The structure of the minimum size supertail of a subspace partition

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Abstract Let \( V = V(n, q) \) denote the vector space of dimension \( n \) over the finite field with \( q \) elements. A subspace partition \( \mathcal{P} \) of \( V \) is a collection of nontrivial subspaces of \( V \) such that each nonzero vector of \( V \) is in exactly one subspace of \( \mathcal{P} \). For any integer \( d \), the \( d \)-supertail of \( \mathcal{P} \) is the set of subspaces in \( \mathcal{P} \) of dimension less than \( d \), and it is denoted by \( \text{ST} \). Let \( \sigma_q(n, t) \) denote the minimum number of subspaces in any subspace partition of \( V \) in which the largest subspace has dimension \( t \). It was shown by Heden et al. that \( |\text{ST}| \geq \sigma_q(d, t) \), where \( t \) is the largest dimension of a subspace in \( \text{ST} \). In this paper, we show that if \( |\text{ST}| = \sigma_q(d, t) \), then the union of all the subspaces in \( \text{ST} \) constitutes a subspace under certain conditions.

Keywords Vector space partition · Subspace partition · Partial \( d \)-spread

Mathematics Subject Classification 51E14 · 51E23

1 Introduction

Let \( V = V(n, q) \) denote a vector space of dimension \( n \) over the finite field with \( q \) elements. We use the term \( d \)-subspace to refer to a subspace of dimension \( d \). For any subspace \( U \) of \( V \), we let \( U^* \) denote the set of nonzero vectors in \( U \). A subspace partition \( \mathcal{P} \) of \( V \), also known as a vector space partition, is a collection of nontrivial subspaces of \( V \) such that...
each vector of $V^*$ is in exactly one subspace of $P$ (e.g., see Heden [12] for a survey). The study of subspace partitions originated from the general problem of partitioning a finite (not necessarily abelian) group into subgroups that only intersect at the identity element (e.g.; see Zappa [21] for a survey). Subspace partitions can be used to construct translation planes, error-correcting codes, orthogonal arrays, and designs (e.g., see [1–3,7,10,17–19]).

Suppose that there are $m$ distinct dimensions, $d_1 < d_2 < \cdots < d_m$, that occur in a subspace partition $P$, and let $n_d$ denote the number of $d$-subspaces in $P$. Then the expression $[d_1^{nd_1}, \ldots, d_m^{nd_m}]$ is called the type of $P$. The general problem in this area is to find necessary and sufficient conditions for the existence of a subspace partition of $V$ of a given type (e.g., see [4,6,8,9,13,19] for the solution of some special cases). Two obvious necessary conditions for the existence of a subspace partition of type $[d_1^{nd_1}, \ldots, d_m^{nd_m}]$ are the packing condition

$$\sum_{i=1}^{m} n_{d_i}(q^{d_i} - 1) = q^n - 1,$$

and the dimension condition

$$\begin{aligned}
&n \geq d_i + d_j \quad \text{if } n_{d_i} + n_{d_j} \geq 2 \text{ and } i \neq j; \quad \text{and} \\
n \geq 2d_i \quad \text{if } n_{d_i} \geq 2.
\end{aligned}$$

To the best of our knowledge, there are not many other known necessary conditions for the existence of a subspace partition $P$ of $V$. Heden and Lehmann [13] derived some necessary conditions (see Lemma 10) by essentially counting in two ways tuples of the forms $(\sigma_1,\ldots,\sigma_t)$ in which the largest subspace has dimension $\sigma_t$.

Let $\mathcal{P}$ be a subspace partition of $V = V(n, q)$ of type $[d_1^{nd_1}, \ldots, d_m^{nd_m}]$. For any integer $d$ such that $d_1 < d \leq d_m$, the $d$-supertail of $\mathcal{P}$ is the set of subspaces in $\mathcal{P}$ of dimension less than $d$, and it is denoted by $ST$. The size of a subspace partition $\mathcal{P}$ is the number of subspaces in $\mathcal{P}$. For $1 \leq t < n$, let $\sigma_q(n, t)$ denote the minimum size of any subspace partition of $V$ in which the largest subspace has dimension $t$. The exact value of $\sigma_q(n, t)$ is given by the following theorem (see André [1] and Beutelspacher [3] for $n \pmod{t} \equiv 0$, and see [14,20] for $n \pmod{t} \neq 0$).

**Theorem 1** Let $n, k, t, and r$ be integers such that $0 \leq r < t, k \geq 1, and n = kt + r$. Then

$$\sigma_q(n, t) = \begin{cases} 
q^{kt} - 1 & \text{for } r = 0, \\
q^t - 1 & \text{for } r \geq 1 \text{ and } 3 \leq n < 2t, \\
q^{t+r} \sum_{i=0}^{k-2} q^{it} + q^{\lfloor \frac{t+r}{2} \rfloor} + 1 & \text{for } r \geq 1 \text{ and } n \geq 2t.
\end{cases}$$

The following theorem of Heden et al. [15] generalizes a theorem of Heden [11, Theorem 1].

**Theorem 2** Let $\mathcal{P}$ be a subspace partition of $V(n, q)$ of type $[d_1^{nd_1}, \ldots, d_m^{nd_m}]$ and let $2 \leq s \leq m$. If $ST$ is $d_s$-supertail of $\mathcal{P}$, then

$$|ST| \geq \sigma_q(d_s, d_{s-1}).$$
If equality holds in (3), then Theorem 2 has the following interesting corollary (see [15]).

**Corollary 3** If $|ST| = \sigma_q(d_1, d_s-1)$ and $d_s \geq 2d_{s-1}$, then the union of the subspaces in $ST$ forms a $d_s$-subspace.

Note that the crucial part of the conclusion of Corollary 3 is that the set of all points covered by the subspaces in $ST$ is a subspace. (In general, the $d_s$-supertail of a subspace partition of $V(n, q)$ need not be a subspace.) One outstanding question that remains is whether the conclusion of Corollary 3 holds for $d_{s-1} < d_s < 2d_{s-1}$. For the special case when $ST$ is a simple tail, Heden [11, Theorem 3] proved the following theorem.

**Theorem 4** Let $P$ be a subspace partition of $V(n, q)$ of type $[d_1^{p_1}, \ldots, d_m^{p_m}]$. If $ST$ is the tail of $P$ (i.e., $ST$ is the set of $d_1$-subspaces) such that $|ST| = q^{d_1} + 1$ and $d_2 < 2d_1$, then $ST$ is a $d_1$-spread (i.e., a subspace partition consisting of $d_1$-subspaces).

In this paper we prove the following generalization of Theorem 4.

**Theorem 5** Let $P$ be a subspace partition of $V(n, q)$ of type $[d_1^{p_1}, \ldots, d_m^{p_m}]$. Let $2 \leq s \leq m$, and suppose $ST$ is a $d_s$-supertail of $P$ such that $|ST| = \sigma_q(d_s, d_{s-1})$ and $d_s < 2d_{s-1}$. Furthermore, assume that one of the following conditions holds.

(i) $s - 1 \leq 2$, that is $ST$ contains subspaces of at most 2 different dimensions.
(ii) $d_s = 2d_{s-1} - 1$.
(iii) All the subspaces in $P \setminus ST$ have the same dimension.

Then the union of the subspaces in $ST$ forms a subspace $W$. Moreover,

(a) $s - 1 = 1$, $n_1 = q^{d_1} + 1$, and dim $W = 2d_1$, or
(b) $s - 1 = 2$, $n_1 = q^{d_2}, n_2 = 1$, and dim $W = d_1 + d_2$.

The following result is a consequence of Theorem 2, Corollary 3, and Theorem 5.

**Corollary 6** Let $P$ be a subspace partition of $V(n, q)$ of type $[d_1^{p_1}, \ldots, d_m^{p_m}]$ and let $2 \leq s < m$. Let $ST$ be the $d_s$-supertail of $P$, let $\hat{ST}$ be its $d_{s+1}$-supertail, and assume that $|ST| = \sigma_q(d_s, d_{s-1})$.

(i) If $2 \leq s \leq 3$, $d_s < 2d_{s-1}$, and $d_{s+1} < 2d_s$, then

$$|\hat{ST}| \geq \sigma_q(d_{s+1}, d_s) + \sigma_q(d_s, d_{s-1}).$$

(ii) If $d_s \geq 2d_{s-1}$, or if $s = 3$ and $d_3 = d_2 + d_1$, then

$$|\hat{ST}| \geq \sigma_q(d_{s+1}, d_s) + \sigma_q(d_s, d_{s-1}) - 1.$$

**Remark 7** Note that the condition $s = 3$ in Corollary 6(ii) is equivalent to saying that $ST$ contains subspaces of two different dimensions, namely $d_1$ and $d_2$.

Theorem 5 can be viewed as a special case of the general question of determining nontrivial conditions under which a set of points of a projective space forms a subspace. We conjecture that Theorem 5 holds in all cases, and not just if (i), (ii), or (iii) holds.

The rest of the paper is organized as follows. In Sect. 2, we gather some known results that we shall use in Sect. 3 to first establish some auxiliary results, and then to prove our main results, i.e., Theorem 5 and Corollary 6. Finally, we include some supporting lemmas in the Appendix section.
# 2 Preliminaries

Let \( n \) be a positive integer and let \( q \) be a prime power. Set \( \Theta_0 = 0 \). For any integer \( i \) such that \( 1 \leq i \leq n \), let

\[
\Theta_i = \frac{q^i - 1}{q - 1}
\]

denote the number of points (i.e., 1-subspaces) in an \( i \)-subspace.

We will need the following elementary results (Propositions 8 and 9).

**Proposition 8** The number of hyperplanes containing a given \( d \)-subspace of \( V(n, q) \) is \( \Theta_{n-d} \).

**Proposition 9** If \( U \) is a subset of \( V = V(n, q) \) containing \( \Theta_d \) points and contained in precisely \( \Theta_{n-d} \) hyperplanes, then \( U \) is a \( d \)-subspace of \( V \).

To state the next lemmas, we need the following definitions. For \( n \geq 2 \), let \( \mathcal{P} \) be a subspace partition of \( V = V(n, q) \) of type \([d_1^{n_{d_1}}, \ldots, d_m^{n_{d_m}}]\). For any hyperplane \( H \) of \( V \), let \( b_{H,x} \) be the number of \( x \)-subspaces in \( \mathcal{P} \) that are contained in \( H \) and set \( b_H = [b_{H,d_1}, \ldots, b_{H,d_m}] \). Define

\[
\mathcal{B} = \{ b_H : H \text{ is a hyperplane of } V \}.
\]

For any \( b \in \mathcal{B} \), let \( s_b \) denote the number of hyperplanes \( H \) of \( V \) such that \( b_H = b \).

We will use Lemmas 10 and 11 by Heden and Lehmann [13].

**Lemma 10** Let \( \mathcal{P} \) be a subspace partition of \( V(n, q) \), and let \( \mathcal{B} \) and \( s_b \) be as defined earlier. If \( \mathcal{P} \) contains two different subspaces, one of dimension \( d \) and another of dimension \( d' \), with \( 1 \leq d, d' \leq n - 2 \), then

(i) \( \sum_{b \in \mathcal{B}} s_b = \Theta_n \),

(ii) \( \sum_{b \in \mathcal{B}} b_d s_b = n_d \Theta_{n-d} \),

(iii) \( \sum_{b \in \mathcal{B}} \binom{b_d}{2} s_b = \binom{n_d}{2} \Theta_{n-2d} \),

(iv) \( \sum_{b \in \mathcal{B}} b_d b_{d'} s_b = n_d n_{d'} \Theta_{n-d-d'} \).

**Lemma 11** Let \( \mathcal{P} \) be a subspace partition of \( V(n, q) \). If \( H \) is a hyperplane of \( V \), then

\[
|\mathcal{P}| = 1 + \sum_{i=1}^{m} b_{H,d_i} q^{d_i}.
\]

We will also use the following lemma due to Heden et al. [14].

**Lemma 12** Let \( \mathcal{P} \) be a subspace partition of \( V(n, q) \) of type \([d_1^{n_{d_1}}, \ldots, d_m^{n_{d_m}}]\) and let \( 2 \leq s \leq m \). If \( ST \) is a \( d_s \)-supertail of \( \mathcal{P} \) and \( H \) is a hyperplane of \( V \), then

\[
\sum_{i=1}^{s-1} (n_{d_i} - b_{H,d_i}) q^{d_i} = c_H \cdot q^{d_s},
\]

where \( c_H = q^{n-d_s} - \sum_{i=s}^{m} (n_{d_i} - b_{H,d_i}) q^{d_i-d_s} \) is a nonnegative integer.
Finally, we will need the following lemma due to Herzog and Schönheim [17], and independently Beutelspacher [3] and Bu [6].

**Lemma 13** Let $n$ and $d$ be integers such that $1 \leq d \leq n/2$. Then $V(n, q)$ admits a partition with one subspace of dimension $n - d$ and $q^{n-d}$ subspaces of dimension $d$.

### 3 Auxiliary results and the proof of the main theorem

In this section, we use $\mathcal{H}$ to denote the set of all hyperplanes of $V$.

**Lemma 14** Let $\mathcal{P}$ be a subspace partition of $V = V(n, q)$ of type $[d_1^{n_1}, \ldots, d_m^{n_m}]$, where $1 \leq d_1 < \ldots < d_m$. Assume that $2 \leq s \leq m$, and let $ST$ be a $d_s$-supertail of $\mathcal{P}$ such that $|ST| = \sigma_q(d_s, d_{s-1})$ and $d_s < 2d_{s-1}$. Then $d_s \leq d_{s-1} + d_1$.

**Proof** Suppose that $d_s > d_{s-1} + d_1$. Let $U, W \in ST$ be such that $\dim U = d_{s-1}$ and $\dim W = d_1$. Let $B_W$ be a basis of $W$, $B_U$ a basis of $U$, and consider a basis $B$ of $V$ obtained by extending $B_U \cup B_W$ is a subspace of $V$ such that $\dim V' = n-d_1$, $U \subseteq V'$, and $V' \cap W = \emptyset$. Now let $\mathcal{P}'$ be the subspace partition induced by $\mathcal{P}$ in $V'$, and let $ST' = \{A \cap V' \neq \{0\}: A \in ST\}$, where $\emptyset$ denotes the zero vector. Let $X'$ be a subspace in $\mathcal{P}' \setminus ST'$ that has a minimum possible dimension. Since $X' = X \cap V'$ for some $X \in \mathcal{P} \setminus ST$ and $\dim X \geq d_s > d_{s-1} + d_1$, it follows that $\dim X' \geq \dim X - d_1 > d_{s-1}$. Moreover, the subspace $U' = U \cap V' = U$ is in $ST'$ and $W' = W \cap V' = \{0\}$. Thus, $ST'$ is a supertail of $\mathcal{P}'$ with highest dimension $d_{s-1} = \dim U'$ and of size $|ST'| \leq |ST \setminus \{W\}| \leq |ST| - 1 = q^{d_{s-1}}$. This contradicts the fact that $|ST'| \geq \sigma_q(\dim X', d_{s-1}) = q^{d_{s-1}} + 1$. The lemma follows. □

**Lemma 15** Let $\mathcal{P}$ be a partition of $V(n, q)$ with supertail $ST$ consisting of subspaces of dimensions at most $t$. For any $H \in \mathcal{H}$, let $\beta_H = \sum_{i \leq t} b_{H,i} q^i$ and let $\beta_0 = \min_{H \in \mathcal{H}} \beta_H$. Then $|ST| \geq \beta_0 + 1$.

**Proof** Suppose that $|ST| \leq \beta_0$. Then, applying the definition of $\beta_H$, implies that

$$\Theta_n|ST| \leq \Theta_n \beta_0 = \sum_{H \in \mathcal{H}} \beta_0 \leq \sum_{H \in \mathcal{H}} \beta_H = \sum_{H \in \mathcal{H}} \sum_{i=1}^t b_{H,i} q^i,$$

and thus with the use of Lemma 10, we obtain

$$\Theta_n|ST| \leq \sum_{i=1}^t q^i \left(\sum_{H \in \mathcal{H}} b_{H,i}\right) \leq \sum_{i=1}^t q^i (n_i \Theta_{n-i}) \leq \sum_{i=1}^t n_i (\Theta_n - \Theta_i) \leq \Theta_n|ST| - \sum_{i=1}^t n_i \Theta_i,$$

which is a contradiction since $\sum_{i=1}^t n_i \Theta_i$ is the number of points in $ST$, and this number is positive. □

**Lemma 16** Let $\mathcal{P}$ be a partition of $V(n, q)$ with a $d$-supertail $ST$ such that $t$ is the maximum dimension of any subspace in $ST$ and $d < 2t$. For any $H \in \mathcal{H}$, let $\beta_H = \sum_{i=d_1}^t b_{H,i} q^i$ and $\beta_0 = \min_{H \in \mathcal{H}} \beta_H$. If $ST$ has size $q^t + 1$, then $\beta_0 = q^t$. Moreover, there exists an integer $c_0$ such that

$$\sum_{i=1}^t n_i \Theta_i = \frac{c_0 q^d - 1}{q - 1}.$$
Proof Let \( a \) be the minimum dimension of any subspace in \( ST \). First, suppose that for some \( H \in \mathcal{H} \), we have \( \beta_H < q^t \). Then by Lemma 12, there exists an integer \( c_H \) such that \( \sum_{i=a}^{t} (n_i - b_{H,i})q^i = c_Hq^d \). Hence, we have

\[
\sum_{i=a}^{t} n_i q^{i-a} = c_Hq^{d-a} + \beta_Hq^{-a}.
\]

Since \( \beta_H < q^t \), we obtain \( \beta_Hq^{-a} < q^{t-a} \). Then, it follows from [15, Proof of Proposition 6(Case 2)] that \(|ST| > q^t + 1\). This is a contradiction and thus \( \beta_H \geq q^t \) for all \( H \in \mathcal{H} \).

By Lemma 15, \( \beta_0 \leq |ST| - 1 = q^t \), and thus \( \beta_0 = q^t \).

Now by Lemma 12, there exists an integer \( c_0 \) such that \( \sum_{i=a}^{t} n_i q^{i-a} = c_0q^d \). Since \( \beta_0 = q^t \) and \( \sum_{i=a}^{t} n_i = |ST| = q^t + 1 \), it follows after some arithmetic that

\[
\sum_{i=a}^{t} n_i \Theta_i = \frac{c_0q^d - 1}{q - 1}.
\]

Let \( \mathcal{P} \) be a partition of \( V(n, q) \) with a \( d \)-supertail \( ST \). If \( ST \) has type \( [t^{q^t+1}] \), then we recall that it follows from Heden [11, Theorem 3] that the union of the subspaces in \( ST \) is a \( 2t \)-subspace. The following lemma is an extension of that result.

**Lemma 17** Let \( \mathcal{P} \) be a partition of \( V(n, q) \) with a \( d \)-supertail \( ST \) of type \( [t^q, a^q] \), i.e., \( ST \) contains one subspace of dimension \( t \) and \( q^t \) subspaces of dimension \( a \), where \( t > a \). Then the union of the subspaces in \( ST \) forms a \( t + a \)-subspace.

Proof Recall that \( \mathcal{H} \) denotes the set of all hyperplanes of \( V \). Let \( H \in \mathcal{H} \) be any hyperplane. It follows from Lemma 12 that there exists an integer \( c_H \geq 0 \) such that

\[
(n_t - b_{H,t})q^t + (n_a - b_{H,a})q^a = c_Hq^d.
\]

Thus,

\[
b_{H,a} = q^t + (1 - b_{H,t})q^{t-a} - c_Hq^{d-a},
\]

where \( 0 \leq b_{H,a} \leq q^t \) and \( b_{H,t} \in \{0, 1\} \). Let \( A \) be the set of \( a \)-subspaces in \( ST \), and let \( \alpha_i \) denote the number of hyperplanes in \( V \) that contain exactly \( i \) members of \( A \). If \( \alpha_i \neq 0 \), then there exists a hyperplane \( H \in \mathcal{H} \) that contains exactly \( b_{H,a} = i \) members from \( A \). Thus, it follows from (5) that

\[
\alpha_i \neq 0 \Rightarrow q^{t-a} \text{ divides } i.
\]

Define the integers \( x, y, \) and \( z \) as follows:

\[
x = \sum_{i=q^{t-a}}^{q^t} i\alpha_i, \quad y = \sum_{i=q^{t-a}}^{q^t} \left(\begin{array}{c} i \\ 2 \end{array}\right) \alpha_i, \quad z = \sum_{i=q^{t-a}}^{q^t} \alpha_i.
\]

Then it follows from Lemma 10 that

\[
x = \sum_{i=q^{t-a}}^{q^t} i\alpha_i = n_a \Theta_{n-a}
\]
and
\[ y = \sum_{i=q^{t-a}}^{q^t} \binom{i}{2} \alpha_i = \binom{n_a}{2} \Theta_{n-2a}. \]  
(8)

Since \( \sum_{i=0}^{q^t} \alpha_i = |\mathcal{H}| = \Theta_n \), it follows by (6) and the definition of \( z \) that
\[ z = \sum_{i=q^{t-a}}^{q^t} \alpha_i = \Theta_n - \alpha_0. \]  
(9)

Using (7)–(9) and the fact that \( n_a = q^t \) and \( \Theta_i = (q^i - 1)/(q - 1) \), we obtain
\[ \sum_{i=q^{t-a}}^{q^t} \alpha_i(i - q^{t-a})(i - q^t) \]
\[ = 2y + (1 - q^{t-a} - q^t)x + q^{2t-a}z \]
\[ = q^t(q^t - 1)\Theta_{n-2a} + (1 - q^{t-a} - q^t)q^t\Theta_{n-a} + q^{2t-a}(\Theta_n - \alpha_0) \]
\[ = \Theta_{n+t-a} - \Theta_{n+t-2a} - q^{2t-a}\alpha_0. \]  
(10)

Note that
\[ (i - q^{t-a})(i - q^t) \begin{cases} 
0 & \text{if } i = q^{t-a}, \\
< 0 & \text{if } q^{t-a} < i < q^t, \\
0 & \text{if } i = q^t. 
\end{cases} \]  
(11)

Thus, it follows from (10) and (11) that
\[ \sum_{i=q^{t-a}}^{q^t} \alpha_i(i - q^{t-a})(i - q^t) = \Theta_{n+t-a} - \Theta_{n+t-2a} - q^{2t-a}\alpha_0 \leq 0, \]  
(12)

and it follows from (12) that
\[ \alpha_0 \geq \Theta_{n-t} - \Theta_{n-t-a}. \]  
(13)

Furthermore, since \( b_{H,t} \in \{0, 1\} \), it follows from (5) that for any hyperplane \( H \), we have
\[ b_{H,a} = 0 \Rightarrow b_{H,t} = 1. \]  
(14)

Hence, if \( W_t \) is a \( t \)-subspace and \( W_a \) is an \( a \)-subspace in the supertail \( ST \), then each of the \( \alpha_0 \) hyperplanes that contain no \( a \)-subspace is a hyperplane that contains \( W_t \) but not \( W_a \). As there are \( \theta_{n-t} - \theta_{n-t-a} \) such hyperplanes, it follows that
\[ \alpha_0 = \theta_{n-t} - \theta_{n-t-a}. \]

Since we considered any \( a \)-subspace of \( ST \), the argument shows that the \( \theta_{n-t-a} \) hyperplanes that contain some \( W_t \) must indeed contain all the \( a \)-spaces of \( ST \). Thus all the subspaces of the supertail must be contained in the intersection \( T \) of these \( \theta_{n-t-a} \) hyperplanes. Moreover, since \( \theta_{n-t-a} \) hyperplanes intersect in a subspace of dimension at most \( t + a \), it follows that \( T \) has dimension \( t + a \) and is thus partitioned by the supertail.

We are now ready to prove our main theorem.
Proof of Theorem 5 Let \( \mathcal{P} \) be a partition of \( V(n, q) \) with a \( d_s \)-supertail \( ST \) of size \( |ST| = \sigma_q(d_s, d_{s-1}) = q^{d_{s-1}} + 1 \). To simplify the notation, we set \( d = d_s \) and \( t = d_{s-1} \). Let \( k \) and \( r_d \) be integers such that \( k \geq 1, n = k d + r_d \), and \( 1 \leq r_d < d \). Recall that if \( d \geq 2t \), then Corollary 3 holds. So we may assume that \( r_t = d - t \) satisfies \( 0 < r_t < t \).

Since \( \mathcal{P} \) contains subspaces of dimensions \( d \) and \( t \), it follows that \( n \geq d + t \). We now show that \( n \geq 2d \). By way of contradiction, assume that \( n < 2d \). Then the dimension condition (see (2) in Sect. 1) implies that \( \mathcal{P} \) contains at most one \( d \)-subspace. Thus, \( \mathcal{P} \) contains exactly one \( d \)-subspace, \( Y \), and its \( d \)-supertail is \( ST = \mathcal{P} \setminus \{ Y \} \). Since \( n \geq d + t \), \( |ST| = q^t + 1 \) and the maximum dimension of any subspace in \( ST \) is (by definition) \( t \), it follows that

\[
q^t + 1 = |ST| \geq \frac{\sum_{X \in ST} |X^*|}{|V(t, q)^*|} = \frac{|V(n, q)^*| - |Y^*|}{|V(t, q)^*|} = \frac{(q^n - 1) - (q^d - 1)}{q^t - 1} \geq q^d, \quad (15)
\]

which is a contradiction since \( d > t \) and \( q \geq 2 \). Thus, a \( d \)-supertail of a partition \( \mathcal{P} \) of \( V(n, q) \) cannot be of minimum size \( \sigma_q(d, t) = q^t + 1 \) if \( n < 2d \). So we may assume that \( n \geq 2d \), i.e., \( k \geq 2 \) (which we will use in the proof of part (iii) below). We now prove the theorem for each of the three conditions (i), (ii), and (iii) stated in the theorem.

(i) Suppose the supertail \( ST \) contains subspaces of at most two different dimensions \( d_1 = a \) and \( d_{s-1} = t \) such that \( t > a \). Since \( n_t \) denotes the number of \( i \)-subspaces, we have

\[
n_t + n_a = |ST| = q^t + 1, \quad (16)
\]

where \( n_t > 0 \) and \( n_a \geq 0 \). Moreover, since \( d = t + r_t \), Lemma 16 yields

\[
n_t \Theta_t + n_a \Theta_a = \frac{c_0 q^d - 1}{q - 1}, \quad (17)
\]

where \( c_0 \) is a positive integer. Since \( \Theta_i = (q^i - 1)/(q - 1) \), it follows from (16) and (17) that

\[
n_t(q^{t-a} - 1) = q^t(c_0 q^{r_i-a} - 1) + q^{t-a} - 1 \Rightarrow n_t = \frac{q^t(c_0 q^{r_i-a} - 1)}{q^{t-a} - 1} + 1.
\]

Since \( \gcd(q^t, q^{t-a} - 1) = 1 \), the above equation implies that \( q^{t-a} - 1 \) divides \( c_0 q^{r_i-a} - 1 \). Hence \( n_t = q^t \cdot x + 1 \), where \( x = \frac{c_0 q^{r_i-a} - 1}{q^{t-a} - 1} \) is either 0 or 1 since \( n_t \leq q^t + 1 \). If \( x = 0 \), then \( n_t = 1 \) and \( n_a = q^t \). In this case, it follows from Lemma 17 that the union of the subspaces in \( ST \) is a subspace of dimension \( t + a = d_{s-1} + a \). If \( x = 1 \), then \( n_t = q^t + 1 \) and \( n_a = 0 \). In this case, \( ST \) contains only subspaces of dimension \( t = d_{s-1} \) and Theorem 4 implies that \( ST \) is a \( d_{s-1} \)-spread.

(ii) If \( t - r_t = 1 \), then it follows from Lemma 14 that the smallest dimension in \( ST \) is \( d_1 \geq r_t = t - 1 \). Thus \( ST \) contains subspaces of at most two different dimensions, namely \( t \) and \( t - 1 \). Now the main theorem follows from Theorem 4 and part (i) above.

(iii) Recall that \( d = d_s, t = d_{s-1}, r_t = d - t \), and \( n \geq 2d \). Moreover, \( k \) and \( r_d \) are integers such that \( k \geq 2, n = k d + r_d \), and \( 1 \leq r_d < d \). Let \( \ell = q^{kd} \sum_{i=0}^{k-2} q^{id} \), and let \( \mathcal{P} \) be a partition of \( V(n, q) \) with a \( d \)-supertail \( ST \) of minimum size \( q^t + 1 \). Then \( \mathcal{P} \setminus ST \) must be a partial \( d \)-spread. Note that

\[
\ell = q^{kd} \sum_{i=0}^{k-2} q^{id} \Rightarrow \ell q^d = \frac{q^d(q^{n-d} - q^{d-a})}{q^d - 1}. \quad (18)
\]
Let $\mu_q(n, d)$ denote the maximum number of $d$-subspaces in any partial $d$-spread of $V(n, q)$. Then the upper bound given by Drake-Freeman [7, Corollary 8] implies that

$$\mu_q(n, d) < \ell q^d + \frac{q^{r_d} + q^{r_d - 1}}{2} + 1. \quad (19)$$

Since $n = kd + r_d$ and the maximum dimension in $P$ is $d$, it follows from Theorem 1 that

$$|P| \geq \ell q^d + q^{\lfloor \frac{d + r_d}{2} \rfloor} + 1. \quad (20)$$

By definition of $P$ and $ST$, it follows that

$$|P| = |P \setminus ST| + |ST| = n_d + q^t + 1. \quad (21)$$

Hence, (19), (20), and (21) yield

$$q^{\lfloor \frac{d + r_d}{2} \rfloor} - q^t \leq n_d - \ell q^d < \frac{q^{r_d} + q^{r_d - 1}}{2} + 1. \quad (22)$$

Since $0 \leq r_d < d$, $d = t + r_t$, and $1 \leq r_t < t$, it follows that

$$-q^d < q^{\lfloor \frac{d + r_d}{2} \rfloor} - q^t \quad \text{and} \quad \frac{q^{r_d} + q^{r_d - 1}}{2} < q^d. \quad (23)$$

Thus, (22) and (23) yield

$$-q^d < n_d - \ell q^d < q^d. \quad (24)$$

Next, it follows from Lemma 16 that there exists some integer $c_0$ such that the number of vectors in $\bigcup_{X \in ST} X$ is equal to

$$\sum_{i=a}^t n_i (q^i - 1) = c_0 q^d - 1, \quad (25)$$

where $a$ is the smallest dimension of a subspace in $ST$. Since $P \setminus ST$ is a partial $d$-spread of $V(n, q)$, it follows from (25) and from counting in two ways the number of nonzero vectors in $V(n, q)$ that

$$n_d(q^d - 1) + c_0 q^d - 1 = q^n - 1 \Rightarrow n_d(q^d - 1) = q^d(q^{n-d} - c_0). \quad (26)$$

Hence, (26) implies that $q^d$ divides $n_d$. Thus $q^d$ divides $n_d - \ell q^d$. Now the second inequality in (24) implies that $n_d - \ell q^d = 0$, i.e.

$$|P \setminus ST| = n_d = \ell q^d. \quad (27)$$

Since $n_d = \ell q^d$ and $d = t + r_t$, it follows from the first inequality in (22) that

$$q^{\lfloor \frac{d + r_d}{2} \rfloor} - q^t \leq n_d - \ell q^d = 0 \Rightarrow r_d \leq t - r_t$$

Since $d = t + r_t$, $|ST| = q^t + 1$, $n_d = \ell q^d$, and $r_d \leq t - r_t$, it follows from Lemma 19 (see the Appendix) that $W = \bigcup_{X \in ST} X$ is a subspace of dimension $d + r_d = t + r_t + r_d \leq 2t$. If $r_d = t - r_t$, then $\dim W = 2t$. Then $ST$ is a subspace partition of $W$ into $t$-subspaces only, since otherwise, counting the nonzero vectors in $W$ in two different ways, i.e.,

$$q^{2t} - 1 = |W^*| = \sum_{X \in ST} |X^*| < |ST| \cdot |V(t, q)^*| = q^{2t} - 1,$$
yields a contradiction. We remark that if \( r_d = t - r_t \), then \( \frac{d + r_d}{2} = t \); thus the subspace partition \( \mathcal{P} \) must be of type \( [d^t q^d, r_t^{t+1}] \) and of minimum size (i.e., \( |\mathcal{P}| = \sigma_q(n, d) \)). Such partitions are known to exist and are discussed in [16] (in particular, see Theorem 2).

Finally, if \( r_d < t - r_t \), then \( \dim W = t + r_t + r_d < 2t \) and it follows from the dimension condition (see (2) in Sect. 1) that \( ST \) is a subspace partition of \( W \) with exactly one \( t \)-subspace and with other subspaces of dimension at most \( r_t + r_d \). Since \( \dim W = t + r_t + r_d < 2t \) and \( |ST| = q^t + 1 \), counting in two ways the nonzero vectors in \( W \) yields

\[
q^{t+r_t+r_d} - 1 = |W^*| = \sum_{X \in ST} |X^*| 
= |V(t, q^t)| + (|ST| - 1) \cdot |V(r_t + r_d, q)| = q^{t+r_t+r_d} - 1. \tag{28}
\]

If some subspace in \( ST \) has dimension less than \( r_t + r_d \), then inequality in (28) becomes strict and we obtain a contradiction. Thus, \( ST \) contains one \( t \)-subspace and \( q^t \) subspaces of dimension \( r_t + r_d \). In this case, we also remark that if \( r_d = t - r_t - 1 \), then the subspace partition \( \mathcal{P} \) is of type \( [d^t q^d, t^1, (t - 1)q^t] \) and of minimum size (e.g., see [16, Theorem 4] for the existence of such partitions). However, if \( r_d < t - r_t - 1 \), then the resulting subspace partitions are not necessarily of minimum size. For instance, if \( n = 34, k = 3, d = 11, r_d = 1, t = 7, \) and \( r_t = 4 \), we can apply (several times) Lemma 13 to construct a subspace partition of \( V(34, q) \) of type \( [11q^3 + q^6, 7^1, 5^7] \) and size \( q^{23} + q^{12} + q^7 + 1 \), which is larger than \( \sigma_q(34, 11) = q^{23} + q^{12} + q^6 + 1 \).

We are now ready to prove Corollary 6.

**Proof** Let \( \mathcal{P} \) be a subspace partition of \( V = V(n, q) \) of type \( [d_1^{n_1}, \ldots, d_m^{n_m}] \). Let \( ST \) be the \( d_s \)-supertail of \( \mathcal{P} \), let \( \hat{ST} \) be its \( ds \) \(-\)supertail, and assume that \( ST \) has size \( \sigma_q(d_s, ds - 1) \). We shall prove the following statements:

(i) If \( 2 \leq s \leq 3, d_s < 2d_{s-1}, \) and \( d_{s+1} < 2d_s \), then

\[
|\hat{ST}| \geq \sigma_q(d_{s+1}, d_s) + \sigma_q(d_s, ds - 1).
\]

(ii) If \( d_s \geq 2d_{s-1} \), or if \( s = 3 \) and \( d_3 = d_2 + d_1 \), then

\[
|\hat{ST}| \geq \sigma_q(d_{s+1}, d_s) + \sigma_q(d_s, ds - 1) - 1.
\]

By using the hypotheses of Corollary 3 and Theorem 5, we can infer that \( W = \bigcup_{X \in ST} X \) is a subspace of \( V \) and \( \dim W \geq d_s \). Thus, it follows from the definitions of \( \mathcal{P} \) and \( ST \) that

\[
\mathcal{P}' = \{X \in \mathcal{P} : X \notin ST\} \cup \{W\} \tag{29}
\]

is a subspace partition of \( V \). Since \( \mathcal{P} \) contains a \( d_s \)-subspace and \( \dim X < d_s \) for any \( X \in ST \), it follows from the definition in (29) that \( \mathcal{P}' \) also contains a \( d_s \)-subspace \( Y \). Let \( \hat{ST}' \) be the \( d_s-1 \)-supertail of \( \mathcal{P}' \) and let \( h \) be the the largest dimension of a subspace in \( \hat{ST}' \). Since \( Y \in \hat{ST}' \), it follows that \( d_s = \dim Y \leq h < d_{s+1} \). If \( W \notin \hat{ST}' \), then \( h = \dim Y = d_s \), and \( \hat{ST} \) is the disjoint union of \( \hat{ST}' \) and \( ST \). Thus,

\[
|\hat{ST}| = |\hat{ST}'| + |ST| \geq \sigma_q(d_{s+1}, d_s) + \sigma_q(d_s, ds - 1). \tag{30}
\]

Observe that \( W \notin \hat{ST}' \) if and only if \( \dim W \geq d_{s+1} \). In this case, \( d_{s+1} \leq \dim W < 2d_s \).

If \( W \in \hat{ST}' \), then \( h = \dim W < d_{s+1} \), and \( \hat{ST} \) is the disjoint union of \( \hat{ST}' \setminus \{W\} \) and \( ST \). Thus,

\[
|\hat{ST}| \geq (|\hat{ST}'| - 1) + |ST| \geq \sigma_q(d_{s+1}, \dim W) + \sigma_q(d_s, ds - 1) - 1. \tag{31}
\]
If \( d_s < 2d_{s-1} \) and \( d_{s+1} < 2d_s \), then \( \dim W < d_{s+1} < 2\dim W \), and it follows from Theorem 1 that
\[
\sigma_q(d_{s+1}, \dim W) = q^{\dim W} + 1 > q^{d_s} + 1 = \sigma_q(d_{s+1}, d_s).
\] (32)
Thus, from (31) and the strict inequality of (32), we obtain that
\[
|\hat{ST}| \geq \sigma_q(d_{s+1}, d_s) + \sigma_q(d_s, d_{s-1}).
\] (33)
Now part (i) of the corollary follows from (30), (32), and (33).

Finally, \( \dim W = d_s \) if \( d_s \geq 2d_{s-1} \) (by Corollary 3), or if \( d_s = d_1 + d_{s-1} \) (by Theorem 5). Thus part (ii) of the corollary follows from (31).

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Appendix

The following two lemmas (Lemmas 18 and 19) are direct adaptations of [16, Lemma 7 and Proposition 1]. For the sake of completeness, we repeat their proofs here. In the following, let \( D \) denote the family of \( d \)-subspaces in a partition \( P \) of \( V = V(n, q) \) with minimum size \( d \)-supertail \( ST \). Let \( \alpha_i \) denote the number of hyperplanes in \( V \) that contain exactly \( i \) members of \( D \).

Lemma 18 Let \( n, k, d, \) and \( r_d \) be integers such that \( k \geq 2, n = kd + r_d, d = t + r_t, 1 \leq r_d < d, \) and \( 1 \leq r_t < t \). Let \( \ell = q^{r_d} \sum_{i=0}^{k-2} q^{i d} \), and let \( P \) be a partition of \( V = V(n, q) \) whose largest subspace dimension is \( d \) and such that \( n_d = \ell q^d \). Assume, furthermore that \( P \) has a \( d \)-supertail \( ST \) of minimum size \( |ST| = q^t \) and with largest subspace dimension \( t \). Then, the following conclusions hold.

(a) If \( \alpha_i \neq 0 \), then \( \delta = \ell - q^{r_d} \leq i \leq \ell \).
(b) The extremal case \( \alpha_\delta \neq 0 \) occurs if there exists a hyperplane \( H \) of \( V \) such that all members of \( ST \) are subspaces of \( H \).
(c) The extremal case \( \alpha_\ell \neq 0 \) occurs if there exists a hyperplane \( H \) of \( V \) such that the number of non-zero vectors in \( \bigcup_{X \in ST} (X \cap H) \) equals \( q^{d+r_d-1} - 1 \).

Proof For any subspace \( U \) of \( V \) and any hyperplane \( H \) of \( V \), let \( B_H(U) \) denote the set of all points (i.e., 1-subspaces) of \( U \) that are not in \( H \). Then elementary linear algebra arguments yield
\[
|B_H(U)| = \begin{cases} 
0 & \text{if } U \subseteq H, \\
q^{\dim U-1} & \text{otherwise}.
\end{cases}
\] (34)
If \( \alpha_i \neq 0 \) then there is at least one hyperplane \( H \) containing \( i \) members of \( D \). From (34) we then get that
\[
(n_d - i)q^{d-1} \leq |B_H(V)| = q^{n-1}
\]
and thus, since \( n = kd + r_d \) and \( \ell = q^{r_d} \sum_{i=0}^{k-2} q^{i d} \), we obtain
\[
i \geq n_d - q^{(k-1)d+r_d} = \ell q^d - q^{(k-1)d+r_d} = \ell - q^{r_d}.
\] (35)
We now show by contradiction that \( i \leq \ell \) for all \( \alpha_i \neq 0 \). Assume that \( i \geq \ell + 1 \) for some \( H \). Since \( D \) denotes the set of members of \( P \) that have dimension \( d \), we have \( |P \setminus D| = q^t + 1 \).
Then it follows from Lemma 11 and the fact that $d > t$ that

$$ |P| \geq i \cdot q^d + 1 \geq (\ell + 1)q^d + 1 > \ell q^d + q^t + 1 = |D| + |P \setminus D| = |P|, $$

which is a contradiction. Now conclusion (a) of the theorem follows from (35) and (36).

Next, we can infer from the analysis leading to (35) that the case $\alpha_i \neq 0$ with $\delta = \ell - q^{rd}$ occurs if all members of $ST$ are contained in some hyperplane $H$. This proves conclusion (b). Finally, if $\alpha_i \neq 0$ occurs, then by definition of $\alpha_i$, there exists a hyperplane $H$ of $V$ that contains exactly $\ell$ subspaces of $\mathcal{D}$. Let $\mathcal{D}' \subseteq \mathcal{D}$ be the set containing those $\ell$ subspaces. Since $\mathcal{P}$ is a subspace partition of $V$, counting the nonzero vectors of $H$ in two ways yields

$$ |H^*| = \left| \bigcup_{X \in \mathcal{D}'} (X \cap H)^* \right| + \left| \bigcup_{X \in \mathcal{D}' \setminus \mathcal{D}'} (X \cap H)^* \right| + \left| \bigcup_{X \in ST} (X \cap H)^* \right|, $$

where $(X \cap H)^*$ is the set of nonzero vectors in $X \cap H$. Since $\mathcal{D}$ contains $\ell q^d$ subspaces of dimension $d$, $\dim H = kd + rd - 1$, and $\dim(X \cap H) = d - 1$ for all $X \in (\mathcal{D} \setminus \mathcal{D}')$, it follows from (37) that

$$ \left\| \bigcup_{X \in ST} (X \cap H)^* \right\| = (q^{kd+rd-1} - 1) - (\ell q^d - 1) - (\ell q^d - \ell)(q^{d-1} - 1) = q^{d+rd-1} - 1, $$

where we used the fact that $\ell = q^d \cdot \sum_{i=0}^{k-2} q^{id} = q^d \cdot \frac{q^{(k-1)d} - 1}{q^d - 1}$. This proves conclusion (c). $\square$

**Lemma 19** Let $n$, $k$, $d$, and $rd$ be integers such that $k \geq 1$, $n = kd + rd$, $d = d + t$, $r_d < d$, and $1 \leq r_i < t$. Let $\ell = q^d \cdot \sum_{i=0}^{k-2} q^{id}$, and let $\mathcal{P}$ be a partition of $V(n, q)$ whose largest subspace dimension is $d$ and such that $n_d = \ell q^d$. Assume, furthermore that $\mathcal{P}$ has a $d$-supertail $ST$ of minimum size $|ST| = q^t + 1$ and with largest subspace dimension $t$. Then the union of the subspaces in $ST$ is itself a $d + rd$-subspace.

**Proof** We again let $\mathcal{D}$ denote the family of $d$-subspaces in $\mathcal{P}$, and we let $\alpha_i$ denote the number of hyperplanes in $V = V(n, q)$ that contain exactly $i$ members of $\mathcal{D}$. It is trivial that $\alpha_i \geq 0$ for all $i$; a fact that we will use later for all $i$. From Lemma 18, we know that

$$ \alpha_i \neq 0 \implies \delta = \ell - q^{rd} < i \leq \ell. $$

We first define the integers $x$, $y$, and $z$ by

$$ x = \sum_{i=\delta}^{\ell} i \alpha_i, \quad y = \sum_{i=\delta}^{\ell} \binom{i}{2} \alpha_i, \quad \text{and} \quad z = \sum_{i=\delta}^{\ell} \alpha_i. $$

Each member of $\mathcal{D}$ is a $d$-subspace and is thus contained in exactly $(q^{(k-1)d+rd} - 1)/(q - 1)$ hyperplanes. By double counting incidences $(H, U)$, for $H \in \mathcal{H}$ with $U \in \mathcal{D}$ and $U \subseteq H$, we obtain

$$ x = \sum_{i=\delta}^{\ell} i \alpha_i = n_d \cdot \Theta_{(k-1)d+rd}. $$

Any two members of $\mathcal{D}$ are contained in $(q^{(k-2)d+rd} - 1)/(q - 1)$ hyperplanes. Thus, by double counting incidences, we get

$$ y = \sum_{i=\delta}^{\ell} \binom{i}{2} \alpha_i = \binom{n_d}{2} \Theta_{(k-2)d+rd}. $$
Furthermore, by counting the number of hyperplanes in \( V \), we obtain
\[
z = \sum_{i=\delta}^{\ell} \alpha_i = \Theta_{kd + rd}.
\] (40)

Observe that (38), (39), and (40) imply that the constants \( x, y \) and \( z \) are independent of the particular choice of subspace of \( P \) with \( n_d = \ell q^d \) and minimum size \( d \)-supertail. Moreover,
\[
\sum_{i=\delta}^{\ell} \alpha_i (i - \delta)(i - \ell) = 2y + x - (\delta + \ell)x + \delta \ell z.
\] (41)

Also note the following facts that we shall use later:
\[
(i - \delta)(i - \ell) \begin{cases} = 0 & \text{if } i = \delta, \\ < 0 & \text{if } \delta < i < \ell, \\ = 0 & \text{if } i = \ell. \end{cases}
\] (42)

Since \( n = kd + rd, d = t + r_1 \), and \( 1 \leq rd \leq t - r_1 \), we can use Lemma 13 to construct a partition \( P_0 \) of \( V(n, q) \) with \( \ell q^d \) subspaces of dimension \( d \), one \( t \)-subspace, and \( q^{r_1} \) subspaces of dimension \( r_d + r_1 \). (Note that if \( r_d = t - r_1 \), \( P_0 \) has type \( [d; \ell q^d, t^{q^{r_1} + 1}] \).)

In order to show that the right side of (41) is equal to zero, we consider the partition \( P_0 \). From the construction of the partition \( P_0 \), it follows that the points in the \( d \)-supertail constitute a \( d + rd \)-subspace \( W \). Any hyperplane \( H \in \mathcal{H} \) either contains \( W \) or intersects \( W \) in \((q^{\dim W - 1} - 1)\) non-zero vectors. These are the two extremal cases discussed in parts (b) and (c) of Lemma 18. So for the partition \( P_0 \), we have \( \alpha_i = 0 \) for \( \delta < i < \ell \). Then, it follows from (42) that the left side of (41) is equal to zero. Thus, we obtain from (41) that for any partition \( P \),
\[
\sum_{i=\delta+1}^{\ell-1} \alpha_i (i - \delta)(i - \ell) = 0.
\]
As \( \alpha_i \geq 0 \), we may thus conclude from the equation above and (42) that
\[
\delta < i < \ell \implies \alpha_i = 0.
\]

Hence, we can now use (38) and (40) (or refer to the partition \( P_0 \), which must have the same solution \( \alpha_\delta \) and \( \alpha_\ell \) to these two equations) to calculate \( \alpha_\delta \) (and \( \alpha_\ell \)). We then get that
\[
\alpha_\delta = \Theta_{(k-1)d}.
\] (43)

Let \( \gamma = q^{(k-1)d+rd} \). Since
\[
\ell = q^{rd} \sum_{i=0}^{k-2} q^{id} = q^{rd} \frac{q^{(k-1)d} - 1}{q^d - 1},
\]
it follows that
\[
|D| - \gamma = \ell q^d - q^{(k-1)d+rd} = \ell - q^{rd} = \delta.
\] (44)

Let \( \mathcal{H}_0 \) denote the set of all hyperplanes of \( V \) that intersect \( \gamma \) members of \( D \). Since a hyperplane of \( V \) either contains a given subspace or intersects it, it follows from (44) that
\( \mathcal{H}_0 \) can be also defined as the set of all hyperplanes of \( V \) that contain \( \delta \) members of \( \mathcal{D} \). Thus it follows from the definition of \( \alpha_i \) that \( \alpha_\delta = |\mathcal{H}_0| \). Let

\[
W = \bigcap_{H \in \mathcal{H}_0} H \quad \text{and} \quad R = \bigcup_{X \in ST} X.
\]

Since \( \mathcal{P} \) is a subspace partition of \( V \) and \( \mathcal{D} \) contains \( \ell q^d \) subspaces of dimension \( d \), the number of non zero vectors in \( R \) is

\[
|R^*| = |V^*| - \sum_{U \in \mathcal{D}} U^* = q^n - 1 - \ell q^d (q^d - 1) = q^{d+r_d} - 1.
\]

Moreover, it follows from Lemma 18(b) that

\[
R \subseteq \bigcap_{H \in \mathcal{H}_0} H = W,
\]

and it follows from (43) that

\[
\dim W \leq n - (k - 1)d = d + r_d.
\]

Thus, it follows from (45)–(47) that \( R = W \) is a \( d + r_d \)-subspace. \( \square \)

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