Quasi-static magnetohydrodynamic turbulence at high Reynolds number

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Abstract. We analyse the anisotropy of turbulence in an electrically conducting fluid submitted to a uniform magnetic field, for low magnetic Reynolds number, using the quasi-static approximation. In the linear limit, the kinetic energy of velocity components normal to the magnetic field decays faster than the kinetic energy of the component along the magnetic field (Moffatt, 1967). However, numerous numerical studies predict a different behaviour, wherein the final state is characterised by dominant horizontal energy. We investigate the corresponding nonlinear phenomenon using Direct Numerical Simulations (DNS) and spectral closures based on Eddy Damping Quasi-Normal Markovian (EDQNM) models. The initial temporal evolution of the decaying flow indicates that the turbulence is very similar to the so-called “two-and-a-half-dimensional” flow (Montgomery & Turner, 1982) which explains the observations in numerical studies. EDQNM models confirm this statement at higher Reynolds number.

1. Introduction

In most geophysical and astrophysical flows, turbulence is affected by forces that distort significantly some of its scales in an anisotropic manner, such as the Lorentz force arising from the presence of an external magnetic field in a conducting fluid. The response of initially isotropic turbulence to a static magnetic field is documented in pioneering theoretical works (Moffatt, 1967), many numerical (Vorobev et al., 2005; Burattini et al., 2008; Favier et al., 2010) and experimental studies (Alemany et al., 1979). One of the main properties of this kind of flow is the suppression of the three-dimensional motion due to anisotropic linear Joule dissipation, leading to a flow without variations in the direction of the imposed magnetic field. In the linear regime, this final state is characterised by $< u_3^2 > \sim 2 < u_1^2 >$, where $u_3$ is the velocity component parallel to the imposed magnetic field whereas $u_1$ are the perpendicular ones (Moffatt, 1967). However, recent numerical simulations of low magnetic Reynolds number turbulence show that the horizontal kinetic energy is dominant, at least at large scales (Vorobev et al., 2005). We present results of DNS and spectral closures based on EDQNM models in order to analyse this nonlinear phenomenon, which is up to now considered as a restoration of isotropy, but still has to be elucidated.
2. Governing equations, numerical methods and parameters

We consider initially isotropic homogeneous turbulence in an incompressible conducting fluid. The fluid is characterised by a kinematic viscosity $\nu$, a density $\rho$ and a magnetic diffusivity $\eta = (\sigma \mu_0)^{-1}$; $\sigma$ is the electrical conductivity, $\mu_0$ the magnetic permeability. The magnetic Prandtl number $Pr_M = \nu/\eta$ is very small in our study. The flow is submitted to a uniform vertical magnetic field $B_0$ scaled as Alfvén speed as $B_0 = B/\sqrt{\rho \mu_0}$. Within the quasi-static approximation, which implies very low magnetic Reynolds numbers (Knaepen et al., 2004), the Navier-Stokes equations become

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} = \frac{1}{\rho} \nabla p + \nu \nabla^2 \mathbf{u} + \frac{B_0^2}{\eta} \Delta^{-1} \frac{\partial^2 \mathbf{u}}{\partial z^2} + \mathbf{F}$$  \hspace{1cm} (1)$$

where $\mathbf{F}$ is the rotational part of the Lorentz force, $\Delta^{-1}$ is the inverse of the Laplacian operator and $z$ the vertical coordinate, along the direction of $B_0$. A pseudo-spectral method is used to solve equation (1). The velocity field is computed in a cubic box of side $2\pi$ with periodic boundary conditions using $512^3$ Fourier modes. A spherical $2/3$-truncation of Fourier modes is used to avoid aliasing and the time scheme is third-order Adams-Bashforth. We also use hereafter a fully axisymmetric version of the Eddy Damped Quasi-Normal Markovian model (EDQNM2 in the following), in which isotropy is broken by nonlinear terms in connection with waves. It has already been successfully applied to rotating or stably stratified turbulent flows, including a comparison with DNS (Cambon et al., 1997). Different degrees of complexity are existent in EDQNM models and, for clarity, we only present results from the so-called EDQNM2 in the following (see Sagaut & Cambon (2008) for details). The EDQNM2 simulations are initialised.

![Figure 1](image-url)

**Figure 1.** Volume rendering of the enstrophy ($\omega_m^2$ is about 25% of the maximum value). The resolution is $512^3$. (a) Initial isotropic condition. (b) Final state for $N = 5$ (the vertical correlation length is roughly one tenth of the numerical box). (c) Vertical plane extracted from figure (b). (d) Horizontal plane extracted from figure (b).
Figure 2. (a) Kinetic energy $K(t)$ versus dimensionless time $t^*$. (b) Ratio between horizontal and vertical kinetic energies versus dimensionless time $t^* = tu_0/l_0$. The dotted lines correspond to linear simulations (neglecting the advection term in equation (1)). The triangles correspond to the EDQNM2 results whereas the continuous lines correspond to nonlinear DNS results.

with the exact initial kinetic energy spectrum obtained from the DNS pre-computation. In contrast to DNS, the wave number discretization used here is logarithmic, thereby improving the sampling of the large scales with respect to DNS. The minimum and maximum wave numbers solved are the same as in DNS. Hereafter, DNS results are plotted with lines only, EDQNM2 results are plotted with lines and symbols.

An initially isotropic turbulent velocity field is created by a hydrodynamic simulation with large-scale forcing in order to reach a quasi-steady state (see figure 1(a)). At the end of this pre-computation stage, the $\text{rms}$ velocity is $u_0 = 0.81$ and the integral scale $l_0 = 0.25$ yielding $Re = u_0l_0/\nu \simeq 333$. This rather low value, considering the resolution, is a consequence of our specific choice of a small initial integral scale $l_0$, in order to lift partially the numerical confinement constraint. The corresponding turbulent flow field is used as initial state for two different MHD simulations. In all of them the magnetic Reynolds number $R_M \simeq 0.1$, so that the quasi-static approximation is justified (Knaepen et al., 2004). Two different amplitudes of the imposed magnetic field are chosen, which correspond to two values of the interaction parameter: $N = (B_0^2l_0)/(\eta u_0) = 1$ and $N = 5$. For reference, we also compute the equivalent linear simulations, setting $\mathbf{u} \cdot \nabla \mathbf{u} = 0$ in equation (1). The quasi-static MHD simulations are freely decaying to avoid spurious effects of a forcing scheme on the development of anisotropy.

3. Results

First of all, we carefully consider the confinement existing in DNS due to periodic boundary conditions and increasing velocity correlations. It is found that the EDQNM2 model is free from any confinement effects since there is no explicit use of the physical space. However, some care is necessary when considering DNS results at large times and at large interaction parameters. The following results are obtained from simulations where the effects of the periodic boundaries are negligible (we stop the simulations when the velocity correlation length is becoming of the same order as the numerical box size). Let us first consider the global kinetic energy $K$ as it decays with time. On figure 2(a), we compare the results from DNS and EDQNM2 in different cases. It is found that the EDQNM2 model is able to reproduce the anisotropic additional
Ohmic dissipation and therefore correctly predict the decay of kinetic energy, which is stronger than in the isotropic hydrodynamic case. The linear simulations undestimate the dissipation, as expected from the lack of direct cascade of kinetic energy. It is interesting to notice that the discrepancy between linear and nonlinear simulations decreases as N increases, indicating that the linear Ohmic dissipation is becoming dominant at large N.

The linear prediction $\langle u_\perp^2 \rangle / \langle u_\parallel^2 \rangle \sim 0.5$ is then checked on figure 2(b). Initially, this quantity decays towards one half as expected from linear theory (Moffatt, 1967). However, after a few turnover times, this ratio increases for both nonlinear DNS and EDQNM2. It was shown that this is not due to a restoration of isotropy but to a nonlinear phenomenon linked to the particular quasi-two-dimensional structure of the flow (for details, see Favier et al. (2010)). Figure 2(b) shows that EDQNM2 reproduces this departure from the linear prediction, although with a time lag and a smaller amplitude.

As expected, the flow becomes invariant in the direction of $B_0$ (see for example figure 1(c)). This is due to the additional Ohmic dissipation in equation (1). In spectral space, it corresponds to a concentration of kinetic energy in modes such that $k \perp B_0$. In order to observe this spectral anisotropy, we compute the angular energy spectra. Instead of averaging over spherical shells as for isotropic turbulence, the average is done over rings with constant polar angle $\theta$. The comparison between DNS and EDQNM2 is presented on figures 3. The kinetic energy is clearly dominant near the equator whereas the kinetic energy near the pole is strongly dissipated. However, the flow is far from being purely two-dimensional since the vertical velocity is non-zero. The vertical component acts like a passive scalar, advected by the two-dimensional horizontal flow. We therefore observe a direct cascade of vertical energy and an inverse cascade of horizontal kinetic energy, both in DNS and EDQNM2. The different dynamics for the horizontal and vertical component of the velocity can be seen on figure 4(a). We present here the energy spectra at the equator (i.e. for horizontal wave vectors only) using a poloidal/toroidal decomposition. Classical scalings of the covariance spectrum of a passive scalar in 2D flow are observed at very high Reynolds number, only achievable using spectral closures.

Finally, we also compare DNS and EDQNM2 closure of two-dimensional turbulence with three components. All variables are invariant in the vertical direction but the vertical velocity is non-zero. This is the asymptotic state expected in quasi-static MHD turbulence. Again, we plot the energy spectra for both horizontal and vertical components on figure 4(b). The agreement is excellent and we observe the different dynamics between the two components. The
horizontal component acts like 2D turbulence, with a reduction of the direct cascade of kinetic energy and a reduction of dissipation whereas the vertical component acts like a passive scalar. These results suggest that the final state of quasi-static MHD turbulence is dominated by a separation of dynamics between longitudinal and transverse velocity components, which explains the discrepancy between observations, numerical experiments and linear analysis.

4. Conclusions

In this paper, we have investigated the dynamics and the detailed anisotropy of magnetohydrodynamic turbulence in the quasi-static approximation at small magnetic Reynolds number, using Direct Numerical Simulations and a two-point statistical closure of EDQNM type. In essence, such closures consider statistical averages, which is a key advantage when considering turbulent flows, for two reasons: first, only one simulation is required to obtain averaged results, in contrast with the large number of realisations needed in DNS; secondly, the obtained averages are smoother functions than in DNS, all the more if one considers high order moments. In terms of computational cost, isotropic EDQNM computations are thousands of times less costly than equivalent DNS. The axisymmetric anisotropic EDQNM2 model abandons one symmetry with respect to the isotropic context, thus the convolution integral is an order of magnitude more expensive.

We have compared results of the EDQNM2 closure model with those of $512^3$ DNS. Comparisons involve kinetic energy and energy spectra. Starting with initial conditions of isotropic turbulence, the results indicate that the flow dynamics becomes closer to a two-dimensional three components state. The dynamics is not significantly altered at higher Reynolds numbers reached with the closure model. However, asymptotic scaling behaviour appears only very slowly. We conclude by noting that rotating turbulence bears strong similarities with QS MHD turbulence. In both case, a transition from 3D to 2D structures is observed. This transition originates from the linear Joule dissipation term in QS MHD, but from nonlinear interactions dominated by cubic transfer terms, when solid body rotation acts.
Therefore, QS MHD turbulence may eventually become fully two-dimensional, whereas complete two-dimensionalization cannot be achieved in rotating turbulence in absence of additional phenomena.

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