Spin-triplet superconductivity has been found in several materials including $Sr_2RuO_4$ and uranium-based compounds (UPT$_3$, $UBGe_2$, $UGe_2$, $URuGe_2$). While the phenomenology of the triplet state is well known (it is similar to that of $p$-wave-superfluid $He^3$), the microscopic origin of the triplet pairing in each of these materials and its relation to the band structure are the subject of intense investigation. The symmetry of the order parameter of the result-}

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The attention to spin-orbit coupling in the elec-

For a cubic center-symmetric two-component lattice the electron spin coupling to the inversion symmetry breaking sublattice displacement $w$ can be described phe-

Here, $\hat{a}^\dagger(\hat{a})$ and $\hat{f}^\dagger(\hat{f})$ stand for creation(annihilation) operators of phonons and electrons, $l_\zeta$, $\omega_\zeta$ and $m_\zeta$ are the polarisation vector, frequency and reduced mass of the optical phonon $\zeta$, indices $\mu, \nu$ identify the electron in the Kramers doublet for each momentum (spin for a simple conduction band), and the parameter $\alpha$ is a material constant. In a cubic crystal, we neglect a small difference between the LO and TO phonon frequencies, thus $\omega_\zeta \approx \omega_0$. In a less symmetric crystal, $\hat{V}_\zeta = \sum_{k=x,y,z} \hat{A}^\dagger_k(\hat{p}; \zeta; \hat{q})$, where $\hat{A}^\dagger_k(\hat{p}; \zeta; \hat{q}) = A^\dagger_k(\hat{p}; \zeta; \hat{q})$ and $A_k(-\hat{p}; \zeta; \hat{q}) = -A^\dagger_k(\hat{p}; \zeta; \hat{q})$, thus, $\hat{V}_\zeta(-\hat{p}; \zeta; \hat{q}) = -\hat{V}_\zeta(\hat{p}; \zeta; \hat{q})$, $\hat{V}_\zeta(\hat{p}; -\hat{q}) = \hat{V}_\zeta(\hat{p}; \hat{q})$. For example, in a layered crystal with a square lattice where we neglect the electron motion perpendicular to the $x$-$y$ planes, we take into account two invariants analogous to Eq. (11): $\alpha_{\perp \omega}\langle p^x \hat{\sigma}^y - p^y \hat{\sigma}^x \rangle$ and $\alpha_{\parallel \omega}\langle \hat{\sigma}^z \rangle$, to distinguish between the electron coupling to optical phonons polarized perpendicu-

A spin-dependent e-ph coupling, Eq. (2) and a virtual polarisation of the crystalline unit cell (10) result in the effective spin-dependent interaction between electrons on

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the opposite sides of the Fermi surface,
\[ \mathcal{U} = -\sum_{\mathbf{p}_1, \mathbf{p}_2} M^{\alpha\beta}_{\gamma\delta} \hat{f}^\dagger_{\mathbf{p}_1} \hat{f}_{\mathbf{p}_2} \hat{f}^\dagger_{-\mathbf{p}_1} \hat{f}_{-\mathbf{p}_2} \, \delta_{\beta\gamma}, \]  
(4)

\[ M^{\alpha\beta}_{\gamma\delta} = \sum_{\zeta} \omega_{\zeta} \frac{V_{\alpha\beta}(\mathbf{p}_1 + \mathbf{p}_2)}{\omega^2_{\zeta} - (\varepsilon_{\mathbf{p}_2} - \varepsilon_{\mathbf{p}_1})^2} \left( \mathbf{p}_1 - \mathbf{p}_2 \right) \cdot \left( \mathbf{p}_1 + \mathbf{p}_2 \right). \]  
(5)

In Eq. (4), we neglect the dispersion of optical phonon modes, \( \omega_{\zeta}(q) \approx \omega_{\zeta} \) and take into account the quasiparticle states close to the Fermi surface \( \mathcal{p} = |\mathbf{p}| \approx p_F \), with \( |\varepsilon_{\mathbf{p}_2}| \leq \omega_{\zeta} \), where \( \varepsilon_{\mathbf{p}} \) is the quasiparticle energy calculated with respect to the Fermi level. The effective electron-electron (e-e) interaction, \( \mathcal{U} \) is spin- and momentum-dependent, therefore, it may allow for attraction in the spin-triplet channel and odd pairing.

To determine what triplet phase can be formed in a metal with the Cooper interaction in Eqs. (4,5), one needs to identify particle-particle channels in which the electron-electron interaction is attractive. The classification of pairing channels and the resulting triplet order parameter is usually based upon the symmetry of the electronic Hamiltonian. Following a procedure developed for \( p \)-wave superfluidity in He\( ^3 \) [2,7], we determine operators of singlet (\( \hat{b} \)) and triplet (\( \hat{t}_s \)) electron pairs,
\[ \hat{b}(\mathbf{p}) = i\sigma^y \hat{f}_{\mathbf{p}} \hat{f}^\dagger_{-\mathbf{p}}; \quad \hat{t}_s(\mathbf{p}) = i(\sigma^y \sigma^x)_{\mu\nu} \hat{f}_{\mathbf{p}} \hat{f}^\dagger_{\mathbf{p}}, \]  
(6)

In terms of these operators [17], the Hamiltonian \( \mathcal{U} \) in Eq. (4) can be written down in a separable form,
\[ \mathcal{U} = -\sum_{\mathbf{p}_1, \mathbf{p}_2} T^{s_1 s_2} t^\dagger_{s_2}(\mathbf{p}_1) t_{s_2}(\mathbf{p}_2) + \hat{C} \hat{b}^\dagger(\mathbf{p}_1) \hat{b}(\mathbf{p}_2), \]  
(7)

which includes the interaction both in the singlet channel [18], with the constant \( \hat{C} = M^{\alpha\beta}_{\gamma\delta} \sigma^y_{\alpha\beta} \sigma^y_{\delta\gamma} \) and in the triplet channel, with
\[ T^{s_1 s_2} = M_{\gamma\delta}^{\alpha\beta} \left( \sigma^y_{\alpha\beta} \sigma^y_{\gamma\delta} \right) \phi_{s_1}^{s_2}, \]  
(8)

Below, we limit the analysis to the case of \( p \)-wave pairing (excluding higher-order angular harmonics) described by the tensor order parameter,
\[ B_{m s} = \sum_{\mathbf{p}} n_m (\hat{t}_s(\mathbf{p})); \quad n = \frac{\mathbf{p}}{\mathbf{p}}, \]  
(9)

and focus it on two model examples: (i) a 3D metal with a cubic lattice and a spherical Fermi surface and (ii) a layered material with a square in-plane lattice and a cylindrical Fermi surface. In such symmetric systems, triplet pairs can be characterised by well-defined quantum number related to irreducible tensor representations of the corresponding symmetry group.

(i). In the case of a model metal with a cubic lattice and a spherical Fermi surface the symmetry is the full rotational group \( \text{SO}_3 \). The in-volvent of the spin-orbit coupling in the formation of the pair interaction assumes that the spin and orbital degrees of freedom of the Cooper pair are not independent, so that its total spin \( S = 1 \) and orbital angular momentum, \( L = 1 \) sum into the total angular momentum, \( J = 0, 1, 2 \). The tensor structure of the order parameter for these values of \( J \) is the following:
\[ B^{J=1}_{m s} = \delta_{m s} D e^{i\phi}, \] where \( B \) is a scalar; \( B^{J=1}_{m s} = \epsilon^{m s l} D_l \), where \( D \) is a complex vector; and \( B^{J=2}_{m s} \) is a traceless symmetric tensor.

Thus, decompose e-e interaction in Eq. (9) into three channels corresponding to different angular momenta of the pair. Using the relation of the matrix \( M_{\gamma\delta}^{\alpha\beta} \) with the e-ph coupling \( V_{\alpha\beta} \) in Eq. (2), we find that
\[ T^{s_1 s_2} = g_{s_0} \sum_{m_1, m_2} n_1 m_1 n_2 m_2 C^{s_1 s_2}_{m_1 m_2}, \]  
\[ G^{s_1 s_2}_{m_1 m_2} = \kappa_0 \delta_{m_1 s_1} \delta_{m_2 s_2} + \kappa_1 \epsilon_{m_1 s_1} \epsilon_{m_2 s_2} + \kappa_2 \left[ \epsilon_{m_1 m_2} \epsilon_{s_{1 s_{2}} s_{m_1 s_{2}}} - \frac{2}{3} \delta_{m_1 s_1} \delta_{m_2 s_2} \right]. \]  

The first term in the matrix \( G^{s_1 s_2}_{m_1 m_2} \) is a product of projectors onto the \( J = 0 \) paired state; the second projects onto the states with \( J = 1 \), whereas the last term selects only the states with \( J = 2 \). Interaction in each channel is described by the corresponding constant,
\[ g_J = \kappa_J g_{s_0}, \quad g_{s_0} \sim (\alpha/\omega_0)^2 p_F^2/2m_o, \]  
(10)

where the scale of the interaction depends on the strength of the original electron-phonon interaction for electrons near the Fermi surface, frequency \( \omega_0 \) and reduced mass \( m_o \) of the optical phonon (here we disregard a small difference between longitudinal and transverse optical phonon modes in a cubic crystal). Interaction in channels with different \( J \)'s differs by the numerical factors,
\[ \kappa_0 = -\frac{4}{3}; \quad \kappa_1 = \frac{1}{2}; \quad \kappa_2 = -\frac{1}{2}. \]  
(11)

According to Eq. (11), the positive sign of \( g_J \) corresponds to attraction. Since only one of the effective coupling constants \( g_J = g_{s_0} \kappa_J \) is positive, we conclude that in the cubic system Cooper interaction due to the spin-orbit-assisted e-ph coupling leads to the condensate of pairs with the total angular momentum \( J = 1 \).

By selecting a pairing channel with \( J = 1 \) we do not fully determine the order parameter \( B_{m s} \) in the lowest energy triplet state, yet. In \( B_{m s}^{J=1} = \epsilon^{m s l} D_l \), \( D = De^{i\phi} (I + iY')/\sqrt{I^2 + (Y')^2} \) is a complex vector [6, 10] parametrized using two perpendicular vectors, \( I \) and \( Y' \), \( I^2 \geq (Y')^2 \). To find a favourable state, we minimize the condensate energy,
\[ \mathcal{E} = \sum_{\mathbf{p} \mu} (|\varepsilon_{\mathbf{p}}| - E_{\mathbf{p} \mu}) + \sum_{m s} g |B_{m s}|^2, \]  
(12)
as a function of $\mathbf{D}$, which includes finding the relative size of two vectors $\mathbf{l}$ and $\mathbf{l}'$ and the absolute value of $D$. The expression for $E$ in Eq. 12 can be formally obtained using the Hubbard-Stratonovich transformation which splits the electron-electron interaction in Eq. 7 using the field $B_{ms}$ and, then, by integrating out fermionic degrees of freedom. Equation 13 is the self-consistency equation for the minimum of $E$, and

$$E_{p\mu} = \sqrt{\varepsilon_p^2 + |d_p|^2 - \mu |(d_p \times d_p^*)|}$$

(13)

describes the spectrum of excitations [19] in the phase with order parameter $B_{ms}$, where three components of the vector $\mathbf{d}_p$ are defined as

$$d_p^* = g \sum_m n_m B_{ms}.$$ 

(14)

The vector product $\mathbf{r}^2 \mathbf{d}_p \times \mathbf{d}_l$ determines the spin quantization axis and spin splitting for Bogolyubov quasiparticles for each direction $\mathbf{n} = \mathbf{p}/p$ on the Fermi sphere and $\mu = \pm 1$ denotes the polarization of the quasiparticle spin with respect to such an axis. In Eqs. 12, 13, $\varepsilon_p$ is the quasiparticle energy in the normal state calculated with respect to the Fermi level, and $g$ is the interaction constant in the attraction channel (here, $g = g_1 \equiv \kappa g_{so}$).

In the context of pairing with $J = 1$ and $B_{ms} = e^{\text{msI}} D$, two possible extrema of the function $E$ can be found using the analysis of the degeneracy space of the order parameter 19. For fixed $|\mathbf{D}| = D$, $E(\mathbf{D})$ takes extremal values on vectors $\mathbf{D}$ that cover minimal surfaces under the application of $SO_3$ rotations. Two such possibilities are $\mathbf{D}_0 = e^{i\phi} D$ where $\mathbf{l}$ is a real unit vector (that’s, $\mathbf{l}' = 0$), and $\mathbf{D}_1 = e^{i\phi} D$ where $\mathbf{l}$ and $\mathbf{l}'$ are two perpendicular unit vectors. For each of these two cases we find the value of $D$ by minimizing the ground state energy $E$ in Eq. 12. The results 20 are shown in Table I indicating that the phase with $\mathbf{D}_0 = e^{i\phi} D$ and an anisotropic gap vanishing along the direction $\mathbf{l}$ (since $d = gD|\mathbf{n} \times \mathbf{l}|$) is energetically favorable.

Although the example of a cubic crystal with highly symmetric Fermi surface may be too abstract, nevertheless, it illustrates that the spin-orbit-assisted e-ph coupling can generate triplet pairing between electrons and that the suggested pairing mechanism is highly prescriptive for the structure of the resulting order parameter.

(ii). In a layered material with a square-symmetric lattice, the spin-orbit-assisted e-ph coupling is also able to generate triplet pairing with definite phases. For simplicity (and with the reference to an almost cylindrical $\gamma$-sheet in the Fermi surface of Sr$_2$RuO$_4$ [11]) we consider below a metal with a cylindrical Fermi surface. Similarly to the BCS theory, the effective attraction $\mathcal{U}$ in Eqs. 14 is point-like, so that only electrons from the same layer can form a pair. Thus, we shall consider only two-dimensional order parameters (the interlayer tunneling can be treated using the Lawrence-Doniach approach [14]). In this case, triplet $p$-wave pairs can be characterized by the component $J_z$ of the total angular momentum in the direction perpendicular to the layers, which is the sum of the projection $L_z = \pm 1$ of their orbital angular momentum and $S_z = \pm 1$ of the pair spin. For $J_z = \pm 1$ and $J_z = \pm 2$, the quantum number $J_z$ fully determines the order parameter, whereas for $J_z = 0$, pairs should be additionally classified according to whether they are symmetric ($s$) or antisymmetric ($a$) with respect to in-plane $x \leftrightarrow y$ reflection, for which we reserve notations $J_z = 0^s$ and $J_z = 0^a$, respectively.

The second line in Table II summarizes tensor structure, $B_{ms} = P_{ms}^J e^{i\phi} D$ of the order parameter for phases with $J_z = 0^s, 0^a, \pm 1, \pm 2$, where matrices $P_{ms}^J$ form the orthogonal basis which realise irreducible representations of $J_z$, and where $D^2 = \sum_{ms} |B_{ms}|^2$.

Using matrices $P_{ms}^J$, the effective interaction $\mathcal{U}$ can be written down in the separable form, Eq. 7 with

$$T^{s\_s} = \sum_{m_1 m_2 J_z} \left( \kappa J_z g_{so} + \kappa J_z g_{so} \right) n_{m_1}^* n_{m_2}^* P_{s_{m_1}^J}^* P_{s_{m_2}^J}^*$$

$$g_{so}^* \sim \left( \alpha \omega / \sqrt{2} \right)^2 p_F / 2 m_c, \quad \kappa = \perp, \parallel,$$

which is determined by couplings [3] to optical phonons polarized across ($\perp = \perp$) and parallel ($\parallel = \parallel$) to the layer. The values of coefficients $\kappa J_z$ and $\kappa J_z$ are listed in Table I11 which shows that positive coupling constants (determining the attraction in the corresponding Cooper channel) may appear only for electron pairs in the following channels: (a) $J_z = 0^s$ where $g_{so} = \kappa_0 g_{so}^* + \kappa_0 g_{so}^* = 2 g_{so}^* - g_{so}^*$, and (b) $J_z = \pm 1$ where $g_{so} = \kappa_{s\_s} g_{so}^* + \kappa_{s\_s} g_{so}^* = g_{so}^* - g_{so}$. This indicates that, within the proposed mechanism, only two triplet phases are possible.

To determine which phase is realized we need to compare the relative strength of the electron coupling to the in- and out-of-plane sublattice vibrations. If $g_{so}^* < \frac{3}{2} g_{so}$, so that $g_{so}^* > g_{so}$, the ground state with $J_z = 0^s$ is realized. This phase is characterized by the vector order parameter $\mathbf{D}$ and an isotropic gap,

$$d_{\mathbf{p}} = \frac{g D}{\sqrt{2}} \left( \begin{array}{c} n_y \\ n_x \\ 0 \end{array} \right), \quad E_p = \sqrt{\varepsilon_p^2 + \Delta^2}, \quad \Delta = 2 \omega_{1\_2} e^{-2/\nu g}, \quad g = 2 g_{so}^* - g_{so}.$$ 

(15)
If $\omega_0 > \frac{1}{2} g_{\omega_0}$, so that $g_{\pm 1} > g_{0s}$ (in terms of microscopic parameters, for a stronger coupling to the in-plane lattice displacement, $\alpha_{||}/(\sqrt{m_{\perp}} \omega_{||}) > \sqrt{2} \alpha_{\perp}/(\sqrt{m_{\perp}} \omega_{\perp})$), then the ground state belongs to the phase with $J_s = \pm 1$ which is similar to the Anderson-Brinkmann-Morel (ABM) phase discussed in the theory of superfluid $He^3$. This phase also has a circularly isotropic gap:

$$d_{\pm 1} = \frac{g D}{\sqrt{2}} \left( \begin{array}{cc} 0 & \frac{E_p}{\sqrt{\epsilon_p^2 + \Delta^2}} \\ \nu x \pm i \nu y & \Delta = 2 \omega_{||} e^{-2/\nu g} \\ 0 & g = g_{\omega_0} - g_{\omega_0} \end{array} \right)$$

In conclusion, we have shown that the spin-orbit-assisted e-ph coupling can generate triplet pairing with a definite tensor structure of the $p$-wave order parameter in each type of a crystal. For example, in a layered metal where conducting planes have square lattice structure, the proposed mechanism of triplet pairing leads to two possible phases in Eqs. (13) and (14), one of which (ABM) is actually considered as the most plausible candidate for the triplet order parameter in superconducting $Sr_2RuO_4$ that has the layered perovskite structure with body-centred tetragonal group symmetry. On the one hand, the latter coincidence may be purely accidental, so that a detailed analysis of the phonon spectrum and the spin-orbit-assisted e-ph coupling strength in each particular material based upon first-principle calculations is needed.

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### Table II: Triplet pairing order parameter for electrons with cylindrical Fermi surface in a layered material with square lattice and the interaction constants in the corresponding $J_s$ channels due to the spin-orbit-assisted e-ph coupling.

| $J_s = \pm 2$ | $J_s = \pm 1$ (ABM) | $J_s = 0^+$ | $J_s = 0^-$ |
|--------------|-----------------|-------------|-------------|
| $P^{\pm 2}_{\pm m} = \frac{1}{2} (\delta_{\pm m} + i \delta_{m}) (\delta_{\pm m} + i \delta_{m})$ | $P^{\pm 1}_{\pm m} = \frac{1}{\sqrt{2}} (\delta_{\pm m} + i \delta_{m}) \delta_{\pm m}$ | $P^{0^+}_{\pm m} = \frac{1}{\sqrt{2}} (\delta_{\pm m} \delta_{\pm m} + \delta_{m} \delta_{m})$ | $P^{0^+}_{\pm m} = \frac{1}{\sqrt{2}} (\delta_{\pm m} \delta_{\pm m} - \delta_{m} \delta_{m})$ |
| $\varphi_{\pm 2} = 0$ | $\varphi_{\pm 1} = 1$ | $\varphi_{0^+} = 2$ |
| $\varphi_{0^+} = -1$ | $\varphi_{0} = -1$ |

Here $\sqrt{2} D = 2^{3/2} (\omega_{||} / g) e^{-2/\nu g}$ was found from minimizing $\mathcal{E}$ in Eq. (12) with respect to $D$.

### References

1. A. P. Mackenzie and Y. Maeno, Rev. Mod. Phys. 75, 657 (2003) and refs. therein.
2. R. Joynt and L. Taillefer, Rev. Mod. Phys. 74, 235 (2002).
3. S. S. Saxena et al., Nature 406, 587 (2000).
4. H. R. Ott et al., Phys. Rev. Lett. 52, 1915 (1984).
5. F. Hardy and A. D. Huxley, Phys. Rev. Lett. 94, 247006 (2005).
6. A. J. Leggett, Rev. Mod. Phys. 47, 331 (1975).
7. P. W. Anderson and W. F. Brinkmann, in The Physics of Liquid and Solid Helium, eds. K. H. Bennemann and J. B. Ketterson, Wiley (1978).
8. M. Sigrist and T. M. Rice, J. Phys. Condens. Matt. 7, 643 (1995); D. F. Agterberg, T. M. Rice, M. Sigrist, Phys. Rev. Lett. 78, 3374 (1997); M. E. Zhitomirsky and T. M. Rice, ibid 87, 057001 (2001); T. M. Rice, Nature 396, 627 (1998).
9. D. L. Cox and M. B. Maple, Physics Today 48(2), 32 (1995); R. Roussev and A. J. Millis, Phys. Rev. B 63, 140504 (2001); Z. Wang, W. Mao, and K. Bedell, Phys. Rev. Lett. 87, 257001 (2001).
10. I. I. Mazin and D. Singh, Phys. Rev. Lett. 79, 733 (1997); ibid 82, 4324 (1999).
11. A. Shick and W. Pickett, Phys. Rev. Lett. 86, 300 (2001). 
12. T. R. Kirkpatrick and D. Belitz, Phys. Rev. Lett. 92, 037001 (2004).
13. L. P. Gor'kov and E. I. Rashba, Phys. Rev. Lett. 87, 037004 (2001); P. A. Frigeri et al., Phys. Rev. Lett. 92, 097001 (2004); K. V. Samokhin, E. S. Zijlstra, and S. K. Bose, Phys. Rev. B 69, 094514 (2004).
14. E. Bauer et al., Phys. Rev. Lett. 92, 02703 (2004).
15. We symmetrize $\mathcal{V}_e$ to take into account non-commutativity of the momentum $\mathbf{p}$ and the coordinate-dependent $\mathbf{w}(\mathbf{x})$.
16. J. B. Ketterson and S. N. Song, Superconductivity, Cambridge UP, 1998.
17. The singlet operator $\hat{b}$ is an even function of $\mathbf{p}$, while the triplet $\hat{t}_s$ are odd. Thus, the e-ph coupling generates an effective electron-electron interaction in the triplet channel, due to the asymmetry $V^c(-\mathbf{p}, \mathbf{q}) = -V^c(\mathbf{p}, \mathbf{q})$ and $V^c(\mathbf{p}, \mathbf{q}) = V^c(\mathbf{p}, \mathbf{q})$ of its spin-orbit-assisted part. The latter property is specific to optical phonons. The acoustic e-ph coupling has a different symmetry, $V^c(\mathbf{p}, \mathbf{q}) = -V^c(\mathbf{p}, \mathbf{q})$ and cannot generate the effective interaction in the $p$-wave channel.
S-wave interaction in Eq. (7) slightly changes the usual BCS coupling. This may be ignored if the latter is less than the Coulomb repulsion. In the $p$-wave channel Coulomb interaction is suppressed, due to screening.

V.I. Fal’ko and I.S. Shapiro, Zh. Eksp. Teor. Fiz. 91, 1194 (1986) [Sov. Phys. JETP 64, 706 (1986)]

$\nu$ is the Fermi density of states in one spin component.

Z.Q. Mao, et al, Phys. Rev. B 63, 144514 (2001)

The sign of the isotope effect observed in Ref. [21] depended on the quality of the structure and it is not clear whether it was related to the change of the frequency of the optical phonon mode upon the isotopic substitution. The inverse sign of the isotope effect in the purest samples in Ref. [21] may have been the result of pair breaking by the annealing-induced disorder and differently diffusing light and heavy isotopes.