MONOTONICITY OF SOLUTIONS FOR A CLASS OF NONLOCAL MONGE-AMPERE PROBLEM

YAHUI NIU

School of Mathematics and Statistics and Hubei Key Laboratory of Mathematical Sciences, Central China Normal University, Wuhan, 430079, China
Department of Mathematical Sciences, Yeshiva University, New York, NY, USA

(Communicated by Wenxiong Chen)

Abstract. In this paper, we consider nonlinear problems involving nonlocal Monge-Ampère operators. By using a sliding method, we establish monotonicity of positive solutions for nonlocal Monge-Ampère problems both in an infinite slab and in an upper half space. During this process, an important idea we applied is to estimate the singular integrals defining the nonlocal Monge-Ampère operator along a sequence of approximate maximum points. It allows us to assume weaker conditions on nonlinear terms. Another idea is to employ a generalized average inequality which plays an important role and greatly simplify the process of the sliding.

1. Introduction. In this paper, we consider nonlinear equations involving the nonlocal Monge-Ampère operator

$$\mathcal{D}_s^\theta u(x) = f(u(x)), \quad x \in \Omega$$

with

$$\mathcal{D}_s^\theta u(x) = \inf_{A \in \mathcal{A}} \left\{ \text{PV} \int_{\mathbb{R}^n} \frac{u(y) - u(x)}{|A^{-1}(y-x)|^{n+2s}} dy \right\},$$

where $0 < s < 1$, $\Omega \subset \mathbb{R}^n$ is an unbounded domain, PV stands for the Cauchy principal value, $\mathcal{A} = \{ A \mid A \text{ is } n \times n \text{ symmetric positive definite matrix, } \det A = 1, \lambda_{\min}(A) \geq \theta > 0 \}$, in which $\lambda_{\min}(A)$ is the smallest eigenvalue of matrix $A$. We prove the monotonicity of solutions for equation (1.1) in two different domains.

The classical Monge-Ampère equation

$$\det D^2 u(x) = f(u(x)), \quad \text{in } \Omega$$

describes the values of the determinant of the Hessian of a convex function $u$. It has many well known applications to differential geometry and mass transportation. It also plays a central role in the regularity theory for elliptic equations in non-divergence form, in part because it can be expressed as the extreme operator

$$\left( \det D^2 u(x) \right)^{\frac{1}{n}} = \inf \{ a_{ij} \partial_{ij} u(x), \det \{ a_{ij} \} = 1, \{ a_{ij} \} > 0 \}.$$
From this, we can deduce that
\[
\left( \det D^2 u(x) \right)^{\frac{1}{n}} = \inf \{ \Delta [u \circ A](x), \det A = 1, A > 0 \},
\]
where \( u \circ A \) means the composition of the linear transformation \( x \to Ax \) with the function \( u \). In [6], Caffarelli and Charro first extended this definition to the fractional order case for any \( s \in (0, 1) \) by
\[
F[u](x) = \inf \{ -(-\Delta)^s [u \circ A](x), \det A = 1, A > 0 \}
\]
and set up a relatively simple framework of global solutions with prescribed data at infinity and under global barriers. In a key estimate, they showed that the operator remains strictly elliptic, which allows one to apply known regularity results for uniformly elliptic operators and deduce that solutions are classical.

Instead of taking the infimum over the linear functionals with determinant one, Caffarelli and Silvestre in [8] introduced another definition of nonlocal Monge-Ampère operator by taking the infimum over all measure preserving transformations. Specifically, for every point \( x \) in the domain of a convex function \( u \), we write
\[
u_x(y) = u(x) + y - u(x) - y \cdot \nabla u(x).
\]
Then, the Monge-Ampère operator can be expressed alternatively as
\[
\left( \det D^2 u(x) \right)^{\frac{1}{n}} = \inf \{ \Delta [\nu_x \circ \phi](0) : \text{for all measure preserving } \phi \text{ s.t. } \phi(0) = 0 \}.
\]
For any \( s \in (0, 1) \), they defined the fractional order Monge-Ampère-like operator using this theory and write
\[
MA_s u(x) = \inf \{ -(-\Delta)^s [\nu_x \circ \phi](0) : \text{for all measure preserving } \phi \text{ s.t. } \phi(0) = 0 \}. \tag{1.4}
\]
They proved that a global problem involving this operator has \( C^{1,1} \) solutions in the whole space. In the second order case, there is no difference between the approaches used in [6, 8], but for nonlocal equations, the operators defined by (1.3) and (1.4) will be significantly different. See [8] for more details.

Inspired by the above results, we are interested in the monotonicity of solutions for nonlocal Monge-Ampère equations. In order the infimum in (1.3) can be realized, in this article, we investigate two problems involving operator (1.2), which is a special case of the definition (1.3).

First, we consider
\[
\begin{align*}
D^\alpha u(x) &= f(u(x)), & x \in \Omega, \\
u(x) &= \varphi(x), & x \in \mathbb{R}^n \setminus \Omega,
\end{align*} \tag{1.5}
\]
where \( \Omega = \{ x = (x_1, \cdots, x_n) \in \mathbb{R}^n \mid 0 < x_n < M \} \) is an unbounded slab. We assume that \( u \) is monotone in the complement of \( \Omega \):

(H) : For any points \( x = (x', x_n), y = (x', y_n), z = (x', z_n) \in \mathbb{R}^n \), with \( y, z \in \Omega^c \), \( y_n < x_n < z_n \), we have
\[
\varphi(y) < u(x) < \varphi(z), \quad \text{if } x \in \Omega,
\]
and
\[
\varphi(y) \leq \varphi(x) \leq \varphi(z), \quad \text{if } x \in \Omega^c.
\]
We also assume that
\[
u(x', M) - \nu(x', 0) \geq \epsilon_0 > 0, \quad \forall x' \in \mathbb{R}^{n-1}. \tag{1.6}
\]
Our first result is as follows:
Theorem 1.1. Assume $u \in L_2(\mathbb{R}^n) \cap C^{1,1,1}_1(\Omega) \cap C(\Omega)$ is a bounded solution of (1.5), satisfying (H) and (1.6). $f$ is Lipschitz continuous.

Then $u$ is strictly increasing with respect to $x_n$ in $\Omega$ and depends on $x_n$ only.

Next, we turn to the monotonicity of solutions for (1.1) in the upper half space. Specifically, we consider

$$
\begin{cases}
D^\theta_s u(x) = f(u(x)), & x \in \mathbb{R}^n_+,
0 < u(x) < h, & x \in \mathbb{R}^n_+,
u(x) = 0, & x \in \mathbb{R}^n \backslash \mathbb{R}^n_+,
\end{cases}
$$

and obtain

Theorem 1.2. Suppose $u \in L_2(\mathbb{R}^n) \cap C^{1,1,1}_1(\mathbb{R}^n_+)$ is an uniformly continuous solution of (1.7), $u \rightharpoonup h$ uniformly as $x_n \to +\infty$.

$f$ is continuous on $[0, h]$ and nondecreasing in $[h - \delta, h]$ for some $\delta > 0$.

Then $u$ is strictly increasing with respect to $x_n$ in $\mathbb{R}^n_+$ and depends on $x_n$ only.

From the definition (1.2), one can see that the operator $D^\theta_s$ is closely related to the well-known fractional Laplacian:

$$
-(-\Delta)^s u(x) = C_{n,s} \text{PV} \int_{\mathbb{R}^n} \frac{u(y) - u(x)}{|y - x|^{n+2s}} dy,
$$

where $C_{n,s}$ is a normalization constant. Notice that $-C_{n,s}^{-1}(-\Delta)^s u(x)$ belongs to the class of operators over which the infimum in the definition of $D^\theta_s u(x)$ is taken, from which we can infer that

$$
D^\theta_s u(x) \leq -C_{n,s}^{-1}(-\Delta)^s u(x).
$$

The fractional Laplacian has attracted much attention recently due to its various applications. The non-locality of the fractional Laplacian makes it difficult to be investigated. To circumvent this, Caffarelli and Silvestre in [7] introduced the extension method that reduced this nonlocal problem into a local one in higher dimensions. One can also reduce a fractional equation into an integral equation as in [12], then applying the method of moving planes in integral forms or regularity lifting to obtain the monotonicity, symmetry and regularity of the solutions.

However, the application of this two methods needs extra conditions to be imposed on the solutions. Moreover, they do not work for nonlinear nonlocal operators, such as the fractional $p$-Laplacian(see [13] for details). Thanks to the works of Chen and Li et al. in [9, 10, 11], direct methods of moving planes are introduced for the fractional Laplacian and fractional $p$-Laplacian without going through extensions or integral equations, which have been applied to obtain symmetry, monotonicity, and non-existence of solutions for various semi-linear equations involving these nonlocal operators.

Except for the moving plane method, another approach to study the monotonicity of solutions for partial differential equations is a sliding method. It was developed by Berestycki and Nirenberg in [3, 4, 5] to investigate the symmetry, monotonicity and uniqueness of solutions involving the regular Laplacian (see [1, 2, 14] for more applications). The essential ingredients are different forms of maximum principles. The main idea lies in comparing values of the solution to the equation at two different points, between which one point is obtained from the other by sliding the domain in a given direction, and then the domain is slide back to a critical position. While in the method of moving planes, one point is the reflection of the other.
Recently, Wu and Chen in [17] introduced a direct sliding method to establish monotonicity of solutions for the fractional Laplacian. It enlightens us to prove our results by a similar idea.

The following **maximum principle** plays an important role in the process of sliding in unbounded domains.

**Theorem 1.3** (Maximum principle in unbounded domains). Let $D$ be an open set in $\mathbb{R}^n$, possibly unbounded and disconnected, satisfying

$$\liminf_{k \to \infty} \frac{|(B_{2k+1} \setminus B_{2k})(q) \cap \bar{D}^c|}{|B_{2k+1}(q) \setminus B_{2k}(q)|} \geq c_0, \quad \forall q \in \mathbb{R}^n$$

(1.8)

for some $c_0 > 0$, $r > 0$. Let $v_1, v_2 \in \mathcal{L}_2(\mathbb{R}^n) \cap C^{1,1}_{loc}(D)$, $v = v_1 - v_2$ is bounded from above and lower semi-continuous in $\bar{D}$, satisfying

$$\begin{align*}
\begin{cases}
\mathcal{D}_sv_1(x) - \mathcal{D}_sv_2(x) + c(x)v(x) \geq 0, & \text{at the points in } D \text{ where } v \geq 0, \\
v(x) \leq 0, & x \in \mathbb{R}^n \setminus D,
\end{cases}
\end{align*}$$

(1.9)

where $c(x) \leq 0$. Then $v(x) \leq 0$ in $D$.

In order to prove a maximum principle in an unbounded domain, one used to assume that the solutions vanish near infinity, then the maximum value can be achieved and one can derive contradictions at the maximum points. However, in Theorem 1.3, without imposing any asymptotic conditions on the solution $v$, the maximum value of $v$ may not be achieved, and a maximizing sequence may tend to infinity.

Berestycki, Caffarelli and Nirenberg in [2] proved a similar maximum principle involving the regular Laplacian, Díaz, Soave and Valdinoci in [14] extended this result to the fractional Laplacian. In both papers, they assumed the exterior cone condition that the complement of $D$ contains an infinite open cone $\Sigma$.

Here, we used a method introduced in [15, 16], which enable us to assume a weaker condition (1.8) than the exterior cone condition. We estimate the singular integral defining $\mathcal{D}_sv_1(x) - \mathcal{D}_sv_2(x)$ along a sequence of approximate maximum points to derive a contradiction if $\sup_D v(x) > 0$. This method can also be used to deal with other difficulties caused by the unbounded slab and upper half space when we conduct the sliding process.

Another important tool we use is a generalize average inequality as follows:

**Lemma 1.4.** Suppose $v_1, v_2 \in \mathcal{L}_2(\mathbb{R}^n) \cap C^{1,1}_{loc}(\Omega)$, and $x_0$ is a maximum point of $v_1 - v_2$ in $\Omega$. Then for any $r > 0$, we have

$$-\frac{\bar{C}}{C_\theta} r^{2s} (\mathcal{D}_sv_1(x_0) - \mathcal{D}_sv_2(x_0)) + \bar{C} \int_{B_\frac{r}{2}(x_0)} \frac{r^{2s}}{|x_0 - y|^{n+2s}} (v_1 - v_2)(y) dy$$

$$\geq (v_1 - v_2)(x_0),$$

(1.10)

where $\bar{C}$ satisfies

$$\bar{C} \int_{B_{\frac{r}{2}}(x_0)} \frac{r^{2s}}{|x_0 - y|^{n+2s}} dy = 1$$

(1.11)

and $C_\theta$ is a constant, which depends on the smallest eigenvalue $\theta$ of $A \in \mathcal{A}$.

**Remark 1.** This inequality can become an effective tool in analyzing nonlocal Monge-Ampère equations, for instance, it can be conveniently applied to prove maximum principle, here we briefly describe as follows:
Proposition 1. Assume $\Omega \in \mathbb{R}^n$ is bounded. Suppose $v_1, v_2 \in L_2(\mathbb{R}^n) \cap C^{1,1}_{loc}(\Omega)$ satisfies
\[
\begin{cases}
D^\theta_+ v_1(x) - D^\theta_+ v_2(x) \geq 0, & \text{in } \Omega, \\
v_1(x) - v_2(x) \leq 0, & x \in \mathbb{R}^n \setminus \Omega.
\end{cases}
\] (1.12)
Then
\[v_1(x) - v_2(x) \leq 0 \text{ in } \Omega.\] (1.13)

If (1.13) does not hold, there exists $x_0 \in \Omega$ such that
\[(v_1 - v_2)(x_0) = \max_{\mathbb{R}^n} (v_1 - v_2)(x) > 0.
\]
From (1.10) and (1.12), we have
\[\bar{C} \int_{B^c_r(x_0)} \frac{r^{2s}}{|x_0 - y|^{n+2s}} (v_1 - v_2)(y) dy \geq (v_1 - v_2)(x_0).
\]
Note that (1.11) holds, so
\[(v_1 - v_2)(x_0) \geq \bar{C} \int_{B^c_r(x_0)} \frac{r^{2s}}{|x_0 - y|^{n+2s}} (v_1 - v_2)(y) dy \geq (v_1 - v_2)(x_0).
\]
As a result,
\[(v_1 - v_2)(y) \equiv (v_1 - v_2)(x_0) > 0 \text{ in } B^c_r(x_0), \forall r > 0,
\]
which contradicts (1.12). Thus we obtain (1.13).

For the case of $\Omega$ is unbounded, one can refer to the proof of Theorem 1.3.

In section 2, we prove the monotonicity in a slab. In section 3, we prove the maximum principle and the monotonicity on an upper half space. In the Appendix, we prove Lemma 1.4.

2. The monotonicity in a slab. In this section, we prove Theorem 1.1.

We first introduce some notation. For any $\tau \geq 0$, define
\[u^\tau(x) = u(x + \tau e_n)
\]
on the set $\Omega^\tau = \Omega - \tau e_n$ which is obtained by sliding $\Omega$ downward a distance $\tau$ parallel to the $x_n$-direction, where $e_n = (0, \cdots, 0, 1)$.

Set
\[D^\tau = \Omega^\tau \cap \Omega \neq \emptyset, \text{ for any } 0 \leq \tau < M.
\]
Denote
\[\omega^\tau(x) = u(x) - u^\tau(x), \quad x \in D^\tau.
\]
As $u^\tau$ satisfies equation (1.5) in $\Omega^\tau$, so $\omega^\tau(x)$ satisfies
\[D^\theta_+ u(x) - D^\theta_+ u^\tau(x) = c^\tau(x) \omega^\tau(x), \quad x \in D^\tau,
\]
where
\[c^\tau(x) = \frac{f(u(x)) - f(u^\tau(x))}{u(x) - u^\tau(x)}
\]
is bounded.

By the outer monotone condition (H) and the definition of $\omega^\tau$, we have
\[\omega^\tau(x) \leq 0, \quad x \in (D^\tau)^c.\] (2.1)
We will prove the theorem by proving that
\[\omega^\tau(x) \leq 0, \quad x \in D^\tau, \forall 0 < \tau < M.\] (2.2)
We divided the proof into two steps:

Step 1: we prove for \( \delta > 0 \) small, there holds
\[
\omega^\tau(x) \leq 0, \quad x \in D^\tau, \quad \forall M - \delta < \tau < M. \tag{2.3}
\]
Under this condition, \( D^\tau \) is a narrow region, Step 1 is equivalent to a narrow region principle, which provides a starting position to the sliding. Then we decrease \( \tau \) and slide \( \Omega^\tau \) upward as long as \( \omega^\tau(x) \leq 0 \) in \( D^\tau \) holds until its limited position.

Step 2: Define
\[
\tau_0 = \inf\{\tau \mid \omega^\tau(x) \leq 0, \quad x \in D^\tau, \quad 0 < \tau < M\}.
\]
We will prove \( \tau_0 = 0 \) to deduce (2.2).

Proof. Step 1: Suppose (2.3) is not true, then there exists \( \{x_k\} \subset D^\tau \), such that
\[
\omega^\tau(x_k) \to \sup_{D^\tau} \omega^\tau(x) > 0, \tag{2.4}
\]
then, there exists \( \epsilon_k > 0 \) small, \( \bar{x}_k \in B_{r_k}(x_k) \) such that
\[
\omega^\tau(\bar{x}_k) + \epsilon_k \psi_k(\bar{x}_k) = \max_{\mathbb{R}^N} \{\omega^\tau(x) + \epsilon_k \psi_k(x)\} > 0,
\]
where \( r_k = \frac{1}{2} \text{dist}(x_k, \partial D^\tau) \), \( \psi_k(x) = \varphi\left(\frac{x-x_k}{r_k}\right) \),
\[
\varphi(x) = \begin{cases} \epsilon e^{-|x|^2}, & |x| < 1, \\ 0, & |x| \geq 1. \end{cases}
\]
Since
\[
\omega^\tau(\bar{x}_k) + \epsilon_k \psi_k(\bar{x}_k) \geq \omega^\tau(x_k) + \epsilon_k \psi_k(x_k)
\]
and \( \psi_k(\bar{x}_k) \leq \psi_k(x_k) \), we obtain
\[
\omega^\tau(\bar{x}_k) \geq \omega^\tau(x_k).
\]
then it follows from (2.4) that
\[
\lim_{k \to \infty} \omega^\tau(\bar{x}_k) > 0. \tag{2.5}
\]
On one hand, by a fact that \( \inf a + \inf b \leq \inf(a + b) \), we have
\[
\begin{aligned}
&\mathcal{D}^\theta_s[u + \epsilon_k \psi_k](\bar{x}_k) - \mathcal{D}^\theta_s u^\tau(\bar{x}_k) \\
= &\inf_{A \in \mathcal{A}} \left\{ PV \int_{\mathbb{R}^n} \frac{[u + \epsilon_k \psi_k](y) - [u + \epsilon_k \psi_k](\bar{x}_k)}{|A^{-1}(y - \bar{x}_k)|^{n+2s}} \, dy \right\} - \mathcal{D}^\theta_s u^\tau(\bar{x}_k) \\
\geq &\inf_{A \in \mathcal{A}} \left\{ PV \int_{\mathbb{R}^n} \frac{u(y) - u(\bar{x}_k)}{|A^{-1}(y - \bar{x}_k)|^{n+2s}} \, dy \right\} - \mathcal{D}^\theta_s u^\tau(\bar{x}_k) \\
&\quad + \epsilon_k \inf_{A \in \mathcal{A}} \left\{ PV \int_{\mathbb{R}^n} \frac{\psi_k(y) - \psi_k(\bar{x}_k)}{|A^{-1}(y - \bar{x}_k)|^{n+2s}} \, dy \right\} \\
= &\mathcal{D}^\theta_s(u(\bar{x}_k)) - \mathcal{D}^\theta_s u^\tau(\bar{x}_k) + \epsilon_k \mathcal{D}^\theta_s \psi_k(\bar{x}_k) \\
= &f(u(\bar{x}_k)) - f(u^\tau(\bar{x}_k)) + C\varepsilon_k = c^\tau(\bar{x}_k) \omega^\tau(\bar{x}_k) + C\varepsilon_k. \tag{2.6}
\end{aligned}
\]
On the other hand, by the definition of infimum, for any sequence \( \varepsilon_k \to 0 \), there exists \( A_k \in \mathcal{A} \) such that
\[
\mathcal{D}^\theta_s u^\tau(\bar{x}_k) = \inf_{A \in \mathcal{A}} \left\{ PV \int_{\mathbb{R}^n} \frac{u^\tau(y) - u^\tau(\bar{x}_k)}{|A^{-1}(y - \bar{x}_k)|^{n+2s}} \, dy \right\}
\]
From (2.6) and (2.8), we obtain

\[ \lambda \text{ therefore, combined it with } \lambda_{\min}A \geq \theta \text{ and (2.1), we have} \]

\[
\begin{align*}
\mathcal{D}^\theta_u[u + \epsilon_k \psi_k](\bar{x}_k) - \mathcal{D}^\theta_u(\bar{x}_k) \\
= \inf_{A \in A} & \left\{ PV \int_{\mathbb{R}^n} \frac{[u + \epsilon_k \psi_k](y) - [u + \epsilon_k \psi_k](\bar{x}_k)}{|A^{-1}(y - \bar{x}_k)|^{n+2s}}dy \right\} \\
- \inf_{A \in A} & \left\{ PV \int_{\mathbb{R}^n} \frac{u^\tau(y) - u^\tau(\bar{x}_k)}{|A^{-1}(y - \bar{x}_k)|^{n+2s}}dy \right\} \\
\leq PV \int_{\mathbb{R}^n} \frac{[u + \epsilon_k \psi_k](y) - [u + \epsilon_k \psi_k](\bar{x}_k)}{|A^{-1}(y - \bar{x}_k)|^{n+2s}}dy - PV \int_{\mathbb{R}^n} \frac{u^\tau(y) - u^\tau(\bar{x}_k)}{|A^{-1}(y - \bar{x}_k)|^{n+2s}}dy + \epsilon_k \\
\leq C_\theta \int_{(D^\tau)^c} \frac{\omega^\tau(y) - \omega^\tau(\bar{x}_k)}{|y - \bar{x}_k|^{n+2s}}dy + \epsilon_k \\
\leq - C_\theta [\omega^\tau + \epsilon_k \psi_k](\bar{x}_k) \int_{(D^\tau)^c} \frac{1}{|y - \bar{x}_k|^{n+2s}}dy + \epsilon_k \\
\leq - C_\theta [\omega^\tau + \epsilon_k \psi_k](\bar{x}_k) \frac{1}{r_k^{2s}} + \epsilon_k \leq - C_\theta [\omega^\tau + \epsilon_k \psi_k](\bar{x}_k) \frac{1}{\delta^{2s}} + \epsilon_k.
\end{align*}
\]  

From (2.6) and (2.8), we obtain

\[ \epsilon_k - \tilde{C}_\theta [\omega^\tau + \epsilon_k \psi_k](\bar{x}_k) \frac{1}{\delta^{2s}} \geq c^\tau(\bar{x}_k)\omega^\tau(\bar{x}_k) - C \epsilon_k, \]

then

\[ \epsilon_k + C \epsilon_k - \tilde{C}_\theta \psi_k(\bar{x}_k) \frac{1}{\delta^{2s}} \epsilon_k \geq \left( c^\tau(\bar{x}_k) + \tilde{C}_\theta \frac{1}{\delta^{2s}} \right) \omega^\tau(\bar{x}_k). \]

(2.9)

Taking \( \delta \) small such that \( c^\tau(\bar{x}_k) + \tilde{C}_\theta \frac{1}{\delta^{2s}} > 0 \), letting \( k \to \infty \), then (2.9) contradicts (2.5) since \( \epsilon_k \to 0 \). Thus we finished the proof step 1.

Step 2. Suppose \( \tau_0 > 0 \). We will prove there exists a small \( \epsilon > 0 \) such that

\[ \omega^\tau(x) \leq 0, \quad x \in D^\tau, \forall \tau \in (\tau_0 - \epsilon, \tau_0) \]

(2.10)

to derive a contradiction.

First, by the continuity of \( \omega^\tau \) with respect to \( \tau \), we have

\[ \omega^{\tau_0}(x) \leq 0 \quad \text{in } D^{\tau_0} \]

(2.11)

and by \( (H) \), we have

\[ \omega^{\tau_0}(x) < 0 \quad \text{on } \partial D^{\tau_0}. \]

Therefore,

\[ \omega^{\tau_0}(x) < 0 \quad \text{in } D^{\tau_0}. \]

(2.12)

Furthermore, there exists a \( C_0 < 0 \) such that

\[ \omega^{\tau_0}(x) \leq C_0 < 0 \quad \text{in } D^{\tau_0}. \]

(2.13)
where \( \psi_k(x) = \varphi(x - x_k) \),
\[
\varphi(x) = \begin{cases} 
\frac{cx}{|x|^{d+1}}, & |x| < 1, \\
0, & |x| \geq 1.
\end{cases}
\]

In addition, by
\[
[\omega^{T_0} + \epsilon_k \psi_k](\bar{x}_k) \geq [\omega^{T_0} + \epsilon_k \psi_k](x_k)
\]
and \( \psi_k(\bar{x}_k) \leq \psi_k(x_k) \), we have
\[
0 > \omega^{T_0}(\bar{x}_k) \geq \omega^{T_0}(x_k) \to 0.
\]
Therefore
\[
\omega^{T_0}(\bar{x}_k) \to 0, \quad \text{as } k \to \infty.
\]
As a result, similar to (2.6), by the continuity of \( f \), we have
\[
\mathcal{D}_x^\theta [u + \epsilon_k \psi_k](\bar{x}_k) - \mathcal{D}_x^\theta u^{T_0}(\bar{x}_k)
\]
\[
= \inf_{A \in A} \left\{ \text{PV} \int_{\mathbb{R}^n} \frac{[u + \epsilon_k \psi_k](y) - [u + \epsilon_k \psi_k](\bar{x}_k)}{|A^{-1}(y - \bar{x}_k)|^{n+2s}} dy \right\} - \mathcal{D}_x^\theta u^{T_0}(\bar{x}_k)
\]
\[
\geq \mathcal{D}_x^\theta (u(\bar{x}_k)) - \mathcal{D}_x^\theta u^{T_0}(\bar{x}_k) + \epsilon_k \mathcal{D}_x^\theta \psi_k(\bar{x}_k)
\]
\[
= f(u(\bar{x}_k)) - f(u^{T_0}(\bar{x}_k)) + C \epsilon_k \to 0, \quad \text{as } k \to \infty.
\]
(2.14)

Then, applying Lemma 1.4 to function \( \omega^{T_0} + \epsilon_k \psi_k \) at the maximum point \( \bar{x}_k \), take \( r = 1 \), we obtain from (1.10) that
\[
- \frac{C}{C_0} \int_{\mathbb{R}^n} \omega^{T_0}(\bar{x}_k) \left[ \mathcal{D}_x^\theta [u + \epsilon_k \psi_k](\bar{x}_k) - \mathcal{D}_x^\theta u^{T_0}(\bar{x}_k) \right] dy
\]
\[
\geq [\omega^{T_0} + \epsilon_k \psi_k](\bar{x}_k).
\]
Combining it with (2.1), (2.11), (2.13) and (2.14), we have
\[
\int_{B_1^c(0)} \frac{1}{|z|^{n+2s}} \omega^{T_0}(x + \bar{x}_k) dz = \int_{B_1^c(\bar{x}_k)} \frac{1}{|x - y|^{n+2s}} \omega^{T_0}(y) dy \to 0, \quad \text{as } k \to \infty.
\]
As a result,
\[
\omega^{T_0}(x + \bar{x}_k) \to 0 \quad \text{in } B_1^c(0), \quad \text{as } k \to \infty.
\]
(2.15)

Denote
\[
u_k(x) = u(x + \bar{x}_k), \quad \nu_k^{T_0}(x) = u^{T_0}(x + \bar{x}_k),
\]
\[
\omega_k^{T_0}(x) = \omega^{T_0}(x + \bar{x}_k) - u_k^{T_0}(x) - \nu_k(x).
\]
Since \( u \) is uniformly continuous, by Arzelà-Ascoli theorem, there exists \( u_\infty \) such that
\[
u_k \to u_\infty \quad \text{as } k \to \infty
\]
Then by (2.15), correspondingly, we have
\[
u_k^{T_0}(x) - u_\infty(x) = 0 \quad \text{in } B_1^c(0).
\]
(2.16)

Therefore,
\[
\nu_\infty(x', x_n) = u_\infty(x', x_n + T_0) = \cdots = u_\infty(x', x_n + sT_0) \in B_1^c(0).
\]

(2.16)

There must exist \( s_0 \in \mathbb{N} \) such that \((s_0 - 1)T_0 < M \leq s_0T_0\), so, by the monotone condition \((H)\) and the uniformly monotonic condition \((1.6)\), we have
\[
u_\infty(x', s_0T_0) \geq u_\infty(x', M) > u_\infty(x', 0).
\]
Taking \( x_n = 0 \), \( s = s_0 \) in (2.16), we get a contradiction. Thus (2.12) is correct.
By the continuity of $\omega^\tau$ with respect to $\tau$, we can obtain (2.10), therefor $\tau_0 = 0$. we finished the proof of step 2.

It follows that

$$\omega^\tau(x) \leq 0, \ x \in D^\tau, \ \forall \ 0 < \tau < M.$$  

In fact

$$\omega^\tau(x) < 0, \ x \in D^\tau, \ \forall \ 0 < \tau < M.$$  

(2.17)

For each $0 < \tau < M$, if there exists a $x_0 \in D^\tau$ such that $\omega^\tau(x_0) = 0$, $x_0$ is a maximum point of $\omega^\tau$ in $\mathbb{R}^n$, then by the definition of infimum, for any $\varepsilon_k \to 0$, there exists $A_k \in A$,

$$D_\nu^\theta u(x_0) - D_\nu^\theta u^\tau(x_0) = \inf_{A \in A} \left\{ PV \int_{\mathbb{R}^n} \frac{u(y) - u(x_0)}{|A^{-1}(y-x_0)|^{n+2s}} dy \right\} - \inf_{A \in A} \left\{ PV \int_{\mathbb{R}^n} \frac{u^\tau(y) - u^\tau(x_0)}{|A^{-1}(y-x_0)|^{n+2s}} dy \right\} \leq PV \int_{\mathbb{R}^n} \frac{\omega^\tau(y) - \omega^\tau(x_0)}{|A^{-1}_k(y-x_0)|^{n+2s}} dy + \varepsilon_k.$$  

For any $A \in A$, we deduce from $\det A = 1$, $\lambda_{\min}(A) \geq \theta > 0$ that

$$\lambda_{\max}(A) \leq \theta^{1-n},$$  

therefore

$$D_\nu^\theta u(x_0) - D_\nu^\theta u^\tau(x_0) \leq C_{\theta,n} PV \int_{\mathbb{R}^n} \frac{\omega^\tau(y) - \omega^\tau(x_0)}{|y-x_0|^{n+2s}} dy + \varepsilon_k$$  

letting $\varepsilon_k \to 0$, we have

$$D_\nu^\theta u(x_0) - D_\nu^\theta u^\tau(x_0) < 0,$$  

while,

$$D_\nu^\theta u(x_0) - D_\nu^\theta u^\tau(x_0) = f(u(x_0)) - f(u^\tau(x_0)) = 0,$$  

this is impossible. Thus, we obtain the strictly incremental properties of $u$ in $\Omega$ about $x_0$.

Furthermore, we say that $u$ is independent of $x'$. In fact, for arbitrary vector $\nu = (\nu_1, \nu_2, \ldots, \nu_n)$ with $\nu_n > 0$, if we replace $u^\tau$ in the above process by $u(x + \tau \nu)$, we can derive that

$$u(x + \tau \nu) \geq u(x), \ \forall \tau > 0, \ x \in \mathbb{R}^n.$$  

If we replace $\nu$ by $\tilde{\nu} = (-\nu_1, -\nu_2, \ldots, -\nu_{n-1}, \nu_n)$, we have

$$u(x + \tau \tilde{\nu}) \geq u(x), \ \forall \tau > 0, \ x \in \mathbb{R}^n.$$  

Therefore,

$$u(x + \tau \nu) \geq u(x) \geq u(x - \tau \tilde{\nu}), \ \forall \tau > 0, \ x \in \mathbb{R}^n.$$  

(2.18)

Letting $\nu_n \to 0$ in (2.18), we have

$$u(x + \tau \nu) = u(x), \ \forall \tau > 0, \ x \in \mathbb{R}^n,$$  

which holds for any $\nu$ with $\nu_n = 0$, which implies that

$$u(x) = u(x_n) \text{ in } \Omega.$$  

(2.19)

We complete the proof of Theorem 1.1. 

□
3. Maximum principle and the monotonicity on an upper half space. In this section, we first prove Theorem 1.3 and then use it to prove Theorem 1.2.

**Proof of Theorem 1.3.** If \( v(x) \leq 0 \) in \( D \) does not hold, then there exists \( x_k \in D \) such that
\[
v(x_k) \to \sup_{x \in \mathbb{R}^n} v(x) > 0, \text{ as } k \to \infty.
\]
For \( r_k > 0 \), there exists \( \varepsilon_k > 0 \) small, \( \bar{x}_k \in B_{r_k}(x_k) \) such that
\[
[v + \varepsilon_k \Phi_k](\bar{x}_k) = \max_{\mathbb{R}^n} [v + \varepsilon_k \Phi_k](x) =: B > 0,
\]
where \( \Phi_k(x) = \phi(\frac{x - x_k}{r_k}) \), \( \phi(x) = \begin{cases} ce^{-\frac{|x|}{r_k}}, & |x| < 1, \\ 0, & |x| \geq 1, \end{cases} \phi(x) \in C_0^\infty(B_1(0)). \)

Noticed that
\[
[v + \varepsilon_k \Phi_k](\bar{x}_k) \geq [v + \varepsilon_k \Phi_k](x_k),
\]
and \( \Phi_k(\bar{x}_k) \leq \Phi_k(x_k) \), so we have
\[
v(\bar{x}_k) \geq [v + \varepsilon_k \Phi_k](x_k) - \Phi_k(\bar{x}_k) \geq v(x_k) \geq 0. \tag{3.1}
\]

Then we consider \( v + \varepsilon_k \Phi_k \) on its maximum points \( \bar{x}_k \).

On one hand, by \( \inf(a + b) \geq \inf(a) + \inf(b) \), (1.9) and (3.1), we have
\[
D_\varepsilon^\theta[v_1 + \varepsilon_k \Phi_k](\bar{x}_k) - D_\varepsilon^\theta v_2(\bar{x}_k)
= \inf_{A \in A} \left\{ PV \int_{\mathbb{R}^n} \frac{v_1 + \varepsilon_k \Phi_k(y) - v_1 + \varepsilon_k \Phi_k(\bar{x}_k)}{|A^{-1}(y - \bar{x}_k)|^{n+2s}} dy \right\} - D_\varepsilon^\theta v_2(\bar{x}_k)
\geq \inf_{A \in A} \left\{ PV \int_{\mathbb{R}^n} \frac{v_1(y) - v_1(\bar{x}_k)}{|A^{-1}(y - \bar{x}_k)|^{n+2s}} dy \right\} - D_\varepsilon^\theta v_2(\bar{x}_k)
+ \varepsilon_k \inf_{A \in A} \left\{ PV \int_{\mathbb{R}^n} \frac{\Phi_k(y) - \Phi_k(\bar{x}_k)}{|A^{-1}(y - \bar{x}_k)|^{n+2s}} dy \right\}
= D_\varepsilon^\theta v_1(\bar{x}_k) - D_\varepsilon^\theta v_2(\bar{x}_k) + C \varepsilon_k \geq -c(\bar{x}_k)v(\bar{x}_k) + C \varepsilon_k \geq 0, \text{ as } k \to \infty. \tag{3.2}
\]

On the other hand, by (2.7) and
\[
|y - \bar{x}_k| \leq |y - x_k| + |x_k - \bar{x}_k| \leq 2|y - x_k|,
\]
we have
\[
D_\varepsilon^\theta[v_1 + \varepsilon_k \Phi_k](\bar{x}_k) - D_\varepsilon^\theta v_2(\bar{x}_k)
\leq PV \int_{\mathbb{R}^n} \frac{v + \varepsilon_k \Phi_k(y) - [v + \varepsilon_k \Phi_k](\bar{x}_k)}{|A^{-1}(y - \bar{x}_k)|^{n+2s}} dy + C \varepsilon_k
\leq C_\theta \int_{D^c \setminus B_{r_k}(x_k)} \frac{[v + \varepsilon_k \Phi_k](y) - [v + \varepsilon_k \Phi_k](\bar{x}_k)}{|y - \bar{x}_k|^{n+2s}} dy + C \varepsilon_k
\leq - C_\theta [v + \varepsilon_k \Phi_k](\bar{x}_k) \int_{D^c \setminus B_{r_k}(x_k)} \frac{1}{|y - \bar{x}_k|^{n+2s}} dy + C \varepsilon_k
\leq - C_\theta B \int_{D^c \setminus B_{r_k}(x_k)} \frac{1}{|y - x_k|^{n+2s}} dy, \text{ as } i \to \infty. \tag{3.3}
\]

By (1.8), there exists \( j_0 \in \mathbb{N}^+ \) large enough such that
\[
\int_{D^c \setminus B_{r_k}(x_k)} \frac{1}{|y - x_k|^{n+2s}} dy
\]
Denote this contradicts (3.2). Thus we complete the proof of Theorem 1.3. For step 2: Decreasing \( \tau \)
and \( \tau \)
step 1: For \( \tau \)
define \( x \)
we will prove \( \tau \)
there exists a sequence \( \{ \)
Proof of Theorem 1.2
And thus we have \( u \)
Combining with (3.3), we obtain

\[
\sum_{j=0}^{\infty} \int_{(B_{2j+1} \setminus B_{2j} \cap D^c)} \frac{1}{|y-x|^{n+2s}} dy \geq \sum_{j=0}^{\infty} \int_{(B_{2j+1} \setminus B_{2j} \cap D^c)} \frac{1}{|y-x|^{n+2s}} dy \geq \sum_{j=0}^{\infty} \left| \frac{(B_{2j+1} \setminus B_{2j} \cap D^c)}{2^{j+1} r_k^{n+2s}} \right| \geq c_0 \sum_{j=0}^{\infty} \frac{|(B_{2j+1} \setminus B_{2j} \cap D^c)|}{2^{j+1} r_k^{n+2s}} = \frac{c_1}{2^{j_0} r_k^{2s}}.
\]

Combining with (3.3), we obtain

\[
D_0^\theta [v_1 + \varepsilon_k \Phi_k] - D_0^\theta v_2(\bar{x}_k) \leq \frac{C_0 B C_1}{2^{j_0} r_k^{2s}},
\]
this contradicts (3.2). Thus we complete the proof of Theorem 1.3.

Next, before the proof of Theorem 1.2, we first give some notations and outline. Denote

\[
u^\tau(x) = u(x + \tau e_n), \text{ where } e_n = (0, \cdots, 0, 1)
\]
and

\[
w^\tau(x) = u(x) - u^\tau(x)
\]
satisfies

\[
w^\tau(x) \leq 0 \quad \text{in } \mathbb{R}^n \setminus \mathbb{R}_+^n.
\]

(3.4)

Outline of the proof: We will use sliding method to prove the monotonicity of \( u \)
and divide the proof into two steps.

step 1: For \( \tau \) large, we prove \( w^\tau(x) \leq 0 \) in \( \mathbb{R}^n \). Specifically, since \( u \equiv h \) uniformly as \( x_n \rightarrow +\infty \), for \( \delta > 0 \), there exists a \( M_0 > 0 \) such that for \( x_n \geq M_0 \), \( u(x', x_n) \in [h - \delta, h] \) in which \( f \) is nondecreasing. Thus we will prove

\[
w^\tau(x) \leq 0, \quad \text{in } \mathbb{R}^n \text{ for } \tau \geq M_0.
\]

step 2: Decreasing \( \tau \) as long as \( w^\tau(x) \leq 0 \) in \( \mathbb{R}^n \) holds to its limited position. If we define

\[
\tau_0 = \inf \{ \tau \mid w^\tau(x) \leq 0, \ x \in \mathbb{R}^n, \ 0 < \tau < M_0 \},
\]
we will prove \( \tau_0 = 0 \). Then we obtain

\[
w^\tau(x) \leq 0, \quad \text{in } \mathbb{R}^n \text{ for } \tau > 0.
\]

And thus we have \( u \) is nondecreasing with respect to \( x_n > 0 \).

Proof of Theorem 1.2. step 1: For \( \tau \geq M_0 \), if \( w^\tau(x) \leq 0 \) in \( \mathbb{R}^n \) does not hold, there exists a sequence \( \{ x^k \} \subset \mathbb{R}^n_+ \) such that

\[
w^\tau(x^k) \rightarrow \sup_{\mathbb{R}^n} w^\tau(x) =: S > 0.
\]

We want to apply Theorem 1.3 to function \( w^\tau(x) - \frac{S}{2} \).
Since \( \tau \geq M_0 \), \( u^\tau(x) \in [h - \delta, h] \). Set

\[
D = \{ x \in \mathbb{R}^n \mid w^\tau(x) - \frac{S}{2} > 0 \}.
\]

For \( x \in D \), we have

\[
u(x) \geq u^\tau(x) \geq h - \delta.
\]
By the monotonicity of $f$, 

$$c(x) = -\frac{f(u(x)) - f(u^\tau(x))}{u(x) - u^\tau(x)} \leq 0.$$ 

As a result, $w^\tau(x) - \frac{S}{2}$ satisfies 

$$\begin{cases} 
\mathcal{D}_x^\theta u(x) - \mathcal{D}_x^\theta u^\tau(x) + c(x)(w^\tau(x) - \frac{S}{2}) \geq 0, & x \in D, \\
-\frac{S}{2} \leq 0, & x \in \mathbb{R}^n \setminus D, 
\end{cases}$$ 

By Theorem 1.3, we have 

$$w^\tau(x) - \frac{S}{2} \leq 0 \text{ in } \mathbb{R}^n,$$

which contradicts (3.6). Thus we finished the proof of Step 1.

Here we provide an alternative proof which is an application of the general average inequality (Lemma 1.4), and this idea can be applied to other problems. As $w^\tau(x) \leq 0$ for $x \in \partial \mathbb{R}^n_+$, note (3.6), so $x^k$ is away from $\partial \mathbb{R}^n_+$, without loss of generality, assume $\text{dist}(x^k, \partial \mathbb{R}^n_+) > 1$. Thus there exists $0 < \epsilon_k \to 0$, $\hat{x}_k \in B_1(x^k)$ such that 

$$[w^\tau + \epsilon_k \psi_k](\hat{x}_k) = \max_{\mathbb{R}^n}[w^\tau + \epsilon_k \psi_k](x) > 0,$$ 

(3.7)

where $\psi_k(x) = \varphi(x - x^k)$, $\varphi(x) = \begin{cases} 
\epsilon e^{\frac{|x|^2}{1 - \epsilon^2}}, & |x| < 1, \\
0, & |x| \geq 1.
\end{cases}$

Since 

$$[w^\tau + \epsilon_k \psi_k](\hat{x}_k) \geq [w^\tau + \epsilon_k \psi_k](x^k)$$

and $\psi_k(\hat{x}_k) \leq \psi_k(x^k)$, we have 

$$w^\tau(\hat{x}_k) \geq w^\tau(x^k).$$ 

(3.8)

So, for $\tau \geq M_0$, 

$$u(\hat{x}_k) \geq u^\tau(\hat{x}_k) \geq h - \delta.$$ 

This means $u(\hat{x}_k)$, $u^\tau(\hat{x}_k)$ are all in the nondecreasing interval of $f$.

As a result, combining with $\inf (a(x) + b(x)) \geq \inf a(x) + \inf b(x)$ we have 

$$\mathcal{D}_x^\theta[u + \epsilon_k \psi_k](\hat{x}_k) - \mathcal{D}_x^\theta u^\tau(\hat{x}_k)$$

$$= \inf_{A \in A} \left\{ PV \int_{\mathbb{R}^n} \frac{u(x) - u(\hat{x}_k)}{|A^{-1}(x - \hat{x}_k)|^{n+2s}} dy \right\} - \mathcal{D}_x^\theta u^\tau(\hat{x}_k)$$

$$\geq \inf_{A \in A} \left\{ PV \int_{\mathbb{R}^n} \frac{u(x) - u(\hat{x}_k)}{|A^{-1}(x - \hat{x}_k)|^{n+2s}} dy \right\} - \mathcal{D}_x^\theta u^\tau(\hat{x}_k)$$

$$+ \epsilon_k \inf_{A \in A} \left\{ PV \int_{\mathbb{R}^n} \frac{\psi_k(x) - \psi_k(\hat{x}_k)}{|A^{-1}(x - \hat{x}_k)|^{n+2s}} dy \right\}$$

$$= f(u(\hat{x}_k)) - f(u^\tau(\hat{x}_k)) + C \epsilon_k \geq C \epsilon_k.$$ 

(3.9)

Now, applying Lemma 1.4 to the function $w^\tau + \epsilon_k \psi_k$ at $\hat{x}_k$, we obtain 

$$- \frac{C}{C_\theta} r^{2s} \left( \mathcal{D}_x^\theta[u + \epsilon_k \psi_k](\hat{x}_k) - \mathcal{D}_x^\theta u^\tau(\hat{x}_k) \right) + \hat{C} \int_{B_2(\hat{x}_k)} \frac{1}{|x - \hat{x}_k|^{n+2s}} [w^\tau + \epsilon_k \psi_k](x) dx$$

$$\geq [w^\tau + \epsilon_k \psi_k](\hat{x}_k).$$
Combining it with (3.7), (3.9) and (1.11), we have
\[
\|\omega^n + \epsilon_k \psi_k\|_{L^1}(\bar{x}_k) - Cr^{2s} \epsilon_k \\
\geq C \int_{B_r^c(\bar{x}_k)} \frac{\|\omega^n + \epsilon_k \psi_k\|_1(y)}{|x_k - y|^{n+2s}} dy - Cr^{2s} \epsilon_k \\
= C \int_{B_r^c(\bar{x}_k)} \frac{\|\omega^n + \epsilon_k \psi_k\|_1(y)}{|y|^{n+2s}} dy - Cr^{2s} \epsilon_k \geq [\omega^n + \epsilon_k \psi_k](\bar{x}_k).
\] (3.10)

By (3.6) and (3.8),
\[
\omega^n(\bar{x}_k) \to S > 0, \quad \text{as } k \to \infty.
\]
So, if we set \(\omega^n(\bar{x}_k) := \lim_{k \to \infty} \omega^n(y + \bar{x}_k)\), letting \(\epsilon_k \to 0\) in (3.10), we obtain
\[
\omega^n(\bar{x}_k) = S > 0, \quad \forall \ y \in B_r^c(0), \quad \forall \ r > 0.
\]
This contradicts \(\omega^n(\bar{x}_k) \leq 0\) for \(y\) large since \(\omega^n(y) \leq 0\) in \(\mathbb{R}^n \setminus \mathbb{R}^n_+\). So we have finished the proof of Step 1.

step 2: We just need to prove \(\tau_0\) as defined in (3.5) equals to 0.

If \(\tau_0 > 0\), we will prove there exists a small \(\epsilon > 0\) such that
\[
\omega^n(x) \leq 0, \quad x \in \mathbb{R}^n_+, \quad \forall \ \tau \in (\tau_0 - \epsilon, \tau_0],
\] (3.11)
which contracts the definition of \(\tau_0\).

Firstly, by the continuity of \(\omega^n\) with respect to \(\tau\), we have
\[
\omega^n(\tau_0) \leq 0 \quad \text{in } \mathbb{R}^n_+.
\] (3.12)

We claim that
\[
\sup_{\mathbb{R}^n \times (0, M_0 + 1]} \omega^n(\tau) < 0.
\] (3.13)

If not,
\[
\sup_{\mathbb{R}^n \times (0, M_0 + 1]} \omega^n(\tau) = 0.
\]
So there exists a sequence \(\{x^k\} \subset \mathbb{R}^n \times (0, M_0 + 1]\) such that \(\omega^n(x^k) \to 0\), as \(k \to \infty\).

There exists \(0 < \epsilon_k \to 0\), \(\bar{x}_k \in B_1(x^k)\) such that
\[
[w^n + \epsilon_k \psi_k](\bar{x}_k) = \max_{\mathbb{R}^n}[w^n + \epsilon_k \psi_k](x) \geq 0,
\]
where \(\psi_k(x) = \varphi(x - x^k), \varphi(x) = \begin{cases} \epsilon x^{1-1}, & |x| < 1, \\ 0, & |x| \geq 1. \end{cases}\)

Since
\[
0 \geq w^n(x^k) \geq w^n(\bar{x}_k) \to 0
\]
and \(f\) is continuous, we have
\[
f(u(\bar{x}_k)) - f(u^n(\bar{x}_k)) \to 0.
\]
Applying it to (3.9), we obtain
\[
P^n_k[w^n + \epsilon_k \psi_k](\bar{x}_k) \geq f(u(\bar{x}_k)) - f(u^n(\bar{x}_k)) + C \epsilon_k \to 0.
\]
On the other hand, applying Lemma 1.4 to the function \(w^n + \epsilon_k \psi_k\) at \(\bar{x}_k\), take \(r = 1\), processed as (2.15), by (3.4) and (3.12), we have
\[
u_\infty(x) = u_\infty(x + r \epsilon_n) = \cdots = u_\infty(x + N r \epsilon_n) \text{ in } B_r^c(0),
\]
where \(N \in \mathbb{N}\).
While, if we note \( \lim_{k \to \infty} (\mathbb{R}^n_k - x^k) = (\mathbb{R}^n_\infty) \), for some \( x \in \mathbb{R}^n \setminus (\mathbb{R}^n_\infty) \) and \( N \) large enough,
\[
  u_\infty(x) = 0, \quad u_\infty(x + N \tau e_n) \text{ close to } h,
\]
this is impossible. We have proved the claim (3.13).

Secondly, we can deduce from (3.13) that
\[
  \sup_{\mathbb{R}^n \times (0, M_0 + 1]} \omega^\tau(x) < 0, \quad \forall \tau \in (\tau_0 - \epsilon, \tau_0].
\]
In order to prove (3.11), we are left to prove
\[
  \sup_{\mathbb{R}^n \times (M_0 + 1, +\infty)} \omega^\tau(x) \leq 0, \quad \forall \tau \in (\tau_0 - \epsilon, \tau_0].
\]
Processed similar to Step 1 of Theorem 1.2, except for \( 0 < \tau < M_0 \) here, however, for \( x^k \in (M_0 + 1, +\infty) \), \( \bar{x}^k \in (M_0, +\infty) \), which can ensure that \( u(\bar{x}^k), u^n(\bar{x}^k) \in [h - \delta, h] \) in which \( f \) is monotonic. We can also deduce a contradiction and finished the proof of Step 2. Thus we obtain
\[
  w^\tau(x) \leq 0, \quad \text{in } \mathbb{R}^n \text{ for } \tau > 0.
\]
Furthermore, similar to the proof of (2.17), we have
\[
  w^\tau(x) < 0, \quad \text{in } \mathbb{R}^n_+ \text{ for } \tau > 0.
\]
Thus, \( u \) is strictly increasing in \( x_n > 0 \).

By a similar analysis process to (2.19), we have
\[
  u(x) = u(x_n) \quad \text{in } \mathbb{R}^n_+.
\]
The proof of Theorem 1.2 is finished. \( \square \)

**Appendix.** Here, we prove Lemma 1.4.

**Proof.** For \( A \in \mathcal{A} \), since \( \det A = 1 \), \( \lambda_{\min}(A) \geq \theta > 0 \), we can deduce that \( \lambda_{\min}(A) \leq \theta^{1-n} \). Thus, by the definition of infimum, for any sequence \( \epsilon_j \to 0 \), there exists \( A_j \in \mathcal{A} \) such that
\[
  D^\theta v_2(x_0) = \inf_{A \in \mathcal{A}} \left\{ PV \int_{\mathbb{R}^n} \frac{v_2(y) - v_2(x_0)}{|A^{-1}(y - x_0)|^{n+2s}} dy \right\} \geq PV \int_{\mathbb{R}^n} \frac{v_2(y) - v_2(x_0)}{|A_j^{-1}(y - x_0)|^{n+2s}} dy - \epsilon_j
\]
\[
  \geq C_{\theta 2} PV \int_{\mathbb{R}^n} \frac{v_2(y) - v_2(x_0)}{|y - x_0|^{n+2s}} dy - \epsilon_j = - \frac{C_{\theta 2}}{C_{n,s}} (-\triangle)^s v_2(x_0) - \epsilon_j.
\]
On the other hand, for this sequence \( A_j \),
\[
  D^\theta v_1(x_0) = \inf_{A \in \mathcal{A}} \left\{ PV \int_{\mathbb{R}^n} \frac{v_1(y) - v_1(x_0)}{|A^{-1}(y - x_0)|^{n+2s}} dy \right\} \leq PV \int_{\mathbb{R}^n} \frac{v_1(y) - v_1(x_0)}{|A_j^{-1}(y - x_0)|^{n+2s}} dy
\]
\[
  \leq C_{\theta 1} PV \int_{\mathbb{R}^n} \frac{v_1(y) - v_1(x_0)}{|y - x_0|^{n+2s}} dy = - \frac{C_{\theta 1}}{C_{n,s}} (-\triangle)^s v_1(x_0),
\]
where \( C_{\theta 1}, C_{\theta 2} \) are constants with respect to the eigenvalue of \( A \in \mathcal{A} \).

As a result, there exists a constant \( C_{\theta} \), such that
\[
  D^\theta_s v_1(x_0) - D^\theta_s v_2(x_0) \leq - \frac{C_{\theta}}{C_{n,s}} (-\triangle)^s [v_1 - v_2](x_0) + \epsilon_j. \quad (3.14)
\]
By the result of Lemma 4.1 in [17], for any \( r > 0 \), we have
\[
\frac{\bar{C}}{C_{n,s}} s^{2s} (-\Delta)^s [v_1 - v_2] (x_0) + \bar{C} \int_{B_\varepsilon (x_0)} s^{2s} \frac{(v_1 - v_2)(y)dy}{|x_0 - y|^{n+2s}} 
\geq (v_1 - v_2)(x_0),
\]
where \(\bar{C}\) satisfies (1.11). Combining this result with (3.14), and letting \(\varepsilon_j \to 0\), we proved (1.10).

REFERENCES
[1] H. Berestycki, L. Caffarelli and L. Nirenberg, Inequalities for second-order elliptic equations with applications to unbounded domains, I, Duke Math. J., 81 (1996), 467–494.
[2] H. Berestycki, L. Caffarelli and L. Nirenberg, Monotonicity for elliptic equations in unbounded Lipschitz domains, Commun. Pure Appl. Math., 50 (1997), 1089–1111.
[3] H. Berestycki and L. Nirenberg, Monotonicity, symmetry and anti-symmetry of solutions of semilinear elliptic equations, J. Geom. Phys., 5 (1988), 237–275.
[4] H. Berestycki and L. Nirenberg, Some qualitative properties of solutions of semi-linear elliptic equations in cylindrical domains, Analysis, et cetera, Academic Press, Boston, MA, (1990), 115–164.
[5] H. Berestycki and L. Nirenberg, On the method of moving planes and the sliding method, Bol. Soc. Brasil. Mat. (N.S.), 22 (1991), 1–37.
[6] L. Caffarelli and F. Charro, On a fractional Monge-Ampère operator, Ann. PDE., 1 (2015), 47pp.
[7] L. Caffarelli and L. Silvestre, An extension problem related to the fractional Laplacian, Commun. PDE., 32 (2007), 1245–1260.
[8] L. Caffarelli and L. Silvestre, A nonlocal Monge-Ampère equation, Commun. Anal. Geom., 24 (2016), 307–335.
[9] W. Chen and C. Li, Maximum principle for the fractional p-Laplacian and symmetry of solutions, Adv. Math., 335 (2018), 735–758.
[10] W. Chen, C. Li, and Y. Li, A direct method of moving planes for the fractional Laplacian, Adv. Math., 308 (2017), 404–437.
[11] W. Chen, C. Li and P. Ma, The Fractional Laplacian, World Scientific Publishing Co, 2019.
[12] W. Chen, C. Li and B. Ou, Classification of solutions for an integral equation, Commun. Pure Appl. Math., 59 (2006), 330–343.
[13] W. Chen and S. Qi, Direct methods on fractional equations, Disc. Cont. Dyna. Sys., 39 (2019), 1269–1310.
[14] S. Dipierro, N. Soave and E. Valdinoci, On fractional elliptic equations in Lipschitz sets and epigraphs: regularity, monotonicity and rigidity results, Math. Ann., 369 (2017), 1283–1326.
[15] Z. Liu and W. Chen, Maximum principles and monotonicity of solutions for fractional p-equations in unbounded domains, arXiv:1905.06493.
[16] L. Wu and W. Chen, The sliding method for the fractional p-Laplacian, Adv. Math., 361 (2020), 106933.
[17] L. Wu and W. Chen, Monotonicity of solutions for fractional equations with De Giorgi type nonlinearities, arXiv:1905.09999.

Received April 2020; revised June 2020.
E-mail address: yahuniu@163.com