Observability of Boolean multiplex control networks

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Boolean multiplex (multilevel) networks (BMNs) are currently receiving considerable attention as theoretical arguments for modeling of biological systems and system level analysis. Studying control-related problems in BMNs may not only provide new views into the intrinsic control in complex biological systems, but also enable us to develop a method for manipulating biological systems using exogenous inputs. In this article, the observability of the Boolean multiplex control networks (BMCNs) are studied. First, the dynamical model and structure of BMCNs with control inputs and outputs are constructed. By using of Semi-Tensor Product (STP) approach, the logical dynamics of BMCNs is converted into an equivalent algebraic representation. Then, the observability of the BMCNs with two different kinds of control inputs is investigated by giving necessary and sufficient conditions. Finally, examples are given to illustrate the efficiency of the obtained theoretical results.

Human Genome Project, which is an international scientific research project with the goal of determining the sequence of nucleotide base pairs1, inspired a new view of biology called the systems biology. Instead of investigating individual genes, proteins or cells, systems biology studies the behavior and relationships of all cells, proteins, DNAs and RNAs in the same biological system, called a cellular network2. The Boolean Networks (BNs) as a powerful tool in describing, analyzing, and simulating the cellular networks, has been most widely used3–17. From decades ago, Kauffman put forward the theory which can describe the net of cell and gene using BNs4. And he made expatiation about the relationship between BNs and gene as well as life5,6. Because the construction and evolutionary process of cell and gene can be revealed very well by BNs, BNs turn into a hot topic concerned by biologists, physicists and neuroscientists. Huang, S. et al. talked about the Boolean modeling and analyzing of biological system10,11. Aldana, M. et al. studied the topological structure of BNs7. Akutsu, T. et al. and Albert, R. et al. considered the dynamic features of BNs12,13. More detailedly and recently, Lu, J. et al. analyzed the synchronization problem of master-slave probabilistic BNs18.

In recent years accompany with the development of biology, control of biological system becomes into a hot topic19–29. As to the research of genetic regulatory networks (GRNs), one of the major goals is to carry out the therapeutic intervention strategies for diseased targets30,31. Correspondingly, Boolean control networks (BCNs) as a theoretical branch of the above studies provide an efficient way to investigate the control of GRNs based on abstract models. So the interests to the BCNs are increasingly going up. The application of BCNs includes not only GRNs11, but also other various fields, such as man-machine dynamic game33 and internal combustion engines34,35. Recently, based on semi-tensor product (STP) proposed by Cheng, D. et al.36, many basic problems for BCNs have been investigated, for example, realization23, controllability24,26, optimal control15,33, etc.

Observability of a system is a structural property. It is also a fundamental concept in control theory and systematic science and, not surprisingly, it has found many applications in systems biology. As early as 1976, Cobelli et al. studied controllability, observability and structural identifiability of biological compartmental systems of any structure37. In evolutionary dynamics, observability is the key to study whether the genetic process itself can be recovered from measurements of phenotypic characteristics38. Observability analysis is a necessary preliminary step to the design of observers, that is, systems that provide an estimate of the complete internal state based on measurements of the inputs and outputs39. There have been many studies on the observability of BCNs in recent years. Cheng, D. et al. have investigated the controllability and observability of BCNs24. Li, F. et al. have studied the observability of time-delayed BCNs25. Laschov, D. et al. have considered the observability of BNs through a graph-theoretic approach39. Zhang, K. et al. have proposed a unified approach to determine all the four types of observability of BCNs in the literature40.

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From the view of systems biology, the analysis in system-level of biological regulation needs to consider the interactions of genes on a holistic level, rather than the independent characteristics of isolated parts of an organism. To understand the intricate variability of biological systems, where many hierarchical levels and interactions coexist, a new level of description is required. Thereupon, multiplex networks as an extension of complex networks were firstly proposed by Mucha in 2010, which is composed of several layered networks interrelated with each other shown in Fig. 1. The previous description implicitly assumes that all biochemical signals are equivalent and then collapses information from different pathways. Actually, in cellular biochemical networks, many different signaling channels do work in parallel. Not only in cellular biochemical, multiplex networks have been applied to the natural, social, and information sciences. As an old concept, multi-layer social networks have been studied from 1975. Transportation systems are natural candidates for a multi-layer network representation. In a recent paper, a two-layer structure has been created by merging the world wide port and airport networks. In multiplex networks, each layer could have particular features and dynamical processes. Between layers, interconnections are represented by some special nodes on behalf of different roles participating in multiple layers of interactions. The final states of those common nodes at each time are determined by all involved layers, which is different from the traditional sense of coupling.

Recent years more and more researchers studied the BMNs. For example, Xu, M. et al. investigated the synchronizability of two-layer networks. Cozzo talked about the stability of BMNs. Luo et al. studied the controllability of BMNs. Zhong, J. et al. studied controllability problem for multi-level Boolean control networks. But when it comes to the observability problem of BMNs, to our best knowledge, there have been even no results, because there are many differences between BMNs and single-layer BNs. Even for the degenerated BMNs, their observability are different from the single-layer BNs, for example the BCNs studied by Cheng, D. et al. and Li, F. et al. Because even when the number of layer is one, our system still has holistic states, which have logical relationship with the states in basic layers. From above discussions, we can know that a study of the observability of the BMNs is meaningful and challenging.

In this article, by following this stream of research, we first address and characterize observability of BCNs defined on multiple topological layers. Based on the model of Boolean multiplex networks presented by Cozzo et al., we introduce the input controls and the outputs. The model of BMNs are changed into algebraic representation using STP tools. We consider the observability of BMNs, following the standard formulation of the observability problem in systems theory, namely, we assume that the BN structure is known and that the goal is to infer the initial condition based on an output sequence. To clearly show the results of our research, we gave observable and unobservable examples in the final part of our essay.

The rest of this article is organized as follows. In Section II, we introduce the dynamic structure of BMNs. In Section III, some concepts and properties of the STP are introduced, and we change our network into discrete form using STP tool. In Section IV, necessary and sufficient conditions for the observability of the BMNs are obtained. Examples are given to show the effectiveness of the obtained results in Section V. Finally, a brief summary is given in Section VI.

Model of BMCNs

In this section, we introduce the model of BMCNs. For multiplex networks, different from the single-layer model, some nodes exist in multiple layers, the states which on different layers evolve independently of each other. The multiplex network we defined has \( N \) nodes per layer and \( K \) layers, and the number of total different nodes is \( n = N \cdot K \). For example in Fig. 1, we have that \( N = 4, K = 2, n = 5 \). For statement ease, we define some related notions.

- \( \mathcal{D} \) is the set of \([0, 1]\).
- \( a_{ij} \in \mathcal{D}, \) and \( a_{ij} = 1 \) if node \( i \) in the \( l \) layer and 0 otherwise. The layers set of node \( i \) is \( \mathcal{L}(i) = \{l \in \{1,2,\ldots,K\} \mid a_{il} = 1 \} \) which refers the set of \( l \) which has \( a_{il} = 1 \).
- \( \gamma_{lj} \in \mathcal{D} \), and \( \gamma_{lj} = 1 \) if node \( j \) is the incoming neighbors of node \( i \) in the \( l \) layer. The incoming neighbors set of node \( i \) in the \( l \) layer is \( \Gamma_{in}(i)(l) = \{ j_{1}, j_{2}, \ldots, j_{l} \} \) which refers the set of \( j \) which has \( \gamma_{lj} = 1 \). And we set \( \Gamma_{in}(i) = \bigcup_{l=1}^{K} \Gamma_{in}(i)(l) \).

In Fig. 1, we have that the layers set of node 1 is \( \mathcal{L}(1) = \{1,2\} \), and \( a_{11} = 1 \) and \( a_{12} = 1 \), the layers set of node 2 is \( \mathcal{L}(2) = \{1\} \), and \( a_{21} = 1 \) and \( a_{22} = 0 \). The incoming neighbors set of node 1 in layer 1 is \( \Gamma_{in}(1)(1) = \{j_{1}, j_{2}\} \) and \( \Gamma_{in}(1)(2) = \{j_{1}\} \).

In each layer, for the specific \( l \in \{1,2,\ldots,K\} \), if \( a_{il} = 1 \), we assume \( x_i^{l}(t) \) represents the state of node \( i \) on layer \( l \) at time \( t \), then the update dynamics of state \( x_i^{l} \) can be described as

\[
\begin{align*}
  x_i^{l}(t+1) &= f_i^{l}(x_{\in \Gamma_{in}(i)(l)}(t)), \quad i=1,2,\ldots,n. 
\end{align*}
\]  

(1)

where \( x_i^{l}(t) \in \mathcal{D}, f_{i}^{l} \) is the updating rule of node \( i \) on layer \( l \).

Furthermore, assume \( \bar{x}_{i}(t), i \in \{1,2,\ldots,n\} \) represents the holistic states of node \( i \) at time \( t \), which means the global states of \( x_i^{1}(t), x_i^{2}(t),\ldots,x_i^{n}(t) \), see Fig. 2. \( \bar{x}_{i}(t+1) \) is influenced by \( x_i^{1}(t+1), x_i^{2}(t+1),\ldots,x_i^{n}(t+1) \), so we describe it as

\[
\begin{align*}
  \bar{x}_{i}(t+1) &= \bar{f}(x_{i}^{1}(t+1), x_{i}^{2}(t+1),\ldots,x_{i}^{n}(t+1)), \quad i=1,2,\ldots,n.
\end{align*}
\]  

(2)

where \( . \) is the canaliszing function.
When considering control-related problems for BMNs, based on above system structure, we introduce the \( m \)-dimensional control \( u(t), u_2(t), \ldots, u_m(t) \in D \) as the inputs of the system, correspondingly, then we have the outputs \( y_1(t), y_2(t), \ldots, y_p(t) \in D \), then the BMCNs can be described as
\[
x_i^j(t + 1) = f_i^j \left( x_i^j(t), u_1(t), u_2(t), \ldots, u_m(t) \right),
\]
and
\[
\bar{x}_i(t + 1) = f_i \left( x_i^1(t + 1), x_i^2(t + 1), \ldots, x_i^m(t + 1), u_1(t), u_2(t), \ldots, u_m(t) \right), \quad i = 1, 2, \ldots, n,
\]
where \( f_i \) is the canalizing function of node \( i \) with the controls \( u_1(t), u_2(t), \ldots, u_m(t) \), see Fig. 3.

In finally, the output dynamics of the BMCNs are given by the following equation
\[
y_j(t) = h_j \left( \bar{x}_1(t), \bar{x}_2(t), \ldots, \bar{x}_n(t) \right), \quad j = 1, 2, \ldots, p,
\]
where \( h_j \) is the output function.

**Remark 1.** The BMCNs are not the simple superposition of \( K \) single-layer BCNs. Because we have the holistic states which are affected by corresponding states in each layers. Between the holistic states with the states in basic layers, we have the canalizing functions \( f_i \) which determine the value of the holistic states. Even when the number of layer is one, our system still has holistic states, which have logical relationship with the states in basic layers through the canalizing functions. So it is still different from the one layer BNs.

**Algebraic representation of BMCNs**

In this section, we introduce some concepts and properties, changing our BMCNs into algebraic representation.

**Concepts and properties of the semi-tensor product.** In this subsection, some concepts and properties of the STP will be briefly introduced\(^{36}\).

**Definition 1.**\(^{36}\)
- Let \( X \) be a row vector of dimension \( np \), and \( Y = [y_1, y_2, \ldots, y_p]^T \) be a column vector of \( p \) dimension. Then we split \( X \) into \( p \) equal-size blocks as \( X_1, X_2, \ldots, X_p \), which are \( 1 \times n \) rows. Define the STP, denoted by \( \otimes \), as
  \[
  X \otimes Y = \sum_{i=1}^{p} X_i^j y_j \in \mathbb{R}^n,
  \]
- Let \( A \in \mathbb{R}^{m \times n} \) and \( B \in \mathbb{R}^{p \times q} \). If either \( n \) is a factor of \( p \), say \( n = pt \), and denote it as \( A \prec_j B \), or \( p \) is a factor of \( n \), say \( n = pt \), and denote it as \( A \succ_j B \), then we define the STP of \( A \) and \( B \), denoted by \( C = A \otimes B \), as the following: \( C \) consists of \( m \times q \) blocks as \( C = (C_{ij}) \) and each block is
  \[
  C_{ij} = A^i \otimes B_j, \quad i = 1, 2, \ldots, m, \quad j = 1, 2, \ldots, q,
  \]
  where \( A^i \) is the \( i \)-th row of \( A \) and \( B_j \) is the \( j \)-th column of \( B \).
  And here we give some fundamental properties of the STP in the following\(^{36}\):

**Lemma 1.**\(^{36}\) Assume \( A \succ_j B \), then (where \( \otimes \) is the Kronecker product, \( I_i \) is the identity matrix).
\[
A \otimes B = A (B \otimes I_j).
\]
Assume \( A \prec_j B \), then
\[
A \otimes B = (A \otimes I_j) B.
\]

**Lemma 2.**\(^{36}\) Assume \( A \in M_{m \times n} \) is given.
- Let \( Z \in \mathbb{R}^l \) be a row vector. Then
  \[
  A \otimes Z = Z \otimes (I_z \otimes A).
  \]
- Let \( Z \in \mathbb{R}^l \) be a column vector. Then
  \[
  Z \otimes A = (I_z \otimes A) \otimes Z.
  \]

It is easy to find out that STP of matrix can be seen as a generalization of conventional matrix product. All the fundamental properties of conventional matrix product, such as distributive rule, associative rule, remain true. So we can omit the symbol \( \otimes \).
Here we defined some notions for statement ease.

- \( \Delta_n = \{ \delta^i_n | i = 1, 2, \ldots, n \} \), where \( \delta^i_n \) denotes the \( i \)-th column of the identify matrix \( I_n \).
- Assume a matrix \( A = [\delta^i_1, \delta^i_2, \ldots, \delta^i_n] \), where \( 1 \leq i_1, i_2, \ldots, i_n \leq n \) are positive integer constants. We can briefly denote it as \( A = \delta_{[i_1, i_2, \ldots, i_n]} \).
- A matrix \( A \in \mathbb{R}^{m \times n} \) is called a logical matrix if the columns of \( A \), denoted by \( \text{Col}(A) \), satisfy \( \text{Col}(A) \subset \Delta_m \).

And the set of \( m \times n \) logic matrices is denoted by \( \mathcal{L}_{m \times n} \).

Then we define a swap matrix \( W_{[m,n]} \in \mathcal{L}_{mn \times mn} \), which is constructed in the following way: label its columns by \((11, 12, \ldots, m1, m2, \ldots, mn)\) and its rows by \((11, 21, \ldots, m1, n2, n, \ldots, mn)\). And its element in the position \((i, j, (i, j))\) is assigned as

\[
\omega_{(i,j),(i,j)} = \delta_{ij}^{(i,j)} = \begin{cases} 1, & \text{if } i = j, \\ 0, & \text{otherwise}. \end{cases}
\]

And we denote \( W_{[m,n]} = W_{[m,n]} \) when \( m = n \).

**Lemma 3.** Let \( X \in \mathbb{R}^m \) and \( Y \in \mathbb{R}^n \) be two columns. Then

\[
W_{[m,m]} \ast X \ast Y = Y \ast X \quad \text{and} \quad W_{[n,m]} \ast Y \ast X = X \ast Y.
\]

For the logical function with \( n \) arguments \( f : \mathcal{D}^n \rightarrow \mathcal{D} \), we can convert it into an algebraic function using the STP of matrices. A logical domain, denoted by \( \mathcal{D} \), is defined as \( \mathcal{D} = \{ T = 1, F = 0 \} \). We identify each element in \( \mathcal{D} \) with a vector as \( T \sim \delta_2^1, F \sim \delta_2^2 \) and \( \Delta = \{ \delta_2^1, \delta_2^2 \} \sim \mathcal{D} \). Based on this, we have

**Lemma 4.** Any logical function \( f(x_1, x_2, \ldots, x_n) \) with logical arguments \( x_1, x_2, \ldots, x_n \in \Delta \), can be expressed in a multi-linear form as

\[
f(x_1, x_2, \ldots, x_n) = M_f \ast x_1 \ast x_2 \ast \ldots \ast x_n,
\]

where \( M_f \in \mathbb{R}^{2 \times 2^n} \) is unique, which is called the structure matrix of logical function \( f \).

And here we give another lemma:

**Lemma 5.** Assume \( X_n = x_1x_2 \ldots x_n = \wedge_{i=1}^n x_i \), then

\[
X_n^2 = \Phi_n X_n,
\]

where

\[
\Phi_n = \prod_{i=1}^n I_2^{x_i-1} \otimes \left( I_2 \otimes W_{[2,2^{x_i-1}]} \right) M_i.
\]

Here \( M_i = \delta_{[1,4]} \), which is power-reducing matrix and it can be verified that \( \mathbb{F}^2 = M, P, \forall P \in \Delta \).

Based on the above properties of STP, we put forward an obvious proposition.

**Proposition 1.** For each \( i \in \{ 1, 2, \ldots, n \} \), if \( \Gamma_{x_i(i)} = \{ i_1, j_2, \ldots, j_r \} \), \( ( r \leq n ) \), we can find a matrix \( R \), such that

\[
R_i x_1 x_2 \ldots x_n = x_{i_1} x_{i_2} \ldots x_{i_r}.
\]

**Algebraic structure of the BMCNs.** In this subsection, we change our BMCNs into discrete version using STP tool. To express it more clearly, here we give some description of variables.

- \( x(t) = \wedge_{i=1}^n x_i(t) \) means the state of layer \( l \).
- \( u(t) = \wedge_{i=1}^n u_i(t) \) means the control,
- \( x(t) = \wedge_{i=1}^n x^i(t) \) means the state of all layers.

In the following step we will change the given BMCNs (3)-(4) into algebraic representation, as we will find out the algebraic relation between \( x(t + 1) \) and \( x(t) \) as well as the algebraic relation between \( x(t + 1) \) and \( x(t) \).

At the first place, we will find out the algebraic relation between \( x(t + 1) \) and \( x(t) \). Using lemma 4 and proposition 1, for each logical rule \( f_j^i \), we can find its structure matrix \( M_j^i \), so we obtain that

\[
\begin{align*}
x_i'(t + 1) &= M_i^j u_i(t) x_i(t) \ldots u_m(t) x_m(t) x_j^i(t) x_j^i(t) \ldots x_j^i(t) \\
&= M_i^j u_i(t) x_i(t) \ldots u_m(t) R^j \wedge_{u_i=1} x_i(t) \\
&= M_i^j u(t) R x_i(t) \\
&= M_i^j u(t) x_i(t)
\end{align*}
\]

where \( x(t) = \wedge_{u_i=1} x_i(t) \), \( u(t) = \wedge_{i=1}^m u_i(t) \), and \( M_i^j = M_i^j (I_2^n \otimes R^j) \).

Hence
Figure 1. Schematic of multiple networks with two layers. Here $K=2$ means that the system is a two layers network. $N=4$ means that there are four nodes in each layer. And $n=5$ means that there are five total different nodes in the system. $x_1^1$, $x_2^1$, $x_3^1$, $x_4^1$ are the states in layer 1, and $x_1^2$, $x_2^2$, $x_3^2$, $x_4^2$ are the states in layer 2.

\[
x^l(t+1) = \kappa_{l-1} x^l(t+1) \\
= x^l_1(t+1) x^l_2(t+1) \ldots x^l_{n_l}(t+1) \\
= \tilde{M}_i^l u(t)x^l_1(t) \tilde{M}_i^l u(t)x^l_2(t) \ldots \tilde{M}_i^l u(t)x^l_{n_l}(t) \\
= \tilde{M}_i^l (I^{m+N} \otimes \tilde{M}_i^l) \Phi_{m+N} u(t)x^l_1(t) \tilde{M}_i^l u(t)x^l_2(t) \ldots \tilde{M}_i^l u(t)x^l_{n_l}(t) \\
= \ldots \\
= \tilde{M}_i^l (I^{m+N} \otimes \tilde{M}_i^l) \Phi_{m+N} (I^{m+N} \otimes \tilde{M}_i^l) \\
\times \Phi_{m+N} \ldots (I^{m+N} \otimes \tilde{M}_i^l) \Phi_{m+N} u(t)x^l_1(t) \tilde{M}_i^l u(t)x^l_2(t) \ldots \tilde{M}_i^l u(t)x^l_{n_l}(t) \\
\triangleq L^l u(t)x^l(t),
\]

with $L^l = \tilde{M}_i^l \kappa_{n_l} \{(I^{m+N} \otimes \tilde{M}_i^l) \Phi_{m+N}\}$. And we have defined that $x(t) = \kappa_{n_l} x^l(t)$, so we obtain that

\[
x(t+1) = x^l(t+1) x^2(t+1) \ldots x^K(t+1) \\
= L^1 u(t)x^1(t) L^2 u(t)x^2(t) \ldots L^K u(t)x^K(t) \\
= L^1 (I^{m+N} \otimes L^2) W_{m+2m+N} \Phi_m u(t)x^1(t)x^2(t) \ldots L^K u(t)x^K(t) \\
= \ldots \\
= L^1 (I^{m+N} \otimes L^2) W_{m+2m+N} \Phi_m \ldots (I^{m+N} \otimes L^K) \\
\times W_{m+2m+N} \Phi_m u(t)x^1(t)x^2(t) \ldots x^K(t) \\
\triangleq Lu(t)x(t),
\]

where $L = L^1 \kappa_{K-2} \{(I^{m+N} \otimes L^1) W_{m+2m+N} \Phi_m\}$.

Subsequently, we will find out the algebraic relation between $\tilde{x}(t+1)$ and $x(t)$. Using the similar steps above, the algebraic representation of (4) can be obtained as
\[ \bar{x}_i(t + 1) = \bar{M}_i u(t) x_i(t) \cdots u_m(t) x_i^m(t + 1) x_i^m(t + 1) \]
\[ = \bar{M}_i u(t) R_i x_i^1(t + 1) x_i^K(t + 1) \]
\[ = \bar{M}_i u(t) R_i \bar{M}_i u(t) x_i^1(t) \bar{M}_i u(t) x_i^2(t) \cdots \bar{M}_i u(t) x_i^K(t) \]
\[ = \bar{M}_i (I^{2^n} \otimes \bar{M}_i^d) \Phi_m u(t) x_i^1(t) \bar{M}_i u(t) x_i^2(t) \cdots \bar{M}_i u(t) x_i^K(t) \]
\[ = \bar{M}_i (I^{2^n} \otimes \bar{M}_i^d) \Phi_m (I^{2^n+\chi} \otimes \bar{M}_i^d) W_{[2^n,2^{n+\chi}]} \Phi_m \]
\[ \times u(t) x_i^1(t) x_i^2(t) \cdots \bar{M}_i u(t) x_i^K(t) \]
\[ = \bar{M}_i (I^{2^n} \otimes \bar{M}_i^d) \Phi_m [K_{i=2}^{K} \{ (I^{2^n+\chi-1} \otimes \bar{M}_i^d) W_{[2^n,2^{n+\chi-1}]} \Phi_m \}] u(t) x_i^1(t) x_i^2(t) \cdots x_i^K(t) \]
\[ = \bar{L}_i u(t) x_i(t), \]

where \( \bar{L}_i = \bar{M}_i (I^{2^n} \otimes \bar{M}_i^d) \Phi_m [K_{i=2}^{K} \{ (I^{2^n+\chi-1} \otimes \bar{M}_i^d) W_{[2^n,2^{n+\chi-1}]} \Phi_m \}] \) is the structure matrix of logical function \( \bar{f}_i \). And we have that \( x(t) = \kappa_{i=1}^{K} x_i(t) \). So we obtain that

\[ \bar{x}(t + 1) = \bar{L}_i u(t) x(t) \bar{x}_i(t + 1) \bar{x}_j(t + 1) \cdots \bar{x}_p(t + 1) \]
\[ = \bar{L}_i u(t) x(t) \bar{L}_i u(t) x(t) \bar{L}_i u(t) x(t) \cdots \bar{L}_i u(t) x(t) \]
\[ = \bar{L}_i (I^{2^n+\chi} \otimes \bar{L}_i) \Phi_{m+\chi} u(t) x(t) \bar{L}_i u(t) x(t) \cdots \bar{L}_i u(t) x(t) \]
\[ = \bar{L}_i (I^{2^n+\chi} \otimes \bar{L}_i) \Phi_{m+\chi} (I^{2^n+\chi} \otimes \bar{L}_i) \Phi_{m+\chi} (I^{2^n+\chi} \otimes \bar{L}_i) \Phi_{m+\chi} \cdots (I^{2^n+\chi} \otimes \bar{L}_i) \Phi_{m+\chi} u(t) x(t) \]
\[ \triangleq \bar{L} u(t) x(t), \]

where \( \bar{L} = \bar{L}_i (I^{2^n+\chi} \otimes \bar{L}_i) \Phi_{m+\chi} (I^{2^n+\chi} \otimes \bar{L}_i) \Phi_{m+\chi} \cdots (I^{2^n+\chi} \otimes \bar{L}_i) \Phi_{m+\chi} \).

Means that

\[ \bar{x}(t + 1) = \bar{L} u(t) x(t). \]

Similarly, by letting \( y(t) = y_1(t) y_2(t) \cdots y_p(t) \), we obtain the algebraic expression of the output dynamics (5) as follows:

\[ \bar{y}(t + 1) = \bar{L} y(t). \]
\[ y(t) = H\bar{x}(t). \] (10)

where \( H = H \cup_{j=1}^p ((I_H \otimes H_j)\Phi_j) \), here \( H_j \) are the structure matrices of \( h_j, j=1, 2, ..., p \). Here we give an example to illustrate this process.

**Example 1.** Consider following two-layer BMCN, with \( N = 2, K = 2, n = 4 \) and \( m = 4 \)

\[
\begin{align*}
&l = 1: \\
&x_1^1(t+1) = u_1(t) \leftrightarrow x_1^3(t), \\
&x_1^4(t+1) = u_2(t) \leftrightarrow x_1^4(t).
\end{align*}
\]

\[
\begin{align*}
&l = 2: \\
&x_2^3(t+1) = u_1(t) \leftrightarrow x_2^3(t), \\
&x_2^4(t+1) = u_2(t) \leftrightarrow x_2^4(t).
\end{align*}
\]

and we have that

\[
\begin{align*}
\bar{x}_1(t+1) &= \neg x_1^1(t+1), \\
\bar{x}_2(t+1) &= x_1^1(t+1), \\
\bar{x}_3(t+1) &= \neg x_2^3(t+1), \\
\bar{x}_4(t+1) &= x_2^3(t+1),
\end{align*}
\]

where \( \neg, \lor, \land, \Rightarrow \) and \( \leftrightarrow \) represent the logical functions of negation, disjunction, conjunction, implication, and equivalence, respectively. Based on Lemma 4, we obtain the corresponding structure matrices of those logical operators, as given in Table 1.

Define \( \bar{x}(t) = \cup_{j=1}^4 \bar{x}_j, u(t) = \cup_{j=1}^4 u_j(t) \). Then we calculate the control-depending network transition matrix of system.
Table 1. Structure matrices of some basic logical functions.

| \( f(x_1,x_2) \) | \(~x_1\) | \( x_1 \lor x_2\) | \( x_1 \land x_2\) | \( x_1 \rightarrow x_2\) | \( x_1 \leftrightarrow x_2\) |
|----------------|--------|-----------------|-----------------|-----------------|-----------------|
| \( M_f \)     | \( M_{\overline{f}}[2,1] \) | \( M_f[1,1,1,2] \) | \( M_f[1,2,2,2] \) | \( M_f[1,2,1,1] \) | \( M_f[1,2,2,1] \) |

\[
l = 1: \begin{cases} 
    x_1^1(t + 1) = M_{\mu_1}(t)x_1^1(t), \\
    x_2^1(t + 1) = M_{\mu_2}(t)x_2^1(t).
\end{cases}
\]

\[
l = 2: \begin{cases} 
    x_1^2(t + 1) = M_{\mu_3}(t)x_1^2(t), \\
    x_2^2(t + 1) = M_{\mu_4}(t)x_2^2(t).
\end{cases}
\]

and we have that

\[
\begin{align*}
\tilde{x}_1(t + 1) &= M_{\mu_1}x_1^1(t + 1), \\
\tilde{x}_2(t + 1) &= x_2^1(t + 1), \\
\tilde{x}_3(t + 1) &= M_{\mu_2}x_2^2(t + 1), \\
\tilde{x}_4(t + 1) &= x_2^2(t + 1).
\end{align*}
\]

Then

\[
x(t + 1) = x_1^1(t + 1)x_2^2(t + 1)
\]

\[
= x_1^1(t + 1)x_2^1(t + 1)x_2^1(t) + x_1^1(t + 1)x_2^2(t + 1)x_2^2(t)
\]

\[
= M_{\mu_1}(t)x_1^1(t)M_{\mu_2}(t)x_2^1(t)M_{\mu_3}(t)x_2^1(t)M_{\mu_4}(t)x_2^2(t)
\]

\[
= \ldots
\]

\[
= \delta_{1234}[1,2,3,4,5,6,7,8,9,10,11,12,13,14,15,16,2,1,4,3,6,5,8,7,10,9,12,11,10,13,15,16,14,12,7,6,5,3,4,1,2,16,15,14,13,12,11,10,9,8,7,6,5,4,3,2,1]
\]

\[
\Delta \rightarrow Lu(t)x(t)
\]

\[
\tilde{x}(t + 1) = \tilde{x}_1(t + 1)\tilde{x}_2(t + 1)\tilde{x}_3(t + 1)\tilde{x}_4(t + 1)
\]

\[
= M_{\mu_1}x_1^1(t + 1)x_2^1(t + 1)x_2^1(t + 1)x_2^2(t + 1)x_2^2(t + 1)
\]

\[
= \ldots
\]

\[
= \delta_{1234}[11,12,9,10,15,16,13,14,3,4,1,2,1,8,7,6,5,9,10,11,12,13,14,15,16,1,2,3,4,5,6,7,8,10,9,12,11,14,13,15,16,10,9,8,7,6,5,3,4,1,2,3,4,1,2,7,8,5,6,12,11,10,9,16,15,14,13,4,3,2,1,8,7,6,5,3,12,14,15,16,9,10,11,12,5,6,7,8,9,10,12,11,14,13,16,15,10,9,12,11,6,5,8,7,2,1,4,3,15,16,13,14,11,12,9,10,7,8,5,6,3,4,1,2,16,15,14,13,12,11,10,9,8,7,6,5,4,3,2,1]
\]

\[
\Delta \rightarrow Lu(t)x(t)
\]

Here we have found out the algebraic relation between \( x(t + 1) \) and \( x(t) \) as well as the algebraic relation between \( \tilde{x}(t + 1) \) and \( x(t) \). Furthermore, we assume that

\[
\begin{align*}
    y_1(t) &= \neg \tilde{x}_1(t), \\
    y_2(t) &= \neg \tilde{x}_2(t), \\
    y_3(t) &= \neg \tilde{x}_3(t), \\
    y_4(t) &= \neg \tilde{x}_4(t).
\end{align*}
\]
Then, according to properties of STP, we obtain the matrix expression of output, as follows

\[ y(t) = y_1(t) y_2(t) y_3(t) y_4(t) \]

\[ = M_1 \tilde{x}(t) M_2 \tilde{x}(t) M_3 \tilde{x}(t) M_4 \tilde{x}(t) \]

\[ = \delta_1 [16, 15, 14, 13, 12, 11, 10, 9, 8, 7, 6, 5, 4, 3, 2, 1] \tilde{x}(t) \]

\[ \triangleq H \tilde{x}(t). \]

**Observability of BMCNs**

In this section, we will analyze and characterize the observability of the BMCNs, with two different types of controls. We first provide some definitions as follows.

Consider the BMCN (3)-(4) with output dynamics (5). For any initial state \( x(0) \), and control input sequence \( u = \{u(0), u(1), \ldots, u(s)\} \), the holistic trajectory at time \( t \) is denoted by \( \tilde{x}(t; u, x(0)) \). Output trajectory at time \( t \) denote by \( y(t; u, x(0)) \).

**Definition 2.** The BMCN (3)-(5) is observable if there exists a finite control sequence \( u = \{u(0), u(1), \ldots, u(s)\} \), with \( s > 0 \), such that for any \( \delta_i, \delta_j \in \Delta_{2^{2m}} \), with \( i \neq j \), we have \( y(t; u, \delta_i) \neq y(t; u, \delta_j) \) for some \( t \in [1, s] \).

In other words, there exists a control input sequence for which the initial state can be uniquely determined from the knowledge of the output sequence.

**Remark 2.** Our definition is motivated by the definition of observability for BCNs proposed in Laschov , et al. [39], which is different from the one proposed by Cheng, D. et al. [24]. In Cheng, D. et al. [24], a BCN is said to be observable if the initial state can be uniquely determined from the knowledge of the control inputs (which may depend on the initial state) and the outputs.

We consider two kinds of controls. The first is that the controls are determined by certain logical rules, which called the input networks.

\[ u_j(t + 1) = g_j(u_j(t), u_{j-1}(t), \ldots, u_m(t)), \]

\[ u_j(t + 1) = g_j(u_j(t), u_{j+1}(t), \ldots, u_m(t)), \]

\[ u_n(t + 1) = g_n(u_n(t), u_{n+1}(t), \ldots, u_m(t)). \]

(11)

where \( g_j: \Delta_{2^m} \rightarrow \Delta, i = 1, 2, \ldots, m \), are logical function. According Lemma 4, we know that the input network (11) can be expressed as

\[ u_j(t + 1) = G_j u(t), \]

where \( G_j, j = 1, \ldots, m \) are the structure matrix of logical function \( g_n \), respectively. Then,

\[ u(t + 1) = G_1 u(t) G_2 u(t) \ldots G_m u(t) \triangleq G u(t). \]

(12)

with \( G = G_1 \otimes \cdots \otimes G_m \).

**Theorem 1.** Consider (3)-(5) (or equivalently (8)-(10)) with input network control (11). The system is observable if and only if there exists finite time \( s, s > 0 \), such that \( \text{rank}(O_{k^s}) = 2^{2m} \), for some \( \delta_k \in \Delta_{2^{2m}} \), where

\[ O_{k} = \left( \begin{array}{c}
H L & H L G (I_{2^{m}} \otimes L) \Phi_m \\
H L G^2 (I_{2^{m}} \otimes LG) (I_{2^{2m}} \otimes L) (I_{2^{m}} \otimes \Phi_m) \Phi_m \\
\vdots \\
H L G^{m-1} (I_{2^{m}} \otimes LG^{m-2}) \cdots (I_{2^{m-2}} \otimes L) (I_{2^{l-2}} \otimes \Phi_m) \cdots \Phi_m
\end{array} \right). \]

**Proof.** By considering the input network, put together (8)-(9) with (12), we can obtain the system

\[ \begin{aligned}
u(t + 1) &= G u(t), \\
x(t + 1) &= L u(t)x(t), \\
\tilde{x}(t + 1) &= L u(t)x(t), \\
y(t) &= H \tilde{x}(t).
\end{aligned} \]

(13)

A straightforward computation shows, we calculate the output \( \{y(0), y(1), \ldots, y(s)\} \) as follows...
\[ y(1) = H\tilde{x}(1) = H\tilde{L}u(0)x(0), \]
\[ y(2) = H\tilde{x}(2) = H\tilde{L}u(1)x(1) = H\tilde{L}G_{u}(0)Lu(0)x(0) = H\tilde{L}G(I_{2^{m}} \otimes L) \]
\[ \times \Phi_m u(0)x(0), \]
\[ y(3) = H\tilde{x}(3) = H\tilde{L}u(2)x(2) = H\tilde{L}G^2u(0)Lu(1)x(1) = H\tilde{L}G^2u(0) \]
\[ \times LG_{u}(0)Lu(0)x(0) = H\tilde{L}G^2(I_{2^{m}} \otimes LG) \]
\[ \times (I_{2^{m}} \otimes L)u(0)u(0)x(0) \]
\[ = H\tilde{L}G^2(I_{2^{m}} \otimes LG)(I_{2^{m}} \otimes L)u(0)\Phi_m u(0)x(0) = H\tilde{L}G^2(I_{2^{m}} \otimes LG) \]
\[ \times (I_{2^{m}} \otimes L)(I_{2^{m}} \otimes \Phi_m)u(0)u(0)x(0) \]
\[ = H\tilde{L}G^2(I_{2^{m}} \otimes LG)(I_{2^{m}} \otimes L)(I_{2^{m}} \otimes \Phi_m)\Phi_m u(0)x(0), \]
\[ \vdots \]
\[ y(s) = H\tilde{x}(s) = H\tilde{L}G^{s-1}(I_{2^{m}} \otimes LG^{s-2}) \ldots (I_{2^{m-1}} \otimes GL)(I_{2^{m-1}} \otimes L) \]
\[ \times (I_{2^{m-1}} \otimes \Phi_m) \ldots (I_{2^{m}} \otimes \Phi_m)\Phi_m u(0)x(0). \]

Hence, in the matrix form, we obtain

\[ (y(1)^T, y(2)^T, \ldots, y(s)^T)^T = O_u(0)x(0). \tag{14} \]

From the solution structure of the system of linear algebraic equations, we know that for some initial control input \( u(0) = \delta_{2^m}^K \), the system of linear equations \( (14) \) with \( 2^{NK} \)-dimension unknown vector \( x(0) \) has a unique solution if and only if the system matrix \( O_u(0) \delta_{2^m}^K \) has rank \( 2^{NK} \). That is, for some initial control input \( u(0) = \delta_{2^m}^K \), the initial state \( x(0) \) is uniquely determined by the knowledge of the output sequence \( y(1), y(2), \ldots, y(s) \) if and only if

\[ \text{rank}(O_u(0) \delta_{2^m}^K) = 2^{NK}. \tag{15} \]

This completes the proof.

**Remark 3.** From the proof of above theorem, we obtain that for some \( \delta_{2^m}^K \in \Delta_{2^m} \), if the matrix \( O_u(0) \delta_{2^m}^K \) has full column rank, means that \( \text{rank}(O_u(0) \delta_{2^m}^K) = 2^{NK} \), then the initial state \( x(0) \) can be reconstructed by the left inverse of \( (O_u(0) \delta_{2^m}^K) \) operation on output sequence \( y(1), y(2), \ldots, y(s) \).

In the following, we consider the case when the controls are free Boolean sequence. Precisely, \( m \) controls are described as \( u(t) = \kappa_{m}^{j_{m-1}}u_{j(t)} \) and freely designed.

**Theorem 2.** Consider \((3)-(4)\) and \((5)\) (or equivalently \((8), (9)\) and \((10)\)), with a free Boolean sequence control. The system is observable if and only if there exists a finite control sequence \( u(0) = \delta_{2^m}^K, u(1) = \delta_{2^m}^K, \ldots, u(s) = \delta_{2^m}^K \) with \( i_{0}, i_{1}, \ldots, i_{s} \in \{1, \ldots, 2^{m}\} \) such that \( \text{rank}(O_f) = 2^{NK} \), where

\[
O_f = \begin{bmatrix}
H\tilde{L}\delta_{2^m}^K \\
H\tilde{L}(I_{2^{m}} \otimes L)\delta_{2^m}^K \delta_{2^{m-1}}^K \\
H\tilde{L}(I_{2^{m}} \otimes L)(I_{2^{m}} \otimes L)\delta_{2^m}^K \delta_{2^m}^K \delta_{2^{m-1}}^K \\
\vdots \\
H\tilde{L}(I_{2^{m}} \otimes L)(I_{2^{m}} \otimes L) \ldots (I_{2^{m-1}} \otimes L) \delta_{2^m}^K \delta_{2^m}^K \delta_{2^m}^K \delta_{2^m}^K \delta_{2^m}^K \\
\end{bmatrix}.
\]

**Proof.** If the controls come from a free Boolean sequence, the system is that

\[
\begin{align*}
x(t + 1) &= Lu(t)x(t), \\
\tilde{x}(t + 1) &= \tilde{L}u(t)x(t), \\
y(t) &= H\tilde{x}(t). \tag{16}
\end{align*}
\]

If free control inputs \( [u(0), u(1), \ldots, u(s)] \) are given, then a straightforward computation shows the following:

\[
\begin{align*}
y(1) &= H\tilde{x}(1) = H\tilde{L}u(0)x(0), \\
y(2) &= H\tilde{x}(2) = H\tilde{L}u(1)x(1) = H\tilde{L}u(1)Lu(0)x(0) = H\tilde{L}(I_{2^{m}} \otimes L)u(1)u(0)x(0), \\
y(3) &= H\tilde{x}(3) = H\tilde{L}u(2)x(2) = H\tilde{L}u(2)Lu(1)x(1) = H\tilde{L}(I_{2^{m}} \otimes L)u(2)u(1)x(1) \\
&= H\tilde{L}(I_{2^{m}} \otimes L)u(2)u(1)Lu(0)x(0) = H\tilde{L}(I_{2^{m}} \otimes L) \\
&\quad \times (I_{2^{m}} \otimes L)u(2)u(1)u(0)x(0), \\
\vdots \\
y(s) &= H\tilde{x}(s) = H\tilde{L}(I_{2^{m}} \otimes L)(I_{2^{m}} \otimes L) \ldots (I_{2^{m-1}} \otimes L) \\
&\quad \times u(s - 1) \ldots u(2)u(1)u(0)x(0).
\end{align*}
\]
Hence, in the matrix form, we obtain
\[
(y(1)^T, y(2)^T, \ldots, y(s)^T)^T = O_J x(0).
\] (17)

As a similar analysis discussed in the proof of Theorem 1, we know that for a given free control inputs \(u(0), u(1), \ldots, u(s)\), the system of linear equations (17) with \(2^NK\)-dimension unknown vector \(x(0)\) has a unique solution if and only if the system matrix \(O_J\) has rank \(2^NK\). That is, for a given free control inputs \(u(0), u(1), \ldots, u(s)\), the initial state \(x(0)\) is uniquely determined by the knowledge of the output sequence \((y(1), y(2), \ldots, y(s))\) if and only if \(\text{rank}(O_J) = 2^{NK}\). Furthermore, as mentioned in Remark 3, the initial state \(x(0)\) can exactly calculated as \(x(0) = (O_J^2 O_J^{-1} O_J^T (y(1)^T, y(1)^T, \ldots, y(s)^T)^T)\). This completes the proof.

**Remark 4.** The observability in this paper is the observability of \(x(0) = \bigoplus_{i=1}^{K} x_i(0)\) which is the initial states of all layers in the initial time. Boolean control network (3)-(4) is observable if for the initial state \(x(0)\) is uniquely determined by the knowledge of the outputs \((y(1), y(2), \ldots, y(s))\). Based on the initial state \(x(0) \in \Delta_{s, NK}\), we can easily obtain the holistic states \(\tilde{x}_1(0), \tilde{x}_2(0), \ldots, \tilde{x}_{s}(0)\) through the canalizing function \(\tilde{J}_i (i = 1, 2, \ldots, n)\). So the holistic states \(\tilde{x}_1(0), \tilde{x}_2(0), \ldots, \tilde{x}_{s}(0)\) are also observable.

**Examples**

In this section, we will give some examples to illustrate our results. Example 2 is an observable case and Example 3 is an unobservable case.

**Example 2.** (Continue to Example 1) Consider the two-layer BMCN given in Example 1. Assume that the control inputs are determined by the following input network

\[
\begin{align*}
u_1(t + 1) &= -u_1(t), \\
u_2(t + 1) &= -u_2(t), \\
u_3(t + 1) &= -u_3(t), \\
u_4(t + 1) &= -u_4(t).
\end{align*}
\] (18)

Then we obtain that
\[
G = \delta_{16} [16, 15, 14, 13, 12, 11, 10, 9, 8, 7, 6, 5, 4, 3, 2, 1],
\]

If we take \(u(0) = \delta_{16}^{16}\), By calculation, we have
\[
\begin{align*}y(1) &= H_L a(0)x(0) = \delta_{16} [11, 12, 9, 10, 15, 16, 13, 14, 3, 4, 1, 2, 7, 8, 5, 6] x(0), \\
y(2) &= H_L G (I_{25} \otimes L) \Phi_{u0}(0)x(0) = \delta_{16} [11, 12, 9, 10, 15, 16, 13, 14, 3, 4, 1, 2, 7, 8, 5, 6] x(0), \\
y(3) &= H_L G^2 (I_{25} \otimes L) (I_{25} \otimes L) (I_{25} \otimes \Phi_{u0})(0)x(0) \\
&= \delta_{16} [6, 5, 8, 7, 2, 1, 4, 3, 14, 13, 16, 15, 10, 9, 12, 11] x(0).
\end{align*}
\]

Then we have that
\[
\begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
\]

\[
O_J \delta_{16}^{16} =
\begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
\]
And we can obtain that \( \delta_{c16}^{NK} = 2^{NK} = 16 \). Then from Theorem 1, we know that the system is observable under the input network (18).

**Example 3.** Consider the following two-layer BMCN, with \( N = 2, K = 2, n = 3 \) and \( m = 1 \)

\[
\begin{align*}
I &= 1: \begin{cases} 
  x^1_1(t + 1) &= u_1(t) \lor x^1_2(t), \\
  x^2_1(t + 1) &= u_1(t) \land x^1_1(t). 
\end{cases} \\
I &= 2: \begin{cases} 
  x^1_2(t + 1) &= u_1(t) \land x^1_2(t), \\
  x^2_2(t + 1) &= u_1(t) \lor x^2_1(t).
\end{cases}
\end{align*}
\]

and we have that

\[
\begin{align*}
\dot{x}_1(t + 1) &= u_1(t) \land (x^1_1(t + 1) \lor x^2_1(t + 1)), \\
\dot{x}_2(t + 1) &= u_1(t) \land x^1_2(t), \\
\dot{x}_3(t + 1) &= u_1(t) \lor x^2_2(t + 1).
\end{align*}
\]

Then we calculate the control-depending network transition matrix of system.

\[
\begin{align*}
I &= 1: \begin{cases} 
  x^1_1(t + 1) &= M_{d1}u_1(t)x^1_1(t), \\
  x^2_1(t + 1) &= M_{d2}u_1(t)x^2_1(t).
\end{cases} \\
I &= 2: \begin{cases} 
  x^1_2(t + 1) &= M_{d1}u_1(t)x^2_2(t), \\
  x^2_2(t + 1) &= M_{d2}u_1(t)x^2_1(t).
\end{cases}
\end{align*}
\]

and we have that

\[
\begin{align*}
\dot{x}_1(t + 1) &= M_{u1}(t)M_{d1}x^1_1(t + 1)x^1_2(t + 1), \\
\dot{x}_2(t + 1) &= M_{u1}(t)x^1_2(t + 1), \\
\dot{x}_3(t + 1) &= M_{d2}u_1(t)x^2_2(t + 1).
\end{align*}
\]

Then

\[
x(t + 1) = x^1(t + 1)x^2(t + 1) = x^1_1(t + 1)x^1_2(t + 1)x^2_1(t + 1)x^2_2(t + 1) = M_{d1}u_1(t)x^1_1(t)M_{u1}(t)x^1_2(t)M_{d2}u_1(t)x^2_1(t)M_{d2}u_1(t)x^2_2(t).
\]

Furthermore, we assume that

\[
\begin{align*}
y_1(t) &= -\dot{x}_1(t), \\
y_2(t) &= -\dot{x}_2(t), \\
y_3(t) &= -\dot{x}_3(t).
\end{align*}
\]

Then, according to properties of STP, we obtain the matrix expression of output, as follows
Now, we analyze the observability of this system, based on Theorem 2. We can calculate that while \( x(0) = \delta_{16}^1 \),

\[
x_{16}^1(1) = Lu(0)x(0) = LW_{16,2}x(0)u(0) = \delta_{16}[1, 5, 1, 5, 1, 5, 1, 5, 3, 7, 3, 7, 7, 7, 7, 7, 8, 8, 15, 15, 16, 16, 16, 7, 7, 8, 8, 15, 15, 16, 16]W_{16,2}\delta_{16}u(0) = \delta_{16}[5, 7]u(0),
\]

\[
\tilde{x}_{16}^1(1) = \tilde{L}u(0)x(0) = \tilde{L}W_{16,2}x(0)u(0) = \delta_{8}[1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 3, 1, 3, 1, 3, 3, 7, 7, 7, 7, 8, 7, 8, 7, 8, 7, 8, 7, 8, 8]W_{16,2}\delta_{16}u(0) = \delta_{8}[1, 7]u(0).
\]

And while \( x(0) = \delta_{16}^2 \), we have that

\[
x_{16}^2(1) = Lu(0)x(0) = LW_{16,2}x(0)u(0) = \delta_{16}[1, 5, 1, 5, 1, 5, 1, 5, 3, 7, 3, 7, 3, 7, 3, 7, 7, 7, 8, 8, 15, 15, 16, 16, 16, 7, 7, 8, 8, 15, 15, 16, 16]W_{16,2}\delta_{16}u(0) = \delta_{16}[5, 7]u(0),
\]

\[
\tilde{x}_{16}^2(1) = \tilde{L}u(0)x(0) = \tilde{L}W_{16,2}x(0)u(0) = \delta_{8}[1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 3, 1, 3, 1, 3, 3, 7, 7, 7, 7, 8, 7, 8, 7, 8, 7, 8, 7, 8, 8]W_{16,2}\delta_{16}u(0) = \delta_{8}[1, 7]u(0).
\]

Then, by induction, we easily obtain that, for any \( s > 0 \), and free control input \( u = \{u(0), u(1), \ldots, u(s)\} \), \( \tilde{x}(t, u, \delta_{16}^1) = \tilde{x}(t, u, \delta_{16}^1) \), and furthermore, \( y(t, u, \delta_{16}^1) = H\tilde{x}(t, u, \delta_{16}^1) = H\tilde{x}(t, u, \delta_{16}^1) = y(t, u, \delta_{16}^1) \). That implies, for any \( s > 0 \), and free control input \( u = \{u(0), u(1), \ldots, u(s)\} \)

\[
\mathcal{O}_f(u, s)\delta_{16}^1 = \mathcal{O}_f(u, s)\delta_{16}^1.
\]

So the linear homogeneous equation

\[
\mathcal{O}_f \dot{x} = 0
\]

has the non-zero solution. Then we obtain that for arbitrary \( s > 0 \), we still have that \( \text{rank}(\mathcal{O}_f) < 2^{NK} \), by Theorem 2, the system is unobservable.

**Conclusions**

In this paper, input controls were introduced into BMNs. By means of STP approach, the above logical dynamics has been converted into an algebraic form and the observability of dynamics is discussed. Firstly, we gave the theorem about the observability of whole dynamic system. Subsequently, the observability of each node in the special layer has been proved. Finally, we put forward some examples to illustrate our results.

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Additional Information

Author Contributions

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Author Contributions

Wei Wang put forward the idea, Yuhu Wu and Jingxue Xu demonstrated the results, Xi-Ming Sun pointed examples. All authors reviewed the manuscript.

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