ON HYDROMAGNETIC STRESSES IN ACCRETION DISK BOUNDARY LAYERS

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Received 2011 November 18; accepted 2012 March 17; published 2012 May 3

ABSTRACT

Detailed calculations of the physical structure of accretion disk boundary layers, and thus their inferred observational properties, rely on the assumption that angular momentum transport is opposite to the radial angular frequency gradient of the disk. The standard model for turbulent shear viscosity satisfies this assumption by construction. However, this behavior is not supported by numerical simulations of turbulent magnetohydrodynamic (MHD) accretion disks, which show that angular momentum transport driven by the magnetorotational instability (MRI) is inefficient in disk regions where, as expected in boundary layers, the angular frequency increases with radius. In order to shed light on physically viable mechanisms for angular momentum transport in this inner disk region, we examine the generation of hydromagnetic stresses and energy density in differentially rotating backgrounds with angular frequencies that increase outward in the shear-layer framework. We isolate the modes that are unrelated to the standard MRI and provide analytic solutions for the long-term evolution of the resulting shearing MHD waves. We show that, although the energy density of these waves can be amplified significantly, their associated stresses oscillate around zero, rendering them an inefficient mechanism to transport significant angular momentum (inward). These findings are consistent with the results obtained in numerical simulations of MHD accretion disk boundary layers and challenge the standard assumption of efficient angular momentum transport in the inner disk regions. This suggests that the detailed structure of turbulent MHD accretion disk boundary layers could differ appreciably from those derived within the standard framework of turbulent shear viscosity.

Key words: accretion, accretion disks – instabilities – magnetohydrodynamics (MHD) – turbulence

1. INTRODUCTION

Basic arguments suggest that the angular frequency \(\Omega(r)\) of an accretion disk surrounding a weakly magnetized star must attain a maximum value, \(\Omega_{\text{max}} \equiv \Omega(r_\text{b})\), and decrease inward (or at least remain constant; see Medvedev 2004) to match the angular frequency of the star at the stellar radius \(\Omega_\star\); see, e.g., Frank et al. 2002; Hartmann 2009; Armitage 2010). The inner disk region, where \(r < r_\text{b}\), and \(d\Omega/dr \geq 0\), is referred to as the accretion disk boundary layer. Standard accretion disk theory (Shakura & Sunyaev 1973) predicts that half of the energy released in the accretion process takes place in this region, estimated to be a fraction of the stellar radius. The spectrum of the radiated energy depends on the detailed properties of this layer (Narayan & Popham 1993; Popham & Narayan 1995; Popham et al. 1996; Popham & Sunyaev 2001). Thus, understanding the various processes that determine its properties (see, e.g., Piro & Bildsten 2004; Balsara et al. 2009; Inogamov & Sunyaev 2010) is of fundamental importance.

Most detailed calculations for determining the structure of the boundary layer rely on effective models for turbulent angular momentum transport. These models are usually built as a turbulent version of the Newtonian viscous stress between fluid layers in a differentially rotating laminar flow (Landau & Lifshitz 1959), and thus assume a linear relationship between the stress and the angular frequency gradient (Lynden-Bell & Pringle 1974). This assumption, however, seems at odds with the properties of magnetohydrodynamic (MHD) turbulence revealed by numerical simulations of accretion disks (Abramowicz et al. 1996; Armitage 2002; Steinacker & Papaloizou 2002; Pessah et al. 2008) which show that angular momentum transport is inefficient in regions of the disk where \(d\Omega/dr > 0\), which are stable to the standard magnetorotational instability (MRI; see Balbus & Hawley 1991, 1998).

Motivated by the need of a deeper understanding of the behavior of an MHD fluid in a differentially rotating background that deviates from a Keplerian profile, we study the dynamics of MHD waves in configurations that are stable to the standard MRI. Employing the shearing-sheet framework, we show that transient amplification of shearing MHD waves can generate magnetic energy without leading to a substantial generation of hydromagnetic stresses. We discuss the implications of these findings.

2. ASSUMPTIONS AND LOCAL MODEL FOR MHD DISK

We focus our attention on a subsonic, weakly magnetized fluid for which the ram pressure and magnetic pressure remain small compared to their thermal counterpart. As a first approximation, we thus consider a differentially rotating fluid with angular frequency \(\Omega = \Omega(r)\xi \hat{z}\) and constant background density \(\rho_0\). This is a reasonable assumption in light of the results presented by Armitage (2002) and Steinacker & Papaloizou (2002), who carried out numerical simulations of boundary layers of unstratified accretion disks and found that the density fluctuations throughout the simulations are in general quite small.

We work in the framework of the shearing-sheet approximation (Hill 1878; Goldreich & Tremaine 1978; Narayan et al. 1987; Hawley et al. 1995), where the equations describing an incompressible MHD fluid in a corotating frame are given by

\[
\partial_t \mathbf{v} + (\mathbf{v} \cdot \nabla) \mathbf{v} = -2\Omega_0 \hat{z} \times \mathbf{v} + 2q\Omega_0^2 x \hat{x} - \frac{\nabla P}{\rho_0} + \frac{(\mathbf{B} \cdot \nabla) \mathbf{B}}{4\pi \rho_0} + \nu \nabla^2 \mathbf{v}, \tag{1}
\]

\[
\partial_t \mathbf{B} + (\mathbf{v} \cdot \nabla) \mathbf{B} = (\mathbf{B} \cdot \nabla) \mathbf{v} + \eta \nabla^2 \mathbf{B}. \tag{2}
\]
Here, \( v(x, t) \) and \( B(x, t) \), with \( \nabla \cdot v = \nabla \cdot B = 0 \), stand for the velocity and magnetic fields; \( v \) and \( \eta \) denote the kinematic viscosity and resistivity; and \( \Omega_0 \equiv \Omega(r_0) \) is the corotating angular frequency at a fiducial radius \( r_0 \). The first and second terms on the right-hand side of Equation (1) correspond to the Coriolis and tidal forces, respectively. The local pressure \( P \) can be found by the divergence-less condition of the velocity field. Recalling that the local density \( \rho_0 \) is assumed to be constant, and in order to simplify notation, hereafter we redefine the symbols denoting the pressure and magnetic field in such a way that and \( \kappa \equiv \rho_0 \Omega_b \). The leading order background velocity is\( \hat{U}_1(x) = -q \Omega_0 \hat{r} \hat{y} \), where the shear parameter \( q \) is given by
\[
q \equiv -\frac{d \ln \Omega}{d \ln r} \Bigg|_{r_0}. \tag{5}
\]
The homogeneous background magnetic field is, in general, a function of time and it evolves according to the induction Equation (2), i.e., \( \partial_t B_0 = -q \Omega_b \hat{r} \hat{y} \).

The substitution of Equations (3) and (4) into (1) and (2) leads to a nonlinear system for the dynamical evolution of the perturbations \( u(x, t) \) and \( b(x, t) \). However, as pointed out in Goodman & Xu (1994), all the nonlinear terms in the resulting equations vanish identically if we consider the evolution of a single Fourier mode.\(^5\) In this case, we obtain
\[
\begin{align*}
\partial_t u - q \Omega_0 \hat{r} \hat{y} b &= B_0 \cdot \nabla v + v \nabla^2 u - 2 \Omega_0 \hat{z} \times u - \nabla P, \tag{6} \\
\partial_t b + q \Omega_0 \hat{r} \hat{y} u &= B_0 \cdot \nabla u + \eta \nabla^2 b. \tag{7}
\end{align*}
\]
The “semi-Lagrangian” time derivative \( \partial_t \equiv \partial_t + U_1 \cdot \nabla \) accounts for advection by the shearing background. The shearing component in the Coriolis term cancels out the tidal force. We remark that Equations (6) and (7) are not just linearized equations, they remain valid even if the amplitude of the perturbations is not small compared to the background values, and they are exact as long as a single Fourier mode is considered. Under these conditions, it is sensible to study the evolution of \( u(x, t) \) and \( b(x, t) \) for a long time.

In order to solve Equations (6) and (7), it is convenient to work in Fourier space. The \( x \)-dependence of the “semi-Lagrangian” time derivative can be removed by employing a shearing coordinate system \((x', y', z', t')\) in which \( \partial_t \equiv \partial_t \) (Goldreich & Lynden-Bell 1965). A single mode with a fixed “shearing” wavenumber \( \kappa \) is thus given by
\[
\begin{align*}
u(k, \omega) &= 2 \text{Re}\{ \hat{u}_k(t) \exp(i k' \cdot x') \}, \tag{8} \\
b(k, \omega) &= 2 \text{Re}\{ \hat{b}_k(t) \exp(i k' \cdot x') \}, \tag{9}
\end{align*}
\]
where \( k' \cdot x' = k(t) \cdot x = (k'_x + q \Omega_0 t k'_z) x + k'_y y + k'_z z \).

Substituting the ansatz (8) and (9) into Equations (6) and (7) leads to a set of equations for the Fourier amplitudes
\[
\begin{align*}
d_t \hat{u} - q \Omega_0 \hat{r} \hat{y} \hat{b} &= i \omega A \hat{b} - v k^2 \hat{u} - 2 \Omega_0 \hat{z} \times \hat{u} - i k \hat{P}, \tag{10} \end{align*}
\]
\[
\begin{align*}
d_t \hat{b} + q \Omega_0 \hat{r} \hat{y} \hat{u} &= i \omega A \hat{u} - \eta k^2 \hat{b}. \tag{11}
\end{align*}
\]
Here, we have replaced \( \partial_t \) by \( d_t \) and omitted the subscripts \( \kappa \)' in the Fourier coefficients in order to simplify the notation. We have also introduced the time-independent Alfvén frequency \( \omega_A \equiv B_0(t) \cdot k(t) \) (see also Balbus & Hawley 1992a).

The pressure term can be eliminated from Equation (10) using the solenoidal character of the velocity field, which implies \( d_t k = i k d_t + i q \Omega_0 \hat{r} \hat{y} \). This leads to
\[
\hat{P} = -\frac{2 \Omega_0}{k^2} ((q - 1) k_x \hat{u}_x + k_z \hat{u}_z), \tag{12}
\]
which is independent of \( \hat{u}_z \). Because we are interested in the transport of angular momentum along the radial direction, this decoupling allows us to solve separately for the \( x \)- and \( y \)-components.

3. DYNAMICAL EVOLUTION OF MRI-STABLE MODES

The dynamical evolution of the modes with \( k = k_x \hat{z} \) is quite simple; they grow exponentially if \( k_x^2 \omega_A^2 \leq 2q \Omega_0^2 \) (Balbus & Hawley 1992b; Pessah et al. 2006). Thus, a Keplerian disk (with \( q = 3/2 \)) can exhibit exponential growth but a shear profile with \( q < 0 \) only supports stable oscillations. In order to isolate the interesting dynamics that could arise from modes that are not associated with the MRI, we thus focus on modes with \( k_x = 0 \). Taking the curl of the momentum equation, it is easy to verify that the Coriolis term does not play a role in the equation for the vorticity, and hence it has no effect on the dynamics of the system (Lithwick 2007). This shows explicitly that the standard MRI is absent in our analysis.

We choose the origin of time so that \( k_x(t) = 0 \) is initially zero. In other words, we use \( k_x(t) \) to define our time coordinate,
\[
\tau \equiv k_x(t)/k_x \equiv q \Omega_0 t, \tag{13}
\]
so the divergence-less conditions become
\[
\tau \hat{u}_x + \hat{u}_y = \tau \hat{b}_x + \hat{b}_y = 0. \tag{14}
\]
Assuming \( v, \eta \ll \omega_A/k^2 \), we can work in the ideal limit and neglect viscosity and resistivity. The \( x \)-components of the MHD equations then become
\[
\begin{align*}
d_t \left[ \begin{array}{c} \hat{u}_x \\
\hat{b}_x \end{array} \right] &= \begin{pmatrix} -\Gamma & i \omega \\ i \omega & 0 \end{pmatrix} \left[ \begin{array}{c} \hat{u}_x \\
\hat{b}_x \end{array} \right], \tag{15} \end{align*}
\]
where all the temporal dependence is contained in the factor
\[
\Gamma(\tau) \equiv 2 \tau/(\tau^2 + 1), \tag{16}
\]
and
\[
\omega \equiv \omega_A/q \Omega_0 \tag{17}
\]
is the dimensionless Alfvén frequency. The linear system (15) can be recast into one equation as
\[
\begin{align*}
d_t^2 \hat{b}_x + \Gamma(\tau) d_t \hat{b}_x + \omega^2 \hat{b}_x &= 0, \tag{18}
\end{align*}
\]
\(^5\) Formally speaking, we consider two modes with wavenumbers \( k \) and \(-k \). However, because the functions under consideration are real, the Fourier coefficients satisfy \( \hat{f}_{-k} = \hat{f}_{k}^* \).
where the dependence on the parameters $q$, $\Omega_0$, and $\omega_A$ is only
through the combination $p(\omega_A/q\Omega_0)^2$. This second-order
differential equation for $\hat{b}_z$ is identical to Equation (2.20) in Balbus &
Hawley (1992a) when $k_z = 0$. In this case, the perturbations in the
$z$-coordinate decouple from the perturbations in the
perpendicular direction (see also their Equation (2.19)).

Unfortunately, Equation (18) does not have an analytical
solution. However, if we consider the limit $\tau^2 \gg 1$, it reduces
to a spherical Bessel equation, which possesses as solutions

\[
\hat{u}_z = S_j(\omega \tau) + C_1(\omega \tau),
\]

\[
= -\frac{C + S\omega \tau}{\omega^2 \tau^2} \cos(\omega \tau) + \frac{S - C\omega \tau}{\omega^2 \tau^2} \sin(\omega \tau),
\]

\[
\hat{b}_z = -iS_j(\omega \tau) - C_1(\omega \tau),
\]

\[
= \frac{iC}{\omega} \cos(\omega \tau) - \frac{iS}{\omega} \sin(\omega \tau),
\]

where $j_n(x)$ and $y_n(x)$, with $n = 0, 1$, are spherical Bessel
functions of the first and second kind, respectively; and $S$ and
$C$ are complex constants determined by the initial conditions.
Using the divergence-less conditions in Equation (14), the
$y$-components are simply

\[
\hat{u}_y = -\tau [S_j(\omega \tau) + C_1(\omega \tau)],
\]

\[
= \frac{C + S\omega \tau}{\omega^2 \tau^2} \cos(\omega \tau) - \frac{S - C\omega \tau}{\omega^2 \tau^2} \sin(\omega \tau),
\]

\[
\hat{b}_y = i\tau [S_j(\omega \tau) + C_1(\omega \tau)],
\]

\[
= \frac{iC}{\omega} \cos(\omega \tau) + \frac{iS}{\omega} \sin(\omega \tau).
\]

Because the pressure in Equation (12) is independent of $\hat{u}_z$, the
exact solutions (for all time) are Alfvén waves,

\[
\hat{u}_z = C' \cos(\omega \tau) + S' \sin(\omega \tau),
\]

\[
\hat{b}_z = iC' \sin(\omega \tau) - iS' \cos(\omega \tau),
\]

where $C'$ and $S'$ are some other complex constants. We note
that, although the direction of the dimensionless time $\tau$ depends
on the sign of the shear parameter $q$, the combination $\omega \tau \equiv \omega_A t$
is insensitive to it. Hence, given the same initial conditions, $\hat{u}_z$
and $\hat{b}_z$ are symmetric, while $\hat{u}_y$ and $\hat{b}_y$ are antisymmetric, in
the shear parameter $q$.

We demonstrate the accuracy of these analytical approximations
in Figure 1, which shows both the numerical and analytical
solutions for $\text{Im}[\hat{b}_y]$ and $\text{Im}[\hat{b}_z]$ with $q = -3/2$ and
$\omega_A = \Omega_0$ as an example. The initial conditions are set at
$\omega \tau = \omega_A t = -20$ by choosing $C_+ = 1$ and $S_- = 0$. The
numerical solutions, shown with thick solid lines, result from
integrating Equation (18) with the definition (16) and using
the divergence-less conditions (14). The dotted lines in the two
panels are obtained by setting $C = C_+$ and $S = S_-$ in
the analytical approximations (20) and (22). These solutions are in-
distinguishable for $\tau \lesssim -1$. As expected, the approximations
break down for $\tau \gtrsim 0$. This is precisely where the numerical
solutions change their amplitudes significantly. The analytical
expressions (20) and (22) are again in excellent agreement with
the numerical solutions for $\tau \gtrsim 1$, provided that their ampli-
tudes are given by $C = C_+ = 0.285$ and $S = S_+ = -1.896$.
These constants are found by requiring that both the numerical
and analytical solutions match for $\omega \tau = \omega_A t \gg 1$ (in practice,
we set $\omega \tau = 20$).

Even though our analytical approach cannot predict the change
in amplitude close to $\tau \approx 0$, the solutions that we obtain
are a very good approximation to the numerical results as
long as $\tau^2 > 1$. We could, in principle, obtain the coefficients
$C_+$ and $S_+$ by an asymptotic matching technique similar to the
one employed in Heinemann & Papaloizou (2009). However,
the analytical solution near $\tau \approx 0$ contains special functions
that are too complicated to be useful. More importantly, as we
show below, the most interesting features of the solutions are
independent of the precise values of these constants.

4. LATE TIME STRESS AND ENERGY

Given the solutions (19)-(24) for the Fourier amplitudes, we
obtain the (mean) total stress $T_{xy} \equiv \langle u_x u_y - b_x b_y \rangle$ and (mean)
early while the energy density asymptotes to a non-vanishing, time-time stress oscillates around zero with decreasing amplitude, 5 Here, we assume 4 We do not include the stress and energy generated by the time-dependent contributions due to any stable Alfvén wave initially present (see Equations (23) and (24)).

Energy density \( E \equiv \langle u^2 + b^2 \rangle / 2 \) of the fluctuating fields, \(^4\) where the brackets stand for the spatial average (see, e.g., Pessah et al. 2006). Because these solutions are only valid for early/late times, we can approximate the total stress and energy density up to first order in \( 1 / \omega \tau \) as

\[
T_{\Sigma} \approx -\frac{2}{\omega^2 \tau}[\langle |S|^2 \rangle - \langle |C|^2 \rangle \cos(2\omega \tau) + \langle S^* C + SC^* \rangle \sin(2\omega \tau)],
\]

\[
E \approx -\frac{1}{\omega^2 \tau}[\langle |S|^2 \rangle - \langle |C|^2 \rangle \sin(2\omega \tau) + \langle S^* C + SC^* \rangle \cos(2\omega \tau)]
\]

\[
+ \frac{1}{\omega^2 \tau} \langle |S|^2 \rangle - \langle |C|^2 \rangle + |S|^2 + |C|^2,
\]

where the asterisk denotes complex conjugation. Using these expressions, it is easy to see that the energy balance equation \( \dot{E} = q \Omega T_{\Sigma} \) is also satisfied up to order \( 1 / \omega \tau \).

In Figure 2, we illustrate the numerical solutions for the stress \( T_{\Sigma}(t) \) and energy \( E(t) \), given by the thin and thick solid lines, together with the analytical approximation for the energy. The latter has been obtained by substituting the two pairs of constants, \( C_+ = 1 \) and \( C_- = 0 \), and \( S_+ = 0.285 \) and \( S_- = -1.896 \), into Equation (26). It is thus clear that the late-time stress oscillates around zero with decreasing amplitude, while the energy density asymptotes to a non-vanishing, time-independent value. The expression for the energy density at early/late times in terms of the constants \( S_\pm \) and \( C_\pm \) is given by\(^3\)

\[
E_{\pm} \equiv \lim_{t \to \pm \infty} E(t) = \frac{|S_{\pm}|^2 + |C_{\pm}|^2}{\omega^2}.
\]

\(^3\) Here, we assume \( S = C = 0 \), and thus avoid the uninteresting contributions due to any stable Alfvén wave initially present (see Equations (23) and (24)).

Therefore, the energy gain via swing amplification, \( E_+ / E_- \), is in general a function of the ratio \( \omega = \omega_0 / \Omega \) and the initial conditions. However, it is possible to obtain conclusions that are independent of the latter.

The dependence of the energy gain on the initial conditions for \( \omega^2 = 1 \) is shown in Figure 3. The horizontal axis describes how the initial energy is distributed between the \( j_3 \) modes and the \( y_3 \) modes, while the different lines show the phase difference in the corresponding initial amplitudes. When \( \arg(S_- / C_-) = \pi / 2 \) or \( 3\pi / 2 \), the \( j_3 \) and y3 modes are completely out of phase and evolve independently. This results in an energy gain that is linear in the initial amplitudes (thick solid line). We have found that the dependence of the energy gain on the phase difference between the constants determining the initial conditions is weaker if \( \omega \) decreases below unity. In this case, all the different curves converge to the thick line corresponding to \( \arg(S_- / C_-) = \pi / 2 \). At the same time, as \( \omega \) decreases below unity, this line gets steeper, providing thus a larger energy gain. This justifies referring to the \( y_3 \) and \( j_3 \) as the “growing” and “decaying” modes, respectively.

We illustrate the dependence of the energy gain on the shear parameter in Figure 4 (because the results depend only on \( \omega^2 \), we only show the positive domain in the horizontal axis). In the limit of weak shear, there are only pure Alfvén waves and there is no net energy gain. The dashed line shows that the energy gain tends to the value \( E_+ / E_- = 10 / \omega^2 \) as \( 1 / \omega \gg 1 \), which provides a good description of the numerical results for strong shear. The asymptotic behavior is insensitive to the initial conditions as long as the growing mode is excited, i.e., \( C_+ \neq 0 \).

5. DISCUSSION

5.1. Summary and Connection to Previous Work

We have employed the shearing-sheet framework to study the dynamical evolution of MHD waves in weakly magnetized differentially rotating backgrounds which are stable to the MRI. While the fact that these waves can be transiently amplified is widely appreciated, our motivation to study them, as well as the results that we obtained, concern dynamical aspects that have
not received as much attention. This is whether these shearing
MHD waves can play a significant role in the transport of angular
momentum in regions of the disk where the MRI is inefficient,
such as the accretion disk boundary layer.

The equations that we have solved are similar to those
presented in Balbus & Hawley (1992a), who provided numerical
solutions showing that transient amplification of MHD waves is
a general outcome for the modes with wavevectors that are not
solutions showing that transient amplification of MHD waves is
Fourier mode is considered (Goodman & Xu 1994).

Figure 4. Filled circles represent the value of the energy gain $E_+ / E_-$
for different values of the shear, parameterized via $q\Omega_0 / \omega_\lambda = 1 / \omega_\lambda$, obtained via
numerical integration using the initial conditions $C_v = 1$ and $S_v = 0$, i.e.,
only the growing mode is excited. In the limit of weak shear, there are only
pure Alfvén waves and thus there is no net energy gain. The dashed line shows
the function $10q^2 \Omega_0 / \omega_\lambda^2$, which is in good agreement with the
numerical results for strong shear. This asymptotic behavior is independent of
the initial conditions as long as $C_v \neq 0$.

5.2. Implications

The importance of understanding the relationship between
the stress and the radial gradient in angular frequency resides in
that this dependence plays a key role when modeling the inner
structure of an accretion disk surrounding a weekly magnetized star (see, e.g., Popham & Narayan 1995; Popham et al. 1996).

In the steady state, the constant inward flux of angular
momentum at any given radius $r$ is given by $J = Ml − 2\pi r^2 H T_{\phi\phi}$ where $M$ stands for the accretion rate, $l$ is the specific
angular momentum, $H$ is the disk height, and $T_{\phi\phi}$ (denoted by $T_{\phi\phi}$ in our analysis) is the component of the stress responsible for the
flux of azimuthal momentum across the radial direction. Thus, the angular momentum flux has two contributions: $J_{\text{matter}} = Ml$, which accounts for the flux of angular momentum due to mass accretion, and $J_{\text{stress}} = −2\pi r^2 H T_{\phi\phi}$, which accounts for the
flux of angular momentum due to the stress acting on the fluid elements constituting the disk. Under the reasonable assumption that the disk must be Keplerian well beyond the boundary layer, i.e., for $r \gg r_h$, and that the stress should vanish in the absence
of shear, we must have $J = Ml(r_h) > 0$, i.e., a slowly rotating star accretes mass and angular momentum.

In order for this picture to be self-consistent, the stress must
satisfy $T_{\phi\phi}(r_h < r < r_b) \leq 0$. In the standard accretion disk model this requirement is satisfied by assuming that the stress is linearly proportional to the local shear $T_{\phi\phi} \sim −d\Omega / dr$. With this model for the stress, and some supplementary assumptions, it is possible to solve for the radial dependence of $\Omega(r)$ and
determine the structure of the disk (see, e.g., Popham & Narayan 1991). However, this assumption, broadly adopted in the framework of enhanced turbulent disk viscosity, does not seem to be supported by the modern paradigm, in which angular momentum transport is due to MHD turbulence driven by the
MRI. Indeed, both local numerical simulations of shearing-boxes with non-Keplerian shear profiles (Pessah et al. 2008; Snellman et al. 2009) and global disk simulations with a rigid inner boundary (Armitage 2002; Steinacker & Papaloizou 2002) suggest that angular momentum transport is inefficient if $d\Omega / dr > 0$. This suggests that the detailed structure of accretion disk boundary layers resulting from the interaction of an MHD disk with a weakly magnetized star could differ appreciably from those derived within the standard turbulent shear viscosity, where the direction of angular momentum transport is always opposite to the angular frequency gradient.

5.3. Final Remarks

It is worth mentioning explicitly that the shearing-sheet
framework that we have employed is inherently limited to
address the conditions expected in the accretion disk boundary
layer. For example, the absence of a hard-inner boundary could prevent Kelvin–Helmholtz instabilities from operating (see, e.g.,

\begin{equation}
E_+ / E_- \approx 10 \left( \frac{q\Omega_0}{\omega_\lambda} \right)^2 \quad \text{for} \quad q\Omega_0 \gg \omega_\lambda, \quad (28)
\end{equation}

(see Equation (27) and Figure 4) and it is thus insensitive to the
sign of the shear parameter $q$.

An important result of this study is that while the energy of
these MHD waves can be significantly amplified, their net
associated stresses oscillate around zero (see Equation (25) and

\footnote{For an analysis of non-axisymmetric spiral waves when only a strong,
vertical background magnetic field is considered, see Tagger et al. (1992).}
Balsara et al. 2009). However, these instabilities do not seem to play a predominant role in the global MHD simulations of Armitage (2002) and Steinacker & Papaloizou (2002). Moreover, because we have assumed a constant background density, our analysis precludes the possibility of buoyant modes or convective instabilities. Whether these instabilities, and the turbulence they could drive, transport angular momentum inward or outward in Keplerian disks has been long debated (Ryu & Goodman 1992; Cabot 1996; Stone & Balbus 1996; Lesur & Ogilvie 2010). To our knowledge, these convective instabilities have not been studied in differentially rotating backgrounds with angular frequencies increasing outward; and speculating about their role goes beyond the scope of the present work.

In spite of the simplifications of our analytical study, the explicit solutions that we have found can provide physical insight and help elucidate transport processes in the inner disk regions close to a weakly magnetized accreting star. The current availability of powerful parallel codes (e.g., Stone et al. 2008) which are already being used to study the hydrodynamics of accretion disk boundary layers (Belyaev & Rafikov 2012) holds the promise that a more detailed understanding of MHD boundary layers will soon be possible.

M.E.P. is grateful to the Knud Højgaard Foundation for its generous support. C.K.C. is supported by a NORDITA fellowship. We thank Tobias Heinemann, John Wettlaufer, and Jim Stone for useful discussions.

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