A singular integrable equation from short capillary-gravity waves

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Introduction. The nonlinear and dispersive propagation of surface waves in an ideal incompressible fluid (depth $h$, density $\sigma$), under the action of gravity $g$ and surface tension $T$, is a classical subject of investigation in mathematical physics \cite{1,2,3}. In this letter we derive and study a new integrable model equation from asymptotic dynamics of a short capillary-gravity wave, namely

$$u_{xt} = \frac{3g(1-3\theta)}{2vh}u - \frac{1}{2}u_{xx}u - \frac{1}{4}u_x^2 + \frac{3h^2}{4v}u_{xx}u_x^2. \quad (1)$$

Here $u(x,t)$ is the fluid velocity on the surface, $x$ and $t$ are space and time variables, subindices mean partial derivatives, $\theta = (T/\sigma h^2)g$ is the dimensionless Bond number and $\nu = (3T/\sigma h)^{1/2}$.

The dynamics of surface waves in an ideal fluid obeys complicated nonlinear and dispersive equations. To simplify them, multiscale asymptotic methods can be employed. Most of the resultant asymptotic models represent, for large $t$, balance between weak nonlinearities and linear dispersion. For instance the long-wave dynamics of a low amplitude initial profile on a shallow dispersive fluid are well known nowadays. The models extend from the oldest Boussinesq systems or the ubiquitous Korteweg-deVries \cite{4} to the more recent Camassa-Holm equation \cite{5} with nonlinear dispersion. In contrast, hardly anything is known about asymptotic models for nonlinear and dispersive dynamics of short-waves. For the most part short waves have been studied in connection with modulation of short-wave trains \cite{6,7,8,9}.

In this paper we derive the model \cite{11} in the short-wave regime of surface waves, we prove that it is integrable and show that it leads to unusual special solutions that develop singular behavior in finite time.

The short-wave limit. To define a short wave (wave length $l$, wave number $k = 2\pi/l$) one needs to compare $l$ to an underlying space scale. We use the unperturbed depth $h$ as the natural reference, and thus consider $h = \mathcal{O}(1)$ and $kh = \mathcal{O}(1/\epsilon)$, \quad (2)

where $\epsilon$ is the parameter of the asymptotic expansions.

Multiscale asymptotic methods are strongly based on the dispersion relation $\omega(k)$ and on the associated phase velocity $v_p$ and group velocity $v_g$. The short-wave limit (2) is meaningful if and only if those two velocities possess a finite limit. Then $v_p$ and $v_g$ allow to define asymptotic variables and to handle the nonlinear regime \cite{10,11}.

For the usual linearization of the Euler equations (with surface tension), the linear dispersion relation

$$\omega(k) = [k(g + T/\sigma)\tanh(kh)]^{1/2} \quad (3)$$

yields in the short-wave limit $v_p \sim (Tk/\sigma)^{1/2} \to \infty$. This not only prevents us from defining asymptotic variables but also infinite dispersion cannot be compensated by weak nonlinearities. We found that the solution to this problem is to employ the Green-Nagdhi conditions of linearization.

The basic model. Green, Laws and Nagdhi \cite{12, 13, 14} developed alternative reductions of the Euler equations leading to models having dispersion relations with good behavior in the short-wave limit, as demonstrated in \cite{10}. They used three main hypothesis, namely non irrotational fluid flow, motion in vertical columns and non-Archimedian pressure condition. For seek of completeness, we derive hereafter the model in a simple manner and include surface tension.

Let the particles of the fluid be identified in a fixed rectangular Cartesian system of center $O$ and axes $(x,y,z)$ with $Oz$ the upward vertical direction. We assume translational symmetry in $y$ and we will only consider a sheet of fluid in the $xz$ plane. This fluid sheet is moving on a rigid bottom at $z = 0$ and its upper free surface is $z = S(x,t)$. The continuity equation and the Newton
equations (in the flow domain) read
\[ u_x + w_z = 0, \]
\[ \sigma(u_t + uu_x + wu_z) = -p^*_x, \]
\[ \sigma(w_t + uw_x + wu_z) = -p^*_z - g\sigma \]
where \(p^*(x, z, t)\) is the pressure and \((u, w)\) the vectorial velocity.

The kinematic and dynamic boundary conditions read
\[ w = 0 \text{ at } z = 0, \]
\[ S_t + uS_x - w = 0 \text{ at } z = S(x, t), \]
\[ p^* = p_0 - \frac{T S_{xx}}{(1 + S_x^2)^{2}} \text{ at } z = S(x, t). \]
The columnar-flow hypothesis consists in assuming that \(u\) does not dependent on \(z\), hence from (1) and (7)
\[ u = u(x, t), \quad w = -zu_x . \]

The integration of (3) over \(z\) from 0 to \(S(x, t)\) then gives
\[ \sigma S(u_t + uu_x) = -p_x + T[(1 + S_x^2)^{-\frac{3}{2}}]_x, \]
\[ p(x, t) = \int_0^{S(x, t)} p^*(x, z, t) dz - p_0 S(x, t). \]

Now we multiply (3) by \(z\) and integrate over it to get
\[ \frac{\sigma S^3}{3}(-u_{xt} - uu_{xx} + u_x^2) = p + \frac{TSS_{xx}}{(1 + S_x^2)^{\frac{3}{2}}} - \frac{g\sigma S^2}{2} . \]

Finally, elimination of \(p\) between (11) and (13) gives, together with (5) and (10), the extension of the Green-Nagdhi system to non-zero surface tension
\[ S(u_t + uu_x) = \frac{1}{3} \left[ S^3(u_{xx} + uu_{xx} - u_x^2) \right]_x - gSS_x + (T/\sigma)S \left[S_{xx}(1 + S_x^2)^{-\frac{3}{2}}\right]_x, \]
\[ S_t + (uS)_x = 0 . \]
This constitutes our basic model.

Asymptotic model for short capillary-gravity waves. In contrast with shallow water theories with dispersion (Boussinesq type equations), this model incorporates finite dispersion both in the long-wave and in the short-wave limits. Indeed the linear dispersion relation is
\[ \Omega^2 = k^2 gh + (Th/\sigma)k'/[1 + (hk)^2/3]. \]
Hence the phase velocity is bounded in the short-wave limit as we have
\[ \frac{\Omega}{k} \sim \left( \frac{3T}{\sigma h} \right)^{1/2} + O \left( \frac{1}{k^2 h^2} \right). \]

This allows to define asymptotic variables
\[ \zeta = (1/\epsilon)(x - vt), \quad \tau = ct. \]

With the power series \(u = e^2(u_0 + e^2u_2 + \ldots)\) and \(S = h + e^2(S_0 + e^2S_2 + \ldots)\), the basic system (11) (13) leads to an equation for \(u_0(\zeta, \tau)\) which, in the laboratory variable, becomes our main equation (11).

Lax pair and finite-time singularities. After appropriate rescalings of the variables, one can bring equation (11) into the form
\[ u_{xt} = u - uu_{xx} - \frac{1}{2} u_x^2 + \frac{\lambda}{2} u_{xx} u_x^2, \]
\(\lambda\) being expressed in terms of the physical parameters of equation (11). The corresponding Lagrangian is:
\[ L = \frac{1}{2} u_x u_t + \frac{1}{2} u^2 + \frac{1}{2} u_{xx}^2 - \frac{\lambda}{24} u_x^4. \]

Equation (11) is integrable with Lax pair (in usual notations):
\[ L = \frac{\partial}{\partial x} + i\sqrt{E} F \sigma_3 + \frac{1}{2} F^3 u_{xxx} \sqrt{1 - \lambda}, \]
\[ M = -\frac{1}{2} \left( u - \frac{1}{2} \lambda u_x^2 \right) F \sigma_3 - \frac{i}{\sqrt{E}} \frac{1}{4} \frac{u_{xxx}}{F} \sigma_3 + \frac{1}{4\sqrt{E}} \frac{1}{F} \sigma_2 . \]
where \(\sigma\) are the usual Pauli matrices, \(E\) the “eigenvalue” and
\[ F^2 = 1 - 2u_x + \lambda u_x^2. \]

One of its most remarkable properties is that with \(F\) one builds the first non-trivial conserved quantity for all \(\lambda:\)
\[ F_t = \left[ \left( u - \frac{\lambda}{2} u_x^2 \right) F \right]_x, \]

and through the change of function from \(u(x, t)\) to
\[ g(y, t) = \frac{1}{\sqrt{1 - \lambda}} \text{Arctanh} \frac{u_{xx} \sqrt{1 - \lambda}}{1 - u_{xx}}, \]

with
\[ y = \int_x^y F dx, \]
one finds that \(g\) satisfies the sinh-Gordon equation
\[ g_{yt} = \frac{1}{\sqrt{1 - \lambda}} \text{sinh} \sqrt{1 - \lambda} g. \]

This is valid for \(\lambda < 1\) and for \(u_{xx}\) small enough so that \(F\) is real. If \(u_{xx}\) is large, a similar change leads to the cosh-Gordon equation, and if \(\lambda > 1\) one obtains the sine-Gordon equation. Finally, for \(\lambda = 1\), one obtains for \(g(y, t) = u_{xx}/(1 - u_{xx})\) the equation of a free field in light-cone coordinates.

Whatever the value of \(\lambda\), it follows from the change of variables \((x, t) \leftrightarrow (y, t)\) that a regular \(g(y, t)\) can give back a singular, multivalued \(u(x, t)\) if the change from \(y\) back to \(x\) is not one-to-one. This happens when \(|y|\) is
large enough, forcing $u_{xx}$ to infinity and a change of sign of $F$ in the equation for $y$. We give an example of this in fig. 1, where two solutions for $u$ are plotted from breather solutions of the sine-Gordon equation, one (dashed curve) with an amplitude just below the singularity threshold, so that $u$ is still regular and single-valued, the other (solid curve) with an amplitude above the singularity threshold, which displays a swallowtail behavior.

For $\lambda = 0$, equation (19) was already discussed in [15, 16, 17, 18, 19, 20], and it contains peakons. These peakons, which are solitons, and their scattering are qualitatively very easy to study via the change of variables to the sinh-Gordon equation, where they correspond to singular solutions obtained by simple analytic continuations of the sine-Gordon multisoliton solutions.

For any value of $\lambda$, the equation (19) has the symmetry $x \rightarrow ax, \ t \rightarrow t/a, \ u \rightarrow u^{\lambda}$, for arbitrary real $a$, which is just the Lorentz group in 1+1 dimensions. Although the Lagrangian (20) does not have the appropriate covariance property to give a Lorentz invariant action, the system, being integrable, has an infinite set of Lagrangians (and derived Hamiltonians and Poisson brackets) leading to the same equation of motion (19), and one of them, built with the invariant field $u_{xx}$, leads to an invariant action.

**Benjamin-Feir instability.** The Benjamin-Feir instability results from resonant interaction of an initial monochromatic wave with side-band modes produced by nonlinearity. This instability, which modulates the carrier envelope, is well described by the nonlinear Schrödinger asymptotic limit.

Following the standard approach [3] we can show that a Stokes wave train of equation (10) is unstable if

$$\theta < \frac{3}{10}. \quad (28)$$

namely any slight deformation of the plane wave experiences an exponential growth. In the case of water at room temperature ($T = 0.074 \ N m^{-1}, \ \sigma = 10^3 \ kg m^{-3}$), we obtain that a short wave train is unstable for a depth $h > 0.49 \ cm$.

Last but not least, the value $\theta = 3/10$ corresponds to $\lambda = 1$ in (19). Precisely

$$\theta < 0.3 \Rightarrow \lambda > 1. \quad (29)$$

**Comments.** The occurrence of singular (even multivalued) solutions in an equation derived in a hydrodynamics framework is interesting, especially since $u$ is the deviation of the free surface from equilibrium. The actual observability of the present singular behaviors would require a more detailed analysis of the validity of the short-wave approximation near the singular points which goes beyond the scope of this paper. In particular the inclusion of viscosity, which acts strongly over small scales, will affect the short-waves dynamics and alter these singular behaviors.

The Green-Naghdi equations can be improved systematically toward higher depths [21]. In the linear limit, these improvements give the higher $(N,N)$ Padé approximants of the Euler dispersion relation [3], the present case $[10]$ corresponding to $N = 1$. In particular they always lead to a finite phase velocity in the short-wave limit, which grows quickly as the order of the approximation grows to try to mimic the behavior of the exact formula.
What is remarkable is that the short-wave asymptotics of these improvements lead to exactly the same integrable equation (4) except for different numerical coefficients in front of the physical quantities $g, h, \theta$ and $v$. Hence we hope that at least some the main features of the singular behavior of the solutions correspond to the actual physics of these water waves in arbitrary depth.

Another point is the existence of a peakon solution in the case $\lambda = 0$. In the rescaling leading from equation (4) to equation (19), this value of $\lambda$ is obtained only for $\theta = 1/3$, where the rescaling is singular for the $x, t$ and $u$ variables themselves, so that the whole asymptotics must be reconsidered from the start. This value $\theta = 1/3$ in the Euler equation leads to a dispersionless system for small $k$ only and it is a peculiar feature of the Green-Naghdi equations to be dispersionless for all $k$ (hence for large $k$) for that value of $\theta$. This large $k$ feature is not inherited by the improvements of [21].

Equation (4) has a Lorentz invariance, and its quantum version promises to exhibit new features not shared by the existing relativistic systems in 1+1 dimension. This new relativistic integrable system, with just one massive bosonic field and a local classical equation of motion, is quite intriguing. In particular, the change of variable which transforms it into sine- or sinh-Gordon mixes the space-time variable $x$ and the field $u_{xx}$ and furthermore requires the equation of motion to be satisfied. Hence, it cannot be quantum mechanically equivalent to sine- or sinh-Gordon. For example, it is not parity invariant (parity in the laboratory frame, $x+t \to -x-t$, $x-t \to x-t$). From this follows an unusual $S$-matrix, in which there are two phase-shifts, one for the left moving particle and one for the right moving one. The quantum field theory and mathematical structures following from this are worth a detailed study for themselves in separate publications. This could also have some analogy in general relativity, where space-time is, through the metric, a dynamical variable and where one can go from a choice of space-time parametrization to another one by a change which can involve the metric itself.

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