Breaking of vortex lines - a new mechanism of collapse in hydrodynamics

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A new mechanism of the collapse in hydrodynamics is suggested, due to breaking of continuously distributed vortex lines. Collapse results in formation of the point singularities of the vorticity field $|\mathbf{\Omega}|$. At the collapse point, the value of the vorticity blows up as $(t_0-t)^{-1}$, where $t_0$ is a collapse time. The spatial structure of the collapsing distribution approaches a pancake form: contraction occurs by the law $l_1 \sim (t_0-t)^{3/2}$ along the "soft" direction, the characteristic scales vanish like $l_2 \sim (t_0-t)^{1/2}$ along two other ("hard") directions. This scenario of the collapse is shown to take place in the integrable three-dimensional hydrodynamics with the Hamiltonian $\mathcal{H} = \int |\mathbf{\Omega}| dr$. Most numerical studies of collapse in the Euler equation are in a good agreement with the proposed theory.

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I. INTRODUCTION

Collapse in hydrodynamics of an ideal incompressible fluid, as a process of singularity formation in a finite time, is one of the central problems in the theory of developed hydrodynamic turbulence. The classical examples of such type spectra are the Phillips spectrum for water-wind waves [2] and the Kadomtsev-Petviashvili spectrum for acoustic turbulence [3]. In the first case white caps – waves of water surface – play a role of singularities, and in the second case these are density breaks (or shocks).

The question about collapse in hydrodynamics is an old problem. For example, in 1981 P.Saffman considered collapse as one of the most important problems in hydrodynamics (see also papers [8] and references therein), probably already L.Richardson and A.N.Kolmogorov understood an importance of this problem. In spite of so long a history of the question, there is no deep understanding of the nature of a collapse in hydrodynamics, even though there are many numerical simulations testifying to the collapse existence collected by now. As far as theory is concerned, essential results are absent, and, moreover, there is no common agreement about collapse as a subject for incompressible hydrodynamics (see, e.g., Sec. 7.8 of the book of U.Frisch and references therein). In the theory, the only appreciable exclusion is the work of V.E.Zakharov, 1988 [9] (more detailed publication has appeared in 1999 [10]), where the consistent theory of collapse for two anti-parallel vortex filaments of small thickness was developed in quasi-two-dimensional approximation, when a flow is almost two-dimensional with a slow dependence with respect to the third coordinate (see also the paper [10]). A significant progress in studying hydrodynamic collapse has been achieved in numerical simulations of the Euler equation. Many numerical experiments testify that the value of the vorticity $|\mathbf{\Omega}|$ becomes infinite in isolated points in a finite time. As follows from the papers of Kerr [10], Grauer, Marliani and Germashevsky [11], Pelz [12], Boratav and Pelz [13], $|\mathbf{\Omega}|$ grows at the collapse point like $(t_0-t)^{-1}$, where $t_0$ is a collapse time. According to [10], [14], a spatial scale of the collapsing distribution contracts as $(t_0-t)^{1/2}$. In the recent paper by Kerr [17], an anisotropy of the collapsing region has been reported. The data processing gave two scales, one of them being contracted as the root $l_1 \sim (t_0-t)^{1/2}$, and another one as $l_2 \sim t_0 - t$. It should be noted that the initial flow either possessed a definite symmetry or it was close to a symmetric flow in most numerical simulations. As a result, several singularities arose simultaneously. For example, the evolution of two anti-parallel vortex tubes was studied in [10]. The collapse here is caused by the Crow instability [14] leading at the nonlinear stage to the vortex reconnection. Symmetry of the flow makes the collapse to happen in two symmetric points.

In the present paper, we suggest a new mechanism of the singularity formation connected with breaking of continuously distributed vortex lines. This mechanism is not related to any symmetry of the initial vorticity. The collapse itself is possible in one separate point. Probably, just this type of collapse has been observed in the recent numerical experiment [15].

The mechanism suggested can be naturally incorporated into the classical catastrophe theory [16]. From this point of view, collapse can be considered as caustic formation for a solenoidal field. It is not so easy to understand, how a collapse arises in the Eulerian description. First, this is connected with a hidden symmetry of the Euler equation, i.e., the relabeling symmetry (for more details see the reviews [18], [19]). This symmetry generates the conservation law for the Lagrangian invariant – the so-called Cauchy invariant, which is expressed through the velocity curl and the Jacoby matrix of a mapping from the Eulerian variables to the Lagrangian ones, and by this reason this invariant occurs very nonlocal in terms of the velocity field. On the one hand, the Cauchy invariant is known as invariant which characterizes the property of frozenness of vortex lines into a fluid. On the other hand, all known conservation laws for vorticity, such as the Kelvin’s and Ehret’s theorems, conservation of the topological Hopf invariant, being a measure of the flow knottiness, are a simple consequence of the Cauchy invariant constancy. The frozenness of vortex lines means that fluid particles are pasted to a given vortex line and never leave it. A destruction of frozenness is possible only due to viscosity, i.e., beyond ideal hydrodynamics. Therefore, as a next natural step in the vortex motion description, a mixed Lagrangian-Eulerian description has been introduced, where the main object is a vortex line [20], [21]. Each vortex line in this description is labeled by a two-dimensional Lagrangian marker, while the third coordinate serves as a parameter determining the curve. This representation, which we called as the vortex line representation, is a key-point in the description of hydrodynamic collapse which can be considered as a process of a caustic formation for the solenoidal vorticity field.

The paper is organized as follows: in Sec.II we introduce
the vortex line representation and explain its meaning. In Sec.III we consider the 3D integrable hydrodynamic model introduced in our previous paper [20]. The Hamiltonian of this model is unusual, it is expressed through the absolute value of the vorticity $|\Omega|$ 

$$H = \int |\Omega| dr. \quad (1.1)$$

The given model can be integrated by means of combination of the vortex line representation and the inverse scattering transform. By applying the vortex line representation, the Hamiltonian is decomposed into a sum of Hamiltonians of non-interacting vortex lines. Dynamics of each vortex line is described by the integrable one-dimensional Landau-Lifshitz equation for a Heisenberg ferromagnet or by its gauge-equivalent - the nonlinear Schrödinger equation. Thus, the integrable hydrodynamics represents a hydrodynamics of free vortex lines. As for hydrodynamics of free particles - hydrodynamics of dust with a null pressure (see, e.g. [22]), for the 3D integrable hydrodynamics typical singularities are also caustics. For hydrodynamics of dust, density turns into infinity at caustics. Unlike the dust density, being a scalar characteristic, vorticity is a vector solinoid field. Therefore, the latter imposes some restrictions on a spatial structure near singularity. As it is shown in Sec.IV, the singularity structure, being very anisotropic, turns into a pancake form. The spatial collapsing distribution at $t \to t_0$ leads to quasi-two-dimensional. Along the "soft" direction a more rapid compression takes place as $l_1 \sim (t_0 - t)^{3/2}$, and along two other "hard" directions $l_2 \sim (t_0 - t)^{1/2}$. At the collapse time, the vorticity vector lies in the pancake plane and its value $|\Omega|$ blows up like $(t_0 - t)^{-1}$. This behavior corresponds to the general situation. The degenerated case is considered in Sect. V, where we consider collapse for topologically nontrivial axi-symmetric distribution of vorticity in the form of the so-called Hopf mapping when any two vortex lines are linked once with each other. In this case two eigen-values of the Jacobian matrix vanish simultaneously in the collapse point. As its sequence, vorticity occurs to have more strong singularity: $|\Omega| \sim (t_0 - t)^{-2}$. And we conclude in Sec. VI with discussion of numerical experiments on collapse observation in the Euler equation and their correspondence to the proposed theory.

II. VORTEX LINES REPRESENTATION OF HYDRODYNAMICS

Let us consider the equations of the hydrodynamic type

$$\frac{\partial \Omega}{\partial t} = \text{curl} \left[ \text{curl} \frac{\partial H}{\partial \Omega} \right], \quad (2.1)$$

where $\mathcal{H}\{\Omega\}$ is the Hamiltonian of a system, $\Omega(r,t) = \text{curl} \, \mathbf{p}(r,t)$ is the generalized vorticity, $\mathbf{p}$ is the canonical momentum. The vector field

$$\mathbf{v} = \text{curl}(\delta H/\delta \Omega) \quad (2.2)$$

is nothing else but the fluid velocity. By the definition $\text{div} \, \mathbf{v} = 0$, i.e., we deal with an incompressible fluid. If the Hamiltonian coincides with the kinetic energy of the fluid

$$\mathcal{H} = \int \frac{p^2}{2} dr = \int \frac{\Omega_1(r_1) \cdot \Omega_2(r_2)}{8\pi|r_1 - r_2|} dr_1 dr_2$$

then the expression (2.2) yields the usual relation $\Omega = \text{curl} \, \mathbf{v}$ between velocity $\mathbf{v}$ and vorticity $\Omega$, while Eq.(2.1) transforms into the Euler equation for vorticity

$$\frac{\partial \Omega}{\partial t} = \text{curl}[\mathbf{v} \times \Omega], \quad \text{div} \, \mathbf{v} = 0. \quad (2.3)$$

The important property of the equation (2.1) is the frozenness of vorticity into the substance, i.e. all Lagrangian fluid particles at $t > 0$ remain at their own vortex line. After that, it is natural to introduce a mixed Lagrangian-Eulerian description when each vortex line is labeled by its own Lagrangian marker $\nu$, which lies in a fixed two-dimensional manifold $\mathcal{N}$, while a parameter $s$ along the line has a meaning of an Eulerian variable. In such vortex line representation vorticity is expressed as follows [20]

$$\Omega(r,t) = \int_{\mathcal{N}} \Omega_0(\nu)d^2\nu \int \delta(r - R(s,\nu,t)) \frac{\partial R}{\partial s} ds. \quad (2.3)$$

Here, the closed curve $r = R(s,\nu,t)$ corresponds to each vortex line $\nu$, so that $R_s$ is its tangent vector. The quantity $\Omega_0(\nu)$, with $R_s$ being its tangent vector. The fixed function $\Omega_0(\nu)$ is the strength of vortex loop. However, without loss of generality, this function can be put equal to the unity. This can be achieved by both an appropriate re-definition of labels $\nu$ and changing the vortex orientation to the opposite one for those lines from the manifold $\mathcal{N}$, for which $\Omega_0(\nu) < 0$. Therefore, in the next section we will omit the multiplier $\Omega_0(\nu)$ in front of $d^2\nu$ in the corresponding formulae.

The generalization of Eq. (2.3) to the case of arbitrary topology of the vortex lines is served by the formula:

$$\Omega(r,t) = \int \delta(r - R(a,t)) / \Omega_0(a) \nabla_a R(a,t) d^3a, \quad (2.4)$$

where $\Omega_0(a)$ is the Cauchy invariant, characterizing the frozenness property. The div$\mathbf{v}_a \Omega_0(a) = 0$ condition guarantees automatically incompressibility for the field $\Omega(r,t)$: div$\mathbf{\Omega}(r) = 0$. In the expression (2.4), the vector

$$\mathbf{b} = (\Omega_0(a) \nabla_a) R(a,t) \quad (2.5)$$

is a tangent vector to the vortex line at the point

$$r = R(a,t). \quad (2.5)$$

In the representation (2.3), an arc-length of line of the initial field $\Omega_0(a)$ can serve as the parameter $s$.

After integrating (2.4) over $a$-variables, the vector $\Omega(r,t)$ is expressed through the Jacobian $J$ of the mapping (2.5)

$$J = \text{det}[\partial R/\partial a]$$

and the Cauchy invariant $\Omega_0(a)$:

$$\Omega(R) = \frac{1}{J}(\Omega_0(a) \nabla_a) R(a). \quad (2.6)$$

It is important to emphasize, that in this expression the Jacobian is not to be equal to the unity: $J \neq 1$. Nevertheless, it does not contradict to the incompressibility condition for the fluid.
As it was shown in [20,21], the equations of motion for vortex lines can be obtained directly from the equation of frozenness [21]

\[ \left\{ \{\Omega(a)\nabla_a\} R(a,t) \right\} \times \{ R_t(a,t) - v(R(a,t),t) \} = 0. \quad (2.7) \]

This equation describes the transverse dynamics of vortex line: obviously any motion along a curve leaves the curve unchanged. In particular, this helps to understand why there are no restrictions imposed on the value of the Jacobian \( J \) changed. In particular, this helps to understand why there are no restrictions imposed on the value of the Jacobian \( J \)

As it was shown in [20,21], Eq.(2.7) can be written in the Hamiltonian form

\[ \left\{ \{\Omega(a)\nabla_a\} R(a,t) \right\} \times \{ R_t(a) \} = \frac{\delta H\{\Omega(R)\}}{\delta R(a)} \bigg|_{\Omega_0}. \quad (2.8) \]

This equation describes a motion of vortex lines in systems with an arbitrary Hamiltonian that depends on \( R \) through the \( \Omega(r, t) \) only.

It is useful also to keep in mind that the expressions for such important characteristics of the system as its momentum \( P = \int p \, dr \) and angular momentum \( M = \int [r \times p] \, dr \), being transformed by integration by parts to a form, where, instead of \( p \), the vorticity \( \Omega \) is employed, and being then rewritten in terms of vortex lines, have the form

\[ P \sim \frac{1}{2} \int [r \times \Omega] \, dr = \int_\mathcal{N} d^2\nu \frac{1}{2} \int [R \times R_s] \, ds \quad (2.9) \]

\[ M \sim \frac{1}{3} \int [r \times [r \times \Omega]] \, dr = \int_\mathcal{N} d^2\nu \frac{1}{3} \int [R \times [R \times R_s]] \, ds \quad (2.10) \]

The ~--sign in these relations means that equalities take place up to integrals over surface with infinitely large radius. Hence one can see that the momentum and the angular momentum are composed of momenta and angular momenta of each vortex line, the momentum of a closed line being equal to the oriented area of a surface tightened on the vortex loop.

It is easy to verify that uniform shift \( R_0 \) of \( R \) does not change the momentum, while the angular momentum is subjected to the well known transformation

\[ M \rightarrow M' = [R_0 \times P] + M. \quad (2.11) \]

III. INTEGRABLE HYDRODYNAMICS

In this and two next sections, we will show how and why collapse is possible in 3D integrable hydrodynamics. This model was introduced in our previous paper [21]. The Hamiltonian of this model is expressed through the absolute value of \( \Omega(r, t) \)

\[ H = \int [\Omega(r)] \, dr, \quad (3.1) \]

and the equation of motion coincides with the frozenness equation [21] with velocity

\[ v = \text{curl} \, \vec{\tau} \]

where \( \vec{\tau} = (\Omega/\Omega) \) is the unit tangent vector along the vortex line. Assuming all the lines closed, choosing the labeling by such a way so that \( \Omega(s) = 1 \), and substituting the representation (2.3) into (1.1), it is easy to see that the Hamiltonian is decomposed as a sum of Hamiltonians for the vortex lines

\[ H(R) = \int d^2\nu \int |\partial R/\partial s| \, ds. \quad (3.2) \]

Here, the integral over \( s \) is the length of the vortex line with the index \( \nu \). The equation of motion for the vector \( R(\nu, s) \), in accordance with the Eq. (3.2), is local in these variables – it doesn’t contain an interaction with other vortices:

\[ [R_\nu \times R_s] = [\vec{\tau} \times [\vec{\tau} \times \vec{\tau}]]. \quad (3.3) \]

By this reason, not only the total energy, momentum, and angular momentum are conserved, but also the corresponding geometrical invariants for each vortex loop: its length

\[ \mathcal{H}(\nu) = \int |R_\nu| \, ds, \]

the oriented area spanned on the vortex loop which coincides with its momentum\( s \) momentum

\[ P(\nu) = \frac{1}{2} \int [R(\nu) \times R_s(\nu)] \, ds, \]

and its angular momentum

\[ M(\nu) = \frac{1}{3} \int [R(\nu) \times [R(\nu) \times R_s]] \, ds. \]

It is important to pay attention to the following fact: The equation (3.3) is invariant with respect to changes \( s \rightarrow \delta(s, t) \). Therefore, it can be solved for \( R_\nu \), up to a shift along the vortex line – the transformation leaves the vorticity \( \Omega \) unchanged. This means that to find the vorticity \( \Omega \) it is enough to have one solution of the equation

\[ |R_\nu \times R_s| = [\vec{\tau} \times [\vec{\tau} \times \vec{\tau}]] + \beta R_s \quad (3.4) \]

which follows from Eq.(3.3) for some choice \( \beta \). This leads to an equation for \( \vec{\tau} \) as a function of the filament length \( l \)

\[ \vec{\tau} = \frac{\partial \vec{\tau}}{\partial l}, \]

It is worth to notice that this property is common for all systems with the Hamiltonians of the type \( H = \int F(\tau, (\nabla \tau), (\nabla^2 \tau, \ldots)) \Omega \, dr \). To explain the idea of collapse of vortex lines, we have chosen the simplest example [21], which has a physical meaning.
\[ (dl = |R_s|ds) \text{ and time } t \text{ (by choosing a new value } \beta = 0), \]
which reduces to the integrable one-dimensional (1D) Landau-Lifshits equation for a Heisenberg ferromagnet
\[ \frac{\partial \vec{\tau}}{\partial t} = \left[ \vec{\tau} \times \frac{\partial^2 \vec{\tau}}{\partial t^2} \right]. \]  
(3.5)

This equation, in its turn, is gauge equivalent to the 1D nonlinear Shrödinger equation \[ i\psi_t + \psi_{xx} + (1/2)|\psi|^2 \psi = 0 \]  
(3.6)

and, for instance, can be reduced to the NLSE by means of the Hasimoto transformation \[ \psi(l, t) = \kappa(l, t) \cdot \exp \left[ i \int^t \chi(l, t) dl \right], \]
where \( \kappa(l, t) \) is a curvature and \( \chi(l, t) \) the line torsion.

The system under consideration has direct relation to hydrodynamics. As it is known \[ [23, 24], \]
the local induction approximation for a thin vortex filament, under assumption of smallness of the filament width to the characteristic longitudinal scale, leads to the Hamiltonian \[ (3.3) \]
but only for a single separate line. The essence of this approximation is in in replacing the logarithmic interaction law by a delta-functional one. When the widths of the filaments are small comparable with distances between them, in the same approximation, the Hamiltonian of vortex lines transforms into the sum of the Hamiltonians of independent vortex loops, yielding in a "continuous" limit the Hamiltonian \[ (3.1). \]

By such a way, we have the model of 3D integrable hydrodynamics of free vortex filaments. In this model, each vortex is a nonlinear object with its own internal dynamics. As we will see later, already in the framework of this model, a singularity formation is possible. Singularities appear in this model as a result of intersection of vortex lines that is analogous to the phenomenon of wave breaking in gas-dynamics.

**A. Stationary vortices**

Let us consider now the simplest solution of Eq.(3.3), i.e., a stationary propagation of a closed vortex line: \( R_s = V \equiv \text{const.} \)
In this case the velocity \( V \) is determined from solution of the equation
\[ [R_s \times V] = [\vec{\tau} \times [\vec{\tau} \times \vec{\tau}]]. \]  
(3.7)

It is easy to check that this equation follows from the variational principle
\[ \delta (H(\nu) - V \cdot P(\nu)) = 0, \]  
(3.8)
i.e., any solution of \[ (3.7) \]
represents a stationary point of the Hamiltonian for a fixed momentum \( P(\nu) \). The equation \[ (3.7) \]
can be simply integrated, being rewritten in terms of the bi-normal \( b \) and the curvature \( \kappa \) of the line as follows
\[ [\vec{\tau} \times V] \kappa [\vec{\tau} \times b], \]  
(3.9)
that gives
\[ V = \kappa b. \]  
(3.10)

A constant value of the velocity \( V \) in this expression implies constancy of the curvature \( \kappa \), i.e. the vortex line must be a ring of radius \( r = 1/\kappa \) and
\[ V = 1/r. \]  
(3.11)

The direction of the ring motion is perpendicular to its plane. It is interesting to note that the exact answer to the velocity of a thin (with width \( d \ll r \) vortex ring in ideal hydrodynamics \[ (27) \]
coincides with Eq.(3.11) up to the logarithmic accuracy that just differs the considering model from the Euler equation.

Stationary solutions \[ (3.10) \] in the form of rings are remarkable within this model, because they are stable, moreover, they are stable in the Lyapunov’ sense. Remind, that momentum \( P \) of a closed vortex line is its oriented surface spanned on the loop:
\[ P = Sn. \]

where \( S \) is the surface value, \( n \) its normal. Inasmuch as the Hamiltonian of a vortex loop coincides with its length, a maximum of the momentum, or, that is the same as a maximum of surface \( S \) is obviously attained, for fixed length, at the perfect circle. Just this proves stability of the vortex ring solution \[ (3.10) \] in the Lyapunov sense.

**IV. COLLAPSE**

The solution \[ (3.10), (3.11) \] enables us to construct the simplest mappings \( R = R(\nu, s, t) \).

Let all vortex lines be circle-shaped and oriented in the same direction, for instance, along \( z \)-axis. We will see further that collapse in our model is a purely local phenomenon. Therefore, it is sufficient to consider some vortex tube (which can be imagined as a torus) to find a mapping. Let vortex rings be distributed continuously inside the tube. We label each vortex line by the two-dimensional parameter \( \nu \), which values coincide with coordinate of some cross section of the tube at \( t = 0 \). We will use the ring arc-length as longitudinal parameter \( \nu \), which can be imagined as a torus \( (ds = rd\phi, \phi \) is the polar angle around \( z \)-axis). Then, with the help of \[ (3.10) \], the desired mapping can be written as follows
\[ R = R_0(\nu) + r(\nu)\cos \phi e_x + r(\nu)\sin \phi e_y + V(\nu)t e_z. \]  
(4.1)

In this formula \( e_x, e_y, e_z \) are unit vectors along the corresponding axes.

It can be easily verified for this mapping that the Jacobian is a linear function of time
\[ J = \frac{\partial (X, Y, Z)}{\partial (\nu_1, \nu_2, s)} = J_0(\nu, s) + A(\nu, s)t. \]  
(4.2)

Here \( A(\nu, s) \) is a coefficient linearly dependent on the velocity derivatives with respect to \( \nu \) and \( J_0 \) the initial value of Jacobian.

Dependence \( J \) \[ (4.2) \] on time means that for every fixed pair \( \nu \) and \( s \) there exists such a moment of time \( t = \tilde{t}(\nu, s) \) \( (t > 0, \) or \( t < 0) \), when Jacobian is equal to zero: \( J(\nu, s, t) = 0 \). Denote as \( \tilde{t}_0 \) the minimal value of \( t = \tilde{t}(\nu, s) \) at \( t > 0 \). And let this minimum be attained at some point \( a = a_0 \) (here we
denote a point \((\nu_1, \nu_2, s)\) as \(a\). It is evident that at \(t = t_0\)
\[
\left(\frac{\partial J}{\partial a}\right)_{a = a_0} = 0
\]
and
\[
\nabla_a J(a, t)|_{a = a_0} = 0,
\]
(4.3) since
\[
\nabla_a J(a, t)|_{a = a_0} + \frac{\partial J(a, t)}{\partial t} \frac{\partial t}{\partial a}|_{a = a_0} = 0.
\]
It is clear also that at \(t = t_0\) the tensor of second derivatives of \(J\) against \(a\),
\[
2\gamma_{ij} = \frac{\partial^2 J}{\partial a_i \partial a_j},
\]
will be positive definite at the point \(a = a_0\). Hence, it is easy to define a behavior of the Jacobian in a small vicinity of \(a = a_0\). Expansion of \(J\) near this point (in a typical situation) at \(t \to t_0\) is as follows
\[
J(a, t) = \alpha(t_0 - t) + \gamma_{ij} \Delta a_i \Delta a_j + \ldots,
\]
where
\[
\alpha = - \frac{\partial J(a, t)}{\partial t} \bigg|_{t = t_0, a = a_0} > 0, \quad \Delta a = a - a_0.
\]
Those are the leading contributions to the Jacobian expansion.

Geometrically, the above expansion corresponds to a sufficiently simple picture. The (hyper-) surface \(J = J(a, t)\) deforms with time in such a manner, that its minimum reaches the (hyper-) plane \(J = 0\) at \(t = t_0\), where two surfaces touch each other. Obviously, for smooth mappings in a typical case, this touching takes place in one separate point. In a degenerated situation, touching is possible in a few points simultaneously, or even at a curve. A case, when two eigenvalues of the Jacoby matrix \(J\) tend to zero simultaneously at the collapse point, will be regarded also as degenerated (such an example will be considered in the next section). In that case one should keep the next terms in the Jacobian expansion along the corresponding directions. We would like to repeat that all these cases can not be considered as typical ones.

In accordance with Eq. (2.6), the equality \(J = 0\) at the singular point means the formation of a singularity for the vorticity at the moment \(t = t_0\):
\[
\Omega(r, t) = \frac{\Omega_0(\nu) R_s}{\alpha(t_0 - t) + \gamma_{ij} \Delta a_i \Delta a_j}.
\]
(4.5)
It is important, that the numerator in this fraction – the tangent vector of a vortex line – does not vanish at the point \(a_0\) due to its geometrical meaning. Therefore, the vorticity at the singular point blows up as \((t_0 - t)^{-1}\), and the characteristic size of the collapsing distribution in \(a\)-coordinates decreases as \(\sqrt{t_0 - t}\).

The above type of collapse arises as a result of vortex line breaking when one vortex overtakes another. For flows of the general type (without symmetries) a singularity must arise at the first time always in one separate point.

As we will see further, for the given type of collapse, the dependence (4.3) for \(\Omega(r, t)\), derived for the particular initial distribution, is actually the general answer, which can be applied not only to the integrable hydrodynamics but also to the whole family (2.1) of hydrodynamic systems, of course, under the condition that they admit such (quasi-) inertial regime of collapse. What necessary conditions should satisfy the Hamiltonian of some particular system, in order to exhibit such regime? Now the answer to this question has been unknown that provides a wide field for future investigations, both theoretical and numerical.

\section*{A. Non-stationary vortices}

Let us consider now the integrable hydrodynamics in a more general case when closed vortex lines are not circles. In this situation, it is possible to map each vortex contour to some vortex ring. It is then natural to introduce the (mean) direction \(n\), as well as the mean area \(S = \pi r_0^2\), with the help of the expression for momentum of vortex line, \(P = nS\). The position \(R_0\) of the ring center changes linearly in time. A corresponding mean velocity of the motion of closed line must be directed along the momentum, in order to satisfy the conservation law for angular momentum. The mean velocity value \(V_0\), generally speaking, is a function of those fundamental integrals of motion, which are independent on the origin choice – the Hamiltonian (i.e. the length \(L\)), the momentum, and the projection on the vector \(n\) of the angular momentum. It is clear also that increasing the contour sizes in \(\lambda\) times should result in decreasing the velocity in \(\lambda\) times, so that
\[
V_0 = \frac{8}{\pi^2 L^3} \cdot U \left( \frac{16 \pi^2 P^2}{L^3} \cdot \frac{(M \cdot P)^2}{L^5} \right).
\]
(4.6)
Explicit dependence of the \(U(\xi, \eta)\)-function has been unknown now. Its first argument can be considered as a measure of a “crumpleness” of the vortex line and changes in the limits \(0 \leq \xi \leq 1\), and the second argument determines a measure of a spirality of the line. In a more or less reasonable approximation, which doesn’t lead to excessive errors, one can suppose that \(U(\xi, \eta) \sim U(1, 0) = 1\).

After introducing the mean characteristics, the mapping \(R(\nu, s, t)\) can be represented as the sum:
\[
R(\nu, s, t) = R(\nu, s, t) + \delta r(\nu, s, t),
\]
(4.7)
where the mean value \(R(\nu, s, t)\) is given by the relation
\[
\tilde{R}(\nu, s, t) = R_0 + r_0 \cos \phi \cdot e'_x + r_0 \sin \phi \cdot e'_y, \quad R_0 = V_0 n.
\]
(4.8)
Here angular parameter \(\phi' = 2\pi s/L\) is proportional to the arc-length \(s\), and unit vectors \(e'_x, e'_y\) lie in the plane, perpendicular to the local \(z'\)-axis, directed along the \(n\). The relationship between \(r_0\) and \(V_0\) is given by the Eq. (4.6).

The vector function \(\delta r(\nu, s, t)\) describes deviations (generally speaking, not small) from the mean value \(R(\nu, s, t)\).

The separation of the mean and oscillatory motions, introduced by means of (4.4), (4.8), (4.4) for each vortex contour,
shows that the mapping $R = R(\nu, s, t)$ at each fixed value $a = (\nu, s)$ is a linear function of time with nonlinear oscillations, which are described by the Landau-Lifshits equation \(^{(3)}\) or its gauge equivalent \(^{(4)}\). The linear mean dependence reflects the fact that the model under consideration is a model of free vortices. The collapse thus arises as a result of a running of one vortex into another. Due to a continuous distribution of vortex lines, their “density” increases infinitely in some point.

A similar situation takes place in the model for large scale structure formation in the Universe studied by V.I.Arnold, Ya.B.Zeldovich and S.F.Shandarin \(^{(22)}\). In the basement of this model, the suggestion lies about initial dust-like distribution of masses, when their behavior can be described by the zero-pressure hydrodynamic equations

$$\rho_t + \text{div} \rho \nu = 0, \quad \frac{\partial \nu}{\partial t} = \nu_t + (\nu \nabla) \nu = 0. \tag{4.9}$$

The integration of this system in the Lagrangian variables gives that i) all fluid particles being free move with a constant velocity

$$r = a + \nu(a)t, \quad \tag{4.11}$$

and ii) the density $\rho$ is expressed through the initial value and the Jacobian of the mapping \(^{(4.11)}\) as follows

$$\rho(r, t) = \frac{\rho_0(a)}{J}. \tag{4.12}$$

In the framework of this model, appearance of large scale structures is connected with the breaking phenomenon resulting in density singularities, due to the Jacobian vanishing for the mapping \(^{(4.11)}\). In a typical situation, these structures have a pancake shape and can be considered as peculiar galaxies. The formula, analogous to the Eq. \(^{(4.12)}\), takes place, as we have seen above, for the vorticity $\Omega$, see the Eq. \(^{(2.6)}\). However, there is a difference from Eq. \(^{(4.12)}\), connected with the vector nature of the $\Omega$ field.

In the given task we will be interested in a geometric structure of a singularity at $t \to t_0$, but $t < t_0$, i.e., in some sense, at the initial stage of collapse, but not at the developed stage, which has the meaning for astrophysics applications, but remains rather unclear for the incompressible hydrodynamics, when viscosity is necessary to be taken into account at small scales.

**B. Structure of the singularity**

Let us consider in more details the structure of the collapsing region in a typical situation. First, it is clear from the above discussion that the vorticity distribution near singularity will be given by the former expression \(^{(4.12)}\). Second, the main features of the singularity will be determined by the Jacobian, that is the denominator of \(^{(4.13)}\). Numerator $$(\Omega_0(a)(\nabla_a)R (\text{ tangent vector to the lines})$$ can be taken at the point $a = a_0$, $t = t_0$ and considered as a constant.

According to Eq. \(^{(4.13)}\) the Jacobian expansion contains positive definite symmetric matrix $\gamma_{ij}$ of its second derivatives taken at the point $a = a_0$, $t = t_0$. At $t < t_0$, this matrix is assumed to be non-degenerate: all its eigenvalues are positive, and the matrix itself can be diagonalized. Hence it follows immediately that compression along all principal axes in $a$-space will have the same law: $L_i \sim \sqrt{t_0 - t}$. Therefore, near the singular point, vorticity $\Omega$ will have the self-similar asymptotics

$$\Omega(r, t) = \frac{\Omega_0(a) R_s}{(t_0 - t)(\alpha + \gamma_i \eta_i \eta_j)}, \tag{4.13}$$

where $\eta = \Delta a/\sqrt{t_0 - t}$ are self-similar variables in $a$-space. However, the equation \(^{(4.13)}\) does not mean, that compression in $r$-space will be the same.

When the Jacobian takes zero value, one of the eigenvalues of the Jacobi matrix becomes equal to zero. This eigenvalue (denote it as $\lambda_1$), as it easily to see, coincides with the Jacobian \(^{(4.14)}\) at a small vicinity of the collapse point up to the almost constant multiplier $\lambda_2 \cdot \lambda_3$.

Represent now the Jacoby matrix $J$ as a decomposition over eigenvectors of direct $\langle J|\psi^{(n)} \rangle = \lambda_n \langle \psi^{(n)} \rangle$ and conjugated $\langle \psi^{(n)}|J = \lambda_n < \psi^{(n)} \rangle$ spectral problems

$$J_{ik} \equiv \frac{\partial x_k}{\partial a_i} = \sum_{n=1}^{3} \lambda_n \psi_i^{(n)} \psi_k^{(n)}. \tag{4.14}$$

Here two sets of eigenvectors of direct and conjugated problems are mutually orthogonal

$$\langle \psi^{(n)}|\psi^{(m)} \rangle = \delta_{nm}.$$

In a small vicinity of the collapse point eigenvalues $\lambda_{2,3}$ can be considered as constants, while

$$\lambda_1 \equiv \frac{J}{\lambda_2 \lambda_3} = (\lambda_2 \lambda_3)^{-1} [\alpha(t_0 - t) + \gamma_i a_i a_j].$$

Here, for simplicity, we have placed the origin at the point $a = a_0$. As for the eigenvectors, they also can be considered constant.

Let us decompose the vectors $x$ and $\nabla_a$ in Eq. \(^{(4.14)}\) through the corresponding bases, denoting their appropriate projections as $X_n$ and $A_n$:

$$X_n = \langle x|\psi^{(n)} \rangle > \frac{\partial}{\partial A_n} = \langle \psi^{(n)}|\nabla_a \rangle >.$$

In this case the vector $a$ is written in terms of $A_n$ as follows

$$a_{\alpha} = \sum_{n} \psi_\alpha^{(n)} \psi^{(n)} A_n.$$

As a result, Eq. \(^{(4.14)}\) can be rewritten in the form

$$\frac{\partial X_1}{\partial A_1} = \tau + \Gamma_{mn} A_m A_n, \tag{4.15}$$
$$\frac{\partial X_2}{\partial A_2} = \lambda_2, \quad \frac{\partial X_3}{\partial A_3} = \lambda_3. \tag{4.16}$$

Here, the matrix

$$\Gamma_{mn} = \gamma_{\alpha \beta} (\lambda_2 \lambda_3)^{-1} \langle \psi_\alpha^{(n)}|\psi^{(m)} \rangle \langle \psi^{(n)}|\psi^{(m)} \rangle \langle \psi^{(m)}|\psi^{(n)} \rangle \langle \psi^{(n)}|\psi^{(m)} \rangle,$$

and parameter $\tau = \alpha(t_0 - t)/(\lambda_2 \lambda_3)^{-1}$ is assumed to be small. It follows from here immediately that size reduction along the second ($X_2$) and the third ($X_3$) directions is the same as in
the auxiliary a-space, i.e., $\sim \sqrt{r}$, but along the “soft” direction $X_1$ behaves like $r^{3/2}$. Respectively, in terms of new self-similar variables $\xi_1 = X_1/r^{3/2}$, $\xi_2 = X_2/r^{1/2}$, $\xi_3 = X_3/r^{1/2}$, integration of the system gives for $\xi_2$ and $\xi_3$ a linear dependence on $\eta$, while for $\xi_1$ it a cubic dependence

$$
\xi_1 = (1 + \Gamma_{ij}\eta_i\eta_j)\eta_1 + \frac{1}{2}\Gamma_{ij}\eta_i\eta_j^2 + \frac{1}{3}\Gamma_{ij}\eta_i^3, \quad i, j = 2, 3 \quad (4.17)
$$

Together with $\xi_1$, the relations (4.17) and (4.18) determine implicit dependence $\Omega(r, t)$. The presence of two different self-similarities shows, that the spatial vorticity distribution becomes strongly flattened in the first direction, taking a pancake form as $t \rightarrow t_0$.

The direction of the field $\Omega$ can be found from the incompressibility condition $\text{div } \Omega = 0$. It is easy to see that in the leading order (as $t \rightarrow t_0$) the gradient of $J$ is determined by the soft direction $e_1$:

$$
\nabla J \approx r^{-3/2} \frac{\partial J}{\partial \xi_1} e_1.
$$

Contributions from two other directions are small in the parameter $\tau$.

Hence, it follows that vector lines of the field $\Omega$ lie in the pancake plane that is in agreement with transversality of motion of vortex lines (compare with Eq. (2.7)).

**V. EXAMPLE OF COLLAPSE IN THE DEGENERATED CASE**

In the previous section we have considered collapse for a non-degenerated situation, when the only one eigenvalue of the Jacoby matrix tends to zero at the touching point. Now we shall consider an example of collapse, when two eigenvalues of the Jacoby matrix vanish simultaneously at the collapse point. We will examine an initial vorticity distribution with the nontrivial topology of vortex lines with linking number $N = 1$. This special distribution is the so-called Hopf mapping. There are several ways how to construct the corresponding field $\Omega$. We will keep here the approach of the paper [28].

Following to that work, let us represent the field $\Omega$ through the n-field ($n^2 = 1$)

$$
\Omega_n(r) = \frac{1}{32} \epsilon_{\alpha\beta\gamma}(n \cdot [\partial_\beta n \times \partial_\gamma n]), \quad (5.1)
$$

where the n-field is supposed to be a smooth function of coordinates tending to the constant value $e$ at the infinity $r \rightarrow \infty$.

It is easy to check that in accordance with Eq. (5.1) each point $n = n_0$ on the unit sphere $S^2$ defines a closed vortex line. Indeed, parameterization of the unit vector $n$ through the spherical angles $\theta$ and $\varphi$ allows to write down the field $\Omega$ as follows

$$
\Omega = \frac{1}{16} [(\nabla \varphi \times \nabla \cos \theta)], \quad (5.2)
$$

so that the variables $\varphi$ and $\cos \theta$ play the role of the Clebsch variables. Thus, each vortex line coincides in this case with the intersection of two surfaces $\varphi = \text{const}$ and $\cos \theta = \text{const}$, i.e. a closed vortex line is the pcurlyp of a point on the sphere $S^2$. By the definition, the Hopf degree $N$ of mapping $\mathbb{R}^3 \rightarrow S^2$ is called an integer number of linkages between arbitrary two vortex lines - pcurlyp of two points on the unit sphere. The Hopf mapping (with $N = 1$) is given by the next relation:

$$
(n \cdot \sigma) = U(e \cdot \sigma)U^\dagger, \quad U = \frac{1 + i(a \cdot \sigma)}{1 - i(a \cdot \sigma)}, \quad (5.3)
$$

where $\sigma$ are the Pauli matrices.

Expressing the vector $n$ from here and substituting the result into Eq. (5.1), one can get (see [28])

$$
\Omega_0(a) = \frac{e(1 - a^2) + 2a(e \cdot a) + 2[e \times a]}{(1 + a^2)^3}, \quad (5.4)
$$

As it was shown in [28], all the flux lines of this field are circles, and each line is linked once with another line. By this reason, the singularity formation is inevitable.

It is worth to note that the field (5.4) has no singular points in the whole space and its absolute value depends only on the absolute value $|a|$. The unit vector $\tilde{r}$ is defined everywhere:

$$
\tilde{r}(a) = \frac{e(1 - a^2) + 2a(e \cdot a) + 2[e \times a]}{(1 + a^2)} \quad (5.5)
$$

The velocity of each ring is connected with the binormal $b(\nu)$ and the radius $r(\nu)$ by the relation

$$
V(\nu) = b(\nu)/r(\nu).
$$

The radii $r(\nu)$ of rings and their orientations $b(\nu)$ are integrals of motion. Only positions of centers of rings can change, and this motion occurs with a constant velocity for each ring.

In the given problem, instead of the variables $\nu$ and $s$, it is convenient to use directly the variables $a$, in which the mapping $R(a, t)$ is written as

$$
R(a, t) = a + tV(a), \quad (5.6)
$$

where the velocity $V(a)$ is expressed through the unit tangent vector $\tilde{r}$ by means of the relation

$$
V(a) = [\tilde{r}(a) \times (\tilde{r}(a)\nabla a)\tilde{r}(a)]. \quad (5.7)
$$

Hence one can find the expression for the Jacobian

$$
J(a, t) = \det \left| \frac{\partial V(a)}{\partial a} \right| \approx \frac{1}{16} \left| \nabla \varphi \times \nabla \cos \theta \right|, \quad (5.8)
$$

It should be noted that the velocity of each ring is constant along the vortex line. Therefore the determinant of the matrix $\partial V(a)/\partial a$, which is the coefficient in front of $t^3$ in the previous expression, is equal to zero identically. As a result, $J$ has a quadratic dependence on time $t$ only:

$$
J = 1 + tc_1(a) + t^2c_2(a).
$$

Singularity formation corresponds to the case $J \rightarrow 0$ at some point. The collapse moment of time $t_0$ will be determined by the coefficients $c_1$ and $c_2$ so that $t_0$ will be a minimal positive root of the equation

$$
\min_a J(a, t_0) \equiv J_{\min}(t_0) = 0.
$$
Calculations by means of Eq-s (5.5), (5.7) and (5.8) lead to the following expressions:

$$V(a) = \frac{-2}{(1 + a^2)^2} \left( (e \times a)(1 - a^2) + 2a(a \cdot e) - 2a^2 \right) ,$$  \hspace{1em} (5.9)

$$J(a, t) = \frac{(1 + a^2)^3 - 8t a_3 (1 + a^2)^2 + 4t (1 + a_3^2 - a_1^2 - a_2^2)}{(1 + a^2)^3} ,$$ \hspace{1em} (5.10)

where $a_3$ is the projection of the vector $a$ to the axis $e$.

Analyzing the latter expression one can show that at $t < 1$ a minimum of the Jacobian is attained on the symmetry axis, i.e., at $a_1 = 0, a_2 = 0$. In this case

$$J|_{axis} = \frac{(1 - a_1^2)^2 + 2(t - 1)^2}{(1 + a_3^2)^2} ,$$  \hspace{1em} (5.11)

hence it becomes clear that the singularity takes place at $t_0 = 1, a_3 = 1$, and the Jacobian tends to zero with the quadratic asymptotics. Thus, in this example

$$|\Omega|_{max} \sim (t_0 - t)^{-2} .$$

It can be easily verified also that the Jacoby matrix has two zero eigenvalues at the touching point which eigenvectors lie in the plane orthogonal to $e$. In the collapse vicinity the field $\Omega$ is directed along the vector $e$. Compression along this direction is linear in time $l_3 \sim (t_0 - t)$, but in the perpendicular plane it is more fast, with the law $l_{1,2} \sim (t_0 - t)^{3/2}$. As a result, the singularity structure occurs strongly stretched along the anisotropy axis.

### CONCLUDING REMARKS

Considering the hydrodynamic model with the Hamiltonian $\mathcal{H} = \int |\Omega| dr$, we have arrived at the conclusion that each vortex line in the given system moves independently of other lines. Just this property makes possible to form singularity in a finite time for the generalized vorticity $\Omega(r, t)$ from smooth initial data. A typical singularity of this kind looks like an infinite condensation of the vortex lines near some point. Thus, the collapse in the integrable hydrodynamics has purely inertial origin. If one will assume that this type of collapse is possible also in the Euler hydrodynamics, then the asymptotics of the vorticity near the singularity point (the point of vortex “overturning”) will be given by Eq-s (4.5) or (1.13) in a non-degenerated situation. That is already clear from the general consideration. Namely, the curl of the velocity will blow up as $(t_0 - t)^{-1}$. Exactly this dependence for vorticity near a singular point has been observed in practically all numerical simulations of the Euler equation, including the above cited: \cite{1}, \cite{7}, \cite{1}. \cite{2}, \cite{3}, \cite{2}.

However, not all the simulations are regarded to numerical integration of the Euler equations for continuous distributions of vorticity. The first numerical experiments \cite{3} relate to investigation of collapse for two anti-parallel vortex filaments which, as it was shown by Crow \cite{1}, are linearly unstable with respect to transverse perturbations (about the development of this direction see also \cite{2}). The theory of collapse for two thin vortex filaments, as the nonlinear stage of the Crow instability, was developed by V.E.Zakharov \cite{7}, \cite{8} (see also \cite{2}). The conclusions of this theory are in a good agreement with the numerical experiments up to the distances comparable with a core size of filaments. This theory predicts decreasing distance between vortex filaments as $\sqrt{t_0 - t}$. For smaller distances the cores of vortices loose their round shape. They become flat, and the process of attraction between vortices becomes more slow \cite{9}, \cite{10}. The same tendency was observed also in the numerical experiments of Kerr \cite{11}, \cite{12}, the most advanced (in our opinion) for the problem of reconnection, and where, unlike \cite{11}, collapse of two anti-parallel vortices was simulated with continuously distributed vorticity. Besides a natural contraction of the minimal distance between the distributed vortices, these simulations have shown, for the first time, the formation of two singularities in two symmetric points. While approaching a collapse moment in time, the explosive growth of the maximal value of vorticity was observed with the law $(t_0 - t)^{-1}$. According to the recent publications of Kerr \cite{12}, the analysis of numerical data gave two distinguished scales, one scale being contracted as the square root: $l_1 \sim (t_0 - t)^{1/2}$, and another as the first power of time: $l_2 \sim t_0 - t$.

In the work of Grauer, Marliani and Germanesewsky \cite{1}, the successful attempt was undertaken to observe collapse for the initial condition not possessing a low symmetry. The initial vorticity was concentrated in the vicinity of a cylinder, and was modulated over the angle in such a way so that the simplest symmetries were absent. In the present time this experiment has the best spatio-temporal resolution. In this simulation appearance of a separate collapsing region was observed with the vorticity growth at the center as $(t_0 - t)^{-1}$.

Thus, the results of the numerical simulations for ideal hydrodynamics go along with the concept of collapse as a process of caustic formation for the solenoidal field. There is an agreement for behavior of the vorticity maximum. Concerning the spatial structure of the collapsing domain, we can make only a qualitative agreement. The results of the papers \cite{3}, \cite{1} (some contraction of the vortex core was observed at the initial stage of reconnection neglecting viscosity) and also the results of Kerr seem to support our theory. We would like to pay attention to the numerical results of the paper \cite{12} where for short-time dynamics for the initial conditions in the form of the Taylor-Green vortex and for random initial conditions it was observed formation of thin vortex layers with high vorticity that also support our predictions.

All these results allow one to say that the scenario presented here looks like very plausible.

We would like to repeat once more, that if such scenario takes place then behavior of the vorticity near of singular point is defined by Eq-s (1.2) or (1.13). The structure of this domain should be highly anisotropic: in one of two directions perpendicular to the vorticity, there is more rapid contraction ($\sim \tau^{1/2}$) than with respect to other directions ($\sim \tau^{1/2}$). The spatial distribution becomes close to the two-dimensional one familiar to a tangential discontinuity. The velocity of flow in this region can be approximated with a good accuracy by the linear dependence

$$v_\perp \sim \Omega_{max} X_1 ,$$

i.e. the flow looks like a shear flow. The given type of col-
lapse, according to the classification of [32], belongs to a weak collapse: the energy captured into singularity (with account of the viscosity $\nu$) tends to zero as $\nu \to 0$. It is interesting to note that the dissipation rate $\sim \int \Omega^2 \, d\mathbf{r}$ from the collapsing region also vanishes when $\nu \to 0$.

One should note once more that unlike the model of free vortices considered here, in the Euler hydrodynamics vortex lines interact pairwise in accordance with the Hamiltonian

$$\mathcal{H}_{Euler} = \frac{1}{8\pi} \int \int \frac{\mathbf{R} (\nu, s) \cdot \mathbf{R} (\mu, \xi)}{\mathbf{R} (\nu, s) - \mathbf{R} (\mu, \xi)} \, d^2 \nu ds \, d^2 \mu \xi. \quad (5.12)$$

From the viewpoint of investigation of the collapse problem it is very important that the interaction function (i.e. the Green function for the Laplace operator) has the singularity for equal arguments $\mathbf{R} (\nu, s) \to \mathbf{R} (\mu, \xi)$. If this singularity would be absent, i.e. if the interaction function would be completely regular, then any initial vortex line distribution equivalent in the sense of iso-vorticity to a smooth field $\Omega_0 (\mathbf{a})$, even very singular distribution, would produce a sufficient smooth velocity field $\mathbf{v} (\mathbf{r})$. In a smooth velocity field, an initial singularity of the generalized vorticity could not disappear in subsequent moment of time, it could be only transported and deformed by the fluid flow. As far as the equations of motion for an inviscid substance are time-reversible, it follows from this, that the formation of singularity from a smooth initial data also would not possible. So, the existence and possible types of collapse of vortex lines, in the systems with the quadratic on $\Omega$ Hamiltonians, depend on the asymptotics of the interaction function $G(\mathbf{r}_1, \mathbf{r}_2)$ at $\mathbf{r}_1 \to \mathbf{r}_2$. So, for a better understanding of the collapse problem in hydrodynamics, it is reasonable to investigate such systems, for which the interaction function has the asymptotics $G \sim |\mathbf{r}_1 - \mathbf{r}_2|^{-q}$, and the exponent $q$ is not necessary equal to the unity.

What is the influence of viscosity on the structure of collapsing region? How does the given type of collapse have effect on turbulent spectra? That are only few of the most important questions, which need investigation. It would be interesting to verify numerically our hypothesis about possibility of (quasi-)inertial collapse in ideal hydrodynamics, both in Eulerian variables and in the vortex line representation.

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