Effective action of conformal spins on spheres with multiplicative and conformal anomalies

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Abstract

Two multiplicative anomalies are evaluated for the determinant of the conformal higher spin propagating operator on spheres given by Tseytlin. One holds for the decomposition of the higher derivative product into its individual second order factors and the other applies to its complete linear factorization. Using this last factorization, I also calculate the determinants explicitly in terms of the Riemann $\zeta$-function, for both even and odd dimensions. In the latter case there is, of course no multiplicative anomaly. The conformal anomaly is also found for arbitrary spin and dimension and exhibited for the eight- and ten-spheres.

Keywords: determinants, conformal spins, anomalies

(Some figures may appear in colour only in the online journal)

1. Introduction

The propagation operator of conformal higher spin, (CHS), fields on spheres has been shown, remarkably, to be a product of standard, second order operators, at least for spins 0, 1, and 2 (I only discuss bosonic fields here) and by conjecture for higher spins, [1]. The effective action and Casimir energy are computed for such fields and shown to obey certain identities, when summed over the spin. For this purpose $\zeta$-function regularization was employed (as well as an alternative cut-off technique). In this process it was assumed that the logdet of the product was the sum of the logdets of the individual factors. Although this is a natural thing to do, it is formally incorrect for even-dimensional spaces, due to the existence, in general, of a
It is my intention to investigate this question for the specific case of the sphere, \(S^d\), for even \(d\). The cases of the Einstein universe \(\mathbb{R} \times S^{d-1}\) or the generalised torus, \(S^1 \times S^{d-1}\), are, perhaps, more interesting, but I leave these for another time. I also calculate some of the logdets themselves and the conformal anomalies to complete the picture.

Cognola et al., [2], have recently analysed the multiplicative anomaly on spheres for a single second order factor. I will make contact with their results and I can also conveniently refer to this paper for a physicist’s introduction to the topic as well as for an alternative mathematical approach.

My aim here is to be as general as I can regarding dimension and spin, consonant with producing formulae which can be programmed easily and computed automatically.

2. The propagation operator, logdet and general form of defect

In order to get on, I will just cite the CHS propagation operator for transverse, traceless symmetric fields describing spin \(s\) given by Tseytlin, [1]. This is (I change the notation slightly to suit me)

\[
V_s \equiv \prod_{i=1}^{s} \left( -\Delta^2 + M^2_{s,i} \right) \equiv \prod_{i=1}^{s} D^i_s, \tag{1}
\]

where \(\Delta^2\) is the usual covariant Laplace operator on transverse traceless fields and \(M_{s,i}\) is a mass parameter given by,

\[
M^2_{s,i} = s - (i - 2)(i + d - 3) \quad i = 1, 2, \ldots, s. \tag{2}
\]

As an old example, the \(d = 4\) spin 2 case is

\[
V_2 = \left( -\Delta^2 + 4 \right) \left( -\Delta^2 + 2 \right).
\]

Of course, other fields, such as ghosts, are involved when computing the physical effective actions or partition functions, but this sector will not concern me here.

The definition of the defect, \(\delta\), is

\[
\log \det \left( \prod_i D^i_s \right) = \sum_i \log \det D^i_s + \delta_i(D^1_s, \ldots, D^s_s), \tag{3}
\]

\(i\) running from 1 to \(s\). The expressions for the logdets of the individual (second order) factors are assumed known. If one had to undertake their calculation, then this might occasion a different definition of \(\delta\). This is expanded on in section 5.

The computation of \(\delta\) was presented in [3] for spins 0 and 1/2 and this is now extended to higher spins.

I write the operator \(D^i_s\) in the form\(^2\),

\[
D^i_s = L^2 - a^2_i,
\]

which fixes the operator \(L^2\), \(a_i\) being a constant chosen so that the \(\zeta\)-function associated with \(L^2\) (equivalently \(L\)) is easily managed. This \(\zeta\)-function is formally determined by the sum

\(\footnote{The term ‘anomaly’ means, here, the violation of an expectation.}

\(\footnote{In the case where the operators \(D_i\) are more general than those for CHS on spheres, this is a restriction and not always possible.}
\[ Z_d(\sigma) = \sum_{m} \frac{1}{\sigma_m} \tag{4} \]

over the eigenvalues, \( \lambda_m \), of the linear \( L \), including repeats. I discuss the best choice of \( \alpha_i \) later. These quantities all depend on the spin, \( s \).

The formal expression for the defect derived in [3] is independent of the nature of the operators, \( D_i^j \) and so I can quote it here as relevant for an operator of the type (1) and a \( \zeta \)-function, (4).

\[
\delta(D_1^j, \ldots, D_s^j) = \frac{s - 1}{2s} \sum_{r=1}^{d/2} \left( \frac{H(r - 1) N_{2r}(d)}{r} \sum_{j=1}^{s} \alpha_{2r}^j \right) - \frac{1}{2s} \sum_{r=1}^{d/2} \frac{1}{r} \sum_{i=1}^{d/2-r} \frac{N_{2r+2i}(d)}{t} \sum_{i<j}^{s} \alpha_{2r}^i \alpha_{2r}^j. \tag{5} \]

In this formula \( H \) is the harmonic series, \( H(r) = \sum_{n=1}^{r} 1/n (H(0) = 0) \), and \( N \) is the residue at a pole of the \( \zeta \)-function, (4). That is

\[
Z_d(\sigma + r) \rightarrow \frac{N_r(d)}{\sigma} + R_r(d) \quad \text{as} \quad \sigma \rightarrow 0. \tag{6} \]

The \( N_{2r} \) are related to the heat-kernel coefficients of the second order operator, \( L^2 \), making them locally computable. Some could be zero. \( N \) and \( R \) will depend on the parameters in the eigenvalues, \( \lambda_m \).

Only the positive poles of \( Z_d(\sigma) \) enter into the derivation of (5) which therefore holds for \( \sigma > d \) by a general theorem.

Incidentally it was shown in [3] and, more generally, in [4], that it is sufficient to give the defect for just two typical factors. For shortness I define \( \delta(i, j) \equiv \delta_2(D_i^j, D_j^i) \), with \( i < j \). The required relation is then

\[
\delta(D_1^i, D_2^j, \ldots, D_s^j) = \frac{2}{s} \sum_{i<j}^{s} \delta(i, j), \tag{7} \]

assuming all the operators are distinct and that the logdets are defined using the relevant \( \zeta \)-functions.

This is useful for a general distribution of the \( \alpha_i \). In the present case, \( \alpha_i \) is a specific function of \( i \) and one might hope to proceed further as in [5–7].

### 3. Choice of \( \alpha_i \)

In order to decide on the form of the \( \alpha_i \), it is sufficient to bring in the standard expressions for the eigenvalues of the (transverse, traceless) rank-\( s \) tensor Laplacian, \( -\Delta^2_\perp \). These are

\[
\mu(n, s, d) = (n + s)(n + s + d - 1) - s, \quad n = 0, 1, 2, \ldots, \tag{8} \]

expressed in eigenlevel form with degeneracies

\[
d(n, s, d) = g(s, d) \frac{(n + 1)(n + 2s + d - 2)(2n + 2s + d - 1)(n + s + d - 3)!}{(d - 1)! (n + s + 1)!}, \tag{9} \]
where \( g(s, d) \) is the spin degeneracy
\[
g(s, d) = \frac{(2s + d - 3)(s + d - 4)!}{(d - 3)! s!},
\]
which is not so important just now. I therefore drop it briefly and put it back later.

The first thing to do is to complete the square in \( \mu \), (8), as a quadratic in \( n \). Defining a half-integer by \( n' = n + s + (d - 1)/2 \), this gives, changing arguments appropriately
\[
\mu(n', s, d) = n'^2 - s - \left( \frac{d - 1}{2} \right)^2.
\]  

(This is an old manipulation. In four dimensions it can be found in Allen, [8], and for higher dimensions in Camporesi and Higuchi, [9].)

The \( \zeta \)-function of the \( i \)th factor in the propagating product, (1), is constructed from the mass shifted eigenvalues \( \mu + M_{ii}^2 \) and so, looking at (9) and (2), the appropriate choice of the constant \( a_i \) is
\[
a_i^2 = s + \left( \frac{d - 1}{2} \right)^2 - M_{ii}^2
\]
\[
= \left( \frac{d - 1}{2} \right)^2 + (i - 2)(i + d - 3)
\]
\[
= \left( i + \frac{d - 5}{2} \right)^2,
\]
which is a perfect square independent of the spin, \( s \).

The ‘simple’ \( \zeta \)-function, (4), then reads
\[
Z_d(\sigma) = \sum_{n=0}^{\infty} d(n, s, d) \frac{n!}{n'^{\sigma}}.
\]  

where \( n' \) is a function of the spin.

It is the poles of this \( \zeta \)-function that are needed to compute the defect (5). Expanding, in traditional fashion, the degeneracy in powers of \( n' \) gives a sum of Hurwitz \( \zeta \)-functions, as often occurs, cf Copeland and Toms, [10], Vardi [11]. Since all the poles have unit residue the offset relating \( n' \) and \( n \) is not relevant and the overall residues are easy to find from just the polynomial expansion of the degeneracy, which has a factor of \( n' \).

To formalise this, the expansion is written, for even \( d \),
\[
d(n, s, d) = n' \frac{\text{g}(s, d)}{2^{d-1}(d - 1)!} \left( (2n')^2 - (2s + d - 3)^2 \right) \prod_{p=0}^{d/2-3} \left( (2n')^2 - (1 + 2p)^2 \right)
\]
\[
\equiv n' \sum_{l=0}^{d/2-1} C_l(s, d) n'^{2l}.
\]

(I have reintroduced the spin degeneracy and an empty product is unity.) This is an odd polynomial in \( n' \). The \( \zeta \)-function, (11), then becomes

\[3\] Some history of scalar sphere determinants was given in [12].
and so the residue at $\sigma = r$ in (6) is $N_{2r}(d) = C_{r-1}(s, d)$, $r$ ranging from 1 to $d/2$, as expected. This quantity is readily found. It can be expressed in terms of central factorial coefficients but I will not use this.

4. Computation of the defect

Although it is possible to work with general spin and dimension, it is, perhaps, easier to proceed dimension by dimension. This allows the sums in (5) to be done explicitly the process being easily programmed. The results are polynomials in the spin. All defects correctly vanish at $s = 0$ and $s = 1$, the first because the product is empty and the second because there is only one factor. The zeros at negative $s$ are kinematic, arising from the spin degeneracy.

The actual polynomials rapidly become unwieldy. I give only the expressions for $d = 4$ and $d = 6$,
\[ \delta_s = \frac{(s - 2) s (s^2 - 1) (2 s + 1) (4 s - 7)}{540}, \quad d = 4 \]
\[ = -s (s^2 - 1) (s + 1) (s + 2) \left( 4 s^2 - 1 \right) (2 s + 3) \left( s^2 + 21 s - 4 \right), \quad d = 6. \]

These include the spin degeneracy and are plotted in figure 1, for a continuous spin. Figure 2 is for \( d = 8 \).
I note that the defect vanishes in four dimensions for spin-two.

5. Calculation of the logdets. Linear factorization

The calculation of the defect is a local one, in contrast to that of the logdets. In this section I wish to start again with the express aim of finding an explicit form for the non-local part of logdet \( V_s \), (1). As intimated before, this might lead to a different (but related) defect, as will become clear. The method I employ involves the resolution of \( D_i \) into linear factors which is reflected in the structure of the eigenlevels, \( \mu \). From (9) and (10),
\[ \mu(n', s, d) = (n' + \alpha_i)(n' - \alpha_i) \]
or
\[ D_i' = (L + \alpha_i)(L - \alpha_i). \] (13)

Although one could deal with each second order factor separately in (3) and use (5), it is better to define another defect, \( \delta_i(\alpha) \), by
\[ \log \det \left( \prod_i L^2 - \alpha_i^2 \right) = \sum \left( \log \det (L + \alpha_i) + \log \det (L - \alpha_i) \right) + \delta_i(\alpha), \] (14)

which arises upon a complete linear factorization of \( V_s \). The defect part can then be read off immediately from the results in [5]. I will write it out later, but I wish first to look at the non-local part in (14). This is not as easy, formally, as in [5] due to the structure of the degeneracy, (12). The calculation applies to both even and odd dimensions.

The relevant \( \zeta \)-function, \( \zeta_d(\sigma, \alpha_i) \), that arises from the factorization, (14), has the denominator \( (n' + \alpha_i)^r = (s + (d - 1)/2 + \alpha_i)^r \).

One requires the combination \( \zeta_d(\sigma, \alpha_i) + \zeta_d(\sigma, -\alpha_i) \) and I note that
\[ s + (d - 1)/2 + \alpha_i = s + i + d - 3 \]
\[ s + (d - 1)/2 - \alpha_i = s - i + 2, \] (15)

which are positive integers, if \( s \) is.
\( \zeta_d(\sigma, \pm \alpha_i) \) is now a sum of Hurwitz \( \zeta \)-functions
\[ \zeta_d(\sigma, \alpha_i) = \sum_{l=0}^{d-1} A_l(s, d, i) \zeta_H(\sigma - l, s + i + d - 3) \]
\[ \zeta_d(\sigma, -\alpha_i) = \sum_{l=0}^{d-1} B_l(s, d, i) \zeta_H(\sigma - l, s - i + 2), \] (16)
where $A$ and $B$ are coefficients in the expansion of the degeneracy
\[
d(n, s, d) \equiv \sum_{j=0}^{d-1} A_j (s, d, i) (n + s + i + d - 3)^j
\equiv \sum_{j=0}^{d-1} B_j (s, d, i) (n + s - i + 2)^j.
\]

(17)

$A$ and $B$ are related by,
\[
B_j = -(-1)^{j+d} A_j.
\]

Then, differentiating (16) at 0 and adding, according to (14),
\[
\sum_{i=1}^{s} \left( \zeta''_d (0, \alpha_i) + \zeta''_d (0, -\alpha_i) \right)
= \sum_{j=0}^{d-1} \sum_{i=1}^{s} A_j (s, d, i) \left( \zeta''_d (-l, s + i + d - 3) - (-1)^{j+d} \zeta''_d (-l, s - i + 2) \right).
\]

(18)

There is no point in writing out the formal expressions for various spins and dimensions. It is best to proceed directly to numerical computation.

Machine evaluation yields the values, for spin-2, of (18) of

$-18.590, -9.953, -73.053, -53.439, -150.284$ for $d = 4, 5, 6, 7, 8$ respectively.

There remains to give the multiplicative anomalies which must be added to these quantities, according to (14). In odd dimensions, this is zero while, for even $d$, the general expression is
\[
\delta_s (d) = -\sum_{r=1}^{d/2} \frac{1}{r} \left( \sum_{j=1}^{r} a_{2r}^j \right) H_r (d) N_{2r} (d)
+ \frac{1}{2s} \sum_{r=1}^{d/2} \frac{d/2-r}{r} \sum_{t=1}^{d/2-r} \alpha_t^2 \alpha_{j+1}^2 N_{2r+2t} (d).
\]

In this formula $H_r$ is related to the harmonic series by,
\[
H_r (r) = H (2r - 1) - \frac{1}{2s} H (r - 1),
\]

for $s = 1$ (one factor), $H_1$ is the odd harmonic series, $H_O, N$ is the residue at the pole of the $\zeta$-function, (11), as in (6).

Machine evaluation produces the following expressions for the defect, modulo the spin degeneracy, $\delta_s (d) \equiv \delta_s (\alpha) / g (s, d)$, with the $\alpha$ distribution, (10),
\[
\delta_s (4) = \frac{1}{2160} s (2s - 1) (2s + 1) \left( 32 s^2 + 60 s + 33 \right)
\delta_s (6) = s \left( \frac{s^6}{945} + \frac{53 s^5}{5400} + \frac{197 s^4}{5400} + \frac{49 s^3}{720} + \frac{837 s^2}{12960} + \frac{391 s}{14400} + \frac{127}{54360} \right).
\]
6. The conformal anomaly

A calculation more involved than that of the (partial) effective action (‘partition function’) is that of the (local) conformal anomaly for the entire field content. I will take this anomaly as being proportional to the value of the \( \zeta \)-function of the operator, (1), evaluated at zero, \( \zeta_i(\sigma)|_{\sigma=0} \), minus the corresponding quantity for the ghost fields, and any other contributions, such as extra zero modes, to the complete field theory. The content of this latter is given by Tseytlin, [1] section 3.2, where there is a treatment of the conformal anomaly using the standard heat-kernel coefficients. In the present approach, I deal with spheres \textit{ab initio} along the lines of Tseytlin in [13]. My treatment differs from his only in a more commodious computation of the \( \zeta \)-functions, I believe. In this regard, the method used in [13] extends the early work of Allen, [8].

I employ the kinematics for spin-\( s \) on \( d \)-spheres given in [13] \S 2 and compute the \( \zeta(0) \) for the combination of fields exhibited in [13] equation (1.2), not forgetting to allow for the extra zero modes introduced by the transformation to transverse fields. It is necessary also to note that for a product of \( s \) second order differential operators, the scaling quantity multiplying \( \log L \) is \( s \zeta(0) \), in general terms. This is the \( B_{s}^f \) coefficient of [1].

Some calculational details are given in an appendix but, here, I simply say that the same ingredients as in the previous paragraphs, e.g. setting \( \sigma \) to zero in (16), are easily programmed and yield the following results, beyond \( B_{s}^f \), [13].

\[
B_{8}^s = -\frac{\nu_8(s)}{31\,752\,000}\left(150s^8 + 3000s^7 + 24\,615s^6 + 106\,725s^5 + 261\,123s^4 + 351\,855s^3 + 225\,042s^2 + 31\,710s - 14\,560\right)
\]

\[
B_{10}^s = -\frac{\nu_{10}(s)}{6286\,896\,000}\left(190s^{10} + 6650s^9 + 99\,945s^8 + 843\,360s^7 + 4379\,820s^6 + 14\,399\,910s^5 + 29\,563\,605s^4 + 35\,558\,040s^3 + 21\,079\,904s^2 + 2785\,888s - 1292760\right).
\]

(19)

The \( \nu \)s are the physical degrees of freedom of the spin-\( s \) fields

\[
\nu_8(s) = \frac{(s + 1)(s + 2)^2(s + 3)^2(s + 4)}{72} \sim s^6 + \frac{(s + 1)(s + 2)(s + 3)^2(s + 4)^2(s + 5)(s + 6)}{2880} \sim s^8 + .
\]

I could not write \( B_8 \) or \( B_{10} \), in terms of \( \nu_8 \) or \( \nu_{10} \).

Just as in 4 and 6 dimensions, the expressions in (19) agree precisely with those obtained using the formulae of Tseytlin, [13], based on the holographic technique of Giombi \textit{et al.}, [14].

\footnote{In fact such special cases are actually an important input to the form of the general coefficient used in [1].}

\footnote{I am grateful to a referee for this comforting information.}
7. A single factor

It is of some interest to calculate the logdet for just one typical second order operator in (1), i.e. $L^2 - \alpha^2$ which appears in the logdet term on the right-hand side of (3).

The spin-0 case (which actually is not included in (1)) has been discussed already in [15] and, with the same method, more particularly in [3]. The multiplicative anomaly, corresponding to the linear factorization, $L^2 - \alpha^2 = (L + \alpha)(L - \alpha)$, has also been computed by Cognola et al., [2], using Wodzicki’s residue. The expressions have also been obtained likewise, in [2], for spins 1 and 2, in four dimensions. The effective actions were found as well.

Using the spectral technique of the previous sections, I firstly list the multiplicative anomaly, $\delta_\alpha(d)$, for any spin and dimensions 4 and 6, again modulo the spin degeneracy

$$\delta_\alpha(4) = \frac{2\alpha^2(2s + 1)(12s^2 + 12s - 8\alpha^2 + 3)}{3(s - 2)(s - 1)s(s + 5)}$$

$$\delta_\alpha(6) = \frac{\alpha^2(2s + 3)(480\alpha^2s^2 - 180s^2 + 1440\alpha^2s - 540s - 368\alpha^2 + 1200\alpha^2 - 405)}{360(s - 2)(s - 1)s(s + 9)}.$$

The non-local part of the logdet is given by the single term in the sum over $i$ obtained by setting $i = \alpha + (d - 5)/2$ in (18). The general structure is clear but I give some explicit computations for any spin but only for dimension 4 since higher dimensions are best left on the computer. I find

$$\zeta^i(0, \alpha) + \zeta^i(0, -\alpha) = \frac{\alpha}{12} \left( 8s^3 + 12s^2 + \left( 6 - 8\alpha^2 \right)s - 4\alpha^2 + 1 \right) \left( \zeta^0_0(w^+) - \zeta^0_0(w^-) \right)$$

$$- \frac{1}{12} \left( 8s^3 + 12s^2 + \left( 6 - 24\alpha^2 \right)s - 12\alpha^2 + 1 \right) \left( \zeta^1_1(w^+) + \zeta^1_1(w^-) \right)$$

$$- (2s + 1) \left( \alpha \left( \zeta^2_{-2}(w^+) - \zeta^2_{-2}(w^-) \right) - \frac{1}{3} \left( \zeta^3_{-3}(w^+) - \zeta^3_{-3}(w^-) \right) \right).$$

where $\zeta^i_{-j}(w) = \zeta^i_{-j}(-L, w)$ and $w^+ = s + (d - 1)/2 \pm \alpha$, a function of $s$.

For $s = 1$, $s = 2$ and $d = 4$, these results agree with those found by Cognola et al., [2].

8. Conclusion and comments

My results are presented, in the first instance, only as mathematical statements since the defect, being a local object, should be removable, physically, by renormalization although this expectation remains to be justified and there may be other reasons for ignoring it.

Furthermore, the expressions derived refer only to one sector of the CHS theory. The complete formula, [1], including ghosts and, for $d > 4$, extra massive fields, is more complicated. The conformal anomaly for this complete system has, however, been evaluated. There is no multiplicative anomaly for the conformal anomaly, and so the computations of this in, e.g., [1] are safe. Our results confirm Tseytlin’s and go a bit beyond, in regard to the dimension of the sphere.

As a purely manipulative point, I was not able to write the $\zeta$-function, (4), in terms of Barnes $\zeta$-functions, as can be done for spins zero and a half. Nor have I tried to use the hemisphere split. This would have formal advantages.
There are no anomalies in odd dimensions and the logdets in this case would be relevant for an \(F\)-theorem.

The extension of this analysis to fermionic fields remains to be made. Tseytlin, [1], gives the basic setup and calculation of field theory quantities. For massive fields (which arise in the product) there are multiplicative anomaly issues, regarding the ‘squaring’ of the Dirac-like operator, [16], which should be addressed irrespective of any physical import.

The method used in [2] for the anomaly is an \textit{ab initio} application of the Wodzicki residue. It might be of interest to note, [3], that a more explicit result of Wodzicki’s, (cf Kassel, [17], §6.6.) can also be employed\(^6\).

Combining the single factor expressions in section 7 according to (3) provides another (only algebraically different) way of deriving the complete logdet of the product as given in section 5. Also, a numerical treatment of (3) is possible and will be detailed at another time.

\textbf{Appendix. The conformal anomaly}

Some details of the calculation of the conformal anomaly in section 6 are given.

The conformal anomaly (equivalent to a heat-kernel coefficient) for a (diffentially) unconstrained propagation operator is \(\zeta_c(0)\) in terms of the corresponding \(\zeta\)-function, \(\zeta_\sigma(\sigma)\). There is no correction for zero modes if this \(\zeta\)-function is defined, as here, as a sum over all modes, including zero and negative ones, [1, 13, 18]\(^7\). For example, Allen’s \(\zeta\)-function is effectively defined in this way. However the conformal anomaly for a constrained operator, with \(\zeta\)-function, \(\zeta_c(\sigma)\), is equal to \(\zeta_c(0) - \mathcal{N}\), where \(\mathcal{N}\) is the number of zero modes introduced by the transition from unconstrained to constrained fields. For the field system of interest here, Tseytlin, [13] gives \(\mathcal{N}\) as the dimension of a particular group representation. For the transverse operator, \(\Delta_s\), in (1),

\[\mathcal{N}^\star_d = (2s + d - 4)(2s + d - 3)(2s + d - 2) \frac{(s + d - 4)!}{s!(s - 1)!} \frac{(s + d - 3)!}{d!(d - 2)!}.\]

The total field content and the dynamics are summarised in [13], section 1, and in [1] section 3, via the expression for the partition function (equivalent when exponentiated to the effective action)

\[Z^S_d = \prod_{i=1}^{s} \left( \frac{\det(D^i_\cdot)}{\det(D^i_i)} \right)^{1/2} \times \prod_{j=3-d/2}^{0} \left( \frac{\det(D^j_\cdot)}{\det(D^j_j)} \right)^{-1/2}.\]

The operator \(D^j_\cdot\) is defined in (1) and (13). The structure of the last term, due to ‘extra’ massive modes, is conjectured, [1].

Then the total conformal anomaly for this set of fields and ghosts is

\[B_d^\star = \sum_{j=1}^{j} \left( B_d^\star_j - B_d^\star_j \right) \mid_{\mathcal{N}^\star_d - 1} + \sum_{j=3-d/2}^{0} B_d^\star_j,\]

\(6\) Indeed Kassel refers to a relevant manuscript by Bost on this precise subject, but I have not been able to trace it.

\(7\) This can be achieved by adding a sufficiently large quantity, say \(M^2\), to the operator to make these modes ‘normal’, setting \(\sigma\) to zero and then removing \(M^2\).
where, in terms of the constrained $\zeta$-function, $\zeta^{	ext{sd}}_d(\sigma), \equiv \text{Tr} \left( D^\text{sd}_d \right)^{-\sigma}$,

$$B^\text{sd}_d = \zeta^\text{sd}_d(0) - \mathcal{N}_d^\text{sd}.$$ 

The values of the $\zeta$-functions at the origin are given by, see [15]\textsuperscript{8},

$$\zeta^\text{sd}_d(0) = \frac{1}{2} \left( \zeta_d(0, \alpha_i) + \zeta_d(0, -\alpha_i) \right),$$

in terms of the $\zeta$-functions of the linear factorization, (13). These are easily computed from (16) and yield the exhibited formulae.

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\textsuperscript{8} This allows for the multiplicative factor of $s$ for the anomaly coefficient.