THE EXCHANGE GRAPH AND VARIATIONS OF THE RATIO OF THE TWO SYMANZIK POLYNOMIALS

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Abstract. Correlation functions in quantum field theory are calculated using Feynman amplitudes, which are finite dimensional integrals associated to graphs. The integrand is the exponential of the ratio of the first and second Symanzik polynomials associated to the Feynman graph, which are described in terms of the spanning trees and spanning 2-forests of the graph, respectively.

In a previous paper with Bloch, Burgos and Fresán, we related this ratio to the asymptotic of the Archimedean height pairing between degree zero divisors on degenerating families of Riemann surfaces. Motivated by this, we consider in this paper the variation of the ratio of the two Symanzik polynomials under bounded perturbations of the geometry of the graph. This is a natural problem in connection with the theory of nilpotent and SL2 orbits in Hodge theory.

Our main result is the boundedness of variation of the ratio. For this we define the exchange graph of a given graph which encodes the exchange properties between spanning trees and spanning 2-forests in the graph. We provide a description of the connected components of this graph, and use this to prove our result on boundedness of the variations.

1. Introduction

Feynman amplitudes in quantum field theory are described as finite dimensional integrals associated to graphs. A Feynman graph \((G, p)\) consists of a finite graph \(G = (V, E)\), with vertex and edge sets \(V\) and \(E\), respectively, together with a collection of external momenta \(p = (p_v)_{v \in V}\), \(p_v \in \mathbb{R}^D\), such that \(\sum_{v \in V} p_v = 0\). Here \(\mathbb{R}^D\) is the space-time endowed with a Minkowski bilinear form.

One associates to a Feynman graph \((G, p)\) two polynomials in the variables \(Y = (Y_e)_{e \in E}\). Denote by \(ST\) the set of all the spanning trees of the graph \(G\). (Recall that a spanning tree of a connected graph is a maximal subgraph which does not contain any cycle. It has precisely \(|V| - 1\) edges.) The first Symanzik \(\psi_G\), which depends only on the graph \(G\), is given by the following sum over the spanning trees of \(G\):

\[
\psi_G(Y) := \sum_{T \in ST} \prod_{e \notin T} Y_e.
\]

A spanning 2-forest in a connected graph \(G\) is a maximal subgraph of \(G\) without any cycle and with precisely two connected components. Such a subgraph has precisely \(|V| - 2\) edges. Denote by \(SF_2\) the set of all the spanning 2-forests of \(G\). The second Symanzik polynomial \(\phi_G\), which depends on the external momenta as well, is defined by

\[
\phi_G(p, Y) := \sum_{F \in SF_2} q(F) \prod_{e \notin F} Y_e.
\]

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Here $F$ runs through the set of spanning 2-forests of $G$, and for $F_1$ and $F_2$ the two connected components of $F$, $q(F)$ is the real number $-\langle p_{F_1}, p_{F_2} \rangle$, where $p_{F_1}$ and $p_{F_2}$ denote the total momentum entering the two connected components $F_1$ and $F_2$ of $F$, i.e.,

$$p_{F_1} := \sum_{v \in V(F_1)} p_v \quad \text{ and } \quad p_{F_2} := \sum_{u \in V(F_2)} p_u.$$  

The Feynman amplitude associated to $(G, p)$ is a path integral on the space of metrics (i.e., edge lengths) on $G$ with the action given by $\bar{\phi}_G/\psi_G$. It is given by

$$I_G(p) = C \int_{[0,\infty]^E} \exp(-i \phi_G/\psi_G) \, d\pi_G,$$

for a constant $C$, and the volume form $d\pi_G = \psi_G^{-D/2} \prod_E dY_e$ on $\mathbb{R}_+^E$, c.f. [6, Equation (6-89)].

Motivated by the question of describing Feynman amplitudes as the infinite tension limit of bosonic string theory, in [1] we proved results describing the ratio of the two Symanzik polynomials; a natural problem arising from [1] is to consider the variation of $\phi_G/\psi_G$ obtained by perturbation of the geometry of the graph, in a sense that we describe below. In order the state the theorem, we need to recall the determinantal representation of the two Symanzik polynomials. We refer to [1] where the discussion below appears in more detail.

1.1. **Determinantal representation of the Symanzik polynomials.** Let $G = (V, E)$ be a finite connected graph on the set of vertices $V$ of size $n$ and with the set of edges $E = \{e_1, \ldots, e_m\}$ of size $m$. Denote by $h$ the genus of $G$, which is by definition the integer $h = m - n + 1$.

Let $R$ be a ring of coefficients (that we will later assume to be either $\mathbb{R}$ or $\mathbb{Z}$), and consider the free $R$-module $R^E \simeq R^m = \{ \sum_{i=1}^m a_i e_i \mid a_i \in R \}$ of rank $m$ generated by the elements of $E$. For any element $a \in R^E$, we denote by $a_i$ the coefficient of $e_i$ in $a$.

Any edge $e_i$ in $E$ gives a bilinear form of rank one $\langle \cdot, \cdot \rangle_i$ on $R^m$ by the formula

$$\langle a, b \rangle_i := a_i b_i.$$

Let $\underline{y} = \{y_i\}_{e_i \in E}$ be a collection of elements of $R$ indexed by $E$, and consider the symmetric bilinear form $\alpha = \langle \cdot, \cdot \rangle_{\underline{y}} := \sum_{e_i \in E} y_i \langle \cdot, \cdot \rangle_i$. In the standard basis $\{e_i\}$ of $R^E$, $\alpha$ is the diagonal matrix with $y_i$ in the $i$-th entry, for $i = 1, \ldots, m$. We denote by $Y := \text{diag}(y_1, \ldots, y_m)$ this diagonal matrix.

Let $H \subset R^E$ be a free $R$-submodule of rank $r$. The bilinear form $\alpha$ restricts to a bilinear form $\alpha|_H$ on $H$. Fixing a basis $B = \{\gamma_1, \ldots, \gamma_r\}$ of $H$ over $R$, and denoting by $M$ the $r \times m$ matrix with row vectors $\gamma_i$ written in the standard basis $\{e_i\}$ of $R^E$, the restriction $\alpha|_H$ can be identified with the symmetric $r \times r$ matrix $MYM^T$ so that for two vectors $c, d \in R^r \simeq H$ with $a = \sum_{j=1}^r c_j \gamma_j$ and $b = \sum_{j=1}^r d_j \gamma_j$, we have

$$\alpha(a, b) = cYM^T d.$$  

The Symanzik polynomial $\psi(H, \underline{y})$ associated to the free $R$-submodule $H \hookrightarrow R^E$ is defined as

$$\psi(H, \underline{y}) := \det(MYM^T).$$
Note that since the coordinates of $MYM^T$ are linear forms in $y_1, \ldots, y_m$, $\psi(H, y)$ is a homogeneous polynomial of degree $r$ in $y_i$s.

For a different choice of basis $B' = \{\gamma'_1, \ldots, \gamma'_r\}$ of $H$ over $R$, the matrix $M$ is replaced by $PM$ where $P$ is the $r \times r$ invertible matrix over $R$ transforming one basis into the other. So the matrix of $\alpha|_H$ in the new basis is given by $PTYT^T P^T$, and the determinant gets multiplied by an element of $R^{\times 2}$. It follows that $\psi(H, y)$ is well-defined up to an invertible element in $R^{\times 2}$. In particular, if $R = \mathbb{Z}$, the quantity $\psi(H, y)$ is independent of the choice of the basis and is therefore well-defined.

From now on, we fix an orientation on the edges of the graph. We have a boundary map $\partial : R^E \rightarrow R^V$, $e \mapsto \partial^+(e) - \partial^-(e)$, where $\partial^+$ and $\partial^-$ denote the head and the tail of $e$, respectively. The homology of $G$ is defined via the exact sequence

$$0 \rightarrow H_1(G, R) \rightarrow R^E \xrightarrow{\partial} R^V \rightarrow R \rightarrow 0.$$  

The homology group $H = H_1(G, R)$ is a submodule of $R^m$ free of rank $h$, the genus of the graph $G$, for any ring $R$. In particular, fixing a basis $B$ of $H_1(G, \mathbb{Z})$, the polynomial $\psi_G(y) := \psi(H, y)$ is independent of the choice of $B$. Writing $M$ for the $h \times m$ matrix of the basis $B$ in the standard basis $\{e_i\}$ of $R^E$, one sees that $\psi_G(y) = \det(MYM^T)$.

It follows from the Kirchhoff’s matrix-tree theorem [7] that

$$\psi_G(Y) = \sum_{T \in ST} \prod_{e \notin T} Y_e,$$

which is the form of the first Symanzik polynomial given at the beginning of this section.

The exact sequence (1.1) yields an isomorphism

$$R^E/H \simeq R^{V,0},$$

where $R^{V,0}$ consists of those $x \in R^V$ whose coordinate sum to zero.

Let now $p \in R^{V,0}$ be a non-zero element, and let $\omega$ be any element in $\partial^{-1}(p)$. Denote by $H_\omega = \partial^{-1}(R.p) = H + R.\omega$, and note that $H_\omega$ is a free $R$-module of rank $h + 1$ which comes with the basis $B_\omega = B \cup \{\omega\}$.

The second Symanzik polynomial of $(G, p)$ is

$$\phi_G(p, y) = \psi(H_\omega, y)$$

for the element $\omega \in R^E$ with $p = \partial(\omega)$. The polynomial $\phi_G(p, y)$ is homogeneous of degree $h + 1$ in $y_i$’s, which is as noted in [1], independent of the choice of the element $\omega \in \partial^{-1}(p)$.

Writing $N$ for the $(h + 1) \times m$ matrix for the the basis $B_\omega$ in the standard basis of $R^E$, we see that

$$\phi_G(p, y) = \det(NYN^T).$$

The definition can be extended to $p \in \mathbb{R}^D$ using the Minkowski bilinear form on $\mathbb{R}^D$, as discussed in [1].
We have the following expression for the second Symanzik polynomial, c.f. e.g. to [3], or Section 3
\[
\phi_G(p, y) = \sum_{F \in SF_2} q(F) \prod_{e \in E(F)} y_e,
\]
which is the form of the second Symanzik polynomial given previously.

1.2. Statement of the main theorem. Let \( U \) be a topological space and \( y_1, \ldots, y_m : U \to \mathbb{R}_{>0} \) be \( m \) continuous functions. Let \( p \in (\mathbb{R}^V)^{0,0} \) be a fixed vector, and let \( \psi_G(y) : U \to \mathbb{R}_{>0} \) and \( \phi_G(p, y) : U \to \mathbb{R}_{>0} \) be the real-valued functions on \( U \) defined by the first and second Symanzik polynomials.

Notation. We will use the following terminology in what follows: for two real-valued functions \( F_1 \) and \( F_2 \) defined on \( U \), we write \( F_1 = O_F(F_2) \) if there exist constants \( c, C > 0 \) such that \( |F_1(s)| \leq c|F_2(s)| \) on all points \( s \in U \) which verify \( y_1(s), \ldots, y_m(s) \geq C \).

Let \( A : U \to \text{Mat}_{m \times m}(\mathbb{R}) \) be a matrix-valued map taking at \( s \in U \) the value \( A(s) \). Assume that \( A \) verifies the following two properties
(i) \( A \) is a bounded function, i.e., all the entries \( A_{i,j} \) of \( A \) take values in a bounded interval \([-C, C]\) of \( \mathbb{R} \), for some positive constant \( C > 0 \).
(ii) The two matrices \( M(Y + A)M^T \) and \( N(Y + A)N^T \) are invertible.

One might view the contribution of \( A \) as a perturbation of the standard scalar product on the edges of the graph given by the (length) functions \( y_1, \ldots, y_m \), which can be further regarded as changing the geometry of the graph, seen as a discrete metric space. The main result of this paper is the following.

Theorem 1.1. Assume \( A : U \to \text{Mat}_{m \times m}(\mathbb{R}) \) verifies the condition (i) and (ii) above. The difference \[
\frac{\det(N(Y + A)N^T)}{\det(M(Y + A)M^T)} - \frac{\det(NYN^T)}{\det(MYM^T)} \]
is \( O(1) \).

To prove this theorem, using Cauchy-Binet formula and some elementary observations, we are led to consider a graph which encodes the exchange properties between the spanning trees and 2-forests in the graph that we call the exchange graph \( G \), see Definition 2.3. As our first result, we give in Theorem 2.12 a classification of the connected components of this graph. This classification theorem combined with further combinatorial arguments are then used in Section 3 to prove Theorem 1.1.

We note that a similar result has been proved using different tools in a recent paper of Burgos, de Jong and Holmes [2] in the setting of what is called normlike functions. The perturbations in [2] are required to be symmetric for the method to work, though, strictly speaking, the result in [2] is more general and goes beyond the case of graphs. In comparison, the methods in this paper are purely combinatorial and the results on the exchange graph might be of independent interest.

We now explain an application of Theorem 1.1 from [1], c.f. Theorem 1.2 below, discussed in more detail in Section 4.

1.3. Boundedness of variation of the Archimedean height pairing. Let \( C_0 \) be a stable curve of genus \( g \) over \( \mathbb{C} \), and with dual graph \( G = (V, E) \) which has genus \( h = |E| - |V| + 1 \), \( h \leq g \).

Consider the versal analytic deformation \( \pi : \mathcal{C} \to S \) of \( C_0 \), where \( S \) is a polydisc of dimension \( 3g - 3 \). The total space \( \mathcal{C} \) is regular and we let \( D_e \subset S \) denote the divisor parametrizing
those deformations in which the point associated to $e$ remains singular. The divisor $D = \bigcup_{e \in E} D_e$ is a normal crossings divisor whose complement $U = S \setminus D$ can be identified with $(\Delta^s)^E \times \Delta^{3g-3-|E|}$. Assume that two collections of sections of $\pi$ are given, which we denote by $\sigma_1 = (\sigma_{l,1})_{l=1,\ldots,n}$ and $\sigma_2 = (\sigma_{l,2})_{l=1,\ldots,n}$. Since $C$ is regular, the points $\sigma_{l,i}(0)$ lie on the smooth locus of $C_0$. Consider two fixed vectors $p_1 = (p_{l,1})_{l=1}^n$ and $p_2 = (p_{l,2})_{l=1}^n$ with $p_{l,i} \in \mathbb{R}^D$ which each satisfy the conservation of momentum. We obtain a pair of relative degree zero $\mathbb{R}^D$-valued divisors

$$\mathcal{A}_s = \sum_{l=1}^n p_{l,1}\sigma_{l,1}, \quad \mathcal{B}_s = \sum_{l=1}^n p_{l,2}\sigma_{l,2}.$$ 

Assume further that $\sigma_1$ and $\sigma_2$ are disjoint on each fiber of $\pi$. To any pair $\mathcal{A}, \mathcal{B}$ of degree zero (integer-valued) divisors with disjoint support on a smooth projective complex curve $C$, one associates a real number, the archimedean height defined on $U$ when $\omega$ by integrating a canonical logarithmic differential $C$ zero (integer-valued) divisors with disjoint support on a smooth projective complex curve $C$. Properties between spanning 2-forest and spanning trees in a graph $G$. We are particularly interested in the “exchange properties” between spanning 2-forest and spanning trees in a graph $G$. To make this precise,
we will define a new graph $\mathcal{H}$ that we call the exchange graph of $G$. First we need to define an equivalence relation on the set of spanning 2-forests of $G$.

**Definition 2.1.**
- For a spanning 2-forest $F$ of a graph $G$, we denote by $\mathcal{P}(F) = \{X, Y\}$ the partition $V = X \cup Y$ of the vertices into the vertex sets $X$ and $Y$ of the two connected components of $F$.
- For any partition $\mathcal{P}$ of $V$, we denote by $E(\mathcal{P})$ the set of all edges in $G$ which connect two vertices lying in two different elements of $\mathcal{P}$.
- Two 2-forests $F$ and $F'$ are called (vertex) equivalent, and we write $F \sim_v F'$, if $\mathcal{P}(F) = \mathcal{P}(F')$.

The following proposition is straightforward.

**Proposition 2.2.** The following statements are equivalent for $F,F' \in \mathcal{SF}_2$:
1. $F$ and $F'$ are not (vertex) equivalent.
2. there exists an edge $e \in F'$ such that $F \cup \{e\}$ is a tree.

**Notation.** In what follows, for a spanning subgraph $G'$ of $G = (V,E)$ and $e \in E \setminus E(G')$, we simply write $G' + e$ to denote the spanning subgraph of $G$ with the edge set $E(G') \cup \{e\}$. For an edge $e \in E(G')$, we write $G' - e$ for the spanning subgraph of $G$ with the edge set $E(G') \setminus \{e\}$.

**Definition 2.3.** The exchange graph $\mathcal{H} = \mathcal{H}_G = (\mathcal{V}, \mathcal{E})$ of $G$ is defined as follows. The vertex set $\mathcal{V}$ of $\mathcal{H}$ is the disjoint union of two sets $\mathcal{V}_1$ and $\mathcal{V}_2$, where

$$\mathcal{V}_1 := \{ (F,T) \mid F \in \mathcal{SF}_2(G), T \in \mathcal{ST}(G), E(F) \cap E(T) = \emptyset \},$$

and

$$\mathcal{V}_2 := \{ (T,F) \mid T \in \mathcal{ST}(G), F \in \mathcal{SF}_2(G), E(F) \cap E(T) = \emptyset \}.$$ 

There is an edge in $\mathcal{E}$ connecting $(F,T) \in \mathcal{V}_1$ to $(T',F') \in \mathcal{V}_2$ if there is an edge $e \in E(T)$ such that $F' = T - e$ and $T' = F + e$.

**Definition 2.4.** If $(T,F)$ and $(F',T')$ are adjacent in $\mathcal{H}$ and $F' = T - e$, we say $(F',T')$ is obtained from $(T,F)$ by pivoting involving the edge $e$.

Our aim in this section is to describe the connected components of $\mathcal{H}$.

First note that there is no isolated vertex in $\mathcal{H}$: consider a spanning tree $T$ and a spanning 2-forest $F$ of $G$ with disjoint sets of edges. Let $\mathcal{P}(F) = \{X,Y\}$, be the vertex sets of the two connected components of $F$. By connectivity of $T$, there is an edge $e$ of $T$ which joins a vertex of $X$ to a vertex of $Y$. It follows that $T' = F + e$ and $F' = T - e$ are spanning tree and 2-forest in $G$, respectively, and $(F,T) \in \mathcal{V}_1$ is connected to $(T',F') \in \mathcal{V}_2$.

Let now $\mathcal{H}_0 = (\mathcal{V}_0, \mathcal{E}_0)$ be a connected component of $\mathcal{H}$. Write $\mathcal{V}_0 = \mathcal{V}_{0,1} \cup \mathcal{V}_{0,2}$ with $\mathcal{V}_{0,i} \subset \mathcal{V}_i$, for $i = 1,2$. Note that both $\mathcal{V}_{0,1}$ and $\mathcal{V}_{0,1}$ are non-empty. Let $(F,T) \in \mathcal{V}_{0,i}$. Let $G_0 = (V,E_0)$ be the spanning subgraph of $G$ having the edge set $E_0 = E(T) \cup E(F)$. By definition of the edges in $\mathcal{H}$, and connectivity of $\mathcal{H}_0$, we have for all $(A,B) \in \mathcal{V}_0$, $E(A) \cup E(B) = E(G_0)$. We refer to $G_0$ as the spanning subgraph of $G$ associated to the connected component $\mathcal{H}_0$ of $\mathcal{H}$.

**Notation.** For a subset $X \subseteq V$ of the vertices of a (multi)graph $G = (V,E)$, we denote by $G[X]$ the induced graph on $X$: it has vertex set $X$ and edge set all the edge of $E$ with both end-points lying both in $X$.
**Definition 2.5.** Let $X \subset V$ be a subset of vertices of $G_0$. We say $X$ is saturated with respect to $G_0$ if the induced subgraph $G_0[X]$ has precisely $2|X| - 2$ edges. A saturated component $X$ of $G_0$ is a maximal subset of $G$ for inclusion which is saturated with respect to $G_0$.

Let $\mathcal{H}_0$ be a connected component of $\mathcal{H}$ with associated spanning subgraph $G_0$.

**Lemma 2.6.** Let $X$ be a saturated subset of $G_0$. Then for all vertices $(A, B) \in \mathcal{Y}_0$, $X$ is connected in both $A$ and $B$, i.e., the induced graphs $A[X]$ and $B[X]$ are disjoint trees on the vertex set $X$.

**Proof.** Both $A[X]$ and $B[X]$ are free of cycles. Since $G_0[X]$ has precisely $2|X| - 2$ edges, and $A[X]$ and $B[X]$ are disjoint, both $A[X]$ and $B[X]$ are trees on vertex set $X$.  

For any edge $e$ of $A$ which lie in $X$, $B + e$ has a cycle. Similarly, for any edge $e$ of $B$ which lie in $X$, $A + e$ has a cycle. It follows that pivoting in $G_0$ do not involve any edge in $X$, and so, by connectivity of $\mathcal{H}_0$, for any pair $(A', B') \in \mathcal{Y}_0$, we have $A'[X] = A[X]$ and $B'[X] = B[X]$.  

**Lemma 2.7.** For two different saturated components $X$ and $X'$ of $G_0$, we have $X \cap X' = \emptyset$.

**Proof.** If $X \cap X' \neq \emptyset$, the set $X \cup X'$ is connected in $G_0$. By maximality of $X$ and $X'$, this implies, $X = X'$ which is not possible since $X \neq X'$.  

As a corollary, the saturated components $X_1, \ldots, X_r$ of $G_0$ form a partition of $V$. In addition, there exist for any $j = 1, \ldots, r$, two disjoint trees $T_{j,1}$ and $T_{j,2}$ with vertex set $S_j$ so that for any pair $(A, B) \in \mathcal{Y}_0$, we have $A[X_j] = T_{j,1}$ and $B[X_j] = T_{j,2}$.

We now give another characterization of the saturated components of $G_0$ in terms of the connected component $\mathcal{H}_0$ of $\mathcal{H}$.

Define two equivalence relations $\simeq_1$ and $\simeq_2$ on the set of vertices $V$ of $G_0$ as follows. For two vertices $u, v \in V$,

- we say $u \simeq_1 v$ if for any $(F, T) \in \mathcal{Y}_{0,1}$, both vertices $u$ and $v$ lie in the same connected component of $T \setminus E(\mathcal{P}(F))$.

Similarly,

- we say $u \simeq_2 v$ if for any $(T, F) \in \mathcal{Y}_{0,2}$, both vertices $u$ and $v$ lie in the same connected component of $T \setminus E(\mathcal{P}(F))$.

**Lemma 2.8.** Let $F$ be a spanning 2-forest in $G_0$. Let $T$ be a spanning tree of $G_0$. Suppose two vertices $u, v \in V$ are in two different connected components of $T \setminus E(\mathcal{P}(F))$. There exists an edge $e \in E(\mathcal{P}(F)) \cap E(T)$ such that $u$ and $v$ are not connected in $T - e$.

**Proof.** Denote by $S_u$ and $S_v$ the two connected components of $T \setminus E(\mathcal{P}(F))$ which contain $u$ and $v$, respectively. There is a path joining $S_u$ to $S_v$ in $T$. Since $S_u \neq S_v$, it contains an edge $e \in E(\mathcal{P}(F))$. For such an edge $e$, $u$ and $v$ are not connected in $T - e$.  

The previous lemma allows to prove the following claim.

**Claim 2.9.** The two equivalence relations $\simeq_1$ and $\simeq_2$ are the same.

**Proof.** Suppose for the sake of a contradiction that this is not the case. By symmetry, let $u, v \in V$ be two vertices with $u \simeq_1 v$ but $u \not\simeq_2 v$. This implies the existence of $(T, F) \in \mathcal{Y}_{0,2}$ such that $u, v$ belong to two different connected components of $T \setminus E(\mathcal{P}(F))$. Applying the previous lemma, we infer the existence of an edge $e \in E(T) \cap E(\mathcal{P}(F))$ such that $u$ and $v$
are not connected in $T - e$. Pivoting involving $e$ gives a pair $(F', T')$ such that $u$ and $v$ lie in two different connected components of $F'$. In particular, it follows that $u \not\sim_1 v$, which is a contradiction. This proves the claim. \hfill $\square$

We denote by $\sim$ the equivalence relation on vertices induced by $\sim_1$. We have actually proved the following

**Proposition 2.10.** W properties are equivalent for $u, v \in V$:

1. we have $u \not\sim v$.
2. there exists $(F, T) \in \mathcal{V}_{0,1}$ such that $u$ and $v$ lie in different connected components of $F$.
3. there exists $(T', F') \in \mathcal{V}_{0,2}$ such that $u, v$ lie in two different connected components of $F'$.

Denote by $\mathcal{P}_\sim$ the partition of $V$ induced by the equivalence classes of $\sim$.

**Proposition 2.11.** The partition $\mathcal{P}_\sim$ coincides with the partition of $V$ into saturated components of $G_0$.

_Proof._ Any two vertices in a saturated component of $G_0$ are clearly equivalent. Thus, in order to prove the proposition, it will be enough to show that each element in $\mathcal{P}_\sim$ saturated with respect to $G_0$. Let $X \subset V$ be an element of $\mathcal{P}_\sim$, and consider two vertices $a, b \in X$. Let $(T_0, F_0) \in \mathcal{V}_0$ be a vertex of $\mathcal{H}_0$, and let $P$ be the unique path in $T_0$ joining $a$ and $b$. We claim that $P$ is contained in $X$. To see this, first note that there is no edge $e \in E(\mathcal{P}(F))$ in the path $P$: otherwise, $a$ and $b$ would lie in two different connected components of the 2-forest $T_0 - e$, which is not possible by Proposition 2.10. By definition of the edges in $\mathcal{H}$, and by connectivity of $\mathcal{H}_0$, this shows that for any $(T, F) \in \mathcal{V}_0$, we have $P$ is included in $T$. By the definition of the equivalence relation $\sim$, we infer that $X$ contains the path $P$. This shows that $T_0[X]$ is connected. Similarly, the induced graph $F_0[X]$ is connected. Since $E(F_0) \cap E(T_0) = \emptyset$, we infer that $X$ is a saturated set with respect to $G_0$. \hfill $\square$

We can now state the main result of this section.

**Theorem 2.12.** Let $G$ be a multigraph.

1. The graph $\mathcal{H}$ is connected if and only if the following two conditions hold:
   
   (i) the edge set of $G$ can be partitioned as $E(G) = E(T) \sqcup E(F)$ for a spanning tree $T$ and a spanning 2-forest $F$ of $G$.  
   
   (ii) any non-empty subset $X$ of $V$ saturated with respect to $G_0$ consists of a single vertex.

2. More generally, there is a bijection between the connected components $\mathcal{H}_0$ of $\mathcal{H}$ and the pair $(G_0; \{T_{1,1}, T_{1,2}, \ldots, T_{r,1}, T_{r,2}\})$ where
   
   (i) $G_0$ is a spanning subgraph of $G$ which is a disjoint union of a spanning tree $T$ and a spanning forest $F$ of $G$.
   
   (ii) denoting the maximal subsets of $V$ saturated with respect to $G_0$ by $X_1, \ldots, X_r$, then $T_{j,1}$ and $T_{j,2}$ are two disjoint spanning trees on the vertex set $X_j$, and $E(G_0[X_j]) = E(T_{j,1}) \sqcup E(T_{j,2})$, for $j = 1, \ldots, r$.

Under this correspondence, the vertex set of $\mathcal{H}_0$ consists of all the vertices $(A, B) \in \mathcal{V}$ which verify $E(A) \cup E(B) = E(G_0)$, and for all $j = 1, \ldots, r$, $A[X_j] = T_{j,1}$ and $B[X_j] = T_{j,2}$.

Before giving the proof of this theorem, we make the following remark.
Remark 2.13. Let \( G \) be a graph whose edge set is a disjoint union of the edges of a spanning tree and a spanning 2-forest, and with the property that there is no saturated subset of size larger than two. The graph \( G \) might contain spanning trees \( T \) with the property that \( G \setminus E(T) \) is not a spanning 2-forest. In a sense, Theorem 1.2 concerns smaller number of spanning trees of \( G \), and the theorem does not seem to follow from the well-known connectivity of edge-exchanges for spanning trees.

\[
\begin{align*}
\text{Figure 1. Example of a graph } G, \text{ on the left, which is a disjoint union of a spanning tree and a spanning 2-forest, in which all saturated components are singletons. Note that } G \text{ contains a spanning tree } T, \text{ given on the right, with a complement which is not a spanning 2-forest.}
\end{align*}
\]

The rest of this section is devoted to the proof of this theorem.

To prove part (1) of the theorem, suppose \( H \) is connected (and so non-empty). Then (i) obviously holds. To prove (ii), let \( X_1, \ldots, X_r \) be all the different maximal subsets of vertices which are saturated with respect to \( V \), and assume for the sake of a contradiction, and without loss of generality that \( |X_1| > 1 \). Let \( T_{1,1}, T_{1,2} \) be the trees on vertex set \( X_j \) associated to \( H \). For any \((A, B) \in V\) let \((A', B')\) be defined by \( A' = A - E(T_{1,1}) + E(T_{1,2}) \) and \( B' = B - E(T_{1,2}) + E(T_{1,1}) \). Since pivoting only involves edges which are neither in \( T_{1,1} \) nor in \( T_{1,2} \), this shows that \((A', B')\) is not a vertex of \( V \). This is a contradiction, since \( A' \) and \( B' \) have the same number of edges as \( A \) and \( B \), respectively, both are without cycles, and \( E(G_0) = E(A') \cup E(B') \).

We now prove the other direction. Suppose both (i) and (ii) in (1) hold. Since any vertex \((F, T)\) in \( \mathcal{H}_1 \) is connected to a vertex of \( \mathcal{H}_2 \), it will be enough to prove that any two vertices \((T, F), (T', F') \in \mathcal{H}_2 \) are connected by a path in \( \mathcal{H} \).

We prove this proceeding by induction on the integer number

\[
r = \text{diff}(T, T') := |E(T) \setminus E(T')|.
\]

- If \( r = 0 \), then \( T = T' \), and the claim trivially holds.
- Assuming the assertion for \( r \), we prove it for \( r + 1 \). So let \( v = (T, F), v' = (T', F') \in \mathcal{H}_2 \) be two vertices with \( |E(T) \setminus E(T')| = r + 1 \). For the sake of a contradiction, assume that \( v \) and \( v' \) (Note that this is ) are not connected in \( \mathcal{H} \). Denote by \( \mathcal{H}_0 \) the connected component of \( \mathcal{H} \) which contains \( v \).

We claim

(1) There is no edge \( e \) in \( E(T) \setminus E(T') \) with \( F + e \in ST(G) \). Similarly, there is no edge \( e \) in \( E(T') \setminus E(T) \) with \( F + e \in ST(G) \).

Otherwise, suppose \( e \in E(T) \setminus E(T') \) is such that \( F + e \) is a spanning tree of \( G \). There exists \( e' \in E(T') \setminus E(T) \) such that \( T'' = T - e + e' \) is a spanning tree of \( G \). This follows from the exchange properties for the spanning trees of \( G \). (Spanning trees of \( G \) form the basis of the graphic matroid on the ground set \( E \).) The complement \( T'' \) in \( G \) is \( F'' := F + e - e \).
Since $F + e$ is a spanning tree of $G$, and $e' \in F$, the subgraph $F''$ is a spanning 2-forest of $G$, and thus $v'' = (T', F'') \in \mathcal{F}$. By definition, $(T', F)$ and $(F + e, T - e)$ are adjacent in $\mathcal{H}$. Moreover, $(F + e, T - e)$ and $v''$ are adjacent in $\mathcal{H}$. Since $\text{diff}(T', T'') = r$, by hypothesis of the induction, $(T'', F'')$ and $(T', F')$ are connected by a path in $\mathcal{H}$. Thus $(T, F)$ and $(T', F')$ are connected, which is a contradiction to the assumption we made. This proves our claim (I).

As a consequence of (I) we now show that

(II) We have $F \sim_v F'$, i.e., the two partitions $\mathcal{P}(F)$ and $\mathcal{P}(F')$ of $V$ coincide.

Let $\mathcal{P}(F) = \{X, Y\}$ and $\mathcal{P}(F') = \{X', Y'\}$, and suppose for the sake of contradiction that the two partitions are not equal. The partition $\mathcal{P}(F)$ (resp. $\mathcal{P}(F')$) induces a partition of both $X'$ and $Y'$ (resp. $X$ and $Y$). One of these four induced partitions has to be non-trivial: by this we mean that, without loss of generality, we can assume that $Z := X \cap X'$ and $W := X \cap Y'$ are both non-empty. Since $F[X]$ is connected, there is an edge $e = \{u, v\} \in F$ with $u \in Z$ and $v \in W$. This edge does not belong to $F'$ since it joins a vertex in $X'$ to a vertex in $Y'$. Therefore, $e \in T'$, and thus $e \in E(T') \setminus E(F)$. Moreover, $F' + e$ is a spanning tree, which is a contradiction to (I). This proves our claim (II).

Let $\mathcal{P}(F) = \mathcal{P}(F') = \{X, Y\}$. Denote by $\mathcal{P}_X$ the partition of $X$ given by the vertex sets of the connected components of $T[X]$. Also, denote by $\mathcal{P}_X'$ the partition of $X$ induced by the connected components of $T'[X]$. Similarly, define $\mathcal{P}_Y$ and $\mathcal{P}_Y'$. Let $E(\mathcal{P}_X)$ (resp. $E(\mathcal{P}_Y)$) be the set of all edges $e$ of $G$ with end-points in two different members of $\mathcal{P}_X$ (resp. $\mathcal{P}_Y$), respectively. Similarly, define $E(\mathcal{P}_X')$ and $E(\mathcal{P}_Y')$.

We now claim.

(III) The intersections $E(T') \cap E(\mathcal{P}_X)$, $E(T') \cap E(\mathcal{P}_Y)$, $E(T) \cap E(\mathcal{P}_X')$, $E(T) \cap E(\mathcal{P}_Y')$ are all empty.

Otherwise, without loss of generality, suppose there is an edge $e' \in T'$ with $e' \in E(\mathcal{P}_X)$. Since $e'$ joins two different connected components of $T[X]$, we have $e' \in F$. The graph $T + e'$ has a cycle, which, once again since $e'$ joins two different connected components of $T[X]$, must include an edge $e \in E(\mathcal{P}(F))$. Since $\mathcal{P}(F) = \mathcal{P}(F')$, we have $e \in E(T')$.

Let $v_1 = (F_1, T_1)$ with $F_1 = T - e$ and $T_1 = F + e$, and $v_2 = (F_2, T_2)$ with $T_2 = F_1 + e'$ and $F_2 = T_1 - e'$. By choices of $e$ and $e'$, both $v_1$ and $v_2$ are vertices in $\mathcal{H}$, and $v, v_1, v_2$ forms a path of length two. We have $\text{diff}(T_2, T') = \text{diff}(T, T') = r + 1$.

Since $F_2$ contains $e$, the edge $e$ lies entirely in one of the two connected components of $F_2$, and so $\mathcal{P}(F_2) \neq \mathcal{P}(F)$. Since by our assumption, $v$ and $v'$ are not connected in $\mathcal{H}$, $v''$ and $v'$ are not connected in $\mathcal{H}$. Applying the above reasoning to $v''$ and $v'$, we must have by (II) that $\mathcal{P}(F_2) = \mathcal{P}(F')$. Since $\mathcal{P}(F') = \mathcal{P}(F)$, this gives a contradiction. This proves our claim (III).

As an immediate corollary of (III), we get

(IV) We have $\mathcal{P}_X = \mathcal{P}_X'$ and $\mathcal{P}_Y = \mathcal{P}_Y'$.

Indeed, since $E(T') \cap E(\mathcal{P}_X) = \emptyset$, any subset $Z'$ of $X$ with $T'[Z']$ connected, should be entirely included in an element of $\mathcal{P}_X$. This in particular, when applied to each $Z' \in \mathcal{P}_X'$, shows the existence of $Z \in \mathcal{P}_X$ with $Z' \subseteq Z$, which shows that $\mathcal{P}_X'$ is a refinement of $\mathcal{P}_X$. By symmetry, we get that $\mathcal{P}_X$ is a refinement of $\mathcal{P}_X'$. Thus, $\mathcal{P}_X = \mathcal{P}_X'$. The equality $\mathcal{P}_Y = \mathcal{P}_Y'$ follows similarly.
As a corollary, we get

(V) The equality \( E(P_X) \cup E(P_Y) = E(P'_X) \cup E(P'_Y) \) holds

To see this, note that by (III), \( E(P_X) \cup E(P_Y) \) and \( E(P'_X) \cup E(P'_Y) \) are subsets of \( E(F) \cap E(F') \), we thus get the equality of the two sets from (IV).

By the definition of \( \mathcal{H} \), all the vertices \( v_2 = (T_2, F_2) \) of \( \mathcal{H} \) at distance 2 from \( (T, F) \) are of the form \( T_2 = T - e_1 + e_2 \) and \( F_2 = F - e_1 + e_2 \) for \( e_1 \in E(P(F)) \) and \( e_2 \in E(P_X) \cup E(P_Y) \).

Indeed, \( F + e_1 \in ST(G) \) implies \( e_1 \in E(P(F)) \). Similarly, \( T - e_1 + e_2 \in ST(G) \) implies \( e_2 \in E(P_X) \cup E(P_Y) \).

By (II), \( E(P(F)) = E(P'(F')) \), and by (V), \( E(P_X) \cup E(P_Y) = E(P'_X) \cup E(P'_Y) \). Thus, for such a vertex \( v_2 \), the pair \( v'_2 = (T'_2, F'_2) \) defined by \( T'_2 = T' - e_1 + e_2 \) and \( F'_2 = F' + e_1 - e_2 \) is also a vertex of \( \mathcal{H} \) at distance two from \( v' \). In addition, we have \( \text{diff}(T_2, T'_2) = \text{diff}(T, T') = r + 1 \).

Since by our assumption, \( v \) and \( v' \) are not connected in \( \mathcal{H} \), any pair of vertices \( v_2 \) and \( v'_2 \) as above are not connected in \( \mathcal{H} \).

Since \( T \neq T' \), there is an edge \( e_* \in E(T') \setminus E(T) \). For any choice of \( e_1, e_2 \) as above, we have \( e_* \neq e_1, e_2 \), and thus we must have \( e_* \in E(T'_2) \setminus E(T_2) \).

Applying the same reasoning to the pair \( v_2 \) and \( v'_2 \), and proceeding inductively on \( k \), we infer that for any vertex \( v_{2k} = (T_{2k}, F_{2k}) \) of \( \mathcal{H} \) obtained from \( (T, F) \) by an ordered sequence of pivoting involving edges \( e_1, e_2, \ldots, e_{2k-1}, e_{2k} \), the pair \( v'_{2k} = (T'_{2k}, F'_{2k}) \) obtained from \( v' \) by pivoting involving the same ordered sequence of edges \( e_1, e_2, \ldots, e_{2k-1}, e_{2k} \) is a vertex of \( \mathcal{H} \), and we have by (I)-(V):

- \( P(F_{2k}) = P(F'_{2k}) = \{X_{2k}, Y_{2k}\} \) (with \( X_{2k} \) and \( Y_{2k} \) depending on the sequence of edges \( e_1, \ldots, e_{2k} \)),
- \( E(P_{X_{2k}}) \cup E(P_{Y_{2k}}) = E(P'_{X_{2k}}) \cup E(P'_{Y_{2k}}) \),
- \( \text{diff}(v_{2k}, v'_{2k}) = r + 1 \), and \( v_{2k} \) and \( v'_{2k} \) are not connected in \( \mathcal{H} \).
- \( e_* \in E(T'_{2k}) \setminus E(T_{2k}) \)

To get a contradiction, note that all the vertices of \( \mathcal{H} \) appear among the set of vertices \( v_{2k} \), and we have \( e_* \in E(T'_{2k}) \setminus E(T_{2k}) \subset E(F_{2k}) \). It follows that the two end-points of \( e_* \) are in the same equivalence class of \( \sim \). Since \( P_\sim \) coincides with the partition of \( \mathcal{V} \) into saturated components of \( G_0 \), this leads to a contradiction to the assumption that all the saturated components are singletons. This final contradiction proves the step \( r + 1 \) of our induction and finishes the proof of the first part of our theorem.

Part (2) follows directly from part (1): contract all the edges lying in a saturated component in \( G_0 \) in order to get the graph \( \widetilde{G}_0 \). One verifies that in \( \widetilde{G}_0 \), all the saturated components are singleton, and the edges of \( \widetilde{G}_0 \) are a disjoint union of the edges of a spanning tree and a spanning 2-forest. Thus by part (1), the graph \( \mathcal{H}_{\widetilde{G}_0} \) is connected. There is an isomorphism from \( \mathcal{H}_0 \) to \( \mathcal{H}_{\widetilde{G}_0} \) which sends a pair \( (A, B) \) in \( \mathcal{Y}_0 \) to the pair \( (\widetilde{A}, \widetilde{B}) \) in \( \mathcal{H}_{\widetilde{G}_0} \) obtained by contracting all the edges in the trees \( T_{j,1}, T_{j,2} \), for \( j = 1, \ldots, r \).

3. PROOF OF THEOREM 1.1

For an \( r \times t \) matrix \( X \), and subsets \( I \subset \{1, \ldots, r\} \) and \( J \subset \{1, \ldots, t\} \) with \( |I| = |J| \), we note by \( X_{I,J} \) the square \( |I| \times |J| \) submatrix of \( X \) with row and columns in \( I, J \), respectively. If \( r \leq t \), and \( I = \{1, \ldots, r\} \) and \( J \subset \{1, \ldots, t\} \), we simply write \( X_J \) instead of \( X_{I,J} \).
We use the notation of the introduction: choosing a basis \( \gamma_1, \ldots, \gamma_h \) for \( H_1(G, \mathbb{Z}) \), we denote by \( M \) the \( h \times m \) matrix of the coefficients of \( \gamma_i \) in the standard basis \( \{ e_i \} \) of \( \mathbb{R}^m \). Similarly, for the element \( \omega \in \mathbb{R}^E \) in the inverse image \( \partial^{-1}(\omega) \) of the external momenta vector \( \mathbf{p} = (\mathbf{p}_e) \), we denote by \( H_\omega \) the \((h+1)\)-dimensional vector subspace of \( \mathbb{R}^m \) generated by \( \omega \) and \( H_1(G, \mathbb{R}) \). The space \( H_\omega \) comes with a basis consisting of \( \gamma_1, \ldots, \gamma_h, \omega \), and we denote by \( N \) the \((h+1) \times m \) matrix of the coefficients of this basis in the standard basis \( \{ e_i \} \) of \( \mathbb{R}^m \).

By Cauchy-Binet formula, we have

\[
\det(NYN^T) = \sum_{I,J \subseteq \{1, \ldots, m\}} \det(N_I) \det(Y_{I,J}) \det(N_J),
\]

which, using that \( Y \) is diagonal, can be further reduced to

\[
\det(NYN^T) = \sum_{I,J \subseteq \{1, \ldots, m\}} \det(N_I)^2 y^I,
\]

where as usual, we pose \( y^I := \prod_{i \in I} y_i \). Similarly, we have

\[
\det(MYM^T) = \sum_{I \subseteq \{1, \ldots, m\}} \det(M_I)^2 y^I.
\]

For a subgraph \( F \) in \( G \), by an abuse of the notation, we write \( F^c \) (instead of \( E \setminus E(F) \)) for the set of edges of \( G \) not in \( F \).

**Lemma 3.1.**

1. For a subset \( I \subseteq \{1, \ldots, m\} \) of size \( h \), we have \( \det(M_I) \neq 0 \) if and only if \( I = T^c \) for a spanning tree \( T \) of \( G \). In this case, we have \( \det(M_I)^2 = 1 \).

2. For a subset \( I \subseteq \{1, \ldots, m\} \) of size \( h+1 \), we have \( \det(M_I) \neq 0 \) if and only if \( I = F^c \) for a spanning 2-forest \( F \) of \( G \). In this case, we have \( \det(N_I)^2 = q(F) = (\sum_{v \in X} \mathbf{p}_v)(\sum_{v \in X} \mathbf{p}_v) \), where \( X, Y \) denotes the partition of \( V \) given by \( F \).

**Proof.** These facts are folklore. Here we only prove (2), part (1) has a similar proof.

Denote by \( e_{i_1}, \ldots, e_{i_{h+1}} \) the \((h+1)\) edges of \( I \). Developing \( \det(N) \) with respect to the last row (which corresponds to the coefficients of \( \omega \)), we have

\[
\det(N_I) = \sum_{j=1}^{m} (-1)^j \omega(e_i) \det(M_{I \setminus \{e_{i_j}\}}).
\]

From the first part, it follows that \( \det(N_I) = 0 \) if none of \( I - e_{i_j} \) is the complement set of edges of a spanning tree, i.e., if \( I \) is not of the form \( F^c \) for a spanning 2-forest of \( G \). So suppose now that \( I = F^c \), denote by \( X, Y \) the partition of \( V \) induced by \( F \), and without loss of generality, let \( e_{i_1}, \ldots, e_{i_r} \) be the set of all the edges in \( E(P(F)) \). We can assume that \( e_i \)'s are all oriented from \( X \) to \( Y \). Let \( T_j = F \cup \{ e_{i_j} \} \) the spanning tree \( F \cup \{ e_{i_j} \} \) for \( j = 1, \ldots, r \). It follows that

\[
\det(N_I) = \sum_{j=1}^{r} (-1)^j \omega(e_i) \det(M_{T_j}).
\]
Since $\partial(\omega) = p$, and the edges $e_{i_1}, \ldots, e_{i_r}$ are oriented from $X$ to $Y$, it follows that

$$\sum_{j=1}^{r} \omega(e_{i_j}) = \sum_{v \in X} p_v.$$

So the lemma follows once we prove that $(-1)^j \det(M_{\{e_{i_j}\}})$ takes the same value for all $j = 1, \ldots, j$. By symmetry, it will be enough to prove $\det(M_{T_1}) + \det(M_{T_2}) = 0$. By multilinearity of the determinant with respect to the columns we see that $\det(M_{T_1}) + \det(M_{T_2}) = \det(P)$ where $P$ is the $h \times h$ matrix with the first column equal to the sum of the first columns of $M_{T_1}^c$ and $M_{T_2}^c$, and the $j$'th column equal to the $j$'th column of $M_{T_1}^c$ (which is the same as that of $M_{T_2}^c$), for $j \geq 1$. So it is enough to show that $\det(P) = 0$. The subgraph $F \cup \{e_{i_1}, e_{i_2}\}$ has a cycle $\gamma$ which contains $e_{i_1}, e_{i_2}$ from $F^c$ and all the other edges are in $F$. Writing $\gamma$ as a linear combination $\sum_{j=1}^{h} \gamma_j$ of the cycles $\gamma_j$, we show that $(a_1, \ldots, a_h)P = 0$. The first coefficient of $(a_1, \ldots, a_h)P$ is zero since the cycle $\gamma$ has $e_{i_1}$ and $e_{i_2}$ with different signs. All the other coordinates of $(a_1, \ldots, a_h)P$ are zero since the only edges of $\gamma$ in $F^c$ are $e_{i_1}$ and $e_{i_2}$. \hfill $\Box$

**Remark 3.2.** The proof of the above lemma shows the following useful property. Suppose that $I$ and $J$ are the complement of the edges of two (vertex-)equivalent 2-forests $F_1 \sim_e F_2$ inducing the partition of $V = X \cup Y$, and $e \in E(\{X,Y\})$ is an edge with one end-point in each of $X$ and $Y$, so both $T_1 = F_1 \cup \{e\}$ and $T_2 = F_2 \cup \{e\}$ are spanning trees. Then

$$\frac{\det(N_I)}{\det(M_{T_1}^c)} = \frac{\det(N_J)}{\det(M_{T_2}^c)} = \pm \sum_{e \in E(X,Y)} \omega(e),$$

where $e$ in the above sum runs over all the oriented edges from $X$ to $Y$. In particular, we have

$$\det(N_I) \det(N_J) = q(F_1) \det(M_{T_1}) \det(M_{T_2}) = q(F_2) \det(M_{T_1}) \det(M_{T_2}).$$

From Lemma 3.1 we infer that in the sum (3.2) (resp. 3.2) above describing $\det(M_{YM^c})$ (resp. $\det(N_{YN^c})$), the only non-zero terms correspond to subsets $I$ which are complements of the edges a spanning tree (resp. spanning 2-forest) of $G$.

Consider the set-up of Theorem 1.1 as in the introduction, where $U$ is a topological space and $y_1, \ldots, y_m : U \to \mathbb{R}_{>0}$ are $m$ continuous functions. Denote by $Y$ the diagonal matrix-valued function on $U$ given by $Y(s) = \text{diag}(y_1(s), \ldots, y_m(s))$. Let $p \in (\mathbb{R})^{k,0}$ be a fixed vector.

Define two real-valued functions $f_1$ and $f_2$ on $U$ by

$$f_1(s) := \det(M_{YM^c}) = \sum_{T \in S_T^I} y(s)^I,$$

and

$$f_2(s) := \det(N_{YN^c}) = \sum_{F \in S_{F_2}^I} q(F)y(s)^I,$$

at each point $s \in U$. Note that $f_1(s) = \phi(y(s))$, for $\phi$ the first Symanzik polynomial, and $f_2(s) = \psi_C(\omega, y(s))$, for $\psi$ the second Symanzik polynomial of the graph $G$.

Let now $A : U \to \text{Mat}_{m \times m}(\mathbb{R})$ be a matrix-valued map taking at $s \in U$ the value $A(s)$. Assume that $A$ verifies the two properties
(i) $A$ is a bounded function, i.e., all the entries $A_{i,j}$ of $A$ take values in a bounded interval $[-C,C]$ of $\mathbb{R}$, for some positive constant $C > 0$. 
(ii) The two matrices $M(Y + A)M^r$ and $N(Y + A)N^r$ are invertible.

Define real-valued functions $g_1, g_2$ on $U$ by $g_1(s) := \det(M(Y + A)M^r)$ and $g_2(s) = \det(N(Y + A)N^r)$, we have by Cauchy-Binet formula,

$$g_1 = \sum_{T_1, T_2 \in ST, I = F_1^T, J = F_2^T} \det(M_I) \det(Y + A)_{I,J} \det(M_J),$$

$$g_2 = \sum_{F_1, F_2 \in SF_2, I = F_1^c, J = F_2^c} \det(N_I) \det(Y + A)_{I,J} \det(N_J).$$

To prove Theorem 1.1 we must show that $g_2/g_1 - f_2/f_1 = O_{\tilde{\gamma}}(1)$ on $U$. Observe first that

**Claim 3.3.** There exist constants $c_1, c_2, C > 0$ such that

$$c_1 f_1(s) < g_1(s) < c_2 f_1(s),$$

for all points $s \in U$ with $y_1(s), \ldots, y_m(s) \geq C$.

**Proof.** By assumption, all the coordinates of $A$ are bounded functions on $U$. Developing the determinant $\det(Y + A)_{I,J}$ as a sum (with $\pm$ sign) over permutations of the products of entries of $(Y + A)_{I,J}$, one observes that each term in the sum is the product of a bounded function with a monomial in the $y_j$’s for indices $j$ in a subset of $I \cap J$. For $I \neq J$, these terms become $o(y^I)$. Also for $I = J$, all the terms but the unique one coming from the product of the entries on the diagonal which gives $y^I$ are $o(y^I)$. Since $f_1 = \sum_{T \in ST} y^{T^c}$, the assertion follows. □

Therefore, in order to prove Theorem 1.1 it will be enough to show that

$$g_2 f_1 - g_1 f_2 = O_{\tilde{\gamma}}(f_1^r).$$

In considering the terms in $g_2 f_1 - g_1 f_2$ it will be convenient to define the bipartite graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, a variation of the exchange graph introduced in the previous section. The vertex set $\mathcal{V}$ of $\mathcal{G}$ is partitioned into two sets $\mathcal{V}_1$ and $\mathcal{V}_2$ with

$$\mathcal{V}_1 := \left\{(F_1, F_2, T) \mid F_1, F_2 \in SF_2, T \in ST\right\},$$

and

$$\mathcal{V}_2 := \left\{(T_1, T_2, F) \mid T_1, T_2 \in ST, F \in SF_2\right\}.$$ 

There is an edge between $(F_1, F_2, T) \in \mathcal{V}_1$ and $(T_1, T_2, F) \in \mathcal{V}_2$ in $\mathcal{G}$ iff there is an edge $e \in E$ such that $T = F + e$, and $F_1 = T_1 - e$ and $F_2 = T_2 - e$.

**Definition 3.4.** If $(F_1, F_2, T) \in \mathcal{V}_1$ and $(T_1, T_2, F) \in \mathcal{V}_2$ are adjacent in $\mathcal{G}$, we say $(T_1, T_2, F) \in \mathcal{V}_1$ is obtained from $(F_1, F_2, T)$ by pivoting involving the edge $e$ (with $E(T) \setminus E(F) = \{e\}$).

Define two weight functions $\xi, \zeta : \mathcal{V} \to C^0(U, \mathbb{R})$ on the vertices of $\mathcal{G}$ as follows. For $(F_1, F_2, T) \in \mathcal{V}_1$, let

$$\xi(F_1, F_2, T) := \det(Y + A)_{F_1^T, F_2^T} y^{T^c},$$

$$\zeta(F_1, F_2, T) := \det(N_{F_1^T}) \det(N_{F_2^T}) \xi(F_1, F_2, T),$$
and for \((T_1, T_2, F) \in \mathcal{V}_2\), define
\[
\xi(T_1, T_2, F) := \det(Y + A)_{T_1, T_2} y^{T^e}, \\
\zeta(T_1, T_2, F) := \det(M_{T_1}) \det(M_{T_2}) q(F) \xi(T_1, T_2, F).
\]

Note that we have
\[
(3.7) \quad g_2 f_1 = \sum_{(F_1, F_2, T) \in \mathcal{V}_1} \zeta(F_1, F_2, T),
\]
and
\[
(3.8) \quad g_1 f_2 = \sum_{(T_1, T_2, F) \in \mathcal{V}_2} \zeta(T_1, T_2, F).
\]

We have the following

**Claim 3.5.**

- For any \((F_1, F_2, T) \in \mathcal{V}_1\), we have
  \[
  \xi(F_1, F_2, T) = O_y(f_1^{F_1 \cap F_2} y^{T^e}).
  \]
- For any \((T_1, T_2, F) \in \mathcal{V}_2\), we have
  \[
  \xi(T_1, T_2, F) = O_y(y^{T_1 \cap T_2} y^{F^e}).
  \]
- For two adjacent vertices \((F_1, F_2, T) \in \mathcal{V}_1\) and \((T_1, T_2, F) \in \mathcal{V}_2\), we have
  \[
  \xi(F_1, F_2, T) = \xi(T_1, T_2, F) + O_y(f_1^2).
  \]

**Proof.** The first two assertions are straightforward. To prove the last one, let \(e\) be the unique edge in \(T \setminus F\). We have
\[
\det(Y + A)_{F_1, F_2} = y_e \det(Y + A)_{T_1, T_2} + O_y(y^{T^e}),
\]
Multiplying both sides by \(y^{T^e}\) gives
\[
\xi(F_1, F_2, T) = \xi(T_1, T_2, F) + O_y(y^{T_1} y^{T^e}) = O_y(f_1^2).
\]

\(\square\)

**Definition 3.6.**

- A tuple \((F_1, F_2, T) \in \mathcal{V}_1\) is called special if \(F_1 \neq v\ F_2\).
- A tuple \((T_1, T_2, F) \in \mathcal{V}_2\) is called special if there exists either \(e \in E(T_1) \setminus E(T_2)\) or \(e \in E(T_2) \setminus E(T_1)\) such that \(F + e\) is a spanning tree.

The following observations are crucial for the proof of our theorem. They show that connected components of \(\mathcal{G}\) which contain special vertices have only "light weight" vertices.

**Claim 3.7.**

1. For any special vertex \(v\) in \(\mathcal{V}\), we have
   \[
   \xi(v) = O_y(f_1^2).
   \]
2. For any vertex \(v \in \mathcal{V}\) connected by a path in \(\mathcal{G}\) to a special vertex \(v\), we have
   \[
   \xi(v) = O_y(f_1^2).
   \]
Claim 3.5.

Proof. If \( \mathfrak{m} = (F_1, F_2, T) \in \mathfrak{V}_1 \), then since \( F_1 \not\subset F_2 \), there exists an edge \( e \in F_2 \) such that \( T_1 = F_1 + e \) is a tree. In this case, we have \( F_1^c \cap F_2^c = (F_1 \cup F_2)^c \subset T_1^c \), and so we have by Claim 3.5,

\[
\xi(F_1, F_2, T) = \mathcal{O}_2(y^{F_1 \cap F_2} y^{T^c}) = \mathcal{O}_2(y^{T_1} y^{T^c}) = \mathcal{O}(f_1^2).
\]

Similarly, if \( \mathfrak{m} = (T_1, T_2, F) \in \mathfrak{V}_2 \), let \( v = \det(v \in G \in V, T = F + e \text{ is a spanning tree. Since } e \notin T_2 \text{, applying Claim 3.5, we get}

\[
\xi(T_1, T_2, F) = \mathcal{O}_2(y^{T_1 \cap T_2} y^{F^c}) = \mathcal{O}_2(y^{T_2} y^{F^c}) = \mathcal{O}(f_1^2).
\]

(2) This follows from (1) and the third assertion in Claim 3.5.

\[\square\]

Definition 3.8. Let \( p \in \mathbb{R}^{V,0} \) be the vector of external momenta. For any vertex \( u = (T_1, T_2, F) \in \mathfrak{V}_2 \), define \( q(u) := q(F) \). For any vertex \( v = (F_1, F_2, T) \in \mathfrak{V}_1 \), with \( F_1 \sim_v F_2 \), define \( q(v) := q(F_1) = q(F_2) \).

We have the following useful property.

Claim 3.9. For two adjacent vertices \( v \in \mathfrak{V}_1 \) and \( u \in \mathfrak{V}_2 \) with \( v \) non-special, we have

\[q(u)\zeta(v) = q(v)\zeta(u) + \mathcal{O}_2(f_1^2).\]

Proof. Let \( v = (F_1, F_2, T) \in \mathfrak{V}_1 \) and \( u = (T_1, T_2, F) \in \mathfrak{V}_2 \), and let \( e \) be an edge in \( E \) with \( T = F + e \), \( T_1 = F_1 + e \) and \( T_2 = F_2 + e \). By assumption, we have \( F_1 \sim_v F_2 \). We already noted that

\[\xi(v) = \xi(u) + \mathcal{O}_2(f_1^2).\]

Multiplying both sides of this equation by \( \det(N_{T_1}) \det(N_{T_2}) q(F) \), and using Equation (3.3), \( \det(N_{T_1}) \det(N_{T_2}) = \det(M_{T_1}) \det(M_{T_2}) q(F_1) \), gives the result.

\[\square\]

As immediate corollary of the above claims, we get

Corollary 3.10. Let \( G \) be a connected component of \( \mathfrak{G} \). If \( G \) contains a special vertex, then for any vertex \( v \in \mathfrak{V}(G) \), we have

\[\zeta(v) = \mathcal{O}_2(f_1^2).\]

Let \( G \) be a connected component of \( \mathfrak{G} \) which does not contain any special vertex. There exists \( \zeta \) such that for any vertex \( \mathfrak{m} \) of \( \mathfrak{G} \), we have

\[\zeta(\mathfrak{m}) = q(\mathfrak{m})\zeta + \mathcal{O}_2(f_1^2).\]

Proof. The first assertion follows from Claim 3.7. To prove the second part, let \( v \) be a vertex of \( G \), and choose \( \zeta \) so that \( \zeta(v) = q(v)\zeta \). The assertion now follows from Claim 3.9 and the connectivity of \( G \).

\[\square\]

The following proposition finally allows us to prove Theorem 1.1.

Proposition 3.11. Let \( G = (V, E) \) be a connected component of \( \mathfrak{G} \) with vertex set \( V = V_1 \cup V_2 \), with \( V_i = V \cap \mathfrak{V}_i \). Suppose that \( G \) does not contain any special vertex. Then we have

\[\sum_{u \in V_1} q(u) = \sum_{w \in V_2} q(w).\]

We will give the proof of this proposition in the next section. Let us first explain how to deduce Theorem 1.1 assuming this result.
Proof of Theorem 1.1. We have to show that \( g_2 f_1 - g_1 f_2 = O_2(f_1^2) \). Let \( \mathcal{G}_1 = (\mathcal{V}_1, \mathcal{E}_1), \ldots, \mathcal{G}_N = (\mathcal{V}_N, \mathcal{E}_N) \) be all the connected components of \( \mathcal{G} \). For each \( i = 1, \ldots, N \), denote by \( \mathcal{V}_{i,1}, \mathcal{V}_{i,2} \) the intersection of \( \mathcal{V}_i \) with \( \mathcal{V}_1 \) and \( \mathcal{V}_2 \) respectively. Using Equations (3.7) and (3.8), we can write

\[
g_2 f_1 - g_1 f_2 = \sum_{v \in \mathcal{V}_1} \zeta(v) - \sum_{u \in \mathcal{V}_2} \zeta(u) = \sum_{i=1}^N \left( \sum_{v \in \mathcal{V}_{i,1}} \zeta(v) - \sum_{u \in \mathcal{V}_{i,2}} \zeta(u) \right).
\]

For each \( 1 \leq i \leq N \), we have the following two possibilities. Either, \( \mathcal{G}_i \) contains a special vertex, in which case we have \( \zeta(w) = O_2(f_1^2) \) for all \( w \in \mathcal{V}(G_i) \). In particular,

\[
\sum_{v \in \mathcal{V}_{i,1}} \zeta(v) - \sum_{u \in \mathcal{V}_{i,2}} \zeta(u) = O_2(f_1^2).
\]

Or \( \mathcal{G}_i \) does not contain any special vertex, in which case, applying Corollary 3.10 and Proposition 3.11 we must have

\[
\sum_{v \in \mathcal{V}_{i,1}} \zeta(v) - \sum_{u \in \mathcal{V}_{i,2}} \zeta(u) = \zeta \left( \sum_{v \in \mathcal{V}_{i,1}} q(v) - \sum_{u \in \mathcal{V}_{i,2}} q(u) \right) + O_2(f_1^2)
\]

Thus, \( g_2 f_1 - g_1 f_2 = O_2(f_1^2) \) and the theorem follows. \( \square \)

3.1. Proof of Proposition 3.11. Recall that for a partition \( P \) of \( V \) into sets \( X_1, \ldots, X_k \), we denote by \( E(P) \) the set of all edges in \( G \) with end-points lying in two different sets among \( X_i \)s. For a spanning 2-forest \( F \), the partition of \( V \) into the vertex sets of the two connected components of \( F \) is as before denoted by \( P(F) \).

Let \( \mathcal{G} \) be a connected component of \( \mathcal{G} \) which does not contain any special vertex. Let \( \mathcal{V} = \mathcal{V}_1 \cup \mathcal{V}_2 \) be the vertex set of \( \mathcal{G} \) with \( \mathcal{V}_i \subset \mathcal{V}_i \), for \( i = 1, 2 \). We will give a complete description of the structure of \( \mathcal{G} \) using the structure theorem we proved for the exchange graph, which in particular allows to prove Proposition 3.11.

Define equivalence relations \( \equiv_1, \equiv_2, \equiv_3 \) on the set of vertices \( V \) of \( G \) as follows. For two vertices \( u, v \in \mathcal{V} \),

- we say \( u \equiv_1 v \) if for any \( (T_1, T_2, F) \in \mathcal{V}_2 \), both vertices \( u \) and \( v \) lie in the same connected component of \( T_1 \setminus E(P(F)) \).

Similarly,

- we say \( u \equiv_2 v \) if for any \( (T_1, T_2, F) \in \mathcal{V}_2 \), both vertices \( u \) and \( v \) lie in the same connected component of \( T_2 \setminus E(P(F)) \).

And finally,

- we say \( u \equiv_3 v \) if for any \( (F_1, F_2, T) \in \mathcal{V}_1 \), both vertices \( u \) and \( v \) lie in the same connected component of \( T \setminus E(P(F_1)) \).

Note that since \( \mathcal{G} \) does not contain any special vertex, we have \( F_1 \sim_v F_2 \) for all \( (F_1, F_2, T) \in \mathcal{V}_1 \). In particular, \( T \setminus E(P(F_1)) = T \setminus E(P(F_2)) \).

The following statements are analogous to the statements of Lemma 2.8 and Claim 2.9 for the exchange graph.
**Lemma 3.12.** Let $F$ be a spanning 2-forest in $G$. Let $T$ be a spanning tree of $G$. Suppose two vertices $u, v \in V$ are in two different connected components of $T \setminus E(P(F))$. There exists and edge $e \in E(P(F)) \cap E(T_1)$ such that $u$ and $v$ are not connected in $T - e$.

**Proof.** Denote by $S_u$ and $S_v$ the two connected components of $T \setminus E(P(F))$ which contain $u$ and $v$, respectively. There is a path joining $S_u$ to $S_v$ in $T$. Since $S_u \neq S_v$, it contains an edge $e \in E(P(F))$. For such an edge $e$, $u$ and $v$ are not connected in $T - e$. □

The previous lemma allows to prove the following claim.

**Claim 3.13.** The three equivalence relations $\equiv_1, \equiv_2, \equiv_3$ are the same.

**Proof.** To prove that $\equiv_1$ and $\equiv_2$ are the same, suppose for the sake of a contradiction that $u \equiv_1 v$ but $u \not\equiv_2 v$ for two vertices $u$ and $v$ in $V$. This implies the existence of $(T_1, T_2, F) \in \mathcal{V}_2$ such that

- the two vertices $u$ and $v$ are both in $X$ with $P(F) = \{X, X^c\}$.
- $u$ and $v$ are in the same connected component of $T_1[X]$, and they are in two different connected components of $T_2[X]$.

Applying the previous lemma, there exists an edge $e \in E(T_2) \cap E(P(F))$ such that $u$ and $v$ lie in two different connected components of $F_2 = T_2 - e$. Since $(T_1, T_2, F)$ is not special, and $e \in E(P(F))$, we have $e \in T_1$. In particular, $u, v$ are in the same connected component of $F_1 = T_1 - e$. We have proved that $P(F_1) \neq P(F_2)$, i.e., the tuple $(F_1, F_2, T)$ obtained from $(T_1, T_2, F)$ by pivoting involving $e$ is special. This contradicts the assumption on $G$ (that it does not contain special vertices), and proves our claim.

We now prove that $\equiv_1$ and $\equiv_3$ are similar. Suppose for the sake of a contradiction that this is not the case. Let $u, v \in V$ be two vertices with $u \equiv_3 v$ but $u \not\equiv_1 v$ (the other case $u \not\equiv_3 v$ but $u \equiv_1 v$ has a similar treatment that we omit). This implies the existence of $(T_1, T_2, F) \in \mathcal{V}_2$ such that $u, v$ belong to two different connected components of $T_1 \setminus E(P(F))$. Applying the previous lemma, we infer the existence of an edge $e \in E(T_1) \cap E(P(F))$ such that $u$ and $v$ are not connected in $T_1 - e$. Pivoting involving $e$ gives a tuple $(F_1, F_2, T)$ such that $u$ and $v$ lie in two different connected components of $F_1$. In particular, it follows that $u \not\equiv_3 v$, which is a contradiction. This proves the claim. □

We denote by $\equiv$ the equivalence relation on vertices induced by $\equiv_i$. As in Remark 2.10, we have the following

**Remark 3.14.** Note that if $u$ and $v$ are two vertices with $u \not\equiv v$, there exists $(F_1, F_2, T) \in \mathcal{V}_1$ such that $u$ and $v$ lie in different connected components of $F_i$. Similarly, there exists $(T_1, T_2, F) \in \mathcal{V}_2$ such that $u, v$ lie in different connected components of $F$.

Denote by $P_\equiv = \{X_1, \ldots, X_n\}$ the partition of $V$ induced by the equivalence classes $X_i$ of $\equiv$. Note that pivoting in $G$ only involves edges in $E \setminus E(P_\equiv)$, i.e., which are not contained in any $X_1, \ldots, X_n$. By connectivity of $G$, it follows that for each $i$, there are three trees $T_{i,1}, T_{i,2}, T_{i,3}$ on the vertex set $X_i$ such that for any $(F_1, F_2, T) \in \mathcal{V}_1$ and any $(T_1, T_2, F) \in \mathcal{V}_2$, we have

$$T_1[X_i] = F_1[X_i] = T_{i,1}, \ T_2[X_i] = F_2[X_i] = T_{i,2}, \ T[X_i] = F[X_i] = T_{i,3}.$$

In other words, the subtrees $T_{i,1}, T_{i,2}, T_{i,3}$ are the ”constant” part of the elements in $G$. We now prove
Claim 3.15. For any \((T_1, T_2, F) \in \mathcal{V}_2\), we have

\[ T_1 \setminus \left( \bigcup_{i=1}^{n} E(\tau_{i,1}) \right) = T_2 \setminus \left( \bigcup_{i=1}^{n} E(\tau_{i,2}) \right). \]

In other words, the edges of \(T_1\) and \(T_2\) outside \(X_i\)'s are the same.

Proof. Let \(e = \{u, v\}\) be an edge of \(T_1\) with \(u\) and \(v\) lying in two different equivalence classes \(X_i\) and \(X_j\). By Remark 3.14, there exists \((T'_1, T'_2, F') \in \mathcal{V}_1\) such that \(u, v\) belong to two different sets of the partition \(\mathcal{P}(F)\). By connectivity of \(\mathcal{G}\), since \(e \notin F'\), we have \(e \in T'_1\). Since \((T'_1, T'_2, F')\) is non-special, we infer \(e \in T'_2\). By connectivity of \(\mathcal{G}\), and the way the edges are defined (which requires pivoting involving the same edge for the two trees in the vertices of \(\mathcal{V}_2\)), we must have \(e \in T_2\), and the claim follows. \qed

Let \(v = (T_1, T_2, F) \in \mathcal{V}_2\). Let

\[
\begin{align*}
E_{1,2}(v) &:= E(T_1) \cap E(\mathcal{P}_\equiv) = E(T_2) \cap E(\mathcal{P}_\equiv), \\
E_3(v) &:= E(F) \cap E(\mathcal{P}_\equiv).
\end{align*}
\]

Obviously, we have

\[
E(T_1) = E_{1,2}(v) \cup \bigcup_{i=1}^{n} E(\tau_{i,1}), \quad E(T_2) = E_{1,2}(v) \cup \bigcup_{i=1}^{n} E(\tau_{i,2}), \quad \text{and} \quad E(F) = E_3(v) \cup \bigcup_{i=1}^{n} E(\tau_{i,1}).
\]

Define the multiset

\[
E_\mathcal{G} := E_{1,2}(v) \cup E_3(v).
\]

By the definition of the edges in the graph \(\mathcal{G}\), and connectivity of (the connected component) \(\mathcal{G}_v\), \(E_\mathcal{G}\) is independent of the choice of \(v \in \mathcal{V}_2\). In addition, if \(u \in \mathcal{V}_1\), we define \(E_{1,2}(u) = E(F_1) \cap E(\mathcal{P}_\equiv)\), and \(E_3(u) = E(T) \cap E(\mathcal{P}_\equiv)\), we should have \(E_\mathcal{G} = E_{1,2}(u) \cup E_3(u)\).

Define an (auxiliary) multigraph \(G_0 = (V_0, E_0)\) obtained by contracting each equivalence class \(X_i\) to a vertex \(x_i\) and having the multiset of edges \(E_0 = E_\mathcal{G}\). More precisely, \(G_0\) has the vertex set \(V_0 = \{x_1, \ldots, x_n\}\), and an edge \(\{x_i, x_j\}\) for any edge \(e = \{u, v\}\) in the multiset \(E_\mathcal{G}\) which joins a vertex \(u \in X_i\) to a vertex \(v \in X_j\). By an abuse of the notation, we identify \(E_0\) with \(E_\mathcal{G}\).

Each \(v = (T_1, T_2, F) \in \mathcal{V}_2\) gives a pair \((T_v, F_v)\) that we denote by \(\pi(v)\) consisting of a spanning tree \(T_v\) of \(G_0\) with edges \(E_{1,2}(v)\) and a spanning 2-forest \(F_v\) of \(G_0\) with edge set \(E_3(v)\). As a multiset, we have \(E_0 = E(T_v) \cup E(F_v)\). Similarly, each \(u = (F_1, T_2, F) \in \mathcal{V}_1\) gives a pair \(\pi(u) = (F_v, T_v)\) consisting of a spanning 2-forest \(F_v\) and a spanning tree \(T_v\) of \(G_0\) with edge sets \(E_{1,2}(u)\) and \(E_3(u)\), respectively.

We will describe \(\mathcal{G}\) in terms of the multigraph \(G_0\). Let \(\mathcal{H}_0 = (\mathcal{Y}_0, \mathcal{E}_0)\) be the exchange graph associated to the multigraph \(G_0\) as in Section 2. Recall that the vertex set \(\mathcal{Y}_0\) of \(\mathcal{H}_0\) is the disjoint union of two sets \(\mathcal{Y}_{0,1}\) and \(\mathcal{Y}_{0,2}\), where

\[
\mathcal{Y}_{0,1} := \left\{(F, T) \mid F \in \mathcal{SF}_2(G_0), T \in \mathcal{ST}(G_0), \ E(F) \cup E(T) = E_0\right\},
\]

and

\[
\mathcal{Y}_{0,2} := \left\{(T, F) \mid T \in \mathcal{ST}(G_0), F \in \mathcal{SF}_2(G_0), \ E(F) \cup E(T) = E_0\right\}.
\]
There is an edge in $E_0$ connecting $(F, T) \in \mathcal{V}_{0,1}$ to $(T', F') \in \mathcal{V}_{0,2}$ if $(T', F')$ is obtained from $(F, T)$ by pivoting involving an edge $e \in E_0$, i.e., if $F = T' - e$ and $F' = T - e$.

With this notation, we get an application $\pi : \mathcal{V} \to \mathcal{V}_0$. By what we have proved so far, it is clear that $\pi$ is injective. By the definition of edges in $\mathfrak{G}$ and $\mathcal{H}_0$, $\pi$ induces a homomorphism of graphs $\pi : \mathfrak{G} \to \mathcal{H}_0$. In addition, any pivoting in $\mathcal{H}_0$ involving an edge $e \in E_0 = E_G$ can be lifted to pivoting involving the same edge $e$ in $\mathfrak{G}$. This proves that $\pi$ induces an isomorphism onto (its image) a connected component of $\mathcal{H}_0$.

**Proposition 3.16.** The exchange graph $\mathcal{H}_0$ is connected. As a consequence, the projection map $\pi$ is an isomorphism.

**Proof.** By the discussion preceding the proposition, we only need to show that $\mathcal{H}_0$ is connected. Since the multigraph $G_0$ is disjoint union of a spanning tree and a spanning forest, this latter statement follows from the first part of Theorem 2.12 by observing that the only saturated non-empty subset of vertices of $G_0$ are singletons. To see this, note that no pivoting involves the edge set of a saturated subset of vertices. So for a saturated subset of vertices $S$ of $V_0$, all the components $X_i$ of the partition $\mathcal{P}_\equiv$ associated to the vertices $x_i \in S$ must lie in the same equivalence class of $\equiv$. However, $X_i \in \mathcal{P}_\equiv$ are already all the equivalence classes of $\equiv$, so we must have $|S| = 1$. □

We can now prove Proposition 3.11.

**Proof of Proposition 3.11.** Let $\mathfrak{G} = (\mathcal{V}, \mathcal{E})$ be a connected component of $\mathfrak{G}$. Let $G_0$ be the multigraph we associated to $\mathfrak{G}$, and $\pi : \mathfrak{G} \to \mathcal{H}_0$ be the isomorphism constructed above.

For $(F, T) \in \mathcal{V}_{0,1}$, we have $(T, F) \in \mathcal{V}_{0,2}$, and by definition, we have

$$q(\pi^{-1}(F, T)) = q(\pi^{-1}(T, F)).$$

Since $\pi$ is an isomorphism, it follows that

$$\sum_{u \in V_2} q(u) = \sum_{(T,F) \in \mathcal{V}_{0,2}} q(\pi^{-1}(T,F)) = \sum_{(F,T) \in \mathcal{V}_{0,1}} q(\pi^{-1}(F,T)) = \sum_{u \in V_1} q(u),$$

and the proposition follows. □

The proof of Theorem 1.1 is now complete.

### 4. Proof of Theorem 1.2

In this section we explain how to derive Theorem 1.2 from Theorem 1.1. The presentation here is heavily based on the results and notations of [1], to which we refer for the missing details.

First we recall the set-up. Let $\Delta$ be a small open disc around the origin in $\mathbb{C}$, and denote by $\Delta^* = \Delta \setminus \{0\}$ the punctured disk. Let $S = \Delta^{3g-3}$. Let $C_0$ be a stable curve of arithmetic genus $g$, and let $G = (V, E)$ be the dual graph of $C_0$. Denote by $h \leq g$ the genus of $G$, so we have $h = |E| - |V| + 1$. The versal analytic deformation of $C_0$ over $S$ is denoted by $\pi : C \to S$. The fibers of $\pi$ are smooth outside a normal crossing divisor $D = \bigcup_{e \in E} D_e \subset S$, which has irreducible components indexed by the set of edges of $G$ (which are in bijection with the singular points of $C_0$). Let $U$ be the complement of the divisor $D$ in $S$, that we identify with $U = (\Delta^*)^E \times \Delta^{3g-3-|E|}$. Let

$$\bar{U} := \mathbb{H}^E \times \Delta^{3g-3-|E|} \longrightarrow U.$$
be the universal cover of $U$. The projection map $\tilde{U} \to U$ is given by $z_e \mapsto \exp(2\pi iz_e)$ in the first factors corresponding to the edges of $G$, and is the identity on the remaining factors.

Suppose that we have two collections

$$\sigma_1 = \{\sigma_{1,l}\}_{l=1,\ldots,n}, \quad \sigma_2 = \{\sigma_{2,l}\}_{l=1,\ldots,n}$$

of sections $\sigma_{i,l}: S \to \mathcal{C}$ of $\pi$, for $1 \leq l \leq n$ and $i = 1, 2$. By regularity of $\mathcal{C}$, these sections cannot pass through double points of $C_0$, and for each $l$, $\sigma_{i,l}(S) \cap C_0$ lies in a unique irreducible component $X_{v_l}$ of $C_0$, which corresponds to a vertex $v_l$ of the dual graph $G$. We assume that the sections $\sigma_{1,l}$ and $\sigma_{2,l}$ are distinct on $C_0$, which implies, after shrinking $S$ if necessary, that $\sigma_1$ and $\sigma_2$ are disjoint as well.

Let $\mathbf{p}_1 = \{\mathbf{p}_{1,l}\}_{l=1}^n \in (\mathbb{R}^D)^{n,0}$ and $\mathbf{p}_2 = \{\mathbf{p}_{2,l}\}_{l=1}^n \in (\mathbb{R}^D)^{n,0}$ be two collections of external momenta satisfying the conservation law. Using the labelings of sections and the external momenta, we associate each marked point $\sigma_{i,l}$ with $\mathbf{p}_{i,l} \in \mathbb{R}^D$, and denote by $\mathbf{p}_1^G = (\mathbf{p}_{v,1}^G)$ and $\mathbf{p}_2^G = (\mathbf{p}_{v,2}^G)$ the restriction of $\mathbf{p}_1$ and $\mathbf{p}_2$ to the graph $G$: for each vertex $v$ of $G$, the vector $\mathbf{p}_{v,i}^G$ is the sum of all the momenta $\mathbf{p}_{i,l}$ with $v_l = v$. In this way, at any point $s \in S$, we get two $\mathbb{R}^D$-valued degree zero divisors on the curve $C_s$ that we denote by $\mathfrak{A}_s$ and $\mathfrak{B}_s$: they are defined by

$$\mathfrak{A}_s := \sum_{l=1}^n \mathbf{p}_{1,l} \sigma_{1,l}(s), \quad \mathfrak{B}_s := \sum_{l=1}^n \mathbf{p}_{2,l} \sigma_{2,l}(s).$$

This gives us the real valued function on $U$ which sends the point $s$ of $U$ to $\langle \mathfrak{A}_s, \mathfrak{B}_s \rangle$, where $\langle \ldots \rangle$ denotes the archimedean height pairing between $\mathbb{R}^D$-valued degree zero divisors, see the introduction and $[1]$ for the definition and the extension to $\mathbb{R}^D$-valued divisors defined by means of the given Minkowski bilinear form.

We are interested in understanding the behaviour of the function $s \mapsto \langle \mathfrak{A}_s, \mathfrak{B}_s \rangle$ close to the origin $0 \in S \setminus U$. This can be carried out using the nilpotent orbit theorem in Hodge theory, c.f. $[1]$. We can reduce to the case where the external momenta are all integers, and in this case, the divisors $\mathfrak{A}_s$ and $\mathfrak{B}_s$ having integer coefficients at any point $s$, the archimedean height pairing between $\mathfrak{A}_s$ and $\mathfrak{B}_s$ can be described in terms of a birextension mixed Hodge structure, c.f. $[5,1]$. Denoting by $H_{\mathfrak{B}_s,\mathfrak{A}_s}$ the birextension mixed Hodge structure associated to the pair $\mathfrak{A}_s$ and $\mathfrak{B}_s$, the family $H_{\mathfrak{B}_s,\mathfrak{A}_s}$ fit together into an admissible variation of mixed Hodge structures. An explicit description of the period map for the variation of the birextension mixed Hodge structures $H_{\mathfrak{B}_s,\mathfrak{A}_s}$ was obtained in $[1]$. We briefly recall this now.

Fix base points $s_0 \in U$ and $\bar{s}_0 \in \tilde{U}$ lying above $s_0$, and choose a symplectic basis

$$a_1, \ldots, a_g, b_1, \ldots, b_g \in H_1(C_{s_0}, \mathbb{Z}) = A_0 \oplus B_0.\,$$

Shrinking $S$ if necessary, the inclusion $C_{s_0} \hookrightarrow \mathcal{C}$ gives a surjective specialization map

$$sp: H_1(C_{s_0}, \mathbb{Z}) \to H_1(C, \mathbb{Z}) \simeq H_1(C_0, \mathbb{Z}).$$

Denote by $A \subset H_1(C_{s_0}, \mathbb{Z})$ the subspace spanned by the vanishing cycles $a_e$, one for each $e \in E$. We have the exact sequence

$$0 \rightarrow A \rightarrow H_1(C_{s_0}, \mathbb{Z}) \xrightarrow{sp} H_1(C_0, \mathbb{Z}) \rightarrow 0,$$

and we define $A' = A + sp^{-1}(\bigoplus_{v \in V} H_1(X_v, \mathbb{Z})) \subset H_1(C_{s_0}, \mathbb{Z})$. We have

$$H_1(C_{s_0}, \mathbb{Z})/A' \simeq H_1(G, \mathbb{Z}).$$
Changing the symplectic basis if necessary, we suppose that the space of vanishing cycles $A$ is generated by $a_1, \ldots, a_h \in A$, and $b_1, \ldots, b_h$ generate $H_1(C_{s_0}, \mathbb{Z})/A' \simeq H_1(G, \mathbb{Z})$ as in (4.2).

For $i = 1, 2$, let $\Sigma_{i,s} = \{ \sigma_{1,i}(s), \ldots, \sigma_{n,i}(s) \}$, and set $\Sigma_s = \Sigma_{1,s} \cup \Sigma_{2,s}$ and $\Sigma = \bigcup_s \Sigma_{i,s}$. By choosing loops that do not meet the points in $\Sigma_{s_0}$, we lift the classes $a_j$ and $b_j$, $j = 1, \ldots, g$ to elements of $H_1(C_{s_0} \setminus \Sigma_{s_0}, \mathbb{Z})$. By an abuse of the notation, we denote by $a_j$ and $b_j$ these new classes as well. This symplectic basis can be spread out to a basis

$$a_{1,\tilde{s}}, \ldots, a_{g,\tilde{s}}, b_{1,\tilde{s}}, \ldots, b_{g,\tilde{s}}$$

of $H_1(C_s \setminus \Sigma_{s}, \mathbb{Z})$, for any $s \in U$ and any $\tilde{s} \in \tilde{U}$ over $s$. The elements $a_{i,\tilde{s}}$ only depend on $s$ and not on $\tilde{s}$; we will also denote them by $a_{i,s}$, and if there is no risk of confusion, we drop $\tilde{s}$, and use simply $a_i$ and $b_i$.

In addition, we have a collection of 1-forms $\{ \omega_{i,s} \}_{i=1,\ldots,g}$ on $\pi^{-1}(U) \subset C$ such that the forms $\{ \omega_{i,s} := \omega_{i}|_{C_s} \}_{i=1,\ldots,g}$, for each $s \in U$, are a basis of the holomorphic differentials on $C_s$ and

$$\int_{a_{i,s}} \omega_{j,s} = \delta_{i,j}. \tag{4.3}$$

The period matrix for the curve $C_s$ is given by $(\int_{b_{i,s}} \omega_{j,s})$.

Choose now an integer valued 1-chain $\gamma_{\mathfrak{B}_{s_0}}$ on $C_{s_0} \setminus \Sigma_{1,s_0}$ with $\mathfrak{B}_{s_0}$ as boundary. Adding a linear combination of the $b_j$ if necessary, we further assume that

$$\langle a_i, \gamma_{\mathfrak{B}_{s_0}} \rangle = 0. \tag{4.4}$$

We spread the class $[\gamma_{\mathfrak{B}_{s_0}}] \in H_1(C_{s_0} \setminus \Sigma_{1,s_0}, \Sigma_{2,s_0}, \mathbb{Z})$ of $\gamma_{\mathfrak{B}_{s_0}}$ to classes $\gamma_{\mathfrak{B}_{s}}$.

Similarly, we obtain a 1-form $\omega_{\mathfrak{A}}$ on $\pi^{-1}(U) \setminus \Sigma_1$ such that each restriction $\omega_{\mathfrak{A},s} := \omega_{\mathfrak{A}|_{C_s}}$ is a holomorphic form of the third kind with residue $\mathfrak{A}_s$. Adding to $\omega_{\mathfrak{A}}$ a linear combination of the $\omega_i$ if needed, we can suppose that $\omega_{\mathfrak{A}}$ is normalized so that

$$\int_{a_{i,s}} \omega_{\mathfrak{A},s} = 0, \quad i = 1, \ldots, g. \tag{4.5}$$

Denote by $\text{Row}_g(\mathbb{C}) \simeq \mathbb{C}^g$ and $\text{Col}_g(\mathbb{C}) \simeq \mathbb{C}^g$ the $g$-dimensional vector space of row and column matrices, and let

$$\tilde{X} := \mathbb{H}_g \times \text{Row}_g(\mathbb{C}) \times \text{Col}_g(\mathbb{C}) \times \mathbb{C}.$$ 

We have the following description of the period map from $[\mathfrak{A}]$.

**Proposition 4.1** ([\mathfrak{A}]). *The period map of the variation of mixed Hodge structures $H_{\mathfrak{B}_s,\mathfrak{A}_s}$ is given by*

$$\tilde{\Phi}: \tilde{U} \longrightarrow \tilde{X}$$

$$\tilde{s} \longmapsto \left( (\int_{b_{i,s}} \omega_{j,s})_{i,j} , (\int_{\gamma_{\mathfrak{B}_{s},\tilde{s}}} \omega_{j,s})_j, (\int_{\gamma_{\mathfrak{A}_{s},\tilde{s}}} \omega_{i,s})_i, \int_{\gamma_{\mathfrak{B}_{s},\tilde{s}}} \omega_{\mathfrak{A},s} \right).$$

We now explain the action of the logarithm of monodromy map $N_e$ for $e \in E$, c.f. $[\mathfrak{A}]$.

As before, each vanishing cycle $a_e \in H_1(C_{s_0}, \mathbb{Z})$ for $e \in E$ can be lifted in a canonical way to a cycle $a_e$ in $H_1(C_{s_0} \setminus \Sigma_{s_0}, \mathbb{Z})$. 


where the matrices \( \tilde{W} \) and \( \tilde{Z} \) are given as follows. The choice of the path \( \tilde{\gamma} \) is obtained by counting the number of times with sign that \( \tilde{\gamma} \) crosses the vanishing cycle \( a_e \). Similarly, \( \omega_{\alpha} \) gives a preimage \( \omega_1 \) for \( p_1^G \) in \( \mathbb{C}^E \) whose \( e \)-th component, for \( e \in E(G) \), is given by \( \int_{a_e} \omega_\alpha \).

With these preliminaries, we can now state the expression of the height pairing in terms of the period map. Let us separate the variables which correspond to the edges of the graph
\[ s_e = \begin{cases} \exp(2\pi i z_e), & \text{for } e \in E, \\ z_e, & \text{for } e \notin E. \end{cases} \]

The following expression for the height pairing function is obtained in [1].

**Proposition 4.2 ([1]).** There exists \( h_0 > 0 \) and a holomorphic map \( \Psi_0 : U \to \overline{X} \),
\[
\Psi_0(s) = (\Omega_0(s), W_0(s), Z_0(s), \rho_0(s)),
\]
such that introducing
\[
y_e = \text{Im}(z_e) = -\frac{1}{2\pi} \log |s_e|,
\]
the height pairing is given by
\[
\langle A_s, B_s \rangle = -2\pi \text{Im}(\rho_0) - \sum_{e \in E} 2\pi y'_e \Gamma_e \overline{t} \mathbf{P}_1 + \\
2\pi \left( \text{Im}(W_0) + \sum_{e \in E} y'_e \mathbf{P}_2 \widetilde{W}_e \right) \cdot \left( \text{Im}(\Omega_0) + \sum_{e \in E} y'_e M_e \right)^{-1} \cdot \left( \text{Im}(Z_0) + \sum_{e \in E} y'_e \mathbf{Z}_e t \mathbf{P}_1 \right),
\]
where \( y'_e = y_e - h_0 \).

We have

**Theorem 4.3.** There exists a bounded function \( h : U \to \mathbb{R} \) such that after shrinking the radius of \( \Delta \) if necessary, we can write the height pairing as
\[
\langle A_s, B_s \rangle = -\sum_{e \in E} 2\pi y_e \mathbf{P}_2 \Gamma_e \overline{t} \mathbf{P}_1 + 2\pi \left( \sum_{e \in E} y_e \mathbf{P}_2 W_e \right) \left( \sum_{e \in E} y_e M_e \right)^{-1} \left( \sum_{e \in E} y_e Z_e \mathbf{P}_1 \right) + h(s).
\]

This theorem was proved in [1] using normlike functions in the terminology of [2, Section 3.1]. We now give a proof based on Theorem 1.1.

Since \( \rho_0 \) is a holomorphic function on \( S = \Delta^{3g-3} \), after shrinking the radius of \( \Delta \) if necessary, we can assume that \( \text{Im}(\rho_0) \) is bounded. In addition, since \( h_0 \) is constant, the difference between \( y'_e \mathbf{P}_2 \Gamma_e \overline{t} \mathbf{P}_1 \) and \( y_e \mathbf{P}_2 \Gamma_e \mathbf{P}_1 \) is constant for each \( e \). So we only need to prove that the third term in the right hand side of equation (4.10) is, up to a bounded function, equal to the second term in the right hand side of (4.11).

We treat first the case \( g = h \) and explain later how to reduce to this case.

First, we can reduce to the case where \( \mathbf{p}_1 \) are real valued, c.f. [1]. Using the bilinearity of the right hand side term in (4.11), we can reduce to the case \( \mathbf{p}_1 = \mathbf{p}_2 \).

Let \( H = H_1(G, \mathbb{R}), \omega \in \mathbb{R}^{3g-3} \) given by \( \mathbf{p} \), and \( H_\omega \supset H \) as in Section 1.2. Let \( \alpha = \sum_{e} y_e \langle \cdot, \cdot \rangle_e \) the bilinear form on \( \mathbb{R}^E \).
For a matrix of the form,

\[ T = \begin{pmatrix} M & W \\ W^t & S \end{pmatrix} \]

where \( M \) is an invertible \( h \times h \) matrix, \( W \) is a (column) vector of dimension \( h \), and \( S \) is a scalar, we have the formula \((\det M)M^{-1} = \text{adj}(M)\), where \( \text{adj}(M) \) is the matrix of minors. This gives

\[ \frac{\det T}{\det M} = -W^t W^{-1} M^{-1} W + S. \]

Using these observations, the expression on the right hand side of \( (4.11) \) is the ratio \( 2\pi \det(\alpha|_{H\omega})/\det(\alpha|_H) \), for the basis of \( H \) (resp.) given by \( B = \{b_1, \ldots, b_h\} \) (resp. \( B_\omega = \{b_1, \ldots, b_h, \omega\} \)). Similarly, the expression on the right hand side of Proposition 4.2 at any point \( s \) of \( U \) is the ratio \( 2\pi \det(\alpha|_{H\omega} + \beta(s)|_{H\omega})/\det(\alpha|_H + \beta(s)|_H) \) for a bilinear form \( \beta(s) \) on \( H_\omega \) (given by \( W_0, Z_0, \Omega_0, h_0, \) and \( \Gamma, W_e, Z_e, M_e \)), calculated using the basis \( B \) and \( B_\omega \) of \( H \) and \( H_\omega \).

By boundedness of \( W_0, Z_0, \Omega_0 \), and \( h_0, \beta(s) \) lies in a compact subset of the space of bilinear forms on \( H_\omega \). Fixing a complement \( H' \) to \( H_\omega \); i.e., \( H_\omega + H' = \mathbb{R}^m \), and extending \( \beta(s) \) trivially (by zero) to \( \mathbb{R}^m \), we can assume that \( \beta(s) \) is the restriction to \( H_\omega \) of a bilinear form \( \bar{\beta}(s) \) on \( \mathbb{R}^m \), and that \( \bar{\beta}(s) \) lie in a compact subset of the space of bilinear forms on \( \mathbb{R}^m \) for \( s \in U \).

Let \( M \) (resp. \( N \)) be the \( h \times m \) (resp. \( (h+1) \times m \)) matrix of the coefficients of the basis \( B \) (resp. \( B_\omega \)) in the standard basis of \( \mathbb{R}^m \). Let \( Y = \text{diag}(y_1, \ldots, y_m) \) be the diagonal \( m \times m \) matrix of \( \alpha \) in the standard basis of \( \mathbb{R}^m \).

Let \( A : U \to \text{Mat}_{m \times m}(\mathbb{R}) \) be the matrix-valued map taking at \( s \in U \) the value \( A(s) \) the matrix of the bilinear form \( \beta(s) \) in the standard basis of \( \mathbb{R}^m \).

Theorem 4.3 in the case \( g = h \) now follows from Theorem 1.2 which is the statement that the difference \( \det(N(Y + A)N^\tau)/\det(M(Y + A)M^\tau) - \det(NY N^\tau)/\det(MY M^\tau) \) is \( O_2(1) \).

We now explain how to reduce the general case \( g > h \) to the case treated above.

Let \( \mathcal{W} := \text{Im}(W_0) - \sum_{e \in E} y_e \hat{p}_2 \hat{W}_e \), and write \( \mathcal{W} = (\mathcal{W}_1, \mathcal{W}_2) \) with \( \mathcal{W}_1 \) the vector of the \( h \) first coordinates. Similarly, write \( \mathcal{Z} := \text{Im}(Z_0) + \sum_{e \in E} y_e \hat{Z}_e \hat{p}_1 \), and write \( \mathcal{Z} = (\mathcal{Z}_1, \mathcal{Z}_2) \) with \( \mathcal{Z}_1 \) the first vector of the \( h \) first coordinates.

Let \( \mathcal{M} = \text{Im}(\Omega_0) + \sum_{e \in E} y_e \hat{M}_e \), and write

\[ \mathcal{M} = \begin{pmatrix} \mathcal{M}_{11} & \mathcal{M}_{12} \\ \mathcal{M}_{21} & \mathcal{M}_{22} \end{pmatrix}. \]

It will be enough to prove

**Proposition 4.4.** We have

\[ \mathcal{W} \mathcal{M}^{-1} \mathcal{Z} - \left( \sum_{e \in E} y_e \hat{p}_2 \hat{W}_e \right) \left( \sum_{e \in E} y_e \hat{M}_e \right)^{-1} \left( \sum_{e \in E} y_e \hat{Z}_e \hat{p}_1 \right) = O_2(1). \]

**Proof.** Let \( \mathcal{N} = \mathcal{M}^{-1} \), and write

\[ \mathcal{N} = \begin{pmatrix} \mathcal{N}_{11} & \mathcal{N}_{12} \\ \mathcal{N}_{21} & \mathcal{N}_{22} \end{pmatrix}. \]
with $\mathcal{N}_1$ and $\mathcal{N}_2$ square matrices of size $h \times h$ and $(g-h) \times (g-h)$, respectively. Writing
$$\mathcal{W}_1 \mathcal{N}_1 \mathcal{W}_2 = \mathcal{W}_1 \mathcal{N}_1 \mathcal{W}_2 + \mathcal{W}_1 \mathcal{N}_2 \mathcal{W}_2 + \mathcal{W}_2 \mathcal{N}_1 \mathcal{W}_2 + \mathcal{W}_2 \mathcal{N}_2 \mathcal{W}_2,$$
in order to prove Claim 4.4 we prove
$$\mathcal{W}_1 \mathcal{N}_1 \mathcal{W}_2 = \left( \sum_{e \in E} y_e \mathbf{p}_e W_e \right) \left( \sum_{e \in E} y_e M_e \right)^{-1} \left( \sum_{e \in E} y_e Z_e \mathbf{p}_1 \right) = O_{\frac{g}{2}}(1),$$
and
$$\mathcal{W}_1 \mathcal{N}_2 \mathcal{W}_2 = O_{\frac{g}{2}}(1), \quad \mathcal{W}_2 \mathcal{N}_1 \mathcal{W}_2 = O_{\frac{g}{2}}(1), \quad \mathcal{W}_2 \mathcal{N}_2 \mathcal{W}_2 = O_{\frac{g}{2}}(1).$$
For $y_1, \ldots, y_m$ large enough, since $\mathcal{M}_2, \mathcal{M}_1, \mathcal{M}_2$ are bounded, we have the following expressions:
$$\mathcal{N}_1 = (\mathcal{M}_1 - \mathcal{M}_2 \mathcal{M}_1^{-1} \mathcal{M}_2)^{-1}, \quad \mathcal{N}_2 = (\mathcal{M}_2 - \mathcal{M}_2 \mathcal{M}_1^{-1} \mathcal{M}_1)^{-1}$$
$$\mathcal{N}_2 = -\mathcal{M}_2 \mathcal{M}_1^{-1} \mathcal{M}_2 - \mathcal{M}_1 \mathcal{M}_1^{-1} \mathcal{M}_2, \quad \text{and}$$
$$\mathcal{N}_2 = -\mathcal{M}_2 \mathcal{M}_1^{-1} \mathcal{M}_2 - \mathcal{M}_1 \mathcal{M}_1^{-1} \mathcal{M}_2.$$

Note that $\mathcal{M}_2(s) = \Omega_0^2(s)$ for $s \in U$, and by our assumption on $U$, the matrices $\mathcal{M}_2^{-1}(s)$ lies in a compact set for $s \in U$. Thus, $\mathcal{N}_1 = \mathcal{M}_2(s) + \sum_e y_e M_e$ for an $h \times h$ matrix-valued map $\mathcal{A}$ on $U$ taking values in a compact set provided that $y_1, \ldots, y_m$ are large. It follows from the case in the result $g = h$ that
$$\mathcal{W}_1 \mathcal{N}_1 \mathcal{W}_2 = \left( \sum_{e \in E} y_e \mathbf{p}_e W_e \right) \left( \sum_{e \in E} y_e M_e \right)^{-1} \left( \sum_{e \in E} y_e Z_e \mathbf{p}_1 \right) = O_{\frac{g}{2}}(1).$$

The boundedness of the other three quantities can be proved similarly. For example, to treat the term $\mathcal{W}_1 \mathcal{N}_2 \mathcal{W}_2$, we observe first that $\mathcal{C} = \mathcal{M}_2(\mathcal{M}_2 - \mathcal{M}_1 \mathcal{M}_2^{-1} \mathcal{M}_1)^{-1}$ lies in a bounded compact set provided that $y_1, \ldots, y_m$ are large enough. We have
$$\mathcal{W}_1 \mathcal{N}_2 \mathcal{W}_2 = -\mathcal{W}_1 \mathcal{M}_1^{-1} \mathcal{C} = -\mathcal{W}_1 \mathcal{M}_1^{-1} (\mathcal{C}_1 - \mathcal{C}_2),$$
with $\mathcal{C}_2 = \sum_{e \in E} y_e Z_e \mathbf{p}_1$ and $\mathcal{C}_1 = \mathcal{C} + \mathcal{C}_2$.

Applying the result in the case $g = h$, we have for both the quantities for $k = 1, 2$
$$\mathcal{W}_1 \mathcal{N}_1 \mathcal{C}_k = \left( \sum_{e \in E} y_e \mathbf{p}_e W_e \right) \left( \sum_{e \in E} y_e M_e \right)^{-1} \left( \sum_{e \in E} y_e Z_e \mathbf{p}_1 \right) = O_{\frac{g}{2}}(1).$$
Taking now their difference shows what we wanted to prove. \qed

This finishes the proof of Theorem 4.3 To prove Theorem 1.2 we remark that by [1], the expression on the right hand side of Theorem 4.3 is precisely the right hand side term in Theorem 1.2.

References

[1] O. Amini, S. Bloch, J. I. Burgos Gil, J. Fresan, Feynman amplitudes and limits of heights, Izvestiya: Mathematics 80 (2016), no. 5, special issue in honor of J-P. Serre, to appear.
[2] J. I. Burgos Gil, R. de Jong, D. Holmes, Singularity of the biextension metric for families of abelian varieties, preprint arxiv:1604.00686.
[3] S. Chaiken, A combinatorial proof of the all minors matrix tree theorem, SIAM J. Algebraic Discrete Methods 3 (1982), no. 3, 319–329.
[4] R. J. Eden, P. V. Landshoff, D. I. Olive, and J. C. Polkinghorne, The Analytic S-Matrix, Cambridge University Press, London-New York-Ibadan, 1966.
[5] R. Hain, Biextensions and heights associated to curves of odd genus, Duke Math. J. 61 (1990), no. 3, 859–898.
[6] C. Itzykson and J.-B. Zuber, Quantum Field Theory, International Series in Pure and Applied Physics, McGraw-Hill International, New York, 1980.
[7] G. Kirchhoff, Über die Auflösung der Gleichungen, auf welche man bei der Untersuchung der linearen Vertheilung galvanischer Ströme geführt wird, Annalen der Physik 148 (1847), no. 12, 497–508.

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