The Borel $C_2$-equivariant $K(1)$-local sphere

William Balderrama

University of Illinois at Urbana-Champaign

https://faculty.math.illinois.edu/~balderr2/

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Intro

Big goal

Compute the homotopy groups of the sphere spectrum $S$.

Recent technique

Compute the homotopy groups of other sphere spectra: Motivic spheres, equivariant spheres, synthetic spheres, ... .

Goal of this talk

Describe a $v_1$-periodic portion of $S_{C_2}$.

Everything will be 2-completed.

Outline

1. Intro and motivation;
2. Prior work;
3. The $v_1$ stuff.
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Lay of the land

General goal

Compute the homotopy ring \( \pi_\ast, \ast \mathbb{S}_{C_2} \), where \( \mathbb{S}_{C_2} \) = unit of \( \mathbb{S}p_{C_2} \).

Definition and grading

Bigraded homotopy: \( \pi_{s,c} := \pi_{c+(s-c)\sigma} \), with \( \sigma = \) sign rep, so

1. \( \pi_{s,c} \mathbb{S}_{C_2} = \operatorname{colim}_{n \to \infty} \pi_0 \operatorname{Map}^{C_2}(S^{(s+(c-s)\sigma)+n(1+\sigma)}, S^n(1+\sigma)) \);
2. Have diagonal embedding \( \pi_s \to \pi_{s,s} \).

Some basic elements

1. Inclusion of poles \( S^0 \to S^\sigma \) gives “Euler class” \( \rho \in \pi_{-1,0} \);
2. \( \eta \in \pi_1 \mathbb{S} \) gives diagonal element \( \mu \in \pi_{1,1} \mathbb{S}_{C_2} \);
3. \( \eta: S^{1+2\sigma} = S(\mathbb{C}^2) \to \mathbb{C}P^1 = S^{1+\sigma} \) equivariant, so \( \eta \in \pi_{1,0} \mathbb{S}_{C_2} \);
   1. Fixed points: \(-\eta^{C_2} = 2: S^1 \to S^1 \), so \( \eta \) not nilpotent;
   2. Homotopy fixed points of \( \eta: \mathbb{S}^{\sigma} \to \mathbb{S}^0 \) gives transfer \( P_1^\infty \to \mathbb{S} \).
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Compute the homotopy ring \( \pi_{\ast,\ast}S_{C_2} \), where \( S_{C_2} = \text{unit of } \text{Sp}C_2 \).

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Compute the homotopy ring $\pi_{*,*}S_{C_2}$, where $S_{C_2}$ = unit of $Sp_{C_2}$.

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Bigraded homotopy: $\pi_{s,c} := \pi_{c+(s-c)\sigma}$, with $\sigma = \text{sign rep}$, so
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Some basic elements
1. Inclusion of poles $S^0 \to S^\sigma$ gives “Euler class” $\rho \in \pi_{-1,0}$;
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Borel $G$-equivariant homotopy theory

Borel equivariant stems

1. Simpler equivariant category: $\mathcal{S}p_{BG} = \text{Fun}(BG, \mathcal{S}p)$;
2. Homotopy $RO(G)$-graded; e.g. if $\nu(X) = X$ with trivial action,
   \[
   \pi_V \nu(X) = \pi_0 \text{Map}_{BG}(S^V, X) = \pi_0 F(\text{Th}(V), X).
   \]
3. Forgetful functor $u: \mathcal{S}p_G \to \mathcal{S}p_{BG}$ has fully faithful right adjoint $b: \mathcal{S}p_{BG} \to \mathcal{S}p_G$, giving Borel completion $(-)^\wedge_h = b \circ u$.

Tate fracture square (Greenlees-May 1995)

For $X \in \mathcal{S}p_G$, there is a Cartesian square

\[
\begin{array}{ccc}
X & \longrightarrow & \Phi_G X \\
\downarrow & & \downarrow \\
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The Segal conjecture

Tate fracture and $\rho$ fracture

Tate fracture for $C_2$ is even simpler: there is an equivalence

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\begin{align*}
X & \longrightarrow \Phi_{C_2} X \\
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X^\wedge_{h} & \longrightarrow X^{tC_2}
\end{align*} \cong
\begin{align*}
X & \longrightarrow X[\rho^{-1}] \\
\downarrow & \\
X^\wedge_\rho & \longrightarrow X^\wedge[\rho^{-1}]
\end{align*}
$$

Lin’s theorem (1980)

The map $\Phi_{C_2} S_{C_2} \rightarrow (S_{C_2})^{tC_2}$ is an equivalence.

Corollary (Segal conjecture for $C_2$)

The map $S_{C_2} \rightarrow (S_{C_2})^\wedge_h$ is an equivalence, and thus

$$\pi_{s,c} S_{C_2} = \pi_c D(P_s^{\infty}_{-c}).$$

These use our implicit 2-completion.
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These use our implicit 2-completion.
One reason to care about $\mathbb{S}_{C_2}$: the root invariant

The root invariant (Mahowald, 1967 in some form)

Lin’s theorem: $\operatorname{colim}_j D(P_{\infty}^{-j}) = \"D(P_{\infty}^{-\infty})\" = \mathbb{S}$. This gives AHSS

$$E_1 = \pi_* \mathbb{S}[\tau^{\pm 1}] \Rightarrow \pi_* \mathbb{S}.$$  

The root invariant of $\alpha \in \pi_* \mathbb{S}$ is those $R(\alpha) \subset \pi_* \mathbb{S}$ that detect $\alpha$.

A reinterpretation

Where $S_{\text{Root}} = \text{filtered spectrum } \cdots \rightarrow D(P_{\infty}^n) \xrightarrow{\rho} D(P_{\infty}^{n-1}) \rightarrow \cdots$,

1. Segal conj.: $\pi_*,* \mathbb{S}_{C_2} = \pi_*,* S_{\text{Root}}$;
2. Lin’s theorem: $S_{\text{Root}}[\rho^{-1}] = \mathbb{S}[\rho^{\pm 1}]$; easy obs: $S_{\text{Root}}/(\rho) = \mathbb{S}[\tau^{\pm 1}]$;
3. $\rho$-BSS $\pi_* \mathbb{S}[\tau^{\pm 1}, \rho] \Rightarrow \pi_*,* S_{\text{Root}} = \pi_*,* \mathbb{S}_{C_2}$ deforms the AHSS.

Computing $R(\alpha)$ via $\pi_*,* \mathbb{S}_{C_2}$ (cf. Bruner-Greenlees 1995)

1. Write $\rho^N \alpha = \rho^N + n \beta$ in $\pi_*,* \mathbb{S}_{C_2}$ for $n$ maximal;
2. Where $\varphi: \mathbb{S}_{C_2} \rightarrow \mathbb{S}_{C_2}/(\rho) \approx \mathbb{S}$, get $\varphi(\beta) \in R(\alpha)$. 

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One reason to care about $\mathbb{S}_{C_2}$: the root invariant

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Past calculations

First systematic calculations

Araki-Iriye (1982) study $\pi_{s,c}S_{C_2}$ for $s \leq 8$ with EHP methods. Tools:

1. LES $\cdots \rightarrow \pi_sS \rightarrow \pi_{s,c}S_{C_2} \rightarrow \pi_{s-1,c}S_{C_2} \rightarrow \pi_{s+c-1}S \rightarrow \cdots$;

2. LES $\cdots \rightarrow \pi_{c+1}S \rightarrow \lambda_{s,c} \rightarrow \pi_{s,c}S_{C_2} \rightarrow \pi_cS \rightarrow \cdots$.

Reinterpreting in other terms

1. First LES: associated to $S_{C_2} \xrightarrow{\rho} S_{C_2} \rightarrow C(\rho) \simeq S[\tau^{\pm 1}]$;

2. Second LES: associated to $S_{C_2} \rightarrow S_{C_2}[\rho^{-1}] \simeq S[\rho^{\pm 1}] \rightarrow S_{C_2}/(\rho^\infty)$; can identify $\lambda_{s,c} = \pi_{c-1}P_{c-s-1}^\infty$.

More recent calculations

Adams spectral sequence and motivic methods.
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**More recent calculations**

Adams spectral sequence and motivic methods.
The motivic-equivariant context

A bunch of categories

\[
\begin{array}{c}
\text{Sp}_\mathbb{R} \xrightarrow{\text{Be}} \text{Sp}_\mathbb{C} \xrightarrow{\text{Be}} \text{Sp} \\
\text{Sp}_{BC_2} \xrightarrow{t} \text{Sp} \\
\text{Sp}_{C_2} \xrightarrow{\Phi} \text{Sp}
\end{array}
\]

All of these categories have Adams spectral sequences.

Some names associated to Adams SS computations

1. **C**: Dugger, Isaksen, Wang, Xu, ... (2009 - );
2. **R**: Belmont, Dugger, Guillou, Isaksen, ... (2015 - );
3. **C_2**: Guillou, Hill, Isaksen, Ravenel, ... (2019 - forthcoming);
4. **BC_2**: Understudied, but cf. Lin-Davis-Mahowald-Adams (1979).
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\(\mathbb{R}\)-motivic and \(C_2\)-equivariant homotopy

The \(\mathbb{R}\)-motivic Steenrod algebra

1. Have Steenrod algebra \(A_{\mathbb{R}}\) over \(\mathbb{F}_2[\tau, \rho]\) (Voevodsky 2003);
2. Have Adams SS \(H^*(A_{\mathbb{R}}) \Rightarrow \pi_{*,*}S_{\mathbb{R}}\). Main tools:
   1. \(A_{\mathbb{R}}/(\rho) = A_\mathbb{C}\), giving BSS \(H^*(A_\mathbb{C})[\rho] \Rightarrow H^*(A_{\mathbb{R}})\) (cf. Hill 2009);
   2. \(\text{Iso } Sq^0 : H^*(A_{\mathbb{R}})[\rho^{-1}] \simeq H^*(A) \otimes \mathbb{Z}[\rho^{\pm 1}]\) (Dugger-Isaksen 2017);
Belmont-Isaksen (2020) compute \(\pi_{s,c}S_{\mathbb{R}}\) for \(c \leq 11\) or so.

What happened to the \(C_2\)-stems?

\(\pi_{*,*}S_{\mathbb{R}}\) is a good approximation to \(\pi_{*,*}S_{C_2}\):

1. \(\pi_{*,*}S_{\mathbb{R}} \simeq \pi_{*,*}S_{C_2}\) in a range (Belmont-Guillou-Isaksen 2020);
2. \(H^*(A_{\mathbb{R}})\) is a summand of \(H^*(A_{C_2})\) (\(A_{C_2}\): Hu-Kriz 2001);
3. \(\pi_{*,*}S_{C_2}\) is the “\(\tau\)-periodization” of \(\pi_{*,*}S_{\mathbb{R}}\) (Behrens-Shah 2019).

Time to stare at some charts.

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$\mathbb{R}$-motivic and $C_2$-equivariant homotopy

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\( \tau \)-periodicity

### Classical James periodicity

Where \( \gamma(n) = n^{\text{th}} \) Radon-Hurwitz number, have

\[
P_{m}^{n+m} \simeq \sum_{-k}^{-k2\gamma(n)} P_{n+k2\gamma(n)}^{n+m+k2\gamma(n)}.\]

### Observation

The cofiber \( C(\rho^{n+1}) \) is built from the spectra \( P_{m}^{m+n} \).

### Theorem (Behrens-Shah 2019)

1. There are maps \( \tau^{k2\gamma(n)} : \Sigma^{0,k2\gamma(n)} S_{R}/(\rho^{n+1}) \rightarrow S_{R}/(\rho^{n+1}) \);
2. This Betti realizes to an equivalence \( \tau^{k2\gamma(n)} : C(\rho^{n+1}) \rightarrow C(\rho^{n+1}) \);
3. For \( X \in \text{Sp}_{R}^{\text{cell}} \), have \( \text{Be}(X)^{\wedge}_{\rho} = X^{\wedge}_{\rho}[\tau^{-1}] \).
\(\tau\)-periodicity

Classical James periodicity

Where \(\gamma(n) = n'\)th Radon-Hurwitz number, have

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P_{n+m} \cong \sum_{-k2\gamma(n)} P_{n+m+k2\gamma(n)}.
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1. There are maps \(\tau^k 2^\gamma(n) : \Sigma^{0,k2^\gamma(n)} S_R / (\rho^{n+1}) \rightarrow S_R / (\rho^{n+1})\);
2. This Betti realizes to an equivalence \(\tau^k 2^\gamma(n) : C(\rho^{n+1}) \rightarrow C(\rho^{n+1})\);
3. For \(X \in Sp^\text{cell}_R\), have \(\text{Be}(X)^\wedge_\rho = X^\wedge_\rho [\tau^{-1}]\).
The Borel $C_2$-equivariant $K(1)$-local sphere

**Im $J$-elements**

The $\mathbb{R}$-motivic charts make apparent:

1. Im $J$-type elements make up a lot of $\pi_\ast,\ast\mathbb{S}_{C_2}$;
2. There is exotic behavior: e.g. $16\sigma \neq 0$.

**Detecting Im $J$**

1. Classically, Im $J$-type elements are detected by $\mathbb{S}_{K(1)}$;
2. Where $\pi_{s,c}\nu(X) = \pi_c F(P_{s-c}^\infty, X)$, have $\pi_{s,c}\mathbb{S}_{C_2} = \pi_{s,c}\nu(\mathbb{S})$.

So one is immediately led to study $\nu(\mathbb{S}_{K(1)})$.

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A complete description of the ring $\pi_\ast,\ast\nu(\mathbb{S}_{K(1)})$. 
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Some chromatic homotopy

General approach

The chromatic approach seems good for computing equivariant stems:

1. For a finite group $G$, have $\nu_G$ with $\pi_V \nu_G(X) = \pi_0 F(\text{Th}(V), X)$;
2. Segal conjecture (Carlsson 1984): $\pi_* \nu_G(S) = \text{completion of } \pi_* S_G$; so $\pi_* S_G$ is well-represented in the various $\pi_* \nu_G(S_{K(n)})$.

Main tool

Have $\nu_G(S_{K(n)}) = \nu(\mathcal{E}_n)^{hG_n}$, so there is an HFPSS

$$\mathbb{H}^*(G_n; \pi_* \nu_G(\mathcal{E}_n)) \Rightarrow \pi_* \nu_G(S_{K(n)}).$$

Benefits of approach

By $K(n)$-local Tate vanishing, have self-duality

$$F(\text{Th}(V), S_{K(n)}) \simeq L_{K(n)} \text{Th}(-V).$$

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Approach

Computation of $\pi_\ast,\ast\nu(S_{K(1)})$ proceeds via the HFPSS’s

$$H^\ast(C_2; \pi_\ast,\ast\nu(KU)) \Rightarrow \pi_\ast,\ast\nu(KO)$$

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Basic features of calculation

1. Few differentials: just from $H^\ast(C_2; \pi_\ast KU) \Rightarrow \pi_\ast KO$;
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Goal

Go over some highlights.

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Complex $K$-theory

Recall $\pi_{s,c}(KU) = KU^{-c}P_{s-c}^\infty$.

Where everything starts

Have $\beta \in \pi_{2,2}(KU)$, $\rho \in \pi_{-1,0}(KU)$, $\tau^2 \in \pi_{0,2}(KU)$, and

1. $\pi_{*,*}(KU) = \mathbb{Z}_2[\beta^{\pm 1}, \rho, \tau^{\pm 2}]/(\rho^3 = 2\rho\beta^{-1}\tau^2)$;

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Here $\tau^2 = \text{Thom class of } -2\sigma$.

Corollary

1. $KU^*P_{2n+1}^\infty \simeq \pi_*KU$ equivariantly, so $P_{2n+1}^\infty \in \text{Pic}^0(\text{Sp}_K(1))$;

2. $P_{2n}^\infty = 2$-cell complex, attaching map with e-invariant $\frac{1}{2}$.

Example

1. $\rho\beta^{-1}\tau^2 = \text{Hurewicz image of } -\eta \in \pi_{1,0}\mathbb{S}_{C_2}$;

2. Follows that $P_1^\infty \simeq \mathbb{S}$ realized by the transfer.
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The homotopy fixed point spectral sequence

Have $H^*(\mathbb{Z}/(2)\{\psi^{-1}\}; \pi_\ast,\ast \nu(KU)) \Rightarrow \pi_\ast,\ast \nu(KO)$. Differentials from:

1. Can show $\pi_{0,0} \nu(KU)$ must consist of permanent cycles;
2. Classic differential $d(\beta^2) = \mu^3$ where $\mu = "\text{nonequivariant } \eta"$.

Corollary

By looking at when generator of $K^0 P_{2n+1}^\infty$ is a permanent cycle, learn

$$P_{2n+1}^\infty \simeq \begin{cases} S & n \equiv 0, 3 \pmod{4} \\ T & n \equiv 1, 2; \pmod{4} \end{cases}$$

with $T = \text{exotic element of } \text{Pic}(\text{Sp}_K(1))$.

More comments

1. By $(P_1^2 \simeq S^1/(2)) \to (P_1^\infty \simeq S) \to (P_3^\infty \simeq T)$, get "$T = \ast$";
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William Balderrama (UIUC)
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$$H^*(\mathbb{Z}\{\psi^k\};\pi_{*,*}\nu(KO)) \Rightarrow \pi_{*,*}.$$  

This collapses into the short exact sequences

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An important subring: (most of) the Milnor-Witt 0-stem

$R = H^0(\mathbb{Z}\{\psi^k\};\pi_{*0}\nu(KO))$ generated by $\omega_a, \eta_a$, with:

1. $\omega_0 = \rho$ and $\eta_0 = -\eta$;
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3. $\omega_a + b \omega_c = \omega_a \omega_{b+c}$ etc., $\omega_0^2 \eta_a = 2\eta_a$, $\eta_0^2 \omega_a = 2\eta_a$, $\eta_0^3 \omega_a = \omega_0^3 \omega_{a+1}$.

This “sees” the root invariants $R(2^n)$. 

William Balderrama (UIUC)  Borel Im J  February 23, 2021
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Hidden multiplicative extensions

HFPSS collapses, but there are many hidden extensions.

### Controlling indeterminacy

1. $\pi_{*,*} \nu(KO)$ is $\tau^4$-periodic, but $\tau^4 \notin \pi_{*,*}$;
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### Tools for resolving hidden extensions

1. $\tau$-periodicity to reduce to computations in $\pi_\ast S$;
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A sample of $\pi_{*,*}$

### Multiplicative generators

$$\omega_a \in \pi_{8a-1,0}, \quad \eta_a \in \pi_{8a+1,0}, \quad \tau^{2b} h \in \pi_{0,2b} \quad (b \neq 0)$$

$$\tau^{4b} \mu_a \in \pi_{8a+1,4b+1}, \quad \tau^{4b} \zeta_a \in \pi_{8a+3,4b+1}$$

$$\rho_{a,b} \in \pi_{8a-1,4b-1}, \quad \xi_{a,b} \in \pi_{8a+3,4b-1}.$$ 

### Notation

$$\pi_{4a-1} S = \mathbb{Z}_2/(2^{ja})$$ and $u_{a,b} = \frac{2^{ja}}{2^{ja-b}} \frac{k^{2b} - k^{2a}}{k^{2a} - 1} \in \mathbb{Z}^*_2$ (indep. of $k$ mod $2^{jb}$).

### Some fun relations

1. Have $\pi_{8a-1,4b-1} = \mathbb{Z}_2 \{ \rho_{a,b}, \omega_0 \eta_0 \rho_{a,b} \}$ mod:

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