HILBERT SERIES OF FIBER CONES OF IDEALS WITH ALMOST MINIMAL MIXED MULTIPLICITY

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Dedicated to Bill Heinzer on the occasion of his sixtieth birthday

Abstract. Let \((R, m)\) be a Cohen-Macaulay local ring and \(I\) be an \(m\)-primary ideal. We introduce ideals of almost minimal mixed multiplicity which are analogues of ideals studied by J. Sally in [13]. The main theorem describes the Hilbert series of fiber cones of these ideals.

1. Introduction

Throughout this paper \(R\) will denote a \(d\)-dimensional Noetherian local ring with unique maximal ideal \(m\) and \(I\) will denote an \(m\)-primary ideal. The fiber cone of \(I\), \(F(I)\), is defined to be the graded ring \(\oplus_{n \geq 0} I^n/mI^n\). The fiber cone is the homogeneous coordinate ring of the fiber over the closed point \(m\) in the blowup of \(\text{Spec } R\) along the subscheme \(\text{Spec } (R/I)\). The fiber cone is in the class of rings called the blowup rings associated with \(I\). Recently it has been investigated by a number of researchers ([2], [3], [4], [5], [6], [7], [14], [15]).

In the last few decades the Rees algebra and the associated graded ring of an ideal (defined below) have been investigated by many researchers. However the fiber cone of an ideal has not been studied much. The fiber cones are useful in a number of situations. For example B. Singh characterized prime ideals \(p\) that can be permissible centers of blowing up in terms of the Hilbert series of the fiber cone of \(p\) ([16]). Fiber cones turn out to be useful in understanding evolutions([8], [7]). In the latter case the Cohen-Macaulay property of the fiber cone is useful while in the former case one needs to have prior knowledge of the Hilbert series of the fiber cone.

Our objective in this paper is to calculate Hilbert series of fiber cones of ideals of almost minimal mixed multiplicity introduced below. These ideals are analogues for \(m\)-primary ideals

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of ideals studied by J. Sally in [13]. The knowledge of Hilbert series of $F(I)$ provides us useful information about the number of generators of all powers of $I$ and more importantly in view of the main theorem of [8] this knowledge helps in detecting its Cohen-Macaulay property.

Let $\mu(I)$ denote the minimum number of generators of an ideal $I$. The Hilbert series of $F(I)$ is defined by $H(F(I), \lambda) := \sum_{n=0}^{\infty} \mu(I^n)\lambda^n$. In order to state the main theorem of this paper we recall the necessary definitions first. Let $\ell(.)$ denote length. It was proved in [8] that for large values of $r$ and $s$, the function $\ell(R/m^r I^s)$ is given by a polynomial $P(r, s)$ of total degree $d$ in $r$ and $s$. The polynomial $P(r, s)$ can be written in the form:

$$P(r, s) = \sum_{i+j \leq d} e_{ij} \binom{r+i}{i} \binom{s+j}{j},$$

where $e_{ij}$ are certain integers. When $i + j = d$, we put $e_{ij} = e_j(m|I)$ for $j = 0, 1, \ldots, d$. In this case these are called the mixed multiplicities of $m$ and $I$. It is known that $e_0(m|I) = e(m)$ and $e_d(m|I) = e(I)$ where $e(.)$ denotes the Hilbert-Samuel multiplicity [11]. The other mixed multiplicities too can be shown to be Hilbert-Samuel multiplicities of certain systems of parameters ([17] and [12]). We recall an important result from [12]. In this important paper Rees introduced joint reductions to calculate mixed multiplicities. A set of elements $x_1, \ldots, x_d$ is a called a joint reduction of a set of ideals $I_1, \ldots, I_d$ if $x_i \in I_i$ for $i = 1, \ldots, d$ and there exists a positive integer $n$ so that

$$\left[ \sum_{j=1}^{d} x_j I_1 \cdots \hat{I}_j \cdots I_d \right] (I_1 \cdots I_d)^{n-1} = (I_1 \cdots I_d)^n.$$

Rees proved that if $R/m$ is infinite, then joint reductions exist and $e_j(m|I)$ is the multiplicity of any joint reduction of the multisets of ideals consisting of $j$ copies of $I$ and $d - j$ copies of $m$. We shall denote such a multisets of ideals by $(m^{d-j}|I^j)$.

It has been proved in [8] that if $R$ is Cohen-Macaulay then $e_{d-1}(m|I) \geq \mu(I) - d + 1$. We say that $I$ has minimal mixed multiplicity if $e_{d-1}(m|I) = \mu(I) - d + 1$. We calculated the Hilbert series of $F(I)$ when $I$ has minimum mixed multiplicity and showed in [8] that for these ideals $F(I)$ is Cohen-Macaulay if and only if the reduction number of $I$ is at most one. See the next section for the definition of reduction number of an ideal.

**Definition 1.1.** Let $(R, m)$ be a Cohen-Macaulay local ring of dimension $d$. An $m$-primary ideal $I$ of $R$ is said to have almost minimal mixed multiplicity if $e_{d-1}(m|I) = \mu(I) - d + 2$.

**Definition 1.2.** Let $J = (x, a_1, a_2, \ldots, a_{d-1})$ be a joint reduction of $(m|I^{d-1})$. Define $r_J(m|I)$ to be the smallest integer $n$, if it exists, so that $m I^n = x I^n + (a_1, a_2, \ldots, a_{d-1}) m I^{n-1}$. The smallest of all $r_J(m|I)$ where $J$ is varying is denoted by $r(m|I)$. If there is no such integer we say that $r_J(m|I)$ is infinite and we write $r(m|I) = \infty$. 
Let $\gamma(I)$ and $\phi(I)$ denote the depths of the ideals generated by elements of positive degree in $G(I) := \bigoplus_{n \geq 0} I^n/I^{n+1}$ and $F(I)$ respectively. We can now state the main theorem of this paper:

**Theorem 1.3.** Let $(R,m)$ be a Cohen-Macaulay local ring of dimension $d$ with infinite residue field. Let $I$ be an $m$-primary ideal of almost minimal mixed multiplicity. Let $\gamma(I) \geq d - 1$ and $\phi(I) \geq d - 2$. Then the Hilbert series of $F(I)$ is given by

$$H(F(I), \lambda) = \begin{cases} 
\frac{1 + (\mu(I) - d)\lambda}{(1 - \lambda)^d} & \text{if } r(m|I) = \infty \\
\frac{1 + (\mu(I) - d)\lambda + \lambda r(m|I)}{(1 - \lambda)^d} & \text{if } r(m|I) \text{ is finite.}
\end{cases}$$

By the results proved in [5], the following corollary is easily deduced.

**Corollary 1.4.** Under the conditions of the above theorem, $F(I)$ is Cohen-Macaulay if and only if either $r(I) \leq 1$ or $r(I) = 2$ and $\ell(I^2/JI^2 + mI^2) = 1$ for some (and hence all) minimal reduction $J$ of $I$.

The paper is organized as follows: In Section 2 we characterize ideals of minimal mixed multiplicity. The main theorem requires a different approach in dimension one and therefore we have proved it in Section 3. In Section 4, we apply basic results about mixed multiplicities of ideals to find the Hilbert series of fiber cones of ideals of almost minimal mixed multiplicity in the two dimensional case. In Section 5 the main theorem is proved for all Cohen-Macaulay local rings of dimension $\geq 2$. In section 6 we provide some examples to illustrate the main theorem. Finally we answer a question about Cohen-Macaulay property of fiber cones raised in [4].

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2. Preliminaries

In this section we prove some preliminary results which will be used in the subsequent sections. We begin by characterizing ideals of almost minimal mixed multiplicity.
Proposition 2.1. Let \((R, m)\) be a Cohen-Macaulay local ring with infinite residue field. Suppose \(I\) is an \(m\)-primary ideal. Then \(I\) has almost minimal mixed multiplicity if and only if for any joint reduction \((x, a_1, a_2, \ldots, a_{d-1})\) of \((m|I^{d-1})\),

\[ \ell(mI/xI + (a_1, a_2, \ldots, a_{d-1})m) = 1. \]

**Proof:** From the proof of Theorem 2.4 of [5] we have

\[ \ell \left( \frac{R}{xI + (a_1, \ldots, a_{d-1})m} \right) - \ell \left( \frac{R}{I} \right) = d - 1 + e_{d-1}(m|I). \]

Hence

\[ \ell \left( \frac{mI}{xI + (a_1, \ldots, a_{d-1})m} \right) + \mu(I) = d - 1 + e_{d-1}(m|I). \]

The proposition is clear from the above equation.

Lemma 2.2. Let \((R, m)\) be a Cohen-Macaulay local ring of positive dimension \(d\). Let \(I\) be an \(m\)-primary ideal of \(R\) having almost minimal mixed multiplicity. Then for any joint reduction \((x, a_1, \ldots, a_{d-1})\) of \((m|I^{d-1})\),

\[ \ell(mI^n/(a_1, \ldots, a_{d-1})mI^{n-1} + xI^n) \leq 1 \text{ for all } n \geq 1. \]

**Proof:** Put \(J = (a_1, \ldots, a_{d-1})\). Since \(\mu(I) = e_{d-1}(m|I) + d - 2\), by Proposition 2.1 \(\ell(Jm/xI + Jm) = 1\). Hence there exists \(y \in m\) and \(b \in I\) such that \(mI = Jm + xI + (yb)\) and \(mI^n \subseteq Jm + xI\). We claim that for all \(n \geq 1\), \(mI^n = JmI^{n-1} + xI^n + (yb^n)\). If \(n = 1\), we are done by Proposition 2.1. If \(n > 1\), then by induction hypothesis

\[ mI^n = (mI^{n-1})I = (JmI^{n-2} + xI^{n-1} + (yb^{n-1}))I = JmI^{n-1} + xI^n + (yb^{n-1})I. \]

Since

\[ (yb^{n-1})I \subseteq bmI^{n-1} = (JmI^{n-2} + xI^{n-1} + (yb^{n-1}))b \subseteq JmI^{n-1} + xI^n + (yb^n), \]

\[ mI^n = JmI^{n-1} + xI^n + (yb^n). \]

Since \(m(yb) \subseteq Jm + xI\),

\[ m(yb^n) = m(yb)(b^{n-1}) \subseteq (Jm + xI)(b^{n-1}) \subseteq JmI^{n-1} + xI^n. \]

Hence \(\ell(mI^n/JmI^{n-1} + xI^n) \leq 1\).

**Notation:** Let \(I\) be an ideal of a ring \(R\). For an element \(a \in I\) let \(a^*\) (resp. \(a^o\)) denote its residue class in \(I/mI\) (resp. \(I/I^2\)).
Lemma 2.3. Let \((R, m)\) be a Cohen-Macaulay local ring of dimension \(d\) and \(I\) an \(m\)-primary ideal of \(R\). Suppose \(\gamma(I) \geq d - 1\) and \(\phi(I) \geq d - 2\). Then there exists a joint reduction \((x, a_1, a_2, \ldots, a_{d-1})\) of \((m|I^{[d-1]}\)) such that \(a_1^*, \ldots, a_{d-1}^*\) and \(a_1, \ldots, a_{d-2}\) are regular sequences in \(G(I)\) and \(F(I)\) respectively.

Proof: By Lemma 1.2 of [12] and its consequences, we can find the desired joint reduction by avoiding the contractions to \(R\) of the associated primes of \((t^{-1}, m)\) and \((t^{-1})\) in \(R[It, t^{-1}]\) at each stage and using Lemma 5.2.

3. Main theorem in dimension one

In this section we will prove the main theorem for one dimensional Cohen-Macaulay local rings. We first recall the concept of reduction of ideals from [10]. Let \((R, m)\) be a local ring with infinite residue field. An ideal \(J\) of \(R\) is called a reduction of an ideal \(I\) of \(R\) if \(J \subset I\) and there exists an integer \(n\) such that \(JI^n = I^{n+1}\). If \(J\) is the smallest such ideal then it is called a minimal reduction of \(I\). All minimal reductions of \(I\) are generated by the same number of elements which is called the analytic spread of \(I\) and it is equal to the Krull dimension of \(F(I)\). The reduction number \(r_J(I)\) of an ideal \(I\) with respect to a minimal reduction \(J\) is the smallest integer \(n\) such that \(JI^n = I^{n+1}\). The reduction number \(r(I)\) of \(I\) is the minimum of \(r_J(I)\) where \(J\) ranges over all the minimal reductions of \(I\).

Throughout this section \((R, m)\) will denote a Cohen-Macaulay local ring of dimension one with infinite residue field.

Lemma 3.1. Let \((x)\) be a minimal reduction of \(m\). Then

\[
H(F(I), \lambda) = \frac{e(R)}{1 - \lambda} - H\left(\frac{mR[It]}{xR[It]}, \lambda\right).
\]

Proof: Since \((x)\) is a minimal reduction of \(m, x\) is a nonzerodivisor. Therefore \(\ell(xR/xI^n) = \ell(R/I^n)\) for all \(n \geq 0\). Hence for all \(n \geq 0\),

\[
\mu(I^n) = \ell\left(\frac{R}{xR}\right) + \ell\left(\frac{xR}{xI^n}\right) - \ell\left(\frac{mR[It]}{xR[It]}\right) - \ell\left(\frac{R}{I^n}\right) = e(R) - \ell\left(\frac{mI^n}{xI^n}\right).
\]

Therefore

\[
H(F(I), \lambda) = \sum_{n=0}^{\infty} \mu(I^n) \lambda^n = \sum_{n \geq 0} \left[ e(R) - \ell\left(\frac{mI^n}{xI^n}\right) \right] \lambda^n = \frac{e(R)}{1 - \lambda} - H\left(\frac{mR[It]}{xR[It]}, \lambda\right).
\]
Proposition 3.2. Suppose \( \mu(I) = e(R) \). Then

\[
H(F(I), \lambda) = \frac{1 + (e(R) - 1)\lambda}{1 - \lambda}.
\]

**Proof:** We have \( \mu(I) = e(R) \) if and only if \( mI = xI \) by (4). Hence \( mI^n = xI^n \) for all \( n \geq 1 \). Therefore by Lemma 3.1,

\[
H(F(I), \lambda) = \frac{e(R)}{1 - \lambda} - \ell \left( \frac{m}{xR} \right) = \frac{e(R)}{1 - \lambda} - e(R) + 1 = \frac{1 + (e(R) - 1)\lambda}{1 - \lambda}.
\]

Theorem 3.3. Suppose \( \mu(I) = e(R) - 1 \). Then the Hilbert series of \( F(I) \) is given by

\[
H(F(I), \lambda) = \begin{cases} 
\frac{1 + (e(R) - 2)\lambda}{1 - \lambda} & \text{if } r(m|I) = \infty, \\
\frac{1 + (e(R) - 2)\lambda + \lambda^s}{1 - \lambda} & \text{if } r(m|I) = s.
\end{cases}
\]

**Proof:** Let \((x)\) be a minimal reduction of \( m \). Apply Lemma 3.1. If \( \ell(mI^n/xI^n) = 1 \) for all \( n \geq 1 \) then

\[
H(F(I), \lambda) = \frac{e(R)}{1 - \lambda} - \ell \left( \frac{m}{xR} \right) - \sum_{n \geq 1} \ell \left( \frac{mI^n}{xI^n} \right) \lambda^n
\]

\[
= \frac{e(R)}{1 - \lambda} - (e(R) - 1) - \sum_{n \geq 1} \lambda^n
\]

\[
= \frac{1 + (e(R) - 1)\lambda}{1 - \lambda} - \frac{\lambda}{1 - \lambda}
\]

\[
= \frac{1 + (e(R) - 2)\lambda}{1 - \lambda}.
\]

Now let \( \ell(mI^n/xI^n) = 0 \) for some \( n \). Let \( s = \min\{n|mI^n = xI^n\} \). Hence by Lemma 2.2 \( \ell(mI^n/xI^n) = 1 \) for all \( n = 1, 2, \ldots, s - 1 \). Therefore

\[
H(F(I), \lambda) = \frac{e(R)}{1 - \lambda} - \ell \left( \frac{m}{xR} \right) - \sum_{n=1}^{s-1} \ell \left( \frac{mI^n}{xI^n} \right) \lambda^n
\]

\[
= \frac{e(R)}{1 - \lambda} - (e(R) - 1) - \lambda(1 + \lambda + \ldots + \lambda^{s-2})
\]

\[
= \frac{1 + (e(R) - 2)\lambda + \lambda^s}{1 - \lambda}.
\]
4. The main theorem in dimension two

The techniques of mixed multiplicities can be illustrated only in rings of dimension two or higher. The main theorem will be proved by induction on the dimension of $R$. Throughout this section $(R, m)$ will denote a Cohen-Macaulay local ring of dimension two with infinite residue field.

Let $I$ and $J$ be ideals in $R$. Let $x, y$ be elements in $R$. Consider the sequence

\[
0 \rightarrow \frac{R}{(I : y) \cap (J : x)} \xrightarrow{\psi} \frac{R}{I} \oplus \frac{R}{J} \xrightarrow{\phi} \frac{(x, y)}{xI + yJ} \rightarrow 0
\]

where $\phi(a', b') = (ax + by)'$ and $\psi(r') = ((-ry)', (rx)')$. Here primes denote the residue classes.

Lemma 4.1. If $x, y$ a regular sequence in $R$, then the sequence (2) is exact.

Proof: Let $(a', b') \in \text{Ker} \phi$. Then $ax + by = xi + yj$ for some $i \in I$ and $j \in J$. Hence $x(a - i) = y(j - b)$. Since $x, y$ is a regular sequence, there exists $t \in R$ such that $a = i - yt$ and $b = j + xt$. Therefore $(a', b') = ((-yt)', (xt)') = \psi(t')$ which implies that $\text{Ker} \phi = \text{Im} \psi$. It is easy to see that $\psi$ is injective.

Lemma 4.2. Let $(x, a)$ be a joint reduction of the set of ideals $(m, I)$. Then

\[
H(F(I), \lambda) = \frac{1 + (e_1(m|I) - 1)\lambda}{(1 - \lambda)^2} - \frac{1}{1 - \lambda} \sum_{n=1}^{\infty} \ell \left( \frac{m I^n}{x I^n + a m I^{n-1}} \right) \lambda^n
\]

\[
+ \frac{1}{1 - \lambda} \sum_{n=1}^{\infty} \ell \left( \frac{(m^{n-1} : x) \cap (I^n : a)}{I^{n-1}} \right) \lambda^n.
\]

Proof: Put $y = a$, $J = m I^{n-1}$ and $I = I^n$ in (2) to get

\[
\ell \left( \frac{x, a}{x I^n + a m I^{n-1}} \right) - \ell \left( \frac{R}{I^n} \right) - \ell \left( \frac{R}{m I^{n-1}} \right) + \ell \left( \frac{R}{(I^n : a) \cap (m I^{n-1} : x)} \right) = 0.
\]

Hence

\[
\ell \left( \frac{I^n}{m I^n} \right) - \ell \left( \frac{I^{n-1}}{m I^{n-1}} \right) = e_1(m|I) - \ell \left( \frac{m I^n}{x I^n + a m I^{n-1}} \right) + \ell \left( \frac{(m I^{n-1} : x) \cap (I^n : a)}{I^{n-1}} \right).
\]
Therefore
\[(1 - \lambda)H(F(I, \lambda) = \sum_{n=1}^{\infty} \left[ \mu(I^n) - \mu(I^{n-1}) \right] \lambda^n = 1 + \sum_{n=1}^{\infty} \left[ e_1(m|I) - \ell \left( \frac{mI^n}{xI^n + amI^{n-1}} \right) + \ell \left( \frac{mI^{n-1} : x \cap (I^n : a)}{I^{n-1}} \right) \right] \lambda^n = 1 + \sum_{n=1}^{\infty} \left[ \frac{1 + (e_1(m|I) - 1)\lambda}{1 - \lambda} - \frac{1}{1 - \lambda} \right] \lambda^n = 1 + \frac{(e_1(m|I) - 2)\lambda}{(1 - \lambda)^2}.
\]

Theorem 4.3. Suppose \( \mu(I) = e_1(m|I) \) and \( \gamma(I) \geq 1 \). Then

\[H(F(I), \lambda) = \begin{cases} \frac{1 + (\mu(I) - 2)\lambda}{(1 - \lambda)^2} & \text{if } r(m|I) = \infty, \\ 1 + \lambda(\mu(I) - 2) + \lambda^s & \text{if } r(m|I) = s. \end{cases} \]

Proof: By Lemma 2.3 there exists a joint reduction \((x, a)\) of \((m, I)\) such that \(a^o\) is a nonzerodivisor in \(G(I)\). Hence \((I^n : a) = I^{n-1}\) for all \(n \geq 0\). By Lemma 2.2 \(\ell(mI^n/xI^n + amI^{n-1}) \leq 1\) for all \(n \geq 1\). If \(\ell(mI^n/xI^n + amI^{n-1}) = 1\) for all \(n \geq 1\) then by Lemma 4.2,

\[H(F(I), \lambda) = \frac{1 + (e_1(m|I) - 1)\lambda}{1 - \lambda} - \frac{1}{1 - \lambda} \sum_{n=1}^{\infty} \lambda^n = \frac{1 + (e_1(m|I) - 2)\lambda}{(1 - \lambda)^2}.
\]

If \(r(m|I) = s\) then by Lemma 4.2,

\[H(F(I), \lambda) = \frac{1 + (e_1(m|I) - 1)\lambda}{1 - \lambda} - \frac{1}{1 - \lambda} \sum_{n=1}^{s-1} \lambda^n = \frac{1 + (e_1(m|I) - 2)\lambda + \lambda^s}{(1 - \lambda)^2}.
\]

5. The main theorem in dimension \(\geq 3\)

In this section we prove the main theorem in Cohen-Macaulay local rings of dimension at least 3. We do this by going modulo a regular element whose initial form in \(F(I)\) and \(G(I)\) is simultaneously regular. This preserves all the hypotheses and we invoke the result in dimension 2.

Lemma 5.1. Let \((R, m)\) be a local ring. Let \(I\) be an ideal of \(R\) and \(a \in I \setminus mI\). Then \(a^o\) is a nonzerodivisor in \(F(I)\) if and only if \((mI^{n+1} : a) \cap I^n = mI^n\) for all \(n \geq 0\).
Proof: Suppose $a^*$ is a nonzerodivisor in $F(I)$. If $b \in (mI^{n+1} : a) \cap I^n$ then $ba \in mI^{n+1}$. Thus $b^*a^* = 0$. Since $a^*$ is a nonzerodivisor, $b^* = 0$. Hence $b \in mI^n$.

Conversely suppose $(mI^{n+1} : a) \cap I^n = mI^n$ for all $n \geq 0$. As $0 : a^*$ is a homogeneous ideal, it is enough to show that it contains no nonzero homogeneous element. Let $b \in I^n$ and $b^*a^* = 0$. Then $ba \in mI^{n+1}$. Hence $b \in (mI^{n+1} : a) \cap I^n = mI^n$, which implies that $b^* = 0$.

Lemma 5.2. Let $I$ be an ideal of a local ring $(R,m)$ and $a \in I \setminus mI$ be a nonzerodivisor. Suppose $a^o$ is a nonzerodivisor in $G(I)$ and $b \in I^n$. Then the map

$$\phi : F(I)/a^*F(I) \longrightarrow F(I/aR), \quad \phi((b + mI^n)^\dagger) = b + (mI^n + aR)$$

is an isomorphism.

Proof: As $a^o$ is a nonzerodivisor in $G(I)$, $aR \cap I^n = aI^{n-1}$ for all $n \geq 0$. Therefore $mI^n + aI^{n-1} = mI^n + (aR \cap I^n)$. Thus

$$F(I/aR) \simeq \bigoplus_{n=0}^{\infty} (I^n + aR)/(mI^n + aR) \simeq \bigoplus_{n=0}^{\infty} I^n/(mI^n + (I^n \cap aR)) \simeq F(I)/a^*F(I).$$

Lemma 5.3. As per the notation in the above lemma, let $a^*$ and $a^o$ be nonzerodivisors in $G(I)$ and $F(I)$ respectively. Then

$$H(F(I/aR), \lambda) = (1 - \lambda)H(F(I), \lambda).$$

Proof: This is clear by the isomorphism in Lemma 5.2.

Lemma 5.4. Let $J = (x, a_1, a_2, \ldots, a_{d-1})$ be a joint reduction of $(m|I^{d-1})$. Suppose $r_J(m|I)$ is finite. If $a_1^*$ is a nonzerodivisor in $F(I)$ and $a_1^0$ is a nonzerodivisor in $G(I)$ then $r_J(m|I) = r_J(m'|I')$. Here $'$ denotes the residue class modulo $(a_1)$.

Proof: Suppose $mI^n + a_1 R = xI^n + (a_2, \ldots, a_{d-1})mI^{n-1} + a_1 R$. Then for any $z \in mI^n$ there exist $b \in I^n$; $c_2, c_3, \ldots, c_{d-1} \in mI^{n-1}$ and $r \in R$ such that $z = xb + a_2c_2 + \ldots + a_{d-1}c_{d-1} + a_1r$.

Hence $ra_1 \in (mI^n \cap a_1 R) \subset I^n \cap a_1 R = a_1 I^{n-1}$, as $a_1^0$ is a nonzerodivisor. Thus $r \in I^{n-1}$. As $a^*$ is a nonzerodivisor, $r \in (mI^n : a_1) \cap I^{n-1} = mI^{n-1}$. Thus $z \in xI^n + (a_1, a_2, \ldots, a_{d-1})mI^{n-1}$.

Theorem 5.5. Let $(R,m)$ be a Cohen-Macaulay local ring of dimension $d \geq 2$. Let $I$ be an $m$–primary ideal of $R$ with almost minimal mixed multiplicity. If $\gamma(I) \geq d-1$ and $\phi(I) \geq d-2$ then
Proposition 5.6. Let \( I \) are Cohen-Macaulay. It is easy to see that 
\[ L = (1 + (\mu(I) - d)\lambda)/(1 - \lambda)^d \]
if \( r(m|I) = \infty \),
\[ 1 + (\mu(I) - d)\lambda + \lambda^s/(1 - \lambda)^d \]
if \( r(m|I) = s \).

Proof: Apply induction on \( d \). The \( d = 2 \) case has been proved in Theorem 4.2. Suppose \( d \geq 3 \). By Lemma 2.3 there exists a joint reduction \( (x, a_1, a_2, \ldots, a_3) \) of \( (m|I^{[d-1]}) \) where \( a_1^*, a_2^*, \ldots, a_{d-1}^* \) is a regular sequence in \( G(I) \) and \( a_1^*, a_2^*, \ldots, a_{d-2}^* \) is a regular sequence in \( F(I) \).

Put \( J = (a_1, a_2, \ldots, a_{d-1})R, M = (a_1^*, a_2^*, \ldots, a_{d-2}^*)G(I), K = (a_1^*, a_2^*, \ldots, a_{d-2}^*)F(I) \) and 
\[ L = (a_1, a_2, \ldots, a_{d-2})R. \]

By repeated use of Lemma 5.2, we get 
\[ F(I)/K \cong F(I'). \]
Moreover \( e_{d-1}(m|I) = e_1(m'|I') \) by 4 and 
\[ \mu(I') = \mu(I) - d + 2 = e_{d-1}(m|I) = e_1(m'|I'). \]

Thus \( I' \) has almost minimal mixed multiplicity. As \( a_1^*, a_2^*, \ldots, a_{d-2}^* \) is a regular sequence in \( F(I) \), 
\[ H(F(I), \lambda)(1 - \lambda)^{d-2} = H(F(I)/K, \lambda). \]
The result follows by Theorem 4.3.

Proposition 5.6. Let \((R, m)\) be a Cohen-Macaulay local ring of dimension \( d \geq 2 \). Let \( I \) be an \( m \)-primary ideal of \( R \). Suppose that \((x, a_1, \ldots, a_{d-1})\) is a joint reduction of \((m|I^{[d-1]})\) satisfying the conditions (i) \( mI^2 = xI^2 + JmI \), where \( J = (a_1, \ldots, a_{d-1}) \) and (ii) \( a_1^*, \ldots, a_{d-1}^* \) is a regular sequence in \( G(I) \). Then depth \( F(I) \geq d - 1 \).

Proof: By Theorem 2.8 of [3], it is enough to show that for all \( n \geq 2 \), \( J \cap mI^n = mJI^{n-1} \). But \( mI^n = xI^n + J^{n-1}mI \) for all \( n \geq 2 \). Hence \( J \cap mI^n = (J \cap xI^n) + J^{n-1}mI \). It remains to show that \( J \cap xI^n \subseteq mJI^{n-1} \). Let \( y \in J \cap xI^n \). Then \( y = xi \) for some \( i \in I^n \). Hence \( i \in (J : x) \cap I^n = JI^{n-1} \). Hence \( y \in mJI^{n-1} \).

6. Examples

Example 6.1. Let \( k \) be a field and \( R = k[[x, y, z]] \) be the power series ring. Put \( I = m^3 \) where \( m \) is the unique maximal ideal of \( R \). Then \( r(I) = 2 \), \( \mu(I) = 10 \) and \( e_2(m|I) = 9 \). Thus \( \mu(I) = e_2(m|I) + d - 2 \). It is easily seen that the fiber cone and the associated graded rings of \( I \) are Cohen-Macaulay. It is easy to see that \( r(m|m^3) = 2 \). The Hilbert series of \( F(I) \) is

\[
H(F(I), \lambda) = \sum_{n \geq 0} \left( \frac{3n + 2}{2} \right) \lambda^n = \sum_{n \geq 0} \left[ 9 \binom{n + 2}{2} - 9 \binom{n + 1}{1} + 1 \right] \lambda^n = \frac{1 + 7\lambda + \lambda^2}{(1 - \lambda)^2}
\]
Example 6.2. This example shows that for the main theorem in dimension 2 we need the depth hypothesis of $G(I)$. Let $R = k[x, y]_m$, where $k$ is an infinite field, $x$ and $y$ are indeterminates and $m = (x, y)$. Let $I = (x^4, x^3y, xy^3, y^4)$. Then $\mu(I) = 4 = e_1(m|I)$ and $F(I) = k[x^4, x^3y, xy^3, y^4]$. Note that for all $n \geq 2$, $I^n = m^{4n}$. Hence

$$H(F(I), \lambda) = 1 + 4\lambda + \sum_{n=2}^{\infty} (4n + 1)\lambda^n = \frac{1 + 2\lambda + 2\lambda^2 - \lambda^3}{(1 - \lambda)^2}.$$ 

As the numerator of the Hilbert series has a negative coefficient, $F(I)$ is not Cohen-Macaulay. Hence $\text{depth } F(I) = 1$. Since $G(I) \subseteq \text{ann}(x^2y^2)$, $\text{depth } G(I) = 0$.

Example 6.3. Let $t$ be an indeterminate and $R = k[[t^4, t^5, t^6, t^7]]$. Consider the ideal $I = (t^4, t^5, t^6)$. Then $(t^4)I^2 = I^3$ and $\mu(I) = e(R) - 1$. As

$$t^7I \subset I^2 = (t^8, t^9, t^{10}, t^{11}) = t^4(t^4, t^5, t^6, t^7)$$

the initial form of $t^7$ in $G(I)$ is a zerodivisor. Thus the depth of $G(I)$ is zero. It is easy to see that $r(m|I) = 2$. The Hilbert series of $F(I)$ is

$$H(F(I), \lambda) = 1 + 3\lambda + \sum_{n=2}^{\infty} 4\lambda^n + \ldots = \frac{1 + 2\lambda + \lambda^2}{(1 - \lambda)}.$$ 

It follows from [5] that $F(I)$ is Cohen-Macaulay.

Example 6.4. In this example we answer the question 3.7 of [4] partly. The question asks whether the fiber cones of all $m$-primary ideals in a one dimensional Cohen-Macaulay local ring of multiplicity 3 are Cohen-Macaulay? The answer is no. By Proposition 3.5 of [4] it is enough to consider ideals I minimally generated by 3 elements. By Proposition 2.3 of [4], it follows that for such ideals $F(I)$ is Cohen-Macaulay if and only if $r(I) \leq 1$. Consider the semigroup ring $R = k[[t^3, t^7, t^{11}]]$ and the ideal $I = (t^6, t^7, t^{11})$. Then $e(R) = \mu(I) = 3$. However $r(I) = 2$. Thus $F(I)$ is not Cohen-Macaulay.

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