A remark on gapped domain walls between topological phases

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Abstract

We give a mathematical definition of a gapped domain wall between topological phases and a gapped boundary of a topological phase. We then provide answers to some recent questions studied by Lan, Wang and Wen in condensed matter physics based on works of Davydov, Müger, Nikshych and Ostrik. In particular, we identify their tunneling matrix and a coupling matrix of Rehren, and show that their conjecture does not hold.

1 Introduction

The modern subfactor theory was initiated by Jones [24] and deep connections to chiral conformal field theory have been known. (See [17] for a general subfactor theory and [25] for connections to conformal field theory.) A natural language for studying such connections is that of a tensor category. A unitary fusion category [16] is particularly natural for subfactor theory and a unitary modular tensor category [2], [38] is particularly natural for the operator algebraic study of chiral conformal field theory [28].

Recently, studies of topological phases, condensation, gapped domain walls, and gapped boundaries have been made in the language of modular tensor categories [1], [20], [21], [30], [31], [32], [33]. The mathematical structures appearing in these studies are the same as those appearing in subfactors and chiral conformal field theory. One dictionary between α-induction for subfactors and condensation has been supplied by us in [31, page 440].

In this note, we further give a correspondence between studies of subfactors and chiral conformal field theory and those of gapped domain walls and gapped boundaries. In particular, we answer some question raised in [33] and show that a conjecture in [33] does not hold.

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2 Preliminaries

We start with a unitary modular tensor category $C$ as in [2], [16], [38]. Each object $\lambda$ in $C$ has $\dim \lambda \in \mathbb{R}_+$ and $C \text{ has dim} \ C = \sum_\lambda (\dim \lambda)^2$, where $\lambda$ is a simple object in $C$. As in [1], [20], [21], [31], [32], [33], a topological phase is represented by a unitary modular tensor category. Note that we have the additive central charge of $C$ in $\mathbb{Q}/8\mathbb{Z}$ as after Lemma 5.25 in [14].

The following definition was introduced in [14, Definition 5.1], where $D^{\text{opp}}$ means the modular tensor category with the braiding reversed.

**Definition 2.1** We say that unitary modular tensor categories $C$ and $D$ are Witt equivalent if $C \boxtimes D^{\text{opp}}$ is equivalent to the quantum double of some unitary fusion category.

The following is due to [35, Theorem 6.1], [36, Theorem 4.9].

**Definition 2.2** A $\mathbb{Q}$-system $(\theta, v, w)$ in a modular tensor category $C$ is a triple of an object $\theta$ in $C$, $v \in \text{Hom}(\text{id}, \theta)$, $w \in \text{Hom}(\theta, \theta^2)$ satisfying the following identities.

\[
(v^* \times 1)w = (1 \times v^*)w \in \mathbb{R}_+,
\]

\[
(w \times 1)w = (1 \times w)w.
\]

We say a $\mathbb{Q}$-system $(\theta, v, w)$ is irreducible when $\text{Hom}(\text{id}, \theta)$ is 1-dimensional. We say a $\mathbb{Q}$-system $(\theta, v, w)$ is local if we have $w = \varepsilon(\theta, \theta)w$, where $\varepsilon$ denotes the braiding of $C$.

An irreducible local $\mathbb{Q}$-system is the same as a condensable algebra in [31, Definition 2.6]. If we have an irreducible local $\mathbb{Q}$-system $(\theta, v, w)$ in a modular tensor category $C$, we can realize $C$ as a category of endomorphisms of a type III factor $N$ and apply the machinery of the $\alpha$-induction as in [36], [39], [3], [6], [7], [9], [10], [11]. The study of condensation in the setting of a modular tensor category is parallel to the study of $\alpha$-induction as in the table in [31, page 440] supplied by us.

If we have an irreducible local $\mathbb{Q}$-system $(\theta, v, w)$ in a modular tensor category $C$, we have an extension modular tensor category $\tilde{C}$ as the ambichiral system in the sense of [11] page 741] based on $\alpha$-induction. (It was proved to be a modular tensor category in [11, Theorem 4.2].) If a modular tensor category $C$ is a representation category of a completely rational local conformal net $\{A(I)\}$ in the sense of [28], then an irreducible local $\mathbb{Q}$-system produces an extension $\{\tilde{A}(I)\}$ of $\{A(I)\}$, and $\tilde{C}$ is the representation category of $\{\tilde{A}(I)\}$. In [29, Definition 1.1], a local $\mathbb{Q}$-system is called a commutative associative algebra $\mathcal{A}$ in a modular tensor category $C$, and the extension modular tensor category $\tilde{C}$ here is denoted by $\mathcal{C}_A^{\text{loc}}$ in [29].

The following is introduced in [14, Definition 4.6].
**Definition 2.3** A $Q$-system $(\theta, v, w)$ in a unitary modular tensor category $\mathcal{C}$ is called *Lagrangian* if we have $(\dim \theta)^2 = \dim \mathcal{C}$.

Note that this condition is equivalent to the one that the extension category $\tilde{\mathcal{C}}$ arising from $(\theta, v, w)$ is trivial in the sense $\dim \tilde{\mathcal{C}} = 1$.

The following has been proved in [14, Section 4]. Here our “quantum double” is often called the Drinfeld center in literature, and has been studied in [22] well in the setting of the Longo-Rehren subfactor [36].

**Theorem 2.4** A unitary modular tensor category has an irreducible local Lagrangian $Q$-system if and only if it is the quantum double of some unitary fusion category.

Since we are interested in only unitary categories, the “if” part of the above theorem follows from the description of the Longo-Rehren subfactor in [22], and the “only if” part follows from [8, Corollary 3.10].

### 3 Gapped domain walls and $Q$-systems

Based on arguments in [33], we make the following definition.

**Definition 3.1** For two unitary modular tensor categories $\mathcal{C}$ and $\mathcal{D}$, an irreducible Lagrangian local $Q$-system in $\mathcal{C} \boxtimes \mathcal{D}^{opp}$ is called a *domain gapped wall* between $\mathcal{C}$ and $\mathcal{D}$. If $\mathcal{D}$ is trivial in the sense $\dim \mathcal{D} = 1$, we say the $Q$-system is a *gapped boundary* of $\mathcal{C}$.

Such a $Q$-system is of the form $(\bigoplus Z_{\lambda,\mu} \lambda \boxtimes \mu, v, w)$, where $\lambda$ and $\mu$ label the simple objects of $\mathcal{C}$ and $\mathcal{D}$, respectively. The unitary modular tensor categories here represent *topological phases*. In [33], the matrix $Z$ is called a *tunneling matrix*. Note that a gapped domain wall between $\mathcal{C}$ and $\mathcal{D}$ is the same as a gapped boundary of $\mathcal{C} \boxtimes \mathcal{D}^{opp}$. This is compatible with the “folding trick” in [30] mentioned in [33]. The above definition is also compatible with the works [20], [21], [32]. (See explanations on condensation in [31]. Also see earlier works [3], [34] for the abelian phases.)

If $\mathcal{C}$ and $\mathcal{D}$ are the representation categories of local conformal nets $\{A(I)\}$, $\{B(I)\}$, respectively, a domain gapped wall between them is the situation considered in [37]. In the context of [37], the matrix $Z$ was called a *coupling matrix*.

We prepare a simple lemma.

**Lemma 3.2** Let $(\theta, v, w)$ be a local $Q$-system in a modular tensor category $\mathcal{C}$. For simple objects $\lambda, \mu \in \mathcal{C}$, we have $\langle \theta, \lambda \mu \rangle \geq \langle \theta, \lambda \rangle \langle \theta, \mu \rangle$, where $\langle \lambda, \mu \rangle = \dim \text{Hom}(\lambda, \mu)$.

**Proof.** The object $\theta$ can be regarded as a dual canonical endomorphism of a subfactor $N \subset M$. Let $\theta = \bar{\iota}$ for $\iota : N \hookrightarrow M$. Then we have the following by the Frobenius reciprocity.

\[
\langle \bar{\iota}, \lambda \mu \rangle = \langle \iota \bar{\mu}, \iota \lambda \rangle,
\langle \bar{\iota}, \lambda \rangle = \langle \iota, \iota \lambda \rangle,
\langle \bar{\iota}, \mu \rangle = \langle \iota, \iota \bar{\mu} \rangle.
\]

These imply the desired inequality. \qed
We now have the following result corresponding to the main result in [33] where the three conditions below are given.

**Theorem 3.3** Let \( (\bigoplus Z_{\lambda\mu}, \mu, v, w) \) be a gapped boundary between \( \mathcal{C} \) and \( \mathcal{D} \). Let 
\[ \mathcal{T}^C, N^C \text{ [resp. } \mathcal{T}^D, N^D \text{]} \] be the \( S \)-matrix, the \( T \)-matrix, the fusion rule coefficients of \( \mathcal{C} \) [resp. \( \mathcal{D} \)]. We then have the following.

1. \( Z_{\lambda\mu} \in \mathbb{N} \).
2. \( \mathcal{T}^C Z = Z \mathcal{T}^D, \mathcal{T}^C Z = Z \mathcal{T}^D \).
3. \( Z_{\lambda\mu} Z_{\lambda'\mu'} \leq \sum_{\lambda'',\mu''} (N_C)_{\lambda''\lambda} Z_{\lambda''\mu''} (N_D)_{\mu\mu''} \).

**Proof.** Condition (1) is obvious.

By Theorem 2.4, we know that \( \mathcal{C} \) and \( \mathcal{D} \) are Witt equivalent. Then [15, Proposition 3.7, Corollary 3.8] specifies the form of the \( Q \)-system. Then [8, Theorem 6.5] implies the desired commutativity (2).

Since \( Z_{\lambda''\mu''} = \langle \lambda''(\bar{\lambda}'' \boxtimes \text{id}), \text{id} \boxtimes \mu'' \rangle \), we have
\[
\sum_{\mu''} Z_{\lambda''\mu''} (N_D)_{\mu\mu''} = \langle \theta(\lambda''(\bar{\lambda''} \boxtimes \text{id}), \text{id} \boxtimes \mu') = \langle \theta(\text{id} \boxtimes \bar{\mu}'' \bar{\mu}), \lambda''(\lambda' \boxtimes \text{id}) \rangle.
\]

Then the right hand side of (3) is equal to the following.
\[
\langle \theta(\text{id} \boxtimes \bar{\mu}'' \bar{\mu}), \lambda''(\lambda' \boxtimes \text{id}) \rangle = \langle \theta, \lambda''(\lambda' \boxtimes \mu') \rangle.
\]

Since the left hand side of (3) is equal to \( \langle \theta, \lambda \boxtimes \mu \rangle \langle \theta, \lambda' \boxtimes \mu' \rangle \), Lemma 3.2 gives the desired inequality. \( \square \)

In the case where \( \mathcal{C} \) and \( \mathcal{D} \) are the representation categories of completely rational local conformal nets, condition (2) was obtained by Müger. (See [27, Theorem 3.1].)

The first half of Question (Q1) in [33] asks when we have a gapped domain wall between \( \mathcal{C} \) and \( \mathcal{D} \). As we have seen, the answer is given by Theorem 2.4. That is, we have one if and only if \( \mathcal{C} \) and \( \mathcal{D} \) are Witt equivalent. This necessity was first pointed out in [19, page 560].

The second half of Question (Q1) in [33] asks how many gapped domain walls we have between \( \mathcal{C} \) and \( \mathcal{D} \). This is a finite combinatorial problem on \( Q \)-systems as shown in [23] since we have only finitely many matrices \( Z_{\lambda\mu} \) for a given \( \mathcal{C} \) and \( \mathcal{D} \), but counting in general can be highly complicated, and we have no easy general methods.

It is conjectured in [33] that these three conditions are sufficient to have a gapped domain wall, but the example in [13] shows that this is not true. The examples in [13] shows that the charge conjugation modular invariant of the quantum double of some finite group does not have a corresponding \( Q \)-system structure. Note that the charge conjugation modular invariant for \( \mathcal{C} = \mathcal{D} \) obviously satisfies (1) and (2) above, and it is also easy to see that it satisfies (3). So this is an example of a matrix \( Z \) satisfying (1), (2) and (3) above,
but it does not correspond to any gapped domain wall. (Note that the identity matrix in [13] corresponds to the charge conjugation matrix here due to a different convention.) Our formulation in terms of a $Q$-system is the same as the one in terms of a condensable algebra in the sense of [31], so the hope in [33] to have a condensable algebra from some data satisfying (1), (2) and (3) does not work. From various classification results related to modular invariants in [18], [26], [27], it looks impossible to have a sufficient condition for a gapped domain wall between $\mathcal{C}$ and $\mathcal{D}$ to exist simply by working on the tunneling matrix $(Z_{\lambda\mu})$, and it is expected we have many other examples of matrices $Z$ satisfying (1), (2) and (3) which do not correspond to gapped domain walls. Also as remarked in [33] based on [12], the matrix $(Z_{\lambda\mu})$ does not have enough information to recover the $Q$-system. This is also compatible with a well-known fact in the study of $Q$-systems. Also see [4, Example 5.8].

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