Gravitational waves in $f(R)$ gravity power law model

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Abstract: We investigate the different polarization modes of gravitational waves in $f(R)$ gravity power law model in de Sitter space. It is seen that the massive scalar field polarization mode exists in this model. The mass of the scalar field depends highly on the background curvature and the power term $n$. However, we found that the model does not exhibit a massive scalar mode for $n = 2$ and instead, it shows a breathing mode in addition to the tensor plus cross modes. Thus, mass of the scalar field is found to vary with $n$ within the range $1 \leq n \leq 2$.

Keywords: Modified gravity; Power law model; Gravitational waves

1. Introduction

Inspired purely by intellectual curiosity, the first modification of Einstein’s gravity was attempted by Weyl in 1919 [1] and then by Eddington in 1923 [2] with the introduction of higher-order curvature invariants in the gravitational action. Due to the lack of experimental motivations, these types of modifications were ignored until 1962. During 1962 for the first time, it was realized that modification of gravitational action could have some merits. In fact, the gravity from general relativity (GR) is not renormalizable, and thus, it is not possible to quantize it according to conventional methods. In 1962, Utiyama and DeWitt showed that the renormalization of gravity at one loop is possible if one modifies the Einstein–Hilbert action by higher-order curvature terms [3]. Later, a number of drawbacks of GR have been observed including its inability to explain cosmic acceleration. To overcome these drawbacks of GR, several new theories have been proposed suggesting further modifications of GR. Among them, $f(R)$ gravity is a prominent one which keeps itself in stand against the challenges and problems it faced till now [4–6]. One of the important features of $f(R)$ gravity is its less complexity compared to other modified theories of gravity [7].

After around 100 years of Einstein’s prediction of existence of gravitational waves (GWs), the Laser Interferometer Gravitational Wave Observatory (LIGO) Scientific Collaboration announced the detection of GWs for the first time on September 14, 2015. This first GW event was named as GW150914 [8]. This observation was later followed by many other events including GW151226 [9], GW170104 [10], GW170814 [11] and GW170817 [12]. These events opened new directions in testing GR and modified theories of gravity. Therefore, the study of GWs in modified gravity will play a very important role in modifying GR and on the predictions of GR. The experimental results obtained from the sector of GWs can effectively constrain the modified gravity models to a fair extent and can even contribute to the bottom-up approach in forming or proposing modified gravity theories and models in an efficient way.

In the metric formalism of $f(R)$ gravity, the GWs have other modes of polarizations besides the tensor modes of polarization found in GR. Generation and possibilities of detection of GWs in $f(R)$ gravity were studied by C. Corda using the Starobinsky model in [13] and using a model of the form $f(R) = R + R^{-1}$ in [14]. These studies indicate the existence of massive mode of GW radiation [15] in $f(R)$ theory in metric formalism. In a recent study, the propagating degrees of freedom of GWs in $f(R)$ gravity are found to be 3 [16], and in another recent study, the Starobinsky model [17] showed that there exists a mixed state of massless breathing mode and massive scalar mode of polarization besides the usual tensor modes in the model. Thus, the Newman–Penrose (NP) formalism cannot be applied in that study due to the existence of massive scalar modes of GW polarization. In our study, we will restrict ourselves to the power law model of $f(R)$ gravity given by

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The action of $f(R) = \alpha R^n$, where $\alpha$ is an arbitrary constant and $n$ is a real number. This model has been pursued in [18] to study the late time acceleration in the Universe. It is already established that realistic $f(R)$ gravity models may unify inflation, radiation dominance, matter dominance and dark energy [19–23]. In Refs. [24–26], the power law model was discussed for the different special cases along with $n = -1$ and $n = 3/2$, in which $n = 3/2$ case is conformally equivalent to Liouville field theory. In another paper [27], the model was used for invoking the chameleon mechanism to work within the solar system and in which it was shown that for this to happen, the value of $n$ is required to be very close to 1. The result rendered from the work implied that the model is a poor candidate for a realistic alternative to dark energy. The cosmological dynamics of the model also has been studied in Ref. [25]. It is to be noted that although the model used in our case is already ruled out in solar system tests [28], we have noticed that in several studies [27, 29–31], the model has been used as a toy model. These studies have motivated us to use this model for its simple form and more importantly because of the existence of massless scalar mode of polarization of GWs even in de sitter space. Also, we think that a wide range of studies with this sort of new result will definitely help to decisively rule out or accept the model from future astrophysical/cosmological observations.

In this work, we would like to test the model in the GW regime to find out the possibility of its viability and the drawbacks. Another important point of this model is that it is the only model in $f(R)$ theories in the metric formalism which can give pure massless breathing mode as a special case. Here, we will study the model with the special case of $n = 2$ along with the stability and polarization modes.

The paper is organized as follows. In the next section, the variation of mass of the scalar field in de Sitter space with the power term $n$ is studied. In the third section, the stability of the model is studied in de Sitter space. The fourth section contains a study toward the equivalence of the theory with the scalar–tensord field theory and the required conditions for such equivalence. The tensor and the scalar polarization modes are studied with the help of geodesic deviation in the fifth section and with the help of Newman–Penrose formalism in the sixth section. In section seven, we have discussed a possible way to check the validity of the model experimentally. In the last section, we conclude the paper with a very brief discussion of the results and the future aspects of the model in such type of studies.

2. Scalar and tensor fields from the model

The action of $f(R)$ gravity is given by

\[ S = \frac{1}{2\kappa} \int d^4x \sqrt{-\bar{g}} f(R), \]  

where $f(R)$ is a function of Ricci curvature $R$. The vacuum field equations obtained from this action are given by

\[ f'(R)R_{\mu \nu} - \frac{1}{2} f(R) g_{\mu \nu} - \nabla_\mu \nabla_\nu f'(R) + g_{\mu \nu} \Box f'(R) = 0, \]  

where $\Box = g^{\mu \nu} \nabla_\mu \nabla_\nu$ and the prime over $f(R)$ denotes the derivative with respect to $R$. Taking trace of vacuum field Eq. (2), we obtain

\[ f'(R) + 3 \Box f'(R) - 2f(R) = 0. \]  

Now, we assume that there is a propagating GWs in the spacetime, which will perturb the metric around its background value. Since the perturbation is usually very small, hence if we consider the background metric as $\bar{g}_{\mu \nu}$ then to the first order of perturbation value $h_{\mu \nu}$, we may express the spacetime metric as

\[ g_{\mu \nu} = \bar{g}_{\mu \nu} + h_{\mu \nu}, \]  

where $|h_{\mu \nu}| << |\bar{g}_{\mu \nu}|$.

In view of this perturbation, expanding the Ricci tensor and the Ricci scalar up to the first order of $h_{\mu \nu}$, we may write

\[ R_{\mu \nu} \simeq \bar{R}_{\mu \nu} + \delta R_{\mu \nu} + O(h^2) \]
\[ = \bar{R}_{\mu \nu} - \frac{1}{2} (\nabla_\mu \nabla_\nu h - \nabla_\mu \nabla_\nu h_{\mu \nu} - \nabla_\nu \nabla_\mu h_{\mu \nu}) \]
\[ + \Box h_{\mu \nu} + O(h^2) \]  

and

\[ R \simeq \bar{R} + \delta R + O(h^2) \]
\[ = \bar{R} - \Box h + \nabla_\mu \nabla_\nu h_{\mu \nu} - \bar{R}_{\mu \nu} h^{\mu \nu} + O(h^2), \]

where $\bar{R}$ is the de Sitter curvature. Thus, due to this perturbation the trace Eq. (3) can be rewritten as

\[ 3f''(\bar{R}) \Box \delta R + [f''(\bar{R}) \bar{R} - f'(\bar{R})] \delta R = 0. \]  

Using our model

\[ f(R) = \alpha R^n, \]  

Eq. (7) can be expressed in the following compact form:

\[ (\Box - m^2) \delta R = 0, \]  

where

\[ m^2 = \frac{(2 - n)}{3(n - 1)} \bar{R}. \]

Equation (9) is the conventional form of the scalar field equation (Klein–Gordon) with $\delta R$ as the scalar field, and hence, here, $m$ can be identified as the mass of this scalar field. Moreover, since this scalar field corresponding to $\delta R$
is due to ripple in spacetime, so the field can be considered as the quantized massive gravitational field.

From expression (10), we see that the mass term of the scalar field depends highly on the exponent term $n$ and the background curvature $\bar{R}$. This indicates that near the massive objects like neutron stars, etc., where the background curvatures are very large, the mass of the scalar form of gravitational field will increase, and consequently, this will slow down the propagation speed of the massive mode of GWs in such regions according to this particular model. Further, for a massive scalar field or massive mode of GWs $n$ should lie in between 1 and 2 beyond which it will result in tachyonic instabilities. When $n = 1$, the model will give the Einstein’s case. When $n$ starts to increase from 1 to 2, mass of the field decreases monotonically, and finally, for $n = 2$, the massive scalar mode of polarization will vanish, i.e., for the value $n = 2$, model will give a massless scalar field. The variation of the scalar field mass with respect to $n$ for the unit value of the background curvature is shown in Fig.1. One interesting observation from the mass term is that when $n$ approaches to 1, the mass of the scalar field increases very rapidly and for $n = 1$, the mass becomes infinite. Thus, for $n = 1$ the scalar mode vanishes and only the massless spin-2 modes propagate.

The tensor field Eq. (2) in vacuum for the power law model (8) takes the form:

$$nR^{a-1}R_{\mu\nu} - \frac{1}{2} \bar{g}_{\mu\nu} R^a - n\nabla_\mu \nabla_\nu R^{a-1} + n \bar{g}_{\mu\nu} \Box R^{a-1} = 0.$$  

(Perturbing this field equation around the de Sitter curvature $\bar{R}$ [see Eq. (5)], we may write,

$$n\bar{R}R_{\mu\nu} + n(n - 1)\bar{R}R_{\mu\nu} \delta R - \frac{1}{2} \bar{g}_{\mu\nu} \bar{R}^2 - \frac{1}{2} n \bar{g}_{\mu\nu} \bar{R} \delta R + nR \delta R_{\mu\nu} - n(n - 1)\nabla_\mu \nabla_\nu (\delta R) + n(n - 1)\bar{g}_{\mu\nu} \Box \delta R = 0.$$  

(12)

In the ideal case, i.e., when $R = \bar{R}$, the de Sitter curvature should satisfy the above field equation. Thus, for the ideal case the above equation gives

$$n\bar{R}_{\mu\nu} = \frac{1}{2} \bar{g}_{\mu\nu} \bar{R}.$$  

(13)

Trace of this equation gives

$$n\bar{R} = 2\bar{R},$$

which can be written as

$$n\bar{R}^2 - 2\bar{R}^2 = 0.$$  

(14)

Again the trace of Eq. (12) is,

$$n\bar{R}^2 - 2\bar{R}^2 + n(n - 2)\bar{R} \delta R + 3n(n - 1) \Box \delta R = 0.$$  

(15)

where $m^2 = \frac{(2 - n)}{3(n - 1)} \bar{R}$. This equation is identical to Eq. (9). For $n = 2$, above equation reduces to,

$$\Box \delta R = 0.$$  

(16)

This is the equation of massless scalar field for the case of $n = 2$. This is an interesting result, which shows that if the de Sitter curvature has to satisfy the field Eq. (12), then model (8) has to take the value $n = 2$; consequently, the massive scalar mode of polarization vanishes, and a pure massless scalar mode of polarization (also known as breathing mode of polarization) is obtained.

Thus, Eq. (16) can be treated as a stability point for the model in de Sitter space which identically satisfies the stability condition discussed in the following section without modifying the Hubble constant in de Sitter space.

Again, since $n = 2$ in model (8) (we’ll refer this model with $n = 2$ as pure $\bar{R}^2$ model) makes the scalar field massless and independent of the background curvature $\bar{R}$, so for simplicity, we take this advantage to choose $\bar{R} = 0$ and perturb the tensor field equation for pure $\bar{R}^2$ model [see Eq. (2)] to give

$$R \delta R_{\mu\nu} - \frac{1}{4} \bar{R}^2 \bar{g}_{\mu\nu} - \partial_\mu \partial_\nu \bar{R} + \bar{g}_{\mu\nu} \Box \bar{R} = 0.$$  

(17)

Now, we define a parameter,
Inflation may write spacetime having zero background curvature. The cosmological constant is added to the action corresponding to a term with $n = 0$ [32, 33]. But for any positive value of $n$, Minkowski space is a solution without any cosmological constant [33]. The stability condition for this model from [33] reads

$$\frac{2 - n}{n(n - 1)} \geq 0.$$  

(26)

This condition gives $1 < n \leq 2$ and $n < 0$. From Eq. (10), it can be easily seen that the first stability condition, i.e., $1 < n \leq 2$ agrees well with the non-tachyonic range of the scalar field. $n > 2$ and $n < 1$ in Eq. (10) will make $m^2$ negative resulting in tachyonic instabilities. However, the second stability condition obtained from Eq. (26), i.e., $n < 0$ in Eq. (10), can also result in tachyonic instabilities. To avoid such tachyonic instabilities in our toy model, we would not consider the situation satisfied by the later condition. Therefore, the model is considered to be stable in the range $1 < n \leq 2$ only. For $n = 2$, it is identically satisfied without imposing any constraints on the Hubble parameter in de Sitter space and also shows that for any values of $n$, the Minkowski space is stable [33]. Thus, our previous results are supported by these inferences.

### 4. Equivalence with the scalar tensor theory

A straightforward way to study $f(R)$ gravity is to see its equivalence with the scalar–tensor theory (STT) [34]. This is reasonable because as like $f(R)$ theories, STTs also have two types of polarization modes, tensor modes and scalar mode. By introducing a scalar field $\phi \equiv R$, the general form of action (1) can be given as

$$S = \frac{1}{2\kappa} \int d^4x \sqrt{-g} \left[ \psi(\phi)R - V(\phi) \right],$$  

(27)

where $\psi(\phi) = f'(\phi)$ and $V(\phi) = \phi f''(\phi) - f(\phi)$. Now, differentiating Eq. (27) with respect to $\phi$, we find

$$f''(R)(R - \phi) = 0.$$  

(28)

This shows that when $f''(R) \neq 0$, we must have $R = \phi$. Thus, $f''(R) \neq 0$ is a very important condition required to be satisfied in order to compare the $f(R)$ theory with the STT. The pure $R^2$ model satisfies this condition. In metric formalism, the STT with Brans–Dicke parameter $\omega = 0$ is equivalent to the $f(R)$ gravity.

Again action (27) can be written as

$$S = \frac{1}{2\kappa} \int d^4x \sqrt{-g} \left[ \psi(\phi)R - V(\phi) \right],$$  

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where $\psi(\phi) = f'(\phi)$ and $V(\phi) = \phi f''(\phi) - f(\phi)$. Now, differentiating Eq. (27) with respect to $\phi$, we find

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\[ S = \frac{1}{2\kappa} \int d^4x \sqrt{-g} [f(\phi) + f'(\phi)(R - \phi)]. \]  
(29)

The field equation obtained from the above action is given as
\[ G_{\mu\nu} = \frac{1}{f'(\phi)} \left[ \nabla_{\mu} \nabla_{\nu} f'(\phi) - g_{\mu\nu} \Box f'(\phi) + \frac{1}{2} g_{\mu\nu} f'(\phi) - \phi f'(\phi) \right]. \]

Trace of the above equation is
\[ \Box f' = \frac{2}{3} f'(\phi) - \frac{1}{3} \partial_{\phi} f'(\phi). \]

For the pure \( R^2 \) model, Eq. (30) gives
\[ G_{\mu\nu} = \phi^{-1} (\partial_{\mu} \phi \partial_{\nu} \phi - g_{\mu\nu} \phi - \frac{1}{4} g_{\mu\nu} \phi^2). \]

And from the trace of this equation, we get,
\[ \Box \phi = 0. \]

This equation is similar to Eq. (16) which gives a massless breathing mode of polarization. In Minkowski space \( \Box \phi \equiv \Box \phi \), since \( \Box \phi = 0 \) in this space, so the solution of this equation can be written as
\[ \partial_{\phi} \phi = \phi_0 \exp(ip_{\mu}x^{\mu}) + c.c., \]

where \( \tilde{g}_{\mu\nu} p^\mu p^\nu = 0 \). In case of nonzero background, a damping factor can appear in the solution. But the effect of this damping factor is very negligible, and we can safely neglect the term [35]. Hence, this equation is valid in both de Sitter spacetime and Minkowski spacetime. Now, combining the solutions for the massless tensor modes and the scalar mode, and considering that the GW is traveling along the \( z \)-axis with the speed of light \( c = 1 \), we get the solution for \( h_{\mu\nu} \) as
\[ h_{\mu\nu} = \tilde{h}_{\mu\nu}(t - z) - \tilde{g}_{\mu\nu} \partial_{\phi}(t - z). \]

Hence, for this solution we may take \( q_{\mu} = \Omega \{1, 0, 0, 1\} \) and \( p_{\mu} = \omega \{1, 0, 0, 1\} \). It should be noted that for \( n \neq 2 \) and \( n \neq 1 \), the above solution will contain a mixed state of massive scalar mode and breathing mode besides the tensor modes of polarization [17].

5. Geodesic deviation and polarization modes

The de Sitter spacetime can be expressed as a hypersurface in the host pseudo-Euclidean space with metric \( \eta_{AB} = (+1, -1, -1, -1) \) with points in Cartesian coordinates \( \chi^A \) satisfying \( \eta_{AB} \chi^A \chi^B = -l^2 \), where \( l \) is the de Sitter length parameter. The four-dimensional stereographic coordinates \( x^\mu \) are obtained through a stereographic projection from the de Sitter hypersurface into a target Minkowski spacetime [36]. Taking into account the local transitivity properties of spacetime, we consider a family of modified de Sitter geodesics that are able to connect any two points in spacetime and they are satisfied by the equation [37, 38]
\[ \frac{dU_{\mu}}{ds} - \Lambda_{\mu\rho} U_{\nu} U^\nu = 0, \]

(36)

where \( s \) is an affine parameter, \( u^\mu \) is the usual 4-velocity of particles, and \( \Lambda_{\mu\rho} \) is the Christoffel connection. \( U^\mu = \tilde{v}^\mu u^\nu \) is the anholonomic 4-velocity, which is related to the usual 4-velocity of particles via the Killing vector \( \tilde{e}_x \). The geodesic deviation equation obtained from Eq. (36) can be written as
\[ \frac{D^2 V^\mu}{Ds^2} \delta \lambda = R^\mu_{\nu\rho\sigma} U^\nu U^\rho \eta^\sigma + \frac{D}{Ds} \left[ (\eta^\mu u^\rho - \eta^\rho u^\mu) \nabla_{\nu} \tilde{e}_x \right], \]

(37)

where \( \frac{D}{Ds} = u^\sigma \nabla_\sigma \) represents the covariant derivative with respect to \( s \), \( V^\mu = \tilde{v}^\mu u^\nu \), \( \eta^\mu = \frac{\partial \tilde{x}^\mu}{\partial x^\nu} \delta \lambda = v^\mu \delta \lambda \) and \( \lambda \) is some parameter such that for each \( \lambda \) is constant, \( x^\nu = x^\nu(s, \lambda) \) will represent the equation of a geodesic in Eq. (36). This equation shows that geodesic deviation in de Sitter spacetime contains translational (first term on RHS) as well as rotational (second term on RHS) terms [39]. Since the second term on RHS is model independent as it does not contain Riemannian tensor term, the rotational effect is a generic property of the spacetime only. That is why, in Newtonian limit with \( l \to \infty \) we can safely neglect the second term on RHS and thus can have an approximated and simplified form of geodesic deviation equation with only first term on the RHS. Now to see the geodesic deviation in our model, we first substitute solution (34) in Eq. (32), which gives,
\[ R_{\mu\nu} = -\frac{1}{4} \tilde{g}_{\mu\nu} \partial_{\phi} \delta \phi - p_{\mu} p_{\nu}. \]

(38)

For the wave traveling along the \( z \) direction, the nonzero components of \( R_{\mu\nu} \) are \( R_{t\nu} \), \( R_{z\nu} \) and \( R_{\nu\nu} \). Now, to linear order we may write,
\[ R_{\mu\nu} \approx \frac{1}{2} \left( h_{\mu\nu,\phi\phi} + h_{\mu\phi,\nu\phi} - h_{\mu\phi,\nu\phi} - h_{\nu\phi,\mu\phi} \right). \]

This can be further simplified as
\[ R_{\mu\nu} = -\frac{1}{2} \left( h_{\mu\nu} + h_{\nu\mu} \right). \]

(39)

Thus, using only the scalar part of Eq. (35) we can write the simplified and approximated geodesic deviation equation from Eq. (37) in the limit \( l \to \infty \) as
\[ \ddot{x} = \frac{1}{2} \left( \delta_{ij} \ddot{\phi} - \ddot{\phi} \delta_{ij} \right) x', \]  

which gives,

\[ \ddot{x} = \frac{1}{2} \delta_{ij} \ddot{\phi} x, \]  

\[ \ddot{y} = \frac{1}{2} \delta_{ij} \ddot{\phi} y, \]  

\[ \ddot{z} = 0. \]  

Thus, the geodesic equations show that there is no longitudinal component of the GW polarization for the pure R^2 model. So, for this model besides the tensor modes there exists only a scalar mode which is massless and pure in nature, known as the breathing mode. Again, from Eq. (35) considering only the scalar part we may write,

\[ \ddot{\phi} = -\omega^2 \delta \phi. \]  

Using this equation in Eq. (41), we have,

\[ \ddot{x} + \frac{1}{2} \omega^2 \delta \phi x = 0, \]  

which for a GW propagating along z-axis takes the form:

\[ \ddot{x} + \frac{1}{2} \phi_0 \omega^2 \cos \omega(t - z)x = 0. \]  

The solution of this Eq. (46) gives the time variation of deviation of x at some fixed z. This equation is a special form of the well-known Mathieu’s equation [40, 41], and the solution is graphically shown in Fig. 2 for two different set of parameters. It is seen that the time period of geodesic deviation depends highly on \( \omega \). The results are identical along the y direction also as clear from Eq. (42).

6. Polarization modes with Newman–Penrose formalism

The Newman–Penrose (NP) formalism [42] can be used to find out the different polarization modes of GWs in a model. However, one major drawback of NP formalism is that it is only applicable to null waves. In f(R) theory metric formalism, usually GWs have massive longitudinal mode of polarization due to which the NP formalism fails and shows deviated results [17]. But in the particular case of pure R^2 model, we have already seen that the scalar field is massless and thus this allows us to use the NP formalism in the study of polarization modes of GWs in the theory. In NP formalism, a GW is described with the help of six amplitudes \( \{\psi_2, \psi_3, \psi_4, \phi_{22}\} \) representing six polarization modes in a particular coordinate system or frame [42]. All these amplitudes are defined as [43]:

\[ \psi_2 = -\frac{1}{6} R_{ztzt}, \]
\[ \psi_3 = -\frac{1}{2} R_{ztzt} + \frac{1}{2} i R_{zstz}, \]
\[ \psi_4 = -R_{ztzt} + R_{zstz} + 2iR_{zstz}, \]
\[ \phi_{22} = -R_{ztzt} - R_{zstz}. \]  

Each of the complex amplitudes \( \psi_3 \) and \( \psi_4 \) is actually equivalent to two real amplitudes [43]. In Brans–Dicke theory, the massless scalar field appears as the breathing

Fig. 2 Geodesic deviation along x for a fixed z. The top plot is for \( \omega = 1.5, \phi_0 = 0.02 \) and \( z = 1 \), and the bottom is for \( \omega = 1, \phi_0 = 0.2 \) and \( z = 1 \). All parameters are in arbitrary units.
mode showing \( \phi_{22} = -R_{xxt} - R_{yty} \neq 0 \). For our model, these amplitudes are found as
\[
\psi_2 = 0,
\psi_3 = 0,
\psi_4 = \tilde{h}_{xx} - \tilde{h}_{yy} + i(2 \tilde{h}_{xy}),
\phi_{22} = -2 \delta \phi.
\] (48)

These results show that our model, i.e., pure \( R^2 \) model, gives three modes of massless polarization: two for tensor modes and one for the breathing mode as found earlier, and according to Lorentz-invariant \( E(2) \) classification of plane waves, the model results in GWs of class \( N_3 \) and all modes are independent of the observer [43]. It should be mentioned that a similar study was done in Ref. [44] for the same toy model in Palatini formalism. In this case, using the NP formalism one can obtain,
\[
\psi_2 = 0,
\psi_3 = 0,
\psi_4 = \tilde{h}_{xx} - \tilde{h}_{yy} + i(2 \tilde{h}_{xy}),
\phi_{22} = 0.
\] (49)

This shows that in Palatini formalism, the model \( f(R) = \alpha R^2 \) gives only tensor modes of polarization of GWs and there is no scalar mode of polarization or scalar degrees of freedom.

7. Pulsar timing arrays and correlation of polarization modes

Pulsar timing arrays (PTAs) can be used to detect the polarization modes of GWs. A good number of works have been going on in this field, and PTAs are found to be effective in the detection of extra polarization modes to be present in GWs. Hence, they can be used as a tool to test modified theories of gravity [45–47].

The presence of GWs disturbs the null geodesic of the signals from pulsars. Due to this reason, the time of arrival of pulsar signal changes. So, by tracing the changes in time of arrival of the radio signals from the pulsars, it is possible to detect the GWs. For the mathematical treatment of the PTA procedure, let us consider a PTA detecting radio signals in the regime of GWs. In case of GWs from the pure \( R^2 \) model, the information about the source is carried by the three polarization amplitudes, viz. \( h_+(t), h_\times(t) \) and \( h_\phi(t) \), where the first two stand for the Einstein or tensor modes and the third one stand for scalar (breathing) mode of polarization. Thus, the GW signal from a source for the pure \( R^2 \) model can be written as
\[
h_{ij}(t, \Omega) = e^{+}_\Omega(h_+(t, \Omega) + e^{-}_\Omega h_\times(t, \Omega) + e^{0}_\Omega h_\phi(t, \Omega)).
\] (50)

Here, \( e^{+}_\Omega \) and \( e^{-}_\Omega \) are the polarization tensors as given by,
\[
e^{+}_\Omega = \hat{m}_i \hat{m}_j - \hat{n}_i \hat{n}_j,
\]
\[
e^{-}_\Omega = \hat{m}_i \hat{n}_j + \hat{n}_i \hat{m}_j.
\]

Hence, these polarization tensors are uniquely defined once the unit vectors \( \hat{m} \) and \( \hat{n} \) used to describe the principal axes of wave are specified. \( \Omega \) is the direction of GW propagation given by \( \hat{m} \times \hat{n} \). As per our previous convention, we set \( \Omega = \hat{z} \) and consider a null vector \( S^\rho \) that points to the solar system barycenter from the pulsar in Minkowski spacetime. In perturbed spacetime vector, \( S^\mu \) will change to another vector \( \sigma^\mu \) as given by,
\[
\sigma^\mu = S^\mu - \frac{1}{2} \eta^{\mu
u} h_{\rho \nu} S^\rho + \delta \phi \eta^{\mu \nu} b_{\rho \nu} S^\rho,
\] (51)

where the second part of this equation is due to tensor modes of perturbation and the third part is due to the scalar mode of perturbation in spacetime with \( b_{\rho \nu} \) being a unit breathing mode matrix having two nonzero unit components \( b_{11} \) and \( b_{22} \), obtained by the application of transverse condition to this mode of GWs. It is to be noted that for this mode, we cannot apply the traceless condition as the application of this condition to this mode will retain only the tensor modes or GR modes [44]. Now, if we define \( S^\mu \) as \( S^\mu = v(1, -x, -\beta, -\gamma) \), where \( x, \beta \) and \( \gamma \) are direction cosines, and \( v \) is the frequency of the radio pulses from the source, then from Eq. (51), we may write,
\[
\sigma^t = v,
\]
\[
\sigma^x = -v \left[ x \left( 1 - \frac{1}{2} \beta h_+ - \delta \phi \right) - \frac{\beta}{2} h_\times \right],
\]
\[
\sigma^\phi = -v \left[ \beta \left( 1 + \frac{1}{2} \beta h_+ + \delta \phi \right) - \frac{\beta}{2} h_\times \right],
\] (52)
and \( \sigma^\gamma = -v \gamma \).

The radio pulses from the pulsar follow a null geodesic through spacetime. The geodesic equation of the pulses with the affine parameter \( \lambda \) is
\[
\frac{dv}{d\lambda} = - v^2 x \beta \dot{h}_\times + \frac{1}{2} v^2 \left[ \beta^2 (\dot{h}_+ - \delta \phi) - 2x^2 (\dot{h}_+ + \delta \phi) \right].
\] (53)

Using this equation in the geodesic deviation equation, we can have the frequency shift at solar system barycenter as given by,
where \( z(t, \Omega) = |v(t) - v_0| \)
\[
= \frac{1}{2(1 + \gamma)} \left[ x^2(\Delta h_+ + \Delta \delta \phi) - \beta^2(\Delta h_+ - \Delta \delta \phi) \right]
+ \frac{\alpha \beta}{(1 + \gamma)} \Delta h_x,
\]
(54)

where \( v(t) \) is the frequency observed at solar system barycenter. Thus, it is seen that the presence of GWs will give rise to a frequency shift and these shifts can be observed with the help of PTAs. It is also clear that the presence of massless breathing mode results in a contribution of a shift in the frequency besides a contribution from the usual tensor (GR) modes. Following [45], for stochastic GW background with normalized frequency \( f \), the correlation function for GR modes and breathing mode of polarization are calculated. This function for tensor modes is found as
\[
C^{+, \times}(\theta) = \zeta^{GR}(\theta) \int_0^\infty \frac{|h^{+\times}_s|^2}{24\pi^2f^3} df,
\]
(55)

where
\[
\zeta^{GR}(\theta) = \frac{3}{4} \frac{1 - \cos \theta}{\log \left( \frac{1 - \cos \theta}{2} \right)}
+ \frac{1}{2} \frac{1 - \cos \theta}{8} + \frac{\delta(\theta)}{2},
\]
and \( \theta \) is the angular separation between two pulsars. For the scalar modes, it is
\[
C^b(\theta) = \zeta^b(\theta) \int_0^\infty \frac{|h^b_s|^2}{12\pi^2f^3} df,
\]
(56)

where
\[
\zeta^b(\theta) = \frac{1}{8} \left[ \cos \theta + 3 + 4 \delta(\theta) \right].
\]

The correlation coefficients \( \zeta^{GR} \) and \( \zeta^b \) as a function of \( \theta \) are plotted in Fig.3. It is seen that provided the cases in [45] hold good, PTAs can effectively distinguish between the breathing mode and tensor modes present in GWs. It needs to mention that the correlation versus angular separation curve for the tensor modes of polarizations was first obtained by Hellings and Downs in 1983, and hence, it is usually known as Hellings–Downs (HD) curve [48, 49].

The calculations of correlation functions are model independent, but for different polarization modes, there are different correlation functions [47]. Since our toy model \( f(R) = 2R^n \) gives only tensor plus, cross and massless breathing modes of polarization of GWs, we have included the correlation functions for tensor modes and massless breathing mode only. This provides an experimental way to test the viability of the toy model. However, in other \( f(R) \) gravity models (in metric formalism), apart from the tensor plus and cross modes there exists a scalar polarization mode which is a mixed state of massive longitudinal and massless transverse breathing mode [17, 44, 50]. Therefore, in those models, the experimental viability can be checked by considering three correlation functions for tensor modes, massless transverse breathing mode and massive longitudinal mode. In the near future, with increased sensitivity and sufficient data, the experimentally obtained correlation functions from pulsar timing array data can hopefully help us to distinguish between different polarization modes which will provide us a way to check the viability of the toy model.

8. Conclusions

In this work, we have used the \( f(R) \) gravity power law model to study the polarization modes of gravitational waves in de Sitter spacetime. We have found that the field equations in the de Sitter spacetime show the existence of a massive scalar mode with a mass term \( m^2 = \frac{2 - n}{(n - 1)} R \). The mass term varies widely with the exponent term \( n \) and the background curvature or the de Sitter curvature, and the mass of the scalar field becomes zero when background curvature is zero or \( n = 2 \). Later using the stability condition for the theory in de Sitter spacetime, we have seen that for a constant curvature \( R \), the theory has stable solutions for \( n = 2 \). It has been observed that for this particular
case of the model, the massive longitudinal mode of polarization vanishes. Thus, this is the only case in $f(R)$ theory in metric formalism treatment where the third scalar mode is a pure breathing mode and there is no massive longitudinal mode present. To validate this result, we have studied the geodesic deviations for the pure $R^2$ model explicitly and later, we used the NP formalism to confirm the validity of our result. The absence of the massive scalar field in this model allows us to use the NP formalism, which is a powerful tool to check the polarization contents of null GWs. The results from the analysis show the absence of massive polarization mode of the GWs in the theory, which establish that the polarization modes and the mass of the scalar field in $f(R)$ gravity are model dependent. Another important result obtained from this work is that the scalar field is independent of the background curvature in pure $R^2$ model of $f(R)$ gravity and evidently chameleonic behavior is not observed here. The absence of massive longitudinal mode makes this pure $R^2$ model different from the other $f(R)$ gravity models, and hence, a more detail study in this model is required. This model passes the cosmological bounds from GW170817 [51].

In a recent study [52], it was reported that due to the screening mechanism of the atmosphere, it might be difficult to observe the scalar mode of polarization at ground-based GWs detectors. However, this can be possible only if the model shows chameleonic behavior which makes the mass of the scalar field background curvature dependent. In case of power law model, Eq. (10) shows the background curvature dependency of the additional scalar field associated with the theory. The screening effect of the atmosphere due to chameleonic behavior can be measured if it is possible to detect the scalar modes of such a theory both at ground-based detectors and space-based detector. But the pure $R^2$ model does not have any massive longitudinal mode of polarization, and the associated scalar field in this case is massless in nature. Due to this reason, the atmospheric screening cannot be seen in this case.

The results of this study are also supported by another very recent study [53], in which the authors studied the GWs in teleparallel gravity explicitly. Their study shows that there are three degrees of freedom associated with teleparallel gravity and the third one is a scalar degrees of freedom. There are two scalar modes of polarizations, viz. massive longitudinal mode and massless breathing mode in a mixed state in the theory. The authors have also mentioned that according to dynamics $f(T, B) \equiv f(R)$. Thus, $f(R)$ gravity shares the same results for polarization modes with $f(T, B)$ gravity [17, 44, 53]. In this study, we have shown that for a particular case, the mixing of longitudinal mode and breathing mode might not be there and the massive longitudinal mode vanishes. Similar results can be expected from $f(T, B)$ gravity also, where there would be only three polarization modes.

It is worth to mention that $f(R)$ gravity with extra degrees of freedom can affect the cosmological dynamics [54, 55]. Existence of extra polarization modes of GWs can impose significant effect on the stochastic cosmological background. The amplitude of GWs generated from inflation also depends on the choice of $f(R)$ gravity model used for the study [54]. In Ref. [55], using $f(R)$ gravity power law model, it was shown that the amplitude evolution of the tensor mode of GWs depends on the cosmological background. However, it will be very premature to comment on the cosmological background due to variation of polarization modes of GWs. We believe a further study will shed more light on this.

The pure $R^2$ model can be extended by adding other terms to the action which can account for the missing part giving rise to mixed polarization states of GWs besides tensor modes of polarization [17, 44]. Those extended models can be constrained with the experimental results obtained so far [56]. It is expected that future experiments can provide better constraints to the $f(R)$ gravity models using which the existence of extra polarization modes can be hopefully confirmed, which in turn give options to test the reliability of $f(R)$ gravity in the modifications and extensions of GR. Extensions like non-minimal matter field coupling and other modified gravity theories can be included as the future aspects of this type of works, which can be tested with future experiments on GWs.

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