The concept of Existence
and the derivation of a Quantum Force

C. COSTA

PUC-Rio, CP 38071, Rio de Janeiro, RJ 22452-970, Brazil

Abstract

We define a new dynamical variable, the relative existence $e$, in terms of space and time. Taking it as a generalized positional coordinate, we show that for conservative systems the canonically conjugated momentum is identified as the classical force. Applying Wilson-Sommerfeld-Bohr’s quantum conditions for a conditionally periodic motion, we derive an expression for the quantum force, $F = \hbar k f_e$, where $k$ is the wave number and $f_e$ is the characteristic frequency of the system. Applying Dirac’s method to the Poisson Brackets involving existence and force, we obtain the uncertainty relation $\Delta e \Delta F \geq \hbar / 2$.

The force quantization may have already been observed in stimulated bichromatic optical force experiments, used to deflect, decelerate and manipulate laser-cooled atomic beams.

Key words: Foundations of Quantum Mechanics, Quantum Force, Quantum Optics, Bichromatic Optical Force
PACS: 03.65.Ta, 32.80.-t, 42.50.Xa
1 Introduction

Based on Planck-Einstein relation for the quantized energy of a photon, $E = h\nu$, and the results of Rutherford experiments on the scattering of alpha particles by matter, Niels Bohr was able to propose the model for the hydrogen atom, defining discontinuous stationary states, so that emission or absorption of radiation corresponds to transitions between such quantized states[1]. The discrete energy levels can be calculated by means of a set of quantum conditions to which the canonical variables $q_i$ and $p_i$ of classical mechanics were to be subjected[2]. Even filled with limitations and being a mix of classical and quantum concepts, Bohr’s “old quantum theory” opened up the way to the Quantum Mechanics.

In this paper we propose a new dynamical variable, related to space and time, which we call relative existence. Regarding existence as a canonical positional variable, we follow wilson-Sommerfeld-Bohr’s prescription, looking after its canonical conjugate momentum and writing the corresponding quantum conditions. We find that for conservative systems, like the simple harmonic oscillator, the variable conjugate to existence is the classical force, so that the quantum conditions lead to the expression for a quantum force. Applying Dirac’s method of quantization[3], it is possible to write the existence-force uncertainty relation. Finally we discuss the possibility that force quantization may have already been manifested in the stimulated bichromatic optical force[4,5,6,7], measured as a deflection or deceleration of atomic beams subjected to a field of two counterpropagating short laser pulses, in laser cooling and atom manipulation experiments [8,9,10].

2 Dynamical variables: defining relative existence

In classical mechanics, the one-dimensional acceleration $a$ is defined as the derivative of velocity $v$ with respect to time $t$, $a = dv/dt$, so that $v = v_0 + \int a \, dt$. Similarly, velocity is defined as the derivative of the position $x$ with respect to time, $v = dx/dt$, so that $x = x_0 + \int v \, dt$.

We now define a new variable, say $e$, enabling us to write the position as the derivative of $e$ with respect to time,

$$x = \frac{de}{dt}, \quad \text{so that:}$$

$$e = e_0 + \int x \, dt. \quad (1)$$

(2)
This variable is related to a body occupying some place in space, extended over the time, and we suggest to name it “relative existence”, or simply existence. Even not moving, the body has some degree of existence. As it moves or it accelerates, more and more intense is the accumulation of existence, as for the mass in special relativity. The absolute value of the variation of existence shall be called experience \( X_p \equiv |e - e_0| = |\Delta e| \), in a fashion similar to the displacement, which is regarded as the absolute value of the variation of position, or to the impulse, the variation of momentum. We shall therefore speak of particles being created at particular space-time coordinates and evolving their existences, accumulating experience more or less rapidly, depending on their behavior.

From definition (2) we obtain the relativistic equation for the existence, in terms of Lorentz Transformations. Let the object denoted by 0 move with respect to the reference system 1, with velocity \( v_{01} \) at the position \( x_{01} \), then for a second system 2, moving with relative velocity \( u_{21} \) with respect to 1, we have:

\[
\begin{align*}
v_{02} &= \gamma^0 \left(1 + \frac{v_{01}u_{12}}{c^2}\right)^{-1} (v_{01} + u_{12}), \\
x_{02} &= \gamma^1 \left(1 + \frac{v_{01}u_{12}}{c^2}\right)^0 \left(x_{01} + (ct_{01}) \frac{u_{12}}{c}\right), \\
ct_{02} &= \gamma^1 \left(1 + \frac{v_{01}u_{12}}{c^2}\right)^0 \left(ct_{01} + (x_{01}) \frac{u_{12}}{c}\right), \\
\epsilon_{02} &= \gamma^2 \left(1 + \frac{v_{01}u_{12}}{c^2}\right)^1 (\epsilon_{01} + \epsilon_{12});
\end{align*}
\]

where \( \gamma = (1 - u^2/c^2)^{-1/2} \) is the Lorentz factor, and the equations are written in a way to stress the relationship between the powers of different factors.

### 3 Canonically conjugate variable: force

Let us assume that the existence may be regarded as a generalized coordinate \( q \):

\[ q = e, \]

(7)

then, the generalized velocity is

\[ \dot{q} = \frac{de}{dt} \equiv x. \]

(8)
In terms of the Lagrangian $\mathcal{L}$ of the system, the generalized momentum $p_e$, conjugate to $q = e$, is

$$p_e = \frac{\partial \mathcal{L}}{\partial \dot{q}} = \frac{\partial \mathcal{L}}{\partial x}. \quad (9)$$

For conservative systems the Lagrangian is given by $\mathcal{L} = T - V(x)$, where $T$ is the kinetic energy, with no explicit dependence on the position $x$, and $V(x)$ is the potential energy. In this case,

$$p_e = \frac{\partial \mathcal{L}}{\partial x} = -\frac{\partial V(x)}{\partial x} \equiv F, \quad (10)$$

meaning that we may identify the canonical conjugate to the existence $e$ as the classical force $F$.

The Simple Harmonic Motion (SHM) easily illustrates equation (10). Hook’s law for a spring acting upon a particle of mass $m$ gives:

$$F = -kx = -(m\omega^2)x, \quad (11)$$

where $k = m\omega^2$ is the effective spring constant and $\omega$ is the harmonic oscillator angular frequency. Newton’s Law then implies the equation of motion given by $\ddot{x} = -\omega^2 x$, with the known solution of the form $x(t) = x_o \cos(\omega t + \phi)$. Assuming $e_o = 0$ at $t = 0$, the time evolution of the existence is

$$e(t) = -\frac{1}{\omega^2} \dot{x}(t). \quad (12)$$

Since $V(x) = \frac{1}{2} kx^2 = \frac{1}{2} m\omega^2 x^2$, we can write the Lagrangean for the SHM,

$$\mathcal{L}(x, \dot{x}) = T - V = \frac{1}{2} m\dot{x}^2 - \frac{1}{2} m\omega^2 x^2, \quad (13)$$

in terms of the generalized coordinates $q = e$ and $\dot{q} = x$:

$$\mathcal{L}(e, x) = \frac{1}{2} m\omega^4 e^2 - \frac{1}{2} m\omega^2 x^2. \quad (14)$$

The canonical conjugate momentum is therefore

$$p_e \equiv \frac{\partial \mathcal{L}}{\partial \dot{q}} = \frac{\partial \mathcal{L}}{\partial x} = -m\omega^2 x, \quad (15)$$
which is readily recognized as the classical force, eq.(11).

Moreover, the generalized force $Q_e$ is, here,

$$Q_e \equiv \frac{\partial L}{\partial q} = \frac{\partial L}{\partial e} = m\omega^4 e,$$

(16)

so that the Euler-Lagrange equation,

$$\frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}} \right) = \frac{\partial L}{\partial q}, \quad \text{or:} \quad \frac{d}{dt} (p_e) = Q_e,$$

(17)

is verified for the existence-force conjugates, in view of equations (15), (16) and (12):

$$\frac{d}{dt} (p_e) = -m\omega^2 \dot{x} = -m\omega^2 (-\omega^2 e) = m\omega^4 e = Q_e.$$

(18)

Similarly, it is straightforward to verify that the Hamilton Canonical Equations also hold.

4 Bohr’s quantum conditions: deriving the quantum force

For any physical system in which the coordinates are periodic function of time, there exist a quantum condition for each coordinate. These are the Wilson-Sommerfeld-Bohr quantum conditions, constraining the classical action integral $I_k$ according to[2]:

$$I_k = \oint p_k dq_k = n_k \hbar,$$

(19)

where the quantum number $n_k$ is an integer, $p_k$ is the momentum associated to the generalized coordinate $q_k$, and each integral is taken over a full period of this coordinate.

Applied to the circular motion with a central field force, corresponding to the angle-angular momentum conjugate variables ($\theta - L$), Bohr obtained the angular momentum quantization for the Hydrogen atom as

$$L = n \frac{\hbar}{2\pi} = n \hbar,$$

(20)
and derived thereafter the quantization of atomic orbits and the correct energies for the line-spectra. Regarding the time-energy pair \((t - E)\) as conjugates, eq.(19) leads directly to Planck-Einstein photon energy quantization,

\[ E = h\nu. \tag{21} \]

For the position-momentum conjugates \((x - p)\), we obtain the matter wave relation of De Broglie,

\[ p = \frac{h}{\lambda} = \hbar k. \tag{22} \]

Considering eq.(19) applied to the existence-force conjugates \((e - F)\), we have

\[ \oint F \, de = n \hbar. \tag{23} \]

Inspired by equations (20), (21) and (22), and following an ansatz similar to De Broglie’s, we may write

\[ F \oint dx = F \oint xd\tau = F \lambda \tau_e, \tag{24} \]

where \(\lambda\) is the associated wavelength (so that \(k = 2\pi/\lambda\)) and \(\tau_e = 1/f_e\) is the characteristic time of the system, with characteristic frequency \(f_e\). We therefore write:

\[ \tilde{F} \equiv \frac{n}{\lambda \tau_e} = n \hbar k f_e. \tag{25} \]

This is to be interpreted as an expression for the force quantization, so that the unit quantum force is \(\tilde{F} = \hbar k f_e\), where we used the tilde in \(\tilde{F}\) to remark its quantized nature and to distinguish from the classical counterpart \(F\).

5 Dirac’s quantum conditions: uncertainty relation

Poisson Brackets (PB) of two state variables \(a(q, p)\) and \(b(q, p)\) are defined as:

\[ \{a(q, p), b(q, p)\} \equiv \left( \frac{\partial a}{\partial q} \frac{\partial b}{\partial p} - \frac{\partial a}{\partial p} \frac{\partial b}{\partial q} \right). \tag{26} \]

The PB between the canonical coordinates and momenta, \(q\) and \(p\), is simply
\[ \{q, p\} = 1. \]
In Dirac’s point of view[3], the PB’s are the classical counterpart of the commutation rule of quantum operators, establishing a rule for the classical-quantum connection. Dirac defined the general quantum conditions, or commutation relations, for any two quantum quantities $a$ and $b$:

$$[a, b] = ab - ba \equiv \pm i \hbar \{a, b\},$$

(27)

which for the canonical conjugates reduces to Dirac’s fundamental quantum condition, $[q, p] = \pm i \hbar$. It is possible to show[11] that whenever the fundamental condition holds, one can write the corresponding Heisenberg uncertainty relation $\Delta q \Delta p \geq \hbar/2$.

Applied to the existence-force canonical conjugates $(e - F)$, we obtain equivalent relations:

$$\{e, F\} = 1,$$

(28)

$$[e, F] = \pm i \hbar,$$

(29)

$$\Delta e \Delta F \geq \frac{\hbar}{2}.$$

(30)

Take for example the SHM described before. Eq.(28) is readily satisfied, as for any canonical conjugate pair. To verify eq.(29), we just write the relations obtained classically:

$$q = e = \frac{\dot{x}}{-\omega^2} = \frac{p}{-m\omega^2},$$

$$p_e = F = m\ddot{x} = -m\omega^2 x,$$

(31)

and notice that in this particular case, the quantum mechanical counterparts may give:

$$[e, F] = \left[\frac{p}{-m\omega^2}, -m\omega^2 x\right]$$

$$= [p, x] = -i \hbar.$$  

(32)

From similar reasoning we conclude that, for the SHM, we may also have $\Delta e \Delta F = \Delta p \Delta x \geq \hbar/2$, and the limiting condition (30) is contemplated.
6 Experimental evidence: bichromatic optical force

We shall look for an experimental manifestation of quantum force effects. The quantum force may act under resonant conditions; and interaction of photons with atoms may be the primary source to produce such conditions. Ultra-short laser pulses may deal with small amounts of momentum transfer; and laser-cooled atoms may be suitable to investigate such exchanges minimizing non-quantum effects (like Doppler broadening). Considering all these, we are led to a series of experiments involving the stimulated bichromatic optical force on atoms \[4,5,6,7\], used in laser cooling and atom manipulation experiments \[8,9,10\].

Excited atoms return to their ground states either by spontaneous decay or stimulated emission. Spontaneous decay results in the dissipative radiative force, whose magnitude saturates at

\[
F_{rad} = \frac{\hbar k \Gamma}{2},
\]  

(33)

where \(\Gamma = 1/\tau\) is the natural linewidth, related to the excited state lifetime \(\tau\). Doppler cooling of two-level atoms has been achieved by an incoherent sequence of absorption followed by spontaneous emission, and the forces in the process are therefore limited to \(F_{rad}\). Laser cooling in multilevel atoms is achieved by the rapid coherent sequence of absorption followed by stimulated emission, through the use of multiple beams of monochromatic light, taking advantage of the dipole force working on the atoms. It can be shown that such sub-Doppler cooling forces are similarly limited to a magnitude of \(\hbar k \Gamma/2\). There has been the theoretical prediction\[12\] that a force in a field of two counterdirected short laser pulses, each containing two frequencies, could generate a force greater than the spontaneous one. From 1994 to 1997 this force has been demonstrated experimentally as deflection or deceleration of molecular \(\text{Na}_2\)[4], atomic \(\text{Na}\)[5] and atomic \(\text{Cs}\) beams\[6,7\]. Later on, experiments producing and measuring the so-called bichromatic optical force on atomic \(\text{Rb}\)[8] and \(\text{He}\)[9] were performed and the mechanism was studied in more detail \[10\].

The stimulated optical force is interpreted in terms of photon transfer induced by optical \(\pi\)-pulses\[6\]: the light field can be viewed as two counterpropagating trains of resonant beat pulses. Each light beam contains the two frequencies \(\omega \pm \delta\), detuned from the atomic resonance \(\omega\) by \(\pm \delta\) (hence the difference is \(2\delta\)). The produced beat frequency is \(1/T = \delta/\pi\), with the amplitude-modulation period \(T \ll \tau\). Their equal intensities \(I\) are chosen so that the so-called \(\pi\)-pulse condition is fulfilled: the Rabi frequency associated with a single monochromatic traveling wave is set to \(\Omega_R = \pi \delta/\Gamma\), where \(\Omega_R = \Gamma \sqrt{I/2I_s}\), with saturation frequency \(I_s = \pi hc/(3\lambda^3\tau)\). Under such condition, the probability of absorp-
tion (or of stimulating emission from the excited atom) is unity. The effect of each beat pulse is to invert the atomic population. The pulse trains can thus force an atom to cycle between its lower and upper state in such a way that it repeatedly absorbs photons from one wave and emits them into the other one. A total sharp momentum of $2\hbar k$ is thus transferred by one set of double $\pi$-pulses to every atom and the transfer rate is given by the beat frequency, so that in principle the corresponding optical force is $2\hbar k/T = 2\hbar k\delta/\pi$, much larger than $F_{rad}$. In practice, the appropriate optical force is achieved for the laser intensity under the condition $\Omega R \simeq \delta$, rather than $\Omega R = \pi\delta/4$, taking into account geometrical limitations that determine the laser beam size.

We now consider the stimulated light force experience in the context of the quantum force derived in eq.(25). For a two-level atomic system $\lambda$ is just the atomic transition wavelength. Under the $\pi$-pulse condition, the system cycles between these two states, until it eventually decays. The characteristic “existence-time” $\tau_e$ may not be just the upper state lifetime $\tau$, because atoms are either in excited or in ground state, and are being subjected to intense scattering light field. Let the characteristic existence frequency $f_e = 1/\tau_e$ be identified with $\Gamma/2$, the maximum scattering rate for a saturated optical transition with natural linewidth $\Gamma = 1/\tau$. We are therefore led to write the expression for the quantum force as

$$\tilde{F} \equiv \hbar k \Gamma/2.$$  

(34)

In spite of the equivalence between equations (33) and (34), the first equation sets a fundamental limit over continuous radiative forces, while the second one sets the quantum unit of interaction exchange.

Stimulated optical force experiments may have already seen the manifestation of such force quantization. The early work from Nölle et al.[5] displays the spatial atom distribution measured for the deflection of a thermal Na atomic beam. First with a single pulse train, the fitted deflection force for is given by $F = (0.8 \pm 0.1)\tilde{F}$, for most of the laser intensity range investigated, respecting the limit given by eq.(33). Switching a reflection mirror on, so that the $\pi$-pulse condition is fulfilled, the deflection force yields $F = (1.9 \pm 0.2)\tilde{F}$ and $F = (3.2 \pm 0.2)\tilde{F}$, for two different laser intensities, in accordance to eq.(34).

In both articles of Ref.[8], the deflection of a Rb atomic beam is measured and calculated for different setup parameters (the relative phase of the pulses of the counter-propagating light beams and the laser intensity). The corresponding bichromatic optical force is then displayed as a function of the atoms velocity $v$. The optical force magnitude is given in units of $\hbar k \Gamma$ (the authors use the notation $\gamma$ instead of $\Gamma$) and we must multiply their values by a factor two in order to get them in units of $\tilde{F} = \hbar k \Gamma/2$. Preliminary assessment suggests that the peak values of measured force are indeed about 1, 2, 3 and 4
units of $\tilde{F}$, indicating the manifestation of the quantum force, in the domain of stimulated bichromatic optical force experiments. In addition, these curves clearly present peaks for different values of the velocity, although sometimes they do overlap and become unresolved. We are tempted to assign a resonance pattern to those curves, meaning that the quantization of the force is correlated to discrete values of atomic beam velocity - in a fashion similar to the quantization of Bohrs orbits due to angular momentum quantization. A numerical correlation of the force quantization leading to velocity quantization, for this particular system, is still to be determined. One possibility is raised by the analysis of the acceleration power of the light field, in these standing wave field schemes, obtained by the stimulated conversion of photons from one frequency component into the other. According to Söding et al.[6], the acceleration power is limited by

$$ F v = \hbar \Omega R \Gamma / 2. \quad (35) $$

The quantum force requirement yields $\tilde{F} v = n \nu h k \Gamma / 2$, which together with eq.(35), in the saturated laser field domain, suggests a condition like

$$ v = \frac{1}{n} \frac{\Omega R}{k}. \quad (36) $$

Such correlation shall yet be further investigated, and confronted to the experimental data.

7 Conclusion

We have proposed the definition of a new dynamical variable, the existence $e$, entangling space and time, with the interpretation that any particle occupying certain position in space, as time goes by, exists, accumulating experience. Besides any kinematical consideration, interesting consequences arise when we assume the existence as a generalized canonical coordinate, obtaining force as the associated canonical momentum. Wilson-Sommerfeld-Bohr’s quantum conditions enabled us to derive the expression for a quantum force, and Dirac’s quantum prescription imposed an Heisenberg-like limitation to it. A few results from the simple harmonic oscillator are used to illustrate the existence-force relationship.

We argue that the recent measurement of stimulated bichromatic optical forces, observed as deflection of laser cooled atomic beams subjected to short counter-propagating $\pi$-pulses of laser, having magnitude several larger then the maximum limit obtainable by radiative forces, may contain indication of
the force quantization, in terms of multiples of $\bar{F} = \hbar k \Gamma / 2$. It may be not surprising that the force in these experiments is actually quantized, since the momentum transfer between the laser field and the atomic system is performed coherently in a stimulated fashion.

Future experiments in the field of bichromatic optical forces deserves to be carefully proposed with the purpose to investigate these conjectures on quantum force in closer detail. A possible correlation between quantized force and discrete values of atomic velocity resonant with the optical field momentum transfer is also to be verified.

Acknowledgements

The author wishes to thank C. Salles for support and encouragement.

References

[1] N. Bohr, Philosophical Magazine 26 (1913) 1.
[2] N. Bohr, On the quantum theory of line spectra, in: B.L. van der Waerden (Ed.), Sources of Quantum Mechanics, Dover, New York, 1968.
[3] P.A.M. Dirac, Proc. Royal Soc. A 109 (1925) 642; P.A.M. Dirac, The Principles of Quantum Mechanics, fourth ed., Clarendon, Oxford, 1976.
[4] V.S. Voitsekhovich, et al., JETP Lett. 59 (1994) 408.
[5] B. Nölle, H. Nölle, J. Schmand and H.J. Andrä, Europhys. Lett. 33 (1996) 261. Note that in this work $\gamma \equiv 1/(2\tau) = \Gamma / 2$.
[6] J. Söding, et al., Phys. Rev. Lett. 78 (1997) 1420. As in Ref.[5] $\gamma = \Gamma / 2$.
[7] A. Goepfert, et al., Phys. Rev. A 56 (1997) R3354.
[8] M.R. Williams, F. Chi, M.T. Cashen and H. Metcalf, Phys. Rev. A 60 (1999) R1763; idem, Phys. Rev. A 61 (2000) 023408.
[9] M.T. Cashen and H. Metcalf, Phys. Rev. A 63 (2001) 025406; M. Cashen, et al., Phys. Rev. A 64 (2001) 063411.
[10] H.J. Metcalf and P. van der Straten, J. Opt. Soc. America B 20 (2003) 887; M.T. Cashen and H. Metcalf, J. Opt. Soc. America B 20 (2003) 915.
[11] Claude Cohen-Tannoudji, Bernard Diu and Frank Laloë, Quantum Mechanics, Wiley, New York, 1977.
[12] A.P. Kazantsev, Sov. Phys. JETP 39 (1974) 784.