One-dimensional time-Floquet photonic crystal

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Abstract

Using the Floquet Hamiltonian derived based on the time-dependent perturbation theory, we investigated the quasienergy bands of a one-dimensional time-Floquet photonic crystal (PC). The PC contains two alternating layers with different permittivities in the static case which are labeled as A and B. The permittivity of layer A is modulated periodically in time. We considered two cases, first the modulation function is a function of time only and second it is a function of both time and space through an unique combination. In the former case, although the permittivity of the whole medium is space-time-modulated, the quasienergy bands are symmetric because inversion symmetry is preserved. Different from the space-time-modulated medium often discussed previously, e.g. a homogeneous medium with the permittivity space-time-modulated according to a wavelike function, where either only exception points (EPs) or only quasienergy gaps generate, the coupling between the positive (negative) and positive (negative) bands in the time-Floquet PC results in quasienergy gaps, while the coupling between the positive and negative bands leads to pairs of EPs, enabling the coexistence of quasienergy gaps and EPs, when the modulation is on the real part of the permittivity. In the latter case, the quasienergy bands become asymmetric since time-reversal and inversion symmetries are simultaneously broken. The coupling between the positive (negative) and positive (negative) bands still results in quasienergy gaps, while the coupling between the positive and negative bands leads to quasienergy gaps at a small modulation speed and pairs of EPs at a high modulation speed.

1. Introduction

Time-Floquet systems with parameters periodically modulated in time provide a versatile platform to realize various exotic physical phenomena in both quantum [1–5] and classical physics [6–18], thus have attracted a growing interest in recent years. Periodically modulating the parameters in time breaks time-reversal symmetry, enabling the time-Floquet system to support topologically nontrivial states that are topologically trivial in the static case, i.e. the Floquet topological states [19–21], and to possess asymmetric band structures [22]. Recently, the Floquet topological states in various classical wave systems were theoretically studied [23–27] and experimentally observed [28–32], and several types of nonreciprocal devices based on the time-modulated elements were designed [33–35]. The periodic time-modulation also brings about discrete translational symmetry in time, which makes the frequency can be converted through adding or subtracting an integer times of the modulation frequency [36], in analogy to the wave vector conversion of the classical waves by a lattice structure. On the other hand, the time modulation requires an external driving in principle, which could make the total energy of the system not necessarily conservative thus introduce non-Hermiticity to the system [37–40], even though there is no gain or loss material used. The coexistence of the non-reciprocity, frequency conversion and non-Hermiticity in the time-Floquet system enables the realization of various attractive applications, such as the broadband nonreciprocal wave amplification [41–44] and frequency-selective wave filtering [45].

In our previous work [37], we have studied the quasienergy bands of a homogeneous medium with a time-periodic complex permittivity. Here, we extended the study to the case that the periodic
space-time-modulation is applied to a one-dimensional (1D) photonic crystal (PC) with two alternating layers A and B. We will show that both quasienergy gaps and exceptional points (EPs) are caused by the time modulation if the medium is modulated partially. While the permittivity of layer \( B \) \( \varepsilon_b \) is static, the permittivity of layer \( A \) is time-modulated as \( \varepsilon_a + \delta(z,t) \), where \( \varepsilon_a \neq \varepsilon_b \) is the static part and \( \delta(z,t) \) is the time-periodic modulation function of variables time \( t \) and spatial coordinate \( z \). We calculated and analyzed the quasienergy bands using the Floquet Hamiltonian which is derived from the Maxwell’s equations based on the time-dependent perturbation theory. Comparing with other analytical techniques such as the plane wave expansion method [17, 33, 41], the Floquet Hamiltonian method enables us to study the coupling between the modes of different frequency orders. And by using the reduced Floquet Hamiltonian, a close investigation of the EPs in the time-Floquet system is attainable [37]. In the vanishing \( \delta \) limit, the bands of the 1D PC in the static case copy themselves and shift up and down in the quasienergy space by \( n \Omega \) to generate the bands of order \( n \), where \( n \) is an integer and \( \Omega \) is the modulation frequency. Bands of different orders cross and form degenerate points. Here, we call the bands with positive frequencies in the static case \((n=1)\) order and band \( -n \) of 0th (0th) order are located at \( \Omega/2 \) \((-\Omega/2)\) in the quasienergy space, similar to the case of a homogeneous medium with time-periodic real permittivity [37].

We also investigated the case that \( \delta(z,t) = \delta_c \cos(\Omega t - \beta z) \), where \( \beta \) is the spatial modulation frequency. Inversion symmetry of the time-Floquet PC is broken, therefore, the quasienergy bands are asymmetric when \( \beta \neq 0 \). The degenerate points formed by two positive (negative) bands are still lifted to open quasienergy gaps when \( \delta \) is real. However, the degenerate point formed by a positive and a negative bands is lifted to open a quasienergy gap when the modulation speed \( \epsilon_m = \Omega/\beta \) is small, but spawns into a pair of EPs when \( \epsilon_m \) is large for a real \( \delta \), which is similar to the case that the space-time-modulation function is imposed to the permittivity of a homogeneous medium [37].

2. Formulation of the Floquet Hamiltonian

The unit cell of the 1D time-Floquet PC we consider is schematically shown in figure 1, which contains two dielectric layers. The static relative permittivities of the A and B layers are \( \varepsilon_a \) and \( \varepsilon_b \), respectively. Consider the permittivity of the A layer is modulated periodically in time, the time-dependent relative permittivity of the PC is expressed as

\[
\varepsilon(z,t) = \varepsilon(z) + \varepsilon_m(z,t) = \begin{cases} 
\varepsilon_a + \delta(z,t), & n\Lambda \leq z \leq n\Lambda + d_a \\
\varepsilon_b, & n\Lambda + d_a \leq z \leq (n+1)\Lambda,
\end{cases}
\]

where \( d_a, d_b \) are the thickness of the A and B layers of the unit cell, \( \Lambda = d_a + d_b \) is the lattice constant of the PC, and \( \delta(z,t) = \delta(z,t + T) \) with \( T \) being the modulation period. We first consider the modulation function \( \delta(z,t) \) is constant of \( z \). For the simplest case, we let \( \delta(z,t) = \delta_c \cos(\Omega t) \), where \( \Omega = 2\pi/T \).

For the sake of mathematical simplicity, we set the permittivity \( \varepsilon_0 \), permeability \( \mu_0 \) and light speed \( c \) in vacuum as unity. Also we consider all the materials are nonmagnetic, namely the relative permeability is 1.
Without loss of generality, let us electric field to be polarized along the x direction. According to the Maxwell’s equations, we can obtain the Schrödinger-like equation as

\[ A \psi = i \partial_t [(B + \tilde{B}) \psi], \]

where

\[ A = \begin{pmatrix} 0 & -i \partial_z \\ -i \partial_z & 0 \end{pmatrix}, \quad B_0 = \begin{pmatrix} \varepsilon_z(z) & 0 \\ 0 & 1 \end{pmatrix}, \quad B_t = \begin{pmatrix} \varepsilon_m(z, t) & 0 \\ 0 & 0 \end{pmatrix}, \quad \psi = \begin{pmatrix} E_x \\ H_y \end{pmatrix}. \]

For the Bloch wavenumber \( q \), the time-Floquet solution can be written as \[ \tilde{\psi} = e^{i q z} \tilde{\varphi}(z, t) \] where \( Q \) is the quasi-energy and \( \tilde{\varphi}(z, t) = \varphi(z, t + T) \) is the time-periodic function. We use the eigenmodes of the static bands (\( \delta = 0 \)) to expand \( \tilde{\varphi}_q \) as

\[ \tilde{\varphi}_q(z, t) = \sum_{j = -\infty}^{\infty} \sum_{m = -\infty}^{\infty} c_m[j] e^{i m \Omega t} = \sum_{j = -\infty}^{\infty} \sum_{m = -\infty}^{\infty} c_m[j, m] q, \]

where \( c_m \) is the expansion coefficient, \( [j]_q \) is the normalized eigenmode of the \( j \)th band in the static case, \( m \) is the order of the frequency and \( [j, m]_q = [j]_q e^{i m \Omega t} \). Substituting equation (4) into equation (2), we obtain

\[ A e^{i q z - i q t} \tilde{\varphi}_q = i \partial_t [(B_0 + B_t) e^{i q z - i q t} \tilde{\varphi}_q] = \sum_{j = -\infty}^{\infty} \sum_{m = -\infty}^{\infty} c_m[j] e^{i m \Omega t} \]

\[ \times \left(\begin{array} {c} \varepsilon_z(z) \\ 0 \end{array} \right) \left(\begin{array} {c} \varepsilon_m(z, t) \\ 0 \end{array} \right) + Qe^{i q z - i q t} (B_0 + B_t) \tilde{\varphi}_q. \]

Eliminating \( e^{i q z - i q t} \) on both sides and using equation (4), equation (5) becomes

\[ A_q \sum_{j = -\infty}^{\infty} \sum_{m = -\infty}^{\infty} c_m[j] e^{i m \Omega t} - i \partial_t \left[ \sum_{j = -\infty}^{\infty} \sum_{m = -\infty}^{\infty} c_m[j] e^{i m \Omega t} \left(\begin{array} {c} \varepsilon_z(z) \\ 0 \end{array} \right) \left(\begin{array} {c} \varepsilon_m(z, t) \\ 0 \end{array} \right) \right] \]

\[ = Q \left( B_0 + B_t e^{i \Omega t} + B_t e^{-i \Omega t} \right) \sum_{j = -\infty}^{\infty} \sum_{m = -\infty}^{\infty} c_m[j] e^{i m \Omega t}, \]

where

\[ A_q = e^{-i q z + i Q t} A e^{i q z - i \Omega t}, \quad B_t = \begin{pmatrix} \varepsilon_m(z) & 0 \\ 0 & 0 \end{pmatrix}. \]

Inner product \( [j, n]_q = [j]_q e^{-i n \Omega t} \) from left on both sides and using the orthonormality of the static eigenmodes \( \langle j | A_q j \rangle = \omega_j \delta_{jj}, \langle j | B_0 j \rangle = \delta_{jj} \), we obtain

\[ \sum_{j = -\infty}^{\infty} \sum_{m = -\infty}^{\infty} c_m[j] \omega_j \delta_{jj} \delta_{nn} + n \Omega (\delta_{nn+1} + \delta_{nn-1}) \Delta_j \]

\[ = Q \sum_{j = -\infty}^{\infty} \sum_{m = -\infty}^{\infty} c_m[j] \delta_{jj} \delta_{nn} + (\delta_{nn+1} + \delta_{nn-1}) \Delta_j], \]

where \( \omega_j \) is the eigenfrequency of the \( j \)th band in the static case, \( \delta_{jj} \) is the Kronecker delta function, and \( \Delta_j = \langle j | B_j j \rangle \). In the matrix form, equation (8) is rewritten as

\[ H_1 \phi = Q H_2 \phi, \]

where
where \( \mathbf{\phi} = (\ldots, c_j-1, c_j0, c_j1, \ldots)^T \), and

\[
\hat{H}_1 = \begin{pmatrix}
-\Omega \Delta & \hat{H}_0 - \Omega I & -\Omega \Delta \\
0 & \hat{H}_0 & 0 \\
\Omega \Delta & \hat{H}_0 + \Omega I & \Omega \Delta \\
\end{pmatrix},
\]

\[
\hat{H}_2 = \begin{pmatrix}
\Delta & I & \Delta \\
\Delta & I & \Delta \\
\Delta & I & \Delta \\
\end{pmatrix}.
\]

In equation (10), \( H_{\theta \theta} = \omega \delta \theta \) and \( I \) is the identity matrix. Then the quasi-energy \( Q \) is obtained as the eigenvalues of the Floquet Hamiltonian \( \hat{H}_F = \hat{H}_2^{-1} \cdot \hat{H}_1 \).

In the static case, the bands are symmetric because time-reversal symmetry is preserved, namely \( \omega_j(q) = \omega_j(-q) \). If we choose the center of the A or B layer as the origin, then the static PC possesses inversion symmetry, which leads to \( \varepsilon(z) = \varepsilon(-z) \). Assume the normalized electromagnetic fields of the static mode \( \omega = \varepsilon(z) \) are \( E_\varepsilon(z), H_j(z) \), and note that the electric field is a vector while the magnetic field is a pseudovector, the normalized electromagnetic fields of the static mode \( \omega = \varepsilon(z) \) are then obtained as \( \hat{O}_j E_\varepsilon(z) = -\hat{E}_x(-z), \hat{O}_j H_j(z) = \hat{H}_j(-z) \), where \( \hat{O}_j \) is the inversion operator. Then \( \Delta \theta \) at \( -q \) is calculated as

\[
\Delta \theta(-q) = \delta \int_0^\Lambda [-\hat{E}_x^{(0)}(z)] [-\hat{E}_x^{(0)}(z)] \varepsilon(z) \, dz
= -\delta \int_0^\Lambda \hat{E}_x^{(0)}(z) \hat{E}_x^{(0)}(z) \varepsilon(-z) \, dz
= \delta \int_0^\Lambda \hat{E}_x^{(0)}(z) e^{i\varepsilon \Lambda} \hat{E}_x^{(0)}(z') e^{-i\varepsilon \Lambda} \varepsilon(z') \, dz' = \Delta \theta(q).
\]

In equation (11), we have used the Bloch theorem \( E_\varepsilon(z + \Lambda) = E_\varepsilon(z) e^{i\varepsilon \Lambda} \). Therefore, when the wavenumber is changed from \( q \) to \( -q \), the block matrices \( \hat{H}_0 \) and \( \hat{\Delta} \) are invariant, and so is the Floquet Hamiltonian \( \hat{H}_F \). This guarantees that the calculated quasienergy bands are symmetric, i.e. \( Q(q) = Q(-q) \).

### 3. Coupling between the positive and positive bands

Consider that \( \varepsilon_a = 1, \varepsilon_b = 100, d_a = 0.9A, d_b = 0.1A \). The static band dispersion and the corresponding eigenmodes of the PC can be calculated using the transfer matrix method [46], see details in the appendix. Here, we choose such a large contrast on the permittivities and filling ratios in order to construct some relatively isolated bands, and then the evolution of the bands is much clearer. The band diagram in the static case is shown in figure 2(a), where the positive (\( \omega > 0 \)) and negative (\( \omega < 0 \)) bands are represented by the black and red solid lines, respectively. The band index is indicated on each band. Note that the bands \( \pm n \) are mirror symmetric to each other with respect to the zero frequency line.

When the time-modulation is introduced, in the limit of vanishing \( \delta \), the Floquet bands can be obtained by copying the static bands and shifting them up and down in the quasienergy space by integer multiples of \( \Omega \). Here, we call the bands obtained by shifting the static bands up (down) by \( n\Omega \) the bands of \( n \)th (\( -n \)th) order. And we call the bands from shifting the positive (negative) static bands also the positive (negative) bands. In figure 2(b), only the 0th, 1st and \( -1 \)st order bands are shown, which are represented by the black, red and blue solid lines, respectively, and the positive and negative bands are represented by the solid and dashed lines. Because the quasienergy bands are symmetric as we have proved in previous section, we only plotted the bands for \( q > 0 \). The bands of different orders can cross each other and form degenerate points which are noted as A–F in figure 2(b). Points A, B, D, and E are formed by bands differing by an order 1, while points C and F are formed by bands differing by an order 2.

For a finite \( \delta \), the Floquet bands can be numerically calculated using equation (9) by truncating the band order and the band index to finite numbers. Because of the nonzero matrix \( \Delta \), there are couplings...
between the bands of different orders, which will lift the degenerate points to generate quasienergy gaps, EPs or a mixture of the two. From equations (9) and (10), we can find that the coupling are vanishing small when the order difference of the two bands is greater than 1.

In the vicinity of the degenerate point formed by bands $|j, 1\rangle$ and $|l, 0\rangle$ (point A for example), the two bands are far away from the rest (the bands of higher orders, not shown in figure 2(b), may be close to them, however, the coupling strength between the higher order band and either of the two bands is negligible. Hence, the evolution of the two bands near the degenerate point can be well described by a two-band model that involves these two bands only. According to equations (9) and (10), the reduced Hamiltonian for this two-band model is expressed as:

$$H_{\text{red}} = \begin{pmatrix} \omega_l & 0 \\ \Omega \Delta_{jl} & \omega_j + \Omega \end{pmatrix}^{-1} \begin{pmatrix} \omega_l - \Omega \Delta_{jl} & -\Delta_{jl}(\omega_l + \Omega) \\ -\Delta_{jl}(\omega_j - \Omega) & \omega_j + \Omega \end{pmatrix}. \quad (12)$$

Note that $\omega_j + \Omega \approx \omega_j$, the quasienergies determined by $H_{\text{red}}$ are calculated as

$$Q_{\pm} = \frac{1}{2(1 - \Delta_{jl}\Delta_j)} [2\omega_l + \Delta_{jl}\Delta_j(\omega_j - \omega_l) \pm \sqrt{\Delta_{jl}\Delta_j(\omega_j - \omega_l)^2 + 4\omega_l\omega_j}]. \quad (13)$$

Since $|\Delta_{jl}\Delta_j| \ll 1$ when $\delta$ is small,

$$Q_{\pm} \approx \omega_l \mp \sqrt{\omega_l\omega_j}, \quad (14)$$

When $\omega_l$ and $\omega_j$ have the same sign, $\sqrt{\omega_l\omega_j}$ is real. For pure real $\delta$, $B_\delta$ is Hermitian, then

$$\Delta_{jl} = \langle l | B_\delta^* j \rangle = \langle j | B_\delta^* l \rangle^* = \Delta_{jl}^*.$$ Thus $\sqrt{\Delta_{jl}\Delta_j} = |\Delta_{jl}|$ is positive real. Therefore, there is a quasienergy gap with gap size given by

$$\delta Q = Q_+ - Q_- \approx 2|\Delta_{jl}|\sqrt{\omega_l\omega_j}. \quad (15)$$

In figure 3(a), we plotted the quasienergy bands evolved from the bands $|2, 0\rangle$, $|2, 1\rangle$, $|3, -1\rangle$, $|3, 0\rangle$, when the time-modulated permittivity becomes $\delta = 0.1$ and the modulation frequency fulfills $\Omega\Lambda/c = 0.5$. As we can see, the degenerate points A and B are lifted and quasienergy gaps open. At point A, we numerically obtain $\omega_2\Lambda/c \approx 3.06, \omega_3\Lambda/c \approx 3.56, \Delta_{23} \approx -1.91 \times 10^{-2} + 0.587 \times 10^{-2}i$ using equations (A1)–(A8). According to equation (15), the gap size lifted from point A is approximated as $\delta Q\Lambda/c \approx 0.12$, which is very close to the result 0.11 obtained from the full Floquet Hamiltonian $H_\delta$ (see figure 3(a)). There is also a band gap opening at the point C, as shown in figure 3(b). However, because the coupling between the bands differing by an order 2 is much weaker, the gap size is several orders smaller.

To verify our results, we introduce the numerical simulations based on the finite-difference time-domain (FDTD) method. In the simulations, we consider that the 1D time-Floquet PC contains 40 periods and each
Figure 3. (a) The quasienergy bands for $\delta = 0.1$ and $\Omega \Lambda / c = 0.5$. Here, we only showed the bands evolved from $|2, 0\rangle$, $|2, 1\rangle$, $|3, -1\rangle$ and $|3, 0\rangle$. The bands in the static case are shown by the blue dashed semi-transparent lines. The red circles are determined by picking out the positions of the peaks of $|h(q, \omega)|$ as shown in figure 4. (b) The zoomed-in view of a piece of band in (a) (as marked by the blue dotted rectangle).

Figure 4. The function $|h(q, \omega)|$ defined in equation (16) is plotted as a function of frequency $\omega$ for $q \Lambda / c = 0.2$. In simulations, $t_1 = 175\Lambda / c$ and $t_2 = 250\Lambda / c$. The four peaks marked by the arrows denote the four Floquet modes.

period of the PC is uniformly divided into 200 grids. The time interval is $\Delta t = \Delta x / 2c$, where $\Delta x = \Lambda / 200$ is the length of the grid. A pulsed line source of the form exp$[-0.5(t - t_0)^2/\tau^2]$ $\delta(z - z_0)$ is considered, where $z_0 < 0$ is the coordinate of the line source, and $t_0 = \tau = 10000\Delta t = 25\Lambda / c$. The complex amplitude of the plane wave inside the PC for a specific wavenumber $q$ and frequency $\omega$ can be obtained by the Fourier transform

$$h(q, \omega) = \frac{1}{(t_2 - t_1)L} \int_{t_1}^{t_2} dt e^{i\omega t} \int_0^L E_z(z, t) e^{-i q z} dz,$$

where $z = 0$ and $z = L = 40\Lambda$ is the left and right ends of the PC, $t_1$ and $t_2$ are the starting and ending time for the measurement.

In figure 4, we plotted $|h(q, \omega)|$ as a function of the frequency $\omega$ which ranges from $2.5c/\Lambda$ to $4c/\Lambda$ for a given wavenumber $q = 0.2\pi / \Lambda$. As we can see four peaks (marked by the red arrows) are shown, which correspond to the frequencies of four Floquet modes of the PC. For different Bloch wavenumbers, we found out the frequencies of the Floquet modes by picking up the peak positions of $|h(q, \omega)|$ and showed the results in figure 3(a) by the red circles. It is seen that the results by the Floquet Hamiltonian method (lines) agree with the FDTD simulation results (circles) very well, indicating the accuracy of our Floquet Hamiltonian method.

When $\delta$ is purely imaginary, $\tilde{B}_r$ is anti-Hermitian, leading to $\Delta \mathbf{q} = \langle \tilde{B}_r | \mathbf{j} \rangle = - \langle \tilde{B}_r | \mathbf{j} \rangle = - \langle \mathbf{j} | \tilde{B}_r \rangle^\dagger = - \Delta \mathbf{q}^\dagger$. Therefore, the quasienergies in the vicinity of the degenerate point become a complex conjugate...
pair in between a pair of EPs. According to equation (13), the EPs are located at

\[ \Delta_0 \Delta_\nu (\omega_j - \omega_l)^2 + 4\omega_j \omega_l = -|\Delta_0|^2 (\omega_j - \omega_l)^2 + 4\omega_j \omega_l = 0. \] (17)

Figure 5 shows the quasienergy bands near the degenerate points A and B when \( \delta = 0.1i \). The bands near the degenerate points coalesce and form a pair of EPs, which are noted as \((K_1, K_2)\) and \((K_3, K_4)\). From equation (17) we see that the wavenumbers for the pair of EPs depend on the frequencies \( \omega_l, \omega_j \) and the \( \Delta_0 \). Thus although the degenerate points A and B are corresponding to the same wavenumber, the wavenumbers of the two pairs of EPs spawned from the two degenerate points are different, as shown in figure 5(b).

4. Coupling between the positive and negative bands

When \( \omega_l \) and \( \omega_j \) have opposite signs, \( \sqrt{\omega_j/\omega_l} \) becomes pure imaginary. If \( \delta \) is pure real, since \( \Delta_0 = \Delta_\nu \), according to equation (13), the quasienergies become a complex conjugate pair in between a pair of EPs.

In figure 6, we showed the quasienergy bands in the vicinity of the degenerate point D by the black circles when \( \delta = 0.2 \) and \( \Omega \Lambda/c = 0.5 \). A pair of EPs form at \( Q = \Omega/2 \). This is very similar to the case of a time-modulated homogeneous medium [35]. Since \( \omega_j = -\omega_l \) and \( |\Delta_{l-1}| < 1 \), according to equation (12), the quasienergies near the point D are given by

\[ Q_n \approx \frac{\Omega}{2} \pm \frac{i}{2} \sqrt{|\Delta_{l-1}|^2 \Omega^2 - (\Omega - 2\omega_l)^2}. \] (18)

Therefore, the EPs are located near \( \omega_1 = \frac{\Omega}{2}(1 \pm |\Delta_{l-1}|) \). Using equations (A1)–(A8), we obtained \( |\Delta_{l-1}| \approx 0.0076 \). Then approximately, the wavenumbers for the two EPs are corresponding to \( 2\omega_1/\Omega = 0.9924 \) and \( 2\omega_3/\Omega = 1.0076 \) of the static band, which are marked by the red arrows in figure 6(a) and very close to the wavenumbers of the two EPs. The positive (negative) value of the imaginary part of the quasienergy denotes the amplification (decay) rate of the fields in time [37]. From equation (18), we see that the maximum imaginary part of the quasienergy is about \( \frac{\Omega}{2} |\Delta_{l-1}| \Omega \approx 0.0019i/c \Lambda \), agrees with the results as shown in figure 6(b).

When the modulation frequency is \( \Omega \Lambda/c = 4, 3, 0 \) and \(-1, 1\) intersects, see figure 7(a). If \( \delta \) is pure real, the bands near the degenerate point also coalesce and form a pair of EPs, as shown in figures 7(b) and (c). However, the EPs are no longer fixed to \( Q = \pm \Omega/2 \). According to equation (13), the locations of the EPs are determined by

\[ |\Delta_{l-1}|^2 (\omega_3 - \omega_{l-1})^2 + 4\omega_3 \omega_{l-1} = |\Delta_{l-1}|^2 (\omega_3 + \omega_{l-1})^2 - 4\omega_3 \omega_{l-1} = 0. \] (19)

To show the field amplification behavior incurred by the positive imaginary part of the quasienergy in the broken phase region between the pair of EPs, we numerically studied the transmission field of the PC using the FDTD method. Similarly, the PC has 40 periods with left and right ends at \( z = 0 \) and \( z = L \), and the pulsed line source has the same form \( \exp[-0.5(t - t_0)/\tau^2] \delta(z - z_0) \) with \( t_0 = \tau = 25 \Lambda/c \). The complex

![Figure 5](image-url)  
Figure 5. (a) The real and (b) imaginary parts of the quasienergy bands for \( \delta = 0.1i \) and \( \Omega \Lambda/c = 0.5 \). The red and green dots denote the two pairs of EPs. The bands in the static case are shown by the blue dashed semi-transparent lines.
amplitude of the transmission field for a specific frequency $\omega$ in a time period $t_1 \sim t_2$ can be calculated according to

$$f(\omega) = \frac{1}{(t_2 - t_1)} \int_{t_1}^{t_2} E_i(z_1, t)e^{i\omega t} dt,$$

(20)

where $z_1 > L$ is the coordinate of the arbitrarily chosen measuring point. For $t_1 = 425\Lambda/c, t_2 = 675\Lambda/c$ and $t_1 = 475\Lambda/c, t_2 = 725\Lambda/c$, $|f(\omega)|$ as functions of the frequency $\omega$ are shown by the black and blue solid lines in figure 8, respectively. In the range of $-1.5c/\Lambda \leq \omega \leq 4.0c/\Lambda$, there are three peaks for each line which are located at $\omega = \pm 0.329c/\Lambda$ and $\omega = 3.677c/\Lambda$. From figure 7, we see that these frequencies are just the quasienergies of three pairs of EPs. The first and second pairs are generated from the upper and lower degenerate points as shown in figure 7(a), and the third pair is mirror symmetric to the second pair with respect to zero quasienergy line. The amplitudes of the incident field for $|\omega| > 0.2c/\Lambda$ is much less than $10^{-3}$. Therefore, the fields of the three frequencies are greatly amplified when passing through the 1D Floquet PC. By comparing the black and blue solid lines, we can find the fields are amplified as time grows.

When $\delta$ is purely imaginary, it is easy to know that the degenerate points formed by the bands differing by an order 1 will be lifted to open quasienergy gaps. To realize a time-modulation on the imaginary part of the permittivity, we can use the absorbing materials evenly mixed with gain materials, such as phosphors, and temporally changing the gain effect (for example, temporally tuning the intensity of the pumping light) or temporally alternating the intrinsic loss of the absorbing material (for example, using graphene as the absorbing material and modulating its conductivity via electro-optic modulation [36]). In figure 9(a), we showed the quasienergy bands in the vicinity of point D when $\delta = 0.2i$, and a quasienergy gap is clearly

Figure 6. (a) The black circles represent the real part of the quasienergy bands and the blue solid and dashed lines denote the static bands. (b) The imaginary parts of the quasienergy bands. Here $\delta = 0.2, \Omega\Lambda/c = 0.5$.

Figure 7. (a) The quasienergy bands in the limit of $\delta \to 0$ for $\Omega\Lambda/c = 4$. (b) The real and (c) the imaginary parts of a piece of quasienergy bands (marked by the green dashed rectangle in (a)) when $\delta = 0.1$. 

$$Q\Lambda/c = 4$$

$$\Omega\Lambda/c = 4$$
Figure 8. The function $|f(\omega)|$ defined in equation (20) as function of the frequency $\omega$. The peaks marked by the arrows show the frequencies of the amplified fields. As we can see that the two frequencies of the peaks are just corresponding to the frequencies of the degenerate points in figure 7 as labeled by the black arrows.

Figure 9. The quasienergy gaps for a purely imaginary $\delta$ (a) and a complex valued $\delta$ (b). The blue dashed and semi-transparent lines denote the bands in the static case ($\delta = 0$). In (b), only the real part of the quasienergy is shown.

5. Asymmetric band structures

As discussed previously, the quasienergy bands are symmetric when the time-modulation function is in the form of $\delta(z, t) = \delta g(t)$. To achieve asymmetric band structures, we now consider that $\delta(z, t)$ depends on both $t$ and $z$. For the simplest case, we assume $\delta(z, t) = \delta \cos(\Omega t - \beta z)$, where $\beta$ is the spatial modulation frequency. Then the modulation speed is defined as $c_m = \Omega / \beta$, which is in units of $c$. For convenience, we define a new variable $u = t - z/c_m$, and use the following transformations,

$$ (z, u) = (z, t - z/c_m), (\partial_z, \partial_t) = (\partial_z - \partial_u/c_m, \partial_u). $$
The time-Floquet solution is then expressed as

\[
\tilde{\psi}_q = e^{i\omega t-q^*} \tilde{\varphi}_q = e^{i\omega t-q^*} \sum_{j=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} c_{jm}(j, m)^{(u)} = e^{i\omega t-q^*} \sum_{j=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} c_{jm}(j, m)^{(u)} e^{im\Omega u},
\]  

(22)

where \( \tilde{q} = q - q' = q - Q/c_m \). Substituting equations (21) and (22) into equation (2), we arrive at

\[
A\tilde{\psi} + C\tilde{\varphi} = i\partial_u [(B_0 + B_1)\tilde{\varphi}],
\]  

(23)

where \( A, B_0, B_1 \) are defined in equation (3), and

\[
C = \frac{1}{c_m} \begin{pmatrix} 0 & i\partial_u \\ i\partial_u & 0 \end{pmatrix}
\]  

(24)

Then equation (23) is rewritten as

\[
Ae^{i\omega t-q^*}q^+ + Ce^{i\omega t-q^*}q^+ = e^{i\omega t-q^*}Q[(B_0 + B_1)\tilde{\varphi}] + e^{i\omega t-q^*}i\partial_u [(B_0 + B_1)\tilde{\varphi}].
\]  

(25)

Multiply \( e^{-i\omega t+iu} \) and inner product \( \langle l, n \rangle_q^u = \langle l | e^{-iu}\rangle_q^u \) from left, and note that

\[
\langle l, n \rangle_q^u e^{-i\omega t}Ae^{-i\omega t}j, m \rangle_q^u = \langle l, n \rangle_q^u e^{-i\omega t}Ae^{-i\omega t}j, m \rangle_q^u + Q/c_m \langle l, \sigma_x, j \rangle_q^u \Delta_{mn} = Q/c_m \Gamma_{j\delta mn},
\]  

(26)

where

\[
\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \Gamma_{j\delta mn} = \langle l, \sigma_x, j \rangle_q^u 
\]  

then we obtain

\[
\sum_{j=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} c_{jm}[\omega\delta_{j\delta mn} - n\beta\delta_{j\delta mn}\Gamma_{j\delta mn} + n\Omega(\delta_{j\delta mn} + \delta_{mn+1}\Delta_{j} + \delta_{mn-1}\Delta_{j})]
\]  

\[
\sum_{j=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} c_{jm}(\delta_{j\delta mn} + \delta_{mn+1}\Delta_{j} + \delta_{mn-1}\Delta_{j}).
\]  

(27)

The concrete formulas for calculating \( \Gamma_{j\delta mn} \) can be obtained by using the transfer matrix method, see equation (A6), (A9) and (A11) in the appendix. In a matrix form, equation (27) is written as

\[
\tilde{H}_t\tilde{\phi} = QH_2\tilde{\phi},
\]  

(28)

where \( H_2 \) is defined in equation (10), and

\[
\tilde{H}_t = \begin{pmatrix}
\ddots & \ddots & \ddots & \ddots \\
-\Omega^\Delta & H_0 + \beta \Gamma - \Omega I & -\Omega^\Delta & \ddots \\
-\Omega^\Delta & H_0 & -\Omega^\Delta & \ddots \\
\Omega^\Delta & \Omega^\Delta & H_0 - \beta \Gamma + \Omega I & \Omega^\Delta \\
\ddots & \ddots & \ddots & \ddots
\end{pmatrix}.
\]  

(29)

The quasienergies are obtained as the eigenvalues of the Floquet Hamiltonian \( \tilde{H}_F = H_z^{-1} \cdot \tilde{H}_t \).

Similarly, the normalized electromagnetic fields of the static modes \( (\omega, q) \) and \( (\omega, -q) \) can be given by
asymmetric and the gap sizes in the degenerate point formed by bands due to the simultaneous broken of both the time-reversal and inversion symmetries. In fact, when there is no inversion center that satisfies \( \varepsilon(z - z_c, t) = \varepsilon(z_c - z, t) \). Therefore, the asymmetric band structure is due to the simultaneous broken of both the time-reversal and inversion symmetries.

In figures 10(a) and (b), we showed the quasienergy bands for \( \beta = 0 \) and \( \beta = \Omega \), respectively, considering the coupling between the positive and positive bands. When \( \beta = 0 \), the quasienergy bands are symmetric and band gaps open symmetrically about \( q = 0 \). However, when \( \beta \neq 0 \), the bands become asymmetric and the gap sizes in the \( q < 0 \) and \( q > 0 \) regions are unequal, see figure 10(b).

According to equation (29), the reduced Hamiltonian for the quasienergy bands in the vicinity of the degenerate point formed by bands \( |j, 1 \rangle \) and \( |l, 0 \rangle \) is expressed as

\[
\tilde{H}_{\text{red}} = \begin{pmatrix} 1 & \Delta_{\beta}^{-1} \omega_j \\ \Omega \Delta_{\beta} & \Omega \omega_j + \Omega - \beta \Gamma_j \end{pmatrix}.
\] (31)

For a small \( \delta \), note that \( \Delta_{\beta} = \Delta_{\beta\delta} \), the eigenvalues of \( \tilde{H}_{\text{red}} \) read

\[
Q_{\pm} \approx \frac{1}{2(1 - |\Delta_{\beta}|^2)} [(\Omega + \omega_j + \omega_l - \beta \Gamma_j) \pm \sqrt{(\Omega + \omega_j - \omega_l - \beta \Gamma_j)^2 + 2\chi |\Delta_{\beta}|^2}],
\] (32)

where

\[
\chi = \Omega(\omega_l - \omega_j) + 2\omega_j \omega_l + \beta \Gamma_j (\Omega - 2\omega_j) - \Omega^2.
\] (33)

From equation (30), we obtain

\[
\Gamma_{\beta} = -4 \int_0^\Lambda S^{(0)}_{\beta} \, dz,
\] (34)

where \( S^{(0)}_{\beta} = \frac{1}{\Omega} \text{Re}(E^{(0)}_{\beta} \tilde{H}^{(0)}_{\beta}) \) is the Poynting vector of the mode of the static band \( j \). Therefore, \( \Gamma_{\beta} \) is definitely real. When \( \omega_j \) is positive, consider that the energy flux is always parallel to the wave vector, \( \Gamma_{\beta} \) is negative for \( q > 0 \) and positive for \( q < 0 \). Therefore, according to equation (32), for the same band \( Q(q) > Q(-q) \) when \( q > 0 \), which agrees with the numerical results as shown in figure 10(b).

According to equation (32), for a small \( \delta \) (so that \( |\Delta_{\beta}| \) is small), the quasienergy gap opens near the point that fulfills

\[
\omega_j(q) - \omega_j(q) + \Omega - \beta \Gamma_j(q) \approx 0.
\] (35)
Substituting equation (35) into equation (32), the quasienergies are approximated as

\[ q = \pm \frac{\beta}{\omega } \approx -0.14 \pi / \Lambda \text{ and } q = 0.295 \pi / \Lambda, \] respectively, which are very close to the points marked by the red arrows in figure 10(b).

For bands [2, 1] and [3, 0], by calculation, this condition is fulfilled at \( q = -0.14 \pi / \Lambda \) and \( q = 0.295 \pi / \Lambda \), respectively, which are very close to the points marked by the red arrows in figure 10(b).

Substituting equation (35) into equation (32), the quasienergies are approximated as

\[ Q_{\pm} \approx \frac{1}{1 - |\Delta_j|^2} [\omega_j \pm \sqrt{\omega_j (\omega_j - \Omega) |\Delta_j|}]. \tag{36} \]

When both \( \omega_j \) and \( \omega_j \) are positive, \( \omega_j - \Omega \approx \omega_j > 0 \), ensuring the quasienergy to be real. From equation (36), we find the gap size is approximated as \( 2 \sqrt{\omega_j (\omega_j - \Omega) |\Delta_j|} \).

We also investigated the quasienergy bands due to the coupling between the band 1 and the band \(-1\). For \( \beta = 5\Omega \) and \( \beta = 0.5\Omega \), the quasienergy bands are shown in figures 11(a) and (b), respectively. Here we used \( \Omega \lambda / c = 0.5 \) and \( \delta = 0.5 \). The quasienergy bands of the both cases are asymmetric. And similar to the case of a space-time modulated homogeneous medium [37], quasienergy gaps will open for the small modulation speed, see figure 11(a), while pairs of EPs will form for the high modulation speed, see figure 11(b).

When \( \omega_l > 0, \omega_j < 0 \), the quasienergy become complex when \( \omega_l - \Omega < 0 \), according to equation (36). Because \( \partial \omega / \partial q (q < 0) > 0 \) (the group velocity is positive), the energy flux is positive for a negative wavenumber \( q \). Thus, \( \Gamma_j (q < 0) < 0 \) according to equation (34). When \( \beta \) is very large, \( \omega_l - \Omega \approx \omega_j - \beta \Gamma_j \approx -\beta \Gamma_j > 0 \) for \( q < 0 \). Then the quasienergy bands are always real. However, when \( \beta \) is small, \( \omega_l - \Omega \approx \omega_j - \beta \Gamma_j \approx \omega_j < 0 \), making the bands near the originally degenerate point complex.

6. Conclusions

In summary, we studied the quasienergy bands of the 1D time-Floquet PC which contains two alternating layers with one layer static and the other time-periodically modulated on the permittivity, using the Floquet Hamiltonian derived from the Maxwell’s equations based on the time-dependent perturbation theory. The permittivities of the two layers are different even in the absence of the time-modulation. When the modulation function is a function of time \( t \) only, the permittivity of the PC as a whole medium is space-time-modulated. However, the quasienergy bands are still symmetric because inversion symmetry is still preserved. For a small modulation strength, we used the \( 2 \times 2 \) reduced Hamiltonian to analyze the evolution of the quasienergy bands in the vicinity of the degenerate points formed by bands differing by an order 1. It is shown that when the time-modulation is on the real part of the permittivity, the degenerate point formed by two positive bands is lifted to open a quasienergy gap, while that formed by a positive and a negative bands spawns into a pair of EPs.

We also investigated the case that \( t \) and \( z \) are not separable variables in the modulation function. We showed that the quasienergy bands are asymmetric when inversion symmetry is broken. When the modulated permittivity is pure real, the coupling between the positive and positive bands still results in quasienergy bands. However, the coupling between the positive and negative bands can lead to quasienergy gaps for a small modulation speed and EP pairs for a high modulation speed, which is similar to the case of a homogeneous medium with permittivity uniformly space-time-modulated according to a sine/cosine function.

Figure 11. (a) The quasienergy bands for (a) \( \beta = 5\Omega \) and (b) \( \beta = 0.5\Omega \). In (b), only the real part of the quasienergy bands is shown. Other parameters: \( \Omega \lambda / c = 0.5, \delta = 0.5 \).
Comparing with the homogeneous medium with permittivity uniformly space-time-modulated [15–18, 35–37, 40, 43], the quasienergy bands of the one-dimensional time-Floquet PC we investigated here are much more complicated and contain more abundant physics. Different from the homogeneous medium which has only one positive and one negative bands, the one-dimensional PC possesses an infinite number of positive and negative bands. Therefore, the interactions can take place not only between the positive and negative bands but also between the positive and positive bands, leading to the coexistence of quasienergy gaps and EPs in a single band diagram. Moreover, we see that the interaction between band $-1$ and band $1$ of the one-dimensional photonic crystals is very similar to the interaction between the positive and negative bands of the homogeneous medium, whether the modulation function is a function of time only, or the modulation function is a function of both time and space, such as the EPs form at $Q = \pm \Omega/2$ when $\beta = 0$ and EPs form for a high modulation speed $\Omega/\beta$ while quasienergy gaps generate for a small modulation speed.

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**Data availability statement**

The data generated and/or analysed during the current study are not publicly available for legal/ethical reasons but are available from the corresponding author on reasonable request.

**Appendix Formulas of $\Delta_{ij}$ and $\Gamma_{ij}$**

The photonic band of AB layered lattice in the static case can be calculated using the transfer matrix method as [45]

$$\cos(q\Lambda) = \cos(k_4d_a) \cos(k_4d_b) - \frac{1}{2} \left( \frac{z_a}{z_b} + \frac{z_b}{z_a} \right) \sin(k_4d_a) \sin(k_4d_b),$$  

(A1)

where $k_i = \sqrt{\varepsilon_i \omega / c}$, $(i = a$ or $b)$ is the wavenumber in the $i$th layer, and $z_i = \sqrt{\mu_i / \varepsilon_i}$ is the impedance of the $i$th layer. For nonmagnetic materials ($\mu_i = 1$), the eigenfield is given by [43]

$$E_x(z) = \begin{cases} t_{12} e^{i k_a z} + t_{11} e^{-i k_a z}, & \text{A layer} \\ s_{11} e^{i k_b z} + s_{12} e^{-i k_b z}, & \text{B layer} \end{cases},$$  

$$H_y(z) = \begin{cases} n_a (t_{12} e^{i k_a z} - t_{11} e^{-i k_a z}), & \text{A layer} \\ n_b (s_{11} e^{i k_b z} - s_{12} e^{-i k_b z}), & \text{B layer} \end{cases},$$  

(A2)

where $n_i = \sqrt{\varepsilon_i}$, $(i = a, b)$ is the refractive index of the $i$th layer, and

$$t_{11} = e^{iq\Lambda} - e^{-ik_a d_a} \cos(k_4d_a) + \frac{i}{2} \left( \frac{n_b}{n_a} + \frac{n_a}{n_b} \right) \sin(k_4d_a),$$  

$$t_{12} = e^{-ik_b d_b} \left( \frac{n_b}{n_a} - \frac{n_a}{n_b} \right) \sin(k_4d_b),$$  

$$s_{11} = e^{ik_b d_b} - e^{-ik_b d_b},$$  

$$s_{12} = e^{ik_b d_b} - e^{-ik_b d_b},$$  

(A3)

According to the definition,

$$\Delta_{ij} = \left< |B_{i,j}|^2 \right> = \frac{\delta \vartheta}{\sqrt{\vartheta(t_{ij})}},$$  

(A4)

where

$$\vartheta = \int_0^\Lambda E_x^{(i)} E_x^{(i)} dz, t_i = \int_0^\Lambda E_x^{(i)} \varepsilon(z) E_x^{(i)} dz + \int_0^\Lambda H_y^{(i)} \mu(z) H_y^{(i)} dz,$$

(A5)
with $E_{q}^{(i)}$, $H_{p}^{(i)}$ being the electromagnetic fields of the $i$th static band. Substituting equation (A2) into equation (A5), we obtain

$$\epsilon_{i} = \epsilon_{o}(\epsilon_{a}^{(i)} + \epsilon_{b}^{(i)}) + \epsilon_{a}(\epsilon_{a}^{(i)} + \epsilon_{b}^{(i)})$$

(A6)

where

$$\epsilon_{a}^{(i)} = d_{a}(|t_{11}^{(i)}|^{2} + |t_{12}^{(i)}|^{2}) + \frac{2}{k_{a}} \sin(k_{a}d_{v})Re(t_{12}^{(i)}e^{ik_{a}(d_{v} + \Lambda)})$$

$$\epsilon_{b}^{(i)} = d_{b}(|s_{11}^{(i)}|^{2} + |s_{12}^{(i)}|^{2}) + \frac{2}{k_{b}} \sin(k_{b}d_{v})Re(s_{12}^{(i)}e^{ik_{b}(d_{v} + \Lambda)})$$

(A7)

$$\epsilon_{a}^{(i)} = d_{a}(|t_{11}^{(i)}|^{2} + |t_{12}^{(i)}|^{2}) - \frac{2}{k_{a}} \sin(k_{a}d_{v})Re(t_{12}^{(i)}e^{ik_{a}(d_{v} + \Lambda)})$$

$$\epsilon_{b}^{(i)} = d_{b}(|s_{11}^{(i)}|^{2} + |s_{12}^{(i)}|^{2}) - \frac{2}{k_{b}} \sin(k_{b}d_{v})Re(s_{12}^{(i)}e^{ik_{b}(d_{v} + \Lambda)})$$

and

$$\vartheta = \frac{\tau_{1} + \tau_{2} + \tau_{3}}{(k_{a}^{(i)} - k_{b}^{(i)})(k_{a}^{(i)} + k_{b}^{(i)})}$$

$$\tau_{1} = -i(k_{a}^{(i)} - k_{b}^{(i)})[t_{11}^{(i)}(t_{11}^{(i)} - t_{12}^{(i)})(t_{11}^{(i)} + t_{12}^{(i)}) - i(k_{a}^{(i)}(t_{11}^{(i)} + t_{12}^{(i)}) + t_{12}^{(i)})$$

$$(A8)$$

$$\tau_{2} = i(k_{a}^{(i)} + k_{b}^{(i)})[t_{11}^{(i)}(t_{11}^{(i)} - t_{12}^{(i)})(t_{11}^{(i)} + t_{12}^{(i)}) - i(k_{a}^{(i)}(t_{11}^{(i)} + t_{12}^{(i)}) + t_{12}^{(i)})$$

$$\tau_{3} = i(k_{a}^{(i)} - k_{b}^{(i)})[t_{11}^{(i)}(t_{11}^{(i)} - t_{12}^{(i)})(t_{11}^{(i)} + t_{12}^{(i)}) - i(k_{a}^{(i)}(t_{11}^{(i)} + t_{12}^{(i)}) + t_{12}^{(i)})$$

Using equation (A2), we have

$$\vartheta_{n} = \int_{n}^{A} \frac{E_{q}^{(i)}H_{p}^{(i)}dz}{(k_{a}^{(i)} - k_{b}^{(i)})(k_{a}^{(i)} + k_{b}^{(i)})} = \frac{1}{(k_{a}^{(i)} - k_{b}^{(i)})(k_{a}^{(i)} + k_{b}^{(i)})}$$

(A9)

where

$$\xi_{1} = \int_{n}^{A} k_{a}^{(i)}H_{p}^{(i)}dz$$

$$\xi_{2} = \int_{n}^{A} k_{a}^{(i)}H_{p}^{(i)}dz$$

$$\xi_{3} = \int_{n}^{A} k_{a}^{(i)}H_{p}^{(i)}dz$$

(A10)

$$\xi_{4} = \int_{n}^{A} k_{a}^{(i)}H_{p}^{(i)}dz$$

Then the matrix element $\Gamma_{ij}$ is obtained as

$$\Gamma_{ij} = \frac{(\vartheta_{n} + \vartheta_{n}^{*})}{\sqrt{\nu_{i}\nu_{j}}}$$

(A11)

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