B₀-VPG Representation of AT-free Outerplanar Graphs

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Abstract. A \(k\)-bend path is a non-self-intersecting polyline in the plane made of at most \(k+1\) axis-parallel line segments. \(B_k\)-VPG is the class of graphs which can be represented as intersection graphs of \(k\)-bend paths in the same plane. In this paper, we show that all AT-free outerplanar graphs are \(B_0\)-VPG, i.e., intersection graphs of horizontal and vertical line segments in the plane. Our proofs are constructive and give a polynomial time \(B_0\)-VPG drawing algorithm for the class. In fact, we show the existence of a \(B_0\)-VPG representation for a superclass of AT-free graphs namely linear outerplanar graphs which we define as the class of subgraphs of biconnected outerpaths. Outerpaths are outerplanar graphs which admit a planar drawing whose weak dual is a path.

Following a long line of improvements, Gonçalves, Isenmann, and Pennarun [SODA 2018] showed that all planar graphs are \(B_1\)-VPG. Since there are planar graphs which are not \(B_0\)-VPG, characterizing \(B_0\)-VPG graphs among planar graphs becomes interesting. Chaplick et al. [WG 2012] had shown that it is NP-complete to recognize \(B_k\)-VPG graphs within \(B_{k+1}\)-VPG. Hence recognizing \(B_0\)-VPG graphs within \(B_1\)-VPG is NP-complete in general, but the question is open when restricted to planar graphs. There are outerplanar graphs and AT-free planar graphs which are not \(B_0\)-VPG. This piqued our interest in AT-free outerplanar graphs.

Keywords: Outerplanar graphs . AT-free . \(B_0\)-VPG . Connectivity augmentation . Outerpath . Linear outerplanar graph . Graph drawing.

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A preliminary version of this work was presented at the 8th International Conference on Algorithms and Discrete Applied Mathematics (CALDAM) 2022 [22]. In this extended version, we show that a proper superclass of AT-free outerplanar graphs, which we name as linear outerplanar graphs, are \(B_0\)-VPG. We believe that this new class is interesting in its own right.

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1 Introduction

A $k$-bend path is a simple path in a two-dimensional grid with at most $k$ bends. Geometrically, it is a non-self-intersecting polyline in the plane made of at most $k + 1$ axis-parallel (horizontal or vertical) line segments. Vertex intersection graphs of Paths on a Grid (VPG) (resp., $B_k$-VPG) is the class of graphs which can be represented as intersection graphs of (resp., $k$-bend) paths in a two-dimensional grid. The bend number of a graph $G$ in VPG is the minimum $k$ for which $G$ is in $B_k$-VPG. One motivation to study $B_k$-VPG graphs comes from VLSI circuit design where the paths correspond to wires in the circuit. A natural concern in VLSI design is to reduce the number of bends in each path (wire) in the representation. A second motivation is that certain algorithmic tasks become easier when restricted to $B_1$-VPG or $B_0$-VPG graphs (cf. [26]).

Several results, some of which we describe in Section 1.3, concern the classification of planar graphs as $B_k$-VPG graphs. Following up on a series of results and conjectures by various authors, Gonçalves, Isenmann, and Pennarun in 2018 showed that all planar graphs are $B_1$-VPG [20]. This is tight since many simple planar graphs like the 4-wheel, 3-sun, and triangular prism, to name a few, are not $B_0$-VPG. This makes the question of characterizing $B_0$-VPG planar graphs very appealing. Characterizing $B_0$-VPG outerplanar graphs will be a good step in this direction since some of the structures that forbid a planar graph from being $B_0$-VPG are also present among outerplanar graphs. Outerplanar graphs were known to be $B_1$-VPG [7] before the same was shown for planar graphs. Chaplick et al. [9] showed that it is NP-complete to decide whether a given graph $G$ is in $B_k$-VPG even when $G$ is guaranteed to be in $B_{k+1}$-VPG. Hence recognizing $B_0$-VPG graphs within $B_1$-VPG is NP-complete in general, but the question is open when restricted to planar graphs or outerplanar graphs.

This article is an outcome of our effort to characterize $B_0$-VPG outerplanar graphs. One can see from the geometry that the closed neighborhood of every vertex in a $B_0$-VPG graph is an interval graph [18]. We strengthen this necessary condition (Proposition 14) by identifying adjacent vertices which are forced to be represented by collinear segments in any $B_0$-VPG drawing. But this is still not sufficient to characterize $B_0$-VPG outerplanar graphs (Figure 6). However, we were able to show that, if the outerplanar graph itself is AT-free, then it is $B_0$-VPG (Theorem 13). We cannot extend this result to AT-free planar graphs since we have examples of AT-free planar graphs, like the 4-wheel and triangular prism, which are not $B_0$-VPG.

While it is relatively easy to find a $B_0$-VPG drawing for biconnected AT-free outerplanar graphs, handling cutvertices turns out to be more challenging. Rather than trying to join $B_0$-VPG drawings of individual blocks, we found it easier to embed the given graph as an induced subgraph of a biconnected outerpath.

**Definition 1 (Outerpath)** An outerpath is an outerplanar graph which admits a planar embedding whose weak dual is a path.

Note that outerpaths need not be biconnected. All the five graphs in Figure 3.(a) are outerpaths.

Our proof has essentially two parts with biconnected outerpaths forming the bridge between the two. The first part is a structural result which shows that any AT-free outerplanar graph can be realized as an induced subgraph of a biconnected outerpath. The second part is a $B_0$-VPG drawing procedure for biconnected outerpaths. Both the parts have a potential to be generalized. Since $B_0$-VPG is easily seen to be hereditary class, the result naturally extends to all induced subgraphs of biconnected outerpaths. This prompted us to name this class (Definition 2) and study it on its own merit.
Definition 2 (Linear Outerplanar Graph) An outerplanar graph is linear if it is a subgraph of a biconnected outerpath.

Figure 1: A Linear Outerplanar Graph

We give a complete characterization of this class in Theorem 7. Though the characterization seems technical, it is very easy to visualize and gives a poly-time recognition algorithm. As a pleasant surprise, we also discover that every graph in this class can be realized both as an induced subgraph as well as a spanning subgraph of (different) biconnected outerpaths. Figure 1 shows an example of a linear outerplanar graph.

The second part of our proof, the drawing procedure for biconnected outerpaths, can also be extended to a larger class of graphs than biconnected outerpaths, but this we set aside for a future work.

1.1 Organization

After recalling some standard graph theoretic terminology in Section 1.2 and a brief literature review in Section 1.3, we layout our proofs in three sections. Section 2 has the $B_0$-VPG drawing procedure for biconnected outerpaths. Section 3 contains the characterization of linear outerplanar graphs. In Section 4, we prove that all $AT$-free outerplanar graphs are linear thereby completing the proof of the titular result. We conclude with Section 5, where we describe some necessary conditions for the existence of a $B_0$-VPG representation. This may help in characterizing $B_0$-VPG outerplanar graphs.

1.2 Terminology

The closed neighborhood $N[v]$ of a vertex $v$ in a graph $G$ is the set containing $v$ and and its neighbors in $G$. A set of three independent vertices is called an asteroidal triple (AT) when there exists a path between each pair of them containing no vertex from the closed neighborhood of the third vertex. An AT-free graph is a graph which does not have an AT.

A plane graph is an embedding of a planar graph in the plane with no crossing edges. Let $G$ be a plane graph. The dual of $G$ is a graph that has a vertex for each face of $G$ and an edge between two of its vertices when the corresponding faces of $G$ share an edge. The weak dual of $G$ is obtained from its dual by removing the vertex corresponding to the outer face. An edge of $G$ incident to the outer face of $G$ is called a boundary edge and its endpoints are called boundary neighbors of each other. The remaining edges of $G$ are called internal edges. A leaf face is a face with at most one internal edge. A planar graph is outerplanar if it has a plane embedding in which all the vertices are incident on the outer face. Such an embedding is called an outerplane embedding (drawing).
Remark 1 The terminology of faces, duals, weak duals, boundary edges and internal edges will be applied directly to an outerplanar graph assuming an outerplane drawing. Since each non-trivial block of an outerplanar graph has a unique outerplane drawing, these notions do not depend on the choice of the outerplane drawing.

Let $G$ be an outerplanar graph. The weak dual of $G$ is a forest [16] and we denote it by $\mathcal{T}_G$. Further, we call $\mathcal{T}_G$ a linear forest if each component in $\mathcal{T}_G$ is a path.

Let $G$ be a graph. A $k^+$-vertex in $G$ is a vertex having at least $k$ neighbors in $G$. A leaf edge is an edge having one endpoint of degree one. A subgraph $H$ of $G$ is spanning if $V(H) = V(G)$, and induced if $E(H) = \{xy \mid x,y \in V(H), xy \in E(G)\}$. A graph induced by a subset $S$ of the vertices of $G$ is denoted by $G[S]$. A subset of vertices in a graph is called a separator if its removal increases the number of components of the graph. A vertex $x$ is a cutvertex if $\{x\}$ is a separator. A graph is $k$-connected if it does not have a separator of size smaller than $k$. A graph is said to be connected (resp. biconnected) if it is 1-connected (resp. 2-connected). A block of a graph is a maximal biconnected subgraph of the graph. A trivial block is a block containing at most two vertices.

A graph $G$ is $H$-free if $G$ does not contain an induced subgraph isomorphic to $H$. A graph $G$ is said to be $H$-minor-free if it does not contain a minor isomorphic to $H$. We use $C_k$ to denote the simple cycle on $k$ vertices. A cycle on $k$ vertices $x_0, \ldots, x_{k-1}$ where each $x_i$ is adjacent to $x_{i+1}$ (addition is modulo $k$) can also be denoted as $x_0, \ldots, x_{k-1}, x_0$. A $C_4$ together with an additional vertex $v$ adjacent to all the vertices of $C_4$ is called a 4-wheel. A triangular prism is the complement of $C_6$. An interval graph is an intersection graph of a set of intervals on $\mathbb{R}$.

1.3 Literature

The class $B_k$-VPG was introduced by Asinowski et al. in 2012 [3]. Nevertheless, these graphs were previously studied in various forms. One such class of them is grid intersection graphs (GIG) which are bipartite graphs that can be represented as intersection graphs of horizontal and vertical line segments in the plane where no two parallel line segments intersect [21]. This class is equivalent to bipartite $B_0$-VPG graphs [3]. String graphs are intersection graphs of curves/strings in the plane. Asinowski et al. [3] showed that they are equivalent to VPG graphs and hence the NP-completeness of the recognition of VPG graphs follows from that of string graphs [24, 28]. The recognition problem for $B_0$-VPG graphs is NP-complete [25, 3]. $B_0$-VPG characterizations are known for block graphs [2], split graphs, chordal bull-free graphs, chordal claw-free graphs [18] and cocomparability graphs [27].

Segment intersection graphs are intersection graphs of line segments in the plane. Chalopin and Gonçalves in 2009 showed that every planar graph is a segment intersection graph [8], confirming a conjecture of Scheinerman from 1984 [29]. One way to refine the class of segment intersection graphs is to restrict the number of directions permitted for the segments. If the number of directions is limited to $k$, the resulting graph is called $k$-DIR. When $k = 2$, we rediscover $B_0$-VPG [3]. It is known that bipartite planar graphs are 2-DIR [21, 12, 14], and triangle-free planar graphs are 3-DIR [13]. West conjectured that any planar graph is 4-DIR [30] which was recently refuted by Gonçalves in 2020 [19]. Before the celebrated result by Gonçalves et al. that planar graphs are $B_3$-VPG [20], there were several results on $B_3$-VPG representation of planar graphs. Since 2-DIR graphs are equivalent to $B_0$-VPG, bipartite planar graphs are $B_0$-VPG. In [3], Asinowski et al. showed that planar graphs are $B_3$-VPG and conjectured that this is tight. Disproving this conjecture, Chaplick and Ueckerdt proved that planar graphs have a $B_2$-VPG representation [10]. This adds to the appeal for characterizing $B_0$-VPG planar graphs.
Outerpaths have many geometric representations like balanced circle-contact representations [1], geometric simultaneous embeddings with a matching [6], and partial geometric simultaneous embeddings with another outerpath [15]. All these representations will extend to linear outerplanar graphs because of Theorem 7 and Remark 3. Babu et al. provides an algorithm to augment outerplanar graphs of pathwidth \( p \) to biconnected outerplanar supergraphs of pathwidth \( \mathcal{O}(p) \) [4]. Connectivity augmentation of outerplanar graphs using minimum number of additional edges is studied in [17, 23]. Barát et al. have characterized the graphs with pathwidth at most two [5] and our class is a strict subclass of that.

### 2 B\(_0\)-VPG Representation of Biconnected Outerpaths

It’s drawing time! In this section, we show that every biconnected outerpath is B\(_0\)-VPG (Theorem 3). The proof of Theorem 3 is constructive and it draws a B\(_0\)-VPG representation for any biconnected outerpath (cf. Figure 2 for example). Since B\(_0\)-VPG is easily seen to be a hereditary graph class (closed under induced subgraphs), and since every linear outerplanar graph can be represented as an induced subgraph of a biconnected outerpath (Theorem 7), it follows that all linear outerplanar graphs are B\(_0\)-VPG. Furthermore, since we show in Section 4 that all AT-free outerplanar graphs are linear (Lemma 5), the main result of this article follows.

**Theorem 3** Every biconnected outerpath is B\(_0\)-VPG.

**Proof:** Let \( G \) be a biconnected outerpath with \( n \) faces labeled \( F_1, \ldots, F_n \) such that the weak dual of \( G \) is the path \( F_1, \ldots, F_n \). For each \( i \in [n - 1] \), the edge shared by \( F_i \) and \( F_{i+1} \) is denoted by \( e_i \). For notational convenience, we set \( e_n \) to be any boundary edge of \( F_n \). For each \( i \in [n] \), let \( G_i \) denote the induced subgraph of \( G \) restricted to the faces \( F_1, \ldots, F_i \).

In a B\(_0\)-VPG drawing \( D_i \) of \( G_i \), we call a non-point horizontal (resp., vertical) line segment \( l \) in \( D_i \) extendable from a point \( p \in l \) if at least one of the two infinite horizontal (resp., vertical)
open rays starting at \( p \) (but not containing \( p \)) does not intersect any other line segment of \( D_i \). A point segment \( l \) is said to be extendable from its location \( p \) if it is extendable from \( p \) both as a horizontal and a vertical line segment. An edge \( xy \) in \( G_i \) is said to be extendable in \( D_i \) if the line segments \( l_x \) and \( l_y \) representing the vertices \( x \) and \( y \) are extendable from a common point \( p \in l_x \cap l_y \) either in the same direction or in orthogonal directions. Finally a \( B_0 \)-VPG drawing \( D_i \) is said to be extendable if \( e_i \) is extendable and whenever \( F_i \) is a triangle, the vertex of \( F_i \) not incident to \( e_{i-1} \) is represented by a point segment. Thus, in particular, if \( D_i \) is extendable, the segments representing the endpoints of the edge \( e_i \) can be extended in orthogonal directions.

If \( F_1 \cong C_3 \), then representing all the three vertices as point segments at the same point gives an extendable \( B_0 \)-VPG drawing \( D_1 \) of \( C_3 \). If the length of \( F_1 \) is 4 or more, then we can represent \( F_1 \) as the intersection graph of line segments laid out on the boundary of an axis-parallel rectangle with the segments representing the endpoints of \( e_1 \) being orthogonal (and hence sharing only a corner of the rectangle). This is an extendable \( B_0 \)-VPG drawing \( D_1 \) of \( C_3 \). Let \( D_i, i < n \), be an extendable \( B_0 \)-VPG drawing of \( G_i \). From \( D_i \), we construct an extendable \( B_0 \)-VPG drawing \( D_{i+1} \) of \( G_{i+1} \) as follows.

**Case 1 (length of \( F_{i+1} \) is 4 or more).** Let \( F_{i+1} = v_0, \ldots, v_k, v_0 \), with \( e_i = v_kv_0 \) and \( e_{i+1} = v_jv_{j+1}, j < k \). Since \( D_i \) is extendable, the edge \( v_kv_0 \) is extendable in \( D_i \). Extend the line segments \( l_k \) and \( l_0 \) (representing \( v_k \) and \( v_0 \) respectively) in orthogonal directions to two points \( q_k \) and \( q_0 \) outside of the bounding box of \( D_i \). Let \( q \) be the intersection point of the perpendiculars to \( l_k \) and \( l_0 \) at \( q_k \) and \( q_0 \) respectively. Represent the path \( v_1, \ldots, v_{k-1} \) by adjacent segments \( l_1, \ldots, l_{k-1} \) on the two line segments from \( q_0 \) to \( q \) and \( q \) to \( q_k \) such that \( l_1 \) contains \( q_0 \), \( l_{k-1} \) contains \( q_k \), and \( l_j \) and \( l_{j+1} \) are orthogonal segments sharing the point \( q_0 \) if \( j = 0 \), \( q_k \) if \( j = k - 1 \), or \( q \) otherwise. This gives the drawing \( D_{i+1} \). It is clear that the new segments added in this stage do not intersect any other line segments in \( D_i \) except \( l_0 \) and \( l_k \). It is easy to verify that the edge \( e_{i+1} = v_jv_{j+1} \) is extendable. Hence \( D_{i+1} \) is extendable.

**Case 2 \((F_{i+1} \cong C_3)\).** Let \( F_{i+1} = a, b, c, a \), with \( e_i = ca \) and \( e_{i+1} = ab \). Since \( D_i \) is extendable, the edge \( ca \) is extendable in \( D_i \) from a point \( p \). If the line segments \( l_c \) and \( l_a \) are extendable in the same direction, then extend them to a point \( q \) outside the bounding box of \( D_i \) and represent \( b \) by a point segment \( l_b \) at \( q \) to obtain \( D_{i+1} \). If \( l_c \) and \( l_a \) are extendable only in orthogonal directions, then neither of them is a point segment. Hence \( F_i \not\cong C_3 \) and hence the vertices \( c \) and \( a \) have no common neighbor in \( G_i \). So the point \( p \) is not contained in any line segment of \( D_i \) other than \( l_c \) and \( l_a \). Represent \( b \) by a point segment \( l_b \) at \( p \) to get \( D_{i+1} \). In both the subcases, it is clear that the new segments added in this stage do not intersect any other line segments in \( D_i \) except \( l_c \) and \( l_a \). It is easy to check that the line segment \( l_b \), the point segment \( l_b \), and also the edge \( ab \) are extendable from \( p \) in \( D_{i+1} \). Since \( ab \) is extendable and \( b \) is represented by a point segment, \( D_{i+1} \) is extendable.

Repeating the above construction \( n - 1 \) times gives a \( B_0 \)-VPG drawing \( D_n \) of \( G_n = G \). \( \square \)

\( B_0 \)-VPG is easily seen to be a hereditary graph class (that is, closed under induced subgraphs). Thus it follows from Theorem 3 that every induced subgraph of a biconnected outerpath is \( B_0 \)-VPG. By the end of the next section, we extend this to every subgraph (not necessarily induced).
Figure 3: Examples of outerplanar graphs which are (a) not block-safe (b) not cut-safe. In (b), $C_v$ has four big components. These components even after removal of $v$ have a $3^+$-vertex or a face which is indicated by blue.

3 Characterization of Linear Outerplanar Graphs

In a preliminary version [22] of this paper, we showed that every AT-free outerplanar graph can be identified as an induced subgraph of a biconnected outerpath. After that work, triggered by a question from Mathew C. Francis, we realized that the induced subgraphs of biconnected outerpaths can be more exotic than AT-free outerplanar graphs, and that this class deserves to be studied on its own. Our definition of linear outerplanar graphs in [22] was a technical choice made for the proof. That definition was more restrictive than the one here (Definition 2).

While the class of linear outerplanar graphs will inherit the rich collection of drawings and geometric representations available for biconnected outerpaths, the structure of a linear outerplanar graph is harder to describe than that of a biconnected outerpath. This section aims to do that. We first build the necessary terminology for stating and proving the characterization theorem (Theorem 7). Recall that the terminology of faces, duals, weak duals, boundary edges and internal edges will be used assuming an outerplane drawing.

Let $v$ be a cutvertex in a graph $G$. For every component of $C$ of $G \setminus v$, the subgraph of $G$ induced on $V(C) \cup \{v\}$ is called a component incident to $v$. The set of components incident to $v$ is denoted by $C_v$. A component $C$ incident to $v$ is said to be incident to a block $B$ if $v$ is in $B$ and $C$ does not contain $B$.

**Definition 4** Let $v$ be a cutvertex in an outerplanar graph $G$ and $C \in C_v$. We call $C$ small for $v$ if $C \setminus v$ is a path and big for $v$ otherwise. Further, when $C$ is small for $v$, we call it a tail at $v$ if $C$ (including $v$) is a path.
Definition 5 (Cut-safety) A cutvertex $v$ in an outerplanar graph $G$ is said to be safe if $C_v$ contains at most two big components. The graph $G$ is said to be cut-safe if every cutvertex in $G$ is safe.

A set of at most two boundary edges of a block $B$ in an outerplanar graph is called antipodal either when $B$ is a single face or when the edges belong to different leaf faces of $B$.

Definition 6 (Block-safety) A nontrivial block $B$ in an outerplanar graph is called safe if there exist two antipodal edges $a_0b_0$ and $a_1b_1$ in $B$ and the set of components incident to $B$ can be partitioned into $C_0$ and $C_1$ such that

1. every component in $C_i$ ($i \in \{0, 1\}$) is incident to either $a_i$ or $b_i$, and
2. at most one component in $C_i$ ($i \in \{0, 1\}$) is incident to $a_i$ and it (if present) is a tail.

An outerplanar graph $G$ is said to be block-safe if every nontrivial block in $G$ is safe. The edges $a_0b_0$ and $a_1b_1$ are called terminal edges of $B$ in $G$. A terminal edge is denoted as an ordered pair $(x, y)$ where $x = a_i$ and $y = b_i$. The components in $C_i$ are said to be associated to the terminal edge $(a_i, b_i)$.

Theorem 7 (Characterization) An outerplanar graph $G$ is linear if and only if $G$ is cut-safe, block-safe and the weak dual of $G$ is a linear forest. Moreover, if $G$ is linear, then it can be realized both as an induced subgraph and as a spanning subgraph of (different) biconnected outerpaths.

Remark 2 Note that the class of linear outerplanar graphs and outerpaths are incomparable. There are outerpaths which are not linear (cf. Figure 3.(a)) and linear outerplanar graphs which are not outerpaths (cf. Figure 4.(a)).

3.1 Proof of Theorem 7 (Necessity)

We first prove that if an outerplanar graph $G$ is linear, then $G$ is cut-safe, block-safe and $\mathcal{T}_G$ is a linear forest. Our strategy is to look at the edges of $G$ which are forced to be internal edges in any biconnected outerplanar supergraph of $G$ and then use the fact that the internal edges in a biconnected outerpath have a natural linear order. It is easy to see that every internal edge of $G$, at least one edge in each face of $G$ (unless $G$ itself is a cycle), and all but at most two edges incident to any vertex of $G$ will all be internal edges in any biconnected outerplanar supergraph of $G$. We can say a bit more about the edges incident to a cutvertex of $G$ in a nontrivial block (Lemma 1) using the simple observation below. The extension of this simple observation to biconnected outerpaths (Observation 9) is a key to the rest of this section.

Observation 8 If $uv$ is an internal edge in a biconnected outerplanar graph $G$, then $G \setminus \{u, v\}$ has exactly two components. Moreover, both these components contain exactly one boundary neighbor each of $u$ and $v$.

Lemma 1 If an outerplanar graph $G$ is an induced subgraph of a biconnected outerplanar graph $G'$ then for any cutvertex $v$ in a nontrivial block $B$ of $G$, one of the two boundary edges incident to $v$ in $B$ is an internal edge in $G'$. 
Proof: Let $uv$ and $vw$ be the two boundary edges of $B$ incident to $v$ and let $x$ be a neighbor of $v$ outside $B$ in $G$. If both $uv$ and $vw$ are boundary edges in $G'$, then $vx$ is an internal edge in $G'$. But $u$ and $w$ are in the same component of $G \setminus \{v, x\}$ and hence of its supergraph $G' \setminus \{v, x\}$. This contradicts Observation 8. □

For any three subsets $X, Y, Z$ of the vertices of a graph, $X$ separates $Y$ and $Z$ if every path between $Y$ and $Z$ contains a vertex from $X$.

Observation 9 Let $u_0v_0, u_1v_1, u_2v_2$ be three distinct (but not necessarily disjoint) internal edges of a biconnected outerpath $G$. Then the endpoints of one of them, say $\{u_i, v_i\}$, separate $\{u_j, v_j\}$ from $\{u_k, v_k\}$ in $G$, where $\{i, j, k\} = \{0, 1, 2\}$.

Lemma 2 If an outerplanar graph $G$ is linear, then $T_G$ is a linear forest.

Proof: Let $G$ be a subgraph of a biconnected outerpath $G'$. If $T_G$ is not a linear forest, then $G$ has a face $f$ with at least three internal edges. These remain internal edges in $G'$ that violate the separation property in Observation 9. □

Lemma 3 If an outerplanar graph $G$ is linear, then $G$ is cut-safe.

Proof: Let $G$ be a subgraph of a biconnected outerpath $G'$. Suppose $G$ has a cutvertex $v$ such that $C_v$ contains three big components $C_0, C_1, C_2$. For each $i \in \{0, 1, 2\}$, since $C_i \setminus v$ is not a path, it either contains a face or a $3^+$-vertex. In either case, $C_i \setminus v$ will contribute an internal edge $e_i$ to $G'$. But $e_0, e_1, e_2$ violate Observation 9. □

Lemma 4 If an outerplanar graph $G$ is linear, then $G$ is block-safe.

Proof: Let $G'$ be an edge-minimal biconnected outerpath which is a supergraph of $G$. If $G$ itself is a biconnected outerpath, we are done. Otherwise, picture any nontrivial block $B$ of $G$ in $G'$. Let $E'$ be the set of boundary edges of $B$ which become internal edges in $G'$. Since $T_{G'}$ is a path, it is easy to see that $E'$ is antipodal. By Lemma 1, every cutvertex of $B$ in $G$ is an endpoint of an edge in $E'$. Let $E' \in \{e_0\}$ if it is singleton, and $\{e_0, e_1\}$ otherwise. The proof will be complete if we can partition the set of components $C_B$ incident to $B$ into $C_i$, $i \in \{0, 1\}$ and label the endpoints of $e_i$ as $a_i$ and $b_i$ respecting the last condition in Definition 6.

By Observation 8, we get exactly two components in $G' \setminus V(e_i)$. Let $G_i$ be the subgraph of $G'$ induced by $e_i$ and the component of $G' \setminus V(e_i)$ that does not contain any vertex of $B$. Let $C_i$ denote the components in $C_B$ that are captured by $G_i$ in $G'$. Consider the leaf face $f_i$ of $G_i$ containing the edge $e_i$. Since $f_i$ is not part of $B$, at least one edge $e'_i$ of $f_i$ is missing from $G$. By edge-minimality of $G$, this is a boundary edge of $G'$ and hence $G_i$. Let $P_i$ denote the (possibly trivial) path in $f_i \setminus e'_i$ between $e_i$ and $e'_i$ that does not contain an internal edge (if any) of $G_i$. We label the endpoint of $e_i$ which meets $P_i$ as $a_i$ and the other as $b_i$. If $P_i$ is trivial then there is no component in $C_i$ incident to $a_i$. If $P_i$ is not trivial, at most one component in $C_i$, and that too a tail which is a subpath of $P_i$, is incident to $a_i$. It is easy to see that this labeling satisfies Definition 6. □
Figure 4: (a) A block-safe and cut-safe outerplanar graph $G$ such that $T_G$ is a linear forest. Terminal edges are shown oriented and cutvertices for Action 1 are highlighted. (b) Intermediate graph $G_3$. Faces created by the two cases of Action 1 are marked 1a and 1b respectively. (c) Final biconnected outerpath $G'$. Faces created by Action 2 are marked 2a if both the components merged are small and 2b otherwise. Since all the connectors are two-length paths, $G$ is an induced subgraph of $G'$.

3.2 Proof of Theorem 7 (Sufficiency)

Now we prove the other direction of Theorem 7. Let $G$ be a cut-safe and block-safe outerplanar graph such that $T_G$ is a linear forest. We will construct a biconnected outerpath $G'$ which contains $G$ as a subgraph. A connector between two nonadjacent boundary vertices $u$ and $v$ of a plane graph $H$ is either an edge $uv$ or a two-length path $(u, w, v)$ such that $w \notin V(H)$ drawn through the outer face. Hence the resultant graph is planar. In the following construction, if every connector used is an edge (resp. two-length path), then $G$ is contained as a spanning (resp. induced) subgraph of $G'$. The construction of $G'$ is done in two phases. Each phase consists of repeated applications of a single action.

**Definition 10** For a cutvertex $v$, a small component $C \in C_v$ is called a maximal small component if $C$ is not a subgraph of a small component incident to another cutvertex.

**Action 1 (Tuck the tails)** Let $v$ be a cutvertex in $G$ and $C \in C_v$ be a maximal small component associated to a terminal edge $(a_i, b_i)$ of a nontrivial block $B$. (a) If $v = a_i$ (in which case $C$ is a tail at $a_i$), add a connector from the leaf of $C$ to $b_i$. This merges the block $B$ and the component $C$ into a new block $B'$. Designate the remaining terminal edge of $B$ and the last edge $(v', b_i)$ of
the connector as the two terminal edges of \(B'\). (b) If \(v = b_1\), add a connector between \(v\) and each endvertex of the path \(C \setminus v\) which is not already adjacent to \(v\) to form a block \(B'\) containing \(C\) (but not \(B\)). If both the endvertices of \(C \setminus v\) were adjacent to \(v\), then \(B' = C\). Designate \((u,v)\) and \((w,v)\) as the terminal edges of \(B'\), where \(uv\) and \(wv\) are the boundary edges of \(B'\) incident to \(v\). If \(B'\) is a single edge \(uv\), designate \((u,v)\) as both the first and second terminal edge of \(B'\).

Let us call the resulting graph \(G_2\). In Case (a), the weak dual \(T_{B'}\) of \(B'\) is a path formed by extending \(T_B\) with a leaf edge corresponding to the dual of \(a_ib_i\). The new terminal edge in \(B'\) is in the new leaf face and hence the two terminal edges of \(B\) (not \(B\) the new leaf face and hence the two terminal edges of \(B\)) extending \(T\) of \(B\) to \(G\) obtain a graph \(C\). Change to \(B\) and \(B\) terminal edge of \(B\) is simple since every internal edge of \(B\) is safe with the given terminals. Next we argue that \(G_2\) is cut-safe. If both \(u \neq v\) be a cutvertex in \(G_2\) then \(C_u\) remains small for \(i\) which continues to be a terminal edge in \(B\). Since \(C\) is a maximal small component, \(C_u\) does not contain \(C\) and hence \(C_u\) remains unaffected (and hence small) in \(G_2\). So \(u\) remains safe in \(G_2\).

We perform Action 1 once for each maximal small component incident to a cutvertex in \(G\) to obtain a graph \(G_3\). Note that for each cutvertex \(v\) in \(G_3\), each small component in \(C_v\) is a block. Also, each component incident to a nontrivial block \(B\) in \(G_3\) is incident at the second vertex of a terminal edge of \(B\). For each trivial block \(B = uv\) which is not a pendant edge of \(G_3\), assign \((u,v)\) to be the first and \((v,u)\) to be the second terminal edge of \(B\). Associate every component incident to \(B\) at \(u\) (resp. \(v\)) with terminal edge \((v,u)\) (resp. \((u,v)\)).

**Action 2 (Bond with your sibling)** Let \(v\) be a cutvertex in \(G_3\) and let \(C_0\) and \(C_1\) be two components from \(C_v\), chosen prioritizing small components over big ones. For \(i \in \{0,1\}\), let \(B_i\) be the block in \(C_i\) which contains \(v\), and \((u_i,v)\) be a terminal edge of \(B_i\). Add a connector from \(u_0\) to \(u_1\). This merges \(B_0\) and \(B_1\) into a new block \(B\). The remaining terminal edges, one each from \(B_0\) and \(B_1\), are designated as the terminal edges of \(B\).

Let us call the resulting graph \(G_4\). The weak dual \(T_B\) of \(B\) is a path obtained by connecting two leaf vertices of \(T_{B_0}\) and \(T_{B_1}\) through the dual vertex corresponding to the new bounded face. The new terminal edges of \(B\) are antipodal in \(B\). Every cutvertex other than \(v\) in \(B_i\) \((i \in \{0,1\})\) is contained in the second terminal edge of \(B_i\) which continues to be a terminal edge in \(B\). If \(C_i\) is small for an \(i \in \{0,1\}\), then \(B_i = C_i\) and the second terminal edge of \(B_i\) has \(v\) as its second vertex. If both \(C_0\) and \(C_1\) are big for \(v\), since \(v\) is safe in \(G_3\), there are no other components in \(C_v\). Hence \(v\) is no longer a cutvertex in \(G_4\). Hence all the cutvertices of \(B\) are contained in its terminal edges and \(B\) is safe. Now we argue that \(G_4\) is cut-safe. The difference from \(C_v\) in \(G_3\) to \(C_v\) in \(G_4\) is that \(C_0\) and \(C_1\) gets merged into a single component. If both \(C_0\) and \(C_1\) are small components, then one can easily check that the new component is \(B\) itself and it is small for \(v\). Hence the number of big components in \(C_v\) does not increase and hence \(v\) remains safe in \(G_4\) or ceases to be a cutvertex. Let \(u \neq v\) be a cutvertex in \(G_4\) and let \(C_u\) be a small component incident to \(u\). Since \(C_u\) is a block, it does not contain the cutvertex \(v\) and hence remains unaffected (and thus small) in \(G_4\). So \(u\) remains safe in \(G_4\).

We repeat Action 2 as long as there is a cutvertex. Let \(G'\) denote the resulting biconnected graph. Since each action preserves cut-safety, block-safety and linearity of the weak dual, \(G'\) is a biconnected outerpath. This completes the proof of Theorem 7.
Remark 3 From the above construction, one can check that each connector added reduces the number of blocks at least by one. Hence the total number of new vertices added is less than the number of blocks in the original graph $G$.

Remark 4 Checking the cut-safety, block-safety and the linearity of weak dual of an outerplanar graph can be done in polynomial time. It can also be verified that the construction of $G'$ from $G$ can be done in polynomial time.

By Theorem 7, any linear outerplanar graph can be realized as an induced subgraph of a biconnected outerpath. This together with Theorem 3 gives the following corollary.

Corollary 11 Every linear outerplanar graph is $B_0$-VPG.

4 Linearity of AT-free outerplanar graphs

By Corollary 11, in order to prove that AT-free outerplanar graphs are $B_0$-VPG, it is enough to show that they are linear. Lemma 5 in this section asserts the same. Note that one may suspect that all AT-free outerplanar graphs can be realized as induced subgraphs of biconnected AT-free outerplanar graphs. But this is incorrect. For example, let $G$ be a $C_5$ together with a pendant vertex. While $G$ is AT-free outerplanar, it is easy to see that any biconnected outerplanar graph $G'$, containing $G$ as an induced subgraph, is not AT-free.

The following observation is helpful in proving Lemma 5.

Observation 12 Let $B$ be a nontrivial block of an outerplanar graph $G$. Every leaf face $f$ of $B$ contains a vertex $u$ of degree two in $G[B]$ which is not incident to any other bounded face of $B$.

Justification: A face has at least three vertices and at most two vertices of a leaf face can be shared by another face. Hence all the remaining vertices are of degree two in $G[B]$ and are not incident to any other bounded face of $B$.

Lemma 5 AT-free outerplanar graphs are linear.

Proof: Let $G$ be an AT-free outerplanar graph. If $T_G$ is not a linear forest, then there exist three internal edges in one face $f$ of $G$ sharing with faces, say, $f_1$, $f_2$, $f_3$. One can verify that we can choose one vertex $v_i$ for $i \leq 3$ from each of $f_i \setminus f$ to form an AT.

If $G$ is not cut-safe, then there exists a cut-vertex $v$ where $C_v$ has more than two big components, say, $C_1, C_2, C_3$. That is, $C_i \setminus \{v\}$ for $1 \leq i \leq 3$ is not a path and thus it has either a $3^+$-vertex $v_i$ or a face $f_i$. If all the neighbors of $v_i$ are adjacent to $v$, then it is easy to see that $C_i$ has $K_{2,3}$ as subgraph. Similarly if all the vertices of $f_i$ are adjacent to $v$, then one can verify that $C_i$ has $K_4$ as minor. Outerplanar graphs do not contain $K_{2,3}$ or $K_4$ as a minor [11]. Thus in both cases, there exists a vertex $v'_i \in C_i \setminus \{v\}$ nonadjacent to $v$ in $C_i$. It is easy to see that $\{v'_1, v'_2, v'_3\}$ forms an AT.

It remains to check whether $G$ is block-safe. Consider any nontrivial block $B$ of $G$. Since $T_G$ is a linear forest, $B$ either has exactly two leaf faces $f_1, f_2$ or $B$ itself is a face $f_1$. In the former case, Observation 12 guarantees that $G[B]$ has two vertices $u_1$ and $u_2$ of degree two in $f_1$ and $f_2$ respectively. For any cut-vertex $v$ in $B$, we denote an arbitrary neighbor of $v$ outside $B$ as $v'$. If there exists three cutvertices in $B$, say, $v_1, v_2, v_3$, then by using the path along the outer cycle (through the boundary) of $B$, one can verify that $\{v'_1, v'_2, v'_3\}$ is an AT. Thus there can be at most two cutvertices in $B$. If $B$ itself is a face $f_1$, we can choose arbitrary boundary edges $a_0b_0$ and $a_1b_1$
of $B$ such that $b_0$ and $b_1$ are the only cutvertices of $B$. These two edges are antipodal in $B$, and thus $B$ satisfies the conditions of the terminal edges as per Definition 6, thereby showing that $B$ is safe. Hence we assume in the rest of the proof that $B$ contains more than one face. If one of the cutvertices, say $v$, is neither incident to $f_1$ nor $f_2$, then the vertices $u_1$ and $u_2$, together with $v'$ form an AT. Hence the cutvertices incident to $B$, if any, must lie in the leaf faces of $B$. Moreover, we intend to associate each cutvertex $v$ to a different leaf face of $B$ containing $v$. If this is not possible, that is, both the cutvertices, say $v_1, v_2$ are not incident to one of the leaf faces, say $f_2$, then $\{v_1', v_2', u_2\}$ forms an AT. Thus we conclude that there exist at most two cutvertices incident to $B$, and they can be associated to different leaf faces of $B$ containing them. We can choose arbitrary boundary edges $a_0b_0$ and $a_1b_1$ of $B$, one from each leaf face such that $b_0$ and $b_1$ are the only cutvertices of $B$. Clearly those edges are antipodal and their endpoints meet the conditions of the terminal edges as per Definition 6. Thus $B$ is safe.

Lemma 5 together with Corollary 11 establish the main result.

**Theorem 13** Every AT-free outerplanar graph is $B_0$-VPG.

### 5 Concluding Remarks

We have showed that all linear outerplanar graphs are $B_0$-VPG. However, it is easy to see that linearity is not necessary for $B_0$-VPG outerplanar graphs. For example, planar bipartite graphs, and hence outerplanar bipartite graphs are $B_0$-VPG [21]. But outerplanar bipartite graphs can be far from being linear, in the sense that their weak duals can be trees with arbitrarily large degrees for internal nodes.

One can see from a $B_0$-VPG drawing that the closed neighborhood of every vertex is an interval graph [18]. Next, we strengthen this necessary condition by identifying adjacent vertices which are forced to be represented by collinear segments in any $B_0$-VPG drawing. An induced $C_4$ has essentially a unique $B_0$-VPG representation [3]. A $C_4$ together with exactly one chord is called a **diamond**, and the chord is called the **diamond diagonal**. A diamond diagonal can only be drawn as the intersection of collinear line segments in every $B_0$-VPG representation of the diamond [18]. In any $B_0$-VPG representation of an odd-cycle, an odd number of edges has to be represented as the intersection of collinear line segments. It is also easy to verify that the binary relation collinearity on the set of line segments is an equivalence relation. We combine these observations to obtain the following.

**Proposition 14** If a graph $G$ is $B_0$-VPG, then there exists a subgraph $H$ of $G$ containing all diamond diagonals of $G$ such that,

(a) for every component $C$ of $H$, the subgraph of $G$ induced by the closed neighborhood $N[C]$ of $C$ is an interval graph, and

(b) the minor of $G$ obtained by contracting every component of $H$ in $G$ is a bipartite $B_0$-VPG graph.

**Proof:** Since $G$ is $B_0$-VPG, there exists a $B_0$-VPG representation $D$ of $G$. Let $H$ be the spanning subgraph of $G$ in which two vertices are adjacent if and only if the corresponding line segments in $D$ are intersecting and collinear. Since any diamond diagonal of $G$ is represented by the intersection of collinear line segments in $D$, their endpoints are adjacent in $H$ too. Thus for proving (a), it remains to show that the closed neighborhood of any component of $H$ is an interval graph. Let $C$
be a component of $H$ and let $D_C$ be the drawing induced by the segments of $D$ that represent the vertices in $N[C]$. In $D_C$, if we restrict the segments that represent the vertices in $N[C] \setminus C$ to point intervals at their intersection point with a vertex in $C$, we obtain an interval representation for the subgraph of $G$ induced on $N[C]$. To obtain the minor $G_H$ of $G$ in (b), we contract every edge of $H$. Each component $C$ of $H$, therefore becomes a branch set of $G_H$, and can be represented as a line segment obtained as the union of all the segments representing vertices in $C$. This gives a $B_0$-VPG representation of $G_H$. It is easily verified that it is a bipartite graph since there are no collinear intersections. \hfill $\square$

**Remark 5** Note that since $G_H$ is bipartite, $H$ contains an odd number of edges from each odd cycle of $G$.

Figure 5: A few outerplanar graphs which are not $B_0$-VPG. The blue edges represent forced collinear edges. The green edges in (d) and (e) represent the edges arbitrarily chosen in an odd cycle to become collinear in a $B_0$-VPG drawing. The 4-cycles in (e) resemble petals and hence the name. The red double edges in (f) and (g) represent $C_4$ edges where the $C_4$ (petal) is not drawn to avoid cluttering.

Proposition 14 shows that the outerplanar graphs in Figure 5 are not $B_0$-VPG. Nonetheless, the above proposition is not sufficient to characterize $B_0$-VPG even among outerplanar graphs. Figure 6 is such an example. There are two key challenges for an outerplanar graph to have a $B_0$-VPG representation. One is the issue of small cycles (of length 3 and 4) while the other is the issue of how cycles branch out (which is captured by the structure of the weak dual). In this work, we mainly focus on the case where the second challenge is absent, and hence we restricted to outerplanar graphs whose weak dual is a linear forest. We have addressed the case when the first challenge is absent in a future work which is under preparation. We hope that these two results together with Proposition 14 will shed some light on the characterization question on $B_0$-VPG outerplanar graphs.
Figure 6: An outerplanar graph which is not $B_0$-VPG. This shows that the conditions in Proposition 14 was not sufficient. The red double edges represent $C_4$ edges where the $C_4$ is not drawn to avoid cluttering. The blue edges are forced to be collinear. Hence the bridge has to be realized as an orthogonal intersection and this prevents a non-crossing drawing of the two 6-cycles.

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References

[1] M. J. Alam, D. Eppstein, M. T. Goodrich, S. G. Kobourov, S. Pupyrev, et al. Balanced Circle Packings for Planar Graphs. In International Symposium on Graph Drawing, pages 125–136. Springer, 2014. doi:10.1007/978-3-662-45803-7_11.

[2] L. Alcón, F. Bonomo, and M. P. Mazzoleni. Vertex Intersection Graphs of Paths on a Grid: Characterization Within Block Graphs. Graphs and Combinatorics, 33(4):653–664, 2017. doi:10.1007/s00373-017-1791-6.

[3] A. Asinowski, E. Cohen, M. C. Golumbic, V. Limouzy, M. Lipshteyn, and M. Stern. Vertex Intersection Graphs of Paths on a Grid. J. Graph Algorithms Appl., 16(2):129–150, 2012. doi:https://dx.doi.org/10.7155/jgaa.00253.

[4] J. Babu, M. Basavaraju, L. S. Chandran, and D. Rajendraprasad. 2-Connecting outerplanar graphs without blowing up the pathwidth. Theoretical Computer Science, 554:119–134, 2014. doi:10.1016/j.tcs.2014.04.032.

[5] J. Barát, P. Hajnal, Y. Lin, and A. Yang. On the structure of graphs with path-width at most two. Studia Scientiarum Mathematicarum Hungarica, 49(2):211–222, 2012. doi: 10.1556/sscmath.49.2012.2.1200.

[6] S. Cabello, M. J. van Kreveld, G. Liotta, H. Meijer, B. Speckmann, and K. Verbeek. Geometric Simultaneous Embeddings of a Graph and a Matching. J. Graph Algorithms Appl., 15(1):79–96, 2011. doi:https://dx.doi.org/10.7155/jgaa.00218.

[7] D. Catanzaro, S. Chaplick, S. Felsner, B. V. Halldórsson, M. M. Halldórsson, T. Hixon, and J. Stacho. Max point-tolerance graphs. Discrete Applied Mathematics, 216:84–97, 2017. doi:10.1016/j.dam.2015.08.019.

[8] J. Chalopin and D. Gonçalves. Every planar graph is the intersection graph of segments in the plane. In Proceedings of the forty-first annual ACM symposium on Theory of computing, pages 631–638, 2009. doi:10.1145/1536414.1536500.
[9] S. Chaplick, V. Jelínek, J. Kratochvíl, and T. Vyskočil. Bend-Bounded Path Intersection Graphs: Sausages, Noodles, and Waffles on a Grill. In *International Workshop on Graph-Theoretic Concepts in Computer Science*, pages 274–285. Springer, 2012. doi:10.1007/978-3-642-34611-8_28.

[10] S. Chaplick and T. Ueckerdt. Planar Graphs as VPG-Graphs. In *International Symposium on Graph Drawing*, pages 174–186. Springer, 2012. doi:10.1007/978-3-642-36763-2_16.

[11] G. Chartrand and F. Harary. Planar permutation graphs. In *Annales de l'IHP Probabilités et statistiques*, volume 3, pages 433–438, 1967.

[12] J. Czyzowicz, E. Kranakis, and J. Urrutia. A simple proof of the representation of bipartite planar graphs as the contact graphs of orthogonal straight line segments. *Information Processing Letters*, 66(3):125–126, 1998. doi:10.1016/S0020-0190(98)00046-5.

[13] N. de Castro, F. J. Cobos, J. C. Dana, A. Márquez, and M. Noy. Triangle-free Planar Graphs and Segment Intersection Graphs. *J. Graph Algorithms Appl.*, 6(1):7–26, 2002. doi:https://www.doi.org/10.7155/jgaa.00043.

[14] H. De Fraysseix, P. Ossona de Mendez, and J. Pach. Representation of planar graphs by segments. *Intuitive Geometry*, 63:109–117, 1991.

[15] W. Evans, V. Kusters, M. Saumell, and B. Speckmann. Column Planarity and Partial Simultaneous Geometric Embedding. In *International Symposium on Graph Drawing*, pages 259–271. Springer, 2014. doi:10.1007/978-3-662-45803-7_22.

[16] H. J. Fleischner, D. P. Geller, and F. Harary. Outerplanar graphs and weak duals. *The Journal of the Indian Mathematical Society*, 38:215–219, 1974.

[17] A. García, F. Hurtado, M. Noy, and J. Tejel. Augmenting the Connectivity of Outerplanar Graphs. *Algorithmica*, 56(2):160–179, 2010. doi:10.1007/s00453-008-9167-1.

[18] M. C. Golumbic and B. Ries. On the Intersection Graphs of Orthogonal Line Segments in the Plane: Characterizations of Some Subclasses of Chordal Graphs. *Graphs and Combinatorics*, 29(3):499–517, 2012. doi:10.1007/s00373-012-1133-7.

[19] D. Gonçalves. Not all planar graphs are in PURE-4-DIR. *Journal of Graph Algorithms and Applications*, 24(3):293–301, 2020. doi:https://dx.doi.org/10.7155/jgaa.00533.

[20] D. Gonçalves, L. Isenmann, and C. Pennarun. Planar graphs as L-intersection or L-contact graphs. In *Proceedings of the Twenty-Ninth Annual ACM-SIAM Symposium on Discrete Algorithms*, pages 172–184. SIAM, 2018. doi:10.1137/1.9781611975031.12.

[21] I. B.-A. Hartman, I. Newman, and R. Ziv. On grid intersection graphs. *Discrete Mathematics*, 87(1):41–52, 1991. doi:10.1016/0012-365X(91)90069-E.

[22] S. Jain, S. K. Pallathamadam, and D. Rajendraprasad. B_0-VPG Representation of AT-free Outerplanar Graphs. In *Conference on Algorithms and Discrete Applied Mathematics*, pages 103–114. Springer, 2022. doi:10.1007/978-3-030-95018-7_9.

[23] G. Kant. Augmenting Outerplanar Graphs. *Journal of Algorithms*, 21(1):1–25, 1996. doi:10.1006/jagm.1996.0034.
[24] J. Kratochvíl. String graphs. II. Recognizing string graphs is NP-hard. *Journal of Combinatorial Theory, Series B*, 52(1):67–78, 1991. doi:10.1016/0095-8956(91)90091-w.

[25] J. Kratochvíl and J. Matoušek. Intersection Graphs of Segments. *Journal of Combinatorial Theory, Series B*, 62(2):289–315, 1994. doi:10.1006/jctb.1994.1071.

[26] S. Mehrabi. Approximation Algorithms for Independence and Domination on B1-VPG and B1-EPG Graphs. *arXiv preprint arXiv:1702.05633*, 2017. doi:10.48550/arXiv.1702.05633.

[27] S. K. Pallathumadam and D. Rajendraprasad. Characterization of B0-VPG Cocomparability Graphs and a 2D Visualization of their Posets. *Order*, 39:465–484, 2022. doi:10.1007/s11083-021-09589-w.

[28] M. Schaefer, E. Sedgwick, and D. Štefankovič. Recognizing string graphs in NP. *Journal of Computer and System Sciences*, 67(2):365–380, 2003. doi:10.1016/S0022-0000(03)00045-X.

[29] E. R. Scheinerman. *Intersection classes and multiple intersection parameters of graphs*. Princeton University, 1984.

[30] D. West. Open problems. *SIAM J. Discrete Math. Newslett.*, 2:10–12, 1991.