On smooth hypersurfaces containing a given subvariety

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Abstract. We show when a nonsingular closed subvariety \( Y \) of a nonsingular affine real variety \( X \) is contained in a nonsingular hypersurface. We also solve this problem in a holomorphic case.

1. Introduction

Let \( X^n \) be a nonsingular affine variety (of dimension \( n \)) over an algebraically closed field \( k \). It was proved in \([4]\) (see also \([7]\)) that any closed nonsingular subvariety \( Y^r \subset X^n \) with \( n \geq 2r + 1 \) is contained in a nonsingular hypersurface. Recall that a subvariety \( H \subset X \) is a hypersurface if its ideal \( I(H) \subset k[X] \) is generated by a single polynomial. Our aim is to prove similar results in the real algebraic case and in the complex analytic case.

Let \( X \) be a real algebraic variety (as in \([1]\)). We say that a subvariety \( H \) of \( X \) is a hypersurface if the ideal

\[
I(H) = \{ f \in \mathcal{R}(X) \mid f|_H = 0 \}
\]

is principal. Here \( \mathcal{R}(X) \) denotes the ring of regular functions on \( X \). We show:

**Theorem 1.1.** Let \( X^n \) be a nonsingular real affine variety and let \( Y^r \) be a closed nonsingular subvariety of \( X \). Assume that either

(i) \( 2r + 1 \leq n \), or
(ii) \( 2r = n \) and \( Y \) has no compact connected components.

Then there exists a nonsingular hypersurface \( H \) in \( X \) with \( Y \subset H \).

If \( X = \mathbb{R}^n \), point (ii) can be extended:
Theorem 1.2. Let $Y^r$ be a compact nonsingular subvariety of $\mathbb{R}^{2r}$. If $Y$ is orientable (as $C^\infty$ manifold), then there exists a nonsingular hypersurface $H$ in $\mathbb{R}^{2r}$ with $Y \subset H$.

We also have the following holomorphic version of Theorem 1.1:

Theorem 1.3. Let $X^n$ be a Stein manifold and let $Y^r$ be a closed submanifold of $X$. If $(3r + 1)/2 \leq n$, then there exists a nonsingular hypersurface $H$ in $X$ with $Y \subset H$.

By definition, in the complex analytic case, $H$ is a hypersurface in $X$, if the ideal of all holomorphic functions on $X$ vanishing on $H$ is principal.

2. Real case

In this section we work with affine real varieties (as in [1]). For the sake of completeness, we record the following fact.

Lemma 2.1. Let $Y \subset X$ be nonsingular affine real varieties and let $s$, $g_1$, ..., $g_r$ be regular functions on $X$ such that $d_y s \neq 0$ for every $y \in Y$ and $I(Y) = (g_1, ..., g_r)$. Then the hypersurface $V(s + \sum_{i=1}^{r} \beta_i g_i^2)$ in $X$ is nonsingular for generic $\beta_i \in \mathbb{R}$.

Proof. Note that $0 \in \mathbb{R}$ is a regular value of the regular function $X \times \mathbb{R}^r \ni (x, (\beta_1, ..., \beta_r)) \mapsto s(x) + \sum_{i=1}^{r} \beta_i g_i(x)^2 \in \mathbb{R}$.

Hence the assertion follows from the parametric transversality theorem (see [5], p. 68).

We make use of Lemma 2.1 in the following:

Theorem 2.2. Let $Y \subset X$ be nonsingular real affine varieties. Then $Y$ is contained in some nonsingular hypersurface $V(f) \subset X$ if and only if the normal bundle of $Y$ contains a one-dimensional trivial summand i.e.,

$$N_{X/Y} = T \oplus E^1,$$

where $E^1$ denotes a trivial line bundle.

Proof. Assume that there is a nonsingular hypersurface $H = V(f) \subset X$ which contains $Y$. We have

$$TY \subset TH \subset TX,$$

and hence

$$N_{X/Y} = N_{H/Z} \oplus N_{X/H}|_Y.$$

However, the normal bundle of the nonsingular hypersurface $H = V(f)$ is trivial (in fact the class of $f$ is a generator of the conormal bundle of $H$).
Conversely, assume that
\[ N_{X/Y} = T \oplus E^1. \]
Hence also
\[ N^*_{X/Y} = T^* \oplus E^1. \]
This means that the conormal bundle \( N^*_{X/Y} \) has a nowhere vanishing section \( s \in \Gamma(Y, N^*_{X/Y}) \). But \( \Gamma(Y, N^*_{X/Y}) = I(Y)/I(Y)^2 \), where \( I(Y) \subset \mathcal{R}(X) \) denotes the ideal of the subvariety \( Y \). Hence \( s \) corresponds to a regular function \( s \in I(X) \) such that the class of \( s \) is \( s \). Take a point \( a \in Y \) and a system of local coordinates \((u_1, \ldots, u_n)\) at \( a \) such that \( Y \) is described by the local equations \( u_1 = 0, \ldots, u_t = 0 \) (\( t = \text{codim} Y \) near \( a \)). Since \( u_1, \ldots, u_t \) freely generate the bundle \( N^*_{X/Y} \) near the point \( a \), we have
\[ s = \sum_{i=1}^t \alpha_i u_i, \quad (1) \]
where \( \alpha_i \in \mathcal{R}(U_a) \) (\( U_a \) denotes some Zariski open neighborhood of \( a \) in \( Y \)). Since the section \( s \) nowhere vanishes, the \( \alpha_i \) do not vanish simultaneously at any point of \( U_a \). Let us compute the derivative \( d_y s \) at the point \( y \in U_a \).

After shrinking \( X \) and \( Y \) we can assume that (1) holds in \( Y \). Moreover we can assume that all \( \alpha_i \) are defined on \( X \). We have
\[ s = \sum_{i=1}^t \alpha_i u_i \mod I(Y)^2, \]
hence there are regular functions \( f_j, h_j \in I(Y), \ j = 1, \ldots, m \), such that
\[ s = \sum_{i=1}^t \alpha_i u_i + \sum_{j=1}^m f_j h_j. \]
Now we easily see that for \( y \in Y \) we have
\[ d_y s = \sum_{i=1}^t \alpha_i d_y u_i. \]
Since \( d_y u_i, i = 1, \ldots, n \), are linearly independent and not all \( \alpha_i \) vanish at \( y \), we have \( d_y s \neq 0 \). Let \( I(Y) = (g_1, \ldots, g_r) \). By Lemma 2.1, the hypersurface \( V(s + \sum_{i=1}^r \beta_i g_i^2) \) is nonsingular for generic \( \beta_i \in \mathbb{R} \). Hence we can take \( f = s + \sum_{i=1}^r \beta_i g_i^2, \)

In the sequel we need the following:

**Lemma 2.3.** Let \( X \) be a nonsingular real affine variety and let \( F \) be a real algebraic vector bundle on \( X \). If \( \text{rank} \ F > \dim X \), then \( F \) has a one-dimensional trivial summand i.e.,
\[ F = T \oplus E^1. \]
Proof. The category of algebraic real vector bundles on $X$ is equivalent with the category of finitely generated projective $\mathcal{R}(X)$-modules ([1], p. 305). Hence Lemma 2.3 is a special case of Serre’s splitting theorem (see [10]). □

Lemma 2.4. Let $X$ be a nonsingular real affine variety and let $F$ be an algebraic real vector bundle on $X$ with rank $F = \dim X$. If $X$ has no compact connected components, then $F$ has a one-dimensional trivial summand.

Proof. Let $d = \dim X$ and let $f : X \to G_d(\mathbb{R}^n)$ be a regular map with $F \cong f^* \Gamma_d(\mathbb{R}^n)$, where $G_d(\mathbb{R}^n)$ is the Grassmannian of $d$—dimensional vector subspaces of $\mathbb{R}^n$ and $\Gamma_d(\mathbb{R}^n)$ is the tautological vector bundle on $G_d(\mathbb{R}^n)$. By Hironaka’s theorem on resolutions of singularities, we may assume that $X$ is an open subvariety of a compact nonsingular variety $\overline{X}$. We regard $f : \overline{X} \to G_d(\mathbb{R}^n)$ as a rational map. According to Hironaka’s theorem on resolution of points of indeterminacy, there exists a regular map $\pi : X' \to \overline{X}$ such that $\pi$ is the composite of finitely many blowups with nonsingular centers, $\pi$ induces a biregular isomorphism between $\pi^{-1}(X)$ and $X$, and the rational map $f \circ \pi : X' \to G_d(\mathbb{R}^n)$ is actually regular. In particular, $F' := (f \circ \pi)^* \Gamma_d(\mathbb{R}^n)$ is an algebraic vector bundle on $X'$. In order to simplify notation, we identify $X$ with $\pi^{-1}(X)$. Then $F'_{|X} = F$. It suffices to find an algebraic section $s : X' \to F'$ with $Z(s) \subset X' \setminus X$, where $Z(s) = \{ x \in X' | s(x) = 0 \}$.

Case 1. Suppose that $\dim X \geq 2$. We may assume that for each connected component $C$ of $X'$, the set $C \setminus X$ is infinite (if necessary, blow up $X'$ at a point in $C \setminus X$). Let $u : X' \to F'$ be a $C^\infty$ section transverse to the zero section. Then the zero locus $Z(u)$ of $u$ is a finite set. Clearly, there exists a $C^\infty$ diffeomorphism $h : X' \to X'$, homotopic to the identity map, such that $h^{-1}(Z(u)) \subset X' \setminus X$. The pullback section $h^*u : X' \to h^*F'$ is transverse to the zero section and $Z(h^*u) = h^{-1}(Z(u))$. Since $h$ is homotopic to $id_X$, the vector bundles $h^*F'$ and $F'$ are $C^\infty$ isomorphic. Consequently, we can find a smooth section $v : X' \to F'$ which is transverse to the zero section and satisfies $Z(v) \subset X' \setminus X$. By [1, pp. 309, 321], $v$ can be approximated in the $C^\infty$ topology by an algebraic section $s : X' \to F'$ with $s|_{Z(v)} = 0$. If $s$ is sufficiently close to $v$, then $Z(s) = Z(v) \subset X' \setminus X$.

Case 2. Suppose that $\dim X = 1$. Each connected component of $X'$ is diffeomorphic to a circle. Thus there exists a smooth section $v : X' \to F'$ which is transverse to the zero section and satisfies $Z(v) \subset X' \setminus X$. Now we get $s$ as in Case 1. □

Proof of Theorem 1.1. Point (i) follows from Theorem 2.2 and Lemma 2.3, whereas (ii) is a consequence of Theorem 2.2 and Lemma 2.4. □

Proof of Theorem 1.2. If $Y'$ is orientable (as $C^\infty$ manifold), then the normal bundle of $Y'$ in $\mathbb{R}^{2r}$ has a nowhere zero $C^\infty$ section [11], and hence in view of [1, p.309], it has an algebraic one-dimensional summand. Consequently, $Y'$ is contained in a nonsingular hypersurface in $\mathbb{R}^{2r}$.

The orientability of $Y'$ is essential here. Indeed, we have:
Example 2.5. Let $f : \mathbb{P}^2(\mathbb{R}) \to \mathbb{R}^4$ be an algebraic embedding given by the formula

$$f((x_1 : x_2 : x_3)) = \frac{1}{x_1^2 + 2x_2^2 + 3x_3^2}(x_1^2 + x_2^2 + x_3^2, x_1x_2, x_1x_3, x_2x_3).$$

It is easy to see that $Y = f(\mathbb{P}^2(\mathbb{R}))$ is a nonsingular affine variety. According to [11], the normal bundle of any embedding of $\mathbb{P}^2(\mathbb{R})$ in $\mathbb{R}^4$ does not have a nowhere vanishing $C^\infty$ section. According to Theorem 2.2, $Y$ is not contained in any nonsingular hypersurface in $\mathbb{R}^4$.

In the general case the orientability of $Y$ does not help. Let $X^{2m}$ be a nonsingular variety and let $Y^m$ be a nonsingular orientable subvariety of $X^{2m}$. We show that in general there does not exist a nonsingular hypersurface $H \subset X^{2m}$, such that $Y^m \subset H$. Indeed we have:

Example 2.6. Let $m$ be an even number and let $S^m$ be an $m$-dimensional sphere. Let $F = T S^m$ be the tangent bundle of the sphere $S^m$. Now let $X^{2m}$ denote the total space of this vector bundle. Then $S^m \subset X^{2m}$ (as the zero-section) and $N_{X^{2m}/S^m} \cong F$. It is well-known that the bundle $F$ does not have a one-dimensional trivial summand. In particular $S^m$ is not contained in any nonsingular hypersurface in $X^{2m}$ (see Theorem 2.2).

3. Holomorphic case

Now we prove a similar result in the holomorphic setting. Let $X$ be a Stein manifold with the sheaf $\mathcal{O}$ of holomorphic functions. If $Y$ is a Stein submanifold of $X$, then the ideal $I(Y) = \{ f \in \mathcal{O}(X) : f|_Y = 0 \}$ is finitely generated (see [3], Theorem 7.5.4). If $I(Y)$ is principal we say that $Y$ is a hypersurface. For Stein manifolds, Lemma 2.1 takes the following form:

Lemma 3.1. Let $Y \subset X$ be Stein manifolds and let $s, g_1, \ldots, g_r$ be holomorphic functions on $X$ such that $d_is \neq 0$ for every $y \in Y$ and $I(Y) = (g_1, \ldots, g_r)$. Then the hypersurface $V(s + \sum_{i=1}^{r} \beta_i g_i^2)$ in $X$ is nonsingular for generic $\beta_i \in \mathbb{C}$.

Proof. Let $f : X \to \mathbb{C}$ be a holomorphic function. Then $0 \in \mathbb{C}$ is a regular value of $f$ if and only if $0 \in \mathbb{R}^2 \cong \mathbb{C}$ is a regular value of $f$ regarded as real $C^\infty$ map. This allows us to conclude arguing as in the proof of Lemma 2.1. 

We have the following counterpart of Theorem 2.2, with a completely analogous proof:

Theorem 3.2. Let $Y \subset X$ be Stein manifolds. Then $Y$ is contained in some nonsingular Stein hypersurface $V(f) \subset X$ if and only if the normal bundle of $Y$ contains a one-dimensional trivial summand i.e.,

$$N_{X/Y} = T \oplus E^1,$$

where $E^1$ denotes a trivial line bundle.
It is well-known that an $n$-dimensional Stein manifold has the homotopy type of a (real) $n$-dimensional CW complex (see [8]). Complex vector bundles on such CW complexes have the following nice property:

**Theorem 3.3.** ([6], p. 111) Let $Y$ be an $r$-dimensional CW complex and let $F$ be a complex vector bundle on $Y$ of rank $k$. If $r \leq 2k - 1$, then $F$ has a one-dimensional trivial summand.

**Proof of Theorem 1.3.** We will make use of Grauert’s theorem on the Oka principle for vector bundles which says that on Stein spaces the holomorphic and topological classifications coincide (see for example [2]). Therefore we can use the topological theory of complex vector bundles. Moreover, since every $n$-dimensional Stein manifold has a homotopy type of a (real) $n$-dimensional CW complex, if we study vector bundles on $X$, we can assume that $X$ itself is a $n$-dimensional CW complex.

The normal bundle $F = N_{X/Y}$ has rank $n - r$. By Theorem 3.3 it has a trivial one-dimensional summand if $r \leq 2(n - r) - 1$, i.e., if $(3r + 1)/2 \leq n$. ⊓⊔

Now we show that the assumption $(3r + 1)/2 \leq n$ is sharp.

**Example 3.4.** Consider the variety $\Gamma^5 = \{(x, y) \in \mathbb{C}^3 \times \mathbb{C}^3 : \sum_{i=1}^{3} x_i y_i = 1\}$. By the Raynaud Theorem (see [9]) the holomorphic vector bundle $F$ on $\Gamma$ given by the unimodular row $(x_1, x_2, x_3)$ is not free. Since every stably trivial line bundle is trivial and rank $F = 2$, we see that the vector bundle $F$ does not have a one-dimensional trivial summand.

Now let $X^7$ denote the total space of this vector bundle. Then $\Gamma \subset X$ (as the zero-section) and $N_{X/\Gamma} \cong F$. As we observed the bundle $F$ does not have a one-dimensional trivial summand. In particular $\Gamma$ is not contained in any nonsingular hypersurface in $X$ (see Theorem 3.2). Hence the inequality $(3r + 1)/2 \leq n$ cannot be replaced by the inequality $(3r + 1)/2 \leq n + 1$.

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**References**

[1] Bochnak, J., Coste, M., Roy, M.F.: Real Algebraic Geometry. Springer, Berlin (2005)
[2] Forster, O.: Topologische Methoden in der Theorie Steinischer Räume. Actes du Congres International des Mathematiciens, Nice, 1970, Tome 2, pp. 613–618, Gauthiers-Villars, Paris (1971)
[3] Forstneric, F.: Stein Manifolds and Holomorphic Mappings. Springer, Berlin (2011)
[4] Greco, S., Valabrega, P.: On the singular locus of a complete intersection through a variety in projective space. Bollettino U.M.I. VI-D 113-145 (1983)
[5] Guillemin, V., Pollack, A.: Differential Topology. Prentice-Hall Inc., Englewood Cliffs (1974)
[6] Husemoller, D.: Fibre Bundles. Springer, New York (1994)
[7] Kleiman, S., Altman, A.: Bertini theorems for hypersurface sections containing a subscheme. Commun. Algebra 7(8), 775–790 (1979)
[8] Milnor, J.: Morse Theory, Annals of Mathematics Studies, vol. 51. Princeton University Press, Princeton (1963)
[9] Raynaud, M.: Modules projectifs universels. Invent. Math. 6, 1–26 (1968)
[10] Serre, J.P.: Modules projectifs et espaces fibres a fibre vectorielle. Sem. Dubreil-Pisot, No. 23, Paris (1957/1958)
[11] Whitney, H.: On the Topology of Differentiable Manifolds. Lectures in Topology. University of Michigan Press, pp. 101–141 (1941)