A note on degree distribution in plane-oriented recursive trees

Panpan Zhang

Abstract. In this note, we investigate the degree profile of nodes in plane-oriented recursive trees. More precisely, we determine the probability mass function of the degree of a node with a fixed label. We also look into the moments of the degree random variable, and compute the exact expectation and variance. Phase transitions of the asymptotic expectation and variance are discovered and discussed briefly.

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1. Introduction

In graph theory, a (mathematical) tree refers to a connected graph which has no cycles; see [2, page 24] for a complete definition. A random recursive tree (non-planar rooted tree) with \( n \) given vertices (nodes) is a labeled tree with label set such that there exists an increasing unique path (label from any path from the root to leaf) from the root (the node labeled with 1) to the node labeled with \( k \) for all \( 2 \leq k \leq n \). This class of uniform recursive trees appears in the literature of the late 1960s, and has found applications in several areas, such as the spread of epidemics [14], family trees [15], and the pyramid scheme [3].

In this note, we consider a class of nonuniform random recursive trees—plane-oriented recursive trees (PORTs). A plane-ordered recursive tree is a tree in which descendants of each vertex are ordered. At time \( n \geq 1 \), we denote the structure of a PORT \( T_n \), i.e., a PORT consisting of \( n \) nodes. The tree \( T_n \) is obtained by starting with a single node (i.e., root). At each timestamp \( n \geq 2 \), a node labeled with \( n \) joins into the tree; the probability of the newcomer (the node labeled with \( n \)) adjacent to the node labeled with \( i \), \( 1 \leq i \leq n - 1 \), of \( T_{n-1} \) is proportional to the degree of the recruiter (i.e., the node labeled with \( i \)). We refer the readers to [16] for a more detailed demonstration of the dynamic growth of PORTs. The key feature of PORTs is that a parent node with higher degrees is more attractive to the newcomers, which coincides with a manifestation of the economic principles “the rich get richer” and “success breeds success.”

1Department of Statistics, University of Connecticut, Storrs, CT 06269, U.S.A.
2In graph theory, the degree of a node is the number of edges incident with the node.
Some pioneering research on PORTs traced back to the late 1980s and early 1990s. The exact and asymptotic moments of two degree profile random variables, the number of nodes of a fixed degree and the degree of a fixed node, are investigated in [16]. The distribution of the depth of nodes is determined in [11]. The exact and asymptotic distribution of leaves (terminal vertices) in PORTs and subtrees (branches) are studied by [12]. The asymptotic average of internal path length is characterized by [4]. More recently, PORTs caught researchers’ attention due to a notable stimulating paper [1], in which a more general network model—preferential attachment networks—was introduced. PORTs are a special class of preferential attachment networks, of which the network index equals one. The joint asymptotic distribution of nodes of different outdegree\(^3\) in PORTs is shown to be normal in [10]; a similar result is given in [5]. One of the most significant research papers on PORTs is [9], in which several limit results, such as the limit distribution of the normalized degree and the asymptotic approximation to the expected width, are derived.

Particularly in this note, we investigate the exact distribution of the degree of nodes in PORTs. Knowing the degree of nodes is helpful to depict the structure of trees. The first two moments of the degree random variable that we are going to look into have been calculated by [16]. However, the exact probability mass function is lacking. We explicitly derive the probability mass function of the degree profile random variable of interest, and prove our results via an elementary mathematical approach—two-dimensional mathematical induction.

The rest of this note is organized as follows: Some definitions and mathematical notations that will be used throughout this note are introduced in Section 2. In Section 3 we derive the probability mass function of the degree profile random variable of interest. In Section 4, we compute the exact and asymptotic first two moments of the degree variable, and realize phase transitions of the asymptotic expectation, as well as the asymptotic variance. Some concluding remarks are given at the end of the note.

2. Notation

The random variable of prime interest is the degree of the node with a fixed label, say \(j\), for \(1 \leq j \leq n\), in \(T_n\), denoted by \(D_{n,j}\). Most of our results are given in terms of gamma functions, \(\Gamma(\cdot)\); see a classic text [6, page 47] for its definition and fundamental properties. For a nonnegative integer \(z\), the double factorial of \(z\) is \(z!! = \prod_{i=0}^{\lfloor z/2 \rfloor - 1} (z - 2i)\), with the interpretation of \(0!! = 1\). We also introduce the Kronecker delta function with two variables \(s\) and \(t\), denoted by \(\delta_{s,t}\). The delta function \(\delta_{s,t}\) equals 1 for \(s = t\); 0 otherwise. The little \(o\) notation defines a relation between two real-valued functions \(f(x)\) and \(g(x)\). We have \(f(x) = o(g(x))\) equivalent to \(\lim_{x \to \infty} \frac{f(x)}{g(x)} = 0\) provided that \(g(x) \neq 0\). Generalized hypergeometric

\(^3\)The outdegree of a node is the number of edges emanating out of the node.
functions are defined in terms of Pochhammer symbols of rising factorials; that is,
\[ pF_q(a_1, \ldots, a_p; b_1, \ldots, b_q; z) = \sum_{s=0}^{\infty} \frac{\langle a_1 \rangle_s \cdots \langle a_p \rangle_s z^s}{\langle b_1 \rangle_s \cdots \langle b_q \rangle_s s!}. \]

3. Degree Distribution

In this section, we determine the probability mass function of \( D_{n,j} \), for \( 1 \leq j \leq n \). We separate the case of \( j \geq 2 \) and \( j = 1 \) for clarity. When \( j = 1 \), the random variable \( D_{n,j} = D_{n,1} \) refers to the degree of the root of \( T_n \). The root is the originator of the tree, so it has no parent. The root is the only node in the tree that has indegree \( 0 \).

**Theorem 1.** For a fixed \( 2 \leq j \leq n \), we have
\[
P(D_{n,j} = d) = \frac{\Gamma(d) \Gamma\left(j - \frac{1}{2}\right) \sum_{i=0}^{d-1} \frac{(-1)^i \Gamma(n-1-\frac{i}{2})}{\Gamma(i+1)\Gamma(d-i)\Gamma(j-1-\frac{i}{2})}}{\Gamma\left(n - \frac{1}{2}\right)}, \tag{1}
\]
for \( d = 1, 2, \ldots, n - j + 1 \).

**Proof.** We prove Theorem 1 by a two-dimensional induction on \( n \geq j \) and \( d \geq 1 \). The proof progresses in the style of filling an infinite lower triangular table, in which the rows are indexed by \( n \) and the columns are indexed by \( d \). A graphic interpretation of the method can be found in [19, page 69]. We initialize the first column and the diagonal of the table to be the basis of the induction. The event of \( \{D_{n,j} = 1\} \) for all \( n \geq j \) is that the node labeled with \( j \) is never chosen as a parent of any newcomer from its first appearance in the tree till time \( n \). Thus, we have
\[
P(D_{n,j} = 1) = \frac{2j - 2}{2j - 1} \times \frac{2j}{2j + 1} \times \cdots \times \frac{2n - 4}{2n - 3} = \frac{\Gamma(n-1)\Gamma\left(j - \frac{1}{2}\right)}{\Gamma\left(n - \frac{1}{2}\right)\Gamma(j - 1)}.
\]

On the other hand, the event of \( \{D_{n,j} = n - j + 1\} \) for all \( n \geq j \) is that the node labeled with \( j \) is never chosen as a parent for all newcomers from time \( j + 1 \) till \( n \). It follows that
\[
P(D_{n,j} = n - j + 1) = \frac{1}{2j - 1} \times \frac{2}{2j + 1} \times \cdots \times \frac{n - j}{2n - 3} = \frac{\Gamma(n-j+1)\Gamma\left(j - \frac{1}{2}\right)}{2^{n-j}\Gamma\left(n - \frac{1}{2}\right)}.
\]

Then, we assume that Equation (1) holds for all \( d \) up to row \( (n-1) \) in the table. Noticing that the degree of the node labeled with \( j \) increases at most by one at each timestamp, we have
\[
P(D_{n,j} = d) = \frac{d - 1}{2n - 3} P(D_{n-1,j} = d - 1) + \frac{2n - 3 - d}{2n - 3} P(D_{n-1,j} = d).
\]

\[\text{The indegree of a node is the number of edges heading into the node}\]
\[
\begin{align*}
&= \frac{d - 1}{2n - 3} \frac{\Gamma(d - 1) \Gamma \left( j - \frac{1}{2} \right)}{\Gamma \left( n - \frac{3}{2} \right)} \sum_{i=0}^{d-2} \frac{(-1)^i \Gamma(n - 2 - \frac{i}{2})}{\Gamma(i+1) \Gamma(d-1-i) \Gamma(j-1-\frac{i}{2})} \\
&\quad + \frac{2n - 3 - d}{2n - 3} \frac{\Gamma(d) \Gamma \left( j - \frac{1}{2} \right)}{\Gamma \left( n - \frac{3}{2} \right)} \sum_{i=0}^{d-1} \frac{(-1)^i \Gamma(n - 2 - \frac{i}{2})}{\Gamma(i+1) \Gamma(d-i) \Gamma(j-1-\frac{i}{2})} \\
&= \frac{\Gamma(d) \Gamma \left( j - \frac{1}{2} \right)}{\Gamma \left( n - \frac{1}{2} \right)} \left[ \frac{1}{2} \sum_{i=0}^{d-2} \frac{(-1)^i \Gamma(n - 2 - \frac{i}{2})}{\Gamma(i+1) \Gamma(d-1-i) \Gamma(j-1-\frac{i}{2})} \right. \\
&\quad \left. + \frac{(n - d - 3)}{2} \sum_{i=0}^{d-1} \frac{(-1)^i \Gamma(n - 2 - \frac{i}{2})}{\Gamma(i+1) \Gamma(d-i) \Gamma(j-1-\frac{i}{2})} \right] \\
&= \frac{\Gamma(d) \Gamma \left( j - \frac{1}{2} \right)}{\Gamma \left( n - \frac{1}{2} \right)} \left[ \sum_{i=0}^{d-2} \left( n - 2 - \frac{i}{2} \right) \frac{(-1)^i \Gamma(n - 2 - \frac{i}{2})}{\Gamma(i+1) \Gamma(d-1-i) \Gamma(j-1-\frac{i}{2})} \right. \\
&\quad \left. + \frac{(n - d - 3)}{2} \sum_{i=0}^{d-1} \frac{(-1)^d \Gamma(n - \frac{d - 3}{2})}{\Gamma(d) \Gamma(j - \frac{d - 3}{2})} \right].
\end{align*}
\]

This is equivalent to Equation (1) stated in the theorem. \(\square\)

The probability mass function of \(D_{n,j}\) is given by the summation of an alternating sequence. We implement the strategy of dividing the total sum into two parts: a partial sums of odd indices and a partial sum of even indices. We respectively evaluate the two partial sums to obtain an alternative interpretation of the probability mass function of \(D_{n,j}\). The result is given in terms of generalized hypergeometric functions, presented in the next corollary.

**Corollary 1.** For a fixed \(2 \leq j \leq n\), we have

\[
\Pr(D_{n,j} = d) = \frac{\Gamma(d) \Gamma \left( j - \frac{1}{2} \right)}{\Gamma \left( n - \frac{1}{2} \right)} \left( \frac{\Gamma(n - 1)_{3}F_{2} \left( \frac{2-d}{2}, \frac{1-d}{2}, 2-j; \frac{1}{2}, 2-n; 1 \right)}{\Gamma(d) \Gamma(j - 1)} \right. \\
- \left. \frac{\Gamma(n - \frac{3}{2})_{3}F_{2} \left( \frac{3-d}{2}, \frac{2-d}{2}, \frac{5}{2} - j; \frac{3}{2}, \frac{5}{2} - n; 1 \right)}{\Gamma(d - 1) \Gamma(j - \frac{3}{2})} \right).
\]
The first generalized hypergeometric function in the result stated in Corollary 1 can be further simplified for small choices of $j$. We present the probability mass functions of $D_{n,j}$ for $j = 2, 3$ as follows:

$$
P(D_{n,2} = d) = \frac{1}{(2n - 3)\Gamma \left( \frac{n}{2} \right)} \left[ \sqrt{\pi} \left( n - \frac{3}{2} \right) \Gamma(n - 1) 
- (d - 1)\Gamma \left( n - \frac{3}{2} \right) \right]
\frac{3F2}{\frac{3}{2} \frac{2 - d}{2} \frac{1}{2} \frac{3}{2} \frac{5}{2} - n; 1};
$$

$$
P(D_{n,3} = d) = \frac{3}{(2n - 3)\Gamma \left( \frac{n}{2} \right)} \left[ \sqrt{\pi} \left( n - \frac{3}{2} \right) \right]
\frac{d^2 - 3d + 2n - 2}{4} \Gamma(n - 2)
- (d - 1)\Gamma \left( n - \frac{3}{2} \right) \left[ \frac{3F2}{\frac{3}{2} \frac{2 - d}{2} \frac{1}{2} \frac{3}{2} \frac{5}{2} - n; 1} \right].
$$

Simplifications for the probability mass function of $D_{n,j}$ for higher values of $j$ are also available, done in a similar manner.

At the end of this section, we look at the degree distribution of the root of a PORT. For $j = 1$, the probability mass function of $D_{n,j}$ (i.e., $D_{n,1}$) cannot be directly derived from Equation (1). Noticing that the only structural difference between root and other nodes is that the root has indegree 0 (versus each of the other nodes has indegree 1), we remedy the problem by substituting $d$ in Equation (1) by $d + 1$, and then letting $j = 1$. Under such setting, we find that the probability mass function of $D_{n,1}$ can be substantially simplified to the following neat and closed form.

**Theorem 2.** The probability mass function of the root of a PORT is

$$
P(D_{n,1} = d) = \frac{d(2n - d - 3)!}{2^{n-d-1}(n - d - 1)!(2n - 3)!!}, \tag{2}
$$

for $d = 1, 2, \ldots, n - 1$.

**Proof.** Recall Equation (1), and set $j = 1$. Replacing $d$ with $d + 1$ in the equation, we have

$$
P(D_{n,1} = d) = \frac{\Gamma(d + 1)\Gamma \left( \frac{1}{2} \right) \sum_{i=0}^{d} \frac{(-1)^i\Gamma(n - \frac{1}{2})}{\Gamma(i + 1)\Gamma(d + 1 - i)\Gamma(j - \frac{1}{2})}}{\Gamma \left( n - \frac{1}{2} \right)}.
$$

Reimplementing the strategy of writing the total sum into partial sums with respect to odd indicies and even indicies, we apply the *Euler’s reflection formula* to gamma functions, and obtain

$$
P(D_{n,1} = d) = \frac{\Gamma(d + 1)\Gamma \left( \frac{1}{2} \right) \sum_{i \text{ is even}}^{d} \frac{\Gamma(n - 1 - \frac{1}{2})}{\Gamma(i + 1)\Gamma(d + 1 - i)\Gamma(j - 1 - \frac{1}{2})}}{\Gamma \left( n - \frac{1}{2} \right)}.
$$
\[-\sum_{\substack{i \text{ is odd} \\ 0 \leq i \leq d}} \frac{\Gamma\left(n - 1 - \frac{i}{2}\right)}{\Gamma(i + 1)\Gamma(d + 1 - i)\Gamma\left(j - 1 - \frac{i}{2}\right)} \]
\[= \frac{\Gamma(d + 1)\Gamma\left(\frac{j}{2}\right)}{\Gamma\left(n - \frac{1}{2}\right)} \frac{2^{d+1}\Gamma(d + 1 - n)}{4^n\Gamma(d)\Gamma(d + 3 - 2n)\cos(n\pi)} \]
\[= \frac{d(2n - d - 3)!}{2^{2n-d-1}(n - d - 1)!\Gamma(2n - d - 3)!!} \frac{(2n - 3)!!}{d(2n - d - 3)!!} \]
\[= \frac{d(2n - d - 3)!(2n - d - 3)!!}{2^{2n-d-1}(n - d - 1)!(2n - 3)!!} \]

We would like to point out that the probability mass function of $D_{n,1}$ in Theorem 2 is not novel, and it has been determined in [18]. However, the proof provided in [18] requires massive computation for simplification. The proof of Theorem 2 given in this note is much more concise and succinct.

4. Moments

Neither the probability mass function of $D_{n,j}$ in Theorem 1 nor in Corollary 1 can be used directly to calculate the moments of $D_{n,j}$. In this section, we appeal to a two-color Pólya urn model for moment computations.

To begin with, we give a brief introduction of Pólya urn models. A two-color Pólya urn scheme is an urn containing balls of two different colors (say white and blue). At each point of discrete time we draw a ball from the urn at random, observe its color and put it back in the urn, then execute some ball additions according to predesignated rules: If the ball withdrawn is white, we add $a$ white balls and $b$ blue balls; otherwise the ball withdrawn is blue, in which case we add $c$ white balls and $d$ blue balls. The dynamics of the urn can thus be represented by the following replacement matrix
\[
\begin{pmatrix}
a & b \\
c & d
\end{pmatrix}
\]
in which the rows from top to bottom are indexed by white and blue, and the columns from left to right are also indexed by white and blue. We refer the interested readers to a classic textbook [13].

Let $W_n$ be the degree of the node labeled with $j$ at time $n$, and $B_n$ be the total degree of all the other nodes. At time $n + 1$, if the node labeled with $j$ is selected, $W_n$ increases by one, and $B_n$ also increases by one\footnote{This is contributed by the edge incident to the node labeled with $n + 1$}; if any node other than the one labeled with $j$ is sampled, $B_n$ increases by two. This dynamic can be represented by the following replacement matrix
\[
\begin{pmatrix}
1 & 1 \\
0 & 2
\end{pmatrix}
\]
As the replacement matrix above is triangular, the Pólya urn associated with such replacement matrix is known as triangular Pólya urn. Triangular urns are well studied, and the moments of white balls are explicitly characterized in [17, Theorem 3.1], which is exploited directly to get the following theorem.

**Theorem 3.** For a fixed $1 \leq j \leq n$ and $n \geq 2$, we have

$$
\mathbb{E}[D_{n,j}] = \frac{\Gamma(n)\Gamma\left(j - \frac{1}{2}\right)}{\Gamma\left(n - \frac{1}{2}\right)\Gamma(j)} - \delta_{j,1},
$$

$$
\mathbb{V}ar[D_{n,j}] = -\frac{\Gamma^2(n)\Gamma^2\left(j - \frac{1}{2}\right)}{\Gamma^2\left(n - \frac{1}{2}\right)\Gamma^2(j)} - \frac{\Gamma(n)\Gamma\left(j - \frac{1}{2}\right)}{\Gamma\left(n - \frac{1}{2}\right)\Gamma(j)} + \frac{4n - 2}{2j - 1}.
$$

We discover that when $n$ is large, both $\mathbb{E}[D_{n,j}]$ and $\mathbb{V}ar[D_{n,j}]$ experience phase transitions. To compute the asymptotic expectation and variance, we apply the Stirling’s approximation to the expectation and variance of $D_{n,j}$ in Theorem [3]. As $n \to \infty$, we have

$$
\mathbb{E}[D_{n,j}] \sim \frac{\Gamma \left(j - \frac{1}{2}\right)}{\Gamma(j)} n^{1/2}, \quad (3)
$$

$$
\mathbb{V}ar[D_{n,j}] \sim \left(\frac{4}{2j - 1} - \frac{\Gamma^2 \left(j - \frac{1}{2}\right)}{\Gamma^2(j)}\right) n - \frac{\Gamma \left(j - \frac{1}{2}\right)}{\Gamma(j)} n^{1/2}. \quad (4)
$$

We keep the second highest order term (i.e., the term that involves $n^{1/2}$) in the asymptotic variance of $D_{n,j}$ because it makes a contribution when $j$ grows in the linear phase (with respect to $n$). We reapply the Stirling’s approximation to Equations (3) and (4), and obtain the next corollary.

**Corollary 2.** As $n \to \infty$, we have

$$
\mathbb{E}[D_{n,j}] \sim \begin{cases} 
\frac{\Gamma \left(j - \frac{1}{2}\right)}{\Gamma(j)} n^{1/2}, & \text{for fixed } j, \\
\left(\frac{4}{2j - 1} - \frac{\Gamma^2 \left(j - \frac{1}{2}\right)}{\Gamma^2(j)}\right) n, & \text{for } j \to \infty,
\end{cases}
$$

and

$$
\mathbb{V}ar[D_{n,j}] \sim \begin{cases} 
\frac{2}{\sqrt{n}}, & \text{for } j \to \infty, j = o(n), \\
\frac{1}{\theta} - \frac{1}{\sqrt{\theta}} & \text{for } j/n = \theta, 0 < c < 1.
\end{cases}
$$

We would like to point out that an equivalent form of $\mathbb{E}[D_{n,j}]$ has been developed in [16, Theorem 7], in which $\mathbb{V}ar[D_{n,j}]$, however, is given in terms of a sum of binomial coefficients. Closed forms of both $\mathbb{E}[D_{n,j}]$ and $\mathbb{V}ar[D_{n,j}]$ are given explicitly in Theorem [3] herein.

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*We refer the readers to [3, (18.21)] for the Stirling’s formula.*
5. CONCLUDING REMARKS

In this section, we add some concluding remarks and propose some future work. In this note, we develop an elementary method to determine the degree distribution of a node with a fixed degree in PORTs. The probability mass function of $D_{n,j}$ is given in terms of a sum of alternating sequences, and an alternative is given by generalized hypergeometric functions. An interesting property to consider in our future work is the local limit property of $D_{n,j}$, i.e., the asymptotic behavior of $D_{n,j}$ as $n$ goes to infinity whereas $j$ is fixed. In combinatorial mathematics and complex analysis, a common method to approximate an asymptotic summation of alternating sequences is the Mellin transform, known as the Rice’s integral method discussed in [7].

We also compute the exact and asymptotic moments of $D_{n,j}$ in this note. Instead of determining moments directly from the distribution function, we exploit a versatile probabilistic model, Pólya urn models, in our derivation. This approach provides us a new perspective to study statistical properties of random tree models and structures, which may be of broad applicability.

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