Pseudo maximal submodules

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Abstract. “Let R be a commutative ring with identity and β be left unitary R – module”. We introduced in this paper the concept of pseudo maximal submodule which was a new generalization of the concept of maximal submodule, several basic properties, of this concept are given. Furthermore, characterization of this concept in the class of multiplication modules are introduced. “Moreover the pseudo Jacobson radical of modules and submodules are investigated”.

Keywords: Pseudo maximal submodules, quasi-essential submodules, Multiplication modules, Pseudo Jacobson radical of modules.

1. Introduction
Throughout this research all rings are commutative with identity and all R – modules are left unitary. A Proper submodule W of an R – modules β is called maximal if F is a submodule of β with W ⊆ F, implies that F = β, that is there is no proper submodule of β containing W properly [1]. The concept of maximal submodule was generalized recently to almost maximal submodules, Weak maximal submodule. Nearly maximal submodules, 2–maximal submodules see [2,3,4,5]. In this note we introduced the concept pseudo maximal submodule as another generalization of maximal submodule, where a proper submodule W of an R – module β is said to be a pseudo maximal if whenever A is a quasi-essential submodule of β with W ⊆ A β, implies that A = β. A submodule H of an R – module β is called quasi-essential if H∩Q ≠ (0) for every non-Zero quasi prime submodule Q of β [6] , where a proper submodule Q of an R – module β is called a quasi-prime, if rsb ∈ Q for r, s ∈ R, b ∈ β, implies that either rb ∈ Q or sb∈ Q [7] . “Many basic proprieties, examples and characterization are introduced It is well known that [Wβ J = {x∈β :x∈ W } is a submodule of β for each ideal J of R with W ⊆ [Wβ J] [8]”. A quasi prime radical of submodule W of an R – module β denoted by Q Pradβ(W) is defined as the intersection of all quasi prime submodule of β which contain W [9]. Recall that an R – module β is a multiplication if every submodule W of β is of the form W = Iβ for some ideal I of R [10].” Recall that an R – module β is faithful if ann (β) = (0) [1]”. “Finally we remark that all R – modules under our study contain quasi prime submodules.”
2. Basic properties of pseudo maximal submodules

In this part of this paper we will know the concept of pseudo maximal submodule and give many basic properties of this concept.

2.1. Definition

A proper submodule $W$ of an $R$-module $\beta$ is said to be a pseudo maximal, if whenever a quasi-essential submodule $A$ of $\beta$ with $W \subseteq A \subseteq \beta$, implies $A = \beta$. That is there is no proper a quasi-essential submodule of $\beta$ containing $W$ properly.

An ideal $\mathfrak{I}$ of a ring $R$ is said to be pseudo maximal if $\mathfrak{I}$ is a pseudo maximal $R$-module of an $R$-module $R$.

2.2. Remark

It is clear that every maximal submodule is a pseudo maximal but not conversely, the following example shows that:

Consider the $Z$-module $2Z \oplus 2Z$ the submodule $4Z \oplus 0$ is a pseudo maximal but not maximal submodule because the only submodule of $2Z \oplus 2Z$ which contain $4Z \oplus 0$ properly is $2Z \oplus 0$, which is not quasi-essential submodule of $2Z \oplus 2Z$ (since $(2Z \oplus 0) \cap (0 \oplus 2Z) = (0)$ for $0 \oplus 2Z$ is a quasi-prime submodule $Z \rightarrow$ module $2Z \oplus 2Z$. Thus $4Z \oplus 0$ is a pseudo maximal.

But $4Z \oplus 0$ is not maximal because $4Z \oplus 0 \subset 2Z \oplus 0 \subset 2Z \oplus 2Z$.

2.3. Proposition

Let $\beta$ be an $R$-module $W$ and $F$ be proper submodules of $\beta$ with $W \subseteq F$ and $W$ is a pseudo maximal submodule of $\beta$. Then $F$ is a pseudo maximal submodule of $\beta$.

2.3.1. Proof. Assume that $F$ is not a pseudo maximal submodule of $\beta$, That is there exist a quasi-essential submodule $Q$ of $\beta$ such that $F \subset Q \subseteq \beta$. But $W \subseteq F$, implies that $W \subseteq Q \subseteq \beta$, it follows that $W$ is not pseudo maximal submodule of $\beta$ which is a contradiction. Hence $F$ is pseudo maximal submodule of $\beta$.

“The following corollaries are direct consequence of proposition (2.3)”

2.4. Corollary

Let $W, F$ be proper submodules of an $R$-module $\beta$ with $W \cap F$ is a pseudo maximal submodule of $\beta$. Then both $W$ and $F$ are pseudo maximal submodules of $\beta$.

2.5. Corollary

Let $W, F$ be a pseudo maximal submodules of an $R$-module $\beta$. Then $W + F$ is pseudo maximal submodule of $\beta$.

2.6. Corollary

Let $\beta$ be an $R$-module and $W$ is a pseudo maximal submodule of $\beta$, then $[W \beta] J$ is a pseudo maximal submodule of $\beta$ for each ideal $J$ of $R$.

The following proposition shows the behavior of a pseudo maximal submodule under $R$-homomorphism.
2.7. Proposition
Let $\mathcal{F}: \beta_1 \to \beta_2$ be an $R$ - epimorphism and $W$ is pseudo maximal submodule of $\beta_1$ with $\ker \mathcal{F} \subseteq \text{Q Prad}_\beta (W) \subseteq W$. Then $\mathcal{F}(W)$ is pseudo maximal submodule of $\beta_2$.

2.7.1. Proof.
It is clear that $\mathcal{F}(W)$ is a proper submodule of $\beta_2$. Assume that $\mathcal{F}(W)$ is not pseudo maximal submodule of $\beta_2$, that is there exist a quasi-essential submodule $F$ of $\beta_2$ such that $\mathcal{F}(W) \subseteq F \subseteq \beta_2$, it follows that $\mathcal{F}^{-1}(F) \subseteq \beta_1$. Since $\mathcal{F}$ is an epimorphism and $\ker \mathcal{F} \subseteq \mathcal{F}(\text{Q Prad}(W)) \subseteq W$, then $W \subseteq \mathcal{F}^{-1}(F) \subseteq \beta_1$. But $F$ is a quasi-essential submodule of $\beta_2$, then by [6, prop.(2.3)] $\mathcal{F}^{-1}(F)$ is a quasi-essential submodule of $\beta_1$. That is $W$ is not pseudo maximal submodule of $\beta_1$ which is a contradiction. Hence $\mathcal{F}(W)$ is a pseudo maximal submodule of $\beta_2$.

“The following corollary is a direct Consequence of proposition (2.7)”.

2.8. Corollary
Let $W$ be a pseudo maximal submodule of an $R$ - module $\beta$, and $F$ is a sub module of $W$ with $F \subseteq \text{Q Prad}(\beta)$. Then $\frac{W}{F}$ is a pseudo maximal submodule of $\frac{\beta}{F}$.

Before we give the next results we need to introduce the following definition.

2.9. Definition
An $R$ - module $\beta$ is called a pseudo semi simple if $\beta$ has no proper quasi-essential submodule of $\beta$.

That is if a submodule $W$ of $\beta$ is a quasi-essential of $\beta$ then $W = \beta$.

2.10. Proposition
Let $\beta$ be an $R$ - module. Then the zero submodule of $\beta$ is a pseudo maximal sub module of $\beta$ if and only if $\beta$ is a pseudo semi simple $R$ - module.

2.10.1. Proof. ($\Rightarrow$) suppose that the zero submodule of $\beta$ is a pseudo maximal submodule of $\beta$, then $\beta$ has no proper quasi-essential submodule. That is $\beta$ is a pseudo semi simple $R$-module.

($\Leftarrow$) suppose that $\beta$ is a pseudo semi simple $R$ - module then $\beta$ has no proper quasi-essential submodule. That is every submodule of $\beta$ is a pseudo maximal. Thus zero submodule is a pseudo maximal submodule of $\beta$.

“The following corollary is a direct application of proposition (2.10).”

2.11. Corollary
Let $\beta$ be an $R$ - module, then the following statements are equivalent:

1- $\beta$ is a pseudo semi simple $R$ - module.
2- Zero submodule of $\beta$ is a pseudo maximal submodule.
3- Every proper submodule of $\beta$ is a pseudo maximal submodule of $\beta$.

We need to introduce the following definition.
2.12. **Definition**
An $R - \text{module } \beta$ is called $\mathcal{P}m - \text{module}$, if every proper submodule of $\beta$ is pseudo maximal. And a ring $R$ is $\mathcal{P}m - \text{ring}$ if every proper ideal of $R$ is a pseudo maximal ideal of $R$.

2.13. **Examples**
1- The $Z - \text{module } Z_{15}$ is $\mathcal{P}m - \text{module}$.
2- The $Z - \text{module } Z$ is not $\mathcal{P}m - \text{module}$ because the submodule (6) is not a pseudo maximal submodule of $Z$.

2.14. **Proposition**
Every pseudo semi simple $R - \text{module}$ is $\mathcal{P}m - \text{module}$.

2.14.1. **Proof**: Direct from corollary (2.11).

2.15. **Proposition**
Let $\beta$ be an $R - \text{module}$ with every proper quasi-essential submodule of $\beta$ is essential, and $\beta = \beta_1 \oplus \beta_2$ where $\beta_1, \beta_2$ are modules such that $\text{ann } \beta_1 + \text{ann } \beta_2 = R$. If $\beta_1$ and $\beta_2$ are $\mathcal{P}m - \text{module}$, then $\beta$ is $\mathcal{P}m - \text{module}$.

2.15.1. **Proof**: Let $W$ be a proper submodule of $\beta$, and $F$ be a submodule of $\beta$ such that $W \subseteq F \subseteq \beta$, where $F$ is a quasi-essential submodule of $\beta$, since $\text{ann } \beta_1 + \text{ann } \beta_2 = R$ then $W = W_1 \oplus W_2$ for some submodules $W_1$ of $\beta_1, W_2$ of $\beta_2$ and $F = F_1 \oplus F_2$ for some submodules $F_1$ of $\beta_1, F_2$ of $\beta_2$ [11]. If both $W_1, W_2$ are proper submodules of $\beta_1, \beta_2$ respectively then we have $W_1 \oplus W_2 \subseteq F_1 \oplus F_2 \subseteq \beta_1 \oplus \beta_2$, where $F_1 \oplus F_2$ is a quasi-essential submodule of $\beta_1 \oplus \beta_2$ thus by [6, prop (1.11)] $F_1$ is a quasi-essential of $\beta_1$ and $F_2$ is a quasi-essential of $\beta_2$. But both $\beta_1$ and $\beta_2$ are $\mathcal{P}m - \text{module}$, then $F_1 = \beta_1$ and $F_2 = \beta_2$, it follows that $F = F_1 \oplus F_2 = \beta_1 \oplus \beta_2 = \beta$, implies that $\beta$ is a $\mathcal{P}m - \text{module}$.

If $W_1$ is a proper submodule of $\beta_1$ and $W_2 = \beta_2$ then $W = W_1 \oplus \beta_2$ but $\beta_1$ is $\mathcal{P}m - \text{module}$, then $F = \beta_1$, hence $F = F_1 \oplus F_2 = \beta_1 \oplus \beta_2 = \beta$. Thus $\beta$ is $\mathcal{P}m - \text{module}$.

Similarly if $W_2$ is a proper submodule of $\beta_2$ and $W_1 = \beta_1$, that is $W = \beta_1 \oplus W_2$.

2.16. **Proposition**
Every multiplication $R - \text{module}$ contains a pseudo maximal submodule.

2.16.1. **Proof**: Let $\beta$ be a multiplication $R - \text{module}$, then $\beta$ contain a maximal submodule [12]. Thus by remark (2.2) $\beta$ contain a pseudo maximal submodule.

It is well known a cyclic $R - \text{module}$ is a multiplication [10].

We get the following direct consequence of proposition (2.16).

2.17. **Corollary**
Every cyclic $R - \text{module}$ contains a pseudo maximal submodule.

2.18. **Proposition**
Let $\beta$ be a non zero multiplication $R - \text{module}$ with only one pseudo maximal submodule which contain all proper submodules of $\beta$. if $W$ is non zero pseudo maximal submodule of $\beta$. Then $W$ is a quasi-essential submodule of $\beta$.

2.18.1. **Proof**: Let $Q$ be a quasi-prime submodule of $\beta$ such that $Q \cap W = (0)$. But $\beta$ is multiplication it follows that $Q$ contained in some maximal [12] (hence pseudo
maximal) submodule of $\beta$. But $\beta$ has only one pseudo maximal submodule which is $W$. Hence $Q \subseteq W$, it follows that $Q = (0)$. That is $W$ is a quasi-essential submodule of $\beta$.

2.19. Remark
If $W$ is a pseudo maximal submodule of $\beta$ then $[W, \beta]$ need not to be a pseudo maximal submodule of $\beta$. The following example shows that Consider the $Z - \text{module}$ $Z_{15}$, the submodule $<O>$ is a pseudo maximal submodule of $Z_{15}$, but $[<O>, Z_{15}] = Z_{15}$ is not pseudo maximal ideal of the $Z - \text{module}$ $Z$.

2.20. Proposition
Let $\beta$ be a faithful multiplication $R - \text{module}$ and $W$ is a proper submodule of $\beta$. Then $W$ is a pseudo maximal submodule of $\beta$ if and only if $[W, \beta]$ is a pseudo maximal ideal of $R$.

2.20.1. Proof. ($\rightarrow$) suppose $W$ is a pseudo maximal submodule of $\beta$, and let $[W, \beta] \subseteq J \subseteq R$ where $J$ is a quasi-essential ideal of $R$. It follows that $[W, \beta] \beta \subseteq J \beta \subseteq R \beta = \beta$. Since $\beta$ is a multiplication then $W = [[W, \beta] \beta] \beta$ and since $J$ quasi-essential ideal of $R$, then by [6, prop.3.1] $J \beta$ is quasi-essential submodule of $\beta$. That is $W \subseteq J \beta \subseteq \beta$. But $W$ is a pseudo maximal then $J \beta = \beta = R \beta$. But $\beta$ is faithful finitely generated then by [12] $J = R$, hence $[W, \beta] \beta$ is a pseudo maximal ideal of $R$.

($\leftarrow$) suppose that $[W, \beta] \beta$ is a pseudo maximal ideal of $R$ and let $W \subseteq F \subseteq R \beta$ where $F$ is a quasi-essential submodule of $\beta$, then $F = J \beta$ for some quasi-essential ideal $J$ of $R$.

$[W, \beta] \beta \subseteq J \beta \subseteq R \beta$. Since $\beta$ is faithful finitely generated then $[W, \beta] \beta \subseteq J \subseteq R$. But $[W, \beta] \beta$ is a pseudo maximal ideal of $R$, then $J = R$, it follows that $J \beta = R \beta = \beta$.

That is $F = \beta$. Hence $W$ is a pseudo maximal submodule of $\beta$.

It is well known cyclic $R - \text{module}$ is finitely generated we get the following corollary.

2.21. Corollary
Let $\beta$ be a faithful cyclic $R - \text{module}$ and $W$ is a proper submodule of $\beta$. Then $W$ is a pseudo maximal submodule of $\beta$ if and only if $[W, \beta]$ is a pseudo maximal ideal of $R$.

We need to recall the following lemma before we introduce the next propositions.

2.22. Lemma
“Let $\beta$ be a finitely generated multiplication $R - \text{module}$ and $I, J$ are ideals of $R$. Then $I \beta \subseteq J \beta$ if and only if $I \subseteq J + \text{ann} (\beta)$ [13, Coro of Theo(9)].”

2.23. Proposition
Let $\beta$ be a finitely generated faithful multiplication $R - \text{module}$ and $J$ is a pseudo maximal ideal of $R$. Then $J \beta$ is a pseudo maximal submodule of $\beta$.

2.23.1. Proof. Suppose that $J$ is a pseudo maximal ideal of $R$, and let $J \beta \subseteq W \subseteq \beta$ where $W$ is quasi-essential submodule of $\beta$. But $\beta$ is multiplication then $W = J \beta$ for some quasi-essential ideal of $R$. That is $J \beta \subseteq J \subseteq R \beta$. Since $\beta$ is a finitely generated multiplication, then by lemma (2.22) $J \beta \subseteq J + \text{ann} (\beta) \subseteq R$. But $\beta$ is faithful, implies
that \( \text{ann} (\beta) = (0) \), thus \( J \subset J \subseteq R \). But \( J \) is a pseudo maximal ideal of \( R \) it follows that \( J = R \). Hence \( J \beta = R \beta = \beta \), it follows that \( W = \beta \).
That is \( J \beta \) is pseudo maximal submodule of \( \beta \).

2.24. Corollary
Let \( \beta \) be a faithful cyclic \( R \) module and \( I \) is a pseudo maximal ideal of \( R \). Then \( I \beta \) is a pseudo maximal submodule of \( \beta \).

Recall that an \( R \) module \( \beta \) is cancellation if \( I \beta = J \beta \) for any ideals \( I, J \) of \( R \), implies that \( I = J \) [14].

“It is well – known if \( \beta \) is a finitely generated faithful multiplication, then \( \beta \) is cancellation [14, prop (3.1)].”

2.25. Proposition
Let \( \beta \) be a faithful finitely generated multiplication \( R \) module and \( W \) is proper submodule of \( \beta \). Then the following statement are equivalent:

1- \( W \) is a pseudo maximal submodule of \( \beta \).
2- \( [W_{\beta}] \) is a pseudo maximal ideal of \( R \).
3- \( W = I \beta \) for some pseudo maximal ideal of \( R \).

2.25.1. Proof. (1) \( \rightarrow \) (2) follows by proposition (2.20). (2) \( \rightarrow \) (3) suppose that \( [W_{\beta}] \) is a pseudo maximal ideal of \( R \), and \( W \) is a sub module of \( \beta \). Since \( \beta \) is a multiplication, then \( W = [W_{\beta}] \beta \) that is \( W = [W_{\beta}] \beta = I \beta \) where \( I = [W_{\beta}] \beta \) is pseudo maximal ideal of \( R \).

(3) \( \rightarrow \) (2) Suppose that \( W = I \beta \) for some pseudo maximal ideal \( I \) of \( R \). That is \( W = [W_{\beta}] \beta = I \beta \). But \( \beta \) is faithful finitely generated multiplication, then \( \beta \) is cancellation, that is \( [W_{\beta}] \beta = I \) a pseudo maximal ideal of \( R \).

The following corollary is a direct consequence of proposition (2.25).

2.26. Corollary
Let \( \beta \) be a faithful cyclic \( R \) module, and \( W \) is a proper submodule of \( \beta \). Then the following statements are equivalent:

1- \( W \) is a pseudo maximal submodule of \( \beta \).
2- \( [W_{\beta}] \) is a pseudo maximal ideal of \( R \).
3- \( W = I \beta \) for some pseudo maximal ideal of \( R \).

2.27. Proposition
Let \( \beta \) be a faithful finitely generated multiplication \( R \) module. Then \( \beta \) is \( \mathcal{P}m \) module if and only if \( R \) is \( \mathcal{P}m \) ring.

2.27.1. Proof. \( (\rightarrow) \) suppose that \( \beta \) is a \( \mathcal{P}m \) module, and \( I \) be a proper ideal of \( R \). Since \( \beta \) is multiplication then \( W = I \beta \), but \( \beta \) is \( \mathcal{P}m \) module then \( W \) is a pseudo maximal submodule of \( \beta \). Thus by proposition (2.25) \( J \) is a pseudo maximal ideal of \( R \). Therefore \( R \) is \( \mathcal{P}m \) ring.

\( (\leftarrow) \) let \( R \) be \( \mathcal{P}m \) ring , and \( W \) is a proper submodule of \( \beta \). Since \( \beta \) is a multiplication then \( W = I \beta \) for some ideal \( J \) of \( R \). But \( R \) is \( \mathcal{P}m \) ring , then \( I \) is
pseudo maximal ideal of $R$. Thus by proposition (2.25) $W$ is a pseudo maximal submodule of $\beta$. Hence $\beta$ is a $Pm$-module.

“We introduce the definition of pseudo maximal Jacobson radical of modules”.

2.28. Definition
Let $\beta$ be an $R$-module, the pseudo Jacobson radical of $\beta$ denoted by $PJ(\beta)$, and defined by $PJ(\beta) = \{ \{W: W \text{ is a pseudo maximal submodule of } \beta \} \text{ if there is no pseudo maximal submodule in } \beta \text{ we say that } PJ(\beta) = \beta \}$. $PJ(R)$ is the intersection of all pseudo maximal ideals of $R$. It is clear that $PJ(\beta) \subseteq J(\beta)$.

2.29. Proposition
If $\beta$ is a pseudo semi simple $R$-module then $PJ(\beta) = (0)$.

2.29.1. Proof. Let $\beta$ be a pseudo semi simple $R$-module, then by corollary (2.11) every proper submodule of $\beta$ is a pseudo maximal. In particular, zero submodule of $\beta$ is a pseudo maximal sub module of $\beta$. Thus $PJ(\beta) = (0)$.

2.30. Remark
It is clear that, if $\beta$ is a $Pm$-module then $PJ(\beta) = (0)$.

2.31. Proposition
Let $\beta$ be a faithful finitely generated and multiplication $R$-module then $PJ(\beta) = \cap \{ I : I is a pseudo maximal ideal of \} R \}$. 

2.31.1. Proof. Put $W = \cap \{ I : I is a pseudo maximal ideal of R \}$. If $PJ(\beta) = \beta$ then $W \subseteq PJ(\beta)$. Thus we assume that $PJ(\beta) \neq \beta$ and $H$ is a pseudo maximal submodule of $\beta$. But $\beta$ is a faithful multiplication module then by Proposition (2.20) $[H_R \beta]$ is a pseudo maximal ideal of $R$. But $W \subseteq [H; \beta]$ and by definition of $PJ(\beta)$, $W \subseteq PJ(\beta)$. Now let $I$ be a pseudo maximal ideal of $R$. Since $\beta$ is faithful finitely generated multiplication then by Proposition (2.23) $I \beta$ is a pseudo maximal submodule of $\beta$. Thus $PJ(\beta) \subseteq I \beta$, hence $PJ(\beta) \subseteq W$. Therefore $W = PJ(\beta)$.

2.32. Corollary
Let $\beta$ be a faithful finitely generated multiplication to $R$-module. Then $PJ(R) \beta \subseteq PJ(\beta)$.

2.32.1. Proof. Let $I$ be a pseudo maximal ideal of $R$, then $PJ(R) \subseteq I$. It follows that $PJ(R) \beta \subseteq I$. But $\beta$ is a faithful finitely generated multiplication then by Proposition (2.31) $PJ(R) \beta \subseteq PJ(\beta)$.

2.33. Proposition
Let $\beta$ be a faithful finitely generated and multiplication $R$-module, and $PJ(R)$ is a pseudo maximal ideal of $R$. Then $PJ(R)\beta = PJ(\beta)$.

2.33.1. Proof. Since $PJ(R)$ is a pseudo maximal ideal of $R$ and $\beta$ is a faithful finitely generated and multiplication then by Proposition (2.23) $PJ(R) \beta$ is a pseudo maximal
2.34. Corollary
Let $R$ be a ring contains only one pseudo maximal ideal and $\beta$ be a faithful finitely generated multiplication an $R$– module. Then $\text{PJ}(\beta) = \text{PJ}(R) \beta$.

2.34.1. Proof. Follows by Proposition (2.31).

Now we introduce the definition of pseudo Jacobson radical of submodule of $\beta$.

2.35. Definition
Let $W$ be a submodule of an $R$– module $\beta$. The $\text{PJ}(W)$ is the pseudo Jacobson radical of $W$ defined as the intersection of all pseudo maximal submodules of $\beta$ containing $W$. And $\text{PJ}(I)$ where $I$ is an ideal of $R$ is the intersection of all pseudo maximal ideals of $R$ containing $I$.

The next result gives some basic properties of pseudo Jacobson radical of submodule.

2.36. Proposition
Let $\beta$ be an $R$– module and $W$, $F$ are submodule of $\beta$. Then the following statements hold.

1- $W \subseteq \text{PJ}(W)$.
2- $\text{PJ}(\text{PJ}(W)) = \text{PJ}(W)$.
3- $\text{PJ}(W \cap F) \subseteq \text{PJ}(W) \cap \text{PJ}(F)$.

2.36.1. Proof. It is easy so we omitted

2.37. Proposition
Let $\beta$ be a $\mathcal{P}m$– module , and $F$ be a submodule of $\beta$. Then $\text{PJ}(F) = F$.

2.37.1. Proof. From definition $\text{PJ}(F) = \cap \{V: V$ is a pseudo maximal submodule of $\beta$ with $F \subseteq V \}$. But $\beta$ is $\mathcal{P}m$– module, it follows that $F$ is a pseudo maximal submodule of $\beta$. Thus $\text{PJ}(F) = F$.

References

[1] Goodearl K R 1976 Ring Theory: Nonsingular Rings and Modules 33 (Utah CRC Press).
[2] Hadi I M, Ali R K 2008 On almost maximal submodules Ibn Al-Haitham Journal for pure and Applied Sciences 1 pp 190-97.
[3] Shwkaea M R 2012 Weak maximal ideals and weak submodules M.Sc. Thesis Univ. of Tikrit.
[4] Ahmed M A 2016 Nearly maximal submodules Basrah journal of science 34(2) pp 1-6.
[5] Mohammed Akram S & Sallman Mohammad D 2017 2-Maximal submodules and related concepts J. of Univ. of Anbar for Pure science 11(3) pp 102-09.
[6] Mohammadali Haibat K 2006 Quasi-essential submodules Al-Faith journal 2(27) pp 199-211.
[7] Abdul-Razak H M 1999 Quasi-prime modules and quasi-prime submodules MD Thesis Univ. of Baghdad.
[8] Sharpe T, Sharpe D W and Vámoss P 1972 Injective modules 62 Cambridge University Press.
[9] Zeina S K 2012 Generalization for some concept of prime submodules M.Sc. Thesis Univ. of Tikrit.
[10] Barnard A 1981 Multiplication modules J. ALGEBRA. 71(1) pp 174-78.
[11] Abbas M S 1990 On fully stable modules Ph. D. Thesis *Univ. of Baghdad.*
[12] El-Bast Z A & Smith P P 1988 Multiplication modules *Communications in Algebra* 16(4) pp 755-79.
[13] Smith P F 1988 Some remarks on multiplication modules *Archiv der Mathematik* 50(3) pp 223-35.
[14] Mijbass A S 1992 On cancellation modules M.Sc. Thesis *Univ. of Baghdad.*