Homotopy theory of net representations

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Abstract

The homotopy theory of representations of nets of algebras over a (small) category with values in a closed symmetric monoidal model category is developed. We illustrate how each morphism of nets of algebras determines a change-of-net Quillen adjunction between the model categories of net representations, which is furthermore a Quillen equivalence when the morphism is a weak equivalence. These techniques are applied in the context of homotopy algebraic quantum field theory with values in cochain complexes. In particular, an explicit construction is presented that produces constant net representations for Maxwell $p$-forms on a fixed oriented and time-oriented globally hyperbolic Lorentzian manifold.

Keywords: Net of algebras, net representation, algebraic quantum field theory, homotopy theory, gauge theory, BRST/BV formalism, Maxwell $p$-forms

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1 Introduction and summary

Historically quantum field theory on Minkowski spacetime was invented in the form of operators acting on a Hilbert space. This structure combines two related, but different, pieces of data, namely (1) the algebra of canonical commutation relations (CCR), witnessing the quantum nature of the theory, and (2) a distinguished choice of representation for the (abstract) CCR algebra, whose role is to draw a bridge between the CCR algebra and its familiar implementation by operators acting on a Hilbert space. One of the fundamental achievements of the algebraic approach to quantum field theory [HK64] was to recognize and isolate the two different and complementary roles played by the data (1) and (2). On the one hand, the general axioms of quantum field theory can be postulated abstractly, at the algebraic level; on the other hand, the (typically many) inequivalent representations provide incarnations of a CCR algebra corresponding to different states of the physical system.

It is remarkable that the viewpoint of the algebraic approach to quantum field theory paved the way to vast generalizations, enlarging both the domain and the target categories of a quantum field theory. For instance, algebraic quantum field theories in the sense of Haag and Kastler [HK64] were originally defined on the directed set of causally convex open subsets of Minkowski spacetime [Dim80]. More generally, locally covariant quantum field theories [BFV03] are defined on the category of all oriented and time-oriented globally hyperbolic Lorentzian manifolds, while locally covariant conformal field theories [Pin09, CRV21, BGS21] are defined on a similar category, however with orientation and time-orientation preserving conformal, instead of isometric, open embeddings whose image is causally convex.

As far as the target category is concerned, while the prime example is certainly the category of complex vector spaces, perturbative algebraic quantum field theory [HW01, BF00, BDF09] replaces the field of complex numbers with a ring of formal power series. Furthermore, the construction of quantum gauge theories through the Batalin-Vilkovisky formalism, which is in the center of our interest, forces one to work in the category of cochain complexes [Hol08, FR12, FR13].

From a conceptual point of view the Batalin-Vilkovisky formalism crucially relies on the notion of quasi-isomorphisms (namely the weak equivalences in the standard model structure on the category of cochain complexes). For instance, it is by means of quasi-isomorphisms that the Batalin-Vilkovisky formalism introduces its auxiliary fields, which are eventually responsible of the efficacy of this approach. Informally, this means that not only isomorphic, but more generally, weakly equivalent cochain complexes should be regarded as “being the same” for physical purposes. This, however, brings along the complication that any physically relevant construction one performs has to respect weak equivalences too. This is the starting point, and one of the main motivations, of the homotopy algebraic quantum field theory programme [BSW19a, BS19a, BBS20, BFS21].

So far the above mentioned homotopy algebraic quantum field theory programme focused mainly on datum (1) of a quantum field theory, i.e. the CCR algebras, as mentioned above. The main purpose of the present paper is to investigate also datum (2), the representations of the CCR algebras and, in particular, their homotopy theory.

To define representations of a quantum field theory, we adopt the framework of nets of algebras, see e.g. [RV12a, RV12b]. In view of the variety of domain $\mathcal{C}$ and target $\mathcal{M}$ categories considered in the literature, some of which are listed in the previous paragraph, we take $\mathcal{C}$ to be any (small) category and $\mathcal{M}$ to be any complete and cocomplete closed symmetric monoidal category. From this general perspective the category of nets of algebras is simply the category of functors $\mathfrak{A}$ from $\mathcal{C}$ to the category of monoids in $\mathcal{M}$ with morphisms given by natural transformations. Furthermore, a net representation $\mathcal{L}$ of $\mathfrak{A}$ consists of an assignment of left $\mathfrak{A}(c)$-modules
\(L_c\) for all objects \(c \in C\), which is coherent (in a suitable, but straightforward sense) with respect to all morphisms in \(C\).

Our first task is to investigate the assignment to a net of algebras \(\mathfrak{A}\) of the corresponding category of net representations \(\text{Rep}(\mathfrak{A})\). In particular, we are interested in the relation between the categories of net representations \(\text{Rep}(\mathfrak{A})\) and \(\text{Rep}(\mathfrak{B})\) that is determined by a morphism of nets \(\Phi : \mathfrak{A} \to \mathfrak{B}\). This goal is achieved via a suitable change-of-net adjunction \(\text{Crl}_\Phi \dashv \text{Res}_\Phi : \text{Rep}(\mathfrak{B}) \to \text{Rep}(\mathfrak{A})\), inspired by the classical change-of-monoid adjunction for left modules over monoids. As one expects, isomorphisms between nets of algebras are associated with (adjoint) equivalences between the associated categories of net representations. Thus, implementing the principle that, since isomorphic nets of algebras “are the same”, the corresponding categories of net representations must also “be the same”. Note, however, that “being the same” for categories of net representations means being equivalent (but not necessarily isomorphic) as categories.

The above question about the categories of net representations “being the same” when the nets of algebras they are associated with “are the same” becomes more intricate when the target category \(M\) comes endowed with a notion of weak equivalences that is weaker than that of isomorphisms. (Recall from above that this situation is encountered in the contexts of the Batalin-Vilkovisky formalism and of homotopy algebraic quantum field theory, where the target category \(M = \text{Ch}_C\) of cochain complexes has weak equivalences given by quasi-isomorphisms.) The latter induces a notion of weak equivalences between nets of algebras, simply given by natural transformations whose components are weak equivalences in \(M\). As a consequence, one would like to ensure that nets of algebras that “are the same”, i.e. weakly equivalent, correspond to categories of net representations that “are the same” too. It turns out, however, that weakly equivalent nets of algebras may fail to be associated with categories of net representations that are equivalent in the ordinary categorical sense.

Indeed, in Example 2.12 we show that two weakly equivalent nets of algebras implementing the simple Klein-Gordon field (the standard Klein-Gordon net and the one coming from the Batalin-Vilkovisky formalism) correspond to categories of net representations that are manifestly inequivalent in the ordinary categorical sense. Informally speaking, two equivalent mathematical models for the same quantum field theory end up to two inequivalent physical descriptions.

This evident shortcoming is the problem that we address in the first part of this paper, Section 2, namely to find the appropriate replacement of the concept of ordinary categorical equivalence such that the assignment of categories of net representations to nets of algebras becomes “invariant” with respect to weak equivalences of nets of algebras. In other words, the goal is to understand the appropriate notion of weak equivalence between net representations. Achieving this goal solves the concrete issue raised above with the Klein-Gordon field.

It turns out that achieving this goal consists of three steps: (1) Endow the categories of net representations with a suitable model structure (including a notion of weak equivalences between net representations), (2) promote the change-of-net adjunction to a Quillen adjunction between the model categories of net representations from step (1), (3) show that the change-of-net Quillen adjunction from step (2) is also a Quillen equivalence when it is associated with a weak equivalence of nets of algebras. By doing so, the principle that, since weakly equivalent nets of algebras “are the same”, the associated categories of net representations must “be the same” too, is restored by formalizing the notion of “being the same” with the concept of Quillen equivalence, which is in general weaker than the concept of ordinary categorical equivalence.

However, it is noteworthy that the concept of Quillen equivalence between two model categories descends to the concept of ordinary categorical equivalence once we pass to their associated homotopy categories [Hov99, Ch. 1] (i.e. the categories obtained by inverting all weak equivalences). Therefore, the three-step approach described above establishes that the homotopy categories of net representations are actually equivalent in the ordinary categorical sense when the associated nets of algebras are weakly equivalent.
With the homotopy theory of net representations at hand and having applications in the context of homotopy algebraic quantum field theory in mind, we turn to the second part of this paper, namely Section 3, that addresses the problem of constructing explicit net representations for a given net of algebras valued in the target category $M = \text{Ch}_C$ of cochain complexes. Concretely, we consider the net of algebras associated with Maxwell $p$-forms $[HT86; HT92]$. Note that Maxwell $p$-forms correspond to the massless Klein-Gordon field for $p = 0$ and to linear Yang-Mills theory for $p = 1$. For this example we consider a fixed oriented and time-oriented globally hyperbolic Lorentzian manifold $M$, which for simplicity we assume ultra-static and admitting a compact spacelike Cauchy surface. As explained in more detail below, we proceed constructing a net representation by first constructing a representation of the algebra of observables that corresponds to the whole spacetime $M$ and then use it to produce a representation for the net of algebras.

The construction of a representation of the global algebra of observables involves two steps. In the first step we construct a two-point function $\omega_2$ as a cochain map defined on the complex of linear observables on $M$. The fact that $\omega_2$ is a cochain map simultaneously encodes its compatibility both with the action of gauge transformations and with the equation of motion. (Incidentally, for $p = 1$ we observe that $\omega_2$ recovers in degree 0 cohomology the Hadamard two-point function constructed in $[FP03]$ for the electromagnetic vector potential, thus drawing a bridge between the approach we propose and well-established ones.) The second step mimics the first stage of the Gelfand-Naimark-Segal construction to define from the two-point function $\omega_2$ a left module on the global algebra of observables.

Having now a representation of the global algebra of observables we derive a constant (in the sense of Construction 2.6) net representation, which essentially amounts to restricting the global algebra representation to the local algebras. (Constant net representations are similar in spirit to $[RV12a, \text{Sec. 4.2}]$.)

Finally, we exploit the homotopy theory of net representations developed in the first part to present a very explicit description of the data of all constant (in the sense of Construction 2.6) net representations up to weak equivalence in the simplest scenario, i.e. $p = 1$ and $M$ the two-dimensional flat Lorentz cylinder.

Let us now summarize the contents of this paper. Section 2 recalls the categories of nets of algebras and of net representations for any domain category $C$ and target symmetric monoidal category $M$, generalizing the standard concepts in a straightforward way. In passing, we present an adjunction relating representations of a net of algebras and left modules over a single monoid from the net. Constant net representations arise through a specific instance of this adjunction, which we shall use later on to construct a concrete net representation. Next, we move on to illustrate the change-of-net adjunction associated with a morphism of nets, which is manifestly an adjoint equivalence in the case of an isomorphism. As those will be useful tools when we shall later endow the categories of net representations with a model structure, we devote some time to discuss the $M$-tensoring, powering and enriched hom on the category of net representations. Motivated by the fact that the assignment of representation categories to nets of algebras is not invariant in the ordinary categorical sense with respect to weak equivalences of nets of algebras (Example 2.12), we conclude Section 2 by endowing the category of net representations with a model structure (Corollary 2.15) obtained through a standard (right) transfer construction (Theorem 2.13). For this purpose we assume $M$ to be a suitable (in the sense of Set-up 2.15) closed symmetric monoidal model category. This allows us to promote the change-of-net adjunction to a Quillen adjunction, which is furthermore a Quillen equivalence when it is associated with a weak equivalence of nets of algebras (Proposition 2.22). Section 3 focuses on constructing concrete net representations when $C$ is the category of causally convex open subsets of a fixed oriented and time-oriented globally hyperbolic Lorentzian manifold $M$, $M = \text{Ch}_C$ is the closed symmetric monoidal model category of cochain complexes over $C$ and the net of algebras is the one of Maxwell
p-forms. First, we construct the net of algebras of Maxwell p-forms via CCR quantization \((3.18)\) of the complex of linear observables for Maxwell p-forms from \((3.9)\). Then we show that the resulting net of algebras is actually a homotopy algebraic quantum field theory in the sense of \([\text{BSW19a}, \text{BBS20}]\), i.e. it is actually defined on the category of all \(m\)-dimensional oriented and time-oriented globally hyperbolic Lorentzian manifolds and it fulfills both the causality axiom and the homotopy time-slice axiom. Then, restricting this net of algebras to all causally convex open subsets of a fixed oriented and time-oriented globally hyperbolic Lorentzian manifold \(M\), which for simplicity we assume ultra-static and admitting a compact spacelike Cauchy surface, we present a simple construction of a two-point function \(\omega_2\), that is the cochain map we use to define a concrete constant net representation. We conclude with a simple and explicit description of all constant net representations up to weak equivalence for Maxwell 1-forms on the flat Lorentz cylinder. Finally, Appendix \(A\) collects some useful facts and constructions about left modules over a monoid and their homotopy theory.

2 Representation theory for nets of algebras

In this section we first recall the basic concepts of nets of algebras over a (small) category \(C\) internal to a symmetric monoidal category \(M\) and the associated categories of net representations. These concepts are straightforward generalizations of their more familiar analogs in the \(C^*\)-algebraic setting, see e.g. \([\text{RV12a}, \text{RV12b}]\). Then we move on to illustrate how morphisms of nets of algebras induce change-of-net adjunctions between the associated categories of net representations, essentially unfolding the concept of change-of-monoid adjunction from Section \(A\). Finally, assuming that \(M\) is a closed symmetric monoidal model category, we endow the category of nets of algebras and the category of net representations with canonical model structures combining the analogous results for monoids and modules recalled in Appendix \(A.3\) and a well-known transfer theorem for cofibrantly generated model structures. The main goal of the model structures we propose is to obtain a Quillen equivalence between model categories of net representations from a weak equivalence of nets of algebra. This goal is motivated by the simple Example \(2.12\), which shows that the ordinary categorical adjunction underlying the above mentioned Quillen equivalence fails to be an ordinary categorical equivalence for many examples of weak equivalences of nets of algebras that feature in concrete applications. Therefore, the outcome of this section is that, when the concept of “being the same” for nets of algebras is encoded by weak equivalence (for instance in the Batalin-Vilkovisky formalism, see Section \(I\), the correct formalization of “being the same” for the corresponding categories of net representations is provided by the concept of Quillen equivalence and not by the concept of ordinary categorical equivalence. Speaking more loosely, the proposed framework explains that suitable model structures on the categories of net representations are crucial for the assignment of categories of net representations to be “invariant” with respect to weak equivalences of nets of algebras.

2.1 Nets of algebras and their representations

Let \(M = (\mathcal{M}, \otimes, \mathbb{1})\) be a complete and cocomplete closed symmetric monoidal category. Let us briefly recall the concepts of monoids and their left modules in \(M\). (The constructions and structures involving monoids and their left modules that are relevant for the present paper are briefly collected in Appendix \(A\).) A monoid \(A = (A, \mu, \mathbb{1})\) in \(M\) consists of an object \(A \in M\) endowed with two morphisms \(\mu : A \otimes A \to A\) and \(\mathbb{1} : \mathbb{1} \to A\) in \(M\), called multiplication and unit respectively, subject to the usual associativity (\((\mu \otimes \mathbb{1}) \circ (\mathbb{1} \otimes \mu) = \mu(\mathbb{1} \otimes A)\)) and unitality (\((\mu(\mathbb{1} \otimes id_A) = id_A = \mu(id_A \otimes \mathbb{1}))\) axioms. Furthermore, one defines a morphism of monoids \(\varphi : A \to B\) as a morphism \(\varphi : A \to B\) in \(M\) that preserves multiplications (\((\mu_B \circ (\varphi \otimes \varphi) = \varphi \circ \mu_A)\) and units \((1_B = \varphi \circ 1_A)\). \(\text{Mon}(M)\) denotes the category of monoids in \(M\). Taking a monoid \(A \in \text{Mon}(M)\), we recall that a left \(A\)-module \(L = (L, \lambda)\) in \(M\) consists of an object \(L \in M\) endowed with a morphism \(\lambda : A \otimes L \to L\) in \(M\), called left \(A\)-action, subject to the usual axioms
natural transformations as morphisms. Forming tensor products of functors object-wise and concomitant unitality axioms. It follows that, for all monoidal category M.

Furthermore, combining the functoriality of the underlying leads to a particularly concise definition of nets of algebras.

Consider a (small) category C, of a net of algebras as a functor from M.

Remark 2.2. Unpacking this compact definition recovers (a generalization of) the usual concepts of a net of algebras as a functor from C to Mon(M) and of a morphism between nets of algebras as a natural transformation between them. In other words, the categories Mon(Fun(C, M)) and Fun(C, Mon(M)) coincide manifestly, as explained below.

Indeed, \( \mathfrak{A} \in \text{Net}_C^M \) consists of an underlying functor \( \mathfrak{A} \in \text{Fun}(C, M) \) and two natural transformations \( \mu : \mathfrak{A} \otimes \mathfrak{A} \rightarrow \mathfrak{A} \) and \( 1 : 1 \rightarrow \mathfrak{A} \) in \( \text{Fun}(C, M) \), subject to the usual associativity and unitality axioms. It follows that, for all \( c \in C \), the components \( \mu_c : \mathfrak{A}(c) \otimes \mathfrak{A}(c) \rightarrow \mathfrak{A}(c) \) and \( 1_c : 1 \rightarrow \mathfrak{A}(c) \) in M endow \( \mathfrak{A}(c) \in M \) with the structure of a monoid \( \mathfrak{A}(c) \in \text{Mon}(M) \).

Example 2.3. For applications in the context of Lorentzian quantum field theory one often takes the source \( C = \text{Loc}_m \), \( m \geq 2 \), to be the category of oriented and time-oriented \( m \)-dimensional globally hyperbolic Lorentzian manifolds with morphisms given by orientation and time-orientation preserving isometric open embeddings whose image is causally convex. This leads to the generally covariant approach to algebraic quantum field theory [BFV03]. Another option is the source category \( \text{CCO}(M) \) of causally convex open subsets of a fixed \( M \in \text{Loc}_m \) with subset inclusions as morphisms. In this case one obtains nets of algebras in the sense of Haag and Kastler [HK64, DimM]. For these applications the prime example of target \( M = \text{Vec}_C \) is the symmetric monoidal category of vector spaces over \( C \). On the other hand, motivated by the Batalin-Vilkovisky formalism, in homotopy algebraic quantum field theory (see Section 1) one considers as target \( M = \text{Ch}_C \) the symmetric monoidal category of cochain complexes over \( C \).

Using the symmetric monoidal functor category \( \text{Fun}(C, M) \), it is also possible to give a concise definition of net representations.

Definition 2.4. Let \( \mathfrak{A} \in \text{Net}_C^M \) be an \( M \)-valued net of algebras over \( C \). The category \( \text{Rep}(\mathfrak{A}) \) of \( \mathfrak{A} \)-representations is the category \( \text{Mod}_A \) of \( \mathfrak{A} \)-modules in the symmetric monoidal functor category \( \text{Fun}(C, M) \).
**Remark 2.5.** Unpacking this compact definition recovers (a generalization of) the usual concepts of an \(\mathfrak{A}\)-representation and of a morphism between \(\mathfrak{A}\)-representations, see e.g. [RV12a; RV12b]. Indeed, \(\mathcal{L} \in \text{Rep}(\mathfrak{A})\) consists of an underlying functor \(\mathcal{L} \in \text{Fun}(\mathbf{C}, \mathbf{M})\) and of a natural transformation \(\lambda : \mathfrak{A} \otimes \mathcal{L} \to \mathcal{L}\) in \(\text{Fun}(\mathbf{C}, \mathbf{M})\) subject to the usual left \(\mathfrak{A}\)-action axioms. This structure is equivalently encoded by the following data (1-2) and axioms (i-ii):

1. For each \(c \in \mathbf{C}\), a left \(\mathfrak{A}(c)\)-module \(\mathcal{L}_c \in \mathfrak{A}(c)\text{-Mod}\).

2. For each morphism \(\gamma : c_1 \to c_2\) in \(\mathbf{C}\), a morphism of left \(\mathfrak{A}(c_2)\)-modules \(\mathcal{L}_\gamma : \mathfrak{A}(c_2) \otimes \mathfrak{A}(c_1)\mathcal{L}_{c_1} \to \mathcal{L}_{c_2}\) in \(\mathfrak{A}(c_2)\text{-Mod}\), or equivalently a morphism of left \(\mathfrak{A}(c_1)\)-modules \(\overline{\mathcal{L}}_\gamma : \mathcal{L}_{c_1} \to \mathcal{L}_{c_2}|_{\mathfrak{A}(c_1)}\) in \(\mathfrak{A}(c_1)\text{-Mod}\).

(i) For each \(c \in \mathbf{C}\), \(\mathcal{L}_{\text{id}_c} : \mathfrak{A}(c) \otimes \mathfrak{A}(c) \mathcal{L}_c \to \mathcal{L}_c\) in \(\mathfrak{A}(c)\text{-Mod}\) coincides with the canonical isomorphism of \(\mathfrak{A}(c)\)-modules induced by the left \(\mathfrak{A}(c)\)-action or, equivalently, \(\overline{\mathcal{L}}_{\text{id}_c} = \text{id}_{\mathcal{L}_c} : \mathcal{L}_c \to \mathcal{L}_c = \mathcal{L}_c|_{\mathfrak{A}(c)}\) in \(\mathfrak{A}(c)\text{-Mod}\) coincides with the identity.

(ii) For each pair of composable morphisms \(\gamma_1 : c_1 \to c_2\) and \(\gamma_2 : c_2 \to c_3\) in \(\mathbf{C}\), the diagram

\[
\begin{array}{ccc}
\mathfrak{A}(c_3) \otimes \mathfrak{A}(c_2) & \xrightarrow{\mathfrak{A}(c_3) \otimes \mathfrak{A}(c_2) \mathcal{L}_{\gamma_1}} & \mathfrak{A}(c_3) \otimes \mathfrak{A}(c_2) \mathcal{L}_{c_2} \\
\mathfrak{A}(c_3) & \xrightarrow{\mathfrak{A}(c_3) \mathcal{L}_{\gamma_1}} & \mathfrak{A}(c_3) \mathcal{L}_{c_3}
\end{array}
\]

commutes in the category \(\mathfrak{A}(c_3)\text{-Mod}\) of left \(\mathfrak{A}(c_3)\)-modules or, equivalently, the diagram

\[
\begin{array}{ccc}
\mathcal{L}_{c_1} & \xrightarrow{\mathcal{L}_{\gamma_1}} & \mathcal{L}_{c_2}|_{\mathfrak{A}(c_1)} \\
\mathcal{L}_{c_3}|_{\mathfrak{A}(c_1)} & \xrightarrow{\mathcal{L}_{\gamma_2 \gamma_1}} & \mathcal{L}_{c_3}|_{\mathfrak{A}(c_2)}|_{\mathfrak{A}(c_1)}
\end{array}
\]

commutes in the category \(\mathfrak{A}(c_1)\text{-Mod}\) of left \(\mathfrak{A}(c_1)\)-modules.

This presentation matches the usual description of a net representation.

A morphism \(\mathcal{F} : \mathcal{L} \to \mathcal{L}'\) in \(\text{Rep}(\mathfrak{A})\) is a natural transformation that preserves the left \(\mathfrak{A}\)-actions. This structure is equivalent to a morphism of left \(\mathfrak{A}(c)\)-modules \(\mathcal{F}_c : \mathcal{L}_c \to \mathcal{L}'_c\) in \(\mathfrak{A}(c)\text{-Mod}\) for each \(c \in \mathbf{C}\), subject to commutativity of the diagram

\[
\begin{array}{ccc}
\mathfrak{A}(c_2) \otimes \mathfrak{A}(c_1) \mathcal{L}_{c_1} & \xrightarrow{\mathfrak{A}(c_2) \otimes \mathfrak{A}(c_1) \mathcal{F}_c} & \mathfrak{A}(c_2) \otimes \mathfrak{A}(c_1) \mathcal{L}'_{c_1} \\
\mathcal{L}_{c_2} & \xrightarrow{\mathcal{F}_c} & \mathcal{L}'_{c_2}
\end{array}
\]

in the category \(\mathfrak{A}(c_2)\text{-Mod}\) of left \(\mathfrak{A}(c_2)\)-modules or, equivalently, subject to the commutativity of the diagram

\[
\begin{array}{ccc}
\mathcal{L}_{c_1} & \xrightarrow{\mathcal{F}_c} & \mathcal{L}'_{c_1} \\
\mathcal{L}_{c_2} & \xrightarrow{\mathcal{F}_c} & \mathcal{L}'_{c_2}
\end{array}
\]

in the category \(\mathfrak{A}(c_1)\text{-Mod}\) of left \(\mathfrak{A}(c_1)\)-modules, for each morphism \(\gamma : c_1 \to c_2\) in \(\mathbf{C}\).
Construction 2.6. Given a fixed object $\tilde{c} \in \mathcal{C}$, there is a useful construction of $\mathfrak{A}$-representations from left $\mathfrak{A}(\tilde{c})$-modules arising as part of the adjunction
\[(\cdot)_{\tilde{c}} : \text{Rep}(\mathfrak{A}) \xrightarrow{\cong} \mathfrak{A}(\tilde{c})\text{Mod} : (\cdot)^{\tilde{c}}.\] (2.5)

We shall use this construction in Section 3 to construct examples of representations for a net of algebras valued in cochain complexes. The left adjoint functor $(-)_{\tilde{c}}$ sends an $\mathfrak{A}$-representation $L \in \text{Rep}(\mathfrak{A})$ to the $\mathfrak{A}(\tilde{c})$-module $L_{\tilde{c}} \in \mathfrak{A}(\tilde{c})\text{Mod}$ and a morphism of $\mathfrak{A}$-representations $\mathcal{F} : L \to L'$ to the morphism of left $\mathfrak{A}(\tilde{c})$-modules $\mathcal{F}_{\tilde{c}} : L_{\tilde{c}} \to L'_{\tilde{c}}$ in $\mathfrak{A}(\tilde{c})\text{Mod}$. To define the right adjoint functor $(\cdot)^{\tilde{c}} : \mathfrak{A}(\tilde{c})\text{Mod} \to \text{Rep}(\mathfrak{A})$, it is convenient to use the equivalent description of $\mathfrak{A}$-representations from Remark 2.5. $(\cdot)^{\tilde{c}}$ sends a left $\mathfrak{A}(\tilde{c})$-module $L \in \mathfrak{A}(\tilde{c})\text{Mod}$ to the $\mathfrak{A}$-representation $L^{\tilde{c}} \in \text{Rep}(\mathfrak{A})$ consisting of the following data:

1. For each $c \in \mathcal{C}$, the left $\mathfrak{A}(c)$-module
   \[(L^{\tilde{c}})_c := \prod_{\tilde{c} \in \mathcal{C}(c, \tilde{c})} L|_{\mathfrak{A}(c)} \in \mathfrak{A}(c)\text{Mod}.\] (2.6a)

2. For each morphism $\gamma : c_1 \to c_2$ in $\mathcal{C}$, the morphism of left $\mathfrak{A}(c_1)$-modules
   \[(L^{\tilde{c}})^{\gamma} : (L^{\tilde{c}})_{c_1} = \prod_{\tilde{c} \in \mathcal{C}(c_1, \tilde{c})} L|_{\mathfrak{A}(c_1)} \xrightarrow{\gamma} \prod_{\tilde{c} \in \mathcal{C}(c_2, \tilde{c})} L|_{\mathfrak{A}(c_1)} = (L^{\tilde{c}})_{c_2}|_{\mathfrak{A}(c_1)}\] (2.6b)

in $\mathfrak{A}(c_1)\text{Mod}$ defined by sending to the $\tilde{c}_2\gamma$-component of the codomain the $\tilde{c}_2\gamma$-component of the domain.

Furthermore, $(-)^{\tilde{c}}$ sends a morphism of left $\mathfrak{A}(\tilde{c})$-modules $F : L \to L'$ in $\mathfrak{A}(\tilde{c})\text{Mod}$ to the morphism of left $\mathfrak{A}(c)$-modules
\[(F^{\tilde{c}})_c := \prod_{\tilde{c} \in \mathcal{C}(c, \tilde{c})} F|_{\mathfrak{A}(c)} : (L^{\tilde{c}})_c = \prod_{\tilde{c} \in \mathcal{C}(c, \tilde{c})} L|_{\mathfrak{A}(c)} \longrightarrow \prod_{\tilde{c} \in \mathcal{C}(c, \tilde{c})} L'_{\mathfrak{A}(c)} = (L'^{\tilde{c}})_c\] (2.6c)
in $\mathfrak{A}(c)\text{Mod}$ for each $c \in \mathcal{C}$. It is straightforward to confirm that the above construction defines a functor $(-)^{\tilde{c}}$. To check that the latter is right adjoint to $(-)_{\tilde{c}}$, we exhibit the unit $\eta$ and the counit $\varepsilon$ of this adjunction. The unit is the natural transformation $\eta : \text{id}_{\text{Rep}(\mathfrak{A})} \to ((-)^{\tilde{c}})^{\tilde{c}}$ whose component at the $\mathfrak{A}$-representation $L \in \text{Rep}(\mathfrak{A})$ is the morphism of $\mathfrak{A}$-representations $\eta_L : L \to (L^{\tilde{c}})^{\tilde{c}}$ in $\text{Rep}(\mathfrak{A})$ consisting of the left $\mathfrak{A}(c)$-module morphism
\[(\eta^{\tilde{c}})_c := (L^{\tilde{c}})|_{\tilde{c} \in \mathcal{C}(c, \tilde{c})} : L^{\tilde{c}} \longrightarrow \prod_{\tilde{c} \in \mathcal{C}(c, \tilde{c})} L_{\mathfrak{A}(c)} = ((L^{\tilde{c}})^{\tilde{c}})_c\] (2.7)
in $\mathfrak{A}(c)\text{Mod}$ for each $c \in \mathcal{C}$. The counit is the natural transformation $\varepsilon : ((-)^{\tilde{c}})^{\tilde{c}} \to \text{id}_{\mathfrak{A}(\tilde{c})\text{Mod}}$ whose component at the $\mathfrak{A}(\tilde{c})$-module $L \in \mathfrak{A}(\tilde{c})\text{Mod}$ is the morphism of left $\mathfrak{A}(\tilde{c})$-modules
\[\varepsilon_L := \text{pr}_{\text{id}_{\tilde{c}}} : (L^{\tilde{c}})|_{\tilde{c} \in \mathcal{C}(c, \tilde{c})} : L^{\tilde{c}} \longrightarrow \prod_{\tilde{c} \in \mathcal{C}(c, \tilde{c})} L_{\mathfrak{A}(\tilde{c})} \longrightarrow L\] (2.8)
in $\mathfrak{A}(\tilde{c})\text{Mod}$ given by the projection $\text{pr}_{\text{id}_{\tilde{c}}}$ of the $\text{id}_{\tilde{c}}$-component. The triangle identities $(\varepsilon_L)^{\tilde{c}} \eta^{\tilde{c}}_L = \text{id}_{L^{\tilde{c}}}$, for all $L \in \mathfrak{A}(\tilde{c})\text{Mod}$, and $\varepsilon^{\tilde{c}}(\eta_L)^{\tilde{c}} = \text{id}_{L^{\tilde{c}}}$, for all $L \in \text{Rep}(\mathfrak{A})$, are straightforward, which shows that (2.5) is indeed an adjunction.

Recall from Example 2.3 that net of algebras in the sense of Haag and Kastler are defined on the source category $\mathcal{C} = \text{CCO}(M)$ of causally convex open subsets of a fixed object $M \in \text{Loc}_m$.,
Then, for \( \tilde{c} = M \in \text{CCO}(M) \), one recognizes that the right adjoint functor \((-)^M \) in (2.5) is the well-known construction of constant \( \mathcal{A} \)-representations from left \( \mathcal{A} \)(M)-modules over the global algebra of observables \( \mathcal{A}(M) \in \text{Mon}(M) \) of a net of algebras \( \mathcal{A} \in \text{Net}^M_{\text{CCO}(M)} \) over \( \text{CCO}(M) \). This large class of net representations is the one that most frequently occurs in the literature, see [RV12a, Sec. 4.2]. (Note that what we call constant \( \mathcal{A} \)-representations are frequently called Hilbert space representations in the \( C^* \)-setting of algebraic quantum field theory. This name is motivated by the fact that the whole net of algebras is represented on the same Hilbert space. In our framework the latter is replaced by a single left \( \mathcal{A}(M) \)-module \( L \in \mathcal{A}(M)\text{Mod} \), which defines an \( \mathcal{A} \)-representation \( L^M \in \text{Rep}(\mathcal{A}) \) through the adjunction (2.5).)

### 2.1.1 Change-of-net adjunction

In order to compare categories of net representations associated with different nets of algebras we will make extensive use of the change-of-net adjunction, i.e. the change-of-monoid adjunction \( \text{Net}^M \) applied on the symmetric monoidal category \( \text{Net}^M_{\text{CCO}(M)} \). More specifically, given a morphism of nets of algebras \( \Phi : \mathcal{A} \to \mathcal{B} \) in \( \text{Net}^M_{\text{CCO}(M)} \), there is an associated change-of-net adjunction

\[
\mathfrak{Ext}_\Phi : \text{Rep}(\mathcal{A}) \rightleftharpoons \text{Rep}(\mathcal{B}) : \mathfrak{Res}_\Phi.
\] (2.9)

In this case, the extension \( \mathfrak{Ext}_\Phi \) and restriction \( \mathfrak{Res}_\Phi \) functors admit also an explicit description in terms of the more elementary extension and restriction functors of the change-of-monoid adjunction \( \text{Net}^M \) in the underlying category \( M \).

Specifically, the restriction functor \( \mathfrak{Res}_\Phi : \text{Rep}(\mathcal{B}) \to \text{Rep}(\mathcal{A}) \) assigns to a \( \mathcal{B} \)-representation \( \mathcal{M} \in \text{Rep}(\mathcal{B}) \) the \( \mathcal{A} \)-representation \( \mathfrak{Res}_\Phi \mathcal{M} \in \text{Rep}(\mathcal{A}) \) consisting of the following data, see Remark 2.5

1. For each \( c \in C \), the left \( \mathcal{A}(c) \)-module

\[
(\mathfrak{Res}_\Phi \mathcal{M})_c := \mathcal{M}_c|_{\mathcal{A}(c)} \in \mathcal{A}(c)\text{Mod}
\] (2.10a)

obtained restricting the left \( \mathcal{B}(c) \)-module \( \mathcal{M}_c \in \mathcal{B}(c)\text{Mod} \) along \( \Phi_c : \mathcal{A}(c) \to \mathcal{B}(c) \) in \( \text{Mon}(M) \).

2. For each morphism \( \gamma : c_1 \to c_2 \) in \( C \), the morphism of left \( \mathcal{A}(c_1) \)-modules

\[
(\mathfrak{Res}_\Phi \mathcal{M})_\gamma := \mathcal{M}_\gamma|_{\mathcal{A}(c_1)} : (\mathfrak{Res}_\Phi \mathcal{M})_{c_1} \longrightarrow (\mathcal{M}_c|_{\mathcal{B}(c_1)})|_{\mathcal{A}(c_1)} = (\mathfrak{Res}_\Phi \mathcal{M})_{c_2}|_{\mathcal{A}(c_1)}
\] (2.10b)

in \( \mathcal{A}(c_1)\text{Mod} \) defined by restricting the morphism of left \( \mathcal{B}(c_1) \)-modules \( \mathcal{M}_\gamma : \mathcal{M}_{c_1} \to \mathcal{M}_{c_2}|_{\mathcal{B}(c_1)} \) in \( \mathcal{B}(c_1)\text{Mod} \) along \( \Phi_{c_1} : \mathcal{A}(c_1) \to \mathcal{B}(c_1) \) in \( \text{Mon}(M) \) and then by observing that both iterated restrictions on the right hand side coincide with the restriction along the composition \( \mathcal{B}(\gamma) \Phi_{c_1} = \Phi_{c_2} \mathcal{A}(\gamma) : \mathcal{A}(c_1) \to \mathcal{B}(c_2) \) in \( \text{Mon}(M) \), see Remark A.3

Furthermore, \( \mathfrak{Res}_\Phi \) assigns to a morphism \( \mathcal{G} : \mathcal{M} \to \mathcal{M}' \) in \( \text{Rep}(\mathcal{B}) \) the morphism \( \mathfrak{Res}_\Phi \mathcal{G} : \mathfrak{Res}_\Phi \mathcal{M} \to \mathfrak{Res}_\Phi \mathcal{M}' \) in \( \text{Rep}(\mathcal{A}) \) consisting of the morphisms of left \( \mathcal{A}(c) \)-modules

\[
(\mathfrak{Res}_\Phi \mathcal{G})_c := \mathcal{G}_c|_{\mathcal{A}(c)} : (\mathfrak{Res}_\Phi \mathcal{M})_c \longrightarrow (\mathfrak{Res}_\Phi \mathcal{M}')_c
\] (2.10c)

in \( \mathcal{A}(c)\text{Mod} \) obtained restricting the morphism of left \( \mathcal{B}(c) \)-modules \( \mathcal{G}_c : \mathcal{M}_c \to \mathcal{M}'_c \) in \( \mathcal{B}(c)\text{Mod} \) along \( \Phi_c : \mathcal{A}(c) \to \mathcal{B}(c) \) in \( \text{Mon}(M) \), for all \( c \in C \). Using the change-of-monoid adjunction \( \text{Net}^M \) and its compatibility with compositions, see Remark A.3 it is straightforward to check that the data listed above fulfill the axioms from Remark 2.5.

Similarly, the extension functor \( \mathfrak{Ext}_\Phi : \text{Rep}(\mathcal{A}) \to \text{Rep}(\mathcal{B}) \) admits a similar description using the extension functor of the change-monoid adjunction \( \text{Net}^M \) in \( M \). \( \mathfrak{Ext}_\Phi \) assigns to an \( \mathcal{A} \)-representation \( \mathcal{L} \in \text{Rep}(\mathcal{A}) \) the \( \mathcal{B} \)-representation consisting of the following data:
For each \(c \in C\), the left \(\mathfrak{B}(c)\)-module

\[
(\mathfrak{Ext}_L)_{c} := \mathfrak{B}(c) \otimes_{\mathfrak{A}(c)} L_c \in \mathfrak{A}(c)\text{-Mod}
\]  \hspace{1cm} (2.11a)

obtained extending the left \(\mathfrak{A}(c)\)-module \(L_c \in \mathfrak{A}(c)\text{-Mod}\) along \(\Phi_c : \mathfrak{A}(c) \to \mathfrak{B}(c)\) in \(\text{Mon}(M)\).

For each morphism \(\gamma : c_1 \to c_2\) in \(C\), the morphism of left \(\mathfrak{B}(c_2)\)-modules \((\mathfrak{Ext}_L)_{\gamma} : \mathfrak{B}(c_2) \otimes_{\mathfrak{A}(c_1)} (\mathfrak{Ext}_L)_{c_1} \to (\mathfrak{Ext}_L)_{c_2}\) in \(\mathfrak{A}(c_2)\text{-Mod}\) defined by the diagram

\[
\begin{array}{ccc}
\mathfrak{B}(c_2) \otimes_{\mathfrak{A}(c_1)} (\mathfrak{B}(c_1) \otimes_{\mathfrak{A}(c_1)} L_{c_1}) & \xrightarrow{(\mathfrak{Ext}_L)_{\gamma}} & \mathfrak{B}(c_2) \otimes_{\mathfrak{A}(c_2)} L_{c_2} \\
\cong & \cong & \\
\mathfrak{B}(c_2) \otimes_{\mathfrak{A}(c_2)} (\mathfrak{A}(c_2) \otimes_{\mathfrak{A}(c_2)} L_{c_1}) & \xrightarrow{\mathfrak{Ext}_L} & \mathfrak{B}(c_2) \otimes_{\mathfrak{A}(c_2)} L_{c_2}
\end{array}
\]  \hspace{1cm} (2.11b)

in \(\mathfrak{A}(c_2)\text{-Mod}\), where we used that both iterated extensions on the left hand side are naturally isomorphic to the extension along the composition \(\mathfrak{B}(\gamma) \Phi_{c_1} = \Phi_{c_2} \mathfrak{A}(\gamma) : \mathfrak{A}(c_1) \to \mathfrak{B}(c_2)\) in \(\text{Mon}(M)\), see Remark \([A,3]\) and then we post-composed with the extension of the morphism of left \(\mathfrak{A}(c_2)\)-modules \(L_{\gamma} : \mathfrak{A}(c_2) \otimes_{\mathfrak{A}(c_1)} L_{c_1} \to L_{c_2}\) in \(\mathfrak{A}(c_2)\text{-Mod}\) along \(\Phi_{c_2} : \mathfrak{A}(c_2) \to \mathfrak{B}(c_2)\) in \(\text{Mon}(M)\).

Furthermore, \(\mathfrak{Ext}_L\) assigns to a morphism \(\mathcal{F} : \mathcal{L} \to \mathcal{L}'\) in \(\text{Rep}(\mathfrak{A})\) the morphism \(\mathfrak{Ext}_L \mathcal{F} : \mathfrak{Ext}_L \mathcal{L} \to \mathfrak{Ext}_L \mathcal{L}'\) in \(\text{Rep}(\mathfrak{B})\) consisting of the morphisms of left \(\mathfrak{B}(c)\)-modules

\[
(\mathfrak{Ext}_L \mathcal{F})_{c} := \mathfrak{B}(c) \otimes_{\mathfrak{A}(c)} \mathcal{F}_c : (\mathfrak{Ext}_L \mathcal{L})_{c} \longrightarrow (\mathfrak{Ext}_L \mathcal{L}')_{c}
\]  \hspace{1cm} (2.11c)

in \(\mathfrak{A}(c)\text{-Mod}\) obtained extending the morphism of left \(\mathfrak{A}(c)\)-modules \(\mathcal{F}_c : \mathcal{L}_c \to \mathcal{L}'_c\) in \(\mathfrak{A}(c)\text{-Mod}\) along \(\Phi_c : \mathfrak{A}(c) \to \mathfrak{B}(c)\) in \(\text{Mon}(M)\), for all \(c \in C\). Using once again the change-of-monoid adjunction \([A,1]\) and its compatibility with compositions, see Remark \([A,3]\) it is straightforward to check that the above data fulfil the axioms from Definition \([2,3]\).

**Remark 2.7.** Since the change-of-net adjunction is just a special instance of the change-of-monoid adjunction, Remark \([A,3]\) applies. Therefore, given an isomorphism \(\Phi : \mathfrak{A} \to \mathfrak{B}\) in \(\text{Net}_C^\text{M}\), the change-of-net adjunction \(\mathfrak{Ext}_L \Phi \dashv \mathfrak{Res}_\Phi\) from \([2,3]\) is an adjoint equivalence.

### 2.1.2 M-tensoring, powering and enriched hom on \(\text{Rep}(\mathfrak{A})\)

This section is devoted to the description of the \(\mathbb{M}\)-tensoring, powering and enriched hom on the category \(\text{Rep}(\mathfrak{A})\) of representations of a net of algebras \(\mathfrak{A} \in \text{Net}_C^\text{M}\). These concepts will be handy in endowing the category \(\text{Rep}(\mathfrak{A})\) with a model structure in Section \([2,2]\).

Having defined the category of nets of algebras as the category of monoids over the closed symmetric monoidal functor category \(\text{Fun}(C, M)\), one obtains immediately from Appendix \([A,2]\) a \(\text{Fun}(C, M)\)-tensoring on the category \(\text{Rep}(\mathfrak{A})\). Then restricting the latter \(\text{Fun}(C, M)\)-tensoring to constant functors provides an \(\mathbb{M}\)-tensoring on \(\text{Rep}(\mathfrak{A})\). Working out the usual adjunctions leads also to the \(\mathbb{M}\)-powering and enrichment on \(\text{Rep}(\mathfrak{A})\). We present below the relevant constructions exploiting the \(\mathbb{M}\)-tensoring, powering and enriched hom on the category of left modules over a monoid, see Appendix \([A,2]\).

Let us start from the \(\mathbb{M}\)-tensoring

\[
\otimes : \text{Rep}(\mathfrak{A}) \times M \longrightarrow \text{Rep}(\mathfrak{A}).
\]  \hspace{1cm} (2.12a)

\(\otimes\) assigns to an \(\mathfrak{A}\)-representation \(\mathcal{L} \in \text{Rep}(\mathfrak{A})\) and an object \(V \in M\) the \(\mathfrak{A}\)-representation \(\mathcal{L} \otimes V \in \text{Rep}(\mathfrak{A})\) consisting of the following data:

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(1) For each $c \in C$, the left $\mathfrak{A}(c)$-module
\[
(\mathcal{L} \otimes V)_c := L(c) \otimes V \in \mathfrak{A}(c)\text{-Mod}
\]
(2.12b) obtained evaluating the $M$-tensoring $\otimes : \mathfrak{A}(c)\text{-Mod} \times M \to \mathfrak{A}(c)\text{-Mod}$ on $L_c \in \mathfrak{A}(c)\text{-Mod}$ and $V \in M$, see (A.5).

(2) For each morphism $\gamma : c_1 \to c_2$ in $C$, the morphism of left $\mathfrak{A}(c_2)$-modules $(\mathcal{L} \otimes V)_\gamma : \mathfrak{A}(c_2) \otimes \mathfrak{A}(c_1) (\mathcal{L} \otimes V)_{c_1} \to (\mathcal{L} \otimes V)_{c_2}$ in $\mathfrak{A}(c_2)\text{-Mod}$ defined by the diagram
\[
\begin{array}{c}
\mathfrak{A}(c_2) \otimes \mathfrak{A}(c_1) (\mathcal{L} \otimes V) \\
\downarrow \mathfrak{A}(c_2) \otimes \mathfrak{A}(c_1) \otimes V \\
\mathfrak{A}(c_2) \otimes \mathfrak{A}(c_1) \otimes V
\end{array}
\]

\[
\begin{array}{cc}
\mathcal{L} \otimes V)_{\gamma} & \xrightarrow{(\mathcal{L} \otimes V)_{\gamma}} \mathcal{L} \otimes V \\
\mathcal{L} \otimes V)_{\gamma} & \xrightarrow{\Lambda \otimes \text{id}_V} \mathcal{L} \otimes V
\end{array}
\]

(2.12c) in $\mathfrak{A}(c_2)\text{-Mod}$ as the composition of the natural isomorphism of left $\mathfrak{A}(c_2)$-modules from Remark (A.5) and the morphism of $\mathfrak{A}(c_2)$-modules obtained by evaluating the $M$-tensoring $\otimes : \mathfrak{A}(c_2)\text{-Mod} \times M \to \mathfrak{A}(c_2)\text{-Mod}$ on $L : \mathfrak{A}(c_2) \otimes \mathfrak{A}(c_1) L_{c_1} \to L_{c_2}$ in $\mathfrak{A}(c_2)\text{-Mod}$ and $\text{id}_V : V \to V$ in $M$.

Furthermore, $\otimes$ assigns to morphisms $\mathcal{F} : L \to L'$ in $\text{Rep}(\mathfrak{A})$ and $\xi : V \to V'$ in $M$ the morphism $\mathcal{F} \otimes \xi : L \otimes V \to L' \otimes V'$ in $\text{Rep}(\mathfrak{A})$ consisting of the morphisms of left $\mathfrak{A}(c)$-modules
\[
(\mathcal{F} \otimes \xi)_c := \mathcal{F}_c \otimes \xi : (\mathcal{L} \otimes V)_c \longrightarrow (\mathcal{L}' \otimes V')_c
\]

(2.12d) in $\mathfrak{A}(c)\text{-Mod}$ obtained evaluating the $M$-tensoring $\otimes : \mathfrak{A}(c)\text{-Mod} \times M \to \mathfrak{A}(c)\text{-Mod}$ on $\mathcal{F}_c : L_c \to L'_c$ in $\mathfrak{A}(c)\text{-Mod}$ and $\xi : V \to V'$ in $M$, for all $c \in C$. Using the $M$-tensoring for left modules (A.5) and its compatibility with the change-of-monoid adjunction, see Remark (A.5), it is straightforward to check that the above data fulfil the axioms of Remark (2.5) and that the above assignment defines a functor $\otimes$.

The $M$-powering
\[
(-)^{-} : \text{Rep}(\mathfrak{A}) \times M^{\text{op}} \longrightarrow \text{Rep}(\mathfrak{A})
\]
(2.13a) on $\text{Rep}(\mathfrak{A})$ can be described explicitly as follows. $(-)^{-}$ assigns to an $\mathfrak{A}$-representation $L \in \text{Rep}(\mathfrak{A})$ and an object $V \in M$ the $\mathfrak{A}$-representation $L^V \in \text{Rep}(\mathfrak{A})$ consisting of the following data:

(1) For each $c \in C$, the left $\mathfrak{A}(c)$-module
\[
(\mathcal{L}^V)_c := (\mathcal{L}^V)_c \in \mathfrak{A}(c)\text{-Mod}
\]
(2.13b) obtained evaluating the powering $(-)^{-} : \mathfrak{A}(c)\text{-Mod} \times M^{\text{op}} \to \mathfrak{A}(c)\text{-Mod}$ on $L_c \in \mathfrak{A}(c)\text{-Mod}$ and $V \in M$, see (A.7).

(2) For each morphism $\gamma : c_1 \to c_2$ in $C$, the morphism of left $\mathfrak{A}(c_1)$-modules
\[
(\mathcal{L}^V)_\gamma := (\mathcal{L}^V)_\gamma : (\mathcal{L}^V)_{c_1} \longrightarrow (\mathcal{L}^V)_{c_2} \mathfrak{A}(c_1) \otimes V = (\mathcal{L}^V)_{c_2} \mathfrak{A}(c_1)
\]
(2.13c) in $\mathfrak{A}(c_1)\text{-Mod}$ defined as the morphism of $\mathfrak{A}(c_1)$-modules obtained evaluating the powering $(-)^{-} : \mathfrak{A}(c_1)\text{-Mod} \times M^{\text{op}} \to \mathfrak{A}(c_1)\text{-Mod}$ on $L : L_c \to L_{c_2} \mathfrak{A}(c_1)$ in $\mathfrak{A}(c_1)\text{-Mod}$ and $\text{id}_V : V \to V$ in $M$, where we also used the commutative square (A.10) on the right hand side.
Furthermore, \((-\quad\cdot\quad-\quad)\) assigns to morphisms \(F: \mathcal{L} \to \mathcal{L}'\) in \(\text{Rep}(\mathfrak{A})\) and \(\xi: V' \to V\) in \(M\) the morphism \(F^\xi: \mathcal{L}^V \to \mathcal{L}'^{V'}\) in \(\text{Rep}(\mathfrak{A})\) consisting of the morphisms of left \(\mathfrak{A}(c)\)-modules

\[
(F^\xi)_c := (F_c)^\xi : (\mathcal{L}^V)_c \longrightarrow (\mathcal{L}'^{V'})_c
\]

in \(\mathfrak{A}(c)\text{-Mod}\) obtained evaluating the powering \((-\quad\cdot\quad-\quad)\) : \(\mathfrak{A}(c)\text{-Mod} \times M^{op} \to \mathfrak{A}(c)\text{-Mod}\) on \(F_c : \mathcal{L}_c \to \mathcal{L}'_c\) in \(\mathfrak{A}(c)\text{-Mod}\) and \(\xi: V' \to V\) in \(M\), for all \(c \in C\). Using the \(M\)-powering functor for left modules (A.7) and its compatibility with the change-of-monoid adjunction, see Remark A.3, it is straightforward to check that the data listed above fulfil the axioms of Remark 2.5 and that the above assignment defines a functor \((-\quad\cdot\quad-\quad)\).

For each object \(V \in M\), partial evaluations of the \(M\)-tensoring and of the \(M\)-powering on \(\text{Rep}(\mathfrak{A})\) give rise to the adjunction

\[
(\quad \otimes \quad V: \text{Rep}(\mathfrak{A}) \longrightarrow \text{Rep}(\mathfrak{A}) : (-)^V.
\]

We conclude describing the \(M\)-enriched hom

\[
[-\quad,-\quad]_\mathfrak{A}: \text{Rep}(\mathfrak{A})^{op} \times \text{Rep}(\mathfrak{A}) \longrightarrow M
\]

on \(\text{Rep}(\mathfrak{A})\). \([-\quad,-\quad]_\mathfrak{A}\) assigns to \(\mathfrak{A}\)-representations \(\mathcal{L}, \mathcal{L}' \in \text{Rep}(\mathfrak{A})\) the equalizer

\[
\mathcal{E} := \text{lim} \big( \prod_{\mathfrak{A}(c)} [\mathcal{L}_c, \mathcal{L}'_c, \mathfrak{A}(c) \longrightarrow \mathcal{E} \longrightarrow \prod_{\mathfrak{A}(c)} [\mathcal{L}_c, \mathcal{L}'_c, \mathfrak{A}(c)] \big) \in M,
\]

where we used the \(M\)-enriched hom \([-\quad,-\quad]_A : \mathfrak{A}\text{-Mod}^{op} \times \mathfrak{A}\text{-Mod} \to M\) from (A.9). \(\mathcal{E}\) above denotes the morphism in \(M\) defined via the universal property of the product by the diagram

\[
\begin{array}{ccc}
\prod_{\mathfrak{A}(c)} [\mathcal{L}_c, \mathcal{L}'_c, \mathfrak{A}(c)] \ar[rr]^{\mathcal{E}} & & \prod_{\mathfrak{A}(c)} [\mathcal{L}_c, \mathcal{L}'_c, \mathfrak{A}(c)] \\
\pr_{c_1} \downarrow & & \downarrow \pr_{c_1} \\
[\mathcal{L}_{c_2}, \mathcal{L}'_{c_2}, \mathfrak{A}(c_2)] \ar[r] & [\mathcal{L}_{c_2}, \mathfrak{A}(c_1), \mathcal{L}'_{c_2}, \mathfrak{A}(c_1)] \ar[r] & [\mathcal{L}_{c_1}, \mathcal{L}'_{c_1}, \mathfrak{A}(c_1)]
\end{array}
\]

in \(M\), for each \(\gamma: c_1 \to c_2\) in \(C\). Furthermore, \(\mathcal{E}\) above denotes the morphism in \(M\) defined via the universal property of the product by the diagram

\[
\begin{array}{ccc}
\prod_{\mathfrak{A}(c)} [\mathcal{L}_c, \mathcal{L}'_c, \mathfrak{A}(c)] \ar[rr]^{\mathcal{E}} & & \prod_{\mathfrak{A}(c)} [\mathcal{L}_c, \mathcal{L}'_c, \mathfrak{A}(c)] \\
\pr_{c_1} \downarrow & & \downarrow \pr_{c_1} \\
[\mathcal{L}_{c_1}, \mathcal{L}'_{c_1}, \mathfrak{A}(c_1)] \ar[r] & [\mathcal{L}_{c_1}, \mathcal{L}'_{c_1}, \mathfrak{A}(c_1)] \ar[r] & [\mathcal{L}_{c_1}, \mathcal{L}'_{c_1}, \mathfrak{A}(c_1)]
\end{array}
\]

in \(M\), for each \(\gamma: c_1 \to c_2\) in \(C\). The action of \([-\quad,-\quad]_\mathfrak{A}\) on morphisms is defined combining the universal property of the limit in (2.15), the \(M\)-enriched hom \([-\quad,-\quad]_A : \mathfrak{A}\text{-Mod}^{op} \times \mathfrak{A}\text{-Mod} \to M\) in (A.9) and the restriction functor \((-\quad\cdot\quad-\quad)\) : \(B\text{-Mod} \to \mathfrak{A}\text{-Mod}\) in (A.1).

Remark 2.8. Let us provide a more explicit description of the \(M\)-enriched hom \([\mathcal{L}, \mathcal{L}']_\mathfrak{A} \in M\) from (2.15) when the target \(M = \text{Vec}_K\) is the familiar closed symmetric monoidal category of vector spaces over a field \(K\). In this case the vector space \([\mathcal{L}, \mathcal{L}']_\mathfrak{A} \in \text{Vec}_K\) consists of collections \(x = (x_c)_{c \in C}\) of \(\mathfrak{A}(c)\)-linear maps \(x_c : \mathcal{L}_c \to \mathcal{L}'_c\), for all \(c \in C\), such that \(x_{c_2} |_{\mathfrak{A}(c_1)} \gamma = \mathcal{L}'_\gamma x_{c_1} : \mathcal{L}_{c_1} \to \mathcal{L}'_{c_2} |_{\mathfrak{A}(c_1)}\) coincide as \(\mathfrak{A}(c_1)\)-linear maps, for all \(\gamma: c_1 \to c_2\) in \(C\). A similar description holds true for any concrete closed symmetric monoidal category, including the category of cochain complexes \(M = \text{Ch}_K\), which will be used in Section 3.
For each object \( L \in \text{Rep}(\mathfrak{A}) \), partial evaluations of the \( M \)-tensoring and of the \( M \)-enriched hom give rise to the adjunction

\[
\mathcal{L} \otimes (-) : M \longrightarrow \text{Rep}(\mathfrak{A}) : \mathcal{L}, -|_{\mathfrak{A}}.
\]

(2.16)

**Remark 2.9.** Note that the change-of-net adjunction \([1.9]\) is compatible with the adjunction \([2.14]\) in the following sense. Given a morphism \( \Phi : \mathfrak{A} \to \mathfrak{B} \) in \( \text{Net}_{C}^{M} \) and an object \( V \in M \), the diagram of right adjoint functors

\[
\begin{array}{ccc}
\text{Rep}(\mathfrak{B}) & \xrightarrow{\text{Res}_{\Phi}} & \text{Rep}(\mathfrak{A}) \\
(-)^{V} & \downarrow & \downarrow (-)^{V} \\
\text{Rep}(\mathfrak{B}) & \xrightarrow{\text{Res}_{\Phi}} & \text{Rep}(\mathfrak{A})
\end{array}
\]

commutes as a straightforward consequence of their definitions. Therefore, the corresponding diagram of left adjoint functors commutes up to a unique natural isomorphism \( \text{Ext}_{\Phi}(- \otimes V) \cong \text{Ext}_{\Phi}(-) \otimes V \).

\( \square \)

**Remark 2.10.** Also the adjunction \([2.15]\) is compatible with the adjunction \([2.14]\) in the following sense. Given \( \tilde{c} \in C \), \( \mathfrak{A} \in \text{Net}_{C}^{M} \) and \( V \in M \), the diagram of left adjoint functors

\[
\begin{array}{ccc}
\text{Rep}(\mathfrak{A}) & \xrightarrow{(-)_{\tilde{c}}} & \mathfrak{A}(\tilde{c}) \text{ Mod} \\
(-)^{V} & \downarrow & \downarrow (-)^{V} \\
\text{Rep}(\mathfrak{A}) & \xrightarrow{(-)_{\tilde{c}}} & \mathfrak{A}(\tilde{c}) \text{ Mod}
\end{array}
\]

commutes as a straightforward consequence of their definitions.

\( \triangle \)

### 2.2 Model structure for net representations

As anticipated in Section 1, nets of algebras valued in a closed symmetric monoidal model category \( M \) come equipped with the projective model structure recalled below.

**Definition 2.11.** Let \( C \) be a (small) category and \( M \) a closed symmetric monoidal model category. A natural transformation in \( \text{Net}_{C}^{M} \) is called a weak equivalence (fibration) if all its components in \( M \) are weak equivalences (respectively fibrations) and a cofibration if it has the left lifting property with respect to all acyclic fibrations.

It turns out that such weak equivalences between nets of algebras do not induce ordinary categorical equivalences between the associated net representation categories, see Example 2.12 below. The goal of the present section is to repair this shortcoming by endowing net representation categories with suitable model structures (including a notion of weak equivalence between net representations), such that weak equivalences between nets of algebras induce Quillen equivalences between the associated model categories of net representations. In this way, ordinary categorical equivalence is recovered at the level of the homotopy categories of net representations, i.e. after inverting all weak equivalences between net representations.

**Example 2.12** (Ordinary categorical equivalence fails for weakly equivalent nets of algebras). To motivate the introduction of model structures on the categories of net representations and showcase the relevance of the upcoming Proposition 2.22 in the context of algebraic quantum field theory, we examine the Klein-Gordon field of mass \( m \geq 0 \) using two different, yet equivalent, descriptions. More precisely, we shall construct two \( \text{Ch}_{C} \)-valued nets of algebras \( \mathfrak{A} \) and \( \tilde{\mathfrak{A}} \) on \( \text{Loc}_{m} \) describing the Klein-Gordon field and the evident weak equivalence \( \Phi : \tilde{\mathfrak{A}} \to \mathfrak{A} \) in \( \text{Net}_{\text{Loc}_{m}}^{\text{Ch}_{C}} \).
relating them. (𝒜 will denote the standard Klein-Gordon net \[\text{FV15}, \text{Sec. 4.2}\], regarded as a \(\text{Ch}_C\)-valued net of algebras concentrated in degree 0, while \(\tilde{\mathcal{A}}\) will denote the Klein-Gordon net from the Batalin-Vilkovisky formalism \[\text{BBS20}\].) From the explicit description of the associated net representation categories \(\text{Rep}(\mathcal{A})\) and \(\text{Rep}(\tilde{\mathcal{A}})\), it will be manifest that the restriction functor \(\text{Res}_\mathcal{A} : \text{Rep}(\tilde{\mathcal{A}}) \rightarrow \text{Rep}(\mathcal{A})\) is not even essentially surjective and hence it fails to be an ordinary categorical equivalence. This explains why the flexibility of the Batalin-Vilkovisky formalism forces one to endow the net representation categories with suitable model structures, that come along with the more flexible concept of Quillen equivalence. In this way, although being genuinely different in the ordinary categorical sense, the net representation categories \(\text{Rep}(\mathcal{A})\) and \(\text{Rep}(\tilde{\mathcal{A}})\), associated with the two equivalent descriptions \(\mathcal{A}\) and \(\tilde{\mathcal{A}}\) of the Klein-Gordon field become the same in the model categorical sense due to Proposition 2.22.

Let us start from the standard Klein-Gordon net of algebras \(\mathcal{A} \in \text{Net}_{\text{Loc}_c}^{\text{Ch}_c}\). To \(M \in \text{Loc}_m\) it assigns the unital associative differential graded algebra

\[
\mathcal{A}(M) := T_C(V(M))/J \in \text{DGA}_C := \text{Mon}(\text{Ch}_C)
\]  

(2.19a)

that is freely generated over \(\mathbb{C}\) by the cochain complex \(V(M) := C_c^\infty(M)/PC_c^\infty(M) \in \text{Ch}_R\) concentrated in degree 0, modulo the two-sided ideal \(I \subseteq T_C(W(M))\) generated by the canonical commutation relations

\[
[\varphi_1] \otimes [\varphi_2] - [\varphi_2] \otimes [\varphi_1] - i \int_M \varphi_1 G \varphi_2 \text{vol}_M \mathbf{1},
\]

(2.19b)

for all \([\varphi_1], [\varphi_2] \in V(M)^0\). Here \(P := \Box - m^2 : C_c^\infty(M) \rightarrow C_c^\infty(M)\) denotes the Klein-Gordon operator, \(G : C_c^\infty(M) \rightarrow C_c^\infty(M)\) denotes the associated retarded-minus-advanced propagator (which vanishes on \(PC_c^\infty(M)\) and is formally skew-adjoint, see \[\text{BGP07}, \text{Sec. 3.4}\]) and \(\text{vol}_M\) denotes the volume form on \(M\). The push-forward of compactly supported smooth functions along morphisms in \(\text{Loc}_m\) turns the assignment \(M \in \text{Loc}_m \mapsto \mathcal{A}(M) \in \text{DGA}_C\) into a net of algebras \(\mathcal{A} \in \text{Net}_{\text{Loc}_c}^{\text{Ch}_c}\). The Klein-Gordon net \(\tilde{\mathcal{A}} \in \text{Net}_{\text{Loc}_c}^{\text{Ch}_c}\) from the Batalin-Vilkovisky formalism is defined in a similar fashion. Explicitly, to \(M \in \text{Loc}_m\) it assigns the differential graded algebra

\[
\tilde{\mathcal{A}}(M) := T_C(\tilde{V}(M))/I \in \text{DGA}_C
\]  

(2.20a)

that is freely generated by the cochain complex

\[
\tilde{V}(M) := (C_c^\infty(M) \overset{P}{\rightarrow} C_c^\infty(M)) \in \text{Ch}_R,
\]

(2.20b)

resolving the quotient \(V(M)\) and concentrated in degrees \(-1\) and 0, modulo the two-sided ideal \(I \subseteq V(M)\) generated by the (graded) canonical commutation relations

\[
\varphi_1 \otimes \varphi_2 - \varphi_2 \otimes \varphi_1 - i \int_M \varphi_1 G \varphi_2 \text{vol}_M \mathbf{1}, \quad \varphi \otimes \varphi^\dagger - \varphi^\dagger \otimes \varphi, \quad \varphi_1^\dagger \otimes \varphi_2^\dagger + \varphi_2^\dagger \otimes \varphi_1^\dagger,
\]

(2.20c)

for all \(\varphi_1, \varphi_2, \varphi \in \tilde{V}(M)^0\) and \(\varphi^\dagger, \varphi_1^\dagger, \varphi_2^\dagger \in \tilde{V}(M)^{-1}\). (The only non-trivial commutation relations involve 0-cochains.) There is an evident morphism

\[
\Phi : \tilde{\mathcal{A}} \rightarrow \mathcal{A}
\]

(2.21)

in \(\text{Net}_{\text{Loc}_c}^{\text{Ch}_c}\), whose \(M\)-component \(\Phi_M : \tilde{\mathcal{A}}(M) \rightarrow \mathcal{A}(M)\) is defined on generators by \(\Phi_M(\varphi) := [\varphi]\) and \(\Phi_M(\varphi^\dagger) := 0\), for all \(M \in \text{Loc}_m\), \(\varphi \in \tilde{V}(M)^0\) and \(\varphi^\dagger \in \tilde{V}(M)^{-1}\). In \[\text{BBS20}, \text{Rem. 6.20}\] it is shown that \(\Phi\) is a weak equivalence.

Consider now a \(\tilde{\mathcal{A}}\)-representation \(\tilde{\mathcal{L}} \in \text{Rep}(\tilde{\mathcal{A}})\). To each \(M \in \text{Loc}_m\), \(\tilde{\mathcal{L}}\) assigns a left \(\tilde{\mathcal{A}}(M)\)-module \(\tilde{\mathcal{L}}(M) \in \tilde{\mathcal{A}}(M)^\text{Mod}\). One can equivalently encode its left \(\mathcal{A}(M)\)-action \(\lambda_M : \mathcal{A}(M) \otimes \tilde{\mathcal{L}}(M) \rightarrow \tilde{\mathcal{L}}(M)\) in \(\text{Ch}_C\) as a morphism \(\tilde{\lambda}_M : \mathcal{A}(M) \rightarrow [\tilde{\mathcal{L}}(M), \tilde{\mathcal{L}}(M)]\) in \(\text{DGA}_C\). (Recall that the
internal hom \([\tilde{L}(M), \tilde{L}(M)]\) ∈ \text{Ch}_C\) carries a canonical monoid structure whose multiplication is the composition and whose unit is the identity. Below we will denote its differential by \(\partial\).

It follows from (2.20) that \(\lambda_M\) assigns (compatibly with the (graded) canonical commutation relations) to each \(\varphi \in \mathcal{V}(M)^0\) a 0-cocycle \(\tilde{\lambda}_M(\varphi) \in Z^0([\tilde{L}(M), \tilde{L}(M)])\) and to each \(\varphi^1 \in \mathcal{V}(M)^{-1}\) a (-1)-cochain \(\tilde{\lambda}_M(\varphi^1) \in [\tilde{L}(M), \tilde{L}(M)]^{-1}\) such that \(\partial(\tilde{\lambda}_M(\varphi^1)) = \lambda_M(\varphi^1)\).

Compared to an \(\mathfrak{A}\)-representation, an \(\mathfrak{A}\)-representation \(\mathcal{L} \in \text{Rep}(\mathfrak{A})\) consists of fewer data, as explained below. Recalling (2.19), for each \(\mathcal{L} \in \text{Rep}(\mathfrak{A})\) only assigns (compatibly with the canonical commutation relations) to each \([\varphi] \in \mathcal{V}(M)^0\) a 0-cocycle \(\tilde{\lambda}_M([\varphi]) \in Z^0([\mathcal{L}(M), \mathcal{L}(M)])\).

Recall now that the restriction functor \(\mathfrak{Res}_\Phi : \text{Rep}(\mathfrak{A}) \to \text{Rep}(\mathfrak{A})\) simply restricts the left actions, retaining all other data. Given an \(\mathfrak{A}\)-representation \(\tilde{\mathcal{L}} \in \text{Rep}(\tilde{\mathfrak{A}})\) such that \(\tilde{\lambda}_M(\varphi^1) \neq 0 \in [\tilde{L}(M), \tilde{L}(M)]^{-1}\) for some \(M \in \text{Loc}_m\) and \(\varphi^1 \in \mathcal{V}(M)^0\) (for instance take \(\tilde{\mathcal{L}} = \tilde{\mathfrak{A}}\) to be the net of algebras \(\mathfrak{A}\) itself regarded as an \(\tilde{\mathfrak{A}}\)-representation), one immediately realizes that there cannot exist an \(\mathfrak{A}\)-representation \(\mathcal{L} \in \text{Rep}(\mathfrak{A})\) and an isomorphism \(\Phi : \mathfrak{Res}_\Phi \mathcal{L} \to \tilde{\mathfrak{A}}\) in \(\text{Rep}(\tilde{\mathfrak{A}})\). Indeed, by construction of \(\Phi\), the left actions \(\lambda_M \circ \Phi_M\) of \(\mathfrak{Res}_\Phi \mathcal{L}\) are such that \(\lambda_M(\Phi_M(\varphi^1))) = 0 \in [\mathcal{L}(M), \mathcal{L}(M)]^{-1}\) vanishes for all \(M \in \text{Loc}_m\) and \(\varphi^1 \in \mathcal{V}(M)^{-1}\); moreover, any \(\mathfrak{A}\)-representation that is isomorphic to \(\mathfrak{Res}_\Phi \mathfrak{A}\) shares this feature. It follows that \(\mathfrak{Res}_\Phi\) is not essentially surjective and hence \(\mathfrak{E} \rightarrow \mathfrak{Res}_\Phi\) cannot be an ordinary categorical equivalence.

The above example indicates that the categories of net representations associated with weakly equivalent nets of algebras fail to be equivalent in the ordinary categorical sense due to the rigidity of the concept of isomorphism in these categories. This suggests that the concept of isomorphism between net representations needs to be replaced with an appropriate concept of weak equivalence. As we will see in the remainder of this section, after endowing the net representation categories with suitable model structures (including weak equivalences), we shall solve the shortcoming evidenced in Example 2.12 by proving with Proposition 2.22 that \(\mathfrak{E} \rightarrow \mathfrak{Res}_\Phi\) gives rise to a Quillen equivalence when \(\Phi\) is a weak equivalence.

We construct the desired model structures on net representation categories \(\text{Rep}(\mathfrak{A})\), for all \(\mathfrak{A} \in \text{Net}_C^M\), by exploiting (right) transfer techniques for cofibrantly generated model categories. After recalling some preliminaries about transfer of model structures, we will proceed with the above mentioned construction in two steps. Even though our results are certainly well-known to practitioners, it seems that the question of endowing the category of left modules over a monoid in the functor category \(\text{Fun}(\mathcal{C}, M)\) has not received attention so far in the literature (at least when finite coproducts fail to exist in the source category \(\mathcal{C}\), see Remark 2.19).

In preparation for the transfer theorem that we shall recall below, let us introduce some terminology. For a category \(\mathcal{E}\) equipped with a terminal object \(* \in \mathcal{E}\) and with two distinguished classes of morphisms, called weak equivalences and respectively fibrations, one says that an object \(E \in \mathcal{E}\) is fibrant when the unique morphism \(E \to *\) in \(\mathcal{E}\) to the terminal object \(*\) is a fibration. A functorial path object is a triplet \((P, w, f)\) consisting of a functor \(P : \mathcal{E} \to \mathcal{E}\), a natural weak equivalence \(w : \text{id}_E \to P\), i.e. a natural transformation whose components \(w_E : E \to P(E)\) in \(\mathcal{E}\) are weak equivalences, for all \(E \in \mathcal{E}\), and a natural fibration \(f : P \to (-)^{\times 2}\), i.e. a natural transformation whose components \(f_E : P(E) \to E \times E\) in \(\mathcal{E}\) are fibrations, for all \(E \in \mathcal{E}\), such that the diagonal morphism \((\text{id}_E, \text{id}_E) = f_E w_E : E \to E \times E\) in \(\mathcal{E}\) factors as the weak equivalence \(w_E\) followed by the fibration \(f_E\), for all \(E \in \mathcal{E}\).

**Theorem 2.13** [BM03, Sec. 2.5 and Sec. 2.6]. Let \(\mathcal{E}\) be a complete and cocomplete category, \(\mathcal{D}\) a cofibrantly generated model category, whose objects are all fibrant, and \(F \dashv U : \mathcal{E} \to \mathcal{D}\) an adjunction. Define a morphism \(f : E \to E'\) in \(\mathcal{E}\) to be a weak equivalence (fibration) if \(U(f) : U(E) \to U(E')\) in \(\mathcal{D}\) is a weak equivalence (respectively fibration). Then this determines a cofibrantly generated model structure on \(\mathcal{E}\) under the following hypotheses:

(i) \(F\) preserves small objects,
(ii) \( E \) has a functorial path object \((P,w,f)\).

\[\text{Remark 2.14.} \text{ Even though this is not stated explicitly in Theorem 2.13 it is straightforward to realize that, given a set } I \text{ of generating cofibrations (respectively acyclic cofibrations) for } D, F(I) \text{ (respectively } F(J)) \text{ is a set of generating cofibrations (respectively acyclic cofibrations) for } E. \text{ Indeed, by the adjunction } F \dashv U, \text{ the right lifting property (see e.g. [Hov99, Def. 1.1.2]) of a morphism } f : E \to E' \text{ in } E \text{ against any morphism of } F(I) \text{ (respectively } F(J)) \text{ is equivalent to the right lifting property of the morphism } U(f) : U(E) \to U(E') \text{ in } D \text{ against any morphism of } I \text{ (respectively } J). \text{ Since weak equivalences and fibrations in } E \text{ are by definition detected by } U, \text{ it follows that } F(I) \text{ (respectively } F(J)) \text{ is a set of generating cofibrations (respectively acyclic cofibrations) for } E. \triangleq\]

\[\text{Set-up 2.15.} \text{ We assume } M \text{ to be a cofibrantly generated closed symmetric monoidal model category with cofibrant unit } 1, \text{ see [Hov99, Sec. 4.2], that satisfies the monoid axiom, see [SS00, Def. 3.3]. In order to meet the hypotheses of Theorem 2.13 we further assume that all objects of } M \text{ are fibrant and the existence of an interval object } (I,r,b) \text{ in } M, \text{ that consists of a cofibrant object } I \in M \text{ equipped with a factorization of the codiagonal morphism } (\text{id}_1, \text{id}_1) = r b : 1 \sqcup 1 \to 1 \text{ in } M \text{ into a cofibration } b : 1 \sqcup 1 \to I \text{ in } M \text{ followed by a weak equivalence } r : I \to 1 \text{ in } M. \]

\[\text{Let us mention that the category } M = \text{Ch}_K \text{ of cochain complexes over a field } K, \text{ which features in our applications in Section 3 meets all the requirements of Set-up 2.13. For more details see the beginning of Section 3.}\]

\[\text{For a given net of algebras } \mathfrak{A} \in \text{Net}_C^M \text{ and using both Set-up 2.15 and Theorem 2.13 we shall now go through a two-step procedure to transfer a model structure on the category } E := \text{Rep}(\mathfrak{A}) \text{ of } \mathfrak{A}-\text{representations from the product model category } D := \prod_{c \in C} M. \text{ The latter is the model category whose weak equivalences, fibrations and cofibrations are defined component-wise, see [Hov99, Ex. 1.1.6]. Since per hypothesis } M \text{ is cofibrantly generated and its objects are all fibrant, the same holds true for the product model category } \prod_{c \in C} M.\]

\[\text{Step 1} \text{ The first step presents the adjunction that will be used to transfer the model structure from } \prod_{c \in C} M. \text{ Consider the forgetful functor } U : \text{Rep}(\mathfrak{A}) \to \prod_{c \in C} M \text{ that sends an } \mathfrak{A}-\text{representation } L \in \text{Rep}(\mathfrak{A}) \text{ to its underlying collection } (L_c)_c \in \prod_{c \in C} M, \text{ where } L_c \in M \text{ denotes the object of } M \text{ underlying the left } \mathfrak{A}(c) \text{-module } L_c \in \mathfrak{A}(c) \text{Mod}, \text{ for all } c \in C. \text{ We emphasize that } U \text{ preserves and lifts both limits and colimits because those can be computed component-wise in the product category } \prod_{c \in C} M \text{ by endowing the resulting collection with the induced structure of an } \mathfrak{A}-\text{representation. This shows that } \text{Rep}(\mathfrak{A}) \text{ is both complete and cocomplete. Furthermore, } U \text{ is part of the adjunction}\]

\[F : \prod_{c \in C} M \xrightarrow{\sim} \text{Rep}(\mathfrak{A}) : U, \tag{2.22}\]

\[\text{whose left adjoint } F \text{ is defined below. } F \text{ sends a collection } V := (V_c)_c \in \prod_{c \in C} M \text{ to the } \mathfrak{A}-\text{representation } F(V) \in \text{Rep}(\mathfrak{A}) \text{ consisting of the following data, see Definition 2.4:}\]

\[\text{(1) For each } c \in C, \text{ the free left } \mathfrak{A}(c)-\text{module}\]

\[(F(V))_c := \mathfrak{A}(c) \otimes \text{ (}\prod_{\gamma \in C(c,c)} V^\gamma_c \text{)} \in \mathfrak{A}(c) \text{Mod}. \tag{2.23a}\]

\[\text{(2) For each morphism } \gamma : c_1 \to c_2 \text{ in } C, \text{ the morphism of left } \mathfrak{A}(c_2)-\text{modules}\]

\[(F(V))_\gamma : \mathfrak{A}(c_2) \otimes_{\mathfrak{A}(c_1)} (F(V))_{c_1} \cong \mathfrak{A}(c_2) \otimes \text{ (}\prod_{\bar{c}_1 \in C} V^\gamma_{\bar{c}_1} \text{)} \xrightarrow{id \otimes \gamma^*} (F(V))_{c_2} \tag{2.23b}\]

\[\text{in } \mathfrak{A}(c_2) \text{Mod, where } \gamma^* : \prod_{\bar{c}_1 \in C} V^\gamma_{\bar{c}_1} \to \prod_{\bar{c}_1 \in C} V^\gamma_{\bar{c}_1} \text{ in } M \text{ denotes the morphism that sends the } (\bar{c}_1, \gamma_{\bar{c}_1})-\text{component of the domain to the } (\bar{c}_1, \gamma_{\bar{c}_1})-\text{component of the codomain.}\]
One easily checks that these data fulfil the axioms for an \( \mathfrak{A} \)-representation. Furthermore, \( F \) sends a morphism \( \xi := (\xi_c) : V \to V' \) in \( \prod_{c \in C} M \) to the morphism of \( \mathfrak{A} \)-representations \( F(\xi) : F(V) \to F(V') \) in \( \text{Rep}(\mathfrak{A}) \) consisting of the morphism of left \( \mathfrak{A}(c) \)-modules

\[
(F(\xi))_c := \text{id} \otimes \prod_{\tilde{c} \in \mathfrak{A}(\tilde{c},c)} \xi_{\tilde{c}} : (F(V))_c \to (F(V'))_c
\]

(2.23c)

in \( \mathfrak{A}(c)\text{-Mod} \), for each \( c \in C \). It is easy to check that these data indeed define a morphism of a \( \mathfrak{A} \)-representations and that \( F \) as defined above is a functor.

In order to show that (2.22) is an adjunction as claimed, we describe its unit \( \eta \) and its counit \( \varepsilon \). The unit is the natural transformation

\[
\eta : \text{id}_{\prod_{c \in C} M} \longrightarrow UF,
\]

(2.24a)

whose component \( \eta_V : V \to UF(V) \) in \( \prod_{c \in C} M \) at the collection \( V \in \prod_{c \in C} M \) consists of the morphisms in \( M \) defined by

\[
(\eta_V)_c : V_c \cong 1 \otimes V_c \overset{1 \otimes \iota_{c,16}}{\longrightarrow} (UF(V))_c,
\]

(2.24b)

for all \( c \in C \), where \( \iota_{c,16} : V_c \to \prod_{\tilde{c} \in \tilde{C}} V_{\tilde{c}} \) in \( M \) denotes the canonical morphism to the coproduct. The counit is the natural transformation

\[
\varepsilon : FU \longrightarrow \text{id}_{\text{Rep}(\mathfrak{A})},
\]

(2.25a)

whose component \( \varepsilon_L : FU(L) \to L \) in \( \text{Rep}(\mathfrak{A}) \) at the \( \mathfrak{A} \)-representation \( L \in \text{Rep}(\mathfrak{A}) \) consists, for each \( c \in C \), of the morphism of left \( \mathfrak{A}(c) \)-modules \( \varepsilon_L)_c : (FU(L))_c \to L_c \) in \( \mathfrak{A}(c)\text{-Mod} \) defined via the universal property of the coproduct by the diagram

\[
\begin{array}{ccc}
(FU(L))_c & \xrightarrow{(\varepsilon_L)_c} & L_c \\
\downarrow \text{id} \otimes \iota_{c,\tilde{c}} & & \downarrow L_{\tilde{c}} \\
\mathfrak{A}(c) \otimes L_{\tilde{c}} & \longrightarrow & \mathfrak{A}(c) \otimes L_{\tilde{c}} \end{array}
\]

(2.25b)

in \( \mathfrak{A}(c)\text{-Mod} \), for all \( \tilde{c} \in C \) and \( \tilde{c} \in \mathfrak{A}(\tilde{c},c) \). The bottom horizontal arrow denotes the canonical morphism to the coequalizer \( \mathfrak{A}(c) \otimes \mathfrak{A}(\tilde{c}) L_{\tilde{c}} \in \mathfrak{A}(\tilde{c})\text{-Mod} \), see (2.14). The verification of the triangle identities \( U(\varepsilon) \eta_U = \text{id}_U \) and \( \varepsilon_F F(\eta) = \text{id}_F \) is straightforward, thus proving that (2.22) is an adjunction, as claimed.

Remark 2.16. We observe incidentally that the right adjoint functor \( U \) preserves the \( M \)-powerings on \( \text{Rep}(\mathfrak{A}) \) and on \( \prod_{c \in C} M \) respectively. The latter is obtained by acting componentwise with the internal hom of \( M \). It follows then by definition of the \( M \)-powering functor on \( \text{Rep}(\mathfrak{A}) \), see (2.13), that for each \( V \in M \), the diagram of functors

\[
\begin{array}{ccc}
\text{Rep}(\mathfrak{A}) & \xrightarrow{U} & \prod_{c \in C} M \\
\downarrow (-)^V & & \downarrow (-)^V \\
\text{Rep}(\mathfrak{A}) & \xrightarrow{U} & \prod_{c \in C} M
\end{array}
\]

(2.26)
commutes. As a consequence, the diagram of left adjoint functors commutes up to a unique natural isomorphism, i.e. there exists a unique natural isomorphism $F(-) \otimes V \cong F((-) \otimes V)$ between functors from $\prod_{c \in C} M$ to $\text{Rep}(\mathfrak{A})$, where the left hand side displays the $M$-tensoring on $\text{Rep}(\mathfrak{A})$ from (2.12) and the right hand side displays the $M$-tensoring on $\prod_{c \in C} M$ defined component-wise in $C$ by the tensor product of $M$.

**Definition 2.17.** A morphism $F : \mathcal{L} \to \mathcal{L}'$ in $\text{Rep}(\mathfrak{A})$ is a weak equivalence (fibration) if $U(F) : U(\mathcal{L}) \to U(\mathcal{L}')$ in $\prod_{c \in C} M$ is a weak equivalence (respectively fibration), i.e. if $F_c : \mathcal{L}_c \to \mathcal{L}'_c$ is a weak equivalence (respectively fibration) in the underlying model category $M$, for all $c \in C$.

Note that all objects of $\text{Rep}(\mathfrak{A})$ are fibrant because, as previously explained, all objects of $\prod_{c \in C} M$ are fibrant.

**Step 2** The second step checks that the hypotheses of Theorem 2.13 are met. Hypothesis (i), i.e. that $F$ preserves small objects, follows by recalling that, as explained in Step 1, $U$ preserves colimits and $F$ is left adjoint to $U$. We show that also hypothesis (ii) of Theorem 2.13 is met, i.e. we construct a functorial path object $(P, w, f)$ in $\text{Rep}(\mathfrak{A})$. This is achieved by a standard construction that uses the interval object $(I, r, b)$ in $M$, see Set-up 2.15, the closed symmetric monoidal model structure on $M$, the hypothesis that all objects of $M$ are fibrant and the $M$-powering on $\text{Rep}(\mathfrak{A})$ from (2.13). Explicitly, we consider the functor

$$P := (-)^I : \text{Rep}(\mathfrak{A}) \longrightarrow \text{Rep}(\mathfrak{A})$$

and the natural transformations

$$w : \text{id}_{\text{Rep}(\mathfrak{A})} \cong (-)^1 \xrightarrow{(\cdot)^r} P,$$

$$f : P \cong (-)^I 1 \xrightarrow{(-)^{1^{1}} \cong (-)^{x^2}}.$$

By definition of the $M$-powering (2.13), for each $\mathcal{L} \in \text{Rep}(\mathfrak{A})$ and up to the evident isomorphisms, $U(w_{\mathcal{L}}) : U(\mathcal{L}) \to U(\mathcal{L}^I)$ consists of the components $[r, \mathcal{L}_c] : [1, \mathcal{L}_c] \to [I, \mathcal{L}_c]$ in $M$, for all $c \in C$, and $U(f_{\mathcal{L}}) : U(\mathcal{L}'^I) \to U(\mathcal{L} \times \mathcal{L})$ consists of the components $[b, \mathcal{L}_c] : [I, \mathcal{L}_c] \to [\mathcal{L} \times \mathcal{L}], \mathcal{L}_c$ in $M$, for all $c \in C$. (Here $[\cdot, \cdot] : M^{op} \times M \to M$ denotes the internal hom of the closed symmetric monoidal model category $M$.) Since by Set-up 2.15 all objects $V \in M$ are fibrant, $[-, V] : M^{op} \to M$ sends weak equivalences between cofibrant objects in $M$ to weak equivalences in $M$ and cofibrations in $M$ to fibrations in $M$, see [Hov99, Sec. 4.2]. Since $r : I \to 1$ in $M$ is a weak equivalence between cofibrant objects and $b : 1 \sqcup 1 \to I$ in $M$ is a cofibration, recalling Definition 2.17 we conclude that $w_{\mathcal{L}} : \mathcal{L} \to \mathcal{L}'$ in $\text{Rep}(\mathfrak{A})$ is a weak equivalence and $f_{\mathcal{L}} : \mathcal{L}'^I \to \mathcal{L} \times \mathcal{L}$ in $\text{Rep}(\mathfrak{A})$ is a fibration. Furthermore, since by Set-up 2.15 the codiagonal morphism factors as $(\text{id}_1, \text{id}_1) = r \circ b : 1 \sqcup 1 \to 1$ in $M$, it follows that the diagonal morphism factors as $(\text{id}_L, \text{id}_L) = f_{\mathcal{L}} w_{\mathcal{L}} : \mathcal{L} \to \mathcal{L} \times \mathcal{L}$ in $\text{Rep}(\mathfrak{A})$, for all $\mathcal{L} \in \text{Rep}(\mathfrak{A})$. This shows that the functor $P$ and the natural transformations $w$ and $f$ introduced in (2.27) define a functorial path object $(P, w, f)$ in $\text{Rep}(\mathfrak{A})$, hence also hypothesis (ii) of Theorem 2.13 is met. The next corollary summarizes the conclusions so far, taking into account also Remark 2.14.

**Corollary 2.18.** Under the assumptions stated in Set-up 2.12, the notions of weak equivalences and fibrations from Definition 2.17 determine a cofibrantly generated model structure on the category $\text{Rep}(\mathfrak{A})$ of $\mathfrak{A}$-representations. Furthermore, given a set of generating (acyclic) cofibrations $I$ for $M$, $F(\prod_{c \in C} I)$ is a set of generating (respectively acyclic) cofibrations for $\text{Rep}(\mathfrak{A})$.

**Remark 2.19.** When the source category $C$ admits finite coproducts, the same model structures on $\text{Net}_C$ and $\text{Rep}(\mathfrak{A})$ can be more concisely obtained by applying Appendix A.3 to the functor category $\text{Fun}(C, M)$. Indeed, as explained below, when $C$ admits finite coproducts, $\text{Fun}(C, M)$ meets the requirements of Appendix A.3.

Assuming, as in Set-up A.7 that $M$ is a cofibrantly generated closed symmetric monoidal model category that satisfies the monoid axiom and whose objects are all small, one gets the
standard cofibrantly generated projective model structure on $\text{Fun}(C, M)$. Since all objects of $M$ are small, all objects of $\text{Fun}(C, M)$ are small too. Furthermore, if the object-wise symmetric monoidal structure from Section 2.1 and the projective model structure are suitably compatible, $\text{Fun}(C, M)$ becomes a closed symmetric monoidal model category that inherits the monoid axiom from $M$. It only remains to check that the object-wise symmetric monoidal structure and the projective model structure interact nicely, i.e. that the pushout-product axiom holds. The latter follows from the assumption that the source category $C$ admits finite coproducts, see (the arXiv version of) the proof of $\text{Prop. 7.9}[\text{PS18}]$. To the best of our knowledge this is the only proof of the pushout-product axiom in this setting.

Unfortunately, finite coproducts do not exist in all the source categories of interest in the context of algebraic quantum field theory, see Example 2.23. For instance, recalling the category $\text{Loc}_{\text{caus}}$ of globally hyperbolic Lorenztian manifolds, one easily realizes that binary coproducts often fail to exist. On the other hand, finite coproducts exist in the source category $C = \text{CCO}(M)$ of causally convex open subsets of a fixed $M \in \text{Loc}_{\text{caus}}$, where they are given by the causally convex hull $U_1 \bigsqcup U_2 := \overline{U_M(U_1 \cup U_2)} \cap \overline{I_M(U_1 \cup U_2)}$ of $U_1, U_2 \in \text{CCO}(M)$. (Here $I^+_M$ denotes the chronological future/past of a subset in $M$.)

#### Remark 2.20

Combining the above observation on $\text{Rep}(\mathfrak{A})$ with the model structure on categories of left modules from Proposition A.10, one easily recognizes that, for each $\xi \in \mathfrak{A}$, the adjunction $(-)_{\xi} \dashv (-)^\xi : \mathfrak{A}(\mathfrak{C})\text{Mod} \to \text{Rep}(\mathfrak{A})$ from (2.5) is a Quillen adjunction, see [Hov99, Sec. 1.3.1]. This follows from the fact that the right adjoint functor $(-)^\xi$ sends (acyclic) fibrations in $\mathfrak{A}(\mathfrak{C})\text{Mod}$, which are detected in $M$, to (acyclic) fibrations in $\text{Rep}(\mathfrak{A})$, which are detected in $M$ component-wise in $C$. Note that the Quillen adjunction (2.5) is compatible with the $M$-tensoring, powering and enriched hom due to Remark 2.10 (In the language of [Hov99, Def. 4.2.18] the left adjoint $(-)^\xi$ is an $M$-Quillen functor.)

With Corollary 2.18 we have established a model structure on $\text{Rep}(\mathfrak{A})$. We now investigate its compatibility with the $M$-tensoring, powering and enriched hom on $\text{Rep}(\mathfrak{A})$ from Section 2.1.2 i.e. we check whether $\text{Rep}(\mathfrak{A})$ is an $M$-model category in the sense of [Hov99, Def. 4.2.18]. This amounts to showing that the $M$-tensoring (2.12) is a Quillen bifunctor, see [Hov99, Def. 4.2.1]. On account of [Hov99, Cor. 4.2.5] and Corollary 2.18 this is equivalent to proving that the pushout-product morphism $F(\eta) \Box \xi$ is a cofibration (acyclic cofibration) when $F(\eta) \in F(\prod_{\mathfrak{C}} I)$ and $\xi \in I$ are both generating cofibrations (respectively $F(\eta) \in F(\prod_{\mathfrak{C}} J)$ is a generating acyclic cofibration and $\xi \in I$ is a generating cofibration or $F(\eta) \in F(\prod_{\mathfrak{C}} I)$ is a generating cofibration and $\xi \in J$ is a generating acyclic cofibration). Recalling that $F$ preserves the $M$-tensorings on $\prod_{\mathfrak{C}} M$ and on $\text{Rep}(\mathfrak{A})$, see Remark 2.18 and mimicking the construction of the pushout-product morphism in $\text{Mod}(\mathfrak{C})$ from (A.14), one finds that the pushout-product morphism $F(\eta) \Box \xi$ constructed in $\text{Rep}(\mathfrak{A})$ coincides (up to the evident isomorphisms) with the image $F(\eta \Box \xi)$ under $F$ of the pushout-product morphism $\eta \Box \xi$ constructed in $\prod_{\mathfrak{C}} M$. (Here we used also that $F$ preserves colimits because it is a left-adjoint functor.) Since $M$ is a closed symmetric monoidal model category and the pushout-product in $\prod_{\mathfrak{C}} M$ is computed component-wise, $\eta \Box \xi$ in $\prod_{\mathfrak{C}} M$ is a cofibration (respectively acyclic cofibration). Since weak equivalences and fibrations in $\text{Rep}(\mathfrak{A})$ are by definition detected by $U$, whose left adjoint is $F$, it follows that $F(\eta \Box \xi)$ in $\text{Rep}(\mathfrak{A})$ has the left lifting property against all acyclic fibrations (respectively fibrations) in $\text{Rep}(\mathfrak{A})$. This means that $F(\eta \Box \xi)$, and hence also $F(\eta) \Box \xi$, in $\text{Rep}(\mathfrak{A})$ is a cofibration (respectively acyclic cofibration). The conclusions of this paragraph are summarized below.

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1For completeness, we recall here the core of this argument. Since the projective model structure on $\text{Fun}(C, M)$ is cofibrantly generated, it suffices to check the pushout-product axiom on the generating (acyclic) cofibrations $\prod_{\mathfrak{C}} L_{\mathfrak{C}(c_i \to \mathfrak{C})}$, where $c \in C$ is any object and $i$ is any generating (acyclic) cofibration of $M$. Since colimits in $M$ commute with each other and with the tensor product, the existence of finite coproducts in $C$ entails that the pushout-product of $\prod_{\mathfrak{C}(c_1 \to \mathfrak{C})} L_{\mathfrak{C}(c_1 \to \mathfrak{C})}$ and $\prod_{\mathfrak{C}(c_2 \to \mathfrak{C})} L_{\mathfrak{C}(c_2 \to \mathfrak{C})}$ in $\text{Fun}(C, M)$ is given by $\prod_{\mathfrak{C}(c_1 \to \mathfrak{C})} L_{\mathfrak{C}(c_1 \to \mathfrak{C})} \Box_{\mathfrak{C}(c_2 \to \mathfrak{C})}$, where $\Box$ denotes the pushout-product in $M$. (Finite coproducts in $\mathfrak{C}$ are responsible for the natural isomorphisms $C(c_1, -) \times C(c_2, -) \cong C(c_1 \prod_{\mathfrak{C}(c_2 \to \mathfrak{C})} c_2, -)$. This shows that the pushout-product axiom of $M$ entails that of $\text{Fun}(C, M)$.
Example 2.12 by endowing the categories of net representations with suitable model structures model categories of net representations. In other words, we solved the shortcoming evidenced in Ext in Example 2.23.

As an immediate consequence of Proposition 2.22, the weak equivalence \( \Phi : \mathcal{A} \rightarrow \mathcal{B} \) in \( \text{Net}_C^\mathbb{M} \) is a Quillen equivalence, see Definition 2.11. Our aim is to show that in this case the change-of-net adjunction \( \mathcal{E}_\mathcal{M} \dashv \mathcal{R}_{\mathcal{M}} : \text{Rep}(\mathcal{B}) \rightarrow \text{Rep}(\mathcal{A}) \) from (2.3) is a Quillen equivalence, see [Hov99, Def. 4.2.18]. By definition of the model structures on \( \text{Rep}(\mathcal{A}) \) and \( \text{Rep}(\mathcal{B}) \), \( \mathcal{R}_{\mathcal{M}} \) detects weak equivalences and, as a consequence of Set-up 2.15, all net representations are fibrant. Therefore, by [Hov99, Cor. 1.3.16], in order to conclude that the change-of-net adjunction \( \mathcal{E}_\mathcal{M} \dashv \mathcal{R}_{\mathcal{M}} \) is a Quillen equivalence, it suffices to show that the components of its unit \( \mathcal{L} \rightarrow \mathcal{R}_{\mathcal{M}} \mathcal{E}_\mathcal{M} \mathcal{L} \) in \( \text{Rep}(\mathcal{A}) \) are weak equivalences for all cofibrant \( \mathcal{A} \)-representations \( \mathcal{L} \in \text{Rep}(\mathcal{A}) \). For this purpose we need to show that, for each \( c \in \mathcal{C} \), the morphism \( \mathcal{L}_c \rightarrow (\mathcal{R}_{\mathcal{M}} \mathcal{E}_\mathcal{M} \mathcal{L})_c \) in \( \mathcal{A}(c)_{\text{Mod}} \) is a weak equivalence. (Refer to Section A.3 for the model structure on \( \mathcal{A}(c)_{\text{Mod}} \).) Recalling the construction of the change-of-net adjunction (2.3), the latter morphism is just the component \( \mathcal{L}_c \rightarrow (\mathcal{B}(c) \otimes_{\mathcal{A}(c)_{\text{Mod}}} \mathcal{L}_c)|_{\mathcal{A}(c)_{\text{Mod}}} \) of the unit of the change-of-monoid adjunction (A.1) associated with \( \Phi_c : \mathcal{A}(c) \rightarrow \mathcal{B}(c) \) in \( \text{Mon}(\mathcal{M}) \). Since \( \mathcal{L} \in \text{Rep}(\mathcal{A}) \) is cofibrant by hypothesis and \( (-)_c \dashv (-)^c : \mathcal{A}(c)_{\text{Mod}} \rightarrow \text{Rep}(\mathcal{A}) \) is a Quillen adjunction, see Remark 2.20 \( \mathcal{L}_c \in \mathcal{A}(c)_{\text{Mod}} \) is cofibrant too. Therefore, we would be able to conclude if the change-of-monoid adjunction associated with \( \Phi_c \) were a Quillen equivalence, for all \( c \in \mathcal{C} \). This result is achieved by Proposition A.12 under the additional assumption that, for each monoid \( A \in \text{Mon}(\mathcal{M}) \) and each cofibrant left \( A \)-module \( L \), the functor \( (-) \otimes_A L : \text{Mod}_A \rightarrow \text{M} \) sends weak equivalences in \( \text{Mod}_A \) to weak equivalences in \( \text{M} \). (Here \( \otimes_A : \text{Mod}_A \times \text{A} \text{Mod} \rightarrow \text{M} \) denotes the familiar relative tensor product over \( A \) between right and left \( A \)-modules. Note that the model structure on \( \text{Mod}_A \) is completely analogous to the one on \( \mathcal{A} \text{Mod} \) from Proposition A.10.) We emphasize that this additional assumption holds true in many examples, see [SS00, Secs. 4 and 5], including the closed symmetric monoidal model category of cochain complexes \( \mathcal{M} = \text{Ch}_{\mathbb{K}} \) over a field \( \mathbb{K} \), which is briefly recalled at the beginning of Section 3. We summarize below the conclusions of the last two paragraphs.

Proposition 2.22 (Change-of-net as a Quillen adjunction). Under the assumptions stated in Set-up 2.15 let \( \Phi : \mathcal{A} \rightarrow \mathcal{B} \) in \( \text{Net}_C^\mathbb{M} \) be a morphism of nets. Then the change-of-net adjunction \( \mathcal{E}_\mathcal{M} \dashv \mathcal{R}_{\mathcal{M}} : \text{Rep}(\mathcal{B}) \rightarrow \text{Rep}(\mathcal{A}) \) from (2.3) is a Quillen adjunction, which is compatible with the \( \mathcal{M} \)-tensoring, powering and enriched hom, see Remark 2.29. (In the language of [Hov99, Def. 4.2.18] the extension functor \( \mathcal{E}_\mathcal{M} \) is an \( \mathcal{M} \)-Quillen functor.) Furthermore, \( \mathcal{E}_\mathcal{M} \dashv \mathcal{R}_{\mathcal{M}} \) is also a Quillen equivalence and the following additional hypothesis is holds: for each monoid \( A \in \text{Mon}(\mathcal{M}) \) and each cofibrant left \( A \)-module \( L \in \text{A} \text{Mod} \), the relative tensor product \( (-) \otimes_A L : \text{Mod}_A \rightarrow \text{M} \) sends weak equivalences in \( \text{Mod}_A \) to weak equivalences in \( \text{M} \).

Example 2.23. As an immediate consequence of Proposition 2.22, the weak equivalence \( \Phi : \mathcal{A} \rightarrow \mathcal{A} \) in \( \text{Net}_C^{\text{Loc}} \) from Example 2.12 between the two nets of algebras describing the Klein-Gordon field gives rise to a Quillen equivalence \( \mathcal{E}_\mathcal{M} \dashv \mathcal{R}_{\mathcal{M}} : \text{Rep}(\mathcal{A}) \rightarrow \text{Rep}(\mathcal{A}) \) between the associated model categories of net representations. In other words, we solved the shortcoming evidenced in Example 2.12 by endowing the categories of net representations with suitable model structures and by formalizing the notion of “being the same” by the concept of Quillen equivalence. Let us
stress once more that ordinary categorical equivalence is recovered by passing to the associated homotopy categories, i.e. by inverting all weak equivalences between net representations. \(\nabla\)

## 3 Net representations for Maxwell \(p\)-forms

Section \(\mathcal{2}\) developed the homotopy theory of net representations with values in a closed symmetric monoidal model category \(\mathcal{M}\). We now move on to the problem of constructing explicit examples of such net representations. To address this question we take \(\mathcal{M} = \mathcal{Ch}_\mathbb{K}\) to be the closed symmetric monoidal model category of cochain complexes, which is the relevant one for applications in the context of the BV formalism and of homotopy algebraic quantum field theory, see Section \(\mathcal{1}\) As an instructive example we shall consider the net of algebras associated with Maxwell \(p\)-forms \([\text{SS00}, \text{Rem. 3.4 and Sec. 5.}]\). For instance, as interval object one can take the cochain monoidal model structure \([\text{Hov}99, \text{Prop. 4.2.13}]\), which fulfills the assumptions listed in Set-up \(\text{2.3}\) and also \([\text{BMR14, Rem. 1.8.}]\). The monoidal structure on a cofibrantly generated model structure such that all objects are both fibrant and cofibrant, see \([\text{Hov}99, \text{Sec. 2.3}]\) and \([\text{BMR14, Rem. 1.8.}]\). The internal hom \([\mathcal{Ch}_\mathbb{K}^{\mathcal{2}}]\), which we construct and study in detail through Sections \(\mathcal{3.1} \mathcal{3.2} \text{and 3.3}\) and goes through the construction of a two-point function \(\omega_2\), which for our purposes consists of a cochain map: it is the preservation of differentials that encodes the compatibility both with the equation of motion and with the action of gauge transformations. Note that Maxwell 1-forms recover linear Yang-Mills theory, i.e. the electromagnetic vector potential, and in this case our \(\omega_2\) extends (in a sense made precise by Remark \(\mathcal{3.7}\)) a Hadamard two-point function constructed in \([\text{FP03}]\) for the gauge invariant on-shell linear observables of the electromagnetic vector potential.

Before delving into the main subject of this section, let us recall some facts about the closed symmetric monoidal model category \(\mathcal{Ch}_\mathbb{K}\) of cochain complexes over a field \(\mathbb{K}\) (either \(\mathbb{K} = \mathbb{R}\) or \(\mathbb{K} = \mathbb{C}\) for our purposes). A cochain complex \(V \in \mathcal{Ch}_\mathbb{K}\) consists of vector spaces \(V^n\), for all \(n \in \mathbb{Z}\), and of a differential \(d_V\), i.e. a collection of linear maps \(d^n_V : V^n \to V^{n+1}\), for all \(n \in \mathbb{Z}\), such that \(d^{n+1}_V d^n_V = 0\). A cochain map \(f : V_1 \to V_2\) in \(\mathcal{Ch}_\mathbb{K}\) consists of linear maps \(f^n : V^n_1 \to V^n_2\), for all \(n \in \mathbb{Z}\), that preserve the differentials \(f^{n+1}_V d^n_V = d^n_{V_2} f^n\). Given \(V \in \mathcal{Ch}_\mathbb{K}\), \(V[k] \in \mathcal{Ch}_\mathbb{K}\) denotes the \(k\)-shifted cochain complex defined by \(V[k]^n := V^{n+k}\), for all \(n \in \mathbb{Z}\), with differential \(d_{V[k]} := (-1)^k d_V\). The weak equivalences are the quasi-isomorphisms, i.e. the cochain maps inducing isomorphisms in cohomology, the fibrations are the degree-wise surjective cochain maps and the cofibrations are the degree-wise injective cochain maps. This endows \(\mathcal{Ch}_\mathbb{K}\) with a cofibrantly generated model structure such that all objects are both fibrant and cofibrant, see \([\text{Hov}99, \text{Sec. 2.3}]\) and also \([\text{BMR14, Rem. 1.8.}]\). The monoidal structure on \(\mathcal{Ch}_\mathbb{K}\) is given by the tensor product \(V \otimes W \in \mathcal{Ch}_\mathbb{K}\) of the cochain complexes \(V, W \in \mathcal{Ch}_\mathbb{K}\), which is defined degree-wise for all \(n \in \mathbb{Z}\) by

\[
(V \otimes W)^n := \bigoplus_{k \in \mathbb{Z}} V^k \otimes W^{n-k},
\]

with differential given by the graded Leibniz rule \(d := d_V \otimes 1 + 1 \otimes d_W\). The monoidal unit \(\mathbb{K} \in \mathcal{Ch}_\mathbb{K}\) is the ground field regarded as a cochain complex concentrated in degree 0 and the symmetric braiding \(\otimes \cong \otimes V \cong V \otimes V\) on \(\mathcal{Ch}_\mathbb{K}\) is defined by the Koszul sign rule \(v \otimes w \mapsto (-1)^{|v||w|} w \otimes v\). The internal hom \([V, W] \in \mathcal{Ch}_\mathbb{K}\) is defined degree-wise for all \(n \in \mathbb{Z}\) by

\[
[V, W]^n := \coprod_{k \in \mathbb{Z}} \text{Lin}_\mathbb{K}(V^k, W^{n+k}),
\]

where \(\text{Lin}_\mathbb{K}(\cdot, \cdot)\) denotes the vector space of linear maps, with differential \(\partial := [d, -]\) given by the graded commutator with the original differentials. This endows \(\mathcal{Ch}_\mathbb{K}\) with a closed symmetric monoidal model structure \([\text{Hov}99, \text{Prop. 4.2.13}]\), which fulfills the assumptions listed in Set-up \(\text{2.15}\) see \([\text{SS00, Rem. 3.4 and Sec. 5.}]\). For instance, as interval object one can take the cochain complex \(I \in \mathcal{Ch}_\mathbb{K}\) that consists of \(\mathbb{K}\) in degree \(-1\) and \(\mathbb{K} \oplus \mathbb{K}\) in degree 0, with differential defined by \(d(1) = 1 \oplus (-1)\). Furthermore, the additional hypothesis of Proposition \(\text{2.22}\) is met, see \([\text{SS00, Secs. 4 and 5.}]\).
3.1 Cochain complexes of solutions and linear observables

Let $M$ be an oriented and time-oriented globally hyperbolic Lorentzian manifold of dimension $m \geq 2$. We consider the field complex $\mathcal{F}(M) \in \operatorname{Ch}_\mathbb{R}$ of $p$-forms on $M$, for $p < m$, defined by

$$\mathcal{F}(M) := \left( \Omega^0(M) \overset{d}{\longrightarrow} \cdots \overset{d}{\longrightarrow} \Omega^{p+n}(M) \overset{d}{\longrightarrow} \cdots \overset{d}{\longrightarrow} \Omega^p(M) \right). \tag{3.3}$$

(Our convention is that the non-displayed degrees, in this case $n < -p$ and $n > 0$, and differentials vanish.) In degree 0 sit the gauge fields $A \in \mathcal{F}(M)^0 = \Omega^0(M)$, in degree $-1$ sit the ghosts $g^{-1} \in \mathcal{F}(M)^{-1} = \Omega^{-1}(M)$ (gauge transformations) and, more generally, in lower degrees $n < -1$ sit higher ghosts $g^n \in \mathcal{F}(M)^n = \Omega^{p+n}(M)$. For $p = 1$ $\mathcal{F}(M)$ is the well-known complex of linear Yang-Mills theory, with connection 1-forms sitting in degree 0 and gauge transformations sitting in degree $-1$, whose action on connections is determined by the de Rham differential $d$. The dynamics are encoded by the linear differential operator

$$\delta d : \mathcal{F}(M)^0 \to \mathcal{F}(M)^0. \tag{3.4}$$

One can interpret this linear differential operator as the equation of motion arising from the variation of the (formal) action functional

$$S(A) := \frac{1}{2} \int_M A \wedge \star \delta A, \tag{3.5}$$

where $\star$ denotes the Hodge star operator defined by the metric and the orientation of $M$ and $\delta := (-1)^{k+1} d \star$ denotes the codifferential on $\Omega^k(M)$. The action functional $S$ is manifestly gauge-invariant since $\delta d = 0$ and $\delta d$ is formally self-adjoint with respect to the integral pairing $\int(-) \wedge \star(-)$ displayed above. Varying $S$ leads to a section $\delta^* S : \mathcal{F}(M) \to T^* \mathcal{F}(M)$ in $\operatorname{Ch}_\mathbb{R}$ of the “cotangent bundle” $T^* \mathcal{F}(M) \in \operatorname{Ch}_\mathbb{R}$ over $\mathcal{F}(M)$, defined as the product

$$T^* \mathcal{F}(M) := \mathcal{F}(M) \times \mathcal{F}_c(M)^* \tag{3.6}$$

of the cochain complex $\mathcal{F}(M) \in \operatorname{Ch}_\mathbb{R}$, interpreted as the base, and the cochain complex

$$\mathcal{F}_c(M)^* := \left( \Omega^0(M) \overset{-\delta}{\longrightarrow} \cdots \overset{-\delta}{\longrightarrow} \Omega^{p+n}(M) \overset{-\delta}{\longrightarrow} \cdots \overset{-\delta}{\longrightarrow} \Omega^0(M) \right) \in \operatorname{Ch}_\mathbb{R}, \tag{3.7}$$

interpreted as the fiber. (The notation $\mathcal{F}_c(M)^*$ is motivated by the existence of an evaluation pairing $\text{ev} : \mathcal{F}_c(M)^* \otimes \mathcal{F}_c(M) \to \mathbb{R}$ in $\operatorname{Ch}_\mathbb{R}$ against the cochain complex $\mathcal{F}_c(M) \in \operatorname{Ch}_\mathbb{R}$ of compactly supported fields, which is defined by $\text{ev}(\Phi \otimes \varphi) := (-1)^{[n/2]} \int_M \Phi \wedge \star \varphi$, for all $n = 0, \ldots, p$, $\Phi \in (\mathcal{F}_c(M)^*)^n$ and $\varphi \in \mathcal{F}_c(M)^{-n}$, where $[\cdot]$ denotes the floor function.) The derived critical locus of $S$, defined as the homotopy pull-back of $\delta^* S : \mathcal{F}(M) \to T^* \mathcal{F}(M)$ in $\operatorname{Ch}_\mathbb{R}$ along the zero-section of $T^* \mathcal{F}(M) \in \operatorname{Ch}_\mathbb{R}$, determines the solution complex $\mathfrak{Sol}(M) \in \operatorname{Ch}_\mathbb{R}$. Computing the latter explicitly as in [BS19a, Sec. 3.4] leads to the cochain complex

$$\mathfrak{Sol}(M) := \left( \Omega^0(M) \overset{d}{\longrightarrow} \cdots \overset{d}{\longrightarrow} \Omega^{p+n}(M) \overset{\delta d}{\longrightarrow} \Omega^0(M) \overset{\delta}{\longrightarrow} \cdots \overset{\delta}{\longrightarrow} \Omega^0(M) \right) \in \operatorname{Ch}_\mathbb{R}. \tag{3.8}$$

The observable complex $\mathcal{L}(M) \in \operatorname{Ch}_\mathbb{R}$ is defined as the 1-shifted compactly supported analog of $\mathfrak{Sol}(M)$. Explicitly, this reads as

$$\mathcal{L}(M) := \left( \Omega^0(M) \overset{-d}{\longrightarrow} \cdots \overset{-d}{\longrightarrow} \Omega^{p+n}(M) \overset{-\delta d}{\longrightarrow} \Omega^0(M) \overset{-\delta}{\longrightarrow} \cdots \overset{-\delta}{\longrightarrow} \Omega^0(M) \right) \in \operatorname{Ch}_\mathbb{R}. \tag{3.9}$$
The cochains in $\mathcal{L}(M)$ are interpreted as linear functionals on the solution complex $\mathfrak{so}(M)$ via the evaluation pairing

$$\text{ev} : \mathfrak{so}(M) \otimes \mathcal{L}(M) \longrightarrow \mathbb{R}$$

in $\text{Ch}_\mathbb{R}$, whose only non vanishing degree

$$\text{ev}^0 : \bigoplus_{k \in \mathbb{Z}} (\mathfrak{so}(M)^k \otimes \mathcal{L}(M)^{-k}) \longrightarrow \mathbb{R}$$

is defined component-wise for all $k \in \mathbb{Z}$ by

$$\text{ev}^0_k(\Phi \otimes \varphi) := (-1)^{[k/2]} \int_M \Phi \wedge \varphi,$$

for all $\Phi \in \mathfrak{so}(M)^k$ and $\varphi \in \mathcal{L}(M)^{-k}$.

### 3.2 Retarded and advanced trivializations and initial data complex

We equip the observable complex with a Poisson structure following the approach of [BBS20], see also [BMS22] for a related, yet more conceptual, approach. In order to achieve this goal, let us consider the analog of the solution complex with an appropriate support restriction. Explicitly, we write $\Omega^k_{\text{sc}}(M)$ for the vector space of $k$-forms whose support is spacelike compact, i.e. contained in the causal shadow $J^+_M(K) := J^+_M(K) \cup J^-_M(K) \subseteq M$ of some compact subset $K \subseteq M$. Implementing the same support restriction on the solution complex $\mathfrak{so}(M)$, we define the solution complex with spacelike compact support $\mathfrak{so}_{\text{sc}}(M) \in \text{Ch}_\mathbb{R}$.

Since by definition $K \subseteq J_M(K)$, there are obvious inclusions $\Omega^k(M) \subseteq \Omega^k_{\text{sc}}(M)$, for all $k = 0, \ldots, p$. These inclusions assemble to form the cochain map $j : \mathcal{L}(M) \rightarrow \mathfrak{so}_{\text{sc}}(M)[1]$ in $\text{Ch}_\mathbb{R}$ to the 1-shifted solution complex with spacelike compact support. Equivalently, this is a 1-cocycle $j \in [\mathcal{L}(M), \mathfrak{so}_{\text{sc}}(M)]^1$ in the internal hom. As it is shown below, it turns out that $j$ is homotopic to 0 in two inequivalent ways. Explicitly, this means that there exist 0-cochains $\Lambda_{\pm} \in \{\mathcal{L}(M), \mathfrak{so}_{\text{sc}}(M)[1]\}$ in the internal hom, called retarded and advanced trivializations, whose differentials $\partial \Lambda_{\pm} = d_{\mathfrak{so}} \Lambda_{\pm} - \Lambda_{\pm} d_{\mathcal{L}} = j$ trivialize the 1-cocycle $j$. Let us construct these retarded and advanced trivializations $\Lambda_{\pm}$ for Maxwell $p$-forms. To achieve this goal we consider the retarded and advanced Green’s operators $G_{\pm}^{(k)}$ for the d’Alembert operator $\Box := \delta d + d \delta : \Omega^k(M) \rightarrow \Omega^k(M)$ on $k$-forms, $k = 0, \ldots, p$. Recall from [BGP07; Bär15] that a retarded/advanced Green’s operator $G_{\pm}^{(k)} : \Omega^k_{\text{sc}}(M) \rightarrow \Omega^k(M)$ for $\Box$ is a linear map such that, for all $\omega \in \Omega^k_{\text{sc}}(M)$, it holds that (i) $\Box G_{\pm}^{(k)} \omega = \omega$, (ii) $G_{\pm}^{(k)} \Box \omega = \omega$ and (iii) $\text{supp}(G_{\pm}^{(k)} \omega) \subseteq J^\pm_M(\text{supp}(\omega))$. It is well-known that retarded and advanced Green’s operators for $\Box$ exist, are unique and commute both with the de Rham differential $d G_{\pm}^{(k)} = G_{\pm}^{(k+1)} d$ and codifferential $\delta G_{\pm}^{(k)} = G_{\pm}^{(k-1)} \delta$. With these preparations, we define the retarded and advanced trivializations $\Lambda_{\pm} \in \{\mathcal{L}(M), \mathfrak{so}_{\text{sc}}(M)[1\}$ degree-wise by

$$\Lambda_{n}^{\pm} := \begin{cases} G_{\pm}^{(p-n)} \delta, & n = -p, \ldots, -1, \\ G_{n}^{(p)}, & n = 0, \\ d G_{\pm}^{(p-n)}, & n = 1, \ldots, p. \end{cases}$$

(All other components $\Lambda_{n}^{\pm}$, $n \leq -p - 1$ and $n \geq p + 1$, necessarily vanish.) Note that property (iii) ensures that the output of $\Lambda_{\pm}$ has the required support. Furthermore, direct inspection using properties (i) and (ii) and the commutation rules of $G_{\pm}^{(k)}$ with $d$ and $\delta$ shows that $\partial \Lambda_{\pm} = j$.

Taking the difference of the retarded and the advanced trivializations from (3.11) defines the cochain map

$$\Lambda := \Lambda_{+} - \Lambda_{-} : \mathcal{L}(M) \longrightarrow \mathfrak{so}_{\text{sc}}(M)$$

in $\text{Ch}_\mathbb{R}$. ($\Lambda$ is indeed a cochain map because $\partial \Lambda = j - j = 0 \in \{\mathcal{L}(M), \mathfrak{so}_{\text{sc}}(M)[1\}$ vanishes when regarded as a 1-cochain in the internal hom.)

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Remark 3.1. The cochain map $\Lambda$ in (3.12) is a quasi-isomorphism since it induces the isomorphism $H(\Lambda) : H(\mathcal{L}(M)) \to H(\mathfrak{Sol}_{sc}(M))$ in cohomology, which in terms of the de Rham cohomologies of $M$ with compact support $H_{dR, sc}(M)$ and with spacelike compact support $H_{dR, sc}(M)$ reads as

$$
\begin{cases}
H^{p+1+n}_{dR,c}(M) \cong H^{p+1+n}_{dR, sc}(M), & n = -p - 1, \ldots, -1, \\
H^{m-p-1+n}_{dR, c} \cong \frac{\Omega^p_{\mathfrak{Sol}_{sc}}(M) / \delta \Omega^p_{\mathfrak{Sol}_{sc}}}{\delta \Omega^{p-1}_{\mathfrak{Sol}_{sc}}(M)}, & n = 0, \\
H^{m-p-1+n}_{dR, c} \cong \Omega^p_{\mathfrak{Sol}_{sc}}(M), & n = 1, \ldots, p + 1.
\end{cases}
$$

(3.13)

(Note that the Hodge star operator $\star$ has been implicitly used to identify the cohomology of the codifferential $\delta$ with the more familiar de Rham cohomology.) We refer to [Ben16] for the definition of the de Rham cohomologies with spacelike compact support $H^k_{dR, sc}(M)$ of an $m$-dimensional oriented and time-oriented globally hyperbolic Lorentzian manifold $M$ and for the isomorphisms in degrees $n \neq 0$, while we refer to [Ben16] for the isomorphism in degree $n = 0$ between "gauge invariant linear observables modulo equations of motion" $\Omega^p_{\mathfrak{Sol}_{sc}}(M)/\delta \Omega^p_{\mathfrak{Sol}_{sc}}(M)$ and "spacelike compact on-shell fields modulo gauge transformations" $\Omega^p_{\mathfrak{Sol}_{sc}}(M)/\delta \Omega^{p-1}_{\mathfrak{Sol}_{sc}}(M)$. See [BMS22] for a more conceptual proof of the fact that $\Lambda$ is a quasi-isomorphism. △

For any choice of a spacelike Cauchy surface $\Sigma \subseteq M$, let us also consider the compactly supported initial data complex

$$
\mathcal{D}_c(\Sigma) := \left( \frac{(-p)}{\Omega^0(\Sigma)} \delta \Sigma \cdots \delta \Sigma \frac{(-1)}{\Omega^{p-1}(\Sigma)} \delta \Sigma \frac{(0)}{\Omega^p(\Sigma)} \delta \Sigma \frac{(1)}{\delta \Sigma} \delta \Sigma \cdots \delta \Sigma \frac{(p)}{\Omega^p(\Sigma)} \right).
$$

(3.14)

(Here the notation $d\Sigma$ and $\delta \Sigma := (-1)^k \star_{\Sigma}^{-1} d\Sigma \star_{\Sigma}$ is used to emphasize that these differential operators are defined with respect to the geometry of $\Sigma$. The compactly supported initial data complex is related to the spacelike compactly supported solution complex via the initial data map $\text{data} : \mathfrak{Sol}_{sc}(M) \to \mathcal{D}_c(\Sigma)$ in $\text{Ch}_R$ defined degree-wise by

$$
data^n := \begin{cases}
\iota^* & , \quad n = -p, \ldots, -1, \\
(\star_{\Sigma}^{-1} \iota^* \star_{\Sigma} d) & , \quad n = 0, \\
(-1)^n \star_{\Sigma}^{-1} \iota^* & , \quad n = 1, \ldots, p.
\end{cases}
$$

(3.15)

where $\iota : \Sigma \to M$ denotes the embedding of the chosen spacelike Cauchy surface. (All other components $\text{data}^n$, $n \leq -p - 1$ and $n \geq p + 1$, necessarily vanish.)

Remark 3.2. The cochain map $\text{data}$ in (3.15) is a quasi-isomorphism since it induces the cohomology isomorphism $H(\text{data}) : H(\mathfrak{Sol}_{sc}(M)) \cong H(\mathcal{D}_c(\Sigma))$, which in terms of the de Rham cohomology of $M$ with spacelike compact support and the de Rham cohomology of $\Sigma$ with compact support reads as

$$
\begin{cases}
H^{p+1}_c(\Sigma) \cong H^{p+1}_c(\Sigma), & n = -p, \ldots, -1, \\
\Omega^p_{\mathfrak{Sol}_{sc}}(M)/\delta \Omega^{p-1}_{\mathfrak{Sol}_{sc}}(M) \cong \left( \Omega^p(\Sigma)/\delta \Sigma \Omega^{p-1}(\Sigma) \right) \times \Omega^p_{\mathfrak{Sol}_{sc}}(\Sigma), & n = 0, \\
H^{m-p-1+n}_c(\Sigma) \cong \Omega^p_{\mathfrak{Sol}_{sc}}(\Sigma), & n = 1, \ldots, p + 1.
\end{cases}
$$

(3.16)

We refer to [Ben16, Kha16] for the isomorphisms in degrees $n \neq 0$ and to [SDH14] for the isomorphism in degree $n = 0$, where it is stated in the form of the well-posed initial value problem for gauge classes of on-shell Maxwell $p$-forms. This perspective suggests the interpretation of the quasi-isomorphism $\text{data}$ as a refinement of this well-posed initial value problem.

Combining the quasi-isomorphisms $\text{data}$ and $\Lambda$ from Remarks 3.1 and 3.2 we obtain a quasi-isomorphism $\text{data} \circ \Lambda : \mathcal{L}(M) \to \mathcal{D}_c(\Sigma)$, which provides an explicit computation of the cohomology of the observable complex $\mathcal{L}(M)$ for Maxwell $p$-forms in terms of the compactly supported de Rham cohomology of a spacelike Cauchy surface $\Sigma \subseteq M$ and of the initial data for gauge classes of on-shell Maxwell $p$-forms. △
3.3 Poisson structure and quantization

This section is devoted to the construction of a \( \mathbf{Ch}_C \)-valued net of algebras, associated with Maxwell \( p \)-forms, over the category \( \mathbf{Loc}_m \) of oriented and time-oriented \( m \)-dimensional globally hyperbolic Lorentzian manifolds, see Example 2.3. The first step constructs a Poisson structure \( \tau_M : \mathcal{L}(M)^{\wedge 2} \to \mathbb{R} \) in \( \mathbf{Ch}_R \) on the observable complex \( \mathcal{L}(M) \in \mathbf{Ch}_R \), for each \( M \in \mathbf{Loc}_m \). This defines a Poisson complex \( (\mathcal{L}(M), \tau_M) \). The second step quantizes this Poisson complex using canonical commutation relations (CCR). The third step exhibits the net structure. Incidentally, we will observe that the resulting net of algebras is actually a homotopy algebraic quantum field theory, i.e. it fulfills both the Einstein’s causality axiom and the homotopy time-slice axiom, see [BBS20, BSW19a]. Since we will work with the closed symmetric monoidal model category of cochain complexes \( M \), we shall replace the term “monoid” with the more familiar “differential graded algebra”. Accordingly, we shall adopt the familiar notation \( \mathbf{DGA}_C := \mathbf{Mon}(\mathbf{Ch}_C) \).

Combining (3.10) and (3.12) allows us to equip the observable complex \( \mathcal{L}(M) \) with the Poisson structure

\[
\tau_M := -\text{ev}(\Lambda \otimes \text{id}) : \mathcal{L}(M)^{\wedge 2} \to \mathbb{R} \quad (3.17)
\]

in \( \mathbf{Ch}_R \). (The obvious inclusion \( \mathfrak{Sol}_C(M) \to \mathfrak{Sol}(M) \) in \( \mathbf{Ch}_R \) is implicit in the definition above.) Note that \( \tau_M \) is graded anti-symmetric, as implicitly claimed in (3.17). This follows because, with respect to the pairing \( \int_M (-) \wedge *(-) \), the codifferential \( \delta \) is the formal adjoint of the de Rham differential \( d \), \( \Box \) is formally self-adjoint and, as a consequence, also the retarded and advanced Green’s operators are each the formal adjoint of the other.

We quantize the Poisson complex \( (\mathcal{L}(M), \tau_M) \), consisting of the observable complex \( \mathcal{L}(M) \in \mathbf{Ch}_R \) from [3.3] and the Poisson structure \( \tau_M : \mathcal{L}(M)^{\wedge 2} \to \mathbb{R} \) in \( \mathbf{Ch}_R \) from (3.17), applying the CCR quantization functor of [BBS20]. Explicitly, we consider the free differential graded algebra

\[
T_C(\mathcal{L}(M)) := \bigoplus_{m \geq 0} \mathcal{L}(M)^{\otimes m}_C \in \mathbf{DGA}_C 
\]

(3.18a)

generated by the complexification \( \mathcal{L}(M)_C := \mathcal{L}(M) \otimes_{\mathbb{R}} \mathbb{C} \subseteq \mathbf{Ch}_C \) of the observable complex \( \mathcal{L}(M) \in \mathbf{Ch}_R \). The multiplication \( \mu \) is defined by juxtaposition \( \varphi_1 \otimes \cdots \otimes \varphi_k \otimes \psi_1 \otimes \cdots \otimes \psi_m \) of words \( \varphi_1 \otimes \cdots \otimes \varphi_k, \psi_1 \otimes \cdots \otimes \psi_m \) in \( T_C(\mathcal{L}(M)) \) and the unit \( \mathbf{1} \) corresponds to the length 0 word \( 1 \in \mathbb{C} \subseteq T_C(\mathcal{L}(M)) \). Taking the quotient by the two-sided ideal \( I_\tau \subseteq T_C(\mathcal{L}(M)) \) generated by the elements

\[
\varphi_1 \otimes \varphi_2 - (-1)^{|\varphi_1||\varphi_2|} \varphi_2 \otimes \varphi_1 - i\tau_M(\varphi_1, \varphi_2)\mathbf{1}, \quad (3.18b)
\]

for all homogeneous cochains \( \varphi_1, \varphi_2 \in \mathcal{L}(M) \) defines the differential graded algebra

\[
\mathfrak{A}(M) := T_C(\mathcal{L}(M))/I_\tau \in \mathbf{DGA}_C. \quad (3.18c)
\]

**Remark 3.3.** It is straightforward to endow the differential graded algebra \( \mathfrak{A}(M) \in \mathbf{DGA}_C \) with a multiplication reversing \(*\)-involution arising from complex conjugation \( \overline{(-)} \) on \( \mathbb{C} \). (We refer to [Jac12, BSW19b] for a more detailed discussion on reversing \(*\)-monoids in an involutive symmetric monoidal category.) Indeed, one defines \(*\) on the free differential graded algebra \( T_C(\mathcal{L}(M)) \in \mathbf{DGA}_C \) as the unique multiplication reversing graded \( \mathbb{C} \)-antilinear map extending the complex conjugation \( \varphi \mapsto \overline{\varphi} \) on length 1 words \( \varphi \in \mathcal{L}(M)_C \subseteq T_C(\mathcal{L}(M)) \). The graded anti-symmetry of \( \tau \) entails that \( I_\tau \subseteq T_C(\mathcal{L}(M)) \) is a two-sided \(*\)-ideal, therefore \(*\) descends to the quotient \( \mathfrak{A}(M) \in \mathbf{DGA}_C \). In other words, \( \mathfrak{A}(M) \) is a differential graded unital and associative \(*\)-algebra.

We focus now on the construction of the net structure. Recall that differential forms with compact support can be extended by zero along open embeddings \( f : M \to N \). Let us denote the
corresponding extension-by-zero map by \( f_* : \Omega^k(M) \to \Omega^k(N) \). Since all morphisms \( f : M \to N \) in \( \text{Loc}_m \) are in particular open embeddings, one defines the cochain maps

\[
\mathcal{L}(f) : \mathcal{L}(M) \longrightarrow \mathcal{L}(N)
\]

in \( \text{Ch}_\mathbb{R} \) degree-wise by \( \mathcal{L}(f)^n := f_* \) for \( n = -p - 1, \ldots, p \) and \( \mathcal{L}(f)^n := 0 \) else. Together with the assignment of the observable complex \( M \in \text{Loc}_m \mapsto \mathcal{L}(M) \in \text{Ch}_\mathbb{R} \), this defines the functor \( \mathcal{L} : \text{Loc}_m \to \text{Ch}_\mathbb{R} \). The naturality of the de Rham differential \( d \) and of the Hodge star operator * with respect to morphisms in \( \text{Loc}_m \) entails the naturality of the linear differential operator \( \Box : \Omega^k(M) \to \Omega^k(M) \). From the uniqueness of the associated retarded and advanced Green’s operators \( G_\pm^k : \Omega^k(M) \to \Omega^k(M) \), it follows that \( f^* G_\pm^k f_* = G_\pm^k \), for all \( f : M \to N \) in \( \text{Loc}_m \), where \( f^* : \Omega^k(N) \to \Omega^k(M) \) denotes the pull-back of k-forms. This fact entails that also the Poisson structure is natural, i.e. \( \tau_N (\mathcal{L}(f) \otimes \mathcal{L}(f)) = \tau_M, \) for all \( f : M \to N \) in \( \text{Loc}_m \). In other words, one obtains a functor \( (\mathcal{L}, \tau) \) that assigns to each \( M \in \text{Loc}_m \) the Poisson complex \( (\mathcal{L}(M), \tau_M) \) and to each morphism \( f : M \to N \) in \( \text{Loc}_m \) the Poisson structure preserving cochain map \( \mathcal{L}(f) : (\mathcal{L}(M), \tau_M) \to (\mathcal{L}(N), \tau_N) \). The CCR quantization recalled in (3.18) is manifestly functorial, see [BBS20]. As a result, composing the functor \( (\mathcal{L}, \tau) \) and the CCR quantization functor we obtain a functor \( \mathfrak{A} : \text{Loc}_m \to \text{DGA}_\mathbb{C} \), i.e. the net of algebras

\[
\mathfrak{A} \in \text{Net}_{\text{Loc}_m}^{\text{Ch}},
\]

see Remark 2.2, associated with Maxwell p-forms.

**Remark 3.4.** We observe incidentally that the net \( \mathfrak{A} \in \text{Net}_{\text{Loc}_m}^{\text{Ch}} \) constructed above, endowed with the *-involution from Remark 3.3, is actually a homotopy algebraic quantum field theory, see [BBS20, BSW19a, BSW19b]. In other words, this net fulfuls both the Einstein’s causality axiom and the homotopy time-slice axiom. The Einstein’s causality axiom, i.e. the fact that the graded commutator

\[
[\mathfrak{A}(f_1), \mathfrak{A}(f_2)] = 0 : \mathfrak{A}(M_1) \otimes \mathfrak{A}(M_2) \longrightarrow \mathfrak{A}(N)
\]

in \( \text{Ch}_\mathbb{C} \) vanishes for all pairs of morphisms \( f_1 : M_1 \to N \leftarrow M_2 : f_2 \) in \( \text{Loc}_m \) with causally disjoint images, is a straightforward consequence of the support properties of the retarded and advanced Green’s operators that enter the definition of the Poisson structure through the cochain map \( \Lambda \), see (3.11), (3.12) and (3.17).

It is slightly more involved to check the homotopy time-slice axiom, i.e. that the morphism \( \mathfrak{A}(f) : \mathfrak{A}(M) \to \mathfrak{A}(N) \) in \( \text{DGA}_\mathbb{C} \) is a weak equivalence whenever \( f : M \to N \) in \( \text{Loc}_m \) is a Cauchy morphism. Recall first that weak equivalences in \( \text{DGA}_\mathbb{C} \) are just quasi-isomorphisms between the underlying cochain complexes. The fact that the cochain map underlying \( \mathfrak{A}(f) \) is indeed a quasi-isomorphism can be best understood taking a closer look at the cochain map \( \mathcal{L}(f) : \mathcal{L}(M) \to \mathcal{L}(N) \) in \( \text{Ch}_\mathbb{R} \) from (3.19). Recalling that per hypothesis \( f \) is a Cauchy morphism, i.e. the image \( f(M) \subseteq N \) contains a spacelike Cauchy surface \( \Sigma \subseteq f(M) \) of the codomain \( N \), one obtains the commutative diagram

\[
\begin{align*}
\mathcal{L}(M) \xrightarrow{\mathcal{L}(f)} \mathcal{L}(N) \\
\text{data } \Lambda \downarrow \quad \downarrow \text{data } \Lambda \\
\mathfrak{D}_c(\Sigma)
\end{align*}
\]

in \( \text{Ch}_\mathbb{R} \) involving the passage to the initial data complex \( \mathfrak{D}_c(\Sigma) \), see (3.14) and (3.15). (Note that we identified the spacelike Cauchy surface \( \Sigma \subseteq N \) with its preimage in \( M \) via \( f \), which is automatically a spacelike Cauchy surface of \( M \).) As observed in Remarks 3.1 and 3.2, both \( \Lambda \) and \textit{data} are quasi-isomorphisms. The fact that quasi-isomorphisms are closed under composition and fulfill the two-out-of-three property entails that \( \mathcal{L}(f) \) is a quasi-isomorphism too. Summing
up, \( \mathcal{L}(f) : (\mathcal{L}(M), \tau_M) \to (\mathcal{L}(N), \tau_N) \) is a Poisson structure preserving quasi-isomorphism. By [BBS20, Prop. 5.3], see also [BS19a], the CCR quantization functor, which enters the construction of the Maxwell p-forms net \( \mathfrak{A} \) from (3.20), maps Poisson structure preserving quasi-isomorphisms to weak equivalences. Therefore, it follows that \( \mathfrak{A}(f) : \mathfrak{A}(M) \to \mathfrak{A}(N) \) in \( \text{DGA}_C \) is a weak equivalence for all Cauchy morphisms \( f : M \to N \) in \( \text{Loc}_m \). This proves that the Maxwell p-forms net \( \mathfrak{A} \) fulfils also the homotopy time-slice axiom holds and hence it defines a homotopy algebraic quantum field theory in the sense of [BBS20, BSW19a, BS19a]. \( \Box \)

### 3.4 Construction of a net representation

The goal of this section is to construct a representation of the \( \text{Ch}_C \)-valued (Haag-Kastler) net

\[
\mathfrak{A}_M := \mathfrak{A}_{\iota_M} \in \text{Net}_{\text{Ch}_C}(\text{CCO}(M)) \tag{3.23}
\]

on \( \text{CCO}(M) \), see Example 2.3, that describes quantized Maxwell p-form fields on a fixed oriented and time-oriented globally hyperbolic Lorentzian manifold \( M \in \text{Loc}_m \). \( \mathfrak{A}_M \) is obtained by restricting the (generally covariant) net \( \mathfrak{A} \in \text{Net}_{\text{Loc}_m} \) from (3.20) along the functor \( \iota : \text{CCO}(M) \to \text{Loc}_m \) that sends a causally convex open subsets \( U \subset \text{CCO}(M) \) to the oriented and time-oriented globally hyperbolic Lorentzian manifolds \( U \subset \text{Loc}_m \) defined by endowing \( U \) with the restriction of the geometry of \( M \). We shall obtain a representation of the net \( \mathfrak{A}_M \) from a two-point function \( \omega_2 : \mathcal{L}(M) \otimes \mathcal{L}(M) \to \mathbb{C} \) in \( \text{Ch}_C \) on the observable complex \( \mathcal{L}(M) \) from (3.9), i.e. a cochain map whose anti-symmetric part agrees with the Poisson structure \( \tau_M \) from (3.17).

Mimicking the usual construction of a quasi-free state from a two-point function, see e.g. [KM15, Sec. 5.2.4], we shall use \( \omega_2 \) to define a linear functional \( \omega : \mathfrak{A}(M) \to \mathbb{C} \) in \( \text{Ch}_C \). The first step of the Gelfand-Naimark-Segal construction applied to \( \omega \) shall then provide a representation of the global algebra of observables \( \mathfrak{A}(M) = \mathfrak{A}_M(M) \). The latter defines a constant net representation of \( \mathfrak{A}_M \) via Construction 2.6.

**Remark 3.5.** Usually two-point functions are not only required to have anti-symmetric part matching the Poisson structure of interest, but are also required to be bisolutions of the equation of motion that governs the field theoretic model of interest. In our framework the latter requirement is replaced and generalized by the condition that \( \omega_2 \) is a cochain map. Notably, this condition takes also care of the compatibility with the action of gauge transformations. \( \Box \)

In order to simplify the presentation and to better highlight the main features of our construction, we shall assume that \( M \in \text{Loc}_m \) is of the form

\[
M = \mathbb{R} \times \Sigma, \tag{3.24a}
\]

with \( \Sigma \) a *compact* spacelike Cauchy surface, and that the metric \( g \) is ultra-static, i.e.

\[
g = -dt^2 + h, \tag{3.24b}
\]

with \( h \) a Riemannian metric on \( \Sigma \) (constant with respect to \( t \in \mathbb{R} \)).

The explicit construction of the two-point function \( \omega_2 : \mathcal{L}(M) \otimes \mathcal{L}(M) \to \mathbb{C} \) in \( \text{Ch}_{\mathbb{R}} \) partly follows the lines of [BCD17]. First, since \( M = \mathbb{R} \times \Sigma \) is a product, \( k \)-forms on \( M \) decompose into sections of the pullbacks along the projection \( \pi_2 : M \to \Sigma \) of the bundles \( \Lambda^k \Sigma \) of \( k \)-forms and \( \Lambda^{k-1} \Sigma \) of \((k-1)\)-forms over \( \Sigma \). By abuse of notation, we denote the pullback bundles over \( M \) again by \( \Lambda^k \Sigma \) and \( \Lambda^{k-1} \Sigma \), so that the above mentioned decomposition takes the form

\[
\Omega^k(M) = \Gamma(M, \Lambda^k \Sigma) \oplus dt \wedge \Gamma(M, \Lambda^{k-1} \Sigma), \quad \alpha = \alpha_S + dt \wedge \alpha_T. \tag{3.25}
\]

The first summand corresponds to \( k \)-forms on \( M \) having only “spatial legs”, while the second summand corresponds to \( k \)-forms on \( M \) having precisely one “time leg”. We shall use the subscripts \( S \) and \( T \) to refer to the space \( \alpha_S \in \Gamma(M, \Lambda^k \Sigma) \) and respectively time \( \alpha_T \in \Gamma(M, \Lambda^{k-1} \Sigma) \).
parts of a $k$-form $\alpha \in \Omega^k(M)$. Since the ultra-static metric $g = -dt^2 + h$ decomposes into time and space parts, the de Rham differential $d$, the codifferential $\delta$ and the d’Alembert operator $\Box$ acting on $k$-forms on $M$ admit a decomposition compatible with (3.25). Explicitly, denoting with $d_S$ the de Rham differential, with $\delta_S$ the codifferential and with $\Delta := \delta_S d_S + d_S \delta_S$ the Laplace operator, all acting on differential forms on $\Sigma$, and with $\partial_t$ the time-derivative, one finds

\[
\begin{align*}
\bar{d} \alpha &= d_S \alpha_S + dt \wedge (\partial_t \alpha_S - d_S \alpha_T), \\
\bar{\delta} \alpha &= (\partial_t \alpha_T + \delta_S \alpha_S) - dt \wedge \delta_S \alpha_T, \\
\bar{\delta} \bar{d} \alpha &= (\partial_t^2 \alpha_S - d_S \partial_t \alpha_T + \delta_S d_S \alpha_S) + dt \wedge (\delta_S d_S \alpha_T - \delta_S \partial_t \alpha_S), \\
\Box \alpha &= (\partial_t^2 \alpha_S + \Delta \alpha_S) + dt \wedge (\partial_t^2 \alpha_T + \Delta \alpha_T),
\end{align*}
\]

for all $\alpha \in \Omega^k(M)$. Combining (3.25) and (3.26) with the Hodge decomposition for differential $k$-forms on $\Sigma$

\[
\Omega^k(\Sigma) = \mathcal{H}^k(\Sigma) \oplus \mathcal{H}_\perp^k(\Sigma),
\]

into (spatially) harmonic $(\sigma_H \in \mathcal{H}^k(\Sigma) (\Delta \sigma_H = 0)$ and orthogonal $\sigma_\perp \in \mathcal{H}_\perp^k(\Sigma) (\sigma_\perp = d_S \sigma_1 + \delta_S \sigma_2)$ parts, one obtains an explicit formula for the retarded-minus-advanced propagator

\[
G^{(k)} := G^{(k)}_+ - G^{(k)}_- : \Omega^k_c(M) \rightarrow \Omega^k(M)
\]

associated with $\Box : \Omega^k(M) \rightarrow \Omega^k(M)$, see Section 3.2 given by

\[
G^{(k)} \alpha = G \alpha_S + dt \wedge G \alpha_T,
\]

for all $\alpha \in \Omega^k_c(M)$, where

\[
(G \beta)(t, \cdot) := \int_\mathbb{R} dt' (t - t') \beta_H(t', \cdot) + \int_\mathbb{R} dt' \left( \Delta^{\frac{1}{2}} \sin \left( \Delta^{\frac{1}{2}} (t - t') \right) \beta_\perp \right) (t', \cdot),
\]

with $\beta \in \Gamma_c(M, \Lambda^j \Sigma)$, $j = k, k - 1$. (Note that the Laplacian $\Delta$ on the orthogonal part $\mathcal{H}_\perp^k(\Sigma)$ has strictly positive spectrum, hence one can consider $\Delta^{\frac{1}{2}}$, as well as its inverse $\Delta^{-\frac{1}{2}}$. Incidentally, we observe that $G$, is the retarded-minus-advanced propagator for the normally hyperbolic linear differential operator $\partial_t^2 + \Delta : \Gamma(M, \Lambda^j \Sigma) \rightarrow \Gamma(M, \Lambda^j \Sigma)$. (3.28) can be used to compute $\Lambda$ from (3.17) more explicitly, which leads to an explicit formula for the Poisson structure $\gamma_M$ from (3.17) that makes the so-called “positive and negative frequency contributions” manifest.

With these preparations, we construct a two-point function $\omega_2$ by mimicking the usual prescription, which amounts to selecting the “positive frequency contribution” in (3.28). Explicitly, we introduce

\[
\omega_2 : \mathcal{L}(M)^{\otimes 2} \rightarrow \mathbb{C}
\]

in $\mathbf{Ch}_\mathbb{R}$ defining the only non-vanishing component

\[
\omega_2^0 : \bigoplus_{m=-p}^p (\mathcal{L}(M)^m \otimes \mathcal{L}(M)^{-m}) \rightarrow \mathbb{C}
\]

component-wise for all $m = -p, \ldots, p$ by

\[
(\omega_2^0)_m := \begin{cases} (-1)^{\lfloor m/2 \rfloor} W^{(p+m)} (\delta \otimes \text{id}), & m = -p, \ldots, -1, \\ W^{(p)}, & m = 0, \\ (-1)^{\lfloor m/2 \rfloor} W^{(p-m)} (\text{id} \otimes \delta), & m = 1, \ldots, p, \end{cases}
\]

where the linear map

\[
W^{(k)} : \Omega^k_c(M)^{\otimes 2} \rightarrow \mathbb{C},
\]
for $k = 0, \ldots, p$, is defined for all $\alpha, \alpha' \in \Omega^k(M)$ by

$$W^{(k)}(\alpha \otimes \alpha') := W_H(\alpha_S \otimes \alpha'_S) + W_\perp(\alpha_S \otimes \alpha'_S) - W_H(\alpha_T \otimes \alpha'_T) - W_\perp(\alpha_T \otimes \alpha'_T),$$

(3.30b)

with

$$W_H(\beta \otimes \beta') := \frac{1}{2} \int_\mathbb{R} dt \int_\mathbb{R} dt' \langle \beta_H(t, \cdot), i(t - t')\beta_H'(t', \cdot) \rangle,$$

(3.30c)

$$W_\perp(\beta \otimes \beta') := \frac{1}{2} \int_\mathbb{R} dt \int_\mathbb{R} dt' \langle \beta_\perp(t, \cdot), (\bigtriangleup - \bigtriangledown)^{-1} \exp(i\bigtriangleup(t - t')\bigtriangleup_\perp)(t', \cdot) \rangle,$$

(3.30d)

for all $\beta, \beta' \in \Gamma_c(M, \Lambda^j \Sigma)$. Here the pairing $\langle \cdot, \cdot \rangle$ denotes the usual scalar product between $\mathbb{C}$-valued $j$-forms on $\Sigma$ for $j = k, k - 1$.

Now it remains to verify that $\omega_2$ is a cochain map whose anti-symmetric part matches the microlocal spectrum condition, see e.g. [KM15, Sec. 5.3.4]. In particular, this entails that the induced two-point function $\omega_2: \chi(M) \otimes \chi(M) \to \mathbb{C}$ agrees with the Hadamard two-point function of [FP03, Sec. IV.C].

Remark 3.6. Note that in (3.30) one could add any (time-constant) symmetric operator $A_j$ acting on harmonic $j$-forms on $\Sigma$ by replacing $i(t - t')$ with $i(t - t') + A_j$. Since $\Sigma$ is assumed to be compact, harmonic $j$-forms on $\Sigma$ form a finite dimensional Hilbert space. Therefore $A_j$ is just any symmetric matrix.

Remark 3.7. We emphasize that the integral kernel of $W_\perp$ from (3.30) is a bidistribution fulfilling the microlocal spectrum condition, see e.g. [KM15, Sec. 5.3.4]. In particular, this entails that the induced two-point function $H^0(\omega_2) : H^0(\mathcal{L}(M)) \otimes H^0(\mathcal{L}(M)) \to \mathbb{C}$ on the degree 0 cohomology of the observable complex $\mathcal{L}(M)$ fulfills the microlocal spectrum condition. Note that in degree 0 cohomology and for $p = 1$ our construction reproduces gauge-invariant on-shell linear observables for the electromagnetic vector potential. Indeed, the two-point function $H^0(\omega_2) : H^0(\mathcal{L}(M)) \otimes H^0(\mathcal{L}(M)) \to \mathbb{C}$ agrees with the Hadamard two-point function of [FP03, Sec. IV.C].
The next step uses \( \omega_2 \) from (3.35) to define a linear functional

\[
\omega : \mathfrak{A}(M) \rightarrow \mathbb{C}
\]

in \( \text{Ch}_C \) on (the cochain complex underlying) the quantized differential graded algebra \( \mathfrak{A}(M) \) from (3.18). The latter is specified on words of arbitrary length \( m \geq 0 \) (\( m = 0 \) corresponds to the unit \( 1 \in \mathfrak{A}(M) \)) by

\[
\omega(\varphi_1 \otimes \cdots \otimes \varphi_m) := \begin{cases} 
1, & m = 0, \\
0, & m = 2k - 1, k \geq 1, \\
\sum_{\sigma \in P} \text{sign}(\sigma; |\varphi_1|, \ldots, |\varphi_{2k}|) \prod_{i=1}^{k} \omega(\varphi_{\sigma_{2i-1}} \otimes \varphi_{\sigma_{2i}}), & m = 2k, k \geq 1,
\end{cases}
\]

(3.35b)

for all homogeneous cochains \( \varphi_1, \ldots, \varphi_m \in \mathcal{L}(M) \), see [KM15, Sec. 5.2.4]. In the previous formula \( P \) denotes the set of all partitions \( \sigma = (\sigma_1, \ldots, \sigma_k) \) of the ordered set \( \{1 < \ldots < 2k\} \) into \( k \) ordered pairs \( \sigma_i = (\sigma_{i1} < \sigma_{i2}) \), \( i = 1, \ldots, k \). Furthermore, \( \text{sign}(\sigma; |\varphi_1|, \ldots, |\varphi_{2k}|) \) denotes the Koszul sign obtained by permuting the letters \( \varphi_1, \ldots, \varphi_{2k} \) according to the permutation associated with \( \sigma \).

Let us confirm that \( \omega \) is a well-defined cochain map. First, combining (3.34) and the explicit formula in (3.35), it follows that \( \omega \) vanishes on the ideal generated by the canonical commutation relations (3.18) and hence it descends to the quotient \( \mathfrak{A}(M) \). Furthermore, the fact that \( \omega_2 \) is a cochain map entails that \( \omega \) is a cochain map too. (To this end Koszul signs play a crucial role.)

Remark 3.8. Since \( \omega \) is a cochain map, passing to degree 0 cohomology one obtains a linear functional \( H^0(\omega) : H^0(\mathfrak{A}(M)) \rightarrow \mathbb{C} \). It is straightforward to check that the algebra \( H^0(\mathfrak{A}(M)) \) contains the CCR algebra \( \mathfrak{A}^{\text{inv}}(M) \subseteq H^0(\mathfrak{A}(M)) \) associated with the degree 0 cohomology \( H^0(\mathcal{L}(M)) \) of the observable complex, consisting of gauge-invariant on-shell linear observables, endowed with the induced Poisson structure \( H^0(\tau_M) : H^0(\mathcal{L}(M)) \wedge^2 \rightarrow \mathbb{R} \). Recalling Remark 3.7, one observes that the restriction \( \omega^{\text{inv}} : \mathfrak{A}^{\text{inv}}(M) \rightarrow \mathbb{C} \) of \( H^0(\omega) \) is a Hadamard state. In particular, for \( p = 1 \), this produces a Hadamard state on the CCR algebra \( \mathfrak{A}^{\text{inv}}(M) \) of gauge-invariant on-shell linear observables for the electromagnetic vector potential. As we shall explain in Remark 3.10, the algebra \( H^0(\mathfrak{A}(M)) \) is often strictly larger than the algebra \( \mathfrak{A}^{\text{inv}}(M) \). As a consequence \( H^0(\omega) \) is often richer than its restriction \( \omega^{\text{inv}} \).

The second step of our construction of a net representation mimics the first part of the Gelfand-Naimark-Segal construction. This yields a representation of the differential graded algebra \( \mathfrak{A}_M(M) = \mathfrak{A}(M) \in \text{DGA}_C \) induced by \( \omega \) from (3.35). Introducing the subcomplex \( R_\omega \subseteq \mathfrak{A}_M(M) \in \text{Ch}_C \) defined degree-wise for all \( n \in \mathbb{Z} \) by

\[
R_\omega^n := \{ a \in \mathfrak{A}_M(M)^n : \omega(ba) = 0, \forall b \in \mathfrak{A}_M(M)^{-n} \},
\]

(3.36a)

one immediately observes that \( R_\omega \) is a left \( \mathfrak{A}_M(M) \)-ideal. Therefore, the quotient of \( \mathfrak{A}_M(M) \) by \( R_\omega \) defines the left \( \mathfrak{A}_M(M) \)-module

\[
V_\omega := \mathfrak{A}_M(M)/R_\omega \in \mathfrak{A}_M(M)\text{Mod}.
\]

(3.36b)

In other words, \( V_\omega \) defines a representation of the global algebra of observables \( \mathfrak{A}_M(M) \).

Finally, in order to obtain a net representation of \( \mathfrak{A}_M \) we exploit Construction 2.6. Evaluating the right adjoint functor \( (-)^M : \mathfrak{A}_M(M)\text{Mod} \rightarrow \text{Rep}(\mathfrak{A}_M) \) on \( V_\omega \) defines the constant net representation

\[
\mathfrak{V}_\omega := V_\omega^M \in \text{Rep}(\mathfrak{A}_M)
\]

(3.37)

of the net \( \mathfrak{A}_M \in \text{Net}_C(\text{CCO}(M)) \) from (3.29). Explicitly, for each causally convex open subset \( U \in \text{CCO}(M) \), \( \mathfrak{V}_\omega(U) = V_\omega|_{\mathfrak{A}_M(U)} \in \mathfrak{A}_M(U)\text{Mod} \) is just the restriction of the left \( \mathfrak{A}_M(M) \)-module \( V_\omega \) along the morphism \( \mathfrak{A}_M(i_U^M) : \mathfrak{A}_M(U) \rightarrow \mathfrak{A}_M(M) \) in \( \text{DGA}_C \) associated with the inclusion \( i_U^M : U \rightarrow M \) in \( \text{CCO}(M) \).
3.5 2-dimensional example

In this section we shall compute explicitly the global algebra of observables $\mathfrak{A}_M(M)$ of the net $\mathfrak{A}_M \in \text{Net}_{\text{CCO}}(M)$ from (3.23) and describe its constant net representations. We shall do so in the simplest scenario, namely setting $m = 2$, $p = 1$ and choosing as oriented and time-oriented globally hyperbolic Lorentzian manifold the 2-dimensional flat Lorentz cylinder $M \in \text{Loc}_\mathbb{R}$ consisting of the manifold $\mathbb{R} \times S^1$ endowed with the constant metric $g = -dt^2 + d\theta^2$, with the time-orientation determined by the vector field $\partial_t$ and with the counter-clockwise orientation on the unit length circle $S^1$. Concretely, we shall construct a differential graded algebra $A \in \text{DGA}_\mathbb{C}$ that is weakly equivalent to the original one $\mathfrak{A}_M(M)$, but has the advantages of having finitely many generators and trivial differential. These features make its category of left modules $A\text{Mod}$ easier to describe than the category $\mathfrak{A}_M(M)\text{Mod}$, which is Quillen equivalent to it on account of Proposition A.12 but practically less accessible. This fact will allow us to obtain a very explicit description of all the constant net representations of $\mathfrak{A}_M$ up to weak equivalence.

To construct the differential graded algebra $A \in \text{DGA}_\mathbb{C}$, weakly equivalent to $\mathfrak{A}_M(M)$, we proceed in three steps. First, we shall construct a cochain complex $L \in \text{Ch}_\mathbb{R}$ with trivial differential and a quasi-isomorphism $c : L \to \mathcal{L}(M)$ in $\text{Ch}_\mathbb{C}$ to the observable complex $\mathcal{L}(M)$ from (3.9). Second, we shall endow $L$ with the Poisson structure $\tilde{\tau} := \tau_M(c \otimes c)$ induced by $\tau_M$ from (3.17). This defines a new Poisson complex $(L, \tilde{\tau})$ and a quasi-isomorphism $c : (L, \tilde{\tau}) \to (\mathcal{L}(M), \tau_M)$ that by construction preserves the Poisson structures. In the third and last step we shall quantize the Poisson complex $(L, \tilde{\tau})$ by means of the canonical commutation relations, as in (3.18).

Since by [BBS20, Prop. 5.3] the CCR quantization functor preserves weak equivalences, from $c : (L, \tilde{\tau}) \to (\mathcal{L}(M), \tau_M)$ we obtain a differential graded algebra $A \in \text{DGA}_\mathbb{C}$ and a weak equivalence $q : A \to \mathfrak{A}_M(M)$ in $\text{DGA}_\mathbb{C}$, to the original differential graded algebra $\mathfrak{A}_M(M) \in \text{DGA}_\mathbb{C}$.

We consider the cochain complex with vanishing differential

$$L := \left( \begin{array}{c@{}c@{}c@{}c} \mathcal{H}^0(S^1) & 0 & \mathcal{H}^1(S^1) & 0 \\ e_1 & \delta & e_2 & \delta \\ \end{array} \right) \in \text{Ch}_\mathbb{R}, \quad (3.38)$$

which is generated as a graded vector space by the harmonic forms $e_1 := 1$, $a_1 := d\theta$, $a_2 := d\theta$, $e := 1$. Furthermore, choosing any compactly supported function $f \in C^\infty_c(\mathbb{R})$ such that $\int_\mathbb{R} dt \ f = 1$, we define the cochain map

$$c : L \longrightarrow \mathcal{L}(M) \quad (3.39a)$$

in $\text{Ch}_\mathbb{R}$ degree-wise by

$$c^{-1}(e_1) := f(t) \ dt \in \mathcal{L}(M)^{-1}, \quad (3.39b)$$

$$c^0(a_1) := f(t) \ d\theta + \left( \int_\mathbb{R} ds \ s \ f(s) \right) f'(t) \ d\theta, \quad c^0(a_2) \in \mathcal{L}(M)^0 := -f'(t) \ d\theta \in \mathcal{L}(M)^0, \quad (3.39c)$$

$$c^1(e) := f(t) \in \mathcal{L}(M)^1. \quad (3.39d)$$

Recalling (3.9) one easily checks that those on the right hand sides are cocycles in $\mathcal{L}(M)$ and therefore $c$ is a well-defined cochain map, as claimed. Even more, $c$ is a quasi-isomorphism. To check this fact, we consider also the cochain map

$$\tilde{c} : \mathcal{L}(M) \longrightarrow L \quad (3.40a)$$
Remark 3.10. Recalling Remark 3.8, one realizes that the CCR algebra gauge-invariant on-shell linear observables is isomorphic to the subalgebra of has unit length and that is indeed well-defined because only. In contrast, the cohomology and is the differential graded algebra generated by formula for the retarded-minus-advanced propagator structures. Recalling [BBS20, Prop. 5.3], one finds that the CCR quantization functor sends the weak equivalence of differential graded algebras and the equations in construction of be constructed for instance using the decompositions in a way similar to the provide an explicit choice of as it is not needed in the sequel. Let us just mention that it can can

\[ \eta \] is a cochain map. Furthermore, one immediately realizes that \( c \) as defined above is indeed a cochain map. Furthermore, one immediately realizes that \( \tilde{c} = \text{id} \), hence \( H(\tilde{c}) = \text{id} \) in cohomology. On the other hand direct inspection shows also that \( H(c) H(\tilde{c}) = \text{id} \). Therefore \( H(c) : H(L) \to H(L(M)) \) is an isomorphism, i.e. \( c \) is a quasi-isomorphism.

Remark 3.9. The previous argument can be refined by constructing a homotopy \( \eta \) that exhibits \( \tilde{c} \) as a quasi-inverse of \( c \). More in detail, \( \eta \in \{ L(M), L(M) \}^{-1} \) is a \((-1)\)-cochain in the internal hom, whose differential \( \partial \eta = c \tilde{c} - \text{id} \) controls to what extent \( c \tilde{c} \) differs from \text{id}. The homotopy \( \eta \) and the equations \( \tilde{c} \tilde{c} = \text{id} \) and \( c \tilde{c} = \text{id} + \partial \eta \) witness that \( \tilde{c} \) is a quasi-inverse of \( c \). We do not provide an explicit choice of \( \eta \) as it is not needed in the sequel. Let us just mention that it can be constructed for instance using the decompositions in a way similar to the construction of \( \tilde{c} \) in (3.40).

\[ \Delta \]

We now endow the cochain complex \( L \) with the transferred Poisson structure

\[ \tilde{\tau} := \tau_M (c \otimes c) : L^{\wedge 2} \to \mathbb{R}, \] (3.41a)

which reads explicitly as

\[ \tilde{\tau}(e, e^i) = 1, \quad \tilde{\tau}(a_2, a_1) = 1, \quad \tilde{\tau}(a_i, e^{(i)}) = 0. \] (3.41b)

This result follows from the definition of the Poisson structure \( \tau_M \) in (3.17) and the explicit formula for the retarded-minus-advanced propagator \( G^{(i)} \) from (3.28). (One also uses that \( S^1 \) has unit length and that \( f \) is such that \( \int_{\mathbb{R}} dt f = 1 \).)

Summarizing, \( c : (L, \tilde{\tau}) \to (L(M), \tau_M) \) is a quasi-isomorphism that preserves the Poisson structures. Recalling [BBS20, Prop. 5.3], one finds that the CCR quantization functor sends \( c \) to the weak equivalence of differential graded algebras \( q : A \to \mathfrak{A}_M(M) \) in \( \text{DGA}_C \), where

\[ A := T_C(L)/I_{\tilde{\tau}} \in \text{DGA}_C \] (3.42a)

is the differential graded algebra generated by \( L \) and subject to the canonical commutation relations encoded by the two-sided ideal \( I_{\tilde{\tau}} \) generated by

\[ e \otimes e^i + e^i \otimes e - i, \quad a_2 \otimes a_1 - a_1 \otimes a_2 - i, \quad a_i e^{(i)} - e^{(i)} a_i, \] (3.42b)

and \( q \) extends \( c \) from generators. (Compare the above construction of \( A \) to (3.18) and note that \( q \) is indeed well-defined because \( c \) preserves the Poisson structures.)

Remark 3.10. Recalling Remark 3.8 one realizes that the CCR algebra \( \mathfrak{A}^{\text{inv}}(M) \) generated by gauge-invariant on-shell linear observables is isomorphic to the subalgebra of \( \mathfrak{A} \) generated by \( a_1, a_2 \) only. In contrast, the cohomology \( H(\mathfrak{A}_M(M)) \) of the global algebra \( \mathfrak{A}_M(M) \) is isomorphic to \( A \) (seen as a graded algebra, i.e. forgetting its differential). In particular, the degree 0 cohomology algebra \( H^0(\mathfrak{A}_M(M)) \cong A^0 \) is strictly larger than \( \mathfrak{A}^{\text{inv}}(M) \). Indeed, along with the generators \( a_1 \)
and $a_2$, which are also contained in $\mathfrak{A}^{inv}(M)$, $A^0$ has an additional generator $e^1$ that commutes with both $a_1$ and $a_2$. The latter is a composite observable formed by an antifield observable $e^1$ and a ghost observable $e$. Those may be regarded as “gauge-invariant” observables in the broader sense of being cohomologically non-trivial.

While being Quillen equivalent to $\mathfrak{A}_M(M)\text{Mod}$ due to Proposition A.142, the category of left $A$-modules $\mathfrak{A}\text{Mod}$ is considerably easier to describe. As it is always the case, a left $A$-module $V = (V, \nu) \in \mathfrak{A}\text{Mod}$ consists of a cochain complex $V \in \text{Ch}_C$ and of a left $A$-action $\nu : A \otimes V \to V$ in $\text{Ch}_C$, which is the same datum as a morphism $\nu : A \to [V, V]$ in $\text{DGA}_C$ to the internal endomorphism algebra $\nu \subseteq [V, V] \in \text{DGA}_C$. The specific form of $A$, however, allows us to describe $\nu$ in terms of very few concrete data. Since $A = T_C(L)/I_\infty$ is defined as the differential graded algebra that is generated by the $(-1)$-cocycle $e^1 \in L^{-1}$, the 0-cocycles $a_1, a_2 \in L^0$ and the 1-cocycle $e \in L^1$ and that is subject to the canonical commutation relations from (3.42), it follows that the left $A$-action $\nu$ is pinned down by the $(-1)$-cocycle $\nu(e^1) \in [V, V]^{-1}$, the 0-cocycles $\nu(a_1), \nu(a_2) \in [V, V]^0$ and the 1-cocycle $e(\nu(e)) \in [V, V]^1$ in the internal endomorphism algebra $[V, V] \in \text{DGA}_C$. Furthermore, these cocycles must fulfill the relations $\nu(e) \nu(e^1) + \nu(e^1) \nu(e) = i\text{id}_V, \nu(a_2) \nu(a_1) - \nu(a_1) \nu(a_2) = i\text{id}_V$ and $\nu(a_1) \nu(e^1) = \nu(e^1) \nu(a_1)$. Similarly, a left $A$-module morphism $F : V \to V'$ in $\mathfrak{A}\text{Mod}$ consists of a cochain map $F : V \to V'$ in $\text{Ch}_C$ that preserves the cocycles above, i.e. $F(\nu(e)) = \nu'(e^1) F, F(\nu(a_1)) = \nu'(a_1) F, F(\nu(a_2)) = \nu'(a_2) F$ and $F \nu(e) = \nu'(e) F$.

The above very explicit description of left $A$-modules gives us a handle to construct all constant representations of the net $\mathfrak{A}_M$ up to weak equivalence. The latter form the essential image of the total right derived functor of the right Quillen functor $(-)^M : \mathfrak{A}_M(M)\text{Mod} \to \text{Rep}(\mathfrak{A}_M)$ from Construction 2.6 and Remark 2.20. Since $q : A \to \mathfrak{A}_M(M)$ in $\text{DGA}_C$ is a weak equivalence, it follows by Proposition A.142 that $\mathfrak{A}_M(M)\text{Mod} \to A\text{Mod}$ is a Quillen equivalence and hence induces an equivalence of homotopy categories. Combining these facts, the essential image of the total right derived functor $(-)^M$ is equivalent to the homotopy category of $A\text{Mod}$. In other words, left $A$-modules provide all constant representations of the net $\mathfrak{A}_M$ up to weak equivalence. (For the notions of homotopy category associated with a model category and of total left (right) derived functor associated with a left (respectively right) Quillen functor we refer the reader to [Hov99, Ch. 1].)

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Data availability statement

Data sharing not applicable to this article as no datasets were generated or analysed during the current study.

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\footnote{Given any cochain complex $V \in \text{Ch}_C$, the internal hom $[V, V] \in \text{Ch}_C$ carries a natural differential graded algebra structure consisting of the usual internal hom differential complemented with the multiplication given by composition and the unit given by $\text{id}_V$. When endowed with this structure we refer to $[V, V] \in \text{DGA}_C$ as the internal endomorphism algebra of $V \in \text{Ch}_C$.}
A Monoids and modules in a symmetric monoidal category

We collect here the basics of the theory of monoids and of modules over a monoid in a complete and cocomplete closed symmetric monoidal category \( \mathbf{M} = (\mathbf{M}, \otimes, \mathbb{1}) \). Standard textbook references for these topics are Mac78, Bor94, Eti+15. The prime example of a complete and cocomplete closed symmetric monoidal category is the category Vec\(_K\) of vector spaces over a field \( K \). Another important example is the category Ch\(_K\) of cochain complexes, which is recalled at the beginning of Section 3, where it will be extensively used.

**Definition A.1.** A monoid \( A = (A, \mu, 1) \) in \( \mathbf{M} \) consists of an object \( A \in \mathbf{M} \) endowed with two morphisms \( \mu : A \otimes A \to A \) and \( 1 : \mathbb{1} \to A \) in \( \mathbf{M} \), called multiplication and unit respectively, subject to the usual associativity \( \mu(\mu \otimes 1_A) = \mu(1_A \otimes \mu) \) and unitality \( \mu(1 \otimes 1_A) = 1_A = \mu(1_A \otimes 1) \) axioms. Furthermore, one defines a morphism of monoids \( \varphi : A \to B \) as a morphism \( \varphi : A \to B \) in \( \mathbf{M} \) that preserves multiplications \( (\mu_B (\varphi \otimes \varphi) = \varphi \mu_A) \) and units \( (1_B = \varphi 1_A) \). \( \text{Mon}(\mathbf{M}) \) denotes the category of monoids in \( \mathbf{M} \).

**Definition A.2.** Given a monoid \( A \in \text{Mon}(\mathbf{M}) \), a left \( A \)-module \( L = (L, \lambda) \) in \( \mathbf{M} \) consists of an object \( L \in \mathbf{M} \) endowed with a morphism \( \lambda : A \otimes L \to L \) in \( \mathbf{M} \), called left \( A \)-action, subject to the usual axioms \( \lambda(1 \otimes L) = \lambda(\mu \otimes L) = \lambda(L \otimes L) \). Furthermore, one defines a morphism of left \( A \)-modules \( F : L \to L' \) as a morphism \( F : L \to L' \) in \( \mathbf{M} \) that is compatible with the left \( A \)-actions \( \lambda(L \otimes F) = F \lambda \). This defines the category \( \text{AMod} \) of left \( A \)-modules in \( \mathbf{M} \).

One defines the category \( \text{Mod}_A \) of right \( A \)-modules in a similar fashion.

### A.1 Change-of-monoid adjunction

For the constructions recalled below, we refer the reader again to the textbooks Mac78, Bor94, Eti+15. Associated with a morphism \( \varphi : A \to B \) in \( \text{Mon}(\mathbf{M}) \), one has the so-called change-of-monoid adjunction

\[
\mathfrak{Ext}_\varphi := B \otimes (-) : \text{AMod} \rightleftarrows \text{BMod} : (-)|_A = : \mathfrak{Res}_\varphi,
\]

whose right adjoint \( \mathfrak{Res}_\varphi \) and left adjoint \( \mathfrak{Ext}_\varphi \) restrict and respectively extend the module structure along \( \varphi \). Explicitly, the restriction functor

\[
\mathfrak{Res}_\varphi := (-)|_A : \text{BMod} \longrightarrow \text{AMod}
\]

assigns to a left \( B \)-module \( M = (M, \lambda) \in \text{BMod} \) the left \( A \)-module \( M|_A := (M, \lambda(\varphi \otimes id M)) \in \text{AMod} \) defined restricting the left \( B \)-module action \( \lambda : B \otimes M \to M \) along \( \varphi \). Furthermore, \( \mathfrak{Res}_\varphi \) assigns to a morphism of left \( B \)-modules \( G : M \to M' \) in \( \text{BMod} \) the morphism of left \( A \)-modules \( G|_A : M|_A \to M'|_A \) in \( \text{AMod} \) whose underlying morphism in \( \mathbf{M} \) coincides with the one underlying \( G \). The left adjoint to \( \mathfrak{Res}_\varphi \) is the extension functor

\[
\mathfrak{Ext}_\varphi := B \otimes_A (-) : \text{AMod} \longrightarrow \text{BMod}
\]

defined below. \( \mathfrak{Ext}_\varphi \) assigns to a left \( A \)-module \( L = (L, \lambda) \in \text{AMod} \) the relative tensor product \( B \otimes_A L \in \text{BMod} \), defined as the coequalizer

\[
B \otimes_A L := \text{colim} \left( \begin{array}{c}
B \otimes A \otimes L \\
\mu_B(id_B \otimes \varphi) \otimes id_L \\
\lambda \otimes id_B
\end{array} \right) \in \mathbf{BMod}.
\]

Furthermore, \( \mathfrak{Ext}_\varphi \) assigns to a morphism of left \( A \)-modules \( F : L \to L' \) in \( \text{AMod} \) the morphism of left \( B \)-modules \( B \otimes_A F : B \otimes A L \to B \otimes A L' \) in \( \text{BMod} \) defined via the universal property of
the coequalizer by the diagram

\[
\begin{array}{ccc}
B \otimes A \otimes L & \xrightarrow{\mu_B((id_B \otimes \varphi) \otimes id_L)} & B \otimes L \\
\downarrow{\text{id}_B \otimes id_A \otimes F} & & \downarrow{id_B \otimes \lambda} \\
B \otimes A \otimes L' & \xrightarrow{\mu_B((id_B \otimes \varphi) \otimes id_{L'})} & B \otimes L'
\end{array}
\]

(A.3c)

in \(B\text{Mod}\). From the definitions above one easily obtains a bijection

\[
_{B\text{Mod}}(\mathfrak{C}t_{\varphi}, L, M) \cong _{A\text{Mod}}(L, \mathfrak{Res}_{\varphi} M),
\]

which is natural with respect to both \(L \in _{A\text{Mod}}\) and \(M \in _{B\text{Mod}}\), hence proving the adjunction

(A.4).

**Remark A.3.** Given composable morphisms \(\varphi : A \to B\) and \(\psi : B \to C\) in \(\text{Mon}(M)\), one immediately realizes that the restriction \(\mathfrak{Res}_{\psi} : _{C\text{Mod}} \to _{B\text{Mod}}\) followed by the restriction \(\mathfrak{Res}_{\varphi} : _{B\text{Mod}} \to _{A\text{Mod}}\) coincides with the restriction \(\mathfrak{Res}_{\varphi \psi} : _{C\text{Mod}} \to _{A\text{Mod}}\) along the composition \(\psi \varphi\). From the change-of-monoid adjunction \([A.1]\) it then follows that the extension \(\mathfrak{C}t_{\varphi} : _{A\text{Mod}} \to _{B\text{Mod}}\) followed by the extension \(\mathfrak{C}t_{\psi} : _{B\text{Mod}} \to _{C\text{Mod}}\) is naturally isomorphic to the extension \(\mathfrak{C}t_{\psi \varphi} : _{A\text{Mod}} \to _{C\text{Mod}}\) along the composition \(\psi \varphi\). \(\triangleleft\)

**Remark A.4.** Given an isomorphism \(\varphi : A \to B\) in \(\text{Mon}(M)\), the corresponding change-of-monoid adjunction \([A.1]\) is actually an adjoint equivalence. This follows from the straightforward observation that in this case both the unit and the counit of the change-of-monoid adjunction are natural isomorphisms. \(\triangleleft\)

### A.2 M-tensoring, powering and enriched hom on \(\text{AMod}\)

For a given monoid \(A \in \text{Mon}(M)\), the category of left \(A\)-modules \(\text{AMod}\) admits the canonical \(M\)-tensoring

\[
\otimes : \text{AMod} \times M \to \text{AMod},
\]

that assigns to a left \(A\)-module \(L = (L, \lambda) \in \text{AMod}\) and an object \(V \in M\) the left \(A\)-module \(L \otimes V \in \text{AMod}\) consisting of the object \(L \otimes V \in M\) equipped with the left \(A\)-module action \(\lambda \otimes id_V : A \otimes L \otimes V \to L \otimes V\), and to a morphism \(F : L \to L'\) in \(\text{AMod}\) and a morphism \(\xi : V \to V'\) in \(M\) the morphism \(F \otimes \xi : L \otimes V \to L' \otimes V'\) in \(\text{AMod}\) whose underlying morphism

\[
\xi : V \to V'
\]

in \(M\) is the tensor product of the underlying morphism \(F\) in \(M\) with \(\xi\).

Given \(V \in M\), the \(M\)-tensoring \((-) \otimes V : \text{AMod} \to \text{AMod}\) is part of the adjunction

\[
(-) \otimes V : \text{AMod} \rightleftarrows \text{AMod} : (-)^V,
\]

whose right adjoint \((-)^V\) is the partial evaluation of the \(M\)-powering

\[
(-)^V : \text{AMod} \times M^{\text{op}} \to \text{AMod}.
\]

The latter is defined through the internal hom \([-,-] : M^{\text{op}} \otimes M \to M\) of the closed symmetric monoidal category \(M\). Explicitly, \((-)^{-}\) assigns to a left \(A\)-module \(L = (L, \lambda) \in \text{AMod}\) and an object \(V \in M\) the left \(A\)-module \(L^V \in \text{AMod}\) consisting of the internal hom \([V, L] \in M\) and the left \(A\)-module action \(A \otimes [V, L] \to [V, L]\) in \(M\) defined as the adjunct of the morphism

\[
A \otimes [V, L] \otimes V \xrightarrow{id_{\otimes} \text{ev}} A \otimes L \xrightarrow{\lambda} L
\]

in \(M\) with respect to the adjunction \((-) \otimes V \dashv [V, -] : M \to M\). Furthermore, \((-)^{-}\) assigns to a morphism \(F : L \to L'\) in \(\text{AMod}\) and a morphism \(\xi : V' \to V\) in \(M\) the morphism \(F^{\xi} : L^V \to L'^{V'}\) in \(\text{AMod}\) whose underlying morphism is \([\xi, F] : [V, L] \to [V', L']\) in \(M\).
Similarly, given $L \in A\text{Mod}$, the other partial evaluation of the $M$-tensoring $L \otimes (-) : M \to A\text{Mod}$ is part of the adjunction
\[
L \otimes (-) : M \rightleftarrows A\text{Mod} : [L, -]_A,
\]
whose right adjoint $[L, -]_A$ is the partial evaluation of the $M$-enriched hom
\[
[-, -]_A : A\text{Mod}^{\text{op}} \times A\text{Mod} \to M.
\] (A.8a)

The latter assigns to left $A$-modules $L = (L, \lambda), L' = (L', \lambda') \in A\text{Mod}$ the equalizer
\[
[L, L']_A := \text{lim} \left( [L, L'] \xrightarrow{[\lambda, \text{id}]} [A \otimes L, L'] \right) \in M,
\] (A.9b)

where $\lambda'$ denotes the adjunct with respect to the adjunction $(-) \otimes A \otimes L \dashv [A \otimes L, -] : M \to M$ of the morphism
\[
[L, L'] \otimes A \otimes L \xrightarrow{\sim} A \otimes [L, L'] \otimes L \xrightarrow{\text{id} \otimes \text{ev}} A \otimes L' \xrightarrow{\lambda'} L'
\] (A.9c)
in $M$. The action of $[-, -]_A$ on morphisms is defined combining the universal property of the equalizer in (A.9b) and the functoriality both of the tensor product $\otimes$ and of the internal hom $[-, -]$ of $M$.

**Remark A.5.** Note that the adjunction (A.6) is compatible with the change-of-monoid adjunction (A.1) in the following sense. Given a morphism of monoids $\varphi : A \to B$ in $\text{Mon}(M)$ and an object $V \in M$, the diagram of right adjoint functors
\[
\begin{array}{ccc}
A\text{Mod} & \xrightarrow{\text{Res}_\varphi} & A\text{Mod} \\
(-)^V \downarrow & & \downarrow (-)^V \\
B\text{Mod} & \xrightarrow{\text{Res}_\varphi} & A\text{Mod}
\end{array}
\] (A.10)

commutes as a straightforward consequence of their definitions. Therefore, the corresponding diagram of left adjoint functors commutes up to a unique natural isomorphism $\mathcal{E}\text{rt}_\varphi(- \otimes V) \cong (\mathcal{E}\text{rt}_\varphi(-)) \otimes V$.

\[\triangle\]

**Remark A.6.** One easily checks that the functors (A.5), (A.7) and (A.9) form a two-variable adjunction with isomorphisms
\[
A\text{Mod}(L_1 \otimes V, L_2) \cong A\text{Mod}(L_1, L_2^V) \cong M(V, [L_1, L_2]_A),
\] (A.11)
natural in $L_1, L_2 \in A\text{Mod}$ and $V \in M$, given by the adjunctions (A.6), (A.8) and
\[
[-, L]_A : A\text{Mod} \rightleftarrows M^{\text{op}} : L(-),
\] (A.12)

for $L \in A\text{Mod}$. Given a morphism of monoids $\varphi : A \to B$ in $\text{Mon}(M)$, the latter adjunction allows us to describe the interplay between the $M$-enriched hom $[-, -]_A : A\text{Mod}^{\text{op}} \times A\text{Mod} \to M$ on $A\text{Mod}$, the $M$-enriched hom $[-, -]_B : B\text{Mod}^{\text{op}} \times B\text{Mod} \to M$ on $B\text{Mod}$ and the restriction $\mathcal{E}\text{rt}_\varphi : B\text{Mod} \to A\text{Mod}$ along $\varphi$ in terms of the natural transformation
\[
[-, -]_B \to [-, -]_A \circ (\mathcal{E}\text{rt}_\varphi \times \mathcal{E}\text{rt}_\varphi)
\] (A.13)
between functors from $B\text{Mod}^{\text{op}} \times B\text{Mod}$ to $M$. Explicitly, its component for $M, M' \in B\text{Mod}$ is the morphism $[M, M']_B \to [M]_A, [M']_A$ in $M$ defined by the following three-step construction: (1) Consider the $M$-component $M \to M'^{[M, M']_B}$ in $B\text{Mod}$ of the unit of the adjunction $[-, M']_B \dashv M'^{(-)} : M^{\text{op}} \to B\text{Mod}$. (2) Construct from the latter the morphism $M]_A \to M][M', M']_B$ in $A\text{Mod}$ by applying the restriction $\mathcal{E}\text{rt}_\varphi : B\text{Mod} \to A\text{Mod}$ along $\varphi$ and by recalling the commutative square in (A.10). (3) Recalling also the adjunction $[-, M']_A \dashv M']_A(-) : M^{\text{op}} \to A\text{Mod}$, define the morphism $[M]_A, [M']_A \to [M, M']_B$ in the opposite category $M^{\text{op}}$, which is equivalent to the desired morphism $[M, M']_B \to [M]_A, [M']_A$ in $M$. (Note that we used here two different instances of the adjunction displayed in (A.12).)

\[\triangle\]
A.3 Model structure for modules over monoids

We recall from [SS00] some fundamental facts about the model structures on the category \( \textbf{Mon}(M) \) of monoids and on the category \( \textbf{AMod} \) of left \( A \)-modules over a monoid \( A \in \textbf{Mon}(M) \). For this purpose we assume that the symmetric monoidal category \( M \) is endowed with a suitable model structure as explained below.

\textbf{Set-up A.7}. Suppose \( M \) is a cofibrantly generated closed symmetric monoidal model category, see [Hov99, Sec. 4.2], that satisfies the monoid axiom and whose objects are small, see [SS00, Secs. 2 and 3].

\textbf{Example A.8}. The assumptions listed above are met, e.g., by the closed symmetric monoidal category \( \textbf{Mon} \) = \( \text{Ch}_K \) of (unbounded) cochain complexes over a field \( K \) equipped with the projective model structure. In particular, the monoid axiom follows from the fact that all cochain complexes \( V \in \text{Ch}_K \) are cofibrant (and fibrant), see [SS00, Rem. 3.4] and also the beginning of Section 3.

\textbf{Definition A.9}. A morphism \( \varphi : A \to B \) in the category \( \textbf{Mon}(M) \) of monoids in \( M \) is called a weak equivalence (fibration) if the underlying morphism in the model category \( M \) is a weak equivalence (respectively fibration) or a cofibration if it has the left lifting property (see e.g. [Hov99, Def. 1.1.2]) with respect to all acyclic fibrations in \( \textbf{Mon}(M) \), i.e. morphisms that are simultaneously fibrations and weak equivalences.

Furthermore, let \( A \in \textbf{Mon}(M) \) be a monoid in \( M \) and consider the category \( \textbf{AMod} \) of left \( A \)-modules in \( M \). A morphism \( F : L \to L' \) in \( \textbf{AMod} \) is called a weak equivalence (fibration) if the underlying morphism in the model category \( M \) is a weak equivalence (respectively fibration) or a cofibration if it has the left lifting property with respect to all acyclic fibrations in \( \textbf{AMod} \).

It follows from [SS00, Th. 4.1] that the previous definition equips both \( \textbf{Mon}(M) \) and \( \textbf{AMod} \) with cofibrantly generated model structures. We record this fact in the next statement.

\textbf{Proposition A.10} (\( \textbf{Mon}(M) \) and \( \textbf{AMod} \) as model categories). Let \( M \) be as in Set-up A.7 and consider a monoid \( A \in \textbf{Mon}(M) \). With the previous Definition A.9 of weak equivalences, fibrations and cofibrations, the category \( \textbf{Mon}(M) \) of monoids in \( M \) and the category \( \textbf{AMod} \) of left \( A \)-modules become cofibrantly generated model categories.

In a similar fashion one endows the category \( \textbf{Mod}_A \) of right \( A \)-modules with a cofibrantly generated model category structure.

Having established a model structure on \( \textbf{AMod} \), we investigate its compatibility with the \( \textbf{M} \)-tensoring, powering and enriched hom on \( \textbf{AMod} \). Namely we ask whether \( \textbf{AMod} \) is an \( \textbf{M} \)-model category, see [Hov99, Def. 4.2.18]. This amounts to showing that the \( \textbf{M} \)-tensoring \( [\text{A.5}] \) is a Quillen bifunctor, see [Hov99, Def. 4.2.1]. For \( F : L \to L' \) in \( \textbf{AMod} \) and \( \xi : V \to V' \) in \( M \), the pushout-product morphism

\[ F \square \xi : P(F, \xi) \to L' \otimes V' \]

(A.14a)

in \( \textbf{AMod} \) is determined by the commutative diagram

\[ \begin{array}{ccc}
L \otimes V & \xrightarrow{F \otimes \text{id}_V} & L' \otimes V \\
\text{id}_L \otimes \xi & & \downarrow \text{id}_{L'} \otimes \xi \\
L \otimes V' & \xrightarrow{F \otimes \text{id}_V'} & L' \otimes V'
\end{array} \]

(A.14b)

in \( \textbf{AMod} \) and by the universal property of the pushout \( P(F, \xi) := \text{colim}(L \otimes V' \leftarrow L \otimes V \to L' \otimes V) \in \textbf{AMod} \). Let \( I \) (\( J \)) be a set of generating cofibrations (respectively acyclic cofibrations) for \( M \). By [SS00, Th. 4.1 and Lem. 2.3], applying the free left \( A \)-module functor \( A \otimes (-) : M \to \textbf{AMod} \) to \( I \) (\( J \)) determines a set of generating cofibrations \( I_A := A \otimes I \) (respectively acyclic cofibrations \( J_A := A \otimes J \)) for \( \textbf{AMod} \). In order to show that the \( \textbf{M} \)-tensoring is a Quillen bifunctor, it suffices
to check that the pushout-product $F \Box \xi$ is a cofibration (acyclic cofibration) in $\mathbb{A}Mod$ when $F \in I_A$ and $\xi \in I$ (respectively $F \in J_A$ and $\xi \in J$ or $F \in I_A$ and $\xi \in J$), see [Hov99, Cor. 4.2.5]. Since $F \in I_A$ ($F \in J_A$) is of the form $F = \text{id}_A \otimes \eta$, with $\eta \in I$ (respectively $\eta \in J$), one obtains the isomorphism $P(F, \xi) \cong A \otimes P(\eta, \xi)$ in $\mathbb{A}Mod$ and hence the pushout-product morphism $F \Box \xi = A \otimes (\eta \Box \xi)$ can be computed from the pushout-product morphism $\eta \Box \xi$ in $\mathbb{M}$. Since $\mathbb{M}$ is per hypothesis a closed symmetric monoidal model category, fibrations and weak equivalences by definition of the model structure on acyclic cofibration) in $\mathbb{M}$ left Quillen functor since it is left adjoint to the forgetful functor $F$. Proposition A.11 (it is convenient to summarize the conclusion of this paragraph in a proposition. [Bär15] C. Bär, “Green-Hyperbolic Operators on Globally Hyperbolic Spacetimes”, Commun. Math. Phys. 333 (2015), pp. 1585–1615.

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