Nearly Spectral Spaces
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Abstract: We study some natural generalizations of the spectral spaces in the contexts of commutative rings and distributive lattices. We obtain a topological characterization for the spectra of commutative (not necessarily unitary) rings and we find spectral versions for the up-spectral and down-spectral spaces. We show that the duality between distributive lattices and Balbes-Dwinger spaces is the co-equivalence associated to a pair of contravariant right adjoint functors between suitable categories.

Keywords: Spectral space, down-spectral space, up-spectral space, Stone duality, prime spectrum, distributive lattice, commutative ring.

MSC: 54H10, 54F65, 54D35.

1 Introduction

A spectral space is a topological space that is homeomorphic to the prime spectrum of a commutative unitary ring. This type of spaces were topologically characterized by Hochster \([8]\) as the sober, coherent and compact spaces. On the other hand, it is known that a topological space is a spectral space if and only if it is homeomorphic to the prime spectrum of a distributive bounded lattice \([10]\), \([1]\).

Therefore, this notion has two natural generalizations: the first in the context of rings and the second in the context of lattices:

We say that:

1. a topological space is almost-spectral if it is homeomorphic to the prime spectrum of a commutative (not necessarily unitary) ring,
2. a topological space is a Balbes-Dwinger space if it is homeomorphic to the prime spectrum of a distributive (not necessarily bounded) lattice\(^3\).

In Chapter VI of \([4]\), there is a topological characterization of the Balbes-Dwinger spaces (called there Stone spaces). As far as we know, in the literature, there is no topological characterization for the almost-spectral spaces.

Furthermore, there exist generalizations on a topological point of view \([5]\), \([6]\):

3. a topological space is called up-spectral if it is sober and coherent,

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\(^3\)In \([4]\) this type of spaces are called Stone spaces. However, in several other references, for example \([9]\), a Stone space is a compact, Hausdorff and totally disconnected space.
(4) A topological space is called \textit{down-spectral} if it is coherent, compact and every proper irreducible closed set is the closure of a unique point.

It is natural to ask if the notions in (3) and (4) have “spectral versions”, that is, if the corresponding spaces are homeomorphic to prime spectra of some kind of rings or lattices.

In this paper we show that all these topological spaces are particular cases of certain class of topological spaces (named here \textit{RA-spaces}) and we give spectral versions for all of them. In addition, we give a topological characterization of the almost-spectral spaces and a new, simpler, topological characterization of the Balbes-Dwinger spaces.

Actually, we extend the co-equivalence (or duality) between the category of distributive bounded lattices and the category of spectral spaces presented in [4] to a pair of contravariant, adjoint functors between the category of distributive lattices and the category of RA-spaces. By means of this adjunction, all the mentioned types of topological spaces arise naturally and the relationship between them becomes clear. In particular, we can easily deduce the duality between up-spectral and down-spectral spaces studied in [6].

2 Preliminaries

We recall some basic definitions and facts that will be useful in the next sections.

\textbf{Notation 1.} If \( g : X \to Y \) is a function, we denote \( g^* \) the inverse image function defined by
\[
g^*: \wp(Y) \to \wp(X) : B \mapsto g^*(B) = \{ x \in X : g(x) \in B \}.
\]

2.1 Lattice theory notions

A \textit{lattice} is a non empty partially ordered set (or poset) such that every pair of elements \( a, b \) has least upper bound (or join) \( a \vee b \), and greatest lower bound (or meet) \( a \wedge b \). The lattice is \textit{distributive} if \( \vee \) is distributive with respect to \( \wedge \) (equivalently \( \wedge \) is distributive with respect to \( \vee \)). The lattice is \textit{bounded} if it has least (or minimum) and greatest (or maximum) elements, usually denoted \( 0 \) and \( 1 \), respectively. An ideal of a lattice is a non empty lower subset that is closed under finite (non empty) joins. A proper ideal \( I \) is prime if \( a \wedge b \in I \) implies \( a \in I \) or \( b \in I \).

A map \( \alpha : L \to M \) between lattices is a \textit{homomorphism} if for each pair of elements \( a, b \in L \), \( \alpha(a \wedge b) = \alpha(a) \wedge \alpha(b) \) and \( \alpha(a \vee b) = \alpha(a) \vee \alpha(b) \). The homomorphism \( \alpha \) is \textit{proper} if the inverse image of any prime ideal of \( M \) is a prime ideal of \( L \).

The \textit{prime spectrum} of a lattice \( L \) is the set of its prime ideals endowed with the \textit{Zariski (or hull-kernel) topology}, whose basic open sets are the sets
\[
d(a) = \{ I : I \text{ is a prime ideal of } L \text{ and } a \notin I \},
\]
where \( a \in L \). We denote this space by \( \text{spec}(L) \). Actually, \( d : L \to \varphi(\text{spec}(L)) \) is a homomorphism of lattices such that \( d(0) = \emptyset \), when \( L \) has minimum and \( d(1) = \text{spec}(L) \), when \( L \) has maximum. This homomorphism is injective if and only if the lattice \( L \) is distributive. It is known that for each \( a \in L \), \( d(a) \) is a compact subspace of \( \text{spec}(L) \).

### 2.2 Ring theory notions

Similarly, the prime spectrum of a commutative ring \( A \) is defined as the set of its prime ideals endowed with the Zariski (or hull-kernel) topology, whose basic open sets are the sets

\[
D(a) = \{ P : P \text{ is a prime ideal of } A \text{ and } a \notin P \},
\]

where \( a \in A \). In this case the closed sets are

\[
V(I) = \{ P : P \text{ is a prime ideal of } A \text{ and } P \supseteq I \},
\]

where \( I \) is an ideal of \( A \). We denote this space by \( \text{Spec}(A) \), as usual. Notice that \( D : A \to \varphi(\text{spec}(A)) \) is such that for each \( a, b \in A \), \( D(ab) = D(a) \cap D(b) \) and \( D(a + b) \subseteq D(a) \cup D(b) \). It is also known that the basic open sets are compact. Therefore, the prime spectrum of a commutative unitary ring is a compact topological space; however, compactness of \( \text{Spec}(A) \) is not equivalent to existence of identity in \( A \). The following theorem, taken from [2], is useful:

**Theorem 2.** Let \( S \) be a commutative ring.

(i) If \( R \) is a commutative ring such that \( S \) is an ideal of \( R \), then \( \text{Spec}(S) \) is homeomorphic to the open subspace \( V(S)^c \) of \( \text{Spec}(R) \).

(ii) There exists a commutative unitary ring \( Q(S) \) such that \( \text{Spec}(S) \) is homeomorphic to an open-dense subspace of \( \text{Spec}(Q(S)) \).

Another known fact is that for each ideal \( I \) of the ring \( A \) the function

\[
\theta : V(I) \to \text{Spec}(A/I) : P \mapsto P/I
\]

is a homeomorphism [3].

### 2.3 Topological notions

A subset \( F \) of a topological space is an irreducible closed set if \( F \) is a non-empty closed set such that for every pair of closed sets \( G \) and \( H \), \( F = G \cup H \) implies \( F = G \) or \( F = H \). We say that \( U \) is a prime open set if its complement is an irreducible closed set.

A space is called sober if every irreducible closed set is the closure of a unique point.

A space is called coherent if it has a basis of open-compact sets that is closed under finite intersections.

For example, an infinite set \( X \) endowed with the co-finite topology is coherent, but it is not sober since \( X \) is an irreducible closed set that is not the closure
of any point. Notice that, in this example, all proper irreducible closed sets are, in fact, closures of points.

We give then the following definition:

**Definition 1.** A topological space is *almost-sober* if every proper irreducible closed set is the closure of some point.

The following definition is taken from [4].

**Definition 2.** Let $X$ be a topological space. We say that $A \subseteq X$ is *fundamental* if

i) $A$ is a non-empty and open-compact set, or

ii) $A = \emptyset$ and for every non-empty collection $\mathcal{A}$ of non-empty open-compact sets whose intersection is empty, there exists a finite subcollection of $\mathcal{A}$ with empty intersection.

We denote $\mathfrak{S}(X)$ the collection of fundamental subsets of $X$.

Notice that $\emptyset$ is fundamental if every non-empty collection of open-compact sets with the finite intersection property has non-empty intersection.

A map $f : X \to Y$ between topological spaces is *strongly continuous* if it is continuous and the inverse image of a fundamental subset of $Y$ is a fundamental subset of $X$.

Recall that if $X$ is a preordered set, the *Alexandroff (or upper sets) topology* on $X$ is the topology generated by \{\[x \downarrow : x \in X\}\}, where $\downarrow x = \{y \in X : y \geq x\}$.

Notice that $\downarrow x$ is an open-compact set in this topological space, thus, every totally ordered set with its Alexandroff topology is a coherent space.

We present now the topological characterization of the Balbes-Dwinger spaces given in [4]:

**Theorem 3.** A topological space is a Balbes-Dwinger space if, and only if, it is $T_0$, coherent and the following condition is satisfied: For every pair of non-empty collections $\mathcal{A}$ and $\mathcal{B}$ of non-empty open-compact sets such that $\bigcap_{A \in \mathcal{A}} A \subseteq \bigcup_{B \in \mathcal{B}} B$, there exist finite subcollections $\mathcal{A}_1$ of $\mathcal{A}$ and $\mathcal{B}_1$ of $\mathcal{B}$ such that $\bigcap_{A \in \mathcal{A}_1} A \subseteq \bigcup_{B \in \mathcal{B}_1} B$.

### 2.4 Balbes-Dwinger duality

Let $\mathcal{D}_p$ be the category of distributive lattices and proper homomorphisms and let $\mathcal{BD}$ be the category of Balbes-Dwinger spaces and strongly continuous functions. We denote $\mathcal{D}_p^1$ the full subcategory of $\mathcal{D}_p$ whose objects are the distributive bounded lattices and $\mathcal{S}$ the full subcategory of $\mathcal{BD}$ whose objects are the

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4This notion is not taken from the literature. The notions of *semi-sober* and *quasi-sober* are found for example in [6] and [7] respectively, but their meanings are different.

5This definition coincides with the one given in [4] for the bounded Balbes-Dwinger spaces.
spectral spaces. If for each morphism $\alpha$ in $\mathcal{D}_p$ we define $\text{spec}(\alpha) = \alpha^*$ and for each morphism $f$ in $\mathcal{BD}$ we define $\mathfrak{F}(f) = f^*$, we have that $\text{spec} : \mathcal{D}_p \to \mathcal{BD}$ and $\mathfrak{F} : \mathcal{BD} \to \mathcal{D}_p$ are contravariant functors. The following theorem is taken from [1] and is an extension of a result in [4].

**Theorem 4.** The functors $\text{spec} : \mathcal{D}_p \to \mathcal{BD}$ and $\mathfrak{F} : \mathcal{BD} \to \mathcal{D}_p$ are covariant functors. The following theorem is taken from [1] and is an extension of a result in [4].

Theorem 4. The functors $\text{spec} : \mathcal{D}_p \to \mathcal{BD}$ and $\mathfrak{F} : \mathcal{BD} \to \mathcal{D}_p$ are co-equivalences of categories such that $\text{spec} \circ \mathfrak{F} \cong 1_{\mathcal{BD}}$ and $\mathfrak{F} \circ \text{spec} \cong 1_{\mathcal{D}_p}$. The restrictions of these functors to the categories $\mathcal{D}_1^0$ and $\mathfrak{F}$ are also co-equivalences.

In particular, we have that for every distributive lattice $L$, $\mathfrak{F}(\text{spec}(L))$ is isomorphic to $L$ and, for every Balbes-Dwinger space $X$, $\text{spec}(\mathfrak{F}(X))$ is homeomorphic to $X$.

## 3 RA-spaces

We introduce here the notion of RA-space. For each RA-space $X$ we define a map $h_X$ which allows us to characterize some topological properties of $X$. This family of maps will become a natural transformation in Section 6 below.

**Definition 3.** We say that a topological space $X$ is an RA-space if $X$ is coherent and $\mathfrak{F}(X)$ is a sub-lattice of $\mathcal{P}(X)$.

Notice that $\mathfrak{F}(X)$ is not a sub-lattice of $\mathcal{P}(X)$ if, and only if, $\emptyset$ is not fundamental and there exist two non-empty open-compact disjoint sets.

From now on, $X$ will be always an RA-space.

We know that $\mathfrak{F}(\text{spec}(\mathfrak{F}(X)))$ is a Balbes-Dwinger space, hence, by Theorem [1] $\mathfrak{F}(\text{spec}(\mathfrak{F}(X))) \cong \mathfrak{F}(X)$ and thus, $\text{spec}(\mathfrak{F}(X))$ is an RA-space.

The proof of the following proposition is straightforward:

**Proposition 1.** For each $x \in X$ the set $\{F \in \mathfrak{F}(X) : x \notin F\}$ is a prime ideal of $\mathfrak{F}(X)$.

Hence, we have a map

$$h_X : X \to \text{spec}(\mathfrak{F}(X)) : x \mapsto \{F \in \mathfrak{F}(X) : x \notin F\}.$$ 

**Proposition 2.** $h_X$ is a strongly continuous and open on its image function.

**Proof.**

Take $F \in \mathfrak{F}(X) - \{\emptyset\}$.

$x \in (h_X)^*(d(F))$ $\iff h_X(x) \in d(F)$

$\iff F \notin h_X(x)$

$\iff x \in F$.

Thus $(h_X)^*(d(F)) = F$. As $\mathfrak{F}(\text{spec}(\mathfrak{F}(X))) \cong \mathfrak{F}(X)$, $h_X$ is strongly continuous and open over its image.

**Proposition 3.** $h_X$ is injective if and only if $X$ is $T_0$.  

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Proof. It is enough to remark that, since $X$ is coherent, $h_X(x) = h_X(y)$ is
equivalent to $\{x\} = \{y\}$.

\begin{proposition}
$h_X$ is surjective if and only if $X$ is almost-sober.
\end{proposition}

Proof.

1. Suppose that $h_X$ is surjective and we have to see that $X$ is almost-sober.
Let $G$ be a proper irreducible closed set of $X$; by definition $G \neq \emptyset$.
We call $A = X - G$. We have that $A \neq \emptyset$, $A \neq X$ and $A$ is a prime open
set of $X$.
Define $\mathcal{I} = \{ F \in \mathfrak{F}(X) : F \subseteq A \}$. As $X$ is coherent, $\mathcal{I} \neq \emptyset$ because $A \neq \emptyset$
and $\mathcal{I} \neq \mathfrak{F}(X)$ given that $A \neq X$. Since $A$ is a prime open set, $\mathcal{I}$ is a
prime ideal of $\mathfrak{F}(X)$; thus, by the hypothesis, there exists $x \in X$ such
that $h_X(x) = \mathcal{I}$.
We have to see that $G = \{x\}$:
$\subseteq$: If $y \notin \{x\}$ then there exists $F \in \mathfrak{F}(X)$ such that $y \in F$ and $x \notin F$,
thus $y \in F$ and $F \in h_X(x) = \mathcal{I}$. Therefore, $y \in A$ and $y \notin G$.
$\supseteq$: If $y \notin G$ then $y \in A$ and therefore, there exists $F \in \mathcal{I}$ such that $y \in F$;
because $X$ is coherent. Thus, $F \in h_X(x)$ so that $x \notin F$. Hence, $y \notin \{x\}$.

2. Suppose that $X$ is almost-sober. We have to see that $h_X$ is surjective.
Consider $\mathfrak{J} \in \text{spec}(\mathfrak{F}(X))$.

Case 1: $\mathfrak{J} = \{\emptyset\}$. We have that $\emptyset$ is fundamental. As $\mathfrak{J}$ is a prime ideal,
every finite intersection of elements of $\mathfrak{F}(X) - \{\emptyset\}$ is non-empty and
therefore, $\bigcap (\mathfrak{F}(Z) - \{\emptyset\}) \neq \emptyset$. For each $x \in \bigcap (\mathfrak{F}(Z) - \{\emptyset\})$ we have
$h_X(x) = \mathfrak{J}$.

Case 2: $\mathfrak{J} \neq \{\emptyset\}$. Define $A = \bigcup \mathfrak{J} = \bigcup F$. We have that $A \neq \emptyset$. If $A = X$,
consider $H \in \mathfrak{F}(X)$, then $H \subseteq \bigcup F$ and as $H$ is compact, there exist
$F_1, \ldots, F_n \in \mathfrak{J}$ such that $H \subseteq F_1 \cup \ldots \cup F_n$ and $F_1 \cup \ldots \cup F_n \in \mathfrak{J}$, so
that $H \in \mathfrak{J}$. Therefore, $\mathfrak{F}(X) = \mathfrak{J}$ which contradicts that $\mathfrak{J}$ is a prime ideal.
We have to show that $A$ is a prime open set. In fact, let $B, C$ be open sets such that $B \cap C \subseteq A$.
As $X$ is coherent, $B = \bigcup_{i} H_i$ and $C = \bigcup_{j} G_j$, where $H_i, G_j \in \mathfrak{F}(X)$
for each $i$ and each $j$. Thus,

$$B \cap C = \left( \bigcup_{i} H_i \right) \cap \left( \bigcup_{j} G_j \right) = \bigcup_{j} \left( \bigcup_{i} (H_i \cap G_j) \right) \subseteq A,$$

so $H_i \cap G_j \subseteq A$ for each $i$ and each $j$. As $H_i \cap G_j$ is compact, there exist
$F_1, \ldots, F_n \in \mathfrak{J}$ such that $H_i \cap G_j \subseteq F_1 \cup \ldots \cup F_n$ and $F_1 \cup \ldots \cup F_n \in \mathfrak{J}$,
then \( H_i \cap G_j \in \mathfrak{I} \), for each \( i \) and each \( j \). As \( \mathfrak{I} \) is prime, \( H_i \in \mathfrak{I} \) or \( G_j \in \mathfrak{I} \), for each \( i \) and each \( j \). Suppose that \( G_{j_0} \notin \mathfrak{I} \), then \( H_i \cap G_{j_0} \notin \mathfrak{I} \) for each \( i \) and then, \( H_i \in \mathfrak{I} \) for every \( i \); thus, \( B \subseteq A \). Similarly, if \( H_{i_0} \notin \mathfrak{I} \), we have that \( C \subseteq A \). We conclude that \( G = \{x\} \).

\[
F \in h_X(x) \iff x \notin F \iff \overline{\{x\}} \cap F = \emptyset \iff G \cap F = \emptyset \iff F \subseteq A \iff F \in \mathfrak{I}.
\]

Hence, \( h_X(x) = \mathfrak{I} \).

**Corollary 1.** \( h_X \) is a homeomorphism if and only if \( X \) is \( T_0 \) and almost-sober.

### 4 Almost-spectral spaces

In this section we characterize almost-spectral spaces, and show, among other things, that they are precisely the sober Balbes-Dwinger spaces.

**Lemma 1.** If \( f : X \to Y \) is continuous and \( F \) is an irreducible closed set of \( X \) then \( \overline{f(F)} \) is an irreducible closed set of \( Y \).

**Proof.** Let \( H \) and \( K \) be two closed sets of \( Y \) such that \( \overline{f(F)} = H \cup K \). We have that \( F = (F \cap f^{-1}(H)) \cup (F \cap f^{-1}(K)) \) and as \( F \) is irreducible, then \( F \subseteq f^{-1}(H) \) or \( F \subseteq f^{-1}(K) \). Hence \( f(F) \subseteq H \) or \( f(F) \subseteq K \) and thus, \( \overline{f(F)} \subseteq H \) or \( \overline{f(F)} \subseteq K \). Therefore \( \overline{f(F)} \) is irreducible.

This Lemma follows immediately if we work in terms of localic maps or frame homomorphisms (see \[9\]).

**Proposition 5.** If \( X \) is a sober space and \( Z \) is an open subspace of \( X \) then \( Z \) is sober.

**Proof.** Let \( F \) be an irreducible closed set of \( Z \). If \( i : Z \to X \) is the inclusion function then, by Lemma \[11\] \( \overline{i(F)}^X \) is an irreducible closed set of \( X \), where \( \overline{i(F)}^X \) is the closure of \( i(F) \) in \( X \). As \( X \) is sober, there exists \( x \in X \) such that \( F^X = \overline{i(F)}^X = \{x\}^X \). It is clear that \( x \in F \) and hence \( \{x\}^Z = F \), because the uniqueness is a consequence of the \( T_0 \) property of \( Z \).

**Proposition 6.** Every almost-spectral space is sober.

**Proof.** Let \( A \) be a commutative ring. We know, by Theorem \[2\] that \( \text{Spec}(A) \) is an open subspace of \( \text{Spec}(Q(A)) \) and \( \text{Spec}(Q(A)) \) is sober because it is a spectral space.

The following lemma is taken from \[1\]:

**Lemma 2.** A distributive lattice \( L \) has a least element if, and only if, \( \text{spec}(L) \) is a sober space.
Theorem 5. Every almost-spectral space is a sober Balbes-Dwinger space.

Proof. Let \( A \) be a commutative ring and let \( \mathfrak{F} \) be the (distributive) lattice of the open-compact sets of \( \text{Spec}(A) \). Since \( \mathfrak{F} \) has a least element we have that \( \text{spec}(\mathfrak{F}) \) is a Balbes-Dwinger sober space. We have to see that \( \text{spec}(\mathfrak{F}) \) and \( \text{Spec}(A) \) are homeomorphic.

If \( I \) is a prime ideal of \( A \), define \( f(I) = \{ B \in \mathfrak{F} : I \notin B \} \). We have to show that \( f(I) \) is a prime ideal of \( \mathfrak{F} \):

It is clear that \( \emptyset \in f(I) \). As \( I \) is a proper ideal of \( A \), there exists \( a \in A - I \), then \( I \subset D(a) \) which is an open-compact set of \( \text{Spec}(A) \). Hence \( f(I) \neq \mathfrak{F} \). If \( B, C \in f(I) \) we have that \( I \notin B \cup C \), then \( B \cup C \in f(I) \). It is clear that \( I \notin C \), therefore \( C \in f(I) \). Consider now \( B, C \in \mathfrak{F} \) such that \( B \cup C \in f(I) \). We have that \( I \notin B \cap C \) then \( I \notin B \) or \( I \notin C \), thus \( B \in f(I) \) or \( C \in f(I) \).

Let \( J \) be a prime ideal of \( \mathfrak{F} \). We have to see that \( W = \bigcup_{B \in J} B \) is a prime open set of \( \text{Spec}(A) \). As \( J \) is proper, there exists \( B \in \mathfrak{F} \) such that \( B \notin J \).

Suppose that \( B \subseteq W \). As \( B \) is compact, there exist \( B_1, \ldots, B_n \in J \) such that \( B \subseteq B_1 \cup \cdots \cup B_n \), then \( B \in J \). We conclude that \( W \neq \text{Spec}(A) \). Let \( S \) and \( T \) be open sets of \( \text{Spec}(A) \) such that \( S \cap T \subseteq W \). There exist \( X, Y \subseteq A \) such that \( S = \bigcup_{x \in X} D(x) \) and \( T = \bigcup_{y \in Y} D(y) \). Thus, \( D(xy) = D(x) \cap D(y) \subseteq W \), for all \( x \in X \) and for all \( y \in Y \). As \( D(xy) \) is compact, there exist \( B_1, \ldots, B_n \in J \) such that \( D(xy) \subseteq B_1 \cup \cdots \cup B_n \), therefore \( D(x) \cap D(y) = D(xy) \in J \). As \( J \) is prime, \( D(x) \in J \) or \( D(y) \in J \). If \( D(y_0) \notin J \) for some \( y_0 \in Y \) then \( D(x) \in J \) for all \( x \in X \), therefore \( S \subseteq W \). Similarly, if \( D(x_0) \notin J \) for some \( x_0 \in X \), then \( T \subseteq W \). We conclude that \( W \) is a prime open set of \( \text{Spec}(A) \). Hence \( W^c \) is an irreducible closed set of \( \text{Spec}(A) \) and as this space is sober, there exists a unique \( P \in \text{Spec}(A) \) such that \( \{ P \} = W^c \). Define \( g(J) = P \).

Thus, we have the maps \( f : \text{Spec}(A) \to \text{spec}(\mathfrak{F}) \) and \( g : \text{spec}(\mathfrak{F}) \to \text{Spec}(A) \).

Besides:

\[
C \in f(g(J)) \iff g(J) \notin C \iff C \supseteq \overline{g(J)}^+ \iff C \subseteq \bigcup_{B \in J} B \iff C \in J,
\]

where the last equivalence is a consequence of the compactness of \( C \).

On the other hand, as \( \bigcup_{B \in f(I)} B = \bigcup_{I \notin B} B = \overline{I}^c \), we have that \( g(f(I)) = I \).

We need to see that \( f \) is continuous and open. Consider \( K \in \mathfrak{F} \):

\[
I \in f^{-1}(d(K)) \iff f(I) \in d(K) \iff K \notin f(I) \iff I \in K
\]

then \( f^{-1}(d(K)) = K \). We conclude that \( f \) is continuous and open over its image and as the image is \( \text{spec}(\mathfrak{F}) \), \( f : \text{Spec}(A) \to \text{spec}(\mathfrak{F}) \) is a homeomorphism. \( \square \)

Theorem 6. Every open of a spectral space is an almost-spectral space.

Proof. Let \( A \) be a commutative ring with identity and let \( Z \) be an open set of \( \text{Spec}(A) \). We know that there exists \( I \) ideal of \( A \) such that \( Z^c = V(I) \). We have,
by Theorem 2
\[ \text{Spec}(I) \approx (V(I))^c = Z. \]

Therefore \( Z \) is an almost-spectral space. \( \square \)

**Theorem 7.** Let \( Z \) be a topological space. The following statements are equivalent:

(i) \( Z \) is almost-spectral.

(ii) \( Z \) is open-dense of a spectral space.

(iii) \( Z \) is open of a spectral space.

(iv) \( Z \) is a sober Balbes-Dwinger space.

(v) \( Z \) is homeomorphic to the prime spectrum of a distributive lattice with minimum.

**Proof.**

(i)\( \Rightarrow \) (ii): If \( Z \) is almost-spectral there exists a commutative ring \( A \) such that \( Z \approx \text{Spec}(A) \). We know that \( \text{Spec}(A) \approx \text{Spec}_A(U_0(A)) \) which is an open-dense of \( \text{Spec}(U_0(A)) \). (See [2]).

(ii)\( \Rightarrow \) (iii): Trivial.

(iii)\( \Rightarrow \) (i): Theorem 6.

(i)\( \Rightarrow \) (iv): Theorem 5.

(iv)\( \Rightarrow \) (v): Theorem 9 (IV-1) of [4] and Proposition 5.7 of [1].

(v)\( \Rightarrow \) (iii): Let \( L \) be a distributive lattice with \( 0 \) such that \( Z \approx \text{spec}(L) \). We have that \( \text{spec}(Z) \) is an open of \( \text{spec}(\hat{L}) \), where \( \hat{L} = L \uplus \Theta \) and \( \Theta \) is a lattice with only one element. \( \square \)

**Proposition 7.** If \( X \) is spectral then every open subspace of \( X \) is almost-spectral and every closed subspace is spectral.

**Proof.** The first part is consequence of Theorem 6. Let \( Z \) be a closed subspace of \( X \). As \( X \approx \text{Spec}(A) \), where \( A \) is a ring with identity, then \( Z \approx V(I) \approx \text{Spec}(A/I) \), for some ideal \( I \) of \( A \).

Similarly we obtain the following proposition.

**Proposition 8.** If \( Z \) is almost-spectral then every open subspace is almost-spectral and every closed subspace is almost-spectral.

### 5 Up-spectral and down-spectral spaces

In this section we present spectral versions for the up-spectral and down-spectral spaces. As a consequence, we obtain a new topological characterization of the Balbes-Dwinger spaces.

First of all we recall the definition of these kind of topological spaces:

**Definition 4.** A space is **up-spectral** if it is coherent and sober. A space is **down-spectral** if it is \( T_0 \), coherent, compact and almost-sober. (See [6]).

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6If \( L \) and \( M \) are lattices, its *ordinal sum* \( L \uplus M \) is defined by the set \( L \times \{0\} \cup M \times \{1\} \) ordered by: \( (x,i) \leq (y,j) \) if \( x \leq y \) and \( i = j \) or if \( i = 0 \) and \( j = 1 \).
Actually, the notions of up-spectral space and almost-spectral space are equivalent, as the following theorem shows.

**Theorem 8.** Let $Z$ be a topological space. The following statements are equivalent:

(i) $Z$ is up-spectral.

(ii) $Z$ is almost-spectral.

**Proof.** If $Z$ is up-spectral then $Z^\omega$ (the trivial compactification of $Z$) is a spectral space (Proposition 1.5 of [2]). Thus, $Z$ is open of a spectral space and therefore is almost-spectral. Reciprocally, if $Z$ is almost-spectral then $Z$ is a Balbes-Dwinger and sober space. Hence, $Z$ is up-spectral. 

**Corollary 2.** Let $Z$ be a topological space. The following statements are equivalent:

(i) $Z$ is up-spectral.

(ii) $Z$ is homeomorphic to the prime spectrum of a distributive lattice with minimum.

**Theorem 9.** Every Balbes-Dwinger space is almost-sober.

**Proof.** Let $Z$ be a Balbes-Dwinger space. Let $G$ be a proper irreducible closed set of $Z$. Then $A = Z - G$ is a non-empty prime open set of $Z$. So we have that $A = \bigcup_{i \in \Lambda} F_i$ for some collection of non-empty open-compact sets of $Z$. Let $\mathcal{I}$ be the ideal of $\mathfrak{F}(Z)$ generated by $\{F_i\}_{i \in \Lambda}$, $\mathcal{I} = \{F \in \mathfrak{F}(Z) : F \subseteq A\}$. As $A$ is a prime open set, $\mathcal{I}$ is a prime ideal of $\mathfrak{F}(Z)$. Since $Z$ is a Balbes-Dwinger space, there exists $x \in Z$ such that $\mathcal{I} = \{F \in \mathfrak{F}(Z) : x \notin F\}$. It is clear that $G = \{x\}$. 

The following theorem gives an additional and simpler topological characterization for the Balbes-Dwinger spaces.

**Theorem 10.** Let $Z$ be a topological space. The following statements are equivalent:

(i) $Z$ is $T_0$, coherent and almost-sober.

(ii) $Z$ is a Balbes-Dwinger space.

**Proof.** By the previous theorem, (ii) implies (i). Now, let $Z$ be a $T_0$, coherent and almost-sober space. Suppose that there exist $F, G$ non-empty open-compact sets such that $F \cap G = \emptyset$, so $F^c \cup G^c = Z$ and then, $Z$ is not an irreducible set. As $Z$ is $T_0$ and almost-sober, then $Z$ is sober and therefore up-spectral. Hence, by Theorems 7 and 8, $Z$ is a Balbes-Dwinger space.

If there do not exist non-empty open-compact sets $F$ and $G$ such that $F \cap G = \emptyset$, then $\mathfrak{F}(Z)$ is a distributive lattice, because $Z$ is coherent. Therefore, by Theorem 2 (IV) of [1], we have that $\text{spec}(\mathfrak{F}(Z))$ is a Balbes-Dwinger space. On the other hand, as $Z$ is an almost-sober, $T_0$ F-space, then by the Corollary 1, $Z$ and $\text{spec}(\mathfrak{F}(Z))$ are homeomorphic, thus $Z$ is a Balbes-Dwinger space. 

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As a corollary we obtain the spectral version of the down-spectral spaces.

**Corollary 3.** Let $Z$ be a topological space. The following statements are equivalent:

(i) $Z$ is down-spectral.

(ii) $Z$ is a Balbes-Dwinger and compact space.

(iii) $Z$ is homeomorphic to the prime spectrum of a distributive lattice with maximum.

### 6 An extension of the Balbes-Dwinger duality

We denote $\mathcal{GBP}$ the category whose objects are the RA-spaces and whose morphisms are the strongly continuous functions.

**Definition 5.** For each strongly continuous function $f : X \to Y$ between RA-spaces we define $\mathfrak{F}(f) : \mathfrak{F}(Y) \to \mathfrak{F}(X)$ by $\mathfrak{F}(f)(F) = f^*(F)$ and for each proper homomorphism $h : L \to M$ between distributive lattices we define $\text{spec}(h) : \text{spec}(M) \to \text{spec}(L)$ by $\text{spec}(h)(I) = h^*(I)$.

**Lemma 3.** If $f : X \to Y$ is a strongly continuous function between RA-spaces, then $\mathfrak{F}(f) : \mathfrak{F}(Y) \to \mathfrak{F}(X)$ is a proper homomorphism.

**Proof.** Let $\mathfrak{I}$ be a prime ideal of $\mathfrak{F}(X)$. If $\mathfrak{F}(f)^*(\mathfrak{I}) = \mathfrak{F}(Y)$ then for every $F \in \mathfrak{F}(Y)$ we have that $f^*(F) \in \mathfrak{I}$. Thus,

$$X = f^* \left( \bigcup_{F \in \mathfrak{F}(Y)} F \right) = \bigcup_{F \in \mathfrak{F}(Y)} f^*(F) \subseteq \bigcup_{G \in \mathfrak{I}} G.$$

As $\mathfrak{I}$ is a proper ideal of $\mathfrak{F}(X)$, there exists $G_0 \in \mathfrak{F}(X) - \mathfrak{I}$. Then $G_0 \subseteq \bigcup_{G \in \mathfrak{I}} G$ and as $G_0$ is compact, there exist $G_1, \ldots, G_n \in \mathfrak{I}$ such that $G_0 \subseteq \bigcup_{i=1}^n G_i \in \mathfrak{I}$ and therefore $G_0 \in \mathfrak{I}$, which is absurd. The missing details to see that $\mathfrak{F}(f)^*(\mathfrak{I})$ is a prime ideal of $\mathfrak{F}(Y)$ are obtained directly from the definition of $\mathfrak{F}(f)$.

**Lemma 4.** If $h : L \to M$ is a proper homomorphism between distributive lattices, then $\text{spec}(h) : \text{spec}(M) \to \text{spec}(L)$ is a strongly continuous function.

**Proof.** By the Proposition 5.6 of [1] we know that $\text{spec}(h)$ sends open-compact sets to open-compact sets by inverse image. We need to see that if $\emptyset$ is fundamental in $\text{spec}(L)$ then $\emptyset$ is fundamental in $\text{spec}(M)$, but this is equivalent to see that if $L$ has minimum, then $M$ has minimum. (Proposition 5.8 of [1]).

We call 0 the minimum of $L$ and suppose that $M$ has not minimum. If $s = h(0)$, there exists $t \in M$ such that $t < s$. We call $I$ to the ideal generated by $t$ and $F$ to the filter generated by $s$. As $M$ is distributive, there exists a prime ideal $P$ of $M$ such that $F \subseteq P$ and $P \cap F = \emptyset$. As $h$ is a proper homomorphism, then $h^*(P) = \emptyset$ is a prime ideal of $L$, but this is contradictory.
It is easy to check that $\mathfrak{F} : \mathfrak{S}p \to \mathfrak{D}_p$ and $\text{spec} : \mathfrak{D}_p \to \mathfrak{S}p$ are contravariant functors.

The following theorem extends Theorem 4.

**Theorem 11.** The functors $\mathfrak{F}$ and $\text{spec}$ are right adjoint contravariant functors.

*Proof.* Let $M$ be a distributive lattice and let $X$ be a RA-space. If $\alpha : M \to \mathfrak{F}(X)$ is a proper homomorphism, then $\text{spec}(\alpha) : \mathfrak{F}(\mathfrak{F}(X)) \to \text{spec}(M)$ is a strongly continuous function and it is known that $h_X : X \to \text{spec}(\mathfrak{F}(X))$ also is a strongly continuous function (Proposition 2). We have to see that $\lambda_{(M,X)} : [M, \mathfrak{F}(X)]_{D_p} \to [X, \text{spec}(M)]_{\mathfrak{S}p}$ defined by $\lambda_{(M,X)}(\alpha) = \text{spec}(\alpha) \circ h_X$ is a bijective function.

i) $\lambda_{(M,X)}$ is injective:

$\lambda_{(M,X)}(\alpha) = \lambda_{(M,X)}(\beta)$

$\iff \text{spec}(\alpha) \circ h_X = \text{spec}(\beta) \circ h_X$

$\iff \alpha^* \circ h_X = \beta^* \circ h_X$

$\iff [\alpha^*(h_X(x)) = \beta^*(h_X(x)), \forall x \in X]$  

$\iff [\alpha(z) \in h_X(x) \iff \beta(z) \in h_X(x), \forall z \in M, \forall x \in X]$  

$\iff [x \notin \alpha(z) \iff x \notin \beta(z), \forall z \in M, \forall x \in X]$  

$\iff \alpha(z) = \beta(z), \forall z \in M$

$\iff \alpha = \beta.$

ii) $\lambda_{(M,X)}$ is surjective: Let be $\varepsilon : X \to \text{spec}(M)$ a strongly continuous map.

We have that $\mathfrak{F}(\varepsilon) : \mathfrak{F}(\text{spec}(M)) \to \mathfrak{F}(X)$ is a proper homomorphism. Consider the proper homomorphism $d : M \to \mathfrak{F}(\text{spec}(M))$ (Theorem 5.7 of [1]).

$\lambda_{(M,X)}(\mathfrak{F}(\varepsilon) \circ d) = \text{spec}(\mathfrak{F}(\varepsilon) \circ d) \circ h_X = (\mathfrak{F}(\varepsilon) \circ d)^* \circ h_X.$ Let be $x \in X.$

$I \in (\mathfrak{F}(\varepsilon) \circ d)^* \circ h_X(x)$  

$\iff (\mathfrak{F}(\varepsilon) \circ d)(I) \in h_X(x)$  

$\iff x \notin (\mathfrak{F}(\varepsilon) \circ d)(I) = \varepsilon^*(d(I))$  

$\iff \varepsilon(x) \notin d(I)$  

$\iff I \in \varepsilon(x).$

Therefore, $\lambda_{(M,X)}(\mathfrak{F}(\varepsilon) \circ d) = \varepsilon.$

The family $\lambda = \{ \lambda_{(M,X)} \}_{(M,X) \in \mathfrak{D}_p \times \mathfrak{S}p}$ is a natual bijection: Let $g \in [Y, X]_{\mathfrak{S}p}$ and $\alpha \in [M, \mathfrak{F}(X)]_{D_p}$, we need to see that $\lambda_{(M,Y)}(\mathfrak{F}(g) \circ \alpha) = \lambda_{(M,X)}(\alpha) \circ g.$ Take $y \in Y.$

$I \in \lambda_{(M,Y)}(\mathfrak{F}(g) \circ \alpha)(y)$  

$\iff I \in \text{spec}(\mathfrak{F}(g) \circ \alpha) \circ h_Y(y)$  

$\iff I \in (\mathfrak{F}(g) \circ \alpha)^* \circ h_Y(y)$  

$\iff (\mathfrak{F}(g) \circ \alpha)(I) \in h_Y(y)$  

$\iff y \notin (\mathfrak{F}(g) \circ \alpha)(I)$  

$\iff y \notin g^*(\alpha(I))$  

$\iff g(y) \notin \alpha(I)$  

$\iff \alpha(I) \in h_X(g(y))$  

$\iff I \in \alpha^*(h_X(g(y)))$  

$\iff I \in \text{spec}(\alpha)(h_X(g(y))).$

Similarly it is obtained that for $f \in [L, M]_{D_p}$ and $\alpha \in [M, \mathfrak{F}(X)]_{D_p}$, it must $\lambda_{(L,X)}(\alpha \circ f) = \text{spec}(f) \circ \lambda_{(M,X)}(\alpha).$  

$\square$
The co-equivalence of this adjunction is between the categories $\mathcal{D}_p$ and $\mathcal{BD}$.

We introduce here two full subcategories of $\mathcal{D}_p$ and two full subcategories of $\mathcal{BD}$:

| Name   | Objects                                                                 |
|--------|------------------------------------------------------------------------|
| $\mathcal{D}_0$ | Distributive lattices with minimum                                        |
| $\mathcal{D}_1$ | Distributive lattices with maximum                                        |
| $\mathcal{US}$ | Up-spectral spaces = Almost-spectral spaces = Sober Balbes-Dwinger spaces |
| $\mathcal{DS}$ | Down-spectral spaces = Compact Balbes-Dwinger spaces                      |

**Corollary 4.** The following pairs of categories are co-equivalent:

(i) $\mathcal{D}_0$ and $\mathcal{US}$.
(ii) $\mathcal{D}_1$ and $\mathcal{DS}$.

Now, it is clear that the notions of up-spectral space and down-spectral space are mutually dual in the category $\mathcal{BD}$.

\* \* \*

The following diagram summarizes the previous results.

The represented examples in the diagram are:

- $X_1 = \mathbb{Z}$ with the Alexandroff topology.
- $X_2 = \mathbb{Z}^-$ with the Alexandroff topology.
- $X_3 = \mathbb{N}$ with the Alexandroff topology.
- $X_4 = \{1, 2, 3, 4\}$ with the Alexandroff topology.
$X_5 = \mathbb{R}$ with the Alexandroff topology. $X_5$ is not almost-sober because for example, $(-\infty, 3)$ is a proper irreducible closed set that is not the closure of any point.

$X_6 = \{a, b, c\}$ with the topology $\{\emptyset, \{a, b\}, \{a, b, c\}\}$.

$X_7 = (\mathbb{R} \times \{0\}) \cup (\mathbb{R} \times \{1\}) \cup \{(\omega, 0), (\omega, 1)\}$ with the Alexandroff topology obtained from the preorder given by: $(x, i) \leq (y, j)$ if $x, y \in \mathbb{R}$, $x \leq y$ and $i = j$; $(x, i) \leq (\omega, j)$ for all $x \in \mathbb{R}$ and all $i, j$; $(\omega, 0) \leq (\omega, 1)$ and $(\omega, 1) \leq (\omega, 0)$.

$X_7$ is not almost-sober because for example, $\mathbb{R} \times \{0\}$ is a proper irreducible closed set that is not the closure of a point.

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