The rates of convergence for generalized entropy of the normalized sums of IID random variables
Hongfei Cui, Jianqiang Sun and Yiming Ding

Abstract—We consider the generalized differential entropy of normalized sums of independent and identically distributed (IID) continuous random variables. We prove that the Rényi entropy and Tsallis entropy of order $\alpha$ ($\alpha > 0$) of the normalized sum of IID continuous random variables with bounded moments are convergent to the corresponding Rényi entropy and Tsallis entropy of the Gaussian limit, and obtain sharp rates of convergence.

Index Terms—Rényi entropy, Shannon entropy, Tsallis entropy, central limit theorem, rate of convergence.

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I. INTRODUCTION

The Shannon entropy of a random variable $X$ with density $f : \mathcal{R} \to [0, \infty)$ is defined as

$$H(X) = -\int_{\mathcal{R}} f \log f$$

provided that the integral make sense. (We use log to represent the natural logarithm throughout this paper). It is interesting to study the convergence of the normalized sums

$$S_n = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} X_i$$

of independent copies $X$ to the Gaussian limit: the central limit theorem for independent and identically distributed (IID) copies of $X$. Without loss of generality, we suppose $\mathbb{E}(X) = 0$ and $\mathbb{E}(X^2) = 1$. The standard Gaussian distribution is denoted by $G$.

The idea of tracking the central limit theorem using Shannon entropy goes back to Linnik [2], who used it to give a particular proof of the central limit theorem. Barron [3] was the first to prove a central limit theorem with convergence in the Shannon entropy sense. He proved that $H(S_n)$ converges to $H(G) = \log \sqrt{2\pi e}$ if $H(S_n)$ is finite for some $n > 0$. Notice that $H(S_n)$ is finite for some $n > 0$ if for some $s > 0$, $E[|S_n|^s] < \infty$, since

$$H(S_n) \leq \frac{1}{s} \log \mathbb{E}[|S_n|^s] = \frac{1}{s} \log \left( 2\Gamma\left( \frac{s}{2} \right) \right) \leq \frac{1}{s} \log \left( \frac{1}{s} \right) \leq \frac{1}{s} \log \left( \frac{1}{s} \right).$$

where $\Gamma$ is the Gamma function. Artstein et al. [4], Johnson and Barron [5] obtained the rate of convergence

$$|H(S_n) - H(G)| = O\left( \frac{1}{n} \right),$$

provided the density of $X$ satisfies some analytical conditions [6]. Moreover, the conclusion that $H(S_n)$ is increasing to $H(G)$ was obtained by Artstein et al. [7], and some simpler proofs can be found in Tulino and Verdu [8] and Madiman and Barron [9].

Rényi entropy is a generalization of Shannon entropy [10]. It is one of a family of functionals for quantifying the diversity, uncertainty or randomness of a system. The Rényi entropy of order $\alpha$ ($\alpha \in \mathbb{R}$) is defined as

$$R_\alpha(X) = \frac{1}{1-\alpha} \log \int_\mathcal{R} f^{\alpha}(x) dx, \; \alpha \neq 1.$$ 

By L'Hopital’s rule, $R_\alpha(X) \to H(X)$ as $\alpha \to 1$. The Rényi entropy of order 2, $R_2(X)$, is called Collision entropy.

The Rényi entropies are important in ecology and statistics as indices of diversity. Rényi entropies appear also in several important contexts such as information theory, statistical estimation, and quantum entanglement [11], [12], [13]. A class of Rényi entropy estimators for multidimensional densities are given by Leonenko et al. [14].

Another generalization of Shannon entropy is Tsallis entropy [15], which is defined as

$$T_\alpha(X) = \frac{1}{\alpha - 1} \log \left( 1 - \int_\mathcal{R} f^{\alpha}(x) dx \right), \; \alpha \neq 1.$$ 

It is easy to see that $T_\alpha(X) \to H(X)$ as $\alpha \to 1$. Historically, this family of entropies was derived by Havrda and Charvát in 1967 [16]. The Rényi entropy and Tsallis entropy with same order $\alpha$ are related by [17]

$$R_\alpha(X) = \frac{1}{1-\alpha} \log (1 - (\alpha - 1)T_\alpha(X)),$$

which is a one-to-one correspondence between $R_\alpha(X)$ and $T_\alpha(X)$. In fact, $R_\alpha(X)$ is a monotone increasing function of $T_\alpha(X)$, because

$$\frac{dR_\alpha(X)}{dT_\alpha(X)} = \frac{1}{1 - (\alpha - 1)T_\alpha(X)} = \frac{1}{\int_\mathcal{R} f^{\alpha}(x) dx} > 0$$

provided $\int_\mathcal{R} f^{\alpha}(x) dx < \infty$.

Parallel to the Shannon case, we consider the convergence of $R_\alpha(S_n)$ and $T_\alpha(S_n)$ ($\alpha > 0$), and investigate the rates of convergence.

Our main results are the following Theorem.
**Main Theorem** Let $X_1, X_2, \ldots, X_n$ be independent copies of a random variable $X$ with characteristic function $\varphi(t)$. Suppose the following hold:

1. $\mathbb{E}(|X|^k)$ is finite for $k = 1, 2, \ldots$;
2. $|\varphi(t)|^v$ is integrable for some $v \geq 1$.

Then for any $\alpha > 0$, we have

$$\lim_{n \to \infty} R_\alpha(S_n) = R_\alpha(G), \quad \lim_{n \to \infty} T_\alpha(S_n) = T_\alpha(G).$$

Furthermore,

$$|R_\alpha(S_n) - R_\alpha(G)| = \begin{cases} O(n^{-\frac{1}{2}}) & 1 < \alpha < \infty; \\ O(n^{-\frac{1}{2}} + \gamma) & 0 < \alpha \leq 1, 0 < \gamma < \frac{1}{2}; \\ O(n^{-\frac{1}{2} + \gamma}) & 0 < \alpha < 1, \frac{1}{2} < \gamma < \frac{1}{2}; \\ O(n^{-\frac{1}{2} + \gamma}) & 0 < \alpha < 1, \gamma = \frac{1}{2}. \end{cases}$$

$$|T_\alpha(S_n) - T_\alpha(G)| = \begin{cases} O(n^{-\frac{1}{2}}) & 1 < \alpha < \infty; \\ O(n^{-\frac{1}{2} + \gamma}) & 0 < \alpha \leq 1, 0 < \gamma < \frac{1}{2}; \\ O(n^{-\frac{1}{2} + \gamma}) & 0 < \alpha < 1, \frac{1}{2} < \gamma < \frac{1}{2}; \\ O(n^{-\frac{1}{2} + \gamma}) & 0 < \alpha < 1, \gamma = \frac{1}{2}. \end{cases}$$

Although the Rényi entropy and Tsallis entropy can be defined for any real number $\alpha$, we only consider the case $\alpha > 0$. In fact, for $\alpha \leq 0$, one can check that $R_\alpha(G) = \infty$ and $T_\alpha(G) = \infty$.

The bounded moment condition 1) is equivalent to $\mathbb{E}(X^k) < +\infty$ for all positive integer $k$, which is also equivalent to the fact that the characteristic function $\varphi(t)$ admits derivatives of all orders at $t = 0$. It is a local condition imposed on $\varphi(t)$, while the integrability condition 2) assumes a global property of $\varphi(t)$.

As we shall see in Lemma 5, the bounded moment condition 1) implies the existence of $R_\alpha(S_n)$ and $T_\alpha(S_n)$ for all $\alpha > 0$. We shall obtain the rates of convergence via Feller’s expansion of densities. According to Feller’s expansion \[13\] (Lemma 3), the density of $S_n$ is convergent to the density of $G$ uniformly with rate $o(n^{-1/2})$, where the integrability condition 2) is assumed. Hence, the rates of convergence we obtained in Main Theorem are sharp.

Since the moment condition is weaker than the Poincaré inequality condition which was used in the Shannon case \[6\], the rate of convergence for Shannon entropy we obtained is $O(n^{-1/2+\gamma}) (\gamma > 0$ is small) rather than $O(n^{-1})$.

If the moment condition 1) in Main Theorem is replaced by $\mathbb{E}|X|^3 < \infty$, it is shown that $R_\alpha(S_n) \to R_\alpha(G)$ as $n \to +\infty$ for every $\alpha > 1$, and rough rates of convergence are obtained in \[19\].

**II. CONVERGENCE OF RÉNYI ENTROPY AND TSALLIS ENTROPY**

Let $\{Y_n\}$ be a sequence of random variables with density functions $\{p_n(x)\}$ and $Y$ be a random variable with density function $p(x)$. It is interesting to ask whether the Rényi entropy and Tsallis entropy of $\{Y_n\}$ of order $\alpha$ ($\alpha > 0$) are convergent the corresponding entropy of $Y$, provided $Y_n \to Y$ in some sense. The following Theorem 1 claims that if $\{p_n(x)\}$ is uniformly bounded, $\{p_n(x)\}$ is uniformly convergent to $p(x)$, and the $L^2$-norm of $\{p_n(x)\}$ is uniform bounded for every $\alpha > 0$, then the convergence results hold.

For discrete random variables, such kind of continuity is also valid \[20\].

**Theorem 1** Let $\{Y_n\}$ be a sequence of random variables with density functions $\{p_n(x)\}$, $Y$ be a random variable with density function $p(x)$, and $A$ be a subset of $\mathbb{R}$ with zero Lebesgue measure. Suppose the following hold:

1. for any $\varepsilon > 0$, there exists a positive integer $N > 0$ such that $\sup_{x \in \mathbb{R}\setminus A} \{|p_n(x) - p(x)|\} < \varepsilon$ for $n > N$;
2. there exists a finite number $M > 0$ such that $p_n(x) \leq M$ uniformly in $x \in \mathbb{R}\setminus A$ and $n \in \mathbb{N}$;
3. for every $\alpha > 0$, there exists a finite number $M_\alpha > 0$ such that $\int_{\mathbb{R}} p_n^\alpha(x) dx \leq M_\alpha$ uniformly in $n \in \mathbb{N}$.

Then for $\alpha > 0$, we have

$$\lim_{n \to +\infty} R_\alpha(Y_n) = R_\alpha(Y), \quad \lim_{n \to +\infty} T_\alpha(Y_n) = T_\alpha(Y).$$

**Remark 1:**

1) It is interesting to note that one may use Theorem 1 to obtain the convergence results of Rényi entropy and Tsallis entropy for random variables with correlations.

2) Theorem 1 claims the convergence of Rényi entropy and Tsallis entropy for all $\alpha > 0$. One can assume weaker conditions on the densities of $\{X_n\}$ to ensure the convergence for those $\alpha$ belong to some bounded and closed subset of $(0, \infty)$.

3) The condition 1) in Theorem 1 is equivalent to the fact that $\{p_n(x)\}$ converges to $p(x)$ uniformly on $\mathbb{R}\setminus A$ as $n \to +\infty$. Combining with condition 2) in Theorem 1, we know that $p(x) \leq M$. Moreover, in the following Lemma 1 we also obtain that for every $\alpha > 0$, $\int_{\mathbb{R}} p^\alpha(x) dx \leq M_\alpha$.

4) Since we are only interested with the asymptotic behavior of $\{p_n(x)\}$, it is enough to require the uniform boundedness of the $L^2$-norm $(0 < \alpha \leq \infty)$ of $\{p_n(x)\}$ for $n > N_\alpha$, where $N_\alpha$ is a positive integer.

We prepare two Lemmas which are important in the proof of Theorem 1.

**Lemma 1:** Suppose that the conditions in Theorem 1 are satisfied. Then for every $\alpha > 0$, we have

$$\lim_{n \to +\infty} \int_{\mathbb{R}} |p_n^\alpha(x) - p^\alpha(x)| dx = 0. \quad (1)$$

**Proof:** At first, we prove that for every $\alpha > 0$, $\int_{\mathbb{R}} p^\alpha(x) dx \leq M_\alpha$. Since for every $\alpha > 0$ the function $f_\alpha(x) = x^\alpha$ is continuous and $\{p_n(x)\}$ converges to $p(x)$ uniformly on $\mathbb{R}\setminus A$ as $n \to +\infty$, we have that $\{p_n^\alpha(x)\}$ converges to $p^\alpha(x)$ uniformly on $\mathbb{R}\setminus A$ as $n \to +\infty$. By Fatou’s Lemma and condition 3 in Theorem 1, we conclude that for every $\alpha > 0$,

$$\int_{\mathbb{R}} p^\alpha(x) dx = \int_{\mathbb{R}} \liminf_{n \to +\infty} p_n^\alpha(x) dx \leq \liminf_{n \to +\infty} \int_{\mathbb{R}} p_n^\alpha(x) dx \leq M_\alpha.$$
The proof of Lemma 1 is decomposed two cases: \( \alpha > 1 \) and \( 0 < \alpha \leq 1 \).

For the case \( \alpha > 1 \), using Lagrange mean value theorem, we have

\[
\int_{\mathbb{R}} |p_n^\alpha(x) - p^\alpha(x)| \, dx = \alpha \int_{\mathbb{R}} |p_n(x) - p(x)| \xi_n^{\alpha-1}(x) \, dx \\
\leq \alpha \sup_{x \in \mathbb{R} \setminus A} |p_n(x) - p(x)| \int_{\mathbb{R}} \xi_n^{\alpha-1}(x) \, dx.
\]

where \( \min\{p_n(x), p(x)\} \leq \xi_n^{\alpha-1}(x) \leq \max\{p_n(x), p(x)\} \).

Next, we’ll prove that \( \int_{\mathbb{R}} \xi_n^{\alpha-1}(x) \, dx \) is bounded.

\[
\int_{\mathbb{R}} \xi_n^{\alpha-1}(x) \, dx = \int_{\{x: p_n(x) > p(x)\}} \xi_n^{\alpha-1}(x) \, dx + \int_{\{x: p_n(x) \leq p(x)\}} \xi_n^{\alpha-1}(x) \, dx \\
\leq \int_{\{x: p_n(x) > p(x)\}} p_n^{\alpha-1}(x) \, dx + \int_{\{x: p_n(x) \leq p(x)\}} p_n^{\alpha-1}(x) \, dx
\]

Using the condition 3) in Theorem 1, we have \( \int_{\mathbb{R}} \xi_n^{\alpha-1}(x) \, dx \leq 2M_{\alpha-1} < +\infty \) for every \( \alpha \in (1, 1) \). Hence we obtain that

\[
\int_{\mathbb{R}} |p_n^\alpha(x) - p^\alpha(x)| \, dx \leq 2\alpha M_{\alpha-1} \sup_{x \in \mathbb{R} \setminus A} |p_n(x) - p(x)|. \tag{2}
\]

Combining inequality (2) and the condition 1) in Theorem 1, we obtain (1) for \( \alpha > 1 \).

Now we consider the case \( 0 < \alpha \leq 1 \).

The condition 3) in Theorem 1 indicates that for every \( \gamma \in (0, \frac{1}{2}) \), there exists an \( M_{2\gamma} > 0 \) such that \( \int_{\mathbb{R}} p_n^{2\gamma}(x) \, dx \leq M_{2\gamma} \) and \( \int_{\mathbb{R}} p^{2\gamma}(x) \, dx \leq M_{2\gamma} \). It follows that

\[
\int_{\mathbb{R}} |p_n(x) + p(x)|^{2\gamma} \, dx = \int_{\{x: p_n(x) > p(x)\}} (p_n(x) + p(x))^{2\gamma} \, dx + \int_{\{x: p_n(x) \leq p(x)\}} (p_n(x) + p(x))^{2\gamma} \, dx
\]

\[
\leq \int_{\{x: p_n(x) > p(x)\}} (2p_n(x))^{2\gamma} \, dx + \int_{\{x: p_n(x) \leq p(x)\}} (2p(x))^{2\gamma} \, dx
\]

\[
\leq 2^{2\gamma} p_n^{2\gamma} \, dx + \int_{\mathbb{R}} 2^{2\gamma} p(x)^{2\gamma} \, dx
\]

\[
\leq 2^{2\gamma+1} M_{2\gamma} < \infty. \tag{3}
\]

By inequality (3) and the trivial inequality \(|b - c| \leq |b - c|\), we have

\[
\int_{\mathbb{R}} |p_n^\alpha(x) - p^\alpha(x)| \, dx
\]

\[
\leq \int_{\mathbb{R}} |p_n(x) - p(x)|^\alpha \, dx
\]

\[
\leq \int_{\mathbb{R}} |p_n(x) - p(x)|^{\alpha-2\gamma} |p_n(x) + p(x)|^{2\gamma} \, dx
\]

\[
\leq \sup_{x \in \mathbb{R} \setminus A} |p_n(x) - p(x)|^{\alpha-2\gamma} \int_{\mathbb{R}} |p_n(x) + p(x)|^{2\gamma} \, dx
\]

\[
\leq 2^{2\gamma+1} M_{2\gamma} \sup_{x \in \mathbb{R} \setminus A} |p_n(x) - p(x)|^{\alpha-2\gamma}. \tag{4}
\]

Combining inequality (4) and the condition 1) in Theorem 1, we obtain (1) for \( 0 < \alpha \leq 1 \).

Lemma 2: Suppose that the conditions in Theorem 1 be satisfied. Then for any \( \varepsilon > 0 \) there exists a \( \delta > 0 \) such that

\[
|T_\alpha(Y_n) - H(Y_n)| < \varepsilon
\]

for all \( n \in \mathbb{N} \) and all \( \alpha \) satisfying \(|\alpha - 1| < \delta\).

Furthermore, for any \( \varepsilon > 0 \) there exists \( \delta > 0 \) such that

\[
|T_\alpha(Y) - H(Y)| < \varepsilon
\]

for all \( \alpha \) satisfying \(|\alpha - 1| < \delta\).

Proof: The proof is given in Appendix.

It is time to give the proof of Theorem 1.

Proof: We consider two cases: \( \alpha \neq 1 \) and \( \alpha = 1 \).

Suppose \( \alpha \neq 1 \). By Lemma 1 for every \( \alpha > 0 \) there exist \( N > 0 \) such that \( \int_{\mathbb{R}} p_n^\alpha(x) \, dx \geq \frac{1}{2} \int_{\mathbb{R}} p^\alpha(x) \, dx \) for \( n > N \).

Using the inequality \( \log(1 + x) < x \) \( \forall x > 0 \), we have

\[
\frac{|R_\alpha(Y_n) - R_\alpha(Y)|}{1 - |\alpha|} \log \left( \frac{\int_{\mathbb{R}} p_n^\alpha(x) \, dx}{\int_{\mathbb{R}} p^\alpha(x) \, dx} \right)
\]

\[
\leq \max \left\{ \frac{\int_{\mathbb{R}} p_n^\alpha(x) - p^\alpha(x) \, dx}{|1 - \alpha|} \frac{\int_{\mathbb{R}} p^\alpha(x) - p_n^\alpha(x) \, dx}{|1 - \alpha|} \right\}
\]

\[
\leq \frac{2}{1 - |\alpha|} \frac{\int_{\mathbb{R}} p^\alpha(x) - p_n^\alpha(x) \, dx}{|1 - \alpha|} \frac{\int_{\mathbb{R}} p^\alpha(x) - p_n^\alpha(x) \, dx}{1 - \alpha} \frac{1}{2} \int_{\mathbb{R}} p^\alpha(x) \, dx
\]

Combining (5) and Lemma 1, we obtain that for every \( \alpha > 0 \) and \( \alpha \neq 1 \),

\[
\lim_{n \to \infty} |R_\alpha(Y_n) - R_\alpha(Y)| = 0.
\]

It is obvious that

\[
\frac{1}{|\alpha|} \left( \int_{\mathbb{R}} p_n^\alpha(x) \, dx - \frac{1}{|\alpha|} \int_{\mathbb{R}} p^\alpha(x) \, dx \right)
\]

\[
= \frac{1}{|\alpha| - 1} \left( \int_{\mathbb{R}} p_n^\alpha(x) - p^\alpha(x) \, dx \right)
\]

\[
\leq \frac{1}{|\alpha| - 1} \frac{\int_{\mathbb{R}} |p_n^\alpha(x) - p^\alpha(x)| \, dx}{|1 - \alpha|} \frac{\int_{\mathbb{R}} |p_n^\alpha(x) - p^\alpha(x)| \, dx}{1 - \alpha}
\]

\[
\leq \frac{1}{|\alpha| - 1} \frac{\int_{\mathbb{R}} |p_n^\alpha(x) - p^\alpha(x)| \, dx}{|1 - \alpha|} \frac{\int_{\mathbb{R}} |p_n^\alpha(x) - p^\alpha(x)| \, dx}{1 - \alpha}
\]

\[
\leq \frac{1}{|\alpha|} \frac{\int_{\mathbb{R}} |p_n^\alpha(x) - p^\alpha(x)| \, dx}{1 - \alpha}
\]

\[
\leq \frac{1}{|\alpha| - 1} \frac{\int_{\mathbb{R}} |p_n^\alpha(x) - p^\alpha(x)| \, dx}{|1 - \alpha|} \frac{\int_{\mathbb{R}} |p_n^\alpha(x) - p^\alpha(x)| \, dx}{1 - \alpha}
\]

\[
\leq \frac{1}{|\alpha|} \frac{\int_{\mathbb{R}} |p_n^\alpha(x) - p^\alpha(x)| \, dx}{1 - \alpha}
\]

\[
\leq \frac{1}{|\alpha| - 1} \frac{\int_{\mathbb{R}} |p_n^\alpha(x) - p^\alpha(x)| \, dx}{|1 - \alpha|} \frac{\int_{\mathbb{R}} |p_n^\alpha(x) - p^\alpha(x)| \, dx}{1 - \alpha}
\]

\[
\leq \frac{1}{|\alpha|} \frac{\int_{\mathbb{R}} |p_n^\alpha(x) - p^\alpha(x)| \, dx}{1 - \alpha}
\]
We obtain that for every $\alpha > 0$ and $\alpha \neq 1$, 
\[
\lim_{n \to \infty} |T_\alpha(Y_n) - T_\alpha(Y)| = 0. \tag{7}
\]

Now we consider the case $\alpha = 1$.
Remember that for any random variable $X$, $R_1(X) = T_1(X) = H(X)$. By triangular inequality, 
\[
|H(Y_n) - H(Y)| \
\leq |H(Y_n) - T_{\alpha_0}(Y_n)| + |T_{\alpha_0}(Y_n) - T_{\alpha_0}(Y)| \
+ |T_{\alpha_0}(Y) - H(Y)|. \tag{8}
\]

Given any $\varepsilon > 0$, there exists $\delta_1 > 0$ such that $|H(Y_n) - T_{\alpha_0}(Y_n)| < \frac{\varepsilon}{2^n}$, for all $n \in \mathbb{N}$ and all points $x$ satisfying $|\alpha_0 - 1| < \delta_1$ by Lemma 2. Similarly, there exists $\delta_2 > 0$ such that $|T_{\alpha_0}(Y) - H(Y)| \leq \frac{\varepsilon}{2^n}$, all points $x$ satisfying $|\alpha_0 - 1| < \delta_2$.

By (7), we know that for every $\alpha_0 \neq 1$ satisfying $|\alpha_0 - 1| < \min\{\delta_1, \delta_2\}$, there exists an $N > 0$ such that $|T_{\alpha_0}(Y_n) - T_{\alpha_0}(Y)| \leq \frac{\varepsilon}{2^n}$ for all $n > N$.

According to inequality (8), we conclude that for any $\varepsilon > 0$, there exists an $N$ such that $|H(Y_n) - H(Y)| \leq \varepsilon$ for $n \geq N$.

Therefore, $H(Y_n)$ converges to $H(Y)$.

\section{III. PROOF OF MAIN THEOREM}

Lemma 3: (Expansions for densities, Feller [13]) Let $X_1, X_2, \ldots, X_n$ be independent copies of a random variable $X$ with characteristic function $\varphi(t)$, $f_n$ be the density function of normalized sum $S_n$, and $g$ be the density of the standard Gaussian distribution $G$. Suppose that $E|X|^3 = \rho < \infty$, and that $|\varphi|^n$ is integrable for some $v > 1$. Then as $n \to \infty$
\[
f_n(x) - g(x) - \frac{\rho}{6\sqrt{n}}(x^3 - 3x)g(x) = o\left(\frac{1}{\sqrt{n}}\right) \tag{9}
\]
uniformly in $x$.

Remark 2: Since $g(x)$ is the density function of the standard Gaussian distribution $G$, $g(x) \leq \frac{1}{\sqrt{2\pi}} \alpha < 1$. For every $\alpha > 0$, there exists an $M'_\alpha > 0$ such that $\int_{\mathbb{R}} g_\alpha(x)dx \leq M'_\alpha$.

According to Lemma 3 we have
\[
\sup_{x \in \mathbb{R}} \{|f_n(x) - g(x)|\} = o(n^{-\frac{1}{2}}). \tag{10}
\]

Using (10), we have that there exists an $N > 0$ such that $f_n(x) \leq 1$ uniformly for $n > N$ and $x \in \mathbb{R}$. Hence, without loss of generality, we can suppose that $f_n(x) \leq 1$ uniformly in $n \in \mathbb{N}$ and $x \in \mathbb{R}$.

Next, we'll prove that $f_n(x)$ satisfy the condition 3) in Theorem 1.

Lemma 4: (Theorem 2.10 [21]) Let $X_1, X_2, \ldots, X_n$ be independent random variables with zero means, and let $k \geq 2$ and $Z_n = X_1 + X_2 + \ldots + X_n$. Then
\[
E|Z_n|^k = C(k)n^{\frac{k}{2} - 1}\sum_{i=1}^{n} E|X_i|^k,
\]
where $C(k)$ is a positive constant depending only on $k$.

Lemma 5: Let $X_1, X_2, \ldots, X_n$ be independent copies of a random variable $X$ with probability density $f$ and characteristic function $\varphi(t)$. $f_n(x)$ be the density of the normalized sum $S_n$. $E(X) = 0$, and $E(|X|^k) = \rho_k$, where $k = 1, 2, \ldots$ and $\rho_k$ is finite. Then for each positive $\alpha$, there exists an $M''_\alpha > 0$ such that
\[
\int_R f_\alpha^n(x)dx \leq M''_\alpha.
\]

Proof: We distinguish three cases: $\alpha = 1$, $\alpha > 1$ and $0 < \alpha < 1$.

The case $\alpha = 1$ is obvious because $\{f_n\}$ are densities. If $\alpha > 1$, noting that $\sup f_n(x) \leq 1$ in Remark 2 we have
\[
\int_R f_\alpha^n(x)dx = \int_R f_\alpha^{\alpha-1}(x)f_n(x)dx \leq 1.
\]

If $0 < \alpha < 1$, one can choose a positive integer $k$ such that $\frac{1}{k+1} < \alpha$. From Lemma 4 we obtain that
\[
E|S_n|^k = E\left|(X_1 + X_2 + \ldots + X_n)/n^{\frac{k}{2}}\right|^k
\]
\[
= E|Z_n|^k/n^{\frac{k}{2}}
\]
\[
\leq C(k)n^{\frac{k}{2} - 1}\sum_{i=1}^{n} |X_i|^k/n^{\frac{k}{2}}
\]
\[
= C(k)E|X|^k
\]
\[
:= \rho_k < \infty.
\]

On the other hand, noting that $\sup f_n(x) \leq 1$ in Remark 2 we have that
\[
\int_R f_\alpha^n(x)dx
\]
\[
= \int_{-\infty}^{1} f_\alpha^n(x)dx + \int_{|x| \geq 1} f_\alpha^n(x)dx
\]
\[
= \int_{-\infty}^{1} f_\alpha^n(x)dx + \int_{|x| \geq 1} f_\alpha^n(x)|x|^{-k\alpha}dx
\]
\[
\leq 2 + (\int_{|x| \geq 1} f_n(x)|x|^\alpha dx)^{\alpha} (\int_{|x| \geq 1} |x|^{\frac{1}{k-1}\alpha} dx)^{1-\alpha}
\]
\[
\leq 2 + \rho_k^{\alpha} (\int_{|x| \geq 1} |x|^{-\frac{1}{k-1}\alpha} dx)^{1-\alpha},
\]
where the first inequality follows from Hölder inequality.

Since $\int_{|x| \geq 1} |x|^{-\frac{1}{k-1}\alpha} dx$ is finite for $\frac{1}{k-1} < \alpha < 1$, one can find a positive constant $M''_\alpha$ independent of $n$ such that
\[
\int_R f_\alpha^n(x)dx < M''_\alpha.
\]

Lemma 6: If $0 \leq y, x \leq 1$, then for every $\gamma \in (0, \frac{1}{2})$, there exists a $Q_\gamma > 0$ such that
\[
Q_\gamma |x^{1-\gamma} - y^{1-\gamma}| \geq |x^\gamma y^{1-\gamma} \log x - y^\gamma x^{1-\gamma} \log y|.
\]

Proof: The proof is given in Appendix.

\section{Proof of Main Theorem.}

Proof: From Remark 2, we know that $f_n(x)$ satisfy the condition 2) in Theorem 1, and $f_n(x), g(x)$ satisfy the condition 1) in Theorem 1. By Lemma 5 and letting $M_\alpha = \max\{M'_\alpha, M''_\alpha\}$, we obtain that for every $\alpha > 0$
\[
\int_R f_\alpha^n(x)dx, \int_R g_\alpha^n(x)dx \leq M_\alpha
\]
uniformly in $n \in \mathbb{N}$ and
\{f_n(x)\} satisfies the condition 3) in Theorem 1. Hence from Theorem 1 we obtain that

\[
\lim_{n \to \infty} R_\alpha(S_n) = R_\alpha(G), \quad \lim_{n \to \infty} T_\alpha(S_n) = T_\alpha(G).
\]

Next, we’ll study the rates of convergence for \(R_\alpha(S_n)\) and \(T_\alpha(S_n)\).

At first, we consider the case \(\alpha > 0\) and \(\alpha \neq 1\).

Using the inequality \(\log(1 + x) < x\) \((x > 0)\), and inequalities (5), (2), (4), we have

\[
\begin{align*}
|\alpha - 1|\int_R f_n^\alpha(x) - g^\alpha(x)dx \\
\leq 2\alpha M_{\alpha-1}\sup_{x \in R}[f_n(x) - g(x)] \\
\leq \begin{cases} 
\frac{2\alpha M_{\alpha-1}\sup_{x \in R}[f_n(x) - g(x)]}{2 + 1 - \gamma} & \alpha \in (1, +\infty); \\
\frac{2\alpha M_{\alpha-1}\sup_{x \in R}[f_n(x) - g(x)]}{2 + 1 - \gamma} & \alpha \in (0, 1), \gamma \in (0, \frac{1}{2}).
\end{cases}
\end{align*}
\]

Combining (10) and (11), we obtain

\[
|\alpha - 1|\int_R f_n^\alpha(x) - g^\alpha(x)dx \\
\leq \begin{cases} 
O(n^{-\frac{1}{2}}) & \alpha \in (1, +\infty); \\
O(n^{-\frac{1}{2} + \gamma}) & \alpha \in (0, 1), \gamma \in (0, \frac{1}{2}).
\end{cases}
\]

On the other hand, using the inequalities (6), (2), (4), we have

\[
\begin{align*}
|\alpha - 1|\int_R f_n^\alpha(x) - g^\alpha(x)dx \\
\leq \begin{cases} 
\frac{2\alpha M_{\alpha-1}\sup_{x \in R}[f_n(x) - g(x)]}{2 + 1 - \gamma} & \alpha \in (1, +\infty); \\
\frac{2\alpha M_{\alpha-1}\sup_{x \in R}[f_n(x) - g(x)]}{2 + 1 - \gamma} & \alpha \in (0, 1), \gamma \in (0, \frac{1}{2}).
\end{cases}
\end{align*}
\]

Combining (10) and (12), we obtain

\[
|\alpha - 1|\int_R f_n^\alpha(x) - g^\alpha(x)dx \\
\leq \begin{cases} 
O(n^{-\frac{1}{2}}) & \alpha \in (1, +\infty); \\
O(n^{-\frac{1}{2} + \gamma}) & \alpha \in (0, 1), \gamma \in (0, \frac{1}{2}).
\end{cases}
\]

Now we investigate the case \(\alpha = 1\).

Observe that for any random variable \(X\),

\[ R_1(X) = T_1(X) = H(X). \]

In what follows we show the following

\[ |H(S_n) - H(G)| = O(n^{-\frac{1}{2} + \gamma}), \quad \text{for } \gamma \in (0, \frac{1}{2}). \]
It follows that
\[
\int f_n(x)g^{1-\gamma}(x) \log f_n(x)dx - \int g(x)\log g(x)dx \leq n^{\frac{1}{\gamma} - \gamma}Q\int |f^{1-\gamma}(x) - g^{1-\gamma}(x)|dx.
\]
Combining inequalities (3), (4) and equality (10), we have
\[
\int |f^{1-\gamma}(x) - g^{1-\gamma}(x)|dx = O(n^{-\frac{1}{\gamma} + \gamma}).
\]
Hence, there exists a \( C_4 > 0 \) such that \( n^{\frac{1}{\gamma} - \gamma}J_2(n) \leq C_4 \) for \( n > N_3 \).

From the above discussion and inequality (14), there exists constant \( C := M_2C_1C_2 + C_3 + C_4 \) such that
\[
n^{\frac{1}{\gamma} - \gamma} \left| \int f_n(x)\log f_n(x)dx - \int g(x)\log g(x)dx \right| \leq C,
\]
for \( n > \max\{N_1, N_2, N_3\} \).

Hence, (13) is true.

\[\square\]

IV. CONCLUSION AND DISCUSSION

We show the convergence of the normalized sum of IID continuous random variables with bounded moments of all order in the sense of Rényi entropy and Tsallis entropy, and obtain sharp rates of convergence. By using Feller’s expansion and detailed analytical properties of the corresponding densities, we estimate the Rényi entropy and Tsallis entropy directly. The main difficulty lies in the case of Shannon entropy, both on the convergence and rate of convergence, because \( \alpha = 1 \) is a singularity of \( R_\alpha \) and \( T_\alpha \). We circumvent it by obtaining some uniform estimations near \( \alpha = 1 \). Compared with the previous proof for Shannon entropy, our proof is more direct, can be generalized to random vectors in higher dimension, and may be used to consider the convergence of normalized sum of dependent random variables. The bounded moment condition we used is weaker than the Poincaré constant condition in Shannon case [6]. As a result, the rates of convergence is slower than \( O(\frac{1}{n}) \) in Shannon case. It is interesting to consider the convergence and rates of convergence for Rényi divergence as Rényi raised in [10].

V. APPENDIX

Proof of Lemma 2

Proof: Observing that
\[
\frac{\partial}{\partial t}(1 - \int p_n(x)dx) = -\int p_n(x)\log p_n(x)dx
\]
and \( (1 - \int p_n(x)dx) |_{t=1} = 0 \), we have that
\[
1 - \int p_n^2(x)dx = -\int \int p_n(x)\log p_n(x)dx dt.
\]
From equality (15) we have that
\[
\begin{align*}
T_n(Y_n) - H(Y_n) &= \frac{1}{1 - \alpha |\frac{1}{1 - t}|} \int \frac{|J_n(t)|}{t-1} dt \leq \frac{1}{1 - \alpha |\frac{1}{1 - t}|} \int L|t-1| dt \leq L |1 - \alpha|.
\end{align*}
\]

by condition 2) of Theorem 1. From condition 3) in Theorem 1, we know that \( \int p_n(x)dx \leq M_1 \). Letting \( L = B(1 + M_1) \), we obtain that \( |J_n(t)/(1 - t)| \leq L \) for all \( n \in \mathbb{N} \).

Combining with inequality (16), we have that
\[
|R_\alpha(Y_n) - H(Y_n)| \leq \frac{1}{1 - \alpha |\frac{1}{1 - t}|} \int \frac{|J_n(t)|}{t-1} dt \leq L |1 - \alpha|.
\]

where
\[
B = \sup_{x \in \mathcal{A}_n, n \in \mathbb{N}} |p_n^2(x)| < \infty.
\]
Therefore, for any \( \varepsilon > 0 \), there exists a \( \delta = \min\{\frac{1}{2}, \frac{1}{4}\} \), such that \( |T_n(Y_n) - H(Y_n)| < \varepsilon \) uniformly for all \( n \in \mathbb{N} \) and all points \( \alpha \) satisfying \( |\alpha - 1| < \delta \) and \( \frac{3}{4} \leq \alpha \leq \frac{3}{2} \).

The conclusion 2) can be obtained by similar arguments. \( \blacksquare \)

**Proof of Lemma 6**

Proof: We just prove the case: \( 0 \leq y \leq x \leq 1 \), the proof of the case \( 0 \leq x \leq y \leq 1 \) is similar.

Suppose that \( d_\gamma(x) = x^\gamma - (1 - 2\gamma)x^\gamma \log x \). Since \( d_\gamma(x) \) is continuous in \( x \in [0,1] \), for every \( \gamma \in (0, \frac{1}{2}) \), \( d_\gamma(x) \) is bounded in \( x \in [0,1] \). Thus, for every \( \gamma \in (0, \frac{1}{2}) \), there exists a \( Q_\gamma > 0 \) such that \( |d_\gamma(x)| \leq Q_\gamma x^{-\gamma} \).

For \( 0 \leq y \leq x \leq 1 \), we have \( x^{1-\gamma} \geq y^{1-\gamma} \) and

\[
x^{\gamma - 2} y^{1-\gamma} \log x - y^{\gamma - 2} x^{1-\gamma} \log y \geq -x^{\gamma - 2} y^{1-\gamma} (x^{2\gamma - 2} - y^{2\gamma - 2}) \log x \geq 0.
\]

It follows that \( Q_\gamma |x^{1-\gamma} - y^{1-\gamma}| = Q_\gamma (x^{1-\gamma} - y^{1-\gamma}) \) and \( |x^{\gamma - 2} y^{1-\gamma} \log x - y^{\gamma - 2} x^{1-\gamma} \log y| = x^{\gamma - 2} y^{1-\gamma} \log x - y^{\gamma - 2} x^{1-\gamma} \log y \).

Denote

\[
A_\gamma(x) := Q_\gamma (x^{1-\gamma} - y^{1-\gamma}) - (x^{\gamma - 2} y^{1-\gamma} \log x - y^{\gamma - 2} x^{1-\gamma} \log y)
= x^{\gamma - 2} y^{1-\gamma} (Q_\gamma y^{\gamma - 1} - Q_\gamma x^{\gamma - 1}) - x^{2\gamma - 2} \log x + y^{2\gamma - 2} \log y
= x^{\gamma - 2} y^{1-\gamma} (Q_\gamma y^{\gamma - 1} - Q_\gamma x^{\gamma - 1}) - x^{2\gamma - 2} \log x + y^{2\gamma - 2} \log y.
\]

where

\[
F_\gamma(x) := Q_\gamma y^{-1} - Q_\gamma x^{-1} - x^{2\gamma - 1} \log x + y^{2\gamma - 1} \log y.
\]

In what follows, we show that \( A_\gamma(x) \geq 0 \) for \( 0 \leq y \leq x \leq 1 \), which implies the Lemma is true for \( 0 \leq y \leq x \leq 1 \).

Obviously, if \( y = 0 \), \( A_\gamma(x) = Q_\gamma x^{1-\gamma} \geq 0 \). If \( y > 0 \), since \( F_\gamma(y) = 0 \) and

\[
F'_\gamma(x) = (1 - \gamma)Q_\gamma x^{\gamma - 2} + (1 - 2\gamma)x^{2\gamma - 2} \log x - x^{2\gamma - 2}
= x^{\gamma - 2} \{ (1 - \gamma)Q_\gamma - [x^{\gamma} - (1 - 2\gamma)x^{\gamma} \log x] \}
= x^{\gamma - 2} \{ (1 - \gamma)Q_\gamma - d_\gamma(x) \}
\geq x^{\gamma - 2} \frac{Q_\gamma - 2}{2} = 0,
\]

we obtain for every \( \gamma \in (0, \frac{1}{2}) \) and \( y > 0 \), \( F_\gamma(x) \geq 0 \) when \( x \geq y \). As a result, \( A_\gamma(x) \geq 0 \). \( \blacksquare \)

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