A weak comparison principle for solutions of very degenerate elliptic equations

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Abstract

We prove a comparison principle for weak solutions of elliptic quasi-linear equations in divergence form whose ellipticity constants degenerate at every point where \( \nabla u \in K \), where \( K \subseteq \mathbb{R}^N \) is a Borel set containing the origin.

1 Introduction

Let \( K \subseteq \mathbb{R}^N \), \( N \geq 2 \), be a Borel set containing the origin \( O \). We consider a vector function \( A : \mathbb{R}^N \rightarrow \mathbb{R}^N \), \( A \in L^\infty_{\text{loc}}(\mathbb{R}^N) \), such that

\[
\begin{cases}
A(\xi) = 0, & \text{if } \xi \in K, \\
[A(\xi) - A(\eta)] \cdot (\xi - \eta) > 0, & \forall \eta \in \mathbb{R}^N \setminus \{\xi\}, \text{ if } \xi \not\in K,
\end{cases}
\]

where \( \cdot \) denotes the scalar product in \( \mathbb{R}^N \). In this note we prove a comparison principle for Lipschitz weak solutions of

\[
\begin{cases}
-\operatorname{div} A(\nabla u) = g, & \text{in } \Omega, \\
u = \psi, & \text{on } \partial \Omega,
\end{cases}
\]

where \( \Omega \) is a bounded domain in \( \mathbb{R}^N \), \( \psi \in W^{1,\infty}(\Omega) \) and \( g \in L^1(\Omega) \). As usual, \( u \in W^{1,\infty}(\Omega) \) is a weak solution of (1.2) if \( u - \psi \in W^{1,\infty}_0(\Omega) \) and \( u \) satisfies

\[
\int_{\Omega} A(\nabla u) \cdot \nabla \phi dx = \int_{\Omega} g \phi dx, \quad \text{for every } \phi \in C_0^1(\Omega). \tag{1.3}
\]

For weak comparison principle we mean the following: if \( u_1, u_2 \) are two solutions of (1.2) with \( u_1 \leq u_2 \) on \( \partial \Omega \), then \( u_1 \leq u_2 \) in \( \Omega \). Clearly, the weak comparison principle implies the uniqueness of the solution.

It is well known that if \( K \) is the singleton \( \{O\} \), then (1.1) guarantees the validity of the weak comparison principle (see for instance [11] and [18]). For this reason, from now on \( K \) will be a set containing the origin and at least another point of \( \mathbb{R}^N \).

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Our interest in this kind of equations comes from recent studies in traffic congestion problems (see [2] and [3]), complex-valued solutions of the eikonal equation (see [13]–[16]) and in variational problems which are relaxations of non-convex ones (see for instance [4] and [10]).

As an example, we can think to
\[ f : [0, +\infty) \to [0, +\infty) \]

given by
\[ f(s) = \frac{1}{p}(s - 1)^p, \tag{1.4} \]

where \( p > 1 \) and \((\cdot)_+\) stands for the positive part, and consider the functional
\[ I(u) = \int_{\Omega} \left[ f(|\nabla u(x)|) - g(x)u(x) \right] dx, \quad u \in \psi W^{1,\infty}_0(\Omega). \tag{1.5} \]

As it is well-known, (1.3) is the Euler-Lagrange equation associated to (1.5) with
\[ A(\nabla u) = \frac{f'(|\nabla u|)}{|\nabla u|} \nabla u, \tag{1.6} \]

and it is easy to verify that \( A \) satisfies (1.1) with \( K = \{ \xi \in \mathbb{R}^N : |\xi| \leq 1 \} \). It is clear that in this case the monotonicity condition in (1.1) can be read in terms of the convexity of \( f \). Indeed, \( f \) is not strictly convex in \([0, +\infty)\) since it vanishes in \([0, 1]\); however, if \( s_1 > 1 \) then
\[ f((1 - t)s_0 + ts_1) < (1 - t)f(s_0) + tf(s_1), \quad t \in [0, 1], \]

for any \( s_0 \in [0, +\infty) \) and \( s_0 \neq s_1 \); the convexity holds in the strict sense whenever a value greater than 1 is considered.

Coming back to our original problem we notice that, since \( A \) vanishes in \( K \), (1.2) is strongly degenerate and no more than Lipschitz regularity of the solution can be expected. It is clear that if \( g = 0 \), then every function with gradient in \( K \) will satisfy the equation. Besides the papers cited before, we mention [1, 5, 9, 17] where regularity issues were tackled and [6] where it is proven that solutions to (1.2) satisfy an obstacle problem for the gradient in the viscosity sense. Here, we will not specify the assumptions on \( A \) and \( g \) that guarantee the existence of a Lipschitz solution and we refer to the mentioned papers for this interesting issue.

We stress that some regularity may be expected if we look at \( A(\nabla u) \). In [3] and [4] the authors prove some Sobolev regularity results for \( A(\nabla u) \) under more restrictive assumptions on \( A \) and \( g \). We also mention that results on the continuity of \( A(\nabla u) \) can be found in [8] and [17].

In Section 2 we prove a weak comparison principle for Lipschitz solutions of (1.3) by assuming the following: (i) one of the two solutions satisfies a Sobolev regularity assumption on \( A(\nabla u) \); (ii) the Lebesgue measure of the set where \( g \) vanishes is zero. As we shall prove, the former guarantees that the set where \( \nabla u \in K \) and \( g \) does not vanish has measure zero. The latter seems to be optimal for proving our result. Indeed, if we assume that \( g = 0 \), then any Lipschitz function with gradient in \( K \) would be a solution and we can not have a comparison between any two of such solutions. For instance, if we consider \( A \) as in (1.6) with \( f \) given by (1.4), then a simple example of functions that satisfy (1.2) is given by \( u_\sigma(x) = \sigma \text{dist}(x, \partial \Omega) \), with \( \sigma \in [-1, 1] \). Since every \( u_\sigma = 0 \) on \( \partial \Omega \), (1.2) does not have a unique solution and a comparison principle
can not hold. Generally speaking, any region where \( g \) vanishes will be source of problems for proving a comparison principle. We mention that, for \( A \) as in (1.6) and \( g = 1 \), a comparison principle for minimizers of (1.5) was proven in [7].

2 Main result

Before proving our main result, we need the following lemma which generalizes a result obtained in [12] for the p-Laplacian. In what follows, \( |D| \) denotes the Lebesgue measure of a set \( D \subset \mathbb{R}^N \).

**Lemma 2.1.** Let \( u \in W^{1,\infty}(\Omega) \) be a solution of (1.3), with \( A \) satisfying (1.1) and let
\[
Z = \{ x \in \Omega : \nabla u(x) \in K \}.
\]
If \( A(\nabla u) \in W^{1,p}(\Omega) \) for some \( p \geq 1 \), then
\[
|Z \setminus G_0| = 0,
\]
where
\[
G_0 = \{ x \in \Omega : g(x) = 0 \}.
\]
In particular, if \( |G_0| = 0 \) then \( |Z| = 0 \).

**Proof.** Since \( A(\nabla u) \in W^{1,p}(\Omega) \), then the function
\[
\frac{|A(\nabla u)|}{\varepsilon + |A(\nabla u)|} \in W^{1,p}(\Omega),
\]
for any \( \varepsilon > 0 \). Let \( \psi \in C^0_0(\Omega) \), set
\[
\phi(x) = \frac{|A(\nabla u(x))|}{\varepsilon + |A(\nabla u(x))|} \psi(x),
\]
and notice that \( \phi \in L^\infty(\Omega) \cap W_0^{1,p}(\Omega) \). Since \( u \) is Lipschitz continuous and \( A \in L^\infty_0(\mathbb{R}^N) \), we have that \( A(\nabla u) \in L^\infty(\Omega) \). Hence, by an approximation argument, \( \phi \) can be used as a test function in (1.3), yielding
\[
\int_{\Omega} \frac{|A(\nabla u)|}{\varepsilon + |A(\nabla u)|} A(\nabla u) \cdot \nabla \psi dx + \varepsilon \int_{\Omega} \frac{A(\nabla u) \cdot \nabla |A(\nabla u)|}{(\varepsilon + |A(\nabla u)|)^2} dx = \int_{\Omega} \frac{|A(\nabla u)|}{\varepsilon + |A(\nabla u)|} \psi g dx.
\]
It is clear that
\[
\int_{\Omega} \frac{|A(\nabla u)|}{\varepsilon + |A(\nabla u)|} \psi g dx = \int_{\Omega \setminus Z} \frac{|A(\nabla u)|}{\varepsilon + |A(\nabla u)|} \psi g dx,
\]
and that Cauchy-Schwarz inequality yields
\[
\left| \frac{\varepsilon A(\nabla u) \cdot \nabla |A(\nabla u)|}{(\varepsilon + |A(\nabla u)|)^2} \right| \leq |\nabla(|A(\nabla u)|)|
\]
uniformly for $\varepsilon > 0$. Since $\nabla (|A(\nabla u)|) \in L^p(\Omega)$, from (2.3)–(2.6) and by letting $\varepsilon$ go to zero, we obtain from Lebesgue’s dominated convergence theorem that

$$\int_{\Omega} A(\nabla u) \cdot \nabla \psi dx = \int_{\Omega \setminus \bar{Z}} g\psi dx,$$

for any $\psi \in C_0^1(\Omega)$. From (1.3) we have

$$\int_{\Omega} g\psi dx = \int_{\Omega \setminus \bar{Z}} g\psi dx,$$

for any $\psi \in C_0^1(\Omega)$, that is

$$g(x) = 0 \text{ for almost every } x \in Z,$$

which implies (2.2).

Our main result is the following.

**Theorem 2.2.** Let $u_j \in W^{1,\infty}(\Omega)$, $j = 1, 2$, be two solutions of (1.3), with $A$ satisfying (1.1) and $g$ such that $|G_0| = 0$, with $G_0$ given by (2.3). Furthermore, let us assume that $A(\nabla u_j) \in W^{1,p}(\Omega)$ for some $p \geq 1$ and $j \in \{1, 2\}$.

If $u_1 \leq u_2$ on $\partial \Omega$ then $u_1 \leq u_2$ in $\Omega$.

**Proof.** We proceed by contradiction. Let us assume that $U = \{x \in \Omega : u_1 > u_2\}$ is nonempty. Since $u_1$ and $u_2$ are continuous, then $U$ is open and we can assume that it is connected (otherwise we repeat the argument for each connected component). Without loss of generality, we can assume that $A(\nabla u_1) \in W^{1,p}(\Omega)$ and we define $E_1 = \{x \in \Omega : \nabla u_1 \notin K\}$.

Let $\phi = (u_1 - u_2)_+$. Since $u_1 \leq u_2$ on $\partial \Omega$, then $\phi \in W^{1,\infty}(\Omega)$ and (1.3) yields:

$$\int_{U} A(\nabla u_j) \cdot \nabla (u_1 - u_2) dx = \int_{U} g(u_1 - u_2) dx, \quad j = 1, 2.$$

By subtracting the two identities, we have

$$\int_{U} [A(\nabla u_1) - A(\nabla u_2)] \cdot (\nabla u_1 - \nabla u_2) dx = 0. \quad (2.7)$$

We notice that Lemma 2.1 yields $|\{\nabla u_1 \in K\}| = 0$ and thus

$$\int_{U} [A(\nabla u_1) - A(\nabla u_2)] \cdot (\nabla u_1 - \nabla u_2) dx =$$

$$= \int_{U \cap E_1} [A(\nabla u_1) - A(\nabla u_2)] \cdot (\nabla u_1 - \nabla u_2) dx; \quad (2.7)$$

and the monotonicity condition in (1.1) imply that

$$\nabla u_1 = \nabla u_2 \text{ a.e. in } U \cap E_1. \quad (2.8)$$

Since $|\{\nabla u_1 \in K\}| = 0$, we obtain that $\nabla u_1 = \nabla u_2$ a.e. in $U$. Being $u_1 = u_2$ on $\partial U$, we have that $u_1 = u_2$ in $U$, which gives a contradiction.  

It is clear that Theorem 2.2 implies the uniqueness of a solution for (1.2). Moreover, from Theorem 2.2 we also obtain the following comparison principle.
Corollary 2.3. Let \( u_j, j = 1, 2, \) \( A \) and \( g \) be as in Theorem 2.2. If \( u_1 < u_2 \) on \( \partial \Omega \) then \( u_1 < u_2 \) in \( \Omega \).

Proof. Since \( \partial \Omega \) is compact and \( u_1 \) and \( u_2 \) are continuous in \( \Omega \), there exists a constant \( c > 0 \) such that \( u_1 + c \leq u_2 \) on \( \partial \Omega \). Being \( u_1 + c \) a solution of \( (1.3) \), Theorem 2.2 yields \( u_1 + c \leq u_2 \) in \( \Omega \) and, since \( c \) is positive, we conclude.  

References

[1] L. Brasco: Global \( L^\infty \) gradient estimates for solutions to a certain degenerate elliptic equation. Nonlinear Anal., 74 (2011), 516-531.
[2] Brasco L., Carlier G.: On certain anisotropic elliptic equations arising in congested optimal transport: local gradient bounds. Preprint (2012). Available at [http://cvgmt.sns.it/paper/1890/]
[3] Brasco L., Carlier G., Santambrogio F.: Congested traffic dynamics, weak flows and very degenerate elliptic equations. J. Math. Pures Appl., 93 (2010), 652-671.
[4] Carstensen C., Müller S.: Local stress regularity in scalar nonconvex variational problems. SIAM J. Math. Anal., 34 (2002), 495-509.
[5] Celada P., Cupini G., Guidorzi M.: Existence and regularity of minimizers of nonconvex integrals with p - q growth. ESAIM Control Optim. Calc. Var., 13 (2007), 343–358.
[6] Ciraolo G.: A viscosity equation for minimizers of a class of very degenerate elliptic functionals. To appear in Geometric Properties for Parabolic and Elliptic PDE’s, Springer INdAM Series (2013).
[7] Ciraolo G., Magnanini R., Sakaguchi S.: Symmetry of minimizers with a level surface parallel to the boundary. Preprint (2012) [arXiv:1208.5295]
[8] Colombo M., Figalli A.: Regularity results for very degenerate elliptic equations. Preprint (2012). Available at [http://cvgmt.sns.it/paper/1996/]
[9] Esposito L., Mingione G., Trombetti C.: On the Lipschitz regularity for certain elliptic problems. Forum Math. 18 (2006), 263–292.
[10] Fonseca I., Fusco N., Marcellini P.: An existence result for a nonconvex variational problem via regularity. ESAIM Control Optim. Calc. Var. 7 (2002), 69–95.
[11] Gilbarg D., Trudinger N.S.: Elliptic partial differential equations of second order, Springer-Verlag, Berlin-New York, 1977.
[12] Lou H.: On singular sets of local solutions to p-Laplace equations. Chin. Ann. Math. 29B (2008), no. 5, 521-530.
[13] Magnanini R., Talenti G.: On complex-valued solutions to a 2D eikonal equation. Part one: qualitative properties. Nonlinear Partial Differential Equations, Contemporary Mathematics 283 (1999), American Mathematical Society, 203–229.
[14] ______: On complex-valued solutions to a 2D eikonal equation. Part two: existence theorems. SIAM J. Math. Anal. 34 (2003), 805–835.

[15] ______: On complex-valued solutions to a 2D Eikonal Equation. Part Three: analysis of a Backlund transformation. Applic. Anal. 85 (2006), no. 1-3, 249–276.

[16] ______: On complex-valued 2D eikonals. Part four: continuation past a caustic. Milan Journal of Mathematics 77 (2009), no. 1, 1–66.

[17] Santambrogio F., Vespri V.: Continuity in two dimensions for a very degenerate elliptic equation. Nonlinear Anal., 73 (2010), 3832-3841.

[18] Tolksdorf P.: Regularity for a more general class of quasilinear elliptic equations. J. Differential Equations, 51 (1984), 126-150.