THE GAUGE INVARIANT EFFECTIVE POTENTIAL: EQUILIBRIUM AND NON-EQUILIBRIUM ASPECTS

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Abstract

We propose a gauge invariant formulation of the effective potential in terms of a gauge invariant order parameter, for the Abelian Higgs model. The one-loop contribution at zero and finite temperature is computed explicitly, and the leading terms in the high temperature expansion are obtained. The result is contrasted to the effective potential obtained in several covariant gauge-fixing schemes, and the gauge invariant quantities that can be reliably extracted from these are identified. It is pointed out that the gauge invariant effective potential in the one-loop approximation is complex for all values of the order parameter between the maximum and the minimum of the tree level potential, both at zero and non-zero temperature. The imaginary part is related to long-wavelength instabilities towards phase separation. We study the real-time dynamics of initial states in the spinodal region, and relate the imaginary part of the effective potential to the growth rate of equal-time gauge invariant correlation functions in these states. We conjecture that the spinodal instabilities may play a role in non-equilibrium processes inside the nucleating bubbles if the transition is first order.

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I. INTRODUCTION AND MOTIVATION

In this article we are concerned with the effective potential in gauge theories. It has been recognized very early on, that the effective potential is a gauge dependent quantity \[\text{[1]}\] and only a limited amount of information extracted from it is actually physically meaningful. This gauge dependence can be understood from several equivalent points of view. The effective potential can be identified with the generating functional of 1-particle irreducible Green’s functions (the effective action) at zero four momentum transfer, and therefore it is an off-shell quantity. Alternatively, the effective potential is identified with the energy (or free energy) of a particular state (or ensemble) constrained to have an homogeneous expectation value of the scalar field.

The energy, or the free energy, is usually calculated by fixing a particular gauge in the path-integral. In a gauge theory, the (complex) scalar fields transform under gauge transformations and their expectation value in a gauge fixed state or ensemble is obviously a gauge dependent quantity. Despite this shortcoming it has been recognized that certain quantities are gauge independent. Dolan and Jackiw \[\text{[1]}\] recognized that the critical temperature is a gauge invariant quantity and recently Metaxas and Weinberg \[\text{[2]}\] used the Nielsen identities \[\text{[3]}\] to prove that the bubble nucleation rate \textit{at zero temperature} is gauge invariant (we are not aware of a similar proof at finite temperature). The gauge invariance of these quantities can be understood from the fact that they are associated with homogeneous and inhomogeneous extrema of the effective action respectively; these are known to be gauge invariant.

Considerable effort has been devoted to constructing a gauge invariant effective action and effective potential \[\text{[4]}\], of which the background field method of Vilkovisky and Dewitt \[\text{[4]}\] is the most popular \[\text{[5]}\]. These formulations of effective actions are technically formidable and do not readily lend themselves to a manageable formulation of equilibrium or non-equilibrium descriptions. Furthermore, it has recently been pointed out using the pinch-technique, that despite its formal gauge invariance, implementation of the background field method requires a gauge fixing parameter for the fluctuations. This leads to a gauge-parameter dependence in finite parts of self-energies at finite temperature \[\text{[6,7]}\], which in turn leads to a gauge dependence of the thermal renormalization group beta function as discussed in detail by Sasaki \[\text{[7]}\].

Alternative formulations of effective potentials have been offered in terms of a radial and angular decomposition of the complex scalar fields \[\text{[8,9]}\] or alternatively in terms of gauge invariant composite operators \[\text{[10]}\]. There are several shortcomings in the formulation of the effective potential in terms of the radial field variable \[\text{[8,9]}\] or composite operators \[\text{[10]}\]. This variable is understood as the “square root” of a composite operator that requires a \textit{subtraction} to be renormalized; shortcomings of this approach had been already recognized \[\text{[8]}\]. Furthermore in the path integral evaluation there is an ambiguous Jacobian arising from the change of variables to radial and angular fields. This Jacobian has to be incorporated in the perturbative expansion to obtain a consistently renormalized effective action \[\text{[8]}\].

However, even when these technicalities are overcome by some renormalization scheme (such as dimensional regularization), it is conceptually unclear how to interpret symmetry breaking in terms of the radial field. At the operator level, the radial field variable acquires a ground state expectation value \textit{even in the symmetric phase} as can be seen with a simple example of a two-dimensional isotropic harmonic oscillator. Composite bilinear operators...
typically require subtractions to be renormalized, and their expectation value is therefore ambiguous.

Our motivation is to obtain a gauge invariant description of the effective potential (and eventually the effective action) and to use it to provide some preliminary information on the dynamics of non-equilibrium processes during the phase transition in gauge theories. In view of the above discussion and critique of previous approaches, such an enterprise is clearly worthwhile because only a truly gauge invariant description of effective potentials can be considered trustworthy in terms of extracting physical quantities such as supercooling temperature, latent heat and others that are very important in the quantitative description of non-equilibrium features.

Our program for the construction of an effective potential can be summarized in the following steps: a) select the gauge invariant states of the theory, namely, those that are annihilated by the first class constraints (such a selection will not involve any gauge fixing), b) recognize a gauge invariant order parameter that is invariant under local gauge transformations, but transforms non-trivially under the global symmetry that can be spontaneously broken, and c) construct the effective potential for this gauge invariant order parameter. An attempt to establish a finite temperature framework in terms of gauge invariant states has been reported previously [11] within a different context and with a different goal, but to our knowledge it has not been implemented or attempted within the context of the effective potential.

In this article we focus on such a description for the Abelian Higgs model (scalar electrodynamics) and we expect to generalize the procedure and its quantitative implementation to Yang-Mills theories in the near future.

In section II we implement the first step of the program, that is, we select the gauge invariant states and order parameter without fixing a gauge, by requiring that the physical states be annihilated by the first class constraints of the theory which are recognized as the generators of local gauge transformations. Gauge invariant operators are then recognized as those that commute with these constraints, out of which we recognize the proper order parameter. In section III we explicitly construct the one loop effective potential both at zero and non-zero temperature and compare our results with those obtained in popular covariant gauges. From this comparison we establish when the gauge-fixed results lead to physical (gauge independent) predictions. In this section we also argue that the gauge dependence of the usual (gauge fixed) effective potential is not relieved by hard-thermal loop resummation. We also provide the high-temperature expansion of the gauge invariant effective potential and point out that the “cubic” terms which are typically taken as a signal of the strength of a first order transition are in general complex and gauge dependent in fixed gauge path integral calculations of the effective potential.

In section IV we use the gauge invariant effective potential to study the early time behavior of spinodal phase separation and the instabilities associated with the spinodal line in gauge theories. We establish a correspondence between the imaginary part of the one loop gauge invariant effective potential and the rate of growth of correlations in the spinodal region. Two appendices are devoted to some technical details.
II. THE GAUGE INVARIANT DESCRIPTION:

The focus of our study is scalar electrodynamics or the Abelian Higgs model whose Lagrangian density is

\[ \mathcal{L} = -\frac{1}{4} F^{\mu\nu} F_{\mu\nu} + D^\mu \phi^\dagger D_\mu \phi - \lambda (\phi^\dagger \phi - \mu^2)^2 \]  

(2.1)

D_\mu \phi = (\partial_\mu \phi + ieA_\mu \phi)

(2.2)

The description in terms of gauge invariant states and operators is best achieved within the canonical formulation, which begins with the identification of canonical field variables and constraints. These will determine the classical physical phase space and, at the quantum level, the physical Hilbert space.

The canonical momenta conjugate to the scalar and vector fields are given by

\[ \Pi^0 = 0 \]  

(2.3)

\[ \Pi^i = \dot{A}^i + \nabla^i A^0 = -E^i \]  

(2.4)

\[ \pi^\dagger = \dot{\phi} + ie A^0 \phi \]  

(2.5)

\[ \pi = \dot{\phi}^\dagger - ie A^0 \phi^\dagger \]  

(2.6)

The Hamiltonian is therefore

\[ H = \int d^3x \left\{ \frac{1}{2} \Pi^i \dot{\Pi}_i + \pi^\dagger \pi + (\nabla \phi - ieA \phi) \cdot (\nabla \phi^\dagger + ieA \phi^\dagger) + \frac{1}{2} (\nabla \times A)^2 + \lambda (\phi^\dagger \phi - \mu^2)^2 + A_0 \left[ \nabla \cdot \Pi - ie(\pi \phi - \pi^\dagger \phi^\dagger) \right] \right\} \]  

(2.7)

There are several different manners of quantizing a gauge theory, but the one that exhibits the gauge invariant states and operators, originally due to Dirac, begins by recognizing the first class constraints (mutually vanishing Poisson brackets). From here there are several possibilities: i) the constraints become operators in the quantum theory and are imposed onto the physical states, thus defining the physical subspace of the Hilbert space and gauge invariant operators. ii) Introduce a gauge, converting the first class system of constraints into a second class (with non-zero Poisson brackets between the constraints) and introducing Dirac brackets. This is the popular way of dealing with the constraints and leads to the usual gauge-fixed path integral representation [12] in terms of Faddeev-Popov determinants and ghosts.

We will instead proceed with the first possibility that leads to an unambiguous projection of the physical states and operators. Such a method has been previously used by James and Landshoff within a different context [13].

In Dirac’s method of quantization [14] there are two first class constraints which are:

\[ \Pi^0 = \frac{\delta \mathcal{L}}{\delta A^0} = 0 \]  

(2.8)

and Gauss’ law:

\[ \mathcal{G}(\vec{x}, t) = \nabla^i \pi^i - \rho = 0 \]  

(2.9)

\[ \rho = ie \left( \phi \pi - \phi^\dagger \pi^\dagger \right) \]  

(2.10)
with \( \rho \) being the matter (complex scalar) field charge density.

Gauss’ law can be seen to be a constraint in two ways: either because it cannot be obtained as a Hamiltonian equation of motion, or because in Dirac’s formalism, it is the secondary (first class) constraint obtained by requiring that the primary constraint \( (2.8) \) remain constant in time. Quantization is now achieved by imposing the canonical equal-time commutation relations

\[
\begin{align*}
\{ \pi^0(\vec{x}, t), A^0(\vec{y}, t) \} &= -i \delta(\vec{x} - \vec{y}) \\
\{ \pi^i(\vec{x}, t), A^i(\vec{y}, t) \} &= -i \delta^i_j \delta(\vec{x} - \vec{y}) \\
\{ \pi^\dagger(\vec{x}, t), \phi^\dagger(\vec{y}, t) \} &= -i \delta(\vec{x} - \vec{y}) \\
\{ \pi(\vec{x}, t), \phi(\vec{y}, t) \} &= -i \delta(\vec{x} - \vec{y})
\end{align*}
\]

In Dirac’s formulation, the projection onto the gauge invariant subspace of the full Hilbert space is achieved by imposing the first class constraints onto the states. Physical operators are those that commute with the first class constraints. With the above equal-time commutation relations it is straightforward to see that the unitary operator

\[
U_\Lambda = \exp \left\{ i \int [\Pi^0 A + G \Lambda] d^3x \right\}
\]

performs the local gauge transformations. Thus the first class constraints are recognized as the generators of gauge transformations. In particular, Gauss’ law \( (2.9) \) is the generator of time independent gauge transformations. Requiring that the physical states be annihilated by these constraints is tantamount to selecting the gauge invariant states. Consequently operators that commute with the first class constraints are gauge invariant.

In the Schroedinger representation, in terms of wave-functionals, the canonical momenta are represented by hermitian differential operators, and the constraints applied onto the states become functional differential equations that the wave-functionals must satisfy:

\[
\begin{align*}
\frac{\delta}{\delta A_0(\vec{x})} \Psi[A, \phi, \phi^\dagger] &= 0 \\
\left[ \nabla_x \frac{\delta}{\delta A(\vec{x})} - ie \left( \phi(\vec{x}) \frac{\delta}{\delta \phi(\vec{x})} - \phi^\dagger(\vec{x}) \frac{\delta}{\delta \phi^\dagger(\vec{x})} \right) \right] \Psi[A, \phi, \phi^\dagger] &= 0
\end{align*}
\]

The first equation simply means that the Schroedinger wave-functional does not depend on \( A_0 \), whereas the second equation means that the wave-functional is only a functional of the combination of fields that is annihilated by the Gauss’ law functional differential operator. It is a simple calculation to prove that the fields

\[
\begin{align*}
\Phi(\vec{x}) &= \phi(\vec{x}) \exp \left[ ie \int d^3y \vec{A}(\vec{y}) \cdot \vec{\nabla}_y G(\vec{y} - \vec{x}) \right] \\
\Phi^\dagger(\vec{x}) &= \phi^\dagger(\vec{x}) \exp \left[ -ie \int d^3y \vec{A}(\vec{y}) \cdot \vec{\nabla}_y G(\vec{y} - \vec{x}) \right]
\end{align*}
\]

are annihilated by Gauss’ law functional differential equation with \( G(\vec{y} - \vec{x}) \) the Coulomb Green’s function that satisfies
\[ \nabla^2 G(\vec{y} - \vec{x}) = 0 \]  

(2.20)

Furthermore writing the gauge field into transverse and longitudinal components as follows

\[
\vec{A}(\vec{x}) = \vec{A}_L(\vec{x}) + \vec{A}_T(\vec{x})
\]

(2.21)

\[
\vec{\nabla} \times \vec{A}_L(\vec{x}) = 0
\]

(2.22)

\[
\vec{\nabla} \cdot \vec{A}_T(\vec{x}) = 0
\]

(2.23)

it is clear that

\[
\vec{\nabla}_x \frac{\delta}{\delta \vec{A}(\vec{x})} = \vec{\nabla}_x \frac{\delta}{\delta \vec{A}_L(\vec{x})}.
\]

(2.24)

Therefore the “transverse component” \(\vec{A}_T(\vec{x})\) is also annihilated by the Gauss’ law operator. This analysis shows that the wave-functional solutions of the functional differential equations that represent the constraints in the Schroedinger representation are of the form

\[
\Psi[\vec{A}, \phi, \phi^\dagger] = \Psi[\vec{A}_T, \Phi, \Phi^\dagger]
\]

(2.25)

The fields \(\vec{A}_T, \Phi, \Phi^\dagger\) are gauge invariant as they commute with the constraints. The canonical momenta conjugate to \(\Phi, \Phi^\dagger\) are found to be

\[
\Pi(\vec{x}) = \pi(\vec{x}) \exp \left[ -ie \int d^3y \vec{A}(\vec{y}) \cdot \vec{\nabla}_y G(\vec{y} - \vec{x}) \right]
\]

(2.26)

\[
\Pi^\dagger(\vec{x}) = \pi^\dagger(\vec{x}) \exp \left[ ie \int d^3y \vec{A}(\vec{y}) \cdot \vec{\nabla}_y G(\vec{y} - \vec{x}) \right]
\]

(2.27)

The momentum canonical to \(\vec{A}; \vec{\Pi}\) is written in terms of “longitudinal” and “transverse” components

\[
\vec{\Pi}(\vec{x}) = \vec{\Pi}_L(\vec{x}) + \vec{\Pi}_T(\vec{x})
\]

(2.28)

both components are gauge invariant.

In the physical subspace of gauge invariant wave-functionals, matrix elements of \(\vec{\nabla} \cdot \vec{\Pi}\) can be replaced by matrix elements of the charge density \(\rho\). Therefore in all matrix elements between gauge invariant states (or functionals) one can replace

\[
\vec{\Pi}_L(\vec{x}) \rightarrow ie \vec{\nabla}_x \int d^3y G(\vec{x} - \vec{y}) \left( \Phi \Pi - \Phi^\dagger \Pi^\dagger \right)(\vec{y})
\]

(2.29)

Finally in the gauge invariant subspace the Hamiltonian becomes

\[
H = \int d^3x \left\{ \frac{1}{2} \vec{\Pi}_T \cdot \vec{\Pi}_T + \Pi^\dagger \Pi + (\vec{\nabla} \Phi - ie \vec{A}_T \Phi) \cdot (\vec{\nabla} \Phi^\dagger + ie \vec{A}_T \Phi^\dagger) + \frac{1}{2}(\vec{\nabla} \times \vec{A}_T)^2 + \lambda(\Phi^\dagger \Phi - \mu^2)^2 \right\} + \frac{1}{2} \int d^3y \int d^3x \rho(\vec{x}) G(\vec{x} - \vec{y}) \rho(\vec{y})
\]

(2.30)

Clearly the Hamiltonian is gauge invariant, and it manifestly has the global \(U(1)\) gauge symmetry under which \(\Phi\) transforms with a constant phase, \(\Pi\) transforms with the opposite phase and \(\vec{A}_T\) is invariant.
This Hamiltonian is reminiscent of the Coulomb gauge Hamiltonian, but we emphasize that we have not imposed any gauge fixing condition. The formulation is fully gauge invariant, written in terms of operators that commute with the generators of gauge transformations and states that are invariant under these transformations.

There is a definite advantage in this gauge invariant formulation: the (composite) field \( \Phi(\vec{x}) \) is a candidate for a *locally gauge invariant order parameter*. The point to stress is the following: this operator is *invariant* under local gauge transformations generated by the unitary transformation \( U_\Lambda \) given by equation (2.15), that is

\[
U_\Lambda \Phi(\vec{x}) U_\Lambda^{-1} = \Phi(\vec{x})
\]

(2.31)

whereas it transforms as a charged operator under the *global* gauge transformations generated by \( Q = \int d^3 x \rho(\vec{x}) \), that is

\[
e^{i\alpha Q} \Phi(\vec{x}) e^{-i\alpha Q} = e^{ie\alpha} \Phi(\vec{x}).
\]

(2.32)

Because the gauge constraints annihilate the physical states and these constraints are the generators of local gauge transformations, these states are invariant under the local gauge transformations and any operator that *is not* invariant under these local transformations *must* have zero expectation value. The *local* gauge symmetry cannot be spontaneously broken; this result is widely known in lattice gauge theory as Elitzur’s theorem [15]. However, the *global* symmetry generated by the charge \( Q \) can be spontaneously broken and the expectation value of a charged field signals this breakdown.

From this discussion we clearly see that a trustworthy order parameter must be invariant under the local gauge transformations, thus commuting with the gauge constraints, but must transform non-trivially under the global gauge transformation generated by the charge. The field \( \Phi \) fulfills these criteria and is the natural candidate for an order parameter.

### III. THE EFFECTIVE POTENTIAL

#### A. Zero Temperature

We are now in condition to define the gauge invariant effective potential. Consider the gauge invariant state \( |\Psi; \chi\rangle \) such that the expectation value of the gauge invariant order parameter \( \Phi(\vec{x}) \) in this state is nonzero and space-time constant

\[
\frac{\langle \Psi; \chi|\Phi(\vec{x})|\Psi; \chi\rangle}{\langle \Psi; \chi|\Psi; \chi\rangle} = \chi
\]

(3.1)

The effective potential is defined as the minimum of the expectation value of the Hamiltonian density in this state, namely

\[
V_{eff}(\chi) = \frac{1}{\Omega} \min \left\{ \frac{\langle \Psi; \chi|H|\Psi; \chi\rangle}{\langle \Psi; \chi|\Psi; \chi\rangle} \right\}
\]

(3.2)

with \( H \) being the gauge invariant, Hamiltonian given by equation (2.30) and \( \Omega \) the spatial volume [14]. The state \( |\Psi; \chi\rangle \) is chosen to minimize the expectation value of the Hamiltonian subject to the constraint that the expectation value of \( \Phi \) in this state is \( \chi \).
It is convenient to separate the expectation value of $\Phi$ as

$$\Phi(\vec{x}) = \chi + \eta(\vec{x}) \quad (3.3)$$

The one-loop correction (formally of $O(\bar{h})$) to the effective potential is obtained by keeping the quadratic terms in the Hamiltonian.

$$H_q = \Omega V_{cl}(|\chi|^2) + \frac{1}{2} \sum_k \left\{ \vec{\Pi}_T(k) \cdot \vec{\Pi}_T(-k) + \omega_T^2(k) \vec{A}_T(k) \cdot \vec{A}_T(-k) + \Pi_1(k) \Pi_1(-k) + \omega_H^2(k) \eta_1(k) \eta_1(-k) + \Pi_2(k) \Pi_2(-k) \frac{\omega_g^2(k)}{k^2} + \eta_2(k) \eta_2(-k) \omega_g^2(k) \right\} \quad (3.7)$$

where the frequencies are given in terms of the effective masses as

$$\omega_T^2(k) = k^2 + m_T^2 ; \quad m_T^2 = 2e^2|\chi|^2 \quad (3.8)$$
$$\omega_H^2(k) = k^2 + m_H^2 ; \quad m_H^2 = 2\lambda(3|\chi|^2 - \mu^2) \quad (3.9)$$
$$\omega_g^2(k) = k^2 + m_g^2 ; \quad m_g^2 = 2\lambda(|\chi|^2 - \mu^2). \quad (3.10)$$

The last two terms can be brought to a canonical form by a Bogoliubov transformation. Define the new canonical coordinate $Q$ and conjugate momentum $P$ as

$$\Pi_2(k) = \frac{k}{\omega_T(k)} P(k) \quad (3.11)$$
$$\eta_2(k) = \frac{\omega_T(k)}{k} Q(k) \quad (3.12)$$

in terms of which the last term of the Hamiltonian (3.7) becomes a canonical quadratic form with the plasma frequency.
\[ \omega_p^2(k) = \omega_g^2(k) \frac{\omega_T^2(k)}{k^2} = [k^2 + 2\lambda(|\chi|^2 - \mu^2)][k^2 + 2e^2|\chi|^2]/k^2 \]  

(3.13)

There are four physical degrees of freedom. The modes with frequency \( \omega_T(k) \) are the two transverse degrees of freedom, the mode with frequency \( \omega_H(k) \) is identified with the Higgs mode. In absence of electromagnetic interactions \( (e = 0) \) the mode with frequency \( \omega_p(k) \) represents the Goldstone mode whereas in equilibrium, namely at the minimum of the tree level potential, when \( |\chi| = \mu \), it represents the plasma mode which is identified as the screened Coulomb interaction, and the transverse and plasma modes all share the same mass. However, when the expectation value of the order parameter acquires a non-equilibrium value, away from the minimum of the tree level potential (at this order), this collective mode does not describe a particle with a Lorentz covariant dispersion relation. The frequency clearly shows the combination of the Goldstone dispersion relation and the long-range Coulomb interaction typical of a description in terms of the dynamical degrees of freedom.

The lack of manifest Lorentz covariance in the dispersion relation can be understood as follows. Although the complex field \( \phi \) in the original Lagrangian density is a Lorentz scalar, the gauge invariant combination \( \Phi \) given by eq. (2.18) is not, although it is a rotational scalar. A particular Lorentz frame has already been chosen in making the transverse and longitudinal decomposition of the vector potential (2.21). The state of lowest available energy is expected to be Lorentz invariant, but for arbitrary \( \chi \) there is a constraint in the space of functions and these constrained states are not the lowest energy states in the functional space. In a scalar theory these states are manifestly Lorentz invariant, simply because all fields are Lorentz scalars. With vector fields the situation is more complicated and the constrained, gauge invariant states are in general not manifestly Lorentz invariant. However the lowest energy equilibrium states (at this order corresponding to \( \chi = \mu \)) are manifestly Lorentz invariant. We will see in detail in section IV that for \( \chi \neq \mu \) these states are not stationary states of the Hamiltonian, therefore the lack of Lorentz covariance for these states is reconciled with their non-equilibrium evolution.

The quadratic Hamiltonian is now diagonalized in terms of creation and destruction operators for the quanta of each harmonic oscillator. The ground state is the vacuum for each oscillator and is the state of lowest energy. Therefore the one loop \( (O(\hbar)) \) contribution to the effective potential is obtained from the zero point energy of the oscillators. Therefore accounting for the two polarizations of the transverse components we find:

\[ V_{\text{eff}}(|\chi|) = V_{\text{cl}}(|\chi|) + \frac{1}{2} \int \frac{d^3k}{(2\pi)^3} [2\omega_T(k) + \omega_H(k) + \omega_p(k)] \]  

(3.14)

The normalized wave-functional that satisfies (3.1) and gives the minimum expectation value of the Hamiltonian, thus determining effective potential via (3.2) is given by

\[ \Psi[A_T, \Phi^\dagger, \Phi] = N \exp \left\{ -\frac{1}{2} \int d^3x \int d^3y \tilde{A}_T(x) \cdot \tilde{A}_T(y) K_T(x - y) \right\} \times \exp \left\{ -\frac{1}{2} \int d^3x \int d^3y \eta_1(x) \eta_1(y) K_H(x - y) \right\} \times \exp \left\{ -\frac{1}{2} \int d^3x \int d^3y \eta_2(x) \eta_2(y) K_p(x - y) \right\} \]  

(3.15)
\[ N = \Pi_k \left[ \frac{\omega_T^2(k)\omega_H(k)\omega_p(k)}{\pi^4} \right]^{1/2} \quad (3.16) \]

\[ K_T(\vec{x} - \vec{y}) = \int \frac{d^3 k}{(2\pi)^3} \omega_T(k)e^{i\vec{k}(\vec{x} - \vec{y})} \quad (3.17) \]

\[ K_H(\vec{x} - \vec{y}) = \int \frac{d^3 k}{(2\pi)^3} \omega_H(k)e^{i\vec{k}(\vec{x} - \vec{y})} \quad (3.18) \]

\[ K_p(\vec{x} - \vec{y}) = \int \frac{d^3 k}{(2\pi)^3} \frac{k^2\omega_p(k)}{\omega_T(k)}e^{i\vec{k}(\vec{x} - \vec{y})} \quad (3.19) \]

This Gaussian wave-functional is clearly gauge invariant, and it has the correct limits: for \( e = 0 \) (\( \omega_T = k \); \( \omega_p = \omega_g \)) gives the (gauge invariant) wave-functional of free electromagnetism times the Gaussian wave-functional of a complex scalar with the \( U(1) \) global symmetry spontaneously broken, which for \( |\chi|^2 = \mu^2 \) corresponds to the Higgs and a Goldstone mode. Writing the fluctuation fields \( \eta \) renormalized and gauge invariant one-loop effective potential \( \rightarrow \infty \) in the \( \Lambda = 0 \) \( \exp(\text{spontaneously broken, which for} |\chi|^2 = \mu^2 \text{corresponds to the Higgs and a Goldstone mode.}) \) Writing the fluctuation fields \( \eta_1; \eta_2 \) in terms of \( \Phi; \Phi^\dagger \), we clearly see that this wave functional describes a broken symmetry state since under the global \( U(1) \) transformation the wavefunctional is changed into an orthogonal wavefunctional in the infinite volume limit.

The unbroken phase, with \( \chi = 0 \); \( -\mu^2 = m^2 > 0 \) corresponds to the ground state wave-functional for free electromagnetism times the ground-state wave-functional of two free real scalar fields with equal mass \( m\sqrt{2\lambda} \). It is straightforward to see that the expectation value of the radial variable \( \rho = \sqrt{\Phi^\dagger\Phi} \) is different from zero in this phase and cannot be used as an order parameter to signal spontaneous global symmetry breaking as discussed previously.

The \( k^- \) integrals in the effective potential \( V_{eff} \) are performed with an upper momentum cutoff \( \Lambda \). Neglecting a \( \chi \)-independent term proportional to \( \Lambda^4 \) as well as terms that vanish in the \( \Lambda \rightarrow \infty \) limit, and introducing a renormalization scale \( \kappa \) we obtain the following (unrenormalized) expression

\[ V_{eff}(|\chi|) = V_{\alpha}(|\chi|) + \frac{1}{4\pi^2} \left\{ \frac{\Lambda^2}{\kappa^2} \left[ 3m_T^2 + m_H^2 + m_g^2 \right] \right. 
+ \frac{1}{16} \ln \left( \frac{\kappa^2}{4\Lambda^2} \right) \left[ 3m_T^4 + m_H^4 + m_g^4 - 2m_T^2m_g^2 \right] 
+ \frac{1}{16} \left[ 2m_T^4 \ln \left( \frac{m_T^2}{\kappa^2} \right) + m_H^4 \ln \left( \frac{m_H^2}{\kappa^2} \right) + (m_g^2 - m_T^2)^2 \ln \left( \frac{m_T^2 + m_g^2 + 2\sqrt{m_g^2m_T^2}}{\kappa^2} \right) \right] 
+ \frac{1}{32} \left[ 3m_T^4 + m_H^4 + m_g^4 + 6m_g^2m_T^2 \right] \right\} \quad (3.20) \]

The cutoff dependent terms can be absorbed in a renormalization of \( \mu^2 \) (terms of \( \mathcal{O}(\Lambda^2) \) and \( \mathcal{O}(\ln(\Lambda)) \) proportional to \( |\chi|^2 \) and the quartic coupling \( \lambda \) (terms of \( \mathcal{O}(\ln(\Lambda)) \) proportional to \( |\chi|^4 \)). Using this renormalization prescription we find the following result for the renormalized and gauge invariant one-loop effective potential

\[ V_{eff,R}(|\chi|) = \lambda(|\chi|^2 - \mu^2)^2 + \frac{1}{4\pi^2} \left\{ \frac{1}{32} \left[ 3m_T^4 + m_H^4 + m_g^4 + 6m_g^2m_T^2 \right] 
+ \frac{1}{16} \left[ 2m_T^4 \ln \left( \frac{m_T^2}{\kappa^2} \right) + m_H^4 \ln \left( \frac{m_H^2}{\kappa^2} \right) + (m_g^2 - m_T^2)^2 \ln \left( \frac{m_T^2 + m_g^2 + 2\sqrt{m_g^2m_T^2}}{\kappa^2} \right) \right] \right\} \quad (3.21) \]
In this expression $\lambda$ and $\mu$ and all masses are renormalized with the above prescription.

In the region $|\chi|^2 < \mu^2/3$ the Higgs “mass” is purely imaginary, whereas for $|\chi|^2 < \mu^2$, $m_g^2 < 0$. Therefore we see that the logarithmic contributions to the effective potential from Higgs and plasma modes are imaginary (the last two logarithms in eq. 3.21), whereas the contribution of the gauge boson is real. The region in which the effective potential is imaginary is a region of unstable states $\uparrow \uparrow \uparrow \downarrow \uparrow \downarrow \downarrow$, and the imaginary part of the effective potential disguises a non-equilibrium situation whose dynamics will be addressed in section IV. This region of instabilities for homogeneous configurations is known as the spinodal region. In this region the system is unstable to phase separation, and the imaginary part of the effective potential appears as a result of attempting to describe an intrinsically time dependent state as a stationary state via analytic continuation.

B. Finite Temperature:

The finite temperature effective potential is identified with the free energy density under the constraint that the ensemble average of the field is given by a space-time independent configuration $\chi$. That is

$$V_T(\chi) = -T \ln(\text{Tr} \hat{\rho})$$

$$\chi = \frac{\text{Tr} \Phi(\vec{x})\hat{\rho}}{\text{Tr} \hat{\rho}}$$

with $\hat{\rho}$ the ensemble density matrix. In equilibrium and when zero conserved charge is considered the density matrix is given by

$$\hat{\rho} = \exp(-\frac{H}{T})$$

In a gauge theory, however, the trace over states in (3.22) must be defined properly in terms of gauge invariant states. Either the physical states are selected and only these are used in the trace or alternatively a projection operator must be introduced in the definition of the trace $\uparrow \uparrow \uparrow \downarrow \uparrow \downarrow \downarrow$.

In our approach we select the states as those annihilated by the set of first class constraints, which are therefore gauge invariant as described in the previous section.

To one-loop order, we have seen that the Hamiltonian is quadratic in terms of gauge invariant operators that describe the physical degrees of freedom. Therefore, to this order the physical partition function is that of a collection of uncoupled harmonic oscillators for each degree of freedom.

We find the free energy density, which is identified as the finite temperature effective potential, to be ($\beta = 1/T$)

$$\mathcal{F} = V_{\text{eff}}(|\chi|; T) = V_{\text{eff}}(|\chi|; T = 0) + \frac{1}{\beta} \int \frac{d^3k}{(2\pi)^3} \left\{ 2 \ln\left[ 1 - e^{-\beta \omega_T(k)} \right] + \ln\left[ 1 - e^{-\beta \omega_H(k)} \right] + \ln\left[ 1 - e^{-\beta \omega_P(k)} \right] \right\}$$

where $V_{\text{eff}}(|\chi|; T = 0)$ is the zero temperature effective potential given by (3.14), and arises from the zero point energy of the oscillators.

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Just as our gauge invariant approach in terms of gauge invariant operators and functionals allowed us to obtain the ground state constrained wave functional eq. (3.15), similarly we can obtain the density matrix elements in the Schroedinger representation \( \langle \Phi, \Phi^\dagger, A_T' | \hat{\rho} | A_T, \Phi^\dagger, \Phi' \rangle \). This representation for the density matrix is very useful to study non-equilibrium aspects and time dependent phenomena [20], which is the focus of the next section. We find the density matrix elements in the Schroedinger representation to be given by

\[
\langle \Phi, \Phi^\dagger, A_T' | \hat{\rho} | A_T, \Phi^\dagger, \Phi' \rangle = \langle \Phi, \Phi^\dagger | \hat{\rho}_\Phi | \Phi^\dagger, \Phi' \rangle \otimes \langle A_T | \hat{\rho}_A | A_T' \rangle,
\]

where the density matrices are of the harmonic oscillator type, the full expression is given in the appendix I.

C. Large T expansion:

The contributions to the free energy from the transverse and Higgs modes are straightforward to obtain by applying the methods developed by Dolan and Jackiw [1]. However the contribution from the “plasma” mode is non-standard and requires a more detailed analysis which is presented in appendix II. We find the leading high temperature behavior to the finite temperature contribution to the effective potential to be given by

\[
V_{\text{eff},T}(|\chi|) = -\frac{4 \pi^2 T^4}{90} + \frac{T^2}{24} \left[ 3 m_T^2 + m_H^2 + m_g^2 \right] - \frac{T}{12 \pi} \left[ 3 (m_H^2) \frac{\hat{\nu}}{2} + (m_H^2) \frac{\hat{\nu}}{2} + (m_g^2) \frac{\hat{\nu}}{2} \right] + \\
- \frac{1}{64 \pi^2} \left\{ (m_g^2 - m_T^2)^2 \ln \left[ \frac{m_g^2 + m_T^2 + 2 \sqrt{m_g^2 m_T^2}}{T^2} \right] + 2 m_T^4 \ln \left( \frac{m_T^2}{T^2} \right) \right\} + \cdots
\]

(3.27)

where the dots stand for terms of \( O(T^0) \) or smaller. Remarkably, the logarithmic terms cancel similar terms of the zero temperature part and though this feature is well known in the standard cases (with standard dispersion relations for the degrees of freedom), it is a new result for the plasma mode. An important feature of this expression is that the terms linear in T, which are non-analytic, are complex. Whereas the term from the gauge boson mass is real, the terms originating in the Higgs couplings given by the contributions from \( m_H^2 \), \( m_g^2 \) are purely imaginary on the spinodal regions \( |\chi|^2 < \mu^2/3 \) for \( m_H^2 \) and \( |\chi|^2 < \mu^2 \) for \( m_g^2 \). These “cubic” terms are usually identified as those responsible for a first order phase transition and used to compute quantities relevant to the transition [21,22,10]. In particular these terms determine the supercooling temperature and the latent heat when they are taken as the leading indicators for a first order transition. In their study of the electroweak effective potential Anderson and Hall [23] neglected the terms involving the Higgs self-coupling keeping only the contributions from the gauge boson and top quark Yukawa couplings which are gauge invariant and real to one loop. Arguably such an approximation is justified for very weak Higgs couplings. Boyd et.al. [24] recognized that the terms arising from the Higgs sector lead to contributions that are imaginary (even after resummation) precisely as pointed out above. Thus using these terms to compute the latent heat, supercooling temperatures
and even approximate dynamics is at best a crude approximation (even when the imaginary parts are ignored) and at worst disguises other non-equilibrium processes which may be equally important (see section IV).

**D. Comparison with gauge-fixed results:**

In this section we compare our gauge invariant result to the one loop effective potential obtained in the usual standard path-integral representation for several gauge-fixing procedures. The purpose is to contrast our results with the suggestion of Fukuda and Kugo [25] that there is a large class of “good gauges” for which the effective potential is gauge invariant. These authors suggested that covariant gauges (including Landau with zero gauge parameter), \( R_\xi \) and other specific gauge fixing schemes are such “good” choices.

In order to make a distinction from the gauge invariant formulation, we write the original complex field \( \phi \) (not to be confused with the gauge invariant field \( \Phi \)) in the Lagrangian density eq. (2.1) as

\[
\phi(x, t) = \frac{1}{\sqrt{2}} \left[ \hat{\phi}_R(x, t) + i \hat{\phi}_I(x, t) \right] + \varphi \tag{3.28}
\]

and \( \varphi \) is taken as the (complex) expectation value. To distinguish from the gauge invariant case we also introduce the masses

\[
M^2_g = 2\lambda(|\varphi|^2 - \mu^2) \tag{3.29}
\]
\[
M^2_A = 2e^2|\varphi|^2 \tag{3.30}
\]
\[
M^2_H = 2\lambda(3|\varphi|^2 - \mu^2) \tag{3.31}
\]

At this point one would be tempted to identify \( \varphi \) with \( \chi \), because the effective tree level masses seem to be the same as in the gauge invariant case under the replacement \( \chi \rightarrow \varphi \). However we make a distinction between these two expectation values because \( \chi \) is a truly gauge invariant quantity, whereas \( \varphi \) is the expectation value of a gauge transforming field in a fixed gauge.

The one-loop effective potential is given by:

\[
V_1 = \sum_j \frac{g_j}{2\beta} \sum_n \int \frac{d^3k}{(2\pi)^3} \ln \left[ \left( \frac{2\pi n}{\beta} \right)^2 + k^2 + M^2_j \right] = V_{10} + V_{1T}
\]

\[
V_{10} = \sum_j \frac{g_j}{2} \int \frac{d^3k}{(2\pi)^3} \sqrt{k^2 + M^2_j} \tag{3.32}
\]

\[
V_{1T} = \sum_j \frac{g_j}{\beta} \int \frac{d^3k}{(2\pi)^3} \ln \left[ 1 - \exp(-\beta \sqrt{k^2 + M^2_j}) \right] \tag{3.33}
\]

where the sum are over all particles \( j \) with \( g_j \) degrees of freedom (\( g_j < 0 \) for ghosts) and masses \( M_j(|\varphi|) \), and we have used a result given in [1].

The zero temperature contribution is divergent, the \( k \) integrals being performed with an ultraviolet cutoff \( \Lambda \). Discarding a field independent quartic divergence we find the result
\[ V_{10} = \sum_j \frac{g_j}{4\pi^2} \left\{ \frac{M_j^4}{16} \ln \left( \frac{M_j^2}{\Lambda^2} \right) + \frac{\Lambda^2 M_j^2}{4} + \frac{M_j^4}{32} \right\} \]  

(3.34)

The finite temperature contribution can be written as

\[ V_{1T} = \frac{T^4}{2\pi^2} \sum_j g_j I \left( \frac{M_J}{T} \right) \]  

(3.35)

The high temperature expansion of \( I(y) \) is given by \[ I(y) = -\frac{\pi^4}{45} + \frac{\pi^2}{12} y^2 - \frac{\pi}{6} (y^2)^{\frac{3}{2}} - \frac{y^4}{32} \left[ \ln(y^2) - \frac{3}{2} - C \right] + O(y^6), \]

where we defined \( C \equiv 2 \ln(4\pi) - 2\gamma = 3.9076 \). The \( \ln(m^2) \) term cancels against the similar term in the zero temperature contribution \( V_{10} \).

1. Lorentz gauge:

Dolan and Jackiw \[1\] calculate the one loop effective potential both at zero and non-zero temperature in Lorentz gauge, which is ghost free, with gauge parameter \( \alpha \). The scalar determinant is diagonalized by solving

\[ \ln[k^4 + M_g^2 k^2 + \alpha M_T^2 M_g^2] = \ln[(k^2 + M_g^2(\alpha))(k^2 + M_T^2(\alpha))] \]  

(3.37)

The resulting masses and effective degrees of freedom are

\[ M_\pm^2(\alpha) = \frac{1}{2} \left\{ M_g^2 \pm \sqrt{M_g^4 - 4\alpha M_T^2 M_g^2} \right\} \quad g_\pm = 1 \]

\[ M_T^2 = 2\lambda (|\phi|^2 - \mu^2) \quad g_T = 1 \]

\[ M_A^2 = 2e^2 |\phi|^2 \quad g_A = 3 \]

(3.38) \hspace{1cm} (3.39) \hspace{1cm} (3.40)

Thus the zero temperature part of the one-loop effective potential in Lorentz gauge is given by

\[ V_{10LG}(|\phi|; \alpha) = \frac{1}{2} \int \frac{d^3k}{(2\pi)^3} \left[ \Omega_H(k) + 3\Omega_A(k) + \Omega_+(k; \alpha) + \Omega_-(k; \alpha) \right] \]  

(3.41)

\[ \Omega^\pm(k; \alpha) = \sqrt{k^2 + M_\pm^2(\alpha)} \]

\[ \Omega_j(k) = \sqrt{k^2 + M_j^2} \quad ; j = H, A. \]

(3.42) \hspace{1cm} (3.43)

The gauge dependence (dependence on the gauge parameter \( \alpha \)) of the above result is explicit. Even for Landau gauge, that is \( \alpha = 0 \), expression (3.41) describes five degrees of freedom, rather than the four physical degrees of freedom described by the gauge invariant result (3.14). With the same renormalization prescription leading to the gauge invariant result eq. (3.21) we find in Landau gauge the result.
\[ V(\varphi, \alpha = 0) = \lambda(|\varphi|^2 - \mu^2) + \frac{1}{4\pi^2} \left\{ \frac{1}{32} \left[ 3M_A^4 + M_H^4 + M_g^4 \right] + \frac{1}{16} \left[ 3M_A^4 \ln \left( \frac{m_T^2}{\kappa^2} \right) + M_H^4 \ln \left( \frac{M_H^2}{\kappa^2} \right) + M_g^4 \ln \left( \frac{M_g^2}{\kappa^2} \right) \right] \right\} \] (3.44)

which is obviously very different from the gauge invariant result given by eq. 3.21 if \( \varphi \) is identified with the gauge invariant order parameter \( \chi \).

The effective potential (3.41) becomes independent of the gauge parameter \( \alpha \) for the values \( \varphi = 0 ; |\varphi|^2 = \mu^2 \). These are the value of the extrema of the tree level potential. The gauge dependence appears at 1-loop order and is therefore formally of \( \mathcal{O}(\hbar) \) since the extrema of the effective potential will acquire \( \mathcal{O}(\hbar) \) corrections. We identify the values of \( \varphi \) at which the gauge dependence cancels out as the extrema of the effective action to this order. Up to an irrelevant constant the gauge invariant effective potential (3.14) and the one-loop effective potential in general covariant gauge (3.41) are the same for \( |\chi|^2 = \mu^2 \), i.e. at the extrema of the effective action. This equality is a consequence of the known result that the extrema of the effective action are gauge independent. At zero temperature gauge independence at the extrema is also a consequence of the Nielsen identities [3]. Therefore we find that the suggestion of reference [25] is only valid at the extrema of the effective action.

At finite temperature we find that the \( \mathcal{O}(T^2) \) is gauge parameter independent and given by

\[ V_{1T}^{(2)} = \frac{T^2}{24} \left[ 3M_A^4 + M_H^4 + M_g^4 \right]. \] (3.45)

This term coincides with that from the gauge invariant effective potential with the identification \( |\varphi| \equiv |\chi| \). This contribution determines (to this order) the critical temperature which is therefore a gauge independent quantity, whereas the \( \mathcal{O}(T) \) and \( \mathcal{O}(\ln(T)) \) contributions are gauge parameter dependent.

The \( \mathcal{O}(T) \) contribution is given by

\[ V_{1T}^{(1)} = -\frac{T}{12\pi} \left[ 3(M_A^2)^{\frac{3}{2}} + (M_H^2)^{\frac{3}{2}} + (M_+^2(\alpha))^\frac{3}{2} + (M_0^2(\alpha))^\frac{3}{2} \right] \] (3.46)

For \( \alpha = 0, M_+^2 = M_g^2 ; M_H^2 = 0 \) and this \( \mathcal{O}(T) \) term coincides with the \( \mathcal{O}(T) \) contribution from the gauge invariant effective potential given by eq. (3.27) if \( |\varphi| \) is identified with the gauge invariant order parameter \( |\chi| \).

This contribution is of particular importance because it is usually taken as a signal for a first order transition and determines its strength, and sometimes used in phenomenological equations to describe the dynamics [21][22]. In the case of first order phase transitions, this term is sometimes used to compute the latent heat and the supercooling temperature [10]. Clearly, quantities calculated solely from this term would be physically meaningless because of the gauge dependence. Furthermore, this contribution is complex for \( |\varphi|^2 < \mu^2 \); only the contribution from the gauge boson is real and gauge invariant. Therefore our conclusion is that this term is provides the correct gauge invariant \( \mathcal{O}(T) \) contribution only in Landau (\( \alpha = 0 \)) gauge and with the identification \( |\varphi| \equiv |\chi| \). However, this equivalence only holds to leading order in the high temperature expansion and is not a general feature to all orders, as displayed explicitly by higher order finite temperature corrections and also by the zero temperature part.
In \( R_\xi \) gauge the following gauge fixing and ghost terms are added to the Lagrangian density

\[
\mathcal{L}_{GF} = -\frac{1}{2\xi} (\partial_\mu A^\mu + \xi e \varphi \hat{\phi}_I)^2, \quad \mathcal{L}_{FGP} = c^\dagger \left[ -\partial^2 - \xi e^2 \varphi (\varphi + \hat{\phi}_R) \right] c
\] (3.47)

Kastening [26] describes a useful variant called \( \bar{R}_\xi \) gauge, in which \( \varphi \) and \( \hat{\phi}_R \) are treated more symmetrically. Its gauge fixing and ghost terms are:

\[
\mathcal{L}_{GF} = -\frac{1}{2\xi} \left[ \partial_\mu A^\mu + \xi e (\varphi + \hat{\phi}_R) \hat{\phi}_I \right]^2, \quad \mathcal{L}_{FGP} = c^\dagger \left[ -\partial^2 - \xi e^2 (\varphi + \hat{\phi}_R)^2 - \hat{\phi}_I^2 \right] c
\] (3.48)

The ghost term is derived as usual by looking at the response of the gauge fixing functional under a gauge transformation. The corresponding gauge fixing functional for the \( \bar{R}_\xi \) gauge fixing procedure is

\[
f[A^\mu] = \partial_\mu A^\mu + \xi e (\varphi + \hat{\phi}_R - \phi)\]

A piece from the photon determinant cancels off half the ghost contribution, so we will treat the ghosts as having \( g_c = -1 \). In both \( R_\xi \) and \( \bar{R}_\xi \) gauges we find the following masses and degrees of freedom:

\[
M_A^2; \quad g_A = 3
\]

\[M_c^2 = \xi M_A^2; \quad g_c = -1\] (3.50)

\[M_H^2; \quad g_H = 1\] (3.51)

\[M_\xi^2 = M_g^2 + \xi M_A^2; \quad g_\xi = 1\] (3.52)

The degree of freedom with \( M_\xi \) is a (gauge-dependent) linear combination of the Goldstone mode and the vector boson. At the tree level minimum \( |\varphi|^2 = \mu^2 \) when \( M_g^2 = 0 \) this mode cancels the remaining ghost contribution, leaving a \( \xi \) dependent result, calculable just from the photon and the Higgs terms. At the tree level maximum \( \varphi = 0 \) the gauge dependence cancels out completely and the ghost decouples. Away from the minimum, however, the 1-loop effective potential is \( \xi \) dependent.

At finite temperature we find that the \( \mathcal{O}(T^2) \) term is again gauge parameter independent and given by the same expression as in equation (3.45).

However the next, \( \mathcal{O}(T) \), term is explicitly gauge parameter dependent and given by

\[
-\frac{T}{12\pi} \left[ 3(M_A^2)^{3/2} - (\xi M_A^2)^{3/2} + (M_H^2)^{3/2} + (M_g^2 + \xi M_A^2)^{3/2} \right]
\] (3.53)

This term depends on \( \xi \) (unless \( e = 0 \) or \( |\varphi|^2 = \mu^2 \)) and is complex for \( |\varphi| < \mu^2 \). In Landau gauge (\( \xi = 0 \)), it is given by

\[
-\frac{T}{12\pi} \left[ 3(M_A^2)^{3/2} + (M_H^2)^{3/2} + (M_g^2)^{3/2} \right] \quad \text{(Landau)}
\] (3.54)

and coincides with the \( \mathcal{O}(T) \) contribution to the high \( T \) expansion of the gauge invariant effective potential [3.27] if \( \varphi \) is identified with the gauge invariant order parameter \( \chi \).

Since to this order only the \( \mathcal{O}(T^2) \) term enters in the estimate of the critical temperature, we see that \( T_\xi \) is a gauge invariant quantity (to this order). But comparing the \( \mathcal{O}(T) \) term
with that in Lorentz gauges should convince the reader that this term is gauge dependent
and complex in general and one must be very careful in attaching any physical meaning to it,
such as a criterion for a first order phase transition, its strength and the ensuing latent heat
and supercooling temperature. Only in Landau gauge do we find that this term is the same
as in the gauge invariant formulation, although even in this gauge, the higher order finite
temperature corrections and the zero temperature part are gauge parameter dependent.

3. Unitary Gauge

In unitary gauge, even the leading term of $V_{1T}$ (calculated to 1 loop) is incorrect: $[1]

$$\frac{T^2}{24}[3(M_A^2) + M_H^2]$$ (3.55)

In this gauge, higher-loop corrections affect the leading term and must be included in the cal-
culation to obtain the correct answer (and gauge independent) for the $O(T^2)$ term. Arnold,
Braaten and Vokos $[27]$ showed that a 2-loop calculation restores the correct leading term
(3.45).

4. Higher-Loop Resummation

We have seen that the 1-loop effective potential, calculated in various gauge-fixing
schemes, is explicitly gauge parameter-dependent. One might argue that a resummation
of higher-loop diagrams would eliminate this dependence. Here we will show that this is not
the case $[28]$.

Although resummation would naturally be invoked to restore gauge invariance in a partic-
ular calculation, clearly resummation does nothing to restore gauge parameter-independence
at $T = 0$, since higher-loop diagrams are all higher-order in $\lambda$ or $e^2$. Thus it is clear that
the gauge dependence will remain in any calculation that includes the zero temperature
contribution.

At finite (but large) $T$ it is a little less obvious, since there is an additional expansion
parameter: $M/T$ (with $M$ any of the masses). In fact, we have already mentioned that
resummation in unitary gauge does recover the correct, gauge-independent, $T^2$ term. We
now show explicitly in $R_\xi$ and $\bar{R}_\xi$ gauges that the $T$ term remains $\xi$-dependent even after
resummation of hard thermal loops $[28]$.

Expansion Parameters:

In this section we take $e^2 \sim \lambda$, and $M$ can represent $M_H$ or $M_A$. Terms in $V_T$
are described as $O(\alpha^\beta \gamma^\epsilon)$ with respect to the leading $T^2M^2$ term, where $[28][29]

$$\alpha \equiv \lambda T^2/m^2, \quad \beta \equiv \lambda T/m, \quad \gamma \equiv \varphi^2/T^2$$ (3.56)

We will take $\alpha \approx 1$ and $\gamma \approx 1$ but $\beta < 1$, so that $\{e^2, \lambda\} = O(\beta^2)$ and $M/T = O(\beta)$. Our 1-loop high-$T$ expansion is then seen to be an expansion in $\beta$ (not to be confused with
inverse temperature here).

We want to resum to $O(\beta)$ (given by daisy diagrams with hard thermal loops). The
simplest approach is the tadpole method (for a more comprehensive discussion, see ref.
replacing the $M_j$'s by thermal masses, which need only be calculated to $\mathcal{O}(\beta^0)$. We then integrate with respect to $\varphi$, adding constants to taste, to give $V_T$. The leading ($T^2$) term is unaffected, and remains gauge-independent.

**Thermal Masses:**

The $\mathcal{O}(\beta^0)$ thermal masses are unambiguous and gauge-independent. For the scalars we can use the relations $\tilde{M}_H^2 = M_H^2 + V''_1(\varphi)$ and $\tilde{M}_\xi^2 = M_\xi^2 + V'_1(\varphi)/\varphi$ (the former is true to all orders in $\bar{R}_\xi$ gauge). We get

$$\tilde{M}_H^2 = M_H^2 + \frac{T^2}{24}[6e^2 + 8\lambda], \quad \tilde{M}_\xi^2 = M_\xi^2 + \xi M_A^2 + \frac{T^2}{24}[6e^2 + 8\lambda] \quad (3.57)$$

Only the longitudinal part of the photon gets a thermal (electric screening, plasma) mass (given by $\Pi^0_0$) to leading order:

$$\tilde{M}_L^2 = M_A^2 + \frac{e^2 T^2}{3}, \quad \tilde{M}_T^2 = M_A^2 \quad (3.58)$$

With these hard-thermal loop resummed masses, the $\mathcal{O}(\beta)$ term in $R_\xi$ or $\bar{R}_\xi$ gauge becomes

$$-\frac{T}{12\pi} \left[ 2M_A^4 + (1 - \xi^{3/2}) \left( M_A^2 + \frac{e^2 T^2}{3} \right)^{3/2} + \left( M_H^2 + \frac{T^2}{24}[6e^2 + 8\lambda] \right)^{3/2} \\
+ \left( M_\xi^2 + \xi M_A^2 + \frac{T^2}{24}[6e^2 + 8\lambda] \right)^{3/2} \right] \quad (3.59)$$

This term is still $\xi$-dependent. We see that resummation (to $\mathcal{O}(\beta)$) does not render the effective potential gauge parameter-independent.

**IV. NON-EQUILIBRIUM ASPECTS**

The main motivation for studying an effective potential is to address the issue of symmetry breaking and phase transitions accounting for quantum and thermal corrections. By its very definition, the effective potential is an *equilibrium* quantity because the expectation value of the scalar field, that serves as the order parameter, is space-time independent. At zero temperature the quantum wave-functional is taken to be a stationary state, at finite temperature, the calculation is performed with the equilibrium partition function. Thus it is clear that the effective potential is only suitable to describe the equilibrium aspects associated with the phase transition. The only equilibrium states correspond to the extrema of the effective action which for homogeneous configurations coincide with the extrema of the effective potential. Thus quantities like the critical temperature defined as the value at which the minima of the effective potential become a maximum, as well as expectation values of the scalar field that extremize the effective potential are meaningful quantities that are useful to determine whether there is a symmetry breaking phase transition.

For values of the order parameter away from the extrema, the effective potential is simply not a reliable tool to describe the situation and by its very definition it is not meant to be. In using the effective potential to address dynamical issues, like “the rolling down” of the
order parameter towards the equilibrium configuration the hope is that the time evolution is rather slow and the use of the “instantaneous” effective potential is justified as some sort of adiabatic approximation. This expectation, however, conceals the physics of the dynamical processes during the phase transition [18].

In second order phase transitions, if the order parameter is initially very small and the system is evolving from an initial disordered (high temperature) phase with short range correlations towards a final equilibrium broken symmetry state with long range correlations, the dynamical process is that of phase separation and growth of correlated domains. This dynamics is not captured solely by the time evolution of the expectation value of the scalar field, but the growth of fluctuations will be manifest in the time dependence of the correlation functions of this field. During the early stages of a second order phase transition, when the order parameter is very small (near the maximum of the effective potential) small amplitude, long wavelength fluctuations become unstable and grow [30]. The two-point correlation functions of the scalar field reflect this instability and grow exponentially, as the order parameter “rolls down” the potential hill.

If the phase transition is of first order, then there are free energy barriers that the system has to overcome to reach the equilibrium state and the phase transition is driven by nucleation, in which large amplitude configurations become unstable and grow. However, if the first order transition is very weak (that is to say that there is a substantial amount of phase mixing) nucleation and phase separation occur on similar spatial and time scales and the long-wavelength instabilities will still be important.

The information on thermodynamic instabilities that lead to phase separation is contained in the effective potential both at zero and finite temperature in the form of an imaginary part. Weinberg and Wu [17] have shown in a beautiful paper, that the imaginary part of the effective potential in scalar theories determines the growth rate of the scalar field two-point correlation function for Gaussian states centered at zero expectation value (in field space). Boyanovsky and de Vega [18] have studied how this growth of correlations affects the time evolution of the order parameter and concluded that the use of the effective potential to describe the time evolution of the expectation value of the scalar field is not only unwarranted but completely misleading and unreliable whenever the initial value of the expectation value is in the “spinodal region” (the region in the tree-level potential where the second derivative is negative).

A. The Imaginary Part

We notice that in the “classical spinodal” region, where \( m_H^2 < 0 \) or \( 0 < |\chi| < \mu/\sqrt{3} \), the Higgs mode has a band of unstable wave-vectors with imaginary frequencies for \( k^2 < \mu^2 - 3|\chi|^2 \). The plasma mode has a band of unstable wave vectors with imaginary frequencies for \( k^2 < \mu^2 - |\chi|^2 \) in the spinodal region \( 0 < |\chi|^2 < \mu^2 \). This new spinodal line ranges from the maximum to the minima of the tree level potential.

When the gauge coupling is switched off, in the absence of long-range forces, this is recognized as the “spinodal” region for the would be Goldstone modes [20]. However in the presence of long range forces, the instabilities are much more severe for long-wavelength fluctuations, as can be seen from the infrared behavior of the plasma frequency eq. (3.13).
As a consequence of these unstable modes, both the zero and finite temperature effective potential acquire an imaginary part for all values of $|\chi|$ between the maximum and the minima of the tree level potential. Therefore we see that unlike the case of a scalar order parameter in which there is a thermodynamically stable region in the phase diagram (between the classical spinodal and the coexistence line), in this case the unstable region covers the whole of the phase diagram below the coexistence curve.

Thus we here obtain one of the important conclusions of this work: the gauge invariant effective potential to one-loop order is complex from the maximum to the minimum of the tree-level potential. There are no homogeneous configurations corresponding to quasi-equilibrium states away from the minimum. Explicitly, the imaginary part of the gauge invariant effective potential at zero temperature is given by:

$$ Im(V_{eff}) = \pm \frac{1}{4\pi^2} \left\{ \int_0^{k_1(|\chi|)} k^2 dk \sqrt{k_1^2(|\chi|) - k^2} \, \Theta(\mu^2 - 3|\chi|^2) \\
+ \int_0^{k_2(|\chi|)} k dk \sqrt{k_2^2(|\chi|) - k^2 \sqrt{2e^2|\chi|^2 + k^2}} \, \Theta(\mu^2 - |\chi|^2) \right\} \\
k_1^2(|\chi|) = 2\lambda(\mu^2 - 3|\chi|^2); \quad k_2^2(|\chi|) = 2\lambda(\mu^2 - |\chi|^2)$$

where the $\pm$ is determined by the direction of the analytic continuation in the frequencies.

The presence of an imaginary part of the effective potential has been sometimes justified as a failure of the loop expansion and “corrected” by the Maxwell construction which yields a convex free energy. However, just as the Van der Walls loop in the equation of state for liquid-gas systems is a signal of thermodynamic instabilities, the imaginary part of the effective potential signals the presence of similar instabilities in the quantum system.

The flat region of the Maxwell constructed free energy indicates that the system will be found in a coexistence of phases but offers no information on the non-equilibrium processes leading to phase separation. In this respect the effective potential with its imaginary part at least provides some restricted information on dynamical processes. As discussed in references [17,18] this imaginary part determines the decay rate of unstable Gaussian states.

In the one-loop approximation, the Hamiltonian for the modes with wavevectors in the unstable bands correspond to inverted harmonic oscillators for which the analysis of references [17,18,20,31] can be applied. The effective potential (3.14) is complex because the modes corresponding to inverted harmonic oscillators were treated as ordinary harmonic oscillators, i.e. by analytically continuing the zero point energy for these oscillators. The Gaussian wavefunctionals and density matrix have complex kernels for the fields $\eta_1, \eta_2$ reflecting this analytic continuation.

In terms of the shifted fields $\eta_1, \eta_2$ with zero expectation value, the Gaussian wavefunctional (3.13) and the density matrix (see appendix I) are “centered” at the origin in field space, and for the wavevectors in the unstable bands, they are not stationary states of the Hamiltonian.

In this one-loop approximation we can use the results of previous investigations of similar situations [17,18,20,31] to study the early time evolution of initially prepared non-equilibrium states.
B. Early time evolution of non-equilibrium states

From the previous discussion it is clear that the imaginary part of the effective potential conceals a dynamical non-equilibrium situation associated with the instabilities towards phase separation. This is a dynamical situation that must be treated as a time dependent problem. The quantum states that lead to the effective potential (and free energy) are not stationary states.

Since only the Higgs and plasma modes have unstable bands we will focus only on these. Consider the situation in which an initial Gaussian density matrix has been prepared such that the gauge invariant scalar field Φ has initial expectation value in this ensemble given by χ, and that the kernels have positive frequencies. That is, the density matrix in the Schrödinger representation is given by the expression in appendix I but with the following real kernels:

\[ K_H^{(1)}(k, t = 0) = -\frac{1}{2} W_H(k) \text{coth} \left( \frac{W_H(k)}{T} \right) \quad ; \quad K_H^{(2)}(k, t = 0) = \frac{W_H(k)}{\sinh \left( \frac{W_H(k)}{T} \right)} \]  
\[ K_p^{(1)}(k, t = 0) = -\frac{1}{2} W_p(k) \text{coth} \left( \frac{W_p(k)}{T} \right) \quad ; \quad K_H^{(2)}(k, t = 0) = \frac{W_p(k)}{\sinh \left( \frac{W_p(k)}{T} \right)} \]  

\[ W_H^2(k) > 0 \quad , \quad W_p^2(k) > 0 \] (4.4)

The frequencies \( W_H(k) \), \( W_p(k) \) determine the initial conditions on the state as discussed in refs. \[17,18,20,31\]. We can now use the results of references \[17,18,20,31\] to determine the time evolution of this initial state. The early time evolution of the expectation value χ will be determined by the classical equations of motion with small quantum and thermal corrections. However at longer times the growth of the unstable modes makes the one-loop approximation to the dynamics unreliable as in the problem of domain growth in scalar theories \[20,31\]. Clearly, the most important dynamics is described by the evolution of the unstable modes and determined by the time evolution of the kernels. We now use the method discussed in reference \[20\] to obtain the following time evolution of the kernels:

\[ K_H^{(1)}(k, t) = \frac{K_H^{(1)}(k, t = 0)}{|U_k(t)|^2} - i \frac{d \ln(|U_k(t)|^2)}{dt} \] (4.5)
\[ K_H^{(2)}(k, t) = \frac{K_H^{(2)}(k, t = 0)}{|U_k(t)|^2} \] (4.6)
\[ K_p^{(1)}(k, t) = \frac{K_p^{(1)}(k, t = 0)}{|V_k(t)|^2} - i \frac{d \ln(|V_k(t)|^2)}{dt} \] (4.7)
\[ K_p^{(2)}(k, t) = \frac{K_p^{(2)}(k, t = 0)}{|V_k(t)|^2} \] (4.8)

where the mode functions \( U_k(t) = U_{1k}(t) + iU_{2k}(t) \), \( V_k(t) = V_{1k}(t) + iV_{2k}(t) \) obey the following evolution equations \[20\]

\[ \left[ \frac{d^2}{dt^2} + k^2 + 2\lambda(3|\chi_{cl}|^2(t) - \mu^2) \right] U_k(t) = 0 \]
\[ U_{1k}(0) = 1, \ U_{2k}(0) = 0, \ \dot{U}_{1k}(0) = 0, \ \dot{U}_{2k}(0) = W_H(k) \]  
\[ \left[ \frac{d^2}{dt^2} + \left[ k^2 + 2\lambda(|\chi|_c|^2(t) - \mu^2) \right] \left[ k^2 + 2e^2|\chi|_c|^2(t) \right] / k^2 \right] V_k(t) = 0 \]
\[ V_{1k}(0) = 1, \ V_{2k}(0) = 0, \ \dot{V}_{1k}(0) = 0, \ \dot{V}_{2k}(0) = W_P(k) \]  

where \( \chi_{cl} \) is the classical evolution of the “zero mode” \[20]. Obviously the mode functions \( U_k(t) \), \( V_k(t) \) grow almost exponentially for wave-vectors in the respective unstable bands when \(|\chi_{cl}|(t)\) is in the corresponding spinodal regions.

Consider the equal-time two-point correlation functions of the gauge invariant operator \( \eta, \eta^\dagger \):
\[
\langle \eta(\vec{x}, t)\eta^\dagger(\vec{y}, t) \rangle = \int \frac{d^3k}{(2\pi)^3} e^{i\vec{k} \cdot (\vec{x} - \vec{y})} \left\{ \frac{|U_k(t)|^2}{2W_H(k)} \coth \frac{W_H(k)}{T} + \frac{|V_k(t)|^2}{2W_P(k)} \coth \frac{W_P(k)}{T} \right\}
\]

When \( \chi_{cl}(t) \) remains in the spinodal, it is clear that the growth rate of \( \langle \eta(\vec{0}, t)\eta^\dagger(\vec{0}, t) \rangle \) is related to the imaginary part of the effective potential given by eq. (4.1) at zero temperature.

At this point we recognize an important payoff of the gauge invariant description. In order to compute gauge invariant correlation functions from the gauge variant operator \( \phi(\vec{x}, t) \) we would have to append a line integral of the (time dependent) gauge field, with the ensuing path ambiguities and complications. The formulation in terms of gauge invariant order parameters from the start overcomes these difficulties and allows to extract physically meaningful correlation functions that provide dynamical information on non-equilibrium processes.

Just as in the case of a scalar field theory undergoing spinodal decomposition \[31\] this equal-time correlation function will grow exponentially at early times because of the unstable modes. This growth is the hallmark of the dynamics of phase separation \[17,18,30\].

If the phase transition is weakly first order in the sense that a considerable amount of phase separation and mixing occurs during nucleation, then the growth of correlations described by the dynamics of the unstable modes will be also a dominant process.

As noted before, unlike the case of a scalar field transforming under a discrete symmetry, in this case the spinodal region reaches all the way to the minimum of the potential (at least in the one-loop approximation) and the long-wavelength instabilities are enhanced by the long-range forces. This feature provides interesting possibilities if the phase transition is strongly first order and nucleation is the dominant mechanism. Typically after nucleation of a critical bubble, the order parameter inside the bubble is very close to its equilibrium value. If this value is smaller in magnitude to that of the equilibrium configuration, spinodal instabilities may be still present inside the bubble.

If the critical droplet has rather small radius it will have to grow to a size \( \approx 1/(|\chi| - \mu) \), with \(|\chi| \) the value of the order parameter inside the bubble, before any instability develops inside the bubble. Therefore the fact that the spinodal line reaches to the minimum may lead to the tantalizing possibility that for large critical bubbles there may still be non-equilibrium processes inside the bubble if the order parameter inside is smaller in magnitude than the equilibrium value.

Clearly all these possibilities will have to be studied in deeper detail and we expect a strong dependence on the values of the gauge and Yukawa couplings (in the case of fermions).
Furthermore our analysis of the time evolution of the spinodal instabilities only holds at very early times after the preparation of the initial state. At longer times a non-perturbative gauge invariant scheme will have to be used to determine the dynamics. We hope to be able to implement the variational schemes of reference [32] to the gauge invariant formulation.

V. CONCLUSIONS

In this article we have presented a formulation of the effective potential in terms of a gauge invariant order parameter for the case of the Abelian Higgs model. The gauge invariant states of the theory are those annihilated by the first class constraints and gauge invariant operators are those that commute with these constraints. We recognized an order parameter that is invariant under the local gauge transformations but transforms as a charged operator under global phase rotations, its expectation value in the lowest energy state therefore signals the breakdown of the global $U(1)$ symmetry.

We evaluated the one-loop contribution to this gauge invariant effective potential both at zero and non-zero temperature and obtained its high temperature expansion. We found that the effective potential is complex both at zero and non-zero temperature and that the spinodal (thermodynamically unstable) region extends from the maximum to the minimum of the tree level potential (in this approximation). The gauge invariant effective potential was compared to the effective potential obtained in several covariant gauges, we found that the dependence on gauge parameter cancels only at the extrema, but for all other values of the order parameter the gauge fixed effective potentials are gauge parameter dependent at zero and non-zero temperature. In particular in a high temperature expansion only the $\mathcal{O}(T^2)$ contribution is gauge invariant, whereas the $\mathcal{O}(T^3), \mathcal{O}(\ln(T)) \cdots$ depend on the gauge parameter and are complex. In general these cannot be taken as trustworthy quantities to extract information on the strength of the transition and its features, such as latent heat and supercooling temperature, although the contribution of the gauge boson masses (we did not study fermions) are gauge independent and real to this order.

The imaginary part of the one-loop effective potential determines the spinodal lines, which identify the region of long-wavelength instabilities. Unlike the case of a scalar field with discrete symmetries, the spinodal line encompasses all values of the homogeneous order parameter ranging between the maximum and the minimum of the tree level potential in the one-loop approximation.

Non-equilibrium dynamical aspects were then studied in terms of gauge invariant correlation functions. It is in these dynamical situations out of equilibrium that the gauge invariant formulation has the greatest impact, since in general, correlation functions involve non-local line integrals of the gauge field. We obtained the early time behavior of the gauge invariant correlation functions for initial conditions in which the order parameter is in the spinodal region, these correlation functions grow exponentially at early times as a consequence of the instabilities.

We conjectured that these spinodal instabilities may play an important role in first order phase transitions, in that they may be responsible for non-equilibrium dynamics inside the nucleating bubbles. Clearly this possibility will have to be studied further.

There are several important avenues to pursue: higher order calculations and the implementation of variational or Hartree-like approximations in the gauge invariant formulation to
address the long-time behavior and to obtain a more solid understanding of non-equilibrium processes. Furthermore, it is important to generalize the resummation program to the gauge invariant effective potential, and to generalize these new methods and results to non-Abelian gauge theories in particular the electroweak theory. Study on these issues is underway.

VI. APPENDIX I

As shown in section II, in the one-loop approximation the Hamiltonian becomes a sum of independent harmonic oscillators for the gauge invariant normal modes and in this case the explicit form of the density matrix in the Schroedinger representation is available (see [20] and references therein). This form of the density matrix is particularly convenient to study real-time dynamics, since the Liouville equation becomes a functional differential equation, which in this case can be solved explicitly for the time dependence [20].

The *equilibrium* density matrix that describes the system at temperature $T = 1/\beta$ is given in the Schroedinger representation by:

$$\langle \Phi, \Phi^\dagger, A_T | \rho | A_T^\dagger, \Phi' \rangle = N[T] \times$$

$$\exp \left\{ \int d^3x \int d^3y \left[ \tilde{A}_T(\vec{x}) \cdot \tilde{A}_T(\vec{y}) + \tilde{A}_T^\dagger(\vec{x}) \cdot \tilde{A}_T^\dagger(\vec{y}) \right] K_T^{(1)}(\vec{x} - \vec{y}) + \tilde{A}_T(\vec{x}) \cdot \tilde{A}_T(\vec{y}) K_T^{(2)}(\vec{x} - \vec{y}) \right\} \times$$

$$\exp \left\{ \int d^3x \int d^3y \left[ \eta_1(\vec{x}) \eta_1(\vec{y}) + \eta_1'(\vec{x}) \eta_1'(\vec{y}) \right] K_H^{(1)}(\vec{x} - \vec{y}) + \eta_1(\vec{x}) \eta_1(\vec{y}) K_H^{(2)}(\vec{x} - \vec{y}) \right\} \times$$

$$\exp \left\{ \int d^3x \int d^3y \left[ \eta_2(\vec{x}) \eta_2(\vec{y}) + \eta_2'(\vec{x}) \eta_2'(\vec{y}) \right] K_p^{(1)}(\vec{x} - \vec{y}) + \eta_2(\vec{x}) \eta_2(\vec{y}) K_p^{(2)}(\vec{x} - \vec{y}) \right\} \right.$$

(6.1)

$$N[T] = N[0] \Pi_k \left[ \tanh^2 \left( \frac{\omega_T(k)}{T} \right) \tanh \left( \frac{\omega_H(k)}{T} \right) \tanh \left( \frac{\omega_p(k)}{T} \right) \right]^{1/2}$$

(6.2)

$$K_T^{(1)}(\vec{x} - \vec{y}) = -\frac{1}{2} \int \frac{d^3k}{(2\pi)^3} \omega_T(k) \coth \left( \frac{\omega_T(k)}{T} \right) e^{i\vec{k} \cdot (\vec{x} - \vec{y})}$$

(6.3)

$$K_T^{(2)}(\vec{x} - \vec{y}) = \int \frac{d^3k}{(2\pi)^3} \frac{\omega_T(k)}{\sinh \left( \frac{\omega_T(k)}{T} \right)} e^{i\vec{k} \cdot (\vec{x} - \vec{y})}$$

(6.4)

$$K_H^{(1)}(\vec{x} - \vec{y}) = -\frac{1}{2} \int \frac{d^3k}{(2\pi)^3} \omega_H(k) \coth \left( \frac{\omega_H(k)}{T} \right) e^{i\vec{k} \cdot (\vec{x} - \vec{y})}$$

(6.5)

$$K_H^{(2)}(\vec{x} - \vec{y}) = \int \frac{d^3k}{(2\pi)^3} \frac{\omega_H(k)}{\sinh \left( \frac{\omega_H(k)}{T} \right)} e^{i\vec{k} \cdot (\vec{x} - \vec{y})}$$

(6.6)

$$K_p^{(1)}(\vec{x} - \vec{y}) = -\frac{1}{2} \int \frac{d^3k}{(2\pi)^3} \frac{k^2 \omega_p(k)}{\omega_T^2(k) \sinh \left( \frac{k^2 \omega_p(k)}{T \omega_T^2(k)} \right)} e^{i\vec{k} \cdot (\vec{x} - \vec{y})}$$

(6.7)

$$K_p^{(2)}(\vec{x} - \vec{y}) = \int \frac{d^3k}{(2\pi)^3} \frac{\omega_p(k)k^2}{\omega_T^2(k) \sinh \left( \frac{k^2 \omega_p(k)}{T \omega_T^2(k)} \right)} e^{i\vec{k} \cdot (\vec{x} - \vec{y})}$$

(6.8)

The trace of this density matrix gives the gauge invariant one-loop partition function, whose logarithm (divided by $-\beta$) gives the one-loop effective potential at finite temperature.
VII. APPENDIX II

In this appendix we provide the essential ingredients to obtain the high temperature expansion of the contribution of the plasma mode to the finite temperature effective potential.

The first two terms in the free energy density given by eq. (3.25) (the transverse and Higgs modes) are of the usual form with dispersion relations of the form \( \omega(k) = \sqrt{k^2 + m^2} \) and their asymptotic high temperature expansion can be obtained by following the steps described in \[1\]. The third term requires special attention because of the unusual dispersion relation of the plasma mode. This term can be written as

\[
\int \frac{d^3k}{(2\pi)^3} \ln \left[ 1 - e^{-\beta \omega_p(k)} \right] = \frac{T^4}{2\pi^2} I(y_1^2, y_2^2)
\]

\[
I(y_1^2, y_2^2) = \int_0^\infty x^2 dx \ln \left[ 1 - e^{-\frac{1}{2} \sqrt{(x^2+y_1^2)(x^2+y_2^2)}} \right]
\]

\[
y_1^2 = \frac{m_0^2}{T^2}, \quad y_2^2 = \frac{m_0^2}{T^2}
\]

(7.1)

The function \( I(y_1^2, y_2^2) \) can be written as

\[
I(y_1^2, y_2^2) = \int_0^{y_1^2} \int_0^{y_2^2} dadb \frac{d}{da} \frac{d}{db} I(a, b) + I(0, y_2^2) + I(y_1^2, 0) - I(0, 0)
\]

(7.2)

The last three terms are of standard form and can be calculated following the procedures of reference \[1\]. Using the identity (3.11) of reference \[1\] we find

\[
\frac{d}{da} \frac{d}{db} I(a, b) = -\frac{1}{2} \frac{d}{da} \frac{d}{db} \int_0^\Lambda x dx \sqrt{(x^2 + a)(x^2 + b)} + \int_0^\Lambda x^2 dx \sum_{l=1}^\infty \frac{2\pi lx}{(x^2 + a)(x^2 + b) + (2\pi lx)}
\]

(7.3)

where we introduced an upper momentum cutoff to regulate the individual terms. The integral in the first term is the same as the plasma contribution to the zero temperature effective potential in eq. (3.14) but in terms of the variables \( a, b \). This integral has a logarithmic cutoff dependence. The second term does not have infrared divergences for \( a, b = 0 \) and can be expanded in power series of \( a, b \). In this series, the first term, with \( a = b = 0 \) has a logarithmic cutoff dependence that exactly cancels (upon integration on \( a, b \)) that of the first term. The remaining series is both ultraviolet and infrared safe. Finally only the finite terms from the zero temperature effective potential but in terms of the variables \( y_1^2; y_2^2 \) contribute to \( I(y_1^2, y_2^2) \) and we obtain the result quoted in equation (3.27).

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