Two-dimensional temperature oscillations and elastic thermocyclic stresses in sphere and space with spherical cavity under spatially inhomogeneous heat transfer conditions

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Abstract. The problem of steady-state temperature oscillations and thermocyclic stresses in regions bounded from inside or from outside by spherical surface is considered. To find them, the uncoupled quasistatic cyclic thermoelasticity problem under axisymmetric heat transfer conditions has been solved. The given definition corresponds to a solution that depends on two spatial variables. The thermal conductivity problem is solved by method of separation of variables for three types of boundary condition. Solution of the elastic displacement problem is obtained by using thermoelastic potential of displacements and Papkovich-Neuber solution for isothermal deformation.

1. Introduction

Temperature oscillations accompanied by thermal cycle stresses occur during operation in a number of technical devices and structures. Such stresses can cause fatigue damage of their components. Temperature oscillations can be conditionally divided into high frequency and low frequency ones. The former excite elastic thermocyclic stresses which can lead to high-cycle thermal fatigue damage, what is confirmed by the results of experiments [1,2]. The latter excite elastoplastic thermocyclic stresses which can lead to low-cycle thermal fatigue damage. The presented work considers high-frequency temperature oscillations and thermocyclic stresses which are typical for rotor blades of gas turbines, components of internal combustion engines, pipelines of nuclear power plants. A specific feature of high-frequency temperature oscillations is their rapid damping with distance from the body surface, what allows to mark out a conditional surface layer of the material, in which the most significant temperature oscillations and thermocyclic stresses occur. This layer is commonly termed as thermal layer.

Although considerable progress has been made now in refining model approximations of heat and momentum transfer in solids [3,4,5], it is acceptable to use the classical Fourier and Hooke laws in engineering calculations, which provide acceptable accuracy of results [6,7]. With their help classic solutions of cyclic problems of thermal conductivity in various regions were obtained, which can be found in [8,9]. Thermocyclic stresses can be found for these temperature fields using thermoelastic theory methods [10]. The example of calculation of thermocyclic stresses in the elastic half-space is given in [11]. The disadvantage of these solutions is their one-dimensionality. In fact, the heat transfer process takes place under spatially inhomogeneous heat transfer conditions, because of which temperature oscillations and thermocyclic stresses in thermal layer are three-dimensional [12]. Also they are influenced by the shape of the surface.
To calculate temperature oscillations and thermocyclic stresses, it is necessary to find a solution of the uncoupled quasi-static cyclic thermoelasticity problem. In the present work it is obtained for regions bounded by a spherical surface. For greater commonality, all the main types of heat transfer conditions were considered. All these heat transfer conditions were assumed to be axisymmetric. This made it possible to solve the thermoelasticity problem in a two-dimensional definition. The spherical boundary surface was chosen because it has the same shape at all points, at each of which its principal curvatures is equal to the same value. This allows to investigate the effect of spatial inhomogeneity of heat transfer conditions per se. Of all surfaces only the surface of circular cylinder has the same remarkable feature: one of its principal curvatures is equal to zero everywhere, and the other one is the same for all points.

2. Problem definition

We will apply the following definitions for the considered problems: \( i = 1, 2, 3 \) corresponds to the first, second and third boundary value problem of thermal conductivity respectively; \( b = 0, 1 \) corresponds to a space with a spherical cavity and to a spherical solid respectively. In cases when the concrete value \( i \) or \( b \) is not specified, it is considered that \( i = \overline{1,3} \) or \( b = 0,1 \). For ease, we will move on to dimensionless variables

\[
\begin{align*}
\hat{T} &= \frac{T}{\Delta T_w}, \hat{T}_w = \frac{T_w}{\Delta T_w}, \Delta \varepsilon_T^* = \beta \Delta T_w^*, i = 1; \\
\hat{T} &= \frac{\Delta q_w}{\lambda T}, \hat{q}_w = \frac{q_w}{\Delta q_w^*}, \Delta \varepsilon_T^* = \frac{\beta \chi \Delta q_w^*}{\lambda}, i = 2; \\
\hat{T} &= \frac{\Delta T_f}{\Delta T_w^*}, \hat{T}_f = \frac{T_f}{\Delta T_w^*}, \text{Bi} = \frac{\alpha \chi}{\lambda}, \Delta \varepsilon_T^* = \beta \Delta T_w^*, i = 3; \\
\hat{u} &= \frac{1 - \nu}{1 + \nu} \frac{u}{\varepsilon_T^*}; \hat{\sigma} = \frac{1}{E \Delta \varepsilon_T^*} \sigma; r = \frac{r}{\chi}; \hat{R} = \frac{R}{\chi}; \hat{\varepsilon} = \omega t.
\end{align*}
\]

Here \( \hat{T} = \hat{T}(\hat{r}, \eta, \hat{\varepsilon}); \hat{T}_w = \hat{T}_w(\eta, \hat{\varepsilon}); \hat{q}_w = \hat{q}_w(\eta, \hat{\varepsilon}); \hat{T}_f = \hat{T}_f(\eta, \hat{\varepsilon}); \text{Bi} = \text{Bi}(\eta, \hat{\varepsilon}); \hat{u} = \hat{u}(\hat{r}, \eta, \hat{\varepsilon}); \hat{\sigma} = \hat{\sigma}(\hat{r}, \eta, \hat{\varepsilon}); T – solid temperature, K; T_w, \Delta T_w^* – solid surface temperature and its specific range respectively, K; q_w, \Delta q_w^* – density of heat flow across the body surface and its specific range respectively, W/m²; \( \alpha > 0 \) – heat transfer coefficient, W/(m²·K); T_f, \Delta T_f^* – fluid temperature and its specific range respectively, K; \( \chi = \sqrt{a/\omega} \) – specific linear scale of the process, m; \( a – \) thermal diffusivity, m²/s; \( \omega = 2\pi/T \) – angular frequency, rad/s; \( T – \) period, s; \( \lambda – \) thermal conductivity, W/(m·K); \( \beta \) – coefficient of linear thermal expansion, K⁻¹; \( \hat{u} \) – displacement vector, m; \( \nu \) – Poisson ratio; \( \sigma \) – stress tensor, Pa; \( E \) – Young modulus, Pa; \( r \) – radial coordinate, m; \( R \) – radius of boundary spherical surface, m; \( t \) – time, s; \( \eta = \sin \phi \); \( \phi \in [-\pi/2, \pi/2] \) – zenith angle, rad. In a spherical coordinate system \( (r, \phi, \varphi) \) at axisymmetric thermostressed state vector \( \hat{u} \) and tensor \( \hat{\sigma} \) can be presented in the following form

\[
\hat{u} = \hat{u}_r \hat{u}_r + \hat{u}_\eta \hat{u}_\eta + \hat{\sigma} = \begin{pmatrix}
\frac{\partial \sigma_{rr}}{\partial r} & \frac{\partial \sigma_{r \eta}}{\partial r} & 0 \\
\frac{\partial \sigma_{r \eta}}{\partial r} & \frac{\partial \sigma_{\eta \eta}}{\partial r} & 0 \\
0 & 0 & \frac{\partial \sigma_{\varphi \varphi}}{\partial \varphi}
\end{pmatrix},
\]

where \( \hat{u}_r, \hat{u}_\eta \) – local orthogonal unit vectors in the directions of increasing \( r \) and \( \eta \) respectively; \( \varphi \in [0,2\pi] \) – azimuth angle, rad. Thereafter it is considered that \( \eta \in [-1,1], \hat{\varepsilon} > -\infty, \hat{r} \in (\hat{R}, +\infty) \) at \( b = 0 \) and \( \hat{r} \in [0, \hat{R}) \) at \( b = 1 \) unless otherwise specified.

Axisymmetric temperature oscillations and thermocyclic stresses are determined by solution of uncoupled quasistatic cyclic problem of thermoelasticity

\[
\begin{align*}
\frac{\partial \hat{T}}{\partial \hat{t}} &= \frac{1}{\hat{r}^2 \partial \hat{r}^2} \left( \hat{r}^2 \frac{\partial \hat{T}}{\partial \hat{r}} \right) + \frac{1}{\hat{r}^2 \partial \hat{r}^2} \left( 1 - \eta^2 \right) \frac{\partial \hat{T}}{\partial \eta}; \quad (1) \\
\hat{T}(\hat{r}, \eta, \hat{\varepsilon} + 2\pi) &= \hat{T}(\hat{r}, \eta, \hat{\varepsilon}); \quad (2) \\
\hat{T}(\hat{R}, \eta, \hat{\varepsilon}) &= \hat{T}_w, i = 1; \quad (3)
\end{align*}
\]
\[ -\gamma \frac{\partial \bar{T}(\bar{R}, \eta, \hat{t})}{\partial \hat{t}} = \bar{q}_w, \quad i = 2; \]
\[ -\gamma \frac{\partial \bar{T}(\bar{R}, \eta, \hat{t})}{\partial \hat{R}} = \text{Bi}[\bar{T}(\bar{R}, \eta, \hat{t}) - \bar{T}_f], \quad i = 3; \]
\[ \lim_{\hat{r} \to +\infty} \left( \hat{r}^2 \frac{\partial \bar{T}}{\partial \hat{r}} \right) = 0, \quad b = 0; \]
\[ \frac{\partial}{\partial \hat{r}} \left[ \frac{1}{\hat{r}^2} \frac{\partial (\hat{r}^2 \hat{u}_r)}{\partial \hat{r}} + \frac{1}{\hat{r}} \frac{\partial (\hat{u}_\eta \sqrt{1 - \eta^2})}{\partial \eta} \right] - \frac{1}{2(1 - \nu)} \hat{r}^2 \frac{\partial}{\partial \hat{r}} \left[ \sqrt{1 - \eta^2} \frac{\partial (\hat{r} \hat{u}_\eta)}{\partial \hat{r}} - (1 - \eta^2) \frac{\partial \hat{u}_r}{\partial \eta} \right] = \frac{\partial \theta}{\partial \hat{r}}; \]
\[ \frac{\partial}{\partial \eta} \left[ \frac{1}{\hat{r}^2} \frac{\partial (\hat{r}^2 \hat{u}_r)}{\partial \hat{r}} + \frac{1}{\hat{r}} \frac{\partial (\hat{u}_\eta \sqrt{1 - \eta^2})}{\partial \eta} \right] + \frac{1 - 2\nu}{2(1 - \nu)} \frac{\partial (\hat{r} \hat{u}_\eta)}{\partial \hat{r}} - \frac{1 - \eta^2}{1 - \eta^2} \frac{\partial \hat{u}_r}{\partial \eta} = \frac{\partial \theta}{\partial \eta}; \]
\[ \hat{u}_r(\hat{r}, \eta, \hat{t} + 2\pi) = \hat{u}_r(\hat{r}, \eta, \hat{t}); \]
\[ \hat{u}_\eta(\hat{r}, \eta, \hat{t} + 2\pi) = \hat{u}_\eta(\hat{r}, \eta, \hat{t}); \]
\[ \frac{\partial \hat{u}_r(\hat{R}, \eta, \hat{t})}{\partial \hat{r}} + \frac{2\nu}{1 - \nu} \frac{\hat{u}_r(\hat{R}, \eta, \hat{t})}{\hat{R}} + \frac{\nu}{1 - \nu} \frac{\partial \hat{u}_\eta(\hat{R}, \eta, \hat{t})}{\partial \eta} \left( \sqrt{1 - \eta^2} \right) = \theta(\hat{R}, \eta, \hat{t}); \]
\[ \frac{\partial \hat{u}_\eta(\hat{R}, \eta, \hat{t})}{\partial \hat{r}} - \frac{\hat{u}_\eta(\hat{R}, \eta, \hat{t})}{\hat{R}} + \frac{\sqrt{1 - \eta^2}}{\hat{R}} \frac{\partial \hat{u}_r(\hat{R}, \eta, \hat{t})}{\partial \eta} = 0; \]
\[ \lim_{\hat{r} \to +\infty} \hat{u}_r = 0, \quad b = 0; \]
\[ \lim_{\hat{r} \to +\infty} \hat{u}_\eta = 0, \quad b = 0; \]
\[ \hat{u}_r(0, \eta, \hat{t}) = 0, \quad b = 1; \]
\[ \hat{u}_\eta(0, \eta, \hat{t}) = 0, \quad b = 1. \]

Here \( \gamma = (-1)^{b+1}; \)

\[ \theta = \bar{T} - \frac{1}{4\pi} \int_{-1}^{1} \int_{0}^{2\pi} \bar{T} \, d\eta \, d\hat{t}. \]

Conditions (15), (16) follow from law of conservation of momentum because the external mechanical load on the sphere does not act. Functions \( \bar{T}_w, \hat{q}_w, \text{Bi}, \bar{T}_f \) are bounded and periodic:
\[ \bar{T}_w(\eta, \hat{t} + 2\pi) = \bar{T}_w(\eta, \hat{t}), \quad \hat{q}_w(\eta, \hat{t} + 2\pi) = \hat{q}_w(\eta, \hat{t}), \]
\[ \text{Bi}(\eta, \hat{t} + 2\pi) = \text{Bi}(\eta, \hat{t}), \quad \bar{T}_f(\eta, \hat{t} + 2\pi) = \bar{T}_f(\eta, \hat{t}). \]

Besides in a steady-state cyclic process the function \( \hat{q}_w \) satisfies the condition
\[ \int_{\hat{t} + 2\pi}^{1} \int_{-1}^{1} \hat{q}_w \, d\eta \, d\hat{t} = 0. \]

Tensor \( \bar{\sigma} \) components are connected with vector \( \hat{\mathbf{u}} \) components by relations
\[ \bar{\sigma}_{rr} = \frac{1}{1 - 2\nu} \left( (1 - \nu) \frac{\partial \hat{u}_r}{\partial \hat{r}} + 2\nu \frac{\hat{u}_r}{\hat{r}} + \nu \frac{1}{\hat{r}} \frac{\partial (\hat{u}_\eta \sqrt{1 - \eta^2})}{\partial \eta} \right) - (1 - \nu)\theta, \]
\[ \bar{\sigma}_{\eta\eta} = \frac{1}{1 - 2\nu} \left( (1 - \nu) \left( \frac{\sqrt{1 - \eta^2}}{\hat{r}} \frac{\partial \hat{u}_\eta}{\partial \eta} + \frac{\hat{u}_\eta}{\hat{r}} \right) + \nu \frac{1}{\hat{r}} \frac{\partial (\hat{r} \hat{u}_\eta)}{\partial \hat{r}} - \nu \frac{\eta}{\hat{r}} \sqrt{1 - \eta^2} \hat{u}_\eta - (1 - \nu)\theta \right), \]
\[ \bar{\sigma}_{\phi\phi} = \frac{1}{1 - 2\nu} \left( (1 - \nu) \left( \frac{\hat{u}_r}{\hat{r}} - \frac{\eta}{\hat{r} \sqrt{1 - \eta^2}} \hat{u}_\eta \right) + \nu \frac{1}{\hat{r}} \frac{\partial (\hat{r} \hat{u}_\eta)}{\partial \hat{r}} + \nu \frac{\sqrt{1 - \eta^2}}{\hat{r}} \frac{\partial \hat{u}_r}{\partial \eta} - (1 - \nu)\theta \right), \]
\[ \bar{\sigma}_{r\eta} = -\frac{1}{2} \left( \frac{\hat{r}}{\partial \hat{r}} \left( \frac{\partial \hat{u}_\eta}{\partial \eta} \right) + \frac{\sqrt{1 - \eta^2}}{\hat{r}} \frac{\partial \hat{u}_r}{\partial \eta} \right). \]

3. Problem solution
Solution of heat conduction problem (1) – (6) is obtained by means of variable separation method in the form

\[ \hat{T} = \sum_{m=0}^{+\infty} \sum_{n=-\infty}^{+\infty} c_{m,n}^T \psi_{m,n}(\hat{\eta}) e^{i\hat{t}n} , \]

where \( i = \sqrt{-1} ; \) \( \hat{c}_{m,n}^T \) – constant of integration; \( P_m(\eta) \) – Legendre polynomial; \( \psi_{m,n} = \psi_{m,n}(\hat{\eta}) \). Functions \( \psi_{m,n} \) are defined by relations (for \( k \in \mathbb{N} , m \in \mathbb{N}_0 , |n| \in \mathbb{N} \);

\[ \psi_{0,0} = 1, \psi_{k,0} = \hat{r}^{-k-1}, \psi_{m,n} = \frac{\pi}{2\hat{r}\sqrt{|n|}} \left[ \text{ker}_{m+1/2}(\hat{r}\sqrt{|n|}) + i \text{sign}\ n \text{kei}_{m+1/2}(\hat{r}\sqrt{|n|}) \right] , b = 0; \]

\[ \psi_{0,0} = 1, \psi_{k,0} = \hat{r}^k , \psi_{m,n} = \frac{\pi}{2\hat{r}\sqrt{|n|}} \left[ \text{ber}_{m+1/2}(\hat{r}\sqrt{|n|}) + i \text{sign}\ n \text{bei}_{m+1/2}(\hat{r}\sqrt{|n|}) \right] , b = 1. \]

Here ber\( m \hat{r} \), bei\( m \hat{r} \), ker\( m \hat{r} \), kei\( m \hat{r} \) – Kelvin functions; \( \mathbb{N} \) – set of natural numbers; \( \mathbb{N}_0 = \mathbb{N} \cup \{0\} \).

To find the constants of integration \( \hat{c}_{m,n}^T \), property of orthogonality

\[ \int_{-1}^{1} \int_{0}^{2\pi} P_k(\eta) P_m(\eta) e^{-i\eta t} e^{i\hat{t}n} d\eta \ d\hat{t} = \frac{4\pi\delta_{k,m}}{2m+1} \delta_{l,n} , \]

was used, which allows to perform the bounded function \( f = f(\eta, \hat{t}) \) satisfying the condition

\[ f(\eta, \hat{t} + 2\pi) = f(\eta, \hat{t}) \]

in the form of series

\[ f = \sum_{m=0}^{+\infty} \sum_{n=-\infty}^{+\infty} c_{m,n}^f \psi_{m,n}(\eta) e^{i\hat{t}n} , \]

where \( \delta_{m,n} \) – Kronecker delta;

\[ c_{m,n}^f = \frac{2m+1}{4\pi} \int_{0}^{2\pi} \int_{-1}^{1} f(\eta, \hat{t}) d\eta \ d\hat{t} . \]

As a result the following relations are obtained (\( m \in \mathbb{N}_0 , n \in \mathbb{Z} \):

\[ \hat{c}_{m,n}^T = c_{m,n}^T / \psi_{m,n} , i = 1; \]

\[ \hat{c}_{m,n}^T = -\gamma c_{m,n}^T / \psi_{m,n} , i = 2; \]

\[ \hat{c}_{m,n}^T = \hat{c}_{m,n}^f / (\psi_{m,n} + \gamma \psi_{m,n}/\text{Bi}) , i = 3, \text{Bi} = \text{var}; \]

\[ \hat{c}_{m,n}^T = \hat{c}_{m,n}^f / (\psi_{m,n} + \gamma \psi_{m,n}/\text{Bi}) , i = 3, \text{Bi} = \text{const}. \]

Here \( \mathbb{Z} \) – set of integers;

\[ \psi_{m,n} = \psi_{m,n}(\hat{R}) ; \psi_{m,n}' = \frac{d\psi_{m,n}(\hat{R})}{d\hat{R}} ; \]

\[ \Omega_{k,l,m,n} = \frac{2m+1}{4\pi} \sum_{l=0}^{+\infty} \int_{0}^{2\pi} \int_{-1}^{1} P_k(\eta) P_l(\eta) P_m(\eta) d\eta . \]

The infinite system of the linear algebraic equations with the infinite number of unknowns \( \hat{c}_{m,n}^T \) obtained at \( \text{Bi} = \text{var} \) can be solved approximately by a reduction method [13]. A similar approach was developed earlier for half-space at inhomogeneous heat transfer [14]. Using the obtained solution for \( \hat{T} \), it’s easy to identify that

\[ \theta = \sum_{m=0}^{+\infty} \sum_{n=-\infty}^{+\infty} c_{m,n}^T \psi_{m,n}(\hat{\eta}) e^{i\hat{t}n} , \]

where \( c_{0,0}^T = 0, c_{m,n}^T = \hat{c}_{m,n}^T, m \in \mathbb{N}_0 , n \in \mathbb{Z}, m + |n| \neq 0. \)
The solution of the theory of elasticity problem (7) – (16) is found by means of the thermoelastic potential of displacements and Papkovich-Neuber solution for isothermal deformation, which allow to perform the displacements \( \bar{u}_r \) and \( \bar{u}_\eta \) in the form

\[
\begin{align*}
\bar{u}_r &= 4(1 - \nu)Y_r - \frac{\partial}{\partial \hat{r}} (\hat{r}Y_r + \Lambda - \Phi), \\
\bar{u}_\eta &= 4(1 - \nu)Y_\eta - \sqrt{1 - \eta^2} \frac{\partial}{\partial \eta} (\hat{r}Y_r + \Lambda - \Xi).
\end{align*}
\]

(21)

(22)

Here \( \Phi \) – thermoelastic potential of displacements; \( Y_r = Y \cdot \omega_r; \ Y_\eta = Y \cdot \omega_\eta; \ Y \) – harmonic vector whose projections on the axis of the Cartesian coordinate system are harmonic functions; \( \Lambda \) – harmonic function. The thermoelastic potential of displacements gives a particular solution of the inhomogeneous equations (11), (12) and is a particular solution of the equation

\[
\frac{1}{\hat{r}^2} \frac{\partial}{\partial \hat{r}} \left( \hat{r}^2 \frac{\partial \Phi}{\partial \hat{r}} \right) + \frac{1}{\hat{r}^2} \frac{\partial}{\partial \eta} \left[ (1 - \eta^2) \frac{\partial \Phi}{\partial \eta} \right] = \theta.
\]

This solution is found in the form

\[
\Phi = \sum_{m=0}^{+\infty} \sum_{n=-\infty}^{+\infty} c_{m,n} \mu_{m,n} \mu_{m,n} P_m(\eta) e^{i\nu t},
\]

where \( c_{0,0}^{\Phi} = 0, \mu_{0,0} = 1; \ c_{m,n} = -i \mu_{m,n} = \psi_{m,n}, \ m \in \mathbb{N}_0, \ |n| \in \mathbb{N}; \)

\[
\frac{\partial \Phi}{\partial \hat{r}} = \frac{\partial}{\partial \hat{r}} \left( \hat{r}^2 \frac{\partial \Phi}{\partial \hat{r}} \right) = \frac{\partial}{\partial \eta} \left[ (1 - \eta^2) \frac{\partial \Phi}{\partial \eta} \right] = \theta.
\]

(23)

In [15] is shown that \( Y_r \) and \( Y_\eta \) can be performed in the form

\[
Y_r = Y_{\rho} \sqrt{1 - \eta^2} + Y_{\xi,\eta}, \ Y_\eta = Y_{\xi} \sqrt{1 - \eta^2} - Y_{\rho,\eta},
\]

where \( Y_{\rho} e^{i\psi} \) and \( Y_{\xi} \) are harmonic functions. It allowed to obtain the following expressions for \( Y_r \) and \( Y_\eta \):

\[
\begin{align*}
Y_r &= \sum_{m=0}^{+\infty} \sum_{n=-\infty}^{+\infty} (m + 1) \mathcal{A}_{m,n} \hat{r}^{m+1} \mu_{m,n} P_m(\eta) e^{i\nu t}, \ Y_\eta = \sum_{m=0}^{+\infty} \sum_{n=-\infty}^{+\infty} \mathcal{A}_{m,n} \hat{r}^{m+1} \mu_{m,n} P_m(\eta) e^{i\nu t}, \ b = 0;
\end{align*}
\]

\[
\begin{align*}
Y_r &= - \sum_{m=0}^{+\infty} \sum_{n=-\infty}^{+\infty} m \mathcal{A}_{m,n} \hat{r}^{m-1} \mu_{m,n} P_m(\eta) e^{i\nu t}, \ Y_\eta = \sum_{m=0}^{+\infty} \sum_{n=-\infty}^{+\infty} \mathcal{A}_{m,n} \hat{r}^{m+1} \mu_{m,n} P_m(\eta) e^{i\nu t}, \ b = 1.
\end{align*}
\]

Here \( P_m(\eta) \) – associated Legendre function, which is connected with \( P_m(\eta) \) by a relation [16]

\[
P_m(\eta) = \sqrt{1 - \eta^2} \frac{dP_m(\eta)}{d\eta}.
\]

For harmonic function \( \Lambda \) are obtained expansions

\[
\Lambda = - \sum_{m=0}^{+\infty} \sum_{n=-\infty}^{+\infty} B_{m,n} \hat{r}^{m-1} \mu_{m,n} P_m(\eta) e^{i\nu t}, \ b = 0; \ \Lambda = - \sum_{m=0}^{+\infty} \sum_{n=-\infty}^{+\infty} B_{m,n} \hat{r}^{m+1} \mu_{m,n} P_m(\eta) e^{i\nu t}, \ b = 1.
\]

Substitution \( Y_r, Y_\eta, \Lambda, \Phi \) into (21) and (22) gives the following expressions for displacements:

\[
\begin{align*}
\bar{u}_r &= \sum_{m=0}^{+\infty} \sum_{n=-\infty}^{+\infty} \left( a_{m,n}^r \mathcal{A}_{m,n} \hat{r}^{m} \mu_{m,n} + b_{m,n}^r B_{m,n} \hat{r}^{m} \mu_{m,n} + \xi_{m,n}^r \frac{d\mu_{m,n}}{d\hat{r}} \right) P_m(\eta) e^{i\nu t}, \\
\bar{u}_\eta &= \sum_{m=0}^{+\infty} \sum_{n=-\infty}^{+\infty} \left( a_{m,n}^\eta \mathcal{A}_{m,n} \hat{r}^{m} \mu_{m,n} + b_{m,n}^\eta B_{m,n} \hat{r}^{m} \mu_{m,n} + \xi_{m,n}^\eta \frac{d\mu_{m,n}}{d\hat{r}} \right) P_m(\eta) e^{i\nu t}.
\end{align*}
\]

By means of them and the formulas (17) – (20) stresses are defined:

\[
\begin{align*}
\sigma_{rr} &= \sum_{m=0}^{+\infty} \sum_{n=-\infty}^{+\infty} \left( a_{m,n}^{rr} \mathcal{A}_{m,n} \hat{r}^{m} \mu_{m,n} + b_{m,n}^{rr} B_{m,n} \hat{r}^{m} \mu_{m,n} + \xi_{m,n}^{rr} \frac{d\mu_{m,n}}{d\hat{r}} \right) P_m(\eta) e^{i\nu t},
\end{align*}
\]
\[
\begin{align*}
\delta_{\eta \eta} &= \sum_{m=0}^{+\infty} \sum_{n=0}^{+\infty} \left[ \alpha_{mn}^\eta A_{mn} \xi_{mn}^\eta + \beta_{mn}^\eta B_{mn} \xi_{mn}^\eta - \tilde{C}_{mn} \left( \frac{d^2 \mu_{mn}}{d \hat{r}^2} + \frac{1}{\hat{r}} \frac{d \mu_{mn}}{d \hat{r}} \right) \right] p_m(\eta) e^{i n \hat{t}} + \\
&\quad + \sum_{m=0}^{+\infty} \sum_{n=0}^{+\infty} \left( \delta_{mn}^\eta A_{mn} \xi_{mn}^\eta + \delta_{mn}^\eta B_{mn} \xi_{mn}^\eta + \tilde{C}_{mn} \frac{\mu_{mn}}{\hat{r}^2} \right) \eta \frac{p_m(\eta)}{\sqrt{1-\eta^2}} e^{i n \hat{t}}, \\
\delta_{\phi \phi} &= \sum_{m=0}^{+\infty} \sum_{n=0}^{+\infty} \left( \alpha_{mn}^\phi A_{mn} \xi_{mn}^\phi + \alpha_{mn}^\phi B_{mn} \xi_{mn}^\phi + \tilde{C}_{mn} \frac{1}{\hat{r}} \frac{d \mu_{mn}}{d \hat{r}} - \tilde{C}_{mn} \frac{\mu_{mn}}{\hat{r}^2} \right) p_m(\eta) e^{i n \hat{t}} + \\
&\quad + \sum_{m=0}^{+\infty} \sum_{n=0}^{+\infty} \left( \delta_{mn}^\phi A_{mn} \xi_{mn}^\phi + \delta_{mn}^\phi B_{mn} \xi_{mn}^\phi - \tilde{C}_{mn} \frac{\mu_{mn}}{\hat{r}^2} \right) \eta \frac{p_m(\eta)}{\sqrt{1-\eta^2}} e^{i n \hat{t}}, \\
\delta_{\tau \eta} &= \sum_{m=0}^{+\infty} \sum_{n=0}^{+\infty} \left[ \alpha_{mn}^\tau A_{mn} \xi_{mn}^\tau + \beta_{mn}^\tau B_{mn} \xi_{mn}^\tau - \tilde{C}_{mn} \left( \frac{1}{\hat{r}} \frac{d \mu_{mn}}{d \hat{r}} - \frac{\mu_{mn}}{\hat{r}^2} \right) \right] p_m(\eta) e^{i n \hat{t}}.
\end{align*}
\]

Here
\[
\xi_{m}^\eta = \hat{r}^{-m}, \quad \xi_{m}^\phi = \hat{r}^{-m}, \quad \alpha_{m}^\eta = m(m+3-4n), \quad \beta_{m}^\eta = -(m+1), \quad \alpha_{m}^\phi = -m+4-4n, \quad \beta_{m}^\phi = 1,
\]
\[
\xi_{m}^\phi = \hat{r}^{m+1}, \quad \xi_{m}^\tau = \hat{r}^{m-1}, \quad \alpha_{m}^\tau = m(m^2+3m+2n), \quad \beta_{m}^\tau = (m+1)(m+1), \quad \alpha_{m}^\sigma = m(m^2+3m+2n) \leq 0, \quad \beta_{m}^\sigma = 1,
\]
\[
\alpha_{m}^\gamma = -m+4-4n, \quad \beta_{m}^\gamma = 1,
\]
\[
\alpha_{m}^\psi = -(m+1)(m+1)(m-2+4n), \quad \beta_{m}^\psi = m, \quad \alpha_{m}^\delta = m+5-4n, \quad \beta_{m}^\delta = 1,
\]
\[
\alpha_{m}^\zeta = -(m+1)(m+1)(m-2+2n), \quad \beta_{m}^\zeta = m, \quad \alpha_{m}^\nu = -(m+5-4n), \quad \beta_{m}^\nu = 1,
\]
\[
\alpha_{m}^\rho = -(m+1)(m-4m+2-2n), \quad \beta_{m}^\rho = m, \quad \alpha_{m}^\chi = -(m+5-4n), \quad \beta_{m}^\chi = 1,
\]
\[
\alpha_{m}^\kappa = -(m+1)(m-4m+2-2n), \quad \beta_{m}^\kappa = m, \quad \alpha_{m}^\mu = -(m+5-4n), \quad \beta_{m}^\mu = 1.
\]

Constants of integration \(A_{mn}, B_{mn}\), at which the conditions (11) – (16) are satisfied, can be found from the solution of the systems of the linear algebraic equations
\[
\begin{align*}
\alpha_{m}^\eta \xi_{m}^\eta A_{mn} + \beta_{m}^\tau \xi_{m}^\tau B_{mn} &= \tilde{C}_{mn} \mu_{mn} \psi_{mn} - \tilde{C}_{mn} M_{mn} M_{mn}' \mu_{mn} \psi_{mn}, \\
\alpha_{m}^\tau \xi_{m}^\tau A_{mn} + \beta_{m}^\tau \xi_{m}^\tau B_{mn} &= \tilde{C}_{mn} \mu_{mn} \phi_{mn} \left( \frac{M_{mn}}{\hat{R}} - \frac{M_{mn}'}{\hat{R}^2} \right), m \in \mathbb{N}_0, n \in \mathbb{Z}.
\end{align*}
\]

Here
\[
\xi_{m}^\mu = \xi_{m}^\phi (\hat{R}); \quad \xi_{m}^\phi = \xi_{m}^\phi (\hat{R}); \quad \mu_{mn} = \mu_{mn} (\hat{R}); \quad M_{mn}' = \frac{d \mu_{mn}}{d \hat{R}}; \quad M_{mn}'' = \frac{d^2 \mu_{mn}}{d \hat{R}^2}.
\]

It's easy to find that
\[
A_{0,0} = B_{0,0} = 0; \quad A_{0,n} = B_{0,n} = 0, \quad |n| \in \mathbb{N}, b = 0; \quad A_{1,0} = B_{1,0} = 0; \quad |n| \in \mathbb{N}, b = 1.
\]

4. Conclusion
In the course of the undertaken study an axisymmetric solution of the uncoupled quasistatic cyclic thermoelastic problem for regions bounded by a spherical surface is found. The problem is solved for all basic types of heat transfer conditions on the boundary surface. A feature of the found solution is the possibility of taking into account the dependence of heat transfer on time and zenith angle. Obtained dependencies can be used for calculation studies of effect of spatial inhomogeneity of heat transfer conditions on temperature oscillations and thermocyclic stresses in regions bounded by spherical surface.

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