Renormalization of 2PI resummation: a renormalization scheme approach

Antal Jakovác*

Institute of Physics, Budapest University of Technology and Economics, Budafoki út 8, H-1111 Budapest, Hungary

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A practical method is suggested for performing renormalized 2PI resummation at finite temperature using specific momentum dependent renormalization schemes. In this method there is no need to solve Bethe-Salpeter equations for 2PI resummation. We examine the consistency of such schemes in the paper. The proposed method is used to perform a two-loop renormalized 2PI resummation in the finite temperature $\Phi^4$ model.

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I. INTRODUCTION

By using direct perturbation theory (ie. strict power series of some coupling constant) to calculate finite temperature or nonequilibrium correlation functions, one very frequently encounters infrared (IR) sensitive contributions, ie. contributions that become very large (occasionally infinite) when we approach a certain energy scale. These contributions, if they show up consequently in every higher loop level, render the direct perturbation theory non-convergent. Then we have to collect the problematic diagrams and sum them up, before the actual perturbative procedure begins. Since these diagrams are usually also ultraviolet (UV) divergent besides their IR sensitivity, the resummation of these diagrams must be accompanied by a proper renormalization.

In the literature there are several approaches to the renormalization of resummed perturbative series. One of them treats the resummation in the original sense, as sum of diagrams belonging to a certain subset of all Feynman diagrams. One possibility is to apply BPHZ renormalization: then, without the reference to the Lagrangian we can assign a finite (renormalized) value to every Feynman diagram [1], and we define the renormalized value of the sum of diagrams as sum of their BPHZ renormalized value. In principle this method can be applied to all kind of resummations, but technically it can be rather involved. The difficulty is to be able to work out the BPHZ forest for an arbitrary diagram in the relevant subset. This task was performed for the 2PI resummation in various models in [2]-[5] by solving a set of Bethe-Salpeter equations for the four point functions. Diagrammatic analysis can be used also to work out counterterms: such a technique was applied in [6] to renormalize the 2PPI resummation. Diagram by diagram renormalization can be applied for large N bubble resummation [7], too.

Another approach is based on the idea that we can try to modify the generating equation of the resummation in order to obtain finite results. It is not guaranteed to work, since the equation that we use is, in general, not exact, and so we may miss some terms important for UV finiteness. In the Schwinger-Dyson resummation it seems to be an important obstacle, cf. [8]. On the other hand, for Hartee-Fock resummation, such improvement exists, cf. [9]. In the O(N) model the resummation is governed by the $1/N$ order of the diagram. Here we can write up closed forms, at least in the lowest orders, and so one expects that it can be also renormalized in closed form. In [10] the authors present renormalization of the O(N) model up to $O(1/N)$ order, at the physical point.

A third approach uses the fact that the contribution of a diagram depends on the renormalization scheme we use. It may be possible to find a scheme where the IR sensitive part of all diagrams disappears. In this IR safe scheme there is no need for resummation, usual perturbation theory can be used to calculate physical observables. Thereafter we can change the renormalization scheme to a standard one (eg. $\overline{\text{MS}}$) by appropriate change of the renormalized parameters of the Lagrangian (“renormalization group” transformation). At the end of this procedure we obtain the same result in the $\overline{\text{MS}}$ and the IR safe scheme for all quantities up the order (coupling)$^n$ where $n$ is the order of calculation. The difference is the resummation effect: it is responsible for the difference that one scheme is IR safe while the other is not.

These thoughts was exploited in [11] to perform a renormalized momentum independent resummation as a change of renormalization scheme from an IR safe one to $\overline{\text{MS}}$. The IR safe scheme was constructed in a way that if in a perturbative calculation we encounter IR sensitive contributions, then we choose the finite part of the counterterm to cancel this IR sensitive part. This guarantees in a natural way that the calculation is free from IR sensitive

*Electronic address: Antal.Jakovac@cern.ch
contributions. At the end of the calculation we switch back to $\overline{\text{MS}}$ scheme with renormalization scheme transformation, thus we obtain the final result in $\overline{\text{MS}}$ scheme.

This idea is motivated by the “environmentally friendly” renormalization of Ref. [12], where the renormalization scale was used to take into account the environmental dependence.

In [13] the first step was done towards the generalization of the method to momentum dependent resummations. It was observed, that if we do perturbation theory with a propagator, which equals to the $\overline{\text{MS}}$ propagator asymptotically, then all the overall divergences – and so all the counterterms – are the same as in the $\overline{\text{MS}}$ scheme. Since IR sensitivity occurs – by definition – in the IR, we expect that it can be captured by an appropriate modification in the IR region. So if we base our perturbation theory on a propagator that is exact in the IR and $\overline{\text{MS}}$-like in the UV, we can use the $\overline{\text{MS}}$ counterterms for renormalization. A great advantage of this method is that one does not need to solve Bethe-Salpeter equations, which is a central task in other, diagram-based 2PI renormalizations. On the other hand it is not easy to implement the momentum independent resummation in this method.

The purpose of the present paper is to connect the momentum independent and momentum dependent method in a common framework. We will examine the possibility and consistency of a perturbation theory with arbitrarily chosen free Lagrangian, which not necessarily approaches the usual free part ($\frac{1}{2}(\partial \Phi)^2 - \frac{m^2}{2}\Phi^2$) asymptotically. Therefore the generated divergences are also different from the usual case. Still it can be consistent, if the usual process of the renormalization can be accomplished – the conditions for this will be examined in the paper. But once we have a consistent perturbation theory with “any” propagator, then nothing prohibits us to apply a scheme, where the renormalized self energy is zero. In this scheme the free propagator is exact – and this is exactly the effect of 2PI resummation. So we can perform renormalized 2PI resummation with help of a cleverly chosen renormalization scheme.

In this paper we accomplish this strategy for the two-loop 2PI resummation of the $\Phi^4$ theory at finite temperature. The paper will be organized as follows. In Section II we study in detail the feasibility of the renormalized perturbation theory with an arbitrary propagator, and work out the strategy for 2PI resummation. In Section III we perform renormalized 2PI resummation in the quartic scalar model at finite temperature up to two loop level. We close the discussion with Conclusions in Section IV.

II. PERTURBATION THEORY WITH NON-CONVENTIONAL PROPAGATORS

Let us start from a renormalizable Lagrangian, which now means that it is renormalizable in the non-perturbative sense: we can define regularization dependent parameters in the Lagrangian in a way that the exact $n$-point functions are all finite when we remove the regularization. We assume that we split this bare Lagrangian into two pieces: a “free” part and an “interaction part”. The free part should be quadratic in the fields in order to be solvable, but there are also quadratic pieces in the interaction Lagrangian (like the mass counterterms). Here, however, we do not fix the kernel of the quadratic part to the form appearing in the bare Lagrangian: we allow arbitrary choice of the kernel. We subdivide the interaction part, as usual, assigning “orders” to them according to a formal loop parameter. In this way we defined a perturbation theory. The question is: when is the so-defined perturbation theory finite order by order? The counterterms should be defined as the overall divergence of the 1PI diagrams, when “overall divergence” means that all the momenta at all lines are going to the infinity at the same rate. When are these counterterms enough to make finite the complete contribution of all the diagrams at a given order of perturbation theory?

For a consistent renormalization, in fact, we should ensure only two conditions [1]. First, the divergence of an overall divergent diagram should be local: in Fourier space it should be a polynomial of its external momenta. Second, if we subtract all subdivergences of a diagram, the diagram should be overall divergent or finite. The proof that these conditions are satisfied in case of using the usual free propagator was done by Weinberg [14]. But the basic property of the propagator, which was in fact used by the proof, is that the free propagator can be power expanded “around infinity”, i.e.:

$$G(p + k) = G(p) + k^\mu \partial_\mu G(p) + \frac{1}{2} k^\mu k^\nu \partial_\mu \partial_\nu G(p) + \ldots,$$

if $p^\mu \rightarrow \infty$ (which means that all of its components go to infinity with a fixed ratio). This ensures that differentiation with respect to external momenta will decrease the degree of divergence of the diagram, and so – if we differentiate enough times – the contribution will be finite. The asymptotic expansion property of eq. (1), however, is true for a wide class of functions, not just for $1/(p^2 - m^2)$. With any of these functions the Weinberg theorem goes through. Then all of the consequences will hold true, too, which can be summarized in a phrase that the theory can be renormalized by local counterterms with polynomial momentum dependence.

The Weinberg theorem in itself does not guarantee that we need a finite number of counterterms, only that the counterterm method works. We also need the condition that only those operators are divergent that show up in the
The exact propagator in four dimensions picks up only logarithmic corrections to the 1/$k^2$ behavior, according to the Weinberg theorem. Thus it may be a good candidate to be the basic propagator of a perturbation theory.

Once we have fixed the kernel, and so the propagator, too, we can use it for ordinary perturbation theory. In particular, we can do finite temperature calculations. Since temperature effects are exponentially suppressed in the propagator at high momenta, the overall divergence of a diagram (when all the momenta goes in the same rate to particular, we can do finite temperature calculations. Since temperature effects are exponentially suppressed in the propagator at high momenta, the overall divergence of a diagram (when all the momenta goes in the same rate to infinity) is temperature independent. This ensures that the counterterms are temperature independent, and so a zero temperature renormalization is enough for a finite temperature calculation, too.

According to the above analysis, a large class of functions is good to be used as “free” propagators. But which one is worth to be used? The best choice is the one which is optimized for the problem at hand, i.e., where the perturbative series converge the fastest. If we keep the bare Lagrangian intact, then any choice describes the same physics, so each is a resummed version of another. This is similar when we choose the best renormalization scale. To compute the potential, for example, the best choice is to have a scale $\mu \sim 1/r$ where $r$ is the distance between the point charges. Then large logarithmic corrections are suppressed, perturbation series converge well. To keep the bare Lagrangian intact we should tune the parameters of the perturbation theory accordingly: in particular we have to work with the $\mu$-dependent coupling $g(1/r)$. We will use the same strategy: we optimize the kernel for a certain problem, for example for computations at temperature $T_0$. To keep the bare Lagrangian intact we should work with $T_0$ dependent parameters in the perturbation theory ($g(T_0)$). Then we describe the same physics from different points of view.

Next we express this strategy in more formal terms in the $\Phi^4$ model. First we establish the generic strategy, then we examine the relation of perturbation theories with different kernel choices to each other.

### A. Generic strategy

The bare Lagrangian of the theory reads

$$\mathcal{L} = \frac{1}{2} (\partial \Phi_{bare})^2 - m_{bare}^2 \Phi_{bare}^2 - \frac{\lambda_{bare}}{24} \Phi_{bare}^4. \quad (2)$$

In the usual procedure we first change to the renormalized field: $\Phi_{bare}^2 = Z \Phi^2$, then we find

$$\mathcal{L} = \frac{1}{2} (\partial \Phi)^2 - m_R^2 \Phi^2 - \frac{\lambda}{24} \Phi^4 + \frac{\delta Z}{2} (\partial \Phi)^2, \quad (3)$$

where $\delta Z = 1 - Z$, $m_0^2 = Z m_{bare}^2$ and $\lambda_0 = Z^2 \lambda_{bare}$. Then we split $m_0^2 = m_R^2 + \delta m^2$ and $\lambda_0 = \lambda_R + \delta \lambda$ and find

$$\mathcal{L} = \frac{1}{2} (\partial \Phi)^2 - m_R^2 \Phi^2 - \frac{\lambda_R}{24} \Phi^4 + \frac{\delta Z}{2} (\partial \Phi)^2 - \frac{\delta m^2}{2} \Phi^2 - \frac{\delta \lambda}{24} \Phi^4. \quad (4)$$

The above form suggests a separation of the non-interacting and interacting theory as

$$\mathcal{L}_0 = \frac{1}{2} (\partial \Phi)^2 - m_R^2 \Phi^2, \quad \mathcal{L}_I = -\frac{\lambda_R}{24} \Phi^4 + \frac{\delta Z}{2} (\partial \Phi)^2 - \frac{\delta m^2}{2} \Phi^2 - \frac{\delta \lambda}{24} \Phi^4, \quad (5)$$

where we treat $\mathcal{L}_I$ as interactions. It must be emphasized that the values of the counterterms are not fixed beforehand: they are determined order-by-order, and they do depend on the choice of the chosen perturbative scheme. We can prove [1], however, that in the $n$-th order of perturbation theory the values of the counterterms are scheme independent up to $\mathcal{O}(\lambda^n)$.

The above described procedure is not unique. Starting from (3) we can separate the renormalized and counterterms as

$$\mathcal{L} = \frac{1}{2} \Phi K(i\partial)\Phi - \frac{\lambda}{24} \Phi^4 + \frac{1}{2} \Phi \delta K(i\partial)\Phi - \frac{\delta \lambda}{24} \Phi^4. \quad (6)$$

The kernel of the quadratic part (shortly “kernel" in the followings) used here is arbitrary, although for consistency we will require certain properties to be satisfied. To obtain the bare Lagrangian (3), the kernel- and the coupling-counterterms should satisfy

$$Z p^2 - m_0^2 = K(p) + \delta K(p), \quad \lambda_0 = \lambda_R + \delta \lambda. \quad (7)$$
The above form suggests the following free - interaction separation

\[ L_0 = \frac{1}{2} \Phi K \Phi, \quad L_I = -\frac{\lambda p^4}{24} \Phi^4 + \frac{1}{2} \Phi \delta K \Phi - \frac{\delta \lambda}{24} \Phi^4. \]  

(8)

The coupling constant and kernel counterterms are considered to be of higher order as compared to the coupling constant and kernel, respectively. Their values should be determined order by order, just like in the usual case. We should compute the diagrams, determine the infinite parts and then make them disappear by a suitable choice of the infinite part of the counterterms. The propagator should satisfy (1), then the divergences are polynomial in their momenta; it also has to have appropriate asymptotics, then only those operators are divergent that are present also in the bare Lagrangian. To make these divergences disappear, we should choose \( \delta K_{\text{div}} = A p^2 - B \) and \( \delta \lambda = C \), where \( A, B \) and \( C \) are divergent constants. Since from (7) we can write \( \delta K = \delta Z p^2 - \delta m^2 \), so we can identify the infinite parts of \( \delta Z \) and \( \delta m^2 \) as \( A \) and \( B \), respectively. It must be emphasized, that the actual form of \( \delta Z, \delta m^2 \) and \( \delta \lambda \) depends on the choice of the kernel. The logics of their determination is the same as in the usual case when we use the kernel \( p^2 - m^2 \).

### B. Theories with different kernels

The best strategy for perturbation theory would be to use that kernel that is the most appropriate for the description of the given observable. The background of working with different kernels is very similar to using different schemes in the usual perturbation theory. If the bare Lagrangian is the same, they should describe the same physics: now this is ensured by eq. (7). Here the momentum dependence of \( K \) and \( \delta K \) should cancel each other, while \( Z \) and \( m_0^2 \) are determined by the local, divergent part of the kernel counterterm. Similarly, the coupling constant counterterms is determined by the divergences of the four point function.

Although we determine the counterterms order by order, the following statement ensures that in an \( n \)th order calculation the bare parameters of the Lagrangian, as well as any observables are independent on the choice of the kernel, up to \( O(\lambda^n) \). As a consequence, at infinite order, we recover the same bare Lagrangian, and obtain the same value for all observables, with any choice of the kernel. Although the proof is essentially the same as in the standard case (cf. [1], in particular the Chapter “Oversubtractions”) it is worth to state here for our somewhat more generic case.

**Statement** In any regularized perturbation theory with kernel \( K \) and kernel counterterm \( \delta K \) which satisfy (7) with \( m_0^2 = m_R^2 + \delta m^2, \) \( Z = 1 + \delta Z \) and \( \lambda_0 = \lambda_R + \delta \lambda \) where \( m_R^2 \) and \( \lambda_R \) are fixed, and \( \delta m^2, \delta Z \) and \( \delta \lambda \) are determined order by order, the following statements are true in \( n \)-th order of the calculation:

- \( \delta m^2, \delta Z \) and \( \delta \lambda \) are independent on the kernel choice up to order \( \lambda^n \).
- any Greens function is kernel independent up to order \( \lambda^n \).

**proof** First we observe that if we have a well defined renormalized theory with \( K \) and \( \delta K \), then in a theory with \( K' = K - \delta K \) and \( \delta K' = \delta K + \delta K \) the bare Lagrangian is the same, and so, nonperturbatively, we obtain the same results for all observables. In this case (7) is automatically satisfied for \( K' \) if it was true for \( K \) and \( \delta K \). This proves that in an infinite order perturbation theory the result is independent on the actual choice of the kernel. In finite order perturbation theory, however, order-by-order finiteness requires different counterterm structure for each kernels. We should prove that the results are stabilized below the order of the calculation \( O(\lambda^n) \).

At tree level there is no need for counterterms \( \delta K = 0, \delta \lambda = 0 \), and (7) ensures that \( K = p^2 - m_R^2 \equiv K_0 \).

At one loop level we have a one-loop self-energy \( \Sigma_1[K] \), which is only overall divergent. When the regularization is fixed then we can determine the “infinite part” of a diagram, so we write

\[ \Sigma_1[K] = \Sigma_1^{\text{div}}[K] + \Sigma_1^{\text{fin}}[K]. \]  

(9)

In order to ensure finiteness, we should choose the one loop kernel counterterm as \( \delta K_1 = \Sigma_1^{\text{div}}[K] + \delta K_1 \), where \( \delta K_1 \) is finite (but, possibly, momentum dependent). Then (7) tells us that the one-loop kernel \( K_1 \) has to satisfy

\[ K_1 + \Sigma_1^{\text{div}}[K_1] + \delta K_1 = K_0 + \delta Z_{\text{1}} p^2 - \delta m_{\text{1}}^2. \]  

(10)

We then identify \( \delta Z_{\text{1}} p^2 - \delta m_{\text{1}}^2 = \Sigma_1^{\text{div}}[K] \) and \( K_1 = K_0 - \delta K_1 \). This latter equation states that \( K_1 = K_0 + O(\lambda) \), therefore \( \delta Z_{\text{1}} p^2 - \delta m_{\text{1}}^2 = \Sigma_1^{\text{div}}[K_0] + O(\lambda^2) \); this means that up to order \( \lambda \) it is independent of the choice of \( \delta K_1 \), ie. of the choice of the kernel.
What happens at higher order? We write $\delta K_1 = \Sigma^{\text{div}}[K] + \delta K_1$. This means that we have a new quadratic counterterm to the Lagrangian with kernel $\delta K_1$. At higher orders this counterterm should be included in any diagram $\Gamma[K]$. Technically this means that we subsequently replace some propagators by $iG \rightarrow iG \cdot i\delta K_1 \cdot iG$. Since $G = K^{-1}$, this replacement means

$$iG \rightarrow \frac{\partial iG}{\partial K} \delta K_1,$$

in functional manner. So, up to $n$ inclusions into the diagram, it can be formulated as

$$\sum_{\ell=0}^{n} \frac{1}{\ell!} \frac{\partial^{\ell} \Gamma}{\partial K^{\ell}} (\delta K_1)^{\ell}.$$

The zeroth term is $\Gamma$ itself, the $1/\ell!$ is the remnant of the $1/n!$ factor of the $n$th order in the perturbative series. We can write it as an infinite sum minus the difference. But the infinite sum is just the Taylor-series of $\Gamma[K + \delta K_1] = \Gamma[K_0]$. So we find:

$$\sum_{\ell=0}^{\infty} \frac{1}{\ell!} \frac{\partial^{\ell} \Gamma}{\partial K^{\ell}} (\delta K_1)^{\ell} - \sum_{\ell=n+1}^{\infty} \frac{1}{\ell!} \frac{\partial^{\ell} \Gamma}{\partial K^{\ell}} (\delta K_1)^{\ell} = \Gamma[K + \delta K_1](1 - O(\lambda^{n+1})) = \Gamma[K_0](1 - O(\lambda^{n+1})).$$

We see, therefore, that the generated finite counterterm restores the value $\Gamma[K_0]$ up to order $\lambda^n$. This means that any renormalized observable, which is represented by the sum of diagrams $\Gamma$, is the same (kernel-independent) up to $O(\lambda^n)$. Since we did not use any specific property of $\delta K_1$ which would exploit its one loop nature, the above statement can directly be generalized to higher orders. Moreover, the same line of thought can be followed for the coupling constant, when it is momentum independent. This completes the proof of the second part of the statement.

The same proof also applies to the counterterm structure. Then we should use $\Gamma \rightarrow \Sigma^{\text{div}}[K]$. At one loop level the value of $\Sigma^{\text{div}}[K]$ defines the kernel counterterms. At higher ($n$th) loop level the inclusion of $\delta K_1$ finite counterterm will generate divergent contribution, which has to be made vanish by the $n$th order kernel counterterm. So $\delta K_n$ contains the following contribution

$$\delta K_n \rightarrow \frac{1}{n!} \frac{\partial^n \Sigma^{\text{div}}[K]}{\partial K^n} (\delta K_1)^n.$$

In the sum $\sum_{i=1}^{n} \delta K_i$, therefore, we recover the subseries:

$$\sum_{i=0}^{n} \frac{1}{i!} \frac{\partial^{\ell} \Sigma^{\text{div}}[K]}{\partial K^{i}} (\delta K_1)^{\ell} = \Sigma^{\text{div}}[K_0] + O(\lambda^{n+1}).$$

So, although the one loop counterterm depends on the kernel, the higher order counterterm contributions coming from $\delta K_1$ inclusions ensure kernel independence up to $O(\lambda^n)$. Since this line of thought did not depend that we start at one loop level, it is valid for any other finite counterterm, and also the same line of thought applies to the coupling constant renormalization. This completes the proof of the first part of the statement.

**QED**

In practice, the most important question is the comparability of two perturbation theories, without a reference to the bare theory. The above statement claims that this can be ensured if we choose the parameters of scheme “2” as $m_2(m_1, Z_1, \lambda_1), Z_2(m_1, Z_1, \lambda_1), \lambda_2(m_1, Z_1, \lambda_1)$, where $m_1$, $Z_1$ and $\lambda_1$ are the parameters of scheme “1”. These relations can be found either by the requirement that the bare Lagrangian is the same (first part of the above Statement), or requiring that some observables has the same value in the two schemes. This latter strategy is the **matching**, which is a widely used technique in field theory (cf. large mass decoupling [15] or dimensional reduction [16]). It works in our case in a way that we compute three observables from theory “1”, and find those $m_2, Z_2, \lambda_2$ parameters that give the same result for these observables from a calculation in theory “2”. The above statement then ensures that calculating any other observables, the difference between the results of the two schemes is of $O(\lambda^{n+1})$ where $n$ is the order of the computation. The best is of course to choose that type of observables which can be calculated quite reliably from both schemes of perturbation theory.

The area where we want to apply the perturbation theory with modified kernel is to optimize perturbation theory for finite temperature or in non-equilibrium calculations. At each temperatures we can use a kernel, which is optimized
to that temperature. To compare the results at different temperatures, however, we have to match the renormalized parameters. A good set of quantities is the asymptotically large momentum regime observables at $T = 0$, since we expect that these observables should be independent of the details of the kernel at finite momenta, but still sensitive to the values of the renormalized parameters. In practical numerical calculations, of course, we should choose quantities at large, but not asymptotic momentum, and determine the matching procedure there.

### C. 2PI resummation

For optimizing the kernel for a given environment, we can use the condition that the free propagator (the inverse of the kernel) is the exact one. Then at any higher order diagrams the self-energy insertions, which would modify the free – in this case the exact – propagator, should be missing. So we should choose the kernel counterterms $\delta K$ at each perturbative order in a way that it exactly cancels the corresponding self-energy $\Sigma[0](p)$, calculated with the kernel $K$. On the other hand the diagrams with self-energy insertions are exactly the 2-particle reducible diagrams. These are missing in this perturbation theory: therefore we realized 2PI resummation, differently from the technique of Cornwall, Jackiw and Tomboulis [17].

In formula, the total self-energy can be written, at $n$th order, as

$$\Sigma_{\text{tot}}(n)[K](p) = \Sigma(n)[K](p) - \delta K(n)(p), \quad (16)$$

where here (and in the followings) $\Sigma[K](p)$ means the self energy with subdivergences subtracted. So 2PI resummation requires $\Sigma_{\text{tot}}(n)[K](p) = 0$ for all $n$: in this case the exact propagator $G_{\text{ex}}^{-1}(p) = K(p) - \Sigma_{\text{tot}}[K](p) = K(p)$ which is the inverse free propagator. So we should choose

$$\delta K(n)(p) = \Sigma(n)[K](p). \quad (17)$$

Equation (7) implies

$$Zp^2 - m_0^2 = K(p) + \Sigma[K](p) \quad (18)$$

at each perturbative order. It seems to be the usual 2PI equation, but it is not quite so. In this formula the infinite part of $Z$ and $m_0^2$ is chosen in a way that it cancels the infinities of $\Sigma[K](p)$: therefore, implicitly, $Z$ and $m_0$ are functionals of $K$. By our Statement of the previous subsection we can claim that at infinite order this dependence vanishes; but at any finite order it is there.

To see a finite equation we use the fact that $\Sigma[K](p) = A_{\text{div}}p^2 - B_{\text{div}} + \Sigma_{\text{fin}}[K](p)$, since $\Sigma$, by definition, is just overall divergent (subdivergences are subtracted). Then we choose $Z = 1 + A_{\text{div}} + \delta \zeta$ and $m_0^2 = m_R^2 + B_{\text{div}}$, and we arrive at ($\zeta = 1 + \delta \zeta$)

$$\zeta p^2 - m_R^2 = K(p) + \Sigma_{\text{fin}}[K](p), \quad (19)$$

which is a finite equation.

We will denote the kernel optimized by 2PI equations for calculation at temperature $T$ as $K(T)$. We also will denote a diagram or observable $\Gamma$ computed with kernel $K$ and at temperature $T$ as $\Gamma[K,T]$. We remark here that we can use a kernel, optimized to a certain temperature, in computations at other temperatures as well. The optimization temperature is just a notation to label the kernel, while there can be another real temperature in the system. Weinberg theorem ensures that $\Gamma^{\text{div}}[K,T]$, the overall divergent part does not depend on the (real) temperature.

The strategy to solve the finite temperature 2PI equations will consist of two steps.

**Step 1.** We determine the kernel $K^{(0)}$ which is optimized for zero temperature, i.e. the solution of (19) at $T = 0$. The parameters $\zeta$, $m_R^2$, and $\lambda_R$ are determined by using physical observables. In the calculations below we just choose them to be $m_R = m$, $\zeta = 1$ and $\lambda_R = \lambda$, so we should solve:

$$K^{(0)}(p) = p^2 - m^2 - \Sigma_{\text{fin}}[K^{(0)},0](p), \quad (20)$$

We will use the normalization conditions for the self-energy as:

$$\Sigma_{\text{fin}}[K^{(0)},0](p^2 = m^2) = 0, \quad \frac{\partial \Sigma_{\text{fin}}[K^{(0)},0]}{\partial p^2} \bigg|_{p^2 = m^2} = 0. \quad (21)$$

Then at $p^2 = m^2$ we will find a pole of the propagator, with unit residuum.
Step 2. We fix the temperature at $T$, and solve (19) to have $K^{(T)}$. To be consistent with the $T = 0$ calculation we cannot just choose the renormalized parameters ($\zeta_R$, $m_R$ and $\lambda_R$) of the Lagrangian, we will find their value by matching to the $T = 0$ case.

There are two possible strategies here. The first is to find the solution by taking arbitrary finite part for $\Sigma$, but using undetermined values for $\zeta_R$, $m_R$ and $\lambda_R$. Then, using $K^{(T)}$ as a kernel, we compute the zero temperature propagator and 4-point function. Since the kernel was not optimized for $T = 0$ calculation, we do not expect especially good convergence of perturbative series, but we do expect that the asymptotic regime will be well described. So we will choose large momenta (eg. $p = (N m, 0, 0, 0)$ where $N = 20 - 50$) and match $G^{-1}(p) \equiv K(p)$ and $\Gamma^{(4)}(p, 0, -p, 0)$ at these momenta (choosing together 3 matching conditions).

The other possible strategy – which will be followed later – defines the finite part of $\Sigma$ in a way that it is possible to choose $\zeta_R = 1$ and $m_R = m$. Then the relevant equation here reads (cf. (19))

$$K^{(T)}(p) = p^2 - m^2 - \Sigma_{\text{fin}}[K^{(T)}, T](p).$$

(22)

For matching we still use $G^{-1}(p) \equiv K(p)$ and $\Gamma^{(4)}(p, 0, -p, 0)$ at asymptotic momenta. The zero temperature propagator, calculated with the finite temperature kernel reads

$$G^{-1}[K^{(T)}, 0](p) = K^{(T)}(p) - \Sigma[K^{(T)}, 0](p) + \delta K_1(p).$$

(23)

Since 2PI equation (ie. $\Sigma_{\text{tot}} = 0$) yields $\delta K_1(p) = \Sigma[K^{(T)}, T](p)$, we obtain

$$G^{-1}[K^{(T)}, 0](p) = K^{(T)}(p) - \Sigma[K^{(T)}, 0](p) + \Sigma[K^{(T)}, T](p).$$

(24)

The infinite part of $\Sigma[K^{(T)}, 0](p)$ and $\Sigma[K^{(T)}, T](p)$ is the same (overall divergence is temperature independent), so we can write

$$G^{-1}[K^{(T)}, 0](p) = K^{(T)}(p) - \Sigma_{\text{fin}}[K^{(T)}, 0](p) + \Sigma_{\text{fin}}[K^{(T)}, T](p).$$

(25)

Now we use $K^{(T)}(p) + \Sigma_{\text{fin}}[K^{(T)}, T](p) = p^2 - m^2$, then we find

$$G^{-1}[K^{(0)}, 0](p) = p^2 - m^2 - \Sigma_{\text{fin}}[K^{(0)}, 0](p) = \Sigma_{\text{fin}}[K^{(T)}, 0](p).$$

(26)

This should be equal to

$$G^{-1}[K^{(0)}, 0](p) = K^{(0)}(p) = p^2 - m^2 - \Sigma_{\text{fin}}[K^{(0)}, 0](p).$$

(27)

So the asymptotic equality of the two propagators requires that we should define the finite part of the self energy in a way that

$$\Sigma_{\text{fin}}[K^{(0)}, 0](p) = \Sigma_{\text{fin}}[K^{(T)}, 0](p)$$

(28)

shall be true for two asymptotic momenta.

After obtaining the optimal kernel, we can use ordinary perturbation theory to compute other observables.

III. $\Phi^4$ THEORY, TWO LOOP LEVEL, FINITE TEMPERATURE

Now let us test the above ideas in the case of the $\Phi^4$ theory at finite temperature at two loop level. We will use real time formalism in R/A basis [18]. Here the original fields ($\Phi_1$, $\Phi_2$) are replaced by ($\Phi_r$, $\Phi_a$) via the definition

$$\Phi_1 = \Phi_r + \frac{1}{2} \Phi_a, \quad \Phi_2 = \Phi_r - \frac{1}{2} \Phi_a.$$  

(29)

The propagator is a $2 \times 2$ matrix, where, in this basis $G_{aa} = 0$ for the exact Green’s function, and also $G_{ra}^{(2)}(p) = G_{ar}(p)$ is true in the Fourier space. The interaction Lagrangian reads

$$-\mathcal{L}_I = \frac{\lambda}{24} \Phi^2_r \Phi_a + \frac{\lambda}{24} \Phi_r \Phi^3_a.$$  

(30)
A generic kernel is a \( 2 \times 2 \) matrix, but it must respect symmetries valid for any finite temperature field theory. In particular, all the propagators should be derived from the spectral function:

\[
G_{11} = G_{ra} + G_{12}, \quad G_{22} = -G_{ra} + G_{21}, \quad iG_{12}(p) = n(p_0)\varphi(p), \quad iG_{21}(p) = (1 + n(p_0))\varphi(p),
\]

and

\[
G_{ra}(p) = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \frac{\varphi(\omega, p)}{p_0 - \omega + i\epsilon}, \quad \varphi(p) = -2\text{Im} G_{ra}(p).
\]

The retarded self-energy \( G_{ra} \) can be expressed through its own self energy, it will not mix with other components:

\[
G_{ra}^{-1}(p) = G_{ra,0}^{-1}(p) - \Sigma_{ar}(p).
\]

The other self energies form a matrix, and they can be obtained as

\[
\begin{align*}
\text{Re } \Sigma_{11} &= -\text{Re } \Sigma_{22} = \text{Re } \Sigma_{ar}, \\
\text{Im } \Sigma_{11} &= \text{Im } \Sigma_{22} = (1 + 2n_B) \text{Im } \Sigma_{ar}, \\
\Sigma_{12} &= -2in_B \text{Im } \Sigma_{ar}, \\
\Sigma_{21} &= -2i(1 + n_B) \text{Im } \Sigma_{ar}.
\end{align*}
\]

These relations are to be respected when we choose the generic kernel \( K \). We denote the retarded kernel by \( K_{ra} \), the other matrix elements come from (34). The free retarded propagator then simply reads as

\[
G_{ra,0}^{-1}(p) = K_{ra}(p).
\]

The free spectral function can be expressed as

\[
\varphi_0(p) = -2\text{Im } G_{ra,0}(p) = \frac{-2\text{Im } K_{ra}(p)}{(\text{Re } K_{ra}(p))^2 + (\text{Im } K_{ra}(p))^2}.
\]

Causality requires that \( \text{Im } K_{ra}(p) \) must be an odd function of \( p_0 \), and also \( \text{Im } K_{ra}(p) < 0 \) must be true for \( p_0 > 0 \). In the \( \overline{\text{MS}} \) scheme it is satisfied by choosing \( \text{Im } K_{ra}(p) = -\varepsilon \text{sgn} p_0 \) (Landau prescription).

### A. One loop level

At one loop we do not generate momentum dependence for the kernel. This problem has been discussed with the momentum-independent version of the present method in \([11]\), but we include it here for completeness and for elucidate the strategy. The self-energy and the one-loop four point function read:

\[
\Sigma^{(1)}[K](p) = \frac{\lambda}{2} T[K], \quad \Gamma^{(4)}(p, q, k, \ell) = \lambda + \frac{\lambda^2}{2} \left\{ I[K](p + q) + I[K](p + k) + I[K](p + \ell) \right\} + \delta\lambda_1.
\]

where \( T[K] \) is the tadpole, \( I[K] \) is the bubble diagram, computed with kernel \( K \). The tadpole can be expressed as \( G_{11}(x = 0) = G_{12}(x = 0) \); in Fourier space

\[
T[K] = \int \frac{d^4k}{(2\pi)^4} n(k_0)\varphi(k) = \int \frac{d^4k}{(2\pi)^4} (1 + 2n(k_0))\varphi(k).
\]

To compute the bubble diagram one should consult Appendix A 1. For the 2PI summation we choose

\[
\delta K_1 = \frac{\lambda}{2} T[K].
\]

For **Step 1** we have to define its finite part so that the self-energy shall be zero on the mass shell at zero temperature. It results in the definition

\[
T[K] = T_{\text{div}}[K] + T_{\text{fin}}[K], \quad T_{\text{div}}[K] = \int \frac{d^4k}{(2\pi)^4} \varphi(k), \quad T_{\text{fin}}[K] = \int \frac{d^4k}{(2\pi)^4} 2n(k_0)\varphi(k).
\]
With this definition the zero temperature equation to solve is (cf. (20))

$$K^{(0)}(p) = p^2 - m^2,$$

(41)

since the self energy is zero at zero temperature.

For Step 2, with proper finite part for $\Sigma$, we have to solve (22):

$$K^{(T)}(p) = p^2 - m^2 - \frac{\lambda T^2}{2} I_{\text{fin}}[K^{(T)}, T].$$

(42)

In order to be write this equation, we have to ensure that the self energies at zero temperature, calculated with kernels $K^{(0)}$ and $K^{(T)}$ are equal. But now the zero temperature self energy, according to (40), is simply zero, so this requirement is trivially satisfied.

(42) is the same result that was obtained also in [11].

For the coupling constant renormalization we choose for kernel $K$ the counterterm

$$\delta \lambda_1 = -\frac{3\lambda^2}{2} I[K, 0](p = 0),$$

(43)

and define the finite part of the bubble diagram as

$$I_{\text{fin}}[K, T](p, q, k, \ell) = I[K, T](p, q, k, \ell) - I[K, 0](0).$$

(44)

This is of course just a specific choice, we could add a finite term to it in another renormalization scheme. The above choice ensures that the renormalized one loop four point function at zero momentum at zero temperature is $\lambda_R = \lambda$.

We use this equation to write the second matching condition as

$$\lambda + \frac{\lambda^2}{2} \left[ I_{\text{fin}}[K^{(0)}, 0](p + q) + I_{\text{fin}}[K^{(0)}, 0](p + k) + I_{\text{fin}}[K^{(0)}, 0](p + \ell) \right] =$$

$$= \lambda_T + \frac{\lambda_R^2}{2} \left[ I_{\text{fin}}[K^{(T)}, 0](p + q) + I_{\text{fin}}[K^{(T)}, 0](p + k) + I_{\text{fin}}[K^{(T)}, 0](p + \ell) \right],$$

(45)

at asymptotic momenta. In this simple case only the mass is modified, let us denote the mass of the kernel $K$ as $M$.

Then we can explicitly write up these contributions:

$$I[K, 0](0) = \frac{1}{16\pi^2} \left[ -\frac{1}{\varepsilon} + \gamma_E + \ln \frac{M^2}{4\pi^2} \right]$$

$$I_{\text{fin}}[K, 0](p^2 > 4M^2) = \frac{1}{16\pi^2} \left[ \ln \frac{p^2}{M^2} + 2X_+(\ln X_+ - 1) + 2X_-(\ln X_- - 1) \right],$$

(46)

where

$$X_\pm = \frac{1}{2} \left( 1 \pm \sqrt{1 - \frac{4M^2}{p^2}} \right).$$

(47)

For asymptotically large momenta ($p^2 \gg M^2$) we have

$$I_{\text{fin}}[K, T = 0](p) = \frac{1}{16\pi^2} \ln p^2 + O(\text{const.})$$

(48)

Therefore, in leading order in momenta, equation (45) can be written (choosing $p = q = -k = -\ell$) as

$$\lambda + \frac{\lambda^2}{32\pi^2} \ln p^2 = \lambda_R + \frac{\lambda_R^2}{32\pi^2} \ln p^2,$$

(49)

which implies $\lambda_T = \lambda$.

We will use this matching condition also at two loop level for simplicity. But we can also argue that – since the asymptotic parts of the kernels $K^{(0)}$ and $K^{(T)}$ are the same (cf. (28)) – we obtain the same $k$-dependence for the 4-point function at asymptotic momenta using either $K^{(T)}$ or $K^{(0)}$. Then $\lambda_R = \lambda$ should be the result at all order.
B. Two loop analysis

Let us repeat the above analysis at two loop level. From the generic strategy we assume that the kernel is fixed throughout the calculation. The diagrams for the self energy read:

\[ \Sigma^{(2)}[K](p) = \frac{\lambda^2}{6} S[K](p) + \frac{\lambda^2}{4} T[K] I[K](0) - \frac{\lambda \delta K}{2} I[K](0) + \frac{\delta \lambda}{2} T[K], \]

where \( T[K] \) stands for tadpole, \( I[K] \) for bubble, \( S[K] \) for setting sun diagrams. The computation of the setting sun diagram is discussed in Appendix A 2. \( \delta K_1 \) and \( \delta \lambda_1 \) is fixed at one-loop level (cf. (39) and (43)).

To see the overall divergence we split the setting sun diagram as \( S[K](p) = S_{\text{fin}}[K](p) + S_{\text{subdiv}}[K] + S_{\text{overall}}[K](p) \), where

\[ S_{\text{subdiv}}[K] = 3T[K]I[K,0](0) = -2\delta \lambda_1 T[K], \quad S_{\text{overall}}[K](p) = \delta Z sp^2 - \delta m_S^2, \]

and \( \delta Z_S \) and \( \delta m_S \) depend on the kernel and the choice of the renormalization conditions. We find

\[ \Sigma^{(2)}[K](p) = \frac{\lambda^2}{6} S_{\text{fin}}[K](p) + \delta Z sp^2 - \delta m_S^2, \]

We want to make the total self-energy correction disappear in the spirit of the 2PI resummation (cf. (17)), so we should choose \( \delta K_2(p) = \Sigma^{(2)}[K](p) \). To tell the bare parameters of the Lagrangian we should write (cf. eq. (7))

\[ Zp^2 - m_0^2 = K(p) + \delta K_1 + \delta K_2. \]

We write \( m_0^2 = m_R^2 + \delta m_1 + \delta m_2^2 \) and \( Z = 1 + \delta Z_1 + \delta Z_2 \). Then we identify

\[ \delta m_1^2 = -\delta K_1 = \frac{\lambda}{2} T_{\text{div}}, \quad \delta m_2^2 = \delta m_S^2 - \frac{\delta \lambda_1}{6} T[K], \quad \delta Z_1 = 0, \quad \delta Z_2 = -\delta Z_S. \]

To renormalize the theory we should perform the two steps described earlier:

**Step 1** we have to solve the zero temperature 2PI equation (20):

\[ K^{(0)}(p) = p^2 - m^2 - \frac{\lambda^2}{6} S_{\text{fin}}[K^{(0)},0](p), \]

since \( T_{\text{fin}}[K^{(0)},0] = 0 \). We should define the finite part as

\[ S_{\text{fin}}[K^{(0)},0](p^2 = m^2) = 0, \quad \frac{\partial S_{\text{fin}}[K^{(0)},0]}{\partial p^2} \bigg|_{p^2=m^2} = 0. \]

Then the above equation yields a propagator which has a pole at \( p^2 = m^2 \) with unit residuum.

In numerical calculations one has to compute the complete contribution, subtract \( 3T[K^{(0)},0]I[K^{(0)},0](0) \) subdivergence, and subtract an \( ap^2 - b \) function. The value of \( a \) and \( b \) can be determined from eq. (56).

**Step 2** the finite temperature equation to solve is (22):

\[ K^{(T)}(p) = p^2 - m^2 - \frac{\lambda}{2} T_{\text{fin}}[K^{(T)},T] - \frac{\lambda^2}{6} S_{\text{fin}}[K^{(T)},T](p). \]

For the correct renormalization we should require (28):

\[ S_{\text{fin}}[K^{(T)},0](p) = S_{\text{fin}}[K^{(0)},0](p) \quad \text{for asymptotic momenta}. \]

Numerically we should compute the complete setting sun contribution with kernel \( K^{(T)} \) at zero temperature, subtract \( 3T[K^{(T)},0]I[K^{(T)},0](0) \) subdivergence, and finally \( Ap^2 - B \). We determine \( A \) and \( B \) from (58), by requiring its fulfillment at two asymptotic momenta \( p_1 \) and \( p_2 \). Then at finite temperature we should compute the renormalized value of the setting sun diagram, that we first compute the complete diagram with kernel \( K^{(T)} \) at finite temperature, subtract \( 3T[K^{(T)},T]I[K^{(T)},0](0) \) subdivergence, and also \( Ap^2 - B \) with the values determined before. This will cancel all divergences, since, as it was mentioned earlier, the overall divergences do not depend on the temperature.
C. Results

The above strategy can be accomplished by numerical calculations. We used the spatial rotational invariance of the propagators, which allows to represent them at finite temperature as two-dimensional functions, with variables $p_0$ and $|\mathbf{p}|$. Therefore for the integrations we need a 2D lattice, which is mapped with a continuous function to the infinite 2D space. All the results presented here are calculated at $\lambda = 10$. Finite temperature, if not stated otherwise, means here $T = m$.

The 2PI equations are solved by successive approximation, starting from a spectral function near to the free one. From the spectral function one can determine the propagators, with the propagators we calculate the self-energies, and with help of self energies we update the spectral function. Then repeat this algorithm till it converges.

First we discuss the renormalization. In Fig. 1 we can see the figures representing the renormalization process. On

![Fig. 1](image1.png)

FIG. 1: Quality of renormalization: a.) zero temperature self-energy at $\mathbf{p} = 0$; b.) zero temperature setting sun diagram with different kernels – the two lines are almost within line width.

Fig. 1/a there appears the self energy calculated with $K^{(0)}$. It can clearly be seen that at $p_0 = m$ the self energy is $\Sigma = 0$, so there is a pole there. It can also be seen that the derivative with respect to $p_0$ is zero – this ensures the unit residuum. Fig. 1/b concerns the finite temperature renormalization. Here we require that the values of setting sun diagram calculated with kernel $K^{(0)}$ or $K^{(T)}$ at zero temperature, are, asymptotically, the same. As this plot shows, this is fulfilled within line width, although for smaller momenta there can be seen some discrepancy. These two plots show that the theory is renormalized.

The next two plots (Fig. 2) show the spectral function at two different spatial momentum. At zero temperature one

![Fig. 2](image2.png)

FIG. 2: Spectral function at two values of the spatial momentum at zero temperature (dashed line) and at finite ($T = m$) temperature (solid line). The y axis is logarithmic.

can clearly see the quasiparticle peak, and a well separated threshold starting at $3m$ in case of $\mathbf{p} = 0$. The position of the peak as well as the threshold value shifts with the spatial momentum. At finite temperature the quasiparticle peak is still very sharp and narrow (at these values of the coupling), but there is no sign for any threshold behavior, only a broad continuum can be observed.
In the third pair of figures (Fig. 3) the quasiparticle properties can be seen. The first plot shows the spatial momentum dependence of the mass. The momentum dependence is rather weak: in the range $0 < |p| < 15m$ it is less than 10%. This fact, together with the dominant quasiparticle peak may explain the success of the “screened perturbation theory” [19]. In the second plot the quasiparticle width can be seen: it drops[21] about a factor of 4 in the range $0 < |p| < 10m$, as was predicted by perturbative calculations [20].

The last two plots demonstrate the temperature dependence of the quasiparticle properties. In Fig. 4/a, the temperature dependence of the on-shell mass can be seen. In the range $T < 0.5m$ there is hardly any temperature dependence; at $T > 1.2m$ the curve fits well to a second order polynomial, so there $m^2 = m_0^2 + cT^2$ is a good assumption. Fig. 4/b shows the on-shell damping rate as a function of the temperature. The same tendency can be observed as in the on-shell mass case: for small temperatures ($T < 0.5m$) there is hardly any damping can be seen, while for high temperatures ($T > m$) the damping rate can be fitted to a straight line $\Gamma = \Gamma_0 + cT$. In both cases the $T \approx m$ regime is a crossover between the two characteristic behavior.

**IV. CONCLUSION**

The most important message of this paper is that the “renormalization scheme method” is appropriate to perform fully renormalized momentum dependent 2PI resummation at finite temperature. The advantage of this method is that one does not need to solve the Bethe-Salpeter equations. Instead one fixes the renormalization in the finite temperature calculation by matching to the zero temperature results.

Renormalization scheme method is based on the observation that in certain momentum dependent schemes there is no need of doing 2PI resummation, since, by construction, the self energy is zero at each order. We studied in the paper, when such an “optimized” scheme yields a consistent perturbation theory. We examined the relation of two
such schemes, and proved that by appropriate choice of the renormalized parameters, two schemes can give the same result up to $O(\lambda^n)$, where $n$ is the order of the perturbation theory – this is the same result as in the “ordinary” momentum independent schemes. But it also suggests that if we fix a scheme at zero temperature, then at finite temperature the optimized scheme can be related to it by matching.

In particular we suggested in the paper a two-step method for the renormalization. In the first step one has to solve the zero temperature 2PI scheme equations, where we fix its parameters by observable quantities (like quasiparticle mass, residuum and scattering cross section). In the second step we solve the finite temperature 2PI scheme equations, defining the finite part of the self energy in a specific way: we require that the zero temperature self energy, calculated with the zero temperature and the finite temperature optimized kernel, are the same for asymptotic momenta. This definition is the manifestation of the matching procedure.

The so-defined rules are numerically easily implementable, as it was demonstrated in the paper on the example of the $\Phi^4$ theory with two-loop 2PI resummation. The results of this study are summarized in the paper. The most important corollary is that the $\Phi^4$ theory can be well characterized by a thermal mass: the spectral function is dominated by a quasiparticle peak where its spatial momentum dependence can be described by an almost constant mass term.

For future studies, we plan to generalize the renormalization scheme method to non-equilibrium case, and also – which may be even more important and interesting – to systems with specific symmetries (global or gauge).

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APPENDIX A: DIAGRAMS

1. Bubble

We compute the retarded diagram that reads

$$iI[K,T](k) = \int \frac{d^4p}{(2\pi)^4} (iG_{11}(p)iG_{11}(k-p) - iG_{12}(p)iG_{12}(k-p)).$$

(A1)

Using the dispersion relation

$$G_{ra}(k) = \int \frac{ds}{2\pi} \frac{\rho(s,k)}{k_0 - s + i\varepsilon}$$

(A2)

the retarded diagram can be computed from its discontinuity

$$\text{Disc}_{k_0} iI[K,T](k) = \int \frac{d^4p}{(2\pi)^4} (1 + n(p_0) + n(k_0 - p_0))\rho_1(p)\rho_2(k-p),$$

$$I[K,T](k) = \int \frac{ds}{2\pi} \frac{\text{Disc}_s iI[K,T](s,k)}{k_0 - s + i\varepsilon}.$$  

(A3)

At zero temperature

$$n(\omega) = \frac{1}{e^{\beta\omega} - 1} \xrightarrow{\beta\to\infty} -\Theta(-\omega).$$

(A4)

Thus

$$1 + n(p_0) + n(k_0 - p_0) = \Theta(p_0) - \Theta(p_0 - k_0) = \Theta(0 < p_0 < k_0).$$

(A5)

So

$$\text{Disc}_{k_0} iI[K,0](k) = \int \frac{d^4p}{(2\pi)^4} \Theta(0 < p_0 < k_0)\rho_1(p)\rho_2(k-p).$$

(A6)
2. Setting sun diagram

The setting sun diagram can be reproduced from its discontinuity, just like the bubble. The discontinuity reads

\[
\text{Disc } iS[K](k) = \int \frac{d^4p}{(2\pi)^4} \frac{d^4q}{(2\pi)^4} \left[ iG_{21}(p) iG_{21}(q) iG_{21}(\ell) + iG_{12}(q) iG_{12}(\ell) \right] = \int \frac{d^4p}{(2\pi)^4} \frac{d^4q}{(2\pi)^4} \left[ (1 + n_p)(1 + n_q)(1 + n_\ell) - n_p n_q n_\ell \right] \varrho_1(p) \varrho_2(q) \varrho_3(\ell),
\]

(A7)

where \( p + q + \ell = k \). At zero temperatures it falls back to

\[
\text{Disc } iS[K, T = 0](k) = \int \frac{d^4p}{(2\pi)^4} \frac{d^4q}{(2\pi)^4} \varrho_1(p) \varrho_2(q) \varrho_3(\ell) \left[ \Theta(p_0) \Theta(q_0) \Theta(\ell_0) - \Theta(-p_0) \Theta(-q_0) \Theta(-\ell_0) \right].
\]

(A8)

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