DUAL INSTABILITY MEASURES OF A SUBSPACE OF $P^n(K)$ UNDER A SUBGROUP OF $\text{Aut}(K)$

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Abstract. Let $K$ be a commutative field and let $V$ be a subspace of $P^n(K)$. Let $\Gamma$ be a subgroup of $\text{Aut}(K)$ and let $\Gamma$ act on $P^n(K)$ by $\sigma((x_i)_{0 \leq i \leq n}) = (\sigma(x_i))_{0 \leq i \leq n}$ for $\sigma \in \Gamma$ and $(x_i)_{0 \leq i \leq n} \in P^n(K)$. In this paper, we ask ‘how much’ unstable $V$ is under $\Gamma$ by asking how much higher (or lower) dimension the join (or the meet) of $\sigma(V)$ ($\sigma \in \Gamma$) has than $V$, and answer it in terms of the Plücker coordinates of $V$ and the invariant field $k$ of $\Gamma$, through presenting dual ‘irrationality’ measures of $V$ over $k$.

1. Introduction

Let $K$ be a commutative field and let $V$ be a subspace of $P^n(K)$ (= the standard projective space of dimension $n$ over $K$). Let $\Gamma$ be a subgroup of $\text{Aut}(K)$ (= the automorphism group of $K$) and let $\Gamma$ act on $P^n(K)$ by

$$\sigma((x_i)_{0 \leq i \leq n}) = (\sigma(x_i))_{0 \leq i \leq n}$$

for all $\sigma \in \Gamma$ and $(x_i)_{0 \leq i \leq n} \in P^n(K)$. Then $\sigma(V)$ ($\sigma \in \Gamma$) are subspaces of $P^n(K)$ of the same dimension as $V$, and, since $V$ is stable under $\Gamma$ if and only if the join (or the meet) of $\sigma(V)$ ($\sigma \in \Gamma$) coincides with $V$, and hence if and only if the join (or the meet) of $\sigma(V)$ ($\sigma \in \Gamma$) has the same dimension as $V$, it is natural to ask ‘how much’ unstable $V$ is under $\Gamma$ by asking the following (i) or (ii):

(i) How much higher dimension does the join of $\sigma(V)$ ($\sigma \in \Gamma$) have than $V$?

(ii) How much lower dimension does the meet of $\sigma(V)$ ($\sigma \in \Gamma$) have than $V$?

In this paper, we answer these questions in terms of the Plücker coordinates of $V$ and the invariant field of $\Gamma$, which is hereafter denoted by $k$. For each $m$-dimensional subspace $X$ of $P^n(K)$, let $(\ldots, X_{j_0 \cdots j_m}, \ldots)$ denote the Plücker coordinates of $X$ and define the $k$-irrationality degree $\text{Irr}_k$ and the $k$-irrationality codegree $\text{Irr}_k^*$ of $X$ by taking a permutation $j_0 \cdots j_n$ of $0 \cdots n$ such that $X_{j_0 \cdots j_n} \neq 0$ and setting

$$\text{Irr}_k X = \dim_k [X]_{j_0 \cdots j_n} - \dim_k \left( [X]_{j_0 \cdots j_n} \cap k^{m+1} \right),$$

$$\text{Irr}_k^* X = \dim_k [X]^*_{j_0 \cdots j_n} - \dim_k \left( [X]^*_{j_0 \cdots j_n} \cap k^{n-m} \right),$$

where $[X]_{j_0 \cdots j_n}$ denotes the linear span over $k$ of the subset

$$\left\{ \left( \frac{X_{j_0 \cdots j_{s-1}j_{s+1} \cdots j_{m}}}{X_{j_0 \cdots j_m}} \right)_{0 \leq s \leq m} : m + 1 \leq t \leq n \right\}$$

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of $K^{m+1}$, and $\lbrack X \rbrack_{j_0 \cdots j_n}$ denotes the linear span over $k$ of the subset
\[
\left\{ \left\{ \frac{X_{j_{0} \cdots j_{t} \cdots j_{t+1} \cdots j_{m}}}{X_{j_{0} \cdots j_{m}}} \right\} : 0 \leq t \leq m \right\}
\]
of $K^{n-m}$. Then our purpose is to show that $\text{Irr}_k$ and $\text{Irr}_k^*$ are well-defined (that is, for every subspace $X$ of $P^n(K)$, $\text{Irr}_k X$ and $\text{Irr}_k^* X$ are independent of the choice of $j_0 \cdots j_n$) and that $\text{Irr}_k V$ and $\text{Irr}_k^* V$ are the answers to (i) and (ii), respectively, that is,
\begin{align}
\dim \bigvee_{\sigma \in \Gamma} \sigma(V) & = \dim V + \text{Irr}_k V, \\
\dim \bigwedge_{\sigma \in \Gamma} \sigma(V) & = \dim V - \text{Irr}_k^* V,
\end{align}
where $\bigvee$ denotes the join operation and $\bigwedge$ denotes the meet operation.

In the following sections, we actually prove that $\text{Irr}_k$ is well-defined and (5) holds and that $\text{Irr}_k^*$ is well-defined and (6) holds, revealing the ‘duality’ of these two.

2. Duality of (5) and (6)

For each subspace $X$ of $P^n(K)$, let $X^\perp$ denote
\[
\left\{ (x_i)_{0 \leq i \leq n} \in P^n(K) : \forall (a_i)_{0 \leq i \leq n} \in X, \sum_{i=0}^{n} a_i x_i = 0 \right\},
\]
which is a subspace of $P^n(K)$ of dimension $n - 1 - \dim X$ that is hereafter called the dual of $X$—though informally. Then, since the join of subspaces of $P^n(K)$ is the dual of the meet of their duals, we have
\[
\bigvee_{\sigma \in \Gamma} \sigma(V) = \left( \bigwedge_{\sigma \in \Gamma} \sigma(V)^\perp \right)^\perp,
\]
which is equivalent to
\begin{align}
\bigvee_{\sigma \in \Gamma} \sigma(V) & = \left( \bigwedge_{\sigma \in \Gamma} \sigma(V^\perp) \right)^\perp,
\end{align}
because every $\sigma \in \Gamma$ satisfies
\[
\sigma(V)^\perp = \left\{ (x_i)_{0 \leq i \leq n} \in P^n(K) : \forall (a_i)_{0 \leq i \leq n} \in \sigma(V), \sum_{i=0}^{n} a_i x_i = 0 \right\}
= \left\{ (x_i)_{0 \leq i \leq n} \in P^n(K) : \forall (a_i)_{0 \leq i \leq n} \in V, \sum_{i=0}^{n} \sigma(a_i) x_i = 0 \right\}
= \left\{ (x_i)_{0 \leq i \leq n} \in P^n(K) : \forall (a_i)_{0 \leq i \leq n} \in V, \sigma \left( \sum_{i=0}^{n} a_i \sigma^{-1}(x_i) \right) = 0 \right\}
= \left\{ (x_i)_{0 \leq i \leq n} \in P^n(K) : \forall (a_i)_{0 \leq i \leq n} \in V, \sum_{i=0}^{n} a_i \sigma^{-1}(x_i) = 0 \right\}
= \sigma(V^\perp).
\]
Also we have

\[
\end{align}
Proposition 1. For every $m$-dimensional subspace $X$ of $P^n(K)$, for every permutation $j_0 \cdots j_m$ of $0 \cdots n$ such that $X_{j_0 \cdots j_m} \neq 0$, the right-hand side of (2) is equal to the expression obtained by replacing $X$ by $X^\perp$, $j_0 \cdots j_m$, $j_{m+1} \cdots j_n$, $j_0 \cdots j_m$ and $n-m$ by $n - \dim X^\perp = n - (n-1-m) = m+1$ in the right-hand side of (3), that is, to

$$\dim_k \left( [X^\perp]_{j_m+1 \cdots j_{n-m}} \right)^* - \dim_k \left( \left( [X^\perp]_{j_m+1 \cdots j_{n-m}} \cap \k m+1 \right)^* \right),$$

where $[X^\perp]_{j_m+1 \cdots j_{n-m}}$ denotes the linear span over $k$ of the subset

$$\left\{ \frac{X_{j_m+1 \cdots j_{t-1} j_t j_{t+1} \cdots j_n}}{X_{j_m+1 \cdots j_n}} : 0 \leq s \leq m \right\}$$

of $K^{m+1}$.

Proof. By the well-known relation between the Plücker coordinates and the dual Plücker coordinates of a space [3, Chapter VII, § 3, Theorem I], letting $\varepsilon$ and $\delta$ be the Levi-Civita symbol and the generalized Kronecker delta symbol, respectively, we have

$$\frac{X_{j_0 \cdots j_s j_{s+1} \cdots j_m}}{X_{j_0 \cdots j_m}} = \varepsilon_{j_0 \cdots j_s j_{s+1} \cdots j_{s+t} j_{s+t+1} \cdots j_n} \frac{X^\perp_{j_m+1 \cdots j_{t-1} j_t j_{t+1} \cdots j_n}}{X_{j_m+1 \cdots j_n}}$$

for every $s$ with $0 \leq s \leq m$ and every $t$ with $m+1 \leq t \leq n$. Therefore (4) is equal to the subset

$$\left\{ - \frac{X^\perp_{j_m+1 \cdots j_{t-1} j_t j_{t+1} \cdots j_n}}{X_{j_m+1 \cdots j_n}} : 0 \leq s \leq m \right\}$$

of $K^{m+1}$, that is, to the set of additive inverses of the elements of (5), which implies

$$[X]_{j_0 \cdots j_m} = [X^\perp]_{j_m+1 \cdots j_n, j_0 \cdots j_m}$$

and hence the desired equality. \hfill \Box

Proposition 1 is easily seen to imply that if one of Irr$_k$ and Irr$_k^*$ is well-defined, then the other is also well-defined and

$$\text{Irr}_k V = \text{Irr}_k^* V^\perp$$

holds; (7) and (8) imply that (5) is equivalent to

$$\dim \left( \bigcap_{\sigma \in \Gamma} \sigma \left( V^\perp \right) \right) = \dim V + \text{Irr}_k^* V^\perp$$

and hence to

$$\dim \bigcap_{\sigma \in \Gamma} \sigma \left( V^\perp \right) = \dim V^\perp - \text{Irr}_k^* V^\perp.$$
that is, to the equality obtained by replacing $V$ by $V^\perp$ in (6). Therefore, to prove that $\text{Irr}_k$ is well-defined and (5) holds and that $\text{Irr}_k^*$ is well-defined and (6) holds, it is enough to prove one of these two, say, the latter, which we prove in the next section.

3. Proof of (10)

Hereafter a subspace of $P^n(K)$ or $K^{n+1}$ is said to be $k$-rational if it is spanned by a subset of $P^n(k)$ or $k^{n+1}$, respectively, where (and hereafter) $P^n(k)$ denotes the image of $k^{n+1} - \{0\}$ by the canonical surjection $K^{n+1} - \{0\} \to P^n(K)$. Now let $\Gamma$ act on $K^{n+1}$ by (1) for all $\sigma \in \Gamma$ and $(x_i)_{0 \leq i \leq n} \in K^{n+1}$. Then, as is seen—though more or less indirectly—from [1, Chapter II, §8, no. 7, Theorem 1 (i)] or its specialization [2, Chapter V, §10, no. 4, Proposition 6 a]), it holds that

a subspace of $K^{n+1}$ is $k$-rational if and only if it is stable under the action of $\Gamma$ on $K^{n+1}$, which is easily shown to imply that a subspace of $P^n(K)$ is $k$-rational if and only if it is stable under the action of $\Gamma$ on $P^n(K)$, which implies that

\[ \bigcap_{\sigma \in \Gamma} \sigma(V) \text{ is } k\text{-rational} \]

because every $\tau \in \Gamma$ satisfies $\tau \Gamma = \Gamma$ and hence

\[ \tau \left( \bigcap_{\sigma \in \Gamma} \sigma(V) \right) = \bigcap_{\sigma \in \Gamma} \tau \sigma(V) = \bigcap_{\sigma \in \tau \Gamma} \sigma(V) = \bigcap_{\sigma \in \Gamma} \sigma(V). \]

For each subspace $X$ of $P^n(K)$, let $\tilde{X}$ denote the span of $X \cap P^n(k)$ in $P^n(K)$, which is the largest $k$-rational subspace of $P^n(K)$ contained in $X$. Then, since a subspace of $P^n(K)$ is $k$-rational if and only if it is the largest $k$-rational subspace of $P^n(K)$ contained in itself, (10) is equivalent to

\[ \bigcap_{\sigma \in \Gamma} \sigma(V) = \tilde{\bigcap_{\sigma \in \Gamma} \sigma(V)}, \]

which is equivalent to

\[ \bigcap_{\sigma \in \Gamma} \sigma(V) = \tilde{V} \]

because

\[ \left( \bigcap_{\sigma \in \Gamma} \sigma(V) \right) \cap P^n(k) = \bigcap_{\sigma \in \Gamma} (\sigma(V) \cap P^n(k)) = \bigcap_{\sigma \in \Gamma} (\sigma(V) \cap \sigma(P^n(k))) = \bigcap_{\sigma \in \Gamma} \sigma(V \cap P^n(k)) = \bigcap_{\sigma \in \Gamma} (V \cap P^n(k)) = V \cap P^n(k). \]

Also we have
Proposition 2. For every $m$-dimensional subspace $X$ of $P^n(K)$, for every permutation $j_0 \cdots j_n$ of $0 \cdots n$ such that $X_{j_0 \cdots j_n} \neq 0$, the right-hand side of (3) is equal to $m - \dim \tilde{X}$.

Proof. Let $\mu$ be the $k$-linear map $k^{n+1} \rightarrow K^{n-m}$ defined by

$$
\mu((x_i)_{0 \leq i \leq n}) = \left( x_j - \sum_{t=0}^{m} x_{j_t} \frac{X_{j_0 \cdots j_{t-1} j_{t+1} \cdots j_m}}{X_{j_0 \cdots j_m}} \right)_{m+1 \leq s \leq n}.
$$

Then we have

$$
\text{Im} \, \mu = k^{n-m} + [X]_{j_0 \cdots j_n}^* \quad \text{and hence}
$$

$$
\dim_k \ker \mu = n + 1 - \dim_k \text{Im} \, \mu = n + 1 - \dim_k \left( k^{n-m} + [X]_{j_0 \cdots j_n}^* \right) = n + 1 - \left( n - m + \dim_k [X]_{j_0 \cdots j_n}^* - \dim_k \left( k^{n-m} \cap [X]_{j_0 \cdots j_n}^* \right) \right) = m + 1 - \dim_k \left( [X]_{j_0 \cdots j_n}^* \cap k^{n-m} \right).
$$

Let $X'$ and $\tilde{X}'$ denote the subspaces of $K^{n+1}$ such that $X' - \{0\}$ and $\tilde{X}' - \{0\}$ are mapped by the canonical surjection $K^{n+1} - \{0\} \rightarrow P^n(K)$ onto $X$ and $\tilde{X}$, respectively. Then we can easily show that $\tilde{X}'$ is the span of $X' \cap k^{n+1}$ in $K^{n+1}$ and hence

$$
\dim_k \left( X' \cap k^{n+1} \right) = \dim \tilde{X}' = \dim \tilde{X} + 1 = m + 1 - \left( m - \dim \tilde{X} \right).
$$

Therefore, for the above $\mu$ and $X'$, the desired

$$
\dim_k [X]_{j_0 \cdots j_n}^* = \dim_k \left( [X]_{j_0 \cdots j_n}^* \cap k^{n-m} \right) = m - \dim \tilde{X}
$$

is equivalent to

$$
\dim_k \ker \mu = \dim_k \left( X' \cap k^{n+1} \right)
$$

and hence is implied by

$$
\ker \mu = X' \cap k^{n+1},
$$

which certainly holds since

$$
x_j - \sum_{t=0}^{m} x_{j_t} \frac{X_{j_0 \cdots j_{t-1} j_{t+1} \cdots j_m}}{X_{j_0 \cdots j_m}} = 0 \quad (s = m + 1, \ldots, n)
$$

is, as is virtually proved in [3] Chapter VII, §2, the third and fourth paragraphs], a necessary and sufficient condition for $(x_i)_{0 \leq i \leq n} \in P^n(K)$ to be in $X$, and hence for $(x_i)_{0 \leq i \leq n} \in K^{n+1}$ to be in $X'$.

Proposition 2 is immediately seen to imply that $\text{Irr}_k^*$ is well-defined and

$$
(12) \quad \dim \tilde{V} = \dim V - \text{Irr}_k^* V
$$

holds; (11) and (12) imply (9). Therefore we have the desired result.
Remark 1. For each subspace $X$ of $P^n(K)$, let $\overline{X}$ denote $\overline{X}^\perp$, which is the dual of the largest $k$-rational subspace of $P^n(K)$ contained in $X^\perp$, that is, is the smallest $k$-rational subspace of $P^n(K)$ containing $X$. Then we have, dually to (11) and (12),

$$(13) \bigvee_{\sigma \in \Gamma} \sigma(V) = V \text{ and } \dim V = \dim V + \Irr_k V,$$

which, by (7) and (9), are respectively equivalent to

$$\left(\bigcap_{\sigma \in \Gamma} \sigma(V)\right)^\perp = \overline{V}^\perp \text{ and } \dim \overline{V}^\perp = \dim V + \Irr_k^* V,$$

and hence to

$$\left(\bigcap_{\sigma \in \Gamma} \sigma(V)\right)^\perp = \overline{V}^\perp \text{ and } \dim \overline{V}^\perp = \dim V^\perp - \Irr_k^* V,$$

that is, to the equalities obtained by replacing $V$ by $V^\perp$ in (11) and (12).

Remark 2. Since $V$ is $k$-rational if and only if $\overline{V}$ (or $\overline{V}^\perp$) coincides with $V$, and hence if and only if $\overline{V}$ (or $\overline{V}^\perp$) has the same dimension as $V$, (12) and the latter of (13) make it natural to regard $\Irr_k^* V$ and $\Irr_k V$ as ‘$k$-irrationality’ measures of $V$.

Remark 3. We can easily see that every $\sigma \in \Gamma$ satisfies $\overline{\sigma(V)} = \overline{\sigma(V)} = \overline{V}$ as well as $\dim \sigma(V) = \dim V$, which, by (12) and the latter of (13), implies that $\Irr_k^* V$ and $\Irr_k V$ are invariants of $V$ under $\Gamma$.

Remark 4. The part ‘every $\tau \in \Gamma$ satisfies $\ldots$’ of the third sentence of this section, which is valid only under our assumption that $\Gamma$ is a subgroup of $\text{Aut}(K)$, can be replaced by ‘every $\tau \in \Gamma$ satisfies $\tau \Gamma \subseteq \Gamma$ and hence

$$\tau \left(\bigcap_{\sigma \in \Gamma} \sigma(V)\right) = \bigcap_{\sigma \in \Gamma} \tau \sigma(V) = \bigcap_{\sigma \in \tau \Gamma} \sigma(V) \supseteq \bigcap_{\sigma \in \Gamma} \sigma(V),$$

which implies

$$\dim \tau \left(\bigcap_{\sigma \in \Gamma} \sigma(V)\right) = \dim \bigcap_{\sigma \in \Gamma} \sigma(V) \implies \tau \left(\bigcap_{\sigma \in \Gamma} \sigma(V)\right) = \bigcap_{\sigma \in \Gamma} \sigma(V),$$

and hence

$$\tau \left(\bigcap_{\sigma \in \Gamma} \sigma(V)\right) = \bigcap_{\sigma \in \Gamma} \sigma(V),$$

which, as well as all of Section 2 and this section except this part, is valid under the weaker assumption that $\Gamma$ is a subsemigroup of $\text{Aut}(K)$. Therefore our results hold under this weaker assumption, which is equivalent to the original assumption when $\Gamma$ is finite.

References

[1] N. Bourbaki, *Algebra I, Chapters 1–3*, translated from the French, reprint of the 1989 English translation, Elements of Mathematics (Berlin) (Springer, Berlin, 1998).

[2] N. Bourbaki, *Algebra II, Chapters 4–7*, translated from the 1981 French edition by P. M. Cohn and J. Howie, reprint of the 1990 English edition, Elements of Mathematics (Berlin) (Springer, Berlin, 2003).

[3] W. V. D. Hodge and D. Pedoe, *Methods of Algebraic Geometry, Vol. I*, reprint of the 1947 original, Cambridge Mathematical Library (Cambridge University Press, Cambridge, 1994).
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