Elementarily free groups are subgroup separable

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Abstract

Elementarily free groups are the finitely generated groups with the same elementary theory as free groups. We prove that elementarily free groups are subgroup separable, answering a question of Zlil Sela.

Limit groups arise naturally in the study of the set of homomorphisms to free groups and, in the guise of fully residually free groups, have long been studied in connection with the first-order logic of groups: see for example, [11], [12] and [15]. Indeed, limit groups turn out to be precisely the groups with the same existential theory\(^1\) as a free group [15]. The name limit group was introduced by Zlil Sela in his solution to the Tarski Problem (see [22], [23] et seq.), wherein he characterized the finitely generated groups with the same elementary theory\(^2\) as a free group. Such groups are called elementarily free. Sela asked if limit groups were subgroup separable in [20].

A group \(G\) is subgroup separable (or LERF) if any finitely generated subgroup \(H\) and any element \(g \notin H\) can be distinguished in a finite quotient. The generalized word problem is soluble for subgroup separable groups. Historically, it has been of interest in topology because, in certain circumstances, an immersion into a space with subgroup separable fundamental group can be lifted to an embedding in a finite cover (see, for example, [20]).

The first significant result concerning subgroup separability is due to M. Hall in [8], who demonstrated the property for free groups. This was

\(^1\)The existential theory of \(G\) is the set of sentences in the elementary theory that only use the existential quantifier \(\exists\).

\(^2\)The elementary theory of a group \(G\) is the set of first-order sentences that are true in \(G\).
generalized by R. G. Burns [6] and N. S. Romanovskii [16], who showed that a free product of subgroup separable groups is subgroup separable.

These results were all proved using algebraic methods, but a more topological approach was developed by J. Hempel in [10], J. R. Stallings in [25] and P. Scott in [17] (see also [18]); Scott used hyperbolic geometry to prove that surface groups are subgroup separable.

A. M. Brunner, R. G. Burns and D. Solitar [5] extended Scott’s theorem by showing that any amalgamated product of free groups along a cyclic subgroup is subgroup separable. The topological thread was continued by, among others, M. Tretkoff [27] and G. Niblo [14]. More recently, D. Long and A. Reid [13] adapted Scott’s approach to show that geometrically finite subgroups of certain hyperbolic Coxeter groups are separable.

In preparation for our main theorem, we give a purely topological proof of Scott’s theorem (see corollary 3.8), as well as proofs of the results of Hall, Burns and Romanovskii.

**Theorem 0.1 (Theorem 5.13)** Elementarily free groups are subgroup separable.

The proof of theorem 5.13 is most closely prefigured by the work of Rita Gitik and Daniel Wise. Wise [28] classified the graphs of free groups with cyclic edge groups that are subgroup separable. Gitik showed in [7] that if $G$ is subgroup separable, $F$ is free and $f \in F$ has no proper roots then $G \ast \langle f \rangle F$ is also subgroup separable. Section 5 explains how one can combine this result with recent work of Martin Bridson, Michael Tweedale and myself on positive-genus towers to provide a second, less direct proof of theorem 5.13.

This paper is organized as follows. In section 1 we introduce the definition of subgroup separability and some basic properties and formulations. In section 2 we develop our basic tools, the notions of pre-cover and elevation. The fundamental Stallings’ Principle of section 3 gives a criterion for completing a pre-cover to a cover, which we use to give topological proofs of the theorems of Hall, Burns and Romanovskii. We anticipate that this formulation of Stallings’ Principle will prove useful in other contexts. We show how assumptions on homology make it possible to apply Stallings’ Principle and provide a topological proof of Scott’s result that surface groups are subgroup separable. A detailed argument in section 4 shows how to replace infinite pre-covers by finite ones. In section 5 we introduce elementarily free groups and prove that they are subgroup separable.
The arguments of the current article are framed with an attack on the general problem of whether all limit groups are subgroup separable in mind. These techniques are, however, ill-suited to the natural, stronger question of whether all finitely generated subgroups of limit groups are closed in the pro-free topology.

1 Subgroup separability

Definition 1.1 Let $G$ be a group. A subgroup $H \subset G$ is separable if it is an intersection of finite-index subgroups of $G$. Call $G$ subgroup separable if every finitely generated subgroup is separable.

Note that a subgroup $H$ is separable if and only if, for any $g \notin H$, there exists a finite-index subgroup $K$ of $G$ so that $H \subset K$ and $g \notin K$. Here are some other useful algebraic reformulations.

Lemma 1.2 Let $G$ be a group and $H$ a subgroup. The following are equivalent.

1. $H$ is separable.
2. For every $g \notin H$ there exists a finite-index subgroup $K \subset G$ so that $g \notin HK$.
3. For every $g \notin H$ there exists a homomorphism $f : G \to Q$ to a finite group so that $f(g) \notin f(H)$.

Proof. That 1 implies 2 is trivial. Given $K$ as in 2, we can construct a canonical normal subgroup

$$N = \bigcap_{g \in G} K^g.$$

It is of finite index in $G$ because if $g_1, \ldots, g_n$ are coset representatives for $K$, $N = \cap_i K^{g_i}$. Setting $Q = G/N$ shows that 2 implies 3.

To deduce 1 from 3, take $f^{-1}(f(H))$ to be the required finite-index subgroup. □

We will also make use of some equivalent topological notions. We work in the category of combinatorial complexes and combinatorial maps.
Lemma 1.3 Let \((X, x)\) be a based, connected complex, \(H \subset \pi_1(X, x)\) a subgroup and \(p : (X^H, x') \to (X, x)\) the covering corresponding to \(H\). The following are equivalent.

1. \(H\) is separable.

2. For every finite subcomplex \(\Delta \subset X^H\) there exists an intermediate, finite-sheeted covering
\[X^H \to \tilde{X} \to X\]
so that \(\Delta\) embeds in \(\tilde{X}\).

3. For any \(g \notin H\), there exists a finite-sheeted covering
\[\hat{X} \to (X, x)\]
so that, for every \(h \in H\), the end-point of the lift of any (based) representative of \(g\) to \(\hat{X}\) doesn’t coincide with the end-point of the lift of any representative of \(h\).

Proof. We start by showing how 2 follows from 1. Let \(\{\sigma^i_j\}\) be the set of cells of \(\Delta\), partitioned so that \(\sigma^i_j\) and \(\sigma^i_{j'}\) have the same image in \(X\) if and only if \(i = i'\). Let \(x^i_j\) be the barycentre of \(\sigma^i_j\). For each \(i\) and \(j\) fix a continuous path \(\alpha^i_j\) from \(x'\) to \(x^i_j\) and, for each \(i\) and distinct \(j\) and \(j'\), fix a continuous path \(\beta^i_{j,j'}\) from \(x^i_j\) to \(x^i_{j'}\). The concatenation
\[(p \circ \alpha^i_j) \cdot (p \circ \beta^i_{j,j'}) \cdot (p \circ \alpha^i_j)^{-1}\]
defines an element \(g^i_{j,j'} \in \pi_1(X) \setminus H\). Let \(K^i_{j,j'}\) be a finite-index subgroup of \(\pi_1(X)\) containing \(H\) but not \(g^i_{j,j'}\). The finite-sheeted covering \((\hat{X}, \hat{x}) \to (X, x)\) with
\[\pi_1(\hat{X}) = \bigcap_{i,j,j'} K^i_{j,j'}\]
is as required. For suppose \(\Delta\) does not embed. Then the images of two cells, \(\sigma^i_j\) and \(\sigma^i_{j'}\), and hence their barycentres, coincide in \(\hat{X}\). If this happens then their images coincide in \(X\), so \(i = i'\) and \(g^i_{j,j'} \in \pi_1(\hat{X})\), a contradiction.

To see that 2 implies 3, fix a representative loop \(\gamma\) for \(g\) (it doesn’t matter which) and take \(\Delta\) to be the image of the lift of \(\gamma\) to \(X^H\). Then the lift of \(\gamma\) to \(\hat{X}\) is not closed, whereas the lift of any representative of \(h \in H\) to \(\hat{X}\) is closed.
Given $\hat{X}$ as in 3, set $K = \pi_1(\hat{X}) \subset \pi_1(X)$. Then if $g \in hK$ for $h \in H$ it follows that $gh^{-1} \in K$, so the end-point of any lift of a representative of $g$ coincides with the end-point of any lift of a representative of $h$, a contradiction. So 3 implies 1.

It is immediate that the property of being subgroup separable is closed under taking subgroups. The following lemma will also be useful.

**Lemma 1.4** Consider a subgroup separable group $G$. If $G$ is a finite-index subgroup of a group $G'$, then $G'$ is subgroup separable.

*Proof.* Replacing $G$ with the intersection of its conjugates, it can be assumed to be normal in $G'$. Let $H \subset G'$ be a finitely generated subgroup and $\gamma \in G' \setminus H$. If $\gamma \notin HG$ then the result is immediate, so assume $\gamma = hg$ for $h \in H$ and $g \in G \setminus G \cap H$. Since $G$ is subgroup separable and $G \cap H$ is finitely generated, there exists a finite-index normal subgroup $K \subset G$ so that $g \notin (G \cap H)K = G \cap (HK)$. Therefore $\gamma \notin HK$, as required. \hfill \Box

## 2 Elevations and pre-coverings

### 2.1 Graphs of spaces

A *graph of spaces* $\Gamma$ consists of:

1. a set $V(\Gamma)$ of connected spaces, called *vertex spaces*;
2. a set $E(\Gamma)$ of connected spaces, called *edge spaces*;
3. for each edge space $e \in E(\Gamma)$ a pair of $\pi_1$-injective continuous *edge maps*

$$
\partial_\pm^e : e \rightarrow \bigsqcup_{V \in V(\Gamma)} V.
$$

When the edge in question is unambiguous, we often suppress the superscript and refer to $\partial_\pm^e$ simply as $\partial_\pm$.

The associated topological space $|\Gamma|$ is defined as the quotient of

$$
\bigsqcup_{V \in V(\Gamma)} V \sqcup \bigsqcup_{e \in E(\Gamma)} (e \times [-1, +1])
$$

5
obtained by identifying \((x, \pm 1)\) with \(\partial^e_x(x)\) for each edge space \(e\) and every \(x \in e\). We will usually assume that \(|\Gamma|\) is connected. If \(X = |\Gamma|\) we will often say that \(\Gamma\) is a graph-of-spaces decomposition for \(X\), of just that \(X\) is a graph of spaces. The underlying graph of \(\Gamma\) is the abstract graph given by replacing every vertex and edge space of \(\Gamma\) by a point. If \(X\) is the topological space associated to the graph of spaces \(\Gamma\), the underlying graph is denoted \(\Gamma(X)\). Note that there is a natural \(\pi_1\)-surjective map \(\phi: X \to \Gamma(X)\).

Given a graph of spaces \(X\), consider a subgraph \(\Gamma' \subset \Gamma(X)\). The corresponding graph of spaces \(X' = \phi^{-1}(X)\) has a natural inclusion \(X' \hookrightarrow X\) and is called a sub-graph of spaces of \(X\). If \(X\) and \(Y\) are graphs of spaces, a map \(f: X \to Y\) is a map of graphs of spaces if, whenever \(Y'\) is a sub-graph of spaces of \(Y\), the pre-image \(f^{-1}(Y')\) is a sub-graph of spaces of \(X\).

The fundamental group of a graph of spaces is naturally a graph of groups by the Seifert–van Kampen Theorem. For more on graphs of spaces, graphs of groups and Bass–Serre theory see [19] and [24].

### 2.2 Elevations to covers

Elevations are a natural generalization of lifts, and were introduced by Wise in [28].

**Definition 2.1** Consider a continuous map of connected based spaces \(f: (A, a) \to (B, b)\) and a covering \(B' \to B\). An elevation of \(f\) to \(B'\) consists of a connected covering \(p: (A', a') \to (A, a)\) and a lift \(f': A' \to B'\) of \(f \circ p\) so that for every intermediate covering

\[
(A', a') \to (\bar{A}, \bar{a}) \overset{q}{\to} (A, a)
\]

there is no lift \(\bar{f}\) of \(f \circ q\) to \(B'\) with \(\bar{f}(\bar{a}) = f'(a')\).

Elevations \(f'_1: A'_1 \to B'\) and \(f'_2: A'_2 \to B'\) are isomorphic if there exists a homeomorphism \(\iota: A'_1 \to A'_2\), covering the identity map on \(A\), such that

\[
f'_1 = f'_2 \circ \iota.
\]

In practice, we will often abuse notation and refer to just the lift \(f'\) as an elevation of \(f\). The next lemma follows from standard covering-space theory. See, for example, proposition 1.33 of [22].
Lemma 2.2 Fix a lift $b' \in B'$ of $b = f(a)$. Consider the covering $p : (A', a') \to (A, a)$ such that

$$\pi_1(A', a') = f_*^{-1}(\pi_1(B', b')).$$

The composition $f \circ p$ admits a lift $f' : A' \to B'$ to $B'$, which is an elevation of $f$.

The degree of the elevation $f' : A' \to B'$, denoted $\deg(A')$, is the conjugacy class of the subgroup $\pi_1(A') \subset \pi_1(A)$. If $\pi_1(A')$ is of finite index in $\pi_1(A)$ then $f'$ is called finite-degree; otherwise, $f'$ is infinite-degree.

Remark 2.3 Consider a map $f : X \to Y$, a covering $Y' \to Y$ and an elevation $f' : X' \to Y'$ of $f$. Let

$$Y' \to \tilde{Y} \to Y$$

be an intermediate covering. Then there exists a unique elevation $\tilde{f} : \tilde{X} \to \tilde{Y}$ of $f$ such that $X' \to X$ factors through $\tilde{X} \to X$ and

$$\begin{array}{ccc}
X' & \xrightarrow{f'} & Y' \\
\downarrow & & \downarrow \\
\tilde{X} & \xrightarrow{\tilde{f}} & \tilde{Y}
\end{array}$$

commutes, determined by the requirement that $\pi_1(\tilde{X}) = f_*^{-1}(\pi_1(\tilde{Y}))$. We say $f'$ descends to $\tilde{f}$.

Remark 2.4 If $X$ has a graph-of-spaces decomposition $\Gamma$ and $X' \to X$ is a covering space then $X'$ inherits a graph-of-spaces decomposition $\Gamma'$, with vertex spaces the connected components of the pre-images of the vertex spaces of $X$ and edge spaces and maps given by all the elevations of the edge maps to the vertex spaces of $X'$, up to isomorphism.

2.3 Elevations of embeddings

These two lemmas, concerning elevations of embeddings, will prove useful later.
Lemma 2.5 Let $f : X \to Y$ be an embedding and $Y' \to Y$ a connected covering. If $f'_1, f'_2$ are non-isomorphic elevations of $f$ to $Y'$ then their images are disjoint.

Proof. Suppose there exists $y' \in \text{im} f'_1 \cap \text{im} f'_2$. Let $y \in Y$ be the image of $y'$ and set $x = f^{-1}(y) \in X$. By lemma 2.2 both $f'_1$ and $f'_2$ have the same domain, namely the covering space $(X', x') \to (X, x)$ corresponding to $f_*^{-1}\pi_1(Y', y')$. The identity map from $X'$ to itself realizes an isomorphism between $f'_1$ and $f'_2$. \hfill \square

Lemma 2.6 If $f : X \to Y$ is an embedding and $f' : X' \to Y'$ is an elevation then $f'$ is an embedding.

Proof. Suppose $x'_1, x'_2 \in X'$ and $f'(x'_1) = f'(x'_2) = y'$. Let $y \in Y$ be the image of $y'$ and set $x = f^{-1}(y)$. Fix a path $\gamma : [0, 1] \to X'$ from $x'_1$ to $x'_2$. Then $f' \circ \gamma$ represents an element of $\pi_1(Y', y')$. By lemma 2.2 we have that $f_*^{-1}(f' \circ \gamma) \in \pi_1(X', x'_i)$ for either $i$, so in fact $\gamma$ is a loop and $x'_1 = x'_2$. \hfill \square

2.4 Pre-covers

Gitik uses a notion of pre-cover extensively in [7]. Our pre-covers are analogous to hers. Pre-coverings are the graph-of-spaces versions of Stallings’ graph immersions in [25].

Definition 2.7 Let $X$ and $\bar{X}$ be graphs of spaces ($\bar{X}$ is not assumed connected). A pre-covering is a locally injective map $\bar{X} \to X$ that maps vertex spaces and edge spaces of $\bar{X}$ to vertex spaces and edge spaces of $X$ respectively, and restricts to a covering on each vertex space and each edge space. Furthermore, for each edge space $\bar{e}$ of $\bar{X}$ mapping to an edge space $e$ of $X$, the diagram of edge maps

$$
\begin{array}{ccc}
\bar{e} & \xrightarrow{\delta_{\bar{e}}} & V_+ \\
\downarrow & & \downarrow \\
e & \xrightarrow{\delta_e} & V_+ \\
\end{array}
$$

is required to commute. The domain $\bar{X}$ is a pre-cover.

The pre-covering $\bar{X} \to X$ is finite-sheeted if the pre-image of every point of $X$ is finite.
Remark 2.8 All the edge maps of $\bar{X}$ are elevations of edge maps of $X$ to the vertex spaces of $\bar{X}$. An elevation of an edge map of $X$ to a vertex of $\bar{X}$ that isn’t an edge map of $\bar{X}$ is called hanging. If none of the elevations are hanging then $\bar{X}$ is in fact a cover.

We will be interested in ways of completing pre-coverings to genuine coverings in the spirit of Stallings [25]. First, we find a canonical way of doing so that doesn’t preserve finiteness.

Proposition 2.9 Let $X$ be a graph of spaces and $\bar{X} \to X$ a pre-covering with $\bar{X}$ connected. Then $\pi_1(\bar{X})$ injects into $\pi_1(X)$ and, furthermore, there exists a unique embedding $\bar{X} \hookrightarrow \tilde{X}$ into a connected covering $\tilde{X} \to X$ such that $\pi_1(\tilde{X}) = \pi_1(\bar{X})$ and $\tilde{X} \to X$ extends $\bar{X} \to X$. This is called the canonical completion of $\bar{X} \to X$.

Disconnected pre-covers also have canonical completions, given by the disjoint union of the canonical completions of their connected components.

To prove proposition 2.9 we will adapt some ideas from Bass–Serre theory. If $X$ is a graph of spaces, a path $\gamma : I \to X$ is reduced if it can’t be homotoped (relative to its end-points) off any vertex or edge space of $X$ that it intersects. It is clear that every path is homotopic to a reduced path.

Lemma 2.10 If $\bar{X} \to X$ is a pre-covering and $\bar{\gamma} : I \to \bar{X}$ is reduced then the image $\gamma$ of $\bar{\gamma}$ in $X$ is also reduced.

Proof. Suppose $V$ is a vertex of $X$ that $\gamma$ can be homotoped off. Then lifting the homotopy to the pre-images of $V$ in $\bar{X}$, it follows that $\bar{\gamma}$ can be homotoped off all the pre-images of $V$. So $\bar{\gamma}$ isn’t reduced. □

We can now prove proposition 2.9.

Proof of proposition 2.9 Let $p : \bar{X} \to X$ be a connected pre-covering. First we show that $p_* : \pi_1(\bar{X}) \to \pi_1(X)$ is an injection. Let $\bar{\gamma}$ be a reduced loop in $\bar{X}$ and suppose that the image $\gamma$ of $\bar{\gamma}$ in $X$ is null-homotopic. Since $\gamma$ is reduced by lemma 2.10 $\gamma$ is contained in a vertex space $V$ of $X$ and so $\bar{\gamma}$ is contained in a vertex space $\bar{V}$ of $\bar{X}$. But $\bar{V} \to V$ is a covering map, so $\bar{X}$ is null-homotopic.

Let $\tilde{X} \to X$ be the covering corresponding to $p_*\pi_1(\bar{X}) \subset \pi_1(X)$. Then, by standard covering space theory, the map $p$ lifts to a map $\tilde{p} : \bar{X} \to \tilde{X}$. It remains to show that $\tilde{p}$ is injective.
Suppose \(x, y \in \tilde{X}\) with \(\tilde{p}(x) = \tilde{p}(y)\), and let \(\tilde{\gamma} : I \to \tilde{X}\) be a reduced path with end-points \(x\) and \(y\). Then \(\gamma = p \circ \tilde{\gamma}\) is a loop in \(X\). Since \(\tilde{p}\) is \(\pi_1\)-surjective, it can be assumed that \(\gamma\) is null-homotopic. But \(\gamma\) is reduced, so in fact \(\gamma\) is contained in a single vertex space \(V\) of \(X\). Hence, \(\tilde{\gamma} = \tilde{p} \circ \gamma\) is a null-homotopic loop in \(\tilde{V}\), and the homotopy to a point lifts to \(\tilde{V}\). So \(x = y\).

\[\Box\]

### 2.5 Elevations to pre-covers

Proposition 2.9 makes it possible to define elevations to pre-covers.

**Definition 2.11** Let \(f : X \to Y\) be a map of graphs of spaces and \(\bar{Y} \to Y\) a pre-covering. Let \(\tilde{Y} \to Y\) be the canonical completion of \(\bar{Y} \to Y\). Consider any finite, pairwise non-isomorphic collection of elevations \(\tilde{f}_i : \tilde{X}_i \to \tilde{Y}\) of \(f\). For each \(i\), let \(\bar{X}_i = \tilde{f}_i^{-1}(\bar{Y}) \subset \tilde{X}_i\). Note that \(\bar{X} = \cup_i \bar{X}_i\) is a pre-cover of \(X\) since \(f\) is a map of graphs of spaces. Suppose that the pre-covering \(\bar{X} \to X\) is non-empty and can be extended to a connected covering of \(X\). Then the coproduct

\[\tilde{f} = \cup_i \tilde{f}_i|_{\tilde{X}_i} : \tilde{X} \to \tilde{Y}\]

is called an elevation of \(f\) to \(\tilde{Y}\).

When \(\bar{X} \to X\) is a genuine covering of \(X\) the elevation \(\tilde{f}\) is called full.

In most examples of elevations to pre-covers in this article, \(X \cong S^1\) and \(\bar{X}\) consists of a finite collection of closed intervals. In this case, \(\bar{X} \to X\) can be realized as a restriction of the universal covering \(\mathbb{R} \to S^1\).

Given some maps of graphs of spaces \(f_i : X \to Y\) and a pre-covering \(\bar{Y} \to Y\) we will often be interested in a finite set of elevations \(\{\tilde{f}_j : \tilde{X}_j \to \tilde{Y}\}\), where each \(\tilde{f}_j\) is an elevation of some \(f_i\). Such a set is called disjoint if the images of distinct \(\tilde{f}_j\) and \(\tilde{f}_k\) are disjoint.

**Definition 2.12** Consider a map \(f : S^1 \to X\) and a pre-covering \(\bar{X} \to X\) with canonical completion \(\tilde{X}\). Let \(\tilde{f} : \tilde{S}^1 \to \tilde{X}\) be an elevation that arises as a restriction of the coproduct of elevations

\[\tilde{f} = \cup_i \tilde{f}_i : \tilde{S}^1 = \cup_i \tilde{S}_i^1 \to \tilde{X}\]

as in definition 2.11. Consider \(x \in \partial \tilde{S}^1\) and let \(C \subset \tilde{S}^1\) be the closure of the component of the complement of \(\tilde{S}^1\) such that \(x \in \partial C\). The interval \(C\) can be
identified with a subinterval of \([0, \infty)\) so that \(x \equiv 0\). For some unique edge space \(\tilde{e}\) of \(\tilde{X}\) and sufficiently small \(\epsilon > 0\), \(\tilde{f}([0, \epsilon]) \subset \tilde{e} \times [-1, +1]\). We say that \(\tilde{f}\) extends to \(\tilde{e}\) at \(x\). The elevation \(\tilde{f} : \tilde{S}^1 \to \tilde{X}\) is diverse if \(\tilde{f}\) extends to distinct edge spaces of the canonical completion at distinct points \(x, y \in \partial \tilde{S}^1\).

More generally, let \(\{f_i : c_i \to X\}\) be a collection of maps from circles \(c_i\) to \(X\) and consider a set of elevations \(\{\tilde{f}_j : \tilde{c}_j \to \tilde{X}\}\) of the \(f_i\). The set \(\{\tilde{f}_j\}\) is diverse if \(\tilde{f}_j\) and \(\tilde{f}_k\) respectively extend to distinct edge spaces at distinct \(x \in \partial \tilde{c}_j\) and \(y \in \partial \tilde{c}_k\).

We will want to extend the pre-covering \(\tilde{X} \to X\) so that the elevation \(\tilde{f} : \tilde{S}^1 \to \tilde{X}\) extends to a full elevation. The next proposition gives one example when this can be done.

**Proposition 2.13** Suppose \(\tilde{X} \to X\) is a pre-covering such that all hanging elevations of edge maps have simply connected domains. Let \(f : S^1 \to X\) be a combinatorial map and \(\tilde{f} : \tilde{S}^1 \to \tilde{X}\) a diverse elevation of \(f\) to \(\tilde{X}\). Let \(\hat{S}^1 \to S^1\) be a finite-sheeted covering of \(S^1\) and suppose there exists an embedding \(\tilde{S}^1 \hookrightarrow \hat{S}^1\) so that \(\hat{S}^1 \to S^1\) extends \(\tilde{S}^1 \to S^1\). Then there exists a pre-covering \(\hat{X} \to X\) extending \(\tilde{X} \to X\) so that \(\hat{f} : \hat{S}^1 \to \hat{X}\) extends to a full elevation \(f\).

**Proof.** Let \(C\) be the closure of a component of \(\hat{S}^1 \setminus \tilde{S}^1\). Without loss identify \(C \equiv [-1, 1]\). Consider the canonical completion \(\tilde{X} \to X\) of \(\tilde{X} \to X\). There is a unique lift \(\hat{f}_+ : C \to \hat{X}\) of \(f\) to \(\hat{X}\) so that \(\hat{f}_+(1) = \tilde{f}(1)\). Let \(\epsilon > 0\) be maximal such that \(\hat{f}_+((\epsilon - 1))\) lies in a vertex space. Let \(X' \subset \hat{X}\) be the pre-cover consisting of \(\hat{X}\) together with the vertex spaces and edge spaces containing \(\hat{f}_+((\epsilon - 1))\). Let \(\tilde{e}_+\) be the edge space of \(\tilde{X}\) so that \(\hat{f}_+([-1, \epsilon - 1]) \subset \tilde{e}_+ \times [-1, 1]\) and without loss assume that \(\hat{f}_+((\epsilon - 1)) \in \tilde{e}_+ \times \{+1\}\).

Similarly, there is a unique lift \(\hat{f}_- : C \to \hat{X}\) of \(f\) so that \(\hat{f}_-((-1)) = \tilde{f}(-1)\). Then \(\hat{f}_-([-1, \epsilon - 1]) \subset \tilde{e}_- \times [-1, 1]\) for some unique edge space \(\tilde{e}_-\) of \(\tilde{X}\). Without loss, assume \(\hat{f}_-((-1)) \in \tilde{e}_- \times \{-1\}\).
The edge spaces $\tilde{e}^+$ and $\tilde{e}^-$ are both simply connected covers of some edge space $e$ of $X$. There exists some unique covering transformation $\tau : \tilde{e}^+ \rightarrow \tilde{e}^-$ such that, whenever $x \in (e - 1, -1)$ with $\tilde{f}^+(x) \in \tilde{e}^+ \times \{1/2\}$,

$$\tau \circ \tilde{f}^+(x) = \tilde{f}^-(x).$$

Now let $\hat{X}$ be the pre-cover given by $X'$ together with the additional edge space $\tilde{e}^+$; the additional edge maps are $\partial \tilde{e}^+ \circ \tau$.

Since $\tilde{f}$ is diverse, this can be done for every of component of $\hat{S}^1 \setminus \bar{S}^1$. □

**Remark 2.14** Consider a finite, diverse set of elevations $\{\bar{f}_j\}$ to $\bar{X}$ of various maps $f_i : S^1 \rightarrow X$, so that each $\bar{f}_j$ satisfies the conditions of proposition 2.13. Then proposition 2.13 can be applied inductively to produce a pre-cover $\hat{X}$ in which every $\bar{f}_j$ extends to a full elevation.

In the case where the vertex spaces of $X$ are simply connected, note that the construction of proposition 2.13 preserves finiteness.

**Corollary 2.15** Consider a graph of spaces $X$ with simply connected vertex spaces and a combinatorial map $f : S^1 \rightarrow X$. Suppose $\bar{X} \rightarrow X$ is a finite-sheeted pre-covering. Let $\bar{f} : S^1 \rightarrow \bar{X}$ be a diverse elevation of $f$ to $\bar{X}$. Let

$$\hat{S}^1 \rightarrow S^1$$

be a finite-sheeted covering of $S^1$ and suppose there exists an embedding $\bar{S}^1 \hookrightarrow \hat{S}^1$ so that $\hat{S}^1 \rightarrow S^1$ extends $\bar{S}^1 \rightarrow S^1$. Then there exists a finite-sheeted pre-covering $\hat{X} \rightarrow X$ extending $\bar{X} \rightarrow X$ so that $\hat{f} : \hat{S}^1 \rightarrow \hat{X}$ extends to a full elevation

$$\hat{f} : \hat{S}^1 \rightarrow \hat{X}$$

of $f$.

### 2.6 The polygon decomposition

It can be useful to treat compact surfaces with non-empty boundary like graphs. Given any such surface $\Sigma$ there exists a finite collection of disjoint embedded arcs $\alpha_1, \ldots, \alpha_n \subset \Sigma$, with $\partial \alpha_i \subset \partial \Sigma$, so that the surface $V$ obtained by cutting along the $\alpha_i$ is a polygon with $4n$ sides. To see this, observe that for any such $\alpha_i$,

$$\chi(\Sigma \setminus \alpha_i) = \chi(\Sigma) + 1.$$
Taking the $\alpha_i$ as edge spaces and $V$ as a vertex space with the obvious inclusions as edge maps decomposes $\Sigma$ as a graph of spaces. Scott used this decomposition in [17] to show that the fundamental groups of such surfaces (that is, free groups) are subgroup separable. We will call it the *polygon decomposition* of $\Sigma$, and will use it often.

### 2.7 Ensuring diversity

In the light of proposition 2.13 it will be useful to be able to impose diversity on collections of elevations.

**Lemma 2.16** Let $\Sigma$ be a compact surface with non-empty boundary and assume $\chi(\Sigma) < 0$. Endow $\Sigma$ with the polygon decomposition described above. Realize the boundary components as embeddings of circles $\{\delta_i : c_i \to \Sigma\}$. Let $\tilde{\Sigma} \to \Sigma$ be a finite-sheeted pre-covering and fix a finite, disjoint collection $\{\tilde{\delta}_j : \tilde{c}_j \to \Sigma_k\}$ of elevations of the $\delta_i$. Then $\tilde{\Sigma} \to \Sigma$ extends to a finite-sheeted pre-covering $\hat{\Sigma} \to \Sigma$ so that:

1. each component of $\hat{\Sigma}$ is a deformation retract of its pre-image in $\Sigma$;
2. each elevation $\tilde{\delta}_j$ extends to an elevation $\hat{\delta}_j : \hat{c}_j \to \hat{\Sigma}$;
3. the set $\{\hat{\delta}_j\}$ is diverse. Indeed, whenever $x \in \partial\hat{c}_j$ and $y \in \partial\hat{c}_k$ are distinct, $\hat{\delta}_j(x)$ and $\hat{\delta}_k(y)$ lie in distinct vertex spaces of $\hat{\Sigma}$.

**Proof.** Without loss, assume $\tilde{\Sigma}$ is connected. Let $\tilde{\Sigma}$ be the canonical completion of $\Sigma$. Each elevation $\tilde{\delta}_j$ extends canonically to some coproduct of elevations $\hat{\delta}_j : \hat{c}_j \to \hat{\Sigma}$. Consider any pair of connected components $\tilde{b}_j \subset \tilde{c}_j$ and $\tilde{b}_k \subset \tilde{c}_k$. We will show that only finitely many vertex spaces of $\Sigma$ intersect both $\tilde{\delta}_j(\tilde{b}_j)$ and $\tilde{\delta}_k(\tilde{b}_k)$.

Suppose there are infinitely many vertex spaces of $\tilde{\Sigma}$ that $\tilde{\delta}_j(\tilde{b}_j)$ and $\tilde{\delta}_k(\tilde{b}_k)$ both intersect. Let $\phi : \Sigma \to \Gamma(\Sigma)$ be the natural map to the underlying graph of $\Sigma$. Then $\phi \circ \tilde{\delta}_j$ and $\phi \circ \tilde{\delta}_k$ are proper maps $\mathbb{R} \to \Gamma(\Sigma)$ that share infinitely many vertices of $\Gamma(\Sigma)$ in their images. Since $\pi_1(\Gamma(\Sigma))$ is finitely generated, $\phi \circ \hat{\delta}_j(b_j) \cap \phi \circ \hat{\delta}_k(b_k)$ contains the image of some injective ray $[0, \infty) \to \Gamma(\Sigma)$. Since $\chi(\Sigma) < 0$ the vertex space $V$ of the polygon decomposition of $\Sigma$ is a $4n$-sided polygon for $n > 1$. Therefore no disjoint pair of elevations of the $\delta_i$ that enter by the same edge space of $V$ leave by the same edge space of $V$, 

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so it follows that $\tilde{\delta}_j(b_j)$ and $\tilde{\delta}_k(b_k)$ intersect non-trivially. But then $\tilde{\delta}_j|_{b_j}$ and $\tilde{\delta}_k|_{b_k}$ are isomorphic elevations by lemma 2.5.

Therefore, by expanding $\Gamma(\tilde{\Sigma})$ to some finite graph $\Gamma(\hat{\Sigma}) \subset \hat{\Gamma} \subset \Gamma(\tilde{\Sigma})$ and setting $\hat{\Sigma} = p^{-1}(\hat{\Gamma})$ the result follows. □

3 From pre-covers to covers

3.1 Stallings’ Principle

Our strategy for proving subgroup separability is to replace a pre-cover by a finite-sheeted pre-cover, and then complete that pre-cover to a finite-sheeted cover. The first step is described in section 4. For the second step, we use an idea developed by John Stallings in the context of graphs, which we therefore name in his honour. Similar ideas were also used by Hempel [10] and Scott [17].

**Proposition 3.1 (Stallings’ Principle)** Let $\tilde{X} \to X$ be a pre-covering with the following property: for each edge space $e$ of $X$ with edge maps $\partial_\pm : e \to V_\pm$, for each conjugacy class $\mathcal{D}$ of subgroups of $\pi_1(e)$, there exists a bijection between the set of elevations of $\partial_\pm$ to $\tilde{X}$ of degree $\mathcal{D}$ and the set of elevations of $\partial_\mp$ to $X$ of degree $\mathcal{D}$. Then $\tilde{X} \to X$ can be extended to a covering $\hat{X} \to X$, with the same set of vertex spaces as $\tilde{X}$.

**Proof.** For each edge space $e$ of $X$ and for each degree $\mathcal{D}$ there is a bijection between the set of elevations of $\partial_+ \pm$ of degree $\mathcal{D}$ and the set of elevations of $\partial_\mp$ of degree $\mathcal{D}$. So there is a bijection between the set of hanging elevations of $\partial_+ \pm$ of degree $\mathcal{D}$ and the set of hanging elevations of $\partial_\mp$ of degree $\mathcal{D}$. Fix such a bijection. For each hanging elevation $\partial_+ : \tilde{e} \to \tilde{V}_+$ of $\partial_+$ take $\tilde{e}$ as an edge space for $\hat{X}$ and for edge maps take the hanging elevation $\partial_+$ and the corresponding hanging elevation $\partial_-$. The resulting pre-cover, with these additional edge spaces, has no hanging elevations and so is a cover. □

**Theorem 3.2** Let $X$ be any graph of spaces with simply connected vertices (for example, a graph). Then any finite-sheeted pre-cover of $X$ can be extended to a finite-sheeted cover.
Proof. Let \( \bar{X} \) be a finite-sheeted pre-cover of \( X \). Let \( N \) be the maximum number of pre-images in \( \bar{X} \) of a vertex space in \( X \). Create a new pre-cover from \( \bar{X} \) by adding disconnected vertex spaces so that every vertex space of \( X \) has \( N \) pre-images. Now every edge map of \( X \) has \( N \) elevations to the pre-cover, all of the same (trivial) degree; so, by proposition 3.1, the pre-cover can be extended to a cover. \( \square \)

Corollary 3.3 (M. Hall [8]) Finitely generated free groups are subgroup separable.

Proof. Realize a free group \( F \) as the fundamental group of a graph \( X \). Consider a finitely generated subgroup \( H \subset F \), and the corresponding covering \( X^H \to X \). Let \( \Delta \subset X^H \) be a finite subgraph. Expanding \( \Delta \) if necessary, it can be assumed that \( \Delta \) is connected and carries \( \pi_1(X^H) \). But \( \Delta \) is a pre-cover of \( X \), so can be extended to a cover \( \hat{X} \). \( \square \)

Indeed, a similar argument gives the stronger theorem originally proved by Hall.

Corollary 3.4 (M. Hall [8]) If \( F \) is a finitely generated free group and \( H \subset F \) is a finitely generated subgroup then there exists a finite-index subgroup \( F' \subset F \) containing \( H \) so that

\[
F' = H \ast F''
\]

Proof. Again, realize \( F \) as the fundamental group of a graph \( X \) and let \( X^H \to X \) be the covering corresponding to \( H \). Since \( H \) is finitely generated there exists a finite, connected subgraph \( X' \subset X^H \) so that \( \pi_1(X') = H \). But \( X' \to X \) is a pre-covering, so can be extended to a genuine finite-sheeted covering \( \hat{X} \). Since \( X' \) is a subgraph of \( \hat{X} \), \( H \) is a free factor in \( F' = \pi_1(\hat{X}) \). \( \square \)

Stallings’ Principle can also be used to prove that free products of subgroup separable groups are subgroup separable.

Theorem 3.5 (Burns [6] and Romanovskii [16]) If \( G_+ \) and \( G_- \) are subgroup separable then \( G = G_+ \ast G_- \) is subgroup separable.
Proof. Realize $G_\pm = \pi_1(V_\pm, v_\pm)$ and let $X$ be the quotient of

$$V_- \sqcup [-1, +1] \sqcup V_+$$

obtained by identifying $\pm 1$ with $v_\pm$, so $G = \pi_1(X)$. For $H \subset G$ a finitely generated subgroup, let $X^H \to X$ be the corresponding covering and consider a finite subcomplex $\Delta \subset X^H$. Since $H$ is finitely generated there exists a sub-graph of spaces $X' \subset X^H$ with finite underlying graph that carries $H$; enlarging $X'$ if necessary, it can be assumed that $\Delta \subset X'$.

For a vertex space $V'$ of $X'$ covering $V_\pm$ in $X$, let $\Delta_{V'} \subset V'$ be a finite subcomplex containing:

1. the images of any edge spaces of $X'$ adjoining $V'$;
2. $\Delta \cap V'$.

Since $G_\pm$ are subgroup separable and $\pi_1(V')$ is finitely generated there exists an intermediate, finite-sheeted covering

$$V' \to \bar{V} \to V$$

such that $\Delta_{V'}$ embeds in $\bar{V}$. Replace each vertex space $V'$ of $X'$ by the corresponding $\bar{V}$; the edge maps $\partial_\pm' : e \to V'$ descend to maps $\bar{\partial}_\pm : e \to \bar{V}$. These new vertex spaces and edge maps give an intermediate, finite-sheeted, connected pre-covering

$$X' \to \bar{X} \to X$$

into which $\Delta$ embeds.

Each vertex space of $\bar{X}$ covers either $V_+$ or $V_-$. Let

$$P_\pm = \sum_{V \to V_\pm} \deg(V \to V_\pm).$$

Without loss of generality, assume $P_+ \geq P_-$. Consider the pre-cover given by the disjoint union of $\bar{X}$ and $P_+ - P_-$ copies of $V_-$. Any edge space $e$ of $X$ is a point; the edge maps are inclusions $\partial_\pm : e \to V_\pm$ and their elevations have only one possible (trivial) degree. Now $\partial_+$ has $P_+$ elevations to vertex spaces of the pre-cover, and $\partial_-$ has $P_- + (P_+ - P_-)$ elevations. So, by proposition 3.1 the pre-cover $\hat{X}$ can be extended to a finite-sheeted cover $\hat{X}$. It is connected, so the result follows. \qed
Stallings' Principle gives remarkable flexibility in constructing covers of graphs and surfaces with boundary. Here is an example that will prove useful. Note that the degree of an elevation of a map \( S^1 \to X \) is (a conjugacy class of) a subgroup of \( \mathbb{Z} \), and so in particular corresponds uniquely to a non-negative integer.

**Corollary 3.6** Let \( \Sigma \) be a compact surface with non-empty boundary. Realize the boundary components as embeddings of circles

\[
\delta_i : c_i \to \Sigma.
\]

Let \( H \subset \pi_1(\Sigma) \) be a finitely generated subgroup, let \( \Sigma^H \to \Sigma \) be the corresponding covering and let \( \Delta \subset \Sigma^H \) be a compact subcomplex. Let \( \delta_j^H : c_j^H \to \Sigma^H \) be all the finite-degree elevations of the \( \delta_i \) to \( \Sigma^H \). Fix a finite collection of infinite-degree elevations

\[
\epsilon_k^H : c_k^H \to \Sigma^H
\]

of the \( \delta_i \). Then for all sufficiently large positive integers \( d \) there exists an intermediate finite-sheeted covering

\[
\Sigma^H \to \hat{\Sigma} \to \Sigma
\]

so that:

1. \( \Delta \subset \hat{\Sigma} \);
2. the \( \delta_j^H \) and \( \epsilon_k^H \) all descend to distinct elevations to \( \hat{\Sigma}; \)
3. if \( \delta_j^H \) descends to \( \bar{\delta}_j \) then \( \deg(\bar{\delta}_j) = \deg(\delta_j^H) \);
4. if \( \epsilon_k^H \) descends to \( \bar{\epsilon}_k \) then \( \deg(\bar{\epsilon}_k) = d \).

**Proof.** Consider the polygon decomposition of \( \Sigma \), with vertex space the polygon \( V \) and arcs \( \{e\} \) for edge spaces. The cover \( \Sigma^H \) inherits a graph-of-spaces decomposition. Since \( H \) is finitely generated, there exists a subgraph of spaces \( \Sigma' \subset \Sigma^H \) with finite underlying graph such that the inclusion \( \Sigma' \hookrightarrow \Sigma^H \) is \( \pi_1 \)-surjective. Enlarging \( \Sigma' \) if necessary, it may be assumed that:

1. \( \Sigma' \) contains \( \Delta \);
2. $\Sigma'$ contains the images of all the $\delta_j^H$;

3. for each $k$, the intersection $\operatorname{im}e_k^H \cap \Sigma'$ is a (non-empty) arc;

4. the set $\{e'_k\}$ is diverse. (We can ensure this by lemma 2.16.)

Note that $\Sigma'$ is a pre-cover of $\Sigma$ and $\Sigma^H$ is its canonical completion. An elevation
\[ e_k^H : c_k^H \to \Sigma^H \]
restricts to an elevation
\[ e'_k : c'_k \to \Sigma', \]
where $c'_k$ is a compact subarc of $c_k^H \cong \mathbb{R}$. If $d$ is sufficiently large then it is clear that the $d$-fold covering $\tilde{c}_k \to c_i$ extends $e'_k \to c_i$. Since $\{e'_k\}$ is diverse we can apply corollary 2.15 repeatedly to extend $\Sigma' \to \Sigma$ to a pre-covering $\tilde{\Sigma} \to \Sigma$ so that every $e'_k$ extends to a full elevation $\tilde{e}_k$ of degree $d$. Finally, extend $\tilde{\Sigma}$ to a genuine cover $\hat{\Sigma}$ by theorem 3.2. \hfill \square

### 3.2 Homological assumptions

In this subsection we use assumptions on the homology of the edge spaces to make it possible to apply Stallings’ Principle.

**Proposition 3.7** Let $X$ be a graph of spaces with one vertex space $V$. Suppose every edge space $e$ is a circle and that, furthermore, $H_1(e, \mathbb{Z})$ is an infinite direct factor in $H_1(X, \mathbb{Z})$. Let $\tilde{X} \to X$ be a finite-sheeted pre-covering. Then there exist finite-sheeted coverings $\hat{X} \to \tilde{X}$ and $\hat{X} \to X$ and an inclusion $\hat{X} \hookrightarrow \tilde{X}$ so that
\[
\begin{array}{ccc}
\hat{X} & \longrightarrow & \tilde{X} \\
\downarrow & & \downarrow \\
\hat{X} & \longrightarrow & X
\end{array}
\]
commutes.

**Proof.** For each elevation $\tilde{\partial}_\pm : \tilde{e} \to \tilde{V}$ of an edge map $\partial_\pm : e \to V$ of $X$ to a vertex space $\tilde{V}$ of $\tilde{X}$, the degree $\deg(\tilde{e})$ of the elevation can be thought of as a positive integer. Let
\[ M = \prod_{\tilde{e}} \deg(\tilde{e}) \]
where the product ranges over all elevations to $\bar{X}$ of edge maps of $X$. Consider the finite-sheeted covering $X_M \to X$ corresponding to the kernel of the composition

$$\pi_1(X) \to H_1(X) \to H_1(X, \mathbb{Z}/M\mathbb{Z}).$$

Pull this covering back along $\bar{X} \to X$ to give a finite-sheeted covering

$$\hat{X} \to \bar{X}$$

with the property that every elevation of an edge map of $X$ to a vertex space of $\hat{X}$ is of degree $M$.

Consider

$$P = \sum V \deg(\hat{V} \to V)$$

where the sum ranges over all the vertex spaces of $\hat{X}$. Any edge map $\partial\pm : e \to V$ has $P/M$ elevations to $\hat{X}$. Since this is a constant, we can apply Stallings’ Principle to deduce the result. □

This conclusion will be enough to deduce separability. As an application, we give a purely topological proof of Scott’s theorem that the fundamental groups of closed surfaces are subgroup separable (cf. [7] and [28]).

**Corollary 3.8 (Scott [17, 18])** Surface groups are subgroup separable.

**Proof.** Let $\Sigma$ be a closed surface of negative Euler characteristic. (Otherwise the result is easy.) Therefore $\Sigma$ has a non-separating simple closed curve $\gamma$. Note that $\gamma$ is primitive in $H_1(\Sigma)$. Let $\Sigma_0$ be the compact surface with boundary that results from deleting a small cylindrical neighbourhood of $\gamma$. This realizes $\Sigma$ as a graph of spaces, with one vertex space $\Sigma_0$ and one edge space $e \cong S^1$. Consider any finitely generated subgroup $H \subset \pi_1(\Sigma)$, and $\Sigma^H \to \Sigma$ the corresponding covering. Fix a representative of a curve $g \notin H$; the lift of $g$ to $\Sigma^H$ is not closed.

The cover $\Sigma^H$ inherits a graph-of-spaces decomposition from $\Sigma$. Since $H$ is finitely generated, there exists some sub-graph of spaces $\Sigma' \subset \Sigma^H$ with finite underlying graph that carries the whole of $H$. Enlarging $\Sigma'$ if necessary, it can be assumed that $\Sigma'$ contains the lift of $g$. Note that $\Sigma'$ is a pre-cover of $\Sigma$. Our first aim is to replace it by a finite-sheeted pre-cover.

Fix a large positive integer $d$. By corollary 3.6 every vertex space $\Sigma'_0$ of $\Sigma'$ has an intermediate covering

$$\Sigma'_0 \to \bar{\Sigma}_0 \to \Sigma_0$$
in which all the compact edge spaces have unchanged, finite degree, all the
non-compact edge spaces have degree $d$ and, furthermore, the components
of $g$ that cross $\Sigma'_0$ embed in $\Sigma_0$. Replacing each $\Sigma'_0$ by $\Sigma_0$ and assigning the
degree to the new vertex groups gives a finite-sheeted pre-covering $\tilde{\Sigma} \to \Sigma$ containing a lift of $g$ that isn’t closed. Applying proposition 3.7 to $\tilde{\Sigma}$ gives a finite-sheeted covering
\[
\hat{\Sigma} \to \Sigma
\]
that extends to a finite-sheeted covering
\[
\tilde{\Sigma} \to \Sigma.
\]
It follows that the end-point of the lift of $g$ to $\tilde{\Sigma}$ doesn’t coincide with the
end-point of the lift of any $h \in H$; otherwise, $g$ would lift to a closed loop in $\tilde{\Sigma}$. □

We will also want to consider graphs of groups with two vertex spaces.

**Proposition 3.9** Let $X$ be a graph of spaces with two vertex spaces $U$ and $V$. Suppose every edge space $e$ is a circle and that, furthermore, $H_1(e, \mathbb{Z})$ is an
infinite direct factor in $H_1(X, \mathbb{Z})$. Let $\tilde{X} \to X$ be a finite-sheeted, connected
pre-covering. Then there exist finite-sheeted coverings $\hat{X} \to \tilde{X}$ and $\tilde{X} \to X$ and an inclusion $\hat{X} \hookrightarrow \tilde{X}$ so that
\[
\begin{array}{ccc}
\hat{X} & \longrightarrow & \tilde{X} \\
\downarrow & & \downarrow \\
\tilde{X} & \longrightarrow & X
\end{array}
\]
commutes.

**Proof.** As in the proof of proposition 3.7 the degree $\deg(\tilde{e})$ of an elevation
$\tilde{\partial}_e : \tilde{e} \to \tilde{U}$ of an edge map $\partial_e : e \to U$ of $X$ can be thought of as a positive
integer. Let
\[
M = \prod_{\tilde{e}} \deg(\tilde{e})
\]
where the product ranges over all elevations of edge maps of $X$ to vertex
spaces of $\tilde{X}$. Define $X_M \to X$ to be the covering associated to the kernel of the
homomorphism
\[
\pi_1(X) \to H_1(X, \mathbb{Z}/M\mathbb{Z}).
\]
Set $\hat{X} \to \bar{X}$ to be the pullback of $X_M \to X$ along $\bar{X} \to X$. The cover $\hat{X}$ inherits a graph-of-spaces decomposition from $\bar{X}$. Any edge space $\hat{e} \to e$ of $\hat{X}$ has degree $M$.

Let $U$ and $V$ be the two vertex spaces of $X$. Write

$$P = \sum_{\hat{U} \to U} \deg(\hat{U} \to U)$$

and

$$Q = \sum_{\hat{V} \to V} \deg(\hat{V} \to V),$$

where the sums range over all vertex spaces of $\hat{X}$ covering $U$ and $V$ respectively. Without loss of generality, $P \geq Q$. Let

$$\tilde{V} \to V$$

be the covering associated to the kernel of the homomorphism

$$\pi_1(V) \to H_1(V, \mathbb{Z}/M\mathbb{Z}).$$

Let $N = |H_1(V, \mathbb{Z}/M\mathbb{Z})| = \deg(\tilde{V} \to V)$.

Consider the pre-cover of $X$ consisting of the disjoint union of $P - Q$ copies of $\tilde{V}$ and $N$ copies of $\hat{X}$. Any edge map $\partial_{\pm} : e \to U \sqcup V$ of $X$ has $NP/M$ elevations to this pre-cover. Since this is a constant, the pre-cover can be extended to a genuine cover $\tilde{X}$. □

4 Making pre-covers finite-sheeted

As explained at the beginning of section 3, our strategy is to replace pre-covers by finite-sheeted pre-covers and then complete these to genuine finite-sheeted covers. This section is devoted to a theorem that shows, in certain circumstances, how to replace pre-covers by finite-sheeted pre-covers.

Theorem 4.1 Let $X$ be a graph of spaces with two vertex spaces, constructed as follows.

1. One vertex space, $L$, has subgroup separable fundamental group.
2. The other vertex space, $\Sigma$, is a compact surface with non-empty boundary and $\chi(\Sigma) < 0$.

3. Each edge space $e$ is a circle. Of the edge maps:

$$\partial_+: e \to L$$

is an embedding; while

$$\partial_-: e \to \Sigma$$

is required to identify $e$ homeomorphically with a boundary component of $\Sigma$. Each boundary component of $\Sigma$ is identified with at most one edge space of $X$.

If $X' \to X$ is a pre-covering with finite underlying graph and $\Delta \subset X'$ is a finite subcomplex then there exists an intermediate finite-sheeted pre-covering

$$X' \to \bar{X} \to X$$

into which $\Delta$ embeds.

\textbf{Proof.} Consider the polygon decomposition of $\Sigma$, so a covering space $\Sigma' \to \Sigma$ inherits a graph-of-spaces decomposition.

For each vertex $\Sigma'$ of $X'$ that lies above $\Sigma$, let $\Delta_{\Sigma'} \subset \Sigma'$ be a sub-graph of spaces with finite, connected underlying graph that carries $\pi_1(\Sigma')$ so that, furthermore:

1. $\Delta \cap \Sigma' \subset \Delta_{\Sigma'}$;
2. if $e'$ is an incident edge space of $X'$ then $\partial_-(e') \cap \Delta_{\Sigma'}$ is non-empty;
3. the collection $\{\partial_-'\}$ of incident edge maps restricts to a diverse collection of elevations to $\Delta_{\Sigma'}$. (This can be ensured by lemma 2.16)

For each vertex space $L'$ of $X'$ covering $L$ fix a finite, connected subcomplex $\Delta_{L'} \subset L'$ such that:

1. $\Delta \cap L' \subset \Delta_{L'}$;
2. $\partial_+((\partial_-')^{-1}(\Delta_{\Sigma'})) \subset \Delta_{L'}$ whenever

$$L' \xleftarrow{\partial_+} e' \xrightarrow{\partial_-'} \Sigma'$$

is an incident edge space.

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Since the edge maps $\partial_\pm$ of $X$ are embeddings, distinct edge maps of $X'$ have disjoint images by lemma 2.5.

Since $\pi_1(L)$ is subgroup separable, for each vertex $L' \to L$ of $X'$ there exists an intermediate, finite-sheeted covering

$$L' \to \bar{L} \to L$$

so that $\Delta_{L'}$ injects into $\bar{L}$.

To construct $\bar{X}$, we first construct the underlying graph $\bar{\Gamma}$. Let $\Gamma'$ be the underlying graph of $X'$. We will define an equivalence relation on the edges of $\Gamma'$. For each vertex space $L'$ of $X'$ that cover the same edge $e$ of $X$. Let $\partial_+: e \to L$ be an edge map of $e$; suppose $e'_1$ and $e'_2$ both adjoin the same vertex space $L'$ with edge maps $\partial^i_+: e'_i \to L'$. If the edge maps $\partial^i_+$ descend to the same elevation $\bar{\partial}_+: \bar{e} \to \bar{L}$ then we write $e'_1 \sim e'_2$. Let $\bar{\Gamma} = \Gamma'/\sim$. Each vertex of $\bar{\Gamma}$ is labelled $L$ or $\Sigma$, depending on which vertex space of $X$ its pre-images in $\Gamma'$ cover.

Each $L$-vertex $\bar{v}$ of $\bar{\Gamma}$ corresponds to a unique $L$-vertex $v'$ of $\Gamma'$. If the vertex space corresponding to $v'$ is $L'$, then the vertex space of $\bar{v}$ is $\bar{L}$.

Each $\Sigma$-vertex $\bar{u}$ of $\bar{\Gamma}$ corresponds to an equivalence class of finitely many vertices $u'_i$ of $\Gamma'$, with corresponding vertex spaces $\Sigma'_i$. Associate to $\bar{u}$ the pre-covering

$$\Sigma'' = \sqcup_i \Delta_{\Sigma'_i} \to \Sigma.$$ Later, we will complete $\Sigma''$ to a cover $\Sigma$.

Each edge of $\bar{\Gamma}$ corresponds to a finite equivalence class of edges $e'_i$ of $\Gamma'$, all covering the same edge $e$ of $X$. The edge maps of $e$ are $\partial_+: e \to L$ and $\partial_-: e \to \Sigma$, and the edge maps of $e'_i$ are elevations $\partial^i_+: e'_i \to L'$ and $\partial^i_- : e'_i \to \Sigma'_i$. We aim to find suitable edge spaces $\bar{e}$ and edge maps $\bar{\partial}_+: \bar{e} \to \bar{L}$ and $\bar{\partial}_- : \bar{e} \to \bar{\Sigma}$.

By the definition of the equivalence relation on the edges of $\Gamma'$, every $\partial^i_+$ descends to the same elevation

$$\bar{\partial}_+: \bar{e} \to \bar{L}.$$ This gives the edge space and one of the edge maps.

On the other side, consider

$$e'' = \bigcup_i (\partial^i_-)^{-1}(\Delta_{\Sigma'_i}) \subset \sqcup_i e'_i.$$
The restriction
\[ \partial''_\ell = \sqcup_i (\partial_i |_{e''}) \]
is an elevation of \( \partial_\ell \) to \( \Sigma'' \).

There is a natural map \( \iota : e'' \to \bar{e} \), namely the coproduct of the restrictions of the covering maps \( e'_i \to \bar{e} \). The claim is that \( \iota \) is an injection. We can then apply corollary 2.15 to extend \( \Sigma'' \) in such a way that \( \partial''_\ell \) extends to \( \bar{e} \).

Consider \( t_1, t_2 \in e'' \) and suppose that \( \iota(t_1) = \iota(t_2) \). Assume that \( e'' \subset e'' \) is the connected component containing \( t_j \). Since \( e'' = \sqcup_i (\partial_i)_{-1}(\Delta \Sigma') \) it follows that \( \partial_1'(t_j) \in \Delta L' \). Since \( L' \to \bar{L} \) is injective on \( \Delta L' \) it follows that \( \partial_1'(t_1) = \partial_2'(t_2) \). But then \( \partial_1' \) and \( \partial_2' \) are elevations of \( \partial_\ell \) with non-disjoint image, so they are in fact isomorphic and \( e'_1 = e'_2 \). Since \( \partial_1' : e'_1 \to L' \) is an elevation of an injective map it is itself injective, so it now follows that \( t_1 = t_2 \). This proves the claim.

To complete the proof of the theorem, we now apply corollary 2.15 (invoking remark 2.14) to extend every \( \Sigma'' \) so that the elevations \( \partial''_\ell : e'' \to \Sigma'' \) extend to full elevations \( \bar{\partial}_\ell : \bar{e} \to \Sigma'' \). Finally, by theorem 3.2 each pre-cover \( \Sigma'' \) can be extended to a genuine finite-sheeted cover \( \bar{\Sigma} \).

**Corollary 4.2** Let \( X \) be a graph of spaces satisfying the hypotheses of theorem 4.1 and such that, furthermore, for each edge space \( e \) of \( X \), \( H_1(e, \mathbb{Z}) \) is an infinite direct factor of \( H_1(X, \mathbb{Z}) \). Then \( \pi_1(X) \) is subgroup separable.

**Proof.** Fix a base-point \( x \in X \). Let \( H \subset \pi_1(X) \) be a finitely generated subgroup and fix a representative of \( g \in \pi_1(X) \setminus H \). Let \( (X^H, x') \to (X, x) \) be the covering corresponding to \( H \). Since \( H \) is finitely generated, there exists a sub-graph of spaces \( X' \subset X^H \) with finite underlying graph so that \( X' \) contains the (based) lift \( g' \) of the representative of \( g \). By theorem 4.1 there exists a finite-sheeted, intermediate pre-cover
\[(X', x') \to (\bar{X}, \bar{x}) \to (X, x)\]
into which the image of \( g' \) embeds. By proposition 3.9 there exists a finite-sheeted covering
\[\hat{X} \to \bar{X}\]
that extends to a finite-sheeted covering
\[\tilde{X} \to X.\]
As in the proof of corollary 3.8, it follows that the end-point of the lift of the representative of \( g \) to \( \tilde{X} \) doesn’t coincide with the end-point of the lift of any representative of any \( h \in H \).

□

5 Elementarily free groups

5.1 \( \omega \)-residually free towers

Elementarily free groups are an important class of limit groups. For an introduction to the theory of limit groups, see \[2\]. The simplest definition of a limit group is in terms of the property of being \( \omega \)-residually free.

Definition 5.1 Fix a finitely generated free group \( F \) of rank at least 2. A group \( G \) is residually free if, for any \( g \in G \), there exists a homomorphism \( f : G \to F \) so that \( f(g) \neq 1 \). A finitely generated group \( G \) is a limit group, or \( \omega \)-residually free, if for any finite subset \( S \subset G \) there exists a homomorphism \( f : G \to F \) such that \( f|_S \) is injective.

Attempts to solve Tarski’s problems on the elementary theory of free groups have led to a comprehensive structure theory for limit groups, most easily stated in terms of \( \omega \)-residually free towers.

Definition 5.2 A tower space of height 0, denoted \( X_0 \), is a one-point union of finitely many compact graphs, tori, and closed hyperbolic surfaces of Euler characteristic less than -1.

A tower space of height \( h \), denoted \( X_h \), is obtained from a tower space \( X_{h-1} \) of height \( h-1 \) by attaching one of two sorts of blocks.

1. Quadratic block. Let \( \Sigma \) be a connected compact hyperbolic surface with boundary, with each component either a punctured torus or having \( \chi \leq -2 \). Then \( X_h \) is the quotient of \( X_{h-1} \sqcup \Sigma \) obtained by identifying the boundary components of \( \Sigma \) with curves on \( X_{h-1} \), in such a way that there exists a retraction \( \rho : X_h \to X_{h-1} \). The retraction is also required to satisfy the property that \( \rho_* (\pi_1(\Sigma)) \) is non-abelian.

2. Abelian block. Let \( T \) be an \( n \)-torus, and fix a coordinate circle \( c \). Fix a loop \( \gamma \) in \( X_{h-1} \) that generates a maximal abelian subgroup in \( \pi_1(X_{h-1}) \). Then \( X_h \) is the quotient of \( X_{h-1} \sqcup (S^1 \times [0,1]) \sqcup T \) obtained by identifying \( S^1 \times \{0\} \) with \( \gamma \) and \( S^1 \times \{1\} \) with \( c \).
A tower space is called hyperbolic if no tori are used in its construction.

**Definition 5.3** An \((\omega\text{-residually free})\) tower of height \(h\), denoted \(L_h\), is the fundamental group of a tower space of height \(h\).

A tower space \(X_h\) has a natural graph-of-spaces decomposition \(\Gamma_X\) with two vertex spaces, namely \(X_{h-1}\) and the block at height \(h\); the edge spaces are circles. By the Seifert–van Kampen Theorem, towers have a corresponding graph-of-groups decomposition.

The following deep theorem of Sela (see [21]) will, for our purposes, serve as a definition of elementarily free groups.

**Theorem 5.4** A group is elementarily free if and only if it is the fundamental group of a hyperbolic tower space.

Towers are limit groups. Another theorem of Sela [23] and, independently, O. Kharlampovich and A. Myasnikov [12], shows how towers provide a structure theory for limit groups.

**Theorem 5.5** A group is a limit group if and only if it is a finitely generated subgroup of an \((\omega\text{-residually free})\) tower.

One consequence of this theorem is that all abelian subgroups of limit groups are finitely generated. For elementarily free groups, one can do better.

**Lemma 5.6** (Cf. Lemma 2.1 of [22]) Every abelian subgroup of an elementarily free group is cyclic.

A key feature of the definition of a tower is the retraction \(\rho : X_h \to X_{h-1}\). In the abelian case, the retraction simply projects \(T\) onto the coordinate circle \(c\), and thence to \(\gamma\). In both cases, \(\rho\) induces a retraction \(\rho_* : L_h \to L_{h-1}\) on the level of fundamental groups.

We will often use the retraction to pull finite covers back from \(X_{h-1}\) to \(X_h\). It is worth noting that such pullbacks inherit from \(X_h\) a similar graph-of-spaces decomposition.

**Lemma 5.7** Let \(X\) be a complex with a graph-of-spaces decomposition \(\Gamma_X\), such that there is a retraction \(\rho : X \to V\) to a vertex space. Let \(\hat{V} \to V\) be a connected covering of degree \(d\), and let \(\hat{X} \to X\) be the connected covering obtained by pulling back along \(\rho\). Then:
1. \( \hat{X} \to X \) is of degree \( d \) and inherits a graph-of-spaces decomposition \( \hat{\Gamma}_X \);
2. the pre-image of \( V \) in \( \hat{X} \) is a (connected) vertex space of \( \hat{\Gamma}_X \) homeomorphic to \( \hat{V} \);
3. \( \hat{X} \to X \) extends \( \hat{V} \to V \) and \( \hat{X} \) inherits a retraction to \( \hat{V} \) covering \( \rho \).

5.2 Simplifying the gluing maps

The aim is to apply corollary 4.2 to prove that elementarily free groups are subgroup separable. We therefore need to simplify the gluing maps of the tower spaces by passing to finite-sheeted covers so that the hypotheses of corollary 4.2 are satisfied. First, we show how to satisfy the homological conditions.

**Lemma 5.8** If \( X \) is a connected complex with \( L = \pi_1(X) \) residually free and \( \gamma : S^1 \to X \) is a loop in \( X \) then there exists a finite-sheeted covering \( \hat{X} \to X \) so that every elevation of \( \gamma \) to \( \hat{X} \) generates an infinite direct factor in \( H_1(\hat{X}) \).

**Proof.** Fix a base-point in \( X \) and without loss of generality assume that \( \gamma \) is a based loop representing an element of \( L \). Since \( L \) is residually free, there exists a homomorphism \( f : L \to F \) with \( f(\gamma) \neq 1 \). By corollary 3.4 there exists a finite-index subgroup \( F' \subset F \) containing \( f(\gamma) \) such that \( \langle f(\gamma) \rangle \) is a free factor in \( F' \); in particular, \( \langle f(\gamma) \rangle \) is a direct factor in \( H_1(F') \). Let \( \hat{X} \to X \) be the covering corresponding to the subgroup \( f^{-1}(F') \). Every elevation \( \hat{\gamma} \) of \( \gamma \) to \( \hat{X} \) corresponds to a conjugate of a power of \( \gamma \). Since \( f(\gamma) \) is primitive and has infinite order in \( H_1(F') \), it follows that \( \hat{\gamma} \) generates an infinite direct factor in \( H_1(\hat{X}) \). \( \square \)

It remains to ensure that the gluing maps are injective. It will prove useful that cyclic subgroups of towers are separable. This can easily be proved by induction on height, but it also follows from the stronger result that maximal abelian subgroups of limit groups are closed in the pro-free topology.

**Lemma 5.9** Let \( L \) be residually free, \( A \subset L \) a maximal abelian subgroup and \( g \in L \setminus A \). Then there exists a homomorphism \( f : L \to F \) so that \( f(g) \notin f(A) \).

Furthermore, if \( L \) is an elementarily free group, \( A \subset L \) is any cyclic subgroup and \( g \in L \setminus A \) then there exists \( f : L \to F \) with \( f(g) \notin f(A) \).

It follows by theorem 3.3 that \( A \) is separable.

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Proof. Choose some \( a \in A \) such that \([g, a] \neq 1\). Then there exists a homomorphism \( f : A \to F \) so that \( f([g, a]) \neq 1\). In particular, \( f(g) \notin f(A)\).

If \( L \) is elementarily free then by lemma 5.6, \( A \) is contained in some maximal abelian, cyclic subgroup \( A' \subset L \) and we are reduced to the case \( g \in A' \setminus A \). Let \( a' \) generate \( A' \). Then \( f : L \to F \) with \( f(a') \neq 1 \) is as required. \( \square \)

We will use this property to desingularize the gluing curves, but first we need a little geometry of non-positive curvature. For the definition of and general references on CAT(0) geodesic metric spaces see \[3\]. Such spaces are contractible and geodesics are unique (corollary II.1.5 and proposition II.1.4 of \[3\]). A metric space in which every point has a CAT(0) neighbourhood is called non-positively curved. We will make use of the following fact, which follows from the results of pages 229 to 231 of \[3\].

**Lemma 5.10** Let \( \gamma : S^1 \to X \) be a loop in a compact, non-positively curved metric space \( X \). Then \( \gamma \) is freely homotopic to a local geodesic \( \gamma_0 \). The length of \( \gamma_0 \) is minimal in the homotopy class of \( \gamma \). It follows that, if \( l(\gamma) \) denotes the length of \( \gamma \),

\[
l(\gamma^n_0) = |n|l(\gamma_0)
\]

for any integer \( n \).

That tower spaces can be endowed with non-positively curved metrics follows immediately from, for example, proposition II.11.13 of \[3\]. More generally, Emina Alibegovic and Mladen Bestvina \[1\] showed that all limit groups act geometrically on a CAT(0) space with the isolated flats property.

**Lemma 5.11** Let \( X \) be a compact, connected, non-positively curved metric space such that cyclic subgroups of \( \pi_1(X) \) are separable and suppose \( \gamma : S^1 \to X \) is a homotopically non-trivial continuous map. Then, after modifying \( \gamma \) by a homotopy, there exists a finite-sheeted covering \( \hat{X} \to X \) such that every elevation of \( \gamma \) to \( \hat{X} \) is injective.

**Proof.** Since \( X \) is compact we can assume that balls of radius less than some \( \epsilon > 0 \) are CAT(0). After a homotopy, \( \gamma \) can be assumed to be a local isometry by lemma 5.10. Parametrizing by arc length we view \( \gamma \) as a map \([0, l] \to X \) with \( \gamma(0) = \gamma(l) \). Let \( D(\gamma) \subset [0, l] \) consist of those \( t \in [0, l] \) such that there exists \( s < t \) with \( \gamma(s) = \gamma(t) \). Since balls of radius \( \epsilon > 0 \) are CAT(0), if \( \gamma(s) = \gamma(t) \) then either \( s = t \) or \( |s - t| \geq \epsilon \).
Note that $D(\gamma)$ is closed. For, suppose $t_n \in D(\gamma)$ and $t_n \to t$. For each $n$ there exists $s_n \leq t_n - \epsilon$ with $\gamma(s_n) = \gamma(t_n)$. Passing to a subsequence, the $s_n$ converge to some $s \leq t - \epsilon$ with $\gamma(s) = \gamma(t)$.

Set $t_0 = \min D(\gamma)$ and without loss of generality take $\gamma(t_0)$ as a base-point for $\pi_1(X)$. Then $\gamma = \gamma_1 \gamma_2$ where $\gamma_1$ is an embedded closed path. The length of $\gamma_1$ is less than the length of $\gamma$, which is the shortest element of $\{\gamma^n | n \in \mathbb{Z}\}$ by lemma 5.10 so $\gamma_1 \notin \langle \gamma \rangle$. Using that $\langle \gamma \rangle$ is separable, therefore, there exists a finite-sheeted cover $X'$ of $X$ to which $\gamma$ lifts but $\gamma_1$ doesn’t. In particular, $\min D(\gamma') > t_0$. Furthermore, since the ball $B(\gamma'(t_0), \epsilon) \subset X'$ is CAT(0) and isometric to $B(\gamma(t_0), \epsilon) \subset X$ it follows that $\min D(\gamma') > t_0 + \epsilon$. Repeating inductively gives a finite-sheeted covering $X' \to X$ and a lift $\gamma' : S^1 \to X'$ with $D(\gamma') = \{l\}$; that is, $\gamma'$ is an embedding.

Let $\hat{X} \to X$ be the covering defined by

$$\pi_1(\hat{X}) = \bigcap_{g \in \pi_1(X)} \pi_1(X')^g,$$

a finite-sheeted, regular covering. Then every elevation $\hat{\gamma}$ to $\hat{X}$ of $\gamma$ descends to $\gamma'$ after composition with a covering automorphism of $\hat{X}$. Since elevations of embeddings are injective, it follows that $\hat{\gamma}$ is injective. \qed

**Proposition 5.12** Let $X_h$ be a tower space, constructed as above by attaching a quadratic block $\Sigma$ to a space $X_{h-1}$ of height $h-1$. Then, after modifying the gluing maps by a homotopy, there exists a connected covering $Y_h \to X_h$ with an inherited graph-of-spaces decomposition $\Gamma_Y$, with one vertex space given by a connected covering $Y_{h-1} \to X_{h-1}$ and the remaining vertex spaces given by connected coverings $\Sigma_i \to \Sigma$, so that each edge space of $Y_h$ generates an infinite direct factor in $H_1(Y_h)$ and all the gluing maps are injective.

**Proof.** Let $\gamma_i$ be the gluing curves in $X_{h-1}$. By lemma 5.8, for each $i$ there exists a finite-sheeted covering $\hat{X}_i \to X_{h-1}$ so that every elevation of $\gamma_i$ to $\hat{X}_i$ is primitive and of infinite order in homology. By lemma 5.11 for each $i$ there exists a finite-sheeted covering $\hat{X}_i \to X_{h-1}$ so that every elevation of $\gamma_i$ to $\hat{X}_i$ is injective. Let $Y_{h-1} \to X_{h-1}$ be the finite-sheeted covering corresponding to the finite-index subgroup

$$\bigcap_i \pi_1(\hat{X}_i) \cap \bigcap_i \pi_1(\hat{X}_i) \subset \pi_1(X_{h-1}).$$
Then the covering \( Y_h \to X_h \) obtained by pulling \( Y_{h-1} \to X_{h-1} \) back along the retraction is as required. □

We are, finally, in a position to prove the main theorem.

**Theorem 5.13** Every elementarily free group is subgroup separable.

**Proof.** The proof is by induction on height. Let \( X_h \) be a tower space of height \( h \). If \( h = 0 \) then the result follows by corollaries 3.3 and 3.8 and theorem 3.5. For the general case, let \( X_h \) be constructed as in definition 5.2 by gluing a quadratic block \( \Sigma \) to a space \( X_{h-1} \) of height \( h - 1 \). By induction, assume \( \pi_1(X_{h-1}) \) is subgroup separable. By proposition 5.12 there exists a finite-sheeted cover \( Y_h \) of \( X_h \) constructed by gluing surfaces \( \Sigma_i \) to a finite-sheeted cover \( Y_{h-1} \) of \( X_{h-1} \), and such that all the gluing maps are injective and boundary components generate infinite direct factors in homology. Applying corollary 4.2 to each \( \Sigma_i \) in turn it follows that \( \pi_1(Y_h) \), and hence \( \pi_1(X_h) \), is subgroup separable. □

6 A less direct proof

In this section we outline a less direct proof. In essence, we take a more extreme finite-sheeted covering of the tower space that forces the attached quadratic blocks to be of positive genus and then apply Gitik’s theorem. The arguments used by Bridson, Tweedale and myself in [4] amount to the following theorem.

**Theorem 6.1 (Proof of theorem 2.6 of [4])** Let \( \mathcal{C} \) be a class of finitely generated groups with the following properties.

1. All free groups and hyperbolic surface groups lie in \( \mathcal{C} \).
2. \( \mathcal{C} \) is closed under taking subgroups.
3. If \( H \in \mathcal{C} \) and \( H \) is a finite-index subgroup of \( G \) then \( G \in \mathcal{C} \).
4. If \( G_1, G_2 \in \mathcal{C} \) then \( G_1 \ast G_2 \in \mathcal{C} \).
5. Let \( \Sigma \) be a positive-genus, hyperbolic surface with one boundary component and \( G \in \mathcal{C} \). Then

\[
G \ast_{\pi_1(\partial \Sigma)} \pi_1(\Sigma) \in \mathcal{C}.
\]
Then every elementarily free group lies in $\mathcal{C}$.

In [7], Gitik proved the following.

**Theorem 6.2 (Gitik [7])** If $G$ is subgroup separable, $F$ is free and $f \in F$ has no proper roots, then $G \ast \langle f \rangle F$ is subgroup separable.

**Alternative proof of theorem 6.1** Let $\mathcal{C}$ be the class of subgroup separable groups. Property 1 of theorem 6.1 is satisfied by corollaries 3.3 and 3.8. It is immediate from the definition that $\mathcal{C}$ is closed under taking subgroups. Lemma 1.4 gives property 3 and theorem 3.5 gives property 4. Finally, property 5 follows from theorem 6.2. So, by theorem 6.1 the class $\mathcal{C}$ contains all elementarily free groups. $\square$

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