Linear Adjusting Programming in Factor Space

Jing He\textsuperscript{1*}, Qi-Wei Kong\textsuperscript{2}, Ho-Chung Lui\textsuperscript{3}, Hai-Tao Liu\textsuperscript{3}
Yi-Mu Ji\textsuperscript{4}, Hai-Chang Yao\textsuperscript{1}, Mo-Zhengfu Liu\textsuperscript{5}, Jin-Jun Chen\textsuperscript{1}

\textsuperscript{1}School of Software and Electrical Engineering, Swinburne University of Technology, Hawthorn, Australia
\textsuperscript{2}Institute of Information Engineering, Nanjing University of Finance and Economics, Nanjing, China
\textsuperscript{3}Research Center on Fictitious economy & Data Science, Chinese Academy of Sciences, Beijing, China
\textsuperscript{4}School of Computer Science, Nanjing University of Posts and Telecommunications, Nanjing, China

Abstract

The definition of factor space and a unified optimization based classification model were developed for linear programming and supervised learning. Intelligent behaviour appeared in a decision process can be treated as a point $y$, the dynamic state observed and controlled by the agent, moving in a factor space impelled by the goal factor and blocked by the constraint factors. Suppose that the feasible region is cut by a group of hyperplanes, when point $y$ reaches the region’s wall, a hyperplane will block the moving and the agent needs to adjust the moving direction such that the target is pursued as faithful as possible. Since the wall is not able to be represented to a differentiable function, the gradient method cannot be applied to describe the adjusting process. We, therefore, suggest a new model, named linear adjusting programming (LAP) in this paper. LAP is similar as a kind of relaxed linear programming (LP), and the difference between LP and LAP is: the former aims to find out the ultimate optimal point, while the latter just does a direct action in short period. You may ask: Where will a blocker encounter? How can the moving direction be adjusted? Where further blockers may be encountered next, and how should the direction be adjusted again?... If the ultimate best is found, that’s a blessing; if not, that’s fine. We request at least an adjusting should be achieved at the first time. what are the former and latter? possible to be more exact? In place of gradient vector, the projection of goal direction $g$ in a subspace plays a core role in linear adjusting programming. If a hyperplane blocks $y$ going ahead along with the direction $d$, then we must adjust the new direction $d'$ as the projection of $g$ in the blocking plane. If there is only one blocker at a time, it is straightforward to calculate the projection, but how to calculate the projection when there are more than one blocker encountered simultaneously? It is an open problem for LP researchers still (M. Hassan, M. Rehmani, and J. Chen 2019)\textsuperscript{[1]} (P. Wang et al. 2020)\textsuperscript{[2]}. We suggest a projection calculation by means of the Hat matrix in this paper. Linear adjusting programming will attract interest in economic restructuring, financial prediction, and reinforcement learning. It might bring a new light to solve the linear programming problem with a strong polynomial solution.

1 Introduction

Factor space is a mathematical theory proposed in 1982 (P. Z. Wang, M. Sugeno 1982)\textsuperscript{[3]}, which provides a unified mathematical framework for artificial intelligence and data science. All classifications and decisions are made in factor space. Based on that, a unified classification model was
refined in 2020. An intelligent decision process can be described as a point $y$, the dynamic state of an agent (or, observed and controlled by the agent). The agent’s target is represented by a vector $g$, which expels the movement of the point $y$. In practice, the vector $g$ can be the economic restructuring or management policy. Apart from the target, there is a group of hyperplanes limits the boundaries for $y$; each one cuts off half of the space, and they form a feasible convex region. The wall of a feasible region will block the moving trace when $y$ reaches the wall, and the agent needs to adjust its goal direction of $y$. If a hyperplane $\alpha : (\tau, y) = c$ blocks $y$, then we must adjust the new direction $d'$ as the projection of $g$:

$$d' = g \downarrow_{\alpha} = g - \tau * (g, \tau) / (\tau, \tau).$$  \hspace{1cm} (1.1)

But how do you calculate the projection when there are more than one blocker encountered in the same stage point? It is still an open problem for LP researchers (X. Cai et al. 2020)\cite{4} (Y. Wang et al. 2019)\cite{5}.

Since there is no differentiable function to represent the wall, the classical gradient method cannot be applied here, and a new model is proposed in next section. It will be of great significance to economic restructuring, financial prediction, machine learning and reinforcement learning.

Authors were engaged in the research of linear programming. Simplex presented by Dantzig (G. B. Dantzig 2002)\cite{6} is a piece of perfect mathematical art; the only defect is that may rotate along edges. But how can the optimization be fastened? We need geometric description for simplex. The cone-cutting theory was proposed in 2011 (P. Z. Wang 2011)\cite{7}, which intuitively shows that a pivoting performed in the standard simplex tableau is taking a cone-cutting in the dual space. Utilizing the idea of cone-cutting, an algorithm was put forward in 2014 (P. Z. Wang 2014)\cite{8}, named Gradient Falling, which searching the minimum point as a body falling by gravity in dual space. The critical problem is how to calculate the projection of a vector in subspace, and a projection calculation algorithm was given by the author himself. In 2017, Lui improved the algorithm by citing null projection method from Matlab, renamed Gravity Sliding algorithm (P. Z. Wang 2017)\cite{9}.

Inspired by Prof. Peizhang Wang’s original ideas, this paper will employee the Hat projection from statistical learning, it is easier and clearer than Wang’s projection calculation in grading falling algorithm and the null projection in Lui’s gravity sliding algorithm. In this paper, we will use the Hat matrix to do projection calculation, which is more precise than the methods mentioned before.

We propose the linear adjusting programming in Section 3. The Hat projection is introduced in Section 4. The algorithm of linear adjusting programming and an example and the comparison with the Simplex, Episode, Interior Point method are given in Section 5. Conclusions are put in Section 6.

2 Linear Programming

Linear programming (LP) (also known as linear optimization) is a technique to accomplish the best outcome (such as maximum revenue or lowest loss) in a mathematical model.

More formally, linear programming is a method for the optimization of a linear objective function, subject to linear equality and linear inequality constraints. Its feasible region is a convex polytope, which is a set defined as the intersection of finitely many half spaces, each of which is defined by a linear inequality (Dimitris Bertsimas, John N. Tsitsiklis 1997)\cite{10}. Its objective function is a real-valued affine (linear) function defined on this polyhedron. A linear programming
algorithm finds a point in the polyhedron where this function has the smallest (or largest) value if such a point exists (Dimitris Bertsimas, John N. Tsitsiklis 1997)\[10\].

Linear programs are problems that can be expressed in canonical form as (Dimitris Bertsimas, John N. Tsitsiklis 1997)\[10\]

\[
\begin{align*}
\text{maximize} & \quad c^T x \\
\text{subject to} & \quad Ax \leq b \\
& \quad x \geq 0
\end{align*}
\]

(2.1)

where \(x\) represents the vector of variables (to be determined), \(c\) and \(b\) are vectors of (known) coefficients, \(A\) is a (known) matrix of coefficients, and \((\cdot)^T\) is the matrix transpose. The expression to be maximized or minimized is called the objective function \((c^T x\) in this case). The inequalities \(Ax \leq b\) and \(x \geq 0\) are the constraints which specify a convex polytope over which the objective function is to be optimized. In this context, two vectors are comparable when they have the same dimensions. If every entry in the first is less-than or equal-to the corresponding entry in the second then we can say the first vector is less-than or equal-to the second vector.

Standard form is the usual and most intuitive form of describing a linear programming problem. It consists of the following three parts:

A linear function to be maximized \(e.g.\)

\[
f(x_1, x_2) = c_1 x_1 + c_2 x_2
\]

(2.2)

Problem constraints of the following form \(e.g.\)

\[
\begin{align*}
a_{11} x_1 + a_{12} x_2 & \leq b_1 \\
a_{21} x_1 + a_{22} x_2 & \leq b_2 \\
a_{31} x_1 + a_{32} x_2 & \leq b_3
\end{align*}
\]

(2.3)

Non-negative variables \(e.g.\)

\[
\begin{align*}
x_1 & \geq 0 \\
x_2 & \geq 0
\end{align*}
\]

(2.4)

The problem is usually expressed in matrix form, and then becomes:

\[
\begin{align*}
\text{maximize} & \quad \{c^T x \mid Ax \leq b \land x \geq 0\}
\end{align*}
\]

(2.5)

Other forms, such as minimization problems, problems with constraints on alternative forms, as well as problems involving negative variables can always be rewritten into an equivalent problem in standard form. There are several open problems in the theory of linear programming, the solution of which would represent fundamental breakthroughs in mathematics and potentially major advances in our ability to solve large-scale linear programs.

Does LP admit a strongly polynomial-time algorithm?

Does LP admit a strongly polynomial-time algorithm to find a strictly complementary solution?

Does LP admit a polynomial-time algorithm in the real number (unit cost) model of computation?

These closely related set of problems have been cited by Stephen Smale as among the 18 greatest unsolved problems of the 21st century. In Smale’s words, the third version of the problem is “the main unsolved problem of linear programming theory”. While algorithms exist to solve
linear programming in weakly polynomial time, such as the ellipsoid methods and interior-point techniques, no algorithms have yet been found allowing strongly polynomial-time performance in the number of constraints and the number of variables. The development of such algorithms would be of great theoretical interest, and perhaps allow practical gains in solving large LPs as well. Although the Hirsch conjecture was recently disproved for higher dimensions, it still leaves the following questions open (M. Hassan, M. Rehmani, and J. Chen 2020)[11] (P. Wang et al. 2019)[12].

Are there pivot rules which lead to polynomial-time simplex variants?
Do all polytopal graphs have polynomially bounded diameter?

These questions relate to the performance analysis and development of simplex-like methods.

3 Linear Adjusting Programming

Definition A Given an objective vector $g = c$ in a prime factor space $X = R^n$ and a group of constraint hyperplanes $\{\beta_i : (\eta_i, x) = b_i\} (i = 1...m)$, where $c$, $x$ stand for column vectors; $(\eta_i, x)$ stands for the inner product of $\eta_i$ and $x$. The prime linear adjusted programming is denoted as follows:

$$ (AP) : Up\{(c, x) | Ax \leq b; x \geq 0\}. \tag{3.1} $$

which aims get an upper value of $(c, x)$ in the constraint area $\cap \{\beta_i | i = 1...m\}$, where $\beta_i = \{x | (\eta_i, x) \leq c_i\}$.

Definition A’ Given an objective vector $g = b$ in the dual factor space $Y = R^m$ and a group of constraint hyperplanes $\alpha_j = \{y | (\tau_j, y) \geq c_j\} (j = 1...n)$, where $b$, $y$ stand for row vectors, the dual linear adjusted programming is denoted as follows:

$$ (AD) : Lw\{(b, y) | yA \geq c; y \geq 0\}; \tag{3.2} $$

which aims to get a lower value of $(b, y)$ under the constraints of $\cap \{\alpha_j | j = 1...n\}$, where $\alpha_j = \{y | (\tau_j, y) \geq c_j\}$.

Linear adjustment programming is different from linear programming that the purpose of linear programming is to get the ultimate best target, while linear adjusting programming only makes a direct action in a short time. What resistance will come next? How does an agent orient himself? What further resistance may be encountered, and how should the direction be adjusted? If we can find the best, it is a blessing; if not, that’s fine. However, we request the next adjusting is given at least.

For convenience, we just state the linear adjusted programming problem in a dual form (AD).

There are two tasks that need to be overcome in linear adjusted programming: 1. Suppose that the position of the moving point is $y = P_t$ at time $t$, given a direction $d$, calculating the next stage point $P_{t+1}$ where one or several constraint hyperplanes block the moving line. 2. Calculating the projection of $d$ in a subspace.

To perform the first task, we have the following proposition: The next stage point can be calculated as follows:

$$ t_j : = (c_j - (\tau_j, P)/(\tau_j, d) \ (j = 1, \ldots, n) $n

$$ f^* : = \operatorname{Argmin}_j \{t_j | t_j > 0; j = 1, \ldots, n\} \tag{3.3} $$

$$ P_{t+1} : = P_t + t^* d $$

4
The coordinate of a point on the ray moving ray starting from \( P_t \) is \( y = P_t + td (t > 0) \), where \( t \) is a parameter. If a constraint hyperplane \( \alpha_j : (\tau_j, y) = c_j \), does not pass through the stage point \( P_t \), then it meets the ray if and only if

\[
(\tau_j, P_t + td) = c_j \ (t > 0).
\]  

(3.4)

When \( (\tau_j, d) \neq 0 \), we can calculate the parameter \( t \) for \( \alpha_j : \)

\[
t_j = (c_j - (\tau_j, P_t))/ (\tau_j, d).
\]  

(3.5)

Then select the first one who blocks the moving, let \( j^* \) be the index of it, we have that

\[
j^* = \text{Argmin}_j \{ t_j \mid t_j > 0; j = 1, ..., n \}.
\]  

(3.6)

Therefore, 3.3 holds.

4 Hat Projection

Projection is an important concept in linear distance spaces. A projection is a mapping \( P : X \to Y \), where \( Y \) is a subspace of linear distance space \( X \). For any point \( x \) in \( X \), there is one and only one point \( p(x) \) in \( Y \), such that

\[
(\forall y \in Y) \ d(x, p(x)) \leq d(x, y)
\]  

(4.1)

where \( d \) is a distance defined in \( X \) (in subspace \( Y \) also). Projection has following basic properties:

- **Idempotent Law:**
  \[
  (\forall x \in X) \ p(p(x)) = p(x);
  \]  

  (4.2)
  Projection obeys idempotent law, all image \( p(x) \) is its fixed points.

- **Transitivity:**
  If \( Y \subset A \subset X \), then
  \[
  \downarrow^A Y \downarrow_A X = \downarrow Y X;
  \]  

  (4.3)
  Projection relays in decrease subspaces. Where \( \downarrow_A Y \) stands for projection \( p : A \to Y \). \( \downarrow Y \) stands for projection \( P : X \to Y \). We will use the symbol later.

- **Complementarity:**
  Let \( A^\perp \) be the complementary subspace \( A^\perp \) in \( X \), then
  \[
  (\forall x \in X) \ x = \downarrow Y x + \downarrow Y \perp x
  \]  

  (4.4)
  Let \( \sigma = \{ \alpha_j : (\tau_j, y) = 0 \mid j = 1, \ldots, k \} = \{ \alpha_1, \ldots, \alpha_k \} \) be the set of planes blocking the way at a stage point with coefficient matrix \( A = \{ \tau_1, \ldots, \tau_k \} \). For the second task, we need to do projection calculation. Denote that

\[
A = \{ y \mid (\tau_j, y) = 0, \ j = 1, \ldots, k \}.
\]  

(4.5)
which is the subspace paralleling to the intersection of \( \sigma \)-blockers, the dimension of \( A \) is \( m-k \). It is obvious that the complementary subspace consists of all linear combinations of \( \{ \tau_1, \ldots, \tau_m \} \), i.e., \( A^\perp = \{ y \mid y = \lambda_1 \tau_1 + \ldots + \lambda_m \tau_m \} \).

How does the calculation of the projection of a vector \( g \) in a subspace work when \( k>1 \)?

If \( A = A_{m \times k} \) is a full rank matrix with \( m \geq k \), then it is obvious that \( B = A^T A \) must be symmetric and reversible.

**Definition B** Denote \( H = AB^{-1} A^T \), which is called a Hat matrix with respect to \( A \).

**Theorem 4.1.** Hat matrix is projective, i.e., for any \( y \in \mathbb{R}^m \), we have that \( HHy = Hy \).

**Proof.** We have that \( HHy = AB^{-1} A^T AB^{-1} A^T y = AB^{-1} BB^{-1} A^T y = AB^{-1} A^T y = Hy \). \( \square \)

As a transformation, a projective matrix is a mapping \( H: X \to Y \), which satisfies the idempotent Law \[ 4.2 \] and is called the projective matrix also. \( Y \) is the image of \( X \), consists of all fixed points of \( H \), called the stable subspace with respect to \( H \).

**Theorem 4.2.** The stable subspace of Hat matrix \( H \) is \( A^\perp \), i.e., \( Y = A^\perp \).

Set \( C = HA \). For \( j = 1, \ldots, m \), we have that \( C_j = (H_1, \ldots, H_m)^T A_j \).

Since \( HA = AB^{-1} A^T A = AB^{-1} B = A \), i.e., \( HA = A \), we have that \( C = A \) and then \( C_j = A_j \), so that \( A_j = C_j = (H_1, \ldots, H_m)^T A_j = HA_j \). Therefore, \( \tau_j = A_j \) is a fixed point of \( H \). It means that \( A^\perp \subseteq Y \).

If \( 0 \neq x \in A \), since that \( (A_j, x) = 0 \) holds for all \( j = 1, \ldots, k \), we have that \( A^T x = 0 \), and then \( Hx = AB^{-1} A^T x = 0 \). It means that \( x \notin Y \), so that \( \tau \cap Y = \emptyset \). Since the dimension of \( A \) is \( m-k \), the dimension of \( Y \) could not be larger than \( k \). Since the dimension of \( A^\perp \) is \( k \), while \( A^\perp \cap Y = A^\perp \), if \( A^\perp \neq Y \), then the dimension of \( Y \) must be larger than \( k \), this is a contradiction, so that \( A^\perp = Y \).

**Theorem 4.3.** The projection of \( g \) in \( A \) is that:

\[
g \downarrow A = g - Hg \quad (4.6)
\]

Since \( Y \) is the direct sum of \( A^\perp \) and \( A \), according to (0.4), for any \( g \in Y \), we have that

\[
g = g \downarrow A + g \downarrow A^\perp \quad (4.7)
\]

Since \( A^\perp \) is the stable subspace of \( H \), we have that \( g \downarrow A^\perp = Hg \). (0.1) is held.

So far, we have got the formula of projection calculation. Is it consistent with classical formula of projection shown in (4.1) of the introduction? Yes! Suppose that \( A = A_{m \times 1} = \tau = (\tau_1, \ldots, \tau_m)^T \), we have that \( B = A^T A = (\tau, \tau) \), and \( B^{-1} = 1/(\tau, \tau) \), then \( H = AB^{-1} A^T = AA^T/(\tau, \tau) \). Where \( AA^T = R \) is a \( m \times m \) matrix with elements \( r_{ij} = \tau_i \tau_j \). We have that:

\[
Rg = (\tau_1(g, \tau), \ldots, \tau_m(g, \tau))^T
\]

= \((\tau_1, \ldots, \tau_m)^T(g, \tau) = (\tau, \tau)g
\]

\[
Hg = [AA^T/(\tau, \tau)]g = AA^T g/(\tau, \tau)
\]

= \(Rg/(\tau, \tau) = \tau(g, \tau)/(\tau, \tau)
\]

\[
g \downarrow \alpha = g - Hg = g - \tau(g, \tau)/(\tau, \tau)
\]

It is coincident with \[ 1.3 \] of the introduction.
5 Algorithm of Linear Adjusting Programming

Suppose that the feasible region is not empty and a point $P_0$ in $D$ is given.

$$s := 0; d := g; \sigma := empty; \quad (5.1)$$

**Step 1** Determination of the next stage point $P_{s+1}$ by means of (0.3) of the linear adjusting programming;

$$t_j := [c_j - (\tau_j, P_s)] / (\tau_j, d) \quad (j = 1, \ldots, m + n; (\tau_j, d) \neq 0);$$

$$f^* = \text{Argmin}_j \{t_j \mid t_j > 0\};$$

$$P_{s+1} := P_s + t^* d; \text{Stop.}$$

Or, $\sigma := \sigma + \{\alpha_j | j = f^*\} = \{\alpha_1, \ldots, \alpha_k\}; go to Step 2$;

**Step 2** Calculating Hat projection:

$$A := s = \{\alpha_1, \ldots, \alpha_k\};$$

$$d_A^\perp := d - H d;$$

$$s := s + 1; \text{go back Step 1.}$$

**Example:**

**Given (AD):** $Lw \{(b,y) \mid g_A \geq c; y \geq 0\}$

Where constraint vectors are as follows:

- $\tau_1 = (2,0,0,1,1)^T; c_1 = 1; \tau_2 = (-1,1,2,1,0)^T;$
- $\tau_2 = (0,1,0,1,1)^T; c_3 = 2; \tau_4 = (-1,1,1,0,0)^T; c_4 = 3; \tau_5 = (1,1,1,1)^T; c_5 = 4;$
- $\tau_6 = (1,0,0,0,0)^T; c_6 = 0; \tau_7 = (0,1,1,0,0)^T; c_7 = 0; \tau_8 = (0,0,1,1,1)^T; c_8 = 0; \tau_9 = (0,0,0,1,1)^T;$
- $c_9 = 0; \tau_{10} = (0,0,0,0,1)^T; c_{10} = 0; \text{The dual objective vector is } g = -O = (-4,-1,-4,-6,-2).$

And the current point $P_0 = (7,4,7,6,5)$, so let’s do linear adjusting programming in some steps.

$$s := 0; d := g; P := P_0; \sigma := empty; \quad (5.4)$$

**Step 1** Calculating the next stage point:

$$t_j := [c_j - (\tau_j, P_s)] / (\tau_j, d) \quad (j = 1, \ldots, 10);$$

$$f^* = \text{Argmin}_j \{t_j \mid t_j > 0\} = \text{Argmin}_j \{t_4 = 1, \ t_9 = 1\} = \{4,9\}$$

$$\sigma := \sigma + \{4,9\} = \{4,9\}$$

$$P_1 := P_0 + t^*_4 d = (7,4,7,6,5) + (-4,-1,-4,-6,-2) = (3,3,3,0,3)$$

**Step 2** Calculating Hat projection of $g$ in the subspace of $\{\tau_j | j \in \sigma\}$;

$$B = A^T A = \begin{pmatrix} -1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \end{pmatrix}, \quad B^{-1} = \begin{pmatrix} 1/3 & 0 \\ 0 & 1 \end{pmatrix}$$

$$H = A B^{-1} A^T = \begin{pmatrix} -1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{pmatrix}, \quad H g = (-13, -2, -11, 0, -6) / 3$$

$$s := s + 1 = 1; \text{go back Step 1.}$$
Step 1 Determining next stage point:

\[ j^* = \operatorname{Argmin}_j \{ t_j = (c_j - (\tau_j, P)) / (\tau_j, d) \mid t_j > 0 \} = \operatorname{Argmin}_j \{ t_6 = 3/13 \} = 6; \]

\[ P_2 := (3, 3, 3, 0, 3) + 3/13 (-13, -2, -11, 0, -6) = (0, 33/13, 6/13, 0, 21/13); \]

\[ \sigma := \sigma \cup \{ j \mid j = j^* \} = \{ 4, 9, 6 \} \]

Step 2 Calculating Hat projection:

\[ B = AA^T = \begin{pmatrix} -1 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} -1 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 3 & 0 & -1 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix} \]

\[ (B, I) = \begin{pmatrix} 3 & 0 & -1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 & 0 & 1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & -1/2 & 3/3 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 2/3 & 1/3 & 0 & 1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & 1/2 & 0 & 1/2 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1/2 & 0 & 3/2 \end{pmatrix} = (I, B^{-1}) \]

\[ B^{-1} = \begin{pmatrix} 1/2 & 0 & 1/2 \\ 0 & 1 & 0 \\ 1/2 & 0 & 3/2 \end{pmatrix} \]

\[ H = AB^{-1}A^{-T} = \begin{pmatrix} -1 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1/2 & 0 & 1/2 \\ 0 & 1 & 0 \\ 1/2 & 0 & 3/2 \end{pmatrix} \begin{pmatrix} -1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1/2 & 1/2 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \]

(Where the inverse matrix is calculated by elimination method.)

\[ d : = g_A | g - Hg = (0, 3/2, -3/2, 0, -2); \]

\[ s : = s + 1 = 2; \text{go back Step 1.} \]

Step 1 Determining the next stage point:

\[ j^* = \operatorname{Argmin}_j \{ t_j = (c_j - (\tau_j, P_2)) / (\tau_j, d) \mid t_j > 0 \} = \operatorname{Argmin}_j \{ t_1 = t_5 = t_8 = t_9 = 4/13 \} = \{ 1, 5, 8, 9 \} \]

\[ P_3 := (0, 33/13, 6/13, 0, 21/13) + 4/13 (0, 3/2, -3/2, 0, -2) = (0, 3, 0, 0, 1) \]

That is enough, so we stop here. Indeed we have got the dual optimal point of the corresponding LP problem: \( y_1^* = 0, y_2^* = 3, y_3^* = 0, y_4^* = 0, y_5^* = 1. \)

The following are attempts of a linear programming example by using different algorithms. Three algorithms are to be used: Simplex Method, Ellipsoid Method and one of interior point methods.

**Description of the example:**

\( R \) is used to denote the set of real numbers. \( A \) is a \( m \times n \) matrix; \( b \) is a column vector in \( R^m \); \( c \) is a column vector in \( R^n \).

\[ \text{(Primal)} \max \{ c^T x : Ax \leq b, x \geq 0 \}; \]

\[ \text{(Dual)} \min \{ y^T b : y^T A \geq c^T, y \geq 0 \}; \]

\[ \text{(5.10)} \]
Equivalently, the dual problem is $\min\{y^T b : y^T[A I] \geq [c^T 0^T] \}$.

**Given information:**

$$
A = \begin{bmatrix}
2 & -1 & 0 & -1 & 1 \\
0 & 1 & 1 & 1 & 1 \\
0 & 2 & 0 & 1 & 1 \\
1 & 1 & 1 & 0 & 1 \\
1 & 0 & 1 & 0 & 1
\end{bmatrix} ;
\quad
b = \begin{bmatrix}
4 \\
1 \\
2 \\
6 \\
2
\end{bmatrix} ;
\quad
\begin{bmatrix}
c \\
P_0
\end{bmatrix} = \begin{bmatrix}
7 \\
4 \\
6 \\
5
\end{bmatrix} ;
\quad
(5.11)
$$

**Simplex Method (full tableau):**

Consider the primal problem with slack variable $s \geq 0$ such that:

$$
(Primal) \equiv \min\{-c^T x : Ax + Is = b, x \geq 0, s \geq 0\};
\quad
(5.12)
$$

Pivoting rule used: Bland’s rule. (always choose the smallest index when deciding entering and leaving index)

We assume the index order is:

$x_1 < x_2 < x_3 < x_4 < x_5 < s_1 < s_2 < s_3 < s_4 < s_5$.

Notice that $b \geq 0$, a natural basic feasible solution is given by $x=0$, $s=b$, so, we have the initial tableau:

**Initial tableau:**

|   | $x_1^*$ | $x_2$ | $x_3$ | $x_4$ | $s_1$ | $s_2$ | $s_3$ | $s_4$ | $s_5$ |
|---|---------|-------|-------|-------|-------|-------|-------|-------|-------|
| 0 | 0       | -1    | -2    | -3    | -4    | 0     | 0     | 0     | 0     |
| $s_1^*$ | 4      | 2     | -1    | 0     | 1     | 1     | 1     | 0     | 0     |
| $s_2$   | 1      | 0     | 1     | 1     | 1     | 0     | 1     | 0     | 0     |
| $s_3$   | 4      | 0     | 2     | 0     | 1     | 1     | 0     | 0     | 0     |
| $s_4$   | 6      | 1     | 1     | 0     | 1     | 0     | 0     | 0     | 1     |
| $s_5$   | 2      | 1     | 0     | 1     | 0     | 1     | 0     | 0     | 0     |

Table 1: initial tableau

**step 1:**

|   | $x_1$ | $x_2^*$ | $x_3$ | $x_4$ | $x_5$ | $s_1$ | $s_2$ | $s_3$ | $s_4$ | $s_5$ |
|---|-------|---------|-------|-------|-------|-------|-------|-------|-------|-------|
| 2 | 0     | -1.5    | -2    | -3.5  | -3.5  | 0.5   | 0     | 0     | 0     | 0     |
| $x_1$ | 2      | 1       | -0.5  | 0     | -0.5  | 0.5   | 0     | 0     | 0     | 0     |
| $s_2$ | 1      | 0       | 1     | 1     | 1     | 1     | 0     | 1     | 0     | 0     |
| $s_3$ | 4      | 0       | 2     | 0     | 1     | 1     | 0     | 0     | 1     | 0     |
| $s_4$ | 4      | 0       | 1.5   | 1.5   | 0.5   | -0.5  | 0     | 0     | 1     | 0     |
| $s_5$ | 0      | 0       | 0.5   | 0.5   | 0.5   | -0.5  | 0     | 0     | 0     | 1     |

Table 2: step 1

**step 2:**

step 3:

|   | $x_1$ | $x_2$ | $x_3$ | $x_4$ | $x_5^*$ | $s_1$ | $s_2$ | $s_3$ | $s_4$ | $s_5$ |
|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 2 | 0 | 0 | 1 | -3 | 0 | 0 | 0 | 7 |
| 2 | 2 | 1 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 1 |
| 3 | 1 | 0 | 0 | -1 | 0 | 1 | 0 | 0 | 0 | -2 |
| 4 | 4 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 5 | 0 | 0 | 1 | 2 | 1 | 0 | 0 | 0 | 0 | 2 |

*Table 3: step 2*

step 4:

|   | $x_1$ | $x_2$ | $x_3$ | $x_4$ | $s_1$ | $s_2$ | $s_3$ | $s_4$ | $s_5$ |
|---|---|---|---|---|---|---|---|---|---|
| 1 | 5 | 0 | 2 | 2 | 0 | 0 | 0 | 0 | 1 |
| 2 | 2 | 1 | 0 | 0 | 1 | 0 | 0 | 0 | 1 |
| 3 | 1 | 0 | 0 | -1 | 0 | 1 | 0 | 0 | 0 | -2 |
| 4 | 3 | 0 | 1 | -1 | 0 | 0 | 0 | 0 | 1 | 0 |
| 5 | 4 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | -1 |

**Table 4: step 3**

**Conclusion:** we have $x^* = [2 \ 0 \ 0 \ 1 \ 0]^T$ as an optimal solution for the primal problem with the objective value $c^T x = 5$. Note that we only have four steps in total.

**Ellipsoid Method:**
Recall that

$$(Primal) \ max \{ c^T x : A x \leq b, x \geq 0 \} ;$$

$$(Dual) \ min \{ y^T b : y^T A \geq c^T, y \geq 0 \} ;$$

$$A = \begin{bmatrix} 2 & -1 & 0 & -1 & 1 \\ 0 & 1 & 1 & 1 & 1 \\ 0 & 2 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 \end{bmatrix} ; \ b = \begin{bmatrix} 4 \\ 1 \\ 4 \\ 6 \\ 2 \end{bmatrix} ; \ c = \begin{bmatrix} 1 \\ 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} ;$$

(5.13) (5.14)
Ellipsoid method is used to find a feasible solution \([x^*] y^*\) for the following feasible region \(D\):
\[
c^T x = y^T b, Ax \leq b, x \geq 0, y^T A \succeq c^T, y \geq 0
\] (5.15)

By the dual theory of linear programming, \(x^*\) is an optimal solution for the primal problem and \(y^*\) is an optimal solution for the dual problem.

To represent as a polyhedron:
\[
D = \{A_0p \geq b_0\}, \text{where } A_0 = \begin{bmatrix} -A & 0 \\ 0 & A^T \\ I & 0 \\ 0 & I \\ c^T & -b \\ -c^T & b^T \end{bmatrix}, \quad p = \begin{bmatrix} x \\ y \end{bmatrix}, \quad b_0 = \begin{bmatrix} -b \\ c \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix};
\] (5.16)

Since all the entries are integers, we set the upper bound of all entries’ absolute values \(U=6\) by scanning the entries of \(A, b, c\). The dimension of variable is \(n=10\). Note that \(c^T x = y^T b\) ensures that the volume of \(D\) must be zero. In other words, there is no lower bound of the volume of \(D\). Therefore \(b\) should be perturbed in order to give a stopping criterion for the ellipsoid method. Perturbation step:
\[
\frac{1}{c} = 2(n+1)((n+1)U^{n+1} \approx 2.277225151082475 \times 10^{21}, \text{and } b_0 \text{ is perturbed to be } b_0 - \epsilon e \text{ where } e \text{ is the column vector of all ones. } \frac{1}{c} \text{ is the common denominator of } A_0 \text{ and } b_0, \text{so after multiply } \frac{1}{c} \text{ to each entry of } A_0 \text{ and } b_0, \text{we have the upper bound of absolute value of all the entries as } \hat{U} \approx 1.3663335090649485 \times 10^{22}. \text{The estimated number of iterations is } O(n^4 \log(n)). \text{Both the values for entries and the estimated number of iterations needed which is approximately } 5 \times 10^5 \text{are too big for human to calculate.}
\]

Conclusion: for this example the ellipsoid method is apparently not as practical as simplex method and gravity sliding algorithm.

**An interior point method: short-step affine scaling algorithm:**

Consider the dual problem:
\[
(Dual) \min \{y^T b : y^T A \succeq c^T, y \geq 0\} \overset{\text{equivalent}}{\longleftrightarrow} \min \{b^T x : [A^T - I] \begin{bmatrix} y \\ s \end{bmatrix} = c, y \geq 0, s \geq 0\};
\] (5.17)

\[
A = \begin{bmatrix} 2 & -1 & 0 & -1 & 1 \\ 0 & 1 & 1 & 1 & 1 \\ 0 & 2 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 \end{bmatrix}; \quad b = \begin{bmatrix} 4 \\ 1 \\ 4 \\ 6 \\ 2 \end{bmatrix}; \quad c = \begin{bmatrix} 1 \\ 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}; \quad P_0 = \begin{bmatrix} 7 \\ 4 \\ 6 \\ 5 \end{bmatrix};
\] (5.18)

For initialization:

Instead, we solve \(\min \{b^T y + M s_6 : [A^T - I] \begin{bmatrix} y \\ s \end{bmatrix} + (c - [A^T - I] e) s_6 = c, y \geq 0, s \geq 0\}\), where \(M\) is a large number and \(e\) is a column vector with all components equal to one. The initial point for the affine scaling algorithm to start with is \(y_j = 1\) for \(j = 1, 2, ..., 5\) and \(s_j = 1\) for \(j = 1, 2, ..., 6\).
Inputs:

\[
A_0 = \begin{bmatrix}
2 & 0 & 0 & 1 & 1 & -1 & 0 & 0 & 0 & 0 & -2 \\
-1 & 1 & 2 & 1 & 0 & 0 & -1 & 0 & 0 & 0 & -1 \\
0 & 1 & 0 & 1 & 1 & 0 & 0 & -1 & 0 & 0 & 0 \\
-1 & 1 & 1 & 0 & 0 & 0 & 0 & -1 & 0 & 3 \\
1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & -1 & 0
\end{bmatrix}
\]  

(5.19)

\[
c_0 = \begin{bmatrix}
4 & 1 & 4 & 6 & 2 & 0 & 0 & 0 & 0 & 10^4
\end{bmatrix}^T; \text{choose } M = 10^4;
\]  

(5.20)

\[
x^0 = \begin{bmatrix}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1
\end{bmatrix}^T \geq 0; \epsilon = 0.01; \text{choose } \beta = 0.997.
\]  

(5.21)

**Implementation** (by using MATLAB command window):

- **Iteration 1:**

  \[
  X_0 = \text{diag}(x^0); P^0 = (A_0X_0^2A_0^T)^{-1}A_0X_0^2c_0 = \begin{bmatrix}
  -0.28 \\
  -1.51 \\
  -0.14 \\
  2.21 \\
  0.64
  \end{bmatrix} \times 10^3;
  \]

  \[
r^0 = c_0 - A_0^TP^0 = 0.62 -1.21 0.16 1.29 -0.22 -0.28 -1.51 -0.14 2.21 0.64 1.29 \]  

  \[
  \text{Optimality check: ask } r^0 > 0 \text{ and } e^TX_0r^0 < \epsilon ? \text{ Ans: No.}
  \]

  \[
  \text{Unboundedness check: ask } -X_0^2r^0 \geq 0 ? \text{ Ans: No.}
  \]

  **Update of solution:**

  \[
  x^1 = x^0 - \beta \frac{X_0^2r^0}{\|X_0r^0\|} = \begin{bmatrix}
  0.83 \\
  1.33 \\
  0.95 \\
  0.64 \\
  1.06 \\
  1.08 \\
  1.04 \\
  0.39 \\
  0.82 \\
  0.64
  \end{bmatrix}^T
  \]

  **Objective value:**  

  \[
  c_0^Tx^1 = 6.43 \times 10^3.
  \]

- **Iteration 2:**

  \[
  X_1 = \text{diag}(x^1); P^1 = (A_0X_1^2A_0^T)^{-1}A_0X_1^2c_0 = \begin{bmatrix}
  0.32 \\
  -1.16 \\
  -0.48 \\
  2.51 \\
  -0.09
  \end{bmatrix} \times 10^3;
  \]

  \[
r^1 = c_0 - A_0^TP^1 = 0.80 -0.78 -0.10 1.42 0.25 0.32 -1.16 -0.48 2.51 -0.09 1.95 \]  

  \[
  \text{Optimality check: ask } r^1 > 0 \text{ and } e^TX_1r^1 < \epsilon ? \text{ Ans: No.}
  \]

  \[
  \text{Unboundedness check: ask } -X_1^2r^1 \geq 0 ? \text{ Ans: No.}
  \]

  **Update of solution:**

  \[
  x^2 = x^1 - \beta \frac{X_1^2r^1}{\|X_1r^1\|} = \begin{bmatrix}
  0.63 \\
  1.82 \\
  0.99 \\
  0.44 \\
  0.96 \\
  0.95 \\
  2.24 \\
  1.22 \\
  0.25 \\
  0.84 \\
  0.36
  \end{bmatrix}^T.
  \]

  **Objective value:**  

  \[
  c_0^Tx^2 = 3.60 \times 10^3.
  \]

- **Iteration 3:**
\( X_2 = \text{diag}(x^2); P^2 = (A_0 X_2^2 A_0^T)^{-1} A_0 X_2^2 c_0 = \begin{bmatrix} 264 \\ -502 \\ -354 \\ 1657 \\ -435 \end{bmatrix}; \)

\( r^2 = c_0 - A_0^T P^2 = \begin{bmatrix} 1.07 -0.37 -0.21 1.03 0.53 0.26 -0.50 -0.35 1.66 -0.44 5.05 \end{bmatrix}^T \times 10^3; \)

Optimality check: ask \( r^2 > 0 \) and \( e^T X_2 r^2 < \epsilon \)? Ans: No.

Unboundedness check: ask \(-X_2^2 r^2 \geq 0\)? Ans: No.

Update of solution:
\( x^3 = x^2 - \beta \frac{X_2^2 r^2}{\|X_2^2 r^2\|} = \begin{bmatrix} 0.47 \\ 2.30 \\ 1.07 \\ 0.36 \\ 0.77 \\ 0.85 \\ 3.22 \\ 1.43 \\ 0.21 \\ 0.96 \\ 0.10 \end{bmatrix}; \)

Objective value: \( c_0^T x^3 = 1059. \)

- Iteration 4:

\( X_3 = \text{diag}(x^3); P^3 = (A_0 X_3^2 A_0^T)^{-1} A_0 X_3^2 c_0 = \begin{bmatrix} 9.53 \\ -40.16 \\ -46.58 \\ 217.5 \\ -92.2 \end{bmatrix}; \)

\( r^3 = c_0 - A_0^T P^3 = \begin{bmatrix} 254 -37.1 -40.0 175.9 131.2 9.5 -40.7 -46.6 21.8 -92.2 9.32 \end{bmatrix}^T \times 10^3; \)

Optimality check: ask \( r^3 > 0 \) and \( e^T X_3 r^3 < \epsilon \)? Ans: No.

Unboundedness check: ask \(-X_3^2 r^3 \geq 0\)? Ans: No.

Update of solution:
\( x^4 = x^3 - \beta \frac{X_3^2 r^3}{\|X_3^2 r^3\|} = \begin{bmatrix} 0.41 \\ 2.49 \\ 1.11 \\ 0.34 \\ 0.69 \\ 0.85 \\ 3.64 \\ 1.52 \\ 0.20 \\ 1.05 \\ 0.0039 \end{bmatrix}; \)

Objective value: \( c_0^T x^4 = 51.01. \)

- Iteration 5:

\( X_4 = \text{diag}(x^4); P^4 = (A_0 X_4^2 A_0^T)^{-1} A_0 X_4^2 c_0 = \begin{bmatrix} 1.38 \\ 0.16 \\ -0.69 \\ -1.09 \\ 0.81 \end{bmatrix}; \)

\( r^4 = c_0 - A_0^T P^4 = \begin{bmatrix} 1.67 -0.37 1.78 4.33 0.49 1.38 0.16 -0.69 1.09 0.81 9.99965 \end{bmatrix}^T \times 10^3; \)

Optimality check: ask \( r^4 > 0 \) and \( e^T X_4 r^4 < \epsilon \)? Ans: No.

Unboundedness check: ask \(-X_4^2 r^4 \geq 0\)? Ans: No.

Update of solution:
\( x^5 = x^4 - \beta \frac{X_4^2 r^4}{\|X_4^2 r^4\|} = \begin{bmatrix} 0.40 \\ 2.55 \\ 1.06 \\ 0.32 \\ 0.69 \\ 0.82 \\ 3.58 \\ 1.56 \\ 0.20 \\ 1.02 \\ 0.0003 \end{bmatrix}; \)

Objective value: \( c_0^T x^5 = 11.98. \)

- Iteration 6:
\[ X_5 = \text{diag}(x^5); \quad P^5 = (A_0 X_5^2 A_0^T)^{-1} A_0 X_5^2 c_0 = \begin{bmatrix} 1.39 \\ 0.20 \\ -0.57 \\ 0.73 \end{bmatrix}; \]

\[ r^5 = c_0 - A_0^T P^5 = [1.22 \ -0.30 \ 1.93 \ 4.04 \ 0.25 \ 1.39 \ 0.20 \ -0.57 \ 0.73 \ 0.93 \ 1.0001 \times 10^4]^T; \]

Optimality check: ask \( r^5 > 0 \) and \( e^T X_5 r^5 < \epsilon \)? Ans: No.

Unboundedness check: ask \(-X_5^2 r^5 \geq 0\)? Ans: No.

Update of solution:
\[ x^6 = x^5 - \beta \frac{X_5 r^5}{\|X_5 r^5\|} = [0.34 \ 3.15 \ 0.39 \ 0.19 \ 0.65 \ 0.53 \ 2.77 \ 1.99 \ 0.19 \ 0.72 \ 0.00002]^T; \]

Objective value: \( c_0^T x^6 = 8.77 \).

- Iteration 7:

\[ X_6 = \text{diag}(x^6); \quad P^6 = (A_0 X_6^2 A_0^T)^{-1} A_0 X_6^2 c_0 = \begin{bmatrix} 1.32 \\ 0.076 \\ -0.041 \\ 0.27 \end{bmatrix}; \]

\[ r^6 = c_0 - A_0^T P^6 = [0.97 \ -0.03 \ 2.85 \ 3.92 \ -0.0075 \ 1.32 \ 0.076 \ -0.041 \ -0.27 \ 0.73 \ 1.0002 \times 10^4]^T; \]

Optimality check: ask \( r^6 > 0 \) and \( e^T X_6 r^6 < \epsilon \)? Ans: No.

Unboundedness check: ask \(-X_6^2 r^6 \geq 0\)? Ans: No.

Update of solution:
\[ x^7 = x^6 - \beta \frac{X_6 r^6}{\|X_6 r^6\|} = [0.97 \ 3.33 \ 0.13 \ 0.106 \ 0.654 \ 0.31 \ 2.42 \ 2.09 \ 0.19 \ 0.50 \ 0.00002]^T; \]

Objective value: \( c_0^T x^7 = 7.10 \).

- Iteration 8:

\[ X_7 = \text{diag}(x^7); \quad P^7 = (A_0 X_7^2 A_0^T)^{-1} A_0 X_7^2 c_0 = \begin{bmatrix} 1.45 \\ 0.018 \\ 0.0080 \\ 0.356 \\ 0.616 \end{bmatrix}; \]

\[ r^7 = c_0 - A_0^T P^7 = [0.85 \ 0.0022 \ 2.99 \ 3.90 \ -0.078 \ 1.45 \ 0.018 \ 0.0080 \ 1.45 \ 0.018 \ 1.0002 \times 10^4]^T; \]

Optimality check: ask \( r^7 > 0 \) and \( e^T X_7 r^7 < \epsilon \)? Ans: No.

Unboundedness check: ask \(-X_7^2 r^7 \geq 0\)? Ans: No.

Update of solution:
\[ x^8 = x^7 - \beta \frac{X_7 r^7}{\|X_7 r^7\|} = [0.20 \ 3.30 \ 0.071 \ 0.055 \ 0.69 \ 0.15 \ 2.30 \ 2.05 \ 0.17 \ 0.32 \ 0.155 \times 10^4]^T; \]

Objective value: \( c_0^T x^8 = 6.25 \).
Conclusion: At the last iteration, the value for \( r^* \) is almost nonnegative. By observing the values of \( x^k \) in each \( k \)-th iteration, the convergence of the first five entries of \( x^k \) to \( y^* = [0 3 0 0 1]^T \) and the convergence of objective values to the optimal value 5 can be seen easily. However, much more iterations are certainly needed for \( x^k \) to get close enough to the optimal value.

The simplex method is the most convenient, but it has been proven that it is not a polynomial-time algorithm, and it will spin down on the boundary of the feasible region. The advantage is that the method is simple and the disadvantage is that it does not have eyes. Existing simple algorithms are difficult to ensure that they do not descend into the slow orbit of a circle. Our method will avoid this and follow the path of falling body. We can see every step of the simplex method, and they can not see it by themselves. If our descending path passes through the vertex, and the next projection direction also passes through the vertex, then ConeCutting algorithm is used, and ConeCutting algorithm is the same as the simplex method. Hence, we can take advantage of its convenience, when we need to leave the vertex, go back to our method.
6 Conclusions

The model of linear adjusting programming focuses on the adjustment technique in artificial intelligence. The core problem is the projective calculation, which extends the gradient method to more wide areas. Linear adjusting programming is a relaxation of linear programming; the algorithm LAP, indeed, has realized the idea of gradient method in linear programming, which raises new hope for the searching of a strongly polynomial-time algorithm in LP. In the future, a combination of factor space, gravity sliding method, and gradient flow method, namely factor space in triple differential, might be finely developed as the last straw to crack the Smale’s problem with LP.

References

[1] M. Hassan, M. Rehmani, and J. Chen, DEAL: Differentially Private Auction for Blockchain based Microgrids Energy Trading, *IEEE Transactions on Services Computing*, in press, 2019, DOI: 10.1109/TSC.2019.2947471.

[2] P. Wang, J. Huang, Z. Cui, L. Xie and J. Chen, A Gaussian Error Correction Multi-Objective Positioning Model with NSGA-II, *Concurrency and Computation: Practice and Experience*, 32(5): e5464, 2020.

[3] P. Z. Wang, M. Sugeno, Factorial field and the background structure of fuzzy sets, in *Fuzzy Mathematics*, (02): pp. 45-54, 1982.

[4] X. Cai, Y. Niu, S. Geng, J. Zhang, Z. Cui, J. Li and J. Chen, An under-sampled software defect prediction method based on hybrid multi-objective cuckoo search, *Concurrency and Computation: Practice and Experience*, 32(5): e5478, 2020.

[5] Y. Wang, P. Wang, J. Zhang, Z. Cui, X. Cai, W. Zhang and J. Chen, A Novel Bat Algorithm with Multiple Strategies Coupling for Numerical Optimization, *Mathematics*, 7(2), Article Number: 135, 2019.

[6] G. B. Dantzig, “Linear programming,” *Operations Research*, 50(1): pp. 42-47, 2002.

[7] P. Z. Wang, Cone-cutting: A variant representation of pivot in Simplex, in *Information Technology & Decision Making* 10(1): pp.65-82, 2011.

[8] P. Z. Wang, Discussions on Hirsch conjecture and the existence of strongly polynomial-time simplex variants, in *Ann. Data. Sci.*, 1(1): pp. 41–71, 2014, //DOI 10.1007/s40745-014-0005-9.

[9] P. Z. Wang, H.C.Lui, H. T. liu, S.C. Guo, Gravity sliding algorithm for linear programming, in *Ann. Data. Sci.*, 2017, 4(2):193–210.//DOI 10.1007/s40745-017-0108-1

[10] Dimitris Bertsimas, John N. Tsitsiklis, *Introduction to Linear Optimization*, (Athena Scientific, Nashua, 1997).

[11] M. Hassan, M. Rehmani, and J. Chen, Differential Privacy Techniques for Cyber Physical Systems: A Survey, *IEEE Communications Surveys and Tutorials*, 22(1): 746-789, 2020.
[12] P. Wang, F. Xue, H. Li, Z. Cui, L. Xie and J. Chen, A Multi-Objective DV-Hop Localization Algorithm Based on NSGA-II in Internet of Things, Mathematics, 7(2), Article Number: 184, 2019.