McKean–Vlasov SDEs under Measure Dependent Lyapunov Conditions

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Abstract

We prove the existence of weak solutions to McKean–Vlasov SDEs defined on a domain $D \subseteq \mathbb{R}^d$ with continuous and unbounded coefficients under a Lyapunov-type condition. We do not require non-degeneracy of the diffusion coefficient. We work with a class of Lyapunov functions that depend on measures and we propose a new type of integrated Lyapunov condition. The main tool used in the proofs is the concept of a measure derivative due to Lions. An important consequence of having appropriate Lyapunov condition is that we can show existence of solutions to the McKean–Vlasov SDEs on $[0, \infty)$. This leads to a probabilistic proof of existence a stationary solution to the nonlinear Fokker–Planck–Kolmogorov equation. Finally we prove uniqueness under an integrated condition based on a Lyapunov function. This extends the standard monotone-type condition for uniqueness.

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1 Introduction

We will consider either the time interval $I = [0, T]$ for some fixed $T > 0$ or $I = [0, \infty)$. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, $(\mathcal{F}_t)_{t \in I}$ a filtration such that $\mathcal{F}_0$ contains all sets of $\mathcal{F}$ that have probability zero and such the filtration is right-continuous. Let $w = (w_t)_{t \in I}$ be an $\mathbb{R}^d$-valued Wiener process which is an $(\mathcal{F}_t)_{t \in I}$-martingale. We consider the McKean–Vlasov stochastic differential equation (SDE)

$$x_t = x_0 + \int_0^t b(s, x_s, \mathcal{L}(x_s)) \, ds + \int_0^t \sigma(s, x_s, \mathcal{L}(x_s)) \, dw_s, \quad t \in I.$$  \hfill (1.1)

Here we use the notation $\mathcal{L}(x)$ to denote the law of the random variable $x$. The law of such an SDE satisfies a nonlinear Fokker–Planck–Kolmogorov equation (see also [4] and more generally [3]): writing $\mu_t := \mathcal{L}(x_t)$ and $a := \frac{1}{2} \sigma \sigma^*$ we have, for $t \in I$,

$$\langle \mu_t, \varphi \rangle = \langle \mu_0, \varphi \rangle + \int_0^t \langle \mu_s, b(s, \cdot, \mu_s) \partial_x \varphi + \text{tr} (a(s, \cdot, \mu_s) \partial_x^2 \varphi) \rangle \, ds \quad \forall \varphi \in C^2_0(D).$$  \hfill (1.2)

The aim of this article is to study the existence and uniqueness of solutions to the equation (1.1). We will show that a weak solution to (1.1) exists for unbounded and continuous coefficients, provided that we can find an appropriate measure-dependent Lyapunov function which ensures integrability of the equation. This generalises the results of [15] and [17].

The work on SDEs with coefficients that depend on the law was initiated by [24], who was inspired of Kac’s programme in Kinetic Theory [19]. An excellent and thorough account of the general theory of McKean–SDEs and their particle approximations can be found in [31]. Sznitman showed that if the coefficients of (1.1) are globally Lipschitz continuous, a fixed point argument on Wasserstein space can be carried out, and consequently a solution to (1.1) is obtained as the limit of classical SDEs. This means that existence and uniqueness results from classical SDEs allows one to establish existence and uniqueness of (1.1). If Lipschitz continuity does not hold, the fixed point argument typically fails. However, in the setting of SDEs with non-degenerate diffusion coefficient, the regularisation effect of the noise allowed Zvonkin and Krylov [35] and later Veretennikov [33], in a general multidimensional case, to show that the fixed point argument works, assuming only that the drift coefficient is Hölder continuous. This result has been recently generalised to McKean–Vlasov SDEs in [12]. The key step of the proof is to establish smoothness property of the corresponding PDE on $D \times \mathcal{P}(D)$. To go beyond
Hölder continuity one typically, uses a compactness argument to establish the existence of a solution to stochastic differential equations. In the context of McKean–Vlasov SDEs, this has been done by Funaki who was interested in probabilistic representation for Boltzmann equations [15]. Funaki formulated a non-linear martingale problem for McKean–Vlasov SDEs that allowed him to established existence of a solution to (1.1) by studying a limiting law of Euler discretisation. His proof of existence holds for continuous coefficients satisfying a Lyapunov type condition in the state variable $x \in \mathbb{R}^d$ with polynomial Lyapunov functions. Whilst we also assume continuity of the coefficients, we allow for a much more general Lyapunov condition that depends on a measure. Furthermore, Funaki is using Lyapunov functions to establish integrability of the Euler scheme which is problematic if one wants to depart from polynomial functions, [32]. Recently [26] assuming only linear growth condition in space and boundedness in measure argument and non-degeneracy of diffusion obtained existence results for (1.1). This novel result was achieved through use of Krylov’s estimates in [21, Ch. 2, Sec. 2]) in the context of McKean–Vlasov SDEs.

An alternative approach to establishing existence of solutions to McKean–Vlasov equations is to approximate the equation with a particle system (a system of classical SDEs that interact with each other through empirical measure) and show that the limiting law solves Martingale problem. In this approach, one works with laws of empirical laws i.e. on the space $P(P(D))$ and proves its convergence to a (weak) solution of (1.1) by studying the corresponding non-linear martingale problem. We refer to [25] for a general overview of that approach and to [7, 14] and references within for recent results exploring this approach. A general approach to establish the existence of martingale solutions has also been presented in [22]. Here, inspired by [26], we tackle the problem using the Skorokhod representation theorem and convergence lemma [29].

For classical SDEs (equations with no dependence on the law), the lack of sufficient regularity of the coefficients, say Lipschitz continuity, proves to be the main challenge in establishing existence and uniqueness of solutions. Lack of boundedness of the coefficients, typically, does not lead to significant difficulty, provided these are at least locally bounded. In that case one can work with local solutions and the only concern is the possible explosion. The conditions that ensure that the solution does not explode can be formulated by using Lyapunov function techniques as has been pioneered in [20]. The key observation is that if one considers two SDEs with coefficients that agree on some bounded open domain then the solutions if unique also agree until first time the solution leaves the domain, see, for example [30, Ch. 10].

This classical localisation procedure does not carry over, at least directly, from the setting of classical SDEs to McKean–Vlasov SDEs. Indeed, if we stop a classical SDE then until the stopping time the stopped process satisfies the same equation. If we take (1.1) and consider the stopped process $y_t := x_{t \wedge \tau}$, with some stopping time $\tau$, then the equation this satisfies is

$$y_t = y_0 + \int_0^{t \wedge \tau} b(s, y_s, \mathcal{L}(x_s)) \, ds + \int_0^{t \wedge \tau} \sigma(s, y_s, \mathcal{L}(x_s)) \, dw_s, \quad t \in I.$$  

Clearly, even for $t \leq \tau$ this is not the same equation since $\mathcal{L}(x_s) \neq \mathcal{L}(y_s)$. Furthermore, this is not a McKean–Vlasov SDE. This could be problematic if one would like to obtain a solution to McKean–Vlasov SDEs through a limiting procedure of stopped processes. Furthermore, let $D_k \subseteq D_{k+1}$ be a
sequence of nested domains, and consider functions \( \bar{b} \) and \( \bar{\sigma} \) such that \( \bar{b} = b \) and \( \bar{\sigma} = \sigma \) on \( D_k \). The equation
\[
\bar{x}_t = \bar{x}_0 + \int_0^t \bar{b}(s, \bar{x}_s, \mathcal{L}(\bar{x}_s)) \, ds + \int_0^t \bar{\sigma}(s, \bar{x}_s, \mathcal{L}(\bar{x}_s)) \, dw_s, \quad t \in I,
\]
is a McKean–Vlasov SDE, but \( x_t \neq \bar{x}_t \) even for \( t \leq \bar{\tau}^k \), where \( \bar{\tau}^k = \inf \{ t \geq 0 : \bar{x}_t \notin D_k \} \). This implies that if one considers a sequence of SDEs with coefficients that agree on these subdomains, one no longer has monotonicity for the corresponding stopping times. We show that despite these difficulties it still possible to establish the existence of weak solutions to the McKean–Vlasov SDEs (1.1) using the idea of localisation, but extra care is needed.

1.1 Main Contributions

Our first main contribution is the generalisation of Lyapunov function techniques to the setting of McKean–Vlasov SDEs. The coefficients of the equation (1.1), depend on \( (x, \mu) \in D \times \mathcal{P}(D) \) for \( D \subseteq \mathbb{R}^d \). Hence the class of Lyapunov functions considered in this paper also depend on \( (x, \mu) \in D \times \mathcal{P}(D) \). See (2.1). Furthermore, it is natural to formulate the integrated Lyapunov condition, in which the key stability assumption is required to hold only on \( \mathcal{P}(D) \), see (2.2) and Section 1.2 for motivating examples. Note that it is not immediately clear how one can obtain tightness estimates for the particle approximation under the integrated conditions we propose. To work with Lyapunov functions on \( \mathcal{P}(D) \), we take advantage of the recently developed analysis on Wasserstein spaces, and in particular derivatives with respect to a measure as introduced by Lions in his lectures at College de France, see [9] and [10, Ch. 5]. This analysis is presented in the appendix to give the measure derivative in a domain.

Our second main contribution is the probabilistic proof of the existence of a stationary solution to
the nonlinear Fokker–Planck–Kolmogorov (1.2). Furthermore the calculus on the Wasserstein spaces allows to study a type Fokker–Planck–Kolmogorov on \( \mathcal{P}_2(D) \). Indeed, for \( \phi \in C^{(1,1)}(\mathcal{P}_2(D)) \) and \( t \in I \),
\[
\phi(\mathcal{L}(x_t^n)) = \phi(\mathcal{L}(x_0^n)) + \int_0^t \left\langle \mathcal{L}(x_s^n), b(s, \cdot, \mathcal{L}(x_s^n)) \partial_\mu \phi(\mathcal{L}(x_s^n)) + \text{tr} [a(s, \cdot, \mathcal{L}(x_s^n))\partial_y \phi(\mathcal{L}(x_s^n))] \right\rangle ds.
\]
(1.3)
Following the remark by Lions from his lectures at College de France, the equation (1.3) can be interpreted as non-local transport equation on the space of measures. Reader may consult [10, Ch. 5 Sec. 7.4] for further details. Another angle is to notice that while (1.2) gives an equation for linear functionals of the measure, equation (1.3) is an equation for nonlinear functionals of the measure. The existence results obtained in this paper imply existence of a stationary solution to (1.3) in the case where \( b \) and \( \sigma \) do not depend on time.

Finally, we formulate uniqueness results under the Lyapunov type condition and integrated Lyapunov type condition that is required to hold only on \( \mathcal{P}(D) \). This extends the standard monotone type conditions studied in literature e.g [5, 23, 16]. Interestingly, in some special cases we are able to obtain uniqueness only under local monotone conditions. Again we do not require a non-degeneracy
condition on diffusion coefficient. We support our results with the example inspired by Scheutzow \[28\] who has showed that, in general, uniqueness of solution to McKean–Vlasov SDEs does not hold if the coefficients are only locally Lipschitz. Again, we would like to highlight that since classical localisation techniques used in the SDEs seem not to work in our setting, we cannot simply obtain global uniqueness results from local uniqueness and suitable estimates on the stopping times.

1.2 Motivating Examples

Let us now present some example equations to motivate the choice of the Lyapunov condition. Consider first the McKean–Vlasov stochastic differential equation

$$
\begin{align*}
    dx_t &= -x_t \left[ \int_{\mathbb{R}} y^4 \mathcal{L}(x_t)(dy) \right] dt + \frac{1}{\sqrt{2}} x_t dw_t, \quad x_0 \in L^4(\mathcal{F}_0, \mathbb{R}^+) .
    \end{align*}
$$

(1.4)

The diffusion generator for (1.4) is

$$
\begin{align*}
    L(x, \mu)v(x) := \frac{1}{4} x^2 v''(x) - x \left[ \int_{\mathbb{R}} y^4 \mu(dy) \right] v'(x) .
    \end{align*}
$$

(1.5)

It is not clear whether one can find a Lyapunov function such that the classical Lyapunov condition holds i.e. $L(x, \mu)v(x) \leq m_1 v(x)m_2$, for $m_1 < 0$ and $m_2 \in \mathbb{R}$. However, with the Lyapunov function given by $v(x) = x^4$ we can establish that

$$
\begin{align*}
    \int_{\mathbb{R}} L(x, \mu)v(x)\mu(dx) \leq - \int_{\mathbb{R}} v(x)\mu(dx) + 4
    \end{align*}
$$

(1.6)

holds. See Example 2.14 for details. We will see that this is sufficient to establish integrability of (1.4) on $I = [0, \infty).$ See Theorem 2.9 and the condition (2.5).

Another way to proceed, is to directly work with $v(\mu) := \int_{\mathbb{R}} x^4 \mu(dx)$ as Lyapunov function on the measure space $\mathcal{P}_4(\mathbb{R}).$ This requires the use of derivatives with respect to a measure as introduced by Lions in his lectures at College de France, see [9] or Appendix A1. Then

$$
\begin{align*}
    \partial_\mu v(\mu)(y) = 4y^3, \quad \partial_y \partial_\mu v(\mu)(y) = 12y^2, \quad y \in \mathbb{R} .
    \end{align*}
$$

The generator corresponding to the appropriate Itô formula, see e.g. Proposition A.6, is

$$
\begin{align*}
    L^\mu v(\mu) \\
    := \int_{\mathbb{R}} \left( -x \int_{\mathbb{R}} y^4 \mu(dy) \partial_\mu v(\mu)(x) + \frac{1}{4} x^4 \partial_y \partial_\mu v(\mu)(x) \right) \mu(dx) = \int_{\mathbb{R}} \left( -4x^4 \int_{\mathbb{R}} y^4 \mu(dy) + 3x^4 \right) \mu(dx) .
    \end{align*}
$$

We note that this is the same expression as found when $v(x) = x^4$ in (1.5) and we integrate over $\mu$ (and so (1.6) again holds). In this case using the Itô formula for measure derivatives brings no advantages.

1Derivatives with respect to a measure are defined in $\mathcal{P}_2(\mathbb{R})$, and therefore one cannot apply Itô formula to $v(\mu) := \int_{\mathbb{R}} x^4 \mu(dx).$ However, in this paper we will only apply the Itô formula for measures supported on compact subsets of $\mathbb{R}^d$. 5
However the advantage of working with a Lyapunov function on the measure space appears where the dependence on the measure in the Lyapunov function is not linear.

Consider the following McKean–Vlasov stochastic differential equation

\[ dx_t = - \left( \int_{\mathbb{R}} (x_t - \alpha y) \mathcal{L}'(x_t)(dy) \right)^3 dt + \left( \int_{\mathbb{R}} (x_t - \alpha y) \mathcal{L}'(x_t)(dy) \right)^2 \sigma dw_t, \tag{1.7} \]

for \( t \in I, \alpha \) and \( \sigma \) constants and with \( x_0 \in L^4(\mathcal{F}_0, \mathbb{R}) \). Assume that \( m := - (6\sigma^2 - 4 + 4\alpha) > 0 \). Since the drift and diffusion are non-linear functions of the law and state of the process, it is natural to seek a Lyapunov function \( v \in C^2, (1,1) (\mathbb{R} \times \mathcal{P}(\mathbb{R})) \). See Definition A.7. The generator corresponding to the appropriate Itô formula, see e.g. Proposition A.8, is then given by (2.1) and we will show that for the Lyapunov function

\[ v(x, \mu) = \left( \int_{\mathbb{R}} (x - \alpha y) \mu(dy) \right)^4, \]

we have

\[ \int_{\mathbb{R}} (L^d v)(x, \mu)(dx) \mu(dx) \leq m - m \int_{\mathbb{R}} v(x, \mu) \mu(dx). \]

See Example 2.15 for details. Thus the condition (2.5) holds. This is sufficient to establish existence of solutions to (1.4) on \( I = [0, \infty) \) as Theorem 2.9 will tell us.

Regarding our continuity assumptions for existence of solutions to (1.1) we note that we only require a type of joint continuity of the coefficients in \((x, \mu) \in \mathbb{R}^d \times \mathcal{P}(\mathbb{R}^d)\) and that this allows us to consider coefficients where the dependence on the measure does not arise via an integral with respect to the said measure. This could be for example

\[ S_\alpha(\mu) := \frac{1}{\alpha} \int_0^\alpha \inf \{|x| : \mu((-\infty, x]) \geq s|}\] 

for \( \alpha > 0 \) fixed. This quantity is known as the “expected shortfall” and is a type of risk measure. See Example 2.16 for details.

## 2 Existence results

For a domain \( D \subseteq \mathbb{R}^d \), we will use the notation \( \mathcal{P}(D) \) for the space of probability measures over \((D, \mathcal{B}(D))\). We will consider this as a topological space with the topology induced by the weak convergence of probability measures. We will write \( \mu_n \rightharpoonup \mu \) if \( (\mu_n)_n \) converges to \( \mu \) in the sense of weak convergence of probability measures. For \( p \geq 1 \) we use \( \mathcal{P}_p(D) \) to denote the set of probability measures that are \( p \)-integrable (i.e. \( \int_D |x|^p \mu(dx) < \infty \) for \( \mu \in \mathcal{P}_p(D) \)). We will consider this as a metric space with the metric given by the Wasserstein distance with exponent \( p \), see (2.8). Denote by \( C_b(D) \) and \( C_0(D) \) the subspaces of continuous functions that are bounded and compactly supported, respectively.

We use \( \sigma^* \) to denote the transpose of a matrix \( \sigma \) and for a square matrix \( a \) we use \( \text{tr}(a) \) to denote its trace. We use \( \partial_v v \) to denote the (column) vector of first order partial derivatives of \( v \) with respect
to the components of $x$ (i.e. the gradient of $v$ with respect to $x$) and $\partial^2_x v$ to denote the square matrix of all the mixed second order partial derivatives with respect to the components of $x$ (i.e. the Hessian matrix of $v$ with respect to $x$). If $a, b \in \mathbb{R}^d$ then $ab$ denotes their dot product.

Recall that we are using the concept of derivatives with respect to a measure as introduced by Lions in his lectures at Collège de France, see [9]. For convenience, the construction and main definitions are in Appendix A. In particular, see Definition A.7 to clarify what is meant by the space $C^{1,2,(1,1)}(I \times D \times \mathcal{P}(D))$. In short, saying that a function $v$ is in such space means that all the derivatives appearing in (2.1) exist and are appropriately jointly continuous so that we may apply the Itô formula for a function of a process and a flow of measures, see Proposition A.8. The use of such an Itô formula naturally leads to the following form of a diffusion generator. First we note that throughout this paper we assume that for a domain $D \subseteq \mathbb{R}^d$ there is a nested sequence of bounded sub-domains, i.e. bounded, open connected subsets of $\mathbb{R}^d$, $(D_k)_k$ such that $D_k \subset D_{k+1}, \overline{D_k} \subset D$ and $\bigcup_k D_k = D$. For $(t,x) \in I \times D$, $\mu \in \mathcal{P}(D_k)$ for some $k \in \mathbb{N}$ and for some $v \in C^{1,2,(1,1)}(I \times D \times \mathcal{P}_2(D))$ we define $L^\mu = L^\mu(t,x,\mu)$ as

\[
(L^\mu v)(t,x,\mu) := (\partial_t v + \frac{1}{2} \text{tr}(\sigma \sigma^* \partial^2_x v) + b \partial_x v)(t,x,\mu) + \int_{\mathbb{R}^d} \left( b(t,y,\mu) (\partial_t v)(t,x,\mu)(y) + \frac{1}{2} \text{tr}( (\sigma \sigma^*)(t,y,\mu)(\partial_t \partial_y v)(t,x,\mu)(y) ) \right) \mu(dy).
\]

We note that in the case $v \in C^{1,2}(I \times D)$, i.e when $v$ does not depend on the measure, the above generator reduces to

\[
(L^\mu v)(t,x) = (Lv)(t,x) := \left( \partial_t v + \frac{1}{2} \text{tr}(\sigma \sigma^* \partial^2_x v) + b \partial_x v \right)(t,x).
\]

### 2.1 Assumptions and Main Result

We assume that $b : I \times D \times \mathcal{P}(D) \to \mathbb{R}^d$ and $\sigma : I \times D \times \mathcal{P}(D) \to \mathbb{R}^d \times \mathbb{R}^d$ are measurable (later we will add joint continuity and local boundedness assumptions).

We require the existence of a Lyapunov function satisfying either of the following conditions.

**Assumption 2.1** (Lyapunov condition). There is $v \in C^{1,2,(1,1)}(I \times D \times \mathcal{P}_2(D))$, $v \geq 0$, such that

i) There are locally integrable, non-random, functions $m_1 = m_1(t)$ and $m_2 = m_2(t)$ on $I$ such that: for all $t \in I$, all $x \in D$ and all $\mu \in \mathcal{P}(D_k)$, $k \in \mathbb{N}$, we have,

\[
L^\mu(t,x,\mu)v(t,x,\mu) \leq m_1(t)v(t,x,\mu) + m_2(t).
\]

ii) There is $V = V(t,x)$ satisfying for all $\mu \in \mathcal{P}(D_k)$, $k \in \mathbb{N}$, we have,

\[
\int_D V(t,x) \mu(dx) \leq \int_D v(t,x,\mu) \mu(dx) \quad \forall t \in I,
\]

and

\[
V_k := \inf_{s \in I, x \in \partial D_k} V(s,x) \to \infty \text{ as } k \to \infty.
\]
iii) The initial value $x_0$ is $\mathcal{F}_0$-measurable, $\mathbb{P}(x_0 \in D) = 1$ and $\mathbb{E}v(0, x_0, \mathcal{L}(x_0)) < \infty$.

**Assumption 2.2** (Integrated Lyapunov condition). There is $v \in C^{1,2,(1,1)}(I \times D \times \mathcal{P}_2(D))$, $v \geq 0$, such that:

i) There are locally integrable, non-random, functions $m_1 = m_1(t)$ and $m_2 = m_2(t)$ on $I$ such that for all $t \in I$ and for all $\mu \in \mathcal{P}(D_k)$, $k \in \mathbb{N}$, we have,

$$
\int_D L^\mu(t, x, \mu) v(t, x, \mu) \mu(dx) \leq m_1(t) \int_D v(t, x, \mu) \mu(dx) + m_2(t). \tag{2.5}
$$

ii) There is $V = V(t, x)$ satisfying (2.3) and

$$
V_k := \inf_{s \in I, x \in D_k} V(s, x) \to \infty \text{ as } k \to \infty. \tag{2.6}
$$

iii) The initial value $x_0$ is $\mathcal{F}_0$-measurable, $\mathbb{P}(x_0 \in D) = 1$ and $\mathbb{E}v(0, x_0, \mathcal{L}(x_0)) < \infty$.

We make the following observations.

**Remark 2.3.**

i) We have deliberately not specified the signs of the functions $m_1$ and $m_2$.

ii) If (2.2) holds then for $L^{\mu,k} := 1_{x \in D_k} L^\mu$ we have that (2.2) also holds with $L^\mu$ replaced by $L^{\mu,k}$. Indeed, since $v$ is non-negative, then $L^\mu v(t, x, \mu) 1_{x \in D_k} \leq [m_1(t)v(t, x, \mu) + m_2(t)] 1_{x \in D_k}$. On the other hand if only (2.5) holds then, in general, this does not imply that (2.5) holds with $L^\mu$ replaced by $L^{\mu,k}$, unless $\mu \in \mathcal{P}(D_k)$.

Regarding the continuity of coefficients in (1.1) and their local boundedness we require the following.

**Assumption 2.4** (Continuity). Functions $b : I \times D \times \mathcal{P}(D) \to \mathbb{R}^d$ and $\sigma : I \times D \times \mathcal{P}(D) \to \mathbb{R}^d \times \mathbb{R}^d$ are jointly continuous in the last two arguments in the following sense: if $(\mu_n) \in \mathcal{P}(D)$ are such that

$$
\sup_{t \in I} \int_D v(t, x, \mu_n) \mu_n(dx) < \infty
$$

and if $(x_n \to x, \mu_n \Rightarrow \mu)$ as $n \to \infty$ then $b(t, x_n, \mu_n) \to b(t, x, \mu)$ and $\sigma(t, x_n, \mu_n) \to \sigma(t, x, \mu)$ as $n \to \infty$.

**Assumption 2.5** (Local boundedness). There exist constants $c_k \geq 0$ such that for any $\mu \in \mathcal{P}(D)$

$$
\sup_{x \in D_k} |b(t, x, \mu)| \leq c_k \left( 1 + \int_D v(t, y, \mu) \mu(dy) \right),
$$

$$
\sup_{x \in D_k} |\sigma(t, x, \mu)| \leq c_k \left( 1 + \int_D v(t, y, \mu) \mu(dy) \right).
$$
Assumption 2.6 (Integrated growth condition). There exists a constant $c \geq 0$ such that for all $\mu \in \mathcal{P}(D_k)$, $k \in \mathbb{N}$, we have,

$$\int_{D_k} |b(t, x, \mu)| + |\sigma(t, x, \mu)|^p \mu(dx) \leq c \left( 1 + \int_{D_k} v(t, x, \mu) \mu(dx) \right), \quad \forall t \in I.$$ 

Continuity Assumption 2.4 in the measure argument is very weak, but might be hard to verify. In Remark 2.8 in time.

Definition 2.7 (Coefficients, therefore a condition on integrability of the law is natural. As a consequence of the continuity of the coefficients, we have that $\sigma(x, \mu_n) \to \sigma(x, \mu)$ as $n \to \infty$. This will be satisfied in particular if

$$|b(x_n, \mu_n) - b(x, \mu)| + |\sigma(x_n, \mu_n) - \sigma(x, \mu)| \leq \rho(|x - x_n|) + W_{p'}(\mu_n, \mu),$$

for some function $\rho = \rho(x)$ such that $\rho(|x|) \to 0$ as $x \to 0$. We note that this is a common assumption, see e.g. [15]. At this point it may be worth noting that the $p$-Wasserstein distance on $\mathcal{P}(D)$ is

$$W_p(\mu, \nu) := \left( \inf_{\pi \in \Pi(\mu, \nu)} \int_{D \times D} |x - y|^p \pi(dx, dy) \right)^{1/p},$$

where $\Pi(\mu, \nu)$ denotes the set of couplings between $\mu$ and $\nu$ i.e. all measures on $\mathcal{B}(D \times D)$ such that $\pi(B_1, B_2) = \mu(B_1)$ and $\pi(B_2, B_3) = \nu(B_3)$ for every $B \in \mathcal{B}(D)$.

Note that in the case of McKean-Vlasov SDEs it is often useful to think of the solution as a pair consisting of the process $x$ and its law i.e. $(x_t, \mathcal{L}(x_t))_{t \in I}$. The coefficients of the McKean-Vlasov SDE depend on the law of the solution and the main focus of this paper is on equations with unbounded coefficients, therefore a condition on integrability of the law is natural.

Definition 2.7 ($\nu$-integrable weak solution). A $\nu$-integrable weak solution to (1.1), on $I$ in $D$ is

$$(\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_t)_{t \in I}, (\mathcal{L}(x_t))_{t \in I}),$$

where $(\Omega, \mathcal{F}, \mathbb{P})$ is a probability space, $(\mathcal{F}_t)_{t \in I}$ is a filtration, $(w_t)_{t \in I}$ is a Wiener process that is a martingale w.r.t. the above filtration, $(x_t)_{t \in I}$ is an adapted process satisfying (1.1) such that $x \in C(I; D)$ a.s. and finally, for all $t \in I$ we have $\mathbb{E}v(t, x_t, \mathcal{L}(x_t)) < \infty$.

Before we state the main theorem of this paper, we state the conditions on $m_1, m_2$ that allow one to establish the integrability and tightness estimate, which in the case $I = [0, \infty)$ needs to be uniform in time.

Remark 2.8 (On finiteness of $M(t)$). Define $\gamma(t) := \exp \left( - \int_0^t m_1(s) \, ds \right)$ and

$$M(t) := \frac{\mathbb{E}v(0, \tilde{x}_0, \mathcal{L}(\tilde{x}_0))}{\gamma(t)} + \int_0^t \frac{\gamma(s)}{\gamma(t)} m_2(s) \, ds,$$

$$M^+(t) := e_{\tilde{x}_0}(m_1(s))^+ \, ds \left( \mathbb{E}v(0, \tilde{x}_0, \mathcal{L}(\tilde{x}_0)) + \int_0^t \gamma(s) m_2^+(s) \, ds \right).$$

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Note that $M(t) \leq M^+(t)$.

i) If $I = [0, T]$, $m_1$ and $m_2$ are set to 0 outside $I$, leading to

$$
\sup_{t<\infty} \int_0^t \frac{\gamma(s)}{\gamma(t)} m_2(s) ds \leq \int_0^T e^{\int_s^T m_1(r) dr} |m_2(s)| ds < \infty.
$$

ii) If $I = [0, \infty)$ and we have

$$
m_1(t) \leq 0 \ \forall t \geq 0 \text{ and } \int_0^\infty \gamma(s)|m_2(s)| ds < \infty,
$$

then

$$
\sup_{t<\infty} \int_0^t e^{\int_s^t m_1(r) dr} m_2(s) ds \leq \int_0^\infty \gamma(s)|m_2(s)| ds < \infty.
$$

In both of these cases we have $\sup_{t \in I} M(t)$ and $\sup_{t \in I} M^+(t) < \infty$.

**Theorem 2.9.** Let $D \subseteq \mathbb{R}^d$ and assumptions 2.4 and 2.5 hold. Then we have the following.

i) If Assumption 2.1 holds and $\sup_{t \in I} M^+(t) < \infty$, then there exists a $v$-integrable weak solution to (1.1) on $I$.

ii) If Assumptions 2.2 and 2.6 hold and $\sup_{t \in I} M(t) < \infty$, then there exists a $v$-integrable weak solution to (1.1) on $I$.

Additionally,

$$
\sup_{t \in I} \mathbb{E}v(t, x_t, \mathcal{L}(x_t)) < \infty.
$$

We make the following comment. By virtue of Assumption 2.5 we have that under conditions of Theorem 2.9, the $v$-integrable weak solution to (1.1) obtained by the theorem satisfies the forward nonlinear Fokker–Planck–Kolmogorov equation (1.2), where $\mu_t = \mathcal{L}(x_t)$.

### 2.2 Proof of the existence results

We will use the convention that the infimum of an empty set is positive infinity. We extend $b$ and $\sigma$ in a measurable but discontinuous way to functions on $\mathbb{R}^+ \times \mathbb{R}^d \times \mathcal{P}(\mathbb{R}^d)$ by taking

$$
b(t, x, \mu) = \sigma(t, x, \mu) = 0 \text{ if } x \in \mathbb{R}^d \setminus D \text{ or if } t \notin I.
$$

For $t \notin I$ we set $m_1(t) = m_2(t) = 0$. We define

$$
b^k(t, x, \mu) := 1_{x \in D_k} b(t, x, \mu) \text{ and } \sigma^k(t, x, \mu) := 1_{x \in D_k} \sigma(t, x, \mu).
$$
\textbf{Lemma 2.10.} Let Assumption 2.5 hold. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, $(\mathcal{F}_t)_{t \in I}$ a filtration, a Wiener process $\tilde{w}$ and a process $\tilde{x}^k$ that satisfies, for all $t \in I$, 

$$d\tilde{x}^k_t = b^k(t, \tilde{x}^k_t, \mathcal{L}(\tilde{x}^k_t)) dt + \sigma^k(t, \tilde{x}^k_t, \mathcal{L}(\tilde{x}^k_t)) d\tilde{w}_t, \quad \tilde{x}^k_0 = \tilde{x}_0.$$ \hspace{1cm} (2.11)

For $m \leq k$, let $\tilde{\tau}^k_m := \inf\{t \in I : \tilde{x}^k_t \notin D_m\}$.

i) If either Assumption 2.1 or 2.2 hold then for any $t \in I$, 

$$\sup_k \mathbb{E} v(t, \tilde{x}^k_t, \mathcal{L}(\tilde{x}^k_t)) \leq M(t).$$

ii) If either Assumption 2.1 or 2.2 hold then for any $t \in I$, 

$$\mathbb{P}(\tilde{x}^k_t < t) \leq \mathbb{P}(\tilde{x}_0 \notin D_k) + M(t)V^{-1}_k.$$ 

iii) If Assumption 2.1 holds then for any $t \in I$, 

$$\sup_k \mathbb{P}(\tilde{x}^k_t < t) \leq \mathbb{P}(\tilde{x}_0 \notin D_k) + M^+(t)V^{-1}_m.$$ 

iv) If Assumption 2.2 holds then for any $t \in I$, 

$$\sup_k \mathbb{P}(\tilde{x}^k_t \notin D_m) \leq M(t)V^{-1}_m.$$ 

\textit{Proof.} For each $k$, $\mathcal{L}(\tilde{x}^k_t) \in \mathcal{P}_2(D)$ and therefore we can apply Itô's formula from Proposition A.8 to $\tilde{x}^k$, its law and $\gamma$. Thus 

$$\gamma(t)v(t, \tilde{x}^k_t, \mathcal{L}(\tilde{x}^k_t)) = \gamma(0)v(0, \tilde{x}_0, \mathcal{L}(\tilde{x}_0))$$

$$+ \int_0^t \gamma(s)[L^v v - m^v v](s, \tilde{x}^k_s, \mathcal{L}(\tilde{x}^k_s)) ds + \int_0^t \gamma(s)[(\partial_x v)\sigma](s, \tilde{x}^k_s, \mathcal{L}(\tilde{x}^k_s)) d\tilde{w}_s.$$ 

Due to the local boundedness of the coefficients and either Lyapunov condition (2.2) or (2.5) we get 

$$\mathbb{E} \gamma(t)v(t, \tilde{x}^k_t, \mathcal{L}(\tilde{x}^k_t)) \leq \mathbb{E} \gamma(0)v(0, \tilde{x}_0, \mathcal{L}(\tilde{x}_0)) + \int_0^t \gamma(s)m^v_2(s)ds.$$ \hspace{1cm} (2.12)

This proves the first part of the lemma.

For the second part we proceed as follows. Since $\tilde{x}^k_t = \tilde{x}^k_{t \wedge \tilde{\tau}^k_k}$ for all $t \in I$, which implies $\mathcal{L}(\tilde{x}^k_t) = \mathcal{L}(\tilde{x}^k_{t \wedge \tilde{\tau}^k_k})$ for all $t \in I$, we further observe 

$$\mathbb{E} v(t, \tilde{x}^k_t, \mathcal{L}(\tilde{x}^k_t)) = \mathbb{E} v(t, \tilde{x}^k_{t \wedge \tilde{\tau}^k_k}, \mathcal{L}(\tilde{x}^k_{t \wedge \tilde{\tau}^k_k})) \geq \mathbb{E} \left[V \left(t, \tilde{x}^k_{t \wedge \tilde{\tau}^k_k}, I_{0 < \tilde{\tau}^k_k < t} \right) \right] = \mathbb{E} \left[V \left(t, \tilde{x}^k_{t \wedge \tilde{\tau}^k_k}, I_{0 < \tilde{\tau}^k_k < t} \right) \right] \geq V_k \mathbb{P}(0 < \tilde{\tau}^k_k < t).$$
Hence,
\[ \mathbb{P}(\tilde{\tau}_k^k < t) = \mathbb{P}(\tilde{\tau}_k^k < t, \tilde{\tau}_k^k > 0) + \mathbb{P}(\tilde{\tau}_k^k < t, \tilde{\tau}_k^k = 0) \leq \mathbb{P}(0 < \tilde{\tau}_k^k < t) + \mathbb{P}(x_0 \notin D_k) \]
\[ \leq \frac{E_v \left( t, \tilde{x}_t^{\tau_k^k}, \mathcal{L}(\tilde{x}_t^{\tau_k^k}) \right)}{V_k} + \mathbb{P}(\tilde{x}_0 \notin D_k). \]

This completes the proof of the second statement.

To prove the third statement we first note that for \( m > k \) we have \( \mathbb{P}(\tau_m^k < t) = \mathbb{P}(x_0 \notin D_m) \). Thus we may assume that \( m \leq k \). We proceed similarly as above but with the crucial difference that \( \tilde{x}_t^k \) is not equal to \( \tilde{x}_t^{\tau_m^k} \) anymore. Our aim is to apply Itô formula to the function \( v \), the process \((\tilde{x}_t^{\tau_m^k})_{t \in I}\) and the flow of marginal measures \((\mathcal{L}(\tilde{x}_t^{k}))_{t \in I}\). Note that \( \mathcal{L}(\tilde{x}_t^{\tau_m^k}) \neq \mathcal{L}(\tilde{x}_t^{k}) \). Nevertheless the Itô formula A.8 may be applied. After taking expectations this yields

\[ \mathbb{E} \left[ \gamma(t \wedge \tau_m^k) v(t \wedge \tau_m^k, \tilde{x}_t^{\tau_m^k}, \mathcal{L}(\tilde{x}_t^{k})) \right] = \mathbb{E} v(0, \tilde{x}_0, \mathcal{L}(\tilde{x}_0)) + \mathbb{E} \int_0^{t \wedge \tau_m^k} \gamma(s) [L^m v - m_1 v](s, \tilde{x}_s^k, \mathcal{L}(\tilde{x}_s^k)) \, ds. \]

We now use (2.2) to see that

\[ \mathbb{E} \left[ \gamma(t \wedge \tau_m^k) v(t \wedge \tau_m^k, \tilde{x}_t^{\tau_m^k}, \mathcal{L}(\tilde{x}_t^{k})) \right] \leq \mathbb{E} v(0, \tilde{x}_0, \mathcal{L}(\tilde{x}_0)) + \mathbb{E} \int_0^{t \wedge \tau_m^k} \gamma(s)m_2(s) \, ds \]
\[ \leq \mathbb{E} v(0, \tilde{x}_0, \mathcal{L}(\tilde{x}_0)) + \int_0^t \gamma(s)m_2(s) \, ds =: M(t). \]

Then

\[ \inf_{s \leq t} \gamma(s) \mathbb{E} v(t \wedge \tau_m^k, \tilde{x}_t^{\tau_m^k}, \mathcal{L}(\tilde{x}_t^{k})) \leq \mathbb{E} \gamma(t \wedge \tau_m^k) v(t \wedge \tau_m^k, \tilde{x}_t^{\tau_m^k}, \mathcal{L}(\tilde{x}_t^{k})) \leq M(t) \]

and so, following same argument as in the proof of the first part of this lemma,

\[ \mathbb{P}(\tau_m^k < t) \leq \frac{1}{\inf_{s \leq t} \gamma(s) \mathcal{V}_m(\mathcal{L}(\tilde{x}_t^{k}))} + \mathbb{P}(\tilde{x}_0 \notin D_m). \]

We conclude by observing that

\[ \inf_{s \leq t} \gamma(s) \geq e^{-\int_0^t (m_1(s))^+ \, ds}. \]

To prove the fourth statement, first note that for \( m > k \), \( \mathbb{P}(\tilde{x}_t^k \notin D_m) = 0 \) and hence we take \( m \leq k \). Conditions (2.3) and (2.6) imply that

\[ \mathbb{E} v(t, \tilde{x}_t^k, \mathcal{L}(\tilde{x}_t^k)) \geq \int_D V(t, x) \mathcal{L}(\tilde{x}_t^k)(dx) \geq \int_{D \cap D_m} V(t, x) \mathcal{L}(\tilde{x}_t^k)(dx) \geq V_m \mathbb{P}(x_t^k \notin D_m). \]
Remark 2.11. Since we are assuming that \( P(x_0 \in D) = 1 \) we have

\[
\lim_{k \to \infty} P(x_0 \notin D_k) = 1 - \lim_{k \to \infty} P(x_0 \in D_k) = 1 - P \left( \bigcup_k \{x_0 \in D_k\} \right) = 0.
\]

Corollary 2.12. Let Assumption 2.5 hold. Let \((\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})\) be a probability space, \((\tilde{F}_t)_{t \in I}\) a filtration, a Wiener process \(\tilde{w}\) and a process \(\tilde{x}^k\) such that (2.11) holds for all \(t \in I\). Assume that \(\tilde{x}^k \to \tilde{x}\) in \(C(I; \bar{D})\). If either Assumption 2.1 or 2.2 hold then

\[
\sup_{t \in I} \mathbb{E}v(t, \tilde{x}_t, \mathcal{L}(\tilde{x}_t)) \leq \sup_{t \in I} M(t),
\]

where \(M\) is given in (2.9).

Proof. By Fatou’s lemma, continuity of \(v\) and (2.9) we get

\[
\mathbb{E}v(t, \tilde{x}_t, \mathcal{L}(\tilde{x}_t)) \leq \liminf_{k \to \infty} \mathbb{E}v(t, \tilde{x}^k_t, \mathcal{L}(\tilde{x}^k_t)) \leq \sup_{t \in I} M(t).
\]

The results follows if we take supremum over \(t\) and consider Remark 2.8. \(\square\)

Our aim is to use Skorokhod’s arguments to prove the existence of a weak (also known as martingale) solution to the equation (1.1). Before we proceed to the proof of the main Theorem 2.9 we need to establish tightness of the law of the process (2.11).

Lemma 2.13 (Tightness). Let \(\tilde{x}^k\) be the process defined in (2.11).

i) Let Assumptions 2.1 and 2.5 hold and \(\sup_{t \in I} M^+(t) < \infty\), then the law of \((\tilde{x}^k)_k\) is tight on \(C(I; \bar{D})\).

ii) Let Assumptions 2.2, 2.5 and 2.6 hold and \(\sup_{t \in I} M(t) < \infty\), then the law of \((\tilde{x}^k)_k\) is tight on \(C(I; \bar{D})\). Additionally for any \(\epsilon > 0\), there is \(m_\epsilon\) such that for \(m \geq m_\epsilon\)

\[
\sup_k \mathbb{P}(\tau_m^k \in I) \leq \epsilon.
\]

Proof. i) Under the Assumption 2.1 tightness of the law of \((\tilde{x}^k)_k\) on \(C(I; \bar{D})\) follows from the first statement in Lemma 2.10, together with Remarks 2.8 and 2.11. Indeed given \(\epsilon > 0\) we can find \(m_0\) such that for any \(m > m_0\)

\[
\mathbb{P}(\tilde{x}^k_m < \infty) \leq \mathbb{P}(\tilde{x}_0 \notin D_m) + \sup_{t \in I} M(t) V_m^{-1} \leq \epsilon/2 + \epsilon/2,
\]

due to, in particular, our assumption that \(V_m \to \infty\) as \(m \to \infty\).

ii) First we observe that for every \(\ell\) and \((t_1, \ldots, t_\ell)\) in \(I\), the joint distribution of \((\tilde{x}^k_{t_1}, \ldots, \tilde{x}^k_{t_\ell})\) is tight. Indeed, statement iii) in Lemma 2.10 guarantees tightness of the law of \(\tilde{x}^k_t\) for any \(t \in I\). Given \(\epsilon > 0\), for any \(\ell \in \mathbb{N}\) we can find \(m_0\) such that for any \(m > m_0\)

\[
\mathbb{P}(\tilde{x}^k_{t_1} \notin D_m, \ldots, \tilde{x}^k_{t_\ell} \notin D_m) \leq \ell \sup_{t \in I} M(t) V_m^{-1} \leq \epsilon,
\]
due to, the assumption that $V_m \to \infty$ as $m \to \infty$. We will use Skorokhod’s Theorem (see [29, Ch. 1 Sec. 6]). This will allow us to conclude tightness of the law of $(\tilde{x}_k)_k$ on $C(I; D)$ as long as we can show that for any $\varepsilon > 0$
\[
\lim_{h \to 0} \sup_k \sup_{|s_1 - s_2| \leq h} \mathbb{P}(|\tilde{x}_{s_1}^k - \tilde{x}_{s_2}^k| > \varepsilon) = 0.
\]
From (2.11), using the Assumption 2.6, we get, for $0 < s_1 - s_2 < 1$,
\[
\mathbb{E}|\tilde{x}_{s_1}^k - \tilde{x}_{s_2}^k| \leq \int_{s_2}^{s_1} \mathbb{E}|b(r, \tilde{x}_r^k, \mathcal{L}(\tilde{x}_r^k))| \, dr + \left(\mathbb{E} \int_{s_2}^{s_1} |\sigma^k(r, \tilde{x}_r^k, \mathcal{L}(\tilde{x}_r^k))|^2 \, dr\right)^{1/2}
\leq c \int_{s_2}^{s_1} \left(1 + \sup_{t \in I} \mathbb{E}r(r, \tilde{x}_r^k, \mathcal{L}(\tilde{x}_r^k))\right) \, dr + \left(c \int_{s_2}^{s_1} (1 + \sup_{t \in I} \mathbb{E}r(r, \tilde{x}_r^k, \mathcal{L}(\tilde{x}_r^k))) \, dr\right)^{1/2}
\leq c \left(1 + \sup_{t \in I} \mathbb{E}r(t)\right) (s_2 - s_1)^{1/2}.
\]
Markov’s inequality leads to
\[
\sup_k \sup_{|s_1 - s_2| \leq h} \mathbb{P} \left(|\tilde{x}_{s_1}^k - \tilde{x}_{s_2}^k| > \varepsilon\right) \leq c\varepsilon (s_2 - s_1)^{1/2}
\]
which concludes the proof of tightness.

We will now prove the second statement in ii). Note that $C(I, D)$ is open and in $C(I, \bar{D})$. Note also that $C(I, D_k) \subset C(I, D_{k+1})$ and $\bigcup_k C(I, D_k) = C(I, D)$. We know that for any $\varepsilon > 0$ there is a compact set $K_\varepsilon \subset C(I, D)$ such that
\[
\sup_k \mathbb{P}(\tilde{x}_k \notin K_\varepsilon) \leq \varepsilon.
\]
Since $K_\varepsilon \subset C(I, D) = \bigcup_k C(I, D_k)$ is compact and $C(I, D_k)$ are nested and open there must be some $k^*$ such that $K_\varepsilon \subset C(I, D_{k^*})$. But this means that
\[
\mathbb{P}(\tilde{x}_k \notin C(I, D_{k^*})) \leq \mathbb{P}(\tilde{x}_k \notin K_\varepsilon)
\]
and so $\mathbb{P}(\tau^k_m \in I) = \mathbb{P}(\tilde{x}_k \notin C(I, D_{m})) \leq \mathbb{P}(\tilde{x}_k \notin C(I, D_{k^*})) \leq \mathbb{P}(\tilde{x}_k \notin K_\varepsilon) \leq \varepsilon$ for all $m \geq k^*$. \hspace{1cm} \square

**Proof of Theorem 2.9.** Recall that we have extended $b$ and $\sigma$ so that they are now defined on $[0, \infty) \times \mathbb{R}^d \times \mathcal{P}(\mathbb{R}^d)$. Hence we will from now on work with $I = [0, \infty)$. Let us define $t_i^n := \frac{i}{n}$, $i = 0, 1, \ldots$ and $\kappa_n(t) = t_i^n$ for $t \in [t_i^n, t_{i+1}^n)$. Fix $k$. We introduce Euler approximations $x^{k,n}$, $n \in \mathbb{N}$,
\[
x^{k,n}_t = x_0 + \int_0^t b^k \left(s, x^{k,n}_{\kappa_n(s)}, \mathcal{L}(x^{k,n}_{\kappa_n(s)})\right) \, ds + \int_0^t \sigma^k \left(s, x^{k,n}_{\kappa_n(s)}, \mathcal{L}(x^{k,n}_{\kappa_n(s)})\right) \, dw_s.
\]

Let us outline the proof: As a first step we fix $k$ and we show tightness with respect to $n$ and Skorokhod’s theorem to take $n \to \infty$. The second step is then to use Lemma 2.10 and remark 2.11 to show tightness with respect to $k$. Finally we can use Skorokhod’s theorem again to show that (for a subsequence) the limit as $k \to \infty$ satisfies (1.1) (on a new probability space).
First Step. Using standard arguments, we can verify that, for a fixed $k$, the sequence $(x^{k,n})_n$ is tight (in the sense that the laws induced on $C([0, \infty), D)$ are tight). By Prohorov’s theorem (see e.g. [2, Ch. 1, Sec. 5]), there is a subsequence (which we do not distinguish in notation) such that $\mathcal{L}(x^{k,n}) \Rightarrow \mathcal{L}(x^k)$ as $n \to \infty$ (convergence in law).

Hence we may apply Skorokhod’s Representation Theorem (see e.g. [2, Ch. 1, Sec. 6]) and obtain a new probability space $(\hat{\Omega}^k, \hat{\mathcal{F}}^k, \hat{\mathbb{P}}^k)$ where on this space there are new random variables $(\hat{x}^n_0, \hat{x}^{k,n}, \hat{\tilde{w}}^n)$ and $(\hat{x}_0, \hat{x}^k, \hat{\tilde{w}})$ such that

$$\mathcal{L}(\hat{x}^n_0, \hat{x}^{k,n}, \hat{\tilde{w}}^n) = \mathcal{L}(x^n_0, x^{k,n}, w^n) \quad \forall n \in \mathbb{N} \quad \text{and} \quad \mathcal{L}(\hat{x}_0, \hat{x}^k, \hat{\tilde{w}}) = \mathcal{L}(x_0, x^k, w).$$

After taking another subsequence to obtain almost sure convergence from convergence in probability,

$$(\hat{x}^n_0, \hat{x}^{k,n}, \hat{\tilde{w}}^n) \to (\hat{x}_0, \hat{x}^k, \hat{\tilde{w}}) \quad \text{as} \quad n \to \infty \quad \text{in} \quad C([0, \infty), D \times \hat{D} \times \mathbb{R}^d) \quad \text{a.s.}$$

We let

$$\hat{\mathcal{F}}^k := \sigma \{ \hat{x}_0 \} \lor \sigma \{ \hat{x}_s, \hat{\tilde{w}}_s : s \leq t \}.$$ 

and define $\hat{\mathcal{F}}^{k,n}$ analogously. Then $\hat{\tilde{w}}^n$ and $\tilde{w}$ are respectively $(\hat{\mathcal{F}}^k)_{t \geq 0}$ and $(\hat{\mathcal{F}}^k)_{t \geq 0}$-Wiener processes. Define

$$\hat{\tau}^{k,n} := \inf \{ t \geq 0 : \hat{x}^{k,n}_t \notin D_k \} \quad \text{and} \quad \hat{\tau}^k := \inf \{ t \geq 0 : \hat{x}^k_t \notin D_k \}.$$ 

These are respectively $\hat{\mathcal{F}}^{k,n}$ and $\hat{\mathcal{F}}^k$ stopping times. Moreover, due to the a.s. convergence of the trajectories $\hat{x}^{k,n}$ to $\hat{x}^k$ we can see that

$$\lim_{n \to \infty} \hat{x}^{k,n} \geq \hat{x}^k.$$

From the fact that the laws of the sequences are identical we see that we still have the Euler approximation on the new probability space: for $t \geq 0$

$$d\hat{x}^{k,n}_t = b^k \left( t, \hat{x}^{k,n}_{\hat{\tau}^{k,n}(t)}, \mathcal{L}(\hat{x}^{k,n}_{\hat{\tau}^{k,n}(t)}) \right) dt + \sigma^k \left( t, \hat{x}^{k,n}_{\hat{\tau}^{k,n}(t)}, \mathcal{L}(\hat{x}^{k,n}_{\hat{\tau}^{k,n}(t)}) \right) d\hat{\tilde{w}}^n_t.$$

Moreover for all $t \leq \hat{\tau}^{k,n}$ the process $\hat{x}^{k,n}$ satisfies the same equation as above but without the cutting applied to the coefficients:

$$d\hat{x}^{k,n}_t = b \left( t, \hat{x}^{k,n}_{\hat{\tau}^{k,n}(t)}, \mathcal{L}(\hat{x}^{k,n}_{\hat{\tau}^{k,n}(t)}) \right) dt + \sigma \left( t, \hat{x}^{k,n}_{\hat{\tau}^{k,n}(t)}, \mathcal{L}(\hat{x}^{k,n}_{\hat{\tau}^{k,n}(t)}) \right) d\hat{\tilde{w}}^n_t.$$

Using Skorohod’s Lemma, see [29, Ch. 2, Sec. 3], together with the continuity conditions in Assumption 2.4, we can take $n \to \infty$ and conclude that for all $t \leq \hat{\tau}^k$ we have

$$d\hat{x}^k_t = b(t, \hat{x}^k_t, \mathcal{L}(\hat{x}^k_t)) dt + \sigma(t, \hat{x}^k_t, \mathcal{L}(\hat{x}^k_t)) d\hat{\tilde{w}}_t. \quad (2.13)$$

At this point we remark that the process $\hat{x}^k$ is well defined and continuous on $[0, \infty)$ but we only know that it satisfies (2.13) until $\hat{\tau}^k$.

Second Step. Tightness of the law of $(\hat{x}^k)_k$ in $C(I; D)$ follows from Lemma 2.13 and Remark 2.8. From Prohorov’s theorem we thus get that for a subsequence $\mathcal{L}(\hat{x}^k) \Rightarrow \mathcal{L}(\tilde{x})$ as $k \to \infty$ (convergence
in law). From Skorokhod’s Representation Theorem we then obtain a new probability space \((\Omega, \mathcal{F}, \mathbb{P})\) carrying new random variables \((\bar{x}_0^k, \bar{x}^k, \bar{w}^k)\) and \((\bar{x}_0, \bar{x}, \bar{w})\) such that

\[
\mathcal{L}(\bar{x}_0, \bar{x}, \bar{w}) = \mathcal{L}(\bar{x}_0^k, \bar{x}^k, \bar{w}^k),
\]

\[
\mathcal{L}(\bar{x}_0^k, \bar{x}^k, \bar{w}^k) = \mathcal{L}(\bar{x}_0^k, \bar{x}^k, \bar{w}^k) \quad \forall k \in \mathbb{N},
\]

and (after taking a further subsequence to go from convergence in probability to almost sure convergence)

\[
(\bar{x}_0^k, \bar{x}^k, \bar{w}^k) \to (\bar{x}_0, \bar{x}, \bar{w}) \quad \text{as} \quad k \to \infty \quad \text{in} \quad C(I; D \times \bar{D} \times \mathbb{R}^d) \quad \text{a.s.}
\]

Let \(\tau^k_m := \inf\{t : \bar{x}_t^k \notin D_m\}\), \(\tau^k_m := \inf\{t : \bar{x}_t^k \notin D_m\}\) and \(\tau^\infty_m := \inf\{t : \bar{x}_t^k \notin D_m\}\). Since \(\sup_{t \leq \infty} |\bar{x}_t^k - \bar{x}_t| \to 0\) we get \(\tau^k_m \to \tau^\infty_m\) as \(k \to \infty\). Then from Fatou’s Lemma, Remark 2.11 and either part iii) of Lemma 2.10 or part ii) of Lemma 2.13 we have that,

\[
\mathbb{P}(\tau^\infty_m \in I) \leq \liminf_{k \to \infty} \mathbb{P}(\bar{x}_m^k \in I) \leq \sup_{k} \mathbb{P}(\bar{x}_m^k \in I) \to 0 \quad \text{as} \quad m \to \infty.
\] (2.14)

Then the distribution of \(\tau^\infty_m\) converges in distribution, as \(m \to \infty\), to a random variable \(\tilde{\tau}\) with distribution \(\mathbb{P}(\tilde{\tau} \leq T) = 0\) and \(\mathbb{P}(\tilde{\tau} = \infty) = 1\). In general convergence in distribution does not imply convergence in probability. But in the special case that the limiting distribution corresponds to a random variable taking a single value a.s. we obtain convergence in probability (see e.g. [13, Ch. 11, Sec. 1]). Hence \(\tau^\infty_m \to \tilde{\tau}\) in probability as \(m \to \infty\). From this we can conclude that there is a subsequence that converges almost surely.

Since (2.13) holds for \(\bar{x}^k\) we have the corresponding equation for \(\bar{x}^k\) i.e. for \(t \leq \tau^k_m\),

\[
d\bar{x}^k_t = b(t, \bar{x}^k_t, \mathcal{L}(\bar{x}^k_t)) dt + \sigma(t, \bar{x}^k_t, \mathcal{L}(\bar{x}^k_t)) d\bar{w}^k_t.
\] (2.15)

Fix \(m < k'\). We will consider \(k > k'\). Then (2.15) holds for all \(t \leq \inf_{k \geq k'} \tau^k_m\). We can now consider \(x^k_t\) (these all stay inside \(D_m\) for all \(k > k' > m\)) and use dominated convergence theorem for the bounded variation integral and Skorokhod’s lemma on convergence of stochastic integrals, see [29, Ch. 2, Sec. 3], and our assumptions on continuity of \(b\) and \(\sigma\) to let \(k \to \infty\). We thus obtain, for \(t \leq \inf_{k \geq k'} \tau^k_m\),

\[
d\bar{x}_t = b(t, \bar{x}_t, \mathcal{L}(\bar{x}_t)) dt + \sigma(t, \bar{x}_t, \mathcal{L}(\bar{x}_t)) d\bar{w}_t.
\] (2.16)

Now, for each fixed \(m < k'\),

\[
\lim_{k' \to \infty} \inf_{k \geq k'} \tau^k_m = \lim_{k \to \infty} \tau^k_m = \tau^\infty_m.
\]

Finally we take \(m \to \infty\) and since \(\tau^\infty_m \to \tilde{\tau}\) we can conclude that (2.16) holds for all \(t \in I\). The last statement of the theorem follows from Corollary 2.12.
2.3 Examples

Example 2.14 (Integrated Lyapunov condition). Consider the McKean–Vlasov stochastic differential equation (1.4) i.e.

\[ dx_t = -x_t \left[ \int_{\mathbb{R}} y^4 \mathcal{L}(x_t)(dy) \right] dt + \frac{1}{\sqrt{2}} x_t \, dw_t, \quad x_0 = \xi > 0. \]

Then for \( v(x) = x^4 \) we have,

\[ L(x, \mu)v(x) = 3x^4 - 4x^4 \int_{\mathbb{R}} y^4 \mu(dy). \]

We see that the stronger Lyapunov condition (2.2) will not hold with \( m_1 < 0 \) (at least for chosen \( v \), which seems to be a natural choice). However, integrating leads to

\[ \int_{\mathbb{R}} L(x, \mu)v(x)\mu(dx) = 3 \int_{\mathbb{R}} x^4 \mu(dx) - 4 \left( \int_{\mathbb{R}} x^4 \mu(dx) \right)^2 \]

using this we will show that the integrated Lyapunov condition (2.5) holds i.e. that

\[ \int_{\mathbb{R}} L(x, \mu)v(x)\mu(dx) \leq -\int_{\mathbb{R}} v(x)\mu(dx) + 4 \]

is satisfied. To see this we note that \(-x^2 \leq -x + 1\). Moreover, Assumption 2.6 is satisfied. Condition (1.6) allows us to obtain uniform in time integrability properties for \((x_t)\) needed to study e.g. ergodic properties.

Example 2.15 (Non-linear dependence of measure and integrated Lyapunov condition). Consider the McKean–Vlasov stochastic differential equation (1.7) i.e.

\[ dx_t = -\left( \int_{\mathbb{R}} (x_t - \alpha y)\mathcal{L}(x_t)(dy) \right)^3 dt + \left( \int_{\mathbb{R}} (x_t - \alpha y)\mathcal{L}(x_t)(dy) \right)^2 \sigma \, dw_t, \]

for \( t \in I \) and with \( x_0 \in L^4(F_0, \mathbb{R}) \). Assume that \( m := -(6\sigma^2 - 4 + 4\alpha) > 0 \). The diffusion generator given by (2.1) is

\[
(L^\mu v)(x, \mu) = \left( \frac{\sigma^2}{2} \left( \int_{\mathbb{R}} (x - \alpha y)\mu(dy) \right)^4 \partial_x^2 v - \left( \int_{\mathbb{R}} (x - \alpha y)\mu(dy) \right)^3 \partial_x v \right)(x, \mu)
\]

\[ + \int_{\mathbb{R}} \left( \frac{\sigma^2}{2} \left( \int_{\mathbb{R}} (z - \alpha y)\mu(dy) \right)^4 (\partial_z \partial_x v)(t, x, \mu)(z) - \left( \int_{\mathbb{R}} (z - \alpha y)\mu(dy) \right)^3 (\partial_x v)(t, x, \mu)(z) \right) \mu(dz) \]

We will show that for the Lyapunov function

\[ v(x, \mu) = \left( \int_{\mathbb{R}} (x - \alpha y)\mu(dy) \right)^4, \]

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we have

\[ \int_{\mathbb{R}} (L^\mu v)(x, \mu) \mu(dx) \leq m - m \int_{\mathbb{R}} v(x, \mu) \mu(dx). \]

Indeed,

\[ \partial_x v(x, \mu) = 4 \left( \int_{\mathbb{R}} (x - \alpha y) \mu(dy) \right)^3, \quad \partial_x^2 v(x, \mu) = 12 \left( \int_{\mathbb{R}} (x - \alpha y) \mu(dy) \right)^2, \]

\[ \partial_\mu v(x, \mu)(z) = -4\alpha \left( \int_{\mathbb{R}} (x - \alpha y) \mu(dy) \right)^3, \quad \partial_z \partial_\mu v(x, \mu)(z) = 0. \]

Hence

\[ (L^\mu v)(x, \mu) = (6\sigma^2 - 4) \left( \int_{\mathbb{R}} (x - \alpha y) \mu(dy) \right)^6 + 4\alpha \int_{\mathbb{R}} \left[ \left( \int_{\mathbb{R}} (x - \alpha y) \mu(dy) \right)^3 \left( \int_{\mathbb{R}} (x - \alpha y) \mu(dy) \right)^3 \right] \mu(dx). \]

Since we want an estimate over the integral of the diffusion generator we observe that

\[ I := \int_{\mathbb{R}} \int_{\mathbb{R}} \left[ \left( \int_{\mathbb{R}} (z - \alpha y) \mu(dy) \right)^3 \left( \int_{\mathbb{R}} (x - \alpha y) \mu(dy) \right)^3 \right] \mu(dz) \mu(dx) \]

\[ = \int_{\mathbb{R}} \int_{\mathbb{R}} \left( \int_{\mathbb{R}} (z - \alpha y) \mu(dy) \right)^3 \mu(dz) \left( \int_{\mathbb{R}} (x - \alpha y) \mu(dy) \right)^3 \mu(dx) \]

\[ = \int_{\mathbb{R}} \left( \int_{\mathbb{R}} (z - \alpha y) \mu(dy) \right)^3 \mu(dz) \int_{\mathbb{R}} \left( \int_{\mathbb{R}} (x - \alpha y) \mu(dy) \right)^3 \mu(dx) \]

\[ \leq \left( \int_{\mathbb{R}} \left( \int_{\mathbb{R}} (z - \alpha y) \mu(dy) \right)^3 \mu(dz) \right)^2. \]

By Cauchy-Schwarz's inequality we obtain

\[ I \leq \int_{\mathbb{R}} \left( \int_{\mathbb{R}} (x - \alpha y) \mu(dy) \right)^6 \mu(dx). \]

Hence, recalling \( m := -(6\sigma^2 - 4 + 4\alpha) > 0 \) and using the inequality \(-x^6 \leq 1 - x^4\), we obtain that

\[ \int_{\mathbb{R}} (L^\mu v)(x, \mu) \mu(dx) \leq \int_{\mathbb{R}} (6\sigma^2 - 4 + 4\alpha) \left( \int_{\mathbb{R}} (x - \alpha y) \mu(dy) \right)^6 \mu(dx) \leq m - m \int_{\mathbb{R}} v(x, \mu) \mu(dx). \]

Moreover, Assumption 2.6 is readily satisfied.

**Example 2.16** (Dependence on measure without an integral). Let \( \mu \) be a law on \((\mathbb{R}, \mathcal{B}(\mathbb{R}))\) and let \( F_{\mu}^{-1} : [0, 1] \rightarrow \mathbb{R} \) be the generalized inverse cumulative distribution function for this law. Recall that the \( \alpha \)-Quantile is given by

\[ F_{\mu}^{-1}(\alpha) := \inf \{ x \in \mathbb{R} : \mu((-\infty, x]) \geq \alpha \}. \]
Define the Expected Shortfall of \( \mu \) at level \( \alpha \), \( ES_\mu(\alpha) \), as

\[
ES_\mu(\alpha) := \frac{1}{\alpha} \int_0^\alpha F_\mu^{-1}(s) \, ds.
\]

It is easy to see that for fixed \( \alpha \), Expected Shortfall is a Lipschitz continuous function of measure w.r.t \( p \)-th Wasserstein distances for \( p \geq 1 \). Indeed fix \( \mu, \nu \in \mathcal{P}_p(\mathbb{R}) \) and observe that

\[
|ES_\mu(\alpha) - ES_\nu(\alpha)| \leq \frac{1}{\alpha} \int_0^\alpha |F_\mu^{-1}(s) - F_\nu^{-1}(s)| \, ds \leq \frac{1}{\alpha} \int_0^1 |F_\mu^{-1}(s) - F_\nu^{-1}(s)| \, ds = \frac{1}{\alpha} W_1(\mu, \nu) \leq \frac{1}{\alpha} W_p(\mu, \nu).
\]

We consider the following one-dimensional example, based loosely on transformed CIR:

\[
\frac{dx_t}{dt} = \frac{\kappa}{2} \left[ (ES_\mu(x_t)(\alpha) \vee \theta) - \frac{\sigma^2}{4\kappa} x_t^{-1} - x_t \right] dt + \frac{1}{2} \sigma dw_t.
\]

Here \( x_0 \) satisfies \( \mathbb{P}[x_0 > 0] = 1 \) and \( \kappa \theta \geq \sigma^2 \).

Note that by defining \( D := (0, \infty) \) and \( D_k := [\frac{k}{k}, \frac{k}{k}] \), we have boundedness of the coefficients on \( D_k \) and from the above observations and assumptions one can easily verify that the conditions of Theorem 2.9 are satisfied. In particular consider \( v(x) = x^2 + x^{-2} \). Then,

\[
L(x, \mu)v(x) = \frac{\kappa}{2} \left[ (ES_\mu(x)(\alpha) \vee \theta) - \frac{\sigma^2}{4\kappa} x^{-1} - x \right] (2(x - x^{-3})) + \frac{1}{8} \sigma^2 (2 + 6x^{-4})
\]

\[
= \kappa \left[ (ES_\mu(x)(\alpha) \vee \theta) - \frac{\sigma^2}{4\kappa} \right] - \kappa x^2 - \left[ \kappa (ES_\mu(x)(\alpha) \vee \theta) - \sigma^2 \right] x^{-4} + \kappa x^{-2} + \frac{\sigma^2}{4}
\]

\[
\leq \|x\| [ES_\mu(x)(\alpha)] + \kappa \theta + \kappa x^{-2}
\]

\[
\leq \|x\| \int_{\mathbb{R}} x \mu(dx) + \kappa \theta + \kappa x^{-2}
\]

\[
\leq \frac{1}{2} \kappa^2 + \frac{1}{2} \int_{\mathbb{R}} x^2 \mu(dx) + \kappa \theta + \kappa x^{-2}.
\]

Integrating with respect to \( \mu \) we see that condition (2.5) holds. Therefore, due to Theorem 2.9, we have existence of a weak solution to the above McKean–Vlasov equation.

### 3 Uniqueness

In this Section we prove continuous dependence on initial conditions and uniqueness under two types of Lyapunov conditions. For the novel integrated global Lyapunov condition we provide an example that has been inspired by the work of [28] on non-uniqueness of solutions to McKean–Vlasov SDEs.
3.1 Assumptions and Results

Recall that by \( \pi \in \Pi(\mu, \nu) \) we denote a coupling between measures \( \mu \) and \( \nu \). In this section we work with a subclass of Lyapunov functions \( \bar{v} \in C^{1,2}(I \times \mathbb{R}^d) \) that has the properties: \( \bar{v} \geq 0 \), \( \text{Ker } \bar{v} = \{0\} \) and \( \bar{v}(x) = \bar{v}(-x) \) for all \( x \in \mathbb{R}^d \). For this class of Lyapunov functions we define semi-Wasserstein distance on \( \mathcal{P}(D) \) as,

\[
W_v(\mu, \nu) := \left( \inf_{\pi \in \Pi(\mu, \nu)} \int_{D \times D} \bar{v}(x - y) \pi(dx, dy) \right)^{1/2}. \tag{3.1}
\]

Indeed \( W_v \) is a semi-metric and the triangle inequality, in general, does not hold. Note that \( \bar{v} \) does not depend on a measure. For \( (t, x, y) \in I \times D \times D \), \( (\mu, \nu) \in \mathcal{P}(D) \times \mathcal{P}(D) \), we define the generator as follows

\[
L(t, x, y, \mu, \nu)\bar{v}(t, x - y) := \partial_t \bar{v}(t, x - y) + \frac{1}{2} \text{tr}((\sigma(t, x, \mu) - \sigma(t, y, \nu))(\sigma(t, x, \mu) - \sigma(t, y, \nu))^\ast \partial_x^2 \bar{v}(t, x - y)) + (b(t, x, \mu) - b(t, y, \nu)) \partial_x \bar{v}(t, x - y).
\]

**Assumption 3.1** (Global Lyapunov condition). There exist locally integrable, non-random, functions \( g = g(t) \) and \( h = h(t) \) on \( I \), such that for all \( (t, x, \mu) \) and \( (t, y, \nu) \) in \( I \times D \times \mathcal{P}(D) \)

\[
L(t, x, y, \mu, \nu)\bar{v}(t, x - y) \leq g(t)\bar{v}(x - y) + h(t)W_v(\mu, \nu). \tag{3.2}
\]

**Assumption 3.2** (Integrated Global Lyapunov condition). There exists locally integrable, non-random, function \( h = h(t) \) on \( I \), such that for all \( (t, \mu) \) and \( (t, \nu) \) in \( I \times \mathcal{P}(D) \) and for all couplings \( \pi \in \Pi(\mu, \nu) \)

\[
\int_{D \times D} L(t, x, y, \mu, \nu)\bar{v}(t, x - y) \pi(dx, dy) \leq h(t) \int_{D \times D} \bar{v}(x - y) \pi(dx, dy). \tag{3.3}
\]

Theorem 3.3 gives a stability estimate for the solution to (1.1) with respect to initial condition (continuous dependence on the initial conditions).

**Theorem 3.3** (Continuous Dependence on Initial Condition). Let Assumption 2.5 hold. Let \( x^i \), \( i = 1, 2 \) be two solutions to (1.1) on the same probability space such that \( \mathbb{E}\bar{v}(x^1_0 - x^2_0) < \infty \). 

i) If Assumption 3.1 holds then for all \( t \in I \)

\[
\mathbb{E}\bar{v}(x^1_t - x^2_t) \leq \exp \left( \int_0^t \left[ g(s) + h(s) + 2|\bar{v}(s)| \right] ds \right) \mathbb{E}\bar{v}(x^1_0 - x^2_0). \tag{3.4}
\]

ii) If Assumptions 3.2 and either of 2.2 or 2.1 hold and if there are \( p, q \) with \( 1/p + 1/q = 1 \) and a constant \( \kappa \) such that for all \( (t, x, \mu) \) in \( I \times D \times \mathcal{P}(D) \)

\[
|\partial_x \bar{v}(x - y)|^{2p} + |\sigma(t, x, \mu)|^{2q} + |\sigma(t, y, \nu)|^{2q} \leq \kappa (1 + v(t, x, \mu) + v(t, y, \nu)) \tag{3.5}
\]

then for all \( t \in I \)

\[
\mathbb{E}\bar{v}(x^1_t - x^2_t) \leq \exp \left( \int_0^t h(s) ds \right) \mathbb{E}\bar{v}(x^1_0 - x^2_0). \tag{3.6}
\]
First we note that in the case when \( I \) is a finite time interval then the sign of the functions \( g \) and \( h \) plays no significant role. In relation to the study of ergodic SDEs e.g. (18) in [6] we make the following observations. If \( I = [0, \infty) \) and Assumption 3.1 holds then if \( g + h + 2|h| < 0 \) then \( \lim_{t \to \infty} \mathbb{E}\tilde{\nu}(x^1_t - x^2_t)^2 = 0 \). However we see that while the spatial dependence of coefficients can play a positive role for the stability of the equation (if \( g \) is negative) it seems that the measure dependence never has such positive role, regardless of the sign of \( h \). If \( I = [0, \infty) \) and we are in the second case of Theorem 3.3 then negative \( h \) can play a positive role for stability (but unlike the first case we also need the condition (3.5)).

**Proof.** Note that if we are in case ii) then, in the following we set \( g = 0 \) for all \( t \in I \). Let

\[
\varphi(t) = \exp \left( - \int_0^t [g(s) + h(s)] ds \right).
\]

Applying the classical Itô formula to \( \varphi \tilde{\nu}(x^1 - x^2) \) we have that for \( t \in I \)

\[
\begin{align*}
\varphi(t)\tilde{\nu}(x^1_t - x^2_t) &= \tilde{\nu}(x^1_0 - x^2_0) \\
&+ \int_0^t \varphi(s) [L(s, x^1_s, x^2_s, \mathcal{L}(x^1_s), \mathcal{L}(x^2_s))\tilde{\nu}(t, x^1_s - x^2_s) - (g(s) + h(s))\tilde{\nu}(x^1_s - x^2_s)] ds \\
&+ \int_0^t \varphi(s)\partial_x \tilde{\nu}(x^1_s - x^2_s)(\sigma(s, x^1_s, \mathcal{L}(x^1_s)) - \sigma(s, x^2_s, \mathcal{L}(x^2_s)))dw_s.
\end{align*}
\]

(3.7)

**Case i) Assumption 3.1 implies**

\[
\begin{align*}
\varphi(t)\tilde{\nu}(x^1_t - x^2_t) \leq \tilde{\nu}(x^1_0 - x^2_0) + \int_0^t \varphi(s) [h(s)\mathcal{W}_\tilde{\nu}(x^1_s, \mathcal{L}(x^1_s)) - h(s)\tilde{\nu}(x^1_s - x^2_s)] ds \\
&+ \int_0^t \varphi(s)\partial_x \tilde{\nu}(x^1_s - x^2_s)(\sigma(s, x^1_s, \mathcal{L}(x^1_s)) - \sigma(s, x^2_s, \mathcal{L}(x^2_s)))dw_s.
\end{align*}
\]

Define the stopping times \( \{\tau_i^m\}_{m \geq 1}, i = 1, 2 \) and \( \{\tau_m\}_{m \geq 1} \)

\[
\tau_i^m := \inf\{t \in I : x^i_t \notin D_m\}, \quad i = 1, 2 \quad \text{and} \quad \tau_m := \tau_1^m \wedge \tau_2^m.
\]

By Definition 2.7 we know that \( x^i \in C(I; D) \) a.s. and so \( \tau_i^m \nearrow \infty \) a.s. and hence \( \tau_m \nearrow \infty \) a.s. as \( m \to \infty \). The local boundedness of \( \sigma \) ensures that the stochastic integral in the above is a martingale on \([t \wedge \tau_m]\), hence

\[
\begin{align*}
\mathbb{E}[\varphi(t \wedge \tau_m)\tilde{\nu}(x^1_{t \wedge \tau_m} - x^2_{t \wedge \tau_m})] \\
\leq \mathbb{E}[\tilde{\nu}(x^1_0 - x^2_0)] + \mathbb{E} \left[ \int_0^{t \wedge \tau_m} \varphi(s) [h(s)\mathcal{W}_\tilde{\nu}(x^1_s, \mathcal{L}(x^1_s)) - h(s)\tilde{\nu}(x^1_s - x^2_s)] ds \right] \\
\leq \mathbb{E}[\tilde{\nu}(x^1_0 - x^2_0)] + \mathbb{E} \left[ \int_0^{t \wedge \tau_m} \varphi(s) [h(s)\tilde{\nu}(x^1_s - x^2_s)] ds \right],
\end{align*}
\]

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where the last inequality follows from the definition of the semi-Wasserstein distance. Since $\tau_m \nearrow \infty$ as $m \to \infty$, application of Fatou’s Lemma gives

$$E[\varphi(t)\bar{v}(x_t^1 - x_t^2)] \leq E\bar{v}(x_0^1 - x_0^2) + \int_0^t |h(s)| E[\varphi(s)\bar{v}(x_s^1 - x_s^2)] \, ds.$$ 

From Gronwall’s lemma we get (3.4).

Case ii) Taking expectation in (3.7), recalling that in this case $g = 0$ and then using Assumption 3.2 we have

$$E \left[ \varphi(t)\bar{v}(x_t^1 - x_t^2) \right] \leq E\bar{v}(x_0^1 - x_0^2) + E \int_0^t \varphi(s)\partial_x\bar{v}(x_s^1 - x_s^2)(\sigma(s,x_s^1,L(x_s^1)) - \sigma(s,x_s^2,L(x_s^2))) \, dw_s.$$ 

Corollary 2.12 together with (3.5) and local integrability of $g$ and $h$ ensures that stochastic integral in the above expression is a martingale. Indeed

$$\int_0^t \varphi(s)^2 E \left[ |\partial_x\bar{v}(x_s^1 - x_s^2)|^2 |\sigma(s,x_s^1,L(x_s^1)) - \sigma(s,x_s^2,L(x_s^2))|^2 \right] \, ds \leq \int_0^t \varphi(s)^2 E \left[ \frac{1}{p} |\partial_x\bar{v}(x_s^1 - x_s^2)|^{2p} + \frac{1}{q} |\sigma(s,x_s^1,L(x_s^1)) - \sigma(s,x_s^2,L(x_s^2))|^{2q} \right] \, ds \leq c_{p,q} \int_0^t \varphi(s)^2 \kappa (1 + E\nu(s,x_s^1,L(x_s^1)) + E\nu(s,x_s^2,L(x_s^2))) \, ds < \infty.$$ 

Hence

$$\varphi(t)E[\bar{v}(x_t^1 - x_t^2)] \leq E[\bar{v}(x_0^1 - x_0^2)].$$

\[\square\]

**Corollary 3.4.** If the conditions for either case i) or ii) of Theorem 3.3 hold and if $x_0^1 = x_0^2$ a.s. then the solutions to (1.1) are pathwise unique.

**Proof.** If $I = [0,T]$ then uniqueness follows immediately from Theorem 3.3, the fact that $\text{Ker} \ \bar{v} = \{0\}$ and local integrability of $g$ and $h$. If $I = [0,\infty)$ then it is enough to observe that when $x_0^1 = x_0^2$ then, due to Theorem 3.3, uniqueness holds on the interval $[0,s]$, for some $s > 0$ and in particular $x_s^1 = x_s^2$ a.s. Thus we can continue in a recursive manner to obtain uniqueness on the intervals $[ks,(k+1)s]$ for $k \in \mathbb{N}$. \[\square\]

### 3.2 Example due to Schuetzow.

Consider the McKean–Vlasov SDE of the form

$$x_t = x_0 + \int_0^t B(x_s, E[\hat{b}(x_s)]) \, ds + \int_0^t \Sigma(x_s, E[\hat{\sigma}(x_s)]) \, dw_s.$$  

(3.8)
Our study of this more specific form of McKean–Vlasov SDE is inspired by [28], where it has been shown that in the case when $\sigma = 0$ and either of functions $b$ or $\bar{b}$ is locally Lipschitz then uniqueness, in general, does not hold. We will show that if we impose some structure on the local behaviour of the functions then these, together with the integrability conditions established in Theorem 2.9, are enough to obtain unique solution (3.8). To be more specific: we impose local (in the second variable) monotone condition on functions $b$ and $\sigma$, which is weaker than local (in the second variable) Lipschitz condition, and local Lipschitz condition on functions $b$ and $\sigma$.

**Assumption 3.5.**

i) Local Monotone condition: there exists locally bounded function $M = M(x', y', x'', y'')$ such that
$$2(x-y)(B(x, x') - B(y, y')) + |\Sigma(x, x'') - \Sigma(y, y'')|^2 \leq M(x', y', x'', y'')(|x-y|^2 + |x'-y'|^2 + |x''-y''|^2)$$

together with

ii) there exists $\kappa$ such that $\forall(x, y, \mu) \in I \times D \times \mathcal{P}(D)$

$$|\bar{b}(x) + |\bar{\sigma}(x)| \leq \kappa(1 + v(t, x, \mu)).$$

iii) $\forall(x, y, \mu) \in I \times D \times \mathcal{P}(D)$ there exists $\kappa$ such that

$$|\bar{b}(x) - \bar{b}(y)| + |\bar{\sigma}(x) - \bar{\sigma}(y)| \leq \kappa(1 + \sqrt{v(t, x, \mu)} + \sqrt{v(t, y, \mu)})|x - y|.$$

**Theorem 3.6.** If Assumptions 2.2 hold, if $\sup_{t \in I} M(t) < \infty$ and if Assumptions 2.5, 3.5 hold then the solution to (3.8) is unique.

We will need the following observation: if $\pi \in \Pi(\mu, \nu)$ then, due to the theorem on disintegration, (see for example [1, Theorem 5.3.1]) there exists a family $(P_x)_{x \in D} \subset \mathcal{P}(D)$ such that

$$\int_{D \times D} f(x, y) \pi(dx, dy) = \int_D \left( \int_D f(x, y) P_x(dy) \right) \mu(dx)$$

for any $f = f(x, y)$ which is a $\pi$-integrable function on $D \times D$. In particular if $f = f(x)$ then

$$\int_{D \times D} f(x, y) \pi(dx, dy) = \int_D f(x) \left( \int_D P_x(dy) \right) \mu(dx) = \int_D f(x) \mu(dx).$$

**Proof.** Our aim is to show that Assumption 3.1 holds since then uniqueness follows from Corollary 3.4. We know, from Lemma 2.10 that for any $t \in I$ we have $\int_D v(t, x, \mathcal{L}(x_t))(\mathcal{L}(x_t))(dx) \leq \sup_{t \in I} M(t)$ and so it is in fact enough to verify (3.2) for measures $\mu$ such that $\int_D v(t, x, \mu) \mu(dx) \leq \sup_{t \in I} M(t)$. From Assumption 3.5 i), we have

$$2(x-y)(b(x, \mu) - b(y, \nu)) + |\sigma(x, \mu) - \sigma(y, \nu)|^2 \leq M(x', y', x'', y'')[|x-y|^2 + |x'-y'|^2 + |x''-y''|^2],$$

for any $x, x', x'', y, y', y'' \in D$.
where \( x' = \int_D \bar{b}(z)\mu(dz), \ y' = \int_D \bar{b}(z)\nu(dz), \ x'' = \int_D \bar{\sigma}(z)\mu(dz) \) and \( y'' = \int_D \bar{\sigma}(z)\nu(dz) \). We note that each of \(|x'|, |y'|, |x''|\) and \(|y''|\) are in a compact subset of \( \mathbb{R} \), since due to Assumption 3.5 ii) we have

\[
\kappa \left( 1 + \int_D v(t, z, \mu) \mu(dz) \right) + \kappa \left( 1 + \int_D v(t, z, \mu) \nu(dz) \right) \leq 2 \sup_{t \in I} M(t).
\]

As \( M \) maps bounded sets to bounded sets we can choose a constant \( g \) sufficiently large so that \( M(x', y', x'', y'') \leq g \) for all \( \mu, \nu \).

We apply the remark on disintegration to see that

\[
|x' - y'|^2 = \left| \int_D \bar{b}(\bar{x})\mu(d\bar{x}) - \int_D \bar{b}(\bar{y})\nu(d\bar{y}) \right|^2 = \left| \int_{D \times D} (\bar{b}(\bar{x}) - \bar{b}(\bar{y})) \pi(dx, dy) \right|^2.
\]

From Assumption 3.5 iii) we get

\[
|x' - y'|^2 \leq \kappa^2 \int_{D \times D} (1 + \sqrt{v(t, \bar{x}, \mu) + v(t, \bar{y}, \mu)})^2 \pi(d\bar{x}, d\bar{y}) \int_{D \times D} |\bar{x} - \bar{y}|^2 \pi(d\bar{x}, d\bar{y})
\]

\[
\leq 4 \left( \kappa^2 \sup_{t \in I} M(t) \right) \int_{D \times D} |\bar{x} - \bar{y}|^2 \pi(d\bar{x}, d\bar{y}).
\]

Since the calculation for \(|x'' - y''|^2\) is identical we finally obtain

\[
2(x - y)(b(x, \mu) - b(y, \nu)) + |\sigma(x, \mu) - \sigma(y, \nu)|^2 \leq g|x - y|^2 + 8g\kappa^2 \sup_{t \in I} M(t) \int_{D \times D} |\bar{x} - \bar{y}|^2 \pi(d\bar{x}, d\bar{y})
\]

as required to have Assumption 3.1 satisfied.

\[\square\]

4 Invariant Measures

4.1 Semigroups on \( C_b(D) \)

We will establish the existence of a stationary measure for semigroups associated with solutions to (1.1) via the Krylov–Bogolyubov Theorem (see [27, Chapter 7]). Let the conditions of Theorem 2.9 hold with suitable assumptions on \( m_1 \) and \( m_2 \) such that we are within the regime where \( I = [0, \infty). \) For every point \( y \in D \) fix a problem \((x^y_t)_{t \geq 0}\) that is a \( \nu \)-integrable solution to the McKean–Vlasov SDE (1.1) started from \( y \). We then define a semigroup \( (P_t)_{t \geq 0} \) by

\[
P_t \varphi(y) := \mathbb{E}[\varphi(x^y_t)] \text{ for } t \geq 0, \varphi \in C_b(D).
\]

Clearly \( P_t \varphi(y) = \langle \varphi, \mathcal{L}(x^y_t) \rangle \) and if \( \varphi \in C^2_b(D) \) then \( \langle \varphi, \mu_t \rangle := \langle \varphi, \mathcal{L}(x^y_t) \rangle \) is given by (1.2). This means that establishing existence of invariant measure to (1.1) shows that if \( b \) and \( \sigma \) are independent of \( t \) then there is a stationary solution to (1.2).

The two main conditions for Krylov–Bogolyubov’s theorem to hold is that the semigroup is Feller and a tightness condition. As we are not assuming any non-degeneracy of the diffusion coefficient we cannot always guarantee that the semigroup is Feller. See, however, Lemma 4.2 for a partial result.
Theorem 4.1. If the conditions of Theorem 2.9 hold with \( I = [0, \infty) \) (i.e. we have either \( \sup_{t \in [0, \infty)} M^+(t) < \infty \) or \( \sup_{t \in (0, \infty)} M(t) < \infty \)) and the semigroup \((P_t)_{t \geq 0}\) has the Feller property then there exists an invariant measure for \((P_t)_{t \geq 0}\) acting on \(C_b(D)\).

Proof. Fix \( y \in D \) and let \((\mu_t)_{t \geq 0}\) be defined as

\[
\mu_t := \frac{1}{t} \int_0^t \mathbb{P}(x_s^y \in \cdot) \, ds = \frac{1}{t} \int_0^t \mathcal{L}(x_s^y) \, ds.
\]

By Fatou’s Lemma and Lemma 2.13 we know that for any \( \varepsilon > 0 \) there exists sufficiently large \( m_0 \) such that for all \( m > m_0 \) we have, via the non-decreasing property of \( \bar{W} \).

We have, via the non-decreasing property of \( \bar{W} \),

\[
\mu_t(D \setminus D_m) = \frac{1}{t} \int_0^t \mathbb{P}(x_s^y \notin D_m) \, ds < \varepsilon \text{ and hence } (\mu_t)_{t \geq 0} \text{ is tight.}
\]

Therefore, since we are assuming that the Feller property holds, the conclusion now follows from Krylov–Bogolyubov Theorem (see [27, Chapter 7]). \( \square \)

Lemma 4.2. If the assumptions of Theorem 2.9 hold with \( I = [0, \infty) \) along with either Assumption 3.1 or 3.2 and that \( \bar{v} \) is non-decreasing, then the semigroup \((P_t)_{t \geq 0}\) acting on \(C_b(D)\) is Feller.

Proof. For \( \varepsilon > 0 \), by continuity of \( \varphi \) there exists \( \delta_\varphi > 0 \) s.t. \( |x_t^{y_1} - x_t^{y_2}| < \delta_\varphi \implies |\varphi(x_t^{y_1}) - \varphi(x_t^{y_2})| < \varepsilon/2 \).

Then

\[
|P_t \varphi(y_1) - P_t \varphi(y_2)| = |E[\varphi(x_t^{y_1}) - \varphi(x_t^{y_2})]| \leq E[|\varphi(x_t^{y_1}) - \varphi(x_t^{y_2})|]
= \mathbb{E}[|\varphi(x_t^{y_1}) - \varphi(x_t^{y_2})| (\mathbb{1}_{|x_t^{y_1} - x_t^{y_2}| < \delta_\varphi} + \mathbb{1}_{|x_t^{y_1} - x_t^{y_2}| \geq \delta_\varphi})]
\leq \frac{\varepsilon}{2} \mathbb{P}(|x_t^{y_1} - x_t^{y_2}| < \delta_\varphi) + 2|\varphi|_{\infty} \mathbb{P}(|x_t^{y_1} - x_t^{y_2}| \geq \delta_\varphi)
\leq \frac{\varepsilon}{2} + 2|\varphi|_{\infty} \mathbb{P}(|x_t^{y_1} - x_t^{y_2}| \geq \delta_\varphi).
\]

We have, via the non-decreasing property of \( \bar{v} \) (first inequality) and the continuous dependence on initial condition (3.4) and (3.6) (second inequality),

\[
\bar{v}(\delta_\varphi) \mathbb{P}(|x_t^{y_1} - x_t^{y_2}| \geq \delta_\varphi) \leq E[\bar{v}(x_t^{y_1} - x_t^{y_2})] \leq c_t E[\bar{v}(y_1 - y_2)].
\]

By continuity of \( \bar{v} \), for any \( \varepsilon_0 > 0 \) there exists \( \delta_0 \) such that, if \( |y_1 - y_2| < \delta_0, \bar{v}(y_1 - y_2) < \varepsilon_0 \). Therefore, by choosing \( \varepsilon_0 \) small enough such that \( \frac{2|\varphi|_{\infty}}{\delta_0} \bar{v}(y_1 - y_2) < \frac{2|\varphi|_{\infty}}{\delta_0} \varepsilon_0 < \varepsilon/2 \), we have, for \( |y_1 - y_2| < \delta_0, |P_t \varphi(y_1) - P_t \varphi(y_2)| < \varepsilon \). Boundedness of \( P_t \varphi \) is immediate by definition. \( \square \)

4.2 Semigroups on \( C_b(\mathcal{P}_2(D)) \)

Now we consider semigroups acting on functions of measures. Define the semigroup \((\mathcal{P}_t)_{t \geq 0}\) by

\[
\mathcal{P}_t \phi(\mu) = \phi(\mathcal{L}(x_t^y)) \quad \text{for } \phi \in C_b(\mathcal{P}_2(D)) \text{ and } t \geq 0.
\]
Here \( x_t^\mu \) denotes a solution to (1.1) started from \( \mu \). To ensure that \( \mathcal{L}(x_t^\mu) \in \mathcal{P}_2(D) \) we assume that the conditions of Theorem 2.9 hold with \( V \) satisfying \( V(t, x) \geq |x|^2 \). If \( D = \mathbb{R}^d \) then we can apply the chain rule for functions of measures from e.g. [8] or [11] to obtain that for \( \phi \in C^{1,1}(\mathcal{P}_2(D)) \)

\[
\phi(\mathcal{L}(x_1)) - \phi(\mathcal{L}(x_0)) = \int_0^t \left( \mathcal{L}(\mathcal{L}(x_s^\mu)), b(s, \cdot, \mathcal{L}(x_s^\mu)) \right) ds.
\]

(4.2)

In the case that \( D \subseteq \mathbb{R}^d \) we have to assume that there is \( \varepsilon > 0 \) and \( k \in \mathbb{N} \) such that \( V(t, x) \geq |x|^{2+\varepsilon} \) for \( x \in D \setminus D_k \). We consider first \( x^k,\mu \) given by (2.11) started from \( \mu \). By Proposition A.6 we have for \( \phi \in C^{1,1}(\mathcal{P}_2(D)) \) that

\[
\phi(\mathcal{L}(x_t^k)) - \phi(\mathcal{L}(x_0^k)) = \int_0^t \left( \mathcal{L}(\mathcal{L}(x_s^k), b(s, \cdot, \mathcal{L}(x_s^k)), d\mu(\mathcal{L}(x_s^k))) + \text{tr} \left[ a(s, \cdot, \mathcal{L}(x_s^k)) \right] ds. \]

(4.3)

From Lemma 2.10 we get that \( \sup_k \sup_t E|x_t^k|^{2+\varepsilon} < \infty \). Moreover Lemma 2.13 implies, together with Prohorov’s theorem convergence of a subsequence of the laws (and since we know the limit of these is given by (1.1) due to the proof of Theorem 2.9). We thus have convergence \( W_2(\mathcal{L}(x_t^k), \mathcal{L}(x_t)) \to 0 \) as \( k \to \infty \). Due to continuity of coefficients \( b, \sigma \) and since \( \phi \in C^{1,1}(\mathcal{P}_2(D)) \) we can take the limit \( k \to \infty \) in (4.3) to obtain (4.2).

**Theorem 4.3.** Let the conditions of Theorem 2.9 hold with \( I = [0, \infty) \), and \( V(t, x) \geq |x|^2 \) for \( x \in D \setminus D_k \) for some \( k \in \mathbb{N} \). If the semigroup \( (\mathcal{P}_t)_{t \geq 0} \) given by (4.1) is Feller then there exists an invariant measure.

We will need to following fact from [25] to prove this theorem: Let \( S \) be a Polish space and \( (m_t)_{t \geq 0} \) be a family of probability measures on \( \mathcal{P}(S) \) i.e. \( m_t \in \mathcal{P}(\mathcal{P}(S)) \). Define the intensity measure \( I(m_t) \) by

\[
\langle I(m_t), f \rangle = \int_{\mathcal{P}(S)} \langle \nu, f \rangle m_t(d\nu), \quad f \in B(S).
\]

Here \( B(S) \) denotes all the bounded measurable functions from \( S \) to \( \mathbb{R} \). Then \( (m_t)_{t \geq 0} \) is tight if and only if the family of intensity measures \( (I(m_t))_{t \geq 0} \subset \mathcal{P}(S) \) is tight.

**Proof of Theorem 4.3.** We recall that \( \mathcal{P}_2(D) \) with the Wasserstein distance \( W_2 \) is Polish [34, Theorem 6.18]. Fix \( \mu \in \mathcal{P}_2(D) \) and let \( x^\mu \) be a solution to (2.16). We note that with \( \pi_t(\mu, B) := \delta_{\mathcal{L}(x_t^\mu)}(B) \) we have, from (4.1), that

\[
\mathcal{P}_t \phi(\mu) = \phi(\mathcal{L}(x_t^\mu)) = \int_{\mathcal{P}_2(D)} \delta_{\mathcal{L}(x_t^\mu)}(\nu) \phi(\nu) d\nu = \int_{\mathcal{P}_2(D)} \phi(\nu) \pi_t(\mu, d\nu).
\]

Define the family of measures \( (m_t^\mu)_{t \geq 0} \subset \mathcal{P}(\mathcal{P}_2(D)) \) by

\[
m_t^\mu(B) := \frac{1}{t} \int_0^t \pi_s(\mu, B) ds = \frac{1}{t} \int_0^t \delta_{\mathcal{L}(x_s^\mu)}(B) ds, \quad B \in \mathcal{B}(\mathcal{P}_2(D)).
\]
To apply the Krylov–Bogolyubov Theorem we need to show that the family \((m_t)_{t \geq 0}\) is tight. We observe that for all \(f \in B(D)\) we have
\[
\int_{\mathcal{P}(D)} \langle \nu, f \rangle \, m_t^\mu(d\nu) = \int_{\mathcal{P}(D)} \langle \nu, f \rangle \frac{1}{t} \int_0^t \delta_{\mathcal{L}(x_s^\mu)} \, (d\nu) \, ds = \frac{1}{t} \int_0^t \langle \mathcal{L}(x_s^\mu), f \rangle \, ds \equiv \frac{1}{t} \int_0^t \mathcal{L}(x_s^\mu) \, ds, f \rangle.
\]
Therefore \(I(m_t^\mu) = \frac{1}{t} \int_0^t \mathcal{L}(x_s^\mu) \, ds\). It remains to show that family of intensity measures \((I(m_t^\mu))_{t \geq 0}\) in \(\mathcal{P}(D)\) is tight. For \(B \in \mathcal{B}(D)\) we have
\[
I(m_t^\mu)(B) = \frac{1}{t} \int_0^t \langle \mathcal{L}(x_s^\mu), 1_B \rangle \, ds = \frac{1}{t} \int_0^t \mathcal{L}(x_s^\mu)(B) \, ds = \frac{1}{t} \int_0^t \mathbb{P}(x_s^\mu \in B) \, ds.
\]
By Fatou’s Lemma and Lemma 2.13 we know that for any \(\varepsilon > 0\) there exists sufficiently large \(m_0\) such that for all \(m > m_0\) we have \(\sup_{t \in I} \mathbb{P}[x_t^\mu \notin D_m] < \varepsilon\). Therefore \(I(m_t^\mu)(D_m) = \frac{1}{t} \int_0^t \mathbb{P}(x_s^\mu \notin D_m) \, ds < \varepsilon\) and hence \((I(m_t^\mu))_{t \geq 0}\) is tight. \(\square\)

We do not assume non-degeneracy of the diffusion thus, in general, the semigroup \(\mathcal{P}_t\) is not expected to be Feller. However Lemma 4.4 gives a partial result.

**Lemma 4.4.** Let assumptions of Theorem 2.9 hold for \(I = \{0, \infty\}\) along with either Assumption 3.1 or 3.2. Assume further that
\[
W_\varepsilon(\mu, \nu) < \infty \quad \text{for } \mu, \nu \text{ in } \mathcal{P}_\varepsilon(D) := \left\{ \mu \in \mathcal{P}(D) : \int_D v(0, x, \mu) \, \mu(dx) < \infty \right\}.
\]
Then the semigroup \((\mathcal{P}_t)_{t \geq 0}\) acting on \(C_b(\mathcal{P}_\varepsilon(D))\) and defined as in (4.1) is Feller.

Note that here, we are considering a semigroup acting on space of measures possibly different to that previously considered. In the case where \(v\) and \(\bar{v}\) are polynomials is often simple process to have the assumptions of Lemma 4.4 replace the assumption of the Feller property in Theorem 4.3 and \(W_\varepsilon(\mu, \nu) < \infty\) for any \(\mu, \nu \in \mathcal{P}_\varepsilon(D)\) is no longer required.

**Proof.** Fix \(t \in I\) and \(\mu_1, \mu_2 \in \mathcal{P}_\varepsilon(D)\). From the continuous dependence on initial condition, Theorem 3.3, we have
\[
W_\varepsilon(\mathcal{L}(x_t^{\mu_1}), \mathcal{L}(x_t^{\mu_2})) \leq \mathbb{E}[\bar{v}(x_t^{\mu_1} - x_t^{\mu_2})] \leq c_t \mathbb{E}[\bar{v}(x_0^{\mu_1} - x_0^{\mu_2})] = c_t \int_{D \times D} \bar{v}(x - y) \pi(dx, dy).
\]
Taking the infimum over all the possible couplings yields,
\[
W_\varepsilon(\mathcal{L}(x_t^{\mu_1}), \mathcal{L}(x_t^{\mu_2})) \leq c_t W_\varepsilon(\mu_1, \mu_2).
\] (4.4)
Let \(\varepsilon > 0\) be given. For any \(\phi \in C_b(\mathcal{P}_\varepsilon(D))\) there is \(\delta_\phi\) such that \(W_\varepsilon(\mathcal{L}(x_t^{\mu_1}), \mathcal{L}(x_t^{\mu_2})) < \delta_\phi\) implies that \(\left| \phi(\mathcal{L}(x_t^{\mu_1})) - \phi(\mathcal{L}(x_t^{\mu_2})) \right| \leq \varepsilon\). Now take \(\delta := \delta_\phi/c_t\). Then, due to (4.4), if \(W_\varepsilon(\mu_1, \mu_2) \leq \delta\) then \(W_\varepsilon(\mathcal{L}(x_t^{\mu_1}), \mathcal{L}(x_t^{\mu_2})) < \delta_\phi\) and we get \(|P_1\phi(\mu_1) - P_1\phi(\mu_2)| \leq \varepsilon\) as required. \(\square\)
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A Measure derivatives of Lions, associated Itô formula and examples

For the construction of the measure derivative in the sense of Lions we follow the approach from [9, Section 6]. There are three main differences: The first difference is that we define the measure derivative in a domain. More precisely we will define the measure derivative for any measure as long as it has support on $D_k \subset D$ for some $k \in \mathbb{N}$ (recall that $D_k \subset D_{k+1}$ and $\bigcup_k D_k = D$ and every $D_k$ is compact) This is precisely what is needed for the analysis in this paper. The second difference is that we are explicit in making it clear why the measure derivative is independent of the probability space used to realise the measure as well as the random variable used. The third difference is in proving the “Structure of the gradient”, see [9, Theorem 6.5]. Thanks to an observation by Sandy Davie (University of Edinburgh), we can show as part iii) of Proposition A.2 that the measure derivative has the right structure even if it only exists at the point $\mu$ instead of for every square integrable measure, as is required in [9]. The method of Sandy Davie also conveniently results in a much shorter proof.

A.1 Construction of first-order Lions’ measure derivative on $D_k \subset D \subseteq \mathbb{R}^d$

Consider $u : \mathcal{P}_2(D) \to \mathbb{R}$. Here $\mathcal{P}_2(D)$ is a space of probability measures on $D$ that have second moments i.e. $\int_D x^2 \mu(dx) < \infty$ for $\mu \in \mathcal{P}_2(D)$. We want to define the derivative at points $\mu \in \mathcal{P}_2(D)$ such that $\text{supp}(\mu) \subset D_k$. We shall write $\mu \in \mathcal{P}(D_k)$ if $\mu$ is a probability measure on $D$ with support in $D_k$.

Definition A.1 (L-differentiability at $\mu \in \mathcal{P}(D_k)$). We say that $u$ is $L$-differentiable at $\mu \in \mathcal{P}(D_k)$ if there is an atomless\(^2\), Polish probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and an $X \in L^2(\Omega)$ such that $\mu = \mathcal{L}(X)$ and the function $U : L^2(\Omega) \to \mathbb{R}$ given by $U(Y) := u(\mathcal{L}(Y))$ is Fréchet differentiable at $X$. We will call $U$ the lift of $u$.

Clearly $X$ s.t. $\mu = \mathcal{L}(X)$ can always be chosen so that supp($X$) $\subseteq D_k$ for $\mu \in \mathcal{P}(D_k)$. We recall that saying $U : L^2(\Omega; D) \to \mathbb{R}$ is Fréchet differentiable at $X$ with supp($X$) $\subseteq D_k$ means that there exists a bounded linear operator $A : L^2(\Omega) \to \mathbb{R}$ such that for

$$\lim_{\|Y\|_2 \to 0 \atop \text{supp}(X+Y) \subseteq D} \frac{|U(X + Y) - U(X)|}{|Y|^2} - \frac{AY}{|Y|^2} = 0.$$

\(^2\) Given $(\Omega, \mathcal{F}, \mathbb{P})$ an atom is $E \in \mathcal{F}$ s.t. $\mathbb{P}(E) > 0$ and for any $G \in \mathcal{F}, G \subseteq E$, $\mathbb{P}(E) > \mathbb{P}(G)$ we have $\mathbb{P}(G) = 0.$
Note that Since $L^2(\Omega)$ is a Hilbert space with the inner product $(X,Y) := \mathbb{E}[XY]$ we can identify $L^2(\Omega)$ with its dual $L^2(\Omega)^*$ via this inner product. Then the bounded linear operator $A$ defines an element $DU(X) \in L^2(\Omega)$ through

$$(DU(X),Y) := AY \quad \forall Y \in L^2(\Omega).$$

**Proposition A.2.** Let $u$ be $L$-differentiable at $\mu \in \mathcal{P}(D_k)$, with some atomless $(\Omega, \mathcal{F}, \mathbb{P})$, lift $U$ and $X \in L^2(\Omega)$. Let $(\bar{\Omega}, \bar{\mathcal{F}}, \bar{\mathbb{P}})$ be an arbitrary atomless, Polish probability space which supports $\bar{X} \in L^2(\bar{\Omega})$ and on which we have the lift $\bar{U}(Y) := u(\mathcal{L}(Y))$. Then

i) The lift $\bar{U}$ is Fréchet differentiable at $\bar{X}$ with derivative $D\bar{U}(\bar{X}) \in L^2(\bar{\Omega})$.

ii) The joint law of $(X, DU(X))$ equals that of $(\bar{X}, D\bar{U}(\bar{X}))$.

iii) There is $\xi : D_k \to D_k$ measurable such that $\int_{D_k} \xi^2(x)\mu(dx) < \infty$ and $\xi(X) = DU(X)$, $\xi(\bar{X}) = D\bar{U}(\bar{X})$.

Once this is proved we will know that the notion of $L$-differentiability depends neither on the probability space used nor on the random variable used. Moreover the function $\xi$ given by this proposition is again independent of the probability space and random variable used.

**Definition A.3** (L-derivative of $u$ at $\mu$). If $u$ is $L$-differentiable at $\mu$ then we write $\partial_\mu u(\mu) := \xi$, where $\xi$ is given by Proposition A.2. Moreover we have $\partial_\mu u : \mathcal{P}_2(D_k) \times D_k \to D_k$ given by

$$\partial_\mu u(\mu, y) := [\partial_\mu u(\mu)](y).$$

To prove Proposition A.2 we will need the following result:

**Lemma A.4.** Let $(\Omega, \mathcal{F}, \mathbb{P})$ and $(\bar{\Omega}, \bar{\mathcal{F}}, \bar{\mathbb{P}})$ be two atomless, Polish probability spaces supporting $D_k$-valued random variables $X$ and $\bar{X}$ such that $\mathcal{L}(X) = \mathcal{L}(\bar{X})$. Then for any $\epsilon > 0$ there exists $\tau : \Omega \to \bar{\Omega}$ which is bijective, such that both $\tau$ and $\tau^{-1}$ are measurable and measure preserving and moreover

$$|X - \bar{X} \circ \tau|_\infty < \epsilon \quad \text{and} \quad |X \circ \tau^{-1} - \bar{X}|_\infty < \epsilon.$$  

**Proof.** Let $(A_n)_n$ be a measurable partition of $D_k$ such that $\text{diam}(A_n) < \epsilon$. Let

$$B_n := \{X \in A_n\}, \quad \bar{B}_n := \{\bar{X} \in A_n\}.$$  

These form measurable partitions of $\Omega$ and $\bar{\Omega}$ respectively and moreover $\mathbb{P}(B_n) = \bar{\mathbb{P}}(\bar{B}_n)$. As the probability spaces are atomless, there exist $\tau_n : B_n \to \bar{B}_n$ bijective, such that $\tau_n$ and $\tau_n^{-1}$ are measurable and measure preserving. See [18, Sec. 41, Theorem C] for details. Let

$$\tau(\omega) := \tau_n(\omega) \quad \text{if } \omega \in B_n, \quad \tau^{-1}(\bar{\omega}) := \tau_n^{-1}(\bar{\omega}) \quad \text{if } \bar{\omega} \in \bar{B}_n.$$  

The theorem in fact provides the required isomorphism between a measure on separable atomless probability space and the unit interval.
We can see that these are measurable, measure preserving bijections. Now consider $\omega \in B_n$. Then $\tau(\omega) = \tau_n(\omega) \in \tilde{B}_n$. But then $X(\omega) \in A_n$ and $\bar{X}(\tau(\omega)) \in A_n$ too. Hence

$$\|X(\omega) - \bar{X}(\tau(\omega))\| < \epsilon \quad \forall \omega \in \Omega.$$ 

The estimate for the inverse is proved analogously. 

We use the notation $L^2 := L^2(\Omega)$ and $\bar{L}^2 := L^2(\bar{\Omega})$.

**Proof of Proposition A.2, part i.** For any $h > 0$ we have $\tau_h, \tau_h^{-1}$ given by Lemma A.4 measure preserving and such that $|X - \bar{X} \circ \tau_h|_\infty < h$. This means that $|X - \bar{X} \circ \tau_h|_2 < h$ and we have the analogous estimate with $\tau_h^{-1}$. Our first aim is to show that $(DU(X) \circ \tau_h^{-1})_{h \geq 0}$ is a Cauchy sequence in $\bar{L}^2$.

Fix $\epsilon > 0$. Then $\exists \delta > 0$ such that we have

$$|U(X + Y) - U(X) - (DU(X), Y)| < \frac{\epsilon}{2}|Y|_2 \quad \text{for all } |Y|_2 < \delta \quad \text{and } \text{supp}(X + Y) \subseteq D,$n

since $U$ is Fréchet differentiable at $X$. Fix $h, h' < \delta/2$ and consider $|\hat{Y}|_2 < \delta/2$ and $\text{supp}(\hat{X} + \hat{Y}) \subseteq D$. Then, since the maps $\tau_h^{-1}$ are measure preserving, we have

$$(DU(X) \circ \tau_h^{-1}, \hat{Y}) = (DU(X), Y \circ \tau_h).$$

Note that the inner product on the left is in $\bar{L}^2$ but the one on the right is in $L^2$. This will not be distinguished in our notation. Let $\hat{Z}_h := \hat{Y} \circ \tau_h - X + \bar{X} \circ \tau_h$. Then $|\hat{Z}_h|_2 < |\hat{Y}|_2 + |\hat{X} \circ \tau_h - X|_2 < \delta$ and since $\text{supp}(\hat{X} + \hat{Y}) \subseteq D$, we have $\text{supp}(X + Z_h) \subseteq D$. Moreover

$$\begin{align*}
(DU(X) \circ \tau_h^{-1} - DU(X) \circ \tau_h^{-1}, \hat{Y}) &= (DU(X), \hat{Z}_h) - (DU(X), \hat{Z}_{h'}) \\
&+ (DU(X), \bar{X} \circ \tau_h - X) + (DU(X), X - \bar{X} \circ \tau_{h'}) \\
&= + U(X + Z_h) - U(X) + (DU(X), \hat{Z}_h) - [U(X + Z_h) - U(X)] \\
&+ U(X + Z_{h'}) - U(X) - (DU(X), \hat{Z}_{h'}) - [U(X + Z_{h'}) - U(X)] \\
&+ (DU(X), \bar{X} \circ \tau_h - X) + (DU(X), X - \bar{X} \circ \tau_{h'}). 
\end{align*}$$

But as $\tau_h$ is measure preserving and $U$ and $\bar{U}$ only depend on the law, we have

$$U(X + Z_h) = U(\hat{Y} \circ \tau_h + \bar{X} \circ \tau_h) = \bar{U}(\hat{Y} + \bar{X}) = U(X + Z_{h'}).$$

Hence

$$|\langle DU(X) \circ \tau_h^{-1} - DU(X) \circ \tau_h^{-1}, \hat{Y} \rangle| \leq \frac{\epsilon}{2}|Z_{h'}|_2 + \frac{\epsilon}{2}|Z_h|_2 + 2|DU(X)|_2 \max(h, h')$$

$$\leq \epsilon |Y|_2 + \epsilon \max(h, h') + 2|DU(X)|_2 \max(h, h').$$

This means that

$$\begin{align*}
|DU(X) \circ \tau_h^{-1} - DU(X) \circ \tau_{h'}^{-1}|_2 \\
= \sup_{|Y|_2 = \delta/2} |\langle DU(X) \circ \tau_h^{-1} - DU(X) \circ \tau_{h'}^{-1}, \hat{Y} \rangle| / |Y|_2 \\
&\leq \epsilon + (2\epsilon + 4|DU(X)|_2) \max(h, h') \frac{\max(h, h')}{\delta}.
\end{align*}$$

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Since we can choose \( h, h' < \frac{\delta}{2} \) and also \( h, h' < \frac{\epsilon\delta}{4|DU(X)|_2} \) we have the required estimate and see that \((DU(X) \circ \tau_h^{-1})_{h>0}\) is a Cauchy sequence in \( L^2 \). Thus, there is \( \psi \in L^2 \) such that 

\[
DU(X) \circ \tau_h^{-1} \to \psi \quad \text{as} \quad h \searrow 0.
\]

The next step is to show that \( \tilde{U} \) is Fréchet differentiable at \( \tilde{X} \) and \( \psi = D\tilde{U}(\tilde{X}) \). To that end we note that \( \tilde{U}(\tilde{X} + \tilde{Y}) = U(X + Z_h) \) and

\[
(DU(X), \tilde{Y} \circ \tau_h) = (DU(X), Z_h) + (DU(X), X - \tilde{X} \circ \tau_h).
\]

Hence

\[
\begin{align*}
|\tilde{U}(\tilde{X} + \tilde{Y}) - \tilde{U}(\tilde{X}) - (\psi, \tilde{Y})| & = |U(X + Z_h) - U(X) - (DU(X), \tilde{Y} \circ \tau_h) + (DU(X), \tilde{Y} \circ \tau_h) - (\psi, \tilde{Y})| \\
& \leq \varepsilon|Z_h|_2 + |DU(X)|_2h + |DU(X) \circ \tau_h^{-1} - \psi||\tilde{Y}|_2 \leq 4\varepsilon|\tilde{Y}|_2,
\end{align*}
\]

for \( h \) sufficiently small. Thus \( \tilde{U} \) is differentiable at \( \tilde{X} \) and \( \psi = D\tilde{U}(\tilde{X}) \in L^2 \). \( \square \)

**Proof of Proposition A.2, part ii).** We first note that

\[
\mathcal{L}(X \circ \tau_h^{-1}, DU(X) \circ \tau_h^{-1}) = \mathcal{L}(X, DU(X))
\]

since the mapping \( \tau_h^{-1} \) is measure preserving. Moreover

\[
(X \circ \tau_h^{-1}, DU(X) \circ \tau_h^{-1}) \to (\bar{X}, D\bar{U}((\bar{X}))) \text{ in } L^2(\Omega; \mathbb{R}^{2d}) \text{ as } h \searrow 0.
\]

Hence we get that \( \mathcal{L}(X, DU(X)) = \mathcal{L}((\bar{X}, D\bar{U}(\bar{X}))) \). \( \square \)

**Proof of Proposition A.2, part iii).** Note that \( \mu \) is not necessarily atomless. We take \( \lambda \), the translation invariant measure on \( \mathcal{B}(S^1) \), with \( S^1 \) denoting the unit circle. The probability space \((D_k \times S^1, \mathcal{B}(D_k) \otimes \mathcal{B}(S^1), \mu \otimes \lambda)\) is atomless. Let \( L^2 \) denote the space of square integrable random variables on this probability space. The random variable \( \bar{X}(x, s) := x \) is in \( L^2 \) and has law \( \mu \). With the usual lift \( \tilde{U} \) we know, from part i), that \( D\tilde{U}(\bar{X}) \) exists in \( L^2 \).

Let \( \xi(x, s) := D\tilde{U}(\bar{X})(x, s) \). We see that

\[
\mathcal{L}(D\tilde{U}(\bar{X}))(B) = \mu \otimes \lambda(D\tilde{U}(\bar{X}) \in B)
\]

depends only on the law of \( X \) which is \( \mu \). So must \( D\tilde{U}(\bar{X})(x, s) \) not change with \( s \) and thus \( \xi(x, s) = \xi(x) \). Then

\[
1 = \mu \left( x \in D_k : \xi(x) = D\tilde{U}(\bar{X})(x) \right) = \mathbb{P}(\xi(X) = DU(X))
\]

since \( \mathcal{L}(X, DU(X)) = \mathcal{L}(\bar{X}, D\bar{U}(\bar{X})) \) due to part ii). Hence \( \xi(X) = DU(X) \mathbb{P}\text{-a.s.} \). \( \square \)
A.2 Higher-order derivatives

We observe that if \( \mu \) is fixed then \( \partial_\mu u(\mu) \) is a function from \( D_k \to D_k \). If, for \( y \in D_k, \partial_y [\partial_\mu u(\mu)(y)]_j \) exists for each \( j = 1, \ldots, d \) then \( \partial_y \partial_\mu u : \mathcal{P}(D_k) \times D_k \to D_k \times D_k \) is the matrix

\[
(\partial_y \partial_\mu u(y))_j = (\partial_y [\partial_\mu u(\mu)(y)]_j)_{j=1,\ldots,d}.
\]

If we fix \( y \in D_k \) then \( \partial_\mu u(\cdot)(y) \) is a function from \( \mathcal{P}(D_k) \to D_k \). Fixing \( j = 1, \ldots, d \), if \( \partial_\mu u(\cdot)(y)_j : \mathcal{P}(D_k) \to \mathbb{R} \) is L-differentiable at some \( \mu \) then its L-derivative is the function given by part iii) of Proposition A.2, namely

\[
\partial_\mu \partial_\mu u(\cdot)_j = \frac{\partial_\mu (\partial^2_\mu u(\mu,y)_j)}{\partial^2_\mu (\mu,y)} = \left. \frac{\partial^2_\mu u(\mu,y)_j}{\partial^2_\mu (\mu,y)} \right|_{\mu = \mu}.
\]

A.3 Itô formula for functions of measures

Assume we have a filtered probability space \((\Omega, \mathcal{F}, \mathbb{P})\) with filtration \((\mathcal{F}_t)_{t \geq 0}\) satisfying the usual conditions supporting an \((\mathcal{F}_t)_{t \geq 0}\)-Brownian motion \( w \) and adapted processes \( b \) and \( \sigma \) satisfying appropriate integrability conditions. We consider the Itô process

\[
dx_t = b_t dt + \sigma_t dw_t, \quad x_0 \in L^2(\mathcal{F}_0)
\]

which satisfies \( x_t \in D_k \) for all \( t \) a.s.

**Definition A.5.** We say that \( u : \mathcal{P}_2(D) \to \mathbb{R} \) is in \( C^{(1,1)}(\mathcal{P}_2(D)) \) if there is a continuous version of \( y \to \partial_\mu u(\mu)(y) \) such that the mapping \( \partial_\mu u : \mathcal{P}_2(D) \times D \to D \) is jointly continuous at any \((\mu, y)\) s.t. \( y \in \text{supp}(\mu) \) and such that \( y \to \partial_\mu u(\mu, y) \) is continuously differentiable and its derivative \( \partial_y \partial_\mu u : \mathcal{P}_2(D) \times D \to D \times D \) is jointly continuous at any \((\mu, y)\) s.t. \( y \in \text{supp}(\mu) \).

The notation \( C^{(1,1)} \) is chosen to emphasise that we can take one measure derivative which is again differentiable (in the usual sense) with respect to the new free variable that arises. Note that in [11] such functions are called partially \( C^2 \).

**Proposition A.6.** Assume that

\[
\mathbb{E} \int_0^\infty |b_t|^2 + |\sigma_t|^4 \, dt < \infty.
\]

Let \( u \) be in \( C^{(1,1)}(\mathcal{P}_2(D)) \) such that for any compact subset \( K \subset \mathcal{P}_2(D) \)

\[
\sup_{\mu \in K} \int_D \left[ |\partial_\mu u(\mu)(y)|^2 + |\partial_y \partial_\mu u(\mu)(y)|^2 \right] \mu(dy) < \infty.
\]

Then, for \( \mu_t := \mathcal{L}(x_t) \),

\[
u(\mu_t) = u(\mu_0) + \int_0^t \mathbb{E} \left[ b_s \partial_\mu u(\mu_s)(x_s) + \frac{1}{2} \text{tr} \left[ \sigma_s \sigma^*_s \partial_y \partial_\mu u(\mu_s)(x_s) \right] \right] ds.
\]

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Note that since we are assuming that the process $x$ never leaves some $D_k$, we have $\text{supp}(\mu_t) \subset D_k$ for all times $t$. The proof relies on replacing $\mu_t$ by an approximation arising as the empirical measure of $N$ independent copies of the process $x$. For marginal empirical measures there is a direct link between measure derivatives and partial derivatives, see [11, Proposition 3.1]. One can then apply classical Itô formula to the approximating system of independent copies of $x$ and take the limit. This is done in [11, Theorem 3.5].

Proposition A.6 can be used to derive an Itô formula for a function which depends on $(t, x, \mu)$.

**Definition A.7.** By $C^{1,2,1,1}([0, \infty) \times D \times \mathcal{P}_2(D))$ we denote the functions $v = v(t, x, \mu)$ such that $v(\cdot, \cdot, \mu) \in C^{1,2}([0, \infty) \times D)$ for each $\mu$, and such that $v(t, x, \cdot)$ is in $C^{(1,1)}(\mathcal{P}_2(D))$ for each $(t, x)$. Moreover all the resulting (partial) derivatives must be jointly continuous in $(t, x, \mu)$ or $(t, x, \mu, y)$ as appropriate.

Finally, by $C^{2,1,1}(D \times \mathcal{P}_2(D))$ we denote the subspace of $C^{1,2,1,1}([0, \infty) \times D \times \mathcal{P}_2(D))$ of functions $v$ that are constant in $t$.

To conveniently express integrals with respect to the laws of the process taken only over the “new” variables arising in the measure derivative we introduce another probability space $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})$ a filtration $(\tilde{\mathcal{F}}_t)_{t \geq 0}$ and processes $\tilde{\omega}, \tilde{b}, \tilde{\sigma}$ and a random variable $\tilde{x}_0$ on this probability space such that they have the same laws as $\omega, b, \sigma$ and $x_0$. We assume $\tilde{\omega}$ is a Wiener process. Then

$$d\tilde{x}_t = \tilde{b}_t dt + \tilde{\sigma}_t d\tilde{\omega}_t, \quad \tilde{x}_0 \in L^2(\tilde{\mathcal{F}}_0)$$

is another Itô process which satisfies $\tilde{x}_t \in D_k$ for all $t$ a.s. Moreover if we now consider the probability space $(\Omega \times \tilde{\Omega}, \mathcal{F} \otimes \tilde{\mathcal{F}}, \mathbb{P} \otimes \tilde{\mathbb{P}})$ then we see that the processes with and without tilde are independent on this new space.

**Proposition A.8 (Itô formula).** Assume that

$$\mathbb{E} \int_0^\infty |b_t|^2 + |\sigma_t|^4 dt < \infty.\]

Let $v \in C^{1,2,1,1}([0, \infty) \times D \times \mathcal{P}_2(D))$ such that for any compact subset $K \subset \mathcal{P}_2(D)$

$$\sup_{t,x,\mu \in K} \int_D \left[ |\partial_{\mu} v(t, x, \mu)(y)|^2 + |\partial_y \partial_{\mu} v(t, x, \mu)(y)|^2 \right] \mu(dy) < \infty. \quad (A.2)$$

Then, for $\mu_t := \mathcal{L}(\tilde{x}_t)$,

$$v(t, x_t, \mu_t) - v(0, x_0, \mu_0) = \int_0^t \left[ \partial_t v(s, x_s, \mu_s) + \frac{1}{2} \text{tr} \left( \sigma_s \sigma_s^\top \partial^2_{xx} v(s, x_s, \mu_s) \right) \right] ds$$

$$+ \int_0^t b_s \partial_x v(s, x_s, \mu_s) dw_s$$

$$+ \int_0^t \mathbb{E} \left[ b_s \partial_{\mu} v(s, x_s, \mu_s)(\tilde{x}_s) + \frac{1}{2} \text{tr} \left( \tilde{\sigma}_s \tilde{\sigma}_s^\top \partial_y \partial_{\mu} v(s, x_s, \mu_s)(\tilde{x}_s) \right) \right] ds.$$
Here we follow the argument from [8] explaining how to go from an Itô formula for function of measures only, i.e. from Proposition A.6, to the general case. Note that it is possible to assume that \( \tilde{w}, \tilde{b}, \tilde{\sigma} \) and \( \tilde{x}_0 \) have the same laws as \( w, b, \sigma \) as \( x_0 \) above, but in fact this is not necessary. In this paper this generality is needed in the proof of Lemma 2.10.

**Outline of proof for Proposition A.8.** Fix \( (\bar{t}, \bar{x}) \) and apply Proposition A.6 to the function \( u(\mu) := v(\bar{t}, \bar{x}, \mu) \) and the law \( \mu_t := \mathcal{L}(\bar{x}_t) \). Then

\[
\begin{align*}
v(\bar{t}, \bar{x}, \mu_t) - v(\bar{t}, \bar{x}, \mu_0) &= \int_0^t \tilde{E} \left[ \tilde{b}_s \partial_\mu v(\bar{t}, \bar{x}, \mu_s)(\bar{x}_s) + \frac{1}{2} \text{tr} \left[ \tilde{\sigma}_s \tilde{\sigma}_s^* \partial_\mu^2 v(\bar{t}, \bar{x}, \mu_s)(\bar{x}_s) \right] \right] ds \\
&=: \int_0^t M(\bar{t}, \bar{x}, \mu_s) ds.
\end{align*}
\]

We thus see that the map \( t \mapsto v(\bar{t}, \bar{x}, \mu_t) \) is absolutely continuous for all \( (\bar{t}, \bar{x}) \) and so for almost all \( t \) we have \( \partial_t v(\bar{t}, \bar{x}, \mu_t) = M(\bar{t}, \bar{x}, \mu_t) \). Note that for completeness we would need to use the definition of \( C^{1,2,0} \) functions and a limiting argument to get the partial derivative for all \( t \). See the proof of the corresponding Itô formula in [11]. We now consider \( \bar{v} \) given by \( \bar{v}(t, x) := v(t, x, \mu_t) \). Then

\[
\partial_t \bar{v}(t, x) = (\partial_t v)(t, x, \mu_t) + M(t, x, \mu_t).
\]

Using the usual Itô formula we then have

\[
\begin{align*}
\bar{v}(t, x_t) - \bar{v}(0, x_0) &= \int_0^t \left[ \partial_t v(s, x_s, \mu_s) + M(s, x_s, \mu_s) + \frac{1}{2} \text{tr} \left[ \sigma_t \sigma_t^* \partial_\mu^2 v(s, x_s, \mu_s) \right] \right] ds \\
&\quad + \int_0^t \tilde{b}_s \partial_\mu v(s, x_s, \mu_s) \, dw_s.
\end{align*}
\]

\[\square\]

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