A causal model of radiating stellar collapse

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Abstract.
We find a simple exact model of radiating stellar collapse, with a shear-free and non-accelerating interior matched to a Vaidya exterior. The heat flux is subject to causal thermodynamics, leading to self-consistent determination of the temperature $T$. We solve for $T$ exactly when the mean collision time $\tau_c$ is constant, and perturbatively in a more realistic case of variable $\tau_c$. Causal thermodynamics predicts temperature behaviour that can differ significantly from the predictions of non-causal theory. In particular, the causal theory gives a higher central temperature and greater temperature gradient.

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1. Introduction

The problem of constructing physically realistic models for radiating collapsing stars is of major importance to relativistic astrophysics. Such models will necessarily involve complicated numerical techniques for their efficient and reliable solution. It is also useful however to construct simple exact models, which are at least not physically unreasonable. This allows for a more transparent analysis of the main physical effects at play, and it can also serve as a useful check for numerical procedures. In this spirit, we construct a simple exact model which generalizes a previous model [1] by incorporating the physically desirable feature of causality into the heat transport process.

A non-rotating spherically symmetric star that is radiating energy must be matched to a Vaidya exterior spacetime. It is known [2] that the junction conditions imply non-vanishing pressure at the boundary, due to the presence of an energy flux. This is part of the reason for the difficulty in finding explicit analytic forms for reasonable interior solutions. Another difficulty arises from the physics of the energy flux, which reduces to a heat flux in the absence of particle flux. Physically consistent heat flux must
be related to the temperature gradient and four-acceleration via a thermodynamical transport equation. Most models employ a relativistic Fourier equation, but this is non-causal, since it leads to superluminal wave-front velocities, and all its equilibrium states are unstable. It should be replaced by the causal transport equation arising in the transient thermodynamics of Israel and Stewart or the essentially equivalent extended thermodynamics. The aim of this paper is to incorporate the relativistically consistent causal thermodynamics (as opposed to the non-causal theories which are essentially not consistent relativistic theories) in a simple model of a non-rotating radiating star.

The application of causal thermodynamics to radiating stellar collapse has recently been developed via physically detailed models, whose solutions require complicated numerical integrations. We aim instead for a simple exact model as a complement to numerical models with physical detail and complexity. We follow in choosing a very simple Friedmann-like interior, i.e. a fluid that collapses without shearing or accelerating. The behaviour of the energy density and pressure in these models was discussed in, and assessed to be not physically unreasonable. Our model shares this feature. The crucial difference is that employs the Fourier equation to determine the stellar temperature, introducing the undesirable feature of non-causal heat transport. We replace the non-causal transport equation used in by a causal equation, in order to produce a more satisfactory model that is constrained by causality.

As shown in, the relaxational effects introduced by the causal theory can have a significant and in principle observable impact on the temperature, rate of collapse and other properties. Our simple exact model confirms this general point. We find that although the collapse rate is unchanged, owing to the simple Friedmann-like nature of the model, the causal temperature can differ significantly from the non-causal prediction. In particular, the causal temperature has greater central value and gradient. This exact result is in agreement with the perturbative results of, which investigates the response of initially static stars to shear-free perturbations. The properties of our model are established in Section 4, where we find an exact solution for the temperature. In Section 2 we briefly review the Friedmann-like stellar model, and in Section 3 we discuss the causal heat transport equation. Finally, concluding remarks and a perturbative solution for the temperature are given in Section 5.

2. The simple stellar model

In isotropic and comoving coordinates, the non-rotating, non-accelerating and shear-free interior metric is given by

\[ ds^2 = -dt^2 + A(t, r)^2 \left[ dr^2 + r^2 (d\theta^2 + \sin^2 \theta d\phi^2) \right], \]  

(1)
\[ A = \frac{M}{2b} \left[ \frac{1 - b^2 \lambda(t)}{1 - r^2 \lambda(t)} \right] u(t)^2, \]  
(2)

where \( u = (6t/M)^{1/3} \), \( \lambda = a \exp u \), and \( a, b \) and \( M \) are constants. The fluid four-velocity is \( u^\alpha = \delta^\alpha_0 \) (so that \( t \) is comoving proper time), and the four-acceleration \( \ddot{u}_\alpha \equiv u^\beta \nabla_\beta u_\alpha \) vanishes. The fluid volume collapse rate is

\[ \Theta = 3 \frac{\dot{A}}{A}, \]  
(3)

and since the shear vanishes, the collapse rates in the radial and tangential directions both equal \( \dot{A}/A \).

The heat flux (which is the total energy flux, since there is no particle flux relative to \( u^\alpha \)) has the form

\[ q_\alpha = q(t, r) n_\alpha, \]

where \( n_\alpha \) is a unit radial vector, so that \( q \) is a covariant scalar measure of the heat flux \( (q^2 = q^\alpha q_\alpha) \). The other dynamical covariant scalars are the energy density \( \rho \) and isotropic pressure \( p \). The Einstein field equations imply (using units with \( c = 1 = 8\pi G \))

\[ \rho = \frac{12}{M^2 u^4} \left[ \left\{ \frac{2}{u} - \frac{(b^2 - r^2)\lambda}{(1 - b^2\lambda)(1 - r^2\lambda)} \right\}^2 - \frac{4b^2\lambda}{(1 - b^2\lambda)^2} \right], \]  
(4)

\[ p = \frac{4}{M^2 u^4 (1 - b^2\lambda)(1 - r^2\lambda)} \left[ \frac{8}{u} + \frac{5}{1 - r^2\lambda} - \frac{1}{1 - b^2\lambda} - 2 \right] \]
\[ + \frac{16}{M^2 u^4 (1 - b^2\lambda)^2}, \]  
(5)

\[ q = \frac{16br\lambda}{M^2 u^4 (1 - b^2\lambda)(1 - r^2\lambda)}. \]  
(6)

Equations (1)–(6) comprise an exact solution to the Einstein field equations for the interior of the radiating star. This must match smoothly to the exterior Vaidya spacetime across a comoving time-like boundary, which we denote by \( \Sigma \). The Vaidya metric is the unique isotropic null-radiation solution, and is given by

\[ ds^2 = - \left[ 1 - \frac{2m(v)}{R} \right] dv^2 - 2dv dR + R^2 \left( d\theta^2 + \sin^2 \theta d\phi^2 \right), \]  
(7)

where \( m \) represents the Newtonian mass of the gravitating body as measured by an observer at infinity.

Matching of the metrics (1) and (7) gives the junction conditions

\[ (R)_\Sigma = (rA)_\Sigma, \]  
(8)

\[ (p)_\Sigma = (q)_\Sigma, \]  
(9)
\[ [r(rA)]'_{\Sigma} = \left[ \dot{v}(R - 2m) + R\dot{R} \right]_{\Sigma}, \quad (10) \]
\[ m(v) = \left[ \frac{1}{2} r^3 A \dot{A}^2 - r^2 A' - \frac{1}{2} r^3 A^{-1} \dot{A}'^2 \right]_{\Sigma}, \quad (11) \]

where a prime denotes \( \partial / \partial r \) and the boundary is defined by \( r_\Sigma = b = \text{constant} \). It follows that the proper stellar radius (not given in \[ \]) is

\[ r_p(t) = \int_0^b A \, dr = \frac{M}{2b\sqrt{\lambda}} \text{artanh} \left( b\sqrt{\lambda} \right) \left( 1 - b^2\lambda \right) u^2, \]

which is decreasing since \( A \) is decreasing and \( r \) is comoving. The pressure at the centre follows from (5) as

\[ p_0 = \frac{12b^2\lambda}{M^2u^4(1 - b^2\lambda)} \left[ 1 + \frac{8}{3u} + \frac{1}{1 - b^2\lambda} \right], \]

which becomes zero when

\[ 1 + \frac{8}{3u} + \frac{1}{1 - b^2\lambda} = 0. \quad (12) \]

The time taken for the formation of the horizon is a solution of

\[ \frac{2}{u} + \frac{1 + b^2\lambda}{1 - b^2\lambda} = 0, \quad (13) \]

as established in \[ \]. Equations (12) and (13) together with the strong energy conditions place the following restrictions \[ \]

\[ 0 \leq ab^2 \leq (3 - \sqrt{8}) e^{\sqrt{2}}, \]

\[ -\infty \leq u \leq u_H, \]

where \( u_H = -\sqrt{2} \), the time of formation of the horizon, is a solution of (13). The body starts collapsing at \( u = -\infty \) with an infinite radius and zero density, and evolves to \( u = u_H \). Note that the results (12) and (13) are the same as in the corresponding non-causal model of \[ \]. The behaviour of the energy density \( \rho \) and the pressure \( p \) in the model (4)–(6), for the line element (1), has been studied extensively in \[ \]. The energy density is a decreasing function in the interval \( -\infty \leq u \leq u_H \). The pressure gradient is negative in the early stages of collapse but at a later epoch the pressure gradient becomes positive. It was established in \[ \] that the heat flow \( q \) is a monotonically increasing function of \( r \) and \( u \). Thus the behaviour of \( \rho, p \) and \( q \) in the simple Friedmann-like solution is not physically unreasonable.

3. Causal heat transport

We now apply the causal relativistic thermodynamics of Israel and Stewart \[ \] to give physical meaning to \( q_\alpha \). The causal transport equation in the absence of rotation and viscous stress is (see \[ \] for the general case)

\[ \tau h^\alpha_\beta q_\beta + q_\alpha = -\kappa \left( h^\alpha_\beta \nabla_\beta T + T\dot{u}_\alpha \right), \quad (14) \]
where $h_{\alpha\beta} = g_{\alpha\beta} + u_{\alpha}u_{\beta}$ projects into the comoving rest space, $g_{\alpha\beta}$ is the metric, $T$ is the local equilibrium temperature, $\kappa \ (\geq 0)$ is the thermal conductivity and $\tau \ (\geq 0)$ is the relaxational time-scale which gives rise to the causal and stable behaviour of the theory. The transport equation as well as expressions for the thermodynamic coefficients $\kappa$ and $\tau$ may be derived via relativistic kinetic theory using the Grad 14-moment method \[4\].

The non-causal Fourier transport equation has $\tau = 0$, and reduces from an evolution equation to an algebraic constraint on the heat flux. Intuitively, the non-causal behaviour arises because the heat flux is instantaneously brought to zero when the temperature gradient and acceleration are ‘switched off’.

For the metric (1), equation (14) becomes

$$\tau \dot{q} + q = -\frac{\kappa T'}{A}, \quad \text{(15)}$$

since $\dot{u}_{\alpha} = 0$. The very simple form of the transport equation (15) is balanced by the complexity of the equations (4)–(6) and (8)–(11).

For a physically reasonable model, we use the thermodynamic coefficients for radiative transfer \[13, 14, 15, 16\]. In other words, we are considering the situation where energy is carried away from the stellar core by massless particles, moving with long mean free path through matter that is effectively in hydrodynamic equilibrium, and that is dynamically dominant. The thermal conductivity has the form

$$\kappa = \gamma T^3 \tau_c, \quad \text{(16)}$$

where $\gamma \ (\geq 0)$ is a constant and $\tau_c$ is the mean collision time between the massless and massive particles. A detailed analysis in \[3\] for the case of neutrinos generated by thermal emission shows that $\tau_c \propto T^{-3/2}$ to a good approximation. Based on this, we will assume the power-law behaviour

$$\tau_c = \left(\frac{\alpha}{\gamma}\right) T^{-\sigma}, \quad \text{(17)}$$

where $\alpha \ (\geq 0)$ and $\sigma \ (\geq 0)$ are constants, with $\sigma = \frac{3}{2}$ in the case of thermal neutrinos. The mean collision time decreases with growing temperature, as expected, except for the special case $\sigma = 0$, when it is constant. This special case can only give a reasonable model for a limited range of temperature. Following \[3\], we assume that the velocity of thermal dissipative signals is comparable to the adiabatic sound speed, which is satisfied if the relaxation time is proportional to the collision time, i.e.

$$\tau = \left(\frac{\beta \gamma}{\alpha}\right) \tau_c, \quad \text{(18)}$$

where $\beta \ (\geq 0)$ is a constant. We can think of $\beta$ as the ‘causality’ index, measuring the strength of relaxational effects, with $\beta = 0$ giving the non-causal case. (A detailed
Substituting (6) and the thermodynamic equations (14)–(18) into the transport equation (15) leads to an equation for the temperature:

$$\alpha T^3 - \sigma \frac{dT}{ds} + \beta \left( \dot{f} + \frac{1}{2} s \right) T^{-\sigma} + 1 = 0,$$

(19)

where

$$s = \frac{4}{M u^2 (1 - r^2 \lambda)},$$

$$f(t) = -\ln\left[ u^2 (1 - b^2 \lambda) \right].$$

This is the fundamental equation for our simple causal model. It is a radial differential equation for each instant of proper time. The solution of (19) together with (4)–(6) represents a complete model in which all thermo-hydrodynamical quantities are known explicitly, and the model can be compared with its non-causal counterpart, which is the special case $\beta = 0$.

4. Causal temperature

The temperature equation (19) is readily solved exactly in the non-causal case $\beta = 0$, as in [1]. For the more satisfactory relativistic model $\beta > 0$, we have succeeded in integrating (19) exactly only when $\sigma = 0$, i.e. when the mean collision time may be approximated as constant. This can only be reasonable for a limited range of temperature, but it is a useful solution for giving a qualitative idea of the impact of causal relaxational effects, for checking numerical routines, and for generating perturbative solutions in the case of small $\sigma$ (see the following section). Even when $\sigma = 0$, the solution of (19) is not simple. It may be given in the form

$$T^4 = \frac{-16 \beta}{\alpha} \left[ \frac{1}{M u^2 (1 - r^2 \lambda)} \right]^2 - \frac{16}{\alpha} \left( \beta \dot{f} + 1 \right) \left[ \frac{1}{M u^2 (1 - r^2 \lambda)} \right] + F(t),$$

(20)

where $F(t)$ is an integration function, which we can determine as follows. The effective surface temperature of a star is given by [7]

$$\left( T^4 \right)_\Sigma = \left( \frac{1}{r^2 A^2} \right)_\Sigma \left( \frac{L}{4\pi \delta} \right),$$

(21)

where $\delta (> 0)$ is a constant and $L$ is the total luminosity at infinity, which has the form

$$L = \frac{dm}{dv} = \frac{2 b^2 \lambda}{(1 - b^2 \lambda)^2} \left[ \frac{2}{u} + \left( \frac{1 + b^2 \lambda}{1 - b^2 \lambda} \right)^2 \right].$$

(22)
We can evaluate (20) at the comoving boundary $r = b$ to find $F(t)$ with the help of (21) and (22). This yields

$$F(t) = \frac{16\beta}{\alpha} \left[ \frac{1}{Mu^2(1-b^2\lambda)} \right]^2 + \frac{16}{\alpha} \left( \beta \dot{f} + 1 \right) \left[ \frac{1}{Mu^2(1-b^2\lambda)} \right]$$

$$+ \frac{2b^2\lambda}{\pi\delta M^2 u^4(1-b^2\lambda)^2} \left[ \frac{2}{u} + \left( \frac{1+b^2\lambda}{1-b^2\lambda} \right) \right]^2.$$  

(23)

Finally, the temperature has the explicit exact form

$$T^4 = \frac{16\lambda(b^2 - r^2)}{\alpha M(1-b^2\lambda)(1-r^2\lambda)u^2} \left\{ \frac{\beta}{M u^2(1-b^2\lambda)(1-r^2\lambda)} + \beta \dot{f} + 1 \right\}$$

$$+ \frac{2b^2\lambda}{\pi\delta M^2 u^4(1-b^2\lambda)^2} \left[ \frac{2}{u} + \left( \frac{1+b^2\lambda}{1-b^2\lambda} \right) \right]^2.$$  

(24)

The temporal and spatial dependence are specified fully, and together with the expressions (4)–(6), this represents a complete exact model for causal radiating stellar collapse.

The non-causal temperature $\tilde{T}$ is obtained by setting $\beta = 0$ in (24). It is interesting to note that the non-causal and causal temperatures coincide at the surface $r = b$ of the radiating star:

$$T(t, b) = \tilde{T}(t, b).$$  

(25)

However, it is clear from (24) that at all interior points, the causal and non-causal temperatures differ. In particular, we observe that the causal temperature is greater than the non-causal temperature at the centre of the star:

$$T(t, 0) > \tilde{T}(t, 0).$$  

(26)

For small values of $\beta$ the temperature profile is similar to that of the non-causal theory, but as $\beta$ is increased, i.e. as relaxational effects grow, it is clear from (24) that the temperature profile can deviate substantially from that of the non-causal theory.

It follows from (27) and (28) that the causal temperature has a greater average gradient. In fact, we can show that the gradient is greater at each $r$ even when $\sigma > 0$. From the transport equation (15) for non-accelerating collapse, we note that

$$\kappa(T)T' - \kappa(\tilde{T})\tilde{T}' = -(A\tau)\dot{q}.$$  

Using the radiative form (14) for $\kappa$ and the power-law generalization (17) of neutrino transport for $\tau$, this becomes

$$(T^4 - \sigma)' - (\tilde{T}^4 - \sigma)' = - \left( \frac{4 - \sigma}{\alpha} \right) (A\tau)\dot{q}.$$  

(27)

(Note that this is independent of the particular form (13) for $\tau$.) Hence the relative radial gradient of the temperatures is governed by $\dot{q}$ and by the collision-time index $\sigma$. 


It can be shown \cite{1} from (6) that \( \dot{q} > 0 \). Since \( \sigma = \frac{3}{2} \) for thermal neutrino transport \cite{2}, we are justified in assuming that \( \sigma - 4 < 0 \). It follows from (27) that the causal temperature gradient is everywhere greater than that of the non-causal temperature, and that the difference grows with increasing \( \tau \). (Note that this conclusion still holds in the case \( \sigma > 4 \).)

This particular exact result in a simple model is in agreement with the general result of \cite{3}, i.e. that for shear-free perturbations, the causal temperature gradient is greater. Our simple exact model provides non-perturbative support for the perturbative result. As pointed out in \cite{3}, the fact that the causal temperature gradient is greater for a given luminosity, is consistent with the numerical results of \cite{4}, which show that for a given temperature gradient, the causal theory leads to lower luminosity. Thus the numerical non-perturbative, the perturbative, and the exact non-perturbative results are consistent.

5. Concluding remarks

By combining a simple stellar solution with physically consistent causal thermodynamics, we have been able to develop an exact model of radiating stellar collapse, in which it is straightforward to identify the relaxational effects of the causal theory without resort to highly complicated numerical methods. This should be seen as a complement to the physically more realistic and detailed models and their numerical integration \cite{5, 6} and perturbative solution \cite{7, 8}. We showed that the causal temperature decreases radially outward more steeply than the non-causal temperature, regardless of the particular form of the relaxation time \( \tau \), and in agreement with the independent perturbative results of \cite{9}. For the case where \( \tau \propto \tau_c \), i.e. where (18) holds, and assuming constant collision time, i.e. \( \sigma = 0 \) in (17), we found the exact solution (24) of the temperature differential equation (19), using the luminosity to evaluate the constant of integration. This exact solution predicts that the causal temperature coincides with the non-causal temperature at the surface, but differs at all interior points during the collapse, and is greater at the centre.

Our results confirm in a highly simplified but also transparent form, the overall conclusion of the more detailed models, i.e. that causal thermodynamics can introduce fundamentally different behaviour, with potentially significant implications in astrophysics. In this sense, our results are an additional motivation for further study of causal relativistic models of stellar collapse.

For a more realistic model, the mean collision time will grow with decreasing temperature, i.e. we have \( \sigma > 0 \) in the temperature equation (14). Using the \( \sigma = 0 \) solution (24), we can solve perturbatively in the case of small temperature parameter.
Let
\[ T = T_0 + \sigma T_1 + O(\sigma^2), \]
where \( T_0 \) is the zero order (\( \sigma = 0 \)) solution (24). Substituting into (19) and linearizing, we obtain the following ordinary differential equation in \( T_1 \):
\[ \alpha \frac{dT_1}{ds} + (3 - \sigma) \frac{T_1 dT_0}{T_0} ds = 0, \]
which easily integrates to give
\[ T_1 = \varphi T_0^{\sigma-3}, \]
where \( \varphi \) (\( \geq 0 \)) is a constant. Hence we may write
\[ T = T_0 \left[ 1 + \sigma \varphi T_0^{\sigma-4} \right], \tag{28} \]
to first order in \( \sigma \). The effect of \( \sigma \) is to increase \( T \), i.e. to retard the cooling due to heat transport, with the correction being greater in cooler regions (near the surface) and less in hotter regions (near the centre) of the collapsing star, since \( \sigma \ll 1 \) ensures that \( \sigma - 3 < 0 \).

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