CAPTURING POLYTOPAL SYMMETRIES BY COLORING THE EDGE-GRAPH

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ABSTRACT. A general (convex) polytope \( P \subset \mathbb{R}^d \) and its edge-graph \( G_P \) can have very distinct symmetry properties. We construct a coloring (of the vertices and edges) of the edge-graph so that the combinatorial symmetry group of the colored edge-graph is isomorphic (in a natural way) to \( \text{Aut}_{GL}(P) \), the group of linear symmetries of the polytope. We also construct an analogous coloring for \( \text{Aut}_O(P) \), the group of orthogonal symmetries of \( P \).

1. Introduction

In the context of this article, a polytope \( P \subset \mathbb{R}^d \) will always be a convex polytope, that is, \( P \) is the convex hull of finitely many points. A symmetry of \( P \) is a certain transformation of the ambient space that fixes the polytope set-wise. Our focus is specifically on the groups

\[
\text{Aut}_{GL}(P) := \{ T \in \text{GL}(\mathbb{R}^d) \mid TP = P \}, \quad \text{and}
\]

\[
\text{Aut}_O(P) := \{ T \in \text{O}(\mathbb{R}^d) \mid TP = P \},
\]

called the linear resp. orthogonal symmetry group of \( P \).

Initially defined geometrically, one can ask whether it is possible to understand these symmetry groups combinatorially. This could mean to identify a purely combinatorial object \( \mathcal{C} \) whose combinatorial symmetry group \( \text{Aut}(\mathcal{C}) \) is isomorphic to \( \text{Aut}_{GL}(P) \) resp. \( \text{Aut}_O(P) \) in a natural way.

For example, consider the edge-graph \( G_P \) of the polytope. Every, say, linear symmetry \( T \in \text{Aut}_{GL}(P) \) induces a distinct combinatorial symmetry \( \sigma_T \in \text{Aut}(G_P) \) of the edge-graph (see Figure 1). We could state this as follows: the edge-graph is at least as symmetric as the polytope. Usually however, it is strictly more symmetric and is therefore unsuited for “capturing the polytope’s symmetries” in our sense.

In this article we ask whether this can be fixed by coloring the vertices and edges of the edge-graph, thereby encoding further geometric information, and hopefully creating a combinatorial objects that is exactly as symmetric as \( P \) (see Figure 2). As we shall see, this is indeed possible.

This should be surprising for at least two reasons. First, it is established wisdom that the edge-graph of a general polytope in dimension \( d \geq 4 \) carries only very little information about the polytope (a graph can be the edge-graph of several combinatorially distinct polytopes, potentially of different dimensions). Thus, whether the geometric symmetries of \( P \) can be captured by coloring only the edges and vertices

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The clockwise $120^\circ$-rotational symmetry of the hexagon permutes its vertices. This permutation correspond to a combinatorial symmetry $\sigma = (135)(246)$ of the edge-graph. Not every combinatorial symmetry of $G_P$ comes from such a geometric symmetry, e.g. $(123456) \in \text{Aut}(G_P)$. The polygon is therefore strictly less symmetric than its edge-graph.

Various hexagons, and to each a coloring of its edge-graph that gives it “the same symmetries” as the polygon.

of $P$ (instead of, say, also higher dimensional faces) should be at least controversial. Second, the same statement is actually wrong for more general geometric objects (such as graph embeddings, see Example 6.5). In fact, our proof for the existence of these colorings is based on a construction by Ivan Izmestiev [4], which relies heavily on the convexity of $P$. Because of this, it is unclear whether our result generalizes to even some form of non-convex polytopes or polytopal complexes.

Our investigation is in part motivated from a result by Bremner et al. [3]: given a polytope $P \subset \mathbb{R}^d$ with $n$ vertices, the authors construct a coloring of the complete graph $K_n$, so that the symmetry group of the colored graph is isomorphic to $\text{Aut}_{GL}(P)$ (resp. $\text{Aut}_0(P)$); a more precise statement is given in Section 2.1). We can interpret this as follows: if we are allowed to color not only the vertices and edges of $P$, but also other pairs of vertices without a direct counterpart in the polytope’s combinatorics, then “capturing the polytope’s symmetries” is indeed possible. The major result of our article is then that coloring these “non-geometric edges” is not actually necessary.

We reiterate this introduction in a more formal manner.

1.1. Notation and setting. Throughout the text we let $P \subset \mathbb{R}^d$ denote a convex polytope that is full-dimensional (i.e., not contained in any proper affine subspace of $\mathbb{R}^d$) and contains the origin in its interior (i.e., $0 \in \text{int}(P)$).

By $\mathcal{F}_\delta(P)$ we denote the set of $\delta$-dimensional faces of $P$. We assume a fixed enumeration $v_1, \ldots, v_n \in \mathcal{F}_0(P)$ of the polytope’s vertices. In particular, $n$ will always denote the number of the vertices.
The edge-graph of $P$ is the finite simple graph $G_P = (V,E)$ with vertex set $V = \{1,...,n\}$ and edge set $E \subseteq \binom{V}{2}$. We implicitly assume that $i \in V$ corresponds to the vertex $v_i \in \mathcal{F}_0(P)$, and that $ij \in E$ (short for $\{i,j\} \in E$) if and only if $\text{conv}\{v_i,v_j\} \in \mathcal{F}_1(P)$.

The (combinatorial) symmetry group of $G_P$\(^1\) is defined as

$$\text{Aut}(G_P) := \{ \sigma \in \text{Sym}(V) \mid ij \in E \iff \sigma(i)\sigma(j) \in E \} \subseteq \text{Sym}(V)^2,$$

that is, the group of permutations of $V$ that fix the edge set of $G_P$.

A **coloring** of $G_P$ is a map $\mathcal{C}: V \cup E \rightarrow \mathcal{C}$ (it assign colors to both, vertices and edges), where $\mathcal{C}$ denotes an abstract set of colors. The pair $(G_P, \mathcal{C})$ is then a **colored edge-graph** and will be abbreviated by $G_P^\mathcal{C}$. Its combinatorial symmetry group is

$$\text{Aut}(G_P^\mathcal{C}) := \left\{ \sigma \in \text{Aut}(G_P) \mid \begin{array}{l}
\mathcal{C}(i) = \mathcal{C}(\sigma(i)) \text{ for all } i \in V \\
\mathcal{C}(ij) = \mathcal{C}(\sigma(i)\sigma(j)) \text{ for all } ij \in E
\end{array} \right\}.$$

If $\sigma \in \text{Aut}(G_P^\mathcal{C})$, we also say that $\sigma$ **preserves** the coloring $\mathcal{C}$.

The **colored adjacency matrix** of $G_P^\mathcal{C}$ is the matrix $A^\mathcal{C} \in (\mathcal{C} \cup \{0\})^{n \times n}$ with entries

$$A^\mathcal{C}_{ij} := \begin{cases} 
\mathcal{C}(i) & \text{if } i = j \\
\mathcal{C}(ij) & \text{if } ij \in E \\
0 & \text{otherwise}
\end{cases}.$$

Clearly, a coloring is completely determined by the colored adjacency matrix, and we might occasionally use $A^\mathcal{C}$ to define a coloring.

A geometric symmetry $T \in \text{Aut}_{\text{GL}}(P)$ of $P$ maps vertices of $P$ onto vertices of $P$ and thus describes a permutation of the vertex set. Let $\sigma_T \in \text{Sym}(V)$ be the permutation of the vertex set of the edge-graph that permutes its vertices in the same way as $T$ permutes the vertices of $P$. Formally, that is

$$Tv_i = v_{\sigma_T(i)}, \quad \text{for all } i \in V.$$

Since $T$ also maps edges of $P$ onto edges of $P$, also $\sigma_T$ maps edges to edges, and so we see that $\sigma_T$ is a symmetry of the edge-graph, i.e., $\sigma_T \in \text{Aut}(G_P)$. The assignment $T \mapsto \sigma_T$ then defines a group homomorphism $\phi: \text{Aut}_{\text{GL}}(P) \rightarrow \text{Aut}(G_P)$ which we shall call the **natural group homomorphism** of the polytope $P$.

Since $P$ is full-dimensional, its vertices contain a basis of $\mathbb{R}^d$, and it follows that $\phi$ must be **injective**. In general however, $\phi$ is not an isomorphism and $\text{Aut}_{\text{GL}}(P) \neq \text{Aut}(G_P)$, which is a formal way to say that the edge-graph $G_P$ can have many more symmetries than the polytope.

Our approach for rectifying this is to assign a coloring $\mathcal{C}: V \cup E \rightarrow \mathcal{C}$ to the edge-graph $G_P$ with the hope that $\text{Aut}_{\text{GL}}(P) \cong \text{Aut}(G_P^\mathcal{C})$. The natural candidate for the isomorphism between the groups is a colored version of the natural homomorphism:

$$\phi^\mathcal{C}: \text{Aut}_{\text{GL}}(P) \rightarrow \text{Aut}(G_P^\mathcal{C}), \quad T \mapsto \sigma_T
$$

For this to work as desired, we need to check two things:

- First, $\phi^\mathcal{C}$ needs to be well-defined. This is not the case for each coloring: one needs to check that for each $T \in \text{Aut}_{\text{GL}}(P)$ the corresponding permutation $\sigma_T$ is indeed a symmetry of $G_P^\mathcal{C}$ (that is, is in $\text{Aut}(G_P^\mathcal{C})$). Intuitively, this

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\(^1\)For convenience, notions like the symmetry group, colorings, the adjacency matrix, etc. are only introduced for the edge-graph, but it is understood that they apply to more general graphs as well.

\(^2\)Sym($V$) denotes the symmetric group, i.e., the group of permutations of the set $V$.  

amounts to checking that the edge-graph, even after coloring, is still at least as symmetric as $P$.

- Second, $\phi^c$ must have an inverse. If so, then $G^c_P$ is exactly as symmetric as $P$. Providing such an inverse will go as follows: for each $\sigma \in \text{Aut}(G^c_P)$ we need to construct a geometric symmetry $T_\sigma \in \text{Aut}_{GL}(P)$ with

$$T_\sigma v_i = v_{\sigma(i)}, \quad \text{for all } i \in V.$$ 

Since $P$ is full-dimensional, if $T_\sigma$ exists then it is unique. The map $\sigma \mapsto T_\sigma$ is then the desired inverse.

The discussion also applies verbatim to the orthogonal symmetry group $\text{Aut}_O(P)$, and we shall use the same notation $\phi^c : \text{Aut}_O(P) \to \text{Aut}(G^c_P)$ to denote the natural homomorphism in this case.

With this in place, we can formalize “capturing symmetries”:

**Definition 1.1.** A coloring $c : V \cup E \to \mathbb{C}$ of $G_P$ is said to capture the linear (resp. orthogonal) symmetries of $P$ if $\text{Aut}(G^c_P) \cong \text{Aut}_{GL}(P)$ (resp. $\text{Aut}(G^c_P) \cong \text{Aut}_O(P)$), where the isomorphism is realized by the natural homomorphism $\phi^c$.

The main results of this article are explicit constructions for colorings that

- capture linear symmetries (Theorem 4.7).
- capture orthogonal symmetries (Theorem 5.2).

1.2. Overview. In Section 2 we introduce the metric coloring and the orbit coloring, two very natural candidates for capturing certain polytopal symmetries. In this section we do not yet show that either coloring capture linear or orthogonal symmetries, but we establish relevant properties used in the upcoming sections.

In Section 3 we derive a sufficient condition for a coloring of the form $c : V \cup E \to \mathbb{R}$ (the colors are real numbers) to capture linear symmetries. The criterion will be in terms of the eigenspaces of the (colored) adjacency matrix of the edge-graph. We shall call this the “linear algebra criterion”.

In Section 4 we introduce the Izmestiev coloring (based on a construction by Ivan Izmestiev [4]) and we show that it satisfies the “linear algebra criterion” from Section 3. We thereby establish the existence of a first coloring that captures linear symmetries (Theorem 4.7). As a corollary we find that the orbit coloring captures linear symmetries as well (Corollary 4.8).

In Section 5 we show that a combination of the Izmestiev coloring and the metric coloring captures orthogonal symmetries (Theorem 5.2).

2. Two useful colorings

This section is preliminary, in that it introduce two natural colorings of the edge-graph, the metric coloring and the orbit coloring, without establishing either coloring as capturing polytopal symmetries. In fact, this is an open question for the metric coloring (see Question 6.6). The orbit coloring captures polytopal symmetries, but we are not able to show this right away. Both colorings will play a role in the upcoming sections.

Figure 3 shows a polygon and its edge-graph with either coloring applied.
2.1. The metric coloring. Our first coloring is motivated from the previously mentioned construction of Bremner et al. [3] – a coloring of the complete graph $K_n$ that “captures orthogonal symmetries”. In our notation their result reads as follows:

**Theorem 2.1** ([3, Theorem 2]). Given a polytope $P \subset \mathbb{R}^d$ with vertex set $F_0(P) = \{v_1, ..., v_n\}$. Consider the coloring $c$ on the complete graph $K_n$ with

- $c(i) := \|v_i\|^2$, for all $i \in \{1, ..., n\}$,
- $c(ij) := \langle v_i, v_j \rangle$, for all distinct $i, j \in \{1, ..., n\}$.

Then $\text{Aut}(K_n) \cong \text{Aut}_O(P)$.

The strength of this result lies in its immediate applicability: constructing this “complete metric coloring” requires no knowledge of the edge-graph (which is usually hard to come by), but only the vertex coordinates of $P$. In practice, this is probably the best tool for an explicit computation of $\text{Aut}_O(P)$.

From a theoretical and aesthetic perspective however, this construction has the flaw of containing massively redundant data and stepping outside the combinatorial structure of the polytope (we assign color to vertex-pairs that are not edges of the polytope). Naturally, we can ask whether one can get away with coloring fewer of these “non-edges”, ideally only the actual edges of the edge-graph.

Based on this hope, we define the following:

**Definition 2.2.** The **metric coloring** of $G_P$ is the coloring $m : V \cup E \rightarrow \mathbb{R}$ with

- $m(i) := \|v_i\|^2$, for all $i \in V$,
- $m(ij) := \langle v_i, v_j \rangle$, for all $ij \in E$.

Whether the metric coloring captures orthogonal symmetries is an open question (see also Question 6.6). Our reason for introducing it anyway is that in Section 5 the metric coloring will be one ingredient to a coloring that indeed captures orthogonal symmetries.

We close this section with another formulation of Theorem 2.1 that also allows for capturing linear symmetries (in fact, this is closer to the original formulation in [3]). Note that the complete metric coloring of $K_n$ in Theorem 2.1 can also be described

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3This result is primarily based on [2, Proposition 3.1], but we found that its first explicit formulation is in [3].

4If $P$ is given in $\mathcal{H}$-representation, one can apply Theorem 2.1 to compute the orthogonal symmetry group of the dual polytope $P^\circ$, which is identical to $\text{Aut}_O(P)$ as a matrix group.

5A coloring whose colors are real numbers is still a purely combinatorial objects. These numbers are just used for a concise definition and could be replaced by any other finite set of distinguishable values. The only information used from the coloring (in the form of the combinatorial symmetry group of the colored graph) is whether two vertices/edges receive the same or a different color.
by its colored adjacency matrix $A^c = \Phi^\top \Phi$, where $\Phi := (v_1, \ldots, v_n) \in \mathbb{R}^{d \times n}$ is the matrix in which the vertex coordinates of $P$ appear as columns.

**Theorem 2.3** (Another formulation of [3, Theorem 2]). Let $\mathcal{C}$ be a coloring of the complete graph $K_n$ with colored adjacency matrix $A^c$:

1. If $A^c = \Phi^\top \Phi$, then $\text{Aut}(K_n^c) \cong \text{Aut}_O(P)$ (this is exactly Theorem 2.1).
2. If $A^c = \Phi^\dagger \Phi^6$, then $\text{Aut}(K_n^c) \cong \text{Aut}_{\text{GL}}(P)$.

A proof for part (ii) will also follow from the theory developed in Section 3 (see Remark 3.2).

2.2. **The orbit coloring.** The next coloring is motivated from the following consideration: suppose that we are given two vertices $v_i, v_j \in F_0(P)$ in the same orbit w.r.t. $\text{Aut}_{\text{GL}}(P)$, which just means that there is a $T \in \text{Aut}_{\text{GL}}(P)$ with $Tv_i = v_j$. The corresponding combinatorial symmetry $\sigma_T \in \text{Aut}(G_P)$ satisfies $\sigma_T(i) = j$. If now $\mathcal{C} : V \cup E \to \mathcal{C}$ is a coloring that captures linear symmetries, then $\sigma_T$ preserves the coloring $\mathcal{C}$ and we have $\mathcal{C}(j) = \mathcal{C}(\sigma_T(i)) = \mathcal{C}(i)$. We can summarize this as follows: if $\mathcal{C}$ is supposed to capture linear symmetries, then vertices in the same $\text{Aut}_{\text{GL}}(P)$-orbit of $P$ must have the same color in $G_P^c$. With an analogous argument we see that the same holds for edges.

Having identified this first necessary condition for capturing symmetries, we can consider the “simplest” coloring that follows this idea:

**Definition 2.4.** The (linear) orbit coloring $\sigma$ of $G_P$ assigns the same color to vertices (resp. edges) of $G_P$ if and only if the corresponding vertices (resp. edges) of $P$ are in the same $\text{Aut}_{\text{GL}}(P)$-orbit.

An analogous coloring can be defined for orthogonal symmetries, which we shall call the orthogonal orbit coloring of $G_P$, still denoted by $\sigma$. For the sake of conciseness, this section only discusses the (linear) orbit coloring, but all statements carry over to the orthogonal version in the obvious way.

As we shall learn in Section 4 (see Corollary 4.8), the orbit coloring indeed captures linear symmetries. However, this is surprisingly hard to show directly. In fact, our eventual proof of this will “just” use the following:

**Lemma 2.5.** If there is any coloring that captures linear symmetries, then so does the orbit coloring $\sigma$.

**Proof.** Suppose that $\mathcal{C}$ is a coloring that captures linear symmetries, in particular, $\phi^c$ is an isomorphism. Our proof that $\sigma$ captures linear symmetries as well is based on two simple observations:

1. the natural homomorphism $\phi^\sigma$ is well-defined (that is, $G_P^\sigma$ is at least as symmetric as $P$), and
2. $\text{Aut}(G_P^\sigma) \subseteq \text{Aut}(G_P^c)$.

Showing either is straightforward, but for the sake of completeness, both proofs are included below. Now, presupposing both, we can write down the following chain of groups in which the first and the last group are the same:

$$\text{Aut}(G_P^c) \xrightarrow{(\phi^c)^{-1}} \text{Aut}_{\text{GL}}(P) \xrightarrow{\phi^\sigma} \text{Aut}(G_P^\sigma) \xrightarrow{(\iota)} \text{Aut}(G_P^c).$$

$^6\Phi^\dagger \in \mathbb{R}^{n \times d}$ denotes the Moore-Penrose pseudo inverse of $\Phi$, that is, $\Phi \Phi^\dagger = \text{Id}_d$. 

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Since all maps are injective, and the groups are finite, all maps must actually be isomorphisms. Thus, $\phi^\sigma$ is an isomorphism and $\sigma$ captures linear symmetries. This concludes the proof, and it remains to verify (i) and (ii).

Proof of (i): let $T \in \text{Aut}_{\text{GL}}(P)$ be a linear symmetry of $P$ with corresponding combinatorial symmetry $\sigma_T \in \text{Aut}(G_P)$. We need to show that $\sigma_T \in \text{Aut}(G_P^\sigma)$. For this, we observe that for each $i \in V$ the vertices $v_i$ and $v_{\sigma_T(i)} = Tv_i$ belong to the same $\text{Aut}_{\text{GL}}(P)$-orbit of $P$. By the definition of the orbit coloring, $i$ and $\sigma_T(i)$ have then the same color in $G_P^\sigma$. Thus, $\sigma_T$ preserves the vertex colors of $\sigma$. Analogously, one shows that $\sigma_T$ preserves edge colors. Thus, $\sigma_T \in \text{Aut}(G_P^\sigma)$.

Proof of (ii): let $\sigma \in \text{Aut}(G_P^\sigma)$ be a permutation that preserves the orbit coloring. We need to show $\sigma \in \text{Aut}(G_P^\sigma)$. For this, we observe that for all $i \in V$ the vertices $i$ and $\sigma(i)$ have the same color in $G_P^\sigma$, which just means (by Definition 2.4) that $v_i, v_{\sigma(i)} \in F_0(P)$ are in the same $\text{Aut}_{\text{GL}}(P)$-orbit of $P$. Repeating the argument of the introductory paragraph to this section we see that $\varepsilon(i) = \varepsilon(\sigma(i))$. An analogous argument holds for edges. In other words, $\sigma$ preserves the coloring $\varepsilon$, and hence $\sigma \in \text{Aut}(G_P^\sigma)$. 

\[ \sigma \in \text{Aut}(G_P^\sigma). \]

3. A LINEAR ALGEBRA CONDITION FOR CAPTURING SYMMETRIES

For this section, fix a coloring $\varepsilon: V \cup E \to \mathcal{C}$ for which $G_P^\varepsilon$ is at least as symmetric as $P$. Then $\phi^\varepsilon: \text{Aut}_{\text{GL}}(P) \to \text{Aut}(G_P^\varepsilon)$ is well-defined. The goal of this section is to derive a sufficient criterion for $\varepsilon$ to capture linear symmetries.

Recall that this amounts to showing that $\phi^\varepsilon$ is an isomorphism. In other words, the desired criterion must ensure that for each $\sigma \in \text{Aut}(G_P^\varepsilon)$ we can find a linear symmetry $T_\sigma \in \text{Aut}_{\text{GL}}(P)$ with

\[ T_\sigma v_i = v_{\sigma(i)}, \quad \text{for all } i \in V. \]

Let us investigate the difficulties in constructing these transformations.

First, note that we can express (3.1) for all $i \in V$ simultaneously by rewriting it into a single matrix equation as follows:

\[ T_\sigma (v_1, \ldots, v_n) = (v_{\sigma(1)}, \ldots, v_{\sigma(n)}) = (v_1, \ldots, v_n) \Pi_\sigma, \]

where $\Pi_\sigma \in \text{Perm}(n)$ denotes the corresponding permutation matrix\(^7\). If we define $\Phi := (v_1, \ldots, v_n) \in \mathbb{R}^{d \times n}$ as the matrix in which the polytope’s vertices $v_i$ appear as columns, this further compactifies to

\[ T_\sigma \Phi = \Phi \Pi_\sigma. \]

This equation will be our benchmark: every ansatz for how to define the transformations $T_\sigma$ must satisfy (3.2), which is then also sufficient.

Now, if $\Phi$ were invertible, we could just solve (3.2) for $T_\sigma$, satisfying (3.2) “by force”. However, $\Phi \in \mathbb{R}^{d \times n}$ is not a square matrix (since $P$ is full-dimensional, we have $n \geq d + 1$). Instead, one naive hope to still “solve for $T_\sigma$” is to use the Moore-Penrose pseudo inverse of $\Phi$: the unique matrix $\Phi^\dagger \in \mathbb{R}^{n \times d}$ with $\Phi \Phi^\dagger = \text{Id}_d$ (the rows of $\Phi^\dagger$ form a dual basis to the columns of $\Phi$). And so we make the following ansatz:

\[ T_\sigma := \Phi \Pi_\sigma \Phi^\dagger. \]

\(^7\)We chose to define $\Pi_\sigma$ so that on multiplication from left it permutes the rows as prescribed by $\sigma$. We emphasize that this, counter-intuitively, means $(\Pi_\sigma v)_i = v_{\sigma^{-1}(i)}$ for a vector $v \in \mathbb{R}^n$. 
It remains to investigate under which conditions this ansatz satisfies (3.2). We compute

\[
T_\sigma \Phi = \Phi \Pi_\sigma \Phi^\dagger \Phi = \Phi \Pi_\sigma \pi_U,
\]

where \( \pi_U := \Phi^\dagger \Phi \) is the orthogonal projector onto the subspace \( U := \text{span } \Phi^\dagger \subseteq \mathbb{R}^n \). Apparently, to arrive at (3.2), we would need to get rid of the projector \( \pi_U \) on the right side of (3.4). And so we see that one possible sufficient criterion for our construction of the \( T_\sigma \) to work (and thus, for \( c \) to capture linear symmetries) would be \( \Phi \Pi_\sigma \pi_U = \Phi \Pi_\sigma \) for all \( \sigma \in \text{Aut} (G_P^c) \).

This is still a rather cumbersome criterion to apply. The main result of this section is then to reformulate this in terms of the adjacency matrix of \( G_P^c \).

**Theorem 3.1.** Let \( c : V \cup E \to \mathbb{R} \) be a coloring of the edge-graph \( G_P \) so that \( G_P^c \) is at least as symmetric as \( P \). If \( U := \text{span } \Phi^\dagger \) is an eigenspace of the colored adjacency matrix \( A^c \), then \( c \) captures the linear symmetries of \( P \).

**Proof.** Fix a combinatorial symmetry \( \sigma \in \text{Aut} (G_P^c) \).

We use the following well-known (and easy to verify) property of the colored adjacency matrix: if \( \sigma \in \text{Aut} (G_P^c) \), then \( \Pi_\sigma A^c = A^c \Pi_\sigma \).

Now, if \( A^c \) and \( \Pi_\sigma \) commute, then the eigenspaces of \( A^c \) (including \( U \)) are invariant subspaces of \( \Pi_\sigma \), i.e., \( \Pi_\sigma U = U \). Equivalently, \( \Pi_\sigma \) commutes with the projector \( \pi_U \).

This suffices to show that the map \( T_\sigma := \Phi \Pi_\sigma \Phi^\dagger \) satisfies (3.2):

\[
T_\sigma \Phi = \Phi \Pi_\sigma \Phi^\dagger \Phi = \Phi \Pi_\sigma \pi_U = \Phi \pi_U \Pi_\sigma = \Phi (\Phi^\dagger \Phi) \Pi_\sigma = (\Phi^\dagger) (\Phi \Pi_\sigma) = \Phi \Pi_\sigma.
\]

Therefore, the map \( \sigma \mapsto T_\sigma \) defines the desired inverse of \( \phi^c \), and \( c \) captures the linear symmetries of \( P \). \( \square \)

It might not be immediately obvious how Theorem 3.1 is a helpful reformulation of the problem. To apply it we need to construct a matrix \( A^c \) with two very special properties: first, \( A^c \) must be a (colored) adjacency matrix of the edge-graph \( G_P \), that is, it must have non-zero entries only where \( G_P \) has edges. Second, we need to ensure that \( A^c \) has \( U \) as an eigenspace. It is not even clear that these two conditions are compatible.

**Remark 3.2.** Consider the "obvious" matrix \( A^c \) with eigenspace \( U := \text{span } \Phi^\dagger \):

\[
A^c := \Phi^\dagger \Phi.
\]

Of course, this matrix has most likely no zero-entries and is therefore not a colored adjacency matrix of \( G_P \) (except if \( G_P \) is the complete graphs). However, it is exactly the colored adjacency matrix of the complete metric coloring as discussed in Theorem 2.3 (ii).

As it turns out, the proof of Theorem 3.1 makes no use of the fact that the coloring \( c \) is defined on the edge-graph. In fact, we can apply it to the complete graph \( K_n^c \) with colored adjacency matrix \( A^c \). In this way, the "linear algebra criterion" provides an alternative proof of Theorem 2.3 (ii).
4. The Izmestiev Coloring

In this section we introduce a coloring of $G_P$ which satisfies the “linear algebra condition” Theorem 3.1. This coloring is based on a construction by Ivan Izmestiev [4] and we shall call it the Izmestiev coloring.

The coloring is built in a quite unintuitive way. First, we need to recall that for a polytope $P$ with $0 \in \text{int}(P)$ the polar dual $P^o$ is defined as

$$P^o := \{ x \in \mathbb{R}^d \mid \langle x, v_i \rangle \leq 1 \text{ for all } i \in V \}.$$ 

We generalize this notion: for a vector $c = (c_1, \ldots, c_n) \in \mathbb{R}^n$ let

$$(4.1) \quad P^o(c) := \{ x \in \mathbb{R}^d \mid \langle x, v_i \rangle \leq c_i \text{ for all } i \in V \}.$$ 

Then $P^o(1, \ldots, 1) = P^o$ and $P^o(c)$ is obtained from $P^o$ by shifting facets along their normal vectors (see Figure 4).

![Figure 4](image)

**Figure 4.** Several instances of the generalized dual $P^o(c)$ of the cube (the usual polar dual of the cube is the regular octahedron; the second from the left). The polytopes differ by a single facet-defining plane being shifted along its normal vector.

In the following, $\text{vol}(C)$ denotes the relative volume (relative to the affine hull of $C$) of a compact convex set $C \subset \mathbb{R}^d$.

**Theorem 4.1** (Izmestiev [4], Theorem 2.4). For a polytope $P \subset \mathbb{R}^d$ with $0 \in \text{int}(P)$ consider the matrix $M \in \mathbb{R}^{n \times n}$ (which we shall call the Izmestiev matrix of $P$) with components

$$M_{ij} := \frac{\partial^2 \text{vol}(P^o(c))}{\partial c_i \partial c_j} \bigg|_{c=(1,\ldots,1)}.$$ 

(in particular, $\text{vol}(P^o(c))$ is two times continuously differentiable in $c$). $M$ then has the following properties:

1. $M_{ij} < 0$ whenever $ij \in E$.
2. $M_{ij} = 0$ whenever $ij \notin E$ and $i \neq j$.
3. $M$ has a unique negative eigenvalue of multiplicity one.
4. $M\Phi^\top = 0$, where $\Phi = (v_1, \ldots, v_n) \in \mathbb{R}^{d \times n}$ is the matrix introduced in (3.2).
5. $\dim \ker M = d$.

**Remark 4.2.** In the words of [4], the matrix $M$ constructed in Theorem 4.1 is a Colin de Verdière matrix of the edge-graph, that is, a matrix satisfying a certain list of properties, including (i), (ii) and (iii) and the so-called strong Arnold property (for details, see e.g. [6]).

Among the Colin de Verdière matrices, one usually cares about the ones with the largest possible kernel. The dimension of this largest kernel is known as the Colin de Verdière graph invariant $\mu(G_P)$ [6], and Theorem 4.1 (v) then shows that $\mu(G_P) \geq d$. This is not too surprising and was known before. However, the result of Izmestiev
is remarkable for a different reason: it shows that there is a Colin de Verdière matrix whose kernel has dimension exactly $d$ (property ($v$)) and that is compatible with the geometry of $P$ (property ($iv$)).

Remark 4.3. Izmestiev also shows that the matrix $M$ can be expressed in terms of simple geometric properties of the polytope: for $ij \in E$ let $f_{ij} \in \mathcal{F}_{d-2}(P^o)$ be the dual face to the edge $\text{conv}\{v_i, v_j\} \in \mathcal{F}_1(P)$. Then

$$(4.2) \quad M_{ij} = -\frac{\text{vol}(f_{ij})}{\|v_i\|\|v_j\| \sin \angle(v_i, v_j)}.$$ 

Definition 4.4. The Izmestiev coloring $\mathcal{I}: V \cup E \to \mathbb{R}$ of $G_P$ is defined by

$\mathcal{I}(i) := M_{ii}, \text{ for all } i \in V,$

$\mathcal{I}(ij) := M_{ij}, \text{ for all } ij \in E,$

where $M \in \mathbb{R}^{n \times n}$ is the Izmestiev matrix of $P$.

Observation 4.5. Since $M_{ij} = 0$ whenever $ij \notin E$ and $i \neq j$ (by Theorem 4.1 ($ii$)), the colored adjacency matrix $A^3$ of $G_P^3$ is exactly the Izmestiev matrix $M$.

In order to apply the “linear algebra criterion” from Section 3, showing that $\phi^3$ is an isomorphism, we first need to show that $\phi^3$ is well-defined, that is, that $G_P^3$ is at least as symmetric as $P$. This part is relatively straightforward if we use that the Izmestiev matrix is a linear invariant of $P$. We include a proof for completeness:

Proposition 4.6. $G_P^3$ is at least as symmetric as $P$, that is, $\phi^3$ is well-defined.

Proof. Fix a linear symmetry $T \in \text{Aut}_{GL}(P)$ and let $\sigma_T \in \text{Aut}(G_P)$ be the induced combinatorial symmetry of the edge-graph. We need to show that $\sigma_T$ preserves the Izmestiev coloring, that is, $\sigma_T \in \text{Aut}(G_P^3)$.

This requires two ingredients. For the first, one checks that the generalized polar dual $P^o(c)$ (like the usual polar dual) satisfies

$$(TP)^o(c) = T^{-\top}P^o(c),$$

which then gives us

$$(4.3) \quad \text{vol}((TP)^o(c)) = \det(T^{-\top}) \text{vol}(P^o(c)) = \text{vol}(P^o(c)),$$

where we used that $\det(T^{-\top}) = \det(T) = 1$ holds for all linear transformations in a finite matrix group such as $\text{Aut}_{GL}(P)$.

The second ingredient is the following:

$$(4.4) \quad (TP)^o(c) = \{x \in \mathbb{R}^d \mid \langle x, Tv_i \rangle \leq c_i \text{ for all } i \in V\}$$

$$= \{x \in \mathbb{R}^d \mid \langle x, v_{\sigma_T(i)} \rangle \leq c_i \text{ for all } i \in V\}$$

$$= \{x \in \mathbb{R}^d \mid \langle x, v_i \rangle \leq c_{\sigma_T^{-1}(i)} \text{ for all } i \in V\}$$

$$= P^o(\Pi_{\sigma_T}c)^8.$$ 

Putting everything together, we can show $\mathcal{I}(i) = \mathcal{I}(\sigma_T(i))$ for all $i \in V$, and equivalently for edges. We show both at the same time by proving $M_{ij} = M_{\sigma_T(i)\sigma_T(j)}$ for
all \( i, j \in \{1, \ldots, n\} \):

\[
M_{ij} = \frac{\partial^2 \text{vol}(P^0(c))}{\partial c_i \partial c_j} \bigg|_{c=c_0} = \frac{\partial^2 \text{vol}(P^0(\Pi_\sigma c))}{\partial c_{\sigma T(i)} \partial c_{\sigma T(j)}} \bigg|_{c=c_0} = \frac{\partial^2 \text{vol}((TP)^0(c))}{\partial c_{\sigma T(i)} \partial c_{\sigma T(j)}} \bigg|_{c=c_0} = \frac{\partial^2 \text{vol}(P^0(c))}{\partial c_{\sigma T(i)} \partial c_{\sigma T(j)}} \bigg|_{c=c_0} = M_{\sigma T(i) \sigma T(j)},
\]

where we set \( c_0 := (1, \ldots, 1) \in \mathbb{R}^n \).

\[ \square \]

\textbf{Theorem 4.7.} The Izmestiev coloring captures the linear symmetries of \( P \).

\textit{Proof.} By Proposition 4.6, the Izmestiev coloring \( \mathcal{I} \) is at least as symmetric as \( P \), and so we can try to apply the “linear algebra criterion” (Theorem 3.1) to show that \( \mathcal{I} \) captures linear symmetries. That is, we need to show that \( U := \text{span} \Phi^1 \) is an eigenspace of the colored adjacency matrix \( A^3 \) of \( G_2^P \). Recall that \( A^3 \) is exactly the Izmestiev matrix (Observation 4.5), and so we can try to use the various properties of this matrix established in Theorem 4.1.

First, \( U = \text{span} \Phi^1 = \text{span} \Phi^{	op} \) (since the columns of \( \Phi^t \) and \( \Phi^\dagger \) are dual bases of \( U \), and so Theorem 4.1 (iv) can be read as \( U \subseteq \ker A^3 \). Second, we have both \( \dim U = \text{rank} \Phi = d \) (since \( P \) is full-dimensional) and \( \dim \ker A^3 = d \) (by Theorem 4.1 (v)). Comparing dimensions, we thus have \( U = \ker A^3 \).

We conclude that \( U \) is an eigenspace of \( A^3 \) (namely, the eigenspace to eigenvalue 0). The “linear algebra criterion” Theorem 3.1 then asserts that \( \mathcal{I} \) captures the linear symmetries of \( P \). \[ \square \]

By Lemma 2.5, if there is any coloring that captures linear symmetries, then the orbit coloring does so as well:

\textbf{Corollary 4.8.} The orbit coloring captures the linear symmetries of \( P \).

\textbf{Remark 4.9.} A coloring \( c \) is said to be finer than a coloring \( \bar{c} \) if

\[ c(i) = \bar{c}(i) \quad \Rightarrow \quad \bar{c}(i) = \bar{c}(i), \quad \text{for all} \ i, i \in V, \]

\[ c(ij) = \bar{c}(ij) \quad \Rightarrow \quad \bar{c}(ij) = \bar{c}(ij), \quad \text{for all} \ ij, ij \in E. \]

Conversely, \( \bar{c} \) is said to be coarser than \( c \).

It is easy to see that the orbit coloring is the finest coloring that captures linear symmetries, that is, it uses the most colors (consider the argument in the first paragraph of Section 2.2). In contrast, the Izmestiev coloring is in general neither the finest nor the coarsest coloring with this property. Actually determining the coarsest such coloring (i.e., using the fewest colors) seems like a challenging task.

5. Capturing orthogonal symmetries

For this section we consider the orthogonal symmetry group \( \text{Aut}_O(P) \) and all notations without an explicit hint to the kind of symmetry (such as \( \phi^c \) or \( \phi \)) implicitly refer to their orthogonal versions.

Recall the metric coloring \( m: V \cup E \to \mathbb{R} \) (Definition 2.2) with

\[ m(i) = \|v_i\|^2, \quad \text{for all} \ i \in V, \]

\[ m(ij) = \langle v_i, v_j \rangle, \quad \text{for all} \ ij \in E. \]

\[ \text{Recall that} \ \Pi_\sigma \ \text{was defined so that} \ \langle \Pi_\sigma v \rangle_i = v_{\sigma^{-1}(i)} \ \text{for a vector} \ v \in \mathbb{R}^n. \]
As previously mentioned, we consider \( m \) a candidate for capturing orthogonal symmetries, but we are yet unable to prove this (see Question 6.6).

Nevertheless, combining the metric coloring and the Izmestiev coloring allows us to construct a coloring for which we can actually prove this.

**Definition 5.1.** Given two colorings \( c: V \cup E \to \mathcal{C} \) and \( \bar{c}: V \cup E \to \bar{\mathcal{C}} \), the *product coloring* \( c \times \bar{c}: V \cup E \to \mathcal{C} \times \bar{\mathcal{C}} \) is defined by

\[
(c \times \bar{c})(i) := (c(i), \bar{c}(i)), \quad \text{for all } i \in V,
\]

\[
(c \times \bar{c})(ij) := (c(ij), \bar{c}(ij)), \quad \text{for all } ij \in E.
\]

The relevant (and easy to verify) property of the product coloring is

\[
\text{Aut}(G_p^c \times \bar{G}_p^\bar{c}) = \text{Aut}(G_p^c) \cap \text{Aut}(\bar{G}_p^\bar{c}).
\]

In particular, if both \( \phi^c \) and \( \phi^\bar{c} \) are well-defined, then so is \( \phi^{c\times\bar{c}} \).

**Theorem 5.2.** The coloring \( \mathfrak{I} \times m \) captures the orthogonal symmetries of \( P \).

**Proof.** The Izmestiev coloring \( \mathfrak{I} \) is at least as symmetric as \( P \) (we know this for linear symmetries by Proposition 4.6, which include the orthogonal symmetries as a special case). Likewise, the metric coloring \( m \) is at least as symmetric as \( P \) (every orthogonal symmetry preserves norms and inner products, and therefore also the metric coloring). So, since \( \phi^\mathfrak{I} \) and \( \phi^m \) are well-defined, so is \( \phi^{\mathfrak{I}\times m} \).

It remains to show that \( \phi^{\mathfrak{I}\times m} \) has an inverse. For that, fix a \( \sigma \in \text{Aut}(G_P^\mathfrak{I} \times \bar{G}_P^m) \). By (5.1) we have \( \sigma \in \text{Aut}(G_P^\mathfrak{I}) \). By Theorem 4.7 there is a corresponding \( T_\sigma \in \text{Aut}_{GL}(P) \) with \( T_\sigma v_i = v_{\sigma(i)} \) for all \( i \in V \). It remains to show that \( T_\sigma \in O(\mathbb{R}^d) \).

Since \( P \) is full-dimensional, a set \( S \) that contains any vertex \( v_i \) together with its neighbors \( \{v_j \mid ij \in E\} \) spans \( \mathbb{R}^d \), and so it suffices to verify \( \langle T_\sigma v_k, T_\sigma v_\ell \rangle = \langle v_k, v_\ell \rangle \) for every two \( v_k, v_\ell \in S \) to prove the orthogonality of \( T_\sigma \).

Also by (5.1), \( \sigma \) preserves the metric coloring \( m \). The claim then follows via

\[
\langle v_k, v_\ell \rangle = \langle v_{\sigma(k)}, v_{\sigma(\ell)} \rangle = \langle T_\sigma v_k, T_\sigma v_\ell \rangle, \quad \text{for all } v_k, v_\ell \in S,
\]

where we used that \( v_k, v_\ell \in S \) implies \( k = \ell \) or \( k \ell \in E \). \( \square \)

By (the orthogonal version of) Lemma 2.5, if there is any coloring that captures orthogonal symmetries, then so does the orthogonal orbit coloring:

**Corollary 5.3.** The orthogonal orbit coloring captures orthogonal symmetries.

### 6. Outlook, Open Questions and Further Notes

In this article we have shown that the edge-graph of a convex polytope, while generally a very weak representative of the polytope’s geometric nature, still has sufficient structure to let us encode two important types of geometric symmetries: linear and orthogonal symmetries. We achieved this by coloring the vertices and edges of the edge-graph.

The first coloring for which we established that it “captures the polytope’s linear symmetries” was the Izmestiev coloring (Theorem 4.7), based on an ingenious construction by Ivan Izmestiev. But we also found that the orbit coloring, a conceptually very easy coloring, does the job as well (Corollary 4.8). Analogous colorings exist for the orthogonal symmetries as well (Theorem 5.2 and Corollary 5.3).

In the following we briefly discuss various potential generalizations and follow up questions concerning these results. This further highlights the very special structure
of convex polytopes that went into our theorems, emphasizing again that these results are non-trivial to achieve and to generalize.

We also want to mention the following neat consequence for "very symmetric" polytopes:

**Corollary 6.1.** If \( P \subset \mathbb{R}^d \) is vertex- and edge-transitive (i.e., its linear resp. orthogonal symmetry group has a single orbit on vertices and edges), then \( P \) is exactly as symmetric as its edge-graph.

This observation has previously been made in [7, Theorem 5.2]. No classification of simultaneously vertex- and edge-transitive polytopes is known so far, and so this fact might help in the study of this class.

6.1. **Capturing other types of symmetries.** Besides linear and orthogonal symmetries, there are at least two further common groups of symmetries associated with a polytope: the *projective symmetries* and the *combinatorial symmetries* (that is, the symmetries of the face lattice).

We can ask whether those too can be captured by a colored edge-graph:

**Question 6.2.** Is there a coloring \( c : V \cup E \rightarrow \mathcal{C} \) that captures projective resp. combinatorial symmetries:

\[
\text{Aut}(G^P_P) \cong \text{Aut}_{\text{PGL}}(P) \quad \text{resp.} \quad \text{Aut}(G^S_P) \cong \text{Aut}_{\text{Comb}}(P) ?
\]

There might be a general strategy derived from the following (informal) inclusion chain of the symmetry groups:

\[
\text{Aut}_O(P) \subseteq \text{Aut}_{\text{GL}}(P) \subseteq \text{Aut}_{\text{PGL}}(P) \quad \text{"\subseteq"} \quad \text{Aut}_{\text{Comb}}(P).
\]

As it turns out, having solved the coloring problem further to the left in the chain can help to solve the problem further to the right – at least to some degree.

For example, note that every polytope \( P \) can be linearly transformed via a transformation \( T \in \text{GL}(\mathbb{R}^d) \) so that \( \text{Aut}_{\text{GL}}(P) = \text{Aut}_{O}(TP) \). That is, a coloring of \( G^P_P \) that captures the orthogonal symmetries of \( TP \) (which has the same edge-graph) also captures the linear symmetries of \( P \). In still other words, we solved the problem of capturing linear symmetries by making use of our ability to capture orthogonal symmetries.

In our approach, we have not made use of this because we needed to solved the linear case before the orthogonal one. However, this can be of use for capturing projective symmetries. More explicitly, the question is as follows: for every polytope \( P \), is there a projective transformation \( T \in \text{PGL}(\mathbb{R}^d) \) so that \( \text{Aut}_{\text{PGL}}(P) = \text{Aut}_{\text{GL}}(TP) ? \)

The same approach seems doomed for capturing combinatorial symmetries: there are polytopes with combinatorial symmetries that cannot be realized geometrically ([1] discusses the case of a combinatorial symmetry that cannot be made linear; to our knowledge, realizing them as projective symmetries remains to be discussed).

6.2. **Edge-only coloring.** For capturing the symmetries of certain 2-dimensional polytopes it is necessary to color both vertices and edges (cf. Figure 2). But it is unclear whether this is still necessary in higher dimensions.

**Question 6.3.** Is it sufficient to color *only the edges* if \( d \geq 3 \)? That is, is there an edge-only coloring \( c : E \rightarrow \mathcal{C} \) that captures (for example) linear symmetries?
A vertex-only coloring is not always sufficient. For example, in even dimensions exist vertex-transitive neighborly polytopes other than the simplex: e.g. for \(n \geq 6\) we have the following cyclic 4-polytope with \(n\) vertices that is not a simplex:

\[
P := \text{conv} \left\{ \begin{array}{c}
\cos\left(\frac{2\pi i}{n}\right) \\
\sin\left(\frac{2\pi i}{n}\right) \\
\cos\left(\frac{\pi i}{n}\right) \\
\sin\left(\frac{\pi i}{n}\right)
\end{array} \mid i \in \{1, \ldots, n\} \right\}.
\]

The edge-graph of \(P\) is the complete graph \(K_n\), and \(P\) has a single orbit of vertices. Thus, if \(c: V \to \mathcal{C}\) is a vertex-only coloring that captures the symmetries of \(P\), then all vertices of \(K_n\) must receive the same color. But if the edges receive no color, then \(\text{Aut}(K_n^\mathcal{C}) = \text{Sym}(V)\). However, it is known that the linear symmetry group of the cyclic polytope \(P\) other than a simplex is strictly smaller than \(\text{Sym}(V)\) [5].

6.3. Non-convex polytopes and general graph embeddings. Our approach suggests no immediate generalization to non-convex polytopes or various forms of polytopal complexes.

**Question 6.4.** What is the most general geometric setting in which the symmetries can be “captured” by coloring the edge-graph? Does it work for non-convex and/or self-intersecting polytopes? What about more general polytopal complexes?

A vast generalization of polytope skeleta are graph embeddings. For a graph \(G = (V, E)\), a graph embedding is simply a map \(v: V \to \mathbb{R}^d\). There are natural notions of symmetry for such embeddings, and so one might ask whether it is possible to “capture” them by coloring the graph. The following example shows that this is not possible in general:

**Example 6.5.** Consider the complete bipartite graph \(K_{4,4}\) with vertex set \(V_1 \cup V_2 = \{1, 2, 3, 4\} \cup \{5, 6, 7, 8\}\) and an embedding into \(\mathbb{R}^4\) defined as follows:

\[
\begin{align*}
v_1 &= (+1, 0, 0, 0), & v_5 &= (0, 0, +1, 0), \\
v_2 &= (0, +1, 0, 0), & v_6 &= (0, 0, 0, +1), \\
v_3 &= (-1, 0, 0, 0), & v_7 &= (0, 0, -1, 0), \\
v_4 &= (0, -1, 0, 0), & v_8 &= (0, 0, 0, -1).
\end{align*}
\]

One can check that the linear symmetry group of this embedding acts transitively on the vertices as well as the edges. Thus, a coloring \(c\) that is at least as symmetric as the graph embedding must assign the same color to all vertices, and like-wise, the same color to all edges. That is, \(\text{Aut}(K_{4,4}^\mathcal{C}) = \text{Aut}(K_{4,4})\).

However, one can also see that the given embedding has a strictly smaller symmetry group than \(\text{Aut}(K_{4,4})\). For example, \(\sigma := (12) \in \text{Aut}(K_{4,4})\) cannot be realized as a geometric symmetry.

It might be interesting to determine conditions under which “capturing symmetries” is possible even in this very general case.

6.4. The metric coloring. It is yet unknown whether the metric coloring alone can capture orthogonal symmetries (cf. Section 2.1 and Section 5).

**Question 6.6.** Can the metric coloring \(m\) capture orthogonal symmetries?
Any potential affirmative answer to Question 6.6 will need to make use of similar assumptions as the construction of the Izmestiev coloring, namely, convexity and $0 \in \text{int}(P)$, as there are known counterexamples for the other cases (see Figure 5 and Figure 6).

![Figure 5](image1.png)

**Figure 5.** A non-convex shape and two drawings of its edge-graph with metric coloring. The colored edge-graph has more symmetries than the polygon.

![Figure 6](image2.png)

**Figure 6.** A convex polygon $P$ with $0 \notin \text{int}(P)$ (the gray dot indicates the origin) and two drawings of its edge-graph with metric coloring. The colored edge-graph has more symmetries than the polygon.

An interesting special case is the following:

**Question 6.7.** If $P$ is inscribed (i.e., it has all its vertices on a common sphere around the origin) and has all edges of the same length, then is it true that $P$ is as symmetric as its edge-graph, that is, $\text{Aut}_O(P) \cong \text{Aut}(G_P)$?

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