Some Properties of Blow up Solutions in the Cauchy Problem for 3D Navier–Stokes Equations

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Abstract: Up to now, it is unknown an existence of blow up solutions in the Cauchy problem for Navier–Stokes equations in space. The first important property of hypothetical blow up solutions was found by J. Leray in 1934. It is connected with norms in $L^p(R^3)$, $p > 3$. However, there are important solutions in $L^2(R^3)$ because the second power of this norm can be interpreted as a kinetic energy of the fluid flow. It gives a new possibility to study an influence of kinetic energy changing on solution properties. There are offered new tools in this way. In particular, inequalities with an invariant form are considered as elements of latent symmetry.

Keywords: Navier–Stokes equations; blow up solution; dissipation parameter; regular solution

1. Introduction

Description of the incompressible fluid flow motion is that problem which attracts attention of some thousand researchers during the last century. A part of this information can be found, for example, in [1]. We consider the simplest problem. More exactly, the Cauchy problem in space which is described by equations:

$$
\frac{\partial u}{\partial t} + \sum_{i=1}^{3} u_i \frac{\partial u}{\partial x_i} = \nu \Delta u - \nabla P, \quad \text{div} \ u = 0, \ u(0, x) = \varphi(x),
$$

where $u = u(t, x) = (u_1(t, x), u_2(t, x), u_3(t, x))$ is a velocity of fluid flow, $t, x = (x_1, x_2, x_3)$ are time and spatial variables, respectively. Here, $P = P(t, x)$ is a pressure function, $\varphi(x)$ is an initial data, $\nu$ is a coefficient of viscosity. Symbols $\triangle$ and $\nabla$ denote the Laplace operator and gradient operator on spatial variables, respectively. In particular, $\nabla u$ is Jacobi matrix on spatial variables (an application of it shown below).

Well-known classical results by O. A. Ladyzhenskaya and J. Serrin show existence of a time interval $[0, T)$ where solution $u$ is regular on an infinite cylinder $[0, T) \times R^3$. Let $T_* = \max T$ be a maximum of these $T$. The finite mean of $T_*$ also has the interest for applications because it indicates collapse phenomenon. Usually, a solution of problem (1) with finite $T_*$ is called a blow up solution and the fluid flow is considered as a turbulence flow.

For the first time, J. Leray (see [2]) proved that the blow up solution had the following estimate:

$$
\|u(t, \cdot)\|_p = \int_{R^3} |u(t, x)|^p dx \geq C(v, p)(T_* - t)^{3-p}, \quad p > 3,
$$

where a constant $C = C(v, p)$ depends on $v, p$. In the same place it is showed another estimate for irregular solution $u$:

$$
\sup_{x \in R^3} |u(t, x)|^2 \geq \frac{A v}{T_* - t}
$$
with some constant $A$.

After 70 years since the work [3], authors indicated that blow up solution satisfied the following condition:

$$\limsup_{t \to T_*} \|u\|_3^3 = \infty.$$  

It was the first essential advance in this way. The final step is done in [4], where it was proved:

$$\lim_{t \to T_*} \|u\|_3^3 = \infty.$$  \hspace{1cm} (2)

These results can be considered as an extension of Leray’s result for $p = 3$.

Similar ideas for other functional spaces are applied in [5–7].

Close to these applications, different results can be found in [8].

Finally, the author suggests a new design where Leray ideas are not applicable and inequalities with invariant form as elements of a latent symmetry are very useful. Following [9] (formulae (5), (68), (69), (87)), let us define numerical parameters $\lambda$, $\mu$, $\varepsilon$, and number $T_0$ using formulae:

$$l(\varphi) = \|\varphi\|_2 \cdot \|\nabla \varphi\|_2, \quad \lambda = \left(\frac{4\sqrt{3}}{3a_1}\right)^2 \frac{\nu^2}{l(\varphi)} = \frac{\sqrt{3}}{4\nu},$$  \hspace{1cm} (3)

$$\mu = \frac{T_*}{T_0},$$  \hspace{1cm} (4)

$$\|u(T_0, \cdot)\|_2^2 = \|\varphi\|_2^2 (1 - \varepsilon \lambda^2),$$  \hspace{1cm} (5)

$$T_0 = \left(\frac{9}{4}\right)^4 \frac{\nu^3}{\|\nabla \varphi\|_2^2} = \frac{cv^3}{\|\nabla \varphi\|_2^2},$$  \hspace{1cm} (6)

where $\nabla \varphi$ is Jacobian matrix of vector field $\varphi$, $[0, T_0)$ is that time interval (it is not optimal) where every weak solution of (1) is regular and it satisfies condition

$$\|\nabla u(T, \cdot)\|_2^2 \leq \frac{\|\nabla \varphi\|_2^2}{1 - \frac{T}{T_0}}.$$  

If mean $T_*$ is finite, (see [9], Lemma 50, Theorem 7), then these parameters satisfy inequalities $\lambda < 1$, $0 < \varepsilon < 1$ and

$$\frac{1}{4} (\varepsilon + 1^2) < \mu < \lambda^{-4}. \hspace{1cm} (7)$$

Moreover, the kinetic energy of the fluid flow must be close to minimum at moment $T_0$, i.e., the following inequalities are fulfilled:

$$\|\varphi\|_2^2 (1 - \lambda^2) < \|u(T_0, \cdot)\|_2^2 < \|\varphi\|_2^2 \sqrt{1 - \lambda^2}.$$  

In addition, in this case the kinetic energy of turbulence flow (see [9]) must change with restrictions:

$$\|\varphi\|_2^2 \lambda^2 \sqrt{\mu - \frac{T}{T_0}} + \|u(T_*, \cdot)\|_2^2 \leq \|u(t, \cdot)\|_2^2 \leq \|\varphi\|_2^2 \left(1 - \sqrt{\mu} \lambda^2 + \lambda^2 \sqrt{\mu - \frac{T}{T_0}}\right).$$

A weak solution (for a definition see, for example, [1]) $u$ of problem (1) is regular on an interval $(0, T)$ if kinetic energy satisfies the condition:

$$\|u(t, \cdot)\|_2^2 > \|\varphi\|_2^2 \left(1 - \lambda^2 \sqrt{1 - \frac{T}{T_0}}\right). \hspace{1cm} (8)$$

An interpretation of this fact is very simple. If the change of kinetic energy is not too large then solution $u$ of the problem (1) is regular (see [9], Theorem 8).
Preferring inequalities with an invariant form of a’priory estimates, it is natural to compare kinetic energy of the flow with a function 
\[ \rho(t) = \|\varphi\|_2^2 \sqrt{1 - \frac{t}{\mu T_0}}. \]

This comparison is one goal among many others. It permits to give nontrivial estimates for mean \( T_* \). In other words, we can find the time interval where solution \( u \) for the Cauchy problem (1) is regular.

In this way, it appears a number 
\[ s_0 = \frac{\|\varphi\|_2^2}{4\nu\|\nabla \varphi\|_2^2}, \]

It does not depend on any constants from a’priory estimates for the solution of problem (1).

A significance of this number can be explained by the following result: if number \( s_0 \) does not belong to the interval with border points \( T_0, T_0^\lambda - 4 \) then the Cauchy problem (1) has a global regular solution (see [9], Theorem 7). Precisely, this is among the main results in [9] and it is described by Theorem 7 there.

2. Main Alternative

Definition and properties of vector fields class \( C_{6/5, 3/2}^\infty \) are given in [9].

**Theorem 1.** Let \( \varphi \in C_{6/5, 3/2}^\infty \) be an initial data and \( u \) be a blow up solution of the problem (1). Then the following alternative is true:

1. either there exists a mean \( \xi \in (0, T_*) \), \( T_* < \infty \), satisfying condition:
\[ \xi + \frac{\|u(\xi, \cdot)\|_2^2}{4\nu\|\nabla u(\xi, \cdot)\|_2^2} = T_*; \]

2. or for all \( t \in (0, T_*) \), the following inequality is fulfilled:
\[ \|u(t, \cdot)\|_2^2 > \|\varphi\|_2^2 \sqrt{1 - \frac{t}{\mu T_0}} \]

and parameter \( \mu < \lambda^{-2} \) i.e.,
\[ T_* < \frac{\|\varphi\|_2^2}{4\nu\|\nabla \varphi\|_2^2}; \]

3. or for all \( t \in (0, T_*) \), the following inequality is fulfilled:
\[ \|u(t, \cdot)\|_2^2 < \|\varphi\|_2^2 \sqrt{1 - \frac{t}{\mu T_0}}, \]

and parameter \( \mu > \lambda^{-2} \), i.e.,
\[ T_* > \frac{\|\varphi\|_2^2}{4\nu\|\nabla \varphi\|_2^2}. \]

**Proof.** Consider a nonlinear equation:
\[ t + \frac{\|u(t, \cdot)\|_2^2}{4\nu\|\nabla u(t, \cdot)\|_2^2} = T_. \]

If this equation has a root \( \xi \in [0, T_*) \), then we have item (1). Suppose that this equation has no roots. Since a function 
\[ \eta(t) = t + \frac{\|u(t, \cdot)\|_2^2}{4\nu\|\nabla u(t, \cdot)\|_2^2} \]
is continuous (see [9], Lemma 36) then on interval $[0, T_\ast)$ only one of two inequalities is fulfilled: $\eta(t) > T_\ast$, $\eta(t) < T_\ast$. We consider the first. Let us rewrite it in the form
\[
\frac{4v\|\nabla u(t, \cdot)\|_2^2}{\|u(t, \cdot)\|_2^2} < \frac{1}{\mu T_0 - t}
\]
and integrate it over segment $[0, t]$. Then we obtain the inequality from item (2). Assume $t = 0$ in the inequality above. Hence, we have the necessary estimate $\mu < \lambda^{-2}$. The opposite case we consider in the same way. The alternative is proved. \hfill \Box

There are other important properties of the function
\[
\eta(t) = t + \frac{\|u(t, \cdot)\|_2^2}{4v\|\nabla u(t, \cdot)\|_2^2}.
\]

For example, it is true the following statement.

**Lemma 1.** Let $J$ be a subinterval of $[0, T_\ast)$. Then the following are true:

1. functions $\eta = \eta(t)$ and $\beta(t) = \|u(t, \cdot)\|_2\|\nabla u(t, \cdot)\|_2^2$ have general intervals of monotonicity i.e., function $\eta = \eta(t)$ is increasing (decreasing) on an interval $J$ if and only if function $\beta = \beta(t)$ is decreasing (increasing) on this interval;

2. if solution $u$ of the Cauchy problem (1) is a blow up solution then functions $\eta = \eta(t)$ and $\gamma(t) = \|\nabla u(t, \cdot)\|_2^2$ are functions of bounded variation on the interval $[0, T_\ast)$.

**Proof.** Item (1) we obtain if we compare signs of derivatives of these functions. They are opposite. The second item follows from decomposition $\eta = \eta_1(t) + \eta_2(t)$, where
\[
\eta_1(t) = t + \frac{\|u(t, \cdot)\|_2^2}{16cv^8}, \quad \eta_2(t) = \frac{\|u(t, \cdot)\|_2^2}{16cv^8} \left( \frac{4cv^4}{\|\nabla u(t, \cdot)\|_2^2} - \|u(t, \cdot)\|_2^2 \right).
\]

Both functions $\eta_1(t)$, $\eta_2(t)$ are monotonic. Function $\eta_1(t)$ is a decreasing function if solution $u$ is a blow up solution (see [9]). It follows immediately from inequality
\[
\|u(t, \cdot)\|_2^2\|\nabla u(t, \cdot)\|_2^2 > 4cv^4
\]
for all $t$. The second function is a decreasing function as the product of a positive decreasing function and negative increasing function (see inequality above and estimate (77) from [9]) if $u$ is a blow up solution. Therefore, $\eta = \eta(t)$ is a function of bounded variation. The statement for the second function we prove in the same way noting that the difference
\[
\frac{4cv^4}{\|\nabla u(t, \cdot)\|_2^2} - \|u(t, \cdot)\|_2^2
\]
is a monotonic function (see [9] and arguments above). Lemma is proved. \hfill \Box

**Remark 1.** Item (1) from the alternative means that there exists a number $\xi_0 \in (\xi, \eta(\xi))$ where a product $\|u(\xi_0, \cdot)\|_2\|\nabla u(\xi_0, \cdot)\|_2^2$ is not greater than the product $\|u(\xi, \cdot)\|_2\|\nabla u(\xi, \cdot)\|_2^2$.

This fact is proved from the opposite by integration of the corresponding inequality over interval $(\xi, \eta(\xi))$. Taking into account the main result from [4], it can be useful for the estimation of norm $\|u(t, \cdot)\|_2$ because it admits an inequality
\[
\|u(t, \cdot)\|_2^2 \leq k\|u(t, \cdot)\|_2\|\nabla u(t, \cdot)\|_2
\]
(9)
with a some constant \( k \). This inequality is a partial case of so called Gagliardo–Nirenberg inequalities.

**Remark 2.** If item (3) is fulfilled, then solution \( u \) can be extended by zero on the set \( [T_*, \infty) \times \mathbb{R}^3 \). Then the extended solution \( u \) is a weak solution which has a unique irregularity point \( T_* \) (compare with results from [10]).

3. Estimates of Kinetic Energy of the Flow

Let \( \epsilon, 0 < \epsilon < 1 \), be a dissipation parameter of kinetic energy at the moment \( T_0 \) (see (5) and (6)). Denote \( \tau(\epsilon) = \frac{1}{2}(\epsilon + \frac{1}{\epsilon}) \).

**Theorem 2.** Let \( \varphi \in C^{\infty}_{6/5, 3/2} \) be an initial data. Suppose parameter \( \lambda < 1 \) and \( u \) is a solution of the Cauchy problem (1). If \( \epsilon \) is a dissipation parameter from above then the following inequalities are true:

\[
\|\varphi\|_2^2 \left(1 - \lambda^2 + \lambda^2 \sqrt{1 - \frac{T}{T_0}}\right) \leq \|u(t, \cdot)\|_2^2 \leq \|\varphi\|_2^2 \left(1 - \tau(\epsilon)\lambda^2 + \lambda^2 \sqrt{T^2(\epsilon) - \frac{T}{T_0}}\right)
\]

(10)

for \( 0 \leq t \leq T_0 \) and

\[
\|u(t, \cdot)\|_2^2 \geq \|\varphi\|_2^2 \left(1 - \tau(\epsilon)\lambda^2 + \lambda^2 \sqrt{T^2(\epsilon) - \frac{T}{T_0}}\right)
\]

(11)

for \( T_0 \leq t \leq \tau^2(\epsilon)T_0 \).

**Proof.** The inequality from the left hand side of (10) is proved in [9], Lemma 41. Consider a function \( \omega(t) = \|u(t, \cdot)\|_2^2 - \|\varphi\|_2^2 \left(1 - \tau(\epsilon)\lambda^2 + \lambda^2 \sqrt{T^2(\epsilon) - \frac{T}{T_0}}\right) \).

Then from (7) it follows that the function \( \omega \) is defined on segment \( [0, \tau^2(\epsilon)T_0] \) and

\[
\omega(0) = \omega(T_0) = 0.
\]

From the Lagrange theorem there exists a number \( \xi \in (0, T_0) \) satisfying condition \( \omega'(<\xi) = 0 \) or

\[
\|\nabla u(\xi, \cdot)\|_2^2 = \frac{\|\varphi\|_2^2}{\sqrt{T^2(\epsilon) - \frac{T}{T_0}}}
\]

Rewrite this equality in the form:

\[
\tau^2(\epsilon)T_0 = \xi + \frac{c \nu^3}{\|\nabla u(\xi, \cdot)\|_2^2}
\]

Since a function \( \delta(t) = t + \frac{c \nu^3}{\|\nabla u(t, \cdot)\|_2^2} \) is increasing (see [9], Lemma 45), then a difference

\[
\|\nabla u(t, \cdot)\|_2^2 - \frac{\|\varphi\|_2^2}{\sqrt{T^2(\epsilon) - \frac{T}{T_0}}}
\]

is negative on the interval \( [0, \xi) \) and positive on the interval \( (\xi, \tau^2(\epsilon)T_0) \). That is, \( \omega'(t) < 0 \) on the first interval and \( \omega'(t) > 0 \) on the second interval. From (12) we have the necessary statement. \( \Box \)

4. Lower Bounds on the Length of the Regularity Interval

Here, the main result is described by the following statement which is more informative for item (1) of Theorem 1.
Theorem 3. Let \( \varphi \in C^{\infty}_{6/5,3/2} \) be an initial data. Suppose parameter \( \lambda < 1 \) and \( u \) is a blow up solution of the problem (1). If there exists a number \( t_0 \in (0, \tau^2(\varepsilon)T_0) \) satisfying condition

\[
\| u(t_0, \cdot) \|_2^2 = \| \varphi \|_2^2 \sqrt{1 - \frac{t_0}{\mu T_0}}
\]

then parameter

\[
\mu \geq \frac{1}{\lambda^2(2 - \lambda^2)},
\]

that is solution \( u \) is regular on the strip domain \( [0, \frac{T_0}{\lambda^2(2 - \lambda^2)}) \times \mathbb{R}^3 \).

Proof. It is sufficient to verify this estimate for means \( \mu < \frac{1}{\lambda^2} \). Suppose \( t_0 \in (0, T_0) \). Then the left hand side of inequality (10) and the equality from the theorem condition imply:

\[
1 - \sqrt{1 - \frac{t_0}{\mu T_0}} \leq \lambda^2(1 - \sqrt{1 - \frac{t_0}{T_0}}).
\]  

(13)

Obvious simple transformations give an equivalent inequality:

\[
\sqrt{\mu} \lambda^2 \geq \frac{1 + \sqrt{1 - \frac{t_0}{T_0}}}{\sqrt{\mu} + \sqrt{\mu - 1 + z^2}}.
\]

Let \( z = \sqrt{1 - \frac{t_0}{T_0}} \). Then the estimate above can be rewritten by the following inequality:

\[
\sqrt{\mu} \lambda^2 \geq \frac{1 + z}{\sqrt{\mu} + \sqrt{\mu - 1 + z^2}} = f(z), 0 < z < 1.
\]

Since \( \mu > 1 \), then function \( f \) is increasing on segment \([0, 1]\). Therefore,

\[
\sqrt{\mu} \lambda^2 \geq f(0) = \sqrt{\mu} - \sqrt{\mu - 1}.
\]  

(14)

Hence, we obtain the necessary inequality.

Now, assume \( t_0 \in [T_0, \tau^2(\varepsilon)T_0) \). Then, from (11) and the equality from the theorem condition, in the same way we obtain:

\[
\mu \lambda^2 \geq \frac{\tau(\varepsilon) + \sqrt{\tau^2(\varepsilon) - \frac{t_0}{T_0}}}{1 + \sqrt{1 - \frac{t_0}{T_0}}},
\]

Denote \( z = \sqrt{\tau^2(\varepsilon) - \frac{t_0}{T_0}} \). Then \( 0 \leq z \leq \tau(\varepsilon) - \varepsilon \) and the previous inequality can be rewritten in the form:

\[
\sqrt{\mu} \lambda^2 \geq \frac{\tau(\varepsilon) + z}{\sqrt{\mu} + \sqrt{\mu - \tau^2(\varepsilon) + z^2}} = g(z).
\]

Function \( g \) is increasing on the segment \([0, \tau(\varepsilon) - \varepsilon]\) because there is an estimate (7). Therefore,

\[
\sqrt{\mu} \lambda^2 \geq g(0) = \frac{\tau(\varepsilon)}{\sqrt{\mu} + \sqrt{\mu - \tau^2(\varepsilon)}}
\]

Since \( \tau(\varepsilon) > 1 \), then \( g(0) > \frac{1}{\sqrt{\mu} + \sqrt{\mu - 1}} \). Hence, again it follows estimate (14). The theorem is proved. \( \square \)
Remark 3. If \( \mu < \lambda^{-2} \) or \( \lambda^{-2} < \mu < \infty \), then in some neighborhood of point \( t = 0 \) the corresponding inequality from the alternative is fulfilled.

It is sufficient to consider the Cauchy problem with the initial data \( u(t, x) \) and to introduce functions

\[
\mu = \mu(t) = \frac{(\mu T_0 - t)\|\nabla u(t, \cdot)\|_2^4}{cv^3}, \quad \lambda = \lambda(t) = \sqrt{\frac{4cv^4}{\|u(t, \cdot)\|_2^2\|\nabla u(t, \cdot)\|_2^2}}
\]

Since these functions are continuous then the rest is obvious.

Remark 4. If \( u \) is a blow up solution then \( \lambda(t) \to 0 \) when \( t \to T_* \). It follows from (2) and (9).

5. Conclusions

The main results are described by Theorems 1 and 3. Moreover, in these results it is important to notice the controlling change of kinetic energy of the fluid flow. Precisely, this is a new idea which permits to define regularity interval of blow up solution. The sufficient condition for solution regularity is given by the lower bound for kinetic energy in (8). It was proved earlier and the opposite is unknown, i.e., no examples where kinetic energy could be less then \( \|\varphi\|_2^2 \left(1 - \lambda^2 \sqrt{\frac{T}{T_0}}\right) \). Therefore, these two aspects may be perspective for further research.

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References

1. Galdi, G.P. An Introduction to the Mathematical Theory of the Navier–Stokes Equations, Steady Problems, 2nd ed.; Springer: New York, NY, USA, 2011.
2. Leray, J. Sur le mouvement d’un liquide visqueux emplissant l’espace. Acta Math. 1934, 63, 193–248. [CrossRef]
3. Escauriaza, L.; Seregin, G.A.; Sverak, V. \( L^3, \infty \) solutions to the Navier-Stokes equations and backward uniqieness. Uspekhi Matematicheskih Nauk 2003, 58, 3–44.
4. Seregin, G. A certain necessary condition of potential blow up for Navier–Stokes equations. arXiv 2011, arXiv:1104.3615v1.
5. Houamed, H. About some possible blow–up conditions for 3–D Navier–Stokes equations. arXiv 2019, arXiv:1904.12485v1.
6. Tao, T. Quantitative bounds for critically bounded solutions to the Navier–Stokes equations. arXiv 2020, arXiv:1908.04958v2.
7. Chae, D.; Lee J. On the geometric regularity conditions for the 3D Navier–Stokes equations. arXiv 2019, arXiv:1606.08126v2.
8. Campolina, C.S.; Mailybaev, A.A. Fluid dynamics on logarithmic lattices. arXiv 2020, arXiv:2005.14027v1.
9. Semenov, V.I. The 3d Navier-Stokes equations: Invariants, local and global solutions. Axioms 2019, 8, 41. [CrossRef]
10. Caffarelli, L.; Kohn, R.; Nirenberg, L. Partial regularity of suitable weak solutions of the Navier-Stokes equations. Comm. Pure Appl. Math. 1982, 35, 771–831. [CrossRef]