Abstract

In the frame of our approach we constructed the generalized oscillator connected with Krawtchouk polynomials (named Krawtchouk oscillator) and coherent states for this oscillator too. Ours results are compared with analogues ones obtained for another variant of Krawtchouk oscillator in the paper [N.M. Atakishiev and S.K. Suslov, *Difference analogs of the harmonic oscillator*, Teor. Mat. Fiz., 85, 64-73 (1990)] from other point of view. Our definition of coherent states is close to one given in [B. Roy and P. Roy, *Phase properties of a new nonlinear coherent state*, quant-ph/0002043].

1. Introduction. The Krawtchouk polynomials ([1] - [3]) have interesting applications in the quantum optics (for instance see the oscillator model studied in the papers [4], [5]). These polynomials provide an important example of the classical orthogonal polynomials of discrete variable. It is very attractive for some applications that they can be considered as finite-dimensional approximations of the Hermite and Charlier polynomials [3]. In the papers [6] - [11] the authors proposed a new approach to construction of oscillator-like systems (so-called generalized oscillators) connected with given family of orthogonal polynomials. Furthermore, in these papers was introduced and investigated the coherent states for this type oscillators. Now in the frame of our approach we construct the generalized oscillator connected with Krawtchouk polynomials (named Krawtchouk oscillator)

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and coherent states for this oscillator too. Our results are consistent with analogous ones obtained in the papers [4], [5], in which similar oscillator was considered from other point of view.

The most attention in our paper is given to construction of coherent states. Because of the space of the states for Krawtchouk oscillator is finite-dimensional space it is obvious that the annihilation operator cannot have any eigenvalue differ from zero. This means that in this case the standard definition of coherent states as eigenvectors of this operator (the coherent states of Barut - Girardello type) falls. In this connection we generalize the method of the construction of coherent states that goes back to the Glauber’s papers [12]. This approach was used in the case of so-called finite-dimensional (truncated) variant of the standard boson oscillator in the paper [13]. We compare the resulting coherent states for Krawtchouk oscillator with ones considered in the papers [4], [14], [15].

2. Krawtchouk Polynomials. Let us recall the definition of Krawtchouk polynomials ([1] - [3])

\[ K_n(x; p, N) := 2^F_1 \left( \begin{array}{c} -n, -x \\ -N \end{array}; p^{-1} \right) = \sum_{k=0}^{N} \frac{(-n)_k(-x)_k}{k!(N)_k} p^{-k} \]  \tag{1}

where \(0 < p < 1, \quad n = 0, 1, \ldots, N,\) and

\[(a)_0 = 1, \quad (a)_k = a(a + 1) \cdot \ldots \cdot (a + k - 1) = \frac{\Gamma(a + k)}{\Gamma(a)}. \]  \tag{2}

It is convenient for construction of generalized oscillator to renormalize the Krawtchouk polynomials, following [6]

\[ \tilde{K}_n(x; p, N) = \sqrt{\rho(n, p, N)} K_n(x; p, N), \quad n = 0, 1, \ldots, N, \]  \tag{3}

where

\[ \rho(n, p, N) = C_N^n p^n (1 - p)^{N-n}, \quad C_N^n := \frac{N!}{\Gamma(n + 1)\Gamma(N - n + 1)}. \]  \tag{4}
The renormalized Krawtchouk polynomials $\tilde{K}_n$ fulfill recurrent relations with symmetrical Jacobi matrix

$$x\tilde{K}_n(x; p, N) = b_n \tilde{K}_{n+1}(x; p, N) + a_n \tilde{K}_n(x; p, N) + b_n \tilde{K}_{n-1}(x; p, N), \quad n = 0, 1, \ldots, d,$$

(5)

$$\tilde{K}_0(x; p, N) = 1,$$

(6)

where

$$a_n = p(N-n) + n(1-p), \quad b_n = -\sqrt{p(1-p)(n+1)(N-n)}, \quad n = 0, 1, \ldots, N.$$

(7)

The Krawtchouk polynomial $\tilde{K}_n(x; p, N)$ is a solution of the difference equation

$$-n\tilde{K}_n(x; p, N)(x; p, N) = p(N-x)\tilde{K}_n(x; p, N)(x+1; p, N) -$$

$$- [p(N-x) + x(1-p)] \tilde{K}_n(x; p, N)(x; p, N) +$$

$$+ x(1-p) \tilde{K}_n(x; p, N)(x-1; p, N)$$

(8)

We recall also orthogonality relations for Krawtchouk polynomials

$$\sum_{x=0}^{N} \rho(x; p, N) \tilde{K}_m(x; p, N) \tilde{K}_n(x; p, N) = \delta_{m,n};$$

(9)

$$\sum_{x=0}^{N} \rho(n; p, N) \tilde{K}_n(x; p, N) \tilde{K}_n(y; p, N) = \delta_{x,y},$$

(10)

where $0 < p < 1$ and $n, x = 0, 1, \ldots, N$.

3. Krawtchouk Oscillator. Now we describe (following [6]) the construction of Krawtchouk oscillator. Let $\tilde{\mathcal{H}}_{p,N} = \ell^2_{N+1}(\rho(x; p, N))$ be the $N + 1$-dimensional Hilbert space spanned by the orthonormal basis $\{ \tilde{K}_n(x; p, N) \}_{n=0}^{N}$ (with weight function $\rho(x; p, N)$). We call the Krawtchouk oscillator the oscillator-like system in $\tilde{\mathcal{H}}_{p,N}$ defined by generalized opera-
tors of "coordinate" $\tilde{X}$, "momentum" $\tilde{P}$ and quadratic Hamiltonian $\tilde{H}$

$$\tilde{X} : = \text{Re}(X - P),$$

$$\tilde{P} : = -i \text{Im}(X - P),$$

$$\tilde{H} : = \frac{1}{4p(1 - p)} \left( \tilde{X}^2 + \tilde{P}^2 \right).$$ (13)

Here the selfadjoint operators $X$ and $P$ are defined by the action on basis elements $\tilde{K}_n(x; p, N)$ in the Hilbert space $\tilde{H}_{p, N}$

$$X\tilde{K}_n(x; p, N) = b_n \tilde{K}_{n+1}(x; p, N) + a_n \tilde{K}_n(x; p, N) + b_{n-1} \tilde{K}_{n-1}(x; p, N),$$

$$P\tilde{K}_n(x; p, N) = -ib_n \tilde{K}_{n+1}(x; p, N) + a_n \tilde{K}_n(x; p, N) + ib_{n-1} \tilde{K}_{n-1}(x; p, N),$$

$$X\tilde{K}_0(x; p, N) = b_0 \tilde{K}_1(x; p, N) + a_0 \tilde{K}_0(x; p, N),$$

$$P\tilde{K}_0(x; p, N) = -ib_0 \tilde{K}_1(x; p, N) + a_0 \tilde{K}_0(x; p, N),$$

where

$$a_n = p(N - n) + n(1 - p), \quad b_n = -\sqrt{p(1 - p)(n + 1)(N - n)}, \quad n = 0, 1, \ldots, N,$$

The action of creation and annihilation operators

$$\tilde{a}^\pm := \frac{1}{2\sqrt{p(1 - p)}} \left( \tilde{X} \pm i\tilde{P} \right),$$

on the basis states in $\tilde{H}_{p, N}$ are given by the relations

$$\tilde{a}^- \tilde{K}_n(x; p, N) = -\sqrt{n(N - n + 1)} \tilde{K}_{n-1}(x; p, N),$$

$$\tilde{a}^+ \tilde{K}_n(x; p, N) = -\sqrt{(n + 1)(N - n)} \tilde{K}_{n+1}(x; p, N).$$

These operators fulfill commutation relations

$$[\tilde{a}^-, \tilde{a}^+] = (N - 1)I - 2N$$

(22)
where operator $\mathcal{N}$ is defined by
\[
\mathcal{N}\tilde{K}_n(x; p, N) = n\tilde{K}_n(x; p, N).
\] (23)

From the relations (20)-(21) it follows that the eigenvalues of Hamiltonian operator
\[
\tilde{H} = \frac{1}{2}(\tilde{a}^+\tilde{a}^- + \tilde{a}^-\tilde{a}^+)
\] (24)
are given by $(0 \leq n \leq N)$
\[
\tilde{H}\tilde{K}_n(x; p, N) = \lambda_n\tilde{K}_n(x; p, N), \quad \lambda_n = N(n + \frac{1}{2}) - n^2,
\] (25)
so that $\lambda_N = \lambda_0 = \frac{1}{2}N$. Using the results obtained in [7] it can be shown that difference equation (8) for Krawtchouk polynomials $\tilde{K}_n(x; p, N)$ is equivalent to eigenvalue equation for Hamiltonian $\tilde{H}$ in the space $\tilde{H}_{p,N}$. For obtaining the explicit expression of this operator as a difference operator in $\tilde{H}_{p,N}$ it is convenient to compare our variant of Krawtchouk oscillator with one was considered in [4].

4. Comparison of Krawtchouk oscillator with variant considered in the work of Atakishiev and Suslov [4]. In the works [4], [5] Atakishiev and Suslov studied the Krawtchouk oscillator with Hamiltonian
\[
H_{AS} = 2p(1-p)N + \frac{1}{2} + (1-2p)\frac{\xi}{h} - \sqrt{p(1-p)}\left[\alpha(\xi)e^{\frac{\hbar}{h}\partial_\xi} + \alpha(\xi-h)e^{-\frac{\hbar}{h}\partial_\xi}\right],
\] (26)
where
\[
h = \sqrt{2Np(1-p)}, \quad \alpha(\xi) = \sqrt{(1-p)N - \frac{\xi}{h}}. \left(pN + 1 + \frac{\xi}{h}\right).
\] (27)
This operator is defined in the Hilbert space $\mathcal{H}_{AS} = \ell^2(\xi)$ with orthonormal basis consisting of Krawtchouk functions
\[
\Psi_n(\xi) = (-1)^n\sqrt{\binom{n}{N}(\frac{p}{1-p})^n \rho(pN + \frac{\xi}{h}; p, N)K_n(pN + \frac{\xi}{h}; p, N)},
\] (28)
which fulfill the orthogonality relations ($\xi_j = h(j - pN)$)

$$
\sum_{j=0}^{N} \Psi_n(\xi_j)\Psi_m(\xi_j) = \delta_{n,m}, \quad \sum_{n=0}^{N} \Psi_n(\xi_i)\Psi_n(\xi_j) = \delta_{i,j}.
$$

(29)

The Krawtchouk functions are eigenfunctions of Hamiltonian $H_{AS}$ in the space $H_{AS}$

$$
H_{AS}\Psi_n(\xi) = \lambda_n \Psi_n(\xi), \quad \lambda_n = n = \frac{1}{2}, \quad n = 0, 1, \ldots, N.
$$

(30)

The Hamiltonian $H_{AS}$ can be factorized

$$
H_{AS}(\xi) = \frac{1}{2} [A^+, A^-] + \frac{1}{2} (N + 1),
$$

(31)

by ladder operators

$$
A^+(\xi) = (1 - p)e^{-h\partial_\xi}\alpha(\xi) - p\alpha(\xi)e^{h\partial_\xi} + \sqrt{p(1 - p)} \left( (2p - 1)N + \frac{2\xi}{h} \right),
$$

(32)

$$
A^-(\xi) = (1 - p)\alpha(\xi)e^{h\partial_\xi} - pe^{-h\partial_\xi}\alpha(\xi) + \sqrt{p(1 - p)} \left( (2p - 1)N + \frac{2\xi}{h} \right),
$$

(33)

which act on the basis elements according to

$$
A^+(\xi)\Psi_n(\xi) = \sqrt{(n + 1)(N - n)}\Psi_{n+1}(\xi), \quad A^-(\xi)\Psi_n(\xi) = \sqrt{n(N - n + 1)}\Psi_{n-1}(\xi).
$$

(34)

The operators $A^\pm$ and the operator

$$
A_0(\xi) := \frac{1}{2} [A^+(\xi), A^-(\xi)]
$$

(35)

fulfill the commutation relations of the Lee algebra $so(3)$

$$
[A^+(\xi), A^-(\xi)] = 2A_0(\xi), \quad [A_0(\xi), A^\pm(\xi)] = \pm A^\pm(\xi).
$$

(36)
Let us denote by $\tilde{H}_{AS}$ the selfadjoint operator in the Hilbert space $\mathcal{H}_{p,N}$ that unitary equivalent to the selfadjoint operator $H_{AS}$ in the space $\mathcal{H}_{AS}$

$$\tilde{H}_{AS} = T^{-1}H_{AS}T.$$  \hfill (37)

The explicit expression for unitary operator $T : \mathcal{H}_{p,N} \rightarrow \mathcal{H}_{AS}$ is rather cumbersome. For brevity we omit it. It follows from comparing the spectrum of the hamiltonian operators $\tilde{H}$ and $\tilde{H}_{AS}$ that these operators are connected by the relation

$$\tilde{H} = -\left(\tilde{H}_{AS} - \frac{1}{2}\mathbb{1}\right)^2 + N\tilde{H}_{AS}.$$  \hfill (38)

5. Coherent states for Krawtchouk oscillator. We consider the $(N+1)$-dimensional Hilbert space $\mathcal{H}_{p,N}$ as the Fock space $\mathcal{F}$ equipped with the basis states $\{|0\rangle, |1\rangle, \ldots, |N\rangle\}$. Note that in coordinate picture we have $\langle x|n\rangle = \tilde{K}_n(x; p, N)$. We define the coherent states (of the Glauber type) in $\mathcal{F}$ by the relation

$$|z\rangle = \sum_{n=0}^{N} \frac{(z\tilde{a}_+ - \bar{z}\tilde{a}_-)^n}{n!} |0\rangle.$$  \hfill (39)

One can rewrite this definition in the following form (wich is close to the coherent state considered in [13])

$$|z\rangle = \sum_{l=0}^{N} \sum_{n=l}^{\infty} d_{n,l}^{N} \frac{(\sqrt{2}b_{l-1})^l}{n!} \left(\frac{z}{\bar{z}}\right)^{(n-l)}(z)^{-l} |l\rangle.$$  \hfill (40)

where $b_k$ are given by (7) and coefficients $d_{n,l}^{N}$ fulfill the recurrence relations

$$d_{n,l}^{N} = \theta_l d_{n-1,l-1}^{N} + 2b_l^2 \theta_{l+1} d_{n-1,l+1}^{N-1},$$
$$d_{n-1,-1}^{N} = 0, \quad d_{0,0}^{N} = 1, \quad d_{n,n+k}^{N} = 0 \quad \text{for} \quad k > 0.$$  \hfill (41)

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The solution of these recurrence relations has the form

\[
d_{n,l}^N = \frac{1}{(2b_l^{-2})! \sum_{k=0}^N \left( \tilde{\psi}_N^{-2}(x_k) \right) \sum_{k=0}^N \left( \frac{\tilde{\psi}_l(x_k)}{\tilde{\psi}_N(x_k)} \right)^2}, \tag{43}
\]

here \( x_k (k = 0, 1, \ldots, N) \) are the roots of the equation

\[
\tilde{\psi}_{N+1}(x) = 0, \tag{44}
\]

where

\[
\tilde{\psi}_n(x\sqrt{2}) = \left( \sqrt{2b_{n-1}} \right)! \tilde{K}_n^{(0)}(x; p, N), \tag{45}
\]

and polynomials \( \tilde{K}_n^{(0)}(x; p, N) \) fulfill the recurrent relations

\[
x \tilde{K}_n^{(0)}(x; p, N) = b_n \tilde{K}_{n+1}^{(0)}(x; p, N) + b_{n-1} \tilde{K}_{n-1}^{(0)}(x; p, N), \quad n = 0, 1, \ldots, N, \tag{46}
\]

\[
\tilde{K}_0^{(0)}(x; p, N) = 1, \tag{47}
\]

which can be obtained from (45) if we put \( a_n = 0 \).

With help of the relations (43) and (45) we can rewrite the expression for the coherent states in the form

\[
|z\rangle = \frac{1}{\sum_{k=0}^N \left( \tilde{\psi}_N^{-2}(x_k) \right)} \sum_{l=0}^N \left(-\frac{i}{|z|}\right)^l \frac{1}{(\sqrt{2b_{l-1}})!} \left( \sum_{k=0}^N \frac{\tilde{K}_l(x_k; p, N)}{(\tilde{K}_N(x_k; p, N))^2} e^{i|z|x_k} \right)|l\rangle \tag{48}
\]

5. Comparison with other types of coherent states. Let us compare definition of coherent states given above with some other definitions of coherent states for Krawtchouk oscillator: the so-called spin coherent states [4], [14] and phase - type coherent states [15].

The spin coherent states for Krawtchouk oscillator in papers [4], [14] was given in the form

\[
\langle x|\xi\rangle = (1 + |\xi|^2)^{-N/2} \sum_{n=0}^N \sqrt{C_N^m} \xi^n \Psi_n(x), \tag{49}
\]
where $\Psi_n(x)$ defined by (28). It is obvious that these states differ from the states (48).

The phase coherent states suggested in [15] look as

$$|z\rangle = \left(1 + \left| \frac{2\pi z}{N + 1} \right|^2 \right)^{-N/2} \sum_{n=0}^{N} \sqrt{c_n} \left( \frac{2\pi z}{N + 1} \right)^n |\theta_n\rangle,$$  

(50)

where

$$|\theta_n\rangle = (N + 1)^{-\frac{1}{2}} \sum_{k=0}^{N} e^{ik\theta_n} |k\rangle, \quad \theta_n = \theta_0 + \frac{2\pi n}{N + 1}, \quad n = 0, 1, \ldots, N + 1.$$  

(51)

These states are also different from the states (48) given above.

Our coherent states (48) similar to the so called finite-dimensional coherent states suggested in [16],[13] in the framework of the Pegg - Barnet formalism. We plan to consider the main properties of coherent states (48) in other publication.

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