T-odd asymmetries in the radiative two-pion tau decay

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Abstract

In this work, we perform a detailed study of the $\tau^{-} \rightarrow \pi^{-} \pi^{0} \gamma \nu$ decay process within the resonance chiral theory. We pay special attention to the triple-product T-odd asymmetry in the $\tau^{-} \rightarrow \pi^{-} \pi^{0} \gamma \nu$ process. The minimal resonance chiral Lagrangian and the odd-intrinsic parity resonance operators are simultaneously included to calculate the decay amplitudes. Various invariant-mass distributions in the $\pi^{-}\pi^{0}$, $\pi^{-}\gamma$ and $\pi^{0}\gamma$ systems are studied and they reveal different resonance dynamics. We further predict the intriguing nonzero T-odd asymmetry distributions, which may provide useful guidelines for future experimental measurements conducted at the Belle-II and super tau-charm facilities.

1 Introduction

Charge-conjugation and parity violation (CPV) is one of the most important open problems in the Standard Model (SM) and beyond. The typical CPV observables are the decay rate asymmetries from the processes related with the charge conjugations, which have been widely used to establish the CPV in the strange-, beauty- and charm-meson sectors. Up to now the CPV in the lepton sector is yet to be discovered, and its establishment will definitely extend our understanding of the CPV in particle physics. The charge-conjugate decay rate asymmetries in various channels are expected to provide promising opportunities to establish the CPV of the $\tau$ lepton \cite{1-6}. Another distinct way to probe the CPV relies on the nontrivial kinematical measurements of the T-odd quantities. One of such observables is the triple-product asymmetry \cite{7,13}, which can be constructed via $\varepsilon_{\mu\nu\rho\sigma}p_{1}^{\mu}p_{2}^{\nu}p_{3}^{\rho}p_{4}^{\sigma}$, being $\varepsilon_{\mu\nu\rho\sigma}$ Levi-Civita anti-symmetric tensor and $p_{i=1,2,3,4}$ the momenta of the involved particles. Other types of T-odd quantities with spin vectors, which usually requires to measure the polarizations of the final-state particles, can be also constructed in a similar way \cite{4,14}.

The T-odd kinematical variable $\xi \equiv \varepsilon_{\mu\nu\rho\sigma}p_{1}^{\mu}p_{2}^{\nu}p_{3}^{\rho}p_{4}^{\sigma}$ naturally originates from the four-body decay processes. In the rest frame of the decaying particle, without loss of any generality, the quantity $\xi$ can be written as a term that is proportional to $(\vec{p}_{1} \times \vec{p}_{2}) \cdot \vec{p}_{3}$, which gives rise to the name of the triple product for such observables. The triple-product asymmetries have been extensively investigated in the charged and neutral $K_{1,3,} \gamma$ decays, i.e., $K^{+} \rightarrow \pi^{0}l^{+}\nu_{l}\gamma$ and $K^{0} \rightarrow \pi^{-}l^{+}\nu_{\gamma}$ \cite{8,10,12}. It is found that the electromagnetic final-state interactions from the
photon loops in the SM [8,12], rather than the strong interactions from the hadronic loops [10], give the dominant contributions to the triple-product asymmetries in the \( K_{l3\gamma} \) decay processes. There are two main effects that reduce the hadronic contributions in the \( K_{l3\gamma} \) decay processes. First, the nonvanishing hadronic effects in the triple-product asymmetries only enter in the structure dependent (SD) parts of the \( K_{l3\gamma} \) decay matrix elements, which are much suppressed than the inner bremsstrahlung or structure independent (SI) parts. Second, the hadronic contributions starting from the two-pion threshold are reduced to a great extent due to the small kinematical phase space up to the kaon mass. However, these two suppression conditions do not hold in the \( \tau \rightarrow \pi\pi\gamma\nu \tau \) decay. To be more specific, not only the hadronic effects in the SD part but also their effects in the SI part can contribute to the T-odd asymmetric distributions in the radiative two-pion \( \tau \) decays. In addition, there is no phase space suppression for the hadronic contributions in the \( \tau \rightarrow \pi\pi\gamma\nu \tau \) process. Therefore, the hadronic contributions are expected to play significant roles in the triple-product asymmetries in the aforementioned two-pion radiative \( \tau \) decays. As an exploratory study, we focus on the triple-product asymmetry in the \( \tau^- \rightarrow \pi^-\pi^0\gamma\nu \tau \) process that originates from the final-state strong interactions in the SM.

This paper is organized as follows. In Sec. 2, we introduce the relevant resonance chiral Lagrangians and perform the calculations of the decay amplitudes of the \( \tau^- \rightarrow \pi^-\pi^0\gamma\nu \tau \) process. The general discussions on the T-odd asymmetries in the radiative two-pion tau decays are given in detail in Sec. 3. The phenomenological studies, including the determinations of the free couplings, the sensitivity of the \( \tau \rightarrow \pi\pi\gamma\nu \tau \) branching ratios on the photon energy cutoffs, the discussions of the resonance dynamics in various two-particle invariant mass distributions, are carried out in Sec. 4, in which we also give the promising predictions of the nonzero T-odd asymmetry distributions. A short summary and conclusions are then presented in Sec. 5. Essential technical details, including the treatment of phase spaces, and the explicit formulas of the relevant form factors and the kinematical coefficients when evaluating the T-odd asymmetries are relegated to the appendices.

2 Resonance chiral Lagrangians and relevant amplitudes of the radiative two-pion tau decay

The matrix element of the \( \tau^- (P) \rightarrow \pi^- (p_1)\pi^0 (p_2)\nu_\tau (q)\gamma (k) \) decay process can be generally written as [15]

\[
\mathcal{M} = e G_F V_{ud}^* \epsilon^{*\mu}(k) \left\{ F_\nu \bar{u}(q)\gamma^\mu(1 - \gamma_5)(m_\tau + \not{P} - \not{k})\gamma_\mu u(P) + (V_{\mu\nu} - A_{\mu\nu}) \bar{u}(q)\gamma^\nu(1 - \gamma_5)u(P) \right\},
\]

where \( e \) is basic unit of electric charge, \( G_F \) stands for the Fermi constant, \( V_{ud} \) denotes the CKM matrix element and \( \epsilon^{*\mu} \) corresponds to the polarization vector of the photon field. The effects from the bremsstrahlung off the \( \tau \) lepton are encoded in the \( F_\nu \) term, and the contributions from the bremsstrahlung off the \( \pi^- \) and the SD parts are included in \( V_{\mu\nu} \) and \( A_{\mu\nu} \) terms. The amplitude of the bremsstrahlung off the \( \tau \) lepton is governed by the non-leptonic decay process and it is given by

\[
F_\nu = (p_2 - p_1)_{\nu} f_+(t)/(2P \cdot k),
\]
where \( t = (p_1 + p_2)^2 \) and \( f_+(t) \) stands for the vector form factor of the two pions \[16\]. For the vector amplitude \( V_{\mu\nu} \), we take the same form as proposed in Ref. \[15\]

\[
V_{\mu\nu} = f_+(u) \frac{p_{1\mu}}{p_1 \cdot k}(p_1 + k - p_2)_{\nu} - f_+(u) g_{\mu\nu} + \frac{f_+(u) - f_+(t)}{(p_1 + p_2) \cdot k}(p_1 + p_2)_{\mu}(p_2 - p_1)_{\nu} + v_1(g_{\mu\nu}p_1 \cdot k - p_1_{\mu} k_{\nu}) + v_2(g_{\mu\nu}p_2 \cdot k - p_2_{\mu} k_{\nu}) + v_3(p_1 \cdot p_2 \cdot k - p_2 \cdot p_1 \cdot k)p_{1\nu} + v_4(p_1 \cdot p_2 \cdot k - p_2 \cdot p_1 \cdot k)(p_1 + p_2 + k)_{\nu},
\]

with \( u = (P - q)^2 \). For the axial-vector amplitude \( A_{\mu\nu} \), there are four independent form factors \[17\] and for later convenience we parameterize them as

\[
A_{\mu\nu} = i(a_1 \epsilon_{\mu\rho\sigma\beta} p_1^\rho k^\sigma + a_2 \epsilon_{\mu\rho\sigma\beta} p_2^\rho k^\sigma + a_3 p_{1\nu} \epsilon_{\mu\rho\sigma\beta} k^\rho p_1^\beta p_2^\sigma + a_4 p_{2\nu} \epsilon_{\mu\rho\sigma\beta} k^\rho p_1^\beta p_2^\sigma),
\]

where the Schouten’s identity has been used to write this decomposition. It is noted that somewhat different decompositions of the axial-vector amplitudes are used in Refs. \[17,19\].

In the SM, the various transition form factors in Eqs. \(2\), \(3\) and \(4\) are mainly governed by the nonperturbative strong interactions. Although chiral perturbation theory can be used to reliably calculate the form factors in the very low energy region around the thresholds, it becomes inadequate in the energy region where hadron resonances start to contribute. Alternatively, resonance chiral theory (R\(\chi\)T) \[20,21\], which explicitly includes the bare resonance fields in a chiral covariant way, offers a reliable theoretical framework to calculate the form factors both in the low and resonant energy regions. Indeed R\(\chi\)T has been widely employed to investigate many phenomenological processes involving the light-flavor hadron resonances, including the meson-meson scattering \[22-27\], the hadronic \(\tau\) decays \[19,28-31\], the form factors \[32,33\], the hadronic and radiative decay processes of various hadrons \[34-37\], etc. We further proceed the applications of the R\(\chi\)T to calculate the form factors appearing in the \(\tau \to \pi\pi\gamma\nu_\tau\) process.

The minimal interaction operators with the vector and axial-vector resonances in R\(\chi\)T read \[20\]

\[
\mathcal{L}_V = \frac{F_V}{2\sqrt{2}} \langle \hat{V}_{\mu\nu} f_{+}^{\mu\nu} \rangle + i \frac{G_V}{\sqrt{2}} \langle \hat{V}_{\mu\nu} u^\mu u^\nu \rangle,
\]

\[
\mathcal{L}_A = \frac{F_A}{2\sqrt{2}} \langle \hat{A}_{\mu\nu} f_{-}^{\mu\nu} \rangle.
\]

We refer to Ref. \[20\] for further details about the definitions of the basic chiral building tensors \( f_{\pm}^{\mu\nu}, u^\mu \) and the assignments of the flavor contents of the resonance multiplets \( \hat{V} (\hat{A})_{\mu\nu} \). The contributions from the minimal R\(\chi\)T to the \(\tau \to \pi\pi\gamma\nu_\tau\) decay amplitudes have been evaluated in Ref. \[15\]. Later on, a vector-meson-dominant (VMD) model was developed to include the refined effects from the odd-parity-intrinsic interacting vertex with \(\rho\omega\pi\), together with other terms beyond the minimal R\(\chi\)T operators \[38,39\]. Compared to the results in Ref. \[15\], the deviations obtained in the latter two references are overwhelmingly caused by the intermediate \(\omega\) meson. This conclusion is also confirmed by a recent study in Ref. \[18\], which extends the minimal R\(\chi\)T study by including a large amount of higher order odd- and even-intrinsic parity operators from Refs. \[40,41\], respectively.
Although the high-energy behavior constraints help to substantially reduce the unknown couplings associated with the interaction operators, many of them are still undetermined and the authors of Ref. [18] rely on the chiral counting arguments to roughly estimate the order of their magnitudes to proceed with the phenomenological studies. The loosely estimation of the free resonance couplings somewhat hinders the precise phenomenological discussions of the $\tau \rightarrow \pi \pi \gamma \nu_\tau$ decay [18]. In the current work, we try to take a different strategy to present more definite phenomenological discussions. For the odd-intrinsic parity part, we will include the operators with both the $VVP$ and $VJP$ types within the framework of $R\chi T$ [42], being $V$ the vector resonances, $P$ the light pseudo-scalar mesons and $J$ the external sources. In such a way, the additional contributions from the hadronic interaction vertices of the $\rho \pi \gamma$, $\omega \pi \gamma$ and $\omega \rho \pi$ types can be included, compared to the minimal $R\chi T$ calculation in Ref. [15]. For the even parity sector, we will stick to the operators in Eqs. (5) and (6) and refrain from including other types of interaction vertices beyond the minimal $R\chi T$, such as the $a_1 \rho \pi$ ones, which usually introduce many free parameters. Due to the rather broad width of the $a_1$, one does not expect obvious resonance peaks in the $\tau \rightarrow \pi \pi \gamma \nu_\tau$ decays corresponding to the $a_1$ state. To account for the uncertainties of the lack of interaction vertices involving $a_1$ beyond the minimal $R\chi T$, the value of the coupling $F_A$ in Eq. (6) that describes the interaction of the $a_1 \pi \gamma$ will be varied, as shown in details in the phenomenological discussions later. This can be considered as a compensation to estimate the effects of omitting the higher-order hadronic interactions in the even parity sector.

In this work, different from the resonance operator basis used in Ref. [18], we employ the ones proposed in Ref. [42] to include the relevant odd-intrinsic parity operators beyond the minimal ones in the $R\chi T$ framework. The merit is that the resonance Lagrangian in the latter basis has been widely exploited to investigate various physical processes and the relevant resonance couplings are also well determined [35-37], which can provide us valuable inputs for the study of the $\tau \rightarrow \pi \pi \gamma \nu_\tau$ decay. To further take the on-shell approximations of the $VJP$ operators and use the $\omega \rightarrow \pi^0 \pi^0 \gamma$ decay width as an additional input, we are able to make parameter-free predictions for the $\tau \rightarrow \pi \pi \gamma \nu_\tau$ decay. Therefore our study offers complementary results to the phenomenological aspects of the $\tau \rightarrow \pi \pi \gamma \nu_\tau$ process, compared to the ones given in Refs. [15, 18, 38]. Furthermore, another important novelty of our work includes the exploratory discussions of the T-odd asymmetry distributions in the $\tau \rightarrow \pi \pi \gamma \nu_\tau$ process, which can supply an important guide for future experiment measurements.

The odd-intrinsic-parity Lagrangians consist of the $VVP$ and $VJP$ types. Their explicit forms are given by [42]

$$\mathcal{L}_{VVP} = d_1 \varepsilon_{\mu \rho \sigma} \langle \{V^{\mu \nu}, V^{\rho \alpha}\} \nabla_\alpha u^\sigma, + i d_2 \varepsilon_{\mu \rho \sigma} \langle \{V^{\mu \nu}, V^{\rho \alpha}\} \chi_-, + d_3 \varepsilon_{\mu \rho \sigma} \langle \nabla_\alpha V^{\mu \nu}, V^{\rho \alpha}\rangle u^\sigma, + d_4 \varepsilon_{\mu \rho \sigma} \langle \nabla^\sigma V^{\mu \nu}, V^{\rho \alpha}\rangle u_\alpha, \quad (7)$$

and

$$\mathcal{L}_{VJP} = \frac{c_1}{M_V} \varepsilon_{\mu \rho \sigma} \langle \{V^{\mu \nu}, f_+^{\rho \alpha}\} \nabla_\alpha u^\sigma, + \frac{c_2}{M_V} \varepsilon_{\mu \rho \sigma} \langle \{V^{\mu \alpha}, f_+^{\rho \sigma}\} \nabla_\alpha u_\nu, + \frac{i c_3}{M_V} \varepsilon_{\mu \rho \sigma} \langle \{V^{\mu \nu}, f_+^{\rho \alpha}\} \chi_-, + \frac{i c_4}{M_V} \varepsilon_{\mu \rho \sigma} \langle V^{\mu \nu} \nabla_\rho \chi_+, + \frac{c_5}{M_V} \varepsilon_{\mu \rho \sigma} \langle \nabla_\alpha V^{\mu \nu}, f_+^{\rho \alpha}\rangle u^\sigma, + \frac{c_6}{M_V} \varepsilon_{\mu \rho \sigma} \langle \nabla_\alpha V^{\mu \alpha}, f_+^{\rho \sigma}\rangle u_\nu, + \frac{c_7}{M_V} \varepsilon_{\mu \rho \sigma} \langle \nabla^\sigma V^{\mu \nu}, f_+^{\rho \alpha}\rangle u_\alpha. \quad (8)$$
The Feynman diagrams that originate from the minimal R\(\chi\)T Lagrangians in Eqs. \((5)\) and \((6)\) are given in the Appendix of Ref. \([15]\). The relevant Feynman diagrams contributing to the vector and axial-vector form factors can be seen in Figs. \(1\) and \(2\) respectively. We give the explicit expressions of the vector and axial-vector form factors in the Appendix B.

\[
\begin{align*}
\nu_{\tau} & \quad \downarrow \quad \tau^- \\
& \quad \downarrow \quad \pi^{-} \quad \pi^0 \\
& \quad \downarrow \quad \rho^- \\
& \quad \downarrow \quad \omega \\
& \quad \downarrow \quad \gamma \\
(V_a)
\end{align*}
\begin{align*}
\nu_{\tau} & \quad \downarrow \quad \tau^- \\
& \quad \downarrow \quad \pi^{-} \quad \pi^0 \\
& \quad \downarrow \quad \omega \\
& \quad \downarrow \quad \gamma \\
(V_b)
\end{align*}
\begin{align*}
\nu_{\tau} & \quad \downarrow \quad \tau^- \\
& \quad \downarrow \quad \rho^- \\
& \quad \downarrow \quad \omega \\
& \quad \downarrow \quad \gamma \\
(V_c)
\end{align*}

Figure 1: Feynman diagrams contributing to the vector form factors from the resonance chiral Lagrangians in Eqs. \((7)\) and \((8)\).

3 \ T-odd asymmetries in tau decays

The T-odd asymmetries can be constructed from the differential decay widths of the \(\tau \rightarrow \pi\pi\gamma\nu_{\tau}\). After the sum/average of the spins of the particles in the final/initial states in Eq. \((1)\), one can obtain the matrix element squared

\[
\frac{1}{2} \sum_{\text{spins}} |\mathcal{M}|^2 = \hat{M}_0 + \xi \hat{M}_1, \tag{9}
\]

where both \(\hat{M}_0\) and \(\hat{M}_1\) are functions of the scalar products of the momenta of final-state particles, and the T-odd kinematical variable \(\xi\) takes the form

\[
\xi = \varepsilon_{\mu\nu\rho\sigma} \frac{P^\mu k^\nu p_1^\rho p_2^\sigma}{m_\tau^4} \frac{\text{rest frame of } \tau}{\vec{k} \cdot (\vec{p}_1 \times \vec{p}_2) / m_\tau^3}. \tag{10}
\]

It is noted that any even power of \(\xi\) can be expressed in terms of the Lorentz scalar products of the final-state momenta, which further implies that \(\hat{M}_0\) and \(\hat{M}_1\) only depend on \(\xi\) with even powers.

The differential decay width of the \(\tau^- \rightarrow \pi^-\pi^0\gamma\nu_{\tau}\) process with respect to \(\xi\) can be obtained by integrating out other kinematical variables of the invariant amplitude squared in Eq. \((9)\).
The T-odd asymmetry can be defined as

\[ A_\xi = \frac{\Gamma_+ - \Gamma_-}{\Gamma_+ + \Gamma_-}, \]  

(11)

with

\[ \Gamma_+ = \frac{(2\pi)^4}{2m_\tau} \int_{\xi>0} d\Phi (\hat{M}_0 + \xi \hat{M}_1), \quad \Gamma_- = \frac{(2\pi)^4}{2m_\tau} \int_{\xi<0} d\Phi (\hat{M}_0 + \xi \hat{M}_1). \]  

(12)

Clearly, the asymmetry \( A_\xi \) is nonzero only when the function \( \hat{M}_1 \) is nonvanishing. Both the CP conserving final-state strong interactions and the CPV sector can contribute to \( \hat{M}_1 \). One can parameterize the two parts as

\[ \hat{M}_1 = \hat{M}_1^{\text{CP-EVEN}} + \hat{M}_1^{\text{CP-ODD}}, \]  

(13)

which transform under CP conjugation as

\[ \hat{M}_1 \rightarrow \overline{\hat{M}_1} = \hat{M}_1^{\text{CP-EVEN}} - \hat{M}_1^{\text{CP-ODD}}. \]  

(14)

The dominant effects of \( \hat{M}_1^{\text{CP-EVEN}} \) are from the SM and the most significant contributions to \( \hat{M}_1^{\text{CP-ODD}} \) are most likely from the beyond SM sector. One possible way to factor out the different origins of the nonvanishing \( \hat{M}_1 \) is to simultaneously analyze \( A_\xi \) and its counterpart from the charge-conjugate process \( \tau^+ \rightarrow \pi^+\pi^0\gamma\nu_\tau \). We introduce a bar on top of each CP-conjugate quantity throughout. For example, \( \bar{\xi} \), obtained by taking the CP transformation on \( \xi \), changes its sign, due to the fact

\[ \bar{\xi} = (-\vec{k}) \cdot \left[ \left( -\vec{p}_1 \right) \times \left( -\vec{p}_2 \right) \right]/m_\tau^3 = -\xi. \]  

(15)
Similarly one could define the T-odd asymmetry in the charge-conjugate process as
\[ A_\xi = A_\xi - \overline{A_\xi}, \]  
with
\[ \Gamma_+ = \frac{(2\pi)^4}{2m_\tau} \int_{\xi > 0} d\Phi (M_0 + \xi M_1), \quad \Gamma_- = \frac{(2\pi)^4}{2m_\tau} \int_{\xi < 0} d\Phi (M_0 + \xi M_1). \]  

In this work we focus on the asymmetries originated from the kinematical variable of the \( \xi \) term, the CPV effects in the \( \hat{M}_0 \) part can be safely neglected, \( i.e. \), we will take \( \hat{M}_0 = \overline{M}_0 \) throughout. In this case, it is easy to demonstrate that \( \Gamma_+ + \Gamma_- = \Gamma_+ + \Gamma_- \). As mentioned previously, the asymmetries of \( A_\xi \) and \( \overline{A_\xi} \) receive contributions from both the CP-conserving and CPV effects. Then the difference and sum of the asymmetries from the charge-conjugate processes, \( i.e. \),
\[ A_\xi = A_\xi - \overline{A_\xi}, \]  
and
\[ S_\xi = A_\xi + \overline{A_\xi}, \]  
can be used to discern the CPV and CP-conserving dynamics in the \( \tau \to \pi \pi \gamma \nu_\tau \) decays, respectively. This further indicates that if just one charged state of the \( \tau \) is measured, instead of the two processes related with charge conjugations, one needs to subtract the CP-conserving contributions from the T-odd asymmetry \( A_\xi \), in order to determine the CPV strength. As an exploring study, we focus on the T-odd asymmetry caused by the CP-conserving final-state strong interactions in this work. We leave the study of the CPV effects in the \( \xi \) distributions in the \( \tau \to \pi \pi \gamma \nu_\tau \) for a future work. The explicit expressions of \( \hat{M}_0 \) and \( \hat{M}_1 \) can be written in terms of the form factors in Eqs. [2], [3] and [4]
\[ N^{-1} \hat{M}_0 = \sum_f C_{f,a} f_b (Re f_a Re f_b + Im f_a Im f_b) + \sum_{f,l} C_{f,a,l} (Re f_a Re I_l + Im f_a Im I_l) \]  
\[ + \sum_l C_{l,l,j} (Re I_l Re I_j + Im I_l Im I_j), \]  
\[ N^{-1} \hat{M}_1 = \hat{C}_{f,a} f_a (Re f_i Im f_a - Im f_i Re f_a) + \sum_{f,l} \hat{C}_{f,a,l} (Re f_i Im I_l - Im f_i Re I_l) \]  
\[ + \sum_l \hat{C}_{l,l,j} (Re I_l Im I_j - Im I_l Re I_j), \]  
where the normalization factor is \( N = 16m_\tau^2 e^2 G_F^2 V_{ud}^2 \), \( f_{a,b} \) correspond to the pion vector form factors \( f_+(t) \) and \( f_+(u) \) evaluated at different energy variables, \( I_{i,j} \) denote the SD vector and axial-vector form factors \( V_{i=1,2,3,4} \) and \( A_{i=1,2,3,4} \). The remaining terms \( C_{f,a,b}, C_{f,a,l}, C_{l,l,j}, C_{f,a,b}, \hat{C}_{f,a,l}, \hat{C}_{l,l,j} \) are the kinematic factors, which are rather lengthy and are explicitly given in the Appendix. In the \( K \to \pi l \nu_l \gamma \) decay, the bremsstrahlung terms are governed by the \( K \to \pi l \nu_l \) form factors \( f_+ \) and \( f_1 \), both of which are real in the entire physical energy ranges. As a result, the only nonvanishing strong interactions contribute to the T-odd asymmetry \( A_\xi \) through the SD form factors \( V_i \) and \( A_i \), which turn out to be much suppressed, compared.
to the electromagnetic final-state interactions from the photon loops. In contrast, the pion vector form factors entering in the bremsstrahlung terms from the $\tau \rightarrow \pi\pi\nu_\tau\gamma$ process do contain significant nonvanishing imaginary parts. As a result, the final-state strong interactions are expected to provide important contributions to the T-odd asymmetry $A_\xi$ in the radiative two-pion tau decay process, a feature that is rather different from the $K_{\ell 3\gamma}$ decays.

4 Phenomenological discussions

In order to proceed with the phenomenological discussions, we need to first fix the unknown resonance couplings. The high-energy behaviors of the various form factors and Green functions as dictated by QCD, can impose strong constraints to the resonance couplings in the $R\chi T$ relations. Many of the resonance couplings $c_i$ and $d_j$ in Eqs. (8) and (7) have been determined in such a way in Refs. 35, 37, 42. The relevant ones to our study read

$$c_1 + 4c_3 = 0, \quad c_1 - c_2 + c_5 = 0, \quad c_5 - c_6 = \frac{N_CM_V}{64\sqrt{2}\pi^2F_V},$$

$$d_1 + 8d_2 = -\frac{N_eM_V^2}{(8\pi F_V)^2} + \frac{F^2}{4F_V^2}, \quad d_3 = -\frac{N_eM_V^2}{(8\pi F_V)^2} + \frac{F^2}{8F_V^2},$$

with $N_C = 3, F = F_\pi = 0.0924$ GeV, $M_V = 0.775$ GeV. The high energy constraints on the $F_A, F_V$ and $G_V$ are subject to the resonance operators included in the amplitudes and the ones used in Ref. 15 read

$$F_A = F_\pi, \quad F_V = \sqrt{2}F_\pi, \quad G_V = F_\pi/\sqrt{2}. \quad (23)$$

The large $N_C$ study of the partial-wave $\pi\pi$ scattering by including the crossed-channel effects gives an updated KSRF relation $G_V = F_\pi/\sqrt{3}$ [24, 25]. Based on the revised KSRF relation, alternative high energy relations can be obtained [18, 29, 43]

$$F_A = \sqrt{2}F_\pi, \quad F_V = \sqrt{3}F_\pi, \quad G_V = F_\pi/\sqrt{3}. \quad (24)$$

We will take the two sets of high energy constraints in Eqs. (23) and (24) in later study and the uncertainties of such constraints can be considered as compensation of neglecting the higher order terms in the resonance chiral Lagrangian. By taking the above constraints, we are still left with several free parameters that will be fixed by taking the on-shell approximations of the transition vertex involving $\omega\pi$ states. After this procedure, the only remaining unknown parameter is $d_4$ in Eq. (7), and it is found that the $\omega \rightarrow \pi\pi\gamma$ decay width can be used to determine its value. We find two solutions for $d_4$: a negative value of $d_4 = -0.12 \pm 0.05$ and a positive one with $d_4 = 0.82 \pm 0.05$. The detailed discussions can be found in the Appendix B.

In the following numerical discussions, we will take four different combinations of the input resonance couplings, and the deviations resulting from the four situations can be considered as the theoretical uncertainties. In the case of Our-1A, we use the high energy constraints in Eq. (23) and take $d_4 = -0.12$. In the case of Our-2A, we take the same value of $d_4 = -0.12$ as in the situation of Our-1A, but use the alternative constraints in Eq. (24). For the cases of Our-1B and Our-2B, the same value of $d_4 = 0.82$ will be employed and the high energy constraints will be taken from Eq. (23) and Eq. (24), respectively.

In the $\tau\rightarrow \pi^-\pi^0\nu_\tau$ process, there are several types of differential decay widths that are worth in-depth consideration, since they can reveal different kinds of resonance interactions.
For example, the invariant-mass distribution of the $\pi \pi$ system allows one to probe the non-radiative form factor $f_{\pi^+}(t)$, which is dominated by the strong interaction of the $P$-wave $\pi \pi$ system and the $\rho$ resonance. The differential widths of the $\pi^0\gamma$ and $\pi^-\gamma$ provide important environments to study the radiative decay mechanisms of the $\omega$ and $\rho^-$ resonances, respectively. The amplitude of Eq. (1) in our study contains infrared divergence that is caused by the soft photons. A nonvanishing photon-energy cutoff needs to be introduced, which is also usually required by the experimental measurement due to the finite energy resolution of the photon detection. The differential decay width as a function of the photon energy $E_\gamma$ allows us to study the sensitivities of the branching ratios to $E_\gamma$, which can provide important guides for the experimental measurement to implement the proper energy cuts of the photons. In this work, although we use the RAMBO generator [44] to handle the kinematics, the general discussions on the four-body phase space are also provided in the Appendix A.

The dependences of the differential decay widths on the photon energy $E_\gamma$ in the $\tau$ rest frame are given in Fig. 3 where one can clearly see the infrared divergence behavior when $E_\gamma$ approaches to zero. When increasing the photon energy cutoffs, the branching ratios would rapidly decrease, according to Fig. 3. In Table I we give the explicit values of the branching ratios of the $\tau^- \to \pi^-\pi^0\gamma\nu_\tau$ process obtained at several intermediate photon energy cutoffs for all the four scenarios with different resonance parameters as inputs.

![Figure 3: Photon energy distribution in the $\tau^- \to \pi^-\pi^0\gamma\nu_\tau$ process. Notice that the differential width of $\tau^- \to \pi^-\pi^0\gamma\nu_\tau$ is normalized to the decay width of the nonradiative process $\tau^- \to \pi^-\pi^0\nu_\tau$, i.e. $N_0 = 1/\Gamma_{\tau^- \to \pi^-\pi^0\nu_\tau}$. The curve labeled as SI is the result by only taking the inner bremsstrahlung contributions. The curve of CEN stands for the result by taking the same amplitude of Ref. [15]. And our results, labeled as Our-1A, Our-2A, Our-1B and Our-2B, are obtained by using variant resonance couplings as the phenomenological inputs. See the text for details about the meanings of different notations.](image-url)
independent inner bremsstrahlung part when taking small photon energy cutoffs. Decay processes have the good chance to be measured by the Belle-II experiment. \( \tau \rightarrow \pi^{-}\pi^{0}\gamma \nu_{\tau} \) since the total width of the \( \tau \) when taking larger photon energy cutoff at 12. The full results at \( E_{\gamma}^{\text{cut}} = 500 \text{ MeV} \) is for taking the full amplitudes in our study, the second one is obtained by omitting the axial-vector resonance. We distinguish three different situations by separately excluding the effect of the \( \omega \pi \pi \gamma \nu_{\tau} \) vertices and the \( \omega \) resonance. Let’s first analyze the results from the scenarios of Our-1A and Our-2A by taking \( d_{4} = -0.12 \). The full results at \( E_{\gamma}^{\text{cut}} = 100 \) and 300 MeV in these two scenarios are close to the branching ratios predicted in Ref. 15, which are explicitly given in the column labeled as CEN in Table 1. While, our predictions in these two scenarios are different from the result of Ref. 15. The meanings of the notations of Our-1A, Our-2A, Our-1B, Our-2B are explained in the text. For each entry in the last four columns, there are four numbers and they correspond to taking different contributions in the decay amplitudes. The first number is for taking the full amplitudes in our study, the second one is obtained by omitting the SD parts, namely the \( \omega \pi \pi \gamma \nu_{\tau} \) vertices and the axial-vector resonance, i.e. the \( F_{A} \) term of Eq. (3), and the fourth number is derived by neglecting the contributions from both the \( \omega \pi \pi \gamma \nu_{\tau} \) vertices and the axial-vector resonance.

Table 1: Branching ratios of the \( \tau^{-} \rightarrow \pi^{-}\pi^{0}\gamma \nu_{\tau} \) by taking different photon energy cuts \( E_{\gamma}^{\text{cut}} \). All the entries in each column (except the first one) are multiplied by \( 10^{-4} \). The results in the columns labeled by SI and CEN are obtained by taking the structure independent expressions and the full amplitudes from Ref. 15, respectively. The tiny differences between the numbers of Ref. 15 and ours are caused by using the updated physical constants, especially the pion decay constant \( F_{\pi} \). The meanings of the notations of Our-1A, Our-2A, Our-1B, Our-2B are explained in the text. For each entry in the last four columns, there are four numbers and they correspond to taking different contributions in the decay amplitudes. The first number is for taking the full amplitudes in our study, the second one is obtained by omitting the SD parts, namely the \( \omega \pi \pi \gamma \nu_{\tau} \) vertices and the axial-vector resonance, i.e. the \( F_{A} \) term of Eq. (3), and the fourth number is derived by neglecting the contributions from both the \( \omega \pi \pi \gamma \nu_{\tau} \) vertices and the axial-vector resonance.

Let’s first analyze the results from the scenarios of Our-1A and Our-2A by taking \( d_{4} = -0.12 \). The full results at \( E_{\gamma}^{\text{cut}} = 100 \) and 300 MeV in these two scenarios are close to the branching ratios predicted in Ref. 15, which are explicitly given in the column labeled as CEN in Table 1. While, our predictions in these two scenarios are different from the result of Ref. 15, when taking larger photon energy cutoff at \( E_{\gamma}^{\text{cut}} = 500 \) MeV. This is within the expectation, since the total width of the \( \tau^{-} \rightarrow \pi^{-}\pi^{0}\gamma \nu_{\tau} \) process is overwhelmingly dominated by the model-independent inner bremsstrahlung part when taking small photon energy cutoffs \( E_{\gamma}^{\text{cut}} \). The SD contributions in Eqs. (1) and (1) would play more important roles in the \( \tau^{-} \rightarrow \pi^{-}\pi^{0}\gamma \nu_{\tau} \) decay when taking larger photon energy cutoffs. Precisely, the key differences between our study and the one in Ref. 15 are the SD parts, namely the \( v_{i=1,2,3,4} \) and \( a_{i=1,2,3,4} \) form factors in Eqs. (3) and (4). This also indicates that it is helpful to distinguish different hadronic models in the \( \tau^{-} \rightarrow \pi^{-}\pi^{0}\gamma \nu_{\tau} \) process by taking larger photon energy cutoffs, although its branching ratio will be reduced.

In addition, we also further investigate the specific roles played by the intermediate \( \omega \) and \( a_{1} \) resonances. We distinguish three different situations by separately excluding the effect of the \( \omega \pi \pi \gamma \nu_{\tau} \) vertices and the \( a_{1} \) effect, and omitting both of these two contributions. The last three numbers of each entry in the last four columns of Table 1 are derived under the aforementioned three situations, respectively. We only observe mild changes of the four numbers in each entry for the columns Our-1A and Our-2A, indicating that the \( \omega \pi \pi \) interactions and the \( a_{1} \) resonance do not seem to play the decisive roles in the \( \tau \rightarrow \pi \pi \gamma \nu_{\tau} \) process. For the scenarios of Our-1B and Our-2B by using \( d_{4} = 0.82 \), the branching ratios from the full amplitudes, look larger than the results from the scenarios of Our-1A and Our-2A by taking \( d_{4} = -0.12 \). Nevertheless, for a given photon energy cutoff, the orders of magnitudes from different scenarios remain the same. According to the Belle-II estimate 15, around 45 billion pairs of the \( \tau \) leptons will be collected. Hence we anticipate that the radiative two-pion \( \tau \) decay processes have the good chance to be measured by the Belle-II experiment.

Next we discuss several interesting invariant-mass spectra of different two-particle systems.
Figure 4: Invariant-mass distributions of the $\pi\pi$ with $E_\gamma^\text{cut} = 0.3$ GeV. The normalization factor is taken as $N_0 = 1/\Gamma_{\tau^{-}\rightarrow\pi^{-}\pi^{0}\nu_\tau}$. The meanings of different curves are the same as those in Fig. 3.

To be definite, all the two-particle spectra shown below are plotted by taking the cutoff $E_\gamma^\text{cut} = 300$ MeV. The invariant-mass distributions of the $\pi\pi$ system are shown in Fig. 4. We give several different curves in this figure, and the meaning of the labels for the curves is the same as that in Table 1. Comparing with the differences between the SI case and other situations, one can conclude that the bump of the $\rho$ resonance is obviously enhanced in the $\pi\pi$ spectrum when the SD form factors $v_i = 1, 2, 3, 4$ are included. The similarity between the curves of the CEN and Our-1A and Our-2A around the $\rho$ energy region indicates that the inclusion of the odd-intrinsic parity operators in Eqs. (7) and (8) only slightly affects the $\pi\pi$ spectrum in that energy region with $d_4 = -0.12$. Nevertheless, the heights of the curves corresponding to the cases of Our-1B and Our-2B by taking $d_4 = 0.82$ are clearly larger than those of the other cases.

The invariant-mass spectra of the $\pi^{-}\gamma$ and $\pi^{0}\gamma$ systems are given in the left and right panels of Fig. 5, respectively. According to the Feynman diagrams in Figs. 1 and 2, both the narrow $\omega$ and broad $\rho^0$ contribute to the $\pi^{0}\gamma$ distribution, while only the broad $\rho^-$ resonance enters in the $\pi^{-}\gamma$ spectrum. The different resonant contents in the charged $\pi^{-}\gamma$ and neutral $\pi^{0}\gamma$ channels are clearly reflected in their invariant-mass distributions, as shown in Fig. 5. Furthermore, the resonance effects enter the $\pi\gamma$ spectra via the $VVP$ and $VJP$ types of operators, which are absent in the SI and CEN amplitudes. This also explains the smooth curves from the latter two cases in Fig. 5. Similar as the situation in the two-pion distribution in Fig. 4, the heights of the curves from the scenarios of Our-1B and Our-2B are larger than those of other cases. A future experimental measurement on the charged and neutral $\pi\gamma$ distributions will definitely be useful to constrain the anomalous interactions of the $\rho$ and $\omega$ resonances.
Figure 5: Differential decay widths of the $\pi^-\gamma$ (left) and the $\pi^0\gamma$ (right) with $E^\text{cut}_\gamma = 0.3$ GeV. The normalization factor $N_0$ is the inverse of the decay width of $\tau^- \rightarrow \pi^-\pi^0\nu_\tau$, i.e. $N_0 = 1/\Gamma_{\tau^- \rightarrow \pi^-\pi^0\nu_\tau}$. The meanings of different curves are the same as those in Fig. 3.

Figure 6: Distributions of the T-odd asymmetry $A_\xi$ with $E^\text{cut}_\gamma = 0.3$ GeV. The normalization factor is taken as $N_0 = 1/\Gamma_{\tau^- \rightarrow \pi^-\pi^0\nu_\tau}$. The meanings of different curves are the same as those in Fig. 3.
The T-odd asymmetry distributions with respect to the $\xi$ are shown in Fig. 3. The first lesson we can learn from Fig. 3 is that sizable nonvanishing T-odd asymmetry distributions are obtained regardless of taking which amplitudes among the SI, CEN and ours. Nevertheless, it is also clear that different amplitudes or couplings can lead to obviously different curves. On the other hand, this implies that the measurement of the $\xi$ distributions can definitely provide new experimental criteria to discern different hadronic models. To further study the roles of different resonance interactions in the $\xi$ distributions, we distinguish several different situations by excluding/including the effects of the $\omega\rho\pi$ interacting vertices and the $a_1$ resonance in Fig. 4, so that one could discern the relative strengths of the vector (specially the $\omega\rho\pi$ interactions) and axial-vector resonances. Comparing the curves resulting from separately excluding the $\omega\rho\pi$ vertices and the $a_1$ with the full results, though in some cases, such as in Our-1A, the $\xi$ distribution curves are slightly more influenced by the $\omega\rho\pi$ interacting vertices than the $a_1$. Nevertheless, the roles of different resonance interactions are also affected by the input parameters. According to the curves in Fig. 7 loosely speaking the roles of $\omega\rho\pi$ interactions in the $\xi$ distributions seem similar as those of the broad $a_1$ resonance.

In Table 2 we give the integrated rates of the asymmetries shown in Fig. 3 i.e., the values of $A_\xi$. Comparing with the values of $A_\xi \sim 10^{-4}$ for the $K_{l3\gamma}$ decays [5,10,12], the corresponding results of the T-odd asymmetries in the $\tau \rightarrow \pi\pi\gamma\nu_\tau$ are around two orders of magnitudes larger, and hence are very encouraging for the future experimental measurements conducted in Belle-II [45] and the prospective super tau-charm facility [46]. Moreover, different amplitudes with different resonance couplings lead to obviously different shapes of the T-odd asymmetry distributions with respect to the $\xi$ variable. Therefore the precise measurements of the T-odd asymmetry distributions as functions of $\xi$ also provide another quantity to discern different
Table 2: Rates of the T-odd asymmetries with different photon energy cutoffs. All the numbers except those in the first column are multiplied by the factor of $10^{-2}$. The meanings of different notations and the way to obtain the four numbers in each entry in the last four columns are the same as those in Table 1.

| $E_{\text{cut}}$ (MeV) | $A_T$(Our-1A) | $A_T$(Our-2A) | $A_T$(Our-1B) | $A_T$(Our-2B) |
|-------------------------|----------------|----------------|----------------|----------------|
| 100                     | 1.2/1.7/1.0/1.5| 1.4/1.7/1.0/1.3| 1.6/1.7/1.4/1.5| 1.7/1.7/1.4/1.3|
| 300                     | 2.0/3.1/1.5/2.6| 2.4/3.2/1.4/2.1| 2.8/3.1/2.4/2.6| 3.1/3.2/2.3/2.1|
| 500                     | 1.5/2.3/1.0/1.7| 1.8/2.5/0.6/1.3| 2.8/2.3/2.5/1.7| 3.2/2.5/2.4/1.3|

Table 2: Rates of the T-odd asymmetries with different photon energy cutoffs. All the numbers except those in the first column are multiplied by the factor of $10^{-2}$. The meanings of different notations and the way to obtain the four numbers in each entry in the last four columns are the same as those in Table 1.
For a process \( p \to p_1 + p_2 + p_3 + p_4 \), the corresponding phase space integral is

\[
\Phi_4 = \frac{1}{(2\pi)^4} \int \frac{d^3\vec{p}_1}{2E_1} \frac{d^3\vec{p}_2}{2E_2} \frac{d^3\vec{p}_3}{2E_3} \frac{d^3\vec{p}_4}{2E_4} \delta^{(4)}(p - p_1 - p_2 - p_3 - p_4) \]

\[
= \frac{\pi^2}{32(2\pi)^4 m_0^2} \int dS_{12} dS_{123} dS_{34} dS_1 dS_{134} \frac{1}{\sqrt{-\Delta_4(p_1, p_2, p_3, p_4)}},
\]

where

\[
p^2 = m_0^2, p_1^2 = m_1^2, p_2^2 = m_2^2, p_3^2 = m_3^2, p_4^2 = m_4^2,
\]

\[
S_{ij} = (p_i + p_j)^2, S_{ijk} = (p_i + p_j + p_k)^2,
\]

and \( \Delta_4 \) is the Gram determinant \[50\]

\[
\Delta_4(p_1, p_2, p_3, p_4) = \begin{vmatrix} p_1 \cdot p_1 & p_1 \cdot p_2 & p_1 \cdot p_3 & p_1 \cdot p_4 \\ p_2 \cdot p_1 & p_2 \cdot p_2 & p_2 \cdot p_3 & p_2 \cdot p_4 \\ p_3 \cdot p_1 & p_3 \cdot p_2 & p_3 \cdot p_3 & p_3 \cdot p_4 \\ p_4 \cdot p_1 & p_4 \cdot p_2 & p_4 \cdot p_3 & p_4 \cdot p_4 \end{vmatrix}.
\]

The Gram determinant \( \Delta_4 \) can be written as

\[
-\Delta_4 = \hat{a} S_{134}^2 + \hat{b} S_{134} + \hat{c},
\]

with

\[
\hat{a} = -[(S_{123} - S_{12})^2 - 2(S_{123} + S_{12})m_3^2 + m_3^4]/16,
\]

\[
\hat{b} = \{S_{13}[m_0^2(m_3^2 + S_{12} - S_{123}) + S_{12}(m_3^2 - 2m_4^2 - S_{12} + S_{123})] - S_{34}[m_0^2(m_3^2 + S_{12} - S_{12}) + S_{123}(m_3^2 - 2m_1^2 - S_{12} + S_{12})]
\]

\[
+ S_{13} S_{14}(m_3^2 - S_{12} - S_{123}) + S_{12} S_{123}(2m_3^2 - m_1^2 - m_2^2)
\]

\[
+ S_{12}[m_0^2(m_1^2 - m_3^2) - m_3^2(m_3^2 + m_1^2) - m_2^2(m_1^2 + m_3^2 - S_{12})]
\]

\[
+ S_{123}[m_0^2(m_3^2 - m_3^2) - m_0^2(m_3^2 + m_1^2) - m_0^2(m_1^2 + m_3^2 - S_{123})]
\]

\[
+ m_3^2[m_0^2(m_3^2 - m_3^2) + m_2^2(m_3^2 + m_3^2 + 2m_0^2m_2^2 + 2m_1^2m_4^2)]/8,
\]

\[
\hat{b}^2 - 4\hat{a}\hat{c} = (\hat{l} S_{13}^2 + \hat{m} S_{13} + \hat{n}) (\hat{p} S_{34}^2 + \hat{q} S_{34} + \hat{r})/16,
\]

\[
\hat{l} = S_{12},
\]

\[
\hat{m} = S_{12}^2 - (S_{123} + m_3^2 + m_2^2 + m_1^2)S_{12} - (m_3^2 - S_{123})(m_2^2 - m_1^2),
\]

\[
\hat{n}^2 - 4\hat{l} \hat{n} = [S_{12} - (m_2 - m_1)^2][S_{12} - (m_2 + m_1)^2][S_{12} - (m_3 - \sqrt{S_{123}})^2][S_{12} - (m_3 + \sqrt{S_{123}})^2],
\]

\[
\hat{p} = S_{123},
\]

\[
\hat{q} = S_{123}^2 - (S_{12} + m_3^2 + m_0^2 + m_4^2)S_{123} - (m_3^2 - S_{123})(m_0^2 - m_4^2),
\]

\[
\hat{q}^2 - 4\hat{p} \hat{q} = [S_{123} - (m_0 - m_4)^2][S_{123} - (m_0 + m_4)^2][S_{123} - (m_3 - \sqrt{S_{123}})^2][S_{123} - (m_3 + \sqrt{S_{123}})^2].
\]

From the requirements of \(-\Delta_4 \geq 0\) and \(\hat{b}^2 - 4\hat{a}\hat{c} \geq 0\), one can determine the upper and lower limits for the integral variables. Then one can write the phase space integrals in terms of the Lorentz scalar variables

\[
\Phi_4 = \frac{\pi^2}{8(2\pi)^4 m_0^2} \int_{(m_0 - m_3 - m_4)^2}^{(m_0 + m_2)^2} dS_{12} \int_{(m_3 + \sqrt{S_{123}})^2}^{(m_0 - m_4)^2} dS_{123} \int_{S_{34}^+}^{S_{34}^-} dS_{34} \int_{S_{13}^-}^{S_{13}^+} dS_{13} \int_{S_{134}^-}^{S_{134}^+} dS_{134} \sqrt{-\Delta_4},
\]
with

\[ S_{134}^\pm = \frac{-b \pm \sqrt{b^2 - 4\hat{a}\hat{c}}}{2\hat{a}} , \]
\[ S_{13}^\pm = \frac{-\hat{m} \pm \sqrt{\hat{m}^2 - 4\hat{l}\hat{n}}}{2\hat{l}} , \]
\[ S_{34}^\pm = \frac{-\hat{q} \pm \sqrt{\hat{q}^2 - 4\hat{p}\hat{r}}}{2\hat{p}} , \]  

(A.8)

where the definitions of the kinematical variables labeled with hats are given in Eq. (A.6).

The above phase-space integral formulas are valid for general situations with arbitrary masses. For the \( \tau(p) \to \pi^-(p_1)\pi^0(p_2)\nu_\tau(p_3)\gamma(p_4) \) process, we have

\[ m_0^2 = m_\tau^2, \quad m_1 = m_2 = m_\pi^2, \quad m_3 = 0, \quad m_4 = 0 . \]  

(A.9)

By taking the changes of variables [47]

\[ S_{134} = \frac{1}{2\hat{a}} \left[ -b + \sin \left( S_{134} \right) \sqrt{b^2 - 4\hat{a}\hat{c}} \right] , \]
\[ S_{13} = 4\sqrt{-a}S_{13} + m_1^2 , \]  

(A.10)

it is possible to make the \( \frac{1}{\sqrt{-\Delta_{\pi \pi \gamma \nu}(p_1,p_2,p_3,p_4)} } \) factor, which is divergent in the integration boundary, absent in Eq. (A.7). In this way, the four-body phase-space integral of the \( \tau(p) \to \pi^-(p_1)\pi^0(p_2)\nu_\tau(p_3)\gamma(p_4) \) can be cast in a neat form

\[ \Phi_4 = \pi^2 (2\pi)^{-12} \int_{4m_\pi^2}^{m_\tau^2} dS_{12} \int_{S_{12}}^{m_2} dS_{123} \int_0^{(S_{123}-S_{12})(m_\pi^2-S_{123})/S_{123}} dS_{34} \int_{\tilde{S}_{13}}^{\tilde{S}_{13}^+} d\tilde{S}_{13} \int_{-\tilde{S}_{13}}^{\tilde{S}_{13}^+} d\tilde{S}_{13} , \]

with

\[ \tilde{S}_{13}^\pm = \frac{1}{2} \pm \sqrt{\frac{1}{4} - \frac{m_\pi^2}{S_{12}}} . \]  

(A.11)

(B) Form factors and determination of the resonance couplings

Although it is straightforward to calculate the relevant Feynman diagrams using the Lagrangians [5-] [5], additional free parameters will appear in the \( \tau \to \pi\pi\gamma\nu_\tau \), since the high-energy behaviors only impose constraints on some specific combinations of the resonance couplings [22]. This motivates us to apply the on-shell Feynman rules to the anomalous \( J_\omega\pi \) and \( Jp\tau \) contact interacting vertices, being \( J \) the charged or neutral vector source fields. This procedure turns out to be very helpful to reduce the unknown resonance couplings in the \( \tau \to \pi\pi\gamma\nu_\tau \) decay.

The on-shell Feynman rule determined from the Lagrangian [8] for the contact \( \omega\pi\gamma \) vertex takes the form

\[ g_\omega e_{\mu\nu\rho\sigma} k^\mu \]  

(B.1)

where the effective coupling \( g_\omega \) is given by

\[ g_\omega = \frac{\sqrt{2}}{M_V F_V} \left[ (c_2 - c_1 + c_5 - 2c_6)M_\pi^2 + (c_1 + c_2 + 8c_3 - c_5)m_\pi^2 \right] . \]  

(B.2)
According to Ref. [20], the following normalization of the anti-symmetric tensor $\omega_{\mu\nu}$ field

$$<0|\omega_{\mu\nu}|\omega(q)> = \frac{i}{M_{\omega}}(g_{\mu}\epsilon_{\nu} - g_{\nu}\epsilon_{\mu}), \quad (B.3)$$

is used here to derive the on-shell expression of the contact $\omega\pi\gamma$ vertex. The charged anomalous vector current to the $\omega\pi$ transition vertex takes the same form as that in Eq. (B.1), with the obvious replacement of the electric charge by other terms, including the Fermi constant and relevant CKM matrix element, in the definition of the decay amplitude in Eq. (1). Similarly, for the $\rho\pi\gamma$ vertex in the Feynman diagrams of Fig. (2), we also use the on-shell description and it is related to the $\omega\pi\gamma$ one via the $SU(3)$ flavor symmetry, that is $g_{\rho} = g_{\omega}/3$.

With these preparations, we can calculate the Feynman diagrams shown in Figs. 1 and 2 and then determine the various vector and axial-vector form factors in Eqs. (3) and (4). The explicit expressions of form factors $v_1, v_2, v_3, v_4$ are given as follows:

$$v_1^{\text{VVP}} = 2\sqrt{2}g_{\omega}D_\omega[(k + p_2)^2]\left\{-\frac{2\sqrt{2}F_V}{F}D_\rho[(k + p_1 + p_2)^2][4d_3k \cdot p_2 + (d_1 + 8d_2 + 2d_3)m_{\pi}^2 + 2d_3(p_1 \cdot p_2 + k \cdot p_1)] + g_{\omega} - \frac{2F_V}{F}D_\rho[k^2][2d_3k \cdot p_2 + (d_1 + 8d_2)m_{\pi}^2]\right\}, \quad (B.4)$$

$$v_2^{\text{VVP}} = -\frac{2\sqrt{2}g_{\omega}}{M_{\omega}^2}D_\omega[(k + p_2)^2]\left\{-\frac{2\sqrt{2}F_V}{F}D_\rho[(k + p_1 + p_2)^2][4d_3k \cdot p_2 + (d_1 + 8d_2 + 2d_3)m_{\pi}^2 + 2d_3(p_1 \cdot p_2 + k \cdot p_1)] - (d_1 + 8d_2 + d_3 + d_4)m_{\pi}^2(p_1 \cdot p_2 + k \cdot p_1) - 2(d_3 + d_4)k \cdot p_2(p_1 \cdot p_2 + k \cdot p_1) - (d_1 + 8d_2 - d_3 + d_4)m_{\pi}^2(m_{\pi}^2 - M_{\omega}^2 + 2k \cdot p_2) - 2d_3(p_1 \cdot p_2 + k \cdot p_1)^2]\right\}, \quad (B.5)$$

$$v_3^{\text{VVP}} = \frac{2\sqrt{2}g_{\omega}}{M_{\omega}^2}D_\omega[(k + p_2)^2]\left\{-\frac{2\sqrt{2}F_V}{F}D_\rho[(k + p_1 + p_2)^2][4d_3k \cdot p_2 + (d_1 + 8d_2 - d_3 - d_4)m_{\pi}^2 + (d_3 + d_4)M_{\omega}^2 - 2d_3k \cdot p_2]\right\}, \quad (B.6)$$

where the quantity of $D_R$ for the resonance $R$ is defined as

$$D_R(s) = \frac{1}{M_R^2 - s - iM_R\Gamma_R(s)}. \quad (B.7)$$

For the broad $\rho$ resonance, we use the energy dependent decay width [16]

$$\Gamma_\rho(s) = \frac{M_\rho s}{96\pi F_{\pi}^2}\left[\left(1 - \frac{4m_{\pi}^2}{s}\right)^{\frac{3}{2}}\theta(s - 4m_{\pi}^2) + \frac{1}{2}\left(1 - \frac{4m_{K}^2}{t}\right)^{\frac{3}{2}}\theta(s - 4m_{K}^2)\right], \quad (B.8)$$
with \( \theta(x) \) the Heaviside function. For the narrow \( \omega \), we simply take the constant width in Eq. (13.7). For the pion, it is simply given by
\[
D_\pi(s) = \frac{1}{m_\pi^2 - s}.
\]

For the axial-vector form factors, we can obtain the corresponding expressions by evaluating the Feynman diagrams in Fig. 2 and the results read
\[
a_1^{VVP} = \frac{2g_\rho}{F M_V^2} \left\{ 2G_V M_V^2 k \cdot p_2 D_\pi[(k + p_1 + p_2)^2] \{D_\rho[(k + p_1)^2] + D_\rho[(k + p_2)^2]\} 
+ D_\rho[(k + p_1)^2](F_V - 2G_V)(M_V^2 - m_\pi^2 - k \cdot p_2) 
- D_\rho[(k + p_2)^2](F_V - 2G_V)(M_V^2 - m_\pi^2 - k \cdot p_1) \right\},
\]
\[
a_2^{VVP} = \frac{2g_\rho}{F M_V^2} \left\{ -2G_V M_V^2 k \cdot p_1 D_\pi[(k + p_1 + p_2)^2] \{D_\rho[(k + p_1)^2] + D_\rho[(k + p_2)^2]\} 
+ D_\rho[(k + p_1)^2](F_V - 2G_V)(M_V^2 - m_\pi^2 - k \cdot p_2) 
- D_\rho[(k + p_2)^2](F_V - 2G_V)(M_V^2 - m_\pi^2 - k \cdot p_1) \right\},
\]
\[
a_3^{VVP} = \frac{2g_\rho}{F M_V^2} \left\{ -2G_V M_V^2 D_\pi[(k + p_1 + p_2)^2] \{D_\rho[(k + p_1)^2] + D_\rho[(k + p_2)^2]\} 
+ D_\rho[(k + p_1)^2](F_V - 2G_V) \right\},
\]
\[
a_4^{VVP} = \frac{2g_\rho}{F M_V^2} \left\{ -2G_V M_V^2 D_\pi[(k + p_1 + p_2)^2] \{D_\rho[(k + p_1)^2] + D_\rho[(k + p_2)^2]\} 
+ D_\rho[(k + p_2)^2](F_V - 2G_V) \right\}.
\]

The form factors \( a_1, a_2 \) in Ref. [15] can be also given in terms of the new bases introduced in Eq. (10).
\[
a_1^{CEN} = -\frac{1}{8\pi^2 F^2} - \frac{1}{4\pi^2 F^2} D_\pi[(k + p_1 + p_2)^2] k \cdot p_2,
\]
\[
a_2^{CEN} = \frac{1}{8\pi^2 F^2} + \frac{1}{4\pi^2 F^2} D_\pi[(k + p_1 + p_2)^2] k \cdot p_1,
\]
\[
a_3^{CEN} = \frac{1}{4\pi^2 F^2} D_\pi[(k + p_1 + p_2)^2],
\]
\[
a_4^{CEN} = \frac{1}{4\pi^2 F^2} D_\pi[(k + p_1 + p_2)^2].
\]

Finally, we will use \( v_i^{Our} = v_i^{CEN} + v_i^{VVP} \), \( a_i^{Our} = a_i^{CEN} + a_i^{VVP} \) to calculate the various quantities discussed in the Sec. 4. The form factors contributed by the minimal resonance chiral Lagrangians, denoted by \( v_i^{CEN} \) and \( a_i^{CEN} \), have been explicitly given in Ref. [15] and we do not show them again here.

After taking into account the constraints of the high-energy behaviors and the on-shell Feynman rules, there is only one undetermined parameter \( d_4 \), which can be fixed via the \( \omega \to \pi \pi \gamma \) decay width. With the \( VVP \) and \( VJP \) Lagrangians in Eqs. (3) and (7), the only
relevant Feynman diagram to the $\omega \to \pi^0 \pi^0 \gamma$ decay amplitude corresponds to the $\rho^0$ mediated one. Under the on-shell approximation of the $\omega \pi J$ transition vertex, the decay amplitude of the $\omega(q) \to \pi^0(p_1)\pi^0(p_2)\gamma(k)$ process reads

$$T_{\omega \to \pi^0 \pi^0 \gamma} = \frac{2}{F} \left\{ d_1(\epsilon_{\lambda\mu\lambda\sigma}p_{1\nu}p_1^\sigma + \epsilon_{\mu\nu\lambda\sigma}p_{1\delta}p_1^\sigma) + 4d_2m_\pi^2\epsilon_{\mu\nu} \lambda \delta 
+ d_3[\epsilon_{\lambda\mu\lambda\sigma}(k+p_2)_\nu p_1^\sigma - \epsilon_{\mu\nu\lambda\sigma}q_\delta p_1^\sigma] + d_4[\epsilon_{\lambda\mu\lambda\sigma}(k+p_2)^\nu p_1^\sigma - \epsilon_{\mu\nu\lambda\sigma}q^\sigma p_1^\delta] \right\}$$

$$D^\lambda_{\mu\nu\rho\sigma}(k+p_2, M_\omega^2) \epsilon_{\lambda\mu\rho\sigma}(k) \frac{q^\mu\epsilon^\nu(q) - q^\nu\epsilon^\mu(q)}{M_\omega} + \left(p_1 \leftrightarrow p_2\right), \quad (B.14)$$

where $D^\lambda_{\mu\nu\rho\sigma}(k, M_\omega^2)$ is defined as

$$D^\lambda_{\rho}(k, M_\omega^2) = \frac{1}{M_\omega^2 D_{\rho}(k^2)} \left[g^{\mu\rho} g^{\nu\sigma} (M_\omega^2 - k^2) + g^{\mu\rho} k^\nu k^\sigma - g^{\mu\sigma} k^\nu k^\rho - (\mu \leftrightarrow \nu) \right], \quad (B.15)$$

and the propagator $D_{\rho}(t)$ with the energy dependent width is given in Eq. (B.8).

It is then straightforward to calculate the $\omega \to \pi^0 \pi^0 \gamma$ decay width

$$\Gamma_{\omega \to \pi^0 \pi^0 \gamma} = \int_{s-}^{s+} ds \int_{t-}^{t+} dt \frac{1}{(2\pi)^3} \frac{1}{32M_\omega^3} \frac{1}{3} |T_{\omega \to \pi^0 \pi^0 \gamma}|^2, \quad (B.16)$$

where the standard Mandelstam variables are defined as

$$s = (p_1 + p_2)^2, \quad t = (k + p_2)^2, \quad (B.17)$$

and the upper and lower limits of the integrals are

$$t_\pm = \frac{M_\omega^2 + 2m_\pi^2 - s}{2} \pm \sqrt{s^2 - 4m_\pi^2 s(M_\omega^2 - s)} \frac{2}{2s}. \quad (B.18)$$

By using the PDG result of the decay width $[51]$

$$\Gamma_{\omega \to \pi^0 \pi^0 \gamma}^{Exp} = (5.8 \pm 1.0) \times 10^{-5} \text{ MeV}, \quad (B.19)$$

we can determine the values of $d_4$ and two distinct solutions are found

$$d_4 = -0.12 \pm 0.05, \quad (B.20)$$

The impact of both solutions on the $\tau \to \pi \pi \gamma \nu_\tau$ is studied in the phenomenological discussions.

### C Kinematical factors in the invariant amplitudes squared

In Eqs. (20) and (21), we have introduced several types of kinematical factors in different parts of the amplitude squared $|M|^2$. During the calculation, the FeynCalc package $[52]$ is used to crosscheck the following formulas. Here we give the explicit expressions of the various factors for completeness.
First, the relevant Lorentz scalars constructed by the momenta of the particles in the $\tau^{-}(P) \to \pi^{-}(p_1)\pi^{0}(p_2)\nu_{\tau}(q)\gamma(k)$ process are defined as follows

\[ P^2 = m_0^2, \quad q^2 = 0, \quad k^2 = 0, \quad P \cdot p_1 = d m_0^2, \quad P \cdot p_2 = c m_0^2, \]
\[ P \cdot q = f m_0^2, \quad P \cdot k = g m_0^2, \quad p_1 \cdot p_2 = h m_0^2, \quad p_1 \cdot q = j m_0^2, \]
\[ p_1 \cdot k = n m_0^2, \quad p_2 \cdot q = l m_0^2, \quad p_2 \cdot k = m m_0^2, \quad q \cdot k = w m_0^2. \] (C.1)

The effect of isospin violation is ignored in this paper, so we have $p_1^2 = r_1 m_0^2 = p_2^2 = r_2 m_0^2 = r m_0^2$. The kinematical factors introduced in Eqs. (20) and (21) are

\[ C_{f_1 f_1} = \frac{(2d^2 g(j - l)(n + m) + d(-2g^2(j - l)(2h + r_1 + r_2))}{(4g^2(n + m)^2)}, \] (C.2)

\[ C_{f_2 f_2} = \frac{(2d(2h j n m + h n^2 w - 2h l m + h n m w + 2j n m n - j n^2 r_2 + 2j n m^2 - j m^2 r_1 - n^3 l + n^3 w - 2n^2 l m + n^2 l r_2 + 2n^2 m w - n l m^2 + n m^2 w - n m r_1 w + l m^2 r_1 w) + n m(-2h l + h w - r_1 w) - 2n^2 l m + n^2 l r_2 + 2n^2 m w - n m r_1 w + l m^2 r_1 w)}{(2n^2(n + m)^2)}, \] (C.3)
\[ C_{ft} = \frac{1}{4}(2m_{0}^{2} + 4m_{0} + 2m_{0} + 2m_{0} - 2m_{0} - 2m_{0} - 2m_{0} + 2m_{0}) \]

\[ \hat{C}_{ft} = \frac{1}{4}(2m_{0}^{2} + 4m_{0} + 2m_{0} + 2m_{0} - 2m_{0} - 2m_{0} - 2m_{0} + 2m_{0}) \]
\[
\begin{align*}
\check{C}_{V_1} &= -(n + m)m_0^4, \\
\check{C}_{V_2} &= -(hn - n(m + r_2))m_0^6, \\
\check{C}_{V_3} &= (hn + n^2 - mr_1)m_0^6, \\
C_{fa_{Ii}} &= C^F_{fa_{Ii}} + C^{SI}_{fa_{Ii}}, \quad \check{C}_{fa_{Ii}} = \check{C}^F_{fa_{Ii}} + \check{C}^{SI}_{fa_{Ii}}
\end{align*}
\]  

(7)

with \( a = t, u \) and \( I_{i=1,2,3,4} = V_{i=1,2,3,4}, A_{i=1,2,3,4} \). The explicit expressions for these factors are

\[
\begin{align*}
\check{C}^F_{f_1V_1} &= -(4d + g + n + m + w)m_0^2/(2g), \\
\check{C}^F_{f_2V_2} &= -(4c + g + n + m + w)m_0^2/(2g), \\
\check{C}^F_{f_3V_3} &= (-dm + cn + h(n + m) - 2jn + jm + nl - nr_1 - mr_1)m_0^4/(2g), \\
\check{C}^F_{f_4V_4} &= ((2d - 2c + g)(m - n) + n(-2j - n - r_1 + r_2 - w) \\
&\quad + m(m - r_1 + r_2 + w + 2l))m_0^4/(2g),
\end{align*}
\]  

(9)

\[
\begin{align*}
C^F_{f_1V_1} &= -(2d^2w + d(-2cw - 2fn + 2fm + 2g(j - l)) \\
&\quad - j(4gn - 2gm + 2n) - 2ghw + 2gnl + 2gr_1w + 2nl)m_0^2/(2g), \\
C^F_{f_2V_2} &= -(2d + 2fn - 2fm - 2gj + 2gl) \\
&\quad + 2c^2w + 2g(-hw + jm + nl - 2lm + r_2w) + 2jm - 2lm)m_0^2/(2g), \\
C^F_{f_3V_3} &= (4d^2jm - 2d^2lm + c(-2dj(2n + m) + 2dnl + 2fn(r_1 - h) \\
&\quad + (n - m)(2jn - r_1w)) + 2dfhm - 2dfmr_1 - 2dhnw - 2djm \\
&\quad + 2djm^2 + dm_r^2w + dm_r^1w + 2c^2jn + 2fhnm - fn^2r_2 - fm^2r_1 \\
&\quad + g(-2hjn + n(jr_2 + lr_1) + mr_1(j - l))m_0^4/(2g), \\
C^F_{f_4V_4} &= (2d^2m(2j + w) - d(2c(2jn + w(n + m) + 2lm) + 2fnm - 2fm^2 \\
&\quad + 2fmr_1 - 2fmr_2 + 2gm(l - j) + 2hnw + 2jnm - 2jm^2 + 2nml \\
&\quad - 2nr_2w - 2lm^2 - mr_1w + mr_2w) + 2c^2n(2l + w) \\
&\quad + (2fn(n - m + r_1 - r_2) + 2gn(l - j) - 2hnw + 2jn^2 - 2jnm \\
&\quad + 2n^2l - 2nlm - nr_1w + nr_2w + 2m_1w) + 4fhnm - 2fn^2r_2 - 2fm^2r_1 \\
&\quad - 2ghjn - 2ghnw - 2ghlm - 2ghnw - 2gnm + 2gjm^2 + gjmr_1 + gjmr_2 \\
&\quad + 2gn^2l - 2glnm + gntr_1 + gntr_2 + 2gm_2w + 2gm_1w)m_0^4/(2g),
\end{align*}
\]  

(10)

\[
\begin{align*}
\check{C}^{SI}_{f_1V_1} &= 2(h + n + r_1)m_0^2/(n + m), \\
\check{C}^{SI}_{f_2V_2} &= 2(h + n + r_2)m_0^2/(n + m), \\
\check{C}^{SI}_{f_3V_3} &= -(h(n - m) + nr_2 - mr_1)m_0^4/(n + m), \\
\check{C}^{SI}_{f_4V_4} &= -(hn + n^2 - nm - 2mr_1)m_0^2/(n^2 + nm), \\
\check{C}^{SI}_{f_1V_1} &= (2m(h + m) + n^2 + n(m - 2r_2))m_0^2/(n(n + m)), \\
\check{C}^{SI}_{f_2V_2} &= -(hn(n + 3m) + n^3 + n^2(m - r_2) - nmr_1 - 2m^2r_1)m_0^4/(n(n + m)), \\
\check{C}^{SI}_{f_3V_3} &= -(hn(n + 3m) + n^3 + n^2(m - r_2) - nmr_1 - 2m^2r_1)m_0^4/(n(n + m)),
\end{align*}
\]  

(11)
\[ C_{f_{1V_1}}^{SI} = (dhw - 2djn + drw - chw + 2ml - crw) + f(-hn + hm - nr + mr) + g(h + r_1)(j - l) m_0^2/(n + m), \]
\[ C_{f_{1V_2}}^{SI} = (dhw - 2dm + drw - chw + 2clm - crw) + f(-hn + hm - nr + mr) + g(h + r_2)(j - l) m_0^2/(n + m), \]
\[ C_{f_{1V_3}}^{SI} = (h(n - m) + nr - mr)(2dj - dl - cj + f(h - r_1)) m_0^4/(n + m), \]
\[ C_{f_{1V_4}}^{SI} = (d(2j + w) - 2cl - cw - fn + f'm - fr_1 + fr_2 + g(j - l)) \]
\[ \times (h(n - m) + nr - mr)m_0^4/(n + m), \] (C.12)

\[ C_{f_{1V_1}}^{SI} = (d(-hn + jm + n^2l - 2n^2w + nlm - 2nmw + mrw) - 2fhnm \]
\[ + c(hn + jm(n + m) - 2n^2l - mrw) + f(n^2r_2 + f'mr_1 + g(hn(l - j)) \]
\[ + j(-2n^2 - 2nm + mr) + 2r_1 w(n + m - lmr_1)m_0^2/(n + m)), \]
\[ C_{f_{1V_2}}^{SI} = -(dhnm + 2dm^2 - dlm + dnmw + dnrw - dlm^2 + dm^2 w \]
\[ + chnw - cjm + cjm^2 + cm^2w + 2cmw + cmw - cmw \]
\[ + f(2hnw + n^2(-r_2) - m^2r_1) + g(n(-2hw + jm + r_2) + lm - lr_2) \]
\[ + m(h(-j + l - 2w) + jm) + n^2l) m_0^2/(n + m)), \]
\[ C_{f_{1V_3}}^{SI} = (4dhjnm + dhnmw + 2dm^2m - 2dm^2r_2 \]
\[ + 2dnm^2 - 2djm^2r_1 - dnl^2 + dln^2r_2 - dlnmrw \]
\[ + dln^2r_1 - dln^2r_2 - cj(2hnw + n^2 + 2n^2(r_2) - m^2r_1) \]
\[ + 2fhnmw - 2fhnmr_1 - fhnmr_2 + f'mr_1 + f'mr_2 \]
\[ + f'mr_1^2 + g(n + m)(hn - mr_1)m_0^4/(n + m)), \]
\[ C_{f_{1V_4}}^{SI} = (4dhjnm + dhnmw + 2dnm^2 + 2dnm^2 \]
\[ - 2dnm^2r_1 - dnl^2 + dln^2w - dln^2r_2 + dlnmr_1 \]
\[ - dlnmr_1 - 2dln^2r_2 - c(n^2(2l(m - r_2) - w(h - m) + r_2)) \]
\[ + n(h(4l + w) + r_1w) + j(n^3 - nml) + n^3(2l + w) - 2lmr_1) \]
\[ - 4fhn^2m - 2fhnr_1 + 2fhmr_2 + f'mr_1 + f'mr_2 + f'mr_1 \]
\[ + 2fhn^2r_1 + f'mr_1^2 - f'mr_1r_2 + g(hn(j(n + 3m) + nl + 2nw \]
\[ - lm + 2nw) + n^3(j - l)(m - r_2) + n(m(j - r_2) - r_1(l + 2w) \]
\[ - 2m^2r_1(j + w) + n^3(-l)) m_0^4/(n + m)), \] (C.13)

\[ \tilde{C}_{f_{1A_1}}^{F} = -(d - c - f - g)m_0^2/g, \]
\[ \tilde{C}_{f_{1A_2}}^{F} = (d - c + f + g)m_0^2/g, \]
\[ \tilde{C}_{f_{1A_3}}^{F} = -m_0^4/(2g) \times (4dj - 2dl + dm - 2cj - cn + 2f(h - r_1) \]
\[ + g(h - r_1) - hw + 2jn + jm + nl + r_1w), \]
\[ \tilde{C}_{f_{1A_4}}^{F} = m_0^4/(2g) \times (2dl - dm - 2cj + cn + 4cl + 2f(h - r_2) \]
\[ + g(h - r_2) - hw + jm + nl - 2lm + r_2w), \] (C.14)
\[ C_{f,A_1}^F = -wd^2 + f(n - m)d + (gj + cw)d - cgj - 2gjn - jn + gnl + gjm + jm + fg(h - r_1) + w(-gh - h + gr_1 + r_1)m_0^2/g, \]
\[ C_{f,A_2}^F = -(w - c)^2 + gic + f(m - n)c + dcw - dgl + gnl + nl + gjm - 2glm - lm + fg(h - r_2) - ghw - hw + gr_2w + rw_2m_0^2/g, \]
\[ C_{f,A_3}^F = -(2mr_1w + nr_2w - 2ghjn + 2hn + gjmr_1 + gjmr_1 - glmr_1 - glmr_1 - 2glm_1 + gijnr_2 - 2ijnr_2 - f(-2gh^2 + 2cnh + 2dmh - 2nmh + m_2^2r_1 - 2cmr_1 + n^2r_2 - 2dmr_2 + 2gr_1r_2) - 2h^2w + 2r_1r_2w + 2c^2(jn - r_1w) + c(2dlm + 2gr_1 - 2hwj + n + bj = 3, 4) + j(2m^2 - 2mn + 2dm))m_0^4/(2g), \]
\[ C_{f,A_4}^F = -(2mr_2w + 2hn + 2hjmn + glmr_1 + 2mr_1 - gjmr_2 - 2ijnr_2 + glmr_2 + gijnr_2 - f(-2gh + 2cnh + 2dmh - 2nmh + m_2^2r_1 - 2cmr_1 + n^2r_2 - 2dmr_2 + 2gr_1r_2) - 2h^2w + 2r_1r_2w + 2c^2(jn - r_1w) + c(2ln^2 - 2dlm - 2ln + 2hwj + r_2w + 2ghj - 2djm + 2glnr_1 + 4hw + m_0^4w))m_0^4/(2g), \]

\[(C.15)\]

\[ \tilde{C}_{f,A_1}^{SI} = -\tilde{C}_{f,A_2}^{SI} = -\tilde{C}_{f,A_1}^{SI} = (d - c + j - l)m_0^2/(n + m), \]
\[ \tilde{C}_{f,A_2}^{SI} = -(dm - cm + g(n + m) + jm + nw - lm + mw)m_0^2/(n(n + m)), \]
\[ \tilde{C}_{f,A_3}^{SI} = -(d + j)m_0^4, \]
\[ \tilde{C}_{f,A_4}^{SI} = -(c + l)m_0^4, \]

\[(C.16)\]

\[ C_{f,A_1}^{SI} = -C_{f,A_2}^{SI} = (dhw - nl + lm - r_2w) + c(hw + jm - jm - r_1w) + g(-h(j + l) + jr_2 + lr_1)m_0^2/(n + m), \]
\[ C_{f,A_2}^{SI} = (dw(-h - 2m + r_2) + dn(l - 2w) - dlm + cw(r_1 - h) + cj(m - n) + gh(j + l) + g(2n + 2m - r_2) - glmr_1)m_0^2/(n + m), \]
\[ C_{f,A_3}^{SI} = m_0^2(d(nlm - mw) - m(h + m - r_2) + lm) - c(m(nh - r_1) + n^2 + nm) + jm(n - m)) + g(m(h(j + l) + jm - jr_2 - lr_1) + n(mj + ml + nl)))/(n^2 + nm), \]
\[ C_{f,A_4}^{SI} = m_0^4(dhw - dnl + cjm - cr_1w - ghj + glr_1), \]
\[ C_{f,A_4}^{SI} = m_0^4(-dlm + dr_2w - chw + cjm + ghj - gjr_2), \]

\[(C.17)\]
\begin{align}
\dot{C}_{V_1 A_2} &= -\dot{C}_{V_2 A_1} = m_0^4 (g + w), \\
\dot{C}_{V_1 A_3} &= -\dot{C}_{V_3 A_1} = m_0^6 n (d + j), \\
\dot{C}_{V_2 A_3} &= -\dot{C}_{V_1 A_2} = m_0^6 m (d + j), \\
\dot{C}_{V_1 A_4} &= m_0^6 n (c + l), \\
\dot{C}_{V_2 A_4} &= m_0^6 m (c + l), \\
\dot{C}_{V_4 A_1} &= -n (d + c + g + j + l + w) m_0^6, \\
\dot{C}_{V_4 A_2} &= m (d + c + g + j + l + w) m_0^6, \quad \text{(C.18)}
\end{align}

\begin{align}
C_{V_1 A_1} &= -2 n m_0^4 (g j - d w), \\
C_{V_2 A_2} &= -2 m m_0^4 (g l - c w), \\
C_{V_1 A_2} &= -m_0^4 (-d m w - c n w + g j m + g n l), \\
C_{V_2 A_1} &= -m_0^4 (-d m w - c n w + g j m + g n l), \\
C_{V_3 A_1} &= n m_0^6 (-d h w + d n l - c j n + c r_1 w + g h j - g l r_1), \\
C_{V_3 A_2} &= m m_0^6 (-d h w + d n l - c j n + c r_1 w + g h j - g l r_1), \\
C_{V_1 A_4} &= n m_0^6 (d l m - d r_2 w + c h w - c j m - g h l + g j r_2), \\
C_{V_2 A_4} &= m m_0^6 (d l m - d r_2 w + c h w - c j m - g h l + g j r_2), \\
C_{V_4 A_1} &= n m_0^6 (d (-w (h + m + r_2) + n l + l m) + c w (h + n + r_1) \\
&\quad -c j (n + m) + g (h (j - l) + j m + j r_2 - n l - l r_1)), \\
C_{V_4 A_2} &= m m_0^6 (d (-w (h + m + r_2) + n l + l m) + c w (h + n + r_1) \\
&\quad -c j (n + m) + g (h (j - l) + j m + j r_2 - n l - l r_1)), \quad \text{(C.19)}
\end{align}

\begin{align}
\dot{C}_{A_1 A_2} &= -m_0^4 (n + m), \\
\dot{C}_{A_1 A_3} &= m_0^6 (h n + n^2 - r_1 m), \\
\dot{C}_{A_1 A_4} &= m_0^6 (m (n + r_1) - h n), \\
\dot{C}_{A_2 A_3} &= m_0^6 (n (m + r_2) - h m), \\
\dot{C}_{A_2 A_4} &= m_0^6 (h m - r_2 n + m^2), \\
\dot{C}_{A_3 A_4} &= m_0^8 (-2 h n m + r_2 n^2 + m^2 r_1), \quad \text{(C.20)}
\end{align}
\[C_A, \frac{A_1}{A_2} = m_0^8 (dnw + gjn - gr_1 w),\]
\[C_A, \frac{A_2}{A_3} = m_0^8 (cmw + glm - gr_2 w),\]
\[C_A, \frac{A_3}{A_4} = (fr_1 - 2d)(-2hm + r_2 n^2 + m^2 r_1) m_0^8 / 2,\]
\[C_A, \frac{A_4}{A_1} = (fr_2 - 2e)(-2hm + r_2 n^2 + m^2 r_1) m_0^8 / 2,\]
\[C_A, \frac{A_1}{A_2} = m_0^8 (d(-hnw - 2jnm + n^2 l + m r_1 w) + cjo^2 + g (jmr_1 - hjn)),\]
\[C_A, \frac{A_2}{A_3} = (fr_1 - 2d)(-2hm + r_2 n^2 + m^2 r_1) m_0^8 / 2,\]
\[C_A, \frac{A_3}{A_4} = (fr_2 - 2e)(-2hm + r_2 n^2 + m^2 r_1) m_0^8 / 2,\]
\[C_A, \frac{A_1}{A_2} = m_0^8 (d(-hnw - 2jnm + n^2 l + m r_1 w) + cjo^2 + g (jmr_1 - hjn)),\]
\[C_A, \frac{A_2}{A_3} = (fr_1 - 2d)(-2hm + r_2 n^2 + m^2 r_1) m_0^8 / 2,\]
\[C_A, \frac{A_3}{A_4} = (fr_2 - 2e)(-2hm + r_2 n^2 + m^2 r_1) m_0^8 / 2.\]

(C.21)

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