Distance and distance signless Laplacian spread of
connected graphs *

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Abstract For a connected graph $G$ on $n$ vertices, recall that the distance signless Laplacian matrix of $G$ is defined to be $Q(G) = Tr(G) + D(G)$, where $D(G)$ is the distance matrix, $Tr(G) = diag(D_1, D_2, \ldots, D_n)$ and $D_i$ is the row sum of $D(G)$ corresponding to vertex $v_i$. Denote by $\rho_D(G)$, $\rho_{min}(G)$ the largest eigenvalue and the least eigenvalue of $D(G)$, respectively. And denote by $q_D(G)$, $q_{min}(G)$ the largest eigenvalue and the least eigenvalue of $Q(G)$, respectively. The distance spread of a graph $G$ is defined as $S_D(G) = \rho_D(G) - \rho_{min}(G)$, and the distance signless Laplacian spread of a graph $G$ is defined as $S_Q(G) = q_D(G) - q_{min}(G)$. In this paper, we point out an error in the result of Theorem 2.4 in “Distance spectral spread of a graph” [G.L. Yu et al, Discrete Applied Mathematics. 160 (2012) 2474–2478] and rectify it. As well, we obtain some lower bounds on distance signless Laplacian spread of a graph.

Keywords: Distance matrix; Distance signless Laplacian; Spectral spread

1 Introduction

Throughout this article, we assume that $G$ is a simple, connected and undirected graph on $n$ vertices. Let $G = (V(G), E(G))$ be a graph with vertex set $V(G) = \{v_1, v_2, \ldots, v_n\}$ and edge set $E(G)$. We denote by $deg(v_i)$ (simply, $d_i$) the degree of vertex $v_i$, and for $u, v \in V$, we denote by $d_G(u, v)$ the distance between $u$ and $v$ in $G$. Recall that the distance matrix is $D(G) = (d_{ij})$ where $d_{ij} = d_G(v_i, v_j)$. For any $v_i \in V(G)$, the transmission of vertex $v_i$, denoted by $Tr_G(v_i)$ or $D_i$, is defined to be $\sum_{v_j \in V(G), j \neq i} d_G(v_i, v_j)$, which is equal to the row sum of $D(G)$ corresponding to vertex $v_i$. Sometimes, $D_i$ is called the distance degree. Let $Tr(G) = diag(D_1, D_2, \ldots, D_n)$ be the diagonal matrix of vertex transmissions of $G$. The distance signless Laplacian matrix of $G$ is defined as $Q(G) = Tr(G) + D(G)$ (see $\Pi$).

For a nonnegative real symmetric matrix $M$, we denote by $P_M(\lambda) = |\lambda I - M|$ its the characteristic polynomial. Its largest eigenvalue is called the spectral radius of $M$. For

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a graph $G$, the spectral radius of $\mathcal{D}(G)$ and $\mathcal{Q}(G)$, denoted by $\rho^D(G)$ and $q^D(G)$, are also called the distance spectral radius and the distance signless Laplacian spectral radius, respectively. Denote by $\rho^D_{\text{min}}(G)$ and $q^D_{\text{min}}(G)$ the least eigenvalue of $\mathcal{D}(G)$ and the least eigenvalue of $\mathcal{Q}(G)$, respectively. The distance spread of graph $G$ is defined as $S_D(G) = \rho^D(G) - \rho^D_{\text{min}}(G)$, and the distance signless Laplacian spread of graph $G$ is defined as $S_Q(G) = q^D(G) - q^D_{\text{min}}(G)$. Without ambiguity, $S_D(G)$ and $S_Q(G)$ are shortened as $S_D$ and $S_Q$ sometimes.

From [3] [12], we know that the spread of a matrix is a very interesting topic. As a result, in algebraic graph theory, the spread of some matrices of a graph also becomes interesting (see [5], [8], [4]). These cause the interests of the researchers on the problem about the distance concerning the distance spectrum of a graph has been studied extensively recently (see [5], [8], [4]). These cause the interests of the researchers on the problem about the distance spectral spread of a graph ([14], [10]). Motivated by these, in this paper, we proceed to consider the distance and distance signless Laplacian spread of a graph.

In Section 3, we point out an error in the result of Theorem 2.4 in “Distance spectral spread of a graph” [G.L. Yu, etc, Discrete Applied Mathematics. 160 (2012) 2474–2478] and rectify it. In Section 4, some lower bounds on distance signless Laplacian spread of a graph are obtained.

2 Some preliminaries

In this section, we introduce some definitions, notations and working lemmas.

Let $I_p$ be the $p \times p$ identity matrix and $J_{p,q}$ be the $p \times q$ matrix in which every entry is 1, or simply $J_p$ if $p = q$. For a matrix $M$, its spectrum $\sigma(M)$ is the multiset of its eigenvalues.

**Definition 2.1.** Let $M$ be a real matrix of order $n$ described in the following block form

$$M = \begin{pmatrix}
M_{11} & \cdots & M_{1t} \\
\vdots & \ddots & \vdots \\
M_{m1} & \cdots & M_{mt}
\end{pmatrix},$$

(2.1)

where the diagonal blocks $M_{ii}$ are $n_i \times n_i$ matrices for any $i \in \{1, 2, \ldots, t\}$ and $n = n_1 + \ldots + n_t$. For any $i, j \in \{1, 2, \ldots, t\}$, let $b_{ij}$ denote the average row sum of $M_{ij}$, i.e. $b_{ij}$ is the sum of all entries in $M_{ij}$ divided by the number of rows. Then $B(M) = (b_{ij})$ (simply by $B$) is called the quotient matrix of $M$. If in addition for each pair $i, j$, $M_{ij}$ has constant row sum, then $B(M)$ is called the equitable quotient matrix of $M$.

Consider two sequences of real numbers: $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n$, and $\mu_1 \geq \mu_2 \geq \ldots \geq \mu_m$ with $m < n$. The second sequence is said to interlace the first one whenever $\lambda_i \geq \mu_i \geq \lambda_{n-m+i}$ for $i = 1, 2, \ldots, m$.

**Lemma 2.2.** ([8]) Let $M$ be a symmetric matrix and have the block form as (2.1), $B$ be the quotient matrix of $M$. Then the eigenvalues of $B$ interlace the eigenvalues of $M$.

**Lemma 2.3.** ([13]) Let $M$ be defined as (2.7), and for any $i, j \in \{1, 2, \ldots, t\}$, $M_{ii} = l_i J_{n_i} + p_i I_{n_i}$, $M_{ij} = s_{ij} J_{n_i n_j}$ for $i \neq j$, where $l_i, p_i, s_{ij}$ are real numbers, $B = B(M)$ be the quotient matrix of $M$. Then

$$\sigma(M) = \sigma(B) \cup \{\lambda^{[t]} \mid i = 1, 2, \ldots, t\},$$

(2.2)

where $\lambda^{[t]}$ means that $\lambda$ is an eigenvalue with multiplicity $t$. [2]
By Lemma 2.3 we can obtain the distance (signless Laplacian) spectrum of $K_{a,b}$ as follows immediately, where $n = a + b$.

$$
\sigma(D(K_{a,b})) = \left\{ (-2)^{\lfloor n/2 \rfloor}, n - 2 \pm \sqrt{n^2 - 3ab} \right\},
$$
and

$$
\sigma(Q(K_{a,b})) = \left\{ (2n-a-4)^{\lfloor b/2 \rfloor}, (2n-b-4)^{\lfloor a/2 \rfloor}, \frac{5n - 8 \pm \sqrt{9n^2 - 32ab}}{2} \right\}.
$$

Lemma 2.4. ([4]) Let $H_n$ denote the set of all $n \times n$ Hermitian matrices, $A \in H_n$ with eigenvalues $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n$, and $B$ be a $m \times m$ principal matrix of $A$ with eigenvalues $\mu_1 \geq \mu_2 \geq \ldots \geq \mu_m$. Then $\lambda_i \geq \mu_i \geq \lambda_{n-m+i}$ for $i = 1, 2, \ldots, m$.

3 Results on $S_D$ for a bipartite graph

In [14], the authors obtained a lower bound for $S_D(G)$ with the maximum degree $\Delta$ of $G$, but it is found that the result is incorrect when $\Delta \leq |V(G)| - 2$. In this section, we rectify it.

Let $G = (V, E)$ be a graph. For $v_i, v_j \in V$, if $v_i$ is adjacent to $v_j$, we denote it by $v_i \sim v_j$ (simply, $i \sim j$). We let $t_v = \frac{\sum_{v_i \sim v} D_i}{d_v}$ be the average distance degree of $v$ ([14]).

Proposition 3.1. ([14], Theorem 2.4) Let $G$ be a simple connected bipartite graph on $n$ vertices with $S = \sum_{i=1}^{n} D_i$ and maximum degree $\Delta$. Suppose $\text{deg}(v_1) = \text{deg}(v_2) = \ldots = \text{deg}(v_k) = \Delta$. Then

(i) if $\Delta \leq n - 2$, we have

$$
S_D(G) \geq \max_{1 \leq i \leq k} \frac{\sqrt{a_i^2 - 4b_i(\Delta + 1)(n - \Delta - 1)}}{(\Delta + 1)(n - \Delta - 1)},
$$

where $a_i = 2(n - t_{v_i} - 1)\Delta^2 + (S - 2t_{v_i} - 2)\Delta + S$ and $b_i = \Delta^2(2S - t_{v_i}^2 - 2t_{v_i} - 1)$.

(ii) if $\Delta = n - 1$, we have

$$
S_D(G) = \begin{cases} 
0, & \text{if } n = 1; \\
2, & \text{if } n = 2; \\
n + \sqrt{n^2 - 3n + 3}, & \text{if } n \geq 3.
\end{cases}
$$

Let $N(v_i) = \{v_{i_1}, v_{i_2}, \ldots, v_{i_A}\}$ be the neighbors set of $v_i$, and $N[v_i] = N(v_i) \cup \{v_i\}$. In the proof of [3.1], the authors partition $V(G)$ into two parts $N[v_i]$ and $V(G) \setminus N[v_i]$ for some $1 \leq i \leq k$. Corresponding to this partition, $D(G)$ can be written as

$$
D(G) = \begin{pmatrix}
0 & 1 & 1 & \ldots & 1 \\
1 & 0 & 2 & \ldots & 2 \\
1 & 2 & 0 & \ldots & 2 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
1 & 2 & 2 & \ldots & 0 \\
\ast & \ast & \ast & \ldots & \ast
\end{pmatrix}.
$$

3
Then the author presented the quotient matrix of $\mathcal{D}(G)$ as:

$$B_1 = \begin{pmatrix}
\frac{2\Delta^2}{\Delta+1} & \frac{t_{v_i} \Delta + \Delta - 2\Delta^2}{n - \Delta - 1} \\
\frac{t_{v_i} \Delta + \Delta - 2\Delta^2}{n - \Delta - 1} & \frac{s_{-2v_i} \Delta + 2\Delta^2 (\Delta - 1)}{n - \Delta - 1}
\end{pmatrix}.$$  \hspace{1cm} (3.3)

The following example shows that (3.3) is false.

![Graph](image)

**Fig. 3.1. $G_1$**

For the above graph (see Fig. 3.1), it is clear that $\Delta = 3$, $t_{v_1} = \frac{34}{3}$, $S = 84$ and

$$\mathcal{D}(G) = (d_{ij})_{7 \times 7} = \begin{pmatrix}
0 & 1 & 1 & 1 & 2 & 2 & 2 \\
1 & 0 & 2 & 2 & 1 & 1 & 3 \\
1 & 2 & 0 & 2 & 3 & 1 & 1 \\
1 & 2 & 2 & 0 & 3 & 3 & 3 \\
2 & 1 & 3 & 3 & 0 & 2 & 4 \\
2 & 1 & 1 & 3 & 2 & 0 & 2 \\
2 & 3 & 1 & 3 & 4 & 2 & 0
\end{pmatrix},$$ \hspace{1cm} (3.4)

and by (3.3), we have the quotient matrix $B_1 = \begin{pmatrix}
\frac{18}{3} & \frac{19}{3} & \frac{19}{3} \\
\frac{25}{3} & \frac{25}{3} & \frac{25}{3}
\end{pmatrix}$. In fact the quotient matrix can be computed by the definition of the quotient matrix and (3.4) immediately as $B_2 = \begin{pmatrix}
\frac{18}{3} & \frac{25}{3} & \frac{25}{3} \\
\frac{25}{3} & \frac{25}{3} & \frac{25}{3}
\end{pmatrix} \neq B_1$, it is a contradiction.

Noticing the error in (3.3), with the similar technique, we rectify (3.1) as follows.

**Theorem 3.2.** Let $G$ be a simple connected bipartite graph on $n$ vertices, $\Delta$ be the maximum degree of $G$, $S = \sum_{i=1}^{n} D_i$. Suppose that $\deg(v_1) = \deg(v_2) = \ldots = \deg(v_k) = \Delta \leq n - 2$ for some $k$ ($1 \leq k \leq n$). Then

$$S_{\mathcal{D}(G)} \geq \max_{1 \leq i \leq k} \frac{\sqrt{a_i^2 + 4b_i(1 + \Delta)(n - \Delta - 1)}}{(1 + \Delta)(n - \Delta - 1)},$$ \hspace{1cm} (3.5)

where $a_i = (\Delta + 1)(S - 2D_i - 2t_{v_i} \Delta) + 2n\Delta^2$ and $b_i = D_i^2 - 2S\Delta^2 + 2D_i t_{v_i} \Delta + t_{v_i}^2 \Delta^2$.

**Proof.** $V(G)$ is partitioned into two parts which are $N[v_i]$ and $V(G) \setminus N[v_i]$ for some $1 \leq i \leq k$. Corresponding to this partition, $\mathcal{D}(G)$ is written as (3.2) and the quotient matrix $B$ of $\mathcal{D}(G)$ is presented as follow:

$$B = \begin{pmatrix}
\frac{2\Delta^2}{\Delta+1} & \frac{t_{v_i} \Delta + \Delta - 2\Delta^2}{n - \Delta - 1} \\
\frac{t_{v_i} \Delta + \Delta - 2\Delta^2}{n - \Delta - 1} & \frac{s_{-2v_i} \Delta + 2\Delta^2 (\Delta - 1)}{n - \Delta - 1}
\end{pmatrix}.$$  \hspace{1cm} (3.3)

Then

$$P_B(\lambda) = |\lambda I - B| = \lambda^2 - \frac{(\Delta + 1)(S - 2D_i - 2t_{v_i} \Delta) + 2n\Delta^2}{(1 + \Delta)(n - \Delta - 1)} \lambda - \frac{D_i^2 - 2S\Delta^2 + 2D_i t_{v_i} \Delta + t_{v_i}^2 \Delta^2}{(1 + \Delta)(n - \Delta - 1)}.$$
Let $P_B(\lambda) = 0$. It follows that
\[
\lambda_{1,2} = a_i \pm \sqrt{a_i^2 + 4b_i(1 + \Delta)(n - \Delta - 1)} \quad \frac{2(1 + \Delta)(n - \Delta - 1)}{2(1 + \Delta)(n - \Delta - 1)},
\]
where $a_i = (\Delta + 1)(S - 2D_i - 2t_v, \Delta) + 2n\Delta^2$ and $b_i = D_i^2 - 2S\Delta^2 + 2D_it_v, \Delta + t_v^2\Delta^2$. Using Lemma 2.2 gets (3.5). \hfill \Box

\textbf{Remark 3.1}

\begin{figure}[h]
\centering
\begin{tikzpicture}
  \node (v1) at (0,0) {$v_1$};
  \node (v2) at (1,1) {$v_2$};
  \node (v3) at (1,0) {$v_3$};
  \node (v4) at (0,1) {$v_4$};
  \node (v5) at (-1,0) {$v_5$};
  \node (v6) at (-1,-1) {$v_6$};
  \node (v7) at (2,1) {$v_7$};
  \node (v8) at (2,0) {$v_8$};
  \node (v9) at (2,-1) {$v_9$};

  \draw (v1) -- (v2);
  \draw (v3) -- (v4);
  \draw (v5) -- (v6);
  \draw (v7) -- (v8);
  \draw (v9) -- (v1);
\end{tikzpicture}
\caption{$G_2$}
\end{figure}

\begin{table}[h]
\begin{tabular}{|c|c|c|}
\hline
\text{graph} & Theorem 3.2 & approximate value \\
\hline
$G_1$ & $S_D(G) \geq 15.5960$ & $S_D(G) \approx 17.6820$ \\
$G_2$ & $S_D(G) \geq 19.0059$ & $S_D(G) \approx 20.9674$ \\
\hline
\end{tabular}
\caption{Table 3.1.}
\end{table}

By computation with mathematica for graphs $G_1$ and $G_2$ (see Figs. 3.1, 3.2 and Table 3.1), it seems that Theorems 3.2 is useful to evaluate the distance spread of a bipartite graph.

From the proof of Theorem 3.2 and by Lemma 2.2, we have the following corollary immediately.

\textbf{Corollary 3.3.} Let $G$ be a simple connected bipartite graph on $n \geq 3$ vertices with maximum degree $\Delta \leq n - 2$. Suppose that $\deg(v_1) = \deg(v_2) = \ldots = \deg(v_k) = \Delta$, $a_i, b_i$ are defined as Theorem 3.2 for $1 \leq i \leq k$. Then
\begin{enumerate}
\item[(i)] $\rho^p(G) \geq \max_{1 \leq i \leq k} \frac{a_i + \sqrt{a_i^2 + 4b_i(1 + \Delta)(n - \Delta - 1)}}{2(1 + \Delta)(n - \Delta - 1)}$;
\item[(ii)] $\rho^p_{\text{min}}(G) \leq \min_{1 \leq i \leq k} \frac{a_i + \sqrt{a_i^2 + 4b_i(1 + \Delta)(n - \Delta - 1)}}{2(1 + \Delta)(n - \Delta - 1)}$.
\end{enumerate}

\section{On $S_Q$}

In this section, we show some bounds of $S_Q$ for bipartite graphs and some bounds with some parameters.

\subsection{Bounds on $S_Q$ for bipartite graphs}

For a graph $G$, $W(G) = \sum_{1 \leq i < j \leq n} d_{ij}$ is called Wiener index. Thus, $W(G) = \frac{1}{2} \sum_{i=1}^{n} D_i$ and $S = 2W(G)$. Similar to the proof of Theorem 3.2 and Corollary 3.3, we get the following theorem and one corollary in term of Wiener index.
Theorem 4.1. Let $G$ be a simple connected bipartite graph on $n \geq 3$ vertices with maximum degree $\Delta$ and Wiener index $W$. Suppose that $\deg(v_1) = \deg(v_2) = \ldots = \deg(v_k) = \Delta$. Then

(i) if $\Delta \leq n - 2$, then

$$S_Q(G) \geq \max_{1 \leq i \leq k} \frac{a_i^2 + 4b_i(1 + \Delta)(n - \Delta - 1)}{(1 + \Delta)(n - \Delta - 1)},$$

where $a_i = 4(W - D_i - t_i\Delta)(\Delta + 1) + 2n\Delta^2 + nD_i + nt_i\Delta$ and $b_i = 4D_i^2 + 8D_i t_i\Delta + 4t_i^2\Delta^2 - 8W\Delta^2 - 4WD_i - 4Wt_i\Delta$.

(ii) if $\Delta = n - 1$, then $S_Q(G) = \sqrt{9n^2 - 32n + 32}$.

Remark 4.1

By computation with mathematica for graphs $G_1$ and $G_2$ (see Figs. 3.1, 3.2 and Table 4.1), it seems that Theorem 4.1 is useful to evaluate the signless Laplacian distance spread of a bipartite graph.

Corollary 4.2. Let $G$ be a simple connected bipartite graph on $n \geq 3$ vertices, $\Delta \leq n - 2$ be maximum degree of $G$. Suppose that $\deg(v_1) = \deg(v_2) = \ldots = \deg(v_k) = \Delta$ for some $k$ ($1 \leq k \leq n$), $a_i, b_i$ are defined as Theorem 4.1 for $1 \leq i \leq k$. Then

(i) $q^D(G) \geq \max_{1 \leq i \leq k} \frac{a_i + \sqrt{a_i^2 + 4b_i(1 + \Delta)(n - \Delta - 1)}}{2(1 + \Delta)(n - \Delta - 1)}$;

(ii) $q_{\min}^D(G) \leq \min_{1 \leq i \leq k} \frac{a_i + \sqrt{a_i^2 + 4b_i(1 + \Delta)(n - \Delta - 1)}}{2(1 + \Delta)(n - \Delta - 1)}$.

Lemma 4.3. Let $n \geq 4$ and $a$ be positive integers with $2a \leq n$. Then $S_Q(K_{a,n-a}) \geq S_Q(K_{\lceil \frac{n}{2} \rceil, \lceil \frac{n}{2} \rceil})$ with equality if and only if $G \cong K_{\lfloor \frac{n}{2} \rfloor, \lfloor \frac{n}{2} \rfloor}$.

Proof. By (2.4), we have

$$\sigma(Q(K_{a,n-a})) = \left\{ (2n - a - 4)^{[n-a-1]}, (n + a - 4)^{[a-1]}, \frac{5n - 8 \pm \sqrt{9n^2 - 32a(n-a)}}{2} \right\}.$$

It is checked that

$$5n - 8 + \frac{\sqrt{9n^2 - 32a(n-a)}}{2} > 2n - a - 4, \quad 5n - 8 - \frac{\sqrt{9n^2 - 32a(n-a)}}{2} > n + a - 4.$$

Then $q^D(K_{a,n-a}) = \frac{5n - 8 + \sqrt{9n^2 - 32a(n-a)}}{2}$, and

$$q_{\min}^D(K_{a,n-a}) = \left\{ \begin{array}{ll} n + a - 4, & a > 1 \\ \frac{5n - 8 - \sqrt{9n^2 - 32a(n-a)}}{2}, & a = 1. \end{array} \right.$$ 

When $0 < a \leq \frac{n}{2}$, it checked that $f(a) = \frac{5n - 8 + \sqrt{9n^2 - 32a(n-a)}}{2}$ is a decreasing function with respect to $a$, and $g(a) = n + a - 4$ is an increasing function with respect to $a$. Then we have $S_Q(K_{2,n-2}) > S_Q(K_{3,n-3}) > \ldots > S_Q(K_{\lfloor \frac{n}{2} \rfloor, \lfloor \frac{n}{2} \rfloor})$. 

| Graph | Theorem 4.1 | Approximate value |
|-------|-------------|-------------------|
| $G_1$ | $S_Q(G) \geq 15.6400$ | $S_Q(G) \approx 18.6100$ |
| $G_2$ | $S_Q(G) \geq 17.8520$ | $S_Q(G) \approx 21.1870$ |

Table 4.1.
Noting that \( n \geq 4 \), by directly computation, we have
\[
S_Q(K_{1,n-1}) - S_Q(K_{2,n-2}) = \sqrt{9n^2 - 32n + 32} - \frac{(5n - 8 + \sqrt{9n^2 - 64n + 128})}{2} - (n - 2) > 0.
\]
Thus \( S_Q(K_{1,n-1}) > S_Q(K_{2,n-2}) > S_Q(K_{3,n-3}) > \ldots > S_Q(K_{\lceil \frac{n}{2} \rceil, \lfloor \frac{n}{2} \rfloor}) \). This completes the proof.

Let \( G \) be a simple connected bipartite graph on \( n \) vertices. If \( n = 4 \), \( G \) is isomorphic to one of the following three graphs: (1) \( K_{2,2} \), (2) \( P_4 \), (3) \( S_4 \); if \( n = 5 \), \( G \) is isomorphic to one of the following five graphs: (4) \( K_{2,3} \), (5) \( G_5 \), (6) \( G_6 \), (7) \( P_5 \), (8) \( S_5 \) (see Fig. 4.1).

![Graphs](image)

**Fig. 4.1.** \( K_{2,2}-S_5 \)

By direct calculation, we obtain the following two tables.

| \( G \)  | \( q^D \) | \( q^D_{\min} \) | \( S_Q(G) \)  |
|---------|------------|----------------|---------------|
| \( K_{2,2} \) | 8          | 2              | 6             |
| \( P_4 \)   | 10.6056    | 2              | 8.6056        |
| \( S_4 \)   | 9.4641     | 2.5359         | 6.9282        |

Table 4.1

| \( G \)  | \( q^D \) | \( q^D_{\min} \) | \( S_Q(G) \)  |
|---------|------------|----------------|---------------|
| \( K_{2,3} \) | 11.3723    | 3              | 8.3723        |
| \( H_1 \)   | 13.3441    | 3.3113         | 10.0328       |
| \( H_2 \)   | 15.3119    | 3.6075         | 11.7044       |
| \( P_5 \)   | 17.1152    | 3.4385         | 13.6767       |
| \( S_5 \)   | 13.4244    | 3.5756         | 9.8488        |

Table 4.2

Combining Lemma 4.3 and the results in Table 4.1, we get the following corollary.

**Corollary 4.4.** For positive integers \( n \) and \( a \) with \( 2a \leq n \), \( S_Q(K_{a,n-a}) \geq S_Q(K_{\lceil \frac{n}{2} \rceil, \lfloor \frac{n}{2} \rfloor}) \) with equality if and only if \( G \cong K_{\lceil \frac{n}{2} \rceil, \lfloor \frac{n}{2} \rfloor} \).

Comparing the results in Tables 4.1 and 4.2, and checking more graphs with computer, it seems that among bipartite graphs, \( S_Q(K_{\lceil \frac{n}{2} \rceil, \lfloor \frac{n}{2} \rfloor}) \) always has the minimum \( S_Q \). Thus, we propose the following problem for further research.

**Conjecture 4.5.** Let \( G \) be a bipartite graph with \( n \) vertices. Then \( S_Q(G) \geq S_Q(K_{\lceil \frac{n}{2} \rceil, \lfloor \frac{n}{2} \rfloor}) \) with equality if and only if \( G \cong K_{\lceil \frac{n}{2} \rceil, \lfloor \frac{n}{2} \rfloor} \).

**Remark 4.2** In order to prove Conjecture 4.5, maybe it is better to show \( S_Q(G) \geq S_Q(K_{a,n-a}) \) holding for some \( a \) first, and then to using Lemma 4.3 to get the desired result.
4.2 Bound on $S_Q$ with clique number

A clique of a graph $G$ is a subgraph in which any pair of vertices is adjacent, and the clique number $\omega(G)$ (simply, $\omega$) is the number of vertices of the largest clique in $G$. In this subsection, we present a lower bound on $S_Q$ with clique number.

**Theorem 4.6.** Let $G$ be a simple connected graph with $n$ vertices, clique number $\omega \geq 2$ and Wiener index $W$. Suppose that $G_1, G_2, \ldots, G_k$ are all the cliques with order $\omega$, $s_i = \sum_{v_j \in V(G_i)} D_j$ for $1 \leq i \leq k$. Then

(i) if $\omega = n$, then $S_Q(G) = n$;

(ii) if $2 \leq \omega \leq n - 1$, then

$$S_Q(G) \geq \max_{1 \leq i \leq k} \frac{\sqrt{a_i^2 - 4b_i(n - \omega)\omega}}{(n - \omega)\omega},$$

(4.2)

where $a_i = n\omega(1 - \omega) + 4\omega(s_i - W) - ns_i$ and $b_i = 4W(\omega - 1) + 4s_i(W - s_i)$.

**Proof.** (i) If $\omega = n$, then $G \cong K_n$. By direct calculation, we have $q^P(G) = 2n - 2$ and $q^P_{\min}(G) = n - 2$. Thus $S_Q(G) = q^P(G) - q^P_{\min}(G) = n$.

(ii) If $\omega \leq n - 1$, for $1 \leq i \leq k$, suppose $V(G_i) = \{v_1, v_2, \ldots, v_{\omega}\}$. Then $V(G)$ is divided into two parts $V(G_i)$ and $V(G) \setminus V(G_i)$. Corresponding to this partition, the quotient matrix of $Q(G)$ is written as

$$B = \begin{pmatrix} s_i - s_i(\omega - 1) & s_i(\omega - 1) \\ s_i(\omega - 1) & W - 3s_i + s_i(\omega - 1) \end{pmatrix}.$$  

Similar to the proof of Theorem 4.2 solving $P_B(\lambda) = 0$ and using Lemma 2.2 get 1.2).

**Remark 4.3** Recall that a kite $K_{i_n,\omega}$ is the graph obtained from a clique $K_\omega$ and a path $P_{n-\omega}$ by adding an edge between an endpoint of the path and a vertex of the clique. For a kite $G = K_{i_5,3}$, by Theorem 4.6 we have $S_Q(G) \geq 10.6158$. On the other hand, by direct calculation, we obtain $S_Q(G) \approx 11.3395$. This shows that Theorem 4.6 is useful to evaluate the distance signless Laplacian spread of a graph with given clique number.

By Lemma 2.2 and Theorem 4.6 we have

**Corollary 4.7.** Let $G$ be a simple connected graph with $n$ vertices and clique number $\omega$. Suppose that $G_1, G_2, \ldots, G_k$ are all the cliques with order $\omega$, $a_i$, $b_i$ are defined as Theorem 4.6 for $1 \leq i \leq k$. Then

(i) $q^P(G) \geq \max_{1 \leq i \leq k} \left\{ -a_i + \sqrt{a_i^2 - 4b_i(n - \omega)\omega} \right\}$;

(ii) $q^P_{\min}(G) \leq \min_{1 \leq i \leq k} \left\{ -a_i - \sqrt{a_i^2 - 4b_i(n - \omega)\omega} \right\}$.

4.3 Bound on $S_Q$ with diameter

In this subsection, we obtain a lower bound on $S_Q$ of a graph with diameter. In a graph, a path is called a diameter path if its length is equal to the diameter of this graph.

**Theorem 4.8.** Let $G$ be a simple connected graph with $n$ vertices, diameter $d$ and Wiener index $W$. Suppose that $P_1, P_2, \ldots, P_k$ are all the diameter paths, and suppose that $s_i = \sum_{v_j \in V(P_i)} D_j$ for $1 \leq i \leq k$. Then
(i) if \( d = 1 \), then \( S_Q(G) = n; \)
(ii) if \( 2 \leq d \), then
\[
S_Q(G) \geq \max_{1 \leq i \leq k} \frac{\sqrt{a_i^2 - 12b_i(d + 1)(n-1 - d)}}{3(d+1)(n-1 - d)}, \tag{4.3}
\]
where \( a_i = 12(1+d)(s_i - W) - nd(d+1)(d+2) - 3ns_i \) and \( b_i = 4d(d+1)(d+2)W + 12s_i(W-s_i). \)

Proof. (i). If \( d = 1 \), then \( G \cong K_n \). By direct calculation, \( q^D(G) = 2n-2, q^D_{\min}(G) = n - 2 \).
Thus, \( S_Q(G) = n \).
(ii). If \( 2 \leq d \), for \( 1 \leq i \leq k \), we let \( T = \sum_{v_i,v_j \in V(P_i)} d_{sj}. \) Then when \( d \) is even, we have
\[
T = 2(1 + 2 + ... + d) + 2[1 + 1 + 2 + 3 + ... + (d-1)] + ... \\
+ 2[1 + 1 + 2 + 3 + ... + (\frac{d}{2} - 1) + (\frac{d}{2} - 1) + \frac{d}{2} + (\frac{d}{2} + 1)] \\
+ 2[1 + 2 + ... + (\frac{d}{2} - 1) + \frac{d}{2}], \\
= d(d+1) + [(d-1)d + 2 \times 1] + [(d-2)(d-1) + 2 \times 3] + ... \\
+ [(\frac{d}{2} + 1)(\frac{d}{2} + 2) + (\frac{d}{2} - 1)\frac{d}{2} + \frac{d}{2}(\frac{d}{2} + 1)] \\
= 1^2 + 2^2 + 3^2 + ... + d^2 + 1 + 2 + 3 + ... + d \\
= \frac{d(d+1)(d+2)}{4}.
\]
When \( d \) is odd, we have
\[
T = 2(1 + 2 + ... + d) + 2[1 + 1 + 2 + 3 + ... + (d-1)] + ... \\
+ 2(1 + 2 + 3 + ... + \frac{d-1}{2} + \frac{d+1}{2} + \frac{d+1}{2}) \\
= d(d+1) + [(d-1)d + 2 \times 1] + [(d-2)(d-1) + 2 \times 3] + ... \\
+ (\frac{d+1}{2})(\frac{d+1}{2} + 1) + (\frac{d-1}{2})(\frac{d-1}{2} + 1) \\
= 1^2 + 2^2 + 3^2 + ... + d^2 + 1 + 2 + 3 + ... + d \\
= \frac{d(d+1)(d+2)}{4}.
\]
Now \( V(G) \) is partition into two parts which are \( V(P_i) \) and \( V(G) \setminus V(P_i) \). Corresponding to this partition, the quotient matrix of \( Q(G) \) can be written as
\[
B = \begin{pmatrix}
\frac{\frac{d(d+1)(d+2)}{4} + s_i}{d+1} & \frac{s_i - \frac{d(d+1)(d+2)}{4}}{n-d-1} \\
\frac{s_i - \frac{d(d+1)(d+2)}{4}}{n-d-1} & \frac{\frac{d(d+1)(d+2)}{4} + s_i}{d+1}
\end{pmatrix}.
\]

Similar to the proof of Theorem 3.2 solving \( P_B(\lambda) = 0 \) and using Lemma 2.2 get \([1.3]\).

Remark 4.4 For \( G_1 \) shown in Fig. 3.1, then by Theorem 4.8 we have \( S_Q(G) \geq 12.1198 \).
From the Table 4.1, we know that \( S_Q(G_1) \approx 18.6100 \). This shows that Theorem 4.8 is useful to evaluate the distance signless Laplacian spread of a graph with given diameter.

Corollary 4.9. Let \( G \) be a simple connected graph with \( n \) vertices and diameter \( d \). Suppose that the path \( P_1, P_2, ..., P_k \) are all the diameter of \( G \), \( a_i, b_i \) are defined as Theorem 4.8 for \( 1 \leq i \leq k \). Then
(i) \( q^D(G) \geq \max_{1 \leq i \leq k} \left\{ \frac{-a_i + \sqrt{a_i^2 - 12b_i(d + 1)(n-1 - d)}}{6(d+1)(n-1 - d)} \right\}; \)
(ii) \( q^D_{\min}(G) \leq \min_{1 \leq i \leq k} \left\{ \frac{-a_i^2 - 12b_i(d + 1)(n-1 - d)}{6(d+1)(n-1 - d)} \right\}. \)
4.4 Bound On $S_Q$ for cacti with given circumference

A connected graph $G$ is a cactus if any two of its cycles have at most one common vertex. Circumference is the length of the longest cycle of a graph. In this section, we present a lower bound on $S_Q$ of a cactus with given circumference.

**Theorem 4.10.** Let $G$ be a cactus on $n$ vertices with circumference $l$ ($l \geq 3$) and Wiener index $W$. Suppose that cycles $C_1, C_2, ..., C_k$ are all with length $l$, $s_i = \sum_{v_j \in V(C_i)} D_j$ for $1 \leq i \leq k$. Then

$$S_Q(G) \geq \max_{1 \leq i \leq k} \frac{\sqrt{a_i^2 - 16b_i l(n - l)}}{4l(n - l)},$$

where

$$a_i = \begin{cases} t^3n + 4ns_i - 16l(s_i - W), & \text{if } l \text{ is even;} \\ t^3n + 4ns_i - ln - 16l(s_i - W), & \text{if } l \text{ is odd}, \end{cases}$$

and

$$b_i = \begin{cases} 4l^3W - 16s_i(s_i - W), & \text{if } l \text{ is even;} \\ 4(l^3 - l)W - 16s_i(s_i - W), & \text{if } l \text{ is odd}. \end{cases}$$

**Proof.** Corresponding to $C_i$, $V(G)$ is partitioned into two parts $V(C_i)$ and $V(G) \setminus V(C_i)$.

**Case 1:** $l$ is even.

Then for any $v \in V(C_i)$, the sum of distance from vertex $v$ to all other vertices on cycle $V(C_i)$ is $\frac{\ell^3}{4}$. Corresponding to the above partition, the quotient matrix of $Q(G)$ is written as

$$B = \begin{pmatrix} \frac{\ell^3}{4} + \frac{W}{4} & \frac{W}{4} - \frac{\ell^3}{4} \\ \frac{\ell^3 - W}{4} & 4W - 3\ell^3 + \frac{\ell^3}{4} \end{pmatrix}.$$  

Similar to the proof of Theorem 3.2 solving $P_B(\lambda) = 0$ and using Lemma 2.2 get (4.2).

**Case 2:** $l$ is odd.

Then for any $v \in V(C_i)$, the sum of distance from vertex $v$ to all other vertices on cycle $V(C_i)$ is $\frac{\ell^3 - l}{4}$. Corresponding to the above partition, the quotient matrix of $Q(G)$ is written as

$$B = \begin{pmatrix} \frac{\ell^3 - 1}{4} + \frac{W}{4} & \frac{W}{4} - \frac{\ell^3 - 1}{4} \\ \frac{\ell^3 - W}{4} & 4W - 3\ell^3 + \frac{\ell^3}{4} \end{pmatrix}.$$  

Similar to the proof of Theorem 3.2 solving $P_B(\lambda) = 0$ and using Lemma 2.2 get (4.2). \hfill \Box

![Fig. 4.2. $G_4$, $G_4$](image)

**Remark 4.4** Let $G_3$, $G_4$ are as shown in Fig. 4.2. By Theorem 4.10 we have $s_Q(G_3) \geq 11.5$. On the other hand, by direct calculation, we have $s_Q(G_3) \approx 12.8$. By Theorem 4.10 we have $s_Q(G_4) \geq 13.4$. On the other hand, by direct calculation, we have $s_Q(G_4) \approx 16.3$. These two examples show that Theorem 4.10 is useful to evaluate the $S_Q$ of the cacti with given circumference.
Corollary 4.11. Let $G$ be a cactus on $n$ vertices with given circumference $l \geq 3$. Suppose that cycles $C_1, C_2, ..., C_k$ are all with length $l$, $a_i, b_i$ are defined as Theorem 4.10 for $1 \leq i \leq k$. Then

1. $q^D(G) \geq \max_{1 \leq i \leq k} \{-a_i + \sqrt{a_i^2 - 16b_il(n-l)}/8l(n-l)}$;

2. $q^D_{\min}(G) \leq \min_{1 \leq i \leq k} \{-a_i - \sqrt{a_i^2 - 16b_il(n-l)}/8l(n-l)}$.

References

[1] M. Aouchiche, P. Hansen, Two Laplacian for the distance matrix of a graph, Linear Algebra Appl. 493 (2013) 21–33.

[2] M. Aouchiche, P. Hansen, Distance spectra of graphs: A survey, Linear Algebra Appl. 458 (2014) 301–386.

[3] A.T. Balaban, D. Ciubotariu, M. Medeleanu, Topological indices and real number vertex invariants based on graph eigenvalues or eigenvectors, J. Chem. Inf. Comput. Sci. 31 (1991) 517–523.

[4] D.M. Cvetković, M. Doob, H. Sachs, Spectra of Graphs Theory and Application, third ed, Johann Ambrosius Barth Verlag Heidelberg, Leipzig, 1995.

[5] R.L. Graham, H.O. Pollack, On the addressing problem for loop switching, Bell Syst. Tech. J. 50 (1971) 2495–2519.

[6] D.A. Gregory, D. Hershkowitz, S.J. Kirkland, The spread of the spectrum of a graph, Linear Algebra Appl. 332–334 (2001) 23–35.

[7] I. Gutman, M. Medeleanu, On the structure-dependence of the largest eigenvalue of distance matrix of an alkane, Indian J. Chem. A 37 (1998) 569–573.

[8] W.H. Haemers, Interlacing eigenvalues and graph, Linear Algebra Appl. 226–228 (1995) 593–616.

[9] C.R. Johnson, R. Kumar, H. Wolkowicz, Lower bounds for the spread of a matrix, Linear Algebra Appl. 71 (1985) 161–173.

[10] H.Q. Lin, On the least distance eigenvalue and its applications on the distance spread, Discrete Mathematics. 338 (2015) 868–874.

[11] M.H. Liu, B.L. Liu, The signless Laplacian spread, Linear Algebra Appl. 432 (2010) 505–514.

[12] L. Mirsky, The spread of a matrix, Mathematica 3 (1956) 127–130.

[13] M. Yang, L.H. You, J.X. Li, Several spectral radius of connected graphs and strongly connected digraphs with given connectivity, submitted.

[14] G.L. Yu, H.L. Zhang, H.Q. Lin, Y.R. Wu, J.L. Shu, Distance spectral spread of a graph, Discrete Applied Mathematics. 160 (2012) 2474–2478.