POLYADIC ANALOG OF GROTHENDIECK GROUP

STEVEN DUPLIJ

Center for Information Technology (WWU IT), Universität Münster, Röntgenstrasse 7-13
D-48149 Münster, Deutschland

ABSTRACT. We generalize the Grothendieck construction of the completion group for a monoid (being the starting point of the algebraic $K$-theory) to the polyadic case, when an initial semigroup is $m$-ary and the corresponding final class group $K_0$ can be $n$-ary. As opposed to the binary case: 1) there can be different polyadic direct products which can be built from one polyadic semigroup; 2) the final arity $n$ of the class groups can be different from the arity $m$ of initial semigroup; 3) commutative initial $m$-ary semigroups can lead to noncommutative class $n$-ary groups; 4) the identity is not necessary for initial $m$-ary semigroup to obtain the class $n$-ary group, which in its turn can contain no identity at all. The presented numerical examples show that the properties of the polyadic completion groups are considerably nontrivial and have more complicated structure than in the binary case.

CONTENTS

1. INTRODUCTION
2. PRELIMINARIES
3. GROTHENDIECK GROUP OF COMMUTATIVE MONOID
4. $n$-ARY GROUP COMPLETION OF $m$-ARY SEMIGROUP
   4.1. Polyadic direct power construction
   4.2. Equivalence relations for $m$-ary semigroups
   4.3. Polyadic group completion
REFERENCES

E-mail address: douplii@uni-muenster.de; sduplij@gmail.com; https://ivv5hpp.uni-muenster.de/u/douplii.
Date: of start February 19, 2022. Date: of completion May 8, 2022. Corrections: June 14, 2022.
Total: 25 references.
2010 Mathematics Subject Classification. 16E20, 16T25, 17A42, 18F30, 19A99, 20B30, 20F36, 20M17, 20N15.
Key words and phrases. K-theory, completion group, class group, direct product, polyadic semigroup, arity, polyadic group.
1. INTRODUCTION

The Grothendieck construction of the completion group for a monoid is a ground and starting point of the algebraic K-theory (see, e.g. [1978], [1994]). Here we generalize this construction to the polyadic case, when an initial monoid (or semigroup) and a final group are m-ary and n-ary, correspondingly, and we denote such a class polyadic group $K_{0}^{(m,n)}$. As concrete examples, we provide several computations for $K_{0}^{(m,n)}$, including the case, when the arities do not coincide $m \neq n$.

2. PRELIMINARIES

We introduce here briefly the usual notation, for details see [2018]. For a non-empty (underlying) set $G$ the n-tuple (or polyad [1940]) of elements is denoted by $(g_{1}, \ldots, g_{n})$, $g_{i} \in G$, $i = 1, \ldots, n$, and the Cartesian product is denoted by $G^{\times n} \equiv G \times \cdots \times G$ and consists of all such n-tuples. For all elements equal to $g \in G$, we denote n-tuple (polyad) by a power $(g)^{n}$. To avoid unneeded indices we denote with one bold letter $p^{g}$ a polyad for which the number of elements in the n-tuple is clear from the context, and sometimes we will write $g^{n}$. On the Cartesian product $G^{\times n}$ we define a polyadic (or n-ary) operation $\mu^{(n)} : G^{\times n} \to G$ such that $\mu^{(n)} (g) \mapsto h$, where $h \in G$. The operations with $n = 1, 2, 3$ are called unary, binary and ternary.

Recall the definitions of some algebraic structures and their special elements (in the notation of [2018]). A (one-set) polyadic algebraic structure $\mathcal{G}$ is a set $G$ closed with respect to polyadic operations. In the case of one n-ary operation $\mu^{(n)} : G^{\times n} \to G$, it is called polyadic multiplication (or n-ary multiplication). A one-set n-ary algebraic structure $\mathcal{M}^{(n)} = \langle G | \mu^{(n)} \rangle$ or polyadic magma (n-ary magma) is a set $G$ closed with respect to one n-ary operation $\mu^{(n)}$ and without any other additional structure. In the binary case $\mathcal{M}^{(2)}$ was also called a groupoid by Hausmann and Ore [1937] (and Clifford and Preston [1961]). Since the term “groupoid” was widely used in category theory for a different construction, the so-called Brandt groupoid [1927], [1966], Bourbaki [1998] later introduced the term “magma”.

Denote the number of iterating multiplications by $\ell_{\mu}$, and call the resulting composition an iterated product $(\mu^{(n)})^{\ell_{\mu}}$, such that

$$\mu^{(n')}(\mu^{(n)})^{\ell_{\mu}} \overset{\text{def}}{=} \mu^{(n)} \circ (\mu^{(n)} \circ \cdots (\mu^{(n)} \times \text{id}^{\times (n-1)}) \cdots \times \text{id}^{\times (n-1)})^{\ell_{\mu}}, \quad (2.1)$$

where the arities are connected by

$$n' = n_{\text{iter}} = \ell_{\mu} (n - 1) + 1, \quad (2.2)$$

which gives the length of a iterated polyad $(g)$ in our notation $(\mu^{(n)})^{\ell_{\mu}} [g]$.
A polyadic zero of a polyadic algebraic structure \( G \mid \mu^{(n)} \) is a distinguished element \( z \in G \) (and the corresponding 0-ary operation \( \mu_z^{(0)} \)) such that for any \((n-1)\)-tuple (polyad) \( g^{(n-1)} \in G^x(n-1) \) we have
\[
\mu^{(n)} [g^{(n-1)}, z] = z,
\]
where \( z \) can be on any place in the l.h.s. of (2.3). If its place is not fixed it can be a single zero. As in the binary case, an analog of positive powers of an element [Post 1940] should coincide with the number of multiplications \( \ell_\mu \) in the iteration (2.1).

A (positive) polyadic power of an element is
\[
g^{(\ell_\mu)} = (\mu^{(n)})^ {\ell_\mu} [g^{\ell_\mu(n-1)+1}].
\]
We define associativity as the invariance of the composition of two \( n \)-ary multiplications. An element of a polyadic algebraic structure \( g \) is called \( \ell_\mu \)-nilpotent (or simply nilpotent for \( \ell_\mu = 1 \)), if there exist \( \ell_\mu \) such that
\[
g^{(\ell_\mu)} = z.
\]
A polyadic \((n\text{-ary})\) identity (or neutral element) of a polyadic algebraic structure is a distinguished element \( e \) (and the corresponding 0-ary operation \( \mu_e^{(0)} \)) such that for any element \( g \in G \) we have
\[
\mu^{(n)} [g, e^{n-1}] = g,
\]
where \( g \) can be on any place in the l.h.s. of (2.6).

In polyadic algebraic structures, there exist neutral polyads \( n \in G^x(n-1) \) satisfying
\[
\mu^{(n)} [g, n] = g,
\]
where \( g \) can be on any of \( n \) places in the l.h.s. of (2.7). Obviously, the sequence of polyadic identities \( e^{n-1} \) is a neutral polyad (2.6).

A one-set polyadic algebraic structure \( G \mid \mu^{(n)} \) is called totally associative, if
\[
(\mu^{(n)})^ {\ell_2} [g, h, u] = \mu^{(n)} [g, \mu^{(n)} [h], u] = \text{invariant},
\]
with respect to placement of the internal multiplication \( \mu^{(n)} [h] \) in r.h.s. on any of \( n \) places, with a fixed order of elements in the any fixed polyad of \( (2n-1) \) elements \( t^{(2n-1)} = (g, h, u) \in G^{x(2n-1)} \).

A polyadic semigroup \( S^{(n)} \) is a one-set \( S \) one-operation \( \mu^{(n)} \) algebraic structure in which the \( n \)-ary multiplication is associative, \( S^{(n)} = \langle S \mid \mu^{(n)} \rangle \text{ associativity (2.8)} \). A polyadic algebraic structure \( G^{(n)} = \langle G \mid \mu^{(n)} \rangle \) is \( \sigma \)-commutative, if \( \mu^{(n)} = \mu^{(n)} \circ \sigma \), or
\[
\mu^{(n)} [g] = \mu^{(n)} [\sigma \circ g], \quad g \in G^{x_n},
\]
where \( \sigma \circ g = (g_{\sigma(1)}, \ldots, g_{\sigma(n)}) \) is a permutated polyad and \( \sigma \) is a fixed element of \( S_n \), the permutation group on \( n \) elements. If (2.3) holds for all \( \sigma \in S_n \), then a polyadic algebraic structure is commutative. A special type of the \( \sigma \)-commutativity
\[
\mu^{(n)} [g, t^{(n-2)}, h] = \mu^{(n)} [h, t^{(n-2)}, g],
\]
where \( t^{(n-2)} \in G^{x(n-2)} \) is any fixed \((n-2)\)-polyad, is called semicommutativity. If an \( n \)-ary semigroup \( S^{(n)} \) is iterated from a commutative binary semigroup with identity, then \( S^{(n)} \) is semicommutative. A polyadic algebraic structure is called (uniquely) \( i \)-solvable, if for all polyads \( t, u \) and element \( h \), one can (uniquely) resolve the equation (with respect to \( h \)) for the fundamental operation
\[
\mu^{(n)} [u, h, t] = g
\]
where \( h \) can be on any place, and \( u, t \) are polyads of the needed length.
A polyadic algebraic structure which is uniquely $i$-solvable for all places $i = 1, \ldots, n$ is called a $n$-ary (or polyadic) quasigroup $Q^{(n)} = \langle Q \mid \mu^{(n)} \rangle$ [solvability]. An associative polyadic quasigroup is called a $n$-ary (or polyadic) group. In an $n$-ary group $G^{(n)} = \langle G \mid \mu^{(n)} \rangle$ the only solution of (2.11) is called a querelement of $g$ and denoted by $\bar{g}$ [DÖRnte 1929], such that

$$\mu^{(n)}[h, \bar{g}] = g, \quad g, \bar{g} \in G,$$

(2.12)

where $\bar{g}$ can be on any place. Any idempotent $g$ coincides with its querelement $\bar{g} = g$. The unique solvability relation (2.12) in a $n$-ary group can be treated as a definition of the unary (multiplicative) quereoperation

$$\bar{\mu}^{(1)}[g] = \bar{g}.$$

(2.13)

We observe from (2.12) and (2.7) that the polyad

$$n_g = (g^{n-2}\bar{g})$$

(2.14)

is neutral for any element of a polyadic group, where $\bar{g}$ can be on any place. If this $i$-th place is important, then we write $n_{g;i}$. In a polyadic group the Dörnte relations [DÖRnte 1929]

$$\mu^{(n)}[g, n_{h;i}] = \mu^{(n)}[n_{h;j}, g] = g$$

(2.15)

hold true for any allowable $i, j$. In the case of a binary group the relations (2.15) become $g \cdot h \cdot h^{-1} = h \cdot h^{-1} \cdot g = g$.

Using the quereoperation (2.13) one can give a diagrammatic definition of a polyadic group [GLEICHgewicht and GLAZEK 1967]: an $n$-ary group is a one-set algebraic structure (universal algebra)

$$G^{(n)} = \langle G \mid \mu^{(n)}, \bar{\mu}^{(1)} \rangle$$

(2.16)

associativity (2.8), Dörnte relations (2.15)

where $\mu^{(n)}$ is a $n$-ary associative multiplication and $\bar{\mu}^{(1)}$ is the quereoperation (2.13).

3. Grothendieck Group of Commutative Monoid

First, we describe the standard Grothendieck construction (see, e.g. Karoubi 1978, Rosenberg 1994, Weibel 1986) of a commutative group from a commutative semigroup with identity (monoid). We will use multiplicative notation, which will allow us to provide a straightforward “polyadization” according to the arity invariance principle [DUPL] 2021.

Let us have a (binary, arity $m = 2$) commutative monoid $S = S^{(2)} = \langle S \mid \mu = \mu^{(2)} \cong (\cdot) \rangle$ assoc$, where $S$ is the underlying set, and $\mu = \mu^{(2)} : S \times S \to S$ is the (associative) multiplication in $S$. The Cartesian product of two underlying sets $S' = S \times S$ can be endowed with the componentwise multiplication $(a_1, b_1) \cdot (a_2, b_2) = (a_1 \cdot a_2, b_1 \cdot b_2), a_i, b_i \in S$, to define the binary direct product $S' = S^{(2)} = \langle S' \mid \mu' = \mu^{(2)} \cong (\cdot) \rangle$ (which coincides with the direct sum, because of the finite number of factors in the product).

For convenience and conciseness, we introduce the doubles

$$S = \left( \begin{array}{c} a \\ b \end{array} \right) \in S \times S$$

(3.1)

and use vector-like notation for the multiplication (being the Kronecker product of the doubles)

$$S_1 \cdot S_2 = \left( \begin{array}{c} a_1 \\ b_1 \end{array} \right) \cdot \left( \begin{array}{c} a_2 \\ b_2 \end{array} \right) = \left( \begin{array}{c} a_1 \cdot a_2 \\ b_1 \cdot b_2 \end{array} \right), \quad S_i \in S', \quad a_i, b_i \in S.$$
or in “polyadic” notation

$$\mu' [S_1, S_2] = \left( \begin{array}{c} \mu [a_1, a_2] \\ \mu [b_1, b_2] \end{array} \right).$$  \hspace{1cm} (3.3)

The associativity of the direct product $\mu'$ follows immediately from that of $\mu$, because of the componentwise multiplication in (3.3). Since $S'$ is a monoid with the neutral element (identity) $e \in S$, satisfying $\mu [e, a] = \mu [a, e] = a, a \in S$, then the identity of the direct product $S'$ is the double

$$E = \left( \begin{array}{c} e \\ e \end{array} \right),$$  \hspace{1cm} (3.4)

such that

$$\mu' [E, G] = \mu' [G, E] = G \in S \times S.$$  \hspace{1cm} (3.5)

Therefore, $S' = S \times S$ is a commutative monoid, as is $S$. Another associative direct product $S'' = S''^{(2)} = \langle S' \mid \mu'' = \mu_2'' \equiv (\cdot') \rangle$ can be obtained using the “twisted” multiplication of the doubles $\mu''$ defined by

$$S_1 \cdot'' S_2 = \left( \begin{array}{c} a_1 \\ b_1 \end{array} \right) \cdot'' \left( \begin{array}{c} a_2 \\ b_2 \end{array} \right) = \left( \begin{array}{c} a_1 \cdot b_2 \\ a_2 \cdot b_1 \end{array} \right),$$  \hspace{1cm} (3.6)

or

$$\mu'' [S_1, S_2] = \left( \begin{array}{c} \mu_2 [a_1, b_2] \\ \mu_2 [a_2, b_1] \end{array} \right).$$  \hspace{1cm} (3.7)

The neutral element (identity) $E$ in the “twisted” direct product $S''$ coincides with (3.4), and therefore $S'' = S \times S$ is a commutative monoid as well.

The question arises: how to construct a (binary) group corresponding to the binary monoid $S = S_2$ which would reflect its substantial and important properties? The answer was provided by Grothendieck: to consider the equivalence relations and corresponding classes in the direct product $S' \times S'$. Because, indeed on classes one can define the inverse elements which are needed to build a group (in addition to associativity and existence of neutral elements which are sufficient for monoids). Here we briefly reproduce the construction of the Grothendieck group corresponding to the commutative monoid $S'$ sometimes this is called the symmetrization of $S$ [Karoubi [1978]] or the group completion of $S$ [Rosenberg [1994], Weibel [1986]] in multiplicative notation, which will allow us to provide its “polyadization” in a straightforward way.

Let us consider two kinds of equivalence relations on the direct product $S' = S \times S$. The first one ($\sim_1$) is reminiscent of “gauge invariance” (in physical language), because it identifies the doubles with equal “shifts”, such that

$$\left( \begin{array}{c} a_1 \\ b_1 \end{array} \right) \sim_1 \left( \begin{array}{c} a_2 \\ b_2 \end{array} \right) \iff \exists x, y \in S \left( \begin{array}{c} \mu [a_1, x] \\ \mu [b_1, x] \end{array} \right) = \left( \begin{array}{c} \mu [a_2, y] \\ \mu [b_2, y] \end{array} \right), \ a_i, b_i \in S;$$  \hspace{1cm} (3.8)

and we call it the “gauge” shifts. The second equivalence relation ($\sim_2$) uses only one “shift” as follows

$$\left( \begin{array}{c} a_1 \\ b_1 \end{array} \right) \sim_2 \left( \begin{array}{c} a_2 \\ b_2 \end{array} \right) \iff \exists z \in S \left( \mu \circ^2 [a_1, b_2, z] = (\mu) \circ^2 [a_2, b_1, z], \ a_i, b_i \in S;$$  \hspace{1cm} (3.9)

so we call this the “twisted” shift (which was used originally by Grothendieck).

**Assertion 3.1.** Two equivalence relations above coincide $\sim_1 = \sim_2 \equiv \sim$. 
Proof. Let (3.9) holds, then putting \( x = \mu [z, b_2], y = \mu [a_1, z] \), we obtain (3.8). Conversely, from (3.9) with
\[
\mu [x, y] = \mu [a_1, b_2, z] = \mu [\mu [a_1, x], \mu [y, b_2]] = \mu [\mu [y, b_1], \mu [a_2, x]] = \mu [a_2, b_1, z].
\] (3.10)

it follows that
\[
(\mu)^{\circ 2} [a_1, b_2, z] = \mu [\mu [a_1, x], \mu [y, b_2]] = \mu [\mu [y, b_1], \mu [a_2, x]] = (\mu)^{\circ 2} [a_2, b_1, z].
\] (3.11)

The group completion of the commutative monoid \( \mathcal{S} \) is defined as the (binary \( n = 2 \)) group \( \tilde{\mathcal{G}}^{(2)} \) (of isomorphism classes) being the factorization of \( \mathcal{S} \times \mathcal{S} \) by the equivalence relation \( \sim \) given in (3.8) or (3.9). It is called the Grothendieck group, and is usually denoted by \( K_0 (\mathcal{S}) = \tilde{\mathcal{G}}^{(2)} \) (for this concrete case of monoid [ROSENBERG 1994, WEIBEL 1986]).

\[
K_0 (\mathcal{S}) = \mathcal{S} \times \mathcal{S} / \sim.
\] (3.12)

The representatives, “observables” (in physical language), are “gauge invariant” doubles (3.8)
\[
\tilde{\mathcal{G}} = \begin{bmatrix} a \\ b \end{bmatrix}.
\] (3.13)

The multiplication of the representatives \( \tilde{\mathcal{G}} \) in \( K_0 (\mathcal{S}) \) inherits the product of doubles in \( \mathcal{S} \times \mathcal{S} \) (3.6)
\[
\tilde{\mu} [\tilde{\mathcal{G}}_1, \tilde{\mathcal{G}}_2] = \begin{bmatrix} \mu [a_1, a_2] \\ \mu [b_1, b_2] \end{bmatrix},
\] (3.14)

where the elements of the resulting binary factor group \( K_0 (\mathcal{S}) = \tilde{\mathcal{G}}^{(2)} = \tilde{\mathcal{G}} = \langle \{ \tilde{\mathcal{G}} \} \mid \tilde{\mu} \rangle \) (the representative doubles \( \tilde{\mathcal{G}} \) (3.13) and the binary operation \( \tilde{\mu} = \tilde{\mu}^{(2)} \) on the classes are marked by waves). The structure of the Grothendieck group for monoids \( K_0 (\mathcal{S}) \) was considered as the starting example in, e.g., [KAROUBI 1978, ROSENBERG 1994, WEIBEL 1986].

The first equivalence relation \( \sim_1 \) gives the form of the neutral element in \( K_0 (\mathcal{S}) \)
\[
\tilde{E} = \begin{bmatrix} e \\ e \end{bmatrix} \sim \begin{bmatrix} a \\ a \end{bmatrix}, \forall a \in \mathcal{S},
\] (3.15)

where \( e \in \mathcal{S} \) is the identity of the monoid \( \mathcal{S} \). Indeed the shape (3.15) of the neutral element and commutativity of the initial monoid \( \mathcal{S} \) allows us to obtain the inverse element in \( K_0 (\mathcal{S}) \) using the multiplication (3.14) in the following way
\[
\tilde{\mu} \begin{bmatrix} a \\ b \end{bmatrix} \tilde{\mu} \begin{bmatrix} b \\ a \end{bmatrix} = \begin{bmatrix} \mu [a, b] \\ \mu [b, a] \end{bmatrix} = \begin{bmatrix} \mu [a, b] \\ \mu [b, a] \end{bmatrix} \sim \begin{bmatrix} e \\ e \end{bmatrix},
\] (3.16)

or
\[
\tilde{\mu} [\tilde{\mathcal{G}}, \tilde{\mathcal{G}}^{-1}] = \tilde{E},
\] (3.17)

where the unique inverse is
\[
\tilde{\mathcal{G}}^{-1} = \begin{bmatrix} a \\ b \end{bmatrix}^{-1} = \begin{bmatrix} b \\ a \end{bmatrix}.
\] (3.18)

Thus, \( \tilde{\mathcal{G}} = K_0 (\mathcal{S}) \) is indeed a (binary) commutative group (of classes) corresponding to the (binary) monoid \( \mathcal{S}_2 = \mathcal{S}_2 \). Using (3.8) and (3.15), the homomorphism (of monoids) \( \Phi_{SG} : \mathcal{S} \to K_0 (\mathcal{S}) \) can be written as
\[
\Phi_{SG} (a) = \begin{bmatrix} \mu [a, a] \\ a \end{bmatrix}, \forall a \in \mathcal{S},
\] (3.19)
where we did not use the identity \( e \in S \) (this can be important in the “polyadization” below). It follows from (3.15), that \( \Phi_{SG} \) can be written with the identity in the form

\[
\Phi_{SG}(a) = \begin{bmatrix} a \\ e \end{bmatrix}, \quad \forall a \in S, \ e \in S. \tag{3.20}
\]

Using (3.18) we observe that the image of \( \Phi_{SG} \) actually generates a group (being \( K_0(S) \)), because

\[
\begin{bmatrix} a \\ b \end{bmatrix} = \tilde{\mu} \left[ \Phi_{SG}(a), (\Phi_{SG}(b))^{-1} \right], \quad \forall a, b \in S. \tag{3.21}
\]

The universal property can be shown in the following way (see, e.g., ROSENBERG [1994]). Let us consider any commutative (binary) group \( \tilde{G}' = \langle G' \mid \tilde{\mu}' \rangle \) and the group homomorphism \( \Phi_{GG'} : \tilde{G} \to \tilde{G}' \), then there exists the unique homomorphism \( \Phi_{SG'} : S \to \tilde{G}' \), such that

\[
\Phi_{SG'} = \Phi_{GG'} \circ \Phi_{SG}. \tag{3.22}
\]

Indeed, we derive, using (3.20) and (3.21)

\[
\Phi_{GG'} \left( \begin{bmatrix} a \\ b \end{bmatrix} \right) = \Phi_{GG'} \left( \tilde{\mu} \left[ \begin{bmatrix} a \\ e \end{bmatrix}, \left( \begin{bmatrix} b \\ e \end{bmatrix} \right)^{-1} \right] \right) = \tilde{\mu}' \left[ \Phi_{GG'} \left( \begin{bmatrix} a \\ b \end{bmatrix} \right), \left( \Phi_{GG'} \left( \begin{bmatrix} b \\ e \end{bmatrix} \right) \right)^{-1} \right] = \tilde{\mu}' \left[ \Phi_{SG'}(a), (\Phi_{SG'}(b))^{-1} \right]. \tag{3.23}
\]

Conversely, for a given homomorphism \( \Phi_{SG'} : S \to \tilde{G}' \) the group homomorphism \( \Phi_{GG'} : \tilde{G} \to \tilde{G}' \) is uniquely defined by (3.22) with \( \Phi_{SG'} = \Phi_{GG'} \circ \Phi_{SG} \) (3.22).

**Example 3.2.** The simplest example is the commutative monoid of nonnegative integers (natural numbers with the zero \( \mathbb{N}_0 \)) under addition \( S = \langle \mathbb{N}_0 \mid \mu = (+) \rangle \), and \( e = 0 \). The elements of the \( K_0(S) = \langle \mathbb{N}_0 \times \mathbb{N}_0 \mid \tilde{\mu} = (\ )) \rangle \) are doubles of natural numbers \( \begin{bmatrix} n \\ m \end{bmatrix} \in \mathbb{N}_0 \times \mathbb{N}_0 \), and the neutral double is \( \begin{bmatrix} 0 \\ 0 \end{bmatrix} \). The equivalence relation (3.9) becomes

\[
\begin{bmatrix} n_1 \\ m_1 \end{bmatrix} \sim \begin{bmatrix} n_2 \\ m_2 \end{bmatrix} \iff n_1 + m_2 = n_2 + m_1, \quad n_i, m_i \in \mathbb{N}_0, \tag{3.24}
\]

because the monoid \( S \) is cancellative. It follows from (3.24) by scaling, that in \( K_0(S) \) there exist two minimal representatives \( \begin{bmatrix} n \\ 0 \end{bmatrix} \) and \( \begin{bmatrix} 0 \\ m \end{bmatrix} \), and from (3.16) we have

\[
\begin{bmatrix} n \\ 0 \end{bmatrix} \sim \begin{bmatrix} 0 \\ n \end{bmatrix} = \begin{bmatrix} n \\ n \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}. \tag{3.25}
\]

Therefore, the representatives \( \begin{bmatrix} n \\ 0 \end{bmatrix} \) and \( \begin{bmatrix} 0 \\ m \end{bmatrix} \) can be treated as positive and negative integers in \( \mathbb{Z} \), such that there exists the homomorphism \( K_0(\mathbb{N}_0) \to \mathbb{Z} \) defined by

\[
\begin{bmatrix} n \\ m \end{bmatrix} \mapsto n - m \in \mathbb{Z}, \tag{3.26}
\]
which is a bijection due to (3.24). The universal property (3.23) becomes

\[ \Phi_{GG'} \left( \begin{bmatrix} n \\ m \end{bmatrix} \right) = \Phi_{GG'} \left( \begin{bmatrix} n \\ 0 \end{bmatrix} - \begin{bmatrix} m \\ 0 \end{bmatrix} \right) = \Phi_{GG'} \left( \begin{bmatrix} n \\ 0 \end{bmatrix} \right) - \Phi_{GG'} \left( \begin{bmatrix} m \\ 0 \end{bmatrix} \right) = \Phi_{GG'} \left( \Phi_{SG} (n) \right) - \Phi_{GG'} \left( \Phi_{SG} (m) \right) = \Phi_{SG'} (n) - \Phi_{SG'} (m). \quad (3.27) \]

Thus, the Grothendieck group of the monoid \( \mathbb{N}_0 \) of natural numbers with zero is the group of integers \( \mathbb{Z} \) under addition \( K_0 (\mathbb{N}_0) = \mathbb{Z} \).

4. \( n \)-ARY GROUP COMPLETION OF \( m \)-ARY SEMIGROUP

Here we propose the “polyadization” (along the ideas of \cite{2019}) of the group completion concept using the arity invariance principle \cite{2021}. By considering the polyadic algebraic structures, the main differences with the binary case will be the following \cite{2022} (and refs therein):

1) There can exist several associative polyadic direct products (powers) which can be built from one polyadic semigroup.
2) If initial \( m \)-ary semigroup is commutative, the polyadic direct product can be noncommutative.
3) The arity \( n \) of the direct product of \( m \)-ary semigroups can be not equal to \( m \).
4) The neutral element (identity) is not necessary for the \( m \)-ary semigroup and its power, because the polyadic analog of the binary inverse element is the querelement of an \( n \)-ary group, which is defined without the usage of a neutral element, and moreover some \( n \)-ary groups do not contain an identity at all.

4.1. Polyadic direct power construction. Let us consider the \( m \)-ary semigroup \( S^{(m)} = \langle S \mid \mu^{(m)} \mid \text{total assoc} \rangle \), where \( S \) is the underlying set, and \( \mu^{(m)} : S^{\times m} \to S \) is the (totally associative) \( m \)-ary multiplication in \( S \). The Cartesian product of two underlying sets \( S' = S^{\times m} \) can be endowed with an associative \( n \)-ary multiplication in various different ways, and also it can be that \( n \neq m \) \cite{2022}. Again, we introduce the doubles \( S = \left( \begin{array}{c} a \\ b \end{array} \right) \in S \times S \), and the polyadic direct product which is the \( n \)-ary semigroup of doubles \( S^{(n)} = \langle S' \mid \mu^{(n)} \mid \text{total assoc} \rangle \).

There are two possibilities to build the polyadic associative direct product \cite{2022}:

1) The componentwise construction which corresponds to the standard binary direct product (3.3).
2) The noncomponentwise one, which corresponds to the twisted direct product (3.7).

In the first case, the \( n \)-ary semigroup of doubles \( S^{(n)} \) corresponds to the full polyadic external product of \cite{2022}.

**Definition 4.1.** An \( n \)-ary componentwise direct product (power) semigroup of doubles consists of two \( n \)-ary semigroups \( S^{(n)} = S^{(n)} \times S^{(n)} \) (of the same arity)

\[ \mu^{(n)} [S_1, S_2, \ldots, S_n] = \left( \mu^{(n)} [a_1, a_2, \ldots, a_n] \right), \quad S_i = \left( \begin{array}{c} a_i \\ b_i \end{array} \right), \quad a_i, b_i \in S, \quad (4.1) \]

where the (total) polyadic associativity (2.8) of \( \mu^{(n)} \) is governed by the associativity of the constituent semigroup \( S^{(n)} \).

The simplest case is give by

**Definition 4.2.** A full polyadic direct product (power) \( S^{(n)} = S^{(n)} \times S^{(n)} \) is called derived, if the constituent \( n \)-ary semigroup \( S^{(n)} \) is derived, such that its operation \( \mu^{(n)} \) is composition of the binary operations \( \mu^{(2)} \).
In the derived case all the operations in [4.1] have the form (see (2.1)—(2.2))

\[ \mu^{(n)} = (\mu^{(2)})^{(n-1)}, \mu^{(2)} = (\cdot), \quad \mu^{(n)} = (\mu^{(2)})^{(n-1)}, \mu^{(2)} = (\cdot). \] (4.2)

The operations of the derived polyadic direct product can be written as (cf. the binary case (3.3))

\[ \mu^{(n)}_{\text{der}} [S_1, S_2, \ldots, S_n] = S_1 \cdot S_2 \cdot \cdots \cdot S_n = \begin{pmatrix} a_1 \cdot a_2 \cdots a_n \\ b_1 \cdot b_2 \cdots b_n \end{pmatrix}, \] (4.3)

and so it is simply a repition of the binary products (3.3). Therefore, it would be more interesting to consider nonderived polyadic analogs of the direct product which do not come down to the binary ones.

The nonderived version of the polyadic direct product, the hetero (“entangled”) product, was introduced in [DUPLI] [2022] for an arbitrary number of constituents. Here we apply it for two \( m \)-ary semigroups to get the nonderived associative direct product, which can have a different arity \( n \neq m \).

The general structure of the hetero product formally coincides (“reversely”) with the main homomorphism equation [DUPLI] [2018]. The additional parameter which determines the arity \( n \) of the hetero power of the initial \( m \)-ary semigroup is the number of intact elements \( \ell_{\text{id}} \) and number of constituents \( k \).

In our case of \( k = 2 \) multipliers, we have only two possibilities \( \ell_{\text{id}} = 0, 1 \). Thus, we arrive at

**Definition 4.3.** The hetero (“entangled”) power \( \boxtimes \) (square) of the \( m \)-ary semigroup \( S^{(m)} = \langle S | \mu^{(m)} \rangle \) is the \( n \)-ary semigroup defined on the Cartesian power \( S^n = S \times S \), such that \( S^{(n)} = \langle S' | \mu^{(n)} \rangle \),

\[ S^{(n)} = S^{(m)} \boxtimes S^{(m)}, \] (4.4)

and the \( n \)-ary multiplication of doubles \( S_{ij} = \begin{pmatrix} a_i \\ a_j \end{pmatrix} \in S \times S \), \( a_i, a_j \in S \), is given (informally) by

\[ \mu^{(n)} \left( \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_{2n-1} \\ a_{2n} \end{pmatrix}, \ldots, \begin{pmatrix} a_{2n-1} \\ a_{2n} \end{pmatrix} \right) = \begin{pmatrix} \mu^{(m)}[a_1, \ldots, a_m], \\ \mu^{(m)}[a_{m+1}, \ldots, a_{2m}], \\ \mu^{(m)}[a_1, \ldots, a_m], \\ a_{m+1} \end{pmatrix}, \quad \ell_{\text{id}} = 0, \quad n = m, \] (4.5)

where \( \ell_{\text{id}} = 0, 1 \) is the number of intact elements in the r.h.s. of the polyadic direct product. The hetero power arities are connected by the **arity changing formula** (with \( k = 2 \)) [DUPLI] [2018]

\[ n = m - \frac{m - 1}{2} \ell_{\text{id}}, \] (4.6)

with the integer \( \frac{n - 1}{2} \ell_{\text{id}} \geq 1 \).

In the case \( \ell_{\text{id}} = 1 \), the initial and final arities \( m \) and \( n \) are not arbitrary, but “quantized” such that the fraction in (4.6) has to be an integer (see the first row in TABLE 1 of [DUPLI] [2022])

\[ m = 3, 5, 7, \ldots \]
\[ n = 2, 3, 4, \ldots \] (4.7)

The concrete placement of elements and multiplications in (4.5) to obtain the associative \( \mu^{(n')} \) is governed by the associativity quiver technique [DUPLI] [2018].

Thus, the classification of the hetero powers for a nonbinary initial semigroup \( m \geq 3 \) consists of two limiting cases:

1) **Intactless power:** there are no intact elements \( \ell_{\text{id}} = 0 \). The arity of the hetero power reaches its maximum and coincides with the arity of the initial semigroup \( n = m \). Since \( m \geq 3 \), there is no binary hetero product.
2) **Binary power:** the final semigroup is of lowest arity, i.e., binary \( n = 2 \), and it follows from (4.7), that the only possibility is \( m = 3 \).

Let us consider some concrete examples of the hetero powers.

**Example 4.4.** Let \( S^{(3)} = \langle S \mid \mu^{(3)} \rangle \) be a commutative ternary semigroup, then we can construct its square of the doubles \( S = \left( \begin{array}{c} a \\ b \end{array} \right) \in S \times S \) in two ways to obtain the associative hetero power

\[
\mu^{(2)}[S_1, S_2] = \mu^{(2)}[\left( \begin{array}{c} a_1 \\ b_1 \end{array} \right), \left( \begin{array}{c} a_2 \\ b_2 \end{array} \right)] \left( \begin{array}{c} \mu^{(3)}[a_1, b_1, a_2] \\ \mu^{(3)}[a_1, b_2, a_2] \end{array} \right), \quad a_i, b_i \in S. \tag{4.8}
\]

This means that the Cartesian square \( S \times S \) can be endowed with an associative multiplication \( \mu^{(2)} \) in two ways, and therefore \( S^{(2)} = \langle S' \mid \mu^{(2)} \rangle \) is a noncommutative semigroup, being the hetero power \( S^{(2)} = S^{(3)} \times S^{(3)} \). If the ternary semigroup \( S^{(3)} \) has a ternary identity \( e \in S \), then \( S^{(2)} \) has only the left (right) identity \( E = \left( \begin{array}{c} e \\ e \end{array} \right) \in S \times S \), since \( \mu^{(2)}[E, S] = S (\mu^{(2)}[S, E] = S) \), but not the right (left) identity. Thus, \( S^{(2)} \) can be a semigroup only, even in the case when \( S^{(3)} \) is a ternary group.

**Example 4.5.** Take \( S^{(3)} = \langle S \mid \mu^{(3)} \rangle \) to be a commutative ternary semigroup, then the multiplication of the doubles is ternary and is componentwise (as opposed to the componentwise product (4.1))

\[
\mu^{(3)}[S_1, S_2, S_3] = \mu^{(3)}[\left( \begin{array}{c} a_1 \\ b_1 \end{array} \right), \left( \begin{array}{c} a_2 \\ b_2 \end{array} \right), \left( \begin{array}{c} a_3 \\ b_3 \end{array} \right)] = \left( \begin{array}{c} \mu^{(3)}[a_1, b_1, a_3] \\ \mu^{(3)}[a_1, b_2, a_3] \\ \mu^{(3)}[a_1, b_3, a_3] \end{array} \right), \quad a_i, b_i \in S. \tag{4.9}
\]

Nevertheless \( \mu^{(3)} \) is associative (and is described by the Post-like associative quiver [DUPLI 2018]), therefore the hetero power \( S^{(3)} = \langle S' \mid \mu^{(3)} \rangle \) is indeed the noncommutative ternary semigroup \( S^{(3)} = \langle G \times G \mid \mu^{(3)} \rangle \). In this case, as opposed to the previous example, the existence of the ternary identity \( e \) in \( S^{(3)} \) implies the ternary identity in the direct product \( S^{(3)} \) by \( E = \left( \begin{array}{c} e \\ e \end{array} \right) \), such that

\[
\mu^{(3)}[S, E, E] = \mu^{(3)}[E, S, E] = \mu^{(3)}[E, E, S] = S. \tag{4.10}
\]

**Proposition 4.6.** If the initial \( m \)-ary semigroup \( S^{(m)} \) contains an identity, then the hetero square \( n \)-ary semigroup \( S^{(n)} = S^{(m)} \times S^{(m)} \) contains an identity only in the intactless case and the Post-like quiver [DUPLI 2018]. For the binary power \( n = 2 \) only the one-sided identity is possible.

Next we consider more complicated hetero power (“entangled”) constructions with and without intact elements, which is possible for \( m \geq 4 \) only [DUPLI 2018].

**Example 4.7.** Let \( S^{(5)} = \langle S \mid \mu^{(5)} \rangle \) be a 5-ary semigroup, then we construct its 5-ary totally associative hetero square \( \mathcal{S}^{(5)} = \langle G' \mid \mu^{(5)} \rangle \) using the Post-like associative quiver without intact elements. We define the 5-ary multiplication of the doubles by

\[
\mu^{(5)}[S_1, S_2, S_3, S_4, S_5] = \left( \begin{array}{c} \mu^{(5)}[a_1, b_2, a_3, b_4, a_5] \\ \mu^{(5)}[b_1, a_2, a_3, a_4, b_5] \end{array} \right), \quad a_i, b_i \in S. \tag{4.11}
\]

It can be shown that \( \mu^{(5)} \) is totally associative, and therefore \( S^{(5)} = \langle S' \mid \mu^{(5)} \rangle \) is a 5-ary commutative semigroup. If \( S^{(5)} \) has the 5-ary identity \( e \) satisfying

\[
\mu^{(5)}[e, e, e, e, a] = a, \quad \forall a \in S, \quad e \in S, \tag{4.12}
\]
then the hetero power $\mathcal{S}^{n(5)}$ has the 5-ary identity

$$E = \left( \begin{array}{c} e \\ e \end{array} \right), \quad e \in S. \quad (4.13)$$

A more nontrivial example is a hetero product (power) which has a different arity than that of the initial semigroup.

Example 4.8. Let $\mathcal{S}^{(5)} = \langle S \mid \mu^{(5)} \rangle$ be a commutative 5-ary semigroup, then we can construct its ternary associative hetero power $\mathcal{S}^{(3)} = \langle S' \mid \mu^{(3)} \rangle$ using the associative quivers with one intact element (see 4.7) for allowed “quantized” arities. We propose for the triples $S_i$ the following ternary multiplication

$$\mu^{(3)}[S_1, S_2, S_3] = \mu^{(3)}\left( \left( \begin{array}{c} a_1 \\ b_1 \end{array} \right), \left( \begin{array}{c} a_2 \\ b_2 \end{array} \right), \left( \begin{array}{c} a_3 \\ b_3 \end{array} \right) \right) = \left( \begin{array}{c} \mu^{(5)}[a_1, b_2, a_3, b_1, a_2] \\ b_3 \end{array} \right), \quad a_i, b_i \in S. \quad (4.14)$$

It can be seen that $\mu^{(3)}$ is totally associative, and therefore the hetero power of 5-ary semigroup $\mathcal{S}^{(5)} = \langle S \mid \mu^{(5)} \rangle$ is a noncommutative ternary semigroup $\mathcal{S}^{(3)} = \langle S' \mid \mu^{(3)} \rangle$, such that $\mathcal{S}^{(3)} = \mathcal{S}^{(5)} \otimes \mathcal{S}^{(5)}$. If the initial 5-ary semigroup $\mathcal{S}^{(5)}$ has the identity satisfying

$$\mu^{(5)}[e, e, e, e, a] = a, \quad \forall a \in S, \quad e \in S, \quad (4.15)$$

then the ternary hetero power $\mathcal{S}^{(3)}$ has only the left ternary identity satisfying one relation

$$\mu^{(3)}[E, E, S] = S, \quad \forall S \in S \times S, \quad (4.16)$$

and therefore $\mathcal{S}^{(3)}$ is a ternary semigroup with the left identity.

4.2. **Equivalence relations for m-ary semigroups.** Consider of the extension of the equivalence relations (5.3) and (5.3) for m-ary semigroups. On the polyadic direct product (square) $\mathcal{S}^{(m)} \times \mathcal{S}^{(m)}$ we define two kinds of corresponding binary relations.

**Definition 4.9.** The first relation ($\sim_{\text{gauge}}$) is described by the polyadic “gauge” shifts of the doubles $S = \left( \begin{array}{c} a \\ b \end{array} \right) \in S \times S$ in the form

$$\left( \begin{array}{c} a_1 \\ b_1 \end{array} \right) \sim_{\text{gauge}} \left( \begin{array}{c} a_2 \\ b_2 \end{array} \right) \iff \exists x, y \in S \left( \begin{array}{c} \mu^{(m)}[a_1^{m-1}, x] \\ \mu^{(m)}[b_1^{m-1}, x] \end{array} \right) = \left( \begin{array}{c} \mu^{(m)}[a_2^{m-1}, y] \\ \mu^{(m)}[b_2^{m-1}, y] \end{array} \right), \quad \forall a_i, b_i \in S. \quad (4.17)$$

The second relation ($\sim_{\text{twist}}$) is given by the polyadic “twisted” shift

$$\left( \begin{array}{c} a_1 \\ b_1 \end{array} \right) \sim_{\text{twist}} \left( \begin{array}{c} a_2 \\ b_2 \end{array} \right) \iff \exists x \in S \left( \begin{array}{c} (\mu^{(m)})^{\circ 2}[a_1^{m-1}, b_2^{m-1}, z] \\ (\mu^{(m)})^{\circ 2}[a_2^{m-1}, b_1^{m-1}, z] \end{array} \right) = \left( \begin{array}{c} (\mu^{(m)})^{\circ 2}[a_2^{m-1}, b_1^{m-1}, z] \\ (\mu^{(m)})^{\circ 2}[a_2^{m-1}, b_1^{m-1}, z] \end{array} \right), \quad \forall a_i, b_i \in S. \quad (4.18)$$

**Proposition 4.10.** The relations described by the polyadic “gauge” shifts and “twisted” shift coincide

$\sim_{\text{gauge}} = \sim_{\text{twist}} \equiv \sim_{\text{m}}$.

**Proof.** $\sim_{\text{twist}} \Rightarrow \sim_{\text{gauge}}$) Let (4.18) hold, then putting $x = \mu^{(m)}[b_2^{m-1}, z], y = \mu^{(m)}[a_2^{m-1}, y], z]$ we get (4.17).

$\sim_{\text{gauge}} \Rightarrow \sim_{\text{twist}}$) Conversely, if (4.17) takes place, then we can always find $t_1, \ldots, t_{m-2} \in S$, such that $z = \mu^{(m)}[x, y, t_1, \ldots, t_{m-2}]$ and

$$\left( \begin{array}{c} (\mu^{(m)})^{\circ 2}[a_1^{m-1}, b_2^{m-1}, z] \\ (\mu^{(m)})^{\circ 2}[a_1^{m-1}, b_2^{m-1}, z] \end{array} \right) = \mu^{(m)}[\mu^{(m)}[a_1^{m-1}, x], \mu^{(m)}[b_2^{m-1}, y], t_1, \ldots, t_{m-2}]$$

$$= \mu^{(m)}[\mu^{(m)}[a_2^{m-1}, x], \mu^{(m)}[b_1^{m-1}, y], t_1, \ldots, t_{m-2}] = (\mu^{(m)})^{\circ 2}[a_2^{m-1}, b_1^{m-1}, z]. \quad (4.19)$$
Proposition 4.11. The relation $\sim_m$ is the equivalence relation on the set of doubles $\{S\}$, $S \in S \times S$.

Proof. 1) Reflexivity $(1 = 2)$ of (4.17) is obvious, and for $\sim_{\text{twist}}$ it follows from (4.18), becoming the identity $(\mu^{(m)})^{\circ2} [a^{m-1}, b^{m-1}, z] = (\mu^{(m)})^{\circ2} [a^{m-1}, b^{m-1}, z]$.

2) Symmetry $(1 \leftrightarrow 2)$ is evident for $\sim_{\text{gauge}}$, while (4.18) is symmetric with respect to $(1 \leftrightarrow 2)$.

3) Transitivity. Let $S_1 \sim_m S_2$ and $S_2 \sim_m S_3$, and we will prove that $S_1 \sim_m S_3$, $S_i \in S \times S$.

We start from the “gauge”-like relations $S_1 \sim_{\text{gauge}} S_2$ and $S_2 \sim_{\text{gauge}} S_3$ for the first components of the doubles

\begin{equation}
\mu^{(m)} [a_1^{m-1}, x_1] = \mu^{(m)} [a_2^{m-1}, y_1],
\end{equation}

\begin{equation}
\mu^{(m)} [a_2^{m-1}, x_2] = \mu^{(m)} [a_3^{m-1}, y_2], \quad \exists x_i, y_i \in S.
\end{equation}

Then we multiply separately the left hand sides and right hand sides of (4.20), (4.21) together with $(m - 2)$ identities $t_1 = t_2 = t_2 = \ldots, t_{m-2} = t_{m-2}, t_1, \ldots, t_{m-2} \in S$, and derive

\begin{equation}
(\mu^{(m)})^{\circ3} [a_1^{m-1}, x_1, a_2^{m-1}, x_2, t_1, \ldots, t_{m-2}] = (\mu^{(m)})^{\circ3} [a_2^{m-1}, y_1, a_3^{m-1}, y_2, t_1, \ldots, t_{m-2}].
\end{equation}

Denoting $x_3 = (\mu^{(m)})^{\circ2} [a_2^{m-1}, x_1, x_2, t_1, \ldots, t_{m-2}]$ and $y_3 = (\mu^{(m)})^{\circ2} [a_2^{m-1}, y_1, y_2, t_1, \ldots, t_{m-2}]$, we obtain in the form

\begin{equation}
\mu^{(m)} [a_1^{m-1}, x_3] = \mu^{(m)} [a_3^{m-1}, y_3],
\end{equation}

The second components of the doubles can be treated similarly

\begin{equation}
\mu^{(m)} [b_1^{m-1}, x_3] = \mu^{(m)} [b_3^{m-1}, y_3],
\end{equation}

and so we have $S_1 \sim_{\text{gauge}} S_3$.

The “twisted”-like relations $S_1 \sim_{\text{twist}} S_2$ and $S_2 \sim_{\text{twist}} S_3$ have the form

\begin{equation}
(\mu^{(m)})^{\circ2} [a_1^{m-1}, b_2^{m-1}, z_1] = (\mu^{(m)})^{\circ2} [a_2^{m-1}, b_1^{m-1}, z_1],
\end{equation}

\begin{equation}
(\mu^{(m)})^{\circ2} [a_2^{m-1}, b_3^{m-1}, z_2] = (\mu^{(m)})^{\circ2} [a_3^{m-1}, b_2^{m-1}, z_2].
\end{equation}

We multiply separately the left hand sides and right hand sides of (4.25), (4.26) together with $(m - 2)$ identities $t_1 = t_1, t_2 = t_2, \ldots, t_{m-2} = t_{m-2}, t_1, \ldots, t_{m-2} \in S$, to get

\begin{equation}
(\mu^{(m)})^{\circ5} [a_1^{m-1}, b_2^{m-1}, z_1, a_2^{m-1}, b_3^{m-1}, z_2, t_1, \ldots, t_{m-2}] = (\mu^{(m)})^{\circ5} [a_2^{m-1}, b_1^{m-1}, z_1, a_3^{m-1}, b_2^{m-1}, z_2, t_1, \ldots, t_{m-2}],
\end{equation}

which can be written as

\begin{equation}
(\mu^{(m)})^{\circ2} [a_1^{m-1}, b_3^{m-1}, z_3] = (\mu^{(m)})^{\circ2} [a_2^{m-1}, b_1^{m-1}, z_3],
\end{equation}

where $z_3 = (\mu^{(m)})^{\circ3} [a_2^{m-1}, b_2^{m-1}, z_1, z_2, t_1, \ldots, t_{m-2}], z_3 \in S$. Therefore, it follows from (4.28), that $S_1 \sim_{\text{twist}} S_3$.

We next introduce the corresponding equivalence classes.

Definition 4.12. The equivalence class of doubles corresponding to the equivalence relation $\sim_m$ on the $m$-ary semigroup $S^{(m)}$ is the factor

\begin{equation}
\tilde{S} = \left[ \begin{array}{c} a \\ b \end{array} \right]_m = \{S\} / \sim_m, \quad S = \left( \begin{array}{c} a \\ b \end{array} \right) \in S \times S,
\end{equation}

where $\sim_m$ is defined in (4.17) or (4.18).
Next we consider the structure of the equivalence classes (4.29) in some examples.

**Example 4.13** (Negative numbers). Let \( S = \mathbb{N}_- \subset \mathbb{Z} \) be the set of negative (integer) numbers (without zero), which do not form a binary semigroup, obviously. However, we can introduce the ternary multiplication \( \mu^{(3)} [a, b, c] = abc \) (ordinary product in \( \mathbb{Z} \)), such that \( \mathcal{S}^{(3)}_{neg} = \langle \mathbb{N}_- | \mu^{(3)} \rangle \) becomes the ternary semigroup (of negative numbers). The doubles have the form

\[
\mathcal{S}_{neg} = \left( \begin{array}{c} -p \\ -q \end{array} \right) \in \mathbb{N}_- \times \mathbb{N}_-, \quad p, q \in \mathbb{N}. \tag{4.30}
\]

The equivalence relations (4.17) and (4.18) become \((m = 3)\)

\[
\left( \begin{array}{c} -p_1 \\ -q_1 \end{array} \right) \sim_{\text{gauge}} \left( \begin{array}{c} -p_2 \\ -q_2 \end{array} \right) \iff \exists x, y \in \mathbb{N} \left( \begin{array}{c} -p_1^2 x \\ -q_1^2 x \end{array} \right) = \left( \begin{array}{c} -p_2^2 y \\ -q_2^2 y \end{array} \right), \quad p_i, q_i \in \mathbb{N}, \tag{4.31}
\]

and

\[
\left( \begin{array}{c} -p_1 \\ -q_1 \end{array} \right) \sim_{\text{twist}} \left( \begin{array}{c} -p_2 \\ -q_2 \end{array} \right) \iff \exists x, y \in \mathbb{N} \left( \begin{array}{c} -p_1^2 x \\ -q_1^2 x \end{array} \right) = p_2^2 q_2^2 y. \tag{4.32}
\]

For instance, we compute

\[
\left( \begin{array}{c} -1 \\ -2 \end{array} \right) \sim \left( \begin{array}{c} -2 \\ -4 \end{array} \right) \sim \left( \begin{array}{c} -3 \\ -6 \end{array} \right) \ldots; \tag{4.33}
\]

\[
\left( \begin{array}{c} -4 \\ -3 \end{array} \right) \sim \left( \begin{array}{c} -8 \\ -6 \end{array} \right) \sim \left( \begin{array}{c} -12 \\ -9 \end{array} \right) \ldots. \tag{4.34}
\]

In general, the two elements

\[
\left( \begin{array}{c} -p \\ -q \end{array} \right), \left( \begin{array}{c} -kp \\ -kq \end{array} \right) \tag{4.35}
\]

are in the same class, \( k \in \mathbb{N} \). Each equivalence class \( \tilde{\mathcal{S}}_{neg} \) contains the maximal representative \( \left[ \begin{array}{c} -p_{\text{min}} \\ -q_{\text{min}} \end{array} \right] \) corresponding to minimal \( p = p_{\text{min}} \) and \( q = q_{\text{min}} \), together with other elements (4.35) of the class. Thus, for the ternary semigroup of negative integers the structure of the binary equivalence classes in \( \mathcal{S}^{(3)}_{neg} \times \mathcal{S}^{(3)}_{neg} / \sim_3 \) is given by the following maximal representatives

\[
\tilde{\mathcal{S}}_{neg,p>q} = \left\{ \left[ \begin{array}{c} -p_{\text{min}} \\ -q_{\text{min}} \end{array} \right] \right\}, \quad p_{\text{min}} > q_{\text{min}}, \tag{4.36}
\]

\[
\tilde{\mathcal{S}}_{neg,p<q} = \left\{ \left[ \begin{array}{c} -p_{\text{min}} \\ -q_{\text{min}} \end{array} \right] \right\}, \quad p_{\text{min}} < q_{\text{min}}, \quad p_{\text{min}}, q_{\text{min}} \in \mathbb{N}, \tag{4.37}
\]

where \( p_{\text{min}} \neq q_{\text{min}} \), and they are mutually prime \( \gcd (p_{\text{min}}, q_{\text{min}}) = 1 \).

### 4.3. Polyadic group completion

In general, the procedure of group completion consists of two steps:

1) Endow the set of equivalence classes \( \tilde{\mathcal{S}} \) with the multiplication \( \tilde{\mu}^{(n)} \) (in the binary case (3.14)).

2) Find (a polyadic analog of) an inverse and construct a group (as in (3.13)).
It is commonly accepted that the multiplication of classes inherits the product of representatives. So the informal formula of class multiplication coincides with the hetero ("entangled") power \( (\mathcal{S}) \), indeed

\[
\tilde{\mu}_S^{(\tilde{n})}\left[\begin{array}{c}
a_1 \\
a_2 \\
\vdots \\
a_{2n-1} \\
a_{2n}
\end{array}\right] = \left\{ \begin{array}{c}
\mu^{(m)}(a_1, \ldots, a_m), \\
\mu^{(m)}(a_{m+1}, \ldots, a_{2m}), \\
\mu^{(m)}(a_1, \ldots, a_m), \\
\mu^{(m)}(a_{m+1}, \ldots, a_{2m})
\end{array}\right\}_{m}, \quad \ell_{id} = 0, \ \tilde{n} = m, \\
\ell_{id} = 1, \ \tilde{n} = \frac{m+1}{2}, \quad (4.38)
\]

where \( \ell_{id} = 0, 1 \) is the number of intact elements in the r.h.s. of \((4.38)\).

**Definition 4.14.** The \( n \)-ary semigroup of the binary equivalence classes is defined as

\[
\tilde{\mathcal{S}}^{(\tilde{n})} = \langle \{\tilde{S}\} \mid \tilde{\mu}_S^{(\tilde{n})}\rangle. \quad (4.39)
\]

**Remark 4.15.** In general, for a fixed initial semigroup \( \mathcal{S}^{(m)} \), the formulas of the doubles multiplication \((4.3)\) and classes multiplication \((4.38)\) can be different (as two choices of \((4.3)\)), because in one case we multiply concrete elements, while in another case–their representatives. Moreover, the final arities \( n \) and \( \tilde{n} \) need not coincide, if we use different choices in r.h.s. of \((4.3)\) and \((4.38)\). For instance, compare the 5-ary \((4.11)\) and ternary \((4.14)\) multiplications of doubles obtained from the same 5-ary semigroup. This could lead to diverse possible polyadic analogs of the group completions of the same semigroup. Nevertheless, in what follows we will assume that

\[
\tilde{n} = n \quad (4.40)
\]

and choose the same product for elements and their representatives. Therefore, all examples of polyadic direct products in **Subsection 4.1** can be applied for the equivalence class multiplication \((4.38)\) as well.

The polyadic analog of the inverse is the queroperation \((\mathfrak{D} \mathcal{R} \mathfrak{N} T E \, 1929)\), which does not need an identity at all. Therefore, the binary formulas \((5.16)\)–\((5.18)\) do not work for \( m \geq 3 \). So we have

**Assertion 4.16.** The presence of an identity is not necessary for constructing polyadic groups, it can be optional and plays no role, and therefore we consider polyadic group completion not of a polyadic monoid, but of a polyadic semigroup \( S^{(m)} \).

Let us suppose that we can construct the queroperation for any double \( \tilde{S} \) in the \( n \)-ary semigroup \( \tilde{\mathcal{S}}^{(n)} \) of equivalence classes, which means that we define the unary queroperation \( \tilde{\mu}^{(1)} \) (see \((2.13)\))

\[
\tilde{S} = \tilde{\mu}^{(1)}\left[\begin{array}{c}
\tilde{S} \\
\tilde{a} \\
\tilde{b}
\end{array}\right] = \tilde{\mu}^{(1)}\left[\begin{array}{c}
\tilde{a} \\
\tilde{b}
\end{array}\right], \quad (4.41)
\]

which satisfies \((2.12)\)

\[
\tilde{\mu}^{(n)}\left[\tilde{S}^{n-1}, \tilde{S}\right] = \tilde{S}. \quad (4.42)
\]

Then the \( n \)-ary semigroup \( \tilde{\mathcal{S}}^{(n)} \) becomes a polyadic group.

**Definition 4.17.** The \( n \)-ary group completion of the \( m \)-ary semigroup \( \mathcal{S}^{(m)} \) is

\[
K_0^{(m,n)}(\mathcal{S}^{(m)}) = \langle \{\tilde{S}\} \mid \tilde{\mu}^{(n)}, \tilde{\mu}^{(1)}\rangle, \quad (4.43)
\]

where \( \tilde{\mu}^{(n)} \) and \( \tilde{\mu}^{(1)} \) are the \( n \)-ary multiplication of the doubles \((4.38)\) and the queroperation \((4.41)\).

In this notation the standard binary case \((3.12)\) is

\[
K_0(\mathcal{S}) = K_0^{(2,2)}(\mathcal{S}^{(2)}). \quad (4.44)
\]
Remark 4.18. The computation of the group completion $K_0^{(m,n)}$ with arbitrary $m$ and $n$ is possible, if we know both multiplications: of the initial $m$-ary semigroup $\mu^{(m)}$ and of the $n$-ary group of equivalent classes $\tilde{\mu}^{(n)}$ together with its quereoperation $\tilde{\mu}^{(1)}$.

Remark 4.19. As opposed to the binary case (4.44), for a given $m$-ary semigroup there can exist several associative products of doubles $\mu^{(n)}$ and corresponding products of classes $\tilde{\mu}^{(n)}$ (see, e.g. (4.1) and (4.9) or (4.11) and (4.14)), and therefore the polyadic group completion $K_0^{(m,n)}$ is not unique, in general. The number of different $K_0^{(m,n)}$ coincides with the number of distinct associative quivers in the doubles multiplication (4.45) and class products (4.38) having the same arity shape [Duelli 2018].

Remark 4.20. If we consider polyadic groups with the same arity shape of the multiplication of classes $\tilde{\mu}^{(n)}$ and the same quereoperation $\tilde{\mu}^{(1)}$, then we can have a polyadic analog of universality (3.22) for this fixed completion group arity $n$.

Example 4.21. Let us compute $K_0^{(m,m)}$ for the $n$-ary componentwise direct power (4.1) of the $m$-ary semigroup $S^{(m)} = \langle S \mid \mu^{(m)} \rangle$. Taking into account Remark 4.15 and (4.40) we obtain for the equivalence classes doubles $\tilde{S}_i$ the following $m$-ary multiplication

$$\tilde{\mu}^{(m)}_{\text{compw}}(\tilde{S}_1, \tilde{S}_2, \ldots, \tilde{S}_m) = \left( \mu^{(m)}[a_1, a_2, \ldots, a_m], \mu^{(m)}[b_1, b_2, \ldots, b_m] \right)_m, \quad \tilde{S}_i = \left[ \begin{array}{c} a_i \\ b_i \end{array} \right]_m, \quad a_i, b_i \in S,$$

(4.45)

because $n = m$ for the componentwise multiplication (4.1). We resolve the equation for the querelement (4.42) as

$$\tilde{S}^{(1)}_{\text{compw}} = \left[ \begin{array}{c} \tilde{a} \\ \tilde{b} \end{array} \right]_m \sim \left[ \begin{array}{c} \mu^{(m)}[a^{\alpha}, b^{m-\alpha}] \\ \mu^{(m)}[a^{m-2\alpha}, b^{2-\alpha}] \end{array} \right]_m,$$

(4.46)

where $\alpha = 0, 1, 2$. Because all three solutions are in the same equivalence class, we choose as the representative the symmetric choice $\alpha = 1$

$$\tilde{S}^{(1)}_{\text{compw}} = \tilde{\mu}^{(1)}_{\text{compw}}(\tilde{S}) = \left[ \begin{array}{c} \tilde{a} \\ \tilde{b} \end{array} \right]_m = \left[ \begin{array}{c} \mu^{(m)}[a, b^{m-1}] \\ \mu^{(m)}[a^{m-1}, b] \end{array} \right]_m.$$

(4.47)

Therefore, for any $m$-ary semigroup with $m$-ary componentwise direct power we obtain

$$K_0^{(m,m)}(S^{(m)}) = \langle \{ \tilde{S} \mid \tilde{\mu}^{(m)}_{\text{compw}}, \tilde{\mu}^{(1)}_{\text{compw}} \} \rangle.$$

(4.48)

Example 4.22 (Negative numbers (continued)). Let us introduce for the equivalence classes $\tilde{S}_{\text{neg}} = \left[ \begin{array}{c} -p \\ -q \end{array} \right]_3, p, q \in \mathbb{N}, \gcd(p, q) = 1$ (see Example 4.13), their ternary componentwise multiplication (4.45)

$$\tilde{\mu}^{(3)}_{\text{neg}} \left[ \left[ \begin{array}{c} -p_1 \\ -q_1 \end{array} \right]_3, \left[ \begin{array}{c} -p_2 \\ -q_2 \end{array} \right]_3, \left[ \begin{array}{c} -p_3 \\ -q_3 \end{array} \right]_3 \right] = \left[ \begin{array}{c} -p_1p_2p_3 \\ -q_1q_2q_3 \end{array} \right]_3, \quad p_i, q_i \in \mathbb{N}.$$

(4.49)

We derive from (4.42) the querelement

$$\tilde{S}^{(1)}_{\text{neg}} = \tilde{\mu}^{(1)}_{\text{neg}}(\tilde{S}) = \tilde{\mu}^{(1)}_{\text{neg}} \left[ \left[ \begin{array}{c} -p \\ -q \end{array} \right]_3 \right] = \left[ \begin{array}{c} -pq^2 \\ -p^2q \end{array} \right]_3, \quad \forall p, q \in \mathbb{N}, \ \gcd(p, q) = 1.$$

(4.50)

Thus, we obtain the group completion of the negative numbers (with the componentwise multiplication of classes $\tilde{\mu}^{(3)}_{\text{neg}}$)

$$K_0^{(3,3)}(\mathbb{N}_{\text{neg}})_{\text{comp}} = \langle \{ \tilde{S}_{\text{neg}} \mid \tilde{\mu}^{(3)}_{\text{neg}}, \tilde{\mu}^{(1)}_{\text{neg}} \} \rangle.$$

(4.51)
If we choose the ternary noncomponentwise multiplication (4.9), then
\[
\mathcal{\tilde{\mu}}^{(3)}_{\text{neg2}} \left[ \begin{bmatrix} -p_1 \\ -q_1 \\ 3 \\
- q_2 \\ 3 \\
- q_3 \\ 3 \\
\end{bmatrix} \right] = \left[ \begin{bmatrix} -p_1 q_2 p_3 \\ -q_1 p_2 q_3 \\ 3 \end{bmatrix} \right], \quad p_i, q_i \in \mathbb{N},
\]
and the querelement will be different from (4.50)
\[
\mathcal{\tilde{S}}_{\text{neg2}} = \mathcal{\tilde{\mu}}^{(1)}_{\text{neg2}} \left[ \begin{bmatrix} -p \\ -q \\ 3 \end{bmatrix} \right] = \left[ \begin{bmatrix} -pq^2 \\ -q^3 \end{bmatrix} \right], \quad \forall p, q \in \mathbb{N}, \text{ gcd}(p, q) = 1.
\]
Therefore, the second polyadic group completion of the negative numbers (with the noncomponentwise multiplication of classes \(\mathcal{\tilde{\mu}}^{(3)}_{\text{neg2}}\) being different from \(\mathcal{\tilde{\mu}}^{(3)}_{\text{neg}}\)) becomes
\[
\mathfrak{K}^{(3,3)}_{\text{odd}}(\mathbb{N}_-)_{\text{noncomp}} = \left\langle \mathcal{\tilde{S}}_{\text{neg2}} \middle| \mathcal{\tilde{\mu}}^{(3)}_{\text{neg2}}, \mathcal{\tilde{\mu}}^{(1)}_{\text{neg2}} \right\rangle.
\]

Another simple example is the set of odd positive numbers.

**Example 4.23.** Let \(\mathbb{N}_{\text{odd}} = \{2k + 1\}, k \in \mathbb{N}_0\), then it is the ternary semigroup \(\mathcal{S}^{(3)}_{\text{odd}} = \left\langle \mathbb{N}_{\text{odd}} \middle| \mu^{(3)}_{\text{odd}} \right\rangle\), where the multiplication is the ordinary addition
\[
\mu^{(3)}_{\text{odd}} [a, b, c] = a + b + c, \quad a, b, c \in \mathbb{N}_{\text{odd}}.
\]
The doubles (3.1) have the form
\[
\mathcal{S}_{\text{odd}} = \left( \begin{array}{c}
2k_1 + 1 \\
2k_2 + 1
\end{array} \right) \in \mathbb{N}_{\text{odd}} \times \mathbb{N}_{\text{odd}}, \quad k_i \in \mathbb{N}_0.
\]
The equivalence relations (4.17) and (4.18) become \((m = 3)\)
\[
\left( \begin{array}{c}
2k_1 + 1 \\
2k_2 + 1
\end{array} \right) \sim_{\text{gauge}} \left( \begin{array}{c}
2l_1 + 1 \\
2l_2 + 1
\end{array} \right) \iff \exists x, y \in \mathbb{N}_{\text{odd}} \left( \begin{array}{c}
\mu^{(3)}_{\text{odd}} \left[ (2k_1 + 1)^2, x \right] \\
\mu^{(3)}_{\text{odd}} \left[ (2k_2 + 1)^2, x \right]
\end{array} \right) = \left( \begin{array}{c}
\mu^{(3)}_{\text{odd}} \left[ (2l_1 + 1)^2, y \right] \\
\mu^{(3)}_{\text{odd}} \left[ (2l_2 + 1)^2, y \right]
\end{array} \right), \quad k_i, l_i \in \mathbb{N}_0,
\]
and
\[
\left( \begin{array}{c}
2k_1 + 1 \\
2k_2 + 1
\end{array} \right) \sim_{\text{twist}} \left( \begin{array}{c}
2l_1 + 1 \\
2l_2 + 1
\end{array} \right) \iff \exists x, y \in \mathbb{N}_{\text{odd}} \left( \begin{array}{c}
\mu^{(3)}_{\text{odd}} \circ^2 \left[ (2k_1 + 1)^2, (2l_2 + 1)^2, z \right] = \left( \mu^{(3)}_{\text{odd}} \right) \circ^2 \left[ (2k_1 + 1)^2, (2l_2 + 1)^2, z \right].
\end{array} \right)
\]
For instance,
\[
\begin{array}{ccc}
(3) & \sim & (5) \\
(1) & \sim & (3)
\end{array}, \quad \begin{array}{ccc}
(1) & \sim & (3) \\
(3) & \sim & (5)
\end{array}, \quad \begin{array}{ccc}
(3) & \sim & (5) \\
(1) & \sim & (3)
\end{array}, \quad \begin{array}{ccc}
(3) & \sim & (5) \\
(7) & \sim & (3)
\end{array}, \quad \begin{array}{ccc}
(5) & \sim & (7) \\
(1) & \sim & (3)
\end{array}, \quad \begin{array}{ccc}
(3) & \sim & (5) \\
(1) & \sim & (3)
\end{array}, \quad \begin{array}{ccc}
(5) & \sim & (7) \\
(3) & \sim & (9)
\end{array}, \quad \begin{array}{ccc}
(7) & \sim & (11) \\
(3) & \sim & (9)
\end{array}, \quad \begin{array}{ccc}
(1) & \sim & (3) \\
(7) & \sim & (3)
\end{array}, \quad \begin{array}{ccc}
(5) & \sim & (7) \\
(3) & \sim & (9)
\end{array}, \quad \begin{array}{ccc}
(3) & \sim & (5) \\
(5) & \sim & (7)
\end{array}, \quad \begin{array}{ccc}
(5) & \sim & (7) \\
(7) & \sim & (9)
\end{array}, \quad \begin{array}{ccc}
(3) & \sim & (5) \\
(1) & \sim & (3)
\end{array}, \quad \begin{array}{ccc}
(5) & \sim & (7) \\
(7) & \sim & (9)
\end{array}, \quad \begin{array}{ccc}
(1) & \sim & (3) \\
(7) & \sim & (3)
\end{array}, \quad \begin{array}{ccc}
(5) & \sim & (7) \\
(9) & \sim & (11)
\end{array}, \quad \begin{array}{ccc}
(3) & \sim & (5) \\
(7) & \sim & (9)
\end{array}, \quad \begin{array}{ccc}
(1) & \sim & (3) \\
(9) & \sim & (11)
\end{array}, \quad \begin{array}{ccc}
(3) & \sim & (5) \\
(7) & \sim & (9)
\end{array}, \quad \begin{array}{ccc}
(1) & \sim & (3) \\
(9) & \sim & (11)
\end{array}, \quad \begin{array}{ccc}
(3) & \sim & (5) \\
(7) & \sim & (9)
\end{array}, \quad \begin{array}{ccc}
(1) & \sim & (3) \\
(9) & \sim & (11)
\end{array}, \quad \begin{array}{ccc}
(3) & \sim & (5) \\
(7) & \sim & (9)
\end{array}, \quad \begin{array}{ccc}
(1) & \sim & (3) \\
(9) & \sim & (11)
\end{array}, \quad \begin{array}{ccc}
(3) & \sim & (5) \\
(7) & \sim & (9)
\end{array}, \quad \begin{array}{ccc}
(1) & \sim & (3) \\
(9) & \sim & (11)
\end{array}, \quad \begin{array}{ccc}
(3) & \sim & (5) \\
(7) & \sim & (9)
\end{array}, \quad \begin{array}{ccc}
(1) & \sim & (3) \\
(9) & \sim & (11)
\end{array}, \quad \begin{array}{ccc}
(3) & \sim & (5) \\
(7) & \sim & (9)
\end{array}, \quad \begin{array}{ccc}
(1) & \sim & (3) \\
(9) & \sim & (11)
\end{array}, \quad \begin{array}{ccc}
(3) & \sim & (5) \\
(7) & \sim & (9)
\end{array}, \quad \begin{array}{ccc}
(1) & \sim & (3) \\
(9) & \sim & (11)
\end{array}, \quad \begin{array}{ccc}
(3) & \sim & (5) \\
(7) & \sim & (9)
\end{array}, \quad \begin{array}{ccc}
(1) & \sim & (3) \\
(9) & \sim & (11)
\end{array}
\]
In general, two elements
\[
\left( \begin{array}{c}
a \\
b
\end{array} \right), \left( \begin{array}{c}
a + 2 \\
 b + 2
\end{array} \right), \quad a, b \in \mathbb{N}_{\text{odd}},
\]

---

July 1, 2022 at 00:39
are in the same equivalence class $\left[ \begin{array}{c} a \\ b \end{array} \right] \in \mathbb{N}_{odd} \times \mathbb{N}_{odd} / \sim$. The minimal representatives are the doubles
\[
\tilde{S}_{up} = \left[ \begin{array}{c} 2k + 1 \\ 1 \end{array} \right], \quad \tilde{S}_{down} = \left[ \begin{array}{c} 1 \\ 2k + 1 \end{array} \right], \quad k \in \mathbb{N}.
\]
Thus, the structure of the classes is the following
\[
\left\{ \tilde{S} \right\} = \left\{ \tilde{S}_{up} \right\} \cup \left\{ \tilde{S}_{down} \right\}, \quad \left\{ \tilde{S}_{up} \right\} \cap \left\{ \tilde{S}_{down} \right\} = \emptyset.
\]
We do not have an identity in the ternary semigroup $S^{(3)}$, and therefore the representatives (4.61) cannot be mapped to negative and positive numbers, as in the binary case (3.26), and moreover their multiplication is ternary (4.55).

To find the polyadic group structure (i.e. querelements (2.12)), for the ternary multiplication of the classes $\tilde{S}$ we consider two cases:

1) The componentwise multiplication (4.1). We use the ternary multiplication of the classes as ordinary addition
\[
\tilde{\mu}^{(3)}_{odd1} \left[ \begin{array}{c} 2k_1 + 1 \\ 1 \\ 1 \\ 2k_2 + 1 \\ 1 \\ 1 \\ 2k_3 + 1 \\ 1 \end{array} \right] = \left[ \begin{array}{c} 2(k_1 + k_2 + k_3) + 3 \\ 3 \end{array} \right] \sim \left[ \begin{array}{c} 2(k_1 + k_2 + k_3) + 1 \\ 1 \end{array} \right], \quad k_i \in \mathbb{N},
\]
and
\[
\tilde{\mu}^{(3)}_{odd1} \left[ \begin{array}{c} 1 \\ 2k_1 + 1 \\ 1 \\ 1 \\ 2k_2 + 1 \\ 1 \\ 2k_3 + 1 \\ 1 \end{array} \right] = \left[ \begin{array}{c} 3 \\ 2(k_1 + k_2 + k_3) + 3 \end{array} \right] \sim \left[ \begin{array}{c} 1 \\ 2(k_1 + k_2 + k_3) + 1 \end{array} \right], \quad k_i \in \mathbb{N}.
\]
Then, apply the general formula for the querelement (4.47) to our ternary case
\[
\tilde{S}_{compw} = \left[ \begin{array}{c} a \\ b \end{array} \right]_3 = \left[ \begin{array}{c} \mu^{(3)}_{odd1} [a, b] \\ \mu^{(3)}_{odd1} [a^2, b] \end{array} \right]_3,
\]
which gives using the equivalence relations (4.60)
\[
\tilde{S}_{up} = \tilde{\mu}^{(1)}_{odd1} \left[ \begin{array}{c} 2k + 1 \\ 1 \end{array} \right] \sim \left[ \begin{array}{c} 1 \\ 2k + 1 \end{array} \right] = \tilde{S}_{down},
\]
\[
\tilde{S}_{down} = \tilde{\mu}^{(1)}_{odd1} \left[ \begin{array}{c} 1 \\ 2k + 1 \end{array} \right] \sim \left[ \begin{array}{c} 1 \\ 2k + 1 \end{array} \right] = \tilde{S}_{up}, \quad k \in \mathbb{N}.
\]
Thus, for the componentwise ternary group completion of the ternary semigroup of odd numbers $\mathbb{N}_{odd}$ we obtain
\[
K^{(3)}_{0}(\mathbb{N}_{odd})_{compw} = \left\langle \left\{ \tilde{S}_{up} \right\} \cup \left\{ \tilde{S}_{down} \right\} | \tilde{\mu}^{(3)}_{odd1}, \tilde{\mu}^{(1)}_{odd1} \right\rangle.
\]

2) The noncomponentwise Post-like multiplication (4.9). In this case, we denote the multiplication of classes corresponding to the double product (4.34) by $\tilde{\mu}^{(3)}_{odd2}$, then the general formula for the querelement will be different from (4.65) and have the form
\[
\tilde{S}_{ncompw} = \left[ \begin{array}{c} a \\ b \end{array} \right]_3 = \left[ \begin{array}{c} \mu^{(3)}_{odd2} [a^2, b] \\ \mu^{(3)}_{odd2} [a, b^2] \end{array} \right]_3,
\]
which gives (using the equivalence relations (4.60))
\[
\bar{S}_{up} = \mu^{(1)}_{\text{odd2}} \left[ \bar{S}_{up} \right] = \left[ \begin{array}{c} 2k + 1 \\ 1 \end{array} \right] \sim \left[ \begin{array}{c} 2k + 1 \\ 1 \end{array} \right] = \bar{S}_{up}, \quad (4.70)
\]
\[
\bar{S}_{down} = \mu^{(1)}_{\text{odd2}} \left[ \bar{S}_{down} \right] = \left[ \begin{array}{c} 1 \\ 2k + 1 \end{array} \right] \sim \left[ \begin{array}{c} 1 \\ 2k + 1 \end{array} \right] = \bar{S}_{down}, \quad k \in \mathbb{N}. \quad (4.71)
\]

Now, each element is a kind of polyadic reflection obeying \( \bar{S} = \tilde{S} \) (an analog of \( g^{-1} = g \) in binary groups). Thus, the noncomponentwise ternary group completion of the ternary semigroup of odd numbers \( \mathbb{N}_{\text{odd}} \) is
\[
K^{(3,3)}_{0} (\mathbb{N}_{\text{odd}})_{\text{ncompw}} = \left\{ \left\{ \bar{S}_{up} \right\} \cup \left\{ \bar{S}_{down} \right\} | \mu^{(3)}_{\text{odd2}}, \mu^{(1)}_{\text{odd2}} \right\}. \quad (4.72)
\]

Example 4.24. Consider the following 4-ary semigroup of complex matrices
\[
S^{(4)}_{\text{matt}} = \left\{ S_{\text{matt}} \mid \mu^{(4)}_{\text{matt}} \right\}, \quad S_{\text{matt}} = \{a\}, \quad a = \left( \begin{array}{cc} u & 0 \\ 0 & 0 \end{array} \right), \quad u \in \mathbb{C}, \quad (4.73)
\]
\[
\mu^{(4)}_{\text{matt}} [a_1, a_2, a_3, a_4] = a_1 + \varepsilon a_2 + \varepsilon^2 a_3 + a_4, \quad (4.74)
\]
where \( \varepsilon = e^{2\pi i/3}, \varepsilon^3 = 1 \).

We construct the doubles \( S = \left( \begin{array}{c} a \\ b \end{array} \right) \in S_{\text{matt}} \times S_{\text{matt}} \) with the componentwise multiplication (4.1) which does not change the arity. Then we write the “twist-like” equivalence relation (4.19) for our case
\[
\left( \begin{array}{c} a_1 \\ b_1 \end{array} \right) \sim_{\text{twist}} \left( \begin{array}{c} a_2 \\ b_2 \end{array} \right) \iff \exists z \in S \left( \mu^{(4)} \right)^{\circ 2} [a_1^3, b_1^3, z] = \left( \mu^{(4)} \right)^{\circ 2} [a_2^3, b_2^3, z], \quad a_i, b_i, z \in S_{\text{matt}}. \quad (4.75)
\]

It follows from (4.73–4.74), that \( S^{(4)}_{\text{matt}} \) is an idempotent semigroup, because \( \mu^{(4)} [a^4] = a, \forall a \in S_{\text{matt}} \), moreover, \( \mu^{(4)} [a^3, b] = b, \forall a, b \in S_{\text{matt}} \), and so \( \mu^{(4)} \circ 2 [a_1^3, b_2^3, z] = \mu^{(m)} \circ 2 [a_2^3, b_2^3, z] = z \) identically for \( \forall a_i, b_i, z \in S_{\text{matt}} \). Thus, we do not have different equivalence classes at all, and therefore for (4.73)
\[
K^{(4,4)}_{0} (S^{(4)}_{\text{matt}}) = 1, \quad (4.76)
\]
in our multiplicative notation.

Let us consider the case, when the completion group of classes has different arity from that of the initial semigroup, i.e \( K^{(m,n)}_{0} (S^{(m)}) \) with \( n \neq m \).

Example 4.25. The representatives of the residue (congruence) class \( S_{\text{res}} = [[7]]_{10} = \{10k + 7\}, k \in \mathbb{N}_0 \) form a 5-ary semigroup with respect to multiplication (see the arity shape for residue classes \( [[a]]_b \) in DUPLII [2017]). We take
\[
S^{(5)}_{\text{res}} = \left\{ \{10k + 7\} \mid \mu^{(5)}_{\text{res}} \right\}, \quad (4.77)
\]
where 5-ary multiplication \( \mu^{(5)}_{\text{res}} \) is the ordinary product (in \( \mathbb{Z}_+ \)) \( \mu^{(5)}_{\text{res}} [a_1, a_2, a_3, a_4, a_5] = a_1 a_2 a_3 a_4 a_5, \quad a_i = 10k_i + 7, \quad k_i \in \mathbb{N}_0 \).

Now the doubles (3.44) are
\[
S_{\text{res}} = \left( \begin{array}{c} 10k + 7 \\ 10l + 7 \end{array} \right) \in [[7]]_{10} \times [[7]]_{10}, \quad k, l \in \mathbb{N}_0. \quad (4.78)
\]
The equivalence relations \((4.17)\) and \((4.18)\) for \(m = 5\) are
\[
\left( \begin{array}{c} 10k_1 + 7 \\ 10l_1 + 7 \end{array} \right) \sim_{\text{gauge}} \left( \begin{array}{c} 10k_2 + 7 \\ 10l_2 + 7 \end{array} \right) \iff \exists x, y \in S_{\text{res}} \left( \frac{(10k_1 + 7)^4 x}{(10l_1 + 7)^4 x} = \frac{(10k_2 + 7)^4 y}{(10l_2 + 7)^4 y} \right), \quad k_i, l_i \in \mathbb{N}_0, \tag{4.79}
\]
and
\[
\left( \begin{array}{c} 10k_1 + 7 \\ 10l_1 + 7 \end{array} \right) \sim_{\text{twist}} \left( \begin{array}{c} 10k_2 + 7 \\ 10l_2 + 7 \end{array} \right) \iff \exists x, y \in S_{\text{res}} \left( (10k_1 + 7)^4 (10l_1 + 7)^4 z = (10k_2 + 7)^4 (10l_1 + 7)^2 z \right). \tag{4.80}
\]

It follows from the “gauge” shifts \((4.79)\) that the components of the double \((4.78)\) are mutually prime. Some of the equivalence relations are (we present only those with \(k_1 < k_2\))
\[
\begin{align*}
(7) & \sim (77) \sim (147), \\
(17) & \sim (187) \sim (357), \\
(77) & \sim (17) \sim (27) \sim (57) \sim (297) \sim (407) \sim (567) \sim (777), \\
(17) & \sim (187) \sim (357) \sim (87) \sim (147) \sim (177) \sim (377) \sim (637) \sim (767) \sim (776). \\
\end{align*}
\]

We denote the equivalence class by
\[
\tilde{\mathcal{S}} = \mathcal{S}_{\text{res}} = \left[ \begin{array}{c} a \\ b \end{array} \right]_{\text{res}} = \left[ \begin{array}{c} 10k + 7 \\ 10l + 7 \end{array} \right]_{\text{res}} \in S_{\text{res}} \times S_{\text{res}} / \sim, \quad k, l \in \mathbb{N}_0, \tag{4.82}
\]
and we use the minimal representative, when needed and possible. The polyadic group of the equivalence classes can be obtained, if we define the classes multiplication and the querelement. If we choose the 5-ary componentwise multiplication \((4.45)\), then we obtain the group completion of the same arity \(K_0^{(5,5)}(S_{\text{res}}^{(5)})\), as in the previous Example 4.23. The more exotic case is to use for the classes doubles \(\tilde{S}_{\text{res}}\) the changing arity product \((4.14)\), which gives us the ternary group of equivalence classes, which is built from the 5-ary semigroup. Indeed, let \(\tilde{\mu}_{\text{res}}^{(3)}\) be the ternary multiplication of the equivalence classes defined by (cf. \((4.14)\))
\[
\tilde{\mu}_{\text{res}}^{(3)} \left[ \tilde{S}_1, \tilde{S}_2, \tilde{S}_3 \right] = \left[ \begin{array}{c} (10k_1 + 7) (10k_2 + 7) (10k_3 + 7) (10l_1 + 7) (10l_2 + 7) \\ 10l_3 + 7 \end{array} \right]_{\text{res}}, \quad k_i, l_i \in \mathbb{N}_0. \tag{4.83}
\]

The querelement \(\tilde{\mathcal{S}}\) can be determined from the manifest form of the ternary multiplication \(\tilde{\mu}_{\text{res}}^{(3)}\) of the equivalence classes, and it has the general form
\[
\tilde{\mathcal{S}} = \left[ \begin{array}{c} \bar{a} \\ \bar{b} \end{array} \right]_{\text{res}} = \tilde{\mu}_{\text{res}}^{(1)} \left[ \tilde{\mathcal{S}} \right] \sim \left[ \begin{array}{c} a^3b^2 \\ a^2b^3 \end{array} \right]_{\text{res}} \sim \left[ \begin{array}{c} (10k + 7)^3 (10l + 7)^2 \\ (10k + 7)^2 (10l + 7)^3 \end{array} \right]_{\text{res}}, \quad a, b \in S_{\text{res}}, \quad k, l \in \mathbb{N}. \tag{4.84}
\]

Thus, the ternary group of the classes \((4.82)\)
\[
K_0^{(5,3)}(S_{\text{res}}^{(5)}) = \left\langle \mathcal{S}_{\text{res}} \mid \tilde{\mu}_{\text{res}}^{(3)}, \tilde{\mu}_{\text{res}}^{(1)} \right\rangle \tag{4.85}
\]
is the completion group of the 5-ary semigroup \(S_{\text{res}}^{(5)}\).
Acknowledgements. The author is deeply thankful to Joachim Cuntz, Siegfried Echterhoff and Christian Voigt for long-ago explanations of $K$-theory which indeed now re-awakened my former interest to consider higher arities in this promising direction, and grateful to Vladimir Akulov, Mike Hewitt, Dimitrij Leites, Thomas Nordahl, Vladimir Tkach, Raimund Vogl and Alexander Voronov for useful discussions, and valuable help.

REFERENCES

BORCEUX, F. (1994). Handbook of categorical algebra 1. Basic category theory, Vol. 50 of Encyclopedia of Mathematics and its Applications. Cambridge: Cambridge University Press. [Not cited]

BOURBAKI, N. (1998). Elements of Mathematics: Algebra I. Berlin: Springer. [Cited on page 2]

BRANDT, H. (1927). Über eine Verallgemeinerung des Gruppenbegriffes. Math. Annalen 96, 360–367. [Cited on page 2]

BRUCK, R. H. (1966). A Survey on Binary Systems. New York: Springer-Verlag. [Cited on page 2]

CLIFFORD, A. H. AND G. B. PRESTON (1961). The Algebraic Theory of Semigroups, Vol. 1. Providence: Amer. Math. Soc. [Cited on page 2]

DÖRNETE, W. (1929). Untersuchungen über einen verallgemeinerten Gruppenbegriff. Math. Z. 29, 1–19. [Cited on pages 4 and 14]

DUPLJ, S. (2017). Polyadic integer numbers and finite $(m, n)$-fields. $p$-Adic Numbers, Ultrametric Analysis and Appl. 9 (4), 257–281. arXiv:math.RA/1707.00719. [Cited on page 18]

DUPLJ, S. (2018). Polyadic algebraic structures and their representations. In: S. DUPLJ (Ed.), Exotic Algebraic and Geometric Structures in Theoretical Physics, New York: Nova Publishers, pp. 251–308. arXiv:math.RT/1308.4060. [Cited on pages 2, 9, 10, and 15]

DUPLJ, S. (2019). Arity shape of polyadic algebraic structures. J. Math. Physics, Analysis, Geometry 15 (1), 3–56. [Cited on page 8]

DUPLJ, S. (2021). Higher braid groups and regular semigroups from polyadic-binary correspondence. Mathematics 9 (9), 972. [Cited on pages 4 and 8]

DUPLJ, S. (2022). Polyadic analogs of direct product. Universe 8 (4), 230. [Cited on pages 8 and 9]

GLEICHGEGWICHT, B. AND K. GLAZEK (1967). Remarks on $n$-groups as abstract algebras. Colloq. Math. 17, 209–219. [Cited on page 4]

HAUSMANN, B. A. AND Ø. ORE (1937). Theory of quasigroups. Amer. J. Math. 59, 983–1004. [Cited on page 2]

HUNGERFORD, T. W. (1974). Algebra. New York: Springer. [Not cited]

IANCU, L. (1991). On the category of $n$-groups. Bul. stiin. Univ. Baia Mare, Seria B 7 (1/2), 9–14. [Not cited]

KAROUBI, M. (1978). $K$-theory. An introduction. Berlin: Springer-Verlag. [Cited on pages 2, 4, 5, and 6]

LAMBEK, J. (1966). Lectures on Rings and Modules. Providence: Blaisdell. [Not cited]

LANG, S. (1965). Algebra. Reading, Mass.: Addison-Wesley. [Not cited]

MAC LANE, S. (1971). Categories for the Working Mathematician. Berlin: Springer-Verlag. [Not cited]

MICHALSKI, J. (1984a). Free products of $n$-groups. Fund. Math. 123 (1), 11–20. [Not cited]

MICHALSKI, J. (1984b). On the category of $n$-groups. Fund. Math. 122 (3), 187–197. [Not cited]

POST, E. L. (1940). Polyadic groups. Trans. Amer. Math. Soc. 48, 208–350. [Cited on pages 2 and 3]

ROSENBERG, J. (1994). Algebraic $K$-theory and its applications. New York: Springer-Verlag. [Cited on pages 2, 4, 5, 6, and 7]

SCHUCHKIN, N. A. (2014). Direct product of $n$-ary groups. Chebysh. Sb. 15 (2), 101–121. [Not cited]

WEBEL, C. A. (1986). The K-Book. An Introduction to Algebraic $K$-theory. Providence: AMS. [Cited on pages 4, 5, and 6]