Exact Inference with Approximate Computation for Differentially Private Data via Perturbations

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SUMMARY

Differential privacy protects individuals’ confidential information by subjecting data summaries to probabilistic perturbation mechanisms, carefully designed to minimize undue sacrifice of statistical efficiency. When properly accounted for, differentially private data are conducive to exact inference when approximate computation techniques are employed. This paper shows that approximate Bayesian computation, a practical suite of methods to simulate from approximate posterior distributions of complex Bayesian models, produces exact posterior samples when applied to differentially private perturbation data. An importance sampling implementation of Monte Carlo expectation-maximization for likelihood inference is also discussed. The results illustrate a duality between approximate computation on exact data, and exact computation on approximate data. A cleverly designed inferential procedure exploits the alignment between the statistical tradeoff of privacy versus efficiency, and the computational tradeoff of approximation versus exactness, so that paying the cost of one gains the benefit of both.

Some key words: approximate Bayesian computation; expectation-maximization; ignorability; Monte Carlo; privacy-efficiency tradeoff

1. INTRODUCTION

Modern applications of likelihood and Bayesian inference face the challenge of model complexity. Likelihood-free methods, in particular approximate Bayesian computation (ABC), supply practical solutions to large-scale models for which the likelihood is implicit or intractable. ABC samplers produce approximate Monte Carlo samples from Bayesian posteriors that need not possess closed-form expressions. The developments brought vast computational convenience to population genetics (Beaumont et al., 2002), systems biology (Toni et al., 2008), ecology (Wood, 2010), and others fields that utilize stochastic differential equations to model complex dynamic systems.

A caveat of likelihood-free methods lies with the often approximate nature of the Monte Carlo samples they produce. An ideal ABC algorithm works by simulating many parameter-data pairs according to the prior and likelihood specifications, and retaining only those simulated parameters that correspond to simulated data that match exactly with the observed data. ABC exhibits a bias whenever it cannot enforce an exact match between the observed and the simulated data. There are several reasons to why an exact match is often infeasible. When the data is high-dimensional or is continuous, the observed and the simulated data have zero (or near-zero) probability to match exactly, prohibiting sampling via rejection. Moreover, when dimension reduction becomes necessary, there may not exist a low-dimensional sufficient statistic to the likelihood.
One must work with some notion of approximate matching, such as approximate sufficiency (Joyce & Marjoram, 2008), projection and regression (Beaumont et al., 2002; Wegmann et al., 2009), optimization of information and distance criteria (Nunes & Balding, 2010; Barnes et al., 2012; Bernton et al., 2019), and auxiliary likelihoods (Drovandi et al., 2011; Gleim & Pigorsch, 2013); see Blum et al. (2013) and Chapter 5 of Sisson et al. (2018) for reviews. Notwithstanding these efforts, what remains a conceptual challenge to the validity of ABC is that, maneuvers to improve consistency in approximate matching may not be desirable in practice. That is because the overall error in the ABC sample is comprised of both the approximation error and the Monte Carlo error, and a smaller error due to approximate matching usually corresponds to a larger Monte Carlo error. Therefore, it is advisable that the algorithm builds in itself some non-negligible extent of approximation error.

This paper establishes a serendipitous result, that ABC delivers exact posterior inference when applied to differentially private data obtained via perturbation mechanisms. Differential privacy (Dwork et al., 2006) protects individuals' confidential information in a dataset by intentionally exposing data queries to specially designed perturbation mechanisms, to create a randomized appearance with probabilistic guarantees. We show that a rejection ABC algorithm, with kernel and bandwidth chosen to correspond to the perturbation mechanism underlying the privatized data, produces exact inference in the form of independent and identically distributed samples from the true posterior distribution. The result hinges on the noisy ABC concept put forth by Fearnhead & Prangle (2012), and the revelation of Wilkinson (2013) to interpret ABC posteriors as exact under the assumption of model error. A importance sampling implementation for likelihood inference using Monte Carlo expectation-maximization is also discussed. Both results owe to the fact that the tuning elements of the algorithms can be chosen in concordance with the probabilistic perturbation mechanism for differential privacy.

2. DIFFERENTIAL PRIVACY AS A PERTURBATION MECHANISM

Differential privacy aims to protect the confidential information of individual respondents in a dataset \( x \in \mathcal{X} \), without unduly sacrificing accuracy in learning about aggregate features of the underlying population as represented by \( x \). By the confidential information of an individual is meant the following. Consider \( x' \in \mathcal{X} \) a neighboring dataset of \( x \), in the sense that the smaller of the two differs from the larger by missing precisely one respondent. In other words, \( d(x, x') = 1 \) where \( d \) is the Hamming distance appropriately defined on \( \mathcal{X} \). Any observable difference between \( x \) and \( x' \) is the confidential information whether a certain respondent is in, or not in, the dataset.

By an aggregate feature is meant a query \( s: \mathcal{X} \rightarrow \mathbb{R}^p \), a deterministic function of \( x \), such as the sample average, variance, quantiles and so on. Queries are the means through which analysts learn from the dataset. Counting queries, including histograms and contingency tables which are ordered multivariate counts over a partition of \( x \), constitute a most useful class of queries in the practice of differential privacy (Yang et al., 2012; Schein et al., 2019). The 2020 U.S. Census will employ differential privacy as its disclosure avoidance system (Abowd, 2018), and central to the reporting tasks of the Census Bureau are counts and histograms tabulated over geographic levels including states, counties, and enumeration districts.

Differential privacy is realized via a mechanism of perturbation, and queries constitute the fundamental building block of perturbation mechanisms. To say that a query function is differentially private means that it reflects as truthful as possible about the status of \( x \), meanwhile behaves similarly should it be calculated based on any neighboring dataset of \( x \). The notion of differentially privacy is defined in probabilistic terms.
Definition 1 (Differential privacy; Dwork & Smith (2009)). A random function \( S \) is \((\epsilon, \delta)\)-differentially private if for all neighboring datasets \( \{(x, x') : d(x, x') = 1\} \) and all \( A \in \mathcal{B}(\mathbb{R}^p) \),

\[
Pr(S(x') \in A) \leq e^\epsilon \cdot Pr(S(x) \in A) + \delta.
\] (1)

\( S \) is called \( \epsilon \)-differentially private if it is \((\epsilon, 0)\)-differentially private.

Definition 1 requires that, when operating on pairs of neighboring datasets, the random function \( S \) induces pairs of probability measures that are close to each other. The parameters \( \epsilon \) and \( \delta \) are called the privacy loss budget: the smaller the budget, the closer the induced probability measures of \( S(x) \) and \( S(x') \) ought to be. In the extreme case that both are zero, i.e. no privacy loss budget to spare, the two measures must be equal almost surely. Differential privacy requires that the distribution of \( S \) be stable with respect to the small neighborhood around the observable dataset, hence by looking at a single realization of \( S(x) \), denoted as \( s_{dp} \), one cannot discern in a probabilistic sense whether any particular individual is in \( x \).

Differential privacy is a property pertinent to the random function \( S \). The majority of differential privacy mechanisms currently employed in practice are perturbation mechanisms taking form as follows.

Definition 2 (Perturbation mechanism). For a dataset \( x \in \mathcal{X} \) and a deterministic function \( s : \mathcal{X} \to \mathbb{R}^p \), the random function \( S \) is a perturbation mechanism based on \( s \) if

\[
S(x) \mid s(x) \sim \eta_{dp}(\cdot \mid s(x)),
\] (2)

for \( \eta_{dp} \) a known conditional probability distribution satisfying \( \mathbb{E}(S(x) \mid s(x)) = s(x) \). Specifically, \( S \) is an additive perturbation mechanism based on \( s \) if

\[
S(x) = s(x) + hu,
\] (3)

where the noise component \( u \) is a \( p \)-dimensional random variable with known distribution \( \eta \), \( \mathbb{E}(u) = 0 \), and \( h > 0 \) is a bandwidth parameter.

\( S \) is a noisy version of its deterministic counterpart \( s \). The protection of privacy is achieved through randomly perturbing what would otherwise be a deterministic query calculated based on \( x \). Below are examples of three most widely employed differentially private mechanisms, all of which are additive perturbation mechanisms.

Example 1 (\( \epsilon \)-Laplace mechanism; Dwork et al. (2006)). In (3), let \( u \sim \text{Lap}_p(1) \), the \( p \)-dimensional product of independent and identically distributed standard Laplace variables, and \( h = \epsilon^{-1}GS(s) \), where

\[
GS(s) = \sup_{x, x'} \left\{ \|s(x) - s(x')\| : d(x, x') = 1 \right\},
\] (4)

is the global sensitivity of \( s \), with \( \| \cdot \| \) denoting the Euclidean norm. Then, \( S \) is \( \epsilon \)-differentially private.

Example 2 ((\( \epsilon, \delta \))-Laplace mechanism; Nissim et al. (2007)). In (3), let \( u \sim \text{Lap}_p(1) \), \( h = \epsilon^{-1}SS_\xi(t, x) \), and \( \xi = \epsilon \{4(d + \log (2/\delta))\}^{-1} \). where

\[
SS_\xi(s, x) = \sup_{x'} \left\{ e^{-\xi d(x, x')} LS(s, x') : x' \in \mathcal{X} \right\}
\] (5)
is the $\xi$-smooth sensitivity ($\xi > 0$) of $s$ at $x$, and

$$LS(s, x) = \sup_{x'} \{ \| s(x) - s(x') \| : d(x, x') = 1 \}$$  \hspace{1cm} (6)

is the local sensitivity of $s$ at $x$. Then, $S$ is $(\epsilon, \delta)$-differentially private.

Example 3 (Gaussian mechanism; Blum et al. (2005); Nissim et al. (2007)). In (3), let $u \sim N(0, I_p)$ the $p$-dimensional standard multivariate Normal variable, $h = \epsilon^{-1} \sqrt{\frac{2 \log(2/\delta) SS(\xi, t, x)}{2}}$, and $\xi = \epsilon \{ 4 (d + \log (2/\delta)) \}^{-1}$. Then, $S$ is $(\epsilon, \delta)$-differentially private.

The above examples invoke three notions of functional sensitivity, (4)-(6), to capture the idea that certain choices of $s$ may be more revealing of individual information in $x$ than others. The global sensitivity measures the extent to which $s$ varies between all conceivable pairs of neighboring datasets, whether or not realized in the observed sample. For example, the global sensitivity of the counting query is 1. On the other hand, the local sensitivity of $s$ measures its maximum variability among neighboring datasets to a given observed dataset $x$. The smooth sensitivity strikes a balance between the two, by providing an upper bound on the local sensitivity at $x$ in such a way that the bound does not vary too quickly as a function of $x$. It is crucial that the bandwidth of the additive perturbation mechanism is chosen as a function of both the sensitivity of $s$ as well as the privacy budget, that is, $h = h(\epsilon, \delta, \xi)$.

The above examples do not make up an exhaustive list of privacy mechanisms, but are most typical in practice. Differentially private mechanisms based on the generalized product Cauchy (Nissim et al., 2007), correlated multivariate Gaussian (Nikolov et al., 2013) and subgaussian distributions (Dwork & Rothblum, 2016) also fall under the umbrella of additive perturbation mechanisms. More general forms of perturbation mechanisms are often constructed by sequentially compounding additive perturbations or incorporating them in a layered fashion. Additive perturbation shall be regarded as the fundamental building block to perturbation mechanisms.

A primary strength of differential privacy over traditional disclosure avoidance frameworks is that the specification of the perturbation mechanism, $\eta_{dp}$, is entirely public. For additive mechanisms, this specification includes $u$’s distribution $\eta$, bandwidth $h$, and the privacy loss budget $\epsilon$ and $\delta$. Perturbation mechanisms can be correctly accounted for in the probabilistic modeling of privatized data. Despite the necessary sacrifice of statistical efficiency, likelihood and Bayesian models utilizing privatized data can retain validity to a maximal extent the models themselves are entitled. In fact, the very nature of the privatized query makes ABC an appealing choice for posterior computation, whether or not the model itself is sufficiently complex to demand the use of ABC. As will be shown next, when properly tuned according to the parameters of the perturbation mechanism, a simple ABC rejection algorithm guarantees the exactness of draws from the true posterior distribution.

3. Exact posterior inference with differentially private data

In the absence of privacy needs, suppose a Bayesian model is posited based on the noiseless query with $\theta \sim \pi_0(\theta)$ the prior distribution, and $s(\theta) \mid \theta \sim \pi(s \mid \theta)$ the sampling distribution of $s$ (as a function of $x$) given $\theta$. As is common with likelihood-free methods, it is only required that for given values of $\theta$ one can simulate from $\pi(s \mid \theta)$, but otherwise it need not be tractable or available in closed form. The posterior distribution of $\theta$ given $s$ is simply

$$\pi(\theta \mid s) \propto \pi_0(\theta) \pi(s \mid \theta).$$  \hspace{1cm} (7)
However, we do not observe \( s = s(x) \) but rather the differentially privatized query \( s_{dp} \), a single realization of \( S(x) \). The subscript “dp” emphasizes that the observed quantity instantiates the privacy perturbation mechanism \( S \), not the data generation mechanism of \( x \). The joint distribution \( \theta \) and \( s_{dp} \) is \( \pi(\theta, s_{dp}) = \int \pi(\theta, s) \eta_{dp}(s_{dp} \mid s) \, ds \), marginalized over the latent \( s \). This identity holds because the conditional distribution \( \pi(s_{dp} \mid s, \theta) = \eta_{dp}(s_{dp} \mid s) \) is free of \( \theta \), as it is precisely the known specification of the perturbation mechanism (2).

In order to make an intuitive connection with ABC, the ensuing discussion focuses on the case of additive perturbation. That is, assume \( \eta_{dp}(s_{dp} \mid s) = \eta\left((s_{dp} - s) / h\right) \) where \( \eta(\cdot) \) is the density of the additive noise \( u \), and \( h \) is the bandwidth. The posterior distribution of \( \theta \) given \( s_{dp} \) can be written as

\[
\pi(\theta \mid s_{dp}) = \int \frac{\pi(s, s_{dp}, \theta)}{\pi(s_{dp})} \, ds = \frac{\pi_0(\theta) \int \eta\left((s_{dp} - s) / h\right) \pi(s \mid \theta) \, ds}{\int \pi_0(\theta) \int \eta\left((s_{dp} - s) / h\right) \pi(s \mid \theta) \, ds \, d\theta}.
\]

Algorithm 1 presents a standard rejection ABC algorithm whose rejection kernel and bandwidth are set to match with the additive perturbation mechanism employed to generate \( s_{dp} \). The restriction to additivity serves the sole purpose of drawing a direct analogy between the privacy mechanism specification and the tuning parameters of ABC. The main result to be established in Theorem 1 is valid whether or not additivity is imposed.

**Algorithm 1.** Rejection ABC algorithm with differentially private queries.

Input: privatized query \( s_{dp} \) and its corresponding perturbation mechanism, including density \( \eta \) and bandwidth \( h > 0 \); integer \( N > 0 \);

Iterate: for \( i = 1, \ldots, N \):

- step 1, simulate \( \theta_i \sim \pi_0(\theta) \);
- step 2, simulate \( s_i \sim \pi(s \mid \theta_i) \);
- step 3, accept \( \theta_i \) with probability \( c\eta\left((s_{dp} - s_i) / h\right) \) where \( c^{-1} = \max\{\eta(\cdot)\} \), otherwise go to step 1;

Output: a set of parameter values \( \{\theta_i\}_{i=1}^N \).

**Theorem 1.** Algorithm 1 samples independently and identically from the true posterior distribution \( \pi(\theta \mid s_{dp}) \).

**Proof.** Let \( I \) be the indicator of the event that a draw of \( \theta \) is accepted. The joint distribution of all quantities produced by the \( i \)th iteration is \( \tilde{\pi}(\theta, s, I = 1) = \pi_0(\theta) \pi(s \mid \theta) \tilde{\pi}(I = 1 \mid s) \), where \( \tilde{\pi}(I = 1 \mid s) \) is the Bernoulli mass function with proportion parameter \( c\eta\left((s_{dp} - s) / h\right) \). The marginal distribution of an accepted \( \theta \) sample is

\[
\tilde{\pi}(\theta \mid I = 1) = \int \frac{\tilde{\pi}(\theta, s, I = 1)}{\tilde{\pi}(I = 1)} \, ds = \frac{\int \pi_0(\theta) \pi(s \mid \theta) \eta\left((s_{dp} - s) / h\right) \, ds}{\int \pi_0(\theta) \pi(s \mid \theta) \eta\left((s_{dp} - s) / h\right) \, ds \, d\theta},
\]

which is equal to \( \pi(\theta \mid s_{dp}) \) as defined in (8). \( \square \)

Proof of Theorem 1 is essentially the same as that of Theorem 1 of Wilkinson (2013), save for a important conceptual difference. In Wilkinson (2013), \( s_{dp} \) stands for a query that was observed noiselessly, but construed as if subject to additive error. The ABC-induced posterior of \( \theta \) therein, while identical to (8), is not the true posterior of \( \theta \) but that of a “best model input \( \tilde{\theta} \)” (p. 132) given \( s_{dp} \). With \( s_{dp} \) being a privatized query, no pretense is necessary in treating it as observed with error, since it indeed was.
Another way to understand Theorem 1 is that the privacy mechanism plays the role of the random summary statistic in the noisy ABC algorithm of Fearnhead & Prangle (2012). Noisy ABC is calibrated with respect to the joint Bayesian model, whereas ABC typically isn’t. What’s different here is that, the kernel and bandwidth in noisy ABC are merely parameters to fine-tune the tradeoff between approximation error and the Monte Carlo error in the posterior, which in turn controls the efficiency of the sampler. In Algorithm 1, both the kernel and the bandwidth are dictated externally, by the perturbation mechanism and the privacy loss budget. The computational tradeoff and the privacy tradeoff becomes “bundled together”: specifying the parameters of ABC also specifies parameters of the privacy mechanism, and vice versa.

Key to the validity of Theorem 1 is that the differentially private perturbation mechanism is ignorable for \( \theta \). In other words, the unobserved noiseless query \( s \) is sufficient with respect to the complete likelihood \( \pi(s, s_{dp} \mid \theta) \). For the additive perturbation mechanism considered here, the correspondence with classic ABC through the shared choices of kernel density \( \eta \) and bandwidth \( h \) jumps out of the page. As long as ignorability holds, however, the validity of Theorem 1 extends to general perturbation mechanisms that need not be additive. That is, if the perturbation mechanism follows (2), and Step 3 of Algorithm 1 adopts \( \eta_{dp} \) (properly scaled) as the acceptance probability, it outputs samples that are independent and identically distributed according to the true Bayesian posterior of (8). Proof of this strengthened version of Theorem 1 can be found in the supplementary file.

Example 4 illustrates an implementation of Algorithm 1 for inference from a simple Bayesian model of privatized count data.

**Example 4 (Differentially private ABC for count data).** Consider modeling the number of respondents in \( x \) who are in possession of a particular feature. \( s(\cdot) \) is the univariate counting query, and a Bayesian model is used to construct a distributional estimate of its expectation \( \theta \). Posit the following model: \( \theta \sim \text{Gamma}(\alpha, \beta) \) where \( \alpha \) and \( \beta \) are prior hyperparameters, and \( s(x) \mid \theta \sim \text{Pois}(\theta) \). This model is chosen so that a tractable posterior can be obtained for illustration. As is common with general ABC samplers, Algorithm 1 can work with arbitrary choices of prior and likelihood that need not be conjugate.

Suppose the \( \epsilon \)-Laplace mechanism of Example 1 is employed. The global sensitivity of the univariate counting query is \( 1 \), thus for the perturbation mechanism, \( u \sim \text{Lap}_p(1) \) and \( h = \epsilon^{-1} \). The posterior distribution of \( \theta \) given privatized query \( s_{dp} \) takes the form

\[
\pi(\theta \mid s_{dp}) \propto \theta^{\alpha-1}e^{-(\beta+1)\theta} \left[ \frac{\Gamma([s_{dp} \mid \theta_{\epsilon}^+])}{\Gamma(s_{dp})} e^{\theta_{\epsilon}^- - \epsilon s_{dp}} + \frac{\gamma([s_{dp} \mid \theta_{\epsilon}^-])}{\Gamma(s_{dp})} e^{\theta_{\epsilon}^+ + \epsilon s_{dp}} \right],
\]

where \( \theta_{\epsilon}^+ = \theta e^\epsilon \), \( \theta_{\epsilon}^- = \theta e^{-\epsilon} \), \( \lceil \cdot \rceil \) is the ceiling function, and \( \Gamma(s, x) = \int_x^\infty r^{s-1}e^{-r}dr \) is the incomplete Gamma function with \( \Gamma(s) = \Gamma(s, 0) \) and \( \gamma(s, x) = \Gamma(s) - \Gamma(s, x) \).

Figure 1 depicts three posterior densities for visual comparison, using hyperparameters \( \alpha = 25, \beta = 1 \), privacy budget \( \epsilon = 0.2 \), and an observed query \( s_{dp} = 37.4 \). The true posterior (8), normalized via numerical integration, coincides with the differentially private ABC posterior (9) whose density is estimated using \( 10^6 \) draws from Algorithm 1. Both differ substantially from the incorrect posterior (7), which treats \( s_{dp} \) as if observed free of noise. This last density amounts to the Gamma-Poisson conjugate posterior, and appears to underestimate the actual posterior uncertainty associated with \( \theta \).

It follows from the proof of Algorithm 1 that the overall acceptance probability of the algorithm is \( \bar{\pi}(I = 1) = \pi(s_{dp}) / \max \eta(\cdot) \), or the model evidence evaluated at \( s_{dp} \) divided by the modal density of \( \eta \). This means that rejection can be frequent if model evidence is low, such
Algorithm 1 can be adapted to work with a variety of alternative ABC sampling techniques to produce consistent posterior estimates for functions of interest. For example, an importance sampling variation modifies Algorithm 1 as follows. At step 1 of each iteration, sample $\theta_i \sim g(\theta)$, a proposal distribution that is positive wherever the prior $\pi_0(\theta)$ is positive. At step 3, no rejection is performed but instead, $\theta_i$ is assigned a weight $\omega_i = \omega(s_i, \theta_i) = \eta\left((s_{dp} - s_i)/h\right)\pi_0(\theta_i)/g(\theta_i)$. The algorithm returns weighted draws $\{\theta_i, \omega_i\}_{i=1}^N$. For a square-integrable function of interest $a(\theta)$, as $N \to \infty$, the weighted average estimator converges in probability to its posterior expectation given $s_{dp}$ (Liu, 2008):

$$\sum_{i=1}^N \omega_i a(\theta_i) \approx \frac{\mathbb{E}_{g}\left(\omega(\theta, s) a(\theta)\right)}{\mathbb{E}_{g}\left(\omega(\theta, s)\right)} = \mathbb{E}\left(a(\theta) \mid s_{dp}\right),$$

where $\mathbb{E}_g(\cdot)$ is with respect to the joint distribution $g(\theta)\pi(s \mid \theta)$, and $\mathbb{E}(\cdot \mid s_{dp})$ is with respect to the true posterior in (8). The proposal distribution $g(\cdot)$ can be chosen to minimize the variance of the estimator in (11), such as a density that is close in shape to $a(\theta)\pi_0(\theta)$ (Liu, 2008). Further ABC adaptations, such as hybrid importance-rejection sampling (Fearnhead & Prangle, 2012), rejection control (Sisson et al., 2018, ch.4), Markov chain Monte Carlo (Marjoram et al., 2003) and sequential Monte Carlo (Sisson et al., 2007) can be developed likewise, while the consistency result of (11) remains intact.

4. **Exact Likelihood Inference with Monte Carlo Expectation-Maximization**

To parallel Theorem 1 within the framework of likelihood inference, this section discusses a Monte Carlo expectation-maximization (Dempster et al., 1977; Wei & Tanner, 1990) implementation for differentially private data. When a likelihood involves both observed and latent data, expectation-maximization seeks the maximum likelihood estimate of the parameter by iteratively integrating the log likelihood over the conditional predictive distribution of the latent data given the observed data and a current parameter value (the E-step), and maximizing the parameter value over the above integral (the M-step).

In the context of differential privacy, the complete data is $(s, s_{dp})$, in which the latent data is the noiseless query $s$, and the observed data is the privatized query $s_{dp}$. In the case of addi-
tive perturbation, \( s_{dp} = s + hu \) is a linear combination of \( s \) and the noise component \( u \). For the E-step at the \( t \)th iteration, evaluate the expectation of the log likelihood with respect to the conditional predictive distribution of \( s \) given \( s_{dp} \) and the current maximizer \( \theta^{(t)} \):

\[
Q(\theta; \theta^{(t)}) = \mathbb{E}_{s_{dp}} \left( \log L(\theta; s, s_{dp}) \mid s_{dp}, \theta^{(t)} \right) = \mathbb{E} \left( \log \pi(s \mid \theta) \mid s_{dp}, \theta^{(t)} \right) + \text{const.} \tag{12}
\]

The constant term, \( \mathbb{E} \left( \log \eta_{dp}(s_{dp} \mid s) \mid s_{dp}, \theta^{(t)} \right) \), can be ignored. As discussed in Section 3, the privacy mechanism \( \eta_{dp} \) is known and free of \( \theta \), so is the conditional predictive expectation of its log density. One may proceed to the M-step and obtain the maximizer as \( \theta^{(t+1)} = \arg\max_{\theta} Q(\theta; \theta^{(t)}) \), to be used at the E-step of the \((t+1)\)th iteration. The E- and M-steps are repeated as such until convergence.

For models with differentially private data, the noiseless data likelihood and the privacy mechanism are respectively chosen by the data analyst and the data curator, usually not in consultation with one another. Thus in general, one cannot expect the complete data likelihood to come from an exponential family (cf. Park et al., 2017), nor be able to perform both the E- and the M-steps analytically. Monte Carlo implementations of one or both steps are needed. In the simpler scenario that the noiseless data likelihood \( \pi(s \mid \theta) \) is from an exponential family, it admits a sufficient statistic \( b(s) \) to the parameter \( \theta \). Then, \( Q(\theta; \theta^{(t)}) \) in (12) can be written as an explicit function of \( \theta \) and

\[
\mathbb{E} \left( b(s) \mid s_{dp}, \theta^{(t)} \right), \tag{13}
\]

the conditional predictive expectation of \( b(s) \) given \( s_{dp} \) and the current maximizer \( \theta^{(t)} \). However, (13) may not be evaluable in closed form. An importance sampling recipe to consistently estimate it is as follows. At the \( t \)th iteration, for \( i = 1, \ldots, N \),

\[
s_i \sim \pi(s \mid \theta^{(t)}), \quad \omega_i = \eta_{dp}(s_{dp} \mid s_i). \tag{14}
\]

As \( N \to \infty \), the weighted estimator \( \sum_{i=1}^{N} \omega_i b(s_i)/\sum_{i=1}^{N} \omega_i \) converges in probability to (13). The maximizer for the E-step of the next iteration, \( \theta^{(t+1)} \), can be found by maximizing \( Q(\theta; \theta^{(t)}) \), replacing (13) therein with the weighted estimator above. The effective sample size at the \( t \)th iteration is (Liu, 2008, section 2.5.3)

\[
\text{ESS}^{(t)}(N) = N \pi^{2} \left( s_{dp} \mid \theta^{(t)} \right) \mathbb{E}_{s_{dp} \mid \theta^{(t)}}^{-1} \left( \eta_{dp}^{2} \left( s_{dp} \mid s \right) \right),
\]

where the subscript “\( s \mid \theta^{(t)} \)” signifies that the expectation is taken with respect to the current approximation to the noiseless data likelihood, or the proposal distribution.

In the importance sampling algorithm of (14), the \( s_i \)'s are simulated from the current approximation to the analyst’s noiseless data likelihood, and the weights \( \omega_i \)'s are determined by the curator’s privacy mechanism. Similar in spirit to Algorithm 1, the separation of specification allows the algorithm to easily accommodate independently derived choices of data likelihood and privacy mechanisms, and does not require the evaluation or integration of quantities that are non-trivial functions of both. That being said, whenever appropriate, one may choose to modify (14) in a variety of ways to sample from the conditional predictive distribution. For example, with rejection or Markov chain-based samplers, \( s_i \) follows a proposal distribution and \( \omega_i = 1 \) if \( s_i \) is accepted and 0 otherwise (McCulloch, 1997; Booth & Hobert, 1999). If possible, one can also perform importance sampling where \( s_i \sim \pi(s \mid s_{dp}, \theta^{(t-1)}) \), the approximation to the conditional predictive distribution at the previous iteration, and \( \omega_i = \pi(s \mid s_{dp}, \theta^{(t)}) / \pi(s \mid s_{dp}, \theta^{(t-1)}) \) the ratio between the current and previous approximations, thereby reweighting and recycling the multiply-imputed \( s_i \)'s to save computational effort (Quintana et al., 1999).
If the noiseless data likelihood does not come from an exponential family, \( Q(\theta; \theta^{(t)}) \) of (12) may not reduce to a straightforward expression involving \( \theta \) and (13). In this case, the E-step requires an approximation to \( Q(\theta; \theta^{(t)}) \) as a mixture of augmented log likelihoods, constructed as follows. Let \( \{s_i, \omega_i\}_{i=1}^N \) be a possibly weighted sample from the target distribution \( \pi \left( s \mid s_{dp}, \theta^{(t)} \right) \), the \( t \)th approximation to the conditional predictive distribution. For example, \( \{s_i, \omega_i\}_{i=1}^N \) can be an importance sample generated from (14), or one of the alternatives described above. Then,

\[
\hat{Q}(\theta; \theta^{(t)}) = m \sum_{i=1}^N \omega_i \log \pi(s_i \mid \theta) \tag{15}
\]

serves as a consistent approximation to \( Q(\theta; \theta^{(t)}) \). The constant \( m^{-1} = \sum_{i=1}^N \omega_i \) in (15) is consequential to the maximizer in the ensuing M-step as long as the \( \omega_i \)'s do not involve the unknown parameter \( \theta \), which is indeed the case since, again, the perturbation mechanism is ignorable for \( \theta \). Writing \( \lambda_\theta(s) = \nabla_\theta \log \pi(s \mid \theta) \), the observed score function can be approximated at the \( t \)th iteration according to

\[
\mathbb{E} \left( \lambda_\theta(s) \mid s_{dp}, \theta^{(t)} \right) \approx m \sum_{i=1}^N \omega_i \lambda_{\theta}(s_i), \tag{16}
\]

and the observed Fisher information according to

\[
\mathbb{E} \left( -\nabla_\theta \lambda_\theta(s) - \lambda_\theta(s) \lambda_\theta(s)^\top \right) \mid s_{dp}, \theta^{(t)} \right) + \mathbb{E} \left( \lambda_\theta(s) \mid s_{dp}, \theta^{(t)} \right) \mathbb{E} \left( \lambda_\theta(s) \mid s_{dp}, \theta^{(t)} \right)^\top 
\approx m \sum_{i=1}^N \omega_i \left\{ -\nabla_\theta \lambda_\theta(s_i) - \lambda_\theta(s_i) \lambda_\theta(s_i)^\top \right\} + m^2 \sum_{i=1}^N \sum_{j=1}^N \omega_i \omega_j \lambda_\theta(s_i) \lambda_\theta(s_j)^\top. \tag{17}
\]

Derivation of (16) and (17) can be found in the supplementary file, and generally follows from the appendix of Louis (1982). Both the observed score function and Fisher information are useful in accelerating and assessing convergence for Newton-type implementations of the M-step, as well as quantifying inferential uncertainty under the normal approximation to the likelihood (Meilijson, 1989). The approximations given by (16) and (17) rely only on that the first and second derivatives of the noiseless likelihood be evaluable at the simulated \( s_i \)'s.

Example 4 is revisited to demonstrate maximum likelihood estimation for \( \theta \). The noiseless data likelihood is the Poisson density, hence importance sampling of (14) is used to construct estimates for (13) at every iteration of the E-step, followed by an analytical M-step. With \( \theta^{(1)} = 1 \), three stages of iterations were performed with successively more stringent tolerance levels \( |\theta^{(t)} - \theta^{(t-1)}| < 10^{-3}, 10^{-4}, \) and \( 10^{-5} \) and larger Monte Carlo sample size \( (N = 10^3, 10^5, \) and \( 10^7 \)), using the stable maximizer from the last stage as the starting point. This is a crude rule to let \( N \) increase, hence the Monte Carlo error decrease, as \( \theta^{(t)} \) approaches the true MLE. Advanced adaptive techniques, such as the ascent-based modification of Caffo et al. (2005), can be employed achieve better performance. The algorithm converged to the maximizer \( \theta = 37.237 \), with observed Fisher information \( 1.582 \times 10^{-2} \) estimated using (17). If \( s_{dp} \) were erroneously treated as non-privatized, noiseless data, the MLE for \( \theta \) would've been 37.4, and the observed Fisher information would've been \( 2.674 \times 10^{-2} \), or 69% larger than the correct estimate, again displaying an underestimation of inferential uncertainty. The reduction of Fisher information content reflects a statistical efficiency loss induced by the privatization mechanism.
A premise to the applicability of the expectation-maximization algorithm is that the noiseless data likelihood \( \pi(s \mid \theta) \) must be evaluable, to an extent that maximization of the \( Q \) function can be done at least numerically. Hence, the methods discussed here require more restrictive model settings than does ABC. That being said, the vast literature on Monte Carlo expectation-maximization has much to instruct on implementing both the E- and the M-steps with better convergence rates, sampling efficiency, or under computational capacity constraints, for adapting modeling scenarios to differentially private data. The additive perturbation mechanism of (3) is a special instance of a linear mixed effects model, which is particularly suitable for Monte Carlo expectation-maximization and has been studied extensively in the literature (e.g. Wolfinger & O’connell, 1993; McCulloch, 1997; Booth & Hobert, 1999).

5. DISCUSSION

The statistical insight underscored by this paper is that there exists a duality between approximate computation on exact data, and exact computation on approximate data. When no privacy mechanism is involved, the justification of ABC algorithms relies on that in the limit of the bandwidth \( h \to 0 \), the ABC posterior \( \pi_{\text{ABC}}(\theta \mid s) \) approaches the true posterior \( \pi(\theta \mid s) \). However in practice, in order for the algorithm to generate an adequate number of samples, \( h \) cannot be be too small, trading off a larger approximation error with a smaller Monte Carlo error. Differentially private data is approximate data. The perturbation mechanism with which the data were treated serves coincidentally as the attributable cause of the approximation error, except that this induced error is justified by privacy needs. When differentially private data are employed, the Monte Carlo error becomes the sole kind of error that is attributable to the ABC algorithm alone. It vanishes as \( N \to +\infty \), just as any other consistent method of simulation.

The pursuit of differential privacy poses a a direct tradeoff against statistical efficiency (Duchi et al., 2018). But the efficiency-privacy tradeoff as a statistical consideration is interwoven with the approximation-exactness tradeoff as a computational consideration, a sentiment that is shared by explorations of other simulation-based Bayesian computational algorithms with differentially private data, including stochastic gradient Monte Carlo (Wang et al., 2015) and Gibbs sampling (Foulds et al., 2016). For ABC algorithms, to insist on maximal statistical efficiency necessitates computational approximation, whereas an act of data perturbation not only gains differential privacy, but also computational exactness for free. Both the ABC and Monte Carlo expectation-maximization approaches discussed in this paper adapt to differentially private data by the same logic, that is, setting the tuning parameters that usually govern the numerical performance based on the privacy parameters. Tailoring an algorithm according to the generative specifications of the data exploits the alignment between the statistical and computational tradeoffs, hitting two birds with one stone, so to speak.

A subject left untouched is the choice of the query function \( s(\cdot) \). Under no privacy considerations, this paper takes for granted that \( \pi(\theta \mid s) \) as defined in (7) is the ultimate posterior one aims. If \( s \) is not a sufficient statistic for the Bayesian model, replacing the true posterior \( \pi(\theta \mid x) \) with \( \pi(\theta \mid s(x)) \) results in a non-trivial distortion of the target distribution. Two things can may be said about this. First, the query formulation employed by this paper is not out of convenience for ABC, but rather out of necessity for differential privacy. In contrast to other perturbation approaches to private data analysis such as synthetic data, differential privacy is characteristically an output perturbation method (Dwork et al., 2006). Instead of adding noise or resampling the entire database \( x \), only when a statistic \( s(x) \) is queried, is the perturbation mechanism invoked and an \( s \)-specific noise injected. In other words, there’s no such thing as differentially private data, only differentially private queries, and the choice of the query must be decided at the outset of
data analysis. As long as a Bayesian model anticipates differentially private input, ABC delivers exact inference in the sense of Theorem 1, affirming the relevance of the current discussion.

Second, the query formulation is sufficiently general in relation to practical implementations of privacy mechanisms. Definition 2 permits the private query \( s_{\text{dp}} \) to be multi-dimensional, where each dimension can be generated in isolation, in conjunction, or sequentially, as long as the joint distribution \( \eta_{\text{dp}} \) is well-defined. This does not preclude \( s(x) = x \) is the identity function. In the case of additive perturbation, a special distribution family worthy of attention is the elliptical distribution with covariance matrix \( \Sigma \). The density of \( u \) can be written as \( \eta_0(u^\top \Sigma^{-1} u) \), where \( \eta_0 \) is the density of the standard element of the family. The elliptical family covers multivariate normal, t- and Laplace distributions, playing central roles in the most widely employed additive perturbation mechanisms for differential privacy.

While this paper does not advise on query choice, it underscores the doubly crucial impact of the query on both the computational and statistical efficiency of the inference task. In the ABC literature, the choice of \( s \) dominates half of the discussion in achieving better approximation. It is also at the heart of the pursuit by the statistical privacy literature, to deliver best quality information within the bounds of privacy guarantees. An open intersection between the literature of ABC summary statistics as discussed in Section 1, and the literature on constructing efficient differentially private estimators (Lei, 2011; Smith, 2011; Avella-Medina, 2019), is to be explored.

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REFERENCES

ABOWD, J. M. (2018). The us census bureau adopts differential privacy. In Proceedings of the 24th ACM SIGKDD International Conference on Knowledge Discovery & Data Mining. ACM.

AVELLA-MEDINA, M. (2019). Privacy-preserving parametric inference: a case for robust statistics. Tech. rep., Columbia University, New York, NY.

BARNES, C. P., FILIPPI, S., STUMPF, M. P. & THORNE, T. (2012). Considerate approaches to constructing summary statistics for abc model selection. Statistics and Computing 22, 1181–1197.

BEAUMONT, M. A., ZHANG, W. & BALDING, D. J. (2002). Approximate Bayesian computation in population genetics. Genetics 162, 2025–2035.

BERNOTON, E., JACOB, P. E., GERBER, M. & ROBERT, C. P. (2019). Approximate bayesian computation with the wasserstein distance. Journal of the Royal Statistical Society: Series B (Statistical Methodology) 81, 235–269.

BLUM, A., DWORK, C., MCSHERRY, F. & NISSIM, K. (2005). Practical privacy: the SULQ framework. In Proceedings of the twenty-fourth ACM SIGMOD-SIGACT-SIGART symposium on Principles of database systems.

BLUM, M. G., NUNES, M. A., PRANGLE, D., Sisson, S. A. et al. (2013). A comparative review of dimension reduction methods in approximate bayesian computation. Statistical Science 28, 189–208.

BOOTH, J. G. & HOBERT, J. P. (1999). Maximizing generalized linear mixed model likelihoods with an automated monte carlo em algorithm. Journal of the Royal Statistical Society: Series B (Statistical Methodology) 61, 265–285.

CAFFO, B. S., JANK, W. & JONES, G. L. (2005). Ascent-based monte carlo expectation–maximization. Journal of the Royal Statistical Society: Series B (Statistical Methodology) 67, 235–251.

DEMPSTER, A. P., LAIRD, N. M. & RUBIN, D. B. (1977). Maximum likelihood from incomplete data via the em algorithm. Journal of the Royal Statistical Society: Series B (Methodological) 39, 1–22.

DROVANDI, C. C., PETTITT, A. N. & FADEY, M. J. (2011). Approximate bayesian computation using indirect inference. Journal of the Royal Statistical Society: Series C (Applied Statistics) 60, 317–337.

DUCHI, J. C., JORDAN, M. I. & WAINEWRIGHT, M. J. (2018). Minimax optimal procedures for locally private estimation. Journal of the American Statistical Association 113, 182–201.

DWORK, C., MCSHERRY, F., NISSIM, K. & SMITH, A. (2006). Calibrating noise to sensitivity in private data analysis. In Theory of cryptography conference. Springer.
Dwork, C. & Rothblum, G. N. (2016). Concentrated differential privacy. arXiv preprint arXiv:1603.01887.

Dwork, C. & Smith, A. (2009). Differential privacy for statistics: What we know and what we want to learn. Journal of Privacy and Confidentiality 1, 135–154.

Fearnhead, P. & Prangle, D. (2012). Constructing summary statistics for approximate Bayesian computation: semi-automatic approximate Bayesian computation. Journal of the Royal Statistical Society: Series B (Statistical Methodology) 74, 419–474.

Foulds, J., Geumlek, J., Welling, M. & Chaudhuri, K. (2016). On the theory and practice of privacy-preserving bayesian data analysis. In Proceedings of the Thirty-Second Conference on Uncertainty in Artificial Intelligence, UAI'16. Arlington, Virginia, United States: AUAI Press.

Gleim, A. & Pigorsch, C. (2013). Approximate bayesian computation with indirect summary statistics. Tech. rep., University of Bonn, Bonn, Germany.

Joyce, P. & Marjoram, P. (2008). Approximately sufficient statistics and bayesian computation. Statistical applications in genetics and molecular biology.

Lei, J. (2011). Differentially private m-estimators. In Advances in Neural Information Processing Systems.

Liu, J. S. (2008). Monte Carlo strategies in scientific computing. Springer Science & Business Media.

Louis, T. A. (1982). Finding the observed information matrix when using the em algorithm. Journal of the Royal Statistical Society: Series B (Methodological) 44, 226–233.

Marjoram, P., Molitor, J., Plagnol, V. & Tavaré, S. (2003). Markov chain Monte Carlo without likelihoods. Proceedings of the National Academy of Sciences 100, 15324–15328.

McCulloch, C. E. (1997). Maximum likelihood algorithms for generalized linear mixed models. Journal of the American statistical Association 92, 162–170.

Meiluison, I. (1989). A fast improvement to the em algorithm on its own terms. Journal of the Royal Statistical Society: Series B (Methodological) 51, 127–138.

Nikolov, A., Talwar, K. & Zhang, L. (2013). The geometry of differential privacy: the sparse and approximate cases. In Proceedings of the forty-fifth annual ACM symposium on Theory of computing. ACM.

Nissim, K., Raskhodnikova, S. & Smith, A. (2007). Smooth sensitivity and sampling in private data analysis. In Proceedings of the thirty-ninth annual ACM symposium on Theory of computing. ACM.

Nunes, M. A. & Balding, D. J. (2010). On optimal selection of summary statistics for approximate bayesian computation. Statistical applications in genetics and molecular biology.

Park, M., Foulds, J., Choudhary, K. & Welling, M. (2017). Dp-em: Differentially private expectation maximization. In Artificial Intelligence and Statistics.

Quintana, F. A., Liu, J. S. & del Pino, G. E. (1999). Monte carlo em with importance reweighting and its applications in random effects models. Computational statistics & data analysis 29, 429–444.

Schein, A., Wu, Z. S., Schofield, A., Zhou, M. & Wallach, H. (2019). Locally private bayesian inference for count models. In Proceedings of the 36th International Conference on Machine Learning, vol. 97. PMLR.

Sisson, S. A., Fan, Y. & Beaumont, M. (2018). Handbook of approximate Bayesian computation. Chapman and Hall/CRC.

Sisson, S. A., Fan, Y. & Tanaka, M. M. (2007). Sequential Monte Carlo without likelihoods. Proceedings of the National Academy of Sciences 104, 1760–1765.

Smith, A. (2011). Privacy-preserving statistical estimation with optimal convergence rates. In Proceedings of the forty-third annual ACM symposium on Theory of computing. ACM.

Toni, T., Welch, D., Strelkowa, N., Ipsen, A. & Stumpf, M. P. (2008). Approximate bayesian computation scheme for parameter inference and model selection in dynamical systems. Journal of the Royal Society Interface 6, 187–202.

Wang, Y.-X., Fienberg, S. & Smola, A. (2015). Privacy for free: Posterior sampling and stochastic gradient monte carlo. In International Conference on Machine Learning.

Wegmann, D., Leuenberger, C. & Excoffier, L. (2009). Efficient approximate bayesian computation coupled with markov chain monte carlo without likelihood. Genetics 182, 1207–1218.

Wei, G. C. & Tanner, M. A. (1990). A monte carlo implementation of the em algorithm and the poor man’s data augmentation algorithms. Journal of the American statistical Association 85, 699–704.

Wilkinson, R. D. (2013). Approximate Bayesian computation (ABC) gives exact results under the assumption of model error. Statistical applications in genetics and molecular biology 12, 129–141.

Wolfgang, R. & O’Connell, M. (1993). Generalized linear mixed models a pseudo-likelihood approach. Journal of statistical Computation and Simulation 48, 233–243.

Wood, S. N. (2010). Statistical inference for noisy nonlinear ecological dynamic systems. Nature 466, 1102.

Yang, X., Fienberg, S. E. & Rinaldo, A. (2012). Differential privacy for protecting multi-dimensional contingency table data: Extensions and applications. Journal of Privacy and Confidentiality.