Intrinsic Structures of Certain Musielak-Orlicz Hardy Spaces

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Abstract For any \( p \in (0, 1] \), let \( H^{0p}(\mathbb{R}^n) \) be the Musielak-Orlicz Hardy space associated with the Musielak-Orlicz growth function \( \Phi_p \), defined by setting, for any \( x \in \mathbb{R}^n \) and \( t \in [0, \infty) \),

\[
\Phi_p(x, t) := \begin{cases} 
\frac{t}{\log(e + t) + [t(1 + |x|)^p]^{1-p}} & \text{when } n(1/p - 1) \notin \mathbb{N} \cup \{0\}; \\
\frac{t}{\log(e + t) + [t(1 + |x|)^p]^{1-p} \log(e + |x|)^p} & \text{when } n(1/p - 1) \in \mathbb{N} \cup \{0\},
\end{cases}
\]

which is the sharp target space of the bilinear decomposition of the product of the Hardy space \( H^p(\mathbb{R}^n) \) and its dual. Moreover, \( H^{0p}(\mathbb{R}^n) \) is the prototype appearing in the real-variable theory of general Musielak-Orlicz Hardy spaces. In this article, the authors find a new structure of the space \( H^{0p}(\mathbb{R}^n) \) by showing that, for any \( p \in (0, 1] \), \( H^{0p}(\mathbb{R}^n) = H^{00}(\mathbb{R}^n) + H^{p0}_{W_p}(\mathbb{R}^n) \) and, for any \( p \in (0, 1) \), \( H^{0p}(\mathbb{R}^n) = H^p(\mathbb{R}^n) + H^{p0}_{W_p}(\mathbb{R}^n) \), where \( H^p(\mathbb{R}^n) \) denotes the classical real Hardy space, \( H^{00}(\mathbb{R}^n) \) the Orlicz-Hardy space associated with the Orlicz function \( \phi_0(t) := t/ \log(e + t) \) for any \( t \in [0, \infty) \) and \( H^{p0}_{W_p}(\mathbb{R}^n) \) the weighted Hardy space associated with certain weight function \( W_p(x) \) that is comparable to \( \Phi_p(x, 1) \) for any \( x \in \mathbb{R}^n \). As an application, the authors further establish an interpolation theorem of quasi-linear operators based on this new structure.

1 Introduction

The real-variable theory of the classical real Hardy spaces on the Euclidean space \( \mathbb{R}^n \) was initially developed by Stein and Weiss [18] and later by Fefferman and Stein [7]. For any \( p \in (0, \infty) \), the classical real Hardy space \( H^p(\mathbb{R}^n) \) consists of all Schwartz distributions \( f \) such that

\[
f^+ := \sup_{t \in (0, \infty)} |\phi_t \ast f| \in L^p(\mathbb{R}^n),
\]

where \( \phi \) is a function in the Schwartz class, \( \int_{\mathbb{R}^n} \phi(x) \, dx = 1 \) and \( \phi_t(\cdot) := t^{-n} \phi(t^{-1} \cdot) \). As one of the most important function spaces in harmonic analysis, \( H^p(\mathbb{R}^n) \) has many applications in various fields of mathematics (see, for example, [7] [17] [6] [16] and their references). Later, the theory of \( H^p(\mathbb{R}^n) \) was extended to the setting of the weighted Hardy spaces by García-Cuerva [8] and Strömberg, Torchinsky [20], and also to the setting of the Orlicz-Hardy space by Strömberg.
Both of the latter two spaces can be viewed as special cases of more general Musielak-Orlicz Hardy spaces which were first introduced by Ky [13] (see also [23] for a complete survey of the real-variable theory of Musielak-Orlicz Hardy spaces).

The main aim of this article is to try to understand some intrinsic structure of the Musielak-Orlicz Hardy space $H^{\Phi_p}(\mathbb{R}^n)$ associated with the Musielak-Orlicz growth function

$$\Phi_p(x, t) := \begin{cases} \frac{t}{\log(e + t) + [t(1 + |x|^p)]^{1-p}} & \text{when } n(1/p - 1) \notin \mathbb{N} \cup \{0\}, \\ \frac{t}{\log(e + t) + [t(1 + |x|^p)]^{1-p} \log(e + |x|)^p} & \text{when } n(1/p - 1) \in \mathbb{N} \cup \{0\}, \end{cases}$$

where $x \in \mathbb{R}^n$ and $t \in [0, \infty)$ (see [4, 13, 2]). The precise definition of $H^{\Phi_p}(\mathbb{R}^n)$ is as follows. In what follows, we use $S(\mathbb{R}^n)$ and $S'(\mathbb{R}^n)$ to denote the space of all Schwartz functions, equipped with the classical well-known topology, and its dual space, equipped with the weak-* topology.

**Definition 1.1.** Let $p \in (0, 1)$ and $\Phi_p$ be as in (1.1).

(i) For any $f \in S'(\mathbb{R}^n)$ and $m \in \mathbb{N}$, the non-tangential grand maximal function $f_m^\ast$ of $f$ is defined by setting, for any $x \in \mathbb{R}^n$

$$f_m^\ast(x) := \sup_{\varphi \in S_m(\mathbb{R}^n)} \sup_{|y| < t, t \in (0, \infty)} |f \ast \varphi(y)|,$$

where $\varphi(\cdot) := r^n \varphi(\cdot)$ for any $t \in (0, \infty)$, and

$$S_m(\mathbb{R}^n) := \left\{ \varphi \in S(\mathbb{R}^n) : \sup_{|y| \leq m+1} \sup_{x \in \mathbb{R}^n} (1 + |x|)^{m+2(n+1)} |D^n \varphi(x)| \leq 1 \right\}.$$

(ii) The Musielak-Orlicz-Lebesgue space $L^{\Phi_p}(\mathbb{R}^n)$ is defined to be the space of all measurable functions $f$ on $\mathbb{R}^n$ such that

$$\|f\|_{L^{\Phi_p}(\mathbb{R}^n)} := \inf \left\{ \lambda \in (0, \infty) : \int_{\mathbb{R}^n} \Phi_p(x, |f(x)|/\lambda) \, dx \leq 1 \right\} < \infty,$$

With $m$ being the largest integer not greater than $n(1/p - 1)$, the Musielak-Orlicz Hardy space $H^{\Phi_p}(\mathbb{R}^n)$ is defined to be the space of all Schwartz distributions $f \in S'(\mathbb{R}^n)$ such that

$$\|f\|_{H^{\Phi_p}(\mathbb{R}^n)} := \|f_m^\ast\|_{L^{\Phi_p}(\mathbb{R}^n)} < \infty,$$

where $f_m^\ast$ is as in (1.2).

Notice that the non-tangential grand maximal function $f^\ast$ in Definition 1.1 can also be used to characterize the Hardy space $H^p(\mathbb{R}^n)$ when $p \in (0, 1]$. Indeed, one has $\|f\|_{H^p(\mathbb{R}^n)} \sim \|f^\ast\|_{L^p(\mathbb{R}^n)}$ for any $p \in (0, 1]$ and any $f \in S'(\mathbb{R}^n)$ with the equivalent positive constants independent of $f$.

One of main motivations for us to study the aforementioned Musielak-Orlicz Hardy spaces $H^{\Phi_p}(\mathbb{R}^n)$ for any $p \in (0, 1]$ comes from the bilinear decomposition of the product of functions in Hardy space $H^p(\mathbb{R}^n)$ and its dual space. When $p = 1$, Bonami et al. [4] established the
following sharp bilinear decomposition of the product of the Hardy space $H^1(\mathbb{R}^n)$ and its dual space $\text{BMO}(\mathbb{R}^n)$

$$H^1(\mathbb{R}^n) \times \text{BMO}(\mathbb{R}^n) \subset L^1(\mathbb{R}^n) + H^{\text{log}}(\mathbb{R}^n),$$

(1.3) where $H^{\text{log}}(\mathbb{R}^n)$ is just the Musielak-Orlicz Hardy space $H^{\Phi_1}(\mathbb{R}^n)$. The precise meaning of (1.3) is that there exist two bounded bilinear operators $S : H^1(\mathbb{R}^n) \times \text{BMO}(\mathbb{R}^n) \to L^1(\mathbb{R}^n)$ and $T : H^1(\mathbb{R}^n) \times \text{BMO}(\mathbb{R}^n) \to H^{\text{log}}(\mathbb{R}^n)$ such that the product, defined in the sense of $S'(\mathbb{R}^n)$, of any $f \in H^1(\mathbb{R}^n)$ and $g \in \text{BMO}(\mathbb{R}^n)$, denoted by $f \times g$, can be written as

$$f \times g = S(f, g) + T(f, g);$$

Furthermore, there exists a positive constant $C$ such that, for any $(f, g) \in H^1(\mathbb{R}^n) \times \text{BMO}(\mathbb{R}^n)$,

$$\|S(f, g)\|_{L^1(\mathbb{R}^n)} \leq C\|f\|_{H^1(\mathbb{R}^n)}\|g\|_{\text{BMO}(\mathbb{R}^n)}$$

and

$$\|T(f, g)\|_{H^{\text{log}}(\mathbb{R}^n)} \leq C\|f\|_{H^1(\mathbb{R}^n)}\|g\|_{\text{BMO}(\mathbb{R}^n)} + \int_{B(\vec{0}_n, 1)} |g(x)| \, dx,$$

here and hereafter, $\vec{0}_n$ denotes the origin of $\mathbb{R}^n$ and $B(\vec{0}_n, 1)$ the open unit ball of $\mathbb{R}^n$ at $\vec{0}_n$. Moreover, it was proved in [4] that the space $H^{\text{log}}(\mathbb{R}^n)$ is optimal in the sense that it can not be replaced by a smaller vector space. The above result of [4] also answers a conjecture raised in [5]. Based on this result, Ky [13] further developed a general real-variable theory of Musielak-Orlicz Hardy spaces (see also [23] for a complete survey). Thus, the space $H^{\text{log}}(\mathbb{R}^n)$ plays a role as a prototype in the study of the real-variable theory of general Musielak-Orlicz Hardy spaces.

Recently, the result of [4] was extended to the case $p \in (0, 1)$ in [2]. Indeed, when $p \in (0, 1)$, it was proved in [2] that the space $H^{\Phi_p}(\mathbb{R}^n)$ is the optimal function space that is adapted to the bilinear decomposition of the product of elements from the Hardy space $H^p(\mathbb{R}^n)$ and its dual space $\mathcal{C}_{1/p-1}(\mathbb{R}^n)$

$$H^p(\mathbb{R}^n) \times \mathcal{C}_{1/p-1}(\mathbb{R}^n) \subset L^1(\mathbb{R}^n) + H^{\Phi_p}(\mathbb{R}^n).$$

(1.4) The precise meaning of (1.4) is as following: there exist two bounded bilinear operators $S : H^p(\mathbb{R}^n) \times \mathcal{C}_{1/p-1}(\mathbb{R}^n) \to L^1(\mathbb{R}^n)$ and $T : H^p(\mathbb{R}^n) \times \mathcal{C}_{1/p-1}(\mathbb{R}^n) \to H^{\Phi_p}(\mathbb{R}^n)$ such that the product, defined in the sense of $S'(\mathbb{R}^n)$, of any $f \in H^p(\mathbb{R}^n)$ and $g \in \mathcal{C}_{1/p-1}(\mathbb{R}^n)$, denoted by $f \times g$, can be written as

$$f \times g = S(f, g) + T(f, g);$$

and, furthermore, there exists a positive constant $C$ such that, for any $(f, g) \in H^p(\mathbb{R}^n) \times \mathcal{C}_{1/p-1}(\mathbb{R}^n)$,

$$\|S(f, g)\|_{L^1(\mathbb{R}^n)} \leq C\|f\|_{H^p(\mathbb{R}^n)}\|g\|_{\mathcal{C}_{1/p-1}(\mathbb{R}^n)}$$

and

$$\|T(f, g)\|_{H^{\Phi_p}(\mathbb{R}^n)} \leq C\|f\|_{H^p(\mathbb{R}^n)}\|g\|_{\mathcal{C}_{1/p-1}(\mathbb{R}^n)} + \int_{B(\vec{0}_n, 1)} |g(x)| \, dx.$$
It should be mentioned that the study of the bilinear decomposition of the product of elements from the Hardy space $H^p(\mathbb{R}^n)$ and its dual space can help us to improve the boundedness of many nonlinear qualities such as the div-curl product and the weak Jacobian (see [4, 13, 12]) as well as the endpoint boundedness of commutators (see [4, 12, 5]).

Motivated by the aforementioned results of [4, 13, 12], it is interesting to give a better understanding of the structure of the Musielak-Orlicz Hardy space $H^{\Phi_p}(\mathbb{R}^n)$. It is easy to observe that, for any $x \in \mathbb{R}^n$ and $t \in (0, \infty)$, one has

$$
\Phi_p(x, t) \sim \begin{cases} 
\frac{t}{1 + [t(1 + |x|)^p]^{1-p}} & \text{when } n(1/p - 1) \notin \mathbb{N} \cup \{0\}, \\
\frac{t}{\log(e + t) + \log(e + |x|)} & \text{when } n(1/p - 1) \in \mathbb{N}, \\
\frac{t}{(1 + |x|)^{p(1-p)}} & \text{when } p = 1
\end{cases}
$$

with the equivalent positive constants independent of $x$ and $t$. Based on (1.5), for any $x \in \mathbb{R}^n$ and $t \in [0, \infty)$, we consider the Orlicz function

$$
\phi_0(t) := \frac{t}{\log(e + t)}
$$

and the weight function

$$
W_p(x) := \begin{cases} 
\frac{1}{(1 + |x|)^{p(1-p)}} & \text{when } n(1/p - 1) \notin \mathbb{N} \cup \{0\}, \\
\frac{1}{(1 + |x|)^{p(1-p)} \log(e + |x|)^{1/p}} & \text{when } n(1/p - 1) \in \mathbb{N}, \\
\frac{1}{\log(e + |x|)} & \text{when } p = 1.
\end{cases}
$$

Let $H^{\phi}(\mathbb{R}^n)$ and $H^p_{W_p}(\mathbb{R}^n)$ be respectively the Orlicz-Hardy space associated with $\phi_0$ and the weighted Hardy space associated with $W_p$, which are defined in the same way as Definition [1.1(ii)], but with $\|f\|_{L^p(\mathbb{R}^n)}$ therein replaced respectively by

$$
\|f\|_{L^{\phi}(\mathbb{R}^n)} := \inf \left\{ \lambda \in (0, \infty) : \int_{\mathbb{R}^n} \phi_0(|f(x)|/\lambda) \, dx \leq 1 \right\}
$$

and

$$
\|f\|_{L^{W_p}(\mathbb{R}^n)} := \left\{ \int_{\mathbb{R}^n} |f(x)|^p W_p(x) \, dx \right\}^{1/p}.
$$

We refer the reader to [8, 19, 10, 20] for more properties on general Orlicz-Hardy spaces and weighted Hardy spaces.

Recall that, in [1], for any two quasi-Banach spaces $A_0$ and $A_1$, the pair $(A_0, A_1)$ is said to be compatible if there exists a Hausdorff space $X$ such that $A_0 \subset X$ and $A_1 \subset X$. For any compatible pair $(A_0, A_1)$ of quasi-Banach spaces, the sum space $A_0 + A_1$ is defined by setting

$$
A_0 + A_1 := \{ a \in X : \exists a_0 \in A_0 \text{ and } a_1 \in A_1 \text{ such that } a = a_0 + a_1 \}.
$$
equipped with the quasi-norm

\[ ||a||_{A_0 + A_1} := \inf \{ ||a_0||_{A_0} + ||a_1||_{A_1} : a = a_0 + a_1, a_0 \in A_0 \text{ and } a_1 \in A_1 \}. \]

In what follows, we use \( H^{\Phi_0}(\mathbb{R}^n) + H^p_W(\mathbb{R}^n) \) (resp., \( H^1(\mathbb{R}^n) + H^p_W(\mathbb{R}^n) \)) to denote the sum space, defined as in \( [1, 8] \), with \( \mathcal{X} := \mathcal{S}'(\mathbb{R}^n), A_0 := H^{\Phi_0}(\mathbb{R}^n) \) (resp., \( A_0 := H^1(\mathbb{R}^n) \)) and \( A_1 := H^p_W(\mathbb{R}^n) \).

The main result of this article is the following representation of \( H^{\Phi_p}(\mathbb{R}^n) \) for any \( p \in (0, 1) \) as the sum of an (Orlicz-)Hardy and a weighted Hardy spaces.

**Theorem 1.2.** Let \( p \in (0, 1] \). Define \( \Phi_p, \Phi_0 \) and \( W_p \) as in \( (1.1), (1.6) \) and \( (1.7) \), respectively. Then

(i) the space \( H^{\Phi_p}(\mathbb{R}^n) \) and \( H^{\Phi_0}(\mathbb{R}^n) + H^p_W(\mathbb{R}^n) \) coincide with equivalent quasi-norms;

(ii) for any \( p \in (0, 1) \), the space \( H^{\Phi_p}(\mathbb{R}^n) \) and \( H^1(\mathbb{R}^n) + H^p_W(\mathbb{R}^n) \) coincide with equivalent quasi-norms.

The new structure of the Musielak-Orlicz Hardy space \( H^{\Phi_p}(\mathbb{R}^n) \) established in Theorem 1.2 enables us to reduce the study of many properties of \( H^{\Phi_p}(\mathbb{R}^n) \) to the corresponding ones of the Orlicz-Hardy space \( H^{\Phi_0}(\mathbb{R}^n) \) when \( p \in (0, 1) \) (or the Hardy space \( H^1(\mathbb{R}^n) \) when \( p \in (0, 1) \)) and the weighted Hardy space \( H^p_W(\mathbb{R}^n) \), where the latter three kinds of Hardy-type spaces are well studied in various literatures; see, for example, \( [7, 16, 8, 19, 10, 20] \) and their references. A major job in the proof of Theorem 1.2 is decomposing every \( f \in H^{\Phi_p}(\mathbb{R}^n) \) into the sum of two parts, which belongs to the desired sum space. We obtain this decomposition by using the atomic characterization of \( H^{\Phi_1}(\mathbb{R}^n) \) when \( p = 1 \) and the Calderón-Zygmund decomposition of \( H^{\Phi_p}(\mathbb{R}^n) \) when \( p \in (0, 1) \). The main trick is that we use different selection principles in different decompositions and these selection principles are based on the norm estimates for the characteristic functions of the balls, which are established in Section 2.

As an application of Theorem 1.2 we consider a concrete problem of the interpolation of quasilinear operators. Recall that the following definition of quasilinear operators is from \( [9] \).

Let \( T \) be an operator defined on some quasi-Banach space \( A \) and taking values in the set of all complex-valued finite almost everywhere measurable functions on \( \mathbb{R}^n \). Such an operator \( T \) is said to be quasilinear if there exists a positive constant \( C \) such that, for any \( f, g \in A \) and \( \lambda \in \mathbb{C}, \)

\[ |T(f)| = |\lambda||f| \quad \text{and} \quad |T(f + g)| \leq C(\|f\| + |g|). \]

**Theorem 1.3.** Let \( p \in (0, 1] \). Let \( \Phi_p, \Phi_0 \) and \( W_p \) be as in \( (1.1), (1.6) \) and \( (1.7) \), respectively. Assume that \( T \) is a quasilinear operator bounded on \( H^p_W(\mathbb{R}^n) \). Then

(i) if \( T \) is bounded on \( H^{\Phi_0}(\mathbb{R}^n) \), then \( T \) is bounded on \( H^{\Phi_p}(\mathbb{R}^n) \);

(ii) if \( p \in (0, 1) \) and \( T \) is bounded on \( H^1(\mathbb{R}^n) \), then \( T \) is bounded on \( H^{\Phi_p}(\mathbb{R}^n) \).

This article is organized as follows. In Section 2 we establish several technical lemmas which are needed in the proof of Theorems 1.2 and 1.3; Section 3 is devoted to the proof of Theorem 1.2. Finally, using Theorem 1.2, we prove Theorem 1.3 in Section 4.

At the end of this section, we make some convention on the notation. Throughout this article, let \( \mathbb{N} := \{1, 2, \ldots\}, \mathbb{Z}_+ := \mathbb{N} \cup \{0\} \) and \( \mathbb{Z} := \{0, \pm 1, \pm 2, \ldots\} \). For any \( x \in \mathbb{R}^n \) and \( r \in (0, \infty) \), denote
by \( B(x, r) \) the ball with center \( x \) and radius \( r \), that is, \( B(x, r) := \{ y \in \mathbb{R}^n : |x - y| < r \} \). For any ball \( B \subset \mathbb{R}^n \), we always denote by \( c_B \) its center and \( r_B \) its radius and, for any \( \lambda \in (0, \infty) \), by \( \lambda B \) the ball with center \( c_B \) and radius \( \lambda r_B \). For any set \( E \subset \mathbb{R}^n \), \( \chi_E \) denotes its characteristic function.

We use \( C \) to denote a positive constant that is independent of the main parameters involved but whose value may differ from line to line. Constants with subscripts, such as \( C_1 \), do not change in different occurrences. If \( f \leq C g \), we then write \( f \lesssim g \) and, if \( f \leq g \leq f \), we then write \( f \sim g \). For any \( s \in \mathbb{R} \), let \([s]\) be the largest integer not greater than \( s \). We always use \( \alpha \) to denote a multi-index \((\alpha_1, \ldots, \alpha_n)\) with every \( \alpha_i \) being a non-negative integer.

\section{Several technical lemmas}

In this section, we present several technical lemmas which serve as preparations to prove Theorems \[1.2\] and \[1.3\]. To this end, we begin with recalling some notions used in \[13\].

**Definition 2.1.** For any \( p \in (0, \infty) \), an Orlicz function \( \phi \) (which means that \( \phi \) is nondecreasing and satisfies \( \phi(0) = 0, \phi(t) > 0 \) for \( t \in (0, \infty) \) and \( \lim_{t \to \infty} \phi(t) = \infty \)) is said to be of positive lower \( (\text{resp., upper}) \) type \( p \) if there exits a positive constant \( C \) such that, for any \( t \in [0, \infty) \) and \( s \in (0, 1] \) \( (\text{resp., } s \in [1, \infty)) \),

\[
\phi(st) \leq Cs^p \phi(t).
\]

**Definition 2.2.** A function \( \phi : \mathbb{R}^n \times [0, \infty) \to [0, \infty) \) is called a Musielak-Orlicz function if the function \( \phi(x, \cdot) : [0, \infty) \to [0, \infty) \) is an Orlicz function for any \( x \in \mathbb{R}^n \), and the function \( \phi(\cdot, t) \) is a measurable function for any \( t \in [0, \infty) \).

**Definition 2.3.** Let \( \phi \) be a Musielak-Orlicz function. For any given \( p \in (0, \infty) \), the function \( \phi \) is said to be of positive uniformly lower \( (\text{resp., upper}) \) type \( p \) if there exits a positive constant \( C \) such that, for any \( x \in \mathbb{R}^n \), \( t \in [0, \infty) \) and \( s \in (0, 1] \) \( (\text{resp., } s \in [1, \infty)) \),

\[
\phi(x, st) \leq Cs^p \phi(x, t).
\]

**Definition 2.4.** Let \( \phi \) be a Musielak-Orlicz function and \( q \in [1, \infty) \). The function \( \phi \) is said to satisfy the uniformly Muckenhoupt \( A_q(\mathbb{R}^n) \) condition, namely, \( \phi \in A_q(\mathbb{R}^n) \), if

\[
[\phi]_{A_q(\mathbb{R}^n)} := \begin{cases} 
\sup_{t \in (0, \infty)} \sup_{B \subset \mathbb{R}^n} \left[ \frac{1}{|B|} \int_B \phi(x, t) \, dx \right] \left[ \frac{1}{|B|} \int_B \{\phi(x, t)\}^{\frac{1}{q-1}} \, dx \right]^{q-1} & \text{when } q \in (1, \infty), \\
\sup_{t \in (0, \infty)} \sup_{B \subset \mathbb{R}^n} \left[ \frac{1}{|B|} \int_B \phi(x, t) \, dx \right] \left[ \sup_{x \in B} \{\phi(x, t)\}^{-1} \right]^{-1} & \text{when } q = 1
\end{cases}
\]

is finite, where the second suprema are taken over all balls \( B \) of \( \mathbb{R}^n \). Let

\[
A_{\infty}(\mathbb{R}^n) := \bigcup_{q \in [1, \infty)} A_q(\mathbb{R}^n).
\]

**Remark 2.5.** Let \( p \in (0, 1] \), \( \Phi_p \) be as in \[1.1\], \( \phi_0 \) as in \[1.6\] and \( W_p \) as in \[1.7\].
(i) We know (see [13] for the case $p = 1$ and [2] for the case $p \in (0, 1)$) that $\Phi_p$ is a Musielak-Orlicz function of uniformly upper 1 and of uniformly lower type $p$, and belongs to the uniformly Muckenhoupt weight class $A_1(\mathbb{R}^n)$.

(ii) Notice that $\phi_0(t) \sim \Phi_1(0, t)$ and $W_\rho(x) \sim \Phi_p(x, 1)$, where the equivalent positive constants are independent of $x$ and $t$. From these and (i) of this remark, it follows immediately that $\phi_0$ is of upper and lower types 1, and $W_\rho$ belongs to the usual Muckenhoupt weight class $A_1(\mathbb{R}^n)$. In particular, there exists a positive constant $C$ such that, for any ball $B$ in $\mathbb{R}^n$,

$$
\int_B W_\rho(x) \, dx \leq C \inf_{y \in B} W_\rho(y).
$$

For any $p \in (0, 1]$, the next lemma provides an $L^{\Phi_p}(\mathbb{R}^n)$-norm estimate for $\chi_B$, which was proved in [13] when $p = 1$ and [2] when $p \in (0, 1)$.

**Lemma 2.6.** Let $p \in (0, 1]$, $\alpha = 1/p - 1$ and $B = B(c_B, r_B)$ with $c_B \in \mathbb{R}^n$ and $r_B \in (0, \infty)$.

(i) If $p = 1$, then

$$
\|\chi_B\|_{L^{\Phi_1}(\mathbb{R}^n)} \sim \frac{|B|}{\log(e + 1/|B|) + \sup_{x \in B} \log(e + |x|)} \sim \frac{|B|}{\log r_B + \log(e + |c_B|)},
$$

where the equivalent constants are positive and independent of $B$.

(ii) If $p \in (0, 1)$, then

$$
\|\chi_B\|_{L^{\Phi_p}(\mathbb{R}^n)} \sim \Psi_\alpha(B)|B|,
$$

where

$$
\Psi_\alpha(B) := \begin{cases} 
\min \left\{ 1, \left( \frac{r_B}{1 + |c_B|} \right)^{\alpha} \right\} & \text{when } \alpha \notin \mathbb{N}, \\
\min \left\{ 1, \left( \frac{r_B}{1 + |c_B|} \right)^{\alpha} \right\} \frac{1}{\log(1 + |c_B| + r_B)} & \text{when } \alpha \in \mathbb{N}
\end{cases}
$$

and the equivalent constants are positive and independent of $B$.

For any $p \in (0, 1]$ and any ball $B \subset \mathbb{R}^n$, we still need to consider the Orlicz norm $\| \cdot \|_{L^{\Phi_p}(\mathbb{R}^n)}$ and the weighted Lebesgue norm $\| \cdot \|_{L^p_{w_\rho}(\mathbb{R}^n)}$ estimates of the characteristic function $\chi_B$.

**Lemma 2.7.** Let $\phi_1$ be as in (1.1), $W_1$ as in (1.7), and $\phi_0$ as in (1.6). Define $\phi_0^{-1}$ to be the inverse function of $\phi_0$. For any $t \in (0, \infty)$, let

$$
\rho(t) := \frac{t^{-1}}{\phi_0^{-1}(t^{-1})}.
$$

Then, for any ball $B \subset \mathbb{R}^n$,

(i) $\|\chi_B\|_{L^{\Phi_0}(\mathbb{R}^n)} = |B| |\rho(|B|)| \sim \frac{|B|}{\log(e + 1/|B|)}$. 

(ii) \( \|\chi_B\|_{L^1_w(R^n)} \sim \frac{|B|}{\sup_{x \in B} \log(e + |x|)} \)

(iii) \( \|\chi_B\|_{L^1_{\Phi_4}(R^n)} \sim \|\chi_B\|_{L^1_{\Phi_0}(R^n)} + \|\chi_B\|_{L^1_{\Phi_1}(R^n)} \)

where the equivalent constants in (i), (ii) and (iii) are positive and independent of \( B \).

Proof. We first prove (i). Recall that the equivalence \( \|\chi_B\|_{L^1_{\Phi_0}(R^n)} \sim \frac{|B|}{\log(e + |B|)} \) was established in \([22\text{ Lemma 7.13}]\). Thus, to finish the proof of (i), it remains to establish the first equality of (i). Indeed, from the definition of \( \rho \) \([22\text{ Lemma 7.13}]\). Thus, to finish the proof of (i), it remains to establish the first equality of (i).

\[
\int_{R^n} \phi_0 \left( \frac{\chi_B(x)}{|B| \rho(B)} \right) \, dx = \int_{R^n} \phi_0 \left( \phi_0^{-1}(|B|) \chi_B(x) \right) \, dx = \int_B \phi_0 \left( \phi_0^{-1}(|B|) \right) \, dx = 1,
\]

which immediately implies the first equality of (i) and hence (i) holds true.

We now prove (ii). By the fact that \( W_1 \) belongs to the Muckenhoupt weight class \( A_1(R^n) \) \([2.1]\), we know that

\[
\int_{R^n} \phi_0 \left( \frac{\chi_B(x)}{|B| \rho(B)} \right) \, dx = \int_{R^n} \phi_0 \left( \phi_0^{-1}(|B|) \chi_B(x) \right) \, dx = \int_B \phi_0 \left( \phi_0^{-1}(|B|) \right) \, dx = 1,
\]

which immediately implies the first equality of (i) and hence (i) holds true.

Finally, (iii) follows directly from Lemma \([2.6\text{ i}]\) and (i) and (ii) of this lemma. This finishes the proof of Lemma \(2.7\).

\[\square\]

**Lemma 2.8.** Let \( p \in (0, 1) \), \( \Phi_p \) and \( W_p \) be respectively as in \([1.1]\) and \([1.7]\). Then, for any ball \( B \subset R^n \),

\[ \|\chi_B\|_{L^p_{\Phi_p}(R^n)} \sim \|\chi_B\|_{L^p_{W_p}(R^n)} \]

where the equivalent constants are positive and independent of \( B \).

**Proof.** Denote by \( c_B \) the center of \( B \) and \( r_B \) its radius. By Lemma \(2.6\text{ ii})\), we have

\[
\|\chi_B\|_{L^p_{\Phi_p}(R^n)} \sim \left| B \right| \min \left\{ 1, \left( \frac{r_B}{1 + |c_B|} \right)^{n(1/p - 1)} \right\} \quad \text{when } n(1/p - 1) \notin \mathbb{N},
\]

\[
\|\chi_B\|_{L^p_{\Phi_p}(R^n)} \sim \left| B \right| \min \left\{ 1, \left( \frac{r_B}{1 + |c_B|} \right)^{n(1/p - 1)} \right\} \frac{1}{\log(e + r_B + |c_B|)} \quad \text{when } n(1/p - 1) \in \mathbb{N}.
\]

To estimate \( \|\chi_B\|_{L^p_{\Phi_p}(R^n)} \), we consider the following three cases.

**Case (i):** \( |c_B| \geq 2r_B \). In this case, for any \( x \in B \), it is easy to see that \( |x| \sim |c_B| \). By this and \([2.3]\), we conclude that, when \( n(1/p - 1) \notin \mathbb{N} \),

\[
\|\chi_B\|_{L^p_{\Phi_p}(R^n)} = \left\{ \int_B \frac{1}{(1 + |x|)^{n(1-p)}} \, dx \right\}^{1/p} \sim \left\{ \int_B \frac{1}{(1 + |c_B|)^{n(1-p)}} \, dx \right\}^{1/p} \sim \frac{|B|^{1/p}}{(1 + |c_B|)^{n(1/p - 1)}} \sim \|\chi_B\|_{L^p_{\Phi_p}(R^n)},
\]

where the equivalent constants in (i), (ii) and (iii) are positive and independent of \( B \).

Proof. We first prove (i). Recall that the equivalence \( \|\chi_B\|_{L^1_{\Phi_0}(R^n)} \sim \frac{|B|}{\log(e + |B|)} \) was established in \([22\text{ Lemma 7.13}]\). Thus, to finish the proof of (i), it remains to establish the first equality of (i). Indeed, from the definition of \( \rho \) \([22\text{ Lemma 7.13}]\). Thus, to finish the proof of (i), it remains to establish the first equality of (i).

\[
\int_{R^n} \phi_0 \left( \frac{\chi_B(x)}{|B| \rho(B)} \right) \, dx = \int_{R^n} \phi_0 \left( \phi_0^{-1}(|B|) \chi_B(x) \right) \, dx = \int_B \phi_0 \left( \phi_0^{-1}(|B|) \right) \, dx = 1,
\]

which immediately implies the first equality of (i) and hence (i) holds true.

We now prove (ii). By the fact that \( W_1 \) belongs to the Muckenhoupt weight class \( A_1(R^n) \) \([2.1]\), we know that

\[
\int_{R^n} \phi_0 \left( \frac{\chi_B(x)}{|B| \rho(B)} \right) \, dx = \int_{R^n} \phi_0 \left( \phi_0^{-1}(|B|) \chi_B(x) \right) \, dx = \int_B \phi_0 \left( \phi_0^{-1}(|B|) \right) \, dx = 1,
\]

which immediately implies the first equality of (i) and hence (i) holds true.

Finally, (iii) follows directly from Lemma \([2.6\text{ i}]\) and (i) and (ii) of this lemma. This finishes the proof of Lemma \(2.7\).

\[\square\]
as desired. Similarly, when \( n(1/p - 1) \in \mathbb{N} \), we have

\[
\| \chi_B \|_{L_{W_p}^p(\mathbb{R}^n)} = \left\{ \int_B \frac{1}{(1 + |x|)^{p(n(1-p))} \log(e + |x|)^{n(1-p)}} dx \right\}^{1/p} \\
\sim \frac{|B|^{1/p}}{(1 + |c_B|)^{p(1/p-1)} \log(e + |c_B|)} \sim \| \chi_B \|_{L_{\Phi}^p(\mathbb{R}^n)}.
\]

**Case (ii):** \(|c_B| < 2r_B < 1\). In this case, for any \( x \in B \), we have \(|x| \leq |x - c_B| + |c_B| < r_B + |c_B| < 2\). Thus, whenever \( n(1/p - 1) \) is an integer or not, we always have

\[
\inf_{x \in B} W_p(x) \sim 1.
\]

From this and the fact that \( W_p \in A_1(\mathbb{R}^n) \) (see (2.1)), it follows that

\[
\| \chi_B \|_{L_{W_p}^p(\mathbb{R}^n)} = |B|^{1/p} \left[ \frac{1}{|B|} \int_B W_p(x) \, dx \right]^{1/p} \sim |B|^{1/p} \left[ \inf_{x \in B} W_p(x) \right]^{1/p} \sim |B|^{1/p} \sim \| \chi_B \|_{L_{\Phi}^p(\mathbb{R}^n)},
\]

as desired.

**Case (iii):** \(|c_B| < 2r_B \) and \( 2r_B \geq 1 \). In this case, for any \( x \in B \), we have \(|x| \leq |x - c_B| + |c_B| < r_B + |c_B| < 3r_B\), so that \( 1 + |x| \leq r_B \) and hence

\[
\inf_{x \in B} W_p(x) = \begin{cases} 
\inf_{x \in B} \frac{1}{(1 + |x|)^{p(1-p)}} & \text{when } n(1/p - 1) \notin \mathbb{N}, \\
\inf_{x \in B} \frac{1}{(1 + |x|)^{p(1-p)} \log(e + |x|)^{p}} & \text{when } n(1/p - 1) \in \mathbb{N} \\
\sim \frac{1}{|B|^{1-p}} & \text{when } n(1/p - 1) \notin \mathbb{N}, \\
\frac{1}{|B|^{1-p} \log(e + r_B)^{p}} & \text{when } n(1/p - 1) \in \mathbb{N}.
\end{cases}
\]

Consequently,

\[
\| \chi_B \|_{L_{W_p}^p(\mathbb{R}^n)} = |B|^{1/p} \left[ \frac{1}{|B|} \int_B W_p(x) \, dx \right]^{1/p} \sim |B|^{1/p} \left[ \inf_{x \in B} W_p(x) \right]^{1/p} \\
\geq \begin{cases} 
|B| & \text{when } n(1/p - 1) \notin \mathbb{N}, \\
\frac{|B|}{\log(e + r_B)} & \text{when } n(1/p - 1) \in \mathbb{N}
\end{cases} \\
\sim \| \chi_B \|_{L_{\Phi}^p(\mathbb{R}^n)}.
\]

Also, when \( n(1/p - 1) \notin \mathbb{N} \), we have

\[
\| \chi_B \|_{L_{W_p}^p(\mathbb{R}^n)} \leq \int_{|x| < 3r_B} \frac{1}{(1 + |x|)^{p(1-p)}} \, dx \sim \frac{|B|^{1/p}}{(1 + 3r_B)^{p(1/p-1)}} \sim |B| \sim \| \chi_B \|_{L_{\Phi}^p(\mathbb{R}^n)}.
\]
Meanwhile, when $n(1/p - 1) \in \mathbb{N}$, we obtain
\[
\|\chi_B\|_{L^p_w(\mathbb{R}^n)} \leq \left[ \int_{|x| < 3r_B} \frac{1}{(1 + |x|)^{p(1 - p)[\log(e + |x|)]^p}} \, dx \right]^{1/p}.
\]
\[
= \sum_{j=1}^{\infty} \int_{2^{-j+1}r_B < |x| < 2^{-j}r_B} \frac{1}{(1 + |x|)^{p(1 - p)[\log(e + |x|)]^p}} \, dx \right]^{1/p}
\]
\[
= \sum_{j=1}^{\infty} \frac{1}{(1 + 2^{-j}r_B)^{p(1 - p)[\log(e + 2^{-j}r_B)]^p}} \right]^{1/p}
\]
\[
= \sum_{j=1}^{\infty} \frac{1}{j(2^{-j}r_B)^{p(1 - p)[\log(e + r_B)]^p}} \right]^{1/p} \leq \frac{|B|}{\log(e + r_B)} \sim \|\chi_B\|_{L^p_w(\mathbb{R}^n)},
\]
where we used the following estimates:
\[
\frac{\log(e + r_B)}{\log(e + 2^{-j}r_B)} \leq \log(e + 2^j) \leq j.
\]
Altogether, we find that $\|\chi_B\|_{L^p_w(\mathbb{R}^n)} \sim \|\chi_B\|_{L^p_w(\mathbb{R}^n)}$ in the case $|c| < 2r_B$ and $2r_B \geq 1$.

Summarizing the above three cases, we conclude that (ii) holds true. This finishes the proof of Lemma \[2.8\] \[\Box\]

From Lemmas \[2.6\], \[2.7\] and \[2.8\] we deduce some interesting properties on the Musielak-Orlicz Hardy space $H^{p_r}(\mathbb{R}^n)$ for any $p \in (0, 1]$. To be precise, we first recall the following definition of $H^{p_r}$-atoms from \[13\], \[14\].

**Definition 2.9.** Let $p \in (0, 1]$, $\Phi_p$ be as in \[1.1\], $q \in (1, \infty]$ and $s \in \mathbb{Z}_+ \cap [n(1/p - 1), \infty)$.

(I) For each ball $B \subset \mathbb{R}^n$, the space $L^q_{\Phi_p}(B)$ with $q \in [1, \infty]$ is defined to be the set of all measurable functions $f$ on $\mathbb{R}^n$, supported in $B$, such that
\[
\|f\|_{L^q_{\Phi_p}(B)} := \left( \sup_{t \in (0, \infty)} \frac{1}{\Phi_p(B, t)} \int_{\mathbb{R}^n} |f(x)|^q \Phi_p(x, t) \, dx \right)^{1/q}, \quad q \in [1, \infty);
\]
\[
\|f\|_{L^\infty_{\Phi_p}(B)} := \int_{\mathbb{R}^n} \Phi_p(x, t) \, dx.
\]
is finite, where, for any measurable set $E$ and $t \in [0, \infty)$, $\Phi_p(E, t) := \int_{E} \Phi_p(x, t) \, dx$.

(II) A function $a$ is called a $(\Phi_p, q, s)$-atom if there exists a ball $B \subset \mathbb{R}^n$ such that

(i) $\text{supp } a \subset B$;

(ii) $\|a\|_{L^q_{\Phi_p}(B)} \leq \|\chi_B\|_{L^q_{\Phi_p}(\mathbb{R}^n)}^{-1};$

(iii) $\int_{\mathbb{R}^n} a(x) x^\alpha \, dx = 0$ for all $\alpha \in \mathbb{Z}^n_+$ with $|\alpha| \leq s.$
(III) The atomic Musielak-Orlicz Hardy space $H^{\Phi_p,q,s}(\mathbb{R}^n)$ is defined to be the space of all $f \in S'(\mathbb{R}^n)$ satisfying that $f = \sum_j \lambda_j a_j$ in $S'(\mathbb{R}^n)$, where $\{\lambda_j\}_j \subset \mathbb{C}$ and $\{a_j\}_j$ is a sequence of $(\Phi_p, q, s)$-atoms, respectively, associated with balls $\{B_j\}_j$, satisfying

$$\sum_j \Phi_p \left( B_j, \frac{|\lambda_j|}{\|\chi_{B_j}\|_{L^{s.p}(\mathbb{R}^n)}} \right) < \infty.$$  

Moreover, let

$$\Lambda_{\Phi_p}(\{\lambda_j a_j\}_j) := \inf \left\{ \lambda \in (0, \infty) : \sum_j \Phi_p \left( B_j, \frac{|\lambda_j|}{\lambda \|\chi_{B_j}\|_{L^{s.p}(\mathbb{R}^n)}} \right) \leq 1 \right\}.$$  

Then the quasi-norm of $f \in H^{\Phi_p,q,s}(\mathbb{R}^n)$ is defined by setting

$$\|f\|_{H^{\Phi_p,q,s}(\mathbb{R}^n)} := \inf \left\{ \Lambda_{\Phi_p}(\{\lambda_j a_j\}_j) \right\},$$

where the infimum is taken over all the decompositions of $f$ as above.

The following atomic characterization of $H^{\Phi_p}(\mathbb{R}^n)$ follows from a general theory of the atomic characterization of Musielak-Orlicz Hardy spaces established in [13, Theorem 3.1].

**Lemma 2.10.** Let $p \in (0, 1]$, $\Phi_p$ be as in (1.1), $q \in (1, \infty)$ and $s \in \mathbb{Z}_+ \cap \lfloor n(1/p - 1) \rfloor$, $\infty$. Then the spaces $H^{\Phi_p}(\mathbb{R}^n)$ and $H^{\Phi_p,q,s}(\mathbb{R}^n)$ coincide with equivalent quasi-norms.

**Remark 2.11.** Let $p \in (0, 1]$, $q \in (1, \infty)$ and $s \in \mathbb{Z}_+ \cap \lfloor n(1/p - 1) \rfloor$, $\infty$. Assume that $\phi_0$ and $W_p$ are as in (1.6) and (1.7), respectively. Following Definition 2.9(II), if we replace $\Phi_p(x, t)$ therein respectively by $t^p$, $t^p W_p(x)$ and $\phi_0(t)$, then we obtain the definitions of $(p, q, s)$-atoms, $(p, q, s)W_p$-atoms and $(\phi_0, q, s)$-atoms. Correspondingly, we follow Definition 2.9(III) to introduce the atomic Hardy spaces $H^{p,q,s}(\mathbb{R}^n)$, $H^{\phi_0,q,s}(\mathbb{R}^n)$ and $H^{\phi_0,q,s}(\mathbb{R}^n)$ by replacing the quasinorm in (2.4), respectively, by

$$\|f\|_{H^{p,q,s}(\mathbb{R}^n)} := \inf \left\{ \left( \sum_{j \in \mathbb{N}} |\lambda_j|^p \right)^{1/p} \right\},$$

$$\|f\|_{H^{\phi_0,q,s}(\mathbb{R}^n)} := \inf \left\{ \left( \sum_{j \in \mathbb{N}} |\lambda_j|^p \right)^{1/p} \right\}$$

and

$$\|f\|_{H^{\phi_0,q,s}(\mathbb{R}^n)} := \inf \left\{ \Lambda_{\phi_0}(\{\lambda_j a_j\}_j) \right\},$$

where

$$\Lambda_{\phi_0}(\{\lambda_j a_j\}_j) := \inf \left\{ \lambda \in (0, \infty) : \sum_j |B_j| \phi_0 \left( \frac{|\lambda_j|}{\lambda |B_j| \rho(B_j)} \right) \leq 1 \right\}.$$
with $\rho$ as in (2.2). Then, from [13] Theorem 3.1, it also follows that

$$
\begin{aligned}
H^p(\mathbb{R}^n) &= H^{p,q,i}(\mathbb{R}^n) \\
H_{W_p}(\mathbb{R}^n) &= H^{p,q,i}_{W_p}(\mathbb{R}^n) \\
H^0(\mathbb{R}^n) &= H^{0,q,i}(\mathbb{R}^n)
\end{aligned}
$$

with equivalent quasinorms. See also [17, 21, 11, 8, 20] and their references for more discussions on these three kinds of Hardy-type spaces.

From these and Lemmas 2.6, 2.7 and 2.8 we deduce the following proposition, which is the basis to prove Theorem 1.2.

**Proposition 2.12.** Let $p \in (0, 1]$, $q \in (1, \infty)$ and $s \in \mathbb{Z}_+ \cap [[n/(1 - p) - 1], \infty)$. Let $\Phi_p$, $\phi_0$ and $W_p$ be as in (1.1), (1.6) and (1.7), respectively. Then, for any ball $B \subset \mathbb{R}^n$ with center $c_B \in \mathbb{R}^n$ and radius $r_B \in (0, \infty)$, the following assertions are true:

(i) any $(\phi_0, \infty, s)_{W_1}$-atom associated with the ball $B$ is also a $(\Phi_1, \infty, s)$-atom associated with the same ball $B$;

(ii) if $r_B < 1$ and $|c_B| < 1/r_B$, then any $(\Phi_1, \infty, s)$-atom associated with the ball $B$ is also a $(\phi_0, \infty, s)$-atom associated with the same ball $B$;

(iii) if $r_B < 1$ and $|c_B| \geq 1/r_B$, or $r_B \geq 1$, then any $(\Phi_1, \infty, s)$-atom associated with the ball $B$ is also a $(1, \infty, s)_{W_1}$-atom associated with the same ball $B$;

(iv) when $p \in (0, 1)$, any $(\Phi_p, \infty, s)$-atom associated with the ball $B$ is also a $(p, \infty, s)_{W_p}$-atom associated with the same ball $B$, and vice versa.

**Proof.** Let $\chi_B$ be the characteristic function of the ball $B$. By Lemma 2.7(iii), we know that

$$
\|\chi_B\|_{L^{\phi_0}(\mathbb{R}^n)}^{-1} \leq \|\chi_B\|_{L^{\Phi_1}(\mathbb{R}^n)}^{-1}
$$

and

$$
\|\chi_B\|_{L^{W_1}(\mathbb{R}^n)}^{-1} \leq \|\chi_B\|_{L^{\Phi_1}(\mathbb{R}^n)}^{-1}.
$$

By these and the definitions of $(\Phi_1, \infty, s)$-atoms, $(\phi_0, \infty, s)$-atoms and $(1, \infty, s)_{W_1}$-atoms, we know that any $(\phi_0, \infty, s)$-atom or $(1, \infty, s)_{W_1}$-atom associated with the ball $B$ is also a $(\Phi_1, \infty, s)$-atom associated with $B$. Hence, (i) holds true.

Now we show (ii). If $r_B < 1$ and $|c_B| < 1/r_B$, then, for any $x \in B$, it holds true that $|x| \leq |x - c_B| + |c_B| < 1 + 1/r_B$, which implies that

$$
\sup_{x \in B} \log(e + |x|) \leq \log(e + 1 + 1/r_B) \sim \log(e + 1/|B|).
$$

This, together with Lemmas 2.6(i) and 2.7(i), shows that

$$
\|\chi_B\|_{L^{\Phi_1}(\mathbb{R}^n)} \sim \frac{|B|}{\log(e + 1/|B|) + \sup_{x \in B} \log(e + |x|)} \sim \frac{|B|}{\log(e + 1/|B|)} \sim \|\chi_B\|_{L^{\phi_0}(\mathbb{R}^n)}.
$$
Then, by the definitions of \((\Phi_1, \infty, s)\)-atoms and \((\Phi_0, \infty, s)\)-atoms, we know that any \((\Phi_1, \infty, s)\)-atom associated with the ball \(B\) is also a \((\Phi_0, \infty, s)\)-atom associated with \(B\). This finishes the proof of (ii).

To prove (iii), we claim that, if \(1/|c_B| \leq r_B < 1\) or \(r_B \geq 1\), then

\[
\log(e + 1/|B|) + \sup_{x \in B}(e + |x|) = \sup_{x \in B}(e + |x|).
\]

Indeed, if \(r_B \geq 1\), then (2.5) holds true immediately. If \(1/|c_B| \leq r_B < 1\), then

\[
\log(e + 1/|B|) \leq \log(e + |c_B|) \leq \sup_{x \in B}(e + |x|),
\]

whence leading to (2.5). Thus, we conclude that

\[
\|\chi_B\|_{L^q_p(\mathbb{R}^n)} \sim \frac{|B|}{\log(e + 1/|B|) + \sup_{x \in B}(e + |x|)} \sim \frac{|B|}{\sup_{x \in B}(e + |x|)} \sim ||\chi_B||_{L^q_{w_1}(\mathbb{R}^n)}.
\]

Then, applying the definitions of \((\Phi_1, \infty, s)\)-atom and \((1, \infty, s)\)-atom, we see that any \((\Phi_1, \infty, s)\)-atom associated with the ball \(B\) is also a \((1, \infty, s)\)-atom associated with \(B\). This finishes the proof of (iii).

To show (iv), for any \(p \in (0, 1)\), by the definitions of \((\Phi_p, \infty, s)\)-atoms and \((p, \infty, s)\)-atoms as well as Lemma 2.13, we immediately conclude that a function \(a\) on \(\mathbb{R}^n\) is a \((\Phi_p, \infty, s)\)-atom associated with \(B\) if and only if \(a\) is a \((p, \infty, s)\)-atom associated with the same ball \(B\). Thus, (iv) holds true, which completes the proof of Proposition 2.12.

We end this section by recalling the following two lemmas, established in [13], on the Calderón-Zygmund decomposition of the elements of Musielak-Orlicz Hardy spaces.

**Lemma 2.13.** Let \(p \in (0, 1), q \in (1, \infty)\) and \(\Phi_p\) be as in (1.1). Then \(L^q_{\Phi_p(\mathbb{R}^n)} \cap H^{\Phi_p}(\mathbb{R}^n)\) is dense in \(H^{\Phi_p}(\mathbb{R}^n)\).

**Lemma 2.14.** Let \(p \in (0, 1), q \in (1, \infty), s \in \mathbb{Z}_+ \cap [\lfloor n(1/p - 1) \rfloor, \infty)\) and \(\Phi_p\) be as in (1.1). For any \(f \in L^q_{\Phi_p(\mathbb{R}^n)} \cap H^{\Phi_p}(\mathbb{R}^n)\), there exist family \(\{\Lambda_k\}_{k \in \mathbb{Z}}\) of index set with elements of countable numbers, \(\{g^k\}_{k \in \mathbb{Z} \setminus \{0\}}\) and \(\{b^k_i\}_{k \in \mathbb{Z} \setminus \{0\}}\) in \(S'(\mathbb{R}^n)\) such that

(i) for any \(k \in \mathbb{Z}\), \(f = g^k + \sum_{i \in \Lambda_k} b^k_i\) in \(S'(\mathbb{R}^n)\);

(ii) \(f = \sum_{k \in \mathbb{Z}} (g^{k+1} - g^k)\) in \(S'(\mathbb{R}^n)\);

(iii) for any \(k \in \mathbb{Z}\), there exists a family \(\{b^k_i\}_{i \in \Lambda_k} \subset L^\infty(\mathbb{R}^n)\) such that \(g^{k+1} - g^k = \sum_{i \in \Lambda_k} b^k_i\) in \(S'(\mathbb{R}^n)\);

(iv) for any \(k \in \mathbb{Z}\) and \(i \in \Lambda_k\), \(h^k_i\) satisfies

(a) \(\text{supp } h^k_i \subset B^k_i\), where \(B^k_i := 18B^k_i\) and \(\{B^k_i\}_{i \in \Lambda_k}\) is a Whitney covering of \(\Omega_k\)

\[\Omega_k := \{x \in \mathbb{R}^n : f^*(x) > 2^k\},\]

where \(f^*\) denotes the non-tangential maximal function of \(f\) as in (1.2) with \(m\) therein equal to \(\lfloor n(1/p - 1) \rfloor\).
(b) \( \|h^k\|_{L^\infty(\mathbb{R}^n)} \leq c2^k \), where \( c \) is a positive constant independent of \( k, i \) and \( f \).

(c) for any multi-index \( \alpha \) satisfying \( |\alpha| \leq s \), it holds true that \( \int_{\mathbb{R}^n} x^\alpha h^k_i(x) \, dx = 0 \).

3 Proof of Theorem 1.2

Based on the technical lemmas established in Section 2, we now prove Theorem 1.2 by considering two cases: \( p = 1 \) and \( p \in (0, 1) \). We point out that these two cases are based on different selection principles to obtain the desired sum space.

Proof of Theorem 1.2 in the case \( p = 1 \). We first establish the inclusion \( H^{\Phi_0}(\mathbb{R}^n) + H^1_{W_1}(\mathbb{R}^n) \subset H^{\Phi_1}(\mathbb{R}^n) \). For any \( f \in H^{\Phi_0}(\mathbb{R}^n) + H^1_{W_1}(\mathbb{R}^n) \), let \( f_0 \in H^{\Phi_0}(\mathbb{R}^n) \) and \( f_1 \in H^1_{W_1}(\mathbb{R}^n) \) satisfy \( f = f_0 + f_1 \) in \( S'(\mathbb{R}^n) \) and

\[
\|f\|_{H^{\Phi_0}(\mathbb{R}^n)+H^1_{W_1}(\mathbb{R}^n)} \sim \|f_0\|_{H^{\Phi_0}(\mathbb{R}^n)} + \|f_1\|_{H^1_{W_1}(\mathbb{R}^n)}.
\]

Using (1.1), we know that, for any \( x \in \mathbb{R}^n \) and \( t \in (0, \infty) \),

\[
\Phi_1(x, t) \leq \min \left\{ \frac{t}{\log(e + t)}, \frac{t}{\log(e + |x|)} \right\},
\]

which, combined with the grand maximal function characterizations of these Hardy-type spaces, shows that

\[
\|f\|_{H^{\Phi_1}(\mathbb{R}^n)} \leq \|f_0\|_{H^{\Phi_0}(\mathbb{R}^n)} + \|f_1\|_{H^{\Phi_1}(\mathbb{R}^n)} \leq \|f_0\|_{H^{\Phi_0}(\mathbb{R}^n)} + \|f_1\|_{H^1_{W_1}(\mathbb{R}^n)} \sim \|f\|_{H^{\Phi_0}(\mathbb{R}^n) + H^1_{W_1}(\mathbb{R}^n)}.
\]

This immediately implies the inclusion \( H^{\Phi_0}(\mathbb{R}^n) + H^1_{W_1}(\mathbb{R}^n) \subset H^{\Phi_1}(\mathbb{R}^n) \).

We now prove the converse inclusion \( H^{\Phi_1}(\mathbb{R}^n) \subset H^{\Phi_0}(\mathbb{R}^n) + H^1_{W_1}(\mathbb{R}^n) \). Without loss of generality, we may assume that \( f \in H^{\Phi_1}(\mathbb{R}^n) \) and \( \|f\|_{H^{\Phi_1}(\mathbb{R}^n)} = 1 \); otherwise, we use \( \tilde{f} := f/\|f\|_{H^{\Phi_1}(\mathbb{R}^n)} \) to replace \( f \) in the same argument as below.

Let \( s \in \mathbb{Z}_+ \) and \( s \geq n(1/p - 1) \), by Lemma 2.10 we know that there exist \( \{a_j\}_{j \in \mathbb{N}} \) of \( (\Phi_1, \infty, s) \)-atoms and \( \{\lambda_j\}_{j \in \mathbb{N}} \subset \mathbb{C} \) such that

\[
f = \sum_{j \in \mathbb{N}} \lambda_j a_j
\]

in \( H^{\Phi_1}(\mathbb{R}^n) \) and hence in \( S'(\mathbb{R}^n) \), and \( \Lambda_{\Phi_1}(\{\lambda_j a_j\}_{j \in \mathbb{N}}) \leq 2 \). Since \( \Phi_1 \) of \( f \) of uniformly lower type 1 and of uniformly upper type 1, it follows easily that, for any \( x \in \mathbb{R}^n \) and \( s, t \in (0, \infty) \),

\[
\Phi_1(x, st) \sim s\Phi_1(x, t).
\]

By this, the fact \( \Lambda_{\Phi_0}(\{\lambda_j a_j\}_{j \in \mathbb{N}}) \leq 2 \) and [13, Lemma 4.2(i)], we conclude that

\[
1 \geq \sum_j \Phi_1 \left( B_j, \frac{|\lambda_j|}{2\|\chi_{B_j}\|_{L^{\Phi_1}(\mathbb{R}^n)}} \right) = \sum_j \int_{B_j} \Phi_1 \left( x, \frac{|\lambda_j|}{2\|\chi_{B_j}\|_{L^{\Phi_1}(\mathbb{R}^n)}} \right) dx
\]
Thus, by Proposition 2.12, we know that, for any
\[\sum_j |\lambda_j| \int_{B_j} \Phi_1 \left( x, \frac{1}{|x|} \right) dx \sim \sum_j |\lambda_j|.\]

For any \(j \in \mathbb{N}\), assume that \(a_j\) is supported on a ball \(B_j := B(c_j, r_j)\), where \(c_j \in \mathbb{R}^n\) and \(r_j \in (0, \infty)\). Define
\[I_0 := \left\{ j : r_j < 1 \text{ and } |c_j| < \frac{1}{r_j} \right\} \quad \text{and} \quad I_1 := \left\{ j : r_j < 1 \text{ and } |c_j| \geq \frac{1}{r_j}, \text{ or } r_j \geq 1 \right\}.
\]
It is easy to see that \(I_0 \cap I_1 = \emptyset\). We now write the decomposition in (3.1) into
\[f = \sum_{j \in I_0} \lambda_j a_j = \sum_{j \in I_0} \lambda_j a_j + \sum_{j \in I_1} \lambda_j a_j =: f_0 + f_1.
\]
Thus, by Proposition 2.12, we know that, for any \(j \in I_0\), \(a_j\) is a \((\phi_0, \infty, s)\)-atom associated with the ball \(B_j\) and, for any \(j \in I_1\), \(a_j\) is a \((1, \infty, s)\text{-atom associated with } B_j\).

We now show \(f_0 \in H^{\phi_0}(\mathbb{R}^n)\). For any \(j \in I_0\), by \(r_j < 1\) and Lemma 2.7(i), we know that
\[|B_j| \phi_0 \left( \frac{1}{|B_j|} \right) \sim |B_j| \phi_0 \left( \frac{\log(e + 1/|B_j|)}{|B_j|} \right) = \frac{\log(e + 1/|B_j|)}{|B_j|} \leq 1,
\]
which, together with the fact that \(\phi_0\) is of lower type 1, further implies that
\[\sum_{j \in I_0} |B_j| \phi_0 \left( \frac{|\lambda_j|}{\sum_{j \in I_0} |\lambda_j|} \right) \leq \sum_{j \in I_0} \frac{|\lambda_j|}{\sum_{j \in I_0} |\lambda_j|} |B_j| \phi_0 \left( \frac{1}{|B_j|} \right) \leq 1.
\]
From this and (3.2), it follows that
\[\Lambda_{\phi_0} \left( \left\{ \lambda_j : j \in I_0 \right\} \right) = \inf \left\{ \lambda \in (0, \infty) : \sum_{j \in I_0} |B_j| \phi_0 \left( \frac{|\lambda_j|}{\sum_{j \in I_0} |\lambda_j|} \right) \leq 1 \right\} \leq \sum_{j \in I_0} |\lambda_j| \leq 1.
\]
Thus, \(f_0 \in H^{\phi_0}(\mathbb{R}^n)\) and \(\|f_0\|_{H^{\phi_0}(\mathbb{R}^n)} \leq \Lambda_{\phi_0} \left( \left\{ \lambda_j : j \in I_0 \right\} \right) \leq 1 \sim \|f\|_{H^{\phi_1}(\mathbb{R}^n)}\).

For \(f_1\), using the atomic characterization of \(H^1_{W_1}(\mathbb{R}^n)\) stated in Remark 2.11, 3.2 and (3.3), we find that \(f_1 \in H^1_{W_1}(\mathbb{R}^n)\) and
\[\|f_1\|_{H^1_{W_1}(\mathbb{R}^n)} \leq \sum_{j \in I_1} |\lambda_j| \leq 1 \sim \|f\|_{H^{\phi_1}(\mathbb{R}^n)}\.
\]
Thus, we conclude that for any \(f \in H^{\phi_1}(\mathbb{R}^n)\), there exist \(f_0 \in H^{\phi_0}(\mathbb{R}^n)\) and \(f_1 \in H^1_{W_1}(\mathbb{R}^n)\) such that \(f = f_0 + f_1\) in \(S'(\mathbb{R}^n)\) and \(\|f_0\|_{H^{\phi_0}(\mathbb{R}^n)} + \|f_1\|_{H^1_{W_1}(\mathbb{R}^n)} \leq \|f\|_{H^{\phi_1}(\mathbb{R}^n)}\). This finishes the proof of
the converse inclusion \(H^{\phi_1}(\mathbb{R}^n) \subset H^{\phi_0}(\mathbb{R}^n) + H^1_{W_1}(\mathbb{R}^n)\) and hence of Theorem 1.2 in the case \(p = 1\). 

We now turn to the proof of Theorem 1.2 under the case \(p \in (0, 1)\).
**Proof of Theorem 1.2** under the case $p \in (0, 1)$. Let $p \in (0, 1)$. For any $x \in \mathbb{R}^n$ and $t \in (0, \infty)$, using (1.1) and (1.5), we observe that

$$\Phi_p(x, t) \lesssim \min \left\{ \phi_0(t), t^p W_p(x) \right\} \quad \text{and} \quad \phi_0(t) \leq t.$$  

From these observations, we argue as in the case $p = 1$ and can obtain the inclusions

$$H^1(\mathbb{R}^n) + H^0_{W_p}(\mathbb{R}^n) \subset H^{\Phi_p}(\mathbb{R}^n) + H^0_{W_p}(\mathbb{R}^n) \subset H^{\Phi_p}(\mathbb{R}^n),$$

with desired norm estimates.

It remains to prove $H^{\Phi_p}(\mathbb{R}^n) \subset H^1(\mathbb{R}^n) + H^0_{W_p}(\mathbb{R}^n)$. Due to similarity, we only consider the case $n(1/p - 1) \in \mathbb{N}$. Consider first the case $f \in L^2_{\phi_p(s)}(\mathbb{R}^n) \cap H^{\Phi_p}(\mathbb{R}^n)$. Without loss of generality, we may also assume that $\|f\|_{H^{\Phi_p}(\mathbb{R}^n)} = 1$.

Let $q \in (1, \infty)$ and $s \in \mathbb{Z}_+ \cap [(n(1/p - 1)], \infty)$. Applying Lemma 2.14 there exist families $\{\Lambda_k\}_{k \in \mathbb{Z}}$ of index sets, $\{h^k_i\}_{i \in \mathbb{Z}, i \in \Lambda_k}$ of functions in $L^\infty(\mathbb{R}^n)$ and $\{B^k_i\}_{i \in \mathbb{Z}, i \in \Lambda_k}$ of balls such that

\begin{equation}
   f = \sum_{k \in \mathbb{Z}} \sum_{i \in \Lambda_k} h^k_i \quad \text{in} \quad \mathcal{S}'(\mathbb{R}^n).
\end{equation}

Define

\begin{equation}
   E := \left\{ x \in \mathbb{R}^n : f^*(x) < (1 + |x|)^{-n}[\log(e + |x|)]^{-p/(1-p)} \right\},
\end{equation}

where $f^*$ denotes the non-tangential maximal function as in (1.2) with $m := n(1/p - 1)$. For any $k \in \mathbb{Z}$ and $i \in \Lambda_k$, define

$$B^k_{i,E} := B^k_i \cap E \quad \text{and} \quad B^k_{i,E^c} := B^k_i \cap E^c.$$

Let

$$I_0 := \left\{ (k, i) : |B^k_{i,E}| \geq \frac{1}{2}|B^k_i| \right\} \quad \text{and} \quad I_1 := \left\{ (k, i) : |B^k_{i,E^c}| \geq \frac{1}{2}|B^k_i| \right\}.$$

It is easy to see that $I_0 \cap I_1 = \emptyset$ and

$$\sum_{k \in \mathbb{Z}} \sum_{i \in \Lambda_k} = \sum_{(k, i) \in I_0} + \sum_{(k, i) \in I_1}.$$

For any fixed $k_0 \in \mathbb{Z}$, by Lemma 2.14 it holds true that

\begin{equation}
   \sum_{(k_0, i) \in I_0} |B^k_{i,E}| \lesssim 2 \sum_{(k_0, i) \in I_0} |B^k_{i,E}| \lesssim \left| \left\{ x \in E : f^*(x) > 2^{k_0} \right\} \right|.
\end{equation}

Similarly, for any $(k_0, i) \in I_1$, using $W_p \in A_\infty(\mathbb{R}^n)$ (see (2.1)) and $|B^k_{i,E^c}| \geq \frac{1}{2}|B^k_i|$, we obtain

$$\frac{W_p(B^k_{i,E})}{W_p(B^k_{i,E^c})} \lesssim \frac{|B^k_{i,E}|}{|B^k_{i,E^c}|} \lesssim 1.$$
This, combined with the same argument as in (3.6), implies that

\[
(3.7) \quad \sum_{(k, l) \in I_0} W_p \left( B_l^{k_0} \right) \leq \sum_{(k, l) \in I_1} W_p \left( B_l^{k_0} \cap E^c \right) \leq W_p \left( \{ x \in E^c : f^*(x) > 2^{k_0} \} \right).
\]

We split the decomposition in (3.4) into

\[
f = \sum_{(k, l) \in I_0} h_l^k + \sum_{(k, l) \in I_1} h_l^k =: \sum_{(k, l) \in I_0} \chi_{k,l} a_{k,l}^{(0)} + \sum_{(k, l) \in I_1} \chi_{k,l} a_{k,l}^{(1)} =: f_0 + f_1,
\]

where

\[
\begin{align*}
\chi_{k,l}^{(0)} &:= c 2^k |B_l^k|; \\
\chi_{k,l}^{(1)} &:= c 2^k [W_p(B_l^k)]^{1/p}; \\
a_{k,l}^{(0)} &:= h_l^k / \chi_{k,l}^{(0)}; \\
a_{k,l}^{(1)} &:= h_l^k / \chi_{k,l}^{(1)}
\end{align*}
\]

and \(c\) is the same as in (b) of Lemma 2.14(iv) and it is independent of \(k, l\) and \(f\). By Remark 2.11 and Lemma 2.14, it is easy to see that \(a_{k,l}^{(0)}\) is a \((1, \infty, s)\)-atom associated with the ball \(B_l^k\) and \(a_{k,l}^{(1)}\) is a \((p, \infty, s) W_p\)-atom associated with \(B_l^k\). From (3.5) and (3.6), it follows that

\[
\sum_{(k, l) \in I_0} \left| \chi_{k,l}^{(0)} \right| \leq \sum_{k \in \mathbb{Z}} 2^k \left| \{ x \in E : f^*(x) > 2^k \} \right|
\]

\[
\leq \int_E f^*(x) \, dx
\]

\[
\sim \int_E \frac{f^*(x)}{1 + [f^*(x)(1 + |x|)^p]^{1-p}[\log(e + |x|)]^p} \, dx
\]

\[
\leq \int_{\mathbb{R}^n} \Phi_p(x, f^*(x)) \, dx \leq 1,
\]

where the last inequality follows from the assumption \(||f||_{H^{p_0}_W(\mathbb{R}^n)} = 1\). Further, using the atomic characterization of \(H^1(\mathbb{R}^n)\), we know that \(f_0 \in H^1(\mathbb{R}^n)\) and \(||f_0||_{H^1(\mathbb{R}^n)} \leq 1 \sim ||f||_{H^{p_0}_W(\mathbb{R}^n)}\).

For \(f_1\), using (3.7), we find that

\[
\sum_{(k, l) \in I_1} \left| \chi_{k,l}^{(1)} \right|^p \leq \sum_{k \in \mathbb{Z}} 2^{kp} W_p \left( \{ x \in E^c : f^*(x) > 2^k \} \right)
\]

\[
\leq \int_{E^c} [f^*(x)]^p W_p(x) \, dx
\]

\[
\sim \int_{E^c} \frac{f^*(x)}{1 + [f^*(x)(1 + |x|)^p]^{1-p}[\log(e + |x|)]^p} \, dx
\]

\[
\leq \int_{\mathbb{R}^n} \Phi_p(x, f^*(x)) \, dx \leq 1,
\]

Then, using the atomic characterization of \(H^1_W(\mathbb{R}^n)\) in Remark 2.11 we know that \(f_1 \in H^1_W(\mathbb{R}^n)\) and \(||f_1||_{H^1_W(\mathbb{R}^n)} \leq 1 \sim ||f||_{H^{p_0}_W(\mathbb{R}^n)}\).
Summarizing the above estimates gives us that $L^q_{\Phi_p(\cdot, 1)}(\mathbb{R}^n) \cap H^0(\mathbb{R}^n) \subset H^1(\mathbb{R}^n) + H^0_{W_p}(\mathbb{R}^n)$ and, for any $f \in L^q_{\Phi_p(\cdot, 1)}(\mathbb{R}^n) \cap H^0(\mathbb{R}^n)$, there exist $f_0 \in H^1(\mathbb{R}^n)$ and $f_1 \in H^0_{W_p}(\mathbb{R}^n)$ such that $f = f_0 + f_1$ in $S'(\mathbb{R}^n)$ and

(3.8) \hspace{1cm} \|f_0\|_{H^1(\mathbb{R}^n)} + \|f_1\|_{H^0_{W_p}(\mathbb{R}^n)} \lesssim \|f\|_{H^0(\mathbb{R}^n)}.

For a general $f \in H^0(\mathbb{R}^n)$, by Lemma 2.13, there exist $\{f_l\}_{l \in \mathbb{N}} \subset L^q_{\Phi_p(\cdot, 1)}(\mathbb{R}^n) \cap H^0(\mathbb{R}^n)$ such that $f = \sum_{l \in \mathbb{N}} f_l$ in $H^0(\mathbb{R}^n)$ and

\[ \|f_l\|_{H^0(\mathbb{R}^n)} \lesssim 2^{2-l} \|f\|_{H^0(\mathbb{R}^n)} \]

(see also [13] p. 138 for this fact). Applying the previous argument to each $f_l$ with $l \in \mathbb{N}$, we find $f_{l,0} \in H^1(\mathbb{R}^n)$ and $f_{l,1} \in H^0_{W_p}(\mathbb{R}^n)$ such that $f_l = f_{l,0} + f_{l,1}$ in $S'(\mathbb{R}^n)$, and

\[ \|f_{l,0}\|_{H^1(\mathbb{R}^n)} + \|f_{l,1}\|_{H^0_{W_p}(\mathbb{R}^n)} \lesssim \|f_l\|_{H^0(\mathbb{R}^n)} \lesssim 2^{-l} \|f\|_{H^0(\mathbb{R}^n)}. \]

Define $f_0 := \sum_{l \in \mathbb{N}} f_{l,0}$ and $f_1 := \sum_{l \in \mathbb{N}} f_{l,1}$. It follows that $f_0 \in H^1(\mathbb{R}^n)$ and $f_1 \in H^0_{W_p}(\mathbb{R}^n)$, with

\[ \|f_0\|_{H^1(\mathbb{R}^n)} \leq \sum_{l \in \mathbb{N}} \|f_{l,0}\|_{H^1(\mathbb{R}^n)} \leq \|f\|_{H^0(\mathbb{R}^n)} \]

and

\[ \|f_1\|^p_{H^0_{W_p}(\mathbb{R}^n)} \leq \sum_{l \in \mathbb{N}} \|f_{l,1}\|^p_{H^0_{W_p}(\mathbb{R}^n)} \leq \sum_{l \in \mathbb{N}} 2^{-lp} \|f\|^p_{H^0_{W_p}(\mathbb{R}^n)} \lesssim \|f\|^p_{H^0(\mathbb{R}^n)}. \]

Altogether, we obtain $f = f_0 + f_1$ in $S'(\mathbb{R}^n)$ and (3.8). This proves the inclusion $H^0(\mathbb{R}^n) \subset H^1(\mathbb{R}^n)$ and $H^0_{W_p}(\mathbb{R}^n)$ in the case $n(1/p - 1) \in \mathbb{N}$.

The proof of $H^0(\mathbb{R}^n) \subset H^1(\mathbb{R}^n) + H^0_{W_p}(\mathbb{R}^n)$ in the case $n(1/p - 1) \notin \mathbb{N}$ is similar to the previous proof for the case $n(1/p - 1) \in \mathbb{N}$, but now instead of (3.5) we define the set $E$ as follows:

\[ E := \{ x \in \mathbb{R}^n : f^*(x) < (1 + |x|)^{-n} \}. \]

The details are omitted. This finishes the proof of Theorem 1.2 when $p \in (0, 1)$.

Remark 3.1. Let $p \in (0, 1)$ and $B$ be a ball in $\mathbb{R}^n$. For any $s \in \mathbb{Z}_+$ and $s \geq \lfloor n(1/p - 1) \rfloor$, we know from Proposition 2.12(iv) that a function $a$ on $\mathbb{R}^n$ is a $(\Phi_p, \infty, s)$-atom associated with $B$ if and only if $a$ is a $(p, \infty, s)_W$-atom associated with the same ball $B$. On the other hand, Theorem 1.2 implies that $H^0_{W_p} \subsetneq H^0(\mathbb{R}^n)$. However, for any $p \in (0, 1)$, from Lemma 2.8 and the definitions of the dual spaces of $H^0(\mathbb{R}^n)$ and $H^0_{W_p}(\mathbb{R}^n)$ (see [13] Theorem 3.2) for $p \in (n/(n + 1), 1)$ and [14] Theorem 3.5] for general $p \in (0, 1)$), we deduce that

\[ \left[ H^0(\mathbb{R}^n) \right]' = \left[ H^0_{W_p}(\mathbb{R}^n) \right]. \]

This shows that the difference between $H^0(\mathbb{R}^n)$ and $H^0_{W_p}(\mathbb{R}^n)$ is very small.
4 Proof of Theorem 1.3

Applying Theorem 1.2, we prove Theorem 1.3 in this section.

**Proof of Theorem 1.3** We first prove (i). Let $p \in (0, 1]$ and $f \in H^{\Phi_p}(\mathbb{R}^n)$. By Theorem 1.2, we know that there exist $f_0 \in H_\Phi^0(\mathbb{R}^n)$ and $f_1 \in H^{W_p}_p(\mathbb{R}^n)$ such that $f = f_0 + f_1$ in $S'(\mathbb{R}^n)$ and

\[ \|f\|_{H^{\Phi_p}(\mathbb{R}^n)} \sim \|f_0\|_{H_\Phi^0(\mathbb{R}^n)} + \|f_1\|_{H^{W_p}_p(\mathbb{R}^n)}. \]

Since $T$ is quasilinear and bounded on $H_\Phi^0(\mathbb{R}^n)$ and $H^{W_p}_p(\mathbb{R}^n)$, it follows, from (1.1), that

\[ \|T(f)\|_{H^{\Phi_p}(\mathbb{R}^n)} \lesssim \|T(f_0)\|_{H_\Phi^0(\mathbb{R}^n)} + \|T(f_1)\|_{H^{W_p}_p(\mathbb{R}^n)} \leq \|T(f_0)\|_{H^{W_p}_p(\mathbb{R}^n)} + \|T(f_1)\|_{H^{W_p}_p(\mathbb{R}^n)} \]

\[ \lesssim \|f_0\|_{H_\Phi^0(\mathbb{R}^n)} + \|f_1\|_{H^{W_p}_p(\mathbb{R}^n)} \sim \|f\|_{H^{\Phi_p}(\mathbb{R}^n)}, \]

which implies that $T$ is bounded on $H^{\Phi_p}(\mathbb{R}^n)$. This shows (i).

The proof of (ii) is similar to that of (i) by applying the equivalence established in Theorem 1.2 (ii)

\[ H^{\Phi_p}(\mathbb{R}^n) = H^1(\mathbb{R}^n) + H^{W_p}_p(\mathbb{R}^n), \]

the details being omitted. This finishes the proof of Theorem 1.3 \hfill \Box

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