Pricing American Options under Regime-Switching Model with a Crank-Nicolson Fitted Finite Volume Method

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Abstract. A new numerical method for pricing American options under regime-switching model is developed. The original problem is first approximated by a set of nonlinear partial differential equations. After that a novel fitted finite volume method for the spatial discretisation of the nonlinear penalised system of partial differential equations is coupled with the Crank-Nicolson time stepping scheme. It is shown that the discretisation scheme is consistent, stable, monotone and hence convergent. In order to solve nonlinear algebraic systems, we apply an iterative algorithm and show its convergence. Numerical experiments demonstrate the convergence, efficiency and robustness of the numerical method.

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1. Introduction

Since its introduction, the Hamilton regime-switching model [12] attracted wide attention of mathematicians and financial engineers [4,8,13]. According to this model, market may switch from one regime to another, which allows to explain periodic caused by a short-term political or economic uncertainty. Thus econometric analysis shows that the blending the regime switching component and the log-normal dynamics of the stock prices better fits the asset price dynamics [7]. Moreover, regime switching models are intuitively attractive and computationally inexpensive. Therefore, they find numerous applications in
other fields, including electric markets \cite{3}, forestry valuation \cite{6}, and the valuation of stock loans \cite{28}.

The American options pricing problem under regime-switching model (2.1) is more complicated than other options pricing problems \cite{20, 21}. It is based on a system of coupled parabolic partial differential complementarity problems (PDCPs), so that efficient numerical methods are required to solve it. We note that various numerical methods for problem (2.1) have been studied recently. In particular, Yang \cite{25} established stability estimates for finite element and simple lattice methods. Liu \cite{19} developed and employed an efficient tree method for the Heston’s stochastic volatility model. Khaliq et al. \cite{17} introduced a fast numerical scheme, which uses the penalty and exponential time differencing Crank-Nicolson (ETD-CN) methods. Yousuf et al. \cite{27} considered a second-order method based on exponential time differencing approach and studied its stability and convergence. Egorova et al. \cite{9} used a multivariable front-fixing transformation to develop a conditionally stable first order in time and second order in space method. Zhang et al. \cite{29} introduced a fitted finite volume method for the spatial discretisation along with a fully implicit time stepping scheme for PDEs. Subsequently, Zhang and Yang \cite{33} provided a detailed analysis on the power penalty approach to coupled parabolic PDCPs arising in the valuation of American options under regime-switching. Xing and Ma \cite{24} studied the convergence rates of trinomial tree methods (TTMs) and perturbed finite difference methods (PFDMs) for American options under regime-switching model. This work provides a link between probability and deterministic approaches.

The fitted finite volume method was first used in price stock options governed by the standard Black-Scholes equation \cite{22}. Later on it was extended to other types of options — cf. \cite{5, 14, 30–32} and references therein. The method combines finite volume formulations with a fitted approximation technique. It overcomes the drift dominated phenomena and gained popularity in option pricing. The fitted finite volume method in \cite{29} evaluates the American options under regime-switching model and the corresponding PDEs are treated by a fully implicit time stepping scheme. We note that the scheme has the first-order convergence rate and the numerical tests carried out in \cite{29} consider two-regime cases only.

In this work, a numerical method for pricing American options under regime-switching model is studied. Based on a penalty approach, the PDCPs (2.1) are reduced to a set of coupled nonlinear PDEs. Afterwards, fitted finite volume method in space with the Crank-Nicolson method in time is applied to American options under regime-switching model. We show that the numerical scheme is consistent, stable and monotone. This guarantees the convergence of the numerical solution to the viscosity solution of continuous problem. In order to solve the discretised nonlinear system, we construct an iterative method and prove its convergence. A number of numerical experiments carried out for two-, three- and four-regime American options models show the accuracy and robustness of the method proposed.

The paper is organised as follows. In Section 2, we introduce American options under regime-switching model and describe the penalty approach to the problem (2.1). Section 3 deals with the Crank-Nicolson fitted finite volume scheme. The convergence of this numerical scheme is studied in Section 4. In Section 5, we construct an iterative algorithm for
nonlinear systems and establish its convergence. The results of numerical tests are provided in Section 6. The paper ends with some conclusions.

2. A Mathematical Model and the Penalty Approach

We consider a continuous-time Markov chain $\alpha_t$ taking values among $Q$ different regimes, where $Q$ is the total number of states (also known as regimes) considered in the economy. Each state represents a particular regime and is labeled by an integer $i$, $1 \leq i \leq Q$. Hence, the state space of $\alpha_t$ is given by $\mathcal{W} = \{1, 2, \ldots, Q\}$. Let $\mathcal{A} = (a_{ij})_{Q \times Q}$ be the generator matrix of $\alpha_t$. Following [26], we assume that

1. $a_{ij} \geq 0$, if $i \neq j$.
2. $a_{ii} = -\sum_{j \neq i} a_{ij}$ for any $i$.

Note that $a_{ii} \leq 0$ for any $i = 1, \ldots, Q$.

Under the risk-neutral measure, the stochastic process for the underlying asset $S_t$ is

$$\frac{dS_t}{S_t} = r_{\alpha_t} dt + \sigma_{\alpha_t} dB_t,$$

where $\sigma_{\alpha_t}$ is the volatility of the asset $S_t$ and $r_{\alpha_t}$ is the risk-free interest rate [10].

Let $V_i(S, \tau)$ be the value of an American put option with striking price $K$ in regime $i$, where the holder can receive a given payoff $V^*(S)$ at the expiry date $T$. Setting

$$V(S, \tau) := [V_1(S, \tau), V_2(S, \tau), \ldots, V_Q(S, \tau)]^T$$

and introducing the time-reverse transformation $\tau := T - t$, one can express the option pricing problem under regime-switching model as coupled parabolic PDECs — cf. [9, 29]. More exactly, let $\mathcal{L}_i$ be the operator defined by

$$\mathcal{L}_i V(S, \tau) = \frac{\partial V_i}{\partial \tau} - \frac{1}{2} \sigma_i^2 S^2 \frac{\partial^2 V_i}{\partial S^2} - r_i S \frac{\partial V_i}{\partial S} + (r_i - a_{ii})V_i - \sum_{j \neq i} a_{ij} V_j.$$

Then almost everywhere on the set $(0, +\infty) \times (0, T)$ we have

$$\mathcal{L}_i V(S, \tau) \geq 0,$$
$$V(S, \tau) - V^*(S) \geq 0,$$
$$\mathcal{L}_i V(S, \tau) \cdot (V(S, \tau) - V^*(S)) = 0, \quad i = 1, \ldots, Q,$$

and this problem is supplemented by the initial and boundary conditions

$$V_i(S, 0) = V^*(S) = \max\{K - S, 0\},$$
$$V_i(0, \tau) = K, \quad \lim_{S \to +\infty} V_i(S, \tau) = 0.$$
For the sake of simplicity, here we only consider the American put options under two-regime model.

**Problem 1.** Let \((S, \tau) \in (0, +\infty) \times (0, T), V(S, \tau) = [V_1(S, \tau), V_2(S, \tau)]^\top\), and

\[
\mathcal{L}_1 V(S, \tau) = \frac{\partial V_1}{\partial \tau} - \frac{1}{2} \sigma_1^2 S^2 \frac{\partial^2 V_1}{\partial S^2} - r_1 S \frac{\partial V_1}{\partial S} + (r_1 - \alpha_{11}) V_1 + \alpha_{11} V_2,
\]

\[
\mathcal{L}_2 V(S, \tau) = \frac{\partial V_2}{\partial \tau} - \frac{1}{2} \sigma_2^2 S^2 \frac{\partial^2 V_2}{\partial S^2} - r_2 S \frac{\partial V_2}{\partial S} + (r_2 - \alpha_{22}) V_2 + \alpha_{22} V_1
\]

be coupled degenerate parabolic partial differential operators and \(i = 1, 2\). We consider the system

\[
\mathcal{L}_i V(S, \tau) \geq 0, \quad V(S, \tau) - V^*(S) \geq 0, \quad \mathcal{L}_i V(S, \tau) \cdot (V(S, \tau) - V^*(S)) = 0 \tag{2.4}
\]

with the initial and boundary conditions (2.2)-(2.3).

For computational purpose, the underlying asset \(S\) is restricted to the finite region \(I = [0, S_{\text{max}}]\), where \(S_{\text{max}}\) is chosen sufficiently large in order to ensure the accuracy of the method. Thus the boundary conditions (2.3) become

\[
V_i(0, \tau) = K, \quad V_i(S_{\text{max}}, \tau) = 0, \quad i = 1, 2. \tag{2.5}
\]

The PDCPs (2.4) can be transformed into the following nonlinear penalised PDEs.

**Problem 2.**

\[
\mathcal{L}_1 V^\gamma(S, \tau) - \gamma \left[ V^* - V_1^\gamma \right]_+ = 0, \tag{2.6}
\]

\[
\mathcal{L}_2 V^\gamma(S, \tau) - \gamma \left[ V^* - V_2^\gamma \right]_+ = 0, \tag{2.7}
\]

with the initial and boundary conditions

\[
V_i^\gamma(S, 0) = V^*(S), \quad V_i^\gamma(0, \tau) = K, \quad V_i^\gamma(S_{\text{max}}, \tau) = 0, \quad i = 1, 2,
\]

where \(V^\gamma(S, \tau) = [V_1^\gamma(S, \tau), V_2^\gamma(S, \tau)]^\top, \gamma > 1\) is the penalty parameter and \([x]_+ = \max\{0, x\}\) for any \(x\). By adding the penalty term, the above penalty approach is to make the positive parts \([V^* - V_1^\gamma]_+\) in (2.6) and \([V^* - V_2^\gamma]_+\) in (2.7) close to zero if \(\gamma\) becomes sufficiently large. Hence, the complementarity conditions in (2.4) are approximately satisfied.

**Remark 2.1.** Since the diffusion operators \(\mathcal{L}_i, i = 1, 2\) are degenerate, \(\max\{\cdot, \cdot\}\) is non-linear and the payoff functions are non-smooth, Problems 1 and 2 have no classical solution, in general. Therefore, we are looking for viscosity solutions of these problems. Detailed discussions concerning the existence and uniqueness of such solutions for PDEs arising in financial mathematics are presented Refs. [1, 2]. It is well known that the viscosity solution is the correct financially relevant solution.
Definition 2.1 (Continuous Viscosity Solution, cf. Barles [2]). Let $\Omega$ be a domain in $\mathbb{R}^{N_2}$ and $H$ be a continuous function such that 

$$H(x, u, p, N_1) \leq H(x, u, p, N_2), \quad N_1 \geq N_2$$

for any $x \in \Omega$, $u \in \mathbb{R}$, $p \in \mathbb{R}^{N_2}$. A function $u \in C(\Omega)$ is the viscosity solution of the equation 

$$H\left(x, u, Du, D^2u\right) = 0$$

in $\Omega$ if and only if the following conditions hold:

1. If $\varphi \in C^2(\Omega)$ and $x_0 \in \Omega$ is a local maximum point of $u - \varphi$, then 

$$H\left(x_0, u(x_0), D\varphi(x_0), D^2\varphi(x_0)\right) \leq 0.$$

2. If $\varphi \in C^2(\Omega)$ and $x_0 \in \Omega$ is a local minimum point of $u - \varphi$, then 

$$H\left(x_0, u(x_0), D\varphi(x_0), D^2\varphi(x_0)\right) \geq 0.$$

We note that according to [29], the solution $V^\gamma$ of Problem 2 is an approximation of the solution $V$ of Problem 1 in the viscosity sense. The convergence of $V^\gamma$ to $V$ as $\gamma \to \infty$ has been studied by Li et al. [18], Wang et al. [23], Zhang et al. [31, 33], and by many others. Therefore, we focus on the numerical approximation of the viscosity solution of Problem 2. In order to avoid symbol overload, we drop subscript $\gamma$ in $V_i^\gamma$, $i = 1, 2$, but will bear in mind that $V_i$, $i = 1, 2$ are the solutions of Problem 2.

3. The Crank-Nicolson Fitted Finite Volume Discretisation

We now consider the Crank-Nicolson fitted finite volume method for the penalised PDEs (2.6) and (2.7).

3.1. Semi-discretisation

Let us transform (2.6) and (2.7) into the following conservative form:

$$\frac{\partial V_1}{\partial \tau} = \frac{\partial}{\partial S} \left[a_1 S^2 \frac{\partial V_1}{\partial S} + b_1 S V_1\right] - c_1 V_1 - \alpha_{11} V_2 + \gamma [V^* - V_1]_+, \quad (3.1)$$

$$\frac{\partial V_2}{\partial \tau} = \frac{\partial}{\partial S} \left[a_2 S^2 \frac{\partial V_2}{\partial S} + b_2 S V_2\right] - c_2 V_2 - \alpha_{22} V_1 + \gamma [V^* - V_2]_+, \quad (3.2)$$

where 

$$a_1 = \frac{\sigma_1^2}{2} > 0, \quad b_1 = r_1 - \sigma_1^2, \quad c_1 = r_1 - \alpha_{11} + b_1,$$

$$a_2 = \frac{\sigma_2^2}{2} > 0, \quad b_2 = r_2 - \sigma_2^2, \quad c_2 = r_2 - \alpha_{22} + b_2.$$
The fitted finite volume method is based on the self-adjoint form (3.1). We first define two spatial partitions of \( I = [0, S_{\text{max}}] \). Let \( I \) be divided into \( M \) sub-intervals
\[
I_i = (S_i, S_{i+1}), \quad i = 0, 1, \ldots, M - 1
\]
with \( 0 = S_0 < S_1 < S_2 < \ldots < S_M = S_{\text{max}} \). For each \( i = 0, 1, \ldots, M - 1 \), let \( h_i = S_{i+1} - S_i \).
We also set \( S_{i-1/2} := (S_{i-1} + S_i)/2 \) and \( S_{i+1/2} := (S_i + S_{i+1})/2 \) for each \( i = 1, \ldots, M - 1 \).
The intervals \( I_i^* = (S_{i-1/2}, S_{i+1/2}) \), \( i = 0, 1, \ldots, M \) form another partition of \( I = [0, S_{\text{max}}] \) if we define \( S_{-1/2} = S_0 \) and \( S_{M+1/2} = S_M \).

Integrating (3.1) over the interval \( I_i^* \), \( i = 1, \ldots, M - 1 \) yields
\[
\int_{I_i^*} \frac{\partial V_1}{\partial \tau} dS = S \left( a_1 S \frac{\partial V_1}{\partial S} + b_1 V_1 \right) \bigg|_{S_{i-1/2}}^{S_{i+1/2}} - \int_{I_i^*} c_1 V_1 dS
\]
\[- \int_{I_i^*} \alpha_{11} V_2 dS + \gamma \int_{I_i^*} [V^* - V_1]_+ dS. \tag{3.3}
\]
Applying the one-point quadrature rule to all terms except the first one in the right-hand side of (3.3), we obtain
\[
\frac{\partial V_1}{\partial \tau} l_i = S_{i+1/2} \rho(V_1)|_{S_{i+1/2}} - S_{i-1/2} \rho(V_1)|_{S_{i-1/2}} - c_1 V_1 l_i
- \alpha_{11} V_2 l_i + \gamma l_i [V^* - V_1]_+, \tag{3.4}
\]
where \( l_i = S_{i+1/2} - S_{i-1/2} \) is the length of interval \( I_i^* \),
\[
\rho(V_1) = a_1 SV_1' + b_1 V_1 \tag{3.5}
\]
is the weighted flux density associated with \( V_1 \), and \( V_{1,i}, V_{2,i} \) are approximations of \( V_1(S_i, \tau) \) and \( V_2(S_i, \tau) \), respectively.

Since the discretisation of (3.4) and (3.5) is identical to [29], the details are omitted here. Thus the semi-discrete form of the Eq. (2.6) is
\[
\frac{\partial V_1}{\partial \tau} = M_1 V_1 + P_1 (V^* - V_1) - \alpha_{11} V_2 + F_1, \tag{3.6}
\]
where
\[
V_1 = [V_{1,1}, V_{1,2}, \ldots, V_{1,M-1}]^T,
\]
\[
V_2 = [V_{2,1}, V_{2,2}, \ldots, V_{2,M-1}]^T,
\]
\[
V^* = [V^*_1, V^*_2, \ldots, V^*_M]^T,
\]
and \( P_1 \) is the \((M - 1) \times (M - 1)\) diagonal matrix with the entries
\[
P_{1,ii} = \begin{cases} 
\gamma, & \text{if } V_{1,i} < V^*_i, \\
0, & \text{otherwise.}
\end{cases}
\]
Besides, the matrix $M_1$ has the form

$$M_1 = \begin{pmatrix}
\chi_{1,1} & w_{1,1} & \beta_{1,2} & X_{1,2} & w_{1,2} & \ddots & \ddots & \ddots \\
\beta_{1,2} & \chi_{1,2} & w_{1,2} & \ddots & \ddots & \ddots & \ddots & \ddots \\
\ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\
\beta_{1,M-1} & X_{1,M-1} & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\
\end{pmatrix} = (M-1) \times (M-1),$$

where

$$\begin{align*}
\beta_{1,1} &= \frac{S_1}{4l_1} (a_1 - b_1), \\
\chi_{1,1} &= \frac{S_1}{4l_1} (a_1 + b_1) - \frac{b_1 S_3 S_1}{S_2 - S_1} l_i - c_1, \\
\beta_{1,i} &= \frac{b_1 S_{i-1} S_1}{S_i - S_1} l_i, \\
\chi_{1,i} &= -\frac{b_1 S_{i-1} S_1}{S_i - S_1} l_i - \frac{b_1 S_{i+1} S_1}{S_i - S_1} l_i - c_1
\end{align*}$$

for $i = 2, \ldots, M-1$. $F_1 = [\beta_{1,1} V_{1,0}, 0, \ldots, 0, w_{1,M-1} V_{1,M}]^T$ and $\eta_1 = b_1/a_1$.

The semi-discrete form of the Eq. (2.7) can be obtained analogously. Thus we have

$$\frac{\partial V_2}{\partial \tau} = M_2 V_2 + P_2 (V^* - V_2) - \alpha_{22} V_1 + F_2,$$

where $P_2$ is an $(M-1) \times (M-1)$ diagonal matrix with the entries

$$p_{2,ii} = \begin{cases} 
\gamma, & \text{if } V_{2,i} < V^*_i, \\
0, & \text{otherwise}.
\end{cases}$$

The matrix $M_2$ has the form

$$M_2 = \begin{pmatrix}
\chi_{2,1} & w_{2,1} & \beta_{2,2} & X_{2,2} & w_{2,2} & \ddots & \ddots & \ddots \\
\beta_{2,2} & \chi_{2,2} & w_{2,2} & \ddots & \ddots & \ddots & \ddots & \ddots \\
\ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\
\beta_{2,M-1} & X_{2,M-1} & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\
\end{pmatrix} = (M-1) \times (M-1),$$

where

$$\begin{align*}
\beta_{2,1} &= \frac{S_1}{4l_1} (a_2 - b_2), \\
\chi_{2,1} &= \frac{S_1}{4l_1} (a_2 + b_2) - \frac{b_2 S_3 S_1}{S_2 - S_1} l_i - c_2,
\end{align*}$$

and

$$w_{2,1} = \frac{b_2 S_3 S_1}{S_2 - S_1} l_i.$$
The equations (3.6) and (3.7) produce the fully discrete coupled system

\[
\begin{align*}
\beta_{2,i} &= \frac{b_2S_{i-1/2}S_{i-1}^{\eta_2}}{(S_{i-1}^{\eta_2} - S_{i-1}^{\eta_1}) l_i}, \\
\chi_{2,i} &= \frac{-b_2S_{i-1/2}S_{i-1}^{\eta_2}}{(S_{i-1}^{\eta_2} - S_{i-1}^{\eta_1}) l_i} - \frac{b_2S_{i+1/2}S_{i+1}^{\eta_2}}{(S_{i+1}^{\eta_2} - S_{i+1}^{\eta_1}) l_i}, \\
w_{2,i} &= \frac{b_2S_{i+1/2}S_{i+1}^{\eta_2}}{(S_{i+1}^{\eta_2} - S_{i+1}^{\eta_1}) l_i},
\end{align*}
\]

for \(i = 2, \ldots, M - 1\), \(F_2 = [\beta_{2,1}V_{2,0}, 0, \ldots, 0, w_{2,M-1}V_{2,M}]^T\) and \(\eta_2 = b_2/a_2\).

### 3.2. The Crank-Nicolson scheme

Let \(O\) and \(I\) respectively denote the zero and identity matrices of the corresponding order, \(N\) be a positive integer, \(\Delta \tau = T/N\) and \(\tau_n = n\Delta \tau\), \(n = 0, 1, \ldots, N\) the partition of \([0, T]\).

The Crank-Nicolson time-stepping scheme coupled with the fitted finite volume discretisation of the Eqs. (3.6) and (3.7) produces the fully discrete coupled system

\[
\left[ I - \frac{\Delta \tau}{2} S + \Delta \tau P(V^{n+1}) \right] V^{n+1} = \left( I + \frac{\Delta \tau}{2} S \right) V^n + \Delta \tau P(V^{n+1}) \Lambda_n + \Delta \tau F^n,
\]

\(n = 0, 1, \ldots, N - 1\),

where \(V^n = [V_1^n, V_2^n]^T\), \(F^n = [F_1^n, F_2^n]^T\) and \(V^0\) is an initial condition,

\[
S = \begin{pmatrix} M_1 & -\alpha_{11} I \\ -\alpha_{22} I & M_2 \end{pmatrix}_{(2M-2)^2}, \quad P(V^{n+1}) = \begin{pmatrix} P_1(V^{n+1}) & O \\ O & P_2(V^{n+1}) \end{pmatrix}
\]

and the \((M-1) \times (M-1)\) diagonal matrices \(P_1(V^{n+1}), P_2(V^{n+1})\) have the entries

\[
P_1(V^{n+1})_{ii} = \begin{cases} \gamma, & \text{if } V_{1,i}^{n+1} < V_i^*, \\ 0, & \text{otherwise}, \end{cases} \quad P_2(V^{n+1})_{ii} = \begin{cases} \gamma, & \text{if } V_{2,i}^{n+1} < V_i^*, \\ 0, & \text{otherwise}. \end{cases}
\]

We note that in (3.8), the Dirichlet boundary conditions (2.5) at \(S = 0\) and \(S = S_{\text{max}}\) are used along with the payoff function as the initial conditions. Moreover, it follows from [29] that \(S\) is an \(M\)-matrix. Therefore, recalling the definition of \(P(V^{n+1})\), we obtain the following result.

**Theorem 3.1.** If \(r_1 > 0\) and \(r_2 > 0\), then \(I - (\Delta \tau/2)S\) and \(I - (\Delta \tau/2)S + \Delta \tau P(V^{n+1})\) are \(M\)-matrices.

**Remark 3.1.** Theorem 3.1 implies that the numerical scheme (3.8) satisfies the discrete maximum principle. Therefore, the discrete arbitrage inequality holds in option pricing theory.
4. Convergence of Numerical Scheme

In order to study the convergence of the method (3.8), we first show its consistency.

**Lemma 4.1.** The numerical scheme (3.8) is consistent.

**Proof:** From the fitted finite volume discretisation, we can see that the consistency of (3.8) relies on the consistency of \( \rho(V_1) \) and \( \rho(V_2) \). Let \( u \) be a sufficiently smooth function and \( u_h \) the discrete approximation of \( u \) with the mesh parameter \( h = \max\{h_i, \Delta \tau\} \). It is shown in [22, 29] that

\[
\left[ [\rho(u) - \rho_h(u_h)]_{S,i+1/2} \right] = o(h),
\]

where

\[
\rho_h(V_i) = \rho_i(V_i) \quad \text{for all} \quad S \in I_i, \quad i = 0, 1, \ldots, M - 1.
\]

Hence the consistency of the numerical scheme (3.8) follows.

Consider the stability of the numerical scheme (3.8).

**Lemma 4.2.** The numerical scheme (3.8) is stable as \( h \to 0 \), i.e.

\[
\|V^n\|_\infty \leq \|\lambda\|_\infty, \quad n = 1, 2, \ldots, N.
\]

**Proof:** The component form of the Eq. (3.8) is

\[
\begin{align*}
&\left[ 1 - \frac{\Delta \tau}{2} \chi_{1,i} + \Delta \tau \rho \left( V^{n+1}_{1,i} \right) \right] V^{n+1}_{1,i} - \frac{\Delta \tau}{2} \beta_{1,i} V^{n+1}_{1,i-1} - \frac{\Delta \tau}{2} \beta_{1,i} V^{n+1}_{1,i+1} \\
&= \left( 1 + \frac{\Delta \tau}{2} \chi_{1,i} \right) V^n_{1,i} + \frac{\Delta \tau}{2} \beta_{1,i} V^n_{1,i-1} + \frac{\Delta \tau}{2} \beta_{1,i} V^n_{1,i+1} \\
&- \frac{\Delta \tau}{2} \alpha_{11} V^{n+1}_{1,i} - \frac{\Delta \tau}{2} \alpha_{11} V^{n+1}_{1,i} + \Delta \tau \rho \left( V^{n+1}_{1,i} \right) V^n_{1,i} + \Delta \tau F^n_{1,i},
\end{align*}
\]

\[
\begin{align*}
&\left[ 1 - \frac{\Delta \tau}{2} \chi_{2,i} + \Delta \tau \rho \left( V^{n+1}_{2,i} \right) \right] V^{n+1}_{2,i} - \frac{\Delta \tau}{2} \beta_{2,i} V^{n+1}_{2,i-1} - \frac{\Delta \tau}{2} \beta_{2,i} V^{n+1}_{2,i+1} \\
&= \left( 1 + \frac{\Delta \tau}{2} \chi_{2,i} \right) V^n_{2,i} + \frac{\Delta \tau}{2} \beta_{2,i} V^n_{2,i-1} + \frac{\Delta \tau}{2} \beta_{2,i} V^n_{2,i+1} \\
&- \frac{\Delta \tau}{2} \alpha_{22} V^{n+1}_{2,i} - \frac{\Delta \tau}{2} \alpha_{22} V^{n+1}_{2,i} + \Delta \tau \rho \left( V^{n+1}_{2,i} \right) V^n_{2,i} + \Delta \tau F^n_{2,i}.
\end{align*}
\]

The inequalities \( \beta_{1,i} \geq 0, \beta_{2,i} \geq 0, w_{1,i} \geq 0, w_{2,i} \geq 0, \chi_{1,i} \leq 0 \) and \( \chi_{2,i} \leq 0 \) yield

\[
\begin{align*}
&\left[ 1 - \frac{\Delta \tau}{2} \chi_{1,i} + \Delta \tau \rho \left( V^{n+1}_{1,i} \right) \right] |V^{n+1}_{1,i}| \\
\leq &\frac{\Delta \tau}{2} \beta_{1,i} |V^{n+1}_{1,i-1}| + \frac{\Delta \tau}{2} \beta_{1,i} |V^{n+1}_{1,i+1}| - \frac{\Delta \tau}{2} \alpha_{11} |V^{n+1}_{2,i}| + \left( 1 + \frac{\Delta \tau}{2} \chi_{1,i} \right) |V^n_{1,i}| \\
&+ \frac{\Delta \tau}{2} \beta_{1,i} |V^{n+1}_{1,i-1}| + \frac{\Delta \tau}{2} \beta_{1,i} |V^{n+1}_{1,i+1}| - \frac{\Delta \tau}{2} \alpha_{11} |V^{n+1}_{2,i}| + \Delta \tau \rho \left( V^{n+1}_{1,i} \right) |V^n_{1,i}| + \Delta \tau F^n_{1,i},
\end{align*}
\]

\[
\begin{align*}
\leq &\frac{\Delta \tau}{2} \left( \beta_{1,i} + w_{1,i} \right) \|V^{n+1}_{1,i}\|_\infty - \frac{\Delta \tau}{2} \alpha_{11} |V^{n+1}_{2,i}| + \left[ 1 + \frac{\Delta \tau}{2} \left( \chi_{1,i} + \beta_{1,i} + w_{1,i} \right) \right] \|V^n_{1,i}\|_\infty
\end{align*}
\]
\[-\frac{\Delta \tau}{2} a_{11} |V^{n+1}_{2,i}| + \Delta \tau P (V^{n+1}_{1,i})_i \|V^*\|_\infty + \Delta \tau \|P^n_i\|_\infty, \tag{4.1}\]

\[
\left[ 1 - \frac{\Delta \tau}{2} \chi_{1,i} + \Delta \tau P (V^{n+1}_{1,i})_i \right] |V^{n+1}_{2,i}| \leq \frac{\Delta \tau}{2} \beta_{2,i} |V^{n+1}_{2,i-1}| + \frac{\Delta \tau}{2} \alpha_{22} |V^{n+1}_{1,i}| + \left( 1 + \frac{\Delta \tau}{2} \chi_{2,i} \right) |V^{n}_{2,i}| \\
+ \frac{\Delta \tau}{2} \beta_{2,i} |V^{n}_{2,i-1}| + \frac{\Delta \tau}{2} \alpha_{22} |V^{n}_{1,i}| + \Delta \tau P (V^{n+1}_{2,i})_i |V^*_{i}| + \Delta \tau \|P^n_i\|_\infty \leq \frac{\Delta \tau}{2} (\beta_{2,i} + w_{2,i}) \|V^{n+1}_{2,i}\|_\infty - \frac{\Delta \tau}{2} \alpha_{22} |V^{n+1}_{1,i}| + \left[ 1 + \frac{\Delta \tau}{2} (\chi_{2,i} + \beta_{2,i} + w_{2,i}) \right] \|V^n_{2,i}\|_\infty \\
- \frac{\Delta \tau}{2} \alpha_{22} |V^{n+1}_{1,i}| + \Delta \tau P (V^{n+1}_{2,i})_i \|V^*\|_\infty + \Delta \tau \|P^n_i\|_\infty, \tag{4.2}\]

where \(a_{11}, \alpha_{22} < 0\).

Choose \(i \in \{1, \ldots, 2M - 2\}\) such that \(\|V^{n+1}\|_\infty = |V^{n+1}_{i}|\). We estimate (4.1) and (4.2) in the following two cases.

Case 1. Let \(1 \leq l \leq M - 1\). It is easily seen that

\[\|V^{n+1}\|_\infty = \|V^{n+1}_{1}\|_\infty = |V^{n+1}_{1,l}| > \|V^{n+1}_{2,l}\|_\infty \geq |V^{n+1}_{2,l}|.\]

Then, (4.1) becomes

\[
\left[ 1 - \frac{\Delta \tau}{2} \chi_{1,i} + \Delta \tau P (V^{n+1}_{1,i})_i \right] |V^{n+1}_{1,i}| \leq \frac{\Delta \tau}{2} (\beta_{1,i} + w_{1,i}) \|V^{n+1}_{1,i}\|_\infty + \left[ 1 + \frac{\Delta \tau}{2} (\chi_{1,i} + \beta_{1,i} + w_{1,i} - a_{11}) \right] \|V^n_{1,i}\|_\infty \\
+ \Delta \tau P (V^{n+1}_{1,i})_i \|V^*\|_\infty + \Delta \tau \|P^n_i\|_\infty
\]

for all admissible \(i\). Hence,

\[
\left[ 1 - \frac{\Delta \tau}{2} \chi_{1,i} + \Delta \tau P (V^{n+1}_{1,i})_i \right] \|V^{n+1}_{1,i}\|_\infty \leq \frac{\Delta \tau}{2} (\beta_{1,i} + w_{1,i}) \|V^{n+1}_{1,i}\|_\infty + \left[ 1 + \frac{\Delta \tau}{2} (\chi_{1,i} + \beta_{1,i} + w_{1,i} - a_{11}) \right] \|V^n_{1,i}\|_\infty \\
+ \Delta \tau P (V^{n+1}_{1,i})_i \|V^*\|_\infty.
\]

This implies the estimate

\[
\|V^{n+1}_{1}\|_\infty \leq \max \left( \|V^n_{1}\|_\infty, \|V^*\|_\infty \right) \times \left[ 1 + \frac{\Delta \tau}{2} (\chi_{1,i} + \beta_{1,i} + w_{1,i} - a_{11}) + \Delta \tau P (V^{n+1}_{1,i})_i \right] \\
1 + \left( \frac{\Delta \tau}{2} \right) (\chi_{1,i} - \beta_{1,i} - w_{1,i} + a_{11}) + \Delta \tau P (V^{n+1}_{1,i})_i \\
= \max \left( \|V^n_{1}\|_\infty, \|V^*\|_\infty \right) \times \left[ 1 - \frac{\Delta \tau}{2} (\chi_{1,i} + \beta_{1,i} + w_{1,i} - a_{11}) \right] \\
1 + \left( \frac{\Delta \tau}{2} \right) (\chi_{1,i} - \beta_{1,i} - w_{1,i} + a_{11}) + \Delta \tau P (V^{n+1}_{1,i})_i \\
\leq \max \left( \|V^n_{1}\|_\infty, \|V^*\|_\infty \right).
\]
where \(-\chi_{1,l} - \beta_{1,l} - w_{1,l} + \alpha_{11} = r_1 > 0\). Furthermore,

\[
\|V^{n+1}\|_\infty = \|V^{n+1}_1\|_\infty \leq \max\left(\|V^n\|_\infty, \|V^n\|_\infty\right).
\] (4.3)

Case 2. \(M \leq l \leq 2M - 2\). Now we obtain a similar estimate — viz.

\[
\|V^{n+1}\|_\infty = \|V^{n+1}_2\|_\infty \leq \max\left(\|V^n\|_\infty, \|V^n\|_\infty\right).
\] (4.4)

Combining (4.3) and (4.4) yields

\[
\|V^{n+1}\|_\infty \leq \max\left(\|V^n\|_\infty, \|V^n\|_\infty, \|V^n\|_\infty\right) \leq \max\left(\|V^n\|_\infty, \|V^n\|_\infty\right) \\
\leq \cdots \leq \max\left(\|V^0\|_\infty, \|V^\ast\|_\infty\right) = \|V^\ast\|_\infty = \|\Lambda\|_\infty
\]

for all feasible \(n\).

Hence the numerical scheme (3.8) is stable.

Finally, the monotonicity of the numerical scheme (3.8) is given by the following lemma.

**Lemma 4.3.** The numerical scheme (3.8) is unconditionally monotone.

**Proof.** For \(i = 0\) or \(i = M\), the result is trivial. If \(0 < i < M\), Theorem 3.1 implies that 
\(-\left(\Delta \tau / 2\right)S\) and \(I - \left(\Delta \tau / 2\right)S\) are \(M\)-matrices. Hence \([I - \left(\Delta \tau / 2\right)S]V^{n+1}\) is the strictly increasing function of \(V^{n+1}_i\), non-increasing function of \(V^{n+1}_{i+1}\) and \(V^{n+1}_{i-1}\), and \([I - \left(\Delta \tau / 2\right)S]V^n\) is also a strictly increasing function of \(V^n_i\), non-increasing function of \(V^n_{i+1}\) and \(V^n_{i-1}\). On the other hand, it is obvious that \(-V^n_i\) is the decreasing function of \(V^n_i\). Finally, the penalty term \(-\Delta \tau P(V^{n+1})_{i}(\Lambda_i - V^{n+1}_i)\) is the non-decreasing function of \(V^{n+1}_i\). Hence the numerical scheme (3.8) is unconditionally monotone.

**Theorem 4.1.** The solution of the numerical scheme (3.8) converges to the viscosity solution of penalised Eqs. (2.6) and (2.7) as \(h \to 0\).

**Proof.** It was shown in [2] that if the discretisation is consistent, stable and monotone, then the solution of the fully discretised system (3.8) converges to the viscosity solution. According to Lemmas 4.1-4.3, the solution of the numerical scheme (3.8) converges to the viscosity solution of penalised Eqs. (2.6) and (2.7) as \(h \to 0\).

5. Iterative Scheme for Discrete System

In order to effectively solve the nonlinear algebraic system, we propose and analyse an iterative method for (3.8) at each time step — cf. Algorithm 5.1.

Now we show the convergence of this iterative method.

**Theorem 5.1.** At each time step \(n\), the iterative scheme (5.1) generates a sequence of solutions \(\{\tilde{V}^i\}\), starting with any initial guess \(\tilde{V}^0\). If \(h\) is sufficiently small, the sequence monotonically converges to the solution \(V^{n+1}\) of the scheme (3.8) as \(l \to \infty\).
Algorithm 5.1 Iterative scheme for (3.8).

1. Let \( n = 0 \).
2. Set \( l = 0 \) and \( \hat{V}^0 = V^n \), where

\[
\hat{V}^0 = (\hat{V}^0_1, \hat{V}^0_2), \quad \hat{V}^0_1 = V^n_1, \quad \hat{V}^0_2 = V^n_2.
\]

3. Solve

\[
\left[ I - \frac{\Delta \tau}{2} S + \Delta \tau P(\hat{V}^i) \right] \hat{V}^{i+1} = \left( I + \frac{\Delta \tau}{2} S \right) V^n + \Delta \tau P(\hat{V}^i) A + \Delta \tau F^n.
\]

4. If

\[
\max_i \frac{|\hat{V}^{i+1}_i - \hat{V}^i_i|}{\max(1,|\hat{V}^{i+1}_i|)} < \text{tolerance},
\]

then stop. Otherwise, set \( l := l + 1 \) and go to Step 3.

5. Set \( V^{n+1} = \hat{V}^l_1, n = n + 1 \) and go to Step 2.

Proof: First, we show that the iterate \( \{\hat{V}^i\} \) is bounded for any \( l \). Writing out (5.1) in component form, we have

\[
\begin{align*}
&\left[ 1 - \frac{\Delta \tau}{2} \chi_{1,i} + \Delta \tau P(\hat{V}^i_1) \right] \hat{V}^{i+1}_1 - \frac{\Delta \tau}{2} \beta_{1,i} \hat{V}^{i+1}_{1,i-1} - \frac{\Delta \tau}{2} w_{1,i} \hat{V}^{i+1}_{1,i+1} \\
&= \left( 1 + \frac{\Delta \tau}{2} \chi_{1,i} \right) V^n_1 + \frac{\Delta \tau}{2} \beta_{1,i} V^n_{1,i-1} + \frac{\Delta \tau}{2} w_{1,i} V^n_{1,i+1} \\
&- \frac{\Delta \tau}{2} \alpha_{11} \hat{V}^{i+1}_{2,i} - \frac{\Delta \tau}{2} \alpha_{11} V^n_{2,i} + \Delta \tau P(\hat{V}^i_1) V^n_1 + \Delta \tau F^n_{1,i} \\
&\left[ 1 - \frac{\Delta \tau}{2} \chi_{2,i} + \Delta \tau P(\hat{V}^i_2) \right] \hat{V}^{i+1}_2 - \frac{\Delta \tau}{2} \beta_{2,i} \hat{V}^{i+1}_{2,i-1} - \frac{\Delta \tau}{2} w_{2,i} \hat{V}^{i+1}_{2,i+1} \\
&= \left( 1 + \frac{\Delta \tau}{2} \chi_{2,i} \right) V^n_2 + \frac{\Delta \tau}{2} \beta_{2,i} V^n_{2,i-1} + \frac{\Delta \tau}{2} w_{2,i} V^n_{2,i+1} \\
&- \frac{\Delta \tau}{2} \alpha_{22} \hat{V}^{i+1}_{1,i} - \frac{\Delta \tau}{2} \alpha_{22} V^n_{1,i} + \Delta \tau P(\hat{V}^i_2) V^n_2 + \Delta \tau F^n_{2,i},
\end{align*}
\]

It follows from the inequalities \( \beta_{1,i} \geq 0, \beta_{2,i} \geq 0, w_{1,i} \geq 0, w_{2,i} \geq 0, \chi_{1,i} \leq 0 \) and \( \chi_{2,i} \leq 0 \) that

\[
\begin{align*}
&\left[ 1 - \frac{\Delta \tau}{2} \chi_{1,i} + \Delta \tau P(\hat{V}^i_1) \right] \hat{V}^{i+1}_1 \\
\leq &\frac{\Delta \tau}{2} \beta_{1,i} \hat{V}^{i+1}_{1,i-1} + \frac{\Delta \tau}{2} w_{1,i} \hat{V}^{i+1}_{1,i+1} - \frac{\Delta \tau}{2} \alpha_{11} \hat{V}^{i+1}_{2,i} + \left( 1 + \frac{\Delta \tau}{2} \chi_{1,i} \right) V^n_1 \\
&+ \frac{\Delta \tau}{2} \beta_{1,i} V^n_{1,i} + \frac{\Delta \tau}{2} w_{1,i} V^n_{1,i+1} - \frac{\Delta \tau}{2} \alpha_{11} V^n_{2,i} + \Delta \tau P(\hat{V}^i_1) V^n_1 + \Delta \tau F^n_{1,i} \\
\leq &\frac{\Delta \tau}{2} (\beta_{1,i} + w_{1,i}) \|V^n_1\|_\infty - \frac{\Delta \tau}{2} \alpha_{11} \hat{V}^{i+1}_{2,i} + \left[ 1 + \frac{\Delta \tau}{2} (\chi_{1,i} + \beta_{1,i} + w_{1,i}) \right] \|V^n_1\|_\infty
\end{align*}
\]
Following the proof of Lemma 4.2, we show that
\[
\|\vec{V}^{(i+1)}\|_\infty \leq \|\Lambda\|_\infty
\]
for all sufficiently small \( h \). Thus the norms of \( \|\vec{V}\|_\infty \) are uniformly bounded.

Let us now show that the iterates \( \{\vec{V}_i\} \) form a non-decreasing sequence. The first iteration of (5.1) is
\[
\left[ I - \frac{\Delta \tau}{2} S + \Delta \tau P(\vec{V}_i^{(i-1)}) \right] \vec{V}_i = \left( I + \frac{\Delta \tau}{2} S \right) V_i^n + \Delta \tau P(\vec{V}_i^{(i-1)}) \Lambda + \Delta \tau F_i^n,
\]
and it can be written as
\[
\left( I - \frac{\Delta \tau}{2} S \right) \vec{V}_i = \left( I + \frac{\Delta \tau}{2} S \right) V_i^n + \Delta \tau P(\vec{V}_i^{(i-1)}) (\Lambda - \vec{V}_i) + \Delta \tau F_i^n. \tag{5.2}
\]

Subtracting (5.2) from (5.1) yields
\[
\left( I - \frac{\Delta \tau}{2} S \right) \left( \vec{V}_i^{(i+1)} - \vec{V}_i \right) = \Delta \tau \left[ P(\vec{V}_i) - P(\vec{V}_i^{(i-1)}) \right] \Lambda + \Delta \tau \left[ P(\vec{V}_i^{(i-1)}) \vec{V}_i - P(\vec{V}_i) \vec{V}_i^{(i+1)} \right].
\]

Rewriting this equation as
\[
\left[ I - \frac{\Delta \tau}{2} S + \Delta \tau P(\vec{V}_i) \right] \left( \vec{V}_i^{(i+1)} - \vec{V}_i \right) = \Delta \tau \left[ P(\vec{V}_i) - P(\vec{V}_i^{(i-1)}) \right] (\Lambda - \vec{V}_i), \tag{5.3}
\]
we examine the components of \( [P(\vec{V}_i) - P(\vec{V}_i^{(i-1)})](\Lambda - \vec{V}_i) \).

Case 1. \( \vec{V}_i < \Lambda_i \). In this case we have \( P(\vec{V}_i)_i = \gamma \), so that
\[
\left[ P(\vec{V}_i) - P(\vec{V}_i^{(i-1)}) \right] (\Lambda - \vec{V}_i) = [\gamma - P(\vec{V}_i^{(i-1)})] (\Lambda_i - \vec{V}_i)_i \geq 0.
\]
Case 2. \( \hat{V}_i^j \geq \Lambda_i \). In this case we have \( P(\hat{V}_i^j)_{ii} = 0 \), so that

\[
\left[ P(\hat{V}^j) - P(\hat{V}^{j-1}) \right] (\Lambda - \hat{V}^j) = -P(\hat{V}^{j-1})_{ii} (\Lambda_i - \hat{V}_i^j) \geq 0.
\]

Thus,

\[
\left[ P(\hat{V}^j) - P(\hat{V}^{j-1}) \right] (\Lambda - \hat{V}^j) \geq 0 \tag{5.4}
\]

and the Eqs. (5.3), (5.4) give

\[
\left[ I - \frac{\Delta \tau}{2} S + \Delta \tau P(\hat{V}^j) \right] (\hat{V}^{j+1} - \hat{V}^j) \geq 0. \tag{5.5}
\]

Moreover, according to Theorem 3.1, \( I - (\Delta \tau / 2) S + \Delta \tau P(\hat{V}) \) is an \( M \)-matrix. Therefore, the inequality (5.5) and the discrete maximum principle show that \( \hat{V}^{j+1} - \hat{V}^j \geq 0 \), i.e. \( \{ \hat{V}^j \} \) is a non-decreasing sequence. Recalling that this sequence is bounded, we conclude that it converges to the solution of numerical scheme (3.8).

In order to prove uniqueness, we suppose that there are two solutions \( B \) and \( C \), such that

\[
\begin{align*}
\left[ I - \frac{\Delta \tau}{2} S + \Delta \tau P(B) \right] B & = \left( I + \frac{\Delta \tau}{2} S \right) V^n + \Delta \tau P(B) \Lambda + \Delta \tau F^n, \\
\left[ I - \frac{\Delta \tau}{2} S + \Delta \tau P(C) \right] C & = \left( I + \frac{\Delta \tau}{2} S \right) V^n + \Delta \tau P(C) \Lambda + \Delta \tau F^n.
\end{align*}
\]

Therefore,

\[
\left[ I - \frac{\Delta \tau}{2} S + \Delta \tau P(C) \right] (B - C) = \Delta \tau [P(B) - P(C)](\Lambda - B), \tag{5.6}
\]

and analogously to the proof of the inequality (5.5), we obtain

\[
[P(B) - P(C)](\Lambda - B) \geq 0. \tag{5.7}
\]

Since \( I - (\Delta \tau / 2) S + \Delta \tau P(C) \) is an \( M \)-matrix, the relations (5.6), (5.7) yield \( B \geq C \). Changing the positions of \( C \) and \( B \) shows that \( C \geq B \), so that \( B = C \). \( \square \)

Remark 5.1. Theorems 4.1 and 5.1 show that approximate solution obtained by Algorithm 5.1 converges to the solution of the Eqs. (3.1)-(3.2), i.e. to the unique viscosity solution or financially relevant solution.

Remark 5.2. Although here we only consider the Crank-Nicolson fitted finite volume scheme for American put options under two-regime model, other state models can be also treated — cf. numerical experiments below.
6. Numerical Experiments

We want to demonstrate the efficiency of the method under consideration. In what follows, the term “Error” refers to the difference between successive numerical solutions following mesh refinements — i.e.

\[
\text{Error} := \left| \frac{V^\Delta \tau}{\Delta S} - \frac{V^\Delta \tau/2}{\Delta S/2} \right|
\]

where \( V^\Delta \tau \) is the solution on the spatial mesh of size \( \Delta S \) and time mesh of size \( \Delta \tau \). Consequently, “Order” is \( \log_2 \) of the ratio of successive changes in option price as the grid is refined. The penalty parameter \( \gamma \) and the tolerance in Algorithm 5.1 are \( 10^3 \) and \( 10^{-4} \), respectively. All codes are run in MATLAB R2016a with 32.00 GB RAM and 3.20 GHz processor.

**Example 6.1** (Two-regime option model). Following [9,17,29], we consider American put options under two-regime model with the parameters

\[
\sigma = \begin{pmatrix} \sigma_1 \\ \sigma_2 \end{pmatrix} = \begin{pmatrix} 0.8 \\ 0.3 \end{pmatrix}, \quad r = \begin{pmatrix} r_1 \\ r_2 \end{pmatrix} = \begin{pmatrix} 0.1 \\ 0.05 \end{pmatrix},
\]

\[
\mathcal{A} = \begin{pmatrix} -6 \\ 6 \\ -9 \end{pmatrix}, \quad T = 1, \quad K = 9
\]

and set \( S_{\text{max}} = 50 \). The grids are the consecutive uniform partitions of the solution domain \([0,S_{\text{max}}] \times [0,T]\) with the space and time steps \( m \) and \( n \), respectively.

Tables 1 and 2 show the prices of the American put option under two-regime model obtained by using the Crank-Nicolson fitted finite volume scheme at \( \tau = T \) and \( S = 9,12 \). We observe that the accuracy of the numerical solutions improves as the discretisation grid is refined. The convergence order of the Crank-Nicolson scheme is about 2, consistent with the properties of the Crank-Nicolson scheme. Some authors have also used this approach when an analytic solution is not available [11,15,16]. Here, we display surface plots and the optimal exercise boundaries of Regime 2, the option values and Greeks (Delta and Gamma) at \( \tau = T \), cf. Fig. 1. We note that the numerical solution shown in Fig. 1 has no oscillations or kinks — i.e. the new numerical scheme is robust.

| \((m,n)\) | Regime 1 | Regime 2 |
|-----------|----------|----------|
| Value     | Error    | Order    | Value     | Error    | Order    |
| (50,100)  | 1.964018 | -        | 1.873812  | -        | -        |
| (100,200) | 1.969764 | 5.75e-03 | 1.880053  | 6.24e-03 | -        |
| (200,400) | 1.971261 | 1.50e-03 | 1.881683  | 1.63e-03 | 1.9      |
| (400,800) | 1.971652 | 3.91e-04 | 1.882103  | 4.20e-04 | 2.0      |
| (800,1600)| 1.971753 | 1.01e-04 | 1.882212  | 1.09e-04 | 2.0      |
Table 2: Prices of the American put option under two-regime model, \( \tau = T, S = 12 \).

| \((m,n)\) | Regime 1 Value | Error | Order | Regime 2 Value | Error | Order |
|--------|---------------|-------|------|----------------|-------|------|
| (50,100) | 1.174476 | - | - | 1.086208 | - | - |
| (100,200) | 1.178689 | 4.21e-03 | - | 1.090629 | 4.42e-03 | - |
| (200,400) | 1.179797 | 1.11e-03 | 1.9 | 1.091789 | 1.16e-03 | 1.9 |
| (400,800) | 1.180086 | 2.89e-04 | 1.9 | 1.092090 | 3.01e-04 | 2.0 |
| (800,1600) | 1.180161 | 7.50e-05 | 2.0 | 1.092168 | 7.80e-05 | 2.0 |

Figure 1: American put option under two-regime model, \((m,n) = (400,800)\), \( \tau = T \). a) Option value surface and optimal exercise boundary curve of regime 2; b) Option value; c) Delta; d) Gamma.

Example 6.2 (Three-regime option model). We now consider American put options under three-regime model with the parameters

\[
\sigma = \begin{pmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \end{pmatrix} = \begin{pmatrix} 0.2 \\ 0.15 \\ 0.30 \end{pmatrix}, \quad r = \begin{pmatrix} r_1 \\ r_2 \\ r_3 \end{pmatrix} = \begin{pmatrix} 0.02 \\ 0.02 \\ 0.02 \end{pmatrix}, \\
\mathcal{A} = \begin{pmatrix} -3.2 & 0.2 & 3.0 \\ 1.0 & -1.08 & 0.08 \\ 3.0 & 0.2 & -3.2 \end{pmatrix}, \quad T = 1,
\]

where \( K = 100 \). We choose \( S_{\text{max}} = 300 \) and use consecutive uniform partitions of the solution domain \( [0, S_{\text{max}}] \times [0, T] \).
Table 3: Prices of the American put option under three-regime model at the first state of economy.

| (m, n)     | Value | Error | Order | Value | Error | Order | Value | Error | Order |
|------------|-------|-------|-------|-------|-------|-------|-------|-------|-------|
| (150,300)  | 13.823942 | -     | -     | 8.599614 | -     | -     | 5.103559 | -     | -     |
| (300,600)  | 13.830726 | 6.78e-03 | -     | 8.606754 | 7.14e-03 | -     | 5.109238 | 5.70e-03 | -     |
| (600,1200) | 13.832445 | 1.72e-03 | 2.0   | 8.608569 | 1.82e-03 | 2.0   | 5.110692 | 1.45e-03 | 2.0   |
| (1200,2400)| 13.832883 | 4.38e-04 | 2.0   | 8.609033 | 4.64e-04 | 2.0   | 5.111066 | 3.74e-04 | 2.0   |
| (2400,4800)| 13.832996 | 1.13e-04 | 2.0   | 8.609153 | 1.20e-04 | 2.0   | 5.111163 | 9.70e-05 | 2.0   |

Table 4: Prices of the American put option under three-regime model at the second state of economy.

| (m, n)     | Value | Error | Order | Value | Error | Order | Value | Error | Order |
|------------|-------|-------|-------|-------|-------|-------|-------|-------|-------|
| (150,300)  | 12.058554 | -     | -     | 6.512556 | -     | -     | 3.235033 | -     | -     |
| (300,600)  | 12.066628 | 8.07e-03 | -     | 6.521871 | 9.32e-03 | -     | 3.241551 | 6.52e-03 | -     |
| (600,1200) | 12.068673 | 2.05e-03 | 2.0   | 6.524237 | 2.37e-03 | 2.0   | 3.243216 | 1.67e-03 | 2.0   |
| (1200,2400)| 12.069189 | 5.16e-04 | 2.0   | 6.524838 | 6.01e-04 | 2.0   | 3.243642 | 4.26e-04 | 2.0   |
| (2400,4800)| 12.069322 | 1.33e-04 | 2.0   | 6.524992 | 1.54e-04 | 2.0   | 3.243752 | 1.10e-04 | 2.0   |

Table 5: The prices of the American put option at the third state of economy under three-regime model.

| (m, n)     | Value | Error | Order | Value | Error | Order | Value | Error | Order |
|------------|-------|-------|-------|-------|-------|-------|-------|-------|-------|
| (150,300)  | 14.431858 | -     | -     | 9.267855 | -     | -     | 5.723200 | -     | -     |
| (300,600)  | 14.438352 | 6.49e-03 | -     | 9.274547 | 6.69e-03 | -     | 5.728658 | 5.46e-03 | -     |
| (600,1200) | 14.439995 | 1.64e-03 | 2.0   | 9.276248 | 1.70e-03 | 2.0   | 5.730051 | 1.39e-03 | 2.0   |
| (1200,2400)| 14.440413 | 4.18e-04 | 2.0   | 9.276682 | 4.34e-04 | 2.0   | 5.730407 | 3.56e-04 | 2.0   |
| (2400,4800)| 14.440521 | 1.08e-04 | 2.0   | 9.276794 | 1.12e-04 | 2.0   | 5.730500 | 9.33e-05 | 2.0   |

Tables 3-5 show the prices of the American put options under three-regime model obtained by using the Crank-Nicolson fitted finite volume scheme at $\tau = T$ and $S = 90, 100, 110$. We observe that the accuracy of the numerical solutions improves as the discretisation grid is refined, and the Crank-Nicolson scheme has the second-order convergence rate for point-wise estimates. Besides, Fig. 2 provides the surface plot and the optimal exercise boundaries of Regime 3, the option values and Greeks (Delta and Gamma) at $\tau = T$. The stability of the proposed method is evident.

**Example 6.3 (Four-regime option model).** Following [9, 17], we consider American put options under four-regime model with the parameters

$$
\sigma = \begin{pmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \sigma_4 \end{pmatrix} = \begin{pmatrix} 0.9 \\ 0.5 \\ 0.7 \\ 0.2 \end{pmatrix}, \quad r = \begin{pmatrix} r_1 \\ r_2 \\ r_3 \\ r_4 \end{pmatrix} = \begin{pmatrix} 0.02 \\ 0.10 \\ 0.06 \\ 0.15 \end{pmatrix},
$$
Figure 2: American put option under three-regime model, \((m, n) = (300, 600), T = T_\text{al} = 1, K = 9.\)

\[
\mathcal{A} = \begin{pmatrix}
-1 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\
\frac{1}{3} & -1 & \frac{1}{3} & \frac{1}{3} \\
\frac{1}{3} & \frac{1}{3} & -1 & \frac{1}{3} \\
\frac{1}{3} & \frac{1}{3} & \frac{1}{3} & -1
\end{pmatrix}, \quad T = 1, \quad K = 9.
\]

We choose \(S_{\text{max}} = 50\) and consider the consecutive uniform partitions of the solution domain \([0, S_{\text{max}}] \times [0, T]\).

Tables 6 and 7 show the prices of the American put options under four-regime model obtained by using the Crank-Nicolson fitted finite volume scheme at \(T = T_\text{al} = 1, S = 9, 12.\) We observe that the accuracy of the numerical solutions improves as the discretisation grid is refined and the convergence order of the Crank-Nicolson scheme is about 2. Fig. 3 provides surface plot and the optimal exercise boundaries of Regime 4, the option values and Greeks (Delta and Gamma) at \(T = T_\text{al}.\) The stability of the proposed method is evident. Thus the Crank-Nicolson fitted finite volume scheme combined with the penalty method is very effective for pricing American put options under regime-switching model.
Table 6: The prices of the American put option under four-regime model, $\tau = T, S = 9$.

| (m, n) | Regime 1 Value | Regime 1 Order | Regime 2 Value | Regime 2 Order | Regime 3 Value | Regime 3 Order | Regime 4 Value | Regime 4 Order |
|--------|----------------|----------------|----------------|----------------|----------------|----------------|----------------|----------------|
| (50,100) | 2.550682       | -              | 1.573019       | -              | 2.048325       | -              | 0.970242       | -              |
| (100,200) | 2.555529       | -              | 1.580574       | -              | 2.054398       | -              | 0.981635       | -              |
| (200,400) | 2.556850       | 1.9            | 1.582603       | 1.9            | 2.056046       | 1.9            | 0.984585       | 2.0            |
| (400,800) | 2.557189       | 2.0            | 1.583124       | 2.0            | 2.056466       | 2.0            | 0.985341       | 2.0            |
| (800,1600) | 2.557275       | 2.0            | 1.583256       | 2.0            | 2.056572       | 2.0            | 0.985532       | 2.0            |

Table 7: The prices of the American put option under four-regime model, $\tau = T, S = 12$.

| (m, n) | Regime 1 Value | Regime 1 Order | Regime 2 Value | Regime 2 Order | Regime 3 Value | Regime 3 Order | Regime 4 Value | Regime 4 Order |
|--------|----------------|----------------|----------------|----------------|----------------|----------------|----------------|----------------|
| (50,100) | 1.749256       | -              | 0.831264       | -              | 1.256673       | -              | 0.466959       | -              |
| (100,200) | 1.752738       | -              | 0.835932       | -              | 1.260770       | -              | 0.469782       | -              |
| (200,400) | 1.753686       | 1.9            | 0.837168       | 1.9            | 1.261890       | 1.9            | 0.470545       | 1.9            |
| (400,800) | 1.753933       | 1.9            | 0.837486       | 2.0            | 1.262180       | 2.0            | 0.470745       | 1.9            |
| (800,1600) | 1.753996       | 2.0            | 0.837567       | 2.0            | 1.262254       | 2.0            | 0.470796       | 2.0            |

Figure 3: American put option under four-regime model, $(m, n) = (400,800), \tau = T$. a) Option value surface and optimal exercise boundary curve of regime 4; b) Option values; c) Delta; d) Gamma.
7. Conclusions

We developed a new numerical method for pricing American options under regime-switching model. The original problem is first approximated by a set of nonlinear PDEs. After that a novel fitted finite volume method for the spatial discretisation of the nonlinear penalised PDE system is coupled with the Crank-Nicolson time stepping scheme. It is shown that the discretisation scheme is consistent, stable, monotone and hence convergent. In order to solve nonlinear algebraic systems, we apply an iterative algorithm and show its convergence. Numerical experiments demonstrate the convergence, efficiency and robustness of the numerical method. The Crank-Nicolson scheme here delivers more accurate approximate solutions than the fully implicit time stepping scheme in [29].

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