Research Article

A Flexible Global GCRO-DR Method for Shifted Linear Systems and General Coupled Matrix Equations

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In this paper, we mainly focus on the development and study of a new global GCRO-DR method that allows both the flexible preconditioning and the subspace recycling for sequences of shifted linear systems. The novel method presented here has two main advantages: firstly, it does not require the right-hand sides to be related, and, secondly, it can also be compatible with the general preconditioning. Meanwhile, we apply the new algorithm to solve the general coupled matrix equations. Moreover, by performing an error analysis, we deduce that a much looser tolerance can be applied to save computation by limiting the flexible preconditioned work without sacrificing the closeness of the computed and the true residuals. Finally, numerical experiments demonstrate that the proposed method illustrated can be more competitive than some other global GMRES-type methods.

1. Introduction

In the current study, we focus on the solution of sequences of shifted linear systems of the form

\[(A_i - \sigma I)x^\ell_i = b^\ell_i, \text{ for } i = 1, 2, \ldots, s; \ell = 1, 2, \ldots, p_i,\]

(1)

where \(\{A_i\}_{i=1}^s \subset \mathbb{C}^{n \times n}\) are large nonsingular matrices of order \(n\), the right-hand sides \(\{b^\ell_i\}_{i=1}^s \subset \mathbb{C}^n\) are vectors of the length \(n\) and \(\{\sigma^\ell_i\}_{i=1}^s \subset \mathbb{C}\) are called shifts. Such shifted linear systems (1) arise in many realistic applications, for example, electromagnetic scattering [1], QCD [2–5], dynamics of structures [6], complex network [7–9], and model reduction [10].

Note that problem (1) is reduced to the solution of a sequence of linear systems with the form

\[\tilde{A}_i x_i = b_i, \text{ for } i = 1, 2, \ldots, s,\]

(2)

when \(\ell = 1\), where the coefficient matrix \(\tilde{A}_i = A_i - \sigma^\ell I\) and the right-hand sides \(b_i\) change from one system to the next. One approach to more efficient solution of such systems is to exploit Krylov subspace recycling; that is, recycling a judiciously selected subspace of the search space typically reduces the number of iterations substantially. There are several algorithms that take advantage of recycling for subsequent systems as well as for restarts of generalized minimal residual (GMRES) type methods, such as the generalized conjugate residual method with inner orthogonalization (GCRO) [11], GCRO with deflated restarting (GCRO-DR) [12], GMRES with deflated restarting (GMRES-DR) [13] (only for a single system), and GCRO with optimal truncation (GCROT) [14].

It is also noted that, for each \(i\), we substantially solve a family of shifted linear systems with the form

\[(A - \sigma^\ell I)x^\ell = b^\ell, \text{ for } \ell = 1, 2, \ldots, p_i.\]

(3)
Krylov subspace solvers [15] are particularly attractive for the solutions of (3) because of its shift-invariance property [16]. That is, for any shift \( \sigma \), the \( m \)-th Krylov subspace satisfies

\[
K_m(A, b) = K_m(A + \sigma I, \tilde{b}),
\]

as long as \( \tilde{b} = \tilde{b} \). This property has led to many approaches for solving such linear systems (3) simultaneously by generating only one subspace. Some solvers are built upon the (Bi-)Lanczos recurrences [17–21] and some others use the Arnoldi-like iteration [16, 22–28]. However, the shift-invariance property will be lost when using the general preconditioning, such as Jacobi preconditioning, SSOR preconditioning, ILU preconditioning, AMG preconditioning, and AINV preconditioning. Furthermore, when \( [b^j]_{j=1}^p \) are unrelated, we cannot employ the shift-invariance property. It means that all residuals are not simultaneously minimized while maintaining collinearity, for example, the GMRES-type methods [16, 23, 29].

Therefore, we propose an alternative without exploiting the shift-invariance. In fact, the solution of (3) is equivalent to solve a Sylvester equation [30],

\[
AX + XT = B,
\]

where \( T = \text{diag}(\sigma^1, \sigma^2, \ldots , \sigma^p) \in \mathbb{C}^{p\times p} \) and \( B = [b^1, b^2, \ldots , b^p] \in \mathbb{C}^{mp} \). Therefore, (1) can also be reformulated as a family of Sylvester equations with the slowly changing coefficient matrices

\[
A_iX_i + X_iT_i = B_i, \quad i = 1, 2, \ldots , s.
\]

For simplicity of the presentation, a linear operator \( \mathcal{A} \) is defined as follows:

\[
\mathcal{A}: X \mapsto AX + XT.
\]

Applying the linear operator \( \mathcal{A} \), we can rewrite (1) as

\[
\mathcal{A}_i(X_i) = B_i.
\]

Moreover, we also assume that the linear operator \( \mathcal{A} \) is invertible; that is,

\[
\mathcal{A}_i^{-1}(B_i) = X_i.
\]

2. Flexible Global GCRO-DR Method
GCRO-DR is a well-established Krylov subspace solver that paired with a harmonic Ritz vector recycling strategy [12]. GCRO [11] is a nested Krylov method, which uses GCR as the outer solver and the restarted GMRES as an inner method. In [12], Parks et al. propose GCRO-DR that exploits deflated restarting within the framework of GCRO for a family of slowly changing linear equations. Because of its superior numerical behavior, we introduce a global version of GCRO-DR to solve (1). Here, we not only employ the subspace recycling at each cycle but also apply the flexible preconditioning operator at each iteration. The proposed method is referred to as flexible global GCRO-DR (FGL-GCRO-DR).

2.1. Flexible Global Arnoldi Process. We first present a global Arnoldi process used in the flexible setting to create an \( F \)-orthonormal basis for the global Krylov subspace. We define \( \mathcal{M}_j \) as the preconditioning operator used at the \( j \)-th iteration, and the action \( \mathcal{M}_j \) on rectangular matrix \( V \) is denoted by \( \mathcal{M}_j(V) \).

The flexible global Arnoldi process, recursively, generates an \( F \)-orthonormal basis \( \mathcal{V}_{m+1} = [V_1, V_2, \ldots , V_{m+1}] \) of the global Krylov subspace spanned by the rectangular matrices \( [V_1, \mathcal{A}V_1, \ldots , \mathcal{A}^mV_1] \). At the end of the \( m \)-th iteration, we obtain a matrix relation (flexible global Arnoldi relation); that is,

\[
A \mathcal{X}_m = \mathcal{Y}_{m+1}(H_m \otimes I_p),
\]

where \( \mathcal{X}_m = [\mathcal{M}_1(V_1), \ldots , \mathcal{M}_m(V_m)] \in \mathbb{C}^{mp} \), \( \mathcal{Y}_{m+1} \in \mathbb{C}^{mn \times (m+1)p} \), and\( \mathcal{X}_m \in \mathbb{C}^{mp} \), and\( \mathcal{Y}_{m+1} \in \mathbb{C}^{mn \times (m+1)p} \).
is an upper Hessenberg matrix.

2.2. The FGL-GCRO-DR Algorithm. Suppose that \( \mathcal{Z}_k = [Z_1, \ldots, Z_k] \) and \( \mathcal{V}_k = [V_1, \ldots, V_k] \in \mathbb{C}^{n \times p} \) are given matrices satisfying

\[
\mathcal{A}[\mathcal{Z}_k, \mathcal{Z}_{k+1}, \ldots, \mathcal{Z}_m] = [V_1, V_2, \ldots, V_{m+1}](H_m \otimes I_p),
\]

where \( Z_i, i = 1, 2, \ldots, k \) is a search vector, recycled from either the previous linear system or the previous cycle, and \( 0 \leq k \leq m \). By minimizing the residual F-norm,

\[
\begin{bmatrix}
    h_{1,1} & h_{1,2} & \cdots & h_{1,m} \\
    h_{2,1} & h_{2,2} & \cdots & \vdots \\
    \vdots & \ddots & \ddots & \vdots \\
    h_{m-1,1} & \cdots & h_{m-1,m} & h_{m,m}
\end{bmatrix} = \begin{bmatrix} H_m \\ H_{m+1,m} \end{bmatrix} \in \mathbb{C}^{(m+1) \times m},
\]

over the search space \( X_0 + \mathcal{R}(\mathcal{V}_k), X_k \) and \( R_k \in \mathbb{C}^{n \times p} \) satisfy

\[
X_k = X_0 + \mathcal{Z}_k\mathcal{V}_k, \quad \mathcal{V}_k = \mathcal{V}_k - \mathcal{A}(\mathcal{V}_k, R_k)\|I_p\|,
\]

where \( X_0, R_k \) are given, \( X_k \) is the approximate solution, \( R_k \) is the residual at the end, and \( R_k \) is the harmonic Ritz vectors.

Since \( R_k \perp \mathcal{R}(\mathcal{V}_k) \), the following flexible global Arnoldi-like relation holds:

\[
\mathcal{A}[\mathcal{Z}_k, \mathcal{Z}_{k+1}, \ldots, \mathcal{Z}_m] = [V_1, V_2, \ldots, V_{m+1}](H_m \otimes I_p).
\]

At the end, we find the approximate solution \( X_m = X_0 + \mathcal{Z}_m(\gamma \otimes I_p) \in \mathbb{C}^{n \times p} \) by minimizing the norm \( \|R_0 - A\mathcal{Z}_m(\gamma \otimes I_p)\|_F \), where \( \gamma \) satisfies

\[
y^* = \arg \min_{y \in \mathbb{C}^n} \|R_0 - \mathcal{A}\mathcal{Z}_m(y \otimes I_p)\|_F
\]

with \( \beta_0 = \langle \mathcal{V}_{m+1}, R_0 \rangle_F \).

To improve the convergence of iterative solvers, the smallest magnitude eigenvalues will be deflated. Following the idea of [12, 32], we use harmonic Ritz vectors with those eigenvalues to construct approximate invariant subspace and then the subspace for next cycles is retained. Firstly, we compute \( k \) harmonic Ritz vectors \( \tilde{v}_k = \mathcal{H}_m(g_k \otimes I_p) \), such that

\[
\mathcal{A}\mathcal{X}_m\mathcal{X}_m^H(\tilde{v}_k \otimes I_p) - \mathcal{X}_m^H\mathcal{X}_m\tilde{v}_k \perp \tilde{v}_k
\]

and each harmonic Ritz pair \( (\tilde{\theta}_i, g_i) \) satisfies

\[
\mathcal{H}_m^H\mathcal{H}_m g_i \perp \tilde{v}_k \}
\]

It is noted that the harmonic residual vectors satisfy

\[
\mathcal{A}\mathcal{X}_m\mathcal{X}_m^H(g_i \otimes I_p) - \mathcal{X}_m^H\mathcal{X}_m g_i \in \mathcal{R}(\mathcal{V}_{m+1}).
\]
Observe that the residual $R_m = R_0 - \mathcal{A}L_m(y^* \otimes I_p) = \mathcal{Z} m (y^* \otimes I_p) \in R(\mathcal{Z} m) \setminus R(\mathcal{A}L_m)$. So, the residual and the harmonic residuals are in the same $p$-dimensional

$$
\mathcal{A}L_m(G_k \otimes I_p) = [\mathcal{Y} m(G_k \otimes I_p) R_m] \cdot [\text{diag}(\theta_1, \ldots, \theta_k)] \otimes I_p,
$$

(21)

where $a(\cdot; i)$ satisfies $\mathcal{A}L_m(g_i \otimes I_p) = \mathcal{A}L_{m+1}(g_i \otimes I_p) = a(1, i) R_m$, $1 \leq i \leq k$, and $G_k = [g_1, \ldots, g_k] \in C_{m \times k}$.

Due to the relations (10) and (21) can be also expressed as

$$
\mathcal{Y} m(G_k \otimes I_p) = [\mathcal{Y} m(G_k \otimes I_p) R_m] \cdot [\text{diag}(\theta_1, \ldots, \theta_k)] \otimes I_p,
$$

(22)

Next, we compute the reduced QR-factorization of $H_m G_k = QR$ with $Q \in C_{m \times (m-1) \times k}$ and $R \in C_{k \times k}$ and then define $\mathcal{Z}_k$, $\mathcal{Y}_k$ in $C_{m \times k}$ as

$$
\mathcal{Z}_k = \mathcal{Z} m(G_k R^{-1} \otimes I_p),
$$

$$
\mathcal{Y}_k = \mathcal{Y} m(Q \otimes I_p),
$$

(23)

such that $\mathcal{A} \mathcal{Z}_k = \mathcal{Y}_k$. Since $R_m$ is $F$-orthogonal to the subspace $\mathcal{A} \mathcal{Z}_m$ and $\mathcal{Y}_{m+1}$ is $F$-orthonormal, we have

$$
\langle \mathcal{Y}_k, R_m \rangle_F = \langle \mathcal{Y} m(Q \otimes I_p)(R \otimes I_p), R_m \rangle_F
$$

$$
= \langle \mathcal{Y} m(H_m G_k \otimes I_p), R_m \rangle_F
$$

$$
= \langle \mathcal{A} \mathcal{Z} m(G_k \otimes I_p), R_m \rangle_F
$$

$$
= 0,
$$

$$
\langle \mathcal{Y}_k, \mathcal{Y}_k \rangle_F = \langle \mathcal{Y} m(Q \otimes I_p), \mathcal{Y} m(Q \otimes I_p) \rangle_F
$$

$$
= Q F Q = I_k.
$$

(24)

Therefore, at the end of the loop, an update $\mathcal{Z}_k$ is generated, scaled such that $\mathcal{Y}_k = \mathcal{A} \mathcal{Z}_k$ has $F$-orthonormal columns; the previous global Krylov subspace basis is dropped, and then we restart. Details of the FGL-GCRO-DR method are summarized in Algorithm 2.

In addition, the major advantage of FGL-GCRO-DR is that it can retain important spectral information generated during the solution of the $i$-th linear system and then exploit such information to accelerate convergence of the iterations when solving the subsequent $i+1$-th system. Then we detail how to recycle subspace when solving sequences of slowly changing linear systems (8) as follows (Algorithm 3).

2.3. Comments on the FGL-GCRO-DR Method. To achieve better efficiency of the linear system solvers, we not only employ the subspace recycling at each cycle but also apply the flexible preconditioning operator at each iteration. As we know, the variable preconditioning can be constructed as an inexact subspace method. If the matrix $\mathcal{A}$ is difficult to invert, we will resort to several steps of iterative method as flexible preconditioning, so we obtain in some way an approximate result,

$$
\tilde{Z}_i = \mathcal{A}^{-1} V_j,
$$

(25)

at $j$-th (inner) iteration for the equation $\mathcal{A} Z_j = V_j$, $j = 1, \ldots, m$. In such a situation, what are the properties of the true residual obtained by using the flexible preconditioning? We will address this question in this section. Following [33, 34], we set

$$
P_j = V_j - \mathcal{A} \tilde{Z}_j, \quad j = 1, \ldots, m,
$$

(26)

to be the residual that results after the iterative solver has been terminated; then

$$
\tilde{Z}_j = \mathcal{A}^{-1} V_j - \mathcal{A}^{-1} P_j, \quad j = 1, \ldots, m.
$$

(27)

Therefore, the perturbation in linear operation with $\mathcal{A}$ can be written explicitly as

$$
\mathcal{A} \tilde{Z}_j = \mathcal{A} \mathcal{A}^{-1} V_j - \mathcal{A} \mathcal{A}^{-1} P_j
$$

$$
= \left( I + P_j \frac{V_j^H}{\|V_j\|_F^2} \right) V_j
$$

$$
= (I - \mathcal{E}_j) V_j.
$$

(28)

The perturbation implies that the error computed has the bound

$$
\|\mathcal{E}_j\| \leq \frac{P_j}{\|V_j\|_F}
$$

(29)

In fact, the solution is obtained directly from the approximation subspace $R(\tilde{Z}_m)$; that is,

$$
X_m = \tilde{Z} m(\tilde{y}^* \otimes I_p),
$$

(30)

indicating that the perturbation $\mathcal{E}_j$ does not affect the computation of the true residual. In other words, in exact arithmetic

$$
R_m = R_0 - \mathcal{A} \tilde{Z}_m(\tilde{y}^* \otimes I_p)
$$

$$
= \mathcal{Y} m+1(\beta e_1 \otimes I_p) - \mathcal{Y} m+1(H_m \tilde{y}^* \otimes I_p)
$$

$$
= \mathcal{Y} m+1(\beta e_1 - H_m \tilde{y}^* \otimes I_p),
$$

(31)
\textbf{Algorithm 1:} The FGL-GCRO-DR method.

\begin{verbatim}
\begin{algorithm}
\caption{FGL-GCRO-DR for solving sequences of slowly changing linear systems.}
\end{algorithm}
\end{verbatim}
and is orthogonal to $\mathcal{R}(\mathcal{V}_{m+1})$. As a consequence, the perturbation of the linear operations only influences the goodness of $\mathcal{R}(\mathcal{V}_m)$ and the accuracy of the approximate solution. Therefore, we deduce that, in the flexible setting, a dynamic stopping criterion or a much looser tolerance can be employed to save computation by limiting the flexible preconditioning work without sacrificing the closeness of the computed and the true residuals.

2.4. The Relationship Between Two Flexible Global Methods with Subspace Recycling. As we know, FGCRO-DR [32] and FGMRES-DR [35] are two important methods in the general category of augmented (deflated) GMRES-type methods, which choose harmonic Ritz vectors as augmented (deflated) subspace. They emphasize on the solution of linear system $Ax = b$ which has only one equation. The relationship between FGCRO-DR and FGMRES-DR has been well described by [32]. It is shown that they are algebraically equivalent as long as a collinearity condition is satisfied. In fact, FGL-GCRO-DR is also closely related to the FGL-GMRES-DR, in their global versions, the FGL-GCRO-DR method is also closely related to the FGL-GMRES-DR approach in algebraical analysis, but their numerical behavior will be slightly different. Details will be shown in Section 3.

Furthermore, we compare the main calculation cost and storage capacity for each cycle of FGL-GCRO-DR and FGL-GMRES-DR in Table 1. In practical applications, $n \gg m, k$. Here, $opM$ and $opA$ denote floating-point calculation counts for the preconditioning application and the linear operations, respectively.

The computational cost of harmonic Ritz vectors and "min$[y - H_n y]_2$" have not been induced in Table 1 since the cost is approximately $O(p^3)$, much more less than $n. c = (opM + opA)(m - k) + 2(m + 1)kp + p + 2mkp + (m - k)(2m + 2k + 6) np$ is referred to as the total cost of FGL-GCRO-DR. One sees that the time complexities of two global algorithms are the same, that is, $O((m - k)n^2p)$, but FGL-GMRES-DR requires slightly more computation per cycle than FGL-GCRO-DR in terms of computational cost.

2.5. Application to the General Coupled Matrix Equations. FGL-GCRO-DR substantially belongs to the family of the global Krylov subspace method, which can be also applied to solve the general coupled matrix equations of the form

$$\sum_{j=1}^{l} \left(A_{i,j}X_{i,j}B_{i,j}k_{i,j} + A_{i,j}X_{i,j}B_{i,j}k_{i,j} + \cdots + A_{i,j}X_{i,j}B_{i,j}k_{i,j}\right) = D_i, \quad i = 1,2,\ldots,l,$$  \hspace{1cm} (32)

including the (coupled) Sylvester and the (coupled) Lyapunov matrix equations. The literature on solving (32) is particularly rich and still growing [36–53], due to the prominent role of such matrix equations in stability theory [54], perturbation analysis [55], control theory [56, 57], and so forth. In particular, the global subspace solvers perform better than other approaches for solving (32), especially for the coefficient matrices which are large and sparse; for more details, see [44–53, 58]. Therefore, we are interested in exploiting the FGL-GCRO-DR method for solving (32). To do so, the linear operator can be defined as follows:

$$\mathcal{F}: \times^{m+1} \rightarrow \times^{n \times p \times m},$$

$$X = (X_1, X_2, \ldots, X_l) \rightarrow \mathcal{F}(X) = (a_i(X), a_i(X), \ldots, a_i(X)),$$  \hspace{1cm} (33)

where $a_i = \sum_{k=1}^{l} \sum_{j=1}^{l} A_{i,j}X_{i,j}B_{i,j}k_{i,j}, \quad i = 1,2,\ldots,l$. Exploiting the operator $\mathcal{F}$, we can rewrite (32) as

$$\mathcal{F}(X) = D,$$  \hspace{1cm} (34)

where $D = (D_1, D_2, \ldots, D_l)$. The effectiveness of FGL-GCRO-DR for the solution of (32) is shown in Section 3.

3. Numerical Experiments

This section presents some numerical examples to illustrate the effectiveness of our algorithm for solving shifted linear system (1) and matrix equations (8) and compare the performance with the global GMRES $(m)$ approach (referred to as GL-GMRES $(m)$) [31, 50, 59] and GL-GMRES-DR [60], with or without preconditioning. In each example, the flexible preconditioning is dominated by five steps of the global full GMRES [50].

We consider three numerical illustrations here. The first two experiments are sequences of linear systems arising from lattice QCD [2–4, 61] and the finite difference scheme of the convection-diffusion-reaction equation [62], respectively. The last one is the general coupled Sylvester matrix equations. The comparisons are mainly made in two aspects: the execution time in seconds ($T$ (s), for short) and number of matrix-vector products (mvps, for short).

All the numerical experiments are carried out on MATLAB 2017b. We choose $X_0 = 0_{n \times p}$, $m = 20$, and $k = 10$. The stopping criteria are $\|B_i - a_i X_i p\|/\|B_i p\| \leq 10^{-8}$, $i = 1,2,\ldots,s$ for all the approaches, or when mvps exceed the set maximum of matrix-vector multiplications (MAXIT, for short). Moreover, the symbol $\dagger$ represents that the solver does not achieve the stopping criteria within MAXIT and RHSs $= \text{rand}(n, p)$.

**Example 1.** We first consider two groups of complex Lattice QCD matrices from Matrix Market (please refer to http://math.nist.gov/Matrix-Market for details). One group has seven $3172 \times 3172$ matrices and the other has seven $49152 \times 49152$ matrices. For each matrix $D_i$, a critical parameter $k_i$ is provided such that $I - k_iD_i$ with $0 \leq k_i \leq k_i$ is positive-real. For
Input: $A \in \mathbb{C}^{m \times n}$, $B \in \mathbb{C}^{n \times p}$. Choose $m$, $k$, $\text{tol}$ and initial guesses $X_0$.
Output: $X_m$ such that $(\|R_m\|/\|R_0\|) \leq \text{tol}$

1. $R_0 = B - AX_0$, $\gamma = \|R_0\|$, $V_1 = R_0/V_0$
2. Apply Algorithm 2 to generate $\widetilde{F}_{m}, \widetilde{F}_{m+1}, H_m$ such that $\widetilde{F}_m = \widetilde{F}_{m+1}(H_m \otimes I_p)$
3. Solve $\min_{\gamma} \|H_m\|_2$ for $\gamma$. Set $X_0 = X_0 + \widetilde{F}_m(\gamma \otimes I_m)$, $R_0 = B - AX_0$
4. while $\|R_m\| > \|R_0\| \times \text{tol}$ do
5. Compute the $k$ smallest eigenpairs $(\theta_i, g_i)$ of $(H_m + h_{m+1,m}h_{m+1,m}H_m, H_m, c_{m}e_{m}^H)$.
6. $Q_R = \left[ \begin{array}{cc} P_k & \gamma - H_m\gamma \end{array} \right]_{1:k}$
7. $V_k = \widetilde{F}_{m+1}Q$
8. $Z_k = \widetilde{F}_mQ$
9. $H_k = Q_k^H H_k Q (1: m, 1: k)$
10. Apply $m-k$ steps of Algorithm 2 with the starting rectangular matrix $V_{k+1} = R_k/\|R_k\|$, such that $\widetilde{F}_{m+k} = \widetilde{F}_{m+k+1}(H_{m+k} \otimes I_p)$ with $\widetilde{F}_j = \mathcal{F}(\widetilde{V}_j)$
11. $\gamma^* = \arg\min_{\gamma} \|H_m\|_2$ with $\gamma = \langle \gamma \rangle$
12. $X_0 = X_{m+1} + Z_m(\gamma \otimes I_m)$
13. $R_m = B - AX_m$
14. endwhile

**Algorithm 4: FGL-GMRES-DR.**

| Computation of | FGL-GMRES-DR $(m, k)$ | FGL-GMRES-DR $(m, k)$ |
|---------------|------------------------|------------------------|
| $V_{k+1}$ $(\cdot, 1: (k+1)p)$ | $2(m+1)(k+1)p$ | $2n(m+1)kp + np$ |
| $Z_k$(\cdot, 1: $kp$) | $2nk^2p$ | $2nk^2p$ |
| $Z_m$(\cdot, $(k+2)p + 1: (m+1)p)$ | $opA(m-k) + (m-k)(2n+2k+5)p$ | $opA(m-k) + (m-k)(2n+2k+6)p$ |
| $Z_m$(\cdot, $(k+1)p + 1: mp)$ | $opA(m-k)$ | $opA(m-k)$ |
| Total cost | $C + (m + k - 1)p$ | $C$ |
| Memory requirement | $(2m + 3)p$ | $(3m + 3)p$ |

Each set, we take $A_i = (k_i + 10^{-3})I - D_i$ as the sequence matrix, and $T_j = [0.01, 0.02, 0.03, 1, 2, 3, 4]$. The numerical computation is carried out with MAXIT $= 1000$. The corresponding numerical results are reported in Tables 2 and 3. In addition, we also depict the residual histories for linear systems solved using iterative algorithms with or without flexible preconditioning.

As seen from Tables 2 and 3 and Figures 1 and 2, it is found that using deflation and flexible preconditioning is conducive to improving the convergence rate. FGL-GCRO-DR has a significant decrease in terms of mvps and T(s), which are similar to FGL-GMRES-DR, but performs slightly better. The GL-GMRES method is less effective and does not converge in some test problems. The above results show that FGL-GCRO-DR has the potential to improve the convergence, compared with other global Krylov subspace solvers.

**Example 2.** For the second case, we assess the performance of global solvers on three-dimensional separable model convection-diffusion problem in the unit cube with the homogeneous Dirichlet boundary conditions [62]:

\[
-\varepsilon \Delta u + v^T \nabla u - ru = f, \quad \text{in} \; \Omega, \tag{35}
\]

where $\varepsilon = 1$ and $r = [0, 100, 400, 600, 800, 1000]$. For obtaining the sequence matrices, we set $v = [0.1 \cdot k_i, 0.5 \cdot i \cdot k_i, k_i]$ with $k_i = 1, 2, \ldots, 5$ and $i = \sqrt{\varepsilon}$ and then use the central finite difference scheme to discretize the convection-diffusion equation with the mesh size $h = 0.025$ or $h = 0.02$. Thus, two groups of complex matrices are obtained. One group has five $59319 \times 59319$ matrices and the other has five $117649 \times 117649$ matrices. As the definition of MAXIT, we choose MAXIT $= 500$.

The numerical results reported in Tables 4 and 5 and Figures 3 and 4 show that FGL-GMRES is slightly superior to FGL-GCRO-DR when the mesh size $h = 0.025$. However, as the number of grid points increases, FGL-GCRO-DR enjoys the significant decrease in the number of mvps and T(s). Meanwhile, according to Tables 4 and 5, we can see that the convergence is speeded up remarkably by considering the flexible preconditioning. Moreover, even the size of grid points increases, and the average number of mvps for FGL-GCRO-DR does not increase.

**Example 3.** For the last test case, we are interested in solving the general coupled Sylvester matrix equations [45, 47]

\[
\begin{align*}
AX - YB &= E, \\
CX - YD &= F,
\end{align*}
\tag{36}
\]

where
\[
A = \text{diag} \left( \text{diag}(\text{rand}(n)) + 2 \right) + \text{triu}(\text{rand}(n, n), 1),
\]
\[
B = \text{diag} \left( \text{diag}(\text{rand}(n)) + 1.75 \right) + \text{triu}(\text{rand}(n, n), 1),
\]
\[
C = -\text{diag} \left( \text{diag}(\text{rand}(n)) + 2 \right) + \text{triu}(\text{rand}(n, n), 1),
\]
\[
D = \text{diag} \left( \text{diag}(\text{rand}(n)) + 1.5 \right) + \text{triu}(\text{rand}(n, n), n),
\]
\[
E = \text{rand}(n, n),
\]
\[
F = \text{rand}(n, n).
\]

Here, we consider \( n = 50, 60, 70, 80, 90, 100 \) different orders. As the definition of \( \text{MAXIT} \), we choose \( \text{MAXIT} = 500 \). The corresponding numerical results of \( \text{mvps} \) and \( T(s) \) are reported in Table 6 and the residual histories are depicted in Figures 5 and 6.

As seen from Table 5 and Figures 5 and 6, it is noted that the flexible deflated GMRES-type methods (FGL-GCRO-DR and FGL-GMRES-DR) are found to be efficient in terms of performance test problems of different orders, while GL-GMRES \((m)\) and FGL-GMRES \((m)\) fail to converge in the given steps with or without variable preconditioning, while the parameter \( n \) is greater than 80. The convergence is improved remarkably by considering the eigenvalue deflation technique and the flexible preconditioning. One should note that FGL-GCRO-DR enjoys the significant decrease in the number of \( \text{mvps} \) and \( T(s) \) and performs slightly better than FGL-GMRES-DR method. In addition, when \( n \) increases, the average number of \( \text{mvps} \) and \( T(s) \) for FGL-GCRO-DR do not increase.
Figure 1: The “−” line and the “+−” line denote the convergence curves for the global method and the flexible global method for solving the $s$ linear equations with $n = 3072$ in sequence, respectively.
Figure 2: The “—” line and the “+—” line denote the convergence curves for the global method and the flexible global method for solving the linear equations with $n = 49152$ in sequence, respectively.

Table 4: Performance comparisons for 59319 convection-diffusion problem.

| Problems | Method | GL-GCRO-DR | FGL-GCRO-DR | GL-GMRES ($m$) | FGL-GMRES ($m$) | GL-GMRES-DR | FGL-GMRES-DR |
|----------|--------|-------------|-------------|----------------|----------------|-------------|--------------|
| Matrix 1 | T (s)  | 1.735e+01  | 29          | 340            | 40             | 170         | 40           |
|          | mvps   | 2.781e+01  | 1.735e+01   | 3.693e+01      | 1.693e+01      | 2.811e+01   | 1.826e+01    |
| Matrix 2 | T (s)  | 1.766e+01  | 32          | 340            | 40             | 190         | 40           |
|          | mvps   | 2.655e+01  | 1.766e+01   | 3.664e+01      | 1.686e+01      | 3.115e+01   | 1.824e+01    |
| Matrix 3 | T (s)  | 1.713e+01  | 31          | 340            | 40             | 180         | 40           |
|          | mvps   | 2.907e+01  | 1.713e+01   | 3.718e+01      | 1.688e+01      | 2.946e+01   | 1.828e+01    |
| Matrix 4 | T (s)  | 1.818e+01  | 34          | 340            | 40             | 190         | 40           |
|          | mvps   | 3.095e+01  | 1.818e+01   | 3.646e+01      | 1.680e+01      | 3.120e+01   | 1.826e+01    |
| Matrix 5 | T (s)  | 1.747e+01  | 32          | 340            | 40             | 200         | 40           |
|          | mvps   | 2.747e+01  | 1.747e+01   | 3.658e+01      | 1.675e+01      | 3.271e+01   | 1.822e+01    |
Table 5: Performance comparisons for 117649 convection-diffusion problem.

| Problems | GL-GCRO-DR | FGL-GCRO-DR | GL-GMRES \( (m) \) | FGL-GMRES \( (m) \) | GL-GMRES-DR | FGL-GMRES-DR |
|----------|-------------|-------------|---------------------|---------------------|-------------|-------------|
| Matrix 1 | mvps        | 209         | 35                  | 480                 | 60          | 250         | 40          |
|          | \( T \) (s)| 7.486e + 01 | \( 3.630e + 01 \)    | 1.018e + 02         | 4.998e + 01 | 8.223e + 01 | 3.604e + 01 |
| Matrix 2 | mvps        | 218         | 36                  | \( \dagger \)       | 60          | 240         | 50          |
|          | \( T \) (s)| 7.660e + 01 | \( 4.006e + 01 \)    | \( \dagger \)       | 4.988e + 01 | 7.876e + 01 | 4.536e + 01 |
| Matrix 3 | mvps        | 219         | 42                  | \( \dagger \)       | 60          | 240         | 50          |
|          | \( T \) (s)| 7.665e + 01 | \( 4.490e + 01 \)    | \( \dagger \)       | 5.013e + 01 | 8.117e + 01 | 4.541e + 01 |
| Matrix 4 | mvps        | 242         | 41                  | 480                 | 60          | 270         | 50          |
|          | \( T \) (s)| 7.664e + 01 | \( 4.396e + 01 \)    | 1026e + 02          | 4.977e + 01 | 8.795e + 01 | 4.545e + 01 |
| Matrix 5 | mvps        | 264         | 44                  | 460                 | 60          | 230         | 50          |
|          | \( T \) (s)| 8.102e + 01 | \( 4.351e + 01 \)    | 9.859e + 01         | 4.985e + 01 | 7.514e + 01 | 4.534e + 01 |

Figure 3: The “−” line and the “+−” line denote the convergence curves for the global method and the flexible global method for solving the linear equations with \( n = 59319 \) in sequence, respectively.
Figure 4: The “−” line and the “+−” line denote the convergence curves for the global method and the flexible global method for solving the $s$ linear equations with $n = 117649$ in sequence, respectively.

Table 6: Performance comparisons for the general coupled Sylvester matrix equations.

| Problems | Method       | GL-GCRO-DR | FGL-GCRO-DR | GL-GMRES ($m$) | FGL-GMRES ($m$) | GL-GMRES-DR | FGL-GMRES-DR |
|----------|--------------|------------|-------------|----------------|----------------|--------------|--------------|
| $n = 50$ | mups         | 134        | 37          | 239            | 37             | 130          | 29           |
|          | $T$ (s)      | $4.060e-01$| $1.336e-01$ | $2.988e-01$    | $1.396e-01$    | $9.266e-01$  | $3.421e-01$  |
| $n = 60$ | mups         | 217        | 41          | $\dagger$      | 59             | 173          | 46           |
|          | $T$ (s)      | $6.551e-01$| $2.165e-01$ | $\dagger$      | 2.195e-01      | $1.668e+00$  | 7.521e-01    |
| $n = 70$ | mups         | 243        | 53          | $\dagger$      | 115            | 232          | 61           |
|          | $T$ (s)      | $1.042e+00$| $3.929e-01$ | $\dagger$      | 6.168e-01      | $3.112e+00$  | 1.354e+00    |
| $n = 80$ | mups         | 507        | 100         | $\dagger$      | 507            | 334          | 138          |
|          | $T$ (s)      | $2.661e+00$| $8.868e-01$ | $\dagger$      | 2.661e+00      | $6.143e+00$  | 3.894e+00    |
| $n = 90$ | mups         | 507        | 100         | $\dagger$      | $\dagger$      | 503          | 175          |
|          | $T$ (s)      | $3.294e+00$| $1.109e+00$ | $\dagger$      | $\dagger$      | 1.206e+01    | 6.228e+00    |
| $n = 100$| mups         | $\dagger$  | 166         | $\dagger$      | $\dagger$      | $\dagger$    | 228          |
|          | $T$ (s)      | $2.326e+00$| $\dagger$  | $\dagger$      | $\dagger$      | $\dagger$    | 9.962e+00    |
Figure 5: The convergence curves for the global methods and the flexible global methods for solving general coupled Sylvester matrix equations.
Figure 6: The convergence curves for the global methods and the flexible global methods for solving general coupled Sylvester matrix equations.
4. Conclusions
In this study, a new flexible global GCRO-DR method is proposed for addressing sequences of shifted linear systems and general coupled matrix equations. The proposed method does not require the right-hand sides to be related and can also be compatible with general preconditioning. By performing an error analysis, we deduce that a much looser tolerance can be applied to save computation by limiting the flexible preconditioned work without sacrificing the closeness of the computed and the true residuals. Meanwhile, we also propose a global version of the FGMRES-DR method and then analyze the relationship between two flexible global methods. Finally, numerical experiments show that FGL-GCRO-DR significantly reduces the mvsps, saves the runtime, and enjoys faster convergence than some other global GMRES-type approaches on tough problems.

As future work, we plan to investigate the numerical properties of FGL-GCRO-DR on large-scale realistic problems. Of interest are applications related to, for example, model adaptation in Gaussian process models [63] or stochastic finite element methods [64] in three dimensions where variable preconditioning using inexact solvers has to be usually considered. Another extension of this work is a distributed-memory parallel implementation and testing using MPI + OpenMP and MPI + CUDA. Investigation will focus on implementing, testing, and comparing the runtime of the introduced FGL-GCRO-DR on CPUs and GPUs with respect to other existent similar methods.

Data Availability
The data will be made available upon reasonable request via e-mail to the corresponding author.

Conflicts of Interest
The authors declare that they have no conflicts of interest.

Authors’ Contributions
All authors collaborated in carrying out the study and all of them read and approved the final manuscript.

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