Using the Mark Weighted Correlation Functions to Improve the Constraints on Cosmological Parameters

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Abstract

We used the mark weighted correlation functions (MCFs), \( W(s) \), to study the large-scale structure of the universe. We studied five types of MCFs with the weighting scheme \( \rho^\alpha \), where \( \rho \) is the local density, and \( \alpha \) is taken as \(-1, -0.5, 0, 0.5, \) and \( 1 \). We found that different MCFs have very different amplitudes and scale dependence. Some of the MCFs exhibit distinctive peaks and valleys that do not exist in the standard correlation functions. Their locations are robust against the redshifts and the background geometry; however, it is unlikely that they can be used as “standard rulers” to probe the cosmic expansion history. Nonetheless, we find that these features may be used to probe parameters related with the structure formation history, such as the values of \( \sigma_8 \) and the galaxy bias. Finally, after conducting a comprehensive analysis using the full shapes of the \( W(s) \)’s and \( W_{\Delta}(\mu) \)’s, we found that combining different types of MCFs can significantly improve the cosmological parameter constraints. Compared with using only the standard correlation function, the combinations of MCFs with \( \alpha = 0, 0.5, 1 \) and \( \alpha = -0.5, 0.5, 1 \) can improve the constraints on \( \Omega_m \) and \( \omega \) by \( \approx 30\% \) and \( 50\% \), respectively. We find highly significant evidence that MCFs can improve cosmological parameter constraints.

Unified Astronomy Thesaurus concepts: Large-scale structure of the universe (902); Cosmological parameters from large-scale structure (340)

1. Introduction

The discovery of cosmic acceleration (Riess et al. 1998; Perlmutter et al. 1999) implies either the existence of a “dark energy” component in our universe or the breakdown of general relativity on cosmological scales. The theoretical explanation and observational probes of cosmic acceleration have attracted tremendous attention, and are still far from being well understood or accurately measured (Weinberg 1989; Li et al. 2011; Yoo & Watanabe 2012; Weinberg et al. 2013).

On scales of a few hundred megaparsecs (Mpc) the spatial distribution of galaxies forms a distinct, very complicated filamentary motif known as the “cosmic web” (Bardeen et al. 1986; de Lapparent et al. 1986; Tegmark et al. 2004; Huchra et al. 2012; Guzzo et al. 2014). The distribution and clustering properties of galaxies in the cosmic web encodes a huge amount of information on the expansion and structure growth history of the universe. In the next decade, several large-scale surveys, including DESI,5 EUCLID,6 LSST,7 WFIRST,8 and CSST (Gong et al. 2019), will begin operations and map out an unprecedented large volume of the universe with extraordinary precision. It is essential that we develop powerful tools that can comprehensively and reliably infer the cosmological parameters from large-scale structure (LSS).

The most widely adopted LSS analysis method is still the two-point correlation function (2pCF) or power spectrum measurements, which are sensitive to the geometric and structure growth history of the universe (Kaiser 1987; Ballinger et al. 1996; Eisenstein et al. 1998; Blake & Glazebrook 2003; Seo & Eisenstein 2003). These methods have achieved tremendous success when applied to a series of galaxy redshift surveys such as the 2 degree Field Galaxy Redshift Survey (Colless et al. 2003), the 6 degree Field Galaxy Survey (Beutler et al. 2012), the WiggleZ survey (Blake et al. 2011a, 2011b), and the Sloan Digital Sky Survey (York et al. 2000; Eisenstein et al. 2005; Percival et al. 2007; Anderson et al. 2012, 2014; Sánchez et al. 2012, 2013, 2017; Samushia et al. 2014; Ross et al. 2015; Alam et al. 2017; Beutler et al. 2017; Chuang et al. 2017). The main limitation of these methods are that they are only sensitive to the Gaussian part of the density field, while both the structure formation process or some primordial conditions can introduce non-Gaussian features in the LSS.

Ongoing research seeks to go beyond the two-point statistics and include methods such as three-point statistics (Sabiu et al. 2016; Slepian et al. 2017), four-point statistics (Sabiu et al. 2019), cosmic voids (Ryden 1995; Lavaux & Wandelt 2012), deep learning (Ravanbakhsh et al. 2017; Mathuriya et al. 2018), and so on. While many of them have proven useful, here we investigate another statistical tool, namely the mark weighted correlation function (MCF; Beisbart & Kerscher 2000; Beisbart et al. 2002; Gottlöber et al. 2002; Sheth & Tormen 2004; Sheth et al. 2005; Skibba et al. 2006; White & Padmanabhan 2009; White 2016; Satpathy et al. 2019; Massara et al. 2020; Philcox et al. 2020), which is simpler and computationally easier compared than the statistics mentioned above.

By weighting each galaxy using a “mark” that depends on its local density, the MCFs provide density-dependent clustering
information from the sample that is useful for data mining. The weights can be set to be proportional to the positive or negative power of the density, to allow the statistics to place more emphasis on dense or undense regions, where the clustered structures and the redshift space distortions (RSDs) are physically very different. It is expected that in this manner we can obtain more information from the data compared with using the traditional 2pcf, which treats all galaxy pairs equally regardless of differences in their physical properties and environments.

This paper is arranged as follows. Section 2 outlines the parameters of the data sets we use. Section 3 presents the methods used for the principle and operation of marked correlation functions. Section 4 explains the clustering statistics of different number densities. Section 5 discusses in more detail the various parameters to test whether the standard ruler persists. In Section 6 we present our general conclusions.

2. Data

The analysis in this work relies on the large N-body simulation: BigMultiDark\(^9\) (BigMD), and also a series of fast simulations generated using COmoving Lagrangian Acceleration (COLA).

The Multiverse simulations are a set of cosmological N-body simulations designed to study how variations in cosmological parameters affect the clustering and evolution of cosmic structures. Among them, the BigMD simulation is produced using 3840\(^3\) particles in a volume of (2.5 \(h^{-1}\) Gpc)\(^3\), assuming a ΛCDM cosmology with \(\Omega_m = 0.307115\), \(\Omega_b = 0.048206\), \(\sigma_8 = 0.8288\), \(n_s = 0.9611\), and \(H_0 = 67.77 \text{ km s}^{-1} \text{ Mpc}^{-1}\) (Klypin et al. 2016). The initial conditions, based on primordial Gaussian fluctuations, are generated using the Zel’dovich approximation at \(z_{ini} = 100\). Its large volume and huge number of particles make this an ideal simulation for the small-scale dynamics. Compared with the other fast simulation algorithms in the market, COLA performs better allowing for robust tracking of substructure. Both halos and subhalos are included in the analysis. To ensure the comparability, we maintain a halo number density \(\bar{n} = 0.001 \text{ \(h^{-1}\) Mpc}^{-3}\) in all simulations.

3. Methodology

The MCF is a simple extension of the standard configuration space 2pCF by assigning a mark to each object. Following (White 2016), we use the local density as the mark, and weight each halo by

\[
\text{weight} = \rho_{\text{hab}},
\]

which is a simpler expression than that proposed in White (2016). Here \(\rho_{\text{hab}}\) is the density estimated using its \(n_{\text{NB}}\) nearest neighbors,

\[
\rho_{\text{hab}}(r) = \sum_{i=1}^{n_{\text{NB}}} W_{\text{ki}}(r - r_i, h_W),
\]

where \(\rho_{\text{hab}}(r)\) is the number density at position \(r\), and \(W_{\text{ki}}\) is the smoothing kernel, for which we choose the third-order B-spline functions having nonzero value within a sphere of radius \(2 h_W \text{ \(h^{-1}\) Mpc}\) (Gingold & Monaghan 1977; Lucy 1977). We adopt an adjustable radius of the smoothing kernel to ensure that the kernel always includes \(n_{\text{NB}}\) nearest neighbor halos within \(2 h_W\).

The value of \(n_{\text{NB}}\) determines the smoothing scale that we applied to the sample. Figure 1 shows the probability distribution function (PDF) of the log density and \(h_W\) in the constructed fields when using \(n_{\text{NB}} = 30\) and \(300\), respectively. Since the values of \(h_W\) depend on the local density, they are not a constant number under a given \(n_{\text{NB}}\). Here we find the central and 1σ width of \(h_W\) is \(8.3 \pm 3.7, 20.1 \pm 4.7 \text{ \(h^{-1}\) Mpc}\) if using \(n_{\text{NB}} = 30, 300\), respectively. Very roughly, the central value scales as \(\propto (n_{\text{NB}})^{0.4}\). A larger \(n_{\text{NB}}\) decreases both the mean and the variance of \(\log_{10} \rho\).

In the MCF, the objects in the high and low dense regions are assigned different weights. Figure 2 shows the weights of some halos distributed in a \(200 \times 200 \times 20 \text{ \(h^{-1}\) Mpc}^3\) slice,
Here we use
\[ \alpha_{11} \]  
Figure 2. Halos selected from a 200 \ \times \ \text{BigMD sample. Their weights, as represented by the circle size, are determined by } \rho^2, \text{ where } \alpha = 1, -1. The \alpha = 1 \text{ scheme puts significantly more emphasis on the objects in a dense environment, while the other scheme does the opposite.}

selected from the \( z = 0.102 \) BigMD snapshot. While \( \alpha = 1 \) assign significantly larger weights to the objects in dense environment, the \( \alpha = -1 \) strategy does the opposite. From the dense to undense regions, the clustering patterns and RSDs vary dramatically, so we expect very different results for MCFs when using the two weighting strategies.

Apart from the weight that is assigned to each halo, the computational procedure to measure the MCF is exactly the same as that used for the standard 2pCFs. We use the most commonly adopted Landy–Szalay estimator:

\[
W(s, \mu) = \frac{WW - 2WR + RR}{RR},
\]  
where \( WW \) is the weighted number of galaxy–galaxy pairs, \( WR \) denotes the galaxy-random pairs, and \( RR \) denotes the number of random–random pairs. They are separated by a distance defined by \( s \pm \Delta s \) and \( \mu \pm \Delta \mu \), where \( s \) is the distance between the pair and \( \mu = \cos(\theta) \), with \( \theta \) being the angle between the line joining the pair and the line of sight (LOS) direction.\(^{11}\) For the random samples, we always use 10\( \times \) more particles than the data samples, and fix the weights of all particles to be 1.

Compared to the tradition CF, which is defined as \[ \xi(r) = \langle \delta(x)\delta(x+r) \rangle, \] the MCF takes the form of

\[
W(r) = \langle \delta(x)\rho_{\text{mab}}(x)\delta(x+r)\rho_{\text{mab}}(x+r) \rangle.
\]  
Notice the difference between \( \delta \) and \( \rho_{\text{mab}} \). The latter one is the smoothed density field, while the former is the contrast of the pointlike density \( \rho \). Effectively, \( \rho \) is the special case of \( \rho_{\text{mab}} \) with \( n_{\text{NB}} = 1. \)

4. A Glance at the Weighted CF

In what follows we present the MCFs measured from the BigMD halos, distributed in the redshift range of \( 0 < z < 1.45 \). To guarantee the comparability of objects at different redshifts, and also to maintain a uniform smoothing scale, from each sample we select a number of the most massive halos to build up a subsample having a constant number density \( \bar{n} = 10^{-3}(h^{-1}\text{Mpc})^3 \) for all samples.

4.1. \( W(s) \) Measurements

Figure 3 shows the MCFs as functions of clustering scale, i.e., the monopole \( W(s) \).\(^{12}\) They are computed by ignoring the \( \mu \)-dependence in Equation (3) when counting the weighted number of pairs. We show the results using \( \alpha = -1, -0.5, 0, 0.5, \) and 1, at the redshifts of 0, 0.51, 1.0, and 1.45, respectively. In all plots, we use \( n_{\text{NB}} = 30 \).

A significant dependence on the weighting scheme is detected when comparing the MCFs using different \( \alpha \). A larger \( \alpha \) assigns more weight to the dense, clustered region, thus resulting in stronger correlation (higher magnitude). The clustering patterns in dense and undense regions are different from each other, so the shape of MCFs is also sensitive to \( \alpha \).

As shown in the figure, when using \( \alpha = -1, -0.5, 0, 0.5, \) and 1, \( s^4W(z = 0) \) peaks at \( s \approx 2, 4, 17, 17, 18 \) \( h^{-1}\text{Mpc} \), with amplitudes of 16, 23, 80, 180, 400 \( (h^{-1}\text{Mpc})^4 \), respectively. The \( \alpha = 1 \) result has a peak magnitude 5 times stronger than the \( \alpha = 0 \) case, while the latter is again 5 times stronger if compared with the \( \alpha = -1 \) case; if comparing the clustering amplitude on the BAO scale, then the \( \alpha = 1 \) case is 2/4/15/100 times stronger than \( \alpha = 0.5/0/-0.5/-1 \) cases, respectively. The statistical error also increases with the

\(^{11}\) Here we use \( s \) instead of \( r \) because the statistics is usually performed using the redshift space positions; due to the RSDs they are related with each other via \( s = r + v/(\mu H) \).

\(^{12}\) Here we do not study the higher-order multipoles, since the \( \mu \)-dependence is studied in the next section using another statistical quantity.
decreasing of $\alpha$. For the $\alpha = -1, -0.5, 0, 0.5, 1$ results, the BAO peak is not very detectable, possibly due to the large noise therein. By enforcing $n = 10^{-3}$ at all redshifts both the clustering amplitude and the shape remain similar at all redshifts. Compared with the low-redshift result we find the BAO peak at higher redshift is more prominent, because there the smearing effect from the peculiar velocity and the nonlinear structure formation is less significant.

The shape of the MCF is changing persistently when we tune the value of $\alpha$. Several distinctive features, including a sharp peak (around $5-10\ h^{-1}\text{Mpc}$) in the $\alpha = 1, -0.5, -1$ results, and a valley (around $15\ h^{-1}\text{Mpc}$) in the $\alpha = -0.5, -1$ results, are detected. We will discuss their origins, implications, and uses in the coming sections. Finally, a quick check presented in Figure 4 shows that for most cases COLA achieves $\lesssim 10\%$ accuracy within the clustering range considered here (Ma et al. 2020).
A relative large discrepancy is detected at the $s \lesssim 20 \, h^{-1}\text{Mpc}$ regime in the $\alpha = 0.5$ and 1 cases. This consistency may be resolved by measures such as increasing the time steps or enhancing the resolution of the simulations, but we will not study it in detail.

### 4.2. $W_{\Delta}(\mu)$ Measurements

The RSDs in high-density and low-density regions are quite different. So we expect different anisotropic clustering features in the different MCFs. In what follows, we study the $\mu$-dependence of the MCFs. By integrating $W(s, \mu)$ along the $s$ direction, we define

$$W_{\Delta}(\mu) \equiv \int_{s_{\text{min}}}^{s_{\text{max}}} W(s, \mu) ds,$$

as well as its normalized version

$$\hat{W}_{\Delta}(\mu) \equiv \frac{W_{\Delta}(\mu)}{\int_{0}^{\mu_{\text{max}}} W_{\Delta}(\mu) d\mu}.\quad (6)$$

These two quantities describe the difference in the clustering strength in different directions w.r.t. the LOS. They have been used to quantify the RSDs and the Alcock–Paczynski (AP) distortions in the tomographic AP method (Li et al. 2014, 2015, 2016, 2018, 2019; Park et al. 2019; Zhang et al. 2019).

Figure 5 shows the measured $W_{\Delta}(\mu)$ at redshifts of $z = 0, 0.51, 1.145$, using $\alpha = -1, -0.5, 0, 0.5, 1$ and an integral range $s \in (6, 40), (2, 10), (10, 30) \, h^{-1}\text{Mpc}$, respectively. In all curves, we see a sharp peak near $1 - \mu = 0.1$, which is produced by the small-scale, nonlinear finger-of-god (FOG) effect (Jackson 1972). Also, we see a slope in the range of $1 - \mu \gtrsim 0.1$, as a consequence of the Kaiser effect (Kaiser 1987).

The amplitude of $W_{\Delta}(\mu)$ is enhanced if we tune down $s_{\text{min}}$ and include more small-scale clustering into the integration. In doing this, we also enhance the leftmost peak since FOG is stronger on smaller clustering scales.

Similar to what we found with $W(s)$, the $W_{\Delta}(\mu)$ has a larger amplitude and smaller statistical noise when using a larger value of $\alpha$. On the other hand, we do not detect any “violent” changes in the shape of $W_{\Delta}(\mu)$ when tuning the value of $\alpha$. However, this does not necessarily mean that the information encoded in these different $W_{\Delta}(\mu)$ are all the same. We will revisit this issue.

### 4.3. Distinctive Features in $W(s)$

In the $W(s)$ curves there are several distinctive peaks and valleys that do not exist in the standard 2pCFs. In what follows, we briefly discuss their possible origins.

#### 4.3.1. Sharp Peak

In many plots of $s \hat{W}(s)$ there exists a sharp peak located around $5 \sim 10 \, h^{-1}\text{Mpc}$ (see Figure 3). This means that on that scale there exists a large number of clustering pairs.

In all plots we use the weight $\rho_{30}$, whose smoothing scale is $\approx 8 \, h^{-1}\text{Mpc}$. That smoothing produces a correlation on that scale, so it is not surprising to see a peak on the corresponding scale. However, comparing with a random sample smoothed in the same way shows that the the amplitude of the peak also heavily depends on the intrinsic clustering property of the sample.

A comparison between the measurements on different directions implies that this sharp clustering peak has something to do with the FOG effect, which produces a $\sim 5 \sim 10 \, h^{-1}\text{Mpc}$ spike-like structure along the LOS. As shown in the left panel of Figure 6, the peak along the LOS direction is far more prominent than what is found in the transverse direction. The other panel of the figure shows that the heights and locations of the peak are rather insensitive to the redshift.

#### 4.3.2. Distinctive Valley

In addition to the peak we also detect a valley located at $s \approx 15 \, h^{-1}\text{Mpc}$ in the $\alpha = -1, -0.5$ cases. In particular, the $\alpha = -1$ case has both the peak and the valley, producing an unusual “S”-like shape.

Figure 7 shows the valleys in the $\alpha = -1$ MCFs. Equation (4) shows that the features in the MCF should be highly related with the difference between the local density $\rho$ and its smoothed counterpart $\rho_{30}$. In the redshift range of $\sim 0 \sim 1.5$, the location and the strength of this valley-like feature remain rather robust.

In contrast to what we see for the peak, we find that the valley looks rather similar in both the LOS and transverse directions, leading us to believe that that this valley has little or nothing to do with RSD effects.

### 5. Implications for Cosmological Analysis

In this section we discuss the implications of the MCFs to the cosmological analysis. In the first part, we report our attempt to utilize the peak and the valley as standard rulers. In the second part, we adopt a more comprehensive approach using the full shape of the MCFs.

#### 5.1. Usability of the Distinctive Features as Standard Rulers

The distinctive peaks and valleys as discovered in the $\alpha \neq 0$ MCFs are not found in the standard 2pCF. A remarkable feature is that the locations of these peaks and valleys are rather robust against the redshift. This inspires us to consider using them as “standard rulers” to probe the expansion history.

In galaxy surveys, the angular positions and redshifts of each galaxy are converted to 3D positions using the redshift-distance relation $r(z)$ adopted in an assumed cosmology. Wrongly adopted cosmology parameters lead to the following distortions of length in the directions parallel and perpendicular to the LOS:

$$\alpha_{r}(z) = \frac{H_{\text{true}}(z)}{H_{\text{wrong}}(z)},$$

$$\alpha_{A}(z) = \frac{D_{A,\text{wrong}}(z)}{D_{A,\text{true}}(z)},$$

where “true” and “wrong” denote the values of quantities in the true and incorrectly assumed cosmologies, respectively. This leads to two effects in the wrong cosmology,

1. The changes in the size of structures, known as the “volume effect.” This changes the BAO peak location, shifts the clustering patterns (Li et al. 2017), and changes the sizes of structures in the density field (Park & Kim 2010).
2. Changes in the shape of structures, known as the Alcock–Paczynski (AP) distortion (Alcock & Paczynski 1979; Ballinger et al. 1996). For an incomplete list of the methods based on this effect and their applications to the data see the literature (Ryden 1995; Matsubara & Suto 1996; Outram et al. 2004; Marinoni & Buzzi 2010; Blake et al. 2011b; Lavaux & Wandelt 2012; Li et al. 2014, 2016; Alam et al. 2017; Mao et al. 2017; Ramanah et al. 2019).

In what follows, we mainly test the feasibility of using the distinctive features to probe the “volume effect.” To mimic the effect, we take Equation (7) to convert the sample into

\[
\frac{1}{2} \int_{\mu_{\text{min}}}^{\mu_{\text{max}}} W(s, \mu) \, d\mu,
\]

where the integration ranges are (6, 40), (2, 10), and (10, 30) h⁻¹ Mpc, respectively. In all plots, there is a sharp peak near \(1 - \mu = 0.1\) as produced by the FOG effect, and a slope in the range of \(1 - \mu \geq 0.1\) as created by the Kaiser effect. The shape and the amplitude depend on the value of \(\alpha\).

Figure 5. Anisotropic clustering in the MCFs. The figures show \(W_s = \int_{\mu_{\text{min}}}^{\mu_{\text{max}}} W(s, \mu) \, d\mu\), where the integration ranges are (6, 40), (2, 10), and (10, 30) h⁻¹ Mpc, respectively. In all plots, there is a sharp peak near \(1 - \mu = 0.1\) as produced by the FOG effect, and a slope in the range of \(1 - \mu \geq 0.1\) as created by the Kaiser effect. The shape and the amplitude depend on the value of \(\alpha\).

Figure 6. In the \(\alpha = 1\) MCF, we find a sharp peak around \(s = 5\) h⁻¹ Mpc. A comparison of the measurements in different directions implies that this peak has something to do with the FOG effect (left panel). Its location and position are insensitive to the redshift (right panel).

Figure 7. The \(\alpha = -1\) MCF has an unusual “S”-like shape. It bears a valley-like feature on scales of \(\sim 15\) h⁻¹ Mpc, arising from the difference between the pointlike field \(\rho\) and the smoothed field \(\rho_{30}\). Its shape and the strength remain robust against redshift.
Note that, in doing this we just “reobserve” the simulation using the $r(z)$s of the new cosmologies, without running new simulations. That is exactly what one is doing when conducting the BAO or AP analysis on the observational data.

The comparability of samples requires them to have the same smoothing scale. Thus, when using a wrong background, we change the lower halo-mass cut to maintain a constant number density $\bar{n} = 1 \times 10^{-3} (h^{-1} \text{Mpc})^{-3}$.

In the following subsections, we test the feasibility of using the $\alpha = -1$ and $\alpha = 1$ MCFs, respectively.

### 5.1.1. The Background

When adopting an incorrect expansion history for the background, we expect a redshift-evolution of the CFs, determined by $(\alpha(z)^2 r(z))^{1/3}$ (see Li et al. 2017).

However, regardless of the strong volume effect in the two extremely incorrect cosmologies considered here, we do not detect any significant change in the scale of the MCFs. Figure 8 shows that in the wrong cosmological backgrounds, the locations of the peaks or valleys remain the same as their fiducial values. The conclusion is unchanged when we try using $n_{\text{NB}} = 10, 30$, and 300.

Consider the valleys of the $n_{\text{NB}} = 30$ measurements. While in the fiducial cosmology the valley is at $s \approx 15 - 16 h^{-1} \text{Mpc}$, in the $\Omega_m = 0.1$ wrong cosmology it still shows up at $s \approx 16 h^{-1} \text{Mpc}$. For comparison, in this cosmology the comoving length is artificially rescaled by a rate of $\geq 20\%$ at $z \geq 0.6$, so we expect the valley to appear near $18 - 19 h^{-1} \text{Mpc}$.

While they are insensitive to the background, the locations of the peaks or valleys are rather sensitive to the choice of smoothing scale. When changing $n_{\text{NB}}$ from 30 to 10/300, the location of the valley is shifted to $26/13 h^{-1} \text{Mpc}$, respectively.

In the $\alpha = 1$ MCFs, again we find that the locations of the peaks are rather insensitive to the background change (see Figure 9). Moreover, this finding appears robust against changes in redshift in the wrong cosmology we chose. This means that it is impossible to make use of their redshift evolution as a signal to identify the wrong cosmologies.¹³

Here we point out that, actually, this FOG-related pattern has been detected in other statistics. Fang et al. (2019) reported a detection of a peak around $\sim 3 h^{-1} \text{Mpc}$ in the $\beta$-skeleton statistics. In Appendix A, we report that the peak in that statistics cannot be used to conduct cosmological analysis, either.

### 5.1.2. Dependence on Bias and $\sigma_8$

Though rather insensitive to the background, these features do have some dependence on the bias and $\sigma_8$. Figure 10 shows the MCFs measured in five sets of COLA simulations, with the $\Lambda$CDM parameters of $(\Omega_m, 10^9 h, \sigma_8) = (0.2, 2.1, 0.5557)$, $(0.31, 2.0, 0.7965)$, $(0.31, 2.1, 0.8161)$, $(0.31, 2.29, 0.8523)$, and $(0.46, 2.1, 1.0576)$. Clearly, when adopting a smaller $\sigma_8$, the locations of the peaks shift toward small scales.

Basically, a smaller $\sigma_8$ leads to a smaller peak scale, except that the $\sigma_8 = 0.8/0.82/0.85$ curves in the $\alpha = 1$, $n_{\text{NB}} = 300$ case do not precisely obey this order. Possibly, because $n_{\text{NB}} = 300$ corresponds to a smoothing scale much larger than $8 h^{-1} \text{Mpc}$, here $\sigma_8$ cannot precisely describe what is

²¹ Not only are the locations of the peaks/valleys insensitive to the background change, but we find their heights are rather insensitive to the background. The reason is that by maintaining the same number density in all backgrounds, we are selecting objects with different biases; the change in the bias counteracts the effect of the background alteration on the clustering strength.

Figure 8. Comparing the $\alpha = -1$ MCFs in backgrounds of the fiducial cosmology and an extremely wrong cosmology. Regardless of the dramatic difference in the background geometry, there is little difference in the locations of the peaks or the valleys. So it is unlikely these features can be used to probe the geometry of the universe.

Figure 9. Comparing the $\alpha = 1$ MCFs in different backgrounds. This conclusion is similar to what we found in Figure 8.

(\Omega_m, w)_{\text{wrong}} = (0.1, -1), (0.3071, -1.5). 

(8)
happening. Although the basic trend is still correct, some complexities arise if we carefully investigate the details.

Meanwhile, we also find they have some dependence on the halo bias. Figure 11 shows the MCFs of three subsamples of BigMD $z = 0$ halos, distributed in different mass ranges (we keep $n = 10^{-3} \, (h^{-1} \text{Mpc})^{-3}$ in all subsamples). The valleys are more affected compared with the peaks.

In summary, our analysis shows that these peaks and valleys cannot be used as “standard rulers” to probe the geometry of the universe. But when changing the parameters related with the structure formation, we do observe shifts in the peaks or valleys. So these features may be useful for probing parameters related with the structure formation, e.g., the values of $\sigma_8$ and the halo/galaxy bias.

Apart from $\hat{W}(s)$, the $\hat{W}_{\Delta}(\mu)$s are also sensitive to $\sigma_8$. We investigate this sensitivity further in Appendix B.

5.2. Using the Full Shape of the MCFs

In what follows, we take a more comprehensive approach and use the full shape of the marked CF to predict the cosmological parameters. Figure 12 shows the correlation coefficients of $\hat{W}(s)$ and $\hat{W}_{\Delta}(\mu)$. They are estimated using the 150 COLA simulations. Among all MCFs, the $\alpha = -1$ case has the weakest correlation with the others. The negative correlations are the consequence of the normalization.

We choose the $\Omega_m = 0.3071$, $\omega = -1$ cosmology as the fiducial geometrical background, and define a statistical function to distinguish the other backgrounds from it,

$$\chi^2 = (p_{\text{fiducial}} - p_{\text{target}}) \cdot \text{Cov}^{-1} \cdot (p_{\text{fiducial}} - p_{\text{target}}),$$

where $p$ denotes $\hat{W}(s)$ or $\hat{W}_{\Delta}(\mu)$. Considering that the number of mocks is not too high compared with the binning number of the $W$s, we use the formula suggested by Hartlap et al. (2007) to correct the bias in the estimated covariance matrix.

The MCFs of the halo catalogs embedded in the backgrounds of the incorrect cosmologies are obtained using the following coordinate transforms (see Li et al. 2018 for details):

$$s_{\text{target}} = s_{\text{fiducial}} \sqrt{\alpha^2 + (1 - \mu^2)},$$

$$\mu_{\text{target}} = \mu_{\text{fiducial}} \alpha,$$

This is much more efficient compared with converting the samples into the different backgrounds and remeasuring the MCFs. A caveat is that Equations (10) and (11) do not capture the change in the values of the weights. That definitely happens, since the Alcock–Paczynski effect nonuniformly distorts the geometry, so the set of $n_{\text{NB}}$ nearest neighbors can differ from one cosmology to next. In Appendix C we check this caveat and show that if we neglect this issue it introduces only a minor effect, thus Equations (10) and (11) are deemed precise enough for this proof-of-concept study.

Equations (10) and (11) only consider the background information of the cosmologies. A more comprehensive analysis should involve the information of the structure growth, but that would require many more numerical simulations.

5.3. Constraints from $\hat{W}(s)$

The conditional constraints on $\Omega_m$ and $\omega$ (fixing one of them as the fiducial and constraining the other one) using the full shape of $\hat{W}_{\Delta}(\mu)$ are presented in the lower panel of Figure 13. In the plots, we use the clustering range of $s \in (5, 50) \, h^{-1} \text{Mpc}$, divided into 15 bins. Including the results on larger scales does not further enhance the power of constraints.

We find the $\alpha = 0$ results lead to the tightest constraints among all cases considered. Also, combining different MCFs can improve the constraint. Consider the $\omega = -0.4$ cosmology: compared with the fiducial cosmology, it is disfavored by $\chi^2 = 7.3/5.8/5.5/1.4$ when using the $\alpha = 0/0.5/ -0.5/1$ MCF, so the $\alpha = 0.5/ -0.5/1$ result is $20%/24%/81%$ worse than the $\alpha = 0$ result. Combining the $\alpha = 0$ and $\alpha = 1$ MCFs, we get a $17%$ improvement compared with only using the $\alpha = 0$ MCF. If we combine the $\alpha = 0/0.5/1$ MCFs together, the $\chi^2$ is then enlarged to 15, a $\approx 100%$ improvement compared with only using the $\alpha = 0$ MCF.

The $\chi^2$ of the $\alpha = 0, 0.5, 1$ combination is very close to the summation of the $\chi^2$s using the three MCFs separately. This
means that the cosmological information carried by the three MCFs is not strongly overlapping. This is important for the MCF statistics, meaning that we can significantly improve the cosmological constraints by combining different MCFs.

5.4. Constraints from $\hat{W}_{\Delta}(\mu)$

The conditional constraints on $\Omega_m$ and $w$ using the full shape of $\hat{W}_{\Delta}(\mu)$ are presented in the lower panel of Figure 13, where the integration range of $s$ is taken as $(6, 40)h^{-1}\text{Mpc}$, and we use the shape of $\hat{W}_{\Delta}(\mu)$ in the range of $\mu \in (0, 0.97)$, divided into 12 bins. Remarkably, the constraints derived using the $\hat{W}_{\Delta}(\mu)$s are much more powerful than those derived using the $\hat{W}(s)$s.

Similarly, we find $\alpha = 0$ achieves the best performance, and the results can be improved by combining different MCFs. In Table 1, we list the $\chi^2$s of the $w = -0.4$ cosmology using different $\alpha$’s or their combinations. Compared with using $\alpha = 0$ MCF, using $\alpha = 0, 0.5, 1$ and $\alpha = 0, -1, -0.5, 0.5, 1$ can improve the $\chi^2$ by 116% and 285%, respectively.

Figure 14 shows the constraints in the 2D $\Omega_m$–$w$ parameter space. The directions of degeneracy using different $\alpha$’s are identical to each other, so combining the different MCFs does not help with breaking the degeneracy. But by doing this we manage to shrink the contour size. Very roughly, compared with the $\alpha = 0$ MCF, the $\alpha = 0, 0.5, 1$, and $\alpha = 0, -1, -0.5, 0.5, 1$ combinations can improve the constraints on the parameters by $\approx 30\%$ and $50\%$, respectively.

6. Conclusion

We performed a detailed analysis on the MCFs for which the objects are weighted by $\rho^\alpha$. In this analysis, we considered five different MCFs, i.e., $\alpha = -1, -0.5, 0, 0.5, 1$, and characterize their scale and angular dependence using $W(s)$ and $W_{\Delta}(\mu)$. When studying the scale dependence of the MCFs, i.e., the $W(s)$s, we find the different MCFs have very different amplitudes and scale dependence. Especially, we found distinctive peaks and valleys in some $\alpha \neq 0$ MCFs, on scales around $\approx 5$ and $15 h^{-1}\text{Mpc}$, depending on the smoothing scale that we adopted to estimate the density. Their origin and properties are studied in detail. One particular point of interest is that the locations of these features are rather invariant with redshift.

In studying the possibilities of using the MCFs in cosmological analysis, we find the locations of the peaks or

![Figure 12](link)

Figure 12. Correlation coefficients of $\hat{W}(s)$ and $\hat{W}_{\Delta}(\mu)$, computed using the 150 sets of COLA simulations. The $\alpha = -1$ MCF has the weakest correlation with the others.

![Figure 13](link)

Figure 13. Conditional constraints on $\Omega_m$ and $w$, derived using $W_{\Delta}(\mu)$ and $\hat{W}_{\Delta}(\mu)$. Estimation is based on the $n = 10^3 (h^{-1}\text{Mpc})^3$ halo samples of the 150 COLA samples having a box size of $(512h^{-1}\text{Mpc})^3$. Among all MCFs, the $\alpha = 0$ MCF has the largest statistical power. We can largely improve the statistical power by combining different MCFs.

![Figure 14](link)

Figure 14. Constraints in the 2D $\Omega_m$–$w$ parameter space. The directions of degeneracy using different $\alpha$’s are identical to each other, so combining the different MCFs does not help with breaking the degeneracy. But by doing this we manage to shrink the contour size. Very roughly, compared with the $\alpha = 0$ MCF, the $\alpha = 0, 0.5, 1$, and $\alpha = 0, -1, -0.5, 0.5, 1$ combinations can improve the constraints on the parameters by $\approx 30\%$ and $50\%$, respectively.
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Table 1

| $\alpha$ | 0   | -1  | -0.5 | 0.5  | 1   | 0, -1 | 0, -0.5 | 0, 0.5 | 0.1 | 0, 0.5, 1 | 0, -1, -0.5, 0.5, 1 |
|----------|-----|-----|------|------|-----|-------|---------|-------|----|---------|---------------------|
| $\chi^2$ | 21.4| 8.1 | 19.8 | 14.9 | 6.9 | 29.2 | 45.0    | 40.0  | 30.8| 46.2    | 82.3                |
| $\chi^2_{\alpha=0}$ | 1   | 0%  | -62% | -7%  | -51% | -68% | 36%     | 111%  | 87% | 44%     | 116%                |

Figure 14. 68.3% CL constraints on $\Omega_m$ and $w$, derived using $\hat{W}_{\Delta s}(\mu)$. Different MCFs have the same direction of degeneracy. Compared with only using one kind of MCF, the two combinations can significantly reduce the constrained area.

valleys are rather insensitive to the background geometry. Thus, it is unlikely that they can be utilized as “standard rulers” to probe the geometry. However, their locations are affected by the value of $\sigma_8$ and the galaxy bias, so they could be useful for the determination of these parameters.

Finally, we studied the power of the different MCFs in distinguishing the different cosmologies, by using the full shape of the $\hat{W}(s)$s and the $\hat{W}_{\Delta s}(\mu)$ s. We find they have a similar direction of degeneracy in constrained $\Omega_m$ and $w$, while the $\alpha = 0$ MCF, corresponding to the standard CF, has the strongest power in distinguishing the background of the different cosmologies. Also, the constraint can be further improved by combining the different MCFs together. In particular, compared with the $\alpha = 0$ $W_{\Delta s}(\mu)$, the $\alpha = 0$, 0.5, 1 and $\alpha = 0$, -1, -0.5, 0.5, 1 combinations achieve $\approx 30\%$ and 50% improvement in reducing the constrained area, respectively.

The reason why MCF can improve the constrain is easy to understand. The dense and underdense regions have very different clustering patterns and RSD features. The many MCFs provide different weighting schemes of the clustering information according to their local density. By using them together, we can separate the regions with different patterns and extract more clustering information.

While previous works regarding the MCF mainly focused on modified gravity theories, our work suggests that MCFs could be useful for probing any parameter that is related with the expansion and structure growth history. By using the MCFs, we can enlarge the obtained information by 3–4 more times. MCF are also computationally efficient compared with the high-order statistics, like the 3pCF.

Philcox et al. (2020) used perturbation theory to study the marked power spectrum using perturbation theory, and found that the mark introduces a significant coupling between small-scale non-Gaussianities and large-scale clustering. This explains why using this statistics can provide additional information, and further support the findings of this work.

We find that the statistical quantity $\hat{W}_{\Delta s}(\mu)$ is more powerful than the $\hat{W}(s)$ in constraining the cosmological parameters. It may be possible to use $\hat{W}_{\Delta s}(\mu)$ instead of just $\hat{\xi}_{\Delta s}(\mu)$ in the tomographic AP method to improve the performance. However, we leave this issue for future works.

While in our analysis the $\alpha = 0$ MCF has the strongest power in distinguishing the background of the different cosmologies, Massara et al. (2020) found that a marked power spectrum can better constrain cosmological parameters than the power spectrum itself. This difference may be due to two reasons. (1) By using tens of thousands of simulations, Massara et al. (2020) built an emulator to capture both the expansion history and structure formation of the universe. In contrast, we just “reobserve” one simulation using different backgrounds to study the effect of the expansion history. It is very possible that the sensitivity of the MCFs to the structure formation is more important than its dependence to the expansion history, but we do not have it quantified in this simple treatment. We need to conduct a more comprehensive study in future analyses. (2) While Massara et al. (2020) used the power spectrum as the statistical discriminator, we used $W_{\Delta s}(\mu)$, the dependence of clustering strength on the direction. The two statistical quantities are physically quite different, and it is follows that the results derived using them are also different.

There are still many issues regarding MCF that are important but that we chose not to address in the present work. Although we have shown that MCFs encode a lot of information, we did not detail specific methods to extract them. In particular, we did not check whether the MCFs are useful for improve the measuring of the BAO peaks. In studying the different weighting schemes we only explore the restricted range of $-1 \leq \alpha \leq 1$. Finally, we only considered the halo number density as the weight, while it is possible to use features computed directly on the connectivity graph of the halo distribution. Those graph features are related to topological characteristics of the cosmic web (J. F. Suarez-Perez et al. 2020, in preparation), features that are in turn naturally correlated to $\sigma_8$ and the halo-galaxy bias.

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Appendix A

Usability of Peaks in the $\beta$-skeleton Statistics

The $\beta$-skeleton is a novel statistical tool proposed in Fang et al. (2019) to study the cosmic web. In this statistic, the "spikes" produced by the FOG lead to a peak in the histogram of connections near 2.5 $h^{-1}$ Mpc, which is rather robust to the redshift. The origin and the properties of this peak are very close to the peak we found in the $\alpha = 1, 0.5, -1$ W(s).

We find that the peak in the $\beta$-skeleton statistics also cannot be used to probe the cosmic expansion history. Figure A1 shows the distribution of the lengths and directions of the connections in the $\beta = 3$ web, measured in three different backgrounds. It is clear that the locations of the peaks are rather insensitive to the background geometry.

The full shape of the histograms of the lengths and directions shows some cosmological dependence. We will discuss its usability in depth.

In other works we have found that the entropy and complexity of the $\beta$-skeleton graph actually correlate with $\sigma_8$ (D. A. Torres-Guarin et al. 2020, in preparation), suggesting that the graph is more sensitive to the global tracer topology than the more geometrical influence of $\Omega_m$ and $w$.

Appendix B

$W_{\Delta_8}(\mu)$ in the Five COLA Simulations

In Section 5.1.2 we only discussed the $W(s)$ measured in simulations with different $\sigma_8$. Here we present the results of $W_{\Delta_8}(\mu)$.

As shown in Figure B1, the $1 - \mu \lesssim 0.1$ part, where the FOG should dominate, has a strong dependence on the value of $\sigma_8$. A larger $\sigma_8$ results in a stronger FOG effect, and thus a sharper peak.

Figure B1. The $W_{\Delta_8}(\mu)$ is measured in the five sets of COLA simulations with different values of $\sigma_8$. The leftmost part of the curve is dominated by the FOG effect. There the amplitude is significantly enhanced if using a large $\sigma_8$.

Appendix C

Accuracy of the Approximately Estimated MCFs

Figure C1 shows $W(s)$ and $W_{\Delta_8}(\mu)$ in the fiducial cosmology ($\Omega_m, w = (0.3071, -1)$, and in the background of a wrong cosmology ($\Omega_m, w = (0.5, -1)$). The results in the wrong background are computed in two ways, the precise measurement obtained by constructing the sample in the wrong background and then remeasuring the MCFs, and also the approximate results inferred using Equations (10) and (11) we can estimate the wrong cosmology results in a fast speed while still maintaining accuracy.

The $1 - \mu \gtrsim 0.1$ part, dominated by the Kaiser effect, seems to have a similar shape with different $\sigma_8$.

Figure C1. $W(s)$ and $W_{\Delta_8}(\mu)$ of the BigMD $z = 0.5$ halos, measured in the backgrounds of the fiducial cosmology and a wrong one. Using Equations (10) and (11) we can estimate the wrong cosmology results in a fast speed while still maintaining accuracy.

The 1 - $\mu \gtrsim 0.1$ part, dominated by the Kaiser effect, seems to have a similar shape with different $\sigma_8$.
