RANDOM WALKS THAT AVOID THEIR PAST CONVEX HULL

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Submitted December 9, 2001, accepted in final form January 29, 2003

AMS 2000 Subject classification: 60K35. Secondary: 60G50, 52A22, 91B28
Keywords: self-avoiding walk, random walk, convex sets

Abstract
We explore planar random walk conditioned to avoid its past convex hull. We prove that it escapes at a positive lim sup speed. Experimental results show that fluctuations from a limiting direction are on the order of $n^{3/4}$. This behavior is also observed for the extremal investor, a natural financial model related to the planar walk.

Figure 1: 300 steps of the rancher
1 Introduction

Consider the following walk in \( \mathbb{R}^d \). Let \( x_0 = 0 \), and given the past \( x_0, \ldots, x_n \) let \( x_{n+1} \) to be uniformly distributed on the sphere of radius 1 around \( x_n \) but conditioned so that the step segment \( x_n x_{n+1} \) does not intersect the interior of the convex hull of \( \{x_0, \ldots, x_n\} \). We will call this process the rancher’s walk or simply the rancher. The name comes form the planar case: a frontier rancher who is walking about and at each step increases his ranch by dragging with him the fence that defines it, so that the ranch at any time is the convex hull of the path traced until that time. This paper studies the planar case of the process.

Since the model provides some sort of “repulsion” of the rancher from his past, it can be expected that the rancher will escape faster than a regular random walk. In fact, he has positive lim sup speed.

Theorem 1 There exists a constant \( s > 0 \) such that \( \lim \sup \|x_n\|/n \geq s \) a.s.

Simulations suggest that in fact \( \|x_n\|/n \) converges a.s. to some fixed \( s \approx 0.314 \).

In Section 2 we discuss simulations of the model. In particular, we consider how far is the ranch after \( n \) steps from a straight line segment. The experiments suggest that the farthest point in the path from the line \( ox_n \) connecting its end-points is at a distance of order \( n^{3/4} \).

In Section 3 we discuss a related one-dimensional model that we call the extremal investor. This models the fluctuations in a stock’s price, when it is subject to market forces that depend on the stock’s best and worst past performance in a certain simple way. As a result, the relation between the stock’s history and its drift is similar to the same relation for the rancher. Simulations for the critical case of this process yield the same exponent \( 3/4 \), distinguishing it from one-dimensional Brownian motion where the exponent equals 1/2.

The rancher’s walk falls into the large category of self-interacting random walks, such as reinforced, self-avoiding, or self-repelling walks. These models are difficult to analyze in general. The reader should consult [1], [2], [6], [4], and especially the survey papers [5], [3] for examples.

2 Simulations: scaling limit and the exponent 3/4

Unlike the self-avoiding walk, the rancher is not difficult to simulate in nearly linear time. At any given time we only need to keep track of the convex hull of the random walk’s trace so far. If the points on the boundary of the convex hull are kept in cyclic order, updating the convex hull is a matter of finding the largest and smallest elements in a cyclic array, which is monotone on each of the two arcs connecting the extreme values.

With at most \( n \) point on the hull, one can update it in order \( \log n \) time, giving a running time of order \( n \log n \) for \( n \) steps of the walk. In fact, the number of points defining the convex hull is much smaller than \( n \), and the extremal elements tend to be very close in the array to the previous point of the walk. This and the actual running times suggest that the theoretical running time is close to linear.

Figure 2: A million step sample of the rancher
In our simulations $\|x_n\|/n$ appears to converge to some fixed $s \approx 0.314$. Assuming this is the case, the rancher’s walk is similar to the random walk in the plane conditioned to always increase its distance from the origin. Since the distance is linear in $n$ and the step size is fixed, the angular change is of order $n^{-1}$. If the signs of the angular change were independent this would imply the following.

**Conjecture 1 (Angular convergence)** The process $x_n/\|x_n\|$ converges a.s.

The difficulty in our case is that the angular movements are positively correlated: if a step has a positive angular component, then subsequent steps have a drift in the same direction. Our simulations suggest that these correlations are not strong enough to prevent angular convergence, and we conjecture that this is in fact the case. This is observed in Figure 2, showing a million-step sample of the rancher’s walk. Still larger simulations yield a picture indistinguishable from a straight line segment.

The scaled path of the rancher’s walk appears to converge to a straight line segment, and it is natural to ask how quickly this happens. If we assume positive speed and angular convergence, then each step has a component in the eventual (say horizontal) direction and a component in the perpendicular (vertical) direction.

If the vertical components were independent, the vertical movement would essentially be a simple one-dimensional random walk. Since the horizontal component increase linearly, the path is then roughly the graph of a one-dimensional random walk.

To test this, we measured a related quantity, the width $w_n$ of the path at time $n$, defined as the distance of the farthest point on the path from the line $ox$. Under the above assumptions, one would guess that $w_n$ should behave as the maximum up to time $n$ of the absolute value of a one-dimensional random walk with bounded steps, and have a typical value of order $n^{1/2}$.

Our simulations, however, show an entirely different picture. Figure 3 is a log base 10 plot of 500 realizations of $w_n$ on independent processes. $n$ ranges from a thousand to a million steps equally spaced on the log scale. The slope of the regression line is 0.746 (SE 0.008). A regression line on the medians of 1000 measurements of walks of length $10^3, 10^4, 10^5, 10^6$ gave a value of .75002 (SE 0.002). Based on these simulations, we conjecture that $w_n$ behaves like $n^{3/4}$. To put it rigorously in a weak form:

![Figure 3: The dimensions of the ranch](image-url)
Conjecture 2 (The exponent 3/4) For every $\varepsilon > 0$, as $n \to \infty$ we have
\[ P \left[ n^{3/4-\varepsilon} < w_n < n^{3/4+\varepsilon} \right] \to 1. \]

It is also feasible that if the path is scaled by a factor of $n^{3/4}$ in the vertical axis and by $n$ in the horizontal axis (parallel to the segment $ox_n$) then the law of the path would converge to some random function. The result of such asymmetric scaling is seen in Figure 4. In the next section we introduce a model that appears closely related.

3 The extremal investor

Stock or portfolio prices are often modeled by exponentiated random walk or Brownian motion. In the simplest discrete-time model, the log stock price, denoted $x_n$, changes every time by an independent standard Gaussian random variable.

One's decision whether to invest in, say, a mutual fund is often based on past performance of the fund. Mutual fund companies report past performance for periods ending at present; the periods are often hand-picked to show the best possible performance. The simplest such statistic is the overall best performance over periods ending in the present. In terms of log interest rate it is given by
\[ r_{\text{max}}^n = \max_{m<n} \frac{x_n - x_m}{n - m}, \]
that is the maximal slope of lines intersecting the graph of $x_n$ in both a past point and the present point.

A more cautious investor also looks at the worst performance $r_{\text{min}}^n$, given by (1) with a min, and makes a decision to buy, sell or hold accordingly, influencing the fund price. In the simplest model, which we call the extremal investor model, the change in the log fund price given the present is simply a Gaussian with standard deviation 1 and expected value given by a fixed influence parameter $\alpha$ times the average of $r_{\text{max}}^n$ and $r_{\text{min}}^n$:
\[ x_{n+1} = x_n + \alpha \frac{r_{\text{max}}^n + r_{\text{min}}^n}{2} + \text{standard Gaussian}. \]

This process is related to the rancher in two dimensions, since the future behavior of $x_n$ is influenced through the shape of the convex hull of the graph of $x_n$ at the tip. For $\alpha = 1$ the drift of the rancher starting with the convex hull has the same direction as the expected next step for the stock value.
Figure 5: The extremal investor process for $\alpha = 1$ and its convex hull

Let $w_n$ denote the greatest distance between $x_n$ and the linear interpolation from time zero to the present (assume $x_0 = 0$):

$$w_n = \max_{m \leq n} \left| x_m - \frac{m}{n} \right|.$$

The following analogue of Conjecture 2 is consistent with our simulations:

**Conjecture 3 (The exponent 3/4 for the extremal investor)** Let $\alpha = 1$. For every $\varepsilon > 0$ as $n \to \infty$ we have

$$P\left[n^{3/4-\varepsilon} < w_n < n^{3/4+\varepsilon}\right] \to 1.$$

A moment of thought shows that for $\alpha > 1$, $x_n$ will blow up exponentially, so $\alpha_c = 1$ is the critical parameter. For $\alpha < 1$ the behavior of $w_n$ seems to be governed by an exponent between $1/2$ and $3/4$ depending on $\alpha$. For $\alpha < 1$ the $x_n/n$ seems to converge to 0, but in the case that $\alpha = 1$, it appears that $x_n/n$ converges a.s. to a random limit.

## 4 Proof of Theorem 1

Denote $\{x_n; n \geq 0\}$ the rancher’s walk. Define the **ranch** $R_n$ as the convex hull of $\{x_0, \ldots, x_n\}$. Since $x_n$ is always on the boundary and $R_n$ is convex, the angle of $R_n$ at $x_n$ is always in $[0, \pi]$. Denote this angle by $\gamma_n$ (as in Figure 6).

The idea of the proof is to find a set of times of positive upper density in which the expected gain in distance is bounded away from 0. There are two cases where the expected gain in distance can be small. First, if $\gamma_n$ is close to 0, the distribution of the next step is close to uniform on the unit circle. Second, when $\gamma_n$ is close to $\pi$, the next step is uniformly distributed on roughly a semicircle. If in addition the direction to the origin is near one of the end-points of the semicircle then the expected gain in distance is small.

We now introduce further notation used in the proof. Set $s_n = \|x_{n+1}\| - \|x_n\|$. Note that since the direction of the $n$th step is uniformly distributed on an arc not containing the direction of origin, $E s_n \geq 0$. For three points $x, y, z$, let $xyz$ denote the angle in the range $(0, 2\pi]$. The
angle $\alpha_n x_{n+1} - \pi$ is denoted by $\beta_n$, so that $\beta_n \in [-\pi, \pi)$. Thus $\beta = 0$ means that the walker moved directly away from $o$, $\beta > 0$ means that the walker moved counterclockwise.

Let $C$ be the boundary of the smallest closed disk centered at $o$ containing the ranch $R_n$. Consider the half-line starting from $x_n$ that contains the edge of $R_n$ incident to and clockwise from $x_n$. Let $y_n$ denote the intersection of this half line and $C$. Let $\alpha_n$ denote the angle $\pi - \alpha_n y_n$, and let $\alpha'_n$ denote the analogous angle in the counterclockwise direction. Let $d_n$ be the distance between $C$ and $x_n$. It follows from these definitions that $\alpha_n + \alpha_n' + \gamma_n = 2\pi$ and that $\beta_n$ has uniform distribution on $[-\alpha_n, \alpha'_n]$.

**Proof of Theorem 1.** We find a set of times of positive upper density in which $E_{s_n}$ is positive and bounded away from 0. If $\gamma_n \in [\varepsilon, \pi - \varepsilon]$, then $E_{s_n}$ is bounded from below by some function of $\varepsilon$. Thus we need only consider the times when $\gamma_n < \varepsilon$ or when $\gamma_n > \pi - \varepsilon$, where $\varepsilon > 0$ will be chosen later to satisfy further constraints.

In the case $\gamma_n < \varepsilon$, the rancher is at the tip of a thin ranch, so a single step can make a large change in $\gamma$, thus we look at two consecutive steps. With probability at least a quarter $\beta_n \in [\pi/4, 3\pi/4]$. In that case $\gamma_{n+1}$ is bounded away from both 0 and $\pi$, and then $E_{s_{n+1}}$ is bounded away from 0. If $\beta_n$ is not in $[\pi/4, 3\pi/4]$, we use the bound $E_{s_{n+1}} > 0$. Combining these gives a uniform positive lower bound on $E_{s_{n+1}}$.

If $\gamma_n$ is close to $\pi$, then we are in a tighter spot: it could stay large for several steps, and $E_{s_n}$ may remain small. The rest of the proof consists of showing that at a positive fraction of time the angle $\gamma_n$ is not close to $\pi$.

If $d_n < D$, where $D$ is some large bound to be determined later, then with probability at least half $\gamma_{n+1} < \pi - 1/(2D)$, thus if we take $\varepsilon \leq 1/(2D)$, it suffices to show that a.s. the Markov process $\{(R_n, x_n)\}$ returns to the set $A = \{(R, x)|d < D\}$ at a set of times with positive upper density.

To show this, we use a martingale argument; it suffices to exhibit a function $f(R_n, x_n)$ bounded from below, so that the expected increase in $f$ given the present is negative and bounded away from zero when $(R_n, x_n) \notin A$, and is bounded from above when $(R_n, x_n) \in A$. The sufficiency of the above is proved in Lemma 1 below; there take $A_n$ to be the event $(R_n, x_n) \in A$, and $X_n = ||x_n||$. We now proceed to construct a function $f$ with the above properties.
The standard function that has this property is the expected hitting time of $A$. We will try to guess this. The motivation for our guess is the following heuristic picture. When the angle $\alpha$ is small, it has a tendency to increase by a quantity of order roughly $1/d$, and $d$ tends to decrease by a quantity of order $\alpha$. This means that $d$ performs a random walk with downward drift at least $1/d$, but this is not enough for positive recurrence. So we have to wait for a few steps for $\alpha$ to increase enough to provide sufficient drift for $d$; the catch is that in every step $\alpha$ has a chance of order $\alpha$ to decrease, and the same order of chance to decrease to a fraction of its size. So $\alpha$ tends to grow steadily and collapse suddenly. If the typical size is $\alpha_*$, then it takes order $1/\alpha_*$ time to collapse. During this time it grows by about $1/(d\alpha_*)$, which should be on the order of the typical size $\alpha_*$, giving $\alpha_* = d^{-1/2}$. This suggests that the process $d$ has drift of this order, so the expected hitting time of 0 is of order $d^{3/2}$. A more accurate guess takes into account the fact that if $\alpha$ is large, the hitting time is smaller.

Define the functions $f_1(d) = d^{3/2}$, and $f_2(d, \alpha) = -((cd^{1/2}) \wedge (\alpha d))$, where $c$ is a constant to be chosen later. Define $f(d, \alpha, \alpha') = f_1(d) + f_2(d, \alpha) + f_2(d, \alpha')$. Since $A$ is defined by some bound on $d$ it is clear that if $(R_n, x_n) \in A$, then $f(d_n, \alpha_n, \alpha'_n)$ can only increase by a bounded amount (this is true for each of the terms). Since $\alpha, \alpha' \leq \pi$, $f$ is bounded from below. To conclude the proof we need to show that if $(R_n, x_n) \notin A$, then the expected change in $f(d_n, \alpha_n, \alpha'_n)$ is negative and bounded away from zero.

First we consider the expected change in $f_1$. All expected values will be conditional on the information available at time $n$. To simplify notation, assume $x_n$ is on the positive $X$-axis. This may be done since the process is invariant to rotations of the plane. We first bound the expected decrease $d_n - d_{n+1}$.

$$-E \Delta d = E\|x_{n+1}\| - \|x_n\| \geq E x_{n+1,1} - x_{n,1}$$  \hspace{1cm} (2)

Recall that $\beta_n = \alpha_n x_{n+1} - \pi$ is uniformly distributed in $[-\alpha_n, \alpha'_n]$. Thus the RHS is simply

$$\frac{1}{\alpha_n + \alpha'_n} \int_{-\alpha_n}^{\alpha'_n} \cos \beta \, d\beta \geq \frac{\sin \alpha_n + \sin \alpha'_n}{2\pi}$$

Since $|\Delta d| \leq 1$ and outside $A$ we have $d > D$, we have

$$(d + \Delta d)^{3/2} = d^{3/2} + \frac{3}{2} d^{1/2} \Delta d + O(d^{-1/2})$$

We can therefore bound

$$E \Delta f_1 \leq -\frac{3}{4\pi} (\sin \alpha_n + \sin \alpha'_n) d^{1/2} + o(1),$$  \hspace{1cm} (3)

where here (and later) $o(1)$ denotes a quantity that is arbitrarily small if $D$ is taken sufficiently large ($D$ will be fixed later so that these terms is small enough).

We now proceed to bound the expected change in $f_2(d_n, \alpha_n)$; denote this change by $\Delta f_2$. We break up $\Delta f_2$ into important and unimportant parts:

$$\Delta f_2 = (cd^{1/2}_n \wedge \alpha_n d_n - cd^{1/2}_n \wedge \alpha_{n+1} d_n) + (cd^{1/2}_n \wedge \alpha_{n+1} d_n - cd^{1/2}_{n+1} \wedge \alpha_{n+1} d_n) + (cd^{1/2}_{n+1} \wedge \alpha_{n+1} d_n - cd^{1/2}_{n+1} \wedge \alpha_{n+1} d_{n+1})$$.
The second term is bounded above by $cd_{n+1}^{1/2} - d_{n+1}^{1/2} = o(1)$. The third term is non-positive unless $cd_{n+1}^{1/2} > \alpha_{n+1} d_n$, implying that $\alpha_{n+1} < \frac{1}{2}d_{n+1}$, and then this term is at most $\alpha_{n+1}|\Delta d| = o(1)$. Thus important increase can only come from the first term. We therefore denote

$$z = (cd_{n+1}^{1/2} + \alpha_{n}d_n - cd_{n}^{1/2} - \alpha_{n+1}d_n),$$

and consider three cases given by the following events, which depend on the value of $\beta = \beta_n$:

$$B_1 = \{ \beta \in (-\alpha_n, 0) \}, \quad B_2 = \{ \beta \in [0, \pi - \alpha_n] \}, \quad B_3 = \{ \beta \in (\pi - \alpha_n, \alpha_n'] \}.$$

As we will see in detail, the contribution of the first and last cases is small. If $\alpha_n$ is small enough then the second also has a small contribution, while if $\alpha_n$ is large the negative expected change of $f_1$ offsets any positive change in $f_2$.

**Event $B_1$:** $\beta \in [0, \pi - \alpha_n]$ (equivalently, $x_{n+1}$ is on the side opposite of $R_n$ for the lines $ox_n$ and $x_ny_n$). In this case the rancher moves sufficiently away from the ranch, so that $\alpha$ increases:

$$\Delta \alpha = \frac{\alpha x_n y_n - \alpha x_{n+1} y_{n+1}}{\alpha x_n y_n - \alpha x_{n+1} y_n} \geq \frac{\alpha x_n y_n - \alpha x_{n+1} y_n}{\alpha x_n y_n - \alpha x_{n+1} y_n} = x_n \alpha x_{n+1} + x_{n+1} y_n x_n \geq x_n + y_n x_n \geq 0. \quad (4)$$

All inequalities follow from our assumption $B_2$. The equality follows from the fact that the angles in the quadrangle $x_n y_n x_{n+1}$ add up to $2\pi$.

We now compute the last angle in (4) using a simple identity in the triangle $x_n y_n x_{n+1}$, and the value of the angle $y_n x_n x_{n+1}$:

$$\|x_{n+1} - y_n\| \sin(x_{n+1} y_n x_n) = \|x_n - x_{n+1}\| \sin(y_n x_n x_{n+1}) = \sin(\beta_n + \alpha_n). \quad (5)$$

A byproduct of (4) is that $B_2$ implies $z \leq 0$. If $\alpha_n < cd_n^{-1/2}$, then a better bound is possible:

$$\|x_{n+1} - y_n\| \leq 1 + \|x_n - y_n\| \leq 1 + \|x_n - p\| = 1 + d_n (\cos \alpha_n)^{-1} = d_n (1 + o(1)),$$

where the point $p$ is the intersection of the line $x_n y_n$ and the tangent to $C$ perpendicular to the ray $ox_n$. We can then conclude from (4) and (5) that

$$\Delta \alpha \geq x_n \alpha x_{n+1} x_n \geq \frac{\sin(\beta_n + \alpha_n)}{d_n} (1 - o(1)). \quad (6)$$

The criterion $\alpha_n < cd_n^{-1/2}$ guarantees that the cutoff at $cd^{-1/2}$ does not apply, and so (6) implies $z \leq -\sin(\beta + \alpha_n)(1 - o(1))$. Therefore

$$\mathbb{E}[z; \ B_2] \leq -(\alpha_n + \alpha_n')^{-1} \int_0^{\pi - \alpha_n} \sin(\beta - \alpha_n) d\beta + o(1) \leq -2\pi^{-1} + o(1),$$

since we assumed $\alpha_n < cd_n^{-1/2}/2$. For larger $\alpha_n$ we only need $z \leq 0$.

**Event $B_3$:** $\beta_n > \pi - \alpha_n$. If $\alpha_{n+1} \geq cd_n^{-1/2}$, then $z \leq 0$ because of the cutoff at $cd^{1/2}$. However, $R_{n+1}$ has an edge $x_{n+1} x_n$, and clearly $\alpha_{n+1} > \pi - \beta$. Thus the probability of $B_3$ and $\{z > 0\}$ is at most

$$\mathbb{P}[0 < z \text{ and } B_3] \leq \mathbb{P}[0 < \pi - \beta < cd_n^{-1/2}] \leq cd_n^{-1/2} \pi^{-1}.$$

Since $z \leq cd^{1/2}$ always holds, this gives

$$\mathbb{E}[z; \ B_3] \leq \frac{c^2}{\pi}.$$
Event $B_1$: $\beta < 0$. We can bound $\alpha_{n+1}$ below by $\beta + \alpha_n$ as follows. First, note that $\alpha_{n+1} = \pi - ox_{n+1} y_{n+1} \geq \pi - ox_{n+1} y_n$. Also $\beta + \alpha_n = y_n x_n x_{n+1} = \pi - x_n x_{n+1} y_n - x_{n+1} y_n x_n$, since the angles of a triangle add to $\pi$. We can split $x_n x_{n+1} y_n = x_n x_{n+1} o + ox_{n+1} y_n$. Putting these together we get $\alpha_{n+1} = \beta + \alpha_n + x_n x_{n+1} + x_{n+1} y_n x_n$, and since the latter two angles are small and positive, $\alpha_{n+1} > \beta + \alpha_n$. Therefore

$$
P[0 < z \text{ and } B_1] \leq P[\beta + \alpha_n < cd_n^{-1/2}] \leq cd_n^{-1/2} \pi^{-1},$$

and as in case $B_3$:

$$
E[z; B_1] \leq c^2 / \pi.
$$

Summarizing the cases $B_1$, $B_2$, $B_3$ we get the bound

$$
E \Delta f_2 (d, \alpha) \leq 2c^2 \pi^{-1} + o(1),
$$

and if $\alpha < cd^{-1/2}/2$, then

$$
E \Delta f_2 (d, \alpha) \leq 2c^2 \pi^{-1} - 2\pi^{-1} + o(1).
$$

Of course, the same bounds hold for $f_2 (d, \alpha')$.

We now summarize our estimates on all the components of $\Delta f$.

- If $\alpha_n < cd_n^{-1/2}/2$, then
  $$
  E \Delta f = E \Delta f_1 + E \Delta f_2 (d, \alpha) + E \Delta f_2 (d, \alpha')
  \leq 0 + (2c^2 - 2) \pi^{-1} + 2c^2 \pi^{-1} + o(1) = \frac{4c^2 - 2}{\pi} + o(1),
  $$
  which is negative for large $D$ if $c < 2^{-1/2}$. The same bound holds if $\alpha'_n < cd_n^{-1/2}/2$.

- If at least one of $\alpha_n, \alpha'_n$ is in $[cd_n^{-1/2}/2, \pi - cd_n^{-1/2}/2]$, then using (3), we get
  $$
  E \Delta f \leq -3/8 \pi^{-1} c + o(1) + 4c^2 \pi^{-1} + o(1) = \frac{4c^2 - 3c/8}{\pi} + o(1),
  $$
  which is negative for large $D$ if $c < 3/32$.

- If $\alpha_n, \alpha'_n > \pi - cd_n^{-1/2}/2$, and $d_n > D$ is large enough, then we have seen that the two step drift $E d_{n+2} - d_n < -c_1$ for some $c_1 > 0$. Thus in this case $E \Delta f_1 \leq -c_1 d_n^{1/2} + o(1)$, while the drift of $f_2$ is uniformly bounded.

Putting the three cases together shows that if we take $0 < c < 3/32$ then for large enough $D$ the function $f$ satisfies the requirements of Lemma 1. \hfill \Box

For the following technical lemma, we use the notation $\Delta_m a_n = a_{n+m} - a_n$, and $\Delta a_n = \Delta_1 a_n$.

**Lemma 1** Let $\{ (X_n, f_n, A_n) \}$ be a sequence of triples adapted to the increasing filtration $\{ F_n \}$ (with $F_0$ trivial) so that $X_n, f_n$ are random variables and $A_n$ are events satisfying the following:

...
There exist positive constants $c_1, c_2, c_3, c_4$, and a positive integer $m$, so we have a.s. for all $n$

\[ |\Delta X_n| \leq 1, \]
\[ \mathbb{E}[\Delta X_n \mid \mathcal{F}_n] \geq 0, \]  
\[ \mathbb{E}[\Delta_m X_n \mid \mathcal{F}_n, A_n] > c_1, \]  
\[ f_n > -c_2, \]  
\[ \Delta f_n 1(A_n) < c_3, \]  
\[ \mathbb{E}[\Delta f_n \mid \mathcal{F}_n, A_n^n] < -c_4. \]

Then for some positive constant $c_5$ we have

\[ \limsup X_n/n > c_5 \text{ a.s.} \]  

**Proof.** Let $G_n = \sum_{i=0}^{n-1} 1_{A_i}$, and let $G_{n,k} = \sum_{i=0}^{n-1} 1_{A_{m+k}}$, $0 \leq k < n$. First we show that the $m + 1$ processes

\[ \{c_1 G_{n,k} - X_{mn+k}\}_{n \geq 0}, \quad 0 \leq k < m, \]
\[ \{f_n - c_3 G_n + c_4 (n - G_n)\}_{n \geq 0} \]

are supermartingales adapted to $\{\mathcal{F}_{mn+k}\}_{n \geq 0}$, $0 \leq k < m$, $\{\mathcal{F}_n\}_{n \geq 0}$, respectively. For the first $m$ processes fix $k$, and note that $\mathbb{E}[c_1 (G_{n+1,k} - G_{n,k}) | \mathcal{F}_{mn+k}] = c_1 1(A_{mn+k})$. Consider

\[ \mathbb{E}[c_1 (G_{n+1,k} - G_{n,k}) | \mathcal{F}_{mn+k}] + \mathbb{E}[-(X_{mn+k} + 1 - X_{mn+k}) | \mathcal{F}_{mn+k}] \]

If $A_{mn+k}$ happens, then the first term equals $c_1$, and the second is less than $-c_1$ by (8). If $A_{mn+k}$ does not happen, then the first term equals 0 and the second is non-positive by (7). Putting these two together shows that (12) are supermartingales. For the last process, consider

\[ \mathbb{E}[\Delta f_n | \mathcal{F}_n] + \mathbb{E}[-c_3 G_n | \mathcal{F}_n] + \mathbb{E}[c_4 (1 - \Delta G_n) | \mathcal{F}_n]. \]

If $A_n$ happens, then the first term is less than $c_3$ by (9), the second term equals $-c_3$, and the last equals 0. If $A_n$ does not happen, then the first term is less than $-c_4$ by (10), the second term equals 0, and the third equals $c_4$. In both cases we get that the process (13) is a supermartingale.

It follows from the supermartingale property that for some $c > 0$ and all $n \geq 0$ we have

\[ \mathbb{E} X_{mn+k} \geq c_1 \mathbb{E} G_{n,k} - c, \quad 0 \leq k < m, \]
\[ \mathbb{E} G_n \geq c_4 (c_3 + c_4) n - c. \]

Since $G_{mn} = G_{n,0} + \ldots + G_{n,m-1}$, it follows from (15) that for some $c_6 > 0$ and all large $n$ there is $k = k(n)$, so that $\mathbb{E} G_{n,k} > c_6 n$. Then for some $c_7 > 0$ we have $\mathbb{E} X_{mn+k} > c_7 n$ by (14). As a consequence, for $Y_n = \max\{X_{mn}, \ldots, X_{mn+m-1}\}$ we have $\mathbb{E} Y_n > c_7 n$.

Thus for some $c_8 < 1$ we have $\mathbb{E}(1 - Y_n/(mn)) < c_8$ for all large $n$. Since $X_n \leq X_0 + n$, we have $Y_n \leq X_0 + mn + m - 1 = mn + c_9$ and therefore $1 - (Y_n - c_9)/(mn) \geq 0$. Fatou’s lemma then implies

\[ \mathbb{E} \liminf (1 - Y_n/(mn)) = \mathbb{E} \liminf (1 - (Y_n - c_9)/(mn)) \leq c_8, \]

for some $c_10 \in (c_8, 1)$ Markov’s inequality gives $\mathbb{P}(\limsup (1 - Y_n/(mn)) < c_8/c_10) > 1 - c_10$. So for some $c_5 > 0$,

\[ \mathbb{P}(\limsup X_n/n > c_5) > 1 - c_10. \]
but we can repeat this argument while conditioning on the \( \sigma \)-field \( \mathcal{F}_t \) to get

\[
P(\limsup_{n} X_n/n > c_5 \mid \mathcal{F}_t) > 1 - c_{10}
\]

so letting \( t \to \infty \) by Lévy’s 0-1 law we get (11).

\[\square\]

5 Further open questions and conjectures

**Conjecture 4** Theorem 1 holds with \( \liminf \) instead of \( \limsup \).

**Conjecture 5** The speed \( \lim \|x_n\|/n \) exists and is constant a.s. This could follow from some super-linearity result on the rancher’s travels.

**Question 6** What is the scaling limit of the asymmetrically normalized path?

**Question 7** What is the behavior in higher dimensions? Is the \( \limsup \) (or even \( \liminf \)) speed still positive? If not, is \( \|x_n\| = O(\sqrt{n}) \) or is it significantly faster than a simple random walk? What about convergence of direction?

**Question 8** If longer step sizes are allowed what happens when the tail is thickened? Are there distributions which give positive speed without convergence of direction?

Acknowledgments. The authors thank the anonymous referees for their helpful comments on previous versions of this paper.

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