The Trajectory PHD Filter for Jump Markov System Models and Its Gaussian Mixture Implementation

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Abstract—Based on the recently proposed set of trajectories, the trajectory probability hypothesis density (TPHD) filter is capable of producing trajectory estimates in first principle without adding labels or tags. In this paper, we propose a robust TPHD filter to track maneuvering targets through compatible the jump Markov system (JMS) model that the highly dynamic targets movement switches between multiple models, referred as JMS-TPHD filter. Firstly, we extend the concept of JMS to set of trajectories and derive the recursion for the proposed filter. Then, we develop the linear Gaussian Mixture (LGM) implementation of JMS-TPHD recursion and also consider the L-scan approximations of the implementations for computational efficiency. Finally, in a challenging multiple maneuvering targets tracking scenario, the simulation results demonstrate the performance of the JMS-TPHD filter.

Index Terms—trajectory probability hypothesis density (TPHD) filter, jump Markov system (JMS), set of trajectories, maneuvering target, Gaussian mixture

I. INTRODUCTION

Multiple maneuvering target tracking (MMTT) not only contains the main problems in multi-target tracking (MTT): data correlation uncertainty, detection uncertainty, noise and clutter, but also confronts the motion uncertainty, for estimating the time-varying number of targets and their individual states from a set of observations. Therefore, MMTT is a very challenging research field and has attracted numerous works [3]–[6], [9]–[16].

The jump Markov system (JMS) approach, in which the target state is augmented with an additional motion model variable and the model evolves with time according to a finite state Markov chain [2], is a popular approach for maneuvering targets tracking [1]–[3]. To track multiple maneuvering targets, the JMS approach has been combined with the traditional MTT algorithms such as the joint probabilistic data association (JPDA) [4], [5] and the multiple hypothesis tracking (MHT) [6]. In addition, as an advanced tool for MTT, the random finite set (RFS) approach [7], [8] also has been adopted to formulate JMS extensions of PHD [9], GM-PHD [10], CPHD [11], [12], multi-Bernoulli [13], LMB [14] and GLMB [15], [16] filters.

However, the above-mentioned filters [4]–[6], [9]–[16] are not “tracker” as their filtering output is target point trace rather than trajectory, although they can generate the trajectory by labels or tags. Thus, the recent principle approach of forming trajectories is attractive. To date, there are two major solution paradigms have been emerged: multi-target state sequence posterior [17] and set of trajectories/trajectory RFS [18]–[20]. In the formulation of multi-target state sequence posterior, the multi-scan generalized labeled multi-Bernoulli (GLMB) filter [17] shows the excellent multi-target tracking performance comparing with the GLMB filter [21], an analytic solution to the multi-target Bayes filter. On the other hand, the trajectory RFS approach, which is computationally efficient, establishes complete trajectory RFS model and presents the Bayesian multi-trajectory recursion. Subsequently, there has been some trackers originated from the trajectory RFS approach: the TPHD/TCPHD [18], the trajectory MB (TM-B) [22], the trajectory PMB (TPMB) [23] and the trajectory PMBM (TPMBM) [20] filters.

Considering the trajectory RFS approach, the TPHD filter [18] is capable to propagate a Poisson multi-trajectory density through the filtering recursion using Kullback-Leibler divergence (KLD) minimisations. The closed-form solution of the filter is also presented in [18] based on the linear Gaussian model. However, in the standard TPHD filter, all targets are assumed to adopt a single motion model, which is obviously invalid to MMTT, whose motion obeys the jump Markov system (JMS) model, i.e., the highly dynamic targets movement switches between multiple models.

In this paper, we generalize the concept of JMS to the trajectory RFS formulation. Combined with the JMS model, we present a new TPHD filter to robustly track the trajectories of maneuvering targets at each time step, named JMS-TPHD filter. Besides deriving the JMS-TPHD recursion, we also develop the LGM implementation in which case the JMS-TPHD filter can be implemented in an analytic closed form. To reduce the computational burden caused by the increase in the length of the trajectory over time in the implementation, the L-scan approximation technique [18] is employed that propagates the joint density of the last L time of trajectories and keeps the rest unaltered. Finally, the simulation results verify the accurate trajectory tracking performance of the JMS-TPHD filter in a multiple maneuvering targets tracking scenario.

This paper is organized as follows. Section II provides the background materials including the trajectory RFS, Bayesian
multi-trajectory recursion and TPHD filter. The JMS, JMS-TPHD recursion, LGM implementation and L-scan approximation are described in Sections III. Section IV exhibits the simulation results and the conclusions are drawn in Section V.

II. BACKGROUND

In this section, we briefly review trajectory RFS, Bayesian multi-trajectory recursion and the TPHD filter [18], which are the background materials of this paper to track multiple maneuvering targets.

A. Trajectory RFS

From the trajectory state modeling [18], [19], a single trajectory kinematic state is represented as a variable \( X = (\beta, x^{1:l}) \), where \( \beta \) is the birth time, \( l \) is the length and \( x^{1:l} = (x^1, \cdots, x^l) \) denotes the continuous states sequence, \( x \in \mathbb{R}^{n_x} \) is the single target kinematic state. Then, the trajectory state space at time \( k \) can be denoted as

\[
T_k = \bigcup_{(\beta, l) \in J_k} \{ \beta \} \times \mathbb{R}^{n_x},
\]

where \( J_k = \{ (\beta, l) : 0 \leq \beta \leq k, 1 \leq l \leq k - \beta + 1 \} \), \( \cup \) denotes disjoint set union.

Different from the straightforward multi-target state sequence \( x^{1:l} \) [17], the trajectory state additionally considers the birth time and the length of the trajectory. This means that given \( X \in T_k \), the trajectory state density can be expressed as a conditional probability

\[
p(X) = p(x^{1:l}|(\beta, l)) P(\beta, l),
\]

where \( (\beta, l) \in J_k \). Consequently, the integral of trajectory state density can be written as

\[
\int p(X) dX = \sum_{(\beta, l) \in J_k} P(\beta, l) \int p(x^{1:l}|(\beta, l)) dx^{1:l}.
\]

Similar to the set of targets [7], the set of trajectories at time \( k \) is defined as

\[
X_k = \{ X = (\beta, x^{1:l}) \in T_k \},
\]

the corresponding state space is \( \mathcal{F}(T_k) \), where \( \mathcal{F} \) denotes the set of all finite subsets.

Poisson Trajectory RFS: For a Poisson trajectory RFS \( X \), its cardinality distribution \( |X| \) is Poisson with mean \( \lambda_\tau \), and the elements of \( X \) are independently and identically distributed (i.i.d.) according to the probability density \( \tilde{v}(\cdot) \), where \( \int \tilde{v}(X) dX = 1 \).

The multi-trajectory density for a Poisson trajectory RFS is given by [18]

\[
\pi(X) = \pi(|X_1, \cdots X_n|) = e^{-\lambda_\tau} \lambda_\tau^n \prod_{i=1}^n \tilde{v}(X_i),
\]

and its cardinality distribution is

\[
\rho(n) = \frac{1}{n!} e^{-\lambda_\tau} \lambda_\tau^n.
\]

1In the tracking system, the trajectory length refers to the number of continuous scanning frames of same target rather than the physical length with a definite unit.

B. Bayesian Multi-trajectory Recursion

Let \( \pi_{k-1}(X_{k-1}) \) represent the multi-trajectory posterior density at time \( k - 1 \). Then, the multi-trajectory posterior density \( \pi_k(X_k) \) can be calculated via Bayesian multi-trajectory recursion as follows [19]

\[
\pi_{k|k-1}(X_k) = \int f_{k|k-1}(X_k|X_{k-1}) \pi_{k-1}(X_{k-1}) \delta X_{k-1},
\]

\[
\pi_k(X_k) = \frac{g_k(z_k|X_k) \pi_{k|k-1}(X_k)}{\int g_k(z_k|X_k) \pi_{k|k-1}(X_k) \delta X_k},
\]

where \( \pi_{k|k-1}(X_k) \) is the predicted multi-trajectory density at time \( k \), \( z_k \) is the set of measurements produced by the trajectory RFS \( X_k \) at time \( k \), \( f_{k|k-1}(|\cdot|) \) is the multi-trajectory transition kernel and \( g_k(\cdot|\cdot) \) is the trajectory-measurement likelihood density.

C. TPHD Filter

The probability hypothesis density (PHD) [7], [8] that represents the first-order statistical moment of multi-trajectory density \( \pi(X) \), is defined as [18]

\[
D_\pi(X) = \int \pi \{ X \} \cup \delta X,
\]

where the set integral [7] of a given real-valued function \( \partial(\cdot) \) for trajectory RFS \( X \) is

\[
\int \partial(X) \delta X = \sum_{n=0}^{\infty} \frac{1}{n!} \int \partial(\{X_1, \cdots, X_n\}) dX_{1:n}.
\]

Combining (5), (9) and (10), the PHD of Poisson multi-trajectory density is derived as \( D_\nu(X) = \lambda_\nu \tilde{v}(X) \) [7].

Instead of propagating the best Poisson approximation for the multi-trajectory density straightforwardly, the TPHD filter recursively propagates its posterior intensity (PHD), in the sense of minimizing the KLD. The prediction and update of TPHD filter are briefly introduced in following Theorem 1 and Theorem 2, respectively. More details can be found in [18].

Theorem 1. Given the posterior PHD \( D_{k|k-1}(X) \) at time \( k - 1 \), the predicted PHD \( D_{k|k-1}(X) \) at time \( k \) as follows

\[
D_{k|k-1}(X) = D_{\gamma,k}(X) + D_{\zeta,k}(X),
\]

where

\[
D_{\gamma,k}(\beta, x^{1:l}) = D_{\gamma_\tau}(k, x^1),
\]

\[
D_{\zeta,k}(\beta, x^{1:l}) = P_S(x^{l-1}) t(x^l|x^{l-1} D_{k-1}(\beta, x^{l-1})),
\]

where \( \gamma_\tau(\cdot) \) is the Poisson multi-target density of new born trajectories, the subindex \( \tau \) represents the density of target RFS, \( P_S(x^{l-1}) \) is the surviving probability of a trajectory and \( t(\cdot|x^{l-1}) \) is the state transition density\(^2\) of a surviving trajectory.

\(^2\)The state transition density of a trajectory is equal to that of the target at the last moment of the same trajectory, i.e., \( t(x^1|x^{l-1}) = t(x^l|x^{l-1}) \).

Besides, there are some other similar designations, such as \( \ell(\cdot|x^l) \).
As shown in (13), to keep the trajectory information, the TPHD filter does not integrate out past states, while the PHD filter [24] does.

Theorem 2. Given the predicted PHD $D_{k|k-1}(X)$, the updated PHD $D_k(X)$ at time $k$ is

$$D_k(X) = D_k\left(\beta, x^{1;l}\right) = D_{k|k-1}\left(\beta, x^{1;l}\right) \times$$

$$\left(1 - P_D(x^l) \sum_{z \in \mathcal{O}} \kappa(z) + \int P_D(x^l) f(z|x^l) D_{k|k-1}(x^l) dx^l\right),$$

(14)

where $P_D(x^l)$ is detection probability of a trajectory, $f(\cdot|x^l)$ is the measurement likelihood function of a trajectory and $\kappa(\cdot)$ is the intensity of Poisson clutter. $D_{k|k-1}(x^l)$ denotes the PHD of the targets at time $k$, which is defined as [18]

$$D_{k|k-1}(x^l) = \sum_{\beta=1}^k \int D_{k|k-1}(\beta, x^{1;l}) dx^{1;l-1}.$$  

(15)

In this paper, we only present the result of alive trajectories, which means that $l = k - \beta + 1$ in Theorem 1 and Theorem 2. In addition, a more general case are described in [18], in which the dead trajectories are also considered.

Although the standard TPHD filter has better tracking performance than the cognate PHD filter [24] in theory and can estimate the trajectory state in principle without labels or tags, it only considers the ideal situation that the target motion model is single. Therefore, the TPHD filter is unavailable for maneuvering target tracking whose dynamic targets motion switches between multiple models.

III. JMS-TPHD FILTER

In this section, by adopting the classical JMS model, we propose a new TPHD filter to track robustly maneuvering targets, referred as JMS-TPHD filter. The JMS model of trajectory RFS is described in Section III-A. We then derive the JMS-TPHD recursion in Section III-B. Finally, the linear Gaussian mixture implementation and the L-scan computationally efficient approximation is developed in Section III-C, III-D, respectively.

For multiple maneuvering target tracking, we indicate an additional variable $o \in \mathcal{O}$ as the label of motion model, where $\mathcal{O}$ represents the discrete space of all possible modes. Thus, the single trajectory state is defined as an augmented vector $\bar{X} = (X, o) = (\beta, x^{1;l}, o^l) \in \mathcal{T} \times \mathcal{O}$, where the mode $o$ means the motion model of the trajectory at current time. The augmented trajectory RFS is denoted as

$$\bar{X} = \{\bar{X} = (\beta, x^{1;l}, o^l) \in \mathcal{T} \times \mathcal{O}\}.$$  

(16)

As a result, the tracking algorithm based on the augmented trajectory RFS can estimate the trajectory state sequence and the single frame motion model of maneuvering targets at each time step.

A. Jump Markov System

In the JMS, the motion models of all targets change with time according to finite-state first order Markov chain. Let $v(o^l|o^{l-1})$ denotes the model switch probability from motion model $o^{l-1}$ to motion model $o^l$. Then, the sum of the switch probabilities of all possible motion model given motion model adds up to 1, i.e., $\sum_{o^l \in \mathcal{O}} v(o^l|o^{l-1}) = 1$.

Considering the real application scenario in which the motion model switching and the state transition are independent, the transition probability of augmented trajectory state is denoted as

$$f(\bar{X}|\bar{X}') = f(X, o|X', o') = t(x^{1;l}|x^{1;l-1}, o^l) v(o^l|o^{l-1}),$$

where $\bar{X}' = (X', o') = (\beta, x^{1;l-1}, o^{l-1})$. Similarly, the measurement likelihood function is generally independent of motion model, therefore, we express the trajectory-measurement likelihood function as

$$g(z|\bar{X}) = g(z|X, o) = g(z|X).$$

(18)

From the TPHD filter, the state transition function and measurement likelihood function are equal to that of the target at the last moment of the same trajectory, shown as (13),(14). Consequently, we can abbreviate equations (17) and (18) as

$$\begin{cases}
    f(\bar{X}|\bar{X}') = t(x^{1;l}|x^{1;l-1}, o^l) v(o^l|o^{l-1}) \\
    g(z|\bar{X}) = f(z|x^l) \ell(z|x^l)
\end{cases}.$$  

(19)

Note that the key of this abbreviation (19) is that the motion model switching is required to satisfy the first order Markov chain, that is, $v(o^l|o^{l-1}) = v(o^l|o^{l-1})$. Meanwhile, it is conventional that the state transition conforms to the first order Markov process [25] based on the target kinematics.

B. The JMS-TPHD Filter Recursion

Combined with JMS model, the standard TPHD is improved to the JMS-TPHD filter for multiple maneuvering target tracking and the recursive details of the JMS-TPHD filter are described in following Propositions 1 and 2.

**Prediction:** Compared with the TPHD prediction, the JMS-TPHD prediction step still adopts the assumptions P1-P3 in [18], but the kinematic state requires to be replaced with augmented state and the state transition density is augmented with motion model as (19).

**Proposition 1.** Given the posterior PHD $D_{k-1}(\bar{X}) = D_{k-1}(\beta, x^{1;l-1}, o^{l-1})$ at time $k - 1$, the predicted PHD $D_{k|k-1}(\bar{X}) = D_{k|k-1}(\beta, x^{1;l}, o^l)$ at time $k$ is given by

$$D_{k|k-1}(\bar{X}) = D_{\gamma,k}(\bar{X}) + D_{\zeta,k}(\bar{X}),$$

(20)

where

$$D_{\gamma,k}(\bar{X}) = D_{\gamma}(k, x^l, o^l),$$

(21)

$$D_{\zeta,k}(\bar{X}) = P_S(x^l, o^l) \times \sum_{o^{l-1} \in \mathcal{O}} t(x^l|x^{l-1}, o^l) v(o^l|o^{l-1}) D_{k-1}(\bar{X}).$$

(22)
Similar to the TPHD filter, the predicted PHD contains the PHD of the newborn augmented trajectories and the PHD of the surviving augmented trajectories. As shown by the summation in (22), the JMS-TPHD prediction does not retain the motion model information as the augmented trajectory just takes into account the single frame motion model, though the past states of trajectory is kept.

**Update:** Compared with the TPHD update, the JMS-TPHD update step similarly adopts the assumptions U1-U3 in [18]. However, besides the kinematic state is replaced with augmented state, the integral on state space is modified to be divided into the integral on trajectory state space and the summation on discrete motion model space.

**Proposition 2.** Given the predicted PHD $D_{k|k-1}(\bar{x})$ at time $k$, the posterior PHD $D_k(\bar{x}) = D_k(\bar{x}; x^{1:t}, o^t)$ at time $k$ is given by

$$D_k(\bar{x}) = D_{\text{mis},k}(\bar{x}) + D_{\text{det},k}(\bar{x}), \quad (23)$$

where

$$D_{\text{mis},k}(\bar{x}) = (1 - P_D(x^t, o^t)) D_{k|k-1}(\bar{x}), \quad (24)$$

$$D_{\text{det},k}(\bar{x}) = D_{k|k-1}(\bar{x}) \sum_{\xi \in \mathcal{X}_k} P_D(x^t, o^t) \ell(z|x^t), \quad (25)$$

$$\varepsilon = \sum_{\omega \in \mathcal{O}} \int P_D(x^t, o^t) \ell(z|x^t) D_{k|k-1}(x^t, o^t) dx^t, \quad (26)$$

$$D_{k|k-1}(x^t, o^t) = \sum_{\beta = 1}^k \int D_{k|k-1}(\beta, x^{1:t}, o^t) dx^{1:t-1}. \quad (27)$$

Similar to the TPHD filter, the update PHD consists of the missed augmented trajectories PHD and the detected augmented trajectories PHD, refer to (23). Since the trajectory state is augmented with the single frame motion model, the JMS-TPHD update step can not only estimate the trajectory state sequence, same as TPHD filter, but also can update the motion model at current time.

Proposition 1 and 2 show how the posterior intensity of augmented Poisson trajectory RFS is propagated with time in the JMS-TPHD filter, using the JMS model. Compared with the TPHD filter [18], the JMS-TPHD filter can additionally each frame motion model of maneuvering targets. Furthermore, compared with the MM-PHD filter [9], [10] for multiple maneuvering target tracking, the JMS-TPHD filter has higher target tracking accuracy which is the inherent advantage of the filters based trajectory RFS.

**C. LGM Implementation**

In this subsection, we present the analytic closed-form solution for the JMS-TPHD filter based on linear Gaussian model, named as LGM-JMS-TPHD filter. In the LGM-JMS-TPHD filter, the PHD is represented as a Gaussian mixture (GM) form and the GM form remains unchanged in the recursion process.

1) **Linear Gaussian Model for Augmented Trajectory RFS:**

According to the common linear Gaussian model [24], we can express the PHD of an augmented trajectory as a Gaussian density

$$N(\bar{X}; \bar{\beta}_k, m_k(o), U_k(o)) = \delta_{\beta_k}(\beta) \delta_k(l) \times N(x^{1:t}; m_k(o), U_k(o)),$$ \quad (28)

where $l_k = \text{dim}(m_k(o)/n_x)$, $n_x$ denotes the dimension of target state vector, $m_k(o) \in \mathbb{R}^{k(0)n_x}$ and $U_k(o) \in \mathbb{R}^{k(0)n_x \times k(0)n_x}$ are the mean and covariance of the augmented trajectory Gaussian density. Then, with the same spirit as the Gaussian sum filter [26], the PHD for augmented trajectory RFS can be written as a GM form, such as the new born trajectory RFS:

$$D_{\gamma,k}(\bar{x}) = \sum_{j = 1}^{J_{\gamma,k}} \omega_{\gamma,k}^j(o^k) N(\bar{x}; k, m_{\gamma,k}^j(o^k), U_{\gamma,k}^j(o^k)); \quad (29)$$

where $J_{\gamma,k} \in \mathbb{N}$ is the number of Gaussian components for a single motion model $o^k$, $\omega_{\gamma,k}^j$ is the weight of the $j$th component.

In addition, in the linear Gaussian model, we also assume that, i. The survival probability and detection probability are constants, $P_S(x, \omega) = P_S$, $P_D(x, \omega) = P_D$; ii. Both the transition density and measurement likelihood are linear Gaussian,

$$t(x^{[j]}; F(o), Q(o)), \quad s(z|x^{[j]}) = N(z; Hx^{[j]}, R). \quad (30)$$

where $F \in \mathbb{R}^{n_x \times n_x}$ is the transition matrix, $Q \in \mathbb{R}^{n_x \times n_x}$ is the covariance matrix of process noise and $F$, $Q$ depend on the mode of target. $H \in \mathbb{R}^{n_x \times n_x}$ is the measurement matrix and $R \in \mathbb{R}^{n_x \times n_x}$ is the covariance matrix of measurement noise.

2) **LGM-JMS-TPHD filter:** Applying the linear Gaussian model for augmented trajectory RFS, we develop the LGM implementation of the JMS-TPHD filter, that is the LGM-JMS-TPHD filter.

**Proposition 3.** Suppose the PHD $D_{k-1}(\bar{x})$ of the augmented trajectory RFS at time $k - 1$ has the GM form

$$D_{k-1}(\bar{x}) = \sum_{j=1}^{J_{k-1}(o^{k-1})} \omega_{k-1}^j(o^{k-1}) \times N(\bar{x}; \beta_{k-1}^j(o^{k-1}), m_{k-1}^j(o^{k-1}), U_{k-1}^j(o^{k-1})), \quad (31)$$

Then, the predicted PHD $D_{k|k-1}(\bar{x})$ at time $k$ is

$$D_{k|k-1}(\bar{x}) = D_{\gamma,k}(\bar{x}) + D_{\zeta,k}(\bar{x}), \quad (32)$$

where
\begin{equation}
D_{C,k}(\bar{X}) = 
\sum_{j=1}^{J_{k-1}(o^{k-1})} \omega_{j,k-1}(o^k)
\times \mathcal{N} \left( \bar{X}; \bar{\beta}_{j,k-1}^T, m_{j,k-1}^T(o^k), U_{j,k-1}(o^k) \right),
\end{equation}

\begin{equation}
\omega_{j,k-1}(o^k) = \sum_{a_{k-1} \in \mathbb{A}} P_{SV}(o^k_{a_{k-1}})^{-1} \omega_{j,k-1}^a(o^k_{a-1}),
\end{equation}

\begin{equation}
m_{j,k-1}(o^k) = \sum_{a_{k-1} \in \mathbb{A}} \omega_{j,k-1}(o^k_{a_{k-1}}) \left[ m_{j,k-1}^a(o^k_{a_{k-1}}) F(o^{k-1}) \right],
\end{equation}

\begin{equation}
U_{j,k-1}(o^k) = \sum_{a_{k-1} \in \mathbb{A}} \omega_{j,k-1}(o^k_{a_{k-1}}) \left[ U_{j,k-1}^a(o^{k-1}) U_1^j + U_2(o^k) \right],
\end{equation}

where $m_{j,k}^{[a]}$ and $U_{j,k}^{[a]}$ denote the parts of the mean and the covariance matrix of the $j$th component for time step $a$ [20], and $U_{j,k}^{[a;b,c;\cdots]}$ denotes the part the covariance matrix with rows for time steps $a$ to $b$ and columns for time steps $c$ to $d$.

**Proposition 4.** Suppose the predicted PHD $D_{k-1}(\bar{X})$ at time $k$ has the GM form as (32). Then, the posterior PHD $D_k(\bar{X})$ at time $k$ is

\begin{equation}
D_k(\bar{X}) = (1 - P_D) D_{k-1}(\bar{X}) + \sum_{z \in \mathbb{T}_k} D_{det,k}(\bar{X}; z),
\end{equation}

where

\begin{equation}
D_{det,k}(\bar{X}; z) = \sum_{j=1}^{J_{k-1}(o^k)} \omega_{j,k}(o^k; z)
\times \mathcal{N} \left( \bar{X}; \bar{\beta}_{j,k}^T, m_{j,k}^T(o^k), U_{j,k}(o^k) \right),
\end{equation}

\begin{equation}
\omega_{j,k}(o^k; z) = P_{DF} \omega_{j,k-1}(o^k),
\end{equation}

\begin{equation}
q_{k}(o^k; z) = \left( z; Hm_{k}^{[i]}(o^k), HP_{j,k}^{[i]}(o^k) H^T + R \right),
\end{equation}

\begin{equation}
m_{j,k}(o^k) = m_{j,k-1}^a(o^k) + K \left( z - Hm_{k}^{[i]}(o^k) \right),
\end{equation}

\begin{equation}
U_{j,k}(o^k) = U_{j,k-1}^a(o^k) - KH_{j,k-1}^T(o^k),
\end{equation}

\begin{equation}
K = U_{j,k-1}^a(o^k) H^T \left( HU_{j,k-1}^a(o^k) H^T + R \right)^{-1}.
\end{equation}

As the consequence of Propositions 1 and 2, Propositions 3 and 4 describe detailedly how the GM components are analytically propagated with time.

**D. L-Scan approximation**

A common issue of the LGM implementation is that the number of Gaussian components increases [24] as time progresses. Hence, to limit Gaussian components, the measurements can be removed by gating [27] before update step and the pruning and absorption techniques [18] with the pruning threshold $\Gamma_p$, absorption threshold $\Gamma_a$ and the maximum number of components $J_{\max}$ is adopted after update step in the LGM-JMS-TPHD filter.

In addition, the lengths of the trajectories increase with time, thereupon the computational cost of a single filter recursion continues to increase, until it is not computationally feasible to implement the proposed filters directly. To resolve this problem, the $L$-scan approximations [18], which propagate the joint density of the last $L$ time steps and keep the density before $L$ time steps independent, is considered. Take the covariance matrix (46) with $l_k > L$ as an example

\begin{equation}
U_k \approx \text{diag}(\tilde{U}_k^{2+1}, \ldots, \tilde{U}_k^{k-L+1:k}),
\end{equation}

can be divided into two parts: $\tilde{U}_k^j$ denotes the covariance matrix of the target state at time $i$ and $\tilde{U}_k^{k-L+1:k}$ denotes the joint covariance matrix of the last $L$ time steps. In the $L$-scan approximation implementation, the joint covariance matrix $\tilde{U}_k^{k-L+1:k}$ for the $L$-scan window is iterated continuously and the independent target covariance matrix outside the $L$-scan window is unchanged and ingored, i.e., it does not occupy computing and storage.

For more implementation details for the pruning and absorption and the $L$-scan approximation techniques, please refer to [18], which extra provides the specific pseudocode.

**IV. Simulation Results**

In this section, we demonstrate the performance of the proposed IMS-TPHD filter with LGM implementation in a challenging multiple maneuvering targets tracking scenario. The metric for trajectory RFS based on linear programming in [28] with parameters $p = 2, c = 10$ and $\gamma = 1$ is used to evaluate the performance.

Consider a two-dimensional surveillance area of $10000m \times 10000m$ with the duration is $K = 60s$ and a total of 5 targets appeared during the tracking duration. Targets 1, 2, 3, 4 and 5 enter the scene at times $k = 1, 5, 10, 10s$ and targets 1, 2, 3, 4 exit the scene at times $k = 40s, 40s, 50s, 50s$, respectively. The motion of each target randomly switches among three possible motion models, where model 1 is a constant velocity (CV) model, model 2 is a coordinated turn (CT) model with a counterclockwise turn rate of $10^s$ and model 3 is also a CT model with a clockwise turn rate of $10^s$. The standard deviation of the process noise of the three models is $\sigma_p = 5m/s^2$. The state transition matrices for the CV and CT models as follows

\begin{equation}
F_{CV} = I_2 \otimes \begin{bmatrix} 1 & T \\ 0 & 1 \end{bmatrix},
\end{equation}

\begin{equation}
F_{CT} = \begin{bmatrix} 1 & \sin(\theta T) / \theta & 0 - (1 - \cos(\theta T)) / \theta \\ \cos(\theta T) & 0 & -\sin(\theta T) \\ 0 & (1 - \cos(\theta T)) / \theta & \sin(\theta T) / \theta \end{bmatrix},
\end{equation}

\begin{equation}
Q_{CV} = Q_{CT} = \sigma_p^2 I_2 \otimes \begin{bmatrix} T^4 / 4 & T^3 / 2 \\ T^3 / 2 & T^2 \end{bmatrix}.
\end{equation}
where ⊗ is the Kronecker product and $T = 1 \text{s}$ is the sampling interval.

The Poisson birth intensity is a Gaussian mixture (29) with parameters: $J_{\gamma,k} = 5, \omega_{\gamma,k}(\eta^k) = 0.2p(\eta^k), m_{\gamma,k}^1 = [2000; 0; 1000; 0], m_{\gamma,k}^2 = [1000; 0; 5000; 0], m_{\gamma,k}^3 = [1500; 0; 6000; 0], m_{\gamma,k}^4 = [8500; 0; 4000; 0], m_{\gamma,k}^5 = [6000; 0; 6000; 0]$ and $U_{\gamma,k} = diag([10; 10; 10; 10])^2$. The initial model distribution is $p(\theta) = [0.4, 0.3, 0.3]$ and the switching between three models obeys the following transition probability matrix (TPM):

$$
\nu(\theta|\theta') = \begin{bmatrix}
0.8 & 0.1 & 0.1 \\
0.1 & 0.8 & 0.1 \\
0.1 & 0.1 & 0.8 \\
\end{bmatrix}.
$$

In addition, the standard deviations of the measurement noise is $\sigma_m = 10 \text{m}$, and the clutter obeys Poisson distribution with clutter rate $\lambda_c = 60$. The survival probability and detection probability are $P_S = 0.99, P_D = 0.98$, respectively. The region and the trajectories of ground truths is shown in Fig. 1.

In the LGM implementation of JMS-TPHD filter, the parameters of the pruning and absorption techniques are $\Gamma_p = 10^{-5}, \Gamma_a = 4$ and $J_{\text{max}} = 30$. Figure. 2 shows four exemplar outputs of the LGM-JMS-TPHD filter, in which the filter provides an estimate of the set of alive trajectories at 4 representative time step. Obviously, the JMS-TPHD filter is capable to estimate the alive trajectories with high accuracy.

Then, we implement the $L$-scan approximation of the proposed filter with $L \in \{1, 2, 5, 10\}$ and use the metric for trajectory RFS to evaluate its performance by Monte Carlo simulation with 500 runs. Similar to the relationship between the $L$-scan GMTPHD/GMTCPHD and GMTPHD/GMTCPHD in the Section VI-E of [18], the $L$-scan LGM-JMS-TPHD filter with $L = 1$ also has same computations as the MM-PHD [10]. In other words, the $1$-scan LGM-JMS-TPHD filter has same trajectory tracking performance as the MM-PHD [10] filter assuming that the estimated states of same target in the MM-PHD can be connected to form trajectories.

The root mean square (RMS) trajectory errors [18] for the
L-scan JMS-TPHD filter are shown in Fig. 3. As expected, increasing $L$ can improve tracking performance of trajectories and the performance of JMS-TPHD (L-scan with $L \geq 2$) is generally better than MM-PHD (L-scan with $L = 2$). Meanwhile, the performance for $L \leq 5$ is similar to the JMS-TPHD filter without L-scan approximation. Fig. 3 shows the cardinality estimation over 500 MC simulations. We can observe that the cardinality estimations of $L \in \{1, 2, 5, 10, \infty\}$ are equal. According to the metric for trajectory RFS, the JMS-TPHD filter can be decomposed intuitively into localization error and cardinality error, it is revealed that the $L$ changes the overall trajectory tracking performance by affecting the localization error.

Moreover, the average running time of 500 MC simulations with Matlab implementation on the processor: Intel(R) Core(TM) i7-10770K CPU @ 3.80GHz, are shown in Table I. Obviously, the run time of the JMS-TPHD filter increases with $L$ in the L-scan implementation and the non L-scan implementation has the longest running time. Comparing different tracking durations, it can be seen that the longer the tracking duration period, the greater the difference in algorithm running time between different $L$. On the other hand, the results of the two tracking durations implys that if in the filter applications, the selection of $L$ is related to the size of the tracking duration, that is, when the tracking duration is short, the selection is biased towards tracking performance, and when the duration is long, the selection is biased towards the computational efficiency.

V. Conclusion

The JMS model has proven to be an effective tool for multiple maneuvering target tracking. With the JMS model, this paper proposes a new algorithm based on TPHD filter for tracking the trajectories of multiple maneuvering targets, named as JMS-TPHD. The recursion of JMS-TPHD filter is derived and the analytic closed-form is developed with linear Gaussian mixture implementation. To reduce computational burden in the implementation, we introduce the pruning and absorption, and the L-scan approximation techniques. The trajectories tracking performance of the JMS-TPHD filter is demonstrated by the simulation results, based on the metric for trajectory RFS.

TABLE I

| Duration | L-Scan |
|----------|--------|
| $K = 60s$ | 1.2271, 1.2489, 1.2922, 1.3813, 1.6051, 1.8966, 2.9456, 3.0925, 3.1261, 3.2597, 3.9325, 4.7340, 44.2777 |
| $K = 200s$ | 2.9456, 3.0925, 3.1261, 3.2597, 3.9325, 4.7340, 44.2777 |

REFERENCES

[1] R. Mahler, “On multi-target jump-Markov filters,” in Proc. 15th Int. Conf. Inf. Fusion, 2012, pp. 149156.

[2] H. A. P. Blom and Y. Bar-Shalom, “The interacting multiple model algorithm for systems with Markovian switching coefficients,” in IEEE Transactions on Automatic Control, vol. 33, no. 8, pp. 780-783, 1988.

[3] A. Doucet, N. J. Gordon and V. Krishnamurthy, “Particle filters for state estimation of jump Markov linear systems,” in IEEE Transactions on Signal Processing, vol. 49, no. 3, pp. 613-624, 2001.

[4] H. A. P. Blom and E. A. Bloem, “Combining IMM and JPDA for tracking multiple maneuvering targets in clutter,” Proceedings of the Fifth International Conference on Information Fusion, FUSION 2002, (IEEE Cat.No.02EX9597), Annapolis, MD, USA, 2002, pp. 705-712.

[5] J. K. Tugnait, “Tracking of multiple maneuvering targets in clutter using multiple sensors, IMM, and JPDA coupled filtering,” in IEEE Transactions on Aerospace and Electronic Systems, vol. 40, no. 1, pp. 320-330, 2004.

[6] W. Koch, “Fixed-interval retrodiction approach to Bayesian IMM-MHT for maneuvering multiple targets,” in IEEE Transactions on Aerospace and Electronic Systems, vol. 36, no. 1, pp. 2-14, 2000.

[7] R. Mahler, Statistical Multisource-Multitarget Information Fusion, MA, Norwood:Artech House, 2007.

[8] R. Mahler, “Multi-target Bayes filtering via first-order multi-target moments,” IEEE Transactions on Aerospace and Electronic Systems, vol. 39, no. 4, pp. 1152-1178, 2003.

[9] K. Punithakumar, T. Kirubarajan and A. Sinha, “Multiple-model Probability Hypothesis Density filter for tracking maneuvering targets,” in IEEE Transactions on Aerospace and Electronic Systems, vol. 44, no. 1, pp. 87-98, 2008.

[10] S. A. Pasha, B. Vo, H. D. Tian and W. Ma, “A Gaussian Mixture PHD Filter for Jump Markov System Models,” in IEEE Transactions on Aerospace and Electronic Systems, vol. 45, no. 3, pp. 919-936, 2009.

[11] R. Georgescu and P. Willett, “The Multiple Model PHD Filter,” in IEEE Transactions on Signal Processing, vol. 60, no. 4, pp. 1741-1751, 2012.

[12] J. Sun, and D. Li, “Multiple model CPHD filter for tracking maneuvering targets,” Applied Mechanics and Materials, vols. 556-562, pp. 3238-3241, 2014.

[13] D. Dunne and T. Kirubarajan, “Multiple Model Multi-Bernoulli Filters for Maneuvering Targets,” in IEEE Transactions on Aerospace and Electronic Systems, vol. 49, no. 4, pp. 2679-2692, 2013.

[14] S. Reuter, A. Scheel and K. Dietmayer, “The multiple model labeled multi-Bernoulli filter,” 2015 18th International Conference on Information Fusion (Fusion), Washington, DC, 2015, pp. 1574-1580.

[15] Y. G. Punchhiowa, B. Vo, B. Vo and D. Y. Kim, “Multiple Object Tracking in Unknown Backgrounds With Labeled Random Finite Sets,” in IEEE Transactions on Signal Processing, vol. 66, no. 11, pp. 3040-3055, 2018.

[16] W. Yi, M. Jiang and R. Hoseinnezhad, “The Multiple Model VoVo Filter,” in IEEE Transactions on Aerospace and Electronic Systems, vol. 53, no. 2, pp. 1045-1054, 2017.

[17] B. N. Vo, B. T. Vo, “A Multi-Scan Labeled Random Finite Set Model for Multi-Object State Estimation,” in IEEE Transactions on Signal Processing, vol. 67, no. 19, pp. 4948-4963, 2019.

[18] Á. F. García-Fernández and L. Svensson, “Trajectory PHD and CPHD Filters,” in IEEE Transactions on Signal Processing, vol. 67, no. 22, pp. 5702-5714, 2019.

[19] Á. F. García-Fernández, L. Svensson and M. R. Morelande, “Multiple Target Tracking Based on Sets of Trajectories,” in IEEE Transactions on Aerospace and Electronic Systems, vol. 56, no. 3, pp. 1685-1707, 2020.

[20] K. Granström, L. Svensson, Y. Xia, J. Williams and Á. F. García-Fernández, “Poisson Multi-Bernoulli Mixture Trackers: Continuity Through Random Finite Sets of Trajectories,” in Proc. Int. Conf. Inform. Fusion, Cambridge, UK, 2018.

[21] B. T. Vo, B. N. Vo, “Labeled random finite sets and multi-object conjugate priors,” IEEE Trans. Signal Process, vol. 61, no. 13, pp. 3460-3475, 2013.

[22] Á. F. García-Fernández, L. Svensson, J. L. Williams, Y. Xia and K. Granström, “Trajectory multi-Bernoulli filters for multi-target tracking based on sets of trajectories,” in 2020 IEEE 23rd International Conference on Information Fusion (FUSION), 2020, pp. 1-8.

[23] Á. F. García-Fernández, L. Svensson, J. L. Williams, Y. Xia and K. Granström, “Trajectory Poisson Multi-Bernoulli Filters,” in IEEE Transactions on Signal Processing, vol. 68, pp. 4933-4945, 2020.

[24] B. N. Vo and W. K. Ma, “The Gaussian Mixture Probability Hypothesis Density Filter,” in IEEE Transactions on Signal Processing, vol. 54, no. 11, pp. 4091-4104, 2006.

[25] L. Xu, Y. Liang, Z. Duan and G. Zhou, “Route-Based Dynamics Modeling and Tracking With Application to Air Traffic Surveillance,” in IEEE Transactions on Intelligent Transportation Systems, vol. 21, no. 1, pp. 209-221, 2020.
[26] H. W. Sorenson and D. L. Alspach, “Recursive Bayesian estimation using Gaussian sum,” *Automatica*, vol. 7, pp. 465-479, 1971.

[27] T. Kurien, “Issues in the design of practical multitarget tracking algorithms,” in *Multitarget -Multisensor Tracking: Advanced Applications*, Y. Bar-Shalom, Ed. Norwood, MA, USA: Artech House, 1990.

[28] A. F. García-Fernández, A. S. Rahmathullah and L. Svensson, “A Metric on the Space of Finite Sets of Trajectories for Evaluation of Multi-Target Tracking Algorithms,” in *IEEE Transactions on Signal Processing*, vol. 68, pp. 3917-3928, 2020.