HANKEL DETERMINANT OF SECOND ORDER FOR SOME CLASSES OF ANALYTIC FUNCTIONS

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Abstract. Let $f$ be analytic in the unit disk $D$ and normalized so that $f(0) = z + a_2 z^2 + a_3 z^3 + \cdots$. In this paper, we give upper bounds of the Hankel determinant of second order for the classes of starlike functions of order $\alpha$, Ozaki close-to-convex functions and two other classes of analytic functions. Some of the estimates are sharp.

1. Introduction and preliminaries

Let $A$ denote the family of all analytic functions in the unit disk $D := \{z \in \mathbb{C} : |z| < 1\}$ satisfying the normalization $f(0) = 0 = f'(0) - 1$.

A function $f \in A$ is said to be starlike of order $\alpha$, $0 \leq \alpha < 1$, if, and only if

$$\Re\left[zf'(z)f''(z)\right] > \alpha \quad (z \in D).$$

We denote this class by $S^*(\alpha)$. If $\alpha = 0$, then $S^* \equiv S^*(0)$ is the well-known class of starlike functions.

By $C(\alpha)$, $-\frac{1}{2} \leq \alpha < 1$, we denote the class Ozaki close-to-convex functions consisting of functions $f \in A$ for which

$$\Re\left[1 + zf''(z)f'(z)\right] > \alpha \quad (z \in D).$$

The special case of this class, when $\alpha = -1/2$ was introduced by Ozaki in 1941 ([7]) and it is a subclass of the class of close-to-convex functions. This, general form of the class, was introduced in [4] by Kargar and Ebadian. We note that for $\alpha = 0$ we have the class of convex functions.

More about this class one can find in [2] and [11].

Similarly, by $G(\alpha)$ $0 < \alpha \leq 1$, we denote the class of functions $f \in A$ for which

$$\Re\left[1 + zf''(z)f'(z)\right] < 1 + \frac{1}{2\alpha} \quad (z \in D).$$

Ozaki in [7] introduced the class $G(1)$ and proved that functions in $G(1)$ are univalent in the unit disk. Later, Umezawa in [13], Sakaguchi in [9] and R. Singh and S. Singh in [10] showed, respectively, that functions in $G(1)$ are convex in one direction, close-to-convex and starlike.

Nunokawa in [5] considered the more general class $G(\alpha)$ and proved that it is subclass of the class of strongly starlike functions of order $\alpha$, i.e., if $f \in G(\alpha)$, then

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Theorem 1. Then we have the next sharp estimation:

\[ |zf'(z)/f(z)| < \alpha \pi/2 \text{ for all } z \in \mathbb{D}. \]  
This general class is extensively studied by Obradović et al. in [6].
All previous mentioned classes are classes of univalent functions in the unit disc.

2. Main results

In this paper we will give the upper bound estimates for the Hankel determinant of second order for the previous given classes. Some of the estimates are sharp.

Definition 1. Let \( f \in \mathcal{A} \). Then the \( q \)th Hankel determinant of \( f \) is defined for \( q \geq 1 \), and \( n \geq 1 \) by

\[
H_q(n) = \begin{vmatrix}
a_n & a_{n+1} & \cdots & a_{n+q-1} \\
a_{n+1} & a_{n+2} & \cdots & a_{n+q} \\
\vdots & \vdots & & \vdots \\
a_{n+q-1} & a_{n+q} & \cdots & a_{n+2q-2}
\end{vmatrix}.
\]

Thus, the second Hankel determinant is \( H_2(2) = a_2a_4 - a_3^2 \).
Namely, we have

Theorem 1. Let \( f(z) = z + a_2z^2 + a_3z^3 + \cdots \) belongs to the class \( \mathcal{S}^*(\alpha) \), \( 0 \leq \alpha < 1 \). Then we have the next sharp estimation:

\[ |H_2(2)| = |a_2a_4 - a_3^2| \leq (1 - \alpha)^2. \]

Proof. From the definition of the class \( \mathcal{S}^*(\alpha) \), we have

\[
\frac{zf'(z)}{f(z)} = \alpha + (1 - \alpha)\frac{1 + \omega(z)}{1 - \omega(z)} = 2\alpha - 1 + 2(1 - \alpha)\frac{1}{1 - \omega(z)},
\]
where \( \omega \) is analytic in \( \mathbb{D} \) with \( \omega(0) = 0 \) and \( |\omega(z)| < 1, z \in \mathbb{D} \).

From (1) we obtain

\[
f'(z) = \left[1 + 2(1 - \alpha)(\omega(z) + \omega^2(z) + \cdots)\right] \cdot \frac{f(z)}{z}.
\]
If we put \( \omega(z) = c_1z + c_2z^2 + \cdots \), and compare the coefficients on \( z, z^2, z^3 \) in the relation (2) then, after some calculations, we obtain

\[
a_2 = 2(1 - \alpha)c_1, \\
a_3 = (1 - \alpha)(c_2 + (3 - 2\alpha)c_1^2), \\
a_4 = \frac{2}{3}(1 - \alpha)(c_3 + (5 - 3\alpha)c_1c_2 + (2\alpha^2 - 7\alpha + 6)c_1^3).
\]

By using the relation (3), after some simple computations, we obtain

\[
H_2(2) = \frac{4}{3}(1 - \alpha)^2\left(c_1c_3 + \frac{1}{2}c_1^2c_2 - \frac{1}{4}(4\alpha^2 - 8\alpha + 3)c_1^4 - \frac{3}{4}c_2^2\right).
\]

From the last relation we have

\[
|H_2(2)| \leq \frac{4}{3}(1 - \alpha)^2\left(|c_1||c_3| + \frac{1}{2}|c_1|^2|c_2| + \frac{1}{4}|4\alpha^2 - 8\alpha + 3||c_1|^4 + \frac{3}{4}|c_2|^2\right).
\]

For the function \( \omega(z) = c_1z + c_2z^2 + \cdots \) (with \( |\omega(z)| < 1, z \in \mathbb{D} \) the next relations is valid (see, for example [8] expression (13) on page 128):

\[
|c_1| \leq 1, \quad |c_2| \leq 1 - |c_1|^2, \quad |c_3(1 - |c_1|^2) + c_1c_2^2| \leq (1 - |c_1|^2)^2 - |c_2|^2.
\]
We may suppose that $a_2 \geq 0$, which implies that $c_1 \geq 0$ and instead of relations \([5]\) we have the next relations
\[
0 \leq c_1 \leq 1, \quad |c_2| \leq 1 - c_1^2, \quad |c_3| \leq 1 - c_1^2 - \frac{|c_2|^2}{1 + c_1}.
\]
By using \([4]\) for $c_1$ and $c_3$, from \([4]\) we have
\[
|H_2(2)| \leq \frac{4}{3} (1 - \alpha)^2 \left[ c_1 (1 - c_1^2) + \frac{3 - c_1}{4 (1 + c_1)} |c_2|^2 \right. \\
+ \left. \frac{1}{2} c_1^2 |c_2| + \frac{1}{4} |4 \alpha^2 - 8 \alpha + 3 c_1^4 \right].
\]
By using $|c_2| \leq 1 - c_1^2$, from \([7]\) after some calculations we obtain
\[
|H_2(2)| \leq \frac{4}{3} (1 - \alpha)^2 \left( \frac{3}{4} - \frac{3 - |4 \alpha^2 - 8 \alpha + 3|}{4 c_1^4} \right) \leq (1 - \alpha)^2,
\]
since $3 - |4 \alpha^2 - 8 \alpha + 3| \geq 0$ for $0 \leq \alpha < 1$. The equality in the last step is valid for $c_1 = 0$. The function $f_\alpha$, defined by the condition
\[
\frac{z f'_\alpha(z)}{f_\alpha(z)} = \alpha + (1 - \alpha) \frac{1 + z^2}{1 - z^2},
\]
(i.e where $\omega(z) = z^2$, $c_2 = 1$ and $c_1 = 0$ for $i \neq 2$) shows that the result of the theorem is sharp. \(\square\)

**Theorem 2.** *Let $f(z) = z + a_2 z^2 + a_3 z^3 + \cdots$ belongs to the class $C(\alpha)$, $-\frac{1}{2} \leq \alpha < 1$. Then we have the next estimations:
\[
|H_2(2)| \leq \left\{ \begin{array}{ll}
\frac{1 - \alpha)^2 (2 \alpha + 6)}{48 (1 + \alpha)}, & \text{if } -\frac{1}{2} \leq \alpha \leq 0 \\
\frac{(1 - \alpha)^2 (17 \alpha^2 - 36 \alpha + 36)}{144 (\alpha^2 - 2 \alpha + 2)}, & \text{if } 0 \leq \alpha < 1
\end{array} \right.
\]
*Proof.* We will use the same method as in the proof of Theorem \([4]\). From the definition of the class $C(\alpha)$, similarly as in \([4]\) we have
\[
(z f'(z))' = (1 + 2 (1 - \alpha) (\omega(z) + \omega^2(z) + \cdots)) f'(z),
\]
where $\omega$ is analytic in $D$ with $\omega(0) = 0$ and $|\omega(z)| < 1, z \in D$.

If we put $\omega(z) = c_1 z + c_2 z^2 + \cdots$, and compare the coefficients on $z$, $z^2$, $z^3$ in the relation \([5]\) then, after some simple calculations, we obtain
\[
a_2 = (1 - \alpha) c_1, \\
a_3 = \frac{1}{2} (1 - \alpha) \left[ c_2 + (3 - 2 \alpha) c_2^2 \right], \\
a_4 = \frac{1}{6} (1 - \alpha) \left[ c_3 + (5 - 3 \alpha) c_1 c_2 + (2 \alpha^2 - 7 \alpha + 6) c_1^3 \right].
\]
Now, by using \([4]\) we have, after some transformations,
\[
H_2(2) = \frac{1}{6} (1 - \alpha)^2 \left[ c_1 c_3 + \frac{3 - \alpha}{3} c_1^2 c_2 - \frac{1}{3} (2 \alpha^2 - 3 \alpha) c_1^4 - \frac{2}{3} c_2^2 \right].
\]
From the previous relation we have
\[
|H_2(2)| \leq \frac{1}{6} (1 - \alpha)^2 \left( |c_1||c_3| + \frac{3 - \alpha}{3} |c_1|^2 |c_2| + \frac{1}{3} |2 \alpha^2 - 3 \alpha||c_1|^4 + \frac{2}{3} |c_2|^2 \right).
\]
As in the proof of Theorem 1 we may suppose that \( c_1 \geq 0 \). In that case the relations (6) are valid and by using the inequality for \( c \) from (11) we have

\[
|H_2(2)| \leq \frac{1}{6}(1-\alpha)^2 \left( c_1(1-c_1^2) + \frac{2-c_1}{3(1+c_1)}|c_2|^2 + \frac{3-\alpha}{3}c_1^2|c_2|^2 + \frac{1}{3}|2\alpha^2 - 3\alpha|c_1^4 \right).
\]

From here, by using \(|c_2| \leq 1 - c_1^2\), we have (after some transformations):

\[
|H_2(2)| \leq \frac{1}{18}(1-\alpha)^2 \left( 2 + (2-\alpha)c_1^2 - (4-\alpha - |2\alpha^2 - 3\alpha|c_1^4) \right). \tag{12}
\]

For \(-\frac{1}{2} \leq \alpha \leq 0\), from (12) we obtain

\[
|H_2(2)| \leq \frac{1}{18}(1-\alpha)^2 \left( 2 + (2-\alpha)c_1^2 - 2(1+\alpha)(2-\alpha)c_1^4 \right) \leq \frac{(1-\alpha)^2(5\alpha + 6)}{48(1+\alpha)},
\]

because the function in the brackets attains its maximum for \( c_1^2 = \frac{1}{4(1+\alpha)} \). For the case when 0 \( \leq \alpha < 1 \) we use the same method.

**Remark 1.**

(i) Sokol and Thomas in [12] studied the second Hankel determinant for \( \delta \)-convex functions of order \( \beta \) (\( \delta \in \mathbb{R}, 0 \leq \beta < 1 \)) of functions \( f \in A \) such that

\[
\text{Re} \left[ (1-\delta)\frac{zf''(z)}{f(z)} + \delta \left( 1 + \frac{zf''(z)}{f(z)} \right) \right] > \beta \quad (z \in \mathbb{D}),
\]

and for \( \delta = 0 \) and \( \delta = 1 \) received the same results as those given in Theorem 4 and Theorem 5.

(ii) As a special case of Theorem 2 for \( \alpha = -1/2 \) and \( \alpha = 0 \) we receive that for a function \( f \in A \), the following implications hold:

\[
\text{Re} \left[ 1 + \frac{zf''(z)}{f(z)} \right] > \frac{1}{2} \quad (z \in \mathbb{D}) \quad \Rightarrow \quad |H_2(2)| \leq \frac{21}{64},
\]

and

\[
\text{Re} \left[ 1 + \frac{zf''(z)}{f(z)} \right] > 0 \quad (z \in \mathbb{D}) \quad \Rightarrow \quad |H_2(2)| \leq \frac{1}{8}.
\]

The second implication is the same as the one in Theorem 4.2.8 on page 63 from [11] where it is also shown that it is sharp.

**Theorem 3.** Let \( f(z) = z + a_2z^2 + a_3z^3 + \cdots \) belongs to the class \( G(\alpha) \), 0 < \( \alpha \leq 1 \). Then we have the next estimation:

\[
|H_2(2)| \leq \frac{\alpha^2}{144} \left( \frac{17}{4} - \frac{\alpha}{4 + \alpha^2} \right).
\]

**Proof.** From the definition of the class \( G(\alpha) \) we can write

\[
1 + \frac{zf''(z)}{f(z)} = 1 + \frac{1}{2} \alpha - \frac{\alpha}{2} \frac{1 + \omega(z)}{1 - \omega(z)} \left( 1 + \alpha - \frac{1}{1 - \omega(z)} \right),
\]

where \( \omega \) is analytic in \( \mathbb{D} \) with \( \omega(0) = 0 \) and \( |\omega(z)| < 1, z \in \mathbb{D} \). The last relation we can write in the form of

\[
(13) \quad (zf'(z))' = \left[ 1 - \alpha(\omega(z) + \omega^2(z) + \cdots) \right] f'(z).
\]
Putting $\omega(z) = c_1 z + c_2 z^2 + \cdots$ in (13) and comparing the coefficients on $z$, $z^2$, $z^3$, after some simple calculations, we obtain

\[ a_2 = -\frac{\alpha}{2} c_1, \]
\[ a_3 = -\frac{\alpha}{6} [c_2 + (1 - \alpha)c_1^2], \]
\[ a_4 = -\frac{\alpha}{24} [2c_3 + (4 - 3\alpha)c_1c_2 + (\alpha^2 - 3\alpha + 2)c_1^3]. \]

(14)

From (14) we have, after some transformations,

\[ H_2(2) = \frac{\alpha^2}{144} [6c_1c_3 + (4 - \alpha)c_1^2c_2 - (\alpha^2 + \alpha - 2)c_1^4 - 4c_2^2], \]

and from here

\[ |H_2(2)| \leq \frac{\alpha^2}{144} [6|c_1||c_3| + (4 - \alpha)|c_1|^2|c_2| - (\alpha^2 + \alpha - 2)|c_1|^4 + 4|c_2|^2]. \]

As in the proof of previous two theorems, we may suppose that $c_1 \geq 0$. In that case the relations (6) are valid and by using the inequality first for $c_3$, after that for $c_2$, from (15) we have (we omit the details):

\[ |H_2(2)| \leq \frac{\alpha^2}{144} [4 + (2 - \alpha)c_1^2 - (4 + \alpha^2)c_1^4]. \]

(16)

For $c_1^2 = \frac{2 - \alpha}{2144 + \alpha^2}$ the function in the brackets in (16) has its maximum, and after calculation we have the statement of the theorem.

Especially for $\alpha = 1$ we obtain the next implication

\[ \text{Re} \left[ 1 + \frac{zf''(z)}{f'(z)} \right] < \frac{3}{2} \ (z \in \mathbb{D}) \quad \Rightarrow \quad |H_2(2)| \leq \frac{9}{320}. \]

□

In their paper [1] Bello and Opoola considered the class $S^*(q)$ of functions $f \in A$ satisfying the condition

\[ \frac{zf'(z)}{f(z)} < \sqrt{1 + z^2} + z \equiv q(z), \]

They find that $|H_2(2)| \leq \frac{49}{92}$. In the next theorem we give the sharp result.

**Theorem 4.** Let $f(z) = z + a_2 z^2 + a_3 z^3 + \cdots$ belongs to the class $S^*(q)$. Then we have the next sharp estimation:

\[ |H_2(2)| \leq \frac{1}{4}. \]

**Proof.** First, by the definition of the class $S^*(q)$, we have that

\[ \frac{zf'(z)}{f(z)} = \sqrt{1 + \omega^2(z)} + \omega(z), \]

where $\omega$ is analytic in $\mathbb{D}$ with $\omega(0) = 0$ and $|\omega(z)| < 1$, $z \in \mathbb{D}$. Under the same notations as in previous three theorems, the authors in [1] obtained that

\[ H_2(2) = \frac{1}{3} \left( c_1 c_3 + \frac{1}{4} c_1^2 c_2 - \frac{7}{16} c_1^4 - \frac{3}{4} c_2^2 \right). \]
If we apply the same method as in previous three cases, we easily obtain that
\[ |H_2(2)| \leq \frac{1}{3} \left( \frac{3}{4} - \frac{1}{4} c_1^2 - \frac{1}{16} c_4^4 \right) \leq \frac{1}{4}. \]
The result is the best possible as the function \( f_\varphi \) defined by the condition
\[ \frac{zf_\varphi'(z)}{f_\varphi(z)} = \sqrt{1 + z^4 + z^2} \]
shows (i.e for \( \omega(z) = z^2 \)).

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