On the equivalence of the rational
Calogero–Moser system to free particles

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Abstract

The canonical transformation and its unitary counterpart which relate the rational Calogero–Moser system to the free one are constructed.

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1 Introduction

The Calogero model and its variations [1] [2] [3] [4] attract recently much attention. They appear to be of some relevance in various areas of theoretical physics like quantum Hall effect [5], fractional statistics [6], two-dimensional gravity [7] and QCD [8], soliton theory [9] and the Seiberg–Witten theory [10].

A variety of advanced theoretical tools has been used to gain a deeper understanding of the structure of Calogero–type models, including the inverse scattering [4] [11] and the \( r \)-matrix methods [12], \( W \)-algebra techniques [13], Dunkl’s operators [14] and others.

The rational Calogero–Moser model is the simplest example of this class of solvable systems. Its scattering properties resemble those of a free particle system [4] [6]. On the classical level, the final configuration of particle momenta coincides with the initial one, up to a permutation, and there is no position shift due to the scattering, while on the quantum level the scattering phase does not depend on energy. When coupled to the external harmonic potential the rational Calogero–Moser model exhibits the same (up to a common shift) energy spectrum and degeneracy as the collection of noninteracting harmonic oscillators [1]. The latter property has been explained recently [15] by explicit construction of relevant similarity transformations relating both systems. For the pure rational Calogero–Moser model the equivalence to the free one was indicated and shown, in an indirect way, both for the classical and quantum case, by Polychronakos [3]. However the explicit form of the relevant similarity transformations cannot be obtained by taking the \( \omega = 0 \) limit of the transformation in [15], as the latter seems to have an essential singularity at this point.

In the present note we construct a canonical transformation which maps the Hamiltonian of the classical Calogero–Moser system to the Hamiltonian of free particles. We then proceed to quantise this construction and thus obtain a unitary transformation relating the quantum Calogero–Moser and free particle systems. The construction is based on the use of the \( sl(2,\mathbb{R}) \) symmetry characteristic for the rational Calogero–Moser models [16]. The presented transformation not only relates both Hamiltonians but it also transforms the set of symmetric dynamical variables (the analogs of sym-
metrised action-angle variables) into their free counterparts. Furthermore it provides a simple proof of the commutation relations conjectured in [3].

2 The classical case

Consider the $N$-particle rational Calogero–Moser model described by the Hamiltonian

$$
H_{CM} = \sum_{i=1}^{N} \frac{p_i^2}{2} + \frac{g}{2} \sum_{i \neq j}^{N} \frac{1}{(q_i - q_j)^2},
$$

(1)

where $p_i, q_i$ are the canonical variables of the phase space $\Gamma_N$, and $g$ is a coupling constant. It is well known [16] [17] that many properties of the Calogero–Moser system can be understood in terms of the representation theory of the Lie algebra $sl(2, \mathbb{R})$.

Define the following three functions on the phase space $\Gamma_N$

$$
T_+ \equiv \frac{1}{\omega} \left( \sum_{i=1}^{N} \frac{p_i^2}{2} + \frac{g}{2} \sum_{i \neq j}^{N} \frac{1}{(q_i - q_j)^2} \right) = \frac{1}{\omega} H_{CM},
$$

(2a)

$$
T_- \equiv \omega \sum_{i=1}^{N} \frac{q_i^2}{2},
$$

(2b)

$$
T_0 \equiv \frac{1}{2} \sum_{i=1}^{N} q_i p_i,
$$

(2c)

where $\omega$ is a fixed, nonzero but otherwise arbitrary, frequency. Using the canonical Poisson brackets on $\Gamma_N$, $\{q_i, p_j\} = \delta_{ij}$, it is easy to see that

$$
\{T_0, T_\pm\} = \pm T_\pm, \quad \{T_+, T_-\} = -2T_0.
$$

(3)

Therefore $T_0, T_\pm$ generate the $sl(2, \mathbb{R})$ (Poisson) algebra. Note that since the Poisson brackets (3) do not depend on the coupling constant $g$, also the functions

$$
\tilde{T}_+ \equiv \frac{1}{\omega} \sum_{i=1}^{N} \frac{p_i^2}{2}, \quad \tilde{T}_- \equiv T_- , \quad \tilde{T}_0 \equiv T_0
$$

(4)

span the $sl(2, \mathbb{R})$ algebra.

Next, consider the following one-parameter family of transformations

$$
q_k \rightarrow e^{i\lambda T_1} * q_k \equiv \sum_{n=0}^{\infty} \frac{(i\lambda)^n}{n!} \{T_1, \ldots \{T_1, q_k\}\ldots\},
$$

$$
p_k \rightarrow e^{i\lambda T_1} * p_k \equiv \sum_{n=0}^{\infty} \frac{(i\lambda)^n}{n!} \{T_1, \ldots \{T_1, p_k\}\ldots\},
$$

(5)
where \( T_1 = \frac{i}{2}(T_+ + T_-) \). These are well-defined, real canonical transformations on \( \Gamma_N/S_q \), where \( S_q \subset \Gamma_N \) consists of points \((q, p)\) such that \( q_i = q_j \) for at least one pair of indices \( i \neq j \). In fact, eqs. (3) simply define the time evolution \((t \equiv \lambda \omega)\) generated by the Hamiltonian of the Calogero–Moser model coupled to the external harmonic force characterized by the frequency \( \omega \)

\[
H_C \equiv -\frac{2i}{\omega}T_1 = \sum_{i=1}^{N} \left( \frac{p_i^2}{2} + \frac{\omega^2 q_i^2}{2} \right) + \frac{g}{2} \sum_{i \neq j = 1}^{N} \frac{1}{(q_i - q_j)^2}.
\]  

(6)

In the following the model corresponding to the Hamiltonian \( H_C \) in (6) will be called the Calogero model (it differs trivially from the original Calogero model [4]).

On the other hand it follows immediately from the Poisson brackets (3) that the canonical transformation (3) rotates the space spanned by \( T_0, T_\pm \) around the axis \( T_1 \) by an angle \( \lambda \). Therefore we obtain

\[
\omega T_+(q, p) \to (e^{i\pi T_1} * T_+)(q, p) = \omega T_-(q, p) = \omega \tilde{T}_-(q, p).
\]  

(7)

Now, one can use the canonical transformation generated by \( \tilde{T}_1 = \frac{i}{2}(\tilde{T}_+ + \tilde{T}_-) \) to rotate \( \tilde{T}_- \) back to \( \tilde{T}_+ \)

\[
\omega \tilde{T}_-(q, p) \to \left( e^{-i\pi \tilde{T}_1} * \tilde{T}_- \right)(q, p) = \omega \tilde{T}_+(q, p) = \sum_{i=1}^{N} \frac{p_i^2}{2}.
\]  

(8)

This is again a well-defined canonical transformation which is a time transformation by \( t = -\frac{\pi}{2\omega} \) generated by the harmonic oscillators Hamiltonian. It can be described explicitly by the following formulae:

\[
q_k \to -\frac{1}{\omega}p_k, \quad p_k \to \omega q_k.
\]  

(9)

Thus we conclude that the canonical transformation

\[
q_k \to e^{-i\pi \tilde{T}_1} * \left( e^{i\pi T_1} * q_k \right), \quad p_k \to e^{-i\pi \tilde{T}_1} * \left( e^{i\pi T_1} * p_k \right)
\]  

(10)

transforms the Calogero–Moser model into the free particle theory. Since this transformation does not depend explicitly on time, both systems are equivalent to each other.
When applied to the Hamiltonian (1), the transformation (10) results effectively in setting \( g = 0 \). We devote the rest of this section to showing that this formal statement remains true for a rather wide class of observables. It is well known [4] [6] that if one restricts oneself to the observables which are symmetric functions on the phase space, the dynamics of the Calogero–Moser system is completely described by the following set of dynamical variables. Begin with the integral of motion in the Hénon form [18] [19]

\[
I_N \equiv e^{-\frac{1}{2} \sum_{i \neq j} \frac{1}{|q_i - q_j|^2} \frac{\partial^2}{\partial p_i \partial p_j}} \prod_{k=1}^{N} p_k. \tag{11}
\]

The full set of mutually commuting independent integrals of motion can be then obtained by taking the successive Poisson brackets of \( I_N \) with \( \sum q_i \) [19], i.e.,

\[
I_{N-n} \equiv \frac{1}{n!} \left\{ \sum_{i=1}^{N} q_i, \ldots, \left\{ \sum_{i=1}^{N} q_i, I_N \right\}, \ldots \right\}, \quad n = 0, 1, \ldots, N - 1. \tag{12}
\]

More generally, one defines [6] [16] [17]

\[
I_{m,n} \equiv \frac{1}{2^m m!} \frac{1}{(N - m - n)!} \left\{ \sum_{i=1}^{N} q_i^2, \ldots, \left\{ \sum_{i=1}^{N} q_i^2, \left\{ \sum_{i=1}^{N} q_i, \ldots, \left\{ \sum_{i=1}^{N} q_i, I_N \right\}, \ldots \right\} \right\} \right\}, \quad 1 \leq m + n \leq N. \tag{13}
\]

The functions \( I_{m,n} \) obey [6]

\[
\dot{I}_{m,n} = m I_{m-1,n+1}. \tag{14}
\]

This implies that all the \( I_{m,n} \) have a polynomial time dependence. Actually, not all functions \( I_{m,n} \) are needed for the description of the dynamics of the system. Modulo particle permutations, the system is fully described by the following \( 2N \) quantities

\[
I_n \equiv I_{0,n}, \quad J_n \equiv I_{1,n-1} = \frac{1}{2} \left\{ \sum_{i=1}^{N} q_i^2, J_n \right\}. \tag{15}
\]

By eqs. (14), the functions \( I_n \) are constants of motion, while the functions \( J_n \) depend linearly on time. The functions \( J_n \) can be used to construct new \( N - 1 \) independent
integrals of motion which do not depend explicitly on time, thus showing the maximal superintegrability of the rational Calogero–Moser systems [20] [21]. Moreover, again up to the particle permutation, they determine the asymptotic form of the particle motion, i.e. they fix the trajectory.

We now show that the canonical transformation (10) applied to the quantities (15) amounts to putting $g = 0$, i.e. the transformed observables read

$$I_n(g = 0) = \sum_{i_1,\ldots,i_n = 1}^{N} \prod_{k=1}^{n} p_{i_k}, \quad J_n(g = 0) = \sum_{j = 1}^{N} q_j \left( \sum_{i_1 < \cdots < i_{n-1} \neq j}^{N} \prod_{k=1}^{n-1} p_{i_k} \right).$$

(16)

It should be, however, stressed that, for the reasons which will be explained below, this conclusion does not apply to $I_{m,n}$ for $m \geq 2$.

The first observation to make is that once it is proven that the transformed $I_N$ has the form stated in (14), i.e., $I_N \rightarrow \prod_{k=1}^{N} p_k$, the relations for all other $I_n$ will follow. This can be seen as follows. From the fact that the centre-of-mass motion is insensitive to the internal forces it is clear that $\sum_{k=1}^{N} q_k$ is invariant under (14). The Poisson bracket is invariant under the canonical transformations, therefore

$$I_{N-n} \rightarrow \frac{1}{n!} \left\{ \sum_{i=1}^{N} q_i, \ldots, \left\{ \sum_{i=1}^{N} q_i, \prod_{k=1}^{N} p_k \right\} \ldots \right\} = I_{N-n}(g = 0),$$

(17)

provided $I_N$ transforms as stated.

In order to find the transformation properties of $I_N$ consider the Calogero model given by the Hamiltonian $H_C$ in (3). It has been shown in [17] that the functions $\phi_k(q,p)$, $k = -\frac{N}{2}, -\frac{N}{2}, \ldots, \frac{N}{2}$ given by

$$\phi_k = \frac{1}{N!} \sum_{n=0}^{N} (N-n)! C_{N-n} \left( \frac{N}{2}, k \right) \left( \omega \sum_{i=1}^{N} q_i \frac{\partial}{\partial p_i} \right)^n I_N,$$

(18)

where the coefficients $C_n \left( \frac{N}{2}, k \right)$ are defined via the relation

$$(x + i)^{N+k} (x - i)^{N-k} = \sum_{n=0}^{N} C_n \left( \frac{N}{2}, k \right) x^n,$$

(19)

have the following simple time behaviour

$$\phi_k(t) = e^{2i\omega t} \phi_k(0).$$

(20)
Since the canonical transformation (3) corresponds to the time shift, the relation (20) implies that
\[
\phi_k \rightarrow e^{i\pi T_1} \ast \phi_k = e^{ik\pi} \phi_k. \tag{21}
\]

The idea of deriving of transformation properties of function \( I_N \) is to express it in terms of \( \phi_k \) using eqs. (18), and then to use the simple form (21) of the transformation of \( \phi_k \).

To implement this idea observe that the identities
\[
(2x)^N = ((x + i) + (x - i))^N = \sum_{k=-N/2}^{N/2} \left( \frac{N}{2} + k \right) (x + i)^{N/2+k}(x - i)^{N/2-k}, \tag{22a}
\]
\[
(2i)^N = ((x + i) - (x - i))^N = \sum_{k=-N/2}^{N/2} \left( \frac{N}{2} + k \right) (x + i)^{N/2+k}(x - i)^{N/2-k}(-1)^{N/2-k}, \tag{22b}
\]
together with (19) imply that
\[
\sum_{k=-N/2}^{N/2} \left( \frac{N}{2} + k \right) C_n \left( \frac{N}{2}, k \right) = 2^N \delta_{n,N}, \tag{23a}
\]
\[
\sum_{k=-N/2}^{N/2} \left( \frac{N}{2} + k \right) e^{ik\pi} C_n \left( \frac{N}{2}, k \right) = (-2)^N \delta_{n,0}. \tag{23b}
\]

The definition of \( \phi_k \) (18) together with eqs. (23a), (23b) lead to
\[
2^{-N} \sum_{k=-N/2}^{N/2} \left( \frac{N}{2} + k \right) \phi_k = I_N \tag{24}
\]
and
\[
2^{-N} \sum_{k=-N/2}^{N/2} \left( \frac{N}{2} + k \right) e^{ik\pi} \phi_k = \left( -1 \right)^N \frac{\omega N!}{N} \left( \sum_{i=1}^{N} q_i \frac{\partial}{\partial p_i} \right)^N I_N = \]
\[
e^{-\frac{\omega}{2} \sum_{i \neq j}^{1} q_i q_j \frac{\partial^2}{\partial q_i \partial q_j}} \left( -\omega \right)^{N} \left( \sum_{i=1}^{N} q_i \frac{\partial}{\partial p_i} \right)^N \prod_{k=1}^{N} p_k = (-\omega)^N \prod_{k=1}^{N} q_k \tag{25}
\]

Since a canonical transformation is a linear transformation, combining (21) with (24) and (25) and then using eqs. (5) we finally obtain
\[
I_N \rightarrow e^{-i\pi T_1} \ast \left( e^{i\pi T_1} \ast I_N \right) = \prod_{k=1}^{N} p_k, \tag{26}
\]
as stated. This completes the proof of the transformation rules (16) for all the $I_n$. On the other hand, due to the relation
\[
\frac{1}{2} \sum_{i=1}^{N} q_i^2 = \frac{1}{\omega} T_-, \tag{27}
\]
we have
\[
\frac{1}{2} \sum_{i=1}^{N} q_i^2 \rightarrow \left( \sum_{i=1}^{N} q_i^2 + \frac{g}{2} \sum_{i,j=1}^{N} \frac{1}{(p_i - p_j)^2} \right) \tag{28}
\]
and
\[
\left\{ \frac{1}{2} \sum_{i=1}^{N} q_i^2, I_n \right\} \rightarrow \left\{ \sum_{i=1}^{N} q_i^2 + \frac{g}{2} \sum_{i,j=1}^{N} \frac{1}{(p_i - p_j)^2}, I_n(g = 0) \right\} = \\
= \left\{ \sum_{i=1}^{N} \frac{q_i^2}{2}, I_n(g = 0) \right\}, \tag{29}
\]
which, together with (15), proves the assertion for $J_n$. Notice that this proof does not apply to $I_{nm}$ with $m \geq 2$. In fact, $\left\{ \sum q_i^2, I_n(g = 0) \right\}$ depends on the $q_i$ and the term $\frac{g}{2} \sum_{i \neq j} \frac{1}{(p_i - p_j)^2}$ cannot be neglected when taking successive Poisson brackets. In order to understand why this is the case consider the motion of the Calogero–Moser particles in the limit $t \rightarrow -\infty$, say. Due to the repulsive character of the internal forces the particles are well-separated and move freely in this limit, i.e., $p_i \simeq p_i^-, q_i \simeq p_i^- t + a_i^-$. Moreover, one can put $g = 0$ when calculating $I_n$ and $J_n$ from the asymptotic data. Therefore we conclude that the canonical transformation (10) transforms the motion of the Calogero–Moser particles into free motion which coincides, up to the particle permutation, with $t \rightarrow -\infty$ asymptotics of the Calogero–Moser trajectory. Note in passing that the property $p_i^- \neq p_j^-, i \neq j$ implies that the image of the transformation (10) is $\Gamma_N/S_p$, where, $S_p$ is defined in the same way as $S_q$ with $q_i$ replaced by $p_i$.

It is now easy to understand why the reasoning concerning the form of transformed $I_n$ and $J_n$ does not work for $I_{nm}$ with $m \geq 2$. The trajectory of the $i$-th Calogero–Moser particle can be written as
\[
q_i(t) = p_i^- t + a_i^- + O\left(\frac{1}{t}\right) \tag{30}
\]
where the term $O(\frac{1}{t})$ depends on $p_i^-, a_i^-$ and $g$ in general. In the product $q_i(t)q_j(t)$ the term linear in $t$, when multiplied by the $O(\frac{1}{t})$ term produces additional nonvanishing $g$-dependent contribution, absent in the free particles case. For $J_n$ this argument is no longer applicable. Although still we have

$$p_i(t) = p_i^- + O\left(\frac{1}{t}\right)$$

(31)

in the coefficient in front of $t$ all the $O(\frac{1}{t})$ contributions cancel because this coefficient is an exact integral of motion.

Another way of looking at this phenomenon is to realise that the (nonlinear) algebraic relations between $I_{nm}$, $m \geq 2$ and the $I_n$ and $J_n$ contain $g$ explicitly.

We conclude this section by computing the canonical transformation (10) and its action on functions $I_{mn}$ in the case of the 2-particle Calogero–Moser model. Separating the centre-of-mass motion

$$q \equiv q_1 - q_2, \quad p \equiv \frac{1}{2}(p_1 - p_2)$$

$$X \equiv \frac{1}{2}(q_1 + q_2), \quad \Pi \equiv p_1 + p_2,$$

(32)

one obtains

$$T_+ = \frac{1}{\omega} \left( \frac{\Pi^2}{4} + \left( p^2 + \frac{g}{q^2} \right) \right), \quad T_0 = \frac{1}{2}X\Pi + \frac{1}{2}qp, \quad T_- = \omega X^2 + \frac{\omega}{4}q^2. \quad (33)$$

Obviously, $X$ and $\Pi$ are invariant under (10). For the relative coordinates we solve first the equations of motion for the Hamiltonian $H_C = p^2 + \frac{q}{q^2} + \frac{\omega^2}{4}q^2$. We obtain

$$q^2(t) = \frac{2E}{\omega^2} \left( 1 + \frac{\omega}{E} q(0)p(0) \sin 2\omega t + \left( \frac{\omega^2q^2(0)}{2E} - 1 \right) \cos 2\omega t \right) \quad (34a)$$

$$q(t)p(t) = \frac{E}{\omega^2} \left( \frac{\omega^2}{E} q(0)p(0) \cos 2\omega t + \left( \frac{\omega^2q^2(0)}{2E} - 1 \right) \sin 2\omega t \right) \quad (34b)$$

$$E = p^2(0) + \frac{g}{q^2(0)} + \frac{\omega^2}{4}q^2(0). \quad (34c)$$

Since $q(0) > 0$, $(q(0) < 0$ resp.) implies $q(t) > 0$, $(q(t) < 0$ resp.), the canonical transformation (10) comes out as

$$q' = \frac{pq \text{sgn}(q)}{\sqrt{p^2 + \frac{g}{q^2}}}, \quad p' = \text{sgn}(q) \sqrt{p^2 + \frac{g}{q^2}}, \quad (35)$$
and is well-defined for \( q \neq 0 \). The inverse transformation reads
\[
q = \text{sgn}(p') \sqrt{q' + \frac{g}{p'^2}}, \quad p = \frac{q' p'^2}{\sqrt{g + q'^2 p'^2}}
\] (36)
and is defined for \( p' \neq 0 \). Both transformations do not depend on \( \omega \).

In order to check that (35) and (36) transform the Calogero–Moser system into a free one we write out the explicit solutions of the Calogero–Moser equations of motion
\[
q(t) = \text{sgn}(q(0)) \sqrt{\frac{g + (2Et + q(0)p(0))^2}{E}}
\]
\[
p(t) = \text{sgn}(q(0)) \frac{(2Et + q(0)p(0))}{\sqrt{g + (2Et + q(0)p(0))^2}}
\]
\[
E \equiv p'^2(0) + \frac{g}{q'^2(0)}
\] (37)
The asymptotic form for \( q(t) \) (\( t \to -\infty \)) reads
\[
q(t) = -\text{sgn}(q(0)) \left( 2\sqrt{Et} + \frac{q(0)p(0)}{\sqrt{E}} + \frac{g}{2\sqrt{E(2Et + q(0)p(0))}} \right) + O\left(\frac{1}{t^2}\right)
\] (38)
i.e.
\[
p^- = -\text{sgn}(q(0)) \sqrt{E}, \quad a^- = -\text{sgn}(q(0)) \frac{q(0)p(0)}{\sqrt{E}}
\] (39)
On the other hand, inserting (37) into the right-hand side of (35), one obtains
\[
q'(t) = \text{sgn}(q(0)) \left( 2\sqrt{Et} + \frac{q(0)p(0)}{\sqrt{E}} \right)
\] (40)
which coincides with the asymptotic form (38), (39), up to the permutation \( 1 \leftrightarrow 2 \).

Finally we calculate \( I_n \), and \( J_n \) explicitly
\[
I_1 = \Pi = p'_1 + p'_2, \quad I_2 = \frac{\Pi^2}{4} - p^2 - \frac{g}{q'^2} = p'_1 p'_2
\]
\[
J_1 = 2X = q'_1 + q'_2, \quad J_2 = X \Pi - qp = q'_1 p'_2 + q'_2 p'_1.
\] (41)
Notice, however, that
\[
I_{20} = q_1 q_2 = X^2 - \frac{q^2}{4} = X'^2 - \frac{q'^2}{4} - \frac{g}{p'^2} = q'_1 q'_2 - \frac{g}{p'^2} \neq I_{20}(g(0)).
\] (42)
In order to explain the appearance of the additional contribution it is sufficient to calculate the asymptotic form of \( q^2(t) \). From eq. (38) one finds that there is an extra
O(1) piece coming from the product of the first and the third terms on the right-hand side of eq. (38). It coincides exactly with what is needed.

To conclude this section we show that, in the general case, the canonical transformation (10), which we denote by $C(\omega)$, does not depend on $\omega$. Notice that $C(\omega) \circ C^{-1}(\omega')$ leaves $I_n(g = 0)$ and $J_n(g = 0)$ invariant. Therefore it is simply equal to a permutation of particles and, depending continuously on $g$, must be an identity.

3 The quantum case

In this section we construct a unitary operator which transforms the $N$-particle quantum Calogero–Moser system into a system of free particles. Let $q_i, p_i$ be the canonical Heisenberg operators, $[q_i, p_i] = i\hbar \delta_{ij}, i,j = 1,2,\ldots,N$. Then the operators $T_0, T_\pm$ ($\tilde{T}_0, \tilde{T}_\pm$ resp.), given by

$$
T_+ = \frac{1}{\omega \hbar} H_{CM}, \quad \tilde{T}_+ = \frac{1}{\omega \hbar} H_{CM}(g = 0)
$$

$$
T_- = \frac{\omega}{2 \hbar} \sum_{i=1}^N q_i^2 = \tilde{T}_-
$$

$$
T_0 = \frac{1}{4 \hbar} \sum_{i=1}^N (q_i p_i + p_i q_i) = \tilde{T}_0,
$$

where $H_{CM}$ is the quantum Calogero-Moser Hamiltonian, generate the $su(1, 1)$ algebra, i.e. $[T_0, T_\pm] = \pm iT_\pm$, $[T_-, T_+] = 2iT_0$. The unitary transformation corresponding to the canonical transformation (10) is provided by the operator

$$
U = e^{-i\pi \tilde{T}_1} e^{i\pi T_1},
$$

where $T_1 = -\frac{1}{2}(T_+ + T_-)$ and $\tilde{T}_1 = -\frac{1}{2}(\tilde{T}_+ + \tilde{T}_-)$. $U$ is a well-defined unitary operator, which describes the time evolution ($t = \frac{\pi}{2\omega}$) generated by the self adjoint Hamiltonian of the Calogero system followed by the time evolution ($t = -\frac{\pi}{2\omega}$) of the set of $N$ harmonic oscillators. Actually, $U$ is simply the time evolution operator for the Calogero model in the interaction picture, calculated for $t = \frac{\pi}{2\omega}$.

Now we prove that $U$ has the same properties as its classical counterpart. First note that:
(i) $I_n$, $J_n$ and $\phi_k$ can be defined by “naive” quantization of the relevant classical formulae; in fact, there is no ordering problem here;

(ii) the transformation properties of $I_n$ and $J_n$ can be derived along the same lines as in Section 2 once the formula (20) is shown to hold also in the quantum case.

To show that eq. (20) is valid in the quantum case too, we use the normal ordering technique of Ref. [22]. For any (analytic) function $f(q, p)$ of canonical variables $q_i, p_i$, we define the operator $:f(q, p):$, which is obtained by replacing the canonical variables in the power series expansion of $f$ by the relevant Heisenberg operators with all the $q_i$ preceding the $p_i$. The product of two such normally ordered operators can be expressed as a normally ordered operator via [22]:

\[
: f(q, p) \cdot : g(q, p) : = : f \left( q, p - i\hbar \frac{\partial}{\partial r} \right) g(r, p) \bigg|_{r=q} : \left( q - i\hbar \frac{\partial}{\partial r}, p \right) f(q, r) \bigg|_{r=p}.
\] (45)

Consider the Heisenberg equations for $: \phi_k(q, p) :$ with $\phi_k(q, p)$ given by (18)

\[
i \hbar \frac{d}{dt} : \phi_k(q, p) : = \left[ : \phi_k(q, p) : , H_C : \right]
\]

\[
= : H_C \left( q - i\hbar \frac{\partial}{\partial r}, p \right) \phi_k(q, r) \bigg|_{r=q} : - \left[ H_C \left( q, p - i\hbar \frac{\partial}{\partial r}, \right) \phi_k(r, p) \bigg|_{r=p}.\right.
\] (46)

The right-hand side of (46) reads

\[
: \left( \sum_i \frac{p_i^2}{2} + \frac{g}{2} \sum_{i \neq j} \frac{1}{(q_i - q_j) - i\hbar \left( \frac{\partial}{\partial q_i} - \frac{\partial}{\partial p_j} \right)} \right)^2 + \frac{\omega^2}{2} \sum_i \left( q_i - i\hbar \frac{\partial}{\partial p_i} \right)^2 \phi_k(q, p) : + \left( \frac{p_i - i\hbar \frac{\partial}{\partial q_i}}{2} \right)^2 + \frac{g}{2} \sum_{i \neq j} \frac{1}{(q_i - q_j)^2} + \frac{\omega^2}{2} \sum_i \phi_k(q, p) : \bigg|_{r=p}.\right.
\] (47)

Expand (47) in the powers of $\hbar$. The first nonvanishing term comes out as

\[
i \hbar : \left\{ \phi_k(q, p), H_C \right\} : ,
\] (48)

so that the time evolution of $: \phi(q, p) :$ is given by the normally-ordered formula (20), provided that all the other terms in the expansion of (47) vanish. To show that this is indeed the case, first note that the expansion terminates on $\hbar^2$ terms. Indeed, for
\( n > 2, \left( \frac{\partial}{\partial p_i} - \frac{\partial}{\partial p_j} \right)^n \) necessarily contains terms of the form \( \frac{\partial^{k+1}}{\partial p_i \partial p_j} \) with \( k \geq 2 \) or \( l \geq 2 \) while \( \phi_k(q, p) \) is multilinear in \( p \). The order \( \hbar^2 \) contribution to (46) reads
\[
\hbar^2 : 3g \sum_{i,j=1, i \neq j}^N \frac{1}{(q_i - q_j)^4} \frac{\partial^2 \phi_k(q, p)}{\partial p_i \partial p_j} + \frac{1}{2} \sum_{i=1}^N \frac{\partial^2 \phi_k(q, p)}{\partial q_i^2}.
\]
Using the explicit form of \( \phi_k(q, p) \), one checks easily that both terms cancel each other.

Finally, we show that \( U \) does not depend on \( \omega \). Arguing in a similar way as in Section 2 we conclude that
\[
U(\omega) U^{-1}(\omega') = e^{i \sigma(\omega, \omega') I}. \tag{49}
\]
Taking the adjoint of eq. (49) one obtains
\[
\sigma(\omega, \omega') = -\sigma(\omega', \omega). \tag{50}
\]
On the other hand the identity
\[
(U(\omega) U^{-1}(\omega'')) (U(\omega'') U^{-1}(\omega')) = e^{i \sigma(\omega, \omega') I} \tag{51}
\]
together with the continuity of \( \sigma \) imply
\[
\sigma(\omega, \omega'') + \sigma(\omega'', \omega') = \sigma(\omega, \omega'). \tag{52}
\]
The general solution to (50) and (52) reads
\[
\sigma(\omega, \omega') = \rho(\omega) - \rho(\omega'). \tag{53}
\]
Apart from \( \omega \), the function \( \rho \) can depend on \( g \) and \( \hbar \) only. However, \( \rho \) is dimensionless, therefore the dimensional analysis yields that \( \rho \) cannot depend on \( \omega \). Therefore \( \sigma(\omega, \omega') \equiv 0 \) and \( U(\omega) \) is \( \omega \)-independent.

We complete the paper by illustrating the above discussion by the reference to the two-particle quantum Calogero–Moser model. Passing to the relative coordinates and ignoring the centre-of-mass motion one easily finds the normalised eigenfunctions of the Calogero Hamiltonian \( H_C = p^2 + \frac{g}{q^2} + \frac{\omega^2}{4} q^2 \), and the corresponding energy levels (for definiteness, we consider the fermionic case)
\[
\phi_{n}^{(a)} = \text{sgn}(q) \sqrt{\frac{n!}{\Gamma(a + n + 1)}} \left( \frac{\omega}{2\hbar} \right)^{\frac{a+1}{2}} |q|^{a+\frac{1}{2}} L_n^a \left( \frac{\omega q^2}{2\hbar} \right) e^{-\frac{\omega q^2}{4}},
\]
\[
E_n = \hbar \omega (2n + a + 1), \tag{54}
\]
where $a \equiv \frac{1}{2} \sqrt{1 + \frac{4a}{\hbar^2}}$, and $L_n^a$ are the Laguerre polynomials. The propagator

$$K^{(a)}(q, q', t) = \sum_{n=0}^{\infty} \phi_n^{(a)}(q) \phi_n^{(a)}(q') e^{-\frac{iE_n}{\hbar}t}$$

(55)
can be calculated using the well-known properties of the Laguerre polynomials [23, pp. 1037–1039]

$$K^{(a)}(q, q', t) = \frac{1}{2} \text{sgn}(qq') \left( \frac{\omega}{2\hbar} e^{\frac{\pi}{2}(a+1)i} \right) |qq'|^{\frac{1}{2}} \exp\left\{ \frac{i\omega}{\pi n} \left( \frac{q^2 + q'^2}{\cot \omega t} \right) \right\} J_a \left( \frac{\omega}{2\hbar \sin \omega t} |qq'| \right),$$

(56)

where $J_a$ is the Bessel function of the first kind. The kernel of the unitary transformation (44) can be rewritten as

$$U(q, q') = \int_{-\infty}^{\infty} dq'' K^{(\frac{1}{2})}(q, q'', t = -\frac{\pi}{2\omega}) K^{(a)}(q'', q', t = \frac{\pi}{2\omega}).$$

(57)

This can be developed further with the help of eq. (56) and the explicit form of $J_{\frac{a}{2}}(x) = \sqrt{\frac{2}{\pi x}} \sin x$

$$U(q, q') = \left( \frac{1 + i}{4\sqrt{\pi}} \right) e^{-\frac{i\pi a}{4}} |qq'|^{\frac{1}{2}} \text{sgn}(qq') \int_{-\infty}^{\infty} d\lambda |\lambda|^\frac{1}{2} \sin(|\lambda q|) J_a(|\lambda q'|).$$

(58)

In order to check the correctness of the above expression compute

$$\int_{-\infty}^{\infty} dq' U^{-1}(q, q') \phi(q') = \int_{-\infty}^{\infty} dq' \overline{U(q', q)} \phi(q'),$$

(59)

where $\phi(q) = \sin(kq)$ is a free fermionic wave function. A simple computation gives

$$\int_{-\infty}^{\infty} dq' U^{-1}(q, q') \phi(q') = \frac{(1 - i)}{2} \sqrt{\pi} e^{\frac{i\pi a}{2}} \text{sgn}(q) |kq|^\frac{1}{2} J_a(k |q|).$$

(60)

The right-hand side is a fermionic solution to the Schrödinger equation for the Calogero–Moser model, corresponding to the energy $E = \hbar^2 k^2$.

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References

[1] F. Calogero, J. Math. Phys. 10 (1969), 2191, 2197
    F. Calogero, J. Math. Phys. 12 (1971), 419
    F. Calogero, G. Marchioro, J. Math. Phys. 15 (1974), 1425

[2] B. Sutherland, J. Math. Phys. 12 (1971), 246, 251
    B. Sutherland, Phys. Rev. A4 (1971), 2019
    B. Sutherland, Phys. Rev. A5 (1972)

[3] J. Moser, Adv. Math. 16 (1975), 1

[4] M.A. Olshanetsky and A.M. Perelomov, Phys. Rep. 71 (1981), 313
    M.A. Olshanetsky and A.M. Perelomov, Phys. Rep. 94 (1983), 313

[5] N. Kawakami, Phys. Rev. Lett. 71 (1993), 275

[6] A.P. Polychronakos, Nucl. Phys. B324 (1989), 597

[7] J. Andric, A. Jevicki and M. Levine, Nucl. Phys. 215 (1983), 307
    A. Jevicki, Nucl. Phys. 376 (1992), 75

[8] J.A. Minahan and A.P. Polychronakos, Phys. Lett. 312 (1993), 155
    J.A. Minahan and A.P. Polychronakos, Phys. Lett. 336 (1994), 288

[9] H. Airault, H. McKean and J. Moser, Comm. Pure Appl. Math. 30 (1977), 95
    I.M. Krichever, Funct. Anal. i Pril. 12 (1978), 76

[10] E. D’Hoker and D.H. Phong, Nucl. Phys. B513 (1998) 405

[11] H. Ujino, K. Hikami and M. Wadati, J. Phys. Soc. Japan 61 (1992), 3425

[12] O. Babelon and C-M. Viallet, Phys. Lett. B237 (1990), 411
    J. Avan and M. Talon, Phys. Lett. B303 (1993), 33

[13] K. Hikami and M. Wadati, J. Phys. Soc. Japan 62 (1993), 4203
    K. Hikami and M. Wadati, Phys. Rev. Lett. 73 (1994), 1191
    H. Ujino and M. Wadati, J. Phys. Soc. Japan 63 (1994), 3585
[14] L. Brink, T.H. Hanson and M.A. Vasiliev, Phys. Lett. B286 (1992), 109

[15] N. Gurappa and P.K. Panigrahi, Equivalence of the Calogero–Sutherland Model to Free Harmonic Oscillators, cond-mat/9710035

[16] G. Barucchi and T. Regge, J. Math. Phys. 18 (1977), 1149
    S. Wojciechowski, Phys. Lett. A64 (1977), 273
    P.J. Gambardella, J. Math. Phys. 16 (1975), 1172

[17] C. Gonera and P. Kosiński, Calogero model and sl(2, R) algebra, to be published

[18] K. Sawada and T. Kotera, J. Phys. Soc. Japan 39 (1975), 1614

[19] S. Wojciechowski, Lett. N. Cim. 18 (1977), 101

[20] S. Wojciechowski, Phys. Lett. A95 (1983), 279

[21] C. Gonera, Phys. Lett. A237 (1998), 365

[22] C. Gonera, J. Math. Phys. 39 (1998), 4759

[23] I.S. Gradshteyn and I.M. Ryzhik, Table of integrals, series and products, Academic Press, New York and London, 1965