The Effects of Orbital Motion on LISA Time Delay Interferometry

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Abstract. In an effort to eliminate laser phase noise in laser interferometer spaceborne gravitational wave detectors, several combinations of signals have been found that allow the laser noise to be canceled out while gravitational wave signals remain. This process is called time delay interferometry (TDI). In the papers that defined the TDI variables, their performance was evaluated in the limit that the gravitational wave detector is fixed in space. However, the performance depends on certain symmetries in the armlengths that are available if the detector is fixed in space, but that will be broken in the actual rotating and flexing configuration produced by the LISA orbits. In this paper we investigate the performance of these TDI variables for the real LISA orbits. First, addressing the effects of rotation, we verify Daniel Shaddock’s result that the Sagnac variables \( \alpha(t) \), \( \beta(t) \), and \( \gamma(t) \) will not cancel out the laser phase noise, and we also find the same result for the symmetric Sagnac variable \( \zeta(t) \). The loss of the latter variable would be particularly unfortunate since this variable also cancels out gravitational wave signal, allowing instrument noise in the detector to be isolated and measured. Fortunately, we have found a set of more complicated TDI variables, which we call \( \Delta \)-Sagnac variables, one of which accomplishes the same goal as \( \zeta(t) \) to good accuracy. Finally, however, as we investigate the effects of the flexing of the detector arms due to non-circular orbital motion, we show that all variables, including the interferometer variables, \( X(t) \), \( Y(t) \), and \( Z(t) \), which survive the rotation-induced loss of direction symmetry, will not completely cancel laser phase noise when the armlengths are changing with time. This unavoidable problem will place a stringent requirement on laser stability of \( \sim 5 \text{Hz}/\sqrt{\text{Hz}} \).

1. Introduction

Space gravitational wave detectors employing Michelson laser interferometry between free-flying spacecraft differ in many ways from their laboratory counterparts. Among these differences is the fact that the interferometer arms may not be maintained at equal lengths. The laser phase noise that cancels in laboratory detectors when the signals in the two equal arms are subtracted from each other will not cancel when the signals in the unequal arms of the spaceborne detectors are subtracted. However, methods have been developed \[ \Pi \] that allow the various one-way signals from the two arms to be combined to produce several variables that are void of laser phase noise. The process of combining one-way signals from the various arms of the constellation of three spacecraft has been named time delay interferometry, or TDI. Previous work has demonstrated how these
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Figure 1. The detector geometry and numbering convention for spacecraft, armlengths, and signals.

variables perform in the limit that the arms of the interferometer are fixed in space, but Daniel Shaddock [2] has pointed out that one set of variables, the so-called Sagnac variables, will not cancel out laser phase noise when the detector is rotating. In this paper we will look at all TDI variables to see how they perform when the armlengths are not direction-symmetric, as a result of the rotation of the arms in the plane of the detector, and are not time-translation symmetric, as a result of non-circular orbits for the end masses of the detector.

2. Canceling Laser Phase Noise in Static Detectors

Let us begin by showing how one may find variables that cancel laser phase noise in the limit that the detector is static. For reasons that will soon be clear, we choose notation that follows that of Hellings [3], as shown in Figure 1. Three spacecraft, numbered 1, 2, and 3, are at the vertices of a roughly equilateral triangle, with laser tracking signals being continuously passed in both directions along each of the arms thus formed. The one-way laser signal sent from spacecraft 2 and received at spacecraft 1 at time $t$ is labeled $y_{21}(t)$. The distance traveled by the signal (in units of time) is labeled $L_{21}$. The signal passing in the opposite direction along the same arm is labeled $y_{12}(t)$, traveling a distance $L_{12}$. Each laser signal $y_{ij}$ is formed by beating the received laser light with light from the local laser, the change in the phase of the beat signal being sensitive to the amplitude of plane gravitational waves passing through the detector. If we write the phase noise in the laser on spacecraft $i$ as $\phi_i(t)$, then the laser phase noise in a one-way
signal will be given by
\[ y_{ij} = \phi_i(t - L_{ij}) - \phi_j(t). \] (1)

Since the laser phase noise is many orders of magnitude larger than the gravitational wave signal that is expected to be seen in the detector, this noise must be eliminated.

There are actually several families of TDI variables that have been found \[4\] that eliminate laser phase noise in a static detector. The first one we will consider is the family of Sagnac variables in which each one-way signal appears once. We write
\[ \sigma(t) = y_{12}(t - \lambda_1) + y_{23}(t - \lambda_2) + y_{31}(t - \lambda_3) - y_{13}(t - \lambda_4) - y_{32}(t - \lambda_5) - y_{21}(t - \lambda_6). \] (2)

The reason for the name ‘Sagnac’ for this variable is that it represents the difference between the phase of a signal passed counter-clockwise around the triangle, starting with spacecraft 1 (the first line of Eq. 2), and one passed clockwise around the triangle (the second line), starting at the same point. The time delays \(\lambda_i\) are to be adjusted to eliminate laser phase noise. To see how this is done, let each term in Eq. 2 be expanded using Eq. 1 one line per term, to give
\[
\sigma(t) = \phi_1(t - \lambda_1 - L_{12}) - \phi_2(t - \lambda_1) \\
+ \phi_2(t - \lambda_2 - L_{23}) - \phi_3(t - \lambda_2) \\
+ \phi_3(t - \lambda_3 - L_{31}) - \phi_1(t - \lambda_3) \\
- \phi_1(t - \lambda_4 - L_{13}) + \phi_3(t - \lambda_4) \\
- \phi_3(t - \lambda_5 - L_{32}) + \phi_2(t - \lambda_5) \\
- \phi_2(t - \lambda_6 - L_{21}) + \phi_1(t - \lambda_6).
\] (3)

If the laser phase noise is to cancel exactly, we must have
\[
\phi_1(t - \lambda_1 - L_{12}) - \phi_1(t - \lambda_3) - \phi_1(t - \lambda_4 - L_{13}) + \phi_1(t - \lambda_6) = 0 \quad (4a)
\]
\[
-\phi_2(t - \lambda_1) + \phi_2(t - \lambda_2 - L_{23}) + \phi_2(t - \lambda_5) - \phi_2(t - \lambda_6 - L_{21}) = 0 \quad (4b)
\]
\[
-\phi_3(t - \lambda_2) + \phi_3(t - \lambda_3 - L_{31}) + \phi_3(t - \lambda_4) - \phi_3(t - \lambda_5 - L_{32}) = 0 \quad (4c)
\]

There are two ways in which the four terms in Eq. 4a for example, may be made to cancel. One must either have \(\lambda_1 - L_{12} = \lambda_3\) and \(\lambda_4 - L_{13} = \lambda_6\) or \(\lambda_1 - L_{12} = \lambda_4 - L_{13}\) and \(\lambda_3 = \lambda_6\). There are thus two equations for the \(\lambda_i\) arising out of Eq. 4a and there are two possibilities for the form of these two equations. From Eqs. 4, therefore, we will have eight \((2 \times 2 \times 2)\) possible sets of six equations for the six unknowns \(\lambda_i\). The six equations, however, will always possess one degeneracy in them, corresponding to the freedom in the origin of \(t\). Thus, Sagnac variables will be found as solutions to any of the eight possible sets of six equations for five unknowns.

When the possible ways of satisfying Eqs. 4 are studied, with the simplification \(L_{ij} = L_{ji}\) that is appropriate to a static detector, there appear only four solutions. One of these is found by setting \(\lambda_6\) to zero and solving for the remaining \(\lambda_i\), giving
\[ \lambda_1 = L_{13} + L_{23} \]
corresponding to TDI variable $\alpha(t)$ of Armstrong, Estabrook, and Tinto. Two other solutions may be found from Eq. 5 by simultaneously permuting $1 \rightarrow 2 \rightarrow 3$ and $4 \rightarrow 5 \rightarrow 6$ on the left hand side and $1 \rightarrow 2 \rightarrow 3$ on the right. A single permutation gives TDI variable $\beta(t)$ and a further permutation gives $\gamma(t)$. The final possible solution to Eqs. 4, arbitrarily choosing $\lambda_6 = L_{23}$, is

$$
\begin{align*}
\lambda_1 &= L_{13} \\
\lambda_2 &= L_{12} \\
\lambda_3 &= L_{23} \\
\lambda_4 &= L_{12} \\
\lambda_5 &= L_{13} \\
\lambda_6 &= L_{23}
\end{align*}
$$

corresponding to the so-called symmetric Sagnac variable $\zeta(t)$. The reason for the existence of solutions to six equations for five unknowns is that symmetries arise in the equations due to the fact that the direction symmetry $L_{ij} = L_{ji}$ reduces from six to three the number of independent constants in the six equations arising out of Eqs. 4.

A second class of TDI variables consists of the so-called interferometer variables, formed by choosing two arms out of the three and using the four one-way signals going up and down these arms, with each signal appearing twice. For example, if we choose the two arms radiating from spacecraft 1, we would form

$$
X(t) = y_{12}(t - \lambda_1) - y_{13}(t - \lambda_2) + y_{21}(t - \lambda_3) - y_{31}(t - \lambda_4) \\
+ y_{13}(t - \lambda_5) - y_{12}(t - \lambda_6) + y_{31}(t - \lambda_7) - y_{21}(t - \lambda_8)
$$

When Eq. 7 is expanded using Eq. 1, one line per term, we write the laser phase noise in $X(t)$ as

$$
X(t) = \phi_1(t - \lambda_1 - L_{12}) - \phi_2(t - \lambda_1) \\
- \phi_1(t - \lambda_2 - L_{13}) + \phi_3(t - \lambda_2) \\
+ \phi_2(t - \lambda_3 - L_{21}) - \phi_1(t - \lambda_3) \\
- \phi_3(t - \lambda_4 - L_{31}) + \phi_1(t - \lambda_4) \\
+ \phi_1(t - \lambda_5 - L_{13}) - \phi_3(t - \lambda_5) \\
- \phi_1(t - \lambda_6 - L_{12}) + \phi_2(t - \lambda_6) \\
+ \phi_3(t - \lambda_7 - L_{31}) - \phi_1(t - \lambda_7) \\
- \phi_2(t - \lambda_8 - L_{21}) + \phi_1(t - \lambda_8).
$$
As in Eqs. 4, we again have three conditions for the identical vanishing of the three laser phase noises, but the counterpart of Eq. 4a, the \( \phi_1 \) equation, contains the eight occurrences of \( \phi_1 \) that appear in Eq. 8. The cancellation by pairs in this equation may occur 24 different ways, so there will be 96 \((24 \times 2 \times 2)\) different sets of eight equations for the seven unknown \( \lambda_i \). Whether \( L_{ij} = L_{ji} \) or \( L_{ij} \neq L_{ji} \), the only unique, nontrivial solution to these sets of equations is

\[
\begin{align*}
\lambda_1 &= L_{13} + L_{31} + L_{21} \\
\lambda_2 &= L_{12} + L_{21} + L_{31} \\
\lambda_3 &= L_{31} + L_{13} \\
\lambda_4 &= L_{21} + L_{12} \\
\lambda_5 &= L_{31} \\
\lambda_6 &= L_{21} \\
\lambda_7 &= 0 \\
\lambda_8 &= 0,
\end{align*}
\]

(9)

corresponding to the \( X(t) \) interferometer variable found by Armstrong, Estabrook, and Tinto [4]. If the two arms are chosen differently, so that Eq. 8 is modified by the permutations \( 1 \to 2 \to 3 \), then the single solution each time will correspond to \( Y(t) \) and then to \( Z(t) \).

Besides the interferometer variables of Eq. 7 there are other ways to combine the six possible one-way links into eight-term combinations. These have been named the beacon, monitor, and relay variables and are described in the appendix of Estabrook, Tinto, and Armstrong [5].

3. The Effects of Rigid Rotation

All of the variables discussed above have the property that laser phase noise exactly cancels out when the armlengths are direction-symmetric, that is, when \( L_{ij} = L_{ji} \). When these variables were derived, this direction symmetry was always assumed. Indeed, the notation used in the defining papers [4] gave a single subscript to each armlength (so, for example, \( L_1 \) was the length of the arm across from spacecraft 1, the length we have called \( L_{23} \)). However, there is a problem with the assumption of direction symmetry. This is that the constellation for the LISA mission [6] will be rotating once per year in the plane of the detector, due to the individual orbits of each end spacecraft. Since this is the case, it will not be true that \( L_{ij} = L_{ji} \), as we shall now show.

Consider spacecraft 1 and 2 in Figure 2. Since the constellation is rotating in the clockwise direction, the signal sent from spacecraft 1 to spacecraft 2 at time \( t \) will have to aim ahead of the position of spacecraft 2 at time \( t \). If we define the length of the arm between spacecraft 1 and 2 to be \( L_{12} \) in the limit of no rotation, then the actual distance traveled by the signal from spacecraft 1 to spacecraft 2 will be \( L_{12} < L_{12} \). Similarly, the signal from spacecraft 2 to spacecraft 1 will have to lead spacecraft 1 in its motion and
will therefore travel a distance \( L_{21} > L_{12} \). When the constellation is rotating, we will therefore not have \( L_{ij} = L_{ji} \), and so the laser-canceling variables will not work. Indeed, since these variables were all derived with the assumption \( L_{ij} = L_{ji} \), it is not even clear whether \( L_{ij} \) or \( L_{ji} \) should be used in the definitions.

In what follows, we will investigate the families of laser-canceling variables for the case of direction asymmetry (\( L_{ij} \neq L_{ji} \)), but in the limit of rigid rotation where the armlengths are constant in time. If exact cancellation is not possible, we will estimate the size of the residual laser phase noise and calculate the requirement for laser frequency stability that this uncanceled noise will impose on the LISA laser stabilization system.

### 3.1. Sagnac-type Variables

When \( L_{ij} \neq L_{ji} \), none of the eight possible sets of six equations for five unknowns represented in Eqs. 4 have solutions. There is no Sagnac-type or symmetric-Sagnac-type variable that will exactly cancel laser phase noise. To see how much noise is left over, let us set the \( \lambda_i \) in Eqs. 4 so as to cancel as many terms as possible and then evaluate the power spectral density of the noise that will remain. We begin with an \( \alpha \)-like variable by arbitrarily setting \( \lambda_6 = 0 \). Then \( \lambda_3 = 0 \) will allow the second and last terms in Eq. 4 to cancel. Setting \( \lambda_5 = L_{21} \) will then allow the last two terms in Eq. 4b to cancel, and so on. Working this way, we are able to cancel all terms but two. We are left with

\[
\alpha(t) = \phi_1(t - L_{12} - L_{23} - L_{31}) - \phi_1(t - L_{13} - L_{32} - L_{21})
\]

(10)

When \( L_{ij} \neq L_{ji} \), these two terms do not cancel, and there remains an unavoidable laser phase noise in the Sagnac variable \( \alpha(t) \). Similar results apply for \( \beta(t) \) and \( \gamma(t) \).

How big is the error produced by these two terms? First, let us note that \( L_{12} + L_{23} + L_{31} \) is the total time around the constellation in the counter-clockwise
direction and that $L_{13} + L_{32} + L_{21}$ is the total time in the clockwise direction. Even if the triangle were perfectly rigid, the times of flight would not be the same because of the rotation. Let us write the time of flight around the triangle in the limit of no motion as $L_{\text{tot}}$, and the times of flight in the counter-clockwise and clockwise directions as $L_{\text{tot}} + \Delta L_{-}$ and $L_{\text{tot}} + \Delta L_{+}$, respectively, when the triangle is rotating. Since the LISA constellation rotates in a clockwise direction (seen from the ecliptic pole), $\Delta L_{-}$ will always be negative and $\Delta L_{+}$ will always be positive. If we now expand the phase noise in Eq. (10) in a Taylor series, we find

$$\alpha(t) = \phi_1(t - L_{\text{tot}} - \Delta L_{-}) - \phi_1(t - L_{\text{tot}} - \Delta L_{+})$$

$$= \phi_1(t - L_{\text{tot}}) - \dot{\phi}_1 \Delta L_{-} - \phi_1(t - L_{\text{tot}}) + \dot{\phi}_1 \Delta L_{+}$$

$$= (\Delta L_{+} - \Delta L_{-}) \dot{\phi}_1$$

(11)

The difference $\Delta L_{+} - \Delta L_{-}$ is the Sagnac time shift for signals circulating around a closed path and is given by

$$\Delta L_{+} - \Delta L_{-} = \Delta L_{\text{Sagnac}} = \frac{4A \Omega}{c^2} = \frac{\sqrt{3} L 22\pi}{c^2 T}$$

(12)

where $A$ is the area enclosed by the light path, $\Omega$ is the angular velocity of the rotating light path, $L$ is a typical armlength, and $T$ is the period of the rotation of the detector. For the LISA mission parameters, this time difference is $\Delta L_{\text{Sagnac}} = 10^{-4}$ s. The relationship between residual noise power in the $\alpha(t)$ variable and the laser frequency noise power is found from Eq. (11) to be

$$S_\alpha = S_\nu (\Delta L_{\text{Sagnac}})^2$$

(13)

so, for a requirement of $\sqrt{S_\alpha} = 10^{-5}$ cycles/$\sqrt{\text{Hz}}$, we derive a laser stability requirement of

$$\sqrt{S_\nu} = \frac{\sqrt{S_\alpha}}{\Delta L_{\text{Sagnac}}} = \frac{10^{-5} \text{cycles}/\sqrt{\text{Hz}}}{10^{-4} \text{s}} = 0.1 \text{ Hz}/\sqrt{\text{Hz}}.$$  

(14)

This is a very difficult requirement to satisfy.

Beginning again with Eq. (3) let us try to find a $\zeta$-like variable that cancels as much laser phase noise as possible. Setting $\lambda_6 = L_{31}$ and $\lambda_5 = L_{32}$, we cancel the first terms in the last two lines of Eq. (3) Continuing in this way, we may choose the remaining $\lambda_i$ so as to cancel all $\phi_2$ and $\phi_3$ terms. We are left with

$$\zeta(t) = \phi_1(t - L_{31} - L_{12}) - \phi_1(t - L_{21} - L_{13}) + \phi_1(t - L_{23}) - \phi_1(t - L_{32}).$$

(15)

If we write each armlength as $L_{ij} = L_{ij} + \Delta L_{ij}$, where $L_{ij} = L_{ji}$ is the armlength in a co-rotating system, and expand $\phi_1$ in a Taylor series, Eq. (15) becomes

$$\zeta(t) = \phi_1(t - L_{31} - L_{12}) - (\Delta L_{31} + \Delta L_{12}) \dot{\phi}_1 - \phi_1(t - L_{21} - L_{13}) +$$

$$(\Delta L_{21} + \Delta L_{13}) \dot{\phi}_1 + \phi_1(t - L_{23}) - \Delta L_{23} \dot{\phi}_1 - \phi_1(t - L_{32}) + \Delta L_{32} \dot{\phi}_1.$$  

(16)

Collecting terms, we have

$$\zeta(t) = \left[ (\Delta L_{13} + \Delta L_{32} + \Delta L_{21}) - (\Delta L_{12} + \Delta L_{23} + \Delta L_{31}) \right] \dot{\phi}_1$$

(17)
This is the same Sagnac path difference as for $\alpha(t)$, so the uncanceled laser phase noise in $\zeta(t)$ will be the same as that derived for $\alpha(t)$, $\beta(t)$, and $\gamma(t)$, with the same resulting requirement for laser frequency stability that we derived in Eq. 14. Since this requirement is not likely to be satisfied, we must draw the conclusion that the $\zeta(t)$ variable will simply not be available. This would be particularly unfortunate, since $\zeta(t)$ has the interesting property that low-frequency-limit gravitational-wave signals exactly cancel out when this combination is formed \[14\] so that it would have provided the potentially useful capability of differentiating between the power produced by instrument noise, which would still remain in $\zeta(t)$, and that produced by a gravitational wave background, which would go to zero in $\zeta(t)$ but not in the other variables. The variable $\zeta(t)$ will not accomplish this goal for a rotating detector, but we have found a more complicated variable that will accomplish the same goal to good accuracy. This $\Delta \zeta(t)$ variable is discussed in the next section.

3.2. $\Delta$-Sagnac-type Variables

The form of the residual noise for the $\alpha$ variable, given in Eq. 10 led us to wonder whether differences of two $\alpha$-like combinations might be able to cancel out the laser noise that remained. We have thus defined what we have called $\Delta$-Sagnac variables, formed by using each of the six one-way links twice, for a total of twelve terms. When the general form for such a variable is expanded using Eq. 14 for the laser phase noise, we find that there are 13824 ($= 24 \times 24 \times 24$) possible sets of 12 equations for the 11 time delays. There are a total of 60 unique, nontrivial solutions to these sets of equations. An example of one of these $\Delta$-Sagnac variables is

$$\Delta \alpha(t) = y_{12}(t - L_{31} - L_{23}) + y_{23}(t - L_{31}) + y_{31}(t)$$
$$- y_{13}(t - L_{21} - L_{32}) - y_{32}(t - L_{21}) - y_{21}(t)$$
$$- y_{12}(t - L_{21} - L_{32} - L_{13} - L_{31} - L_{23}) - y_{23}(t - L_{21} - L_{32} - L_{13} - L_{31})$$
$$- y_{31}(t - L_{21} - L_{32} - L_{31}) + y_{13}(t - L_{31} - L_{23} - L_{12} - L_{21} - L_{32})$$
$$+ y_{32}(t - L_{31} - L_{23} - L_{12} - L_{21}) + y_{21}(t - L_{31} - L_{23} - L_{12}).$$

(18)

Most importantly for the goal of isolating instrument noise in the detectors, we have found one $\Delta$-Sagnac variable that recovers the insensitivity to low-frequency gravitational waves that characterized $\zeta(t)$. This is

$$\Delta \zeta(t) = y_{13}(t - L_{21} - L_{12} - L_{32}) + y_{32}(t - L_{12} - L_{23} - L_{31}) + y_{21}(t - L_{23} - L_{12} - L_{32})$$
$$- y_{12}(t - L_{32} - L_{13} - L_{21}) - y_{23}(t - L_{21} - L_{12} - L_{32}) - y_{31}(t - L_{32} - L_{23} - L_{12})$$
$$- y_{13}(t - L_{31} - L_{12}) - y_{32}(t - L_{13} - L_{31}) - y_{21}(t - L_{32} - L_{13})$$
$$+ y_{12}(t - L_{31} - L_{13}) + y_{23}(t - L_{31} - L_{12}) + y_{31}(t - L_{13} - L_{32}).$$

(19)

The $\zeta$-like insensitivity to gravitational waves of this variable may be seen by the fact that, in the limit $L_{ij} = L_{ji}$, it may be written

$$\Delta \zeta(t) = \zeta(t - L_{12} - L_{23}) - \zeta(t - L_{13}),$$

(20)
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so that \( \Delta \zeta(t) \) only differs from this combination of \( \zeta(t) \) terms by the small amount arising from the asymmetry in armlengths, or by only a few parts in \( 10^{-6} \) of the gravitational wave signals in each \( y_{ij} \). The subtraction in Eq. 20 will affect both the signal and the noise in the \( \zeta \) terms, so the signal-to-noise ratio for \( \Delta \zeta(t) \) will be the same as that for \( \zeta(t) \). Thus, we are able to recover a diagnostic TDI variable, at the cost of requiring slightly more telemetered data from each spacecraft to allow the complicated twelve-term variable to be constructed.

3.3. Interferometer-type Variables

In contrast to the cases of the Sagnac and symmetric Sagnac variables, the interferometer variables \( X(t) \), \( Y(t) \), and \( Z(t) \), will provide output for LISA in which laser phase noise is exactly eliminated, even when the detector arms are rotating, though we will need to differentiate between \( L_{ij} \) and \( L_{ji} \) in the proper places. The solution displayed in Eq. 9 is valid even when \( L_{ij} \neq L_{ji} \), yielding a form for \( X(t) \) given by

\[
X(t) = y_{12}(t - L_{31} - L_{13} - L_{21}) - y_{13}(t - L_{21} - L_{12} - L_{31}) + y_{21}(t - L_{31} - L_{13}) - y_{31}(t - L_{21} - L_{12}) + y_{13}(t - L_{31}) - y_{12}(t - L_{21}) + y_{31}(t) - y_{21}(t) \tag{21}
\]

Essentially, this form for \( X(t) \) is found from the old form in Armstrong, Estabrook, and Tinto [4] by substituting \( 2L_{ij} \rightarrow L_{ij} + L_{ji} \). If we begin with Eq. 21 and permute indices according to \( 1 \rightarrow 2 \rightarrow 3 \), we find valid redefinitions for the other interferometer variables \( Y(t) \) and \( Z(t) \) as well.

Using procedures like that leading to Eq. 21 we have also found redefinitions of the beacon, monitor, and relay variables that will succeed in exactly canceling laser phase noise, even if the detector is rotating.

4. The Effects of Flexing

The direction symmetry that is broken by rotation is a discrete symmetry, but pure rigid rotation still preserves the important continuous symmetry, \( L_{ij}(t + \tau) = L_{ij}(t) \) for any value of \( \tau \). When this symmetry is preserved, then the eight equations for seven unknowns that must be satisfied to eliminate phase noise in Eq. 7 will contain only the four constants, \( L_{12}, L_{21}, L_{13}, \) and \( L_{31} \). These few constants create a subtle symmetry in the equations that permits the \( X(t) \) solution to exist. Unfortunately, the LISA orbits do not allow for time-translation symmetry. The non-circularity of the two-body orbits and the perturbations of the orbits by the planets produce a complicated flexing of the arms in the detector. As a result, there is a time dependence in the armlengths, and Eq. 4 for the laser phase noise in a one-way link should be written

\[
y_{ij} = \phi_i(t - L_{ij}(t)) - \phi_j(t). \tag{22}
\]

To see how this asymmetry complicates the equations for laser phase noise cancellation, let us rewrite Eq. 8 taking the time dependence in the \( L_{ij} \) explicitly into account

\[
X(t) = \phi_1(t - \lambda_1 - L_{12}^{-1}) - \phi_2(t - \lambda_1)
\]
where the notation $L_{ij}^{k}$ denotes $L_{ij}(t - \lambda_k)$. There are eight different constants in Eq. 23 and the orbits are complicated enough that there appears to be no orbital symmetry relating them to each other. As a result, there are no solutions to any of the 96 possible sets of equations that must be satisfied in order to cancel the $\phi_i$ terms identically.

To evaluate how much noise may be canceled and how much noise remains, let us cancel as many terms in Eq. 23 as we can. We start by setting $\lambda_7 = \lambda_8 = 0$ and proceed from bottom to top in Eq. 23. We end up with a definition of $X(t)$ given by

$$X(t) = y_{12}(t - L_{31} - L_{13}^{(1)} - L_{21}^{(2)}) - y_{13}(t - L_{21} - L_{12}^{(1)} - L_{31}^{(2)}) + y_{21}(t - L_{31} - L_{13}^{(1)})$$

$$- y_{11}(t - L_{21} - L_{12}^{(1)}) + y_{13}(t - L_{31}) - y_{12}(t - L_{21}) + y_{31}(t) - y_{21}(t),$$

where $L_{21} = L_{21}(t)$, $L_{31} = L_{31}(t)$, $L_{12}^{(1)} = L_{12}(t - L_{31})$, $L_{13}^{(1)} = L_{13}(t - L_{31})$, $L_{21}^{(2)} = L_{21}(t - L_{31} - L_{13}^{(1)})$, and $L_{31}^{(2)} = L_{31}(t - L_{21} - L_{12}^{(1)})$. With this definition, all but two terms in Eq. 23 cancel, leaving

$$X(t) = \phi_1(t - L_{31} - L_{13}^{(1)} - L_{21}^{(2)} - L_{12}^{(3)}) - \phi_3(t - L_{21} - L_{12}^{(1)} - L_{31}^{(2)} - L_{13}^{(3)}),$$

where $L_{12}^{(3)} = L_{12}(t - L_{31} - L_{13}^{(1)} - L_{21}^{(2)})$ and $L_{13}^{(3)} = L_{13}(t - L_{21} - L_{12}^{(1)} - L_{31}^{(2)})$. To evaluate the size of the noise that remains in Eq. 25 let us estimate the time dependence in the armlengths using $L_{ij}^{(n)} = L_{ij}(t) - nV_{ij}L$, where $V_{ij} \approx V_{ji}$ is the rate of change of the armlength in seconds per second and $L$ is a typical one-way light time. Expanding each $\phi_1$ term in Eq. 25 in a Taylor series, we find

$$X(t) = \phi_1(t - L_{31} - L_{13} - L_{21} - L_{12}) + (V_{13}L + 5V_{12}L)\dot{\phi}_1$$

$$- \phi_1(t - L_{21} - L_{12} - L_{31} - L_{13}) - (V_{12}L + 5V_{13}L)\dot{\phi}_1,$$

Cancelling and combining, we arrive at

$$X(t) = 4L(V_{12} - V_{13})\dot{\phi}_1,$$

giving a requirement on laser frequency noise of

$$\sqrt{S_x} = \frac{\sqrt{S_X}}{4L(V_{12} - V_{13})} = 5Hz/\sqrt{Hz},$$

where, in the last step, we have used $V_{12} - V_{13} = (10m/s)/c$ and $L = 16.7$ s. The current science requirement for LISA specifies a laser frequency stability of $30Hz/\sqrt{Hz}$. We conclude that this requirement needs to be strengthened by at least a factor of 6 and probably 10.
Similar results apply to all TDI variables — those that survive the breaking of direction symmetry will be useful, even though they do not survive the breaking of time-translation symmetry, but only if the laser frequency stability requirement in Eq. 28 is satisfied.

5. Summary

A general property of all TDI variables is that the conditions governing the canceling of laser phase noise involve \( n \) equations for the \( n - 1 \) time delays. In order for a solution to exist, the interferometer geometry must possess certain symmetries. When the discrete symmetry \( L_{ij} = L_{ji} \) is broken by rotation, the interferometer variables survive, but the Sagnac variables are lost. However, the loss of a discrete symmetry can be overcome by defining more complicated TDI combinations. On the other hand, when the continuous symmetry \( L_{ij}(t) = L_{ij}(t + \tau) \) is broken, as is the case for the LISA orbits, there are no TDI variables that completely cancel laser phase noise.

Thus, the first conclusion of this paper is that, without a breakthrough in laser stability that would allow the requirement in Eq. 14 to be realized with some margin, the Sagnac variables, \( \alpha(t), \beta(t), \) and \( \gamma(t) \), and the symmetric Sagnac variable \( \zeta(t) \), will be dominated by laser phase noise and will not be of use in LISA data analysis. On the other hand, in the limit of rigid rotation, the \( \Delta \)-Sagnac variables, especially the \( \Delta \zeta(t) \) variable that allows the detector to go “off-source”, are still available, as are the simple interferometer variables, \( X(t), Y(t), \) and \( Z(t) \).

Our second conclusion is that laser phase noise will dominate the LISA sensitivity window for all variables, because of the breaking of time-translation symmetry, unless a way is found to reduce laser frequency stability to that calculated in Eq. 28. In this case the variables that survive the direction asymmetry will all be useful for data analysis. If the laser stability requirement proves too difficult, an alternative would be to reduce the LISA armlengths, simultaneously reducing \( L \) and \( V_{ij} \). This would ease the laser stability requirement, though the resulting reduction in the response of the detector to gravitational waves is likely to reduce the overall sensitivity of the detector, due to other noise sources.

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