Actions of metric groups and continuous logic

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Abstract We study expressive power of continuous logic in classes of metric groups defined by properties of their actions. For example we consider properties non-OB, non-FH and non-FR.

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1 Introduction

Hereditary properties of basic classes of topological groups studied in measurable and geometric group theory have deserved more attention of researches in recent investigations, see [13], [14], [21]. This is mainly connected with the tendency of study of such notions as amenability or property (T) of Kazhdan outside the class of locally compact groups, see [10], [12], [17] and [29].

From this point of view it is natural to verify the behaviour of these classes under logical constructions. Moreover this task looks quite attractive because some logical constructions, for example ultraproducts, have become common in group theory.

On the other hand typical classes of groups studied in geometric group theory are non-axiomatizable. For example let us consider the following well-known classes of topological groups.

Definition 1.1 • $G \in \text{FH}$ if any continuous affine isometric action of $G$ on a real Hilbert space has a fixed point;
• $G \in \mathbf{FR}$ if any continuous isometric action of $G$ on a real tree has a fixed point;

• $G \in \mathbf{OB}$ if for any continuous isometric action of $G$ on a metric space all orbits are bounded.

We remind the reader that an action of $G$ on a metric space $X$ is called continuous if it is continuous as a 2-argument function $G \times X \rightarrow X$. When the action is isometric this is equivalent to the condition that for any $x \in X$ the map $g \rightarrow gx$ is continuous.

Discrete groups of the class $\mathbf{OB}$ are called strongly bounded. Y. de Cornulier have proved in [6] that they are contained in $\mathbf{FH}$ and $\mathbf{FR}$. Moreover it is also shown in [6] that for any finite perfect group $F$ and an infinite $I$ the power $F^I$ is strongly bounded.

Since $F^I$ is locally finite, any its countable subgroup has cofinality $\omega$, i.e. is the union of a strictly increasing $\omega$-chain of proper subgroups. Since such groups are outside of $\mathbf{FH} \cup \mathbf{FR}$, any countable elementary subgroup $H$ of $F^I$ witnesses the non-axiomatizability of $\mathbf{OB}$, $\mathbf{FH}$, $\mathbf{FR}$ and non-$\mathbf{OB}$, non-$\mathbf{FH}$, non-$\mathbf{FR}$.

Although this argument is carried out in the discrete case, it can be applied in many other situations, for example in continuous logic. Thus we see that the basic logic constructions involving elementary equivalence look foreign to the properties defined above.

On the other hand note that the properties we look at are formulated in the language of $G$-actions. Thus in order to adapt the situation to the logic approach let us consider the following definition and the corresponding question after it.

**Definition 1.2** Let $\mathcal{K}$ be a class of (first-order/continuous) structures. We say that $\mathcal{K}$ is logically analyzable if there is a family of (first-order/continuously) axiomatizable classes $\mathcal{K}_\alpha$, $\alpha \in I$, in expanded languages with possibly new sorts, so that

• for each $\alpha$ reducts of structures of $\mathcal{K}_\alpha$ to the language of $\mathcal{K}$ belong to $\mathcal{K}$,

• every $G \in \mathcal{K}$ has an expansion $\hat{G}$ belonging to one of these classes $\mathcal{K}_\alpha$.

We now formulate the main question of the paper.

**Analyzeability Question.** Let $\mathcal{K}$ be one of the classes $\mathbf{OB}$, $\mathbf{FH}$, $\mathbf{FR}$, non-$\mathbf{OB}$, non-$\mathbf{FH}$, non-$\mathbf{FR}$ or any other class of groups. Is $\mathcal{K}$ logically analyzable?

Note that we may consider some other kinds of axiomatizability in this question, for example the $L_{\omega_1,\omega}$-version of continuous logic.
This idea is connected with papers [20], [22], [25] and [28] where the following Ph. Hall’s notion is investigated. A class of discrete groups $\mathcal{K}$ is called \textbf{bountiful} if for any pair of infinite groups $G \leq H$ with $H \in \mathcal{K}$ there is $K \in \mathcal{K}$ such that $G \leq K \leq H$ and $|G| = |K|$. Generalizing some logical observations from [20] it is easy to see that if $\mathcal{K}$ is logically analyzable then $\mathcal{K}$ is bountiful (see Section 1.1 for a precise argument).

When one considers topological groups, the definition of bountiful classes should be modified by replacing cardinality of groups by density character, i.e. the smallest cardinality of a dense subset of $K$. Moreover it is natural to replace the subgroup $G$ in the definition by a set. As a result we formulate the definition as follows.

\textbf{Definition 1.3} A class of topological groups $\mathcal{K}$ is called \textbf{bountiful} if for any infinite group $H \in \mathcal{K}$ and any $C \subseteq H$ there is $K \in \mathcal{K}$ such that $C \subseteq K \leq H$ and the density character of $C$ coincides with the density character of $K$.

Proposition 1.7(a) in the final part of this section shows that properties OB, FH and FR are not bountiful. So, as we have already mentioned, it is easy to see that these classes are not logically analyzable. On the other hand part 1.7(b) of this proposition states that properties non-OB, non-FH and non-FR are bountiful, i.e. we may conjecture that the classes non-OB, non-FH and non-FR are logically analyzable.

The main results of our paper confirm this conjecture under some uniform continuity assumptions. They are presented in Sections 3 and 4. As a consequence we obtain bountifulness of some uniform versions (i.e. subclasses) of non-FH and non-FR in the form which is more precise than the statements of Proposition 1.7(b) below.

\textbf{Novelty of the approach.} When we apply logical methods to properties involving group actions the basic problem which we face is axiomatization of the action. Typically unbounded metric spaces are considered in continuous logic as many-sorted structures of $n$-balls of a fixed point of the space ($n \in \omega$). Section 15 of [2] contains nice examples of such structures.

If the action of a bounded metric group $G$ is isometric and preserves these balls we may consider the action as a sequence of binary operations where the first argument corresponds to $G$. In such a situation one just fixes a sequence of continuity moduli for $G$ (for each $n$-ball). We will see in Section 2 that this approach works well for the negation of property (T) (non-(T)) in the class of locally compact groups.

The situation dramatically changes when the action does not preserve $n$-balls. For example this happens when we study properties FH/non-FH (or FR/non-FR), where affine actions on Hilbert spaces appear (or non-elliptic actions on unbounded trees). In Section 3 we present a new approach to such situations. Using geometric properties of Hilbert spaces and real trees we
introduce sequences of ternary predicates and show that under some natural assumptions on the action, continuity moduli for these predicates can be defined. This is the crucial element of the paper. It allows us to axiomatize classes of actions which we consider, see Theorems 3.12 and 3.17.

In Section 4 we slightly simplify the circumstances. Replacing non-OB by some uniform versions of it we arrive at a situation where instead of adding new sorts one just adds two continuous predicates to the signature. This trick can be also applied to non-Roelcke bounded groups, non-Roelcke precompact groups and non-(OB)_k-groups (see [24]).

Uniform continuity. Actions of metric groups which can be analyzed by tools of continuous logic must be uniformly continuous for each sort appearing in the presentation of the space by metric balls. This slightly restricts the field of applications of our results. On the other hand note that in the case of discrete groups we do not lose generality and moreover our methods become more powerful. In Section 4.B we analyze some other properties of discrete groups, for example FA.

Remark 1.4 We mention paper [14] where related questions were studied in the case of locally compact groups. It is proved in [14] that for any locally compact group G, the entire interval of cardinalities between ℵ_0 and w(G), the weight of the group, is occupied by the weights of closed subgroups of G. We remind the reader that the weight of a topological space (X, τ) is the smallest cardinality which can be realized as the cardinality of a basis of (X, τ). If the group G is metric, the weight of G coincides with the density character of G. This yields the following version of the Löwenheim-Skolem Theorem (see Section 1.1).

Let G be a locally compact group which is a continuous structure. Then for any cardinality κ < density(G) there is a closed subgroup H < G such that density(H) = κ and H is an elementary substructure of G.

In the rest of this introduction we briefly remind the reader some preliminaries of continuous logic. Since we want to make the paper available for non-logicians these preliminaries can look too tedious for specialists. On the other hand we inform the reader that all necessary algebraic terms will be defined in the introductionary parts of corresponding sections.

1.1 Continuous structures

We fix a countable continuous signature

\[ L = \{d, R_1, \ldots, R_k, \ldots, F_1, \ldots, F_l, \ldots\}. \]
Let us recall that a \textit{metric L-structure} is a complete metric space \((M, d)\) with \(d\) bounded by 1, along with a family of uniformly continuous operations on \(M\) and a family of predicates \(R_i\), i.e. uniformly continuous maps from appropriate \(M^{k_i}\) to \([0, 1]\). It is usually assumed that to a predicate symbol \(R_i\) a continuity modulus \(\gamma_i\) is assigned so that when \(d(x_j, x'_j) < \gamma_i(\varepsilon)\) with \(1 \leq j \leq k_i\) the corresponding predicate of \(M\) satisfies

\[ |R_i(x_1, \ldots, x_j, \ldots, x_{k_i}) - R_i(x_1, \ldots, x'_j, \ldots, x_{k_i})| < \varepsilon. \]

It happens very often that \(\gamma_i\) coincides with \(id\). In this case we do not mention the appropriate modulus. We also fix continuity moduli for functional symbols. Each classical first-order structure can be considered as a complete metric structure with the discrete \([0, 1]\)-metric.

By completeness continuous substructures of a continuous structure are always closed subsets.

Atomic formulas are the expressions of the form \(R_i(t_1, \ldots, t_r), d(t_1, t_2)\), where \(t_i\) are terms (built from functional \(L\)-symbols). In metric structures they can take any value from \([0, 1]\). \textit{Statements} concerning metric structures are usually formulated in the form

\[ \phi = 0 \]

(called an \textit{L-condition}), where \(\phi\) is a \textit{formula}, i.e. an expression built from 0,1 and atomic formulas by applications of the following functions:

\[ x/2, x - y = \max(x - y, 0), \quad \min(x, y), \quad \max(x, y), \quad |x - y|, \]

\[ -(x) = 1 - x, \quad x + y = \min(x + y, 1), \quad \sup_x \text{ and } \inf_x. \]

A \textit{theory} is a set of \(L\)-conditions without free variables (here \(\sup_x \text{ and } \inf_x\) play the role of quantifiers).

It is worth noting that any formula is a \(\gamma\)-uniformly continuous function from the appropriate power of \(M\) to \([0, 1]\), where \(\gamma\) is the minimum of continuity moduli of \(L\)-symbols appearing in the formula.

The condition that the metric is bounded by 1 is not necessary. It is often assumed that \(d\) is bounded by some rational number \(d_0\). In this case the (truncated) functions above are appropriately modified.

We sometimes replace conditions of the form \(\phi - \varepsilon = 0\) where \(\varepsilon \in [0, d_0]\) by more convenient expressions \(\phi \leq \varepsilon\).

A tuple \(\bar{a}\) from \(M^n\) is \textit{algebraic} in \(M\) over \(A\) if there is a compact subset \(C \subseteq M^n\) such that \(\bar{a} \in C\) and the distance predicate \(\text{dist}(\bar{x}, C)\) is definable (in the sense of continuous logic, \([2]\)) in \(M\) over \(A\). Let \(\text{acl}(A)\) be the set of all elements algebraic over \(A\). In continuous logic the concept of algebraicity is parallel to that in traditional model theory (see Section 10 of \([2]\)).
Axiomatizability in continuous logic. When one considers classes axiomatizable in continuous logic it is usually assumed that all operations and predicates are uniformly continuous with respect to some fixed continuity moduli. Suppose that \( C \) is a class of metric \( L \)-structures. Let \( \text{Th}^c(\mathcal{C}) \) be the set of all \( L \)-conditions without free variables which hold in all structures of \( \mathcal{C} \). It is proved in [2] (Proposition 5.14 and Remark 5.15) that every model of \( \text{Th}^c(\mathcal{C}) \) is elementary equivalent to some ultraproduct of structures from \( \mathcal{C} \).

Metric groups. Below we always assume that our metric groups are continuous structures with respect to bi-invariant metrics (see [2]). This exactly means that \((G, d)\) is a complete metric space and \( d \) is bi-invariant. Note that the continuous logic approach takes weaker assumptions on \( d \). Along with completeness it is only assumed that the operations of a structure are uniformly continuous with respect to \( d \). Thus it is worth noting here that any group which is a continuous structure has an equivalent bi-invariant metric. See [15] for a discussion concerning this observation.

Hilbert spaces in continuous logic. We treat a Hilbert space over \( \mathbb{R} \) exactly as in Section 15 of [2]. We identify it with a many-sorted metric structure
\[
\left( \{B_n\}_{n \in \omega}, 0, \{I_{mn}\}_{m<n}, \{\lambda_r\}_{r \in \mathbb{R}}, +, -, \langle \rangle \right),
\]
where \( B_n \) is the ball of elements of norm \( \leq n \), \( I_{mn} : B_m \to B_n \) is the inclusion map, \( \lambda_r : B_m \to B_{km} \) is scalar multiplication by \( r \), with \( k \) the unique integer satisfying \( k \geq 1 \) and \( k - 1 \leq |r| < k \); furthermore, \(+, - : B_n \times B_n \to B_{2n} \) are vector addition and subtraction and \( \langle \rangle : B_n \to [-n^2, n^2] \) is the predicate of the inner product. The metric on each sort is given by \( d(x, y) = \sqrt{\langle x - y, x - y \rangle} \). For every operation the continuity modulus is standard. For example in the case of \( \lambda_r \) this is \( \frac{1}{|r|} \). Note that in this version of continuous logic we do not assume that the diameter of a sort is bounded by 1. It can become any natural number.

Stating existence of infinite approximations of orthonormal bases (by a countable family of axioms, see Section 15 of [2]) we assume that our Hilbert spaces are infinite dimensional. By [2] they form the class of models of a complete theory which is \( \kappa \)-categorical for all infinite \( \kappa \), and admits elimination of quantifiers.

This approach can be naturally extended to complex Hilbert spaces,
\[
\left( \{B_n\}_{n \in \omega}, 0, \{I_{mn}\}_{m<n}, \{\lambda_c\}_{c \in \mathbb{C}}, +, -, \langle \rangle_{\text{Re}}, \langle \rangle_{\text{Im}} \right).
\]
We only extend the family \( \lambda_r : B_m \to B_{km}, r \in \mathbb{R} \), to a family \( \lambda_c : B_m \to B_{km}, c \in \mathbb{C} \), of scalar products by \( c \in \mathbb{C} \), with \( k \) the unique integer satisfying \( k \geq 1 \) and \( k - 1 \leq |c| < k \).
We also introduce \(Re\)- and \(Im\)-parts of the inner product.

If we remove from the signature of complex Hilbert spaces all scalar products by \(c \in \mathbb{C} \setminus \mathbb{Q}[i]\), we obtain a countable subsignature

\[
\{(B_n)_{n \in \omega}, 0, \{I_{mn}\}_{m < n}, \{\lambda_c\}_{c \in \mathbb{Q}[i]}, +, -, \langle \rangle_{Re}, \langle \rangle_{Im}\},
\]

which is dense in the original one:
- if we present \(c \in \mathbb{C}\) by a sequence \(\{q_i\}\) from \(\mathbb{Q}[i]\) converging to \(c\), then the choice of the continuity moduli of the restricted signature still guarantees that in any sort \(B_n\) the functions \(\lambda_{q_i}\) form a sequence which converges to \(\lambda_c\) with respect to the metric

\[
\sup_{x \in B_n} \{|f^M(x) - g^M(x)| : M \text{ is a model of the theory of Hilbert spaces}\}
\]

Löwenheim-Skolem theorem and bountifulness. The following theorem is one of the main tools of this paper.

Löwenheim-Skolem Theorem. (\cite{2}, Proposition 7.3) Let \(\kappa\) be an infinite cardinal number and assume \(|L| \leq \kappa\). Let \(M\) be an \(L\)-structure and suppose \(A \subseteq M\) has density \(\leq \kappa\). Then there exists a substructure \(N \subseteq M\) containing \(A\) such that density \((N) \leq \kappa\) and \(N\) is an elementary substructure of \(M\), i.e. for every \(L\)-formula \(\phi(x_1, ..., x_n)\) and \(a_1, ..., a_n \in N\) the values of \(\phi(a_1, ..., a_n)\) in \(N\) and in \(M\) are the same.

In the case of discrete \(\{0, 1\}\)-metric this theorem becomes the standard Löwenheim-Skolem theorem.

Remark 1.5 Let \(L\) be a continuous signature. Following Section 4.2 of \cite{3} we define a topology on \(L\)-formulas relative to a given continuous theory \(T\). For \(n\)-ary formulas \(\phi\) and \(\psi\) of the same sort set

\[
d^T_x(\phi, \psi) = \sup\{|\phi(\bar{a}) - \psi(\bar{a})| : \bar{a} \in M, M \models T\}.
\]

The function \(d^T_x\) is a pseudometric. The language \(L\) is called separable with respect to \(T\) if for any tuple \(\bar{x}\) the density character of \(d^T_x\) is countable. By Proposition 4.5 of \cite{3} in this case for every \(L\)-model \(M \models T\) the set of all interpretations of \(L\)-formulas in \(M\) is separable in the uniform topology. By Theorem 4.6 of \cite{3} if in the formulation of the Löwenheim-Skolem theorem we replace the assumption \(|L| \leq \kappa\) by the condition that \(L\) is separable with respect to \(Th^c(M)\) then the statement the Löwenheim-Skolem theorem also holds for \(\kappa = \aleph_0\).

As we have already noticed the \(\mathbb{Q}[i]\)-subsignature of the language of Hilbert spaces is dense in the standard signature. Thus the original language of Hilbert spaces is separable with respect to the theory of Hilbert spaces. In particular we may apply the approach of \cite{3} in this important case.
The following corollary of the Löwenheim-Skolem theorem is obvious. It in particular states that if a class \( \mathcal{K} \) of first-order/continuous-metric structures is logically analyzable, then \( \mathcal{K} \) is bountiful.

**Corollary 1.6** Let \( \mathcal{K} \) be a class of first-order/continuous-metric structures. Assume that there is a family of classes \( \mathcal{K}_\alpha, \alpha \in I, \) in expanded languages with possibly new sorts, so that

- every \( G \in \mathcal{K} \) has an expansion \( \hat{G} \) belonging to one of these classes \( \mathcal{K}_\alpha, \)
- the reduct of any elementary substructure of such \( \hat{G} \) belongs to \( \mathcal{K}. \)

Then \( \mathcal{K} \) is bountiful.

The following proposition was already mentioned in the introduction as a kind of motivation of this paper. It can be also considered as a demonstration of our method in the easiest form. The part of the proof concerning property \( \text{FR} \) can be slightly unclear for an inexperienced reader. Some helpful preliminaries can be found in the beginning of Section 3.

**Proposition 1.7** (a) Classes of discrete members of \( \text{OB}, \text{FH} \) and \( \text{FR} \) are not bountiful.
(b) Classes of discrete members of non-\( \text{OB} \), non-\( \text{FH} \) and non-\( \text{FR} \) are bountiful.

Moreover statement (a) and (b) also hold in the case of metric groups which are continuous structures.

**Proof.** (a) This follows from the theorem of Y. de Cornulier that for any finite perfect group \( F \) and an infinite \( I \) the power \( F^I \) is strongly bounded and the fact that any countable subgroup of \( F^I \) has cofinality \( \omega \), i.e. does not belong to \( \text{FH} \cup \text{FR} \), see [1], [6] and [21]. In particular such a subgroup has an isometric action on a metric space with unbounded orbits.

(b) Let us consider the case of groups which are first-order structures of the class non-\( \text{OB} \). Let \( C \subseteq H \) with \( H \in \text{non-OB} \). Take an action of \( H \) on a metric space \( X \) with an unbounded orbit. Extend \( C \) by countably many elements witnessing this unboundedness. By the Löwenheim-Skolem theorem there is a subgroup \( K \) of \( H \) which is an elementary substructure of \( H \) in the expanded (by constants) language such that \( C \subseteq K \) and \( |C| = |K| \). Since this subgroup contains the distinguished constants its action on \( X \) has unbounded orbits.

The case of non-\( \text{FH} \) is identical. Indeed, by Theorem 2.2.9 of [1] property non-\( \text{FH} \) is equivalent to existence of an affine isometric action of \( G \) on a real Hilbert space so that all (equivalently some) orbits are unbounded.

In the case of non-\( \text{FR} \) one should use the result of [27] that each isometry of a real tree is elliptic (i.e. fixes a point) or hyperbolic (i.e. acts as a non-trivial translation of some line) and every action of a group of elliptic isometries on
a tree has a fixed point or a fixed end. We remind the reader that an end is an equivalence class of half-lines under the equivalence relation of having a common half-line. The latter statement is a straightforward generalization of Exercise 2 of Section 6.5 from [26] (see Section 1 of [7] for the case of \( \mathbb{R} \)-trees). When \( H \) has a hyperbolic isometry \( h \) all orbits of \( \langle h \rangle \) are unbounded. Thus we apply the argument above arranging that \( K \) has a hyperbolic isometry too. When \( H \) consists of elliptic isometries and does not have a global fixed point, then find a half-line, say \( L \), representing the fixed end and take a cofinal sequence \( c_1, c_2, \ldots \) on it. For each \( c_i \) choose \( h_i \in H \) which does not fix \( c_1, \ldots, c_i \). Then any subgroup of \( H \) containing all \( h_i \) does not have a global fixed point. Indeed, if \( c \) is such a point, then consider segments \( [c, c_n] \). All of them are decomposed into a segment belonging to \( L \) and the unique bridge from \( c \) to \( L \). Thus there is \( n_0 \) such that for all \( n > n_0 \) segments \( [c, c_n] \) contain \( c_{n_0} \). This contradicts the assumption that \( c_{n_0} \) is not fixed by \( h_n \) with \( n > n_0 \).

This argument also works in the case of continuous logic. Since first-order structures can be viewed as continuous ones there is no need for a continuous version of (a). We only mention that when one wants to have a non-discrete examples witnessing (a) one can have this by adding a compact group as a direct summand. □

2 Non-amenability vs negation of (T)

A. Introduction. It is well-known that closed subgroups of amenable locally compact groups are amenable. This in particular implies that the class of amenable locally compact groups which are continuous metric structures is bountiful. Indeed applying the continuous Löwenheim-Skolem theorem we see that for any infinite metric group \( H \) with a subset \( C \subseteq H \) where \( H \) is amenable and locally compact there is an elementary submodel \( K \) of \( H \) such that \( C \subseteq K \) and the density character of \( C \) coincides with the density character of \( K \). Since \( K \) is closed in \( H \) it is amenable too.

Remark 2.1 It is worth noting that the class of all discrete amenable groups is not axiomatizable: there are locally finite countable groups having elementary extensions containing free groups.

Since non-compact amenable locally compact groups do not satisfy property (T) of Kazhdan the argument above suggests that the class of non-(T) locally compact groups is bountiful too. Corollary 2.6 below is a confirmation of a uniform version of this suggestion. The main result of this section Theorem 2.5 shows that in the context of continuous logic the class of locally compact groups with property non-(T) is logically analyzable.

We apply methods announced in the introduction. The case of non-(T) is relatively easy, because we only need to consider group actions on Hilbert
spaces which preserve \( n \)-balls of 0. It can be considered as a warm up before more difficult cases in Section 3. In the rest of part A of this section we give necessary algebraic definitions.

Let a topological group \( G \) have a continuous unitary representation on a complex Hilbert space \( H \). A closed subset \( Q \subseteq G \) has an almost \( \varepsilon \)-invariant unit vector \( v \) in \( H \) if

there exists \( v \in H \) such that \( \sup_{x \in Q} \| x \circ v - v \| < \varepsilon \) and \( \| v \| = 1 \).

A closed subset \( Q \) of the group \( G \) is called a Kazhdan set if there is \( \varepsilon > 0 \) with the following property: for every unitary representation of \( G \) on a Hilbert space where \( Q \) has an almost \( \varepsilon \)-invariant unit vector there is a non-zero \( G \)-invariant vector. If the group \( G \) has a compact Kazhdan subset then it is said that \( G \) has property (T) of Kazhdan.

Proposition 1.2.1 of [1] states that the group \( G \) has property (T) of Kazhdan if and only if any unitary representation of \( G \) which weakly contains the unit representation of \( G \) in \( \mathbb{C} \) has a fixed unit vector.

By Corollary F.1.5 of [1] the property that the unit representation of \( G \) in \( \mathbb{C} \) is almost contained in a unitary representation \( \pi \) of \( G \) (this is denoted by \( 1_G \prec \pi \)) is equivalent to the property that for every compact subset \( Q \) of \( G \) and every \( \varepsilon > 0 \) the set \( Q \) has an almost \( \varepsilon \)-invariant unit vector with respect to \( \pi \).

The following example shows that in the first-order logic property (T) is not elementary.

**Example 2.2** Let \( n > 2 \). According Example 1.7.4 of [1] the group \( SL_n(\mathbb{Z}) \) has property (T). Let \( G \) be a countable elementary extension of \( SL_n(\mathbb{Z}) \) which is not finitely generated. Then by Theorem 1.3.1 of [1] the group \( G \) does not have (T).

**B. Unitary representations in continuous logic.** In order to treat analyzability question in the class of locally compact groups satisfying some uniform version of property non-(T) we need the preliminaries of continuous model theory of Hilbert spaces from Section 1.1. Moreover since we want to consider unitary representations of metric groups \( G \) in continuous logic we should fix continuity moduli for the corresponding binary functions \( G \times B_n \to B_n \) induced by \( G \)-actions on metric balls of the corresponding Hilbert space.

This is why we have to consider uniformly continuous versions of the notion of a Kazhdan set. We define it as follows.

**Definition 2.3** Let \( G \) be a metric group of diameter \( \leq 1 \) which is a continuous structure in the language \( (d, \cdot^{-1}, 1) \). Let \( F = \{F_1, F_2, \ldots\} \) be a family of continuity moduli for the \( G \)-variables of continuous function \( G \times B_i \to B_i \).
We call a closed subset $Q$ of the group $G$ an $\mathcal{F}$-Kazhdan set if there is $\varepsilon$ with the following property: every $\mathcal{F}$-continuous unitary representation of $G$ on a Hilbert space with almost $(Q, \varepsilon)$-invariant unit vectors also has a non-zero invariant vector.

It is clear that for any family of continuity moduli $\mathcal{F}$ a subset $Q \subset G$ is $\mathcal{F}$-Kazhdan if it is Kazhdan. We will say that $G$ has property $\mathcal{F}$-non-(T) if $G$ does not have a compact $\mathcal{F}$-Kazhdan subset.

To study such actions in continuous logic let us consider a class of many-sorted continuous metric structures which consist of groups $G$ together with metric structures of complex Hilbert spaces $(d, \cdot, -1, 1) \cup \{B_n\}_{n \in \omega}, \{I_{mn}\}_{m<n}, \{\lambda_c\}_{c \in \mathbb{C}}, [...].$

Such a structure also contains a binary operation $\circ$ of an action which is defined by a family of appropriate maps $G \times B_m \rightarrow B_m$ (in fact $\circ$ is presented by a sequence of functions $\circ_m$ which agree with respect to all $I_{mn}$). When we add the obvious continuous sup-axioms that the action is linear and unitary, we obtain an axiomatizable class $\mathcal{K}_{GH}$. Given unitary action of $G$ on $H$ we denote by $A(G, H)$ the member of $\mathcal{K}_{GH}$ which is obtained from this action.

When we fix continuity moduli, say $\mathcal{F} = \{F_1, F_2, \ldots\}$, for the $G$-variables of the operations $G \times B_m \rightarrow B_m$ we denote by $\mathcal{K}_{GH}(\mathcal{F})$ the corresponding subclass of $\mathcal{K}_{GH}$.

**Definition 2.4** The class $\bigcup \{\mathcal{K}_\delta(\mathcal{F}) : \delta \in (0, 1) \cap \mathbb{Q}\}$. Let $\mathcal{K}_\delta(\mathcal{F})$ be the subclass of $\mathcal{K}_{GH}(\mathcal{F})$ axiomatizable by all axioms of the following form

$$\sup_{x_1, \ldots, x_k \in G} \min_{v \in B_m} \sup_{x \in \bigcup x_i K_\delta} \max(\| x \circ v - v \| - \frac{1}{n}, |1 - \| v \| |) = 0,$$

where $k, m, n \in \omega \setminus \{0\}$ and $K_\delta = \{g \in G : d(1, g) \leq \delta\}$.

It is easy to see that the axiom of Definition 2.4 implies that each finite union $\bigcup_{i=1}^k g_i K_\delta$ has an almost $\frac{1}{n}$-invariant unit vector in $H$. To see that it can be written by a formula of continuous logic note that $\sup_{x \in \bigcup x_i K_\delta}$ can be replaced by $\sup_x$ with simultaneous inclusion of the quantifier-free part together with $\max(\delta - d(x, x_i) : 1 \leq i \leq k)$ into the corresponding min-formula.

In fact the following theorem shows that the class of $\mathcal{F}$-non-(T) locally compact groups satisfies the conditions of Corollary 1.6.

**Theorem 2.5** Let $\mathcal{F} = \{F_1, F_2, \ldots\}$ be a family of continuity moduli for $G$-variables of continuous function $G \times B_i \rightarrow B_i$.

(a) In the class of all unitary $\mathcal{F}$-representations of locally compact metric groups the condition of almost containing the unit representation $1_G$ coincides
with the condition of having expansions of the form \( A(G, H) \) which are members of \( \bigcup \{ K_\delta(\mathcal{F}) : \delta \in (0, 1) \cap \mathbb{Q} \} \).

(b) If for every compact subset \( Q \) of a locally compact metric group \((G, d)\) and every \( \varepsilon > 0 \) there is an expansion of \( G \) to a structure from \( K_{GH}(\mathcal{F}) \) with a \((Q, \varepsilon)\)-almost invariant unit vector but without non-zero invariant vectors, then \( G \) has a unitary \( \mathcal{F} \)-representation which almost contains the unit representation \( 1_G \) but does not fix any vector of norm 1. Moreover any elementary substructure of the corresponding structure \( A(G, H) \) is of the form \( A(G_0, H_0) \), where \( G_0 \leq G, H_0 \leq H \), and also almost contains the unit representation \( 1_{G_0} \) but does not fix any vector of norm 1.

Proof. (a) Let \( G \) be a locally compact metric group and let the ball \( K_\delta = \{ g \in G : d(g, 1) \leq \delta \} \subseteq G \) be compact. If a unitary \( \mathcal{F} \)-representation of \( G \) almost contains the unit representation \( 1_G \), then considering it as a structure \( A(G, H) \) we see that this structure belongs to \( K_\delta(\mathcal{F}) \).

On the other hand if some structure of the form \( A(G, H) \) belongs to \( K_\varepsilon(\mathcal{F}) \), then assuming that \( \varepsilon \leq \delta \) we easily see that the corresponding representation almost contains \( 1_G \). If \( \delta < \varepsilon \), then \( K_\varepsilon \) may be non-compact. However since \( K_\delta \subseteq K_\varepsilon \) any compact subset of \( G \) belongs to a finite union of sets of the form \( xK_\varepsilon \). Thus the axioms of \( K_\varepsilon(\mathcal{F}) \) state that the corresponding structure \( A(G, H) \) defines a representation almost containing \( 1_G \).

(b) Choose \( \delta > 0 \) so that the \( \delta \)-ball \( K = \{ g \in G : d(g, 1) \leq \delta \} \) in \( G \) is compact. To see that the group \( G \) has a required expansion in \( K_\delta(\mathcal{F}) \) we apply the following standard argument (see Proposition 1.2.1 from [1]). For every finite union \( \bigcup g_i K \) and every \( n \) fix a unitary representation \( \pi_{n, g} \) of \( G \) without non-zero invariant vectors and with a unit vector which is \( \frac{1}{n} \)-invariant with respect to \( \bigcup g_i K \). Then the direct sum of these representations almost contains \( 1_G \). Indeed, since every compact subset of \( G \) is contained in some finite union \( \bigcup g_i K \) we see that for every compact subset \( Q \subset G \) and every \( \varepsilon > 0 \) the representation has a \((Q, \varepsilon)\)-almost invariant unit vector. It is clear that there are no non-zero \( G \)-invariant vectors. Let us denote by \( M \) the corresponding structure from \( K_{GH}(\mathcal{F}) \).

To see the last assertion of part (b) note that since the condition \( d(g, 1) \leq \delta \) defines a totally bounded complete subset in any elementary extension of \( G \), the set \( K \) above is a definable subset of \( acl(\emptyset) \).

Let \( M_0 \preceq M \) and \( G_0 \) be the sort of \( M_0 \) corresponding to \( G \). Since the existence of an invariant unit vector can be written by a continuous formula we see that \( M_0 \) does not have such a vector.

It remains to verify that for any compact subset \( D \subset G_0 \) and any \( \varepsilon > 0 \) the representation \( M_0 \) always has a \((D, \varepsilon)\)-almost invariant unit vector. To see this note that since \( G_0 \prec G \) and \( K \) is compact and algebraic, the ball \( \{ g \in G_0 : d(g, 1) \leq \delta \} \subset G_0 \) is a compact neighbourhood of \( 1 \) which coincides with \( K \). In particular \( D \) is contained in a finite union of sets of the form \( gK \). The rest follows from the conditions that \( M_0 \in K_\delta(\mathcal{F}) \) and \( G_0 \prec G \). □
C. Comments. In Theorem 2.5 we cannot axiomatize the class of unitary $F$-representations $A(G,H)$ without fixed unit vectors (it cannot be done in continuous logic). Thus the definition of logical analyzability is satisfied for property $F$-non-$\langle T \rangle$ in a slightly weaker form.

On the other hand applying Corollary 1.6 we obtain the following corollary.

Corollary 2.6 Locally compact metric groups which have property $F$-non-$\langle T \rangle$ form a bountiful class.

It is an open question if the statement of Corollary 2.6 holds for metric groups which are not locally compact.

Remark 2.7 The author thinks that the following question is basic in this topic:

Is property $\langle T \rangle$ bountiful in the class of all metric groups?

Analyzing typical examples of groups with Kazhdan property $\langle T \rangle$ (for example in [21]) it seems likely that bountifulness of $\langle T \rangle$ is connected with the following question:

Does an elementary substructure of a discrete group with property $\langle T \rangle$ also have property $\langle T \rangle$?

It is natural to consider this question in the case of linear groups, where property $\langle T \rangle$ and elementary equivalence are actively studied, see [8] and [4].

Remark 2.8 Since in the locally compact case non-compact groups with property $\langle T \rangle$ are not amenable the following question seems related to Remark 2.7:

Is the class of all non-amenable metric groups bountiful?

One of definitions of non-amenability says that a topological group is non-amenable if there is a locally convex topological vector space $V$ and a continuous affine representation of $G$ on $V$ such that some non-empty invariant convex compact subset $K$ of $V$ does not contain a $G$-fixed point ([1], Theorem G.1.7).

This statement cannot be expressed in logic because the notion of locally convex topological vector spaces is not logically formalizable. On the other hand it is easy to see that if we restrict ourself just by linear representations on normed/metric vector spaces we obtain a bountiful property which is stronger than non-amenability. We call this property strong non-FP.

Considering in continuous logic linear $G$-representations on metric vector spaces $(V,d)$ we fix continuity moduli for the corresponding binary functions.
$G \times B_n \to B_n$ induced by the action on metric balls $B_n = \{v \in V : d(0, v) \leq n\}$. Since the action is not necessary isometric, we now need continuity moduli for $B_n$-variables too. Thus we define an uniform version strong non-FP as follows.

**Definition 2.9** Let $G$ be a metric group of a bounded diameter which is a continuous structure in the language $(d, \cdot, -1, 1)$. Let $F = \{F_1, F_2, \ldots\}$ be a family of continuity moduli for continuous functions $G \times B_i \to B_i$.

We say that $G$ has the strong $F$-non-FP property if there is a metric vector space $V$ and an $F$-continuous linear representation of $G$ on $V$ such that some non-empty invariant convex compact subset $K$ of $V$ does not have $G$-fixed points.

To realize our approach in this case we should consider the class of many-sorted continuous metric structures which consists of groups $G$ together with metric structures of metric vector spaces

$$(G, d, \cdot, -1, 1) \cup (\{B_n\}_{n \in \omega}, \{I_{mn}\}_{m < n}, d_0, \{\lambda_r\}_{r \in \mathbb{R}}, +, -) \cup (K, d, I)$$

and the new sort $K$ corresponding to convex compact subspace of $V$. It is mapped by $I$ into $B_1$ so that $I$ preserves the metric.

We use the property that when $K$ is compact for every natural $n$ there is a number $k_n$ such that any subset of $K$ of size $k_n$ contains a pair of distance $< \frac{1}{n}$. Express this property by a continuous formula, say $\phi_n$. Note that any many-sorted structure as above which satisfies some family of the form $\{\phi_n : n \in \omega \setminus \{0\}\}$, has the sort $K$ algebraic from the point of view of continuous logic, [2]. This implies that any elementary substructure has $K$ as a compact sort. Now it is easy to verify that the strategy of Theorem 2.5 and Corollary 2.6 works in this context too. In particular we have:

*The class of metric groups with the strong $F$-non-FP property is bountiful.*

**Remark 2.10** Let $G$ be a metric group of diameter 1 which is a continuous structure in the language $(d, \cdot, -1, 1)$. Let $F$ be a continuity modulus for continuous functions $G \times B \to B$ where $B$ is a metric space of diameter 1.

We say that $G$ is **F-non-extremely amenable** if there is a compact metric space $B$ of diameter 1 and an $F$-continuous action of $G$ on $B$ which does not have a $G$-fixed point in $B$. Using the arguments above it is easy to see that the following statement holds.

*The class of metric groups which are F-non-extremely amenable is bountiful.*

### 3 Unbounded actions

In this section we consider actions which do not preserve $n$-balls of metric spaces. This situation is more complicated than the one of Section 2.
The following material is standard. [9]. Let $(X,d)$ be a metric space. It is called **pointed** if we fix a point from $X$. A **geodesic path** joining $x \in X$ to $y \in X$ is an isometric map $\rho$ from some closed interval $[0,l] \subset \mathbb{R}$ to $X$ such that $\rho(0) = x$ and $\rho(l) = y$. Let $[x,y] = \rho([0,l])$. The space $(X,d)$ is called **uniquely geodesic** if there is exactly one geodesic path joining $x$ and $y$ for each $x,y \in X$. Note that the Hilbert space $H$ is uniquely geodesic. A uniquely geodesic space is called an **$\mathbb{R}$-tree** if for any $x,y,z$ the condition $[y,x] \cap [x,z] = \{x\}$ implies $[y,x] \cup [x,z] = [y,z]$. A subset $S \subseteq X$ is **convex** if $(\forall x,y \in S)[x,y] \subseteq S$.

Assume that $X$ is an $\mathbb{R}$-tree. Then a convex subset is also called a **sub-tree**. Given $x,y,z \in X$, there is a unique element $c \in [x,y] \cap [y,z] \cap [z,x]$, called the **median** of $x,y,z$. When $c \notin \{x,y,z\}$, the subtree $[x,y] \cup [x,z] \cup [y,z]$ is called a **tripod**. A **line** is a convex subset containing no tripod and maximal for inclusion. The **betweenness** relation $B$ of $X$ is the ternary relation $B(x;y,z)$ defined by $x \in (y,z)$. For any line $L$ and a point $a \in L$ the relation $\neg B(a;y,z)$ is an equivalence relation on $L \setminus \{a\}$. An equivalence class of this relation together with $a$ is called a **half-line**.

We say that half-lines $L_1$ and $L_2$ are equivalent if $L_1 \cap L_2$ contains a half-line. An **end** is an equivalence class of this relation.

We consider complete uniquely geodesic metric spaces as a many-sorted metric structures of $n$-balls of pointed spaces

$$((B_n)_{n \in \omega}, 0, \{I_{mn}\}_{m<n}, d),$$

where 0 denotes the fixed point and other symbols are interpreted in the natural way.

It is shown in [5] that the class of pointed real trees is axiomatizable in continuous logic by axioms of 0-hyperbolicity and the approximate midpoint property. We remind the reader that a metric space $(X,d)$ has the **approximate midpoint property** if for any $x,y \in X$ and any rational $\varepsilon > 0$ there exists $z \in X$ such that

$$|d(x,z) - d(x,y)/2| \leq \varepsilon \text{ and } |d(y,z) - d(x,y)/2| \leq \varepsilon.$$  

The space $(X,d)$ is called **0-hyperbolic** if for any $x,y,z,w \in X$ and any rational $\varepsilon > 0$

$$\min((x \cdot y)_w, (y \cdot z)_w) \leq (x \cdot z)_w + \varepsilon,$$

where

$$(x \cdot y)_w = \frac{1}{2}(d(x,w) + d(w,y) - d(x,y)) \text{ (Gromov product)}.$$  

We now give necessary information concerning isometric actions of groups on real trees. This is taken from [7], [26] and [27].
Let $T$ be a real tree. Assume that $G$ has an isometric action on an $\mathbb{R}$-tree $T$. We say that $g \in G$ is \textbf{elliptic} if it has a fixed point, and \textbf{hyperbolic} otherwise.

**Lemma 3.1** Let $G$ be a group with an isometric action on an $\mathbb{R}$-tree $T$.

- If $g \in G$ is elliptic, its set of fix points $T^g$ is a closed convex subset.
- If $g$ is hyperbolic, there exists a unique line $L_g$ preserved by $g$; moreover, $g$ acts on $L_g$ by a translation.
- If $g$ is hyperbolic, then for any $p \in T$, $[p, g(p)]$ meets $L_g$ and $[p, g(p)] \cap L_g = [q, g(q)]$ for some $q \in L_g$.
- If $g$ and $h \in G$ are elliptic and $T^g \cap T^h = \emptyset$, then $gh$ is hyperbolic.

When $g$ is hyperbolic, $L_g$ is called the \textbf{axis} of $g$.

### 3.1 Actions and continuity moduli

**A. Assumptions.** Let a metric group $(G, \cdot, -1, d)$ act on $(X, d)$ by isometries, where $(X, d)$ is a uniquely geodesic space. When we consider this situation we always take the following assumptions. They are necessary to present the situation in continuous logic.

We firstly fix a point $0 \in X$, present $X$ as the union of $n$-balls of 0, and assume the existence of a function $g_{th} : \mathbb{N} \times [0, 1) \to \mathbb{N}$ such that for every natural $m$ if $g \in G$ satisfies $d(1, g) = \delta < 1$, then $g$ takes the ball $B_m$ into the ball $B_{g_{th}(n, \delta)}$. We will assume that $g_{th}$ is increasing with respect to the first argument. Note that when $(G, d)$ is discrete with with the $\{0, 1\}$-metric we can take $g_{th}(n, \delta) = \text{id}(n)$.

Note that for any pair $m < n$ the action defines a partial map $G \times B_m \to B_n$ which is uniformly continuous with respect to the $B_m$-argument. In order to present this action in continuous logic we assume that this function is uniformly continuous with respect to the $G$-argument too. To do this in a formal way let us define ternary predicates

$$
\circ_{mn}(g, x, y) : G \times B_m \times B_m \to [0, m + n] , m \leq n \in \mathbb{N},
$$

as follows.

**Definition 3.2**

$$
\circ_{mn}(g, x, y) = \text{length}([g \circ x, y] \cap B_n), \text{ where } x \in B_m \text{ and } y \in B_m.
$$

In this formula the length is defined with respect to the metric of $X$. Note that for $x, y \in B_m$ with $g \circ x \in B_n$ we have $\circ_{mn}(g, x, y) = d(g \circ x, y)$. We now formulate our basic assumption.
Definition 3.3 Let $\text{ort} : \mathbb{N} \to \mathbb{N}$ be a function such that for any $m, n \in \mathbb{N}$ with $\text{ort}(m) < n$ there is a continuity modulus for the $G$-variable of the predicate $\circ_{mn}$.

We fix such a modulus and denote it by $\gamma_{G}^{m,n}$.

Remark 3.4 It is worth mentioning here that the assumption of uniform continuity of the unrestricted function $\circ : G \times X \to X$ strongly trivialize the situation. Indeed, assume that $X$ is an $\mathbb{R}$-tree and $g \in G$ acts on $X$ with non-fixed points. If $g$ is elliptic, fixes $x_0$ and takes some $x$ to $y \neq x$, then a half-line starting at $x_0$ and containing $x$ is taken by $g$ to a half-line containing $y$. If the first half-line is unbounded the set of values $d(z, g \circ z)$ for elements $z$ of this half-line is unbounded too. A similar argument works in the case of hyperbolic $g$. In this case one should consider half-lines not cofinal with the axis of $g$. We see that for non-discrete $(G, d)$ no condition of the form $d(1, g) < \delta$ bounds the set of values $d(z, g \circ z)$ if $X$ has infinitely many pairwise nonequivalent unbounded half-lines. In particular the action is not uniformly continuous as a binary function.

We now discuss the assumptions of existence of functions $\text{gth}$ and $\text{ort}$ in the case of $\mathbb{R}$-trees and in the case of real $H$ separately. We will see below that in the former case some natural geometric condition guarantees that the identity function $\text{id}$ works for $\text{ort}$. In the latter case function $\text{id}$ is not relevant. However we will give a completely satisfactory answer in this case: the only assumption that all partial maps $G \times B_m \to B_n$ defined by the action on $H$ are uniformly continuous implies the existence of a function $\text{ort}$ satisfying Definition 3.3.

B. The case of $\mathbb{R}$-trees. Let a metric group $(G, \cdot, ^{-1}, d)$ act on $(X, d)$ by isometries, where $(X, d)$ is a pointed $\mathbb{R}$-tree. For any pair $m < n$ let $\tilde{\gamma}_{G}^{m,n}$ be a continuity modulus of the $G$-variable of the partial function $G \times B_m \to B_n$.

In Definition 3.3 we formulate a condition which guarantees the assumption of Definition 3.3 in the case $\text{ort} = \text{id}$. This will be proved in Lemma 3.7. In Remark 3.8 we will see that these conditions are equivalent.

Definition 3.5 We say that the action of $G$ on $X$ has hyperbolic continuity moduli if for any $h \in G$, any pair $m < n$ and any rational $q > 0$ if $x \in B_m$ and $y \in h(B_m) \setminus B_n$ then

$$\forall g \in G(d(1, g) \leq \tilde{\gamma}_{G}^{m,n}(q) \to |\text{length}([x, y] \cap B_n) - \text{length}([x, g \circ y] \cap B_n)| \leq q).$$

In fact this property links the metric of $G$ with the metric of $X$. For an illustration consider the case when $X$ is a countable simplicial tree of finite valancy and $G \leq \text{Aut}(X)$ is under a metric defining the pointwise convergence topology. Then the property of hyperbolic continuity moduli
Remark 3.6 The property of hyperbolic moduli follows from the condition that for any \( q > 0 \) there exists \( \delta > 0 \) such that

\[
\forall g \in G \forall y \in X (d(1,g) \leq \delta \wedge y \notin B_n \rightarrow \text{length}(\{y, g \circ y\} \cap B_n) \leq q).
\]

Indeed correcting \( \tilde{\gamma}^{m,n}_G \) if necessary we can replace \( \delta \) in this formula by \( \tilde{\gamma}^{m,n}_G(q) \).

Now to verify the condition of Definition 3.5 take the median, say \( c \), of the triple \( x, y, g \circ y \). If \( c \notin B_n \) then

\[
\text{length}(\{x, y\} \cap B_n) = \text{length}(\{x, g \circ y\} \cap B_n).
\]

If \( c \in B_n \) then

\[
|\text{length}(\{x, y\} \cap B_n) - \text{length}(\{x, g \circ y\} \cap B_n)| \leq \\
|\text{length}(\{y, c\} \cap B_n) + \text{length}(\{c, g \circ y\} \cap B_n)|,
\]

and the latter value is not greater than \( q \).

Lemma 3.7 Assume that a group \( G \) acts on \( X = \bigcup B_n \) by isometries and the action \( \circ \) has hyperbolic continuity moduli \( \tilde{\gamma}^{m,n}_G \) on \( G \). Let \( A(G, X) \) be the corresponding many-sorted metric structures of \( n \)-balls of 0

\[
(G, \cdot, d) \cup (\{B_n\}_{n \in \omega}, 0, \{I_{mn}\}_{m<n}, d)
\]

extended by the predicates \( \circ_{mn} \), \( m < n \), as above.

Then the predicates \( \circ_{mn} \) have continuity moduli \( \gamma^{m,n}_G = \tilde{\gamma}^{m,n}_G \) for the \( G \)-variable and \( \text{id} \) for variables of \( X \).

Proof. Let us consider the first \( B_{m} \)-variable. Assume \( d(x, x') < \varepsilon \). Then

\[
d(g \circ x, g \circ x') < \varepsilon.
\]

If \( g \circ x, g \circ x' \in B_n \) then obviously \( |d(g \circ x, y) - d(g \circ x', y)| < \varepsilon \).

Let \( g \circ x' \notin B_n \) and let \( c \) be the median of the triple \( y, g \circ x, g \circ x' \). Then \( [y, g \circ x] = [y, c] \cup [c, g \circ x] \) and \( [y, g \circ x'] = [y, c] \cup [c, g \circ x'] \). If \( [y, c] \) contains a point of distance \( n \) from 0 then \( \circ_{mn}(g, x, y) = \circ_{mn}(g, x', y) \). If \( [y, c] \) does not contain a point of distance \( n \) from 0 then

\[
|\circ_{mn}(g, x, y) - \circ_{mn}(g, x', y)| \leq |d(g \circ x, c) - d(g \circ x', c)| < \varepsilon.
\]

The case of the second \( B_m \)-variable is similar.

In the case of the variable of \( G \) assume that \( d(g, g') \leq \tilde{\gamma}^{m,n}_G(\varepsilon) \). Then for \( g \circ x, g' \circ x \in B_n \) we have \( d(g \circ x, g' \circ x) \leq \varepsilon \) and by the triangle inequality,

\[
|\circ_{mn}(g, x, y) - \circ_{mn}(g', x, y)| \leq \varepsilon.
\]
If \( g' \circ x \not\in B_n \) then

\[
| \circ_{mn} (g, x, y) - \circ_{mn}(g', x, y) | \leq | \text{length}([y, g' \circ x] \cap B_n) - \text{length}([y, g(g')^{-1} \circ (g' \circ x)] \cap B_n) |
\]

and the latter value does not exceed \( \varepsilon \) by hyperbolicity of continuity moduli. The remaining case is similar. □

Remark 3.8 Note that Lemma 3.7 also holds in the opposite direction. Assume that a group \( G \) acts on \( X = \bigcup B_n \) by isometries and the predicates \( \circ_{mn}, m < n \), have continuity moduli \( \gamma_m^n \) which are also continuity moduli of functions \( G \times B_m \to B_n \) defined by the action \( \circ \). Then these continuity moduli satisfy the condition of hyperbolicity from Definition 3.5.

In terms of Definition 3.5 to see this statement it suffices to consider the value

\[
| \circ_{mn} (h, h^{-1}(y), x) - \circ_{mn}(gh, h^{-1}(y), x) |.
\]

Lemma 3.7 shows that typical \( G \)-spaces which are real trees can be considered as continuous metric structures with respect to some functions \( gth \) and \( ort \). Indeed we can define \( ort(m) = m \). In Section 3.2 we demonstrate that this approach works for logical analyzability of the class non-\( \text{FR} \) with respect to actions with some \( gth \) and \( ort \) defined. In particular

- in Theorem 3.12 we prove that the class of structures \( A(G, X) \) as in Lemma 3.7 (and with fixed \( gth \) and \( ort \)) is axiomatizable,

- the statement of Proposition 1.7(b) concerning non-\( \text{FR} \) can be extended to a statement of logical analyzability, see Theorem 3.15.

Does this approach work in the case of isometric actions on \( \mathcal{H} \)? It is easy to see that the definition \( ort(m) = m \) cannot be justified in the case of \( \mathcal{H} \). We now discuss this problem and show how our approach should be modified in this case.

C. The case of Hilbert spaces. Assume that a metric group \( (G, \cdot, ^{-1}, d) \) acts on the real Hilbert space \( \mathcal{H} \) by isometries with respect to the metric induced by \( \| \| \). It is well-known that such isometries are affine transformations ([H], Chapter 2).

We start with the description why in the case of Hilbert spaces the ternary predicates \( \circ_{mn} \) defined above can lose continuity moduli for some \( m < n \). Let us fix \( k, l, m \in \mathbb{N} \setminus \{0\} \) and \( \varepsilon > 0 \). Let \( n = m + 1 \). Take a point \( y_1 \) of norm \( m \) and a point \( y_2 \in B_n \setminus B_{n-\frac{\varepsilon}{4}} \). Assume that \( d(y_1, y_2) = l \). Let \( g \in G \) satisfy \( g \circ 0 \notin B_n \), \( d(g \circ 0, y_2) = \varepsilon \) and \( y_2 \in [0, g \circ 0] \). Let \( g' \circ 0 = y_2 \). Assuming
that $k$, $l$ and $m$ are sufficiently large (in particular the $n$-sphere bounding $B_n$ is close to be flat in some domain containing $y_1$ and $y_2$) we can arrange that $|\circ_{mn}(g,0,y_1) - \circ_{mn}(g',0,y_1)|$ is close to 1, while $d(g \circ 0, g' \circ 0) = \varepsilon$. In particular we do not have a method of defining continuity moduli in this case.

The following geometric lemma shows how the function $\text{ort}$ should be defined.

**Lemma 3.9** For every natural number $m$ there exists a number $n > m$ such that the following statements hold.

(a) For any positive $\varepsilon < \frac{1}{2}$ if $v, v_1, v_2 \in H$ satisfy $\min(\|v_1\|, \|v_2\|) \geq n$, $\|v\| \leq m$, and $\|v_1 - v_2\| < \varepsilon$, then the distance between the points of the $\|\|\|\|n$-norm $n$, say $v_1'$ and $v_2'$, belonging correspondingly to the segments $[v, v_1]$ and $[v, v_2]$ is smaller than $2\varepsilon$.

(b) For any positive $\varepsilon < \frac{1}{2}$ if $v, v_1, v_2 \in H$ satisfy $\max(\|v_1\|, \|v_2\|) \leq m$, $\|v\| > n$, and $\|v_1 - v_2\| < \varepsilon$, then the distance between the points of the $\|\|\|n$-norm $n$, say $v_1'$ and $v_2'$, belonging correspondingly to the segments $[v, v_1]$ and $[v, v_2]$ is smaller than $2\varepsilon$.

(c) For any positive $\varepsilon < \frac{1}{2}$ if $v, v_1, v_2 \in H$ satisfy $\|v_1\| \geq n$, $\|v_2\| \leq m$, $\|v\| = n$, $v \in [v_1, v_2]$ and $v_1 \in B_{n+\varepsilon}$, then $\|v_1 - v\| \leq 2\varepsilon$.

**Proof.** We prove (a) and (b) simultaneously. Let us choose $n$ so that for any $u_1, u_2 \in B_m$ and any $v_1, v_2 \notin B_n$ with $d(v_1, v_2) \leq \frac{1}{2}$ the angle $\alpha$ between $\overrightarrow{u_1v_1}$ and $\overrightarrow{u_2v_2}$ is sufficiently small. In particular these vectors are close to be collinear to $0\overrightarrow{v_1}$ and $0\overrightarrow{v_2}$. Then for $v_1', v_2'$ of the norm $n$ so that $v_1' \in [u_1, v_1]$ and $v_2' \in [u_2, v_2]$, the vector $\overrightarrow{v_1'v_2'}$ is close to be orthogonal both to $\overrightarrow{u_1v_1}$ and $\overrightarrow{u_2v_2}$. In particular we may assume that when $d(u_1, u_2) \leq \varepsilon \leq \frac{1}{2}$ and $d(v_1, v_2) \leq \varepsilon \leq \frac{1}{2}$ we have $d(v_1', v_2') \leq 2\varepsilon$.

Statement (c) follows from the fact that $\overrightarrow{v_1v_2}$ is close to be collinear with $\overrightarrow{v_10}$.

Let us fix a function $\text{ort} : \mathbb{N} \to \mathbb{N}$ which finds for every $m$ a number $n$ as in the formulation of Lemma 3.9. Notice that this function only depends on geometric properties of $H$. Let $\text{ort}^H$ be the minimal one. We consider ternary predicates

$\circ_{mn}(g, x, y) : G \times B_m \times B_m \to [0, n + m]$, $m \in \omega$, $\text{ort}^H(m) \leq n$,

as in Definition 3.2. Let $A(G, H)$ be the corresponding many-sorted metric structures of $n$-balls of $0$

$$(G, \cdot, d) \cup (\{B_n\}_{n \in \omega}, \{I_{mn}\}_{m < n}, d).$$

extended by the predicates $\circ_{mn}$ for $\text{ort}^H(m) < n$.  

20
Lemma 3.10  Let a group $G$ act on $H = \bigcup B_n$ by affine isometries. Assume that for any $m, n$ with $\text{ort}^H(m) < n$ there is a continuity modulus $\gamma_{G,m,n}$ for the $G$-variable of the partial function $G \times B_m \to B_n$ induced by the action. Let $A(G, H)$ be the corresponding structure with all $\omega_{mn}$ where $\text{ort}^H(m) < n$. Then the predicates $\omega_{mn}$ have continuity moduli $8\gamma_{G,m,n}$ for the $G$-variable and $3\text{id}$ for variables of $H$.

Proof. The proof is similar to Lemma 3.7. Let us consider the first $B_m$-variable. Assume $d(x, x') < \varepsilon < \frac{1}{2}$. Then $d(g \circ x, g \circ x') < \varepsilon$. If $g \circ x, g \circ x' \in B_n$ then obviously $|d(g \circ x, y) - d(g \circ x', y)| < \varepsilon$. Let $g \circ x' \notin B_n$ and let $x'' \in [y, g \circ x']$ be the point of distance $n$ from 0. Then in the case $g \circ x \in B_n$ Lemma 3.9(c) and the triangle inequality guarantee that $d(g \circ x, x'') \leq 3\varepsilon$, i.e. $|d(g \circ x, y) - d(x'', y)| < 3\varepsilon$. When $g \circ x \notin B_n$ choose $x \in [y, g \circ x]$ of distance $n$ from 0. Then by Lemma 3.9(a) we have $|d(x, y) - d(x'', y)| < 2\varepsilon$ which in fact is equivalent to $|\omega_{mn}(g, x, y) - \omega_{mn}(g, x', y)| < 2\varepsilon$.

The case of the $B_m$-variable $y$ is similar.

In the case of the variable of $G$ assume that $d(g, g') \leq 8\gamma_{G,m,n}(\varepsilon)$. If $g \circ x, g' \circ x \in B_n$ then $d(g \circ x, g' \circ x) \leq \varepsilon$ and by the triangle inequality,

$$|\omega_{mn}(g, x, y) - \omega_{mn}(g', x, y)| \leq \varepsilon.$$ 

Assume that neither $g \circ x$ nor $g' \circ x$ belongs to $B_n$. The map $g'g^{-1}$ takes $[y, g \circ x]$ to $[g'g^{-1} \circ y, g' \circ x]$. Let $y' \in [y, g \circ x]$ and $y'' \in [g'g^{-1} \circ y, g' \circ x]$ be elements of norm $n$. Replacing $y'$ by $y''$ if necessary we may assume below that $(g'g^{-1})^{-1} \circ y'' \in B_n$. Then $\|y'' - (g'g^{-1})^{-1} \circ y''\| \leq \varepsilon$ and $\|y - g'g^{-1} \circ y\| \leq \varepsilon$. Since $n - \|(g'g^{-1})^{-1} \circ y''\| \leq \varepsilon$ applying arguments as in Lemma 3.9 we easily see that $\|y' - (g'g^{-1})^{-1} \circ y''\| \leq 2\varepsilon$ and $\|y' - y''\| \leq 3\varepsilon$. In particular the difference between $\|y - y'\|$ and $\|g'g^{-1} \circ y - y''\|$ does not exceed $5\varepsilon$. Applying Lemma 3.9(b) we see that the length of the $B_n$-part of the segment $[y, g' \circ x]$ does not differ from $\|g'g^{-1} \circ y - y''\|$ more than $3\varepsilon$. In particular it does not differ from $\|y - y'\|$ more than $8\varepsilon$. By definition of $\omega_{mn}$ we have

$$|\omega_{mn}(g, x, y) - \omega_{mn}(g', x, y)| \leq 8\varepsilon.$$ 

The case when $|\{g \circ x, g' \circ x\} \cap B_n| = 1$ can be arranged by similar arguments. \hfill \Box

3.2 Non-$\mathbb{F}\mathbb{R}$-actions

Let us fix continuity moduli $\gamma_{G,m,n}$ and consider the class of continuous metric structures which are unions of continuous groups $(G, \cdot, -1, d)$ together with many-sorted metric structures of $n$-balls of pointed $\mathbb{R}$-trees

$$X = (\{B_n\}_{n \in \omega}, 0, \{I_{mn}\}_{m < n}, d), \text{ where } I_{mn} : B_m \to B_n,$$
and ternary predicates

\[ \circ_{mn}(g, x, y) : G \times B_m \times B_m \to [0, m + n], \quad m \leq n \in \omega, \]

where \( \gamma_{G}^{m,n} \) is the continuity modulus with respect to \( G \) and the continuity moduli with respect to \( B_m \) are equal to \( \text{id} \).

We now introduce an axiomatizable class \( \mathcal{K}_{iso}(\text{gth}, \gamma_{G}^{m,n}) \) which contains all structures of the form \( A(G, X) \) from Lemma 3.7 which correspond to actions with a fixed function \( \text{gth} : \mathbb{N} \times [0, 1) \to \mathbb{N} \) (when we assume that \( \text{ort} = \text{id} \)). We will see in Theorem 3.12 that the axioms below guarantee that all members of \( \mathcal{K}_{iso}(\text{gth}, \gamma_{G}^{m,n}) \) are obtained in this way. This is a confirmation that our approach works well!

Since our formulas are becoming quite complicated we will not mention below the domains of quantifiers \( \inf \) and \( \sup \). Usually this is clear from the sorts of the predicates appearing in the formula.

**Definition 3.11** Let \( \mathcal{K}_{iso}(\text{gth}, \gamma_{G}^{m,n}) \) be the class of structures of the language above which satisfy axioms of metric groups with invariant metrics for the sort \( G \), the axioms of pointed real trees and the following three groups of axioms of the relation \( \circ_{mn} \).

1(a) The value \( g \circ x \) is eventually defined inside some \( B_n \).

\[ \sup_{g \sup_{x} \min(\circ_{ms}(g, x, 0))} - n, \delta - d(1, g)) = 0, \]

where \( \delta \in [0, 1) \cap \mathbb{Q} \) and \( \text{gth}(m, \delta) \leq n < s \).

1(b) Correctness.

\[ \sup_{g \sup_{x,y} \min(| \circ_{ms}(g, x, y) - \circ_{ms}(g, I_{mn'}(x), I_{mn'}(y))|, \delta - d(1, g)) = 0, \]

where \( \delta \in [0, 1) \cap \mathbb{Q}, \ m < m', \ \text{gth}(m, \delta) < s < t \) and \( \text{gth}(m', \delta) < t \).

1(c) Approximation of the value \( g \circ x \) by points \( z \) (when \( g \circ x \in B_m \)).

\[ \sup_{g \sup_{x} \min(m - \circ_{mn}(g, x, 0), \Phi(g, x)) = 0, \]

where

\[ \Phi(g, x) = \inf_{z} \max(\circ_{mn}(g, x, z), \sup_{u} \min(|d(u, z) - \circ_{mn}(g, x, u)|, m - d(0, u))). \]

2(a) The predicate \( \circ_{mn} \) measures the distance inside \( B_m \). Triangle inequalities involving \( g \circ x \).

\[ \sup_{g \sup_{x,y_1,y_2} \min(d(y_1, y_2) - (\circ_{mn}(g, x, y_1) + \circ_{mn}(g, x, y_2)), m - \circ_{mn}(g, x, 0)) = 0. \]

\[ \sup_{g \sup_{x,y_1,y_2} \min(\circ_{mn}(g, x, y_1) - (d(y_1, y_2) + \circ_{mn}(g, x, y_2)), m - \circ_{mn}(g, x, 0)) = 0. \]

2(b) A version of the triangle inequality when the action \( \circ \) is isometric.

\[ \sup_{g \sup_{x_1,x_2} \min(\sup_{y} d(x_1, x_2) - (\circ_{mn}(g, x_1, y) + \circ_{mn}(g, x_2, y)), m - \circ_{mn}(g, x_1, 0), m - \circ_{mn}(g, x_2, 0)) = 0. \]

2(c) The action \( \circ \) is isometric.

\[ \sup_{g \sup_{x_1,x_2} \min(\inf_{y} ((\circ_{mn}(g, x_1, y) + \circ_{mn}(g, x_2, y)) - d(x_1, x_2)), m - \circ_{mn}(g, x_1, 0), \]

...
\[
m^{\cdot}(\circ_{mn}(g, x_2, 0)) = 0.
\]

3(a) The neutral element acts trivially. 
\[\sup_{x,y}(|\circ_{mn}(1, x, y) - d(x, y)|) = 0.\]
3(b) The action of \(g^{-1}\) inside \(B_m\).
\[\sup_{g} \sup_{x,y} \min(|\circ_{mn}(g, x, y) - \circ_{mn}(g^{-1}, y, x)|, m^{\cdot} \circ_{mn}(g, x, 0), m^{\cdot} \circ_{mn}(g^{-1}, y, 0)) = 0.\]
3(c) When \(g'(x) = z\) and \(g(z) = y\), then \(gg'(x) = y\).
\[\sup_{x,y,g,g'} \min(\Phi(g, g', x, y), m^{\cdot} \circ_{mn}(g^{-1}, y, 0), m^{\cdot} \circ_{mn}(g', x, 0), m^{\cdot} \circ_{mn}(gg', x, 0)) = 0,\]
where \(\Phi(g, g', x, y) = \sup_{z}(\circ_{mn}(gg', x, y) - (\circ_{nm}(g^{-1}, y, z) + \circ_{nm}(g', x, z))).\]

**Theorem 3.12** (a) Assume that a group \(G\) acts on \(X = \bigcup B_n\) by isometries according to a function \(gth\) and the action \(\circ\) has hyperbolic continuity moduli \(\{\gamma_{G}^{m,n} : m < n \in \omega\}\) on \(G\). Let \(A(G, X)\) be the corresponding structure defined as in Lemma 3.7. Then \(A(G, X) \in K_{iso}(gth, \gamma_{G}^{m,n}).\)

(b) Any structure of the class \(K_{iso}(gth, \gamma_{G}^{m,n})\) is of the form \(A(G, X)\) for an appropriate isometric action of \(G\) with hyperbolic continuity moduli \(\gamma_{G}^{m,n}\) for the sort \(G\).

**Proof.** (a) The first part of the theorem is straightforward. It uses Lemma 3.7.

(b) Let a continuous structure \(M\) belong to \(K_{iso}(gth, \gamma_{G}^{m,n})\). Let \(X = \bigcup B_n\), where \(B_n\) are sorts of balls of \(M\). Let \(g \in G\) and \(x \in B_m\). Using axioms 1(a,b) we can find \(m < n\) large enough such that \(\circ_{ms}(g, x, 0) \leq m\) for all \(s > n\).

Using axiom 1(c) choose in \(B_m\) a sequence \(u_1, u_2, \ldots, u_i, \ldots\), such that for \(n\) and \(s\) as above,
\[
\max(\circ_{ms}(g, x, u_{i+1}), \min(|d(u_i, u_{i+1}) - \circ_{ms}(g, x, u_i)|) < \frac{1}{2^{i+1}}.
\]
By completeness of \(B_m\) there is \(\hat{u} \in B_m\) which is the limit of the sequence \(u_1, u_2, \ldots, u_i, \ldots\). Using axiom 1(c) and the choice of \(u_i\) it is easy to see that \(\circ_{ms}(g, x, \hat{u}) = 0\). Define the value \(g \circ x\) to be \(\hat{u}\). By axioms 1(b,c) this value is defined in a unique way. This procedure defines a binary function \(\circ : G \times X \rightarrow X\).

By axioms 3(a - c) the function \(\circ\) is an action of \(G\) on \(X\). By axioms 2(a) it is easy to see that when \(x, y, u \in B_m\) and \(g \circ x = u\) then \(\circ_{mn}(g, x, y) = d(u, y)\). Using axioms 2(b,c) we have that the action is isometric.

Since \(\circ_{mn}\) has continuity modulus \(\gamma_{G}^{m,n}\) on \(G\) we easily obtain that the action \(\circ\) also has the \(G\)-continuity modulus \(\gamma_{G}^{m,n}\) for the partial function \(G \times B_m \rightarrow B_m\). To see that continuity moduli \(\gamma_{G}^{m,n}\) are hyperbolic, note that when \(x \in B_m\) and \(y \in h(B_m) \setminus B_n\), the implication
\[
\forall g \in G(d(1, g) \leq \gamma_{G}^{m,n}(q)) \rightarrow |\text{length}([x, y] \cap B_n) - \text{length}([x, g \circ y] \cap B_n)| \leq q).
\]
can be rewritten as
\[ \forall g \in G(d(1, g) \leq \gamma_{G}^{m,n}(q) \rightarrow |\circ_{mn}(h, h^{-1}(y), x) - \circ_{mn}(gh, h^{-1}(y), x)| \leq q). \]

We now conclude that \( M \) is of the form \( A(G, X) \) for an appropriate isometric action of \( G \) with hyperbolic continuity moduli \( \gamma_{G}^{m,n} \). □

We apply this theorem for groups which are non-\( \text{FR} \). Let us start with the following definition which modifies non-\( \text{FR} \) by taking attention to the corresponding continuity moduli.

**Definition 3.13** Let \( G \) be a metric group of a bounded diameter which is a continuous structure in the language \((d, \cdot, -^1, 1)\).

We say that \( G \) has property non-\( \text{FR} \) in the class \( K_{\text{iso}}(\text{gth}, \gamma_{G}^{m,n}) \) if it has an isometric action on a \( \mathbb{R} \)-tree \( X \) without fixed points and the action can be presented as a structure \( A(G, X) \in K_{\text{iso}}(\text{gth}, \gamma_{G}^{m,n}) \) for some point \( 0 \in X \).

If a group is non-\( \text{FR} \) in \( K_{\text{iso}}(\text{gth}, \gamma_{G}^{m,n}) \), then it is non-\( \text{FR} \). Note that a metric group \((G, d)\) satisfies property non-\( \text{FR} \) in \( K_{\text{iso}}(\text{gth}, \gamma_{G}^{m,n}) \) if there is a continuous isometric action of \( G \) on a real tree \( X \) and a natural number \( s > 0 \) such that the corresponding structure \( A(G, X) \) belongs to \( K_{\text{iso}}(\text{gth}, \gamma_{G}^{m,n}) \) and satisfies all statements of the following form:

\[ \sup_{v \in B_{m}} \inf_{g} \left( \frac{1}{s} - \circ_{mn}(g, v, v) \right) = 0 \], where \( m, n \in \omega \) and \( m < n \).

(saying that each element of \( B_{m} \) is moved by some element of \( G \) by approximately \( \frac{1}{s} \)). Let us denote it by \( \Theta_{m,n,s} \). The following proposition shows that this is the only reason to be non-\( \text{FR} \). It does not use any logic.

**Proposition 3.14** If a group \( G \) acts on a real tree \( X \) by isometries without fixed points then there is a natural number \( s \) such that for any \( m \) each element of \( B_{m} \) is moved by some element of \( G \) by a distance greater than \( \frac{1}{s} \).

**Proof.** If \( G \) has a hyperbolic element \( g \) of hyperbolic length \( r \) (i.e. \( g \)-shifts all points of its axis \( L \)), then it is easy to see by Lemma 3.1 that any element of \( X \) is moved by \( g \) at the distance \( \geq r \). Thus \( s \) can be chosen so that \( \frac{1}{s} < r \).

Consider the case when \( G \) consists of elliptic elements. Since \( G \) does not fix any point, by a well-known argument \( G \) fixes an end \((20, \text{Section 6.5, Exercise 2})\). Let \( L_{0} \) be the half-line starting from 0 which represents this end and let \( v_1, \ldots, v_i, \ldots \) be a cofinal \( \omega \)-sequence in \( L_{0} \) with \( d(v_i, v_{i+1}) \geq 1 \). Then we may assume that \( G \) is the union of a strictly increasing chain of stabilizers \( G_i \) of \( v_i \).

Having \( m \) find \( j \) with \( v_{j-1} \notin B_m \) (thus \( v_j \notin B_m \)). Since any arc linking \( v_j \) with an element from \( B_m \) must contain \( v_{j-1} \), we see that if \( g \in G_j \) fixes
a point of $B_m$ then it fixes $v_{j-1}$. Since $G_j \neq G_{j-1}$ the group $G_j$ contains an element $g$ which does not fix any element of $B_m$. Since $d(v_j, v_{j-1}) \geq 1$ any point of $B_m$ can be taken by $g$ at a distance greater than 1. Thus we define $s = 1$. □

We now prove logical analyzability of the case of non-$\text{F}_\mathbb{R}$ in $K_{iso\mathbb{R}}(\text{gth}, \gamma_{G}^{m,n})$.

**Theorem 3.15** (a) Every group with property non-$\text{F}_\mathbb{R}$ in $K_{iso\mathbb{R}}(\text{gth}, \gamma_{G}^{m,n})$ has an expansion to a structure $A(G, X)$ which belongs to the subclass of $K_{iso\mathbb{R}}(\text{gth}, \gamma_{G}^{m,n})$ axiomatizable by the family $\{\Theta_{s,m,n} : m < n, m, n \in \omega\}$ for some fixed $s$.

(b) The class of structures $A(G, X)$ witnessing property non-$\text{F}_\mathbb{R}$ in $K_{iso\mathbb{R}}(\text{gth}, \gamma_{G}^{m,n})$ is bountiful.

**Proof.** Statement (a) follows from Proposition 3.14. The last statement follows from the Löwenheim-Skolem theorem and statement (a). □

Note that Theorem 3.15 states a stronger property than just bountifulness of the class of groups with property non-$\text{F}_\mathbb{R}$ in $K_{iso\mathbb{R}}(\text{gth}, \gamma_{G}^{m,n})$. Statement 3.15(b) can be considered as a version of the corresponding case of Proposition 1.7(b) in a precise form.

### 3.3 Non-FH-actions

We introduce an axiomatizable class $K_{iso\mathbb{H}}(\text{gth}, \text{ort}_H, \gamma_{G}^{m,n})$ of structures of the form $A(G, X)$ in the case of Hilbert spaces. We will see in Theorem 3.17 that the axioms guarantee that all members of $K_{iso\mathbb{H}}(\text{gth}, \text{ort}_H, \gamma_{G}^{m,n})$ are obtained in the way described in Lemma 3.10. After this we show logical analyzability of property non-$\text{FH}$.

In some sense the result of this section looks stronger than the results of Section 3.2. Contrary to the previous case we do not have any additional assumptions on continuity moduli of partial functions $G \times B_m \to B_n$.

**Definition 3.16** Let $K_{iso\mathbb{H}}(\text{gth}, \text{ort}_H, \gamma_{G}^{m,n})$ be a class of continuous metric structures which are unions of continuous groups $(G, \cdot, \gamma_{G}^{-1}, d)$ together with many-sorted metric structures of $n$-balls of real $H$

$$H = \left(\{B_n\}_{n \in \omega}, 0, \{I_{mn}\}_{m < n}, d, \{\lambda_r\}_{r \in \mathbb{R}}, +, -, \langle \rangle\right),$$

where $I_{mn} : B_m \to B_n$, and ternary predicates

$$\circ_{mn}(g, x, y) : G \times B_m \times B_m \to [0, n + m], m, n \in \omega, \text{ort}_H(m) < n$$

where $\gamma_{G}^{m,n}$ is the continuity modulus with respect to $G$ and the continuity moduli with respect to $B_m$ are equal to $3id$. The class $K_{iso\mathbb{H}}(\text{gth}, \text{ort}_H, \gamma_{G}^{m,n})$
is axiomatizable by the axioms of groups with an invariant metric for the sort $G$, the axioms of pointed real Hilbert spaces and the three groups of axioms of the relation $\circ_{mn}$ from Definition 3.11 where it is always assumed that $\text{ort}^H(m) < n$ and for $x, y \in H$ the value $\| x - y \|$ is denoted by $d(x, y)$.

The following theorem is a version of Theorem 3.12 in the case of isometric actions on $H$.

**Theorem 3.17** (a) Assume that a group $G$ acts on $H = \bigcup B_n$ by affine isometries according to a function $gth$. Let $A(G, H)$ be the corresponding structure defined as in Lemma 3.10. Assume that the predicates $\circ_{mn}$ for $\text{ort}^H(m) < n$, have continuity moduli $\{ \gamma^m_n : m < n \in \omega \}$ for $G$-variables and $3id$ for $B_m$-variables. Then $A(G, H) \in K_{isoH}(gth, \text{ort}^H, \gamma^m_n)$.

(b) Any structure of the class $K_{isoH}(gth, \text{ort}^H, \gamma^m_n)$ is of the form $A(G, H)$ for an appropriate isometric action of $G$ on $H$.

**Proof.** (a) This part is straightforward.

(b) The proof of this part repeats the corresponding place of the proof of Theorem 3.12(b). □

We now define an appropriate versions of non-FH.

**Definition 3.18** Let $G$ be a metric group of a bounded diameter which is a continuous structure in the language $(d, \cdot, -^1, 1)$.

(a) We say that $G$ satisfies **uniform property non-FH** if there is a function $gth : \mathbb{N} \times [0, 1) \to \mathbb{N}$ and there is an affine isometric action $\circ$ of $G$ on real $H$ without fixed points and the action $\circ$ corresponds to $gth$ and has the property that for any $m, n$ with $\text{ort}^H(m) < n$ there is a continuity modulus for the $G$-variable of the partial function $G \times B_m \to B_n$ induced by $\circ$.

(b) We say that $G$ has **property non-FH** in the class $K_{isoH}(gth, \text{ort}^H, \gamma^m_n)$ if it has an affine isometric action on $H$ without fixed points and the action can be presented as a structure $A(G, H) \in K_{isoH}(gth, \text{ort}^H, \gamma^m_n)$.

It is clear that the property of Definition 3.18(b) implies uniform non-FH and the latter implies non-FH. We start logical analysis of these properties with the following remark.

In the case of affine isometric actions on the real Hilbert space there is an obvious version of Proposition 3.14:

If a group $G$ acts on $H$ by isometries without fixed points then for any $m$ there is a natural number $s$ such that each element of $B_m$ is moved by some element of $G$ by a distance greater than $\frac{1}{s}$.

Indeed since $G$ does not fix any point, each orbit of $G$ is unbounded (Proposition 2.2.9 of [1]). Thus there is $g \in G$ so that $B_m \cap g(B_m) = \emptyset$. In particular there is $s \in \mathbb{N}$ such that $\frac{1}{s} \leq \| g \circ v - v \|$ for all $v \in B_m$. 26
We now see that if a metric group \((G, d)\) satisfies property non-FH in the class \(K_{iso}(\text{gth}, \text{ort}^H, \gamma^m_G)\) then there is a \(\gamma_G\)-continuous affine isometric action of \(G\) on \(H\) and a function \(m \to s(m)\) such that the corresponding structure \(A(G, H)\) satisfies all statements of the following form:

\[
\inf_g \sup_{v \in B_m} \left( \frac{1}{s(m)} \circ_{mn} (g, v, v) \right) = 0,
\]

(saying that there is an element of \(G\) which takes each element of \(B_m\) by approximately \(\frac{1}{s(m)}\)). Let us denote it by \(\Theta_{m,n}^{H,s}\).

The following theorem gives logical analyzability of uniform property non-FH.

**Theorem 3.19** (a) Every group with property non-FH in \(K_{iso}(\text{gth}, \text{ort}^H, \gamma^m_G)\) has an expansion to a structure \(A(G, H)\) which belongs to the subclass of \(K_{iso}(\text{gth}, \text{ort}^H, \gamma^m_G)\) axiomatizable by the family \(\{\Theta_{m,n}^{H,s} : \text{ort}^H(m) < n, m, n \in \omega\}\) for some function \(s : \mathbb{N} \to \mathbb{N}\).

(b) Every group with uniform property non-FH has an expansion to a structure \(A(G, H)\) which belongs to the subclass of some \(K_{iso}(\text{gth}, \text{ort}^H, \gamma^m_G)\) axiomatizable by the family \(\{\Theta_{m,n}^{H,s} : \text{ort}^H(m) < n, m, n \in \omega\}\) for some function \(s : \mathbb{N} \to \mathbb{N}\).

(c) The class of structures \(A(G, H)\) witnessing uniform property non-FH is bountiful.

**Proof.** Statement (a) follows from the above consequence of Proposition 2.2.9 of [1]. Statement (b) follows from (a), Lemma 3.10 and Theorem 3.17.

The last statement follows from the Löwenheim-Skolem theorem and statements (a) and (b). \(\Box\)

### 4 Around non-OB

In paragraph A below we uniformize property non-OB in order to make it logically analyzable. Unfortunately when the topology is discrete the class we obtain coincides with the class of all infinite groups. Since property OB is not empty in the class of discrete infinite groups we consider this class in paragraph B using a completely different approach. In fact we apply the standard version of \(L_{\omega_1 \omega}\). These arguments are very close to the paper [20], which was the starting point of the subject.

**A. Uniformization.** Property OB is defined in Definition [11]. In this section we study some modifications of OB in the case of metric groups which are continuous structures.

It is known (see Section 1.4 of [24]) that for Polish groups property OB is equivalent to the property that for any open symmetric \(V \neq \emptyset\) there is a
finite set $F \subseteq G$ and a natural number $k$ such that $G = (FV)^k$. Thus when $G$ is non-OB, there is an non-empty open $V$ such that for any finite $F$ and a natural number $k$, $G \neq (FV)^k$. Note that for such $F$ and $k$ there is a real number $\varepsilon$ such that some $g \in G$ is $\varepsilon$-distant from $(FV)^k$. Indeed, otherwise $(FV)^k V$ would cover $G$.

This suggests that the following property can be considered as a substitute for the complement to OB.

**Definition 4.1** A metric group $G$ is called uniformly non-OB if there is an open symmetric $V \neq \emptyset$ such that for any finite $F$ and a natural number $k$, there is $g \in G$ which is $\varepsilon$-distant from $(FV)^k$.

It is clear that all infinite discrete groups are uniformly non-OB. In order to verify that this property is logically analyzable we will consider metric groups with two distinguished grey subsets of them, i.e. two unary predicates denoted by $P$ and $Q$. In some sense the definition below states that the nullset of $Q$ is the complement of the nullset of $P$.

**Definition 4.2** Let $K_{\text{grey}}$ be the class of all continuous metric structures $\langle G, P, Q \rangle$ such that $G$ is a group with a bi-invariant $[0,1]$-metric, $P : G \to [0,1]$ and $Q : G \to [0,1]$ are unary predicates on $G$ with continuity moduli $\text{id}$ and the following axioms are satisfied:

\[
\begin{align*}
\sup_x |P(x) - P(x^{-1})| &= 0, & \inf_x |P(x) - 1/2| &= 0, \\
\sup_x |Q(x) - Q(x^{-1})| &= 0, & \inf_x |Q(x) - 1/2| &= 0, \\
Q(1) &= 0, & \sup_x \min(P(x), Q(x)) &= 0, \\
\sup_x \min(\varepsilon - Q(x), \inf_y (\max(d(x,y), 2\varepsilon, \varepsilon - P(y)))) &= 0, & \text{for all rational } \varepsilon \in [0, \frac{1}{2}].
\end{align*}
\]

Note that the last axiom implies that any neighbourhood of an element from the nullset of $Q$ contains an element with non-zero $P$.

**Definition 4.3** For any natural $m$ and $k$ and any rational $\varepsilon$ let $\theta(m, k, \varepsilon)$ be the following formula:

\[
\sup_{x_1, \ldots, x_m} \inf_x \sup_{y_1, \ldots, y_k} \min(P(y_1), \ldots, P(y_n), (\varepsilon - \min_{w \in W_{m,k}} (d(x, w)))) = 0,
\]

where $W_{m,k}$ consists of all words of the form $x_{i_1}y_1x_{i_2}y_2\ldots x_{i_k}y_k$.

It is worth noting that in the definition the expression $\theta(m, k, \varepsilon)$ formally is not a formula. On the other hand it is easy to see that it can be written by a formula of continuous logic. It states that for any finite set $F = \{x_1, \ldots, x_m\}$ there is $x$ which is $\varepsilon$-distant from $(F \cdot \{y : P(y) > 0\})^k$.

The following theorem shows logical analyzability of uniform non-(OB).
Theorem 4.4  (a) A metric group $G$ is uniformly non-OB if and only if there is a function $s : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ such that $G$ has an expansion from $\mathcal{K}_{\text{grey}}$ which satisfies all conditions $\theta(m, k, \frac{1}{s(m, k)})$.

(b) The class of uniformly non-OB metric groups is bountiful.

Proof. If $G$ is a uniformly non-OB-group, then find an open symmetric $V$ such that $1 \in V$ and for any natural numbers $m$ and $k$ there is a real number $\varepsilon$ such that for any $m$-element subset $F \subseteq G$ there is $g \in G$ which is $\varepsilon$-distant from $(FV)^k$. We interpret $Q(x)$ by $d(x, V)$ and $P(x)$ by $d(x, G \setminus V)$ (possibly normalizing them to satisfy the axioms of $\mathcal{K}_{\text{grey}}$). Then observe that $(G, P, Q) \in \mathcal{K}_{\text{grey}}$ and for any natural numbers $m$ and $k$ there is a rational number $\varepsilon$ so that $\theta(m, k, \varepsilon)$ holds in $(G, P, Q)$.

Assume that there is a function $s$ such that $G$ has an expansion as in statement (a). To verify uniform non-OB take the complement of the nullset of $P(x)$ as an open symmetric subset $V$. This proves statement (a).

Statement (b) follows from (a) and the Löwenheim-Skolem theorem for continuous logic. □

The following definition from [24] gives several versions of property OB.

Definition 4.5 Let $G$ be a topological group.

(1) The group $G$ is called bounded if for any open $V$ containing 1 there is a finite set $F \subseteq G$ and a natural number $k > 0$ such that $G = FV^k$.

(2) The group $G$ is Roelcke bounded if for any open $V$ containing 1 there is a finite set $F \subseteq G$ and a natural number $k > 0$ such that $G = V^kFV^k$.

(3) The group $G$ is Roelcke precompact if for any open $V$ containing 1 there is a finite set $F \subseteq G$ such that $G = VFV$.

(4) The group $G$ has property $(\text{OB})_k$ if for any open symmetric $V \neq \emptyset$ there is a finite set $F \subseteq G$ such that $G = (FV)^k$.

It is worth noting that by Section 1.10 of [24] in the case of $\sigma$-locally compact groups (i.e., $\sigma$-compact locally compact) all these properties coincide with property OB. Applying our method of uniformization of the corresponding negations of these properties one can obtain bountiful classes of metric groups which are:

- uniformly non-bounded;
- uniformly non-Roelcke bounded;
- uniformly non-Roelcke precompact;
- uniformly non-($\text{OB})_k$.  

The following definition from [24] gives several versions of property OB.
B. Discrete groups. An abstract group $G$ is Cayley bounded if for every generating subset $U \subseteq G$ there exists $n \in \omega$ such that every element of $G$ is a product of $n$ elements of $U \cup U^{-1} \cup \{1\}$. If $G$ is a Polish group then $G$ is topologically Cayley bounded if for every analytic generating subset $U \subset G$ there exists $n \in \omega$ such that every element of $G$ is a product of $n$ elements of $U \cup U^{-1} \cup \{1\}$. It is proved in [23] that for Polish groups property OB is equivalent to topological Cayley boundedness together with uncountable topological cofinality: $G$ is not the union of a chain of proper open subgroups.

Let us consider the abstract (discrete) case. Since in this case we do not need continuous logic, our considerations become simpler.

A group is strongly bounded if it is Cayley bounded and cannot be presented as the union of a strictly increasing chain $\{H_n : n \in \omega\}$ of proper subgroups (has cofinality $> \omega$). It is a discrete version of OB, which is also called Bergman’s property, [3]. It is known that strongly bounded groups have property FA, i.e. any action on a simplicial tree fixes a point.

As we already know the class of strongly bounded groups is not bountiful. The corresponding arguments given in Introduction can be also applied to property FA.

It is shown in [6], that strongly bounded groups have property FH. It can be also deduced from [6] that strongly bounded groups have property FR that every isometric action of $G$ on a real tree has a fixed point (since such a group acting on a real tree has a bounded orbit, all the elements are elliptic and it remains to apply cofinality $> \omega$). It is now clear that the bountiful class of groups having free isometric actions on real trees (or on real Hilbert spaces) is disjoint from strong boundedness.

In the following Proposition we apply the standard version of $L_{\omega_1 \omega}$.

**Proposition 4.6** The following classes of groups are reducts of axiomatizable classes in $L_{\omega_1 \omega}$:

1. The complement of the class of strongly bounded groups;
2. The class of groups of cofinality $\leq \omega$;
3. The class of groups which are not Cayley bounded;
4. The class of groups presented as non-trivial free products with amalgamation (or HNN-extensions);
5. The class of groups having homomorphisms onto $\mathbb{Z}$.

All these classes are bountiful. The class of groups which do not have property FA is bountiful too.

**Proof.** (1) We use the following characterization of strongly bounded groups from [6].

A group is strongly bounded if and only if for every presentation of $G$ as $G = \bigcup_{n \in \omega} X_n$ for an increasing sequence $X_n$, $n \in \omega$,
with \( \{1\} \cup X_n^{-1} \cup X_n \cdot X_n \subset X_{n+1} \) there is a number \( n \) such that \( X_n = G \).

Let us consider the class \( K_{nb} \) of all structures \( \langle G, X_n \rangle_{n \in \omega} \) with the first-order axioms stating that \( G \) is a group, \( \{X_n\} \) is a sequence of unary predicates on \( G \) defining a strictly increasing sequence of subsets of \( G \) with

\[
\{1\} \cup X_n^{-1} \cup X_n \cdot X_n \subset X_{n+1}
\]

and with the following \( L_{\omega_1 \omega} \)-axiom:

\[
(\forall x)(\bigvee_{n \in \omega} x \in X_n).
\]

By the Löwenheim-Skolem theorem for countable fragments of \( L_{\omega_1 \omega} \) ([19], p.69) any subset \( C \) of such a structure is contained in an elementary submodel of cardinality \( |C| \) (the countable fragment which we consider is the minimal fragment containing our axioms). This proves bountifulness in case (1).

(2) The case groups of cofinality \( \leq \omega \) is similar.

(3) The class of groups which are not Cayley bounded is a class of reducts of all groups expanded by an unary predicate \( \langle G, U \rangle \) with an \( L_{\omega_1 \omega} \)-axiom stating that \( U \) generates \( G \) and with a system of first-order axioms stating that there exists an element of \( G \) which is not a product of \( n \) elements of \( U \cup U^{-1} \cup \{1\} \). The rest is clear.

(4) The class of groups which can be presented as non-trivial free products with amalgamation is the class of reducts of all groups expanded by two unary predicates \( \langle G, U_1, U_2 \rangle \) with first-order axioms that \( U_1 \) and \( U_2 \) are subgroups and with \( L_{\omega_1 \omega} \)-axioms stating that \( U_1 \cup U_2 \) generates \( G \) and a word in the alphabet \( U_1 \cup U_2 \) is equal to 1 if and only if this word follows from the relators of the free product of \( U_1 \) and \( U_2 \) amalgamated over \( U_1 \cap U_2 \). The rest of (4) is clear.

(5) Groups having homomorphisms onto \( \mathbb{Z} \) can be considered as reducts of structures in the language \( \langle \cdot, ... U_{-n}, ..., U_0, ..., U_m, ... \rangle \), where predicates \( U_t \) denote preimages of the corresponding integer numbers.

To see that the class of groups without \( \text{FA} \) is bountiful, take any infinite \( G \) which is not \( \text{FA} \). It is well-known ([26], Section 6.1) that such a group belongs to the union of the classes from statements (2),(4) and (5). Thus \( G \) has an expansion as in one of the cases (2),(4) or (5). Now applying the Löwenheim-Skolem theorem, for any \( C \subset G \) we find a subgroup of \( G \) of cardinality \( |C| \) which contains \( C \) and does not satisfy \( \text{FA} \). □

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