A remark for meeting times in large random regular graphs

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Abstract

We revisit the meeting times of bivariate random walks on large discrete tori studied by Cox and Spitzer in [5] and consider closely related meeting times of bivariate Markov chains on large finite sets subject to certain mild asymmetry. The main result explicitly characterizes the asymptotic distributions of these meeting times by the spectra of the transition kernels and applies to large random regular graphs of degree at least three.

Keywords: Meeting times of Markov chains, Voter models, the Kesten-McKay law

Mathematics Subject Classification (2000): 60J27, 60K35, 60F99

1 Introduction

Let $q$ be an irreducible, reversible transition kernel defined on a finite nonempty set $E$ with size $N \geq 1$, and $\pi$ be the associated unique stationary distribution. We assume in addition that $q$ has zero trace so that $\sum_x q(x,x) = 0$ and write $M_{x,y}$ for the first nonnegative time that two independent rate-1 $q$-Markov chains starting from points $x$ and $y$ meet. In this note, we study mainly the meeting time $M_{U,U'}$ for such a bivariate Markov chain in the limit of large $N$, where the initial condition $(U,U')$ is distributed as a product of stationarity.

Our particular interests in the meeting times $M_{U,U'}$ arise from the program of investigating universal diffusive limits of voter models and their perturbations on large finite sets [5, 15, 6, 7, 8]. The diffusive limits are explicitly determined up to the time scaling constants $E[M_{U,U'}]$, and their validation requires the Aldous-Brown type conditions for the closely related almost exponentiality of $M_{U,U'}/E[M_{U,U'}]$ (cf. [2, Chapter 3]). In particular, our recent investigation in [8] manages to understand a prediction in [14] for diffusion approximations of certain voter model perturbations on large random $k$-regular graphs for $k \geq 3$, since [14] uses physics methods distant from the related probability methods developed thus far. Although the drift and noise coefficients in some of the diffusions in [14] are now proven precise by [8, Theorem 4.7], a comparison of the time scaling constants for the diffusions in [14] and [8, Theorem 4.7] further leads to the question whether we have

\[ E[M_{U,U'}] \sim \frac{N(k-1)}{2(k-2)} \quad \text{as } N \to \infty. \]

(1.1)

See [8, Remark 4.10 and Question 4.11] for further details.

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The two theorems of Cox and Spitzer in [5, Theorems 4 and 7] for meeting times of random walks on large discrete tori, proven around thirty years ago, give examples where explicit limiting distributions of meeting times can be obtained. The proofs for both theorems in [5] start with the exact factorizations of Laplace transforms of meeting times into ratios of Green functions of random walks. In dimensions \( d \geq 3 \), Cox’s theorem in [5, Theorem 4] obtains the explicit asymptotic distributions of the meeting times \( M_{x,y} \) for all pairs of points \((x,y)\) with distances at least of the order \( o(N^{1/d}) \). The proof of [5, Theorem 4] studies the involved Green functions by considering the random walks run separately below the growing time thresholds \( N^{2/d} \log N^{1/d} \) to use local convergences of discrete tori in the sense of \([13, 3]\) and central-limit-theorem type estimates and above the same time thresholds to use mixing of the random walks. The more subtle case of \( d = 2 \) is also handled in that proof.

The main result of this note is Theorem 2.1. It obtains the explicit asymptotic distributions of \( M_{U,U'} \) on spatial structures subject to mild asymmetry and can be strengthened to get explicit time scaling constants \( E[M_{U,U'}] \) for the diffusion approximations mentioned above (Corollary 2.3). Corollary 2.4 then obtains an extension of the theorem of Spitzer in [5, Theorem 7] for the asymptotic distributions of the meeting times \( M_{U,V} \), where the initial condition of the underlying bivariate Markov chain is replaced by \((U, V)\) and is randomized by the distribution \( \pi(x)q(x, y) \). For these results, we work with a sequence of transition kernels \((E_n, q^{(n)})\) such that \#\(E_n = N_n \to \infty\). Among other appropriate conditions, we assume mild asymmetry of the kernels so that the return probabilities are mostly constant in the limit: for some constants \( q^{(\infty)}, \ell \),

\[
\lim_{n \to \infty} \pi^{(n)} \{ x \in E_n; q^{(n),\ell}(x, x) \neq q^{(\infty),\ell} \} = 0, \quad \forall \ell \in \mathbb{Z}_+.
\]

Large random regular graphs of degree at least three satisfy the conditions.

In the context of discrete tori, the Laplace transforms of \( M_{U,U'} \) admit simple exact solutions in terms of the spectra of the transition kernels (cf. the proof of Theorem 2.1). Moreover, for discrete tori in dimensions \( d \geq 3 \) to which Theorem 2.1 applies, Cox’s theorem provides much more detailed results since it considers all the meeting times starting from most pairs of points sufficiently apart as recalled above. On the other hand, the proof of Theorem 2.1 uses arguments different from the proof of Cox’s theorem in [5] from drawing links to Green functions of the underlying bivariate Markov chains to calculating asymptotic values of the involved Green functions. We handle the possible lack of exact factorizations of meeting-time Laplace transforms under (1.2) by the absolute continuity of the laws of the Markov chains at first meeting with respect to the stationary distributions (Proposition 2.2); then the asymptotic values of the involved Green functions are obtained by using the spectra of the kernels. The latter step allows for a greater generality of transition kernels in obtaining the asymptotics of \( M_{U,U'} \). It can circumvent the need to quantify growing thresholds between the time permitting local convergences of spatial structures and the time for enough mixing of the chains.

Theorem 2.1 is applied to explicitly evaluate the limiting distributions of \( M_{U,U'} \) on large random \( k \)-regular graphs for fixed \( k \geq 3 \). By the Kesten-McKay law \([11, 13]\) for the spectra of infinite \( k \)-regular trees to which the large random graphs converge locally to, we prove in Proposition 2.5 that the limit in (1.1) is indeed precise.

2 Approximate factorizations of the meeting times

We begin with the setup for the proof of Theorem 2.1. Let \( \{X^x; x \in E\} \) and \( \{Y^x; x \in E\} \) be independent systems of independent rate-1 Markov chains driven by a transition kernel \((E, q)\) satisfying the conditions specified in Section 1. From now on, the first meeting time \( M_{x,y} \) will be referred as the first meeting time of \( X^x \) and \( Y^y \). The initial conditions for these meeting times will be taken from the triplet \((U, U', V)\) taking values in \( E \times E \times E \) and satisfying

\[
\mathbb{P}(U = x, U' = y) \equiv \pi(x)\pi(y) \quad \text{and} \quad \mathbb{P}(U = x, V = y) \equiv \pi(x)q(x, y),
\]
and we assume that \((U, U', V)\) is independent of the two systems \(\{X^x; x \in E\}\) and \(\{Y^x; x \in E\}\). In addition, we set

\[
G_\lambda(x, y) = G_\lambda(y, x) \overset{\text{def}}{=} \int_0^\infty e^{-\lambda t} \mathbb{P}(X_t^x = Y_t^y) dt, \quad x, y \in E, \lambda > 0,
\]

and

\[
\langle f, g \rangle \overset{\text{def}}{=} \sum_{y \in E} f(y)g(y). \quad (2.1)
\]

**Theorem 2.1.** Let \((E_n, q^{(n)})\) be a sequence of irreducible reversible transition kernels with \(N_n = \#E_n \to \infty\). Assume that they have zero traces and satisfy the following three conditions:

1. For every \(\ell \in \mathbb{Z}_+\), the limit (1.2) holds for some constant \(q^{(\infty), \ell} \in \mathbb{R}_+\).
2. Writing \(\sigma(q^{(n)})\) for the spectrum of \(q^{(n)}\), we can find \(\delta \in (0, 1)\) such that
   \[
   \sigma(q^{(n)}) \setminus \{1\} \subseteq (-\delta, \delta) \text{ for all large } n \geq 1.
   \]
3. We have
   \[
   \lim_{n \to \infty} \sum_{x, y \in E^{(n)}} |\pi^{(n)}(x) - \pi^{(n)}(y)| = 0.
   \]

Then the infinite series \(\sum_{\ell} q^{(\infty), \ell}\) is convergent and, for every fixed \(\lambda_0 \in (0, \infty)\),

\[
\lim_{n \to \infty} \sup_{\lambda \in [\lambda_0, \infty)} \left| N_n \sum_{x \in E_n} \pi^{(n)}(x)^2 - E^{(n)}[e^{-\lambda M_{U, U}/N_n}] \left( \frac{1}{\lambda} + \sum_{\ell=0}^\infty \frac{q^{(\infty), \ell}}{(\lambda + 2)^{\ell+1}} \right) \right| = 0. \quad (2.2)
\]

**Proof.** First we show that the infinite series \(\sum_{\ell} q^{(\infty), \ell}\) is convergent. Note that condition (3) implies that for all small \(\varepsilon > 0\), we can find \(N_\varepsilon\) large and \(x_\varepsilon^{(n)} \in E_n\) for all \(n \geq N_\varepsilon\) such that

\[
\sum_{y \in E_n} |\pi^{(n)}(y)/\pi^{(n)}(x_\varepsilon^{(n)}) - 1| \leq \varepsilon.
\]

Hence,

\[
\limsup_{n \to \infty} \left( \frac{\pi^{(n)}_{\max}}{\pi^{(n)}_{\min}} \right) < \infty, \quad (2.3)
\]

where \(\pi^{(n)}_{\max} = \max_{x \in E_n} \pi^{(n)}(x)\) and \(\pi^{(n)}_{\min}\) is similarly defined. This implies that \(\lim_{n \to \infty} \pi^{(n)}_{\max} = 0\), and we get

\[
\sum_{\ell=0}^\infty q^{(\infty), \ell} = \sum_{\ell=0}^\infty \lim_{n \to \infty} [q^{(n), \ell}(x, x) - \pi^{(n)}(x)] \leq \sum_{\ell=0}^\infty \left( \lim_{n \to \infty} \frac{\pi^{(n)}_{\max}}{\pi^{(n)}_{\min}} \right) e^{-(1-\delta)\ell} < \infty, \quad (2.4)
\]

where the first inequality follows from [12, Eq. (12.11)] and the last inequality follows from assumption (2) and (2.3). To save notation in the following, we suppress references to the \(n\)-th transition kernel unless large \(n\) limits are taken.

Now, we consider the following expected Green function:

\[
\mathbb{E}[G_{\lambda/N}(U, U')] = \int_0^\infty e^{-\lambda t/N} \sum_{x, y, z \in E} \pi(x)\pi(y)q_t(x, z)q_t(y, z) dt
\]

\[
= \int_0^\infty e^{-\lambda t/N} \sum_{x, y, z \in E} \pi(x)\pi(z)q_t(x, z)q_t(z, y) dt
\]

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and introduce the error function

\[ s(x, y) = \frac{\pi(z)}{\pi(x)^{1/2}} \left( \int_{0}^{t} e^{-t\lambda/N} e^{t q(z)} dt \right) \]

where the second equality follows from reversibility and the third follows from stationarity. On the other hand, by the strong Markov property of \((X^U, Y^U)\) at \(M_{U,U}\) and the fact that \(X_{M_{U,U}}^U = Y_{M_{U,U}}^U\),

\[ \mathbb{E}[G_{\lambda/N}(U, U')] = \mathbb{E}[e^{-\lambda M_{U,U}/N} G_{\lambda/N}(X_{M_{U,U}}^U, X_{M_{U,U}}^U)] \]

To handle \(G_{\lambda/N}(X_{M_{U,U}}^U, X_{M_{U,U}}^U)\), first we symmetrize \(q\) by considering

\[ A(x, y) = \pi(x)^{1/2} q(x, y) \pi(y)^{-1/2}, \quad x, y \in E, \]

and introduce the error function

\[ \varepsilon_{\lambda/N}(x, y) = \sum_{z \in E} \left( \frac{\pi(z)}{\pi(x)^{1/2} \pi(y)^{1/2}} - 1 \right) \int_{0}^{\infty} e^{-t\lambda/N} e^{t A} e^{t (A-I)}(y, z) ds. \]  

Then \(\sigma(A) = \sigma(q)\) and an orthonormal set of eigenfunctions of \(A\) (with respect to the inner product in (2.1)) can be chosen such that constant function 1/\(\sqrt{N}\) is the eigenfunction associated with the eigenvalue 1 of \(A\). (Cf. the proof of [12, Lemma 12.2].) On the other hand, applying [12, Theorem 20.6] to the right-hand side of (2.8), we deduce that

\[ |\varepsilon_{\lambda/N}(x, y)| \leq \sum_{z \in E} \left| \frac{\pi(z)}{\pi(x)^{1/2} \pi(y)^{1/2}} - 1 \right| \left( \frac{\pi_{\max}}{\pi_{\min}} \right)^{3/2} \left( \frac{2}{\lambda/N + (1 - \delta)} + \frac{N}{\lambda} \pi(z)^2 \right). \]

The use of the error function defined by (2.8) is for the following estimate of \(G_{\lambda/N}(x, y)\):

\[ G_{\lambda/N}(x, y) = \sum_{z \in E} \int_{0}^{\infty} e^{-t\lambda/N} e^{t q(z)} e^{t (A-I)}(y, z) dt \]

\[ = \sum_{z \in E} \frac{\pi(z)}{\pi(x)^{1/2} \pi(y)^{1/2}} \int_{0}^{\infty} e^{-t\lambda/N} e^{t A} e^{t (A-I)}(y, z) dt \]

\[ = \sum_{z \in E} \int_{0}^{\infty} e^{-t\lambda/N} e^{t (A-I)}(y, z) e^{t (A-I)}(y, z) dt + \varepsilon_{\lambda/N}(x, y) \]

\[ \left\langle \delta_x, \frac{1}{N + 2(I-A)^{1/2}} \delta_y \right\rangle + \varepsilon_{\lambda/N}(x, y) \]

\[ = \left\langle \delta_x, \delta_y \right\rangle + \left\langle \delta_x, \sum_{\ell=0}^{\infty} \frac{2\ell A^\ell}{(N+2)^{\ell+1}} I_{\{A<1\}} \delta_y \right\rangle + \varepsilon_{\lambda/N}(x, y), \]

where the first term in the last equality follows since the eigenfunction corresponding to 1 is the constant function \(1/\sqrt{N}\). To use (2.10), we will need the domination bound in (2.13) below.

**Proposition 2.2.** For any \(\lambda \in (0, \infty)\), the function

\[ F_\lambda(x) = \pi(x)^{-1} \mathbb{E}[e^{-\lambda M_{U,U}}; X_{M_{U,U}}^U = x], \quad x \in E, \]

solves the following matrix equation:

\[ (I + B\lambda) F_\lambda = \frac{1 + 2\lambda}{2\lambda} \pi, \]
where \( B_\lambda \) is a symmetric matrix with nonnegative entries defined by

\[
B_\lambda(x, y) = \frac{1}{2} \int_0^\infty e^{-\lambda t} e^{(A-I)(x, y)} (Ae^{(A-I)})(x, y) dt.
\]

In particular,

\[
E\left[e^{-\lambda M_{U,U'}}; X_{M_{U,U'}}^U = x\right] \leq \frac{1 + 2\lambda}{2\lambda} \pi(x)^2. \tag{2.13}
\]

**Proof.** It follows from the stationarity and the independence of \( X_t^U \) and \( Y_t^U \) that

\[
\frac{\pi(x)^2}{\lambda} = \int_0^\infty e^{-\lambda t} \mathbb{P}(X_t^U = x, Y_t^U = x) dt
\]

\[
= \mathbb{E} \left[ e^{-\lambda M_{U,U'}} \mathbb{E} \left[ \int_0^J e^{-\lambda t} dt; X_{M_{U,U'}}^U = x \right] \right]
\]

\[
+ \sum_{a \in E} \mathbb{E} \left[ e^{-\lambda M_{U,U'}}; X_{M_{U,U'}}^U = a \right] \cdot \mathbb{E}[e^{-\lambda J}] \cdot 2 \int_0^\infty e^{-\lambda t} \sum_{b \in E} \frac{q(a, b) q(t, a) q(t, b)}{2} dt
\]

for an exponential random variable \( J \) with mean \( 1/2 \). Here, the second equality follows from a renewal argument for the two-dimensional chain \( (X_t^U, Y_t^U; t \geq M_{U,U'}) \) at its first epoch time. By the definition (2.7) of \( A \) and a division of both sides of the foregoing equality by \( 2\pi(x)/(1+2\lambda) \), (2.12) follows. \( \blacksquare \)

We are ready to complete the proof of the theorem. Recall the constant \( \delta \in (0, 1) \) in assumption (2). For fixed \( \lambda_0 \in (0, \infty) \) and \( L \in \mathbb{N} \), we have

\[
\limsup_{n \to \infty} \sup_{\lambda \in [\lambda_0, \infty]} \left| \sum_{x \in E_n} \mathbb{P}^{(n)}(e^{-\lambda M_{U,U'} / N_n}; X_{M_{U,U'}}^U = x) \right|
\]

\[
\left( \delta_x \sum_{\ell=0}^{\infty} \frac{2^\ell (A^{(n)})^\ell}{(\lambda/N_n + 2)^{\ell+1}} I_{\{A^{(n)} < \ell\} \delta_x} + \varepsilon_{\lambda/N_n}(x, x) - \sum_{\ell=0}^{\infty} \frac{2^\ell q^{(\infty), \ell}}{(\lambda/N_n + 2)^{\ell+1}} \right)
\]

\[
\leq \frac{\delta^{L+1}}{1-\delta} + \sum_{\ell=L+1}^{\infty} q^{(\infty), \ell} + \limsup_{n \to \infty} \sup_{\lambda \in [\lambda_0, \infty]} \sum_{x \in E_n} \frac{N_n \pi^{(n)}(x)^2}{2\lambda} |\varepsilon_{\lambda/N_n}(x, x)| \tag{2.15}
\]

\[
\leq \frac{\delta^{L+1}}{1-\delta} + \sum_{\ell=L+1}^{\infty} q^{(\infty), \ell} + \limsup_{n \to \infty} \sum_{x \in E_n} \frac{N_n \pi^{(n)}(x)^2}{2\lambda}
\]

\[
\times \sum_{z \in E_n} \left| \frac{\pi^{(n)}(z)}{\pi^{(n)}(x)} - 1 \right|^3 \left( \frac{\pi^{(n)}(x)}{\pi^{(n)}(z)} \right)^{3/2} \left( \frac{2}{\lambda_0/N_n + (1-\delta)} + \frac{N_n \pi^{(n)}(z)^2}{\lambda_0} \right) \tag{2.16}
\]

\[
\leq \frac{\delta^{L+1}}{1-\delta} + \sum_{\ell=L+1}^{\infty} q^{(\infty), \ell}. \tag{2.17}
\]

Here, (2.15) uses assumption (2) to bound the tails of the power series in \( A^{(n)} \) with summands indexed by \( \ell \geq L + 1 \). Then we can apply assumption (1) to points \( x \in E_n \) for which \( q^{(n), \ell}(x, x) \) approximate \( q^{(\infty), \ell} \) for all \( 0 \leq \ell \leq L \) and handle the rest of the points \( x \in E_n \) and the error functions \( \varepsilon_{\lambda/N_n}(x, x) \) by (2.13). Also, (2.16) and (2.17) follow from (2.9) and assumption (3), respectively. Since \( \sum_{\ell} q^{(\infty), \ell} < \infty \) by (2.4), passing \( L \to \infty \) in (2.17) shows that the limit superior in (2.14) is equal to zero. If we put this result for (2.14) together with (2.5), (2.6) and (2.10), the required (2.2) follows. The proof is complete. \( \blacksquare \)
We write \( \frac{(W_n)}{n \to \infty} \) for convergence under the \( L_1 \)-Wasserstein distance for probability measures defined on the real line, and \( \mathscr{L}(X) \) for the law of a random variable \( X \). In addition, \( e \) denotes a standard exponential random variable.

**Corollary 2.3.** Under the assumption of Theorem 2.1, it holds that

\[
\mathscr{L} \left( \frac{M_{U,U'}}{N_n} \right) \xrightarrow{n \to \infty} \mathscr{L} \left( \left( \frac{1}{2} \sum_{\ell=0}^{\infty} q^{(\infty),\ell} \right) e \right),
\]

where the law of \( M_{U,U'}/N_n \) is understood to be under \( \mathbb{P}^{(n)} \). In particular,

\[
\lim_{n \to \infty} \frac{\mathbb{E}^{(n)}[M_{U,U'}]}{N_n} = \frac{1}{2} \sum_{\ell=0}^{\infty} q^{(\infty),\ell}.
\]

**Proof.** By assumption (2) of Theorem 2.1 and [2, Theorem 3.23], \( M_{U,U'}/\mathbb{E}^{(n)}[M_{U,U'}] \) converges in distribution to the standard exponential. Moreover, by [1, Theorem 1.4 (b)], the tail distributions of \( M_{U,U'}/\mathbb{E}^{(n)}[M_{U,U'}] \) can be uniformly dominated by a function which decays exponentially; that theorem in [1] is applicable here by assumptions (2) and (3), [12, Theorem 12.3], and [6, Eq. (3.21)]. Hence, the required \( L_1 \)-Wasserstein convergence in (2.18) holds by dominated convergence and Vallender’s characterization of the \( L_1 \)-Wasserstein distance in [16] if we replace the scaling constants \( N_n \) by \( \mathbb{E}^{(n)}[M_{U,U'}] \) for each \( n \). On the other hand, Theorem 2.1 implies that (2.18) under the weaker mode of convergence in distribution holds. It follows that \( \lim_{n \to \infty} \mathbb{E}^{(n)}[M_{U,U'}/N_n \text{ must exist in } (0, \infty) \text{ and is given by (2.19). This is enough for (2.18)}.

**Corollary 2.4.** Under the assumptions of Theorem 2.1, it holds that

\[
\mathscr{L} \left( \frac{M_{U,V}}{N_n} \right) \xrightarrow{n \to \infty} \left( 1 - \frac{1}{\sum_{\ell=0}^{\infty} q^{(\infty),\ell}} \right) \delta_0 + \frac{1}{\sum_{\ell=0}^{\infty} q^{(\infty),\ell}} \mathscr{L} \left( \left( \frac{1}{2} \sum_{\ell=0}^{\infty} q^{(\infty),\ell} \right) e \right).
\]

**Proof.** By (2.19) and the application of [1, Theorem 1.4 (b)] in the proof of Corollary 2.3, we can bound the tail distributions of \( M_{U,V}/N_n \) uniformly by a function which decays exponentially and so again it is enough to show that the required convergence holds under weak convergence. But this follows upon applying (2.19) and [6, Corollary 4.2].

The second main result of this note is the following.

**Proposition 2.5.** Fix an integer \( k \geq 3 \). For a sequence of growing random \( k \)-regular graphs \( \{G_n\} \) where \( G_n \) has degree \( k \geq 3 \), the conclusions of Corollary 2.3 and Corollary 2.4 hold with

\[
\sum_{\ell=0}^{\infty} q^{(\infty),\ell} = \frac{k-1}{k-2}.
\]

**Proof.** For the assumptions of Theorem 2.1, (1) is a well-known property of random regular graphs readily observed in [13], see [9, 4] for the proof of (2), and (3) holds trivially. Moreover, for each \( \ell \in \mathbb{Z}_+ \), \( q^{(\infty),\ell} \) can be represented as the normalized \( \ell \)-th moment \( \int (a/k)^\ell \mu_k(da) \) of the Kesten-Mckay law \( \mu_k \) defined as follows [11, 13]:

\[
\mu_k(da) = \frac{k}{2\pi} \frac{\sqrt{4(k-1)-a^2}}{k^2-a^2}, \quad -2\sqrt{k-1} \leq a \leq 2\sqrt{k-1}.
\]

Hence, \( \sum_{\ell=0}^{\infty} q^{(\infty),\ell} = \int k/(k-a) \mu_k(da) \) and then (2.20) follows from [10, (4.7) and the first equation in (4.8)] with \( z = k, p = k \) and \( q = k-1 \) (see [10, (4.3)] for this choice of parameters).
3 References

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