FAST RELAXATION TIME IN A SPIN MODEL WITH GLASSY BEHAVIOR

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Abstract
We consider a frustrated spin model with a glassy dynamics characterized by a slow component and a fast component in the relaxation process. The slow process involves variables with critical behavior at finite temperature $T_p$ and has a global character like the (structural) $\alpha$-relaxation of glasses. The fast process has a more local character and can be associated to the $\beta$-relaxation of glasses. At temperature $T > T_p$, the fast relaxation follows the non-Arrhenius behavior of the slow variables. At $T \lesssim T_p$, the fast variables have an Arrhenius behavior, resembling the $\alpha - \beta$ bifurcation of fragile glasses. The model allows us to analyze the relation between the dynamics and the thermodynamics.

1. INTRODUCTION
A recent review\cite{1} has pointed out some of the leading questions in the theoretical study of glassforming liquids. Following the fruitful path of theoretical studies on spin models with glassy behavior\cite{2}, here we present results on a frustrated spin model that give an insight on some of these questions, showing relations between the dynamics and the statics of the model.

We consider a model, introduced by Coniglio and coworkers\cite{3,4,5,6,7,8}, with two kinds of variables, coupled to each other. One of them has a finite temperature transition and a slow relaxation, while the other has a transition only at zero temperature in two dimensions (2D) and has a fast relaxation at finite temperature. The model has been shown\cite{6,8} to have glassy behavior and the slow and fast components of the relaxation can be related to the $\alpha$ and $\beta$ processes, respectively, of glasses. Here we show numerical results resembling the $\alpha-\beta$ bifurcation process\cite{1} observed in experiments on fragile liquids\cite{9,10}.

In the following we give an introduction to the problem. In Sec. 2 we introduce the spin model, reviewing some results on its phase diagram and on its glassy behavior. In Sec. 3 we show some numerical results about the bifurcation in this model and the connection
with the thermodynamics. In Sec. 4 we give the conclusions and the perspectives for future work.

1.1 Glasses And The $\alpha$-$\beta$ Bifurcation

The majority of liquids can form a glass if cooled at high enough rate. The glass state is experimentally defined by measuring macroscopic quantities, like the diffusion coefficient or the viscosity. The calorimetric glass transition temperature $T_g$ is defined as the temperature at which the viscosity reaches a value of $10^{12}$sPa and depends on the cooling rate.

The glass transition is considered as a dynamic off-equilibrium phenomenon, but a large amount of work has recently shown that the dynamics of glassy systems is strictly related to its static properties, i.e. to its thermodynamics. The main focus of these studies is the slow component of the dynamics. One of the aim of this paper is to show that also the fast processes can be related to the thermodynamics of the system.

Following the Angell’s classification, glassforming liquids are divided in ‘strong’ and ‘fragile’. Strong liquids are characterized by a relaxation time $\tau(T)$ that increases with decreasing temperature $T$ in a way well described by an Arrhenius law

$$\tau(T) = \tau_\infty \exp[A/(k_B T)]$$

where $\tau_\infty$ is interpreted as a characteristic microscopic relaxation time at infinite $T$ and $A$ is an activation energy (energy barrier) for global rearrangements ($k_B$ is the Boltzmann constant). A strong glassformer is, for example, SiO$_2$.

Fragile liquids are defined as those whose $\tau(T)$ shows a large departure from the Arrhenius law (non-Arrhenius behavior) for $T > T_g$. For $T < T_g$ the relaxation time is well described by an Arrhenius law. Example of fragile glassformers are glycerol and orto-therphenyl. A function widely used to fit data for fragile liquids is the Vogel-Tamman-Fulcher (VTF) law

$$\tau(T) = \tau_\infty \exp[B/(T - T_0)]$$

where $\tau_\infty$ is a hight-T characteristic time, $B$ and $T_0$ are fitting constants. Since $T_0 < T_g$ the diverging behavior is never reached.

An alternative non-Arrhenius behavior is derived in the Mode Coupling Theory (MCT) for $T > T_g$, in which the relaxation time increases as a power law of the temperature,

$$\tau(T) \simeq (T - T_{MCT})^{-\gamma}$$

where $T_{MCT}$ is the MCT transition temperature (with $T_{MCT} < T_g$) and $\gamma$ is a parameter.

The non-Arrhenius behavior can be explained, in a consistent way with the MCT, also by the Adam-Gibbs theory of the excluded volume, in which the relaxation time is

$$\tau = \tau_\infty \exp(C/T S_c)$$

where $\tau_\infty \simeq 10^{-14}$s, $C$ is a constant and $S_c$ is the entropy difference between liquid and crystal – ‘excess’ entropy. Since $S_c$ is, usually, constant below $T_g$, the low-$T$ Arrhenius behavior is recovered.

This phenomenology concerns the structural $\alpha$-relaxation, i.e. the process described by the long-time part of the relaxation functions of the system. For high $T$, the relaxation
functions, like the density-density correlation function, have a simple exponential behavior ('simple liquid' regime)

\[ f(t) = f_0 \exp(t/\tau) , \]

with \( f_0 \) and \( \tau \) depending on \( T \) and where \( t \) is the time.

Decreasing \( T \) the relaxation functions usually show a ‘two-steps’ behavior where a fast relaxation process, associated to the first step, is followed by a slow relaxation process, the second step. The second step is the \( \alpha \)-relaxation and is associated to a macroscopic process involving non-local degrees of freedom (as the global reorganization of the system). Below a characteristic temperature \( T^* > T_g \), the second step is well approximated in most cases by the Kohlrausch-Williams-Watts (KWW) stretched exponential function

\[ f_{\text{KWW}}(t) = f_0 \exp[(t/\tau)^\beta] , \]

where \( f_0 \), \( \tau \) and \( \beta \) depend on \( T \). The KWW function recovers the simple exponential for \( \beta = 1 \), with the parameter \( \beta \) describing the departure from exponentiality. The fit of glassy relaxation functions, usually, shows \( \beta \) decreasing with the temperature, with \( \beta < 1 \) for \( T < T^* \). In many cases, a better approximation of the second step of the relaxation function is given by using the Ogielski\[4 \]stretched exponential function

\[ f_{\text{O}}(t) = t^{-x} f_{\text{KWW}}(t) , \]

with \( x \) depending on \( T \).

The first step (or \( \beta \)-relaxation) is usually associated to any microscopic process occurring at very short time scale. In the MCT for atomic liquids it is associated to the fast diffusion of mobile particles inside ‘cages’ of immobile particles. In molecular liquids it can be associated also to the relaxation of internal degrees of freedom\[15 \].

The \( \beta \)-relaxation presents some non-universal features in fragile liquids\[9 \], and sometimes in intermediate liquid\[10 \], that are not well understood. The \( \beta \)-relaxation time follows the non-Arrhenius \( \alpha \)-relaxation behavior at high \( T \) and shows a crossover (the \( \alpha - \beta \) bifurcation) to an Arrhenius behavior at a temperature \( T_{\alpha - \beta} \simeq T_{\text{MCT}} \). To have an insight on this phenomenon and its connection with other static and dynamic transitions, we consider the spin model presented in the next section.

2. THE SPIN MODEL

We consider a lattice model with, on each lattice site, a Potts variable \( \sigma_i \), with and integer number \( s \) of states \( (\sigma_i = 1, \ldots , s) \), coupled to an Ising spin \( S_i \) with two states \( (S_i = \pm 1) \). The model can be considered as a schematic representation of structural glasses, such as dense molecular glasses, plastic crystal, or orto-therphenyl at low temperature, with orientational degrees of freedom frustrated by geometrical hindrance between non-spherical molecules. The orientational degrees of freedom are represented by the Potts variables and the frustration is modeled by means of ferro/antiferromagnetic interactions for the Ising spins. The model is defined by the Hamiltonian

\[ H_s\{\tau_i, e_{i,j}\} = -2sJ \sum_{\langle i,j \rangle} \delta_{e_{i,j} \tau_i, \tau_j} \]
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Fig. 1  a) A Potts fully frustrated (PFF) model on a square lattice: on each vertex there is variable \( \tau_i \equiv S_i \sigma_i = \pm 1, \pm 2 \ldots \pm s \) with \( s = 4 \). Here we represent the spin \( S_i \) by an open or a full dot (respectively positive and negative, for example) and the orientation \( \sigma_i \) by an arrow pointing in 4 different directions. Ferromagnetic (antiferromagnetic) interactions are represented by full (dotted) lines. b) The analytic value of the transition temperature (in 2D) \( T_c \equiv T_p(q, 0) \), with \( q = 2s \), of the ferromagnetic Potts model (upper line), corresponding to \( X = 0 \), is given together with the 2D Monte Carlo data for the PSG model (circles), corresponding to \( X = 0.5 \), with transition at \( T_p^{PSG} \) and for the PFF model (squares), corresponding to \( X = 1 \), with transition at \( T_p^{PFF} \). The data are fitted with Eq. (10) with \( T_p^{PSG} \) and \( T_p^{PFF} \). The parameters \( a(X) \) are shown in the inset. Errors are smaller than symbols size.

where \( \tau_i \equiv S_i \sigma_i = \pm 1, \pm 2 \ldots \pm s \) has \( q \equiv 2s \) states, the sum is extended over all the nearest neighbor (n.n.) sites, \( J \) is the strength of interaction, \( \epsilon_{i,j} = \pm 1 \) is a quenched variable that represents the sign of the ferro/antiferromagnetic interaction, \( \delta_{n,m} = 0,1 \) is a Kronecker delta.

In the original formulation, the Hamiltonian shows a clearer separations between the two \( (\sigma_i \text{ and } S_i) \) coupled variables:

\[
H_s\{S_i, \sigma_i, \epsilon_{i,j}\} = -sJ \sum_{(i,j)} \delta_{\sigma_i,\sigma_j}(\epsilon_{i,j}S_iS_j + 1) .
\]  

The Eq. (9) shows that the Ising spins interactions are dilut ed by a ferromagnetic Potts model, i.e. two n.n. Ising spins can interact only when the corresponding n.n. Potts variables are in the same state (or orientation).

The model has been proposed in two versions. In the first version, the interaction signs \( \epsilon_{i,j} \) are randomly assigned, giving rise to frustration and disorder. This model can be considered the generalization of an Ising spin glass (SG) – with two states – to a model with 2s states – the Potts SG (PSG).

In the other version, the interactions signs are assigned in a deterministic way and there is an odd number of \( \epsilon_{i,j} = -1 \) (antiferromagnetic interaction) on each lattice cell. In this way the model has frustration and no disorder and is the generalization of the Ising fully frustrated (IFF) model to a 2s-states Potts fully frustrated (PFF) model (Fig. 1.a). In the following we will review some results on the PSG and the PFF model.
Fig. 2 The slow relaxation in the PFF model with $q \equiv 2s = 4$ in 2D. a) The fitting parameter $\beta$ (upper panel) and $\tau$ (lower panel) for the second step of the correlation function of the Potts order parameter $M$ (extrapolated to the thermodynamic limit). The circles are the parameters for the Ogielski stretched exponential form in Eq. (7), the triangles for KWW stretched exponential form in Eq. (6). The arrow shows the estimate of $T_p(4, 1)$ in Eq. (10). The lower panel includes also data for the integral correlation time $\tau_{int} = \lim_{t_{max} \to \infty} \left[ \frac{1}{2} + \sum_{t=0}^{t_{max}} f_M(t) \right]$. Where not shown, the errors are smaller that the symbols size. b) The log-log plot of the autocorrelation time $\tau_0$, for the energy density $E$ (squares) and the Potts order parameter $M$ (circles) at the finite-size transition temperature $T_p(L)$ as function of the size $L = 20, 24, 30, 40, 50$. The autocorrelation time $\tau_0$ is defined as the time (in unit of Monte Carlo steps) at which $f_M(\tau_0, T_p(L)) = 0$ and $f_E(\tau_0, T_p(L)) = 0.3$. Here $T_p(L) \to T_p(4, 1)$ for $L \to \infty$. The positive slopes $z_M$ and $z_E$ show that these times diverge in the thermodynamic limit $L \to \infty$.

2.1 The Thermodynamic Transitions

Both the PFF and the PSG model have two thermodynamic transitions, one associated to the variables $\sigma_i$ and one to the spins $S_i$. At zero temperature (in 2D) there is a transition for the Ising spins $S_i$. The transition is in the IFF universality class for the PFF model, and in the Ising SG class for the PSG model. At finite temperature $T_p$ it has been shown, by means of Monte Carlo (MC) simulations and analytic approaches, that the Potts variables undergo a transition in the universality class of a s-state Potts ferromagnetic model (Fig. 1b).

It is possible to extend numerically the analytic expression of the ferromagnetic Potts transition temperature to the PSG and the PFF case, by introducing the fraction $X$ of elementary frustrated cell in the lattice, with the ferromagnetic Potts model corresponding to $X = 0$, the PSG model to $X = 0.5$ and the PFF model to $X = 1$. The generalized relation is

$$\frac{k_B T_p(q, X)}{a(X)qJ} = \frac{1}{\ln \left[ 1 + \sqrt{a(X)q} \right]}$$

where $q = 2s$ and the parameter $a(X)$ is reported in Fig. 1b. Note that in 3D the transition of the Ising spins is expected at finite $T < T_p$.

2.2 The Glassy Behavior and the connection with the thermodynamics

It has been shown that the PSG and the PFF models have a glassy behavior. For
example in the PFF with \( q = 2s = 2, 4 \) in 2D and 3D the autocorrelation function

\[
\rho_A(t, T) = \frac{\langle A(t, T)A(0, T) \rangle - \langle A(T) \rangle^2}{\langle A(0, T)^2 \rangle - \langle A(T) \rangle^2},
\]

for a global quantity \( A \) (like the total energy \( E \) or the Potts order parameters \( M \)) shows the ‘two-step’ behavior of glasses, with a non-exponential second step (upper panel in Fig.3.a).

These global correlation functions can be considered as measures of the structural relaxation process. Their relaxation times (\( \tau^E_0 \) and \( \tau^M_0 \) in Fig.3.b) diverge, in the thermodynamic limit, at the ordering Potts transition temperature \( T_p(q, 1) \) in Eq.(10). For \( T > T_p \) this diverging relaxation time resembles the \( \alpha \)-relaxation process occurring for \( T > T_g \) in fragile liquids.

In the PSG case the onset \( T^* \) of non-exponentiality coincides with the Griffiths temperature corresponding to the transition temperature \( T_c(q) \equiv T_p(q, 0) \) of the ferromagnetic \( q \)-states Potts model.\(^1\) This finding generalizes the results for the Ising SG\( ^2 \)\( ^3 \). In the Ising SG, indeed, the non-exponential behavior for \( T < T_c \) was related to the presence of unfrustrated regions, due to the randomness, with a size probability distribution that decreases exponentially. At \( T_c \) the ferromagnetic-like correlation length is equal to the characteristic size of each region, giving rise to a non-Gaussian distribution of relaxation times, that is responsible for the non-exponentiality of the global relaxation time.

Due to the lack of randomness, in the PFF model this explanation is not valid (the Griffiths temperature is not defined in this case). Indeed, for the PFF model, the onset of non-exponential relaxation corresponds to the Potts transition temperature, as has been shown numerically\( ^4 \) in 2D for \( q = 2, 4 \) (upper panel in Fig.3.a) and\( ^3 \) in 3D for \( q = 2 \). In these cases the previous argument can be extended considering that the ferromagnetic-like correlation length is now associated to the Potts variables, ordering at \( T_p \).

Summarizing, the results suggest that if the system is disordered \( (X \neq 1) \), then \( T^* \) corresponds to the Potts transition temperature for \( X = 0 \) (i.e. the Griffiths temperature \( T_c \)); if the system is fully frustrated \( (X = 1) \), \( T^* \) corresponds to the Potts transition temperature for \( X = 1 \) (there is no Griffiths temperature in this case). Note that the transition is vanishing for \( X \neq 1 \) (because the Griffiths temperature marks a transition that disappears for vanishing external field) and for the case \( X = 1 \) with \( q = 2 \) (in this case the transition is defined only as a percolation transition, because there are no orientational states). The transition is actually present for the cases with \( X = 1 \) and \( q > 2 \).

Hence the onset of the non-exponentiality is marked by a temperature related, at least in this model, to a thermodynamic transition. The transition, in this case, is due to the Potts (orientational) variables, but, in a more general case, could be associated to any global

\(^*\) For \( T < T_p \), these correlation times decrease with \( T \) (e.g. for \( M \) in lower panel in Fig.3.a), because they are proportional to \( \xi^z \), where the correlation length \( \xi \) is finite for \( T \neq T_p \) \( (z \) is the quantity-dependent dynamical exponent, approximated by the exponents in Fig.3.b). In real glasses, instead, the relaxation time is always increasing for decreasing \( T \). A relaxation time with a more appropriate behavior could be the one associated to the overlap between Potts configurations visited at different times. We will propose an ‘extension’ of the model, in the last section, that should give a monotonic slow relaxation time.

\(^1\) In disordered systems it is possible to show that for finite external field a free energy (Griffiths) singularity arises. In the limit of external field going to zero, the temperature at which this singularity occurs goes to the transition temperature \( T_c \) of the corresponding system with no disorder, and the singularity vanishes. The case with no disorder, corresponding to the \( q \)-states PSG model, is the \( X = 0 \) case (i.e. the ferromagnetic \( q \)-states Potts model) and is \( T_c \equiv T_{p(q, 0)} \).
The \( \beta \)-relaxation in the PSG model in 2D for \( q \equiv 2s = 4 \). a) The normalized correlation function \( f_\chi(t) \) of the nonlinear susceptibility: temperatures are in units of \( J/k_B \); times in units of MC steps. The data are fitted with the forms in Eqs. (5, 6, 7). The Ogielski form in Eq.(7) is the only able to fit the data on 4 decades. Where not shown the error bars are smaller then the symbols size. b) The fitting parameters \( \beta \) and \( \tau \) for the stretched exponential forms in Eqs.(6, 7). For the values of \( x \) see Ref. 6. The onset \( T^* \) of the stretched exponential corresponds to the Griffiths temperature \( T_c \) (marked with \( C \)) above the Potts transition temperature \( T_p \) (marked with \( P \)). The lower panel includes also data for the integral correlation time \( \tau_{\text{int}} = \lim_{t_{\text{max}} \to \infty} \frac{1}{N} \langle \sum_{t_{\text{max}}}^{t_{\text{max}}} f_\chi(t) \rangle \). The lines are only guides for the eyes.

(Structural) ordering. This kind of ordering could be, in principle, due to some internal degrees of freedom non easily detected in real experiments.

3. THE \( \beta \)-RELAXATION

Both PSG and PFF models in 2D show a non-exponential behavior also for the normalized correlation functions \( f_\chi(t) \) of the time-dependent nonlinear susceptibility

\[
\chi_{SG}(t) = \frac{1}{N} \left\langle \left[ \sum_{i=1}^{N} S_i(t + t_0) S_i(t_0) \right]^2 \right\rangle
\]  

(with \( N \) total number of spins, \( t_0 \) equilibration time, \( \chi_{SG}(0) = N \)) where the angular brackets stand for the thermal average and the bar stands for the average over the disorder in the PSG case, and is absent in the PFF case. Note that \( f_\chi(t) \) depends explicitly only on the Ising spins.

In the PSG case, the onset \( T^* \) of non-exponentiality for \( f_\chi \) corresponds to the Griffiths temperature \( T_c \) (Fig.3). In the PFF case it has been shown that \( T^* \) corresponds to the Potts transition temperature. These results seem to have the same interpretation as those for the correlation functions of the quantities depending explicitly on the Potts variables (previous section).

The difference in this case is that the Ising spins have no thermodynamic transition at \( T_p \). Therefore their correlation time \( \tau \), associated to \( f_\chi(t) \), does not diverge at \( T_p \) (lower panel in Fig.3b) and, hence, describes a fast process, respect to the slow dynamics of the Potts variables. This fast \( \tau \) can be considered as a measure of the time needed by the Ising
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Fig. 4 The β-relaxation in the PFF model in 2D with \( q = 4 \). a) The logarithm of the correlation time \( \tau \), associated to \( f_\chi \), for two finite systems with linear sizes \( L = 40, 120 \), plotted against the inverse temperature (\( T \) is in units of \( J/k_B \)). The finite size effect is evident only near the Potts transition temperature \( T_p \) (marked by a vertical line), where the Potts correlation length reach the system size. An Arrhenius behavior in this plot is represented by a straight line. b) The data for \( T > T_p \) plotted against the logarithm of \( T - T_p \) to verify the power law predicted by MCT. The power law is asymptotically satisfied and the slope is an approximate estimate of the power exponent. In both panels the lines are only guides for the eyes.

Spins to minimize the energy locally, on the diluted lattice given by the clusters of ordered Potts variables. This fast (local) relaxation corresponds to the β-process.

The intriguing result in this case is that this correlation time \( \tau \) has a non-Arrhenius behavior for \( T > T_p \) and an Arrhenius behavior for \( T < T_p \) (Fig.3.b and Fig.4). This result resembles the not well understood \( \alpha - \beta \) bifurcation, presented in the introduction, that is seen at a characteristic temperature \( T_{\alpha-\beta} \) in some fragile liquids and intermediate liquids.

Moreover the crossover, from non-Arrhenius to Arrhenius, occurs at \( T_p \) in both models (with or without disorder). Therefore \( T_{\alpha-\beta} \) is separated by the onset of non-exponential relaxation function \( T^* \) – marked by \( T_c \) if there is disorder, and by \( T_p \) if there is no disorder (previous section).

In the experiments the temperature \( T_{\alpha-\beta} \) seems to coincide with the \( T_{MCT} \) of the Mode Coupling Theory, at which the \( \alpha \)-relaxation time diverges as a power law, Eq.(3). Here the bifurcation coincides numerically with \( T_{p*} \), at which the global relaxation time diverges. Indeed, the correlation time \( \tau \) can be fitted with the VTF law, Eq.(2), with \( T_0 = T_p \), or can be (asymptotically) well described by a power law diverging at \( T_p \) (e.g. Fig.4.b).

These results suggest that the bifurcation here is related to the thermodynamic transition of the Potts variables. The fast β-process is driven by the slow dynamics of the global process for \( T \) approaching \( T_p \) from above. The non-Arrhenius behavior of the global process induces a non-Arrhenius behavior in the coupled fast variable and the dominant dynamic process is the one related to the slow variable. As has been shown for supercooled liquids, it is reasonable that in this regime the dominant dynamic process is the relaxation to the lower accessible energy, at the given \( T \), following the instability directions, in the energy landscape, of the visited states, i.e. the activated processes play no role.

For the supercooled liquid case it has been shown that \( T_{MCT} \) corresponds to the temperature below which the average number of instability directions for the visited configuration
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goes to zero. From preliminary results, the same situation appears to be valid here at \( T_p \).

Finally, below \( T_p \) the system is exploring one of the basins of attraction of the Potts variables \( \sigma_i \) and the dynamics of the fast Ising spins is no longer coupled to that of the \( \sigma_i \). The dynamics of the fast variables become activated (Arrhenius) consistently with the lack of instability directions. Indeed, in this case the energy cannot be lowered by following an instability direction, but only by doing an activated process. This is analogous of what has been found for the global relaxation in supercooled liquids\(^\text{11}\), but in our case the focus is on the local \( \beta \)-processes.

Therefore the results here suggest that the \( \alpha - \beta \) bifurcation can be related to the thermodynamics of the systems, and that the Potts transition temperature \( T_p \) of these systems plays the role of the \( T_{MCT} \) of supercooled liquids.

4. CONCLUSION AND PERSPECTIVES

The glassy spin models considered here, the PSG and the PFF model, are characterized by the presence of two coupled variables, whose dynamics decouple when the relaxation time of the slow variable diverges. This happens at the temperature \( T_p \) where the slow variable has a thermodynamic phase transition.

The picture that can be derived is the following. Due to the symmetry breaking of the slow variable for \( T \leq T_p \), the system is attracted in one of the basin of the energy landscape of the slow variable. Inside this basin, the dynamics of the system is mainly due to the fast variable and the dominant dynamical mechanism is the activated process, as consequence of the lack of instability directions for the visited states.

Macroscopically, this is shown by the \( \alpha-\beta \) bifurcation that can be seen in some fragile and intermediate glassforming liquids\(^\text{9,10}\), i.e. by the crossover of the \( \beta \)-relaxation time from a non-Arrhenius behavior to an Arrhenius law at a temperature \( T_{\alpha-\beta} \).

The experiments suggest that \( T_{\alpha-\beta} \approx T_{MCT} \), consistently with the analysis reported here. Indeed, in our models the bifurcation occurs at the temperature which plays the role of \( T_{MCT} \), i.e. \( T_p \), where the slow process has a diverging relaxation time.

Therefore the \( \alpha-\beta \) bifurcation can been related to the thermodynamics, as well as the onset \( T^* \) of non-exponential relaxation. What we can learn from the ‘toy model’ studied here is that, in cases in which there is disorder, \( T^* \) and \( T_{\alpha-\beta} \) do not coincide. The first, indeed, corresponds to the Griffiths temperature \( T_c \) – associated to a real transition only for a non-zero external field coupled to the slow variable – and the second to the temperature \( T_p \), where the slow relaxation time diverges. The two temperatures coincides only in the particular case of a frustrated system with no disorder.

An open questions is the effect of the \( T \)-dependence of the global correlation length \( \xi \). An interesting case is the one in which \( \xi \) does not decrease below the thermodynamic transition temperature. A model with such characteristic is the \( XY \) model, undergoing the Kosterlitz-Thouless-Berezinskii\(^\text{20}\) transition. The resulting Hamiltonian will be

\[
H\{S_i, \phi_i, \epsilon_{i,j}\} = -J \sum_{\langle i,j \rangle} \cos(\phi_i - \phi_j - A_{i,j})(\epsilon_{i,j} S_i S_j + 1)
\]

where \( A_{i,j} \) are constants depending on the gauge (or the external field) and \( \phi_i \in [0, 2\pi] \) are a more realistic representation of the continuous orientational variables.
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