Cohomology of $\text{Aut}(F_n)$ in the $p$-rank two case

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Abstract

For odd primes $p$, we examine $\tilde{H}^*(\text{Aut}(F_{2(p-1)}); \mathbb{Z}_{(p)})$, the Farrell cohomology of the group of automorphisms of a free group $F_{2(p-1)}$ on $2(p-1)$ generators, with coefficients in the integers localized at the prime $(p) \subset \mathbb{Z}$. This extends results in [9] by Glover and Mislin, whose calculations yield $\tilde{H}^*(\text{Aut}(F_n); \mathbb{Z}_{(p)})$ for $n \in \{p-1, p\}$ and is concurrent with work by Chen in [6] where he calculates $\tilde{H}^*(\text{Aut}(F_n); \mathbb{Z}_{(p)})$ for $n \in \{p+1, p+2\}$. The main tools used are Ken Brown’s “normalizer spectral sequence” from [4], a modification of Krstic and Vogtmann’s proof of the contractibility of fixed point sets for outer space in [16], and a modification of the Degree Theorem of Hatcher and Vogtmann in [11].

Key words: cohomology of groups, free groups, outer space, auter space, Farrell cohomology

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1 Introduction

Let $F_n$ denote the free group on $n$ letters and let $\text{Aut}(F_n)$ and $\text{Out}(F_n)$ denote the automorphism group and outer automorphism group, respectively, of $F_n$. In [7] Culler and Vogtmann defined a space on which $\text{Out}(F_n)$ acts nicely called “outer space”. By studying the action of $\text{Out}(F_n)$ on this space, various people have been able to calculate the cohomology of $\text{Out}(F_n)$ in specific cases. In [3], Tom Brady calculated the integral cohomology of $\text{Out}(F_3)$. This remains even today the only complete (nontrivial) calculation of the integral cohomology of $\text{Out}(F_n)$ or $\text{Aut}(F_n)$. More recently, Hatcher in [10] and Hatcher and Vogtmann in [11] have defined a space on which $\text{Aut}(F_n)$ acts nicely called “auter space” and have used this to calculate the cohomology of $\text{Aut}(F_n)$ in specific cases.
A “Degree Theorem” is introduced by Hatcher and Vogtmann in [11] which is a very useful tool for simplifying cohomological calculations concerning $\text{Aut}(F_n)$. For example, they are able to derive linear stability ranges for the integral cohomology of $\text{Aut}(F_n)$ and able to calculate the rational cohomology of $\text{Aut}(F_n)$ in some low dimensional cases. In [14], we modified this degree theorem and used it to calculate the cohomologies of some spaces having to do with $\text{Aut}(F_{2(p-1)})$.

Glover and Mislin [9] calculated the cohomology with coefficients in $\mathbb{Z}_{(p)}$ of $\text{Out}(F_n)$ for $n = p - 1, p, p + 1$. In addition, Chen [5] calculates the integral cohomology of $\text{Out}(F_n)$ for $n = p + 2$ and $\text{Aut}(F_n)$ for $n = p + 1, p + 2$. In each of the above cases, the maximal $p$-subgroups of $\text{Out}(F_n)$ or $\text{Aut}(F_n)$ had $p$-rank one. The case of $\text{Aut}(F_{2(p-1)})$ is the first one where the maximal $p$-subgroups can have a higher $p$-rank, and it is this case which we calculate here.

In this paper we use Brown’s “normalizer spectral sequence” [4], to show

**Theorem 1.1** Let $p$ be an odd prime, and $n = 2(p - 1)$. Then $\hat{H}^t(\text{Aut}(F_n); \mathbb{Z}_{(p)})$

\[
\begin{align*}
\mathbb{Z}/p^2 \oplus \mathbb{Z}/p & \quad t = 0 \\
\mathbb{Z}/p & \quad |t| = kn \neq 0 \\
\mathbb{Z}/p & \quad t = 1 \\
\frac{(3(k-1)}{2} \mathbb{Z}/p & \quad t = kn + 1 > 1 \\
H^{n-1}(\tilde{Q}_{p-1}; \mathbb{Z}/p) \oplus \frac{(3(k-1)}{2} \mathbb{Z}/p & \quad t = kn - 1 > 0 \\
H^{n-1}(Q_{p-1}; \mathbb{Z}/p) & \quad t = -kn - 1 < 0 \\
H^r(Q_{p}; \mathbb{Z}/p) \oplus \sum_{i=0}^{p-1} H^r(\tilde{Q}_i \times Q_{p-1-i}; \mathbb{Z}/p) & \quad t = kn + r, 2 \leq r \leq n - 2
\end{align*}
\]

Here the $(2k - 2)$-dimensional space $Q_k$ is the quotient of the spine of outer space $X_k$ by $\text{Aut}(F_k)$. The $(2k - 1)$- and $(2p - 4)$-dimensional spaces $\tilde{Q}_k$ and $Q_{p}^*$, respectively, are the quotients of contractible spaces (defined in this paper) $\tilde{X}_k$ and $X_{p}^*$ on which $F_k \rtimes \text{Aut}(F_k)$ and $(F_{p-2} \times F_{p-2}) \rtimes (\mathbb{Z}/2 \times \text{Aut}(F_{p-2}))$, respectively, act properly with finite quotient.

Moreover, the submodule of $\hat{H}^t(\text{Aut}(F_n); \mathbb{Z}_{(p)})$ generated by all of the cohomology classes that are explicitly listed above (that is, the ones not listed as coming from quotient spaces) is a subring and is isomorphic as a ring to $\hat{H}^t((\star_{i=1}^{p-1} \Sigma_p) \ast \varsigma_p; \mathbb{Z}_{(p)})$, where $(\star_{i=1}^{p-1} \Sigma_p)$ is the free product of $p - 1$ copies of the symmetric group, $\varsigma_p$ is the fundamental group of the graph of groups pictured in Figure 1, and $((\star_{i=1}^{p-1} \Sigma_p) \ast \varsigma_p$ is the free product of these two groups. All of the cohomology classes that are explicitly listed above, with the exception of the one $\mathbb{Z}/p$ listed in dimension 1, are detected upon restriction to finite sub-
groups. Specifically, they are detected upon restriction to the finite subgroups coming from stabilizers of marked graphs with underlying graphs (see Figure 2) \( R_{2(p-1)}, R_k \vee \theta_{p-1} \vee R_{p-1-k} \) for \( k \in \{1, \ldots, p-2\} \), \( \Omega_{2(p-1)} \), and \( \Psi_{2(p-1)} \).

A quick note about our notation is appropriate here. In general, groups without any additional structure will be written using multiplicative notation (e.g., \( \mathbb{Z}/p \times \mathbb{Z}/p \cong (\mathbb{Z}/p)^2 \)) but modules like cohomology groups will be written using additive notation (e.g., \( \mathbb{Z}/p \oplus \mathbb{Z}/p \cong 2(\mathbb{Z}/p) \)). Hence the case \( t = 0 \) of our main result above should be read as stating that

\[
\hat{H}^0(\text{Aut}(F_{2(p-1)}); \mathbb{Z}/p) \cong \mathbb{Z}/p^2 \oplus (\mathbb{Z}/p \oplus \cdots \oplus \mathbb{Z}/p)
\]

where there are \( p \) copies of \( \mathbb{Z}/p \) in \( (\mathbb{Z}/p \oplus \cdots \oplus \mathbb{Z}/p) \).

Fig. 1. The graph of groups giving \( \varsigma_p \)

For the case of the prime \( p = 3 \), the cohomology groups of all of the relevant quotient spaces are calculated in Appendix C of this paper. This allows us to state concisely what the above result gives us in the case \( p = 3 \). This calculation was also (independently) done by Glover and Henn for both \( \text{Out}(F_4) \) and \( \text{Aut}(F_4) \).

**Corollary 1.2**

\[
\hat{H}^t(\text{Aut}(F_4); \mathbb{Z}/(3)) = \begin{cases} 
\mathbb{Z}/9 \oplus 3(\mathbb{Z}/3) & t = 0 \\
(\lfloor \frac{3k}{2} \rfloor + 2)\mathbb{Z}/3 & |t| = 4k \neq 0 \\
\mathbb{Z}/3 & t = 1 \\
0 & |t| \equiv 1, 2 \ (\text{mod} \ 4) \ and \ t \neq 1 \\
\left\lfloor \frac{3(k-1)}{2} \right\rfloor \mathbb{Z}/3 & |t| = 4k - 1
\end{cases}
\]

Moreover, all of the cohomology classes, with the exception of the one \( \mathbb{Z}/p \) listed in dimension 1, are detected upon restriction to finite subgroups. Specifically, they are detected upon restriction to the finite subgroups coming from stabilizers of marked graphs corresponding to \( R_4, R_1 \vee \theta_2 \vee R_1, \Omega_4 \), and \( \Psi_4 \). Finally, there is a (ring) isomorphism from the Farrell cohomology of \( \text{Aut}(F_4) \) above to that of \( \Sigma_3 * \Sigma_3 * \varsigma_3 \).
In the next section, we introduce some useful spectral sequences, in Section 3 we review the definitions of auter space and its spine, and in Section 4, we compute the elementary abelian $p$-subgroups of $Aut(F_n)$. For Section 5, we adapt methods of Krstic to compute the normalizers of these subgroups, in the section after that we define certain contractible subcomplexes of auter space that these normalizers act on, and in the following section we use the equivariant cohomology spectral sequence associated to the action of the normalizers on these contractible subcomplexes to compute the cohomology of the normalizers. Finally, Theorem 1.1 will be proved in Section 8. Appendix A states what Theorem 1.1 implies about the usual cohomology of $Aut(F_l)$, Appendix B briefly states some results about the cohomology of $Aut(F_l)$ in lower rank cases, and Appendix C contains the proof of Corollary 1.2.

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2 Spectral sequences

Let $G$ be a group acting cellularly on a finite dimensional CW-complex $X$ such that the stabilizer $stab_G(\delta)$ of every cell $\delta$ is finite and such that the quotient of $X$ by $G$ is finite. Further suppose that for every cell $\delta$ of $X$, the group $stab_G(\delta)$ fixes $\delta$ pointwise. Let $M$ be a $G$-module. Recall (see [4]) that the equivariant cohomology groups of the $G$-complex $X$ with coefficients in $M$ are defined by

$$H^*_G(X; M) = H^*(G; C^*(X; M))$$

and that if in addition $X$ is contractible (which will usually, but not always, be the case in this paper) then

$$H^*_G(X; M) = H^*(G; M).$$

In [4] Brown reviews the spectral sequence for equivariant cohomology:

$$\tilde{E}_1^{r,s} = \prod_{[\delta] \in \Delta^r} H^s(stab(\delta); M) \Rightarrow H^{r+s}_G(X; M)$$

(2.1)

where $[\delta]$ ranges over the set $\Delta^r$ of orbits of $r$-simplices $\delta$ in $X$.

If $M$ is $\mathbb{Z}/p$ or $\mathbb{Z}_{(p)}$ then a nice property should be noted about the spectral sequence (2.1). This property will greatly reduce the calculations we need to
go through, and in general will make concrete computations possible. Since each group $\text{stab}(\delta)$ is finite, a standard restriction-transfer argument in group cohomology yields that $|\text{stab}(\delta)|$ annihilates $H^s(\text{stab}(\delta); M)$ for all $s > 0$. (For examples of these sorts of arguments see [1] or [4].) Since all primes not equal to $p$ are divisible in $\mathbb{Z}/p$ or $\mathbb{Z}_p$, this in turn shows that the $p$-part of $|\text{stab}(\delta)|$ annihilates $H^s(\text{stab}(\delta); M)$ for $s > 0$. In particular, if $p$ does not divide some $|\text{stab}(\delta)|$, then this $[\delta]$ does not contribute anything to the spectral sequence (2.1) except in the horizontal row $s = 0$. It follows that if our coefficients are $\mathbb{Z}/p$ or $\mathbb{Z}_p$ then we are mainly just concerned with the simplices $\delta$ which have “$p$-symmetry”.

If $G$ is a group with finite virtual cohomological dimension (vcd) and $M$ is a $G$-module, then Farrell cohomology groups

$$\hat{H}^*(G; M)$$

are defined in [8]. For the basics about Farrell cohomology, along with several useful properties, see [4] or [8].

The equivariant cohomology spectral sequence for Farrell cohomology is given by

$$E_1^{r,s} = \prod_{[\delta] \in \Delta^r} \hat{H}^*(\text{stab}(\delta); M) \Rightarrow \hat{H}^{r+s}_G(X; M). \quad (2.2)$$

and has analogous properties to those listed for spectral sequence (2.1).

Ken Brown [4] introduces another spectral sequence that can be used to calculate $\hat{H}^*(G; \mathbb{Z}_p)$. It involves normalizers of elementary abelian $p$-subgroups of $G$ and hence is often called the “normalizer spectral sequence.” This is not the only alternative available to the standard spectral sequence for equivariant cohomology, as Hans-Werner Henn [13] has created a “centralizer spectral sequence” that involves centralizers of elementary abelian $p$-subgroups; however, Brown’s spectral sequence appears to be the easiest to apply in our situation.

Let $p$ be a prime, $G$ be a group with finite virtual cohomological dimension, $\mathcal{A}$ be the poset of nontrivial elementary abelian $p$-subgroups of $G$, $\mathcal{B}$ be the poset of conjugacy classes of nontrivial elementary abelian $p$-subgroups of $G$, and $|\mathcal{B}|_r$ be the set of $r$-simplices in the realization $|\mathcal{B}|$. Brown’s normalizer spectral sequence is

$$E_1^{r,s} = \prod_{(A_0 \subset \ldots \subset A_r) \in |\mathcal{B}|_r} \hat{H}^*(\bigcap_{i=0}^r N_G(A_i); \mathbb{Z}_p) \Rightarrow \hat{H}^{r+s}(G; \mathbb{Z}_p) \quad (2.3)$$

We will use the above normalizer spectral sequence to calculate the cohomol-
ogy of Aut\( (F_n) \). The cohomology groups of the normalizers, which are required as input into the normalizer spectral sequence, will be computed using equivariant cohomology spectral sequences via their actions on certain fixed point subspaces of auter space.

3 Aut\( (F_n) \) and auter space

We review some basic properties and definitions of the automorphism group Aut\( (F_n) \) of a free group \( F_n \) of rank \( n \) (where \( n = 2(p-1) \) for our work.) Most of these can be found in [7], [11], [18], [19], and [21]. Let \( (R_n, v_0) \) be the \( n \)-leafed rose, a wedge of \( n \) circles. We say a basepointed graph \( (G, x_0) \) is admissible if it has no free edges, all vertices except the basepoint have valence at least three, and there is a basepoint-preserving continuous map \( \phi: R_n \to G \) which induces an isomorphism on \( \pi_1 \). The triple \( (\phi, G, x_0) \) is called a marked graph. Two marked graphs \( (\phi_i, G_i, x_i) \) for \( i = 0, 1 \) are equivalent if there is a homeomorphism \( \alpha: (G_0, x_0) \to (G_1, x_1) \) such that \( (\alpha \circ \phi_0)\# = (\phi_1)\# : \pi_1(R_n, v_0) \to \pi_1(G_1, x_1) \). Define a partial order on the set of all equivalence classes of marked graphs by setting \( (\phi_0, G_0, x_0) \leq (\phi_1, G_1, x_1) \) if \( G_1 \) contains a forest (a disjoint union of trees in \( G_1 \) which contains all of the vertices of \( G_1 \)) such that collapsing each tree in the forest to a point yields \( G_0 \), where the collapse is compatible with the maps \( \phi_0 \) and \( \phi_1 \).

From [10] and [11] we have that Aut\( (F_n) \) acts with finite stabilizers on a contractible space \( X_n \), called the spine of auter space. The space \( X_n \) is the geometric realization of the poset of marked graphs that we defined above. Let \( Q_n \) be the quotient of \( X_n \) by Aut\( (F_n) \). Note that the CW-complex \( Q_n \) is not necessarily a simplicial complex. Since Aut\( (F_n) \) has a torsion free subgroup of finite index [10] and it acts on the contractible, finite dimensional space \( X_n \) with finite stabilizers and finite quotient, Aut\( (F_n) \) has finite vcd. Thus it makes sense to talk about its Farrell cohomology, and to apply the normalizer spectral sequences calculate its cohomology.

For \( n = 0 \), we set \( X_0 = Q_0 = \{ \text{a point} \} \) as a notational convenience.

We will use spectral sequence (2.3) to calculate the cohomology of Aut\( (F_n) \):

\[
E_1^{r,s} = \prod_{(A_0 \subset \cdots \subset A_r) \in |B|_r} \hat{H}^s(\prod_{i=0}^r N_{\text{Aut}(F_n)}(A_i); \mathbb{Z}(p)) \Rightarrow \hat{H}^{r+s}(\text{Aut}(F_n); \mathbb{Z}(p)) \quad (3.1)
\]

In order to use the above spectral sequence, we must classify the elementary abelian \( p \)-subgroups of Aut\( (F_n) \). From Zimmerman’s realization theorem [21] (cf. Culler [6]), any finite subgroup of Aut\( (F_n) \) is realized on a marked graph, where a subgroup \( G \) is said to be realized by a specific marked graph \( \eta :...
$R_n \to \Gamma$ if it is contained in the stabilizer of that marked graph. Smillie and Vogtmann [18] examined the structure of these stabilizers in detail, and we list their results here. Consider a given $r$-simplex

$$(\phi_r, \Gamma_r, x_r) > \cdots > (\phi_1, \Gamma_1, x_1) > (\phi_0, \Gamma_0, x_0)$$

with corresponding forest collapses

$$(H_r \subseteq \Gamma_r), \ldots, (H_2 \subseteq \Gamma_2), (H_1 \subseteq \Gamma_1).$$

For each $i \in 0, 1, \ldots, r$, let $F_i$ be the inverse image under the map

$$\Gamma_r \to \cdots \to \Gamma_{i+1} \to \Gamma_i$$

of forest collapses, of the forest $H_i$. That is, we have

$$F_r \subseteq \cdots \subseteq F_2 \subseteq F_1 \subseteq \Gamma_r.$$ 

The stabilizer of the simplex under consideration is isomorphic to the group $\text{Aut}(\Gamma_r, F_1, \ldots, F_r, x_r)$ of basepointed automorphisms of the graph $\Gamma_r$ that respect each of the forests $F_i$. For example, the stabilizer of a point $(\phi, \Gamma, x_0)$ in $X_n$ is isomorphic to $\text{Aut}(\Gamma, x_0)$. If a marked graph $(\phi, \Gamma, x_0)$ realizes a subgroup $G$, we can think of $\Gamma$ as a graph with an action of $G$ on it. Hence a first step toward calculating the elementary abelian $p$-subgroups of $\text{Aut}(F_n)$ will be finding out which graphs have $p$-symmetry.

4 Conjugacy classes of $p$-subgroups of $\text{Aut}(F_n)$

From [2] and [17], we see that $p^2$ is an upper bound for the order of any $p$-subgroup of $\text{Aut}(F_n)$. (Recall that we set $n = 2(p-1)$ earlier.) Every nontrivial $p$-subgroup of $\text{Aut}(F_n)$ is isomorphic to either $\mathbb{Z}/p$ or $\mathbb{Z}/p \times \mathbb{Z}/p$ (see, for example, Smillie and Vogtmann in [19].)

We want to find all conjugacy classes of (elementary abelian) $p$-subgroups of $\text{Aut}(F_n)$. By Zimmerman’s realization theorem [21], we can do this by analyzing marked graphs with $p$-symmetry, so that the stabilizers of these marked graphs have elements of order $p$. The action of $\text{Aut}(F_n)$ is transitive on marked graphs with the same underlying graph. Since we are only interested in conjugacy classes of subgroups, we will just examine the underlying graphs.

We first calculate which graphs $\Gamma$ with a $\mathbb{Z}/p$ action on them have $\pi_1(\Gamma) \cong F_n$. To simplify our calculations, we will consider only reduced $\mathbb{Z}/p$-graphs, where a $\mathbb{Z}/p$-graph $\Gamma$ is reduced if it contains no $\mathbb{Z}/p$-invariant subforests.
Fig. 2. Some graphs with $p$-symmetry

Some preliminary definitions of a few common graphs are in order. Let $\Theta_{p-1}$ be the graph with two vertices and $p$ edges, each of which goes from one vertex to the other (see Figure 2.) Say the “leftmost vertex” of $\Theta_{p-1}$ is the basepoint. Hence when we write $\Theta_{p-1} \lor R_{p-1}$ then we are stipulating that the rose $R_{p-1}$ is attached to the non-basepointed vertex of $\Theta_{p-1}$, while when we write $R_{p-1} \lor \Theta_{p-1}$ then we are saying that the rose is attached to the basepoint of $\Theta_{p-1}$. Let $\Phi_{2(p-1)}$ be a graph with $3p$ edges $a_1, \ldots, a_p, b_1, \ldots, b_p, c_1, \ldots, c_p$, and $p + 3$ vertices $v_1, \ldots, v_p, x, y, z$. The basepoint is $x$ and each of the edges $a_i$ begin at $x$ and end at $v_i$. The edges $b_i$ and $c_i$ begin at $y$ and $z$, respectively, and end at $v_i$. Note that there are obvious actions of $\mathbb{Z}/p$ on $\Theta_{p-1}$ and $\Phi_{2(p-1)}$, given by rotation, and that these actions are unique up to conjugacy. Let $\Psi_{2(p-1)}$ be the graph obtained from $\Phi_{2(p-1)}$ by collapsing all of the edges $a_i$ to a point. Let $\Omega_{2(p-1)}$ be the graph obtained from $\Phi_{2(p-1)}$ by collapsing either the edges $b_i$ or the edges $c_i$ (the resulting graphs are isomorphic) to a point. Note that the only difference between $\Psi_{2(p-1)}$ and $\Omega_{2(p-1)}$ is where the basepoint is located.

**Lemma 4.1** Let $p$ be an odd prime and $n = 2(p - 1)$, and let $\Gamma$ be a reduced (basepointed) graph with a nontrivial $\mathbb{Z}/p$-action, where $\pi_1(\Gamma) \cong F_n$. Let $e$ be an edge of $\Gamma$ which is moved by the $\mathbb{Z}/p$ action. Then the orbit $e_1, e_2, \ldots, e_p$ of the 1-cell $e$ under the $\mathbb{Z}/p$-action on the CW-complex $\Gamma$ forms either a rose $R_p$ or a $\Theta$-graph $\Theta_{p-1}$.
Proof. Since $\Gamma$ is reduced, the edges $e_i$ have more than one endpoint in common, else they form an invariant subforest. They also cannot form a $p$-gon, else we could take a minimal path from $e_1$ to the basepoint, consider its orbit under $\mathbb{Z}/p$, and find in those edges a star of $p$ edges that could be collapsed. $\square$

Proposition 4.1 Let $p$ be an odd prime and $n = 2(p - 1)$. The only reduced (basepointed) graphs $\Gamma$ with a nontrivial $\mathbb{Z}/p$-action and $\pi_1(\Gamma) \cong F_n$ are $R_n$, $R_k \lor \Theta_{p-1} \lor R_{p-1-k}$, $\Psi_n$, and $\Omega_n$.

Proof. First, suppose that $\Gamma$ has only one nontrivial $\mathbb{Z}/p$-orbit of edges

$$e_1, e_2, \ldots, e_p.$$ 

From the lemma, $\Gamma$ is $R_{2(p-1)}$ with $\mathbb{Z}/p$ rotating $p$ of the leaves and leaving the other $p - 2$ fixed, or $\Gamma$ is of the form $R_k \lor \Theta_{p-1} \lor R_{p-1-k}$ where $\mathbb{Z}/p$ rotates the edges of the $\Theta$-graphs and leaves the roses at either end of the $\Theta$-graph fixed.

Second, suppose $\Gamma$ has more than one nontrivial $\mathbb{Z}/p$-orbit of unoriented edges. Take two distinct orbits $e_1, e_2, \ldots, e_p$ and $f_1, f_2, \ldots, f_p$. As in the previous paragraph, we can use the fact that $\Gamma$ is reduced and Lemma 4.1 to get that the $e_i$ and $f_i$ either form roses or $\Theta$-graphs. If either one of them is a rose, then the rank of $\pi_1(\Gamma)$ is at least $2p - 1$, which is a contradiction. So both form $\Theta$-graphs. Since $\pi_1(\Gamma) \cong F_{2(p-1)}$, it follows that $\Gamma$ is either $\Psi_n$ or $\Omega_n$. $\square$

We now define several $p$-subgroups $A$, $B_k$, $C$, $D$, and $E$ of $Aut(F_n)$. Our goal is to show that these are a complete listing of the distinct conjugacy classes of $p$-subgroups of $Aut(F_n)$.

- There is an action of $\mathbb{Z}/p$ on the rose $R_{2(p-1)}$ given by rotating the first $p$ leaves of the rose. By looking at the stabilizer of a marked graph with underlying graph $R_{2(p-1)}$, this action gives us a subgroup $A \cong \mathbb{Z}/p$ of $Aut(F_{2(p-1)})$. This subgroup is a maximal $p$-subgroup, in the sense that no other $p$-subgroup properly contains it.
- For each $k \in \{0, \ldots, p-1\}$, there is an action of $\mathbb{Z}/p$ on $R_k \lor \Theta_{p-1} \lor R_{p-1-k}$ given by rotating the edges in $\Theta_{p-1}$. This action gives us a subgroup $B_k \cong \mathbb{Z}/p$ of $Aut(F_{2(p-1)})$. If $k \in \{1, \ldots, p-2\}$, then $B_k$ is a maximal $p$-subgroup.
- There is an action of $\mathbb{Z}/p$ on $\Phi_{2(p-1)}$ which gives us a (non-maximal) $p$-subgroup $C \cong \mathbb{Z}/p$ of $Aut(F_{2(p-1)})$.
- There is an action of $\mathbb{Z}/p \times \mathbb{Z}/p$ on $\Omega_{2(p-1)}$ given by having the first $\mathbb{Z}/p$ rotate one of the $\Theta$-graphs in $\Omega_{2(p-1)}$ and having the second $\mathbb{Z}/p$ rotate the other $\Theta$-graph. This action gives us a subgroup $D \cong \mathbb{Z}/p \times \mathbb{Z}/p$ of $Aut(F_{2(p-1)})$. 


This subgroup is maximal among $p$-subgroups, and it contains $B_0$, $B_{p-1}$, and $C$.

- There is an action of $\mathbb{Z}/p \times \mathbb{Z}/p$ on $\Psi_{2(p-1)}$ given by having the first $\mathbb{Z}/p$ rotate one of the $\Theta$-graphs in $\Psi_{2(p-1)}$ and having the second $\mathbb{Z}/p$ rotate the other $\Theta$-graph. This action gives us a subgroup $E \cong \mathbb{Z}/p \times \mathbb{Z}/p$ of $\text{Aut}(F_{2(p-1)})$. This subgroup is maximal among $p$-subgroups, and it contains $B_{p-1}$ and $C$.

**Proposition 4.2** Every $p$-subgroup of $\text{Aut}(F_{2(p-1)})$ is conjugate to one of

\[ A, B_0, \ldots, B_{p-1}, C, D, \text{ or } E. \]

**Proof.** As we asserted earlier, every nontrivial $p$-subgroup $P$ is either $\mathbb{Z}/p$ or $\mathbb{Z}/p \times \mathbb{Z}/p$. From Zimmerman’s realization theorem, this subgroup is realized by an action on a reduced basepointed marked graph $(\eta, \Gamma, \ast)$.

If $P = \mathbb{Z}/p$, then Proposition 4.1 gives us that $\Gamma$ is one of $R_n$, $R_k \lor \Theta_{p-1} \lor R_{p-1-k}$, $\Psi_n$, or $\Omega_n$. If $\Gamma$ is $R_n$ then $P$ is conjugate to $A$. If $\Gamma$ is $R_k \lor \Theta_{p-1} \lor R_{p-1-k}$ then $P$ is conjugate to $B_k$. Finally, note that by collapsing different invariant forests the action of $\mathbb{Z}/p$ on $\Phi_n$ gives a diagonal action of $\mathbb{Z}/p$ on both $\Psi_n$ and $\Omega_n$. Hence if $\Gamma$ is either $\Psi_n$ or $\Omega_n$ then $P$ is conjugate to $C$.

Next, suppose $P = \mathbb{Z}/p \times \mathbb{Z}/p = (\alpha) \times (\beta)$. The first cyclic summand must rotate $p$ edges $e_1, e_2, \ldots, e_p$ of $\Gamma$. Without loss of generality, we may assume that the basepoint $\ast$ is one of the endpoints of each $e_i$. Now if $\beta$ sends all of the $e_i$ to another whole collection $\beta e_i$ (with $\{e_i\}$ disjoint from $\{\beta e_j\}$) then the basepoint $\ast$ must be one of the endpoints of each $\beta^i e_i$ also; therefore, we obtain at least $p^2$ edges emanating from the basepoint $\ast$ which are moved by $\alpha$ and $\beta$. This implies that the rank of $\pi_1(\Gamma)$ is at least $p(p-1)$ (i.e., the best that can happen is that $p$ copies of $\Theta_{p-1}$ are wedged together at the basepoint), which is too large as $p \geq 3$.

So $\beta$ does not send the $e_i$ to another whole collection $\beta e_i$ of edges disjoint from the $e_i$. Without loss of generality, we may assume $(\beta)$ fixes the edges $e_i$ (by replacing $\beta$ with $\beta - \alpha^j$ if necessary.) Hence the collection $\{e_i\}$ is $P$-invariant. Now $\beta$ must rotate $p$ other edges $f_1, f_2, \ldots, f_p$. As $\Gamma$ is reduced, the $e_i$ do not form a subforest. In addition, they do not form a $p$-gon as $\ast$ is an endpoint of each of them. Hence the $e_i$ form either a rose or a $\Theta$-graph by the logic of Lemma 4.1. They do not form a rose, else the existence of the edges $f_i$ forces the rank of $\pi_1(\Gamma)$ to be at least $2p - 1$. Hence the $e_i$ form a $\Theta$-graph $\Theta_{p-1}$. A similar argument shows that the $f_i$ also form a $\Theta_{p-1}$. Hence $\Gamma$ is either $\Psi_n$ or $\Omega_n$. In the former case $P$ is conjugate to $E$ while in the latter case it is conjugate to $D$. □
In order to determine whether or not two different graphs give us the same elementary abelian $p$-subgroup, we need to use work of Krstic in [15] involving Nielsen transformations. More will be said on this in the next section, but for now we briefly recall Krstic’s definition of Nielsen transformations and state the result of Krstic that we need.

**Definition 4.3 (Nielsen transformation)** Let $G$ be a finite subgroup of $\text{Aut}(F_n)$, which is realized by an action of $G$ on a reduced, basepointed graph $\Gamma$. Let $V$ and $E$ be the vertex and oriented edge sets, respectively, of $\Gamma$. Finally, let $i, \tau: E \to V$ be maps which give the initial and terminal points, respectively, of oriented edges. If there are two edges $e$ and $f$ of $\Gamma$ such that:

- $e$ and $f$ are in different orbits (i.e., $f \not\in eG \cup \bar{e}G$),
- $\tau e = \tau f$, and
- $\text{stab}(e) \subseteq \text{stab}(f)$.

then there is an admissible Nielsen transformation $<e, f>$ from $\Gamma$ to a new graph $<e, f>$ in $\Gamma$. The graph $\Gamma' = <e, f>$ has the same vertex and edge sets $V$ and $E$ as $\Gamma'$; however, the map $\tau': E \to V$ which gives the terminal point of an edge is changed as follows. For edges $h$ not in the orbit of $e$, set $\tau'h = \tau h$; but set $\tau'eg = i fg$ for $g \in G$.

If a sequence of Nielsen transformations can change a $G$-graph $\Gamma_1$ into a $G$-graph $\Gamma_2$, then $\Gamma_1$ is said to be Nielsen equivalent to $\Gamma_2$.

We need the following theorem of Krstic, Theorem 2 from [15]:

**Theorem 4.4 (Krstic)** Let the graphs $\Gamma_1$ and $\Gamma_2$ realize the same subgroup $G$ of $\text{Aut}(F_n)$. If $\Gamma_1$ and $\Gamma_2$ are reduced as $G$-graphs then they are Nielsen equivalent, up to an equivariant isomorphism (a basepoint preserving isomorphism.)

**Proposition 4.5** The subgroups $A, B_0, \ldots, B_{p-1}, C, D, E$ are in distinct conjugacy classes. The diagram of subgroups up to conjugacy is

$$
\begin{array}{ccc}
B_0 & \downarrow \\
& D \\
B_{p-1} & \simeq \quad C \\
& E \\
\end{array}
$$

(4.2)

**Proof.** We apply Theorem 4.4. Any two reduced graphs realized by the same subgroup of $\text{Aut}(F_n)$ can be connected up by a sequence of Nielsen transformations. This could only occur for two distinct graphs listed in Proposition
4.1 when \( P = \mathbb{Z}/p \) and the two graphs are \( \Psi_n \) and \( \Omega_n \). In this case, \( P \) is conjugate to the subgroup \( C \) of \( \text{Aut}(F_n) \). \( \square \)

5 Normalizers of \( p \)-subgroups of \( \text{Aut}(F_n) \)

The structure of centralizers of finite subgroups of \( \text{Aut}(F_n) \) is given by Krstic in [15] in some detail. He shows that, for a finite subgroup \( G \) of \( \text{Aut}(F_n) \), an element of the centralizer \( C_{\text{ Aut}(F_n)} (G) \) is a product of Nielsen transformations followed by a centralizing graph isomorphism. Moreover, one of the main propositions of his paper can be used to yield information about the structure of normalizers of finite subgroups, which is what we are interested in here.

We now recall some definitions and theorems from Krstic [15].

**Definition 5.1 (Nielsen isomorphism)** Let \( <e,f> \) be a Nielsen transformation from \( \Gamma \) to \( \Gamma' \) (see Definition 4.3.) A Nielsen isomorphism is the isomorphism between fundamental groupoids:

\[
< e, f > : \Pi(\Gamma) \rightarrow \Pi(\Gamma')
\]

given by \( < e, f > h = h \) if \( h \) is not in the orbit of \( e \) and \( < e, f > x = (ef)x \) for \( x \in G \).

The following is Proposition 4 from [15]:

**Proposition 5.2 (Krstic)** If \( \Gamma_1 \) and \( \Gamma_2 \) are finite basepointed, reduced \( G \)-graphs and

\[
F : \Pi(\Gamma_1) \rightarrow \Pi(\Gamma_2)
\]

is an equivariant groupoid isomorphism which preserves the base vertex, then there exists a product \( T \) of Nielsen transformations and an equivariant isomorphism of basepointed graphs

\[
H : TT\Gamma_1 \rightarrow \Gamma_2
\]

such that \( F = HT \).

Recall that \( \pi_1(\Gamma_i) \) is a sub-groupoid of \( \Pi(\Gamma_i) \). Krstic uses the above theorem to get information about maps between fundamental groups. He shows the following, which is Corollary 1 from [15]:

**Corollary 5.3 (Krstic)** Let \( \Gamma_1 \) and \( \Gamma_2 \) be reduced pointed \( G \)-graphs of rank \( \geq 2 \). Then every equivariant isomorphism

\[
\pi_1(\Gamma_1,*) \rightarrow \pi_1(\Gamma_2,*)
\]
is the restriction of an equivariant isomorphism

$$\Pi(\Gamma_1) \to \Pi(\Gamma_2).$$

For a $G$-graph $\Gamma$ and an automorphism $\phi : G \to G$, let $\phi(\Gamma)$ be the $G$-graph with underlying graph $\Gamma$ and $G$-action given by $xg := x\phi(g)$ where $g \in P$, $x \in V(\Gamma) \cup E(\Gamma)$, and the latter multiplication is given by the standard action on $\Gamma$. In other words, $\phi(\Gamma)$ is just the graph $\Gamma$ with the $G$-action “twisted” by $\phi$.

Realize the finite subgroup $G$ of $Aut(F_n)$ by a marked graph $\eta : R_n \to \Gamma$ where the induced action of $G$ on $\Gamma$ is reduced.

**Proposition 5.4** For every element $\alpha$ of $N_{Aut(F_n)}(G)$ there exists an automorphism $\phi$ of $G$ such that $\alpha$ is realized by a $G$-equivariant isomorphism

$$\pi_1(\Gamma, \ast) \to \pi_1(\phi(\Gamma), \ast).$$

**Sketch of proof.** Define $N'_{Aut(F_n)}(G)$ to be the set of all equivalence classes of pairs $(\phi, \psi)$ where $\phi \in Aut(G)$ and $\psi : \Gamma \to \Gamma$ is a basepoint preserving, continuous surjection of graphs such that

$$\psi_\# : \pi_1(\Gamma, \ast) \to \pi_1(\phi(\Gamma), \ast)$$

is a $G$-equivariant group isomorphism. Two such pairs $(\phi_1, \psi_1)$ and $(\phi_2, \psi_2)$ are equivalent if $\phi_1 = \phi_2$ and

$$(\psi_1)_\# = (\psi_2)_\# : \pi_1(\Gamma, \ast) \to \pi_1(\phi(\Gamma), \ast)$$

on the level of fundamental groups.

Define a group operation *composition* in $N'_{Aut(F_n)}(G)$ in the obvious way by letting

$$(\phi_2, \psi_2) \circ (\phi_1, \psi_1) = (\phi_2 \circ \phi_1, \psi_2 \circ \psi_1).$$

To prove the proposition, it suffices to show that there is a group isomorphism

$$\xi : N_{Aut(F_n)}(G) \to N'_{Aut(F_n)}(G).$$

This isomorphism is defined as follows. Choose a fixed homotopy inverse $\bar{\eta} : \Gamma \to R_n$; i.e., a map such that

$$(\eta \bar{\eta})_\# = 1 \in Aut(\pi_1(\Gamma))$$
and

\[(\tilde{\eta}\eta)_\# = 1 \in Aut(\pi_1(R_n)).\]

If \(\alpha \in N_{Aut(F_n)}(G)\), then \(\alpha\) induces an automorphism \(\phi\) of \(G\) via conjugation:

\[\phi(g) = \alpha g \alpha^{-1}.\]

In addition, since \(\alpha \in Aut(F_n)\) it corresponds to a map \(\bar{\alpha} : R_n \to R_n\), allowing us to set \(\psi = \eta\bar{\alpha}\bar{\eta}\). Now define \(\xi\) by sending \(\alpha\) to the pair \((\phi, \psi)\).

We leave it as an exercise for the reader to verify that \(\xi\) as defined actually is a group isomorphism. \(\square\)

As a result of Proposition 5.4, we obtain the following:

**Proposition 5.5** An element of \(N_{Aut(F_n)}(G)\) is a product of Nielsen transformations followed by a normalizing graph isomorphism; that is, if \((\phi, \psi) \in N_{Aut(F_n)}'(G)\), then there exist a product \(T\) of Nielsen isomorphisms and a graph isomorphism \(H\) such that \(\psi_\# = HT:\)

\[\psi_\# : \pi_1(\Gamma, \ast) \xrightarrow{T} \pi_1(T\Gamma, \ast) \xrightarrow{H} \pi_1(\phi(\Gamma), \ast).\]

**Proof.** Any element \((\phi, \psi)\) of \(N_{Aut(F_n)}'(G)\) induces a \(G\)-equivariant map

\[\pi_1(\Gamma, \ast) \xrightarrow{\psi_\#} \pi_1(\phi(\Gamma), \ast).\]

From Corollary 5.3, \(\psi_\#\) is the restriction of a \(G\)-equivariant map \(F\) between fundamental groupoids:

\[\Pi(\Gamma) \xrightarrow{F} \Pi(\phi(\Gamma)).\]

Now from Proposition 5.2 we know that there exists a product \(T\) of Nielsen transformations starting with the graph \(\Gamma\), and a basepoint preserving graph isomorphism \(H : T\Gamma \to \phi(\Gamma)\) such that \(F\) is the map induced by \(HT:\)

\[F : \Pi(\Gamma) \xrightarrow{T} \Pi(T\Gamma) \xrightarrow{H} \Pi(\phi(\Gamma)).\]

By restricting the above map of fundamental groupoids to one of fundamental groups, we have that \(\psi_\#\) is also the map induced by \(HT\). \(\square\)

We now use Proposition 5.5 to calculate the normalizers of the subgroups \(A, B_0, \ldots, B_{p-1}, D,\) and \(E\) listed in Proposition 4.5. We do this so that their cohomology groups will be easier to calculate in a later section. We will not need to calculate the normalizer of \(C\) in this explicit manner, and later will find its cohomology through more geometric means.
Lemma 5.1

\[ N_{\text{Aut}(F_n)}(A) \cong N_{\Sigma_p}(\mathbb{Z}/p) \times ((F_{p-2} \times F_{p-2}) \times (\mathbb{Z}/2 \times \text{Aut}(F_{p-2}))) \]

where \( \mathbb{Z}/2 \) acts by exchanging the two copies of \( F_{p-2} \) and \( \text{Aut}(F_{p-2}) \) acts diagonally on the two copies of \( F_{p-2} \).

Proof. Write \( F_n = \langle x_1, \ldots, x_n \rangle \) as the free group on the letters \( x_j \). In the above decomposition of \( N_{\text{Aut}(F_n)}(A) \), the group \( N_{\Sigma_p}(\mathbb{Z}/p) \) corresponds to automorphisms of \( F_n \) which permute the first \( p \) letters \( x_1, \ldots, x_p \). In the decomposition, the \( i \)th generator of the first copy of \( F_{p-2} \) corresponds to an automorphism of \( F_n \) which, for \( j \leq p \), sends \( x_j \mapsto x_jx^{-1}_{p+i} \) and which fixes \( x_j \) for \( j > p \). The \( i \)th generator of the second copy of \( F_{p-2} \) corresponds to an automorphism of \( F_n \) which, for \( j \leq p \), sends \( x_j \mapsto x_{p+i}x_j \) and which fixes \( x_j \) for \( j > p \). Next, the \( \mathbb{Z}/2 \) comes from the map which, for \( j \leq p \), sends \( x_j \mapsto x_j^{-1} \) and which fixes the remaining \( x_j \). Finally, the \( \text{Aut}(F_{p-2}) \) comes from automorphisms which fix the first \( p \) generators \( x_1, \ldots, x_p \) and which act on the latter generators by identifying \( F_{p-2} \) with \( \langle x_{p+1}, x_{p+2}, \ldots, x_n \rangle \).

Recall that \( A \) comes from the action of \( \mathbb{Z}/p \) on the first \( p \) petals of the rose \( R_{2(p-1)} \). From Proposition 5.5 any element of \( N_{\text{Aut}(F_n)}(A) \) is induced by a product of Nielsen transformations \( T \) followed by a graph isomorphism

\[ H : TR_{2(p-1)} \rightarrow \phi(R_{2(p-1)}) \]

for some \( \phi \in \text{Aut}(A) \). The only types of Nielsen transformations that are possible are

- Nielsen transformations obtained by pulling either the front ends or the back ends of the first \( p \) petals of the rose uniformly around paths in the last \( p - 2 \) petals. This subgroup is isomorphic to \( F_{p-2} \times F_{p-2} \).
- Nielsen transformations contained entirely in the copy of \( \text{Aut}(F_{p-2}) \) corresponding to graph automorphisms and Nielsen transformations involving the last \( p - 2 \) petals.

Note that for any product \( T \) of the above Nielsen transformations, the \( A \)-graph \( TR_{2(p-1)} \) is exactly the same as the \( A \)-graph \( R_{2(p-1)} \). Hence from Proposition 5.5, any element of \( N_{\text{Aut}(F_n)}(A) \) is induced by a product of Nielsen transformations \( T \) followed by a graph isomorphism

\[ H : TR_{2(p-1)} = R_{2(p-1)} \rightarrow \phi(R_{2(p-1)}) \]

for some \( \phi \in \text{Aut}(A) \). That is, the only “normalizing graph automorphisms” we need to examine are automorphisms of one particular graph \( R_{2(p-1)} \) which are in the normalizer \( N_{\text{Aut}(F_n)}(A) \).
The normalizing graph automorphisms are:

- Those involving just the last \( p - 2 \) petals of the rose. As the action of \( A \) on these petals is trivial, the normalizer of \( A \) contains any graph automorphism involving just those last \( p - 2 \) petals. All of these graph automorphisms are contained in the copy of \( \text{Aut}(F_{p-2}) \) obtained from graph automorphisms and Nielsen transformations involving the last \( p - 2 \) petals.
- Normalizing graph automorphisms of the first \( p \) petals of the rose. This gives a subgroup of \( N_{\text{Aut}(F_n)}(A) \) that is isomorphic to \( N_{\Sigma_p}(\mathbb{Z}/p) \).

We leave it as an exercise for the reader to show that all of the various subgroups of \( N_{\text{Aut}(F_n)}(A) \) now fit together as described in the statement of the lemma. \( \square \)

A final remark about the structure of the subgroup \( N_{\text{Aut}(F_n)}(A) \), described above in the proof of Lemma 5.1, is appropriate here:

**Remark 5.2**

Consider a subgroup \( \langle \omega \rangle \cong \mathbb{Z}/2 \) of \( \text{Aut}(F_p) \) corresponding to the action of \( \mathbb{Z}/2 \) on \( R_p \) given by switching just the first two petals of the rose. Note that

\[
N_{\text{Aut}(F_p)}(\omega) = C_{\text{Aut}(F_p)}(\omega) = \mathbb{Z}/2 \times ((F_{p-2} \times F_{p-2}) \rtimes (\mathbb{Z}/2 \times \text{Aut}(F_{p-2})))
\]

where the action of \( \mathbb{Z}/2 \times \text{Aut}(F_{p-2}) \) on \( F_{p-2} \times F_{p-2} \) in the semidirect product is the same as before (that is, as in \( N_{\text{Aut}(F_n)}(A) \)). Consequently, we see that

\[
N_{\Sigma_p}(\mathbb{Z}/p) \times N_{\text{Aut}(F_p)}(\omega) \cong \mathbb{Z}/2 \times N_{\text{Aut}(F_n)}(A)
\]

and hence

\[
\hat{H}^*(N_{\Sigma_p}(\mathbb{Z}/p) \times N_{\text{Aut}(F_p)}(\omega); \mathbb{Z}(p)) = \hat{H}^*(N_{\text{Aut}(F_n)}(A); \mathbb{Z}(p))
\]

because \( p \geq 3 \) and so the first summand \( \mathbb{Z}/2 \) in \( \mathbb{Z}/2 \times N_{\text{Aut}(F_n)}(A) \) can be ignored when taking Farrell cohomology with \( \mathbb{Z}(p) \) coefficients.

**Lemma 5.3**

\[
N_{\text{Aut}(F_n)}(B_k) \cong N_{\Sigma_p}(\mathbb{Z}/p) \times (F_k \rtimes \text{Aut}(F_k)) \times \text{Aut}(F_{p-1-k})
\]

**Proof.** Write \( F_n = \langle x_1, \ldots, x_n \rangle \) as the free group on the letters \( x_j \). In the above decomposition of \( N_{\text{Aut}(F_n)}(A) \), the group \( N_{\Sigma_p}(\mathbb{Z}/p) \) is obtained as follows. The permutation \((12 \ldots p)\) of \( \Sigma_p \) corresponds to the automorphism of \( F_n \) which fixes \( x_j \) if \( j \leq k \); sends \( x_j \mapsto x_{j+1} \) if \( k + 1 \leq j \leq p + k - 2 \), sends \( x_{p+k-1} \mapsto x_{p+k-1}^{-1}x_{p+k-2}^{-1} \cdots x_{k+1}^{-1} \), and sends \( x_j \mapsto x_{k+1}^{-1}x_jx_{k+1} \) if \( j \geq p + k \).
The transposition \((12)\) of \(\Sigma_p\) corresponds to the automorphism which sends \(x_{k+1} \mapsto x_{k+1}^{-1}\), sends \(x_{k+2} \mapsto x_{k+1}x_{k+2}\), sends \(x_j \mapsto x_{k+1}^{-1}x_j x_{k+1}\) if \(j \geq p + k\), and fixes all other letters \(x_j\). Since \((12)\) and \((12 \ldots p)\) generate \(\Sigma_p\), this suffices to define a copy of \(\Sigma_p\) in \(Aut(F_n)\). Now let \(N_{\Sigma_p}(\mathbb{Z}/p)\) in the above decomposition correspond to the normalizer of \(\mathbb{Z}/p \cong \langle (12 \ldots p) \rangle\) in this copy of the symmetric group.

The \(F_k\) which is being acted upon in the semidirect product in the above decomposition has its \(i\)th generator corresponding to the automorphism of \(F_n\) which sends \(x_j\) to its conjugate \(x_i^{-1}x_j x_i\) if \(j \geq k + 1\) and fixes \(x_j\) otherwise. The \(Aut(F_k)\) is the above decomposition cooresponds to automorphisms of \(F_n\) which fix the last \(n - k\) generators \(x_{k+1}, \ldots, x_n\) and which act on the first \(k\) generators \(x_1, \ldots, x_k\) as \(Aut(F_k)\) indicates.

Finally, the \(Aut(F_{p-1-k})\) in the above decomposition corresponds to automorphism which fix the first \(p - 1 + k\) generators of \(F_n\) and act on the last \(p - 1 - k\) generators by identifying \(Aut(F_{p-1-k})\) with \(Aut((x_{p+k}, x_{p+k+1}, \ldots, x_n))\).

The remaining part of the proof is similar to that for Lemma 5.1, and we only sketch it here, leaving the details for the reader to verify. Also, just for this proof, let \(\Gamma\) denote the \(B_k\)-graph \(R_k \lor \Theta_{p-1} \lor R_{p-1-k}\). Recall that \(B_k\) comes from the action of \(\mathbb{Z}/p\) on the \(\Theta\)-graph in \(\Gamma\).

The normalizer \(N_{Aut(F_n)}(B_k)\) contains four types of Nielsen transformations: ones contained in the subgroup \(Aut(F_k)\) of the normalizer obtained from taking graph automorphisms and Nielsen transformations of the rose \(R_k\); ones contained in the subgroup \(Aut(F_{p-1-k})\) of the normalizer obtained from taking graph automorphisms and Nielsen transformations of the rose \(R_{p-1-k}\), redundant ones – which can be ignored since they are already included in the Nielsen transformations in \(Aut(F_{p-1-k})\) – obtained by pulling the back edges of the \(\Theta\)-graph uniformly around the \(p - 1 - k\) petals of the rose \(R_{p-1-k}\) on the right, and ones corresponding to the \(F_k\) in the decomposition in the statement of the lemma which are obtained by pulling the front edges of the \(\Theta\)-graph uniformly around the \(k\) petals of the rose \(R_k\) on the left.

As was the case for Lemma 5.1 above, for any product \(T\) of any of the above types of Nielsen transformations, the \(B_k\)-graph \(TT\) is exactly the same as the \(B_k\)-graph \(\Gamma\). The normalizing graph automorphisms take one of three forms: automorphisms contained in the subgroup \(Aut(F_k)\) obtained from graph automorphisms and Nielsen transformations involving just the rose \(R_k\), automorphisms contained in the subgroup \(Aut(F_{p-1-k})\), and graph automorphisms of the \(\Theta_{p-1}\) sitting inside of \(\Gamma\) which yield the \(N_{\Sigma_p}(\mathbb{Z}/p)\) in the decomposition of \(N_{Aut(F_n)}(B_k)\). □
Lemma 5.4

\[ N_{\text{Aut}(F_n)}(D) = N_{\Sigma_p}(\mathbb{Z}/p) \times N_{\Sigma_p}(\mathbb{Z}/p). \]
\[ N_{\text{Aut}(F_n)}(E) = (N_{\Sigma_p}(\mathbb{Z}/p) \times N_{\Sigma_p}(\mathbb{Z}/p)) \rtimes \mathbb{Z}/2. \]

**Proof.** The groups \( N_{\text{Aut}(F_n)}(D) \) and \( N_{\text{Aut}(F_n)}(E) \) are the easiest to calculate of all of the normalizers. There are no admissible Nielsen transformations of the reduced \((\mathbb{Z}/p \times \mathbb{Z}/p)\)-graphs \( \Omega_n \) or \( \Psi_n \). It follows that \( N_{\text{Aut}(F_n)}(D) \) and \( N_{\text{Aut}(F_n)}(E) \) are just finite groups consisting of normalizing graph automorphisms of \( \Omega_n \) and \( \Psi_n \), respectively. So we need only examine the graphs \( \Omega_n \) or \( \Psi_n \) and see which graph automorphisms are in the normalizers of the respective \((\mathbb{Z}/p \times \mathbb{Z}/p)\)-actions. Direct examination of these graphs yields that the normalizers are as claimed above. \( \square \)

6 Fixed point sets of the normalizers

Let \( G \) be a finite subgroup of \( \text{Aut}(F_n) \), realized by a reduced graph \( \Gamma \) as in the previous section. Define the **fixed point subcomplex** \( X^G_n \) of \( G \) in the spine \( X_n \) of auter space by

\[ X^G_n = \{ x \in X_n : xg = x \text{ for all } g \in G \}. \]

From [16] and Theorem 12.1 in Part III of [14], we have

**Fact 6.1** The space \( X^G_n \) is a contractible, finite-dimensional complex and \( N_{\text{Aut}(F_n)}(G) \) acts on it with finite quotient and finite stabilizers.

A few spaces related to fixed point subcomplexes will come up so frequently in our work that we will give them special names, \( X^\omega_p, Q^\omega_p, \tilde{X}_m, \) and \( \tilde{Q}_m \). We now define these spaces.

Recall from Krstic and Vogtmann [16] that an edge of a reduced \( G \)-graph \( \Gamma \) is **inessential** if it is contained in every maximal \( G \)-invariant forest in \( \Gamma \), and that there is an \( N_{\text{Aut}(F_p)}(G) \)-equivariant deformation retract of \( X^G_n \) obtained by collapsing all inessential edges of marked graphs in \( X^G_n \).

**Definition 6.2** \( (X^\omega_p, Q^\omega_p) \) Let \( \omega \) be the automorphism of \( F_p \) defined by interchanging the first two basis elements of \( F_p \) (cf. Remark 5.2), and let \( X^\omega_p \) be the subcomplex of \( X_p \) fixed by the subgroup \( \langle \omega \rangle \cong \mathbb{Z}/2 \). Define \( X^\omega_p \) to be the associated \( N_{\text{Aut}(F_p)}(\omega) \)-equivariant deformation retract of \( X^\omega_p \) and let \( Q^\omega_p \) be the quotient of \( X^\omega_p \) by the action of \( N_{\text{Aut}(F_p)}(\omega) \).
From Remark 12.16 of [14] we know that the dimension of $X^\omega_p$ is $2p - 4$. In the next section, we will use $X^\omega_p$ to study the cohomology of $N_{\text{Aut}(F_n)}(A)$. The rest of this section, however, will be devoted to defining the space $\tilde{X}_m$, finding an alternative characterization of this space, and explaining how the space relates to to the normalizer $N_{\text{Aut}(F_n)}(B_m)$.

**Definition 6.3 ($\tilde{X}_m$ and $\tilde{Q}_m$)** Let $m$ be a positive integer. Let $\alpha$ be the automorphism of $F_{m+2}$ defined by:

$$
\begin{align*}
  x_i &\mapsto x_i \text{ for } i \leq m, \\
  x_{m+1} &\mapsto x_{m+2} \\
  x_{m+2} &\mapsto x_{m+2}^{-1}x_{m+1}.
\end{align*}
$$

The subgroup $Q$ generated by $\alpha$ has order 3, and is realized on the graph $G_m \vee \Theta_2$ by rotating the edges of $\Theta_2$ cyclically. Define $\tilde{X}_m$ to be the fixed point set $X^Q_{m+2}$, and let $\tilde{Q}_m$ be the quotient of $\tilde{X}_m$ by $N_{\text{Aut}(F_{m+2})}(Q)$. For $m = 0$, we set $\tilde{X}_0 = \tilde{Q}_0 = \{a \text{ point}\}$ as a notational convenience.

Let $\tilde{\Gamma}$ be the graph $R_m \vee \Theta_2$, where the basepoint of the resulting graph $\tilde{\Gamma}$ is the center of the rose $R_m$. Label the 3 edges of the $\Theta$-graph in $\tilde{\Gamma}$ as $e_1, e_2, e_3$ and orient them so that they begin at the basepoint of $\tilde{\Gamma}$. Construct a specific marked graph $\tilde{\eta} : R_{m+2} = R_m \vee R_2 \to R_m \vee \Theta_{p-1}$ by sending the first rose $R_m$ in $R_m \vee R_2$ to $R_m$ in $R_m \vee \Theta_2$ via the identity map. Then send the second rose $R_2$ in $R_m \vee R_2$ to $\Theta_2$ in $R_m \vee \Theta_2$ by sending the $i$th petal to $e_i * \bar{e}_{i+1}$. As noted above in Definition 6.3, the subgroup $Q$ acts on this marked graph by rotating the edges of $\Theta_2$ cyclically.

We will now study the structure of $\tilde{X}_m$ and $\tilde{Q}_m$ in detail. Recall from Lemma 5.3 that

$$
N_{\text{Aut}(F_n)}(B_k) \cong N\Sigma_3^*(\mathbb{Z}/p) \times (F_k \rtimes \text{Aut}(F_k)) \times \text{Aut}(F_{p-1-k}).
$$

In a similar way, we could use Proposition 5.5 just as Lemma 5.3 did to obtain that

$$
N_{\text{Aut}(F_{m+2})}(Q) \cong \Sigma_3 \times (F_{m} \rtimes \text{Aut}(F_{m})).
$$

The $\Sigma_3 \cong N\Sigma_3(\mathbb{Z}/3)$ comes from normalizing graph automorphisms of the $\Theta_2$, the $F_m$ comes from Nielsen moves done by pulling the back ends of all of the edges $e_i$ uniformly around some loop in the rose $R_m$, and the Aut($F_m$) comes from Nielsen transformations and graph automorphisms concerning just the petals of the rose $R_m$.

Let us examine the Nielsen transformations corresponding to the $F_m$ in a little more detail. Say the petals of the rose are $r_1, \ldots, r_m$. The group $F_m$ is generated by the $m$ Nielsen moves $a_k := \langle \bar{e}_1, r_k \rangle$ where $\langle \bar{e}_1, r_k \rangle$ fixes
the rose and sends $e_j$ to $\bar{r}_k * e_j$ for all $j \in \{1, 2, 3\}$. Observe that the marked graph $\eta \cdot a_k$ sends $R_m$ identically to $R_m$ as before; however, the $i$th petal of the second rose is now sent to $r_k * e_i * e_{i+1} * \bar{r}_k$ (recalling that we want a right action on marked graphs and so $a_k$ acts on $\pi_1(\Gamma)$ by $a_k^{-1}$.) Hence some word

$$w = a_k, \ldots, a_k \in F_m \subset N_{Aut(F_m+2)}(Q)$$

in the letters $a_1, \ldots, a_m$ acts on the marked graph $\eta$ by sending $R_m$ in $R_m \lor R_2$ identically to the rose $R_m$ in $R_m \lor \Theta_2$; however, the second rose $R_2$ is mapped to $R_m \lor \Theta_2$ by conjugating the old way it was mapped by the path $r_k \cdot \cdots \cdot r_k$.

**Proposition 6.4** The fixed point space $\tilde{X}_m$ can be characterized as the realization of the poset of equivalence classes of pairs $(\alpha, f)$, where $\alpha : R_m \to \Gamma_m$ is a basepointed marked graph whose underlying graph $\Gamma_m$ has a special (possibly valence 2) vertex which is designated as $\circ$, $\circ$ may equal the basepoint $*$ of $\Gamma_m$, and $f : I \to \Gamma_m$ is a homotopy class (rel endpoints) of maps from $*$ to $\circ$ in $\Gamma_m$.

**Proof.** By definition, $\tilde{X}_m$ consists of simplices in the spine $X_{m+2}$ that are fixed by $Q$. It is the subcomplex generated by marked graphs (i.e., vertices of $X_{m+2}$) which realize the finite subgroup $Q \subset Aut(F_{m+2})$. From Theorem 4.4, any two vertices in $\tilde{X}_m$ corresponding to reduced marked graphs, are connected to each other by Nielsen moves. In other words, they are connected to each other by Nielsen moves that can be represented by elements of

$$N_{Aut(F_m+2)}(Q) \cong \Sigma_3 \times (F_m \ltimes Aut(F_m)),$$

since the normalizer contains all of the relevant Nielsen transformations. In particular, the Nielsen moves come from $< a_1, \ldots, a_m > \cong F_m$ and the Nielsen moves in $Aut(F_m)$ involving only the petals of the rose $R_m$ in $R_m \lor \Theta_2$.

Hence the reduced marked graphs representing vertices of $\tilde{X}_m$ are of the form

$$\psi = \alpha \lor \beta : R_m \lor R_2 \to R_m \lor \Theta_2$$

where $\alpha : R_m \to R_m$ corresponds to any reduced marked graph representing a vertex of $X_m$ and $\beta : R_2 \to R_m \lor \Theta_2$ sends the $i$th petal to

$$r_k \cdot \cdots \cdot r_k \cdot e_i \cdot \bar{e}_{i+1} \cdot \bar{r}_k \cdot \cdots \cdot \bar{r}_k,$$

and corresponds to some word $a_1, \ldots, a_s$ in $F_m$. We could thus represent the reduced marked graph $\psi$ more compactly as a pair $(\alpha, f)$ where $\alpha : R_m \to R_m$ is any reduced marked graph and $f : I = [0, 1] \to R_m$ is a homotopy class (rel the endpoints $f(0) = f(1) = *)$ of maps representing a path $r_k \cdot \cdots \cdot r_k$ in the rose $R_m$.
By considering the stars, in $\tilde{X}_m$, of reduced marked graphs $(\alpha, f) : R_{m+2} \to R_m \lor \Theta_2$, we will obtain a characterization of all marked graphs representing vertices of $\tilde{X}_m$. Recall that $\mathcal{Q}$ rotates the edges of $\Theta_2$ and leaves everything else fixed.

**Definition 6.5** Let $G$ be a finite subgroup of $\text{Aut}(F_r)$ for some integer $r$. A marked graph

$$\eta^1 : R_r \to \Gamma^1$$

is a $G$-equivariant blowup in the fixed point space $X^G_r$ of a marked graph

$$\eta^2 : R_r \to \Gamma^2$$

if there is a 1-simplex $\eta^1 > \eta^2$ in $X^G_r$. In other words, this happens exactly when we can collapse a $G$-invariant forest in $\Gamma^1$ to obtain $\Gamma^2$. We often abbreviate this and just say that $\eta_1$ is a blowup of $\eta_2$ or that $\eta_2$ can be blown up to get $\eta_1$. If the forest that we collapsed was just a tree, we say that we blew up the corresponding vertex of $\eta_2$ to get $\eta_1$.

**Claim 6.6** Any equivariant blowup of $\Gamma$ is a blowup of $R_m$, with $\Theta_2$ attached at a point.

**Proof.** Suppose $\Gamma$ is obtained from $\Gamma'$ by collapsing an invariant forest $F$. It suffices to show that $F$ is fixed by the action of $\mathcal{Q}$, since then the initial (resp. terminal) vertices of the edges mapping to $e_i$ must be the same.

Suppose $F$ is not fixed by $\mathcal{Q}$, and let $F_0$ be the union of edges of $F$ with non-trivial orbits. Let $v$ be a terminal vertex of $F_0$, and let $e$ and $f$ be edges of $\Gamma'$ terminating at $v$ which are not in $F_0$. If $e$ or $f$ is fixed by $\mathcal{Q}$, then $v$ is fixed; in particular, if $e$ or $f$ is in $F - F_0$, then $v$ is fixed by $\mathcal{Q}$. If $e$ and $f$ are not fixed, then since they are not in $F$, they must map to edges $e_i$ and $e_j$ of $\Gamma$, with $i \neq j$. Since the map is equivariant, some element of $\mathcal{Q}$ takes $e$ to $f$. But this element must then fix their common vertex, $v$, so in all cases $v$ must be fixed by $\mathcal{Q}$. Since all terminal vertices of $F_0$ are fixed, $F_0$ must be fixed, contradicting the definition of $F_0$. □

Hence the only way to blow up $(\alpha, f)$ is by blowing up the $R_m$ part of $R_m \lor \Theta_2$. Think of the resulting marked graph as some $(\hat{\alpha}, \hat{f})$ where

$$\hat{\alpha} : R_m \to \Gamma_m$$

is any marked graph, except that the underlying graph $\Gamma_m$ has one extra distinguished vertex, which we will call $\circ$ and which might have valence 2, aside from the basepoint $\ast$, and where

$$\hat{f} : I \to \Gamma_m$$
is a homotopy class of paths from \( * \) to \( o \) in \( \Gamma_m \). We allow the possibility that \( o \) might just be \( * \). The pair \((\hat{\alpha}, \hat{f})\) is really representing a marked graph \( R_{m+2} \to \Gamma_m \vee \Theta_2 \) as follows. The wedge \( \vee \) connects the point \( o \) of \( \Gamma_m \) to the left hand vertex of \( \Theta_2 \). The basepoint of the resulting graph \( \Gamma_m \vee \Theta_2 \) is whatever the old basepoint \( * \) of \( \Gamma_m \) was. The first \( m \) petals of \( R_{m+2} \) map to \( \Gamma_m \) via \( \hat{\alpha} \) and the \((m+i)\)-th petal of \( R_{m+2} \) first goes around the image of \( \hat{f} \) from \( * \) to \( o \), then goes around \( e_i \bar{e}_{i+1} \) of the \( \Theta \)-graph, and then finally goes back around the image of \( \hat{f} \) in reverse from \( o \) to \( * \).

For \((\hat{\alpha}, \hat{f})\) to be in the star of \((\alpha, f)\) it has to be the case that when the graph \( \Gamma_m \vee \Theta_2 \) is collapsed to \( R_m \vee \Theta_2 \) the marking \( \hat{\alpha} \) collapses to \( \alpha \) and the homotopy class (rel endpoints) of maps \( \hat{f} \) collapses to \( f \). This concludes the proof of Proposition 6.4. □

As in [7], there is an obvious definition of when two of the marked graphs described in Proposition 6.4 are equivalent.

**Definition 6.7** Marked graphs \((\alpha_1, f_1)\) and \((\alpha_2, f_2)\) are equivalent if there is a homeomorphism \( h \) from \( \Gamma_1 \) to \( \Gamma_2 \) which sends \( * \) to \( * \), \( o \) to \( o \), such that

\[
(h\alpha_1)_\# = (\alpha_2)_\# : \pi_1(R_m, *) \to \pi_1(\Gamma^2_m)
\]

and such that the paths

\[
hf_1, f_2 : I \to \Gamma_m
\]

are homotopic rel endpoints.

By recalling how \( N_{\text{Aut}(F_{m+2})}(Q) \) acts on reduced marked graphs and looking at \((\alpha, f)\) as being a marked graph

\[
R_{m+2} \to \Gamma_m \vee \Theta_2
\]

in the star of some reduced marked graph

\[
R_{m+2} \to R_m \vee \Theta_2,
\]

it is direct to prove

**Proposition 6.8** The group

\[
N_{\text{Aut}(F_{m+2})}(Q) \cong \Sigma_3 \times (F_m \rtimes \text{Aut}(F_m))
\]

acts on a marked graph

\[
(\alpha, f) : R_m \sqcup I \to \Gamma_m
\]

in \( \tilde{X}_m \) as follows. The subgroup \( \Sigma_3 \) permutes the edges of the \( \Theta \) graph attached at \( o \), giving a marked graph which is equivalent to the original one. An element
\(a_k, a_{k_2}, \ldots a_k\) in \(F_m = \langle a_1, \ldots, a_m \rangle\) doesn’t change \(\alpha\) at all, but sends the path \(f\) to the path 

\[\alpha(r_{k_2}) * \alpha(r_{k_{-1}}) * \cdots * \alpha(r_1) * f\]

where \(r_i\) is the \(i\)th petal of the rose \(R_m\) in the domain of \(\alpha\). Lastly, an element \(\phi \in \text{Aut}(F_m)\) does not change \(f\) at all and acts on \(\alpha : R_m \to \Gamma_m\) by precomposition: 

\[(\alpha, f) \cdot \phi = (\alpha \circ \phi, f)\].

Just as Proposition 6.4 gives us a simple characterization of \(\tilde{X}_m\), the following remark provides a nice characterization of \(\tilde{Q}_m\):

**Remark 6.1** The quotient space \(\tilde{Q}_m\) of \(\tilde{X}_m\) by \(N_{\text{Aut}(F_{m+2})}(\mathcal{Q})\) can be characterized as the realization of the poset of equivalence classes of basepointed graphs \(\Gamma_m\) which have a special (possibly valence 2) vertex designated as \(\circ\), which may equal the basepoint \(\ast\). Two such graphs \(\Gamma^1_m\) and \(\Gamma^2_m\) are equivalent if there is a homeomorphism 

\[h : \Gamma^1_m \to \Gamma^2_m\]

such that \(h(*) = \ast\) and \(h(\circ) = \circ\). Define the poset structure on these graphs by forest collapses. That is, \(\Gamma^1_m > \Gamma^2_m\) if there is a simplicial map \(g : \Gamma^1_m \to \Gamma^2_m\) such that \(g(*) = \ast\), \(g(\circ) = \circ\), and \(g^{-1}(\text{vertices of } \Gamma^2_m)\) is a subforest of \(\Gamma^1_m\).

In other words, the quotient \(\tilde{Q}_m\) is just the “moduli space of unmarked graphs \(\Gamma_m\) with \(\pi_1(\Gamma_m) \cong F_m\) and where \(\Gamma_m\) has two distinguished points.”

Recall from [11] that the spine \(X_m\) of auter space is a deformation retraction of auter space \(\bar{A}_m\). Similarly, we can construct a space \(\bar{A}_m\) which deformation retracts to \(\tilde{X}_m\). We can then think of \(\tilde{X}_m\) as being the “spine” of \(\bar{A}_m\).

**Definition 6.9** (\(\bar{A}_m\)) Construct an analog \(\bar{A}_m\) of auter space for \(N(\mathcal{Q})\) by considering markings 

\[(\alpha, f) : R_m \coprod I \to \Gamma_m\]

where the edges of \(\Gamma_m\) are assigned lengths which must sum to 1. Just as in [7], the space \(\bar{A}_m\) deformation retracts to its spine \(\bar{X}_m\).

Now from [11] we have \(\dim(X_m) = \dim(Q_m) = 2m - 2\) and \(\dim(\bar{A}_m) = 3m - 3\). As the graph \(\Gamma_m\) in a particular marked graph has possibly one extra vertex \(\circ\) of valence 2, we see that \(\dim(\bar{X}_m) = \dim(\bar{Q}_m) = 2m - 1\) and \(\dim(\bar{A}_m) = 3m - 2\).

Now that we are more familiar with the structure of our spaces \(\tilde{X}_m\) and \(\tilde{Q}_m\), we can proceed with describing how they are related to the normalizers \(N_{\text{Aut}(F_m)}(B_k)\).
Definition 6.10 (Action of $N_{\text{Aut}(F_n)}(B_k)$ on $\tilde{X}_k$) For $k \in \{0, \ldots, p-1\}$, the map from

$$N_{\text{Aut}(F_n)}(B_k) \cong N_{\Sigma_p}(\mathbb{Z}/p) \times (F_k \rtimes \text{Aut}(F_k)) \times \text{Aut}(F_{p-1-k})$$

to

$$N_{\text{Aut}(F_{m+2})}(Q) \cong \Sigma_3 \times (F_m \rtimes \text{Aut}(F_m))$$
given by projection to the second factor followed by inclusion induces an action of $N_{\text{Aut}(F_n)}(B_k)$ on $\tilde{X}_k$. Since $N_{\Sigma_p}(\mathbb{Z}/p) \subset N_{\text{Aut}(F_n)}(B_k)$ acts trivially, the quotient of this action is $\tilde{Q}_k$.

We can also define an action of $N_{\text{Aut}(F_n)}(B_k)$ on $X_{p-1-k}$:

Definition 6.11 (Action of $N_{\text{Aut}(F_n)}(B_k)$ on $X_{p-1-k}$) For $k \in \{0, \ldots, p-1\}$, the map from

$$N_{\text{Aut}(F_n)}(B_k) \cong N_{\Sigma_p}(\mathbb{Z}/p) \times (F_k \rtimes \text{Aut}(F_k)) \times \text{Aut}(F_{p-1-k})$$

to $\text{Aut}(F_{p-1-k})$ given by projection on the third factor induces an action of $N_{\text{Aut}(F_n)}(B_k)$ on $X_{p-1-k}$, with quotient $Q_{p-1-k}$.

From Definition 6.10 and Definition 6.11, there is an induced action of $N_{\text{Aut}(F_n)}(B_k)$ on $\tilde{X}_k \times X_{p-1-k}$. Note that the space $\tilde{X}_k \times X_{p-1-k}$ is a product of posets and can be given a poset structure by saying that

$$(\alpha^1, f^1) \times \beta^1 \geq (\alpha^2, f^2) \times \beta^2$$

if

$$(\alpha^1, f^1) \geq (\alpha^2, f^2) \text{ and } \beta^1 \geq \beta^2.$$ 

This gives $\tilde{X}_k \times X_{p-1-k}$ a simplicial structure. Often we will not use this simplicial structure, however, and just use the cellular structure that comes from products of simplices in $\tilde{X}_k$ and $X_{p-1-k}$.

Because $\tilde{X}_k$ and $X_{p-1-k}$ are contractible from Fact 6.11 and [11], respectively, we see that

Theorem 6.12 The above induced action of $N_{\text{Aut}(F_n)}(B_k)$ on the contractible space $\tilde{X}_k \times X_{p-1-k}$ has finite stabilizers and quotient $\tilde{Q}_k \times Q_{p-1-k}$.

It is interesting to note that the space $\tilde{X}_k \times X_{p-1-k}$ is in fact homeomorphic to the fixed point subcomplex $X^B_{n^k}$. We leave this as a straightforward exercise for the reader, as this fact will not be used in this paper.
7 The cohomology of the normalizers

In this section, we will compute the cohomology of the normalizers of the subgroups $A, B_k, C, D,$ and $E$ listed in Proposition 4.5. First, we list some helpful facts that will allow us to compute these cohomology groups.

The following theorem of Swan’s (see [20] or [1]) is a standard tool for computing the cohomology of groups:

**Theorem 7.1 (Swan)** If $G$ is a finite group with a $p$-Sylow subgroup $P$ that is abelian, then

$$H^*(G; \mathbb{Z}(p)) = H^*(P; \mathbb{Z}(p))^{N_G(P)}.$$ 

In part II of [14] we showed several low dimensional cohomology groups having to do with $\text{Aut}(F_n)$ are zero. The methods used are variants of those used by Hatcher and Vogtmann in [11] and [12] where they use a “degree theorem” to reduce complicated calculations involving $X_n$ to more manageable calculations involving subcomplexes of $X_n$. These results are used in this paper to prove that certain spectral sequences converge at the $E_2$-page. We list the results here in the following fact:

**Fact 7.2**

1. $H^1(Q^\omega_p; \mathbb{Z}/p) = 0$.
2. $H^1(Q_m; \mathbb{Z}/p) = 0$ for $1 \leq m \leq p - 1$.
3. $H^2(Q_{p-1}; \mathbb{Z}/p) = 0$.
4. $H^1(Q_m; \mathbb{Z}/p) = 0$ for $1 \leq m \leq p - 1$.
5. $H^2(Q_{p-1}; \mathbb{Z}/p) = 0$.

We can now calculate the cohomology of the normalizers of the various subgroups $A, B_k, C, D,$ and $E$.

**Lemma 7.1**

$$\hat{H}^t(N_{\text{Aut}(F_n)}(A); \mathbb{Z}(p)) \cong \begin{cases} \mathbb{Z}/p & t \equiv 0 \pmod{n} \\ 0 & t \equiv \pm1 \pmod{n} \\ H^r(Q^\omega_p; \mathbb{Z}/p) & t \equiv r \pmod{n}, \quad 2 \leq r \leq n - 2 \end{cases}$$

**Proof.** We now define an action of $N_{\Sigma_p}(\mathbb{Z}/p) \times N_{\text{Aut}(F_p)}(\omega)$ on the space $X^\omega_p$ by stipulating that $N_{\Sigma_p}(\mathbb{Z}/p)$ acts trivially and that $N_{\text{Aut}(F_p)}(\omega)$ acts in the usual manner on $X^\omega_p$. This in turn defines an action of $N_{\text{Aut}(F_n)}(A)$ on the contractible space $X^\omega_p$. This action has finite stabilizers and quotient $Q^\omega_p$. 

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The equivariant cohomology spectral sequence for this action is

$$E_{1}^{r,s} = \prod_{[\delta] \in \Delta_{p-1}^{r}} \hat{H}^{s}(\text{stab}_{N_{\text{Aut}(F_{p})}(A)}(\delta); \mathbb{Z}_{(p)}) \Rightarrow \hat{H}^{r+s}(N_{\text{Aut}(F_{p})}(A); \mathbb{Z}_{(p)}) \quad (7.2)$$

where $[\delta]$ ranges over the set $\Delta_{p-1}^{r}$ of orbits of $r$-simplices $\delta$ in $X_{p}^{\omega}$.

We claim that each $\text{stab}_{N_{\text{Aut}(F_{p})}(A)}(\delta)$ is the direct sum of $N_{\Sigma_{p}}(\mathbb{Z}/p)$ with a finite subgroup of $N_{\text{Aut}(F_{p})}(\omega)$ that does not have any $p$-torsion. This is because Glover and Mislin showed in [9] that the only $p$-torsion in $\text{Aut}(F_{p})$ comes from stabilizers of marked graphs with underlying graphs $\Theta_{p}$, $\Theta_{p-1} \lor R_{1}$, $\Xi_{p}$, or $R_{p}$, where the graph $\Xi_{p}$ is defined to be the 1-skeleton of the cone over a $p$-gon. Since $\omega$ acts by switching the first two petals of the rose $R_{p}$, $X_{p}^{\omega}$ does not contain any marked graphs with underlying graph $R_{p}$, $\Theta_{p}$, $\Theta_{p-1} \lor R_{1}$, or $\Xi_{p}$. Although $X_{p}^{\omega}$ obviously does contain marked graphs with underlying graph $R_{p}$, these will not worry us as $p$ does not divide the orders of the stabilizers – under the action of just $N_{\text{Aut}(F_{p})}(\omega)$ – of such marked graphs. For example, $N_{\text{Aut}(F_{p})}(\omega)$ does not contain the permutation $(12 \ldots p)$ that rotates the petals of the rose $R_{p}$ because

$$(12 \ldots p) \circ \omega \circ (p \ldots 21)$$

is the permutation $(23)$, which is not equal to $(1)$ or $\omega = (12)$.

Thus for every $[\delta]$, we have

$$\hat{H}^{s}(\text{stab}_{N_{\text{Aut}(F_{p})}(A)}(\delta); \mathbb{Z}_{(p)}) = \hat{H}^{s}(N_{\Sigma_{p}}(\mathbb{Z}/p); \mathbb{Z}_{(p)}) = \hat{H}^{s}(\Sigma_{p}; \mathbb{Z}_{(p)}).$$

The $E_{1}^{r,s}$-page of the spectral sequence is 0 in the rows where $s \neq kn$ and a copy of the cellular cochain complex with $\mathbb{Z}/p$-coefficients of the $(n-2)$-dimensional complex $Q_{p}^{\omega}$ in rows $kn$. It follows that the $E_{2}$-page has the form:

$$E_{2}^{r,s} = \begin{cases} 
\mathbb{Z}/p & r = 0 \text{ and } s = kn \\
H^{s}(Q^{\omega}_{p}; \mathbb{Z}/p) & 1 \leq r \leq n-2 \text{ and } s = kn \\
0 & \text{otherwise}
\end{cases}$$

Hence we see that the spectral sequence converges at the $E_{2}$-page.

That $H^{1}(Q^{\omega}_{p}; \mathbb{Z}/p) = 0$ follows from part 1 of Fact 7.2. □
Lemma 7.3

\[ \hat{H}^t(\mathcal{N}_{\text{Aut}(F_n)}(B_0); \mathbb{Z}(p)) \cong \begin{cases} \mathbb{Z}/p^2 & t = 0 \\ (j + 1)\mathbb{Z}/p & |t| = nj \neq 0 \\ (j - 1)\mathbb{Z}/p & |t| = nj - 1 \\ 0 & |t| \equiv 1 \pmod{n} \\ 0 & t \equiv 2 \pmod{n} \\ H^r(\mathcal{Q}_{p-1}; \mathbb{Z}/p) & t \equiv r \pmod{n}, \\ 3 \leq r \leq n - 2 \end{cases} \]

For \( k \in \{1, \ldots, p - 2\} \),

\[ \hat{H}^t(\mathcal{N}_{\text{Aut}(F_n)}(B_k); \mathbb{Z}(p)) \cong \begin{cases} \mathbb{Z}/p & t \equiv 0 \pmod{n} \\ 0 & t \equiv \pm 1, -2, \pmod{n} \\ H^r(\mathcal{Q}_k \times \mathcal{Q}_{p-1-k}; \mathbb{Z}/p) & t \equiv r \pmod{n}, \\ 2 \leq r \leq n - 3 \end{cases} \]

\[ \hat{H}^t(\mathcal{N}_{\text{Aut}(F_n)}(B_{p-1}); \mathbb{Z}(p)) \cong \begin{cases} \mathbb{Z}/p^2 \oplus \mathbb{Z}/p & t = 0 \\ (2j + 1)\mathbb{Z}/p & |t| = nj \neq 0 \\ 0 & t = nj + 1 > 0 \\ (2j - 2)\mathbb{Z}/p & t = -nj + 1 < 0 \\ (2j - 2)\mathbb{Z}/p \oplus H^{n-1}(\mathcal{Q}_{p-1}; \mathbb{Z}/p) & t = nj - 1 > 0 \\ H^{n-1}(\mathcal{Q}_{p-1}; \mathbb{Z}/p) & t = -nj - 1 < 0 \\ 0 & t \equiv 2 \pmod{n} \\ H^r(\mathcal{Q}_{p-1}; \mathbb{Z}/p) & t \equiv r \pmod{n}, \\ 3 \leq r \leq n - 2 \end{cases} \]

**Proof.** We use the equivariant cohomology spectral sequence (2.2) corresponding to the action, defined in Definition 6.10 and Definition 6.11, of \( \mathcal{N}_{\text{Aut}(F_n)}(B_k) \cong \mathcal{N}_{\text{Aut}(F_n)}(\mathbb{Z}/p) \times (F_k \times \text{Aut}(F_k)) \times \text{Aut}(F_{p-1-k}) \) on the space \( \tilde{X}_k \times X_{p-1-k} \). From Theorem 6.12 the action of \( \mathcal{N}_{\text{Aut}(F_n)}(B_k) \) on the contractible space \( \tilde{X}_k \times X_{p-1-k} \) has finite stabilizers and quotient \( \mathcal{Q}_k \times \mathcal{Q}_{p-1-k} \).

Applying the spectral sequence (2.2) we obtain the following \( E_1 \)-page:

\[ E_1^{r,s} = \prod_{[\delta] \in \Delta^r} \hat{H}^s(\text{stab}_{\mathcal{N}_{\text{Aut}(F_n)}(B_k)}(\delta); \mathbb{Z}(p)) \Rightarrow \hat{H}^{r+s}(\mathcal{N}_{\text{Aut}(F_n)}(B_k); \mathbb{Z}(p)) \quad (7.4) \]

where \([\delta]\) ranges over the set \( \Delta^r \) of orbits of \( r \)-simplices \( \delta \) in \( \tilde{X}_k \times X_{p-1-k} \). Vertices in \( \Delta^0 \) are pairs of unmarked graphs \( (\Gamma_1, \Gamma_2) \) where \( \pi_1(\Gamma_1) \cong F_k \),
\( \pi_1(\Gamma_2) \cong F_{p-1-k} \), the graph \( \Gamma_1 \) has two distinguished points \(*\) and \( \circ \), and the graph \( \Gamma_2 \) has one distinguished point \(*\).

The stabilizer of this vertex of \( \Delta^0 \) under the action of

\[ N_{\text{Aut}(F_n)}(B_k) \cong N_{\Sigma_p}(\mathbb{Z}/p) \times (F_k \rtimes \text{Aut}(F_k)) \times \text{Aut}(F_{p-1-k}) \]

is isomorphic to

\[ N_{\Sigma_p}(\mathbb{Z}/p) \times \text{Aut}(\Gamma_1) \times \text{Aut}(\Gamma_2) \]

where by \( \text{Aut}(\Gamma_1) \) we mean graph automorphisms of \( \Gamma_1 \) that preserve both distinguished points, and \( \text{Aut}(\Gamma_2) \) is the group of graph automorphisms of \( \Gamma_2 \) that preserve its distinguished point.

There are three cases, depending upon what value \( k \) takes.

**CASE 1:** \( k \in \{1, \ldots, p-2\} \). Consider an \( r \)-simplex \( \delta \)

\[ ((\alpha^0, f^0), \beta^0) > ((\alpha^1, f^1), \beta^1) > \cdots > ((\alpha^r, f^r), \beta^r) \]

of \( \tilde{X}_k \times X_{p-1-k} \). Let \( \Gamma^1 \) be the underlying graph of \( (\alpha^0, f^0) \) and let \( \Gamma^2 \) be the underlying graph of \( \beta^0 \). The stabilizer of \( \delta \) is a subgroup of the group

\[ N_{\Sigma_p}(\mathbb{Z}/p) \times \text{Aut}(\Gamma_1) \times \text{Aut}(\Gamma_2) \]

The finite group \( \text{Aut}(\Gamma_1) \) has no \( p \)-torsion since \( k < p-1 \) and so none of the underlying graphs of marked graphs in \( \tilde{X}_k \) have any \( p \)-symmetry. Similarly, the finite group \( \text{Aut}(\Gamma_2) \) has no \( p \)-torsion since \( k > 0 \) and so none of the underlying graphs of marked graphs in \( X_{p-1-k} \) have any \( p \)-symmetry. Thus we have that

\[ \hat{H}^* (\text{stab}(\delta); \mathbb{Z}/p) \cong \hat{H}^*(N_{\Sigma_p}(\mathbb{Z}/p); \mathbb{Z}/p) \cong \hat{H}^*(\Sigma_p; \mathbb{Z}/p) \]

Since the above holds for every simplex \( \delta \), we see that the spectral sequence (7.4) has \( E_2 \) page

\[ E_2^{r,s} = \begin{cases} 
\mathbb{Z}/p & r = 0 \text{ and } s = kn \\
H^r(\tilde{Q}_k \times Q_{p-1-k}; \mathbb{Z}/p) & 1 \leq r \leq n-3 \text{ and } s = kn \\
0 & \text{otherwise}
\end{cases} \]

because the dimension of \( \tilde{Q}_k \times Q_{p-1-k} \) is \( 2p-5 \).

Now apply parts 2 and 4 of Fact 7.2 to conclude that all of the groups \( H^1(\tilde{Q}_k \times Q_{p-1-k}; \mathbb{Z}/p) \) are zero. The lemma now follows for \( k \in \{1, \ldots, p-2\} \).

**CASE 2:** \( k = 0 \). Then the simplices \( \delta \) in spectral sequence (7.4) are all in \( Q_{p-1} \). Since only one graph in \( Q_{p-1} \) has \( p \)-symmetry, namely the graph \( \Theta_{p-1} \), we have
\[
\text{stab}_{N_{\text{Aut}(F_n)(B_k)}}(\delta) = \begin{cases} 
N_{\Sigma p}(\mathbb{Z}/p) \times \Sigma p & \text{if } \delta \text{ has underlying graph } \Theta_{p-1} \\
N_{\Sigma p}(\mathbb{Z}/p) \times H & \text{otherwise, where } H \text{ is a group with } p \nmid |H| 
\end{cases}
\]

Arguments similar to those in Lemma 7.1 show that the \(E_2\) page has the form

\[
E_2^{r,s} = \begin{cases} 
\mathbb{Z}/p^2 \oplus \mathbb{Z}/p & r = 0 \text{ and } s = 0 \\
H^r(Q_{p-1}; \mathbb{Z}/p) & 1 \leq r \leq n - 2, s = kn \\
(j - 1)\mathbb{Z}/p & r = 0 \text{ and } |s| = nj - 1 \\
(j + 1)\mathbb{Z}/p & r = 0 \text{ and } |s| = nj \neq 0 \\
0 & \text{otherwise}
\end{cases}
\]

Because part 3 of Fact 7.2 gives that \(H^2(Q_{p-1}; \mathbb{Z}/p) = 0\), the differentials \(d_2 : E_2^{0,-nj+1} \to E_2^{2,-nj}\) are zero and we see that the spectral sequence converges at the \(E_2\) page. Lastly, by part 2 of Fact 7.2, \(H^1(Q_{p-1}; \mathbb{Z}/p) = 0\) and we are done with the case \(k = 0\).

**CASE 3:** \(k = p - 1\). Then the simplices \(\delta\) in spectral sequence (7.4) are all in \(\tilde{Q}_{p-1}\). Now only two graphs in \(\tilde{Q}_{p-1}\) have \(p\)-symmetry. One is the graph \(\Theta^1_{p-1}\) where both \(\ast\) and \(\circ\) are the left hand vertex of the \(\Theta\)-graph. The other is the graph \(\Theta^2_{p-1}\) where \(\ast\) is the vertex on the left side of the \(\Theta\)-graph and \(\circ\) is the vertex on the right side. Each of these graphs gives a vertex of \(\tilde{Q}_{p-1}\) with \(p\)-symmetry.

We have

\[
\text{stab}_{N_{\text{Aut}(F_n)(B_k)}}(\delta) = \begin{cases} 
N_{\Sigma p}(\mathbb{Z}/p) \times \Sigma p & \text{if } \delta \text{ has underlying graph } \Theta^1_{p-1} \text{ or } \Theta^2_{p-1} \\
N_{\Sigma p}(\mathbb{Z}/p) \times H & \text{otherwise, where } H \text{ is a group with } p \nmid |H| 
\end{cases}
\]

Arguments similar to those in Lemma 7.1 show that the \(E_2\) page has the form

\[
E_2^{r,s} = \begin{cases} 
\mathbb{Z}/p^2 \oplus \mathbb{Z}/p & r = 0 \text{ and } s = 0 \\
H^r(\tilde{Q}_{p-1}; \mathbb{Z}/p) & 1 \leq r \leq n - 1, s = kn \\
(2j - 2)\mathbb{Z}/p & r = 0 \text{ and } |s| = nj - 1 \\
(2j + 1)\mathbb{Z}/p & r = 0 \text{ and } |s| = nj \neq 0 \\
0 & \text{otherwise}
\end{cases}
\]

Let \(j \geq 2\). Now the differential

\[
d_n : E_n^{0,nj-1} \to E_n^{n,n(j-1)} = 0
\]
is necessarily trivial and thus \( \hat{H}^{n_j-1}(N_{\text{Aut}(F_n)}(B_{p-1}); \mathbb{Z}(p)) \) has a filtration with successive terms

\[
E_{2}^{0,n_j-1} = (2j - 2)\mathbb{Z}/p
\]

and

\[
E_{2}^{2p-3,n(j-1)} = H^{2p-3}(\tilde{Q}_{p-1}; \mathbb{Z}/p).
\]

Since

\[
\hat{H}^{n_j-1}(N_{\text{Aut}(F_n)}(B_{p-1}); \mathbb{Z}(p)) = H^{n_j-1}(N_{\text{Aut}(F_n)}(B_{p-1}); \mathbb{Z}(p))
\]

(because \( nj - 1 \) is above the vcd of \( N_{\text{Aut}(F_n)}(B_{p-1}) \)), we can use the Künneth formula for the latter cohomology group to specify the form that the above filtration takes and obtain that

\[
\hat{H}^{n_j-1}(N_{\text{Aut}(F_n)}(B_{p-1}); \mathbb{Z}(p)) = (2j - 2)\mathbb{Z}/p \oplus H^{2p-3}(\tilde{Q}_{p-1}; \mathbb{Z}/p).
\]

The other tricky cohomology group to calculate is

\[
\hat{H}^{-n_j+1}(N_{\text{Aut}(F_n)}(B_{p-1}); \mathbb{Z}(p))
\]

(again, for \( j \geq 2 \)). This can be computed by noting that

\[
d_2 : E_{2}^{0,-n_j+1} \to E_{2}^{2,-n_j}
\]

is zero by part 5 of Fact 7.2 and that

\[
E_{2}^{1,-n_j} = H^{1}(\tilde{Q}_{p-1}; \mathbb{Z}/p) = 0
\]

by part 4 of Fact 7.2. The result follows for the case \( k = p - 1 \). □

**Lemma 7.5**

\[
\hat{H}^t(N_{\text{Aut}(F_n)}(C); \mathbb{Z}(p)) \cong \begin{cases} 
\mathbb{Z}/p^2 \oplus \mathbb{Z}/p & t = 0 \\
\frac{(3k-p-1)}{2} + 1)\mathbb{Z}/p & |t| = kn \neq 0 \\
\frac{(3k-p-1)}{2} - 2)\mathbb{Z}/p & |t| = kn - 1 \\
0 & \text{otherwise}
\end{cases}
\]

**Proof.** The normalizer \( N_{\text{Aut}(F_n)}(C) \) acts on the contractible space \( X_n^C \) with finite stabilizers and finite quotient. Hence we can use the equivariant cohomology spectral sequence (2.2) to calculate the cohomology of the normalizer. This gives us:

\[
E_1^{r,s} = \prod_{[\delta] \in \Delta^r} \hat{H}^s(\text{stab}_{N_{\text{Aut}(F_n)}(C)}(\delta); \mathbb{Z}(p)) \Rightarrow \hat{H}^{r+s}(N_{\text{Aut}(F_n)}(C); \mathbb{Z}(p)) \quad (7.6)
\]

where \([\delta]\) ranges over the set \( \Delta^r \) of orbits of \( r \)-simplices \( \delta \) in \( X_n^C \).
Claim 7.3 The quotient space $\bigcup_r \Delta^r$ has 3 vertices and 2 edges in it. The vertices correspond to marked graphs with underlying graphs $\Phi_n$, $\Omega_n$, and $\Psi_n$. The two edges come from the forest collapses of $\Phi_n$ to $\Omega_n$ or $\Psi_n$. Pictorially, we have

$$
\Phi_n \leftarrow \Omega_n \rightarrow \Psi_n
$$

Proof. From Theorem 4.4, any two vertices (marked graphs) of $X^C_n$ whose underlying graphs are reduced, can be connected by a sequence of Nielsen transformations. The graphs that we obtain from $\Psi_n$ by doing Nielsen moves are all isomorphic to either $\Psi_n$ or $\Omega_n$.

It follows that if $\eta$ is a vertex of $X^C_n$ corresponding to a reduced marked graph, then the underlying graph of $\eta$ is either $\Psi_n$ or $\Omega_n$.

It remains to consider which graphs can be blowups (see Definition 6.5) of $\Psi_n$ or $\Omega_n$. Such a blowup would have a nontrivial $\mathbb{Z}/p$ action on at least $2p$ of its edges. From this, it is not hard to see (using similar methods to those in Proposition 4.1) that the only possibility for the underlying graph of such a blowup is $\Phi_n$. □

Direct examination reveals that the vertex in $\Delta^0$ corresponding to $\Phi_n$ has automorphism group $\Sigma_p \times \mathbb{Z}/2$. In the notation used to define $\Phi_n$ (refer to the text just above Figure 2), the $\Sigma_p$ in $\Sigma_p \times \mathbb{Z}/2$ acts on the collections of edges $\{a_i\}$, $\{b_i\}$, and $\{c_i\}$, respectively, by permuting their indices. On the other hand, the $\mathbb{Z}/2$ in $\Sigma_p \times \mathbb{Z}/2$ fixes the edges $a_i$ and switches the edges $b_i$ with the edges $c_i$. The group $C$ is included in $\Sigma_p \times \mathbb{Z}/2$ as the cyclic group generated by the permutation $(12\ldots p)$ in $\Sigma_p$. Hence the subgroup of normalizing graph automorphisms in $\Sigma_p \times \mathbb{Z}/2$ is

$$
N_{\Sigma_p}(\mathbb{Z}/p) \times \mathbb{Z}/2.
$$

The stabilizer of the vertex in $\Delta^0$ corresponding to $\Phi_n$ has cohomology

$$
\hat{H}^*(N_{\Sigma_p}(\mathbb{Z}/p) \times \mathbb{Z}/2; \mathbb{Z}(p)) = \hat{H}^*(\Sigma_p; \mathbb{Z}(p)).
$$

Similarly, the group of graph automorphisms of $\Omega_n$ is

$$
\Sigma_p \times \Sigma_p.
$$

The group $C$ is included in this as the subgroup generated by

$$
(12\ldots p) \times (12\ldots p).
$$
The stabilizer of the vertex in $\Delta^0$ which corresponds to $\Omega_n$ is the normalizer of $C$ in $\Sigma_p \times \Sigma_p$. This normalizer is

$$(\mathbb{Z}/p \times \mathbb{Z}/p) \rtimes \mathbb{Z}/(p - 1).$$

The generator of $\mathbb{Z}/(p - 1)$ acts diagonally on $(\mathbb{Z}/p \times \mathbb{Z}/p)$ by conjugating the generator of either $\mathbb{Z}/p$ to its $s$-th power for some generator $s$ of $\mathbb{F}_p^\times$.

The cohomology of $(\mathbb{Z}/p \times \mathbb{Z}/p) \rtimes \mathbb{Z}/(p - 1)$ can be calculated in a straightforward way from the cohomology of $\mathbb{Z}/p \times \mathbb{Z}/p$ using Swan’s theorem 7.1 as

$$H^*((\mathbb{Z}/p \times \mathbb{Z}/p) \rtimes \mathbb{Z}/(p - 1); \mathbb{Z}_{(p)}) = H^*(\mathbb{Z}/p \times \mathbb{Z}/p; \mathbb{Z}_{(p)})^\mathbb{Z}/(p - 1).$$

For example, if $z_n, \bar{z}_n \in H^n(\mathbb{Z}/p; \mathbb{Z}/p)$ are the generators corresponding to the first and second $\mathbb{Z}/p$’s in $\mathbb{Z}/p \times \mathbb{Z}/p$, respectively, then we can calculate the cohomology of $(\mathbb{Z}/p \times \mathbb{Z}/p) \rtimes \mathbb{Z}/(p - 1)$ in dimensions $kn - 1 > 0$ as follows. For $i$ ranging from 1 to $k(p - 1) - 1$ elements of the form

$$z_{2i-1} \bar{z}_{2j} - z_{2i} \bar{z}_{2j-1}$$

are both $\mathbb{Z}/(p - 1)$-invariant and are in the kernel of the Bockstein homomorphism. This gives $k(p - 1) - 1$ generators for the cohomology group. So we see that $\tilde{H}^t((\mathbb{Z}/p \times \mathbb{Z}/p) \rtimes \mathbb{Z}/(p - 1); \mathbb{Z}_{(p)}) =$

$$\begin{cases}
\mathbb{Z}/p^2 & t = 0 \\
(k(p - 1) + 1)\mathbb{Z}/p & |t| = kn \neq 0 \\
(k(p - 1) - 1)\mathbb{Z}/p & |t| = kn - 1 \\
0 & \text{otherwise}
\end{cases}$$

Lastly, the group of graph automorphisms of $\Psi_n$ is

$$(\Sigma_p \times \Sigma_p) \rtimes \mathbb{Z}/2.$$
\[(\mathbb{Z}/p \times \mathbb{Z}/p) \times (\mathbb{Z}/2 \times \mathbb{Z}/(p-1))\] can be calculated using Swan’s theorem 7.1 which indicates that

\[H^*((\mathbb{Z}/p \times \mathbb{Z}/p) \times (\mathbb{Z}/2 \times \mathbb{Z}/(p-1)); \mathbb{Z}(p)) = H^*(\mathbb{Z}/p \times \mathbb{Z}/p; \mathbb{Z}(p))^{(\mathbb{Z}/2 \times \mathbb{Z}/(p-1))}.\]

For example, we can calculate the cohomology of \((\mathbb{Z}/p \times \mathbb{Z}/p) \rtimes (\mathbb{Z}/2 \times \mathbb{Z}/(p-1))\) in dimensions \(kn - 1 > 0\) as follows. For \(i\) ranging from 1 to \(k(p-1)/2 - 1\) elements of the form

\[z_{2i-1} \bar{z}_{2j} - z_{2j-1} \bar{z}_{2i} - z_{2i} \bar{z}_{2j-1} + \bar{z}_{2j} \bar{z}_{2i-1}\]

are \(\mathbb{Z}/(p-1)\)-invariant, \(\mathbb{Z}/2\)-invariant, and are in the kernel of the Bockstein homomorphism. This gives \(k\frac{(p-1)}{2} - 1\) generators for the cohomology group.

Hence \(\hat{H}((\mathbb{Z}/p \times \mathbb{Z}/p) \rtimes (\mathbb{Z}/2 \times \mathbb{Z}/(p-1)); \mathbb{Z}(p)) =\)

\[\begin{cases} 
\mathbb{Z}/p^2 & t = 0 \\
\left(\frac{k(p-1)}{2} + 1\right)\mathbb{Z}/p & |t| = kn \neq 0 \\
\left(\frac{k(p-1)}{2} - 1\right)\mathbb{Z}/p & |t| = kn - 1 \\
0 & \text{otherwise}
\end{cases}\]

The two edges in \(\Delta^1\) have stabilizers isomorphic to \(N_{\Sigma_p}(\mathbb{Z}/p)\) or \(N_{\Sigma_p}(\mathbb{Z}/p) \times \mathbb{Z}/2\). We omit the argument here, but the stabilizers of the edges can be found by examining which graph automorphisms in

\[\text{stab}(\Phi_n) = N_{\Sigma_p}(\mathbb{Z}/p) \times \mathbb{Z}/2\]

preserve the relevant forest collapses. In either case, if we take Farrell cohomology with \(\mathbb{Z}(p)\)-coefficients, then both edges have stabilizers whose cohomology is the same as that of the symmetric group \(\Sigma_p\).

Combining all of this into the spectral sequence (7.6) and then applying the differential on the \(E_1\) page, we see that

\[E_2^{r,s} = \begin{cases} 
\mathbb{Z}/p^2 \oplus \mathbb{Z}/p & r = 0, s = 0 \\
\left(\frac{3k(p-1)}{2} - 2\right)\mathbb{Z}/p & r = 0, |s| = kn - 1 \\
\left(\frac{3k(p-1)}{2} + 1\right)\mathbb{Z}/p & r = 0, |s| = kn \neq 0 \\
0 & \text{otherwise}
\end{cases}\]

Thus \(E_2 = E_\infty\) and \(\hat{H}^*(N_{\text{Aut}(F_n)}(C); \mathbb{Z}(p))\) is as stated. \(\square\)

**Lemma 7.7**

\[
\hat{H}^*(N_{\text{Aut}(F_n)}(D); \mathbb{Z}(p)) \cong \hat{H}^*(\Sigma_p \times \Sigma_p; \mathbb{Z}(p)).
\]

\[
\hat{H}^*(N_{\text{Aut}(F_n)}(E); \mathbb{Z}(p)) \cong \hat{H}^*((\Sigma_p \times \Sigma_p) \rtimes \mathbb{Z}/2; \mathbb{Z}(p)).
\]
\[
\begin{cases}
Z/p^2 & t = 0 \\
([k/2] + 1)Z/p & |t| = kn \neq 0 \\
((k - 1)/2)Z/p & |t| = kn - 1 \\
0 & otherwise
\end{cases}
\]

**Proof.** From Lemma 5.4,

\[N_{\text{Aut}(F_n)}(D) = N_{\Sigma_p}(\mathbb{Z}/p) \times N_{\Sigma_p}(\mathbb{Z}/p).\]

and

\[N_{\text{Aut}(F_n)}(E) = (N_{\Sigma_p}(\mathbb{Z}/p) \times N_{\Sigma_p}(\mathbb{Z}/p)) \rtimes \mathbb{Z}/2.\]

The detailed description of the cohomology of \( N_{\text{Aut}(F_n)}(E) \) in the statement of this lemma is then obtained by using Swan’s theorem 7.1. Generators in dimensions \( kn - 1 > 0 \) come from expressions of the form

\[z_{i-1} \bar{z}_{jn} - z_{i-1} \bar{z}_{jn-1}\]

where \( i \) ranges from 1 to \( [(k - 1)/2] \). □

8 **Cohomology of Aut\((F_n)\): Proof of Theorem 1.1**

In this section, we will use the lemmas of the previous section to establish Theorem 1.1.

**Proof of Theorem 1.1.** The cohomology \( \hat{H}^*(\text{Aut}(F_n); \mathbb{Z}(p)) \) will be calculated using the normalizer spectral sequence (3.1), which has \( E_1 \) page

\[E_1^{r,s} = \prod_{(A_0 \subset \cdots \subset A_r) \in |B|_r} \hat{H}^s(\bigcap_{i=0}^r N_{\text{Aut}(F_n)}(A_i); \mathbb{Z}(p)) \Rightarrow \hat{H}^{r+s}((\text{Aut}(F_n); \mathbb{Z}(p))\]

where \( B \) denotes the poset of conjugacy classes of nontrivial elementary abelian \( p \)-subgroups of \( \text{Aut}(F_n) \), and \( |B|_r \) is the set of \( r \)-simplices in \( |B| \). We computed \( |B| \) in Proposition 4.5. It is 1-dimensional, so the above spectral sequence is zero except in the columns \( r = 0 \) and \( r = 1 \).

Recall that the realization \( |B| \) of the poset \( B \) has \( p \) path components. One component just consists of a point corresponding to the subgroup \( A \). In addition, \( p - 2 \) other components are also just points corresponding to the subgroups \( B_k \) for \( k \in \{1, \ldots, p - 2\} \). Finally, the last component is a 1-dimensional simplicial complex corresponding to the subgroups listed in diagram (4.2), which
we duplicate here:

\[
\begin{array}{c}
B_0 \\
\downarrow
\end{array}
\quad
\begin{array}{c}
D \\
\leftrightarrow
\end{array}
\quad
\begin{array}{c}
C \\
\downarrow
\end{array}
\quad
\begin{array}{c}
B_{p-1} \\
\leftrightarrow
\end{array}
\quad
\begin{array}{c}
E \\
\downarrow
\end{array}
\]

We have already calculated (in the lemmas of the previous section) the contributions of all of the vertices in \(|B|\) to the \(E_1\) page in (3.1).

The contribution of a 1-simplex in \(|B|\) can be obtained by taking the cohomology of the intersections of the normalizers of the vertices of the 1-simplex. Note that each of these intersections is a finite group (since each is a subgroup of either the finite group \(N_{Aut(F_n)}(D)\) or \(N_{Aut(F_n)}(E)\) of normalizing graph automorphisms.) In this way, we can calculate the (now just Tate) cohomological contributions of the 1-simplices in (4.2) to be:

\[
\begin{align*}
\hat{H}^*(N_{Aut(F_n)}(D) \cap N_{Aut(F_n)}(B_0); \mathbb{Z}(p)) &= \hat{H}^*(N_{\Sigma_p}(\mathbb{Z}/p) \times N_{\Sigma_p}(\mathbb{Z}/p); \mathbb{Z}(p)) \\
&= \hat{H}^*(\Sigma_p \times \Sigma_p; \mathbb{Z}(p)). \\
\hat{H}^*(N_{Aut(F_n)}(D) \cap N_{Aut(F_n)}(B_{p-1}); \mathbb{Z}(p)) &= \hat{H}^*(N_{\Sigma_p}(\mathbb{Z}/p) \times N_{\Sigma_p}(\mathbb{Z}/p); \mathbb{Z}(p)) \\
&= \hat{H}^*(\Sigma_p \times \Sigma_p; \mathbb{Z}(p)). \\
\hat{H}^t(N_{Aut(F_n)}(D) \cap N_{Aut(F_n)}(C); \mathbb{Z}(p)) &= \hat{H}^t((\mathbb{Z}/p \times \mathbb{Z}/p) \times \mathbb{Z}/(p-1); \mathbb{Z}(p)) \\
&= \begin{cases} \\
\mathbb{Z}/p^2 & t = 0 \\
(k(p-1) + 1)\mathbb{Z}/p & |t| = kn \neq 0 \\
(k(p-1) - 1)\mathbb{Z}/p & |t| = kn - 1 \\
0 & \text{otherwise} \\
\end{cases} \\
\hat{H}^t(N_{Aut(F_n)}(E) \cap N_{Aut(F_n)}(B_{p-1}); \mathbb{Z}(p)) &= \hat{H}^t(N_{\Sigma_p}(\mathbb{Z}/p) \times N_{\Sigma_p}(\mathbb{Z}/p); \mathbb{Z}(p)) \\
&= \hat{H}^*(\Sigma_p \times \Sigma_p; \mathbb{Z}(p)). \\
\hat{H}^t(N_{Aut(F_n)}(E) \cap N_{Aut(F_n)}(C); \mathbb{Z}(p)) &= \hat{H}^t((\mathbb{Z}/p \times \mathbb{Z}/p) \times (\mathbb{Z}/2 \times \mathbb{Z}/(p-1)); \mathbb{Z}(p)) \\
&= \begin{cases} \\
\mathbb{Z}/p^2 & t = 0 \\
(k(p-1)/2 + 1)\mathbb{Z}/p & |t| = kn \neq 0 \\
(k(p-1)/2 - 1)\mathbb{Z}/p & |t| = kn - 1 \\
0 & \text{otherwise} \\
\end{cases}
\end{align*}
\]

We are now ready to compute the \(E_2\) page of the spectral sequence (3.1). The contributions coming from the isolated points of \(|B|\) (i.e., from \(A, B_1, \ldots, B_{p-2}\)) survive unaltered from the \(E_1\) page. The contributions from the 1-dimensional component of \(|B|\) pictured in (4.2) will be what we concentrate on from now on.

First, we compute the values for the \(E_2\) page in rows \(s\) where \(s = nj + k\) with \(2 \leq k \leq 2(p - 2)\). The \(E_1\) page is only nonzero in the column \(r = 0\) for these
rows. Hence the entries in $E_{1}^{0,s}$ necessarily survive to the $E_{2}$ page and from there survive to the $E_{\infty}$ page. This gives us that $\hat{H}^{t}(\text{Aut}(F_{n}); \mathbb{Z}(p))$ is as the proposition claims for $t = nj + k$ with $2 \leq k \leq 2(p - 2)$.

For the rest of our calculations, we will use the fact that the boundary map on the $E_{1}$ page is just the restriction map. From a comment by Brown in [4] on page 286, we know that we can compute these restriction maps (from normalizers of $p$-subgroups to finite subgroups of those normalizers) by looking at the $E_{2}$ pages of the various spectral sequences used to compute the cohomologies of the normalizers (in the lemmas of the previous section.) This will help us to compute, for any row $s$, the value $E_{2}^{0,s}$. As an example of this, consider the copy of $\hat{H}^{*}(N_{\text{Aut}(F_{n})}(B_{0}); \mathbb{Z}(p))$ contained in the column $E_{2}^{0,s}$ of our spectral sequence (3.1). Recall from Lemma 7.3 that the cohomology of $N_{\text{Aut}(F_{n})}(B_{0})$ is

$$
\hat{H}^{t}(N_{\text{Aut}(F_{n})}(B_{0}); \mathbb{Z}(p)) = \begin{cases}
\mathbb{Z}/p^{2} & t = 0 \\
(j + 1)\mathbb{Z}/p & |t| = nj \neq 0 \\
(j - 1)\mathbb{Z}/p & |t| = nj - 1 \\
0 & |t| \equiv 1 \pmod{p} \\
0 & t \equiv 2 \pmod{p} \\
H^{r}(Q_{p-1}; \mathbb{Z}/p) & t \equiv r \pmod{p} \\
& 3 \leq r \leq n - 2
\end{cases}
$$

and this was calculated by looking at a spectral sequence whose $E_{2}$ page (which we now denote with a script $E$ as $\mathcal{E}$ to distinguish it from the spectral sequence (3.1) above) is

$$
\mathcal{E}_{2}^{r,s} = \begin{cases}
\mathbb{Z}/p^{2} & r = 0 \text{ and } s = 0 \\
H^{r}(Q_{p-1}; \mathbb{Z}/p) & 1 \leq r \leq 2(p - 2), s = kn \\
(j - 1)\mathbb{Z}/p & r = 0 \text{ and } |s| = nj - 1 \\
(j + 1)\mathbb{Z}/p & r = 0 \text{ and } |s| = nj \neq 0 \\
0 & \text{otherwise}
\end{cases}
$$

From (4.6) in Chapter X of [4], the restriction map from $\hat{H}^{*}(N_{\text{Aut}(F_{n})}(B_{0}); \mathbb{Z}(p))$ to the ring $\hat{\mathfrak{r}}^{*}(N_{\text{Aut}(F_{n})}(B_{0}); \mathbb{Z}(p))$ deriving from the cohomology of the finite subgroups of $N_{\text{Aut}(F_{n})}(B_{0})$ (see [4] for a definition of this ring) is the canonical surjection coming from the vertical edge homomorphism

$$
\hat{H}^{t}(N_{\text{Aut}(F_{n})}(B_{0}); \mathbb{Z}(p)) \to \mathcal{E}_{2}^{0,t} = \hat{H}^{t}(\Sigma_{p}(\mathbb{Z}/p) \times \Sigma_{p}(\mathbb{Z}/p); \mathbb{Z}(p))
$$

where

$$
\hat{H}^{t}(\Sigma_{p}(\mathbb{Z}/p) \times \Sigma_{p}(\mathbb{Z}/p); \mathbb{Z}(p)) = \begin{cases}
\mathbb{Z}/p^{2} & t = 0 \\
(j + 1)\mathbb{Z}/p & |t| = nj \neq 0 \\
(j - 1)\mathbb{Z}/p & |t| = nj - 1 \\
0 & \text{otherwise}
\end{cases}
$$

Recall that in order to know the column $E_{2}^{0,s}$ of (3.1), we want to calculate
the restriction map

\[
\hat{H}^*(N_{Aut(F_n)}(B_0); \mathbb{Z}_p) \to \hat{H}^*(N_{Aut(F_n)}(B_0) \cap N_{Aut(F_n)}(D))
\]

\[
\hat{H}^*(N_{\Sigma_p}(\mathbb{Z}/p) \times N_{\Sigma_p}(\mathbb{Z}/p); \mathbb{Z}_p)
\]

This restriction map is the vertical edge homomorphism calculated above.

We can similarly look at the $E_2$-pages of the other spectral sequences in section 7 to calculate the restriction maps from the cohomologies of $N_{Aut(F_n)}(B_{p-1})$ and $N_{Aut(F_n)}(C)$ to their finite subgroups

\[
N_{Aut(F_n)}(B_{p-1}) \cap N_{Aut(F_n)}(D), \quad N_{Aut(F_n)}(B_{p-1}) \cap N_{Aut(F_n)}(E),
\]

\[
N_{Aut(F_n)}(C) \cap N_{Aut(F_n)}(D), \quad N_{Aut(F_n)}(C) \cap N_{Aut(F_n)}(E).
\]

In these two cases, the restriction map is not just the vertical edge homomorphism. This is because the $E_1$-pages (and hence the $E_2$-pages) of the spectral sequences used to calculate $N_{Aut(F_n)}(B_{p-1})$ and $N_{Aut(F_n)}(C)$ each had vertical edges with cohomology groups coming from two different underlying marked graphs, namely $\Omega_{2(p-1)}$ and $\Psi_{2(p-1)}$. The cohomological contribution from $\Omega_{2(p-1)}$ is what we are concerned with when calculating the restriction to $N_{Aut(F_n)}(B_{p-1}) \cap N_{Aut(F_n)}(D)$ or $N_{Aut(F_n)}(C) \cap N_{Aut(F_n)}(D)$, while the contribution from $\Psi_{2(p-1)}$ is what we are concerned with when calculating restrictions to intersections involving $N_{Aut(F_n)}(E)$. So the restriction maps in these cases are found by composing the vertical edge homomorphism with a restriction map onto the portions of the $E_2$ page that come from either $\Omega_{2(p-1)}$ or $\Psi_{2(p-1)}$.

On the other hand, the restrictions from $N_{Aut(F_n)}(D)$ and $N_{Aut(F_n)}(E)$ to their subgroups are easy to compute as $N_{Aut(F_n)}(D)$ and $N_{Aut(F_n)}(E)$ are just well known finite groups.

After calculating all of the terms in the column $E_2^{0,s}$ of (3.1) as above, the value $E_2^{1,s}$ is found from an Euler characteristic argument using $E_1^{0,s}$ and $E_1^{1,s}$. That is, if $s \neq 0$ then $E_1^{0,s}$ and $E_1^{1,s}$ are the only nonzero terms on the row $s$ and they are both $\mathbb{F}_p$-vector spaces. As the boundary map goes from $E_1^{0,s} \to E_1^{1,s}$, this yields $dim_{\mathbb{F}_p}(E_1^{0,s}) = dim_{\mathbb{F}_p}(E_1^{1,s}) = dim_{\mathbb{F}_p}(E_2^{0,s}) - dim_{\mathbb{F}_p}(E_2^{1,s})$ by the standard Euler characteristic argument.

Assume $s = -kn + 1 < 0$ (the case $s = kn - 1 > 0$ follows similarly, with the only exception being the additional summand of $H^{n-1}(\mathbb{Q}_p; \mathbb{Z}/p)$ that needs to be dealt with.) Then $E_2^{0,s} = (k + [(k - 1)/2] - 1)\mathbb{Z}/p$. Now $k - 1$ of these $\mathbb{Z}/p$’s come from 0-cocycles that are summations of cocycles from:

- The portion of $\hat{H}^*(N_{Aut(F_n)}(B_{p-1}); \mathbb{Z}_p)$ coming from the graph $\Omega_n$. (In the spectral sequence we used to calculate $\hat{H}^*(N_{Aut(F_n)}(B_{p-1}); \mathbb{Z}_p)$ in part 3. of Lemma 7.3, this was the contribution given by the stabilizer of the graph $\Theta_{p-1}$.)
- $\hat{H}^*(N_{Aut(F_n)}(D); \mathbb{Z}_p)$.
\begin{itemize}
  \item $\hat{H}^s(N_{\text{Aut}(F_n)}(B_0); \mathbb{Z}_{(p)})$.
  \item The portion of $\hat{H}^s(N_{\text{Aut}(F_n)}(C); \mathbb{Z}_{(p)})$ coming from the graph $\Omega_n$ (in the spectral sequence we used to calculate $\hat{H}^s(N_{\text{Aut}(F_n)}(C); \mathbb{Z}_{(p)})$.)
\end{itemize}

The other $[(k - 1)/2]$ of the $\mathbb{Z}/p$’s come from 0-cocycles that are summations of cocycles from:

\begin{itemize}
  \item The portion of $\hat{H}^s(N_{\text{Aut}(F_n)}(B_{p-1}); \mathbb{Z}_{(p)})$ coming from the graph $\Psi_n$. (In the spectral sequence we used to calculate $\hat{H}^s(N_{\text{Aut}(F_n)}(B_{p-1}); \mathbb{Z}_{(p)})$ in part 3. of Lemma 7.3, this was the contribution given by the stabilizer of the graph $\Theta_{p-1}^1$)
  \item $\hat{H}^s(N_{\text{Aut}(F_n)}(E); \mathbb{Z}_{(p)})$.
  \item The portion of $\hat{H}^s(N_{\text{Aut}(F_n)}(C); \mathbb{Z}_{(p)})$ coming from the graph $\Psi_n$ (in the spectral sequence we used to calculate $\hat{H}^s(N_{\text{Aut}(F_n)}(C); \mathbb{Z}_{(p)})$.)
\end{itemize}

This allows us to find that the contribution of the component of $|B|$ corresponding to diagram (4.2) in the row $s = -kn + 1 < 0$ is $E_2^{0,s} = (k + [(k - 1)/2] - 1)\mathbb{Z}/p$. Since $\dim_{F_p}(E_1^{0,s}) - \dim_{F_p}(E_1^{1,s}) =

\begin{align*}
(4(k-1) + \frac{3k(p-1)}{2} + \left\lceil \frac{k-1}{2} \right\rceil - 2) - (3(k-1) + \frac{3k(p-1)}{2} - 1) = k + \left\lceil \frac{k-1}{2} \right\rceil - 1,
\end{align*}

we find that $E_2^{1,s} = 0$.

Assume $s = kn > 0$ or $s = -kn < 0$. Then $E_2^{0,s} = (k + [k/2] + p - 1)\mathbb{Z}/p$. First, observe that $p - 1$ of these $\mathbb{Z}/p$’s come from the points in $|B|$ corresponding to $A, B_1, \ldots, B_{p-2}$. Also, $k - 1$ of these $\mathbb{Z}/p$’s come from summations of cocycles in $H^s(N; \mathbb{Z}_{(p)})$ for $N$ equal to $N_{\text{Aut}(F_n)}(B_0)$, $N_{\text{Aut}(F_n)}(B_{p-1})$, $N_{\text{Aut}(F_n)}(C)$, and $N_{\text{Aut}(F_n)}(E)$ where in all cases the cohomology came from the graph $\Omega_n$ (in the spectral sequences used to calculate the cohomology of the various normalizers.) In addition, $[k/2]$ of the $\mathbb{Z}/p$’s come from 0-cocycles that are summations of cocycles from $\hat{H}^s(N; \mathbb{Z}_{(p)})$ for $N$ equal to $N_{\text{Aut}(F_n)}(B_{p-1})$, $N_{\text{Aut}(F_n)}(C)$, and $N_{\text{Aut}(F_n)}(E)$ where in all cases the cohomology came from the graph $\Psi_n$. Finally, one cocycle comes from summing up cocycles from $H^s(N; \mathbb{Z}_{(p)})$ for $N$ equal to $N_{\text{Aut}(F_n)}(B_0)$, $N_{\text{Aut}(F_n)}(B_{p-1})$, $N_{\text{Aut}(F_n)}(C)$, $N_{\text{Aut}(F_n)}(D)$, and $N_{\text{Aut}(F_n)}(E)$. This allows us to find that the contribution of the component of $B$ corresponding to diagram (4.2) in the row $s = kn > 0$ or $s = -kn < 0$ is $E_2^{0,s} = (k + [k/2] + p - 1)\mathbb{Z}/p$. Note that because $\dim_{F_p}(E_1^{0,s}) - \dim_{F_p}(E_1^{1,s}) =

\begin{align*}
(4k + \frac{(3k + 2)(p-1)}{2} + [k/2] + 5) - (3k + \frac{3k(p-1)}{2} + 5) = k + [k/2] + p - 1,
\end{align*}

we find that $E_2^{1,s} = 0$.

Next, $E_2^{0,0}$ of the normalizer spectral sequence is readily computed to be $\mathbb{Z}/p^2 \oplus$
$p(\mathbb{Z}/p)$, where $p - 1$ of the $\mathbb{Z}/p$'s come from the isolated points of $|\mathcal{B}|$ and $\mathbb{Z}/p^2 \oplus \mathbb{Z}/p$ comes from vertices in the 1-dimensional component of $|\mathcal{B}|$. In addition, $E^{0,1}_2 = \mathbb{Z}/p$ where the $\mathbb{Z}/p$ corresponds to edges of $|\mathcal{B}|$ which give cohomology classes in $E^{0,1}_1$ that are not mapped onto by cohomology classes in $E^{0,0}_1$.

We further illustrate the above calculations by looking at the contribution of diagram (4.2) to the explicitly listed (i.e., not listed at coming from cohomologies of various quotient spaces) cohomology classes in $E^{0,*}_2$. This the same as that contributed by the by the graph of groups listed in the figure below.

![Fig. 3. A graph of groups](image)

In the graph of groups in Figure 3, there are five vertices $v_i$ and five edges $e_i$. The vertex and edge groups are

- $v_1$: $N_{\Sigma_p}(\mathbb{Z}/p) \times \Sigma_p$
- $v_2$: $N_{\Sigma_p}(\mathbb{Z}/p) \times N_{\Sigma_p}(\mathbb{Z}/p)$
- $v_3$: $(N_{\Sigma_p}(\mathbb{Z}/p) \times \Sigma_p)^* \Sigma_p (N_{\Sigma_p}(\mathbb{Z}/p) \times \Sigma_p)$, where the amalgamated $\Sigma_p$ includes into the $\Sigma_p$ on the right in each factor.
- $v_4$: $(N_{\Sigma_p}(\mathbb{Z}/p) \times N_{\Sigma_p}(\mathbb{Z}/p)) \times \mathbb{Z}/2$
- $v_5$: $[(\mathbb{Z}/p \times \mathbb{Z}/p) \times \mathbb{Z}/(p - 1)]^* N_{\Sigma_p}(\mathbb{Z}/p) [((\mathbb{Z}/p \times \mathbb{Z}/p) \times (\mathbb{Z}/2 \times \mathbb{Z}/(p - 1)))$
  where the amalgamated $N_{\Sigma_p}(\mathbb{Z}/p)$ includes diagonally into the factors on both sides.
- $e_1$: $N_{\Sigma_p}(\mathbb{Z}/p) \times N_{\Sigma_p}(\mathbb{Z}/p)$
- $e_2$: $N_{\Sigma_p}(\mathbb{Z}/p) \times N_{\Sigma_p}(\mathbb{Z}/p)$
- $e_3$: $N_{\Sigma_p}(\mathbb{Z}/p) \times N_{\Sigma_p}(\mathbb{Z}/p)$
- $e_4$: $(\mathbb{Z}/p \times \mathbb{Z}/p) \times (\mathbb{Z}/2 \times \mathbb{Z}/(p - 1))$
- $e_5$: $(\mathbb{Z}/p \times \mathbb{Z}/p) \times \mathbb{Z}/(p - 1)$
The vertex groups consist of the explicitly listed contributions from the normalizers of \( B_0, D, B_{p-1}, E, \) and \( C, \) respectively, to the column \( E_2^{0,*} \) of the normalizer spectral sequence. For example, from the action of \( N_{\Sigma_p}(\mathbb{Z}/p) \times \Sigma_p \) on the graph \( \Omega_n, \) we obtain an inclusion of this group into \( N_{\text{Aut}(F_n)}(B_0). \) We similarly obtain two inclusions of \( N_{\Sigma_p}(\mathbb{Z}/p) \times \Sigma_p \) into \( N_{\text{Aut}(F_n)}(B_{p-1}) \) by taking actions on \( \Omega_n \) or \( \Psi_n, \) respectively. This yields (cf Case 3 of the proof of Lemma 7.3) an inclusion of the vertex group for \( v_3 \) into \( N_{\text{Aut}(F_n)}(B_{p-1}). \) Lastly, from the proof of Lemma 7.5, we see that \( N_{\text{Aut}(F_n)}(C) \) is isomorphic to the vertex group for \( e_5. \) The edge groups above are just the intersections of the normalizers of the \( B_0, D, B_{p-1}, E, \) and \( C. \)

Using the property that \( X \ast Y \cong X \) and noting that the cohomology of \( \Sigma_p \) and \( N_{\Sigma_p}(\mathbb{Z}/p) \) are the same, we see that graph of groups in Figure 1 which yielded the group \( \varsigma_p. \) The left hand vertex in the graph of groups in Figure 1 comes from the first half of the amalgamated product decomposition of the vertex group \( v_3, \) and the right hand vertex comes from \( v_4. \) The top edge group comes from the \( \Sigma_p \) being amalgamated over to make \( v_3, \) and the bottom edge group comes from the \( N_{\Sigma_p}(\mathbb{Z}/p) \) being amalgamated over to make \( v_5. \)

For example, we can look at the contribution of Figure 1 to the row \( s = n \) by examining an equivariant spectral sequence for \( \varsigma_p. \) This would have 3 generators in \( E_1^{0,n}, \) two from \( \Sigma_p \times \Sigma_p \) and one from \( (\Sigma_p \times \Sigma_p) \rtimes \mathbb{Z}/2. \) It would also have 2 generators in \( E_1^{1,n}, \) one each from the top edge group in Figure 1 and one from the bottom edge group. The 2 by 3 matrix corresponding to the coboundary map in this row has rank 2, since the first generator of \( E_1^{0,n} \) includes only into the second generator of \( E_1^{1,n} \) while the second generator of \( E_1^{0,n} \) includes diagonally. Since the matrix has a 1-dimensional nullspace, diagram (4.2) contributes one \( \mathbb{Z}/p \) to \( E_2^{0,n} \) and nothing to \( E_2^{1,n} \) is 0. Other rows can be examined similarly.

Finally, the last remark in the statement of the theorem is justified since all of the explicitly listed cohomology classes in the column \( E_2^{0,*} \) coming from isolated points of \( |B| \) are detected upon restriction to the portions of the cohomologies of the normalizers in \( \text{Aut}(F_n) \) of \( A, B_1, \ldots, B_{p-2}, \) coming from the stabilizers of marked graphs corresponding to \( R_{2(p-1)}, R_1 \rtimes \theta_{p-1} \rtimes R_{p-2}, \ldots, R_{p-2} \rtimes \theta_{p-1} \rtimes R_1. \) The explicitly listed classes in \( E_2^{0,*} \) coming from diagram (4.2) all come from the graph of groups \( \varsigma_p \) listed in Figure 1. In the figure, the \( \Sigma_p \times \Sigma_p \) comes from graph automorphisms of \( \Omega_{2(p-1)} \) and the \( (\Sigma_p \times \Sigma_p) \rtimes \mathbb{Z}/2 \) comes from graph automorphisms of \( \Psi_{2(p-1)}. \) The fact that the isomorphism from the “explicit” part of the cohomology of \( \text{Aut}(F_n) \) to that of \( (\ast_{p-1} B_{p}) \ast \varsigma_p \) preserves the ring structure follows from (4.5) (vi) in Chapter X of [4]. □
A The usual cohomology of Aut$(F_{2(p-1)})$

We now state the (only partial) characterization of $H^t(Aut(F_n);\mathbb{Z}_{(p)})$ which results from our calculations in Theorem 1.1, where $n = 2(p-1)$ as before.

**Proposition A.1**

$$H^t(Aut(F_n);\mathbb{Z}_{(p)}) = \begin{cases} \hat{H}^t(Aut(F_n);\mathbb{Z}_{(p)}) & t > 4p-6 \\ H^t(Q_n;\mathbb{Z}_{(p)}) & 0 \leq t \leq 2p-3 \end{cases}$$

**Sketch of proof.** Since the virtual cohomological dimension of Aut$(F_n)$ is $4p-6$, it follows directly (see [4]) that $H^t(Aut(F_n);\mathbb{Z}_{(p)}) = \hat{H}^t(Aut(F_n);\mathbb{Z}_{(p)})$ for $t > 4p-6$ and that the sequence

$$H^m(\Gamma);\mathbb{Z}_{(p)}) \to H^m(Aut(F_n);\mathbb{Z}_{(p)}) \to \hat{H}^m(Aut(F_n);\mathbb{Z}_{(p)}) \to 0$$

is exact for $m = 4p-6$, where $\Gamma$ is any torsion free subgroup of Aut$(F_n)$ of finite index and where the first map is the transfer map.

To establish the final part of the proposition, we can use the equivariant cohomology spectral sequence (2.1) to calculate $H^t(Aut(F_n);\mathbb{Z}_{(p)})$ for $0 \leq t \leq 2p-3$. We want to show that for $0 < s \leq 2p-3$, the rows $E_2^{r,s}$ are all zero, so that $H^s(Aut(F_n);\mathbb{Z}_{(p)}) = H^s(Q_n;\mathbb{Z}_{(p)})$.

Let $X_s$ be the $p$-singular locus of $X_n$, or the set of all simplices in $X_n$ such that $p$ divides $|\text{stab}(\delta)|$. Let $e_{1}^{r,s}$ be the $E_1$-page of the equivariant cohomology spectral sequence for calculating $H^*_\text{Aut}(F_n)(X_s;\mathbb{Z}_{(p)})$. Note that for $s > 0$, $E_1^{r,s} = e_{1}^{r,s}$ and $E_2^{r,s} = e_{2}^{r,s}$. Further note that $X_s$ separates into $p$ disjoint, Aut$(F_n)$-invariant (not necessarily connected) subcomplexes corresponding to the $p$ path components of $|\mathcal{B}|$. (These path components were mentioned in the proof of Theorem 1.1.) So we see that

$$H^*_\text{Aut}(F_n)(X_s;\mathbb{Z}_{(p)}) = \prod_Y \hat{H}^*_\text{Aut}(F_n)(Y;\mathbb{Z}_{(p)})$$

where $Y$ ranges over the $p$ distinct, disjoint, Aut$(F_n)$-invariant subcomplexes. Hence the $E_1$ and $E_2$ pages can be calculated separately for each $Y$. Only one of the subcomplexes $Y$ has dihedral symmetry in it and is relevant to the horizontal rows between 1 and $2p-3$; namely, the subcomplex $Y_A$ which has marked graphs with underlying graph the rose $R_n$ in it. Observe that $Y_A$ is $Aut(F_n) \cdot X_n^A$, several disjoint copies of $X_n^A$ grouped together. Let $\Xi_p$ be the 1-skeleton of the cone over a $p$-gon. Note that $\Xi_p$ has dihedral symmetry. It is easily shown that the only marked graphs with dihedral symmetry are in $Y_A$ and have underlying graph $\Xi_p \vee \Gamma_{p-2}$, where $\Gamma_{p-2}$ is any pointed graph.
with fundamental group isomorphic to $F_{p-2}$, and where the wedge does not necessarily take place at the basepoint.

Now consider a specific path component of $Y_A$, say $X_n^A$. There is an action of $A \cong \mathbb{Z}/p$ on each simplex of $X_n^A$. In [16], Krstic and Vogtmann define an equivariant deformation retract $L_A$ of $X_n^A$ by only keeping the “essential graphs” in $X_n^A$. Among the graphs that are in $X_n^A - L_A$ are all of the ones with dihedral symmetry, along with various other inessential graphs. Define $L'_A$ to be the subcomplex of $X_n^A$ obtained by collapsing the wedge summand $\Sigma_p$ to $R_p$ in graphs with dihedral symmetry. By the same poset lemma used in [16], the collapse from $X_n^A$ to $L'_A$ is an equivariant deformation retraction. Observe that $X_n^A \supset L'_A \supset L_A$.

Denote the $E_1$-page of the equivariant cohomology spectral sequence used to calculate $\hat{H}^*_F(X_n^A; \mathbb{Z}/p)$ by $E_1^{r,s}$. The contribution of the equivariant cohomology of $Y_A$ to $E_1^{r,s}$ is the same as $E_1^{r,s}$. For $0 < s < 2p - 3$ and $s \neq 4k$, the row $E_1^{r,s}$ is all zero. On the other hand, for $0 < s = 4k < 2p - 3$, direct examination reveals that the row $E_1^{r,s}$ is the relative cochain complex $C^*(X_n/N_{Aut(F_n)}(A), L'_A/N_{Aut(F_n)}(A); \mathbb{Z}/p)$. Because the homotopy in the deformation retraction from $X_n^A$ to $L'_A$ is $N_{Aut(F_n)}(A)$-equivariant, the relative cohomology groups

$$\hat{H}^*(X_n/N_{Aut(F_n)}(A), L'_A/N_{Aut(F_n)}(A); \mathbb{Z}/p)$$

are all zero. Hence all of the rows $E_2^{r,s}$ are zero for $0 < s < 2p - 3$, which completes the proof. $\Box$

### B The Farrell cohomology of $Aut(F_l)$ for $l < 2(p-1)$

For $l < p - 1$, $\hat{H}^*(Aut(F_l); \mathbb{Z}/p) = 0$ from an easy spectral sequence argument. We have already remarked that Glover and Mislin’s work in [9] directly implies that

$$\hat{H}^*(Aut(F_{p-1}); \mathbb{Z}/p) = \hat{H}^*(\Sigma_p; \mathbb{Z}/p),$$

$$\hat{H}^*(Aut(F_p); \mathbb{Z}/p) = 3\hat{H}^*(\Sigma_p; \mathbb{Z}/p),$$

and we have noted that Yu Qing Chen’s work in [6] shows that

$$\hat{H}^*(Aut(F_{p+1}); \mathbb{Z}/p) = 4\hat{H}^*(\Sigma_p; \mathbb{Z}/p)$$

and that $\hat{H}^*(Aut(F_{p+2}); \mathbb{Z}/p) = 5\hat{H}^*(\Sigma_p; \mathbb{Z}/p) \oplus \hat{H}^{p-4}(\Sigma_p; \mathbb{Z}/p)$.

In this section, we show how to calculate $\hat{H}^*(Aut(F_l); \mathbb{Z}/p)$ for $p \leq l < n$, where $n = 2(p-1)$. We do this by modifying in a direct manner the arguments
we have made in the previous sections to calculate $\hat{H}^*(Aut(F_{2(p-1)}); \mathbb{Z}(p))$.

As before, one of the elements of $\mathcal{B}$ corresponds to the subgroup $A \cong \mathbb{Z}/p$ of $Aut(F_l)$ obtained by rotating the first $p$ petals of the rose $R_l$. The other elements of $\mathcal{B}$ correspond to subgroups $B_k \cong \mathbb{Z}/p$ for $k \in \{0, \ldots, l-p+1\}$. The subgroup $B_k$ is obtained, as before, by rotating the $p$-edges of the $\Theta$-graph in the middle of $R_k \vee \Theta_{p-1} \vee R_{l-p+1-k}$.

As in Lemma 7.1, we see that

$$N_{Aut(F_l)}(A) \cong N_{\Sigma_p}(\mathbb{Z}/p) \times ((F_{l-p} \times F_{l-p}) \times (\mathbb{Z}/2 \times Aut(F_{l-p})).$$

Let $\langle \omega \rangle \cong \mathbb{Z}/2$ be the subgroup of $Aut(F_{l-p+2})$ corresponding to the action given by switching the first two petals of the rose $R_{l-p+2}$. Accordingly,

$$N_{Aut(F_l)}(A) \cong C_{Aut(F_l)}(A) \cong (F_{l-p} \times F_{l-p}) \times (\mathbb{Z}/2 \times Aut(F_{l-p})).$$

As in Definition 6.2, let $X_{l-p+2}^\omega$ be the fixed point set of $\omega$ in $X_{l-p+2}$. Then let $X_{l-p+2}^\omega$ be the deformation retract of $X_{l-p+2}^\omega$ obtained by collapsing out inessential edges of marked graphs. Finally, define $Q_{l-p+2}^\omega$ to be the quotient of $X_{l-p+2}^\omega$ by $N_{Aut(F_{l-p+2})}^\omega$.

Putting all of this together, and using the same methods as those used in Lemma 7.1, Lemma 7.3 and Theorem 1.1, we see that

$$\hat{H}^t(N_{Aut(F_l)}(A); \mathbb{Z}(p)) = \begin{cases} \mathbb{Z}/p & t \equiv 0 \pmod{n} \\ \hat{H}^*(Q_{l-p+2}^\omega; \mathbb{Z}/p) & t \equiv r \pmod{n}, 2 \leq r \leq 2(l-p) \\ 0 & \text{otherwise} \end{cases}$$

and that for $k \in \{0, \ldots, l-p+1\}$, $\hat{H}^t(N_{Aut(F_l)}(B_k); \mathbb{Z}(p)) =

$$\begin{cases} \mathbb{Z}/p & t \equiv 0 \pmod{n} \\ \hat{H}^*(Q_k \times Q_{p-1-k}; \mathbb{Z}/p) & t \equiv r \pmod{n}, 2 \leq r \leq 2(l-p) + 1 \\ 0 & \text{otherwise} \end{cases}$$

so we have that for $p$ odd and $l \in \{p, \ldots, 2p-3\}$, $\hat{H}^t(Aut(F_l); \mathbb{Z}(p)) =

$$\begin{cases} (l-p+3)\mathbb{Z}/p & t \equiv 0 \pmod{n} \\ \sum_{k=0}^{l-p+1} \hat{H}^*(Q_k \times Q_{l-p+1-k}; \mathbb{Z}/p) & t \equiv r \pmod{n}, 2 \leq r \leq 2(l-p) + 1 \\ \oplus \hat{H}^*(Q_{l-p+2}; \mathbb{Z}/p) & \text{otherwise} \end{cases}$$
The aforementioned results of Glover and Mislin and Chen give more information than our results here, however, as they actually explicitly calculate the cohomology groups
\[ \sum_{k=0}^{t-p+1} H^r(\tilde{Q}_k \times Q_{t-p+1-k}; \mathbb{Z}/p) \oplus H^r(Q_{t-p+2}; \mathbb{Z}/p) \]
that arise in the above formula in their cases.

C On the prime \( p = 3 \): Proof of Corollary 1.2

For the sake of having a concrete example, we calculate the cohomologies of all of the quotient spaces involved in Theorem 1.1 when \( p = 3 \). This was also done, independently, by Glover and Henn.

Proof of Corollary 1.2. Examining Theorem 1.1 reveals that we must show that none of the various groups \( H^r(Q_k; \mathbb{Z}/3) \), \( H^r(Q_k; \mathbb{Z}/3) \), and \( H^r(Q^\omega_3; \mathbb{Z}/3) \) (where \( k = 1, 2 \)) contribute any nonzero cohomology classes.

The groups \( H^r(Q_1; \mathbb{Z}/3) \), \( H^r(Q_2; \mathbb{Z}/3) \), \( H^r(\tilde{Q}_1; \mathbb{Z}/3) \), and \( H^r(\tilde{Q}_2; \mathbb{Z}/3) \) are all zero by Fact 7.2.

For \( H^r(Q^\omega_3; \mathbb{Z}/3) \) (see Definition 6.2 to recall the definitions related to this space), we note that the relevant marked graphs in \( Q^\omega_3 \cong X^A_4/N_{\text{Aut}(F_4)}(A) \) are those listed in component \((A)\) of section 4 in the paper [9] by Glover and Mislin, with the additional complication that a basepoint \(*\) can be added to the graphs in various places. However, most of these graphs have inessential edges (see [16]) under the action of \( N_{\text{Aut}(F_4)}(A) \), and are thus collapsed directly away when we reduce from \( Q^\omega_3 \) to the space \( Q^\omega_3 \). We list the graphs from [9] that give \( X^A_4/N_{\text{Aut}(F_4)}(A) \) here:

- \( R_4 \). The rose has no inessential edges, even when you attach the basepoint \(*\) to the middle of one of the petals.
- \( \Theta_4 \). This \( \Theta \)-graph has no inessential edges, regardless of where the basepoint is attached.
- \( W_3 \vee R_1 \). In our notation, this would be \( \Xi_3 \vee R_1 \). The inessential edges in the “spokes” of the graph \( W_3 \) are collapsed and reduce this graph to \( R_4 \).
- \( \Theta_3 \ast R_1 \). This is \( \Theta_3 \) with a loop \( R_1 \) attached to the middle of one of the edges of the \( \Theta \)-graph. The edge of the \( \Theta \)-graph that the loop is attached to is inessential. (It will be 2 or 3 actual edges in the resulting graph, all of which are inessential, depending upon where the basepoint is placed.) Collapsing the inessential edges yields \( R_4 \).
• $\Theta_2 \circ \text{Y}$. A graph in the shape of a letter Y attached to the graph $\Theta_2$, with the top vertices of the Y attached to one side of the $\Theta$-graph, and the bottom vertex of the Y attached to the other side of the $\Theta$-graph. The bottom edge of the Y is inessential. Collapsing this gives $\Theta_4$.

• $\Theta_2 \ast \ast \Theta_1$. Two $\Theta$-graphs with a line drawn from the left vertex of one to the left vertex of the other, and a line drawn from the right vertex of one to the right vertex of the other. The new lines drawn are inessential edges, and can be collapsed away to yield $\Theta_4$.

So we are left with 4 basepointed graphs. Two come from the rose $R_4$, depending upon where we place the basepoint, and the other two come from $\Theta_4$ in a similar manner. In particular, only $\Theta_4$ (with the basepoint $\ast$ placed in the middle of one of its edges) can contribute a 2-simplex to our complex, and the relevant marked graph only has one maximal subforest (up to an isomorphism of the graph). Hence it contributes exactly two 2-simplices, which join together to form a square. Consequently it is clear that $H^2(Q_3^\omega; \mathbb{Z}/3) = 0$, which is all we needed to show to prove that $Q_3^\omega$ contributes nothing more to our cohomology calculations.

For the final case of considering the contributions of $H^3(\tilde{Q}_2; \mathbb{Z}/3)$, we again use arguments like those in [12] and Proposition 10.3 of [14]. We show that all of the 3-simplices in $\tilde{Q}_2$ can be collapsed away, so that $H^3(\tilde{Q}_2; \mathbb{Z}/3)$ is necessarily zero. The relevant graphs which can give 3-simplices are listed in Figure C.1. For each graph, the filled-in dot is the basepoint, but the open circle $\circ$ is the other “distinguished point” of the graph and indicates where a $\Theta$-graph $\Theta_2$ should be attached.

The first of these graphs has four subforests $\{b,c,e\}$, $\{b,c,d\}$, $\{b,d,e\}$, and $\{c,d,e\}$, each of which gives a 3-dimensional cube. These can be collapsed in a manner similar to that described in Proposition 10.3 in [14]. That is, the cube corresponding to the first subforest $\{b,c,e\}$ has a free minusface obtained by collapsing b. So we can collapse the interior of this cube away
from that face. Then the cube \{b, c, d\} has a free plusface corresponding to \{b, c\} and \{b, d, e\} has a free plusface corresponding to \{b, e\}. Both of these cubes can be collapsed away from those respective plusfaces. This leaves the cube corresponding to \{c, d, e\} with all plusfaces free. Thus we can disregard the first of the graphs in Figure C.1. (For a more detailed description of what plusfaces and minusfaces are, along with several more examples, see Chapter 10 of [14].

The second of the graphs that give 3-simplices contributes 4 cubes, one for each of the subforests \{a, d, e\}, \{a, b, d\}, \{a, c, d\}, and \{a, c, e\}. The cube corresponding to \{a, d, e\} has a free minusface given by collapsing the edge a. So we can disregard this cube. The remaining 3 cubes join together to form a solid 3-ball as described in the proof of Proposition 10.3 of [14], and so can also be collapsed away. □

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