An information theoretic formulation of the Dictionary Learning and Sparse Coding Problems on Statistical Manifolds

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Abstract

In this work, we propose a novel information theoretic framework for dictionary learning (DL) and sparse coding (SC) on a statistical manifold (the manifold of probability distributions). Unlike the traditional DL and SC framework, our new formulation does not explicitly incorporate any sparsity inducing norm in the cost function but yet yields SCs. Moreover, we extend this framework to the manifold of symmetric positive definite matrices, \( P_n \). Our algorithm approximates the data points, which are probability distributions, by the weighted Kullback-Leibler center (KL-center) of the dictionary atoms. The KL-center is the minimizer of the maximum KL-divergence between the unknown center and members of the set whose center is being sought. Further, we proved that this KL-center is a sparse combination of the dictionary atoms. Since, the data reside on a statistical manifold, the data fidelity term can not be as simple as in the case of the vector-space data. We therefore employ the geodesic distance between the data and a sparse approximation of the data element. This cost function is minimized using an accelerated gradient descent algorithm. An extensive set of experimental results show the effectiveness of our proposed framework. We present several experiments involving a variety of classification problems in Computer Vision applications. Further, we demonstrate the performance of our algorithm by comparing it to several state-of-the-art methods both in terms of classification accuracy and sparsity.

1 Introduction

Dictionary learning and sparse coding have found wide applicability in Machine Learning and Computer Vision in recent times. Examples include but are not limited to, image classification [26], image restoration [42] and face recognition [41]. The traditional dictionary learning (DL) and sparse coding (SC) formulation assumes that the input data lie in a vector space, and assumes a linear...
generative model for the data by approximating the data with a sparse linear combination of the dictionary atoms (elements). Thus, the objective function of the DL problem typically has a data fidelity term to minimize the “reconstruction error” in the least squares sense. Sparsity is then enforced on the weights in the linear combination via a tolerance threshold on the ℓ₀-norm of the weight vector. This however leads to an NP-hard problem and the most popular approach for solving this problem (with no convergence guarantees) is the K-SVD based approach [1]. For a fixed dictionary, a convex approximation to the ℓ₀-norm minimization to induce sparsity can be achieved using the ℓ₁-norm constraint on the weight vector [6, 13]. The problem of finding both the dictionary and the sparse-codes however remains to be a hard problem in general. For further discussion on this topic, we refer the readers to [38], where authors provide a provably convergent algorithm for the dictionary recovery problem.

In many application domains however, the data do not reside in a vector space, instead they reside on a Riemannian manifold such as the Grassmannian [9, 8], the hypersphere [37, 29], the manifold of symmetric positive definite (SPD) matrices [16, 36, 43], etc. Generalizing the DL & SC problem from the case of vector space inputs to the case when the input data reside on a Riemannian manifold is however difficult because of the nonlinear structure of Riemannian manifolds [43]. One could consider embedding the Riemannian manifold into a Euclidean space, but a problem with this method is that there does not exist a canonical embedding for a general Riemannian manifold. This motivated researchers [43, 20, 36, 24] to generalize the DL and SC problem to a Riemannian manifold. Though the formulation on a manifold involves a “reconstruction error” term analogous to the vector space case, defining a sparsity inducing constraint on a manifold is nontrivial and should be done with caution. This is because, a Riemannian manifold lacks “global” vector space structure since it does not have the concept of a global origin. Hence as argued in [43], one way to impose the sparsity inducing constraint is via an an affine constraint, i.e., the sparsity constraint is over an affine subspace defined by the tangent space at a data point on the manifold. We now briefly review a few representative algorithms for the DL & SC problem on Riemannian manifolds.

A popular solution to the DL problem is to make use of the tangent spaces, which are linear spaces associated with each point on a Riemannian manifold. This approach essentially involves use of linear approximation in the smooth neighborhood of a point. Guo et al. [17] use a Log-Euclidean framework [3] to achieve a sparse linear representation in the tangent space at the Fréchet mean of the data. Xie et al. [43] developed a general dictionary learning formulation that can be used for data on Riemannian manifolds. In their approach, for the SC problem, authors use the Riemannian Exp. and Log. maps to define a generative process for each data point involving a sparse combination of the Log.-mapped dictionary atoms residing on the manifold. This sparse combination is then realized on the manifold via the Exp.-map. Their formulation is a direct generalization of the linear sparsity condition with the exception of the origin of the linear space being at the data point. Further, they impose an affine
constraint in the form of the weights in the weight vector summing to one. This constraint implies the use of affine subspaces to approximate the data. For fixed weights however, estimating the dictionary atoms is a hard problem and a manifold line search method is used in their approach. In another method involving DL and SC on the manifold of SPD matrices, Cherian et al. [10] proposed an efficient optimization technique to compute the sparse codes.

Several works report the use of kernels to accomplish dictionary learning and sparse coding on Riemannian manifolds [19, 24, 20]. In these, the Riemannian manifold is embedded into the Reproducing Kernel Hilbert Space (RKHS). DL and SC problems are then formulated in the RKHS. RKHS is a linear space, and so it is easier to derive simple and effective solutions for DL and SC. Recently, authors in [15] presented conditions that must be strictly satisfied by geodesic exponential Kernels on general Riemannian manifolds. This important result forces one to exercise caution when designing a kernel based approach for general Riemannian manifolds.

In this work, we present a novel formulation of the DL and SC problems for data residing on a statistical manifold, without explicitly enforcing a sparsity inducing constraint. The proposed formulation circumvents the difficulty of defining a sparsity constraint on a Riemannian manifold. Our formulation is based on an information theoretic framework and is shown to yield sparse codes. Further, we extend this framework to the manifold of SPD matrices. An extensive set of experimental results are presented that demonstrate the competitive performance of our proposed algorithm in comparison to the state-of-the-art.

The rest of the paper is organized as follows: in Section 2, we first present the conventional DL and SC problem formulation in vector spaces and motivate the need for a new formulation of the DL and SC problem on Riemannian manifolds. This is followed by a brief summary of relevant mathematical background on statistical manifolds. Following this, we summarize the mathematical results in this paper and then present the details along with our algorithm for the DL and SC problem. In Section 3, we present several experimental results and comparisons to the state-of-the-art. Finally, in Section 4, we draw conclusions.

2 An Information Theoretic Formulation

In the traditional SC problem, a set of data vectors \( X = \{x_i\}_{i=1}^N \subset \mathbb{R}^n \), and a collection of atoms, \( A = \{a_j\}_{j=1}^r \subset \mathbb{R}^n \), are given. The goal is to express each \( x_i \) as a sparse linear combination of atoms in \( A \). Let, \( A \) be the (overcomplete) dictionary matrix of size \( n \times r \) whose \( i^{th} \) column consists of \( a_i \). Let \( W = [w_1, \ldots, w_N] \) be a \( r \times N \) matrix where each \( w_i \in \mathbb{R}^r \) consists of the coefficients of the sparse linear combination. In the DL and SC problem, the goal is to minimize the following objective function:

\[
\min_{A, w_1, \ldots, w_n} \sum_{i=1}^n \|x_i - Aw_i\|^2 + Sp(w_i),
\]
Here, \( Sp(w_0) \) denotes the sparsity promoting term, which can be either an \( \ell_0 \) norm or an \( \ell_1 \) norm. Since, in the above optimization problem, both the dictionary \( A \) and the coefficient matrix \( W \) are unknown, it leads to a hard optimization problem. As this optimization problem is intractable when the sparsity promoting term is an \( \ell_0 \) norm constraint, most existing approaches use a convex relaxation to this objective using an \( \ell_1 \) norm in place of the \( \ell_0 \) norm constraint when performing the sparse coding. Now, instead of the traditional DL & SC setup where data as well as atoms are vector valued, we address the problem when each data point and the atom are probability densities, which are elements of a statistical manifold. In this paper, we present a novel DL and SC framework for data residing on a statistical manifold. Before delving into the details, we need to introduce some pertinent mathematical concepts from Differential Geometry and Statistical Manifolds.

### 2.1 Statistical Manifolds: Mathematical Preliminaries

Let \( \mathcal{M} \) be a smooth \((C^\infty)\) manifold [7]. We say that \( \mathcal{M} \) is \( n \)-dimensional if \( \mathcal{M} \) is locally Euclidean of dimension \( n \), i.e., locally diffeomorphic to \( \mathbb{R}^n \). Given a \( C^\infty \) vector bundle \( E \rightarrow \mathcal{M} \), a connection \( \nabla \) on \( E \rightarrow \mathcal{M} \) is a map \( \nabla : \Omega^k(\mathcal{M},E) \rightarrow \Omega^{k+1}(\mathcal{M},E) \) satisfying the Leibnitz’s rule, i.e., \( \nabla(fs) = df \otimes s + f \nabla s \), \( \forall s \in \Gamma(\mathcal{M}) \) and \( \forall f \in C^\infty(\mathcal{M}) \). Here, \( \Omega^k(\mathcal{M},E) \) is the smooth \((C^\infty)\) valued \( k \)-form on \( \mathcal{M} \), \( k \geq 0 \), i.e., \( \Omega^k(\mathcal{M},E) = \Gamma(E \otimes \wedge^k T^* \mathcal{M}) \), where \( T^* \mathcal{M} \) is the cotangent bundle, i.e., it is point wise dual to the tangent bundle \( T\mathcal{M} = \bigcup_{p \in \mathcal{M}} T_p \mathcal{M} \), \( \wedge^k T^* \mathcal{M} \) is the \( k \)-th exterior power of \( T^* \mathcal{M} \) and \( \Gamma(\mathcal{F}) \) is the section of the vector bundle \( \mathcal{F} \). We use \( df \) to denote the exterior derivative of the smooth function \( f \). Define a Riemannian metric \( g \) on \( E \) as a field of smoothly varying inner products on \( E_p \), \( \forall p \in \mathcal{M} \). When \( E = T\mathcal{M} \), \( \nabla \) is called an affine connection on \( T\mathcal{M} \). We say that \( \nabla \) respects \( g \) if \( \forall X \in \Gamma(T\mathcal{M}) \), \( s_1, s_2 \in \Gamma(E) \), we have \( X(g(s_1, s_2)) = g(\nabla_X s_1, s_2) + g(s_1, \nabla_X s_2) \) when \( E = T\mathcal{M} \), an affine connection \( \nabla \) is torsion free, if \( \forall X, Y \in \Gamma(T\mathcal{M}) \), \( \nabla_X Y = -\nabla_Y X - [X, Y] = 0 \). Moreover, for \( E = T\mathcal{M} \) an affine connection which respects \( g \) and is torsion free, is called the Levi-Civita connection on \( T\mathcal{M} \). An affine connection \( \nabla^* \) is called a dual connection of \( \nabla \) if \( \forall X, Y \in \Gamma(T\mathcal{M}) \), \( dg(X, Y) = g(\nabla_X Y) + g(X, \nabla^* Y) \). The triple \((M, g, \nabla)\) is called a statistical manifold whenever both \( \nabla \) and \( \nabla^* \) are torsion free.

A point on an \( n \)-dimensional statistical manifold, \( \mathcal{D} \) (from here on, we will use the symbol \( \mathcal{D} \) to denote a statistical manifold unless mentioned otherwise), can be identified with a (smooth) probability distribution function on a measurable topological space \( \Omega \), denoted by \( P(x; \theta) \) [39, 2]. Here, each distribution function can be parametrized using \( n \) real variables \((\theta_1, \ldots, \theta_n)\). So, an open subset \( S \) of a statistical manifold, \( \mathcal{D} \), is a collection of the probability distribution functions on \( \Omega \). And the chart map is the mapping from \( S \) to the parameter space, \( \Theta = \{ \theta \} \subset \mathbb{R}^n \). Let \( \mu \) be a \( \sigma \)-finite additive measure defined on a \( \sigma \)-algebra of subsets of \( \Omega \). Let \( f(x; \theta) \) be the density of \( P(x; \theta) \) with respect to the measure \( \mu \) and assume the densities to be smooth \((C^\infty)\) functions. Now, after giving \( \mathcal{D} \) a topological structure, we can define a Riemannian metric as
follows. Let \( l(x; \theta) = \log f(x; \theta) \), then a Riemannian metric, \( g \) can be defined as 
\[
  g_{ij}(\theta) = E_\theta \left[ \frac{\partial l(x; \theta)}{\partial \theta_i} \frac{\partial l(x; \theta)}{\partial \theta_j} \right],
\]
where \( E_\theta[y] \) is the expectation of \( y \) with respect to \( \theta \). In general, \( g = [g_{ij}] \) is symmetric and positive semi-definite, hence a pseudo-metric. We can make \( g \) positive definite by assuming the functions \( \{\frac{\partial l(x; \theta)}{\partial \theta_i}\}_{i=1}^n \) to be linearly independent. This metric is called the Fisher-Rao metric \([28]\) on \( \mathcal{D} \).

2.2 Summary of the mathematical results

In the next section, we propose an alternative formulation to the DL and SC problem. We first state a couple of theorems as background material that will be used subsequently. Then, we define the objective function for the DL and SC problem posed on a statistical manifold in Section 2.3.1. Our key mathematical results are stated in Theorem 2.4, Corollary 2.4.1 and 2.4.2 respectively. Using these results, we show that our DL & SC framework, which does not have an explicit sparsity constraint, yields sparse coding. Then, we extend our DL and SC framework to the manifold of SPD matrices, \( \mathcal{P}_n \), in Section 2.3.2.

2.3 Detailed mathematical results

In this section, we will first formulate the DL and SC problem for data residing on a statistical manifold. Then, we extend this framework to the manifold of SPD matrices. Let the manifold of densities, hereafter denoted by \( \mathcal{D} \) be the \( n \)-dimensional statistical manifold, i.e., each point on \( \mathcal{D} \) is a probability density. We will use the following notations in the rest of the paper.

- Let \( \mathcal{G} \) be a dictionary with \( r \) atoms \( g_1, \cdots, g_r \), where each \( g_i \in \mathcal{D} \).
- Let \( \mathcal{F} = \{f_i\}_{i=1}^N \subset \mathcal{D} \) be a set of data points.
- And \( w_{ij} \) be nonnegative weights corresponding to \( i^{th} \) data point and \( j^{th} \) atom, \( i \in \{1, \cdots, N\} \) and \( j \in \{1, \cdots, r\} \).

Note that, here we assume that each density \( f \) or \( g \) is parameterized by \( \theta \). The KL divergence \([11]\) between two densities \( f_1 \) and \( f_2 \) on \( \mathcal{D} \) is defined by
\[
  \text{KL}(f_1, f_2) = \int f_1(x) \log \frac{f_1(x)}{f_2(x)} dx
\]
Given a set of densities \( \mathcal{F} = \{f_i\} \), the KL divergence from \( \mathcal{F} \) to a density \( f \) can be defined by
\[
  \text{KL}(\mathcal{F}, f) = \max_i \text{KL}(f_i, f)
\]
We can define the KL-center of \( \mathcal{F} \), denoted by \( f_m(\mathcal{F}) \), by
\[
  f_m(\mathcal{F}) = \arg \min_f \text{KL}(\mathcal{F}, f)
\]
The symmetrized KL divergence, also called the Jensen-Shannon divergence (JSD) \[11\] between two densities \(f_1\) and \(f_2\) is defined by

\[
\text{JSD}(f_1, f_2) = \frac{1}{2} \text{KL}(f_1, f_2) + \frac{1}{2} \text{KL}(f_2, f_1)
\] (5)

In general, given the set \(F = \{f_i\}\), define a mixture of densities, \(f = \sum \alpha_i f_i\), \(\sum \alpha_i = 1, \alpha_i \geq 0, \forall i\). It is evident that the set of \(\{\alpha_i\}\) forms a simplex, which is denoted by \(\Delta\). Then, the JSD of the set \(F\) with the mixture weights \(\{\alpha_i\}\) is defined as

\[
\text{JSD}(\{f_i\}) = H(\sum \alpha_i f_i) - \sum \alpha_i H(f_i)
\] (6)

where \(H(f) = -\int f(x) \log f(x) dx\) is the entropy of the density \(f\). It is easy to see the following lemma.

**Lemma 2.1.** \(\text{JSD}(\{f_i\})\) is concave in \(\{\alpha_i\}\) and JSD attains the minimum at an extreme point of the simplex \(\Delta\).

**Proof.** We refer the reader to [35] for a proof of this lemma. \(\square\)

In [35], it was shown that we can compute the KL-center of \(\mathcal{F}\), i.e., \(f_m(\mathcal{F})\) in Equation (4) using the following theorem.

**Theorem 2.2.** The KL center of \(\mathcal{F}\), i.e., \(f_m(\mathcal{F})\) is given by

\[
f_m(\mathcal{F}) = \sum \hat{\alpha}_i f_i
\]

where \(\hat{\alpha} = \arg\max_{\alpha} \text{JSD}(\{f_i\})\)

**Proof.** We refer the reader to [35] for a proof of this theorem. \(\square\)

Now, given the set of densities \(\mathcal{F} = \{f_i\}_{i=1}^N\) and a set of weights \(\{\alpha_i\}_{i=1}^N\), we can define the weighted KL-center, denoted by \(f_m(\mathcal{F}, \{\alpha_i\})\) as follows:

\[
f_m(\mathcal{F}, \{\alpha_i\}) = \arg\min_{f} \sum \alpha_i \text{KL}(f_i, f)
\] (7)

**Theorem 2.3.** Given \(\mathcal{F}\) and \(\{\alpha_i\}\) as above, \(f_m(\mathcal{F}, \{\alpha_i\}) = \sum \alpha_i f_i\)

**Proof.** For simplicity, assume each \(f_i\) is discrete and can take say \(k\) discrete values \(x_1, \cdots, x_k\). Then, consider minimization of \(\sum_i \text{KL}(f_i, f)\) with respect to \(f\) subject to the constraint that \(f\) is a density, i.e., for the discrete case, \(\sum_j f(x_j) = 1\). By using a Lagrange multiplier \(\lambda\), we want to compute,

\[
\frac{\partial}{\partial f(x_j)} \left\{ \sum_i \alpha_i \text{KL}(f_i, f) + \lambda \left( \sum_j f(x_j) - 1 \right) \right\} = 0, \forall j
\]

\[
\Rightarrow \left( \lambda - \sum_i \frac{\alpha_i f_i(x_j)}{f(x_j)} \right) = 0, \forall j
\]

\[
\Rightarrow f(x_j) = \frac{\sum_i \alpha_i f_i(x_j)}{\lambda}, \forall j
\]
Now, by taking \( \frac{\partial}{\partial \lambda} \left\{ \sum \alpha_i \text{KL}(f_i, f) + \lambda \left( \sum_j f(x_j) - 1 \right) \right\} \) and equate to 0, we get \( f(x_j) = \sum \alpha_i f_i(x_j) \), \( \forall j \). Thus, \( f = \sum \alpha_i f_i \). We can easily extend this to continuous \( f_i \) by replacing summation with integration and obtain a similar result.

### 2.3.1 DL and SC on a statistical manifold

Now, we will formulate the DL and SC problems on a statistical manifold. The idea is to express each data point, \( f_i \), as a sparse weighted combination of the dictionary atoms, \( \{g_j\} \). Given the above hypothesis, our objective function is given by:

\[
\begin{align*}
\text{arg min}_{\Phi^*, W^*} & \quad E = \sum_{i=1}^{N} \text{KL}(f_i, \hat{f}_i) \\
\text{subject to} & \quad w_{ij} \geq 0, \forall i, j \\
& \quad \sum_j w_{ij} = 1, \forall i.
\end{align*}
\]

where, \( \hat{f}_i = \sum_{j=1}^{r} w_{ij} g_j, \forall i \). In the above objective function, \( \hat{f}_i \) is the minimizer of the weighted KL-center of \( \{g_j\}_{j=1}^{r} \) with weights \( \{w_{ij}\}_{j=1}^{r} \). The constraint \( w_{ij} \geq 0 \) and \( \sum_j w_{ij} = 1 \) is to make \( \hat{f}_i \) a probability density. Note that, we can view \( \hat{f}_i \) as a reconstructed density from the dictionary elements \( \{g_j\} \) and weights \( \{w_{ij}\} \). Now, we will prove our key result namely, that the minimization of the above objective function with respect to \( \{w_{ij}\} \) yields a sparse set of weights.

**Theorem 2.4.** Let \( \Phi = \{g_j\} \) and \( W = [w_{ij}] \) be the solution of the objective function \( E \) in Equation 8. Then,

\[
\text{KL}(f_i, g_j) \geq r_{ij}, \forall j \text{ where } r_{ij} = \min_{j} \text{KL}(f_i, g_j)
\]

**Proof.** Consider the random variables \( X_1, \ldots, X_N \) with the respective densities \( f_1, \cdots, f_N \). Moreover, as each dictionary elements \( g_j \) is “derived” from \( \{f_i\} \), hence, we can view each \( g_j \) to be associated with a random variable \( Y_j \) such that \( Y_j = \tilde{g}_j (\{X_i\}) \), i.e., \( Y_j \) is a transformation of random variables \( \{X_i\} \). Now,

\[
E = \sum_{i=1}^{N} \text{KL} \left( f_i, \sum_{j=1}^{r} w_{ij} g_j \right)
\]

\[
= \sum_{i=1}^{N} \left[ \int f_i(x) \log(f_i(x)) dx \right] - \int \left\{ f_i(x) \log \left( \sum_j w_{ij} g_j(x) \right) \right\} dx
\]
Using Jensen’s inequality, we have,

\[ E \leq \sum_{i=1}^{N} \left[ \int \{ f_i(x) \log(f_i(x)) \} dx \right] - \int \left\{ f_i(x) \sum_j w_{ij} \log(g_j(x)) \right\} dx \]

\[ = \sum_{i=1}^{N} E_{X_i} \left[ \log(f_i) - \sum_j w_{ij} \log(g_j) \right] \]

where, \( E_{X}[h(X)] \) is the expectation of \( h(X) \), where \( h(X) \) is a transformation of the random variable \( X \). So,

\[ E \leq \sum_{i=1}^{N} E_{X_i}[\log(f_i)] - \sum_{i=1}^{N} \sum_j w_{ij} E_{X_i}[\log(g_j)] \]

\[ = \sum_{i=1}^{N} \sum_{j=1}^{r} w_{ij} E_{X_i}[\log(f_i)] - \sum_{i=1}^{N} \sum_{j=1}^{r} w_{ij} E_{X_i}[\log(g_j)] \]

\[ = \sum_{i=1}^{N} \sum_{j} w_{ij} E_{X_i}[\log(f_i) - \log(g_j)] \]

So, \( E \leq \sum_{i=1}^{N} \sum_{j} w_{ij} \text{KL}(f_i, g_j) \). And as both \( E \) and \( \sum_{i=1}^{N} \sum_j w_{ij} \text{KL}(f_i, g_j) \) attain minima at the same value (equal to 0), we can minimize \( \sum_{i=1}^{N} \sum_{j} w_{ij} \text{KL}(f_i, g_j) \) instead of \( E \). Using a Lagrange multiplier \( r_i \) for each constraint \( \sum_j w_{ij} = 1 \), and \( \gamma_{ij} \) for each constraint \( w_{ij} \geq 0 \), we get the following function

\[ \sum_j w_{ij} \text{KL}(f_i, g_j) + \sum_{i=1}^{N} r_i (1 - \sum_j w_{ij}) - \sum_{i,j} \gamma_{ij} w_{ij} \]

We minimize the above objective function and add the KKT conditions

\[ \gamma_{ij} w_{ij} = 0 \]

to get,

\[ \text{KL}(f_i, g_j) = \begin{cases} r_i + \gamma_{ij}, & \text{if } w_{ij} = 0 \\ r_i, & \text{if } w_{ij} > 0 \end{cases} \]  

(11)

As each \( \gamma_{ij} \geq 0 \), this concludes the proof. \( \blacksquare \)

A straightforward corollary of the above theorem is as follows:

**Corollary 2.4.1.** The objective function \( E \) is bounded above by \( \sum_{i=1}^{N} r_i \), i.e.,

\[ E \leq \sum_{i=1}^{N} r_i \]
Proof. From Theorem 2.4 we know that, $E \leq \sum_{i=1}^{N} \sum_{j} w_{ij} \text{KL}(f_i, g_j)$. Now, from Equation 11 we can get $\sum_{j} w_{ij} \text{KL}(f_i, g_j) = r_i, \forall i$. Thus the corollary holds.

Moreover, as a consequence of the Theorem some of the $w_{ij}$ are exactly zero and the rest are positive real numbers, which is similar to the $\ell_1$ sparsity, i.e., sparsity induced by the $\ell_1$ norm on $W$. Also, we can see that the dictionary elements, $g_j$, for which the associated weight are positive, are exactly at the same distance $r_i$ from the density $f_i$. Corollary 2.4.1 implies that solving the objective function in Equation 8 yields a “tight cluster” structure around each $f_i$, as minimizing $E$ is equivalent to minimizing each $r_i$.

Corollary 2.4.2. Let, $f_i$ be well approximated by a single dictionary element $g_l$. Moreover, assume that $g_l$ is a convex combination of a set of dictionary atoms, i.e., $g_l = \sum_{k=1}^{r_i} w_{ijk} g_{jk}$. Without loss of generality (WLOG), assume that $w_{ijk} > 0, \forall k$. Let, $r_i = KL(f_i, g_l)$ and $\hat{r}_i = KL(f_i, g_{jk}), \forall k = 1, \ldots, r_1$. Then, $r_i < \hat{r}_i$.

Proof. Using the hypothesis in Theorem 2.4 we have,

$$
\begin{align*}
    r_i &= \int f_i(x) \log(f_i(x))dx - \int f_i(x) \log(g_l(x))dx \\
    &= \int f_i(x) \log(f_i(x))dx - \int f_i(x) \sum_{k=1}^{r_1} w_{ijk} \log(g_{jk})dx \\
    &= \sum_{k=1}^{r_1} w_{ijk} \text{KL}(f_i, g_{jk}) \\
    &= \hat{r}_i
\end{align*}
$$

Hence, $r_i < \hat{r}_i$. Using Corollary 2.4.1 we can see that, in order to represent $f_i$, the objective function is to minimize $r_i$. Thus, using Corollary 2.4.1 we can say that a sparse set of weights, i.e., corresponding to $g_l$, is preferable over a set of non-zero weights, i.e., corresponding to a set of $\{g_{jk}\}$.

By, combining results of Theorem 2.4 and Corollary 2.4.2 it is evident that our proposed formulation yields a sparse set of weights and the achieved sparsity is “similar” to the sparsity induced by the $\ell_1$ norm. In Figure 1 we give a pictorial description of the results of Corollary 2.4.1 and 2.4.2. Here, for data point $f_i$, let $g = \{\hat{g}_j\}$. Using Corollary 2.4.2 we get $r_i < \hat{r}_i$ and using the Corollary 2.4.1 we get $g$ to be the solution to $E$. Moreover using $g$ the SC for $f_i$ is sparse.

Figure 1: Illustrative figure
2.3.2 DL and SC on a manifold of SPD matrices

Let, the manifold of $n \times n$ SPD matrices be denoted by $\mathcal{P}_n$. We will use the following notations throughout the rest of the paper. On $\mathcal{P}_n$,

- $\mathcal{C}$ be a dictionary with $r$ atoms $C_1, \cdots, C_r$, where each $C_i \in \mathcal{P}_n$.
- $\mathcal{X} = \{X_i\}_{i=1}^N \subset \mathcal{P}_n$ be a set of data points.
- $w_{ij}$ be nonnegative weights corresponding to the $i^{th}$ data point and the $j^{th}$ atom, $i \in \{1, \cdots, N\}$ and $j \in \{1, \cdots, r\}$.

We now extend the DL and SC formulation to $\mathcal{P}_n$. Note that, a point, $C \in \mathcal{P}_n$ can be identified with a Gaussian density with zero mean and covariance matrix $C$. Hence, it is natural to extend the information theoretic DL & SC framework from a statistical manifold to $\mathcal{P}_n$. Recall that the symmetrized KL divergence between two densities $f$ and $g$ can be defined by Equation 5. Using the square root of the JSD, one can define a distance between two matrices on $\mathcal{P}_n$. Similar to Equation 7, we can analogously define the symmetrized weighted $KL$ center, denoted by $M_{KL}$, as the minimizer of the sum of weighted squared $KL$ divergences. Given, $\mathcal{X} = \{X_i\}_{i=1}^N$, we can define the symmetrized $KL$-center of $\mathcal{X}$ as follows

$$M_{KL}(\mathcal{X}) = \sqrt{B^{-1}} \sqrt{\sqrt{BA} \sqrt{B^{-1}}}$$

where $A = \frac{1}{N} \sum_i X_i$, $B = \frac{1}{N} \sum_i X_i^{-1}$. We can extend the above result to define the symmetrized weighted $KL$-center by the following Lemma.

**Lemma 2.5.** On $\mathcal{X} = \{X_i\}_{i=1}^N$ with weights $\{w_i\}_{i=1}^N$, the symmetrized weighted $KL$-center, $M_{KL}(\mathcal{X}, \{w_i\})$ is defined as

$$M_{KL}(\mathcal{X}, \{w_i\}) = \sqrt{B^{-1}} \sqrt{\sqrt{BA} \sqrt{B^{-1}}}$$

where $A = \frac{1}{\sum_j w_j} \sum_i w_i X_i$, $B = \frac{1}{\sum_j w_j} \sum_i w_i X_i^{-1}$

Now, analogous to Equation 8 we can define our formulation for DL and SC on $\mathcal{P}_n$ as follows:

$$\arg \min_{C^*, \mathcal{W}} E = \sum_{i=1}^N d^2(X_i, \hat{X}_i)$$

where $\hat{X}_i = M_{KL}(\mathcal{C}, \{w_{ij}\}_{j=1}^r)$

subject to $w_{ij} \geq 0, \forall i, j$

$$\sum_j w_{ij} = 1, \forall i.$$

On $\mathcal{P}_n$, we have used the $GL(n)$-invariant metric, where $GL(n)$ denotes the group of $n \times n$ non-singular matrices. $\mathcal{P}_n$ admits a group action defined by, $\forall h \in$
GL(n), \forall C \in P_n, C[h] = hCh^t. Let, U, V \in T_C P_n, then the GL(n)-invariant metric can be defined as \( <U, V>_C = \text{trace}(C^{-1/2}UC^{-1}VC^{-1/2}) \). The distance induced by the metric is defined as, \( d(C, \hat{C}) = \text{trace}(\text{Log}(C^{-1}\hat{C})) \), here Log is the matrix logarithm. One can use any distance or divergence function for \( d \) in Equation 12, but we have used the GL(n)-invariant metric. Now, we will present an algorithm for DL and SC on \( P_n \), that we call, the information theoretic dictionary learning and sparse coding (iDLSC). We used an “alternating step” optimization procedure, i.e., first learn \( W \) with \( C \) held fixed, and then learn \( C \) with \( W \) held fixed. We used the well known Nestrov’s accelerated gradient descent [5] for the optimization. The algorithm is summarized in the Algorithm block 1.

**Algorithm 1:** The iDLSC algorithm

```
Input: \( \mathcal{X} = \{X_i\}_{i=1}^N \subset P_n, \eta > 0, \epsilon > 0 \)
Output: \( \mathcal{C} = \{C_j\}_{j=1}^r \subset P_n, W = [w_{ij}] \geq 0 \)
1 Initialize \( C \) by using the k-means algorithm on \( P_n \); 
2 Initialize \( W \) randomly using random non-negative numbers from \([0, 1]\); 
3 for \( i = 1, \ldots, N \) do 
4 Normalize the vector \( w(i, \cdot) \) so that it sums to 1 ; 
5 end 
6 flag \( \leftarrow 1; \)
7 Compute objective function, \( E \) using Equation 12 
8 \( E^{old} \leftarrow E; \)
9 iter \( \leftarrow 1; \)
10 \( \lambda(1) \leftarrow 1; \)
11 \( Y^W \leftarrow W; \)
12 \( Y^C \leftarrow C_j, \forall j; \)
13 while flag = 1 do 
14 Do alternating step optimization by alternating between \( C \) and \( W \) using the accelerated gradient descent method [5]; 
15 \( \lambda(\text{iter} + 1) \leftarrow 1 + \sqrt{1 + 4\lambda(\text{iter})^2}; \)
16 \( \gamma(\text{iter}) \leftarrow 1 + \sqrt{1 + 4\lambda(\text{iter})^2}; \)
17 \( nY^W \leftarrow W - \eta * \nabla E / \nabla W; \)
18 \( W \leftarrow (1 - \gamma(\text{iter}))nY^W + \gamma(\text{iter})Y^W; \)
19 Using \( W \) update \( C_j \) using the following steps, \( \forall j; \)
20 \( nY^C \leftarrow \text{Exp}_{nY^C}(\gamma(\text{iter})\text{Log}(nY^C)Y^C); \)
21 \( C_j \leftarrow \text{Exp}_{nY^C}(\gamma(\text{iter})\text{Log}(nY^C)Y^C); \)
22 Recompute the objective function, \( E \) using new \( C \) and \( W \), using Eq 12 
23 if \( |E - E^{old}| < \epsilon \) then 
24 flag \( \leftarrow 0; \)
25 end 
26 iter \( \leftarrow \text{iter} + 1 \)
27 end 
```
3 Experimental Results

In this section, we present experimental results on several real-world datasets demonstrating the effectiveness of our proposed method \textit{iDLSC} compared to the state-of-the-art algorithms on classification using the SCs as features for the classification problem on the manifold of SPDs. We report the classification accuracy to measure the performance in the context of classification experiments. Moreover, we also report a measure of sparsity, denoted by $\varsigma$, which captures the percentage of elements of $W$ that are $\leq 0.01$. All the experimental results reported here were obtained on a desktop with a single 3.33 GHz Intel-i7 CPU with 24 GB RAM. We performed comparisons to four state-of-the-art methods namely, (i) the dictionary learning and sparse coding on Riemannian manifolds (DLM) [43], (ii) Riemannian sparse coding for SPD matrices (Riem-SC) [10], (iii) Sparse coding using kernel defined by symmetric Stein divergence (kStein-SC) [20], (iv) Log-Euclidean sparse coding (LE-DLSC) [17]. For the LE-DLSC, we used the highly cited SPAMS toolbox [27] to do DL and SC on the tangent space.

We tested our algorithm on six publicly available datasets namely, (i) the Brodatz texture data [4], (ii) the Yale ExtendedB face data [23], (iii) the KTH action recognition data [32], (iv) the ETH80 object recognition data [14], (v) the ETHZ person re-identification data [33], (vi) the AT&T face recognition data [30]. The datasets are described below. From each of data set, we first extract $\mathcal{P}_n$ valued features. Then, both \textit{iDLSC} and DLM learn the dictionary atoms and the sparse codes. Whereas, for Riem-SC and kStein-SC, we used k-means on $\mathcal{P}_n$ and used the cluster centers as the dictionary atoms. For the Log-Euclidean sparse coding, we used the Riemannian Inverse Exponential map [7] at the Fréchet mean (FM) of the data and performed a Euclidean DL and SC on the tangent space at the FM. For classification, we used the $\nu-SVM$ [31] on the sparse codes taken as features. The SVM parameters are learned using a cross-validation scheme.

**Brodatz texture data:** This dataset contains 111 texture images. We used the same experimental setup as was used in [34]. Each image is of 256 $\times$ 256 dimension and we first partitioned each image into 64 non-overlapping blocks of size 32 $\times$ 32. From each block, we computed a $5 \times 5$ covariance matrix $FF^t$, summing over the block, where $F = (I, |\partial I|, |\partial I|, |\partial^2 I|, |\partial^2 I|)$. The matrix $FF^t$ is symmetric positive semidefinite. To make this matrix SPD, we add $\sigma I$ to the matrix $FF^t$, where $\sigma$ is a small positive real number. Thus, the covariance descriptor from each image lies on $\mathcal{P}_5$. For this data, we consider each image as a class, resulting in a 111 class classification problem. As DLM is computationally very expensive, this 111 class classification is infeasible for this method, hence we also randomly selected 16 texture images and performed classification on 16 classes. We took the number of dictionary atoms ($r$) to be 555 and 80 for 111 classes and 16 classes respectively.
Yale face data: This YaleExtendedB face dataset contains 16128 face images taken from 28 human subjects with varying pose and illumination conditions. We randomly fixed a pose and for that pose took all the illuminations, leading to 252 face images taken from 28 human subjects. We used a similar type of experimental setup as described in [9]. From each face image, we construct a SIFT descriptor [25] and take the first 4 principal vectors. Thus, each image is identified with a point on the Grassmann manifold of appropriate dimension. And then, inspired by the isometric mapping between the Grassmannian and $\mathbb{P}_n$ [21], we construct the covariance descriptor from the aforementioned principal vectors. Here, we used 84 dictionary atoms.

KTH action recognition data: This dataset contains 6 actions performed by 25 human subjects in different scenarios, outdoor, indoor etc. All videos were captured by static camera with homogeneous background. From each video, we first extract 25 frames, then extract the HOG features [12] from each of the frame. Then, using similar technique as used in [9], we mapped each point on the Grassmannian and then used the isometric mapping described in [21] to map it on to $\mathbb{P}_{10}$. This data contains 150 videos of 6 different actions. We consider each action as a class to perform the classification task. Here, we used 30 dictionary atoms.

ETH80 object recognition data: This dataset contains 8 different objects, each having 10 different instances from 41 different views resulting in 3280 images. We first segment the objects from each image using the provided ground truth. We used both texture and edge features to construct the covariance matrix. For the texture feature, we used three texture filters [22]. The filter bank is $[H_1 H_1^t, H_2 H_2^t, H_3 H_3^t]$, where $H_1 = [1,2,1]^t$, $H_2 = [-1,0,1]^t$, $H_3 = [-1,2,-1]^t$. In addition to the three texture features, we used the image intensity gradient and the magnitude of the smoothed image using Laplacian of the Gaussian filter. We used 40 dictionary atoms for this data.

ETHZ person re-identification data: This dataset contains surveillance images of 122 subjects. We consider 10 images of each subject. Motivated by the experiments in [10], we used the feature vector $F = [Y, |I_x|, |I_y|, |\sin(\theta) + \cos(\theta)|, |H_y|]$, where $Y$ is the pixel intensity in the YCbCr color space, $H_y$ is the y-gradient of pixel hue, and $\theta = \arctan(I_y/I_x)$. Thus, this data is on $\mathbb{P}_5$ and 122 classes. We took $r = 80$ for this data.

AT&T face recognition data: This dataset contains face images of 40 subjects with 10 images per subject. To each image, we apply 18 Gabor filters and combine the filter responses into a covariance matrix of size $18 \times 18$. We used the publicly available code [18] to construct the Gabor filters. We took the number of atoms to be 80.

We compared the performance of iDLSC with Riem-SC and DLM in Table
Table 1: Comparison results between iDLSC, Riem-SC and DLM

| Data       | iDLSC acc. (%) | Riem-SC acc. (%) | DLM acc. (%) | Time(s) |
|------------|----------------|------------------|--------------|---------|
| Brodatz 16 | 96.58          | 95.96            | 68.55        | 539.63  |
| Brodatz 111| 87.01          | 97.98            | 49.90        | 1909.15 |
| Yale face  | 100.00         | 92.85            | 99.60        | 293.91  |
| KTH action | 99.33          | 85.16            | 80.00        | 60.84   |
| ETH80      | 96.10          | 88.07            | 91.46        | 2403.96 |
| ETHZ       | 100.00         | 91.26            | 98.52        | 1589.15 |
| AT&T       | 78.00          | 91.99            | 73.00        | 851.54  |

Table 2: Comparison results between iDLSC, kStein-SC and LE-DLSC

| Data       | iDLSC acc. (%) | kStein-SC acc. (%) | LE-DLSC acc. (%) | Time(s) |
|------------|----------------|-------------------|------------------|---------|
| Brodatz 16 | 96.58          | 90.96             | 98.50            | 5.46    |
| Brodatz 111| 87.01          | 97.98             | 87.30            | 97.04   |
| Yale face  | 100.00         | 92.85             | 90.48            | 127.73  |
| KTH action | 99.33          | 95.16             | 99.80            | 75.43   |
| ETH80      | 96.10          | 88.07             | 96.10            | 1687.38 |
| ETHZ       | 100.00         | 91.26             | 98.52            | 726.56  |
| AT&T       | 78.00          | 91.99             | 73.00            | 208.53  |

1 All of these three methods are intrinsic, i.e., the DL and SC are tailored to the underlying manifold, i.e., \( P_n \). From the table, we can see that iDLSC yields the best sparsity from amongst the three methods. While DLM gives best accuracy for some datasets, one can see that DLM is not computationally efficient. For the Brodatz 111 dataset, DLM did not converge even after several days of runtime. For other datasets, one can see that our method, iDLSC, performs the best in terms of accuracy in a reasonable amount of time. In fact for three of the datasets, namely Yale, ETHZ and ETH80, iDLSC is fastest among the three algorithms. Thus, it is clear that iDLSC outperforms both Riem-SC and DLM both in terms of sparsity and accuracy while yielding the results in a reasonable amount of time as opposed to DLM.

The result of iDLSC in comparison to the two extrinsic methods, kStein-SC and LE-DLSC are shown in the Table 2. The results show that iDLSC outperforms the two extrinsic algorithms in terms of classification accuracy. While LE-DLSC gives best sparsity it fails to achieve good classification accuracy. The poor results from LE-DLSC makes sense as Riemannian Inverse Exponential map is a diffeomorphism only on a small neighbourhood around a data point. Thus, using the Inverse Exponential map to approximate the dataset is not always a good choice.

The comparative results in Table 1 and 2 show that iDLSC outperforms the state-of-the-art algorithms both in terms of achieved sparsity as well as classification accuracy.
4 Conclusions

In this paper, we presented an information theoretic dictionary learning and sparse coding algorithm on statistical manifolds. In the traditional dictionary learning approach on a vector space, the goal is to express each data point as a sparse linear combination of the dictionary atoms. This is typically achieved via the use of a data fidelity term and a term to induce sparsity on the coefficients of the linear combination. In this paper, we proposed an alternative formulation of the DL and SC problem, where we do not have an explicit sparsity constraint in our objective function. Our algorithm, iDLSC expresses each data point, which is a probability distribution, as a weighted KL-center of the dictionary atoms. We presented a proof that our proposed formulation yields sparsity without explicit enforcement. Further, we presented an extension of this formulation to data residing on $\mathbb{P}_n$. A Riemannian accelerated gradient descent algorithm was employed to learn the dictionary atoms and an accelerated gradient descent algorithm was employed to learn the sparse weights in a two stage alternating optimization framework. We have experimented on seven real datasets and compared with four state-of-the-art methods. The experimental results demonstrate the effectiveness of iDLSC in terms of classification accuracy and sparsity.

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