Are highly connected 1-planar graphs Hamiltonian?

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Abstract

It is well-known that every planar 4-connected graph has a Hamiltonian cycle. In this paper, we study the question whether every 1-planar 4-connected graph has a Hamiltonian cycle. We show that this is false in general, even for 5-connected graphs, but true if the graph has a 1-planar drawing where every region is a triangle.

1 Introduction

Planar graphs are graphs that can be drawn without crossings. They have been one of the central areas of study in graph theory and graph algorithms, and there are numerous results both for how to solve problems more easily on planar graphs and how to draw planar graphs (see e.g. [1, 9, 10])). We are here interested in a theorem by Tutte [17] that states that every planar 4-connected graph has a Hamiltonian cycle (definitions are in the next section). This was an improvement over an earlier result by Whitney that proved the existence of a Hamiltonian cycle in a 4-connected triangulated planar graph. There have been many generalizations and improvements since; in particular we can additionally fix the endpoints and one edge that the Hamiltonian cycle must use [16, 12]. Also, 4-connected planar graphs remain Hamiltonian even after deleting 2 vertices [14]. Hamiltonian cycles in planar graphs can be computed in linear time; this is quite straightforward if the graph is triangulated [2] and a bit more involved for general 4-connected planar graphs [5].

There are many graphs that are near-planar, i.e., that are “close” to planar graphs. One such generalization are the 1-planar graphs, which are the graphs that can be drawn with at most one crossing per edge [11]. Naturally one wonders which of the properties of planar graphs carry over to 1-planar graphs. Many results have been developed, for example it is well-known that 1-planar graphs have at most $4n - 8$ edges [13] and are 6-colourable [11] and it has been characterized when 1-planar graphs have a straight-line drawing [15]. See [7] for a recent overview of many existing results for 1-planar graphs.

This paper investigates results on Hamiltonicity in sufficiently connected 1-planar graphs. In particular, we show that every 4-connected triangulated 1-planar graph has a Hamiltonian cycle. This is done via a detour: we show that (with the exception of $K_5$) every triangulated 1-planar 4-connected graph contains a triangulated planar 4-connected graph as a subgraph, and then appeal to Tutte’s theorem. This in particular implies that all the generalizations, such as fixing the endpoints or one edge to be visited, or staying Hamiltonian after deleting two vertices, carry over to triangulated 1-planar 4-connected graphs.
The argument crucially requires that the graph is triangulated: we can easily construct a 4-connected 1-planar graph that does not have a Hamiltonian path (and not even a near-perfect matching, which is a weaker condition). In fact, even 5-connected 1-planar graphs do not always have near-perfect matchings, while the question remains open for 1-planar graphs of higher connectivity.

2 Background

Let \( G = (V, E) \) be a graph with \( n \) vertices and \( m \) edges. We assume familiarity with graph theory (see e.g. [6]) and review only some of the notations below. All graphs in this paper are simple, i.e., no edge connects a vertex with itself and no two edges connect the same pair of vertices. A subgraph of \( G \) is obtained by deleting some vertices (and all their incident edges) and/or deleting some edges. A spanning subgraph is obtained by deleting only edges. A graph \( G \) is called connected if any two vertices \( v, w \) are connected by a path within \( G \). A connected component of \( G \) is a maximal subgraph that is connected. A cutting \( k \)-set is a set \( S \) of at most \( k \) vertices such that \( G \setminus S \) has more connected components than \( G \). Graph \( G \) is called \( k \)-connected if it has no cutting \((k-1)\)-set, and if further it has at least \( k \) vertices. Menger’s theorem states that \( G \) is \( k \)-connected if and only if for any two vertices \( v, w \) there are at least \( k \) paths from \( v \) to \( w \) that are interior vertex-disjoint, i.e., they have no vertex in common except the ends \( v \) and \( w \).

A Hamiltonian path/cycle is a path/cycle in the graph that visits every vertex exactly once. A matching is a set \( M \) of edges such that no two edges in \( M \) have a common endpoint. It is called near-perfect if \( M = \lceil n/2 \rceil \), which is to say, it is as big as any matching can be in an \( n \)-vertex graph. Note that any graph with a Hamiltonian path also has a near-perfect matching, though the reverse is not always true.

A graph \( G \) is called planar if it can be drawn in the plane without crossing. If one particular such drawing \( \Gamma \) is fixed then \( G \) is called plane. The maximal connected regions of \( \mathbb{R} \setminus \Gamma \) are called faces. Graph \( G \) is called triangulated if every face is bounded by a triangle. A triangulated simple planar graph is 3-connected and has a unique planar embedding.

A graph is called 1-planar if it can be drawn in the plane such that every edge has at most one crossing. We assume here that the drawing is good, which means that no edges with a common endpoint cross and no edge crosses itself. If one particular such drawing \( \Gamma \) is fixed then \( G \) is called 1-plane. In a 1-plane graph an edge is called crossed if it has a crossing and uncrossed otherwise. Generalizing, a cycle is called crossed if at least one of its edges is crossed and uncrossed otherwise. The maximal connected regions of \( \mathbb{R} \setminus \Gamma \) are bounded by a sequence of vertices and crossings; these are called the corners of the region. A triangulated 1-plane graph is one where all regions have exactly three corners. Such a graph may have multiple 1-planar embeddings, and not all of them are necessarily triangulated (for example \( K_4 \) has a triangulated drawing without crossing, or a drawing with one crossing that is not triangulated.)

Let \( G \) be a graph with a fixed (planar or 1-planar) drawing \( \Gamma \). A separating cycle of \( G \) is a cycle \( C \) such that the curve traced by \( C \) in \( \Gamma \) does not self-intersect and has at least
one vertex strictly inside and strictly outside. It is easy to see that if the drawing is planar and triangulated, then the graph is \( k \)-connected if and only if it has no separating cycle of length at most \( k - 1 \). To our knowledge no equivalent characterization is known for 1-planar drawing. Obviously, if a 1-planar drawing has a separating uncrossed triangle \( T \), i.e., a separating 3-cycle where no edge is crossed, then the vertices of \( T \) form a cutting 3-set and the graph is not 4-connected. As part of our exploration it follows (see Corollary [1]) that this exactly characterizes when a 1-planar triangulated drawing represents a 4-connected graph.

3 Making 1-planar graphs planar

It is quite obvious that any triangulated 1-plane graph can be made planar by deleting one of each pair of crossing edges. Furthermore, the resulting graph is triangulated. We argue here that if we are more carefully about which of the crossing edges is removed, then we can additionally ensure that no uncrossed separating triangle is created.

**Lemma 1.** Let \( G \) be a triangulated 1-plane graph without uncrossed separating triangle, and assume that \( G \) contains at least 6 vertices. Then \( G \) contains a spanning subgraph \( G' \) that is planar, triangulated, and has no separating triangle.

**Proof.** We proceed by induction on the number of crossings in \( G \). If there is none, then \( G \) itself is planar and triangulated. Any separating triangle \( T \) of \( G \) would be uncrossed by planarity, so none can exist by assumption.

Now assume that \( G \) has a crossing, say \((v_1, v_2)\) crosses \((w_1, w_2)\). Since \( G \) is triangulated, the 4-cycle \( \langle v_1, w_1, v_2, w_3 \rangle \) also exists and consists of uncrossed edges. See Figure 1.

**Case 1:** Assume first that for any vertex \( v_3 \neq v_1, v_2, w_1, w_2 \), either \( \{v_1, v_2, v_3\} \) is not a triangle, or at least one of the edges \((v_1, v_3)\) and \((v_2, v_3)\) is crossed. See Figure 1(a).

In this case, delete edge \((w_1, w_2)\). We claim that the resulting graph 1-plane \( G' \) satisfies the conditions of the lemma, i.e., it is triangulated 1-plane and without uncrossed separating triangles. The result then follows by induction since the spanning subgraph of \( G' \) is also a spanning subgraph of \( G \).

Obviously \( G' \) is 1-plane in the inherited embedding. Deleting \((w_1, w_2)\) results in two regions \( \{v_1, v_2, w_1\} \) and \( \{v_1, v_2, w_2\} \), while all other regions are unchanged, so \( G' \) is again triangulated. Any separating triangle \( T \) of \( G' \) is also a separating triangle of \( G \). If \( T \) were uncrossed in \( G' \) but crossed in \( G \), then \( T \) must include edge \((v_1, v_2)\). So \( T = \{v_1, v_2, v_3\} \) for some \( v_3 \neq v_1, v_2 \), and also \( v_3 \neq w_1, w_2 \) because \( T \) is separating while \( \{v_1, v_2, w_1\} \) and \( \{v_1, v_2, w_2\} \) bound regions in \( G' \). But by case assumption one of \((v_1, v_3)\) and \((v_2, v_3)\) is crossed, contradicting that \( T \) is uncrossed.

**Case 2:** Now assume that there exists a vertex \( v_3 \neq v_1, v_2, w_1, w_2 \) for which edges \((v_1, v_3)\) and \((v_2, v_3)\) exist and are uncrossed in \( G \). See Figure 1(b).

Delete edge \((v_1, v_2)\) from the drawing and call the resulting 1-plane graph \( G' \). We claim that \( G' \) satisfies the conditions of the lemma; the result then follows by induction. As before \( G' \) is triangulated 1-plane. Assume for contradiction that it contains a separating uncrossed
triangle, which necessarily has the form \{w_1, w_2, w_3\} for some vertex \(w_3\), since \((w_1, w_2)\) is the only edge that was crossed in \(G\) but uncrossed in \(G'\).

The drawing of \(G\) contained triangle \(T_v = \{v_1, v_2, v_3\}\), which forms a closed curve. Edge \((w_1, w_2)\) intersects \(T_v\) once at \((v_1, v_2)\) and cannot intersect it again by 1-planarity, hence (up to renaming) \(w_1\) is inside \(T_v\) while \(w_2\) is outside. But there is a path \(\pi = \langle w_1, w_3, w_2 \rangle\) from \(w_1\) to \(w_2\) in \(G\), which also must cross \(T_v\). Since edge \((v_1, v_2)\) cannot be crossed again in a 1-planar graph, and edges \((v_1, v_3)\) and \((v_2, v_3)\) are uncrossed by case assumption, this implies that some vertex of \(\pi\) must be on \(T_v\). But none of \\{\(v_1, v_2, w_1, w_2\)\} coincide since they participate in a crossing. We also know \(w_3 \neq v_1, v_2\) since \\{\(w_1, w_2, v_1\)\} and \\{\(w_1, w_2, v_2\)\} are regions of \(G'\), not separating triangles. Therefore we must have \(w_3 = v_3\).

But then \(w_3 = v_3\) is adjacent to all of \\{\(v_1, v_2, w_1, w_2\)\}. Since \\{\(v_1, v_2, w_1, w_2\)\} form a \(K_4\), this gives a \(K_5\). See also Figure 1(c). Furthermore, all edges incident to \(v_3 = w_3\) in this \(K_5\) are uncrossed in \(G\), by case assumption and since \\{\(w_1, w_2, w_3\)\} is an uncrossed triangle in \(G'\). We claim that this is impossible. Namely, by \(n \geq 6\) at least one region \(R\) of this \(K_5\) contains additional vertices. If \(R\) is incident to \(w_3\), then its boundary is an uncrossed triangle; this would make \(R\) an uncrossed separating triangle, a contradiction. If \(R\) is not incident to \(w_3\), then it is incident to the crossing \(c\) of \((v_1, v_2)\) and \((w_1, w_2)\), and an edge \((a, b)\) of the uncrossed cycle \\{(\(v_1, w_1, v_2, w_3\)\). Since \((a, b)\) is uncrossed, and the part-edges from \(a, b\) to \(c\) cannot be crossed again, the vertices inside \(R\) can be adjacent only to \(a, b\), making \\{\(a, b\)\} a cutting 2-set of \(G\). But one easily convinces oneself that a triangulated simple 1-planar graph is 3-connected (for example because we can delete crossed edges until we obtain a triangulated planar graph as a spanning subgraph), so this is a contradiction.

This implies a few useful results.

**Corollary 1.** Let \(G\) be a 1-plane triangulated graph with \(n \geq 6\). Then \(G\) is 4-connected if and only if \(G\) contains no uncrossed separating triangle.
Proof. If $G$ contains an uncrossed separating triangle $T$, then the vertices of $T$ form a separating 3-set since none of the edges of $T$ are crossed. Therefore $G$ is not 4-connected.

Now assume that $G$ contains no uncrossed separating triangle. By Lemma 1, we can find a subgraph $G^-$ that is planar and triangulated and has no separating triangle. Therefore $G^-$ (and with it its supergraph $G$) is 4-connected.

\textbf{Theorem 1.} Any 4-connected triangulated 1-plane graph $G$ has a Hamiltonian cycle.

\textbf{Proof.} The claim clearly holds for $n \leq 5$, because there are only three 4-connected triangulated 1-plane graphs with $n \leq 5$ ($K_4$, $K_5$ and $K_5 \setminus e$) and all three have Hamiltonian cycles. For $n \geq 6$, a 4-connected triangulated 1-plane graph $G$ has no separating uncrossed triangle, so by Lemma 1 we can find a subgraph $G^-$ that is planar, triangulated and has no separating triangle. Graph $G^-$ is 4-connected and planar, so we can find a Hamiltonian cycle in $G^-$ (hence also $G$).

A 1-planar graph is called an \textit{optimal 1-planar graph} if it has $4n - 8$ edges (the maximum possible number in a 1-planar graph). It is known that every optimal 1-planar graph is 4-connected and triangulated \cite{13} and hence we have:

\textbf{Corollary 2.} Every optimal 1-planar graph has a Hamiltonian cycle.

\subsection{Linear run-time}

Finding a Hamiltonian cycle in a planar triangulated graph can be done in linear time \cite{2}. To find the Hamiltonian cycle of Theorem 1 likewise in linear time, we therefore must argue that the proof of Lemma 1 gives rise to an algorithm that runs in linear time.

Fix a 1-planar graph $G$. As a first step, we compute all triangles in $G$, using the algorithm by Chiba and Nishizeki \cite{4}. To understand its run-time, we need a minor detour. Define the \textit{arboricity} $a(G)$ to be the smallest number $k$ such that the edges of $G$ can be partitioned into $k$ sets, each of which forms a forest. By the Nash-Williams formula \cite{8} we know that $a(G) = \max_H \left\lceil \frac{|E(H)|}{|V(H)| - 1} \right\rceil$, where the maximum goes over all subgraphs $H$ of $G$ with at least 2 vertices. It is known that an $n$-vertex 1-planar simple graph has at most $4n - 8$ edges, except for $n = 2$ where it has at most $4n - 7$ edges. Therefore the arboricity of $G$ is at most 4. The triangle-finding algorithm by Chiba and Nishizeki has run-time $O(a(G)n)$, which therefore is $O(n)$. (This in particular implies that there is only a linear number of triangles.)

We store a list $\mathcal{T}$ of triangles, and cross-link all triangles to the edges that contain them; this takes time $O(|\mathcal{T}|) = O(n)$. Within the same time we can also build for each edge $e$ a list $\mathcal{T}_e$ of triangles that contain $e$. We also assume that from the 1-planar embedding we have a list of all crossings that are cross-linked to the two crossing edges. Now we parse all crossings. For each crossing $c$, look up the edges $e, e'$ involved in it. After possible renaming, $|\mathcal{T}_e| \leq |\mathcal{T}_{e'}|$. For each triangle $T \in \mathcal{T}_e$, check whether any of the other edges of $T$ are crossed. If not, then check whether $T$ consists of $e$ and one endpoint of $e'$. If not, then we have found a separating uncrossed triangle and we delete $e'$ and are done parsing $c$. Otherwise (if none
of the triangles in \( T_e \) are separating uncrossed) we delete \( e \). The run-time to determine this choice is \( O(1 + |T_e|) \).

Deletion of one edge \( \hat{e} \in \{e, e'\} \) consists of updating that the other edge is now uncrossed, and deleting all triangles in \( T_{\hat{e}} \). This takes time \( O(1 + |T_{\hat{e}}|) \). Since \( |T_e| \leq |T_{\hat{e}}| \), the entire run-time for removing one crossing is proportional to the number of deleted triangles plus one. This is overall linear time since there are \( O(n) \) triangles and \( O(n) \) crossings. We conclude:

**Theorem 2.** Given a 4-connected triangulated 1-planar graph \( G \), we can find a 4-connected triangulated planar spanning subgraph of \( G \) in linear time.

**Corollary 3.** Every 4-connected triangulated 1-planar graph \( G \) has a Hamiltonian cycle, and it can be found in linear time.

## 4 Non-hamiltonian 1-planar graphs

In this section, we exhibit some 1-planar graphs that do not have a Hamiltonian path, and in fact, do not even have a near-perfect matching.

### 4.1 Could Theorem [1] be made stronger?

Theorem [1] requires two assumptions on the 1-planar graph: It must be triangulated and it must be 4-connected. The latter is obviously required; one can easily construct planar (hence 1-planar) triangulated graphs that have no Hamiltonian path (and indeed, no near-perfect matching). In fact, there are even non-Hamiltonian 1-plane graphs that are maximal: we cannot add any edge to it without destroying 1-planarity or simplicity.

**Lemma 2.** For any \( N \), there exists a simple maximal 1-plane graph \( G \) with \( n \geq N \) vertices that has no Hamiltonian path.

**Proof.** The following construction was given by Wittnebel [18] and is shown in Figure 2(a); we repeat it here for completeness. Set \( h \geq \max\{6, \frac{N+8}{3}\} \) to be an even number, and let \( H \) be the \( h \)-vertex graph that consists of an \((h - 2)\)-cycle \( C \) plus two more vertices adjacent to all vertices of \( C \). Note that \( H \) has even vertex-degrees, hence its dual graph is bipartite and the \( 2h - 4 \) faces can be coloured as \( h - 2 \) “black” faces and \( h - 2 \) “white” faces such that no two faces of the same colour share an edge.

To obtain \( G \), we replace all white faces by a \( K_4 \) and all black faces by a \( K_6 \). The resulting graph \( G \) has \( h + (h - 2) + 3(h - 2) = 5h - 8 \geq N \) vertices, and Figure 2(a) shows that it is 1-planar. One easily verifies that it is maximal: We could add an edge while staying 1-planar only within two adjacent uncrossed regions. This exists only at the \( K_4 \)'s, and the edges we could add there would be double edges.

Observe that \( G \setminus V(H) \) has \( 2h - 4 \) components (one per face of \( H \)) that all have odd size, and we removed \( h \) vertices. By the “easy” part of the Tutte-Berge formula, [3] any matching of \( G \) has at least \( h - 4 \) unmatched vertices. By \( h \geq 6 \) hence \( G \) has no near-perfect matching and no Hamiltonian cycle.
Figure 2: Two 1-planar graphs that do not have near-perfect matchings or a Hamiltonian path. (a) A triangulated 1-plane graph. (b) A 4-connected 1-planar graph.

It is less obvious that the “triangulated” assumption of Theorem 1 is also required. For planar graphs, 4-connectivity alone is enough to guarantee the existence of a Hamiltonian cycle, but the situation is different for 1-planar graphs.

Lemma 3. For any $N$, there exists a 4-connected 1-planar graph with $n \geq N$ vertices that has no Hamiltonian path. In particular, any matching has size at most $\frac{n+4}{3}$.

Proof. Consider the graph in Figure 2(b), which has been built as follows. Start with a simple planar graph $H$ where every face is a 4-cycle, and where $h := |V(H)| \geq \max\{4, \frac{N+1}{3}\}$. Let $H_s$ be the graph obtained by stellating every face of $H$, i.e., by inserting into every face of $H$ a new vertex that is adjacent to all vertices of the face. Let $G$ be the graph obtained by double-stellating every face of $H$, i.e., by inserting into every face of $H$ two new vertices that are adjacent to all vertices of the face.

Figure 2(b) shows that $G$ is 1-planar. Also, $H$ has $h-2$ faces, hence $G$ has $n = h + 2(h-2) = 3h - 4 \geq N$ vertices. Observe that $G \setminus V(H)$ has $2(h-2)$ components (two per face of $H$) that are singleton vertices. Again using the easy part of the Tutte-Berge formula $\mathbb{E}$, any matching $M$ in $G$ leaves at least $h - 4$ unmatched vertices. So there are at most $2h$ matched vertices and $|M| \leq h = \frac{n+4}{3}$.

It remains to argue that $G$ is 4-connected. Observe first that $H$ is bipartite and has no triangle, in particular it has no separating triangle. Also, since all its faces are 4-cyles and it is bipartite, no two non-consecutive vertices on a face of $H$ are adjacent. Therefore stellating $H$ does not create a separating triangle, and it makes the graph triangulated, which means that $H_s$ is 4-connected. To show that $G$ is 4-connected it suffices to argue that there are four interior vertex-disjoint paths from $v$ to $w$ for any two vertices $v, w$ in $G$. If $v$ and $w$ have been inserted into the same face of $H$, then this is obvious, since they have four common
neighbours. Otherwise, \(v\) and \(w\) existed also as vertices in \(H_s\), and we can find four paths between them already in \(H_s\).

4.2 A 5-connected non-Hamiltonian 1-planar graph

We already exhibited a 4-connected 1-planar graph that does not have a near-perfect matching (and hence no Hamiltonian path). We now show an example that shows that even 5-connectivity is not enough (though the graph is very close to having a perfect matching).

Lemma 4. For any \(N\), there exists a 5-connected 1-planar graph with \(n \geq N\) vertices for which any matching has size at most \(\frac{n-2}{2}\).

Proof. The graph \(G\) is illustrated in Figure 3(d); its construction is not difficult but will be given here via a number of subgraphs \(W, W_s, W', W'_s\) because these are useful for arguing 5-connectivity. Fix an integer \(k \geq \max\{1, (N-2)/40\}\). Common to all constructed graphs are \(2k+1\) cycles \(C_0, \ldots, C_{2k}\), where \(C_0\) and \(C_{2k}\) have 5 vertices each while \(C_1, \ldots, C_{2k-1}\) have 10 vertices each. These cycles are arranged as nested cycles (in the figures they are drawn as horizontal lines on the flat cylinder).

Graph \(W\) is the wall graph, where we add connector-edges between consecutive cycles such that all vertices have degree 3, all faces incident to \(C_0\) or \(C_{2k}\) have degree 5 and all other faces have degree 6. (For \(k = 1\) this is exactly the dodecahedron.) In Figure 3(a) the connector-edges are drawn vertically, except those incident to \(C_0\) or \(C_{2k}\), which are drawn downward-diagonal. If instead we had used upward-diagonals from \(C_0\) and \(C_{2k}\), then we would get a graph \(W'\) that is isomorphic to \(W\), but uses the “other” vertical segments as connector-edges, see Figure 3(c).

Let \(W_s\) be the graph obtained from \(W\) by stellating all faces, see Figure 3(b). Note that \(W_s\) is triangulated and (as one verifies) has no separating 4-cycle; therefore \(W_s\) is 5-connected. The vertices in \(S := V(W_s) \setminus V(W)\) are called the stellation vertices; let \(x_0, x_{2k} \in S\) be the stellation-vertices of the faces bounded by \(C_0\) and \(C_{2k}\). Likewise let \(W'_s\) be the stellation of \(W'\), and let \(S'\) be its stellation-vertices. We use the same vertices \(x_0\) and \(x_{2k}\) to stellate \(C_0\) and \(C_{2k}\), but all other vertices in \(S'\) are different from the ones in \(S\).

Now define \(G\) as follows. It consists of cycles \(C_0, \ldots, C_{2k}\), stellation-vertices \(S \cup S'\) (with \(x_0\) and \(x_{2k}\) added only once), and all edges incident to \(S \cup S'\). Put differently, \(G\) is the union of \(W_s\) and \(W'_s\) after deleting the connector-edges. Figure 3(d) shows that \(G\) is 1-planar. Also, \(W\) has \(20k\) vertices and \(10k+2\) faces, as does \(W'\). Since \(x_0\) and \(x_{2k}\) are added only once, graph \(G\) has hence \(n = 20k + 2(10k+2) - 2 = 40k + 2 \geq N\) vertices. Since the \(20k + 2\) stellation-vertices form an independent set, and there are only \(20k\) other vertices, any matching has at least two unmatched vertices.

It remains to argue that \(G\) is 5-connected. Roughly speaking, this holds because \(W_s\) and \(W'_s\) are 5-connected, and \(G\) contains subdivisions of \(W_s\) and \(W'_s\) as subgraphs (each connector-edge can be replaced by a path through a stellation-vertex of the other wall-graph, see Figure 3(c-d)). Formally, assume for contradiction that \(Q\) is a cutting-4-set, and let \(y, y'\) be two vertices in two distinct components of \(G \setminus Q\). Fix one vertex \(z\) that belongs to one of the cycles \(C_0, \ldots, C_{2k}\), and furthermore \(z \neq y, y'\) and \(z \not\in Q\). We claim that both
Figure 3: (a) The wall $W$. (b) Its stellation $W_s$. (c) The stellation $W'_s$ of the other wall $W'$. Five paths between two vertices are in bold red. (d) A 5-connected 1-planar graph without a near-perfect matching. The corresponding paths are bold red.

$y$ and $y'$ are connected to $z$ in $G \setminus Q$; this is a contradiction since then $y, z, y'$ would all be in one connected component of $G \setminus Q$.

We only show the existence of a path from $y$ to $z$ in $G \setminus Q$, the argument is identical for $y'$. We may also assume that $y \in W'_s$, for if it is only in $W_s$ then a symmetric argument applies. Since $z$ belongs to one of the cycles, also $z \in W'_s$. Since $W'_s$ is 5-connected there are five interior vertex-disjoint paths from $y$ to $z$ in $W'_s$. Because $G$ contains a subdivision of $W'_s$ as a subgraph, these paths in $W'_s$ transfer to five interior vertex-disjoint paths from $y$ to $z$ in $G$. Since $y, z \notin Q$ and $|Q| = 4$, at most four of these paths can be “hit” by $Q$, which means that at least one path exists even in $G \setminus Q$, which makes $y$ connected to $z$ as desired.

Note that the constructed graph is very close to having a near-perfect matching, and we believe that this holds in general.

**Conjecture 1.** Every 5-connected 1-planar graph has a matching of size $\frac{n}{2} - O(1)$. 
5 Conclusion

In this paper, we studied the Hamiltonicity of 4-connected 1-planar graphs, and showed that while in general they do not have a Hamiltonian path, they always have a Hamiltonian cycle if they are 4-connected and triangulated.

Among the most interesting open questions is whether higher connectivity implies Hamiltonicity in all 1-planar graphs. We have not been able to construct a 6-connected 1-planar graph that does not have a near-perfect matching. Do all 6-connected 1-planar graphs have a Hamiltonian cycle? A Hamiltonian path? Or at least a near-perfect matching? How about 7-connected 1-planar graphs?

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