Multi-critical point and unified description of broken-symmetry phases in spin-1/2 anti-ferromagnets on a square lattice

Oğuz Türker and Kun Yang

1 National High Magnetic Field Laboratory and Department of Physics, Florida State University, Tallahassee, Florida 32306, USA

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We show that several distinct broken-symmetry phases in a spin-1/2 anti-ferromagnet on a square lattice with easy-plane anisotropy, including valence bond solid, chiral spin liquid, and the XY-ordered state, can all be accessed by perturbing a multi-critical point with two massless Dirac fermions coupled to a level-one Chern-Simons gauge field. This allows for a unified description of these phases, as well as the phase transitions between them. In a specific phase transition our analysis provide a lattice realization of one of the recently proposed fermion-boson duality, thus lending support to it. We also briefly discuss the relation between our work and the long-sought de-confined criticality in such systems.

I. INTRODUCTION

Two-dimensional (2D) spin-1/2 antiferromagnets can support a large variety of phases, many of them break spin-rotation and/or lattice symmetries. Spin liquids[1], which break none of these symmetries, have been the focus of much recent research activities. They come in many different types as well. One such type, known as chiral spin liquid that breaks time-reversal symmetry, will be of particular relevance to our discussion below. Needless to say quantum phase transitions among all these phases are also of strong interest.

Broken symmetry phases are traditionally described Ginzburg-Landau theory, which is a field theory written in terms of the local order parameter associated with the spontaneously broken symmetry (see, e.g., Ref. [2]). In such descriptions phases with different broken symmetries are described using different order parameter fields, and direct 2nd order transitions between them require fine-tuning. Instead the more generic situation are first order transitions or intermediate phases where both types of orders co-exist. In Ref. [3], Senthil and co-workers argue that such descriptions miss the possibility of de-confined criticality, which is a critical point separating two different broken symmetries facilitating a direct 2nd order transition between them. Such novel quantum criticality can only be captured in a field theory that describe both types of broken symmetries on equal footing. Specifically, they argue that such deconfined critical points separate the Neel ordered and valence bond solid phases of 2D spin-1/2 antiferromagnets, which break spin-rotation and lattice translation symmetry respectively. In the appropriate field theory the two symmetry-breaking order parameters are dual to each other, thus afford a unified description. Numerous attempts have been made to identify such deconfined critical points, with inconclusive outcomes thus far (see Refs. [4] and [5] for recent attempts for the Heisenberg and XY symmetry classes respectively, and references therein).

While it is not our goal to resolve the fate of deconfined criticality, our work is motivated by the line of thoughts that lead to it. To this end we seek to find a field theory that provides a unified description of relevant phases in this description, and beyond. We find by perturbing a theory of two massless Dirac fermions coupled to a single level-one Chern-Simons (CS) gauge field, we can reach the XY-ordered (we only consider 2D spin-1/2 antiferromagnets with easy-plane anisotropy in this paper), VBS, chiral spin liquid, and an Ising Neel state in which the Neel order is along the z-direction despite the easy-plane anisotropy (which is possible in the presence of frustration). Within this description a direct 2nd order transition between the XY ordered phase and the VBS or Ising Neel phase must go through this massless point, which requires fine-tuning.

The remainder of the paper is organized as what follows. In Sec. II we introduce the spin-1/2 XY model and arrive at the multi-critical point by attaching a flux quantum to each hard-core boson that represents an up spin, and perform a mean-field approximation to smear out the flux. This results in two massless Dirac fermions coupled to a level-1 CS field. In Sec. III we discuss the phases that result when various mass terms are added to perturb this critical point. In Sec. IV we discuss how the mass terms responsible for spontaneous lattice symmetry breaking are generated by fermion interactions. Sec. V is devoted to deriving the dual bosonic theory of the multi-critical point, where we also make comparison with the existing theory of de-confined criticality. A brief summary is offered in Sec. VI.

II. MODEL AND COMPOSITE FERMION MEAN-FIELD APPROXIMATION

We consider the following spin-1/2 Hamiltonian on the square lattice:

$$H = - \sum_{\langle ij \rangle} (S_i^x S_j^x + S_i^y S_j^y) + \cdots$$  \hspace{1cm} (1)

$$= H_0 + \cdots,$$  \hspace{1cm} (2)
where \(<ij>\) stands for nearest neighbours, and ellipsis represents generic additional couplings that respect the XY rotation symmetry and all lattice symmetries unless noted otherwise. Note the minus sign means the XY coupling is ferromagnetic instead of anti-ferromagnetic; the two are equivalent under a \(\pi\) rotation along the \(z\)-direction for one of the two sub-lattices. The advantage of considering the ferromagnetic XY coupling is the XY-ordered phase only breaks the \(O(2)\) spin rotation symmetry, but none of the lattice symmetries. This makes the discussion of broken symmetries in various phases simpler below. The antiferromagnetic nature of Eq. (1) is thus hidden in the ellipsis, which include \(S^z\) and further neighbour couplings between XY spins in the same sublattice.

We can map half-spin ladder operators to annihilation and creation operators of the hard-core bosons. Accordingly, the nearest neighbour XY spin coupling in Eq. (1) becomes nearest neighbour boson hopping:

\[
H_0 = -\frac{1}{2} \sum_{<ij>} (b_i^\dagger b_j + b_j^\dagger b_i),
\]

and the ground state has half-filling in the absence of a net magnetization along the \(z\)-direction. We will use the spin and boson representations interchangeably below.

To proceed, we map the hard-core bosons to composite fermions (CFs) attached to a flux quantum by coupling them with pure Chern-Simons theory in lattice, and then make a mean-field approximation to spread out the flux uniformly that results in a \(\pi\) (or half) flux per plaquette [6]. With the gauge choice of Fig. 1 the resultant band Hamiltonian takes the form

\[
h_k = \begin{pmatrix}
0 & \sin k_x + i \sin k_y \\
\sin k_x - i \sin k_y & 0
\end{pmatrix},
\]

in which \(k\) is the lattice momentum. Importantly, we have two Dirac points at \((0, 0)\) and \((\pi, 0)\) where the two bands meet. In the ground state the lower band is filled while the upper band is empty, so the chemical potential coincides with the Dirac points. Thus the low-energy physics of the system at this level of approximation is described by two species of massless Dirac fermions coupled to a single Chern-Simons gauge field:

\[
\mathcal{L} = i \bar{\Psi} \gamma^\mu \partial_\mu \Psi + \mathcal{L}_{CS}[a] + \cdots,
\]

where,

\[
\mathcal{L}_{CS}[a] = \frac{1}{4\pi} e^{\mu\nu\lambda} a_\mu \partial_\nu a_\lambda = a \wedge da,
\]

is the level-1 Chern-Simons term, \(\Psi = (\psi_1, \psi_2)^T\) combines the two Dirac fields [7] where \(\psi_i\) are two component Dirac spinors, the slash notation is defined for general 3-vector \(b_\mu\) as \(\bar{\psi} = \gamma^\mu b_\mu\) where \(\gamma^\mu\) are two by two Dirac matrices obeying the Clifford algebra \(\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}\) where \(\eta^{\mu\nu}\) is the metric of the Minkowski \(2+1\) space-time and \(\eta^{\mu\nu} = \text{diag}(+,-,-)\) which will be used for raising and lowering the indices throughout the paper and \(\{,\}\) is the anti-commutator. \(D_\mu = \partial_\mu - i a_\mu - i A_\mu\) includes coupling to both the dynamic field \(a_\mu\) and background gauge field \(A_\mu\), and ellipsis represents the less relevant terms like Maxwell term of \(\bar{\Psi} \gamma^\mu \partial_\mu \Psi\).

To demonstrate below a variety of interesting phases supported by Eq. (1) can be accessed by perturbing Eq. (4) with various mass terms for the Dirac fermions.

**III. DIRAC MASS TERMS AND CORRESPONDING BROKEN SYMMETRY PHASES**

The most general mass term that couples the two Dirac points takes the form \(\bar{\Psi} M \Psi\), where

\[
M = m_0 \mathbb{1} + m_1 \sigma_1 + m_2 \sigma_2 + m_3 \sigma_3 = m_0 \mathbb{1} + \mathbf{m} \cdot \mathbf{\sigma}
\]

is a two-by-two Hermitian matrix. In the following we discuss how such mass terms can be generated beyond the mean-field Hamiltonian Eq. (4), and what phases they generate once added to the critical theory Eq. (5).

### A. Uniform Mass \(m_0\)

We first consider next nearest neighbor XY coupling:

\[
H_{\text{nnn}} = J' \sum_{<<ij>>} \left( S_i^x S_j^x + S_i^y S_j^y \right)
\]

\[
+ J' \sum_{<<ij>>} \left( b_i^\dagger b_j + b_j^\dagger b_i \right),
\]

where \(<<ij>>\) stands for next nearest neighbors. Within the mean-field approximation and using the gauge choice that hopping between A sublattice sites...
have phase +1 and that between B sublattice sites have phase -1, it results in a term of the form

$$h'_k = 2J' \begin{pmatrix} \cos k_x \cos k_y & 0 \\ 0 & -\cos k_x \cos k_y \end{pmatrix},$$

resulting in a uniform mass term with $m_0 = 2J'$, while $m = 0$.

We now analyse the phases stabilized by $m_0 \neq 0$. Since the fermions are massive, they can be integrated out. This results in a Chern-Simons term $\text{sgn}(m_0)\mathcal{L}_{CS}[A + a]$ whose sign depends on that of $m_0$ or equivalently, $J'$, which needs to be combined with the original Chern-Simons term for $a$ in Eq. (5). We analyse the two cases separately.

- $m_0 < 0$. In this case we have

$$\mathcal{L}_{\text{eff}}[a, A] = \mathcal{L}_{CS}[a] - \mathcal{L}_{CS}[a + A] = -2a \wedge dA - 2\mathcal{L}_{CS}[A].$$

Since the CS coupling of $a$ gets cancelled, we are left with a linear coupling between $a$ and $A$. Further integrating out $a$ yields a constraint $dA = 0$. This corresponds to Meissner response of the hard core bosons, indicating they are in a superfluid phase that spontaneously breaks the U(1) symmetry that corresponds to charge conservation$[8]$. For the original spin-1/2 Hamiltonian Eq. (1) this is the XY ordered phase$[6]$.

- $m_0 > 0$. In this case we have

$$\mathcal{L}_{\text{eff}}[a, A] = \mathcal{L}_{CS}[a] + \mathcal{L}_{CS}[a + A] = 2a \wedge dA + 2\mathcal{L}_{CS}[A].$$

Further integrating out $a$ yields

$$\mathcal{L}_{\text{eff}}[A] = -\frac{1}{2} A \wedge dA + 2\mathcal{L}_{CS}[A] = \frac{1}{2} \mathcal{L}_{CS}[A].$$

This is a fractional quantum Hall response corresponding to the $\nu = \frac{1}{2}$ Laughlin state for bosons$[8]$. In the original spin model this corresponds to the Kalmeyer-Laughlin chiral spin liquid (CSL) state, in which time-reversal symmetry is spontaneously broken. The same result was obtained earlier on the triangular and Kagomé lattices with anti-ferromagnetic nearest neighbor XY coupling only$[9]$.

It should be noted that while the mean-field Hamiltonian (4) suggests that the nearest neighbor XY model is a critical point separating the CSL and XY ordered phases, it is known that its ground state is actually XY ordered. As discussed in Ref. 6, fluctuation effects beyond the mean-field approximation tend to generate a negative $m_0$. We thus need a positive $J'$, which frustrates the XY order, to reduce the magnitude of the dynamically generated negative mass, and eventually drive the system into the CSL phase. It would be very interesting to study the spin-1/2 XY model with the frustrating nearest neighbor $J'$ coupling to see if such a transition exists.

B. Staggered Mass $m_3$

We call $m_3$ in (7) staggered mass because it gives rise to masses of opposite sign to the two Dirac fermions. Interestingly, it comes from a staggered potential coupled to the hardcore boson density (which is equal to the CF density):

$$v = m_3 \sum_i (-1)^i \bar{b}_i b_i = 2m_3 \sum_i (-1)^i S_i^z + \text{const.},$$

and the second equality above indicates it couple to the staggered magnetization along $z$-direction$[6]$, that breaks lattice translation symmetry spontaneously. We call the resultant phase Ising ordered phase (to be distinguished from the XY ordered phase discussed earlier). $m_3$ could also come from an external potential with wave vector $(\pi, \pi)$, which breaks lattice translation symmetry explicitly.

Regardless of its origin, in the presence of $m_3$ the Dirac fermions can again be integrated out. Since they have opposite masses, the CS terms they generate cancel. Further integrating out $a$ with existing CS term thus generates no term involving $A$, indicating the state has no (non-trivial) electromagnetic response. This is thus a Mott insulator state for the hardcore bosons.

Ref. 6 was mainly concerned about the quantum phase transition from the XY ordered to Ising ordered state in the nearest neighbor XXZ model, which is actually a first order transition that occurs at the Heisenberg point. In the presence frustration like that induced by $J'$, XY order gets suppressed and a direct 2nd order transition between them may be possible. Since $m_3$ breaks lattice translation symmetry, it must remain zero at this (putative) critical point. As a result the transition must again be driven through the critical point described by (5), where $m_0$ vanishes and $m_3$ gets turned on simultaneously. This is different from the conclusion of Ref. 6, where the authors assumed the presence of both $m_0$ and $m_3$, resulting in masses $m_0 \pm m_3$ for the two Dirac fermions, and the critical point is reached at $m_0 = m_3$, where only one of the two Dirac fermions become massless:

$$\mathcal{L} = i\bar{\psi} D\psi + \mathcal{L}_{CS}[a] + \cdots,$$

where $\psi$ is the field of this massless Dirac fermion. This same model also dicussed in [10].

From the discussions above it becomes clear that in order for the theory (Eq. (15)) to be relevant, before the XY order is suppressed, the Ising order must be present already, due to either spontaneous or explicit breaking of lattice translation symmetry. We consider the latter for its simplicity. With a staggered lattice potential of the form (Section III B), the unit cell of the square lattice gets doubled, and so does the boson filling from half to one per unit cell. We thus have a standard SF to Mott-Insulator transition in this case, which is described by
we note all components of $m$ to the 2D plane spanned by $m$ while not breaking lattice translation symmetry, but in different ways. On the other hand the masses $m$ break lattice translation symmetry and carry momentum $(\pi,0)$ or $(0,\pi)$. Thus we have here is actually a 4D phase diagram projected down to the familiar O(2) $\phi^4$ theory. Our analysis thus support the recently proposed duality between Eq. (15) and the corresponding Wilson-Fisher fixed point[11].

C. Off-diagonal Masses $m_1$ and $m_2$

As discussed above the staggered mass $m_3$ breaks lattice translation symmetry and carries momentum $(\pi,\pi)$. The off-diagonal mass terms $m_1$ and $m_2$ couple the two Dirac points, and must carry momentum $(\pi,0)$ or $(0,\pi)$. They thus break lattice translation symmetry in a different manner. As we demonstrate below, column valence bond solid (VBS) orders correspond to such symmetry breaking pattern and generates these masses.

VBS order, generated either spontaneously or explicitly, modulates the spin-spin coupling strength. We consider the most important column VBS patterns, which could align along either the x- or y-direction (see Fig. 2). It is immediately clear that they carry momenta $(\pi,0)$ and $(0,\pi)$ respectively. A straightforward calculation yields $m_{2,3} = \delta$ for the patterns of Fig. 2a and b respectively, where $\delta$ is the bond modulation.

In the presence of $m_2$ and/or $m_3$, we can diagonalize the mass matrix Eq. (7) resulting in the same Dirac Hamiltonian as that of $m_3$ mass only. So the system has the same topological properties as well, and is in the Mott insulator phase.

D. Summary

When all masses are present, diagonalizing the mass matrix (7) yields

$$m = m_0 \pm \sqrt{m_1^2 + m_2^2 + m_3^2} = m_0 \pm |m|,$$

where $m = (m_1, m_2, m_3)$. The resultant phase diagram, Fig. 3, looks similar to that of Ref. 8, although what we have here is actually a 4D phase diagram projected down to the 2D plane spanned by $m_0$ and $|m|$. In particular we note all components of $m$ break lattice translation symmetry, but in different ways. On the other hand $m_0$ while not breaking lattice translation symmetry, leads to phases that breaks O(2) spin rotation symmetry or timer-reversal symmetry when it dominates, in the same manner as discussed in Sec. IIIA.

In our description the two perpendicular VBS order parameters give rise to the real and imaginary parts of the off-diagonal Dirac mass in (7). They thus naturally form a complex order parameter, consistent with earlier study[3]. We find they can be further combined with stagger magnetization $m_3$ to form an O(3) order parameter, and they cooperate to enhance the Mott gap; in other word, they are “intertwined”. On the other hand they compete with the uniform mass $m_0$, and such competition leads to various quantum phase transitions. Despite such competition, our analysis suggest all the phases that appear in Fig. 3 naturally appear near each other in a frustrated spin-1/2 model on the square lattice, in the neighborhood of a multi-critical point described by (5).

IV. SPONTANEOUS BREAKING OF LATTICE SYMMETRY

From the perspective of the field theory (5), the massless point for both of the Dirac fermion (the origin in Fig. 3) is multi-critical and the full mass matrix of Eq. (7) must be tuned to zero. For example a direct 2nd order transition from the XY phase to the VBS phase must go through this point, while a more generic situation is going through the co-existing region or a direct 1st order transition. On the other hand the masses $m$ break lattice symmetries. Thus unlike $m_0$, they are not tuning parameters, but are instead generated from (sufficiently) strong interactions that leads to spontaneous symmetry breaking. Accordingly, we consider the following four-Fermi (Gross-Neveu type) interaction

$$\mathcal{L}_{int} = \lambda_0[(\bar{\psi}_1\psi_1)^2 + (\bar{\psi}_2\psi_2)^2] + \lambda_1(\bar{\psi}_1\psi_1)(\bar{\psi}_2\psi_2) + \lambda_2(\bar{\psi}_1\psi_2)(\bar{\psi}_2\psi_1) + \lambda_3(\bar{\psi}_1\psi_1)^2 + \lambda_4(\bar{\psi}_2\psi_2)^2. \quad (17)$$

It is clear that a positive $\lambda_1$ favours $m_3$, while a negative $\lambda_2$ and any $\lambda_3$ favour $m_{1,2}$. We can introduce Hubbard-
Stratonovich fields $\Phi$ to decouple these interactions, resulting in a Yukawa type of coupling

$$L_Y = \bar{\Psi} \Phi \Psi,$$

where

$$\Phi = \phi_0 \mathbb{1} + \phi_1 \sigma_1 + \phi_2 \sigma_2 + \phi_3 \sigma_3. \quad (19)$$

Obviously $\phi = (\phi_1, \phi_2, \phi_3)$ is an order parameter field describing the broken lattice symmetry.

The ordering transition is described by an effective field theory in terms of $\phi$ obtained from integrating out $\Psi$. This can be done under the generic situation of $m_0 \neq 0$. Such a transition, if continuous, takes the system from the XY/CSL phase to a mixed phase where spontaneously broken XY/time-reversal symmetry coexist with spontaneously broken lattice symmetry. A direct continuous transition from the XY phase to the VBS phase, however, again requires fine-tuning $m_0$ to zero; in this case the Dirac fermions are massless and cannot be integrated out perturbatively.

Returning to the generic situation of $m_0 \neq 0$, to determine the order of transition this effective theory describes at mean-field level, we are interested in the sign of the prefactor of the $|\phi|^4$ term. Let us denote this prefactor as $\beta_4$. We can calculate it diagrammatically. For notational simplicity, we focus on the $\phi_3$ term in Eq. (19) as a representative of $\phi$; the conclusions below are general. Also, we assume a uniform $|\phi| = m$.

The Feynman rules we use are given in Fig. 4. The diagrams up to two loops that contribute to $\beta_4$ are shown in Fig. 5[12]. The expansion in number of loops is equivalent to a weak coupling expansion in terms of the coupling constant between fermions and CS gauge field [13], the inverse of the square root of the absolute value of the Chern-Simons level. We do not show this coupling constant explicitly in our calculations for brevity of the notation. We adopt a renormalized perturbation theory approach, in which we replace the bare mass $m_0$ with renormalized mass $m_r$ in the free propagators and compensate this by adding mass and field-strength renormalization counter-terms.

Before the quantitative computation of $\beta_4$, we can have a qualitative discussion on what to expect. The dimension of the $\beta_4$ has an inverse mass dimension, i.e. $[\beta_4] = [m^{-1}]$. On the other hand the only dimensionful free parameter in the theory (and relevant Feynman diagrams that generate $\beta_4$) is the mass $m_r$. We thus expect $\beta_4$ is inversely proportional to the mass. We will calculate this proportionality constant below.

We start with calculating the one-loop contribution $D_1$ which is shown in Fig. 5:

$$D_1 = -\frac{1}{4} \text{tr} \left[ \int \frac{d^3p}{(2\pi)^3} \frac{(i(p - m_r))^4}{(p^2 - m_r^2)^4 + i\epsilon} \right]$$

$$= -\frac{1}{4} \text{tr} \left[ \int \frac{d^3p}{(2\pi)^3} \frac{p^4}{(p^2 - m_r^2)^4 + i\epsilon} \right]$$

$$= -\frac{1}{4} \int \frac{d^3p}{(2\pi)^3} \frac{p^4 + 6p^2m_r^2 + m_r^4}{(p^2 - m_r^2)^4 + i\epsilon}, \quad (20)$$

which, after Wick rotating to Euclidean coordinates $p_E^\mu = p_\mu = (-ip^0, \mathbf{p})$ and $\bar{\phi} = -i\phi^\mu E_\mu$ becomes

$$D_1 = -\frac{1}{2m_r} \int \frac{d^3p_E}{(2\pi)^3} \frac{p_E^4 - 6p_E^2 + 1}{((p_E)^2 + 1)^4}. \quad (21)$$
Using the integral identities given in Appendix A with $n = 4$, $d = 3$ we find,

$$D_1 = -\frac{i}{2} \frac{\Gamma\left(\frac{1}{2}\right)}{(4\pi)^{3/2}\Gamma(4)} \left( \frac{1}{|m_r|} \right) \left[ \frac{15}{4} - \frac{18}{4} + \frac{3}{4} \right],$$

$$= 0.$$  \hspace{1cm} (22)

As a preparation for the calculation of the two-loop diagrams, we first calculate the fermion self-energy, $\Sigma$, up to leading (one-loop) order in gauge coupling:

$$-i\Sigma = -i\Sigma_2 + D_5$$  \hspace{1cm} (23)

where $-i\Sigma_2$ is shown at Fig. 6 and given as

$$-i\Sigma_2 = 2\pi i \int \frac{d^3q}{(2\pi)^3} \frac{\gamma_\mu(\not{q} - m_\tau)\gamma_\nu}{q^2 - m_\tau^2 + i\epsilon} \frac{\epsilon^{\mu\alpha}(p - q)_\alpha}{(p - q)^2 + i\epsilon}$$

and $D_5$ is the corresponding counter term. Eq. (24) has a linear UV divergence. We can remove this by applying Pauli-Villars regularization, which is equivalent to the following substitution [14] in Eq. (24):

$$\frac{1}{(p - q)^2} \rightarrow \frac{1}{(p - q)^2} - \frac{1}{(p - q)^2 - \Lambda^2}$$  \hspace{1cm} (25)

where $\Lambda$ is the cutoff. Next, we use Feynman parameters to bring the denominator of the Eq. (24) in a spherically symmetric form, by using the identity [14]

$$\frac{1}{(q^2 - m_\tau^2)^n(p - q)^2} = \int_0^1 dx \frac{n(1 - x)^{n-1}}{(q - xp)^2 - \Delta^{n+1}},$$  \hspace{1cm} (26)

where $\Delta = -p^2 x(1 - x) + (1 - x)m_\tau^2$ and $n$ is a positive integer. If we substitute Eq. (25) to Eq. (24), apply Eq. (26) with $n = 1$ and change the integration variables as $q - xp \rightarrow k$ we have

$$-i\Sigma_2 = i \int_0^1 dx \int \frac{d^3k}{(2\pi)^2} \gamma_\mu(\not{k} + x\not{p} - m_\tau)\gamma_\nu\epsilon^{\mu\alpha}$$

$$\times (-k + p(1 - x))_\alpha \left[ \frac{1}{k^2 - \Delta^2 + i\epsilon} - \frac{1}{k^2 - \Delta_\Lambda^2 + i\epsilon} \right],$$  \hspace{1cm} (27)

where $\Delta_\Lambda = -p^2 x(1 - x) + (1 - x)m_\tau^2 + \Lambda^2 x$. Next, we use the following identities for two-dimensional gamma matrices:

$$\epsilon^{\mu\nu}\gamma_\mu\not{\gamma}\nu = 2ia^\alpha,$$  \hspace{1cm} (28a)

$$\epsilon^{\mu\alpha}\gamma_\mu\gamma_\nu = -2i\gamma^\alpha.$$  \hspace{1cm} (28b)

The self-energy is then given as

$$-i\Sigma_2 = -\int_0^1 dx \int \frac{d^3k}{2\pi^2} \left( -k^2 + (1 - x)(p^2 x + m_\tau p) \right)$$

$$\times \left[ \frac{1}{k^2 - \Delta^2 + i\epsilon} - \frac{1}{k^2 - \Delta_\Lambda^2 + i\epsilon} \right],$$  \hspace{1cm} (29)

where we have removed the terms that are odd in $k$. Next, we perform a Wick rotation and obtain

$$-i\Sigma_2 = -\frac{2i}{\pi} \int_0^1 dx \int \frac{dk_Ek_E^2}{2\pi} \left( k_E^2 + (1 - x) \right)$$

$$\times \left( -p_E^2 x - im_\tau p_E^2 \right) \left[ \frac{1}{k_E^2} - \frac{1}{k_E^2 + \Delta E} \right] - \frac{1}{k_E^2 + \Delta E_\Lambda},$$  \hspace{1cm} (30)
where $\Delta_E = p_E^2 x (1 - x) + (1 - x) m_r^2$ and $\Delta_{EA} = p_E^2 x (1 - x) + (1 - x) m_r^2 + 1 x^2$. We can evaluate the integral over $k_E$ using

$$I_1 = \int_0^\infty dk_E k_E \left[ \frac{1}{|k_E^2 + \Delta_E|^2} - \frac{1}{|k_E^2 + \Delta_{EA}|^2} \right],$$

$$= \frac{3\pi}{4} \left( \sqrt{\Delta_{EA}} - \sqrt{\Delta_E} \right),$$

and

$$I_2 = \int_0^\infty dk_E k_E^2 \left[ \frac{1}{|k_E^2 + \Delta_E|^2} - \frac{1}{|k_E^2 + \Delta_{EA}|^2} \right],$$

$$= \frac{\pi}{4} \left( \frac{1}{\sqrt{\Delta_E}} - \frac{1}{\sqrt{\Delta_{EA}}} \right).$$

Note that $\lim_{\Delta \to \infty} \Delta_{EA} = 1 x^2$. Finally, after undoing the Wick rotation we have

$$-i \Sigma_2 = -\frac{i}{2} \int_0^1 dx \left[ -3\sqrt{\Delta + \sqrt{x}} \right.$$

$$\left. + \frac{1}{\sqrt{\Delta}} \left(p^2 x (1 - x) + \beta m_r (1 - x) \right) \right],$$

where we clearly see the linear divergence. The counterterms will remove this divergence. We define renormalization conditions as

$$-i \Sigma(\beta = m_r) = 0 \quad (34a),$$

$$-i \frac{d\Sigma}{d\beta}|_{\beta = m_r} = 0, \quad (34b)$$

which fixes the location of the poles and the residue, thus the physical mass [14]. After substituting Eq. (23) to Eq. (34) we have,

$$D_3 = i (p \delta_2 + \delta m_r)$$

$$= i \left( -\frac{1}{2} \text{sgn}(m_r) p + \frac{1}{3} - \frac{3}{2} m_r \right).$$

Next, we calculate $D_2$ which is shown at Fig. 5 and explicitly given as;

$$D_2 = -\frac{1}{4} \text{tr} \left[ \int \frac{d^3p}{(2\pi)^3} \frac{1}{(p^2 - m_r^2 + i\epsilon)^5} (-i \Sigma) \right],$$

we then substitute Eq. (23), Eq. (24) and Eq. (35) to Eq. (36) perform a Wick rotation and let $p_E \to p_E m_r$ which gives;

$$D_2 = \frac{i}{8\pi^2} m_r \int_0^1 dx \int dp_E p_E^2 \left[ (p_E^4 - 10p_E^2 + 5)(-p_E^2) \right.$$

$$\times (1 + \frac{1 - x}{\sqrt{\Delta_0}}) - (5p_E^4 - 10p_E^2 + 1)$$

$$\times \left( -3\sqrt{\Delta_0} + \frac{1 - x}{\sqrt{\Delta_0}} (-p_E^2 x) + 3 \right) \left( \frac{1}{(p_E^2 + 1)^5} \right).$$

where $\Delta_0 = (p_E^2 x + 1)(1 - x)$. It is easy to see that sign of $D_2$ depends on the combination of the sign of $m_r$ and sign of the level of Chern-Simons term and the same is true for all two-loop contributions to $\beta_4$. Evaluate this integral yields

$$D_2 = \frac{i}{64\pi} \frac{1}{m_r^3} \Delta. \quad (38)$$

In preparation for the calculation of $D_3$, we first need to calculate the vertex correction $\Gamma_1$, which is shown at Fig. 6 and explicitly given as

$$\Gamma_1 = 2 \pi \int \frac{d^3q}{(2\pi)^3} \frac{\gamma_{\mu}(i(q - m_r))^2 \gamma_{\nu} e^{\nu\alpha}(p - q)_{\alpha}}{\sqrt{(q^2 - m_r^2)^2 + i\epsilon (p - q)^2 + i\epsilon}}, \quad (39)$$

Here, if we check the superficial degree of divergence of $\Gamma_1$ by counting the net order of $q$, we naively find a logarithmic UV divergence. However, this is not the actual case, because the leading term of the integrand is an odd function of $q$. As a result the naive logarithmically divergent term has zero coefficient, and the integral in Eq. (39) actually converges. We apply Eq. (26) with $n = 2$, change the integration variables as $q - xp \to k$, and obtain

$$\Gamma_1 = -4\pi \int_0^1 dx \int \frac{d^3k}{(2\pi)^3} \frac{\gamma_{\mu}(k + xp - m_r)^2 \gamma_{\nu} e^{\nu\alpha}}{\sqrt{k^2 - \Delta^3 + i\epsilon}} \times (-k + p(1 - x))_\alpha \left[ \frac{1 - x}{k^2 - \Delta^3 + i\epsilon} \right]. \quad (40)$$

After using the Eq. (28) and removing the odd terms in $k$ we have

$$\Gamma_1 = 8\pi \int_0^1 dx \int \frac{d^3k}{(2\pi)^3} \left[ (p(1 - x)(k^2 + x^2 p^2 + m_r^2) - k(k^2 + x^2 p^2 + m_r^2) \right.$$

$$\times \left[ \frac{1 - x}{k^2 - \Delta^3 + i\epsilon} \right], \quad (41)$$

now it is clear that this integral is not divergent, because the term that would produce logarithmic UV divergence is cancelled as a result of the removal of the odd terms. We can further simplify this by making the following substitution

$$k(p \cdot k) \to \frac{1}{3} k^2 p, \quad (42)$$

which is a result of the symmetry of the integral in $k$. Then, we perform a Wick rotation,

$$\Gamma_1 = 8\pi \int_0^1 dx \int \frac{d^3k_E}{(2\pi)^3} \left[ -ip_E(1 - x)(-k_E^2 - x^2 p_E^2 + m_r^2) \right.$$

$$+ \frac{2k_E^2 p_E}{3} + 2m_r(k_E^2 - (1 - x) xp_E^2) \left[ \frac{1 - x}{k_E^2 + \Delta^3} \right]. \quad (43)$$
We can evaluate the integral over \( k_E \) using the integral identities given in Appendix A. We have,

\[
\Gamma_1 = \frac{1}{4} \int_0^1 dx (1-x)^2 \left[ 3i \phi(t)(1-5x/3) + 2m_r \right] \sqrt{\Delta E} + (m_r^2 - x^2 p_E^2) (1-x)(-i \phi(t) - 2m_r x(1-x)p_E^2) \Delta_E^{1/2}.
\]

Finally we can calculate \( D_3 \) which is shown at Fig. 5 and explicitly given as

\[
D_3 = -\frac{1}{4} \text{tr} \left[ \int \frac{d^3p}{(2\pi)^3} \left\{ \frac{(i\phi_m - m_r)^4}{p^2 - m_r^2 + i\epsilon} + \Gamma_1 \right\} \right].
\]

As before we make Wick rotation, let \( p_E \to p_E m_r \) and substitute \( \Gamma_1 \) which gives

\[
D_3 = -\frac{i}{16\pi^2} m_r \int_0^1 dx \int_0^\infty dp_E p_E^2 \left[ -p_E^2 \left( 3 - 5x \Delta_E^{1/2} \right) - (1-x)^2 \Delta_E^{1/2} \left( -4p_E^2 + 4 \right) + \left( \frac{6}{\sqrt{\Delta_0}} - 2(1-x)p_E^2 \right) \left( p_E^4 - 6p_E^2 + 1 \right) \right] \times \frac{1-x}{(p_E^2 + 1)^3}.
\]

Evaluating this integral yields

\[
D_3 = 0.
\]

Next we calculate \( D_4 \). First, we start with \( \Gamma_2 \) which is shown in Fig. 6 and explicitly given as

\[
\Gamma_2 = -2 \int \frac{d^3q}{(2\pi)^3} \gamma_\mu (i\phi_m - m_r)^3 \gamma_\nu e^{\mu\nu\alpha}(p-q)\alpha \Delta_0^{1/2} \left[ (q^2 - m_r^2 + i\epsilon)(p-q)^2 + i\epsilon \right],
\]

which is convergent. First we apply Eq. (26) with \( n = 3 \) and change the integration variables as \( q - xp \to k \), and we have

\[
\Gamma_2 = -\frac{3i}{64\pi^2} \int_0^1 dx \int_0^\infty dp_E p_E^2 \left[ (1-x)^2 \Delta_0^{1/2} \right] \times \left\{ \left( k^2 + x^2 \right) \gamma_\mu (k + x p - m_r)^3 \gamma_\nu e^{\mu\nu\alpha} \right. \times \left. (k - p(1-x)) \Delta_0^{1/2} \right\}.
\]

Next, we simplify the gamma matrix terms by using Eq. (28) and show that

\[
e^{\mu\nu\alpha} \gamma_\mu (k + x p - m_r)^3 \gamma_\nu \left( -k + p(1-x) \right) \Delta_0^{1/2} \left\{ 2i(k^2 + x^2 p^2 + 2xk \cdot p + 3m_r^2) (k + x p - m_r) \right. \times \left. (-k + p(1-x)) \Delta_0^{1/2} \right\}.
\]

then, we apply Eq. (42) and a Wick rotation, so \( \Gamma_2 \) becomes

\[
\Gamma_2 = 12i\pi \int_0^1 dx \int \frac{d^3k_E}{(2\pi)^3} \left( \frac{k_E^2 - p_E^2 x(1-x)}{\Delta_E} \right) \times \left( -k_E^2 - x^2 p_E^2 + 3m_r^2 \right) + \frac{2x}{3} \frac{p_E^2 k_E^2}{\Delta_E} (1 - 2x) - im_r 2xk^2 \phi_E - im_r \phi_E (1-x) \left( 3(-k_E^2 - x^2 p_E^2) + m_r^2 \right) \times \left[ \frac{1-x}{k_F^2 + \Delta_E^2} \right].
\]

We can evaluate the integral over \( k_E \) using the integral identities given in Appendix A. \( \Gamma_2 \) is now

\[
\Gamma_2 = \frac{3i}{16} \int_0^1 dx (1-x)^2 \left[ -\frac{5}{\Delta_0^{1/2}} + \frac{1}{\Delta_0^{1/2}} \left( \frac{5x}{3} \frac{p_E^2 (1-2x)}{(1-x)^2} \right) + \frac{1}{\Delta_0^{1/2}} \left[ -p_E^2 x(1-x) \right] \times \left( -x^2 p_E^2 + 3m_r^2 \right) + im_r \phi_E (1-x) \right] \left( -3x^2 p_E^2 + m_r^2 \right) \left( -3x^2 p_E^2 + m_r^2 \right) \right] \right].
\]

Finally we can calculate \( D_4 \) which is shown at Fig. 5 and explicitly given as

\[
D_4 = -\frac{1}{4} \text{tr} \left[ \int \frac{d^3p}{(2\pi)^3} \left( i\phi_m - m_r \right)^3 \left( \frac{p^2 - m_r^2 + i\epsilon}{p^2 - m_r^2 + i\epsilon} \right) \right].
\]

As before we make Wick rotation, let \( p_E \to p_E m_r \) and substitute \( \Gamma_2 \) which gives

\[
D_4 = -\frac{3i}{64\pi^2} \int_0^1 dx \int_0^\infty dp_E p_E^2 \left[ (1-x)^2 \Delta_0^{1/2} \right] \times \left\{ \left( k^2 + x^2 \right) \gamma_\mu (k + x p - m_r)^3 \gamma_\nu e^{\mu\nu\alpha} \right. \times \left. (k - p(1-x)) \Delta_0^{1/2} \right\}.
\]

If we evaluate this numerically we have

\[
D_4 = -\frac{i}{32\pi} m_r.
\]

Finally, we get \( \beta_4 \) by using Eq. (37), Eq. (46) and Eq. (54) which gives

\[
\beta_4 = -\frac{1}{64\pi} m_r,
\]

which is the main result of this section.
We now discuss three different cases.

- \( m_r > 0 \): This describes a CSL phase. Since \( \beta_4 > 0 \), its transition into the phase with VBS and/or Ising Neel order is first order at meanfield level.
- \( m_r < 0 \): This describes a XY phase. Since \( \beta_4 < 0 \), its transition into the phase with VBS and/or Ising Neel order is second order at meanfield level.
- \( m_r = 0 \): This is our multi-critical point, at which we cannot integrate out the (massless) Dirac fermions perturbatively as done above. One can, nevertheless, perform a non-perturbative calculation of the effective potential\[14\] \( V_{\text{eff}}(\phi_{cl}) \) in terms of \( \phi_{cl} \), which is the vacuum expectation value of \( \phi \) where \( V_{\text{eff}}(\phi_{cl}) \) is minimized. Since the fermion theory is massless and contains no scale, one expects its coupling to \( \phi_{cl} \) generates a scale-invariant term \( |\phi_{cl}|^3 \), which is easy to verify by calculating the change of fermion ground state energy due to \( \phi_{cl} \) that plays the role of a mass. The non-analyticity of such a term originates from the masslessness of the Dirac fermion. Its presence signals non-meanfield behavior of the transition into the phases with broken translation symmetry, even if the theory is analysed at the meanfield level.

V. DUAL DESCRIPTION

The theory of multi-critical point is also discussed at Ref. \[21\], which is mainly done by considering a meanfield approach by considering the dual version of the theory. Thus to have a connection with the literature we also briefly find a dual version of our theory. In Sec. II we started with the lattice spin model given in Eq. (2), then we mapped it to hard-core bosons, and then mapped those hard-core bosons to non-relativistic fermions in lattice with level-1 CS term. Then, we find that the continuum limit this theory is described by two Dirac fermions coupled to level-1 CS term that is given at Eq. (5). In this section, we will apply a bosonization transformation to Eq. (5), which in a sense close the circle of our mappings.

We will use the well-known bosonization conjecture [11, 15-18, 22]. First we have to make several definitions to simplify the notation in following calculations. We closely follow the approach of Ref. [15] in this section. We define the Chern-Simons term and background field coupling as [15]

\[
S_{\text{CS}}[a] = \frac{1}{4\pi} \int d^3x e^{\mu\nu\lambda} a_\mu \partial_\nu a_\lambda, \quad (57a)
\]
\[
S_{\text{BF}}[a, B] = \frac{1}{2\pi} \int d^3x e^{\mu\nu\lambda} a_\mu \partial_\nu B_\lambda, \quad (57b)
\]

where \( a \) is a dynamic gauge field and \( B \) is a background gauge field, note that we use lower case letters for dynamic gauge fields and upper case letters for background gauge fields as before. The actions for material fields are given as

\[
S_{\text{fermion}}[\psi, A] = \int d^3x \bar{\psi}(i\gamma^\mu \partial_\mu - iA_\mu)\psi, \quad (58a)
\]
\[
S_{\text{scalar}}[\phi, A] = \int d^3x [(\partial_\mu - iA_\mu)\phi]^2 - \alpha|\phi|^4, \quad (58b)
\]

where we have an action for a free Dirac fermion coupled to background gauge field and complex Wilson-Fischer (WF) scalar, with coupling constant \( \alpha \) which flows to infinity at WF fixed point and the mass flows to zero [15]. Their partition functions

\[
Z_{\text{fermion}}[A] = \int \mathcal{D}\bar{\psi} \mathcal{D}\psi e^{iS_{\text{fermion}}[\psi, A]}, \quad (59a)
\]
\[
Z_{\text{scalar}}[A] = \int \mathcal{D}\phi \mathcal{D}\phi e^{iS_{\text{scalar}}[\phi, A]}, \quad (59b)
\]

are related by the bosonization conjecture [15]

\[
Z_{\text{fermion}}[A] e^{-\frac{1}{4} S_{\text{CS}}[a]} = \int \mathcal{D}a Z_{\text{scalar}}[\alpha] e^{iS_{\text{CS}}[a] + iS_{\text{BF}}[a, A]}, \quad (60)
\]

We have to clarify the origin of the extra half-level CS term at LHS, which is something purely notational. To understand this, assume for a moment the fermions in LHS are massive. In our notation when we integrate out the fermions of \( Z_{\text{fermion}}[A] \) we do not perform any Pauli-Villars regularization [19], and as a result of that we get a half-level CS term after integrating out the fermions. However, CS term with non-integer level breaks the large gauge invariance [19]. So, to preserve the gauge invariance of the theory, we have to add that extra half-level CS term [20].

Our goal is to obtain Eq. (5) by applying series of manipulations to LHS of Eq. (60). Applying the same manipulations to RHS of Eq. (60) yields the dual (or bosonized) version of Eq. (5).

First, we multiply two copies of LHS of Eq. (60) and integrate it over \( A \). We denote this integration variable as \( \bar{a} \), and we add a coupling with background field \( C \). So, the theory becomes,

\[
S_L = S_\beta[\psi_1, \bar{a}] + S_\beta[\psi_2, \bar{a}] - S_{\text{BF}}[\bar{a}, C] - S_{\text{CS}}[\bar{a}], \quad (61)
\]

which gives the fermionic side of our new duality. Performing the same manipulations to the RHS of Eq. (60) yields

\[
S_R = S_\alpha[\phi_1, a_1] + S_\alpha[\phi_2, a_2] + S_{\text{CS}}[a_1] + S_{\text{CS}}[a_2]
+ S_{\text{BF}}[a_1 + a_2 - C, \bar{a}], \quad (62)
\]

and this is the bosonic side of our new duality. Next, we integrate out \( \bar{a} \) at RHS which gives rise to the constraint

\[
C = a_1 + a_2, \quad (63)
\]
which we solve by introducing a new dynamic field \( b \) as \( a_1 = b \) and \( a_2 = -b + C \). Then \( S_R \) becomes
\[
S_R = S_s[\phi,b+] + S_s[\phi_2, -b + C] + S_{CS}[b] + S_{CS}[-b + C].
\]
(64)

Next, we apply time reversal transformation to both sides by simply changing the signs of the BF and CS terms. We then have,
\[
S'L = S_1[\psi_1, \bar{a}] + S_1[\psi_2, \bar{a}] + S_{BF}[\bar{a}, C] + S_{CS}[\bar{a}].
\]
(65)
The motivation behind this transformation is clear, as the fermionic theory now contains a level-1 CS term as in Eq. (5). Accordingly, the bosonic side of the duality is,
\[
S'_R = S_s[\phi_1, b] + S_s[\phi_2, -b + C] - S_{CS}[b] - S_{CS}[-b + C].
\]
(66)
Next, we let \( C \to -C \) and we add \( S_{CS}[C] \) to the both sides. So, both sides of the duality are given as,
\[
S''_L = S_1[\psi_1, \bar{a}] + S_1[\psi_2, \bar{a}] + S_{CS}[\bar{a} - C],
\]
(67)
notice that for \( C = A \) this is just the action of Eq. (5). And the bosonic side of the duality is,
\[
S''_R = S_s[\phi_1, b] + S_s[\phi_2, -b - C] - 2S_{CS}[b] - S_{BF}[b, C],
\]
(68)
finally we let \( \phi_2 \leftrightarrow \phi_2^* \) and get,
\[
S''_R = S_s[\phi_1, b] + S_s[\phi_2, b + C] - 2S_{CS}[b] - S_{BF}[b, C],
\]
(69)
which concludes the bosonization of Eq. (5). One should note that this is not the only possible duality that one can find. For example we can find different bosonic dual models to our original model by considering time-reversed version of Eq. (60) to the one of the fermionic degrees of freedom in our original model.

VI. SUMMARY AND DISCUSSION

In this paper we provide a unified description of various possible phases supported by a spin-1/2 antiferromagnet with easy-plane anisotropy on the square lattice, including Neel-order states, chiral spin liquid, and valence bond solids. The description is based on two Dirac fermions coupled to a level-1 Chern-Simons gauge field, and the various phases correspond to different combinations of the various Dirac mass terms. All these phases meet at a multi-critical point where the entire Dirac mass matrix vanishes. Within our description a direct continuous transition from the XY-ordered Neel state to the valence bond solid must go through this multi-critical point. In more generic situations there is either an intermediate phase with both orders, or a direct 1st order transition.

The theory of this multi-critical point and its dual description have some similarities to that of the de-confined criticality[3] and its dual description[21]. The main difference is our models contain Chern-Simons couplings, while their models do not. As a result their phase diagram does not contain the chiral spin liquid phase.

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Appendix A: Integral identities

Here we discuss the common integrals we will encounter in the main text [14],

\[
\int \frac{d^dk_E}{(2\pi)^d} \frac{1}{|k_E^2 + \Delta_E|^n} = \frac{1}{(4\pi)^{d/2}} \frac{\Gamma(n - \frac{d}{2})}{\Gamma(n)} \left( \frac{1}{\Delta_E} \right)^{n - \frac{d}{2}}, \quad (A1a)
\]
\[
\int \frac{d^dk_E}{(2\pi)^d} \frac{k_E^2}{|k_E^2 + \Delta_E|^n} = \frac{d}{2(4\pi)^{d/2}} \frac{\Gamma(n - \frac{d}{2} - 1)}{\Gamma(n)} \left( \frac{1}{\Delta_E} \right)^{n - \frac{d+2}{2}}, \quad (A1b)
\]
\[
\int \frac{d^dk_E}{(2\pi)^d} \frac{k_E^2}{|k_E^2 + \Delta_E|^n} = \frac{d(d/2 + 1)}{2(4\pi)^{d/2}} \frac{\Gamma(n - \frac{d}{2} - 2)}{\Gamma(n)} \left( \frac{1}{\Delta_E} \right)^{n - \frac{d+4}{2}}, \quad (A1c)
\]
where $n \in \mathbb{Z}^+$, which can be proved easily by converting the LHS to the Euler integral (Beta function) by substituting $x = \Delta E/(k_E^2 + \Delta E)$.

[1] C. Broholm, R. J. Cava, S. A. Kivelson, D. G. Nocera, M. R. Norman, and T. Senthil, Quantum spin liquids, Science 367, eaa0668 (2020).
[2] S. M. Girvin and K. Yang, Modern Condensed Matter Physics (Cambridge University Press, 2019).
[3] T. Senthil, L. Balents, S. Sachdev, A. Vishwanath, and M. P. A. Fisher, Quantum criticality beyond the landau-ginzburg-wilson paradigm, Phys. Rev. B 70, 144407 (2004).
[4] B. Zhao, J. Takahashi, and A. W. Sandvik, Multicritical deconfined quantum criticality and lifshitz point of a helical valence-bond phase, Phys. Rev. Lett. 125, 257204 (2020).
[5] N. Desai and R. K. Kaul, First-order phase transitions in the square-lattice easy-plane j-q model, Phys. Rev. B 102, 195135 (2020).
[6] A. Lopez, A. G. Rojo, and E. Fradkin, Chern-Simons theory of the anisotropic quantum Heisenberg antiferromagnet on a square lattice, Phys. Rev. B 49, 15139 (1994), arXiv:9401156 [hep-th].
[7] One for each Dirac point, and they each have two components representing the A and B sublattice. Note a rotation is performed on one of the Dirac points so that the two Dirac fields have the same chirality which allows for a unified description in Eq. (5).
[8] M. Barkeshli and J. McGreevy, Continuous transition between fractional quantum hall and superfluid states, Phys. Rev. B 89, 13749 (1993).
[9] M. E. Peskin and D. V. Schroeder, An introduction to quantum field theory (Westview, Boulder, CO, 1995) includes exercises.
[10] V. Shyta, J. van den Brink, and F. S. Nogueira, Deconfined criticality and bosonization duality in easy-plane chern-simons two-dimensional antiferromagnets, Phys. Rev. Lett. 127, 045701 (2021).
[11] T. Senthil, D. T. Son, C. Wang, and C. Xu, Duality between (2+1)d quantum critical points, Physics Reports 827, 1–48 (2019).
[12] There are other possible four external leg and two-loop, irreducible diagrams with one photon propagator. We find those diagrams to be zero under an appropriate regularization scheme.
[13] W. Chen, M. P. Fisher, and Y. S. Wu, Mott transition in an anyon gas, Phys. Rev. B 48, 13749 (1993).
[14] M. E. Peskin and D. V. Schroeder, An introduction to quantum field theory (Westview, Boulder, CO, 1995) includes exercises.
[15] A. Karch, B. Robinson, and D. Tong, More Abelian Dualities in 2+1 Dimensions 10.1007/JHEP01(2017)017 (2016), arXiv:1609.04012.
[16] A. Karch and D. Tong, Particle-vortex duality from 3d bosonization, Phys. Rev. X 6, 031043 (2016).
[17] N. Seiberg, T. Senthil, C. Wang, and E. Witten, A duality web in 2 + 1 Dimensions and Condensed Matter Physics, arXiv:arXiv:1606.01989v2.
[18] J. H. Son, J. Y. Chen, and S. Raghu, Duality web on a 3D Euclidean lattice and manifestation of hidden symmetries, J. High Energy Phys. 2019 (6), arXiv:1811.11367.
[19] D. Tong, Lectures on the gauge theory (2018), URL: http://www.damtp.cam.ac.uk/user/tong/gaugetheory/gt.pdf.
[20] We could have used an alternative notation such that when we integrate out the fermions in $Z_f[A]$ we could have performed Pauli-Villars regularization. In that case, Pauli-Villars regularization would automatically add that extra half-level CS term which would eliminate to necessity of adding it by hand. However, in his paper we don’t use this notation.
[21] C. Wang, A. Nahum, M. A. Metlitski, C. Xu, and T. Senthil, Deconfined Quantum critical points: Symmetries and dualities, Phys. Rev. X 7, 1 (17), arXiv:1703.02426.
[22] Chen, J., Son, J., Wang, C. & Raghu, S. Exact Boson-Fermion Duality on a 3D Euclidean Lattice. Phys. Rev. Lett., 120. 016602 (2018,1), https://link.aps.org/doi/10.1103/PhysRevLett.120.016602.