BSDEs with random default time and their applications to default risk

Shige Peng, Xiaoming Xu†
School of Mathematics, Shandong University, Jinan, 250100, China

Abstract

In this paper we are concerned with backward stochastic differential equations with random default time and their applications to default risk. The equations are driven by Brownian motion as well as a mutually independent martingale appearing in a defaultable setting. We show that these equations have unique solutions and a comparison theorem for their solutions. As an application, we get a saddle-point strategy for the related zero-sum stochastic differential game problem.

Keywords: Backward stochastic differential equation, Random default time, Comparison theorem, Zero-sum stochastic differential game

1 Introduction

Credit risk is a kind of the most fundamental, most ancient and most dangerous financial risk. Particularly in recent years it has been greatly concerned once more. The most extensively studied form of credit risk is the default risk, that is, the risk that a counterparty in a financial contract will not fulfill a contractual commitment to meet her/his obligations stated in the contract. Many people, such as Bielecki, Jarrow, Jeanblanc, Kusuoka and so on, have worked on this subject (see e.g. [2-4, 10, 15-17]).

In a defaultable market, the noise is created by the Brownian motion $B$ as well as a random time $\tau$ which is referred to as a default time. Then the information at time $t$ we can get is of two kinds: one from the assets prices, generated by

---

*This work is partially supported by the National Basic Research Program of China (973 Program) (Grant No. 2007CB814900) (Financial Risk).
†Corresponding author, E-mail: xmxu@mail.sdu.edu.cn
B_t and denoted by \( F_t \), the other from the default time, generated by the default process \( H_t := 1_{\{\tau \leq t\}} \) and denoted by \( H_t \). It should be noted here that the default time \( \tau \) is not an \( F \)-stopping time in general. The filtration we consider is the so-called enlarged filtration \( G := F \vee H \). Then how do we deal with this case? Roughly speaking, we construct a process \( \Gamma \), called the \( F \)-hazard process of \( \tau \), by setting

\[
\Gamma_t = -\ln[1 - P(\tau \leq t)]
\]

where \( P \) is the historical probability measure. Then the process \( M \), defined by

\[
M_t = H_t - \Gamma_t \wedge \tau,
\]

is a \( G \)-martingale independent of \( B \). Assume that \( \Gamma \) is absolutely continuous, then there exists a \( F \)-adapted process \( \gamma \), called the intensity process, such that \( \Gamma_t = \int_0^t \gamma_s ds \). By the well-known Kusuoka’s martingale representation theorem, which states that any \( G \)-square integrable martingale can be represented as the sum of integrals w.r.t \( B \) and \( M \), we know that in a defaultable setting, \( B \) and \( M \) are of great importance.

When studying the utility maximization problem in a defaultable setting, Bielecki et al. [2] and Lim-Quenez [17] conclude that the value function is a solution of a BSDE with a quadratic driver, which we call BSDE with random default time in this paper. Actually this type of BSDEs appears very naturally. For the evaluation/hedging problem, Bielecki et al. studied the PDE approach in [3] (see also [4]), where it is assumed that the defaultable market is complete and the dynamics of the primary assets are linear SDEs driven by both \( B \) and \( M \). Their goal is to replicate a contingent claim \( \xi \) which depends on whether the default event occurs or not. In fact, we know already that the theory of contingent claim valuation in a complete default-free market (see e.g. Black-Scholes [5], Merton [19] and so on) can be expressed in terms of classical BSDEs. Here we will check detailedly in the text that the evaluation/hedging problem of \( \xi \) can be represented as a linear BSDE with random default time \( \tau \) of the following form:

\[
Y_t = \xi + \int_t^T (u_s Y_s + v_s Z_s + w_s 1_{\{\tau > s\}} \gamma_s \zeta_s) ds - \int_t^T Z_s dB_s - \int_t^T \zeta_s dM_s,
\]

which can be solved thanks to the existence of the risk-neutral measure \( Q \) equivalent to the historical probability \( P \). In fact \( Q \) is of the following form:

\[
Q = \exp[\int_0^T \ln(1+w_s) dH_s - \int_0^T w_s 1_{\{\tau > s\}} \gamma_s ds + \int_0^T v_s dB_s - \frac{1}{2} \int_0^T v_s^2 ds + \int_0^T u_s ds].
\]

Then we have \( Y_0 = E_Q[\xi] \) which is called the fair price of \( \xi \). While in general, we do not know the exact values of \( (u, v, w) \) but a set \( \Theta \) they belong to, which will lead to model uncertainty or ambiguity (see e.g. [6, 9] for details). Then in this case, instead of having only one risk-neutral probability measure \( Q \) fixed, we will face an uncertain subset of probability measures \( \{P_\theta : \theta = (u, v, w) \in \Theta\} \). For this situation a robust way to evaluate \( \xi \) is its upper price \( \hat{Y}_0 \) achieved by a
superhedging strategy and \( \hat{Y}_0 \) can be calculated by
\[
\hat{Y}_0 = \sup_{\theta \in \Theta} Y_0^\theta = \sup_{\theta \in \Theta} E_{F_\theta}[\xi],
\]
where \( Y_0^\theta \) is in fact the fair price for \( \xi \) in a fictitious market. In evaluation/hedging problem with this imprecise knowledge of the risk-neutral measure, we will face a nonlinear BSDE with random default time (the general form):
\[
Y_t = \xi + \int_t^T g(s, Y_s, Z_s, \zeta_s) ds - \int_t^T Z_s dB_s - \int_t^T \zeta_s dM_s.
\]
(1)

It is worth noting that for the calculation of the upper price \( \hat{Y}_0 \), the generator \( g \) is given by
\[
g(s, y, z, \zeta) = \sup_{(u_s, v_s, w_s) \in \Theta} (u_s y + v_s z + w_s \zeta),
\]
which can be easily seen from Section 4.

We are interested in the problem of the existence and uniqueness of a solution for (1), that is, whether there exists a unique triple of \( \mathcal{G} \)-adapted processes \((Y, Z, \zeta)\) satisfying (1).

It is well-known that, in the framework of Brownian filtration, the general form of BSDE was firstly studied by Pardoux-Peng \[20\]. Since then, the theory of BSDEs has been studied with great interest. One of the achievements of this theory is the comparison theorem. It is due to Peng \[22\] and then generalized by Pardoux-Peng \[21\], El Karoui et al. \[9\]. It allows to compare the solutions of two BSDEs whenever we can compare the terminal conditions and the generators. These results are applied widely to default-free markets. For example, BSDE was firstly applied to the problem of zero-sum stochastic differential games by Hamadene-Lepeltier \[11\]. From then on, BSDEs were linked with the game problems closer and closer (see e.g. \[8, 12\]).

In this paper, we show that under proper assumptions, BSDE (1) has a unique solution. Besides we also establish a comparison theorem. It should be noted here that, the comparison theorem needs one more condition for the generator than the existence and unique theorem, which is different from the classical case. As an application, we deal with a zero-sum stochastic differential game problem, which can also be seen as a utility maximization problem under model uncertainty. For the game, we assume that there are two players \( J_1 \) and \( J_2 \) whose advantages are antagonistic. The dynamics of the controlled system is
\[
X_t = x_0 + \int_0^t (b(s, X_{s-}, u_s, v_s) + c(s, X_{s-}, u_s, v_s)1_{\{\tau > s\}} \gamma_s) ds + \int_0^t \sigma(s, X_{s-}) dB_s + \int_0^t \kappa(s, X_{s-}) dM_s.
\]
The player $J_1$ (resp. $J_2$) chooses a control $u$ (resp. $v$). The object of $J_1$ (resp. $J_2$) is to minimize (resp. maximize) the cost functional $J^{u,v}$. In this paper, we show that there exists a saddle point $(u^*, v^*)$ such that $J(u^*, v^*) \leq J(u^*, v^*) \leq J(u, v^*)$ for each $(u, v)$.

The paper is organized as follows: in Section 2, we list some notations and assumptions we will use. In Section 3, we will first start with a simple model following [3], which in fact implies the new idea, that is, BSDE with random default time, for credit risk modeling. Then we prove an existence and uniqueness result for BSDEs with random default time and also establish a comparison theorem. In the last section (i.e., Section 4), we solve a zero-sum stochastic differential game problem in a defaultable setting as an application of the study of the previous section. For reader’s convenience we present some basic results in the Appendix.

### 2 Notations and assumptions

Let $\{B_t; t \geq 0\}$ be a $d$-dimensional standard Brownian motion on a probability space $(\Omega, F, P)$ and $(F_t)_{t \geq 0}$ be its natural filtration. Denote by $| \cdot |$ the norm in $\mathbb{R}^m$.

Let $\{\tau_i; i = 1, 2, \ldots, k\}$ be $k$ nonnegative random variables satisfying

$$P(\tau_i > 0) = 1; \quad P(\tau_i > t) > 0, \forall t \in \mathbb{R}^+; \quad P(\tau_i = \tau_j) = 0 \ (i \neq j).$$

For each $i$, we introduce a right-continuous process $\{H^i_t; t \geq 0\}$ by setting $H^i_t := 1_{\{\tau_i \leq t\}}$ and denote by $\mathbb{H}^i = (\mathcal{H}^i_t)_{t \geq 0}$ the associated filtration $\mathcal{H}^i_t = \sigma(H^i_s: 0 \leq s \leq t)$.

Just as in the general reduced-form approach, for fixed $T > 0$, there are two kinds of information: one from the assets prices, denoted by $\mathbb{F} = (\mathcal{F}_t)_{0 \leq t \leq T}$, and the other from the default times $\{\tau_i; i = 1, 2, \ldots, k\}$, denoted by $\{\mathbb{H}^i; i = 1, 2, \ldots, k\}$ from the above. The enlarged filtration considered is denoted by $\mathbb{G} = (\mathcal{G}_t)_{0 \leq t \leq T}$ where $\mathcal{G}_t = \mathcal{F}_t \vee \mathcal{H}^1_t \vee \mathcal{H}^2_t \vee \ldots \vee \mathcal{H}^k_t$, which indicates that each $\tau_i$ is a $\mathbb{G}$-stopping time but not necessarily an $\mathbb{F}$-stopping time in the general case.

Now we assume the following (see [16]):

(A) There exist $\mathbb{F}$-adapted processes $\gamma^i \geq 0 \ (i = 1, 2, \ldots, k)$ such that

$$M^i_t := H^i_t - \int_0^t 1_{\{\tau_i > s\}} \gamma^i_s ds \ (i = 1, 2, \ldots, k)$$

are $\mathbb{G}$-martingales under $P$.

(H) Every $\mathbb{F}$-local martingale is a $\mathbb{G}$-local martingale.

It should be mentioned that (H) is a very general and essential hypothesis in the area of enlarged filtration (see [28]).
The following are just for the sake of simplicity:

(i) notations of vectors:

\[ H_t := (H^1_t, H^2_t, \ldots, H^k_t)', \quad M_t := (M^1_t, M^2_t, \ldots, M^k_t)', \]
\[ 1_{\{\tau > t\}} \gamma_t := (1_{\{\tau_1 > t\}} \gamma^1_t, 1_{\{\tau_2 > t\}} \gamma^2_t, \ldots, 1_{\{\tau_k > t\}} \gamma^k_t)', \]

where \((\cdot)'\) is the transpose;

(ii) notations of sets:

- \(L^2(G_T; \mathbb{R}^m) := \{\xi \in \mathbb{R}^m \mid \xi \text{ is a } G_T\text{-measurable random variable such that} \ E|\xi|^2 < +\infty\};\)
- \(L^2_G(0, T; \mathbb{R}^m) := \{\varphi : \Omega \times [0, T] \rightarrow \mathbb{R}^m \mid \varphi \text{ is progressively measurable and} \ E\int_0^T |\varphi_t|^2 dt < +\infty\};\)
- \(S^2_G(0, T; \mathbb{R}^m) := \{\varphi : \Omega \times [0, T] \rightarrow \mathbb{R}^m \mid \varphi \text{ is progressively measurable and} \ E[\sup_{0 \leq t \leq T} |\varphi_t|^2] < +\infty\};\)
- \(L^2_{G\tau}(0, T; \mathbb{R}^{m \times k}) := \{\varphi : \Omega \times [0, T] \rightarrow \mathbb{R}^{m \times k} \mid \varphi \text{ is progressively measurable and} \ E\int_0^T |\varphi_t|^2 1_{\{\tau > t\}} \gamma_t dt := E\int_0^T \sum_{j=1}^m \sum_{i=1}^k |\varphi_{ji,t}|^2 1_{\{\tau_i > t\}} \gamma^i_t dt < +\infty\}.\)

3 **BSDE with random default time**

This section discusses BSDEs with random default time of the general form. We start by analyzing the following example in a defaultable financial market.

### 3.1 An example

At the beginning, we assume that the defaultable market is complete and arbitrage free, that is to say, any \(G_T\)-measurable random variable is a tradable contingent claim.

In the remaining part of this subsection, following Bielecki et al. [3], we will with a markovian set-up. For simplicity, we assume that here \(k = 1\), the density \(\gamma\) is a constant, the trading occurs on the interval \([0, T]\), and the dynamics of primary assets are

\[ dY^i_t = Y^i_t (\mu_i dt + \nu_i dB_t + \kappa_i dM_t), \quad i = 1, 2, 3, \]

where \(\mu_i, \nu_i, \kappa_i \geq -1\) are constants and the primary assets may be default-free \((\kappa_i = 0)\) or defaultable \((\kappa_i \neq 0)\). Our goal is to replicate a contingent claim of the form

\[ Y_T = 1_{\{\tau \leq T\}} g_1(Y^1_T, Y^2_T, Y^3_T) + 1_{\{\tau > T\}} g_0(Y^1_T, Y^2_T, Y^3_T) = G(H_T, Y^1_T, Y^2_T, Y^3_T), \]
which settles at time $T$. From the completeness of the market, we know that $Y_T$ is replicatable.

Let us now consider a small investor whose actions cannot affect market prices and who can decide at time $t \in [0, T]$ what amount $\theta_i^t$ of the wealth $Y_t$ to invest in the $i$th asset, $i = 1, 2, 3$. Of course, his decisions can only be based on the current information $\mathcal{G}_t$, i.e., the processes $\theta = (\theta^1, \theta^2, \theta^3)^t$ and $\theta^1 = Y - \theta^2 - \theta^3$ are predictable. Following Harrison-Pliska [13], we say a strategy is self-financing if the wealth process satisfies the equality

$$Y_t = Y_0 + \int_0^t \theta^1_s dY^1_s + \int_0^t \theta^2_s dY^2_s + \int_0^t \theta^3_s dY^3_s,$$

or, equivalently, if the wealth process satisfies the linear stochastic differential equation

$$dY_t = \sum_{i=1}^3 \theta^i_t \mu_i dt + \sum_{i=1}^3 \theta^i_t \nu_i dB_t + \sum_{i=1}^3 \theta^i_t \kappa_i dM_t.$$ 

Noting $\theta^1 = Y - \theta^2 - \theta^3$, we have to find a strategy $\theta$ satisfying

\[
\begin{align*}
    dY_t &= [\mu_1 Y_t + \theta^2_t (\mu_2 - \mu_1) + \theta^3_t (\mu_3 - \mu_1)]dt \\
    &\quad + [\nu_1 Y_t + \theta^2_t (\nu_2 - \nu_1) + \theta^3_t (\nu_3 - \nu_1)]dB_t \\
    &\quad + [\kappa_1 Y_t + \theta^2_t (\kappa_2 - \kappa_1) + \theta^3_t (\kappa_3 - \kappa_1)]dM_t, \\
Y_T &= G(H_T, Y^1_T, Y^2_T, Y^3_T).
\end{align*}
\] 

(2)

Let

$$Z_t = \nu_1 Y_t + \theta^2_t (\nu_2 - \nu_1) + \theta^3_t (\nu_3 - \nu_1), \quad \zeta_{1\{\tau > t\}} = \kappa_1 Y_t + \theta^2_t (\kappa_2 - \kappa_1) + \theta^3_t (\kappa_3 - \kappa_1).$$

That is, $\zeta$ is well defined only on $[0, \tau \wedge T]$, in fact, we have $\zeta_{1\{\tau > t\}} dM_t = \zeta_t dM_t$ since $dM_t = 0$ on $[\tau \wedge T, T]$. Then by simple computation we can get $\theta^i = \theta^i(Y, Z, \zeta)$ ($i = 2, 3$) where $\theta^i(\cdot, \cdot, \cdot)$ ($i = 2, 3$) are linear functions of the following form:

\[
\begin{align*}
    \theta^2 := \theta^2(Y, Z, \zeta) &= a_2 Y + b_2 Z + c_2 1_{\{\tau > t\}} \zeta \\
    &\quad - \frac{\kappa_1 (\nu_3 - \nu_1) - \nu_1 (\kappa_3 - \kappa_1)}{(\nu_2 - \nu_1)(\kappa_3 - \kappa_1) - (\kappa_2 - \kappa_1)(\nu_3 - \nu_1)} Y \\
    &\quad + \frac{\kappa_3 - \kappa_1}{(\nu_2 - \nu_1)(\kappa_3 - \kappa_1) - (\kappa_2 - \kappa_1)(\nu_3 - \nu_1)} Z \\
    &\quad - \frac{\nu_3 - \nu_1}{(\nu_2 - \nu_1)(\kappa_3 - \kappa_1) - (\kappa_2 - \kappa_1)(\nu_3 - \nu_1)} 1_{\{\tau > t\}} \zeta,
\end{align*}
\]
\[ \theta^3 := \theta^3(Y, Z, \zeta) = a_3 Y + b_3 Z + c_3 1_{\{\tau > t\}} \zeta \]

\[ := \frac{\kappa_1 (\nu_2 - \nu_1) - \nu_1 (\kappa_2 - \kappa_1)}{(\nu_3 - \nu_1)(\kappa_2 - \kappa_1) - (\kappa_3 - \kappa_1)(\nu_2 - \nu_1)} Y \]

\[ + \frac{\kappa_2 - \kappa_1}{(\nu_3 - \nu_1)(\kappa_2 - \kappa_1) - (\kappa_3 - \kappa_1)(\nu_2 - \nu_1)} Z \]

\[ - \frac{\nu_2 - \nu_1}{(\nu_3 - \nu_1)(\kappa_2 - \kappa_1) - (\kappa_3 - \kappa_1)(\nu_2 - \nu_1)} 1_{\{\tau > t\}} \zeta. \]

Write \( a = \mu_1 + (\mu_2 - \mu_1) a_2 + (\mu_3 - \mu_1) a_3, \ b = (\mu_2 - \mu_1) b_2 + (\mu_3 - \mu_1) b_3, \ c = \frac{1}{\gamma} [(\mu_2 - \mu_1) c_2 + (\mu_3 - \mu_1) c_3], \) then (2) becomes

\[
\begin{aligned}
  dY_t &= (aY_t + bZ_t + c1_{\{\tau > t\}} \gamma \zeta_t)dt + Z_t dB_t + \zeta_t dM_t, \\
  Y_T &= G(H_T, Y_T^1, Y_T^2, Y_T^3),
\end{aligned}
\]

which is just a linear backward stochastic differential equation with random default time \( \tau. \) Suppose that \( c < 1, \) which is in fact a more general condition than that in [3]. Set

\[ Q_t := \exp[\ln(1 - c)H_t + c \int_0^t 1_{\{\tau > s\}} \gamma ds - bB_t - \frac{1}{2} b^2 t - at]. \]

i.e.,

\[
\begin{aligned}
  dQ_t &= -Q_t - (adt + bdB_t + cdM_t), \\
  Q_0 &= 1.
\end{aligned}
\]

Applying Itô’s formula (see Appendix) on \( Q_t Y_t, \) we have

\[
\begin{aligned}
dQ_t Y_t &= cQ_t 1_{\{\tau > t\}} \gamma \zeta_t dt + Q_t (Z_t - bY_t) dB_t + Q_t - (\zeta_t - cY_t) dM_t - cQ_t - \zeta_t dH_t \\
&= Q_t (Z_t - bY_t) dB_t + Q_t - (\zeta_t - cY_t) dM_t,
\end{aligned}
\]

which implies \( Q_t Y_t = E^{\tilde{Q}}[Q_T G(H_T, Y_T^1, Y_T^2, Y_T^3)]. \) In the financial market, \( Y \) will be called the fair price of the contingent claim \( G(H_T, Y_T^1, Y_T^2, Y_T^3). \)

### 3.2 BSDE with random default time

The model mainly discussed in this part is:

\[ Y_t = \xi(H_T) + \int_t^T g(s, Y_s, Z_s, \zeta_s) ds - \int_t^T Z_s dB_s - \int_t^T \zeta_s dM_s. \quad (3) \]

In the defaultable financial market, \( \xi(H_T) \) represents a contingent claim needed to be replicated, settled at time \( T, \) depending on the event whether the default
occurs at time $T$. Trading occurs on the interval $[0, T]$. $Z$ and $\zeta$ represent the information of the hedging strategies, for example, in the linear case (see Subsection 3.1), we can compute the hedging strategies by $Z$ and $\zeta$.

The function $g$ is called the generator of (3). Our object is to find a triple $(Y_t, Z_t, \zeta_t) \in S^2_0(0, T; \mathbb{R}^d) \times L^2_G(0, T; \mathbb{R}^{m \times d}) \times L^2_{G^T}(0, T; \mathbb{R}^{m \times k})$ satisfying (3). For this purpose, we first consider a very simple case: $g(y, z, \zeta)$ is a real valued process that is independent of the variable $(y, z, \zeta)$.

**Lemma 3.1** For a fixed $\xi(H_T) \in L^2(\mathcal{G}_T; \mathbb{R})$ and $g_0(\cdot)$ satisfying

$$E[(\int_0^T g_0(s) ds)^2] < +\infty,$$

there exists a unique triple of processes $(y_t, z_t, \zeta_t) \in L^2_G(0, T; \mathbb{R}) \times L^2_G(0, T; \mathbb{R}^d) \times L^2_{G^T}(0, T; \mathbb{R}^{k})$ satisfying

$$y_t = \xi(H_T) + \int_t^T g_0(s) ds - \int_t^T z_s dB_s - \int_t^T \zeta_s dM_s. \quad (4)$$

If $g_0(\cdot) \in L^2_G(0, T; \mathbb{R})$, then $(y_t, z_t, \zeta_t) \in S^2_0(0, T; \mathbb{R}) \times L^2_G(0, T; \mathbb{R}^d) \times L^2_{G^T}(0, T; \mathbb{R}^{k})$.

We have the following basic estimate:

$$|y_t|^2 + E^{G_t} \int_t^T \frac{\beta}{2} |y_s|^2 + |z_s|^2 + \|\zeta_s\|_\tau^2 e^{\beta(s-t)} ds \leq E^{G_t} |\xi(H_T)|^2 e^{\beta(T-t)} + \frac{2}{\beta} E^{G_t} \int_t^T |g_0(s)|^2 e^{\beta(s-t)} ds, \quad (5)$$

in particular,

$$|y_0|^2 + E \int_0^T \frac{\beta}{2} |y_s|^2 + |z_s|^2 + \|\zeta_s\|_\tau^2 e^{\beta s} ds \leq E |\xi(H_T)|^2 e^{\beta T} + \frac{2}{\beta} E \int_0^T |g_0(s)|^2 e^{\beta s} ds, \quad (6)$$

where $\|\zeta_s\|_\tau := \|\zeta_s 1_{\{\tau > s\}}\sqrt{\tau_s} = (\sum_{i=1}^k |\zeta|^{2} 1_{\{\tau > s\}} \gamma_i^s)^{\frac{1}{2}}$ and $\beta > 0$ is an arbitrary constant.

We also have

$$E[\sup_{0 \leq t \leq T} |y_t|^2] \leq C_T E[|\xi(H_T)|^2] + \int_0^T |g_0(s)|^2 ds, \quad (7)$$

where the constant $C_T$ depends only on $T$.

**Proof.** Define

$$N_t = E^{G_t} [\xi + \int_0^T g_0(s) ds].$$

8
Obviously $N_t$ is a square integrable $\mathcal{G}$-martingale. Thanks to Kusuoka’s martingale representation theorem (see Appendix), there exists a unique pair of adapted process $(z_t, \varsigma_t) \in L^2_\mathcal{G}(0, T; \mathbb{R}^d) \times L^2_{\mathcal{G}}(0, T; \mathbb{R}^k)$ such that

$$N_t = N_0 + \int_0^t z_s dB_s + \int_0^t \varsigma_s dM_s.$$ 

Thus

$$N_t = N_T - \int_t^T z_s dB_s - \int_t^T \varsigma_s dM_s.$$ 

Denote

$$y_t = N_t - \int_0^t g_0(s) ds = N_T - \int_0^t g_0(s) ds - \int_t^T z_s dB_s - \int_t^T \varsigma_s dM_s.$$ 

Since $N_T = \xi + \int_0^T g_0(s) ds$, immediately we get (4).

The uniqueness is a simple consequence of the estimate (6). We only need to prove the priori estimates. To prove (5), we first consider the case where $\xi$ and $g_0(\cdot)$ are both bounded. Since $y_t = E^\mathcal{G}_t[\xi(H_T) + \int_t^T g_0(s) ds]$, thus the process $y$ is also bounded.

From the equation (4), we have

$$dy_s = -g_0(s) ds + z_s dB_s + \varsigma_s dM_s.$$ 

We then apply Itô’s formula to $y_s^2 e^{\beta s}$ (see Example A.1) for $s \in [t, T]$:

$$dy_s^2 e^{\beta s} = e^{\beta s} (\beta y_s^2 - 2y_s g_0(s) + |z_s|^2 + |\varsigma_s|^2 1_{\{\tau > s\}} \gamma_s) ds + 2e^{\beta s} y_s z_s dB_s + e^{\beta s} (2y_s \varsigma_s + |\varsigma_s|^2) dM_s.$$ 

Integrating $s$ from $t$ to $T$ and take conditional expectation with regard to $\mathcal{G}_t$ on both sides, we obtain

$$|y_t|^2 + E^{\mathcal{G}_t} \int_t^T [\beta |y_s|^2 + |z_s|^2 + |\varsigma_s|^2 1_{\{\tau > s\}} \gamma_s] e^{\beta(s-t)} ds = E^{\mathcal{G}_t} |\xi|^2 e^{\beta(T-t)} + E^{\mathcal{G}_t} \int_t^T 2y_s g_0(s) e^{\beta(s-t)} ds \leq E^{\mathcal{G}_t} |\xi|^2 e^{\beta(T-t)} + E^{\mathcal{G}_t} \int_t^T \left[ \frac{\beta}{2} |y_s|^2 + \frac{2}{\beta} |g_0(s)|^2 \right] e^{\beta(s-t)} ds.$$ 

From this it follows (5) and (6).

We now consider the case where $\xi$ and $g_0(\cdot)$ are possibly unbounded. We set

$$\xi^n := (\xi \wedge n) \lor (-n), \quad g_0(s) := (g_0(s) \wedge n) \lor (-n),$$
and
\begin{align*}
y^n_t := \xi^n + \int_t^T g^n_0(s) ds - \int_t^T z^n_s dB_s - \int_t^T \zeta^n_s dM_s.
\end{align*}

Thanks to the boundedness of \( \xi^n, \xi^k, g^n_0 \) and \( g^k_0 \) for each positive integer \( n \) and \( k \), we have
\begin{align*}
|y^n_t|^2 + E^G_t &\int_t^T [\frac{2}{T}|y^n_s|^2 + |z^n_s|^2 + |\zeta^n_s|^2 1_{\{\tau > s\}} \gamma_s] e^{\beta(s-t)} ds \\
&\leq E^G_t |\xi^n|^2 e^{\beta(T-t)} + \frac{2}{\beta} E^G_t \int_t^T |g^n_0(s)|^2 e^{\beta(s-t)} ds
\end{align*}

and
\begin{align*}
E \int_0^T [\frac{2}{T} |y^n_s - y^k_s|^2 + |z^n_s - z^k_s|^2 + |\zeta^n_s - \zeta^k_s|^2 1_{\{\tau > s\}} \gamma_s] e^{\beta s} ds \\
&\leq E |\xi^n - \xi^k|^2 e^{\beta T} + \frac{2}{\beta} E \int_0^T |g^n_0(s) - g^k_0(s)|^2 e^{\beta s} ds.
\end{align*}

The second inequality implies that the processes \( y^n, z^n \) and \( \zeta^n \) are Cauchy sequences in their corresponding spaces. Thus (3) is proved by letting \( n \) tend to \(+\infty\) in (8).

Easily we can get \( g_t \in S^2_G(0, T; \mathbb{R}) \) as (7) is a simple consequence of (6) together with B-D-G inequality applied to (4). \( \square \)

With the above basic estimates, we can now consider the general case of (3).

We assume that \( g(\omega, t, y, z, \varsigma) : \Omega \times [0, T] \times \mathbb{R}^m \times \mathbb{R}^{m \times d} \times \mathbb{R}^{m \times k} \rightarrow \mathbb{R}^m \) satisfies the following conditions:

(a) \( g(\cdot, 0, 0, 0) \in L^2(\mathcal{G}) \); 

(b) the Lipschitz condition: for each \((t, y, z, \varsigma), (\bar{t}, \bar{y}, \bar{z}, \bar{\varsigma}) \in [0, T] \times \mathbb{R}^m \times \mathbb{R}^{m \times d} \times \mathbb{R}^{m \times k}\), there exists a constant \( C \geq 0 \) such that

\[ |g(t, y, z, \varsigma) - g(\bar{t}, \bar{y}, \bar{z}, \bar{\varsigma})| \leq C (|y - \bar{y}| + |z - \bar{z}| + |\varsigma - \bar{\varsigma}| 1_{\{\tau > t\}} \sqrt{\gamma_t}). \]

**Theorem 3.1** Assume that \( g \) satisfies (a) and (b), then for any fixed terminal condition \( \xi(\mathcal{H}_T) \in L^2(\mathcal{G}_T; \mathbb{R}^m) \), BSDE (3) has a unique solution, i.e., there exists a unique triple of \( \mathcal{G}_t \)-adapted processes

\[ (Y_t, Z_t, \zeta_t) \in S^2_G(0, T; \mathbb{R}^m) \times L^2_G(0, T; \mathbb{R}^{m \times d}) \times L^2_G(0, T; \mathbb{R}^{m \times k}) \]

satisfying (3).

**Proof.** First we introduce a norm in \( L^2_G(0, T; \mathbb{R}^m) \times L^2_G(0, T; \mathbb{R}^{m \times d}) \times L^2_G(0, T; \mathbb{R}^{m \times k}) \):
\[ \| (u, v, w) \|_\beta := \{ E \int_0^T (|u_s|^2 + |v_s|^2 + |w_s|^2) e^{\beta s} ds \}^{\frac{1}{2}}. \]
Remark 3.1 In the above theorem, from the conditions that the generator \( g \) satisfies, we know that here \( g \) is independent of the last element \( \varsigma \) after the default occurs, i.e., \( g(t, y, z, \varsigma) \equiv g(t, y, z) \) on \( t \in [\tau \wedge T, T] \). Its financial explanation is that after the default occurs the influence factor on the contingent claim is apart from the defaultable risky part absolutely.
Remark 3.2 The solution of (3) is unique, that is to say, if both \((Y, Z, \zeta)\) and \((\tilde{Y}, \tilde{Z}, \tilde{\zeta}) \in S^2_{G}(0, T; \mathbb{R}^m) \times L^2_{G}(0, T; \mathbb{R}^{m \times d}) \times L^2_{G}(0, T; \mathbb{R}^{m \times k})\) satisfy (3), then

\[
E \int_0^T |Y_t - \tilde{Y}_t|^2 dt = 0, E \int_0^T |Z_t - \tilde{Z}_t|^2 dt = 0, E \int_0^T |\zeta_t - \tilde{\zeta}_t|^2 1_{\tau > t} \gamma_t dt = 0.
\]

Remark 3.3 The uniqueness of \(\{\zeta_t; t \in [0, T]\}\) can be explained in this way: \(\{\zeta_t; t \in [0, T]\}\) is unique, that is to say, \(\{\zeta_t; t \in [0, T]\}\) can only be uniquely determined on the random interval \([0, \tau \wedge T] \cap \{t : \gamma_t \neq 0\}\), i.e., its effective definition domain is just the set \([0, \tau \wedge T] \cap \{t : \gamma_t \neq 0\}\). On the interval \([\tau \wedge T, T]\) \cup \{t : \gamma_t = 0\}\), \(\zeta\) can be arbitrary adapted random process. In fact, this is a direct conclusion of the truth that \(dM_t \equiv 0\) on \([\tau \wedge T, T]\) \cup \{t : \gamma_t = 0\}\), indeed \(M_t \equiv 1 - \int_0^{\tau \wedge T} \gamma_s ds\) on \([\tau \wedge T, T]\).

### 3.3 Comparison theorem for 1-dimensional BSDEs with random default time

Consider the following two 1-dimensional BSDEs with random default time:

\[
Y_t = \xi(H_T) + \int_t^T g(s, Y_s, Z_s, \zeta_s)ds - \int_t^T Z_s dB_s - \int_t^T \zeta_s dM_s, \quad (9)
\]

\[
\bar{Y}_t = \bar{\xi}(H_T) + \int_t^T \bar{g}_s ds - \int_t^T \bar{Z}_s dB_s - \int_t^T \bar{\zeta}_s dM_s. \quad (10)
\]

For the generator function \(g\), we introduce one more assumption:

(c) for each \((t, y, z) \in [0, T] \times \mathbb{R} \times \mathbb{R}^d, (\varsigma, \bar{\varsigma}) \in \mathbb{R}^k \times \mathbb{R}^k, (\varsigma^i - \bar{\varsigma}^i)1_{\{\gamma^i_t > 0\}} \gamma^i_t \neq 0\), the following holds:

\[
g(t, y, z, \varsigma^i) - g(t, y, z, \bar{\varsigma}^i) > 1, \quad \frac{(\varsigma^i - \bar{\varsigma}^i)1_{\{\gamma^i_t > 0\}} \gamma^i_t}{(\varsigma^i - \bar{\varsigma}^i)1_{\{\gamma^i_t > 0\}} \gamma^i_t} > 1,
\]

where \(\varsigma^i = (\varsigma^1, \varsigma^2, \ldots, \varsigma^i, \varsigma^{i+1}, \varsigma^{i+2}, \ldots, \varsigma^k)\) and \(\varsigma^i\) is the \(i\)-th component of \(\varsigma\).

**Theorem 3.2** Suppose \(\xi, \bar{\xi}\) satisfy the same assumptions as in Theorem 3.1. \(g\) satisfies (a)-(c), \(\bar{g}_s \in L^2_G(0, T; \mathbb{R})\). Let \((Y, Z, \zeta), (\bar{Y}, \bar{Z}, \bar{\zeta})\) be the unique solutions of (9), (10) respectively. If

\[
\xi \geq \bar{\xi}, \quad g(t, \bar{Y}_t, \bar{Z}_t, \bar{\zeta}_t) \geq \bar{g}_t, \quad a.e., a.s.,
\]

then

\[
Y_t \geq \bar{Y}_t, \quad a.e., a.s.
\]

Besides, the following holds true (the strict comparison theorem):

\[
Y_0 = \bar{Y}_0 \Leftrightarrow \xi = \bar{\xi}, \quad g(t, \bar{Y}_t, \bar{Z}_t, \bar{\zeta}_t) \equiv \bar{g}_t.
\]
Proof. Let \( \hat{\xi} = \xi - \xi, \hat{Y}_s = Y_s - Y_s, \hat{Z}_s = Z_s - Z_s, \hat{\zeta}_s = \zeta_s - \zeta_s, \hat{g}_s = g(s, \bar{Y}_s, \bar{Z}_s, \bar{\zeta}_s) - \gamma_s \), then we have

\[
\begin{align*}
-d\hat{Y}_s &= (a_s \hat{Y}_s + b_s \hat{\zeta}_s + c_s \hat{g}_s) ds - \hat{Z}_s dB_s - \hat{\zeta}_s dM_s, \\
\hat{Y}_T &= \hat{\xi},
\end{align*}
\]

where

\[
\begin{align*}
a_s &= \begin{cases} 
\frac{g(s, Y_s, Z_s, \zeta_s) - g(s, \bar{Y}_s, \bar{Z}_s, \bar{\zeta}_s)}{Y_s - \bar{Y}_s}, & \text{if } Y_s \neq \bar{Y}_s, \\
0, & \text{if } Y_s = \bar{Y}_s,
\end{cases} \\
b_s &= \begin{cases} 
\frac{g(s, Y_s, Z_s, \zeta_s) - g(s, \bar{Y}_s, \bar{Z}_s, \bar{\zeta}_s)}{Z_s - \bar{Z}_s}, & \text{if } Z_s \neq \bar{Z}_s, \\
0, & \text{if } Z_s = \bar{Z}_s,
\end{cases} \\
c^i_s &= \begin{cases} 
\frac{g(s, \bar{Y}_s, \bar{Z}_s, \bar{\zeta}_s, \gamma_s)}{(\bar{\zeta_s} - \gamma_s) \Gamma(\tau > s) \gamma_s}, & \text{if } (\bar{\zeta_s} - \gamma_s) \Gamma(\tau > s) \gamma_s \neq 0, \\
0, & \text{if } (\bar{\zeta_s} - \gamma_s) \Gamma(\tau > s) \gamma_s = 0.
\end{cases}
\]

Since \( g \) satisfies (b) and (c), thus \( |a_s| \leq C, |b_s| \leq C \) and \( c^i_s > -1 \). Set

\[
Q_s := \exp\left[ \int_0^s (1+c_u) dH_u - \int_0^s c_u \Gamma(\tau > u) \gamma_u du + \int_0^s b_u dB_u - \frac{1}{2} \int_0^s b_u^2 du + \int_0^s a_u du \right],
\]

i.e.,

\[
\begin{align*}
dQ_s &= Q_s(a_s ds + b_s dB_s + c_s dM_s), \\
Q_0 &= 1.
\end{align*}
\]

Applying Itô’s formula on \( Q_s \hat{Y}_s \), we have

\[
dQ_s \hat{Y}_s = -Q_s(c_s \hat{\zeta}_s \Gamma(\tau > s) \gamma_s + \hat{g}_s) ds + Q_s(\hat{Z}_s + b_s \hat{Y}_s) dB_s + Q_s(\hat{\zeta}_s + c_s \hat{Y}_s) dM_s + Q_s(\hat{\zeta}_s + c_s \hat{Y}_s) dH_s
\]

Integrate from \( t \) to \( T \) and take conditional expectation w.r.t \( G_t \) on both sides:

\[
Q_t \hat{Y}_t = E^G_t[Q_T \hat{Y}_T + \int_t^T Q_s \hat{g}_s ds] \geq 0, \text{ a.e., a.s.}
\]

Then \( \hat{Y}_t \geq 0 \) immediately follows. □
Remark 3.4 In the above, the definition of $c_s$ is proper. For simplicity, we only discuss the case $k = 1$. Indeed, for the case when $(\zeta_s - \bar{\zeta}_s)\cdot 1_{\{\tau > s\}}\gamma_s = 0$, we should have the following equality:

$$a_s\hat{Y}_s + b_s\hat{Z}_s + c_s\hat{\zeta}_s \cdot 1_{\{\tau > s\}}\gamma_s + \hat{g}_s$$

$$= g(\tau_s, Y_s, Z_s, \zeta_s) - g(s, Y_s, Z_s, \zeta_s) + g(s, Y_s, Z_s, \bar{\zeta}_s) - \bar{g}_s$$

which will hold if

$$(\zeta_s - \bar{\zeta}_s)\cdot 1_{\{\tau > s\}}\gamma_s = 0 \Rightarrow \zeta_s - \bar{\zeta}_s = 0.$$  

For this we can refer to Remark 3.1 and 3.3 more detailedly, if $1_{\{\tau > t\}} = 0$ then $g(t, Y_t, Z_t, \zeta_t) \equiv g(t, Y_t, Z_t, \bar{\zeta}_t)$, and if $\gamma_s = 0$ then $\zeta$ can be arbitrary and we can choose $\zeta = \bar{\zeta}$.

Remark 3.5 Condition (c) for the generator $g$ is significant for the comparison theorem. In the following we give an example which indicates that the strict comparison theorem will not hold if $g$ does not satisfy (c).

Example 3.1 Suppose that $k=1$, $\xi = H_T$, $\bar{\xi} = 0$, $g(t, y, z, \varsigma) = 1_{\{\tau > t\}}\sqrt{\gamma_t} - 1_{\{\tau > t\}}\sqrt{\gamma_t} + 1, \bar{g}_t = 0$. Clearly $g$ does not satisfy (c). Consider the following two BSDEs:

$$Y_t = H_T + \int_t^T (1_{\{\tau > s\}}\sqrt{\gamma_s} - 1_{\{\tau > s\}}\sqrt{\gamma_s} + 1)\varsigma_s ds - \int_t^T Z_s dB_s - \int_t^T \varsigma_s dM_s, \quad (11)$$

$$\bar{Y}_t = 0 - \int_t^T Z_s dB_s - \int_t^T \varsigma_s dM_s. \quad (12)$$

It is easy to check that $(H_t, 0, 1)$, $(0, 0, 0)$ are the unique solutions of $(11), (12)$, respectively.

Then we have

$$\xi = H_T \geq \bar{\xi} = 0, g(t, Y_t, Z_t, \varsigma_t) = g(t, 0, 0, 0) = 1_{\{\tau > t\}}\sqrt{\gamma_t} \geq 0 = \bar{g}_t,$$

while in the meantime we get

$$Y_0 = \bar{Y}_0 = 0, \text{ but } P(\xi > \bar{\xi}) > 0 \text{ and } (L \times P)(g(t, Y_t, Z_t, \varsigma_t) > \bar{g}_t) > 0,$$

where $L$ denotes Lebesgue measure.

The comparison theorem, which allows us to compare the solutions of two BSDEs with random default time, can ensure the attainability of the upper price of a contingent claim in the evaluation/hedging problem. The main idea can be seen in the next section.
4 Application in zero-sum stochastic differential game problem

We are now going to study the link between the zero-sum stochastic differential games in the defaultable setting and the BSDEs with random default time studied in the previous section. First let us describe the framework of the zero-sum game we consider.

Assume here that $m = d = k$ and $\gamma_t \geq 0$ is bounded. Let $x_0 \in \mathbb{R}^m$ and let $X_t$ be the solution of the following stochastic differential equation:

$$X_t = x_0 + \int_0^t \sigma(s, X_{s-})dB_s + \int_0^t \kappa(s, X_{s-})dM_s,$$

where the mapping $\sigma : [0, T] \times \mathbb{R}^m \rightarrow \mathbb{R}^{m \times m}$ and $\kappa : [0, T] \times \mathbb{R}^m \rightarrow \mathbb{R}^{m \times m}$ satisfy the following assumptions:

(i) for $1 \leq i, j \leq m$, $\sigma_{ij}$ and $\kappa_{ij}$ are progressively measurable;

(ii) for any $(t, x) \in [0, T] \times \mathbb{R}^m$, there exists a constant $C_1 \geq 0$ such that

$$|\sigma(t, x)| + |\kappa(t, x)| \leq C_1(1 + |x|);$$

(iii) for any $(t, x), (t, y) \in [0, T] \times \mathbb{R}^m$, there exists a constant $C_2 \geq 0$ such that

$$|\sigma(t, x) - \sigma(t, y)| + |\kappa(t, x) - \kappa(t, y)| \leq C_2|x - y|;$$

(iv) $\sigma(t, x), \kappa(t, x)$ are invertible and $\sigma^{-1}(t, x), \kappa^{-1}(t, x)$ are bounded.

Then the process $\{X_t; t \in [0, T]\}$ exists and is unique.

Let $U$ (resp. $V$) be a compact metric space and $\mathcal{U}$ (resp. $\mathcal{V}$) be the space of all progressively measurable processes $u = (u_t)_{t \in [0, T]}$ (resp. $v = (v_t)_{t \in [0, T]}$) with values in $U$ (resp. $V$).

Let the drift function $b$ map $[0, T] \times \mathbb{R}^m \times U \times V$ into $\mathbb{R}^m$. Furthermore, $b$ is supposed to satisfy:

(i) $b$ is $\mathcal{B}([0, T] \times \mathbb{R}^m \times U \times V)$-measurable;

(ii) $b(t, x, u, v)$ is bounded for any $(t, x, u, v)$;

(iii) for any $(t, x) \in [0, T] \times \mathbb{R}^m$, $b(t, x, \cdot, \cdot)$ is continuous on $U \times V$.

Now for each $u \in \mathcal{U}$, $v \in \mathcal{V}$, let $L^{u,v}$ be the positive local martingale solution of:

$$
\begin{align*}
\left\{ \begin{array}{l}
    dL_t^{u,v} = L_t^{u,v}(\sigma^{-1}(t, X_{t-})b(t, X_{t-}, u_t, v_t)dB_t + \kappa^{-1}(t, X_{t-})c(t, X_{t-}, u_t, v_t)dM_t), \\
    L_0^{u,v} = 1,
\end{array} \right. 
\end{align*}
$$
where for any \((t, x, u, v), i = 1, 2, \ldots, m\), the \(i\)-th component of \(\kappa^{-1}(t, x)c(t, x, u, v)\) is larger than \(-1\), i.e., \((\kappa^{-1}(t, x)c(t, x, u, v))^i > -1\).

According to the Girsanov Theorem (see Appendix), \(P_{u,v}^a\) defined by \(\frac{dP_{u,v}^a}{dP}\) is a probability measure equivalent to \(P\). Moreover, under \(P_{u,v}^a\), the process \(B_{t}^{u,v} = B_t - \int_{0}^{t} \sigma^{-1}(s, X_{s-})b(s, X_{s-}, u_s, v_s)ds\) is a Brownian motion, the processes \(M_{t}^{u,v} = M_t - \int_{0}^{t} (\kappa^{-1}(s, X_{s-})c(s, X_{s-}, u_s, v_s))\gamma_s^i ds\) \((i = 1, 2, \ldots, m)\) are \(\mathcal{G}\)-martingales orthogonal to each other and orthogonal to \(B_{t}^{u,v}\) and \((X_t)_{0 \leq t \leq T}\) satisfies

\[
\begin{cases}
    dX_t = (b(t, X_{t-}, u_t, v_t) + c(t, X_{t-}, u_t, v_t)1_{\{\gamma > 0\}}\gamma_t)dt \\
    + \sigma(t, X_{t-})dB_{t}^{u,v} + \kappa(t, X_{t-})dM_{t}^{u,v}, \\
    X_0 = x_0.
\end{cases}
\]

It means that \((X_t)_{0 \leq t \leq T}\) is a weak solution for the above stochastic differential equation and it stands for an evolution of a controlled system.

It is well-known that in zero-sum game problems, there are two players \(J_1\) and \(J_2\). We suppose that \(J_1\) (resp. \(J_2\)) chooses a control \(u(t, x) \in U\) (resp. \(v(t, x) \in V\)). Now we introduce two functions \(f : [0, T] \times \mathbb{R}^m \times U \times V \rightarrow \mathbb{R}_+\), satisfying the same assumptions as \(b\), and \(h : \{0, 1\} \times \mathbb{R}^m \rightarrow \mathbb{R}_+\) which is measurable, bounded. Let \(E_{u,v}^a\) denote the expectation w.r.t \(P_{u,v}^a\). Then the cost functional corresponding to \(u \in U\) and \(v \in V\) is given by

\[
J_{u,v} = E_{u,v}[\int_{0}^{T} f(s, X_{s}, u_s, v_s)ds + h(H_T, X_T)],
\]

which is a cost (resp. reward) for \(J_1\) (resp. \(J_2\)).

The object of \(J_1\) (resp. \(J_2\)) is to minimize (resp. maximize) the cost functional. In this zero-sum game problem, we aim at showing the existence of a saddle point, more precisely, a pair \((\tilde{u}^*, \tilde{v}^*)\) such that \(J(\tilde{u}^*, v) \leq J(\tilde{u}^*, \tilde{v}^*) \leq J(u, \tilde{v}^*)\) for each \((u, v) \in U \times V\).

Thus let us define the Hamilton function associated with this game problem as following: \(\forall (t, x, z, \varsigma, u, v) \in [0, T] \times \mathbb{R}^m \times \mathbb{R}^m \times \mathbb{R}^m \times U \times V\),

\[
H(t, x, z, \varsigma, u, v) := z\sigma^{-1}(t, x)b(t, x, u, v) + \varsigma\kappa^{-1}(t, x)c(t, x, u, v)1_{\{\gamma > 0\}}\gamma_t \\
+ f(t, x, u, v).
\]

Here we should pay special attention to the difference between the notations of the Hamilton function \(H(t, \cdot, \cdot, \cdot, \cdot, \cdot)\) and the default process \(H_t\).

Next assume that Isaacs’ condition, which plays an important role in zero-sum stochastic differential game problems, is fulfilled, i.e, for any \((t, x, z, \varsigma) \in \mathbb{R}^m \times \mathbb{R}^m \times \mathbb{R}^m \times U \times V\),
Under the above Isaacs’ condition, through the assumptions above and Benes’s selection theorem (see e.g. [8]), the following holds true (see e.g. [8]).

**Proposition 4.1** There exist two measurable functions $u^*(t, x, z, \varsigma), v^*(t, x, z, \varsigma)$ mapping from $[0, T] \times \mathbb{R}^m \times \mathbb{R}^m \times \mathbb{R}^m$ into $U, V$ respectively such that:

(i) the pair $(u^*, v^*)(t, x, z, \varsigma)$ is a saddle point for the function $H$, i.e,

$$H(t, x, z, \varsigma, u^*(t, x, z, \varsigma), v^*(t, x, z, \varsigma)) \leq H(t, x, z, \varsigma, u, v^*(t, x, z, \varsigma)), \quad \forall u \in U,$$

$$H(t, x, z, \varsigma, u^*(t, x, z, \varsigma), v^*(t, x, z, \varsigma)) \geq H(t, x, z, \varsigma, u^*(t, x, z, \varsigma), v), \quad \forall v \in V;$$

(ii) the function $(z, \varsigma) \rightarrow H(t, x, z, \varsigma, u^*(t, x, z, \varsigma), v^*(t, x, z, \varsigma))$ satisfies (b) and (c), uniformly in $(t, x)$.

Now we introduce two notations just for simplicity:

$$H(t, z, \varsigma) := H(t, X_{t-}, z, \varsigma, u_t, v_t),$$

$$H^*(t, z, \varsigma) := H(t, X_{t-}, z, \varsigma, u^*(t, X_{t-}, z, \varsigma), v^*(t, X_{t-}, z, \varsigma)).$$

Suppose that $J_1$ (resp. $J_2$) has chosen $u \in U$ (resp. $v \in V$). The conditional expected remaining cost from time $t \in [0, T]$ is

$$J^{u,v}_t = E_{u,v}^G \left[ \int_t^T f(s, X_s, u_s, v_s)ds + h(H_T, X_T) \right].$$

It is obvious that $J^{u,v}_0 = J^{u,v}$. The following theorem tells us that the conditional costs can be characterized as solutions of BSDEs with random default time.

**Theorem 4.1** The BSDE with random default time

$$Y_t = h(H_T, X_T) + \int_t^T H(s, Z_s, \zeta_s)ds - \int_t^T Z_sdB_s - \int_t^T \zeta_s dM_s \quad (13)$$

has a unique solution $(Y, Z, \zeta) \in S_0^2(0, T; \mathbb{R}) \times L_0^2(0, T; \mathbb{R}^m) \times L_0^{2,r}(0, T; \mathbb{R}^m)$ which satisfies $Y_t = J^{u,v}_t$. 

17
Proof. Notice that

\[
H(t, z, \varsigma) = H(t, X_{t-}, z, \varsigma, u_t, v_t) = z\sigma^{-1}(t, X_{t-})b(t, X_{t-}, u_t, v_t) + \varsigma\kappa^{-1}(t, X_{t-})c(t, X_{t-}, u_t, v_t)1_{\{\tau > t\}}\gamma_t + f(t, X_{t-}, u_t, v_t).
\]

Then (13) can be transformed to the following:

\[
Y_t = h(H_T, X_T) + \int_t^T f(s, X_s, u_s, v_s)ds - \int_t^T Z_s dB_s - \int_t^T \varsigma_s dB_s - \int_t^T \zeta_s dM_s - \int_t^T \sigma^{-1}(r, X_r) b(r, X_r, u_r, v_r) dr - \int_t^T \kappa^{-1}(r, X_r) c(r, X_r, u_r, v_r) 1_{\{\tau > r\}} \gamma_r dr.
\]

According to the Girsanov Theorem, we can easily obtain

\[
Y_t = E_{u,v}^{G_t}[\int_t^T f(s, X_s, u_s, v_s)ds + h(H_T, X_T)],
\]

i.e., \(Y_t = J_t^{u,v}\). \(\Box\)

Next is the main result of this part.

Theorem 4.2 The BSDE with random default time

\[
Y_t = h(H_T, X_T) + \int_t^T H^*(s, Z_s, \varsigma_s)ds - \int_t^T Z_s dB_s - \int_t^T \zeta_s dM_s.
\] (14)

has a unique solution \((Y, Z, \varsigma) \in S_\beta^2(0, T; \mathbb{R}) \times L^2_\beta(0, T; \mathbb{R}^m) \times L^2_\beta(0, T; \mathbb{R}^m)\) which satisfies

\[
Y_t = J_t^{\tilde{u}^*, \tilde{v}^*},
\]

where \(\tilde{u}^*(t, X_{t-}) = u^*(t, X_{t-}, Z_t, \varsigma_t), \tilde{v}^*(t, X_{t-}) = v^*(t, X_{t-}, Z_t, \varsigma_t)\). Moreover, the pair \((\tilde{u}^*, \tilde{v}^*)\) is a saddle point for the game.

Proof. By Proposition 4.1 (ii) and Theorem 3.1 we can easily get the existence and uniqueness of the solution \((Y, Z, \varsigma)\) to (14). Similarly as in Theorem 4.1 we have \(Y_t = J_t^{\tilde{u}^*, \tilde{v}^*}\).

Next we prove that \((\tilde{u}^*, \tilde{v}^*)\) is a saddle point for the game. The main tool we use is the comparison theorem for BSDEs with random default time.
First let us consider the following equation:

\[ Y_t = h(H_T, X_T) + \int_t^T H(s, X_{s-}, Z_s, \zeta_s, u_s, \tilde{v}^*)ds - \int_t^T Z_s dB_s - \int_t^T \zeta_s dM_s. \tag{15} \]

By Proposition 4.1 (ii), (15) has a unique solution \((Y^u, Z^u, \zeta^u)\) with \(Y_t^u = J_t^{u, \tilde{v}^*}\). By Proposition 4.1 (i), we have \(H(t, X_{t-}, Z_t, \zeta_t, u_t, \tilde{v}^*) \geq H^*(t, Z_t, \zeta_t)\) for each \(u \in \mathcal{U}\). It then follows from the comparison theorem that for each \(t \in [0, T]\), \(J_t^{u, \tilde{v}^*} \geq J_t^{u, \tilde{v}^*}\) a.s., for each \(u \in \mathcal{U}\). Then \(J_t^{u, \tilde{v}^*} \geq J_t^{\tilde{u}^*, \tilde{v}^*}\) for each \(u \in \mathcal{U}\).

In a symmetric way, considering

\[ Y_t = h(H_T, X_T) + \int_t^T H(s, X_{s-}, Z_s, \zeta_s, \tilde{u}^*, \tilde{v}_s)ds - \int_t^T Z_s dB_s - \int_t^T \zeta_s dM_s, \]

we obtain \(J_t^{\tilde{u}^*, \tilde{v}^*} \leq J_t^{\tilde{u}^*, \tilde{v}^*}\) and \(J_t^{\tilde{u}^*, \tilde{v}^*} \leq J_t^{\tilde{u}^*, \tilde{v}^*}\) for each \(v \in \mathcal{V}\).

Therefore \(J_t^{\tilde{u}^*, \tilde{v}^*} \leq J_t^{\tilde{u}^*, \tilde{v}^*} \leq J_t^{u, \tilde{v}^*}\) for each \(u \in \mathcal{U}\) and \(v \in \mathcal{V}\), i.e, the pair \((\tilde{u}^*, \tilde{v}^*)\) is a saddle point for the game. \(\Box\)

**Remark 4.1** From another point of view, the game problem is just a utility maximization problem under model uncertainty, once \(\mathcal{U}\) is regarded as the set that leads to model uncertainty and \(\mathcal{V}\) the set of admissible trading strategies for the investor.

**Remark 4.2** The results can be applied to the control problem (as well as the utility maximization problem) of the existence of an optimal strategy where the diffusions are bounded. For this, we can choose \(f, b\) and \(c\) independent of \(u\). Then we get that \(\tilde{v}^*(t, X_{t-}) = \tilde{v}^*(t, X_{t-}, Z_t, \zeta_t)\) is an optimal strategy for the optimal stochastic control problem, where \(\tilde{v}^*(t, x, z)\) maximizes

\[ H(t, x, z, \zeta, v) = z\sigma^{-1}(t, x)b(t, x, v) + \zeta\kappa^{-1}(t, x)c(t, x_{t-}, v)1_{\{\tau > t\}}\gamma_t + f(t, x, v), \]

and where \((Z_t, \zeta_t)\) is such that \((Y_t, Z_t, \zeta_t)\) is the unique solution of

\[ Y_t = h(H_T, X_T) + \int_t^T H(s, X_{s-}, Z_s, \tilde{v}_s^*)ds - \int_t^T Z_s dB_s - \int_t^T \zeta_s dM_s. \]

It should be noted that in the utility maximization problem, we always choose \(f = 0\) and call \(h\) the utility function. Besides, it is obvious that the value of the utility maximization problem just equals to \(Y_0\).

**Acknowledgements**

The first author thanks Prof. Monique Jeanblanc for very helpful discussions during her visit in Shandong University. The second author is grateful to Shuai Jing for his careful reading and useful suggestions.
Appendix: Some basic results

Let us recall some basic and essential results for this paper. Note that all are in a defaultable setting.

**Theorem A.1 (Itô’s formula).** Let \( X_t \) be an \( m \)-dimensional Itô jump-diffusion process given by
\[
dX_t = b_t dt + \sigma_t dB_t + \kappa_t dM_t,
\]
where \( B_t \) is a \( d \)-dimensional Brownian motion, \( M_t \) is a \( k \)-dimensional martingale (i.e., there are \( k \) default times \( \tau_1, \tau_2, \cdots, \tau_k \)), \( b_t, \sigma_t \) and \( \kappa_t \) are \( \mathbb{G} \)-adapted processes with corresponding dimensions satisfying
\[
E \int_0^T |b_t| dt < +\infty, \quad E \int_0^T |\sigma_t|^2 dt < +\infty, \quad E \int_0^T |\kappa_t|^2 \mathbb{1}_{\{\tau > t\}} \gamma_t dt < +\infty.
\]

Let \( f(t, x) \in C^{1,2}([0, T] \times \mathbb{R}^m; \mathbb{R}) \).
Then the process
\[
Y_t := f(t, X_t)
\]
is again an Itô jump-diffusion process, and it can be given by
\[
dY_t = \frac{\partial f}{\partial t}(t, X_t) dt + \sum_{i=1}^m \frac{\partial f}{\partial x_i}(t, X_t) dX^i_t
\]
\[
+ \frac{1}{2} \sum_{i,j=1}^m \sum_{k=1}^d \frac{\partial^2 f}{\partial x_i \partial x_j}(t, X_t) \sigma_{ik} \sigma_{jk} dt
\]
\[
+ \sum_{j=1}^k \left[ \Delta_j f(t, X_{t-}) - \sum_{i=1}^m \frac{\partial f}{\partial x_i}(t, X_{t-}) \kappa_{ij,t} \right] dH^j_t,
\]
where
\[
\Delta_j f(t, X_{t-}) := f(t, X^1_{t-} + \kappa_{1j,t}, \cdots, X^i_{t-} + \kappa_{ij,t}, \cdots, X^m_{t-} + \kappa_{mj,t}) - f(t, X_{t-}).
\]

The main idea of the proof can be referred to [7], [14] or [23]. Here we only give a sketch of proof for reader’s convenience.

**Sketch of proof.** For the sake of simplicity, we only give the proof for the case when \( m = d = k = 1 \). We know that the jump-diffusion process \( X \) jumps only at \( \tau \) with the jump size \( \kappa_\tau \), thus on \([0, \tau \land T]\) and \((\tau \land T, T]\),
\[
dX_t = dX^c_t = b_t dt + \sigma_t dB_t - \kappa_t \mathbb{1}_{\{\tau > t\}} \gamma_t dt.
\]
Applying the Itô’s formula in the Brownian case, we obtain
\[
Y_t - Y_0 = \int_0^t \left[ \frac{\partial f}{\partial s}(s, X_s) + \frac{1}{2} \sigma_s^2 \frac{\partial^2 f}{\partial x^2}(s, X_s) \right] ds + \int_0^t \frac{\partial f}{\partial x}(s, X_s) dX^c_s,
\]
20
Since $dX_s = dX_s^c$ on $[0, t] \subset [0, \tau \wedge T]$;

$$Y_t - Y_r = \int_r^t [\frac{\partial f}{\partial s}(s, X_s) + \frac{1}{2} \sigma_s^2 \frac{\partial^2 f}{\partial x^2}(s, X_s)] ds + \int_r^t \frac{\partial f}{\partial x}(s, X_s) dX_s^c,$$

since $dX_s = dX_s^c$ on $[r, t] \subset (\tau \wedge T, T]$.

If the default event occurs at the default time $\tau$ with jump size $\kappa_\tau$, then the resulting change in $Y_t$ is given by $f(\tau, X_\tau) - f(\tau, X_{\tau-}) = f(\tau, X_{\tau-} + \delta_\tau) - f(\tau, X_{\tau-})$.

Thus the total change in $Y_t$ can be written as the sum of these two contributions:

$$Y_t - Y_0 = \int_0^t [(\frac{\partial f}{\partial s}(s, X_s) + \frac{1}{2} \sigma_s^2 \frac{\partial^2 f}{\partial x^2}(s, X_s)] ds + \int_0^t \frac{\partial f}{\partial x}(s, X_s) dX_s^c + 1_{\{\tau \leq t\}} [f(\tau, X_\tau) - f(\tau, X_{\tau-})]$$

$$= \int_0^t (\frac{\partial f}{\partial s}(s, X_s) + (b_s - \kappa_s 1_{\{\tau > s\}} \gamma_s) \frac{\partial f}{\partial x}(s, X_s) + \frac{1}{2} \sigma_s^2 \frac{\partial^2 f}{\partial x^2}(s, X_s)] ds$$

$$+ \int_0^t \frac{\partial f}{\partial x}(s, X_s) \sigma_s dB_s + \int_0^t [f(s, X_s) - f(s, X_{s-})] dH_s.$$

Note that for any Borel measurable function $g$, we have $\int_0^t g(s, X_{s-}) ds = \int_0^t g(s, X_s) ds$ since $X_{s-}$ and $X_s$ differ only for at most one value of $s$ (for each $\omega \in \Omega$). By simple computation, we can get

$$Y_t - Y_0 = \int_0^t \frac{\partial f}{\partial s}(s, X_s) ds + \int_0^t \frac{\partial f}{\partial x}(s, X_s) dX_s + \int_0^t \frac{1}{2} \sigma_s^2 \frac{\partial^2 f}{\partial x^2}(s, X_s) ds$$

$$+ \int_0^t [(f(s, X_s) - f(s, X_{s-})) - \kappa_s \frac{\partial f}{\partial x}(s, X_{s-})] dH_s.$$

**Example A.1** As an application of Itô’s formula, we compute $f(t, X_t) = e^{\beta t} X_t^2$. Obviously $f(t, x) = e^{\beta t} x^2 \in C^{1,2}([0, T] \times \mathbb{R}; \mathbb{R})$, and we have

$$\frac{\partial f}{\partial t}(t, x) = \beta e^{\beta t} x^2$$

$$\frac{\partial f}{\partial x}(t, x) = 2 e^{\beta t} x$$

$$\frac{\partial^2 f}{\partial x^2}(t, x) = 2 e^{\beta t}.$$

Hence due to Itô’s formula (A.1), we obtain

$$df(t, X_t) = \beta e^{\beta t} X_t^2 dt + 2 e^{\beta t} X_t dX_t + \frac{1}{2} \sigma_t^2 \cdot 2 e^{\beta t} dt$$

$$+ [(e^{\beta t} X_t^2 - e^{\beta t} X_{t-}^2) - \kappa_t \cdot 2 e^{\beta t} X_{t-}] dH_t$$

$$= \beta e^{\beta t} X_t^2 dt + 2 e^{\beta t} X_t dX_t + \sigma_t^2 \cdot e^{\beta t} dt$$

$$+ [(e^{\beta t}(X_t - X_{t-}) + \kappa_t)^2 - \kappa_t \cdot 2 e^{\beta t} X_{t-}] dH_t$$

$$= e^{\beta t}(\beta X_t^2 + \sigma_t^2 + \kappa_t^2 1_{\{\tau > t\}} \gamma_t) dt + 2 e^{\beta t} X_t dX_t + e^{\beta t} \kappa_t^2 dM_t.$$

21
If we use the formula of integration by parts, first compute \( e^{\beta t}X_t \) as follows:
\[
de^{\beta t}X_t = e^{\beta t}dX_t + X_tde^{\beta t} + d[e^{\beta t}, X_t]
\]
secondly compute \( f(t,X_t) = e^{\beta t}X_t \cdot X_t \):
\[
df(t,X_t) = d(e^{\beta t}X_t \cdot X_t) = e^{\beta t}X_t dX_t + X_t de^{\beta t}X_t + d[e^{\beta t}X_t, X_t]
\]
\[
= e^{\beta t}X_t (b_t dt + \sigma_t dB_t + \kappa_t dM_t)
\]
\[
+ X_t e^{\beta t}[(b_t + \beta X_t) dt + \sigma_t dB_t + \kappa_t dM_t] + e^{\beta t} \sigma_t^2 dt + e^{\beta t} \kappa_t^2 dH_t
\]
\[
= e^{\beta t}(\beta X_t^2 + \sigma_t^2 + \kappa_t^2 1_{\{\tau > t\}} \gamma_t) dt + 2e^{\beta t}X_t dX_t + e^{\beta t}\kappa_t^2 dM_t. \tag{A.3}
\]

From the above, we can find that (A.2) and (A.3) are of the same form. This indicates that the formula of integration by parts is in fact a special case of Itô’s formula, which is well-known already in the Brownian case.

The next theorem is Theorem 2.3 of Kusuoka [16]:

**Theorem A.2 (Martingale Representation Theorem).** Assume that both (A) and (H) hold. Then any \( \mathbb{G} \)-square integrable martingale admits a representation as the sum of a stochastic integral w.r.t the Brownian motion and stochastic integrals w.r.t the martingales \( \{M^i; i = 1, 2, \ldots, k\} \) associated with \( \{\tau_i; i = 1, 2, \ldots, k\} \) respectively.

More precisely, suppose \( (N_t)_{0 \leq t \leq T} \) is a \( \mathbb{G} \)-square integrable martingale. Then there exist \( \mathbb{G} \)-adapted processes \( \mu_s : [0,T] \times \Omega \to \mathbb{R}^d \) and \( \nu^i_s : [0,T] \times \Omega \to \mathbb{R} \) \( (i = 1, 2, \ldots, k) \) such that
\[
E \int_0^T |\mu_s|^2 ds < \infty, \ E \int_0^T |\nu^i_s|^2 \gamma^i_s ds < \infty, \ i = 1, 2, \cdots, k \tag{A.4}
\]
and
\[
N_t = N_0 + \int_0^t \mu_s dB_s + \int_0^t \nu_s dM_s := N_0 + \int_0^t \mu_s dB_s + \sum_{i=1}^k \int_0^t \nu^i_s dM^i_s. \tag{A.5}
\]

**Remark A.1** In fact, in Kusuoka’s martingale representation theorem, the processes \( \mu(\cdot) \) and \( \nu^i(\cdot) \) \( (i = 1, 2, \ldots, k) \) are unique, that is to say, if processes \( \tilde{\mu} : [0,T] \times \Omega \to \mathbb{R}^d \) and \( \tilde{\nu}^i : [0,T] \times \Omega \to \mathbb{R} \) \( (i = 1, 2, \ldots, k) \) also make (A.4) and (A.3) true, then we undoubtedly have
\[
E \int_0^T |\mu_s - \tilde{\mu}_s|^2 ds = 0, \ E \int_0^T |\nu^i_s - \tilde{\nu}^i_s|^2 1_{\{\tau_i > t\}} \gamma^i_s ds = 0, \ i = 1, 2, \cdots, k.
\]
The Girsanov Theorem, stated below, can be referred to Kusuoka [16] (Proposition 3.1) or Bielecki et al. [4] (Proposition 3.2.2).

**Theorem A.3 (Girsanov Theorem).** Let \( Q \) be a probability measure on \( (\Omega, \mathcal{G}_T) \) equivalent to \( P \). If the Radon-Nikodym density \( \eta \) of \( Q \) w.r.t \( P \) is given as follows:

\[
\eta_t := \left. \frac{dQ}{dP} \right|_{\mathcal{G}_t} = 1 + \int_0^t \eta_s (\rho_s dB_s + \kappa_s dM_s) = 1 + \int_0^t \eta_s (\rho_s dB_s + \sum_{i=1}^k \kappa_i^s dM_i^s),
\]

where \( \kappa^i > -1, \ i = 1, 2, \ldots, k. \) Then the process

\[
B^*_t = B_t - \int_0^t \rho_s ds, \ \forall t \in [0,T]
\]

follows a Brownian Motion w.r.t \( \mathcal{G} \) under \( Q \), and the processes

\[
M^i_{t^*} = M^i_t - \int_0^t \kappa^i_s 1_{\{\tau^i > s\}} \gamma^i_s ds, \ i = 1, 2, \ldots, k
\]

are \( \mathcal{G} \)-martingales orthogonal to each other and orthogonal to \( B^* \).

**References**

[1] V.E.Benes, Existence of optimal stochastic control laws, SIAM Journal on Control Optimization 9(3) (1971) 446-472.

[2] T.R.Bielecki, M.Jeanblanc, M.Rutkowski, Hedging of defaultable claims, Paris-Princeton Lecture on Mathematical Finance 2003, Lecture Notes in Mathematics 1847 (2004) 1-132. Springer.

[3] T.R.Bielecki, M.Jeanblanc, M.Rutkowski, PDE approach to valuation and hedging of credit derivatives, Quantitative Finance 5 (2005) 257-270.

[4] T.R.Bielecki, M.Jeanblanc, and M.Rutkowski, Introduction to mathematics of credit risk modeling, Stochastic Models in Mathematical Finance, CIMP-UNESCO-MOROCCO School, Marrakech, Morocco, April 9-20, 2007, pp.1-78.

[5] F.Black, M.Scholes, The pricing of options and corporate liabilities, Journal of Political Economy 81(3) (1973) 637-654.

[6] Z.Chen, L.Epstein, Ambiguity, risk, and asset returns in continuous time, Econometrica 70(4) (2002) 1403-1443.
[7] C. Dellacherie, P. A. Meyer, Probabilities and Potential B: Theory of Martingales, North-Holland, Amsterdam, 1982.

[8] N. El Karoui, S. Hamadene, BSDEs and risk-sensitive control, zero-sum and nonzero-sum game problems of stochastic functional differential equations, Stochastic Processes and Their Applications 107 (2003) 145-169.

[9] N. El Karoui, S. Peng, M. Quenez, Backward stochastic differential equations in finance, Mathematical Finance 7(1) (1997) 1-71.

[10] J. Gregory, J. P. Laurent, I will survive, Risk, June (2003) 103-107.

[11] S. Hamadene, J. P. Lepeltier, Zero-sum stochastic differential games and backward equations, Systems and Control Letters 24 (1995) 259-263.

[12] S. Hamadene, J. P. Lepeltier, Z. Wu, Infinite horizon reflected backward stochastic differential equation and applications in mixed control and game problems, Probability and Mathematical Statistics 19 (1999) 211-234.

[13] M. Harrison, S. R. Pliska, Martingales and stochastic integrals in the theory of continuous trading, Stoch. Process. Appl. 11 (1981) 215-260.

[14] S. He, J. Wang, J. Yan, Semimartingale Theory and Stochastic Calculus, Science Press and CRC Press, 1992.

[15] R. A. Jarrow, F. Yu, Counterparty risk and the pricing of defaultable securities, Journal of Finance 56 (2001) 1765-1799.

[16] S. Kusuoka, A remark on default risk models, Advances in Mathematical Economics 1 (1999) 69-82.

[17] T. Lim, M. C. Quenez, Utility maximization in incomplete markets with default, pdf-file available in arXiv:0811.4715v2 [math.OC] 22 Mar 2009.

[18] R. Mansuy, M. Yor, Random times and enlargements of filtrations in a Brownian setting, Lecture Notes in Mathematics, vol.1873, 2006, Springer.

[19] R. Merton, Theory of rational option pricing, Bell J. Econ. Manage. Sci. 4 (1973) 141-183.

[20] E. Pardoux, S. Peng, Adapted solution of a backward stochastic differential equation, Systems and Control Letters 14 (1990) 55-61.

[21] E. Pardoux, S. Peng, Backward stochastic differential equations and quasilinear parabolic partial differential equations, Lecture Notes in Control and Inform. Sci. 176 (1992) 200-217, Springer, Berlin.
[22] S. Peng, A generalized dynamic programming principle and Hamilton-Jacobi-Bellman equation, Stochastics and Stochastics Reports 38 (1992) 119-134.

[23] P. Protter, Stochastic Integration and Differential Equations, Springer-Verlag, Berlin, second edition, 2004.