Guided subdivision surfaces: modeling, shape and refinability

Kęstutis Karčiauskas\textsuperscript{a} and Jörg Peters\textsuperscript{b}
\textsuperscript{a} Vilnius University \textsuperscript{b} University of Florida

\begin{figure}
\centering
\begin{subfigure}{0.3\textwidth}
\includegraphics[width=\textwidth]{control_net}
\caption{control net}
\end{subfigure}
\begin{subfigure}{0.3\textwidth}
\includegraphics[width=\textwidth]{guide}
\caption{guide}
\end{subfigure}
\begin{subfigure}{0.3\textwidth}
\includegraphics[width=\textwidth]{subdivision_rings}
\caption{subdivision rings}
\end{subfigure}
\begin{subfigure}{0.3\textwidth}
\includegraphics[width=\textwidth]{embossed_detail}
\caption{embossed detail}
\end{subfigure}
\begin{subfigure}{0.3\textwidth}
\includegraphics[width=\textwidth]{catmull克拉克}
\caption{Catmull-Clark}
\end{subfigure}
\begin{subfigure}{0.3\textwidth}
\includegraphics[width=\textwidth]{guided_subdivision}
\caption{guided subdivision}
\end{subfigure}
\caption{Construction and comparison of guided subdivision: (a) input net defining in (c) the irregular subdivision region and a regular (green) bi-3 region; (b) guide surface that does not match the bi-3 spline but defines the shape; (c) six subdivision rings (alternating gold and cyan) capped by a finite polynomial (red) surface cap; (d) embossing exploiting the degrees of freedom in the subdivision rings. (e) Catmull-Clark vs (f) guided subdivision: improvement of the highlight line distribution.}
\end{figure}

Abstract

Converting quad meshes to smooth manifolds, guided subdivision offers a way to combine the good highlight line distributions of recent G-spline constructions with the refinability of subdivision surfaces. Specifically, we present a \(C^2\) subdivision algorithm of polynomial degree bi-6 and a curvature bounded algorithm of degree bi-5. We prove that the common eigen-structure of this class of subdivision algorithms is determined by their guide and demonstrate that the eigenspectrum (speed of contraction) can be adjusted without harming the shape.

Categories and Subject Descriptors (according to ACM CCS):

1. Introduction

The automatic conversion of regular quad meshes into \(C^2\) surfaces of good shape and with built-in refinability addresses challenges of design and engineering analysis. Catmull-Clark subdivision offers refinability but the highlight line distributions of the resulting surfaces are often deficient, see Fig. 1e. A decade-old technique, called guided subdivision [KP07], addresses in principle both shape and refinability when based on a high-quality guide surface. The underlying idea is to sample Hermite data from a well-shaped surface that need not match the smoothness or connectivity requirements of the final output surface. However inheriting the shape of the guide is not straightforward. A careful reparameterization is required since derivatives of the guide surface may not correspond to the natural directions of the subdivision sequence of contracting surface rings.
The proposed $C^2$ subdivision (and its visually identical curvature-bounded counterpart of degree bi-5) leverage the guide shape of a recent G-spline construction, [KP15]. Highlighting the distinction between shape and mathematical smoothness, the derived guide neither matches the surface surrounding the subdivision construction nor does it generate $C^2$ surfaces; yet the derived guided subdivision surface is $C^2$ up to the central point; and it is $C^2$ at the point if the degree is bi-6, and the curvature is bounded at the center if the degree is bi-5, bi-4 or bi-3.

All new (bi-6, bi-5, bi-4 or bi-3) guided subdivision algorithms have a common simple eigenstructure that is determined by the guide. In particular, the eigenspectrum and with it the speed of contraction can be adjusted without harming the shape. For example, the surface rings that make up the subdivision surface can be made to uniformly contract by 1/2 regardless of the valence.

1.1. Related Literature

Starting with [Doo78, CC78] (and [Loo87] for triangulations), subdivision surfaces have become dominant in the animation industry (see e.g. [DKT98, NLMD12]) and have inspired many polyhedral geometry processing algorithms and vice versa [WS04]. Shape control, e.g. via semi-smooth creases, have had a stronger impact than formal smoothness; except in pockets of academia, extensions of Catmull-Clark-subdivision to $C^2$ continuity [Lev06, Zor06, KP07] have largely been ignored.

Two new developments re-kindle the interest in higher-order smooth subdivision: isogeometric computations on surfaces and advances in quad meshing. Following the early lead of [COS00] more recently subdivision surfaces have been used as computational domains, see e.g. [Bar13, PXXZ16, RSAF16, JMPR16]. Quad meshing [BLP’12, VCD’16] has matured over the past decade starting with [ACSD’03, MK04, KNP07] leveraging directional fields [MPZ14, PPM’16]. Such meshes provide natural input for guided subdivision surfaces. The use of guided subdivision surface can be traced back to [Lev06], where the guide is a single polynomial and [KP07] where the guide is allowed to be any piecewise smooth function, piecewise polynomial in particular. The critical ingredients for good shape, measured via uniform curvature and highlight line distribution, are a well-shaped guide and its careful sampling.

2. Definitions and Setup

Analogous to Catmull-Clark-subdivision, we consider the input a network of quadrilateral facets or quads. Nodes where four quads meet are called regular, otherwise irregular nodes. Our focus is on the neighborhood of irregular nodes: a 2-ring of quads surrounding an irregular node that we call a c-net. The 1-ring of a c-net consists of regular nodes and the c-net has $n \neq 4$ sectors. For example the interior nodes of Fig. 1a form a c-net.

The subdivision surface will be expressed piecewise in terms of tensor-product polynomials $p$ of bi-degree $d$ in Bernstein-Bézier (BB) form

$$p(u,v) := \sum_{i=0}^{d} \sum_{j=0}^{d} p_{ij} B_i(u) B_j(v), \quad (u,v) \in \Box := [0..1]^2,$$

where $B_i^d(t) := \binom{d}{i} (1-t)^{d-i} t^i$ are the Bernstein-Bézier (BB) polynomials of degree $d$ and $p_{ij}$ are the BB coefficients [Far02, PBP02]. A central role will be played by the corner jet constructor

$$[f]_{j \times j}^d (u_0, v_0)$$

that expresses the expansion of a function $f$ at $(u_0, v_0)$ up to and including order $j-1$ in $u$ and $j-1$ in $v$ in BB-form of degree bi-$d$, i.e. by $j \times j$ BB-coefficients. Fig. 2a displays four corner jet constructors $[f]_{3 \times 3}^d$ merged to form a bi-5 patch and Fig. 2b displays four corner jet constructors $[f]_{4 \times 4}^d$ merged to form a bi-6 patch by averaging the overlapping BB-coefficients.

Figure 2: (a) bi-5 patch assembled from $3 \times 3$ jets; (b) bi-6 patch assembled by averaging $4 \times 4$ jets.

3. Construction of guided surfaces

The key to good shape is to construct and judiciously reparameterize a guide surface so that the jet constructors can form well-shaped subdivision rings, i.e. a sequence of surface annuli in $\mathbb{R}^3$ that join smoothly leaving an ever small central hole (see Fig. 3).

3.1. Guide surface

In this section, we create a $G^2$ guide surface $g$ of degree bi-5 for filling of multi-sided hole by a series of rings sampled from $g$.

Denote by $L$ the linear shear that maps a unit square to the unit parallelogram with opening angle $\frac{2\pi}{3}$. Abbreviating $c := \cos \frac{2\pi}{3}$ set...
\( \hat{f} := L \) and let \( \hat{f} \) be its reflection across the edge \( v = 0 \), Fig. 4a. Then, along the common boundary \( v = 0 \),

\[
\partial_v \hat{f} + \partial_v \hat{f} = 2c \partial_v \hat{f} = 0 . \tag{1}
\]

\[
\partial_v^2 \hat{f} - \partial_v^2 \hat{f} + 4c \partial_v \partial_v \hat{f} = 4c^2 \partial_v^2 \hat{f} = 0 . \tag{2}
\]

The constraints (1) and (2) are ‘unbiased’ in the sense that exchanging \( f \) and \( \hat{f} \) does not alter the equations. For degree bi-5 polynomials \( f \) and \( \hat{f} \), the equations (1) and (2) yield a system of 12 linear equations in the BB-coefficients \( p_{ij} \) and \( \hat{p}_{ij} \). These equations are solved symbolically, leaving as unconstrained BB-coefficients those marked in Fig. 4b by red or black bullets plus one circled cross.

The interactions between the \( n \) local systems of equations at the irregular point \( p_{00} \) are resolved by selecting six BB-coefficients \( p_{ij} \), \( 0 \leq i + j \leq 2 \) for \( k = 0 \) and so defining a quadratic expansion at the central point; the BB-coefficients \( p_{ij} \), \( 0 \leq i + j \leq 2 \), for \( k > 0 \), are then defined recursively as

\[
p_{00} := p_{00} , \quad p_{10} := p_{10} , \quad p_{20} := p_{20} ;
\]

\[
p_{01} := - p_{01} + 2c p_{10} + 2(1-c)p_{00} ;
\]

\[
p_{11} := - p_{11} + \frac{8c}{3} p_{20} + (2-6c)p_{10} - 2c p_{00} ;
\]

\[
p_{02} := p_{02} - 5c p_{11} + 4c^2 p_{20} + (5c-4)p_{10} + c(9-8c)p_{10} + (4-9c+4c^2)p_{00} .
\]

Assigning, in each local system,

\[
p_{12} := p_{12} - 2c p_{21} + 2c p_{21} + c p_{01} + (3c-4)p_{11} + \frac{4c(4-3c)}{5} p_{20} + (4-\frac{27}{5}c+\frac{4}{5}c^2)p_{10} + \frac{c}{5}(8c-9)p_{00} ,
\]

yields a circulant system of \( n \) linear equations in \( p_{12}^k \) (since \( p_{12}^k | p_{12}^k+1 = p_{12} \) see rotationally-symmetric indexing in Fig. 5). This system has a unique solution (unless \( n = 3 \) when one \( p_{12}^k \) is additionally free to choose). The explicit expressions for \( p_{30}, p_{31}, p_{32}, p_{41}, p_{41}, p_{51} \), \( p_{51} \) are presented in Appendix A.

Denote by \( P \) the set of BB-coefficients unconstrained by the \( G^2 \) constraints. We define \( n^* := n+1 \) whenever the valence \( n \in \{3,6\} \) and \( n^* = n \) otherwise. The elements of \( P \) are shown in Fig. 5: six red bullets for the quadratic expansion, \( 8n+n^* \) black bullets, and, in each sector corner, \( 3 \times 3 \) cyan bullets that do not affect \( G^2 \) continuity between sectors. The elements of \( P \) are pinned down to best approximate the bi-5 surface \( G \) of the c-net as defined in [KP15]. Let \( \sigma : \mathbb{R}^{2 \times n} \rightarrow \mathbb{R}^2 \) be the \( 2\pi/n \)-rotationally symmetric map obtained by applying the algorithm of [KP15] to the control points of the planar characteristic c-net of Catmull-Clark-subdivision shown in Fig. 6a. Fig. 6b, left, shows one sector of \( \sigma \) for \( n = 5 \) and Fig. 6b, right shows \( L^{-1} \circ \sigma \) to be used for sampling the guide \( g \).
the BB-coefficients \( b^k_{ij}, k = 0, \ldots, n - 1, i, j = 0, \ldots, 5 \). The \( b^k_{ij} \) are linear expressions in \( P \) and we assemble them to form \( n \) patches \( b^k \) of degree bi-5. Then \( P \) is the least-squares solution of minimizing the sum of differences between each \( b^k_{ij} \) and its corresponding coefficient of \( G \).

### 3.2. Parameterization and guided surface rings

The guide \( g \) was constructed to facilitate filling the multi-sided hole by a contracting sequence of sampled guided rings. The construction, see Fig. 7, leverages the fact that the characteristic ring \( \chi \) of Catmull-Clark subdivision joins \( C^2 \) to its scaled copy \( \lambda \chi \), where \( \lambda := \frac{1}{16} (c + 5 + \sqrt{(c + 1)(c + 9)}) \) is the subdominant eigenvalue of Catmull-Clark subdivision. Since the shear \( L \) is linear, also \( L^{-1} \circ \chi \) is \( C^2 \)-connected to its scaled copy \( \lambda (L^{-1} \circ \chi) \) (see Fig. 7d). Moreover, the outer second-order Hermite data of \( L^{-1} \circ \chi \) (underlaid gray in Fig. 7a) is determined by the \( C^2 \) prolongation \( L^{-1} \circ (\lambda^{-1} \chi) \) and binary splitting (see Fig. 7c).

### Subdivision Algorithm.

Input: a sector \( g^k \) of guide \( g \).

Output: BB-coefficients of three bi-\( d \) (\( d = 5 \) or \( d = 6 \)) patches in the \( k \)-th sector of the \( l \)th subdivision ring \( x^l_i \), \( l = 0, \ldots. \)

Bi-5 Algorithm: sample \( [g^k \circ (L^{-1} \circ \lambda \chi)]_{3 \times 3} \) at the inner corners marked by bullets in Fig. 7a to obtain the bi-5 coefficients of Fig. 7c underlaid white; sample \( [g^k \circ (\lambda^2 \chi)]_{3 \times 3} \) at the outer corners marked as by circles in Fig. 7b and mid-split the resulting bi-5 data to obtain the bi-5 coefficients of Fig. 7c underlaid gray. The first of these gray sets of BB-coefficients (\( l = 0 \)) is replaced by \( C^2 \) prolongation from the surrounding surface.

The bi-6 algorithm is identical, except that \( [g^k \circ (L^{-1} \circ \lambda \chi)]_{4 \times 4} \) is sampled and assembled according to Fig. 2b.

### Proposition 1

The Subdivision Algorithm fills a multi-sided hole in a spline surface \( C^2 \) up to the central point. If the degree is bi-6, the surface is \( C^2 \); if the degree is bi-5, the surface is \( C^2 \) except for the central point where it is \( C^1 \) and curvature bounded.

**Proof:** Since \( \chi \) is \( C^2 \) and since the same \( G^1 \) and \( G^2 \) constraints (1) and (2) tie together adjacent sectors of \( g \) and tie together adjacent sectors of \( L \), adjacent sectors are \( C^2 \)-connected. Due to prolongation, adjacent rings are \( C^2 \)-connected and so is the first ring to the surrounding spline surface. The limit point characterization follows by the arguments of [KP07, Thm 1] applied to \( (g \circ L^{-1}) \circ \chi \).

Since the guide \( g \) closely approximates the surface \( G \) constructed according to [KP15], the good shape of \( G \) is retained.

#### Efficient Implementation.

Computing \( [g^k \circ (L^{-1} \circ \lambda \chi)]^5_{3 \times 3} \) for the bi-5 Subdivision Algorithm is equivalent to (i) linearly mapping \( S : [0..1]^2 \to [0..\lambda]^2 \) and (ii) sampling \( [[g^k \circ S] \circ (L^{-1} \circ \chi)]^3_{3 \times 3} \) and \( ([g^k \circ S] \circ (L^{-1} \circ \chi)]^3_{3 \times 3} \). In Step (i) applying de Casteljau’s algorithm at \( u = \lambda = v \) yields the BB-coefficients of \( g^k \circ S \) as an affine combination of \( g^k \). This affine \( 6^2 \times 6^2 \) map is tabulated. For (ii), a sector consisting of three patches of degree bi-5 is pre-computed for each valence \( n \) as a table of size \( 3(6^2 \times 6^2) \), by sampling \( [g^k \circ (L^{-1} \circ \chi)]^3_{3 \times 3} \) and \( [g^k \circ (L^{-1} \circ \chi)]^3_{3 \times 3} \) for symbolic \( g^k \). Analogously, for bi-6 subdivision, we pre-compute for each valence \( n \) a table of size \( 3(7^2 \times 6^2) \).

If the storage of the pre-computed data is a concern, one can leverage that the majority of the entries can be derived by \( C^2 \) prolongation from the previous ring and the remainder is defined by two jets in each sector, one of which has diagonal symmetry. For bi-5 subdivision, this reduces the tabulation for each valence by a factor of \( \sim 7 \) from \( 3(6^2 \times 6^2) \) to \( 15 \times 6^2 \).

Computing \( [g^k \circ (L^{-1} \circ \lambda \chi)]^6_{4 \times 4} \) analogously yields an efficient implementation for the bi-6 variant.

![Figure 7: Construction of guided ring.](image)

![Figure 8: \( G^1 \) caps for \( n = 5 \). The cap is surrounded by the last ring produced by guided subdivision (blue).](image)

**A Practical Hybrid.** Being able to cap the subdivision surface after a few steps is useful in practice, already for visualization (see the next section). For Catmull-Clark subdivision the first refinement steps introduce highlight line distortions that no cap can repair. By contrast, guided subdivision retains good shape and enables capping without loss of shape quality. Fig. 8 shows the natural structure of these bi-6 and bi-5 caps (bi-5 caps with one patch per sector lead to reduced quality, hence the split). A prototypical construction, of a bi-6 cap, is detailed in Appendix B. The good shape of this finite hybrid representation makes them aesthetically useful in their own right and the finite number of polynomial surface pieces simplifies their use for downstream applications and to serve as a domain for surface-based computations.
4. Examples

The Subdivision Algorithm is applied to the challenging c-nets presented in Fig. 9. Displaying the surrounding spline surface in green in Fig. 10 for valence \( n = 5 \), Fig. 11 for valences \( n = 6, 7 \), and Fig. 12 for valences \( n = 8, 9 \) and \( n = 3 \) demonstrates that we do not just create good caps for the various \( n \), but caps that transition well from the input data. The finite caps of the hybrid representation are red. 'Zoom' indicates that innermost guided ring and the finite cap are displayed. The zoom demonstrates that even the caps have a calm curvature distribution, quite in contrast to Catmull-Clark-subdivision, where the shape deficiencies are acerbated at each step.

5. Eigen-decomposition

The eigen-decomposition of guided subdivision differs from that of conventional subdivision such as Catmull-Clark, in that it is defined by the guide surface: any guided subdivision inherits the guide’s eigen-decomposition. Therefore the decomposition does not overly involve the specification of large circulant matrices or of characteristic polynomials; and the same analysis applies to guided subdivision constructions of different polynomial degree. If we consider just one bi-5 patch of \( g \) without neighbor-interaction, the scaling by \( \lambda \) to the next-level subdivision ring (Section 3.2) yields, due to the monomial structure, eigenvalues \( \lambda^s \) for total degree \( s = 0, \ldots, 10 \) and eigenfunctions \( u^i v^j \) of tensor-degree \((i, j)\), \( 0 \leq i, j \leq 5 \) (see Fig. 13). The eigen-analysis of a full ring is more complex due to the interaction between neighbors, but retains this basic structure.

The Subdivision Algorithm constructs the \( f \)th surface ring from a vector of BB-coefficients \( P_f \) corresponding to the set \( \mathcal{P} \) of the guide \( g \) restricted to \([0, \lambda_1^2] \). With \( M \) the map so that \( P_f = M P_{f-1} \), the eigen-decomposition determines \( w \) and \( \mu \) so that \( M w = \mu w \). Appendix C shows that all eigenvalues \( \mu \) are of the form \( \lambda^s \) where \( 0 < \lambda < 1 \) is free to choose:

\[
\begin{align*}
\lambda^s, \mu = & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\
\text{multiplicity} = & 1 & 2 & 3 & n^* & 2n & 3n & 2n & n 
\end{align*}
\]
where \( n^* = n + 1 \) when \( n \in \{3, 6\} \) and \( n^* = n \) otherwise. The bi-5 eigenfunctions for \( n = 5 \) shown in Fig. 14 are determined by setting one free parameter \( x_1^3 \) (see Fig. 24) to 1 and all others to 0, completing this determining set to a smooth eigenring according to the Subdivision Algorithm, and generating six scaled copies before filling the center with a (red) cap. For the Subdivision Algorithm, and generating six scaled copies before completing this determining set to a smooth eigenring according to (1) and (2) and the eigendecomposition are analogous to bi-5 case. Corresponding to the unconstrained control points for the guides \( g \) (cf. Fig. 15), we count

\[
\begin{array}{|c|c|}
\hline
\text{degree of } g & \text{eigenfunctions} \\
\hline
\text{bi-5} & 17n + 6 + n^* \\
\text{bi-4} & 9n + 6 + n^* \\
\text{bi-3} & 3n + 6 + n^* \\
\hline
\end{array}
\]

Due to the index-rotational symmetry, for \( s = 3 \ldots 10 \) only 23 (= 17 + 1 + 5) eigenfunctions (besides the constant one for the eigenvalue 1) need to be tabulated. An exception to this symmetry is \( s = 3 \) when \( n = 3, 6 \) and 26, 29 functions are required. Since the eigen-ring \( \ell \) is obtained by multiplying the initial eigen-ring by \( \langle \lambda^s \rangle^\ell \), it suffices to store the initial eigen-ring. The \( \ell \)th surface ring is a linear combination of these \( \langle \lambda^s \rangle^\ell \)-scaled eigen-rings.

6. Discussion

Here we discuss alternatives and options that were not given prominence in the main construction: reducing the degree of the guide (to simplify the eigenstructure), reducing the degree of the overall guided subdivision scheme, changing the speed of the contraction of the subdivision rings and refining functions on a guided subdivision surface.

6.1. Guides of lower degree

The degree of the subdivision surfaces is not strongly coupled to the degree of the guide surface. We can choose guides of degree bi-4 or bi-3 where the enforcement of (1) and (2) and the eigendecomposition are analogous to bi-5 case. Corresponding to the unconstrained control points for the guides \( g \) (cf. Fig. 15), we count
see Fig. 16b; and twice repeating the bi-5 subdivision improves the highlight line distribution to Fig. 16c. The c-net of Fig. 9h, a single off-center spike, is notorious for revealing shortcomings of surface constructions. Yet, the results (see Fig. 16 bottom row) of applying a bi-4 guide (after zero or one step of bi-5 subdivision) are difficult to distinguish from those of the bi-5 guide. By contrast, (full enlargement of) Fig. 16g shows oscillations of the highlight lines for a bi-3 guide even after two bi-5 preprocessing steps.

6.2. Bi-4 and bi-3 guided subdivision surfaces

For some applications, degree bi-5 may be considered a drawback. Leveraging the weak link between shape and smoothness, we introduce $2 \times 2$ macro patches for degree bi-4 and $3 \times 3$ macro patches for degree bi-3 (see Fig. 17) to enforce the $C^2$ prolongation between rings. The resulting surfaces are curvature bounded at the central point and preserve the highlight line distribution of the bi-5 construction well – at the cost of increasing the number of polynomial pieces by a factor of 4, respectively 9.

Notably, for the corresponding hybrid construction, the $G^1$ bi-3 caps are formally only $C^0$-connected to the last guided ring. Yet, the resulting highlight line distribution is without flaw (rightmost zoomed image of Fig. 18b); by contrast, enforcing exact $C^1$-continuity reduces the uniformity of the highlight line distribution.

6.3. Changing the contraction speed

Using the subdominant eigenvalue $\lambda$ of Catmull-Clark subdivision for $\sigma$ implies that the contraction of guided rings becomes slower when the valence increases (Fig. 12b vs c). Using instead the characteristic map of [KP09], the eigenvalue can be set to $\frac{1}{2}$ for all valencies $n > 4$ to yield a uniform contraction speed. Fig. 19 top vs bottom contrasts the characteristic map for Catmull-Clark subdivision and $\lambda$, with the uniform contraction by $\frac{1}{2}$ of [KP09]. Fig. 20a shows the same surface as Fig. 12a while Fig. 20b is constructed with contraction speed $1/2$. The latter has visually identical highlight lines, an observation that holds for all c-nets that we tested, including all of Fig. 9 (The increased contraction is evident from the size of the red caps.) Fig. 12b compares to Fig. 20d. All c-nets of Fig. 9 have a more uniform curvature distribution in the vicinity of the caps when using speed $1/2$.

6.4. Refinement for functions on guided subdivisions surfaces

With the shape of the subdivision surface determined by the c-net via $P$ of the guide, here we define a nested space of refinable functions on the surface. The combinatorial layout of the functions is identical to that of the surfaces. For example, each refinement of the bi-5 construction yields $18n$ new degrees of freedom. In Fig. 21a,
the BB-coefficients determined by $C^2$ extension inwards are underlaid in gray and the 18$n$ free coefficients of the outermost subdivision ring that is no longer influenced by a once-refined set $P$ are shown as brown bullets. Functions corresponding to the markers 1, 2, 3 are displayed in Fig. 22, b, c, d. (Standard bi-5 respectively bi-6 spline subdivision with triple knots can be applied to these rings in subsequent refinements.) In addition to the 18$n$ degrees of freedom, each subdivision step offers $N$ degrees of freedom corresponding to the set $P$. If the identity function is to be represented, the refined set $P$ is obtained from its coarser predecessor Fig. 21b via de Casteljau’s algorithm.

While eigendecomposition can be used to obtain finite expressions for computations on subdivision surfaces, the considerable number of terms make us think that most numerical computations are better served by computing with the hybrid representation after a suitable number of refinement steps.

Figure 21: Structure of refinable functions on guided subdivision surfaces. (a) Degrees of freedom (brown bullets) in the $C^2$ bi-5 ring not influenced by the refined $P$. The BB-coefficients in the gray layers are defined by $C^2$ expansion from the surrounding surface. (b) A guide set $P$ exist at each refinement step.

Figure 22: (a) BB-coefficients of the innermost ring from Fig. 21a and the filling by subdivision. (b), (c), (d) Bi-5 functions $f_j$ with the meaning of the subscripts indicated in Fig. 21a.

7. Conclusion

The new guided subdivision surfaces offer an automatic conversion of quad meshes with irregular vertices into $C^2$ surfaces of good shape and built-in refi nanility. A hybrid surface alternatively uses finitely many polynomial pieces that preserve the shape but are more readily amenable to subsequent computations on the surface. The construction of guided subdivision surfaces is conceptually simple, and has been implemented and tested on challenging examples. The eigen-structure of this class of subdivision algorithms is determined by the guide and fully analyzed. The speed of contraction can be adjusted without harming the shape.

References

[ACSD+03] Alliez P., Cohen-Steiner D., Devillers O., Levy B., Desbrun M.: Anisotropic polygonal remeshing. ACM Trans. Graph. 22, 3 (July 2003), 485–493. 2

[Ba13] Barendrecht P. J.: IsoGeometric Analysis with Subdivision Surfaces. PhD thesis, MS Thesis, Technical University of Eindhoven, 2013. 2

[BLP+12] Bommes D., Levy B., Pietroni N., Puppo E., Silva C., Tarini M., Zorin D.: State of the art in quad meshing. In Eurographics STARS (2012). 2

[CC78] Catmull E., Clark J.: Recursively generated B-spline surfaces on arbitrary topological meshes. Computer Aided Design 10 (1978), 350–355. 2, 7

[COS00] Cirak F., Ortiz M., Schröder P.: Subdivision surfaces: A new paradigm for thin-shell finite-element analysis. Internat. J. Numer. Methods Engrg. 47, 12 (2000), 2039–2072. 2

[DKT98] DeRose T., Kass M., Truong T.: Subdivision surfaces in character animation. In SIGGRAPH ’98: Proceedings of the 25th annual conference on Computer graphics and interactive techniques (New York, NY, USA, 1998). ACM Press, pp. 85–94. 2

[Doo78] Doo D.: A subdivision algorithm for smoothing down irregularly shaped polyhedrons. In Int’l Conf. Interactive Techniques in Computer Aided Design (Bologna, Italy, 1978), IEEE Computer Soc., pp. 157–165. 2

[Far02] Farin G.: Curves and Surfaces for Computer Aided Geometric Design: A Practical Guide. Academic Press, San Diego, 2002. 2

[JMPR16] Jüttler B., Mantzafaris A., Perl R., Rumpf M.: On numerical integration in isogeometric subdivision methods for PDEs on surfaces. Computer Methods in Applied Mechanics and Engineering 302 (2016), 131–146. 2

[KnP07] Kalberer F., Nieser M., Polthier K.: Quadcover - surface parameterization using branched coverings. Comput. Graph. Forum 26, 3 (2007). 2

[KP07] Karčiauskas K., Peters J.: Concentric tesselation maps and curvature continuous guided surfaces. Computer Aided Geometric Design 24, 2 (Feb 2007), 99–111. 1, 2, 4

[KP09] Karčiauskas K., Peters J.: Adjustible speed surface subdivision. Computer Aided Geometric Design. 26 (2009), 962–969. 7

[KP15] Karčiauskas K., Peters J.: Improved shape for multifaceted surface blendings. Graphical Models 82 (2 2015), 87–98. 2, 3, 4

[Lev06] Levin A.: Modified subdivision surfaces with continuous curvature. ACM Trans. Graph 25, 3 (2006), 1035–1040. URL: http://doi.acm.org/10.1145/1141911.1141990. 2

[Loo87] Loop C.: Smooth subdivision surfaces based on triangles. Master’s thesis, Department of Mathematics, University of Utah, 1987. 2

[MK04] Marinov M., Kobelt L.: Direct anisotropic quad-dominant remeshing. In Proceedings of the Computer Graphics and Applications, 12th Pacific Conference (Washington, DC, USA, 2004), PG ’04, IEEE Computer Society, pp. 207–216. URL: http://dl.acm.org/citation.cfm?id=1025128.1026044. 2

[MPZ14] Mylles A., Pietroni N., Zorin D.: Robust field-aligned global parameterization. ACM Trans. Graph. 33, 4 (July 2014), 135:1–135:14. URL: http://doi.acm.org/10.1145/2600107.2600154, doi:10.1145/2600107.2600154. 2

[NLMD12] Niessner M., Loop C., Meyer M., DeRose T.: Feature-adaptive GPU rendering of catmull-clark subdivision surfaces. ACM Trans. Graph. 31, 1 (2012). 2

© 2017 The Author(s)

Computer Graphics Forum © 2017 The Eurographics Association and John Wiley & Sons Ltd.
A. Appendix

Let
\[ d_1 := 4 + 8c + 5c^2, \quad d_2 := 4 + 5c, \quad d_3 := 36 + 45c + 5c^2, \]
\[ d_4 := 20 + 36c + 9c^2 - 15c^3, \quad d_5 := 32 + 48c + 15c^2, \]
\[ d_6 := 16 + 36c + 36c^2 + 15c^3, \quad d_7 := 32 + 64c + 72c^2 + 48c^3 + 15c^4. \]

Then
\[
\begin{align*}
\bar{p}_{30} &= \frac{2}{3} p_{10} + c - \frac{5}{3} c p_{10} + \frac{5}{6} c (p_{21} + p_{22}); \\
\bar{p}_{32} &= \frac{d_2}{d_1} (p_{11} - p_{10}) + \frac{d_7}{3d_1} (p_{21} - p_{22}) - \frac{d_6}{3d_1} (p_{21} - p_{22}); \\
\bar{p}_{31} &= \frac{2c}{15} p_{10} + \frac{8c}{15} p_{10} + \frac{2}{3} p_{11} - \frac{25 + 16c^2}{15c} p_{20}; \\
&\quad + \frac{1}{6c} (p_{21} + \frac{3}{2} p_{21} + \frac{1}{3c} (p_{22} - p_{22}) + \frac{2c}{5} p_{40}; \\
\bar{p}_{41} &= \frac{2c}{15d_1} (c d_4 p_{20} - d_5 d_4 p_{10} + 5d_4 p_{11}) \\
&\quad + \frac{4d_1}{15d_1} p_{20} - \frac{d_5}{d_1} p_{21} - \frac{c(16 + 25c^2)}{6d_1} p_{21} + \frac{d_2}{3d_1} (p_{22} - p_{22}) \\
&\quad + (1 + \frac{3c}{5}) p_{40} + d_2 \frac{d_1}{4d_1} (p_{42} - p_{42}) + \frac{c}{4d_1} (p_{50} - p_{52} - p_{52}); \\
\bar{p}_{51} &= \frac{2c}{3d_1} (c p_{40} - (3c - 5)p_{10} + 5p_{11}) \\
&\quad + \frac{c}{6d_1} (c(16 + 25c^2)p_{20} + (5c - 4)(p_{21} - p_{21}) + 10(p_{22} - p_{22}) - (c p_{40} + \frac{5c}{4d_1} (p_{42} - p_{42}) + (1 + c)p_{50} + \frac{4 + 3c}{4d_1} (p_{52} - p_{52}).
\end{align*}
\]

For \( i = 3, 4, 5 \), we get \( \bar{p}_i \) from \( p_i \) by swapping \( \bar{p} \) with \( \bar{p} \).

B. Appendix: hybrid caps of degree bi-6

The bi-6 caps are internally \( G^1 \) according to
\[ \partial_i \bar{f} + \partial_i f - 2c(1 - u^2) \partial_i \bar{f} = 0. \]
and they are \( C^1 \)-connected to the last guided bi-6 surface ring. With the notations and indexing of Fig. 23a in the guide construction of Section 3.1, the unconstrained coefficient of the local solution to (6) are bullets in Fig. 23a. The interactions between the \( n \) local \( G^1 \) systems of equations at the irregular point \( p_{10} := p_{00} \) are resolved by selecting three BB-coefficients in one sector (red disks in Fig. 23a) to define the tangent plane at the irregular point and define the corresponding BB-coefficients in other sectors recursively as
\[ p_{10} := p_{10}, \quad p_{01} := -p_{01} + 2c p_{10} + 2(1-c)p_{00}. \]
The explicit formulas for the dependent points of the local solution are
\[
\begin{align*}
p_{20} &= \frac{3}{5c} (p_{11} + p_{11}) + \frac{6}{5c} (p_{10} + p_{10}) - \frac{1}{5p_{10}}; \\
p_{30} &= \frac{1}{20} (p_{00} + p_{60}) - \frac{3}{10} (p_{10} + p_{50}) + \frac{3}{4} (p_{20} + p_{40}); \\
p_{40} &= \frac{1}{2} (p_{41} + p_{41}) + \frac{c}{15} (p_{50} - p_{60}); \\
p_{50} &= \frac{1}{2} (p_{51} + p_{51}); \quad p_{60} = \frac{1}{2} (p_{51} + p_{61}); \\
p_{21} &= -p_{21} - \frac{c}{15} p_{00} + \frac{2c}{5} p_{10} + (2 - c)p_{20} + c p_{40} - \frac{2c}{5} p_{50} - \frac{c}{15} p_{60}; \\
p_{31} &= -p_{31} + \frac{1}{10} p_{00} - \frac{3}{2} p_{10} + \frac{3}{2} p_{20} + \frac{3-c}{2} p_{40} + \frac{1-c}{10} p_{50} - \frac{c}{60} p_{60}. 
\end{align*}
\]

A bi-6 reparameterization \( \tilde{\sigma} \) for sampling the guide is rotationally and sector bisection symmetric and the outer BB-coefficients (blue underlay in Fig. 23b) \( C^\infty \)-extend the characteristic ring of Catmull-Clark subdivision. This leaves 14 free parameters that are set to minimize the sum up to fifth derivatives,
\[ \int_0^1 \int_{|s+t| \leq 3.1} h_{s,t}(s,t) (\partial^3_{s,t} f(s,t))^2 dsdt. \]
Applying De Casteljau at \( u = \lambda' = v \) to the sector of the guide and sampling \( \frac{5}{4} \sqrt{\lambda'} \circ \partial_{\lambda\lambda}^4 \) at all four corners of \( \tilde{\sigma} \) to form the bi-6 patch according to Fig. 2b.
implies that the resulting cap \( \hat{g}_k \) inherits the unique quadratic expansion of the guide. \( C_1 \)-extending the last guided ring leaves \( \hat{p}_{121}, \hat{p}_{31}, \hat{p}_{21}, \hat{p}_{31} \) (see Fig. 23a) to be the least squares best fit to the corresponding BB-coefficients of \( \hat{g}_k \) and \( \hat{g}_{k+1} \). As for \( g \) pretabulation simplifies practical computation.

Although the construction is formally only \( G^1 \), it is curvature continuous at the center and this partly accounts for its good shape.

C. Appendix: Eigenanalysis of the subdivision algorithm

Since the central point stays fix, the dominant eigenvalue is 1. Fig. 24 lists the indices of the other unconstrained control points \( \mathcal{P} \) of the guide \( g \) (recall that red bullets labeled 1,...,5 are only unconstrained for sector \( k = 0 \) and that for \( n = 3, 6 \) there is an additional degree of freedom at the location marked by a circled cross in Fig. 4b). With the abbreviations

\[
\begin{align*}
\kappa &:= \begin{cases} 
n + 1, & \text{for } n \in \{3, 6\}, 
 1, & \text{else,}
\end{cases} \\
\tilde{m} &:= 6 + n, \\
\tilde{k} &:= 6 + k, \\
\tilde{k} &:= m + k, \\
\tilde{n} &:= 6n + \tilde{k}, \\
d_1 &:= 1, \ldots, 16,
\end{align*}
\]

we label the \( N := 17n + m - 1 \) elements of \( \mathcal{P} \) as illustrated in Fig. 24.

![Figure 24: Indices for the eigen-analysis.](image)

After application of de Casteljau at \( \lambda \), the linearly-reparameterized bi-5 patches satisfy the unchanged constraints (1) and (2); this was intended and is verified by inspection. The mapping of \( \mathcal{P} \) to its next-level counterpart yields systems of linear eigen-equations \( eq_i^s \) in unknowns \( x_i^s, i, j = 1, \ldots, N \), that form the eigenvectors corresponding to eigenvalues \( \lambda^s \), \( s = 1, \ldots, 10 \). Solving the large and highly underconstrained systems with symbolic entries \( \lambda \) defies the capabilities of Maple, hence requires some careful guesses based on an observed pattern for concrete \( n \) and \( \lambda \). The underconstrained systems are reduced by judiciously setting various \( x_i^s \) to zero and solving, for the specific \( \lambda := \frac{1}{2} \), a subset of (system, variables)-pairs \( (eq_i^s, x_i^s) \). We abbreviate

\[
a : b := a, a + 1, \ldots, b, \quad \kappa(\alpha, \beta) := (m + \alpha n : m + \beta n - 1).
\]

For \( s = 1 : 10 \), we list parameters set to zero, pairs of equations and variables, and the free parameters that will characterize the eigenvectors:

\[
s = \begin{cases} 
x_i^s = 0 & (eq_i^{s,N}, x_i^{s,N}) \\
\end{cases}
\]

\[
\begin{array}{|c|c|c|}
\hline
s & i & j & k \\
\hline
1 & 3 & 1, 2 & \\
2 & 2 & 6 & 3, 5 \\
3 & 5 & m & 6, m - 1 \\
4 & m - 1 & 2n + m & \kappa(0, 2) \\
5 & 2n + m - 1 & (I_2, I_5) & \kappa(2, 4) \cup \kappa(5, 6) \\
6 & 4n + m - 1 & 7n + m & \kappa(4, 7) \\
7 & 7n + m - 1 & (I_5, I_7) & \kappa(7, 8) \cup \kappa(9, 11) \\
8 & 11n + m - 1 & 14n + m & \kappa(11, 14) \\
9 & 14n + m - 1 & 16n + m & \kappa(14, 16) \\
10 & 16n + m - 1 & \kappa(16, 17) \\
\hline
\end{array}
\]

For \( s = 5 \), system indices and variables differ: \( I_5 := \kappa(5, 17) \) and the variables are the union of labels \( I_5 := \kappa(4, 5) \cup \kappa(6, 17) \). For \( s = 7 \), \( I_7 := \kappa(8, 9) \cup \kappa(11, 17) \). The system of equations \( eq_{10}^s \) is not listed since it is satisfied by setting \( x_{10}^{10} = 0 \). The solutions of these systems are substituted into the initial systems with symbolic \( \lambda \) to verify that they solve the equations for any choice of \( \lambda \). This yields explicit formulas for the eigenvectors in terms of the free parameters listed in the right column of (9). The eigenvectors, one per free parameter, span the eigenspace with the eigenvalues listed in Table (5).