label coordinates on $S$). Moreover, we consider its torsion:

$$\ell^A = (m | \nabla_A n),$$ 

(42)

where

$$n := \frac{k}{\|k\|},$$ 

(43)

$$\|k\| = \sqrt{k^a g_{ab} k^b},$$

and $m$ is a vector orthogonal to both $k$ and $S$.

Definition 4. Let $M$ be a Lorentzian four-manifold (a generic curved spacetime) and let $S \subset M$ be a spacelike submanifold homeomorphic with $S^2 \subset \mathbb{R}^3$. We say that $S$ is a rigid sphere if $k = (k^a)$ is spacelike and the following two conditions are satisfied:

$$\|k\| \in \mathcal{M}^4,$$ 

(44)

$$\nabla_A \ell^A \in \mathcal{M}^3.$$ 

(45)

In this paper we limit ourselves to the purely Riemannian 3D-setting. The general, pseudo-Riemannian case will be analyzed in a subsequent paper [1].

Example 3. Rigid spheres in a 4D Minkowski spacetime and in Euclidean three-space.

Let $M_0$ be the flat Minkowski spacetime, i.e. the space $\mathbb{R}^4$ parameterized by the Lorentzian coordinates $(x^a) = (x^0, \ldots, x^3)$ and equipped with the metric $\eta = (\eta_{ab}) = \text{diag}(-1, 1, 1, 1)$ (Greek indices run always from 0 to 3).

Consider in $M_0$ a round sphere, i.e. the 2D submanifold defined by

$$S_{T,R} := \left\{ x \in \mathbb{R}^4 \mid x^0 = T, \sum_{i=1}^{3} (x^i)^2 = R^2 \right\},$$
where the time \( T \in \mathbb{R} \) and the sphere’s radius \( R > 0 \) are fixed. It may be easily verified that the submanifold fulfills the following conditions:

\[
\sqrt{k^a g_{ab} k^b} = \frac{2}{R} \in \mathcal{M}^4, \tag{46}
\]

\[
\nabla_A E^A = 0 \in \mathcal{M}^3, \tag{47}
\]

hence each round sphere \( S_{T,R} \) in Minkowski spacetime \( M_0 \) is a rigid sphere. Using Poincaré symmetry group of \( M_0 \), it is easy to check that there is an eight-parameter family of such spheres. Indeed, fixing the value of \( R \), a seven-parameter family remains left. All of them may be obtained from a single sphere, say \( S_{0,R} \), by the action of the ten-parameter Poincaré group.

Because the three-parameter subgroup of rotations corresponds to internal symmetries of \( S_{0,R} \), we are left with seven parameters only. The parameter \( R \) corresponds to the dilation group. Hence, we have 8 (= 10 – 3 + 1) parameters.

In Euclidean three-space (represented by a slice \( \{x^0 = 0\} \) in \( M_0 \)) the family of rigid spheres reduces to four-parameter family of such spheres, where \( 4 = 3 + 1 \)—three translations plus dilation (or similarity transformations minus rotations \( 4 = 7 - 3 \)). Each round sphere in Euclidean three-space is a rigid sphere because its mean extrinsic curvature \( k = -\frac{2}{R} \in \mathcal{M}^4 \).

3. Existence of rigid spheres in a Riemannian space

Let \( \Sigma \) be a 3D Riemannian manifold. Let \( S \subset \Sigma \) be a two-manifold diffeomorphic to the unit sphere \( S^2 \subset \mathbb{R}^3 \). We consider the following problems: (1) Can we deform \( S \) in such a way that the resulting submanifold becomes a rigid sphere? (2) How many of such deformations exist in a vicinity of \( S \)?

To parameterize these deformations we introduce in a neighborhood of \( S \) a Gaussian system of coordinates \((u, x^A)\). Here, by \((x^A)\), \( A = 1, 2 \), we denote any coordinate system on \( S \), whereas \( u \) is the arc-length parameter along the \( \{x^A = \text{const.}\} \) geodesics starting orthogonally from \( S \). The three-metric takes, therefore, the form

\[
g = du^2 + g_{AB}(u, x^A) \, dx^A \, dx^B. \tag{48}
\]

Suppose, moreover, that coordinates \((x^A) = (\vartheta, \varphi)\) are conformal and equilibrated on \( S \). This means that we have

\[
\hat{g}_{AB} \, dx^A \, dx^B = \psi \cdot (\sigma_{AB} \, dx^A \, dx^B), \tag{49}
\]

where

\[
\hat{g}_{AB} := g_{AB}(0, x^A) \tag{50}
\]

is the induced two-metric on \( S \). \( \sigma \) is the ‘round’ two-metric on the Euclidean unit sphere:

\[
\sigma_{AB} \, dx^A \, dx^B = d\vartheta^2 + \sin^2 \vartheta \, d\varphi^2, \tag{51}
\]

and the function \( \psi \) is dipole-free (\( \psi^4 = 0 \)). Second fundamental form of \( S \) is given by:

\[
\hat{k}_{AB} = -\frac{1}{2} \hat{g}_{AB,u}. \tag{52}
\]

Its trace does not need to belong to the space \( \mathcal{M}^4 \) of mono-dipole-like functions, i.e. the surface \( S \) does not need to be a rigid sphere. We are looking for such deformations of \( S \), for which the resulting surface fulfills already the rigidity condition.

Any deformation of \( S \) which is sufficiently small may be uniquely parameterized by a function \( \tau = \tau(x^A) \), such that the deformed surface \( S_\tau \) is given by:

\[
S_\tau = \{(u,x^A) | u = \tau(x^A)\}. \tag{53}
\]
The surface $S_{\tau}$ carries the induced metric:
\[ g_{S_{\tau}} = [d\tau(x^i)]^2 + g_{AB}(\tau(x^C), x^C) \, dx^A \, dx^B =: g_{AB}(x^C) \, dx^A \, dx^B, \]
(54)
where
\[ g_{AB}(x^C) = (\partial_A \tau)(\partial_B \tau) + g_{AB}(\tau(x^C), x^C). \]
(55)
Here, we use the same coordinate system $(x^A)$, which was previously used for $S$. However, these coordinates do not need to be neither conformally spherical nor equilibrated. To verify that the deformation $\tau$ was successful, i.e. that $S_{\tau}$ is a rigid sphere, we have to pass to an equilibrated system of spherical coordinates, say $\tilde{x}^A$, on $S_{\tau}$. To make this choice unique, we use the north pole: $n := \{0 = \theta \}$, and the 'Gulf of Guinea': $n := \{\theta = \frac{\pi}{2}; \varphi = 0\}$ to get rid of the rotation non-uniqueness (cf theorem 2). This way we obtain an equilibrated version $\tilde{g}_{AB}$ of the metric (55). Finally, we calculate the extrinsic curvature $k$ and check whether or not its wave part $k^w(S_{\tau})$ satisfies condition $k^w(S_{\tau}) = 0$.

The idea of our paper may, therefore, be sketched as follows. We begin with a metric (48) which is of the class $C^{(k,0)}$. The above construction defines a continuous mapping:
\[ C^{(k-1,0)}_{md} \times C^{(k-1,0)}_w \ni (\tau_{md}, \tau^w) \mapsto F(\tau) := k^w \in C^{(k-1,0)}_w. \]
(56)
Indeed, the resulting metric in a neighborhood of $S_{\tau_0}$ is obtained from $g$ and the first derivatives of $\tau$. The function $\tau$ being of the class $C^{(k+1,0)}$, the metric obtained this way is again of the class $C^{(k,0)}$. Due to theorem 2, its equilibrated version $\tilde{g}_{AB}$ is again of the same class. Finally, the extrinsic curvature $k$ is obtained, using first derivatives of this metric. Hence, the result is of the class $C^{(k-1,0)}$ and the entire procedure is continuous.

Now, rigid spheres are those, which satisfy equation:
\[ F(\tau) = 0. \]
(57)
We are going to prove that, for a generic metric $g$, which is sufficiently close to the flat metric, the above equation defines an implicit function:
\[ \mathcal{M}^4 \equiv C^{(k+1,0)}_{md} \ni \tau_{md} \mapsto H(\tau_{md}) \in C^{(k+1,0)}_w, \]
(58)
such that
\[ F(\tau_{md}, H(\tau_{md})) \equiv 0, \]
(59)
or, equivalently, that $S_{\tau_{md}, H(\tau_{md})}$ is a rigid sphere. The main result of our paper follows as:

**Theorem 3.** Generically, there exists a four-parameter family of rigid spheres in a neighborhood of a given two-sphere $S \subset \Sigma$, corresponding to the four-parameter family of mono-dipole functions $\tau_{md}$ on $S$.

### 3.1. Infinitesimal deformations of spheres

To prove existence of the implicit function (59) it is sufficient to show that, given a mono-dipole deformation $\tau_{md}$, the partial derivative of $F$ with respect to the ‘wave-like’ deformation $\tau^w$ is an isomorphism of $C^{(k+1,0)}_w$ onto $C^{(k-1,0)}_w$.

For this purpose, we analyze the infinitesimal, linear version of the construction discussed above. Consider, therefore, a transversal deformation $\tau = \tau(x^i)$ of $S \subset \Sigma$ and a small deformation parameter $\varepsilon$:
\[ S \sim S_{\tau} = \{(u, x^i)| u = \varepsilon \tau(x^i)\}. \]
(60)
Under such transformation the induced metric changes in the following way:
\[ g_{AB} - \hat{g}_{AB} = -2\varepsilon \tau \hat{k}_{AB} + O(\varepsilon^2). \]
(61)
Even if the initial system of coordinates was equilibrated, the transformed metric does not need to be conformally spherical. The non-sphericality of the metric must be, therefore, compensated by a change of coordinates. Its infinitesimal version is described by a tangential (with respect to \( S \)) deformation

\[
\tilde{\xi}^A = x^A - \varepsilon \xi^A.
\]

(62)

Under such coordinate transformation the metric changes as follows:

\[
\tilde{g}_{AB} = g_{AB} - \xi_{(A|B)} + O(\varepsilon^2),
\]

(63)

where the last term represents the Lie derivative of the metric \( g_{AB} \) with respect to the vector field \( -\varepsilon \xi^A \). But, according to (61), the difference between \( \xi_{AB} \) and \( \tilde{g}_{AB} \) is already of the first order in \( \varepsilon \). Hence, if we replace it by the Lie derivative of the metric \( \tilde{g}_{AB} \), the error will be of the second order in \( \varepsilon \). Using the Killing formula for the Lie derivative of the metric, we finally obtain:

\[
\tilde{g}_{AB} = g_{AB} + 2\varepsilon \xi_{(A|B)} + O(\varepsilon^2).
\]

(64)

and the covariant derivative \( \nabla_{(A} \) is taken with respect to the original metric \( \tilde{g}_{AB} \). Hence, we have:

\[
\tilde{g}_{AB} - \tilde{g}_{AB} = -2\varepsilon \tau \tilde{k}_{AB} + 2\varepsilon \xi_{(A|B)} + O(\varepsilon^2).
\]

(65)

Let us decompose the above equation into the trace and the trace-free parts, calculated with respect to \( \tilde{g}_{AB} \) (we omit the terms of order \( \varepsilon^2 \) and higher):

\[
\tilde{g}_{AB} - \tilde{g}_{AB} = (\varepsilon \xi^C \nabla_{(C} \tilde{g}_{AB} - 2\varepsilon \tau \tilde{k}_{AB} + 2\varepsilon \xi_{(A|B)} - \frac{1}{2} \varepsilon \xi^C \nabla_{(C} \tilde{g}_{AB})
\]

(66)

where

\[
\tilde{k}_{AB} := \tilde{k}_{AB} - \frac{1}{2} \tilde{k}_{AB}
\]

(67)

is the traceless part of \( \tilde{k}_{AB} \). We want \( \tilde{g}_{AB} \) to be conformally spherical, i.e. \( \tilde{g}_{AB} = \alpha \cdot \tilde{g}_{AB} \). This implies:

\[
(1 - \varepsilon \tau \hat{k} - \alpha + \varepsilon \xi^C \nabla_{(C} \tilde{g}_{AB} - 2\varepsilon \tau \hat{k}_{AB} + 2\varepsilon \xi_{(A|B)} - \varepsilon \xi^C \nabla_{(C} \tilde{g}_{AB} = 0.
\]

(68)

The trace part of this equation defines uniquely the value of \( \alpha \):

\[
\alpha = 1 - \varepsilon \tau \hat{k} + \varepsilon \xi^C \nabla_{(C} \tilde{g}_{AB}
\]

(69)

whereas the trace-free part reduces to:

\[
\xi_{(A|B} + \xi_{B|A} - \xi^C \nabla_{(C} \tilde{g}_{AB} = 2\varepsilon \tau \hat{k}_{AB}.
\]

(70)

It is convenient to rewrite equation (70) in terms of the ‘round’ unit-sphere geometry \( \sigma_{AB} \).

For this purpose we use the following conventions: components of a vector (i.e. an object having upper indices) are the same in both geometries \( \sigma_{AB} \) and \( g_{AB} = \psi \sigma_{AB} \). Components of a co-vector (lowered indices) are denoted as follows:

\[
\xi^A = \sigma_{AB} \xi^B, \quad \xi_A = \tilde{g}_{AB} \xi^B = \psi \sigma_{AB} \xi^B = \psi \xi^A.
\]

(71)

The covariant derivative with respect to \( \sigma_{AB} \) will be denoted by \( \nabla_{(A} \), e.g. \( \xi_{A|B} \). Equation (70) can be easily rewritten as:

\[
\xi^A_{(A|B} + \xi^B_{B|A} - \xi^C \nabla_{(C} \sigma_{AB} = 2\tau \psi \hat{k}_{AB}.
\]

(72)

The left-hand side of this equation defines a mapping from the space of vector fields on the unit sphere to the space of trace-free rank 2 tensor fields. The kernel of this mapping consists of the dipole fields. The ‘Fredholm alternative’ argument shows that the operator on the left-hand
side defines an isomorphism between the space of dipole-free vector fields on the unit sphere and the space of trace-free rank 2 tensor fields (see also [7]). This isomorphism (in metric $\sigma$) will be denoted by $i_{12}$. Hence, the wave part of $\xi^A$ is implied uniquely by equation (72) (see appendix):

$$\xi^w_A = i^{-1}_{12} \left( \frac{1}{2} \nabla \tilde{k}_{AB} \right),$$

(73)

whereas the dipole part of $\xi^A$, i.e. the field $\xi^d_A$, remains arbitrary.

The above choice of the wave-like component of the tangential deformation $\xi^w_A$ guarantees that the new coordinate system $\tilde{x}^A$ is conformally spherical. We would like it to be also: (1) equilibrated and (2) satisfying conditions related to the two fixed points $n$ and $m$. These conditions mean that the field $\xi$ has to vanish at the north pole $n$ and that its $\psi$-component vanishes at $m$. The above $3 + 3 = 6$ conditions fix uniquely the total dipole-part of the tangential (to $S$) deformation $\xi^A$. This way the continuous mapping which assigns uniquely the tangential deformation $\xi^A$ to its transversal component $\tau$ has been defined.

### 3.2. The infinitesimal change of the extrinsic curvature

Now, we are going to calculate the infinitesimal change of the wave part $k^m$ of the mean curvature $\kappa$, i.e. derivative of the mapping (56) with respect to the ‘wave-like’ deformation $\tau^m$. We have $k = \tilde{g}^{AB} k_{AB}$, where $\tilde{g}^{AB}$ denotes the inverse of the two-metric $g_{AB}$ (whereas $g^{AB}$ denotes the corresponding components of the inverse three-metric.) The simplest way to calculate this change is to use a coordinate system $\omega, x^A$, adapted to the deformed surface:

$$\omega = \sigma - \varepsilon \tau(x^A), \quad \text{i.e.} \ S_\tau = \{ \omega = 0 \},$$

(74)

and the formula:

$$k_{AB} = \frac{1}{\sqrt{g}} \Gamma^{\omega}_{AB}.$$  

(75)

The three-metric $g$ takes now the following form:

$$g = d\sigma^2 + 2\varepsilon \tau_A d\sigma dx^A + g_{AB} dx^A dx^B + O(\varepsilon^2).$$

(76)

This implies $g^{\omega\omega} = 1 + O(\varepsilon^2)$ and, consequently,

$$k_{AB} = \Gamma^{\omega}_{AB} + g^{\omega C} \Gamma_{CAB} + O(\varepsilon^2) = \frac{1}{2} (g_{\omega A|B} + g_{\omega B|A} - g_{AB,\omega}) + O(\varepsilon^2),$$

(77)

where we treat the ‘shift vector’ $g_{\omega A} = \varepsilon \tau_A$ as a covector field on $S_\tau$. The first two terms combine to $\varepsilon \tau_{[AB]}$, whereas the last one: $g_{AB,\omega}(S_\tau)$ can be approximated by the quantity $g_{AB,\omega}(S_\tau) = -2k_{AB}$ plus the derivative of this object. Finally, we have

$$k_{AB} = \kappa_{AB} + \varepsilon \tau k_{AB,\omega} + \varepsilon \tau_{[AB]} + O(\varepsilon^2).$$

(78)

Since the derivative $g_{AB,\omega}$ of the metric $g_{AB}$ is described by $-2k_{AB}$, the derivative of its inverse $\tilde{g}^{AB}$ is described by $+2k^{AB}$. Hence, we have:

$$\tilde{g}^{AB} - g^{AB} = 2\varepsilon \tau k^{AB} + O(\varepsilon^2),$$

(79)

and, consequently:

$$k = \tilde{g}^{AB} k_{AB} = \kappa + \varepsilon \tau \delta_A \kappa + \varepsilon \tau \tau^{[A} + O(\varepsilon^2).$$

(80)

The quantity $\tau \delta_A \kappa + \tau^{[A}$ describes already the second variation of area (see appendix), i.e. the derivative $\nabla \kappa$. However, to calculate the derivative of the mapping (56), we have

\[ First variations of the total mean curvature $k$ is known in the literature as the second variations of area, cf e.g. [3]. See also discussion in the appendix. \]
to select its wave part \( k^w \). For this purpose we have to pass to the conformally spherical, equilibrated coordinates \( \hat{x}^A \), given by formula (62). Infinitesimal change of the scalar function \( k \) with respect to this deformation is given by formula:

\[
\hat{k} = k - \varepsilon \hat{x}^A \hat{k}_A + O(\varepsilon^2).
\]

Hence, we get:

\[
\frac{1}{\varepsilon} (\hat{k} - \hat{\sigma}^A \hat{k}_A) = \tau \partial_u \hat{k}^A + \tau^{|A} \hat{k}_A + O(\varepsilon), \tag{81}
\]

or, equivalently (cf appendix),

\[
\frac{1}{\varepsilon} (\hat{k} - \hat{\sigma}^A \hat{k}_A) = \tau (R_u^w + \hat{k}^{AB} \hat{k}_A) + \tau^{|A} \hat{k}_A + O(\varepsilon), \tag{82}
\]

where \( R_u^w = R(\partial_u, \partial_u) \) is the component of the Ricci tensor.

### 3.3. Proof of the theorem 3

The last formula gives, finally, the value of the derivative of the mapping (56). When restricted to the subspace of wave (i.e. mono-dipole-free) deformations, it gives us:

\[
C_w^{(k+1, \alpha)} \ni \tau \mapsto \left[ \tau (R_u^w + \hat{k}^{AB} \hat{k}_A) + \tau^{|A} \hat{k}_A - \varepsilon \hat{x}^A \hat{k}_A \right]^w \in C_w^{(k-1, \alpha)}. \tag{83}
\]

The above linear operator is, obviously, continuous. In particular, the vector field \( \hat{x}^A \) is given by formula (73), together with the accompanying vanishing conditions at \( n \) and \( m \).

If the space \( \Sigma \) is flat (Euclidean) and \( S \) is a standard (rigid) sphere of radius \( r \), then we have:

\[
\hat{g}_{AB} = r^2 \sigma_{AB}; \quad \hat{k}_{AB} = -r \sigma_{AB}; \quad \hat{k}_A = 0; \quad R_u^w = 0. \tag{84}
\]

Hence, the above operator reduces to:

\[
\tau^w \mapsto \left[ \tau (R_u^w + \hat{k}^{AB} \hat{k}_A) + \tau^{|A} \hat{k}_A - \varepsilon \hat{x}^A \hat{k}_A \right]^w = \frac{1}{r^2} \left[ (\Delta_u + 2) \tau \right]^w
\]

\[
= \frac{1}{r^2} \left( \Delta_u + 2 \right) (\tau^w), \tag{85}
\]

which is obviously an invertible mapping from \( C_w^{(k+1, \alpha)} \) to \( C_w^{(k-1, \alpha)} \). But the mapping (85) depends in a continuous way upon the geometry (metric and curvature) of \( S \). This implies that it remains invertible for sufficiently small deformations of the geometry. This is the case e.g. of a sufficiently big "coordinate sphere" defined as follows:

\[
S_{\hat{\Omega}, l} := \left\{ x \in \mathbb{R}^3 \left| \sum_{i=1}^{3} \left( x^i - \hat{x}^i_0 \right)^2 = l^2 \right. \right\},
\]

in an asymptotically flat \( \Sigma \).

We say, that \( \Sigma \) is asymptotically flat if there is a coordinate chart \( (x^i) \) covering the exterior of a compact domain \( D \subset \Sigma \) and such that

\[
g_{ij} = \delta_{ij} + h_{ij},
\]

where \( h \) vanishes sufficiently fast at infinity. In that case \( \Sigma \setminus D \) admits a four-parameter family of rigid spheres, similarly as in the case of the flat metric.
4. Conclusions

The main technical ingredient of this paper is the intrinsic, coordinate invariant definition of the ‘multipole expansion’ of a function defined on a Riemannian two-manifold, diffeomorphic with \( S^2 \). This enables us to select a finite-dimensional family of ‘rigid spheres’. The dipole part \( k^d \) of the curvature parameterizes the position of the center of such a sphere with respect to the center of mass. In particular, \( k^d = 0 \) corresponds to the spheres, which are centered at the center of mass. Properties of such a foliation have been analyzed in [4]. General topologically spherical coordinates, having property that surfaces \( \{ r = \text{const.} \} \) are rigid, do not admit supertranslations ambiguity at space infinity. This way symmetries of the ‘tangent space at infinity’ reduce to a finite-dimensional one. The 4D version of our results, valid for a generic 4D Lorentzian spacetime, which will be presented in the subsequent paper, will do the same job for the symmetry group of the Scri.

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Appendix

A.1. The dipole part of traceless symmetric part

The kernel of the mapping
\[
\xi_A^\sigma \mapsto \xi_{A;B}^\sigma + \xi_{B;A}^\sigma - \sigma^{CD} \xi_{C;D}^\sigma \sigma_{AB}
\]
defined by the left-hand side of the formula (72) consists of the dipole fields. This is a simple consequence of the following observations.

- In case of the unit sphere the Hodge decomposition \( \xi = d\alpha + \delta\beta + h \) of the covector \( \xi \) on a compact manifold does not contain the harmonic part, i.e. harmonic 1-form \( h \) vanishes \( (dh = 0 = \delta h \) implies \( h = 0 \)). The topology of the unit sphere (triviality of the corresponding cohomology class) cancels the harmonic part and we can always represent \( \xi \) as follows
\[
\xi_A = \alpha_A + \epsilon_{A}^{\hat{A}B} \beta_{\hat{A}B}, \quad \text{(A.1)}
\]
where functions \( \alpha \) and \( \beta \) are defined up to a constant but their gradients are unique.

- The purely dipole covector \( \xi \) simply means that the potentials \( \alpha \) and \( \beta \) are purely dipole functions: \( \alpha = a_i X^i \), \( \beta = b_i X^i \), where \( a_i \), \( b_i \) are real constants.

- Direct computation for dipole functions \( X^i \) enables one to check the following identity: \( X^i_{/AB} = -X^i \sigma_{AB} \), hence for any dipole function \( \alpha \)
\[
\alpha_{/AB} = -\alpha \sigma_{AB}. \quad \text{(A.2)}
\]

- Formulae (A.1) and (A.2) give
\[
\xi_{A;B} = -\alpha \sigma_{AB} - \beta \epsilon_{AB},
\]
hence the traceless symmetric part of \( \xi_{A;B} \) vanishes.
A.2. The isomorphism between covector fields and symmetric traceless tensors on \((S^2, g_{AB})\)

Let us consider the following diagram:

\[
\begin{array}{cccccccc}
V^0_{k+2} \oplus V^0_{k+2} & \xrightarrow{i_{01}} & V^1_{k+1} & \xrightarrow{i_{12}} & V^2_k & \xrightarrow{i_{23}} & V^1_{k-1} & \xrightarrow{i_{30}} & V^0_{k-2} \oplus V^0_{k-2} \\
\downarrow F_l & & \downarrow & & \downarrow & & \downarrow & & \downarrow F_l \\
V^0_{k+2} \oplus V^0_{k+2} & \xrightarrow{i_{01}} & V^1_{k+1} & \xrightarrow{i_{12}} & V^2_k & \xrightarrow{i_{23}} & V^1_{k-1} & \xrightarrow{i_{30}} & V^0_{k-2} \oplus V^0_{k-2}
\end{array}
\]

where the mappings and the spaces are defined as follows:

\[i_{01}(f, g) = f_{iA} + \varepsilon_A^B g_{iB},\]
\[i_{12}(v) = v_{A:B} + v_{B:A} - \sigma_{AB} v_{C:C},\]
\[i_{23}(\chi) = \chi_A^B : B,\]
\[i_{30}(v) = \left(v_{A:P}^C, \varepsilon_{AB} v_{A:B}\right),\]

\[F_l(f, g) = (g, f), \quad \hat{v}_A = \varepsilon_A^B v_B, \quad \hat{\chi}_{AB} = \varepsilon_A^C \chi_{CB},\]

\[V^0_k - \text{scalars on } S^2 \text{ belonging to Hölder space } C^{k,a},\]
\[V^0_k - \text{covectors on } S^2 \text{ belonging to Hölder space } C^{k,a},\]
\[V^0_0 - \text{symmetric traceless tensors on } S^2 \text{ belonging to Hölder space } C^{k,a}.\]

Denote by \(\Delta_\sigma\) the Laplace operator on \(S^2\) and by \(SH^l\) the space of spherical harmonics of degree \(l\), \((f \in SH^l \iff \Delta_\sigma f = -l(l+1)f)\). The following equality

\[i_{10} \circ i_{21} \circ i_{12} \circ i_{01} = \Delta_\sigma (\Delta_\sigma + 2)\]

shows that if we restrict ourselves to the spaces \(V^0_k := V^0_k \oplus [SH^0 \oplus SH^1] = (I - P_{\text{rad}})V^0_k (\Delta_\sigma + 2) V^0_k = V^0_k\) and \(V^l = V^l \oplus [i_{01}(SH^1)] (\Delta_\sigma + l)V^1 = V^l\) then all the mappings in the above diagram become isomorphisms.

A.2.1. Integral operators, generalized Green’s functions. Solution of the Helmholtz equation on a unit sphere \(S^2\):

\[
[\Delta_\sigma + l(l+1)] \Psi_l(n) = \Phi(n), \quad n \in S^2
\]

is given (see e.g. [13]) in terms of the generalized Green’s function \(\tilde{G}_l\) as follows:

\[
\Psi_l(n) = \int_{S^2} \tilde{G}_l(n, n') \Phi(n') \, d^2n'.
\] (A.3)

Here \(n = D(\vartheta, \varphi)\) given by (17) and \(d^2n = d\vartheta d\varphi\). The solution \(\Psi_l(n)\) is automatically orthogonal to the space \(SH^l\) (the kernel of Helmholtz operator \(\Delta_\sigma + l(l+1)\)) because Green’s function is orthogonal to this space. In our case we need to write the inverse of the operator \(\Delta_\sigma (\Delta_\sigma + 2)\) as a double integral with the corresponding kernels \(\tilde{G}_l\) for \(l = 0\) and \(l = 1\). More precisely, the solution \(g\) of the equation \(\Delta_\sigma (\Delta_\sigma + 2) g = f\) (with \(P_{\text{rad}} f = 0\)) is given in the following form:

\[
g(n) = P_n \int_{S^2} \tilde{G}_0(n, n'')
\]

\[
\cdot \left[ \int_{S^2} \tilde{G}_1(n', n'') f(n') \, d^2n' \right] = \frac{1}{4\pi} \int_{S^2 \times S^2} \tilde{G}_1(m, n') f(n') \, d^2n' \, d^2m
\]

\[
= \int_{S^2} \tilde{G}_0(n, n'') \left[ \int_{S^2} \tilde{G}_1(n', n'') f(n') \, d^2n' \right] \, d^2n'',
\] (A.4)
where the projection operator \( P_w \) provides orthogonality\(^{10} \) of \( g \) to the space \( SH^0 \oplus SH^1 \). The generalized Green’s function written in a standard form:

\[
\tilde{G}_1(n, n') = \frac{1}{4\pi} P_l(n \cdot n') \left[ \ln \frac{1 - n \cdot n'}{2} + c_l \right] + \frac{1}{2\pi} \sum_{i=0}^{l-1} \frac{2i + 1}{(l-i)(l+i+1)} P_l(n \cdot n'),
\]

can be simplified as follows (cf [13]):

\[
c_l := \frac{1}{2l+1} - 2 \sum_{i=0}^{l-1} (-1)^{i+i} \frac{2i + 1}{(l-i)(l+i+1)}.
\]

\( Y_{im} \)—spherical harmonics (orthonormal basis in \( SH^1 \)), \( n \cdot n' \in [-1, 1] \) is a scalar product of unit vectors in \( \mathbb{R}^3 \) and \( P_l(x) := \frac{1}{2l!} (x^2 - 1)^l \) is the Legendre polynomial.

**A.3. Second variation of area**

The Gaussian coordinates (48) and the definition of the Riemann tensor gives

\[
R^w_{\alpha\beta\mu \nu} = \hat{k}_{AB,w} + \hat{k}_{BC,w} k_{AB}.
\]

This leads to

\[
k_{AB} = \hat{k}_{AB} + \varepsilon \tau (R^w_{\alpha\beta\mu \nu} - \hat{k}_{AB}) + \varepsilon \tau \varepsilon \varepsilon_{AB} + O(\varepsilon^2).
\]

Taking the trace (and using (79)), we obtain:

\[
k = \hat{k} + \varepsilon \tau (R^w_{\alpha\beta\mu \nu} - \hat{k}_{AB}) + \varepsilon \tau \varepsilon \varepsilon_{AB} + O(\varepsilon^2).
\]

The formulae (80) and (A.7) are equivalent because of the Gauss–Codazzi equations:

\[
2\partial_t \hat{k} = R(g_{\ell\ell}) - R(\hat{g}_{AB}) + \hat{k}_{AB},
\]

\[
2R_{\mu\nu} = R(g_{\ell\ell}) - R(\hat{g}_{AB}) + \hat{k}_{AB} + \hat{k}^2,
\]

where \( R(g_{\ell\ell}) \) and \( R(\hat{g}_{AB}) \) are scalar of curvatures of the three-metric \( g_{\ell\ell} \) and the two-metric \( \hat{g}_{AB} \), respectively. Obviously (80), (A.7) are the first variations of the mean curvature \( k \), which in the literature (see e.g. [3]) are known as the second variations of area. They are usually presented in the following equivalent form:

\[
k - \hat{k} = \frac{\varepsilon}{2} \left[ (R(g_{\ell\ell}) - R(\hat{g}_{AB}) + \hat{k}_{AB} + \hat{k}^2) \tau + 2 \tau \varepsilon \varepsilon_{AB} \right] + O(\varepsilon^2).
\]

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\(^{10}\) The above integral operators do not mix the wave part with the mono-dipole part of a function. This means that \( P_w f = f \) implies \( P_w (G_1 \ast f) = G_1 \ast f \) and \( P_w (G_0 \ast f) = G_0 \ast f \).
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