Product cones in dense pairs

Pantelis E. Eleftheriou

School of Mathematics, University of Leeds, Leeds LS2 9JT, United Kingdom

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Let $M = \langle M, <, +, \ldots \rangle$ be an o-minimal expansion of an ordered group, and $P \subseteq M$ a dense set such that certain tameness conditions hold. We introduce the notion of a product cone in $\tilde{M} = \langle M, P \rangle$, and prove: if $M$ expands a real closed field, then $\tilde{M}$ admits a product cone decomposition. If $M$ is linear, then it does not. In particular, we settle a question from [10].

1 Introduction

Tame expansions $\tilde{M} = \langle M, P \rangle$ of an o-minimal structure $M$ by a set $P \subseteq M$ have received lots of attention in recent literature (cf. [1–4, 6, 7, 12, 14]). One important category is when every definable open set is already definable in $M$. Dense pairs and expansions of $M$ by a dense independent set or by a dense multiplicative group with the Mann Property are of this sort. In [10], all these examples were put under a common perspective and a cone decomposition theorem was proved for their definable sets and functions. This theorem provided an analogue of the cell decomposition theorem for o-minimal structures in this context, and was inspired by the cone decomposition theorem established for semi-bounded o-minimal structures (cf. [8, 9, 15]). The central notion is that of a cone, and, as its definition in [10] appeared to be quite technical, in [10, Question 5.14], we asked whether it can be simplified in two specific ways. In this paper we refute both ways in general, showing that the definition in [10] is optimal, but prove that if $M$ expands a real closed field, then a product cone decomposition theorem does hold.

In § 2, we provide all necessary background and definitions. For now, let us only point out the difference between product cones and cones, and state our main theorem. Let $M = \langle M, <, +, \ldots \rangle$ be an o-minimal expansion of an ordered group in the language $L$, and $\tilde{M} = \langle M, P \rangle$ an expansion of $M$ by a set $P \subseteq M$ such that certain tameness conditions hold (these are listed in § 2). For example, $\tilde{M}$ can be a dense pair (cf. [6]), or $P$ can be a dense independent set (cf. [5]) or a multiplicative group with the Mann Property (cf. [7]). By ‘definable’ we mean ‘definable in $\tilde{M}$’, and by $L$-definable we mean ‘definable in $M$’. The notion of a small set is given in Definition 2.1 below, and it is equivalent to the classical notion of being $P$-internal from geometric stability theory ([10, Lemma 3.11 & Corollary 3.12]). A supercone generalizes the notion of being co-small in an interval (Definition 2.2). Now, and roughly speaking, a cone is then defined as a set of the form

$$h\left(\bigcup_{g \in S} \{g\} \times J_g\right),$$

where $h$ is an $L$-definable continuous map with each $h(g, -)$ injective, $S \subseteq M^m$ is a small set, and $\{J_g\}_{g \in S}$ is a definable family of supercones. In Definition 2.4 below, we call a cone a product cone if we can replace the above family $\{J_g\}_{g \in S}$ by a product $S \times J$. That is, $C$ has the form

$$h(S \times J),$$

with $h$ and $S$ as above and $J$ a supercone. Let us say that $\tilde{M}$ admits a product cone decomposition if every definable set is a finite union of product cones. Our main theorem below asserts whether $\tilde{M}$ admits a product cone decomposition or not based solely on assumptions on $M$. Recall that $M$ is linear if it is an expansion of an ordered group and every definable function is piecewise affine (Definition 3.1).

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The presence of two different notions of cones in this setting, the semi-bounded cones (from [9]) and the current ones, a bounded interval but not on the whole of $\mathbb{R}$. In the ‘intermediate’, semi-bounded case (cf. [9]), where $\mathcal{M}$ defines a field on a bounded interval but not on the whole of $\mathcal{M}$, the answer to [10, Question 5.14] is rather unclear. Indeed, in the presence of two different notions of cones in this setting, the semi-bounded cones (from [9]) and the current ones, the methods in §§ 3.1 & 3.2 do not seem to apply and a new approach is needed.

**Remark 1.2** Theorem 1.1 deals with the two main categories of o-minimal structures; namely, $\mathcal{M}$ is linear or it expands a real closed field. In the ‘intermediate’, semi-bounded case (cf. [9]), where $\mathcal{M}$ defines a field on a bounded interval but not on the whole of $\mathcal{M}$, the answer to [10, Question 5.14] is rather unclear. Indeed, in the presence of two different notions of cones in this setting, the semi-bounded cones (from [9]) and the current ones, the methods in §§ 3.1 & 3.2 do not seem to apply and a new approach is needed.

**Notation** The topological closure of a set $X \subseteq \mathbb{M}$ is denoted by $cl(X)$. Given any subset $X \subseteq \mathbb{M}^n \times \mathbb{M}^n$ and $a \in \mathbb{M}^n$, we write $X_a$ for

$$\{b \in \mathbb{M}^n : (b, a) \in X\}.$$  

If $m \leq n$, then $\pi_m : \mathbb{M}^n \to \mathbb{M}^m$ denotes the projection onto the first $m$ coordinates. We write $\pi$ for $\pi_{n-1}$, unless stated otherwise. A family $\mathcal{J} = \{J_g\}_{g \in S}$ of sets is called definable if $\bigcup_{g \in S} \{g\} \times J_g$ is definable. We often identify $\mathcal{J}$ with $\bigcup_{g \in S} \{g\} \times J_g$.

## 2 Preliminaries

In this section we lay out all necessary background and terminology. Most of it is extracted from [10, § 2], where the reader is referred to for an extensive account. We fix an o-minimal theory $T$ expanding the theory of ordered abelian groups with a distinguished positive element 1. We denote by $L$ the language of $T$ and by $L(P)$ the language $L$ augmented by a unary predicate symbol $P$. Let $\tilde{T}$ be an $L(P)$-theory extending $T$. If $\mathcal{M} = \langle M, <, +, \ldots \rangle \models \tilde{T}$, then $\tilde{\mathcal{M}} = \langle \mathcal{M}, P \rangle$ denotes an expansion of $\mathcal{M}$ that models $\tilde{T}$. By ‘$A$-definable’ we mean ‘definable in $\tilde{\mathcal{M}}$ with parameters from $A$’. By ‘$L_A$-definable’ we mean ‘definable in $\tilde{\mathcal{M}}$ with parameters from $A$’. We omit the index $A$ if we do not want to specify the parameters. For a subset $A \subseteq \mathcal{M}$, we write $dcl(A)$ for the definable closure of $A$ in $\mathcal{M}$, and for an $L$-definable set $X \subseteq \mathbb{M}$, we write $dcl(X)$ for the corresponding pregeometric dimension.

The following definition is taken essentially from [7].

**Definition 2.1** Let $X \subseteq \mathbb{M}^n$ be a definable set. We call $X$ large if there is some $m$ and an $L$-definable function $f : \mathbb{M}^m \to \mathcal{M}$ such that $f(X^m)$ contains an open interval in $\mathcal{M}$. We call $X$ small if it is not large. We call $X$ co-small in a definable set $Y$, if $Y \mathcal{M} \mathcal{N}$ is small.

Consider the following Tameness Conditions (cf. [10]):

(I) $P$ is small.

(II) Every $A$-definable set $X \subseteq \mathbb{M}^n$ is a boolean combination of sets of the form

$$\{x \in \mathbb{M}^n : \exists z \in \mathbb{P}^m \varphi(x, z)\},$$

where $\varphi(x, z)$ is an $L_A$-formula.

(III) (Open definable sets are $L$-definable) For every parameter set $A$ such that $A \setminus P$ is $dcl$-independent over $P$, and for every $A$-definable set $V \subseteq \mathbb{M}$, its topological closure $cl(V) \subseteq \mathbb{M}$ is $L_A$-definable.

*From now on, we assume that every model $\tilde{\mathcal{M}} \models \tilde{T}$ satisfies Conditions (I)-(III) above. We fix a sufficiently saturated model $\tilde{\mathcal{M}} = \langle \mathcal{M}, P \rangle \models \tilde{T}$.*

We next turn to define the central notions of this paper. As mentioned in the introduction, the notion of a cone is based on that of a supercone, which in turn generalizes the notion of being co-small in an interval. Both notions,
supercones and cones, are unions of specific families of sets, which not only are definable, but they are so in a very uniform way.

**Definition 2.2** ([10]) A supercone $J \subseteq M^k$, $k \geq 0$, and its shell $sh(J)$ are defined recursively as follows:

1. $M^0 = \{0\}$ is a supercone, and $sh(M^0) = M^0$.
2. A definable set $J \subseteq M^{n+1}$ is a supercone if $\pi(J) \subseteq M^n$ is a supercone and there are $L$-definable continuous maps $h_1, h_2 : sh(\pi(J)) \to M \cup \{\pm \infty\}$ with $h_1 < h_2$, such that for every $a \in \pi(J)$, $J_a$ is contained in $(h_1(a), h_2(a))$ and it is co-small in it. We let $sh(J) = (h_1, h_2)_{sh(\pi(J))}$.

Abusing terminology, we call a supercone $A$-definable if it is an $A$-definable set and its closure is $L_A$-definable.

Note that, for $k > 0$, $sh(J)$ is the unique open cell in $M^k$ such that $cl(sh(J)) = cl(J)$. That is, $sh(J)$ is the interior of $cl(J)$. In particular, if $J$ is $A$-definable, then all defining maps $h_1, h_2$ used in its recursive definition are $L_A$-definable.

Recall that in our notation we identify a family $\mathcal{J} = \{ J_g \}_{g \in S}$ with $\bigcup_{g \in S} \{ g \} \times J_g$. In particular, $cl(\mathcal{J})$ and $\pi_n(\mathcal{J})$ denote the closure and a projection of that set, respectively.

**Definition 2.3** (Uniform families of supercones [10]) Let $\mathcal{J} = \bigcup_{g \in S} \{ g \} \times J_g \subseteq M^{m+k}$ be a definable family of supercones (so $S \subseteq M^m$, and $J_g \subseteq M^k$, $g \in S$, are supercones). We call $\mathcal{J}$ uniform if there is a cell $V \subseteq M^{m+k}$ containing $\mathcal{J}$, such that for every $g \in S$ and $0 < j \leq k$,

$$cl(\pi_{m+j}(\mathcal{J})_g) = cl(\pi_{m+j}(V)_g).$$

We call such a $V$ a shell for $\mathcal{J}$. Abusing terminology, we call $\mathcal{J}$ $A$-definable, if it is an $A$-definable family of sets and has an $L_A$-definable shell.

In case $S$ is a singleton, $\mathcal{J}$ can be identified with a supercone, and its shell with the shell from Definition 2.2 (after projecting on the last $k$ coordinates).

In particular, if $\mathcal{J}$ is uniform, then so is each projection $\pi_{m+j}(\mathcal{J})$. Moreover, if $V$ is a shell for $\mathcal{J}$, then $\pi_{m+j}(V)$ is a shell for $\pi_{m+j}(\mathcal{J})$. Observe also that if $V$ is a shell for $\mathcal{J}$, then for every $x \in \pi_{m+k-1}(\mathcal{J})$, $\mathcal{J}_x$ is co-small in $V_x$.

A shell for $\mathcal{J}$ need not be unique. Whenever we say that $\mathcal{J}$ is a uniform family of supercones with shell $V$, we just mean that $V$ is a shell for $\mathcal{J}$.

**Definition 2.4** (Cones [10] and product cones) A set $C \subseteq M^n$ is a $k$-cone, $k \geq 0$, if there are a definable small $S \subseteq M^m$, a uniform family $\mathcal{J} = \{ J_g \}_{g \in S}$ of supercones in $M^k$, and an $L$-definable continuous function $h : V \subseteq M^{m+k} \to M^n$, where $V$ is a shell for $\mathcal{J}$, such that

1. $C = h(\mathcal{J})$, and
2. for every $g \in S$, $h(g, -) : V_g \subseteq M^k \to M^n$ is injective.

We call $C$ a $k$-product cone if, moreover, $\mathcal{J} = S \times J$, for some supercone $J \subseteq M^k$. A (product) cone is a $k$-(product) cone for some $k$. Abusing terminology, we call a (product) cone $h(\mathcal{J})$ $A$-definable if $h$ is $L_A$-definable and $\mathcal{J}$ is $A$-definable.

The cone decomposition theorem below (Fact 2.6) is a statement about definable sets and functions. The notion of a ‘well-behaved’ function in this setting is given next.

**Definition 2.5** (Fiber $L$-definable maps [10]) Let $C = h(\mathcal{J}) \subseteq M^n$ be a $k$-cone with $\mathcal{J} \subseteq M^{m+k}$, and $f : D \to M$ a definable function with $C \subseteq D$. We say that $f$ is fiber $L$-definable with respect to $C$ if there is an $L$-definable continuous function $F : V \subseteq M^{m+k} \to M$, where $V$ is a shell for $\mathcal{J}$, such that

$$(f \circ h)(x) = F(x), \text{ for all } x \in \mathcal{J}.$$

We call $f$ fiber $L_A$-definable with respect to $C$ if $F$ is $L_A$-definable.

As remarked in [10, Remark 4.5(4)], the terminology is justified by the fact that, if $f$ is fiber $L_A$-definable with respect to $C = h(\mathcal{J})$, then for every $g \in \pi(\mathcal{J})$, $f$ agrees on $h(g, J_g)$ with an $L_{A_g}$-definable map; namely $F \circ h(g, -)^{-1}$. Moreover, the notion of being fiber $L$-definable with respect to a cone $C = h(\mathcal{J})$, depends on $h$ and $\mathcal{J}$ ([10, Example 4.6]). However, it is immediate from the definition that if $f$ is fiber $L_A$-definable with respect
to a cone $C = h(J)$, and $h(J') \subseteq h(J)$ is another cone (but with the same $h$), then $f$ is also fiber $L_A$-definable with respect to it.

We are now ready to state the cone decomposition theorem from [10].

**Fact 2.6** (Cone decomposition theorem [10, Theorem 5.1])

1. Let $X \subseteq M^n$ be an $A$-definable set. Then $X$ is a finite union of $A$-definable cones.
2. Let $f : X \to M$ be an $A$-definable function. Then there is a finite collection $C$ of $A$-definable cones, whose union is $X$ and such that $f$ is fiber $L_A$-definable with respect to each cone in $C$.

Another important notion from [10] is that of ‘large dimension’, which we recall next. The proof of Theorem 1.1(2) runs by induction on large dimension.

**Definition 2.7** (Large dimension [10]) Let $X \subseteq M^n$ be definable. If $X \neq \emptyset$, the large dimension of $X$ is the maximum $k \in \mathbb{N}$ such that $X$ contains a $k$-cone. The large dimension of the empty set is defined to be $-\infty$. We denote the large dimension of $X$ by $\operatorname{ldim}(X)$.

**Remark 2.8** The tameness conditions that we assume in this paper guarantee that the notion of large dimension is well-defined; namely, the above maximum $k$ always exists ([10, § 4.3]).

### 3 Product cone decompositions

In this section we prove Theorem 1.1.

#### 3.1 The linear case

The following definition is taken from [13].

**Definition 3.1** ([13]) Let $\mathcal{N} = \langle N, +, <, 0, \ldots \rangle$ be an o-minimal expansion of an ordered group. A function $f : A \subseteq N^n \to N$ is called affine, if for every $x, y, x + t, y + t \in A$,

$$f(x + t) - f(x) = f(y + t) - f(y).$$

We call $\mathcal{N}$ linear if every definable $f : A \subseteq N^n \to N$ is piecewise affine, namely if there is a partition of $A$ into finitely many definable sets $B$, such that each $f_B$ is affine.

The typical example of a linear o-minimal structure is an ordered vector space $\mathcal{V} = \langle V, <, +, 0, \{d\}_{d \in D} \rangle$ over an ordered division ring $D$. In general, if $\mathcal{N}$ is linear, then there exists a reduct $\mathcal{S}$ of such $\mathcal{V}$, such that $\mathcal{S} \equiv \mathcal{N}$ (cf. [13] for details). Using this description, it is not hard to see that every affine function has a continuous extension to the closure of its domain.

Assume now that our fixed structure $\mathcal{M}$ is linear.

**Lemma 3.2** Let $h : [a, b] \times [c, d] \to M$ be an $L$-definable continuous function, such that for every $t \in (a, b)$, $h(t, \cdot) : [c, d] \to M$ is strictly increasing. Then

$$h(b, d) - h(b, c) > 0.$$

**Proof.** Let $\mathcal{W}$ be a cell decomposition of $[a, b] \times [c, d]$ such that for every $W \in \mathcal{W}$, $h|_W$ is affine. Since $d - c > 0$, there must be some $W = (f, g) \in \mathcal{W}$, where $I$ is an interval with $\sup I = b$, and $r \in I$, such that the map $\delta(t) := g(t) - f(t)$ is increasing on $[r, b)$. We claim that for every $t \in (r, b)$,

$$h(t, g(t)) - h(t, f(t)) \geq h(r, g(r)) - h(r, f(r)).$$

Indeed, there is $k \geq 0$, such that

$$h(t, f(t) + \delta(t)) - h(t, f(t)) = h(t, f(t) + \delta(r) + k) - h(t, f(t))$$

$$= h(t, f(t) + \delta(r) + k) + h(t, f(t) + \delta(r))$$

$$- h(t, f(t) + \delta(r)) + h(t, f(t))$$
Since the origin is in the closure of \( H \), the origin is in its closure. Since \( 4.27 \). Now let \( t \) may assume that the latter is always strictly increasing. By \([10, Lemma 5.10]\) applied to \( \lim \)

where the inequality holds because \( h(t, -) \) is increasing, and the last equality holds because \( h \) is affine on \( W \). We conclude that

\[
    h(b, d) - h(b, c) = \lim_{t \to b} (h(t, d) - h(t, c)) \geq \lim_{t \to b} (h(t, g(t)) - h(t, f(t))) \geq h(r, g(r)) - h(r, f(r)) \leq 0,
\]

where the first and last inequalities hold because \( h(t, -) \) and \( h(r, -) \) are strictly increasing. \( \Box \)

**Counterexample to product cone decomposition** Let \( S \subseteq M \) be a small set such that 0 is in the interior of its closure (by translating \( P \) to the origin, such an \( S \) exists). Let

\[ X = \bigcup_{a \in S^0} \{a\} \times (0, a). \]

**Claim 3.3** \( X \) is not a finite union of product cones.

**Proof.** First of all, \( X \) cannot contain any \( k \)-cones for \( k > 1 \), since \( \text{ldim}(X) = 1 \), by [10, Lemmas 4.24 & 4.27]. Now let \( H(T \times J) \) be an 1-product cone contained in \( X \), with \( H = (H_1, H_2) : Z \subseteq M^{l^1 + 1} \to M^2 \), such that the origin is in its closure. Since \( H \) is \( L \)-definable and continuous, and for each \( g \in T \), \( H_2(g, -) \) is injective, we may assume that the latter is always strictly increasing. By [10, Lemma 5.10] applied to \( J \), \( f(-) = \pi_1 H(g, -) \) and \( S \), we have

for every \( g \in T \), there is \( a \in S \), such that \( H(g, J) \subseteq \{a\} \times (0, a) \).

By continuity of \( H \), it follows that

for every \( g \in \text{cl}(T) \cap \pi(Z) \), there is \( a \in M \), such that \( H(g, \text{cl}(J)) \subseteq \{a\} \times [0, a] \).

Let \( F : \pi(Z) \to M \) be the \( L \)-definable map given by

\[ F(g) = \pi_1(H(g, \text{cl}(J))). \]

Since the origin is in the closure of \( H(T \times J) \), there must be an affine \( \gamma : (a, b) \to \text{cl}(T) \cap \pi(Z) \) with \( \lim_{t \to b} F(\gamma(t)) = 0 \). Fix any \( [c, d] \subseteq \text{cl}(J) \). Now the map

\[ H_2(\gamma(-), -) : (a, b) \times (c, d) \to M \]

is piecewise affine and hence has a continuous extension \( h \) to \( [a, b] \times [c, d] \). By definition of \( X \),

\[ h(b, c) = h(b, d) = 0. \]

But, by Lemma 3.2,

\[ h(b, d) - h(b, c) > 0, \]

a contradiction. Since \( X \) contains no product cone whose closure contains the origin, \( X \) cannot be a finite union of product cones. \( \Box \)

**3.2 The field case**

We now assume that \( \mathcal{M} \) expands an ordered field. The main idea behind the proof in this case is as follows. By Fact 2.6, it suffices to write every cone as a finite union of product cones. We illustrate the case of a 1-cone \( C = h(J) \), for some \( J = \{J_k\}_{k \in S} \).
Step I (Lemma 3.4). Replace $\mathcal{J}$ by a cone $\mathcal{J}' = \{J'_g\}_{g \in S}$, such that for some fixed interval $I$, each $J'_g$ is contained in $I$ and it is co-small in it. Here we use the field structure of $\mathcal{M}$, so this step would fail in the linear case.

Step II (Lemma 3.5). By [10, Lemma 4.25], the intersection $J = \bigcap_{g \in S} J'_g$ is co-small in $I$. Moreover, if we let $L = S \times J$, then, by [10, Lemma 4.29], we obtain that the large dimension of $\mathcal{J} \setminus L$ is 0.

Step III (Theorem 3.6). Use Steps I and II and induction on large dimension. Here, the inductive hypothesis is only applied to sets of large dimension 0. In general, $\text{ldim}(\mathcal{J} \setminus L) < \text{ldim}(\mathcal{J})$.

To achieve Step I, we first need to make an observation and fix some notation. Using the field operations, one can define an $L_0$-definable continuous $f : M^2 \to M$, such that for every $b, c \in M$,

$$f(b, c, -) : (b, c) \to (0, 1)$$

is a bijection. Similarly, there are $L_0$-definable continuous maps $f_1, f_2 : M^2 \to M$, such that for every $b, c \in M$, the maps

$$f_1(b, -) : (b, +\infty) \to (0, 1)$$

and

$$f_2(c, -) : (-\infty, c) \to (0, 1)$$

are bijections. To give all these maps a uniform notation, we write $f(b, +\infty, x)$ for $f_1(b, x)$, and $f(-\infty, c, x)$ for $f_2(c, x)$. We fix this $f$ for the next proof. Observe that if $J \subseteq (b, c)$ is co-small in $(b, c)$, for $b, c \in M \cup \{\pm \infty\}$, then $f(b, c, J)$ is co-small in $(0, 1)$.

**Lemma 3.4** Let $\mathcal{J} = \bigcup_{g \in S} \{J_g\} \subseteq M^{m+k}$ be an $A$-definable uniform family of supercones, with shell $Z \subseteq M^{m+k}$. Then there are

1. an $A$-definable uniform family $\mathcal{J}' = \{J'_g\}_{g \in S}$ of supercones $J'_g \subseteq M^k$, with shell $\pi(Z) \times (0, 1)^k$,
2. an $L_A$-definable continuous and injective map $F : Z \to M^{m+k}$, such that $F(\mathcal{J}) = \mathcal{J}'$.

**Proof.** For every $g \in \pi_m(\mathcal{J})$, since $J_g$ is a supercone, it follows that $Z_g$ is an open cell. Hence, for every $0 < j \leq k$, there are $L_A$-definable continuous maps $h^1_j, h^2_j : \pi_{m+j-1}(Z) \to M$ such that

$$\pi_{m+j}(Z) = (h^1_j, h^2_j)_{\pi_{m+j-1}(Z)}.$$

We define

$$F = (F_1, \ldots, F_{m+k}) : Z \to M^{m+k},$$

as follows. Let $I = (0, 1)$ and $f$ be the map fixed above. Let $(g, t) \in Z \subseteq M^{m+k}$. If $1 \leq i \leq m$,

$$F_i(g, t) = g_i$$

(the $i$th coordinate of $g$). If $i = m + j$, with $0 < j \leq k$,

$$F_{m+j}(g, t) = f(h^1_j(g, t_1, \ldots, t_{j-1}), h^2_j(g, t_1, \ldots, t_{j-1}, t_j)).$$

Clearly, $F$ is injective, $L_A$-definable and continuous. Let

$$\mathcal{J}' = F(\mathcal{J}).$$

That is, $\mathcal{J}' = \{J'_g\}_{g \in S}$, where for every $g \in S$, $J'_g = F(g, J_g)$. It is not hard to check, by induction on $m$, that for every $0 < m \leq k$, $\pi_{m+j}(\mathcal{J}')$ is an $A$-definable uniform family of supercones with shell $F(Z) = \pi(Z) \times I^m$. □

**Lemma 3.5** Let $\mathcal{J} = \bigcup_{g \in S} \{J_g\} \subseteq M^{m+k}$ be an $A$-definable uniform family of supercones in $M^k$ with shell $Z$, and assume $S \subseteq M^m$ is small. Suppose that $Z = \pi(Z) \times I^j$, where $I = (0, 1)$. Then $\mathcal{J}$ is a disjoint union

$$(S \times J) \cup Y,$$

where $S \times J$ is an $A$-definable uniform family of supercones with shell $Z$, and $Y$ is an $A$-definable set of large dimension $< k$. 

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Proof. By induction on \( k \). For \( k = 0 \), the statement is trivial. We assume the statement holds for \( k \), and prove it for \( k + 1 \). Let \( \pi : M^{m+k+1} \to M^{m+k} \) be the projection onto the first \( m + k \) coordinates. Since \( \pi(J) \) is also an \( A \)-definable uniform family of supercones with shell \( \pi(Z) \), by inductive hypothesis we can write \( \pi(J) \) as a disjoint union

\[
\pi(J) = (S \times T) \cup Y,
\]

where \( T \subseteq M^{k} \) is an \( A \)-definable supercone with \( cl(T) = cl(I^{k}) \), and \( Y \) is an \( A \)-definable set of large dimension \(< k \). By [10, Corollary 5.5], the set \( \bigcup_{t \in T} \{ t \} \times J_t \subseteq J \) has large dimension \(< k + 1 \), and hence we only need to bring its complement \( X \) in \( J \) into the desired form. We have

\[
X = \bigcup_{t \in S \times T} \{ t \} \times J_t.
\]

Define, for every \( a \in T \),

\[
K_{a} = \bigcap_{g \in S} J_{g,a}.
\]

Since each \( J_{g,a} \) is co-small in \( I \), by [10, Lemma 4.25] \( K_{a} \) is co-small in \( I \). Hence, the set

\[
L = \bigcup_{a \in T} \{ a \} \times K_{a}
\]

is a supercone in \( M^{k+1} \). Since \( cl(T) = cl(I^{k}) \) and for each \( a \in T \), \( cl(K_{a}) = cl(I) \), it follows that \( cl(L) = cl(I^{k+1}) \). In particular, \( Z \) is a shell for \( S \times L \). Since \( S \times L \subseteq X \), it remains to prove that \( ldim(X \setminus (S \times L)) < k + 1 \). We have

\[
X \setminus (S \times L) = \bigcup_{(g,a) \in S \times T} \{ (g,a) \} \times (J_{g,a} \setminus K_{a}).
\]

But \( J_{g,a} \setminus K_{a} \) is small, and hence, by [10, Lemma 4.29], the above set has large dimension \( = ldim(S \times T) = k \). \qed

We can now conclude the main theorem of the paper.

**Theorem 3.6** (Product cone decomposition in the field case) Let \( X \subseteq M^{n} \) be an \( A \)-definable set. Then

1. \( X \) is a finite union of \( A \)-definable product cones.

2. If \( f : X \to M \) is an \( A \)-definable function, then there is a finite collection \( C \) of \( A \)-definable product cones, whose union is \( X \) and such that \( f \) is fiber \( L_{A} \)-definable with respect to each cone in \( C \).

**Proof.** (1) By induction on the large dimension of \( X \). Suppose \( ldim(X) = k \). By Fact 2.6, we may assume that \( X \) is a \( k \)-cone. Every 0-cone is clearly a product cone. Now let \( k > 0 \). By induction, it suffices to write \( X \) as a union of an \( A \)-definable product cone and an \( A \)-definable set of large dimension \(< k \). Let \( X = h(J) \) be as in Definition 2.4, and \( Z \subseteq M^{m+k} \) a shell for \( J \).

**Claim** We can write \( X \) as a \( k \)-cone \( h'(J') \), such that for every \( g \in \pi(J') \), \( cl(J') \cap (0,1)^{k} = (0,1)^{k} \).

**Proof of Claim.** Let \( J' \) and \( F : Z \to M^{m+k} \) be as in Lemma 3.4, and define \( h' = h \circ F^{-1} : F(Z) \to M^{n} \). Then

\[
h(J) = hF^{-1}(F(J)) = h'(J')
\]

is as required. \qed

By the claim, we may assume that for every \( g \in S \), \( cl(J') \cap (0,1)^{k} = (0,1)^{k} \). By Lemma 3.5, we have \( J = (S \times J) \cup Y \), where \( J \subseteq M^{k} \) is an \( A \)-definable supercone, and \( ldim Y < k \). Thus \( h(J) = h(S \times J) \cup h(Y) \) has been written in the desired form.

(2) By Fact 2.6, we may assume that \( X \) is a \( k \)-cone and that \( f \) is fiber \( L_{A} \)-definable with respect to it. So let again \( X = h(J) \) with shell \( Z \subseteq M^{m+k} \), and in addition, \( \tau : Z \subseteq M^{m+k} \to M \) with \( J \subseteq Z \), be \( L_{A} \)-definable so that for every \( x \in J \),

\[
(f \circ h)(x) = \tau(x).
\]
By induction on large dimension, it suffices to show that $X$ is the union of a product cone $C$ and a set of large dimension $< k$, such that $f$ is fiber $L_{A^a}$-definable with respect to $C$. Let $X = h'(\mathcal{J}')$ be as in Claim of (1) and $F : Z \to M^{m+k}$ as in its proof. So $h' = h \circ F^{-1} : F(Z) \to M^n$. Define $\tau' : F(Z) \to M^n$ as $\tau' = \tau \circ F^{-1}$. We then have, for every $x' \in \mathcal{J}'$,

$$\tau'(x') = f h'(x') = f h(x) = \tau(x) = \tau F^{-1}(x) = \tau(x),$$

witnessing that $f$ is fiber $L_{A^a}$-definable with respect to $h'(\mathcal{J}')$.

Therefore, we may replace $h$ by $h$ and $\mathcal{J}$ by $\mathcal{J}'$. Now, as in the proof of (1), we can write $h(\mathcal{J})$ as the union of a product cone $h(S \times J)$ and a set of large dimension $< k$. By the remarks following Definition 2.5, $f$ is also fiber $L$-definable with respect to $h(S \times J)$. \hfill \Box

Remark 3.7 From the above proof it follows that in cases where we have disjoint unions in Fact 2.6 (as in [10, Theorem 5.12]), this is also the case in Theorem 3.6.

4 Refined supercones

In this section we answer [10, Question 5.14(1)] negatively. The question asked whether the Structure Theorem holds if we strengthen the notion of a supercone as follows.

Definition 4.1 A supercone $\mathcal{J}$ in $M^k$ is called refined if it is of the form

$$\mathcal{J} = J_1 \times \cdots \times J_k,$$

where each $J_i$ is a supercone in $M$. Let us call a $(k)$-cone $C = h(\mathcal{J})$ a $(k)$-refined cone if $\mathcal{J}$ is refined.

Our result is the following.\footnote{The proof is based on an idea suggested by Hieronymi.}

Proposition 4.2 Assume $\mathcal{M}$ expands a real closed field. Then there is a supercone in $M^2$ which contains no 2-refined cone. In particular, it is not a finite union of refined cones.

Proof. The ‘in particular’ clause follows from [10, Corollaries 4.26 & 4.27]. Now, for every $a \in M$, let

$$J_a = M \setminus (P + aP)$$

and define $\mathcal{J} = \bigcup_{a \in M} \{a\} \times J_a$. Towards a contradiction, assume that $\mathcal{J}$ contains a 2-refined cone. That is, there are supercones $J_1, J_2 \subseteq M$, an open cell $U \subseteq M^2$ with $cl(J_1 \times J_2) = cl(U)$, and an $L$-definable continuous and injective map $f : U \to M^2$, such that $C = f(J_1 \times J_2) \subseteq \mathcal{J}$. We write $X = f(U)$, and for each $a \in M$, $X_a \subseteq M$ for the fiber of $X$ above $a$. Suppose $C$ is $A$-definable.

By saturation, there is $a \in M$ which is dcl-independent over $A \cup P$, and further $g, h \in P$ which are dcl-independent over $a$.

So

$$\dim(g, h, a) = 3.$$

By assumption, there are $(p, q) \in U \setminus (J_1 \times J_2)$, such that

$$f(p, q) = (a, g + ha).$$

Observe that $a \in dcl(p, q)$. Also, one of $p, q$ must be in $dcl(AP)$. Indeed, we have $p \not\in J_1$ or $q \not\in J_2$. If, say, the former holds, then $p \in \pi(U) \setminus J_1$. Since the last set is $A$-definable and small, we obtain by [10, Lemma 3.11], that $p \in dcl(AP)$.

We may now assume that $p \in dcl(AP)$. If we write $f = (f_1, f_2)$, we obtain

$$f_2(p, q) = g + hf_1(p, q). \tag{2}$$

Since $a$ is dcl-independent over $A \cup P$, there must be an open interval $I \subseteq M$ of $p$, such that for every $x \in I$,

$$f_2(x, q) = g + hf_1(x, q).$$
Viewing both sides of the equation as functions in the variable $f_1(x, q)$, and taking their derivatives with respect to it, we obtain:

$$\frac{\partial f_2(x, q)}{\partial f_1(x, q)} = f_1(x, q) + h.$$  

Evaluated at $p$, the last equality gives $h \in \text{dcl}(p, q)$. By (2), also $g \in \text{dcl}(p, q)$. All together, we have proved that $g, h, a \in \text{dcl}(p, q)$. It follows that

$$\dim(g, h, a) \leq \dim(p, q) \leq 2,$$

a contradiction. \hfill $\Box$

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