Dynamical and qKZ Equations Modulo $p^n$: an Example

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Abstract—We consider an example of the joint system of dynamical differential equations and qKZ difference equations with parameters corresponding to equations for elliptic integrals. We solve this system of equations modulo any power $p^n$ of a prime integer $p$. We show that the $p$-adic limit of these solutions as $n \to \infty$ determines a sequence of line bundles, each of which is invariant with respect to the corresponding dynamical connection, and that the sequence of line bundles is invariant with respect to the corresponding qKZ difference connection.

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1. INTRODUCTION

Let $z = (z_1, z_2)$, and let

$$\Phi(t; z; \lambda, \mu) = t^{-\lambda}(t - z_1)^{-\mu}(t - z_2)^{-\mu},$$

where $\lambda, \mu$ are rational numbers. Consider the column vector

$$I^{(C)}(z; \lambda, \mu) = \int_C \Phi(t; z; \lambda, \mu) \ dt,$$

where $C \subset \mathbb{C} - \{0, z_1, z_2\}$ is a contour on which the integrand takes its initial value when $t$ encircles $C$. As a function of $z$, the vector $I^{(C)}(z; \lambda, \mu)$ extends to a multi-valued analytic function on $\{a \in \mathbb{C}^2 \mid a_1 a_2 (a_1 - a_2) \neq 0\}$.

The function $I^{(C)}(z; \lambda, \mu)$ satisfies the differential and difference equations

$$z_1 \frac{\partial I}{\partial z_1}(z; \lambda, \mu) = \left(\begin{array}{cc} -\lambda - \mu & -\mu \\ 0 & 0 \end{array}\right) + \frac{\mu z_1}{z_1 - z_2} \left(\begin{array}{cc} -1 & 1 \\ 1 & -1 \end{array}\right) I(z; \lambda, \mu),$$

$$z_2 \frac{\partial I}{\partial z_2}(z; \lambda, \mu) = \left(\begin{array}{cc} 0 & 0 \\ -\mu & -\lambda - \mu \end{array}\right) + \frac{\mu z_2}{z_2 - z_1} \left(\begin{array}{cc} -1 & 1 \\ 1 & -1 \end{array}\right) I(z; \lambda, \mu),$$

$$I(z; \lambda + 1, \mu) = \left[\begin{array}{cc} \frac{\lambda + \mu}{z_1} & \frac{\mu}{z_1} \\ \frac{\mu}{z_2} & \frac{\lambda + \mu}{z_2} \end{array}\right] I(z; \lambda, \mu).$$

If $\lambda, \mu, \lambda + 2\mu \notin \mathbb{Z}_{>0}$, then all solutions of these equations are given by the integrals $I^{(C)}(z; \lambda, \mu)$ (with different choices of $C$) up to multiplication by a scalar 1-periodic function of $\lambda$.

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Up to a gauge transformation, Eqs. (2) and (3) are the simplest example of the trigonometric KZ differential equations and dynamical difference equations. They are also the simplest example of the dynamical differential equations and qKZ difference equations up to a gauge transformation. Equations (2) and (3) are the equivariant quantum differential equations and qKZ difference equation associated with the cotangent bundle of projective line. The family of functions $I^{(C)}(z; \lambda, \mu)$, labeled by contours $C$, are the hypergeometric solutions of these equations constructed in [1]. In particular, see the integral $I^{(C)}(z; \lambda, \mu)$ (gauche transformed) in [1, Section 7.4]. We call Eqs. (2) the dynamical differential equations and Eq. (3) the qKZ difference equation. This pair of equations in enumerative geometry is discussed, for example, in [2]–[4].

In this paper, we discuss polynomial solutions of Eqs. (2) and (3) modulo $p^n$, where $p$ is a prime integer and $n$ is a positive integer. We also discuss the $p$-adic limit of these solutions as $n \to \infty$.

More precisely, we consider the following problem. For $\lambda_0 \in \mathbb{Q}$, let $\Lambda(\lambda_0) = \{\lambda_0 + l \mid l \in \mathbb{Z}\}$ be the arithmetic sequence with initial term $\lambda_0$ and step 1. For a positive integer $\ell$, let $\Lambda(\lambda_0, \ell) = \{\lambda_0 + l \mid l \in \mathbb{Z}, |\lambda_0 + l| < \ell\}$ be an interval of the sequence $\Lambda$.

**Problem.** Let $p$ be a prime integer, $\lambda_0, \mu_0 \in \mathbb{Q}$, and $\ell \in \mathbb{Z}_{>0}$. For any $n \in \mathbb{Z}_{>0}$, find a sequence of column vectors

$$\{I(z; \lambda; \ell; n) = (I_1(z; \lambda; \ell; n), I_2(z; \lambda; \ell; n)) \mid \lambda \in \Lambda(\lambda_0, \ell)\}$$

such that

(i) the coordinates of these vectors are polynomials in $z$ with integer coefficients;

(ii) each of the vectors $I(z; \lambda; \ell; n)$ satisfies modulo $p^n$ the differential equations (2) with parameter $\mu = \mu_0$;

(iii) this sequence of vectors satisfies modulo $p^n$ the difference equation (3) with parameter $\mu = \mu_0$.

We may also require that the vectors are functorial in the following sense. If $\lambda \in \Lambda(\lambda_0, \ell_0)$ for some $\ell_0$, then the vector $I(z; \lambda; \ell; n)$ does not depend on $\ell$ for $\ell \geq \ell_0$. Having a solution $\{I(z; \lambda; \ell; n)\}$ of this problem, we may study the $p$-adic limit of the vectors as $n \to \infty$.

In this paper, we construct a solution of this problem for

$$\lambda_0, \mu_0 = \left(\frac{1}{2}, \frac{1}{2}\right).$$

We also describe the $p$-adic limit of our solution. It turns out that the limit is not a solution of Eqs. (2) and (3) over the $p$-adic field, as one may naively think, but a line bundle invariant with respect to the dynamical connection defined by Eqs. (2) and invariant with respect to the discrete qKZ connection defined by Eq. (3), see Theorem 6. Note that there is no such line bundle if we consider the same differential and difference equations over the field of complex numbers, see Section 5.

The choice of parameters in (5) corresponds to elliptic integrals in (1). This choice is technically, arithmetically easier than the choice of an arbitrary pair $(\lambda_0, \mu_0)$ of rational numbers, although a similar construction can be performed for a wide class of parameters $(\lambda_0, \mu_0)$.

Quantum differential equations and associated qKZ difference equations, as well as their solutions, are a mathematical structure with applications in representation theory, algebraic geometry, and theory of special functions, to name a few. It would be of interest to study how the properties of these equations and their solutions are reflected in the solutions of the same equations modulo powers of a prime integer and in their $p$-adic limits.

In Section 2, we reformulate Eqs. (2) and (3) for $\lambda_0, \mu_0 = (1/2, 1/2)$; see Eqs. (7) and (8). In Section 3, we solve Eqs. (7) and (8) modulo a power $p^s$ of an odd prime integer $p$. The construction of these solutions is a version of the constructions in [5], [6]. The constructed solutions are called the $p^s$-hypergeometric solutions. We prove Dwork-type congruences for these solutions in Section 4. Using these congruences, we describe the $p$-adic limit of our solutions as $s \to \infty$ in Section 5.

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2. EQUATIONS FOR \((\lambda_0, \mu_0) = (1/2, 1/2)\)

Denote

\[
H_1(z; \lambda) = \begin{bmatrix}
-\lambda - 1 & -1 \\
0 & 0
\end{bmatrix} + \frac{z_1}{z_1 - z_2} \begin{bmatrix}
-1 & 1 \\
1 & -1
\end{bmatrix},
\]

\[
H_2(z; \lambda) = \begin{bmatrix}
0 & 0 \\
-1 & -\lambda - 1
\end{bmatrix} + \frac{z_2}{z_2 - z_1} \begin{bmatrix}
-1 & 1 \\
1 & -1
\end{bmatrix},
\]

\[
K(z; \lambda) = \begin{bmatrix}
\frac{\lambda + 1}{z_1} & \frac{1}{z_1} \\
\frac{z_1 + \lambda}{z_2} & \frac{z_1 + \lambda}{z_2}
\end{bmatrix}.
\]

The substitution \((\lambda, \mu) \to (\lambda/2, 1/2)\) transforms system (2), (3) into the following system of equations for a column vector \(I(z; \lambda)\):

\[
2z_i \frac{\partial I(z; \lambda)}{\partial z_i} = H_i(z; \lambda)I(z; \lambda), \quad i = 1, 2; \quad I(z; \lambda + 2) = K(z; \lambda)I(z; \lambda).
\]

Denote

\[
D_i(\lambda) := 2z_i \frac{\partial}{\partial z_i} - H_i(z; \lambda), \quad i = 1, 2.
\]

3. SOLUTIONS MODULO POWERS OF \(p\)

3.1. Notation

In this paper, \(p\) is an odd prime integer. In this paper, we consider system (7), (8) for the values of \(\lambda\) from the arithmetic sequence of odd integers, \(\Lambda := 1 + 2\mathbb{Z}\).

Given a positive integer \(s\), denote

\[
\Lambda_s = \{\lambda \in 1 + 2\mathbb{Z} \mid -p^s < \lambda < p^s\},
\]

an interval of the arithmetic sequence \(\Lambda\).

- For \(\lambda \in \Lambda_s\), we have \(0 < (p^s - \lambda)/2 < p^s\).
- For a positive integer \(e\), if \(s > e\) and \(\lambda \in \Lambda_e\), then \(|(p^s - \lambda)/2|_p > p^{-e}\), where \(|x|_p\) denotes the \(p\)-adic norm of a rational number \(x\).

For a polynomial \(f(t)\), denote by \(\{f(t)\}_s\) the coefficient of \(t^{p^s-1}\) in \(f(t)\). For a function \(g(z)\), denote by \(\text{grad}_z g\) the column gradient vector \((\partial g/\partial z_1, \partial g/\partial z_2)\).

3.2. Solutions

For \(\lambda \in \Lambda_s\), define the master polynomial

\[
\Phi_s(t; z; \lambda) := t^{(p^s - \lambda)/2}(t - z_1)^{(p^s - 1)/2}(t - z_2)^{(p^s - 1)/2}.
\]

Define the column vector

\[
\Psi_s(t; z; \lambda) = (\Psi_{s,1}(t; z; \lambda), \Psi_{s,2}(t; z; \lambda)) := \Phi_s(t; z; \lambda) \begin{bmatrix}
1/\left(t - z_1\right) \\
1/\left(t - z_2\right)
\end{bmatrix}.
\]

The coordinates of \(\Psi_s\) are polynomials in \(t, z\) with integer coefficients. We denote

\[
I_s(z; \lambda) = (I_{s,1}(z; \lambda), I_{s,2}(z; \lambda)) := \{\Psi_s(t; z; \lambda)\}_s,
\]

the coefficient of \(t^{p^s-1}\) in \(\Psi_s(t; z; \lambda)\), and

\[
T_s(z; \lambda) := \{\Phi_s(t; z; \lambda)\}_s.
\]
the coefficient of $t^p - 1$ in $\Phi_s(t; z; \lambda)$. For every $\lambda \in \Lambda_s$, the functions $I_{s,1}, I_{s,2}, T_s$ are polynomials in $z_1, z_2$ with integer coefficients.

We have

$$\frac{1 - p^s}{2} I_s(z; \lambda) = \text{grad}_z T_s(z; \lambda).$$

(9)

**Theorem 1.** Let $s \in \mathbb{Z}_{>0}$, $\lambda \in \Lambda_s$, $i = 1, 2$. Then the vector $I_s(z; \lambda)$ satisfies the congruence

$$\mathcal{D}_i(\lambda) I_s(z; \lambda) \equiv 0 \pmod{p^s}.$$  

(10)

**Proof.** We have

$$\frac{\partial \Phi_s}{\partial t} = \left(\frac{\partial^2}{\partial t^2} - \frac{p^s - 1}{2(t - z_1)} + \frac{p^s - 1}{2(t - z_2)}\right) \Phi_s,$$

(11)

Also

$$- \frac{\partial \Phi_{s,1}}{\partial t} = \Phi_s \left(- \frac{p^s - \lambda}{2t(t - z_1)} - \frac{p^s - 3}{2(t - z_1)^2} - \frac{p^s - 1}{2(t - z_1)(t - z_2)}\right)$$

$$- \Phi_s \left(\frac{p^s - \lambda}{2z_1} \left[\frac{1}{t - z_1} - \frac{1}{t - z_1}\right] - \frac{1}{2(t - z_1)^2} - \frac{1}{2z_1 - 2} \left[\frac{1}{t - z_1} - \frac{1}{t - z_2}\right]\right)\Phi_s.$$

Hence

$$\frac{\partial \Phi_s}{\partial t} = \Phi_s \left(- \frac{p^s - 3}{2(t - z_1)^2} = \frac{\partial}{\partial t} \left(\frac{\Phi}{t - z_1}\right)\right)$$

$$+ \Phi_s \left(\frac{p^s - \lambda}{2z_1} \left[\frac{1}{t - z_1} - \frac{1}{t - z_1}\right] + \frac{p^s - 1}{2z_1 - 2} \left[\frac{1}{t - z_1} - \frac{1}{t - z_2}\right]\right)$$

$$- \frac{\partial}{\partial t} \left(\frac{\Phi_s}{t - z_1}\right) + \frac{\partial}{\partial z_1} \left(- \frac{1}{z_1} \frac{\partial \Phi_s}{\partial t}\right)$$

$$+ \frac{\partial}{\partial z_1} \left(\frac{2p^s - \lambda - 1}{2z_1} \frac{1}{t - z_1} + \frac{p^s - 1}{2z_1 - 2} \left[\frac{1}{t - z_1} - \frac{1}{t - z_2}\right]\right).$$

We have

$$\left\{\frac{\partial}{\partial t} \left(\frac{\Phi_s}{t - z_1}\right)\right\}_{s} \equiv 0, \quad \left\{\frac{\partial \Phi_s}{\partial t}\right\}_{s} \equiv 0, \quad \frac{2p^s - \lambda - 1}{2} \equiv -\frac{\lambda + 1}{2}, \quad \frac{2p^s - 1}{2} \equiv -\frac{1}{2}$$

(12)

modulo $p^s$. Hence

$$2z_1 \frac{\partial I_{s,1}}{\partial z_1} \equiv - (\lambda + 1) I_{s,1} - I_{s,2} + \frac{z_1}{z_1 - z_2} (I_{s,1} - I_{s,2}) \pmod{p^s}.$$  

We also have

$$\frac{\partial \Psi_{s,2}}{\partial z_1} = - \frac{p^s - 1}{2} \frac{\Phi_s}{(t - z_1)(t - z_2)} = - \frac{p^s - 1}{2} \frac{\Phi_s}{z_1 - z_2} \left[\frac{1}{t - z_1} - \frac{1}{t - z_2}\right].$$

Hence

$$2z_1 \frac{\partial I_{s,2}}{\partial z_1} \equiv \frac{z_1}{z_1 - z_2} (I_{s,1} - I_{s,2}) \pmod{p^s}.$$  

This proves Eq. (10) for $i = 1$. Equation (10) for $i = 2$ can be proved in a similar way.
\textbf{Theorem 2.} Let \( s > e \) be positive integers, and let \( \lambda, \lambda + 2 \in \Lambda_e \). Then the vector \( I_s(z; \lambda) \) satisfies the congruence
\[
I(z; \lambda + 2) \equiv K(z; \lambda) I(z; \lambda) \pmod{p^{s-e}}.
\]

\textbf{Proof.} Equation (11) can be written as
\[
- \frac{\Phi_s}{t} = - \frac{2}{p^s - \lambda} \frac{\partial \Phi_s}{\partial t} + \frac{p^s - 1}{p^s - \lambda} \left( \frac{1}{t - z_1} + \frac{1}{t - z_2} \right) \Phi_s.
\]
Hence
\[
\Psi_{s,1}(z; \lambda + 2) = \frac{\Phi_s(z; \lambda)}{t(t - z_1)} = - \frac{\Phi_s(z; \lambda)}{z_1} \left[ \frac{1}{t - z_1} \right] = \frac{\Phi_s(z; \lambda)}{z_1(t - z_1)} - \frac{2}{(p^s - \lambda)z_1} \frac{\partial \Phi_s}{\partial t} + \frac{p^s - 1}{p^s - \lambda} \left( \frac{1}{z_1(t - z_1)} + \frac{1}{z_1(t - z_2)} \right) \Phi_s(z; \lambda).
\]

By (12), the term
\[
\left\{ \frac{2}{(p^s - \lambda)z_1} \frac{\partial \Phi_s}{\partial t} \right\}_s
\]
is divisible at least by \( p^{s-e} \). Hence Eq. (14) implies
\[
I_{s,1}(z; \lambda + 2) \equiv \frac{\lambda + 1}{z_1 \lambda} I_{s,1}(z; \lambda) + \frac{1}{z_1 \lambda} I_{s,2}(z; \lambda) \pmod{p^{s-e}}.
\]
In a similar way, we obtain
\[
I_{s,2}(z; \lambda + 2) \equiv \frac{\lambda + 1}{z_2 \lambda} I_{s,2}(z; \lambda) + \frac{1}{z_2 \lambda} I_{s,1}(z; \lambda) \pmod{p^{s-e}}.
\]
The proof of Theorem 2 is complete. □

\textbf{3.3. Formulas for} \( I_{s,1}, I_{s,2}, T_s \)

\textbf{Lemma 1.} For \( \lambda \in \Lambda_s \), we have
\[
T_s(z; \lambda) = (-1)^{(p^s-\lambda)/2} \sum_{k+\ell = \frac{p^s-\lambda}{2}} \left( \frac{p^s-1}{2} \right) \left( \frac{p^s-1}{2} \right) \frac{z_1^k z_2^\ell}{k! \ell!},
\]
\[
I_{s,1}(z; \lambda) = (-1)^{(p^s-\lambda)/2-1} \sum_{k+\ell = \frac{p^s-\lambda}{2} - 1} \left( \frac{p^s-1}{2} - 1 \right) \left( \frac{p^s-1}{2} \right) \frac{z_1^k z_2^\ell}{k! \ell!},
\]
\[
I_{s,2}(z; \lambda) = (-1)^{(p^s-\lambda)/2-1} \sum_{k+\ell = \frac{p^s-\lambda}{2} - 1} \left( \frac{p^s-1}{2} - 1 \right) \left( \frac{p^s-1}{2} \right) \frac{z_1^k z_2^\ell}{k! \ell!}.
\]
□

\textbf{3.4.} \( p \)-\textit{ary Representations}

For \( \lambda \in \Lambda_s \), we have the \( p \)-\textit{ary representations}
\[
\frac{p^s - \lambda}{2} = w_0(\lambda) + w_1(\lambda)p + \cdots + w_{s-1}(\lambda)p^{s-1},
\]
\[
\frac{-\lambda}{2} = w_0(\lambda) + w_1(\lambda)p + \cdots + w_{s-1}(\lambda)p^{s-1} + \frac{p - 1}{2} p^s + \frac{p - 1}{2} p^{s+1} + \cdots
\]
for some integers \( w_i(\lambda), \ 0 \leq w_i(\lambda) \leq p - 1 \). Denote \( w_i(\lambda) = (p - 1)/2 \) for \( i \geq s \).
For \( \lambda \in \Lambda \), denote by \( W(\lambda) \) the set of all distinct integers \( w_i(\lambda) \) in the \( p \)-ary representation of \(-\lambda/2\). The set \( W(\lambda) \) has at most \( p \) elements.

For example, \(-1/2 = (1 + p + \ldots)(p - 1)/2 \) and \( W(1) = \{(p - 1)/2\} \), while
\[
\frac{1}{2} = \frac{p + 1}{2} + \frac{p - 1}{2}(p + p^2 + \ldots)
\]
and \( W(-1) = \{(p + 1)/2, (p - 1)/2\} \).

For \( w = 0, 1, \ldots, p - 1 \), let \( h(z; w) \) (resp. \( g_1(z; w) \), resp. \( g_2(z; w) \)) be the coefficient of \( t^{p-1} \) in \( t^w(t - z_1)(p-1)/2(t - z_2)(p-1)/2 \) (resp. in \( t^w(t - z_1)(p-1)/2 - 1(t - z_2)(p-1)/2 \), resp. in \( t^w(t - z_1)(p-1)/2 - 2(t - z_2)(p-1)/2 \).

### Lemma 2

For \( \lambda \in \Lambda_\ell \), we have
\[
T_s(z; \lambda) \equiv \prod_{i=0}^{s-1} h(z^{p^i}; w_i(\lambda)) \pmod{p}.
\] (15)

The polynomial \( T_s(z; \lambda) \) is nonzero modulo \( p \).

#### Proof.

We have \((p^s - 1)/2 = (1 + p + \cdots + p^{s-1})(p - 1)/2 \). Then
\[
\Phi_s(t; z; \lambda) \equiv \prod_{i=0}^{s-1} (t^{p^i})^{w_i(\lambda)}(t^{p^i} - z_1^{p^i})^{(p-1)/2}(t^{p^i} - z_2^{p^i})^{(p-1)/2} \pmod{p}.
\]
This implies (15). To prove the second statement of the lemma, it suffices to verify that the polynomial \( h(z; w) \) is nonzero modulo \( p \) for \( w = 0, 1, \ldots, p - 1 \). Indeed, there exist nonnegative integers \( k, \ell \) such that \( w = k + \ell \) and \( k, \ell \leq (p - 1)/2 \). Then the coefficient of \( z_1^k z_2^\ell \) in \( h(z; w) \) equals \((-1)^w \frac{p-1}{k} \frac{p-1}{2} \) and is nonzero modulo \( p \) by the Lucas Theorem.

#### Lemma 3

Let \( \lambda \in \Lambda_\ell \) and \( j = 1, 2 \). Then
\[
I_{s,j}(z; \lambda) \equiv g_j(z; w_0(\lambda)) \prod_{i=1}^{s-1} h(z^{p^i}; w_i(\lambda)) \pmod{p}.
\]
If \( \lambda \) is not divisible by \( p \), then \( I_{s,j}(z; \lambda) \) is nonzero modulo \( p \).

#### Proof.

If \( \lambda \) is not divisible by \( p \), then \( w_0(\lambda) > 0 \). Then \( g_j(z; w_0(\lambda)) \) is nonzero modulo \( p \). Also, the polynomials \( h(z; w_i(\lambda)) \) are nonzero modulo \( p \).

### 4. DWORK-TYPE CONGRUENCES

In this section, we apply results from [6] to obtain congruences relating the polynomials \( I_s(z; \lambda) \) and \( T_s(z; \lambda) \) in \( z \) for different \( s \). This type of congruences was originated by B. Wark in [7]; see also [8], [9]. For examples of applications of such congruences, see [10]–[12].

A congruence \( F(x) \equiv G(x) \pmod{p^s} \) for two polynomials in some variables \( x \) with integer coefficients is understood as the divisibility by \( p^s \) of all coefficients of \( F(x) - G(x) \).

Let \( F_1(x), F_2(x), G_1(x), G_2(x) \) be polynomials such that \( F_2(x), G_2(x) \) are both nonzero modulo \( p \). Then the congruence \( F_1(x)/F_2(x) \equiv G_1(x)/G_2(x) \) modulo \( p^s \) is understood as the congruence
\[
F_1(x)G_2(x) \equiv G_1(x)F_2(x) \pmod{p^s}.
\]

Recall the master polynomial
\[
\Phi_s(t; z; \lambda) \equiv t^{(p^s - \lambda)/2}(t - z_1)^{(p^s - 1)/2}(t - z_2)^{(p^s - 1)/2},
\]
in particular,
\[
\Phi_1(t; z; 1) = t^{(p-1)/2} (t - z_1) (t - z_2)^{(p-1)/2}. 
\]
For \( \lambda \in \Lambda_r \), we have
\[
\Phi_s(t; z; \lambda) = \Phi_e(t; z; \lambda) \Phi_1(t; z; 1)^{p^e + p^{e+1} + \cdots + p^{s-1}}. 
\]
(16)
In particular, we have
\[
\Phi_{s-e}(t; z; 1) = \Phi_1(t; z; 1)^{1 + p + \cdots + p^{s-e-1}}. 
\]
For \( \lambda \in \Lambda_r \) and \( s > e \), the Newton polytope of \( \Phi_s(t; z; \lambda) \) with respect to the variable \( t \) is the interval \( [(p^s - \lambda)/2, (p^s - \lambda)/2 + p^s - 1] \).
(17)
For a positive integer \( k \), the point \( kp^s - 1 \) lies in this interval only if \( k = 1 \).
Recall that \( T_s(z; \lambda) = \{ \Phi_s(t; z; \lambda) \}_s \) is the coefficient of \( t^{p^s-1} \) in \( \Psi_s(t; z; \lambda) \).
For \( \lambda \in \Lambda_r \), the polynomial \( T_s(z; \lambda) \) is nonzero modulo \( p \).
(18)
by Lemma 9.
In [6], certain congruences were proved for a sequence of polynomials like \( \Phi_s(z; \lambda) \), \( s \geq e \), with properties like (16), (17), and (18). In the case of the polynomials \( \Phi_s(z; \lambda) \), \( s \geq e \), the congruences in [6] say the following.
For \( j \in \{1, 2\} \), denote \( D_j = \partial/\partial z_j \).

**Theorem 3.** Let \( e \in \mathbb{Z}_{>0} \) and \( \lambda \in \Lambda_r \).

(i) For \( j \in \{1, 2\} \) and \( s > e \), we have
\[
\frac{D_j(T_s(z; \lambda))}{T_s(z; \lambda)} \equiv \frac{D_j(T_{s-1}(z; \lambda))}{T_{s-1}(z; \lambda)} \pmod{p^{s-e}}. 
\]
(ii) For \( i, j \in \{1, 2\} \) and \( s > e \), we have
\[
\frac{D_i(D_j(T_s(z; \lambda)))}{T_s(z; \lambda)} \equiv \frac{D_i(D_j(T_{s-1}(z; \lambda)))}{T_{s-1}(z; \lambda)} \pmod{p^{s-e}}. 
\]
Statements (i)-(ii) are special cases of [6, Theorems 2.8 and 2.9].

**Corollary 1.** Let \( e \in \mathbb{Z}_{>0} \), \( \lambda \in \Lambda_r \), \( s > e \), and \( i, j \in \{1, 2\} \). Then
\[
\frac{I_{s,j}(z; \lambda)}{T_s(z; \lambda)} \equiv \frac{I_{s-1,j}(z; \lambda)}{T_{s-1}(z; \lambda)} \pmod{p^{s-e}}, 
\]
(19)
\[
\frac{\partial I_{s,j}}{\partial z_i}(z; \lambda) \equiv \frac{\partial I_{s-1,j}}{\partial z_i}(z; \lambda) \pmod{p^{s-e}}. 
\]
(20)
The corollary follows from formula (9) and Theorem 3.

**Theorem 4.** Let \( e \in \mathbb{Z}_{>0} \). Let \( \lambda, \lambda + 2 \in \Lambda_r \) and \( s > 2e \). Then
\[
\frac{I_s(z; \lambda + 2)}{T_s(z; \lambda)} \equiv \frac{I_{s-1}(z; \lambda + 2)}{T_{s-1}(z; \lambda)} \pmod{p^{s-2e}}. 
\]
(21)

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Proof. Using formulas (19) and (20) and the fact that \( \lambda \in \Lambda_e \), we obtain
\[
\frac{1}{\lambda} \left[ \begin{array}{ccc} 1+1/z_1 & 1/z_2 & z_1 \\ 1/z_2 & \lambda+1/z_2 & z_2 \\ 1/z_2 & \lambda+1/z_2 & \lambda+1 \\
\end{array} \right] I_s(z; \lambda) = \frac{1}{\lambda} \left[ \begin{array}{ccc} 1+1/z_1 & 1/z_2 & z_1 \\ 1/z_2 & \lambda+1/z_2 & z_2 \\ 1/z_2 & \lambda+1/z_2 & \lambda+1 \\
\end{array} \right] I_s(z; \lambda) \quad \text{(mod} \ p^{s-2e}) \].
\]

Congruence (13) implies the congruences
\[
\frac{I_s(z; \lambda + 2)}{T_s(z; \lambda)} \equiv \frac{1}{\lambda} \left[ \begin{array}{ccc} 1+1/z_1 & 1/z_2 & z_1 \\ 1/z_2 & \lambda+1/z_2 & z_2 \\ 1/z_2 & \lambda+1/z_2 & \lambda+1 \\
\end{array} \right] I_s(z; \lambda) \quad \text{(mod} \ p^{s-e}) ,
\]
\[
\frac{I_{s-1}(z; \lambda + 2)}{T_{s-1}(z; \lambda)} \equiv \frac{1}{\lambda} \left[ \begin{array}{ccc} 1+1/z_1 & 1/z_2 & z_1 \\ 1/z_2 & \lambda+1/z_2 & z_2 \\ 1/z_2 & \lambda+1/z_2 & \lambda+1 \\
\end{array} \right] I_{s-1}(z; \lambda) \quad \text{(mod} \ p^{s-e-1}) .
\]

These three congruences imply congruence (21). \( \square \)

5. CONVERGENCE

5.1. Unramified Extensions of \( \mathbb{Q}_p \)

We fix an algebraic closure \( \overline{\mathbb{Q}_p} \) of \( \mathbb{Q}_p \). For every \( m \), there exists a unique unramified extension of \( \mathbb{Q}_p \) in \( \overline{\mathbb{Q}_p} \) of degree \( m \), denoted by \( \mathbb{Q}_p^{(m)} \). This can be obtained by attaching to \( \mathbb{Q}_p \) a primitive root of 1 of order \( p^m - 1 \). The norm \( | \cdot |_p \) on \( \mathbb{Q}_p \) extends to a norm \( | \cdot |_p \) on \( \mathbb{Q}_p^{(m)} \). Let
\[
\mathbb{Z}_p^{(m)} = \{ a \in \mathbb{Q}_p^{(m)} \mid |a|_p \leq 1 \}
\]
denote the ring of integers in \( \mathbb{Q}_p^{(m)} \). The ring \( \mathbb{Z}_p^{(m)} \) has the unique maximal ideal
\[
\mathbb{M}_p^{(m)} = \{ a \in \mathbb{Q}_p^{(m)} \mid |a|_p < 1 \}
\]
such that \( \mathbb{Z}_p^{(m)}/\mathbb{M}_p^{(m)} \) is isomorphic to the finite field \( \overline{\mathbb{F}}_p^m \).

For every \( t \in \mathbb{F}_p^m \), there exists a unique \( \tilde{t} \in \mathbb{Z}_p^{(m)} \) that is a lift of \( t \) such that \( \tilde{p}^m = \tilde{t} \). The element \( \tilde{t} \) is called the Teichmüller lift of \( t \).

5.2. Domain \( \mathcal{D}_B \)

For \( t \in \mathbb{F}_p^m \) and \( r > 0 \), denote
\[
D_{t,r} = \{ a \in \mathbb{Z}_p^{(m)} \mid |a - \tilde{t}|_p < r \} .
\]
We have the partition
\[
\nu_p^{(m)} = \bigcup_{t \in \mathbb{F}_p^m} D_{t,1} .
\]
Recall that \( z = (z_1, z_2) \). For \( B(z) \in \mathbb{Z}[z] \), define
\[
\mathcal{D}_B = \{ a \in (\mathbb{Z}_p^{(m)})^2 \mid |B(a)|_p = 1 \} .
\]
Let \( \bar{B}(z) \) be the projection of \( B(z) \) into \( \mathbb{F}_p[z] \subset \mathbb{F}_p^m[z] \). Then \( \mathcal{D}_B \) is a union of unit polydiscs,
\[
\mathcal{D}_B = \bigcup_{t_1, t_2 \in \mathbb{F}_p^m, B(t_1, t_2) \neq 0} D_{t_1,1} \times D_{t_2,1} .
\]
Lemma 4. For any nonnegative integer $k$, we have
\[
\{ a \in (\mathbb{Z}_p^m)^2 \mid |B(a^{p^k})|_p = 1 \} = \bigcup_{t_1, t_2 \in \mathbb{F}_p^m} D_{t_1, 1} \times D_{t_2, 1}
\]
\[
= \bigcup_{t_1, t_2 \in \mathbb{F}_p^m} D_{t_1, 1} \times D_{t_2, 1} = \mathcal{D}_B.
\]

\[\square\]

Lemma 5 [13, Lemma 6.1]. Let $\bar{B}(z) \in \mathbb{F}_p[z]$ be a nonzero polynomial of degree $d$, and let $d + 1 < p^m$. Then the set $\{ a \in (\mathbb{F}_p^m)^2 \mid B(a) \neq 0 \}$ is nonempty. Moreover, there exist at least
\[
\frac{p^{2m} - 1}{p^m - 1} (p^m - 1 - d) + 1
\]
points of $(\mathbb{F}_p^m)^2$ where $\bar{B}(z)$ is nonzero.

5.3. Domains of Convergence

Recall the polynomials $T_s(z; \lambda)$, $I_{s, 1}(z; \lambda)$, and $I_{s, 2}(z; \lambda)$ as well as the polynomials $h(z; w)$, $g_1(z; w)$, and $g_2(z; w)$. For $\lambda \in \Lambda$, denote
\[
H(z; \lambda) = \prod_{w \in W(\lambda)} h(z; w),
\]
\[
\mathcal{D}^{(m)}(\lambda) = \{ a \in (\mathbb{Z}_p^m)^2 \mid |H(a; \lambda)|_p = 1 \}.
\]
Let $\lambda \in \Lambda$ be not divisible by $p$ (that is, $w_0(\lambda) > 0$). For $j = 1, 2$, denote
\[
G_j(z; \lambda) = g_j(z; w_0(\lambda)) \prod_{w \in W(\lambda)} h(z; w),
\]
\[
\mathcal{D}_s^{(m)}(\lambda) = \{ a \in (\mathbb{Z}_p^m)^2 \mid |G_1(a; \lambda)|_p = 1 \text{ or } |G_2(a; \lambda)|_p = 1 \}.
\]
Denote $\mathcal{C}^{(m)} = \{ a \in (\mathbb{Z}_p^m)^2 \mid a_1a_2 \neq 0 \}$. For $\lambda \in \Lambda$ divisible by $p$, denote
\[
\mathcal{D}_s^{(m)}(\lambda) = \mathcal{D}^{(m)}(\lambda) \cap \mathcal{D}_s^{(m)}(\lambda + 2) \cap \mathcal{C}^{(m)}.
\]
Clearly, for any $\lambda \in \Lambda$, we have
\[
\mathcal{D}_s^{(m)}(\lambda) \subset \mathcal{D}^{(m)}(\lambda)
\]
and
\[
\bigcap_{\lambda \in \Lambda} \mathcal{D}_s^{(m)}(\lambda) \supset \{ a \in (\mathbb{Z}_p^m)^2 \mid |a_1a_2h(a; 0) \prod_{w=1}^{p-1} h(a; w)g_1(a; w)g_2(a; w)|_p = 1 \}.
\]

Lemma 6. If $m \geq 3$, then $\bigcap_{\lambda \in \Lambda} \mathcal{D}_s^{(m)}(\lambda)$ is nonempty.

Proof. We have $\deg_z h(z; w) = w$ and $\deg_z g_1(z; w) = \deg_z g_2(z; w) = w - 1$. Hence the polynomial in (22) has degree $(3p^2 - 7p + 8)/2 < p^3 - 1$. The lemma follows from Lemma 5.

\[\square\]

Theorem 5. For $e \in \mathbb{Z}_{>0}$ and $\lambda \in \Lambda_e$, we have the following statements.

(i) The sequence $(I_s(z; \lambda)/T_s(z; \lambda))_{s \geq e}$ of column vectors, whose entries are rational functions in $z$ regular on $\mathcal{D}^{(m)}(\lambda)$, uniformly converges on $\mathcal{D}^{(m)}(\lambda)$ as $s \to \infty$ to a vector whose entries are analytic functions on $\mathcal{D}^{(m)}(\lambda)$. The vector will be denoted by $\mathcal{I}(z; \lambda) = (\mathcal{I}_1(z; \lambda), \mathcal{I}_2(z; \lambda))$. 

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(ii) The sequence \((I_s(z; \lambda + 2)/T_s(z; \lambda))\) of column vectors whose entries are rational functions in \(z\) regular on \(D^{(m)}(\lambda)\), uniformly converges on \(D^{(m)}(\lambda)\) as \(s \to \infty\) to a vector, whose entries are analytic functions on \(D^{(m)}(\lambda)\). The vector will be denoted by \(\tilde{I}(z; \lambda + 2) = (\tilde{I}_1(z; \lambda + 2), \tilde{I}_2(z; \lambda + 2))\).

(iii) For \(j = 1, 2\), the sequence
\[
\left(\frac{\partial I_s(z; \lambda)}{\partial z_j}/T_s(z; \lambda)\right)_{s \geq e}
\]
of column vectors, whose entries are rational functions in \(z\) regular on \(D^{(m)}(\lambda)\), uniformly converges on \(D^{(m)}(\lambda)\) as \(s \to \infty\) to a vector whose entries are analytic functions on \(D^{(m)}(\lambda)\). The vector will be denoted by \(I^{(i)}(z; \lambda) = (I^{(i)}_1(z; \lambda), I^{(i)}_2(z; \lambda))\).

Proof. Parts (i), (ii), and (iii) follow from the congruences in (19), (20), and (21), respectively.

5.4. Relations between Limit Vectors

Lemma 7. Let \(\lambda \in \Lambda\). We have the following equations on \(D^{(m)}(\lambda)\):
\[
\frac{\partial T}{\partial z_i}(z; \lambda) = I^{(i)}(z; \lambda) - \frac{1}{2} I_1(z; \lambda) I(z; \lambda), \tag{23}
\]
\[
I^{(i)}(z; \lambda) = H_i(z; \lambda) I(z; \lambda), \tag{24}
\]
\[
\frac{\partial T}{\partial z_i}(z; \lambda) = \left(H_i(z; \lambda) - \frac{1}{2} I_1(z; \lambda)\right) I(z; \lambda), \tag{25}
\]
\[
\tilde{I}(z; \lambda + 2) = K(z; \lambda) I(z; \lambda). \tag{26}
\]

Proof. Differentiating the congruence in (19) with respect to \(z_i\) we obtain
\[
\frac{\partial I_s}{\partial z_{si}}/T_s \equiv \frac{\partial I_{s-1}}{\partial z_{si}}/T_{s-1} \equiv \frac{\partial I_{s-1}}{\partial z_{si}}/T_{s-1} \equiv \frac{\partial I_{s-1}}{\partial z_{si}}/T_{s-1} \equiv \frac{\partial I_{s-1}}{\partial z_{si}}/T_{s-1} \quad (\text{mod } p^{s-e}).
\]
This congruence gives Eq. (23) as \(s \to \infty\). Equation (24) follows from the congruences in (10) after dividing by \(T_s(z; \lambda)\) and taking the limit \(s \to \infty\). Equation (25) follows from Eqs. (23) and (24). Equation (26) follows from congruence (13) after dividing by \(T_s(z; \lambda)\) and taking the limit as \(s \to \infty\).

5.5. Limit Vectors

The vector function \(\tilde{I}(z; \lambda + 2) = (\tilde{I}_1(z; \lambda + 2), \tilde{I}_2(z; \lambda + 2))\) is defined on \(D^{(m)}(\lambda)\), and the vector function \(I(z; \lambda + 2) = (I_1(z; \lambda + 2), I_2(z; \lambda + 2))\) is defined on \(D^{(m)}(\lambda + 2)\), see Theorem 5.

Lemma 8. The vector functions \(\tilde{I}(z; \lambda + 2)\) and \(I(z; \lambda + 2)\) are proportional on the intersection \(D^{(m)}(\lambda) \cap D^{(m)}(\lambda + 2)\); i.e.,
\[
\tilde{I}_1(z; \lambda + 2) I_2(z; \lambda + 2) - \tilde{I}_2(z; \lambda + 2) I_1(z; \lambda + 2) = 0.
\]

Proof. To obtain \(\tilde{I}(z; \lambda + 2)\), we divide the vector \(I_s(z; \lambda + 2)\) by \(T_s(z; \lambda)\) and take the limit as \(s \to \infty\). To obtain \(I(z; \lambda + 2)\), we divide the same vector \(I_s(z; \lambda + 2)\) by \(T_s(z; \lambda + 2)\) and take the limit as \(s \to \infty\). Hence the limits are proportional.

Lemma 9. Let \(e \in \mathbb{Z}_{>0}\) and \(\lambda \in \Lambda_e\).

(i) Assume that \(a \in D^{(m)}(\lambda)\). Then \(|T_s(a; \lambda)|_p = 1\) for any \(s \geq e\).
Assume that $\lambda$ is not divisible by $p$ and $a \in \mathfrak{D}_s^{(m)}(\lambda)$. Then there exists a $j \in \{1, 2\}$ such that $|I_s,j(a; \lambda)|_p = 1$ for any $s \geq e$.

**Proof.** The lemma follows from Lemmas 2 and 3.  

**Lemma 10.**

(i) Let $\lambda \in \Lambda$ be not divisible by $p$, and let $a \in \mathfrak{D}_s^{(m)}(\lambda)$. Then there exists a $j \in \{1, 2\}$ such that $|I_s,j(a; \lambda)|_p = 1$.

(ii) Let $\lambda + 2 \in \Lambda$ be not divisible by $p$, and let $a \in \mathfrak{D}^{(m)}(\lambda) \cap \mathfrak{D}_s^{(m)}(\lambda + 2)$. Then there exists a $j \in \{1, 2\}$ such that $|I_s,j(a; \lambda + 2)|_p = 1$.

(iii) Let $\lambda \in \Lambda$ be divisible by $p$, and let $a \in \mathfrak{D}_s^{(m)}(\lambda)$. Then the vector $I(a; \lambda)$ is nonzero.

**Proof.** Under the assumptions of part (i), there exists a $j \in \{1, 2\}$ such that $|G_j(a; \lambda)|_p = 1$. Then $|I_{s,j}(a; \lambda)|_p = |T_{s,j}(a; \lambda)|_p = 1$ for all large $s$ by Lemmas 2, 3, 4, and 9. The proof of (i) is complete.

Under the assumptions of part (ii), there exists a $j \in \{1, 2\}$ such that $|G_j(a; \lambda + 2)|_p = 1$. Then $|I_{s,j}(a; \lambda + 2)|_p = |T_{s,j}(a; \lambda)|_p = 1$ for all large $s$ by Lemmas 2, 3, 4, and 9. The proof of (ii) is complete.

To prove part (iii), consider Eq. (26),

$$\tilde{I}(a; \lambda + 2) = \frac{1}{\lambda} \left[ \frac{\lambda + 1}{a_1} \frac{1}{\bar{a}_2} \right] I(a; \lambda),$$

(27)

which holds for $a \in \mathfrak{D}^{(m)}(\lambda)$. In view of (ii), the vector $\tilde{I}(a; \lambda + 2)$ is nonzero for the intersection $a \in \mathfrak{D}^{(m)}(\lambda) \cap \mathfrak{D}_s^{(m)}(\lambda + 2)$. Since $a \in \mathfrak{D}^{(m)}(\lambda) \cap \mathfrak{D}_s^{(m)}(\lambda + 2) \cap \mathfrak{C}^{(m)}$, we have $a_1a_2 \neq 0$. Hence the matrix in (27) is well defined, and so $I(a; \lambda)$ is nonzero.  

5.6. Invariant Line Bundle

Denote $\mathcal{W} = (\mathbb{Q}_p^{(m)})^2$. The differential operators

$$\mathcal{D}_i(\lambda) = \frac{\partial}{\partial z_i} - H_i(z; \lambda), \quad i = 1, 2,$$

define a connection on the trivial bundle $\mathcal{W} \times (\mathbb{Z}_p^{(m)})^2 \times \Lambda \rightarrow (\mathbb{Z}_p^{(m)})^2 \times \Lambda$ called the dynamical connection.

For any $\lambda \in \Lambda$, we have a map of local sections of the bundle $\mathcal{W} \times (\mathbb{Z}_p^{(m)})^2 \times \{\lambda\} \rightarrow (\mathbb{Z}_p^{(m)})^2 \times \{\lambda\}$ to local sections of the bundle $\mathcal{W} \times (\mathbb{Z}_p^{(m)})^2 \times \{\lambda + 2\} \rightarrow (\mathbb{Z}_p^{(m)})^2 \times \{\lambda + 2\}$ defined by the formula

$$\tau : s(z) \mapsto K(z; \lambda) s(z).$$

We call the operator $\tau$ the qKZ discrete connection on the trivial bundle

$$\mathcal{W} \times (\mathbb{Z}_p^{(m)})^2 \times \Lambda \rightarrow (\mathbb{Z}_p^{(m)})^2 \times \Lambda.$$

The dynamical and qKZ connections are compatible. Namely, for $\lambda \in \Lambda$ and a local section $s(z)$ of $\mathcal{W} \times (\mathbb{Z}_p^{(m)})^2 \times \{\lambda\} \rightarrow (\mathbb{Z}_p^{(m)})^2 \times \{\lambda\}$, we have

$$\mathcal{D}_1(\lambda)(\mathcal{D}_2(\lambda)s(z)) = \mathcal{D}_2(\lambda)(\mathcal{D}_1(\lambda)s(z)),$$

$$\tau(\mathcal{D}_1(\lambda)s(z)) = \mathcal{D}_1(\lambda + 2)(\tau s(z)), \quad i = 1, 2.$$

Denote

$$\mathfrak{D}^{(m)}[\Lambda] := \bigcup_{\lambda \in \Lambda} \mathfrak{D}_s^{(m)}(\lambda) \times \{\lambda\} \subset (\mathbb{Z}_p^{(m)})^2 \times \Lambda.$$
For any \((a, \lambda) \in \mathcal{D}^{(m)}[\Lambda]\), the vector \(I(a, \lambda)\) is nonzero by Lemma 10.

For any \((a, \lambda) \in \mathcal{D}^{(m)}[\Lambda]\), let \(L_{(a,\lambda)} \subset \mathcal{W}\) be the one-dimensional vector subspace generated by the vector \(I(a, \lambda)\). Then

\[
\mathcal{L} := \bigcup_{(a, \lambda) \in \mathcal{D}^{(m)}[\Lambda]} \mathcal{L}_{(a, \lambda)} \times \{(a, \lambda)\} \rightarrow \mathcal{D}^{(m)}[\Lambda]
\]

is an analytic line subbundle of the trivial bundle \(\mathcal{W} \times \mathcal{D}^{(m)}[\Lambda] \rightarrow \mathcal{D}^{(m)}[\Lambda]\).

**Theorem 6.** The line bundle \(\mathcal{L} \rightarrow \mathcal{D}^{(m)}[\Lambda]\) is invariant with respect to the dynamical and \(qKZ\) connections. More precisely,

(i) if \(s(z)\) is a local section of \(\mathcal{L}\) over \(\mathcal{D}^{(m)}_{*}(\lambda) \times \{\lambda\}\), then \(D_{i}(\lambda)s(z)\), \(i = 1, 2\), are local sections of \(\mathcal{L}\) over \(\mathcal{D}^{(m)}_{*}(\lambda) \times \{\lambda\}\) as well;

(ii) if \(s(z)\) is a local section of \(\mathcal{L}\) over \((\mathcal{D}^{(m)}_{*}(\lambda) \cap \mathcal{D}^{(m)}_{*}(\lambda + 2)) \times \{\lambda\}\), then \(\tau s(z)\), is a local section of \(\mathcal{L}\) over \((\mathcal{D}^{(m)}_{*}(\lambda) \cap \mathcal{D}^{(m)}_{*}(\lambda + 2)) \times \{\lambda + 2\}\).

**Proof.** Let \((a, \lambda) \in \mathcal{D}^{(m)}_{*}(\lambda) \times \{\lambda\}\). Let \(c(z)\) be a scalar analytic function at \(a\). Consider the local section \(c(z)I(z; \lambda)\) of \(\mathcal{L}\) at \((a, \lambda)\). Then

\[
D_{i}(\lambda)(c(z)I(z; \lambda)) = -cH_{i}I + \frac{\partial c}{\partial z_{i}}I + \frac{\partial c}{\partial z_{i}}I
\]

\[
= -cH_{i}I + c\left(\frac{1}{2}I_{i}I - \frac{1}{2}I_{i}I\right) + \frac{\partial c}{\partial z_{i}}I
\]

\[
= -cH_{i}I + c\left(H_{i}I - \frac{1}{2}I_{i}I\right) + \frac{\partial c}{\partial z_{i}}I
\]

\[
= -\frac{c}{2}I_{i} + \frac{\partial c}{\partial z_{i}}I.
\]

Here we have used Lemma 7. Clearly, the last expression is a local section of \(\mathcal{L}\) at \((a, \lambda)\). The proof of (i) is complete.

By the definition of \(\tau\), we have \(\tau(c(z)I(z; \lambda)) = c(z)K(z; \lambda)I(z; \lambda)\). We also have the equality

\[
c(z)K(z; \lambda)I(z; \lambda) = c(z)\tilde{I}(z; \lambda + 2),
\]

which holds on \(\mathcal{D}^{(m)}(\lambda)\) by Lemma 7. The vectors \(\tilde{I}(z; \lambda + 2)\) and \(I(z; \lambda + 2)\) are proportional on \(\mathcal{D}^{(m)}(\lambda) \cap \mathcal{D}^{(m)}(\lambda + 2)\) by Lemma 8. For the smaller set \(\mathcal{D}^{(m)}_{*}(\lambda) \cap \mathcal{D}^{(m)}(\lambda + 2)\), the initial vector \(I(z; \lambda)\) and the resulting vector \(\tilde{I}(z; \lambda + 2)\) are both nonzero by Lemmas 9 and 10. This proves (ii).

\[\square\]

**5.7. Special Points**

**Lemma 11.** The points \((z_{1}, z_{2}; \lambda) = (0, 1; 1), (1, 0; 1), (1, 1; 1)\) belong to \(\mathcal{D}^{(m)}_{*}(1) \times \{1\} \subset \mathcal{D}^{(m)}[\Lambda]\).

**Proof.** A straightforward calculation gives \(T_{s}(0, 1; 1) = (-1)^{(p^{s}-1)/2}\), \(I_{s,1}(0, 1; 1) = (-1)^{(p^{s}-3)/2}\), and \(I_{s,2}(0, 1; 1) = (-1)^{(p^{s}-3)/2}(p^{s} - 1)/2\). Hence \(\tilde{I}(0, 1; 1) = (-1, 1/2)\). In a similar way, we obtain \(\tilde{I}(1, 0; 1) = (1/2, -1)\) and \(\tilde{I}(1, 1; 1) = (-1/2, -1/2)\). These vectors are nonzero modulo \(p\).

\[\square\]
By Lemma 11, the analytic vector function \( I(z; 1) \) is nonzero at \((0, 1; 1)\), and its values generate the line bundle \( \mathcal{L} \) over a neighborhood of the point \((z_1, z_2) = (0, 1)\) in \( \mathcal{D}_s^{(m)}(1) \). Over a neighborhood of the point \((z_1, z_2) = (0, 1)\), the same line bundle can be defined differently. Consider the family of elliptic curves \( X(z) \) defined by the equation \( y^2 = t(z_1 - t)(z_2 - t) \). If the parameter \((z_1, z_2)\) is close to \((0, 1)\), then the curve \( X(z) \) has a vanishing cycle denoted by \( C_{0,1} \). The vector function

\[
I^{(C_{0,1})}(z; 1) := \int_{C_{0,1}} \left( \frac{1}{(t-z_1)y}, \frac{1}{(t-z_2)y} \right) dt
\]

(28)

is holomorphic at \((z_1, z_2) = (0, 1)\), solves dynamical equations (7) for \( \lambda = 1 \), and \( I^{(C)}(0, 1; 1) \neq 0 \). The values of \( I^{(C)}(z; 1) \) generate a line bundle denoted by \( \mathcal{L}_{0,1} \) over a neighborhood of \((z_1, z_2) = (0, 1)\). The line bundle \( \mathcal{L}_{0,1} \) is invariant with respect to the dynamical connection. The dynamical connection for \( \lambda = 1 \) does not have other invariant proper nontrivial subbundles near \((z_1, z_2) = (0, 1)\), because other solutions of Eqs. (7) for \( \lambda = 1 \) at \((z_1, z_2) = (0, 1)\) include log \( z_1 \). Hence our line bundle \( \mathcal{L} \) coincides with the line bundle \( \mathcal{L}_{0,1} \) over a neighborhood of \((0, 1) \subset \mathcal{D}_s^{(m)}(1)\).

Similarly, the elliptic curve \( X(z) \) has a vanishing cycle denoted by \( C_{1,0} \) if the parameter \((z_1, z_2)\) is close to \((1, 0)\). The values of the nonzero vector function

\[
I^{(C_{1,0})}(z; 1) := \int_{C_{1,0}} \left( \frac{1}{(t-z_1)y}, \frac{1}{(t-z_2)y} \right) dt
\]

generate a line bundle denoted by \( \mathcal{L}_{1,0} \) over a neighborhood of \((z_1, z_2) = (1, 0)\). Our line bundle \( \mathcal{L} \) coincides with the line bundle \( \mathcal{L}_{1,0} \) over a neighborhood of \((z_1, z_2) = (1, 0) \subset \mathcal{D}_s^{(m)}(1)\).

Also, the elliptic curve \( X(z) \) has a vanishing cycle denoted by \( C_{1,1} \) if the parameter \((z_1, z_2)\) is close to \((1, 1)\). Then the values of the nonzero vector function

\[
I^{(C_{1,1})}(z; 1) := \int_{C_{1,1}} \left( \frac{1}{(t-z_1)y}, \frac{1}{(t-z_2)y} \right) dt
\]

generate a line bundle \( \mathcal{L}_{1,1} \) over a neighborhood of \((z_1, z_2) = (1, 1)\). Our line bundle \( \mathcal{L} \) coincides with the line bundle \( \mathcal{L}_{1,1} \) over a neighborhood of \((1, 1) \subset \mathcal{D}_s^{(m)}(1)\).

Thus, our global line bundle \( \mathcal{L} \) extends over the field \( \mathbb{Q}^{(m)} \) the three local line bundles \( \mathcal{L}_{0,1}, \mathcal{L}_{1,0}, \mathcal{L}_{1,1} \), each defined by integrals over the cycles vanishing at different places. This \( p \)-adic phenomenon was observed by B. Dwork in a different context in [7]. The corresponding global line bundle was called in [7] the \( p \)-cycle.

The operator \( \tau \) of the qKZ difference connection identifies solutions of the dynamical equations (7) with parameter \( \lambda \) with solutions of dynamical equations with parameter \( \lambda + 2 \). Hence our line bundle \( \mathcal{L} \) over a neighborhood of the point \((0, 1) \subset \mathcal{D}_s^{(m)}(1 + 2k), k \in \mathbb{Z}\), corresponds to the line bundle generated by the vector function

\[
I^{(C_{0,1})}(z; 1 + 2k) = \int_{C_{0,1}} \left( \frac{1}{t^k(t-z_1)y}, \frac{1}{t^k(t-z_2)y} \right) dt.
\]

(29)

The vector functions

\[
I^{(C_{1,0})}(z; 1 + 2k) = \int_{C_{1,0}} \left( \frac{1}{t^k(t-z_1)y}, \frac{1}{t^k(t-z_2)y} \right) dt,
\]

\[
I^{(C_{1,1})}(z; 1 + 2k) = \int_{C_{1,1}} \left( \frac{1}{t^k(t-z_1)y}, \frac{1}{t^k(t-z_2)y} \right) dt.
\]

play similar roles in neighborhoods of points \((1, 0)\) and \((1, 1)\), respectively.
5.8. Monodromy

For an odd integer $\lambda$, the differential operators $D_i(\lambda), i = 1, 2$, define a flat dynamical connection on the trivial bundle $\mathbb{C}^2 \times \mathbb{C}^2 \to \mathbb{C}^2$. The flat sections of the connection have the form

$$I^{(C)}(z; \lambda) = \int_C t^{-\lambda/2}(t - z_1)^{-1/2}(t - z_2)^{-1/2}\left(\frac{1}{t - z_1}, \frac{1}{t - z_2}\right) dt,$$

see (1). The monodromy of this connection does not depend on the choice of the odd integer $\lambda$ and is isomorphic to the monodromy of the Gauss-Manin connection on the bundle with fibers being the first homology groups of elliptic curves $X(z)$ of the family defined by the equation $y^2 = t(t - z_1)(t - z_2)$. It is classically known that this monodromy is irreducible. Hence the dynamical connection defined by $D_i(\lambda), i = 1, 2$, over the field of complex numbers has no invariant line subbundles. Thus, the presence of our line subbundle $L \to \mathcal{O}^{(m)}[\Lambda]$ invariant with respect to the dynamical and qKZ connections is a specific $p$-adic feature.

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