Wavefunction Collapse and Conservation Laws

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Abstract

It is emphasized that the collapse postulate of standard quantum theory can violate conservation of energy-momentum and there is no indication from where the energy-momentum comes or to where it goes. Likewise, in the Continuous Spontaneous Localization (CSL) dynamical collapse model, particles gain energy on average. In CSL, the usual Schrödinger dynamics is altered so that a randomly fluctuating classical field interacts with quantized particles to cause wavefunction collapse. In this paper it is shown how to define energy for the classical field so that the average value of the energy of the field plus the quantum system is conserved for the ensemble of collapsing wavefunctions. While conservation of just the first moment of energy is, of course, much less than complete conservation of energy, this does support the idea that the field could provide the conservation law balance when events occur.

1 The Collapse Postulate and Nonconservation of Energy-Momentum

It is not generally appreciated that the collapse postulate of standard quantum theory (SQT) can violate the geometric conservation laws, e.g., the conservation of energy and momentum.\[1\]

In quantum theory, conservation of a physical quantity represented by the operator $Q$ is the statement that the probability distribution of $Q$'s eigenvalues $q$, $\langle q|\psi, t\rangle^2$, is constant in time. For the Schrödinger evolution of a statevector, conservation is guaranteed by the vanishing of the commutator of $Q$ with the Hamiltonian. This ensures that $\langle \psi, t|F(Q)|\psi, t\rangle = \langle \psi, 0|F(Q)|\psi, 0\rangle$, where $F$ is an arbitrary function of $Q$: the constancy of $\langle q|\psi, t\rangle^2$ immediately follows. (It will be useful for later purposes to note here that this is also equivalent to constancy of the expectation value of arbitrary integer powers of $Q$, which in turn is equivalent to constancy of the expectation value of the generating function $G \equiv \exp iaQ$.)

However, according to SQT, the Schrödinger evolution is only part of a statevector’s evolution. Under some circumstances (such as measurement) the “collapse” postulate is to be applied: when the states $|\psi_i\rangle$ become sufficiently
“macroscopically distinct,” the statevector $|\psi, T\rangle = \sum_i \alpha_i |\psi_i\rangle$ is to be replaced by one of the orthonormal statevectors $|\psi_i\rangle$ with probability $|\alpha_i|^2$. Conservation of momentum is conserved in collapse are a set of measure zero. G lapse. These generating functions are, respectively, generating functions for energy and momentum to be unchanged b y the col-
to be orthogonal), then the collapse postulate says that

“sufficiently” distinctly localized (e.g., gaussian-like peaks which are nar-
well-separated, but with small wiggles in the right places in the tails so as to
to be orthogonal), then the collapse postulate says that $|\psi\rangle$ is to be replaced by one of the $|\phi\rangle$ with probability $|\alpha_i|^2$. However, we shall now point out that the class of distinctly localized pairs of states $\{ |\Phi_1\rangle, |\Phi_2\rangle \}$ for which energy-momentum is conserved in collapse are a set of measure zero.

Conservation of energy-momentum requires the expectation value of the generating functions for energy and momentum to be unchanged by the collapse. These generating functions are, respectively, $G_E \equiv \exp ia(H_{cm} + H_{int}) \equiv G_{cm}G_{int}$ ($H_{cm} \equiv P^2/2M$ where $P$ is the cm momentum operator and $M$ is the pointer mass; $H_{int}$ is the Hamiltonian of the internal degrees of freedom) and $G_P \equiv \exp ibP$.

Then, $\langle \psi | G_P | \psi \rangle = \sum_{i=1}^2 |\alpha_i|^2 \langle \Phi_i | G_P | \Phi_i \rangle$ is implied by momentum conservation in collapse, i.e., $\alpha_1^* \alpha_2 \langle \Phi_1 | G_P | \Phi_2 \rangle + \alpha_1 \alpha_2^* \langle \Phi_2 | G_P | \Phi_1 \rangle = 0$. Since the collapse postulate is to be applied to the states $|\Phi_i\rangle$ because of their macroscopic distinctness and regardless of the amplitudes $\alpha_i$, the phase of $\alpha_1 \alpha_2^*$ is arbitrary, so $\langle \Phi_2 | \exp ibP | \Phi_1 \rangle = 0$. Multiplying this by $\exp -ibp$ and integration over $b$ gives the momentum projection operator $\delta(P - p)$, with the result

$$\langle \Phi_2 | p | \Phi_1 \rangle = 0$$

(1)

The similar argument for energy implies that $\langle \Phi_2 | G_{cm} | \Phi_1 \rangle \langle \phi | G_{int} | \phi \rangle = 0$. However, if Eq. (1) is satisfied then the first factor vanishes:

$$\langle \Phi_2 | G_{cm} | \Phi_1 \rangle = \int dp \langle \Phi_2 | p | \Phi_1 \rangle e^{iaP^2/2M} = 0.$$ 

Therefore, (1) implies not only momentum conservation but also energy conserva-
tion in collapse as well.

The condition (1) is very stringent: most (all but a set of measure zero) superpositions of macroscopically distinct states will not satisfy it. But, actually,
of those states which do satisfy (1) only a measure zero set of them will be macroscopically distinct. For, in order that (1) be satisfied, there must be at least one range of momentum $p_1 \leq p \leq p_2$ over which $\langle p|\Phi_2 \rangle = 0$ and

$$\langle p|\Phi_1 \rangle = \Theta(p_2 - p)\Theta(p - p_1)\langle p|\Phi \rangle$$

($\Theta$ is the step function). However, each such range of momentum contributes a piece to the wavefunction $\langle x|\Phi_1 \rangle$ that has infinite $\langle x^2 \rangle$:

$$\langle x^2 \rangle = \int_{-\infty}^{\infty} dp \langle \Phi_1|p\rangle \left\{ \frac{d}{dp} \langle \Phi_1|p\rangle \right\}^2 = \int_{-\infty}^{\infty} dp \left| \frac{d}{dp} \langle \Phi_1|p\rangle \right|^2$$

$$= \langle p_1|\Phi \rangle^2 \int_{-\infty}^{\infty} dp \delta^2(p - p_1) + \langle p_2|\Phi \rangle^2 \int_{-\infty}^{\infty} dp \delta^2(p - p_2) + \cdots = \infty$$

Another way to see this is to note that such a momentum space wave function segment contributes an asymptotic $x^{-1}$ behavior to the position space wavefunction $\langle x|\Phi_1 \rangle$, since integrating the identity

$$e^{ipx}\Phi(p) = \frac{d}{dp} \left\{ e^{ipx} \frac{\Phi(p)}{ix} - \frac{\Phi'(p)}{(ix)^2} + \frac{\Phi''(p)}{(ix)^3} - \cdots \right\}$$

results in

$$\int_{p_1}^{p_2} dp e^{ipx} \langle p|\Phi \rangle = e^{ip_2x} \frac{\langle p_2|\Phi \rangle}{ix} [1 + O\left( \frac{1}{x} \right)] - e^{ip_1x} \frac{\langle p_1|\Phi \rangle}{ix} [1 + O\left( \frac{1}{x} \right)]$$

(2)

This demonstration that the wavefunctions are not well localized assumes that $\langle p_1|\Phi \rangle^2 \neq 0$. A set of measure zero of these wavefunctions has zeros at the $p_i$’s. Then, a similar argument shows that $\langle x^4 \rangle$ is infinite provided that also $\langle p_i|\Phi \rangle' \neq 0$, etc. We see therefore that one can construct a superposition of distinctly localized wavefunctions which conserve energy-momentum in collapse if the zeros at the $p_i$’s are of large enough order. An example is if the momentum space form is

$$\langle p|\Phi_1 \rangle \sim \Theta(p_2 - p)\Theta(p - p_1)(p_2 - p)^n(p - p_1)^n \exp -ipb$$

(where $n$ is large) and similarly for $\langle p|\Phi_2 \rangle$ (with nonoverlapping $p_i$ ranges, and a quite different mean position $b$).

We conclude that all but a set of measure zero of wavefunctions $\langle x|\Phi_i \rangle$ which are macroscopically distinct do not conserve energy-momentum under the collapse rule.

To push this a bit further, given the desideratum of energy-momentum conservation, one could imagine a collapse rule which is not of the usual type,
in that it would not require collapse of $|\Phi\rangle = \alpha_1|\Phi_1\rangle + \alpha_2|\Phi_2\rangle$ for arbitrary \(\alpha_i\), but only for one set of \(\alpha_i\), provided energy-momentum is conserved. Then one cannot invoke the arbitrariness of phases used to obtain (1). Instead, one may without loss of generality absorb phases in the $|\Phi_i\rangle$ so that \(\alpha_1\) is real and \(\alpha_2\) is imaginary. Then, (1) is replaced by the more general energy-momentum conserving requirement

$$\langle \Phi_2|p\rangle \langle p|\Phi_1\rangle = \langle \Phi_1|p\rangle \langle p|\Phi_2\rangle$$

This can be satisfied by wavefunctions which obey (1) with the consequences presented above but, in addition, it can be satisfied if the momentum space wavefunctions have identical phases:

$$\langle p|\Phi_i\rangle = R_i(p)e^{i\theta(p)}$$

\((R_i, \theta \text{ are real})\). Certainly this is a set of measure zero but, more than that, it is hard to see how such wavefunctions could be of the macroscopically distinct type one desires. To see this, consider that the criterion $\langle \Phi_2|\exp iPb|\Phi_1\rangle = \langle \Phi_1|\exp iPb|\Phi_2\rangle$ is, in position space,

$$\int dx \Phi_2^*(x)\Phi_1(x + b) = \int dx \Phi_2^*(x)\Phi_1^*(x - b)$$

Thus the absolute magnitude of the overlap integral of one wavefunction with the other translated by the distance \(b\) is the same for translation to the right or left. Since the integral $= 0$ for \(b = 0\) (the wavefunctions are orthogonal) and vanishes for infinite \(b\), it must have a maximum for some translation to the right and for an equal translation to the left. It is hard to reconcile that behavior with distinctly localized wavefunctions whose overlap integral one would think would grow to a maximum for translation in one direction as the peaks of the wavefunctions are brought to overlap, but which would diminish for translation in the other direction as the peaks are moved further away from each other.

So, we again conclude that all but a set of measure zero of wavefunctions which are macroscopically distinct do not conserve energy-momentum, even under this relaxed collapse rule.

Of course, the above example is rather specialized. A practical scheme for getting a pointer into a superposition of two spatially separated states could very well entail different internal states for the two pointer positions and entail correlation with a system external to the pointer as in a measurement. However, that a similar difficulty exists in realistic cases can be seen as follows.

Consider the measurement of whether a particle is in a given region (again, for simplicity we will discuss just a one dimensional world: the argument is easily extended to three dimensions) so that the apparatus will either register “in the region” (1) or “outside the region” (2). The apparatus may be as complicated as needed, it may not make a perfect measurement, it may disturb
the particle: we just suppose that the final state of apparatus and particle at time $T$ is a superposition of two orthogonal macroscopically distinguishable states $|Ψ, T⟩ = α_1 |χ_1⟩ + α_2 |χ_2⟩$. We also suppose, going backwards in time, that $\exp iH(T − T_0)|χ_i⟩ = |Φ_i⟩|Γ⟩|γ⟩$, where $|Φ_i⟩$ is a state of the particle at initial time $T_0$ and $|Γ⟩, |γ⟩$ are respectively the initial apparatus cm and internal statevectors, so

$$|Ψ, T_0⟩ = [α_1 |Φ_1⟩ + α_2 |Φ_2⟩]|Γ⟩|γ⟩$$

Now, suppose we adopt the collapse postulate to collapse the final statevector to states $|χ_i⟩$ and that energy-momentum is conserved. In particular, this means that $⟨χ_2|\exp ib(P + P_{cm})|χ_1⟩ = 0$, where $P_{cm}$ is the apparatus momentum operator (which only acts on the apparatus cm statevector) and $P$ is the particle’s momentum operator. Since $[P + P_{cm}, H] = 0$, this means that

$$⟨Φ_2|e^{ibP}|Φ_1⟩⟨Γ|e^{ibP_{cm}}|Γ⟩⟨γ|γ⟩ = 0$$

For a reasonable initial apparatus cm wavefunction (such as a gaussian), the second scalar product (of $|Γ⟩$ with itself translated by a distance $b$) does not vanish for any finite $b$. This means that $⟨Φ_2|e^{ibP}|Φ_1⟩ = 0$, which implies that (1) and therefore (2) are satisfied by the initial particle wavefunction.

Thus, the hypothesis that energy-momentum is conserved in the collapse to the states $|χ_i⟩$ requires (apart from the caveat about a set of measure zero discussed previously) that the apparatus measures one infinite $⟨x^2⟩$ state of the particle as being inside the region and another infinite $⟨x^2⟩$ state as being outside the region. This could not be the case for a reasonably designed (even somewhat imperfect) apparatus. One must conclude that the hypothesis is untenable and that energy-momentum will generally not be conserved if the collapse postulate is applied.

### 2 A Collapse Model and Nonconservation of Energy-Momentum

Dynamical collapse models replace the collapse postulate of SQT by a modified Schrödinger equation which describes wavefunction collapse as a continuous physical process. The hope is that there really is such a process and that construction of phenomenological models and investigation of their experimental consequences will contribute to its confirmation. The first such models were designed to produce, as final states, just the results of the collapse postulate. Thus their conservation law violation is just of the type illustrated above. However, these models have the unsatisfactory feature that the onset of collapse and the “preferred basis” (final states of collapse) are put in by hand, for each application. The Spontaneous Localization (SL) model of Ghirardi,
Rimini and Weber showed how to overcome this, although its method of achieving collapse is not via a modified Schrödinger equation, and it has the unsatisfactory feature of violating particle exchange symmetry. These last problems are overcome, and the good features of earlier models and SL are retained in the nonrelativistic Continuous Spontaneous Localization (CSL) model\[6, 9, 10, 11\]. However, SL and CSL introduce a new mechanism for conservation law violation.

In CSL a randomly fluctuating classical field \( w(x,t) \) interacts with the particle number density (or mass density or energy density) operator to produce collapse toward its spatially localized eigenstates (this resolves the preferred basis problem). CSL possesses the SL feature that the collapse interaction is always "on" (this resolves the collapse onset problem). The collapse of a many–particle state in a superposition of widely separated clumps to one of the clumps is rapid, but even a single isolated particle continually undergoes collapse, a narrowing of its wavefunction, albeit slowly. This narrowing means that its energy increases.

The predicted increase of particle energy due to collapse has been the focus of experimental tests\[12, 13, 14\] (in lieu of the more difficult direct tests of macroscopic interference\[15, 16, 17\]). These have suggested that the coupling between \( w(x,t) \) and the particle number density operator is proportional to the particle's mass, i.e., that \( w(x,t) \)'s coupling is to mass density (or energy density), with its suggestive overtones of a connection between collapse and gravity\[18, 19, 20, 21, 22\].

This violation of the conservation of particle energy has also been the focus of criticism of the CSL model\[23, 24\]. As has been emphasized in section 1, this criticism also deserves to be applied to SQT plus the collapse postulate. However, here I shall show a way to define energy for the complete system of classical field plus quantized particles so that its expectation value is constant, not for any individual statevector \( |\psi, t\rangle_w \) evolving under its particular field \( w(x,t) \), but for the ensemble of collapsing wavefunctions \( \{ |\psi, t\rangle_w \} \) (with the correct probability for the occurrence of each \( w \), as given by CSL). This is as it should be: the ensemble of collapsed states and their associated fields \( w \) describe the realized physical states in nature, and it is this ensemble which should satisfy the conservation law.

Of course this is far from complete conservation of energy: that would require conservation of all powers of the energy, not just the first. Perhaps conservation of the energy expectation value is as much as one might expect from a model where the collapse–causing field isn’t quantized. However, this result does suggest that the hitherto unaccounted for violation of energy conservation, by SQT plus the collapse postulate, involved in describing the occurrence of physical events, may be accounted for in a dynamical collapse model as due to an energy exchange with a collapse–causing field.

It should be mentioned that a number of authors have suggested models where collapse takes place toward energy eigenstates (rather than toward energy density eigenstates as in CSL)\[25, 26, 27\]. In such models energy is conserved but the resulting collapsed states may not be the states seen in nature: macroscopic
superpositions of spatially separated states of the same energy result from the dynamics of these models. This seems to miss the point of a collapse model which is, roughly speaking, “what you see (in nature) is what you get (from the theory)".

For explanatory ease, a simpler CSL model than described above shall be employed. In this model the collapse is toward the eigenstates of a single operator $A$ rather than nonrelativistic CSL’s collapse toward the eigenstates of the mass density or energy operator (actually an infinity of commuting operators, one at each point of space). This uses a fluctuating classical quantity $w(t)$ which only depends upon $t$. Section 3 contains a review of this CSL formalism. Section 4 presents the expression for the energy associated with $w(t)$ plus the quantum system, and gives the proof of conservation of the ensemble mean energy. In section 5, expressions are given separately for the quantum system’s mean energy and $w(t)$’s mean energy, showing how a change of the former is at the expense of the latter. In conclusion, section 6 contains a simple example and also sketches how to apply this to the full nonrelativistic CSL model and to the other geometric conservation laws such as momentum conservation.

3 CSL

Consider the statevector evolution

$$|\psi, T\rangle_w \equiv T e^{-\frac{i}{\hbar} \int_0^T dt [w(t) - 2\lambda A(t)]^2} |\psi, 0\rangle \quad (3)$$

($T$ is the time-ordering operator). This is in the “collapse interaction picture” where the operator $A(t) \equiv \exp(iH_A t) A \exp-(iH_A t)$ evolves according to the usual Schrödinger dynamics and the statevector evolves only due to collapse dynamics.

In addition to (3), we need to know the probability density for each $w(t)$:

$$P_T(w) \equiv w \langle \psi, T | \psi, T \rangle_w . \quad (4)$$

The probability that $w(t)$ lies between $w(t)$ and $w(t) + dw(t)$ for each $t$ in the range $(0, T)$ is

$$Dw P_T(w) \equiv \prod_{t=0}^{t=T} \frac{dw(t)}{\sqrt{2\pi\lambda/dt}} P_T(w) . \quad (5)$$

(In expressions like (5), $t$ may be thought of taking on closely spaced discrete values: $P_T(w)$ is a functional of $w(t)$ for all $0 \leq t \leq T$.)

Since (3) is a nonunitary evolution it does not preserve statevector norm (which is perfectly all right since, in a collapse theory, the direction of a statevector in Hilbert space is all that is needed to describe the associated physical reality) so (4) says that statevectors which have largest norm are most probable.

Eqs. (3) and (4) comprise the CSL model discussed here. To see how they work, neglect the unitary evolution (set $H_A = 0$) and set $|\psi, 0\rangle = \sum_i c_i |a_i\rangle$ ($A|a_i\rangle = a_i |a_i\rangle$). Eqs. (3) and (4) respectively become

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\[ |\psi, T\rangle_w = \sum_i c_i |a_i\rangle e^{-\frac{1}{\hbar} \int_0^T dt |w(t) - 2\lambda a_i|^2} \]

\[ \mathcal{P}_T(w) = \sum_i |c_i|^2 e^{-\frac{1}{\hbar} \int_0^T dt |w(t) - 2\lambda a_i|^2} . \]

First, suppose \( w(t) = 2\lambda a_j \). Then, as \( T \to \infty \),

\[ |\psi, T\rangle_w \to c_j |a_j\rangle + \sum_{i \neq j} c_i |a_i\rangle e^{-\lambda T |a_j - a_i|^2} \]

\[ \mathcal{P}_T(w) \to |c_j|^2 + \sum_{i \neq j} |c_i|^2 e^{-2\lambda T |a_j - a_i|^2} . \]

In this case the statevector asymptotically "collapses" to \( |a_j\rangle \). More generally, only if \( w(t) = 2\lambda a_j + w_0(t) \), where \( w_0(t) \) is a sample white noise function with zero drift \( \langle \mathcal{P}(w_0) = \exp \left( -\frac{1}{2\lambda} \int dt w_0(t)^2 \right) \rangle \) will there be a non–negligible probability for large \( T \). In that case, each of these \( |\psi, T\rangle_w \to |a_j\rangle \) for large \( T \) and the total probability of these \( w(t) \)'s is \( \int Dw \mathcal{P}_T(2\lambda a_j + w_0(t)) \to |c_j|^2 \). This is, of course, the same result as would be obtained by applying the collapse postulate to the original statevector.

The density matrix follows from (3) and (4):

\[ \rho(T) = \int Dw \mathcal{P}_T(w) |\psi, T\rangle_w \langle \psi, T| = \int Dw |\psi, T\rangle_w \langle \psi, T| \]

\[ = T e^{-\frac{1}{\hbar} \int_0^T dt [A(t) \otimes 1 - 1 \otimes A(t)]^2} \rho(0) \]  

(6)

We are employing the notation \( (X \otimes Y)Z = XY \) and \( T \) time–orders operators to the left of \( \rho(0) \equiv |\psi, 0\rangle \langle \psi, 0| \) and time–reverse orders operators to the right of \( \rho(0) \) so, for example, Eq. (3) in a less compact notation is

\[ \rho(T) = \rho(0) - \frac{\lambda}{2} T \int_0^T dt [A(t), [A(t), \rho(0)]] + ... \]

The collapse behavior in the previous example is easy to see in the density matrix (6),

\[ \rho(t) = \sum_i \sum_j c_i^* c_j e^{-\frac{1}{\hbar} T |a_i - a_j|^2} |a_i\rangle \langle a_j| , \]

whose off–diagonal elements vanish as \( T \to \infty \).
4 Energy Expectation Conservation

Although $w(t)$ is a classical field, in order to put it on a par with the quantized quantity $A(t)$ we introduce the functional differential operators $\Pi(t) \equiv i^{-1} \frac{\delta}{\delta w(t)}$ (which act on functionals of $w(t)$ to the right and left respectively) and $\Pi(t) \equiv \frac{1}{2}(\Pi(t) + \Pi(t))$, all of which are conjugate operators to $w(t)$ (i.e., $[w(t), \Pi(t')] = i\delta(t - t')$, etc.). We define an energy operator for the classical field

$$H_w = \frac{1}{2} \int_{-\infty}^{\infty} dt [\dot{w}(t)\Pi(t) + \Pi(t)\dot{w}(t)].$$  \hspace{1cm} (7)

It is readily verified that

$$[\frac{d^n}{dt^n} w(t), H_w] = i\frac{d^{n+1}}{dt^{n+1}} w(t), \quad [\frac{d^n}{dt^n} \Pi(t), H_w] = i\frac{d^{n+1}}{dt^{n+1}} \Pi(t)$$

so $H_w$ is the time–translation generator for $w(t)$ and $\Pi(t)$.

The expectation value of the field energy for the state $|\psi, T\rangle_w$ is defined to be

$$\langle H_w(T) \equiv \frac{w\langle \psi, T|H_w|\psi, T\rangle_w}{w\langle \psi, T|\psi, T\rangle_w}$$  \hspace{1cm} (8)

which is real and which vanishes at $T = 0$ ($|\psi, 0\rangle$ does not depend upon $w$, so $\Pi(t)|\psi, 0\rangle = 0$). It should be emphasized that, in spite of the notation, (8) is not a Hilbert space expectation value of $H_w$: the statevector (3) is a vector in $A$’s Hilbert space but a functional of $w(t)$. For example, in (8) we cannot replace $\Pi(t)$ by $\Pi(t)$ as we shall be able to do in Eq. (11) et. seq. below. (An analogy is that $\psi^*(x)(i\psi(x)/dx) \neq (i\psi(x)/dx)^*\psi(x)$ but the integrals of both sides are equal.)

We also introduce the interaction energy $V \equiv 2\lambda\Pi(0)A$. The Schrödinger picture’s constant energy operator $H \equiv H_0 + V$ ($H_0 \equiv HA + H_w$) becomes $H(T) \equiv H_0 + 2\lambda\Pi(T)A(T)$ ($V(T) = \exp(iH_0T)V\exp(-iH_0T)$) in the interaction picture. The expectation value of the total energy for the state $|\psi, T\rangle_w$ is

$$\langle H(w, T) \equiv \frac{w\langle \psi, T|H(T)|\psi, T\rangle_w}{w\langle \psi, T|\psi, T\rangle_w}.$$  \hspace{1cm} (9)

We now prove the constancy of the expectation value of the ensemble energy

$$\Pi(T) = \int Dw \langle \psi, T|\psi, T\rangle_w H(w, T) = \int Dw \langle \psi, T|H(T)|\psi, T\rangle_w.$$  \hspace{1cm} (10)

Eq. (10) is now a scalar product in a Hilbert space in which $H_w$ and $\Pi$ act, and so we may set $\Pi(t) = \Pi(t) = \frac{1}{2}\Pi(t)$. Since $[\exp -i \int dt B(t)\Pi(t)]f[\int dt w(t)] = \int Dw \langle \psi, T|H(T)|\psi, T\rangle_w.$
\[
f[\int dt(w(t) - B(t))] \text{because } \Pi(t) \text{ is the translation operator for } w(t), \text{we may write the statevector (3) as}
\]
\[
|\psi, T⟩_w = T e^{-i\lambda \int_0^T dtA(t)\Pi(t)} |\psi, 0⟩ e^{-\frac{i}{2} \int_0^T dt w^2(t)}
\]
\[
\equiv e^{iH_A T} e^{-iHT} |\psi, 0⟩ e^{-\frac{i}{4} \int_0^T dt w^2(t)}. \quad (11)
\]

The last step invokes the well known representation of the interaction picture time evolution operator. Since \( H(T) = \exp(iH_0 T) \exp(-iH_0 T) \), putting (11) into (10) results in
\[
\bar{H}(T) = \int Dw e^{-\frac{i}{12} \int_0^T dt w^2(t)} \langle \psi, 0 | e^{iHT} H e^{-iHT} |\psi, 0⟩ e^{-\frac{i}{4} \int_0^T dt w^2(t)}
\]
\[
= \int Dw e^{-\frac{i}{12} \int_0^T dt w^2(t)} \langle \psi, 0 | [H_A + H_w + 2\lambda A \Pi(0)] |\psi, 0⟩ e^{-\frac{i}{4} \int_0^T dt w^2(t)}
\]
\[
= \langle \psi, 0 | H_A |\psi, 0⟩ = \bar{H}(0) \quad (12)
\]
where the integrals \( \int Dw \{1, \dot{w}(t)w(t), w(0)\} e^{-(1/2\lambda) \int_0^T dt w^2(t)} = \{1, 0, 0\} \) were used in the last step.
Thus the energy expectation value is independent of time.

5 Expectation Values of Field and System Energies

Here we give expressions for the separate pieces that make up \( \bar{H}(T) \).

The system energy expectation may immediately be found from (11):
\[
\bar{H}_A(T) = \text{Tr} H_A \rho(T) = \text{Tr} H_A T e^{-\frac{i}{4} \int_0^T dt [A(t) \otimes 1 - 1 \otimes A(t)]^2} |\psi, 0⟩ \langle \psi, 0|. \quad (13)
\]
Next we show that \( \bar{V}(T) = 0 \). Since
\[
V(T) |\psi, T⟩_w = 2\lambda A(T) \frac{1}{i} \delta \frac{1}{\delta w(T)} T e^{-\frac{i}{4} \int_0^T dt [w(t) - 2\lambda A(t)]^2} |\psi, 0⟩
\]
\[
= iA(T) |w(T) - 2\lambda A(T)| |\psi, T⟩_w
\]
it is straightforward to perform the integral
\[
\bar{V}(T) = \int Dw |\psi, T⟩ iA(T) [w(T) - 2\lambda A(T)] |\psi, T⟩_w = 0
\]
(because \( T \) is the largest time, \( A(T) \) is at the outside of the time ordering of each statevector, so that the average value of \( w(T) \) is \( 2\lambda A(T) \)).
Last we calculate $\bar{H}_w(T)$:

\[
\bar{H}_w(T) = \int Dw w\langle \psi, T|\frac{1}{2} \int_0^\infty dt [\dot{w}(t) \bar{H}(t) + \bar{H}(t) \dot{w}(t)]|\psi, T\rangle_w
\]

\[
= -\frac{1}{4\lambda t} \int Dw \int_0^T dt \dot{w}(t) \left\{ w(\psi, T)|T[w(t) - 2\lambda A(t)]e^{-\frac{1}{2} \int_0^T dt [w(t) - 2\lambda A(t)]^2} \right\}
\]

\[
= \frac{1}{2i} \int Dw \int_0^T dt \dot{w}(t) \left\{ w(\psi, T)|T[A(t)e^{-\frac{1}{2} \int_0^T dt [w(t) - 2\lambda A(t)]^2} \right\}
\]

(\(T_R\) is the time-reversal operator). To perform the functional integral over \(w\) in (14), i.e., to find the mean value of \(\dot{w}(t)\), we first find the mean value of \(w(t)\)

\[
\int Dw w(\psi, T|\psi, T)_w = \text{Tr} \lambda [A(t) \otimes 1 + 1 \otimes A(t)] e^{-\frac{1}{2} \int_0^T [A(t) \otimes 1 - 1 \otimes A(t)]^2} |\psi, 0\rangle\langle \psi, 0|
\]

and so (14) becomes

\[
\bar{H}_w(T) = \text{Tr} \lambda \frac{T}{2i} \int_0^T dt [\hat{A}(t) \otimes 1 + 1 \otimes \hat{A}(t)][A(t) \otimes 1 - 1 \otimes A(t)] \times e^{-\frac{1}{2} \int_0^T [A(t') \otimes 1 - 1 \otimes A(t')]^2} |\psi, 0\rangle\langle \psi, 0|.
\]

The time ordering of \(A(t)\) and \(\hat{A}(t)\) must be done carefully. We note that the integrals in our expressions are of Stratonovich form, e.g.,

\[
\int_{t=0}^T \dot{w}(t)w(t) = \sum_{t=0}^T (\Delta t)^{-1} [w(t + \Delta t) - w(t)][w(t + \Delta t) + w(t)]
\]

\[
= \sum_{t=0}^T [w^2(t + \Delta t) - w^2(t)] = w^2(T + \Delta t) - w^2(0)
\]

rather than, say, Ito integrals (e.g., replacing \([w(t + \Delta t) - w(t)][w(t + \Delta t) + w(t)]\) in the above sum by \([w(t + \Delta t) - w(t)][2w(t)]\) which will not give a result depending only on the endpoint values of \(w\)). Therefore
We next note, since the $A$–terms in (16) have upper time $t + \Delta t$, that the time-ordering and trace operations in (13) yield

$$\mathcal{T}[\dot{A}(t) \otimes 1 + 1 \otimes \dot{A}(t)][A(t) \otimes 1 - 1 \otimes A(t)]$$

$$\equiv (\Delta t)^{-1}[(A(t + \Delta t) - A(t)) \otimes 1 + 1 \otimes (A(t + \Delta t) - A(t))] \times$$

$$\frac{1}{2}[(A(t + \Delta t) + A(t)) \otimes 1 - 1 \otimes (A(t + \Delta t) + A(t))]
$$

$$= (2\Delta t)^{-1}\left\{(A^2(t + \Delta t) - A^2(t)) \otimes 1 - 1 \otimes (A^2(t + \Delta t) - A^2(t))
$$

$$+ 2[A(t) \otimes A(t + \Delta t) - A(t + \Delta t) \otimes A(t)]\right\}.
$$

(16)

Putting (17) into (15) gives the desired result:

$$\text{Tr} e^{-\frac{i}{\hbar} \int_{t}^{T} dt' [A(t') \otimes 1 - 1 \otimes A(t')]} = 1$$

so $t$ may be set as the upper limit of the exponential's integral in (13). Then the $A$–terms in (16) have the largest times in (the thus-modified) (13), so that the time-ordering and trace operations allow one to apply $B \otimes C = CB$ to (16). The $A^2$ terms in (16) cancel, leaving

$$\mathcal{T}[\dot{A}(t) \otimes 1 + 1 \otimes \dot{A}(t)][A(t) \otimes 1 - 1 \otimes A(t)]
$$

$$= (\Delta t)^{-1}[A(t + \Delta t), A(t)] = [\dot{A}(t), A(t)] = i[A(t), [A(t), H_A]].
$$

(17)

Putting (13) into (16) gives the desired result:

$$\mathcal{H}_w(T) = \text{Tr} \frac{\lambda}{2} \int_{0}^{T} dt [A(t), [A(t), H_A]] e^{-\frac{i}{\hbar} \int_{0}^{T} dt' [A(t') \otimes 1 - 1 \otimes A(t')]} |\psi, 0\rangle \langle \psi, 0|.
$$

(18)

6 Concluding Remarks

It can immediately be seen from (13) and (18) that

$$\frac{d}{dT} \tilde{H}_A(T) = -\frac{d}{dT} \tilde{H}_w(T)
$$

$$= -\frac{\lambda}{2} \text{Tr} \mathcal{T}[A(T), [A(T), H_A]] e^{-\frac{i}{\hbar} \int_{T}^{0} dt' [A(t') \otimes 1 - 1 \otimes A(t')]} |\psi, 0\rangle \langle \psi, 0|.
$$

(19)

For a simple example, consider a single free particle moving in one dimension where collapse is toward a position eigenstate $|2\rangle$, i.e., $A = x$ and $A(t) = x + (p/m)t$. Since $[x(t), p^2/2m] = -\hbar^2/2m$, it follows from (19) that

$$\frac{d}{dT} \tilde{H}_A(T) = -\frac{d}{dT} \tilde{H}_w(T) = \frac{\lambda \hbar^2}{2m}.$$
so, from the initial values of (13), (18),

\[ \bar{H}_A(T) = \langle \psi, 0 | H_A | \psi, 0 \rangle + \frac{\lambda \hbar^2}{2m} T \]

\[ \bar{H}_w(T) = -\frac{\lambda \hbar^2}{2m} T. \]

Thus, since collapse narrows wavefunctions (toward eigenstates of position), the ensemble average particle energy steadily increases at the expense of a steadily decreasing ensemble average field energy.

It is straightforward to apply the results given here to nonrelativistic CSL, by replacing \((t)\) by \((x, t)\), e.g., the expressions now contain \(w(x, t), \Pi(x, t), A(x, t)\) (the particle number density operator smeared by a Gaussian of width \(\approx 10^{-5}\) cm), \(dx dt\), etc. One may also readily obtain conservation of the expectation values of the other geometric conservation laws such as momentum

\[ \mathcal{P}_w = \frac{1}{2} \int_{-\infty}^{\infty} dx dt [\nabla w(x, t) \Pi(x, t) + \Pi(x, t) \nabla w(x, t)]. \]

To conclude, we reiterate that this calculation gives a bit of support to the suggestion that conservation law violation obtained in applying SQT’s collapse postulate could be overcome in a dynamical collapse theory.
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