Classically and Quantum Stable Emergent Universe in a Jordan-Brans-Dicke Theory

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Abstract

The study of Emergent Universe models is based on the assumption that the universe emerged from a past eternal Einstein Static (ES) state towards an inflationary phase and then evolves into a hot big bang era. These models are appealing since they provide specific examples of nonsingular (geodesically complete) inflationary universes. However, it has been pointed out by Mithani-Vilenkin [1–5] that certain Emergent Universe scenarios which have a classically stable ES state could experience a semiclassical instability and collapse. In this paper, we investigate the classical and quantum stability of the ES regime of Emergent Universes within the framework of Jordan-Brans-Dicke theory. We demonstrate that when considering these models, it is possible to have both, classical and semiclassical stability of the ES state without addressing the instability highlighted by Mithani-Vilenkin.

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1. INTRODUCTION

The standard cosmological model [6–8] and the inflationary paradigm [9–12] have successfully provided a description of our universe [6–8]. However, despite its great success, a series of significant questions remain. Among these questions is the inquiry into whether the universe possesses a definite origin, marked by an initial singularity, or if, in contrast, it extends infinitely to the past, implying an eternal existence without a discernible beginning. In this context, spacetime singularity theorems have been developed in the framework of inflationary theory, showing that the universe necessarily had a beginning. According to these theorems the existence of an initial singularity remains inevitable even if inflation takes place [13–17].

The search for cosmological models without initial singularities has led to the development of the so-called Emergent Universes models (EU) [18, 19]. These models do not satisfy the geometrical assumptions of the spacetime singularity theorems mentioned above.

The Emergent Universe refers to models in which the universe emerges from a past eternal Einstein Static (ES) state, inflates, and then evolves into a hot big bang era. The EU is an attractive scenario since it avoids the initial singularity and provides a smooth transition towards an inflationary period.

The original proposal for the emergent universe [18, 19] was developed in the context of General Relativity, then, the past eternal static period suffer from classical instabilities associated with the instability of Einstein’s static universe. The ES solution is unstable to homogeneous perturbations, as was early discussed by Eddington in Ref. [20] and more recently studied in Refs. [21–24]. The instability of the ES solution ensures that any perturbation, no matter how small, rapidly force the universe away from the static state, thereby aborting the EU scenario. This classical instability is possible to cure by going away from General Relativity. Several models have been developed in this regard [25–90], in particular the Jordan-Brans-Dicke (JBD) models, where it has been found that contrary to general relativity, a static universe could be classically stable against homogeneous perturbations, and also stable against anisotropic and inhomogeneous perturbations, see Refs. [91–95]. In this context, Mithani-Vilenkin in Refs. [1]–[5] have shown that certain static universe solutions which are classically stable could be unstable under a semiclassical quantum gravity effect and collapse.
In this work, we show that EU models constructed in the context of a JBD theory, with a self interacting potential and matter content corresponding to a scalar field \cite{91, 92, 94} could have a stable ES regimen, which can have both classical stability and do not suffer the quantum instability pointed out by Mithani-Vilenkin. This is a similar result to the case studied in Refs. \cite{57, 75}, for emergent universes derived from scale invariant two measures theories.

The Jordan-Brans-Dicke \cite{96} theory is a class of models in which the effective gravitational coupling evolves with time. The strength of this coupling is determined by a scalar field, the so-called Brans-Dicke field, which tends to the value $G^{-1}$, the inverse of the Newton’s constant. The origin of Brans-Dicke theory is in Mach’s principle according to which the property of inertia of material bodies arises from their interactions with the matter distributed in the universe. In modern context, Brans-Dicke theory appears naturally in supergravity models, Kaluza-Klein theories and in all the known effective string actions \cite{97, 103}.

The paper is organized as follows. In Sect. II we study the Hamiltonian formalism for a JBD theory following the ADM decomposition. In particular we consider the minisuperspace approach, compatible with the Friedmann-Robertson-Walker metric. In Sect. III by following a Hamiltonian approach, we study static universe solutions for Emergent Universes in the context of a JBD theory. We study the stability of these static solution under homogeneous and isotropic perturbations. In Sect. IV we constructed the Wheeler-DeWitt (WDW) equation in order to study the semiclassical stability of the static universe. We use the WKB approximation and the potential associated with the WDW equation to analyze the possibility of a semiclassically collapse of the static universe solution. In Sect. V we consider specific examples. In Sect. VI we summarize our results.

II. HAMILTONIAN FORMALISM FOR A JORDAN-BRANS-DICKE THEORY

We consider the following JBD action for a self-interacting potential and matter, given by \cite{104}\footnote{\cite{104}}

\[
S = \int d^4x \sqrt{-g} \left[ -\frac{1}{2} \Phi R + \frac{1}{2} \omega \partial_\mu \Phi \partial^\mu \Phi - V(\Phi) + L_m(\psi) \right],
\]  

(1)

\footnote{\cite{104}}
where $L_m(\psi)$ denote the Lagrangian density of the matter content

$$L_m(\psi) = \frac{1}{2} \partial_{\mu} \psi \partial^{\mu} \psi - U(\psi),$$

(2)

$\mathcal{R}$ is the Ricci scalar curvature, $\Phi$ is the JBD field, $\omega$ is the JBD parameter, $V(\Phi)$ is the potential associated to the JBD field, $\psi$ is the inflaton field and $U(\psi)$ its effective potential.

We use the minisuperspace approximation [105], which is appropriated for our model, where we are going to consider that the universe is homogeneous, isotropic and closed during the ES regimen, see [91, 92]. Then, the metric is given by the following expression

$$ds^2 = N^2(t) dt^2 - a(t)^2 \left( \frac{dr^2}{1 - r^2} + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2 \right).$$

(3)

By evaluating the JBD action, Eq. (1), in the metric Eq. (3) we obtain the following Lagrangian

$$L = -2\pi^2 a^3 N U(\psi) - 2\pi^2 a^3 N V(\Phi) + \frac{\pi^2 a^3 \dot{\Phi}^2 \omega}{N \Phi} + \frac{\pi^2 a^3 \dot{\psi}^2}{N} - \frac{6\pi^2 (a^2 \dot{\Phi})}{N}$$

\begin{equation}
+ 6\pi^2 a N \Phi - \frac{6\pi^2 (a \ddot{a}^2 \Phi)}{N}.
\end{equation}

(4)

The Hamiltonian of the model is given by

$$H = P_a \dot{a} + P_\Phi \dot{\Phi} + P_\psi \dot{\psi} - L,$$

(5)

where the conjugate momenta are given by

$$P_a = \frac{\partial L}{\partial \dot{a}} = -\frac{6\pi^2 a \left( 2\dot{a} \Phi + a \dot{\Phi} \right)}{N},$$

(6)

$$P_\Phi = \frac{\partial L}{\partial \dot{\Phi}} = \frac{2\pi^2 a^2 \left( \dot{\Phi} \omega - 3\dot{a} \Phi \right)}{N \Phi},$$

(7)

$$P_\psi = \frac{\partial L}{\partial \dot{\psi}} = \frac{2\pi^2 a^3 \dot{\psi}}{N},$$

(8)

then, we obtain

$$H = -\frac{N \omega}{12\pi^2 (3 + 2\omega) \Phi a} \left[ P_a^2 - \frac{6\Phi^2 P_\Phi^2}{\omega a} + \frac{6 \Phi P_a P_\Phi}{\omega a} - \frac{3 \Phi P_\psi^2 (3 + 2\omega)}{\omega a^2} + \frac{72\pi^4 \Phi^2 (3 + 2\omega) a^2}{\omega}$$

$$- \frac{24\pi^4 \Phi a^4 (3 + 2\omega) V(\Phi)}{\omega} - \frac{24\pi^4 \Phi a^4 (3 + 2\omega) U(\psi)}{\omega} \right] = N \mathcal{H}.$$

(9)
The classical Hamiltonian constraint is \( \mathcal{H} = 0 \). The classical field equations are given by,

\[
\dot{P}_a = -\frac{\partial \mathcal{H}}{\partial a}, \quad \dot{P}_\Phi = -\frac{\partial \mathcal{H}}{\partial \Phi}, \quad \dot{P}_\psi = -\frac{\partial \mathcal{H}}{\partial \psi},
\]

which we can write as follows

\[
\left( \frac{\dot{a}}{a} \right)^2 + \frac{1}{a^2} + \frac{\dot{\Phi}}{\Phi} = \frac{\rho}{3\Phi} + \frac{\omega}{6} \left( \frac{\dot{\Phi}}{\Phi} \right)^2 + \frac{V}{3\Phi}, \tag{10}
\]

\[
\frac{2\ddot{a}}{a} + H^2 + \frac{1}{a^2} + \frac{\ddot{\Phi}}{\Phi} + 2H \frac{\dot{\Phi}}{\Phi} + \frac{\omega}{2} \left( \frac{\dot{\Phi}}{\Phi} \right)^2 - \frac{V}{\Phi} = -\frac{P}{\Phi}, \tag{11}
\]

\[
\ddot{\Phi} + 3H \dot{\Phi} = \frac{\rho - 3P}{2\omega + 3} + \frac{2}{2\omega + 3} [2V - \Phi V'], \tag{12}
\]

\[
\ddot{\psi} + 3H \dot{\psi} = -\frac{\partial U(\psi)}{\partial \psi}, \tag{13}
\]

where \( \rho = \frac{\dot{\psi}^2}{2} + U(\psi) \), \( P = \frac{\dot{\psi}^2}{2} - U(\psi) \) and \( V' = \frac{dV(\Phi)}{d\Phi} \).

The equations (10)-(13), determine the classical evolution of the JBD model described by the action Eq. (1).

III. STATIC UNIVERSE SOLUTION AND ITS CLASSICAL STABILITY

In this section we review, by following a Hamiltonian approach, the static universe solution discussed in Refs. [91, 92].

The static universe solution in a JBD theory is characterized by the conditions \( a = a_0 = \text{Constant}, \dot{a} = 0 = \ddot{a} \) and \( \Phi = \Phi_0 = \text{Constant}, \dot{\Phi} = 0 = \ddot{\Phi} \), see Ref. [91].

Following the scheme of the model in Ref. [91], we are going to consider that the matter potential \( U(\psi) \) is flat, that is \( U(\psi) = U_0 = \text{Constant} \). This is justified, since usually in the EU models it is considered that during the static regimen the inflaton field \( \psi \) is rolling in the flat section of its potential, see Ref. [91]. Then, we can notice that during the static regimen the momentum \( P_\psi \) is conserved. Then we can write the Hamiltonian constraint as follows

\[
\mathcal{H} = -\frac{\omega}{12\pi^2(3 + 2\omega)} \frac{\Phi}{a} \left[ P_a^2 - \frac{6\Phi^2 P_\Phi^2}{\omega a} + \frac{6\Phi P_a P_\Phi}{\omega a} + U(a, \Phi) \right] = 0, \tag{14}
\]

where we have defined the following effective potential
\[ U(a, \Phi) = -\frac{3 P_\psi^2 \Phi(2\omega + 3)}{a^2 \omega} + \frac{72\pi^4 a^2 \Phi^2 (2\omega + 3)}{\omega} - \frac{24\pi^4 a^4 U_0' \Phi(2\omega + 3)}{\omega} \]

In the effective potential \( U(a, \Phi) \) we have included the term proportional to \( P_\psi \) given that this momentum is conserved.

From the Hamiltonian constraint, Eq. (10), and the Hamilton’s equations we obtain that the effective potential \( U(a, \Phi) \) satisfies the following conditions in order to have a static universe solution at \( a = a_0 \) and \( \Phi = \Phi_0 \) in a JBD theory

\[ U(a_0, \Phi_0) = 0, \]  
\[ \frac{\partial U}{\partial a}(a_0, \Phi_0) = 0, \]  
\[ \frac{\partial U}{\partial \Phi}(a_0, \Phi_0) = 0. \]

The conditions (16, 17, 18) are satisfied if the following equations are fulfilled

\[ V_0' = \frac{3}{a_0^2}, \]  
\[ P_\psi^2 = 8\pi^4 a_0^4 \Phi_0, \]  
\[ U_0 + V_0 = \frac{2\Phi_0}{a_0^2}. \]

In order to study the stability of the static solution described above, against small homogeneous and isotropic perturbations, we study the Hamiltonian Eq. (9) near the static solution. In order to do this, we consider small perturbations around the static solution for the scale factor and the JBD field. We set

\[ a(t) = a_0[1 + \varepsilon(t)], \]  
\[ \Phi(t) = \Phi_0[1 + \beta(t)], \]

where \( \varepsilon \ll 1 \) and \( \beta \ll 1 \) are small perturbations.

The Hamiltonian Eq. (9) near the static solution is given by
\[ \tilde{H}(\varepsilon, \beta) = -\frac{N\omega}{12\pi^2a_0^3\Phi_0(2\omega + 3)} \left[ P_\varepsilon^2 - \frac{6P_\varepsilon^2}{\omega} + \frac{6\varepsilon P_\varepsilon}{\omega} - \frac{a_0^4\Phi_0^2(144\pi^4(2\omega + 3))}{\omega} \varepsilon \beta \right] \] 

\[ -\frac{a_0^6\Phi_0^3(12\pi^4(2\omega + 3))V_0''}{\omega} \beta^2 - \frac{a_0^4\Phi_0^2(288\pi^4(2\omega + 3))}{\omega} \varepsilon^2, \]

where \( V_0'' = (d^2V(\Phi)/d\Phi^2)_{\Phi=\Phi_0} \). The conjugate momenta associated to \( \varepsilon \) and \( \beta \) are given by

\[ P_\varepsilon = -\frac{6\pi^2a_0^3\Phi_0}{N} \varepsilon \] \[ P_\beta = \frac{2\pi^2a_0^3\Phi_0}{N} \beta \]

We can write Eq. (24) as follow

\[ \tilde{H} = \frac{1}{2} Q_{ab} q^a q^b + \frac{1}{2} P_{ab} p_a p_b, \quad a, b = 1, 2, \] 

where \( q_a = (\varepsilon, \beta) \), \( p_a = (P_\varepsilon, P_\beta) \). The 2 × 2 constant matrices \( P \) and \( Q \) are given by

\[ P = \frac{N}{2\pi^2a_0^3\Phi_0(2\omega + 3)} \begin{pmatrix} \frac{\omega}{2} & \frac{1}{2} \\ \frac{1}{2} & -1 \end{pmatrix} \]

\[ Q = N\pi^2a_0\Phi_0 \begin{pmatrix} 24 & 6 \\ 6 & a_0^2\Phi_0 V_0'' \end{pmatrix} \]

By using the Hamilton equation in Eq. (27) we obtain

\[ \frac{dq^a}{dt} = \frac{\partial \tilde{H}}{\partial p_a} = P_{ab} p_b. \]

We take the derivative of this expression obtaining

\[ \frac{d^2q^a}{dt^2} = P_{ab} \frac{dp_b}{dt} = -P_{ab} Q_{bc} q^c, \]

where we have used that \( \frac{dp_b}{dt} = \frac{\partial H}{\partial q^b} = -Q_{bc} q^c \). From Eq. (31) we notice that the eigenvalues of the matrix \( \Lambda = PQ \) are the square of the frequencies for the small oscillations around the static solution. The eigenvalues of \( \Lambda \) are

\[ \lambda_{\pm}^2 = \frac{N^2}{(3 + 2\omega)a_0^2} \left[ a_0^2\Phi_0 V_0'' - 2(3 + 2\omega) \pm \sqrt{[a_0^2\Phi_0 V_0'']^2 + 4a_0^2\Phi_0 V_0''(3 + 2\omega) + 8\omega(3 + 2\omega)} \right]. \]
The static solution is stable if \( \lambda^2 > 0 \). Then assuming that the parameter \( \omega \) satisfies \((3 + 2\omega) > 0\), we find that the following inequalities must be satisfied in order to have a stable static solution

\[
0 < a_0^2 \Phi_0 V_0'' < \frac{3}{2},
\]

\[
-\frac{3}{2} < \omega < -\frac{1}{4} \left[ \sqrt{9 - 6a_0^2 \Phi_0 V_0''} + (3 + a_0^2 \Phi_0 V_0'') \right].
\]

These inequalities restrict the parameters of the model, as discussed in Refs. [91, 92]. The first imposes a condition on the JBD potential, specifically for its first and second derivatives: \( 0 < V_0'' < V_0'/\Phi_0 \). The second inequality restricts the values of the JBD parameter.

We can note that, contrary to General Relativity, the static universe solution could be stable against homogeneous and isotropic classical perturbations.

In the next section we turn our attention to possible quantum instability of this static solution, similar to the ones discussed in Refs. [1–5].

IV. WHEELER-DEWITT EQUATION

In the context of quantum theory, the universe could be described by a wave function \( \Psi(a, \Phi) \), the conjugate momenta \( P_a \) and \( P_\Phi \) become operators \( P_a \to \hat{P}_a \), \( P_\Phi \to \hat{P}_\Phi \). In this context the Hamiltonian constraint (14) is replaced by the Wheeler-DeWitt (WDW) equation \( \mathcal{H}\Psi(a, \Phi) = 0 \), see [106]. Then, we have

\[
\left[ \hat{P}_a^2 - \frac{6\Phi^2 \hat{P}_\Phi^2}{\omega a} + \frac{6\Phi \hat{P}_a \hat{P}_\Phi}{\omega a} + U(a, \Phi) \right] \Psi(a, \Phi) = 0.
\]

In order to obtain Eq. (35), we have used the minisuperspace approximation, which is appropriate for our model where the universe is homogeneous, isotropic and closed during the ES regimen [105]. Also in equation (35), we have not included the terms related to the ambiguity in the ordering of the non-commuting factors in the Hamiltonian. These terms do not affect the wave function in the semiclassical regimen and usually in the study of semiclassical stability of EU these terms are not included, see [15, 57].

With the purpose of study the possibility of a semiclassical collapse of the static universe \((a \to 0)\) we study the properties of the effective potential \( U(a, \Phi) \), Eq. (15), near \( a \approx 0 \) and
$\Phi \approx 0$. In this study, in order to set the parameters of the model and of the JBD potential $V(\Phi)$, we take into account the classical stability conditions discussed in the previous section.

In the WKB approximation the probability of collapse through quantum tunneling from the static solution could be estimated by the expression\textsuperscript{11},

$$P \sim e^{-2S},$$

where the WKB tunneling action is given by

$$S = \int \sqrt{U(a, \Phi)} \ dl.$$ \hspace{1cm} (37)

The differential $dl$ refers to the possible ways that may exist to get from the stable point $(a_0, \Phi_0)$ to $(0, \Phi)$ or $(a, 0)$. As a first approach to the problem we will consider the case in which the JBD field is maintained at the equilibrium point ($\Phi = \Phi_0$) and it is the scale factor, the variable that is tunneled from $a = a_0$ to $a = 0$. In this case we have that Eq. (37) becomes:

$$S = \int_{\epsilon}^{a_0} \sqrt{U(a, \Phi_0)} \ da,$$ \hspace{1cm} (38)

where we are going to consider the limit $\epsilon \rightarrow 0$ at the end. We can notice that near $a \approx 0$ the only relevant term in the effective potential $U(a, \Phi)$ is the first term in Eq. (15). Then we have

$$S \approx \int_{\epsilon}^{a_0} \sqrt{-\frac{3 P_\psi^2 \Phi_0 (2\omega + 3)}{a^2 \omega}} \ da = \sqrt{-\frac{3 P_\psi^2 \Phi_0 (2\omega + 3)}{\omega}} \ln(a_0/\epsilon).$$ \hspace{1cm} (39)

Therefore we obtain that the probability of collapse through quantum tunneling from the static solution is given by

$$P \sim \left(\frac{\epsilon}{a_0}\right) \sqrt{\frac{(-12 P_\psi^2 \Phi_0 (2\omega + 3))/\omega}{}}.$$ \hspace{1cm} (40)

We can note that in the limit $\epsilon \rightarrow 0$, $P$ vanish.

On the other hand, if we calculate the probability of tunneling of the JBD field from the equilibrium point $\Phi_0$ to zero ($\Phi = 0$), but when the scale factor remain in the equilibrium point ($a = a_0$), we have that

$$S = \int_{\epsilon}^{\Phi_0} \sqrt{U(a_0, \Phi)} \ d\Phi.$$ \hspace{1cm} (41)

In this case the relevant term in the effective potential $U(a, \Phi)$ is the fourth term in Eq. (15). Then we have

$$S = \int_{\epsilon}^{\Phi_0} \sqrt{-\frac{24 \pi^4 a_0^4 \Phi (2\omega + 3) V(\Phi)}{\omega}} \ d\Phi.$$ \hspace{1cm} (42)
If we assume that near $\Phi \sim 0$ the JBD potential behaves as $V(\Phi) \sim \frac{C}{\Phi^\alpha}$, where $C$ and $\alpha$ are positive constants, we can write

$$S = \int_{\epsilon}^{\Phi_0} \sqrt{- \frac{24\pi^4a_0^4(2\omega + 3)}{\omega} \frac{C}{\Phi^\alpha}} d\Phi$$

$$= \left( \frac{-2}{\alpha - 3} \right) \sqrt{- \frac{24\pi^4a_0^4(2\omega + 3)C}{\omega} \left[ \Phi_0^{(3-\alpha)/2} - \epsilon^{(3-\alpha)/2} \right]}.$$

Then, the probability of tunneling of the JBD field from the equilibrium point $\Phi_0$ to zero ($\Phi_0 \to 0$) is given by

$$P \sim \exp \left\{ - \frac{D}{\epsilon^{(\alpha-3)/2}} \right\},$$

where $D$ is a constant given by $D = \frac{8\pi^2}{(\alpha-3)} \sqrt{- \frac{6C\alpha^4(3+2\omega)}{\omega}}$. In the limit $\epsilon \to 0$, $P$ vanish for $\alpha > 3$.

Therefore, we can conclude that in the context of a JBD theory it is possible to have a static universe solution ES, which is stable under classical perturbations, similar to those studied in [91, 92], but also, it does not suffer from the quantum instability pointed out by Mithani-Vilenkin [1–5], if the JBD potential behaves as $V(\Phi) \sim \frac{C}{\Phi^\alpha}$, with $C$ a positive constant and $\alpha > 3$, when $\Phi$ is near to zero.

In this way, we have that it is possible to obtain a past eternal Static Universe in a JBD theory, in the context of the EU scheme, depending upon the characteristics of the JBD potential and the Brans-Dicke parameter.

V. SPECIFIC EXAMPLES

As a first example we will consider the model studied in Refs. [91, 92]. In these works, the parameters of the JBD model were adjusted to satisfy the classical equilibrium conditions of the static Universe solution at $a = a_0$ and $\Phi = \Phi_0$. Particularly, these works consider a JBD potential similar to the following.

$$V(\Phi) = V_0 + A(\Phi - \Phi_0) + \frac{1}{2}B(\Phi - \Phi_0)^2.$$  (46)
FIG. 1: The plots were obtained by using the JBD potential, Eq. (46), and by considering $a_0 = 5.4$, $\Phi_0 = 0.9$ and $\omega = -1.45$. In the plots (a) and (b) we assume that the value of the $\Phi$ field is fixed and the scale factor $a$ is varied.

The constants in the potential (46) were fixed by consider the classical stability conditions of the ES solution discussed in the previous section. Then we have

$$V_0 = \frac{2\Phi_0}{a_0^2} - U_0,$$

(47)

$$A = \frac{3}{a_0^2},$$

(48)

$$B = \frac{X}{a_0^2\Phi_0}.$$  \hspace{1cm} (49)

The dimensionless parameter $X$, satisfy $0 < X < 3/2$ and the JBD parameter requires

$$-\frac{3}{2} < \omega < -\frac{\sqrt{3}}{4} \sqrt{3 - 2X} - \frac{1}{4}(3 + X).$$

(50)

In order to plot the effective potential $U(a, \Phi)$ we take the following values for the parameters in the JBD potential $\Phi_0 = 0.9$, $a_0 = 5.4$, $X = 1$ and $\omega = -1.45$, where units are such that $8\pi G = 1$ and $c = \hbar = 1$. These particular parameters satisfy all the classical equilibrium conditions of the static Universe solution discussed previously, for the static Universe solution at $a = a_0$ and $\Phi = \Phi_0$.

In Figs. (1) (2) it is plot the effective potential $U(a, \Phi)$ where it is consider the JBD potential (46) and the parameters discussed above. In particular in Fig. (1) it is consider $\Phi = \Phi_0$ fixed and we plot $U(a, \Phi_0)$ as a function of $a$. We notice that the effective potential has a local minimum at $a_0 = 5.4$ which is classically stable, as we expect. In the left panel
FIG. 2: The plots were obtained by using the JBD potential, Eq. (46), and by considering $a_0 = 5.4$, $\Phi_0 = 0.9$ and $\omega = -1.45$. In figures (a) and (b), we consider the case where the value of the scale factor remains constant at $a = a_0$, while the JBD field $\Phi$ is varied.

It is plot the effective potential near the equilibrium point $a \sim a_0$ and in the right panel it is consider $a$ in a wider region. We can notice that the quantum tunneling of the scale factor from $a = a_0$ to $a = 0$ is not possible, as we expect given the results of the previous section, see Eq. (40). In Fig. (2), we consider $a = a_0$ fixed and we plot $U(a_0, \Phi)$ as a function of $\Phi$. We can notice that the effective potential has a local minimum at $\Phi = \Phi_0 = 0.9$ which is classically stable, as we expect. However, from Fig. (2)-(a), it can be observed that there exist the possibility for the JBD field to tunnel through the finite-sized barrier from $\Phi = \Phi_0$ to $\Phi = 0$. Then, it can be concluded that the static universe solution discussed in Ref. [91], which is classically stable, may collapse ($\Phi \to 0$) via quantum tunneling. However, this situation can be easily remedied by including a term in the JBD potential that corrects its behavior as the JBD field approaches zero, in accordance with the previous section’s discussion.

As a second example, we consider a model similar to the one developed in Ref. [91], but where the JBD potential has been corrected to incorporate the term discussed in the previous section, which prevents the decay $\Phi \to 0$.

$$V(\Phi) = A(\Phi - \Phi_0) + \frac{1}{2}B(\Phi - \Phi_0)^2 + \frac{C}{\Phi^6},$$

(51)

this potential takes into account the classical stability conditions discussed in Section III but also, the stability conditions under quantum tunneling discussed in the previous Section.
FIG. 3: The plots were obtained by using the Brans-Dicke potential (51), and by considering $a_0 = 5.4$, $\Phi_0 = 0.9$ and $\omega = -1.45$. In the plots (a) and (b) we assume that the value of the field is fixed at $\Phi = \Phi_0$ and the scale factor $a$ is varied.

The parameters $V_0$, $A$, $B$ are given by

\[ V_0 = \frac{C}{\Phi_0^6}, \]
\[ A = \frac{3}{a_0^2} + \frac{6C}{\Phi_0^7}, \]
\[ B = \frac{X}{a_0^2\Phi_0} - \frac{42C}{\Phi_0^8}. \]

Similar to the previous example, the dimensionless parameter $X$ satisfy $0 < X < 3/2$ and the JBD parameter requires $-\frac{3}{2} < \omega < -\frac{\sqrt{3}}{4} \sqrt{3 - 2X} - \frac{1}{4}(3 + X)$. For this case we consider $\Phi_0 = 0.9$, $a_0 = 5.4$, $C = 0.0002$, $X = 1$ and $\omega = -1.45$, these numerical values satisfy the classical stability conditions, Eqs. (33)-(34) mentioned in section III. Units are such that $8\pi G = 1$ and $c = \hbar = 1$. The effective potential $U(a, \Phi)$ is plotted in Figs. (3, 4), taking into account the JBD potential (51) and the parameters discussed above.

In Fig. (3) it is is consider $\Phi = \Phi_0$ fixed and we plot $U(a, \Phi_0)$ as a function of the scale factor $a$. Similar to the first example, we can notice that the effective potential has a local minimum at $a_0 = 5.4$ which is classically stable and the quantum tunneling of the scale factor from $a_0$ to $a = 0$ is not possible, as we expect given the results of previous section.

In Fig. (4) it is is consider $a = a_0$ fixed and we plot $U(a_0, \Phi)$ as a function of the JBD field $\Phi$. We can note that in this case the effective potential has a local minimum at $\Phi_0 = 0.9$ which is classically stable and the quantum tunneling of the JBD field from $\Phi = \Phi_0$ to $\Phi = 0$
FIG. 4: The plots were obtained by using the JBD potential (51), considering $a_0 = 5.4$, $\Phi_0 = 0.9$ and $\omega = -1.45$. In figures (a) and (b), we consider the case where the value of the scale factor remains constant at $a = a_0$, while the JBD field $\Phi$ is subjected to variations. It is not possible, since the potential diverges when $\Phi \to 0$, which prevents the system from decaying from $\Phi = \Phi_0$ to $\Phi = 0$, in conformity with what was discussed in the previous section.

Therefore, we can conclude that within the context of a Jordan-Brans-Dicke theory, a static universe solution can be achieved. This solution remains stable under classical perturbations, akin to those examined in previous studies [91, 92]. Also, this solution is not subject to the quantum instability identified by Mithani-Vilenkin [1–5].

VI. CONCLUSIONS

In this paper, we study the classical and quantum stability of the Einstein Static (ES) universe solution in the context of a JBD theory. This study is motivated by the Emergent Universe scenario, where it is considered that the universe emerges from a past eternal static state towards an inflationary regime and then evolves into a hot big bang era. We focus on examining the stability of the static universe solution in the context of a JBD model, with a particular interest in a class of instabilities identified by Mithani-Vilenkin as a semiclassical quantum gravity effect [1–5], which can arise even in a static universe solution that is classically stable.

In particular, in this work, we present an explicit construction of a classically stable ES
universe, in the context of a JBD model, that is not subject to the quantum instability highlighted by Mithani-Vilenkin.

In the first part of the paper, Sect. II we study the Hamiltonian formalism for a JBD theory following the ADM decomposition. In particular we consider the minisuperspace approach, compatible with the Friedmann-Robertson-Walker metric. Then, we proceed to review the Static Universe solution discussed in Refs. [91, 92], by employing a Hamiltonian approach, which is suitable for the semiclassical quantum gravity calculation in the subsequent sections. In particular, we examine the stability of these Static Universe solutions under homogeneous and isotropic perturbations. We can notice that, in contrast to General Relativity, in a JBD theory Static Universe solutions could be stable against homogeneous and isotropic classical perturbations, as was discussed in Refs. [91, 92]. In Sect. IV by using the Hamiltonian found in the previous sections, we constructed the Wheeler-DeWitt (WDW) equation associated to this model, in order to study the semiclassical stability of the Static Universe solution. We consider the WKB approximation and the effective potential associated with the WDW equation to analyze the probability of the semiclassical collapse. We found that if we impose a extra condition to the JBD potential $V(\Phi)$ for values of $\Phi$ near zero, the ES solution does not suffer from the quantum instability pointed out in Refs. [1-4]. In Sect. V we consider specific examples.

In summary, in this work we show that in the context of a JBD theory it is possible to have a static universe solution that is classically and semiclassically stable without suffering the quantum instability pointed out by Mithani-Vilenkin towards collapse.

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