Calculating hard probe radiative energy loss beyond soft-gluon approximation: how valid is the approximation?

Bojana Blagojevic,1 Magdalena Djordjevic, and Marko Djordjevic2
1Institute of Physics Belgrade, University of Belgrade, Belgrade, Serbia
2Faculty of Biology, Institute of Physiology and Biochemistry, University of Belgrade, Belgrade, Serbia
(Dated: April 23, 2018)

The soft-gluon approximation, which implies that radiated gluon carries away a small fraction of initial parton’s energy, is a commonly used assumption in calculating radiative energy loss of high momentum partons traversing QGP created at RHIC and LHC. While soft-gluon approximation is convenient, different theoretical approaches reported significant radiative energy loss of high $p_{T}$ partons, thereby questioning its validity. To address this issue, we relaxed the soft-gluon approximation within DGLV formalism. The obtained analytical expressions are quite distinct compared to the soft-gluon case. However, numerical results for the first order in opacity fractional energy loss lead to small differences in predictions for the two cases. The difference in the predicted number of radiated gluons is also small. Moreover, the effect on these two variables has an opposite sign, which when combined results in almost overlapping suppression predictions. Therefore, our results imply that, contrary to the commonly held doubts, the soft-gluon approximation in practice works surprisingly well in DGLV formalism. Finally, we also discuss generalizing this relaxation in the dynamical QCD medium, which suggests a more general applicability of the conclusions obtained here.

PACS numbers: 12.38.Mh; 24.85.+p; 25.75.-q

I. INTRODUCTION

One of the main assumptions in the radiative energy loss calculations of energetic parton (in the further text referred to as jet) traversing the QGP medium, is the soft-gluon approximation which assumes that radiated gluon carries away a small portion of initial jet energy, i.e. $x = \omega/E \ll 1$, where $E$ is the energy of initial jet and $\omega$ is the radiated gluon energy.

Such assumption was widely used in various energy loss models: i) in multiple soft scattering based ASW model [1,2]; ii) BDMP-S [3,4] and BDMS-Z [5,6]; iii) in opacity expansion based GLV model [7,8] and iv) in multi-gluon evolution based HT approach [9,10], etc. These various energy loss models predict a significant medium induced radiative energy loss, questioning the validity of the soft-gluon approximation. To address this issue, a finite $x$ (or large $x$ limit) was introduced in some of these models [11,12] or their extensions [13]. However, introduction of finite $x$ lead to different conclusions on the importance of relaxing the soft-gluon approximation. To address this issue, a finite $x$ (or large $x$ limit) was introduced in some of these models [11,12] or their extensions [13]. However, introduction of finite $x$ lead to different conclusions on the importance of relaxing the soft-gluon approximation. To address this issue, a finite $x$, or large $x$, limit was introduced in some of these models [11,12] or their extensions [13].

The soft-gluon approximation was also used in the development of our dynamical energy loss formalism [15–17], specifically in its radiative energy loss component. This formalism was comprehensively tested against angular averaged nuclear modification factor $R_{AA}$ data, where we obtained robust agreement for wide range of probes [17,20], centralities [21] and beam energies [20,22], including clear predictions for future experiments [23,24]. This might strongly suggest that our energy loss formalism can well explain the jet-medium interactions in QGP, making this formalism suitable for the tomography of QCD medium.

However, the soft-gluon approximation obviously breaks-down for: i) intermediate momentum ranges ($5 < p_{T} < 10$ GeV) where the experimental data are most abundant and with the smallest error-bars, and ii) gluon energy loss, since due to the color factor of 9/4 gluons lose significantly more energy compared to quark jets, therefore questioning the reliability of our formalism in such cases. Due to this, and for precise predictions, it became necessary to relax the soft-gluon approximation, and consequently test its validity in dynamical energy loss formalism.

This paper presents our first step toward this goal. Since the dynamical energy loss is computationally very demanding, we will, in this study, start with relaxing this approximation on its simpler predecessor, i.e. DGLV [25] formalism. Within this, we will concentrate on gluon jets, since, due to their color factor, the soft-gluon approximation has the largest impact for this type of partons. For the gluon jets, we perform the radiative energy loss calculation, to the first order in the number of scattering centers (opacity), where we consider that the radiation of one gluon is induced by one collisional interaction with the medium.

Our calculation is done within the pQCD approach for a finite size, optically thin QCD medium and since it is technically demanding – it will be divided in several steps: i) First, the calculation will be done in the simplest case of massless gluons in the system of static scattering centers [26] within GLV, ii) Then it will be extended to the gluons with the effective mass [27], which presents expansion of DGLV [25] toward larger loss of jet energy via radiated gluon, and iii) Finally, we will discuss the impact of finite $x$ on the radiative energy loss, when
In that manner we will assess the validity of the soft-gluon assumption for gluon jets, and this will also provide an insight into whether or not a finite $x$ has to be implemented in quark-jet radiative energy loss calculations within our formalism. Namely, if the relaxation of the soft-gluon approximation only slightly modifies gluon-jet radiative energy loss, then even smaller modification would be expected in quark-jet case, thus making this relaxation redundant. Otherwise, if the effect of a finite $x$ appears to be significant in gluon-jet case, then the relaxation in quark-jet case may also be required, and would represent an important future task.

Secondly, as stated above, the relaxation of the soft-gluon approximation is needed in order to extend the applicability of our model [17] towards intermediate momentum region. Thus, the other benefit of this relaxation would be to extend the $p_\perp$ range in which our predictions are valid.

The sections are organized as follows: In section II, we provide the theoretical framework. In section III, we outline the computation of the zeroth order in opacity gluon-jet radiative energy loss in static QCD medium, beyond soft-gluon approximation, in the cases of both massless and massive gluons. For $x \ll 1$ the results from [9, 25] are reproduced.

Section IV contains concise description of relaxing the soft-gluon approximation in calculating the first order in opacity radiative energy loss for massless gluon jet in static QCD medium. In a limit of very small $x$ result from [9] is recovered.

In section V we explain the computation of the first order in opacity gluon-jet energy loss in static QCD medium, with effective gluon mass [27] included, and beyond soft-gluon approximation. This presents an extension of the calculations from [25] toward finite $x$, so that results from [25] can be recovered in $x \ll 1$. The detailed calculations corresponding to sections III - V are presented in the Appendices C-J.

In section VI we outline the numerical estimates based on our beyond soft-gluon calculations for gluon jet and the comparison with our previous results from [25], i.e. the results with soft-gluon approximation. Particularly, we investigate the effect of finite $x$ on gluon-jet fractional radiative energy loss, number of radiated gluons, differential radiative energy loss, single gluon radiation spectrum and gluon suppression [28]. Conclusions and outlook are presented in section VII.

II. THEORETICAL FRAMEWORK

In this work, we concentrate on relaxing soft-gluon approximation in calculating the first order in opacity radiative energy loss of high $p_\perp$ eikonal gluon jets within (GLV) DGLV [25] formalism. That is, we assume that high $p_\perp$ gluon jet is produced inside a ”thin” finite QGP medium at some initial point $(t_0, z_0, \mathbf{x}_0)$, and that the medium is composed of static scattering centers [26]. Therefore, we model the interactions in QGP assuming a static (Debye) colored-screened Yukawa potential, whose Fourier and color structure acquires the following form (9, 25, 26):

$$V_n = V(q_n) e^{i q_n \cdot x_n} = 2 \pi \delta(q_n^0) v(q_n) e^{-i q_n \cdot x_n} \times T_{a_n}(R) \otimes T_{a_n}(n),$$

where $x_n$ denotes time-space coordinate of the $n^{th}$ scattering center, $\mu$ is Debye screening mass, $\alpha_s = g_s^2/4\pi$ is strong coupling constant, while $T_{a_n}(R)$ and $T_{a_n}(n)$ denote the generators in SU($N_c = 3$) color representation of gluon jet and target (scattering center), respectively.

For consistency with [9, 25], we use the same notation for 4D vectors (e.g. momenta), which is described in detail in Appendix A and proceed throughout using Light-cone coordinates. The same Appendix contains algebra manipulation and identities for SU($N_c$) generators, as well as the Feynman rules, used in these calculations.

The approximations that we assume throughout the paper are stated in Appendix B.

The small transverse momentum transfer elastic cross section for interaction between gluon jet and target parton in GW approach [8, 26] is given by:

$$\frac{d\sigma_{el}}{d^2q_1} = \frac{C_2(G)C_2(T)}{d_G} \frac{|v(0, q_1)|^2}{(2\pi)^2},$$

where $q_1$ corresponds to transverse momentum of exchanged gluon, $C_2(G)$ represents Casimir operator in adjoint representation (G) of gluons SU($N_c = 3$) with dimension $d_G = 8$, whereas $C_2(T)$ denotes Casimir operator in target (T) representation.

Since this formalism assumes optically ”thin” plasma, the final results are expanded in powers of opacity, which is defined as the mean number of collisions in the medium: $L/\lambda = N\sigma_{el}/A_\perp$ [9], where $L$ is the thickness of the QCD medium, $\lambda$ is a mean free path, while $N$ denotes the number of scatterers (targets) in transverse area $A_\perp$. Note that, we restrict our calculations to the first order in opacity, which is shown to be the dominant term (29, 30).

III. ZEROTH ORDER RADIATIVE ENERGY LOSS

To gradually introduce technically involving beyond soft-gluon calculations, we first concentrate on massless gluons traversing static QCD medium.

We start with $M_0$ Feynman diagram, which corresponds to the source $J$ that produces off-shell gluon with momentum $p+k$, that further, without interactions with
QCD medium, radiates on-shell gluon with momentum \( k \) and emerges with momentum \( p \). We will further refer to these two outgoing gluons as the radiated (\( k \)) and the final (\( p \)) gluon. Note that, both in this and consecutive sections that involve interactions with one and two scattering centers, we consistently assume that initial jet propagates along the longitudinal \( z \) axis. The detailed calculation of \( M_0 \) for finite \( x \) in massless case is presented in Appendix C with all assumptions listed in Appendix B.

We also assume that gluons are transversely polarized particles and although we work in covariant gauge, we can choose any polarization vector for the external on-shell gluons \([14]\), so in accordance with \([9, 14, 25]\) we choose \( \nu^\mu = [0, 2, 0] \) (i.e. \( \epsilon(k) \cdot k = 0, \epsilon(k) \cdot n = 0 \) and \( \epsilon(p) \cdot p = 0, \epsilon(p) \cdot n = 0 \)). Likewise, we assume that the source has also the same polarization as real gluons \([14]\) (i.e. \( \epsilon(p + k) \cdot (p + k) = 0, \epsilon(p + k) \cdot n = 0 \)). Thus, for massless gluon’s momenta we have:

\[
p + k = [E^+, E^-, 0], \quad k = [x E^+, \frac{k^2}{x E^+}, k],
\]

\[
p = [(1 - x)E^+, \frac{p^2}{(1 - x)E^+}, p],
\]

where \( E^+ = p^0 + k^0 + p_z + k_z, \) \( E^- = p^0 + k^0 - p_z - k_z \) and due to 4-momentum conservation:

\[
p + k = 0.
\]

The polarization vectors read:

\[
\epsilon_i(k) = [0, 2\epsilon_i, \frac{k}{x E^+}, \epsilon], \quad \epsilon_i(p) = [0, 2\epsilon_i \cdot p, \frac{p}{(1 - x)E^+}, \epsilon],
\]

\[
\epsilon_i(p + k) = [0, 0, \epsilon],
\]

where \( i = 1, 2 \), and we also make use of Eq. (5). So, the amplitude that gluon jet, produced at \( x_0 \) inside QCD medium, radiates a gluon of color \( c \) without final state interactions reads:

\[
M_0 = J_c(p + k)\epsilon^{i(p + k) x_0}(-2 ig_s(1 - x + x^2) \epsilon \cdot \frac{k}{k^2 + m_g^2(1 - x + x^2)}(T^c)_{da}.
\]

The radiation spectrum is obtained when Eq. (7) is substituted in:

\[
d^3 N_0 d^3 N_j \approx \text{Tr} \langle |M_0|^2 \rangle \frac{d^3 \tilde{p} d^3 \tilde{k}}{(2\pi)^3 2p^0 (2\pi)^3 2\omega},
\]

where \( \omega = k_0 \), and where \( d^3 N_j \) reads:

\[
d^3 N_j = d_G |J(p + k)|^2 \frac{d^3 \tilde{p} J}{(2\pi)^3 2E_j}.
\]

Here \( E_j = E + p_0 + k_0 \) and \( \tilde{p} J \) denotes energy and 3D momentum of the initial gluon jet, respectively. The jet part can be decoupled by using the equality:

\[
\frac{d^3 \tilde{p} d^3 \tilde{k}}{(2\pi)^3 2p^0 (2\pi)^3 2\omega} = \frac{d^3 \tilde{p} J d\tilde{k}}{(2\pi)^3 2E_j (2\pi)^3 2x(1 - x)},
\]

which is obtained by substituting \( p_z, k_z \rightarrow p'_z, xE \). Finally, energy spectrum acquires the form:

\[
x d^3 N^{(0)}_g \frac{d^3 N^{(0)}_j}{dx d\tilde{k}^2} = \frac{\alpha_s}{\pi} \frac{C_2(G) k^2}{k^2 + m_g^2(1 - x + x^2)^2} \times \frac{\epsilon \cdot k}{1 - x},
\]

which recovers the well-known Altarrelli-Parisi \([31]\) result.

We now briefly concentrate on generating result in finite temperature QCD medium, since in \([27]\), it was shown that gluons in finite temperature QGP can be approximated as a massive transverse plasmons with mass \( m_\rho = \mu/\sqrt{2} \), where \( \mu \) is the Debye mass. In this case, \( M_0 \) amplitude becomes:

\[
M_0 = J_c(p + k) e^{i(p + k) x_0}(-2 ig_s(1 - x + x^2) \epsilon \cdot \frac{k}{k^2 + m_g^2(1 - x + x^2)}(T^c)_{da}.
\]

leading to:

\[
x d^3 N^{(0)}_g \frac{d^3 N^{(0)}_j}{dx d\tilde{k}^2} = \frac{\alpha_s}{\pi} \frac{C_2(G) k^2}{k^2 + m_g^2(1 - x + x^2)^2} \times \frac{\epsilon \cdot k}{1 - x},
\]

where \( \omega = k_0 \), and where \( d^3 N_j \) reads:

\[
d^3 N_j = d_G |J(p + k)|^2 \frac{d^3 \tilde{p} J}{(2\pi)^3 2E_j},
\]

Here \( E_j = E + p_0 + k_0 \) and \( \tilde{p} J \) denotes energy and 3D momentum of the initial gluon jet, respectively. The jet part can be decoupled by using the equality:

\[
\frac{d^3 \tilde{p} d^3 \tilde{k}}{(2\pi)^3 2p^0 (2\pi)^3 2\omega} = \frac{d^3 \tilde{p} J d\tilde{k}}{(2\pi)^3 2E_j (2\pi)^3 2x(1 - x)},
\]

which is obtained by substituting \( p_z, k_z \rightarrow p'_z, xE \). Finally, energy spectrum acquires the form:

\[
x d^3 N^{(0)}_g \frac{d^3 N^{(0)}_j}{dx d\tilde{k}^2} = \frac{\alpha_s}{\pi} \frac{C_2(G) k^2}{k^2 + m_g^2(1 - x + x^2)^2} \times \frac{\epsilon \cdot k}{1 - x},
\]

IV. FIRST ORDER RADIATIVE ENERGY LOSS IN MASSLESS CASE

In accordance with \([25]\), we compute the first order in opacity radiative energy loss of gluon jet for finite \( x \) starting from the expression:

\[
d^3 N^{(1)}_g d^3 N_j = \left( \frac{1}{d_T} \text{Tr} \langle |M_1|^2 \rangle + \frac{2}{d_T} \text{Re} \text{Tr} \langle M_2 M_0^* \rangle \right) \frac{d^3 \tilde{p}}{(2\pi)^3 2p^0} \frac{d^3 \tilde{k}}{(2\pi)^3 2\omega},
\]

where \( M_0 \) corresponds to the diagram without final state interactions with QCD medium, introduced in previous
section, $M_1$ is the sum of all diagrams with one scattering center, $M_2$ is the sum of all diagrams with two scattering centers in the contact-limit case, while $d_T$ denotes the dimension of the target color representation (for pure gluon medium $d_T = 8$). In obtaining the expression for differential energy loss, we again incorporate Eqs. (9) [10] in Eq. (14).

The assumption that initial jet propagates along $z$-axis, takes the following form in the two cases stated below:

1. One interaction with QCD medium ($M_1$):

$$p + k - q_1 = [E^+ - q_{1z}, E^- + q_{1z}, 0],$$

where $p + k - q_1$ corresponds to the initial jet, while $k$ and $p$ retain the same expressions as in Eq. (4), with the distinction that now $p ≠ -k$, since due to 4-momentum conservation, the following relation holds:

$$q_1 = p + k;$$

The rest of the notation is the same as in Eq. (4).

2. Two interactions with QCD medium ($M_2$):

$$p + k - q_1 - q_2 = [E^+ - q_{1z} - q_{2z}, E^- + q_{1z} + q_{2z}, 0],$$

where $p + k - q_1 - q_2$ corresponds to the initial jet and $q_i$ = [q_{iz}, -q_{iz}, q_i] to exchanged gluons, $i = 1, 2$ with $q^{0}_i = 0$, while $p$, $k$ retain the same expressions as in Eq. (4). Also, due to 4-momentum conservation, the following relation between gluon transverse momenta holds:

$$p + k = q_1 + q_2,$$

which in the contact-limit case (when $q_1 + q_2 = 0$) reduces to $p + k = 0$.

Note that Eq. (10) has to be satisfied for $M_1$ diagrams in order to claim that initial jet propagates along $z$-axis, i.e. for $M_1$ diagrams $p + k$ is different from 0. This is an important distinction between the calculations presented in our study, and the calculations done within SCET formalism (see e.g. [14]), where $p + k = 0$ was used in calculation of both $M_1$ and $M_2$ diagrams, though the assumption that initial jet propagates along $z$-axis was used in that study as well.

The transverse polarization vectors $\epsilon_i(k)$ and $\epsilon_i(p)$ for both: $M_1$ and $M_2$ amplitudes are given by the same expression as in the previous section (with an addition that in $M_1$ case: $p ≠ -k$, as discussed above), while $\epsilon$ for initial jets consistently has the same form as in Eq. (9), i.e. $\epsilon_i(p + k - q_1) = [0, 0, \epsilon_i]$ for $M_1$ amplitudes, and $\epsilon_i(p + k - q_1 - q_2) = [0, 0, \epsilon_i]$ for $M_2$ amplitudes.

The detailed calculation of the remaining 10 Feynman diagrams, under the approximations stated in Appendix [6], contributing to the first order in opacity radiative energy loss, is given in Appendices [D] [H], whereas thorough derivation of the single gluon radiation spectrum beyond soft-gluon approximation in massless case is given in Appendix [I] and reads (energy loss expression can be straightforwardly extracted by using $dE^{(1)}(x)/dx ≡ \omega dN^{(1)}_g/dx ≈ xEdN^{(1)}_g/dx$):

$$\frac{dN^{(1)}_g}{dx} = \frac{C_2(G)\alpha_s}{\pi} \int \frac{d^2q_1}{\pi} \frac{\mu^2}{(q^2_1 + \mu^2)^2} \int dk^2 \times$$

$$\times \left\{ \frac{(k - q_1)^2}{(\frac{4x(1-x)E}{L})^2 + (k - q_1)^2} \left( 2 - \frac{k \cdot (k - q_1)}{k^2} - \frac{(k - q_1) \cdot (k - xq_1)}{(k - xq_1)^2} \right) + \frac{k^2}{(\frac{4x(1-x)E}{L})^2 + k^4} \left( 1 - \frac{k \cdot (k - xq_1)}{(k - xq_1)^2} \right) \right\},$$

where we assumed a simple exponential distribution $2 e^{-2ax}$ of longitudinal distance between the gluon-jet production site and target rescattering site, emerging as $(\frac{2x(1-x)E}{L})^2$ in the denominators of the integrand. Note that, Eq. (19) reduces to massless case of Eq. (11) from [25] in the $x \to 0$ limit, as expected.

It is straightforward to show that our result is symmetric under the exchange of radiated (k) and final (p) gluon, as expected beyond soft-gluon approximation, and due to inability to distinguish between these two identical gluons.
V. GLUON RADIATIVE ENERGY LOSS IN
FINITE TEMPERATURE QCD MEDIUM

Next, we note that in ultra-relativistic heavy ion collisions, finite temperature QCD medium is created, that modifies the gluon self energies, and can consequently significantly influence the radiative energy loss results. It is therefore essential to include finite temperature effects in gluon radiative energy loss calculations beyond soft-gluon approximation, which is the main goal of this section. To address this issue, we note that in [27], it was shown that gluons can be approximated as massive transverse plasmons with effective mass $m_g$ (for gluons with the hard momenta $k \gg T$) equal to its asymptotic value. The assumption of initial jet propagating along $z$-axis, for massive case, leads to the following form of momenta, in the three cases stated below:

1. No interaction with QCD medium ($M_0$):

$$
egin{align*}
p + k &= [E^+, E^-, 0], \\
k &= [xE^+, (k^2 + m_g^2/x^2), k], \\
p &= [(1−x)E^+, p^2 + m_g^2/(1−x)E^+, p],
\end{align*}
$$

(20)

where Eq. (5) holds;

2. One interaction with QCD medium ($M_1$):

$k$ and $p$ retain the same expressions as in Eq. (20), with addition that (as in the previous section) Eq. (10) holds due to conservation of momentum, while initial jet has the momentum of a same form as in Eq. (15).

3. Two interactions with QCD medium ($M_2$):

$p$, $k$ have the same expressions as in Eq. (20). Also, due to 4-momentum conservation Eq. (18) holds and in the contact-limit case reduces to $p+k=0$, while initial jet momentum retains the same form as in Eq. (17).

The transverse polarization vectors remain the same as in the massless case.

We retain all approximation from the previous section, which are reviewed in Appendix [B] and recalculate the same 11 diagrams from Appendices [C-H] also beyond soft-gluon approximation. The overview of all intermediate results is contained in Appendix [J]. Thus, Eq. (19) in the massive case acquires more complex form given by:

$$
\frac{dN_g}{dx} = C_2(G)\alpha_s L \frac{(1−x+x^2)^2}{x(1−x)} \int \frac{d^2q_1}{\pi} \frac{\mu^2((k−q_1)^2+\chi)}{(k^2+\mu^2)^2} \int dk^2 \times
\begin{align*}
&\left\{ \frac{(k−q_1)^2+\chi}{(4x(1−x)E)^2+((k−q_1)^2+\chi)^2} \right\}^2 \left( \frac{k^2}{k^2+\chi} \right) \\
&+ \frac{(k−q_1)^2+\chi}{(4x(1−x)E)^2+(k^2+\chi)^2} \left( \frac{k^2}{k^2+\chi} \right)^2
\end{align*}
$$

(21)

where $\chi = m_g^2(1−x+x^2)$. It can easily be verified that, in the soft-gluon limit, we recover Eq. (11) from [25] (note that for gluon jet $M \equiv m_g$, so that the term $M^2x^2$ from [25] should be neglected), and that in the massless limit Eq. (11) reduces to Eq. (19).

To our knowledge, this result presents the first introduction of effective gluon mass beyond-soft-gluon approximation radiative energy loss. Additionally, we again verified that single gluon radiation spectrum is symmetric to substitution of $p$ and $k$ gluons, as necessary (see the previous section and Appendix [J]). Furthermore, note that the analytical form of Eq. (21) is quite different from the corresponding expression with the soft-gluon approximation (Eq. (11) from [25]). In the next section, we will evaluate the extent of numerical differences to which these two different analytical expressions lead.

In particular, we are interested in what is the effect of finite $x$ on gluon fractional radiative energy loss ($\frac{\Delta E}{E}$), number of radiated gluons ($N_g$) and on the suppression ($R_{AA}$). We accordingly note that $\frac{dE}{dx} \approx \omega \frac{dN_g}{dx} \approx xE\frac{dN_g}{dx}$ from which we can further straightforwardly numerically evaluate $\frac{\Delta E}{E}$, as well as the number of radiated gluons ($N_g$).

VI. NUMERICAL RESULTS

We next assess how the relaxation of soft-gluon approximation modifies gluon-jet energy loss to the 1st order in opacity. We consequently compare the predictions based on the results derived in this paper, with the one obtained in the soft-gluon limit from [25] (applied to gluons) - the comparison is done for gluons with effective mass $m_g = \mu/\sqrt{2}$, where $\mu = \sqrt{4\pi\alpha_s(1+n_f/6)T}$ and $n_f = 3$ is the number of the effective light-quark flavor. For all figures, we assume constant strong coupling $\alpha_s$. 

FIG. 1: The effect of relaxing the soft-gluon approximation on integrated variables to the 1st order in opacity of DGLV formalism, as a function of $p_\perp$. The top left panel compares gluon’s fractional radiative energy loss without (the solid curve) and with (the dashed curve) soft-gluon approximation. The top right panel represents relative change of the radiative energy loss when the soft-gluon approximation is relaxed with respect to the soft-gluon limit. The bottom left panel compares number of radiated gluons without (the solid curve) and with (the dashed curve) soft-gluon approximation, whereas the bottom right panel provides a percentage of radiated gluon number change when soft-gluon approximation is relaxed.

Quantitative assessment of relaxing the soft-gluon approximation on these two variables can be observed from the two right panels of Fig. 1. We see that finite values of $x$ slightly increase fractional radiative energy loss by maximum of $\sim 3\%$ up to $p_\perp \approx 10$ GeV compared to $sg$ case. Afterwards, the difference between $bsg$ and $sg$ $\Delta E^{(1)}_\perp E$ steeply decreases towards 0%. Additionally, finite $x$ also decreases number of radiated gluons for a small amount (up to 5%) compared to $sg$ case for very low transverse momenta. Further the relative difference reaches a peak of $-2\%$ also at $p_\perp \approx 10$ GeV, and for higher transverse momenta remains nearly constant somewhat below $-2\%$. Consequently, the overall conclusion from Fig. 1 is that the effect on both variables is small and with opposite signs.

The effect of finite $x$ value is further assessed on the differential radiative energy loss ($\frac{dE^{(1)}_\perp}{dx}$), and on single gluon radiation spectrum ($\frac{dN^{(1)}_g}{dx}$) and it’s relative change. These effects are shown as a function of $x$ on Fig. 2 for higher $p_\perp$.

αs = $\frac{g^2}{4\pi} = 0.3$, and use $L = 5$ fm, $\lambda = 1$ fm, $T = 300$ MeV, to mimic standard LHC conditions.

The top left panel of Fig. 1 shows comparison of the fractional radiative energy loss $\Delta E^{(1)}_\perp$, for calculations beyond the soft-gluon approximation, and with the soft-gluon approximation, as a function of initial jet transverse momentum ($p_\perp$). More specifically, the curve corresponding to beyond soft-gluon approximation ($bsg$) case is obtained from Eq. (21) multiplied by $x E$ and integrated over $x$, while the curve corresponding to soft-gluon approximation ($sg$) case is obtained by numerically integrating Eq. (11) from [25]. These two curves almost overlap, even converge towards one another at higher $p_\perp$. Note that, the upper limit of $x$ integration is equal to $1/2$ instead of 1, in order to avoid double counting. The upper limits of integration for $|k|$ and $|q_1|$, determined kinematically, are $2x(1-x)E$ and $\sqrt{4ET}$, respectively [25].

The bottom left panel of Fig. 1 presents comparison of number of radiated gluons in $bsg$ and $sg$ cases. These two curves also nearly overlap, with a slight disagreement at higher $p_\perp$. Quantitative assessment of relaxing the soft-gluon approximation on these two variables can be observed from the two right panels of Fig. 1. We see that finite values of $x$ slightly increase fractional radiative energy loss by maximum of $\sim 3\%$ up to $p_\perp \approx 10$ GeV compared to $sg$ case. Afterwards, the difference between $bsg$ and $sg$ $\Delta E^{(1)}_\perp E$ steeply decreases towards 0%. Additionally, finite $x$ also decreases number of radiated gluons for a small amount (up to 5%) compared to $sg$ case for very low transverse momenta. Further the relative difference reaches a peak of $-2\%$ also at $p_\perp \approx 10$ GeV, and for higher transverse momenta remains nearly constant somewhat below $-2\%$. Consequently, the overall conclusion from Fig. 1 is that the effect on both variables is small and with opposite signs.
FIG. 2: The effect of relaxing the soft-gluon approximation on differential variables to the 1st order in opacity of DGLV formalism, as a function of $x$. The comparison of: i) differential gluon radiative energy loss ($dE^{(1)}/dx$); ii) single gluon radiation (spectrum) distribution in momentum fraction ($dN^{(1)}_g/dx$) between $bsg$ (the solid curve) and $sg$ (the dashed curve) case, for different values of initial jet transverse momenta (5 GeV, 10 GeV, 50 GeV, as indicated in panels) is shown in the first and second column, respectively. The relative change of the single gluon radiation spectrum with respect to soft-gluon limit is shown in the third column.

Different values of initial jet transverse momentum $p_\perp$; $bsg$ curves for $dE^{(1)}_g/dx$ are obtained by numerically integrating Eq. (21) multiplied by $xE$ over $|k|$ and $q_1$, whereas $sg$ curves correspond to Eq. (11) in [25]. From Fig. 2 we observe a small difference between $bsg$ and $sg$ results for $x \lesssim 0.3$ (roughly up to 0.4), i.e. for smaller $x$, as expected. We also recognize $x \approx 0.3$ as a “cross-over” value, below which differential radiative energy loss and single gluon radiation spectrum are somewhat lower in $bsg$ compared to $sg$ case, and above which the opposite is true. At high value of $x$, i.e. $0.4 < x \lesssim 0.5$, the differences between our $bsg$ differential radiative energy loss and previously obtained $sg$ [25] ascend to notable values ($\sim 50\%$) and increases with increasing $p_\perp$.

To investigate the effect of relaxing the soft-gluon approximation on the single gluon radiation spectrum in more detail, the third column is added in Fig. 2 (see also Fig. 3), showing relative change of $dN^{(1)}_g/dx$. This quantitative estimation (difference smaller than 10% for $x \lesssim 0.4$) is in agreement with the previous discussion. In particular, at higher $x$ values, there is a notably larger spectra in $bsg$ compared to $sg$ case, and this difference enhances (up to 60% at $p_\perp = 50$ GeV) with increasing $p_\perp$. Never-
FIG. 3: The effect of relaxing the soft-gluon approximation on \( \frac{dN^{(1)}}{dx} \) for different \( p_{\perp} \) values. The relative change of the single gluon radiation spectrum with respect to soft-gluon case, calculated to the 1\(^{\text{st}}\) order in opacity of DGLV formalism, for different values of initial \( p_{\perp} \) (as indicated in the legend) is depicted as a function of \( x \). The curves fade as transverse momentum increases.

theless, for both variables: \( \left( \frac{dE^{(1)}}{dx} \text{ and } \frac{dN^{(1)}}{dx} \right)_{\text{bsg}} \) and \( \text{sg} \) cases lead to similar results for \( x \lesssim 0.4 \).

The effect of relaxing the soft-gluon approximation on single gluon radiation spectrum for different transverse momentum values of initial gluon jet is further addressed in Fig. 3. We observe that a notable, that is, tenfold increase of \( p_{\perp} \) leads to a modest increase (less than 25\%) of \( \frac{dN^{(1)}}{dx} \) in \( \text{bsg} \) compared to \( \text{sg} \) case. Note that the same dependence is obtained for \( \left( \frac{dE^{(1)}}{dx} \right)_{\text{bsg}} / \left( \frac{dE^{(1)}}{dx} \right)_{\text{sg}} - 1 \) (since \( \frac{dE^{(1)}}{dx} \sim x \frac{dN^{(1)}}{dx} \), so that \( x \) cancels when taking the relative ratio). Therefore, we conclude that the relaxation of the soft-gluon approximation has nearly the same effect on \( \frac{dN^{(1)}}{dx} \) and \( \frac{dE^{(1)}}{dx} \) (across the whole \( x \) region) independently on \( p_{\perp} \) of the initial jet.

Although we showed that relaxing the soft-gluon approximation has small numerical impact on both integrated \( (\frac{\Delta E^{(1)}}{E}, N^{(1)}_g, \text{ across the whole } x \text{ region}) \) and differential \( (\frac{dE^{(1)}}{dx}, \frac{dN^{(1)}}{dx}) \), up to \( x \approx 0.4 \) variables, the difference between \( \text{bsg} \) and \( \text{sg} \) cases can go up to 10\% (and with different signs), and moreover can be quite large for \( x > 0.4 \). This, therefore, leads to a question, how the relaxation of the soft-gluon approximation affects predictions for measured observables, such as the angular averaged nuclear modification factor \( R_{AA} \) \[18, 19\]. Comparing \( R_{AA} \) with and without soft-gluon approximation allows assessing how adequate is this approximation in obtaining reliable numerical predictions.

To that end, we next concentrate on generating the predictions for bare gluon \( R_{AA} \), based only on radiative energy loss, with and without soft-gluon approximation.

\( R_{AA} \) is defined as the ratio of the quenched \( A + A \) spectrum to the \( p + p \) spectrum, scaled by the number of binary collisions \( N_{bin} \):

\[
R_{AA}(p_{\perp}) = \frac{dN_{AA}/dp_{\perp}}{N_{bin}dN_{pp}/dp_{\perp}}.
\]

In order to obtain gluon quenched spectra, we use generic pQCD convolution \[33\]:

\[
\frac{E_i d^3\sigma(q)}{dp_i^3} = \frac{E_i d^3\sigma(q)}{dp^3_i} \otimes P(E_i \rightarrow E_f),
\]

where \( \frac{E_i d^3\sigma(q)}{dp_i^3} \) denotes initial gluon spectrum, which is computed according to \[34, 35\], while \( P(E_i \rightarrow E_f) \) denotes radiative energy loss probability, which includes multi-gluon \[24\] and path-length \[33\] fluctuations. The path-length distributions for \( 0 - 5\% \) most central collisions are implemented as described in \[19\]. Note that we omitted fragmentation and decay functions, because we are considering the parton’s quenching, as we are primarily interested in how the relaxation of the soft-gluon approximation in energy loss affects \( R_{AA} \). Therefore, we will also investigate how the initial gluon distribution influences \( R_{AA} \).

Therefore, the left panel of Fig. 4 compares \( R_{AA} \) predictions with and without soft-gluon approximation accounted, while the percentage change arising from relaxing the approximation is given by the right panel of Fig. 4 as a function of the final \( p_{\perp} \). We observe that this relaxation barely modifies \( R_{AA} \), in particular the relative change drops to somewhat less than \(-1\%\) at \( p_{\perp} \approx 10 \text{ GeV} \) and further rises to the constant value of \(2\%\), with increasing \( p_{\perp} \). This very good agreement (with even smaller differences compared to previously studied variables) between \( \text{bsg} \) and \( \text{sg} R_{AA} \) raises questions of: \( i) \) why relaxing the soft-gluon approximation has negligible effect on \( R_{AA} \) and \( ii) \) why the large discrepancy observed in Figs. 3 for high \( x \) values does not lead to larger difference in \( R_{AA} \)?

Regarding \( i) \) above, we argue that this pattern is expected, as it is well-known that in suppression calculations both \( \frac{\Delta E^{(1)}}{E} \) and \( N^{(1)}_g \) non-trivially affect the \( R_{AA} \). Namely, by comparing the two right panels of Fig. 4 with the right panel of Fig. 4 we observe that relaxing the soft-gluon approximation has opposite effects on \( \frac{\Delta E^{(1)}}{E} \) and \( N^{(1)}_g \), while their interplay is responsible for the negligible effect on \( R_{AA} \) - i.e. the effect on \( R_{AA} \) is qualitatively a superposition of the effects on \( \frac{\Delta E^{(1)}}{E} \) and \( N^{(1)}_g \).

To answer \( ii) \) above, it is convenient to recall that suppression of gluon jet (see Eq. \[25\]) depends not only on the energy loss probability, but also on the initial gluon distribution. In order to intuitively interpret the role of the initial gluon distribution, we refer to a descriptive Fig. 5 which represents its dependence on initial transverse momentum. The concept considered is the following: Some parent gluon with unknown initial momentum traverses QGP, looses its energy by gluon bremsstrahlung, and emerges with final
FIG. 4: The effect of relaxing the soft-gluon approximation on gluon nuclear modification factor $R_{AA}$ versus $p_{\perp}$. The suppression of gluon jet beyond soft-gluon approximation (the solid curve) is compared to soft-gluon $R_{AA}$ (the dashed curve) as a function of transverse momentum in the left panel. The right panel quantifies the effect and expresses it in percentage.

FIG. 5: The role of initial gluon distribution in constraining relevant $x$ region. The solid black curve represents initial gluon distribution as a function of $p_{\perp}$ at the LHC [34, 35]. The dot-dashed gray line marks the final gluon transverse momentum, while dotted arrows link parent gluons, that lost momentum fraction equal to $x$, with their corresponding initial transverse momenta. The arrows fade as $x$ increases, as indicated in the legend.

The main theoretical goal of this paper was to investigate what effect relaxing of the soft-gluon approximation has on radiative energy loss, and consequently on suppression, which depends only on initial distribution and energy loss of high-momentum parton in QGP. Particularly we chose high $p_{\perp}$ gluon, as due to the color factor of $9/4$ compared to the quarks, this assumption affects gluons the most. To this end, we analytically calculated all Feynman diagrams contributing to the first order in opacity correlated with $x$. From Fig. 5 we infer that, due to the exponentially decreasing initial gluon momentum distribution, the initial gluon corresponding to $x = 0.1$ has the highest probability to be the parent one, and as $x$ increases the probability sharply decreases (i.e. for $x \gtrsim 0.4$ it diminishes for 2 orders of magnitude compared to the $x = 0.1$ case). This makes only the region below the crossover noted above ($x \lesssim 0.3$) relevant for generating the predictions, and also explains why the large inconsistency between $bsg$ and $sg$ $\frac{dE}{dx}$ (or equivalently $\frac{dN^{(1)}_{g}}{dx}$) curves from Figs. (2, 3) at higher $x$ does not affect $R_{AA}$. Additionally, the effect of relaxing the soft-gluon approximation on $\frac{dN^{(1)}_{g}}{dx}$ and $\frac{dE^{(1)}}{dx}$ is practically insensitive to initial transverse momentum (see Fig. 3), which is the reason why finite $x$ affects equivalently gluon $R_{AA}$ regardless of its transverse momentum, as observed in Fig. 4.

Finally, we also recalculated our finite $x$ results, when running coupling $\alpha_s(Q^2)$, as defined in [36], instead of constant value $\alpha_s = 0.3$, is introduced in radiative energy loss formula. The obtained predictions lead to the same conclusions as obtained above (and are consequently omitted), which supports the generality of the obtained results.

VII. CONCLUSIONS AND OUTLOOK

The main theoretical goal of this paper was to investigate what effect relaxing of the soft-gluon approximation has on radiative energy loss, and consequently on suppression, which depends only on initial distribution and energy loss of high-momentum parton in QGP. Particularly we chose high $p_{\perp}$ gluon, as due to the color factor of $9/4$ compared to the quarks, this assumption affects gluons the most. To this end, we analytically calculated all Feynman diagrams contributing to the first order in opac-
ity radiative energy loss beyond soft-gluon approximation, first within GLV [9] (massless case), and later within DGLV [24] (massive case), formalism, and numerically predicted: fractional and differential energy loss, number of radiated gluons, single gluon radiation spectrum and gluon’s suppression. Unexpectedly we obtained that, although the analytic results significantly differ from the corresponding soft-gluon results, the numerical predictions are nearly indistinguishable, i.e. within few per-cents. We then explained that, due to exponentially decreasing initial gluon distribution, only $x < 0.3$ region effectively contributes to the integrated variable predictions. We also showed that negligible suppression change is due to an interplay between the finite $x$ effects on i) fractional energy loss and ii) number of radiated gluons, that have opposite sign. The presented comparisons are done under the assumption of fixed strong coupling constant, but also tested with running coupling leading to the same conclusions. Since we showed that gluon quenching in QCD medium composed of static scattering centers is not affected by the soft-gluon assumption, quark radiative energy loss is even less likely to be notably altered, though this remains to be further tested.

This, to our knowledge, presents the first opportunity to assess the effect of relaxing the soft-gluon approximation on radiative energy loss within DGLV formalism. Some other radiative energy loss formalisms, which also imply static scatterers, generated their results on a finite $x$. However, contrary to the conclusions derived for these formalisms, where significant difference in the radiative energy loss was obtained, we found that relaxing soft-gluon approximation brings negligible change to the results. Consequently, this implies that, within DGLV formalism, there is no need to go beyond the soft-gluon approximation.

Based on the results of this paper, we also expect that the soft-gluon approximation can be reliably applied to the dynamical energy loss formalism, as implicitly suggested by the previous robust agreement [17, 20, 22] of our theoretical predictions with a comprehensive set of experimental data. In particular, the effective cross section $v(q)$ (which corresponds to interaction between the jet and exchanged gluon) [37] does not depend on $x$, so introduction of finite $x$ will not affect this term. We also expect that the rest of the energy loss expression (i.e. $f(k, q, x)$, which corresponds to interaction between the jet and radiated gluon [37] will be modified in the similar manner as in the static case, since when $x \to 0$, these two expression coincide. However, relaxing the soft-gluon approximation in dynamical energy loss model is out of the scope of this paper, and this claim still remains to be rigorously tested in the future.

Acknowledgments: This work is supported by the European Research Council, grant ERC-2016-COG: 725741, and by the Ministry of Science and Technological Development of the Republic of Serbia, under project numbers ON171004 and ON173052.

Appendix A: NOTATIONS AND USEFUL FORMULAS

In this paper we used the following notation for vectors, in consistency with both [9, 24]:

- $\vec{p}$ denotes momentum 3D vector
- $p$ denotes transverse momentum 2D vector
- $p_z$ denotes component of momentum vector along the initial jet
- $p = (p^0, p_z, \vec{p}) = [p^+, p^-, \vec{p}]$ denotes momentum 4D vector in Minkowski and Light Cone coordinates, respectively, where $p^+ = p^0 + p_z$ and $p^- = p^0 - p_z$.

For simplicity, we here consider QCD medium consisting of static partons and model the interactions of the gluon jet with the medium via static color-screened Yukawa potential, whose Fourier and color structure acquires the following form [9, 26]:

$$V_n = V(q_n) e^{i q_n \cdot x_n} = 2 \pi \delta(q_n^0) v(q_n) e^{-i q_n \cdot \vec{x}_n} \times T_{as}(R) \otimes T_{an}(n),$$

$$v(q_n) = \frac{4 \pi \alpha_s}{q^2_n + \mu^2},$$

where $x_n$ denotes space-time coordinate of the $n^{th}$ scatterer (target), $T_{as}(R)$ and $T_{an}(n)$ denote generators in $SU(N_c = 3)$ color representation of jet and target, respectively, while $\mu$ is Debye screening mass and $\alpha_s = g^2/4 \pi$ is strong coupling constant. In the following lines we will briefly display the identities and algebra that $SU(N_c = 3)$ generators meet:

$$\text{Tr}(T^a(n)) = 0,$$

$$\text{Tr}(T^a(i)T^b(j)) = \delta_{ij} \delta^{ab} \frac{C_2(i) d_i}{d_G},$$

where $d_G = 8$ is the dimension of the adjoint representation ($G$). We assume that all target partons are in the same $d_T$ dimensional representation ($T$) with Casimir operator $C_2(T)$, while the gluon jet is in the adjoint representation ($G$), with Casimir operator $C_2(G)$.

In $SU(N_c = 3)$ color algebra, the following identities hold as well:

$$[T^a, T^b] = if^{abc}T^c,$$

while in the adjoint representation we have:

$$(T^a)^{ab} = if^{abc},$$

$$T^a(G)T^a(G) = C_2(G)I,$$
where $I$ denotes identity matrix of dimension $d_G$ and the $SU(N_c = 3)$ structure constants $f^{abc}$ are completely antisymmetric to indices permutations, which we frequently apply. In the adjoint representation the following equalities also stand:

$$C(G) = C_2(G) = N_c = 3,$$  \hspace{1cm} (A8)

$$\text{Tr}(T^a(G)T^a(G)) = d_GC_2(G).$$ \hspace{1cm} (A9)

And finally, in our computations we frequently make use of the fact that trace is invariant under cyclic permutations and that generators are Hermitian matrices.

### Appendix B: ASSUMPTIONS

Throughout the paper we assume that initial gluon jet propagates along the $z$-axis, i.e. has transverse momentum equal to zero, while radiated gluon carries away a finite rate $x$ of initial gluon longitudinal momentum and energy, and final gluon emerges with momentum $p$. Therefore, instead of assuming soft-gluon approximation ($x \ll 1$), as it was done in [9, 25], we allow $x$ to acquire finite non-zero values, thus relaxing the soft-gluon approximation.

Since we are calculating radiative energy loss within the (GLV) DGLV formalism apart from abandoning the soft-gluon approximation, the following assumptions remain:

- **The soft-rescattering approximation.** Consistently with [9, 25] we assume that partons energies and longitudinal momenta are high compare to their transverse momenta, which disables the radiated and the final gluon to digress much from the initial longitudinal direction (the eikonal approximation).

$$E^+ \sim (1 - x)E^+ \sim xE^+ \gg |p|, |k|, |q_i|,$$  \hspace{1cm} (B1)

- **The first order approximation.** The gluon-jet radiative energy loss is calculated up to the first order in opacity expansion, as argued in [9, 29, 30].

- **Scattering centers distribution and ensemble average.** We consider that all scattering centers $x_i$ are distributed with the same transversely homogeneous density:

$$\rho(\vec{x}) = \frac{N}{A_\perp \bar{\rho}(z)},$$  \hspace{1cm} (B2)

where $\int dz \bar{\rho}(z) = 1$ and also that impact parameter (i.e. relative transverse coordinate) $b = \vec{x}_i - \vec{x}_0$ alters within a large transverse area $A_\perp$ compared to the interaction area $\frac{1}{\nu^2}$. Therefore, the ensemble average over the scattering center locations reduces to an impact parameter average:

$$\langle ... \rangle = \int \frac{d^2b}{A_\perp} ...,$$  \hspace{1cm} (B3)

which in our case is mainly used in the following form:

$$\left\langle e^{-i(q_i + q_j)b} \right\rangle = \frac{(2\pi)^2}{A_\perp} \delta^2(q_i + q_j).$$  \hspace{1cm} (B4)
We also assume that the energy of initial hard probe is large compared to the potential screening scale:

\[ E^+, (1 - x) E^+, x E^+ \gg \mu. \]  

(B5)

Next, we assume that the distance between the source \( J \) and the scattering centers is large relative to the interaction length:

\[ z_i - z_0 \gg \frac{1}{\mu}, \]  

(B6)

then, that source current varies slowly with momentum:

\[ J(p + k - q) \approx J(p + k), \]  

(B7)

and that the source current can be written explicitly in terms of polarization vector:

\[ J_\mu(p + k - q) \equiv J_\mu(p + k + q) = \epsilon_\mu(p + k - q) \approx J_\mu(p + k) \epsilon_\mu(p + k - q). \]  

(B8)

In the following sections first we assume that gluons are massless (GLV) in order to make the comprehensive derivations more straightforward and easier to follow, but later we recalculate all the results with gluon mass included (DGLV) (Appendix J).

Appendix C: Gluon jet \( M_0 \)

First we calculate gluon-jet radiation amplitude to emit a gluon, carrying a finite fraction \( x \) of initial jet energy, with momentum, polarization and color \((k, \epsilon(k), c)\) and without interactions with the medium \( M_0 \).

We assume that initial gluon \((p + k)\) propagates along \( z \)-axis. By using \( M_0 \) amplitude as an example, we will implement the aforementioned assumptions in order to acquire momentum and polarization expressions. Thus, the initial gluon 4-momentum reads:

\[ p + k = (p^0 + k^0, p_z + k_z, 0), \]

\[ p + k = [E^+, E^-, 0], \]  

(C1)

where \( E^+ = p^0 + k^0 + p_z + k_z \) and \( E^- = p^0 + k^0 - p_z - k_z \).

Assuming massless (real) gluons for simplicity, the momentum vectors of the radiated \( k \) and the final \( p \) gluons acquire the following form:

\[ k^2 = 0 \Rightarrow k = [x E^+, \frac{k^2}{x E^+}, k], \]  

(C2)

\[ p^2 = 0 \Rightarrow p = [(1 - x) E^+, \frac{p^2}{(1 - x) E^+}, p]. \]  

(C3)

We also assume that gluons are transversely polarized particles. Although we work in covariant gauge, we can choose any polarization vector for the external on-shell \( M_0 \) gluons, so in accordance with [9, 14, 25] we choose \( n^\mu = [0, 2, 0, 0] \), as stated above:

\[ \epsilon(k) \cdot k = 0, \quad \epsilon(k) \cdot n = 0, \quad \epsilon(k)^2 = -1, \]

\[ \epsilon(p) \cdot p = 0, \quad \epsilon(p) \cdot n = 0, \quad \epsilon(p)^2 = -1, \]  

(C4)

while we assume that the source has also the physical polarizations as the real gluons [14]:

\[ \epsilon(p + k) \cdot (p + k) = 0, \quad \epsilon(p + k) \cdot n = 0, \quad \epsilon(p + k)^2 = -1. \]  

(C5)

Using Eqs. (C2, C5) we can now obtain the following expressions for the gluon polarizations:

\[ \epsilon_i(k) = [0, \frac{2 \epsilon_i \cdot k}{x E^+}, \epsilon_i], \quad \epsilon_i(p) = [0, \frac{2 \epsilon_i \cdot p}{(1 - x) E^+}, \epsilon_i], \]

\[ \epsilon_i(p + k) = [0, 0, \epsilon_i], \]  

(C6)

where \( i = 1, 2 \) counts for polarization vectors. Note that the 4-momentum is conserved, which leads to the relation:

\[ p + k = 0, \]  

(C7)

that we implement in Eqs. (C3, C6) in order to ensure that everything is expressed in terms of \( k \). Also, \( E^+ = 2E, E^- = \frac{k^2}{x(1 - x) E^+}, \) where \( E = p^0 + k^0 \) is the energy of initial jet.

Using the notation from Fig. (C) we obtain:

\[ \text{FIG. 6: Zeroth order diagram that includes no interaction with the QCD medium, and contributes to gluon radiation amplitude to the first order in opacity } L/\lambda. \]
\[ M_0 = \epsilon_\sigma(p)\epsilon_\rho(k)g_s f^{acd}(g^{\mu\sigma}(2p+k)\rho + g^{\rho\sigma}(-p-2k)\sigma + g^{\sigma\rho}(-p+k)\mu) \times \frac{-i\delta_{\alpha\alpha'}g_{\mu\nu'}}{(p+k)^2 + i\epsilon} iJ_a(p+k)\epsilon^{i(p+k)x_0} \times \epsilon^{\mu'}(p+k) \approx J_a(p+k)\epsilon^{i(p+k)x_0}(-2g_s)(1-x+x^2)\frac{\epsilon \cdot k}{k^2} f^{acd} \]

\[ = J_a(p+k)\epsilon^{i(p+k)x_0}(-2i\epsilon)(1-x+x^2)\frac{\epsilon \cdot k}{k^2} (T^c)_{da}. \]  

(C8)

Eq. (C8) after summation by using Eq. (A12) gives:

\[ \langle |M_0|^2 \rangle = |J(p+k)|^2 (4g_s^2) \frac{C_2(G)d_G}{k^2} (1-x+x^2)^2. \]  

(C9)

Next we substitute the Eq. (C9) in:

\[ d^3N_g^{(0)} d^3N_J \approx \text{Tr} \langle |M_0|^2 \rangle \frac{d^3\vec{p}}{(2\pi)^32p^0} \frac{d^3\vec{k}}{(2\pi)^32k^0}. \]  

(C10)

Note that, contrary to the soft-gluon approximation [25], now \( p, \) denoting the momentum of the final gluon jet, is not approximately equal to the momentum of initial gluon jet (i.e. the radiated gluon can carry away a substantial amount of the initial jet energy and longitudinal momentum). Thus instead of Eq. (C11) throughout this paper we use the general one:

\[ d^3N_J = d_G|J(p+k)|^2 \frac{d^3\vec{p}_J}{(2\pi)^32E_J}, \]  

(C12)

where \( E_J = E \) and \( \vec{p}_J \) denotes energy and 3D momentum of the initial gluon jet, respectively. Knowing that the substitution of variables \( (p_z, k_z \rightarrow p'_z, xE) \) gives:

\[ \frac{d^3\vec{p}}{(2\pi)^32p^0} \frac{d^3\vec{k}}{(2\pi)^32k^0} = \frac{d^3\vec{p}_J}{(2\pi)^32E_J} \frac{dxdk^2}{(2\pi)^32x(1-x)}, \]  

(C13)

and by substituting Eqs. (C9, C12, C13) in Eq. (C10), for radiation spectrum we now obtain:

\[ \frac{xd^3 N_g^{(0)}}{dxdk^2} = \frac{\alpha_s}{\pi} \frac{C_2(G)(1-x+x^2)^2}{k^2(1-x)}, \]  

(C14)

which recovers well-known Altarelli-Parisi result [31] and for \( x \ll 1 \) reduces to the massless soft-gluon limit of Eq. (9) from [25]. We have also checked that, the same result can be obtained by directly implementing polarization vectors (Eq. (C6)) in Eq. (C8), instead of using Eq. (A12) when averaging.

Appendix D: Diagrams \( M_{1,1,0}, M_{1,0,0}, M_{1,0,1} \)

In this section we provide a detailed calculations of Feynman amplitudes, corresponding to gluon-jet interaction with one scattering center, which are depicted in Fig. 7. Again for consistency, we assume that initial jet \( (p + k - q) \) propagates along z-axis. Throughout this section momentum and polarization vector for initial gluon read:

\[ p + k - q_1 = [E^+ - q_{1z}, E^- + q_{1z}, 0], \]  

(D1)

\[ \epsilon_i(p + k - q_1) = [0, 0, 0], \]  

(D2)

where \( q_1 = [q_{1z}, -q_{1z}, q_1], \) with \( q_1^0 = 0, \) denotes the momentum of exchanged gluon, while \( p, k \) and corresponding polarization vectors retain the same expression as in Eqs. (C2, C3, C6), with the distinction that the following relation between gluon transverse momenta, due to 4-momentum conservation, holds:

\[ q_1 = p + k. \]  

(D3)

1. Computation of \( M_{1,1,0} \) diagram

We chose to start with thorough derivation of the expression for \( M_{1,1,0} \) amplitude, simply because it has no counterpart regarding the symmetry under \( (p \leftrightarrow k, x \leftrightarrow (1-x), c \leftrightarrow d) \) substitutions, and it provides all necessary steps for calculating the remaining two amplitudes from this chapter, apart from having one less singularity compared to the amplitudes \( M_{1,0,0} \) and \( M_{1,0,1} \). Thus, using the notation from the left diagram of Fig. 7, we write:
and assumed that $J$ varies slowly with momentum $q_1$, i.e. Eq. (B7). The longitudinal momentum transfer integral:

$$I_1(p, k, q_1, z_1 - z_0) = \int \frac{dq_{1z} v(q_{1z}, q_1) e^{-iq_{1z}(z_1 - z_0)}}{2\pi ((p + k - q_1)^2 + i\epsilon)((p + k - q_1)^2 + i\epsilon)},$$

has to be performed in the lower half-plane of the complex plain, since $z_1 > z_0$. In order to determine the pole arising from potential, we rewrite Eq. (A2) in a more appropriate form:

$$v(q_n) = \frac{4\pi\alpha_s}{(q_{nz} + i\mu_{n\perp})(q_{nz} - i\mu_{n\perp})},$$

where $\mu_{n\perp}^2 = \mu^2 + q_n^2$, with $n$ denoting the corresponding scattering center.

Aside from the pole originating from Eq. (D7) ($q_{1z} = -i\mu_{1\perp}$), there is also a singularity emerging from the gluon propagator:

$$\bar{q}_1 = -\frac{k^2}{xe^+} - \frac{p^2}{(1 - x)e^+ + i\epsilon}$$

$$= -\frac{k^2}{2\omega} - \frac{x}{1 - x} \frac{(k - q_1)^2}{2\omega} - i\epsilon,$$

where $\omega = k_0 \approx \frac{xe^+}{2}$. The residue around the pole at \(\bar{q}_1\) is computed as (the negative sign is due to the clock-wise orientation of the closed contour in the complex plain):

$$Res(\bar{q}_1) \approx -v(-\frac{k^2}{xe^+} - \frac{p^2}{(1 - x)e^+}, q_1) i \frac{k^2}{x e^+} e^{i\frac{k^2}{xe^+}(z_1 - z_0)}$$

$$= -v(-\frac{k^2}{2\omega} - \frac{x}{1 - x} \frac{(k - q_1)^2}{2\omega}, q_1) i \frac{k^2}{x e^+} e^{i\frac{k^2}{2\omega}(k - q_1)^2/2}(z_1 - z_0).$$

FIG. 7: Three diagrams, corresponding to interaction with one static scattering center, that contribute to gluon-jet radiation amplitude to the first order in opacity $L/\lambda$. $z_1$ denotes longitudinal coordinate of the interactions with one scattering center. Crossed circle represents scatterer that exchanges 3D momentum $\vec{q}_1$ with the jet. Note that, all three diagrams assume equivalently ordered Latin and Greek indices as indicated by the first diagram. Remaining labeling is the same as in Fig. 6.
The pole originating from the potential ($q_{1z} = -i\mu_{1\perp}$) does not contribute to the longitudinal integral, since residue around that pole is exponentially suppressed due to Eq. (D6), i.e. $\mu(z_1 - z_0) = \mu \gg 1$ (and $\mu \approx \mu_{1\perp}$):

$$Res(-i\mu_{1\perp}) \approx -i \frac{4\pi\alpha_s}{(2\pi)^2} e^{-\mu_{1\perp}(z_1 - z_0)} \rightarrow 0,$$

where we assumed that $E^+ \gg \mu$ and soft-rescattering approximation.

This makes only $\bar{q}_1$ singularity relevant for calculating longitudinal integral. Therefore $I_1$ coincides with Eq. (D9), i.e.:

$$I_1(p, k, q_1, z_1 - z_0) \approx -v(-\frac{k^2}{x E^+} - \frac{p^2}{(1 - x) E^+}, q_1) \frac{i}{E^+} e^{i \frac{p^2}{x E^+} + (\frac{x}{2} + \frac{1}{2}) x q_1^2 (z_1 - z_0)} \approx -v(0, q_1) \frac{i}{E^+} e^{i \frac{p^2}{x E^+} + (\frac{x}{2} + \frac{1}{2}) x q_1^2 (z_1 - z_0)},$$

where we use eikonal approximation (i.e. for a finite $x$: $\frac{k^2}{(2\pi)^2} \ll 1$ and $\frac{p^2}{(1 - x) E^+} \ll 1$). Finally, $M_{1,1,0}$ amplitude reads:

$$M_{1,1,0} = J_a(p + k)e^{i(p+k)x_0}(-i)(1 - x + x^2)f^{abcd}f_{a1b}T_{a1} \int \frac{d^2q_1}{(2\pi)^2} v(0, q_1)e^{-iq_1 \cdot b_1(-2ig_s)} \frac{e \cdot ((1 - x)k - xp)^2}{(1 - x)k - xp} \times \frac{e \cdot (k - xq_1)}{(k - xq_1)^2}$$

where we denoted $b_1 \equiv x_1 - x_0$. In this subsection, we constantly make use of Eq. (D3) in the following form:

$$p^2 = (k - q_1)^2,$$

and also manipulate with $SU(N_c = 3)$ structure constants by using Eqs. (A5) (A6). Note from Fig. 7 that, as expected, $M_{1,1,0}$ is symmetric under the substitutions: ($p \leftrightarrow k, x \leftrightarrow (1 - x), c \leftrightarrow d$), where the symmetry can be straightforwardly verified by implementing these substitutions in the first two lines of Eq. (D12).

2. Computation of $M_{1,0,0}$ and $M_{1,0,1}$ diagrams

Applying the same procedure as in the previous subsection, we proceed with calculating $M_{1,0,0}$. Note that the order of the color and Dirac indices denoting vertices is the same for all three diagrams in Fig. 7 and are therefore omitted in the last two diagrams.

$$M_{1,0,0} \approx J_a(p + k)e^{i(p+k)x_0}f^{abcd}f_{a1b}T_{a1}(-i)(1 - x + x^2)k^2 \int \frac{d^2q_1}{(2\pi)^2} e^{-iq_1 \cdot b_1(2g_s)} \frac{e \cdot k}{x} I_2,$$

(D14)
where:

\[
I_2(p, k, q_1, z_1 - z_0) = \int \frac{dq_1}{2\pi} \int \frac{dv(q_1, q_1)}{2\pi} e^{-i q_1 (z_1 - z_0)} \times \frac{1}{(p + k - q_1)^2 + i\epsilon} \times \frac{1}{(p - q_1)^2 + i\epsilon}.
\]

In order to calculate the previous integral, due to \(z_1 > z_0\)

\[
I_2(p, k, q_1, z_1 - z_0) \approx \frac{i x}{E+p k^2} v(0, q_1) \left( e^{i \left( \frac{k^2}{E+p k^2} + \frac{p^2}{(1-x)E^2} \right) (z_1 - z_0)} - e^{i \left( \frac{p^2-k^2}{(1-x)E^2} \right) (z_1 - z_0)} \right)
\]

leading to:

\[
M_{1,0,0} = J_a (p + k) e^{i(p+k)x_0} (-i)(1 - x + x^2) f^db a f^{adm} T_{a_1} \int \frac{d^2 q_1}{(2\pi)^2} v(0, q_1) e^{-i q_1 \cdot b_1} \frac{1}{2i\epsilon} \times \left( e^{i \left( \frac{k^2}{E+p k^2} + \frac{p^2}{(1-x)E^2} \right) (z_1 - z_0)} - e^{i \left( \frac{p^2-k^2}{(1-x)E^2} \right) (z_1 - z_0)} \right)
\]

\[
= J_a (p + k) e^{i(p+k)x_0} (-i)(1 - x + x^2) (T^{01} T^c)_{da} T_{a_1} \int \frac{d^2 q_1}{(2\pi)^2} v(0, q_1) e^{-i q_1 \cdot b_1} \frac{1}{2i\epsilon} \times \left( e^{i \frac{1}{E+p k^2} (k^2 + \frac{p^2}{1-x}) (z_1 - z_0)} - e^{-i \frac{1}{E+p k^2} (k^2 + \frac{p^2}{1-x}) (z_1 - z_0)} \right).
\]

By applying similar procedure for \(M_{1,0,1}\) we obtain:

\[
M_{1,0,1} = J_a (p + k) e^{i(p+k)x_0} (-i)(1 - x + x^2) f^db a f^{adm} T_{a_1} \int \frac{d^2 q_1}{(2\pi)^2} v(0, q_1) e^{-i q_1 \cdot b_1} \frac{1}{2i\epsilon} \times \left( e^{i \left( \frac{k^2}{E+p k^2} + \frac{p^2}{(1-x)E^2} \right) (z_1 - z_0)} - e^{i \left( \frac{p^2-k^2}{(1-x)E^2} \right) (z_1 - z_0)} \right)
\]

\[
= J_a (p + k) e^{i(p+k)x_0} (-i)(1 - x + x^2) [T^c, T^{a_1}]_{da} T_{a_1} \int \frac{d^2 q_1}{(2\pi)^2} v(0, q_1) e^{-i q_1 \cdot b_1} \frac{1}{2i\epsilon} \times \left( e^{i \frac{1}{E+p k^2} (k^2 + \frac{p^2}{1-x}) (z_1 - z_0)} - e^{-i \frac{1}{E+p k^2} (k^2 + \frac{p^2}{1-x}) (z_1 - z_0)} \right).
\]

Notice from Fig. 7 that \(M_{1,0,1}\) and \(M_{1,0,0}\) are symmetric under the following substitutions: \((p \leftrightarrow k, x \leftrightarrow (1-x), c \leftrightarrow d)\); it can be straightforwardly verified that Eqs. (D17)(D18) are symmetric under these substitutions.

Appendix E: Diagram \(M_{2,2,0}\)

Next we concentrate on the diagrams containing two interactions with the static scattering centers, since they also contribute to the gluon radiative energy loss to the first order in opacity, when multiplied by \(M_5^0\). There are seven such diagrams, that we gather into four groups,
each of which contains two (or one) diagrams symmetric under \((p \leftrightarrow k, x \leftrightarrow (1-x), c \leftrightarrow d)\) substitutions.

For consistency the initial gluon jet (with momentum \(p + k - q_1 - q_2\)) propagates along \(z\)-axis, i.e.:

\[
p + k - q_1 - q_2 = [E^+ - q_{1z} - q_{2z}, E^- + q_{1z} + q_{2z}, 0],
\]

(E1)

\[
\epsilon_i(p + k - q_1 - q_2) = [0, 0, \epsilon_i],
\]

(E2)

where \(q_i = [q_{iz}, -q_{iz}, q_{iz}], i = 1, 2\) with \(q_1^0 = 0\) denote momenta of exchanged gluons, while \(p, k\) and corresponding polarizations retain the same expressions as in Eqs. (C2), (C3), (C6), with distinction that, due to 4-momentum conservation, the following relation between gluon transverse momenta holds:

\[
p + k = q_1 + q_2.
\]

(E3)

Again, from seven diagrams we chose one model diagram \(M_{2,2,0}\), based on the same reason as in Appendix [F] for thorough derivation of the final amplitude expression. From Fig. 8 where gluon jet after two consecutive interactions with scattering centers radiates a gluon with momentum \(k\), we observe that there are two limiting cases that we consider.

Using the notation from Fig. 8 we write:

\[
M_{2,2,0} = \int \frac{d^4q_1}{(2\pi)^4} \frac{d^4q_2}{(2\pi)^4} \epsilon^*_\sigma(p) \epsilon^*_\rho(k) g_\sigma f^{ecd}(g^{\epsilon\sigma}(2p + k)^\rho + g^{\rho\epsilon}(-p - 2k)^\sigma + g^{\alpha\sigma}(-p + k)^\epsilon) -i\delta_{\epsilon\epsilon'} g_{\epsilon'\rho} (p + k)^2 + i\epsilon \times
\]

\[
\times f^{ba2} \left(g^{\rho(0)}(p + k - 2q_2)^\epsilon + g^{\epsilon\rho'}(-2p - 2k + q_2)^0 + g^{\epsilon0}(p + k + q_2)^\rho\right) T_{a2} V(q_2) e^{iq_{2x}z} -i\delta_{\rho\rho'} g_{\rho\rho'} (p + k - q_2)^2 + i\epsilon \times
\]

\[
\times f^{ab1} \left(g^{\rho(0)}(p + k - q_2 - q_1 - q_2)^0 + g^{\epsilon\rho'}(-2p - 2k + q_1 + q_2)^0 + g^{\epsilon0}(p + k + q_2)^\rho\right) T_{a1} V(q_1) e^{iq_1x_1} \times
\]

\[
\times \frac{-i\delta_{\epsilon\epsilon'} g_{\epsilon'\rho'} (p + k - q_1 - q_2)^2 + i\epsilon \times
\]

\[
\times \frac{-iJ_{a1}(p + k - q_1 - q_2)^\rho (p + q_1 - q_2)^\epsilon (p + k - q_1 - q_2) e^{i(p + k - q_1 - q_2)x_0}}{(p + k - q_1 - q_2)^2 + i\epsilon}
\]

\[
\approx iJ_{a}(p + k)e^{i(p + k)x_0} f^{ecd} f^{ba2} f^{ab1} T_{a2} T_{a1} (1 - x + x^2)(-i) \int \frac{d^2q_1}{(2\pi)^2} (-i) \int \frac{d^2q_2}{(2\pi)^2} (2ig_{a}) e \cdot ((1 - x)k - x p) \times
\]

\[
\times e^{-i_{q_1} b_{1}} e^{-i_{q_2} b_{2}} (E^+)^2 \frac{d\sqrt{q_1} q_2}{\sqrt{2\pi} \sqrt{2\pi}} (2\pi)^2 \left((p + k - q_1 - q_2)^2 + i\epsilon\right) \left((p + k - q_2)^2 + i\epsilon\right).
\]

(E4)

where \(b_i \equiv x_i - x_0\), \(i = 1, 2\) denote transverse impact parameters. We used Eq. (13) and assumed that \(J\) varies slowly with momentum \(q_i\), i.e. \(J(p + k - q_1 - q_2) \approx J(p + k)\).

Regarding the longitudinal \(q_{1z}\) integral, we introduce a new variable: \(q_z = q_{1z} + q_{2z}\) throughout this, and the following sections involving Feynman amplitudes which include interactions with two scattering centers. Therefore, we rewrite the exponent in the following manner:

\[
e^{-i_{q_1}(z_1 - z_0)} e^{-i_{q_2}(z_2 - z_1)} = e^{-i_{q_1}(z_1 - z_0)} e^{-i_{q_2}(z_2 - z_1)}.
\]

Rewriting \(q_1z\) longitudinal integral in terms of \(q_z\), i.e. changing the variables, we obtain:

\[
I_2(p, k, q_1, q_2, z_1 - z_0) = \int \frac{d q_z}{2\pi} \frac{v(q_z - q_{2z}, q_1)}{(p + k - q_1 - q_2)^2 + i\epsilon}
\]

(E5)

Again, due to \(z_1 > z_0\), the contour must be closed in the lower half-plane of complex \(q_z\) plane, so additional minus sign arises from the negative orientation of the contour and also we neglect the pole at \(q_z = -i\mu_{11} + q_{2z}\), since it is exponentially suppressed due to Eq. (B6). Thus, only one pole, originating from the gluon propagator, contributes to the first longitudinal integral:

\[
\bar{q} = \frac{-k^2}{xE^+} - \frac{p^2}{(1-x)E^+} - i\epsilon
\]

\[
= \frac{-k^2}{2\omega} - \frac{x}{1-x} \frac{(k - q_1 - q_2)^2}{2\omega} - i\epsilon,
\]

(E6)

where we used, as well as throughout the Appendices F and H the relation between transverse momenta Eq. (E3). The residue at Eq. (E6) then gives:
FIG. 8: Feynman diagram $M_{2,2,0}$ and its contribution to the first order in opacity gluon-jet radiative energy loss: contact-limit $M_{2,2,0}$, $z_i$, where $i = 1, 2$, denotes longitudinal coordinate of the interactions with the consecutive scattering centers (or in the contact limit $z_1 = z_2$). Crossed circles represent scatterers that exchange 3D momentum $q_i$ with the jet, which in contact-limit case merge into one gridded ellipse. Note that, all the following figures assume equivalently ordered Latin and Greek indices as in this figure. Remaining labeling is the same as in Figs. 6, 7.

Next we need to solve the remaining $q_z$ longitudinal momentum transfer integral:

$$I_2(p, q_1, q_2, z_1 - z_0) \approx -v(-q_2z - \frac{k^2}{2\omega} - \frac{x}{1-x} \frac{(k - q_1 - q_2)^2}{2\omega}) \cdot \frac{i}{E^+} \cdot \frac{i}{E^+} e^{\frac{-i}{\epsilon} \frac{1}{(k^2 + \chi_1^2)(k - q_1 - q_2)^2}(z_1 - z_0)}. \tag{E7}$$

Since $z_2 > z_1$ again we close the contour below the real $q_z$ axis and thus obtain:

$$I_3(p, q_1, q_2, z_2 - z_1) \approx -v(0, q_1)v(0, q_2) \cdot \frac{i}{E^+} \cdot \frac{i}{E^+} e^{\frac{-i}{\epsilon} \frac{1}{(k^2 + \chi_1^2)(k - q_1 - q_2)^2}(z_2 - z_1)}. \tag{E10}$$

In the special case of contact limit, i.e. when $z_1 = z_2$, instead of Eq. (E8) we need to calculate the following $q_z$ integral:

$$I_5^c(p, q_1, q_2, 0) = \int \frac{dq_2z}{2\pi} \frac{v(q_2z, q_2)}{(p + k - q_2)^2 + i\epsilon} \times v(-q_2z - \frac{k^2}{2\omega} - \frac{x}{1-x} \frac{(k - q_1 - q_2)^2}{2\omega}), \tag{E11}$$

Now, the contributions from Yukawa singularities ($q_2z = -i\mu_{1\perp}, q_2z = -i\mu_{2\perp}$) are not negligible and need to be included together with Eq. (E9). By choosing the same integration contour we obtain:

$$I_5^c(p, k, q_1, q_2, 0) \approx -i \cdot \frac{v(0, q_1)v(0, q_2)}{E^+} \cdot \frac{(4\pi\alpha_s)^2}{2} \cdot \frac{1}{\mu_2^2 - \mu_1^2} \cdot \frac{1}{\mu_2^2 - \mu_1^2} \cdot \frac{i}{E^+}, \tag{E12}$$

which is exactly $\frac{1}{7}$ of the strength of Eq. (E10). Note that, in previous calculations we applied soft-rescattering
approximation and also assumed $E^+ \gg \mu_{i,\perp}$, $i = 1, 2$. Finally, contact limit of this amplitude reads:

$$M_{2,2,0}^c = -iJ_a(p+k)e^{i(p+k)z_0}f^{ecd}f^{bca}_2f^{abq}_3T_{a_2}\bar{T}_{a_1}(1-x+x^2)(-i)\int \frac{d^2q_1}{(2\pi)^2} v(0, q_1)v(0, q_2)e^{-i(q_1+q_2)\mathbf{b}_1} \times$$

$$\times \frac{1}{2f_{q_a}^2} \frac{\epsilon \cdot ((1-x)\mathbf{k} - \mathbf{x}p)}{((1-x)\mathbf{k} - \mathbf{x}p)^2} e^{i\frac{p^2}{2(1-x)p^2}(z_1-z_0)}$$

$$= -J_a(p+k)\epsilon_i(p+k)x_0(T\pi^a_2T^a_1)_{da}T_{a_2}\bar{T}_{a_1}(1-x+x^2)(-i)\int \frac{d^2q_1}{(2\pi)^2} v(0, q_1)v(0, q_2)e^{-i(q_1+q_2)\mathbf{b}_1} \times$$

$$\times \frac{1}{2f_{q_a}^2} \frac{\epsilon \cdot (k - x(q_1 + q_2))}{(k - x(q_1 + q_2))^2} e^{\frac{i}{2(1-x^2)}(k^2 + i\pi_1(k-q_1-q_2)^2)(z_1-z_0)},$$

(E13)

where we applied Eq. [E3] and manipulated with $SU(N_c = 3)$ structure constants by using Eqs. [A5, A6]. Also we assumed that $x_1 = x_2$, since diagrams with two different center will not contribute to the final result due to Eq. [A3, A4].

Note from Fig. 8 that $M_{2,2,0}$ is symmetric under the substitutions: $(p \leftrightarrow k, x \leftrightarrow (1-x), c \leftrightarrow d)$, which can be straightforwardly verified by implementing these substitutions in the first two lines of Eq. (E13).

Appendix F: Diagrams $M_{2,0,3}$ and $M_{2,0,0}$

Next we consider $M_{2,0,3}$ diagram, where the radiated gluon suffers two consecutive interactions with the QCD medium (the first row of Fig. 9).

Note that the order of the color and Dirac indices denoting vertices is the same for all the remaining diagrams containing two interactions with the scatterers as in Fig. 8 and therefore omitted onward.

$$M_{2,0,3} = \int \frac{d^4q_1}{(2\pi)^4} \frac{d^4q_2}{(2\pi)^4} \epsilon_1^\ast(k) f^{ecda} \left( g^{\xi^0}(k-2q_2)^\rho + g_\rho^\xi(-2k+q_2)^0 + g^{\xi^0}(k+q_2)^\xi \right) T^a_2 V(q_2) e^{iq_2\mathbf{z}_2} \frac{-i\delta_{ee'}g^{e'e'}_{\xi\xi'}}{(k-q_2)^2 + i\epsilon} \times$$

$$\times f^{bea}_1 \left( g^{\epsilon^0}(k-2q_1-q_2)^\epsilon + g^{\xi'}(-2k+q_1 + 2q_2)^0 + g^{\xi^0}(k+q_1-q_2)^\xi \right) T^a_1 V(q_1) e^{iq_1\mathbf{z}_1} \frac{-i\delta_{\nu\nu'}g_{\nu\nu'}}{(k-q_1-q_2)^2 + i\epsilon} \times$$

$$\times \epsilon_2^\ast(p) g_{\nu\rho}f^{abdl} \left( g^{\mu'}(-2p+k+q_1+q_2)^\sigma + g^{\mu\sigma}(-2p+k+q_1+q_2)^\nu + g^{\sigma'}(p-k+q_1+q_2)^\mu \right) \times$$

$$\times \frac{-i\delta_{\rho\sigma}g_{\mu\mu'}}{(p+k-q_1-q_2)^2 + i\epsilon} J_a'(p+k-q_1-q_2)e^{i(p+k-q_1-q_2)x_0} \approx iJ_a(p+k)\epsilon^{i(p+k)x_0}f^{ecda}f^{bea}_1f^{abq}_3T_{a_2}\bar{T}_{a_1}(1-x+x^2)\left(1-x+x^2\right)(-i)\int \frac{d^2q_1}{(2\pi)^2}(-i)\int \frac{d^2q_2}{(2\pi)^2}(2i\sigma_3) \epsilon \cdot p e^{-i\mathbf{q}_1\cdot \mathbf{b}_1} e^{-i\mathbf{q}_2\cdot \mathbf{b}_2} \times$$

$$\times \int \frac{dq_{1z}}{2\pi} \frac{dq_{2z}}{2\pi} E^+k^+v(q_{1z}, q_1)v(q_{2z}, q_2)e^{-i\xi_{1z}(z_1-z_0)}e^{-iq_{2z}(z_2-z_0)}$$

$$\times \frac{1}{((p+k-q_1-q_2)^2 + i\epsilon)((k-q_1-q_2)^2 + i\epsilon)}.$$  

(F1)

Next, again by changing the variables $q_{1z} \rightarrow q_z = q_{1z} + \mathbf{q}_{2z}$, we define the following integral:

$$I_2(p, k, q_1, \bar{q}_2, z_1 - z_0) = \int \frac{dq_z}{2\pi} \frac{q_z - q_{2z}, q_1}{((p+k-q_1-q_2)^2 + i\epsilon)((k-q_1-q_2)^2 + i\epsilon)}.$$  

(F2)
Again, as explained in the previous section, we close the contour in lower half-plane, and since $\mu(z_1 - z_0) \gg 1$ the pole at $q_z = -i\mu_{1,\perp} + q_{2z}$ is again exponentially suppressed. Therefore the remaining $q_z$ singularities originating from gluon propagators are:

\[ \bar{q}_1 = -\frac{k^2}{x E^+} - \frac{p^2}{(1 - x) E^+} - i\epsilon = -\frac{k^2}{2\omega} - \frac{x}{1 - x} \frac{(k - q_1 - q_2)^2}{2\omega} - i\epsilon, \]
\[ \bar{q}_2 = -\frac{k^2}{x E^+} + \frac{p^2}{x E^+} - i\epsilon = -\frac{k^2}{2\omega} + \frac{(k - q_1 - q_2)^2}{2\omega} - i\epsilon. \]  

After performing the integration, i.e. summing the residues at these two poles, $I_2$ now reads:

\[ I_2(p, k, q_1, \bar{q}_2, z_1 - z_0) \approx v(-q_{2z}, q_1) \frac{i(1 - x)}{E^+(k - q_1 - q_2)^2} \left( e^{\frac{i}{\pi\omega}(k^2 + \frac{1}{2\pi}(k - q_1 - q_2)^2)(z_1 - z_0)} - e^{\frac{i}{\pi\omega}(k^2 - (k - q_1 - q_2)^2)(z_1 - z_0)} \right). \]  

The remaining integral over $q_{2z}$ is:

\[ I_3(p, k, q_1, q_2, z_2 - z_1) = \int \frac{dq_{2z}}{2\pi} \frac{v(q_{2z}, q_2) e^{-iq_{2z}(z_2 - z_1)}}{(k - q_2)^2 + i\epsilon} \times v(-q_{2z}, q_1), \]  

and since we are interested only in the contact-limit case

\[ I_3^c(p, k, q_1, q_2, 0) = \int \frac{dq_{2z}}{2\pi} \frac{v(q_{2z}, q_2)v(-q_{2z}, q_1)}{(k - q_2)^2 + i\epsilon}, \]  

(i.e. $z_1 = z_2$), we need to calculate:

\[ I_3^c(p, k, q_1, 0, 0) = \int \frac{dq_{2z}}{2\pi} \frac{v(q_{2z}, q_2)v(-q_{2z}, q_1)}{(k - q_2)^2 + i\epsilon}. \]
which gives:
\[ I_2^y(p, k, q_1, q_2, 0) \approx -v(0, q_1)v(0, q_2) \frac{i}{2\pi E^+}, \quad (F7) \]
which can readily be shown to represent exactly \( \frac{1}{2} \) of the strength of the well-separated limit Eq. (F5), as for \( M_{2,2,0} \) amplitude. The contact limit of this amplitude reduces to:

\[
M_{2,0,3}^c = iJ_a(p+k)e^{i(p+k)x_0} f^{e_2a_2} f^{b_1a_1} f^{a_2d_2} T_{a_2} T_{a_1} (1 - x + x^2)(-i) \int \frac{d^2q_1}{(2\pi)^2} (-i) \int \frac{d^2q_2}{(2\pi)^2} v(0, q_1)v(0, q_2)e^{-i(q_1+q_2)-b_1} \times \]
\[
\frac{1}{2} (2ig_{\gamma}) \frac{{\varepsilon \cdot k}}{k^2} \left( e^{i\left( \frac{k^2}{2\pi} + \frac{\nu}{2\pi}\right)(z_1-z_0)} - e^{i\left( \frac{(k^2+\nu^2)}{2\pi}\right)(z_1-z_0)} \right)
\]
\[
= J_a(p+k)e^{i(p+k)x_0} [(V^c, T^{a_2}, T^{a_1})_{d_2} T_{a_2} T_{a_1} (1 - x + x^2)(-i) \int \frac{d^2q_1}{(2\pi)^2} (-i) \int \frac{d^2q_2}{(2\pi)^2} v(0, q_1)v(0, q_2)e^{-i(q_1+q_2)-b_1} \times \]
\[
\frac{1}{2} (2ig_{\gamma}) \frac{{\varepsilon \cdot (k-q_1-q_2)}}{(k-q_1-q_2)^2} \left( e^{\frac{i}{\pi\gamma}(k^2+\nu^2)(z_1-z_0)} - e^{-\frac{i}{\pi\gamma}(k^2-(k-q_1-q_2)^2)(z_1-z_0)} \right).
\] (F8)

Proceeding in the same manner, for \( M_{2,0,0}^c \) amplitude (the second row of Fig. [9]) we obtain:

\[
M_{2,0,0}^c = iJ_a(p+k)e^{i(p+k)x_0} f^{e_2a_2} f^{b_1a_1} f^{a_2d_2} T_{a_2} T_{a_1} (1 - x + x^2)(-i) \int \frac{d^2q_1}{(2\pi)^2} (-i) \int \frac{d^2q_2}{(2\pi)^2} v(0, q_1)v(0, q_2)e^{-i(q_1+q_2)-b_1} \times \]
\[
\frac{1}{2} (2ig_{\gamma}) \frac{{\varepsilon \cdot k}}{k^2} \left( e^{i\left( \frac{k^2}{2\pi} + \frac{\nu}{2\pi}\right)(z_1-z_0)} - e^{i\left( \frac{(k^2+\nu^2)}{2\pi}\right)(z_1-z_0)} \right)
\]
\[
= J_a(p+k)e^{i(p+k)x_0} (V^{a_2} T^{a_1} T^{c_1})_{d_2} T_{a_2} T_{a_1} (1 - x + x^2)(-i) \int \frac{d^2q_1}{(2\pi)^2} (-i) \int \frac{d^2q_2}{(2\pi)^2} v(0, q_1)v(0, q_2)e^{-i(q_1+q_2)-b_1} \times \]
\[
\frac{1}{2} (2ig_{\gamma}) \frac{{\varepsilon \cdot k}}{k^2} \left( e^{\frac{i}{\pi\gamma}(k^2+\nu^2)(z_1-z_0)} - e^{-\frac{i}{\pi\gamma}(k^2-(k-q_1-q_2)^2)(z_1-z_0)} \right).
\] (F9)

From Fig. [9] we infer that \( M_{2,0,3}^c \) and \( M_{2,0,0}^c \) are symmetric under the following substitutions: \((p \leftrightarrow k, x \leftrightarrow (1-x), c \leftrightarrow d)\), which can be straightforwardly verified by implementing these substitutions in Eqs. (F8, F9).

**Appendix G: Diagrams \( M_{2,0,1} \) and \( M_{2,0,2} \)**

Here we consider the case when both initial gluon jet and radiated gluon interact with one scattering center.
Since we define the following integral: \( I \approx -i\mu_{1\perp} + q_{2z} \). Therefore the remaining \( q_{z} \) singularities originating from gluon propagators are:

\[
\begin{align*}
I_2(p, k, q_1, q_2, z_1 - z_0) = & \int \frac{d^4q_2}{2\pi^2} \frac{v(q_2 - q_2, q_1) e^{-iq_{z}(z_1 - z_0)}}{k - q_1} \times \frac{1}{(k - q_1)^2 + i\epsilon}. \\
& \left( q_1 - \frac{k^2}{2\omega} - \frac{x (k - q_1 - q_2)^2}{2\omega} - i\epsilon, \right. \\
& \left. q_2 - \frac{k^2}{2\omega} + \frac{(k - q_1)^2}{2\omega} + q_{2z} - i\epsilon. \right)
\end{align*}
\]

Since \( z_1 > z_0 \) we must close the contour in lower half-plane, and since \( \mu(z_1 - z_0) \gg 1 \) again we neglect the pole at \( q_z = -i\mu_{1\perp} + q_{2z} \). Therefore the remaining \( q_z \) singularities originating from gluon propagators are:

\[
\begin{align*}
I_2(p, k, q_1, q_2, z_1 - z_0) &= \int \frac{d^4q_2}{2\pi^2} \frac{v(q_2 - q_2, q_1) e^{-iq_{z}(z_1 - z_0)}}{k - q_1} \times \frac{1}{(k - q_1)^2 + i\epsilon}. \\
& \left( q_1 - \frac{k^2}{2\omega} - \frac{x (k - q_1 - q_2)^2}{2\omega} - i\epsilon, \right. \\
& \left. q_2 - \frac{k^2}{2\omega} + \frac{(k - q_1)^2}{2\omega} + q_{2z} - i\epsilon. \right)
\end{align*}
\]

Summing the residues gives:

\[
\begin{align*}
I_2(p, k, q_1, q_2, z_1 - z_0) &\approx e^{i\frac{k^2}{2\omega}(z_1 - z_0)} \times \frac{v(-q_{2z} - \frac{k^2}{2\omega} - \frac{x (k - q_1 - q_2)^2}{2\omega}, q_1) e^{i\frac{1}{2\omega} (k - q_1 - q_2)^2 (z_1 - z_0)}}{E^+ k^+ (q_{2z} + \frac{(k - q_1)^2}{2\omega} + \frac{x (k - q_1 - q_2)^2}{2\omega})}. \\
& \times \left( v(-q_{2z} - \frac{k^2}{2\omega} - \frac{x (k - q_1 - q_2)^2}{2\omega}, q_1) e^{i\frac{1}{2\omega} (k - q_1 - q_2)^2 (z_1 - z_0)} - v(-q_{2z} - \frac{k^2}{2\omega}, q_1) e^{-i(q_{2z} + \frac{(k - q_1)^2}{2\omega}) (z_1 - z_0)} \right).
\end{align*}
\]

The remaining \( q_{2z} \) integral is:
where the singularity on \( q_{2z} \) real axis: \( q_{2z} = -\frac{(k-q_1)^2}{2\omega} - \frac{x}{2\omega} (k-q_1-q_2)^2 \equiv -a, (a > 0) \) has to be avoided by taking Cauchy principal value of \( I_3 \) according to the Fig. 11 i.e.:

\[
I_3 \equiv I_{PV} = I_B - I_C - I_D,
\]

where \( I_B = -2\pi i \sum_i \text{Res}(I_3(\tilde{q}_i)) \), with \( i \) counting the poles in the lower-half plane. Additionally \( I_C = 0 \), and it’s straightforward to show, that after the following substitution \( q_{2z} = -a + re^{i\varphi} \), where \( r \to 0 \), also \( I_D = 0 \). Therefore, principal value of \( I_3 \) reduces to \( I_B \), i.e. \( -2\pi i \sum_i \text{Res}(I_3(\tilde{q}_i)) \).

In the well-separated case Eq. (G5) poles originating from Yukawa potentials \( q_{2z} = \frac{-k}{2\omega} - \frac{x}{1-x} (k-q_1-q_2)^2 - i\mu_{1\perp} \) and \( q_{2z} = -i\mu_{2\perp} \) are again exponentially suppressed \( \left( e^{-\mu_{1\perp}(z_2-z_0,1)} \to 0, i = 1, 2 \right) \) and therefore can be neglected, so only the pole from the propagator survives \( q_{2z} = \frac{1}{2\omega} \left( \frac{(k-q_1)^2}{(k-q_1-q_2)^2} - \frac{x}{2\omega} \right) \). However, since we are interested only in the contact-limit case (i.e. \( z_1 = z_2 \)), instead of Eq. (G5) we need to calculate the principal value of the following integral:

\[
I_3^c(p, k, q_1, q_2, z_1 - z_0) = \int \frac{dq_{2z}}{2\pi} \frac{1}{q_{2z} + \frac{(k-q_1)^2}{2\omega} + \frac{x}{1-x} \frac{(k-q_1-q_2)^2}{2\omega}} \times \left( \frac{(\frac{k^2}{2\omega} + \frac{x}{1-x} (k-q_1-q_2)^2)(z_1-z_0)}{(p-q_{2z})^2 + i\epsilon} \right)^{\frac{v(q_{2z}, q_2) - \frac{x}{2\omega} \frac{(k-q_1-q_2)^2}{2\omega}}{v(q_{2z}, q_2)}} - \frac{e^{-i\varphi q_{2z}(z_1-z_0)} e^{-\frac{1}{2\omega}(k-q_1)^2-k^2)(z_1-z_0)}}{(p-q_{2z})^2 + i\epsilon} \),
\]

which again reduces to the sum of residsa, with \(-a\) effectively not being a pole (Fig. 11). Particularly, for the second term in the bracket of Eq. (G7), only the propagator pole survives, while for the first term in the bracket all three poles have to be accounted, although residues at poles from potentials sum to the order of \( \mathcal{O}(x(z_1-z_0)^2) \), and thus can be neglected compared to the remaining residue.

Finally, in the contact-limit case we obtain:

\[
M_{2,0,1}^c \approx -i J_a(p+k) e^{i(p+k)\varphi_0} \int f_{e\varphi d2} f_{ac} f_{bc} T_{a2} T_{a1}(1-x+x^2)(-i) \int \frac{d^2q_1}{(2\pi)^2} \frac{v(0, q_1) v(0, q_2) e^{-i(q_1+q_2)\cdot b_1}}{x} \times (2i\epsilon_s) \frac{e^{-\frac{1}{2\omega}(k-q_2)^2}}{(k-q_1)^2} \left( e^{\left( \frac{k^2}{2\omega} + \frac{x}{1-x} (k-q_1-q_2)^2 \right)(z_1-z_0)} - e^{\left( \frac{k^2}{2\omega} + \frac{x}{1-x} (k-q_1-q_2)^2 \right)(z_1-z_0)} \right) \right),
\]

Notice that, contrary to the previous three amplitudes that also included two scattering centers, in Eq. (G8) no
factor $\frac{1}{2}$ when comparing to well-separated limit appears. Proceeding in the same manner, for $M_{2,0,2}^c$ we obtain:

$$M_{2,0,2}^c \approx iJ_\alpha(p+k)e^{i(p+k)x_0} f^{cba} f^{dbd} T_aT_bT_e(1-x+x^2)(-i) \int \frac{d^2q_1}{(2\pi)^2} (-i) \int \frac{d^2q_2}{(2\pi)^2} v(0,q_1)v(0,q_2)e^{-i(q_1+q_2)b_1} \times$$

$$\times \left( \frac{2i\epsilon_1}{\epsilon_1} \left( \begin{array}{c} (p-q_1) \\ (p-q_1)^2 \end{array} \right) - \epsilon_1^{(\frac{k^2}{\pi^2} \frac{q_1^2}{\pi^2} \frac{k^2+q_1^2}{p^2+q_1^2})} \right)$$

$$= J_\alpha(p+k)e^{i(p+k)x_0} (T_{a_1}[T^{a_1},T^{a_2}])_d aT_bT_e(1-x+x^2)(-i) \int \frac{d^2q_1}{(2\pi)^2} (-i) \int \frac{d^2q_2}{(2\pi)^2} v(0,q_1)v(0,q_2)e^{-i(q_1+q_2)b_1} \times$$

$$\times \left( \frac{2i\epsilon_1}{\epsilon_1} \left( \begin{array}{c} (k-q_2) \\ (k-q_2)^2 \end{array} \right) - \epsilon_1^{(\frac{k^2}{\pi^2} \frac{q_2^2}{\pi^2} \frac{k^2+q_2^2}{p^2+q_2^2})} \right). \quad \text{(G9)}$$

As for $M_{2,0,1}^c$ amplitude, no factor of $\frac{1}{2}$ appears. From well-separated analogon of Fig. 10 we could infer that $M_{2,0,1}$ and $M_{2,0,2}$ are symmetric under the following substitutions: $(p \leftrightarrow k, x \leftrightarrow (1-x), c \leftrightarrow d)$, which can readily be verified by implementing these substitutions in the first two lines of either of the two Eqs. (G8-G9) and by using structure constant asymmetry. Note that, in Eq. (G9) we applied Eq. (E3). Also, since in contact-limit case these two diagrams are topologically indistinct, we need to either omit one of them in order to avoid over counting, or to include both, but multiply each by a factor $\frac{1}{2}$ (we will do the latter).

### Appendix H: Diagrams $M_{2,1,0}$ and $M_{2,1,1}$

The contact-limit case of the remaining two diagrams is presented in Fig. 12. These diagrams correspond to the case when one interaction with the scattering center located at $x_1$ occurs before and the other interaction at the same place occurs after the gluon has been radiated.

In the light of time-ordered perturbation theory from [8] [39] these two diagrams are identically equal to zero, since $\int_{t_1}^{t_2} dt ... = 0$, but for the consistency we will provide a brief verification of this argument.
\[
M_{2,1,0} = \int \frac{d^4 q_1}{(2\pi)^4} \frac{d^4 q_2}{(2\pi)^4} \epsilon^*(p) \bar{f} c d a_2 \left( g^{\xi_0}(p - 2q_2)^\sigma + g^{\xi^0}(p + q_2)^0 \right) T_{a_2} V(q_2) e^{i q_2 z_x} \frac{-i\delta_{\epsilon\epsilon'}g_{\epsilon\epsilon'}}{(p - q_2)^2 + i\epsilon} \times \\
\times \epsilon^*(p) g_{a_1} f^b c \left( g^{\xi}(2p + k - 2q_2)^\sigma + g^{\nu}(p - 2k + q_2)^\nu + g^{\mu}(p + q_2)^\mu \right) \frac{-i\delta_{\mu\nu}g_{\mu\nu'}}{(p + k - q_2)^2 + i\epsilon} \times \\
\times f^{a'b'} \left( g^{\mu_0}(p + k - q_2)^\mu_0 + g^{\nu_0}(p + q_2)^\nu_0 + g^{\nu_0}(p + q_1 - q_2)^\nu_0 \right) T_{a_1} V(q_1) e^{i q_1 z_x} \times \\
\times \frac{-i\delta_{\epsilon'a'}g_{\epsilon'a'}}{(p + k - q_1 - q_2)^2 + i\epsilon} iJ_a'(p + k - q_1 - q_2) e^{i\epsilon'(p + k - q_1 - q_2)x_0} \\
\approx iJ_a(p + k) e^{i(p + k)x_0} f e d a_2 f c e b a_1 T_{a_2} T_{a_1} \left( 1 - x + x^2 \right) \left( -i \right) \int \frac{d^2 q_1}{(2\pi)^2} \left( -i \right) \int \frac{d^2 q_2}{(2\pi)^2} \left( 2i g_{a_1} \epsilon \cdot (k - x q_1) e^{(p + k - q_1 - q_2)x_0} \times \\
\times e^{-i q_2 \cdot b_2(E^+)^2} \int \frac{dq_{22}}{2\pi} \frac{v(q_{22}, q_2) e^{-i q_2 (z_2 - z_1)}}{(p + k - q_2)^2 + i\epsilon}) \left( -i \right) \int \frac{dq_{21}}{2\pi} \frac{v(q_{21}, q_2) e^{-i q_2 (z_2 - z_1)}}{(p - q_2)^2 + i\epsilon)} \times \\
\approx J_a(p + k) e^{i(p + k)x_0} f e d a_2 f c e b a_1 T_{a_2} T_{a_1} \left( 1 - x + x^2 \right) \left( -i \right) \int \frac{d^2 q_1}{(2\pi)^2} \left( -i \right) \int \frac{d^2 q_2}{(2\pi)^2} \left( 2i g_{a_1} \epsilon \cdot (k - x q_1) e^{(p + k - q_1 - q_2)x_0} \times \\
\times e^{-i q_2 \cdot b_2(E^+)^2} \int \frac{dq_{22}}{2\pi} \frac{v(q_{22}, q_2) e^{-i q_2 (z_2 - z_1)}}{(p + k - q_2)^2 + i\epsilon}) \left( -i \right) \int \frac{dq_{21}}{2\pi} \frac{v(q_{21}, q_2) e^{-i q_2 (z_2 - z_1)}}{(p - q_2)^2 + i\epsilon)} \times \\
\times e^{-i \frac{1}{2\omega} (k^2 + i\epsilon x) (k - q_1 - q_2)^2 (z_1 - z_2) \right). \]

In the contact-limit case there are four \( q_{2z} \) poles of the above integral in the lower half-plane: \(-k^2_{2z} - \)

\[x \frac{(k - q_1 - q_2)^2}{2\omega} + \frac{q_1^2}{2\omega} - i\epsilon, \quad x - \frac{2\omega}{k^2} (k - q_1 - q_2)^2 - \frac{q_1^2}{2\omega} - \frac{2\omega}{k^2} (k - q_1 - q_2)^2 - \frac{q_1^2}{2\omega} - i\epsilon, \]

\(-i\mu_{1\perp} \) and \(-i\mu_{2\perp} \), which give:

\[ M_{2,1,1} = iJ_a(p + k) e^{i(p + k)x_0} (T_{a_2} T_{a_1})_{d_2} T_{a_2} T_{a_1} \left( 1 - x + x^2 \right) \left( -i \right) \int \frac{d^2 q_1}{(2\pi)^2} \left( -i \right) \int \frac{d^2 q_2}{(2\pi)^2} \left( 2i g_{a_1} \epsilon \cdot (k - x q_1) e^{(p + k - q_1 - q_2)x_0} \times \\
\times \left( i g_{a_1} \epsilon \cdot (k - x q_1) \right) e^{\frac{1}{2\omega} (k^2 + i\epsilon x) (k - q_1 - q_2)^2 (z_1 - z_2)} \right) \mu_{1\perp}^2 + \mu_{1\perp} \mu_{2\perp} + \mu_{2\perp}^2 \times \\
\times \frac{(k - x q_1)^2}{x(1 - x) E^+ (\mu_{1\perp} + \mu_{2\perp})}. \]

where the residues at first two poles (i.e. originating from the gluon propagators) cancel each other exactly, leading to the result Eq. [H2] that is suppressed by a factor of \( \mathcal{O}\left( \frac{(k - x q_1)^3}{x^3 E^+ (\mu_{1\perp}^2 + \mu_{2\perp}^2)} \right) \) compared to the all previous amplitudes (note that \( x \) is finite), as in the case of soft-gluon approximation [9, 24].

The same conclusion applies to \( M_{2,1,1}^c \) amplitude, which can be straightforwardly verified by repeating the analogous procedure as for \( M_{2,1,0}^c \) and by the fact that these two amplitudes are symmetric (see Fig. [12] to the
exchange \((p \leftrightarrow k, x \leftrightarrow (1-x), c \leftrightarrow d)\).

**Appendix I: Calculation of radiative energy loss**

In this section we provide concise outline of calculating the first order in opacity radiative energy loss. We start with the equation:

\[
d^3 N^{(1)}_g d^3 N_J = \left( \frac{1}{dT} \text{Tr} \langle |M_1|^2 \rangle + \frac{2}{dT} \text{Re} \text{Tr} \langle M_2 M_0^* \rangle \right) \times \frac{d^3 \mathbf{p}}{(2\pi)^3 2\rho^0} \frac{d^3 \mathbf{k}}{(2\pi)^3 2\omega^0}, \tag{I1}
\]

Next, we summarize contact limits of all diagrams that contain two scattering centers from Appendices \[E\] and \[F\]. We also used the definition of commutator, the fact that trace is invariant under the cyclic permutations, Eq. [A.3] (with \(i = j\) and \(d_i = d_T\)) and the relation \(E^+ \approx 2E\). We verified that this result is also symmetric under the substitutions: \((p \leftrightarrow k, x \leftrightarrow (1-x), c \leftrightarrow d)\) when written in terms of structure constants.

The final results from Appendix \[D\] yield:

\[
M_1 = M_{1,1,0} + M_{1,0,0} + M_{1,0,1} = J_0(p + k) e^{\ell(p+k)x_0(1 - x + x^2) T_a}(-i) \int \frac{d^2 q_1}{(2\pi)^2} v(0, q_1) e^{-i q_1 \cdot b_1} (2i g_a) \times \\
\times \left\{ \left( \frac{\epsilon \cdot (k - q_1)}{(k - q_1)^2} \right) \text{Tr}(T^c, T^{a_1})_1 + \frac{\epsilon \cdot (k - q_1)}{(k - q_1)^2} (T^{a_1} T^c)_2 \right\} \frac{\epsilon \cdot k}{k^2} \left( (k^2 - (k - q_1)^2)(z_1 - z_0) \right) \}
\tag{I2}
\]

leading to:

\[
\frac{1}{dT} \text{Tr} \langle |M_1|^2 \rangle = N[J(p + k)^2(4\pi)^2] \frac{1}{A\perp} (1 - x + x^2)^2 \int \frac{d^2 q_1}{(2\pi)^2} v(0, q_1)^2 \frac{C_2(T)}{d\ell} \left\{ \frac{\epsilon \cdot (k - q_1)}{(k - q_1)^2} \text{Tr}(T^c)^2(T^{a_1})^2 \right\} + \\
+ 2\alpha \left( \frac{\epsilon \cdot (k - q_1)}{(k - q_1)^2} - \frac{\epsilon \cdot k}{k^2} \right) \frac{\epsilon \cdot (k - q_1)}{(k - q_1)^2} - \frac{\epsilon \cdot k}{k^2} \frac{\epsilon \cdot (k - q_1)}{(k - q_1)^2} \frac{2 \cos \left( \frac{x(1 - x)E^+}{2} \right)(z_1 - z_0)}{x(1 - x)^2} + \\
+ 2\left( \frac{\epsilon \cdot k}{k^2} \text{Tr}(T^c)^2(T^{a_1})^2 \right) - \frac{\epsilon \cdot (k - q_1)}{(k - q_1)^2} \text{Tr}(T^c T^{a_1} T^c) \frac{\epsilon \cdot k}{k^2} \right\} \frac{\epsilon \cdot k}{k^2} \left( (k^2 - (k - q_1)^2)(z_1 - z_0) \right) \}
\tag{I3}
\]

where the number of scattering centers \(N\) comes from summation over scattering centers Eqs. [B2, B3], then \(\alpha \equiv \text{Tr}((T^c)^2(T^{a_1})^2) - \text{Tr}(T^c T^{a_1} T^c)\), and we also used the definition of commutator, the fact that trace is invariant under the cyclic permutations, Eq. [A.4] (with \(i = j\) and \(d_j = d_T\)) and the relation \(E^+ \approx 2E\). We verified that this result is also symmetric under the substitutions: \((p \leftrightarrow k, x \leftrightarrow (1-x), c \leftrightarrow d)\) when written in terms of structure constants.

Next, we summarize contact limits of all diagrams that contain two scattering centers from Appendices \[E\] and \[F\] and then take their ensemble average according to Eqs. [B2, B3, B4] in order to obtain \(M_2\):
\(M_2 = M_{2,2,0} + M_{2,0,3} + M_{2,0,0} + \frac{1}{2}(M_{2,0,1} + M_{2,0,2}) = \frac{1}{2}N\sigma_d(p + k)e^{i(p+k)x_0}(-2tg_\perp)\frac{1}{A_\perp}(1 - x + x^2)T_{a_2}T_{a_1}\times \)

\[
\times \int \frac{d^2q_1}{(2\pi)^2}|v(0, q_1)|^2 \left\{ (\frac{\epsilon \cdot k}{k^2})^2 \left\{ e^{\frac{k^2}{2\pi} (z_1 - z_0)} \left\{ [T^c, T^{a_2}]_{da} + [T^{a_2}T^a, T^c]_{da} \right\} - \left\{ [T^c, T^{a_2}]_{da} - [T^{a_2}T^a, T^c]_{da} \right\} \right\} - \frac{\epsilon \cdot (k - q_1)}{(k - q_1)^2} \left\{ e^{\frac{(k - q_1)^2}{2\pi} (z_1 - z_0)} - e^{\frac{k^2}{2\pi} (z_1 - z_0)} \right\} \right\} \right\}.
\]

(14)

Then, by multiplying the previous expression by \(M^2_0\), we obtain:

\[
\frac{2}{dT} \text{Re Tr} \langle M_2M_0^* \rangle = N|J(p + k)|^2(4g_d^2)\frac{1}{A_\perp}(1 - x + x^2)^2 \int \frac{d^2q_1}{(2\pi)^2}|v(0, q_1)|^2 \frac{C_2(T)}{dG} \times \]

\[
\times \left\{ \left\{ (\frac{\epsilon \cdot k}{k^2})^2 \left\{ 2\alpha \cos \left( \frac{k^2}{x(1-x)}E^+(z_1 - z_0) \right) - 2\alpha - \text{Tr}((T^c)^2(T^{a_1})^2) \right\} - \frac{2\alpha \epsilon \cdot k}{k^2} \left\{ \frac{\epsilon \cdot (k - q_1)}{(k - q_1)^2} \left\{ \cos \left( \frac{k^2}{x(1-x)}E^+(z_1 - z_0) \right) - \cos \left( \frac{k^2 - (k - q_1)^2}{x(1-x)}E^+(z_1 - z_0) \right) \right\} \right\} \right\},
\]

(15)

which can easily be verified to be symmetric to the exchange \((p \leftrightarrow k, x \leftrightarrow (1-x), c \leftrightarrow d)\), when written in terms of structure constants. By summing the expressions Eqs. (13) and (15) we obtain:

\[
\frac{1}{dT} \text{Tr} \langle |M_1|^2 \rangle + \frac{2}{dT} \text{Re Tr} \langle M_2M_0^* \rangle = N\sigma_d|J(p + k)|^2(4g_d^2)\frac{C_2(T)}{dG} \frac{C_2(G)}{A_\perp}(1 - x + x^2)^2 \int \frac{d^2q_1}{(2\pi)^2}|v(0, q_1)|^2 \times \]

\[
\times \left\{ \left\{ (1 - \cos \left( \frac{k^2}{x(1-x)}E^+(z_1 - z_0) \right) \right\} \left\{ \frac{\epsilon \cdot k}{k^2} - \frac{\epsilon \cdot (k - xq_1)}{(k - xq_1)^2} \right\} \left\{ \frac{\epsilon \cdot k}{k^2} + \left\{ \frac{\epsilon \cdot (k - xq_1)}{(k - xq_1)^2} \right\} \left\{ \frac{\epsilon \cdot k}{k^2} \right\} \right\} \right\} \right\}.
\]

(16)

which in the soft-gluon approximation coincides with massless limit of Eq. (82) from [25] and where we used the following equalities that are valid in adjoint representation: \(\text{Tr}(T^cT^{a_1}T^cT^{a_1}) = \frac{1}{2}C_2^2(G)d_G = \alpha = \frac{1}{2}\text{Tr}((T^c)^2(T^{a_1})^2)\), which follow from Eqs. (A1) and the commutator definition.

Since we are considering optically "thin" QCD plasma, it would be convenient to expand energy loss in powers of opacity, which is defined by the mean number of collisions in QCD medium [9]:

\[
\tilde{n} = \frac{L}{\lambda} = \frac{N\sigma_d}{A_\perp},
\]

(17)

where the small transverse momentum transfer elastic cross section between the jet and the target partons is taken from GW model (Eq. (6) from [9]), which in our case reads:

\[
\frac{d\sigma_{el}}{d^2q_1} = \frac{C_2(G)C_2(T)}{dG} \frac{|v(0, q_1)|^2}{(2\pi)^2}.
\]

(18)

Combining the Eqs. (17) and (18) we obtain:

\[
\tilde{L} = \frac{N}{4\pi c_G} \frac{C_2(G)C_2(T)}{dG} \left( 4\pi \alpha_s \right)^2 \frac{1}{\mu^2}.
\]

(19)

Next we incorporate Eq. (19) in Eq. (16), substitute obtained expression in Eq. (11), keeping in mind that \(\tilde{p}\) is 3D momentum of a final jet, and that we need to apply Eqs. (C12) and (C13). The single gluon radiation spectrum in the first order in opacity then becomes:
\[
\frac{dN_g^{(1)}}{dx} = \frac{C_2(G)\alpha_s L (1 - x + x^2)^2}{\lambda} \int \frac{d^2q_1}{\pi} \frac{\mu^2}{(q_1^2 + \mu^2)^2} \int \frac{d^2k}{\pi} \left\{ -\frac{\epsilon \cdot (k - q_1)}{(k - q_1)^2} \left( \frac{\epsilon \cdot k}{k^2} + \frac{\epsilon \cdot (k - xq_1)}{(k - xq_1)^2} - 2\frac{\epsilon \cdot (k - q_1)}{(k - q_1)^2} \right) \times \int dz_1 \left( 1 - \cos \left( \frac{(k - q_1)^2}{x(1 - x)E^+} (z_1 - z_0) \right) \right) \frac{2}{L} e^{-\frac{2(z_1 - z_0)}{L}} \right\},
\]

and the differential radiative energy loss \( \frac{dE^{(1)}}{dx} \),

\[
\omega \frac{d^3N_g^{(1)}}{dx} \approx xE \frac{d^3N_g^{(1)}}{dx},
\]

acquires the form:

\[
\frac{dE^{(1)}}{dx} = \frac{C_2(G)\alpha_s L E (1 - x + x^2)^2}{\lambda} \int \frac{d^2q_1}{\pi} \frac{\mu^2}{(q_1^2 + \mu^2)^2} \int \frac{d^2k}{\pi} \left\{ -\frac{\epsilon \cdot (k - q_1)}{(k - q_1)^2} \left( \frac{\epsilon \cdot k}{k^2} + \frac{\epsilon \cdot (k - xq_1)}{(k - xq_1)^2} - 2\frac{\epsilon \cdot (k - q_1)}{(k - q_1)^2} \right) \times \int dz_1 \left( 1 - \cos \left( \frac{(k - q_1)^2}{x(1 - x)E^+} (z_1 - z_0) \right) \right) \frac{2}{L} e^{-\frac{2(z_1 - z_0)}{L}} \right\},
\]

where we also assumed a simple exponential distribution \( 2 e^{-\frac{2(z_1 - z_0)}{L}} \) between the scattering centers (as in [25]). So we finally obtain:

\[
\frac{dN_g^{(1)}}{dx} = \frac{C_2(G)\alpha_s L (1 - x + x^2)^2}{\lambda} \int \frac{d^2q_1}{\pi} \frac{\mu^2}{(q_1^2 + \mu^2)^2} \int d^2k \times
\]

\[
\frac{\left( \frac{k \cdot (k - q_1)^2}{L} \right)^2 + k^4 \left( 1 - \frac{k \cdot (k - xq_1)^2}{(k - xq_1)^2} \right) + \left( \frac{1}{(k - xq_1)^2} - \frac{1}{k^2} \right) \right\},
\]

which is symmetric to the exchange of \( p \) and \( k \) gluons, and:

\[
\frac{dE^{(1)}}{dx} = \frac{C_2(G)\alpha_s L E (1 - x + x^2)^2}{\lambda} \int \frac{d^2q_1}{\pi} \frac{\mu^2}{(q_1^2 + \mu^2)^2} \int d^2k \times
\]

\[
\frac{\left( \frac{k \cdot (k - q_1)^2}{L} \right)^2 + k^4 \left( 1 - \frac{k \cdot (k - xq_1)^2}{(k - xq_1)^2} \right) + \left( \frac{1}{(k - xq_1)^2} - \frac{1}{k^2} \right) \right\},
\]
which in soft-gluon approximation reduces to massless limit of Eq. (84) from [27].

Appendix J: Diagrams and radiative energy loss in finite T QCD medium

Next we recalculate the results from Appendices C-H when the gluon mass \( m_g = \sqrt{\mu^2} \) is included, i.e. gluon propagator has the following form [27]:

- gluon propagator with mass \( m_g \) in Feynman gauge:

\[
a_{\mu a} \frac{p}{k} b_{\nu b} = -i\delta_{ab} P_{\mu\nu} \frac{1}{p^2 - m_g^2 + i\epsilon}, \quad (J1)
\]

where \( P_{\mu\nu} \), given by Eq. (12) from [27] (specifically \( P_{\mu\nu} = \eta_{\mu\nu} - \frac{p_\mu p_\nu}{m^2} - \frac{q_\mu q_\nu}{m^2} - \eta_{\mu\nu} p_\rho p_\rho(n) - \eta_{\mu\nu} q_\rho q_\rho(n) \)), represents the transverse projector. Note that, since the transverse projectors act directly or indirectly on transverse polarization vectors one may immediately replace \( P_{\mu\nu} \) with \( g_{\mu\nu} \) in gluon propagators, in order to facilitate the calculations.

This observation is obvious for off-shell gluon propagator, whereas the derivation for the remaining internal gluon lines is straightforward.

Consistently throughout this section, initial jet propagates long z-axis, 4-momentum is conserved and minus Light cone coordinate of \( p \) and \( k \) momenta acquire an additional term \( +m_g^2 \) in the numerator compared to massless case (Appendices C-H), due to relations \( k^2 = p^2 = m_g^2 \), while the polarizations remain the same.

We provide only the final expressions for all 11 Feynman diagrams beyond soft-gluon approximation, when the gluon mass is included, since its derivation is similar to the case of massless gluons and in order to avoid unnecessary repetition (Appendices C-H).

Thus, for \( M_0 \) we obtain:

\[
M_0 = J_a (p + k) e^{i(p+k)z_0} (-2ig_a)(1 - x + x^2) \times \frac{\epsilon \cdot k}{k^2 + m_g^2 (1 - x + x^2)} (T^c)_{da}. \quad (J2)
\]

The expression for \( M_{1.1,0} \) now reads:

\[
M_{1.1,0} = J_a (p + k) e^{i(p+k)z_0} (-i)(1 - x + x^2)(T^c T^{a_1})_{da} T_{a_1} \int \frac{d^2 q_1}{(2\pi)^2} v(0, q_1) e^{-iq_1 \cdot b_1} \times
\]

\[
\frac{\epsilon \cdot (k - xq_1)}{(k - xq_1)^2 + m_g^2 (1 - x + x^2)} e^{\frac{i}{\epsilon} (k^2 + \frac{q_1^2}{q_1^2} (k - q_1)^2 + \chi)} (z_1 - z_0), \quad (J3)
\]

which differs from Eq. (D12) in the term \( \chi \equiv m_g^2 (1 - x + x^2) \), which now appears in the denominator and in exponent, accompanying the squared transverse momentum.

Further on, we will use the shorthand notation \( \chi \).

Similarly, for \( M_{1,0,0} \) and \( M_{1,0,1} \) we obtain, respectively:

\[
M_{1,0,0} = J_a (p + k) e^{i(p+k)z_0} (-i)(1 - x + x^2)(T^{a_1} T^c)_{da} T_{a_1} \int \frac{d^2 q_1}{(2\pi)^2} v(0, q_1) e^{-iq_1 \cdot b_1} \times
\]

\[
\frac{\epsilon \cdot k}{k^2 + \chi} \left( e^{\frac{i}{\epsilon} (k^2 + \frac{q_1^2}{q_1^2} (k - q_1)^2 + \chi)} (z_1 - z_0) - e^{\frac{i}{\epsilon} (k^2 - (k - q_1)^2)} (z_1 - z_0) \right), \quad (J4)
\]

\[
M_{1,0,1} = J_a (p + k) e^{i(p+k)z_0} (-i)(1 - x + x^2)(T^c T^{a_1})_{da} T_{a_1} \int \frac{d^2 q_1}{(2\pi)^2} v(0, q_1) e^{-iq_1 \cdot b_1} \times
\]

\[
\frac{\epsilon \cdot (k - q_1)}{(k - q_1)^2 + \chi} \left( e^{\frac{i}{\epsilon} (k^2 + \frac{q_1^2}{q_1^2} (k - q_1)^2 + \chi)} (z_1 - z_0) - e^{\frac{i}{\epsilon} (k^2 - (k - q_1)^2)} (z_1 - z_0) \right), \quad (J5)
\]

Proceeding in the similar manner, we obtain the following expressions for contact-limit diagrams which include
interactions with two scattering centers:

\[ M_{2,2,0}^c = - J_a(p+k) e^{i(p+k) x_0} T^c T^{a_2} T^{a_1} (1 - x^2)(-i) \int \frac{d^2 \mathbf{q}_1}{(2\pi)^2} (-i) \int \frac{d^2 \mathbf{q}_2}{(2\pi)^2} v(0, \mathbf{q}_1) v(0, \mathbf{q}_2) e^{-i(q_1+q_2) \cdot b_1} \times \]
\[ \frac{1}{2} (2\pi g_s) \left( \frac{\epsilon \cdot (k - x(q_1 + q_2))}{(k - x(q_1 + q_2))^2 + \chi} \right) e^{\frac{1}{2}\pi\omega x} (k^2 + \frac{1}{\omega} (k-q_1-q_2)^2 + \frac{1}{\omega} x)(z_1-z_0), \] (J6)

\[ M_{2,0,3}^c = J_a(p+k) e^{i(p+k) x_0} [T^c, T^{a_2}] T^{a_1} (1 - x^2)(-i) \int \frac{d^2 \mathbf{q}_1}{(2\pi)^2} (-i) \int \frac{d^2 \mathbf{q}_2}{(2\pi)^2} v(0, \mathbf{q}_1) v(0, \mathbf{q}_2) e^{-i(q_1+q_2) \cdot b_1} \times \]
\[ \frac{1}{2} (2\pi g_s) \left( \frac{\epsilon \cdot (k - q_1 - q_2)}{(k - q_1 - q_2)^2 + \chi} \right) e^{\frac{1}{2}\pi\omega x} (k^2 + \frac{1}{\omega}(k-q_1-q_2)^2 + \frac{1}{\omega} x)(z_1-z_0), \] (J7)

\[ M_{2,0,0}^c = J_a(p+k) e^{i(p+k) x_0} (T^{a_2} T^{a_1} T^c) d T_T^a T_T^a (1 - x^2)(-i) \int \frac{d^2 \mathbf{q}_1}{(2\pi)^2} (-i) \int \frac{d^2 \mathbf{q}_2}{(2\pi)^2} v(0, \mathbf{q}_1) v(0, \mathbf{q}_2) e^{-i(q_1+q_2) \cdot b_1} \times \]
\[ \frac{1}{2} (2\pi g_s) \left( \frac{\epsilon \cdot k}{k^2 + \chi} \right) e^{\frac{1}{2}\pi\omega x} (k^2 + \frac{1}{\omega}(k-q_1-q_2)^2 + \frac{1}{\omega} x)(z_1-z_0), \] (J8)

\[ M_{2,0,1}^c = J_a(p+k) e^{i(p+k) x_0} (T^{a_2} [T^c, T^{a_1}]) d T_T^a T_T^a (1 - x^2)(-i) \int \frac{d^2 \mathbf{q}_1}{(2\pi)^2} (-i) \int \frac{d^2 \mathbf{q}_2}{(2\pi)^2} v(0, \mathbf{q}_1) v(0, \mathbf{q}_2) e^{-i(q_1+q_2) \cdot b_1} \times \]
\[ \frac{1}{2} (2\pi g_s) \left( \frac{\epsilon \cdot (k - q_1)}{(k - q_1)^2 + \chi} \right) e^{\frac{1}{2}\pi\omega x} (k^2 + \frac{1}{\omega}(k-q_1-q_2)^2 + \frac{1}{\omega} x)(z_1-z_0), \] (J9)

\[ M_{2,0,2}^c = J_a(p+k) e^{i(p+k) x_0} (T^{a_1} [T^c, T^{a_2}]) d T_T^a T_T^a (1 - x^2)(-i) \int \frac{d^2 \mathbf{q}_1}{(2\pi)^2} (-i) \int \frac{d^2 \mathbf{q}_2}{(2\pi)^2} v(0, \mathbf{q}_1) v(0, \mathbf{q}_2) e^{-i(q_1+q_2) \cdot b_1} \times \]
\[ \frac{1}{2} (2\pi g_s) \left( \frac{\epsilon \cdot (k - q_2)}{(k - q_2)^2 + \chi} \right) e^{\frac{1}{2}\pi\omega x} (k^2 + \frac{1}{\omega}(k-q_1-q_2)^2 + \frac{1}{\omega} x)(z_1-z_0), \] (J10)

The amplitudes $M_{2,1,0}^c$ and $M_{2,1,1}^c$ are omitted as they are suppressed compared to the remaining amplitudes. After adding Eqs. (J3), (J4), (J5), we obtain:
\[ \frac{1}{d_T} \text{Tr} \langle |M_1|^2 \rangle = N|J(p + k)|^2(4g_s^2) \frac{1}{A_\perp} (1 - x + x^2)^2 \int \frac{d^2q_1}{(2\pi)^2} |v(0, q_1)|^2 \frac{C_2(T)}{d_G} \left\{ \left( \frac{\epsilon \cdot (k - xq_1)}{(k - xq_1)^2 + \chi} \right) \frac{C_2(G)}{d_G} \left( \frac{\epsilon \cdot (k - xq_1)}{(k - xq_1)^2 + \chi} \right) \right\} \]

\[ + 2\alpha \left( \frac{\epsilon \cdot (k - q_1)}{(k - q_1)^2 + \chi} - \frac{\epsilon \cdot k}{k^2 + \chi} - \frac{\epsilon \cdot (k - xq_1)}{(k - xq_1)^2 + \chi} \right) \left( \frac{\epsilon \cdot (k - q_1)}{(k - q_1)^2 + \chi} - \frac{\epsilon \cdot k}{k^2 + \chi} - \frac{\epsilon \cdot (k - xq_1)}{(k - xq_1)^2 + \chi} \right) \]

\[ + 2 \left( \frac{\epsilon \cdot k}{k^2 + \chi} \text{Tr}((T^c)^2(T^{a_1})^2) - \frac{\epsilon \cdot (k - xq_1)}{(k - xq_1)^2 + \chi} \text{Tr}(T^cT^{a_1}T^cT^{a_1}) \right) \frac{\epsilon \cdot k}{k^2 + \chi} \]

\[ - 2\alpha \left( \frac{\epsilon \cdot (k - q_1)}{(k - q_1)^2 + \chi} - \frac{1}{2} \frac{\epsilon \cdot k}{k^2 + \chi} - \frac{1}{2} \frac{\epsilon \cdot (k - xq_1)}{(k - xq_1)^2 + \chi} \left( \frac{2}{2} \frac{\epsilon \cdot k}{k^2 + \chi} \right) \frac{2}{2} \frac{\epsilon \cdot (k - xq_1)}{(k - xq_1)^2 + \chi} \frac{2}{2} \frac{\epsilon \cdot k}{k^2 + \chi} \right) \]

\[ + \left( \frac{\epsilon \cdot (k - q_1)}{(k - q_1)^2 + \chi} - \frac{1}{2} \frac{\epsilon \cdot k}{k^2 + \chi} - \frac{1}{2} \frac{\epsilon \cdot (k - xq_1)}{(k - xq_1)^2 + \chi} \left( \frac{2}{2} \frac{\epsilon \cdot k}{k^2 + \chi} \right) \frac{2}{2} \frac{\epsilon \cdot (k - xq_1)}{(k - xq_1)^2 + \chi} \frac{2}{2} \frac{\epsilon \cdot k}{k^2 + \chi} \right). \]

Likewise, after adding Eqs. [J6]-[J10], we obtain:

\[ \frac{2}{d_T} \text{Re} \text{Tr} \langle M_2M_0^* \rangle = N|J(p + k)|^2(4g_s^2) \frac{1}{A_\perp} (1 - x + x^2)^2 \int \frac{d^2q_1}{(2\pi)^2} |v(0, q_1)|^2 \frac{C_2(T)}{d_G} \times \]

\[ \times \left\{ \left( \frac{\epsilon \cdot k}{k^2 + \chi} \right) \left( \frac{2}{2} \frac{\epsilon \cdot k}{k^2 + \chi} \right) \left( \frac{2}{2} \frac{\epsilon \cdot k}{k^2 + \chi} \right) \right\} \]

\[ \times \left\{ \left( 1 - \cos \left( \frac{k^2 + \chi}{x(1 - x)E^+} (z_1 - z_0) \right) \right) \left( \frac{\epsilon \cdot k}{k^2 + \chi} - \frac{\epsilon \cdot (k - xq_1)}{(k - xq_1)^2 + \chi} \right) \frac{\epsilon \cdot k}{k^2 + \chi} + \left( \frac{\epsilon \cdot (k - xq_1)}{(k - xq_1)^2 + \chi} - \frac{\epsilon \cdot k}{k^2 + \chi} \right) \frac{\epsilon \cdot (k - xq_1)}{(k - xq_1)^2 + \chi} \right\} \]

\[ + \left( 1 - \cos \left( \frac{(k - q_1)^2 + \chi}{x(1 - x)E^+} (z_1 - z_0) \right) \right) \left( \frac{2}{2} \frac{\epsilon \cdot (k - q_1)}{(k - q_1)^2 + \chi} - \frac{\epsilon \cdot k}{k^2 + \chi} - \frac{\epsilon \cdot (k - xq_1)}{(k - xq_1)^2 + \chi} \right). \]

leading to:

\[ \frac{1}{d_T} \text{Tr} \langle |M_1|^2 \rangle + \frac{2}{d_T} \text{Re} \text{Tr} \langle M_2M_0^* \rangle = N d_G |J(p + k)|^2(4g_s^2) \frac{1}{A_\perp} (1 - x + x^2)^2 \int \frac{d^2q_1}{(2\pi)^2} |v(0, q_1)|^2 \frac{C_2(T)}{d_G} \frac{C_2(G)}{d_G} \left\{ \left( \frac{\epsilon \cdot k}{k^2 + \chi} \right) \left( \frac{2}{2} \frac{\epsilon \cdot k}{k^2 + \chi} \right) \right\} \]

\[ \times \left\{ \left( 1 - \cos \left( \frac{k^2 + \chi}{x(1 - x)E^+} (z_1 - z_0) \right) \right) \left( \frac{\epsilon \cdot k}{k^2 + \chi} - \frac{\epsilon \cdot (k - xq_1)}{(k - xq_1)^2 + \chi} \right) \frac{\epsilon \cdot k}{k^2 + \chi} + \left( \frac{\epsilon \cdot (k - xq_1)}{(k - xq_1)^2 + \chi} - \frac{\epsilon \cdot k}{k^2 + \chi} \right) \frac{\epsilon \cdot (k - xq_1)}{(k - xq_1)^2 + \chi} \right\} \]

\[ + \left( 1 - \cos \left( \frac{(k - q_1)^2 + \chi}{x(1 - x)E^+} (z_1 - z_0) \right) \right) \left( 2 \frac{\epsilon \cdot (k - q_1)}{(k - q_1)^2 + \chi} - \frac{\epsilon \cdot k}{k^2 + \chi} - \frac{\epsilon \cdot (k - xq_1)}{(k - xq_1)^2 + \chi} \right). \]

In the soft-gluon approximation the previous expression coincides with Eq. (82) from [25] (note that contrary to the cited paper, we here consider gluon jet, so that \( M \) no longer denotes heavy quark mass, but instead \( M \equiv m_q \) and therefore the term \( M^2 x^2 \) is also negligible). If we further apply the same procedure as in Appendix 4 and again assume the simple exponential distribution \( \frac{2}{\pi} e^{-\frac{x^2}{1 + x^2}} \) between the scattering centers, we obtain:
which is symmetric to the exchange of $p$ and $k$ gluons, and which for $m_g \to 0$ coincides with Eq. (12). Also:

\[
\frac{dE^{(1)}}{dx} = \frac{C_2(G)\alpha_s L}{\lambda} \frac{2(m^2 - \mu^2)}{2} \int \frac{d^2 \mathbf{q}_1}{\pi} \frac{\mu^2}{(\mathbf{q}_1^2 + \mu^2)^2} \int d\mathbf{k}^2 \times 
\times \left\{ \frac{(k - q_1)^2 + \chi}{(4\pi \lambda x)^2 + ((k - q_1)^2 + \chi)^2} \left( \frac{2(k - q_1)^2}{(k - q_1)^2 + \chi} - \frac{k \cdot (k - q_1)}{k^2 + \chi} - \frac{(k - q_1) \cdot (k - x q_1)}{(k - x q_1)^2 + \chi} \right) + \frac{k^2}{(k - q_1)^2 + \chi} \right\}, \tag{J15}
\]

which, in soft-gluon approximation, reduces to Eq. (84) from [29], and which for $m_g \to 0$ coincides with our massless beyond soft-gluon approximation expression Eq. (13).

Further, we display the beyond soft-gluon approxima-
tion expressions needed for numerical evaluation of the corresponding variables. So, the number of radiated gluons to the first order in opacity for gluons with effective mass $m_g$ and for finite $x$ reads:

\[
N^{(1)}_g = \frac{C_2(G)\alpha_s L}{\lambda} \frac{2}{\int_0^x (1 - x + x^2)^2} \int \frac{d^2 \mathbf{q}_1}{\pi} \frac{\mu^2}{(\mathbf{q}_1^2 + \mu^2)^2} \int d\mathbf{k}^2 \times 
\times \left\{ \frac{(k - q_1)^2 + \chi}{(4\pi \lambda x)^2 + ((k - q_1)^2 + \chi)^2} \left( \frac{2(k - q_1)^2}{(k - q_1)^2 + \chi} - \frac{k \cdot (k - q_1)}{k^2 + \chi} - \frac{(k - q_1) \cdot (k - x q_1)}{(k - x q_1)^2 + \chi} \right) + \frac{k^2}{(k - q_1)^2 + \chi} \right\}. \tag{J16}
\]

Similarly, the fractional radiative energy loss is obtained after numerically integrating the following expression:

\[
\frac{\Delta E^{(1)}}{E} = \frac{C_2(G)\alpha_s L}{\lambda} \frac{2}{\int_0^x (1 - x + x^2)^2} \int \frac{d^2 \mathbf{q}_1}{\pi} \frac{\mu^2}{(\mathbf{q}_1^2 + \mu^2)^2} \int d\mathbf{k}^2 \times 
\times \left\{ \frac{(k - q_1)^2 + \chi}{(4\pi \lambda x)^2 + ((k - q_1)^2 + \chi)^2} \left( \frac{2(k - q_1)^2}{(k - q_1)^2 + \chi} - \frac{k \cdot (k - q_1)}{k^2 + \chi} - \frac{(k - q_1) \cdot (k - x q_1)}{(k - x q_1)^2 + \chi} \right) + \frac{k^2}{(k - q_1)^2 + \chi} \right\}. \tag{J17}
\]
[1] U.A. Wiedemann, Nucl. Phys. B 588, 303 (2000).
[2] C.A. Salgado and U.A. Wiedemann, Phys. Rev. D 68, 014008 (2003).
[3] N. Armesto, C.A. Salgado and U.A. Wiedemann, Phys. Rev. D 69, 114003 (2004).
[4] R. Baier, Y.L. Dokshitzer, A.H. Mueller, S. Peigne and D. Schiff, Nucl. Phys. B 483, 291 (1997).
[5] R. Baier, Y.L. Dokshitzer, A.H. Mueller, S. Peigne and D. Schiff, Nucl. Phys. B 484, 265 (1997).
[6] B.G. Zakharov, JETP Lett. 63, 952 (1996).
[7] B.G. Zakharov, JETP Lett. 65, 615 (1997).
[8] M. Gyulassy, P. Levai and I. Vitev, Nucl. Phys. B 571, 197 (2000).
[9] M. Gyulassy, P. Levai and I. Vitev, Nucl. Phys. B 594, 371 (2001).
[10] X.N. Wang and X.F. Guo, Nucl. Phys. A 696, 788 (2001).
[11] A. Majumder and M. Van Leeuwen, Prog. Part. Nucl. Phys. A 66, 41 (2011).
[12] L. Apolinario, N. Armesto and C.A. Salgado, Phys. Lett. B 718, 160-168 (2012).
[13] B.W. Zhang and X.N. Wang, Nucl. Phys. A 720, 429-451 (2003).
[14] G. Ovanesyan and I. Vitev, JHEP 1106, 080 (2011); Phys. Lett. B 706, 371 (2012).
[15] M. Djordjevic and U. Heinz, Phys. Rev. Lett. 101, 022302 (2008).
[16] M. Djordjevic, Phys. Rev. C 80, 064909 (2009).
[17] M. Djordjevic and M. Djordjevic, Phys. Lett. B 734, 286 (2014).
[18] D. d’Enterria and B. Betz, Lect. Notes Phys. 785, 285 (2010).
[19] A. Dainese, Eur. Phys. J. C 33, 495 (2004).
[20] M. Djordjevic and M. Djordjevic, Phys. Rev. C 90, 034910 (2014).
[21] M. Djordjevic, M. Djordjevic and B. Blagojevic, Phys. Lett. B 737, 298 (2014).
[22] M. Djordjevic, Phys. Rev. Lett. 112, 042302 (2014).
[23] M. Djordjevic and M. Djordjevic, Phys. Rev. C 92, no. 2, 024918 (2015).
[24] M. Djordjevic, B. Blagojevic and L. Zivkovic, Phys. Rev. C 94, no. 4, 044908 (2016).
[25] M. Djordjevic and M. Gyulassy, Nucl. Phys. A 733, 265-298 (2004).
[26] M. Gyulassy and X.N. Wang, Nucl. Phys. B 420, 583 (1994).
[27] M. Djordjevic and M. Gyulassy, Phys. Rev. C 68, 034914 (2003).
[28] J.D. Bjorken: FERMILAB-PUB-82-059-THY, 287-292 (1982).
[29] M. Gyulassy, P. Levai and I. Vitev, Phys. Lett. B 538, 282 (2002).
[30] S. Wicks (2008), arXiv:0804.4704
[31] G. Altarelli and G. Parisi, Nucl. Phys. B 126, 298 (1977).
[32] B. Betz and M. Gyulassy, Phys. Rev. C 86, 024903 (2012).
[33] S. Wicks, W. Horowitz, M. Djordjevic and M. Gyulassy, Nucl. Phys. A 784, 426 (2007).
[34] Z.B. Kang, I. Vitev and H. Xing, Phys. Lett. B 718, 482 (2012).
[35] R. Sharma, I. Vitev and B.W. Zhang, Phys. Rev. C 80, 054902 (2009).
[36] R.D. Field, Applications of Perturbative QCD (Perseus Books, Cambridge, Massachusetts, 1995).
[37] M. Djordjevic and M. Djordjevic, Phys. Lett. B 709, 229-233 (2012).
[38] J. Ellis, TikZ-Feynman: Feynman diagrams with TikZ, arXiv:hep-ph/1601.05437 (2016).
[39] M. Gyulassy, P. Levai and I. Vitev, Nucl. Phys. B 661, 637c (1999).