Analysis of the symmetry group and exact solutions of the dispersionless KP equation in $n + 1$ dimensions

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Abstract

The Lie algebra of the symmetry group of the $(n + 1)$-dimensional generalization of the dispersionless Kadomtsev–Petviashvili equation is obtained and identified as a semi-direct sum of a finite dimensional simple Lie algebra and an infinite dimensional nilpotent subalgebra. Group transformation properties of solutions under the subalgebra $sl(2, \mathbb{R})$ are presented. Known explicit analytic solutions in the literature are shown to be actually group-invariant solutions corresponding to the infinite-dimensional part of the symmetry group for a specific choice of the arbitrary group functions.

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1 Introduction

The $(n + 1)$-dimensional generalization of the dispersionless Kadomtsev–Petviashvili equation (in short, it will be referred to as the dKP$_n$ equation)

$$E(x, \vec{y}, t) = (u_t + uu_x)_x + \Delta_{\perp} u = 0, \quad \Delta_{\perp} = \sum_{i=1}^{n-1} \partial^2_{y_i}, \quad n \geq 2, \quad x \in \mathbb{R}, \quad \vec{y} \in \mathbb{R}^{n-1},$$

(1.1)
where $u$ is a real-valued function depending on $n+1$ variables, $u = u(x, y_1, \ldots, y_{n-1}, t)$, was studied some time ago from several perspectives. This equation describes the propagation of weakly nonlinear quasi one dimensional waves in $n+1$ dimensions. Eq. (1.1), which is the $x$-dispersionless limit of the $n+1$ dimensional generalization of the Kadomtsev–Petviashvili (KP) equation is comprised of two competing terms; nonlinear term $uu_x$ and the $\Delta_\perp$ term (the Laplacian in the transverse variables), describing diffraction in the transversal $(n-1)$-dimensional hyperplane and includes as particular cases the integrable Riemann equation for $n = 1$ (dKP$_1$), the integrable dispersionless KP (dKP), also called the Khokhlov–Zabolotskaya (KZ) equation, for $n = 2$ (dKP$_2$) [1] and the nonintegrable KZ equation for $n = 3$ (dKP$_3$).

The potential Khokhlov–Zabolotskaya equation or Lin–Reissner–Tsien equation

$$2v_{xt} + v_x v_{xx} + v_{yy} = 0$$

is related to the dKP$_2$ equation with the transformation $u = v_x$ and the scaling $t \to 2t$. The dKP$_3$ equation describes the propagation of a confined three-dimensional sound beam in a slightly non–linear medium without absorption or dispersion. We refer to [2] and [3] for more details on physical motivation.

We organize this paper as follows. In Section 2, we obtain the Lie point symmetry algebra $\mathfrak{g}$ of dKP$_n$ equation (1.1) and identify its Lie algebraic structure. We write it in a suitable basis with commutation relations. In Section 3 we study group transformations mapping solutions amongst themselves which allow simple solutions like constants or more nontrivially $\vec{y}$-independent solutions, namely $(t, x)$-dependent solutions of dKP$_1$, to be mapped to new solutions of all independent variables $(t, x, \vec{y})$ depending on parameters and arbitrary functions of time and derive solutions invariant under infinite dimensional subalgebras for appropriately chosen arbitrary functions figuring in the subalgebra, by reducing the dKP$_n$ equation (1.1) to the dKP$_1$ equation for which the Cauchy problem is implicitly solved for some given initial data. The obtained group invariant solutions are then compared with the existing exact solutions, confirming that they all coincide.

2 Lie point symmetries

In this Section we apply the classical Lie algorithm to find the Lie algebra $\mathfrak{g}$ of the symmetry group or simply the symmetry algebra of Eq. (2.14). We can write a general element of $\mathfrak{g}$ as a vector field

$$X = \tau \frac{\partial}{\partial t} + \xi \frac{\partial}{\partial x} + \sum_{i=1}^{n-1} \xi_i \frac{\partial}{\partial y_i} + \eta \frac{\partial}{\partial u},$$

(2.1)

where $\tau$, $\xi$, $\xi_1$, $\xi_2$, \ldots, $\xi_{n-1}$ and $\eta$ are the coefficients of the vector field defined on the jet space $J(\mathbb{R}^{n+1}, \mathbb{R})$ which has the local coordinates $(t, x, y_1, \ldots, y_{n-1}, u)$. The algorithm consists of requiring that the second prolongation $\text{pr}^{(2)}X$ of the vector field $X$ to the second order jet space $J^{(2)}(\mathbb{R}^{n+1}, \mathbb{R})$ of the independent and dependent
variables and all derivatives of the dependent variable up to order two with dimension $2(n + 2)(n + 3)$ should annihilate Eq. (2.14) on its solution surface

$$\text{pr}^{(2)} X(E) \big|_{E=0} = 0,$$

where $E = 0$ is the equation under study.

Applying the condition (2.2) and splitting the resulting expression with respect to the linearly independent derivative terms we obtain an overdetermined system of determining equations (first order linear PDEs). We solve this determining system and express the general element of the symmetry algebra as a theorem:

**Theorem 1.** The Lie symmetry algebra $\mathfrak{g}$ of (2.14) for $n > 2$ is realized by vector fields (2.1) with coefficients given by

$$\xi = \frac{\tau_2}{n + 7} \left[ 2(5 - n)x t - 3 \sum_{i=1}^{n-1} y_i^2 \right] + (2\sigma - \tau_1) x - \frac{1}{2} \sum_{i=1}^{n-1} F'_i(t) y_i + F_u(t),$$

$$\tau = \tau_2 t^2 + \tau_1 t + \tau_0,$$

(2.3)

$$\bar{\xi} = \begin{pmatrix} \bar{x}_1 \\ \bar{x}_2 \\ \vdots \\ \bar{x}_{n-1} \end{pmatrix} = \begin{pmatrix} \delta & c_{12} & c_{13} & c_{1,n-1} \\ -c_{12} & \delta & c_{23} & c_{2,n-1} \\ -c_{13} & -c_{23} & \delta & c_{3,n-1} \\ \vdots & \vdots & \vdots & \vdots \\ -c_{1,n-1} & -c_{2,n-1} & \cdots & \delta \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_{n-1} \end{pmatrix} + \begin{pmatrix} F_1(t) \\ F_2(t) \\ \vdots \\ F_{n-1}(t) \end{pmatrix},$$

with

$$\delta = \frac{12\tau_2 t}{n + 7} + \sigma,$$

$$\eta = \frac{1}{n + 7} \left[ -4(n + 1)\tau_2 t + 2(n + 7)(\sigma - \tau_1) \right] u + \frac{2(5 - n)}{n + 7} \tau_2 x - \frac{1}{2} \sum_{i=1}^{n-1} F''_i(t) y_i + F'_u(t),$$

where $\tau_2, \tau_1, \tau_0, \sigma, c_{ij} = -c_{ji}$ for $i = 1, \ldots, n-1$ and $j = 1, \ldots, n-1$ are arbitrary constants, and $F'_i(t)$, for $i = 1, \ldots, n-1$ are arbitrary functions of time. $\bar{\xi}$ can be written in the form $\bar{\xi} = C \bar{y} + \delta I + \bar{F}(t)$, $\bar{y} = (y_1, y_2, \ldots, y_{n-1})^T$, $\bar{F}(t) = (F_1(t), \ldots, F_{n-1}(t))^T$, $I$ is the identity matrix of size $(n - 1) \times (n - 1)$ and $C$ is a skew-symmetric matrix of the same size.

A basis for the symmetry algebra $\mathfrak{g}$ is given by the following vector fields

$$T = \frac{\partial}{\partial t}, \quad \bar{D} = t \frac{\partial}{\partial t} - x \frac{\partial}{\partial x} - 2u \frac{\partial}{\partial u},$$

(2.4)

$$\bar{C} = t^2 \frac{\partial}{\partial t} + \frac{1}{n + 7} \left[ 2(5 - n)x t - 3|\bar{y}|^2 \right] \frac{\partial}{\partial x} + \frac{12t}{n + 7} \bar{y} \cdot \frac{\partial}{\partial \bar{y}} + \frac{2}{n + 7} \left[ (5 - n)x - 2(n + 1)tu \right] \frac{\partial}{\partial u},$$

(2.5)

$$+ \frac{2}{n + 7} \left[ (5 - n)x - 2(n + 1)tu \right] \frac{\partial}{\partial u},$$

(2.6)
The commutation relations among other elements are

\[ D_0 = 2x \frac{\partial}{\partial x} + \vec{y} \cdot \frac{\partial}{\partial y} + 2u \frac{\partial}{\partial u}, \quad (2.7) \]

\[ R_{ij} = y_i \frac{\partial}{\partial y_j} - y_j \frac{\partial}{\partial y_i}, \quad i < j, \quad i, j = 1, 2, \ldots, n - 1, \quad (2.8) \]

\[ X(F_n) = F_n(t) \frac{\partial}{\partial x} + F'_n(t) \frac{\partial}{\partial u}, \quad Y(F_i) = F_i(t) \frac{\partial}{\partial y_i} - \frac{1}{2} F'_i(t) y_i \frac{\partial}{\partial x} - \frac{1}{2} F''_i(t) y_i \frac{\partial}{\partial u}, \quad i = 1, \ldots, n - 1, \quad (2.9) \]

where

\[ \vec{y} = (y_1, y_2, \ldots, y_{n-1}), \quad |\vec{y}|^2 = \sum_{j=1}^{n-1} y_j^2, \quad \vec{y} \cdot \frac{\partial}{\partial \vec{y}} = \sum_{j=1}^{n-1} y_j \frac{\partial}{\partial y_j}. \]

Here \( T, \hat{D}, D_0, C \) and \( R_{ij} \) correspond to time translations, dilations, projective transformations, and rotations on the hyperplane \( y_iy_j \)-plane, respectively. The commutation relations among \( T, \hat{D}, C \) are

\[ [T, \hat{D}] = T, \quad [T, C] = 2\hat{D} + D_0, \quad [\hat{D}, C] = C. \quad (2.10) \]

Taking into account that \( D_0 \) commutes with \( T, \hat{D}, C \), together with change of basis

\[ D = \hat{D} + \frac{D_0}{2} = t \frac{\partial}{\partial t} + \frac{1}{2} \vec{y} \cdot \frac{\partial}{\partial \vec{y}} - u \frac{\partial}{\partial u}. \quad (2.11) \]

we see that \( \{T, D, C\} \) forms a \( \text{sl}(2, \mathbb{R}) \) subalgebra with commutation relations

\[ [T, D] = T, \quad [T, C] = 2D, \quad [D, C] = C. \quad (2.12) \]

The commutation relations among other elements are

\[ [D_0, X(G)] = -2X(G), \quad [D_0, Y(F_i)] = -Y(F_i), \quad [X(G), Y(F_i)] = 0, \quad \text{(2.13a)} \]

\[ [X(G), T] = -X(G'), \quad [X(G), D] = -X(tG'), \quad [X(G), C] = -X(t^2G'), \quad \text{(2.13b)} \]

\[ [Y(F_i), T] = -Y(F_i'), \quad [Y(F_i), D] = Y(F_i) \frac{F_i}{2} - tF'_i, \quad [Y(F_i), C] = Y(tF_i - t^2F'_i), \quad \text{(2.13c)} \]

\[ [Y(F_i), Y(G_j)] = -\frac{1}{2} \delta_{ij} X(F_iG_j' - F'_iG_j), \quad \text{(2.13d)} \]

\[ [R_{ij}, X(G)] = 0, \quad [R_{ij}, Y(F_k)] = -\delta_{kj} Y_j(F_i), \quad \text{(2.13e)} \]

where we have defined

\[ Y_j(F_i) \equiv F_i(t) \frac{\partial}{\partial y_j} - \frac{1}{2} F'_i(t) y_j \frac{\partial}{\partial x} - \frac{1}{2} F''_i(t) y_j \frac{\partial}{\partial u}, \quad Y_k(F_k) \equiv Y(F_k). \]

Rotations \( R_{ij} \) commute with \( T, D, C \).

The point Lie symmetry algebra \( \mathfrak{g} \) of Eq. (1.1) is infinite-dimensional and can be written as a semi-direct sum Lie algebra (Levi decomposition), \( \mathfrak{g} = S \oplus R \), where \( S = \{ \text{sl}(2, \mathbb{R}) \oplus \text{so}(n-1) \} \) is the finite-dimensional semisimple part, namely Levi factor of \( \mathfrak{g} \) and

\[ R = \{ X(F_n), Y(F_1), \ldots, Y(F_{n-1}), D_0 \} \]
is its infinite-dimensional radical (nonnilpotent ideal). Here so\((n - 1)\) is the \((n - 1)(n - 2)/2\)-dimensional algebra of rotations. The subalgebra \(R\) has the structure of a centerless Kac-Moody algebra.

In the special case when \(n = 2\), the equation is completely integrable and the structure of the symmetry algebra \(\mathfrak{g}\) is exceptionally different than \(n > 2\). A basis for \(\mathfrak{g}\) is given by

\[
T(f) = f \partial_t + \frac{1}{6} (2xf' - y^2f'') \partial_x + \frac{2y}{3} f' \partial_y + \frac{1}{6} (-4uf' + 2xf'' - y^2f''') \partial_u, \tag{2.14a}
\]

\[
X(g) = g \partial_x + g' \partial_u, \tag{2.14b}
\]

\[
Y(h) = h \partial_y - \frac{1}{2} yh' \partial_x - \frac{1}{2} yh'' \partial_u, \tag{2.14c}
\]

\[
D = 2x \partial_x + y \partial_y + 2u \partial_u, \tag{2.14d}
\]

where \(f, g, h\) are arbitrary smooth functions of the time variable \(t\), and the prime denotes derivative with respect to \(t\). The commutation relations are

\[
[T(f_1), T(f_2)] = T(f_1f'_2 - f'_1f_2), \tag{2.15a}
\]

\[
[D, X(g)] = -2X(g), \quad [X(g), Y(h)] = 0 \tag{2.15b}
\]

\[
[D, Y(h)] = -Y(h), \quad [X(g), T(f)] = X\left(\frac{1}{3} f'g - f'g'\right), \tag{2.15c}
\]

\[
[D, T(f)] = 0, \quad [Y(h), T(f)] = Y\left(\frac{2}{3} f'h - f'h'\right), \tag{2.15d}
\]

\[
[X(g_1), X(g_2)] = 0, \quad [Y(h_1), Y(h_2)] = \frac{1}{2} X(h_1h'_2 - h'_1h_2). \tag{2.15e}
\]

The symmetry algebra \(\mathfrak{g}\) can be written as a semi-direct sum Lie algebra (Levi decomposition), \(\mathfrak{g} = S \oplus R\), where \(S = \{T(f)\}\) is the (infinite-dimensional) semisimple part, also called Levi factor of \(\mathfrak{g}\) and \(R = \{X(g), Y(h), D\}\) is its radical. \(S\) is a simple Lie algebra, i.e. it has no nontrivial ideal. The radical \(R\) (maximal solvable ideal) is actually nonnilpotent. The algebra \(S\) can be identified as a centerless Virasoro algebra (the Witt algebra). The algebra \(R\) is a subalgebra of a centerless Kac-Moody algebra \([5]\). The vector fields \(T(f)\) generate a Lie algebra isomorphic to \(\text{Vect}(\mathbb{R})\) of real vector fields on \(\mathbb{R}\). If the parameter \(t\) is compactified, we can replace \(\text{Vect}(\mathbb{R})\) by the Witt algebra \(\text{Vect}(S^1)\). The presence of \(S\) implies that the equation is invariant under arbitrary reparametrizations of time.

This typical symmetry structure arises in completely integrable evolutionary equations in \(2 + 1\)-dimensions like KP equation \([6, 7]\), three-wave resonance equations \([8]\), Davey-Stewartson system \([9]\) and a few others (see [10] [11] for a further discussion). Indeed, this is the case here. Eq. (2.14) is known to enjoy all the features of integrability. When \(n > 2\) its integrability is destroyed and the Virasoro algebra reduces to its finite dimensional subalgebra, which is \(\text{sl}(2, \mathbb{R})\). The infinite-dimensional subalgebra \(R\) remains intact.

We comment that in this case \(\text{sl}(2, \mathbb{R})\) subalgebra can be embedded into the centerless Virasoro algebra (the Witt algebra) generated by the algebra of vector fields
\[ L_n = T(t^{n+1}) : n \in \mathbb{Z} \], whose elements satisfy the commutation relations
\[ [L_m, L_n] = (n - m)L_{m+n}. \]

Group-invariant solutions of (2.14) can be found in Ref. [12, 13]. We remark that in [14] the following variable coefficient version was studied
\[ (u_t + p(t)uu_x)_x + \sigma(t)u_{yy} = 0. \tag{2.16} \]

The conditions on the coefficients were established to ensure the transformability to its constant coefficient case. The Virasoro structure only survives under this equivalence.

### 3 Group Transformations and exact solutions

The subgroups corresponding to \( T \) and \( D \) are easy to obtain. They are simply translations in \( t \): \( t \rightarrow t + t_0 \) and dilations: \( (t, x, y, u) \rightarrow (\lambda t, x, \lambda^{1/2} y, \lambda^{-1} u) \), \( \lambda > 0 \), where \( t_0, \lambda \) are the group parameters. The one-parameter projective symmetry group generated by the subalgebra \( \{C\} \) (the flow of the vector field \( C \)), \( G_\varepsilon : (\tilde{t}, \tilde{x}, \tilde{y}, \tilde{u}) = \exp(\varepsilon C). (t, x, y, u) \) is obtained by integrating the vector field \( C \) as follows:
\[
\begin{align*}
\frac{d\tilde{t}}{d\varepsilon} &= \tilde{t}^2, \\
\frac{d\tilde{x}}{d\varepsilon} &= (n + 7)^{-1}[2(5 - n)\tilde{t}\tilde{x} - 3|\tilde{y}|^2], \\
\frac{d\tilde{y}}{d\varepsilon} &= 12(n + 7)^{-1}\tilde{t}\tilde{y}, \\
\frac{d\tilde{u}}{d\varepsilon} &= 2(n + 7)^{-1}(5 - n)\tilde{x} - 2(n + 1)\tilde{u},
\end{align*}
\tag{3.1}
\]

subject to the initial conditions
\[
\begin{align*}
\tilde{t}(t, x, y; \varepsilon)|_{\varepsilon=0} &= t, \\
\tilde{x}(t, x, y; \varepsilon)|_{\varepsilon=0} &= x, \\
\tilde{y}(t, x, y; \varepsilon)|_{\varepsilon=0} &= y, \\
\tilde{u}(t, x, y; \varepsilon)|_{\varepsilon=0} &= u.
\end{align*}
\]

We find
\[
\begin{align*}
\tilde{t} &= \frac{t}{1 - \varepsilon t}, \\
\tilde{x} &= (1 - \varepsilon t)^{-\alpha} \left( x - \frac{3\varepsilon |\tilde{y}|^2}{(n + 7)(1 - \varepsilon t)} \right), \\
\tilde{y}_i &= (1 - \varepsilon t)^{-12/(n+7)} y_i, \\
\tilde{u}(t, x, y; \varepsilon) &= (1 - \varepsilon t)^{\beta} \left\{ u + \alpha \left( \frac{\varepsilon x}{1 - \varepsilon t} - \frac{3\varepsilon^2 |y|^2}{2(n + 7)(1 - \varepsilon t)^2} \right) \right\},
\end{align*}
\tag{3.2}
\]

where \( \varepsilon \) is the group parameter and \( \alpha, \beta \) are defined by
\[
\alpha = \frac{2(5 - n)}{n + 7}, \quad \beta = \frac{4(n + 1)}{n + 7}, \quad \alpha + \beta = 2.
\]

The Lie group of local point transformations (3.2) transforms a solution \( u(t, x, y) \) into a new solution \( \tilde{u}(\tilde{t}, \tilde{x}, \tilde{y}) = G_\varepsilon \circ u(t, x, y) \) (see formula (3.25)).
In the special case of \( n = 2 \) the symmetry transformations can be expressed as

\[
\tilde{t} = T(t), \quad \tilde{x} = T'(t)^{1/3}x - \frac{1}{6}T''(T')^{-2/3}y^2, \quad \tilde{y} = T'(t)^{2/3}y,
\]

\[
\tilde{u} = T'(t)^{-2/3}\left\{ u + \frac{T''(t)}{3T'(t)}x - \frac{1}{6}\left\{ \{T; t\} + \frac{1}{6} \left( \frac{T''}{T'} \right)^2 \right\} y^2 \right\},
\]

(3.3)

where \( T(t) \) is an arbitrary smooth function of \( t \) and \( \{T; t\} \) is the Schwarzian derivative of \( T \)

\[
\{T; t\} = \frac{T'''}{T'} - \frac{3}{2} \left( \frac{T''}{T'} \right)^2.
\]

A group parameter \( \lambda \) can be introduced into (3.3) through the transformation

\[
\tilde{t} = T(t; \lambda) = F^{-1}(F + \lambda)(t), \quad F(t) = \int^t ds \frac{1}{f(s)},
\]

with the property \( T(t; 0) = t \) and \( T_3(t; 0) = f(t) \). Here \( f(t) \) is the \( t \)-coefficient of \( T(f) \) in (2.11) and \( F^{-1}(t) \) is the inverse to \( F(t) \). The corresponding infinite-dimensional symmetry group (Virasoro group) is the exponentiation \( \exp(\lambda T(f)) \) of the vector filed \( T(f) \). For the particular choice \( f(t) = t^2 \), we have \( T(t; \lambda) = t/(1 - \lambda t) \) and the formula (3.3) coincides with the projective transformation (3.2) for \( n = 2, (\alpha, \beta) = (2/3, 4/3) \). Taking \( f(t) = t, 1 \) we have \( T(t; \lambda) = e^\lambda t \) and \( T(t; \lambda) = t + \lambda \), generating dilation and translation symmetries, respectively.

If \( T(t) \) in (3.3) is restricted to the Moebius transformations \( T(t) = (at + b)/(ct + d) \), \( ad - bc = 1 \) for which \( \{T; t\} = 0 \), and \( T''/T' = -2c/(c + d) \), then the transformation formula (3.3) is reduced to the \( \text{SL}(2, \mathbb{R}) \) invariance group of the equation depending on three group parameters \( a, b, c \):

\[
\tilde{t} = \frac{at + b}{ct + d}, \quad \tilde{x} = (ct + d)^{-2/3}x + \frac{c}{3} (ct + d)^{-5/3}y^2, \quad \tilde{y} = (ct + d)^{-4/3}y,
\]

\[
\tilde{u} = (ct + d)^{4/3}\left[ u - \frac{2cx}{3(ct + d)} - \frac{c^2y^2}{9(ct + d)^2} \right].
\]

(3.4)

The vector field \( D_0 \) generates the dilation group \( (t, x, y, u) \rightarrow (t, \lambda^2 x, \lambda^2 y, \lambda^2 u) \). The vector field \( X(F_n) \) generates the group transformations

\[
\tilde{t} = t, \quad \tilde{x} = x + \lambda F_n(t), \quad \tilde{y}_i = y_i, \quad \tilde{u} = u + \lambda F'_n(t).
\]

(3.5)

Physically it is a transformation to a frame moving with an arbitrary acceleration in the \( x \) direction. For \( F_n = \text{const} \) we have a translation and for \( F_n = t \) a Galilei transformation. The solutions depending on the arbitrary function \( F_n(t) \) are obtained from the reduced equation

\[
\Delta_{\perp} F(t, \bar{y}) = -\frac{F''_n(t)}{F_n(t)}
\]

by the transformation formula

\[
u = \frac{F'_n(t)}{F_n(t)} x + F(t, \bar{y}).
\]
The vector fields \(Y(F_i(t))\) generate the transformations

\[
\tilde{t} = t, \quad \tilde{x} = x + \sum_{i=1}^{n-1} (\lambda_i y_i F'_i - \lambda_i^2 F_i F'_i), \quad \tilde{y}_i = y_i - 2\lambda_i F_i, \quad \tilde{u} = u + \sum_{i=1}^{n-1} (\lambda_i F''_i y_i - \lambda_i^2 F_i F''_i),
\]

where \(\lambda_i, i = 1, \ldots, n-1\) are the group parameters.

In particular, the choice \(F_i(t) = t\) leads to the transformation

\[
\tilde{t} = t, \quad \tilde{x} = x + \sum_{i=1}^{n-1} (\lambda_i y_i F'_i t - \lambda_i^2 F_i F'_i t), \quad \tilde{y}_i = y_i - 2\lambda_i F_i, \quad \tilde{u} = u.
\]

This is the transformation used in [2] to construct group invariant solutions, namely those mapped into themselves by the subgroup (3.7). The corresponding infinitesimal generators has the form of quasi-rotation

\[
Y_j(t) = 2t \frac{\partial}{\partial y_j} - y_j \frac{\partial}{\partial x}, \quad j = 1, 2, \ldots, n-1
\]

admitting the invariant \(\xi = x + (4t)^{-1}|\vec{y}|^2\) (a paraboloid). Invariants \(t, u\) and \(\xi\) suggest that solutions invariant under \(Y_j(t)\) are achieved via the transformation \(u = v(t, \xi)\), where \(v\) satisfies

\[
v_t + \frac{n-1}{2t} v + vv_\xi = 0,
\]

which is equivalent to the Riemann-Hopf equation \(q_\tau + qq_\xi = 0\) by the change of variable \(v = a(t)q(\tau, \xi)\) with \(a(t)\) and \(\tau(t)\) appropriately chosen (see [2]). In some sense, the authors of [2] rotated the general solution of \(u_t + uu_x = 0\) into solution of (1.1). For the sake of completeness, we reproduce the invariant (implicit) solution found in [2]

\[
u = \begin{cases} 
  t^{-\frac{n-1}{2}} F \left( x + \frac{1}{4t} \sum_{i=1}^{n-1} y_i^2 - \frac{2ut}{3-n} \right), & n \neq 3, \\
  t^{-1} F \left( x + \frac{1}{4t} \sum_{i=1}^{n-1} y_i^2 - u t \ln t \right), & n = 3,
\end{cases}
\]

where \(F\) is an arbitrary function of one argument.

One can of course use more general invariants \(t\) and

\[
\xi = x + \frac{1}{4} \sum_{i=1}^{n-1} \frac{F'_i}{F_i} y_i^2, \quad v = u + \frac{1}{4} \sum_{i=1}^{n-1} \frac{F''_i}{F_i} y_i^2 \equiv u - R(t, \vec{y})
\]

(3.11)

to perform a similar reduction. Substitution of the symmetry ansatz

\[
u = v(t, \xi) + R(t, \vec{y})
\]

(3.12)

into (1.1) gives the invariant equation

\[
(v_t + vv_\xi)\xi + (\xi_t + (\nabla\xi)^2 + R)v_\xi \xi + (\Delta_\perp \xi) v_\xi + \Delta_\perp R = 0.
\]

(3.13)
The coefficient of $v\xi$ vanishes and the reduced equation becomes

$$(v_t + vv + A(t)v)\xi + B(t) = 0,$$  \hspace{1cm} (3.14)\]

where we have defined

$$A(t) = \Delta_{\perp} = \frac{1}{2} \sum_{i=1}^{n-1} \frac{F'_i}{F_i}, \quad B(t) = \Delta_{\perp} R = -\frac{1}{2} \sum_{i=1}^{n-1} \frac{F''_i}{F_i}.$$  

The arbitrary functions $F_i(t)$ can always be arranged such that $B(t) = 0$. We assume that this is the case here. Eq. (3.14) can then be integrated once to give

$$v_t + vv + A(t)v + g(t) = 0,$$  \hspace{1cm} (3.15)\]

where $g(t)$ is an arbitrary function. The transformation

$$v = a(t)q(\tau, \xi), \quad a(t) = \exp \left[ -\int A(t) dt \right], \quad \tau = \int a(t) dt$$  \hspace{1cm} (3.16)\]

can be used to set $A(t)$ equal to zero in (3.15). Furthermore, $g(t)$ can be removed by a time dependent translation of $\xi$ and $v$ (see Eq. (3.23)). The reduced equation is again $q_{\tau} + qq = 0$.

The special choice $F_i(t) = t - t_i$ with $t_i$ being arbitrary constants leads to

$$\xi = x + \frac{1}{4} \sum_{i=1}^{n-1} \frac{y_i^2}{t - t_i}, \quad A(t) = \frac{1}{2} \sum_{i=1}^{n-1} \frac{1}{t - t_i}, \quad B(t) = 0$$  \hspace{1cm} (3.17)\]

and the invariant equation

$$v_t + vv + A(t)v = 0.$$  \hspace{1cm} (3.18)\]

The functions $a(t)$ and $\tau(t)$ in the transformation (3.16) are found from the relations

$$a(t) = \prod_{i=1}^{n-1} (t - t_i)^{-1/2}, \quad \tau(t) = a(t)$$  \hspace{1cm} (3.19)\]

so that $v = a(t)q(\tau, \xi)$ takes Eq. (3.18) to $q_{\tau} + qq = 0$.

In particular, for $t_i = 0, i = 1, 2, \ldots, n-1$, we have $A(t) = (n - 1)/(2t)$, $B(t) = 0$, $a(t) = t^{-(n-1)/2}$ and Eq. (3.9) is recovered. Different special choices of $F_i$ lead to different reductions and invariant solutions.

Motivated by the form of the group invariant solutions of [2], the authors of [15] used an ansatz to study reductions of the generalized DKPn equation

$$(u_t + u^N u_x)_x + \frac{1}{2} \Delta_{\perp} u = 0, \quad N \in \mathbb{Z}^+$$  \hspace{1cm} (3.20)\]

to the generalized Riemann–Hopf equation

$$U_T + U^N U_X + mU = g(t),$$  \hspace{1cm} (3.21)\]
where \( m \) is some constant and \( g(t) \) an arbitrary function. Up to the change of variables 
\((t, x, u) \rightarrow (\tau(t), x, a(\tau)u(\tau, x))\), this reduction coincides with the symmetry reduction corresponding to the subalgebras \( Y_i(t - t_i) \). Indeed, contrary to the comment in \([15]\),
the exact solutions obtained therein stem from the invariance obtained by restricting the arbitrary functions \( F_i(t) \) figuring in the subalgebras of the symmetry algebra of the equation. In this setting, in the particular case \( N = 1 \) and \( n = 2, 3, t_i = 0 \) \((i \in \{1, 2\})\), the solutions invariant under \( (3.8) \) were recovered as already discussed before. Also, the solution constructed in \([15]\) with the choice \( N = 1, n = 3 \) (with the notation of the present paper) \( t_1 < 0, t_2 > 0, g(t) = 0 \) is a group invariant (similarity or self-similar) solution.

We note that Eq. \((3.20)\) still remains invariant under the symmetry group generated by the vector field \( (3.8) \), but not under \( Y(F_i) \) unless \( F_i'' = 0 \), and this invariance has been used in Ref. \([3]\) to construct group invariant solutions of \((3.20)\) by reducing to the generalized Riemann–Hopf equation \( u_t + uu_x = 0 \) having an implicit general solution. The fact that the group invariant solutions can be expressed implicitly depending on an arbitrary function allows to solve a Cauchy problem for a small initial condition.

Finally, the generators \( R_{ij} \) correspond to the invariance of the transversal Laplace operator \( \Delta_\perp \) under rotations. The invariant solutions will be of the form \( u = v(t, x, |\vec{y}|) \).

The symmetry group \((3.2)\) can be used to transform trivial solutions like a constant solution or \( \vec{y} \) independent solutions to other solutions depending on all variables. \( \vec{y} \) independent solutions satisfy the quasilinear Riemann–Hopf equation

\[
    u_t + uu_x = f(t),
\]

where \( f \) is an arbitrary integration function to be determined from the initial condition. The following point transformation transforms away \( f(t) \)

\[
    \tilde{t} = t, \quad \tilde{x} = x + \lambda(t), \quad \tilde{u} = u + g(t)
\]

with \( \lambda, g \) satisfying \( \tilde{\lambda} - f = 0 \) and \( \lambda - g = 0 \). So the general solution \( u(t, x) \) of the first order nonlinear PDE \( u_t + uu_x = 0 \) (also known as inviscid Burgers’ equation) determined implicitly by

\[
    u = F(\xi) = F(x - tu)
\]

with \( F \) being arbitrary differentiable function of the characteristic coordinate \( \xi \), will generate new solutions of the form \( u(t, x, y_i) \) in implicit form depending on the parameter \( \varepsilon \) and arbitrary function \( F \). We recall that the solution \( u \) breaks at \( t = t_b \) on the characteristic \( \xi_b \), where \( t_b = -1/F'(\xi_b) > 0 \). In other words, if \( F'(\xi_b) < 0 \) the solution suffers a gradient catastrophe type of singularity.

From \((3.2)\) we can state that if \( U(T, X, \vec{Y}) \) is a solution of \((1.1)\), then so is

\[
    u(t, x, \vec{y}) = (1 + \varepsilon t)^{-\beta} U(T, X, \vec{Y}) - \frac{(n - 5)}{(n + 7)^2} (1 + \varepsilon t)^{-2} \frac{2(n + 7)(1 + \varepsilon t)x + 3\varepsilon^2 |\vec{y}|^2}{2},
\]

where \( n \) is some positive integer and \( g(t) \) an arbitrary function.
\[ T = \frac{t}{1 + \varepsilon t}, \quad X = (1 + \varepsilon t)^{-\alpha} \left( x + \frac{3\varepsilon |\vec{y}|^2}{(n + 7)(1 + \varepsilon t)} \right), \quad \vec{Y} = (1 + \varepsilon t)^{-12/(n+7)} \vec{y}. \]  

(3.26)

In the special case \( n = 5 \) (\( \alpha = 0, \beta = 2 \)), the above solution formula substantially simplifies to the form

\[ u = (1 + \varepsilon t)^{-2} U(T, X, \vec{Y}), \quad T = \frac{t}{1 + \varepsilon t}, \quad X = x + \frac{\varepsilon}{4(1 + \varepsilon t)} |\vec{y}|^2, \quad \vec{Y} = \frac{\vec{y}}{1 + \varepsilon t}. \]

Action of the \( SL(2, \mathbb{R}) \) symmetry group on solutions can be formulated as follows:

\[ u(t, x, \vec{y}) = (ct + d)^{-\beta} U(T, X, \vec{Y}) - \frac{(n - 5)}{(n + 7)^2} (ct + d)^{-2} \left[ 2(n + 7)(ct + d)x + 3c^2 |\vec{y}|^2 \right], \]

\[ T = \frac{at + b}{ct + d}, \quad X = (ct + d)^{-\alpha} \left( x + \frac{3c|\vec{y}|^2}{(n + 7)(ct + d)} \right), \quad \vec{Y} = (ct + d)^{-12/(n+7)} \vec{y}, \]

(3.27)

where \( ad - bc = 1 \), \( a, b, c \) are the group parameters and \( \alpha, \beta \) (\( \beta > 0 \)) were defined in (3.2). We comment that this holds for any value of \( n \). For \( n = 2 \) there is a more general formula involving an arbitrary function rather than parameters (see the symmetry group (3.3)).

Restriction of the arbitrary function \( f(t) \) of (2.14) to a quadratic function results in this formula. This type of formulas (3.27) play an important role in demonstrating existence of the blow-up profiles for some appropriate Cauchy problem using some simple solutions \( U \) (like \( \vec{y} \) independent solutions among many others).

On the other hand the scaling symmetry \( D_0 \) implies that if \( u(t, x, \vec{y}) \) solves (1.1), then so does \( \tilde{u}(t, x, \vec{y}) = \lambda^2 u(t, \lambda^{-2} x, \lambda^{-1} \vec{y}), \lambda > 0 \). With further application of this invariance we can obtain new solutions depending on four parameters \( (a, b, c, \lambda) \) from known solutions.

References

[1] E. A. Zabolotskaya and R. V. Khokhlov. Quasi-plane waves in the nonlinear acoustics of confined beams. Sov. Phys. Acoustics, 15:35–40, 1969.

[2] S V Manakov and P M Santini. On the dispersionless Kadomtsev–Petviashvili equation in \( n+1 \) dimensions: exact solutions, the cauchy problem for small initial data and wave breaking. Journal of Physics A: Mathematical and Theoretical, 44(40):405203, 2011.

[3] F Santucci and P M Santini. On the dispersionless Kadomtsev–Petviashvili equation with arbitrary nonlinearity and dimensionality: exact solutions, long-time asymptotics of the cauchy problem, wave breaking and shocks. Journal of Physics A: Mathematical and Theoretical, 49(40):405203, 2016.
[4] F. Schwarz. Symmetries of the Khokhlov-Zabolotskaya Equation. Comment on: "Towards the Conservation Laws and Lie Symmetries for the Khokhlov-Zabolotskaya Equation in Three Dimensions". *J. Phys. A: Math. and Gen.*, 20:1613–1614, 1987.

[5] F. Güngör. Infinite-dimensional symmetries of a two-dimensional generalized Burgers equation. *J. Math. Phys.*, 51:083504:1–12, 2010.

[6] D. David, N. Kamran, D. Levi, and P. Winternitz. Subalgebras of loop algebras and symmetries of the Kadomtsev–Petviashvili equation. *Phys. Rev. Let.*, 55(20):2111–2113, 1985.

[7] D. David, N. Kamran, D. Levi, and P. Winternitz. Symmetry reduction for the Kadomtsev–Petviashvili equation using a loop algebra. *J. Math. Phys.*, 27(5):1225–1237, 1986.

[8] L. Martina and P. Winternitz. Analysis and applications of the symmetry group of the multidimensional three-wave resonant interaction problem. *Annal. Phys.*, 196:231–277, 1989.

[9] B. Champagne and P. Winternitz. On the infinite dimensional symmetry group of the Davey–Stewartson equation. *J. Math. Phys.*, 29:1–8, 1988.

[10] F. Güngör. On the Virasoro structure of symmetry algebras of nonlinear partial differential equations. *SIGMA*, 2:14–, 2006.

[11] P. Basarab-Horwath, F. Güngör, and C. Özemir. Infinite-dimensional symmetries of a general class of variable coefficient evolution equations in 2+1 dimensions. *Journal of Physics: Conference Series*, 474(1):012010, 2013.

[12] J. C. Ndogmo. Group-invariant solutions of a nonlinear acoustics model. *J. Phys. A: Math. and Theor.*, 41:485201, 2008.

[13] J. C. Ndogmo. Symmetry properites of a nonlinear acoustics model. *Nonlinear Dym.*, 55:151–167, 2009.

[14] F. Güngör and C. Özemir. Lie symmetries of a generalized Kuznetsov–Zabolotskaya–Khokhlov equation. *J. Math. Anal. and Appl.*, 423(1):623 – 638, 2015. arxiv.org/pdf/1402.1941.

[15] A.M. Kamchatnov and M.V. Pavlov. On exact solutions of nonlinear acoustic equations. *Wave Motion*, 67:81 – 88, 2016.