Controlled transfer of quantum amplitude via modulation of a potential barrier: numerical study in a model of SQUID

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We numerically integrate the time-dependent Schrödinger equation in a single-degree-of-freedom model of SQUID with a variable potential barrier between the basis flux states. We find that linear superpositions of the basis states, with relatively little residual excitation, can be formed by pulsed modulations of the barrier, provided the pulse duration exceeds the period of small oscillations of the flux. Two pulses applied in sequence exhibit strong interference effects, which we propose to use for an experimental determination of the decoherence time in SQUIDs.

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I. INTRODUCTION

Ability to manipulate linear superpositions of quantum states is expected to lead to significant advances in information processing, the paradigm that became known as quantum computing [1]. In addition, if the basis states are macroscopically distinct, one may be able to gain useful insights into how quantum coherence is destroyed (or maintained) in the macroscopic world [2].

Coherent superpositions of macroscopically distinct states have been obtained in two recent experiments [3,4] on superconducting quantum interferometers (SQUIDs). The SQUIDs were in the regime where the basis states, corresponding to different values of the magnetic flux, were separated by a potential barrier. In the first experiment [3], a coherent superposition was obtained by exciting a SQUID with a pulse of microwave radiation, thus bringing the system closer to the top of the potential barrier separating the basis states. In the second experiment, the barrier was low by design [4].

For quantum information processing, on the other hand, it is essential that the system can be excited and subsequently deexcited so as to produce a controlled superposition of stable, low-lying states. The energy of these states, after deexcitation, should be much lower than the height of the potential barrier separating them, to ensure that no unwanted transitions occur afterwards. The protocols used in refs. [3,4] do not allow for such deexcitation.

It is clear that a system with a fixed low-height barrier will not be able to satisfy the above requirement. Thus, it is natural to look at systems with variable potential barriers: lowering the barrier, by some external means, will correspond to excitation, while restoring it to the original height, to deexcitation. The combined effect of the two operations will be writing a stable quantum superposition to the device.

While at present it is far from obvious that SQUIDs will become the underlying technology for quantum computers, these systems do allow modulation of the barrier height (which in this case is the Josephson coupling energy) by an external agent, such as electric current or magnetic field. Indeed, the height of the barrier was adjustable in the experiment of ref. [3].

The aim of the present work was to find, through numerical integrations, if modulation of the potential barrier in such a system, by a pulse of current or field, can be useful for obtaining controlled superpositions of basis states, and what the required durations of pulses may be. As a model, we used the reduced, single-degree-of-freedom, description of a SQUID with the parameters adopted from ref. [3].

Our main result is that rather clean transfers of the quantum amplitude in such a system are possible within a few tens of picoseconds. (In Sect. 3 we present a measure of how “clean” the transfer is.) This time scale determines the switching time, \( t_0 \), of the qubit. The number of useful operations that the qubit can perform is limited by the ratio \( t_d/t_0 \), where \( t_d \) is the decoherence time, as well as by another factor, which we discuss in Sect. 3.

Decoherence [2] is an intrinsically many-body effect and cannot be studied in the single-degree-of-freedom model. That model, however, helps to identify interference phenomena that can be used for an experimental measurement of the decoherence time in SQUIDs. In Sect. 4, we propose to use, for that purpose, interference between two consecutive pulses. This method measures

† Our method is different from that recently proposed in ref. [5], where one changes the biasing flux through a SQUID, while keeping the Josephson energy fixed (and small).
II. THE MODEL

We start with the usual model of a SQUID as a system with a single degree of freedom, corresponding to the magnetic flux through the loop. Fluxes are conveniently measured in units of the flux quantum $\Phi_0$. If we denote the total flux in these units as $y$, and the biasing flux by $\bar{y}$, the Schrödinger equation for the wave function $\Psi(y, t)$ will read

$$i\hbar \partial_t \Psi = \left[ -\frac{E_C}{\pi^2} \partial_y^2 + E_L(y - \bar{y})^2 - E_J(t) \cos(2\pi y) \right] \Psi,$$

(1)

where $E_C$ is the charging energy due to the junction’s capacitance $C$: $E_C = e^2/2C$; $E_L$ is the magnetic field energy, due to the inductance $L$: $E_L = \Phi_0^2/2L$; and $E_J$ is the Josephson coupling energy. In our case, the latter depends on time, causing a variation in the height of the potential barrier separating states with different values of the flux.

In this paper, we only present results for the case when the SQUID is biased by exactly half of the flux quantum:

$$\bar{y} = 1/2,$$

(2)

although results for an arbitrary bias can be obtained similarly. In the case (2), the potential in eq. (1) is symmetric about the point $y = 1/2$; in particular, it has two degenerate minima. Accordingly, we introduce a shifted variable:

$$x = y - 1/2.$$

(3)

We also define a new time variable $\tau$ via

$$t = \hbar \tau.$$

(4)

Frequencies, with respect to this new time, are measured in energy units, for which we use degrees Kelvin. Therefore one unit of time $\tau$, $\Delta\tau = 1$, corresponds to

$$\Delta t = \frac{\hbar}{k_B \times 1 \text{ K}} = 7.64 \text{ ps}$$

(5)

of the physical time. Finally, we shift the potential down by a constant equal to the unperturbed Josephson energy $E_0$. This corresponds to using the wave function $\psi = \Psi \exp(iE_0 t/\hbar)$. Eq. (1) then leads to

$$i\partial_\tau \psi = \left[ -\frac{E_C}{\pi^2} \partial_x^2 + E_L x^2 + E_J(\tau) \cos(2\pi x) - E_0 \right] \psi.$$  

(6)

We use parameters of the SQUID described in ref. (3): $E_C = 9 \text{ mK}$, $E_L = 645 \text{ K}$, $E_0 = 76 \text{ K}$. The time-dependent Josephson energy is taken in the form

$$E_J(\tau)/E_0 = 1 - A \exp[-(\tau - \tau_c)^2/\tau_0^2].$$

(7)

This corresponds to a Gaussian (in time) pulse of current through the circuit that controls the Josephson coupling. If the time $\tau_c$, corresponding to the center of the pulse, is noticeably larger than the duration of the pulse, $\tau_0$, the initial value of $E_J$, $E_J(0)$, is practically indistinguishable from $E_0$. This initial value corresponds to a potential barrier in (3) high enough for the tunneling between different flux states to be inefficient. Accordingly, we can talk about the left and right ground states, which are the lowest states in each well. Near $\tau = \tau_c$, however, the barrier is lower, and it disappears altogether. At large times, the barrier comes back to its original height.

III. NUMERICAL RESULTS

We numerically integrated eq. (1), with a Gaussian pulse of the form (7). In the preparation of each run, we started by relaxing the system to its true ground state wavefunction (which is a superposition of the left and right ground states). Then we selected (and normalized) the left half of that wavefunction as our initial state. After that we numerically followed the system’s evolution.

The magnitude of the pulse, $A$, and the duration $\tau_0$ were varied, with the goal to achieve either a complete transfer of the amplitude between the left and right wells, or a superposition state. We concentrated primarily on
values of $\tau_0$ large enough for the transition to be adiabatic, so that in the end the system was not significantly excited beyond the left and right ground states.

Throughout the course of the evolution, we monitored a number of quantities. One of these was the probability $P_L$ to find the system in the left well after the pulse was completed. It is shown in Fig. 1 for a range of values of $A$ and $\tau_0$. For values of $A$ larger than the critical value (8), we observe that $P_L$ is quasiperiodic with respect to $\tau_0$, and the period decreases as $A$ increases. We interpret this by noting that for such values of $A$, the potential during the pulse has a minimum at $x = 0$, about which the system can oscillate. Duration of the pulse will determine in what phase of these oscillations the system will be decoupled.

Theoretically, we expect oscillations of $P_L$, due to tunneling back and forth between the two wells, even when $A$ is well below the critical value. However, the corresponding timescale is too long for such oscillations to show up in the figure.

Fig. 1 indicates that multiple choices of the parameters may lead to the same final $P_L$. The presence of decoherence in any realistic system suggests that one should choose the smallest possible $\tau_0$, so as to achieve the maximal number of quantum memory switches before the device decoheres. On the other hand, a too small $\tau_0$ will result in overexcitation, and after many such switches the control of the system will be lost. (One may contemplate reducing excitation through cooling; however, any cooling apparatus is likely to become a significant source of decoherence.) Thus, in addition to the quality factor $Q = t_d/t_0$, where $t_d$ is the decoherence time, we introduce a “fidelity factor”

$$F = (E_b - E_g)/\Delta E,$$  \hspace{1cm} (9)

where $E_b$ is the energy at the top of the unperturbed potential barrier, $E_g$ is the ground state energy, and $\Delta E$ is the increase in energy after one switch. Decreasing $\tau_0$ to achieve a larger $Q$ leads, via the time-energy uncertainty, to an increase in $\Delta E$ and a smaller $F$, not unlike how increasing the clock rate in ordinary computers leads to overheating.

To study the “fidelity” of transitions in more detail, we now focus on two values of $A$: $A = 0.53$ and 0.59 (corresponding to two sections of Fig. 1). Both are close to the threshold (8). However, the first is below that threshold, so tunneling effects are present. In Fig. 2 we show the corresponding probabilities $P_L$ together with the rescaled final energy $E$ of the system. In our case, $E_b = 0$ (because of the way the potential has been shifted), and $E_g$ is negative ($E_g = -41.1$), so the plotted quantity $E/|E_g|$ is simply related to the fidelity parameter (8):

$$\frac{E}{|E_g|} = \frac{1}{F} - 1.$$  \hspace{1cm} (10)

For example, for $A = 0.59$ and $\tau_0 = 5$, we obtain $P_L = 0.004$ (so the amplitude is almost completely transferred to the right) and $E = -39.8$, corresponding to $F \sim 30$. Increasing $\tau_0$ to $\tau_0 = 35$ ($P_L = 0.09$) results in $F \sim 100$. We also observe that, perhaps contrary to intuition, for short pulses the fidelity is higher when the magnitude is slightly above the threshold (8), rather than slightly below.

![FIG. 2. Probability $P_L$, as a function of the pulse duration, for $A = 0.59$ (solid line) and $A = 0.53$ (long dashes). The ratio of the final state energy to the magnitude of the ground state (initial) energy is shown by short dashes and dots, respectively. This ratio is a measure of the excitation energy supplied to the system by the pulse.](image)

Finally, Fig. 3 illustrates how the transition develops in time. In this case, it is predominantly an overbarrier transition, rather than tunneling. In the full many-body theory of the SQUID, such overbarrier motion of the flux will be associated with dissipation and decoherence. It remains to see if the time required for the transition can be made sufficiently short for these effects not to present a problem.

![FIG. 3. A few profiles of the probability density, showing how the transition occurs for $A = 0.59$ and $\tau_0 = 5.1$. The maximum of $|\psi|^2$ moves from left to right as time increases.](image)
IV. TWO-PULSE INTERFEROMETRY

We now discuss the behavior of the system under two consecutive pulses and the possibility to use such two-pulse sequences for an experimental determination of the decoherence time. Here we discuss decoherence after a transition, as opposed to decoherence during it, mentioned above. In its general outline, the method is similar to the two-pulse method used in studies of magnetic resonance \[7\]: in either case, the final state depends on the phase accumulated by some part of the wavefunction during the interval between the pulses. We now describe our method in more detail.

Suppose that the system is originally in its left ground state, which we call \(\psi_{L0}\). The first pulse will transfer part of the amplitude to the right well. Because the transition inevitably introduces some excitation, the state on the right will not be the precise right ground state, \(\psi_{R0}\), and the state on the left will not remain the precise left ground state. For simplicity, and during this discussion only, let us assume that only the admixture of the first excited states, \(\psi_{L1}\) and \(\psi_{R1}\), is substantial, while higher excited states can be neglected. Then, the state after the first pulse is

\[
\psi = c_{L0}\psi_{L0} + c_{L1}\psi_{L1} + c_{R0}\psi_{R0} + c_{R1}\psi_{R1}. \tag{11}
\]

Time evolution of this state is given by

\[
\psi(t) = [c_{L0}\psi_{L0} + c_{R0}\psi_{R0} + (c_{L1}\psi_{L1} + c_{R1}\psi_{R1})\exp(-i\omega t)]\exp(-iE_g t/\hbar), \tag{12}
\]

where \(\omega\) is the angular frequency for transitions between the ground and the first excited states. (The potential is assumed left-right symmetric.)

\[
P'_L(t) = |\psi^*_{L0} U\psi(t)|^2 + |\psi^*_{L1} U\psi(t)|^2,
\]

which in general is an oscillating function of \(t\) with frequency \(\omega\).

The amplitude of such oscillations can be quite large, as we show in Fig. 4. This figure is based on numerical integration and does not involve the simplifying assumption \[11\]. In practice, the system is coupled to the environment, and this coupling introduces a certain amount of decoherence. Suppose that evolution of the environment depends only on which well the system is in. We then consider two histories of the environment, \(\chi_L\) and \(\chi_R\), and instead of eq. \[12\] obtain

\[
\psi(t) = \{c_{L0}\psi_{L0}\chi_L(t) + c_{R0}\psi_{R0}\chi_R(t)
\]

\[
+ [c_{L1}\psi_{L1}\chi_L(t) + c_{R1}\psi_{R1}\chi_R(t)]\exp(-i\omega t)\}
\]

\[
\times \exp(-iE_g t/\hbar).
\]

The probability to observe the system in the left well after the second pulse is now

\[
P'_L(t) = \sum_n \left[|\psi^*_{L0}\chi_n U\psi(t)|^2 + |\psi^*_{L1}\chi_n U\psi(t)|^2\right],
\]

where \(\chi_n\) is a complete system of states of the environment. The oscillating dependence of \(P'_L\) on time is now modulated by the coherence factor \(|\chi_R(t)\chi_L(t)|^2\), which decreases in time. Note that even when this coherence factor becomes essentially zero, the oscillations of \(P'_L\) remain, due to nonzero matrix elements, such as \(\psi^*_{L0} U\psi_{L1}\), between the states in the same well.

Thus, due to decoherence, the oscillations of \(P'_L\), such as those shown in Fig. 4, will acquire an envelope. By fitting the envelope with an exponential

\[
f(\Delta t) = a_1 + a_2 \exp(-\Delta t/t_d)
\]

(or, perhaps, some other function that may emerge on theoretical grounds), one may be able to extract the decoherence time \(t_d\).

\[1\] For a review, see A. Ekert and R. Jozsa, Rev. Mod. Phys. 68, 733 (1996).
\[2\] For a review, see W. H. Zurek, Phys. Today 44, 36 (October 1991).
\[3\] J. R. Friedman et al., Nature, 406, 43 (2000).
\[4\] C. H. van der Wal et al., Science, 290, 773 (2000).
\[5\] P. Silvestrini and L. Stodolsky, Phys. Lett. A 280, 17 (2001).
\[6\] C. Cosmelli et al., Phys. Rev. Lett. 82, 5357 (1999).
\[7\] N. F. Ramsey, Phys. Rev. 100, 1191 (1955).