COXETER POLYNOMIALS OF SALEM TREES

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Abstract. We explicitly calculate the Coxeter polynomial of a family of Salem trees and find the limit of the spectral radius of their Coxeter transformations. We also prove a relation about multiplicities of eigenvalues of Coxeter transformations of joins of trees.

1. Introduction

In [7] Piroska Lakatos proved a result about the spectral radius of Coxeter transformations of noncyclotomic starlike trees (which she called wild stars). Let $S^{(0)}_{p_1, \ldots, p_k}$ denote the wild star consisting of $k$ paths of length $p_1, \ldots, p_k$ and one branching point. Lakatos [7] proved that the limit of the spectral radius of the Coxeter transformations of $S^{(0)}_{p_1, \ldots, p_k}$ as $p_1, \ldots, p_k \to \infty$ is $k - 1$. The aim of this paper is to generalize that result to the case where instead of $k$ paths we have $i$ Dynkin diagrams of type $D_k$ and $k - i$ paths. In addition our line of proof is different from the one in [7].

In this paper we consider simple graphs (i.e. graphs without multiple edges and loops) which we call (simply laced) Coxeter graphs. The set of vertices of the graph $\Gamma$ will be denoted by $V(\Gamma) = \{v_1, v_2, \ldots, v_n\}$ and the edges of $\Gamma$ by $E(\Gamma)$, where $(v_i, v_j) \in E(\Gamma) \iff$ the vertices $v_i, v_j$ are connected with an edge. The adjacency matrix of $\Gamma$ is the $n \times n$ symmetric 0-1 matrix $A$, with $A_{i,j} = 1$ if and only if $(v_i, v_j) \in E(V)$. The characteristic polynomial $\chi_{\Gamma}(x)$ of $\Gamma$ is that of $A$.

Let $V$ be an $n$-dimensional real vector space with basis $\{e_1, \ldots, e_n\}$. The Coxeter reflections $\sigma_i$ are the transformations defined by

$$\sigma_i(e_j) = e_j - (2\delta_{i,j} - A_{j,i})e_i,$$

where $\delta_{i,j}$ is the Kronecker delta. The group $W$ generated by the Coxeter reflections $\{\sigma_1, \sigma_2, \ldots, \sigma_n\}$ is known as the Coxeter group of $\Gamma$ and it has presentation

$$W = \langle \sigma_1, \sigma_2, \ldots, \sigma_n : (\sigma_i\sigma_j)^{M_{i,j}} = 1 \rangle,$$

where $M_{i,j}$ is the Coxeter matrix defined by $M_{i,i} = 1$ for all $i = 1, 2, \ldots, n$ and $M_{i,j} = A_{i,j} + 2$ for all $i \neq j$. The product of a permutation of the generators $\sigma_i$ is of particular interest and is called Coxeter transformation or Coxeter element. These elements were first studied by Coxeter in [3] for the finite reflection groups, where he showed that their eigenvalues have remarkable properties.

In this paper we will be concerned only with Coxeter graphs that are trees. This class of graphs has the property that all their Coxeter elements are conjugate in $W$ (see [2]) and therefore we can speak about the Coxeter polynomial of $\Gamma$ which is the characteristic polynomial of a Coxeter element. We will denote this polynomial by $\Gamma(x)$. Another important property of trees is that they are bipartite, i.e. the set of

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their vertices, \(V(\Gamma)\), can be partitioned into two subsets \(V_1, V_2\) with the property \(v_1, v_2 \in V_1\) or \(v_1, v_2 \in V_2\) \(\implies (v_1, v_2) \not\in E(\Gamma)\).

Let \(p(x)\) be a monic polynomial with integer coefficients. We denote the set of its zeros \(\{z \in \mathbb{C} : p(z) = 0\}\) by \(Z(p)\) and the maximum value of the set \(\{|z| : z \in Z(p)\}\) by \(\rho(p)\). The reciprocal polynomial of \(p(x)\) is the polynomial \(p^*(x) = x^n p \left( \frac{1}{x} \right)\), where \(n = \deg(p)\). The polynomial \(p(x)\) is called reciprocal polynomial if it is its own reciprocal. With the polynomial \(p(x)\) we associate the polynomial \(f(x) = x^n p \left( x + \frac{1}{x} \right)\), which is reciprocal. The sets \(Z(p)\) and \(Z(f)\) are related in the following way. Suppose \(r \in Z(p)\) is real and let \(z = \frac{r + \sqrt{r^2 - 4}}{2}\). Then \(z \in Z(f)\) and if \(|r| \leq 2 \implies |z| = 1\) while if \(|r| > 2 \implies z \in \mathbb{R}\) and \(|z| > 1\). Suppose now that the polynomial \(p(x)\) is irreducible. If \(\rho(p) = 1\) then the polynomial \(p\) is called cyclotomic. If \(\rho(p)\) is the only zero of \(p(x)\) with modulus larger than 1, then if \(\frac{1}{\rho(p)} \not\in Z(p)\) it is called Pisot polynomial. The corresponding values \(\rho(p)\) are called Salem and Pisot numbers respectively. It is not so hard to see that cyclotomic and Salem polynomials are reciprocal. A Coxeter graph \(\Gamma\) is called cyclotomic graph if all the roots of its Coxeter polynomial are on the unit disk or equivalently its Coxeter polynomial is a product of cyclotomic polynomials. It is called a Salem graph if its Coxeter polynomial has only one root outside the unit circle or equivalently its Coxeter polynomial is a product of a Salem and cyclotomic polynomials.

![Figure 1. The graphs \(S_{p_1, \ldots, p_k}^{(i)}\)](image)

For \(k\) Coxeter graphs \(\Gamma_1, \Gamma_2, \ldots, \Gamma_k\) their join on the vertices \(v_i \in V(\Gamma_i)\) is the graph obtained by adding a new vertex and joining that to \(v_i\) for all \(i = 1, 2, \ldots, k\). In [8] it was shown that if a noncyclotomic tree is the join of cyclotomic trees then it is a Salem tree. For \(k \in \mathbb{N}, p_1, \ldots, p_k \in \mathbb{N}\) and \(i \in \{0, 1, 2, \ldots, k\}\) consider the graph \(S_{p_1, \ldots, p_k}^{(i)}\) which is the join of the Dynkin diagrams \(D_{p_1}, \ldots, D_{p_i}\) and \(A_{p_{i+1}}, \ldots, A_{p_k}\), as shown in fig. 1. For particular values of \(i, p_i\) the graphs \(S_{p_1, \ldots, p_k}^{(i)}\) give rise to well known graphs. For \(k = 2, i = 0\) we obtain the Dynkin diagrams \(A_{p_1+p_2+1}\), for \(k = 3, i = 0, p_1 = 1, p_2 = 2\) we obtain the graphs \(E_{p_3+4}\), for \(k = 3, i = 0, p_1 = 2\) the affine Dynkin diagram \(E_6^{(i)}\), for \(k = 3, i = 1, p_2 = p_3 = 1\) the affine Dynkin diagrams \(D_{p_1+2}^{(i)}\) and many others. The polynomial \(S_{1,2,6}^{(0)}(x)\) is the well known Lehmer’s polynomial which is conjectured to have the smallest Mahler measure among the monic integer polynomials (see for instance [9]). In this paper we will mainly concerned with the case \(k = 3\) and prove three theorems about the Coxeter polynomials \(S_{p,q,r}^{(i)}(x)\). In theorem 3 we explicitly calculate the Coxeter polynomials \(S_{p,q,r}^{(i)}(x)\) for \(i = 0, 1, 2, 3\). In theorem 4 we show that the limits
For the case of the Salem trees. It follows from their characterization that the cyclotomic cases of Theorem 3.

\[
\lim_{\rho \to \infty} \rho \left( S_{p,q,r}^{(i)} \right), \lim_{\rho \to \infty} \rho \left( S_{p,q,r}^{(i)} \right) \text{ and } \lim_{\rho \to \infty} \rho \left( S_{p,q,r}^{(i)} \right) \text{ are Pisot numbers and also that } \lim_{\rho \to \infty} \rho \left( S_{p,q,r}^{(i)} \right) = 2 \text{ for all } i = 0, 1, 2, 3. \text{ Pirosla Lakatos in [7] showed that for } k \in \mathbb{N}, \lim_{p_1,\ldots,p_k \to \infty} \rho \left( S_{p_1,\ldots,p_k}^{(i)} \right) = k - 1. \text{ In theorem E we generalize that result by showing that for all } i \in \{0, 1, \ldots, k\} \text{ the } \lim_{p_1,\ldots,p_k \to \infty} \rho \left( S_{p_1,\ldots,p_k}^{(i)} \right) = k - 1.

**Theorem 1.** For \( i \leq 2 \) the Coxeter polynomial \( S_{p,q,r}^{(i)}(x) \) of the Coxeter graph \( S_{p,q,r}^{(i)} \), is given by the formula

\[
\frac{x - 1}{x + 1} S_{p,q,r}^{(i)}(x) = x^{r+2} F_{p,q}^{(i)}(x) - \left( F_{p,q}^{(i)} \right)^* (x),
\]

where the polynomials \( F_{p,q}^{(i)} \) are

\[
F_{p,q}^{(0)}(x) = x^{p+q} - A_{p-1}(x)A_{q-1}(x),
\]

\[
F_{p,q}^{(1)}(x) = x^{p+q-2}(x - 1) - (x^{p-2} + 1)A_{q-1}(x) \text{ and }
\]

\[
F_{p,q}^{(2)}(x) = x^{p+q-4}(x - 1)^2 - (x^{p-2} + 1)(x^{q-2} + 1).
\]

The Coxeter polynomial \( S_{p,q,r}^{(3)}(x) \) is given by the formula

\[
\frac{1}{x + 1} S_{p,q,r}^{(3)}(x) = x^r F_{p,q}^{(3)}(x) + \left( F_{p,q}^{(3)} \right)^* (x),
\]

with \( F_{p,q}^{(3)}(x) = F_{p,q}^{(2)}(x) \).

**Theorem 2.** The spectral radius \( \rho \left( S_{p,q,r}^{(i)} \right) \), of the Coxeter transformation of \( S_{p,q,r}^{(i)} \), satisfies

1. \( \lim_{\rho \to \infty} \rho \left( S_{p,q,r}^{(i)} \right) = \rho \left( F_{p,q}^{(i)} \right) \text{ and } \rho \left( F_{p,q}^{(i)} \right) \text{ is a Pisot number for } i = 0, 1, 2, \]
2. \( \lim_{\rho \to \infty} \rho \left( S_{p,q,r}^{(i)} \right) = \rho \left( F_{p,q}^{(i-1)} \right) \text{ for } i = 1, 2, 3, \]
3. \( \lim_{\rho \to \infty} \rho \left( S_{p,q,r}^{(i)} \right) = \rho \left( x^{r+2} - 2x^{r+1} + 1 \right) \text{ for } i = 0, 1, 2, \]
4. \( \lim_{\rho \to \infty} \rho \left( S_{p,q,r}^{(i)} \right) = \rho \left( x^p - 2x^{p-1} - 1 \right) \text{ for } i = 1, 2, 3 \text{ and } \]
5. \( \lim_{\rho \to \infty} \rho \left( S_{p,q,r}^{(i)} \right) = 2 \text{ for all } i = 0, 1, 2, 3. \)

**Theorem 3.** For \( k, p_1, \ldots, p_k \in \mathbb{N} \) and all \( i \in \{0, 1, \ldots, k\} \)

\[
\lim_{p_1,\ldots,p_k \to \infty} \rho \left( S_{p_1,\ldots,p_k}^{(i)} \right) = k - 1.
\]

**Remark 1.** James McKee and Chris Smyth in [8] gave a characterization of all Salem trees. It follows from their characterization that the cyclotomic cases of \( S_{p_1,\ldots,p_k}^{(i)} \) are those for \( k = i = 2 \) or \( k = 3, i = 0, p_1 = p_2 = p_3 = 2 \) or \( k = 3, i = 0, p_1 = 1, p_2 = p_3 = 3 \) or \( k = 3, i = 0, p_1 = 1, p_2 = 2, p_3 = 5 \) and subgraphs of these. For all the other cases, \( S_{p_1,\ldots,p_k}^{(i)} \) are Salem trees.

**Example 1.** For the case of the Dynkin diagrams, theorem 4 gives

\[
D_n(x) = S_{1,1,n-3}^{(0)}(x) = \frac{1}{x - 1} (x^{n-1}(x^2 - 1) + x^2 - 1) = x^n + x^{n+1} + x + 1.
\]
For the case of the $D_n^{(1)}$ affine Dynkin diagrams theorem 4 gives

$$D_n^{(1)}(x) = S_{n-2,1,1}^{(1)}(x) = \frac{x+1}{x-1} \left(x^3(x^{n-2} - x^{n-3} - x^{n-4} - 1) + x^{n-2} + x^2 + x - 1\right) = \left(x^{n-2} - 1\right)(x-1)(x+1)^2.$$  

For the case of the $E_n$ diagrams it gives

$$E_n(x) = S_{1,2,n-4}^{(0)}(x) = \frac{1}{x-1} \left(x^n-2(x^3 - x - 1) + x^3 + x^2 - 1\right).$$

All these agree with the known formulas of Coxeter polynomials of the Dynkin and $E_n$ diagrams given in [4] and [5].

We will also prove the following theorem concerning joins of Coxeter graphs.

**Theorem 4.** Let $\Gamma$ be the join of the Coxeter graphs $\Gamma_i$, $i = 1, 2, \ldots, n$. Suppose that $z$ is a zero of $\Gamma_i(x)$ with multiplicity $m_i$. Then $z$ is a zero of the Coxeter polynomial $\Gamma(x)$ with multiplicity at least

$$\min\{m - m_i : i = 1, 2, \ldots, n\}$$

where $m = m_1 + m_2 + \ldots + m_n$.

Theorem 4 generalizes a theorem due to V.F. Kolmykov (see [10]) which says that if $\Gamma$ is the join of the Coxeter graphs $\Gamma_1, \Gamma_2$ and $z$ is a root of the Coxeter polynomials $\Gamma_1(x)$ and $\Gamma_2(x)$ then $\Gamma(z) = 0$. According to [11] if $z \neq \pm 1$ then $m_i \in \{0, 1\}$ and therefore in that case theorem 4 can be found in [5] where the authors have proved that $z$ is a root of $\Gamma(x)$ with multiplicity at least $m - 1$. For $z = \pm 1$ however, $z$ can be a zero of $\Gamma(x)$ with multiplicity less than $m - 1$. For example consider the join $\Gamma$ of the affine Dynkin diagrams $D_4^{(1)}$ as shown in fig. 2. The polynomials $\Gamma(x)$ and $D_4^{(1)}(x)$ both have 1 as a zero with multiplicity 2.

![Figure 2. The join of two $D_4^{(1)}$ graphs](image)

For the convenience of the reader we include all theorems that will be used, in several cases with proofs, thus making this paper self-contained. This is done in section 2. In section 3 we prove the theorems 1 to 4.

## 2. Preliminaries

In this section we state and prove some results that we will need for the proof of theorems 1, 2, 3 and 4. The following proposition is due to Subbotin-Sumin and the proof we present here is from [10].

**Proposition 1.** Let $e = (v_1, v_2) \in E(\Gamma)$ be a splitting edge of the Coxeter graph $\Gamma$ that splits it to the Coxeter graphs $\Gamma_1$ and $\Gamma_2$. Assume that $v_1 \in V(\Gamma_1)$ and $v_2 \in V(\Gamma_2)$. Then

$$\Gamma(x) = \Gamma_1(x)\Gamma_2(x) - x\tilde{\Gamma}_1(x)\tilde{\Gamma}_2(x)$$

where $\tilde{\Gamma}_i$ denotes the subgraph of $\Gamma_i$ with vertex set $V(\Gamma_i) \setminus \{v_i\}$.  


Proof. We enumerate the vertices of $\Gamma$ as $V(\Gamma_1) = \{u_1, u_2, \ldots , u_k\}$ and $V(\Gamma_2) = \{v_{k+1}, v_{k+2}, \ldots , v_{k+m}\}$, where $v_1 = u_k$ and $v_2 = u_{k+1}$. Let $\hat{e} = \hat{e}_1 \cup \hat{e}_2$ be a basis for the vector space $V$, where $\hat{e}_1 = \{e_1, e_2, \ldots , e_k\}$ is a basis of $V_1$ and $\hat{e}_2 = \{e_{k+1}, e_{k+2}, \ldots , e_{k+m}\}$ is a basis of $V_2$. Also let $\sigma_i$ be the Coxeter reflections corresponding to $u_i$. Therefore $R_1 = \sigma_1 \sigma_2 \ldots \sigma_k$ is a Coxeter transformation of $\Gamma_1$, $R_2 = \sigma_{k+1} \sigma_{k+2} \ldots \sigma_{k+m}$ is a Coxeter transformation of $\Gamma_2$ and $R_1 R_2$ is a Coxeter transformation of $\Gamma$. If we represent $R_1$, $R_2$ and $R$ as matrices with respect to the basis $\hat{e}$ we have
\[
R = R_1 R_2 = \begin{pmatrix} Q_1 & E_{k,1} & I_k & 0_{k, m} \\ 0_{m,k} & I_m & E_{1,k} & Q_2 \end{pmatrix},
\]
where $Q_i$ are the transformations $R_i$ restricted to $V_i$, $E_{i,j}$ is the matrix with all entries zero except the $i,j$ entry which is $1$ and $0_{i,j}$ is the $i \times j$ zero matrix. The Coxeter polynomial of $\Gamma$ is then given by
\[
\Gamma(x) = \det(R - xI_{k+m}) = \det \begin{pmatrix} Q_1 + E_{k,k} - xI_k & E_{k,1}Q_2 \\ E_{1,k} & Q_2 - xI_m \end{pmatrix}.
\]
Subtracting the $k + 1$th row from the $k$th row we obtain
\[
\Gamma(x) = \det \begin{pmatrix} Q_1 - xI_k \\ E_{1,k} \\ xE_{k,1} \\ Q_2 - xI_m \end{pmatrix}.
\]
Expanding the determinant with respect to the $k$th row we deduce that
\[
\Gamma(x) = \Gamma_1(x) \Gamma_2(x) - x \Gamma_1(x) \hat{\Gamma}_2(x).
\]

The following lemma says that the eigenvalues of a bipartite graph are symmetric around 0.

**Lemma 1.** If $\lambda$ be an eigenvalue of the adjacency matrix $A$ of $\Gamma$ then $-\lambda$ is an eigenvalue of $A$.

**Proof.** Enumerate the vertices of $\Gamma$ such that its adjacency matrix has the form
\[
A = \begin{pmatrix} 0 & B \\ B^T & 0 \end{pmatrix}.
\]
Suppose that $\begin{pmatrix} x \\ y \end{pmatrix}$ is an eigenvector of $A$ with eigenvalue $\lambda$. Then $\begin{pmatrix} -x \\ y \end{pmatrix}$ is an eigenvector of $A$ with eigenvalue $-\lambda$. \hfill \Box

The following proposition can be found in [1].

**Proposition 2.** The Characteristic polynomial $\chi_\Gamma(x)$ of the graph $\Gamma$ and the Coxeter polynomial $\Gamma(x)$ are related in the following way
\[
\Gamma(x) = x^2 \chi_\Gamma \left( \sqrt{x} + \frac{1}{\sqrt{x}} \right),
\]
where $n$ is the degree of $\chi_\Gamma(x)$.

**Proof.** Let $V(\Gamma) = \{v_1, v_2, \ldots , v_k, v_{k+1}, \ldots v_{k+m}\}$ be the vertices of $\Gamma$ enumerated such that $i, j \leq k$ or $i, j > k \Rightarrow (v_i, v_j) \notin \mathcal{E}(\Gamma)$. Let $\sigma_i$ be the Coxeter reflections associated to $v_i$, i.e. if $\hat{e} = \{e_1, e_2, \ldots , e_{k+m}\}$ is a basis of the vector space $V$ then $\sigma_i(e_j) = e_j - (2\delta_{i,j} - A_{i,i})e_i$. Then with respect to the basis $\hat{e}$ the Coxeter reflection $\sigma_i$ is given by the matrix where its $i$th row is the $i$th row of the matrix $A - I$ and its $j$th row is the $j$th row of the identity matrix $I$. We see at once that $\sigma_i^2 = I$ for
all $i$ and that for $i,j \leq k$ or $i,j > k \Rightarrow \sigma_i \sigma_j = \sigma_i + \sigma_j - I$. Therefore we obtain the following relations

$$\sigma_1 \sigma_2 \ldots \sigma_k = \sigma_1 + \sigma_2 + \ldots + \sigma_k - (k - 1)I,$$

$$\sigma_{k+1} \sigma_{k+2} \ldots \sigma_{k+m} = \sigma_{k+1} + \sigma_{k+2} + \ldots + \sigma_{k+m} - (m - 1)I.$$ 

Let’s denote by $C_1$ the transformation $\sigma_1 \sigma_2 \ldots \sigma_k$ and by $C_2$ the transformation $\sigma_{k+1} \sigma_{k+2} \ldots \sigma_{k+m}$. It follows that $C_1^2 = I$, $C_2^2 = I$ and $C_1 + C_2 = A$. We thus get

$$2I + C_1 C_2 + C_2 C_1 = (C_1 + C_2)^2 = A^2.$$ 

If $e^{z_1}, e^{z_2}, \ldots$ are the roots of the Coxeter polynomial of the graph $\Gamma$ then $2 + e^{z_1} + e^{-z_1}, 2 + e^{z_2} + e^{-z_2}, \ldots$ are the roots of $\chi_A^2(x)$. Since $\Gamma$ is bipartite the polynomial $\chi_\Gamma(x)$ is of the form

$$\chi_\Gamma(x) = (x - r_1)(x - r_1^2)(x - r_2) \ldots = (x^2 - r_1^2)(x^2 - r_2^2)\ldots$$

where $r_i^2 = 2 + e^{z_i} + e^{-z_i}$. It follows that

$$x^2 \chi_\Gamma \left( \frac{\sqrt{x} + 1}{\sqrt{x} - 1} \right) = (x^2 + (2 - r_1^2) x + 1) \left( x^2 + (2 - r_2^2) x + 1 \right) \ldots =$$

$$(x^2 - (e^{z_1} + e^{-z_1}) x + 1) \left( x^2 - (e^{z_2} + e^{-z_2}) x + 1 \right) \ldots =$$

$$(x - e^{z_1}) (x - e^{-z_1}) \cdots (x - e^{z_k}) \cdots = \Gamma(x).$$

We immediately get the following corollary.

**Corollary 1.** The Coxeter polynomial of $\Gamma$ is reciprocal.

The next lemma is due to Hoffman and Smith (see [6]).

**Lemma 2.** If $k, p_1, \ldots, p_k \in \mathbb{N}$, $0 \leq i \leq k$ and $p_j' > p_j$ for some $1 \leq j \leq k$ then

1. $\rho \left( S_{p_1, \ldots, p_j, \ldots, p_k}^{(i)} \right) \leq \rho \left( S_{p_1, \ldots, p_j', \ldots, p_k}^{(i)} \right)$ if $j > i$ and
2. $\rho \left( S_{p_1, \ldots, p_j, \ldots, p_k}^{(i)} \right) \geq \rho \left( S_{p_1, \ldots, p_j', \ldots, p_k}^{(i)} \right)$ if $j \leq i$.

Equality can happen if and only if the graph $S_{p_1, \ldots, p_j', \ldots, p_k}^{(i)}$ is cyclotomic.

We will also need the following lemma.

**Lemma 3.** Suppose that $f_n(x) = x^g(x) + h(x)$ is a sequence of functions such that $g, h$ are continuous, for all $n \in \mathbb{N}$ $f_n(z_n) = 0$ and that $\lim_{n \to \infty} z_n = z_0$. If $|z_0| > 1$ then $g(z_0) = 0$ while if $|z_0| < 1$ then $h(z_0) = 0$.

**Proof.** Suppose that $|z_0| > 1$. Since $h$ is continuous and $|g(z_n)| = \left| \frac{h(z_n)}{z_n} \right|$ it follows that $\lim_{n \to \infty} |g(z_n)| = 0$. Using $|g(z_0)| - |g(z_n)| \leq |g(z_0) - g(z_n)|$ we conclude that $g(z_0) = 0$. The proof for the case $|z_0| < 1$ is similar. \(\square\)

3. **Proof of Theorems**

In this section we prove the theorems [4] to [4].

**Proof of theorem [4]** For simplicity of notation, we will write $u_j, v_j, w_j$ instead of $v_{1,j}, v_{2,j}, v_{3,j}$ respectively. Applying proposition [1] to the splitting edge $(t, u_1)$ of the graph $S_{p,q,r}^{(0)}$ we see that

$$S_{p,q,r}(x) = A_p(x)A_{q+r+1}(x) - xA_{p-1}(x)A_q(x)A_r(x).$$

The Coxeter polynomial $A_m(x)$ can be easily calculated using proposition [4]. It satisfies the recurrence

$$A_m(x) = A_{m-1}(x) + x(A_{m-1}(x) - A_{m-2}(x))$$
and is given by the formula \( A_n(x) = x^n + x^{n-1} + \ldots + x + 1 \). Therefore
\[
(x - 1)^3 S_{p,q,r}^{(0)}(x) = x^{p+q+r+4} - 2x^{p+q+r+3} + x^{p+r+2} + x^{q+r+2} - x^{r+2} + x^{p+q+r+2} - x^{p+r+2} + 2x - 1 \Rightarrow
\]
\[
(x - 1)^2 S_{p,q,r}^{(0)}(x) = x^{p+r+2}(x - 1) - x^{r+2}(x^q - 1)A_{p-1}(x) + x^2A_{p-1}(x)A_{q-1}(x) - 1 = x^{r+2}F_{p,q}^{(0)}(x) - \left( F_{p,q}^{(0)}(x) \right)^* \cdot (x).
\]
The proof for \( i = 1, 2 \) is done similarly by applying proposition \( 3 \) to the splitting edges \((u_{p-2}, u_p)\) and using the formula for \( S_{p,q,r}^{(i-1)}(x) \).

For the Coxeter polynomial \( S_{p,q,r}^{(3)} \) we apply proposition \( 3 \) to the splitting edge \((w_{r-2}, w_r)\) to obtain
\[
S_{p,q,r}^{(3)}(x) = (x + 1)S_{p,q,r-1}^{(2)} - x(x + 1)S_{p,q,r-3}^{(2)}.
\]
Therefore
\[
\frac{x - 1}{(x + 1)^3} S_{p,q,r}^{(3)}(x) = \frac{x - 1}{(x + 1)^2} S_{p,q,r-1}^{(2)} - \frac{x - 1}{(x + 1)^2} S_{p,q,r-3}^{(2)} = x^{r+1}F_{p,q}^{(2)}(x) - \left( F_{p,q}^{(2)}(x) \right)^* (x) - x^rF_{p,q}^{(2)}(x) + x \left( F_{p,q}^{(2)}(x) \right)^* (x) \Rightarrow
\]
\[
\frac{1}{(x + 1)^3} S_{p,q,r}^{(3)}(x) = x^rF_{p,q}^{(2)}(x) + \left( F_{p,q}^{(2)}(x) \right)^* (x).
\]

**Remark 2.** For the case \( i = 1 \) we could have applied proposition \( 4 \) to the splitting edge \((u_{p-2}, u_p)\) and use that \( S_{p,q,r}^{(0)} = S_{q,r,p}^{(0)} \) to obtain
\[
\frac{1}{(x + 1)} S_{p,q,r}^{(1)}(x) = x^pF_{q,r}^{(0)}(x) + \left( F_{q,r}^{(0)}(x) \right)^* (x).
\]
Similarly by noting that \( q, r \) are interchangeable in \( S_{p,q,r}^{(1)} \) and \( p, q \) are interchangeable in \( S_{p,q,r}^{(2)} \), proposition \( 4 \) applied to the splitting edge \((v_{q-2}, v_q)\) gives
\[
\frac{1}{(x + 1)^2} S_{p,q,r}^{(2)}(x) = x^pF_{q,r}^{(1)}(x) + \left( F_{q,r}^{(1)}(x) \right)^* (x).
\]

**Proof of theorem 2.** From theorem 1 and lemma 8, we see that in order to prove (i), it is enough to show that the sequence \( \rho(S_{p,q,r}^{(i)}) \) is convergent. From lemma 8, it follows that for \( i = 0, 1, 2 \) the sequence \( \rho(S_{p,q,r}^{(i)}) \) is increasing. Since the polynomial \( S_{p,q,r}^{(i)}(x) = x^{r+2}F(x) + G(x) \) where \( F, G \) are monic polynomials, the sequence \( \rho(S_{p,q,r}^{(i)}) \) is also bounded. For, if \( M \) is large enough such that the polynomials \( F, G \) are positive for all \( x \geq M \), then \( z < M \) for all \( z \in Z \cdot S_{p,q,r}^{(i)} \).

We now prove that \( \rho(F_{p,q}^{(i)}) \) is a Pisot number (cf. lemma 4.3 in [8]). Let \( \epsilon > 0 \) be small enough and \( r \) be large enough such that \( \rho(S_{p,q,r}^{(i)}) > 1 + \epsilon \) and \( |x^{r+2}F_{p,q}^{(i)}(x)| > \left| F_{p,q}^{(i)}(x) \right|^* (x) \) for every \( |x| = 1 + \epsilon \). From Rouche’s theorem it follows that the polynomial \( F_{p,q}^{(i)}(x) \) has only one root, let’s say \( z_0 \), outside the unit circle. If \( z_0 \) was a Salem number then we would have \( F^*(z_0) = 0 \Rightarrow S_{p,q,r}(z_0) = 0 \).
We conclude that \( \lim_{V} \) where the polynomial easily get similar formulas when for all \( i \) where \( 2 \) we deduce from lemma 3 that \( \lim_{p \to \infty} S_{p,q,r}^{(0)}(x) = x \). Therefore \( \lim_{p \to \infty} S_{p,q,r}^{(0)}(x) = x \). The other cases of (iii) and (iv) are done similarly.

It remains to prove (v). Let’s denote by \( \ell_p \) the \( \lim_{q,r} \rho \left( S_{p,q,r}^{(0)}(x) \right) \). The polynomial \( H(x) = x^{p+2} - 2x^{p+1} + 1 \) is decreasing in \( \left( 1, \frac{2p+2}{p+2} \right) \), increasing in \( \left( \frac{2p+2}{p+2}, 2 \right) \), \( H(1) = 0 \) and \( H(2) = 1 \). Therefore the only root of \( H \) outside the unit circle is \( \ell_p \) and it satisfies \( \frac{2p+2}{p+2} < \ell_p < 2 \). We conclude that \( \lim_{p,q,r} \rho \left( S_{p,q,r}^{(0)}(x) \right) = 2 \). The cases \( i = 1, 2, 3 \) are similar.

**Proof of Theorem 3.** Proposition 3 applied to the splitting edge \((t, v_{k,1})\) gives

\[
(x - 1)S_{p_1,\ldots,p_k}^{(i)}(x) = x^{p_k+1}F(x) - F^*(x)
\]

where

\[
F(x) = S_{p_1,\ldots,p_{k-1}}^{(i)}(x) - D_{p_1}(x)\ldots D_{p_k}(x)A_{p_{i+1}}(x)\ldots A_{p_{k-1}}(x),
\]

for all \( i \in \{0, 1, \ldots, k - 1\} \). Therefore \( \lim_{p_k} \rho \left( S_{p_1,\ldots,p_k}^{(i)}(x) \right) = \rho(F(x)) \). We easily get similar formulas when \( i = k \) and inductively we see that

\[
\lim_{p_2,\ldots,p_k} \rho \left( S_{p_1,\ldots,p_k}^{(i)}(x) \right) = \rho(G(x))
\]

where the polynomial \( G(x) \) is given by

\[
G(x) = \begin{cases} 
x^{p_1} - (k - 1)x^{p_1-1} - k + 2, & \text{if } i \neq 0, 
x^{p_1+1} - (k - 1)x^{p_1} + k - 2, & \text{if } i = 0.
\end{cases}
\]

We conclude that \( \lim_{p_1,p_2,\ldots,p_k} \rho \left( S_{p_1,\ldots,p_k}^{(i)}(x) \right) = k - 1 \).

**Proof of Theorem 4.** Let’s denote by \( \Gamma_{(k)} \) the join of the graphs \( \Gamma_i \) at the vertices \( v_i \in V(\Gamma_i) \), \( i = 1, 2, \ldots, k \). The graph \( \Gamma_{(n)} \) looks like the one in fig. 3. Applying

![Figure 3. The join of the graphs \( \Gamma_i \)](image-url)
proposition to the splitting edge \((t, v_n)\) we obtain

\[
\Gamma_n(x) = \Gamma_{n-1}(x)\Gamma_n(x) - x\Gamma_1(x)\Gamma_2(x)\ldots\Gamma_{n-1}(x)\tilde{\Gamma}_n(x),
\]

where with \(\tilde{\Gamma}_k\) we denote the subgraph of \(\Gamma_k\) with vertices \(V(\Gamma_k) \setminus \{v_k\}\). Inductively we get

\[
\Gamma_n(x) = (x + 1)\Gamma_1(x)\Gamma_2(x)\ldots\Gamma_n(x) - x(P_1(x) + \ldots + P_n(x)),
\]

where \(P_k\) is the polynomial \(\Gamma_1(x)\ldots\tilde{\Gamma}_k(x)\ldots\Gamma_n(x)\). The theorem follows. \(\square\)

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