The Hydrogen Atom within a pseudo-complex Quantum Mechanics, involving a minimal length

P. O. Hess\textsuperscript{1,2}

\textsuperscript{1} Instituto de Ciencias Nucleares, Universidad Nacional Autónoma de México, Circuito Exterior, C.U., A.P. 70-543, 04510 México D.F., Mexico

\textsuperscript{2} Frankfurt Institute for Advanced Studies, Johann Wolfgang Goethe Universität, Ruth-Moufang-Str. 1, 60438 Frankfurt am Main, Germany

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Abstract

The hydrogen atom is investigated, within a pseudo-complex extension of the coordinates and momenta, which introduces a minimal length scale \( l \) and results into a non-commutative Quantum Mechanics. After resuming the pseudo-complex extension of Quantum Mechanics, the modified energies of the hydrogen atom are deduced, producing corrections of the order of the square of the minimal length scale. Using the Lamb Shift, we obtain an upper boundary for the minimal length scale \( l \), orders of magnitude more restrictive than former estimations.

1 Introduction

At small distances, for example the Planck length \( l_p = 10^{-33}\text{cm} \), one expects that Quantum Mechanics is affected by a minimal length scale \( l > l_p \), i.e., that the commutation relations between coordinates and momenta may change such that even coordinates do not commute. To my knowledge, the first to propose a non-commutative Quantum Mechanics was H. S. Snyder [1]. Since then, many more contributions
where published, where I mention only the definition of the Moyal product \[2\]. Other mainstream theories, as the String Theory \[3\] and Quantum Loop Theory \[4\], have as an essential structure a minimal length, either in terms of the dimension of a string or the granulation of space-time. The resulting theory is in general quite involved.

Any novel idea to ease the handling of a minimal length scale is of importance and here I will propose such a new path: In \[5\] a proposal was presented on how to obtain a non-commutative Quantum Mechanics. The novelty consists in using a standard quantization rule in an 8-dimensional space, extending algebraically the coordinates to so-called pseudo-complex coordinates (see Section 2), doubling the dimension of the space to 8. The physical 4-dimensional space is then represented by a 4-dimensional subspace within this larger space. The attractive part of the pseudo-complex extension is its simplicity, as it uses the standard quantization rule in the 8-dimensional space, and the introduction of an effective length scale parameter, simulating a granulation of space, without having to give up Lorentz symmetry in the larger space.

In the present contribution the Hydrogen Atom is investigated within the pseudo-complex extension of Quantum Mechanics \[5\] and an upper limit of the minimal length scale is obtained. In Section 2 the pseudo-complex Quantum Mechanics in a flat space will be resumed and in Section 3 this theory will be applied to the Hydrogen atom. In Section 4 Conclusions will be drawn.

For a better understanding of the results, in the Appendix we resume the dimensions in natural units \((\hbar = c = 1)\) of the variables, parameters and operators used.

## 2 Pseudo-complex Quantum Mechanics of the Hydrogen Atom

The dynamical symmetry group of the hydrogen atom is \(SO(4)\), whose generators are the sum of the three angular momentum and three Runge-Lenz vectors \[6\] in a three dimensional space, depending on the coordinates \(x_i\) and momenta \(p_j\) \((i, j = 1, 2, 3)\). As shown in \[6\] these six operators can be redefined in terms of an operator \(L_{ij}\) where now the indices run from 1 to 4. This operator is represented in terms of a combination of four abstract coordinates and their conjugate momenta in a 4-dimensional space and an example will be given further below.
This has to be extended such that the theory contains a minimal length.

In [5] a proposal was presented for a quantization in flat Cartesian space with a minimal length parameter. The main ingredient is the definition of pseudo-complex coordinates and momenta \((i,j,k = 1,2,3,4,\) being the indices for the Cartesian coordinate in an abstract 4-dimensional space [6])

\[
\begin{align*}
X_i &= x_i + Il_i, \\
P_i &= p_i^+ + Il_i^-
\end{align*}
\]

with \(\sigma_\pm = \frac{1}{2} (1 \pm I)\) where in addition an alternative representation of the coordinates in terms of the zero-divisor components \(X_i^\pm (P_i^\pm)\) is given. Due to the property of \(\sigma_\pm^2 = 1\) and \(\sigma_+ \sigma_- = 0\) the coordinates (momenta) imply a so-called zero divisor. The coordinates \(x_i\) and \(y_i\) form an 8-dimensional space. The operator \(I\) has the property \(I^2 = 1\), which is the reason to call the coordinates and momenta pseudo-complex [8]. The standard definition and meaning of the four coordinates is explained in [6]. The reason to use pseudo-complex coordinates is explained in [8, 9], where it is shown that only these coordinates represent a viable extension from real ones, because otherwise the theory contains ghost and/or tachyon solutions.

In [5] is is shown that the pseudo-complex term of the length element \(d\omega^2 = g_{ij}dx_i dx_j\) is given by (in Cartesian components) \(\delta_{ij}dx_i dy_j\), which has to be set to zero in order that the length element is a real quantity and where \(\delta_{ij}\) is the Cartesian metric. The solution is \(u_i = \frac{dx_i}{d\tau}\) [8], where \(\tau\) is the proper time. The dimension of \(y_i\) in natural units is therefore 1. A similar consideration can be done, as M. Born did [8], defining in the momentum space a length element and in analogy to the pseudo-imaginary part of the length element in the \(x\)-space and using (1) one obtains that the units of \(p_i^\mu\) are \(\text{GeV}^2\). This is also the reason why in (1) the pseudo-imaginary part has the factor of \(l\), whose unit is \(\text{GeV}^{-1}\), such that the units of \(P_i\) remains \(\text{GeV}\). Analogue to the coordinates the \(p_i^\mu\) has to be the eigen-time derivative of \(p_i^\mu\), i.e., \(p_i^\mu = \frac{dp_i^\mu}{d\tau}\).

The condition that the length element \(d\omega^2\) is real, reduces again the 8-dimensional space to a 4-dimensional one. Thus, the physical space is a 4-dimensional subsurface embedded in the 8-dimensional pseudo-complex space.

The coordinates \(X_i (X_i^\pm)\) and momenta \(P_i (P_i^\pm)\) are quantized in
the standard way, namely

\[ [X_i, P_j] = i\delta_{ij} \rightarrow \quad \left[ X_i^\pm, P_j^\pm \right] = i\delta_{ij}, \quad \left[ X_i^\pm, P_j^\pm \right] = 0 \] \quad (2)

Using (2), it is shown in [5] that the resulting commutation relations between standard coordinates \((x_i)\) and momenta \((p_j)\) are (here rewritten in terms of Cartesian coordinates)

\[ [x_i, x_j] = -i\hbar \delta_{ij}, \quad \left[ x_i, p_j \right] = i\hbar \delta_{ij} \] \quad (3)

With the help of the Appendix it can be shown that the dimensions are maintained. The relations in (3) are similar to those obtained by H. S. Snyder [1].

The generators of the pseudo-complex SO(4) group, \(SO_{pc}(4)\), are given by \((i, j = 1, 2, 3, 4)\)

\[ L_{ij} = X_i P_j - X_j P_i = L_{ij}^+ \sigma_+ + L_{ij}^- \sigma_- \, , \quad L_{ij}^\pm = X_i^\pm P_j^\pm - X_j^\pm P_i^\pm \] \quad (4)

These pseudo-complex generators satisfy the commutation relations

\[
\begin{align*}
[L_{ij}, L_{kl}] &= i(\delta_{jk} L_{qi} + \delta_{qj} L_{ik} + \delta_{ik} L_{jq} + \delta_{jq} L_{ki}) \\
[L_{ij}^\pm, L_{kl}^\pm] &= i(\delta_{jk} L_{qi}^\pm + \delta_{qj} L_{ik}^\pm + \delta_{ik} L_{jq}^\pm + \delta_{jq} L_{ki}^\pm) \\
[L_{ij}^\pm, L_{kl}^-] &= 0
\end{align*}
\] \quad (5)

Thus, the quantization rules are the same as in standard Quantum Mechanics, with the difference that it is now in an 8-dimensional space. However, this is no longer the case when the real and pseudo-imaginary part of \(L_{ij}\) are considered separately. As can be seen by (3) the algebra in these components are non-commutative [5]. In this way, the non-commutative behavior of the coordinates is represented by the much simpler commutation relations in the components of the zero-divisor.

Before discussing the Hydrogen Atom in the pseudo-complex formulation, we have to define various operators and relations: Using [1] and (4), we express the \(L_{ij}^\pm\) in terms of operators which depend on \(x_i\), \(y_j\) and their momenta, i.e.,

\[
\begin{align*}
L_{ij}^\pm &= \left(L_{ij}^x + L_{ij}^y\right) \pm \hbar \left(L_{ij}^{xy} + L_{ij}^{yx}\right) \\
L_{ij}^x &= x_i p_j^x - x_j p_i^x \, , \quad L_{ij}^y = y_i p_j^y - y_j p_i^y \\
L_{ij}^{xy} &= x_i y_j^x - x_j y_i^x \, , \quad L_{ij}^{yx} = y_i p_j^x - y_j p_i^x \, .
\end{align*}
\] \quad (6)
Though, for the operators the structure look like the generators of $SO(4)$ groups, they are not [5]. The reason lies in the non-commutative behavior of the coordinates and momenta. Using natural units (see the Appendix), the units of the operators in (6) are $[L_{ij}^x] = 1$, $[L_{ij}^y] = \text{GeV}^2$, $[L_{ij}^{xy}] = \text{GeV}$.

The real (pseudo-imaginary) part is obtained by taking the sum (difference) of the plus and minus zero-divisor component of the generators, which leads to

$$L_{ij}^R = \frac{1}{2} \left( L_{ij}^+ + L_{ij}^- \right) = L_{ij}^x + l^2 L_{ij}^y , \quad L_{ij}^I = \frac{1}{2} \left( L_{ij}^+ - L_{ij}^- \right) = l \left( L_{ij}^{xy} + L_{ij}^{yx} \right) ,$$

(7)

where $R$ refers to the real and $I$ to the pseudo-imaginary part. By construction the $L_{ij}^R$ and $L_{ij}^I$, the pseudo-real and pseudo-imaginary part of $L_{ij} = L_{ij}^R + L_{ij}^I$, form an algebra $SO_R(4)$ and $SO_I(4)$ respectively. Note, that $[L^R] = 1$ and $[L^I] = \text{GeV}$. The same hold for the operators $M_{ij}^R$ and $M_{ij}^I$.

Following [6], one defines the operators

$$L^2 = L_1^2 + L_2^2 + L_3^2 , \quad M^2 = M_1^2 + M_2^2 + M_3^2 ,$$

(8)

with $L_1 = L_{23}$, $L_2 = L_{13}$, $L_3 = L_{12}$, $M_1 = L_{14}$, $M_2 = L_{24}$, $M_3 = L_{34}$. The $SO_{pc}(4)$ has two pseudo-complex Casimir operators, where we need only the relevant one for our purpose, namely [6]

$$C^R = \frac{1}{2} \left( L_R^2 + M_R^2 \right) \rightarrow 2k_R(k_R + 1) ,$$

(9)

where on the right its eigenvalue is listed, with $k_R$ is integer or half-integer.

### 3 The Hydrogen Atom as an Example

According to [6], the Hamiltonian of the Hydrogen Atom is given by (in natural units)

$$H = -\frac{\mu e^2}{4 \left( C^x + \frac{1}{2} \right)} ,$$

(10)

with $C^x$ the operator which for $l = 0$ is a Casimir operator of the $SO(4)$ group [6]. The $e$ is the elementary charge and $\mu$ is the reduced
mass of the proton-electron system, i.e., approximately the electron mass \( m_e \). The units are \( [\mu] = \text{GeV}, [e^2] = 1 \). The \( C^x \) can be expressed as
\[
C^x = \frac{1}{2} (L_z^2 + M_z^2).
\]
For \( l > 0 \) the \( C^x \) operator is in general not a Casimir operator of a \( SO(4) \) group.

In order to express the Hamiltonian in terms of the \( L \) and \( M \) operators, we use (7) and the same for the \( M^x_{ij} \) and \( M^R_{ij} \). I.e., we have
\[
L^x_{ij} = L^R_{ij} - l^2 L^y_{ij} \quad \text{and} \quad M^x_{ij} = M^R_{ij} - l^2 M^y_{ij}.
\]
The \( C^x \) operator is written, up to the leading term of corrections in the minimal length, as
\[
C^x \approx C^R - \sum_{\mu \nu} \left[ (L^R \cdot L^\mu) + (M^R \cdot M^\nu) \right].
\]
Substituting this into (10) and expanding again in leading order in \( l^2 \), we arrive at
\[
H \approx -\frac{\mu e^2}{4(C^R + \frac{1}{2})} \left( 1 + l^2 \frac{[L^R \cdot L^y] + (M^R \cdot M^y)}{4(C^R + \frac{1}{2})} \right).
\]
Note, that the units of \( l^2 \) is GeV\(^{-2} \) and that the factor is of the units GeV\(^2 \) (see section 2), i.e., the total unit of the product is 1. What is not included here is the correction due to the Lamb shift. The correction calculated here is only due to the minimal length correction, i.e., the \( l \)-dependent contribution should be smaller than the one of the Lamb shift, otherwise it should have been measured already.

The operator \( 4 \left( C^R + \frac{1}{2} \right) \) in the denominator has the eigenvalue \( 2(2kR + 1)^2 \). The expression in the numerator of the \( l^2 \) dependent term is also of the same order. Its matrix elements can be calculated noting that its suffices to neglect there the contribution of \( l^2 \), any contribution due to the minimal scale is of higher order. Thus, the correction is estimated as \( l^2 \otimes O(1) \). The Lamb shift is of the order of 1000Hz, corresponding to about \( \Delta E \approx 4 \times 10^{-9} \text{eV} \), while the absolute value of the energy of the ground state is about \( 13 \text{eV} \). Assuming the factor of \( l^2 \) in (12) is of the order of 1GeV\(^2 \), we obtain for the correction the parenthesis in (12) \( l^2 \text{GeV}^2 \approx \frac{\Delta E}{13} \text{GeV}^2 \approx 3 \times 10^{-10} \text{GeV}^{-2} \), which corresponds to a length parameter of about \( 1.7 \times 10^{-5} \text{GeV}^{-1} \), or \( 3.5 \times 10^{-6} \text{fm} = 3.5 \times 10^{-19} \text{cm} \). In other words, in order that the effect of the length parameter shows up in the measurement of the order of the Lamb shift, it has to have this value. Of course we do not see it yet, which leads to an upper estimate \( l < 3.5 \times 10^{-19} \text{cm} \).

This is a more stringent restriction as obtained in [10], within the Born-Infeld theory, where an upper limit for the maximal acceleration
of $A_m > 10^{22} \frac{m}{sec^2}$ was obtained. The minimal length is related to this maximal acceleration via $l = (A_m)^{-1}$, which with $c = 1$ translates to an $l$ of $10^{-7}$ cm within the Born-Infeld theory.

4 Conclusions

The motivation of this contribution was to obtain an estimation for the upper limit of a minimal length in the context of the Hydrogen atom, using pseudo-complex Quantum Mechanics, which provides this minimal length. The formulation differs from [5] in that the explicit dynamical symmetry $SO(4)$ of the hydrogen atom is exploited. The space considered is flat and the definitions follow the one of [6].

For $l = 0$, the Hamiltonian of the Hydrogen atom depends on a second order Casimir operator of $SO_x(4)$, where the $x$ indicates that it is defined in the coordinates space $x_k$ ($k = 1, 2, 3, 4$). When $l > 0$, the corresponding operator in terms of $x_k$ and $p_k$ is no longer a Casimir operator, but can be expressed in terms of the one in $SO_R(4)$, where $R$ refers to the pseudo-real space part.

After having presented the essential mathematics of the $SO_{pc}(4)$ structure and its relation to the $(x_k, y_k)$ space, the Hamilton operator of the Hydrogen atom was investigated, with the result that the corrections to the level energies of the Hydrogen atom is of the order of $l^2$. Comparing to the Lambshift an upper limit of $l \leq 3.5 \times 10^{-19}$ cm was obtained, the best one up to now.

Finally, I want to stress that the use of a length parameter, which is unaffected by a Lorentz contraction, is of a quite practical use, because Lorentz-invariance can be maintained, simplifying a lot not only calculations but also the investigation of the effects of a minimal length. This description might be still useful as an effective description of the granulation of space, in case it results that there is a minimal physical length which violates Lorentz-invariance.

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Appendix: Units

The natural units are used, defined as: $\hbar = c = 1$. In what follows, only approximate numerical values for some constants are given, because exact values are not of interest here. The condition $c = 1 \approx 3 \times 10^8 \frac{\text{m}}{\text{sec}}$ tells us that time has units of length and its value is $1\text{sec} = 3 \times 10^8 \text{m}$. From there, one obtains a relation of length with energy: $1\hbar c \approx 200\text{MeVfm}$, namely $1\text{fm} = 10^{-13} \text{cm}$ $1\text{fm} \approx 5\text{GeV}^{-1}$.

Using that $1eV \approx 1.6 \times 10^{-19} \frac{\text{kg m}^2}{\text{sec}^2}$, we obtain the kg in units of GeV: $1\text{kg} \approx 6 \times 10^{26} \text{GeV}$.

For the pc-coordinates $X_\mu = x_\mu + lIy_\mu$ the units are $[X_\mu] = [x_\mu] = [l] = \text{GeV}^{-1}$, $[y_\mu] = 1$ and for the pc-momenta $P_\mu = p_\mu + lIp_\mu$ $[P_\mu] = [p_\mu] = \text{GeV}$, $[p_\mu] = \text{GeV}^2$.

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