Adaptive $\alpha$ Significance Level for Linear Models

D. Vélez$^1$, M.E. Pérez$^2$, and L. R. Pericchi$^2$

$^1$University of Puerto Rico, Río Piedras Campus, Statistical Institute and Computerized Information Systems, Faculty of Business Administration, 15 AVE Universidad STE 1501, San Juan, PR 00925-2535, USA

$^2$University of Puerto Rico, Río Piedras Campus, Department of Mathematics, Faculty of Natural Sciences, 17 AVE Universidad STE 1701, San Juan, PR 00925-2537, USA

Abstract

We put forward an adaptive alpha that decreases as the information grows, for hypothesis tests in which nested linear models are compared. A less elaborate adaptation was already presented in Pérez and Pericchi (2014) for comparing general i.i.d. models. In this article we present refined versions to compare nested linear models. This calibration may be interpreted as a Bayes-non-Bayes compromise, and leads to statistical consistency, and most importantly, it is a step forward towards statistics that leads to reproducible scientific findings.

Keywords: p-value calibration; Bayes factor, linear model; likelihood ratio; adaptive alpha; PBIC
1 Motivation

It is clear that obtaining a \( p \)-value lower than 0.05 no longer opens the doors for publication, but now statisticians must provide alternatives to scientists. One of the most important problems in statistics and in science as a whole, is to provide statistical measures of evidence that lead to reproducible scientific findings. In this article, we propose an adaptive alpha level for linear models that depend on the design matrices, the difference in dimension between the models and the "effective" sample size. The adaptive alpha, mimics the behaviour of a natural Bayes Factor, but uses the familiar concepts of Significance Hypothesis Testing. In Section 2, we present the basic derivation. The sampling distribution of minus \( n - 1 \) likelihood ratio is presented in Section 3. In Section 4, a condition is established for equivalence between tail probabilities and Bayes factor. We develop an adaptive alpha equivalent to a Bayes Factor in Section 5 and calibration strategies are discussed in Section 6. In Section 7, examples are presented and finally in Section 8 conclusions are advanced.

2 Basic Derivation

Consider the linear regression model \( \mathbf{y} = \mathbf{X}\beta + \mathbf{\epsilon} \), where \( \mathbf{y} \) represents the \( n \)-dimensional random vector of response variables, \( \mathbf{X} \) is the \( n \times k \) matrix of non-stochastic explanatory variables (for simplicity, here we assume that \( \mathbf{X} \) is a full rank matrix), \( \beta \) is a \( k \)-dimensional vector of regression parameters, \( \mathbf{\epsilon} \) is an \( n \)-dimensional vector of standard normal errors, i.e. \( \mathbf{\epsilon} \sim N(0, \sigma^2 \mathbf{I}_n) \), and \( \sigma \) is the standard deviation of the error, \( \sigma > 0 \).

We denote with \( M \) the full model whose matrix form is given by:
\[
\begin{bmatrix}
y_1 \\
y_2 \\
\vdots \\
y_n \\
\end{bmatrix}
= 
\begin{bmatrix}
1 & x_{12} & x_{13} & \cdots & x_{1k} \\
1 & x_{22} & x_{23} & \cdots & x_{2k} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
1 & x_{n2} & x_{n3} & \cdots & x_{nk} \\
\end{bmatrix}
\begin{bmatrix}
\beta_1 \\
\beta_2 \\
\vdots \\
\beta_k \\
\end{bmatrix}
+ 
\begin{bmatrix}
\epsilon_1 \\
\epsilon_2 \\
\vdots \\
\epsilon_n \\
\end{bmatrix}
\]

where \( \epsilon_i \sim N(0, \sigma^2) \), with \( 1 \leq i \leq n \).

Now suppose that we want to perform pairwise model comparisons between nested generic sub-models \( M_i \) and \( M_j \) from \( M \), where \( M_j \) is a sub-model having \( j (\leq k) \) regression coefficients, with \( M_i \) nested to \( M_j \). Formally, we want to test the hypothesis

\[
H_i : \text{Model } M_i \text{ \textit{versus} } H_j : \text{Model } M_j,
\]

in other words, we are comparing the following two nested linear models

\[
M_i : y = X_i \delta_i + \epsilon_i, \quad \epsilon_i \sim N(0, \sigma_i^2 I_n)
\]

and

\[
M_j : y = X_j \beta_j + \epsilon_j, \quad \epsilon_j \sim N(0, \sigma_j^2 I_n).
\]

So the Bayes Factor is:

\[
B_{ij}(y) = \frac{\int f(y | X_i \delta_i, \sigma_i^2 I_n) \pi^N(\delta_i, \sigma_i) d\delta_i d\sigma_i}{\int f(y | X_j \beta_j, \sigma_j^2 I_n) \pi^N(\beta_j, \sigma_j) d\beta_j d\sigma_j}.
\]

The construction of the adaptive alpha is based on \( B_{ij}(y) \), without explicit assessment of prior distributions by the user. Instead we will use well estab-
lished statistical practices to directly construct summaries of evidence.

1. **Approximation to Bayes factors under regularity conditions**: Laplace’s asymptotic method, under regularity conditions, gives the following approximation (see for example [Berger and Pericchi 2001]):

\[
B_{ij} = \frac{f(y|X_i\hat{\delta}_i, S^2_i I_n)|\hat{I}_i|^{-1/2}}{f(y|X_j\hat{\beta}_j, S^2_j I_n)|\hat{I}_j|^{-1/2}} \cdot \frac{(2\pi)^{i/2} \pi^N(\hat{\delta}_i, S_i)}{(2\pi)^{j/2} \pi^N(\hat{\beta}_j, S_j)},
\]

where \( \hat{\delta}_i, S^2_i, \hat{\beta}_j, S^2_j \) are MLE’s at the parameters and \( \hat{I}_i, \hat{I}_j \) are the observed information matrices respectively for \( M_i \) and \( M_j \). Since the first factor typically goes to \( \infty \) or to 0 as the sample size accumulates, but the second factor stays bounded, it is useful to rewrite (1) as:

\[
-2 \log(B_{ij}) = -2 \log\left(\frac{f(y|X_i\hat{\delta}_i, S^2_i I_n)}{f(y|X_j\hat{\beta}_j, S^2_j I_n)}\right) - 2 \log\left(\frac{|\hat{I}_j|^{1/2}}{|\hat{I}_i|^{1/2}}\right) + C.
\]

2. **Likelihood ratio**: The likelihood ratio can be written as:

\[
\frac{f(y|X_i\hat{\delta}_i, S^2_i I_n)}{f(y|X_j\hat{\beta}_j, S^2_j I_n)} = \left(\frac{S_j^2}{S_i^2}\right)^{\frac{n}{2}} = \left(\frac{y' (I - H_j)y}{y' (I - H_i)y}\right)^{\frac{n}{2}}
\]

See Appendix 1 for derivations.

3. **The Fisher information matrix**: The Observed Fisher Information Matrix (OFIM) with \( i \) adjustable parameters is

\[
\hat{I}_i(\hat{\delta}_i) = \frac{1}{S^2_i} \cdot X_i'X_i.
\]

Returning to equation (2) and using (3) and (4) we have
\[-2 \log(B_{ij}) = -(n-1) \log \left( \frac{y^t(I - H_j)y}{y^t(I - H_i)y} \right) - \log \left( \frac{|X_j^t X_j|}{|X_i^t X_i|} \right) + C. \tag{5}\]

The constant $C$ depends on the prior assumptions and does not go to zero, but it is of lesser importance as the sample size grows.

### 2.1 Sampling distribution of the likelihood ratio

Under $H_0$, the sampling distribution of $\frac{y^t(I - H_j)y}{y^t(I - H_i)y}$ is a beta distribution,

$$\frac{y^t(I - H_j)y}{y^t(I - H_i)y} \sim \text{Beta} \left( \frac{n-j}{2}, \frac{q}{2} \right) \tag{6}$$

where $q = j - i$ (see Casella et al., 2009, Corollary 1).

**Theorem 1.**

\[-(n-1) \log \left( \frac{y^t(I - H_j)y}{y^t(I - H_i)y} \right) \sim Ga \left( \frac{q}{2}, \frac{n-j}{2} \right) \tag{7}\]

**Proof.** Let $Z = -(n-1) \log(Y)$ and $Y \sim \text{Beta} \left( \frac{n-j}{2}, \frac{q}{2} \right)$, then

$$F_Z(z) = P(Z \leq z) = P(Y \geq e^{-\frac{z}{n-1}}) = 1 - F_Y(e^{-\frac{z}{n-1}}) = 1 - \frac{1}{n-1} e^{-\frac{z}{n-1}} f_Y(e^{-\frac{z}{n-1}}).$$

Thus

$$f_Z(z) = \frac{1}{n-1} \frac{\Gamma \left( \frac{n-j}{2} + \frac{q}{2} \right) \Gamma \left( \frac{n-j}{2} \right) \Gamma \left( \frac{q}{2} \right)}{\Gamma \left( \frac{q}{2} \right) \Gamma \left( \frac{n-j}{2} \right)} e^{-\left(\frac{n-j}{2(n-1)}\right)^2} (1 - e^{-\frac{z}{n-1}})^{\frac{q}{2} - 1}.$$
but \( \Gamma(n + \alpha) \approx \Gamma(n)n^\alpha \) (see Abramowitz and Stegun [1970] eq. 6.1.46), so,

\[
f_Z(z) = \frac{1}{n-1} \frac{(n-j)\frac{n-j}{2}}{\Gamma\left(\frac{n-j}{2}\right)} e^{-\left(\frac{n-j}{2(n-1)}\right)z} \left(1 - e^{\frac{z}{n-j}}\right)^{\frac{n-j}{2}-1}
\]

\[
= \frac{(n-j)\frac{n-j}{2}}{\Gamma\left(\frac{n-j}{2}\right)} e^{-\left(\frac{n-j}{2(n-1)}\right)z} \left(\frac{n-j}{2} - \frac{n-j}{2} e^{-\frac{z}{n-j}}\right)^{\frac{n-j}{2}-1}
\]

\[
= \frac{(n-j)\frac{n-j}{2}}{\Gamma\left(\frac{n-j}{2}\right)} e^{-\left(\frac{n-j}{2(n-1)}\right)z} \left(\frac{n-j}{2(n-1)}\right)^{\frac{n-j}{2}-1} + O(n^{-2})
\]

hence \( Z \sim Ga\left(\frac{n-j}{2}, \frac{n-j}{n-1}\right) \).

**Remark 1.** Theorem 1 is consistent with the Wilks Theorem, that is

\[
Ga\left(\frac{n-j}{2}, \frac{n-j}{n-1}\right) \rightarrow \chi^2(q) \text{ as } n \rightarrow \infty,
\]

see for example Casella and Berger (2001), Theorem 10.3.3.

### 2.2 Condition for the adaptive \( \alpha \) to be approximately equivalent (yield the same decision) to a Bayes factor

If we denote by \( g_{n,\alpha}(q) \) the quantile of the test statistic of (7) corresponding to a tail probability \( \alpha \), using (5) and (7) we can make an important departure from classical hypothesis testing: instead of fixing the tail probability (and the quantile) as in significance testing, we let the quantile vary according to the following rule

\[
\left[\begin{array}{c}
g_{\alpha,(X_i,X_j,n)}(q) = g_{n,\alpha}(q) + \log\left(\frac{|X_j^tX_j|}{|X_i^tX_i|}\right)
\end{array}\right]
\]

(8)
Then the Bayes factor will converge to a constant (and $g_{\alpha(X_i,X_j,n)}(q)$ will replace the fixed quantile). Note that (8) establishes an approximate equivalence between Bayes Factor and adaptive significance levels.

### 3 Adaptive alpha for linear models

In order to establish the asymptotic correspondence between $\alpha$ levels and Bayes factor we need the following asymptotic expansion for the upper tail for large $Ga(q, \frac{n-1}{2}) = g_n(q)$,

$$1 - F(g_n(q)) = 1 - Pr(g_n(q)) \approx g_n(q)^{q-1} \exp\left\{ - \frac{n-j}{2(n-1)} \cdot g_n(q) \right\} \left( \frac{2(n-1)}{n-j} \right)^{q/2-1} \Gamma\left( \frac{q}{2} \right), \quad (9)$$

see Richter and Schumacher (2000).

Now we equate the significance level $\alpha$ to the approximate upper tail probability in (9):

$$\alpha \approx \frac{g_{n,\alpha}(q)^{q-1} \exp\left\{ - \frac{n-j}{2(n-1)} \cdot g_{n,\alpha}(q) \right\}}{\left( \frac{2(n-1)}{n-j} \right)^{q/2-1} \Gamma\left( \frac{q}{2} \right)}.$$

If we replace the fixed quantile $g_{n,\alpha}(q)$ by $g_{\alpha(X_i,X_j,n)}(q)$ as in (8), the following result is obtained:

$$\alpha_{(b,n)}(q) = \frac{\left[ g_{n,\alpha}(q) + \log(b) \right]^{q-1}}{b^{n-j} \cdot \left( \frac{2(n-1)}{n-j} \right)^{q/2-1} \Gamma\left( \frac{q}{2} \right)} \times C_\alpha, \quad (10)$$

where $b = \frac{|X'jX|}{|X_iX_j|}$. This is the simple (approximate) calibration we have been looking for, and defines the linear adaptive $\alpha_{(b,n)}$ levels and also the corresponding adaptive quantiles, which are suitable for constructing adaptive
testing intervals for any \( q \). Note that we still need to assign a value the constant \( C_\alpha \) in (10); this will be discussed in next section.

Remark 2. Note that the adaptive significance level \( \alpha_{(b,n)} \) depends exclusively on the design matrices, the sample size \( n \) and the difference of dimension between the models being compared. This makes its value sensitive to the contribution that each predictor variable can give in the design matrix, such as the correlation that may exist between the variable that enters and those that are already in the design matrix.

Remark 3. The derivation in this section will be further refined in next section along the lines to the Prior Based Bayes Factor and the Effective Sample Size Bayarri et. al (2019).

4 Strategies to select the calibration constant \( C_\alpha \)

We now introduce some strategies for choosing the constant \( C_\alpha \), which are simple enough for fast implementation in practice. Different strategies could be developed, providing alternative calibrations.

1. The strategy of a simple approximation

The simplest approximation in (1), which is implicit in the BIC approximation, comes from assuming priors \( \pi^N(\beta_j, S_j), \pi^N(\delta_i, S_i) \) to be \( N((\beta_j, \sigma_j)|(\beta_j, S_j), \hat{I}_j^{(-1)}) \), \( N((\delta_i, \sigma_i)|(\delta_i, S_i), \hat{I}_i^{(-1)}) \) respectively, where \( \hat{I}_k = \hat{I}_k/n^* \), with \( \hat{I}_k \) being the observed Fisher Information matrix, but noting that \( n^* \) is the Effective Sample Size, mentioned before. This leads to a \( C = 1 \) in (5) and then a \( C_\alpha = \exp \left\{ -\frac{n-J}{2(n-1)} \cdot g_{n,\alpha}(q) \right\} \) in (10).

2. The strategy of a minimal balanced experiment: The one-way Layout
We suppose that \( m \) group of observations are available, with \( n_k \) observations in the \( k \)-th group, and that

\[
y_{kh} \sim N(\mu_k, \sigma^2), \quad k = 1, \ldots, m, \quad h = 1, \ldots, n_k,
\]

independently, given \( \mu_1, \ldots, \mu_m, \sigma^2 \). We shall denote by \( M_i \) the model which sets \( \mu_1 = \cdots = \mu_m \), and by \( M_j \) the model which allows \( \mu_1 \neq \cdots \neq \mu_m \). Note that \( j = m, i = 1 \) and the matrices \( X_i, X_j \), are easily identified. We have that \( q = m - 1 \), thus (10) is reduced to

\[
\alpha(n_k, q) = \left[ g_{n,\alpha}(q) + \log((\prod_{k=1}^{q+1} n_k)/n) \right]^{q/2-1} C_{\alpha},
\]

where \( n = \sum_{k=1}^{m} n_k \). For the minimal balanced experiment, \( n_k = 2 \), for each group and \( n = 2m = 2(q+1) \) then,

\[
C_{\alpha} = \alpha \cdot \frac{(2^q/(q+1))^{\frac{q+1}{2(q+1)}} \Gamma\left(\frac{q}{2}\right)}{[g_{n,\alpha}(q) + \log(2^q/(q+1))]^{\frac{q}{2}-1}},
\]

where \( \alpha \) is the desired level for the minimal sample. The case \( m = 2 \) is of particular interest since \( q = 1 \), then the calibration constant \( C_{\alpha} \) is:

\[
C_{\alpha} = \alpha \cdot \sqrt{\pi \cdot g_{n,\alpha}(1)}.
\]

3. The strategy based in PBIC

A major improvement, over BIC type approximations, has been recently introduced in \cite{Bayarri et al. (2019)} and termed Prior Based Information Criterion (PBIC). The improvement is due to: i) "the sample size" \( n \) is replaced by a much more precise "effective sample size" \( n^e \) (for i.i.d. observations \( n^e = n \), but not for non-i.i.d. observations), and ii) the effect of the prior is retained in the final expression,
on which a flat tailed non-normal prior is employed. This strategy consists in replacing in (5) the constant $C$ that depends on the prior assumptions by

$$C = 2 \sum_{m_i=1}^{q_i} \log \left(1 - e^{-v_{m_i}} \right) - 2 \sum_{m_j=1}^{q_j} \log \left(1 - e^{-v_{m_j}} \right),$$

where $v_{m_l} = \frac{\hat{\xi}_{m_l}}{[d_{m_l}(1+n_{m_l})]}$ with $l = i, j$ corresponding to the Model $M_i$ and $M_j$ respectively, see Bayarri et al. (2019). Here $n_{m_l}^e$ refers to the effective sample size (called TESS, see Berger et al. (2014)). Hence

$$\alpha(b,n)(q) = \left[ g_{n,\alpha}(q) + \log(b) + C \right]^{\frac{q}{2}-1} \times C_{\alpha},$$

and

$$C_{\alpha} = \exp \left\{ -\frac{n-j}{2(n-1)} (g_{n,\alpha}(q) + C) \right\}. $$

5 Example:

5.1 Balanced One Way Anova

Suppose we have $k$ groups with $r$ observations each, for a total sample size of $kr$ and let $H_0 : \mu_1 = \cdots = \mu_k = \mu \ vs \ H_1 : \text{At least one } \mu_i \text{ different.}$ Then the design matrices for both models are:
\( \mathbf{X}_1 = \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix}, \quad \mathbf{X}_k = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \\ 0 & 0 & \cdots & 1 \end{pmatrix}, \quad b = \frac{\mathbf{X}_t^\top \mathbf{X}_k}{\mathbf{X}_t^\top \mathbf{X}_1} = k^{-1}r^{k-1}, \)

and the adaptive alpha for linear model in accordance with (10) is

\[
\alpha(k, r) = \left[ g_{r, \alpha}(k - 1) - \log(k) + (k - 1) \log(r) \right]^{k-3/2} \frac{4^{r-1} \Gamma \left( \frac{k-1}{2} \right)}{k^{r-1/2}} C\alpha.
\]

Here, the number of replicas \( r \) is The Effective Sample Size (TESS) see remark 3.

We will use initially the strategy of selecting \( C\alpha \) by fixing the sample size for a designed experiment, as suggested in Pérez and Pericchi (2014), allowing us to compare our adaptive \( \alpha \) for linear models with the simpler version suggested there. The experiments were designed using an effect size of \( f = 0.25 \) (\( f = \frac{\mu_1 - \mu_2}{\sigma} \)), which according to Cohen (1988) represents a medium effect size. We fixed \( \alpha = 0.05 \) and the power at 0.8. The sample sizes obtained were \( r_0 = 64 \), 40 and 26 for \( k = 2, 5 \) and 10, respectively. The results shown Table 1 evidence that both corrections for \( \alpha \) yield very similar results, with the significance level decreasing steadily with the number of replicates.
Table 1: Adaptive $\alpha$ for linear model vs. Adaptive $\alpha$

| $k$ | $r$ | 2 | 5 | 10 | 2 | 5 | 10 |
|-----|-----|---|---|----|---|---|----|
| 50  | 0.057 | 0.0327 | $3.6 \times 10^{-3}$ | 0.058 | 0.0333 | $3.8 \times 10^{-3}$ |
| 100 | 0.038 | 0.0087 | $2.2 \times 10^{-4}$ | 0.038 | 0.0093 | $2.4 \times 10^{-4}$ |
| 500 | 0.016 | 0.0004 | $3.1 \times 10^{-7}$ | 0.015 | 0.0005 | $3.4 \times 10^{-7}$ |
| 1000| 0.011 | 0.0001 | $1.8 \times 10^{-8}$ | 0.010 | 0.0001 | $2.0 \times 10^{-8}$ |

Table 2: Adaptive alpha for linear model for each strategies

| $k$ | Minimal sample | Simple Calibration | PBIC Calibration |
|-----|----------------|--------------------|-----------------|
| $r$ | $\bar{k}$     | $\bar{k}$         | $\bar{k}$       |
| 4   | 0.0523        | 0.0360             | 0.0283          |
| 10  | 0.0342        | 0.0235             | 0.0159          |
| 50  | 0.0130        | 0.0090             | 0.0061          |
| 100 | 0.0087        | 0.0060             | 0.0041          |
| 500 | 0.0035        | 0.0024             | 0.0017          |
| 1000| 0.0024        | 0.0017             | 0.0011          |

The table shows how the three different strategies decrease the adaptive alpha as the effective sample size grows. It is reassuring that the different strategies yields comparable results, with the strategy based on PBIC being somewhat more drastic (in this case but not in general, see sect.7.2) in its penalization for higher samples.

We now proceed to present a simulation that shows how our methodology for decreasing alpha works precisely to improve the scientific inference and interpretation. Inspired by an example in Sellke et al. (2001) we perform the following experiment:

For Normal data with standard deviation one, with an ANOVA Model of two groups, half of the data are generated from the null with mean zero, and half from the alternative with effect $f = 0.25$. It is counted how many of the p-values lies between $.05 - \epsilon$ and 0.05, with $\epsilon$ in this example equal to 0.04. That is the data are significant but close to 0.05., or in terms of the
terminology between one star * and two stars **. What is the proportion of them generated by \( H_0 \)? The usual (flawed) interpretation is that only about 5% are generated from \( H_0 \). However, the reality is that much more are false positives. We count how many of them are generated from \( H_0 \) and present the proportion in the first column of Table 3, on which we move the number of replicas \( r \). The proportion is not monotonic with \( r \), but always far higher that 5%. In the limit, for \( r = 1000 \), all are generated from \( H_0 \). No wonder for large sample sizes and fixed confidence levels, most null hypothesis are rejected! On the other hand, in the second column, the alpha is corrected via PBIC, and somewhat unstable (due to the small \( \epsilon \)), it decreases steadily giving a much more reliable measure of control of Type I error.

| \( r \) | 2-group | 2-group |
|---|---|---|
| 10 | 39.06% | 34.18% |
| 50 | 21.43% | 8.57% |
| 100 | 15.73% | 3.07% |
| 500 | 39.04% | 0.22% |
| 1000 | 97.15% | 0.11% |

Table 3: % p-value around of the significance

### 5.2 Linear Regression Model

Consider linear regression model \( M_j : y_v = \beta_1 + \beta_2 x_{v2} + \cdots + \beta_j x_{vj} + \epsilon_v \) with \( 1 \leq v \leq n \) and \( 2 \leq j \leq k \), then

\[
|X_j^T X_j| = n(n - 1)^{(j-1)} \prod_{i=2}^{j} s_i^2 |R_j|
\]
where $s_i^2$ and $R_j$ is the variance and the correlation matrix of the predictors in model $M_j$ respectively, so in the adaptive alpha in (10)

$$b = (n - 1)^{-i} \left( \prod_{t=i+1}^j s_t^2 \right) |R_{j-i} - R_{ij}^i R_{ij}^{-1} R_{ij}|,$$

here $R_{ij}$ is the correlation matrix between predictors of the models $M_j$ that are not in $M_i$ with predictors of the model $M_i$, and $R_{j-i}$ is the correlation matrix of the predictors of the models $M_j$ that are not in $M_i$, see Appendix 2 for more detail.

The following dataset is taken from [http://academic.uprm.edu/eacuna/datos.html](http://academic.uprm.edu/eacuna/datos.html).

We want to find the best linear model to explain the average mileage per gallon (mpg) of the vehicles according to four predictor variables:

- vol: Cabin capacity in cubic feet
- hp: Engine power
- sp: Maximum speed (mph)
- wt: Vehicle weight (100 lb)

The Figure 1 shows the association between the four predictor variables and the response variable mpg.

Clearly we can see that the predictor variable wt is the one with the strongest linear association with mpg and that vol has a poor linear association with mpg. Note that sp and hp seem to be highly correlated, but neither of them with mpg.

To study the correction of our adaptive alpha to the significance and the effect that the variance and the correlation of the predictors have on these, we want to compare the following models:

1. $H_0 : M_2 : (\text{mpg}=\beta_1 + \beta_2 wt_i + \epsilon_i) \ vs \ H_1 : M_3 : (\text{mpg}=\beta_1 + \beta_2 wt_i + \beta_3 sp_i + \epsilon_i)$
2. $H_0 : M_2 : (\text{mpg}=\beta_1 + \beta_2 wt_i + \epsilon_i) \ vs \ H_1 : M_3 : (\text{mpg}=\beta_1 + \beta_2 wt_i + \beta_3 hp_i + \epsilon_i)$
3. $H_0 : M_2 : (\text{mpg}=\beta_1 + \beta_2 wt_i + \epsilon_i) \ vs \ H_1 : M_3 : (\text{mpg}=\beta_1 + \beta_2 wt_i + \beta_3 vol_i + \epsilon_i)$
Figure 1: Matrix plot of the mileage data; response variable: mpg, predictor variables: vol, hp, sp, wt.

Table 4: Effects of the variance of the predictor and its correlation with the variable already included in the model on the calibration of the significance.

In each one of the these test we can to rewrite (11) as:

\[ b = (n-1)s_3^2(1 - \rho_{23}^2), \]

where \( s_3^2 \) is the variance of the entering predictor in model \( M_3 \) and \( \rho_{23} \) is the correlation between wt y the new predictor in \( M_3 \). The data consist in a sample size of \( n = 82 \), the Table II show the correction of the significance (\( \alpha = 0.05 \)) through our adaptive alpha using simple and PBIC calibration noticing the effect that generates the variance and the correlation.

For test 1, the output in R of the p-value of the F-test is 0.03245, which
using the significance $\alpha = 0.05$, would reject $M_2$ (accept $M_3$). However, following our suggested adaptive alpha both simple and PBIC calibrations, $M_2$ is not rejected since both are smaller than 0.03245.

For test 2, the p-value of the F-test is 0.1661, which under any circumstances accepts $M_2$.

For test 3, the p-value of the F-test is 0.6482, which under any circumstances accepts $M_2$.

6 Final comments, questions and some answers

1. The adaptive $\alpha$ provides guidance for adjusting significance to the sample size. The Linear Model version incorporates not only the sample size and the difference of dimensions, but also the information provided by the predictors or the design, and particularly their correlations, correcting for co-linearity.

2. The adaptive $\alpha$ is simple to use, and gives equivalent results than a sensible Bayes Factor, like Bayes Factors with Intrinsic Priors, but easy to understand and to be employed by practitioners, even by those who are not trained in sophisticated Bayesian Statistics. We hope that this development will give tools to the practice of Statistics.

3. The results exposed here, make use of state of the art large sample approximations of Bayes Factors like the PBIC and can be coupled with recent sensible base thresholds like $\alpha = 0.005$, [Benjamin et al. (2017)](#).

References

Abramowitz, M. and I. Stegun (1970). *Handbook of Mathematical Functions.* Washington, D.C.: National Bureau of Standards.
Bayarri, M. J., J. O. Berger, W. Jang, S. Ray, L. R. Pericchi, and I. Visser (2019). Prior-based bayesian information criterion. *Statistical Theory and Related Fields* 3(1), 2–13.

Benjamin, D., J. Berger, M. Johannesson, B. Nosek, E.-J. Wagenmakers, R. Berk, K. Bollen, B. Brembs, L. Brown, C. Camerer, D. Cesarini, C. Chambers, M. Clyde, T. Cook, P. De Boeck, Z. Dienes, A. Dreber, K. Easwaran, C. Efferson, E. Fehr, F. Fidler, A. Field, M. Forster, E. George, R. Gonzalez, S. Goodman, E. Green, D. Green, A. Greenwald, J. Hadfield, L. Hedges, L. Held, T. Hua Ho, H. Hoijtink, D. Hruschka, K. Imai, G. Imbens, J. Ioannidis, M. Jeon, J. Jones, M. Kirchler, D. Laibson, J. List, R. Little, A. Lupia, E. Machery, S. Maxwell, M. McCarthy, D. Moore, S. Morgan, M. Munafó, S. Nakagawa, B. Nyhan, T. Parker, L. Pericchi, M. Perugini, J. Rouder, J. Rousseau, V. Savalei, F. Schönbrodt, T. Sellke, B. Sinclair, D. Tingley, T. Van Zandt, S. Vazire, D. Watts, C. Winship, R. Wolpert, Y. Xie, C. Young, J. Zinman, and V. Johnson (2017). Redefine statistical significance. *Nature Human Behavior*.

Berger, J., M. Bayarri, and L. Pericchi (2014). The effective sample size. *Econometric Reviews* 33(1-4), 197–217.

Berger, J. and L. Pericchi (2001). Objective bayesian methods for model selection: Introduction and comparison. In *Model selection*, pp. 135–207. Institute of Mathematical Statistics.

Casella, G. and R. Berger (2001). *Statistical Inference* (2nd ed.). Duxbury Resource Center.

Casella, G., J. Girón, L. Martínez, and E. Moreno (2009, junie). Consistency of bayesian procedures for variable selection. *The Annals of Statistics* 37(3), 1207–1228.

Cohen, J. (1988). *Statistical Power Analysis for the Behavioral Sciences* (2nd ed.). Psychology Press.
Pérez, M. E. and L. R. Pericchi (2014). Changing statistical significance with the amount of information: The adaptive alfa significance level. *Statistics and Probability Letters* 85, 20–24.

Richter, W.-D. and J. Schumacher (2000, sept). Asymptotic expansions for large desviation probabilities of noncentral generalized chi-square distributions. *Multivariate Analysis* 75, 184–218.

**Appendix 1 The likelihood ratio**

Define

\[ r(y|(X_i, X_j)) = \frac{f(y|X_i\hat{\delta}_i, S_i^2 I_n)}{f(y|X_j\hat{\beta}_j, S_j^2 I_n)} \]

we will perform the calculations for the hypothesis test

\[ H_0 : \text{Model } M_i \ \text{versus} \ H_1 : \text{Model } M_j. \]

Indeed, for model \( M_i \)

\[ L(y|X_i, \sigma_i^2, \delta_i) = \frac{1}{(2\pi)^{n/2}(\sigma_i^2)^{n/2}} \exp \left\{ -\frac{1}{2\sigma_i^2} (y - X_i\delta_i)^t (y - X_i\delta_i) \right\}. \]

Since the MLE of \( \delta_i \) is \( \hat{\delta}_i = (X_i^t X_i)^{-1} X_i^t y \) and the MLE of \( \sigma_i^2 \) is \( S_i^2 = \frac{y^t(1-H_i)y}{n} \), where \( H_i = X_i(X_i^t X_i)^{-1} X_i^t \)

\[ \sup_{\Omega_0} L(y|X_i, \sigma_i^2, \delta_i) = \frac{1}{(2\pi)^{n/2}(S_i^2)^{n/2}} \exp \left\{ -\frac{n}{2} \right\}. \]

For model \( M_j \)

\[ L(y|X_j, \sigma_j^2, \beta_j) = \frac{1}{(2\pi)^{n/2}(\sigma_j^2)^{n/2}} \exp \left\{ -\frac{1}{2\sigma_j^2} (y - X_j\beta_j)^t (y - X_j\beta_j) \right\}. \]
Since MLE of $\beta_j$ is $\hat{\beta}_j = (X_j^T X_j)^{-1} X_j^T y$ and the MLE of $\sigma_j^2$ is $S_j^2 = \frac{y^T (I - H_j)y}{n}$

$$
\sup_{\Omega} L(y|X_j, \sigma_j^2, \beta_j) = \frac{1}{(2\pi)^{n/2} (S_j^2)^{n/2}} \exp\left\{ -\frac{n}{2} \right\}.
$$

Thus the likelihood ratio is

$$
r(y|(X_i, X_j)) = \frac{\sup_{\Omega_i} L(y|X_i, \sigma_i^2, \alpha_i)}{\sup_{\Omega_j} L(y|X_j, \sigma_j^2, \beta_j)} = \left(\frac{S_j^2}{S_i^2}\right)^{\frac{n}{2}} = \left(\frac{y^T (I - H_j)y}{y^T (I - H_i)y}\right)^\frac{n}{2}.
$$

**Appendix 2 An expression for $b$ in (10)**

Consider linear regression model $M_j : y_v = \beta_1 + \beta_2 x_{v2} + \cdots + \beta_j x_{vj} + \epsilon_v$ with $1 \leq v \leq n$ and $2 \leq j \leq k$, then

$$
X_j = \begin{bmatrix}
1 & x_{12} - \bar{x}_2 & \cdots & x_{1j} - \bar{x}_j \\
1 & x_{22} - \bar{x}_2 & \cdots & x_{2j} - \bar{x}_j \\
\vdots & \vdots & \ddots & \vdots \\
1 & x_{n2} - \bar{x}_2 & \cdots & x_{nj} - \bar{x}_j
\end{bmatrix}
$$

and

$$
X_j^T X_j = \begin{bmatrix}
n & 0 & 0 & \cdots & 0 \\
0 & (n-1)s_2^2 & (n-1)s_2s_3\rho_{23} & \cdots & (n-1)s_2s_j\rho_{2j} \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & (n-1)s_2s_j\rho_{2j} & (n-1)s_3s_j\rho_{2j} & \cdots & (n-1)s_j^2
\end{bmatrix}
$$

then

$$
|X_j^T X_j| = n(n-1)^{j-1} \begin{vmatrix}
s_2^2 & s_2s_3\rho_{23} & \cdots & s_2s_j\rho_{2j} \\
s_2s_3\rho_{23} & s_3^2 & \cdots & s_3s_j\rho_{3j} \\
\vdots & \vdots & \ddots & \vdots \\
s_2s_j\rho_{2j} & s_3s_j\rho_{3j} & \cdots & s_j^2
\end{vmatrix}.
$$
note that row $l$ and column $l$ are multiplied by $s_l$, using properties of the determinants

$$|X_j^tX_j| = n(n - 1)^{j-1}s_2^2s_3^2 \cdots s_j^2 = n(n - 1)^{j-1}\prod_{l=2}^{j}s_l^2|R_j|$$

in the other hand,

$$R_j = \begin{bmatrix} 1 & \rho_{23} & \cdots & \rho_{2j} \\ \rho_{23} & 1 & \cdots & \rho_{3j} \\ \vdots & \vdots & \ddots & \vdots \\ \rho_{2j} & \rho_{3j} & \cdots & 1 \end{bmatrix} = \begin{bmatrix} R_i & R_{ij} \\ R_i & R_{j-i} \end{bmatrix}$$

where

$$R_{ij} = \begin{bmatrix} \rho_{2j+1} & \rho_{3j+1} & \cdots & \rho_{i+1} \\ \rho_{2j+2} & \rho_{3j+2} & \cdots & \rho_{i+2} \\ \vdots & \vdots & \ddots & \vdots \\ \rho_{2j} & \rho_{3j+2} & \cdots & \rho_{ij} \end{bmatrix}$$

and

$$R_{j-i} = \begin{bmatrix} 1 & \rho_{i+2i+1} & \cdots & \rho_{ji+1} \\ \rho_{i+1i+2} & 1 & \cdots & \rho_{ji+2} \\ \vdots & \vdots & \ddots & \vdots \\ \rho_{i+1j} & \rho_{i+2j} & \cdots & 1 \end{bmatrix}.$$  

Now since $X_j$ is a full rank matrix, it can be seen that

$$|R_j| = |R_i||R_{j-i} - R_{ij}^tR_i^{-1}R_{ij}|$$

thus

$$b = \frac{|X_j^tX_j|}{|X_i^tX_i|} = (n - 1)^{j-i}\left(\prod_{l=i+1}^{j}s_l^2\right)|R_{j-i} - R_{ij}^tR_i^{-1}R_{ij}|$$