An inverse scattering problem for short-range systems in a time-periodic electric field.

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Abstract
We consider the time-dependent Hamiltonian $H(t) = \frac{1}{2} p^2 - E(t) \cdot x + V(t, x)$ on $L^2(\mathbb{R}^n)$, where the external electric field $E(t)$ and the short-range electric potential $V(t, x)$ are time-periodic with the same period. It is well-known that the short-range notion depends on the mean value $E_0$ of the external electric field. When $E_0 = 0$, we show that the high energy limit of the scattering operators determines uniquely $V(t, x)$. When $E_0 \neq 0$, the same result holds in dimension $n \geq 3$ for generic short-range potentials. In dimension $n = 2$, one has to assume a stronger decay on the electric potential.

1 Introduction.
In this note, we study an inverse scattering problem for a two-body short-range system in the presence of an external time-periodic electric field $E(t)$ and a time-periodic short-range potential $V(t, x)$ (with the same period $T$). For the sake of simplicity, we assume that the period $T = 1$.
The corresponding Hamiltonian is given on $L^2(\mathbb{R}^n)$ by:

\begin{equation}
H(t) = \frac{1}{2} p^2 - E(t) \cdot x + V(t, x),
\end{equation}
where \( p = -i \partial_x \). When \( E(t) = 0 \), the Hamiltonian \( H(t) \) describes the dynamics of the hydrogen atom placed in a linearly polarized monochromatic electric field, or a light particle in the restricted three-body problem in which two other heavy particles are set on prescribed periodic orbits. When \( E(t) = \cos(2\pi t) E \) with \( E \in IR^n \), the Hamiltonian describes the well-known AC-Stark effect in the \( E \)-direction [7].

In this paper, we assume that the external electric field \( E(t) \) satisfies:

\[
(A_1) \quad t \rightarrow E(t) \in L^1_{loc}(IR; IR^n) , \quad E(t + 1) = E(t) \text{ a.e}.
\]

Moreover, we assume that the potential \( V \in C^\infty(IR \times IR^n) \), is time-periodic with period 1, and satisfies the following estimations:

\[
(A_2) \quad \forall \alpha \in IN^n, \forall k \in IN, |\partial_t^k \partial_x^\alpha V(t, x)| \leq C_{k,\alpha} < x >^{-\delta-|\alpha|}, \text{ with } \delta > 0,
\]

where \( < x > = (1 + x^2)^{1/2} \). Actually, we can accommodate more singular potentials (see [10], [11], [12] for example) and we need \( (A_2) \) for only \( k, \alpha \) with finite order. It is well-known that under assumptions \( (A_1) - (A_2) \), \( H(t) \) is essentially self-adjoint on \( S(IR^n) \) the Schwartz space, [16]. We denote \( H(t) \) the self-adjoint realization with domain \( D(H(t)) \).

Now, let us recall some well-known results in scattering theory for time-periodic electric fields. We denote \( H_0(t) \) the free Hamiltonian:

\[
(1.2) \quad H_0(t) = \frac{1}{2} p^2 - E(t) \cdot x ,
\]

and let \( U_0(t, s) \), (resp. \( U(t, s) \)) be the unitary propagator associated with \( H_0(t) \), (resp. \( H(t) \)) (see section 2 for details).

For short-range potentials, the wave operators are defined for \( s \in IR \) and \( \Phi \in L^2(IR^n) \) by:

\[
(1.3) \quad W^\pm(s) \Phi = \lim_{t \rightarrow \pm \infty} U(s, t) U_0(t, s) \Phi.
\]

We emphasize that the short-range condition depends on the value of the mean of the external electric field:

\[
(1.4) \quad E_0 = \int_0^1 E(t) \, dt .
\]

**The case \( E_0 = 0 \).**

By virtue of the Avron-Herbst formula (see section 2), this case falls under the category of two-body systems with time-periodic potentials and this case was studied by Kitada and Yajima ([10], [11]), Yokoyama [22].

We recall that for a unitary or self-adjoint operator \( U, H_c(U), H_{ac}(U), H_{sc}(U) \) and \( H_p(U) \) are, respectively, continuous, absolutely continuous, singular continuous and point spectral subspace of \( U \).

We have the following result ([10], [11], [21]):
Theorem 1

Assume that hypotheses $(A_1), (A_2)$ are satisfied with $\delta > 1$ and with $E_0 = 0$. Then:
(i) the wave operators $W^{\pm}(s)$ exist for all $s \in \mathbb{R}$.
(ii) $W^{\pm}(s + 1) = W^{\pm}(s)$ and $U(s + 1, s) W^{\pm}(s) = W^{\pm}(s) U_0(s + 1, s)$.
(iii) $\text{Ran } (W^{\pm}(s)) = \mathcal{H}_{ac}(U(s + 1, s))$ and $\mathcal{H}_{sc}(U(s + 1, s)) = \emptyset$.
(iv) the purely point spectrum $\sigma_p(U(s + 1, s))$ is discrete outside $\{1\}$.

Comments.

1 - The unitary operators $U(s + 1, s)$ are called the Floquet operators and they are mutually equivalent. The Floquet operators play a central role in the analysis of time periodic systems. The eigenvalues of these operators are called Floquet multipliers. In [5], Galtbayar, Jensen and Yajima improve assertion (iv) : for $n = 3$ and $\delta > 2$, $\mathcal{H}_p(U(s + 1, s))$ is finite dimensional.

2 - For general $\delta > 0$, $W^{\pm}(s)$ do not exist and we have to define other wave operators. In ([10], [11]), Kitada and Yajima have constructed modified wave operators $W^{\pm}_{HJ}$ by solving an Hamilton-Jacobi equation.

- The case $E_0 \neq 0$.

This case was studied by Moller [12] : using the Avron-Herbst formula, it suffices to examine Hamiltonians with a constant external electric field, (Stark Hamiltonians) : the spectral and the scattering theory for Stark Hamiltonians are well established [2]. In particular, a Stark Hamiltonian with a potential $V$ satisfying $(A_2)$ has no eigenvalues [2]. The following theorem, due to Moller, is a time-periodic version of these results.

Theorem 2

Assume that hypotheses $(A_1), (A_2)$ are satisfied with $\delta > \frac{1}{2}$ and with $E_0 \neq 0$. Then:
(i) the Floquet operators $U(s + 1, s)$ have purely absolutely continuous spectrum.
(ii) the wave operators $W^{\pm}(s)$ exist for all $s \in \mathbb{R}$ and are unitary.
(iii) $W^{\pm}(s + 1) = W^{\pm}(s)$ and $U(s + 1, s) W^{\pm}(s) = W^{\pm}(s) U_0(s + 1, s)$.

The inverse scattering problem.

For $s \in \mathbb{R}$, we define the scattering operators $S(s) = W^{++}(s) W^-(s)$. It is clear that the scattering operators $S(s)$ are periodic with period 1.

The inverse scattering problem consists to reconstruct the perturbation $V(s, x)$ from the scattering operators $S(s), s \in [0, 1]$.

In this paper, we prove the following result :
Theorem 3
Assume that $E(t)$ satisfies $(A_1)$ and let $V_j, j = 1, 2$ be potentials satisfying $(A_2)$. We assume that $\delta > 1$ (if $E_0 = 0$), $\delta > \frac{3}{2}$ (if $E_0 \neq 0$ and $n \geq 3$), $\delta > \frac{3}{4}$ (if $E_0 \neq 0$ and $n = 2$). Let $S_j(s)$ be the corresponding scattering operators.

Then :

$$\forall s \in [0, 1], \ S_1(s) = S_2(s) \iff V_1 = V_2.$$ 

We prove Theorem 3 by studying the high energy limit of $[S(s), p]$, (Enss-Weder’s approach [4]). We need $n \geq 3$ in the case $E_0 \neq 0$ in order to use the inversion of the Radon transform [6] on the orthogonal hyperplane to $E_0$. See also [15] for a similar problem with a Stark Hamiltonian.

We can also remark that if we know the free propagator $U_0(t, s), s, t \in I\mathbb{R}$, then by virtue of the following relation :

$$(1.5) \quad S(t) = U_0(t, s) \ S(s) \ U_0(s, t),$$

the potential $V(t, x)$ is uniquely reconstructed from the scattering operator $S(s)$ at only one initial time.

In [21], Yajima proves uniqueness for the case of time-periodic potential with the condition $\delta > \frac{n}{2} + 1$ and with $E(t) = 0$ by studying the scattering matrices in a high energy regime. In [20], for a time-periodic potential that decays exponentially at infinity, Weder proves uniqueness at a fixed quasi-energy.

Note also that inverse scattering for long-range time-dependent potentials without external electric fields was studied by Weder [18] with the Enss-Weder time-dependent method, and by Ito for time-dependent electromagnetic potentials for Dirac equations [8].

2 Proof of Theorem 3.

2.1 The Avron-Herbst formula.

First, let us recall some basic definitions for time-dependent Hamiltonians. Let $\{H(t)\}_{t \in I\mathbb{R}}$ be a family of selfadjoint operators on $L^2(I\mathbb{R}^n)$ such that $S(I\mathbb{R}^n) \subset D(H(t))$ for all $t \in I\mathbb{R}$.

Definition.

We call propagator a family of unitary operators on $L^2(I\mathbb{R}^n)$, $U(t, s), t, s \in I\mathbb{R}$ such that :

1. $U(t, s)$ is a strongly continuous fonction of $(t, s) \in I\mathbb{R}^2$.
2. $U(t, s) \ U(s, r) = U(t, r)$ for all $t, s, r \in I\mathbb{R}$.
3. $U(t, s) (S(I\mathbb{R}^n)) \subset S(I\mathbb{R}^n)$ for all $t, s \in I\mathbb{R}$.
4. If $\Phi \in S(I\mathbb{R}^n)$, $U(t, s)\Phi$ is continuously differentiable in $t$ and $s$ and satisfies :

$$i \frac{\partial}{\partial t} U(t, s) \Phi = H(t) \ U(t, s) \Phi, \quad i \frac{\partial}{\partial s} U(t, s) \Phi = -U(t, s) \ H(s) \Phi.$$
To prove the existence and the uniqueness of the propagator for our Hamiltonians $H(t)$, we use a generalization of the Avron-Herbst formula close to the one given in [3]. In [12], the author gives, for $E_0 \neq 0$, a different formula which has the advantage to be time-periodic. To study our inverse scattering problem, we use here a different one, which is defined for all $E_0$. We emphasize that with our choice, $c(t)$ (see below for the definition of $c(t)$) is also periodic with period 1; in particular $c(t) = O(1)$.

The basic idea is to generalize the well-known Avron-Herbst formula for a Stark Hamiltonian with a constant electric field $E_0$, [2]; if we consider the Hamiltonian $B_0$ on $L^2(\mathbb{R}^n)$,

$$B_0 = \frac{1}{2}p^2 - E_0 \cdot x ,$$

we have the following formula :

$$e^{-itB_0} = e^{-i\frac{E_0^2}{4}t^3} e^{itE_0 \cdot x} e^{-i\frac{E_0^2}{2}p^2} e^{-it\frac{E_0^2}{2}} .$$

In the next definition, we give a similar formula for time-dependent electric fields.

**Definition.**

We consider the family of unitary operators $T(t)$, for $t \in \mathbb{R}$ :

$$T(t) = e^{-ia(t)} e^{-ib(t) \cdot x} e^{-ic(t) \cdot p} ,$$

where :

$$b(t) = -\int_0^t (E(s) - E_0) \, ds - \int_0^1 \int_0^t (E(s) - E_0) \, ds \, dt .$$

$$c(t) = -\int_0^t b(s) \, ds .$$

$$a(t) = \int_0^t \left( \frac{1}{2} b^2(s) - E_0 \cdot c(s) \right) \, ds .$$

**Lemma 4**

The family $\{ H_0(t) \}_{t \in \mathbb{R}}$ has an unique propagator $U_0(t, s)$ defined by :

$$U_0(t, s) = T(t) e^{-i(t-s)B_0} T^*(s) .$$

**Proof.**

We can always assume $s = 0$ and we make the following ansatz :

$$U_0(t, 0) = e^{-ia(t)} e^{-ib(t) \cdot x} e^{-ic(t) \cdot p} e^{-itB_0} .$$
Since on the Schwartz space, $U_0(t, 0)$ must satisfy :

\begin{equation}
(2.8) 
\frac{i}{\partial t} U_0(t, 0) = H_0(t) U_0(t, 0),
\end{equation}

the functions $a(t)$, $b(t), c(t)$ solve :

\begin{equation}
(2.9) \begin{cases}
\dot{b}(t) = -E(t) + E_0, \quad \dot{c}(t) = -b(t), \quad \dot{a}(t) = \frac{1}{2} b^2(t) - E_0 \cdot c(t).
\end{cases}
\end{equation}

We refer to [3] for details and [12] for a different formula. □

In the same way, in order to define the propagator corresponding to the family $\{H(t)\}$, we consider a Stark Hamiltonian with a time-periodic potential $V_1(t, x)$, (we recall that $c(t)$ a is $C^1$-periodic function) :

\begin{equation}
(2.10) \quad B(t) = B_0 + V_1(t, x) \quad \text{where} \quad V_1(t, x) = e^{ic(t) \cdot p} V(t, x) e^{-ic(t) \cdot p} = V(t, x + c(t)).
\end{equation}

Then, $B(t)$ has an unique propagator $R(t, s)$, (see [16] for the case $E_0 = 0$ and [12] for the case $E_0 \neq 0$). It is easy to see that the propagator $U(t, s)$ for the family $\{H(t)\}$ is defined by :

\begin{equation}
(2.11) \quad U(t, s) = T(t) R(t, s) T^*(s).
\end{equation}

**Comments.**

Since the Hamiltonians $H_0(t)$ and $H(t)$ are time-periodic with period 1, one has for all $t, s \in IR$ :

\begin{equation}
(2.12) \quad U_0(t + 1, s + 1) = U_0(t, s), \quad U(t + 1, s + 1) = U(t, s).
\end{equation}

Thus, the wave operators satisfy $W^\pm(s + 1) = W^\pm(s)$.

### 2.2 The high energy limit of the scattering operators.

In this section, we study the high energy limit of the scattering operators by using the well-known Enss-Weder’s time-dependent method [4]. This method can be used to study Hamiltonians with electric and magnetic potentials on $L^2(\mathbb{R}^n)$ [1], the Dirac equation [9], the N-body case [4], the Stark effect ([15], [17]), the Aharonov-Bohm effect [18].

In [13], [14] a stationary approach, based on the same ideas, is proposed to solve scattering inverse problems for Schrödinger operators with magnetic fields or with the Aharonov-Bohm effect.
Before giving the main result of this section, we need some notation.

- \( \Phi, \Psi \) are the Fourier transforms of functions in \( C_0^\infty(\mathbb{R}^n) \).
- \( \omega \in S^{n-1} \cap \Pi_{E_0} \) is fixed, where \( \Pi_{E_0} \) is the orthogonal hyperplane to \( E_0 \).
- \( \Phi_{\lambda,\omega} = e^{i\sqrt{\lambda}x \cdot \omega} \Phi, \Psi_{\lambda,\omega} = e^{i\sqrt{\lambda}x \cdot \omega} \Psi \).

We have the following high energy asymptotics where \( \langle , \rangle \) is the usual scalar product in \( L^2(\mathbb{R}^n) \):

**Proposition 5**

Under the assumptions of Theorem 3, we have for all \( s \in [0, 1] \),

\[
\langle [S(s), p] \Phi_{\lambda,\omega}, \Psi_{\lambda,\omega} \rangle = \lambda^{-\frac{3}{2}} \left( \int_{-\infty}^{+\infty} \partial_x V(s, x + t\omega) \, dt \right) \Phi, \Psi >+o\left( \lambda^{-\frac{3}{2}} \right)
\]

**Comments.**

Actually, for the case \( n = 2 \), \( E_0 \neq 0 \) and \( \delta > \frac{3}{4} \), Proposition 5 is also valid for \( \omega \in S^{n-1} \) satisfying \( |\omega \cdot E_0| < |E_0| \), (see ([18], [15]).

Then, Theorem 3 follows from Proposition 5 and the inversion of Radon transform ([6] and [15], Section 2.3).

**Proof of Proposition 5.**

For example, let us show Proposition 5 for the case \( E_0 \neq 0 \) and \( n \geq 3 \), the other cases are similar. More precisely, see [18] for the case \( E_0 = 0 \), and for the case \( n = 2 \), \( E_0 \neq 0 \), see ([17], Theorem 2.4) and ([15], Theorem 4).

**Step 1.**

Since \( c(t) \) is periodic, \( c(t) = O(1) \). Then, \( V(t, x) \) is a short-range perturbation of \( B_0 \), and we can define the usual wave operators for the pair of Hamiltonians \( (B(t), B_0) \):

\[
\Omega^\pm(s) = s - \lim_{t \to \pm \infty} R(s, t) e^{-i(t-s)B_0}.
\]

Consider also the scattering operators \( S_1(s) = \Omega^{+*}(s) \Omega^-(s) \). By virtue of (2.6) and (2.11), it is clear that :

\[
S(s) = T(s) S_1(s) T^*(s).
\]

Using the fact that \( e^{-ib(s) \cdot x} p e^{ib(s) \cdot x} = p + b(s) \), we have :

\[
[S(s), p] = [S(s), p + b(s)] = T(s) [S_1(s), p] T^*(s).
\]

Thus,

\[
\langle [S(s), p] \Phi_{\lambda,\omega}, \Psi_{\lambda,\omega} \rangle = \langle [S_1(s), p] T^*(s) \Phi_{\lambda,\omega}, T^*(s) \Psi_{\lambda,\omega} \rangle.
\]
In other hand,
(2.17) \[ T^*(s) \Phi_{\lambda,\omega} = e^{i\sqrt{\lambda}x \cdot \omega} e^{ic(s) \cdot (p+\sqrt{\lambda} \omega)} e^{ib(s) \cdot x} e^{ia(s) \cdot \Phi}. \]
So, we obtain :
(2.18) \[ < [S(s), p] \Phi_{\lambda,\omega}, \Psi_{\lambda,\omega} > = < [S_1(s), p] f_{\lambda,\omega}, g_{\lambda,\omega} >, \]
where
(2.19) \[ f = e^{ic(s) \cdot p} e^{ib(s) \cdot x} \Phi \text{ and } g = e^{ic(s) \cdot p} e^{ib(s) \cdot x} \Psi. \]
Clearly, \( f, g \) are the Fourier transforms of functions in \( C_0^\infty(\mathbb{R}^n) \).

• Step 2 : Modified wave operators.
Now, we follow a strategy close to [15] for time-dependent potentials. First, let us define a free-modified dynamic \( U_D(t, s) \) by :
(2.20) \[ U_D(t, s) = e^{-i(t-s)B_0} e^{-i \int_0^{t-s} V_1(u+s, up'+\frac{1}{2}u^2E_0) \, du}, \]
where \( p' \) is the projection of \( p \) on the orthogonal hyperplane to \( E_0 \).
We define the modified wave operators :
(2.21) \[ \Omega_D^\pm(s) = s - \lim_{t \to \pm \infty} R(s, t) \, U_D(t, s). \]
It is clear that :
(2.22) \[ \Omega_D^\pm(s) = \Omega^\pm(s) e^{-ig^\pm(s, p')} , \]
where
(2.23) \[ g^\pm(s, p') = \int_0^{\pm \infty} V_1(u + s, up' + \frac{1}{2}u^2E_0) \, du. \]
Thus, if we set \( S_D(s) = \Omega_D^{+*}(s) \Omega_D^-(s) \), one has :
(2.24) \[ S_1(s) = e^{-ig^+(s, p')} S_D(s) e^{ig^-(s, p')} \]

• Step 3 : High energy estimates.
Denote \( \rho = \min (1, \delta) \). We have the following estimations, (the proof is exactly the same as in ([15], Lemma 3) for time-independent potentials).

Lemma 6
For \( \lambda >> 1 \), we have :
(i) \[ \| \left( V_1(t, x) - V_1(t, (t-s)p' + \frac{1}{2}(t-s)^2E_0) \right) U_D(t, s) e^{ig^\pm(s, p')} f_{\lambda,\omega} \| \]
\[ \leq C \left( 1 + \left| (t-s)\sqrt{\lambda} \right| \right)^{-\frac{1}{2}-\rho}. \]
(ii) \[ \| \left( R(s, t) \Omega_D^\pm(s) - U_D(t, s) \right) e^{ig^\pm(s, p')} f_{\lambda,\omega} \| = O \left( \lambda^{-\frac{1}{2}} \right), \text{ uniformly for } t, s \in \mathbb{R}. \]
• Step 4.

We denote $F(s, \lambda, \omega) = \langle [S_1(s), p] f_{\lambda, \omega}, g_{\lambda, \omega} \rangle$. Using (2.24), we have:

$$F(s, \lambda, \omega) = \langle [e^{-ig^+(s,p')} S_D(s) e^{ig^-(s,p')}, p] f_{\lambda, \omega}, g_{\lambda, \omega} \rangle$$

$$= \langle [S_D(s), p] e^{ig^-(s,p')} f_{\lambda, \omega}, e^{ig^+(s,p')} g_{\lambda, \omega} \rangle$$

$$= \langle [S_D(s) - 1, p - \sqrt{\lambda}] e^{ig^-(s,p')} f_{\lambda, \omega}, e^{ig^+(s,p')} g_{\lambda, \omega} \rangle$$

$$= \langle (S_D(s) - 1) e^{ig^-(s,p')}(pf)_{\lambda, \omega}, e^{ig^+(s,p')} g_{\lambda, \omega} \rangle$$

$$- \langle (S_D(s) - 1) e^{ig^-(s,p')}(pg)_{\lambda, \omega} \rangle$$

$$= F_1(s, \lambda, \omega) - F_2(s, \lambda, \omega).$$

First, let us study $F_1(s, \lambda, \omega)$. Writing $S_D(s) - 1 = (\Omega_D^+(s) - \Omega_D^-(s))^\ast \Omega_D^-(s)$ and using

$$\Omega_D^+(s) - \Omega_D^-(s) = i \int_{-\infty}^{+\infty} R(s,t) \left( V_1(t,x) - V_1(t,(t-s)p' + \frac{1}{2}(t-s)^2 E_0) \right) U_D(t,s) \, dt,$$

we obtain:

$$S_D(s) - 1 = -i \int_{-\infty}^{+\infty} U_D(t,s)^\ast \left( V_1(t,x) - V_1(t,(t-s)p' + \frac{1}{2}(t-s)^2 E_0) \right) R(t,s) \Omega_D^-(s) \, dt.$$

Thus,

$$F_1(s, \lambda, \omega) = -i \int_{-\infty}^{+\infty} < R(t,s) \Omega_D^-(s) e^{ig^-(s,p')}(pf)_{\lambda, \omega},$$

$$\left( V_1(t,x) - V_1(t,(t-s)p' + \frac{1}{2}(t-s)^2 E_0) \right) U_D(t,s) e^{ig^+(s,p')} g_{\lambda, \omega} > dt$$

$$- i \int_{-\infty}^{+\infty} < U_D(t,s) e^{ig^-(s,p')}(pf)_{\lambda, \omega},$$

$$\left( V_1(t,x) - V_1(t,(t-s)p' + \frac{1}{2}(t-s)^2 E_0) \right) U_D(t,s) e^{ig^+(s,p')} g_{\lambda, \omega} > dt$$

$$+ R_1(s, \lambda, \omega),$$

where:

$$R_1(s, \lambda, \omega) = -i \int_{-\infty}^{+\infty} < \left( R(t,s) \Omega_D^-(s) - U_D(t,s) \right) e^{ig^-(s,p')}(pf)_{\lambda, \omega},$$

$$\left( V_1(t,x) - V_1(t,(t-s)p' + \frac{1}{2}(t-s)^2 E_0) \right) U_D(t,s) e^{ig^+(s,p')} g_{\lambda, \omega} > dt.$$

By Lemma 6, it is clear that $R_1(s, \lambda, \omega) = O(\lambda^{-1})$. Thus, writing $t = \frac{s}{\sqrt{\lambda}} + s$, we obtain:
\[(2.28) \quad F_1(s, \lambda, \omega) = -\frac{i}{\sqrt{\lambda}} \int_{-\infty}^{+\infty} \left< U_D\left(\frac{\tau}{\sqrt{\lambda}} + s, s\right) e^{i\gamma(s, \omega')}(pf)_{\lambda, \omega}\right> \]

\[
\left( V_1\left(\frac{\tau}{\sqrt{\lambda}} + s, x\right) - V_1\left(\frac{\tau}{\sqrt{\lambda}} + s, \frac{\tau^2}{2\lambda} E_0\right) \right) U_D\left(\frac{\tau}{\sqrt{\lambda}} + s, s\right) e^{i\gamma(s, \omega')} g_{\lambda, \omega} > d\tau + O(\lambda^{-1}).
\]

Denote by \(f_1(\tau, s, \lambda, \omega)\) the integrand of the (R.H.S) of (2.28). By Lemma 6 (i),

\[(2.29) \quad |f_1(\tau, s, \lambda, \omega)| \leq C (1 + |\tau|)^{-\frac{3}{2} - \rho}.
\]

So, by Lebesgue’s theorem, to obtain the asymptotics of \(F_1(s, \lambda, \omega)\), it suffices to determine \(\lim_{\lambda \to +\infty} f_1(\tau, s, \lambda, \omega)\).

Let us denote :

\[(2.30) \quad U^{\pm}(t, s, p') = e^{i \int_{t}^{\pm\infty} V_1(u + s, up' + \frac{1}{2} u^2 E_0) \, du}.
\]

We have :

\[(2.31) \quad f_1(\tau, s, \lambda, \omega) = e^{-i\frac{\tau}{\sqrt{\lambda}} B_0} U^-\left(\frac{\tau}{\sqrt{\lambda}}, s, p'\right) (pf)_{\lambda, \omega},
\]

\[
\left( V_1\left(\frac{\tau}{\sqrt{\lambda}} + s, x\right) - V_1\left(\frac{\tau}{\sqrt{\lambda}} + s, \frac{\tau^2}{2\lambda} E_0\right) \right) e^{-i\frac{\tau}{\sqrt{\lambda}} B_0} U^+\left(\frac{\tau}{\sqrt{\lambda}}, s, p'\right) g_{\lambda, \omega} > .
\]

Using the Avron-Herbst formula (2.2), we deduce that :

\[(2.32) \quad f_1(\tau, s, \lambda, \omega) = e^{-i\frac{\tau}{2\sqrt{\lambda}} p^2} U^-\left(\frac{\tau}{\sqrt{\lambda}}, s, p'\right) (pf)_{\lambda, \omega},
\]

\[
\left( V_1\left(\frac{\tau}{\sqrt{\lambda}} + s, x + \frac{\tau^2}{2\lambda} E_0\right) - V_1\left(\frac{\tau}{\sqrt{\lambda}} + s, \frac{\tau^2}{2\lambda} E_0\right) \right) e^{-i\frac{\tau}{2\sqrt{\lambda}} p^2} U^+\left(\frac{\tau}{\sqrt{\lambda}}, s, p'\right) g_{\lambda, \omega} > .
\]

Then, we obtain :

\[(2.33) \quad f_1(\tau, s, \lambda, \omega) = e^{-i\frac{\tau}{2\sqrt{\lambda}} (p + \sqrt{\lambda} \omega)^2} U^-\left(\frac{\tau}{\sqrt{\lambda}}, s, p' + \sqrt{\lambda} \omega\right) pf,
\]

\[
\left( V_1\left(\frac{\tau}{\sqrt{\lambda}} + s, x + \frac{\tau^2}{2\lambda} E_0\right) - V_1\left(\frac{\tau}{\sqrt{\lambda}} + s, \frac{\tau^2}{2\lambda} (p' + \sqrt{\lambda} \omega) + \frac{\tau^2}{2\lambda} E_0\right) \right)
\]

\[
e^{-i\frac{\tau}{2\sqrt{\lambda}} (p + \sqrt{\lambda} \omega)^2} U^+\left(\frac{\tau}{\sqrt{\lambda}}, s, p' + \sqrt{\lambda} \omega\right) g > .
\]

Since

\[(2.34) \quad e^{-i\frac{\tau}{2\sqrt{\lambda}} (p + \sqrt{\lambda} \omega)^2} = e^{-i\frac{\tau}{2\sqrt{\lambda}} p^2} e^{-i\tau \omega \cdot p} e^{-i\frac{\tau}{2\sqrt{\lambda}} (p + \sqrt{\lambda} \omega)^2},
\]
we have

\( f_1(\tau, s, \lambda, \omega) = < e^{-i \frac{\tau^2}{2\lambda}} U^{-}\left(\frac{\tau}{\sqrt{\lambda}}, s, p' + \sqrt{\lambda}\omega\right) pf, \)

\[
\left( V_1\left(\frac{\tau}{\sqrt{\lambda}} + s, x + \tau\omega + \frac{\tau^2}{2\lambda} E_0\right) - V_1\left(\frac{\tau}{\sqrt{\lambda}} + s, \frac{\tau}{\sqrt{\lambda}}(p' + \sqrt{\lambda}\omega) + \frac{\tau^2}{2\lambda} E_0\right) \right)
\]

\[ e^{-i \frac{\tau^2}{2\lambda}} U^+\left(\frac{\tau}{\sqrt{\lambda}}, s, p' + \sqrt{\lambda}\omega\right) g > . \]

Since \(| V_1(u + s, u(p' + \sqrt{\lambda}\omega) + \frac{1}{2}u^2E_0)) | \leq C (u^2 + 1)^{-\delta} \in L^1(\mathbb{R}^+, du)\), it is easy to show (using Lebesgue's theorem again) that:

\( s - \lim_{\lambda \to +\infty} U^\pm\left(\frac{\tau}{\sqrt{\lambda}}, s, p' + \sqrt{\lambda}\omega\right) = 1 . \)

Then,

\( \lim_{\lambda \to +\infty} f_1(\tau, s, \lambda, \omega) = < pf, (V_1(s, x + \tau\omega) - V_1(s, \tau\omega)) g > . \)

So, we have obtained:

\( F_1(s, \lambda, \omega) = -\frac{i}{\sqrt{\lambda}} < pf, \left( \int_{-\infty}^{+\infty} (V_1(s, x + \tau\omega) - V_1(s, \tau\omega)) d\tau \right) g > + o\left(\frac{1}{\sqrt{\lambda}}\right), \)

In the same way, we obtain

\( F_2(s, \lambda, \omega) = -\frac{i}{\sqrt{\lambda}} < \Phi, \left( \int_{-\infty}^{+\infty} (V_1(s, x + \tau\omega) - V_1(s, \tau\omega)) d\tau \right) pg > + o\left(\frac{1}{\sqrt{\lambda}}\right), \)

so

\( F(s, \lambda, \omega) = F_1(s, \lambda, \omega) - F_2(s, \lambda, \omega) \)

\( = \frac{1}{\sqrt{\lambda}} < \Phi, \left( \int_{-\infty}^{+\infty} \partial_x V_1(s, x + \tau\omega) d\tau \right) g > + o\left(\frac{1}{\sqrt{\lambda}}\right). \)

Using (2.19) and \( \partial_x V(s, x + \tau\omega) = e^{-ic(s-p)} \partial_x V_1(s, x + \tau\omega) e^{ic(s-p)} \), we obtain:

\( F(s, \lambda, \omega) = \frac{1}{\sqrt{\lambda}} < \Phi, \left( \int_{-\infty}^{+\infty} \partial_x V(s, x + \tau\omega) d\tau \right) \Psi > + o\left(\frac{1}{\sqrt{\lambda}}\right). \)
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