A DEFINABLE \((p, q)\)-THEOREM FOR NIP THEORIES

ITAY KAPLAN

Abstract. We prove a definable version of Matoušek’s \((p, q)\)-theorem in NIP theories. This answers a question of Chernikov and Simon. We also prove a uniform version.

The proof builds on a proof of Boxall and Kestner who proved this theorem in the distal case, utilizing the notion of locally compressible types which appeared in the work of the author with Bays and Simon.

1. Introduction

The aim of this paper is to prove a model-theoretic definable version of Matoušek’s \((p, q)\)-theorem in combinatorics [Mat04] under the assumption that the theory is NIP.

We begin by recalling Matoušek’s theorem (a version of which was first proved for convex sets by Alon and Kleitman [AK92]). The statement uses the notion of the (dual) VC-dimension of set systems. For the definitions see Section 2 below.

Definition 1.1. Say that a set system \((X, F)\) has the \((p, q)\)-property for \(q \leq p < \omega\) if for any \(F \subseteq F\) of size \(|F| = p\) there is some \(F_0 \subseteq F\) such that \(|F_0| = q\) and \(\bigcap F_0 \neq \emptyset\).

Fact 1.2 (The \((p, q)\)-theorem). [Mat04] There exists a function \(N_{pq} : \mathbb{N}^2 \rightarrow \mathbb{N}\) such that for any \(q \leq p < \omega\), if \((X, F)\) is a finite set system with the \((p, q)\)-property such that every \(s \in F\) is nonempty and \(\text{VC}^*(F) < q\), then there is \(X_0 \subseteq X\) of size \(|X_0| = N_{pq}(p, q)\) such that \(X_0 \cap s \neq \emptyset\) for all \(s \in F\).

Model theoretically, this implies that if \(\phi(x, y)\) is NIP then for any \(\text{VC}^*(\phi) < q \leq p\) and \(n := N_{pq}(p, q)\), if \(B\) is a finite set of \(y\)-tuples such that \(\{\phi(x, b) \mid b \in B\}\) has the \((p, q)\)-property then there are \(n\) elements \(a_0, \ldots, a_{n-1}\) such that for all \(b \in B\) there is some \(i < n\) for which \(\phi(a_i, b)\) holds.

This theorem turned out to be tremendously useful in the model-theoretic study of NIP. For instance, it was instrumental in the proof of the uniform definability of types over finite sets (UDTFS) in NIP theories by Chernikov and Simon [CS15], in their study of definably amenable NIP groups [CS18] and more recently in the proof that honest definitions exist uniformly for NIP formulas [BKS22].

In order to phrase a definable version of the \((p, q)\)-theorem, we use the following definition.

Definition 1.3. Let \(M\) be a structure. Say that a pair of formulas \((\phi(x, y), \psi(y))\) over \(M\) has the \((p, q)\)-property if \(F := \{\phi(x, b) \mid b \in \psi(M)\}\) is a family of nonempty sets with the \((p, q)\)-property: for every choice of distinct \(p\) elements \(F\), some \(q\) of them have a nonempty intersection.

It is not hard to see that for any structure \(M\), a pair \((\phi(x, y), \psi(y))\) of formulas over \(M\) has the \((p, q)\)-property for some \(p \geq q > \text{VC}^*\) iff for all \(b \in \psi(C)\), \(\phi(x, b)\) does not divide over \(M\) (this was proved in [Sim15], Lemma 2.4], but see also...
In light of this, Section 2 formulates the following corollary of the
(p,q)-theorem (see also [Sim14, beginning of Section 2]).

Fact 1.4. Suppose that T is NIP and that M ⊨ T. Assume that φ(x, y) and ψ(y)
are formulas over M and that (φ, ψ) has the (p,q)-property for VC*(φ) < q ≤ p.
Then there are sets W0, . . . , Wn−1 ⊆ S^n(M) for n := Npq(p,q) such that \( \bigcup_{i<n} W_i = \{ p \in S^n(M) \mid \psi(y) ∈ p \} \) and for each i < n, \( \{ φ(x,b) \mid tp(b/M) ∈ W_i \} \) is consistent.

For example consider the family \( F \) of rays in DLO (i.e., Th(Q,<)): sets defined
by x > a or x < a. It is easy to see that the dual VC-dimension of \( F \) is 1 (given
by any two rays, if they intersect, then their union is everything). In the context of
Fact 1.4, \( F \) can be formalized by setting φ(x, y, z1, z2) = ((z1 = z2 → x > y) ∧ (z1 \neq z2 → x < y)) and ψ(y, z1, z2) = (y = y). Then (φ, ψ) has the (3,2) property: every
three rays must intersect. Given any model M, let W0 be the set of types of pairs
(a,a) over M and W1 be the set of types of pairs (a,b) over M where a \neq b. In
other words, we cover \( F \) by positive and negative rays.

In this paper we will prove a definable version of this fact. Here is the statement,
especially taken from [Sim14, below Fact 2.2]:

Theorem 1.5. Suppose that T is NIP and that M ⊨ T and that φ(x, y), ψ(y) are formulas
over M. Assume that (φ, ψ) has the (p,q)-property for VC*(φ) < q ≤ p. Then
there are formulas ψ0(y), . . . , ψn−1(y) over M such that ψ(y) is equivalent to the
disjunction \( \bigvee_{i<n} ψ_i(y) \) and for each i < n, \( \{ φ(x,b) \mid b ∈ ψ_i(M) \} \) is consistent.

This is \( \text{Corollary 3.6} \) In the example above this is illustrated by taking \( ψ_0 = (z_1 = z_2) \) and \( ψ_1 = (z_1 \neq z_2) \).

The above theorem follows by compactness from the following theorem.

Theorem 1.6. Suppose that T is NIP and that M ⊨ T. Suppose that φ(x, y) is a
formula over M and b ∈ ℂ(φ) is such that φ(x,b) does not fork over M. Then there
is a formula ψ(y) ∈ tp(b/M) such that \( \{ φ(x,c) \mid ℂ ⊨ ψ(c) \} \) is consistent.

This is Theorem 3.3 The idea of the proof is to generalize the argument of
[Boxall and Kestner] who proved this theorem in the case when T is distal
(a subclass of NIP theories introduced by Simon in [Sim14]). Their proof uses
the fact that T is distal only once, in [Boxall and Kestner, Proposition 4.1]. We generalize this
proposition to the NIP case by using the notion of locally compressible types from
[BKS22], and being more careful with the choice of the strict Morley sequence.

We also consider a uniform version of Theorem 1.5, i.e., varying the model M.

Theorem 1.7. Suppose that T is NIP, and that φ′(x,y,z), ψ′(y,z) are two for-

mulae without parameters. Then for any q ≤ p < ω there is n < ω and formulas
ψ0(y,w), . . . , ψn−1(y,w) such that the following hold.

Suppose that M ⊨ T and c ∈ Mω. Let φ(x,y) = φ′(x,y,c) and ψ(y) = ψ′(y,c).
If (φ, ψ) has the (p,q)-property and VC*(φ(x,y)) < q then for some d0, . . . , dn−1 ∈ Mω, ψ(y,c) is equivalent to the disjunction \( \bigvee_{i<n} ψ_i(y,d_i) \) and for each i < n, the
set \( \{ φ(x,b) \mid b ∈ ψ_i(M,d_i) \} \) is consistent.

This is Theorem 4.1 We deduce a uniform version related to the more re

fined notion of VC-density (used in Matoušek’s original formulation of Fact 1.2
see Fact 4.6 in [Corollary 4.9] answering positively a version of [AG22, Questions
3.7(2)] where T is assumed to be NIP, without restrictions on the VC-codensity.

A short history of the problem. Theorem 1.6 was first posed as a question
by Chernikov and Simon in [CS13, Problem 29] even before the relation to the
(p, q)-theorem was noticed. It was later conjectured by Simon [Sim15b, Conjecture 5.1].

This was settled in the following cases:

- In [Sim14], Simon proved it for dp-minimal theories with small or medium directionality (a notion which measures the number of coheirs, see [KS14]).
- In [SS14], Simon and Starchenko prove a stronger version of this conjecture for a large class of dp-minimal theories (e.g., those with definable Skolem functions), namely that every such formula $\phi(x, b)$ belongs to a definable type.
- In [Sim15b], Simon generalized the first item to NIP theories (still constraining the directionality). In both papers [Sim14, Sim15b] there are very interesting discussions of this problem and related results.
- In [BK18], Boxall and Kestner proved the conjecture for distal theories.
- In [Rak21], Rakotonarivo proved the conjecture for certain dense pairs of geometric distal structures.
- In [AG22], Andújar-Guerrero proved a special case of a stronger conjecture, [Sim15b, Conjecture 2.15] which assumes only that $\phi$ is NIP, namely the case in which the VC-codensity of $\phi$ is less than 2.

**Structure of the paper.** In Section 2 we go over all the basic notions involved in the proof, including NIP and forking. In Section 3 we prove Theorem 1.6. In Section 4 we prove Theorem 1.7 and in Section 4.1 we review the notion of VC-density and deduce a variant of Theorem 4.1 corresponding to this notion. We conclude in Section 5 with some questions and final thoughts.

**Acknowledgements.** I would like to thank Martin Bays and Pierre Simon for many useful discussions and for their comments on previous versions of this paper. I would like to thank Pierre Simon in particular for reminding me of the direct proof of Lemma 3.1.

I would also like to thank Pablo Andújar-Guerrero for his comments on a previous version, for encouraging me to relate the results to VC-density, and for pointing out Lemma 4.11.

2. Preliminaries

Our notation is standard and is the same as in [BKS22, Section 2]. The only exception is that we use the notation $\mathcal{C}$ for the monster model.

In the following subsections we will recall the basic definitions and facts we will use in the rest of the paper.

2.1. VC-dimension and NIP.

**Definition 2.1** (VC-dimension). Let $X$ be a set and $\mathcal{F} \subseteq \mathcal{P}(X)$. The pair $(X, \mathcal{F})$ is called a set system. We say that $A \subseteq X$ is shattered by $\mathcal{F}$ if for every $S \subseteq A$ there is $F \in \mathcal{F}$ such that $F \cap A = S$. A family $\mathcal{F}$ is said to be a VC-class on $X$ if there is some $n < \omega$ such that no subset of $X$ of size $n$ is shattered by $\mathcal{F}$. In this case the VC-dimension of $\mathcal{F}$, denoted by $\text{VC}(\mathcal{F})$, is the smallest integer $n$ such that no subset of $X$ of size $n + 1$ is shattered by $\mathcal{F}$.

If no such $n$ exists, we write $\text{VC}(\mathcal{F}) = \infty$.

**Fact 2.2.** [Sim15b, Lemma 6.3] Suppose $\mathcal{F}$ is a VC-class on $X$. Let $\mathcal{F}^* = \{s \in \mathcal{F} \mid x \in s \mid x \in X\} \subseteq \mathcal{P}(\mathcal{F})$ be the dual of $\mathcal{F}$. Then $\mathcal{F}$ is a VC-class iff $\mathcal{F}^*$ is, and moreover $\text{VC}^*(\mathcal{F}) := \text{VC}(\mathcal{F}^*) < 2^{\text{VC}(\mathcal{F}) + 1}$. 

**Definition 2.3.** Suppose $T$ is an $L$-theory and $\phi(x,y)$ is a formula. Say that $\phi(x,y)$ is NIP if for some/every $M \models T$, the family $\{\phi(M,a) \mid a \in M^y\}$ is a VC-class. Otherwise, $\phi$ is IP (IP stands for “Independence Property” while NIP stands for “Not IP”).

Let $\text{VC}(\phi)$ be the VC-dimension of $\{\phi(M,a) \mid a \in M^y\}$, where $M$ is any (some) model of $T$. Note that this definition depends on the partition of variables.

Let $\phi^{opp}$ be the partitioned formula $\phi(y,x)$ (it is the same formula with the partition reversed). Let $\text{VC}^*(\phi) = \text{VC}(\phi^{opp})$ be the dual VC-dimension of $\phi$ (this definition agrees with the one in Fact 2.2 below).

The (complete first-order) theory $T$ is NIP if all formulas are NIP. A structure $M$ is NIP if $\text{Th}(M)$ is NIP.

**Remark 2.4.** NIP formulas are especially well-behaved in the presence of indiscernible sequences. Indeed, for $\phi(x,y)$ NIP and an indiscernible sequence $(a_i)_{i<\omega}$ of $y$-tuples there is no $b \in \mathcal{C}^x$ such that $C = \{ \phi(b, a_i) \leftrightarrow \phi(b, a_i+1) \}$ for all $i < 2 \text{VC}^*(\phi) + 1$ (see [Adl08, Proposition 3] or [Sim15a, Lemma 2.7]). The maximal such $l$ is called the alternation rank of $\phi$.

From this one deduces the following facts.

**Fact 2.5.** Let $\phi(x,y)$ be an NIP formula. Suppose that $I = (a_i)_{i<\omega}$ is an indiscernible sequence of $y$-tuples.

1. [Sim15a, Proposition 2.8] For any $b \in \mathcal{C}^x$ there is some $n < \omega$ and $\epsilon < 2$ such that $\phi(b, a_i) \epsilon$ holds for all $i > n$.

2. (“Lowness”) [Sim15a, Lemma 2.2] There is some $n < \omega$ such that if $\{ \phi(x, a_i) \mid i < n \}$ is consistent, then so is $\{ \phi(x, a_i) \mid i < \omega \}$. In fact, $n$ can be chosen to be $\text{VC}^*(\phi) + 1$.

### 2.2. Locally compressible types.

**Definition 2.6.** Fix a formula $\phi(x,y)$, $k < \omega$ and a parameter set $A \subseteq \mathcal{C}^y$.

- $p \in S_\phi$ is $k$-compressible if for any finite $A_0 \subseteq A$ there is $A_1 \subseteq A$ with $|A_1| \leq k$ such that $p|_{A_1} \vdash p|_{A_0}$.
- $S_{\phi,k}(A) \subseteq S_\phi(A)$ is the space of $k$-compressible $\phi$-types.

**Definition 2.7.** Suppose $\phi(x,y)$ is a formula, $B \subseteq \mathcal{C}^y$ and $p_0(x), \ldots, p_{n-1}(x) \in S_\phi(B)$.

The rounded average of $p_0(x), \ldots, p_{n-1}(x) \in S_\phi(B)$ is the following (possibly inconsistent) collection of formulas:

$$\text{rAvg}(p_i \mid i < n) = \{ \phi(x,b)^\epsilon \mid b \in B, \epsilon < 2, \text{Maj}_{i < n} \{ \phi(x,b)^\epsilon \in p_i(x) \} \},$$

where Maj be the majority rule Boolean operator: for truth values $0, \ldots, n-1$, let

$$\text{Maj}_{i < n} p_i = \bigvee_{I_0 \subseteq \omega \text{ with } |I_0| > n/2} \bigwedge_{i \in I_0} p_i.$$

**Fact 2.8.** [RKS23, Theorem 5.17] Let $T$ be any theory. Let $\phi(x,y)$ be an NIP formula. Then there exist $n$ and $k$ depending only on $\text{VC}(\phi)$ such that for $A \subseteq \mathcal{C}^y$, any $p \in S_\phi(A)$ is the rounded average of $n$ types in $S_{\phi,k}(A)$.

### 2.3. Forking and NIP.

We recall the definition of forking in any theory $T$ and its behavior under NIP.

**Definition 2.9 (Forking).** Fix a set $A$.

A formula $\phi(x,b)$ divides over $A$ if there is some $k < \omega$ such that $\phi(x,b)$ $k$-divides: for some sequence $(b_i)_{i<\omega}$ such that $b_i \equiv_A b$, $\{ \phi(x,b_i) \mid i < \omega \}$ is $k$-inconsistent.
A formula $\psi(x)$ forks over $A$ if it implies a finite disjunction of dividing formulas over $A$.

A partial type $\pi(x)$ divides/forks over $A$ if it implies a dividing/forking formula over $A$.

For more on the basic properties of forking and dividing, see e.g., [CK12, Section 2].

2.3.1. Forking and the $(p,q)$-theorem. We now turn to the $(p,q)$-theorem (Fact 1.2) and its connection to non-forking. Recall Definition 1.3 from the introduction.

Applying Fact 1.2, we get directly the following.

**Lemma 2.10.** Suppose that $M \models T$ and that $\phi(x,y), \psi(y)$ are formulas over $M$. Suppose that $(\phi, \psi)$ has the $(p,q)$-property for $VC^*(\phi) < q \leq p$ (in particular, $\phi$ is NIP). Let $n = N_{pq}(p,q)$. Then for every finite set $C \subseteq \psi(M)$, there are $a_0, \ldots, a_{n-1} \in M^2$ such that for any $c \in C$, $\phi(a_i, c)$ holds for some $i < n$.

The following was proved in [Sim15b], Lemma 2.4 over models, but the proof there omitted the (necessary) condition that the $(M, b) \neq \emptyset$ for all $b \in \psi(M)$, so we repeat it briefly here.

**Lemma 2.11.** Suppose that $M \models T$ and that $\phi(x,y), \psi(y)$ are formulas over some set $A \subseteq M$ with $\phi$ NIP. Then the following are equivalent:

1. For all $b \in \psi(C)$, $\phi(x,b)$ does not divide over $A$.
2. For all $0 < q$ there is some $p \geq q$ such that $(\phi, \psi)$ has the $(p,q)$-property.
3. $(\phi, \psi)$ has the $(p,q)$-property for some $VC^*(\phi) < q \leq p$.

**Proof.** Note that the statement “$(\phi, \psi)$ has the $(p,q)$-property” is elementary, so it is true in $M$ if it is true in $C$.

For (1) implies (2), note that (1) implies immediately that for all $b \in \psi(C)$, $\phi\langle C, b \rangle$ is nonempty. Fix $q > 0$. If for every $p \geq q$, $(\phi, \psi)$ does not have the $(p,q)$-property then by compactness and Ramsey there is an $A$-indiscernible sequence $(b_i)_{i<\omega}$ of elements from $\psi(C)$ witnessing that $\phi(x,b_0)$ divides over $A$.

(2) implies (3) is trivial.

Assume (3). Suppose that for some $b \in \psi(C)$, $\phi(x,b)$ divides over $A$. Then there is an $A$-indiscernible sequence $(b_i)_{i<\omega}$ starting with $b_0 = b$ such that $\phi(x,b_i) \models i < \omega$ is inconsistent. By Remark 2.1, it is already $VC^*(\phi) + 1 \leq q$-inconsistent. Note that for each $i < j < \omega$, $\phi\langle C, b_j \rangle \neq \phi\langle C, b_i \rangle$ since otherwise by indiscernibility, $(\phi\langle C, b_i \rangle)_{i<\omega}$ is constant and thus all these sets will be empty. Together, we get that $\phi\langle C, b_i \rangle$ are elements $i < p$ contradicts the $(p,q)$-property.

Finally, recall the following corollary of “lowness” of NIP formulas (Fact 2.5).

**Fact 2.12.** [Sim15b, Corollary 2.3] Suppose that $\phi(x,y)$ is a NIP formula (without parameters), and that $\phi(x,b)$ does not divide over a model $M$. Then there is a formula $\psi(y) \in \text{tp}(b/M)$ such that for all $b' \in \psi(C)$, $\phi(x,b')$ does not divide over $M$.

Together with Lemma 2.11, we get that $(\phi, \psi)$ has the $(p,q)$-property for $q := VC^*(\phi) + 1$ and some $p \geq q$.

2.3.2. Forking in NIP and NTP$_2$. The basic property of non-forking in NIP is the equivalence with Lascar-invariance, which over models translates to invariance.

**Fact 2.13.** [Sim15b, Proof of Proposition 5.21] If $\phi(x,y)$ is NIP and $p(x) \in S_\phi(C)$ is a global $\phi$-type, then $p$ is $M$-invariant.

It follows that if $\phi(x,c)$ does not fork over $M$, and $B \subseteq C^0$ is a set of realizations of $\text{tp}(c/M)$ then $\{\phi(x,b) \mid b \in B\}$ does not fork over $M$ (and is in particular consistent).
When \( T \) is NIP or even NTP\(_2 \) — a larger class which contains also simple theories (for the definition, see [CK12, Definition 2.27]), we have that:

**Fact 2.14.** [CK12] Assume that \( T \) is NTP\(_2 \) and let \( M \models T \). Then forking equals dividing over \( M \): if \( \phi(x,y) \) is a formula over \( M \) and \( \phi(x,b) \) forks over \( M \), then it divides over \( M \).

One of the main tools used to show this was the existence of global strictly invariant types since Morley sequences in them are universal witnesses for dividing.

**Definition 2.15.** A global type \( w(y) \in \text{S}(\mathcal{C}) \) is strictly invariant over a model \( M \) if \( w(y) \) is invariant over \( M \) and for any set \( C \supseteq M \), if \( a \models w|_C \) then \( \text{tp}(C/\mathcal{M}a) \) does not fork over \( M \).

When \((I,\prec)\) is a linear order and \((b_i)_{i \in I} \) is a Morley sequence of an \( M \)-strictly invariant type \( w \) over \( M \) (see e.g., [BKS22, Section 2.2] for the definition of a Morley sequence), we call it a strict Morley sequence over \( M \).

**Fact 2.16.** [CK12, Lemma 3.14] Assume that \( T \) is NTP\(_2 \), \( M \models T \) and \( \phi(x,y) \) a formula over \( M \). Suppose that for some \( b \in \mathcal{C}^I \), \( \phi(x,b) \) divides over \( M \).

Then, if \((b_i)_{i \in \omega} \) is a strict Morley sequence over \( M \), then \( \{\phi(x,b_i) \mid i < \omega\} \) is inconsistent.

We will need a strict invariant type which is also a coheir. The existence of such a type is implied by [CK12, Proposition 3.7 (1)], applied to coheir independence, using the fact that forking over models implies quasi-dividing in NTP\(_2 \), see [CK12, Corollary 3.13] which is precursor to Fact 2.14. We state this explicitly:

**Fact 2.17.** Assume that \( T \) is NTP\(_2 \) and let \( M \models T \). Then for any type \( q(y) \in \text{S}(M) \) there is a global type \( w(y) \in \text{S}(\mathcal{C}) \) extending \( q(y) \) such that:

- \( w \) is finitely satisfiable in \( M \).
- If \( C \supseteq M \) and \( a \models w|_C \) then \( \text{tp}(C/\mathcal{M}a) \) does not fork over \( M \).

### 3. The proof of Theorem 1.6

We start with the following lemma.

**Lemma 3.1.** Suppose that \( \phi(x,y) \) is NIP. Then there is some \( k \) depending only on \( \text{VC}(\phi) \) such that if \( I := (a_i)_{i < \omega} \) is an \( \emptyset \)-indiscernible sequence of \( y \)-tuples such that \( p := \{\phi(x,a_i) \mid i < \omega\} \) is consistent, then there is a \( k \)-compressible type \( q \in \text{S}_{\phi,k}(I) \) such that for some \( n < \omega \), for all \( i > n \), \( \phi(x,a_i) \in q \).

**Proof.** By [Fact 2.8], \( p \) is the rounded average of \( n \) types in \( \text{S}_{\phi,k}(A) \) for some \( k, n \) which depend only on \( \text{VC}(\phi) \). By [Fact 2.5], each of these types must be eventually positive or eventually negative. However, since \( p \) is eventually positive (in fact all the instances of \( \phi \) in \( p \) are positive), and \( p \) is the rounded average of these types, at least one (and in fact the majority) of them has to be eventually positive.

Here is a more direct argument which does not use [Fact 2.8] (this argument was used in its proof\(^1\)). The idea is to take a \( \phi \)-type which first alternates maximally and then is constantly true. More precisely, let \( m = 2 \text{VC}(\phi) + 3 \). Let \( l < m \) be maximal such that there is some \( q_0(x) \in \text{S}_{\phi}(\{a_i \mid i < m\}) \) such that \( q_0 \vdash \phi(x,a_i) \leftrightarrow \neg \phi(x,a_{i+1}) \) for \( i < l \) and \( q_0(x) \vdash \phi(x,a_i) \) for \( i \in [l, m) \).

Note that \( \text{tp}_{[a_m]} \) is an example of such a type with \( l = 0 \) so that such an \( l \) and \( q_0 \) exist. By [Remark 2.3], \( l \leq 2 \text{VC}(\phi) < m - 2 \).

Let \( q = q_0 \cup \{\phi(x,a_i) \mid l \leq i\} \). Then \( q \) consistent and moreover \( (m-1) \)-compressible.

---

\(^1\)We thank Pierre Simon for reminding us of the existence of a more direct proof.
Indeed, let $l < n < \omega$. We show that $q_{\mid a_{<l} A} \cup \left\{ a_{a_{>l}} \right\} \upharpoonright q_{\mid a_{<n}}$. Otherwise, by indiscernibility, $q_{\mid a_{<l} A} \cup \left\{ \neg \phi(x, a_{l+1}) \right\}$ is consistent, contradicting the choice of $l$. Finally, note that the left-hand side of this implication is consistent by indiscernibility (and since $q_0$ is consistent).

The following is a generalization of [BK18, Proposition 4.1].

**Proposition 3.2.** Suppose that $T$ is NIP, $M \models T$, $\phi(x, y)$ is a formula over $M$, $b \in M$ and that $\phi(x, b)$ does not fork over $M$. Let $q(y) = \text{tp}(b/M)$.

Then there is a formula $\theta(x, z)$ over $M$ and a type $r(z) \in S(M)$ such that:

1. For any $c \models r$, $\theta(x, c)$ does not fork over $M$.
2. For any finite set $B$ of realizations of $q$, there is some $c \models r$ such that $C/Ma \models \forall x \theta(x, c) \rightarrow \phi(x, b)$ for any $b' \in B$.

**Proof.** By [Fact 2.17] there is a global type $w(y) \in S(C)$ extending $q(y)$ such that:

- $w$ is finitely satisfiable in $M$.
- If $C \supseteq M$ and $a \models w|_{C}$ then $\text{tp}(C/Ma)$ does not fork over $M$.

Let $J = (a_i)_{i \in \mathbb{Z}}$ be a Morley sequence of $w$ over $M$, i.e., $a_i \models w|_{M a_{<l}}$ for all $i \in \mathbb{Z}$. For $i < \omega$, let $b_i = a_{-i}$ and let $I = (b_i)_{i < \omega}$. Note that for all $i < \omega$:

1. $\text{tp}(b_{<i}/M_{>i})$ is finitely satisfiable in $M$ (by transitivity of coher independence).
2. $\text{tp}(b_{<i}/M_{b_i})$ does not fork over $M$.

Let $p = \left\{ \phi(x, b_i) \mid i < \omega \right\} \in S_\omega(I)$. As $I$ is an $M$-indiscernible sequence, $b_0 \equiv_M b$ and as $\phi(x, b)$ does not fork over $M$, $p$ is consistent.

By [Lemma 3.1] applied to $p$, there is some $k < \omega$ and some $r \in S_\phi(I)$ such that $r$ is eventually positive: for some $s < \omega$, $\phi(x, b_i) \in r$ for all $i > s$. (In fact, the direct proof of [Lemma 3.1] allows us to set $s = 2 VC^*(\phi) + 2$.)

Thus, for some $A \subseteq \omega$ of size $|A| = k$, $r|_{\{b_i \mid i \in A\}} \models r|_{b_{<i+k+1}}$. In particular, for some $s < j \leq s + k + 1$, $j \notin A$.

Write $A = A_0 \cup A_1$ where $A_0 = A \cap j$ and $A_1 = A \setminus A_0$. By indiscernibility and the choice of $s$, we get that $r|_{\{b_i \mid i \in A_0\}} \cup \{ \phi(x, b_j) \mid j < i \} \models \phi(x, b_j)$ where $t = |A_1|$.

Let $u = (u_i)_{i \in A_0}$ be a tuple of variables of the sort $y$, and let $\theta_0(x, u) = \bigwedge_{i \in A_0} \phi(x, u_i).$ Let $z = (z_i)_{i < t}$ be a sequence of variables of the sort $y$ and let $\theta_1(x, z) = \bigwedge_{i < t} \phi(x, z_i).$ We claim that letting $r(z) = \text{tp}(b_{<i}/M)$, there is some $m \in M^n$ such that the pair $r(z)$ and $\theta_0(x, z) := \theta_0(x, m) \land \theta_1(x, z)$ satisfies the conclusion of the proposition.

By "lowness" [Fact 2.5(3)], there is some number $f < \omega$ such that for any $m \in M^n$, if $\{ \theta_0(x, m) \land \phi(x, b_i) \mid i < f \}$ is consistent, then $\{ \theta_0(x, m) \land \phi(x, b_i) \mid i < \omega \}$ is consistent (in fact $f$ could be chosen to be $VC^*(\phi) + 1$). Fix such an $m \in M^n$.

Since strict Morley sequences are universal witnesses for dividing [Fact 2.10] and $J$ is a strict Morley sequence over $M$, $\theta_0(x, m) \land \phi(x, b_0)$ does not divide over $M$. Since forking equals dividing over $M$ by [Fact 2.14] and non-forking over models is the same as invariance by [Fact 2.13] it follows that $\theta_0(x, b_{<i})$ does not fork over $M$.

Recall that by [Item (i)] $\text{tp}(b_{j} \mid j \in A_0/M_{>j})$ is finitely satisfiable in $M$. Thus, there is $m \in M^n$ such that:

- $\theta_0(x, b_{j+1} \mid j \leq i \leq j+1+i) \models \phi(x, b_j),$
- $\theta_0(x, m) \land \bigwedge_{j+i \leq j+1+i+1} \phi(x, b_i)$ is consistent.

Thus, by the previous paragraph and indiscernibility, letting $\theta(x, z) = \theta_0(x, z)$, $\theta(x, b_{<i})$ does not fork over $M$.

By indiscernibility, it follows that $\theta(x, b_{1 \leq i < i+1+1}) \models \phi(x, b_0).$ Recall that by [Item (ii)] $\text{tp}(b_{1 \leq i < i+1+1}/M_{b_0})$ does not fork over $M$. Now, given a finite set $B$
realization of \(q(y)\), by extension, there is some \(c \models r(z)\) such that \(\text{tp}(c/MB_0)\) does not fork over \(M\). As non-forking over models is the same as invariance by \(\text{Fact 2.13}\) we get that \(\theta_m(x,c) \models \phi(x,b')\) for all \(b' \in B\) as required. \(\square\)

We now deduce \(\text{Theorem 1.6}\), which we repeat below. The proof is exactly the same as in \(\text{BK18, after Proposition 4.1}\), but we break it down for use in \(\text{Section 3}\).

**Proposition 3.3.** Let \(M\) be any structure and let \(\phi(x,y), \psi(y), \theta(x,z), \zeta(z)\) be formulas over \(M\). Suppose that:

1. \(\theta(x,z)\) is NIP.
2. \((\theta, \zeta)\) has the \((p,q)\)-property for some \(\text{VC}^*(\theta(x,z)) < q \leq p\) and let \(n = N_{pq}(p,q)\).
3. For every set \(B \subseteq \psi(M)\) such that \(|B| \leq n\), there is some \(c \in \zeta(M)\) such that \(M \models \forall x(\theta(x,c) \rightarrow \phi(x,b))\) for all \(b \in B\).

Then \(\{\phi(x,b) \mid b \in \psi(M)\}\) is consistent.

**Proof.** By \(\text{Lemma 2.10}\) we get that for any finite subset \(C \subseteq \zeta(M)\) there are \(a_0, \ldots, a_{n-1} \in M^C\) such that for all \(c \in C\), \(\theta(a_i,c)\) holds for some \(i < n\). By compactness, there are \(a_0, \ldots, a_{n-1} \in \mathcal{C}^C\) such that for all \(c \in \zeta(M)\), \(\theta(a_i,c)\) holds for some \(i < n\).

We claim that for some \(i < n\), \(\phi(a_i, b')\) holds for all \(b' \in \psi(M)\). Otherwise, for each \(i < n\) there is \(b_i \in \psi(M)\) such that \(\neg \phi(a_i, b_i)\). Let \(B = \{b_i \mid i < n\}\), and let \(c\) be as in \(3\). By the previous paragraph, for some \(i < n\), \(\theta(a_i,c)\) holds, and thus \(\phi(a_i, b')\) for all \(b' \in B\) and in particular \(\mathcal{C} \models \phi(a_i, b_i)\), a contradiction. \(\square\)

**Theorem 3.4** (Theorem 1.6). Suppose that \(T\) is NIP and that \(M \models T\). Suppose that \(\phi(x,y)\) is a formula over \(M\) and \(b \in \mathcal{C}^y\) is such that \(\phi(x,b)\) does not fork over \(M\). Then there is a formula \(\psi(y) \in \text{tp}(b/M)\) such that \(\{\phi(x,c) \mid \mathcal{C} \models \psi(c)\}\) is consistent.

**Proof.** Let \(q(y) = \text{tp}(b/M)\). By \(\text{Proposition 3.2}\) we get some \(\theta(x,z)\) and \(r(z) \in S(M)\) as in there. As \(\theta(x,c)\) does not fork over \(M\) for all \(c \models r\), by \(\text{Fact 2.12}\) there is a formula \(\zeta(z)\) in \(r(z)\) such that for all \(c' \models \zeta(z)\), \(\theta(x,c')\) does not divide over \(M\).

By \(\text{Lemma 2.11}\) there is some \(p > q := \text{VC}^*(\theta(x,z)) + 1\) such that \((\theta, \zeta)\) has the \((p,q)\)-property.

Let \(n := N_{pq}(p,q)\). By compactness and the choice of \(\theta\) (and the fact that \(\zeta \models r\)), there is a formula \(\psi(y) \in q\) such that for any set \(B \subseteq \psi(M)\) of size \(|B| \leq n\), there is some \(c \in \zeta(M)\) such that \(M \models \forall x(\theta(x,c) \rightarrow \phi(x,b'))\) for any \(b' \in B\).

Applying \(\text{Proposition 3.3}\) we are done. \(\square\)

**Corollary 3.5.** Under the same assumptions as in \(\text{Theorem 3.4}\), there is a global type \(p(x)\) and a formula \(\psi(y) \in \text{tp}(b/M)\) such that \(p(x) \supseteq \{\phi(x,c) \mid \mathcal{C} \models \psi(c)\}\) and \(p\) is a non-forking heir over \(M\).

**Proof.** By \(\text{Theorem 3.3}\) there is a formula \(\psi(y) \in \text{tp}(b/M)\) such that \(\Gamma(x) := \{\phi(x,b) \mid b \in \psi(M)\}\) is consistent. Let \(p_0(x) \in S(M)\) extend \(\Gamma(x)\) to a complete type over \(M\). By \(\text{CK12 Proposition 3.7 (2)}\) applied to coheir independence, there is a non-forking heir \(p(x) \in S(\mathcal{C})\) extending \(p_0\). It follows from the heir property that \(p(x)\) contains \(\{\phi(x,c) \mid \mathcal{C} \models \psi(c)\}\). \(\square\)

The following was already essentially stated in \(\text{Sim14, beginning of Section 2}\):

**Corollary 3.6** (Theorem 1.5). Suppose that \(T\) is NIP, \(M \models T\) and that \(\phi(x,y), \psi(y)\) are formulas over \(M\). Assume that \((\phi, \psi)\) has the \((p,q)\)-property for \(\text{VC}^*(\phi) < q \leq p\). Then there are formulas \(\psi_0(y), \ldots, \psi_{n-1}(y)\) over \(M\) such that \(\psi(y)\) is equivalent to the disjunction \(\bigvee_{i<n} \psi_i(y)\) and for each \(i < n\), \(\{\phi(x,b) \mid b \in \psi_i(M)\}\) is consistent.
Proof. By Lemma 2.11 for every \( b \in \psi(C) \), \( \phi(x, b) \) does not fork over \( M \). Theorem 3.4 implies that for every \( b \in \psi(C) \) there is a formula \( \psi_b(y) \in \text{tp}(b/M) \) such that \( \{ \phi(x, b) \mid b \in \psi_b(M) \} \) is consistent. By compactness, finitely many such formulas cover all of \( \psi(y) \) as required. \( \square \)

4. UNIFORMITY

Now we want to give a uniform version of Corollary 3.6.

**Theorem 4.1** Suppose that \( T \) is NIP, and that \( \phi'(x, y, z), \psi'(y, z) \) are two formulas without parameters. Then for any \( q \leq p < \omega \) there is \( n < \omega \) and formulas \( \psi_0(y, w), \ldots, \psi_{n-1}(y, w) \) such that the following hold.

Suppose that \( M \models T \) and \( c \in M^x \). Let \( \phi(x, y) = \phi'(x, y, c) \) and \( \psi(y) = \psi'(y, c) \). If \( (\phi, \psi) \) has the \((p, q)\)-property and \( \text{VC}^* (\phi(x, y)) < q \) then for some \( d_0, \ldots, d_{n-1} \in M^y, \psi(y) \) is equivalent to the disjunction \( \bigvee_{i<n} \psi_i(y, d_i) \) and for each \( i < n \), the set \( \{ \phi(x, b) \mid b \in \psi_i(M, d_i) \} \) is consistent.

**Remark 4.2.** Based on Lemma 2.11 one could ask if the following holds:

Suppose that \( T \) is NIP, and that \( \phi(x, y, z), \psi(y, z) \) are two formulas (without parameters). Then there is \( n < \omega \) and formulas \( \psi_0(y, w_0), \ldots, \psi_{n-1}(y, w_{n-1}) \) such that for any \( M \models T \) and any \( c \in M^x \), if \( \phi(x, b, c) \) does not fork over \( M \) for any \( b \in \psi(M, c) \) then for some \( d_0, \ldots, d_{n-1} \in M \), \( \psi(y, c) \) is equivalent to the disjunction \( \bigvee_{i<n} \psi_i(y, d_i) \) and for each \( i < n \), the set \( \{ \phi(x, b) \mid b \in \psi_i(M, d_i) \} \) is consistent.

However, consider the following example. Let \( L = \{ E \} \) and let \( T \) say that \( E \) is an equivalence relation such that for every \( n < \omega \) there is exactly one class with \( n \) elements. Then \( T \) is a complete stable theory. Let \( \psi(y, z) = (y E z) \) and \( \phi(x, y, z) = (x = y) \). Suppose that \( \psi_i \) for \( i < n \) are as above. Fix some model \( M \). Let \( c \) be such that \( n < |[c]_E| < \omega \). Then \( (x = b) \) does not fork over \( M \) for any \( b \in E c \). By the choice of \( \psi_i \) for \( i < n \), there are \( d_0, \ldots, d_{n-1} \) as above. But then there must be some \( i < n \) such that \( 2 \leq |\psi_i(M, d_i)| \) and \( \{ x = b \mid b \in \psi_i(M, d_i) \} \) is inconsistent.

**Remark 4.3.** Before giving the proof, consider a special case, where \( T \) is pseudofinite. In that case, the formulas \( \psi_i \) can be taken to be of the form \( \phi^{\text{opp}}(y, w) \land \psi(y) \) and \( n = \text{N}_{\text{pr}}(p, q) \). The reason is that given any \( M, c \) as above, there are \( d_i \in M^x \) for \( i < n \) such that for all \( b \in \psi(M, c) \), \( \phi(d_i, b) \) holds for some \( i < n \). Indeed, if not, then one can formulate the negation in a first-order way, expressing the fact that there is a set system of nonempty sets of dual VC-dimension strictly bounded by \( q \) satisfying the \((p, q)\)-property, without \( n \) elements such that any set contains one of those elements, but then this must be true in a finite structure, contradicting Fact 1.2.

**Proof of Theorem 4.1.** First note that we may assume that all models of \( T \) are infinite as otherwise we would be done by Remark 4.3.

By the usual coding tricks (see e.g., [Sh69, Theorem II.2.12(1)]) and Lemma 2.5), it is enough to find formulas \( \psi_i(y, w) \) for \( i < n \) as in the statement such that for every \( M, c \) some subset of these formulas satisfies the statement of the theorem. Indeed, if we find such formulas, then we can replace each \( \psi_i \) by \( \psi_i'(y, w, s, t_0, \ldots, t_{n-1}) = ((s = t_i) \land \psi_i) \). Since all our models have size at least \( 2 \), it is easy to see that the new formulas \( \psi_i' \) satisfy the requirements.

Suppose this is not true and fix \( \phi(x, y, z), \psi(y, z), q \) and \( p \) as above. For every choice of formulas \( \Psi := \{ \psi_0(y, w), \ldots, \psi_{n-1}(y, w) \} \) (with \( w \) being any tuple of variables) there is a model \( M_\Psi \) and \( c_\Psi \in M_\Psi^y \) witnessing that the conclusion does not hold: the pair \( (\phi(x, y, c_\Psi), \psi(y, c_\Psi)) \) has the \((p, q)\)-property and \( \text{VC}^* (\phi(x, y, c_\Psi)) < q \), but for all \( s \leq n \) and \((d_i)_{i \in s}\) from \( M_\Psi^y \), either \( \psi(y, c) \) is not equivalent to the
disjunction $\bigvee_{i \in s} \psi_i(y,d_i)$ or for some $i \in s$, the set $\{ \phi(x,b) \mid b \in \psi_i(M,d_i) \}$ is inconsistent.

Let $\mathcal{U}$ be an ultrafilter on the family $X$ of finite sets of formulas of the form $\xi(y,w)$ (for varying $w$) extending the filter generated by $\{ \Psi' \in X \mid \Psi \subseteq \Psi' \}$ for any $\Psi \in X$. Let $M^*$ be the ultraproduct $\prod_{\Psi \in X} M_{\Psi}/\mathcal{U}$. Finally, let $c^* \in M^*$ be the class of $(c_{\Psi})_{\Psi \in X}$.

Note that $(\phi(x,y,c^*), \psi(y,c^*))$ has the $(p,q)$-property and $\text{VC}^*(\phi(x,y,c^*)) < q$ since this is expressible in first-order. By Lemma 2.11 it follows that for any $b \in \psi(C,c^*)$, $\phi(x,b,c^*)$ does not fork over $M^*$. Now we want to apply Corollary 3.6 but being consistent is not first-order expressible, so we have to be a bit more careful.

By compactness and the proof of Theorem 3.4 there are finitely many formulas $\psi_i(y,w), \theta_i(x,u,v), \zeta_i(u,v)$ without parameters, and $d_i \in (M^*)^w$, $e_i \in (M^*)^v$ for $i < n$ such that:

1. $\psi(y,c^*)$ is equivalent to the disjunction $\bigvee \psi_i(y,d_i)$.
2. For each $i < n$, $(\theta_i(x,u,e_i), \zeta_i(u,e_i))$ has the $(p_i,q_i)$-property for some VC$^*(\theta_i(x,u,e_i)) < q_i \leq p_i$ and let $n_i = N_{p_i q_i}(p_i,q_i)$
3. For every set $B \subseteq \psi_i(M^*,d_i)$ of size $\leq n_i$ there is some $c \in \zeta_i(M^*,e_i)$ such that $M^* \vDash \forall x (\theta_i(x,c,e_i) \rightarrow \phi(x,b,c^*))$ for all $b \in B$.

Let $\Psi = \{ \psi_0, \ldots, \psi_{n-1} \}$. By Łoś’s theorem, there is some $\Psi' \supseteq \Psi$ such that in $M'_\Psi$, there are $d'_i, e'_i$ for which the above is true replacing $c^*$ by $c_{\Psi'}$. Then by applying Proposition 3.3 in $M_{\Psi'}$ we get that $\{ \phi(x,b,c_{\Psi'}) \mid b \in \psi_i(M_{\Psi'},d'_i) \}$ is consistent for all $i < n$, contradicting the choice of $M_{\Psi'}$. 

4.1. A uniform version related to VC-density. Matoušek’s original formulation of Fact 1.2 was in terms of VC-density and not in terms of (the less refined notion of) VC-dimension. We will explain this notion and prove a variant of Theorem 4.1 related to this notion.

Let $(X,F)$ be a set system. For any $n < \omega$, let $\pi_F(n) = \max\{ |F \cap A| \mid A \subseteq X, |A| \leq n \}$ where $F \cap A = \{ F \cap A \mid F \in F \}$. Similarly, define $\pi_F^*$ as $\pi_F^*(x)$ (see Fact 2.2).

Definition 4.4 (VC-density). Suppose that $(X,F)$ is a set system. The VC-density of $F$, denoted by $\text{vc}(F)$, is $\limsup_{n \rightarrow \infty} \frac{\log \pi_F(n)}{\log n}$ $\in \mathbb{R} \cup \{ \infty \}$. The VC-codensity of $F$, denoted by $\text{vc}^*(F)$, is defined as $\text{vc}^*(F)$ (see Fact 2.2).

In other words, $\text{vc}(F)$ is the infimum of all non-negative real numbers $r$ such that $\pi_F(n) = O(n^r)$.

Fact 4.5. By the Sauer-Shelah lemma (see e.g., [Sim15a, Lemma 6.4]), either $\pi_F(n) = 2^n$ for all $n < \omega$, which happens exactly when $F$ is not a VC-class, or $\pi_F(n) = O(n^{\text{VC}(F)})$. Thus, $\text{vc}(F) \leq \text{VC}(F)$.

Here is Matoušek’s formulation of Fact 1.2.

Fact 4.6. [Mat04, Theorem 4] Let $(X,F)$ be a set system and suppose that $F$ is a VC-class such that $\text{vc}^*(F) < q \leq p$ for some $q, p < \omega$, or even just $\pi_F^*(n) = o(n^q)$. Then there is some number $N$ such that for any finite $\mathcal{G} \subseteq F$, if $\mathcal{G}$ has the $(p,q)$-property, then for some $X_0 \subseteq X$ of size $N$, $X_0 \cap F \neq \emptyset$ for any nonempty $F \in \mathcal{G}$.

Remark 4.7. To see why this formulation implies the one found in Fact 1.2, see Remark 7.

In the context of a complete $\mathcal{L}$-theory $T$, for a formula $\phi(x,y)$ define $\text{vc}(\phi)$ as $\text{vc}(\phi(F))$, where $F = \{ \phi(M,a) \mid a \in M^p \}$ for some/model $M \models T$ (even though the VC-density of $\phi$ is not determined by a sentence, it is determined by a partial
type describing $\pi_x(n)$ for each $n < \omega$). Similarly, define $vc^*(\phi)$ as the VC-codensity of $F$, i.e., $vc^*(\phi) = vc(\phi^{pp})$. In addition, let $\pi_\phi = \pi_F$ and $\pi_\phi^* = \pi_F^* (= \pi^{pp})$.

There are many theories where the VC-density of formulas have been computed, see [ADH+16, ADH+13]. For example, [ADH+16, Theorem 1.1] states that in weakly-o-minimal theories, the VC-density of a formula $\phi(x, y)$ is bounded by $|y|$.

**Remark 4.8.** A naive attempt at generalizing [Theorem 4.1] is the following statement:

Suppose that $T$ is NIP, and $\phi'(x, y, z)$, $\psi'(y, z)$ are two formulas without parameters. Then for any $q \leq p < \omega$ there is $n < \omega$ and formulas $\psi_0(y, w), \ldots, \psi_{n-1}(y, w)$ such that the following hold.

Suppose that $M \models T$ and $c \in M^2$. Let $\phi(x, y) = \phi'(x, y, c)$ and $\psi(y) = \psi'(y, c)$. If $(\phi, \psi)$ has the $(p, q)$-property and $vc^*(\phi(x, y)) < q$ then for some $d_0, \ldots, d_{n-1} \in M^w$, $\psi(y)$ is equivalent to the disjunction $\bigvee_{i<n} \psi_i(y, d_i)$ and for each $i < n$, the set $\{ \phi(x, b) \mid b \in \psi_i(M, d_i) \}$ is consistent.

However, this is false. Indeed, let $T$ as be as in [Remark 4.2] Let $\psi'(y, z) = (y E z)$ and $\phi'(x, y, z) = (x = y) \land \psi'(y, z)$. Let $q = p = 1$. Suppose that $\psi_i(y, w)$ for $i < n$ are as in the statement. Let $M \models T$, let $c$ be such that $[c]_E$ is finite and of size $> n$ and let $\phi, \psi$ be as above. Then $vc^*(\phi(x, y)) = 0$ since $\{ \phi(M, a) \mid a \in M \}$ is finite.

Clearly, the $(1, 1)$-property holds for $(\phi, \psi)$. Thus, there are $d_i$ for $i < n$ as above, and we get the same contradiction as in [Remark 4.2].

A more sensible version is the following, which we will deduce from [Theorem 4.1]

**Corollary 4.9.** Suppose that $T$ is NIP, and that $\phi(x, y)$, $\psi'(y, z)$ are two formulas without parameters. Additionally, assume that $vc^*(\phi) < q \leq p < \omega$, or even just that $\pi_\phi^*(n) = o(n^q)$. Then there is $n < \omega$ and formulas $\psi_0(y, w), \ldots, \psi_{n-1}(y, w)$ such that the following hold.

Suppose that $M \models T$ and $c \in M^2$. Let $\psi(y) = \psi'(y, c)$. If $(\phi, \psi)$ has the $(p, q)$-property then for some $d_0, \ldots, d_{n-1} \in M^w$, $\psi(y)$ is equivalent to the disjunction $\bigvee_{i<n} \psi_i(y, d_i)$ and for each $i < n$, the set $\{ \phi(x, b) \mid b \in \psi_i(M, d_i) \}$ is consistent.

**Remark 4.10.** [Corollary 4.9] answers positively a question of Andújar-Guerrero [AG22, Questions 3.7(2)] where $T$ is assumed to be NIP, without restrictions on the VC-codensity.

To prove this we will need the following observation, which was pointed out to us by Pablo Andújar-Guerrero. It is an improvement of [Lemma 2.11].

**Lemma 4.11.** Suppose that $(X, F)$ is a set system. Suppose that $\pi_F^*(n) = o(n^q)$ and $q \leq p < \omega$. Then for any $0 < q' < \omega$ there is $p' < \omega$ such that for any finite $\mathcal{G} \subseteq F$, if $\mathcal{G}$ has the $(p, q)$-property then $\mathcal{G}$ has the $(p', q')$-property.

**Proof.** Let $N$ be as in [Fact 4.6] Given $q' < \omega$, let $p' = N \cdot (q' - 1) + 2$. The lemma follows by the choice of $N$ and the pigeonhole principle (it is $+2$ and not $+1$ to allow for the empty set to be in $\mathcal{G}$).

**Remark 4.12.** The proof of [Lemma 4.11] allows one to improve [Lemma 2.11]: one can improve (2) by expressing $p$ as in the proof, and (3) can be improved by replacing $VC^*(\phi)$ by $vc^*(\phi)$.

**Proof of Corollary 4.9.** Let $q' = VC^*(\phi) + 1$ and let $p'$ be given by [Lemma 4.11] for the family $F := \{ \phi(M, a) \mid a \in M^n \}$. Now apply [Theorem 4.1] with $\phi' := \phi$, $\psi'$, $q'$ and $p'$ to get $\psi_0(y, w), \ldots, \psi_{n-1}(y, w)$ as in there. Then these formulas satisfy the desired conclusion: indeed, if $c \in M^2$ and $\psi(y) := \psi'(y, c)$ are such that $(\phi, \psi)$ have the $(p, q)$-property, then by the choice of $p'$, it has the $(p', q')$-property, and hence there are $d_0, \ldots, d_{n-1}$ as required.

□
5. Final thoughts and questions

The following question remains open.

*Question 5.1.* In [Theorem 3.4](#) is it enough to assume that $\phi(x,y)$ is NIP instead of the whole theory?

A positive answer is conjectured in [Sim15b, Conjecture 2.15].

*Remark 5.2.* Essentially, there are two places where the proof of [Theorem 3.4](#) uses the fact that $T$ is NIP. The first one is the existence of strict non-forking global types which are coheirs [Fact 2.17], but this holds assuming that $T$ is NTP$_2$ instead of NIP. The other place is in the very last part of the proof of Proposition 3.2 where we used that non-forking is the same as invariance over models [Fact 2.13]. However, this would still work if the implication formula $\forall x(\theta_m(x,z) \rightarrow \phi(x,y))$ was NIP. Thus, the theorem is still true provided that $T$ is NTP$_2$ and all formulas of the form $\xi(z,y) = \forall x(\bigwedge_{i<\omega} \phi(x,z_i) \rightarrow \phi(x,y))$ are NIP. In fact, a close inspection of the proof yields a bound on the length of the tuple $z$ in terms of VC($\phi$).

The following is natural in light of [Theorem 4.1](#).

*Problem 5.3.* Suppose that $T$ is NIP, and fix two formulas $(\phi'(x,y,z), \psi'(y,z))$. Find an effective bound on the function mapping $(p,q)$ for $p \geq q$ to $n$ as in [Theorem 4.1](#) and explore how this function depends on $(\phi', \psi')$.

Now we want to discuss when general theories satisfy the conclusion of [Theorem 3.4](#).

In the next definition, due to Adler [Adl08], Card denotes the class of all cardinals.

**Definition 5.4.** We say that non-forking is bounded if there is a function $f : \text{Card} \rightarrow \text{Card}$ such that for any set $C$ and every $p \in S(C)$, the number of global non-forking extensions of $p$ is bounded by $f(|C| + |T|)$.

**Fact 5.5.** [Adl08, Corollary 38] Non-forking is bounded iff non-forking is equivalent to Lascar-non-splitting, that is: a global type $p$ does not fork over a model $M$ iff for any formula $\phi(x,y)$ and any $c,d$ such that $c \equiv_M d$, $\phi(x,c) \in p$ iff $\phi(x,d) \in p$.

Thus, from [Fact 2.13](#) we get that:

**Fact 5.6.** If $T$ is NIP then non-forking is bounded.

Say that non-forking is strongly bounded if whenever $M \models T$, if $\phi(x,b)$ does not fork over $M$, then there is some $\psi(y) \in \text{tp}(b/M)$ such that $\{\phi(x,b') \mid b' \in \psi(M)\}$ is consistent.

Justifying the name, we have:

**Proposition 5.7.** If non-forking is strongly bounded then non-forking is bounded.

*Proof.* Suppose that non-forking is not bounded. Then there is a model $M \models T$, a formula $\phi(x,y)$ over $M$ and $c,d \in M^\#$ such that $\phi(x,c) \land \neg \phi(x,d) \in p$ while $c \equiv_M d$. Recall that $c,d$ have Lascar distance at most 2: for some $e$, both $(c,e)$ and $(d,e)$ start an indiscernible sequence (by e.g., letting $e$ realize a global coheir extension of $tp(c/M)$ over $M\text{cd}$). So there is an $M$-indiscernible sequence $(c_i)_{i<\omega}$ such that $\phi(x,c_0) \land \neg \phi(x,c_i)$ does not fork over $M$. Let $q(y_0,y_1) = \text{tp}(c_0,c_1/M)$ and let $\psi(x,y_0,y_1) = \phi(x,y_0) \land \neg \phi(x,y_1)$. Then clearly $(c_0,c_1),(c_1,c_2) \equiv q$ but $\psi(x,c_0,c_1), \psi(x,c_1,c_2)$ is inconsistent, and in particular there is no formula $\psi(y) \in q$ witnessing strong boundedness.

**Corollary 5.8.** If non-forking is strongly bounded and is NTP$_2$, then $T$ is NIP.

*Proof.* This follows directly from [CK12, Theorem 4.3].
Example 5.9. There is an IP (and thus TP$_2$) theory $T$ with strongly bounded non-forking. Indeed, by [CKS16, Corollary 5.24], there is such a theory where a global type $p$ does not fork over a model $M$ iff $p$ is finitely satisfiable in $M$.

Now, suppose that $\phi(x, b)$ does not fork over a model $M$. Then $\phi(M, b) \neq \emptyset$, so let $m \in M$ be such that $\phi(m, b)$ holds. Let $\psi(y) = \phi(m, y)$. Then clearly the set \[ \{ \phi(x, b') \mid M \models \psi(b') \} \] is consistent (realized by $m$) as required.

Question 5.10. Suppose that non-forking is bounded. Is it also strongly bounded?

References

[ADH+13] Matthias Aschenbrenner, Alf Dolich, Deirdre Haskell, Dugald Macpherson, and Sergei Starchenko. Vapnik-Chervonenkis density in some theories without the independence property, II. *Notre Dame J. Form. Log.*, 54(3-4):311–363, 2013.

[ADH+16] Matthias Aschenbrenner, Alf Dolich, Deirdre Haskell, Dugald Macpherson, and Sergei Starchenko. Vapnik-Chervonenkis density in some theories without the independence property, I. *Trans. Amer. Math. Soc.*, 368(8):5889–5949, 2016.

[Adl08] Hans Adler. An introduction to theories without the independence property, 2008. available in the author’s website, http://www.logic.univie.ac.at/~adler/docs/nip.pdf.

[AG22] Pablo Andújar Guerrero. Definable $(\omega, 2)$-theorem for families with VC-codensity less than 2, 2022. arXiv:2205.13665.

[AK92] Noga Alon and Daniel J. Kleitman. Piercing convex sets and the Hadwiger-Debrunner $(p, q)$-problem. *Adv. Math.*, 96(1):103–112, 1992.

[BK18] Gareth Boxall and Charlotte Kestner. The definable $(P, Q)$-theorem for distal theories. *J. Symb. Log.*, 83(1):123–127, 2018.

[BKS22] Martin Bays, Itay Kaplan, and Pierre Simon. Density of compressible types and some consequences. *J. Eur. Math. Soc. (JEMS)*, 2022. accepted.

[CK12] Artem Chernikov and Itay Kaplan. Forking and dividing in NTP$_2$ theories. *J. Symbolic Logic*, 77(1):1–20, 2012.

[CKS16] Artem Chernikov, Itay Kaplan, and Saharon Shelah. On non-forking spectra. *J. Eur. Math. Soc. (JEMS)*, 18(12):2821–2848, 2016.

[CS15] Artem Chernikov and Pierre Simon.Externally definable sets and dependent pairs II. *Trans. Amer. Math. Soc.*, 367(7):5217–5235, 2015.

[CS18] Artem Chernikov and Pierre Simon. Definably amenable NIP groups. *J. Amer. Math. Soc.*, 31(3):609–641, 2018.

[Gui12] Vincent Guingona. On uniform definability of types over finite sets. *J. Symbolic Logic*, 77(2):499–514, 2012.

[KS14] Itay Kaplan and Saharon Shelah. Examples in dependent theories. *J. Symb. Log.*, 79(2):585–619, 2014.

[Mat04] Jiří Matoušek. Bounded VC-dimension implies a fractional Helly theorem. *Discrete Comput. Geom.*, 31(2):251–255, 2004.

[Rak21] Tsinjo Odilon Rakotonarivo. The definable $(p, q)$-theorem for dense pairs of certain geometric structures. PhD thesis, Stellenbosch University, 2021.

[She90] S. Shelah. *Classification theory and the number of nonisomorphic models*, volume 92 of *Studies in Logic and the Foundations of Mathematics*. North-Holland Publishing Co., Amsterdam, 1990.

[Sim13] Pierre Simon. Distal and non-distal NIP theories. *Ann. Pure Appl. Logic*, 164(3):294–318, 2013.
[Sim14] Pierre Simon. Dp-minimality: invariant types and dp-rank. *J. Symb. Log.*, 79(4):1025–1045, 2014.

[Sim15a] Pierre Simon. *A guide to NIP theories*, volume 44 of *Lecture Notes in Logic*. Association for Symbolic Logic, Chicago, IL; Cambridge Scientific Publishers, Cambridge, 2015.

[Sim15b] Pierre Simon. Invariant types in NIP theories. *J. Math. Log.*, 15(2):1550006, 26, 2015.

[SS14] Pierre Simon and Sergei Starchenko. On forking and definability of types in some DP-minimal theories. *J. Symb. Log.*, 79(4):1020–1024, 2014.

Einstein Institute of Mathematics, Hebrew University of Jerusalem, 91904, Jerusalem Israel.

*Email address: kaplan@math.huji.ac.il*