A counterexample for the conjecture of finite simple groups

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Abstract

In this note we provide some counterexamples for the conjectures of finite simple groups, one of the conjectures said "all finite simple groups $G$ can be determined using their orders $|G|$ and the number of elements of order $p$, where $p$ the largest prime divisor of $|G|$".

Keywords: Finite Simple Groups, Classification Theorem, Quantitative Characterization.

AMS Mathematics Subject Classification(2010): 20D05 20D60.

For a finite group, the order of group and the element orders are two of the most important basic concepts. Let $G$ be a finite group and $\pi_e(G)$ be the set of element orders of $G$. In 1987, the author posed the following conjecture[1]:

**Conjecture.** Let $G$ be a group and $S$ a finite simple group. Then $G \cong S$ if and only if (a) $\pi_e(G) = \pi_e(S)$, and (b) $|G| = |S|$.

That is, for all finite simple groups we may characterize them using only their orders and the sets of their element orders (briefly, "two orders").

From 1987 to 2003, the authors of [1-7] proved that this conjecture is correct for all finite simple groups except $B_n$, $C_n$ and $D_n$ ($n$ even). In 2009, the authors of [8] proved that this conjecture is correct for $B_n$, $C_n$ and $D_n$ ($n$ even). Thus, this conjecture is proved and become a theorem, that is, all finite simple groups can determined by their "two orders".

**Question 1.** Weaken the condition of "two orders", characterize all finite simple groups.
Assume that \(|G| = p_1^{\alpha_1}p_2^{\alpha_2}...p_t^{\alpha_t}|, p_1 < p_2 < ... < p_t\) and \(\pi_e(G) = \{1, ..., m_3, m_2, m_1\}, 1 < ... < m_3 < m_2 < m_1\). Whether we may use the part of “order of group” (i.e. the order of Hall-subgroup of \(G\)) or the maximum part of \(\pi_e(G)\) (i.e. \(m_1, \{m_2, m_1\}\), or \(\{m_3, m_2, m_1\}\)) to characterize all finite simple groups?

**Remark.** A remarkable result of [9] shows the three largest element orders \(\{m_3, m_2, m_1\}\) determine the characteristic of simple groups of Lie type of odd characteristic.

**Result 1.**[10, Theorem] Let \(G\) be a group and \(p > 3\) a prime. Suppose that \(|G| = |PSL(2, p)| = (p - 1)p(p + 1)/2\) and \(m_1(G) = m_1(PSL(2, p)) = p, p \neq 7\). Then \(G \cong PSL(2, p)\). For \(p = 7\), if \(|G| = 168\) and \(\{m_2, m_1\} = \{4, 7\}\), then \(G \cong PSL(2, 7)\).

**Question 2.** Compare to \(\pi_e(G)\), consider other (conjugated) invariant quantitative sets of \(G\) and discuss their influence on \(G\).

Let \(G\) be a group and \(nse(G) = \{m_k|k \in \pi_e(G)\}\), where \(m_k\) denote the number of elements of order \(k\) in \(G\).

**Result 2.**[11, Theorem] A group \(G\) (which is not assumed finite) is isomorphic to \(A_i\) if and only if \(nse(G) = nse(A_i), i = 4, 5, 6\).

**Result 3.**[12, Main Theorem] Let \(G\) be a group and \(S\) a simple \(K_4\)-group (i.e. \(|\pi(S)| = 4\)). Then \(G \cong S\) if and only if the following hold: (1) \(|G| = |S|\), (2) \(nse(G) = nse(S)\).

**Result 4.**[13] If \(G\) is a group such that \(nse(G) = nse(PSL(2, q))\), where \(q \in \{7, 8, 11, 13\}\), then \(G \cong PSL(2, q)\).

In [14], the author took a new perspective and show that some families of simple groups are basically determined just by the number of elements of order \(p\), where \(p\) is the largest prime divisor of the order of the group. Then the author posed the following two conjectures:

**Conjecture C.** Let \(S\) be a finite simple group and \(p\) the largest prime divisor of \(|S|\). If \(G\) is a finite group with the same number of elements of order \(p\) as \(S\) and \(|G| = |S|\), then \(G \cong S\).

**Conjecture E.** Let \(S\) be a finite simple group that is not isomorphic to \(L_2(q)\), where \(q\) is a Mersenne prime and let \(p\) be the largest prime divisor of \(|S|\). If a finite group \(G\) is generated by elements of order \(p\) and \(G\) has the same number of elements of order \(p\) as \(S\), then \(G/Z(G) \cong S\).
In this note we provide some counterexamples for the above conjectures. We construct the counterexample of Conjecture C firstly.

**Counterexample 1.** Let $S = A_8 \cong L_4(2)$. Then $|S| = 2^6 \cdot 3^2 \cdot 5.7$ and $p = 7$ is the largest prime. Since $\pi_e(A_8) = \{1, 2, 3, ..., 7, 15\}$, we have $|C_S(< a >)| = 7$, where $o(a) = 7$.

From $N_S(< a >)/C_S(< a >) \cong$ a subgroup of $\text{Aut}(\langle a \rangle)$ and $|S : N_S(< a >)| \equiv 1 (mod 7)$, we have $S$ contains $2^7 \cdot 3^2 \cdot 5$ elements of order 7.

Let $G = L_3(4)$. From $|G| = 2^6 \cdot 3^2 \cdot 5.7$, $p = 7$ and $\pi_e(L_3(4)) = \{1, 2, 3, 4, 5, 7\}$, we have $G$ contains $2^7 \cdot 3^2 \cdot 5$ elements of order 7 similarly. $L_3(4)$ is not isomorphic to $A_8$.

**Counterexample 2.** Let $S = O_7(3)$ and $G = S_6(3)$. We have $|S| = |G| = 2^9 \cdot 3^9 \cdot 5.7.13$ and $p = 13$ is the largest prime. Since $\pi_e(O_7(3)) = \{1, 2, 3, ..., 10, 12, 13, 14, 15, 18, 20\}$ and $\pi_e(S_6(3)) = \{1, 2, 3, ..., 10, 12, 13, 14, 15, 18, 20, 24, 30, 36\}$, we may give same conclusion as Counterexample 1.

**Counterexample 3.** Let $S = L_2(7)$ and $G = ([Z_2 \times Z_2 \times Z_2]Z_7)Z_3$, a 2-Frobenius group. We have $|S| = |G| = 2^3 \cdot 3.7$ and $p = 7$ is the largest prime. Since $\pi_e(L_2(7)) = \{1, 2, 3, 4, 7\}$ we have $|C_S(< a >)| = 7$, where $o(a) = 7$. We have $S$ contains $2^4 \cdot 3$ elements of order 7. Similarly $G$ contains $2^4 \cdot 3$ elements of order 7, it is also a counterexample for Conjecture C.

For Conjecture E, $A_8$ and $L_3(4)$ is also a counterexample. Put $S = A_8$ and $G = L_3(4)$, then 7 is the largest prime divisor of $|A_8| = |L_3(4)|$. We get the number of elements of order 7 is $2^7 \cdot 3^2 \cdot 5$ in $S$ by calculation. The order of subgroup generated by all elements of order 7 greater than $2^7 \cdot 3^2 \cdot 5$, and great than the orders of all maximal subgroups of $A_8$(see [15]). Thus $A_8$ is generated by elements of order 7. By the same argument, $L_3(4)$ has the same number of elements of order 7 as $A_8$, but $L_3(4)$ is not isomorphic to $A_8$, we get the counterexample. Moreover, $O_7(3)$ and $S_6(3)$ is also a counterexample for Conjecture E.

However, if we avoid these same order and non-isomorphic simple groups, is the conjecture E hold? This is a question that we can be further studied.

Related the number of elements of same order Prof. J.G. Thompson posed the following open problem[16].

**Definition.** Groups $G_1$ and $G_2$ are called same order type groups, if $\pi_e(G_1) = \pi_e(G_2)$, and
for all $k \in \pi_e(G_1)$, the number of elements of order $k$ in $G_1$ and $G_2$ are equal. In particular, $nse(G_1) = nse(G_2)$.

In 1987, J.G. Thompson gave the following nonsolvable example of same order type groups which are not isomorphic$^{[16]}$.

$$G_1 = 2^4 : A_7 \text{ and } G_2 = L_3(4) : 2$$
both are maximal subgroups of $M_{23}$.

In the above example, $|G_1| = |G_2| = 2^7 \cdot 3^2 \cdot 5 \cdot 7$, $\pi_e(G_1) = \pi_e(G_2) = \{1, 2, 3, 4, 5, 6, 7, 8, 14\}$ and $nse(G_1) = nse(G_2) = \{1, 435, 2240, 6300, 8064, 6720, 5040, 5760\}$. $p = 7$ is the largest prime and the number of elements of order $p$ is 5760.

"This example shows that there are finite groups $G_1, G_2$ of the same order type which do not have the same set of composition factors"$^{[16]}$

**Open Problem.$^{[16]}$** Suppose $G_1, G_2$ are finite groups of same order type. Suppose also that $G_1$ is solvable. Is it true that $G_2$ is also necessarily solvable?

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