Introduction and the Statement.

In [A] S. Akbulut obtained an example of the exotic manifold cutting off the contractible submanifold from the standard manifold and regluing it via nontrivial involution of the boundary.

In these notes we give a proof of a decomposition theorem stated below. It generalizes the example of Akbulut.

Another proof of the theorem was independently obtained by C.L. Curtis, M.H. Freedman, W.C. Hsiang, and R. Stong in [CHFS].

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Throughout these notes all maps and manifolds are smooth and immersions are in general position (or do their best if they have to obey some extra conditions). We also make the convention that if a star appears in place of a subindex, we consider a union of all objects in the family, where the index substituted by the star runs over its range. For example, $D_* \overset{\text{def}}{=} \bigcup D_i$.

**Theorem.** Let $U$ be a smooth, 5-dimensional, simply-connected $h$-cobordism with $\partial U = M_1 \sqcup (-M_2)$. Let $f : M_1 \to M_2$ be the homotopy equivalence induced by $U$.

1. There are decompositions

   $$M_1 = M_1^* \Sigma W_1, \quad M_2 = M_2^* \Sigma W_2$$

   such that $\text{in}_{2,1}^{-1} \circ \text{in}_{1,*} = f_* : H_2(M_1) \to H_2(M_2)$. Here $\text{in}_{2,*}$, $\text{in}_{1,*}$ are the maps induced in the second homology by embeddings of $M$ into $M_1$ and $M_2$ respectively, and $W_1$, $W_2$ are smooth, compact, contractible 4-manifolds, and $\Sigma = \partial W_1 = \partial W_2 = \partial M$.

2. These decompositions may be chosen so that $W_1$ is diffeomorphic to $W_2$.

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In fact, it can be seen from the proof that the whole cobordism can be decomposed into two subcobordisms, each is a product cobordism and one is diffeomorphic to $D^4$ (as a smooth manifold, without any additional structure).

We will also need the following

**Definition.** We say that two collections $\{S_i\}_{i=1}^n$, $\{P_i\}_{i=1}^n$ of oriented 2-spheres immersed in oriented 4-manifold are algebraically dual if $\langle[S_i], [P_j]\rangle = \delta_{ij}$. Here, $[S]$ is a homology class of immersed sphere $S$ and $\langle\cdot, \cdot\rangle$ is the intersection form in the second homology of 4-manifold.

They are geometrically dual if they are algebraically dual and, moreover, card$(S_i \cap P_j) = \delta_{ij}$.

**Proof of Theorem.**

First, observe that $U$ has a handlebody with no 1- and 4-handles. Let $N$ be the middle level of $U$ between 2- and 3-handles. Then we have two diffeomorphisms

\[
\phi_1 : M_1 \# S^2_{11} \times S^2_{12} \# \ldots \# S^2_{n1} \times S^2_{n2} \to N
\]

\[
\phi_2 : M_2 \# S^2_{11} \times S^2_{12} \# \ldots \# S^2_{n1} \times S^2_{n2} \to N.
\]

By enriching the handlebody of $U$ by 2-3 canceling pairs of handles and choosing $\phi_1$ and $\phi_2$, we can assume that $(\phi_2^{-1} \circ \phi_1)_*|_{H_2(M_i)} = f_*$ and $(\phi_2^{-1} \circ \phi_1)_*[S^2_{ij}] = [S^2_{ij}]$, $i = 1, \ldots, n$, $j = 1, 2$.

Then the two embeddings $c_1 : \bigcup S^2_{ij} \to N$, $c_2 : \bigcup S^2_{ij} \to N$, representing cores of products of spheres in decompositions $\phi_1$ and $\phi_2$, are homotopic. We have two algebraically dual collections of embedded 2-spheres $\{S_i = c_1(S^2_{ij}) \subset N\}_{i=1}^n$, $\{P_i = c_2(S^2_{ij}) \subset N\}_{i=1}^n$.

and surgery along $\{P_i\}$ gives $M_1$; along $\{S_i\}$ gives $M_2$. Put $V_0 = \overline{N_d}(S_s \cup P_s)$, the closed regular neighborhood of $S_s \cup P_s$ in $N$.

Here we give a sketch of the rest of the construction and than work out the details.

1. Manifold $V_0$ has a free fundamental group and its second homology are generated by classes of spheres $S_i$ and $P_i$, $i = 1, \ldots, n$. We enlarge $V_0$ to obtain the bigger manifold $V_1$ so that collections of spheres $\{S_i\}$ and $\{P_i\}$ have geometrically dual collections of immersed spheres inside of $V_1$ and the properties mentioned above are satisfied.

2. We add 1-handles and essential 2-handles to $V_1$ so that the fundamental group vanishes and no new two dimensional homology classes appear.

3. Surgery of the manifold obtained in step 2 along collections $\{S_i\}$ and $\{P_i\}$ gives two contractible submanifolds $W_1$ and $W_2$ of $M_1$ and $M_2$, respectively, and $\overline{M_1 \setminus W_1} \cong \overline{M_2 \setminus W_2}$.

The intersection points of embedded spheres $S_i$ and $P_j$ can be grouped in pairs of points of opposite sign with one extra point of positive sign when $i = j$, which we refer to as wedge points. Consider a disjoint collection of Whitney circles $l_1, l_2, \ldots, l_m$ in $S_s \cup P_s$, one for each pair of points considered above. Push these circles to the boundary of the neighborhood of $S_s \cup P_s$ in $N$ and call this new circles $l'_1, l'_2, \ldots, l'_n$ as in. Each of $l'_i$ is contractible in $N \setminus S_s$,
A DECOMPOSITION OF SMOOTH SIMPLY-CONNECTED H-COBORDANT 4-MANIFOLDS.

as well as in $N \setminus P_*$, since the latter two are simply-connected manifolds. Thus, for each $l_i$ we can find two immersed disks $D_i$ and $E_i$ with $\partial D_i = \partial E_i = l_i$, coinciding along the collar of the boundary, and satisfying the following condition:

$$D_* \cap S_* = \emptyset, \quad E_* \cap P_* = \emptyset. \quad (1)$$

Now we shall show that it is possible to find disks $D'_i$ and $E'_i$ so that they are homotopic rel(collar of the boundary) and the condition similar to (1) is satisfied.

The union of $D_i$ and $E_i$ with appropriate orientations gives us a class $[D_i \cup E_i]$ in the homology of $N$, which splits into the direct sum $H_2(N) = H_2(M_1) \oplus <[S_i], [P_i]>$. So, we can write $[D_i \cup E_i] = a + \sum \beta_k[S_k] + \sum \gamma_k[P_k]$, where $a \in H_2(M_1)$. Using the fact that $\pi_1(M_1) = \{1\}$ and the decomposition $\phi$, we can realize class $a$, considered as a class in $H_2(N)$, by an immersed sphere $A$ in $N$ disjoint from $S_*$. Classes $\sum \beta_k[S_k]$ and $\sum \gamma_k[P_k]$ can be realized by the immersed spheres $B$ and $C$ in $N$ disjoint from $S_*$ and $P_*$, respectively. Now taking the connected sums $D'_i = D_i \# (-A) \# (-B)$, $E'_i = E_i \# (-C)$ (‘-’ here denotes reversal of orientation) ambiently along carefully chosen paths, we obtain discs $D'_i$ and $E'_i$ satisfying property similar to (1) and homotopic rel(collar of the boundary).

Consider homotopy $F_t : \bigsqcup D_i^2 \to N$ rel(collar of the boundary), where $F_0(\bigsqcup D_i^2) = D'_i$ and $F_1(\bigsqcup D_i^2) = E'_i$. It can be also viewed as a homotopy of the union of the disks and spheres $S_*$, $P_*$, where the spheres stay fixed during the homotopy.

**Lemma.** Homotopy $F$ can be perturbed, fixing ends, $S_* \cup P_*$ and the collar of the boundary of disks in the disks, to homotopy $F'$ satisfying the following property: $F'|_{[0, \frac{1}{2}]}$ can be decomposed in a sequence of simple homotopies each of which is either a cusp or finger move; analogously, $F'|_{[\frac{1}{2}, 1]}$ can be decomposed in a sequence of inverse cusp moves and Whitney tricks.

For definitions of cusp, finger move, inverse cusp move and Whitney trick see [FQ, K, GM].

Proof of Lemma is given in the next section.

Assume, now, that $F$ has the property provided by Lemma above. Consider $M_* = \bigcup_i M_i = F_\uparrow(\bigcup D_i^2)$. Each of $M_i$ may intersect both $S_*$ and $P_*$, but it is homotopic rel(collar of the boundary) to the disk with interior disjoint from $S_*$, via a homotopy increasing number of intersection points. This means that the intersection points of $M_*$ and $S_*$ can be grouped in pairs so that for each pair there is an embedded Whitney disk $X_k$ with interior disjoint from the rest of the picture. The same argument shows that there are disks $Y_k$ for intersection points of $M_*$ and $P_*$, see figure 1.
Using embedded disks from collection \{X_k\} we can push all disks \(M_i\) off \(S_\ast\), producing the new disks \(M'_i\). Each \(M'_i\) has an interior disjoint from \(S_\ast\) and its boundary is a Whitney circle for a pair of points of opposite sign in \(S_\ast \cap P_\ast\). Applying an immersed Whitney trick to \(P_\ast\) we eliminate all the intersections with \(S_\ast\), except those required by algebraic conditions and produce the collection \(\{S_i^\perp\}\) geometrically dual to \(\{S_i\}\). Note that, since disks \(M'_i\) are, in general, immersed and may have non-trivial relative normal bundle in \(N\), we may have to introduce (self-)intersection points to spheres \(\{S_i^\perp\}\) during this process. In the same way, using disks from collection \(\{Y_i\}\) we can obtain the immersed Whitney disks \(M''_i\) disjoint from \(P_\ast\). They allow us to create a collection \(\{P_i^\perp\}\) of immersed spheres dual to \(\{P_i\}\). Note, that all described homotopies can be performed in a regular neighborhood of \(M_\ast \cup X_\ast \cup Y_\ast\).

Thus, one can see that the manifold \(V_1 = V_0 \cup \text{Nd}_N(M_\ast \cup X_\ast \cup Y_\ast)\) has the following properties:

1. \(H_2(V_1)\) is generated by homology classes of embedded spheres \(S_i, P_i\).
2. The collection of spheres \(\{S_i\}\) has the geometrically dual collection \(\{S_i^\perp\}\) of immersed spheres in \(V_1\). Similarly, collection of spheres \(\{P_i\}\) has geometrically dual collection \(\{P_i^\perp\}\).
3. \(\pi_1(V_1)\) is a free group.

Consider a handlebody of \(N\) starting from \(V_1\). Put \(V_2 = V_1 \cup (\text{union of } 1\text{-handles})\). \(V_2\) still satisfies properties similar to (1), (2), (3) for \(V_1\).

Let \(g_1, \ldots, g_l\) be free generators of \(\pi_1(V_2)\). If we fix paths from a basepoint to attaching spheres of 2-handles then they represent elements of \(\pi_1(\partial V_2)\), say \(h_1, \ldots, h_L\). Since \(V_2 \cup (\text{all } 2\text{-handles})\) is a simply connected manifold, each \(g_i\) has a lift \(\tilde{g}_i\) in \(\pi_1(\partial V_2)\), such that it belongs to the normal subgroup of \(\pi_1(\partial V_2)\) generated by \(h_1, \ldots, h_L\). In other words \(\tilde{g}_i = (\alpha_1 h_{i1}^{\pm 1} \alpha_1^{-1})(\alpha_2 h_{i2}^{\pm 1} \alpha_2^{-1}) \ldots (\alpha_k h_{ik}^{\pm 1} \alpha_k^{-1})\). Duplicating 2-handles, equipping them with appropriate orientations and choosing paths joining attaching circles of 2-handles to the basepoint we may write \(\tilde{g}_i = h'_1 h'_2 \ldots h'_k\), where \(h'_1, h'_2, \ldots, h'_k\) are elements of \(\pi_1(\partial V_2)\) obtained from attaching spheres of new handles with new paths to the basepoint. Now, handles
in the decomposition of \( \tilde{g}_i \) are all distinct and we can slide the first handle over all others along paths joining feet of the handles to the basepoint. The attaching circle of the resulting handle, call it \( G_i \), is freely homotopic to \( g_i \) in \( V_2 \). Adding such \( G_i \)'s to \( V_2 \) for each generator of \( \pi_1(V_2) \) we obtain the simply-connected manifold \( V_3 \). Since \( g_1, \ldots, g_l \) are free generators of the fundamental group, we do not create any additional two-dimensional homology classes.

Surgery of \( V_3 \) along collections of embedded spheres \( \{S_i\} \) and \( \{P_i\} \) gives two contractible sub-manifolds \( W_1 \) and \( W_2 \) of \( M_1 \) and \( M_2 \), respectively. Put \( M \triangleq N \setminus V_3 \cong M_1 \setminus W_1 \cong M_2 \setminus W_2 \). So we have the decompositions:

\[
M_1 = M\natural_\Sigma W_1, \quad M_2 = M\natural_\Sigma W_2
\]

The property of induced maps in the homology stated in Theorem is obvious. The first part of Theorem is proved.

Proof of the second part is based on the

**Fact.** If \( W_1, W_2 \) are homotopy balls built in the proof of the first part of Theorem and \( S^4 \) is a 4-dimensional sphere with standard smooth structure, then

\[
W_1\natural_\Sigma W_1 \cong S^4, \quad W_1\natural_\Sigma W_2 \cong S^4.
\]

Denote the boundary connected sum by ‘\( \natural \)’. Then we can write

\[
M_1 = M\natural_\Sigma W_1 \cong (M\natural_\Sigma W_1)\natural(W_1\natural_\Sigma W_2) \cong (M\natural W_1)\natural_\Sigma(W_1\natural W_2)
\]

and

\[
M_2 = M\natural_\Sigma W_2 \cong (M\natural_\Sigma W_2)\natural(W_1\natural_\Sigma W_1) \cong (M\natural W_1)\natural_\Sigma(W_2\natural W_1).
\]

See figure 2.

In order to proof Fact we will use the art of Kirby calculus. First, we build a handlebody of \( V_3 \).
Consider $V'_0$, a closed neighborhood of $S_* \cup P_* \cup (\text{arcs joining wedge points in } S_i \cap P_i \text{ and } S_{i+1} \cap P_{i+1})$. If there were no intersections between $S_*$ and $P_*$, except a wedge point for each pair $S_i, P_i$, then the handlebody would look like as shown on figure 3.

![Figure 3](image-url)

Introducing a pair of intersection points of opposite signs corresponds to the move in Kirby calculus shown in figure 4.

Here, we introduce two 1-handles, one corresponding to a Whitney circle, another to an accessory circle of a newly introduced pair of intersections. The handlebody of $V'_0$ is obtained by applying several moves shown in figure 4 to the picture in figure 3. Again, if disks $M_i$ were embedded and disjoint from $S_* \cup P_*$, then to obtain the handlebody of $V_0 \cup \text{Nd}(M_*)$ one has to attach 2-handles to Whitney circles for each pair of intersection points of $S_*$ and $P_*$. Then we have to introduce intersections between $M_*$ and $S_* \cup P_*$ and self-intersections of $M_*$. Corresponding moves are shown on figures 4 and 5.

![Figure 4](image-url)

Introducing a self-intersection corresponds to adding one 1-handle to the picture. Addition of disks from collections $\{X_i\}, \{Y_i\}$ corresponds to attaching 2-handles to the circles linked

![Figure 5](image-url)

![Figure 6](image-url)
Once to the 1-handles corresponding to the Whitney circles, and unlinked from other 1-handles. They may link each other according to the fact that boundaries of $X$’s and $Y$’s are not necessarily disjoint. This phenomenon is illustrated on figure 2. Then, as in previous steps, add 1-handles to the picture reflecting intersections of $X_*$ and $Y_*$, as in figure 6.

To obtain the handlebody of $V_2$ we have to attach several 1-handles. The link on figure 6 shows all possible phenomena which can occur. Thus, the handlebody of $V_2$ has 1-handles of six types coming from

1. Whitney circles of intersections of $S_*$ and $P_*$;
2. accessory circles of intersections of $S_*$ and $P_*$;
3. Whitney circles of intersections of $M_*$ and $S_* \cup P_*$;
4. accessory circles of intersections of $M_*$ and $S_* \cup P_*$;
5. Whitney and accessory circles of self-intersections of $M_*$;
6. Extra 1-handles.

For each 1-handle of type 1 and 3 there is a 2-handle attached to the circle linked to this 1-handle geometrically once and algebraically zero times to other 1-handles.

The attaching circles $h_1, \ldots, h_l$ of other 2-handles are homotopic to free generators of $\pi_1(V_2)$. As generators we may choose the cores $g_1, \ldots, g_l$ of 1-handles of type 2, 4, 5 and 6.

Consider the homotopy $F : S^1 \times I \to V_2, F(\cdot, 0) = g_i, F(\cdot, 1) = h_i$. We can make it disjoint from $S_* \cup M^* \cup X_* \cup Y_*$. First, intersections of the image of the homotopy with $X_*$ and $Y_*$ can be turned into intersections with $M_*$ by pushing them toward the boundary of $X_* \cup Y_*$. Then, in the same way, we can avoid intersections with $M_*$ by cost of new intersections with $P_*$. Finally, intersections with $S_*$ can be removed using the geometrically dual collection of immersed spheres $\{S_i^\perp\}$. Call this new homotopy $F'$ and let $\{x_1, x_2, \ldots, x_k\} = F'^{-1}(P_*)$. If we remove a small neighborhood of a union of disjoint arcs joining each $x_i$ to the point on $S^1 \times \{0\}$ from $S^1 \times I$, then restriction of $F'$ on this set is a homotopy of $h_i$ to the curve $g'_i$ which is a band connected sum of $g_i$ and meridians of $P_*$. This homotopy is disjoint from $S_* \cup P_* \cup M_* \cup X_* \cup Y_*$ and can be pushed to the boundary of $V_2$.

We obtain the handlebody of $W_1$ by attaching 2-handles to $h_1, \ldots, h_l$ and surgering $S_*$, which corresponds to putting dots on the circles representing $S_*$ in Kirby calculus.

Here we summarize all the information about the handlebody of $W_2$. For 2-handles we use same notation as for the corresponding object in construction of $W_1$ above.
1. Surgery of $S_\ast$.

2. Whitney circles of intersections of $S_\ast$ and $P_\ast$.

3. Whitney circles of intersections of $M_\ast$ and $S_\ast \cup P_\ast$.

4. Accessory circles of all intersections, Whitney circles of (self-)intersections of $M_\ast$, Whitney circles of intersections of $X_\ast$ and $Y_\ast$, extra 1-handles.

$P_\ast$; attaching circles go geometrically once through corresponding 1-handles $S_\ast$.

$M_\ast$; attaching circles go geometrically once through corresponding 1-handles from the left entry of the table.

$X_\ast$, $Y_\ast$; attaching circles go geometrically once through corresponding 1-handles from the left entry of the table.

Extra 2-handles $H_\ast$; attaching circles go geometrically once through corresponding 1-handles from the left entry of the table and homotopic to band connected sum of 1-handles with meridians of $P_\ast$.

Taking a double of $W_1$ corresponds to attaching zero-framed 2-handles to the meridians of existent 2-handles, and as many 3-handles as there are 1-handles in the handlebody of $W_1$.

Remember that attaching circles $h_i$ of 2-handles $H_i$ are homotopic to $g'_i$. Sliding $H_\ast$ over dual handles $H'_*\ast$ we can obtain handles $H'_*\ast$ attached to $g'_*$. Now $g'_1, \ldots, g'_l$ may be linked to the meridians of handles $P_\ast$, $M_\ast$, $X_\ast$ and $Y_\ast$, but situation can be improved by sliding handles dual to $P_\ast$, $M_\ast$, $X_\ast$ and $Y_\ast$ over handles dual to $H'_*\ast$. Further we may slide $H'_*\ast$ over handles dual to $P_\ast$ to obtain handles attached to meridians of 1-handles from the fourth row of the table, so they can be cancelled. The framings of $H'_\ast$ result in a twist of attaching circles of remaining 2-handles.

We undo this twist using dual handles.

With the next step we unlink handles $X_\ast$ and $Y_\ast$ from each other using dual 2-handles and cancel them with 1-handles of type 3. Now $M_\ast$ are attached to meridians of 1-handles of type 1 and also can be cancelled. After cancelling $P_\ast$ with 1-handles from surgery of $S_\ast$ we end up with unknotted, unlinked, zero framed handles and the same number of 3-handles, which is clearly $S^4$.

We need only small changes in our construction to prove the second diffeomorphism in our Fact. The handlebody of $W_2$ differs from $W_1$ by putting dots on the attaching circles of $P_\ast$ rather than $S_\ast$. Thus, the handlebody of $W_1 \sharp W_2$ is obtained from $W_1$ by attaching
A decomposition of smooth simply-connected $H$-cobordant 4-manifolds.

2-handles to meridians of 1-handles, coming from surgered $S_i$’s, and to meridians of all 2-handles, except $P_i$’s. Using the same trick we can make the homotopy of $h_*$ to $g_*$ disjoint from $P_* \cup M_* \cup X_* \cup Y_*$, rather than $S_* \cup M_* \cup X_* \cup Y_*$. Applying the same procedure to $H_*$ gives us 2-handles $G''_*$ attached to the band connected sum of meridians of 1-handles corresponding to generators of $\pi_1(V_2)$ and meridians of $S_*$. But now 1-handles corresponding $S_*$ have dual 2-handles, so sliding $H_*$ over them gives handles dual to 1-handles generating $\pi_1(V_2)$. So we may apply the same procedure to simplify the handlebody and end up with $S^4$. This finishes the proof of Fact and the second part of Theorem.

Proof of Lemma.

It is a simple consequence of singularity theory that a homotopy of a surface in 4-manifold can be decomposed (after small perturbation) in a sequence of finger moves, cusps, Whitney tricks and inverse cusp moves. In our case we have to consider, in addition, a finger move along a path in one of the disks joining (self-)intersection of the disks with a point on its boundary. The inverse of this move is a Whitney trick with a Whitney circle intersecting the boundary of the disk. We have to show that it is possible to reorder these simple homotopies so that those increasing number of intersections come first. This is obvious after the following consideration: cusp birth happens in a small neighborhood of the point in the surface and we can assume that part of the surface in this neighborhood stays fixed during the part of homotopy preceding this cusp birth, so we can push this cusp up to the beginning of the homotopy. A finger move can be localized in the neighborhood of an embedded arc joining two points in the surface, thus we may apply the same argument and this finishes the proof of Lemma.

References

[A] Akbulut, S., A Fake compact contractible 4-manifold. J. Diff. Geom Vol. 33 (1991) pp. 335-356.
[CH0] Curtis, C.L., Hsiang, W.C., Elementary notes on h-cobordant simply-connected smooth 4-manifolds. Preprint
[CH1] Curtis, C.L., Hsiang, W.C., A decomposition theorem for h-cobordant simply-connected smooth 4-manifolds. Preprint
[CHFS] Curtis, C.L., Freedman M.H., Hsiang W.C., Stong, R. A decomposition theorem for h-cobordant smooth simply-connected compact 4-manifolds. preprint.
[FQ] Freedman M.H., Quinn F.F. Topology of 4-manifolds. Princeton, N.J. : Princeton University Press, 1990.
[GM] Guillou L., Marin A. (editors), A la Recherche de la Topologie Perdue. Birkhäuser, 1986.
[K] Kirby, R.C., The Topology of 4-manifolds. Lecture Notes in Mathematics 1374, Springer-Verlag, 1989.
[Wall1] Wall, C.T.C., Diffeomorphisms of 4-manifolds. J. London Math. Soc. Vol. 39 (1964), pp. 130–140.
[Wall2] Wall, C.T.C., On simply-connected 4-manifolds. J. London Math. Soc. Vol. 39 (1964), pp. 141–149.
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