Pseudo-Killing Spinors, Pseudo-supersymmetric $p$-branes, Bubbling and Less-bubbling AdS Spaces

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ABSTRACT

We consider Einstein gravity coupled to an $n$-form field strength in $D$ dimensions. Such a theory cannot be supersymmetrized in general, we nevertheless propose a pseudo-Killing spinor equation and show that the AdS$\times$Sphere vacua have the maximum number of pseudo-Killing spinors, and hence are fully pseudo-supersymmetric. We show that extremal $p$-branes and their intersecting configurations preserve fractions of the pseudo-supersymmetry. We study the integrability condition for general $(D, n)$ and obtain the additional constraints that are required so that the existence of the pseudo-Killing spinors implies the Einstein equations of motion. We obtain new pseudo-supersymmetric bubbling AdS$_5 \times S^5$ spaces that are supported by a non-self-dual 5-form. This demonstrates that non-supersymmetry conformal field theories may also have bubbling states of arbitrary droplets of free fermions in the phase space. We also obtain an example of less-bubbling AdS geometry in $D = 8$, whose bubbling effects are severely restricted by the additional constraint arising from the integrability condition.
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1 Introduction

The original proposal of the AdS/CFT correspondence \cite{1,2,3} is under the framework of supergravity or superstring. Supersymmetry provides a powerful organization to test the correspondence. The holographic principle underlying the AdS/CFT correspondence, however, goes beyond supersymmetry. Application of the AdS/CFT correspondence in non-supersymmetric gauge theories has been a flourishing research topic since the inception of the AdS/CFT correspondence.

The AdS/CFT correspondence connects the strongly-coupled gauge theory in the boundary of the anti-de Sitter spacetime (AdS) with classical gravity in the bulk. The key to apply the correspondence is to construct the appropriate classical gravity background that is dual to the boundary field theory which is typically impossible to study on its own due to its strong-coupling nature. The construction of the bulk geometry belongs to the conventional subject of general relativity. There is a significant advantage of supersymmetry which enables one to construct a large class of BPS solutions that preserve a certain fraction of the maximum number of the supersymmetry of the theory. For simple supergravities, it was demonstrated that the assumption of the existence of a Killing spinor, a defining property of BPS solutions, enables one to derive all the supersymmetric solutions of the theory \cite{4}. Employing the same technique, the most general $1/2$-BPS solutions in type IIB supergravity supported by the self-dual 5-form field strength was constructed in \cite{5}. These solutions, called LLM bubbling AdS spaces, describe smooth geometries asymptotic to $\text{AdS}_5 \times S^5$ and can be reduced to solutions of certain linear Laplace equation with two-dimensional bubble-like boundary conditions. These smooth geometries are dual to arbitrary droplets of free fermions in phase space of the dual conformal field theory. Such solutions are very unlikely to be found without the aid of supersymmetry.

One motivation of this paper is to investigate whether it is possible to construct such highly non-trivial bubbling geometry in a theory that cannot be supersymmetrized. This will resolve the issue whether arbitrary free-fermion droplets can also arise in a non-supersymmetric gauge theory. The bubbling AdS spaces obtained in type IIB supergravity and M-theory involve very simple field configurations: gravity coupled with the self-dual 5-form or 4-form field strengths respectively. Indeed such a system provides the simplest origin of the cosmological constant in lower dimensions. We thus consider Einstein gravity coupled to an $n$-form field strength in general $D$ dimensions. For specific cases, namely $(D,n) = (11,4), (10,5), (6,3), (5,2), (4,2)$, the system becomes (part of) the bosonic action of a supersymmetric theory. (To be precise, in 10 and 6 dimensions, supersymmetry requires
that the 5-form and 3-form be self-dual.) In general, however, the \((D, n)\) system cannot be made to or part of a supersymmetry theory. Our goal is to investigate whether such an intrinsically non-supersymmetric system may nevertheless admit solutions with characteristics of BPS solutions in supergravities. For simple cohomogeneity-one classes, it was known that even the Schwarzschild-AdS black hole can be solved by the super-potential method in which the second-order differential equations can be successfully reduced to a set of first-order equations via a super potential \([6, 7]\). (See also \([8]\).) This suggests that certain non-supersymmetric systems may exhibit characteristics of supersymmetry; they are pseudo-supersymmetric. Pseudo-supersymmetry for gravity coupled to scalars were introduced in \([9]\). Pseudo-supersymmetry for de Sitter “supergravity” was discussed in \([10, 11]\).

In section 2, we introduce a pseudo-Killing spinor equation for the \((D, n)\) system. It involves one \((n - 1)\)-gamma structure and one \((n + 1)\)-gamma structure,

\[
D_M \epsilon + \frac{\tilde{\alpha}}{(n - 1)!} \Gamma^{M_1 \cdots M_{n-1}} F_{M_1 \cdots M_{n-1}} \epsilon + \frac{\tilde{\beta}}{n!} \Gamma^{M_1 \cdots M_n} F_{M_1 \cdots M_n} \epsilon = 0, \tag{1.1}
\]

with two constant parameters \((\tilde{\alpha}, \tilde{\beta})\) to be determined, as follows. The \((D, n)\) system admits two AdS×Sphere vacua with the \(n\)-form carries either the electric or the magnetic fluxes. We can fix the value of \(\tilde{\alpha}\) and \(\tilde{\beta}\) up to a relative sign, by requiring that the vacua have the maximum number of allowed pseudo-Killing spinors. These pseudo-Killing spinors are tensor products of real Killing spinors in the AdS spaces and spheres. Note that the overall sign of \((\tilde{\alpha}, \tilde{\beta})\) can be absorbed into the \(n\)-form field strength \(F\). We then obtain the explicit pseudo-Killing spinors for the AdS×Sphere vacua in section 3.

In section 4, we obtained both the electric and magnetic extremal \(p\)-branes for the \((D, n)\) system. We show, by explicit construction, that the existence of the pseudo-Killing spinors for these solutions fixes the relative sign of \((\tilde{\alpha}, \tilde{\beta})\). This thus fully determines the pseudo-Killing spinor equation. It turns out that the \(p\)-branes all preserve half of the maximum number of the pseudo-Killing spinors of the vacua. It should be emphasized that in the special cases mentioned above for which the system can be indeed supersymmetrized, the pseudo-Killing spinors become real Killing spinors of the supersymmetric theory. The corresponding brane solutions are the previously-known BPS \(p\)-branes in supergravities. In section 5, we obtain a large class of pair-wise intersecting \(p\)-branes, and construct the corresponding pseudo-Killing spinors.

Even in a supersymmetric theory, the existence of Killing spinor of a bosonic configuration does not always imply that it would satisfy the equations of motion. Additional constraints have to be imposed, e.g. the Killing vector constructed from the Killing spinor.
has to be time-like or null. In section 6, we investigate the integrability condition of the pseudo-Killing spinor equation for generic \((D, n)\). We obtain additional algebraic quadratic constraints on the form fields. These constraints, although satisfied by the vacua and \(p\)-branes discussed earlier, give rise to severe restrictions on possible solutions one may have. For \(n = 2, 3, 4, 5\), there exist critical dimensions in each case, corresponding precisely to relevant supergravities, where such a restriction vanishes.

For \(n = 5\), the critical dimension is 10 and all the constraints vanish if the 5-form is self-dual. The resulting theory is then the \(SL(2, R)\) singlet of type IIB supergravity. If we relax the requirement that the 5-form be self-dual, there are extra constraints from the integrability condition. However, we find that these conditions can be satisfied provided that the non-vanishing components of the 5-form are restricted to lie in a sub-manifold of seven dimensions. This property enables us to construct explicitly a new pseudo-supersymmetric bubbling geometry in the \((D, n) = (10, 5)\) system. The result is presented in section 7. The 5-form is not self-dual and cannot be made so by adjusting parameters in the solution. Thus the system is intrinsically non-supersymmetric and cannot be embedded in type IIB supergravity. Our construction demonstrates that bubbling AdS geometries are not uniquely possessed by supergravities, and consequently, some strongly-coupled non-supersymmetric conformal gauge theory in four-dimensions can also have bubbling states, \(i.e\). the arbitrary droplets of free fermions in the phase space.

In section 8, we construct analogous configurations in \((D, n) = (8, 4)\) to examine the effect of additional constraints on the bubbling nature of the asymptotic \(AdS_4 \times S^4\) geometries. Although the pseudo-Killing spinor equation still leads to a linear Laplace equation, an additional non-linear constraint has to be imposed from the integrability condition. This implies that linear supposition of the solutions, the key to the bubbling effect, is no longer applicable, and the geometry becomes less bubbling. Although we are unable to obtain the most general solution for the non-linear system, we argue that it is not unreasonable to expect that additional less-bubbling solutions beyond the vacua may still exist, which may reveal some specific droplets of free fermions that are allowed in the phase space.

We conclude our paper in section 9. Appendices A and B are detail construction of the bubbling and less-bubbling geometries presented in sections 7 and 8 respectively.
2 The theory and pseudo-Killing spinors

The theory we consider throughout this paper is the Einstein-Hilbert action coupled to an $n$-form field strength. The Lagrangian is given by

$$\mathcal{L} = \sqrt{-g} \left( R - \frac{1}{2n!} F_{(n)}^2 \right),$$  \hspace{1cm} (2.1)

where $F_{(n)} = dA_{(n-1)}$. The Bianchi identity and the equation of motion for the $n$-form are

$$dF_{(n)} = 0 \quad \rightarrow \quad \partial_M F_{M_1M_2\ldots M_n} = 0,$$
$$d*F_{(n)} = 0 \quad \rightarrow \quad \frac{1}{\sqrt{-g}} \partial_M F^{M_1M_2\ldots M_n} = 0.$$  \hspace{1cm} (2.2)

The Einstein equations of motion are given by

$$R_{MN} = \frac{1}{2(n-1)!} \left( F_{MN}^2 - \frac{n-1}{n(D-2)} F_{MJ}^2 g_{MN} \right).$$  \hspace{1cm} (2.3)

The system admits AdS$_n \times S^{D-n}$ and AdS$_{D-n} \times S^n$ vacuum solutions, carrying electric and magnetic fluxes respectively. The AdS$_n \times S^{D-n}$ solution is given by

$$ds_D^2 = ds_{AdS_n}^2 + d\Sigma_{D-n}^2, \quad F_{(n)} = \frac{2\lambda d}{\sqrt{\Delta}} \omega_{AdS_n},$$  \hspace{1cm} (2.4)

where $ds_{AdS_n}^2$ and $d\Sigma_{D-n}^2$ are metrics for AdS$_n$ and $S^{D-n}$ respectively, with

$$R_{\mu\nu} = -(n-1)\lambda^2 g_{\mu\nu}, \quad R_{ij} = \tilde{d}\bar{\lambda}^2 g_{ij}, \quad \lambda d = \tilde{\lambda} \tilde{d},$$  \hspace{1cm} (2.5)

and $\omega_{AdS_n}$ is the volume form for $ds_{AdS_n}^2$. Note that for convenience, we have introduced

$$d = n - 1, \quad \tilde{d} = D - n - 1, \quad \Delta = \frac{2\tilde{d} \tilde{\lambda}}{D-2}.$$  \hspace{1cm} (2.6)

The AdS$_{D-n} \times S^n$ solution is given by

$$ds_D^2 = ds_{AdS_{D-n}}^2 + d\Sigma_n^2, \quad F_{(n)} = \frac{2\lambda d}{\sqrt{\Delta}} \omega_{S^n},$$  \hspace{1cm} (2.7)

where

$$R_{\mu\nu} = -\tilde{d}\bar{\lambda}^2 g_{\mu\nu}, \quad R_{ij} = d\lambda^2 g_{ij}, \quad \lambda d = \bar{\lambda} \tilde{d}.$$  \hspace{1cm} (2.8)

We now introduce a spinor $\hat{\epsilon}$ that satisfies the following equation

$$D_M \hat{\epsilon} + \frac{\tilde{\alpha}}{(n-1)!} \Gamma_{M_1\ldots M_{n-1}}^{M_1\ldots M_n} F_{M_1M_2\ldots M_{n-1}} \hat{\epsilon} + \frac{\tilde{\beta}}{n!} \Gamma_M^{M_1\ldots M_n} F_{M_1\ldots M_n} \hat{\epsilon} = 0,$$  \hspace{1cm} (2.9)

where $D_M$ is covariant derivative defined by

$$D_M \hat{\epsilon} \equiv \partial_M \hat{\epsilon} + \frac{1}{2} (\omega_M)^a_b \Gamma_a \hat{\epsilon}.$$  \hspace{1cm} (2.10)
The constant parameters \((\tilde{\alpha}, \tilde{\beta})\) are to be determined. For vanishing \(F\), the equation (2.9) defines the standard Killing spinors in Ricci-flat backgrounds. Just as the standard Ricci-flat Killing spinors which may exist in a non-supersymmetric theory, the definition of our generalized Killing spinor does not have to depend on whether the theory (2.1) can be supersymmetrized or not. However, we wish that there exist solutions of (2.1) that admit the generalized Killing spinors. In particular, we impose a condition that (2.9) gives rise to the maximum number of generalized Killing spinors for both the AdS\(\times\)Sphere vacua, which are tensor products of real Killing spinors in the AdS and sphere spaces. In section 3, we show by explicit construction, that existence of maximum number of allowed such spinors in the AdS\(\times\)Sphere vacua enables us to fix the parameters \(\tilde{\alpha}\) and \(\tilde{\beta}\), given by

\[
\tilde{\alpha} = i^{(n+1)/2} \frac{\sqrt{\Delta}}{4d}, \quad d\tilde{\alpha} + \tilde{d}\beta = 0, \tag{2.11}
\]

To be precise, the parameters \((\tilde{\alpha}, \tilde{\beta})\) can be fixed by this consideration up to a relative sign. (The overall sign of \((\tilde{\alpha}, \tilde{\beta})\) can be absorbed in to \(F\).) The given relative sign in the above equation is actually fixed later in section 4, by requiring that extremal \(p\)-branes also admit the generalized Killing spinors.

Since the theory (2.1) are generically non-supersymmetric, we shall call the spinors satisfying (2.9) pseudo-Killing spinors. In special cases such as \((D, n) = (11, 4), (10, 5), (6, 3), (5, 2), (4, 2)\), the pseudo-Killing spinors become real ones. (Appropriate chirality conditions must be imposed in even dimensions.) In this paper, we construct solutions of the theory (2.1) with pseudo-Killing spinors. We call these solutions pseudo-supersymmetric.

### 3 Pseudo-Killing spinors in AdS\(\times\)Sphere

In section 2, we introduce the concept of pseudo-Killing spinors for the theory (2.1) in arbitrary dimensions. We remarked that the parameters \(\tilde{\alpha}\) and \(\tilde{\beta}\) in (2.9) are fixed, up to a relative sign, so that the AdS\(\times\)Sphere solutions (2.4) and (2.7) have the maximum number of pseudo-Killing spinors. In this section, we demonstrate this by obtaining the pseudo-Killing spinors explicitly. The analysis for pseudo-Killing spinors in AdS\(\times\)Sphere vacua resembles that for real Killing spinors in such vacua in supergravities [12]. The full spacetime indices \((M, N, \cdots)\) are now split into \((\mu, \nu, \cdots)\) and \((i, j, \cdots)\) which are indices for the AdS and sphere respectively. Since the decomposition of the \(D\)-dimensional gamma matrices into those of \(n\) and \((D - n)\) dimensions depends on whether \(n\) and \((D - n)\) are odd or even numbers, there are four cases [12] to consider.
Case 1: $(n, D - n) = (\text{even, odd})$

In this case, the gamma matrices can be decomposed as follows

$$\hat{\Gamma}_\mu = \Gamma_\mu \otimes 1, \quad \hat{\Gamma}_i = \gamma \otimes \Gamma_i, \quad (3.1)$$

where $\gamma$ is the chirality operator formed from the product of the $\Gamma_\mu$ matrices, satisfying $\gamma^2 = 1$. Thus we may have

$$\gamma = i^{(n-2)/2} \Gamma_0 \cdots \Gamma_{n-1}. \quad (3.2)$$

We find that the pseudo-Killing spinor in the AdS$_n \times S^{D-n}$ vacuum is given by

$$\hat{\epsilon} = \epsilon \otimes \eta, \quad (3.3)$$

where $\epsilon$ and $\eta$ are real Killing spinors in the AdS$_n$ and $S^{D-n}$ respectively, satisfying

- AdS$_n$: $D_\mu \epsilon = \frac{i}{2} \gamma \Gamma_\mu \epsilon,$
- $S^{D-n}$: $D_i \eta = \pm \frac{i}{2} \gamma \Gamma_i \eta.$ \quad (3.4)

Explicit construction of Killing spinors in AdS space-times and spheres can be found in [13, 12].

For the AdS$_{D-n} \times S^n$ vacuum, the gamma matrix decomposition is given by

$$\hat{\Gamma}_\mu = \Gamma_\mu \otimes \gamma, \quad \hat{\Gamma}_i = 1 \otimes \Gamma_i, \quad (3.5)$$

with

$$\gamma = i^{n/2} \Gamma_{12} \cdots \Gamma_{D-n}. \quad (3.6)$$

The pseudo-Killing spinor again takes the form (3.3), but with

- AdS$_{D-n}$: $D_\mu \epsilon = \frac{\lambda}{2} \Gamma_\mu \epsilon,$
- $S^n$: $D_i \eta = \pm \frac{\lambda}{2} \gamma \Gamma_i \eta.$ \quad (3.7)

Case 2: $(n, D - n) = (\text{odd, even})$

The exercise is analogous to the previous one, and we shall present just the results

- AdS$_n \times S^{D-n}$: $\hat{\Gamma}_\mu = \Gamma_\mu \otimes \gamma, \quad \hat{\Gamma}_i = 1 \otimes \Gamma_i,$
  $$\gamma = i^{(D-n)/2} \Gamma_{12} \cdots (D-n), \quad i^{(n+1)/2} \Gamma_0 \cdots (n-1) = -1,$$
  $$D_\mu \epsilon = \frac{\lambda}{2} \Gamma_\mu \epsilon, \quad D_i \eta = \frac{\lambda}{2} \gamma \Gamma_i \eta;$$
- AdS$_{D-n} \times S^n$: $\hat{\Gamma}_\mu = \Gamma_\mu \otimes 1, \quad \hat{\Gamma}_i = \gamma \otimes \Gamma_i$, 
  with

$$\gamma = i^{(n+1)/2} \Gamma_0 \cdots \Gamma_{n-1}. \quad (3.7)$$
\[ \gamma = i^{(D-n)/2} \Gamma^{\hat{1} \hat{2} \cdots (D-n)}, \quad i^{[(n+1)/2]} \hat{\Gamma}^{\hat{1} \hat{2} \cdots \hat{n}} = -i, \]
\[ D_{\mu} \epsilon = \frac{i \lambda}{2} \gamma \Gamma_{\mu} \epsilon, \quad D_{i} \eta = \frac{i \lambda}{2} \Gamma_{i} \eta. \]  

**Case 3:** \((n, D - n) = (\text{even, even})\)

There are two ways to decompose the gamma matrices. The first is given by
\[ \hat{\Gamma}_{\mu} = \Gamma_{\mu} \otimes 1, \quad \hat{\Gamma}_{i} = \gamma \otimes \Gamma_{i}. \]  
In this case, the pseudo-Killing spinors for the electric \(\text{AdS} \times \text{Sphere}\) metrics take the same form as those given in case 1; those for the magnetic solutions take the same form as those given in case 2.

Alternatively, we can decompose the gamma matrix as follows
\[ \hat{\Gamma}_{\mu} = \Gamma_{\mu} \otimes \gamma, \quad \hat{\Gamma}_{i} = 1 \otimes \Gamma_{i}. \]  
The pseudo-Killing spinors then take the same form as those given in case 2 for the electric solutions and case 1 for the magnetic solutions.

**Case 4:** \((n, D - n) = (\text{odd, odd})\)

In this case, the gamma matrices are decomposed as follows
\[ \hat{\Gamma}_{\mu} = \sigma_{1} \otimes \Gamma_{\mu} \otimes 1, \quad \hat{\Gamma}_{i} = \sigma_{2} \otimes 1 \otimes \Gamma_{i}, \]  
where \(\sigma_{1}\) and \(\sigma_{2}\) are Pauli matrices. For the electric \(\text{AdS}_{n} \times S^{(D-n)}\) solution, the pseudo-Killing spinors take the form as
\[ \hat{\epsilon} = 1 \otimes \epsilon \otimes \eta, \]  
where
\[ D_{\mu} \epsilon = \frac{\lambda}{2} \Gamma_{\mu} \epsilon, \quad D_{i} \eta = \frac{i \lambda}{2} \Gamma_{i} \eta. \]  
For the magnetic \(\text{AdS}_{D-n} \times S^{n}\) solution, the pseudo-Killing spinors take the same form, but with \(\lambda\) and \(\hat{\lambda}\) switched.

Thus we have demonstrated that the \(\text{AdS} \times \text{Sphere}\) vacuum solutions of (2.1) admit the maximum number of the pseudo-Killing spinors, defined by (2.9). It should be remarked however that the existence of the pseudo-Killing spinors for these solutions does not fix the relative sign of the parameters \(\alpha\) and \(\beta\). The relation \(d\alpha - d\beta = 0\), instead of the one given in (2.11), works equally well, provided that the orientations of the Killing spinors in AdS and spheres are adjusted appropriately. However, the relative sign choice in (2.11) can be fixed by requiring that extremal \(p\)-brane solitons also admit pseudo-Killing spinors. We shall discuss this in the next section.
4 Pseudo-supersymmetric $p$-branes

4.1 Electric branes

The Lagrangian (2.1) admits electrically-charged $(n-2)$-brane, for which the full spacetime $(x^M)$ is split into the $d = n - 1$ dimensional world volume with coordinates $x^\mu$ and $(D - d)$ dimensional transverse space with coordinates $y^m$. The solution is given by

$$ds^2 = e^{2A}dx^\mu dx_\mu + e^{2B}dy^i dy^i,$$
$$F_{(n)} = \frac{2}{\Delta} dH^{-1} \wedge dt \wedge dx^1 \wedge \cdots dx^{n-2},$$

(4.1)

where $H$ is a harmonic function in the flat transverse space $ds_{D-d}^2 = dy^i dy^i$. (See, e.g. [14].)

We now calculate the pseudo-Killing spinor for this bosonic $p$-brane background. A convenient choice for the metric in (4.1) is given by

$$e^\mu = e^A dx^\mu, \quad e^i = e^B dy^i.$$

(4.2)

Here we use barred letters to denote the tangent flat indices. The non-vanishing components of the corresponding spin connection are given by

$$\omega^{\bar{\mu}}_i = e^{-B} \partial_i A e^{\bar{\mu}}, \quad \omega_{ij} = e^{-B} (\partial_j B e^{\bar{i}} - \partial_i B e^{\bar{j}}).$$

(4.3)

Thus the covariant derivative on the spinor $\epsilon$ is given by

$$D\epsilon = d\epsilon + \frac{1}{2} \partial_i A e^{\bar{\mu}} \Gamma^{\bar{\mu}}_i \epsilon + \frac{1}{2} \partial_i B e^{\bar{i}} \Gamma^{\bar{i}}_j \epsilon.$$

(4.4)

The non-vanishing components of the field strength are given by

$$F_{ij\cdots\mu_{n-1}} = -\frac{2}{\Delta} H^{-1} \partial_i H \epsilon_{\bar{\mu}_1 \cdots \bar{\mu}_{n-1}}.$$

(4.5)

Substituting these into the generalized Killing spinor equation (2.9), we have

$$(\partial_i \epsilon - \frac{1}{2} \partial_i A \Gamma \epsilon) + \frac{1}{2} \Gamma^{\bar{i}\bar{j}} \partial_j B (1 - \Gamma) \epsilon = 0,$$
$$\partial_{\bar{\mu}} \epsilon + \frac{1}{2} \partial_i A \Gamma^{\bar{\mu}}_i (1 - \Gamma) \epsilon = 0,$$

(4.6)

(4.7)

where

$$\Gamma \equiv \frac{i^{[(n+1)/2]}}{(n-1)!} \epsilon_{\bar{\mu}_1 \cdots \bar{\mu}_{n-1}} \Gamma^{\bar{\mu}_1 \cdots \bar{\mu}_{n-1}}.$$

(4.8)

Thus the generalized Killing spinor is given by

$$\epsilon = e^{\frac{1}{2} A} \epsilon_0.$$

(4.9)
where $\epsilon_0$ is a constant spinor, satisfying the projection

$$\Gamma \epsilon_0 = \epsilon_0 .$$

(4.10)

Owing to this projection, the number of pseudo-Killing spinors is half of the maximum number possessed by the vacua discussed in the previous section. Thus the extremal $p$-branes are half pseudo-BPS solutions.

It should be pointed out that if we had $d\tilde{\alpha} - \tilde{\beta} = 0$, instead of the relation given in (2.11), the $(1 - \Gamma)$ factor in (4.6) would become $(1 + \Gamma)$, and hence there could be no pseudo-Killing spinors. Thus the existence of pseudo-Killing spinors for the extremal $p$-branes determines the relative sign choice between $\tilde{\alpha}$ and $\tilde{\beta}$.

4.2 Magnetic branes

The Lagrangian (2.1) also admits magnetically-charged $(D - n - 2)$-brane, for which the full spacetime $(x^M)$ is split into the $\tilde{d} = D - n - 1$ dimensional world volume with coordinates $x^\mu$ and $(D - \tilde{d})$ dimensional transverse space with coordinates $y^m$. The solution is given by

$$ds^2 = e^{2A} dx^\mu dx_\mu + e^{2B} dy^i dy^i, \quad e^{2A} = H^{-\frac{\tilde{d}}{2}}, \quad e^{2B} = H^{\frac{\tilde{d}}{2}},$$

(4.11)

$$F^{(n)} = \frac{2}{\sqrt{\Delta}} * dH^{-1} \wedge dt \wedge dx^1 \wedge \cdots dx^{\tilde{d}-1},$$

where $*$ denotes the Hodge dual and $H$ is a harmonic function in the flat transverse space $ds^2_{D-\tilde{d}} = dy^i dy^i$. As in the electric branes, the pseudo-Killing spinors can also be obtained. They take the same form as (4.9) and (4.10), except that now the projection operator is given by

$$\Gamma \equiv \frac{i^{[n/2]}}{(n-1)!} \epsilon_{i_1 \cdots i_{n-1}} \Gamma^{i_1 \cdots i_{n-1}} .$$

(4.12)

5 Pseudo-supersymmetric intersecting branes

For the right conditions, the $p$-brane solutions obtained in the previous section can intersect with each other. Here we shall consider only the harmonic intersection where each ingredient component is described by its harmonic function in a straightforward way. Harmonic intersections of supersymmetric $p$-branes was obtained in [15, 17, 16]. We shall present in the following subsections the pair-wise intersections, from which all possible multi-intersections can be built straightforwardly.
5.1 Electric/electric intersection

The full spacetime is split into four categories: $k$-dimensional overall world volume $x^\mu$, $2(d-k)$-dimensional relative spaces $(u^\alpha, v^\beta)$ and the remaining $(D-2d+k)$-dimensional overall transverse space $y^i$. The ansatz is given by

$$ds^2 = (H_1 H_2)^{-\frac{2}{d}} dx^\mu dx_\mu + H_1^{-\frac{2}{d}} H_2^2 du^\alpha du^\alpha + H_1^2 H_2^{-\frac{2}{d}} dv^\beta dv^\beta + (H_1 H_2)^{\frac{2}{d}} dy^i dy^i,$$

$$F_{(n)} = \frac{2}{\sqrt{\Delta}} (\omega H_1^{-1} \wedge d^{(k)} x \wedge d^{(d-k)} u + dH_2^{-1} \wedge d^{(k)} x \wedge d^{(d-k)} v),$$

where $H_1$ and $H_2$ are harmonic functions of the $y$ space, namely

$$\Box y H_1 = 0, \quad \Box y H_2 = 0. \quad (5.2)$$

This ansatz satisfies the full set of equations of motion of the Lagrangian (2.1) provided that

$$\frac{d^2}{D-2} = k = \text{positive integers}. \quad (5.3)$$

Applying this to M-theory or type IIB supergravity, we conclude that the M2/M2 intersection gives rise to a black hole ($k = 1$) and D3/D3 intersection gives rise to a string ($k = 1$). Although there is an infinite number of solutions to (5.3), the dimensions $D$ rise rapidly. The next example is $(D, d, k) = (14, 6, 3)$, corresponding to the 5/5-intersection in 16 dimensions that gives rise to a 2-brane.

The pseudo-Killing spinors can also be calculated easily, which satisfy the following two projections

$$\bar{\Gamma} \epsilon = \epsilon, \quad \hat{\Gamma} \epsilon = \epsilon, \quad (5.4)$$

where

$$\bar{\Gamma} = \Gamma^{01\cdots(k-1)} \Gamma^{k\cdots(d-k-1)} \equiv \bar{\Gamma}, \quad \hat{\Gamma} = \Gamma^{01\cdots(k-1)} \Gamma^{(d-k)\cdots(2d-k-1)} \equiv \hat{\Gamma}. \quad (5.5)$$

For $\bar{\Gamma}$ and $\hat{\Gamma}$ to have common eigenvalues, they must commute, which implies that

$$d - k = \text{even}. \quad (5.6)$$

5.2 Electric/magnetic intersection

The full spacetime is split into four categories: $\tilde{k}$-dimensional overall world volume $x^\mu$, $(d + \tilde{d} - 2\tilde{k})$-dimensional relative spaces $(u^\alpha, \nu^\beta)$ and the remaining $(\tilde{k} + 2)$-dimensional overall transverse space $y^i$. The indices $\alpha$ and $\beta$ run $(d - \tilde{k})$ and $(\tilde{d} - \tilde{k})$ values. The ansatz is given by

$$ds^2 = H_1^{-\frac{2}{d}} H_2^{-\frac{2}{d}} dx^\mu dx_\mu + H_1^{-\frac{2}{d}} H_2^2 du^\alpha du^\alpha + H_1^2 H_2^{-\frac{2}{d}} dv^\beta dv^\beta + H_1^2 H_2^\frac{2}{d} dy^i dy^i.$$
\[ F_{(n)} = \frac{2}{\sqrt{\Delta}} \left( dH_1^{-1} \wedge d^{(k)} x \wedge d^{(d-k)} u + *dH_2^{-1} \wedge d^{(k)} x \wedge d^{(d-k)} v \right), \]  

(5.7)

where \( H_1 \) and \( H_2 \) are harmonic functions of the \( y \) space. The ansatz satisfies the full set of equations of motion of the Lagrangian \([22]\) provided that

\[ \frac{d \tilde{d}}{D-2} = \hat{k} = \text{positive integers}. \]  

(5.8)

Applying this to M-theory, we conclude that the M2/M5 intersection gives rise to a string \((\hat{k} = 2)\). There are two independent projection gamma matrix operators associated with the electric and magnetic brane components for pseudo-Killing spinors. These two projectors have to commute, yielding to the following additional condition

\[ \tilde{k} = \text{even}. \]  

(5.9)

### 5.3 Magnetic/magnetic intersection

The full spacetime is split again into four categories: \( \tilde{k} \)-dimensional overall world volume \( x^\mu \), \( 2(\tilde{d} - \tilde{k}) \)-dimensional relative spaces \((u^\alpha, v^\alpha)\) and the remaining \((D - 2\tilde{d} + \tilde{k})\)-dimensional overall transverse space \( y^i \). The ansatz is given by

\[
\begin{align*}
    ds^2 &= (H_1 H_2)^{-\frac{2}{D-2}} dx^\mu dx_\mu + H_1^{-\frac{2}{D-2}} H_2^\frac{2}{D-2} du^\alpha du^\alpha + H_2^{-\frac{2}{D-2}} H_1^\frac{2}{D-2} dv^\alpha dv^\alpha + (H_1 H_2)^\frac{D}{D-2} dy^i dy^i, \\
    F_{(n)} &= \frac{2}{\sqrt{\Delta}} * \left( dH_1^{-1} \wedge d^{(k)} x \wedge d^{(d-k)} u + dH_2^{-1} \wedge d^{(k)} x \wedge d^{(d-k)} v \right),
\end{align*}
\]

(5.10)

where \( H_1 \) and \( H_2 \) are harmonic functions of the \( y \) space. The ansatz satisfies the full set of equations of motion of the Lagrangian \([22]\) provided that

\[ \frac{d \tilde{d}}{D-2} = \hat{k} = \text{positive integers}. \]  

(5.11)

Applying this to M-theory, we conclude that the M5/M5 intersection gives rise to a 3-brane \((k = 4)\). As in the previous examples, the existence of pseudo-Killing spinors requires an additional condition

\[ \hat{d} - \hat{k} = \text{even}. \]  

(5.12)

It should be remarked that since \( k + \tilde{k} = d \) and \( \hat{k} + \tilde{k} = \tilde{d} \). The conditions for the possibility of harmonic intersection for the electric/electric, electric/magnetic and magnetic/magnetic cases are either satisfied for all or none. The same is the true for the existence of the pseudo-Killing spinors for these solutions. For the intersections with pseudo-supersymmetry, the preserved fraction of the pseudo-Killing spinors is \( \frac{1}{4} \).

Having obtain the pair-wise intersections of the electric and magnetic \( p \)-branes, it is straightforward to construct all possible multi-intersections.
6 Integrability conditions

In the previous sections, we introduced pseudo-Killings spinors for the theory (2.1) and then obtained large classes of solutions including $p$-branes and intersecting $p$-branes that preserve a certain fraction of pseudo-Killing spinors. In these solutions, the equations of motion are solved without making use of the pseudo-Killing spinor equations, but we verify that pseudo-Killing spinors exist in these backgrounds.

The purpose of introducing pseudo-Killings spinors is to help us to obtain new solutions that may otherwise not be possible to construct. The pseudo-Killing spinor equations can reduce the second-order equations to those of the first order. However, the success of obtaining pseudo-supersymmetric $p$-branes in earlier sections should not give a wrong impression that the existence of a pseudo-Killing spinor of a bosonic ansatz is sufficient for it to satisfy the full set of equations of motion. It is advantageous to obtain the integrability condition for the pseudo-Killing spinor equation. The pseudo-Killing spinor equation together with the integrability condition enables us to construct complicated new solutions without having to verify explicitly the complicated Einstein equations of motion. We start with discussion for general $(D, n)$ in the following subsection and then go on to specific cases in the subsequent subsection.

6.1 General analysis

Here, we change the notation and express the pseudo-Killing spinor equation as

$$D_M \eta + b \left( \Gamma_M^{M_1M_2...M_n} - a \delta_M^{M_1} \Gamma^{M_2...M_n} \right) F_{M_1M_2...M_n} \eta = 0.$$  \hspace{1cm} (6.1)

The constants $(a, b)$ can be read off from (2.9) and (2.11), but are left arbitrary for now. The integrability condition is given by

$$D_{[N} D_{M]} \eta = T_{NM}^{(1)} + T_{NM}^{(2)},$$  \hspace{1cm} (6.2)

where $T^{(1)}$ and $T^{(2)}$ are anti-symmetric tensors proportional to the $n$-form field strength in linear and quadratic fashions respectively,

$$T_{NM}^{(1)} = -b \left( \Gamma_{[M}^{M_1M_2...M_n} - (a + n) \delta_M^{M_1} \Gamma^{M_2...M_n} \right) \left( \nabla_N \right) F_{M_1M_2...M_n} \eta,$$

$$T_{NM}^{(2)} = b^2 \left[ (-1)^n \Gamma_{MN}^{M_1M_2...M_n} \Gamma^{N_1N_2...N_n} \right. \left. - (a + 3n)(-1)^n \Gamma_{[MN}^{M_1M_2...M_n} \Gamma^{N_1N_2...N_n} \right. \left. - (a + n) \Gamma_{[M}^{N_1N_2...N_n} \Gamma^{M_1M_2...M_n} \Gamma^{N_2...N_n} \right. \left. - 2(n - 1)(a + n)(-1)^n \delta_M^{M_1} \delta_N^{M_2} \Gamma^{M_3...M_n} \Gamma^{N_1N_2...N_n} \right].$$
\[ + (a + n)^2 \delta_M^N \delta_N^M \Gamma^{M_1 M_2 \ldots M_n} \Gamma^{N_2 \ldots N_n} \] \[ F_{M_1 M_2 \ldots M_n} F_{N_1 N_2 \ldots N_n} \eta, \quad (6.3) \]

Contracting (6.2) with \( \Gamma^N \), the left-hand side gives

\[ 4\Gamma^N D_{[N} D_{M]} \eta = R_{MN} \Gamma^N. \quad (6.4) \]

The right-hand side gives much more complicated expressions

\[ \Gamma^N T^{(1)}_{NM} = \frac{b}{2(n+1)} dF_{NM} M_2 \ldots M_n \left( \Gamma^N M_1 M_2 \ldots M_n \frac{1}{n+1} a \delta_M^N \Gamma^{M_1 M_2 \ldots M_n} \right) \eta \]
\[ + ((-1)^n D b/2 (d + F) M_2 \ldots M_n \left( n \Gamma_{M_2 \ldots M_n} - (n - 1) a \delta_M^N \Gamma^{M_3 \ldots M_n} \right) \eta \]
\[ - \frac{b}{2} \left( \frac{1}{n+1} a - D + n + 1 \right) \nabla_M F_{M_1 M_2 \ldots M_n} \Gamma_{M_1 M_2 \ldots M_n} \eta, \quad (6.5) \]

and

\[ \Gamma^N T^{(2)}_{NM} = b^2 \left[ \sum_{k=0}^{[n/2]} \frac{(-1)^k (n+1) n-k (n-1)!}{(n-2k)! (2k+1)!} \frac{1}{2} \eta \right] \]
\[ \times \delta^{M_1}_{N_1} \ldots \delta^{M_{n-2k}}_{N_{n-2k}} \Gamma^{M_{n-2k+1} \ldots M_n N_1 N_{n-2k+1} \ldots N_n} F_{M_1 M_2 \ldots M_n} F_{N_1 N_2 \ldots N_n} \]
\[ + \sum_{k=0}^{[n/2]} \frac{(-1)^k (n-2) (n-1-k) (n-1)!}{(n-2k-1)! (2k)! (2k+1)!} \]
\[ \times \left( (2k + n - 1) a^2 + 4kn + 2n(n + 2k - D + 1) \right) \]
\[ \times \delta^{M_2}_{N_2} \ldots \delta^{M_{n-2k}}_{N_{n-2k}} \Gamma^{M_{n-2k+1} \ldots M_n N_1 N_{n-2k+1} \ldots N_n} F_{M_1 M_2 \ldots M_n} F_{N_1 N_2 \ldots N_n} \]
\[ + \sum_{k=0}^{[n/2]} \frac{(-1)^k (n+1) n-k (n-1)!}{(n-2k-2)! (2k+1)! (2k+2)!} \]
\[ \times \left( (2k + 2) a^2 + 2n(2n-D)a - 2n^2 (k+1) \right) \]
\[ \times \delta^{M_2}_{N_2} \ldots \delta^{M_{n-2k}}_{N_{n-2k}} \Gamma^{M_{n-2k+1} \ldots M_n N_1 N_{n-2k+1} \ldots N_n} F_{M_1 M_2 \ldots M_n} F_{N_1 N_2 \ldots N_n} \eta. \quad (6.6) \]

The vanishing of the last term in (6.5) requires that

\[ a = \frac{n(D - n - 1)}{n - 1}. \quad (6.7) \]

Note that this condition fixes the relative sign and ratio between \( \hat{\alpha}, \hat{\beta} \) in (2.9) and (2.11).

Assemble the three structures (6.4), (6.5) and (6.6), the term with single gamma matrix is

\[ \left[ R_{MN} + 4b^2 n! (-1)^{1/2} (n+1)^n \left( (D-n-1) g_{MN} F^2 + \left( \frac{1}{n} a^2 + n(n-D+1) F^2_{MN} \right) \right) \Gamma^N \right] \eta. \quad (6.8) \]

The factor in the square-bracket is related to the Einstein’s equation of motion provided that

\[ b^2 = (-1)^{1/2} (n+1)^n \frac{n - 1}{8(D - n)(D - n - 1)(n)!^2} \quad (6.9) \]

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Note that the pseudo-Killing spinor equation (6.11) with the constants \((a, b)\) determined by (6.7) and (3.9) above is precisely the same one given in (2.9) and (2.11), determined by examining the vacuum and \(p\)-brane solutions. The integrability condition now reduces to

\[
\left[ R_{MN} + \frac{n-1}{2(D-2)(n-1)} g_{MN} F^2 - \frac{1}{2(n-1)!} F_{MN}^2 \right] \Gamma^N \eta \\
- \frac{2b}{n+1} d F_{NM_1 M_2 \ldots M_n} \left( \frac{1}{n-1} \right) \delta^N_{MN} \Gamma^{M_1 M_2 \ldots M_n} \eta \\
- (-1)^n D n b (dF)_{M_2 \ldots M_n} \left( \Gamma^M M_2 \ldots M_n - (D - n - 1) \delta^M_2 \Gamma^M_3 \ldots M_n \right) \eta \\
- \frac{1}{2(D-2)(D-n-1)} \sum_{k=1}^{[\frac{n}{2}]} \frac{(-1)^k}{(n-2k)!} \left[ (2k - n + 1) D + (n^2 - 4k - 1) \right] \\
\times \delta^{M_1 \ldots M_{n-2k}}_{N_1 \ldots N_{n-2k}} \Gamma^{M_{n-2k+1} \ldots M_n N_{n-2k+1} \ldots N_n} F_{M_1 \ldots M_n} F^{N_1 \ldots N_n} \eta \\
- \frac{1}{2(D-n-1)(n-1)} \sum_{k=1}^{[\frac{n}{2}]} \frac{(-1)^{k+1}}{(n-2k-1)!(2k+1)!} \left[ (2k - n + 1) D + (n^2 - 4k - 1) \right] \\
\times \delta^{M_2 \ldots M_{n-2k}}_{N_2 \ldots N_{n-2k}} \Gamma^{M_{n-2k+1} \ldots M_n N_1 N_{n-2k+1} \ldots N_n} F_{M M_2 \ldots M_n} F^{N_1 N_2 \ldots N_n} \eta \\
+ \frac{D-2n}{2(D-2)(D-n-1)(n-1)} \sum_{k=0}^{[\frac{n}{2}]-1} \frac{(-1)^k}{(n-2k-2)!(2k+1)!} \left[ (2k - n + 3) D + (n^2 - 4k - 5) \right] \\
\times \delta^{M_2 \ldots M_{n-2k-1}}_{N_2 \ldots N_{n-2k-1}} \Gamma^{M_{n-2k} \ldots M_n N_1 N_{n-2k} \ldots N_n} F_{M M_2 \ldots M_n} F^{N_1 N_2 \ldots N_n} \eta = 0 . \quad (6.10)
\]

Thus we see that even if we substitute the full set of equations of motion (2.2,2.3), there are still many terms that are not vanishing. These terms are quadratic in \(F\), giving rise to additional algebraic conditions for a bosonic configuration to satisfy the equations of motion. A careful analysis shows that for the vacuum or \(p\)-brane solutions discussed earlier, these algebraic constraints indeed vanish, as we would have expected.

Generically, in order for these extra contraints to vanish, we must have, for all possible \(k > 0\), that

\[
D = \frac{n^2 - 4k - 1}{n - 2k - 1} . \quad (6.11)
\]

This clearly cannot be satisfied in general. We may consider the case where the above condition is satisfied for the lowest value of \(k\), namely \(k = 1\). Then we have

\[
D = \frac{n^2 - 5}{n - 3} = n + 3 + \frac{4}{n - 3} . \quad (6.12)
\]

The only integer solutions are \((n, D) = (4, 11), (5, 10), (7, 11)\). The third case is Hodge dual to the first, and hence we shall not consider it further.

It is worth mentioning that simplification occurs when \(D = 2n\) and \(n\) is odd. We can further impose the self-duality condition, the integrability condition becomes

\[
\left[ R_{MN} - \frac{1}{2(n-1)!} F_{MN}^2 \right] \Gamma^N \eta
\]

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In this section, we examine the integrability condition case by case for significantly supersymmetric imposed. As one would have expected, these constraints are satisfied in the case of pseudo-

\[ n = 5, \ 4, \ 3, \ 2 \]

In this section, we examine the integrability condition case by case for \( n = 5, 4, 3, 2 \).

\( n = 5 \)

The integrability condition for \( n = 5 \) in arbitrary dimensions is given by

\[
\left[ R_{MN} - \frac{1}{2(4!)} F_{MN}^2 + \frac{2}{(5!)(D - 2)} g_{MN} F^2 \right] \Gamma^N \eta
\]

\[
- \frac{1}{6!} \sqrt{\frac{2}{(D-6)(D-2)}} dF_{NM_1M_2M_3M_4M_5} \left( \Gamma_M^{M_1M_2M_3M_4M_5} \Gamma_M^{M_1M_2M_3M_4M_5} - \frac{5}{3} (D - 6) \delta_M^{M_1M_2M_3M_4M_5} \right) \eta
\]

\[
- \frac{1}{4!} \left[ \frac{2}{(D-6)(D-2)} (+d * F)_{M_1M_2M_3M_4} \left( \Gamma_M^{M_1M_2M_3M_4} - \frac{20}{3} \delta_M^{M_1M_2M_3M_4} \right) \eta
\]

\[
+ \frac{1}{4(4!)} \left[ - 5 \delta_M^{M_1M_2M_3M_4M_5N_1N_2N_3N_4N_5} F_{M_1M_2M_3M_4M_5} \eta \right] = 0 \ . \quad (6.13)
\]

Thus, we see that in general dimensions, the equations of motion are necessary conditions for the integrability condition, but not sufficient. Additional constraints have to be imposed. As one would have expected, these constraints are satisfied in the case of pseudo-supersymmetric \( p \)-branes and intersecting branes discussed in the previous sections.

The integrability condition (6.14) clearly implies that \( D = 10 \) is the critical dimension for \( n = 5 \), where many of the constraints vanish. For \( D = 10 \), the integrability simplifies significantly

\[
\left[ R_{MN} - \frac{1}{2(4!)} F_{MN}^2 + \frac{2}{(5!)(D - 2)} g_{MN} F^2 \right] \Gamma^N \eta
\]

\[
- \frac{i}{4(6!)} dF_{NM_1M_2M_3M_4M_5} \left( \Gamma_M^{M_1M_2M_3M_4M_5} \Gamma_M^{M_1M_2M_3M_4M_5} - \frac{20}{3} \delta_M^{M_1M_2M_3M_4M_5} \right) \eta
\]

\[
- \frac{i}{4(4!)} (+d * F)_{M_1M_2M_3M_4} \left( \Gamma_M^{M_1M_2M_3M_4} - \frac{20}{3} \delta_M^{M_1M_2M_3M_4} \right) \eta
\]

\[
+ \frac{1}{4(4!)(5!)} \left[ - 5 \delta_M^{M_1M_2M_3M_4M_5N_1N_2N_3N_4N_5} F_{M_1M_2M_3M_4M_5} \eta \right] = 0 \ . \quad (6.14)
\]
True Killing spinors arise when the 5-form is self-dual, for which the theory becomes part of type IIB supergravity. In this case, we have

\[ \epsilon^{[M N M_2 M_3 M_4 M_5} F_{M_1[M_2 M_3 M_4 M_5} F^{M_1} N_2 N_3 N_4 N_5] \sim F^{[M M_2 M_3 M_4 M_5} F^{N]M_2 M_3 M_5} = 0, \]  

(6.16)
as well as

\[ \Gamma^{M_2 M_3 M_4 M_5 N_1 N_2 N_3 N_4 N_5} F_{M M_2 M_3 M_4 M_5} F_{N N_2 N_3 N_4 N_5} \eta = -5! F_{M M_2 M_3 M_5} F_{N M_2 M_3 M_4} \Gamma^{N} \eta. \]  

(6.17)

Here we have imposed the chirality condition

\[ \Gamma \eta = \eta, \quad \Gamma = \Gamma^{012...9}. \]  

(6.18)

After imposing the chirality condition, the integrability condition becomes

\[ 0 = \left[ R_{M N} - \frac{1}{4!} F_{M N}^2 \right] \Gamma^N \eta \]

\[ - \frac{i}{4(6!)} dF_{N M_1 M_2 M_3 M_4 M_5} \left( \Gamma_M^{N M_1 M_2 M_3 M_4 M_5} - \frac{20}{3} \delta_M^N \Gamma_{M_1 M_2 M_3 M_4 M_5} \right) \eta \]

\[ - \frac{i}{4(4!)^2} (d* F)_{M_1 M_2 M_3 M_4} \left( \Gamma_M^{M_1 M_2 M_3 M_4} - 4 \delta_{M_1}^M \Gamma_{M_2 M_3 M_4} \right) \eta. \]  

(6.19)

This result was obtained in [18, 19]. Thus for bosonic ansatz with \( dF = 0 = d* F \), the existence of a Killing spinor \( \eta \) that gives rise to a time-like Killing vector, namely

\[ \zeta^M = \bar{\eta} \Gamma^M \eta, \quad \zeta^M \zeta_M < 0, \]  

(6.20)
implies the Einstein equations of motion. It is of interest to point out that the 9-gamma structure, which is dual to single-gamma (6.17), doubles the contribution of the energy-momentum tensor associated with the self-dual 5-form.

For our non-supersymmetric theory where the 5-form is not self-dual, if the extra conditions

\[ F_{M [M_2 M_3 M_4 M_5} F^{M_1} N_2 N_3 N_4 N_5] = 0, \quad F_{M [M_2 M_3 M_4 M_5} F_{N_1 N_2 N_3 N_4 N_5]} = 0. \]  

(6.21)
are satisfied, then the pseudo-Killing spinor equation plus \( dF = d* F = 0 \) will also automatically imply the Einstein equations of motion. The conditions (6.21) can be easily satisfied if the non-vanishing components of the 5-form are restricted to a sub-manifold with dimensions less than or equal to seven. This enables us to construct new pseudo-symmetric bubbling AdS spaces in section 7.
we find that the integrability condition becomes
\[
\left[ R_{MN} - \frac{1}{12} \left( \frac{F_{MN}^2}{4(D-2)} g_{MN} F^2 \right) \right] \Gamma^N \eta
- \frac{1}{2(5!)} \sqrt{\frac{6}{(D-5)(D-2)}} dF_{M_L M_M M_N M_s} \left( \Gamma_M^{N M_1 M_2 M_3 M_4} - \frac{2}{3} (D-5) \delta_M^{[N} \Gamma^{M_1 M_2 M_3 M_4]} \right) \eta
- \frac{1}{2 \sqrt{6(D-5)(D-2)}} (d * F)_{M_1 M_2 M_3} \left( \Gamma_M^{M_1 M_2 M_3} - (D-5) \delta_M^{M_1} \Gamma_{M_2 M_3} \right) \eta
- \frac{b^2}{3} (D + 7) F_{M_1 M_2 M_3 M_4} F_{N_1 N_2 N_3 N_4} \left( 3 \Gamma_M^{M_1 M_2 M_3 M_4 N_1 N_2 N_3 N_4} - 4 (D-8) \delta_M^{M_1} \Gamma_{M_2 M_3 N_1 N_2 N_3 N_4} \right) \eta
+ \frac{6 b^2}{3} (D - 11)(D - 2) F_{M_1 M_2 M_3 M_4} F_{N_1 N_2 N_3 N_4} \left( 2 (D - 8) \delta_M^{M_1} \Gamma_{M_2 N_1 N_2} - 3 \Gamma_M^{M_1 M_2 N_1 N_2} \right) \eta = 0.
\] (6.22)

In general, the requirement of the vanishing of the last three terms gives some additional algebraic constraints that are outside of the equations of motion.

It is clear that the critical dimension for \( n = 4 \) is \( D = 11 \), for which the last two terms vanish. The third last term can now be expressed as

\[
\text{The third last term} = \frac{1}{72} * (F \wedge F)_{M_5 M_6 M_7} \left[ \Gamma_M^{M_5 M_6 M_7} - 6 \delta_M^{M_5} \Gamma_{M_6 M_7} \right].
\] (6.23)

To derive the above identity at \( D = 11 \), we have used

\[
\Gamma^0 \Gamma^1 \ldots \Gamma^{10} = 1, \quad \Gamma_{M_1 \ldots M_k} = \frac{(-1)^{k(k-1)/2}}{(11-k)!} \epsilon_{N_1 \ldots N_{(11-k)} M_1 \ldots M_k} \Gamma^{N_1 \ldots N_{(11-k)}}.
\] (6.24)

Thus the integrability condition for \( D = 11 \) becomes \[20\] \[21\]
\[
0 = \left( R_{MN} - \frac{1}{2(3!)} F_{MN}^2 + \frac{1}{6(4!)} g_{MN} F^2 \right) \Gamma^N \eta
- \frac{1}{6!} dF_{M_L M_M M_N M_s} \left( \Gamma_M^{N M_1 M_2 M_3 M_4} - 10 \delta_M^{[N} \Gamma^{M_1 M_2 M_3 M_4]} \right) \eta
- \frac{1}{36} * (d * F - \frac{1}{2} F \wedge F)_{M_1 M_2 M_3} \left( \Gamma_M^{M_2 M_3 M_4} - 6 \delta_M^{M_2} \Gamma_{M_3 M_4} \right) \eta.
\] (6.25)

This integrability condition implies that the proper equation of motion for the 4-form is
\[
d * F = \frac{1}{2} F \wedge F.
\] (6.26)

The origin of the self-interaction of the 4-form is the FFA term
\[
\mathcal{L}_{FFA} = \frac{1}{6} F \wedge F \wedge A
\] (6.27)
of eleven-dimensional supergravity. Thus we see that for \( n = 4 \) in \( D = 11 \), the pseudo-Killing spinor is promoted to become the real Killing spinor of \( D = 11 \) supergravity.
For $D \neq 11$, the consistency between the existence of a pseudo-Killing spinor and the equations of motion requires additional constraints associated with the vanishing of the last three terms in $\text{(6.22)}$. The AdS×Sphere vacua, the $p$-brane and intersecting $p$-branes discussed earlier satisfy these constraints. In section 8, we shall consider a specific example in $D = 8$ and demonstrate that the effect of these constraints is to restrict severely the bubbling nature of the AdS geometry.

$n = 3$:

We find that the integrability condition is given by

$$
\left[ R_{MN} - \frac{1}{4} F_{MN}^2 + \frac{1}{6(D-2)} g_{MN} F^2 \right] \Gamma^N \eta \\
- \frac{1}{2} \sqrt{\frac{1}{(D-4)(D-2)}} dF_{N M_1 M_2 M_3} \left( \Gamma^M_{N M_1 M_2 M_3} - 2(D-4) \delta^{[N}_{M} \Gamma^{M_1 M_2 M_3]} \right) \eta \\
- \frac{1}{2} \sqrt{\frac{1}{(D-4)(D-2)}} (-1)^D (d \ast F)_{M_1 M_2} \left( \Gamma^M_{M_1 M_2} - (D-4) \delta^M_{M_1} \Gamma^M_{M_2} \right) \eta \\
+ 7 b^2 F_{M_1 [M_2 M_3} \Gamma_{N_2 N_3]} \left[ \Gamma^M_{M_2 M_3 N_2 N_3} - (D-6) \delta^M_{M_2} \Gamma^M_{M_3 N_2 N_3} \right] \eta \\
- 12 b^2 (D-2) \Gamma^M_{M_2 M_3 N_2 N_3} F_{M_1 M_2 M_3} F_{N_1 N_2 N_3} \eta = 0.
$$

(6.28)

In the above derivation, we have used

$$
F_{M_1 [M_2 M_3} \Gamma_{N_2 N_3]} = F_{M_1 M_2 [M_3} \Gamma_{N_2 N_3]}.
$$

(6.29)

The requirement of the vanishing of the three terms in the last two lines in $\text{(6.28)}$ gives some additional algebraic constraints. The critical dimension is $D = 6$, for which the three-gamma term in the last two lines of $\text{(6.28)}$ vanishes. In addition, we may impose chirality on the Killing spinor, namely

$$
\Gamma \eta = -\eta, \quad \Gamma = -\Gamma^0 \Gamma^1 \cdots \Gamma^5.
$$

(6.30)

We then have

$$
- 12 b^2 (D-2) \Gamma^M_{M_2 M_3 N_1 N_2 N_3} F_{M_1 M_2 M_3} F_{N_1 N_2 N_3} \eta = - \frac{1}{2} F_{M_1 M_2 M_3} (d \ast F)_{N_1 M_2 N_3} \Gamma^N \eta.
$$

(6.31)

Here we have used

$$
\Gamma \Gamma_{M_1 \ldots M_k} = -\frac{(-1)^{k(k-1)}}{6-k)!} e_{N_1 \ldots N_{(6-k)} M_1 \ldots M_k} \Gamma^{N_1 \ldots N_{(6-k)}}.
$$

(6.32)

If the 3-form is self-dual, we have

$$
e^{M N M_2 M_3 N_2 N_3} F_{M_1 [M_2 M_3} \Gamma_{N_2 N_3]} = 8 F^{[M_{M_2 M_3} \Gamma^N_{M_2 M_3} = 0.
$$

(6.33)
Then the integrability condition becomes

\[
0 = \left[ R_{MN} - \frac{1}{2} F_{MN}^2 \right] \Gamma^N \eta - \frac{1}{4(D-2)} g_{MN} F^2 \eta - \frac{1}{4\sqrt{2}} \left( \ast d \ast F \right)_{M_1 M_2} \left( \Gamma_M^{N M_1 M_2 M_3} - 4 \delta_M^{[N} \Gamma^{M_1 M_2 M_3]} \right) \eta
\]

This is precisely the integrability condition for \( \mathcal{N} = (1,0) \) supergravity in six dimensions, studied in [22]. The spinor \( \eta \) is promoted to be a real Killing spinor. Bubbling \( \text{AdS}_3 \times S^3 \) solution were constructed in [23], where an additional axion has to be turned on in the \( T^2 \) direction. Note that the 5-gamma structure, which is dual to a single-gamma term, doubles the contribution of the energy-momentum tensor associated with the self-dual 3-form.

\( n = 2 \):

The integrability condition is

\[
\left[ R_{MN} - \frac{1}{2} F_{MN}^2 + \frac{1}{4} g_{MN} F^2 \right] \Gamma^N \eta - \frac{i}{3! \sqrt{2(D-3)(D-2)}} dF_{NM_1 M_2} \left( \Gamma_M^{N M_1 M_2 M_3} - 3(D-3) \delta_M^{[N} \Gamma^{M_1 M_2]} \right) \eta
\]

\[
- \frac{i}{\sqrt{2(D-3)(D-2)}} \left( \ast d \ast F \right)_{M_1} \left( \Gamma_M^{M_1} - (D-3) \delta_M^{M_1} \right) \eta
\]

\[
+ \frac{i (D-1)}{4! (D-3)(D-2)} (F \wedge F)_{M_1 M_2 N_1 N_2} \left( \Gamma_{M_1 M_2 N_1 N_2} - 2(D-4) \delta_{M_1}^{M_2} \Gamma^{M_2 N_1 N_2} \right) \eta = 0.
\]

Two critical dimensions arise in this case. One is \( D = 4 \), for which the terms in the last line vanish. The integrability condition is then precisely that for \( \mathcal{N} = 2 \), \( D = 4 \) supergravity, namely

\[
\left[ R_{MN} - \frac{1}{2} F_{MN}^2 + \frac{1}{4} g_{MN} F^2 \right] \Gamma^N \eta - \frac{i}{12} dF_{NM_1 M_2} \left( \Gamma_M^{N M_1 M_2 M_3} - 3 \delta_M^{[N} \Gamma^{M_1 M_2]} \right) \eta
\]

\[
- \frac{i}{2 \ast (d \ast F)_{M_1} \left( \Gamma_M^{M_1} - \delta_M^{M_1} \right) \eta = 0.
\]

Another critical dimension is \( D = 5 \), corresponding to \( D = 5 \), \( \mathcal{N} = 2 \) supergravity, for which the integrability condition was studied in [4]. In \( D = 5 \), we have

\[
i \Gamma^0 \Gamma^1 \cdots \Gamma^4 = 1, \quad \Gamma_{M_1 \cdots M_k} = i\frac{(-1)^{k(k-1)/2}}{(5-k)!} \epsilon_{N_1 \cdots N_{(5-k)} M_1 \cdots M_k} \Gamma^{N_1 \cdots N_{(5-k)}}.
\]

The integrability condition can be further simplified, namely

\[
\left[ R_{MN} - \frac{1}{2} F_{MN}^2 + \frac{1}{4} g_{MN} F^2 \right] \Gamma^N \eta
\]

\[
- \frac{i}{12 \sqrt{3}} dF_{NM_1 M_2} \left( \Gamma_M^{N M_1 M_2 M_3} - 6 \delta_M^{[N} \Gamma^{M_1 M_2]} \right) \eta
\]

\[
- \frac{i}{2 \sqrt{3}} \left[ \ast (d \ast F + \frac{1}{\sqrt{3}} F \wedge F) \right]_{M_1} \left( \Gamma_M^{M_1} - 2 \delta_M^{M_1} \right) \eta = 0.
\]
This is very much like the case in $D = 11$, and the equation of motion for the 2-form is given by
\[ d \ast F = -\frac{1}{\sqrt{3}} F \wedge F. \] (6.39)
The origin of this self-interaction of the 2-form is the FFA term that needs to be augmented to the Lagrangian
\[ \mathcal{L}_{FFA} = \frac{1}{3\sqrt{3}} F \wedge F \wedge A. \] (6.40)

7 Pseudo-supersymmetric bubbling AdS in $D = 10$

As we have shown in the previous section, the integrability conditions for the generic pseudo-Killing spinor equation imply that additional constraints have to be imposed for a bosonic configuration with pseudo-Killing spinors to satisfy the Einstein equations of motion. For extremal $p$-branes, these extra conditions are indeed satisfied. In this section, we explore the possibility of constructing bubbling AdS geometry based on the pseudo supersymmetry. The supersymmetric bubbling AdS geometries in type IIB supergravity and M-theory were constructed in [5]. The example we consider in this section is $n = 5$ with $D = 10$. Of course, if the 5-form is self-dual, the system is part of type IIB supergravity, and was discussed in [5]. We shall instead consider an intrinsically non-supersymmetric theory where the 5-form is not self-dual. The Lagrangian is given by (2.1). We are looking for solutions with the $SO(4) \times SO(4)$ isometry, and the most general ansatz is given by
\[ ds^2 = g_{\mu\nu} dx^\mu dx^\nu + e^H d\Omega_3^2 + e^{\tilde{H}} d\tilde{\Omega}_3^2, \]
\[ F_5 = \frac{1}{2} F_{\mu\nu} dx^\mu \wedge dx^\nu \wedge \Omega_3 + \frac{1}{2} \tilde{F}_{\mu\nu} dx^\mu \wedge dx^\nu \wedge \tilde{\Omega}_3. \] (7.1)
The general pseudo-Killing spinor equation is given in section 2; specializing to our specific case, we have
\[ 0 = D_M \eta + \frac{i}{960} \left( \Gamma^M_{M_1 M_2 M_3 M_4 M_5} - 5 \delta^M_{M_1} \Gamma^{M_2 M_3 M_4 M_5} \right) F_{M_1 M_2 M_3 M_4 M_5} \eta \]
\[ = D_M \eta + \frac{i}{960} \Gamma^{M_1 M_2 M_3 M_4 M_5} \Gamma_M F_{M_1 M_2 M_3 M_4 M_5} \eta. \] (7.2)
Note that this pseudo-Killing spinor equation takes the exact form as the real one associated with the self-dual 5-form.

As was demonstrated in section 6, when the 5-form is not self-dual, additional constraints (6.21) have to be imposed. The simplest way to satisfy the conditions (6.21) is to restrict the non-vanishing components of the 5-form so that they all lie only in 7 space-time directions. This can be achieved by setting either $F_{\mu\nu}$ or $\tilde{F}_{\mu\nu}$ to zero. Without loss of generality we may
set \( \tilde{F}_{\mu\nu} = 0 \), and then the non-vanishing components of \( F_{(5)} \) span on 4-dimensional space-time with \( g_{\mu\nu}dx^\mu dx^\nu \) and the \( S^3 \) with \( d\Omega_3^2 \), totalling 7 directions. This ansatz is different from the bubbling AdS\(_5\) in [5] where the self-duality of the 5-form is required by type IIB supergravity and the \( F \) and \( \tilde{F} \) are both non-vanishing. Consequently, in the LLM solution, the non-vanishing components of the self-dual 5-form lie in all 10 space-time directions.

We can now proceed and construct the new bubbling AdS\(_5\) solution supported by the non-self-dual 5-form. The detail construction is given in appendix \( \text{A} \). Here we shall just present the solution:

\[
\begin{align*}
\text{ds}^2 &= -h^{-2}(dt + V_i dx^i)^2 + h^2(dy^2 + dx^i dx^i) + ye^H d\Omega_3^2 + ye^{-H}d\tilde{\Omega}_3^2 \\
F_{(5)} &= dB_t \wedge (dt + V) + h^2 e^{3H} *_3 d\Phi \wedge d\Omega_3, \quad B_t = \beta y^2 e^{2H}, \\
\Phi &= -\alpha y^2 e^{-2H}, \quad D = \frac{1}{2} \alpha \beta \tanh H ,
\end{align*}
\]

(7.3)

where \( \alpha, \beta = \pm 1 \), depending on the detail structure of the pseudo-Killing spinor, and the basic function \( D \) satisfies

\[
\partial_i^2 D + y \partial_y \left( \frac{1}{y} \partial_y D \right) = 0 .
\]

(7.4)

To avoid the singularities at \( y = 0 \), we must impose the following boundary condition

\[
D(y = 0) = \pm \frac{1}{2}.
\]

(7.5)

As in LLM case, the general solution of (7.4) is given by

\[
\begin{align*}
D(x_1, x_2, y) &= \frac{y^2}{\pi} \int_D \frac{D(x'_1, x'_2, 0) dx'_1 dx'_2}{[(x - x')^2 + y^2]^2} = -\frac{1}{2\pi} \int_{\partial D} dl' n_i' \frac{x_i - x'_i}{[(x - x')^2 + y^2]^2} + \sigma , \\
V_i(x_1, x_2, y) &= \frac{\epsilon_{ij}}{\pi} \int_D \frac{D(x'_1, x'_2, 0) dx'_1 dx'_2}{[(x - x')^2 + y^2]^2} = \frac{\epsilon_{ij}}{2\pi} \int_{\partial D} dx'_j \frac{1}{(x - x')^2 + y^2}.
\end{align*}
\]

(7.6)

The new solution we obtained shares the same characteristic properties as the LLM solution. The solution is completely fixed by the boundary condition (7.5) on the \( y = 0 \) two-dimensional plane. The plus or minus choice indicates the collapsing of either \( S^3 \) or \( \tilde{S}^3 \) respectively on the \( y = 0 \) boundary. One important difference is that the 5-form field strength given in (7.3) can never be self-dual in our set up. In particular, the 5-form carries either electric or magnetic fluxes for the AdS\(_5\)×S\(_5\) geometry when the boundary condition is in the shape of a round disc with either \( S^3 \) or \( \tilde{S}^3 \) shrinking inside respectively. Thus neither the theory nor the solution can be embedded in type IIB supergravity.

For the LLM solution, the corresponding BPS states have simple field theoretical description in terms of free fermions [24, 25]. The smooth geometric configurations are dual
to the arbitrary droplets of free fermions in the phase space \([5]\). In our case, the dual field theory of the AdS boundary is an intrinsically non-supersymmetric gauge theory; nevertheless, our construction suggests that arbitrary free-fermion droplets can also exist in a non-supersymmetric Yang-Mills theory. The origin of these bubbling states may lie in the \(SO(6)\) global symmetry of the boundary conformal field theory, which is the same as that of the four-dimensional \(N = 4\) super-conformal Yang-Mills theory.

8 Pseudo-supersymmetric less-bubbling AdS in \(D = 8\)

The critical dimension for \(n = 4\) is \(D = 11\), as demonstrated in section 6. For other dimensions, there can be no supersymmetry, and the existence of the pseudo-Killing spinor of a bosonic ansatz does not necessarily imply that Einstein’s equations of motion are satisfied. Additional constraints have to be imposed. In this section, we investigate the effect of these constraints on bubbling AdS solutions, by constructing an explicit solution for \(D = 8\). As can be seen from \([6:22]\) that the constraints implies that the following condition

\[
F_{[MN}^{\phantom{[MN}} RS F_{PQ]RS} = 0.
\]

The detail construction can be found in appendix B. Here we shall simply present the solution:

\[
\begin{align*}
\text{ds}^2 &= -h^{-2} (dt + V_i dx^i)^2 + h^2 (dy^2 + dx^i dx^i) + ye^H d\Omega_5^2 + ye^{-H} d\tilde{\Omega}_5^2, \\
F &= (dB_t \wedge (dt + V) + h^2 e^{2H} e_3 d\Phi) \wedge \Omega_{(2)},
\end{align*}
\]

where

\[
\begin{align*}
h^{-2} &= 2y \cosh H, \\
B_t &= -\frac{2m}{\sqrt{3}} y^2 e^{\frac{3}{2}H}, \\
\Phi &= -\frac{2}{\sqrt{3}} y^2 e^{-\frac{3}{2}H}, \\
D &= -m \tanh H.
\end{align*}
\]

Here \(m = \pm 1\) and the basic function \(D\) satisfies

\[
\begin{align*}
\partial_t^2 D + y \partial_y \left( \frac{1}{y} \partial_y D \right) &= 0, \\
(\partial_t D)^2 + (\partial_y D)^2 &= y^{-2} (1 - D^2)^2.
\end{align*}
\]

To avoid the singularities at \(y = 0\), we must impose the following boundary condition

\[
D(y = 0) = \pm 1.
\]
The solution for (8.4) is given in (7.6). Note that the additional equation (8.5) is a consequence of imposing the condition (8.1). Note that the constraint (8.5) is invariant under $D \leftrightarrow 1/D$.

The non-linearity of (8.5) implies that we can no longer generate new solutions by superposing the known solutions. This severely restricts the bubbling effect that is associated with (8.4). We have tested boundary conditions for many shapes, such as the disc, rectangle, ring, belt and multi-discs. The only solutions we have found analytically are the AdS$^4 \times S^4$ and pp-wave solutions, which correspond to the boundary conditions in the shapes of a disc and half-filled plane respectively. On the other hand, the constraint (8.5) is clearly consistent with the boundary condition (8.6). It is quite possible that less-bubbling solutions beyond the vacuum solution may exist. These solutions would correspond to some specific free-fermion droplets in the phase space of the dual gauge theory.

9 Conclusions

In this paper we consider Einstein gravity coupled to an $n$-form field strength in general $D$ dimensions. We introduce the pseudo-Killing spinor equation for such a system. In the special case when the system becomes (part of) the bosonic Lagrangian of a supergravity theory, the pseudo-Killing spinors become real Killing spinors. We show by explicit construction, for AdS$\times$Sphere vacuum solution, extremal $p$-branes and intersecting $p$-branes, pseudo-Killing spinors behave just like real Killing spinor in supergravities. The vacua have the maximum number of pseudo-Killing spinors whilst $p$-branes and intersecting $p$-branes have fractions of pseudo-Killing spinors.

We study the integrability condition of the pseudo-Killing spinor equation. We find that additional constraints have to be imposed so that the bosonic ansatz with pseudo-Killing spinors can automatically satisfy the Einstein equations of motion. For $n = 5, 4, 3, 2$, there exist critical dimensions for which the additional constraints vanish; these corresponds to relevant supergravities. However, in some non-supersymmetric cases, additional constraints can nevertheless be satisfied by appropriate ansatz, leading to the construction of new non-trivial pseudo-supersymmetric solutions with pseudo-Killing spinors. Thus, even though the pseudo-Killing spinors we introduced in this paper are not consistent at the full level with those non-supersymmetric theories, they provide a useful technique for reducing second-order Einstein equations to the first-order system, for a large number of special cases of ansatze that can circumvent the additional constraints. Even in the case when the additional
constraints cannot be circumvented in a trivial way, the technique can still provide non-trivial solutions. These solutions are unlikely to be found by considering equations of motion only.

A concrete example we present is the new bubbling AdS geometry in $D = 10$ supported by a non-self-dual 5-form. Although the solution exhibit almost identical features of the previously-constructed LLM solution, it is intrinsically non-supersymmetric and cannot be embedded in type IIB supergravity in any limit of parameters of the solution. This demonstrates that bubbling states of arbitrary free-fermion droplets in the phase space can exist also in non-supersymmetric boundary conformal field theory. The origin of these states are not due to the supersymmetry, but likely to be associated with the $SO(6)$ global symmetry, which is shared by both the supersymmetric and non-supersymmetric conformal field theories. The bubbling geometry we constructed preserves half of pseudo-supersymmetry. It is of interest to investigate whether there are bubbling geometry with less pseudo-supersymmetry, analogous to the $\frac{1}{4}$ and $\frac{1}{8}$-BPS solutions constructed in \cite{20}.

We also present a concrete example of $(D, n) = (8, 4)$ for which the additional constraint can not be circumvent. This additional constraint gives rise to non-linear differential equation on the basic function of the would-be bubbling geometry. The non-linearity implies that we cannot construct new solutions by superposing the known solutions. This severely restricts the bubbling nature of the geometry. Free-fermion droplets in the dual gauge theory can no longer arbitrary. Nevertheless there may still exist less bubbling configurations that go beyond the vacuum solution. How to solve this non-linear equation remains an open problem.

Thus our introduction of pseudo-Killing spinors to an intrinsically non-supersymmetric system can help us to construct new solutions that are unlikely to be found by examining only the equations of motion. This technique enlarges the possibility of constructing more non-trivial bulk gravity backgrounds. Applying the AdS/CFT correspondence, our results show that an intrinsically non-supersymmetric conformal field theory may have pseudo-supersymmetric states with characteristics of BPS states of superconformal field theory.

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A  Detail derivation of the new bubbling AdS$_5$ solution

In this appendix, we provide detail derivation of the new bubbling AdS$_5 \times S^5$ geometry, presented in section 7. The Lagrangian is given by (2.1), but specialized to $n = 5$ and $D = 10$. This is an intrinsically non-supersymmetric theory since we do not require that the 5-form be self dual. The pseudo-Killing spinor equation is given by (7.2). Of course, they become real Killing spinors if we impose the self-duality for the 5-form. The ansatz with the SO(4) × SO(4) isometry is given by (7.1). Note that we use $\mu$, $a$ and $\tilde{a}$ to denote indices for $ds^2_4$, $d\Omega^2_3$ and $d\tilde{\Omega}^2_3$ respectively.

To proceed, we decompose the gamma matrices as follows

$$
\Gamma_{\mu} = \gamma_{\mu} \otimes 1 \otimes 1 \otimes 1, \quad \Gamma_a = \hat{\gamma} \otimes \sigma_1 \otimes \sigma_a \otimes 1, \quad \Gamma_{\tilde{a}} = \hat{\gamma} \otimes \sigma_2 \otimes 1 \otimes \sigma_{\tilde{a}} \quad (A.1)
$$

where

$$
\hat{\gamma} = i\epsilon_{0123} = -\frac{1}{4!} i\epsilon_{\mu_1\mu_2\mu_3\mu_4} \gamma^{\mu_1\mu_2\mu_3\mu_4}, \quad \epsilon_{0123} = 1, \quad \sigma_3 = -i\sigma_1\sigma_2. \quad (A.2)
$$

Here the hatted indices are the vielbein indices. The $\sigma_i$'s are the Pauli matrices and the hatted indices in Gamma matrices are the vielbein indices. The $\Gamma_{11}$ in this decomposition is given by

$$
\Gamma_{11} = \hat{\gamma} \sigma_3. \quad (A.3)
$$

We now perform the reduction on $S^3 \times S^3$ by introducing two component spinors $\chi_{\pm}$ and $\tilde{\chi}_{\pm}$ which obey the equations

$$
\nabla'_a \chi_\alpha = \alpha \frac{i}{2} \sigma_{\tilde{a}} \chi_\alpha, \quad \nabla'_a \tilde{\chi}_\beta = \tilde{\beta} \frac{i}{2} \sigma_{\tilde{a}} \tilde{\chi}_\beta. \quad (A.4)
$$

Note that $(\alpha, \tilde{\beta})$ are $\pm 1$ when they appear in the equations as numbers and $\pm$ as indices of the Killing spinors $\chi_\pm$ and $\tilde{\chi}_\pm$. The $\nabla'$ is the covariant derivative on a unit sphere. It is related to covariant derivatives in the sphere directions in our ansatz as

$$
\nabla_a = \nabla'_a - \frac{1}{4} \Gamma^\mu_a \partial_\mu H, \quad \nabla_{\tilde{a}} = \nabla'_{\tilde{a}} - \frac{1}{4} \Gamma^\mu_{\tilde{a}} \partial_\mu H. \quad (A.5)
$$

We may expand the spinor $\eta$ over basis of the real Killing spinors on the two spheres

$$
\eta = \sum_{\alpha, \tilde{\beta} = \pm} \epsilon_{\alpha\tilde{\beta}} \otimes \chi_\alpha \otimes \tilde{\chi}_\beta. \quad (A.6)
$$

All pseudo-Killing spinors can be written as a linear combination of chiral and anti-chiral spinors, defined by

$$
\Gamma_{11} \eta = \lambda \eta, \quad \lambda = \pm 1 \quad (A.7)
$$

26
Since the $\Gamma_{11}$ commutes with the Killing spinor equation, the chiral and anti-chiral Killing spinors are independent of each other. For simplicity, we do not use any subscript on $\eta$ to distinguish the two types of pseudo-Killing spinors, but instead just let the unspecified parameter $\lambda$ to indicate the chirality of the spinor. By contrast, in type IIB supergravity with the self-dual 5-form, the chirality is specified. The expression in (7.2) involving the 5-form becomes

$$
\frac{i}{960} \Gamma_{M_1 M_2 M_3 M_4 M_5} F_{M_1 M_2 M_3 M_4 M_5} = \frac{1}{96} (e^{-\frac{1}{2} H} F_{\mu \nu} \Gamma^{\mu \nu} \epsilon_{a b c} \Gamma_{a b c} + e^{-\frac{1}{2} H} \tilde{F}_{\mu \nu} \Gamma^{\mu \nu} \epsilon_{\bar{a} \bar{b} \bar{c}} \Gamma_{\bar{a} \bar{b} \bar{c}})
$$

$$= -\frac{1}{16} (e^{-\frac{1}{2} H} F_{\mu \nu} \gamma^{\mu \nu} \tilde{\gamma} \otimes \sigma_1 \otimes 1 \otimes 1 + e^{-\frac{1}{2} H} \tilde{F}_{\mu \nu} \gamma^{\mu \nu} \tilde{\gamma} \otimes \sigma_2 \otimes 1 \otimes 1). \quad (A.8)
$$

The pseudo-Killing spinor equations in ($\mu, a, \bar{a}$) directions are given by

$$\sum_{\alpha, \beta} \left[ \nabla_\rho \epsilon_{\alpha \beta} \otimes \chi_\alpha \otimes \bar{\chi}_\beta - \frac{1}{16} (e^{-\frac{1}{2} H} F_{\mu \nu} \gamma^{\mu \nu} \tilde{\gamma} \gamma_{\rho} \sigma_1 \epsilon_{\alpha \beta} \otimes \chi_\alpha \otimes \bar{\chi}_\beta
+ e^{-\frac{3}{2} H} \tilde{F}_{\mu \nu} \gamma^{\mu \nu} \tilde{\gamma} \gamma_{\rho} \sigma_2 \epsilon_{\alpha \beta} \otimes \chi_\alpha \otimes \bar{\chi}_\beta) = 0,
\right.$$

$$\sum_{\alpha, \beta} \left[ \frac{i\alpha}{2} e^{-\frac{1}{2} H} (e_{\alpha \beta} \otimes \sigma_\alpha \chi_\alpha \otimes \bar{\chi}_\beta) - \frac{1}{4} \partial_\mu H (\gamma^{\mu \nu} \gamma_{\rho} \sigma_1 \epsilon_{\alpha \beta} \otimes \sigma_\alpha \chi_\alpha \otimes \bar{\chi}_\beta) - \frac{1}{16} (e^{-\frac{1}{2} H} F_{\mu \nu} \gamma^{\mu \nu} \epsilon_{\alpha \beta} \otimes \sigma_\alpha \chi_\alpha \otimes \bar{\chi}_\beta + i e^{-\frac{3}{2} H} \tilde{F}_{\mu \nu} \gamma^{\mu \nu} \epsilon_{\alpha \beta} \otimes \sigma_\alpha \chi_\alpha \otimes \bar{\chi}_\beta) = 0,
\right.$$

$$\sum_{\alpha, \beta} \left[ \frac{i\beta}{2} e^{-\frac{1}{2} H} (e_{\alpha \beta} \otimes \sigma_\alpha \chi_\alpha \otimes \bar{\chi}_\beta) - \frac{1}{4} \partial_\mu H (\gamma^{\mu \nu} \gamma_{\rho} \sigma_2 \epsilon_{\alpha \beta} \otimes \sigma_\alpha \chi_\alpha \otimes \bar{\chi}_\beta) - \frac{1}{16} (i e^{-\frac{1}{2} H} F_{\mu \nu} \gamma^{\mu \nu} \sigma_3 \epsilon_{\alpha \beta} \otimes \sigma_\alpha \chi_\alpha \otimes \bar{\chi}_\beta + e^{-\frac{3}{2} H} \tilde{F}_{\mu \nu} \gamma^{\mu \nu} \epsilon_{\alpha \beta} \otimes \sigma_\alpha \chi_\alpha \otimes \bar{\chi}_\beta) = 0. \quad (A.9)
$$

It is easy to see that the spinor basis $\chi_\alpha \otimes \bar{\chi}_\beta$ in the decomposition (A.6) are independent. Thus there are four independent possible solutions depending on the discrete choices of the $\pm 1$ values of the parameters $\alpha$ and $\beta$. However, they cannot be simultaneous solutions. If we choose one set of $(\alpha, \beta)$ values to obtain pseudo-Killing spinors, the other three choices cannot lead to any solutions. Thus we can drop the sum notation in the equations in (A.9).

Furthermore, we shall drop the subscripts for $\epsilon_{\alpha \beta}$ for convenience from now on. It should be understood that the parameters $(\alpha, \beta)$ now denote one, but unspecified choice. This leads to

$$\left\{ \begin{array}{l}
\nabla_\rho - \frac{1}{16} (e^{-\frac{1}{2} H} F_{\mu \nu} \gamma^{\mu \nu} \tilde{\gamma} \gamma_{\rho} \sigma_1 + e^{-\frac{3}{2} H} \tilde{F}_{\mu \nu} \gamma^{\mu \nu} \tilde{\gamma} \gamma_{\rho} \sigma_2) \right) \epsilon = 0,
\quad i \alpha e^{-\frac{1}{2} H} \tilde{\gamma} \sigma_1 + \frac{1}{2} \partial_\mu H \gamma^\mu - \frac{1}{8} (e^{-\frac{1}{2} H} F_{\mu \nu} \gamma^{\mu \nu} \tilde{\gamma} \sigma_1 - e^{-\frac{3}{2} H} \tilde{F}_{\mu \nu} \gamma^{\mu \nu} \tilde{\gamma} \sigma_2) \epsilon = 0,
\quad i \beta e^{-\frac{1}{2} H} \tilde{\gamma} \sigma_2 + \frac{1}{2} \partial_\mu H \gamma^\mu + \frac{1}{8} (e^{-\frac{1}{2} H} F_{\mu \nu} \gamma^{\mu \nu} \tilde{\gamma} \sigma_2 - e^{-\frac{3}{2} H} \tilde{F}_{\mu \nu} \gamma^{\mu \nu} \tilde{\gamma} \sigma_1) \epsilon = 0. \quad (A.10)
\end{array} \right.$$

As was demonstrated in section 6, the existence of pseudo-Killing spinors is not sufficient for the ansatz to satisfy the Einstein equations of motion. Additional constraints (6.21)
have to be satisfied. For our ansatz, the constraints become

\[ F_{\mu_1\mu_2} \tilde{F}^{\mu_1\nu_2} = 0, \quad F_{\mu_1[\mu_2} \tilde{F}^{\mu_1\nu_2]} - \tilde{F}_{\mu_1[\mu_2} F^{\nu_1\nu_2]} = 0, \quad F_{[\mu_1\mu_2} \tilde{F}^{\nu_1\nu_2]} = 0. \] (A.11)

These constraints are difficult to solve except for two special cases: one is that \( F_{(2)} \) is self-dual, and the other is that one of the \( F_{(2)} \) and \( \tilde{F}_{(2)} \) vanishes. The former case was discussed in [5]. We shall focus on the later case. Let \( F_{(2)} = dB \) and \( \tilde{F}_{(2)} = 0 \), the pseudo-Killing spinor equations now become significantly simpler:

\[
\begin{align*}
\left[ \nabla_\mu - \frac{1}{16} e^{-\frac{3}{2}H} F_{\mu\nu} \gamma^{\mu\nu} \gamma_0 \gamma_{\sigma} \right] \epsilon &= 0, \quad (A.12) \\
\left[ \partial_\mu H \gamma^\mu - \frac{1}{8} e^{-\frac{3}{2}H} F_{\mu\nu} \gamma^{\mu\nu} \gamma_{\sigma} \right] \epsilon &= 0, \quad (A.13) \\
\left[ \alpha e^{-\frac{3}{2}H} \gamma \sigma_2 + \frac{1}{2} \partial_\mu \tilde{H} \gamma^\mu + \frac{1}{8} e^{-\frac{3}{2}H} F_{\mu\nu} \gamma^{\mu\nu} \gamma \sigma_1 \right] \epsilon &= 0. \quad (A.14)
\end{align*}
\]

We are now in the position to derive the solution using the standard G-structure technique. Let us define the following real spinor bilinears

\[ f_1 = i \bar{\epsilon} \sigma_1 \epsilon, \quad f_2 = i \bar{\epsilon} \sigma_2 \epsilon, \quad K_\mu = \bar{\epsilon} \gamma_\mu \epsilon, \quad L_\mu = \bar{\epsilon} \gamma_\mu \gamma_\epsilon, \quad Y_{\mu\nu} = \bar{\epsilon} \gamma_{\mu\nu} \sigma_1 \epsilon, \] (A.15)

as well as the complex spinor bilinears

\[ L_\mu^c = \bar{\epsilon}^c \gamma_\mu \sigma_1 \epsilon, \quad Y_{\mu\nu}^c = \bar{\epsilon}^c \gamma_{\mu\nu} \epsilon. \] (A.16)

where \( \bar{\epsilon} = \epsilon^+ \gamma^0 \) and \( \bar{\epsilon}^c = \epsilon^t C \). We choose the basis where \( \gamma_2 \) and \( \sigma_2 \) are antisymmetric while other \( \gamma_\mu \)’s and \( \sigma_1 \) are symmetric, thus \( C = \gamma_2 \sigma_1 \). From (7.34) we have the following relations between the bi-spinors

\[
\begin{align*}
\nabla_\mu f_1 &= \frac{1}{8} e^{-\frac{3}{2}H} \epsilon_{\mu}^{\nu\rho\sigma} F_{\nu\rho} K_{\sigma}, \quad (A.17) \\
\nabla_\mu f_2 &= \frac{1}{4} \lambda e^{-\frac{3}{2}H} F_{\mu\nu} K^\nu, \quad (A.18) \\
\nabla_\mu K_\nu &= -\frac{1}{8} e^{-\frac{3}{2}H} \left( \epsilon_{\mu\nu}^{\rho\lambda} F_{\rho\lambda} f_1 + 2 \lambda F_{\mu\nu} f_2 \right), \quad (A.19) \\
\nabla_\mu L_\nu &= \frac{1}{8} e^{-\frac{3}{2}H} \left( g_{\mu\nu} F_{\lambda\rho} Y^{\lambda\rho} + 2 F_{\mu\rho} Y_{\rho\nu} + 2 F_{\nu\rho} Y_{\rho\mu} \right), \quad (A.20) \\
\nabla_\mu L_\nu^c &= -\frac{1}{8} e^{-\frac{3}{2}H} \left( g_{\mu\nu} F_{\lambda\rho} Y^{\lambda\rho}_c + 2 F_{\mu\rho} Y_{\rho\nu}^c + 2 F_{\nu\rho} Y_{\rho\mu}^c \right). \quad (A.21)
\end{align*}
\]

From (A.19), we have \( \nabla_\mu K_\nu = 0 \). Thus \( K = K^\mu \partial_\mu \) is a Killing vector. Also the 1-forms \( L = L_\mu dx^\mu \) and \( L^c = L^c_\mu dx^\mu \) are (locally) exact. Using Fierz identities we obtain the following two identities

\[ K \cdot L = 0, \quad L^2 = -K^2 = f_1^2 + f_2^2. \] (A.22)

Thus the Killing vector \( K \) is time-like.
Since \( L = L_\mu dx^\mu \) is (locally) exact, we can choose a coordinate \( y \) such that
\[
d y = L_\mu dx^\mu. \tag{A.23}
\]
We then choose the other three coordinates in the subspace orthogonal to \( y \)
\[
ds^2 = h^2 dy^2 + g_{\alpha\beta} dx^\alpha dx^\beta, \quad h^{-2} = f_1^2 + f_2^2. \tag{A.24}
\]
Using the relation
\[
0 = K^\mu L_\mu = K^y L_y = K^y, \tag{A.25}
\]
we find that \( K^\alpha \) is a vector in three dimensional space spanned by \( x^\alpha \). Since \( K \) is time-like, we can choose the time \( t \) as the coordinate along \( K^\alpha \), i.e. \( K^\alpha = \partial/\partial t \). We find
\[
ds^2 = -h^{-2}(dt + V_i dx^i)^2 + h^2(dy^2 + \tilde{g}_{ij} dx^i dx^j) \tag{A.26}
\]
where \( i, j = 1, 2 \).

The equation of motion for the 5-form is \( d*_{10} F_{(5)} = 0 \). For our ansatz, this becomes
\[
d(e^{\frac{3}{2}(\tilde{H}-H)}*_4 dB) = 0. \tag{A.27}
\]
Let us write the components of the gauge field \( B \) as
\[
B_\mu dx^\mu = B_t dt + B_\alpha dx^\alpha = B_t(dt + V_i dx^i) + (B_\alpha - B_t V_\alpha) dx^\alpha = B_t(dt + V_i dx^i) + \hat{B}, \tag{A.28}
\]
then the components of the field strength and its dual are given by
\[
F = dB_t \wedge (dt + V) + (dB_t + B_t dV), \quad *_4 F = h^2 *_3 dB_t + h^{-2} (dt + V) \wedge *_3 (dB_t + B_t dV), \tag{A.29}
\]
where \( *_3 \) is the Hodge dual in \( ds_3^2 = dy^2 + \tilde{g}_{ij} dx^i dx^j \). The equation (A.27) now becomes
\[
d *_3 \left[h^{-2} e^{\frac{3}{2}(\tilde{H}-H)}(dB_t + B_t dV)\right] = 0. \tag{A.30}
\]
Thus, locally we can introduce a dual potential \( \Phi \):
\[
d \wedge dB_t + B_t dV = h^2 e^{\frac{3}{2}(\tilde{H}-H)} *_3 d\Phi. \tag{A.31}
\]
The time component of (A.27) leads to the equation:
\[
d \left[V \wedge d\Phi + h^2 e^{\frac{3}{2}(\tilde{H}-H)} *_3 dB_t \right] = 0. \tag{A.32}
\]
From the equation (A.18) and the fact that \( B_i \) is independent of \( t \), we find
\[
\partial_\mu f_2 = \frac{1}{4} \lambda e^{-\frac{3}{2} H} \partial_\mu B_t. \tag{A.33}
\]
By using (A.13), we find
\[
\partial_\mu B_t = F_{\mu\nu} K^\nu = \frac{1}{4} \epsilon[\gamma_\mu, F] \epsilon
= \epsilon[\gamma_\mu, 2i\alpha e^H - e^\frac{3}{2} H \partial_\nu H \gamma^\nu \gamma \sigma_1] \epsilon = 2\lambda e^\frac{3}{2} H \partial_\mu H f_2
\]  
(A.34)

Thus
\[
\partial_\mu f_2 = \frac{1}{2} f_2 \partial_\mu H \implies f_2 = c_2 e^\frac{3}{2} H, \quad B_t = \lambda c_2 e^{2H}.
\]  
(A.35)

From the equation (A.17), we find
\[
\partial_\mu f_1 = \frac{1}{8} e^{-\frac{3}{2} H} \epsilon_\mu^\rho \epsilon F_{\rho\nu} K_\sigma = \frac{1}{4} e^{-\frac{3}{2} H} (\ast_4 F)_{\mu\nu} K^\nu = -\frac{1}{4} e^{-\frac{3}{2} H} \partial_\mu \tilde{H} f_1.
\]  
(A.36)

By using (A.14), we have
\[
\epsilon_\mu^\rho \epsilon F_{\rho\nu} K^\sigma = \frac{i}{2} F_{\mu\rho} \epsilon_\gamma^\gamma \{\gamma_\mu, \gamma^\nu \} \epsilon
= i \epsilon \{\gamma_\mu, -4\beta e^{\frac{3}{2} H} \gamma \sigma_3 + 2e^{\frac{3}{2} H} \partial_\nu \tilde{H} \gamma^\nu \sigma_1 \} \epsilon = 4e^{\frac{3}{2} H} \partial_\mu \tilde{H} f_1.
\]  
(A.37)

Thus
\[
\partial_\mu f_1 = \frac{1}{2} f_1 \partial_\mu \tilde{H} \implies f_1 = c_1 e^\frac{3}{2} H, \quad \Phi = -c_1 e^{2H}.
\]  
(A.38)

From (A.13) and (A.14), we find
\[
\gamma^\mu \partial_\mu (H + \tilde{H}) \epsilon = -2i(\alpha e^{-\frac{H}{2}} \gamma \sigma_1 + \beta e^{-\frac{H}{2}} \gamma \sigma_2) \epsilon
\]  
(A.39)

Thus
\[
c_1 \partial_\mu (H + \tilde{H}) e^{\frac{3}{2} H} = \partial_\mu (H + \tilde{H}) f_1 = \frac{1}{2} \partial_\mu (H + \tilde{H}) \epsilon \{\gamma_\mu, \gamma^\nu \} \sigma_1 \epsilon
= -i \epsilon \{\gamma_\mu, i \alpha e^{-\frac{H}{2}} \gamma - a \beta e^{-\frac{H}{2}} \gamma \} \epsilon = 2\alpha e^{-\frac{H}{2}} L_\mu,
\]
\[
c_2 \partial_\mu (H + \tilde{H}) e^{\frac{3}{2} H} = \partial_\mu (H + \tilde{H}) f_2 = \frac{1}{2} \partial_\mu (H + \tilde{H}) \epsilon \{\gamma_\mu, \gamma^\nu \} \sigma_2 \epsilon
= -i \epsilon \{\gamma_\mu, a \alpha e^{-\frac{H}{2}} + i \beta e^{-\frac{H}{2}} \gamma \} \epsilon = 2\beta e^{-\frac{H}{2}} L_\mu.
\]  
(A.40)

By changing the overall sign of the 5-form flux and an appropriate rescaling of the Killing spinor, we can set
\[
c_1 = \alpha.
\]  
(A.41)

It follows that
\[
e^{\frac{3}{2} (H + \tilde{H})} = y + c_3, \quad c_2 = \beta.
\]  
(A.42)

By an appropriate shift of y, we can set $c_3 = 0$. Now we find that
\[
h^{-2} = -K^2 = f_1^2 + f_2^2 = e^H + e^\tilde{H}.
\]  
(A.43)
The equations $K^t = 1$ and $L_y = 1$ imply that $\epsilon^t \epsilon = h^{-1}$ and $\epsilon^t \gamma^0 \gamma^3 \gamma^3 \epsilon = h^{-1}$ respectively. Thus we find

$$0 = \epsilon^t \left[ 1 - \gamma^0 \gamma^3 \right] \epsilon = \frac{1}{2} \epsilon^t \left[ 1 - \gamma^0 \gamma^3 \right]$$

$$\epsilon^t \left[ 1 - \gamma^0 \gamma^3 \right] \epsilon = \frac{1}{2} \epsilon^t \left[ 1 - \gamma^0 \gamma^3 \right]$$

(A.44)

It follows

$$\left[ 1 - \gamma^0 \gamma^3 \right] \epsilon = 0, \quad \text{or} \quad \left[ 1 - \gamma^0 \gamma^3 \right] \epsilon = 0$$

(A.45)

Substituting (A.42) back to (A.39), we find

$$\left( \sqrt{1 + e^{H - \tilde{H}} \gamma^3 \sigma_1} + \imath \alpha \gamma - \lambda \beta e^{H - \tilde{H}} \right) \epsilon = 0$$

$$\Rightarrow \left( e^{-2 \lambda \beta \gamma^3 \sigma_1} + \imath \alpha \gamma^3 \sigma_1 \right) \epsilon = 0.$$}

(A.46)

where $\sinh(2 \xi) = e^{\frac{1}{2}(H - \tilde{H})}$. It implies that the pseudo-Killing spinor has the form

$$\epsilon = e^{\lambda \beta \gamma^3 \sigma_1} \epsilon_1, \quad (1 + \imath \alpha \gamma^3 \sigma_1) \epsilon_1 = 0.$$

(A.47)

Inserting (A.47) into the expression (A.38) for $f_1$ gives

$$\alpha e^{\frac{1}{2} \tilde{H}} = i \epsilon^t e^{- \lambda \beta \gamma^3 \sigma_1} \xi \sigma_1 e^{\lambda \beta \gamma^3 \sigma_1} \xi \epsilon_1 = i \epsilon^t \gamma^0 \sigma_1 \epsilon_1.$$}

(A.48)

Thus

$$\epsilon_1 = e^{\frac{1}{2} \tilde{H}} \epsilon_0, \quad \epsilon_0^t \gamma^0 \sigma_1 \epsilon_0 = \imath \alpha,$$

$$\epsilon^t \epsilon = e^{\frac{1}{2} \tilde{H}} \epsilon_0 (\cosh 2 \xi + \sinh 2 \xi \gamma^3 \sigma_1) \epsilon_0$$

$$= e^{\frac{1}{2} \tilde{H}} (\cosh 2 \xi \epsilon_0 \epsilon_0 - \imath \sinh 2 \xi \epsilon_0 \gamma^3 \sigma_1 \epsilon_0) = h^{-1} \epsilon_0 \epsilon_0.$$}

(A.49)

From the above expressions for Killing spinor we find

$$L_c^t = e^t \gamma_2 \gamma_0 \epsilon = e^t \gamma_3 \sigma_1 \epsilon = 0$$

$$L_{11}^t = e^t \gamma_2 \gamma_1 \epsilon = i \epsilon^t \epsilon = i h^{-1} \epsilon_0 \epsilon_0$$

$$L_{22}^t = e^t \gamma_2 \gamma_2 \epsilon = \epsilon^t \epsilon = h^{-1} \epsilon_0 \epsilon_0$$

$$L_{33}^t = e^t \gamma_2 \gamma_3 \epsilon = e^t \gamma_0 \sigma_1 \epsilon = 0$$

$$L^c = L_c^t e^t dx^0 = (\epsilon_0^t \epsilon_0) (i \epsilon_0^t + e^t) dx^i$$}

(A.50)

Where $\tilde{c}$ is the vielbein of the metric $\tilde{g}_{ij} = \tilde{c}^\epsilon c_i c_j$ and $c_i^\epsilon = h \tilde{c}_i^\epsilon$ is the full vielbein for the four dimensional metric in the directions 1,2. The equation $dL^c = 0$ implies that $\epsilon_0$ is independent of the time $t$. Then we can make it as a constant spinor by setting the phase of $\epsilon_0$ to zero via a local Lorentz rotation in the $(x_1, x_2)$-plane. Under this gauge choice, the
equation \( dL = 0 \) further implies that the vielbeins \( \tilde{e}^i \) are independent of \( y \) and that the two-dimensional metric is flat. So we may choose coordinates such that \( \tilde{g}_{ij} = \delta_{ij} \).

From (A.19) we find that
\[
\begin{align*}
d\left[ h^{-2}(dt + V) \right] &= -dK = \frac{1}{2} e^{-\frac{3}{2}H} (f_1 \ast_4 F + a f_2 F) \\
&= \frac{1}{2} \alpha e^{\frac{H-\tilde{H}}{2}} \left[ h^2 \ast_3 dB_t + e^{\frac{3}{2}(H-\tilde{H})} (dt + V) \wedge d\Phi \right] \\
&\quad + \frac{1}{2} \lambda \beta e^{-H} \left[ dB_t \wedge (dt + V) + h^2 e^{\frac{3}{2}(H-\tilde{H})} \ast_3 d\Phi \right].
\end{align*}
\]
(A.51)

It follows that
\[
\begin{align*}
h^{-2} dV &= \frac{1}{2} h^2 \left[ \alpha e^{\frac{H-\tilde{H}}{2}} \ast_3 dB_t + \lambda \beta e^{\frac{H-\tilde{H}}{2}} \ast_3 d\Phi \right] \\
&= \lambda \alpha \beta h^2 y \ast_3 d(H - \tilde{H}).
\end{align*}
\]
(A.52)

Let
\[
H_{\pm} = \frac{1}{2}(H \pm \tilde{H}), \quad D = \frac{1}{2} \lambda \alpha \beta \tanh H_-,
\]
we find
\[
dV = 2 \lambda \alpha \beta \left( e^H + e^{\tilde{H}} \right)^{-2} y \ast_3 dH_- = y^{-1} \ast_3 dD.
\]
(A.54)

Finally, the consistency condition \( d^2 V = 0 \) implies
\[
\partial_i^2 D + y \partial_y \left( \frac{1}{y} \partial_y D \right) = 0.
\]
(A.55)

Thus the solution is completely solved up to (A.55), which is the Laplace equation. This basic equation is identical to that of the LLM bubbling solution in type IIB supergravity. We summarize the solution in (7.3) and (7.4). The \( H \) in (7.3) should be \( H_- \) as the logical consequence of our derivation, but we remove the subscript in (7.3) for the stylistic reason. We also dropped \( \lambda \) since it appears always together with \( \beta \).

**B Detail derivation of the less-bubbling AdS_4 solution**

In this section, we give the detail derivation of the less-bubbling solution presented in section 8. As shown in section 6, the critical dimension for \( n = 4 \) is \( D = 11 \), corresponding to M-theory. For other dimensions, additional constraints have to be imposed in order for the existence of the pseudo-Killing spinors to imply the equations of motion. We shall consider the case with \( D = 8 \). The pseudo-Killing spinor equation is given by
\[
0 = D_M \eta + \frac{1}{96 \sqrt{3}} \left( \Gamma_M^{M_1M_2M_3M_4} - 4 \delta_M^{M_1} \Gamma^{M_2M_3M_4} \right) F_{M_1M_2M_3M_4} \eta
= D_M \eta + \frac{1}{96 \sqrt{3}} \Gamma_M^{M_1M_2M_3M_4} \Gamma_M^* F_{M_1M_2M_3M_4}. \quad (B.1)
\]
The integrability condition can be found in section 6. We begin with the following ansatz

\[ ds^2 = g_{\mu\nu} dx^\mu dx^\nu + e^H d\Omega^2_2 + e^{\tilde{H}} d\tilde{\Omega}^2_2, \]

\[ F_{(4)} = \frac{1}{2} F_{\mu\nu} dx^\mu \wedge dx^\nu \wedge \Omega_{(2)} + \frac{1}{2} \tilde{F}_{\mu\nu} dx^\mu \wedge dx^\nu \wedge \tilde{\Omega}_{(2)}, \] (B.2)

which has the \( SO(3) \times SO(3) \) isometry. The gamma matrices can be decomposed as follows

\[ \Gamma_\mu = \gamma_\mu \otimes 1 \otimes 1, \quad \Gamma_a = \hat{\gamma} \otimes \sigma_a \otimes 1, \quad \Gamma_{\tilde{a}} = \hat{\gamma} \otimes \sigma_{\tilde{a}} \otimes 1. \] (B.3)

where

\[ \hat{\gamma} = i\gamma_{0123} = -\frac{1}{4!} i\epsilon_{\mu_1\mu_2\mu_3\mu_4} \gamma^{\mu_1\mu_2\mu_3\mu_4}, \quad \epsilon_{0123} = 1, \quad \sigma_3 = i\sigma_1 \sigma_2. \] (B.4)

where the hatted indices are the velbein indices and the \( \sigma \)'s with hatted subscripts are Pauli matrices. We now perform the reduction on \( S^2 \times S^2 \) and introduce two component spinors \( \chi^\pm \) and \( \tilde{\chi}^\pm \) on the two \( S^2 \), which obey the equations

\[ \nabla'_{a} \chi^{\pm} = \alpha \frac{i}{2} \sigma_a \chi^{\pm}, \quad \nabla'_{\tilde{a}} \tilde{\chi}^{\pm} = \beta \frac{i}{2} \sigma_{\tilde{a}} \tilde{\chi}^{\pm}. \] (B.5)

Note that when \( \alpha \) and \( \beta \) appear as the indices for \( \chi^{\pm} \) and \( \tilde{\chi}^{\pm} \) respectively, they denote \( \pm \); when they appear as numbers, they are \( \pm 1 \). The \( \nabla' \) is the covariant derivative defined on a unit 2-sphere. It is related to the covariant derivative in the sphere directions of our ansatz as

\[ \nabla_a = \nabla'_{a} - \frac{1}{4} \Gamma^\mu_a \partial_\mu H, \quad \nabla_{\tilde{a}} = \nabla'_{\tilde{a}} - \frac{1}{4} \Gamma_{\tilde{a}}^\mu \partial_\mu \tilde{H}, \] (B.6)

The covariance under the \( SU(2) \) transformations ensures that we can take, without loss of generality, that

\[ \chi^{-} = i\sigma_3 \chi^{+}, \quad \tilde{\chi}^{-} = i\sigma_3 \tilde{\chi}^{+}. \] (B.7)

Now we expand the spinor \( \eta \) over basis of the tensor product of the real Killing spinors on the two \( S^2 \),

\[ \xi = \sum_{\alpha,\beta = +, -} \epsilon_{\alpha\beta} \otimes \chi_\alpha \otimes \tilde{\chi}_\beta. \] (B.8)

The expression in (B.1) involving the 4-form becomes

\[
\begin{align*}
\frac{1}{96\sqrt{3}} \Gamma^{M_1 M_2 M_3 M_4} F_{M_1 M_2 M_3 M_4} &= \frac{1}{16\sqrt{3}} (e^{-H} F_{\mu\nu} \Gamma^{\mu\nu} \epsilon_{ab} \Gamma^{ab} + e^{-\tilde{H}} \tilde{F}_{\mu\nu} \Gamma^{\mu\nu} \epsilon_{\tilde{a}\tilde{b}} \Gamma^{\tilde{a}\tilde{b}}) \\
&= \frac{1}{8\sqrt{3}} (e^{-H} F_{\mu\nu} \gamma^{\mu\nu} \otimes i\sigma_3 \otimes 1 + e^{-\tilde{H}} \tilde{F}_{\mu\nu} \gamma^{\mu\nu} \otimes 1 \otimes i\sigma_3)
\end{align*}
\] (B.9)

The pseudo-Killing spinor equation implies

\[
\sum_{\alpha,\beta} \left[ \nabla^{\rho} \epsilon_{\alpha\beta} \otimes \chi_\alpha \otimes \tilde{\chi}_\beta + \frac{1}{8\sqrt{3}} (e^{-H} F_{\rho\sigma} \gamma^{\rho\sigma} \epsilon_{\alpha\beta} \otimes i\sigma_3 \chi_\alpha \otimes \tilde{\chi}_\beta \right]
\]

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Great simplification occurs if we take $\tilde{F}_{(2)} = 0$, in which case, $\{\epsilon_{++}, \epsilon_{--}\}$ decouple from $\{\epsilon_{--}, \epsilon_{++}\}$. We can rewrite the above equations as

$$\nabla_\rho \left( \frac{\epsilon_{++}}{\tilde{\gamma}_{\epsilon_{--}}} \right) + \frac{1}{8\sqrt{3}} e^{-H} F_{\mu\nu}\tilde{\gamma}^{\mu\nu} \tilde{\gamma}_\rho \left( \tilde{\gamma}_{\epsilon_{--}} \right) = 0, \quad (B.13)$$

$$\left( e^{-\frac{\mu}{2}} - \frac{1}{2} \partial_\mu H \tilde{\gamma}^\mu \tilde{\gamma} \right) \epsilon_{++} + \frac{1}{4\sqrt{3}} e^{-H} F_{\mu\nu}\tilde{\gamma}^{\mu\nu} \tilde{\gamma}_\rho \left( \tilde{\gamma}_{\epsilon_{--}} \right) = 0, \quad (B.14)$$

$$\left( e^{-\frac{\mu}{2}} + \frac{1}{4\sqrt{3}} e^{-H} F_{\mu\nu}\tilde{\gamma}^{\mu\nu} \tilde{\gamma} \right) \epsilon_{--} - \frac{1}{2} \partial_\mu H \tilde{\gamma}^\mu \tilde{\gamma}_{\epsilon_{--}} = 0, \quad (B.15)$$
\[ \nabla_\rho \left( \epsilon_{+-} \hat{\gamma} \epsilon_{--} \right) + \frac{1}{8\sqrt{3}} e^{-H} F_{\mu\nu} \hat{\gamma} \gamma^{\mu\nu} \epsilon_{+-} = 0, \quad (B.16) \]

\[ \left( i e^{-\frac{H}{2}} - \frac{1}{2} \partial_\mu H \gamma^{\mu} \right) \left( \epsilon_{+-} \hat{\gamma} \epsilon_{--} \right) + \frac{1}{4\sqrt{3}} e^{-H} F_{\mu\nu} \hat{\gamma} \gamma^{\mu\nu} \epsilon_{+-} = 0, \quad (B.17) \]

\[ \left( - e^{-\frac{\tilde{H}}{2}} + \frac{1}{4\sqrt{3}} e^{-H} F_{\mu\nu} \gamma^{\mu\nu} \right) \left( \epsilon_{+-} \hat{\gamma} \epsilon_{--} \right) - \frac{1}{2} \partial_\mu \tilde{H} \hat{\gamma} \epsilon_{+-} = 0. \quad (B.18) \]

Now it is easy to see that there are in fact four independent combinations

\[ \epsilon_{++} \pm \hat{\gamma} \epsilon_{+-}, \quad \epsilon_{+-} \pm \hat{\gamma} \epsilon_{--}. \quad (B.19) \]

Any one but only one of the above four independent combinations can be the pseudo-Killing spinor. We can thus choose one combination and drop the subscript, giving rise to the following equations.

\[ \left[ \nabla_\rho + \frac{1}{8\sqrt{3}} e^{-H} F_{\mu\nu} \hat{\gamma} \gamma^{\mu\nu} \right] \epsilon = 0, \quad (B.20) \]

\[ \left[ i e^{-\frac{H}{2}} - \frac{1}{2} \partial_\mu H \gamma^{\mu} + \frac{1}{4\sqrt{3}} e^{-H} F_{\mu\nu} \right] \epsilon = 0, \quad (B.21) \]

\[ \left[ m e^{-\frac{\tilde{H}}{2}} + \frac{1}{4\sqrt{3}} e^{-H} F_{\mu\nu} \gamma^{\mu\nu} \hat{\gamma} - \frac{1}{2} \partial_\mu \tilde{H} \hat{\gamma} \right] \epsilon = 0, \quad (B.22) \]

where \( m = \pm 1 \).

We are now in the position to proceed to derive the bubbling solution. As in the previous example in appendix A, we start by defining the following real spinor bilinears

\[ f_1 = i \epsilon e, \quad f_2 = \bar{\epsilon} \hat{\gamma} e, \quad K_\mu = \bar{\epsilon} \gamma_\mu e, \quad L_\mu = \bar{\epsilon} \gamma_\mu \hat{\gamma} e, \quad Y_{\mu\nu} = \bar{\epsilon} \gamma_{\mu\nu} e, \quad (B.23) \]

as well as the complex spinor bilinears

\[ L^c_\mu = \bar{\epsilon} e^{c} \gamma_\mu e, \quad Y^c_{\mu\nu} = \bar{\epsilon} e^{c} \gamma_{\mu\nu} e, \quad (B.24) \]

where \( \bar{\epsilon} = \epsilon^{+} \hat{\gamma}^0 \) and \( \epsilon^c = \epsilon^{\dagger} \gamma_2 \). We find

\[ \nabla_\mu f_1 = - \frac{1}{4\sqrt{3}} e^{-H} \epsilon_\mu^{\nu\rho\sigma} F_{\nu\rho} K_\sigma, \quad (B.25) \]

\[ \nabla_\mu f_2 = \frac{1}{2\sqrt{3}} e^{-H} F_{\mu\nu} K^\nu, \quad (B.26) \]

\[ \nabla_\mu K_\nu = \frac{1}{4\sqrt{3}} e^{-H} \left( \epsilon_{\mu\nu}^{\lambda\rho} F_{\lambda\rho} f_1 - 2F_{\mu\nu} f_2 \right), \quad (B.27) \]

\[ \nabla_\mu L_\nu = - \frac{1}{4\sqrt{3}} e^{-H} \left( g_{\mu\nu} F_{\lambda\rho} Y^{\lambda\rho} + 2F_{\mu\rho} Y^\rho_\nu + 2F_{\nu\rho} Y^\rho_\mu \right), \quad (B.28) \]

\[ \nabla_\mu L^c_\nu = \frac{1}{4\sqrt{3}} e^{-H} \left( g_{\mu\nu} F^{\lambda\rho} Y^{c\lambda\rho}_{\lambda\rho} + 2F^c_{\mu\rho} Y^c_{\rho\nu} + 2F^c_{\nu\rho} Y^c_{\rho\mu} \right). \quad (B.29) \]
Thus $K^\mu$ is a Killing vector while the form $L = L_\mu dx^\mu$ and $L^c = L^c_\mu dx^\mu$ are (locally) exact. Using Fierz identities we have

$$K \cdot L = 0, \quad L^2 = -K^2 = f_1^2 + f_2^2. \tag{B.30}$$

Since $L = L_\mu dx^\mu$ is (locally) exact, we can choose a coordinate $y$ through

$$dy = L_\mu dx^\mu \tag{B.31}$$

The other three coordinates can be chosen to lie in the subspace orthogonal to $y$ such that

$$ds^2 = h^{-2}(dt + V_i dx^i)^2 + h^2(dy^2 + \tilde{g}_{ij} dx^i dx^j), \quad h^{-2} = L^2 = f_1^2 + f_2^2. \tag{B.32}$$

Let us now look at the Killing vector $K^\mu$. Using the relation

$$0 = K^\mu L_\mu = K^y L_y = K^y \tag{B.33}$$

we find that $K^\alpha$ is a vector in the three-dimensional space spanned by $x^\alpha$. Choosing one of the coordinates along $K^\alpha$ (we shall call it $t$), the most general metric of the four-dimensional subspace is given by

$$ds^2 = -h^{-2}(dt + V_i dx^i)^2 + h^2(dy^2 + \tilde{g}_{ij} dx^i dx^j) \tag{B.34}$$

where $i, j = 1, 2$. The signature is thus determined by the fact that the Killing vector $K$ is time-like.

The equation of motion for the 4-form, $d \ast_4 G^{(4)} = 0$, implies that

$$d(e^{\hat{H} - H} \ast_4 dB) = 0 \tag{B.35}$$

Let us split the vector field $B$ as

$$B_\mu dx^\mu = B_t dt + B_\alpha dx^\alpha = B_t(dt + V_i dx^i) + (B_\alpha - B_t V_\alpha) dx^\alpha \equiv B_t(dt + V_i dx^i) + \hat{B}. \tag{B.36}$$

Then its field strength $F = dB$ is

$$F = dB_t \wedge (dt + V) + (d\hat{B} + B_t dV), \quad *_4 F = h^2 *_3 dB_t + h^{-2}(dt + V) \wedge *_3 (d\hat{B} + B_t dV), \tag{B.37}$$

where $*_3$ is defined in $ds^2 = dy^2 + \tilde{g}_{ij} dx^i dx^j$. Then (B.35) implies

$$d *_3 \left[ h^{-2} e^{\hat{H} - H} (d\hat{B} + B_t dV) \right] = 0. \tag{B.38}$$
This means that locally we can introduce a dual potential \( \Phi \):

\[
d\hat{B} + B_t dV = h^2 e^{\hat{H} - \hat{H}} \ast_3 d\Phi \tag{B.39}
\]

Thus the vector field \( B \) can be described by its time component \( B_t \) together with the potential \( \Phi \). The time component of (B.35) leads to the equation:

\[
d \left[ V \wedge d\Phi + h^2 e^{\hat{H} - \hat{H}} \ast_3 dB_t \right] = 0 \tag{B.40}
\]

As shown in section 6 and remarked in section 8, there is an additional constrain (8.1) from the integrability condition. For our ansatz, it implies that \( F \wedge F = 0 \), which gives rise to

\[
dB_t \wedge \ast_3 d\Phi = 0 \Rightarrow \partial \mu B_t \partial^\mu \Phi = 0. \tag{B.41}
\]

From the equation (B.26) and the fact that \( B_i \) is independent of \( t \), we find

\[
\partial \mu f_2 = \frac{1}{2\sqrt{3}} e^{-H} \partial \mu B_t, \quad \text{i.e.} \quad df_2 = \frac{1}{2\sqrt{3}} e^{-H} dB_t. \tag{B.42}
\]

By using (B.21), we find

\[
\partial \mu f_2 = \frac{1}{2} f_2 \partial \mu H \quad \Rightarrow \quad f_2 = c_2 e^{\frac{1}{\sqrt{3}} H}, \quad B_t = \frac{2}{\sqrt{3}} c_2 e^{\frac{1}{\sqrt{3}} H}. \tag{B.43}
\]

From the equation (B.25), we find

\[
\partial \mu f_1 = -\frac{1}{4\sqrt{3}} e^{-H} e^{\mu \rho \sigma} F_{\nu \rho} K^\sigma = -\frac{1}{2\sqrt{3}} (\ast_4 F)_{\mu \nu} K^\nu = \frac{1}{2\sqrt{3}} e^{-\hat{H}} \partial \mu \Phi. \tag{B.44}
\]

By using (B.22), we have

\[
\epsilon_{\mu \nu \rho \sigma} F^{\nu \rho} K^\sigma = \frac{i}{2} F_{\nu \rho} \bar{\epsilon} \gamma \{ \gamma_{\mu}, \gamma^{\nu} \} e = -2\sqrt{3} e^{H} \partial \mu \tilde{H} f_1. \tag{B.45}
\]

Thus

\[
\partial \mu f_1 = \frac{1}{2} f_1 \partial \mu \tilde{H} \quad \Rightarrow \quad f_1 = c_1 e^{\frac{1}{\sqrt{3}} \tilde{H}}, \quad \Phi = \frac{2}{\sqrt{3}} c_1 e^{\frac{1}{\sqrt{3}} \tilde{H}}. \tag{B.46}
\]

By choosing the overall sign of the five-form flux and an appropriate rescaling of the Killing spinor, we may set

\[
c_1 = -1. \tag{B.47}
\]

From (B.21) and (B.22)

\[
\gamma^\mu \partial \mu (H + \tilde{H}) e = (2mc^{-\frac{4}{3}} - 2i\tilde{\gamma} e^{-\frac{4\tilde{H}}{3}}) e. \tag{B.48}
\]
Thus
\[- \partial_\mu (H + \bar{H}) e^{\frac{1}{2} H} = \partial_\mu (H + \bar{H}) f_1 = \frac{i}{2} \partial_\nu (H + \bar{H}) \epsilon \{ \gamma_\mu, \gamma_\nu \} \epsilon = -\frac{i}{2} \epsilon [\gamma_\mu, 2 \gamma_2 e^{-\frac{\bar{H}}{2}} - 2i \gamma_2 e^{-\frac{H}{2}}] \epsilon = -2e^{-\frac{H}{2}} L_\mu, \]
\[c_2 \partial_\mu (H + \bar{H}) e^{\frac{3}{2} H} = \partial_\mu (H + \bar{H}) f_2 = \frac{1}{2} \partial_\nu (H + \bar{H}) \epsilon \bar{\gamma} \{ \gamma_\mu, \gamma_\nu \} \epsilon = \frac{1}{2} \epsilon \bar{\gamma} [\gamma_\mu, 2 \gamma_2 e^{-\frac{\bar{H}}{2}} - 2i \gamma_2 e^{-\frac{H}{2}}] \epsilon = -2me^{-\frac{H}{2}} L_\mu. \] (B.50)

It follows that
\[e^{\frac{1}{2} (H + \bar{H})} = y + c_3, \quad c_2 = -m. \] (B.51)

By an appropriate shift of $y$, we can set $c_3 = 0$. Now we find that
\[h^{-2} = f_1^2 + f_2^2 = e^H + e^{\bar{H}}. \] (B.52)

The equations $K^t = 1$ and $L_y = 1$ imply that $e^t \epsilon = h^{-1}$ and $e^t \gamma^0 \gamma^3 \epsilon = h^{-1}$ respectively. Thus we find
\[0 = e^t \left[ 1 - \gamma^0 \gamma^3 \right] \epsilon = \frac{1}{2} e^t \left[ 1 - \gamma^0 \gamma^3 \right] \epsilon \] (B.53)

It follows
\[\left[ 1 - \gamma^0 \gamma^3 \right] \epsilon = 0, \quad \text{or} \quad \left[ 1 - i \gamma^1 \gamma^2 \right] \epsilon = 0 \] (B.54)

Substituting (B.51) back to (B.49), we find
\[\left( \sqrt{1 + e^{H - \bar{H}} \gamma^3 + i \gamma - m e^{\frac{H - \bar{H}}{2}}} \right) \epsilon = 0 \]
\[\Rightarrow \left( e^{-2m \gamma^3 \epsilon + i \gamma^\intercal \gamma} \right) \epsilon = 0. \] (B.55)

where $\sinh 2\xi = e^{\frac{1}{2} (H + \bar{H})}$. It implies that the Killing spinor has the form
\[\epsilon = e^{m \gamma^3 \epsilon} \epsilon_1, \quad (1 + i \gamma^3 \bar{\gamma}) \epsilon_1 = 0, \] (B.56)

Inserting (B.56) into the expression (B.47) for $f_1$ gives
\[e^{\frac{1}{2} H} = i \epsilon_1 e^{-m \gamma^3} e^{m \gamma^3} \epsilon_1 = i \epsilon_1^\circ \gamma^0 \epsilon_1. \] (B.57)

Thus
\[\epsilon_1 = e^{\frac{1}{2} \bar{H}} \epsilon_0, \quad \epsilon_0^\circ \gamma^0 \epsilon_0 = -i, \]
\[e^t \epsilon = e^{\frac{1}{2} H} e^t (\cosh 2 \xi + \sinh 2 \xi \gamma^3) \epsilon_0 \]
\[= e^{\frac{1}{2} \bar{H}} (\cosh 2 \xi \epsilon_0 \epsilon_0 - i \sinh 2 \xi \epsilon_0^\circ \gamma^2 \epsilon_0) = h^{-1} \epsilon_0 \epsilon_0. \] (B.58)

From the above expressions for Killing spinor we find
\[L_0^c = e^t \gamma^2 \gamma^0 \epsilon = e^{c_0} \gamma^3 \epsilon = 0. \]
\[ L^c_1 = e^t \gamma^c_1 \epsilon = i e^t = i \hbar^{-1} e_0 \epsilon_0 \]
\[ L^c_2 = e^t \gamma^c_2 \epsilon = e^t = h^{-1} e_0 \epsilon_0 \]
\[ L^c_3 = e^t \gamma^c_3 \epsilon = \bar{e}^c_\gamma \gamma^c_0 \epsilon = 0 \]
\[ L^c = L^c_0 \epsilon^\rho_\mu dx^\mu = (i\tilde{e}^1_c \epsilon + \bar{e}^2_c \epsilon) dx \]

(B.59)

Where \( \tilde{e}^c_i \) is the vielbein of the metric \( \tilde{g}_{ij} = \tilde{e}^c_i \tilde{e}^c_j \) and \( e^i = \hbar \tilde{e}^i \) is the full vielbein for the four dimensional metric in the directions 1, 2. The equation \( dL^c = 0 \) implies that \( \epsilon_0 \) is independent of the time \( t \). Then we can make it to be a constant spinor by setting the phase of \( \epsilon_0 \) to zero via a local Lorentz rotation in the \((x_1, x_2)\)-plane. Under this gauge choice, the equation \( dL = 0 \) further implies that the vielbeins \( \tilde{e}^c_i \) are independent of \( y \) and that the two dimensional metric is flat. So we choose coordinates such that \( \tilde{g}_{ij} = \delta_{ij} \).

From (B.27) we find that

\[ d[h^{-2} (dt + V)] = -dK = \frac{1}{\sqrt{3}} e^{-H} (f_1 *_4 F - f_2 F) \]
\[ = \frac{1}{\sqrt{3}} e^{-H} \left[ h^2 *_3 dB_t + e^{H-H} (dt + V) \wedge d\Phi \right] 
\[ = \frac{-m}{\sqrt{3}} e^{-H} \left[ dB_t \wedge (dt + V) + h^2 e^{H-H} *_3 d\Phi \right]. \]

(B.60)

It follows that

\[ h^{-2} dV = \frac{1}{\sqrt{3}} h^2 \left[ e^{H-H} *_3 dB_t - m e^{H-H} *_3 d\Phi \right] \]
\[ = -mh^2 y *_3 d(H - \tilde{H}) \]

(B.61)

Let

\[ H_\pm = \frac{1}{2} (H \pm \tilde{H}), \quad D = -\frac{1}{2} m \tanh H_- \]

(B.62)

we find

\[ dV = -2m \left( e^H + e^{\tilde{H}} \right)^{-2} y *_3 dH_- = y^{-1} *_3 dD. \]

(B.63)

Finally, the consistency condition \( d^2 V = 0 \) implies

\[ \partial^2_i D + y \partial_y \left( \frac{1}{y} \partial_y D \right) = 0. \]

(B.64)

The additional condition is

\[ \partial_\mu B_t \partial^\mu \Phi = 0 \Rightarrow \partial_\mu H \partial^\mu \tilde{H} = 0 \Rightarrow (\partial_t D)^2 + (\partial_y D)^2 = y^{-2} (1 - D^2)^2 \]

(B.65)

We can now assemble the final solution, and we present the result in section 8. Note that we remove the subscript in the notation \( H_- \) in the presentation of the solution. The discussion of the properties of the solution can be found there.
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