Quantum state verification (QSV) is the task of relying on local measurements only to verify that a given quantum device does produce the desired target state. Up to now, certain types of entangled states can be verified efficiently or even optimally by QSV. However, given an arbitrary entangled state, how to design its verification protocol remains an open problem. This study presents a systematic strategy to tackle this problem by considering the locality of what it initiates as the choice-independent measurement protocols, whose operators can be directly achieved when they are homogeneous. Taking several typical entangled states as examples, this study shows the explicit procedures of the protocol design using standard Pauli projections, demonstrating the superiority of the method for attaining better QSV strategies. Moreover, the framework can be naturally extended to other tasks such as the construction of entanglement witnesses, and even parameter estimation.

1. Introduction

With the rapid development of quantum techniques, numerous applications are moving toward practicality, such as quantum computing, quantum communication, and quantum metrology. Meanwhile, a fundamental task in all of these applications, i.e., quantum characterization is becoming more and more crucial alongside. The standard tool of quantum characterization of large quantum systems extremely tricky. Hence, attention has been turned to nontomographic methods among which quantum state verification (QSV) particularly stands out due to its unconditionally high efficiency, which is at least quadratically better than other methods. In the last few years, certain types of entangled quantum states and processes have been verified efficiently or even optimally using local measurements only; see Ref. [43] for a recent review.

As the task of QSV is to verify a quantum device that is supposed to produce the target state $|\psi\rangle$, but in fact states $\sigma_1, \sigma_2, \ldots, \sigma_N$ might be emitted. To accomplish this mission, one can randomly perform some pass-fail tests $\{\Omega, 1 - \Omega\}$, which detect any bad state $\sigma_i$ with fidelity $\langle \sigma_i | \psi \rangle \leq 1 - \epsilon$. Then, a verification protocol can be constructed as $\Omega = \sum \mu_i \Omega_i$, where $\mu_i$ denotes a probability distribution.

Importantly, these tests should be suitably designed such that the target state can always pass, i.e., $\Omega | \psi \rangle = | \psi \rangle$, $\forall i$. The maximal probability that a bad state passes the protocol is $1 - \nu(\Omega)$, where $\nu(\Omega) \approx 1 - \lambda_i(\Omega)$ denotes the spectral gap between the largest and the second largest eigenvalues of $\Omega$. Thus, in order to gain a confidence level $1 - \gamma$, the protocol $\Omega$ requires

$$N \geq \frac{\ln \gamma^{-1}}{\ln \{1 - \nu(\Omega)\epsilon^{-1}\}} \approx \frac{1}{\nu(\Omega)} \epsilon^{-1} \ln \gamma^{-1}$$

(1)

copies of the state to verify $| \psi \rangle$ within infidelity $\epsilon$.

Similarly, other characterization methods including tomography can also be performed probabilistically with the measurement settings following a probability distribution. The unknown state can then be reconstructed by tomography once the expectation values of all the measurement settings (or observables) are obtained, the success of which demands that the measurement outcomes and the settings must be matched. QSV, instead, owns a distinct feature that the measurement protocol can be regarded as a black box as we only need to know the numbers of “pass” and “fail” outcomes. The one-to-one correspondence between the measurement outcomes and the settings is not necessary in QSV. Hence, we dub such measurement protocols as being choice-independent.

In general, constructing an efficient verification protocol for a target state with only local measurements is difficult. Nevertheless, as the verification protocols are choice-independent, it is likely to start from an overall perspective of the verification protocol $\Omega$. In this way, the problem can be converted to check whether $\Omega$ can be realized locally or not. Coincidentally, in the
study of QSV, the so-called homogeneous protocols\(^{(24,25)}\) emerge, of which \(\Omega\) can be directly written down. The structure of homogeneous verification protocols is highly symmetric such that they have the best performance in the adversarial scenario\(^{(24,25)}\).

Indeed, several types of entangled states have already been efficiently verified by local homogeneous measurements\(^{(16,20,22,27,32)}\). Therefore, the equivalence between protocol design and the locality of homogeneous measurements offers us a possible way to convert this problem to the checking of the locality of homogeneous protocols directly by checking whether they can be local or not.

### Figure 1
Schematic illustration of the protocol design. Instead of searching for valid measurement settings then protocols among numerous possible local measurements, one can start with homogeneous measurement protocols directly by checking whether they can be local or not.

#### 2. Locality of Measurement Protocols

From a more general perspective, consider an arbitrary measurement protocol \(\Pi\), which can be decomposed into

\[
\Pi = \sum_i \mu_i \Pi_i
\]

where \(\Pi_i\)s are individual measurement settings and \(\mu_i\) is a probability distribution. We would like to find out under what circumstances \(\Pi\) is local. Without loss of generality, assume that we can realize an ensemble of \(s\)-outcome positive operator-valued measures (POVMs) \(\{M_1, \ldots, M_s\}\), where \(\sum_{i=1}^s M_i = 1\). For an \(n\)-partite quantum system, the protocol \(\Pi\) is called local if all the measurement settings are local, such that

\[
\Pi_{\mu_1, \ldots, \mu_s} = \sum_j z_{ij} M_1^{M_1} \otimes \cdots \otimes M_s^{M_s}
\]  

(3)

where the sum is taken over \(j = j_1, \ldots, j_s \in \{1, \ldots, s\}^n\), and the parameters \(z_{ij}\) are either 0 or 1 that tell us that outcomes \(j_1, \ldots, j_s\) of the measurement setting \(\Pi\) correspond to the “pass” instances. More generally, we can let \(z_{ij} \in [0, 1]\) if some outcomes are allowed to pass the test with probability \(0 < z_{ij} < 1\).

By combining all the measurement settings, we get the decomposition of a measurement protocol with the form

\[
\Pi = \sum_{i,j} p_{ij}^{(\Pi)} M_1^{M_1} \otimes \cdots \otimes M_s^{M_s}
\]  

(4)

where \(p_{ij}^{(\Pi)} := \mu_{ij} z_{ij}\) is called the quasi-probability distribution, as \(\sum_{i,j} p_{ij}^{(\Pi)}\) is typically not equal to 1. With this, we have the following theorem for the locality of measurement protocols.

**Theorem 1.** A measurement protocol is local iff the quasi-probability distribution \(p_{ij}\) satisfies the following two constraints under the representation of Equation (4).

- **Positivity**
  \[
  \min_{i,j} \{p_{ij}^{(\Pi)}\} \geq 0
  \]  

(5)

- **Completeness**
  \[
  S(\Pi) := \sum_i \max_j \{p_{ij}^{(\Pi)}\} \leq 1
  \]  

(6)

**Proof.** For a measurement protocol \(\Pi\) as in Equation (2), the probability distribution should satisfy (1) positivity \(\mu_i \geq 0, \forall i\); and (2) completeness \(\sum_i \mu_i = 1\). From Equation (4), the quasi-probability distribution is given by \(p_{ij} = \mu_{ij} z_{ij}\) with \(z_{ij} \in [0, 1]\), which then leads to the two constraints straightforwardly.

Our consideration here can be naturally extended to two other perspectives. First, one can consider local operations and classical communication (LOCC) for the measurement settings, also
known as adaptive measurements\cite{19-21,26} see Appendix A for the corresponding discussions. Second, the decomposition of Π as in Equation (4) with finite local measurements can be generalized to the infinite scenario using continuous local measurements; see Appendix B for detailed discussions.

3. Homogeneous QSV Protocols with Finite Local Projections

A first application of the previous discussion on the locality of measurement protocols is QSV. To verify a target pure state |ψ⟩, a homogeneous QSV protocol takes on the general form

\[ \Omega_{\text{Hom}} = (1 - \nu)|ψ⟩⟨ψ| + \nu |ϕ⟩⟨ϕ| \]

where 0 < \nu \leq 1. All eigenvalues of the homogeneous protocol are 1 – \nu except the largest one that is the unity. The parameter \nu is exactly the spectral gap, thus 1/\nu gives the scaling of the verification efficiency for homogeneous protocols.

We note that the identity 1 is a trivial measurement, which can be considered as no measurement at all or an arbitrary measurement whose outcomes must be accepted. Thus, for the limit \nu = 0, the protocol \Omega_{\text{Hom}} = 1 must be local. On the other hand, the protocol becomes \Omega_{\text{Hom}} = |ψ⟩⟨ψ| for \nu = 1 that cannot be realized locally since the target state |ψ⟩ is assumed to be entangled. Hence, in general, the homogeneous QSV protocol as in Equation (7) is the convex combination of the identity 1 and the projection |ψ⟩⟨ψ|, and the locality of \Omega_{\text{Hom}} can be interpreted as finding the local ball around 1 with the maximal value of the parameter \nu representing the radius.

An arbitrary measurement protocol for n-qubit systems can always be expanded with the Pauli representation uniquely as

\[ \Pi = \frac{1}{2^n} \sum_i c_i \sigma_{i_1} \otimes \cdots \otimes \sigma_{i_n} \]

where the coefficients are \[ c_i = \text{tr}(\Pi \sigma_{i_1} \otimes \cdots \otimes \sigma_{i_n}) \]

with \[ \sigma_i = \sigma_{i_1} \cdots \sigma_{i_n} \in \{0, 1, 2, 3\}^\otimes n \], and \[ \sigma_1 = 1 \]. Expanding the Pauli operators with a finite set of measurements \{M_{i_1}^1, \ldots, M_{i_n}^1\}_{i_n=1}^N, we have

\[ \sigma_i = \tilde{t}_{i_n} \cdot \tilde{M} \]

where

\[ \tilde{t}_{i_n} = \left[ t_{i_n}^{(1,1)}, \ldots, t_{i_n}^{(1,1)}, \ldots, t_{i_n}^{(1,1)}, \ldots, t_{i_n}^{(1,1)} \right] \]

and

\[ \tilde{M} = [M_{i_1}^1, \ldots, M_{i_m}^1, \ldots, M_{i_n}^1]^T \]

Then the Pauli representation can be converted to

\[ \sigma_{i_1} \otimes \cdots \otimes \sigma_{i_n} = \tilde{t}_{i_1} \otimes \cdots \otimes \tilde{t}_{i_n} \cdot \tilde{M}^{\otimes n} \]

Thus, we get the quasi-probability distribution

\[ p_{i_1 \cdots i_n} := \left( \frac{1}{2^n} \sum_i c_i \tilde{t}_{i_1} \otimes \cdots \otimes \tilde{t}_{i_n} \right)_{i_1 \cdots i_n} \]

Notice that due to the flattening vector \[ \tilde{M} \], the quasi-probability \[ p_{i_1 \cdots i_n} \] is the flattening 1D vector from the previous \[ p_{ij} \] as defined in Theorem 1. Such a quasi-probability distribution is a linear function, then for the homogeneous QSV protocol, we have

\[ p_i(\Omega_{\text{Hom}}) = (1 - \nu)p_i(1) + \nu p_i(\psi) \]

where \[ p_i(1) \] and \[ p_i(\psi) \] are the quasi-probability distributions of the identity and the projection on the target state, respectively.

The transformation between Pauli operators and the finite set of measurements \{M_{i_1}^1, \ldots, M_{i_m}^1\}_{i_m=1}^N requires these measurements to constitute a complete set of bases in the Hilbert space. In the following, we consider the standard Pauli projections, which are easy to realize in experiments.

4. Homogeneous QSV Protocols with Pauli Projections

The Pauli projections \[ \{P^0, P^1\}_{i=1}^N \] form an overcomplete set of bases for qubit systems, thus the representation is not unique. One possible transformation is \[ P_i^j = \frac{1}{2} \left[ I + (-1)^i \sigma_j \right] \] for \[ i = 1, 2, 3 \]. As for \[ \sigma_0 = I \], we specifically choose the symmetric form \[ I = \frac{1}{2} \sum_i P_i^i \], i.e.,

\[ T_{\text{Pauli}} = \begin{bmatrix} T_0 & T_1 & T_2 \\ T_1 & T_0 & -T_2 \\ T_2 & -T_1 & T_0 \end{bmatrix} = \begin{bmatrix} 1/3 & 1/3 & 1/3 & 1/3 & 1/3 \\ 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 & 1 \end{bmatrix} \]

With such a transformation, the following corollary can be generated from Theorem 1; see Appendix C for the proof.

\textbf{Corollary 1}. Considering a homogeneous QSV protocol \[ \Omega_{\text{Hom}} \] for the target state |ψ⟩ as defined in Equation (7), it is local under Pauli projections if the following two constraints are satisfied:

\begin{itemize}
  \item \textbf{Positivity} \[ \nu \leq \frac{1}{1 - 3 \min\{p_i(\psi)\}} \leq \frac{1}{2^n - 2^{1-n} + 1} \]
  \item \textbf{Completeness} \[ S(\psi) \leq 1 \]
\end{itemize}

Two remarks are in order. First, for the positivity constraint, we note that the first inequality in Equation (14) gives the radius of the local ball; while the second one is obtained by considering all possible target states. In other words, for the homogeneous QSV protocol, any target state can be verified with an efficiency no more than \[ O(2^n) \] as long as the completeness constraint is satisfied. Second, here we focus on the locality of homogeneous QSV protocols with local Pauli projections only, and it is reasonable to expect that more measurements should improve the efficiency of the protocols. However, this is not true as the efficiency is still bounded by \[ O(2^n) \] even infinite continuous local projections or multi-outcome measurements are considered; see Appendixes B.2 and B.3 for more details.
5. Applications

By employing Pauli projections, here we consider the verification of several typical entangled states with our method. More detailed analyses can be found in Appendix D.

(i) Bell state $|\Phi^+\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$. By using our method, a homogeneous QSV protocol using only local Pauli measurement settings can be designed with an efficiency of $1/\nu = 3$. Clearly, such a protocol is not optimal as the best one has an efficiency of $1/\nu = 3/2$. The reason lies in that a specific quasi-probability distribution is chosen in our method, where the identity operation exists in each measurement setting. Hence, with an appropriate revision process, our protocol can be improved to give exactly the optimal efficiency.

(ii) Three-qubit GHZ state $|\text{GHZ}_1\rangle = \frac{\sqrt{3}}{2}(|000\rangle + |111\rangle)$. With our method, the local homogeneous QSV protocol designed has an efficiency of $1/\nu = 17/4$. It is worse than that in Ref. [16] which is $1/\nu = 7/4$. However, with an additional revision process, a better efficiency of $1/\nu = 5/3$ can be obtained. Moreover, one can achieve the optimal efficiency of $1/\nu = 3/2$ with a proper choice of the transformation $T_{\text{Pauli}}$. Note that all stabilizer states can be verified by QSV protocols constructed with their stabilizers, which are in the Pauli group[16] thus our method is able to give the local homogeneous QSV protocols for all stabilizer states with the quasi-probability distribution based on the Pauli representation.

(iii) Three-qubit W state $|W_1\rangle = \frac{1}{\sqrt{3}}(|001\rangle + |010\rangle + |100\rangle)$. In this case, our method is not able to give a local homogeneous QSV protocol as the quasi-probability distribution has $S(W_1) = 1.40(7)$ which violates the completeness constraint. Even with a revision process, the constraint $\text{tr}(\Omega|\psi\rangle\langle\psi|) = 1$ for QSV cannot be satisfied for all the settings. Note, however, that this only means $|W_1\rangle$ cannot be verified by any local homogeneous protocol using the Pauli projections, but with other local projections it might be possible. On the other hand, a valid inhomogeneous protocol can be achieved by properly choosing the settings. Such a protocol has an efficiency of $1/\nu = 13/3$, which is better than that of the previous inhomogeneous protocol with Pauli projections.[26]

More importantly, if we allow LOCC for the verification protocol, we are able to obtain a local homogeneous QSV protocol by modifying the completeness constraint. The efficiency is given by $1/\nu = 2$, which is better than the $1/\nu = 3$ reported in Ref. [26]. It is slightly worse than that of the nearly optimal homogeneous protocol of $1/\nu = 8/5$.[12], which however, requires much more complex local measurement settings.

6. Extended Applications

The abstraction of choice-independent measurement protocols can be naturally extended to other tasks concerning only the protocol operators instead of the specific settings, such as the entanglement witness for detecting entanglement. Moreover, with appropriate modifications, the universality of choice-independent measurement protocols enables its extension to the task of parameter estimation including fidelity, entanglement, and so on.

6.1. Construction of Entanglement Witness

An entanglement witness $W$ is defined if for every separable state $\rho_{\text{sep}}$, one has $\text{tr}(W\rho_{\text{sep}}) \geq 0$; and for some entangled state $\rho_{\text{ent}}$, $\text{tr}(W\rho_{\text{ent}}) < 0$. Witnesses for detecting entanglement are typically of the form

$$W = \kappa \mathbb{1} - |\psi\rangle\langle\psi|$$

where $|\psi\rangle$ is the entangled state to be detected. The parameter $\kappa$ is the square of the maximal Schmidt coefficient of $|\psi\rangle$ when all bipartitions are considered.[44]

We can associate entanglement witnesses with the homogeneous protocols as

$$W = \left( \kappa + \frac{1-\nu}{\nu} \right) \mathbb{1} - \frac{1}{\nu} \Omega_{\text{Hom}}$$

Hence, to determine whether a state $|\psi\rangle$ is entangled is equivalent to verifying if the target state is $|\psi\rangle$ within the infidelity $\epsilon = 1 - \kappa$. Such a relation transforms the witnesses from the formation of observables to the construction of choice-independent measurement protocols. This equivalence improves the estimation of shot noise from $1/\sqrt{N}$ (statistical mean error) to $1/N$ (error of hypothesis testing).

6.2. Parameter Estimation

Considering choice-independent measurement protocols, one finds that homogeneous QSV can also be regarded as fidelity estimation,[27] i.e.,

$$F = \langle \psi | \sigma | \psi \rangle = \frac{\text{tr}(\Omega_{\text{Hom}} \sigma)}{\nu} - (1 - \nu)$$

with standard deviation

$$\Delta F = \frac{\sqrt{(1 - F)(F + \nu - 1)}}{N} \leq \frac{1}{2\nu \sqrt{N}}$$

It shows that the number of copies required is $N \sim O(\epsilon^2)$, which is worse than that of verification. Nevertheless, one can directly achieve the value of fidelity rather than a bound. In addition, performing fidelity estimation only needs to know the frequency of pass instances rather than the number of successive ones, which is much more robust in experiments. Moreover, considering entanglement quantified by witness operators,[45] the local protocol for entanglement estimation can be similarly designed as being choice-independent as well.

For a homogeneous protocol $\Omega_{\text{Hom}}$, the positivity constraint can always be achieved with a proper $\nu$. If the completeness constraint is violated, we can consider the measurement protocol

$$\tilde{\Omega}_{\text{Hom}} = \frac{\Omega_{\text{Hom}}}{S(\Omega_{\text{Hom}})}$$

$$\Delta F = \frac{\sqrt{(1 - F)(F + \nu - 1)}}{N} \leq \frac{1}{2\nu \sqrt{N}}$$

$$\text{tr}(W\rho_{\text{ent}}) < 0.$$
Now the passing probability of the target state is given by \(1/S_\Omega\), so it cannot be used for verification. However, the task of estimation is immune to this problem, as we only need to add a corresponding factor of scaling. Exemption of the completeness constraint enables our method to give a local measurement protocol for arbitrary estimation tasks that is comparable to the optimal one.

7. Conclusion

We have proposed a systematic strategy to design QSV protocols for arbitrary entangled pure states. By initiating the concept of choice-independent measurement protocols, we have successfully converted the original problem to the checking of the locality of homogeneous protocols. By formalizing the locality of measurement protocols, we have derived the corresponding constraints for local measurements, LOCC, as well as infinite continuous measurements, respectively. Then, for the homogeneous QSV protocols whose operators can be directly written down for arbitrary pure states, we acquired the corresponding constraints for them being local. Specifically, we demonstrated the explicit procedures of the protocol design using Pauli projections, for verifying Bell states, stabilizer states and \(W\) states. For certain cases, our method has achieved the best strategies to date.

Furthermore, the discussions on the locality of measurement protocols can be applied to more tasks, such as the construction of entanglement witness. Finally, we have shown that all these tasks can be converted to parameter estimation. In this case, the local measurement protocols can be directly given, as the constraints of local protocols for these tasks can always be satisfied.

Appendix A: Local Operations and Classical Communication

In the main text, local measurements are employed for the construction of measurement protocols. Here, we extend our scheme by considering local operations and classical communication (LOCC), also known as adaptive measurements. For an \(n\)-partite quantum system, a measurement protocol with adaptive measurements can be written as

\[
\Pi = \sum \mu_i \Pi_i
\]

where \(\Pi_i\) represent adaptive measurement settings constructed by \(n\) local s-outcome POVMs \(\{M_{i1}, \ldots, M_{iS}\}\) with different measurement orders. It can be understood as

\[
\Pi_i = P_i \left\{ \sum q_{ij_1,ij_2,\ldots,ij_{n-1}} q_{ij_1,ij_2,\ldots,ij_{n-1}} M_{i1} \otimes \cdots \otimes M_{in} \right\}
\]

The setting is adaptively measured from \(M_{i1}\) to \(M_{in}\) with the probability distribution \(q_{ij_1,ij_2,\ldots,ij_{n-1}}\) based on previous outcomes, and all possible measurement orders are considered with the permutation operator \(P_i \} \). The parameter \(z_{\{\}}\) is either 0 or 1 that tells us which outcomes of the final measurement correspond to “pass” instances, and it can be generally considered as \(z_1 \in [0, 1]\). Denoting them compactly as \(q_{i,ij}\), we have

\[
\Pi = \sum_{ij} p_{ij} M_{i1} \otimes \cdots \otimes M_{in}
\]

where \(p_{ij} := \sum \mu_i q_{i,ij}\).

Theorem 2. A measurement protocol can be realized with \(\text{LOCC}\) if the quasi-probability distribution \(p_{ij}\) satisfies the following two constraints under the representation of Equation (A3),

- Positivity \(\min \{p_{ij}(\Pi)\} \geq 0\) (A4)
- Completeness \(S(\Pi) := \sum \max \{p_{ij}(\Pi)\} \leq s^{n-1}\) (A5)

Proof. For the measurement protocol \(\Pi\) as in Equation (A1), the probability distribution should satisfy (1) positivity \(\mu_i \geq 0\) \(\forall k\); and (2) completeness \(\sum_i \mu_i = 1\). Then the positivity constraint in Equation (A4) can be directly achieved. For the completeness constraint, considering only one adaptive measurement setting, there are \(\sum_i q_{ij_1,ij_2,\ldots,ij_{n-1}} = 1\) and \(\sum q_{ij_1,ij_2,\ldots,ij_{n-1}} = 1\). Then we have \(\sum_i \mu_i q_{ij_1,ij_2,\ldots,ij_{n-1}} \leq 1\), and thus \(\sum q_{ij_1,ij_2,\ldots,ij_{n-1}} q_{ij_1,ij_2,\ldots,ij_{n-1}} \leq s\). With \(z_1 \in [0, 1]\), \(\forall j\), for each adaptive measurement setting, we have \(\sum \max \{q_{ij_1,ij_2,\ldots,ij_{n-1}}(\Pi)\} \leq s^{n-1} \cdot 1\). Finally, for the quasi-probability distribution \(p_{ij} = \sum \mu_i q_{ij}\), one can deduce \(\sum \max \{p_{ij}(\Pi)\} \leq s^{n-1}\).

Appendix B: Extension to Infinite Continuous Local Measurements

B.1. Locality of Measurement Protocols Based on Infinite Continuous Local Measurements

Considering \(n\)-qubit systems, the decomposition of \(\Pi\) as in Equation (4) with finite local measurements can be generalized to the infinite scenario using continuous local projections over the Bloch sphere, such that

\[
\Pi = \int d\omega_1 \cdots d\omega_n w_{\omega_1,\ldots,\omega_n}(\Pi) P_{\omega_1} \otimes \cdots \otimes P_{\omega_n}
\]

where \(\beta\) denotes the integral over \(n\) Bloch spheres. The local operator \(P_{\omega} = \frac{1}{2}(1 + \vec{r} \cdot \vec{s})\) is the projection onto the pure state located at the unit vector \(\vec{r}\), and \(\vec{s} = (s_1, s_2, s_3)\) are the Pauli matrices. Then, one obtains the following theorem.

Theorem 3. A measurement protocol for \(n\)-qubit systems is local iff the quasi-probability distribution \(w_{\omega_1,\ldots,\omega_n}(\Pi)\) satisfies the following two constraints under the diagonal \(P\)-representation of Equation (B1),

- Positivity \(\min \{w_{\omega_1,\ldots,\omega_n}(\Pi)\} \geq 0\) (B2)
- Completeness \(S(\Pi) \leq 2^n\) (B3)

\(S(\Pi)\) is the integral over the envelope surface of all \(2^n\) quasi-probability distributions \(\{w_{\omega_1,\ldots,\omega_n}(\Pi)\}_{\omega \in [0, 1)^{n+1}}\), i.e.,

\[
S(\Pi) := \int d\omega_1 \cdots d\omega_n \tilde{w}_{\omega_1,\ldots,\omega_n}(\Pi)
\]

where \(\tilde{w}_{\omega_1,\ldots,\omega_n}(\Pi) = \max_{\omega \in [0, 1)^{n+1}} \{w_{\omega_1,\ldots,\omega_n}(\Pi)\}\), with \(\omega_0 := \vec{r}, \tilde{r}_i := \vec{r}_i, \text{ and } \tilde{r}_{i0} + \tilde{r}_{i1} = 0\).
The probability distribution \( w(\Pi) \) is not unique for the measurement protocol \( \Pi \). Considering the spherical harmonics expansion of \( w(\Pi) \) with order-0 and order-1 components only, a unique representation for \( n \)-qubit systems can be written as

\[
w_{1 \ldots n}(\Pi) = \frac{1}{(4\pi)^n} \text{tr}[\Pi(1 + 3r_1 \cdot \hat{\sigma}) \otimes \cdots \otimes (1 + 3r_n \cdot \hat{\sigma})]
\]

and higher-order spherical harmonics do not change \( \Pi \).

The representation of \( w(\cdot) \) in Equation (B6) is a linear function of operators, then \( w(\Omega_{\text{Hom}}) \) of the homogeneous QSV protocol is given by

\[
w(\Omega_{\text{Hom}}) = (1 - v)w(1) + vw(\psi)
\]

where \( w(1) \) and \( w(\psi) \) are the quasi-probability distributions of the identity and the projection on the target state, respectively.

**Corollary 2.** Considering a homogeneous QSV protocol \( \Omega_{\text{Hom}} \) for the target state \( |\psi\rangle \) as defined in Equation (B5), it is local if the following two constraints are satisfied:

- **Positivity**
  \[
  v \leq \frac{1}{1 - (2\pi)^n} \min_{\forall \psi} \{w(\psi)\} \leq \frac{1}{2^{n+1}} + 1
  \]  

- **Completeness**
  \[
  S(\psi) \leq 2^n
  \]

**Proof.** From Equation (B6), we directly get

\[
w(1) = 2^n/(4\pi)^n
\]

by considering the symmetry of the quasi-probability distribution. Then, the positivity constraint of Equation (B2) in Theorem 3 is translated here for \( \Omega_{\text{Hom}} \) such that

\[
\min_{\forall \psi} \{w(\Omega_{\text{Hom}})\} = (1 - v)2^n/(4\pi)^n + v \min_{\forall \psi} \{w(\psi)\} \geq 0
\]

from which the inequality \( v \leq 1/[1 - (2\pi)^n] \min_{\forall \psi} \{w(\psi)\} \) can be deduced. Next, since the eigenvalues of \((1 + 3\hat{\sigma})\) are 4 and \(-2\), one obtains

\[
\min_{\forall \psi} \{w(\psi)\} = [4^{n-1} \times (-2)]/(4\pi)^n
\]

then the second inequality in Equation (B8) follows.

**B.2. Homogeneous QSV Protocols with Infinite Local Projections**

We revisit the homogeneous QSV protocol in its general form as in Equation (7),

\[
\Omega_{\text{Hom}} = (1 - v)1 + v|\psi\rangle \langle \psi|, \quad (0 < v \leq 1)
\]  

for a target pure state \( |\psi\rangle \). We consider the locality of \( \Omega_{\text{Hom}} \) with infinite local projections by following Theorem 3. Note that the quasi-probability distribution \( w(\Pi) \) is not unique for the measurement protocol \( \Pi \). Considering the spherical harmonics expansion of \( w(\Pi) \) with order-0 and order-1 components only, a unique representation for \( n \)-qubit systems can be written as

\[
w_{1 \ldots n}(\Pi) = \frac{1}{(4\pi)^n} \text{tr}[\Pi(1 + 3r_1 \cdot \hat{\sigma}) \otimes \cdots \otimes (1 + 3r_n \cdot \hat{\sigma})]
\]

and higher-order spherical harmonics do not change \( \Pi \).

The representation of \( w(\cdot) \) in Equation (B6) is a linear function of operators, then \( w(\Omega_{\text{Hom}}) \) of the homogeneous QSV protocol is given by

\[
w(\Omega_{\text{Hom}}) = (1 - v)w(1) + vw(\psi)
\]

where \( w(1) \) and \( w(\psi) \) are the quasi-probability distributions of the identity and the projection on the target state, respectively. Hence, we have the corollary below for homogeneous QSV protocols.

**Corollary 2.** Considering a homogeneous QSV protocol \( \Omega_{\text{Hom}} \) for the target state \( |\psi\rangle \) as defined in Equation (B5), it is local if the following two constraints are satisfied:

- **Positivity**
  \[
  v \leq \frac{1}{1 - (2\pi)^n} \min_{\forall \psi} \{w(\psi)\} \leq \frac{1}{2^{n+1}} + 1
  \]

- **Completeness**
  \[
  S(\psi) \leq 2^n
  \]

**Proof.** From Equation (B6), we directly get

\[
w(1) = 2^n/(4\pi)^n
\]

by considering the symmetry of the quasi-probability distribution. Then, the positivity constraint of Equation (B2) in Theorem 3 is translated here for \( \Omega_{\text{Hom}} \) such that

\[
\min_{\forall \psi} \{w(\Omega_{\text{Hom}})\} = (1 - v)2^n/(4\pi)^n + v \min_{\forall \psi} \{w(\psi)\} \geq 0
\]

from which the inequality \( v \leq 1/[1 - (2\pi)^n] \min_{\forall \psi} \{w(\psi)\} \) can be deduced. Next, since the eigenvalues of \((1 + 3\hat{\sigma})\) are 4 and \(-2\), one obtains

\[
\min_{\forall \psi} \{w(\psi)\} = [4^{n-1} \times (-2)]/(4\pi)^n
\]

then the second inequality in Equation (B8) follows.

Finally, the completeness constraint of Equation (B3) in Theorem 3 is transformed here for \( \Omega_{\text{Hom}} \) as

\[
S(\Omega_{\text{Hom}}) = (1 - v)2^n + vS(\psi) \leq 2^n
\]

Since \( 0 < v \leq 1 \), we have \( S(\psi) \leq 2^n \), which is the completeness constraint for homogeneous QSV protocols.

For the positivity constraint, we note that the first inequality in Equation (B8) gives the radius of the local ball; while the second one is obtained by considering all possible target states. In other words, using the homogeneous QSV protocol, any target state can be verified with an efficiency no more than \( O(2^n) \) as long as the completeness constraint is satisfied. Moreover, the upper bound of this complexity depends on the structure of the Hilbert space, such that more complex measurements like the multi-outcome POVMs will not improve the upper bound; see Appendix B3 below.

**B.3. Multi-Outcome Measurements**

Here, we modify Corollary 2 to the multi-outcome scenario by considering s-outcome rank-1 POVMs as \( \{P'_{1}, \ldots, P'_{s}\} \), where the measurements \( P'_{i} = \frac{1}{s}(I + \hat{r}_{i} \cdot \hat{\sigma}) \) with \( \sum s_{i} = 1 \). Then we have the following corollary for QSV.
Corollary 3. Considering a homogeneous QSV protocol $\Omega_{\text{Hom}}$ as defined in Equation (B5) for the target state $|\psi\rangle$, it is local under s-outcome rank-1 POVMs ($s \geq 2$) if the following two constraints are satisfied:

- **Positivity**
  \[
  \nu^{(i)} \leq \frac{1}{1 - (4\pi/s)^a \min\{w^{(i)}(\psi)\}} \leq \frac{1}{2^{n-1} + 1} \tag{B14}
  \]

- **Completeness**
  \[
  S^{(i)}(\psi) \leq s^n \tag{B15}
  \]

Proof. For the POVMs $\{P'_1, \ldots, P'_n\}$, we generalize Corollary 2 by following the remark of Theorem 3. Being equivalent to Equation (B1), one has

\[
\Pi = \int dV_1 \cdots dV_n w^{(i)}_{i_1 \cdots i_n} (\Pi) P'_1 \otimes \cdots \otimes P'_n \tag{B16}
\]

where the quasi-probability distribution is $w^{(i)}(\Pi) = (s/2)^n w(\Pi)$, and we have

\[
w^{(i)}(1) = s^n / (4\pi)^n \tag{B17}
\]

The positivity constraint is generalized to

\[
\min_{\nu} \{w^{(i)}(\Omega_{\text{Hom}})\} = (1 - \nu)(1 - (4\pi/s)^a) / (4\pi) + \nu \min \{w^{(i)}(\psi)\} \geq 0 \tag{B18}
\]

and the inequality $\nu^{(i)} \leq 1 / [1 - (4\pi/s)^a \min \{w^{(i)}(\psi)\}]$ can be deduced. Note that the eigenvalues of $(1 + 3\xi - d)$ are 4 and $-2$, one obtains

\[
\min \{w^{(i)}(\psi)\} = \left[4^{n-1} \cdot (-2) / (4\pi)^n \right] = -(1/2) (1/s)^{(n/2)} \tag{B19}
\]

then the second inequality in Equation (B14) follows.

With the completeness constraint of Equation (B3) in Theorem 3, we have

\[
S^{(i)}(\Omega_{\text{Hom}}) = (1 - \nu) \int \left[s^n / (4\pi)^n\right] dV + \nu \int S^{(i)}(\psi) \leq s^n \tag{B20}
\]

Since $0 < \nu \leq 1$, we have $S^{(i)}(\psi) \leq s^n$, which is the completeness constraint.

Appendix C: Proof of Corollary 1

Proof. For the identity 1, all the coefficients of the Pauli representation are 0 except for $c_0 = 2^n$, thus

\[
p_i(1) = 3^{-n}, \forall i \tag{C1}
\]

Using the positivity constraint of Equation (5) in Theorem 1 on the homogeneous protocol $\Omega_{\text{Hom}}$, one has

\[
\min \{p_i(\Omega_{\text{Hom}})\} = (1 - \nu) 1 / 2^{n} + \nu \min \{p_i(\psi)\} \geq 0 \tag{C2}
\]

and the first inequality in Equation (14) is deduced.

Considering the transformation $T_{\text{Pauli}}$ with the form of Equation (13), from Equation (11) we get

\[
\min \{p_{i_1 \cdots i_n}(\psi)\} = \frac{1}{2^n} \left(\sum a(\psi)\right)_{i_1 \cdots i_n} \tag{C3}
\]

where $h$ is the Hamming weight of the string $i_1 \cdots i_n$. Since $1 \leq c_{x_1 \cdots x_n} \leq 1$, the minimal value of $p_i(\psi)$ is obtained when $|\psi\rangle = 1$ such that $(q_a(\psi)) < 0$, $\forall a$, except for $c_0 = 1$. Thus we have

\[
\min \{p_i(\psi)\} = \frac{1}{2^n} \left(C_n 1 - \frac{1}{3} - \frac{1}{3} \cdots - \frac{1}{3} \right) \tag{C4}
\]

and the second inequality in Equation (14) is derived.

Using the positivity constraint of Equation (6) in Theorem 1 on the homogenous protocol $\Omega_{\text{Hom}}$, we have

\[
S(\Omega_{\text{Hom}}) = (1 - \nu) \sum_{i=1}^{n} 1 / 2^n + \nu S(\psi) \leq 1 \tag{C6}
\]

Since $0 < \nu \leq 1$, $S(\psi) \leq 1$ follows.

Appendix D: Additional Details on the Applications

Here, we present more details on the protocol design for verifying Bell states, stabilizer states, including CHZ states, and W states. From numerical results to concrete realizations, we also show additional procedures on how to improve all the results.

D.1. Bell States

For the first example, we consider the Bell state $|\Phi^+\rangle = \frac{1}{\sqrt{2}} (|00\rangle + |11\rangle)$. Its optimal QSV protocol is precisely homogeneous such that

\[
\Omega_{\text{Bell}} = \Omega_{\text{Hom}}(\Phi^+) = \frac{1}{3} (P_{XX}^+ + P_{YY}^- + P_{ZZ}^+) = \frac{1}{3} \left|\Phi^+\right\rangle\langle\Phi^+| \tag{D1}
\]

where $X$, $Y$, and $Z$ are the Pauli operators, and the superscripts $+$ and $-$ indicate the projections onto the eigenspaces with eigenvalues $+1$ and $-1$, respectively. The verification efficiency is given by $1/\nu = 1/2$.

As shown by Corollary 1, the constraints are directly related to the quasi-probability distribution $p(\Phi^+)$ of the target state. With the transformation of Equation (13), it is

\[
p(\Phi^+) = \frac{1}{36} \begin{bmatrix}
10 & -8 & -8 & 10 \\
1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1
\end{bmatrix} \tag{D2}
\]

\[
\begin{bmatrix}
10 & -8 & -8 & 10 \\
1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 \\
10 & -8 & -8 & 10
\end{bmatrix} \tag{D2}
\]
where \( p_j \) represents the coefficient for the local Pauli projection \( P_{(-y_1 \otimes \cdots \otimes y_n)} \). One notices that as an entangled state, some coefficients of the quasi-probability distribution \( p(\Phi^+) \) are negative under local measurements. However, it does satisfy \( S(\Phi^+) = 1 \), meaning that the Bell state \( |\Phi^+\rangle \) can be verified with the homogeneous protocol using local Pauli measurement settings only. Since the radius of the local ball is given by \( \nu = \sqrt{1 - 3^m \min(p(\Phi^+))} = \frac{1}{3} \), then the quasi-probability distribution of the homogeneous protocol is

\[
p(\Omega_{\text{Hom}}(\Phi^+)) = \frac{1}{12}
\begin{bmatrix}
2 & 0 & 0 & 2 \\
1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 \\
0 & 2 & 2 & 0 \\
1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 \\
2 & 0 & 0 & 2
\end{bmatrix}
\]  

(D3)

where all the coefficients are nonnegative, and the verification efficiency is \( \frac{1}{\nu} = 3 \).

Furthermore, one notices that \( \sum p_j P_{(-y_1 \otimes \cdots \otimes y_n)} = I \), \( \forall i \). Then, some settings of the above protocol, such as the second row in Equation (D3), indicate that the homogeneous protocol \( \Omega_{\text{Hom}} \) contains null operations 1, hence can be improved. We add a revision process to delete the additional null operation \( a1 \) with an appropriate scaling factor \( a \), thus, in general we have

\[
\Omega'_{\text{Hom}}(\Phi^+) = \frac{\Omega_{\text{Hom}}(\Phi^+)}{1 - a} = \frac{1 - \nu - a}{1 - a} I + \frac{\nu}{1 - a}|\psi\rangle\langle\psi|
\]

\[
= (1 - \nu) I + \nu|\psi\rangle\langle\psi| \quad \text{with} \quad \nu = \frac{\nu}{1 - a},
\]  

(D4)

Hence, with the revision process, the improved protocol \( \Omega'_{\text{Hom}} \) for the Bell state is exactly the same as the optimal one, for which we have \( a = 1/2 \), and the quasi-probability distribution is

\[
p(\Omega'_{\text{Hom}}(\Phi^+)) = \frac{1}{3}
\begin{bmatrix}
1 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 1 & 1 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
1 & 0 & 0 & 1
\end{bmatrix}
\]  

(D5)

\[\frac{1}{2} \leq \theta = \min\{\nu, \theta_5, \theta_6\} = \frac{3^m + 2^n - 1}{2^n}. \]

Corollary 4. For the \( n \)-qubit GHZ state, there exists a homogeneous QSV protocol using Pauli projections with the transformation matrix as defined in Equation (13), of which we can achieve the efficiency

\[
\frac{1}{\nu} = \frac{3^m + 2^n - 1}{2^n}.
\]  

Proof. Using the three-qubit GHZ state as an example, we can find the coefficients of the Pauli representation with the following property

\[
\sum_{\sigma_1 \cdots \sigma_n} c_{\sigma_1 \cdots \sigma_n} \sigma_1 \cdots \sigma_n = \sigma_0 \otimes \cdots \otimes \sigma_0.
\]

For the \( n \)-qubit GHZ state, there exists a homogeneous QSV protocol using Pauli projections with the transformation matrix as defined in Equation (13), for which we have the following corollary.

Since \( \sigma_2 = i\sigma_1 \sigma_2 \), for the \( n \)-qubit GHZ state, besides the coefficient \( c_{(1)0^n} \) of the identity operator, only the coefficients \( c_{(1)0^n} \) and \( c_{(0,1,0^n)} \) with even Hamming weight are 1 and others are zero. Thus, with Equation (11) and the transformation Equation (13), for \( i_1 \cdots i_n \in \{1, 2, 3, 4, 5, 6\}^{10^n} \), we have

\[
p_{i_1 \cdots i_n}(\text{GHZ}_3) = \begin{cases} 
\frac{1}{2^n}, & i_1 \cdots i_n \in \{1, 2, 3, 4, 5, 6\}^{10^n} \\
\frac{1}{2^n} \pm 1, & i_1 \cdots i_n \in \{1, 2\}^{10^n} \text{ or } i_1 \cdots i_n \in \{5, 6\}^{10^n} 
\end{cases}
\]

(D9)

Combining Corollary 1, we have

\[
\frac{1}{\nu} = 1 - 3^m \min(p(\text{GHZ}_3)) = 1 - 3^{m-1} \frac{1}{2^n} = \frac{3^m + 2^n - 1}{2^n}
\]

(D10)

Furthermore, this efficiency can be improved to \( 1/\nu = 5/3 \) with an additional revision process. This is better than that in Ref. [16], and slightly

\[\Box\]
worse than the optimal one. The quasi-probability distribution is

\[
p(\Omega_{\text{Hom}}(\text{GHZ}_n)) = \frac{1}{20}\begin{bmatrix}
3 & 0 & 0 & 3 & 0 & 3 & 3 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 3 & 3 & 0 & 3 & 0 & 3 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 1 & 1 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
\]

\[
p(\Omega_{\text{Hom}}(W_3)) = \frac{1}{222}\begin{bmatrix}
12 & 0 & 0 & 0 & 0 & 0 & 0 & 12 \\
6 & 6 & 0 & 0 & 0 & 0 & 6 & 6 \\
24 & 5 & 0 & 17 & 0 & 17 & 24 & 5 \\
6 & 0 & 6 & 0 & 0 & 6 & 0 & 6 \\
6 & 0 & 0 & 6 & 6 & 0 & 0 & 6 \\
1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\
24 & 4 & 5 & 0 & 17 & 0 & 24 & 17 \\
1 & 1 & 0 & 1 & 1 & 0 & 0 & 0
\end{bmatrix}
\]

Furthermore, with the fact that all stabilizer states can be verified by QSV protocols constructed with their stabilizers that are in the Pauli group, our method is able to give the local homogeneous QSV protocols for all stabilizer states with the quasi-probability distribution based on the Pauli representation. This can be shown numerically such that we have checked all the graph states up to five qubits (which are equivalent to stabilizer states).

\[
\Omega(\text{W}_3) = \frac{1}{\sqrt{13}}\begin{bmatrix}
9 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 11 & 1 & 0 & 1 & 0 & 0 & 0 \\
0 & 1 & 11 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 9 & 0 & 0 & 0 & 0 \\
0 & 1 & 1 & 0 & 11 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 9 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 6 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 6
\end{bmatrix}
\]

D.3. W States

W states (or more generally, Dicke states) have been efficiently verified in our previous work \cite{26} using only local Pauli-Z and Pauli-X measurements. The efficiency is $1/\nu = n - 1$ for $n \geq 4$ ($1/\nu = 3$ for $n = 3$) with adaptive measurements and is worsened by a factor of $2$ with nonadaptive measurements. In addition, Li et al.\cite{26} proposed a nearly optimal protocol with the efficiency of $1/\nu = 8/5$, which is also homogeneous. However, besides the Pauli-X and Pauli-Z measurements, their protocol requires an additional projection on $(|000\rangle - |111\rangle)/\sqrt{2}$ as well as certain symmetrization procedures.

Consider the three-qubit W state $|W_3\rangle = \frac{1}{\sqrt{6}}(|001\rangle + |010\rangle + |100\rangle)$. Unfortunately, one has $S(|W_3\rangle) = 1.40(7)$ by using our method, which violates the completeness constraint. In turn, the revision process cannot make the constraint be satisfied either. The revised quasi-probability distribution is

\[
\Omega'(\text{W}_3) = \frac{1}{13}\begin{bmatrix}
9 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 11 & 1 & 0 & 1 & 0 & 0 & 0 \\
0 & 1 & 11 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 9 & 0 & 0 & 0 & 0 \\
0 & 1 & 1 & 0 & 11 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 9 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 6 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 6
\end{bmatrix}
\]

with $S(\Omega_{\text{Hom}}(\text{W}_3)) = 1.19(8) > 1$.

On the other hand, since we have the quasi-probability distribution, further analysis is still meaningful. We find that not only the operator $\Omega_{\text{Hom}}(\text{W}_3)$ does not satisfy the constraints, but some of the measurement settings do not fulfill $\text{tr}(\Omega(|W_3\rangle\langle W_3|)) = 1$. Then, we pick out the settings that do satisfy $\text{tr}(\Omega(|W_3\rangle\langle W_3|)) = 1$, and use them to construct a verification protocol with a uniform probability distribution. Hence, we have
D.4. Different Choices of the Transformation $T_{\text{Pauli}}$

With the specific choice of the transformation between the Pauli operators $\{\sigma_0, \sigma_1, \sigma_2, \sigma_3\}$ and the Pauli projections $\{P^0, P^1\}$ as in Equation (13)

$$T_{\text{Pauli}} := \begin{bmatrix} t_0 & t_1 & t_2 & t_3 \\ 1/3 & 1/3 & 1/3 & 1/3 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad (D14)$$

we get Corollary 1 as well as all the results of the previous applications. As mentioned in the main text, the choice of $t_0$ is arbitrary such that

$$1 = P^0_1 + P^1_1 = P^0_2 + P^1_2 = P^0_3 + P^1_3 = \sum_i a_i (P^0_i + P^1_i) \quad (D15)$$

where $\sum_i a_i = 1$. Thus, in general, one has

$$t_0 = [a_1, a_2, a_3, a_3] \quad (D16)$$

Obviously, Corollary 1 is not valid anymore with a different transformation. Reconsidering Theorem 1, we have

$$\min\{p_i | \Omega_{\text{Hom}}(W_i)\} = \min\{1-v\} p_i(1) + vp_i(\psi) \geq 0 \quad (D17)$$

with $p_i(\psi) = (\mathbb{S}^2)_{i_1(1) \ldots i_n(1)}$. Under such a circumstance, it is difficult to give a general bound for $v$. However, one finds that, the revision process in Equation (D4) does not require $a > 0$. Hence, we get

$$\Omega_{\text{Hom}}(\psi) = \frac{\langle \psi | | \psi \rangle - a}{1 - a} = \frac{-a}{1 - a} 1 + \frac{1}{1 - a} | \psi \rangle \langle \psi | = (1 - v) 1 + v | \psi \rangle \langle \psi | \quad \text{with} \quad v = \frac{1}{1 - a}. \quad (D18)$$

More importantly, for the three-qubit $W$ state $|W_4\rangle = \frac{1}{\sqrt{2}}(|001\rangle + |100\rangle)$, with the transformation $t_0 = [0 0 0 0 1 1]$, we have the quasi-probability distribution

$$\rho_{\Omega_{\text{Hom}}(W_4)} = \frac{1}{12} \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 2 & 0 & 1 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 2 & 0 & 1 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 2 & 0 & 1 & 0 & 1 & 2 \end{bmatrix} \quad (D19)$$

with $S(\Omega_{\text{Hom}}(W_4)) = 7/6 > 1$. However, considering adaptive measurements as in Theorem 2, the constraint can be satisfied such that $S(\Omega_{\text{Hom}}(W_3)) = 7/6 < 4$. Therefore, we are able to get a homogeneous QSV protocol for $|W_4\rangle$ using local Pauli projections as

$$\Omega_{\text{Hom}}(W_3) = \sum_k 3 \rho_k \{ P^+_Z \left[ \begin{array}{c} |0\rangle \langle 0| + |1\rangle \langle 1| \end{array} \right] + P^+_Y \left[ \begin{array}{c} |0\rangle \langle 0| + |1\rangle \langle 1| \end{array} \right] \} \quad (D20)$$

where $P^+_k = \{ (R^k)_{i_1(1) \ldots i_n(1)} \}$. Under such a circumstance, it is difficult to give a general bound for $v$. However, one finds that, the revision process in Equation (D4) does not require $a > 0$. Hence, we get

$$\Omega_{\text{Hom}}(\psi) = \frac{\langle \psi | \psi \rangle - a}{1 - a} = \frac{-a}{1 - a} 1 + \frac{1}{1 - a} | \psi \rangle \langle \psi | = (1 - v) 1 + v | \psi \rangle \langle \psi | \quad \text{with} \quad v = \frac{1}{1 - a}. \quad (D18)$$

Taking the three-qubit GHZ state $|\text{GHZ}_3\rangle = \frac{1}{\sqrt{2}}(|000\rangle + |111\rangle)$ as an example, with the transformation $t_0 = [0 0 0 0 1 1]$, our method along with the revision process is able to give a local homogeneous protocol with a better efficiency of $1/v = 3/2$, which happens to be the optimal one.\[27\]

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**Conflict of Interest**

The authors declare no conflict of interest.
Data Availability Statement

The data that support the findings of this study are available from the corresponding author upon reasonable request.

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quantum benchmarking, quantum information, quantum measurement, quantum verification