EXISTENCE RESULTS FOR FRACTIONAL DIFFERENTIAL EQUATIONS IN PRESENCE OF UPPER AND LOWER SOLUTIONS

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ABSTRACT. In this paper, we study some existence results for fractional differential equations subject to some kind of initial conditions. First, we focus on the linear problem and we give an explicit form of solutions by reduction to an integral problem. We analyze some properties of the solutions to the linear problem in terms of its coefficients. Then we provide examples of application of the mentioned properties. Secondly, with the help of this theory, we study the nonlinear problem subject to initial value conditions. By using the upper and lower solutions method and the monotone iterative algorithm, we show the existence and localization of solutions to the nonlinear fractional differential equation.

1. Introduction. In the past years, much attention has been devoted to the study of fractional differential equations due to the fact that they have many applications in a broad range of areas such as physics, viscoelasticity, porous media, and other fields, where important phenomena can be modeled by fractional differential equations. We refer the reader to [6, 10, 24] and the references therein, for some applications. Concerning the study of the existence of solutions for fractional differential equations, in the last few decades, many different analytical techniques and methods have been applied such as iterative methods [21, 28], the Mawhin continuation theorem [21], the upper and lower solutions method [15, 25], or the use of fixed point theory [5].

In this paper, we study some existence results for fractional differential equations subject to initial value conditions by using the method of upper and lower solutions and the monotone iterative method. In fact, the upper and lower solutions method is a powerful tool to prove existence results both for initial and boundary value
problems. This technique has been widely applied to study the existence of multiple solutions for initial and boundary value problems associated to ordinary differential equations, and it is also applicable to the fractional case. In detail, the upper and lower solutions method consists on assuming the existence of a lower solution, $f$, and an upper solution $g$ to the problem of interest, which are generally well-ordered, allowing to prove the existence of a solution lying between them and, hence, this way, we have information not only about the existence of a solution, but also about its location. In relation with this, the monotone iterative technique provides a constructive method to prove existence results in a closed set delimited by the upper and lower solutions. Both methods are useful to study the properties of the solutions to nonlinear problems. The monotone iterative method is useful for the study of nonlinear equations and systems because it allows to reduce the problem to solve sequences of related linear differential equations. Specifically, if the nonlinear system is unwieldy, either too difficult or impossible to solve explicitly, then the monotone method may be useful to derive conclusions.

Recently, the study of the existence, the approximation of solutions, and the application of iterative techniques for systems of fractional differential equations has attracted some attention, see, for instance, [26] for more details.

More precisely, in this paper, we start by considering the linear fractional equation

$$
\left( D_{\alpha}^{m} - \lambda \sum_{i=1}^{m-1} A_i D_{\alpha_i}^{\eta_i} - \lambda A_0 \right) y(t) = f(t),
$$

(1)

where $A_i$ are constants for all $i = 0, \ldots, m-1$, $\alpha_i$ are positive real numbers, $i = 1, \ldots, m$, $\alpha_i = [\alpha_i] + \gamma_i$, $[\alpha_i] \in \mathbb{N}$, $\gamma_i \in [0,1)$, $\eta_i = [\alpha_i] + 1$, $i = 1, \ldots, m-1$, $0 < \alpha_{i-1} < \alpha_i$, for $i = 2, \ldots, m$, and $f$ is a continuous function on the interval $[0, b]$. If $\alpha = [\alpha]$, then $\gamma = 0$ and $D^{\alpha} = D^{[\alpha]}$, where $[\alpha]$ means the integer part of $\alpha$. The operator $D^{\alpha}$ represents the Riemann-Liouville fractional derivative.

Secondly, we analyze some properties of the solutions to the linear differential problem in terms of its coefficients. With the help of this theory, we study the nonlinear problem subject to initial value conditions by assuming that the nonlinearity depends on the solution and its fractional derivatives. So, by using the upper and lower solutions method and the monotone iterative technique, we prove the existence of solutions to the equation of interest. The approach we follow contains some new features with respect to others presented in [2, 9]. Firstly, we compute the solution to the linear problem, and we analyze the properties of the solution in terms of its coefficients. In fact, this part was not studied in [9]. Secondly, the novelty in our approach with respect to others present in the literature is that we are able to ensure the existence of solution to a problem by means of the upper and lower solutions for another problem subject to initial conditions. Moreover, we develop the monotone iterative technique to prove some existence results for the nonlinear problem.

The structure of this paper is displayed as follows. In the next Section, we state some preliminary lemmas and definitions. Section 3 is devoted to establish the existence of positive solutions for the linear equation (1) by converting the mentioned equation into an integral equation. We also analyze some properties of the solutions to equation (1) in terms of its coefficients and we give examples of application of the mentioned properties. On the other hand, Section 4 of this paper is consecrated to study the nonlinear problem subject to initial value conditions, proving the existence of solution by using the upper and lower solutions method.
The Riemann-Liouville fractional integral of order $\alpha > 0$ for a measurable function $f: (0, +\infty) \to \mathbb{R}$ is defined as

$$I^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s) \, ds, \quad t > 0,$$

where $\Gamma$ is the Euler Gamma function, provided that the right-hand side is pointwise defined on $(0, +\infty)$.

**Definition 2.2.** The Riemann-Liouville fractional derivative of order $\alpha > 0$ for a measurable function $f: (0, +\infty) \to \mathbb{R}$ is defined as

$$D^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} \left( \frac{d}{dt} \right)^n \int_0^t (t-s)^{n-\alpha-1} f(s) \, ds = \left( \frac{d}{dt} \right)^n I^{n-\alpha} f(t),$$

provided that the right-hand side is pointwise defined on $\mathbb{R}^+$. 

Here $n = \lfloor \alpha \rfloor + 1$, and $\lfloor \alpha \rfloor$ denotes the integer part of the real number $\alpha$. We will use the spaces $\mathcal{I}_0([0, b])$ and $\mathcal{I}_n([0, b])$. First, $\mathcal{I}_0([0, b])$ denotes the space of locally bounded functions $f$ on $(0, b]$, with $f \in L^1([0, b])$. On the other hand, $\mathcal{I}_n([0, b]) = \{ f, f^{(n)} \in \mathcal{I}_0([0, b]) \}, n \in \mathbb{N}$.

In the following Lemma, we recall some properties of the fractional operators. For more details, see [2] and [8, p.74]. In the statements iii)–vii), we refer to some of the constants introduced in problem (1), namely $\alpha_i$ are positive real numbers, $i = 1, \ldots, m$, $\alpha_i = [\alpha_i] + \gamma_i$, $[\alpha_i] \in \mathbb{N}$, $\gamma_i \in [0, 1)$, $0 < \alpha_{i-1} < \alpha_i$, for $i = 2, \ldots, m$.

**Lemma 2.3.** The following properties are satisfied:

i) If $\alpha > 0$, $\beta > 0$, then $I^\alpha I^\beta f(x) = I^{\alpha+\beta} f(x)$, a.e, for $f \in L^1(0, b)$.

ii) If $\alpha > \beta > 0$, then $D^\beta I^\alpha f(x) = I^{\alpha-\beta} f(x)$ a.e , for $f \in L^1(0, b)$. In particular, when $\alpha = \beta$, we have

$$D^\beta I^\alpha f(x) = f(x), \text{ a.e},$$

for $f \in L^1(0, b)$.

iii) For $y \in L^1(0, b)$, we have

$$D^{\alpha_i} y(t) = D^{\alpha_m} I^{\alpha_m - \alpha_i} y(t).$$

Now, take the case when $\alpha_m - \alpha_{m-1} > 1$.

iv) Assume that $y$ is bounded on $[0, b]$, then we have

$$D^{\alpha_m} y(t) = D^{[\alpha_m]+1} I^{1-\gamma_m} y(t).$$

v) There exists a function $u(t)$ such that

$$I^{1-\gamma_m} y(t) = I^{[\alpha_m]} u(t), \quad u(t) = D^{\alpha_m-1} y(t).$$

vi) Using (iv) and (v), we have

$$D^{\alpha_m} y(t) = DD^{[\alpha_m]} I^{[\alpha_m]} y(t) = Du(t).$$

The last Section is devoted to develop the monotone method to solve the nonlinear equation.

2. Preliminary results. For the convenience of the reader, we present here the necessary definitions and properties about fractional calculus theory. We refer the reader to [9, 23, 1, 3, 2] for more details.

**Definition 2.1.** The Riemann-Liouville fractional integral of order $\alpha > 0$ for a measurable function $f: (0, +\infty) \to \mathbb{R}$ is defined as

$$I^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s) \, ds, \quad t > 0,$$

where $\Gamma$ is the Euler Gamma function, provided that the right-hand side is pointwise defined on $(0, +\infty)$.
vii) For \( i = 0, \ldots, m - 1 \), we have
\[
D^{\alpha_i} y(t) = I^{\beta_i} u(t),
\]
where \( \beta_i = \alpha_m - \alpha_{m-1} + \alpha_{m-1} - \alpha_i - 1 \). We note that \( \beta_i > 0 \) for all \( i = 0, \ldots, m - 1 \).
viii) If \( \alpha > 0 \), \( m \in \mathbb{N} \), and \( D^{\alpha} y \), \( D^{\alpha+m} y \) exist, then
\[
D^m D^{\alpha} y = D^{\alpha+m} y. \tag{2}
\]

The following result for the convergence of a sequence in a partially ordered Banach space will be also useful to our procedure. For details, we refer to [13] and the references therein.

**Lemma 2.4.** [13] Let \( E \) be a partially ordered Banach space, and \( \{x_n\} \subset E \) a monotone sequence and relatively compact set, then \( \{x_n\} \) is convergent.

3. **Linear problem: The case** \( \alpha_m - \alpha_{m-1} \geq 1 \). In this section, we study the following linear equation in the case \( \alpha_m - \alpha_{m-1} \geq 1 \)
\[
\left( D^{\alpha_m} - \lambda \sum_{i=0}^{m-1} A_i D^{\alpha_i} - \lambda A_0 \right) y(t) = f(t), \tag{3}
\]
where \( A_i \) are constants for all \( i = 0, \ldots, m - 1 \), \( \alpha_i \) are positive real numbers, \( i = 1, \ldots, m \), \( \alpha_i = [\alpha_i] + \gamma_i, [\alpha_i] \in \mathbb{N}, \gamma_i \in [0,1), \eta_i = [\alpha_i] + 1, i = 1, \ldots, m - 1, 0 < \alpha_{i-1} < \alpha_i \), for \( i = 2, \ldots, m \), and \( f \) is a continuous function on the interval \([0,b]\). If \( \alpha = [\alpha] \), then \( \gamma = 0 \) and \( D^{\alpha} = D^{[\alpha]} \), where \([\alpha]\) means the integer part of \( \alpha \). Suppose also that \( \alpha_0 = 0 \). The operator \( D^{\alpha} \) is the Riemann-Liouville fractional derivative given by Definition 2.2.

We will give the explicit expression of the solution and some comparison results for equation (3). By using some properties of fractional differential operators, we will consider the transformation into a Volterra’s integral equation of second type (for more details see [2]).

Hereinafter, equation (3) is written as
\[
\left( D^{\alpha_m} - \lambda \sum_{i=0}^{m-1} A_i D^{\alpha_i} \right) y(t) = f(t), 0 \leq t \leq b. \tag{4}
\]

Using the properties of fractional derivatives in Lemma 2.3, the equation (4) changes its analytical form into
\[
Du(t) - \lambda \sum_{i=0}^{m-2} A_i I^{\beta_i} u(t) - \lambda A_{m-1} I^{\beta_{m-1}} u(t) = f(t). \tag{5}
\]

Therefore, we get
\[
D(u(t) - \lambda \sum_{i=0}^{m-2} A_i \int_0^t \int_0^\tau (\tau - s)^{\beta_i-1} u(s) ds d\tau
- \lambda A_{m-1} \int_0^t I^{\beta_{m-1}}(\tau) u(\tau) d\tau - \int_0^t f(\tau) d\tau - c) = 0,
\]
where \( c \) is an arbitrary constant.
Hence, we obtain
\[
\begin{align*}
  u(t) - \lambda \sum_{i=0}^{m-2} A_i \int_0^t \int_0^\tau (\tau - s)^{\beta_i - 1} u(s) ds d\tau \\
  - \lambda A_{m-1} \int_0^t I^{\beta_{m-1}}(\tau) u(\tau) d\tau = \int_0^t f(\tau) d\tau + c.
\end{align*}
\]

Interchanging the order of integration in the previous equation and using Dirichlet’s formula, we have
\[
\begin{align*}
  u(t) - \lambda \sum_{i=0}^{m-1} A_i \int_0^t u(s) \int_s^t A_i \frac{(\tau - s)^{\beta_i - 1}}{\Gamma(\beta_i)} d\tau ds = \int_0^t f(\tau) d\tau + c. \\
\end{align*}
\]

By transposing the variables \(\tau\) and \(s\), we obtain
\[
\begin{align*}
  u(t) - \lambda \sum_{i=0}^{m-1} A_i \int_0^t u(\tau) \int_\tau^t A_i \frac{(s - \tau)^{\beta_i - 1}}{\Gamma(\beta_i)} d\sigma d\tau = \int_0^t f(\tau) d\tau + c. \\
\end{align*}
\]

Moreover, we get
\[
\begin{align*}
  u(t) = F(t) + \lambda \int_0^t K(t, \tau) u(\tau) d\tau,
\end{align*}
\]
where \(F(t) = \int_0^t f(s) ds + c\).

We remark that equation (8) is the Volterra’s integral equation of the second kind. The kernel \(K(t, \tau)\) is given by
\[
\begin{align*}
  K(t, \tau) = \sum_{i=0}^{m-1} \int_\tau^t A_i \frac{(s - \tau)^{\beta_i - 1}}{\Gamma(\beta_i)} ds.
\end{align*}
\]

Then, we have
\[
\begin{align*}
  K(t, \tau) = \sum_{i=0}^{m-1} A_i \frac{(t - \tau)^{\beta_i}}{\Gamma(\beta_i + 1)}.
\end{align*}
\]

In the rest of the paper, when the variables \((t, \tau)\) appear, we always consider the domain
\[
T(0, b) = \{(t, \tau) \in \mathbb{R}^2 : t \in [0, b], \tau \in [0, t]\}.
\]

In the following Lemma, we give the explicit expression for the kernel.

**Lemma 3.1.** The iterated kernel \(K_n\) is given by the expression
\[
\begin{align*}
  K_n(t, \tau) = \sum_{i_1=0}^{m-1} \cdots \sum_{i_n=0}^{m-1} A_{i_1} \times \cdots \times A_{i_n} \frac{A_n}{\Gamma(\beta_{i_1} + \cdots + \beta_{i_n} + n)} (t - \tau)^{\beta_{i_1} + \cdots + \beta_{i_n} + n - 1}, \text{ for all } n \in \mathbb{N}.
\end{align*}
\]

**Proof.** The iterated kernels \(K_1, K_2, ..., K_n\) are defined inductively by the relation
\[
\begin{align*}
  K_n(t, \tau) = \int_\tau^t K(t, s) K_{n-1}(s, \tau) ds.
\end{align*}
\]
For \( n = 2 \), we have
\[
K_2(t, \tau) = \int_\tau^t K(t, s)K(s, \tau)ds
= \int_\tau^t \sum_{i=0}^{m-1} \frac{A_i}{\Gamma(\beta_i + 1)}(t-s)^{\beta_i} \sum_{j=0}^{m-1} \frac{A_j}{\Gamma(\beta_j + 1)}(s-\tau)^{\beta_j}ds
= \sum_{i,j=0}^{m-1} \frac{A_i A_j}{\Gamma(\beta_i + \beta_j + 1)} \int_\tau^t (t-s)^{\beta_i}(s-\tau)^{\beta_j}ds.
\]
Using the fact that, for each \( a, b \geq 0 \) and \( p, q > 0 \), we have
\[
\int_a^b (b-\theta)^p(\theta-a)^q d\theta = (b-a)^{p+q+1} \frac{\Gamma(p+1)\Gamma(q+1)}{\Gamma(p+q+2)},
\]
finally, we get
\[
K_2(t, \tau) = \sum_{i,j=0}^{m-1} \frac{A_i A_j}{\Gamma(\beta_i + \beta_j + 2)}(t-\tau)^{\beta_i + \beta_j + 1}.
\]
Then, (11) holds for \( n = 2 \). Now, assume that (11) holds for each \( n \geq 2 \) and prove that, for all \( n+1 \), we have
\[
K_{n+1}(t, \tau) = \int_\tau^t K(t, s)K_n(s, \tau)ds
= \sum_{i_1=0}^{m-1} \cdots \sum_{i_{n+1}=0}^{m-1} \frac{A_i \cdots A_i}{\Gamma(\beta_i + \cdots + \beta_{i_{n+1}} + n)} \int_\tau^t (t-s)^{\beta_{i_{n+1}}}ds
= \sum_{i_1=0}^{m-1} \cdots \sum_{i_{n+1}=0}^{m-1} \frac{A_i \cdots A_i A_i}{\Gamma(\beta_i + \cdots + \beta_{i_{n+1}} + n)} \int_\tau^t (t-s)^{\beta_{i_{n+1}}}ds
= \sum_{i_1=0}^{m-1} \cdots \sum_{i_{n+1}=0}^{m-1} \frac{A_i \cdots A_i A_i}{\Gamma(\beta_i + \cdots + \beta_{i_{n+1}} + n+1)} \int_\tau^t (t-s)^{\beta_{i_{n+1}}}ds
= \sum_{i_1=0}^{m-1} \cdots \sum_{i_{n+1}=0}^{m-1} \frac{A_i \cdots A_i A_i}{\Gamma(\beta_i + \cdots + \beta_{i_{n+1}} + n+1)}(t-\tau)^{\beta_{i_{n+1}}+1}
\]
Therefore, (11) holds for any \( n \geq 2 \).

\( \square \)

**Theorem 3.2.** [2] Suppose that, in Equation (4), the function \( f \in L_0([0,b]) \) is continuous on \([0,b]\). Then the Volterra’s integral equation (8) has one and only one family of bounded solutions given by
\[
u(t, c_1) = F(t) + \lambda \int_0^t R(t, \tau, \lambda)F(\tau)d\tau, \quad 0 \leq t \leq b,
\]
where \( c_1 \) is an arbitrary constant, and the resolvent Kernel \( R \) is given by the series
\[
R(t, \tau, \lambda) = K(t, \tau) + \sum_{n=1}^\infty \lambda^n K_n(t, \tau),
\]
which is convergent for all values of \( \lambda \), \( t \) and \( \tau \), where

\[
K_n(t, \tau) = \sum_{i_1=0}^{m-1} \cdots \sum_{i_n=0}^{m-1} \frac{A_{i_1} \times \cdots \times A_{i_n}}{\Gamma(\beta_{i_1} + \cdots + \beta_{i_n} + n)} (t - \tau)^{\beta_{i_1} + \cdots + \beta_{i_n} + n - 1}.
\]

**Proof.** ([19]). The function \( f \) is in \( \mathcal{I}_0([0, b]) \), then the function \( F(t) = \int_0^t f(s)ds + c \) is bounded on \([0, b]\) and integrable. The kernel \( K(t, \tau) \) is bounded and integrable on the triangle \( T(0, b) \). Then we can apply the Theorem of Volterra’s integral equation. This completes the proof. \( \square \)

We recall some properties for alternating series that will be useful in our reasoning.

**Lemma 3.3.** Consider the alternating series \( \sum_{n \geq 0} u_n \), with \( u_{2n}u_{2n+1} \leq 0 \), for every \( n \in \mathbb{N} \cup \{0\} \), and such that \( \{|u_n|\} \) is monotonic decreasing and convergent to 0. Then the sum of the series lies between \( u_0 \) and \( u_0 + u_1 \).

**Proof.** Suppose that \( u_0 \leq 0 \). If we represent by \( S_l = \sum_{n=0}^{l} u_n \) the \( l \)-th term of the sequence of partial sums of the series \( \sum_{n \geq 0} u_n \), then the subsequence of odd terms

\[
S_{2l+1} = (u_0 + u_1) + (u_2 + u_3) + \cdots + (u_{2l} + u_{2l+1})
\]

is monotonic decreasing since all the terms in parentheses are nonpositive. Besides,

\[
S_{2l+1} = u_0 + (u_1 + u_2) + \cdots + (u_{2l-1} + u_{2l}) + u_{2l+1} \geq u_0,
\]

since the other terms are nonnegative. Hence \( \{S_{2l+1}\} \to L \), as \( l \to \infty \), with \( L \geq u_0 \). Also, \( S_{2l+2} = S_{2l+1} + u_{2l+2} \to L \). Note that the sum of the series is such that \( L \geq u_0 \). Besides, \( L \leq S_1 = u_0 + u_1 \).

Suppose that \( u_0 \geq 0 \). In this case,

\[
S_{2l+1} = (u_0 + u_1) + (u_2 + u_3) + \cdots + (u_{2l} + u_{2l+1})
\]

is monotonic increasing since all the terms in parentheses are nonnegative. Besides,

\[
S_{2l+1} = u_0 + (u_1 + u_2) + \cdots + (u_{2l-1} + u_{2l}) + u_{2l+1} \leq u_0,
\]

since the other terms are nonpositive. Hence \( \{S_{2l+1}\} \to L \), as \( l \to \infty \), with \( L \leq u_0 \). Also, \( S_{2l+2} = S_{2l+1} + u_{2l+2} \to L \). Note that the sum of the series is such that \( L \leq u_0 \). Besides, \( L \geq S_1 = u_0 + u_1 \). \( \square \)

In the next Lemma, we will give some sufficient conditions in order to know the sign of the sum of the series \( \sum_{n=0}^{\infty} \lambda^n K_n(t, \tau) \), redefine \( K_0 := K \).

**Lemma 3.4.** The sign of the resolvent \( R(t, \tau, \lambda) \) is given by the sign of the parameters \( \lambda \) and \( A_i \), \( i = 0, \ldots, m - 1 \).
Proof. We distinguish three cases, the last two take into account the specified criterion for alternating series applied to \( \sum_{n \geq 0} u_n \), where \( u_n = \lambda^n K_n(t, \tau) \):

Case 1. If \( \lambda \geq 0 \) and \( A_i \geq 0 \), for all \( i = 0, \ldots, m - 1 \), then \( u_n \geq 0 \) (and, hence, the resolvent is nonnegative).

Case 2. Suppose that \( \lambda < 0 \), \( A_i > 0 \), for all \( i = 0, \ldots, m - 1 \), and that \( |u_n| \) is decreasing and tending to cero. If, moreover, \( \lambda \in [-1, 0) \), then

\[
u_0 = \sum_{i=0}^{m-1} \frac{A_i}{\Gamma(\beta_i + 1)} (t - \tau)^{\beta_i} \geq 0
\]

and \( u_0 + u_1 = (1 + \lambda) K(t, \tau) = (1 + \lambda) u_0 \geq 0 \) and, therefore, the resolvent is nonnegative.

Case 3. If \( \lambda > 0 \), \( A_i < 0 \), for all \( i = 0, \ldots, m - 1 \), and that \( |u_n| \) is decreasing and tending to cero. Then

\[
u_0 = \sum_{i=0}^{m-1} \frac{A_i}{\Gamma(\beta_i + 1)} (t - \tau)^{\beta_i} \leq 0,
\]

and \( u_0 + u_1 = (1 + \lambda) K(t, \tau) = (1 + \lambda) u_0 \leq 0 \) and, therefore, the resolvent is nonpositive.

In the following Tables, we show sets of hypotheses that imply the nonnegativity or nonpositivity of the solution to (8).

| \( f \geq 0 \) on \([0, b]\) | \( f \geq 0 \) on \([0, b]\) |
|---|---|
| \( c \geq 0 \) | \( c \geq 0 \) |
| \( \lambda \geq 0 \) | \( \lambda < 0 \) |
| \( R \geq 0 \) | \( R \leq 0 \) |

Table 1. Sufficient conditions for the nonnegativity of solutions

| \( f \leq 0 \) on \([0, b]\) | \( f \leq 0 \) on \([0, b]\) |
|---|---|
| \( c \leq 0 \) | \( c \leq 0 \) |
| \( \lambda \geq 0 \) | \( \lambda \geq 0 \) |
| \( R \geq 0 \) | \( R \leq 0 \) |

Table 2. Sufficient conditions for the nonpositivity of solutions

In consequence, taking into account the cases considered in the proof of Lemma 3.4, we have the following comparison principles for equation (13).

**Theorem 3.5.** Suppose that the hypotheses of Theorem 3.2 are satisfied. The following properties hold:

- If \( \lambda \geq 0 \), \( c \geq 0 \), \( A_i \geq 0 \), \( i = 0, \ldots, m - 1 \), and \( f \geq 0 \) on \([0, b]\), then \( u(t, c_1) \geq 0 \) on \([0, b]\), and, therefore, \( y(t, c_1) \geq 0 \) on \([0, b]\).
- If \( \lambda \geq 0 \), \( c \leq 0 \), \( A_i \geq 0 \), \( i = 0, \ldots, m - 1 \), and \( f \leq 0 \) on \([0, b]\), then \( u(t, c_1) \leq 0 \) on \([0, b]\), and, therefore, \( y(t, c_1) \leq 0 \) on \([0, b]\).
- If \( \lambda > 0 \), \( c \leq 0 \), \( A_i < 0 \), for all \( i = 0, \ldots, m - 1 \) (with \( |u_n| \) is decreasing and tending to cero), and \( f \leq 0 \) on \([0, b]\), then \( u(t, c_1) \leq 0 \) on \([0, b]\), and, therefore, \( y(t, c_1) \leq 0 \) on \([0, b]\).
Suppose that \( \alpha_m - \alpha_m - 1 > 1 \) and \( f(t) \in L_2([0, b]) \) is continuous on \([0, b]\). Then equation (1) admits a family of solutions \( y(t, c_1) = \int_0^t u(t, c_1) \) in \( L_2(\alpha_m, b) \) where \( \alpha_m = [\alpha_m] \); and \( y(t, c_1) \) is the unique solution to the Volterra’s integral equation. In addition, suppose that \( y(t, c_1) \) is such that

\[
\int_0^t (t-s)^{\alpha_m-1} f(s) ds + c t^{\alpha_m-1} \geq \frac{A|\lambda|}{\alpha_m(1-|\lambda|)} \int_0^t (t-s)^{\alpha_m} f(s) ds + \frac{A|\lambda| c}{\alpha_m(1-|\lambda|)} t^{\alpha_m},
\]

where \( A \) is such that

\[
\sum_{i_1=0}^{m-1} \cdots \sum_{i_n=0}^{m-1} \frac{A_{i_1} \cdots A_{i_n}}{\Gamma(\beta_{i_1} + \cdots + \beta_{i_n} + n)} (t - \tau)^{\beta_{i_1} + \cdots + \beta_{i_n} - 1} \leq A.
\]

Then equation (1) admits a nonnegative solution.

Proof. The existence of the solution \( y \) is shown by [1], since the Volterra’s integral equation (8) is the same as that given in Reference [1].

Now, let us show the nonnegativity of the solution. Applying \( I^{\alpha_m-1} \) to equation (13), we obtain

\[
I^{\alpha_m-1} y(t, c_1) = I^{\alpha_m-1} F(t) + \lambda I^{\alpha_m-1} h(t),
\]

where \( h(t) = \int_0^t R(t, \tau, \lambda) F(\tau) d\tau \), hence

\[
|I^{\alpha_m-1} h(t)| = \left| \frac{1}{\Gamma(\alpha_m - 1)} \int_0^t (t-s)^{\alpha_m-2} \int_0^s R(s, \tau, \lambda) \left[ \int_t^\tau f(u) du + c \right] d\tau ds \right|.
\]

Using Fubini’s theorem, we get

\[
|I^{\alpha_m-1} h(t)| = \left| \frac{1}{\Gamma(\alpha_m - 1)} \int_0^t (t-s)^{\alpha_m-2} \left[ \int_0^s \int_u^s R(s, \tau, \lambda) d\tau du + c \int_0^s R(s, \tau, \lambda) d\tau ds \right] ds \right|
\]

\[
\leq \frac{1}{\Gamma(\alpha_m - 1)} \int_0^t (t-s)^{\alpha_m-2} \left[ \int_0^s |f(u)| \int_u^s |R(s, \tau, \lambda)| d\tau du + c \int_0^s |R(s, \tau, \lambda)| d\tau ds \right] ds.
\]

Consider the hypothesis

\[
\sum_{i_1=0}^{m-1} \cdots \sum_{i_n=0}^{m-1} \frac{A_{i_1} \cdots A_{i_n}}{\Gamma(\beta_{i_1} + \cdots + \beta_{i_n} + n)} (t - \tau)^{\beta_{i_1} + \cdots + \beta_{i_n} - 1} \leq A,
\]

then, for the resolvent \( R(s, \tau, \lambda) \), we have

\[
|R(s, \tau, \lambda)| \leq \frac{A}{1-|\lambda|} =: \tilde{A}.
\]

Therefore, we get

\[
|I^{\alpha_m-1} h(t)| \leq \frac{\tilde{A}}{\Gamma(\alpha_m - 1)} \int_0^t (t-s)^{\alpha_m-2} \int_0^s |f(u)| (s-u) du ds + \frac{\tilde{Ac}}{\Gamma(\alpha_m - 1)} \int_0^t (t-s)^{\alpha_m-2} ds.
\]
By Fubini’s theorem, and the changes of variable $s = u + r(t - u)$ and $s = rt$, we have

$$|I_{\alpha_m}^{-1} h(t)| \leq \frac{\tilde{A}}{\Gamma(\alpha_m - 1)} \int_0^t |f(u)| \int_u^t (t - s)^{\alpha_m - 2}(s - u)dsdu + \frac{\tilde{A}c}{\Gamma(\alpha_m - 1)} \int_0^t (t - s)^{\alpha_m - 2}s ds$$

$$= \frac{\tilde{A}}{\Gamma(\alpha_m - 1)} \int_0^t |f(u)| \int_0^1 (t - u)^{\alpha_m - 2}(1 - r)^{\alpha_m - 2}r(t - u)(t - u)drdu + \frac{\tilde{A}c}{\Gamma(\alpha_m - 1)} \int_0^1 (1 - r)^{\alpha_m - 2}r dr$$

$$= \frac{\tilde{A}}{\Gamma(\alpha_m - 1)} \int_0^t |f(u)|(t - u)^{\alpha_m} \int_0^1 (1 - r)^{\alpha_m - 2}r drdu + \frac{\tilde{A}c}{\Gamma(\alpha_m - 1)} \int_0^1 r^{\alpha_m} \frac{(\alpha_m - 1)\Gamma(2)}{\Gamma(\alpha_m + 1)} du$$

Then, we obtain

$$I_{\alpha_m}^{-1} h(t) \geq -\frac{\tilde{A}}{\Gamma(\alpha_m + 1)} \int_0^t |f(u)|(t - u)^{\alpha_m} du - \frac{\tilde{A}c}{\Gamma(\alpha_m + 1)} t^{\alpha_m}.$$  

On the other hand, by Fubini’s theorem, we have

$$I_{\alpha_m}^{-1} F(t) = \frac{1}{\Gamma(\alpha_m - 1)} \int_0^t (t - s)^{\alpha_m - 2} \left[ \int_0^s f(u)du + c \right] ds$$

$$= \frac{1}{\Gamma(\alpha_m - 1)} \int_0^t f(u) \int_u^t (t - s)^{\alpha_m - 2}dsdu + \frac{c}{\Gamma(\alpha_m - 1)} \int_0^t (t - s)^{\alpha_m - 2}ds$$

$$= \frac{1}{\Gamma(\alpha_m)} \int_0^t (t - u)^{\alpha_m - 1} f(u)du + \frac{c}{\Gamma(\alpha_m)} t^{\alpha_m - 1}.$$  

Finally, we get

$$I_{\alpha_m}^{-1} u(t, c_1) \geq -\frac{|\lambda|\tilde{A}}{\Gamma(\alpha_m + 1)} \int_0^t |f(u)|(t - u)^{\alpha_m} du - \frac{\tilde{A}|\lambda|c}{\Gamma(\alpha_m + 1)} t^{\alpha_m} + \frac{1}{\Gamma(\alpha_m)} \int_0^t (t - u)^{\alpha_m - 1} f(u)du + \frac{c}{\Gamma(\alpha_m)} t^{\alpha_m - 1}. \quad (15)$$

Now, using the hypothesis (14), we have that $y(t, c_1)$ is a nonnegative solution to (1).
Theorem 3.7. Suppose that \( \alpha_m - \alpha_{m-1} > 1 \) and \( f(t) \in \mathcal{I}_0([0,b]) \) is continuous on \([0,b] \). Moreover, suppose that there exists a constant \( h > 1 \) such that

\[
\int_0^t (t-s)^{\alpha_m-2} u^+(s) ds \geq h \int_0^t (t-s)^{\alpha_m-2} u^-(s) ds, \quad t \in [0,b],
\]

where \( u^+ = \max \{u,0\} \) and \( u^- = \max \{-u,0\} \).

Then \( y(t) \geq 0 \), for all \( t \in [0,b] \).

Proof. First, for simplicity, we consider \( y(t) = I^{\alpha_m-1}u(t) \), for \( t \in [0,b] \), then, using the hypothesis (16), we have

\[
y(t) = I^{\alpha_m-1}u(t) = \frac{1}{\Gamma(\alpha_m-1)} \int_0^t (t-s)^{\alpha_m-2} u(s) ds
\]

\[
= \frac{1}{\Gamma(\alpha_m-1)} \int_0^t (t-s)^{\alpha_m-2} u^+(s) ds - \frac{1}{\Gamma(\alpha_m-1)} \int_0^t (t-s)^{\alpha_m-2} u^-(s) ds
\]

\[
\geq \frac{h}{\Gamma(\alpha_m-1)} \int_0^t (t-s)^{\alpha_m-2} u^+(s) ds - \frac{1}{\Gamma(\alpha_m-1)} \int_0^t (t-s)^{\alpha_m-2} u^-(s) ds
\]

\[
\geq \frac{(h-1)}{\Gamma(\alpha_m-1)} \int_0^t (t-s)^{\alpha_m-2} u^-(s) ds \geq 0.
\]

This completes the proof. \( \square \)

4. Examples.

4.1. Example. Consider the linear fractional equation

\[
D^2 y(t) - 3D^\alpha y(t) - y(t) = 2, \quad 0 < t \leq b,
\]

where \( 0 < \alpha < 1 \) and \( b > 0 \). Equation (17) has the form of equation (1) with \( m = 2 \), \( \alpha_2 = 2, \alpha_1 = \alpha \). Here, \( \lambda = 1, A_0 = 1, A_1 = 3, \beta_0 = 1 \) and \( \beta_1 = 1 - \alpha \). It is easy to see that the kernel \( K(t,\tau) \) is given by

\[
K(t,\tau) = \frac{1}{2} (t-\tau) + \frac{3}{\Gamma(2-\alpha)} (t-\tau)^{1-\alpha},
\]

the solution to the Volterra’s integral equation (8), \( u(t,c_1) \), is given by

\[
u(t,c_1) = F(t) + \int_0^t R(t,\tau,1) F(\tau) d\tau.
\]

Then, by Theorem 3.5, we conclude that the solution \( u \) is positive for all \( t \in [0,b] \).

See Figure 1.

On the other hand, we consider (1) in the case when \( \lambda > 0, A_i > 0 \) for all \( i = 0, \ldots, m - 1 \) and \( f(t) \leq 0 \) for all \( t \in [0,b] \). Then, by Theorem 3.5, \( u(t,c_1) \leq 0 \).

See Figure 2.

5. Nonlinear problem. It is well known that the method of upper and lower solutions coupled with the monotone iterative technique offers theoretical as well as constructive existence results in a closed set that is generated, respectively, by the upper and lower solutions. In this Section, we use the first method to ensure the existence of solution for a nonlinear fractional problem.

We consider the following nonlinear fractional differential equation

\[
D^{\alpha_m} y(t) = f(t, y(t), D^{\alpha_1} y(t), \ldots, D^{\alpha_{m-1}} y(t)),
\]

(20)
with \( f \) continuous, subject to the initial conditions
\[
\begin{align*}
[D^{\alpha_m-1}y]^{(i)}(0^+, c_1) &= 0, \quad i = 0, \ldots, [\alpha_m] - 1, \\
[D^{\alpha_m-1}y]^{([\alpha_m])}(0^+, c_1) &= c,
\end{align*}
\] (21)
where \( \alpha_m - \alpha_{m-1} \geq 1 \), and \( D^{\alpha_m} \) denotes the Riemann-Liouville derivative of order \( \alpha_m \).

**Definition 5.1.** A lower solution to the problem (20)-(21) is a function \( v \in C([0, 1]) \) satisfying that
\[
\begin{align*}
D^{\alpha_i}v(t) &\leq f(t, v(t), D^{\alpha_1}v(t), \ldots, D^{\alpha_m-1}v(t)), \\
[D^{\alpha_m-1}v]^{(i)}(0^+, c_1) &= 0, \text{ for } i = 0, \ldots, [\alpha_m] - 1, \\
[D^{\alpha_m-1}v]^{([\alpha_m])}(0^+, c_1) &\leq c.
\end{align*}
\]
Similarly, an upper solution to (20)-(21) is a function \( w \in C([0, b]) \) satisfying similar expressions, but with the inequalities reversed.

For \( v, w \in C([0, b]) \) with \( v \leq w \), \( D^{\alpha_i}v \leq D^{\alpha_i}w \), for \( i = 0, \ldots, m-1 \), we introduce the truncation function \( p_i : [0, b] \times \mathbb{R} \to \mathbb{R} \), \( i = 0, \ldots, m-1 \), given by
\[
p_i(t, u) = \max\{D^{\alpha_i}v(t), \min\{u, D^{\alpha_i}w(t)\}\} = \begin{cases} 
D^{\alpha_i}v(t), & \text{if } u \leq D^{\alpha_i}v(t), \\
u, & \text{if } D^{\alpha_i}v(t) \leq u \leq D^{\alpha_i}w(t), \\
D^{\alpha_i}w(t), & \text{if } u \geq D^{\alpha_i}w(t),
\end{cases}
\]
for \( t \in [0, b] \) and \( u \in \mathbb{R} \). Here, we take \( \alpha_0 = 0 \), and \( p := p_0 \).
If $D^{\alpha_i}v, D^{\alpha_i}w$ are continuous for $i = 0, \ldots, m - 1$, the function $t \to p_i(t, u(t))$ is continuous on $[0, b]$, for $u \in C([0, b]), i = 0, \ldots, m - 1$.

Now, we consider the truncated problem

$$D^{\alpha_m}y(t) - \lambda \sum_{i=0}^{m-1} A_i D^{\alpha_i}y(t) = f(t, p(t, y(t)), p_1(t, D^{\alpha_1}y(t)), \ldots, p_{m-1}(t, D^{\alpha_{m-1}}y(t))) - \lambda \sum_{i=0}^{m-1} A_i p_i(t, D^{\alpha_i}y(t)),$$

with the initial conditions (21). We denote by $\sigma(t)$ the right-hand side in (22), which should formally be denoted by $\sigma_y(t)$, due to its dependence on the function $y$.

$$\sigma(t) := f(t, p(t, y(t)), p_1(t, D^{\alpha_1}y(t)), \ldots, p_{m-1}(t, D^{\alpha_{m-1}}y(t))) - \lambda \sum_{i=0}^{m-1} A_i p_i(t, D^{\alpha_i}y(t)).$$

Using the relation between $u$ and $y$ given in Section 3, $\sigma(t)$ is given by

$$\sigma(t) = f(t, p(t, \lambda^{\alpha_{m-1}}u(t, c_1)), p_1(t, D^{\alpha_1} \lambda^{\alpha_{m-1}}u(t, c_1)), \ldots, p_{m-1}(t, D^{\alpha_{m-1}} \lambda^{\alpha_{m-1}}u(t, c_1))) - \lambda \sum_{i=0}^{m-1} A_i p_i(t, D^{\alpha_i} \lambda^{\alpha_{m-1}}u(t, c_1)).$$

Using (ii) of Lemma 2.3, we can write $\sigma$ (remarking its dependence on $u$) as

$$\sigma_u(t) := f(t, p(t, \lambda^{\alpha_{m-1}}u(t, c_1)), p_1(t, D^{\alpha_1} \lambda^{\alpha_{m-1}}u(t, c_1)), \ldots, p_{m-1}(t, D^{\alpha_{m-1}} \lambda^{\alpha_{m-1}}u(t, c_1))) - \lambda \sum_{i=0}^{m-1} A_i p_i(t, D^{\alpha_i} \lambda^{\alpha_{m-1}}u(t, c_1)).$$

Next, we define an operator as follows

$$T_c : C([0, b]) \to C([0, b]),$$

$$u \to T_c(u),$$

given by

$$[T_c u](t) = \int_0^t \sigma_u(s) ds + c_1 + \lambda \int_0^t R(t, \tau, \lambda) \left[ \int_0^\tau \sigma_u(q) dq + c_1 \right] d\tau. \quad (23)$$

Such a notation for the operator is a way to remark the dependence of its definition on the initial conditions given in (21).

**Lemma 5.2.** The operator $T_c$ is completely continuous.

**Proof.** First, let us prove that $T_c$ is continuous. In fact, for $u_0$ fixed in $C([0, b])$, we have

$$\left| [T_c u(t)] - [T_c u_0(t)] \right| = \int_0^t \left| \sigma_u(s) - \sigma_{u_0}(s) \right| ds + \lambda \int_0^t R(t, \tau, \lambda) \left[ \int_0^\tau \left| \sigma_u(q) - \sigma_{u_0}(q) \right| dq \right] d\tau.$$


Recall that
\[
|\sigma_u(s) - \sigma_{\bar{u}_0}(s)| = \left| f(s, p_0(s, I^{\alpha_m - \alpha_1 - 1} u(s)), ..., p_m(s, I^{\alpha_m - \alpha_1 - 1} u(s))) \\
- f(s, p_0(s, I^{\alpha_m - \alpha_1 - 1} u_0(s)), ..., p_m(s, I^{\alpha_m - \alpha_1 - 1} u_0(s))) \\
- \lambda \sum_{i=0}^{m-1} A_i \left[ p_i(t, I^{\alpha_m - \alpha_i - 1} u(t)) - p_i(t, I^{\alpha_m - \alpha_i - 1} u_0(t)) \right] \right| \tag{24}
\]

By hypothesis, \(f\) is continuous on \([0, b] \times \mathbb{R}^m\). Let
\[
\gamma_0 = \min_{t \in [0, b]} \{v(t)\}, \eta_0 = \max_{t \in [0, b]} \{v(t)\},
\]
and
\[
\gamma_i = \min_{t \in [0, b]} \{D^{\alpha_i} v(t)\}, \eta_i = \max_{t \in [0, b]} \{D^{\alpha_i} w(t)\},
\]
then \(f\) is uniformly continuous on \([0, b] \times [\gamma_0, \eta_0] \times \prod_{i=1}^{m-1} [\gamma_i, \eta_i]\). Besides, taking
\(
\hat{B} = \max_{i=0, ..., m-1} b^{\alpha_m - \alpha_1 - 1}, \quad \hat{K} = \max_{i=0, ..., m-1} \frac{1}{\Gamma(\alpha_m - \alpha_i)}
\)
we can prove that
\[
|I^{\alpha_m - \alpha_i - 1} u(s) - I^{\alpha_m - \alpha_i - 1} u_0(s)| = |I^{\alpha_m - \alpha_i - 1}[u - u_0](s)|
\]
\[
= \left| \frac{1}{\Gamma(\alpha_m - \alpha_i - 1)} \int_0^t (t - s)^{\alpha_m - \alpha_i - 2} |u - u_0|(s) ds \right|
\]
\[
\leq \frac{1}{\Gamma(\alpha_m - \alpha_i - 1)} \int_0^t (t - s)^{\alpha_m - \alpha_i - 2} |u(s) - u_0(s)| ds
\]
\[
\leq \|u - u_0\| \frac{1}{\Gamma(\alpha_m - \alpha_i)} b^{\alpha_m - \alpha_i - 1}
\]
\[
\leq \hat{B} \hat{K} \|u - u_0\| \to 0, \quad \text{as} \quad u \to u_0,
\]
and we also know that
\[
|p_i(s, x(s)) - p_i(s, x_0(s))| \leq |x(s) - x_0(s)|
\]
\[
= \|x - x_0\| \to 0, \quad \text{as} \quad x \to x_0.
\]
Hence, expression (24) tends to 0 uniformly for \(s \in [0, b]\). Indeed, given \(\varepsilon > 0\), there exists \(\tilde{\delta} > 0\) such that \(\|u - u_0\| \leq \tilde{\delta}\) implies \(|\sigma_u(s) - \sigma_{\bar{u}_0}(s)| < \varepsilon\) uniformly in \(s\). Therefore, we have
\[
||T_c u|| (t) - [T_c u_0](t)| \leq \varepsilon \left[ b + |\lambda| \int_0^t |R(t, \tau, \lambda)| d\tau \right] \to 0, \quad \text{as} \quad u \to u_0.
\]
Then \(T_c\) is continuous.

Now, let \(B\) be bounded in \(C([0, b])\), then there exists a positive constant \(M > 0\) such that \(\|u\| \leq M\), for all \(u \in B\). Recall that \(\{\sigma_u : u \in B\}\) is bounded since \(f\) is continuous on \([0, b] \times [\gamma_0, \eta_0] \times \prod_{i=1}^{m-1} [\gamma_i, \eta_i]\) and, then, \(f\) is bounded on that set. Then, there exists \(\theta > 0\) such that \(\|\sigma_u\| \leq \theta\) for all \(u \in B\). Hence, for \(u \in B\), we have
\[
||T_c u||(t) \leq \theta b + |c_1| + \frac{\theta}{2} b^2 + |c| b := E, \quad \text{where} \quad \mathbf{R} \text{ is a bound for the resolvent.}
\]
Then \(T_c(B)\) is bounded. In fact, we can prove that \(T_c(C[0, b])\) is bounded.
Now, let us prove that $T_c$ is equicontinuous. For any $t_1, t_2 \in [0, b]$ with $t_2 < t_1$, we have

$$
|T_c u(t_1) - T_c u(t_2)| = \left| \int_{t_2}^{t_1} \sigma_u(s) \, ds + \lambda \int_0^{t_1} R(t_1, \tau, \lambda) \left[ \int_0^\tau \sigma_u(q) \, dq + c_1 \right] \, d\tau 
- \lambda \int_0^{t_2} R(t_2, \tau, \lambda) \left[ \int_0^\tau \sigma_u(q) \, dq + c_1 \right] \, d\tau \right|
\leq \left| \int_{t_2}^{t_1} \sigma_u(s) \, ds \right| + |\lambda| \left| \int_{t_2}^{t_1} R(t_1, \tau, \lambda) \left[ \int_0^\tau \sigma_u(q) \, dq + c_1 \right] \, d\tau \right|
+ |\lambda| \left| \int_0^{t_2} \left[ R(t_1, \tau, \lambda) - R(t_2, \tau, \lambda) \right] \left[ \int_0^\tau \sigma_u(q) \, dq + c_1 \right] \, d\tau \right|.
$$

Note that $\left| \int_{t_2}^{t_1} \sigma_u(s) \, ds \right| \to 0$ and $\left| \int_{t_2}^{t_1} R(t_1, \tau, \lambda) \left[ \int_0^\tau \sigma_u(q) \, dq + c_1 \right] \, d\tau \right| \to 0$ as $|t_2 - t_1| \to 0$, and

$$
\left| \int_0^{t_2} [R(t_1, \tau, \lambda) - R(t_2, \tau, \lambda)] d\tau \right| \leq \int_0^{t_2} |R(t_1, \tau, \lambda) - R(t_2, \tau, \lambda)| d\tau
\leq \int_0^{t_2} |K(t_1, \tau) - K(t_2, \tau)| d\tau + \int_0^{t_2} \sum_{n=1}^\infty \lambda^n |K_n(t_1, \tau) - K_n(t_2, \tau)| d\tau.
$$

Next,

$$
\int_0^{t_2} |K(t_1, \tau) - K(t_2, \tau)| d\tau
\leq \sum_{i=0}^{m-1} \frac{|A_i|}{\Gamma(\beta_i + 2)} \int_0^{t_2} [(t_1 - \tau)^\beta_i - (t_2 - \tau)^\beta_i] \, d\tau
= \sum_{i=0}^{m-1} \frac{|A_i|}{\Gamma(\beta_i + 2)} \left[ -(t_1 - t_2)^{\beta_i + 1} + t_1^{\beta_i + 1} - t_2^{\beta_i + 1} \right],
$$

and, by Fatou’s lemma,

$$
\int_0^{t_2} \sum_{n=1}^\infty \lambda^n |K_n(t_1, \tau) - K_n(t_2, \tau)| \, d\tau
\leq \int_0^{t_2} \sum_{n=1}^\infty |\lambda|^n \left| \sum_{i=1}^{m-1} \sum_{i_n=0}^{m-1} \frac{A_{i_1} \times \cdots \times A_{i_n}}{\Gamma(\beta_{i_1} + \cdots + \beta_{i_n} + n)} \right| \left[ (t_1 - \tau)^{\beta_{i_1} + \cdots + \beta_{i_n} + n - 1} - (t_2 - \tau)^{\beta_{i_1} + \cdots + \beta_{i_n} + n - 1} \right] \, d\tau
\leq \sum_{n=1}^\infty |\lambda|^n \left| \sum_{i=1}^{m-1} \sum_{i_n=0}^{m-1} \frac{A_{i_1} \times \cdots \times A_{i_n}}{\Gamma(\beta_{i_1} + \cdots + \beta_{i_n} + n)} \right| \left[ (t_1 - t_2)^{\beta_{i_1} + \cdots + \beta_{i_n} + n - 1} - (t_2 - t_2)^{\beta_{i_1} + \cdots + \beta_{i_n} + n - 1} \right] \, d\tau
= \sum_{n=1}^\infty |\lambda|^n \left| \sum_{i=1}^{m-1} \sum_{i_n=0}^{m-1} \frac{A_{i_1} \times \cdots \times A_{i_n}}{\Gamma(\beta_{i_1} + \cdots + \beta_{i_n} + n)} \right| \left[ \frac{-(t_1 - t_2)^{\beta_{i_1} + \cdots + \beta_{i_n} + n}}{\beta_{i_1} + \cdots + \beta_{i_n} + n} + \frac{t_1^{\beta_{i_1} + \cdots + \beta_{i_n} + n}}{\beta_{i_1} + \cdots + \beta_{i_n} + n} - \frac{t_2^{\beta_{i_1} + \cdots + \beta_{i_n} + n}}{\beta_{i_1} + \cdots + \beta_{i_n} + n} \right].
$$
Theorem 5.5. \[
= \sum_{n=1}^{\infty} \lambda^n \sum_{i_1=0}^{m-1} \cdots \sum_{i_n=0}^{m-1} \frac{A_{i_1} \times \cdots \times A_{i_n}}{\Gamma(\beta_{i_1} + \cdots + \beta_{i_n} + n + 1)}
\times \left[ t_1^{\beta_{i_1} + \cdots + \beta_{i_n} + n} - t_2^{\beta_{i_1} + \cdots + \beta_{i_n} + n} - (t_1 - t_2)^{\beta_{i_1} + \cdots + \beta_{i_n} + n} \right],
\]
and both expressions tend to 0 when \(|t_2 - t_1|\) tends to 0. Then, we have that the operator \(T\) admits a fixed point \(u \in C([0, b])\).

Proof. Using the previous Lemma, we have that the operator \(T\) is completely continuous. In addition, if \(u = \delta T_c u\), for \(\delta \in (0, 1)\), then
\[
\|u\| = \|\delta T_c u\| = |\delta| \sup_{t \in [0, b]} |[T_c u(t)]| \leq |\delta| E < E.
\]
And so, \(T_c(B)\) is equicontinuous. Consequently, by Ascoli’s theorem, we conclude that \(T_c(B)\) is relatively compact in \(C([0, b])\). Then, \(T_c\) is completely continuous. \(\square\)

Lemma 5.3. The operator \(T_c\) admits a fixed point \(u \in C([0, b])\).

Proof. Suppose also that \(y(\tau, \lambda) \leq w(\tau, \lambda)\), for all \(\tau, \lambda \in [0, b]\), and the fact that \(T_c(B)\) is relatively compact in \(C([0, b])\). Then, \(T_c\) is completely continuous. \(\square\)

Lemma 5.4. If \(u\) is a solution to the integral equation (25) for some constant \(c \in \mathbb{R}\), then \(y\) is a solution to the truncated problem (22)-(21).

Proof. The proof is immediate from the definition of the operator \(T_c\) and the fact that \(y(t, c_1) = T^{m-1} u(t, c_1)\).

Theorem 5.5. Assume that \(\alpha_m - \alpha_i \in \mathbb{N}, \forall i = 1, \ldots, m - 1\). Suppose also that \(v, w \in C([0, b])\) are, respectively, lower and upper solutions to (20)-(21) such that \(D^{\alpha_j} v, D^{\alpha_j} w \in \mathcal{T}_0([0, b])\) for all \(i = 0, \ldots, m\), and \(v \leq w\), \(D^{\alpha_i} v \leq D^{\alpha_i} w\), for \(i = 1, \ldots, m - 1\). Moreover, assume that there exist constants \(\lambda > 0, A_i > 0\), for \(i = 0, \ldots, m - 1\), such that
\[
f(t, v(t), \ldots, D^{\alpha_{m-1}} v(t)) - \lambda \sum_{i=0}^{m-1} A_i D^{\alpha_i} v(t)
\leq f(t, z(t), \ldots, D^{\alpha_{m-1}} z(t)) - \lambda \sum_{i=0}^{m-1} A_i D^{\alpha_i} z(t),
\]
and

\[
f(t, w(t), \ldots, D^{\alpha_{m-1}}w(t)) - \lambda \sum_{i=0}^{m-1} A_i D^{\alpha_i}w(t) \geq f(t, z(t), \ldots, D^{\alpha_{m-1}}z(t)) - \lambda \sum_{i=0}^{m-1} A_i D^{\alpha_i}z(t), \quad (27)
\]

for \( t \in [0, b] \), and function \( z \) with \( v \leq z \leq w \), and \( D^{\alpha_j}v \leq D^{\alpha_j}z \leq D^{\alpha_j}w \), for every \( i = 1, \ldots, m-1 \). Then there exists a solution to the problem (20)-(21) in the functional interval \([v, w]\).

**Proof.** Consider the solution \( y \) obtained for problem (22)-(21), and prove that \( v \leq y \leq w \), and \( D^{\alpha_j}v \leq D^{\alpha_j}y \leq D^{\alpha_j}w \) for all \( j = 1, \ldots, m-1 \). Let us prove that \( y - v \geq 0 \), that is, \( y \geq v \), so consider the following problem

\[
\begin{cases}
D^{\alpha_m}(y - v)(t) - \lambda \sum_{i=0}^{m-1} A_i D^{\alpha_i}(y - v)(t) = g(t), \\
[D^{\alpha_m-1}(y - v)]^{(i)}(0^+, c_1) = 0, \text{ for } i = 0, \ldots, [\alpha_m] - 1, \\
[D^{\alpha_m-1}(y - v)]^{(\alpha_m)}(0^+, c_1) \geq c,
\end{cases}
\quad (28)
\]

where

\[
g(t) = f(t, p(t, y(t)), \ldots, p_{m-1}(t, D^{\alpha_{m-1}}y(t))) - \lambda \sum_{i=0}^{m-1} A_i p_i(t, D^{\alpha_i}y(t)) - D^{\alpha_m}v(t) + \lambda \sum_{i=0}^{m-1} A_i D^{\alpha_i}v(t),
\]

which is nonnegative by (26). In order to apply Theorem 3.2, let us verify that the function \( g(t) \) is in \( I_0([0, b]) \).

In fact, recall that the function \( f \) is continuous on \([0, b] \times \mathbb{R}^m \). Analogously, the truncations \( p_i \) are continuous for all \( i = 0, \ldots, m-1 \). Using also the fact that \( D^{\alpha_i}v \in I_0([0, b]) \) for all \( i = 0, \ldots, m \), we conclude that \( g(t) \) is in \( I_0([0, b]) \). Therefore, by the same manner as in Section 2, problem (28) is reduced to a Volterra’s integral equation. Hence, by Theorem 3.5, we can affirm that \( y - v \geq 0 \), that is, \( y \geq v \).

On the other hand, consider the following problem

\[
\begin{cases}
D^{\alpha_m}(w - y)(t) - \lambda \sum_{i=0}^{m-1} A_i D^{\alpha_i}(w - y)(t) = h(t), \\
[D^{\alpha_m-1}(w - y)]^{(i)}(0^+, c_1) = 0, \text{ for } i = 0, \ldots, [\alpha_m] - 1, \\
[D^{\alpha_m-1}(w - y)]^{(\alpha_m)}(0^+, c_1) \geq c,
\end{cases}
\quad (29)
\]

where

\[
h(t) = D^{\alpha_m}w(t) - \lambda \sum_{i=0}^{m-1} A_i D^{\alpha_i}w(t) + \lambda \sum_{i=0}^{m-1} A_i p_i(t, D^{\alpha_i}y(t)) - f(t, p(t, y(t)), \ldots, p_{m-1}(t, D^{\alpha_{m-1}}y(t))),
\]

which is nonnegative by (27). Using the fact that \( D^{\alpha_i}w \) are in \( I_0([0, b]) \), for all \( i = 0, \ldots, m \), and \( p_i \) are continuous, for all \( i = 0, \ldots, m \), by Theorem 3.5, we get \( w \geq y \) on \([0, b]\).
Now, let us prove that $D^{\alpha_j}y - D^{\alpha_j}v \geq 0$ and $D^{\alpha_j}w - D^{\alpha_j}y \geq 0$ for all $j = 1, \ldots, m - 1$. Using relation (2), we obtain, since $\alpha_m - \alpha_1 \in \mathbb{N}$,

$$D^{\alpha_m-\alpha_1}[D^{\alpha_1}y - D^{\alpha_1}v](t) - \lambda \sum_{j=1}^{m-1} A_j D^{\alpha_j-\alpha_1}[D^{\alpha_1}y - D^{\alpha_1}v](t)$$

$$= D^{\alpha_m}y(t) - D^{\alpha_m}v(t) - \lambda \sum_{j=1}^{m-1} A_j D^{\alpha_j}y(t) + \lambda \sum_{j=1}^{m-1} A_j D^{\alpha_j}v(t)$$

$$= D^{\alpha_m}y(t) - \lambda \sum_{j=0}^{m-1} A_j D^{\alpha_j}y(t) + \lambda \sum_{j=1}^{m-1} A_j D^{\alpha_j}v(t)$$

$$= f(t, y(t), \ldots, p_{m-1}(t, D^{\alpha_{m-1}}y(t))) - \lambda \sum_{j=0}^{m-1} A_j p_j(t, D^{\alpha_j}y(t)) + \lambda \sum_{j=1}^{m-1} A_j D^{\alpha_j}v(t).$$

Now let

$$k(t) = f(t, p(t, y(t)), \ldots, p_{m-1}(t, D^{\alpha_{m-1}}y(t))) - \lambda \sum_{j=0}^{m-1} A_j p_j(t, D^{\alpha_j}y(t)) + \lambda \sum_{j=1}^{m-1} A_j D^{\alpha_j}v(t),$$

which is nonnegative by condition (26) and the inequality

$$\lambda \sum_{j=0}^{i-1} A_j D^{\alpha_j}y(t) \geq \lambda \sum_{j=0}^{i-1} A_j D^{\alpha_j}v(t)$$

(for $i = 1$, it was already proved that $y \geq v$, and when proving the inequality for $i$, we can assume it true by recurrence for $j = 0, \ldots, i - 1$).

Consider now the following fractional problem

$$\begin{cases}
D^{\alpha_m-\alpha_1}[D^{\alpha_1}y - D^{\alpha_1}v](t) - \lambda \sum_{j=0}^{m-1} A_j D^{\alpha_j-\alpha_1}[D^{\alpha_1}y - D^{\alpha_1}v](t) = k(t), \\
[D^{\alpha_m+\alpha_1-1}(y - v)]^{(i)}(0^+, c_1) = 0, \text{ for } i = 0, \ldots, [\alpha_m - \alpha_1] - 1, \\
[D^{\alpha_m+\alpha_1-1}(y - v)]^{([\alpha_m - \alpha_1])}(0^+, c_1) = 0.
\end{cases} \quad (30)$$

As we have previously proved, we apply Theorem 3.5, and then we obtain that $D^{\alpha_j}y \geq D^{\alpha_j}v$ for all $j = 1, \ldots, m - 1$.

In the same manner, we obtain that $D^{\alpha_j}w \geq D^{\alpha_j}y$ for all $j = 1, \ldots, m - 1$. This completes the proof.

6. Monotone method. Recall the nonlinear problem

$$D^{\alpha_m}y(t) = f(t, y(t), D^{\alpha_1}y(t), \ldots, D^{\alpha_{m-1}}y(t)) \quad (31)$$
subject to the initial conditions

\[
\begin{cases}
[D^{\alpha_m-1}y]^{(i)}(0^+, c_1) = 0, & i = 0, \ldots, [\alpha_m] - 1, \\
[D^{\alpha_m-1}y]^{(\lfloor \alpha_m \rfloor)}(0^+, c_1) = c.
\end{cases}
\] (32)

Next, consider the following problem

\[
D^\alpha y(t) - \lambda \sum_{i=0}^{m-1} A_i D^{\alpha_i} y(t) = f(t, y(t), D^{\alpha_1} y(t), \ldots, D^{\alpha_{m-1}} y(t)) - \lambda \sum_{i=0}^{m-1} A_i D^{\alpha_i} y(t),
\] (33)

for suitable constants \(\lambda\) and \(A_i, i = 0, \ldots, m - 1\).

**Theorem 6.1.** Assume that \(\alpha_m - \alpha_i \in \mathbb{N}, \forall i = 1, \ldots, m - 1\). Suppose that there exist \(v, w\), respectively, lower and upper solutions for problem (31)-(32) such that \(D^\alpha v \leq D^\alpha w\) are in \(I_\alpha([0, b])\) and \(v \leq w, D^\alpha v \leq D^\alpha w, \text{ for } i = 0, \ldots, m - 1\). Suppose also that there exist constants \(\lambda > 0, A_i > 0, \text{ for } i = 0, \ldots, m - 1\), such that

\[
f(t, x(t), \ldots, D^{\alpha_{m-1}} x(t)) - \lambda \sum_{i=0}^{m-1} A_i D^{\alpha_i} x(t) \leq f(t, z(t), \ldots, D^{\alpha_{m-1}} z(t)) - \lambda \sum_{i=0}^{m-1} A_i D^{\alpha_i} z(t),
\] (34)

for \(t \in [0, b]\), and functions \(x, z\) with \(v \leq x \leq z \leq w\), and \(D^\alpha v \leq D^\alpha x \leq D^\alpha z \leq D^\alpha w\), for every \(i = 1, \ldots, m - 1\).

Then, there exist monotone sequences \(\{v_n\}, \{w_n\}\) such that \(v_0 := v, w_0 := w\), and converging uniformly to the extremal solutions of (31)-(32) in

\[
[v_0, w_0] := \left\{ x \in C([0, b]) : v_0 \leq x \leq w_0, D^\alpha v \leq D^\alpha x \leq D^\alpha w, i = 0, \ldots, m - 1 \right\}.
\]

**Proof.** Let \(\eta \in [v_0, w_0]\), and consider the following problem

\[
\begin{cases}
D^\alpha y(t) - \lambda \sum_{i=0}^{m-1} A_i D^{\alpha_i} y(t) = \sigma_\eta(t), \\
[D^{\alpha_{m-1}} y]^{(i)}(0^+, c_1) = 0, & i = 0, \ldots, [\alpha_m] - 1, \\
[D^{\alpha_m-1} y]^{(\lfloor \alpha_m \rfloor)}(0^+, c_1) = c,
\end{cases}
\] (35)

where

\[
\sigma_\eta(t) := f(t, \eta(t), D^{\alpha_1} \eta(t), \ldots, D^{\alpha_{m-1}} \eta(t)) - \lambda \sum_{i=0}^{m-1} A_i D^{\alpha_i} \eta(t).
\]

Since \(v_0, w_0\) are, respectively, upper and lower solutions to the initial value problem (31)-(32) and using condition (34), we have that \(w_0, v_0\) are, respectively, upper and lower solutions to the problem (35). Indeed,

\[
D^\alpha w_0(t) - \lambda \sum_{i=0}^{m-1} A_i D^{\alpha_i} w_0(t) = f(t, w_0(t), \ldots, D^{\alpha_{m-1}} w_0(t)) - \lambda \sum_{i=0}^{m-1} A_i D^{\alpha_i} w_0(t),
\]

\[
\leq f(t, v_0(t), \ldots, D^{\alpha_{m-1}} v_0(t)) - \lambda \sum_{i=0}^{m-1} A_i D^{\alpha_i} v_0(t) \leq f(t, v(t), \ldots, D^{\alpha_{m-1}} v(t)) - \lambda \sum_{i=0}^{m-1} A_i D^{\alpha_i} v(t).
\]

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On the other hand, we have

\[
D^{\alpha_m}w_0(t) - \lambda \sum_{i=0}^{m-1} A_i D^{\alpha_i}w_0(t) \geq f(t, w_0(t), \ldots, D^{\alpha_m-1}w_0(t)) - \lambda \sum_{i=0}^{m-1} A_i D^{\alpha_i}w_0(t)
\]

\[
\geq f(t, \eta(t), \ldots, D^{\alpha_m-1}\eta(t)) - \lambda \sum_{i=0}^{m-1} A_i D^{\alpha_i}\eta(t).
\]

Then, using Theorem 5.5, we can affirm that problem (35) has a solution \(y_\eta\) which is the unique solution obtained by passing to Volterra’s equation mentioned in Section 2.

Now, consider the mapping \(A\) such that

\[
A : [v_0, w_0] \to C([0, b]),
\]

\[
\eta \to A\eta,
\]

given by \([A\eta](t) = I^{\alpha_m-1}u_\eta(t)\), where \(u_\eta(t) = F(t, c_1) + \lambda \int_0^t R(t, \tau, \lambda)F(\tau)d\tau,\)

\[F(t) = \int_0^t g_\eta(s)ds + c_1,\]

\[g_\eta(t) = f(t, \eta(t), \ldots, D^{\alpha_m-1}\eta(t)) - \lambda \sum_{i=0}^{m-1} A_i D^{\alpha_i}\eta(t) - D^{\alpha_m}v_0(t) + \lambda \sum_{i=0}^{m-1} A_i D^{\alpha_i}v_0(t).\]

Now, let us prove that:

(i) \(v_0 \leq Av_0\) and \(v_0 \geq Aw_0\).

(ii) \(D^{\alpha_i}v_0 \leq D^{\alpha_i}Av_0\) and \(D^{\alpha_i}w_0 \geq D^{\alpha_i}Aw_0\), \(\forall i = 1, \ldots, m - 1\).

(iii) \(A[v_0, w_0] \subseteq [v_0, w_0]\).

(iv) \(A\) is a monotone operator on \([v_0, w_0]\), in the sense that, for \(\eta_1, \eta_2 \in [v_0, w_0]\) with \(\eta_1 \leq \eta_2\), and \(D^{\alpha_i}\eta_1 \leq D^{\alpha_i}\eta_2\), for every \(i = 1, \ldots, m - 1\), we have \(A\eta_1 \leq A\eta_2\) and \(D^{\alpha_i}A\eta_1 \leq D^{\alpha_i}A\eta_2\), for every \(i = 1, \ldots, m - 1\).

To prove (i), we set \(v_1 = Av_0\), where \(v_1\) is the solution to (35) with \(\eta = v_0\).

Now, consider the following problem

\[
\begin{align*}
D^{\alpha_m}(v_1 - v_0)(t) & - \lambda \sum_{i=0}^{m-1} A_i D^{\alpha_i}(v_1 - v_0)(t) = g_{v_0}(t), \\
[D^{\alpha_m-1}(v_1 - v_0)]^{[i]}(0^+, c_1) & = 0, \text{ for } i = 0, \ldots, [\alpha_m] - 1, \\
[D^{\alpha_m-1}(v_1 - v_0)]^{[\alpha_m]}(0^+, c_1) & \geq c,
\end{align*}
\]

(36)

where

\[
g_{v_0}(t) = f(t, v_0(t), \ldots, D^{\alpha_m-1}v_0(t))
\]

\[
- \lambda \sum_{i=0}^{m-1} A_i D^{\alpha_i}v_0(t) - D^{\alpha_m}v_0(t) + \lambda \sum_{i=0}^{m-1} A_i D^{\alpha_i}v_0(t).
\]

Then, using Theorem 3.5, we have \(v_0 \leq v_1\). In the same manner, we get \(w_0 \geq w_1\), then (i) holds.

Concerning the property (ii) on the inequalities on the fractional derivatives, since \(\alpha_m - \alpha_i \in \mathbb{N}\), for \(i = 1, \ldots, m - 1\), we have

\[
D^{\alpha_m-\alpha_i}[D^{\alpha_i}Av_0 - D^{\alpha_i}v_0(t)] = \lambda \sum_{j=0}^{m-1} A_j D^{\alpha_j-\alpha_i}[D^{\alpha_i}Av_0 - D^{\alpha_i}v_0(t)]
\]

for \(i = 1, \ldots, m - 1\).
\[
D^{\alpha_i}(A v_0) (t) - D^{\alpha_i} v_0 (t) - \lambda \sum_{j=0}^{m-1} A_j D^{\alpha_j} (A v_0) (t) + \lambda \sum_{j=0}^{m-1} A_j D^{\alpha_j} v_0 (t)
= f(t, v_0(t), \ldots, D^{\alpha_i-1} v_0 (t))
- \lambda \sum_{j=0}^{m-1} A_j D^{\alpha_j} v_0 (t) - D^{\alpha_i} v_0 (t) + \lambda \sum_{j=0}^{m-1} A_j D^{\alpha_j} v_0 (t) \geq 0.
\]

For the initial conditions, we get
\[
D^{\alpha_i-1} [D^{\alpha_i} A v_0 - D^{\alpha_i} v_0] (0^+, c_1) = 0,
\]
and
\[
D^{\alpha_i-1} [D^{\alpha_i} A v_0 - D^{\alpha_i} v_0] (0^+, c_1) = 0.
\]

Therefore, we get \(D^{\alpha_i} A v_0 \geq D^{\alpha_i} v_0\). Similarly, we obtain that \(D^{\alpha_i} A w_0 \leq D^{\alpha_i} w\), for \(i = 1, \ldots, m - 1\).

To prove (iii), we take \(y \in [v_0, w_0]\), and check that \(A y \in [v_0, w_0]\). Indeed, the inequality \(v_0 \leq A y \leq w_0\) can be obtained similarly to (i) by replacing \(v_1\) (and \(w_1\)) by \(A y\). Similarly, we can replace \(A v_0\) (and \(A w_0\)) by \(A y\) in (ii), and we conclude. Both procedures include the application of condition (34).

Now, let us prove (iv), so let \(\eta_1, \eta_2 \in [v_0, w_0]\) be such that \(\eta_1 \leq \eta_2\), and \(D^{\alpha_i} \eta_i \leq D^{\alpha_i} \eta_2\), for every \(i = 1, \ldots, m - 1\). We denote \(y_1 := A \eta_1\) and \(y_2 := A \eta_2\). Now, let
\[
D^{\alpha_i}(y_2 - y_1) (t) - \lambda \sum_{i=0}^{m-1} A_i D^{\alpha_i} (y_2 - y_1) (t) = h(t),
\]
where
\[
h(t) := f(t, \eta_2(t), \ldots, D^{\alpha_i-1} \eta_2 (t))
- f(t, \eta_1(t), \ldots, D^{\alpha_i-1} \eta_1 (t)) - \lambda \sum_{i=0}^{m-1} A_i D^{\alpha_i} (\eta_2 - \eta_1) (t),
\]
subject to
\[
\begin{align*}
\left( D^{\alpha_i-1} (v_1 - v_0) \right) (0^+, c_1) &= 0, \text{ for } i = 0, \ldots, [\alpha_m] - 1, \\
\left( D^{\alpha_i-1} (v_1 - v_0) \right) (0^+, c_1) &\geq 0.
\end{align*}
\]

Using the hypotheses, we get that \(h(t) \in \mathcal{I}_0 ([0, b])\), so hypothesis (34) and Theorem 3.5 ensures that \(y_2 \geq y_1\). The procedure shown in (ii) allows to deduce the corresponding inequalities for the fractional derivatives. Therefore, the operator \(A\) is nondecreasing.

Define now the sequences \(v_n = A w_{n-1}\), \(w_n = A w_{n-1}\), \(n \in \mathbb{N}\), and we have
\[
v = v_0 \leq v_1 \leq \cdots \leq v_n \leq w_n \leq \cdots \leq w_1 \leq w_0 = w \text{ on } [0, b].
\]

Now, let us prove that the operator \(A\) is continuous and relatively compact. In fact, firstly, for \(\eta, \xi\) in \([v_0, w_0]\), we define
\[
d(\eta, \xi) := \max \left\{ \sup_{t \in [0, b]} |\eta(t) - \xi(t)|, \sup_{t \in [0, b]} |D^{\alpha_i} \eta(t) - D^{\alpha_i} \xi(t)|, i = 1, \ldots, m - 1 \right\}.
\]

So, the type of continuity we derive is very restrictive, in the sense that the operator is continuous assuming as codomain a space where functions have more regularity than simply the elements in \(C([0, b])\). So, let us prove that the operator
\( A \) is continuous at a fixed \( \eta_0 \). Take \( \varepsilon > 0 \) fixed and choose \( \beta > 0 \) such that 
\[
\beta < \frac{\varepsilon}{2 \lambda \text{max}\{A_i\}}.
\]
Define the set
\[
B = \{(t, x, x_1, \ldots, x_{m-1}) \in \mathbb{R}^{m+1}, \text{ such that } |x - \eta_0(t)| \leq \beta, \quad |x_1 - D^{\alpha_1} \eta_0(t)| \leq \beta, \quad \ldots, \quad |x_{m-1} - D^{\alpha_{m-1}} \eta_0(t)| \leq \beta, \quad t \in [0, b]\}
\]
which is compact. Moreover, \( f \) is uniformly continuous on \( B \).

Given \( \varepsilon > 0 \), \( \exists \delta > 0 \) such that \( |(t, x, x_1, \ldots, x_{m-1}) - (t, y, y_1, \ldots, y_{m-1})| < \delta \), we have \( |f(t, x, x_1, \ldots, x_{m-1}) - f(t, y, y_1, \ldots, y_{m-1})| < \frac{\varepsilon}{\delta} \).

If we choose \( \eta \) with \( d(\eta, \eta_0) < \tilde{\delta} \), then, for all \( i = 1, \ldots, m - 1 \), and \( t \in [0, b] \), we have \( |\eta(t) - \eta_0(t)| < \tilde{\delta}, \ldots, |D^{\alpha_i} \eta(t) - D^{\alpha_i} \eta_0(t)| < \tilde{\delta} \), so, for \( \delta = \text{min}\{\tilde{\delta}, \beta\} \), we deduce that \( (t, \eta(t), \ldots, D^{\alpha_{m-1}} \eta(t)) \in B, (t, \eta_0(t), \ldots, D^{\alpha_{m-1}} \eta_0(t)) \in B \), so that, using the expression of the function \( g_\eta \), we get \( \sup_{t \in [0, b]} |g_\eta(t) - g_{\eta_0}(t)| < \varepsilon \). Moreover, by definition, we have
\[
d((A\eta)(t), (A\eta_0)(t)) = \max \left\{ \sup_{t \in [0, b]} |I^{\alpha_{m-1}} u_\eta(t) - I^{\alpha_{m-1}} u_{\eta_0}(t)|, \sup_{t \in [0, b]} |I^{\alpha_m - \alpha_{i-1}} u_\eta(t) - I^{\alpha_m - \alpha_{i-1}} u_{\eta_0}(t)| \right\}.
\]

It follows that
\[
|I^{\alpha_{m-1}} u_\eta(t) - I^{\alpha_{m-1}} u_{\eta_0}(t)| \leq \frac{b^{\alpha_{m-1}}(1 + b\lambda R)}{\alpha_m \Gamma(\alpha_m - 1)}, \quad \forall i = 1, \ldots, m - 1.
\]

Here, \( R \) is a bound of the resolvent \( R \).

In the same manner, we obtain
\[
|I^{\alpha_m - \alpha_i} u_\eta(t) - I^{\alpha_m - \alpha_i} u_{\eta_0}(t)| \leq \varepsilon \frac{b^{\alpha_{m-1}}(1 + b\lambda R)}{(\alpha_m - 1) \Gamma(\alpha_m - \alpha_i - 2)}, \quad \forall i = 1, \ldots, m - 1.
\]

Therefore, the continuity of \( A \) is guaranteed.

Now, let us prove that \( A \) is relatively compact. Indeed, let \( \tilde{B} \subset [v_0, w_0] \) be a bounded set. Consider \( A(\tilde{B}) \), we have to prove that \( A(\tilde{B}) \) is relatively compact. So let \( \{A(y_n)\} \) be a sequence in \( A(\tilde{B}) \), we have to prove that \( \{A(y_n)\} \) has a convergent subsequence. Following the same procedure as in Section 3, there exists a sequence \( \{u_n\} \) such that \( A(y_n) = I^{\alpha_{m-1}} u_n \), where
\[
u_n(t, c_1) = \int_0^t g_{y_n}(s) ds + c_1 + \lambda \int_0^t R(t, \tau, \lambda) \left( \int_0^s g_{y_n}(q) dq + c_1 \right),
\]
and
\[
g_{y_n}(s) = f(s, y_n(s), D^{\alpha_1} y_n(s), \ldots, D^{\alpha_{m-1}} y_n(s)) - \lambda \sum_{i=0}^{m-1} A_i D^{\alpha_i} y_n(s).
\]

We have that \( \{y_n\} \) is bounded, then there exists \( \tilde{K} > 0 \) such that
\[
d(y_n, 0) = \max \left\{ \sup_{t \in [0, b]} |D^{\alpha_i} y_n(t)| \right\} \leq \tilde{K}, \quad \forall n \in \mathbb{N}.
\]
The set $[0, b] \times [-\bar{K}, \bar{K}]^m$ is compact and $f$ is bounded on $[0, b] \times [-\bar{K}, \bar{K}]^m$, so we obtain that $\{g_{n}\}$ is uniformly bounded on $[0, b]$. Then, we get that $u_n$ is totally bounded with $u_n \in C([0, b])$. Now, let us verify that $\{u_n\}$ is equicontinuous. So, let $t, s \in [0, b]$ with $s < t$, then we have

$$
|u_n(t) - u_n(s)| = \left| \int_s^t g_{n}(q)dq + \lambda \int_0^t R(t, \tau, \lambda) \int_0^\tau g_{n}(q)dq - \lambda \int_0^s R(t, \tau, \lambda) \int_0^\tau g_{n}(q)dqdt \right| .
$$

As we showed in the proof of Lemma 5.2, we obtain $|u_n(t) - u_n(s)| \to 0$, as $|s - t| \to 0$. Then $\{u_n\}$ is uniformly equicontinuous. Therefore, there exists a subsequence $\{u_{n_k}\} \to u$ uniformly, with $u \in C([0, b])$. Moreover, we have $\mathcal{A}y_{n_k} = I^{\alpha-1}u_k$, which is uniformly convergent to $I^{\alpha-1}u$, as $k \to +\infty$. In addition, we obtain that $D^{\alpha_i}\mathcal{A}y_{n_k} = I^{\alpha-\alpha_i-1}u_k \to I^{\alpha-\alpha_i-1}u$ uniformly as $k \to +\infty$. In conclusion, we get that $\mathcal{A}y_{n_k}$ converges to $I^{\alpha-1}u$ in the space $[v_0, w_0]$. So, $\mathcal{A}$ is completely continuous.

As $\{v_n\}, \{w_n\} \subset \mathcal{A}([v_0, w_0])$, we get that $\{v_n\}$ and $\{w_n\}$ are relatively compact sets and we know that they are monotone sequences. In view of Lemma 2.4, since the distance is defined from a norm which provides completeness, we can obtain that there exist $v^*$ and $w^*$ in $C([0, b])$ such that

$$
\lim_{n \to \infty} v_n(t) = v^*(t), \quad \lim_{n \to \infty} w_n(t) = w^*(t).
$$

By the continuity of $\mathcal{A}$, we have that $v^* = \mathcal{A}v^*, w^* = \mathcal{A}w^*$. So $v^*, w^*$ are fixed points of $\mathcal{A}$. It is clear that $y$ is a solution to problem (31)-(32) if and only if $y$ is a fixed point of $\mathcal{A}$. Hence, $v^*, w^*$ are solutions to problem (31)-(32) between $v$ and $w$.

We prove that $v^*, w^*$ are, respectively, the minimal and the maximal solutions to the boundary value problem between $v_0$ and $w_0$. Assume that $y \in [v_0, w_0]$ is a solution to problem (31)-(32). We can easily obtain that $(\mathcal{A}v_0)(t) \leq (\mathcal{A}y)(t) \leq (\mathcal{A}w_0)(t)$ by the fact that $\mathcal{A}$ is monotone increasing in $[v_0, w_0]$, that is, $v_1(t) \leq y(t) \leq w_1(t)$. In fact, we have checked that analogous inequalities are valid for the fractional derivatives considered. Doing this repeatedly, we have that $v_n(t) \leq y(t) \leq w_n(t)$, for $n \geq 1$. Then, passing to the limit as $n \to \infty$, we obtain that $v^* \leq y \leq w^*$. Therefore, $v^*, w^*$ are, respectively, the minimal and the maximal solutions to the boundary value problem (31)-(32) between $v$ and $w$.

The proof is complete. □

**Remark 1.** Condition (34) can be rewritten as

$$
f(t, z(t), \ldots, D^{\alpha_{m-1}}z(t)) - f(t, x(t), \ldots, D^{\alpha_{m-1}}x(t)) \geq \lambda \sum_{i=0}^{m-1} A_i (D^{\alpha_i}z(t) - D^{\alpha_i}x(t)),
$$

for $t \in [0, b]$, and functions $x, z$ with $v \leq x \leq z \leq w$, and $D^{\alpha_i}v \leq D^{\alpha_i}x \leq D^{\alpha_i}z \leq D^{\alpha_i}w$, for every $i = 1, \ldots, m - 1$, which is trivially valid if the function

$$
f(t, z_0, \ldots, z_{m-1}) - \lambda \sum_{i=0}^{m-1} A_i z_i
$$

is nondecreasing in the variables

$$(z_0, \ldots, z_{m-1}) \in [v(t), w(t)] \times [D^{\alpha_1}v(t), D^{\alpha_1}w(t)] \times \cdots \times [D^{\alpha_{m-1}}v(t), D^{\alpha_{m-1}}w(t)],$$

for $t \in [0, b]$ fixed.
Therefore, the condition (34) could be weakened by selecting other type of appropriate constants. The role of the sign of the constants is relevant in some intermediate calculations, as well as for the study of the sign of the solutions to auxiliary linear problems. Thus, other cases in Theorems 3.5–3.7 could be used. However, due to the difficult behavior of the functions involved, the use of a different set of constants could make it quite tedious to apply the comparison results. Thus, we have considered the case where the constants $\lambda, A_i, i = 0, \ldots, m - 1$, are all positive, what makes the application of the monotone method simpler.

Analogous considerations could be made concerning conditions (26) and (27).

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