MULTIPLE POSITIVE SOLUTIONS OF A \((p_1, p_2)\)-LAPLACIAN SYSTEM WITH NONLINEAR BCS

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Abstract. Using the theory of fixed point index, we discuss existence, non-existence, localization and multiplicity of positive solutions for a \((p_1, p_2)\)-Laplacian system with nonlinear Robin and/or Dirichlet type boundary conditions. We give an example to illustrate our theory.

1. Introduction

In the remarkable paper [39] Wang proved the existence of one positive solution of following one-dimensional \(p\)-Laplacian equation

\[
(\varphi_p(u'))'(t) + g(t)f(u(t)) = 0, \ t \in (0, 1),
\]
subject to one of the following three pair of nonlinear boundary conditions (BCs)

\[
\begin{align*}
    u'(0) &= 0, \ u(1) + B_1(u'(1)) = 0, \\
    u(0) &= B_0(u'(0)), \ u'(1) = 0, \\
    u(0) &= B_0(u'(0)), \ u(1) + B_1(u'(1)) = 0.
\end{align*}
\]

The results of [39] were extended by Karakostas [23] to the context of deviated arguments. In both cases, the existence results are obtained via a careful study of an associated integral operator combined with the use of the Krasnosel’skii-Guo Theorem on cone compressions and cone expansions.

The Krasnosel’skii-Guo Theorem, more in general, topological methods are a commonly used tool in the study of existence of positive solutions for the \(p\)-Laplacian equation (1.1) subject to different BCs. This is an active area of research, for example, homogeneous Dirichlet BCs have been studied in [1, 5, 16, 25, 31, 37, 43, 47], homogeneous Robin BCs in [31, 43, 47], nonlocal BCs of Dirichlet type in [3, 4, 6, 7, 9, 14, 24, 39, 41, 48] and nonlocal BCs of Robin type in [14, 30, 32, 40, 42, 48].

Here we study the the one-dimensional \((p_1, p_2)\)-Laplacian system

\[
\begin{align*}
    (\varphi_{p_1}(u'))'(t) + g_1(t)f_1(t, u(t), v(t)) &= 0, \ t \in (0, 1), \\
    (\varphi_{p_2}(v'))'(t) + g_2(t)f_2(t, u(t), v(t)) &= 0, \ t \in (0, 1),
\end{align*}
\]

with \(\varphi_{p_i}(w) = |w|^{p_i-2}w\), subject to the nonlinear boundary conditions (BCs)

\[
\begin{align*}
    u'(0) &= 0, \ u(1) + B_1(u'(1)) = 0, \\
    v(0) &= B_2(v'(0)), \ v(1) = 0.
\end{align*}
\]

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The existence of positive solutions of systems of equations of the type (1.2) has been widely studied, see for example [8, 28, 29, 44] under homogeneous Dirichlet BCs and [16, 22, 34, 38, 46] with homogeneous Robin or Neumann BCs. For earlier contributions on problems with nonlinear BCs we refer to [11, 12, 15, 17, 18, 20, 23, 30, 33, 39] and references therein.

We improve and complement the previous results in several directions: we obtain multiplicity results for \((p_1, p_2)\)-Laplacian system subject to nonlinear BCs, we allow different growths in the nonlinearities \(f_1\) and \(f_2\) and we also discuss non-existence results. Finally we illustrate in an example that all the constants that occur in our results can be computed.

Our approach is to seek solutions of the system (1.2)-(1.3) as fixed points of a suitable integral operator. We make use of the classical fixed point index theory and benefit of ideas from the papers [19, 21, 23, 39].

2. The system of integral equations

We recall that a cone \(K\) in a Banach space \(X\) is a closed convex set such that \(\lambda x \in K\) for \(x \in K\) and \(\lambda \geq 0\) and \(K \cap (-K) = \{0\}\). If \(\Omega\) is a open bounded subset of a cone \(K\) (in the relative topology) we denote by \(\overline{\Omega}\) and \(\partial \Omega\) the closure and the boundary relative to \(K\). When \(\Omega\) is an open bounded subset of \(X\) we write \(\Omega_K = \Omega \cap K\), an open subset of \(K\).

The following Lemma summarizes some classical results regarding the fixed point index, for more details see [2, 13].

**Lemma 2.1.** Let \(\Omega\) be an open bounded set with \(0 \in \Omega_K\) and \(\overline{\Omega}_K \neq K\). Assume that \(F: \overline{\Omega}_K \to K\) is a compact map such that \(x \neq Fx\) for all \(x \in \partial \Omega_K\). Then the fixed point index \(i_K(F, \Omega_K)\) has the following properties.

1. If there exists \(e \in K \setminus \{0\}\) such that \(x \neq Fx + \lambda e\) for all \(x \in \partial \Omega_K\) and all \(\lambda > 0\), then \(i_K(F, \Omega_K) = 0\).
2. If \(\mu x \neq Fx\) for all \(x \in \partial \Omega_K\) and for every \(\mu \geq 1\), then \(i_K(F, \Omega_K) = 1\).
3. If \(i_K(F, \Omega_K) \neq 0\), then \(F\) has a fixed point in \(\Omega_K\).
4. Let \(\Omega^1\) be open in \(X\) with \(\overline{\Omega^1} \subset \Omega_K\). If \(i_K(F, \Omega_K) = 1\) and \(i_K(F, \Omega^1_K) = 0\), then \(F\) has a fixed point in \(\Omega_K \setminus \overline{\Omega^1}_K\). The same result holds if \(i_K(F, \Omega_K) = 0\) and \(i_K(F, \Omega^1_K) = 1\).

To the system (1.2)-(1.3) we associate the following system of integral equations, which is constructed in similar manner as in [39], where the case of a single equation is studied.

\[
\begin{align*}
u(t) &= \int_t^1 \varphi_{p_1}^{-1} \left( \int_0^s g_1(\tau)f_1(\tau, u(\tau), v(\tau))\,d\tau \right)\,ds \\
&\quad + B_1 \left( \varphi_{p_1}^{-1} \left( \int_0^1 g_1(\tau)f_1(\tau, u(\tau), v(\tau))\,d\tau \right) \right), \quad 0 \leq t \leq 1, \\
\end{align*}
\]

\[
\begin{align*}
v(t) &= \begin{cases} 
\int_t^1 \varphi_{p_1}^{-1} \left( \int_0^{s_{u,v}} g_2(\tau)f_2(\tau, u(\tau), v(\tau))\,d\tau \right)\,ds \\
&\quad + B_2 \left( \varphi_{p_2}^{-1} \left( \int_0^1 g_2(\tau)f_2(\tau, u(\tau), v(\tau))\,d\tau \right) \right), \quad 0 \leq t \leq \sigma_{u,v}, \\
\int_0^1 \varphi_{p_2}^{-1} \left( \int_0^s g_2(\tau)f_2(\tau, u(\tau), v(\tau))\,d\tau \right)\,ds, & \sigma_{u,v} \leq t \leq 1,
\end{cases}
\end{align*}
\]
where \( \varphi^{-1}_{p_1}(w) = \frac{1}{w^{p_1-1}} \text{sgn} w \) and \( \sigma_{u,v} \) is the smallest solution \( x \in [0, 1] \) of the equation

\[
\int_0^x \varphi^{-1}_{p_1} \left( \int_0^x g_2(\tau) f_2(\tau, u(\tau), v(\tau)) d\tau \right) d\tau + B_2 \left( \varphi^{-1}_{p_2} \left( \int_0^x g_2(\tau) f_2(\tau, u(\tau), v(\tau)) d\tau \right) \right)
= \int_x^1 \varphi^{-1}_{p_2} \left( \int_x^s g_2(\tau) f_2(\tau, u(\tau), v(\tau)) d\tau \right) ds.
\]

By a solution \( \| w \| \) of (1.2)-(1.3), we mean a solution of the system (2.1).

In order to utilize the fixed point index theory we state the following assumptions on the terms that occur in the system (2.1).

(C1) For every \( i = 1, 2, f_i : [0, 1] \times [0, \infty) \times [0, \infty) \to [0, \infty) \) satisfies Carathéodory conditions, that is, \( f_i(\cdot, u, v) \) is measurable for each fixed \( (u, v) \) and \( f_i(t, \cdot, \cdot) \) is continuous for almost every (a.e.) \( t \in [0, 1] \), and for each \( r > 0 \) there exists \( \phi_{i,r} \in L^\infty[0, 1] \) such that

\[
f_i(t, u, v) \leq \phi_{i,r}(t) \quad \text{for } u, v \in [0, r] \text{ and a.e. } t \in [0, 1].
\]

(C2) \( g_1 \in L^1[0, 1], g_1 \geq 0 \) and

\[
0 < \int_0^1 \varphi^{-1}_{p_1} \left( \int_0^1 g_1(\tau) d\tau \right) d\tau < +\infty.
\]

(C3) \( g_2 \in L^1[0, 1], g_2 \geq 0 \) and

\[
0 < \int_0^{1/2} \varphi^{-1}_{p_2} \left( \int_0^{1/2} g_2(\tau) d\tau \right) d\tau + \int_1^{1/2} \varphi^{-1}_{p_2} \left( \int_1^{1/2} g_2(\tau) d\tau \right) d\tau < +\infty.
\]

(C4) For every \( i = 1, 2, B_i : \mathbb{R} \to \mathbb{R} \) is a continuous function and there exist \( h_{i1}, h_{i2} \geq 0 \) such that

\[
h_{i1} v \leq B_i(v) \leq h_{i2} v \quad \text{for any } v \geq 0.
\]

**Remark 2.2.** The condition (2.2) is weaker than the condition

\[
0 < \int_0^1 \varphi^{-1}_{p_1} \left( \int_0^1 g_2(\tau) d\tau \right) d\tau < +\infty.
\]

In fact, for example, the function

\[
g_2(t) = \begin{cases}
\frac{1}{(t-1)^2}, & t \in [0, 1/2], \\
\frac{1}{t^2}, & t \in (1/2, 1],
\end{cases}
\]

satisfies (2.2) but not satisfies (2.3).

**Remark 2.3.** From (C2) and (C3) follow that there exists \( [a_1, b_1] \subset [0, 1] \) such that \( \int_{a_1}^{b_1} g_1(s) ds > 0 \) and there exists \( [a_2, b_2] \subset (0, 1) \) such that \( \int_{a_2}^{b_2} g_2(s) ds > 0 \).

We work in the space \( C[0, 1] \times C[0, 1] \) endowed with the norm

\[
\|(u, v)\| := \max\{|u|_{\infty}, |v|_{\infty}\},
\]

where \( |w|_{\infty} := \max\{|w(t)|, t \in [0, 1]\} \).

Take the cones

\[
K_1 := \{ w \in C[0, 1] : w \geq 0, \text{ concave and nonincreasing}\},
\]

\[
K_2 := \{ w \in C[0, 1] : w \geq 0, \text{ concave}\}.
\]
It is known (see e.g. \cite{39}) that

- for \( w \in K_1 \) we have \( w(t) \geq (1 - t)\|w\|_{\infty} \), for \( t \in [0, 1] \);
- for \( w \in K_2 \) we have \( w(t) \geq \min\{t, 1 - t\}\|w\|_{\infty} \), for \( t \in [0, 1] \).

It follows that the functions in \( K_1 \) are strictly positive on the sub-interval \([a_i, b_i]\) and in particular we have

- for \( w \in K_1 \) we have \( \min_{t \in [0, b_1]} w(t) \geq (1 - b_1)\|w\|_{\infty} \);
- for \( w \in K_2 \) we have \( \min_{t \in [a_2, b_2]} w(t) \geq \min\{a_2, 1 - b_2\}\|w\|_{\infty} \).

In the following we make use of the notations:

\[
    c_1 := 1 - b_1, \quad c_2 := \min\{a_2, 1 - b_2\}.
\]

Consider now the cone \( K \) in \( C[0, 1] \times C[0, 1] \) defined by

\[
    K := \{(u, v) \in K_1 \times K_2\}.
\]

For a \textit{positive} solution of the system \((2.1)\) we mean a solution \((u, v) \in K\) of \((2.1)\) such that \(\|(u, v)\| > 0\). We seek such solution as a fixed point of the following operator \(T\).

Consider the integral operator

\[
    (2.4) \quad T(u, v)(t) := \begin{pmatrix}
    T_1(u, v)(t) \\
    T_2(u, v)(t)
\end{pmatrix},
\]

where

\[
    T_1(u, v)(t) := \int_t^1 \varphi^{-1}_{p_1} \left( \int_0^s f_1(\tau, u(\tau), v(\tau)) d\tau \right) ds + B_1 \left( \varphi^{-1}_{p_1} \left( \int_0^1 g_1(\tau) f_1(\tau, u(\tau), v(\tau)) d\tau \right) \right)
\]

and

\[
    T_2(u, v)(t) := \begin{cases}
    \int_0^t \varphi^{-1}_{p_2} g_2(t, u(\tau), v(\tau)) d\tau ds \\
    + B_2 \left( \varphi^{-1}_{p_2} \left( \int_0^1 g_2(t, u(\tau), v(\tau)) d\tau \right) \right), & 0 \leq t \leq \sigma_{u,v}, \\
    \int_t^1 \varphi^{-1}_{p_2} g_2(t, u(\tau), v(\tau)) d\tau ds, & \sigma_{u,v} \leq t \leq 1,
\end{cases}
\]

From the definitions, for every \((u, v) \in K\) we have

\[
    \max_{t \in [0, 1]} T_2(u, v)(t) = T_2(u, v)(\sigma_{u,v}).
\]

Under our assumptions, we can show that the integral operator \(T\) leaves the cone \( K \) invariant and is compact.

\textbf{Lemma 2.4.} The operator \((2.4)\) maps \( K \) into \( K \) and is compact.

\textit{Proof.} Take \((u, v) \in K\). Then we have \(T(u, v) \in K\). Now, we show that the map \(T\) is compact. Firstly, we show that \(T\) sends bounded sets into bounded sets. Take \((u, v) \in K\) such that
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\[\| (u, v) \| \leq r.\] Then, for all \(t \in [0, 1]\) we have

\[T_1(u, v)(t) = \int_t^1 \varphi_{p_1}^{-1}\left( \int_0^s g_1(\tau) f_1(\tau, u(\tau), v(\tau))\,d\tau \right)\,ds + B_1\left( \varphi_{p_1}^{-1}\left( \int_0^1 g_1(\tau) f_1(\tau, u(\tau), v(\tau))\,d\tau \right) \right)\]

\[\leq \int_t^1 \varphi_{p_1}^{-1}\left( \int_0^s g_1(\tau) \phi_{1, r}(\tau)\,d\tau \right)\,ds + h_1\varphi_{p_1}^{-1}\left( \int_0^1 g_1(\tau) \phi_{1, r}(\tau)\,d\tau \right)\]

\[\leq \int_t^1 \varphi_{p_1}^{-1}\left( \int_0^1 g_1(\tau) \phi_{1, r}(\tau)\,d\tau \right)\,ds + h_1\varphi_{p_1}^{-1}\left( \int_0^1 g_1(\tau) \phi_{1, r}(\tau)\,d\tau \right) < +\infty.\]

We prove now that \(T_1\) sends bounded sets into equicontinuous sets. Let \(t_1, t_2 \in [0, 1]\), \(t_1 < t_2\), \((u, v) \in K\) such that \(\| (u, v) \| \leq r\). Then we have

\[|T_1(u, v)(t_1) - T_1(u, v)(t_2)| = \left| \int_{t_1}^{t_2} \varphi_{p_1}^{-1}\left( \int_0^s g_1(\tau) f_1(\tau, u(\tau), v(\tau))\,d\tau \right)\,ds \right| \leq C_r |t_1 - t_2|.\]

Therefore we obtain \(|T_1(u, v)(t_1) - T_1(u, v)(t_2)| \to 0\) when \(t_1 \to t_2\). By the Ascoli-Arzelà Theorem we can conclude that \(T_1\) is a compact map. In a similar manner we proceed for \(T_2(u, v)\).

Moreover, the map \(T\) is compact since the components \(T_i\) are compact maps.

\[\square\]

### 3. Existence results

For our index calculations we use the following (relative) open bounded sets in \(K\):

\[K_{p_1, p_2} = \{(u, v) \in K : \|u\|_\infty < p_1 \text{ and } \|v\|_\infty < p_2\}\]

and

\[V_{p_1, p_2} = \{(u, v) \in K : \min_{t \in [a_1, b_1]} u(t) < c_1 p_1 \text{ and } \min_{t \in [a_2, b_2]} v(t) < c_2 p_2\}\]

and if \(p_1 = p_2 = p\) we write simply \(K_p\) and \(V_p\). The set \(V_p\) was introduced in [10] as an extension to the case of systems of a set given by Lan [27]. The use of different radii, in the spirit of the paper [21], allows more freedom in the growth of the nonlinearities.

The following Lemma is similar to the Lemma 5 of [10] and therefore its proof is omitted.

**Lemma 3.1.** The sets defined above have the following properties:

- \(K_{c_1 p_1, c_2 p_2} \subset V_{p_1, p_2} \subset K_{p_1, p_2}\).
- \((w_1, w_2) \in \partial V_{p_1, p_2} \iff (w_1, w_2) \in K \text{ and } \min_{t \in [a_1, b_1]} w_i(t) = c_i p_i \text{ for some } i \in \{1, 2\} \text{ and } \min_{t \in [a_2, b_2]} w_j(t) \leq c_j p_j \text{ for } j \neq i\).
- If \((w_1, w_2) \in \partial V_{p_1, p_2}\), then for some \(i \in \{1, 2\}\) \(c_i p_i \leq w_i(t) \leq p_i \text{ for each } t \in [a_i, b_i]\) and \(\|w_i\|_\infty \leq p_i; \text{ moreover for } j \neq i \text{ we have } \|w_j\|_\infty \leq p_j\).

We firstly prove that the fixed point index is 1 on the set \(K_{p_1, p_2}\).
Lemma 3.2. Assume that
\[(1^{1}_{\rho_1, \rho_2})\] there exist \(\rho_1, \rho_2 > 0\) such that for every \(i = 1, 2\)
\[
f^p_{1, \rho_2} < \varphi_{p_1}(m_i)
\]
where
\[
f^p_{1, \rho_2} = \sup \left\{ \frac{f_i(t, u, v)}{\rho_{p_1}^{-1}} : (t, u, v) \in [0, 1] \times [0, \rho_1] \times [0, \rho_2], \right\},
\]
\[
\frac{1}{m_1} = \int_0^1 \varphi_{p_1}^{-1} \left( \int_0^s g_1(\tau) d\tau \right) ds + h_{12} \varphi_{p_1}^{-1} \left( \int_0^1 g_1(\tau) d\tau \right),
\]
\[
\frac{1}{m_2} = \max \left\{ \int_0^{\frac{1}{2}} \varphi_{p_2}^{-1} \left( \int_0^{\frac{1}{2}} g_2(\tau) d\tau \right) ds + h_{22} \varphi_{p_2}^{-1} \left( \int_0^1 g_2(\tau) d\tau \right), \int_0^{\frac{1}{2}} \varphi_{p_2}^{-1} \left( \int_{\frac{1}{2}}^s g_2(\tau) d\tau \right) ds \right\}.
\]

Then \(i_K(T, K_{\rho_1, \rho_2}) = 1\).

Proof. We show that \(\lambda(u, v) \neq T(u, v)\) for every \((u, v) \in \partial K_{\rho_1, \rho_2}\) and for every \(\lambda \geq 1\); this ensures that the index is 1 on \(K_{\rho_1, \rho_2}\). In fact, if this does not happen, there exist \(\lambda \geq 1\) and \((u, v) \in \partial K_{\rho_1, \rho_2}\) such that \(\lambda(u, v) = T(u, v)\).

Firstly we assume that \(\|u\| = \rho_1\) and \(\|v\| \leq \rho_2\).

Then we have
\[
\lambda u(t) = \int_t^1 \varphi_{p_1}^{-1} \left( \int_0^s g_1(\tau) f_1(\tau, u(\tau), v(\tau)) d\tau \right) ds + B_1 \left( \varphi_{p_1}^{-1} \left( \int_0^1 g_1(\tau) f_1(\tau, u(\tau), v(\tau)) d\tau \right) \right) \\
\leq \int_t^1 \varphi_{p_1}^{-1} \left( \int_0^s g_1(\tau) f_1(\tau, u(\tau), v(\tau)) d\tau \right) ds + h_{12} \varphi_{p_1}^{-1} \left( \int_0^1 g_1(\tau) f_1(\tau, u(\tau), v(\tau)) d\tau \right) \\
= \rho_1 \left( \int_t^1 \varphi_{p_1}^{-1} \left( \int_0^s g_1(\tau) \frac{f_1(\tau, u(\tau), v(\tau))}{\rho_{p_1}^{-1}} d\tau \right) ds + h_{12} \varphi_{p_1}^{-1} \left( \int_0^1 g_1(\tau) \frac{f_1(\tau, u(\tau), v(\tau))}{\rho_{p_1}^{-1}} d\tau \right) \right).
\]

Taking \(t = 0\) gives
\[
\lambda u(0) = \lambda \rho_1 \leq \rho_1 \left( \int_0^1 \varphi_{p_1}^{-1} \left( \int_0^s g_1(\tau) f_1^{p_1-p_2} d\tau \right) ds + h_{12} \varphi_{p_1}^{-1} \left( \int_0^1 g_1(\tau) f_1^{p_1-p_2} d\tau \right) \right) \\
= \rho_1 \varphi_{p_1}^{-1} \left( f_1^{p_1-p_2} \right) \left( \int_0^1 \varphi_{p_1}^{-1} \left( \int_0^s g_1(\tau) d\tau \right) ds + h_{12} \varphi_{p_1}^{-1} \left( \int_0^1 g_1(\tau) d\tau \right) \right) \\
= \rho_1 \frac{1}{m_1} \varphi_{p_1}^{-1} \left( f_1^{p_1-p_2} \right).
\]

Using the hypothesis (3.1) and the strictly monotonicity of \(\varphi_{p_1}^{-1}\) we obtain \(\lambda \rho_1 < \rho_1\). This contradicts the fact that \(\lambda \geq 1\) and proves the result.

Now we assume \(\|v\| = \rho_2\) and \(\|u\| \leq \rho_1\).

Then we have
\[
\lambda \rho_2 = \|T_2(u, v)\| = T_2(u, v)(\sigma_{u,v}).
\]
If $\sigma_{u,v} \leq \frac{1}{2}$, we have

$$\lambda_{p_2} = \|T_2(u,v)\|_\infty = T_2(u,v)(\sigma_{u,v})$$

$$= \int_0^{\sigma_{u,v}} \varphi_{p_2}^{-1}\left(\int_s^{\sigma_{u,v}} g_2(\tau)f_2(\tau,u(\tau),v(\tau))d\tau\right)ds + B_2\left(\varphi_{p_2}^{-1} \left(\int_0^{\sigma_{u,v}} g_2(\tau)f_2(\tau,u(\tau),v(\tau))d\tau\right)\right)$$

$$\leq \int_0^{\frac{1}{2}} \varphi_{p_2}^{-1}\left(\int_s^{\frac{1}{2}} g_2(\tau)f_2(\tau,u(\tau),v(\tau))d\tau\right)ds + h_{22} \varphi_{p_2}^{-1}\left(\int_0^{\sigma_{u,v}} g_2(\tau)f_2(\tau,u(\tau),v(\tau))d\tau\right)$$

$$\leq \int_0^{\frac{1}{2}} \varphi_{p_2}^{-1}\left(\int_s^{\frac{1}{2}} g_2(\tau)f_2(\tau,u(\tau),v(\tau))d\tau\right)ds + h_{22} \varphi_{p_2}^{-1}\left(\int_0^{\frac{1}{2}} g_2(\tau)d\tau\right)$$

If $\sigma_{u,v} > \frac{1}{2}$, we have

$$\lambda_{p_2} = \|T_2(u,v)\|_\infty = T_2(u,v)(\sigma_{u,v})$$

$$= \int_0^{\sigma_{u,v}} \varphi_{p_2}^{-1}\left(\int_s^{\sigma_{u,v}} g_2(\tau)f_2(\tau,u(\tau),v(\tau))d\tau\right)ds \leq \int_0^{\frac{1}{2}} \varphi_{p_2}^{-1}\left(\int_s^{\frac{1}{2}} g_2(\tau)f_2(\tau,u(\tau),v(\tau))d\tau\right)ds$$

$$= \rho_2 \int_0^{\frac{1}{2}} \varphi_{p_2}^{-1}\left(\int_s^{\frac{1}{2}} g_2(\tau)f_2(\tau,u(\tau),v(\tau))d\tau\right)ds \leq \rho_2 \varphi_{p_2}^{-1}(f_{p_1,p_2}^{0,1,\rho_2}) \int_0^{\frac{1}{2}} \varphi_{p_2}^{-1}\left(\int_s^{\frac{1}{2}} g_2(\tau)d\tau\right)ds.$$

Then, in both cases, we have

$$\lambda_{p_2} = \|T_2(u,v)\|_\infty = T_2(u,v)(\sigma_{u,v}) \leq \rho_2 \varphi_{p_2}^{-1}(f_{p_1,p_2}^{0,1,\rho_2}) \times$$

$$\max \left\{ \int_0^{\frac{1}{2}} \varphi_{p_2}^{-1}\left(\int_s^{\frac{1}{2}} g_2(\tau)d\tau\right)ds + h_{22} \varphi_{p_2}^{-1}\left(\int_0^{\frac{1}{2}} g_2(\tau)d\tau\right), \int_0^{\frac{1}{2}} \varphi_{p_2}^{-1}\left(\int_s^{\frac{1}{2}} g_2(\tau)d\tau\right)ds \right\}$$

$$= \rho_2 \varphi_{p_2}^{-1}(f_{p_1,p_2}^{0,1,\rho_2}) \frac{1}{m_2}.$$ 

Using the hypothesis (3.1) and the strictly monotonicity of $\varphi_{p_2}^{-1}$ we obtain $\lambda_{p_2} < \rho_2$. This contradicts the fact that $\lambda \geq 1$ and proves the result. $\square$

We give a first Lemma that shows that the index is 0 on a set $V_{\rho_1,\rho_2}$.

**Lemma 3.3.** Assume that:

(1) There exist $\rho_1, \rho_2 > 0$ such that for every $i = 1, 2$

$$(3.2) \quad f_{i, (\rho_1, \rho_2)} > \varphi_{p_i}(M_i),$$

where

$$f_{1, (\rho_1, \rho_2)} = \inf \left\{ \frac{f_1(t,u,v)}{\rho_1^{p_1-1}} : (t,u,v) \in [0,b_1] \times [c_1 \rho_1, \rho_1] \times [0,\rho_2] \right\},$$

$$f_{2, (\rho_1, \rho_2)} = \inf \left\{ \frac{f_2(t,u,v)}{\rho_2^{p_2-1}} : (t,u,v) \in [a_2,b_2] \times [0,\rho_1] \times [c_2 \rho_2, \rho_2] \right\},$$
\[
\frac{1}{M_1} = \int_0^{b_1} \varphi_{p_1}^{-1} \left( \int_0^s g_1(\tau) d\tau \right) ds + h_{11} \varphi_{p_1}^{-1} \left( \int_0^{b_1} g_1(\tau) d\tau \right),
\]

and

\[
\frac{1}{M_2} = \frac{1}{2} \min_{a_2 \leq \nu \leq b_2} \left\{ \int_{a_2}^\nu \varphi_{p_2}^{-1} \left( \int_s^\nu g_2(\tau) d\tau \right) ds + \int_\nu^{b_2} \varphi_{p_2}^{-1} \left( \int_s^\nu g_2(\tau) d\tau \right) ds + h_{21} \varphi_{p_2}^{-1} \left( \int_{a_2}^\nu g_2(\tau) d\tau \right) \right\}.
\]

Then \( i_K(T, V_{\rho_1, \rho_2}) = 0. \)

**Proof.** Let \( e(t) \equiv 1 \) for \( t \in [0, 1] \). Then \((e, e) \in K\). We prove that

\[
(u, v) \neq T(u, v) + \lambda(e, e) \quad \text{for} \quad (u, v) \in \partial V_{\rho_1, \rho_2} \quad \text{and} \quad \lambda \geq 0.
\]

In fact, if this does not happen, there exist \((u, v) \in \partial V_{\rho_1, \rho_2}\) and \( \lambda \geq 0 \) such that 
\((u, v) = T(u, v) + \lambda(e, e)\). We examine the two cases:

Case (1): \( c_1 \rho_1 \leq u(t) \leq \rho_1 \) for \( t \in [0, b_1] \) and \( 0 \leq v(t) \leq \rho_2 \) for \( t \in [0, 1] \).
Thus for \( t \in [0, b_1] \), we have

\[
\rho_1 \geq u(t) = \int_t^1 \varphi_{p_1}^{-1} \left( \int_0^s g_1(\tau) f_1(\tau, u(\tau), v(\tau)) d\tau \right) ds + B_1 \left( \varphi_{p_1}^{-1} \left( \int_0^1 g_1(\tau) f_1(\tau, u(\tau), v(\tau)) d\tau \right) \right) + \lambda \geq \int_t^{b_1} \varphi_{p_1}^{-1} \left( \int_0^s g_1(\tau) f_1(\tau, u(\tau), v(\tau)) d\tau \right) ds + h_{11} \varphi_{p_1}^{-1} \left( \int_0^1 g_1(\tau) f_1(\tau, u(\tau), v(\tau)) d\tau \right) + \lambda \geq \int_t^{b_1} \varphi_{p_1}^{-1} \left( \int_0^s g_1(\tau) f_1(\tau, u(\tau), v(\tau)) d\tau \right) ds + h_{11} \varphi_{p_1}^{-1} \left( \int_0^{b_1} g_1(\tau) f_1(\tau, u(\tau), v(\tau)) d\tau \right) + \lambda \geq \rho_1 \int_t^{b_1} \varphi_{p_1}^{-1} \left( \int_0^s \frac{g_1(\tau, u(\tau), v(\tau))}{\rho_{p_1}^{-1}} d\tau \right) ds + \rho_1 h_{11} \varphi_{p_1}^{-1} \left( \int_0^{b_1} g_1(\tau) \frac{f_1(\tau, u(\tau), v(\tau))}{\rho_{p_1}^{-1}} d\tau \right) + \lambda.
\]

For \( t = 0 \) we obtain

\[
\rho_1 \geq \rho_1 \varphi_{p_1}^{-1} \left( f_1(\rho_1, \rho_2) \right) \left( \int_0^{b_1} \varphi_{p_1}^{-1} \left( \int_0^s g_1(\tau) d\tau \right) ds + h_{11} \varphi_{p_1}^{-1} \left( \int_0^{b_1} g_1(\tau) d\tau \right) \right) + \lambda > \rho_1 \varphi_{p_1}^{-1} \left( f_1(\rho_1, \rho_2) \right) \frac{1}{M_1} + \lambda.
\]

Using the hypothesis \((3.2)\) we obtain \( \rho_1 > \rho_1 + \lambda \), a contradiction.

Case (2): \( 0 \leq u(t) \leq \rho_1 \) for \( t \in [0, 1] \) and \( c_2 \rho_2 \leq v(t) \leq \rho_2 \).
We distinguish three cases:

Case (i) \( 0 < \sigma_{u,v} \leq a_2 \).
Therefore we get

\[
\rho_2 \geq v(\sigma_{u,v}) = T_2(u,v)(\sigma_{u,v}) + \lambda = \int_{\sigma_{u,v}}^{1} \varphi_{\rho_2}^{-1}\left( \int_{\sigma_{u,v}}^{\rho_2} g_2(\tau) f_2(\tau, u(\tau), v(\tau)) d\tau \right) ds + \lambda \\
\geq \int_{a_2}^{b_2} \varphi_{\rho_2}^{-1}\left( \int_{a_2}^{b_2} g_2(\tau) f_2(\tau, u(\tau), v(\tau)) d\tau \right) ds + \lambda \\
= \rho_2 \int_{a_2}^{b_2} \varphi_{\rho_2}^{-1}\left( \int_{a_2}^{b_2} \frac{g_2(\tau) f_2(\tau, u(\tau), v(\tau))}{\rho_2^{\rho_1}} d\tau \right) ds + \lambda \\
\geq \rho_2 \varphi_{\rho_2}^{-1}(f_2(\rho_1, \rho_2)) \left( \int_{a_2}^{b_2} \varphi_{\rho_2}^{-1}\left( \int_{a_2}^{b_2} g_2(\tau) d\tau \right) ds \right) + h_2 \varphi_{\rho_2}^{-1}\left( \int_{a_2}^{b_2} g_2(\tau) f_2(\tau, u(\tau), v(\tau)) d\tau \right) + \lambda \\
\geq \rho_2 \varphi_{\rho_2}^{-1}(f_2(\rho_1, \rho_2)) \int_{a_2}^{b_2} g_2(\tau) d\tau + h_2 \varphi_{\rho_2}^{-1}\left( \int_{a_2}^{b_2} g_2(\tau) d\tau \right) + \lambda \\
\geq \rho_2 \varphi_{\rho_2}^{-1}(f_2(\rho_1, \rho_2)) \frac{1}{M_2} + \lambda.
\]

Using the hypothesis (3.2) we obtain \(\rho_2 > \rho_2 + \lambda\), a contradiction.

Case (ii) \(\sigma_{u,v} \geq b_2\).

\[
\rho_2 \geq v(\sigma_{u,v}) = T_2(u,v)(\sigma_{u,v}) + \lambda = \int_{\sigma_{u,v}}^{\sigma_{u,v}} \varphi_{\rho_2}^{-1}\left( \int_{\sigma_{u,v}}^{\rho_2} g_2(\tau) f_2(\tau, u(\tau), v(\tau)) d\tau \right) ds \\
+ B_2 \varphi_{\rho_2}^{-1}\left( \int_{0}^{\sigma_{u,v}} g_2(\tau) f_2(\tau, u(\tau), v(\tau)) d\tau \right) + \lambda \\
\geq \int_{a_2}^{b_2} \varphi_{\rho_2}^{-1}\left( \int_{a_2}^{b_2} g_2(\tau) f_2(\tau, u(\tau), v(\tau)) d\tau \right) ds + h_2 \varphi_{\rho_2}^{-1}\left( \int_{a_2}^{b_2} g_2(\tau) f_2(\tau, u(\tau), v(\tau)) d\tau \right) + \lambda \\
= \rho_2 \int_{a_2}^{b_2} \varphi_{\rho_2}^{-1}\left( \int_{a_2}^{b_2} g_2(\tau) \frac{f_2(\tau, u(\tau), v(\tau))}{\rho_2^{\rho_1}} d\tau \right) ds + \rho_2 h_2 \varphi_{\rho_2}^{-1}\left( \int_{a_2}^{b_2} g_2(\tau) d\tau \right) + \lambda \\
\geq \rho_2 \varphi_{\rho_2}^{-1}(f_2(\rho_1, \rho_2)) \left( \int_{a_2}^{b_2} \varphi_{\rho_2}^{-1}\left( \int_{a_2}^{b_2} g_2(\tau) d\tau \right) ds \right) + h_2 \varphi_{\rho_2}^{-1}\left( \int_{a_2}^{b_2} g_2(\tau) d\tau \right) + \lambda \\
\geq \rho_2 \varphi_{\rho_2}^{-1}(f_2(\rho_1, \rho_2)) \frac{1}{M_2} + \lambda.
\]
Using the hypothesis \((\ref{3.2})\) we obtain \(\rho_2 > \rho_2 + \lambda\), a contradiction.

Case (iii) \(a_2 < \sigma_{u,v} < b_2\).

\[
2\rho_2 \geq 2v(\sigma_{u,v}) = 2\lambda + 2T_2(u, v)(\sigma_{u,v}) = 2\lambda + \int_{b_2}^{\sigma_{u,v}} g_2(\tau) f_2(\tau, u(\tau), v(\tau)) d\tau \, ds \\
+ B_2 \left( \varphi_{p_2}^{-1} \left( \int_{0}^{\sigma_{u,v}} g_2(\tau) f_2(\tau, u(\tau), v(\tau)) d\tau \right) \right) + \int_{\sigma_{u,v}}^{1} \varphi_{p_2}^{-1} \left( \int_{\sigma_{u,v}}^{s} g_2(\tau) f_2(\tau, u(\tau), v(\tau)) d\tau \right) ds \\
\geq 2\lambda + B_2 \left( \varphi_{p_2}^{-1} \left( \int_{0}^{\sigma_{u,v}} g_2(\tau) f_2(\tau, u(\tau), v(\tau)) d\tau \right) \right) + \int_{\sigma_{u,v}}^{1} \varphi_{p_2}^{-1} \left( \int_{\sigma_{u,v}}^{s} g_2(\tau) f_2(\tau, u(\tau), v(\tau)) d\tau \right) ds \\
+ h_{21} \varphi_{p_2}^{-1} \left( \int_{0}^{\sigma_{u,v}} g_2(\tau) f_2(\tau, u(\tau), v(\tau)) d\tau \right) + \int_{\sigma_{u,v}}^{b_2} \varphi_{p_2}^{-1} \left( \int_{\sigma_{u,v}}^{s} g_2(\tau) f_2(\tau, u(\tau), v(\tau)) d\tau \right) ds \\
= 2\lambda + B_2 \left( \varphi_{p_2}^{-1} \left( \int_{0}^{\sigma_{u,v}} g_2(\tau) f_2(\tau, u(\tau), v(\tau)) d\tau \right) \right) + \int_{\sigma_{u,v}}^{b_2} \varphi_{p_2}^{-1} \left( \int_{\sigma_{u,v}}^{s} g_2(\tau) f_2(\tau, u(\tau), v(\tau)) d\tau \right) ds \\
\geq 2\lambda + B_2 \left( \varphi_{p_2}^{-1} \left( \int_{0}^{\sigma_{u,v}} g_2(\tau) f_2(\tau, u(\tau), v(\tau)) d\tau \right) \right) + \int_{\sigma_{u,v}}^{b_2} \varphi_{p_2}^{-1} \left( \int_{\sigma_{u,v}}^{s} g_2(\tau) f_2(\tau, u(\tau), v(\tau)) d\tau \right) ds \\
\geq 2\lambda + 2\rho_2 \varphi_{p_2}^{-1}(f_{2, (\rho_1, \rho_2)}) \left( \frac{1}{M_1} \right)
\]

Using the hypothesis \((\ref{3.2})\) we obtain \(\rho_2 > \lambda + \rho_2\), a contradiction.

\[\square\]

**Remark 3.4.** We point out that a stronger, but easier to check, hypothesis than \((\ref{3.2})\) is

\[
f_{i, (\rho_1, \rho_2)} > \varphi_{p_i}(\tilde{M}_i),
\]

where

\[
\frac{1}{M_1} = \int_{0}^{b_1} \varphi_{p_1}^{-1} \left( \int_{0}^{s} g_1(\tau) d\tau \right) ds
\]

and

\[
\frac{1}{M_2} = \frac{1}{2} \min_{a_2 \leq \nu \leq b_2} \left\{ \int_{a_2}^{\nu} \varphi_{p_2}^{-1} \left( \int_{0}^{s} g_2(\tau) d\tau \right) ds + \int_{\nu}^{b_2} \varphi_{p_2}^{-1} \left( \int_{\nu}^{s} g_2(\tau) d\tau \right) ds \right\}
\]

In the following Lemma we exploit an idea that was used in \([19, 21]\) and we provide a result of index 0 controlling the growth of just one nonlinearity \(f_i\), at the cost of having to deal with a larger domain. Nonlinearities with different growths were considered for examples in \([35, 36, 45]\).

**Lemma 3.5.** Assume that

\[(\Pi_{\rho_1, \rho_2})^* \text{ there exist } \rho_1, \rho_2 > 0 \text{ such that for some } i \in \{1, 2\} \text{ we have}
\]

\[
f_{i, (\rho_1, \rho_2)}^* > \varphi_{p_i}(\tilde{M}_i),
\]

\[\text{(3.3)}\]
Theorem 3.6. The system (2.1) has at least one positive solution in $K$ if one of the following conditions holds.

(S1) For $i = 1, 2$ there exist $\rho_i, r_i \in (0, \infty)$ with $\rho_i < r_i$ such that $(\Pi_{p_1, p_2})$, [or $(\Pi_{p_1, p_2})^*], (\Pi_{r_1, r_2})$ hold.

(S2) For $i = 1, 2$ there exist $\rho_i, r_i \in (0, \infty)$ with $\rho_i < c_i r_i$ such that $(\Pi_{p_1, p_2})$, $(\Pi_{r_1, r_2})$ hold.

The system (2.1) has at least two positive solutions in $K$ if one of the following conditions holds.

(S3) For $i = 1, 2$ there exist $\rho_i, r_i, s_i \in (0, \infty)$ with $\rho_i < r_i < c_i s_i$ such that $(\Pi_{p_1, p_2})$, [or $(\Pi_{p_1, p_2})^*], (\Pi_{r_1, r_2})$ and $(\Pi_{s_1, s_2})$ hold.

(S4) For $i = 1, 2$ there exist $\rho_i, r_i, s_i \in (0, \infty)$ with $\rho_i < c_i r_i$ and $r_i < s_i$ such that $(\Pi_{p_1, p_2})$, $(\Pi_{r_1, r_2})$ and $(\Pi_{s_1, s_2})$ hold.

The system (2.1) has at least three positive solutions in $K$ if one of the following conditions holds.

(S5) For $i = 1, 2$ there exist $\rho_i, r_i, s_i, \delta_i \in (0, \infty)$ with $\rho_i < r_i < c_i s_i$ and $s_i < \delta_i$ such that $(\Pi_{p_1, p_2})$, [or $(\Pi_{p_1, p_2})^*], (\Pi_{r_1, r_2})$, $(\Pi_{s_1, s_2})$ and $(\Pi_{\delta_1, \delta_2})$ hold.

(S6) For $i = 1, 2$ there exist $\rho_i, r_i, s_i, \delta_i \in (0, \infty)$ with $\rho_i < c_i r_i$ and $r_i < s_i < c_i \delta_i$ such that $(\Pi_{p_1, p_2})$, $(\Pi_{r_1, r_2})$, $(\Pi_{s_1, s_2})$ and $(\Pi_{\delta_1, \delta_2})$ hold.

4. Non-existence results

We now provide some non-existence results for system (2.1).

Theorem 4.1. Assume that one of the following conditions holds.

1. For $i = 1, 2$,

\[ f_i(t, u_1, u_2) < \varphi_{p_i}(m_i u_i) \text{ for every } t \in [0, 1] \text{ and } u_i > 0. \]

2. For $i = 1, 2$,

\[ f_i(t, u_1, u_2) > \varphi_{p_i} \left( \frac{M_i}{c_i} u_i \right) \text{ for every } t \in [a_i, b_i] \text{ and } u_i > 0. \]

3. There exists $k \in \{1, 2\}$ such that (4.1) is verified for $f_k$ and for $j \neq k$ condition (4.2) is verified for $f_j$. 

\[ f_i(t, u_1, u_2) = \inf \left\{ \frac{f_i(t, u, v)}{p_{p_1}} : (t, u, v) \in [a_i, b_i] \times [0, \rho_1] \times [0, \rho_2] \right\}. \]
Then there is no positive solution of the system \( (2.1) \) in \( K \).

Proof. (1) Assume, on the contrary, that there exists \((u, v) \in K\) such that \((u, v) = T(u, v)\) and \((u, v) \neq (0, 0)\). We distinguish two cases.

- Let be \( \|u\|_\infty \neq 0 \). Then we have

\[
u(t) = \int_t^1 \varphi_{p_1}^{-1} \left( \int_0^s g_1(\tau)f_1(\tau, u(\tau), v(\tau))d\tau \right) ds + B_1 \left( \varphi_{p_1}^{-1} \left( \int_0^1 g_1(\tau)f_1(\tau, u(\tau), v(\tau))d\tau \right) \right) < m_1 \int_t^1 \varphi_{p_1}^{-1} \left( \int_0^s g_1(\tau)d\tau \right) ds + m_1h_2\varphi_{p_1}^{-1} \left( \int_0^1 g_1(\tau)d\tau \right) \leq m_1\|u\|_\infty \left( \int_t^1 \varphi_{p_1}^{-1} \left( \int_0^s g_1(\tau)d\tau \right) ds + h_2\varphi_{p_1}^{-1} \left( \int_0^1 g_1(\tau)d\tau \right) \right).
\]

Taking \( t = 0 \) gives

\[
\|u\|_\infty = u(0) < m_1\|u\|_\infty \left( \int_0^1 \varphi_{p_1}^{-1} \left( \int_0^s g_1(\tau)d\tau \right) ds + h_2\varphi_{p_1}^{-1} \left( \int_0^1 g_1(\tau)d\tau \right) \right) = m_1\|u\|_\infty \frac{1}{m_1},
\]
a contradiction.

- Let be \( \|v\|_\infty \neq 0 \).

Reasoning as in Lemma [3.2] we distinguish the cases \( \sigma_{u,v} \leq 1/2 \) and \( \sigma_{u,v} > 1/2 \). In the first case we have

\[
\|v\|_\infty = \|T_2(u, v)\|_\infty = T_2(u, v)(\sigma_{u,v}) = \int_0^{\sigma_{u,v}} \varphi_{p_2}^{-1} \left( \int_0^{\sigma_{u,v}} g_2(\tau)f_2(\tau, u(\tau), v(\tau))d\tau \right) ds + B_2 \left( \varphi_{p_2}^{-1} \left( \int_0^{\sigma_{u,v}} g_2(\tau)d\tau \right) \right) < m_2\|v\|_\infty \left( \int_0^{\sigma_{u,v}} \varphi_{p_2}^{-1} \left( \int_0^{\sigma_{u,v}} g_2(\tau)d\tau \right) ds + h_{22}\varphi_{p_2}^{-1} \left( \int_0^{\sigma_{u,v}} g_2(\tau)d\tau \right) \right) \leq m_2\|v\|_\infty \frac{1}{m_2},
\]
a contradiction.

The proof is similar in the last case \( \sigma_{u,v} > 1/2 \).

(2) Assume, on the contrary, that there exists \((u, v) \in K\) such that \((u, v) = T(u, v)\) and \((u, v) \neq (0, 0)\). We distinguish two cases.
• Let be $\|u\|_\infty \neq 0$. Then, for $t \in [a_1, b_1] = [0, b_1]$, we have

$$ u(t) = \int_t^1 \varphi_{p_1}^{-1} \left( \int_0^s g_1(\tau)f_1(\tau, u(\tau), v(\tau)) d\tau \right) ds + B_1 \left( \varphi_{p_1}^{-1} \left( \int_0^1 g_1(\tau)f_1(\tau, u(\tau), v(\tau)) d\tau \right) \right) $$

$$ \geq \int_t^{b_1} \varphi_{p_1}^{-1} \left( \int_0^s g_1(\tau)f_1(\tau, u(\tau), v(\tau)) d\tau \right) ds + h_{11} \varphi_{p_1}^{-1} \left( \int_0^{b_1} g_1(\tau)f_1(\tau, u(\tau), v(\tau)) d\tau \right) $$

$$ \geq \int_t^{b_1} \varphi_{p_1}^{-1} \left( \int_0^s g_1(\tau)f_1(\tau, u(\tau), v(\tau)) d\tau \right) ds + h_{11} \varphi_{p_1}^{-1} \left( \int_0^{b_1} g_1(\tau)f_1(\tau, u(\tau), v(\tau)) d\tau \right) $$

$$ \geq \frac{M_1}{c_1} \left( \int_t^{b_1} \varphi_{p_1}^{-1} \left( \int_0^s g_1(\tau) g_p(1)(u(\tau)) d\tau \right) ds + h_{11} \varphi_{p_1}^{-1} \left( \int_0^{b_1} g_1(\tau) g_p(1)(u(\tau)) d\tau \right) \right) $$

$$ \geq \frac{M_1}{c_1} \left( \int_t^{b_1} \varphi_{p_1}^{-1} \left( \int_0^s g_1(\tau) g_p(1)(u(\tau)) d\tau \right) ds + h_{11} \varphi_{p_1}^{-1} \left( \int_0^{b_1} g_1(\tau) g_p(1)(u(\tau)) d\tau \right) \right) $$

For $t = 0$ we obtain

$$ u(0) = \|u\|_\infty > M_1 \|u\|_\infty \frac{1}{M_1}, $$

a contradiction.

• Let be $\|v\|_\infty \neq 0$. We examine the case $\sigma_{u,v} \geq b_2$. We have

$$ \|v\|_\infty = v(\sigma_{u,v}) = T(u,v)(\sigma_{u,v}) = \int_{\sigma_{u,v}}^{\sigma_{u,v}} \varphi_{p_2}^{-1} \left( \int_s^{\sigma_{u,v}} g_2(\tau)f_2(\tau, u(\tau), v(\tau)) d\tau \right) ds $$

$$ + B_2 \left( \varphi_{p_2}^{-1} \left( \int_0^{\sigma_{u,v}} g_2(\tau)f_2(\tau, u(\tau), v(\tau)) d\tau \right) \right) $$

$$ \geq \int_{a_2}^{b_2} \varphi_{p_2}^{-1} \left( \int_s^{b_2} g_2(\tau)f_2(\tau, u(\tau), v(\tau)) d\tau \right) ds + h_{21} \varphi_{p_2}^{-1} \left( \int_{a_2}^{b_2} g_2(\tau)f_2(\tau, u(\tau), v(\tau)) d\tau \right) $$

$$ \geq \frac{M_2}{c_2} \|v\|_\infty \left( \int_{a_2}^{b_2} \varphi_{p_2}^{-1} \left( \int_s^{b_2} g_2(\tau) d\tau \right) ds + h_{21} \varphi_{p_2}^{-1} \left( \int_{a_2}^{b_2} g_2(\tau) d\tau \right) \right) $$

$$ \geq \frac{M_2}{c_2} \|v\|_\infty \left( \int_{a_2}^{b_2} \varphi_{p_2}^{-1} \left( \int_s^{b_2} g_2(\tau) d\tau \right) ds + h_{21} \varphi_{p_2}^{-1} \left( \int_{a_2}^{b_2} g_2(\tau) d\tau \right) \right) $$

a contradiction. By similar proofs, the cases $0 < \sigma_{u,v} \leq a_2$ and $a_2 < \sigma_{u,v} < b_2$ can be examined.

(3) Assume, on the contrary, that there exists $(u, v) \in K$ such that $(u, v) = T(u,v)$ and $(u,v) \neq (0,0)$. If $\|u\|_\infty \neq 0$ then the function $f_1$ satisfies either (4.1) or (4.2) and the proof follows as in the previous cases. If $\|v\|_\infty \neq 0$ then the function $f_2$ satisfies either (4.1) or (4.2) and the proof follows as previous cases.

\[ \Box \]

5. An example

We illustrate in the following example that all the constants that occur in the Theorem 3.6 can be computed.

Consider the system

\[ (\varphi_{p_1}(u'))'(t) + g_1(t)f_1(t, u(t), v(t)) = 0, \ t \in (0, 1), \]

\[ (\varphi_{p_2}(v'))'(t) + g_2(t)f_2(t, u(t), v(t)) = 0, \ t \in (0, 1), \]

subject to boundary conditions

\[ u'(0) = 0, \ u(1) = B_1(u'(1)) = 0, \ v(0) = B_2(v'(0)), \ v(1) = 0, \]
where $B_1$ and $B_2$ are defined by:

$$B_1(w) = \begin{cases} 
  w, & w \leq 0, \\
  \frac{w}{7}, & 0 \leq w \leq 1, \\
  \frac{w}{6} + \frac{1}{3}, & w \geq 1,
\end{cases}$$

and

$$B_2(w) = \begin{cases} 
  \frac{w}{7}, & 0 \leq w \leq 1, \\
  \frac{w}{6} + \frac{2}{3}, & w \geq 1.
\end{cases}$$

Now we assume $g_1 = g_2 = 1$. Thus we have

$$\frac{1}{m_1} = \frac{p_1 - 1}{p_1} + h_{12},$$

$$\frac{1}{m_2} = \frac{p_2 - 1}{p_2} \left( \frac{1}{2} \right)^{p_2 - 1} + h_{22} \left( \frac{1}{2} \right)^{p_2 - 1},$$

$$\frac{1}{M_1} = \frac{1}{M_1[0,b_1]} = \frac{p_1 - 1}{p_1} b_1^{p_1 - 1} + h_{11} b_1^{p_1 - 1},$$

and

$$\frac{1}{M_2} = \frac{1}{M_2[a_2,b_2]} = \frac{1}{2} \min_{a_2 \leq \nu \leq b_2} \left( \frac{p_2 - 1}{p_2} \left( (\nu - a_2)^{p_2 - 1} + (b_2 - \nu)^{p_2 - 1} \right) + h_{21}(\nu - a_2)^{-p_2} \right).$$

The choice $p_1 = \frac{3}{2}$, $p_2 = 3$, $b_1 = \frac{2}{3}$, $a_2 = \frac{1}{4}$, $b_2 = \frac{3}{4}$, $h_{11} = 1/6$, $h_{12} = 1/2$, $h_{21} = 1/9$ and $h_{22} = 1/3$ gives by direct computation:

$$c_1 = \frac{1}{3}; c_2 = \frac{1}{4}; m_1 = 1.2; M_1 = 5.78571; m_2 = 2.12132; M_2 = 9.14497.$$

Let us now consider

$$f_1(t,u,v) = \frac{1}{16}(u^4 + t^3v^3) + \frac{27}{50}, \quad f_2(t,u,v) = (tu)^{\frac{1}{2}} + 10v^9.$$

Then, with the choice of $\rho_1 = \rho_2 = 1/20$, $r_1 = 1$, $r_2 = 2/3$, $s_1 = s_2 = 9$, we obtain

$$\inf \left\{ f_1(t,u,v) : (t,u,v) \in [0,\frac{2}{3}] \times [0,\rho_1] \times [0,\rho_2] \right\} = f_1(0,0,0) = 0.54 > \sqrt{M_1\rho_1} = 0.538,$n

$$\sup \left\{ f_1(t,u,v) : (t,u,v) \in [0,1] \times [0,r_1] \times [0,r_2] \right\} = f_1(1,r_1,r_2) = 0.62 < \sqrt{m_1r_1} = 1.095,$n

$$\inf \left\{ f_1(t,u,v) : (t,u,v) \in [0,\frac{2}{3}] \times [c_1s_1,s_1] \times [0,s_2] \right\} = f_1(0,c_1s_1,0) = 5.602 > \sqrt{M_1s_1} = 1.247,$n

$$\sup \left\{ f_2(t,u,v) : (t,u,v) \in [0,1] \times [0,r_1] \times [0,r_2] \right\} = f_2(1,r_1,r_2) = 1.260 < (m_2r_2)^2 = 2,$n

$$\inf \left\{ f_2(t,u,v) : (t,u,v) \in [\frac{1}{4},\frac{3}{4}] \times [0,s_1] \times [c_2s_2,s_2] \right\} = f_2(t,0,c_2s_2) = 14778.9 > (M_2s_2)^2 = 6774.07.$$

Thus the conditions $(I_1^{1,2/3})^*, (I_1^{1,2/3})$ and $(I_0^{9,9})$ are satisfied; therefore the system (5.1)-(5.2) has at least two nontrivial solutions $(u_1,v_1)$ and $(u_2,v_2)$ such that $1/20 < \| (u_1,v_1) \| \leq 1$ and $1 < \| (u_2,v_2) \| \leq 9.$
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