Hypergraph States in $SU(N)_1$, $N$ odd prime, Chern-Simons Theory

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Abstract: Graph states and hypergraph states can be constructed from products of basic operations that appear in $SU(N)_1$. The level-rank dual of a theorem of Salton, Swingle, and Walter implies that these operations can be prepared topologically in the $n$-torus Hilbert space of Chern-Simons theory for $N \neq 5 \mod 4$.

For $SU(N)_1$, $N = 5 \mod 4$, only stabilizer states can be prepared on the $n$-torus Hilbert space, which restricts the construction to graph states.
1 Introduction

The generalized Pauli group and Clifford operators [1–3] are obtained from products of basic operations of SU($N)_1$ Chern-Simons theory for $N$ odd prime. Graph and hypergraph states [4–8] are described as products of such operations, which allow for explicit representations of graph and hypergraph states. For $N$ odd prime, $N \neq 5 \mod 4$, these operations can be obtained topologically in the $n$-torus Hilbert space of Chern-Simons theory as a result of the level rank dual [9] of a theorem of Salton, Swingle, and Walter [10]. Thus graph and hypergraph states of SU($N)_1$, $N \neq 5 \mod 4$, are topological on the $n$-torus Hilbert space of Chern-Simons theory.

For SU($N)_1$, $N = 5 \mod 4$, only graph states are topological, as in this case only stabilizer states are obtained from the $n$-torus Hilbert space. Hypergraph states for SU($N)_1$, $N = 5 \mod 4$, are therefore not topological in the above sense.

In Section 2 the construction of the generalized Pauli group and Clifford operations for SU($N)_1$ is reviewed. Section 3 presents the construction of graph and hypergraph states in terms of basic operations of SU($N)_1$. Section 4 is a discussion of related issues.
SU\((d)\), \(d\) odd prime

We first review the generalized Pauli group and Clifford operations for \(SU(d)\), \(d\) odd prime, following [1–3].

2.1 The \(SU(d)\) Pauli group

Representations of \(SU(d)\) describing qudits are given by a single column Young tableau, with zero, one, \ldots, \((d - 1)\) boxes. The fusion tensor of the theory is

\[
N_{ab}^c; \quad a + b = c \mod d
\]

so that

\[
N |a \rangle |b \rangle = |a \rangle |a + b, \mod d \rangle.
\]

The modular transformation matrix \(S_{ab}\) satisfies

\[
|a \rangle = \sum_{b=0}^{d-1} S_{ab} |b \rangle, \quad a = 0 \ldots d - 1.
\]

For \(\omega\) a primitive \(d\)th root of unity

\[
\omega = \exp\left(\frac{2\pi i}{d}\right)
\]

so that

\[
S^\ast = \frac{1}{\sqrt{d}} \sum_{a=0}^{d-1} \sum_{b=0}^{d-1} \omega^{ab} |a \rangle \langle b|
\]

which is the \(d\)-dimensional generalization of the Hadamard gate.

The Pauli operator \(Z\) is given by

\[
Z_{ac} = \sum_{b=0}^{d-1} S_{ab} \left(S_{b+1,a}^\dagger\right) \delta_{ac}
\]

so that

\[
Z = \sum_{a=0}^{d-1} \omega^a |a \rangle \langle a|.
\]

The Pauli operator \(X\) is obtained from the fusion matrix, since

\[
N_{a,b}^1 |a \rangle = |a + 1, \mod d \rangle
\]

which is identical to

\[
X |a \rangle = |a + 1, \mod d \rangle
\]
The single qudit Pauli group is the collection of operators
\[ \omega^r X^a Z^b; \quad a, b, r \in \mathbb{Z}_d. \] (2.10)

Thus the one-qudit Pauli group is constructed from basic operations of \( SU(d)_1, d \) odd.

The \( n \)-qudit Pauli group is obtained from products of operators of the one-qudit Pauli group. That is
\[ X^a Z^b = X^{a_1} Z^{b_1} \otimes X^{a_2} Z^{b_2} \otimes \cdots \otimes X^{a_n} Z^{b_n} \] (2.11)

The operator \( X^a Z^b \), along with all scalar multiples thereof,
\[ \{ \omega^c X^a Z^b \mid c \in \mathbb{Z}_d \} \] (2.12)
defines the \( n \)-qudit Pauli group.

**2.2 SU(\( d \)) Clifford operators, \( d \) odd**

The necessary gates for the single-qudit Clifford operators are [1–3] i) the QFT gate, Eq. (2.5), and ii) the phase gate
\[ P |j\rangle = \omega^{j(j-1)/2} |j\rangle. \] (2.13)

The multi-qudit Clifford operators are obtained from the generalizations of (2.5) and (2.8) as well as the SUM gate,
\[ C_{\text{SUM}} |i\rangle |j\rangle = N |i\rangle |j\rangle \]
\[ = |i\rangle |i + j, \quad \text{mod} \; d \rangle \] (2.14)

**3 Graph and hypergraph states**

### 3.1 Graph states

There are many equivalent constructions of graph states [4–8]. We follow arxiv:1612.06418 for a definition of qudit graph states. The multigraph is \( G = (V, E) \), with vertices \( V \) and edges \( E \), where an edge has multiplicity \( m_e \in \mathbb{Z}_d \). To \( G \) associate a state \( |G\rangle \) such that to each vertex \( i \in V \), there is a local state
\[ |+\rangle = |p_0\rangle = \frac{1}{\sqrt{d}} \sum_{q=0}^{d-1} |q\rangle \] (3.1)
Recall that the Hadamard gate generalizes to (2.5), so that
\[ S^* |0\rangle = \frac{1}{\sqrt{d}} \sum_{q=0}^{d-1} |q\rangle \]
\[ = |+\rangle = |p_0\rangle. \] (3.2)

To each edge \( e = \{i, j\} \) apply the unitary
\[ Z_{me} = \sum_{q_i=0}^{d-1} |q_i\rangle \langle q_i| \otimes (Z_{m_e})^{q_i} \] (3.3)
to the state
\[ |+\rangle^V = \bigotimes_{i \in V} |+\rangle_i \] (3.4)

The graph state is
\[ |G\rangle = \prod_{e \in E} Z_{me} |+\rangle^V \] (3.5)
\[ = \prod_{e \in E} Z_{me} \bigotimes_{i \in V} |+\rangle_i \] (3.6)

The level-rank dual [9] of Theorem 1 of Salton, Swingle, and Walter [10] for \( d \) odd prime implies that the graph state \( |G\rangle \) can be constructed from topological operations on the \( n \)-torus Hilbert space of Chern-Simons SU(\( d \)) by means of the operations detailed in Section 2. Every stabilizer state is LC equivalent to a graph state, while the Clifford group enables conversion between different multigraphs [4–8].

### 3.2 Hypergraph states

We again follow arxiv:1612.06418 for the construction of qudit multi-hypergraph states. Given a multi-hypergraph \( H = (V, E) \), associate a quantum state \( |H\rangle \), with \( m_e \in \mathbb{Z}_d \) the multiplicity of the hyperedge \( e \). To each vertex \( i \in V \), associate a local state
\[ |+\rangle = \frac{1}{\sqrt{d}} \sum_{q=0}^{d-1} |q\rangle \]
\[ = S^* |0\rangle \] (3.7)

To each hyperedge \( e \in E \), with multiplicity \( m_e \), apply the controlled unitary \( Z_{me} \) to the state
\[ |+\rangle^V = \bigotimes_{i \in V} |+\rangle_i \] (3.8)
The hypergraph state is
\[ |H\rangle = \prod_{e \in E} Z_{m_e}^{e} |+\rangle^V \] (3.9)

The elementary hypergraph state is
\[ |H\rangle = \sum_{q=0}^{d-1} |q\rangle \otimes \left( Z_{m_{\{1\}}}^{e} \right)^q |+\rangle^V \] (3.10)

For \( d \) prime, all \( n \)-elementary hypergraph states are equivalent under SLOCC.

Hypergraph and graph states admit a representation in terms of Boolean functions,
\[ |H\rangle = \sum_{q=0}^{d-1} \omega^{f(q)} |q\rangle \] (3.11)

with \( f : \mathbb{Z}_d^n \rightarrow \mathbb{Z}_d \), where
\[ f(x) = \sum_{\{i_1, \ldots, i_k\} \in E} x_{i_1} \cdots x_{i_k} \] (3.12)

For graph states, \( f(x) \) is quadratic, i.e.
\[ f(x) = \sum_{\{i_1, i_2\} \in E} x_{i_1} x_{i_2} \] (3.13)

while for \( f(x) \) cubic or higher, \(|H\rangle \) is a hypergraph state. Therefore, for quadratic \( f(x) \), one has a representation of stabilizer states, up to LC equivalence. For \( f(x) \) cubic or higher, \(|H\rangle \) represents hypergraph states which contain “magic” states. Examples of magic states are the CCZ state and Toffoli states, constructed from appropriate gates [3, 11–18]. Thus
\[ \text{CCZ } |x_1 x_2 x_3\rangle = \omega^{x_1 x_2 x_3} |x_1 x_2 x_3\rangle \] (3.14)

with
\[ |\text{CCZ}\rangle = \text{CCZ } |+ \otimes^3\rangle \] (3.15)

as an example of a magic hypergraph state. Similarly
\[ |\text{Toff}\rangle = \text{Toff } |+ \otimes^3\rangle \] (3.16)

where the Toffoli gate can be expressed in terms of the fusion matrix (2.2). Explicitly,
\[ \text{Toff } |i, j, k\rangle = N_{i, j, k}^{(i j + k)} = |i, j, i j + k, \text{ mod } d\rangle \] (3.17)
where the Young tableau for \((ij)\) has \(i + j\) vertical boxes, \(\mod d\).

For \(SU(d)_1\), \(d \) odd prime, \(d \neq 5 \mod 4\), both graph and hypergraph states can be obtained from operators which can be constructed from products of topological operations on the \(n\)-torus Hilbert space [10].

For \(SU(d)_1\), \(d = 5 \mod 4\), the topological argument does not apply, since in this case Theorem 1 of Salton et al [10] implies that only stabilizer states can be constructed on the \(n\)-torus Hilbert space. Thus, graph states can be so constructed but not hypergraph states with cubic or higher functions (3.12).

4 Discussion

It was shown above that graph states and hypergraph states for \(SU(d)_1\), \(d \) odd, can be constructed from the basic operations \(N_{ab}^c\), \(S_{ab}\), \(Z\), and \(X\) of \(SU(d)_1\) Chern-Simons theory. For \(d \) odd prime, \(d \neq 5 \mod 4\), these operations can be constructed topologically on the \(n\)-torus Hilbert space of Chern-Simons theory. A subset of hypergraph states are “magic.” For \(d \neq 5 \mod 4\), they are topological in the above sense.

Fliss [19] has studied knot and link states of \(SU(2)_d\) Chern-Simons theory, and has shown that knot and link states are generically magical. However for \(U(1)_d\), magic is absent for all knot and link states. Since \(U(1)_d\) is level-rank dual to \(SU(d)_1\), the knot and link states for this theory also have zero magic [19–21],[22].

There is a great deal of recent interest in magic states [6, 23–25]. One feature that deserves further study is to understand which magic states are topological. For example, universal topological computing is possible for \(SU(2)_3\) [26] and \(SU(3)_2\) [27] Chern-Simons theory. Implicitly this implies that magic states are present in these theories, presumably due to the braiding operations. It would be interesting to make this explicit.

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