AXISYMMETRIC SELF-SIMILAR EQUILIBRIA OF SELF-GRAVITATING ISOTHERMAL SYSTEMS

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ABSTRACT

We show that there are two families of axisymmetric self-similar equilibria of self-gravitating, rotating, isothermal systems: (1) cylindrically symmetric solutions in which the density varies with cylindrical radius as $R^{-2(n+1)}$, with $-1 \leq n \leq 0$ and (2) axially symmetric solutions in which the density varies as $f(\theta)/r^2$, where $r$ is the spherical radius and $\theta$ is the colatitude. The singular isothermal sphere is a special case of the latter class with $f(\theta) = constant$. The axially symmetric equilibrium configurations form a two-parameter family of solutions and include equilibria which are asymmetric with respect to the equatorial plane. The asymmetric equilibria are, however, not force-free at the singular points $r = 0, \infty$, and their relevance to real systems is unclear. For each hydrodynamic equilibrium, we determine the phase-space distribution of the collisionless analog.

Subject headings: galaxies: kinematics and dynamics — galaxies: structure — hydrodynamics — ISM: kinematics and dynamics — stars: formation — stars: kinematics

1. INTRODUCTION

Many astrophysical objects consist of equilibrium configurations in which self-gravity is resisted by pressure (velocity dispersion) and centrifugal forces. A particularly simple example, which is well suited for practical analysis, is the case of an isothermal fluid with a linear pressure-density relation, $p = \rho c_s^2$, where $p$ is the pressure, $\rho$ is the density, and $c_s$ is the sound speed (taken to be a constant). Equilibrium configurations of isothermal systems have been studied as models of galaxies (Toomre 1982; Binney & Tremaine 1987; see also Richstone 1980; Monet, Richstone, & Schechter 1981 for some special cases) and newly formed stars (e.g., Hayashi, Narita, & Miyama 1982; Kiguchi et al. 1987). Toomre’s equilibria have recently been generalized (Evans 1993, 1994; Evans & de Zeeuw 1994) to a nice, analytically simple family of composite galaxy models with logarithmic and power-law potentials. Axisymmetric galaxy models, both self-similar and with a central mass, have been considered by Qian et al. (1995).

Although some analytical solutions have been published previously, no systematic study of isothermal equilibria has been presented so far. In this paper we classify and derive analytically all possible self-similar axisymmetric equilibria of a self-gravitating isothermal system. We find that there are only two distinct classes of equilibria:

1. Cylindrically symmetric equilibria, in which all quantities, such as density, potential, velocity, etc., depend on the cylindrical radius $R$ only.
2. Axially symmetric equilibria, in which the quantities are functions of both the spherical radius $r$ and the colatitude $\theta$.

The cylindrically symmetric solutions are simple: the density varies according to $\rho \propto R^{-2(n+1)}$, and the azimuthal velocity according to $\Omega R \equiv v_\phi = v_{\phi 0} R^{-n}$, where the allowed range of $n$ is $-1 \leq n \leq 0$.

The axially symmetric solutions are more rich. First, the density always varies according to $\rho \propto 1/r^2$ and the rotation curve is flat, $v_\phi = v_{\phi 0} = constant$. Second, these equilibria form a two-parameter family of solutions. One of the parameters, $A$, determines the rotation velocity, $A = v_{\phi 0} R^2/c_s^2$, and the other parameter, $B$, controls the symmetry of the solutions with respect to the equatorial plane. For $B = 1$, the solutions are up-down symmetric. These solutions have been previously obtained by Toomre (1982) and Hayashi et al. (1982) and include the singular isothermal sphere ($A = 2$) and the cold Mestel disk ($A \to \infty$) as limiting cases.

The solutions with $B \neq 1$ are asymmetric with respect to the equatorial plane. This contradicts Lichtenstein’s theorem (Lichtenstein 1933; Wavre 1932), which states that a barotropic, self-gravitating equilibrium must have a plane of symmetry perpendicular to the axis of rotation. The paradox is resolved by noting that the solutions with $B \neq 1$ are not force-free at two singular points, $r = 0$ and $r = \infty$, where Poisson’s equation is ill-defined. External forces have to be applied to the matter at the singularities to hold the system in equilibrium. This is proved rigorously for the case without rotation, $A = 2$, and is likely to be correct also for rotating solutions.

The paper is organized as follows. In § 2, we present a general analysis of the problem. We classify all possible self-similar axisymmetric equilibrium configurations of self-gravitating isothermal systems in § 3, and present analytical expressions for a complete family of cylindrical and axisymmetric solutions. We describe the properties of axisymmetric solutions in § 4. In § 5 we determine the steady state distribution function of collisionless stellar systems, and we summarize the conclusions in § 6.

2. GENERAL CONSIDERATIONS

We consider spherical coordinates $(r, \theta, \phi)$, but also occasionally use the cylindrical radius, $R = r \sin \theta$.

A hydrostatic equilibrium configuration satisfies the momentum equation with vanishing time derivative, and Poisson’s equation. By definition, the $\hat{r}$ and $\hat{\theta}$ components (where the “hats” denote unit vectors) of the velocity vanish in equilibrium. Also, by the condition of axial symmetry, all
derivatives with respect to the toroidal angle \( \phi \) vanish. Thus, we need to consider only the \( \hat{r} \) and \( \theta \) components of the momentum equation and Poisson’s equation for the potential:

\[
\begin{align*}
\frac{v_{\phi}^2}{r} &= \frac{c_s^2}{\rho} \frac{\partial \rho}{\partial r} + \frac{1}{r} \frac{\partial}{\partial \theta} \left( \rho \frac{\partial \phi}{\partial \theta} \right), \\
\frac{v_r^2 \cot \theta}{r} &= \frac{c_s^2}{\rho} \frac{\partial \rho}{\partial r} + \frac{1}{r} \frac{\partial}{\partial \theta} \left( \rho \frac{\partial \phi}{\partial \theta} \right), \\
\nabla_{r,\theta}^2 \phi &= 4\pi G \rho,
\end{align*}
\]

(1a) (1b) (1c)

where

\[
\nabla_{r,\theta}^2 = \frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \frac{\partial}{\partial r} \right) + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial}{\partial \theta} \right)
\]

is the axisymmetric Laplacian operator; \( \phi \) is the gravitational potential; \( v_\phi \) is the toroidal component of the velocity due to rotation, which is, in general, a function of \( r \) and \( \theta \); and \( G \) is Newton’s constant. The radial coordinate and the density are taken to be dimensionless throughout the paper.

We begin with a general treatment of the problem, without any assumption of self-similarity. We take the gas to be isothermal, i.e., the pressure and the density are related as follows:

\[
p = c_s^2 \rho,
\]

(2)

where \( c_s = \text{constant} \) is the sound speed. Since an isothermal system is a special case of a barotropic system, \( p = \rho(\rho) \), the following generic properties immediately follow from the Poincaré-Wave theorem (Tassoul 1978): (i) the angular velocity is constant on cylinders centered on the axis of rotation, i.e., in cylindrical coordinates \((R, \phi, z)\), the velocity \( v_\phi = v_\phi(R) \) is independent of \( z \); (ii) the effective gravity can be derived from an effective potential; and (iii) the effective gravity is normal to the isopycnic (i.e., constant density) surfaces. Tassoul (1978) has derived general relations between \( \rho, \phi, \) and \( v_\phi \) in a rotating barotropic system; we briefly state some of these results.

Eliminating \( v_\phi \) between equations (1a) and (1b), we arrive at the following differential equation:

\[
\left( \frac{\partial}{\partial \ln r} - \frac{\partial}{\partial \ln \sin \theta} \right) (c_s^2 \ln \rho + \phi) = 0.
\]

(3)

Any function of the argument \( \ln (r \sin \theta) \) is a solution of this differential equation. Absorbing the logarithm into the definition of the function, we write the solution:

\[
\phi(r, \theta) = -c_s^2 \ln \rho(r, \theta) + u(r \sin \theta),
\]

(4)

where \( u \) is an arbitrary function of the cylindrical radius \( R = r \sin \theta \). The function \( u \) is related to the rotation velocity as follows:

\[
v_\phi^2(r, \theta) = v_\phi^2(R) = R \frac{\partial u(R)}{\partial R}.
\]

(5)

This relation is obtained upon substituting equation (4) in equation (1a).

Next, we use Poisson’s equation to relate the density and gravitational potential. Substituting equation (4) in equation (1c) and using equation (5) and the identity \( \nabla_{r,\theta}^2 u(r \sin \theta) = \nabla_R^2 u(R) \) (where \( \nabla_R^2 \) is the radial part of a cylindrical Laplacian) we obtain the following equation for the density distribution,

\[
-c_s^2 \nabla_{r,\theta}^2 \ln \rho + \frac{1}{R} \frac{\partial}{\partial R} v_\phi^2 = 4\pi G \rho.
\]

(6)

Given the rotation curve, \( v_\phi(R) \), the solution of equation (6), together with equations (2), (4), and (5), completely determines the solution. To proceed further, we need to make some simplifying assumptions.

3. Self-Similar Solutions

We now look for self-similar solutions of equation (6). Let us assume that \( v_\phi \) is described by a power law in \( R = r \sin \theta \). Then the density distribution is also a power law in \( r \), and we write

\[
\rho = \rho_0 \frac{f(\theta)}{r^2}, \quad v_\phi = v_{\phi 0} \frac{R^2}{r^2},
\]

(7)

where \( f(\theta) \) is an unknown function to be calculated, \( \rho_0 \) and \( v_{\phi 0} \) are normalization constants, and \( a \) and \( n \) are parameters. Substituting equations (7) in equation (6), we obtain

\[
-c_s^2 \left( \frac{a - 1}{\sin \theta} \frac{\partial}{\partial \theta} \sin \theta \frac{\partial}{\partial \theta} \ln f(\theta) \right) - \frac{2nv_{\phi 0}^2}{(r \sin \theta)^{a+2}} = 4\pi G \rho_0 \frac{f(\theta)}{r^2}.
\]

(8)

This equation can be satisfied only when the powers of \( r \) on the various terms match. There are two possible cases:

1. Cylindrically symmetric solutions. In this case, the first term on the left-hand-side of equation (8) vanishes identically and \( 2n + 2 = a \).
2. Axially symmetric solutions. In this case, \( a = 2 \) and \( n = 0 \), and the term proportional to \( v_{\phi 0} \) in equation (8) vanishes. These solutions have flat rotation curves.

3.1. Cylindrical Solutions

In this case, the first term on the left-hand side of equation (8) vanishes identically, so that \( 2n + 2 = a \). This condition forces \( f \) to be \( f(\theta) = (\sin \theta)^{-a} \). The self-similar solution for \( \rho \) in terms of \( R = r \sin \theta \) is easily obtained from equation (8),

\[
\rho(R) = -\frac{nv_{\phi 0}^2}{2\pi G R^{2n+2}}.
\]

(9a)

We notice that this solution is physical only for \( n \leq 0 \); otherwise, the density is negative. The gravitational potential is determined from equation (4). We have,

\[
\phi = -c_s^2 \ln \left( -\frac{nv_{\phi 0}^2}{2\pi G R^{2n+2}} \right) + u(R) + \phi_0,
\]

(9b)

where \( \phi_0 \) is a constant that defines the zero level of the potential and the function \( u(R) \) is obtained by integrating equation (5) for a given power-law rotation curve,

\[
u_{\phi 0} = \begin{cases} 
\frac{v_{\phi 0}^2}{(2n + 2)} R^{2n+2} & \text{if } n \neq 0, \\
\frac{v_{\phi 0}^2 \ln (R)}{} & \text{if } n = 0.
\end{cases}
\]
The solution is well behaved everywhere except at the axial singularity, where the density distribution on the axis, \( n = 1 \), is ill-defined at \( r = 0 \). Therefore, a singular solution, proportional to the Dirac \( \delta \)-function, \( \delta(\theta) \), is allowed on the axis. This additional mass density may be calculated using Gauss’s theorem, equation (10). As a surface of integration, we choose a conical section about the axis with the opening angle \( \Delta \theta \) of a spherical shell of thickness \( \Delta r \). Taking the limit \( \Delta \theta \rightarrow 0 \), and noticing that the radial distance, \( \Delta r \), at \( \theta = 0 \) is the length along the axis, \( \Delta r_{\theta=0} = L \), we obtain the linear mass density of a singular density distribution on the axis,

\[
\mu_s \equiv \frac{\partial M_s}{\partial L} = \frac{c_s^2}{2G} \left( 2 + \frac{v_{\phi s}^2}{c_s^2} - A \right).
\]

We see that \( A \) is related to the rotation velocity and the mass density at the axis. The mass on the axis cannot be negative, hence we have the constraint \( A \leq 2 + v_{\phi s}^2/c_s^2 \). The requirement of regularity of the solution, i.e., the continuity of \( \rho \) and \( \phi \) and their derivatives everywhere, constrains one of the free parameters:

\[
A = 2 + v_{\phi s}^2/c_s^2 \geq 2.
\]

3.2. Axially Symmetric Solutions

The equation for \( \rho \) (eq. [6] with \( x = 2 \) and \( n = 0 \)) reads

\[
2 - \frac{1}{\sin \theta \partial \theta} \left( \sin \theta \frac{\partial}{\partial \theta} \ln f(\theta) \right) = \frac{4\pi \rho \partial}{c_s^2} f(\theta) . \tag{12}
\]

As one can see, the velocity gradient term drops out. That is, the density distribution appears to be independent of the rate of rotation, in apparent contradiction to the Poincaré-Wavre theorem. As we shall see from the careful analysis below, this is not the case.

The nonlinear, second-order differential equation (12) may be solved analytically, and a general two-parameter family of self-similar solutions can be found in a closed and explicit form:

\[
\rho(r, \theta) = \frac{c_s^2}{2\pi G} \frac{A^2}{(r \sin \theta)^2} \frac{B \tan^4 (\theta/2)}{\left[1 + B \tan^4 (\theta/2)\right]^2} , \tag{13a}
\]

where \( A > 0 \) and \( B > 0 \) are two free parameters. We have restricted \( A \) to be positive, because a change of sign of \( A \) is identical to the replacement: \( B \rightarrow 1/B \). The gravitational potential calculated from equation (4) may be written in the form

\[
\phi(r, \theta) = -c_s^2 \ln \left[ \frac{\rho(r, \theta)}{|r \sin \theta| v_{\phi s}^2/c_s^2} \right] + \phi_0 . \tag{13b}
\]

The denominator in the logarithm is the contribution due to rotation, \( \sqrt{u_r \sin \theta} = v_{\phi s} \ln (r \sin \theta) + \text{constant} \), as follows from equation (5).

The solution (eqs. [13a] and [13b]) is smooth and well behaved in the domain \( 0 < \theta < \pi \). However, the differential operator in equation (12) is ill-defined at \( \theta = 0 \). Therefore, a singular solution, proportional to the Dirac \( \delta \)-function, \( \delta(\theta) \), is allowed on the axis. This additional mass density may be calculated using Gauss’s theorem, equation (10).

4. PROPERTIES OF THE AXIALLY SYMMETRIC SOLUTIONS

4.1. General Properties

Figures 1–4 show typical examples of the self-similar solution in equations (13a) and (13b). Contours of equal density for different values of the parameters \( A \) and \( B \) are displayed. As is seen, the parameter \( A \) determines the overall shape of the matter distribution. As \( A \rightarrow 0 \) the configuration tends to the cylindrically symmetric limit, and an axial singularity with \( \mu_s \neq 0 \) is always present (see eq. [14]). Figure 1 shows an example with \( A = 0.7 \). The density does not vanish at infinity at \( \theta = 0, \pi \).

Solutions without an axial singularity, i.e., with \( \mu_s = 0 \), exist for \( A \geq 2 \) (cf. eq. [15]). The nonrotating limit, \( A = 2 \), yields density surfaces which are confocal ellipsoids/spheres, as shown in Figure 2 (see § 4.3). For a nonrotating model with \( A > 2 \), toroidal configurations are obtained, as shown in Figures 3 and 4. Interestingly, as \( A \rightarrow \infty \), the profile flattens and tends to a thin disk configuration, as shown in Figure 4 for \( A = 20 \).

The parameter \( B \) is responsible for up-down asymmetry. For \( B = 1 \), the solutions are symmetric with respect to the equatorial plane. The solutions with \( B > 1 \) are shifted (distorted) upward, and those with \( B < 1 \) are shifted downward. Note that solutions with \( B \) and \( 1/B \) are identical except that they are turned upside-down with respect to

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2 The singularity of the solution at \( \theta = 0 \) is due to the singularity of the differential operator of eq. (8) on the axis.

3 Note that the density distribution depends on the rotation velocity through eq. (14), in agreement with the Poincaré-Wavre theorem.

4 The parameter \( A \) is given by \( A = 2n + 2 \) in Toomre (1982) and by \( A = 2g \) in Hayashi et al. (1982).
Fig. 1.—Isodensity contours for $A = 0.7, B = 1$, and $B = 3$. The horizontal axis corresponds to $R$ and the vertical axis to $z$.

each other. Note also that the solutions with $B \neq 1$ violate Lichtenstein’s theorem (Lichtenstein 1933; Wavre 1932), which proves the existence of an equatorial plane of symmetry. The resolution of this paradox is discussed below (§ 4.3).

4.2. The Finite System Limit

We now determine some global properties, such as the total mass $M$, gravitational energy $W$, and angular momentum $L$ of the axially symmetric self-similar solutions as functions of $A$ and $B$. These quantities diverge unless a cutoff is imposed at large radii. The most natural way to do this is to assume that the system is immersed in a medium with finite pressure, $p_{\text{ext}}$, but vanishing density. As follows from equation (2), the cutoff surface in this case is the iso-

Fig. 2.—Same as Fig. 1, but for $A = 2, B = 1$, and $B = 3$

Fig. 3.—Same as Fig. 1, but for $A = 3, B = 1$, and $B = 3$

Fig. 4.—Same as Fig. 1, but for $A = 20, B = 1$, and $B = 10^4$
density surface with \( \rho_c = \rho_{\text{ex}}/c_s^2 \). The equation for the cutoff surface is
\[
r_c(\theta) = \sqrt{g\Theta(\theta)/\rho_c},
\]
where \( g = c_s^2/(2\pi G) \) and
\[
\Theta(\theta) = \frac{A^2}{\sin^2 \theta} \frac{B \tan^4 (\theta/2)}{[1 + B \tan^4 (\theta/2)]^2}
\]
is the angular part of the function \( \rho(r, \theta) \). Notice that if \( \mu_c \neq 0 \), and \( W \) still diverge because of the infinite mass located on the axis. So we set \( \mu_c = 0 \), which means that we are restricted to solutions with \( A \geq 2 \). The total mass, gravitational potential energy, and angular momentum are defined as follows:
\[
M = \int_V \rho \, dV, \quad W = \frac{1}{2} \int_V \rho \phi \, dV, \quad L = \int_V \rho v \times r \, dV.
\]

The binding energy is then \( E = -(W + K) \), where the kinetic energy is \( K = M v_{\phi 0}^2/2 \). The surfaces of constant density and constant potential do not coincide, because of rotation, unless \( A = 2 \) (the nonrotating case). For this reason, the potential at the cutoff surface is a function of position, \( \phi_c = \phi(r, \theta) \). To get rid of insignificant constants in the expression for \( W \), we redefine the gravitational potential in such a way that the maximum potential at the surface is zero. This constrains the constant \( \phi_0 \). The value of \( \theta_m \), the extremum point of \( \phi \) along \( r_i(\theta) \), is found to satisfy the equation, \( \tan^4 (\theta_m/2) = 1/\theta \). Then we have
\[
\phi_0 = c_s^2 \ln \rho_c - v_{\phi 0}^2 \ln \sqrt{\rho_c} - v_{\phi 0}^2 \ln (A/2).
\]

As shown in § 4.3, the solutions with \( B \neq 1 \) are not force-free; they include an external gravitational potential gradient. We do not include the external potential in our definition of the gravitational energy \( W \). The necessary integrals can be obtained via contour integrations or derived from the entry (3.194.3) of Gradshteyn & Ryzhik (1980) and properties of the beta function. We find that
\[
M = \frac{2\pi g^{3/2}}{\sqrt{\rho_c}} \frac{\pi (A^2 - 4)}{16 \cos (\pi/4) (B^{1/4} + B^{-1/4})}, \quad (19a)
\]
\[
W = -\frac{\pi g^{3/2}}{\sqrt{\rho_c}} \frac{\pi (A - 2)^2}{16 \cos (\pi/4) \left(2 + (A + 2) \left[1 - \frac{1}{2} \ln 2 - \gamma + \pi \frac{\tan \pi}{2} - \psi\left(\frac{1}{2} + \frac{1}{A}\right)\right]\right)} (B^{1/4} + B^{-1/4}), \quad (19b)
\]
\[
L_z = \frac{2\pi g^{3/2}}{12 \sin (\pi/4)} \frac{\pi (A^2 - 1)}{\sqrt{A - 2(B^{1/4} + B^{-1/4})}}, \quad (19c)
\]
where \( g = c_s^2/(2\pi G), \psi(x) = \Gamma'(x)/\Gamma(x) \) is the digamma function, and \( \gamma \equiv -\psi(1) = 0.57721 \ldots \) is Euler’s constant. All three quantities increase with increasing \( A \), i.e., as the rotation velocity \( v_{\phi 0} \) increases. Their dependence as a function of \( B \) is trivial; all are proportional to \( (B^{1/4} + B^{-1/4}) \) and have minima at \( B = 1 \), the symmetric configuration.

We emphasize that the truncation of a system at a finite \( \rho = \rho_c \) breaks the self-similarity of the solutions. Thus, the above equations are approximate and should be used with care. Nevertheless, equations (19a)–(19c) suggest that both the specific angular momentum of a finite configuration, \( L_c/M \), and the gravitational binding energy per unit mass, \( W/M \), are functions only of \( A \) and are independent of \( B \). The kinetic energy per unit mass, \( v_{\phi 0}^2/2 \), is also a function only of \( A \). Finally, if we consider the specific entropy, \( s = S/M \), where \( S \sim -\ln \rho \ln p \), we see that \( S \) differs from the gravitational energy by additive and multiplicative constants only. Thus \( s \) is again independent of \( B \). The question then is, if energy and entropy are indistinguishable and independent of \( B \), what determines which value of \( B \) is selected by a finite isothermal system with a given value of \( L_c/M \)? The answer, as we show in the next section, is the boundary condition, namely, the magnitude of external forces acting on the mass.

4.3. The Nonrotating Limit, \( A = 2 \)

We first prove that nonrotating solutions, \( A = 2 \), consist of nested confocal ellipsoids. The solution (eq. [13a]) reads
\[
\rho(r, \theta) = \frac{2\pi v_{\phi 0}^2}{\pi G (r \sin \theta)^2} \left[1 + B \tan^2 (\theta/2)\right]^{-2/3}.
\]
The equation of an isodensity contour line is obtained by setting \( \rho(r, \theta) = \rho_1 \) constant. Upon straightforward trigonometric simplifications, we obtain
\[
r(\theta) = \left(\frac{2\pi v_{\phi 0}^2}{\pi G \rho_1(1 + B)^2}\right)^{1/2} \left[1 + \frac{(1 - B) \cos \theta}{1 + B}\right]^{-1/2}.
\]
This is the equation of an ellipse with eccentricity \( e \) and semimajor axis \( a \), with the origin located at one of the foci. The three-dimensional isodensity surfaces are the surfaces of revolution about the major axis. Note that the case \( B = 1 \) corresponds to the singular isothermal sphere.

Next, we demonstrate that the solutions with \( B \neq 1 \) are not force-free. Let us consider an equipotentiality which is also an equipotential surface with density \( \rho_1 \). The gravitational potential inside the surface produced by the “outside” mass, \( \rho < \rho_1 \), is linear in the vertical coordinate, \( z \), as shown in the Appendix:
\[
\Phi = -z \sqrt{2\pi G c_s^2} \sqrt{\rho_1 (B^{1/2} - B^{-1/2})} \times 1 - e^2 \left[\frac{1}{2} e \left(\frac{1}{1 + |e|} - 1\right)\right].
\]
As one can see, \( \Phi \propto \sqrt{\rho_1} \). On the other hand, the mass inside the equipotential is \( M = \pi g^{3/2} / \sqrt{\rho_1 (B^{1/2} + B^{-1/2})} \), i.e., \( M \propto 1/\sqrt{\rho_1} \), as follows from equations (19a), (19b), and (19c). The total gravitational force exerted by the “outside” mass on the “inside” mass, \( F_s = -M V \Phi \), is, thus, independent of \( \rho_1 \), i.e., independent of the equipotensity surface chosen:
\[
F_s = -z \frac{2\pi g^{3/2}}{G} \left[1 - \frac{1}{2} e |e| \left(\frac{1}{1 + |e|} - 1\right)\right],
\]
where \( e = -(B - 1)/(B + 1) \). Taking an equipotential surface infinitely close to the origin, one can see that there is a finite gravitational pull, \( F_s \) (upward for \( B > 1 \) and downward for \( B < 1 \)), acting on the matter at the singularity \( r = 0 \). This force is produced by the gravity of all the mass of the system. For the system to remain in equilibrium,
there must be an equal and opposite external force, \(-F_\theta\), applied at \(r = 0\). A similar consideration shows that there must also be an extra force acting at infinity or at the last equidensity surface of a finite system. These external forces are uniquely related to the asymmetry parameter, \(B\), and, hence, determine the structure of the equilibrium. When \(B = 1\), the external forces vanish and the equilibrium configuration is force-free. While this analysis is restricted to the nonrotating case \(A = 2\), we expect that similar results should hold for any \(A\).

As we mentioned earlier, our \(B \neq 0\) asymmetric solutions apparently contradict Lichtenstein’s theorem (Lichtenstein 1933; Wavre 1932), according to which any rotating body for which the angular velocity does not depend on \(z\) (which is true for a self-gravitating isothermal system) always possesses an equatorial plane of symmetry that is perpendicular to the axis of rotation. There are, however, two assumptions (among others) used in the theorem that are violated by our \(B \neq 1\) solutions: (i) no external potential is allowed, and (ii) the system is assumed to be bounded and the density is nowhere infinite. As we have demonstrated, an asymmetric configuration with \(B \neq 1\) exists only when external forces act on the system and pull it apart, in violation of condition (i). Moreover, these forces are applied precisely at the positions where condition (ii) is violated: either \(\rho\) or \(r\) is infinite at these portions.

4.4. The Thin-Disk Limit, \(A \to \infty\)

The solutions with \(A > 2\) have rotation. The systems become flattened as the rotation increases, and they tend to a thin disk as \(A \approx v_\phi^2/c_s^2 \to \infty\). In these solutions the parameter \(B\) determines the “inclination angle.” We define the inclination angle (or the opening angle of the “equatorial cone”), \(\theta_m\), as the angle at which the density is maximum for a fixed radial distance \(r\). In the large-\(A\) limit, we can neglect the \(\sin^2 \theta\) in the denominator in equation (13a). We then find

\[
\theta_m \approx 2 \arccot(B^{1/4}).
\]

(24)

Taking into account that \(\arccot(x) = \pi/2 - \arccot(1/x)\) and expanding the solution (13a) about \(\theta_m\), for small angles \(\theta - \theta_m \equiv S \ll \arccot(B^{1/4})\), we obtain

\[
\rho(r, S) \approx \frac{c_s^2}{8\pi G} \frac{A^2}{r^2} \frac{1 + B^2}{2B} \text{sech}^2 \left[ 5A \left( \frac{1 + B^2}{2B} \right) \right].
\]

(25)

For large \(A\), the \(\text{sech}^2\) function is very sharply peaked. Thus, the height-to-radius ratio of the disk may be estimated as

\[
\frac{H}{R} \approx \Delta S \approx \frac{c_s^2}{v_\phi^2} \left( \frac{2}{B + 1/B} \right).
\]

(26)

In the symmetric case, \(B = 1\), this reduces to the standard result, \(H/R \approx c_s^2/v_\phi^2 \leq 1\). As \(c_s \to 0\) the disk becomes the cold Mestel disk having a flat rotation profile.

5. EQUILIBRIUM OF COLLISIONLESS STELLAR SYSTEMS

There is a unique connection between the density and velocity fields of fluid equilibrium configurations and the distribution functions of stars in their collisionless analogs (Binney & Tremaine 1987). For any stellar distribution function \(f\), the profiles of density, streaming velocity, \(\bar{v}_\phi\), velocity dispersion, etc., are calculated uniquely by taking the moments of \(f\). Similarly, given \(\rho, \bar{v}_\phi\), etc., an equilibrium \(f(x, v)\) may be determined. In general, there is no guarantee that the stellar distribution function so obtained will be physical, i.e., nonnegative over the entire six-dimensional phase space (see, e.g., Binney & Tremaine 1987 for more discussion). Remarkably, however, the distribution function is guaranteed to be positive for isothermal systems: the structure of an equilibrium of a self-gravitating isothermal gas is identical to the structure of a collisionless system of stars. We now determine the distribution function of a stellar system that corresponds to the hydrostatic equilibria found in § 3.2.

We consider here the simple case when the distribution function depends on two classical integrals of motion, the energy and axial component of the angular momentum.\(^5\) Lynden-Bell (1962) and subsequently Hunter & Qian (1973) and Hunter (1975) developed a method to derive distribution functions from the density and mean velocity profiles. We introduce the integrals of energy and angular momentum of a particle as follows:

\[
f = \frac{v_\phi^2 + v_\theta^2 + v_r^2}{2} + \phi(r, \theta) - \phi_0, \quad \mathcal{L}_z = |v_\phi| r \sin \theta,
\]

(27)

where the \(v_i\) are the components of the particle velocity. Since the density profile is insensitive to the direction of rotation, it yields, upon inversion, a distribution function that is even in \(\mathcal{L}_z\), i.e., \(f = f(\mathcal{E}, \mathcal{L}_z) + f(\mathcal{E}, -\mathcal{L}_z)\). Thus there are, in principle, infinitely many distributions that produce identical density profiles and differ by an arbitrary, odd in \(\mathcal{L}_z\), function, \(f_- = f(\mathcal{E}, \mathcal{L}_z) - f(\mathcal{E}, -\mathcal{L}_z)\), determined by the sense of motion of individual stars. For real stellar systems, \(f_-\) can also be determined, given the average rotation profile, \(\bar{v}_\phi\). Henceforth, we focus on the symmetric part of the distribution function. We omit the subscript “+,” since this should not cause any confusion.

To determine \(f\), we need to express the density profile as a function of the cylindrical radius and gravitational potential. From equations (13b) and (15), it follows that

\[
\rho(R, \phi) = R^{4-2} \exp \left[-(\phi - \phi_0)/c_s^2 \right].
\]

(28)

We now observe that \(\rho(R, \phi)\) is independent of \(B\). Hence \(f(\mathcal{E}, \mathcal{L}_z)\) derived from it is also independent of \(B\).\(^6\) Since \(f(\mathcal{E}, \mathcal{L}_z)\) is unique (up to \(f_-\)), the distribution function proposed by Toomre (1982) as the starting point of his analysis for the specific case of \(B = 1\) is, in fact, valid for any value of \(B\), and reads

\[
f(\mathcal{E}, \mathcal{L}_z) = f_0 \mathcal{L}_z^{4-2} \exp \left(-\mathcal{E}/c_s^2 \right),
\]

(29)

where \(f_0\) is a normalization constant. We can now write the phase-space distribution function of a steady state, self-gravitating system of stars, \(f(x, v)\), in an explicit form as follows:

\[
f(r, \theta, v) = f_0 \rho(r, \theta) \exp \left(-\frac{v_\phi^2 + v_\theta^2 + v_r^2}{2c_s^2} + \frac{v_\phi^2}{c_s^2} \ln |v_\phi| \right).
\]

(30)

\(^5\) According to Jeans’s theorem, the distribution function of a steady state, self-gravitating system may be presumed to be a function of at most three isolating integrals. Usually, two of the integrals are the energy and axial angular momentum, while, in general, there is no simple analytical form for the third integral (Binney & Tremaine 1987).

\(^6\) More precisely, it depends on \(B\) via \(\phi(r, \theta)\) entering the definition of \(\mathcal{E}\).
Here $c_s^2 \equiv \bar{v}^2 \bar{v}^2$ is the velocity dispersion of stars, and $|v_{r0}|$ is their mean circular velocity. We again notice a remarkable fact: the parameter $B$ defines the shape of the gravitational potential but does not affect the velocity distribution of stars in this potential. The velocity part of the distribution function coincides with that from Toomre (1982). For a sphere, $A = 2$, it also reproduces the isothermal, isotropic distribution of Evans (1994).

6. DISCUSSION

In this paper, we analytically calculated all possible self-similar, axisymmetric equilibrium states of a self-gravitating, isothermal gas with rotation. We showed that there are two distinct classes of hydrostatic equilibria, namely, cylindrically symmetric equilibria and axially symmetric equilibria. Among the axially symmetric solutions, we found equilibrium states which are asymmetric with respect to the equatorial plane (Figs. 1–4). Such states satisfy Poisson’s equation but are not force-free at the singularities, $r = 0, \infty$.

Are the asymmetric configurations likely to be realized in nature? Since real systems are finite, we should truncate our solutions at some $\rho_i$ and throw away the “outside” mass. For a truncated solution to be valid, the gravitational potential of the discarded “outside” mass has to be replaced by a suitable external potential. As equation (22) shows, the potential must have a linear gradient in $z$ (for a nonrotating solution) of a magnitude determined by $\rho_i$ and $B$. Such a potential may be produced, for instance, by an external object located on the $z$-axis at a distance large compared to the size of the system. Second, to keep the system at rest, another force of equal magnitude and opposite sign must act on the central core. Also, since the force must act on the core alone, it must be of non-gravitational origin. One could imagine, for example, that the core is ionized (by radiation of a central source, for instance), while the material outside the core is neutral. Then, if an external magnetic field threads the ionized core, one could imagine the field pinning the core but having no influence on the neutral outer material. The distortion from equatorial symmetry would then be determined by the strength of the gravitational attraction of the external object and the counterbalancing force from the tension and curvature of the field lines. Although this construction is rather artificial, it demonstrates that asymmetric, non-force-free equilibria may, at least in principle, exist in nature. If they do, and if the systems are isothermal, they will resemble some of the $B \neq 1$ solutions derived in this paper.

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APPENDIX

GRAVITATIONAL POTENTIAL OF A TRUNCATED, NONROTATING SYSTEM

Here we analytically calculate the gravitational potential of the nonrotating, $A = 2$, system truncated at an equidensity (and, hence, equipotential) surface with density $\rho = \rho_i$. Note that, since we cut the system at an equipotential surface, the potential produced by the “inside” mass in the outer space and the potential inside the cavity due to all “outside” mass are of equal magnitude and add up to a constant. The direct evaluation of the gravitational potential via volume integration involves elliptic integrals and turns out to be cumbersome. We use a different approach.

Let us assume, without loss of generality, that $B > 1$. Then, the ellipsoids are shifted upwards, as follows from equation (20) and as plotted in Figure 5. We label each ellipsoid with the quantity

$$\Delta = a \epsilon = \Delta_0 / \sqrt{\rho}$$

where $a$ is the major axis, $\epsilon$ is the eccentricity, and $\Delta_0 = -\sqrt{c_s^2 / 8\pi G (B^{1/2} - B^{-1/2})}$. We now consider two such confocal ellipsoids; quantities referred to the larger ellipsoid are denoted by “primes,” as shown in Figure 5. Let us now fill the space between the two ellipsoids with matter of homogeneous density $\rho$. The gravitational potential in the empty space inside the smaller ellipsoid is equal to the potential inside the large homogeneous ellipsoid, $\Phi_{in}(x)$, less the potential inside the small one, $\Phi_{in}(x)$, having the same density. The gravitational potential in the interior of a homogeneous prolate ellipsoid centered at the origin is known (Binney & Tremaine 1987):

$$\Phi_{in}(x) = -\pi G \rho (b^2 - \sum_{i=1}^{3} A_i x_i^2)$$

where $b = a(1 - \epsilon^2)^{1/2}$ is the minor axis, $x = (x, y, z)$, and

$$I = \sqrt{1 - \epsilon^2} \ln \left( \frac{1 + \epsilon}{1 - \epsilon} \right), \quad A_1 = A_2 = \frac{1 - \epsilon^2}{\epsilon^2} \left[ \frac{1}{1 - \epsilon} - \frac{1}{2 \epsilon} \ln \left( \frac{1 + \epsilon}{1 - \epsilon} \right) \right], \quad A_3 = 2 \frac{1 - \epsilon^2}{\epsilon^2} \left[ \frac{1}{2 \epsilon} \ln \left( \frac{1 + \epsilon}{1 - \epsilon} \right) - 1 \right].$$

Taking into account that our coordinate system is shifted through $\Delta < 0$, we write $\sum_i A_i x_i^2 = A_1 R^2 + A_3 (z + \Delta)^2$. The potential inside the shell bounded by the surfaces $\Delta'$ and $\Delta$ is, thus,

$$\Phi_{inside} = \Phi_{in} - \Phi_{int} = -\pi G \rho \left[ I \left( \frac{1 - \epsilon^2}{\epsilon^2} - A_3 \right) \left( \Delta'^2 - \Delta^2 \right) - 2 A_3 (z + \Delta') \right].$$
Making $\Delta'$ infinitesimally close to $\Delta$, we obtain the contribution to the potential due to the infinitely thin shell of "thickness" $\delta \Delta = \Delta' - \Delta$ and density $\rho = (\Delta_0/\Delta)^2$: \[
\delta \Phi_{\text{inside}} = -2\pi G \left( \frac{\Delta_0}{\Delta} \right)^2 \left[ \left( \frac{1 - \epsilon^2}{\epsilon^2} - A_3 \right) \Delta \delta \Delta - A_3 \Delta' \delta \Delta \right]. \quad (A5)
\]
The gravitational potential inside the truncated system extending from the cutoff surface $\Delta = \Delta_0/\sqrt{\rho_0}$ to infinity is simply the integral over all thin shells with $\Delta > \Delta_c$, i.e., $\Phi_{\text{cav}} = \int \delta \Phi_{\text{inside}}$. The first term in equation (A5) evaluates to a constant, which may be absorbed into $\Phi_0$, the zero level of the potential. The second term yields the coordinate-dependent contribution we are looking for, \[
\Phi_{\text{cav}}(x) = -z \sqrt{\frac{\pi G c_s^2}{2}} \left( B^{1/2} - B^{-1/2} \right) A_3(\epsilon) \sqrt{\rho_c}, \quad (A6)
\]
where $\epsilon = (1 - B)/(1 + B)$ and $A_3(\epsilon) = A_3(-\epsilon)$ is given by equation (A3). We emphasize that the potential is linear in the vertical coordinate, $z$, which corresponds to a homogeneous gravitational field. The strength of this field is determined by $B$ and the cutoff density, $\rho_c$, only. The gravitational acceleration, $g = -\nabla \Phi_{\text{cav}}$, is upward for $B > 1$ and downward for $B < 1$.

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