Eccentric distance sum and adjacent eccentric distance sum index of complement of subgroup graphs of dihedral group

A Abdussakir1, *, E Susanti1, N Hidayati2 and N M Ulya2

1 Department of Mathematics Education, Graduate Program, Universitas Islam Negeri Maulana Malik Ibrahim Malang, Jl. Raya Ir. Soekarno 34 Dadaprejo, Kota Batu 65233, Indonesia
2 Department of Mathematics, Faculty of Science and Technology, Universitas Islam Negeri Maulana Malik Ibrahim Malang, Jl. Gajayana 50, Malang 65144, Indonesia

* sakir@mat.uin-malang.ac.id

Abstract. Let \( G = (V(G), E(G)) \) is a connected simple graph. Let \( ec(v) \) is the eccentricity of vertex \( v \), \( D(v) = \sum_{uv \in E(G)} d(u, v) \) is the sum of all distances from vertex \( v \) and \( deg(v) \) is the degree of vertex \( v \) in \( G \). The eccentric distance sum index of \( G \) is defined as \( \xi^d(G) = \sum_{v \in V(G)} ec(v)D(v) \) and the adjacent eccentric distance sum index of \( G \) is defined as \( \xi^{ae}(G) = \sum_{v \in V(G)} \frac{ec(v)d(v)}{deg(v)} \). For positive integer \( m \) and \( m \geq 3 \), let \( D_{2m} \) be dihedral group of order \( 2m \) and \( N \) is a normal subgroup of \( D_{2m} \). The subgroup graph \( \Gamma_N(D_{2m}) \) of dihedral group \( D_{2m} \) is a simple graph with vertex set \( D_{2m} \) and two distinct vertices \( x \) and \( y \) are adjacent if and only if \( xy \in N \). In the present paper, we compute eccentric distance sum and adjacent eccentric distance sum index of complement of subgroup graph of dihedral group \( D_{2m} \). Total eccentricity, eccentric connectivity index, first Zagreb index, and second Zagreb index of these graphs are also determined.

1. Introduction

The concept of eccentric distance sum index (EDSI) of a connected graph was first introduced by Gupta, Singh, and Madan [1] in 2002. In the same year, Sardana and Madan [2] introduced the concept of adjacent eccentric distance sum index (AEDSI) of a graph. In the beginning, research on EDSI and AEDSI was closely related to the chemical properties and biological activities of molecular structures [1-3]. In subsequent developments, research on EDSI and AEDSI is not only limited to graphs relating to molecular structures. Until now, EDSI and AEDSI have become research topics that are widely studied by researchers on various graphs [4-18].

New concepts of graphs are increasingly being introduced including graphs obtained from an algebraic structure such as group or ring. One of them is the subgroup graph introduced by Anderson, Fasteen, and Lagrange [19] in 2012. If \( G \) is a finite group and \( N \) is a normal subgroup of \( G \), then the subgroup graph \( \Gamma_N(G) \) of \( G \) is a simple and undirected graph with \( V(\Gamma_N(G)) = G \) and \( xy \in E(\Gamma_N(G)) \) if and only if \( xy \in N \). This implies that \( \Gamma_N(G) \) also a simple and undirected graph [20].

In this paper, we determined the EDSI and AEDSI of complement of subgroup graphs of dihedral group. The discussion is specific to the complements of subgroup graphs of dihedral group because the discussed subgroup graphs are not connected.
2. Literature review

Suppose graph $G$ is a simple and connected of order $|V(G)| = p$ and size $|E(G)| = q$. Let $N(u)$ denotes the neighbor of a vertex $u$ in $G$. The degree of a vertex $u$ in $G$ is defined as $\text{deg}(u) = |N(u)|$. The distance $d(v, w)$ between two vertices $v$ and $w$ in $G$ is defined as the length of the shortest $v$-$w$ path in $G$. The sum of all distances from vertex $v$ to any vertices of $G$ is denoted by $D(v)$. Therefore, $D(v) = \sum_{w \in V(G)} d(v, w)$. The eccentricity $ec(v)$ of vertex $v$ is defined by $ec(v) = \max\{d(v, w) : w \in V(G)\}$. Radius $\text{rad}(G)$ of graph $G$ is defined by $\text{rad}(G) = \min\{ec(v) : v \in V(G)\}$ while diameter $\text{diam}(G)$ of graph $G$ is defined by $\text{diam}(G) = \max\{ec(v) : v \in V(G)\}$ [21]. The total eccentricity of graph $G$ is defined as

$$\xi(G) = \sum_{v \in V(G)} ec(v).$$

The eccentric distance sum index (EDSI) of a connected graph $G$ was first introduced by Gupta, Singh, and Madan [1] in 2002 and defined as

$$\xi^d(G) = \sum_{v \in V(G)} ec(v)D(v).$$

In the same year, Sardana and Madan [2] introduced the concept of adjacent eccentric distance sum index (AEDSI) and defined as

$$\xi^{sp}(G) = \sum_{v \in V(G)} \frac{ec(v)D(v)}{\text{deg}(v)}.$$

Other indices based on eccentricity of vertex are eccentric connectivity index, first Zagreb index, and second Zagreb index. Sharma, Goswami, and Madan [22] defined the eccentric connectivity index (ECI) of a connected graph as

$$\xi^c(G) = \sum_{v \in V(G)} ec(v)\text{deg}(v).$$

The first and second Zagreb indices of a connected graph were first introduced by Gutman and Trinajstić [23] and revised by Ghorbani and Hosseinizadeh [24]. For a connected graph $G$, the first Zagreb index is defined as

$$E_1(G) = \sum_{v \in V(G)} (ec(v))^2,$$

while the second Zagreb index is defined as

$$E_2(G) = \sum_{vw \in E(G)} ec(v)ec(w).$$

Let $D_{2m} = \langle r, s \rangle$ is dihedral group of order $2m$ where $m$ is positive integer and $m \geq 3$. If $m$ is odd, the normal subgroup of $D_{2m}$ are $\{1\}, \langle r^d \rangle$ where $d$ divides $m$ and $D_{2m}$ itself. If $m$ is even, the normal subgroups of $D_{2m}$ are $\{1\}, \langle r^d \rangle$ where $d$ divides $m$, $\langle r^2, s \rangle$, $\langle r^2, rs \rangle$ and $D_{2m}$ itself [25]. By taking any normal subgroup $N$ of $D_{2m}$, then subgroup graph $\Gamma_N(D_{2m})$ is an undirected simple graph. When $N = D_{2m}$ we have $\Gamma_{D_{2m}}(D_{2m}) = K_{2m}$. So, $ec(v) = 1, \text{deg}(v) = 2m - 1$ and $D(v) = 2m - 1$ for any vertex $v$ in $\Gamma_{D_{2m}}(D_{2m})$. By direct computation, we obtain
\[ \xi^c \left( \Gamma_{D_{2m}}(D_{2m}) \right) = \xi^d \left( \Gamma_{D_{2m}}(D_{2m}) \right) = 2m(2m - 1), \]
\[ \xi \left( \Gamma_{D_{2m}}(D_{2m}) \right) = \xi^{sv} \left( \Gamma_{D_{2m}}(D_{2m}) \right) = 2m, \]
\[ E_1 \left( \Gamma_{D_{2m}}(D_{2m}) \right) = 2m, \]

and
\[ E_2 \left( \Gamma_{D_{2m}}(D_{2m}) \right) = m(2m - 1). \]

Next, we present our results on the subgroup graphs of the dihedral group for other normal subgroups.

3. Method
To determine the eccentric distance sum and adjacent eccentric distance sum index of the complement of the subgroup graph of dihedral groups, several steps taken in this study are the following.
- Drawing the subgroup graph of dihedral group \( D_{2m} \) for \( m = 3, 4, 5, 6, 7, 8 \).
- Drawing the complement of each subgroup graph in step (1).
- Calculating the eccentric distance sum and adjacent eccentric distance sum index of each complement of the subgroup graph in step (2).
- Formulating conjectures based on the eccentric distance sum and adjacent eccentric distance sum index pattern.
- Reformulating the conjectures as theorems and giving their formal proof.

4. Results
The normal subgroup \( \langle r \rangle \) of \( D_{2m} \) is \( \langle r \rangle = \{1, r, r^2, \ldots, r^{m-1} \} \). We have \( \Gamma_{\langle r \rangle}(D_{2m}) \) is a disconnected graph and \( \Gamma_{\langle r \rangle}(D_{2m}) = 2K_m \). The vertex sets of each component are \( \{1, r, r^2, \ldots, r^{m-1} \} \) and \( \{s, sr, sr^2, \ldots, sr^{m-1} \} \). So, \( \overline{\Gamma_{\langle r \rangle}}(D_{2m}) = K_{m,m} \). Then, \( \deg(v) = m, ec(v) = 2 \) and \( D(v) = 3m - 2 \) for all \( v \in \overline{\Gamma_{\langle r \rangle}}(D_{2m}) \). We obtain the following results by direct computation.

Theorem 3.1 For \( m \geq 3 \), then
(a) \( \xi(\overline{\Gamma_{\langle r \rangle}}(D_{2m})) = 4m \),
(b) \( \xi^c(\overline{\Gamma_{\langle r \rangle}}(D_{2m})) = 4m^2 \),
(c) \( \xi^d(\overline{\Gamma_{\langle r \rangle}}(D_{2m})) = 12m^2 - 8m \),
(d) \( \xi^{sv}(\overline{\Gamma_{\langle r \rangle}}(D_{2m})) = 12m - 8 \),
(e) \( E_1(\overline{\Gamma_{\langle r \rangle}}(D_{2m})) = 8m \), and
(f) \( E_2(\overline{\Gamma_{\langle r \rangle}}(D_{2m})) = 4m^2 \).

Proof
(a) \( \xi(\overline{\Gamma_{\langle r \rangle}}(D_{2m})) = \sum_{v \in \overline{\Gamma_{\langle r \rangle}}(D_{2m})} ec(v) = \sum_{v \in \overline{\Gamma_{\langle r \rangle}}(D_{2m})} 2 = 2(2m) = 4m. \)
(b) \( \xi^c(\overline{\Gamma_{\langle r \rangle}}(D_{2m})) = \sum_{v \in \overline{\Gamma_{\langle r \rangle}}(D_{2m})} ec(v) \deg(v) = \sum_{v \in \overline{\Gamma_{\langle r \rangle}}(D_{2m})} 2m = 2m(2m) = 4m^2. \)
(c) \( \xi^d(\overline{\Gamma_{\langle r \rangle}}(D_{2m})) = \sum_{v \in \overline{\Gamma_{\langle r \rangle}}(D_{2m})} ec(v) D(v) = \sum_{v \in \overline{\Gamma_{\langle r \rangle}}(D_{2m})} 2(3m - 2) = 2m[2(3m - 2)] = 12m^2 - 8m. \)
(d) \( \xi^{sv}(\overline{\Gamma_{\langle r \rangle}}(D_{2m})) = \sum_{v \in \overline{\Gamma_{\langle r \rangle}}(D_{2m})} \frac{ec(v) D(v)}{\deg(v)} \).
\[ = \sum_{v \in V(\Gamma_{(r^2)}(D_{2m}))} \frac{(3m-2)^2}{m} \]
\[ = 2m \left( \frac{(3m-2)^2}{m} \right) \]
\[ = 12m - 8. \]
\[ (c) \ E_1(\Gamma_{(r^2)}(D_{2m})) = \sum_{v \in V(\Gamma_{(r^2)}(D_{2m}))} (ec(v))^2 = \sum_{v \in V(\Gamma_{(r^2)}(D_{2m}))} 2 = 2m(4) = 8m. \]
\[ (f) \ E_2(\Gamma_{(r^2)}(D_{2m})) = \sum_{u \in E(\Gamma_{(r^2)}(D_{2m}))} ec(u)ec(v) = m^2 \cdot 2 \cdot 2 = 4m^2. \]

The following are results for \( m \geq 4 \) and \( m \) is even.

**Theorem 3.2** For \( m \geq 4 \) and \( m \) is even, then

(a) \( \xi(\Gamma_{[r^2]}(D_{2m})) = 4m, \)

(b) \( \xi^c(\Gamma_{[r^2]}(D_{2m})) = 6m^2, \)

(c) \( \xi^d(\Gamma_{[r^2]}(D_{2m})) = 10m^2 - 8m, \)

(d) \( \xi^{av}(\Gamma_{[r^2]}(D_{2m})) = \frac{20m-16}{3}, \)

(e) \( E_1(\Gamma_{[r^2]}(D_{2m})) = 8m, \)

(f) \( E_2(\Gamma_{[r^2]}(D_{2m})) = 12m. \)

**Proof**

For \( m \geq 4 \) and \( m \) is even, \( \{ r^2, r, r^4, \ldots, r^{m-2} \} \) is a normal subgroup. So, the subgroup graph \( \Gamma_{[r^2]}(D_{2m}) = 4K_{m/2}. \) Then, \( \Gamma_{[r^2]}(D_{2m}) = K_{m/2,m/2,m/2,m/2}. \) We obtain \( deg(v) = 3m/2, ec(v) = 2, \) and \( D(v) = (5m - 4)/2 \) for all \( v \in V(\Gamma_{[r^2]}(D_{2m})). \) Therefore

(a) \( \xi(\Gamma_{[r^2]}(D_{2m})) = \sum_{v \in V(\Gamma_{[r^2]}(D_{2m}))} ec(v) = \sum_{v \in V(\Gamma_{[r^2]}(D_{2m}))} 2 = 2(2m) = 4m. \)

(b) \( \xi^c(\Gamma_{[r^2]}(D_{2m})) = \sum_{v \in V(\Gamma_{[r^2]}(D_{2m}))} ec(v)D(v) = 2m(5m - 4) = 10m^2 - 8m. \)

(c) \( \xi^d(\Gamma_{[r^2]}(D_{2m})) = \sum_{v \in V(\Gamma_{[r^2]}(D_{2m}))} ec(v)D(v) = \sum_{v \in V(\Gamma_{[r^2]}(D_{2m}))} \frac{5m-4}{2} = 2m(\frac{5m-4}{2}) = 10m^2 - 8m. \)

(d) \( \xi^{av}(\Gamma_{[r^2]}(D_{2m})) = \sum_{v \in V(\Gamma_{[r^2]}(D_{2m}))} ec(v)D(v) = \sum_{v \in V(\Gamma_{[r^2]}(D_{2m}))} \frac{5m-4}{3m} = 2m(\frac{5m-4}{3m}) = 10m^2 - 16m. \)

(e) \( E_1(\Gamma_{[r^2]}(D_{2m})) = \sum_{v \in V(\Gamma_{[r^2]}(D_{2m}))} (ec(v))^2 = \sum_{v \in V(\Gamma_{[r^2]}(D_{2m}))} 2 = 2m(4) = 8m. \)

(f) \( E_2(\Gamma_{[r^2]}(D_{2m})) = \sum_{u \in E(\Gamma_{[r^2]}(D_{2m}))} ec(u)ec(v) = \frac{6m}{2} \cdot 2 = 12m. \)

For \( m \geq 4 \) and \( m \) is even, the normal subgroup \( \langle r^2, s \rangle \) of dihedral group \( D_{2m} \) is \( \langle r^2, s \rangle = \{ 1, r^2, r^4, \ldots, r^{m-2}, s, sr^2, sr^4, \ldots, sr^{m-2} \}. \) So, \( \Gamma_{(r^2,s)}(D_{2m}) = 2K_{m} \) and we have \( \Gamma_{(r^2,s)}(D_{2m}) = K_{m,m}. \) Then, \( deg(v) = m, ec(v) = 2, \) and \( D(v) = 3m - 2 \) for all \( v \in V(\Gamma_{(r^2,s)}(D_{2m})). \) Hence, we have the following results.
Theorem 3.3 For $m \geq 4$ and $m$ is even, then
(a) $\xi\left(\overline{\Gamma'(r^2,rs)}(D_{2m})\right) = 4m$,
(b) $\xi^c\left(\overline{\Gamma'(r^2,rs)}(D_{2m})\right) = 4m^2$,
(c) $\xi^d\left(\overline{\Gamma'(0^2,0)}(D_{2m})\right) = 12m^2 - 8m$,
(d) $\xi^{cd}\left(\overline{\Gamma'(r^2,rs)}(D_{2m})\right) = 12m - 8$,
(e) $E_1\left(\overline{\Gamma'(r^2,rs)}(D_{2m})\right) = 8m$, and
(f) $E_2\left(\overline{\Gamma'(r^2,rs)}(D_{2m})\right) = 4m^2$.

We can easily observe that $\Gamma'(r^2,rs)(D_{2m}) = 2K_m$, for $m \geq 4$ and $m$ is even. Then, $\overline{\Gamma'(r^2,rs)}(D_{2m}) = K_{m,m}$. This fact brings us to the following results.

Theorem 3.4 For $m \geq 4$ and $m$ is even, then
(a) $\xi\left(\overline{\Gamma'(0^2,0)}(D_{2m})\right) = 4m$,
(b) $\xi^c\left(\overline{\Gamma'(0^2,0)}(D_{2m})\right) = 4m^2$,
(c) $\xi^d\left(\overline{\Gamma'(0^2,0)}(D_{2m})\right) = 12m^2 - 8m$,
(d) $\xi^{cd}\left(\overline{\Gamma'(0^2,0)}(D_{2m})\right) = 12m - 8$,
(e) $E_1\left(\overline{\Gamma'(0^2,0)}(D_{2m})\right) = 8m$.
(f) $E_2\left(\overline{\Gamma'(0^2,0)}(D_{2m})\right) = 4m^2$.

The proof of Theorem 3.3 and 3.4 are similar to the proof of Theorem 3.1.

5. Conclusion
We have determined the formula of total eccentricity, eccentric connectivity index, eccentric distance sum index, adjacent eccentric distance sum index, first Zagreb index, and second Zagreb index of some subgroup graphs of dihedral groups. The eccentric distance sum index and adjacent eccentric distance sum index of other subgroup graphs of dihedral group that are not discussed in this paper still need to be computed in further research.

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