SOME AUTOMORPHISM GROUPS ARE LINEAR ALGEBRAIC

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Abstract. Consider a normal projective variety $X$, a linear algebraic subgroup $G$ of $\text{Aut}(X)$, and the field $K$ of $G$-invariant rational functions on $X$. We show that the subgroup of $\text{Aut}(X)$ that fixes $K$ pointwise is linear algebraic. If $K$ has transcendence degree 1 over $k$, then $\text{Aut}(X)$ is an algebraic group.

1. Introduction

Let $X$ be a projective variety over an algebraically closed field $k$. The automorphism group $\text{Aut}(X)$ has a natural structure of smooth $k$-group scheme, locally of finite type (see [Gro61, p. 268]). But $\text{Aut}(X)$ is not necessarily an algebraic group; equivalently, it may have infinitely many components, e.g. when $X$ is a product of two isogenous elliptic curves. Still, the automorphism group is known to be a linear algebraic group for some interesting classes of varieties including smooth Fano varieties, complex almost homogeneous manifolds (this is due to Fu and Zhang, see [FZ13, Thm. 1.2]) and normal almost homogeneous varieties in arbitrary characteristics (see [Bri19, Thm. 1]; we say that $X$ is almost homogeneous if it admits an action of a smooth connected algebraic group with an open dense orbit). In this note, we obtain a relative version of the latter result:

Theorem 1. Let $X$ be a normal projective variety, $G \subset \text{Aut}(X)$ a linear algebraic subgroup, $K := k(X)^G$ the field of $G$-invariant rational functions on $X$, and $\text{Aut}_K(X)$ the subgroup of $\text{Aut}(X)$ fixing $K$ pointwise. Then $\text{Aut}_K(X)$ is a linear algebraic group as well.

By a theorem of Rosenlicht (see [Ros63], and [BGR17] for a modern proof of a more general result), the rational functions in $K$ separate the $G$-orbit closures of general points of $X$. Thus, $\text{Aut}_K(X)$ is the largest subgroup of $\text{Aut}(X)$ having the same general orbit closures as $G$. Also, note that $K = k$ if and only if $X$ is almost homogeneous; then $\text{Aut}_K(X)$ is just the full automorphism group.

The proof of Theorem 1 is presented in Section 2. As in [FZ13, Bri19], the idea is to construct a big line bundle on $X$, invariant under $\text{Aut}_K(X)$. To
handle our relative situation, we use some tools from algebraic geometry over an arbitrary field, which seem to be unusual in this setting.

As an application of Theorem 1 and its proof, we show that \( \text{Aut}(X) \) is a linear algebraic group under additional assumptions:

**Theorem 2.** Let \( X \) be a normal projective variety, \( G \subseteq \text{Aut}(X) \) a linear algebraic subgroup, and \( K := k(X)^G \). If \( K \) has transcendence degree 1 over \( k \), then \( \text{Aut}(X) \) is an algebraic group. If in addition \( G \) is connected and \( K \) is not the function field of an elliptic curve, then \( G \) is linear.

This is again due (in essence) to Fu and Zhang when \( X \) is a complex manifold and \( K \cong \mathbb{C}(t) \), see [FZ13, Appl. 1.4]. Another instance where this result was known occurs when \( G \) is a torus, say \( T \). Then \( X \) is said to be a \( T \)-variety of complexity one; its automorphism group is explicitly described in [AHHL14], when \( K \cong \mathbb{C}(t) \) again.

The proof of Theorem 2 is presented in Section 3. As a direct consequence of this theorem and the main result of [Pop16], we obtain:

**Corollary 3.** Let \( X \) be a normal projective surface having a nontrivial action of a connected linear algebraic group. Then \( \text{Aut}(X) \) is an algebraic group. Moreover, \( \text{Aut}(X) \) is linear unless \( X \) is birationally equivalent to \( Y \times \mathbb{P}^1 \) for some elliptic curve \( Y \).

For smooth rational surfaces, this result is due to Harbourne (see [Har87, Cor. 1.4]); for ruled surfaces over an elliptic curve, it also follows from the explicit description of automorphism groups obtained by Maruyama (see [Mar71, Thm. 3]). Note that the linearity assumption for the acting group cannot be suppressed, as shown again by the example of a product of two isogenous elliptic curves. Also, there exist (much more elaborate) examples of smooth projective surfaces having a discrete, non-finitely generated automorphism group (see [DO19, Ogu19]).

Theorem 1 may be viewed as a first step towards “Galois theory” for linear algebraic subgroups of \( \text{Aut}(X) \). With this in mind, it would be interesting to characterize the fields of invariants of linear algebraic groups among the subfields \( K \subseteq k(X) \), and the linear algebraic groups which occur as \( \text{Aut}_K(X) \).

In characteristic 0, it is easy to see that a subfield \( K \subseteq k(X) \) is the field of invariants of a connected linear algebraic group if and only if \( K \) is algebraically closed in \( k(X) \), \( k(X) \)-vector space \( \text{Der}_K(k(X)) \) is spanned by global vector fields, and \( X \) is unirational over \( K \).

**Notation and conventions.** The ground field \( k \) is algebraically closed, of arbitrary characteristic. A *variety* is an integral separated scheme of finite type. An *algebraic group* \( G \) is a group scheme of finite type. The *neutral component* \( G^0 \) is the connected component of \( G \) containing the neutral element. We say that \( G \) is a *linear algebraic group* if it is smooth and affine.
2. Proof of Theorem

It proceeds via a sequence of reduction steps and lemmas. We begin with two easy and useful observations:

Lemma 4. (i) $\text{Aut}_K(X)$ is a closed subgroup of $\text{Aut}(X)$.

(ii) If $G$ is connected, then $K$ is algebraically closed in $k(X)$.

Proof. (i) It suffices to show that the stabilizer $\text{Aut}_f(X)$ is closed in $\text{Aut}(X)$ for any nonzero $f \in k(X)$. Let

$$\text{Aut}_f(X) := \{ g \in \text{Aut}(X) \mid g^*(f) = \lambda(g) f \text{ for some } \lambda(g) \in k^* \}. $$

Then $\text{Aut}_f(X) \subset \text{Aut}(f(X)) \subset \text{Aut}(X)$.

Denote by $D_0$ (resp. $D_\infty$) the scheme of zeroes (resp. poles) of $f$, and by $\text{Aut}(X, D_0, D_\infty) \subset \text{Aut}(X)$ the common stabilizer of these subschemes of $X$. We claim that $\text{Aut}_f(X) = \text{Aut}(X, D_0, D_\infty)$. Indeed, the inclusion $\text{Aut}_f(X) \subset \text{Aut}(X, D_0, D_\infty)$ is obvious, and the opposite inclusion follows from the fact that every $g \in \text{Aut}(X, D_0, D_\infty)$ satisfies $\text{div}(g^*(f)) = \text{div}(f)$.

By the claim, $\text{Aut}_f(X)$ is closed in $\text{Aut}(X)$. Moreover, $\text{Aut}_f(X)$ stabilizes the open subset $U := X \setminus (D_0 \cup D_\infty) \subset X$, and $\text{Aut}_f(X)$ is the stabilizer of $f \in \mathcal{O}(U)$. So $\text{Aut}_f(X)$ is closed in $\text{Aut}(f(X))$.

(ii) Let $f \in k(X)$ be algebraic over $K$. Then the stabilizer of $f$ in $G$ is a closed subgroup of finite index, and hence is the whole $G$. So $f \in K$. 

Step 1. We may assume that $G$ is connected.

Proof. Denote by $\pi_0(G) := G/G^0$ the group of components. Then the invariant field $L := k(X)^{G_0}$ is equipped with an action of the finite group $\pi_0(G)$, and $L^{\pi_0(G)} = K$. Thus, $L/K$ is a Galois extension with Galois group a quotient of $\pi_0(G)$. As $G^0$ is connected, $L$ is algebraically closed in $k(X)$. Therefore, $L$ is the algebraic closure of $K$ in $k(X)$, and hence is stable under $\text{Aut}_K(X)$. Since the composition $G \to \text{Aut}_K(X) \to \text{Aut}_K(L)$ is surjective, this yields an exact sequence

$$1 \to \text{Aut}_L(X) \to \text{Aut}_K(X) \to \text{Aut}_K(L) \to 1,$$

and in turn the assertion in view of Lemma 4.

Lemma 5. $\text{Aut}_K(X)^0$ is a linear algebraic group.

Proof. Let $\tilde{G} := \text{Aut}_K(X)^0$. Then $G \subset \tilde{G}$ and $k(X)^{\tilde{G}} = K$. In view of Rosenlicht’s theorem mentioned in the introduction, there exists a dense open $G$-stable subset $U \subset X$ such that the orbit $G \cdot x$ is open in $\tilde{G} \cdot x$ for all $x \in U$. Since $G$ is connected and linear, it follows that the variety $\tilde{G} \cdot x$ is unirational, and hence its Albanese variety $\text{Alb}(\tilde{G} \cdot x)$ is trivial.
We now recall a group-theoretic description of $\text{Alb}(\tilde{G} \cdot x)$. By Chevalley’s structure theorem, $\tilde{G}$ has a largest connected linear normal subgroup $\tilde{G}_{\text{aff}}$ and the quotient $\tilde{G}/\tilde{G}_{\text{aff}}$ is an abelian variety (see e.g. [BSU13, Thm. 1.1.1]). Denote by $H = \tilde{G}_x$ the stabilizer of $x$; then $\tilde{G}/(\tilde{G}_{\text{aff}} H)$ is an abelian variety as well. As the Albanese morphism of $\tilde{G}$ (resp. of $\tilde{G} \cdot x \simeq \tilde{G}/H$) is invariant by $\tilde{G}_{\text{aff}}$, it follows readily that $\text{Alb}(\tilde{G}) = \tilde{G}/\tilde{G}_{\text{aff}}$ and $\text{Alb}(\tilde{G} \cdot x) = \tilde{G}/(\tilde{G}_{\text{aff}} H)$.

Thus, $\tilde{G} = \tilde{G}_{\text{aff}} H$. But since $H$ is affine (see e.g. [BSU13, Lem. 2.3.2]), the natural map $\tilde{G}/\tilde{G}_{\text{aff}} \to \tilde{G}/\tilde{G}_{\text{aff}} H$ is an isogeny. Therefore, $\tilde{G} = \tilde{G}_{\text{aff}}$. \hfill \square

**Step 2.** In view of Step 1 and Lemma 3 we may assume that $G = \text{Aut}_K(X)^0$; then $K$ is algebraically closed in $k(X)$.

We may further assume that there exists a morphism $f : X \to Y$ where $Y$ is a normal projective variety, such that $f$ induces an isomorphism $k(Y) \simeq K$, we have $f_*(\mathcal{O}_X) = \mathcal{O}_Y$, and $\text{Aut}_K(X)$ is isomorphic to a closed subgroup of the relative automorphism group $\text{Aut}_Y(X)$, containing $\text{Aut}_Y(X)^0$.

**Proof.** Choose generators $f_1, \ldots, f_n$ of the field extension $K/k$. This defines a rational map $(f_1, \ldots, f_n) : X \dashrightarrow \mathbb{P}^{n-1}$ and hence a rational map $f : X \dashrightarrow Y$, where $Y$ is a normal projective variety and $f$ induces an isomorphism $k(Y) \simeq K$. The normalization of the graph of $f$ yields a normal projective variety $X'$ equipped with morphisms $f' : X' \to Y$, $g : X' \to X$ such that $g$ is birational and $f' = f \circ g$. Thus, $f'$ also induces an isomorphism $k(Y) \simeq K$. Consider the Stein factorization of $f'$ as $X' \xrightarrow{\psi} Z \xrightarrow{\psi} Y$. Then $\psi$ is finite, and hence so is the extension $k(Z)/k(Y)$. But $k(Z) \subset k(X') \simeq k(X)$, and $k(Y) = K$ is algebraically closed in $k(X)$. Therefore, $\psi$ is birational. As $Y$ is normal, $\psi$ is an isomorphism by Zariski’s Main Theorem. Thus, $f_*(\mathcal{O}_X) = \mathcal{O}_Y$.

By construction, $\text{Aut}_K(X)$ acts on $X'$ and $f'$ is invariant under this action. This yields a morphism

$$u : \text{Aut}_K(X) \to \text{Aut}_Y(X').$$

On the other hand, Blanchard’s lemma (see e.g. [BSU13, Prop. 4.2.1]) yields a morphism $\text{Aut}(X')^0 \to \text{Aut}(X)^0$, which restricts to a morphism $g_* : \text{Aut}_Y(X')^0 \to \text{Aut}_K(X)^0$.

Since $g$ is birational, $g_*$ is the inverse of the restriction to neutral components $u^0_0 : \text{Aut}_K(X)^0 \to \text{Aut}_Y(X)^0$. In particular, the image of $u$ contains $\text{Aut}_Y(X')^0$, and hence is closed in $\text{Aut}_Y(X')$. \hfill \square

**Step 3.** In view of Step 2, we now consider a contraction $f : X \to Y$, i.e., a morphism of normal projective varieties such that $f_*(\mathcal{O}_X) = \mathcal{O}_Y$. We assume that the algebraic group $G := \text{Aut}_Y(X)^0$ is linear, and $f$ induces
an isomorphism \( k(Y) \simeq K \). To prove Theorem \( \text{[1]} \) it suffices to show that \( \text{Aut}_Y(X) \) is a linear algebraic group.

It suffices in turn to construct a big line bundle \( L \) on \( X \) that admits an \( \text{Aut}_Y(X) \)-linearization (see e.g. [KM98, 2.5] for the notion of bigness). Indeed, \( \text{Aut}_Y(X) \) is closed in \( \text{Aut}(X) \) (as follows from Lemma \( \text{[1]} \)), and hence in the stabilizer \( \text{Aut}(X, [L]) \) of the isomorphism class of \( L \) for any \( \text{Aut}_Y(X) \)-linearized line bundle \( L \). If in addition \( L \) is big, then \( \text{Aut}(X, [L]) \) is a linear algebraic group in view of [Bri19, Lem. 2.3]. The desired line bundle \( L \) will be constructed in Step 6, after further preparations.

Denote by \( \eta \) the generic point of \( Y \) and by \( X_\eta \) the generic fiber of \( f \). Then \( X_\eta \) is a projective scheme over \( k(\eta) = K \), equipped with an action of the \( K \)-group scheme \( G_K \) (since \( G \) acts on \( X \) by relative automorphisms). Likewise, the geometric generic fiber \( X_{\bar{\eta}} \) is a projective scheme over \( k(\bar{\eta}) = \bar{K} \), equipped with an action of \( G_{\bar{K}} \) (a linear algebraic group over \( \bar{K} \)).

**Lemma 6.** With the above assumptions, \( X_\eta \) is normal and geometrically integral. Moreover, \( X_{\bar{\eta}} \) is almost homogeneous under \( G_\eta \) and we have \( k(X_{\bar{\eta}}) = k(X) \otimes_K \bar{K} \).

**Proof.** By [Spr89] Lem. IV.1.5, the extension \( k(X)/K \) is separable. As a consequence, the ring \( k(X) \otimes_K \bar{K} \) is reduced (see e.g. [SP19, 10.42.5]). So \( X_\eta \) is geometrically reduced. As \( K \) is algebraically closed in \( k(X) \), the spectrum of \( k(X) \otimes_K \bar{K} \) is irreducible (see e.g. [SP19, 10.46.8]). Hence \( X_\eta \) is geometrically irreducible and the field of rational functions \( \bar{K}(X_{\bar{\eta}}) \) equals \( k(X) \otimes_K \bar{K} \). Thus,

\[ \bar{K}(X_{\bar{\eta}})^{G_{\bar{\eta}}} = (k(X) \otimes_K \bar{K})^{G_{\bar{\eta}}} \subset (k(X) \otimes_K \bar{K})^G = \bar{K}, \]

where \( G \) is identified with its group of \( k \)-rational points. So \( \bar{K}(X_{\bar{\eta}})^{G_{\bar{\eta}}} = k(\bar{\eta}) \). By Rosenlicht’s theorem, it follows that \( X_{\bar{\eta}} \) is almost homogeneous under \( G_{\bar{\eta}} \).

It remains to show that \( X_\eta \) is normal. Consider an open affine subset \( V \) of \( Y \) and an open affine cover \( (U_i) \) of \( f^{-1}(V) \). Then the \( (U_i)_{\eta} \) form an open affine cover of \( X_\eta \) and \( \mathcal{O}((U_i)_{\eta}) = \mathcal{O}(U_i) \otimes_{\mathcal{O}(V)} k(\eta) \) is a localization of \( \mathcal{O}(U_i) \), hence a normal domain. \( \square \)

**Remark 7.** The generic fiber \( X_\eta \) is geometrically normal if \( \text{char}(k) = 0 \). But this fails if \( \text{char}(k) = p > 0 \), as shown by the following example: let \( G = \mathbb{G}_a \) act on \( X = \mathbb{P}^2 \) via \( t \cdot [x : y : z] = [x + ty + t^{2p}z : y : z] \). Then \( K = k(\frac{z}{x}) \) and the \( G \)-orbit closure of \( [x : y : z] \) is singular at \( \infty \) whenever \( z \neq 0 \). Likewise, the geometric generic fiber \( X \times_{\text{Spec}(K)} \text{Spec}(\bar{K}) \) is a singular curve.

**Step 4.** We may further assume that \( X(k(\eta)) \) contains a point \( x_0 \) such that the orbit \( G_{\bar{\eta}} \cdot x_0 \) is open in \( X_{\bar{\eta}} \).

**Proof.** Denote by \( K^* \) the separable closure of \( K \) in \( \bar{K} \). Then \( X(K^*) \) is dense in \( X_{\bar{\eta}} = X_{\bar{K}} \), since the latter is integral. So we may find \( x_0 \in X(K^*) \) such
that $G_{\eta} \cdot x_0$ is open in $X_{\bar{\eta}}$. Then $x_0 \in X(K')$ for some finite Galois extension $K'/K$.

Denote by $\Gamma$ the Galois group of $K'/K$, by $Y'$ the normalization of $Y$ in $K'$, and by $X'$ the normalization of $X$ in $k(X) \otimes_K K'$ (the latter is a field as a consequence of Lemma 6). Then we have a commutative square

\[
\begin{array}{ccc}
X' & \longrightarrow & Y' \\
\downarrow g & & \downarrow h \\
X & \longrightarrow & Y,
\end{array}
\]

where $g, h$ are the quotients by $\Gamma$. Denoting by $\eta'$ the generic point of $Y'$, this yields a commutative square

\[
\begin{array}{ccc}
X'_{\eta'} & \longrightarrow & \eta' \\
\downarrow & & \downarrow \\
X_{\eta} & \longrightarrow & \eta,
\end{array}
\]

where the vertical arrows are quotients by $\Gamma$ again, and $x_0$ is a section of the top horizontal arrow. Since the right vertical arrow is a $\Gamma$-torsor, so is the left vertical arrow and the square (2.2) is cartesian. Therefore,

\[
X'_{\eta'} = X'_{\eta'} \times_{\eta'} \bar{\eta}' = (X_{\eta} \times_{\eta} \eta') \times_{\eta'} \bar{\eta}' = X_{\eta} \times_{\eta} \bar{\eta} = X_{\bar{\eta}}.
\]

In particular, $X'_{\eta'}$ is normal and geometrically integral (Lemma 6).

It follows that $f'_*(\mathcal{O}_{X'}) = \mathcal{O}_{Y'}$ by an argument of Stein factorization as in Step 2. More specifically, $f'$ is the composition

\[
X' \longrightarrow \text{Spec } f'_*(\mathcal{O}_{X'}) =: Z' \longrightarrow Y',
\]

where $Z'$ is a variety and $g'$ is finite. Moreover, $Z'_{\eta'}$ is geometrically integral and the finite morphism $Z'_{\eta'} \rightarrow \eta'$ has a section (since so has $X'_{\eta'} \rightarrow \eta'$). Thus, $g'$ is an isomorphism at $\eta'$. In view of the normality of $Y'$ and Zariski’s Main Theorem, we conclude that $g'$ is an isomorphism.

Next, we construct an isomorphism

\[
\begin{array}{ccc}
\text{Aut}^\Gamma(X') & \longrightarrow & \text{Aut}(X),
\end{array}
\]

where the right-hand side denotes the subgroup of $\Gamma$-invariants in $\text{Aut}(X')$. For any scheme $S$, (2.1) yields a commutative square

\[
\begin{array}{ccc}
X' \times S & \longrightarrow & Y' \times S \\
\downarrow & & \downarrow \\
X \times S & \longrightarrow & Y \times S,
\end{array}
\]
where the vertical arrows are again quotients by \( \Gamma \). Since these quotients are categorical (see e.g. [Mum08, §12, Thm. 1]), every \( \Gamma \)-equivariant automorphism of \( X' \times S \) over \( S \) induces an automorphism of \( X \times S \) over \( S \). This yields a morphism of (abstract) groups

\[
\text{Aut}_S^\Gamma(X' \times S) \longrightarrow \text{Aut}_S(X \times S)
\]

which is clearly functorial in \( S \). Thus, we obtain a morphism of automorphism group schemes \( u : \text{Aut}_X^\Gamma \longrightarrow \text{Aut}_X \) with an obvious notation. The induced morphism of Lie algebras is the natural map \( \text{Der}^\Gamma(\mathcal{O}_{X'}) \rightarrow \text{Der}(\mathcal{O}_X) \) with an obvious notation again. The kernel of this map is contained in \( \text{Der}_{k(X)}(k(X')) \), and hence is zero since \( k(X') \) is separable algebraic over \( k(X) \). Restricting \( u \) to the reduced subscheme of \( \text{Aut}_X^\Gamma \) yields a morphism

\[
v : \text{Aut}_X^\Gamma(X') \longrightarrow \text{Aut}(X).
\]

The set-theoretic kernel of \( v \) is contained in \( \text{Aut}_{k(X)}^\Gamma(k(X')) \), and hence is trivial by Galois theory. Thus, \( v \) is a closed immersion. Moreover, for any \( g \in \text{Aut}(X) \), we have a \( \Gamma \)-equivariant automorphism \( g \otimes \text{id} \) of \( k(X) \otimes_K K' = k(X') \), which stabilizes the normalization of \( \mathcal{O}_X \) and hence yields a lift of \( g \) in \( \text{Aut}_X^\Gamma(X') \). So \( v \) is surjective on \( k \)-rational points. This yields the desired isomorphism \([2, 3]\).

By construction, \( v \) restricts to an isomorphism \( \text{Aut}_X^\Gamma(X') \cong \text{Aut}_Y(X) \). In particular, if \( \text{Aut}_Y(X') \) is linear algebraic, then so is \( \text{Aut}_Y(X) \).

Since \( G = \text{Aut}_Y(X)^0 \) has an open orbit in the general fibers of \( f \), it follows that the connected algebraic group \( G' : = \text{Aut}_Y(X')^0 \) has an open orbit in the general fibers of \( f' \). As a consequence, \( K' = k(X')^{G'} \). Likewise, the orbit \( G'_M \cdot x_0 \) is open in \( X'_M \).

To complete the proof, it remains to show that if \( X' \) admits a big line bundle \( L' \) which is \( \text{Aut}_Y(X') \)-linearized, then \( X \) admits a big line bundle \( L \) which is \( \text{Aut}_Y(X) \)-linearized. This follows from a norm argument. More specifically, let \( M : = \bigotimes_{\gamma \in \Gamma} \gamma^*(L') \). Then \( M \) is a big line bundle on \( X' \) and is \( \text{Aut}_Y(X')^\Gamma \)-linearized. Moreover, \( M = g^*(L) \) for a line bundle \( L \) on \( X \) (the norm of \( M \), see [EGA II.6.5]); we have \( L = g_*(M)^\Gamma \). Thus, \( L \) is \( \text{Aut}_Y(X) \)-linearized in view of the isomorphism \( \text{Aut}_Y(X) \cong \text{Aut}_Y(X')^\Gamma \). Furthermore, we have

\[
H^0(X', M^{\otimes n}) = H^0(X', g^*(L)^{\otimes n}) = H^0(X, L^{\otimes n} \otimes g_*(\mathcal{O}_{X'}))
\]

for any integer \( n \). Since \( g_*(\mathcal{O}_{X'})^\Gamma = \mathcal{O}_X \), it follows that

\[
H^0(X', M^{\otimes n})^\Gamma = H^0(X, L^{\otimes n}).
\]

Thus, the section ring

\[
R(X, L) : = \bigoplus_{n=0}^{\infty} H^0(X, L^{\otimes n})
\]
satisfies $R(X,L) = R(X',M)^\Gamma$. As a consequence, the fraction fields of the domains $R(X,L)$ and $R(X',M)$ have the same transcendence degree over $k$. Since $M$ is big, it follows that $L$ is big as well. \qed

**Step 5.** Let $d := \dim(X) - \dim(Y)$; this is the maximal dimension of the $G$-orbits in $X$. Let $X_0$ denote the set of $x \in X$ such that the orbit $G \cdot x$ has dimension $d$; then $X_0$ is an open $G$-stable subset of $X$. Since $\text{Aut}_Y(X)$ normalizes $G$, it stabilizes $X_0$ as well.

We first assume that $\text{char}(k) = 0$. Denote by $\mathfrak{g}$ the Lie algebra of $G$; then $\text{Aut}_Y(X)$ acts on $\mathfrak{g}$ via its conjugation action on $G$. Consider the Tits morphism (see [Hab74, HO84])

$$\tau_0 : X_0 \longrightarrow \text{Grass}^d(\mathfrak{g}), \quad x \longmapsto \mathfrak{g}_x,$$

where $\mathfrak{g}_x \subset \mathfrak{g}$ denotes the Lie algebra of the stabilizer of $x$, and Grass$^d(\mathfrak{g})$ stands for the Grassmannian of subspaces of $\mathfrak{g}$ of codimension $d$. Then $\tau_0$ is equivariant for the natural actions of $\text{Aut}_Y(X)$. We view $\tau_0$ as a rational map $X \dasharrow \text{Grass}^d(\mathfrak{g})$, and denote by $X'$ the normalization of its graph. Then $X'$ is equipped with morphisms $g : X' \to X$, $\tau' : X' \to \text{Grass}^d(\mathfrak{g})$ such that $g$ restricts to an isomorphism above $X_0$, and $\tau' = \tau_0 \circ g$. Moreover, the action of $\text{Aut}_Y(X)$ on $X$ lifts to an action on $X'$ such that $g$ and $\tau'$ are equivariant. Arguing as at the end of Step 2, one may check that $G \simeq \text{Aut}_Y(X')^0$ and the image of $\text{Aut}_Y(X)$ in $\text{Aut}_Y(X')$ is closed. Since $f \circ g$ is a contraction, this yields a reduction to the case where $\tau_0$ extends to a morphism

$$\tau : X \longrightarrow \text{Grass}^d(\mathfrak{g}).$$

Next, we assume that $\text{char}(k) = p > 0$. For any integer $n > 0$, consider the $n$th iterated Frobenius morphism

$$F_{G/k}^n : G \longrightarrow G^{(p^n)}$$

and its kernel $G_n$. Then $G_n$ is an infinitesimal group scheme, characteristic in $G$, and its formation commutes with base change; see e.g. [SGA3, VIIA.4.1]. (Note that $G_1$ is the infinitesimal group scheme of height 1 corresponding to the Lie algebra of $G$). The stabilizer $\text{Stab}_{G_n}$ of the $G_n$-action on $X$ is a subgroup scheme of $G_{n,X} = G_n \times X$, stable by the diagonal action of $\text{Aut}_Y(X)$. Since $X$ is integral, it has a largest open subset $U = U_n$ over which $\text{Stab}_{G_n}$ is flat, or equivalently locally free. Then $U$ is stable by $\text{Aut}_Y(X)$; moreover, since $\mathcal{O}_{\text{Stab}_{G_n}}$ is a quotient of $\mathcal{O}(G_n) \otimes \mathcal{O}_X$, we obtain a morphism

$$\tau_n : U \longrightarrow \text{Grass}^m \mathcal{O}(G_n), \quad x \longmapsto \text{Stab}_{G_n}(x)$$

for some $m \geq 1$. By construction, $\tau_n$ is $\text{Aut}_Y(X)$-equivariant. As above, we may reduce to the case where $\tau_n$ extends to a morphism

$$\tau : X \longrightarrow \text{Grass}^m \mathcal{O}(G_n).$$
Step 6. Consider again the open $\text{Aut}_Y(X)$-stable subset $X_0 \subset X$ consisting of $G$-orbits of maximal dimension $d$. Replacing $X$ with the normalized blow-up of $X \setminus X_0$, we may assume that there is an effective Cartier divisor $\Delta$ on $X$ such that $X \setminus \text{Supp}(\Delta)$ consists of $G$-orbits of dimension $d$, and $\mathcal{O}_X(\Delta)$ is $\text{Aut}_Y(X)$-linearized; in particular, $\text{Supp}(\Delta)$ is $\text{Aut}_Y(X)$-stable.

Let $\tau$ be the morphism as in (2.4) (if char($k) = 0$) or in (2.5) (if char($k) = p$). Denote by $M$ the Plücker line bundle on the Grassmannian and let $L := \tau^*(M)$. Then $L$ is a line bundle on $X$, equipped with an $\text{Aut}_Y(X)$-linearization.

Lemma 8. With the above notation, the line bundle $L(\Delta)_\eta$ on $X_\eta$ is big (for $n \gg 0$ if char($k$) = $p$).

Proof. Choose $x_0 \in X(k(\eta))$ such that $G_\eta \cdot x_0$ is open in $X_\eta$ (Step 4); then $G_\eta \cdot x_0 = (X_0)_\eta$. Denote by $H$ the stabilizer of $x_0$ in $G_\eta$ and by $\mathfrak{h}$ its Lie algebra. If char($k) = 0$, then we have the equality of normalizers $N_{G_\eta}(H^0) = N_{G_\eta}(\mathfrak{h})$. On the other hand, if char($k) > 0$ then $N_{G_\eta}(H^0) = N_{G_\eta}(H_n)$ for $n \gg 0$, in view of [Bri19, Lem. 3.1]. The assertion follows from this by arguing as at the very end of the proof of Theorem 1 in [Bri19].

We say that a line bundle on $X$ is $f$-big, if its pull-back to the generic fiber $X_\eta$ is big (this notion turns out to be equivalent to that of [KM98, Def. 3.22]). We will need the following observation:

Lemma 9. Let $f : X \to Y$ be a morphism of projective varieties, $L$ a line bundle on $X$, and $M$ a line bundle on $Y$. Denote by $\eta$ the generic point of $Y$. Assume that $L$ is $f$-big and $M$ is ample. Then $L \otimes f^*(M^{\otimes n})$ is big for $n \gg 0$.

Proof. We adapt the argument of [CCP08, Lem. 2.4]. Let $A$ be an ample line bundle on $X$, and $m$ a positive integer. Then $f_*(L^{\otimes m} \otimes A^{-1})$ is a coherent sheaf on $Y$, and $f_*(L^{\otimes m} \otimes A^{-1})_\eta = H^0(X_\eta, L^{\otimes m} \otimes A^{-1})$ (see e.g. [Har77, III.9.4]). Since $L$ is big on $X_\eta$, it follows that $f_*(L^{\otimes m} \otimes A^{-1})_\eta \neq 0$ for $m \gg 0$ by arguing as in the proof of [KM98, Lem. 2.60]. As $M$ is ample on $Y$, we thus have $H^0(Y, f_*(L^{\otimes m} \otimes A^{-1}) \otimes M^{\otimes n}) \neq 0$ for $n \gg m \gg 0$. Equivalently, $H^0(X, L^{\otimes m} \otimes A^{-1} \otimes f^*(M^{\otimes n})) \neq 0$. So $L^{\otimes m} \otimes f^*(M^{\otimes n}) = A \otimes E$ for some effective line bundle $E$ on $X$. Thus, taking $n$ to be a large multiple of $m$ yields the statement in view of [KM98, Lem. 2.60] again.

By Lemmas 8 and 9, $X$ admits a big line bundle which is $\text{Aut}_Y(X)$-linearized. As seen in Step 3, this completes the proof of Theorem 1.

Remark 10. Assume that char($k) = 0$ and consider $f : X \to Y$ as in Step 2. Choose a desingularization $\psi : Y' \to Y$ and denote by $X'$ the normalization of the irreducible component of $X \times_Y Y'$ which dominates $Y$. By arguing as at the end of the proof of Step 2, one shows that $\text{Aut}_Y(X)$ is isomorphic to a closed subgroup of $\text{Aut}_{Y'}(X')$. So we may assume that $Y$ is smooth.
We may further assume that $X$ is smooth, in view of the existence of a canonical desingularization (see [Kol07, Thm. 3.36]; such a desingularization is $\text{Aut}(X)$-equivariant by the argument of [Kol07, Prop. 3.9.1]). Then the generic fiber $X_\eta$ is smooth (by generic smoothness, see e.g. [Har77, III.10.7]); also, $X_\bar{\eta}$ is almost homogeneous under $G_\bar{\eta}$ in view of Lemma 6. By [FZ13, Thm. 1.2], it follows that the anticanonical class $-K_{X_\bar{\eta}}$ is big, hence so is $-K_{X_\eta}$. Also, $\mathcal{O}_X(-K_X)$ is $\text{Aut}(X)$-linearized. Combining this with Lemmas 8 and 9, this yields a shorter proof of Theorem 1, in characteristic zero again.

3. PROOF OF THEOREM 2

Like that of Theorem 1, it goes through a succession of reduction steps.

Step 1. Let $Y$ be the smooth projective curve such that $K = k(Y)$. Then we may assume that the inclusion $k(Y) \subset k(X)$ comes from a contraction $f : X \to Y$, and $G = \text{Aut}_Y(X)^0$.

Proof. The field $k(X)^0/G^0$ is algebraic over $K$, and hence has transcendence degree 1 over $k$. Thus, we may replace $G$ with $G^0$, and hence assume $G$ connected.

Denote by $f : X \dashrightarrow Y$ the rational map associated with the inclusion $k(Y) \subset k(X)$. Arguing as in Step 2 of Section 2, we may further assume that $f$ is a morphism; then it is a contraction.

By Lemma 5 the group $\text{Aut}_Y(X)^0$ is linear; thus, we may also assume that $G = \text{Aut}_Y(X)^0$. □

Step 2. We may further assume that $X$ is not almost homogeneous under $\tilde{G} := \text{Aut}(X)^0$.

Proof. By Blanchard’s lemma again (see [BSU13, Prop. 4.2.1]), there is a unique action of $\tilde{G}$ on $Y$ such that $f$ is equivariant. This yields an exact sequence of group schemes

$$1 \to \tilde{G} \cap \text{Aut}_Y(X) \to \tilde{G} \to \text{Aut}(Y)^0,$$

where $\tilde{G} \cap \text{Aut}_Y(X)$ has neutral component $G$, and hence is affine.

If $X$ is almost homogeneous under $\tilde{G}$, then so is $Y$. Thus, either $Y \simeq \mathbb{P}^1$ or $Y$ is an elliptic curve. In the former case, we claim that every rational map from $X$ to an abelian variety is constant. Indeed, any such map $\varphi : X \dashrightarrow A$ is $G$-invariant. Since $k(X)^G = k(\mathbb{P}^1)$, it follows that $\varphi$ factors through a rational map $\mathbb{P}^1 \dashrightarrow A$, implying the claim.

By this claim, the open orbit of $\tilde{G}$ in $X$ has a trivial Albanese variety. Arguing as in Step 1 of Section 2, it follows that $\tilde{G}$ is linear. Thus, so is $\text{Aut}(X)$ in view of [Bri19, Thm. 1].
On the other hand, if $Y$ is an elliptic curve, then $f$ is the Albanese morphism of $X$, as follows by the above argument. Thus, there is a unique action of $\text{Aut}(X)$ on $Y$ such that $f$ is equivariant. This yields an exact sequence of group schemes

\[(3.1) \quad 1 \longrightarrow N \longrightarrow \text{Aut}(X) \xrightarrow{f^*} \text{Aut}(Y),\]

where the reduced subscheme of $N$ is $\text{Aut}_Y(X)$. In view of Theorem 1), it follows that $N$ is an affine algebraic group. Also, the image of $f_*$ contains $Y = \text{Aut}(Y)^0$, since $X$ is almost homogeneous under $\tilde{G}$. As $\text{Aut}(Y)/\text{Aut}(Y)^0$ is finite, we conclude that $\text{Aut}(X)$ is a nonlinear algebraic group.

**Step 3.** We may further assume that $G = \tilde{G} = \text{Aut}(X)^0$, and $\text{Aut}(X)$ acts on $Y$ so that we still have the exact sequence (3.1); in addition, $Y \cong \mathbb{P}^1$ or $Y$ is an elliptic curve.

**Proof.** By Rosenlicht’s theorem again, the subfield $k(X)^\tilde{G} \subset K$ has transcendence degree 1 over $K$. As $k(X)^G$ is algebraically closed in $k(X)$, it follows that $k(X)^\tilde{G} = K$ and hence $\tilde{G}$ is linear (Lemma 5).

Therefore, we may assume that $G = \text{Aut}(X)^0$. Then $\text{Aut}(X)$ stabilizes $K$, and hence acts on $Y$; this yields (3.1).

Denote by $g$ the genus of $Y$. If $g \geq 2$ then $\text{Aut}(Y)$ is finite, and hence $\text{Aut}(X)$ is a linear algebraic group. So we may further assume that $g \leq 1$. $\square$

**Step 4.** We may further assume that $X$ has an $\text{Aut}(X)$-linearized line bundle $L$ which is $f$-big and $f$-globally generated, i.e., the adjunction map

\[u : f^*f_*(L) \longrightarrow L\]

is surjective.

**Proof.** Arguing as in Steps 4, 5 and 6 of Section 2 we may assume that $X$ has an $\text{Aut}(X)$-linearized, $f$-big line bundle $L$; in addition, $H^0(X, L) \neq 0$. Then $f_*(L)$ is a coherent, torsion-free sheaf on $Y$, and hence is locally free. Thus, so is $f^*f_*(L)$; moreover, $u$ is a morphism of $\text{Aut}(X)$-linearized sheaves. Also, $u \neq 0$ as $L$ has nonzero global sections. Therefore, $u$ yields a nonzero morphism $L^{-1} \otimes f^*f_*(L) \to \mathcal{O}_X$. Its image is the ideal sheaf of a closed subscheme $Z \subset X$, stable by $\text{Aut}(X)$.

Denote by $\pi : X' \to X$ the normalization of the blow-up of $Z$ in $X$ and let $f' := f \circ \pi : X' \to Y$. Then the action of $\text{Aut}(X)$ on $X$ lifts to a unique action on $X'$; moreover, $\pi^*(L)$ is $\text{Aut}(X)$-linearized and we have $f'^*f_\pi^*(L) = \pi^*f^*f_*(L)$. The image of the adjunction map $u' : f'^*f_\pi^*(L) \to \pi^*(L)$ generates an invertible subsheaf $L' \subset \pi^*(L)$, as proved (in essence) in [Har77, Ex. II.7.17.3]. Note that $L'$ is $\text{Aut}(X)$-linearized and $f'$-globally generated.
We claim that $L'$ is $f'$-big, possibly after replacing $L$ with a positive tensor power $L^\otimes m$, and $L'$ with the associated subsheaf $L'_m \subset \pi^*(L^\otimes m)$. Indeed, denoting by $\eta$ the generic point of $Y$, we have

$$f^*_\eta(L^\otimes m) = H^0(X_\eta, L^\otimes m)$$

by [Har77, III.9.4], and likewise,

$$f'_\eta(\pi^*(L^\otimes m)) = H^0(X'_\eta, \pi^*(L^\otimes m)) = H^0(X_\eta, L^\otimes m).$$

Thus, $\pi^*(L)$ is $f'$-big. Moreover, denoting by $i : X_\eta \to X$ and $i' : X'_\eta \to X'$ the inclusions, we obtain

$$i^* f^* f'_* \pi^*(L^\otimes m) = f'_* \pi^*(L^\otimes m_\eta) \otimes O_{X'_\eta} = H^0(X'_\eta, \pi^*(L^\otimes m)) \otimes O_{X'_\eta}.$$  

Since $L'_m$ is the subsheaf of $\pi^*(L^\otimes m)$ generated by the image of $f'^* f'_* \pi^*(L^\otimes m)$, we see that $i^*(L'_m)$ is generated by the image of the natural map

$$H^0(X_\eta, L^\otimes m) \otimes O_{X'_\eta} \to i^* \pi^*(L^\otimes m).$$

As $i^* \pi^*(L^\otimes m)$ is big, this yields the claim.

This claim combined with [Bri19, Lem. 2.1] yields the desired reduction. \(\square\)

**Step 5.** Choose an ample line bundle $M$ on $Y$. Then $L \otimes f^*(M^\otimes n)$ is big for $n \gg 0$ (Lemma 3). We claim that $L \otimes f^*(M^\otimes n)$ is also globally generated for $n \gg 0$.

Indeed, since $M$ is ample, the sheaf $f^*_\eta(L) \otimes M^\otimes n$ is globally generated for $n \gg 0$. Equivalently, the map

$$H^0(Y, f^*_\eta(L) \otimes M^\otimes n) \otimes O_Y \to f^*_\eta(L) \otimes M^\otimes n$$

is surjective. Since $f$ is flat, the map

$$H^0(Y, f^*_\eta(L) \otimes M^\otimes n) \otimes O_X \to f^*(f^*_\eta(L) \otimes M^\otimes n)$$

is surjective as well. As $H^0(Y, f^*_\eta(L) \otimes M^\otimes n) = H^0(X, f^*(f^*_\eta(L) \otimes M^\otimes n))$, the sheaf $f^*(f^*_\eta(L) \otimes M^\otimes n)$ is globally generated. Thus, so is its quotient $L \otimes f^*(M^\otimes n)$, proving the claim.

We may now complete the proof of Theorem 2. Observe that $\text{Aut}(X)$ fixes the numerical equivalence class of $f^*(M^\otimes n)$ (since $\text{Aut}(Y)$ fixes that of $M^\otimes n$) and hence that of $L \otimes f^*(M^\otimes n)$. Thus, $\text{Aut}(X)$ is an algebraic group in view of [Bri19, Lem. 2.2].

Finally, if $Y \cong \mathbb{P}^1$ then $\text{Aut}(X)$ fixes the isomorphism class of $f^*O_{\mathbb{P}^1}(1)$. So $\text{Aut}(X)$ is linear algebraic by [Bri19, Lem. 2.3]. Otherwise, $G$ is connected and $K$ is the function field of an elliptic curve by Step 3.

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