HYPERBOLIC $P(\Phi)_2$-MODEL ON THE PLANE

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Abstract. We study the hyperbolic $\Phi^{k+1}_2$-model on the plane. By establishing coming down from infinity for the associated stochastic nonlinear heat equation (SNLH) on the plane, we first construct a $\Phi^{k+1}_2$-measure on the plane as a limit of the $\Phi^{k+1}_2$-measures on large tori. We then study the canonical stochastic quantization of the $\Phi^{k+1}_2$-measure on the plane thus constructed, namely, we study the defocusing stochastic damped nonlinear wave equation forced by an additive space-time white noise (= the hyperbolic $\Phi^{k+1}_2$-model) on the plane. In particular, by taking a limit of the invariant Gibbs dynamics on large tori constructed by the first two authors with Gubinelli and Koch (2022), we construct invariant Gibbs dynamics for the hyperbolic $\Phi^{k+1}_2$-model on the plane. Our main strategy is to develop further the ideas from a recent work on the hyperbolic $\Phi^{3}_3$-model on the three-dimensional torus by the first two authors and Okamoto (2021), and to study convergence of the so-called enhanced Gibbs measures, for which coming down from infinity for the associated SNLH with positive regularity plays a crucial role. By combining wave and heat analysis together with ideas from optimal transport theory, we then conclude global well-posedness of the hyperbolic $\Phi^{k+1}_2$-model on the plane and invariance of the associated Gibbs measure. As a byproduct of our argument, we also obtain invariance of the limiting $\Phi^{k+1}_2$-measure on the plane under the dynamics of the parabolic $\Phi^{k+1}_2$-model.

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2020 Mathematics Subject Classification. 35L71, 60H15, 35K05, 35R60.
Key words and phrases. nonlinear wave equation; stochastic damped nonlinear wave equation; canonical stochastic quantization; Gibbs measure; stochastic nonlinear heat equation; coming down from infinity.
1. Introduction

1.1. Hyperbolic $\Phi_2^{k+1}$-model. We study the following stochastic damped nonlinear wave equation (SdNLW) forced by an additive space-time white noise, posed on the plane $\mathbb{R}^2$:

$$\partial_t^2 u + \partial_t u + (1 - \Delta)u + u^k = \sqrt{2}\xi,$$

where $k \in 2\mathbb{N} + 1$, $u$ is a real-valued unknown, and $\xi(x,t)$ is a Gaussian space-time white noise on $\mathbb{R}^2 \times \mathbb{R}^+$ with the space-time covariance given by

$$
\mathbb{E}[(\xi(x_1,t_1)\xi(x_2,t_2))] = \delta(x_1 - x_2)\delta(t_1 - t_2).
$$

With $\vec{u} = (u, \partial_t u)$, define the energy $E(\vec{u})$ by

$$
E(\vec{u}) = \frac{1}{2} \int_{\mathbb{R}^2} |\nabla u|^2 dx + \frac{1}{2} \int_{\mathbb{R}^2} (\partial_t u)^2 dx + \frac{1}{k+1} \int_{\mathbb{R}^2} u^{k+1} dx,
$$

where $E(u)$ is given by

$$
E(u) = \frac{1}{2} \int_{\mathbb{R}^2} |(\nabla u)|^2 dx + \frac{1}{k+1} \int_{\mathbb{R}^2} u^{k+1} dx.
$$

Note that the energy $E(\vec{u})$ is precisely the energy (= Hamiltonian) for the (deterministic) nonlinear wave equation (NLW):

$$\partial_t^2 u + (1 - \Delta)u + u^k = 0.
$$

Namely, with $v = \partial_t u$, we can write NLW (1.4) in the following Hamiltonian formulation:

$$
\partial_t \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} \partial u \\ \partial v \end{pmatrix}.
$$

Similarly, we can write SdNLW (1.1) as

$$
\partial_t \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} \partial u \\ \partial v \end{pmatrix} + \begin{pmatrix} 0 \\ -v + \sqrt{2}\xi \end{pmatrix}.
$$

Consider a Gibbs measure $\vec{\rho}$ of the form

$$
\text{“}d\vec{\rho}(\vec{u}) = Z^{-1} e^{-E(\vec{u})} d\vec{u} = d\rho \otimes d\mu_0(\vec{u}),\text{”}
$$

where $\vec{u} = (u, v) = (u, \partial_t u)$, $\rho$ is a $\Phi_2^{k+1}$-measure on $\mathbb{R}^2$, and $\mu_0$ is the (spatial) white noise measure on $\mathbb{R}^2$. By drawing an analogy to the finite-dimensional Hamiltonian dynamics, we expect that the Gibbs measure $\vec{\rho}$ in (1.7) is invariant under the NLW dynamics (1.4) on $\mathbb{R}^2$. 
Moreover, it is easy to see that the white noise measure $\mu_0$ on the second component $v = \partial_t u$ is invariant under the Ornstein-Uhlenbeck dynamics:

$$\partial_t v = -v + \sqrt{2}\xi. \quad (1.8)$$

Thus, by viewing the SdNLW dynamics (1.6) as a superposition of the NLW dynamics (1.5) and the Ornstein-Uhlenbeck dynamics (1.8), we expect the Gibbs measure $\bar{\rho}$ in (1.7) to be invariant under the SdNLW dynamics (1.1). Namely, the SdNLW equation (1.1) is the hyperbolic counterpart of the well-studied stochastic quantization equation in the parabolic setting (= the parabolic $\Phi^{k+1}_d$-model) [22, 41, 55, 56, 42, 36, 81, 37, 2]:

$$\partial_t X + (1 - \Delta)X + X^k = \sqrt{2}\xi, \quad (1.9)$$

under which the $\Phi^{k+1}_d$-measure remains invariant. For this reason, we also refer to (1.1) as the hyperbolic $\Phi^{k+1}_d$-model. The hyperbolic $\Phi^{k+1}_d$-model (1.1) corresponds to the so-called canonical stochastic quantization equation\footnote{The subscript $d$ denotes the dimension of the underlying space.} for the $\Phi^{k+1}_d$-measure, and thus is of importance in mathematical physics. See [82]. Our main goal in this paper is to construct the global-in-time dynamics at the Gibbs equilibrium for the hyperbolic $\Phi^{k+1}_2$-model (1.1) posed on $\mathbb{R}^2$.

1.2. Review of the periodic problem. Following the previous works [76, 38], Gubinelli, Koch, and the first two authors [40] studied SdNLW (1.1) on the two-dimensional torus $\mathbb{T}^2 = (\mathbb{R}/\mathbb{Z})^2$. By introducing a proper renormalization (see (1.25) below; see also [76, 38]), they constructed global-in-time invariant Gibbs dynamics for SdNLW (1.1) on $\mathbb{T}^2$ via the so-called Bourgain’s invariant measure argument [10, 11]. See [76] for the construction of invariant Gibbs dynamics for the deterministic NLW (1.4) on $\mathbb{T}^2$, preceding [40]. See also [71] for the corresponding results (for both SdNLW (1.1) and NLW (1.4)) on a two-dimensional compact Riemannian manifold without boundary.

Given $L > 0$, define a dilated torus $\mathbb{T}_L^2$ by setting

$$\mathbb{T}_L^2 = (\mathbb{R}/L\mathbb{Z})^2.$$

Then, the aforementioned result in [40] applies to SdNLW posed on $\mathbb{T}_L^2$ for any $L > 0$. Our main strategy for studying SdNLW (1.1) on $\mathbb{R}^2$ is then to take a large torus limit $L \to \infty$ of the $L$-periodic SdNLW dynamics on $\mathbb{T}_L^2$. As such, we first provide a review of the $L$-periodic problem on the dilated torus $\mathbb{T}_L^2$ in this subsection (especially since the presentation in [40] is only for the $L = 1$ case).

Let us first introduce some notations. Fix $L > 0$ and set

$$\mathbb{Z}_L^2 = (\mathbb{Z}/L)^2.$$

Given $\lambda \in \mathbb{Z}_L^2$, we set

$$e_{\lambda}^L(x) = \frac{1}{L} e^{2\pi i \lambda \cdot x} \quad (1.10)$$

\footnote{Namely, the ‘Hamiltonian’ stochastic quantization equation given as the Langevin equation with the momentum $v = \partial_t u$. The parabolic $\Phi^{k+1}_d$-model is the stochastic gradient flow for the energy functional $E(u)$ defined in (1.3).}
for $x \in \mathbb{T}_L^2$. Then, $\{e_\lambda\}_{\lambda \in \mathbb{Z}_L^2}$ forms an orthonormal basis of $L^2(\mathbb{T}_L^2)$. We define the Fourier transform $\hat{f}(\lambda)$ of a function $f$ on $\mathbb{T}_L^2$ by

$$\hat{f}(\lambda) = \int_{\mathbb{T}_L^2} f(x) e^{i\lambda(x)} dx, \quad \lambda \in \mathbb{Z}_L^2,$$

with the associated Fourier series expansion

$$f(x) = \sum_{\lambda \in \mathbb{Z}_L^2} \hat{f}(\lambda) e^{i\lambda(x)}.$$

We first go over the construction of the Gibbs measure on $\mathbb{T}_L^2$; see [23, 75] for details (with $L = 1$). See also [19] for the construction of analogous log-correlated Gibbs measures in the one-dimensional setting. Given $s \in \mathbb{R}$, let $\mu_{s,L}$ denote a Gaussian measure on $L$-periodic distributions with the covariance operator $(1 - \Delta_{\mathbb{T}_L^2})^{-s}$, formally defined by

$$d\mu_{s,L}(u) = Z_{s,L}^{-1} \exp \left( -\frac{1}{2} \|u\|_{H^s(\mathbb{T}_L^2)}^2 \right) du$$

where \( \hat{\mu}(\lambda) \), $\lambda \in \mathbb{Z}_L^2$, denotes the Fourier transform of $u$ on $\mathbb{T}_L^2$. We note that $\mu_{1,L}$ corresponds to the massive Gaussian free field on $\mathbb{T}_L^2$, while $\mu_{0,L}$ corresponds to the white noise on $\mathbb{T}_L^2$. We then set

$$\mu_L = \mu_{1,L} \otimes \mu_{0,L}, \quad \text{where} \quad \mu_L = \mu_{1,L}. \quad (1.12)$$

Given $s \in \mathbb{R}$, let

$$\mathbb{H}^s(\mathbb{T}_L^2) = H^s(\mathbb{T}_L^2) \times H^{s-1}(\mathbb{T}_L^2).$$

Then, $\mu_L$ in (1.12) is formally given by

$$d\mu_L(u, v) = Z_{L}^{-1} \exp \left( -\frac{1}{2} \|u, v\|_{\mathbb{H}^1(\mathbb{T}_L^2)}^2 \right) du dv. \quad (1.13)$$

Namely, the measure $\mu_L$ is defined as the induced probability measure under the map:

$$\omega \in \Omega \longmapsto (u(\omega), v(\omega)) \in \mathcal{D}'(\mathbb{T}_L^2) \times \mathcal{D}'(\mathbb{T}_L^2),$$

where $u(\omega) = u_L(\omega)$ and $v(\omega) = v_L(\omega)$ are given by the following Gaussian Fourier series:

$$u(\omega) = u_L(\omega) = \sum_{\lambda \in \mathbb{Z}_L^2} \frac{g_{\lambda\lambda}(\omega)}{\langle \lambda \rangle} e^{i\lambda} \quad \text{and} \quad v(\omega) = v_L(\omega) = \sum_{\lambda \in \mathbb{Z}_L^2} h_{\lambda\lambda}(\omega)e^{i\lambda}. \quad (1.14)$$

Here, $\{g_n, h_n\}_{n \in \mathbb{Z}}$ denotes a family of independent standard complex-valued Gaussian random variables conditioned that $g_{-n} = \overline{g}_n$ and $h_{-n} = \overline{h}_n$, $n \in \mathbb{Z}^2$. From (1.14), it is easy to see that $\mu_L = \mu_{1,L} \otimes \mu_{0,L}$ is supported on $\mathbb{H}^s(\mathbb{T}_L^2) \setminus \mathbb{H}^{0}(\mathbb{T}_L^2)$ for $s < 0$. In view of (1.2) and (1.13), the Gibbs measure $\mu_L$ on $\mathbb{T}_L^2$ is formally given by

$$d\mu_L(u, \partial_t u) = Z_{L}^{-1} e^{-\frac{1}{2} \int_{\mathbb{T}_L^2} u^{k+1} dx} d\mu_L(u, \partial_t u). \quad (1.15)$$

---

3Hereafter, we may drop the inessential factor $2\pi$.  
4We use $\mathcal{Z}_{s,L}$, etc. to denote various normalizing constants, which may vary line by line.
Due to the roughness of the support of $\bar{\mu}_L$, the interaction potential $\int_{T^2_L} u^{k+1} dx$ in (1.15) is not well defined and thus a renormalization is required to give a proper meaning to the expression in (1.15).

Given $N \in \mathbb{N}$, let $P_N$ be the Dirichlet projection onto the frequencies $\{ |\lambda| \leq N \}$. Then, with $u$ as in (1.14), it follows from (1.14), (1.10), and a Riemann sum approximation that

$$\sigma_{N,L} := \mathbb{E}[(P_Nu(x))^2] = \sum_{\lambda \in \mathbb{F}_L^2} \frac{1}{|\lambda|^2} \frac{1}{L^2} = \sum_{n \in \mathbb{Z}^2} \frac{1}{(2\pi)^2} \frac{1}{L^2} \sim \int_{\mathbb{R}^2} \frac{1}{1 + |z|^2} \sim \log N \quad (1.16)$$

for $N \gg 1$, independent of $x \in T^2_L$, which diverges to $\infty$ as $N \to \infty$ (for each fixed period $L > 0$). In particular, $u = \lim_{N \to \infty} P_N u$ is only a distribution and thus, for any integer $\ell \geq 2$, the power $(P_Nu)^\ell$ does not converge to any limit. For each $x \in T^2_L$, we now define the Wick power $:(P_Nu)^\ell(x): = (P_Nu)^\ell(x)_{\mid L}$ by\footnote{Note that the definition (1.17) of the Wick power depends on the period $L > 0$. We, however, suppress the subscript $L$ from the Wick power, when it is clear from the context.}

$$:(P_Nu)^\ell(x): = H_\ell(P_Nu(x); \sigma_{N,L}), \quad (1.17)$$

where $H_\ell(x; \sigma)$ is the Hermite polynomial of degree $\ell \in \mathbb{Z}_{\geq 0} := \mathbb{N} \cup \{0\}$ with a variance parameter $\sigma > 0$. Arguing as in [38, 39, 40], we can show that $(P_Nu)^\ell$ converges, almost surely and in $L^p(\Omega)$ for any finite $p \geq 1$, to a limit, denoted by $u^\ell$ in $H^s(T^2_L)$, $s < 0$. Then, by defining the truncated renormalized potential energy:

$$R^L_N(u) = \frac{1}{k+1} \int_{T^2_L} (P_Nu)^{k+1}(x) \, dx, \quad (1.18)$$

a standard computation shows that $\{ R^L_N(u) \}_{N \in \mathbb{N}}$ converges to some limit, denoted by $R^L(u)$, in $L^p(d\mu_L)$ for any finite $p \geq 1$, as $N \to \infty$. The convergence of the truncated renormalized potential energy $\{ R^L_N(u) \}_{N \in \mathbb{N}}$ together with Nelson’s estimate implies that the truncated density $\{ e^{-R^L_N(u)} \}_{N \in \mathbb{N}}$ converges to the limiting density $e^{-R^L(u)}$ in $L^p(d\mu_L)$ for any finite $p \geq 1$, as $N \to \infty$. Hence, by defining the renormalized truncated Gibbs measure:

$$d\bar{\mu}_{N,L}(u, \partial_t u) = Z^{-1}_{N,L} e^{-R^L_N(u)} d\bar{\mu}_L(u, \partial_t u), \quad (1.19)$$

we then conclude that the renormalized truncated Gibbs measure $\bar{\mu}_{N,L}$ converges in total variation to the limiting Gibbs measure $\bar{\mu}_L$ given by

$$d\bar{\mu}_L(u, \partial_t u) = Z^{-1}_L e^{-R^L(u)} d\bar{\mu}_L(u, \partial_t u) = Z^{-1}_L \exp \left( - \frac{1}{k+1} \int_{T^2_L} u^{k+1}(x) : dx \right) d\bar{\mu}_L(u, \partial_t u). \quad (1.20)$$

Furthermore, for each fixed $0 < L < \infty$, the resulting Gibbs measure $\bar{\mu}_L$ is equivalent\footnote{Namely, $\bar{\mu}_L$ and $\mu_L$ are mutually absolutely continuous.} to the base Gaussian measure $\bar{\mu}_L$. 

5Note that the definition (1.17) of the Wick power depends on the period $L > 0$. We, however, suppress the subscript $L$ from the Wick power, when it is clear from the context.

6Namely, $\bar{\mu}_L$ and $\bar{\mu}_L$ are mutually absolutely continuous.
Remark 1.1. The first marginal of the Gibbs measure \( \tilde{\rho}_L \) (namely, after integrating (1.20) in \( \partial_t u \)) is precisely the \( \Phi^{k+1}_2 \)-measure \( \rho_L \) on \( \mathbb{T}^2_L \), given by

\[
d\rho_L(u) = Z_L^{-1} e^{-R^k(u)} d\mu_L(u) = Z_L^{-1} \exp \left( - \frac{1}{k+1} \int_{\mathbb{T}^2_L} u^{k+1}(x) \, dx \right) d\mu_L(u),
\]

where \( \mu_L = \mu_{1,L} \) is as in (1.12). Note that \( \rho_L \) is the Gibbs measure associated with the energy functional \( E(u) \) in (1.3). The Gibbs measure \( \tilde{\rho}_L \) in (1.20) can then be written as \( \tilde{\rho}_L = \rho_L \otimes \mu_{0,L} \), where \( \mu_{0,L} \) is the (spatial) white noise measure on \( \mathbb{T}^2_L \).

Next, we discuss stochastic dynamics associated with the Gibbs measure \( \tilde{\rho}_L \) in (1.20). This process is known as stochastic quantization [80]. In the parabolic setting, Da Prato and Debussche [22] studied the parabolic \( \Phi^{k+1}_2 \)-model (1.9) on \( \mathbb{T}^2 \), associated with the \( \Phi^{k+1}_2 \)-measure \( \rho_{L=1} \) in (1.21). With the Wick renormalization, they constructed global-in-time invariant dynamics for (1.9) on \( \mathbb{T}^2 \) with the initial data distributed by the \( \Phi^{k+1}_2 \)-measure \( \rho_{L=1} \).

This result is readily applicable to the parabolic \( \Phi^{k+1}_2 \)-model (1.9) posed on the dilated torus \( \mathbb{T}^2_L \) for any \( L > 0 \). In [55], with an intricate use of weighted Besov spaces (see (2.25) below), Mourrat and Weber extended this result to the parabolic \( \Phi^{k+1}_2 \)-model on the plane \( \mathbb{R}^2 \).

We now consider the hyperbolic \( \Phi^{k+1}_2 \)-model on \( \mathbb{T}^2_L \) for fixed \( L > 0 \):

\[
\partial^2_t u + \partial_t u + (1 - \Delta) u + u^k = \sqrt{2} \xi_L
\]

with the Gibbsian initial data distributed by the Gibbs measure \( \tilde{\rho}_L \) in (1.20), where \( \xi_L \) is a space-time white noise on \( \mathbb{T}^2_L \times \mathbb{R}_+ \). In view of the equivalence of the Gibbs measure \( \tilde{\rho}_L \) in (1.20) and the base Gaussian measure \( \tilde{\mu}_L \) in (1.12) for each fixed \( L > 0 \), it suffices to study (1.22) with the Gaussian initial data distributed by \( \tilde{\mu}_L \).

Let \( \Phi = \Phi_L \) be the solution to the linear stochastic damped wave equation on \( \mathbb{T}^2_L \) with the Gaussian initial data distributed by \( \tilde{\mu}_L \) in (1.12):

\[
\begin{aligned}
&\partial^2_t \Phi + \partial_t \Phi + (1 - \Delta) \Phi = \sqrt{2} \xi_L \\
&(\Phi, \partial_t \Phi)|_{t=0} = (\phi_0, \phi_1) \quad \text{with} \quad \text{Law}(\phi_0, \phi_1) = \tilde{\mu}_L.
\end{aligned}
\]

Here, \( \text{Law}(X) \) of a random variable \( X \) denotes the law of \( X \). Define the linear damped wave propagator \( D(t) \) by

\[
D(t) = e^{-\frac{t}{2} \frac{\sqrt{2}}{\sqrt{\frac{3}{4} - \Delta}}} \sin \left( \frac{t \sqrt{\frac{3}{4} - \Delta}}{\sqrt{\frac{3}{4} - \Delta}} \right)
\]

as a Fourier multiplier operator. Then, the stochastic convolution \( \Phi \) defined above can be expressed as

\[
\Phi(t) = (\partial_t D(t) + D(t)) \phi_0 + D(t) \phi_1 + \sqrt{2} \int_0^t D(t - t') dW_L(t'),
\]

where \( W_L \) denotes a cylindrical Wiener process on \( L^2(\mathbb{T}^2_L) \):

\[
W_L(t) = \sum_{\lambda \in \mathbb{Z}^2_L} B_\lambda(t) \xi^\lambda
\]
and \( \{ B_\lambda \}_{\lambda \in \mathbb{Z}^2} \) is defined by \( B_\lambda(t) = \langle \xi_L, 1_{[0,t]} \cdot e^L_{\lambda} \rangle_{T^2_L \times \mathbb{R}^+} \). Here, \( \langle \cdot, \cdot \rangle_{T^2_L \times \mathbb{R}^+} \) denotes the duality pairing on \( T^2_L \times \mathbb{R}^+ \). Then, we see that \( \{ B_\lambda \}_{\lambda \in \mathbb{Z}^2} \) is a family of mutually independent complex-valued Brownian motions conditioned that \( B_{-\lambda} = \overline{B_\lambda}, \lambda \in \mathbb{Z}^2 \). Note that \( \text{Var}(B_\lambda(t)) = t, \lambda \in \mathbb{Z}^2 \).

Let \( N \in \mathbb{N} \). Given \((x, t) \in T^2_L \times \mathbb{R}_+\), we see that \( \Phi_N(x, t) = P_N \Phi(x, t) \) is a mean-zero real-valued Gaussian random variable with variance

\[
\mathbb{E}[\Phi_N^2(x, t)] = \sigma_{N,L} \sim \log N \rightarrow \infty
\]
as \( N \rightarrow \infty \), where \( \sigma_{N,L} \) is as in (1.16). As in (1.17), we define the Wick power \( \Phi_N^\ell(x, t) \): by setting

\[
: \Phi_N^\ell(x, t) := H_\ell(\Phi_N(x, t); \sigma_{N,L}).
\]

Then, \( : \Phi_N^\ell \) converges, almost surely and in \( L^p(\Omega) \) for any finite \( p \geq 1 \), to a limit, denoted by \( \Phi^\ell : \) in \( C(\mathbb{R}_+; H^s(T^2_L)) \), \( s < 0 \).

Given \( N \in \mathbb{N} \), consider the following truncated SdNLW on \( T^2_L \):

\[
\partial_t^2 u_N + \partial_t u_N + (1 - \Delta)u_N + P_N((P_N u_N)^{k}) = \sqrt{2} \xi_L.
\]

(1.26)

Proceeding with the first order expansion (1.25):

\[
u_N = \Phi_N + v_N,
\]

(1.27)

we see that the remainder term \( v_N = u_N - \Phi_N \) satisfies

\[
\partial_t^2 v_N + \partial_t v_N + (1 - \Delta)v_N + \sum_{\ell=0}^{k} \binom{k}{\ell} P_N(\Phi_N^\ell v_N^{k-\ell}) = 0
\]

(1.28)

with the zero initial data. As pointed out above, the power \( \Phi_N^\ell \) does not converge to any limit as \( N \rightarrow \infty \). Furthermore, a triviality is known for (1.28) (at least for \( k = 3 \)). Namely, the solution \( u_N \) to (1.26) tends to 0 as we remove the regularization \( (N \rightarrow \infty) \); see [62]. This triviality result necessitates the use of a renormalization. Therefore, we instead consider the following renormalized version of (1.28):

\[
\partial_t^2 v_N + \partial_t v_N + (1 - \Delta)v_N + \sum_{\ell=0}^{k} \binom{k}{\ell} P_N(\Phi_N^\ell v_N^{k-\ell}) = 0
\]

(1.29)

with the zero initial data. By formally taking a limit as \( N \rightarrow \infty \), we then obtain the limiting equation:

\[
\partial_t^2 v + \partial_t v + (1 - \Delta)v + \sum_{\ell=0}^{k} \binom{k}{\ell} : \Phi^\ell : v^{k-\ell} = 0.
\]

(1.30)

Given the almost sure space-time regularity of the Wick powers \( \{ : \Phi^\ell : \}_{\ell=1}^k \), standard deterministic analysis with the product estimates (Lemma 2.8) and Sobolev’s inequality yields

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7In particular, \( B_0 \) is a standard real-valued Brownian motion.

8Here, we endow the space \( C(\mathbb{R}_+; H^s(T^2_L)) \) with the compact-open topology in time, namely with the topology of uniform convergence on compact intervals.
local well-posedness of \((1.30)\) with the continuous map, sending the enhanced data set to the solution:

\[
(\Phi, \Phi^2, \ldots, \Phi^k) \in \left(C([0, T]; H^{-\varepsilon}(\mathbb{T}_L^2))\right)^{\otimes k}
\]

\[
\longrightarrow (v, \partial_t v) \in C([0, T]; \tilde{H}^{1-\varepsilon}(\mathbb{T}_L^2))
\]

for some small \(\varepsilon > 0\). This deterministic local well-posedness analysis also applies to \((1.29)\), uniformly in \(N \in \mathbb{N}\).

In view of the decomposition \((1.27)\), the renormalized truncated SdNLW \((1.29)\) for \(v_N\) corresponds to the following renormalized truncated SdNLW for \(u_N = \Phi_N + v_N\):

\[
\partial_t^2 u_N + \partial_t u_N + (1 - \Delta) u_N + P_N \left( : (P_N u_N)^k : \right) = \sqrt{2} \xi_L. 
\]

(1.31)

Here, the renormalized nonlinearity in \((1.31)\) is interpreted as

\[
P_N \left( : (P_N u_N)^k : \right) = \sum_{\ell=0}^{k} \left( \begin{array}{c} k \\ \ell \end{array} \right) P_N \left( : \Phi^\ell_N : v_N^{k-\ell} \right).
\]

Then, the aforementioned local well-posedness of \((1.30)\), together with the convergence of the truncated enhanced data set \(\{ : \Phi^\ell_N : \}_{\ell=1}^k\) to the limiting enhanced data set \(\{ : \Phi^\ell : \}_{\ell=1}^k\), implies that \(u_N\) converges almost surely to a stochastic process \(u = \Phi + v\), where \(v\) satisfies \((1.30)\). It is in this sense that we say that the renormalized SdNLW on the dilated torus \(\mathbb{T}_L^2\):

\[
\partial_t^2 u + \partial_t u + (1 - \Delta) u + : u^k : = \sqrt{2} \xi_L
\]

(1.32)
is locally well-posed with the Gaussian initial data distributed by \(\tilde{\mu}_L\), and hence with the Gibbsian initial data in view of the equivalence of the Gibbs measure \(\tilde{\rho}_L\) and the base Gaussian measure \(\tilde{\mu}_L\).

Once the local-in-time dynamics is constructed, Bourgain’s invariant measure argument \([10, 11]\) allows us to construct global-in-time dynamics for the hyperbolic \(\Phi^{k+1}\)-model \((1.32)\) on \(\mathbb{T}_L^2\) and to prove invariance of the Gibbs measure \(\tilde{\rho}_L\) in \((1.20)\). This argument is based essentially on the following three ingredients:

- invariance of the truncated Gibbs measure \(\tilde{\rho}_{N,L}\) in \((1.19)\) under the truncated SdNLW dynamics \((1.31)\), which provides a probabilistic growth bound on the solution \(u_N\), uniformly in \(N \in \mathbb{N}\),

- a PDE approximation argument (analogous to the local well-posedness argument in the current setting),

- convergence (in total variation) of the truncated Gibbs measure \(\tilde{\rho}_{N,L}\) in \((1.19)\) to the limiting Gibbs measure \(\tilde{\rho}_L\) in \((1.20)\).

See \([71]\) for full details of this argument.

1.3. Main result. Our goal in this paper is to extend the construction of the invariant Gibbs dynamics of the hyperbolic \(\Phi^{k+1}\)-model on \(\mathbb{T}_L^2\) described in the previous subsection to the plane \(\mathbb{R}^2\). The main idea is to apply Bourgain’s invariant measure argument ‘in spirit’, where we replace the frequency truncation parameter \(N \to \infty\) by the growing period \(L \to \infty\).
We first state our main result. Given \( s \in \mathbb{R}, \ 1 \leq p < \infty, \) and \( \mu > 0, \) let \( W_{\mu}^{s,p}(\mathbb{R}^2) \) be the weighted Sobolev space defined in (2.22) below. We set
\[
\tilde{W}_{\mu}^{s,p}(\mathbb{R}^2) = W_{\mu}^{s,p}(\mathbb{R}^2) \times W_{\mu}^{s-1,p}(\mathbb{R}^2).
\]
We also set
\[
\tilde{H}_{\text{loc}}^{s}(\mathbb{R}^2) = H_{\text{loc}}^{s}(\mathbb{R}^2) \times H_{\text{loc}}^{s-1}(\mathbb{R}^2).
\]
Here, \( H_{\text{loc}}^{s}(\mathbb{R}^2) \) denotes the class of functions \( u \) belonging to \( H^{s}(K) \) for each compact subset \( K \subset \mathbb{R}^2, \) where the \( H^{s}(K) \)-norm is defined as the restriction norm onto the set \( K; \) see (2.3) below. In the following, we endow \( C(\mathbb{R}^+; \tilde{H}_{\text{loc}}^{s}(\mathbb{R}^2)) \) with the compact-open topology.

Throughout the paper, we assume that random initial data are independent of a stochastic forcing (such as \( \xi_L \) and \( \xi \)). Given \( R > 0, \) let \( B_R = \{ x \in \mathbb{R}^2 : |x| \leq R \} \) denotes the (closed) ball of radius \( R \) centered at the origin and \( C_R \) denote the cone given by
\[
C_R = \{(x,t) \in \mathbb{R}^2 \times \mathbb{R}^{+} : |x| + |t| \leq R \} = \{(x,t) \in \mathbb{R}^2 \times [0,R] : x \in B_{R-t} \}.
\](1.33)

**Theorem 1.2.** Let \( k \in 2\mathbb{N} + 1. \) There exists a subset \( \mathcal{A} = \{ L_j : j \in \mathbb{N} \} \subset \mathbb{N} \) (with \( L_j < L_{j'} \) for \( j < j' \)) such that the following statements hold.

(i) Let \( s < 0, \) finite \( p \geq 1, \) and \( \mu > 0. \) Let \( \tilde{\rho}_j^2 \) be the Gibbs measure on the dilated torus \( T_L^2 \) defined in (1.20). When viewed as a probability measure on the weighted Sobolev space \( \tilde{W}_{\mu}^{s,p}(\mathbb{R}^2), \) the \( L_j \)-periodic Gibbs measure \( \tilde{\rho}_{L_j} \) converges weakly to a limiting Gibbs measure \( \tilde{\rho}_{\infty} \) as \( j \to \infty. \) The limiting Gibbs measure \( \tilde{\rho}_{\infty} \) on \( \mathbb{R}^2 \) can be written as
\[
\tilde{\rho}_{\infty} = \rho_{\infty} \otimes \mu_{0,\infty},
\]
where \( \rho_{\infty} \) is the \( \Phi_2^{k+1} \)-measure on \( \mathbb{R}^2 \) constructed as a limit of the \( L_j \)-periodic \( \Phi_2^{k+1} \)-measure \( \rho_{L_j}, \) and \( \mu_{0,\infty} \) is the white noise measure on \( \mathbb{R}^2. \)

(ii) The hyperbolic \( \Phi_2^{k+1} \)-model on \( \mathbb{R}^2:\)
\[
\partial_t^2 u + \partial_t u + (1 - \Delta) u + : u^k : = \sqrt{2} \xi
\]
(1.34)
is globally well-posed almost surely with respect to the Gibbsian initial data distributed by the Gibbs measure \( \tilde{\rho}_{\infty} \) on \( \mathbb{R}^2, \) constructed in Part (i). Furthermore, the Gibbs measure \( \tilde{\rho}_{\infty} \) is invariant under the resulting hyperbolic \( \Phi_2^{k+1} \)-dynamics on \( \mathbb{R}^2. \)

More precisely, the following statements hold. There exist a sequence \( \{(u_{L_j}, \partial u_{L_j})\}_{j \in \mathbb{N}} \) and a non-trivial stochastic process \( (u, \partial_t u) \) almost surely belonging to \( C(\mathbb{R}^+; \tilde{H}_{\text{loc}}^{-\varepsilon}(\mathbb{R}^2)) \) for any \( \varepsilon > 0 \) such that

- for each \( j \in \mathbb{N}, \) \( (u_{L_j}, \partial u_{L_j}) \) is the global-in-time solution to SdNLW (1.32) on \( T_{L_j}^2 \) with Law \( ((u_{L_j}(0), \partial u_{L_j}(0))) = \tilde{\rho}_{L_j}, \) constructed in [40],
- as \( j \to \infty, \) \( (u_{L_j}(0), \partial u_{L_j}(0)) \) converges almost surely to \( (u(0), \partial u(0)) \), distributed by the limiting Gibbs measure \( \tilde{\rho}_{\infty}, \) in \( \tilde{H}_{\text{loc}}^{-\varepsilon}(\mathbb{R}^2), \)
- given any \( R > 0, \) \( (u_{L_j}, \partial u_{L_j}) \) converges in probability to \( (u, \partial_t u) \) on the cone \( C_R \)
  (more precisely, in \( L^\infty([0,R]; \tilde{H}_{\text{loc}}^{-\varepsilon}(B_{R-t}); \text{see (2.4)}) \)), as \( j \to \infty. \)

\(^9\)We say that \( u \in C(\mathbb{R}^+; \tilde{H}_{\text{loc}}(\mathbb{R}^2)) \) if \( u \in C(\mathbb{R}^+; \tilde{H}_{\text{loc}}(K) \) for each compact subset \( K \subset \mathbb{R}^2. \)
Furthermore, the law of \((u(t), \partial_t u(t))\) for any \(t \in \mathbb{R}_+\) is given by the renormalized Gibbs measure \(\tilde{\rho}_\infty\).

As for the construction of the limiting \(\Phi_{k+1}^\frac{k+1}{2}\)-measure \(\rho_\infty\) on \(\mathbb{R}^2\), see also \[84\] and the references therein\[^{10}\]. Note that the assumption on the almost sure convergence of \((u_{L_j}(0), \partial_t u_{L_j}(0))\) in Theorem 1.2(ii) indeed follows from the weak convergence of \(\tilde{\rho}_{L_j}\) to \(\tilde{\rho}_\infty\) in Theorem 1.2(i) and the Skorokhod representation theorem (Lemma 2.19) and therefore is not an additional assumption.

Stochastic nonlinear wave equations (SNLW) have attracted extensive attention from both applied and theoretical points of view; see \[24\] Chapter 13\[^{11}\] and \[61\] for the references therein. In particular, over the last five years, we have seen a significant progress in the well-posedness theory of SNLW in the singular setting:

\[\partial_t^2 u + \partial_t u + (1 - \Delta)u + \mathcal{N}(u) = \zeta,\]

where the noise \(\zeta\) is primarily taken to be a space-time white noise \(\xi\). Here, \(\mathcal{N}(u)\) denotes a nonlinearity which may be of a power-type \[38, 39, 40, 71, 62, 89, 63, 15, 64, 77, 17\] and trigonometric and exponential nonlinearities \[69, 72, 70, 79\]. We also mention the works \[76, 67, 66, 85, 78, 65\] on the (deterministic) nonlinear wave equations (1.1) with rough random initial data and \[25, 26, 61\] on SNLW with more singular (both in space and time) noises. We point out that the only known well-posedness result up to date for singular SNLW posed on an unbounded domain is the work \[89\] by the second author, where he established \textit{pathwise} global well-posedness of the cubic SNLW on \(\mathbb{R}^2\) with an additive space-time white noise forcing:

\[\partial_t^2 u + (1 - \Delta)u + u^k = \xi,\]

where \(k = 3\). Note that a slight modification of the argument yields pathwise global well-posedness for SdNLW (1.1) on \(\mathbb{R}^2\) when \(k = 3\).

Theorem 1.2 provides the second well-posedness result for singular SNLW on an unbounded domain. Our construction of the global-in-time dynamics, however, is quite different from that in \[89\]. In particular, it is not pathwise but is based on a probabilistic argument, more precisely, on Bourgain’s invariant measure argument ‘in spirit’. We point out that pathwise global well-posedness of SdNLW (1.1) or SNLW (1.35) for the (super-)quintic case \(k \geq 5\) remains a challenging open question even on the torus \(\mathbb{T}^2\), and therefore, the situation for the stochastic wave equation is completely different from SNLH (1.9) on \(\mathbb{R}^2\), where Mourrat and Weber \[55\] proved pathwise global well-posedness of (1.9) on \(\mathbb{R}^2\) for any \(k \in 2\mathbb{N} + 1\).

Thanks to the finite speed of propagation, the argument in \[38, 40\] yields local well-posedness of (1.34) on the ball \(B_R \subset \mathbb{R}^2\) for each \(R > 0\); see Proposition 4.2 below. As \(R \to \infty\), however, the local existence time shrinks to 0. In order to construct a solution to (1.34) on some time interval \([0, \tau]\), uniformly on \(\mathbb{R}^2\), we need to make use of statistical ingredients, and thus, establishing local well-posedness of (1.34), uniformly on \(\mathbb{R}^2\), is essentially as difficult as establishing its global well-posedness, as already observed in \[89\].

On the dilated torus \(\mathbb{T}_L^2\), the Gibbs measure \(\tilde{\rho}_L\) and the base Gaussian measure \(\mu_L\) are equivalent for each finite \(L > 0\). However, this equivalence of \(\tilde{\rho}_L\) and \(\mu_L\) is not uniform when \[^{10}\]See also \[27\] which appeared after the first version of this paper.
[^{11}]Some of the works mentioned below are on SNLW without damping.
$L \to \infty$, since the potential energy $R^L$, defined as the limit of $R^L_N$ in (1.18), grows like $\sim L$ as $L \to \infty$. Indeed from (1.18), (1.17), and Lemma 2.13 with (1.10), we have

$$E_{\mu_L}[(R^L(u))^2] = \lim_{N \to \infty} E_{\mu_L}[(R^L_N(u))^2] = C_k \lim_{N \to \infty} \int_{T^2_L \times T^2_L} \left( \mathbb{E}_{\mu_L}[P_N u(x)P_N u(y)] \right)^{k+1} dxdy = C_k \lim_{N \to \infty} \sum_{\lambda \in \mathbb{Z}^2_L} \frac{1}{|\lambda|^2} \frac{1}{L^{2k}} \sim L^2,$$

where the last step follows from a Riemann sum approximation. See also Remark 1.4(i). This non-uniformity causes difficulty in studying the large torus limit $L \to \infty$. In the one-dimensional case, there is no need for a renormalization in constructing a Gibbs measure, and Bourgain [13] used the Brascamp-Lieb concentration inequality [14, Theorem 5.1] to reduce the relevant analysis to that for the Gaussian case, uniformly in the period $L \gg 1$.

In the current two-dimensional case, due to the use of the renormalization, the log-concavity needed for the Brascamp-Lieb concentration inequality is not available, and thus we need an alternative approach.

In this work, the main statistical control comes from the study on the so-called enhanced Gibbs measures, namely the distributions of the enhanced data sets. This idea played a crucial role in a recent work [64] on the hyperbolic $\Phi^3_3$-model on the three-dimensional torus $T^3$ by the first two authors and Okamoto. Given $R > 0$, our goal is to construct a solution to (1.34) on the cone $C_R$ in (1.33) as a limit of the solution $u_L$ to the $L$-periodic problem (1.32); see (4.2) below. When $L \gg R$, we see that the $L$-periodic spatial white noise $\zeta_L$ defined in (3.53) and the $L$-periodic space-time white noise $\xi_L$ defined in (4.4) agree on the cone $C_R$ with the spatial white noise $\zeta$ on $\mathbb{R}^2$ (see Definition 3.4) and the space-time white noise on $\mathbb{R}^2 \times \mathbb{R}_+$, respectively; see (4.6) below. Therefore, by restricting our attention to the cone $C_R$, we conclude from the finite speed of propagation and the observation above that the difference of the $L$-periodic problem (1.2) for different values of $L \geq 1$ appears only in the first component $u_{0,L}$ of the initial data with Law$(u_{0,L}) = \rho_L$, where $\rho_L$ is the $L$-periodic $\Phi^{k+1}_2$-measure in (1.21). This essentially reduces the convergence problem on the cone $C_R$ of the $L$-periodic problem (4.2) to studying convergence properties of the enhanced Gibbs measure

$$\nu_L = \text{Law}(\Xi_0(u_{0,L})), \quad (1.36)$$
where $\Xi_0(u_{0,L})$ denotes the enhanced data set\footnote{Namely, the Wick powers of the associated linear solution.} emanating from the first component $u_{L|t=0} = u_{0,L}$ of the initial data with $\text{Law}(u_{0,L}) = \rho_L$; see (4.36) and (4.12). As mentioned above, by restricting our attention to the cone $C_R$, when $L \gg R$, the second enhanced data set $\Xi_1(u_1,\xi)$ in (4.13) (see also (4.10)), involving the second component $u_1$ of the initial data and the space-time white noise forcing $\xi$, does not depend on $L$, which simplifies some analysis. As for the second enhanced data set $\Xi_1(u_1,\xi)$, its \textit{mapping properties} (viewing its elements as (random) multiplication operators) play an important role; see Definition 4.1 and Proposition 4.9.

Let us briefly describe four main steps of the proof of Theorem 1.2.

- **Step 1:** Coming down from infinity for the associated SNLH on $\mathbb{R}^2$.
  
  In this first step, we establish coming down from infinity (namely, an estimate on a solution independent of initial data) for SNLH (3.5) in (i) weighted Lebesgue spaces $L_{\mu}^p(\mathbb{R}^2)$ (Proposition 3.1) and (ii) weighted Sobolev spaces $W_{\mu}^{s,p}(\mathbb{R}^2)$ of positive regularities (Proposition 3.3). The coming down from infinity in weighted Lebesgue spaces yields tightness of the $L$-periodic Gibbs measures $\tilde{\rho}_L$, which then allows us to extract a sequence $\{\tilde{\rho}_L\}_{L \in A}$ converging weakly to a limiting Gibbs measure $\tilde{\rho}_\infty$; see Subsection 3.3. On the other hand, the coming down from infinity in weighted Sobolev spaces of positive regularities plays a crucial role in Step 3.

  In recent years, coming down from infinity has been studied in the context of (singular) SNLH; see, for example, [92, 53, 54]. See the introduction in [53] for a further discussion. In the case of weighted Lebesgue spaces (Proposition 3.1), our argument follows closely that in [92] and aims to establish a certain differential inequality (see Lemma 3.2). Due to the use of the weight, however, the argument is more complicated and requires a careful decomposition of the physical space into unit cubes, combined with the Littlewood-Paley decomposition; see (3.16)-(3.26). In the case of weighted Sobolev spaces of positive regularities (Proposition 3.3), our argument is based on a Gronwall-type argument with the coming down from infinity for weighted Lebesgue spaces. We present details in Section 3.

- **Step 2:** Local well-posedness and stability of SdNLW (1.34) on the cone $C_R$.
  
  This second step is entirely deterministic, by viewing initial data $(u_0, u_1)$ and a forcing $\xi$ as given deterministic spatial / space-time distributions, and follows from a slight modification of the local well-posedness argument in [10]. Here, stability refers to that with respect to the enhanced data set $\Xi_0(u_0)$ in (4.12), emanating from the first component $u_0$ of the initial data. See Subsection 4.1 for details.

- **Step 3:** Convergence of the enhanced Gibbs measures.
  
  In this step and Step 4, we restrict our attention to $L \in A \subset \mathbb{N}$ constructed in Step 1. This is due to the non-uniqueness of the limiting Gibbs measure on $\mathbb{R}^2$; see Remark 1.4(ii).

  Our main goal in this step is to prove convergence of $\{\nu_L\}_{L \in A}$ to a natural limit
  \begin{equation}
  \nu_\infty = \text{Law}(\Xi_0(u_0))
  \tag{1.37}
  \end{equation}
  with $\text{Law}(u_0) = \rho_\infty$ constructed in Step 1. Here, the mode of convergence is weak convergence as well as convergence in the Wasserstein-1 metric. The proof is broken into two parts, where we first establish tightness of $\{\nu_L\}_{L \in A}$ (Proposition 4.4) and then show that the limit is indeed unique, given by $\nu_\infty$ in (1.37) (Proposition 4.9).

  The (first) enhanced data set $\Xi_0(u_{0,L})$ consists of the Wick powers: $(S(t)u_{0,L})^\ell : \ell = 1, \ldots, k$, of the linear solution $S(t)u_{0,L} = (\partial_t \mathcal{D}(t) + \mathcal{D}(t))u_{0,L}$ with $\text{Law}(u_{0,L}) = \rho_L$, where

\[
S(t)u_{0,L} = (\partial_t \mathcal{D}(t) + \mathcal{D}(t))u_{0,L}
\]
\(\rho_L\) is the \(L\)-periodic \(\Phi_k^{k+1}\)-measure in (1.21). Hence, in view of the invariance of \(\rho_L\) under the parabolic \(\Phi_k^{k+1}\)-model, instead of studying \(\Xi_0(u_0, L)\), it suffices to study \(\Xi_0(X_{L,1})\), where \(X_L\) is the solution to the \(L\)-periodic SNLH (3.42) (namely, the parabolic \(\Phi_k^{k+1}\)-model) with the Gibbsian initial data \(\text{Law}(X_{0,L}) = \rho_L\). As a result, the argument involves an intricate combination of wave and heat analysis (see, in particular, the proof of Proposition 4.6) as well as the coming down from infinity in weighted Sobolev spaces of positive regularities. See Subsection 4.2 for details.

- **Step 4:** Global well-posedness and invariance of the limiting Gibbs measure.

  We first prove well-posedness of the hyperbolic \(\Phi_k^{k+1}\)-model on the cone \(C_R\) for each \(R > 0\). In view of the global well-posedness of the \(L\)-periodic hyperbolic \(\Phi_k^{k+1}\)-model (4.2), the local well-posedness and stability results established in Step 2 allow us to reduce the problem to estimating the size of the first enhanced data set \(\Xi_0(u_0, L)\), studying the mapping properties of the second enhanced data set \(\Xi_1(u_2, \xi)\), and convergence in the Wasserstein-1 metric of the enhanced Gibbs measure \(\nu_L\) to \(\nu_\infty\) established in Step 3. Invariance of the limiting Gibbs measure \(\tilde{\rho}_\infty\) then follows from the weak convergence of \(\{\tilde{\rho}_L\}_{L \in \mathcal{A}}\) to \(\tilde{\rho}_\infty\) (Theorem 1.2(i)), the convergence in law, as \(D'(\mathbb{R}^2 \times \mathbb{R}_+)\)-valued random variables, of the solution \(u_L, L \in \mathcal{A}\), to the \(L\)-periodic hyperbolic \(\Phi_k^{k+1}\)-model (4.2) to the solution \(u\) to the hyperbolic \(\Phi_k^{k+1}\)-model (4.1) on \(\mathbb{R}^2\), which follows as a corollary of the global well-posedness (Remark 4.12).

In view of the finite speed of propagation, when we work on the cone \(C_R\), the idea of breaking the enhanced data set into two groups \(\Xi_0(u_0, L)\) in (4.12) and \(\Xi_1(u_1, \xi)\) in (4.13), where the latter is independent of \(L \gg R\), seems new but is natural. Let us make a brief comparison to the work [64] on the hyperbolic \(\Phi_3^3\)-model by the first two authors and Okamoto. What is common is the use of the enhanced Gibbs measures (= the laws of the enhanced data sets of the \(L\)-periodic (or frequency-truncated) Gibbs measures), while, in the current paper, we only consider the enhanced Gibbs measure emanating from the first component of the initial data for the reason explained above. In this work, we reduce various problems to those for the associated stochastic heat equation, which leads to an interesting mixture of wave and heat analysis. Lastly, we remark that, in a recent remarkable preprint [17] resolving a challenging open problem on well-posedness of the hyperbolic \(\Phi_3^3\)-model, a mixture of wave and heat analysis also played an important role (but in a different context from our analysis).

**Remark 1.3.** (i) A slight modification of the proof of Theorem 1.2 applies to the deterministic NLW (1.4) on \(\mathbb{R}^2\) with the Gibbsian initial data, thus yielding global well-posedness of (the renormalized version of) (1.4) with \(\text{Law}(u(0), \partial_x u(0)) = \tilde{\rho}_\infty\) and invariance of \(\tilde{\rho}_\infty\) under the resulting dynamics \(^{13}\) See [51] for the one-dimensional case. We also mention [13, 94, 19, 20, 46, 16, 65] for works on the construction of invariant Gibbs dynamics for Hamiltonian PDEs on unbounded domains.

(ii) In this paper, we only consider the defocusing case \(k \in 2\mathbb{N} + 1\). If (a) \(k \in 2\mathbb{N} + 2\) or (b) \(k \in 2\mathbb{N} + 1\) but with the – sign on the potential energy \(\frac{1}{k+1} \int u^{k+1} dx\) in (1.2), namely, the focusing case, then it is known that the associated (renormalized) Gibbs measure on the

\(^{13}\)In a recent preprint [4], Barashkov and Laarne independently obtained global well-posedness and invariance of the Gibbs measure the deterministic NLW (1.4) on \(\mathbb{R}^2\) (with \(k = 3\)).
two-dimensional torus $\mathbb{T}^2$ is not normalizable even with a taming by a power of the Wick-ordered $L^2$-norm; see [73]. See also [18]. We refer interested readers to [48, 74, 63, 64] on the (non-)construction of focusing Gibbs measures on the general $d$-dimensional torus.

When $k = 2$, it is possible to construct the Gibbs measure on $\mathbb{T}^2$ with a taming by a power of the Wick-ordered $L^2$-norm (see [12, 73]) and the associated invariant Gibbs dynamics for the hyperbolic $\Phi^3_2$-model on $\mathbb{T}^2$; see Remark 1.8 in [76]. See also [64] for the three-dimensional case. Due to the non-defocusing nature of the problem, however, we expect a certain triviality phenomenon to take place in taking a large torus limit of the $L^2$-periodic $\Phi^3_2$-measure, just as in the one-dimensional focusing case [81, 91].

(iii) In [88, 90], the second author proved ergodicity of the Gibbs measure for the hyperbolic $\Phi^{k+1}_{d+1}$-model posed on $\mathbb{T}^d$ when $d = 1, 2$. See also [29]. It is of interest to study the corresponding problem on $\mathbb{R}^d$.

Remark 1.4. (i) As mentioned above, the equivalence of the $L$-periodic $\Phi^{k+1}_2$-measure $\rho_L$ and the base Gaussian free field $\mu_L$ on $\mathbb{T}^2$ is not uniform. In fact, it is expected that the limiting $\Phi^{k+1}_2$-measure $\rho_\infty$ on $\mathbb{R}^2$ constructed in Theorem 1.2 (i) and the base Gaussian measure (= the large torus limit of $\mu_L$) are mutually singular. We will address this issue in a forthcoming work.

(ii) The limiting Gibbs measure $\rho_\infty$ constructed in Theorem 1.2 (i) in principle depends on the sequence $L \in A$ and there is no uniqueness statement for the limiting Gibbs measure. See [30, 31, 32]. See also the discussion at the end of Section 2 in [3].

Remark 1.5. Let $\rho_\infty$ be the limiting $\Phi^{k+1}_2$-measure in Theorem 1.2 (i) constructed as a limit of $\{\rho_L\}_{L \in A}$. Consider the parabolic $\Phi^{k+1}_2$-model (3.1) on $\mathbb{R}^2$ with the Gibbsian initial data $\text{Law}(X_0) = \rho_\infty$. Then, it follows from the proof of Proposition 4.6 (see Remark 4.7) that $\rho_\infty$ is invariant under the parabolic $\Phi^{k+1}_2$-dynamics on $\mathbb{R}^2$. While this fact is not difficult to prove, to the authors’ knowledge, it is not explicitly written and thus we decided to mention it here.

2. Preliminary

2.1. Notations. By $A \lesssim B$, we mean $A \leq CB$ for some constant $C > 0$. We use $A \sim B$ to mean $A \lesssim B$ and $B \lesssim A$. We write $A \ll B$, if there is some small $c > 0$ such that $A \leq cB$. We may use subscripts to denote dependence on external parameters; for example, $A \lesssim_\delta B$ means $A \leq C(\delta)B$. We use $a - \varepsilon$ (and $a + \varepsilon$) to denote $a - \varepsilon$ (and $a + \varepsilon$, respectively) for arbitrarily small $\varepsilon > 0$. If this notation appears in an estimate, then an implicit constant is allowed to depend on $\varepsilon > 0$ (and it usually diverges as $\varepsilon \to 0$). We also use the notation $p = \infty$ to denote a sufficiently large number $p \gg 1$, depending on the context. For conciseness of notation, we set $a \vee b = \max(a, b)$.

Throughout this paper, we fix a rich enough probability space $(\Omega, \mathcal{F}, \mathbb{P})$, on which all the random objects are defined. The realization $\omega \in \Omega$ is often omitted in the writing. Given a random variable $X$, we denote by $\text{Law}(X)$ the law of $X$.

In the remaining part of the paper, we only work with real-valued functions / distributions. Given a time-differentiable function, we set $\mathbf{u} = (u, \partial_t u)$.

\footnote{See a recent preprint [83] on the triviality of the $\Phi^4_2$-measure on the plane.}
We define the Fourier transform of a function \( f \) on \( \mathbb{R}^d \) by setting
\[
\hat{f}(\eta) = \mathcal{F}(f)(\eta) = \int_{\mathbb{R}^d} f(x) e^{-2\pi i \eta \cdot x} \, dx
\]
with the inverse Fourier transform given by \( \mathcal{F}^{-1}(f)(\eta) = \hat{f}(-\eta) \).

Let \( s \in \mathbb{R} \) and \( 1 \leq p \leq \infty \). We define the \( L^2 \)-based Sobolev space \( H^s(\mathbb{R}^d) \) by the norm:
\[
\|f\|_{H^s} = \|\langle \nabla \rangle^s f\|_{L^2} = \|\langle \eta \rangle^s \hat{f}(\eta)\|_{L^2},
\]
where \( \langle \cdot \rangle = (1 + |\cdot|^2)^{\frac{s}{2}} \). We also define the \( L^p \)-based Sobolev space \( W^{s,p}(\mathbb{R}^d) \) by the norm:
\[
\|f\|_{W^{s,p}} = \|\langle \nabla \rangle^s f\|_{L^p} = \|\mathcal{F}^{-1}[\langle \eta \rangle^s \hat{f}(\eta)]\|_{L^p},
\]
(2.1)

When \( p = 2 \), we have \( H^s(\mathbb{R}^d) = W^{s,2}(\mathbb{R}^d) \). We recall the following interpolation result which follows from the Littlewood-Paley characterization of Sobolev norms via the square function and Hölder’s inequality; let \( s, s_1, s_2 \in \mathbb{R} \) and \( p, p_1, p_2 \in (1, \infty) \) such that \( s = \theta s_1 + (1 - \theta) s_2 \) and \( \frac{1}{p} = \frac{\theta}{p_1} + \frac{1 - \theta}{p_2} \) for some \( 0 < \theta < 1 \). Then, we have
\[
\|u\|_{W^{s,p}} \lesssim \|u\|_{W^{s_1,p_1}} \|u\|_{W^{s_2,p_2}}^{1 - \theta}. 
\]
(2.2)

Let \( X(\mathbb{R}^d) \) be a function space on \( \mathbb{R}^d \) such as the Sobolev space \( W^{s,p}(\mathbb{R}^d) \) defined above or the weighted Sobolev / Besov spaces defined below. Given a (nice) subset \( K \subset \mathbb{R}^d \), we define the localized version of a function space \( X(\mathbb{R}^d) \) by the restriction norm:
\[
\|f\|_{X(K)} = \inf \{ \|g\|_{X(\mathbb{R}^d)} : f \equiv g \text{ on } K \}. 
\]
(2.3)

In dealing with a space of space-time functions, we often use short-hand notations such as \( L^q_T H^s_x \) for \( L^q([0,T]; H^s(\mathbb{R}^d)) \). Given a space \( X(\mathbb{R}^d) \) of functions on \( \mathbb{R}^d \), we set \( L^\infty([0,R]; X(B_{R-t})) \) by
\[
L^\infty([0,R]; X(B_{R-t})) = \{ u \in \mathcal{D}'(\mathbb{C}_R) : t \in [0,R] \mapsto u(t)\|_{X(B_{R-t})} \text{ is in } L^\infty \},
\]
(2.4)
where \( \mathbb{C}_R \) denotes the interior of the cone \( C_R \) and \( X(B_{R-t}) \) is defined by the restriction norm as in (2.3). Here, \( B_R \subset \mathbb{R}^d \) denotes the (closed) ball of radius \( R \) centered at the origin and \( C_R \) denotes the cone as in (1.33) (but in \( \mathbb{R}^d \times \mathbb{R}_+ \)).

Let \( \mathcal{D}(t) \) denote the linear damped wave propagator defined in (1.23). We then set
\[
\mathcal{S}(t) = \partial_t \mathcal{D}(t) + \mathcal{D}(t). 
\]
(2.5)

Namely, \( u = \mathcal{S}(t)u_0 \) satisfies
\[
\begin{align*}
\partial_t^2 u + \partial_t u + (1 - \Delta) &= 0 \\
(u, \partial_t u)|_{t=0} &= (u_0, 0).
\end{align*}
\]

Let \( p_t \) denote the standard heat kernel on \( \mathbb{R}^2 \) given by \( p_t(x) = \frac{1}{4\pi t} e^{-\frac{|x|^2}{4t}} \). Then, let \( P(t) \) denote the linear heat propagator given by
\[
P(t)f = e^{t(\Delta-1)}f = e^{-t}(p_t * f).
\]
(2.6)

We now introduce the Littlewood-Paley projector, which is adapted to weighted Sobolev and Besov spaces defined in the next subsection. Recall the following definition \[55, \text{Definition 1.1}] \footnote{Definition 2.1 in the arXiv version.}. Given \( \theta_0 \geq 1 \), the Gevrey class \( \mathcal{G}^{\theta_0} \subset C^\infty(\mathbb{R}^d; \mathbb{R}) \) of order \( \theta_0 \) consists of functions \( f \) satisfying the following; given any compact set \( K \subset \mathbb{R}^d \), there exists \( C_K > 0 \) such that
\[ \sup_{x \in K} |\partial^\alpha f(x)| \leq (C_K)|\alpha|^{\alpha + 1} (\alpha!)^{\theta_0} \quad \text{for any multi-index } \alpha. \]

We use \( G_\epsilon^{\theta_0} \) to denote the set of compactly supported functions in \( G^{\theta_0} \).

Let \( \mathbb{Z}_{\geq 0} := \mathbb{N} \cup \{0\} \). Let us now introduce the Littlewood-Paley projector \( Q_k, \ k \in \mathbb{Z}_{\geq 0}, \) defined by Gevrey class multipliers. As seen in [55], such a choice is suitable for weighted spaces with a stretched exponential weight; see (2.10). As in Section 3 of [55], let

\[ \chi_0, \chi_1 \in G_\epsilon^{\theta_0}(\mathbb{R}^d; [0, 1]) \quad (2.7) \]

with

\[ \text{supp } \chi_0 \subset \{ |\xi| \leq \frac{4}{3} \} \quad \text{and} \quad \text{supp } \chi_1 \subset \{ \frac{3}{4} \leq |\xi| \leq \frac{8}{3} \} \]

such that

\[ \sum_{k=0}^{\infty} \chi_k(\xi) \equiv 1 \]

on \( \mathbb{R}^d \), where \( \chi_k(\xi) = \chi_1(2^{1-k}\xi) \) for \( k \geq 2 \). We then define the Littlewood-Paley projector \( Q_k \) by

\[ Q_k(f) = F^{-1}(\hat{\chi_k} \hat{f}) = \eta_k * f, \quad (2.8) \]

where \( \eta_k \) is defined by

\[ \eta_k = F^{-1}(\chi_k) \quad (2.9) \]

for \( k \in \mathbb{Z}_{\geq 0} \). Namely, we have \( \eta_k(x) = 2^{dk} \eta(2^k x) \) for \( k \in \mathbb{N} \), where \( \eta(x) = 2^{-d} \eta_1(2^{-1} x) \). We also recall Proposition 1 in [55]\(^{16}\) that

\[ |\eta(x)| \lesssim e^{-c|x|^\frac{1}{\theta_0}}. \quad (2.10) \]

2.2. Weighted Sobolev and Besov spaces. In this subsection, we introduce weighted Sobolev and Besov spaces and discuss their basic properties. We first recall the definition of the stretched exponential weight introduced in [55]. Let \( \theta_0 \) be as in (2.7), appearing in the definition of the Littlewood-Paley projector \( Q_k \) in (2.8). Given \( 0 < \delta < \frac{1}{\theta_0} (\leq 1) \) and \( \mu > 0 \), we define the weight \( w_\mu(x) \) by setting

\[ w_\mu(x) = e^{-\mu(x)^\delta}. \quad (2.11) \]

It is easy to check that the weight \( w_\mu \) is \( w_{-\mu}\)-moderate in the sense:

\[ w_\mu(x + y) \leq w_{-\mu}(x) w_\mu(y) \quad (2.12) \]

for any \( x, y \in \mathbb{R}^2 \); see [55] (2.6)]. In particular, it follows from (2.12) that

\[ w_\mu(x) \lesssim w_\mu(n) \quad (2.13) \]

for any \( x \in Q_n := n + [-\frac{1}{2}, \frac{1}{2}]^2 \). Let \( 0 < \alpha < 1 \). Then, recall from [55] (3.25)\(^{17}\) that we have

\[ |Q_kw_\mu(x)| \lesssim 2^{-\alpha k}w_\mu(x) \quad (2.14) \]

for any \( x \in \mathbb{R}^2 \) and \( k \in \mathbb{Z}_{\geq 0} \). In the remaining part of the paper, we fix \( 0 < \delta < \frac{1}{\theta_0} \) in (2.11) and we often drop the dependence on \( \delta \) in various estimates.

\(^{16}\)Proposition 2.2 in the arXiv version.

\(^{17}\)(3.24) in the arXiv version.
Let $1 \leq p < \infty$ and $\mu > 0$. Then, the weighted Lebesgue space $L^p_\mu(\mathbb{R}^d)$ is defined by the norm:

$$
\|f\|_{L^p_\mu} = \left( \int_{\mathbb{R}^d} |f(x)|^p w_\mu(x) \, dx \right)^{\frac{1}{p}}.
$$

(2.15)

While it is possible to introduce a weighted Sobolev space by the $L^p_\mu$-norm of $(\nabla)^s f$ (see (2.31) below), we use a slightly different definition by first introducing spatial localization.

Let $\phi : \mathbb{R} \to [0, 1]$ be a smooth even function such that $\phi$ is non-increasing on $\mathbb{R}_+$, $\phi = 1$ on $[-\frac{5}{8}, \frac{5}{8}]$, and $\phi = 0$ on $(-\infty, -\frac{8}{5}] \cup [\frac{8}{5}, \infty)$. Given $j \in \mathbb{Z}_{\geq 0}$, we define $\phi_j$ by

$$
\begin{align*}
\phi_0(x) &= \phi(|x|) & \text{and} & \quad \phi_j(x) &= \phi\left(\frac{|x|}{2^j}\right) - \phi\left(\frac{|x|}{2^{j-1}}\right), & j \in \mathbb{N}.
\end{align*}
$$

(2.16)

Then, it is easy to check that

$$
\sum_{j=0}^{\infty} \phi_j(x) = 1 \quad \text{for any } x \in \mathbb{R}^d,
$$

(2.17)

and

$$
\text{supp } \phi_0 \subset \{|x| \leq \frac{8}{5}\} \quad \text{and} \quad \text{supp } \phi_j \subset \left\{\frac{5}{8} \cdot 2^j \leq |x| \leq \frac{8}{5} \cdot 2^j\right\}, \quad j \in \mathbb{N}.
$$

(2.18)

By convention, we set $\phi_{-1} = 0$. Given $s \geq 0$ and $1 \leq p \leq \infty$, it follows from (2.16) that

$$
\|\phi_j\|_{W^{s,p}(\mathbb{R}^d)} \sim 2^{js/2}.
$$

(2.19)

We also define slightly fattened cutoff functions $\tilde{\phi}_j$, $j \in \mathbb{Z}_{\geq 0}$, by setting

$$
\tilde{\phi}_j = \phi_{j-1} + \phi_j + \phi_{j+1}, \quad j \in \mathbb{N}.
$$

(2.20)

Then, we have

$$
\tilde{\phi}_j \phi_j = \phi_j.
$$

(2.21)

for any $j \in \mathbb{Z}_{\geq 0}$.

Given $1 \leq p < \infty$ and $\mu > 0$, we define the weighted Sobolev space $W^{s,p}_\mu(\mathbb{R}^d)$ by the norm:

$$
\|f\|_{W^{s,p}_\mu} = \left( \sum_{j=0}^{\infty} w_\mu(2^j) \|\phi_j f\|_{W^{s,p}}^p \right)^{\frac{1}{p}},
$$

(2.22)

where $W^{s,p} = W^{s,p}(\mathbb{R}^d)$ denotes the usual $L^p$-based Sobolev space on $\mathbb{R}^d$ defined in (2.1).

When $p = 2$, we set

$$
H^s_\mu(\mathbb{R}^d) = W^{s,2}_\mu(\mathbb{R}^d).
$$

(2.23)

The following embedding follows from the definition (2.22) with Lemma 2.8 and (2.19):

$$
W^{s,p_1}_\mu(\mathbb{R}^d) \subset W^{s,p_2}_\mu(\mathbb{R}^d) \quad \text{for } 1 \leq p_2 \leq p_1 < \infty \text{ and } \mu_2 \geq \mu_1 > 0.
$$

From (2.22) and (2.15) with (2.11) and (2.18), there exists $\alpha_1 > 0$ such that

$$
\begin{align*}
\|f\|_{W^{s,p}_\mu} &\leq \sum_{j=0}^{\infty} w^{1/2}_{2^j} \|\phi_j f\|_{W^{s,p}_\mu} \leq \sum_{j=0}^{\infty} e^{-\frac{\mu}{p} 2^j} \|\phi_j f\|_{W^{s,p}_\mu}, \\
\|\phi_j f\|_{W^{s,p}_\mu} &\leq e^{\frac{\mu}{p}} e^{\frac{\mu}{p} 2^j} \|f\|_{W^{s,p}_\mu}, \\
\|\phi_j f\|_{L^p} &\leq e^{\frac{\mu}{p} 2^j} \|f\|_{L^p_\mu}
\end{align*}
$$

(2.23)
for any $j \in \mathbb{Z}_{\geq 0}$. Similar bounds hold when we replace $\phi_j$ by $\bar{\phi}_j$.

Let $1 \leq q \leq p < \infty$. By applying Sobolev’s inequality (with $\frac{q}{s} \geq \frac{1}{q} - \frac{1}{p}$) and the embedding $L^q(\mathbb{Z}_{\geq 0}) \subset \ell^p(\mathbb{Z}_{\geq 0})$, we have

$$\|f\|_{W^0_{\mu,p}} = \|w_{p}^\mu(2^j)\|_{\ell^p(\mathbb{Z}_{\geq 0})} \lesssim \|w_{p}^\mu(2^j)\|_{\ell^p(\mathbb{Z}_{\geq 0})} = \|f\|_{W^0_{\mu,p}}.$$  

(2.24)

**Remark 2.1.** We point out that when $s = 0$, the $W^0_{\mu,p}$-norm defined in (2.22) is not equivalent to the $L^p_{\mu}$-norm defined in (2.15). See also Remark 2.4 below.

We also recall the definition of the weighted Besov space $B^s_{p,q}(\mathbb{R}^d)$ from [55, Section 3]. Given $s \in \mathbb{R}$, $1 \leq p, q \leq \infty$, and $\mu \geq 0$, we define the weighted Besov space $B^s_{p,q}(\mathbb{R}^d)$ as the completion of $C_c^\infty(\mathbb{R}^d)$ under the norm:

$$\|f\|_{B^s_{p,q}} = \left\|2^{sk}\|Q_k f\|_{L^p_{\mu}(\mathbb{R}^d)}\right\|_{L^q(\mathbb{Z}_{\geq 0})}.$$  

(2.25)

See also [36]. When $\mu = 0$, the weighted Besov space $B^s_{p,q}(\mathbb{R}^d)$ reduces to the usual Besov space $B^s_{p,q}(\mathbb{R}^d)$. We recall the following embeddings for (unweighted) Sobolev and Besov spaces:

$$\|f\|_{B^{s_0}_{p,\infty}} \lesssim \|f\|_{W^{s,\mu}} \lesssim \|f\|_{B^{s_0}_{p,1}} \lesssim \|f\|_{B^{s_0+\varepsilon}_{p,\infty}}$$  

(2.26)

for any $\varepsilon > 0$.

We first establish the following compact embedding for weighted Besov spaces, which plays a crucial role in proving Theorem 1.2(i).

**Lemma 2.2.** Let $1 \leq p < \infty$. Then, given any $s > s'$ and $\mu < \mu'$, the embedding

$$W^s_{\mu,p}(\mathbb{R}^d) \hookrightarrow W^{s'}_{\mu',p}(\mathbb{R}^d)$$

is compact.

**Proof.** Let $\{f_n\}_{n \in \mathbb{N}}$ be a bounded sequence in $W^s_{\mu,p}(\mathbb{R}^d)$. Then it follows from (2.22) and (2.18) that for any $j \in \mathbb{N}$, the sequence $\{\phi_j f_n\}_{n \in \mathbb{N}}$ is a bounded sequence in $W^{s,\mu}(\mathbb{R}^2)$ with a bounded support: $\{|x| \leq \frac{s}{s'} \cdot 2^j\}$, $j \in \mathbb{Z}_{\geq 0}$.

In the following, we implement a diagonal argument. Let $j = 0$. Then, by Rellich’s lemma (with $s' < s$), there exists a subsequence $\{\phi_0 f_{n_k}\}_{k \in \mathbb{N}}$ which is convergent in $W^{s',\mu}(\mathbb{R}^d)$.

Then, for $j \in \mathbb{N}$, by Rellich’s lemma, we can choose a subsequence $\{n^{(j)}_{k}\}_{k \in \mathbb{N}}$ of $\{n^{(j-1)}_{k}\}_{k \in \mathbb{N}}$ such that $\{\phi_j f_{n^{(j)}_{k}}\}_{k \in \mathbb{N}}$ is convergent in $W^{s',\mu}(\mathbb{R}^d)$. Now, consider the sequence $\{\phi_j f_{n^{(j)}_{k}}\}_{k \in \mathbb{N}}$. Then, it is clear from the construction that, for each $j \in \mathbb{Z}_{\geq 0}$, $\phi_j f_{n_k^{(j)}}$ converges to some $F_j$ in $W^{s',\mu}(\mathbb{R}^d)$ as $k \to \infty$.

By setting

$$F = \sum_{j=0}^\infty F_j,$$

we claim that $F \in W^{s',\mu}(\mathbb{R}^d)$ for any $\mu' > \mu$ and that $f_{n_k^{(j)}}$ converges to $F$ in $W^{s',\mu}(\mathbb{R}^d)$ for $\mu' > 2^j \mu$. 

First, note that we have
\[ \| \phi_j f \|_{W^{s',p}} \lesssim \| f \|_{W^{s',p}}, \quad (2.27) \]
uniformly in \( j \in \mathbb{Z}_{\geq 0} \). When \( s' = 0 \), (2.27) follows from Hölder’s inequality. When \( s' < 0 \), it follows from Lemma 2.8 (ii) and (2.19) that
\[ \| \phi_j f \|_{W^{s',p}} \lesssim \| f \|_{W^{s',p}}. \]
When \( s' > 0 \), the fractional Leibniz rule (see Lemma 2.8 (i) below), (2.19), and Sobolev’s inequality yield
\[ \| \phi_j f \|_{W^{s',p}} \lesssim \| f \|_{W^{s',p}}. \]
This proves the bound (2.27). Then, from (2.23), (2.18), (2.27), \( w_{\mu'}(2^j) = w_{\mu}(2^j)w_{\mu'-\mu}(2^j) \), and the uniform bound on the \( W^{s,p}_\mu \)-norm of \( f_{n_k} \), we have
\[ \| F \|_{W^{s',p}_{\mu'}} \leq \sum_{j=0}^{\infty} \| F_j \|_{W^{s',p}_{\mu'}} \leq \sum_{j=0}^{\infty} \sum_{\ell=0}^{\infty} w_{\mu'}(2^\ell) \lim_{k \to \infty} \| \phi_{\ell} \phi_j f_{n_k} \|_{W^{s',p}} \]
\[ \leq \sum_{\ell=0}^{\infty} \sum_{i=-1}^{1} w_{\mu'}(2^\ell) \lim_{k \to \infty} \| \phi_{\ell} f_{n_k} \|_{W^{s,p}} \]
\[ \leq \sum_{\ell=0}^{\infty} w_{\mu'}(2^\ell) \lim_{k \to \infty} \| \phi \|_{W^{s,p}} \sup_{k \in \mathbb{N}} \left( w_{\mu'}(2^\ell) \| \phi \|_{W^{s,p}} \right) \]
\[ < \infty, \]
provided that \( s' < s \) and \( \mu' > \mu \).

Next, we show convergence. First, note that, as a limit of \( \phi_j f_{n_k} \), we have \( F_j(x) = 0 \) whenever \( \phi_j(x) = 0 \). Thus, in view of (2.18), we can write \( F_j = \phi_j G_j \) with a well-defined function \( G_j = 1_{\phi_j \neq 0} : F_j / \phi_j \) and thus, we have
\[ f_{n_k} - F = \sum_{j=0}^{\infty} \phi_j (f_{n_k} - G_j). \quad (2.28) \]
We set \( F_{-1} = G_{-1} = 0 \).

From (2.22), we have
\[ \| f_{n_k} - F \|_{W^{s',p}_{\mu'}} = \sum_{j=0}^{\infty} w_{\mu'}(2^j) \| \phi_j (f_{n_k} - G_j) \|_{W^{s',p}_{\mu'}}. \quad (2.29) \]
Then, from (2.28) with (2.18) followed by (2.27) we have
\[ w_{\mu'}(2^j) \| \phi_j(f_{n_k}(k) - F) \|_{W^{s',p}}^p = w_{\mu'}(2^j) \left\| \phi_j \sum_{i=1}^{j} (f_{n_k}(k) - G_{j+i}) \right\|_{W^{s',p}}^p \]
\[ \lesssim w_{\mu'}(2^j) \left\| \sum_{i=1}^{j} (f_{n_k}(k) - G_{j+i}) \right\|_{W^{s',p}}^p \]
\[ \lesssim \sum_{\ell=j-1}^{j+1} w_{\mu'}(2^{\ell-1}) \| \phi_{\ell} f_{n_k}(k) - G_{\ell} \|_{W^{s',p}}^p \]
\[ = \sum_{\ell=j-1}^{j+1} w_{\mu'}(2^{\ell-1}) \| \phi_{\ell} f_{n_k}(k) - F_{\ell} \|_{W^{s',p}}^p. \]

(2.30)

For each \( j \in \mathbb{Z}_{\geq 0} \), the right-hand side of (2.30) tends to 0 as \( k \to \infty \). Namely, for each \( j \in \mathbb{Z}_{\geq 0} \), the summand on the right-hand side of (2.29) tends to 0 as \( k \to \infty \). On the other hand, in view of the uniform bound on the \( W_{\mu'}^{s',p} \)-norm of \( f_{n_k}(k) \) and the fact that \( F \in W_{\mu'}^{s',p}(\mathbb{R}^d) \), we have
\[ w_{\mu'}(2^j) \| \phi_j(f_{n_k}(k) - F) \|_{W^{s',p}}^p \]
\[ \lesssim w_{\mu'}(2^j) \| \phi_j f_{n_k}(k) \|_{W^{s',p}}^p + w_{\mu'}(2^j) \| \phi_j F \|_{W^{s',p}}^p, \]
which is summable in \( j \). Therefore, by the dominated convergence theorem applied to (2.29), we conclude that \( f_{n_k}(k) \) converges to \( F \) in \( W_{\mu'}^{s',p}(\mathbb{R}^d) \) as \( k \to \infty \). This concludes the proof of Lemma 2.2. \( \square \)

There seems to be no embedding relation analogous to (2.29) for weighted Sobolev spaces \( W_{\mu}^{s,p}(\mathbb{R}^d) \) and weighted Besov spaces. With a slight loss in \( s \) and \( \mu \), however, we have the following embeddings for weighted Sobolev spaces and weighted Besov spaces. We present the proof in Appendix A.

**Lemma 2.3.** Let \( s \in \mathbb{R} \), \( 1 \leq p < \infty \), and \( \mu > 0 \). Then, there exist \( c_1, c_2 > 0 \) such that
\[ \| f \|_{W_{\mu}^{s,p}} \lesssim \| f \|_{W_{\mu'}^{s',p'}} \]
for any \( s' > s \) and \( 0 < \mu' < c_1 \mu \), and
\[ \| f \|_{W_{\mu'}^{s',p'}} \lesssim \| f \|_{W_{\mu}^{s,p}} \]
for any \( 0 < \mu' < c_2 \mu \).

**Remark 2.4.** Another way to define a weighted Sobolev space would be to define a space \( W_{\mu}^{s,p}(\mathbb{R}^2) \) via the norm:
\[ \| f \|_{W_{\mu}^{s,p}} = \| \langle \nabla \rangle^s f \|_{L_{\mu}^p}. \]
(2.31)

When \( s = 0 \), we have \( W_{\mu}^{0,p}(\mathbb{R}^2) = L_{\mu}^p(\mathbb{R}^2) \). Furthermore, this weighted Sobolev space \( W_{\mu}^{s,p}(\mathbb{R}^2) \) is compatible with the weighted Besov space defined in (2.25) in the following sense:
\[ \| f \|_{B_{\mu}^{s,\infty}} \lesssim \| f \|_{W_{\mu}^{s,p}} \lesssim \| f \|_{B_{\mu}^{s,p}}, \]
(2.32)
where the first inequality follows from Young’s inequality with [55, Lemma 1] while the second inequality follows from Minkowski’s inequality. It follows from Lemma 2.4 and (2.32) that

\[ \|f\|_{W^{s-\epsilon,p}_\mu} \lesssim \|f\|_{W^{s,p}_\mu} \lesssim \|f\|_{W^{s+\epsilon,p}_{\mu_2}}. \]

for any \( \epsilon > 0 \) and \( \mu_1 \gg \mu \gg \mu_2 > 0 \).

In this paper, we use the weighted Sobolev space \( W^{s,p}_\mu(\mathbb{R}^2) \) defined in (2.22) since it is more convenient to work with \( W^{s,p}_\mu(\mathbb{R}^2) \) in establishing coming down from infinity in a weighted Sobolev spaces of positive regularities.

See Remark 2.7 for a discussion on the bounded property of the linear damped wave propagator on \( W^{s,2}_\mu(\mathbb{R}^d) \).

2.3. Linear estimates on weighted Sobolev spaces. In this subsection, we establish linear estimates for the linear heat propagator and the linear damped wave propagator on weighted Sobolev spaces.

Lemma 2.5. There exist small \( C_0, c > 0 \) such that for any \( s \geq 0, 1 \leq p < \infty, \mu > 0, \) and \( 0 < \mu' < C_0\mu, \) we have

\[ \|P(t)f\|_{W^{s,p}_\mu(\mathbb{R}^d)} \lesssim t^{-\frac{s}{2}}e^{-ct}\|f\|_{L^p_{\mu'}(\mathbb{R}^d)} \]  

for any \( t > 0 \), where \( P(t) = e^{t(\Delta-1)} \) is as in (2.6).

Proof. We first introduce a cutoff function \( \phi^t_m \) in terms of the self-similar variable \( x \) for the (homogeneous) heat equation \( \partial_t u - \Delta u = 0 \) by setting

\[ \phi^t_m(x) := \phi_m\left(\frac{x}{t^\frac{1}{2}}\right) \]  

for \( m \in \mathbb{Z}_{\geq 0} \). Then, given \( j \in \mathbb{Z}_{\geq 0} \), it follows from (2.17), (2.18), and (2.34) that

\[ \phi^t_j(P(t)f) = e^{-t} \sum_{m, m' = 0}^\infty \phi^t_j((\phi^t_m f) * (\phi^t_{m'} f)) \]

\[ = e^{-t} \sum_{(m, m') \in \Lambda_{t,j}} \phi^t_j((\phi^t_m f) * (\phi^t_{m'} f)), \]

where \( \Lambda_{t,j} \subset (\mathbb{Z}_{\geq 0})^2 \) is given by

\[ \Lambda_{t,j} = \left\{ (m, m') \in (\mathbb{Z}_{\geq 0})^2 : \left\{ \frac{5}{8} \cdot 2^{m'} \cdot 1_{\{m' \geq 1\}} \leq \frac{8}{5} \cdot 2^j + \frac{8}{5} \cdot 2^{m't^\frac{1}{2}} \right. \right. \]

\[ \left. \left. \text{and} \frac{5}{8} \cdot 2^{m't^\frac{1}{2}} \cdot 1_{\{m \geq 1\}} \leq \frac{8}{5} \cdot 2^j + \frac{8}{5} \cdot 2^{m'} \right\}. \]

In view of (2.34), a direct computation shows that

\[ \|\phi^t_m f\|_{L^1} \lesssim \exp(-c_1 4^m), \]

\[ \|\phi^t_m f\|_{W^{1,1}} \lesssim t^{-\frac{1}{2}} \exp(-c_1 4^m) \]

for some \( c_1 > 0 \). Hence, by the interpolation (2.22), we have

\[ \|\phi^t_m f\|_{W^{s,1}} \lesssim t^{-\frac{s}{2}} \exp(-c_1 4^m) \]  

\[ \leq (2.37) \]

\[ \leq (2.37) \]

18Lemma 2.6 in the arXiv version.
for any $t > 0$ and $0 < s < 1$. We point out that (2.37) also holds for any $s \geq 0$. Then, from (2.6), (2.23), (2.35), the fractional Leibniz rule (Lemma 2.8 (i) below), (2.16), Young’s inequality, and (2.37), we have

$$
\|P(t)f\|_{W^s,p} \leq e^{-t} \sum_{j=0}^{\infty} e^{-\frac{\mu}{p} \frac{2^j s}{\delta}} \|\phi_j (P_t * f)\|_{W^s,p}
$$

$$
\leq e^{-t} \sum_{j=0}^{\infty} e^{-\frac{\mu}{p} \frac{2^j s}{\delta}} \sum_{(m,m') \in \Lambda_{t,j}} \|\phi_j\|_{W^{s,\infty}} \|(\phi^t_{m}P_t) * (\phi_{m'}f)\|_{W^s,p}
$$

$$
\lesssim e^{-t} \sum_{j=0}^{\infty} e^{-\frac{\mu}{p} \frac{2^j s}{\delta}} \sum_{(m,m') \in \Lambda_{t,j}} \|\phi^t_{m}P_t\|_{W^{s,1}} \|\phi_{m'}f\|_{L^p}
$$

$$
\lesssim t^{-\frac{s}{2}} e^{-t} \sum_{j=0}^{\infty} e^{-\frac{\mu}{p} \frac{2^j s}{\delta}} \sum_{(m,m') \in \Lambda_{t,j}} e^{-c_1 4^m \|f\|_{L^p} e^{\frac{\mu}{p} a_1 2^{m'} \delta}}.
$$

Hence, it remains to estimate

$$
A_t := \sum_{j=0}^{\infty} e^{-\frac{\mu}{p} \frac{2^j s}{\delta}} \sum_{(m,m') \in \Lambda_{t,j}} e^{-c_1 4^m \|f\|_{L^p} e^{\frac{\mu}{p} a_1 2^{m'} \delta}}.
$$

First, we consider the case $0 < t \lesssim 1$. By summing over $m'$ with (2.36), we have

$$
A_t \lesssim \sum_{j=0}^{\infty} e^{-\frac{\mu}{p} \frac{2^j s}{\delta}} e^{-c_1 4^m \|f\|_{L^p} e^{\frac{\mu}{p} a_1 2^{m'} \delta}} \lesssim 1,
$$

(2.39)

provided that $\mu > c_2 \mu'$. Next, we consider the case $t \gg 1$. Note that $4^m \sim 2^{m\delta} t^{2\delta}$ implies

$$
2^{m\delta} t^{2\delta} \sim t^{\frac{2\delta}{1-\delta}} \ll t
$$

(2.40)

for $t \gg 1$ since $\delta < 1$. By first summing over $m'$ as in (2.39) and then summing over $m$,

$$
A_t \lesssim \sum_{j=0}^{\infty} e^{-\frac{\mu}{p} \left( \frac{2^j s}{\delta} + 4^m \right)} e^{c_1 \frac{2^j \delta}{1-\delta}} \lesssim 1.
$$

Therefore, the desired bound (2.33) follows from (2.38), (2.39), and (2.41) with (2.40). \qed

Next, we establish estimates for the linear damped wave operator.

**Lemma 2.6.** Let $s \in \mathbb{R}$. Then, there exists $C_0 > 0$ such that

$$
\|D(t)f\|_{H^s_{\mu}} \lesssim e^{-t} \|f\|_{H^{s-1}_{\mu}}
$$

(2.42)

and

$$
\|S(t)f\|_{H^s_{\mu}} \lesssim e^{-t} \|f\|_{H^s_{\mu}}
$$

(2.43)

for $\mu > C_0 \mu' > 0$. Here, $D(t)$ and $S(t)$ are as in (1.23) and (2.5).

**Proof.** In view of (2.17), write

$$
\|\phi_j D(t)f\|_{H^s} = \left\| \sum_{\ell=0}^{\infty} \phi_j D(t)(\phi_{\ell} f) \right\|_{H^s}.
$$

(2.44)
In the following, we only consider $j, \ell \geq 1$ but a similar argument holds for the case $j = 0$ or $\ell = 0$. Thanks to the finite speed of propagation and (2.18), we have

$$\text{supp } D(t)(\phi \ell f) \subset \{ \frac{5}{8} \cdot 2^\ell - t \leq |x| \leq \frac{8}{5} \cdot 2^\ell + t \}.$$  

As a result, $\phi_j D(t)(\phi \ell f) \neq 0$ only when

$$\frac{5}{8} \cdot 2^j \leq \frac{8}{5} \cdot 2^\ell + t \quad \text{and} \quad \frac{5}{8} \cdot 2^\ell - t \leq \frac{8}{5} \cdot 2^j.$$  

(2.45)

From the triangle inequality, we have $(\xi)^s \lesssim \langle \xi_1 \rangle^{|s|} \langle \xi_2 \rangle^s$ for $\xi = \xi_1 + \xi_2$. Then, from Young’s inequality on the Fourier side, we have

$$\|fg\|_{H^s} \lesssim \|f\|_{F^s L^1} \|g\|_{H^s},$$  

where the Fourier-Lebesgue norm is defined by

$$\|f\|_{F^s L^1} = \|\hat{\phi}^s f(\xi)\|_{L^1_\xi}.$$  

(2.46)

Note that from (2.16), we have

$$\|\phi_j\|_{F^s L^1} \lesssim 1,$$  

(2.47)

uniformly in $j \in \mathbb{Z}_{\geq 0}$.

Given $j \in \mathbb{Z}_{\geq 0}$ and $t \geq 0$, define $\Gamma_{j, t}$ by

$$\Gamma_{j, t} = \{ \ell \in \mathbb{Z}_{\geq 0} : \ell \text{ satisfies } (2.45) \}.$$  

Then, we deduce from (2.44), (2.46), (1.23), and (2.23), we have

$$\|\phi_j D(t)f\|_{H^s} \lesssim e^{-\frac{t}{2}} \sum_{\ell \in \Gamma_{j, t}} \|\phi \ell f\|_{H^{s-1}} \lesssim e^{-\frac{t}{2}} \left( \sum_{\ell \in \Gamma_{j, t}} e^{\nu' \cdot \ell \delta^j} \right) \|f\|_{H^{s-1}} \lesssim e^{-\frac{t}{2}} e^{c \mu'(2^j + t \delta)} \|f\|_{H^{s-1}},$$  

(2.49)

where the implicit constant is independent of $j \in \mathbb{Z}_{\geq 0}$. Then, from (2.23) and (2.49), we have

$$\|D(t)f\|_{H^s_{\mu}} \lesssim \sum_{j=0}^{\infty} e^{-\frac{t}{2} 2^j \delta} \|\phi_j D(t)f\|_{H^s} \lesssim e^{-\frac{t}{2}} \sum_{j=0}^{\infty} e^{\nu' 2^j} e^{c \mu'(2^j + t \delta)} \|f\|_{H^{s-1}} \lesssim e^{-\frac{t}{2}} e^{c \mu t^j} \|f\|_{H^{s-1}} \lesssim e^{-\frac{t}{4}} \|f\|_{H^{s-1}},$$

provided that $\mu > 2c_0 \mu'$. This proves (2.42). Recalling that

$$S(t) = \frac{1}{2} D(t) + e^{-\frac{t}{2}} \cos \left( t \sqrt{\frac{3}{4} - \Delta} \right),$$

the second bound (2.43) follows from an analogous computation.
Remark 2.7. (i) Let $w_\mu$ be as in (2.11). Then, for any $0 < \delta \leq 2$ and $\mu > 0$, the Fourier transform of the weight $w_\mu$ is positive. Indeed, it follows from Lemma 5 in [28] on completely monotonic functions that

$$w_\mu(x) = \int_0^\infty e^{-\delta(x^2)} d\alpha_\mu(t) = e^{-\delta} \int_0^\infty e^{-\delta t|x|^2} d\alpha_\mu(t),$$

where $\alpha_\mu(t)$ is bounded and non-decreasing and the integral converges for any $x \in \mathbb{R}^2$. Hence, as a superposition of the Gaussians $e^{-\delta t|x|^2}$, we conclude that the Fourier transform of the weight $w_\mu$ is positive.

(ii) Let $W^{s,2}_\mu(\mathbb{R}^d)$ be the weighted Sobolev space defined in (2.31) with $p = 2$. Then, an analogue of Lemma 2.6 also holds for $W^{s,2}_\mu(\mathbb{R}^d)$. Using the positivity of the Fourier transform of the weight $w_\mu$, we have

$$\|D(t)f\|_{W^{s,2}_\mu} = \|\langle \nabla \rangle^s D(t)f\|_{L^2_\mu} = \|\langle \cdot \rangle^s \hat{D(t)}(\xi) \ast \hat{w}_\mu^2(\xi)\|_{L^2_\xi} \leq \|\langle \cdot \rangle^{s-1} \hat{f} \ast \hat{w}_\mu(\xi)\|_{L^2_\xi} = \|f\|_{W^{s-1,2}_\mu}.$$  

A similar computation holds for $S(t)$. Note that, unlike Lemma 2.6, there is no loss in the coefficient $\mu$ of the weight.

2.4. Product estimates. We first state the product estimate on $\mathbb{R}^d$.

Lemma 2.8. (i) Let $s \geq 0$. Suppose that $1 < p_j, q_j \leq \infty$ and $1 \leq r < \infty$ such that $\frac{1}{p_j} + \frac{1}{q_j} = \frac{1}{r}$, $j = 1, 2$. Then, we have

$$\|fg\|_{W^{s,r}(\mathbb{R}^d)} \lesssim \left( \|f\|_{L^{p_1}(\mathbb{R}^d)} \|g\|_{W^{s,q_1}(\mathbb{R}^d)} + \|f\|_{W^{s,q_2}(\mathbb{R}^d)} \|g\|_{L^{q_2}(\mathbb{R}^d)} \right). \quad (2.50)$$

(ii) Let $s > 0$. Suppose that

(iia) $1 < p \leq \infty$ and $1 < q, r \leq \infty$,

(iib) $1 < p = r \leq \infty$ and $q = \infty$, or

(iic) $1 < p = q' \leq \infty$ and $r = 1$

such that $\frac{1}{p} \leq \frac{1}{p} + \frac{1}{q} \leq \frac{1}{r} + \frac{1}{q'}$ and $q, r' \geq p'$. Then, we have

$$\|fg\|_{W^{-s,r}(\mathbb{R}^d)} \lesssim \|f\|_{W^{-s,p}(\mathbb{R}^d)} \|g\|_{W^{s,q}(\mathbb{R}^d)}. \quad (2.51)$$

As for the fractional Leibniz rule (2.50) see [44] [21] for $p_j, q_j < \infty$ and [58] (2.1) (for the one-dimensional case which can be easily extended to higher dimensions); see also [35]. As for the second estimate (2.51), see [38, 5]. The estimates (2.50) and (2.51) indeed hold for wider ranges of indices; see [6] for a further discussion.

Remark 2.9. Note that, given a (nice) subset $K \subset \mathbb{R}^d$, the estimates (2.50) and (2.51) also hold on the localized version of Sobolev spaces $W^{s,p}(K)$ defined in (2.3). Given $f$ and $g$ on $K$, let $f$ and $g$ be extensions onto $\mathbb{R}^d$. Then, from (2.3) and (2.50), we have

$$\|fg\|_{W^{s,r}(\mathbb{R}^d)} \leq \|fg\|_{W^{s,r}(\mathbb{R}^d)} \lesssim \left( \|f\|_{L^{p_1}(\mathbb{R}^d)} \|g\|_{W^{s,q_1}(\mathbb{R}^d)} + \|f\|_{W^{s,p_2}(\mathbb{R}^d)} \|g\|_{L^{q_2}(\mathbb{R}^d)} \right).$$
By taking infima over extensions \( f \) and \( g \), we then obtain
\[
\|fg\|_{W^{s,r}(K)} \lesssim \left( \|f\|_{L^{p_1}(K)}\|g\|_{W^{s,q_1}(K)} + \|f\|_{W^{s,q_2}(K)}\|g\|_{L^{p_2}(K)} \right).
\]

A similar comment applies to all the preliminary estimates presented in this section.

Next, we state a bi-parameter version of the fractional Leibniz rule.

**Lemma 2.10.** Let \( 0 \leq s \leq 1 \), \( 1 < p_1, p_2 \leq \infty \), and \( 1 \leq p < \infty \) with \( \frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2} \). Then, we have
\[
\left\| (\nabla_x)^s (\nabla_y)^s (fg) \right\|_{L^p_{x,y}(\mathbb{R}^{2d})} \lesssim \left( \left\| (\nabla_x)^s (\nabla_y)^s f \right\|_{L^{p_1}_{x,y}(\mathbb{R}^{2d})} \right) \left( \left\| (\nabla_x)^s (\nabla_y)^s g \right\|_{L^{p_2}_{x,y}(\mathbb{R}^{2d})} \right) \quad (2.52)
\]
for any functions \( f = f(x,y) \) and \( g = g(x,y) \) on \( \mathbb{R}^d \times \mathbb{R}^d \).

**Proof.** Let \( 0 \leq s \leq 1 \) and \( 1 < p < \infty \). Let \( f = f(x,y) \) and \( g = g(x,y) \) be functions on \( \mathbb{R}^d \times \mathbb{R}^d \). Then, by the Marcinkiewicz multiplier theorem (Theorem 6.2.4 in [33]), we have
\[
\left\| (\nabla_x)^s (\nabla_y)^s (fg) \right\|_{L^p_{x,y}(\mathbb{R}^{2d})} \lesssim \left\| f \right\|_{L^p_{x,y}(\mathbb{R}^{2d})} + \left\| (\nabla_x)^s (fg) \right\|_{L^p_{x,y}(\mathbb{R}^{2d})} + \left\| (\nabla_y)^s (fg) \right\|_{L^p_{x,y}(\mathbb{R}^{2d})} \quad (2.53)
\]
Indeed, noting that
\[
\text{RHS of (2.53)} \sim \left\| (1 + |\nabla_x|^s)(1 + |\nabla_y|^s)(fg) \right\|_{L^p_{x,y}(\mathbb{R}^{2d})},
\]
the bound (2.53) follows since the multiplier \( m(\eta_1, \eta_2) \), \( \eta_1, \eta_2 \in \mathbb{R}^d \times \mathbb{R}^d \), defined by
\[
m(\eta_1, \eta_2) = \frac{\langle \eta_1 \rangle^s \langle \eta_2 \rangle^s}{(1 + |\eta_1|^s)(1 + |\eta_2|^s)}
\]
is a Marcinkiewicz multiplier.

By Hölder’s inequality and the usual fractional Leibniz rule (Lemma 2.8(i)), the first three terms on the right-hand side of (2.53) are bounded by the right-hand side of (2.52). As for the last term on the right-hand side of (2.53), it follows from the bi-parameter fractional Leibniz rule ([57] (61) on p. 295); see also [58] (3.3)) that
\[
\left\| (\nabla_x)^s |\nabla_y|^s (fg) \right\|_{L^p_{x,y}(\mathbb{R}^{2d})} \lesssim \left\| (\nabla_x)^s |\nabla_y|^s f \right\|_{L^{p_1}_{x,y}(\mathbb{R}^{2d})} \left\| g \right\|_{L^{p_2}_{x,y}(\mathbb{R}^{2d})} + \left\| (\nabla_x)^s |\nabla_y|^s g \right\|_{L^{p_1}_{x,y}(\mathbb{R}^{2d})} \left\| f \right\|_{L^{p_2}_{x,y}(\mathbb{R}^{2d})} + \left\| (\nabla_y)^s |\nabla_x|^s f \right\|_{L^{p_1}_{x,y}(\mathbb{R}^{2d})} \left\| g \right\|_{L^{p_2}_{x,y}(\mathbb{R}^{2d})} + \left\| (\nabla_y)^s |\nabla_x|^s g \right\|_{L^{p_1}_{x,y}(\mathbb{R}^{2d})} \left\| f \right\|_{L^{p_2}_{x,y}(\mathbb{R}^{2d})},
\]
which is once again bounded by the right-hand side of (2.52). This concludes the proof of Lemma 2.10. \( \Box \)

We extend the product estimates in Lemma 2.8 to weighted Sobolev spaces.

**Lemma 2.11.** (i) Let \( s \geq 0 \) and \( \mu > 0 \). Suppose that \( 1 < p_j, q_j \leq \infty \) and \( 1 \leq r < \infty \) such that \( \frac{1}{p_j} + \frac{1}{q_j} = \frac{1}{r} \), \( j = 1, 2 \). Then, we have
\[
\|fg\|_{W^{s,r}(\mathbb{R}^d)} \lesssim \|f\|_{W^{s,q_1}(\mathbb{R}^d)} \|g\|_{W^{s,q_2}(\mathbb{R}^d)} + \|f\|_{W^{s,q_2}(\mathbb{R}^d)} \|g\|_{W^{s,q_2}(\mathbb{R}^d)}. \quad (2.54)
\]
(ii) Let \( s > 0 \) and \( \mu > 0 \), and let \( 1 \leq p, q, r \leq \infty \) be as in Lemma 2.8 (ii). Then, we have
\[
\|fg\|_{W^{s-r, \mu}} \lesssim \|f\|_{W^{s-p, \mu}} \|g\|_{W^{s-q, \mu}}^{2} \delta.
\] (2.55)

Proof. (i) From (2.22), the fractional Leibniz rule (Lemma 2.8 (i)), and Hölder’s inequality (in \( j \)) with \( \frac{1}{r} = \frac{1}{p} + \frac{1}{q} \), (2.20), and (2.11), we have
\[
\|fg\|_{W^{s-r, \mu}} \lesssim \left( \sum_{j=0}^{\infty} w_{\mu}(2^{j}) \|\phi_{j}f\|_{W^{s-r, \mu}}^{r} \right)^{\frac{1}{r}} = \left( \sum_{j=0}^{\infty} w_{\mu}(2^{j}) \|\phi_{j}f\|_{W^{s-p, \mu}}^{r} \right)^{\frac{1}{r}} \lesssim \left( \sum_{j=0}^{\infty} w_{\mu}(2^{j}) \|\phi_{j}f\|_{L^{p,1}}^{r} \|\phi_{j}g\|_{L^{q,1}}^{r} \right)^{\frac{1}{r}}
\[
+ \left( \sum_{j=0}^{\infty} w_{\mu}(2^{j}) \|\phi_{j}f\|_{W^{s-p,2}}^{r} \|\phi_{j}g\|_{L^{q,2}}^{r} \right)^{\frac{1}{r}} \lesssim \|f\|_{W^{0,p,1}} \|g\|_{W^{s,q,1}}^{2} + \|f\|_{W^{s,p,2}} \|g\|_{W^{0,q,2}}^{2},
\]
which yields (2.54).

(ii) By Lemma 2.8 (ii), we have
\[
\|\phi_{j}f \cdot \tilde{\phi}_{j}g\|_{W^{s-r, \mu}} \lesssim \|\phi_{j}f\|_{W^{s-p, \mu}} \|\tilde{\phi}_{j}g\|_{W^{s-q, \mu}}.
\]
Then, (2.55) follows from applying Hölder’s inequality as in Part (i). \( \square \)

2.5. Tools from stochastic analysis. In the following, we review some basic facts on the Hermite polynomials and the Wiener chaos estimate. See, for example, \[47, 60\].

We define the \( k \)th Hermite polynomials \( H_{k}(x; \sigma) \) with variance \( \sigma > 0 \) via the following generating function:
\[
e^{tx - \frac{1}{2} \sigma t^{2}} = \sum_{k=0}^{\infty} \frac{t^{k}}{k!} H_{k}(x; \sigma)
\]
for \( t, x \in \mathbb{R} \) and \( \sigma > 0 \). When \( \sigma = 1 \), we set \( H_{k}(x) = H_{k}(x; 1) \). Then, we have
\[
H_{k}(x; \sigma) = \sigma^{\frac{k}{2}} H_{k}(\sigma^{-\frac{1}{2}} x).
\] (2.56)
It is well known that \( \{ H_{k}/\sqrt{k!} \}_{k \in \mathbb{Z} \geq 0} \) form an orthonormal basis of \( L^{2}(\mathbb{R}; \frac{1}{\sqrt{2\pi}}e^{-x^{2}/2}dx) \). The following identity:
\[
H_{k}(x + y) = \sum_{\ell=0}^{k} \binom{k}{\ell} x^{k-\ell} H_{\ell}(y),
\]
which, together with (2.56), yields
\[
H_{k}(x + y; \sigma) = \sigma^{\frac{k}{2}} \sum_{\ell=0}^{k} \binom{k}{\ell} \sigma^{-\frac{k-\ell}{2}} x^{k-\ell} H_{\ell}(\sigma^{-\frac{1}{2}} y)
\[
= \sum_{\ell=0}^{k} \binom{k}{\ell} x^{k-\ell} H_{\ell}(y; \sigma).
\] (2.57)
Let \((H, B, \mu)\) be an abstract Wiener space. Namely, \(\mu\) is a Gaussian measure on a separable Banach space \(B\) with \(H \subset B\) as its Cameron-Martin space. Given a complete orthonormal system \(\{e_j\}_{j \in \mathbb{N}} \subset B^*\) of \(H^* = H\), we define a polynomial chaos of order \(k\) to be an element of the form \(\prod_{j=1}^{\infty} H_{k_j}(\langle x, e_j \rangle)\), where \(x \in B\), \(k_j \neq 0\) for only finitely many \(j\)'s, \(k = \sum_{j=1}^{\infty} k_j\), \(H_{k_j}\) is the Hermite polynomial of degree \(k_j\), and \(\langle \cdot, \cdot \rangle_{B^*} = B^*\langle \cdot, \cdot \rangle_B\) denotes the \(B-B^*\) duality pairing. We then denote the closure of polynomial chaoses of order \(k\) under \(L^2(B, \mu)\) by \(H_k\). The elements in \(H_k\) are called homogeneous Wiener chaoses of order \(k\). Then, we have the following Ito-Wiener decomposition:

\[
L^2(B, \mu) = \bigoplus_{k=0}^{\infty} H_k.
\]

See Theorem 1.1.1 in [60]. See also [43, 9]. We also set \(H_{\leq k} = \bigoplus_{j=0}^{k} H_j\) for \(k \in \mathbb{N}\).

We now state the Wiener chaos estimate, which is a consequence of Nelson’s hypercontractivity [59]. See, for example, [84, Theorem I.22]. See also [86, Proposition 2.4].

**Lemma 2.12.** Let \(k \in \mathbb{Z}_{\geq 0}\). Then, we have

\[
\left( \mathbb{E}[|X|^p] \right)^\frac{1}{p} \leq (p - 1)^\frac{1}{2} \left( \mathbb{E}[|X|^2] \right)^\frac{1}{2}
\]

for any random variable \(X \in H_k\) and any \(2 \leq p < \infty\).

Next, we recall the following orthogonal property of Wick powers; see [60] Lemma 1.1.1.

**Lemma 2.13.** Let \(Y_1, Y_2\) be two real-valued, mean-zero, and jointly Gaussian random variables with variances \(\sigma_1 = \mathbb{E}[Y_1^2] > 0\) and \(\sigma_2 = \mathbb{E}[Y_2^2] > 0\). Then, for \(k, m \in \mathbb{N} \cup \{0\}\), we have

\[
\mathbb{E}[H_k(Y_1; \sigma_1)H_m(Y_2; \sigma_2)] = \mathbf{1}_{k=m} \cdot k! (\mathbb{E}[Y_1Y_2])^k.
\]

The following lemma allows us to compute the regularity of a stochastic term by testing it against a test function. See [87, Proposition A.3.3] for the proof.

**Lemma 2.14.** Let \(X \in H_{\leq k}\) for some \(k \in \mathbb{N}\). Let \(R \geq 1\). Suppose that there exist \(\sigma \in \mathbb{R}\) and \(2 \leq q < \infty\) such that

\[
\mathbb{E}\left[\|X\varphi\|_{W^{\sigma,q}}^2\right] \leq A^2\|\varphi\|_{W^{\sigma,q}}^2
\]

for any test function \(\varphi \in C_0^\infty\) supported on a ball \(B\) of radius 1 with \(B \subset B_{2R}\), where \(\frac{1}{q'} + \frac{1}{q} = 1\). Then, given any small \(\varepsilon > 0\) and any finite \(p \geq \frac{4}{\varepsilon}\), we have

\[
\mathbb{E}\left[\|X\|^p_{W^{-\frac{p}{2} - \sigma - \varepsilon, \infty}(B_R)}\right] \lesssim_{\sigma, q, R} A^p.
\]

\(^{19}\)This means that \(\langle X, \varphi \rangle \in H_{\leq k}\) for any test function \(\varphi\).
Wasserstein-1 metric and weak convergence.

Let $(X, d)$ be a Polish space (separable complete metric space). Then, the Wasserstein-1 metric for two probability measures $\mu, \nu$ on $X$ is defined by

$$d_{\text{Wass}}(\mu, \nu) = \inf_{p \in \Pi(\mu, \nu)} \int_{X \times X} d(x, y) dp(x, y).$$

(2.58)

Here, $\Pi(\mu, \nu)$ is the set of probability measures $p$ on $X \times X$ whose first and second marginals are given by $\mu$ and $\nu$, namely,

$$\int_{y \in X} dp(x, y) = d\mu(x) \quad \text{and} \quad \int_{x \in X} dp(x, y) = d\nu(y).$$

(2.59)

The Wasserstein-1 metric is also known as the Kantorovich-Rubinstein distance; see the Kantorovich-Rubinstein theorem ([93, Theorem 1.14]) which provides the dual characterization of (2.58). We point out that the infimum in (2.58) is indeed attained; see [93, Theorem 1.3].

In general, convergence in the Wasserstein-1 metric is stronger than weak convergence. We recall the following characterization of convergence in the Wasserstein-1 metric; see [93, Theorem 7.12].

**Lemma 2.15.** Let $(X, d)$ be a Polish space. Given a sequence of probability measures $\{\mu_n\}_{n \in \mathbb{N}}$ on $X$ and a probability measure $\mu$ on $X$, the sequence $\{\mu_n\}_{n \in \mathbb{N}}$ converges to $\mu$ in the Wasserstein-1 metric as $n \to \infty$ if and only if

(i) $\mu_n$ converges weakly to $\mu$ as $n \to \infty$, and

(ii) for some (and thus for any) $x_0 \in X$, we have

$$\lim_{R \to \infty} \limsup_{n \to \infty} \int_{d(x, x_0) \geq R} d(x, x_0) d\mu_n(x) = 0.$$  

(2.60)

**Remark 2.16.** When the metric $d$ on $X$ is bounded, the condition (2.60) trivially holds, and thus convergence in the Wasserstein-1 metric coincides with weak convergence. See also Remark 7.13 (iii) in [93].

Lastly, we recall the Prokhorov theorem and the Skorokhod representation theorem. We first recall the definition of tightness.

**Definition 2.17.** Let $\mathcal{J}$ be a nonempty index set. A family $\{\mu_n\}_{n \in \mathcal{J}}$ of probability measures on a metric space $\mathcal{M}$ is said to be tight if, for every $\varepsilon > 0$, there exists a compact set $K_\varepsilon \subset \mathcal{M}$ such that $\sup_{n \in \mathcal{J}} \mu_n(K_\varepsilon^{c}) \leq \varepsilon$. We say that $\{\mu_n\}_{n \in \mathcal{J}}$ is relatively compact, if every sequence in $\{\mu_n\}_{n \in \mathcal{J}}$ contains a weakly convergent subsequence.

We now recall the following Prokhorov theorem from [8].

**Lemma 2.18 (Prokhorov theorem).** If a sequence of probability measures on a metric space $\mathcal{M}$ is tight, then it is relatively compact. If in addition, $\mathcal{M}$ is separable and complete, then relative compactness is equivalent to tightness.

Lastly, we recall the following Skorokhod representation theorem from [5, Chapter 31].

**Lemma 2.19 (Skorokhod representation theorem).** Let $\mathcal{M}$ be a complete separable metric space (i.e. a Polish space). Suppose that a sequence $\{\mu_n\}_{n \in \mathbb{N}}$ of probability measures on $\mathcal{M}$
converges weakly to a probability measure $\mu$ as $n \to \infty$. Then, there exist a probability space $(\bar{\Omega}, \bar{F}, \bar{\mathbb{P}})$, and random variables $X_n, X : \bar{\Omega} \to \mathcal{M}$ such that
\[
\text{Law}(X_n) = \mu_n \quad \text{and} \quad \text{Law}(X) = \mu,
\]
and $X_n$ converges $\bar{\mathbb{P}}$-almost surely to $X$ as $n \to \infty$.

3. Coming down from infinity

In this section, we study the following stochastic nonlinear heat equation (SNLH) on $\mathbb{R}^2$:
\[
\begin{cases}
\partial_t X + (1 - \Delta)X + :X^k: = \sqrt{2} \xi \\
X|_{t=0} = X_0,
\end{cases}
\tag{3.1}
\]
where $k \in 2\mathbb{N} + 1$ and $\xi$ is a Gaussian space-time white noise on $\mathbb{R}_+ \times \mathbb{R}^2$. Let $Z$ denote the solution to the following linear equation:
\[
\begin{cases}
\partial_t Z + (1 - \Delta)Z = \sqrt{2} \xi \\
Z|_{t=0} = 0.
\end{cases}
\tag{3.2}
\]
Namely, $Z$ is the stochastic convolution, formally given by
\[
Z(t) = \sqrt{2} \int_0^t P(t - t') \xi(dt'),
\]
where $P(t) = e^{t(\Delta - 1)}$ is the linear heat propagator. Then, with the first order expansion
\[
X = Y + Z,
\tag{3.3}
\]
the remainder term $Y = X - Z$ satisfies
\[
\begin{cases}
\partial_t Y + (1 - \Delta)Y + \sum_{\ell=0}^k \binom{k}{\ell} :Z^\ell: Y^{k-\ell} = 0 \\
Y|_{t=0} = X_0,
\end{cases}
\tag{3.4}
\]
where $:Z^\ell:$ denotes the Wick-renormalized power of $Z$. See Section 5 in [55] for a further discussion. We say that $X$ is a solution to (3.1) if it satisfies (3.3) together with (3.2) and (3.4).

Our main goal in this section is to establish coming down from infinity for a solution $Y$ to (3.4) in weighted Sobolev spaces of positive regularities (Proposition 3.3), which will play a crucial role in Subsection 4.2. For this purpose, we consider
\[
\begin{cases}
\partial_t Y + (1 - \Delta)Y + \sum_{\ell=0}^k Z^\ell Y^{k-\ell} = 0 \\
Y|_{t=0} = X_0,
\end{cases}
\tag{3.5}
\]
where $Z^{(0)} = 1$ and $Z^{(\ell)}, \ell = 1, \ldots, k$, are given space-time distributions. We first establish coming down from infinity for a solution $Y$ to (3.5) in weighted Lebesgue spaces in Subsection 3.1. In Subsection 3.2, we then establish coming down from infinity in weighted Sobolev spaces of positive regularities (Proposition 3.3). As a corollary to the coming down from infinity in weighted Lebesgue spaces, we construct a limiting Gibbs measure $\bar{\rho}_\infty$ on the plane $\mathbb{R}^2$ (Theorem 1.2(i)); see Subsection 3.3. We point out that while the construction of a limiting Gibbs measure on $\mathbb{R}^2$ requires only the coming down from infinity in weighted Lebesgue spaces (Proposition 3.1), the coming down from infinity in weighted Sobolev spaces of positive regularity (Proposition 3.3) plays an essential role in the construction of the enhanced Gibbs measure as well as global-in-time dynamics on $\mathbb{R}^2$ presented in Section 4.
In the following discussion, the regularity of the initial data \( X_0 \) in (3.5) does not play an important role (which is precisely the point of coming down from infinity) as long as a solution \( X = Y + Z \) exists. For this reason, we do not precisely state the regularity of the initial data \( X_0 \) in Propositions 3.1 and 3.3. For our application to (3.1) (and its \( L \)-periodic counterpart (3.42)), it suffices to take \( X_0 \in B_{p_0,\infty}^{-\varepsilon,\mu} (\mathbb{R}^2) \) for some \( \varepsilon > 0, p_0 \gg 1, \) and \( \mu > 0 \) so that a global solution is guaranteed to exist; see (3.5). In particular, the initial data distributed by the \( L \)-periodic \( \Phi_k^{k+1} \)-measure \( \rho_L \) in (1.21) almost surely satisfies this regularity condition.

3.1. Coming down from infinity on weighted Lebesgue spaces. In this subsection, our main goal is to prove the following coming down from infinity for a solution \( Y \) to (3.5) in the weighted Lebesgue space \( L^p_\mu (\mathbb{R}^2) \). As compared to the previous work [92] on \( T^2 \), we need to proceed with more care, using a decomposition of the physical space into unit cubes and the Littlewood-Paley decomposition; see (3.16)-(3.26).

**Proposition 3.1.** Let \( k \in 2\mathbb{N} + 1 \) and \( T > 0 \). Let \( Y \) be a solution to (3.5) on the time interval \( [0, T] \). Then, given any finite \( p \geq 1 \) and \( \mu > 0 \), there exist \( \lambda > 1 \), small \( \varepsilon > 0 \), finite \( p_0 \gg 1 \), and \( 0 < \mu_0 < \mu \) such that

\[
\| Y(t) \|_{L^p_\mu} \leq C \left\{ t^{\lambda - 1} p \vee \sum_{\ell = 0}^{k} \| Z^{\ell}(t) \|_{L^p_{B_{p_0, p_0}^{-\varepsilon, \mu}}} \right\} \tag{3.6}
\]

for \( 0 < t \leq T \), where the constant \( C \) is independent of the initial condition \( X_0 \) in (3.5) and \( T > 0 \).

For the later use, we set

\[
C_{Z, p, \mu}(t) = \sup_{0 \leq t' \leq t} \sum_{\ell = 0}^{k} \| Z^{\ell}(t') \|_{L^p_{B_{p_0, p_0}^{-\varepsilon, \mu}}}. \tag{3.7}
\]

Before proceeding to the proof of Proposition 3.1, we first recall the following key lemma; see Lemma 3.8 in [92].

**Lemma 3.2.** Let \( F : [0, T] \rightarrow [0, \infty) \) be a differentiable function. Suppose that there exist \( \lambda > 1 \) and \( c_1, c_2 > 0 \) such that

\[
\partial_t F(t) + c_1 F^\lambda(t) \leq c_2 \tag{3.8}
\]

for any \( 0 \leq t \leq T \). Then, we have

\[
F(t) \leq \frac{F(0)}{(1 + t F^{\lambda - 1}(0)(\lambda - 1) \frac{c_2}{2})^{\lambda^{-1}}} \left( \frac{2 c_2}{c_1} \right)^{\lambda^{-1}} \leq \left\{ t^{-\frac{1}{\lambda - 1}} \left( \frac{\lambda - 1}{2} \right)^{\lambda^{-1}} \right\} \vee \left( \frac{2 c_2}{c_1} \right)^{\lambda^{-1}}
\]

for any \( 0 < t \leq T \).

Lemma 3.2 shows that in order to establish coming down from infinity, it suffices to establish a bound of the form (3.8).

We now present the proof of Proposition 3.1.
Proof of Proposition 3.1. By Hölder’s inequality, it suffices to consider $p \in 2\mathbb{N}$. Fix $p \in 2\mathbb{N}$. Then, using the equation (3.5), we have

$$
\frac{1}{p} \partial_t \|Y(t)\|_{L^p}^p = \langle \partial_t Y(t), Y^{p-1}(t) w_\mu \rangle_{L^2_x} \\
= \left\langle \Delta Y(t) - Y(t) - \sum_{\ell=0}^{k} Z^{(\ell)}(t) Y^{k-\ell}(t), Y^{p-1}(t) w_\mu \right\rangle_{L^2_x}.
$$

(3.9)

For simplicity of notation, we will drop the time dependence in the following computations. From (2.11) with $0 < \delta < 1$, we have

$$
\langle \Delta Y, Y^{p-1} w_\mu \rangle_{L^2} = - \int_{\mathbb{R}^2} \nabla Y \cdot \nabla (Y^{p-1} w_\mu) dx \\
= -(p-1) \int_{\mathbb{R}^2} |\nabla Y|^{2} Y^{p-2} w_\mu dx \\
+ \mu \delta \int_{\mathbb{R}^2} (\nabla Y \cdot x) Y^{p-1}(x)^{\delta-2} w_\mu dx \\
=: -(p-1) K_t + B_0.
$$

(3.10)

Since $p$ is even, we have

$$
\langle -Y, Y^{p-1} w_\mu \rangle_{L^2} = - \int_{\mathbb{R}^2} Y^{p} w_\mu dx \leq 0.
$$

(3.11)

As for the contribution from the last term in (3.9), recalling that $Z^{(0)} = 1$, we write

$$
\left\langle - \sum_{\ell=0}^{k} Z^{(\ell)} Y^{k-\ell}, Y^{p-1} w_\mu \right\rangle_{L^2} \\
= - \int_{\mathbb{R}^2} Y^{p+k-1} w_\mu dx - \sum_{\ell=1}^{k} \int_{\mathbb{R}^2} Y^{p+k-\ell-1} Z^{(\ell)} w_\mu dx \\
=: -L_t - \sum_{\ell=1}^{k} B_\ell.
$$

(3.12)

Since $p \in 2\mathbb{N}$ and $k \in 2\mathbb{N} + 1$, we have $K_t$ and $L_t$ are non-negative. In the following, we control the terms $B_\ell$, $\ell = 0, \ldots, k$, by these non-negative terms. For this reason, we set

$$
M_t = K_t + L_t.
$$

(3.13)

We first treat $B_0$ in (3.10). In view of the fast decay of the weight $w_\mu$, it follows from Hölder’s inequality that

$$
\|Y\|_{L^p_t}^p \leq C_{\mu, \delta} \|Y\|_{L^{p+k-1}_t}^p = C_{\mu, \delta} L_t^{\frac{p}{p+k-1}} \leq C_{\mu, \delta} M_t^{\frac{p}{p+k-1}}.
$$

(3.14)
Then, by recalling $0 < \delta < 1$ and applying Cauchy-Schwarz’s inequality and Young’s inequality with (3.14) and (3.13), we have

\[ |B_0| \leq \mu \delta \int_{\mathbb{R}^2} |\nabla Y| \cdot |Y|^{2-\delta-1} w_\mu dx \]

\[ \leq C_{\mu, \delta} \int_{\mathbb{R}^2} |\nabla Y| |Y|^{\frac{p-2}{2}} \cdot |Y|^\frac{p}{2} w_\mu dx \]

\[ \leq C_{\mu, \delta} R_t \|Y\|_{L_p^p}^p \leq \frac{1}{100} K_t + C_{\mu, \delta} \|Y\|_{L_p^p}^p \]

\[ \leq \frac{1}{100} M_t + C_{\mu, \delta}' \]

Next, we consider $B_\ell$, $\ell = 1, \ldots, k$. Let $Q = \left[-\frac{1}{2}, \frac{1}{2}\right]^2$ be the unit cube in $\mathbb{R}^2$ and set $Q_n = n + Q$. Fix small $\varepsilon > 0$ and $r > 1$ sufficiently close to 1. Then, we have

\[ |B_\ell| = \left| \int_{\mathbb{R}^2} Y^{p+k-\ell-1} Z^{(\ell)} w_\mu dx \right| \leq \sum_{n \in \mathbb{Z}^2} \left| \int_{Q_n} Y^{p+k-\ell-1} Z^{(\ell)} w_\mu dx \right| \]

\[ \leq \sum_{n \in \mathbb{Z}^2} \|Y^{p+k-\ell-1} w_\mu\|_{W^{\varepsilon, r}(Q_n)} \|Z^{(\ell)}\|_{W^{-\varepsilon, r}(Q_n)} \]

where $\frac{1}{r} + \frac{1}{r'} = 1$.

From (2.26) and the Littlewood-Paley decomposition with the uniform boundedness on $L^r(\mathbb{R}^2)$ of the Littlewood-Paley projector $Q_j$, we have

\[ \|Y^{p+k-\ell-1} w_\mu\|_{W^{\varepsilon, r}(Q_n)} \lesssim \|Y^{p+k-\ell-1} w_\mu\|_{B^{2\varepsilon}_{2,\infty}(Q_n)} \]

\[ = \sup_{j \in \mathbb{Z}_{\geq 0}} 2^{2\varepsilon j} \|Q_j (Y^{p+k-\ell-1} w_\mu)\|_{L^r(Q_n)} \]

\[ \leq \sup_{j \in \mathbb{Z}_{\geq 0}} 2^{2\varepsilon j} \sum_{j_1, j_2 = 0}^\infty \|Q_j (Q_{j_1} (Y^{p+k-\ell-1}) Q_{j_2} w_\mu)\|_{L^r(Q_n)} \]

\[ \lesssim \sum_{j_1 \geq j_2 + 2} 2^{2\varepsilon j_1} \|Q_{j_1} (Y^{p+k-\ell-1}) Q_{j_2} w_\mu\|_{L^r(Q_n)} \]

\[ + \sum_{j_1 < j_2 + 2} 2^{2\varepsilon j_2} \|Q_{j_1} (Y^{p+k-\ell-1}) Q_{j_2} w_\mu\|_{L^r(Q_n)} \]

\[ =: \text{I} + \text{II}. \]

As for the first term I, it follows from (2.14) and (2.13) that

\[ \text{I} \lesssim \sum_{j_1 \geq j_2 + 2} 2^{2\varepsilon j_1} 2^{-\frac{1}{2} j_2} w_\mu(n) \|Q_{j_1} (Y^{p+k-\ell-1})\|_{L^r(Q_n)} \]

\[ \lesssim w_\mu(n) \|Y^{p+k-\ell-1}\|_{W^{3\varepsilon, r}(Q_n)}. \]

Similarly, we have

\[ \text{II} \lesssim \sum_{j_1 < j_2 + 2} 2^{(2\varepsilon - \frac{1}{2}) j_2} w_\mu(n) \|Q_{j_1} (Y^{p+k-\ell-1})\|_{L^r(Q_n)} \]

\[ \lesssim w_\mu(n) \|Y^{p+k-\ell-1}\|_{W^{3\varepsilon, r}(Q_n)}. \]
In the following, we estimate \( \|Y^{p+k-\ell-1}\|_{W^{3,\varepsilon}(Q_n)} \). Define \( M_t(Q_n) \) by
\[
M_t(Q_n) = \int_{Q_n} |\nabla Y|^2 Y^{p-2} + Y^{p+k-1} dx. \tag{3.20}
\]
By the interpolation (2.2), we have
\[
\|Y^{p+k-\ell-1}\|_{W^{3,\varepsilon}(Q_n)} \lesssim \|Y^{p+k-\ell-1}\|_{L^r(Q_n)}^{1-3\varepsilon} \|Y^{p+k-\ell-1}\|_{W^{1,r}(Q_n)}^{3\varepsilon}
\sim \|Y^{p+k-\ell-1}\|_{L^r(Q_n)}^{1-3\varepsilon} \|
abla (Y^{p+k-\ell-1})\|_{L^r(Q_n)}^{3\varepsilon} + \|Y^{p+k-\ell-1}\|_{L^r(Q_n)}. \tag{3.21}
\]
By Hölder’s inequality and by choosing \( r > 1 \) sufficiently close to 1, we have
\[
\|Y^{p+k-\ell-1}\|_{L^r(Q_n)} = \|Y\|_{L^{(p+k-\ell-1)r}(Q_n)}^{p+k-\ell-1} \leq M_t(Q_n) \tag{3.22}
\]
for any \( \ell = 1, \ldots, k \). On the other hand, by Hölder’s inequality, we have
\[
\|
abla (Y^{p+k-\ell-1})\|_{L^r(Q_n)} \sim \|
abla Y \cdot |Y|^{\frac{p-2}{2}} Y^{\frac{p}{2}+k-\ell-1}\|_{L^r(Q_n)}
\leq M(Q_n)^{\frac{1}{2}} \|Y^{\frac{p}{2}+k-\ell-1}\|_{L^{\frac{2r}{p+r-2}}(Q_n)}
\leq M(Q_n)^{\frac{1}{2}} \||Y|^{\frac{p}{2}}\|_{L^q(Q_n)}^{1-\frac{p}{2r}} + M_t(Q_n)^{\frac{1}{2}}, \tag{3.23}
\]
where \( q = \frac{2}{p} \cdot \frac{2r}{p+r-2} (\frac{p}{2} + k - \ell - 1) \). By Sobolev’s inequality\(^\text{20}\) the interpolation (2.2), and Young’s inequality followed by Hölder’s inequality, we have
\[
\|Y\|^{\frac{p}{2}}_{L^q(Q_n)} \lesssim \|Y\|_{L^2(Q_n)}^{\frac{p}{2}} + \|Y\|_{L^2(Q_n)}^{\frac{p-2}{2}} \|
abla Y\|_{L^2(Q_n)} \leq M_t(Q_n)^{\frac{1}{2}} + M_t(Q_n)^{\frac{1}{2}} \tag{3.24}
\]
Hence, from (3.21), (3.22), (3.23), and (3.24), we obtain
\[
\|Y^{p+k-\ell-1}\|_{W^{3,\varepsilon}(Q_n)} \lesssim M_t(Q_n)^{1-\theta} + 1 \tag{3.25}
\]
for some \( \theta > 0 \), provided that \( \varepsilon > 0 \) is sufficiently small, where the implicit constant is independent of \( \ell = 1, \ldots, k \).

Hence, putting (3.16), (3.17), (3.18), (3.19), and (3.25) together and applying Young’s inequality with (2.26), we have
\[
|B_\ell| \lesssim \sum_{n \in \mathbb{Z}^2} w_\mu(n) \left( M_t(Q_n)^{1-\theta} + 1 \right) \|Z^{(\ell)}\|_{W^{-\varepsilon,r}(Q_n)}
\leq \varepsilon_0 \sum_{n \in \mathbb{Z}^2} w_\mu(n) M_t(Q_n) + C_{\varepsilon_0} \sum_{n \in \mathbb{Z}^2} w_\mu(n) \|Z^{(\ell)}\|_{W^{-\varepsilon,r}(Q_n)}^{\frac{1}{2}} + C_1 \tag{3.26}
\lesssim \varepsilon_0 M_t + C_{\varepsilon_0} \|Z^{(\ell)}\|_{B_{p_0,p_0}}^{p_0} + C_2
\]
for some small \( \varepsilon_0 > 0 \), finite \( p_0 \gg 1 \), and \( \mu_0 < \mu \). Here, the last step follows from (3.13), (3.20), and (2.13) for the first term, while it follows from (2.25) and (2.13) for the second term.

\(^{20}\)When \( \ell = k \), we have \( q \leq 2 \) for \( r > 1 \) sufficiently close to 1, and thus there is no need to apply Sobolev’s inequality.
Therefore, from (3.10), (3.11), (3.12), (3.15), and (3.26), we obtain
\[
\frac{1}{p} \partial_t \|Y(t)\|_{L^p_t}^p + c_0 M_t \leq \sum_{\ell=0}^{k} \|Z^{(\ell)}(t)\|_{B_{p_0,\mu_0}}^{p_0}.
\]

Finally, in view of (3.14), there exists \(\lambda > 1\) such that
\[
\frac{1}{p} \partial_t \|Y(t)\|_{L^p_t}^p + c_1 \|Y(t)\|_{L^p_t}^p \leq \sum_{\ell=0}^{k} \|Z^{(\ell)}(t)\|_{B_{p_0,\mu_0}}^{p_0}
\]
(3.27) for any \(0 \leq t \leq T\). Finally, the desired bound (3.6) follows from (3.27) and Lemma 3.2. This concludes the proof of Proposition 3.1.

\(\square\)

3.2. Coming down from infinity on weighted Sobolev spaces of positive regularities. In this subsection, we establish the following coming down from infinity for a solution \(Y\) to (3.5) in a weighted Sobolev space of positive regularity.

**Proposition 3.3.** Let \(k \in 2\mathbb{N} + 1\) and \(T > 0\). Let \(Y\) be a solution to (3.5) on the time interval \([0, T]\). Then, given any \(0 < s < 1\), finite \(p \geq 1\), and \(\mu > 0\), there exist \(\lambda, \lambda' > 1\), small \(\theta > 0\), finite \(q_0, q \gg 1\), and \(0 < \mu', \mu_1 < \mu\) such that
\[
\|Y(t)\|_{W^{s,p}_\mu} \leq C(T) \left\{ t^{-\frac{s}{2}} \left( t^{-\frac{1}{(s-1)p}} \vee C_{Z,p,\mu'}(t) \right) \right. \\
+ t^{-\frac{s}{q}} \vee \left( C_{Z,q_0,\mu}(t) \right)^{q_0} + Q_{Z,\theta,q,\mu_1}(t) + 1 \right\}
\]
(3.28)

for \(0 < t \leq T\), where the constant \(C(T)\) is independent of the initial condition \(X_0\) in (3.5). Here, \(C_{Z,p,\mu}\) is as in (3.7) and
\[
Q_{Z,\theta,q,\mu_1}(t) = \sup_{0 \leq t' \leq t} \sum_{\ell=0}^{k} \|Z^{(\ell)}(t')\|_{B_{q,\mu_0,\mu_1}}^q .
\]

(3.29)

The proof of Proposition 3.3 is based on a Gronwall-type argument, utilizing the already established coming down from infinity in weighted Lebesgue spaces (Proposition 3.1). While the idea is straightforward, an actual implementation requires careful analysis via a spatial decomposition (analogous to the proof of Lemma 2.5), for which we find our definition (2.22) of the weighted Sobolev spaces \(W^{s,p}_\mu(\mathbb{R}^2)\) more convenient (than \(W^{s,p}_\mu(\mathbb{R}^2)\) defined in (2.31)).

**Proof.** Fix \(0 < r < t\). Then, from (3.5) and (2.24), we have
\[
\|Y(t)\|_{W^{s,p}_\mu} \leq \|P(t-r)Y(r)\|_{W^{s,p}_\mu} + \sum_{\ell=0}^{k} \int_{r}^{t} \|P(t-t')(Z^{(\ell)}Y^{k-\ell})(t')\|_{W^{s,p}_\mu} dt' \\
\leq \|P(t-r)Y(r)\|_{W^{s,p}_\mu} + \sum_{\ell=0}^{k} \sum_{j=0}^{\infty} e^{-\frac{\mu}{2} 2^{j\delta}} \int_{r}^{t} \|\phi_j P(t-t')(Z^{(\ell)}Y^{k-\ell})(t')\|_{W^{s,p}_\mu} dt' .
\]

(3.30)

Let \(F_{k,\ell} = Z^{(\ell)}Y^{k-\ell}\). We proceed as in the proof of Lemma 2.5 and estimate
\[
\|\phi_j P(t-t')F_{k,\ell}\|_{W^{s,p}_\mu}.
\]
With (2.35), the fractional Leibniz rule (Lemma 2.8(i)), Young’s inequality, and (2.37), we have
\[
\|\phi_j P(t)F_{k,\ell}\|_{W^{s,p}} \lesssim e^{-t} \sum_{(m,m') \in \Lambda_{t,j}} \|\phi_j \left[ (\phi_{m'}^t P_t) \ast (\phi_{m'} F_{k,\ell}) \right]\|_{W^{s,p}} \\
\lesssim e^{-t} \sum_{(m,m') \in \Lambda_{t,j}} \|\phi_j\|_{W^{s,\infty}} \| (\phi_{m'}^t P_t) \ast (\phi_{m'} F_{k,\ell}) \|_{W^{s,p}} \\
\lesssim e^{-t} \sum_{(m,m') \in \Lambda_{t,j}} \|\phi_{m'} F_{k,\ell}\|_{W^{s+\theta,1}} \|\phi_{m'} F_{k,\ell}\|_{W^{-\theta,p}} \\
\lesssim e^{-t} \sum_{(m,m') \in \Lambda_{t,j}} t^{-\frac{\mu}{2} + \frac{\theta}{2}} e^{-c_1 t^m} \|\phi_{m'} F_{k,\ell}\|_{W^{-\theta,p}} 
\] (3.31)
for any $\theta > 0$, where $\phi_{m'}^t$ and $\Lambda_{t,j}$ are as in (2.34) and (2.36). In the following, we choose $\theta > 0$ sufficiently small such that $s + \theta < 1$.

From (2.21), Lemma 2.8(ii) (with finite $q \gg 1$), the fractional Leibniz rule (Lemma 2.8(i) with $q_0 \gg 1$), the interpolation (2.22), and Hölder’s inequality with (2.18), we have
\[
\|\phi_{m'} F_{k,\ell}\|_{W^{-\theta,p}} = \| (\tilde{\phi}_{m'} Y)^{k-\ell} \cdot \phi_{m'} Z^{(l)} \|_{W^{-\theta,p}} \\
\lesssim \| (\tilde{\phi}_{m'} Y)^{k-\ell} \|_{W^{s,p}} \| \phi_{m'} Z^{(l)} \|_{W^{-\theta,q}} \\
\lesssim \| \tilde{\phi}_{m'} Y \|_{L_{t,0}^{\theta}} \| \phi_{m'} Y \|_{W^{\theta,p}} \| \phi_{m'} Z^{(l)} \|_{W^{-\theta,q}} \\
\lesssim \| \tilde{\phi}_{m'} Y \|_{L_{t,0}^{\theta}} \| \phi_{m'} Y \|_{W^{\theta,p}} \| \phi_{m'} Z^{(l)} \|_{W^{-\theta,q}} \\
\times 2^{\frac{1}{p} - \frac{1}{q_0}} (1 - \frac{\theta}{2}) 2^{m'} \| \phi_{m'} Y \|_{L_{t,0}^{\theta}} \| \phi_{m'} Z^{(l)} \|_{W^{-\theta,q}}, 
\] (3.32)
provided that $0 < \theta \leq s$. Then, it follows from (3.32) with (2.23) (for $\tilde{\phi}_{m'}$ defined in (2.20)) that
\[
\|\phi_{m'} F_{k,\ell}\|_{W^{-\theta,p}} \lesssim 2^{\frac{1}{p} - \frac{1}{q_0}} (1 - \frac{\theta}{2}) 2^{m'} \| \phi_{m'} Y \|_{L_{t,0}^{\theta}} \| Y \|_{L_{t,0}^{q}} \| Z^{(l)} \|_{W^{-\theta,q}} \\
\times e^{\frac{\theta}{q} 2^{m'}} \| Y \|_{L_{t,0}^{q}} \| Z^{(l)} \|_{W^{-\theta,q}}. 
\] (3.33)
Note that, given any small $\kappa_0 > 0$, there exists $C(p, q_0, s, \theta, \kappa_0) > 0$ such that
\[
2^{\frac{1}{p} - \frac{1}{q_0}} (1 - \frac{\theta}{2}) 2^{m'} \leq C(p, q_0, s, \theta, \kappa_0) e^{\kappa_0 2^{m'}} 
\] (3.34)
for any $m' \in \mathbb{Z}_{\geq 0}$. Hence, from (3.31) and (3.33) with (3.34), we have
\[
\sum_{j=0}^\infty e^{-\frac{\mu}{p} 2^{j\ell}} \|\phi_j P(t)F_{k,\ell}\|_{W^{s,p}} \\
\lesssim e^{-\frac{\mu}{p} 2^{j\ell}} e^{-c_1 m'} e^{\frac{\theta}{q} 2^{m'}} \sum_{j=0}^\infty \sum_{(m,m') \in \Lambda_{t,j}} e^{-c_1 t^m} \| Y \|_{L_{t,0}^{\theta}} \| Z^{(l)} \|_{W^{-\theta,q}}, 
\]
where \( \tilde{\mu} \) is given by
\[
\tilde{\mu} = \mu \frac{a_1}{q_0} \left( k - \ell - \frac{\theta}{s} \right)^2 + \frac{\mu \theta}{p} 2^\delta + \frac{\mu}{q} + \kappa_0.
\]

By choosing sufficiently large \( q_0, q \gg 1 \), \( 0 < \theta < s \), and sufficiently small \( \kappa_0 > 0 \), we have \( p\tilde{\mu} \ll \mu \). Then, by summing over \( m' \) with (2.36) (as in the proof of Lemma 2.5), then summing over \( m \) (with (2.40); see also (2.41)), and finally summing over \( j \) with \( p\tilde{\mu} \ll \mu \), we obtain
\[
\sum_{j=0}^{\infty} e^{-\frac{2j^2}{p} t} \|\phi_j P(t) F_{k,\ell} \| W^{s,p}_\mu
\]
\[
\leq t^{-\frac{\theta}{2}} e^{-t} \sum_{j,m=0}^{\infty} e^{-c_1 4^m e^{c_0 \tilde{\mu}^2 + 2^m j^2} - \frac{\theta}{2}} \| Y \| L^\theta_{\tilde{\mu},p} \| Z(f) \| W^{s,q}_\mu
\]
\[
\leq t^{-\frac{\theta}{2}} e^{-c_1} \sum_{j,m=0}^{\infty} e^{-c_0 (\tilde{\mu}^2 + 2^m j^2)} \| Y \| L^\theta_{\tilde{\mu},p} \| Z(f) \| W^{s,q}_\mu
\]
\[
\leq t^{-\frac{\theta}{2}} e^{-c_1} \| Y \| L^\theta_{\tilde{\mu},p} \| Z(f) \| W^{s,q}_\mu
\]
for any \( t > 0 \).

Therefore, from (3.30), (3.35), Young’s inequality, and Lemma 2.3 (with some \( 0 < \mu_1 < \mu \), we have
\[
\| Y(t) \| W^{s,p}_\mu \lesssim \| P(t-r) Y(r) \| W^{s,p}_\mu
\]
\[
+ k \int_r^t (t-t')^{-\frac{\theta}{2}} \| Y(t') \| L^\theta_{\tilde{\mu},p} \| Z(f) \| W^{s,q}_\mu dt'
\]
\[
\lesssim \| P(t-r) Y(r) \| W^{s,p}_\mu + \int_r^t (t-t')^{-\frac{\theta}{2}} \| Y(t') \| W^{s,p}_\mu dt'
\]
\[
+ \int_r^t (t-t')^{-\frac{\theta}{2}} \left( \| Y(t') \| L^\theta_{\tilde{\mu},p} + \sum_{l=0}^k \| Z(f) (t') \| B_{q,\tilde{\mu},l} + 1 \right) dt'.
\]
From Proposition 3.1 with (3.7), there exists \( \lambda = \lambda(q_0, \mu) > 1 \) such that
\[
\| Y(t') \| L^\theta_{\tilde{\mu},p} \lesssim (t')^{-\frac{1}{\lambda-1} q_0} \vee C_{Z,q_0,\mu}(t')
\]
\[
\leq r^{-\frac{1}{\lambda-1} q_0} \vee C_{Z,q_0,\mu}(t)
\]
for \( r \leq t' \leq t \). Then, from (3.36), Lemma 2.5 (3.37), and (3.29), we have
\[
\| Y(t) \| W^{s,p}_\mu \lesssim (t-r)^{-\frac{\theta}{2}} \| Y(r) \| L^\theta_{\tilde{\mu}}
\]
\[
+ C(T) \left( r^{-\frac{1}{\lambda-1}} \vee (C_{Z,q_0,\mu}(t))^q_0 + Q_{Z,q,\mu}(t) + 1 \right)
\]
\[
+ \int_r^t (t-t')^{-\frac{\theta}{2}} \| Y(t') \| W^{s,q}_\mu dt'.
\]
Let
\[ B(t, r) = (t - r)^{-\frac{3}{2}} \|Y(r)\|_{L^p_\mu}^2 + C(T) \left( r^{-\frac{1}{2}} \left( t^{-\frac{3}{2}} \|C_{Z, \rho_0, \mu}(t)\|_{\tilde{W}^{s_0, 0}} Q_{X, \rho, \mu}(t) + 1 \right) \right). \]  
\[ (3.39) \]

Then, recalling that \( s + \theta < 1 \), it follows from (3.38) and Cauchy-Schwarz’s inequality that
\[ \|Y(t)\|_{W^{s, p}}^2 \leq CB(t, r)^2 + C(T) \int_r^t \|Y(t')\|_{W^{s, p}}^2 dt'. \]
Hence, by Gronwall’s inequality, we obtain
\[ \|Y(t)\|_{W^{s, p}} \leq C(T)B(t, r) \]  
\[ (3.40) \]
for any \( r \leq t \leq T \). Finally, by choosing \( r = t^\epsilon \), it follows from (3.40), (3.39) and Proposition 3.1 that
\[ \|Y(t)\|_{W^{s, p}} \leq C(T) \left\{ t^{-\epsilon} \left( t^{-\frac{1}{2}} \|C_{Z, \rho_0, \mu}(t)\|_{\tilde{W}^{s_0, 0}} Q_{X, \rho, \mu}(t) + 1 \right) \right\} \]
for some \( \lambda' = \lambda(p, \mu') > 1 \). This concludes the proof of Proposition 3.3. \( \square \)

3.3. Construction of the Gibbs measure \( \tilde{\rho}_\infty \) on the plane. We conclude this section by presenting the proof of Theorem 1.2(i).

Let \( X \) be a global-in-time solution to (3.1) on \( \mathbb{R}^2 \) constructed in [55], satisfying the decomposition \( X = Y + Z \) in (3.3) with \( Y \) and \( Z \) satisfying (3.4) and (3.2), respectively. Fix \( s < 0 \), finite \( p \geq 1 \), and \( \mu > 0 \). Then, by Lemma 2.3 (2.32) with \( s = 0 \), and Proposition 3.1 we have
\[ \|X(t)\|_{W^{s, p}} \lesssim \|Z(t)\|_{W^{s, p}} + \|Y(t)\|_{L^p_\mu} \]
\[ \lesssim \|Z(t)\|_{P^p_{\rho, p}} + \left\{ t^{-\frac{1}{2}} \|C_{Z, \rho_0, \mu}(t)\|_{\tilde{W}^{s_0, 0}} Q_{X, \rho, \mu}(t) \right\}, \]
\[ (3.41) \]
for a suitable choice of parameters \( \mu', \mu'', \lambda, \rho, \varepsilon, \) and \( \rho_0, \mu_0 \), depending on \( p \) and \( \mu \), where the implicit constant is independent of the initial data \( X_0 \).

Given \( L \geq 1 \), let \( \xi_L \) denote the (spatially) \( L \)-periodic space-time white noise on \( \mathbb{R}^2 \times \mathbb{R}_+ \) obtained by first restricting the space-time white noise \( \xi \) on \( \mathbb{R}^2 \times \mathbb{R}_+ \), appearing in (3.1), onto \([ -\frac{L}{2}, \frac{L}{2}]^2 \times \mathbb{R}_+ \) and then extending it periodically (with the spatial period \( L \)) onto \( \mathbb{R}^2 \times \mathbb{R}_+ \). See Section 5 in [55] for a further discussion. See also (4.4) and (4.5) below. We now consider the following SNLH with the \( L \)-periodic space-time white noise:
\[ \begin{cases} \partial_t X_L + (1 - \Delta) X_L + :X_L^k = \sqrt{2} \xi_L \\ X_L|_{t=0} = X_{0, L} \end{cases} \]
\[ (3.42) \]
where the initial data \( X_{0, L} \) is assumed to be \( L \)-periodic. This is the parabolic \( \Phi_2^{k+1} \)-model on \( \mathbb{T}_L^2 \) studied in [22] (with \( L = 1 \)). Write \( X_L \) as
\[ X_L = Y_L + Z_L, \]
\[ (3.43) \]
where $Z_L$ satisfies
\[
\begin{cases}
\partial_t Z_L + (1 - \Delta) Z_L = \sqrt{2} \xi_L \\
Z_L|_{t=0} = 0
\end{cases}
\] (3.44)
and $Y_L$ satisfies
\[
\begin{cases}
\partial_t Y_L + (1 - \Delta) Y_L + \sum_{\ell=0}^{k} \binom{k}{\ell} Z_L^\ell : Y_{L-t}^k = 0 \\
Y_L|_{t=0} = X_{0,L}
\end{cases}
\] (3.45)
Then, by applying Lemma 2.3 and Proposition 3.1 once again we have
\[
\|X_L(t)\|_{W^{s,p}_\mu} \lesssim \|Z_L(t)\|_{W^{s,p}_\mu} + \|Y_L(t)\|_{L^p_\mu} \lesssim \|Z_L(t)\|_{B^{s+\varepsilon,\mu'}_p} + \left\{ t^{-\frac{1}{(s-1)p}} \sqrt{\sum_{\ell=0}^{k} \| : Z_L^\ell(t) : \|_{B^{s+\varepsilon,\mu'}_p}} \right\}
\] (3.46)
where the implicit constant is independent of the initial data $X_{0,L}$ and of the period $L \geq 1$.
Moreover, it follows from [55, Theorem 5.1] that the $p$th moment of the right-hand side of (3.46) (and of (3.41)) is bounded, uniformly in $L \geq 1$.
Let $\rho_L$ denote the $L$-periodic $\Phi^{k+1}_2$-measure on $\mathbb{T}^2_L$, appearing in (1.21). By taking Law$(X_{0,L}) = \rho_L$ in (3.42) (and we assume that $X_{0,L}$ is independent of the noise $\xi_L$), it is known [22] that
\[
\text{Law}(X_L(t)) = \rho_L
\] (3.47)
for any $t \geq 0$. Moreover, from (3.46) and the observation mentioned right after (3.46), we have
\[
\sup_{L \geq 1} \mathbb{E} \left[ \|X_L(1)\|_{W^{s,p}_\mu}^p \right] \lesssim 1.
\] (3.48)
Now, given $s < 0$ and $\mu > 0$, let $s < s' < 0$ and $0 < \mu' \ll \mu$. Then, given $M > 0$, set
\[
K_M = \{ f \in W^{s,p}_\mu(\mathbb{R}^2) : \|f\|_{W^{s',p}_{\mu'}} \leq M \}.
\]
Then, it follows from Chebyshev’s inequality and (3.48) that, given $\varepsilon > 0$, there exists $M > 0$ such that
\[
\rho_L(K_M^c) = \mathbb{P}(\|X_L(1; \omega)\|_{W^{s',p}_{\mu'}} > M) \lesssim M^{-p} < \varepsilon,
\]
uniformly in $L \geq 1$. On the other hand, it follows from Lemma 2.22 that $K_M$ is compact in $W^{s,p}_\mu(\mathbb{R}^2)$. Therefore, from the Prokhorov theorem (Lemma 2.18), we conclude that $\{\rho_L\}_{L \in \mathbb{N}}$ is tight and hence admits a subsequence $\{\rho_L\}_{j \in \mathbb{N}}$ which converges weakly to a limiting $\Phi^{k+1}_2$-measure $\rho_\infty$ on $\mathbb{R}^2$ as $j \to \infty$.
Next, we prove weak convergence of the $L$-periodic white noise measure $\mu_{0,L}$ defined in (1.11) (with $s = 0$) to the limiting white noise measure on $\mathbb{R}^2$. Formally, a (spatial) white noise $\zeta$ on $\mathbb{R}^2$ is a centered Gaussian distribution on $\mathbb{R}^2$ with covariance
\[
\mathbb{E}[\zeta(x_1)\zeta(x_2)] = \delta(x_1 - x_2).
\] (3.49)
The expression (3.49) is merely formal but we can make it rigorous by testing it against a test function.
Definition 3.4. A (spatial) white noise $\zeta$ on $\mathbb{R}^2$ is a family of centered Gaussian random variables $\{\zeta(\varphi) : \varphi \in L^2(\mathbb{R}^2)\}$ such that

$$\mathbb{E}[\zeta(\varphi)]^2 = \|\varphi\|^2_{L^2(\mathbb{R}^2)} \quad \text{and} \quad \mathbb{E}[\zeta(\varphi_1)\zeta(\varphi_2)] = \langle \varphi_1, \varphi_2 \rangle_{L^2(\mathbb{R}^2)}.$$ 

Given $k \in \mathbb{N}$, let $\eta_k$ be as in (2.9). Then, a direct computation with Lemma 2.12 shows

$$\mathbb{E}[\zeta(\eta_k)]^p \leq_p \left( \mathbb{E}[\zeta(\eta_k)^{2p}] \right)^{\frac{p}{2}} = \|\eta_k\|^p_{L^2} \sim 2^{kp} \quad (3.50)$$

for any finite $p \geq 1$. Hence, we conclude from (the time-independent version of) Lemma 9 in [55]22 that the white noise $\zeta$ on $\mathbb{R}^2$ belongs almost surely to $B^{s,p}_v(\mathbb{R}^2)$ for any $s < -1$, $\mu > 0$, $1 \leq p < \infty$, and $1 \leq q \leq \infty$, with the following bound:

$$\mathbb{E}[\|\zeta\|_{B^{s,p}_v}] \leq C(r, s, \mu, p, q) < \infty \quad (3.51)$$

for any finite $r \geq 1$. Then, by Lemma 2.23 we see that the white noise $\zeta$ on $\mathbb{R}^2$ also belongs almost surely to the weighted Sobolev space $W^{s,p}_\mu(\mathbb{R}^2)$ for any $s < -1$, $1 \leq p < \infty$, and $\mu > 0$, with a bound:

$$\mathbb{E}[\|\zeta\|_{W^{s,p}_\mu}] \leq C(r, s, \mu, p) < \infty \quad (3.52)$$

for any finite $r \geq 1$.

Lemma 3.5. Let $\zeta$ be a white noise on $\mathbb{R}^2$ as in Definition 3.4 and set $\mu_{0,\infty} = \text{Law}(\zeta)$. Then, given any $s < -1$, finite $p \geq 1$, and $\mu > 0$, by viewing the $L$-periodic white noise measure $\mu_{0,L}$ defined in (1.14) with $s = 0$ as a measure on $W^{s,p}_\mu(\mathbb{R}^2)$, the sequence $\{\mu_{0,L}\}_{L \in \mathbb{N}}$ converges weakly to $\mu_{0,\infty}$ as $L \to \infty$.

By putting the weak convergence of a subsequence $\{\rho_{L_j}\}_{j \in \mathbb{N}}$ with Lemma 3.5 we conclude that as probability measures on $\tilde{W}^{s,p}_\mu(\mathbb{R}^2) = W^{s,p}_\mu(\mathbb{R}^2) \times W^{s-1,p}_\mu(\mathbb{R}^2)$ with $s < 0$, finite $p \geq 1$, and $\mu > 0$, the $L_j$-periodic Gibbs measure $\tilde{\rho}_{L_j} = \rho_{L_j} \otimes \mu_{0,L_j}$ in (1.20) converges weakly to a limiting Gibbs measure $\tilde{\rho}_{\infty} = \rho_{\infty} \otimes \mu_{0,\infty}$ on $\mathbb{R}^2$ as $j \to \infty$. This proves Theorem 1.2 (i).

Remark 3.6. In view of the embedding between weighted Sobolev spaces and weighted Besov spaces stated in Lemma 2.3 the weak convergence of the $L_j$-periodic Gibbs measure $\tilde{\rho}_{L_j}$ to the limiting Gibbs measure $\tilde{\rho}_{\infty}$ also holds as probability measures on the weighted Besov space $B^{s,p,q}_\mu(\mathbb{R}^2) \times B^{s-1,p,q}_\mu(\mathbb{R}^2)$ for any $s < 0$, finite $p \geq 1$, $1 \leq q \leq \infty$, and $\mu > 0$.

We conclude this section by presenting the proof of Lemma 3.5.

Proof of Lemma 3.5. As we have seen in Section 1 an $L$-periodic white noise on the dilated torus $\mathbb{T}^2_L$ is given by the second Gaussian Fourier series for $v_L$ in (1.14) such that $\text{Law}(v_L) = \mu_{0,L}$. While it is possible to work with $v_L$ in (1.14), by viewing it as an $L$-periodic distribution on $\mathbb{R}^2$, and show that the $L$-periodic white noise measure $\mu_{0,L} = \text{Law}(v_L)$ converges weakly to the white noise measure $\mu_{0,\infty} = \text{Law}(\zeta)$ on $\mathbb{R}^2$, it is more convenient to work with the $L$-periodized version $\zeta_L$ of the white noise $\zeta$. Define $\zeta_L$ on $\mathbb{T}^2_L$ by setting

$$\zeta_L = \sum_{\lambda \in \mathbb{Z}^2_L} \tilde{\zeta}_L(\lambda) e^L_\lambda, \quad (3.53)$$

22Lemma 5.2 in the arXiv version.
where \( e^L_\lambda \) is as in (1.10) and \( \zeta^L_L(\lambda) \) is given by
\[
\zeta^L_L(\lambda) = \int_{[-\frac{L}{2}, \frac{L}{2})^2} \zeta(x)e^L_\lambda(x)dx = \zeta \left( \textbf{1}_{[-\frac{L}{2}, \frac{L}{2})^2} e^L_\lambda \right) \\
:= \zeta \left( \textbf{1}_{[-\frac{L}{2}, \frac{L}{2})^2} \text{Re}(e^L_\lambda) \right) - i\zeta \left( \textbf{1}_{[-\frac{L}{2}, \frac{L}{2})^2} \text{Im}(e^L_\lambda) \right).
\] (3.54)

Then, by viewing \( \zeta^L_L \) as an \( L \)-periodic distribution on \( \mathbb{R}^2 \), we see that \( \text{Law}(\zeta^L_L) = \mu_{0,L} \). Indeed, it follows from Definition 3.4 and the orthonormality of \( \{e^L_\lambda\}_{\lambda \in \mathbb{Z}^2_L} \) on \( T^2_L = [-\frac{L}{2}, \frac{L}{2})^2 \) that \( \{\zeta^L_L(\lambda)\}_{\lambda \in \mathbb{Z}^2_L} \) forms a family of independent standard complex-valued Gaussian random variables conditioned that \( \zeta^L_L(-\lambda) = \zeta^L_L(\lambda), \lambda \in \mathbb{Z}^2_L \). Hence, from (1.14) and (3.53), we conclude that \( \text{Law}(\zeta^L_L) = \text{Law}(v^L_L) = \mu_{0,L} \). Before proceeding further, we point out that by repeating the computation (3.50) for \( \zeta^L_L \), we see that the bounds (3.51) and (3.52) also hold for \( \zeta^L_L \), uniformly in \( L \geq 1 \).

By definition, \( \zeta^L_L \) agrees with \( \zeta \) on \( T^2_L = [-\frac{L}{2}, \frac{L}{2})^2 \). Now, fix \( s < -1, 1 \leq p < \infty \), and \( \mu > 0 \). Then, given finite \( r \geq 1 \), it follows from (3.52) for \( \zeta \) and \( \zeta^L_L \) that
\[
\mathbb{E} \left[ \|\zeta - \zeta^L_L\|_{W^{s,p}(\mathbb{R}^2)}^p \right] = \mathbb{E} \left[ \|\zeta - \zeta^L_L\|_{W^{s,p}(T^2_L)}^p \right] \\
\leq C(r, s, \mu', p) \cdot w_{\mu - \mu'}(c_0L)$
\] (3.55)
for any \( 0 < \mu' < \mu \), where \( c_0 > 0 \) is an absolute constant independent of \( L \geq 1 \) (and the other parameters). Then, summing over \( L \in \mathbb{N} \), we obtain
\[
\mathbb{E} \left( \sum_{L=1}^{\infty} \|\zeta - \zeta^L_L\|_{W^{s,p}(\mathbb{R}^2)}^p \right) = \sum_{L=1}^{\infty} \mathbb{E} \left[ \|\zeta - \zeta^L_L\|_{W^{s,p}(\mathbb{R}^2)}^p \right] \\
\leq \sum_{L=1}^{\infty} w_{\mu - \mu'}(c_0L) < \infty,
\]
since \( \mu > \mu' \). This implies immediately that \( \sum_{L=1}^{\infty} \|\zeta - \zeta^L_L\|_{W^{s,p}(\mathbb{R}^2)}^p \) is finite almost surely, which in particular implies that \( \zeta^L_L \) converges almost surely to \( \zeta \) in \( W^{s,p}(\mathbb{R}^2) \) as \( L \to \infty \) (with \( L \in \mathbb{N} \)). Therefore, we conclude that \( \mu_{0,L} = \text{Law}(\zeta^L_L) \) converges weakly to \( \mu_{0,\infty} = \text{Law}(\zeta) \) as \( L \to \infty \) (with \( L \in \mathbb{N} \)). This conclude the proof of Lemma 3.5 \( \square \)

4. Global well-posedness and invariance of the Gibbs measure

In this section, we present the proof of Theorem 1.2 (ii). Our main goal is to construct global-in-time dynamics for the hyperbolic \( \Phi^{k+1}_2 \)-model on \( \mathbb{R}^2 \) with the Gibbsian initial data:
\[
\begin{cases}
\partial_t^2 u + \partial_t u + (1 - \Delta)u + \cdot u^k; = \sqrt{2}\xi \\
(u, \partial_t u)|_{t=0} = (u_0, u_1) \quad \text{with} \quad \text{Law}(u_0, u_1) = \bar{\rho}_\infty,
\end{cases}
\] (4.1)
where \( \bar{\rho}_\infty = \rho_\infty \otimes \mu_{0,\infty} \) is the Gibbs measure on \( \mathbb{R}^2 \), constructed as a limit of the \( L_j \)-periodic Gibbs measure \( \bar{\rho}_{L_j} = \rho_{L_j} \otimes \mu_{0,L_j} \) in the previous section. For simplicity of notation, we set
\[
A = \{L_j : j \in \mathbb{N}\} \subset \mathbb{N}
\]
and only consider the values of \( L \in A \) in the following, unless otherwise specified.
Consider the $L$-periodic hyperbolic $\Phi^{2k+1}$-model on $\mathbb{R}^2$:
\[
\begin{align*}
\partial_s^2 u_L + \partial_s u_L + (1 - \Delta)u_L + : u_L^k : &= \sqrt{2} \xi_L \\
(u_L, \partial_s u_L)|_{t=0} &= (u_{0,L}, u_{1,L}) \quad \text{with Law}(u_{0,L}, u_{1,L}) = \tilde{\rho}_L,
\end{align*}
\]
(4.2)
where $\xi_L$ is an $L$-periodic space-time white noise (see (4.1) and (4.5) below) and $\tilde{\rho}_L = \rho_L \otimes \mu_{0,L}$ is the $L$-periodic Gibbs measure in (1.20); see also Remark 1.1. As mentioned in Section 1, the Cauchy problem (4.2) is globally well-posed and the Gibbs measure $\tilde{\rho}_L$ is invariant under the resulting dynamics. In the following, we construct a solution to (4.1) as a limit of the solutions $u_L$ to (4.2), $L \in \mathcal{A}$.

Let us first introduce some notations. Let $\Psi$ be the solution to the following linear stochastic damped wave equation:
\[
\begin{align*}
\partial_s^2 \Psi + \partial_s \Psi + (1 - \Delta)\Psi &= \sqrt{2} \xi \\
(\Psi, \partial_s \Psi)|_{t=0} &= (0, 0).
\end{align*}
\]
(4.3)
With $\mathcal{D}(t)$ as in (1.23), we formally have
\[
\Psi(t) = \sqrt{2} \int_0^t \mathcal{D}(t - t') \xi(dt').
\]

Let $L \geq 1$. Given a spatial white noise $\zeta$ on $\mathbb{R}^2$, we let $\zeta_L$ denote the $L$-periodized version of $\zeta$ defined in (3.53) and (3.54). Similarly, let $\xi_L$ be the $L$-periodized version (in space) of the space-time white noise $\xi$. Namely, we define $\xi_L$ on $\mathbb{T}_L^2 \times \mathbb{R}_+ \cong [- \frac{L}{2}, \frac{L}{2})^2 \times \mathbb{R}_+$ by setting
\[
\xi_L = \sum_{\lambda \in \mathbb{Z}_L^2} \hat{\xi}_L(\lambda)e_\lambda,
\]
(4.4)
where $e_\lambda$ is as in (1.10) and $\hat{\xi}_L(\lambda)$ is given by
\[
\hat{\xi}_L(\lambda) = \int_{[- \frac{L}{2}, \frac{L}{2})^2} \xi(x)e_\lambda(x)dx = \xi\left(1_{[- \frac{L}{2}, \frac{L}{2})^2} e_\lambda\right)
\]
(4.5)
with the equality understood in the sense of temporal distributions. Alternatively, $\hat{\xi}_L(\lambda)$ is given as the (distributional) time derivative of
\[
\hat{B}_L(\lambda, t) = \xi\left(1_{[0,1]} \cdot 1_{[- \frac{L}{2}, \frac{L}{2})^2}(x) e_\lambda(x)\right),
\]
where the right-hand side is the action of $\xi$ on a function in $L^2(\mathbb{R}^2 \times \mathbb{R}_+)$. It is then easy to check that $\{\hat{\xi}_L(\lambda)\}_{\lambda \in \mathbb{Z}_L^2}$ forms a family of independent temporal white noises conditioned that $\hat{\xi}_L(-\lambda) = \hat{\xi}_L(\lambda)$, $\lambda \in \mathbb{Z}_L^2$. Compare this with (1.24). By the $L$-periodic extension in space, we then view $\xi_L$ as a space-time distribution on $\mathbb{R}^2 \times \mathbb{R}_+$.

In the following, we construct a solution $u$ to (4.1) on the cone $\mathcal{C}_R$ defined in (1.33) for each $R > 0$ as a limit of the solution $u_L$ to (4.2). Fix $R > 0$. Then, thanks to the finite speed of propagation, we have
\[
\zeta_L = \zeta \quad \text{and} \quad \xi_L = \xi \quad \text{on the cone } \mathcal{C}_R,
\]
(4.6)
provided that $L \geq 2R$. In particular, on the cone $\mathcal{C}_R$, the equation (4.2) agrees (in law) with
\[
\begin{align*}
\partial_s^2 u_L + \partial_s u_L + (1 - \Delta)u_L + : u_L^k : &= \sqrt{2} \xi \\
(u_L, \partial_s u_L)|_{t=0} &= (u_{0,L}, u_{1,L}) \quad \text{with Law}(u_{0,L}) = \rho_L,
\end{align*}
\]
(4.7)
We first introduce a space $W$ such that

$$
\text{Let Definition 4.1.}
$$

and we set

$$
\text{Then, by the independence of}
$$

and thus we can rewrite (4.9) as

$$
\text{and}
$$

We also set

$$
\Phi_0(t) = \Phi_0(u_0)(t) = S(t)u_0 \quad \text{and} \quad \Phi_1(t) = \Phi_1(u_1, \xi)(t) = D(t)u_1 + \Psi(t)
$$

such that

$$
\Phi(u_0, u_1, \xi) = \Phi_0(u_0) + \Phi_1(u_1, \xi).
$$

Then, by the independence of $u_0$ and $(u_1, \xi)$ in (4.1), we have

$$
\Phi^\ell(t) = \Phi^\ell(u_0, u_1, \xi)(t) = \sum_{m=0}^{\ell} \left( \begin{array}{c} \ell \\ m \end{array} \right) \Phi^m_0(u_0)(t) : \Phi^\ell-m_1(u_1, \xi)(t):,
$$

and thus we can rewrite (4.9) as

$$
v(t) = -\sum_{\ell=0}^{k} \sum_{m=0}^{\ell} \left( \begin{array}{c} k \\ \ell \end{array} \right) \left( \begin{array}{c} \ell \\ m \end{array} \right) \int_0^t D(t - t')(\Phi^\ell(t') : \Phi^{\ell-m}_1(u_1, \xi)(t') : v^{k-\ell}(t')) dt'.
$$

In view of (4.11), we define enhanced data sets $\Xi_0$ and $\Xi_1$ by setting

$$
\Xi_0(\phi) = (\Phi_0(\phi), \Phi^2_0(\phi), \ldots, \Phi^k_0(\phi)) = (S(t)\phi, (S(t)\phi)^2, \ldots, (S(t)\phi)^k)
$$

and

$$
\Xi_1(u_1, \xi) = (\Phi_1, \Phi^2_1, \ldots, \Phi^k_1),
$$

where $\Phi_1 = \Phi_1(u_1, \xi)$ is as in (4.10).

We now introduce distances / sizes by which we measure these enhanced data sets.

**Definition 4.1.** Let $k \in 2N + 1$. Let $0 < \varepsilon \ll 1$, finite $p \gg 1$, and $\mu > 0$ (to be chosen later).

(i) We first introduce a space $W$ for the first enhanced data set $\Xi_0(\phi)$ in (4.12) by setting

$$
W = \left( C(\mathbb{R}^+; W^{-\varepsilon,p}(\mathbb{R}^2)) \right)^{\otimes k}.
$$

---

More precisely, recall that the product of independent homogenous Wiener chaoses is a homogenous Wiener chaos of the order given by the sum of the two.
We endow \( \mathcal{W} \) with the following bounded metric:

\[
d_{\mathcal{W}}(f, g) = d_{\mathcal{W}}((f_1, \ldots, f_k), (g_1, \ldots, g_k)) = \sum_{\ell=1}^{\infty} \frac{1}{2^\ell} \left( 1 + \sum_{j=1}^{k} \|f_j - g_j\|_{C([0, 2^\ell]; W^{-\epsilon, p}(\mathbb{R}^2))} \right),
\]

which induces the compact-open topology (in time) on \( \mathcal{W} \). Then, the metric space \( (\mathcal{W}, d_{\mathcal{W}}) \) is a bounded Polish space\(^{24}\).

Given \( R > 0 \), we also set

\[
\mathcal{W}(R) = (L^\infty([0; R]; W^{-\epsilon, p}(B_R)))^\otimes_k
\]

with the norm given by

\[
\|f\|_{\mathcal{W}(R)} = \|(f_1, \ldots, f_k)\|_{\mathcal{W}(R)} = \sum_{j=1}^{k} \|f_j\|_{L^\infty([0, R]; W^{-\epsilon, p}(B_R))}.
\]

(ii) Let \( \Xi_0 = (\Xi_{01}, \ldots, \Xi_{0k}) \in \mathcal{W}(R) \). Given \( R > 0 \), let \( \|\Xi_1\|_{\Xi_0, R} \) denote the smallest constant \( K_1 \geq 1 \) such that the following inequality holds for \( 1 \leq j, \ell \leq k \) with \( j + \ell \leq k \):

\[
\int_0^T \| : \Phi_1(t) : \Xi_0\ell(t) \|_{W^{-2(\ell+1)\epsilon, p}(B_R)} dt \leq K_1 \int_0^T \| \Xi_0\ell(t) \|_{W^{-\epsilon, p}(B_R)} dt
\]

for any \( 0 < T \leq R \).

(iii) Let \( \Xi_0 = (\Xi_{01}, \ldots, \Xi_{0k}) \in \mathcal{W}(R) \). Given \( R > 0 \), let \( K_2, K_3 \geq 1 \) be the smallest constants such that the following inequalities hold for \( 1 \leq j, \ell \leq k \) with \( j + \ell \leq k \):

\[
\int_0^T \| : \Phi_1(t) : \Xi_0\ell(t) \|_{W^{-2(\ell+1)\epsilon, p}(B_R)} dt \leq K_2 \int_0^T \| \Xi_0\ell(t) \|_{W^{-\epsilon, p}(B_R)} dt
\]

and

\[
\int_0^T \| : \Phi_1(t) : (\Xi_0\ell(t) - \Xi_0\ell(t)) \|_{W^{-2(\ell+1)\epsilon, p}(B_R)} dt \leq K_3 \int_0^T \| \Xi_0\ell(t) - \Xi_0\ell(t) \|_{W^{-\epsilon, p}(B_R)} dt
\]

for any \( 0 < T \leq R \). Then, we set \( \|\Xi_1\|_{\Xi_0, \Xi_0, R} = \max(K_1, K_2, K_3) \), where \( K_1 \) is as in Part (ii).

In the remaining part of this paper, we fix small \( \epsilon > 0 \), finite \( p = p(\epsilon) \gg 1 \)\(^{22}\) and \( \mu > 0 \) (to be chosen later), and often drop dependence on these parameters. For simplicity of notation, we set

\[
\epsilon_k = 2(k + 1)\epsilon
\]

in the remaining part of this section. This \( \epsilon_k \) comes from the condition appearing in the proof of Proposition 4.9(i).

\(^{24}\)Recall that the space of continuous functions from a separable metric space \( X \) to another separable metric space \( Y \) with the compact-open topology is separable; see \[52\]. See also the paper \[45\], Corollary 3.3.

\(^{25}\)The condition \( p = p(\epsilon) \gg 1 \) is needed in Propositions \[1.3\] and \[1.9\].
4.1. Local well-posedness and stability of SdNLW. In this subsection, we study local well-posedness and stability of SdNLW (4.1), where we view \( u_0, u_1, \) and \( \xi \) in (4.1) as given deterministic spatial / space-time distributions such that the enhanced data sets \( \Xi_0(u_0) \) and \( \Xi_1(u_1, \xi) \) in (4.12) and (4.13) make sense. In order to emphasize the dependence on \( u_0, u_1, \) and \( \xi \), we may write \( u = u(u_0, u_1, \xi) \). By writing \( u \) as in (4.8), we study the equation (4.11) for the remainder term

\[
v = u - S(t)u_0 - D(t)u_1 - \Psi
\]

(4.20)

Given \( \Xi_0 = (\Xi_{01}, \ldots, \Xi_{0k}) \), we consider (4.11), starting from \( t = t_0 \) with \( (v, \partial_tv)|_{t=t_0} = (v_0, v_1) \), where we replace \( \Phi_0^m(u_0) : \Xi_{0m} \); i.e.,

\[
v(t) = S(t-t_0)v_0 + D(t-t_0)v_1 - \sum_{\ell=0}^k \sum_{m=0}^\ell \binom{k}{\ell} \binom{\ell}{m} \int_{t_0}^t D(t-t') \Xi_{0m}(t') : \Phi_1^\ell(u_1, \xi)(t') : v^{k-\ell}(t') dt'.
\]

(4.21)

Given \( s \in \mathbb{R} \) and \( \tau > 0 \), we set

\[
X_{t_0,R}^s(\tau) = L^\infty([t_0, \max(t_0 + \tau, R)]; H^s(B_{R-1})),
\]

\[
X_{t_0,R}^s(\tau) = X_{t_0,R}^s(\tau) \times X_{t_0,R}^{s-1}(\tau).
\]

Proposition 4.2 (local well-posedness). Let \( R > 0, 0 < t_0 < R, \) and \( (v_0, v_1) \in \dot{H}^{1-\varepsilon_k}(B_{R-t_0}) \), where \( \varepsilon_k = 2(k+1)\varepsilon \) is as in (4.19). Suppose that there exist \( M_0, M_1 \geq 1 \) such that

\[
\|\Xi_0\|_{\mathbb{W}(R)} \leq M_0 < \infty,
\]

\[
\|\Xi_1(u_1, \xi)\|_{\Xi_{0,R}} \leq M_1 < \infty,
\]

(4.22)

where \( \mathbb{W}(R) \) is as in (4.15) and \( \|\Xi_1(u_1, \xi)\|_{\Xi_{0,R}} \) is as in Definition 4.1(ii). Then, there exist small \( \tau = \tau(\|v_0, v_1\|_{\dot{H}^{1-\varepsilon_k}(B_{R-t_0})}, M_0, M_1) > 0 \) and a unique solution \( v \) to (4.21) on \( C_{R} \cap \{t_0 \leq t \leq \min(t_0 + \tau, R)\} \) such that

\[
\|v, \partial_tv\|_{\dot{H}^{1-\varepsilon_k}(B_{R-t})} \leq C_0 \|v_0, v_1\|_{\dot{H}^{1-\varepsilon_k}(B_{R-t_0})}
\]

(4.23)

for some absolute constant \( C_0 > 0 \).

Proof. The proof of Proposition 4.2 follows from a slight modification of Proposition 4.1 in [40] together with Definition 4.1(ii). Denote the right-hand side of (4.21) by \( \Gamma(v) = \Gamma_{\Xi_0,\Xi_1}(u_1, \xi)(v) \), and set \( \bar{\Gamma}(v) = (\Gamma(v), \partial \Gamma(v)) \). Then, by the finite speed of propagation, we have

\[
\Gamma(v)(t) = S(t-t_0)v_0 + D(t-t_0)v_1 - \sum_{\ell=0}^k \sum_{m=0}^\ell \binom{k}{\ell} \binom{\ell}{m} \int_{t_0}^t D(t-t') \Xi_{0m}(t') : \Phi_1^{\ell-m}(u_1, \xi)(t') : v^{k-\ell}(t') dt'.
\]

(4.24)

\footnote{As for \( \xi \), we should really view \( \Psi = \Psi(\xi) \) as a given space-time distribution.}
on the cone $C_R$.

Let $(v_0, v_1)$ be an extension of $(v_0, v_1)$ onto $\mathbb{R}^2$. Given $\tilde{v} = (v, \partial_t v)$ on $\{(x, t) \in C_R : t \in [t_0, \max(t_0 + \tau, R)]\}$, let $\tilde{v} = (v, v')$ be an extension onto $\mathbb{R}^2 \times \mathbb{R}_+$. Similarly, let $\Phi^{m, \ell-m}$ be an extension of $\Xi_{0m} : \Phi^{\ell-m}$ from $B_R \times [0, R]$ to $\mathbb{R}^2 \times [0, R]$. Then, proceeding as in the proof of Proposition 4.1 in [40], it follows from (4.24), Lemma 2.8, and Sobolev’s inequality that

$$
\|\tilde{\Gamma}(v)\|_{X_{1,0}^{1-\varepsilon_k}(\tau)} \lesssim \|(v_0, v_1)\|_{H_{1-\varepsilon_k}(B_R)}
$$

$$
+ \tau^\theta \sum_{\ell=0}^k \sum_{m=0}^\ell c_{k, \ell, m} \|\Phi_{0,1}^{m, \ell-m}\|_{L^p_{[t_0,t_0+\tau]}W^{2-\varepsilon_k,p}(\mathbb{R}^2)} \|v\|_{L^\infty_{[t_0,t_0+\tau]}H_x^{1-\varepsilon_k}(\mathbb{R}^2)}
$$

for some $\theta > 0$ and large $p \gg 1$, where $\varepsilon_k = 2(k+1)\varepsilon$ is as in (4.19) with sufficiently small $\varepsilon > 0$. Then, by taking infima over all the extensions $(v_0, v_1)$, $\tilde{v}$, and $\Phi_{0,1}^{m, \ell-m}$ and then using Definition 4.1(ii) with (4.22), we obtain

$$
\|\tilde{\Gamma}(v)\|_{X_{1,0}^{1-\varepsilon_k}(\tau)} \lesssim \|(v_0, v_1)\|_{H_{1-\varepsilon_k}(B_R)}^k
$$

$$
+ \tau^\theta M_1 \sum_{\ell=0}^k \sum_{m=0}^\ell c_{k, \ell, m} \|\Xi_{0m}\|_{L^p_{[t_0,t_0+\tau]}W^{2-\varepsilon_k,p}(B_R)} \||\tilde{v}|\|_{X_{1,0}^{1-\varepsilon_k}(\tau)}^{k-\ell}
$$

$$
\lesssim \|(v_0, v_1)\|_{H_{1-\varepsilon_k}(B_R)}^k + \tau^\theta M_0 M_1 \left(1 + \|\tilde{v}\|_{X_{1,0}^{1-\varepsilon_k}(\tau)}^k\right).
$$

A similar argument yields the following difference estimate:

$$
\|\tilde{\Gamma}(v_1) - \tilde{\Gamma}(v_2)\|_{X_{1,0}^{1-\varepsilon_k}(\tau)} \lesssim \tau^\theta M_0 M_1 \left(1 + \|\tilde{v}_1\|_{X_{1,0}^{1-\varepsilon_k}(\tau)}^{k-1} + \|\tilde{v}_2\|_{X_{1,0}^{1-\varepsilon_k}(\tau)}^{k-1}\right) \|\tilde{v}_1 - \tilde{v}_2\|_{X_{1,0}^{1-\varepsilon_k}(\tau)},
$$

where $\tilde{v}_j = (v_j, \partial_t v_j)$, $j = 1, 2$. Hence, by taking $\tau = \tau(\|(v_0, v_1)\|_{H_{1-\varepsilon_k}(B_R-t_0)}^k, M_0, M_1) > 0$ sufficiently small, the desired claim follows from a standard contraction argument with (4.25) and (4.26).

**Proposition 4.3** (stability). Suppose that $\Xi_0$, $u_1$, and $\xi$ satisfy (4.22) and that $v = v(\Xi_0, u_1, \xi)$ is a solution to (4.21) (with $t_0 = 0$) on the cone $C_R$ such that

$$
\|\tilde{v}\|_{L^\infty([0,R];H_{1-\varepsilon_k}(B_R-t))} \leq M_2
$$

for some $M_2 \geq 1$. Then, there exists $\delta_* = \delta_*(R, M_0, M_1, M_2) > 0$ such that for any $\tilde{\Xi}_0$ satisfying

$$
\|\Xi_0 - \tilde{\Xi}_0\|_{W(R)} < \delta_*,
$$

$$
\|\Xi_1(u_1, \xi)\|_{\Xi_0, \tilde{\Xi}_0, R} \leq M_1 < \infty,
$$

(4.28)
where \( \Xi_0 = (\Xi_{01}, \ldots, \Xi_{0k}) \) and \( \mathbb{W}(R) \) is as in \((4.15)\), there exists \( \tilde{v} = \tilde{v}(\Xi_0, u_1, \xi) \in L^\infty([0, R]; H^k(B_{R-t})) \), satisfying

\[
\tilde{v}(t) = -\sum_{\ell=0}^k \sum_{m=0}^\ell \binom{k}{\ell} \binom{\ell}{m} \times \int_0^t D(t-t')(\Xi_{0m}(t') : \Phi_1^{\ell-m}(t') : \tilde{\nu}^{k-\ell}(t')) \, dt'
\]

and \((4.34)\) we have

\[
\left\langle \sum_{\ell=0}^k \binom{k}{\ell} \binom{\ell}{m} \times \int_0^t D(t-t')(\Xi_{0m}(t') : \Phi_1^{\ell-m}(t') : \tilde{\nu}^{k-\ell}(t')) \, dt' \right\rangle
\]

such that

\[
\sup_{0 \leq t \leq R} \|\tilde{v}(t) - \tilde{v}(t)\|_{H^{k-1}(B_{R-t})} \leq C_0(R, M_0, M_1, M_2)\|\Xi_0 - \tilde{\Xi}_0\|_{\mathbb{W}(R)},
\]

where \( \tilde{v} = (v, \partial_t v) \), \( \tilde{\nu} = (\nu, \partial_t \nu) \), and \( \varepsilon_k = 2(k+1)\varepsilon \) is as in \((4.19)\).

**Proof.** Note that Proposition \(4.2\) with \((4.22)\) and \((4.28)\) guarantees existence of a solution \( \tilde{v} = \tilde{v}(\Xi_0, u_1, \xi) \) on a short time interval. Namely, it satisfies

\[
\tilde{v}(t) = -\sum_{\ell=0}^k \sum_{m=0}^\ell \binom{k}{\ell} \binom{\ell}{m} \times \int_0^t D(t-t')(\Xi_{0m}(t') : \Phi_1^{\ell-m}(t') : \tilde{\nu}^{k-\ell}(t')) \, dt'
\]

on \( C_R \cap \{0 \leq t \leq \tau\} \) for some \( \tau > 0 \), where \( \Phi_1 = \Phi_1(u_1, \xi) \) is as in \((4.10)\). From \((4.23)\), we have

\[
\|\tilde{v}\|_{L^\infty_t H^{k-1}(B_{R-t})} \leq 1.
\]

As for \( v \), it satisfies \((4.21)\) (with \( t_0 = 0 \)) on \( C_R \) (with an additional cutoff function \( 1_{C_R} \) as in \((4.31)\)).

Given an interval \( I \subset \mathbb{R}_+ \), let

\[
B_I(\tilde{v}, \tilde{\nu}) = \|\tilde{v}\|_{L^\infty(I; H^{k-1}(B_{R-t}))} + \|\tilde{\nu}\|_{L^\infty(I; H^{k-1}(B_{R-t}))}
\]

Then, it follows from \((4.27)\) and \((4.32)\) that

\[
B_{[0, \tau]}(\tilde{v}, \tilde{\nu}) \leq M_2 + C_0' \lesssim M_2
\]

for \( I = [0, \tau] \). Then, proceeding as in the proof of Proposition \(4.2\) with \((4.21)\), \((4.31)\), Lemma \(2.8\) and Sobolev’s inequality with \((4.22), (4.28)\), and \((4.34)\) we have

\[
\|\tilde{v} - \tilde{\nu}\|_{L^\infty_{t_0} H^{k-1}(B_{R-t})} \lesssim \tau^\theta_0 M_1 \sum_{\ell=0}^k \sum_{m=0}^\ell c_{k, \ell, m} B_{[0, \tau]}(\tilde{v}, \tilde{\nu})^{k-\ell-1}
\]

\[
\times \left\{ B_{[0, \tau]}(\tilde{v}, \tilde{\nu}) \|\Xi_0 - \tilde{\Xi}_0\|_{\mathbb{W}(R)} + M_0\|\tilde{v} - \tilde{\nu}\|_{L^\infty_{t_0} H^{k-1}(B_{R-t})} \right\}
\]

\[
\lesssim \tau^\theta_0 M_1 M_2^{k-1} \left\{ M_2\|\Xi_0 - \tilde{\Xi}_0\|_{\mathbb{W}(R)} + M_0\|\tilde{v} - \tilde{\nu}\|_{L^\infty_{t_0} H^{k-1}(B_{R-t})} \right\}
\]
for any $0 < \tau_0 \leq \tau$. Hence, by taking $\tau_0 = \tau_0(M_0, M_1, M_2) > 0$ sufficiently small, we obtain

$$\|\tilde{v} - \bar{v}\|_{L^2_{\tau_0}H^{1+\epsilon_k}(B_{R-1})} \leq C_1\|\Xi_0 - \bar{\Xi}_0\|_{W(R)}.$$ 

By repeating the computation on the second interval $I_2 = [\tau_0, 2\tau_0)$, using (4.21) with $t_0 = \tau_0$ and an analogous formulation for $\tilde{v}$, we have

$$\|\tilde{v} - \bar{v}\|_{L^\infty(I_2; H^{1+\epsilon_k}(B_{R-1}))} \leq C_1(1 + C_2)\|\Xi_0 - \bar{\Xi}_0\|_{W(R)},$$

provided that $\delta_\ast = \delta_\ast(M_2) > 0$ in (4.28) is sufficiently small that

$$B_I(\tilde{v}, \bar{v}) \lesssim M_2$$  \hspace{1cm} (4.35)

holds for $I = I_j$, $j = 1, 2$, where $B_I(\tilde{v}, \bar{v})$ is as in (4.33). Then, by iterating the computation on the $j$th interval $I_j = [(j - 1)\tau_0, j\tau_0)$, we obtain

$$\|\tilde{v} - \bar{v}\|_{L^\infty(I_j; H^{1+\epsilon_k}(B_{R-1}))} \leq C_1 \sum_{i=0}^{j-1} C_2^i \|\Xi_0 - \bar{\Xi}_0\|_{W(R)},$$

provided that $\delta_\ast = \delta_\ast(j, M_2) > 0$ in (4.28) is sufficiently small that (4.35) holds for $I = I_i$, $i = 1, \ldots, j$.

Note that our choice of $\tau_0 = \tau_0(M_0, M_1, M_2)$ does not depend on $\delta_\ast$ in (4.28). Thus, by iterating this argument $\sim \frac{R}{\tau_0}$ many times (which requires us to choose $\delta_\ast(R, M_0, M_1, M_2) > 0$ sufficiently small), we obtain (4.30). This concludes the proof of Proposition 4.3.  \hspace{1cm}  \blacksquare

4.2. Convergence of the enhanced Gibbs measures. Let $\Xi_0(\phi)$ be the first enhanced data set defined in (4.12). By viewing $\Xi_0$ as a map sending $\phi$ to $\Xi_0(\phi)$, we consider the pushforward measure $\nu_L$ of the $L$-periodic $\Phi^{k+1}_2$-measure $\rho_L$ under the map $\Xi_0$. Namely, $\nu_L$ is given by

$$\nu_L = (\Xi_0) \# \rho_L.$$  \hspace{1cm} (4.36)

See also (1.36). In the following, we refer to $\nu_L$ as the enhanced Gibbs measure. In this subsection, we study convergence properties of the enhanced Gibbs measure $\nu_L$, which plays an essential role in establishing global well-posedness of the hyperbolic $\Phi^{k+1}_2$-model on $\mathbb{R}^2$ (as a limit of the $L$-periodic hyperbolic $\Phi^{k+1}_2$-model) and invariance of the Gibbs measure $\bar{\rho}_\infty$ constructed in Theorem 1.2(i).  

**Proposition 4.4.** As probability measures on $\mathbb{W} = (C(\mathbb{R}_+; W_{-\epsilon,p}(\mathbb{R}^2)))^\otimes k$, the family $\{\nu_L\}_{L \geq 1}$ is tight, and thus there exists a sequence $\{\nu_{L_j}\}_{j \in \mathbb{N}}$ of the $L_j$-periodic enhanced Gibbs measures that converges weakly to some limit $\nu_\infty$. Moreover, $\nu_{L_j}$ also converges to the same limit $\nu_\infty$ in the Wasserstein-1 metric.

**Proof.** By the definition (4.36), we have $\nu_L = \text{Law}(\Xi_0(\phi))$ with $\text{Law}(\phi) = \rho_L$. Let $X_L$ be the solution to the $L$-periodic SNLH (3.42) with $\text{Law}(X_{0,L}) = \rho_L$. Then, by the invariance (3.47) of $\rho_L$ under (3.12), we have $\text{Law}(X_L(1)) = \rho_L$, and thus $\nu_L = \text{Law}(\Xi_0(X_L(1)))$, where

$$\Xi_0(X_L(1)) = (\mathcal{S}(t)X_L(1), (\mathcal{S}(t)X_L(1))^2, \ldots, (\mathcal{S}(t)X_L(1))^k :).$$  \hspace{1cm} (4.37)

Hence, tightness of $\{\nu_L\}_{L \geq 1}$ follows once we prove tightness of $\{\text{Law}(\Xi_0(X_L(1)))\}_{L \geq 1}$. Once we have tightness of $\{\nu_L\}_{L \geq 1}$, the Prokhorov theorem (Lemma 2.18) implies that there exists a subsequence of $\{\nu_L\}_{L \geq 1}$ converging weakly to some limit $\nu_\infty$. Moreover, noting that the
metric \( d_W \) on \( W \) defined in (4.14) is bounded, it follows from Lemma 2.15 and Remark 2.16 that the same subsequence of \( \{ \nu_L \}_{L \geq 1} \) converges to the same limit \( \nu_\infty \) in the Wasserstein-1 metric, which shows the second claim in Proposition 4.4. Therefore, we focus on proving tightness of \( \{ \text{Law}(\Xi_0(X_L(1))) \}_{L \geq 1} \) in the following.

By the decomposition (3.13), we have

\[
S(t)X_L(1) = S(t)Y_L(1) + S(t)Z_L(1).
\]

Then, from (2.57), we have

\[
(S(t)Liates_{X_L(1)})^\ell_m = \sum_{m=0}^{\ell^m} (S(t)Liates_{Y_L(1)})^{\ell^m-m} (S(t)Liates_{Z_L(1)})^m.
\]

for \( \ell = 1, \ldots, k \). When \( m \neq 0 \), Lemma 2.11(ii) yields

\[
\| (S(t)Liates_{Y_L(1)})^{\ell^m-m} (S(t)Liates_{Z_L(1)})^m \|_{W^{-\epsilon,p}_p} \lesssim \| (S(t)Liates_{Z_L(1)})^m \|_{W^{-\epsilon,p_1}_p} \| (S(t)Liates_{Y_L(1)})^{\ell^m-m} \|_{W^{-\epsilon,p_2}_p} \]

for \( 1 < p_1, p_2 < \infty \) with \( p_1 \gg 1 \), satisfying \( \frac{1}{p} \leq \frac{1}{p_1} + \frac{1}{p_2} \leq \frac{1}{p} + \frac{\epsilon}{2} \). When \( m = 0 \), we simply use the bound:

\[
\| (S(t)Liates_{Y_L(1)})^\ell \|_{W^{-\epsilon,p}_p} \lesssim \| (S(t)Liates_{Y_L(1)})^\ell \|_{W^{-\epsilon,p_2}_p}
\]

for \( p_2 \geq p \) and apply the fractional Leibniz rule (Lemma 2.11(i)); see (4.58) below.

**Part 1:** We first estimate \( (S(t)Liates_{Z_L(1)})^m \), \( m = 1, \ldots, k \). Recall from (2.5) with (1.23) that

\[
S(t) = \partial_t D(t) + D(t) = e^{-t^2} \cos(t[\nabla]) + \frac{1}{2} e^{-t^2} \frac{\sin(t[\nabla])}{\|\nabla\|},
\]

where

\[
[\nabla] = \sqrt{\frac{3}{4} - \Delta}.
\]

Given \( \lambda \in \mathbb{Z}_L^2 \), we set

\[
\widehat{S(t)}(\lambda) = e^{-t^2} \cos(t[2\pi \lambda]) + \frac{1}{2} e^{-t^2} \frac{\sin(t[2\pi \lambda])}{[2\pi \lambda]},
\]

where \( [x] = \sqrt{\frac{3}{4} + |x|^2} \). From (3.41), we have

\[
S(t)Z_L(1) = \sqrt{2} \int_0^1 S(t)P(1-t')\xi_L(dt'),
\]

where \( \xi_L \) is as in (4.4) and (4.5).
We first consider the case $m = 1$. From (4.48) with (4.4) and (4.5), we have

$$E \left[ \left( \langle \nabla \rangle^{-\varepsilon} \phi_j \mathcal{S}(t) Z_L(1) \right) (x) \right]^2 = \frac{2}{L^2} E \left[ \left| \sum_{n \in \mathbb{Z}^2} \int_{\mathbb{R}^2} \frac{1}{\langle \eta \rangle^2} \hat{S}(t) \left( \frac{n}{\varepsilon} \right) e^{i \varepsilon \eta \cdot \phi_j (\eta - \frac{n}{\varepsilon}) \varepsilon t} d\eta \right|^2 \right]. \tag{4.44}$$

Recall that $\left\{ \hat{\xi}_L (\cdot, \frac{n}{\varepsilon}) \right\}_{n \in \mathbb{Z}^2}$ is a family of independent temporal white noises conditioned that $\hat{\xi}_L (\cdot, \frac{n}{\varepsilon}) = \hat{\xi}_L (\frac{n}{\varepsilon}, n \in \mathbb{Z}^2)$, and thus the temporal stochastic integrals in (4.44) are standard Wiener integrals. Hence, we have

$$\tag{4.45} \left| \int_{\mathbb{R}^2} \frac{1}{\langle \eta \rangle^2} \hat{\phi}_j (\eta - \frac{n}{\varepsilon}) e^{2\pi i \eta \cdot x} d\eta \right| \lesssim \frac{1}{\langle \frac{n}{\varepsilon} \rangle^2} \int_{\mathbb{R}^2} \frac{1}{\langle \eta \rangle^2} |\hat{\phi}_j (\eta - \frac{n}{\varepsilon})| d\eta \lesssim \frac{1}{\langle \frac{n}{\varepsilon} \rangle^2}. \tag{4.46}$$

From the triangle inequality $\langle \nabla \rangle^{-\varepsilon} \lesssim \langle \eta \rangle^{-\varepsilon} \langle \eta - \frac{n}{\varepsilon} \rangle^{-\varepsilon}$ with (2.47) and (2.48), we have

$$\int_{\mathbb{R}^2} \frac{1}{\langle \eta \rangle^2} \hat{\phi}_j (\eta - \frac{n}{\varepsilon}) e^{2\pi i \eta \cdot x} d\eta = \int_{\mathbb{R}^2} G_\varepsilon (x - y) \phi_j (y) e^{2\pi i \frac{n}{\varepsilon} y} dy, \tag{4.47}$$

where $G_\varepsilon$ is the kernel of the Bessel potential $\langle \nabla \rangle^{-\varepsilon}$ of order $\varepsilon$. Recall the following bound on the kernel $G_\varepsilon (x)$ (see [34, Proposition 1.2.5]):

$$|G_\varepsilon (x)| \lesssim \begin{cases} e^{- |\frac{x}{2\varepsilon}|}, & \text{for } |x| \geq 2, \\ |x|^{\varepsilon - 2}, & \text{for } |x| < 2, \end{cases} \tag{4.48}$$

since $0 < \varepsilon < 2$. Thus, it follows from (4.47) and (4.48) with (2.16) that

$$\left| \int_{\mathbb{R}^2} \frac{1}{\langle \eta \rangle^2} \hat{\phi}_j (\eta - \frac{n}{\varepsilon}) e^{2\pi i \eta \cdot x} d\eta \right| \leq \int_{\mathbb{R}^2} |G_\varepsilon (x - y)| |\phi_j (y)| dy \lesssim \int_{|x - y| \geq 2} \frac{1}{|x - y|^{M + 2}} \frac{1}{\langle \frac{n}{\varepsilon} \rangle^M} dy + \int_{|x - y| < 2} \frac{1}{|x - y|^{2 - \varepsilon}} \frac{1}{\langle \frac{n}{\varepsilon} \rangle^M} \frac{1}{\langle \frac{n}{\varepsilon} \rangle^M} dy \leq \frac{1}{\langle \frac{n}{\varepsilon} \rangle^M} \tag{4.49}$$

for any $M \gg 1$, where, in the third step, we used the fact that $\langle \frac{x - y}{\frac{n}{\varepsilon}} \rangle \sim 1$ for $|x - y| \leq 2$. Hence, by interpolating (4.46) and (4.49), we obtain from (4.44), (4.45), and a Riemann sum...
For any \( M \gg 1 \), uniformly in \( L \geq 1 \).

Given any finite \( q \geq p \geq 2 \), it follows from \((2.22)\), Minkowski’s integral inequality, the Wiener chaos estimate (Lemma \( 2.12 \)), and \((4.50)\) that

\[
\| \mathcal{S}(t) Z_L(1) \|_{W^{q,-p}} \leq \left\| w_p(2^j) \| (\nabla)^{-\varepsilon} (\phi_j \mathcal{S}(t) Z_L(1)) (x) \|_{L^q(\Omega)} \right\|_{L^p} \leq q^\frac{1}{2} e^{-\frac{t}{2}} \left\| w_p(2^j) \right\|_{L^p} \leq q^\frac{1}{2} e^{-\frac{t}{2}} \left\| w_p(2^j) \right\|_{L^p} \leq q^\frac{1}{2} e^{-\frac{t}{2}},
\]

uniformly in \( L \geq 1 \).

From \((4.52)\) and the mean value theorem, we have

\[
| \hat{\mathcal{S}}(t+h)(\lambda) - \hat{\mathcal{S}}(t)(\lambda) | \lesssim e^{-\frac{t}{2}} \min (1, |h| \langle \lambda \rangle)^{\kappa}
\]

for any \( \lambda \in \mathbb{Z}^2_L \) and \( 0 \leq \kappa \leq 1 \). Then, by repeating an analogous computation with \((4.52)\), we obtain

\[
\| \mathcal{S}(t+h) Z_L(1) - \mathcal{S}(t) Z_L(1) \|_{W^{q,-p}} \leq q^\frac{1}{2} e^{-\frac{t}{2}} |h|^\kappa
\]

for any \( q \geq 1 \) and \( 0 < \kappa < \varepsilon \), uniformly in \( L \geq 1 \).

Let \( T > 0 \). In view of \((4.51)\) and \((4.52)\), by applying Kolmogorov’s continuity criterion \((5.1 \, \text{Theorem 8.2})\), we see that \( \mathcal{S}(t) Z_L(1) \in C^\alpha([0, T]; W^{q,-p}_\mu(\mathbb{R}^2)) \) for some small \( \alpha > 0 \). Moreover, the \( \alpha \)-Hölder semi-norms of \( \{ \mathcal{S}(t) Z_L(1) : t \in [0, T] \} \) are uniformly (in \( L \)) bounded in \( L^q(\Omega) \) for any finite \( q \geq 1 \). Therefore, by noting that the discussion above holds for any small \( \varepsilon > 0 \) and any \( \mu > 0 \), it follows from the Arzelà-Ascoli theorem with Lemma \( 2.2 \) that the family \( \{ \mathcal{S}(t) Z_L(1) \}_{L \geq 1} \in C([0, T]; W^{q,-p}_\mu(\mathbb{R}^2)) \) is tight\(^{27}\) for any finite \( T \), and hence \( \{ \mathcal{S}(t) Z_L(1) \}_{L \geq 1} \in C(\mathbb{R}_+; W^{q,-p}_\mu(\mathbb{R}^2)) \) is tight.

Remark 4.5. By repeating the argument above for \( e^{\frac{t}{2}} \mathcal{S}(t) Z_L(1) \), we see that \( e^{\frac{t}{2}} \mathcal{S}(t) Z_L(1) \) is almost surely \( \alpha \)-Hölder continuous in time. In particular, it grows at most polynomially in time. From this observation, we conclude that \( \mathcal{S}(t) Z_L(1) \) is almost surely bounded on \( \mathbb{R}_+ \) with values in \( W^{q,-p}_\mu(\mathbb{R}^2) \).

\(^{27}\)Namely, the laws of \( \{ \mathcal{S}(t) Z_L(1) : t \in [0, T] \}_{L \geq 1} \) are tight as probability measures on \( C([0, T]; W^{q,-p}_\mu(\mathbb{R}^2)) \).
We now study the higher powers : $\langle \mathcal{S}(t)Z_L(1)^m \rangle$, $m = 2, \ldots, k$. From Lemma 2.13 and (4.43) with (4.44) and (4.49), we have

$$\mathbb{E}\left[ \langle \mathcal{S}(t)Z_L(1)^m \rangle \right] = m! \left\{ \mathbb{E}[\langle \mathcal{S}(t)Z_L(1)^m \rangle \cdot \langle \mathcal{S}(t)Z_L(1)^m \rangle] \right\}^{m-1} \sum_{n \in \mathbb{Z}^2} e^{2\pi i \frac{n}{L}(y-y')} \prod_{j=1}^{m} \left\{ \left( \frac{\mathcal{S}(t)(n_j)}{2} \right) \right\}^{1-e^{-2(\frac{2\pi n_j}{L})^2}}.$$  

Then, by (4.54) and interpolating (4.46) and (4.49), we have

$$\mathbb{E}\left[ \langle \nabla \rangle^{-\varepsilon} \left( \phi_j : \langle \mathcal{S}(t)Z_L(1)^m \rangle \right) \langle x \rangle \right]^2 = \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \frac{1}{(\eta)^{\varepsilon}} \langle \eta \rangle^{\varepsilon} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \phi_j(y)\phi_j(y') \times \mathbb{E}\left[ \langle \mathcal{S}(t)Z_L(1)^m \rangle \cdot \langle \mathcal{S}(t)Z_L(1)^m \rangle \right] \times e^{-2\pi i y} e^{-2\pi i y'} dy dy' \frac{1}{\langle \eta \rangle^{\varepsilon}} \frac{1}{\langle \eta' \rangle^{\varepsilon}}$$

$$\sim \frac{1}{L^{2m}} \sum_{n \in \mathbb{Z}^2} \left( \int_{\mathbb{R}^2} \frac{1}{\langle \eta \rangle^{\varepsilon}} \phi_j(\eta - \frac{n}{L}) e^{2\pi i \eta x} d\eta \right) \left( \int_{\mathbb{R}^2} \frac{1}{\langle \eta' \rangle^{\varepsilon}} \phi_j(\eta' + \frac{n}{L}) e^{2\pi i \eta' x} d\eta' \right) \prod_{j=1}^{m} \left\{ \left( \frac{\mathcal{S}(t)(n_j)}{2} \right) \right\}^{1-e^{-2(\frac{2\pi n_j}{L})^2}}.$$  

By a Riemann sum approximation, we then obtain

$$\mathbb{E}\left[ \langle \nabla \rangle^{-\varepsilon} \left( \phi_j : \langle \mathcal{S}(t)Z_L(1)^m \rangle \right) \langle x \rangle \right]^2 \lesssim \frac{e^{-mt}}{(\frac{x}{\varepsilon})^M} \int_{(\mathbb{R}^2)^m} \frac{1}{(y_1 + \cdots + y_m)^{2\varepsilon}} \prod_{j=1}^{m} \frac{1}{\langle y_j \rangle^{2}} dy_1 \cdots dy_m$$

$$\lesssim \frac{e^{-mt}}{(\frac{x}{\varepsilon})^M}$$

for any $M \gg 1$, uniformly in $L \geq 1$. 

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28Strictly speaking, in applying Lemma 2.13, we need to go back to the frequency truncated version of the Wick power, just as in (1.25) for the damped wave equation, and remove the frequency truncation.
We now proceed as in (4.51). Namely, given any finite $q \geq p \geq 2$, it follows from (2.22) and Minkowski’s integral inequality, the Wiener chaos estimate (Lemma 2.12), and (4.55) that
\[
\| (S(t)Z_L(1))^m : \|_{W^p - \varepsilon,p} \|_{L^q(\Omega)} \lesssim q^m \left\| w_{\mu}(2^j)(\nabla)^{-\varepsilon}(\phi_j : (S(t)Z_L(1))^m : (x)) \right\|_{L^2(\Omega)} \|_{L^p_\mu}^m
\]
\[
\lesssim q^m e^{-\frac{\mu}{2}t} \left\| w_{\mu}(2^j) \frac{1}{\langle \frac{s}{2} \rangle^m} \right\|_{L^p_\mu} \lesssim q^m e^{-\frac{\mu}{2}t} \| w_{\mu}(2^j)2^{j} \|_{L^p_\mu}
\]
\[
\lesssim q^m e^{-\frac{\mu}{2}t},
\]
uniformly in $L \geq 1$.

We briefly discuss how to handle the time increment $(S(t)Z_L(1))^m : - (S(t)Z_L(1))^m :$. The main idea is to proceed as in the second half of the proof of Proposition 2.1 in [38].

Given $h \in \mathbb{R}$, define the difference operator $\delta_h$ by setting\(^{29}\)
\[
\delta_h F(t) = F(t + h) - F(t).
\]

Then, by Lemma 2.13, we have
\[
\frac{1}{m!} \mathbb{E} \left[ \delta_h : (S(t)Z_L(1))^m (y) : \delta_h : (S(t)Z_L(1))^m (y') : \right] = \left\{ \mathbb{E} \left[ (S(t + h)Z_L(1))(y) \cdot (S(t + h)Z_L(1))(y') \right] \right\}^m
\]
\[
- \left\{ \mathbb{E} \left[ (S(t)Z_L(1))(y) \cdot (S(t + h)Z_L(1))(y') \right] \right\}^m
\]
\[
- \left\{ \mathbb{E} \left[ (S(t + h)Z_L(1))(y) \cdot (S(t)Z_L(1))(y') \right] \right\}^m
\]
\[
+ \left\{ \mathbb{E} \left[ (S(t)Z_L(1))(y) \cdot (S(t)Z_L(1))(y') \right] \right\}^m
\]
\[
= \mathbb{E} \left[ \delta_h(S(t)Z_L(1))(y) \cdot (S(t + h)Z_L(1))(y') \right]
\]
\[
\times \sum_{j=0}^{m-1} \left\{ \mathbb{E} \left[ (S(t + h)Z_L(1))(y) \cdot (S(t + h)Z_L(1))(y') \right] \right\}^{m-j-1}
\]
\[
\times \left\{ \mathbb{E} \left[ (S(t)Z_L(1))(y) \cdot (S(t + h)Z_L(1))(y') \right] \right\}^j
\]
\[
- \mathbb{E} \left[ \delta_h(S(t)Z_L(1))(y) \cdot (S(t)Z_L(1))(y') \right]
\]
\[
\times \sum_{j=0}^{m-1} \left\{ \mathbb{E} \left[ (S(t + h)Z_L(1))(y) \cdot (S(t)Z_L(1))(y') \right] \right\}^{m-j-1}
\]
\[
\times \left\{ \mathbb{E} \left[ (S(t)Z_L(1))(y) \cdot (S(t)Z_L(1))(y') \right] \right\}^j
\]
Note that in each term on the right-hand side above, the difference operator $\delta_h$ appears exactly once. Then, with (4.52), we can repeat the computation above for $(S(t)Z_L(1))^m :$ and obtain
\[
\| \delta_h : (S(t + h)Z_L(1))^m : \|_{W^p - \varepsilon,p} \|_{L^q(\Omega)} \lesssim q^m e^{-\frac{\mu}{2}t} |h|^\kappa
\]
for any $q \geq 1$ and $0 < \kappa < \varepsilon$, uniformly in $L \geq 1$.

\(^{29}\)We assume that $t + h \geq 0$. 
Therefore, applying Kolmogorov’s continuity criterion with (4.56) and (4.57) and arguing as in the $m = 1$ case with Lemma 2.2 and the Arzelà-Ascoli theorem, we conclude that the family $\{(S(t)Z_L(1))^m\}_{t \geq 1}$ in $C(\mathbb{R}_+; W_\mu^{\varepsilon,p}(\mathbb{R}^2))$ is tight.

*Part 2:* Next, we consider $(S(t)Y_L(1))^m$, $m = 1, \ldots, k$. By the fractional Leibniz rule (Lemma 2.11 (i)), (2.24), and Lemma 2.6 we have

$$
\| (S(t)Y_L(1))^m \|_{L_t^\infty W_\mu^{\varepsilon,p}(\mathbb{R}^2)} \lesssim \| S(t)Y_L(1) \|_{L_t^\infty W_\mu^{\varepsilon,p}(\mathbb{R}^2)} \lesssim \| S(t)Y_L(1) \|_{L_t^\infty W_\mu^{1-\frac{2}{mp}+\varepsilon}(\mathbb{R}^2)}
$$

(4.58)

for some $0 < \mu_3 < \mu_2 < \mu_1 < \mu$. Here, we take $\varepsilon > 0$ sufficiently small such that $1 - \frac{2}{mp} + \varepsilon < 1$. It follows from Proposition 3.3 with (3.7) and (3.29), Lemma 2.12 and [55, Theorem 5.1] that, given any finite $q \geq 1$, the $q$th moment of the right-hand side of (4.58) is bounded, uniformly in $L \geq 1$.

A similar computation with (4.52) (but with $\lambda \in \mathbb{R}^2$) yields

$$
\| \delta_h (S(t)Y_L(1))^m \|_{L_t^\infty W_\mu^{\varepsilon,p}(\mathbb{R}^2)} \lesssim \| h \|_k \| Y_L(1) \|_{L_t^\infty W_\mu^{1-\frac{2}{mp}+\varepsilon}(\mathbb{R}^2)}.
$$

(4.59)

Then, as long as $\varepsilon, \kappa > 0$ are sufficiently small such that $1 - \frac{2}{mp} + \varepsilon + \kappa < 1$, Proposition 3.3 with (3.7) and (3.29), Lemma 2.12 and [55, Theorem 5.1] that, given any finite $q \geq 1$, the $q$th moment of the right-hand side of (4.59) is bounded, uniformly in $L \geq 1$. Therefore, applying Kolmogorov’s continuity criterion with (4.58) and (4.59) and arguing as in the $m = 1$ case with Lemma 2.2 and the Arzelà-Ascoli theorem, we conclude that the family $\{(S(t)Y_L(1))^m\}_{L \geq 1}$ in $C(\mathbb{R}_+; W_\mu^{\varepsilon,p}(\mathbb{R}^2))$ is tight.

*Part 3:* Finally, from (4.39), we have

$$
\delta_h : (S(t)X_L(1))^\ell = \sum_{m=0}^\ell \binom{\ell}{m} \delta_h (S(t)Y_L(1))^\ell - m : (S(t+h)Z_L(1))^m :
$$

(4.60)

By applying Lemma 2.11 (ii) as in (4.40), we conclude from (4.51), (4.53), (4.56), (4.57), (4.58), and (4.59) with Kolmogorov’s continuity criterion ([5, Theorem 8.2]), Lemma 2.2 and the Arzelà-Ascoli theorem that the family $\{(S(t)X_L(1))^m\}_{L \geq 1}$ in $C(\mathbb{R}_+; W_\mu^{\varepsilon,p}(\mathbb{R}^2))$ is tight. Namely, as probability measures on $\mathcal{W} = C(\mathbb{R}_+; W_\mu^{\varepsilon,p}(\mathbb{R}^2))$, the family $\{\nu_L\}_{L \in \mathbb{N}}$ is tight.

Let $\mathcal{A}$ be the index set such that Theorem 1.2 (i) holds. Namely, that the $L$-periodic $\Phi_2^{k+1}$-measures $\rho_L$, $L \in \mathcal{A}$ converges weakly to a limiting $\Phi_2^{k+1}$-measure $\rho_\infty$ on $\mathbb{R}^2$. Note that Proposition 4.4 also applies to the family $\{\nu_L\}_{L \in \mathcal{A}}$, showing that there exists a subsequence which converges, weakly and also in the Wasserstein-1 metric, to some limit $\nu_\infty$. The next proposition identifies this limiting probability measure $\nu_\infty$.

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30 Theorem 5.4 in the arXiv version.
31 Theorem 5.4 in the arXiv version.
Proposition 4.6. Let the index set $\mathcal{A} \subset \mathbb{N}$ be as in Theorem 1.2(i). Then, as probability measures on $\mathbb{W} = (C(\mathbb{R}^+; W^{\varepsilon,p}_\mu(\mathbb{R}^2)))^\otimes k$, the entire sequence $\{\nu_L\}_{L \in \mathcal{A}}$ of the $L$-periodic enhanced Gibbs measures converges, weakly and also in the Wasserstein-1 metric, to the unique limit

$$\nu_\infty = (\Xi_0)_0 \mathbf{#} p_\infty. \quad (4.61)$$

Proof. In the following, we only consider the values of $L$ belonging to $\mathcal{A} = \{L_j : j \in \mathbb{N}\} \subset \mathbb{N}$. For simplicity of notation, we drop the subscript and simply write $L \rightarrow \infty$ with the understanding that each $L$ belongs to $\mathcal{A}$.

In view of the tightness of $\{\nu_L\}_{L \in \mathcal{A}}$ (Proposition 4.4) and Lemma 2.15, we only need to show that the limit in law of $\{\nu_L\}_{L \in \mathcal{A}}$ is unique.

Let $X_L = Y_L + Z_L$ be the solution to the $L$-periodic SNLH (3.42), where $Y_L$ and $Z_L$ satisfy (3.45) and (3.44) with the Gibbsian initial data $X_{0,L}$ with $\text{Law}(X_{0,L}) = \rho_L$. Here, we assume that $X_{0,L}$ is independent of the noise $\xi_L$. In view of the weak convergence of $\rho_L$ to $\rho_\infty$ as $L \rightarrow \infty$ (with $L$'s belonging to $\mathcal{A}$) established in Subsection 3.3, it follows from the Skorokhod representation theorem (Lemma 2.19) that there exist a probability measure $\tilde{\mu}$ (on a new probability space) and random variables $\tilde{X}_{0,L}$ with $\text{Law}(\tilde{X}_{0,L}) = \rho_L$, $L \in \mathcal{A}$, such that $\tilde{X}_{0,L}$ converges $\tilde{\mu}$-almost surely to some $\tilde{X}_{0,\infty}$, with $\text{Law}(\tilde{X}_{0,\infty}) = \rho_\infty$, in $B^{-\varepsilon,p}_{\rho_0,\rho_0}(\mathbb{R}^2)$ for any $\varepsilon > 0$, finite $\rho_0 \geq 1$, and $\mu > 0$; see also Remark 3.6. Note that by the independence of $X_{0,L}$ and $\xi_L$, the change of the probability spaces due to the use of the Skorokhod representation theorem (Lemma 2.19) does not affect the noise $\xi_L$ (and hence $Z_L$).

Let $\tilde{X}_L = \tilde{Y}_L + Z_L$ be the solution to (3.42) with the new initial data $\tilde{X}_{0,L}$, where $\tilde{Y}_L$ satisfies (3.45) with the initial data $X_{0,L}$ replaced by $\tilde{X}_{0,L}$. By the invariance of $\rho_L$ under (3.42), we have $\text{Law}(\tilde{X}_L(1)) = \rho_L$. Thus, it suffices to show that

(i) $\Xi_0(\tilde{X}_L(1))$ converges in law to $\Xi_0(\tilde{X}_{\infty}(1))$ as $L \rightarrow \infty$ (with $L \in \mathcal{A}$), where $\tilde{X}_{\infty}$ is the solution to (3.1) with the initial data $\tilde{X}_{0,\infty}$ satisfying $\text{Law}(\tilde{X}_{0,\infty}) = \rho_\infty$, and $\Xi_0(\phi)$ is the enhanced data set defined in (1.12), and

(ii) $\text{Law}(\tilde{X}_{\infty}(1)) = \rho_\infty$. See Remark 4.7 below.

Before proceeding further, we recall the decomposition $\tilde{X}_{\infty} = \tilde{Y}_{\infty} + Z_{\infty}$, where $\tilde{Y}_{\infty}$ is the solution to (3.1) with the initial data $\tilde{X}_{0,\infty}$ and $Z_{\infty}$ is the solution to (3.2). For simplicity of notation, we denote $\tilde{X}_L$, $\tilde{Y}_L$, etc. by $X_L$, $Y_L$, etc. in the remaining part of the proof.

Given $R > 0$, let $B_R \subset \mathbb{R}^2$ denotes the ball of radius $R$ centered at the origin. In view of the tightness of $\{\nu_L\}_{L \in \mathcal{A}}$ (Proposition 4.4), it suffices to show that, given any $t \in \mathbb{R}^+$ and $R > 0$, $(S(t)X_L(1))^f$: converges in probability to $(S(t)X_{\infty}(1))^f$: in $W^{-\varepsilon,p}(B_R)$ as $L \rightarrow \infty$ (with $L \in \mathcal{A}$). Then, for each fixed $t \in \mathbb{R}^+$, this allows us to identify $(S(t)X_{\infty}(1))^f$: with the limit in law of $(S(t)X_L(1))^f$: as a $D'(\mathbb{R}^2)$-valued random variable by testing against a test function $\phi \in D(\mathbb{R}^2) = C^\infty_c(\mathbb{R}^2)$ with $\text{supp} \phi \subset B_R$. From this observation and Proposition 4.4 with the uniqueness of a limit, we then conclude that the entire sequence $\{\Xi_0(X_L(1))\}_{L \in \mathcal{A}}$ converges in law to $\Xi_0(X_{\infty}(1))$ as $L \rightarrow \infty$ (with $L \in \mathcal{A}$).

Remark 4.7. Once we prove (i) in the sense explained above, the claim (ii) above follows immediately. On the other hand, as probability measures on $B^{-\varepsilon,p}_{\rho_0,\rho_0}(\mathbb{R}^2)$, $\rho_L = \text{Law}(X_L(1))$ converges weakly to $\rho_\infty$ as $L \rightarrow \infty$ (with $L \in \mathcal{A}$). On the other hand, we will show that, given $t \in \mathbb{R}^+$, $S(t)X_L(1)$ converges in law to $S(t)X_{\infty}(1)$ as $L \rightarrow \infty$ (with $L \in \mathcal{A}$). With $t = 0$, etc.
this implies that, as probability measures on $\mathcal{D}'(\mathbb{R}^2)$, $\rho_L = \text{Law}(X_L(1))$ converges weakly to $\text{Law}(X_{\infty}(1))$. Therefore, from the uniqueness of a limit, we conclude that $\text{Law}(X_{\infty}(1)) = \rho_{\infty}$.

Note that a similar argument shows that $\text{Law}(X_{\infty}(t)) = \rho_{\infty}$ for any $t > 0$, thus yielding invariance of the limiting $\Phi^{k+1}_2$-measure $\rho_{\infty}$ on $\mathbb{R}^2$ under the dynamics of the parabolic $\Phi^{k+1}_2$-model (3.1) claimed in Remark 1.5.

Fix $t \in \mathbb{R}_+$ and $R > 0$ in the remaining part of the proof. From Lemma 2.28(ii) (see also Remark 2.39 with (4.38) and (4.39), we have

$$
\|:(S(t)X_L(1))^\ell: - (S(t)X_{L'}(1))^\ell:\|_{W^{-\varepsilon,p}(B_R)} \\
\lesssim \sum_{m=0}^{\ell} \left( \ell \choose m \right) \|:(S(t)Z_L(1))^m: - (S(t)Z_{L'}(1))^m:\|_{W^{-\varepsilon,p_1}(B_R)} \\
\times \|:(S(t)Y_L(1))^\ell-m\|_{W^{\varepsilon,p_2}(B_R)} \\
+ \|:(S(t)Z_{L'}(1))^m:\|_{W^{-\varepsilon,p_1}(B_R)} \\
\times \|:(S(t)Y_L(1))^\ell-m - (S(t)Y_{L'}(1))^\ell-m\|_{W^{\varepsilon,p_2}(B_R)}
$$

(4.62)

for any $L, L' \in A$, where $1 < p_1, p_2 < \infty$ with $p_1 \gg 1$, satisfying $\frac{1}{p} \leq \frac{1}{p_1} + \frac{1}{p_2} \leq \frac{1}{p} + \frac{1}{2}$. In the following, we separately study convergence of $(S(t)Y_L(1))^\ell-m$ and $(S(t)Z_{L'}(1))^m$.

**Part 1:** Let us first study the terms involving $Y_L$ and $Y_{L'}$. Proceeding as in (4.58) with the fractional Leibniz rule (Lemma 2.28(i)), Sobolev’s inequality, and the boundedness of $S(t)$ on $H^s(\mathbb{R}^2)$, we have

$$
\|:(S(t)Y_L(1))^\ell-m\|_{L^{\infty}_t W^{\varepsilon,p_2}(B_R)} \lesssim \|S(t)Y_L(1))^\ell-m\|_{L^{\infty}_t W^{\varepsilon,p_2}_x(B_R)} \lesssim \|S(t)Y_L(1))^\ell-m\|_{L^{\infty}_t H^{1-\frac{2}{\ell-m}+\varepsilon}(B_R)} \lesssim \|Y_L(1))^\ell-m\|_{H^{1-\frac{2}{\ell-m}+\varepsilon}(B_R)}
$$

(4.63)

Then, by Hölder’s inequality and (2.26), we have

$$
\|:(S(t)Y_L(1))^\ell-m\|_{L^{\infty}_t W^{\varepsilon,p_2}(B_R)} \leq C(R)\|Y_L(1))^\ell-m\|_{B^{1-\frac{2}{\ell-m}+\varepsilon}_{p_3,\infty}(B_R)}
$$

(4.64)

for any $p_3 \geq 2$. Proposition 3.3 with Theorem 5.1 in [55, 55] controlling the right-hand side of (3.28), shows that, given any finite $q \geq 1$, the $q$th moment of the right-hand side of (4.61) is bounded, uniformly in $L \geq 1$.

A slight modification of (4.63) and (4.64) yields

$$
\|:(S(t)Y_L(1))^\ell-m - (S(t)Y_{L'}(1))^\ell-m\|_{L^{\infty}_t W^{\varepsilon,p_2}(B_R)} \\
\leq C(R) \left( \|Y_L(1))^\ell-m-1\|_{B^{1-\frac{2}{\ell-m}+\varepsilon}_{p_3,\infty}(B_R)} + \|Y_{L'}(1))^\ell-m-1\|_{B^{1-\frac{2}{\ell-m}+\varepsilon}_{p_3,\infty}(B_R)} \right) \\
\times \|Y_L(1)-Y_{L'}(1)|_{B^{1-\frac{2}{\ell-m}+\varepsilon}_{p_3,\infty}(B_R)}
$$

(4.65)

for any $L, L' \geq 1$.

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[32] Theorem 5.4 in the arXiv version.
Given $L \geq 1$, let $Z_L$ and $Y_L$ be the solutions to
\[
\begin{cases}
\partial_t Z_L + (1 - \Delta) Z_L = \sqrt{2} \xi_L \\
Z_L|_{t=0} = X_{0,L}
\end{cases}
\]
and
\[
\begin{cases}
\partial_t Y_L + (1 - \Delta) Y_L + \sum_{\ell=0}^{k} \binom{k}{\ell} Y_L^{k-\ell} = 0 \\
Y_L|_{t=0} = 0,
\end{cases}
\]
respectively. Note that we have $X_L = Y_L + Z_L = Y_L + Z_L$ and that
\[
Y_L(1) - Y_L'(1) = Y_L(1) - Y_L'(1) - P(1)(X_{0,L} - X_{0,L'}).
\]
(4.66)
By the Schauder estimate (Proposition 5 in Theorem 5.4 in the arXiv version. and the almost sure convergence of $X_{0,L}$ in $B_{p_3,\infty}(\mathbb{R}^2)$, we have
\[
\|P(1)(X_{0,L} - X_{0,L'})\|_{B_{p_3,\infty}((\ell-m)p_2+2\varepsilon)} \to 0,
\]
almost surely, as $L, L' \to \infty$ (with $L, L' \in A$).

Next, we study the difference $Y_L(1) - Y_L'(1)$ in (4.66). Write $Z_L'$ as
\[
Z_L' := \sum_{m=0}^{\ell} \binom{\ell}{m} (P(t)X_{0,L})^{\ell-m} : Z_L^m :.
\]
Then, it follows from the almost sure convergence of $P(t)X_{0,L}$ and Theorem 5.1 in the arXiv version. on convergence in probability of $Z_L^m$ to a limit in $C([0,1]; B_{p_4,\infty}(\mathbb{R}^2))$ for some $p_4 \gg 1$, as $L \to \infty$ (with $L \in A$). Now, recall from Theorem 8.1 and Theorem 9.2 in the arXiv version. that the solution map for the following (deterministic) equation:
\[
\begin{cases}
\partial_t Y + (1 - \Delta) Y + \sum_{\ell=0}^{k} \binom{k}{\ell} Y^{k-\ell} = 0 \\
Y|_{t=0} = 0,
\end{cases}
\]
sending $\{Z^{(\ell)} \}_{\ell=0}^{k}$ (with $Z^{(0)} = 1$) to a solution $Y$ is continuous from $C([0,1]; B_{p_4,\infty}(\mathbb{R}^2))^\otimes k$ to $C([0,1]; B_{p_5,\infty}(\mathbb{R}^2))$ for some $p_5 \gg 1$. In view of the aforementioned convergence in probability of $Z_L'$, $\ell = 1, \ldots, k$, the continuous mapping theorem then yields that
\[
\|Y_L(1) - Y_L'(1)\|_{B_{p_3,\infty}((\ell-m)p_2+2\varepsilon)} \to 0
\]
(4.68)
in probability, as $L, L' \to \infty$ (with $L, L' \in A$).

Therefore, putting (4.66), (4.67), and (4.68) together with the aforementioned uniform (in $L$) boundedness of the $B_{p_3,\infty}((\ell-m)p_2+2\varepsilon)(B_R)$-norm of $Y_L(1)$ (see the discussion after (4.64)), we conclude from (4.65) that $(S(t)Y_L(1))^{\ell-m}$ converges in probability to a unique limit in $W^{\epsilon,p_2}(B_R)$ as $L \to \infty$ (with $L \in A$). Moreover, by repeating the argument with $L' = \infty$, we see that the limit is given by $(S(t)Y_{\infty}(1))^{\ell-m}$, where $Y_{\infty}$ is the solution to (3.4) with the Gibbsian initial data $X_{0,\infty}$.

\textsuperscript{33}Proposition 3.11 in the arXiv version.

\textsuperscript{34}Theorem 5.4 in the arXiv version.

\textsuperscript{35}Theorems 8.1 and 9.5 in the arXiv version.
\textbf{Part 2:} In view of \((4.62)\) and the discussion above, it suffices to show that, given any \(R > 0\), there exists \(r \geq 1\) such that
\[
(S(t)Z_L(1))^m : \text{ converges to } (S(t)Z_\infty(1))^m : \quad (4.69)
\]
in \(L'(\Omega; W^{-\varepsilon,p_1}(B_R))\) as \(L \to \infty\), \(m = 1, \ldots, k\). We point out that in showing \((4.69)\), there is no need to restrict \(L\) in \(A\). Indeed, once we prove \((4.69)\), it follows from \((4.62)\), Part 1, and \((4.69)\) (which in particular implies convergence in probability) that \((S(t)X_L(1))^\ell : \text{ converges in law to } (S(t)X_\infty(1))^\ell : \text{ in } W^{-\varepsilon,p}(B_R)\) as \(L \to \infty\) (with \(L \in A\)) for any \(t \in \mathbb{R}_+, R > 0\), and each \(\ell = 1, \ldots, k\).

Hence, it remains to prove \((4.69)\). Fix \(t \in \mathbb{R}_+\) and \(R > 0\). Given \(L, L' \geq 1\), \(x, y \in \mathbb{R}^2\), define \(K_{L,L'}(x, y; t)\) by
\[
K_{L,L'}(x, y; t) = \mathbb{E}[(S(t)Z_L(1))(x)(S(t)Z_{L'}(1))(y)] = K_{L,L'}(y, x; t). \quad (4.70)
\]
Then, from Lemma \((2.13)\) we have
\[
\mathbb{E}\left[(S(t)Z_L(1))^m(x) : (S(t)Z_{L'}(1))^m(y) : \right] = m! \{K_{L,L'}(x, y; t)\}^m. \quad (4.71)
\]
Given a test function \(\varphi \in \mathcal{D}(\mathbb{R}^2)\) supported on a ball \(B\) of radius 1 with \(B \subset B_{2R}\), we set
\[
K_{L,L'}^{m,\varphi}(t) = \mathbb{E}\left[|(S(t)Z_L(1))^m : (S(t)Z_{L'}(1))^m : \varphi|^2\right], \quad (4.72)
\]
where \(\langle \cdot, \cdot \rangle\) denotes the \(\mathcal{D}'-\mathcal{D}\) duality pairing. Then, from \((4.71)\) and Hölder’s inequality, we have
\[
K_{L,L'}^{m,\varphi}(t) = m! \int_{B_{2R}}^2 \left(\{K_{L,L}(x, y; t)\}^m - 2\{K_{L,L'}(x, y; t)\}^m + \{K_{L',L'}(x, y; t)\}^m\right) \varphi(x) \varphi(y) dxdy
\leq \left\| \{K_{L,L}(x, y; t)\}^m - 2\{K_{L,L'}(x, y; t)\}^m \right\|_{L^q(\mathbb{R}^2)} \left\| \varphi \right\|_{L^q}^2
\quad (4.73)
\]
for any \(1 \leq q \leq \infty\) with \(\frac{1}{q} + \frac{1}{q'} = 1\).

Now, set \(K_{\infty,\infty}(x, y; t) = \lim_{L,L' \to \infty} K_{L,L'}(x, y; t)\); see \((4.95)\) below for the precise definition. We then claim that there exists \(c > 0\) such that
\[
|K_{L,L}(x, y; t) - K_{\infty,\infty}(x, y; t)| \lesssim 1 + (- \log |x - y|)_+, \quad (4.74)
\]
for any \(L, L' \gg R + t\) and \(x, y \in B_{2R}\), where \(a_+ = \max(a, 0)\).

For now, we assume \((4.71)\) and prove \((4.69)\). By the triangle inequality and \((4.74)\), we have
\[
|\{K_{L,L}(x, y; t)\}^m - 2\{K_{L,L'}(x, y; t)\}^m + \{K_{L',L'}(x, y; t)\}^m| \\
\leq \left| \{K_{L,L}(x, y; t)\}^m - \{K_{\infty,\infty}(x, y; t)\}^m \right| \\
+ 2\left| \{K_{L,L'}(x, y; t)\}^m - \{K_{\infty,\infty}(x, y; t)\}^m \right| \\
+ \left| \{K_{L',L'}(x, y; t)\}^m - \{K_{\infty,\infty}(x, y; t)\}^m \right| \\
\lesssim \left(1 + (- \log |x - y|)_+\right)^{m-1}(e^{-cL} + e^{-cL'}) \quad (4.75)
\]
Hence, by choosing $q = q(\varepsilon) \gg 1$ and applying Lemma 2.14 with (4.72), (4.73), and (4.75), we obtain
\[
\mathbb{E}\left[\| (S(t)Z_L(1))^m : - : (S(t)Z_{L'}(1))^m \|_{W^{-\varepsilon,p_1}(B_R)}\right] \\
\leq C(R, r) \sup_{\|\psi\|_{L^q} \leq 1} \left( K_{L,L'}(t) \right)^{\frac{2}{\varepsilon}} \\
\lesssim C(R, r) \left( e^{-cL} + e^{-cL'} \right),
\] (4.76)
which tends to zero as $L, L' \to \infty$. This shows that $(S(t)Z_L(1))^m$ converges to some limit in $L^r(\Omega; W^{-\varepsilon,p_1}(B_R))$ as $L \to \infty$. By repeating an analogous argument with $L' = \infty$, we obtain (4.69). See Remark 4.8 below.

The remaining part of the proof is devoted to proving the claim (4.74); see Remark 4.8 below for the case $L' = \infty$. Let $\varphi \in L^2(\mathbb{R}^2)$. Then, from (4.43) with (4.4) and (4.5), we have
\[
\langle S(t)Z_L(1), \varphi \rangle = \sqrt{2\xi} \left( \mathbf{1}_{[0,1]}(t') \mathbf{1}_{[-\frac{1}{2L}, \frac{1}{2L}]} P(1 - t') S(t) \varphi^\text{per}_L \right),
\]
where $\varphi^\text{per}_L$ is the $L$-periodized version of $\varphi$ defined by
\[
\varphi^\text{per}_L(x) = \sum_{n \in \mathbb{Z}^2} \varphi(x + nL). \quad (4.77)
\]
Given $x \in \mathbb{R}^2$, we then have
\[
(S(t)Z_L(1))(x) = \langle S(t)Z_L(1), \delta_x \rangle \\
= \sqrt{2\xi} \left( \mathbf{1}_{[0,1]}(t') \mathbf{1}_{[-\frac{1}{2L}, \frac{1}{2L}]} P(1 - t') S(t) \delta_x^\text{per}_L \right),
\]
where $\delta_x$ denotes the Dirac delta function at $x \in \mathbb{R}^2$, and hence we can rewrite (4.70) as
\[
K_{L,L'}(x,y,t) = 2\mathbb{E}\left[ \xi \left( \mathbf{1}_{[0,1]}(t') \mathbf{1}_{[-\frac{1}{2L}, \frac{1}{2L}]} P(1 - t') S(t) \delta_x^\text{per}_L \right) \right. \\
\times \left. \xi \left( \mathbf{1}_{[0,1]}(t') \mathbf{1}_{[-\frac{1}{2L}, \frac{1}{2L}]} P(1 - t') S(t) \delta_y^\text{per}_{L'} \right) \right] \\
= 2 \int_{\mathbb{R}^2} \int_0^1 \left( \mathbf{1}_{[-\frac{1}{2L}, \frac{1}{2L}]} P(t') S(t) \delta_x^\text{per}_L \right)(z) \\
\times \left( \mathbf{1}_{[-\frac{1}{2L'}, \frac{1}{2L'}]} P(t') S(t) \delta_y^\text{per}_{L'} \right)(z) dt' dz. \quad (4.78)
\]
In the following, we assume $x, y \in B_{2R}$. By the finite speed of propagation, we have
\[
\text{supp } S(t) \delta_x + nL \subset x + nL + B_t. \quad (4.79)
\]
Let $\{\psi^L_j\}_{j \in \mathbb{Z}^2_{\geq 0}}$ be a smooth partition of unity on $\mathbb{R}^2$ such that $\psi^L_0 = 1$ on $B_{\frac{L}{2}}$, $\text{supp } \psi^L_0 \subset B_{\frac{L}{2}}$, and
\[
\text{supp } \psi^L_j \subset \left\{ \max \left( (j - 1), \frac{1}{2L} \right) \leq |z| \leq (j + 1)L \right\}. \quad (4.80)
\]
In particular, for $j = 0$, we set $\psi^L_0(z) = \psi_0(\frac{z}{L})$ for some $\psi_0 \in C^\infty_c(\mathbb{R}^2)$. Here, we choose $\{\psi^L_j\}_{j \in \mathbb{Z}^2_{\geq 0}}$ such that their first and second derivatives are uniformly bounded. Then, we
have
\[ \mathbf{1}_{[-\frac{L}{2}, \frac{L}{2})} P(t') S(t)(\delta_x)_L \perp = e^{-t'} \sum_{j=0}^{\infty} \mathbf{1}_{[-\frac{L}{2}, \frac{L}{2})} \left\{ (\psi_j^L p_{\nu}) \ast (S(t)(\delta_x)_L \perp) \right\}, \]
where \( p_{\nu} \) is as in (2.6). Under the assumption \( L \gg R + t \) and \( x \in B_{2R} \), it follows from (4.77) and (4.79) that
\[ e^{-t'} \mathbf{1}_{[-\frac{L}{2}, \frac{L}{2})} (z) \cdot \left\{ (\psi_0^L p_{\nu}) \ast (S(t)(\delta_x)_L \perp) \right\}(z) = e^{-t'} (\psi_0^L p_{\nu}) \ast (S(t)\delta_x)(z) = e^{-t'} S(t)(\psi_0^L p_{\nu})(z-x). \]
(4.81)

By the boundedness of \( S(t) \) on \( H^s(\mathbb{R}^2) \) and \( \psi_j^L(z) = \psi_0(\frac{z}{L}) \) for some \( \psi_0 \in C_0^\infty(\mathbb{R}^2) \), we have
\[ \|4.81\|_{H^{s-1}} \lesssim e^{-t'} \|\psi_0^L p_{\nu}\|_{H^{s-1}} \]
\[ \lesssim e^{-t'} \left\| \int_{\eta = \eta_1 + \eta_2} L^2 \widehat{\psi}_0(L\eta_1) e^{-4\pi^2 t' |\eta|^2} d\eta \right\|_{L^\infty_{\eta}} \]
\[ \lesssim e^{-t'} \int_{\eta \in \mathbb{R}} \frac{L^2}{(L\eta_1)^{100}} d\eta_1 \lesssim e^{-t'}, \]
(4.82)
uniformly in \( x \in B_{2R} \), \( t \geq 0 \), and \( t' \in [0, 1] \) with \( R + t \ll L \).

For \( j \in \mathbb{N} \), it follows from (4.77) and (4.81) with \( L \gg R + t \) and \( x \in B_{2R} \) that
\[ e^{-t'} \mathbf{1}_{[-\frac{L}{2}, \frac{L}{2})} (z) \cdot \left\{ (\psi_j^L p_{\nu}) \ast (S(t)(\delta_x)_L \perp) \right\}(z) \]
\[ = e^{-t'} \mathbf{1}_{[-\frac{L}{2}, \frac{L}{2})} (z) \sum_{n \in \mathbb{Z}^2} \left\{ (\psi_j^L p_{\nu}) \ast (S(t)\delta_{x+nL}) \right\}(z) \]
\[ = e^{-t'} \mathbf{1}_{[-\frac{L}{2}, \frac{L}{2})} (z) \left\{ \sum_{n \in \mathbb{Z}^2 \setminus \{0\}} S(t)(\psi_j^L p_{\nu})(z-x-nL) \right\} \]
\[ + \mathbf{1}_{|z| \geq \frac{L}{10}} S(t)(\psi_j^L p_{\nu})(z-x), \]
(4.83)
where the second term of the last factor is relevant only for \( j = 1 \). Note that, for \( |z| \geq \frac{L}{10} \) and \( x \in B_{2R} \) with \( L \gg R + t \), we have \( |z-x| \geq \frac{L}{16} \). From the definition of \( \psi_j^L \) and (2.6), we then have
\[ \|4.83\|_{L^2_{\mathbb{R}}} \lesssim je^{-t'} \|\psi_j^L p_{\nu}\|_{L^2_{\mathbb{R}^2}(|z| \geq \frac{L}{10})} \]
\[ \lesssim \begin{cases} \exp \left( -c j^2 t^2 \left| \frac{L}{j} \right| - t' \right), & \text{for } 0 < t' \ll jL, \\ \exp(-c j L), & \text{for } t' \sim jL, \\ \exp(-ct'), & \text{for } t' \gg jL. \end{cases} \]
Thus, we obtain
\[ \|4.83\|_{L^2_{\mathbb{R}}} \lesssim e^{-cjL}, \]
(4.84)
uniformly in \( x \in B_{2R} \), \( t \geq 0 \), and \( t' \in [0, 1] \) with \( R + t \ll L \).

Let \( \chi_L \in C_0^\infty(\mathbb{R}^2; [0, 1]) \) be a smooth cutoff function such that \( \chi_L \equiv 1 \) on \( [-\frac{L}{2}, \frac{L}{2}) \) and \( \text{supp} \chi_L \subseteq \left[ -\frac{(1+\gamma)L}{2}, \frac{(1+\gamma)L}{2} \right] \) for some small \( \gamma > 0 \). We assume that the first and second
derivatives of $\chi_L$ are bounded uniformly in $L \gg 1$. Define $F(z) = F_{j,L,x,t,t'}(z)$ by

$$F(z) = F_{j,L,x,t,t'}(z) = e^{-t'} \chi(z) \cdot \left\{ (\psi_j^L p_{t'}) \ast (\mathcal{S}(t)(\delta_x)_L^{\text{per}}) \right\}(z).$$

Then, a similar computation with the uniform boundedness of the first and second derivatives of $\psi_j^L$ shows that

$$\|F_{j,L,x,t,t'}\|_{L^2} \lesssim \|\Delta z F_{j,L,x,t,t'}\|_{L^2} \lesssim e^{-c j L},$$

uniformly in $x \in B_{2R}$, $t \geq 0$, and $t' \in [0,1]$ with $R + t \ll L$.

In view of the discussion above (in particular, see (4.81)), we write $K_{L,L'}(x, y; t)$ in (4.84) as

$$K_{L,L'}(x, y; t) = K_{L,L'}^{(0)}(x, y; t) + K_{L,L'}^{(1)}(x, y; t),$$

where $K_{L,L'}^{(0)}(x, y; t)$ is defined by

$$K_{L,L'}^{(0)}(x, y; t) = 2 \int_{\mathbb{R}^2} \int_0^1 \left( e^{-t'} (\psi_0^L p_{t'}) \ast (\mathcal{S}(t)(\delta_x)_L^{\text{per}}) \right)(z) \times \left( e^{-t'} (\psi_0^{L'} p_{t'}) \ast (\mathcal{S}(t)(\delta_y)_{L'}^{\text{per}}) \right)(z) dt' dz$$

and $K_{L,L'}^{(1)}(x, y; t) = K_{L,L'}(x, y; t) - K_{L,L'}^{(0)}(x, y; t)$. Then, from (4.78) and (4.87), we have

$$K_{L,L'}^{(1)}(x, y; t) = 2 \sum_{j=1}^\infty \sum_{j'=1}^\infty \int_{\mathbb{R}^2} \int_0^1 \left( 1_{[-\frac{L}{2}, \frac{L}{2}]^2} e^{-t'} (\psi_j^L p_{t'}) \ast (\mathcal{S}(t)(\delta_x)_L^{\text{per}}) \right)(z) \times \left( 1_{[-\frac{L'}{2}, \frac{L'}{2}]^2} e^{-t'} (\psi_{j'}^{L'} p_{t'}) \ast (\mathcal{S}(t)(\delta_y)_{L'}^{\text{per}}) \right)(z) dt' dz$$

$$+ 2 \sum_{j=1}^\infty \int_{\mathbb{R}^2} \int_0^1 \left( 1_{[-\frac{L}{2}, \frac{L}{2}]^2} e^{-t'} (\psi_j^L p_{t'}) \ast (\mathcal{S}(t)(\delta_x)_L^{\text{per}}) \right)(z) \times \left( e^{-t'} (\psi_0^{L'} p_{t'}) \ast (\mathcal{S}(t)(\delta_y)_{L'}^{\text{per}}) \right)(z) dt' dz$$

$$+ 2 \sum_{j'=1}^\infty \int_{\mathbb{R}^2} \int_0^1 \left( e^{-t'} (\psi_0^L p_{t'}) \ast (\mathcal{S}(t)(\delta_y)_{L'}^{\text{per}}) \right)(z) \times \left( 1_{[-\frac{L'}{2}, \frac{L'}{2}]^2} e^{-t'} (\psi_{j'}^{L'} p_{t'}) \ast (\mathcal{S}(t)(\delta_x)_L^{\text{per}}) \right)(z) dt' dz$$

$$=: I(x, y; t) + II(x, y; t) + III(x, y; t).$$

Note that, in view of (4.81), we dropped the cutoff function $1_{[-\frac{L'}{2}, \frac{L'}{2}]^2}$ (and $1_{[-\frac{L}{2}, \frac{L}{2}]^2}$) in II (and in III, respectively). By Cauchy-Schwarz’s inequality with (4.83) and (4.84), we have

$$|I(x, y; t)| \lesssim \sum_{j=1}^\infty \sum_{j'=1}^\infty e^{-c j L} e^{-c j' L'} \lesssim e^{-c(L+L')}.$$  

Next, we estimate II and III in (4.88). Without loss of generality, assume $L \geq L'$. As for II, we first insert $\chi_L$ and then drop the cutoff function $1_{[-\frac{L}{2}, \frac{L}{2}]^2}$ in view of (4.81). Thus,
from (4.82) and (4.85), we have
\[ |\Pi(x, y; t)| = 2 \sum_{j=1}^{\infty} \int_{\mathbb{R}^2} \int_{0}^{1} \left( e^{-t'} \chi_L(z)(\psi_j^{L'} p_{t'}) * (S(t)(\delta_x)_{L'}^{\text{per}}) \right)(z) \times \left( e^{-t'} (\psi_0^{L'} p_{t'}) * (S(t)(\delta_y)_{L'}^{\text{per}}) \right)(z) dt' dz \]
\[ \lesssim \sum_{j=1}^{\infty} \int_{0}^{1} \left\| e^{-t'} \chi_L(z)(\psi_j^{L'} p_{t'}) * (S(t)(\delta_x)_{L'}^{\text{per}}) \right\|_{H_x^2} \times \left\| e^{-t'} (\psi_0^{L'} p_{t'}) * (S(t)(\delta_y)_{L'}^{\text{per}}) \right\|_{H_x^{-1-\varepsilon}} dt' \]
\[ \lesssim e^{-cL}. \]

As for the term III, we can not drop the cutoff function $1_{[-\frac{L}{2}, \frac{L}{2}]^2}$ under the current assumption $L \geq L'$, and thus we need to proceed with more care. We first note that by the mean value theorem, we have
\[ |\mathcal{F}(1_{[-\frac{L}{2}, \frac{L}{2}]^2})(\eta)| \lesssim \frac{(L')^2}{\langle \eta \rangle^2}, \quad (4.91) \]
where $\eta = (\eta^1, \eta^2)$. Then, proceeding as in (4.82) with the triangle inequality $\langle \eta_1 \rangle^\varepsilon \lesssim \langle \eta \rangle^\varepsilon \langle \eta_2 \rangle^\varepsilon \langle \eta_3 \rangle^\varepsilon$, Young’s inequality, and (4.91), we have
\[ \left\| 1_{[-\frac{L}{2}, \frac{L}{2}]^2}(z)S(t)(\psi_j^{L'} p_{t'})(z - x) \right\|_{H_x^{-1-2\varepsilon}} \lesssim \left\| \mathcal{F}(1_{[-\frac{L}{2}, \frac{L}{2}]^2})(\eta_1) \right\|_{L^2} \left\| \tilde{\psi}_0(L\eta_2) e^{-4\pi^2 t'|\eta_1|^2} d\eta_2 d\eta_1 \right\|_{L^2_0} \lesssim \frac{1}{\langle \eta \rangle^{1+\varepsilon}} \int_{\eta=\eta_1+\eta_2+\eta_3} \langle \eta_1 \rangle^{-\varepsilon} \mathcal{F}(1_{[-\frac{L}{2}, \frac{L}{2}]^2})(\eta_1) \right\|_{L^2_0} \times \langle \eta_2 \rangle^\varepsilon L^2 \left\| \tilde{\psi}_0(L\eta_2) \right\|_{L^2_0} \lesssim (L')^2 (t')^{\frac{-8}{7}}. \quad (4.92) \]
Hence, from (4.85) and (4.92), we obtain
\[ \left| III(x, y; t) \right| = 2 \sum_{j=1}^{\infty} \int_{\mathbb{R}^2} \int_{0}^{1} e^{-t'} 1_{[-\frac{L}{2}, \frac{L}{2}]^2}(z)S(t)(\psi_j^{L'} p_{t'})(z - x) \times \left( e^{-t'} \chi_L(z)(\psi_j^{L'} p_{t'}) * (S(t)(\delta_x)_{L'}^{\text{per}}) \right)(z) dt' dz \]
\[ \lesssim (L')^2 e^{-cL'} \int_{0}^{1} (t')^{-\frac{8}{7}} dt' \lesssim e^{-cL'}. \quad (4.93) \]
Therefore, from (4.88), (4.89), (4.90), and (4.93), we obtain
\[ |K^{(1)}_{L,L'}(x, y; t)| \lesssim e^{-cL} + e^{-cL'}. \quad (4.94) \]
Next, we study $K^{(0)}_{L,L'}(x, y; t)$ for $L, L' \gg 1$. Define $K_{\infty, \infty}(x, y; t)$ by

$$K_{\infty, \infty}(x, y; t) = 2 \int_{\mathbb{R}^2} \int_0^1 \left( e^{t'} (1 - \psi_0^L) * (S(t) \delta_x) \right)(z)(P(t') S(t) \delta_y)(z) dt' dz$$

$$= \langle (1 - \Delta)^{-1} (1 - e^{-2\langle 1, \Delta \rangle}) S(t) \delta_x, S(t) \delta_y \rangle_{L^2}.$$  

(4.95)

From (4.87) and (4.95), we have

$$K_{\infty, \infty}(x, y; t) - K_{L, L'}^{(0)}(x, y; t)$$

$$= 2 \int_{\mathbb{R}^2} \int_0^1 \left( e^{-t'} ((1 - \psi_0^L) p_{t'}) * (S(t) \delta_x) \right)(z)$$

$$\times \left( P(t') S(t) \delta_y \right)(z) dt' dz$$

$$+ 2 \int_{\mathbb{R}^2} \int_0^1 \left( e^{-t'} (\psi_0^L p_{t'}) * (S(t) \delta_x) \right)(z)$$

$$\times \left( e^{-t'} ((1 - \psi_0^L) p_{t'}) * (S(t) \delta_y) \right)(z) dt' dz$$

$$= 2 \int_{\mathbb{R}^2} \int_0^1 \left( e^{-t'} S(t)((1 - \psi_0^L) p_{t'}) \right)(z - x)$$

$$\times \left( e^{-t'} S(t) p_{t'} \right)(z - y) dt' dz$$

$$+ 2 \int_{\mathbb{R}^2} \int_0^1 \left( e^{-t'} S(t)(\psi_0^L p_{t'}) \right)(z - x)$$

$$\times \left( e^{-t'} S(t)((1 - \psi_0^L) p_{t'}) \right)(z - y) dt' dz$$

$$= : IV(x, y; t) + V(x, y; t).$$

We only consider the first term IV on the right-hand side of (4.96) since the term V can be treated in a similar manner. Recalling that $\psi_0^L = 1$ on $B_{\frac{L}{16}}$, we have

$$\|e^{-t'} (1 - \psi_0^L) p_{t'}\|_{L^2} \lesssim e^{-\frac{cL}{8} t'} \lesssim e^{-cL},$$

(4.97)

where the second inequality follows from separately estimating the cases (i) $t' \ll L$, (ii) $t' \sim L$, and (iii) $t' \gg L$. Moreover, since $\psi_0^L = 1$ on $B_{\frac{L}{16}}, L \gg R + t$, and $x, y \in B_{2R}$, we have

$$|z - y| \geq |z - x| - |x| - |y| \geq \frac{L}{8} - t - 4R \geq \frac{L}{16}$$

in the domain of integration for IV in (4.96). Under this condition, we have, as in (4.84),

$$\|1_{|z - y| \geq \frac{L}{16}} e^{-t'} S(t) p_{t'} (z - y)\|_{L^2} \lesssim e^{-cL}.$$  

(4.98)

Then, by applying Cauchy-Schwarz’s inequality with (4.97) and (4.98), we obtain

$$|IV(x, y; t)| \lesssim e^{-cL}.$$  

(4.99)

A similar computation yields

$$|V(x, y; t)| \lesssim e^{-cL'}.$$  

(4.100)

Therefore, from (4.96), (4.99), and (4.100), we conclude that

$$|K_{\infty, \infty}(x, y; t) - K_{L, L'}^{(0)}(x, y; t)| \lesssim e^{-cL} + e^{-cL'}.$$  

(4.101)
It remains to estimate \( K_{\infty,\infty}(x, y; t) \). In view of \((4.95)\) with \((4.42)\), we have
\[
K_{\infty,\infty}(x, y; t) = K^t(x - y)
\]
\[
: = \int_{\mathbb{R}^2} \frac{(\hat{S}(t)(\eta))^2}{(2\pi \eta)^2} (1 - e^{-(2\pi \eta)^2}) e^{2\pi i \eta \cdot (x - y)} d\eta.
\]
(4.102)
From \((4.42)\) with \((4.41)\), we have
\[
\left|(\hat{S}(t)(\eta))^2 (1 - e^{-(2\pi \eta)^2}) - e^{-t} \cos^2(t \eta)\right| \lesssim \frac{1}{\eta},
\]
(4.103)
uniformly in \( t \in \mathbb{R}_+ \). Then, by defining \( K^t_1 \) by
\[
K^t_1(z) = e^{-t} \int_{\mathbb{R}^2} \frac{\cos^2(t \eta)}{(2\pi \eta)^2} e^{2\pi i \eta \cdot z} d\eta,
\]
(4.104)
it follows from \((4.102)\) and \((4.103)\) that
\[
|K_{\infty,\infty}(x, y; t) - K^t_1(x - y)| \lesssim 1,
\]
(4.105)
uniformly in \( t \in \mathbb{R}_+ \) and \( x, y \in \mathbb{R}^2 \).

We now estimate \( K^t_1 \). Recall from Appendix B.5 in \([33]\) that the Fourier transform of a radial function \( F(x) = f(|x|) \) on \( \mathbb{R}^2 \) is given by
\[
\hat{F}(\eta) = 2\pi \int_0^\infty f(r) J_0(2\pi r |\eta|) r dr,
\]
(4.106)
where \( J_0 \) is the Bessel function of order zero, satisfying the following asymptotics as \( r \to \infty \):
\[
J_0(r) = \sqrt{\frac{2}{\pi r}} \cos \left( r - \frac{\pi}{4} \right) + O(r^{-\frac{3}{2}}).
\]
(4.107)
See Appendix B.8 in \([33]\) for the asymptotics \((4.107)\). Thus, from \((4.104)\) and \((4.106)\), we have
\[
K^t_1(z) = 2\pi e^{-t} \int_0^\infty \frac{\cos^2(t \eta)}{(2\pi \eta)^2} J_0(2\pi r |z|) r dr.
\]
(4.108)
Recalling that \( J_0 \) is bounded on \( \mathbb{R}_+ \), it follows from \((4.108)\) with \((4.107)\) that
\[
|K^t_1(z)| \lesssim e^{-t} \int_0^{\max(\frac{1}{|z|}, 1)} \frac{1}{\eta} d\eta + \int_{\max(\frac{1}{|z|}, 1)}^\infty \frac{1}{r^{\frac{3}{2}} |z|^\frac{1}{2}} dr
\]
(4.109)
\[
\lesssim 1 + (- \log |z|)_+.
\]
Hence, putting together \((4.86), (4.94), (4.101), (4.105),\) and \((4.109)\), we obtain
\[
|K_{L,L}(x, y; t)| + |K_{\infty,\infty}(x, y; t)| \lesssim 1 + (- \log |x - y|)_+.
\]
(4.110)
Therefore, the claim \((4.74)\) follows from \((4.86), (4.94), (4.101),\) and \((4.110)\). \( \square \)

Remark 4.8. When \( L' = \infty \), we write
\[
K_{L,\infty}(x, y; t) = K_{L,\infty}^{(0)}(x, y; t) + K_{L,\infty}^{(1)}(x, y; t),
\]
(4.111)
where
\[
K_{L,\infty}^{(0)}(x, y; t) = 2 \int_{\mathbb{R}^2} \int_0^1 \left( e^{-t'} (\psi_0^t p(r) * (S(t) \delta_x))(z) \right. \\
\times \left. (P(t') S(t) \delta_y)(z) \right) dt' dz
\]
(4.112)
and

\[ K_{L, \infty}^{(1)}(x, y; t) = 2 \sum_{j=1}^{\infty} \int_{\mathbb{R}^2} \int_0^1 \left( 1_{\left[ -\frac{1}{2}, \frac{1}{2} \right]} e^{-t^r} \phi_j^r \right) \left( \psi_j^r \right) (S(t) (\delta_0^r \mid L)) \right) (z) \times \left( P(t') S(t \delta_0^r) (z) dt' dz. \right) \]

Proceeding as in (4.92), we have

\[ \| 1_{\left[ -\frac{1}{2}, \frac{1}{2} \right]} (P(t') S(t \delta_0^r) (z) \right) \|_{H^{-\infty}_{-\infty}} \lesssim L^2 (t')^{-\frac{\varepsilon}{2}}. \]

Then, proceeding as in (4.93), we obtain

\[ |K_{L, \infty}^{(1)}(x, y; t)| \lesssim e^{-cL}. \]  

From (4.95) and (4.112), we have

\[ K_{\infty, \infty} (x, y; t) - K_{L, \infty}^{(1)} (x, y; t) \]

\[ = 2 \int_{\mathbb{R}^2} \int_0^1 \left( \right) \left( \psi_j^r \right) (S(t) (\delta_0^r \mid L)) \right) (z) \times \left( P(t') S(t \delta_0^r) (z) dt' dz, \right) \]

which is precisely IV(x, y; t) in (4.96). Hence, from (4.99), we have

\[ |K_{\infty, \infty} (x, y; t) - K_{L, \infty}^{(1)} (x, y; t)| \lesssim e^{-cL}. \]  

Therefore, putting (4.111), (4.113), and (4.114) together, we obtain

\[ |K_{\infty, \infty} (x, y; t) - K_{L, \infty} (x, y; t)| \lesssim e^{-cL}. \]  

On the other hand, from (4.74) and (4.115), we have

\[ |K_{L, L} (x, y; t)| + |K_{L, \infty} (x, y; t)| + |K_{\infty, \infty} (x, y; t)| \lesssim 1 + (- \log |x - y|)_{+}, \]

\[ |K_{L, L} (x, y; t) - K_{\infty, \infty} (x, y; t)| \lesssim e^{-cL}. \]  

A computation analogous to (4.76) with Lemma 2.14 (4.75) (with L' = \infty), (4.113), and (4.116) yields

\[ \mathbb{E} \left[ \| (S(t) Z_L(1))_{m'} : (S(t) Z_{\infty}(1))_{m'} : \|_{W^{-\epsilon, p1} (B_r)} \right] \]

\[ \lesssim C(R, r) e^{-cL} \rightarrow 0, \]

as L \to \infty. This proves (4.69).

4.3. Estimates on the enhanced data sets. In this subsection, we establish the following proposition regarding the regularity of the enhanced data sets.

**Proposition 4.9.** (i) Let R > 0. Given small \( \varepsilon > 0 \), there exists finite \( p = p(\varepsilon) \gg 1 \) such that the following holds true. Suppose that random \( k \)-tuple of functions \( \phi_0, \psi_0 \in \mathcal{W}(R) \) are independent of \( \phi_1 = \phi_1(u_1, \xi) \) defined in (4.10), where \( \mathcal{W}(R) \) is as in (4.15). Then, we have

\[ \mathbb{E}_{u_1, \xi, \phi_0, \psi_0} \left[ \| \Xi_1(u_1, \xi) \|_{\Xi_0, \Xi_0, R} \right] \lesssim 1, \]

where the implicit constant is independent of the distributions of \( \Xi_0 \) and \( \Xi_0 \). Here, \( \Xi_1(u_1, \xi) \) is as in (4.13) and \( \| \cdot \|_{\Xi_0, \Xi_0, R} \) is as in Definition 4.1 (iii).
(ii) Let \( \Xi_0(\phi) \) be as in (4.12). Then, given any \( \varepsilon > 0 \), finite \( p \gg 1 \), and \( \mu > 0 \), we have
\[
\sup_{L \in A} E_{pl} \left[ \| \Xi_0(\phi) \|_{(L^\infty_{\mu} W_\infty^{\varepsilon,p})^\otimes k} \right] < \infty. \tag{4.117}
\]

We first state and prove an auxiliary lemma. The proof of Proposition 4.3 is presented at the end of this subsection.

**Lemma 4.10.** Let \( \Phi_1 = \Phi_1(u_1, \xi) \) be as in (4.10), where \( u_1 \) and \( \xi \) denote the independent spatial and space-time white noises as in (4.11). Define \( H(x, y; t) \) by
\[
H(x, y; t) = E[\Phi_1(x, t) \Phi_1(y, t)]. \tag{4.118}
\]
Let \( R > 0 \). Then, given small \( \varepsilon > 0 \) and \( 1 \leq q < \frac{1}{\varepsilon} \), there exists a finite constant \( C(R) > 0 \) such that
\[
\| \langle \nabla \rangle^\varepsilon \langle \nabla \rangle^\varepsilon H(x, y; t) \|_{L^q_{\infty, R}(B_R \times B_R)} \leq C(R), \tag{4.119}
\]
uniformly in \( t \in \mathbb{R}_+ \).

**Proof.** With a slight abuse of notation, it follows from Definition 3.4 and an analogous property for the space-time white noise \( \xi \) that
\[
H(x - y; t) = H(x, y; t)
\]
\[
= \langle D(t) \delta_x, D(t) \delta_y \rangle_{L^2(\mathbb{R}^2)} + \int_0^t \langle D(t - t') \delta_x, D(t - t') \delta_y \rangle_{L^2(\mathbb{R}^2)} dt'.
\]
Then, we have
\[
\langle \nabla \rangle^\varepsilon \langle \nabla \rangle^\varepsilon H(x - y; t) = \langle \nabla \rangle^\varepsilon H(z; t) |_{z = x - y}. \tag{4.120}
\]
Define \( \tilde{H}^t(z) \) by
\[
\tilde{H}^t(z) = \int_{\mathbb{R}^2} \frac{\sin^2(\frac{t}{2} [2 \pi \eta])}{([2 \pi \eta]^2)^2} e^{2 \pi i \eta \cdot z} d\eta.
\]
Then, from (1.23), we have
\[
H(z; t) = e^{-t} \tilde{H}^t(z) + \int_0^t e^{-t'} \tilde{H}^{t'}(z) dt'. \tag{4.121}
\]
In view of (4.120) and (4.121), it suffices to study
\[
\langle \nabla \rangle^\varepsilon \langle \nabla \rangle^\varepsilon \tilde{H}^t(z) = \int_{\mathbb{R}^2} \langle \eta \rangle^2 \sin^2(\frac{t}{2} [2 \pi \eta]) \frac{e^{2 \pi i \eta \cdot z}}{([2 \pi \eta]^2)^2} d\eta.
\]
As the inverse Fourier transform of a radial function, \( \langle \nabla \rangle^\varepsilon \langle \nabla \rangle^\varepsilon \tilde{H}^t \) is radial. Hence, from (4.106), we have
\[
\langle \nabla \rangle^\varepsilon \tilde{H}^t(z) = 2\pi \int_0^\infty \langle r \rangle^2 \sin^2(\frac{t}{2} [2 \pi r]) \frac{J_0(2 \pi r |z|) r}{[2 \pi r]^2} dr,
\]
where \( J_0 \) is the Bessel function of order zero. Using the boundedness of \( J_0 \) on \( \mathbb{R}_+ \) and (4.107), we then have
\[
|\langle \nabla \rangle^\varepsilon \tilde{H}^t(z)| \lesssim \int_0^{\max(\frac{1}{\varepsilon}, 1)} \frac{1}{(r)^{1-2\varepsilon}} dr + \int_0^\infty \frac{1}{\max(\frac{1}{\varepsilon}, 1) r^{\frac{7}{2}-2\varepsilon} |z|^\frac{7}{2}} dr \tag{4.122}
\]
\[
\lesssim 1 + |z|^{-2\varepsilon}.
\]
Therefore, from (4.120), (4.121), and (4.122), we obtain

\[ |(\nabla_x)^\varepsilon \langle \nabla_y \rangle^\varepsilon H(x, y; t)| \lesssim 1 + |x - y|^{-2\varepsilon}, \]

uniformly in \( t \in \mathbb{R}_+ \). Then, by integrating in \( x, y \in B_R \), we obtain (4.119), provided that \( q < \frac{1}{\varepsilon} \).

We are now ready to present the proof of Proposition 4.9.

**Proof of Proposition 4.9.** (i) We only show that

\[ \mathbb{E} \left[ \| \Xi_1(u_1, \xi) \|_{\Xi_0, R} \right] < \infty, \tag{4.123} \]

where \( \| \Xi_1(u_1, \xi) \|_{\Xi_0, R} \) is as in Definition 4.1(iii). A similar argument yields finiteness of the expectations of the constants \( K_2 \) and \( K_3 \) in (4.17) and (4.18), respectively.

Let \( j = 1, \ldots, k \). Given a function \( f \) on \( B_R \subset \mathbb{R}^2 \), let \( \tilde{f} \) an extension of \( f \) onto \( \mathbb{R}^2 \). Given a test function \( \varphi \in C_c^\infty(\mathbb{R}^2) \) supported on a ball \( B \) of radius 1 with \( B \subset B_{2R} \), it follows from Lemma 2.13 that

\[
\mathbb{E} \left[ \left| \left\langle f : \Phi_1^j(t) : \varphi \right\rangle_{L^2(\mathbb{R}^2)} \right|^2 \right]
= \int_{(B_{2R})^2} f(x) \tilde{f}(y) \varphi(x) \varphi(y) \mathbb{E} \left[ : \Phi_1^j(x, t) : : \Phi_1^j(y, t) : \right] dx \, dy
= j! \int_{\mathbb{R}^4} f(x) \tilde{f}(y) \varphi(x) \varphi(y) (H(x, y; t))^j dx \, dy,
\]

where \( H(x, y; t) \) is as in (4.118). Then, given small \( \varepsilon > 0 \), it follows from the duality, Lemma 2.8(ii), the bi-parameter fractional Leibniz rule (Lemma 2.10, and Lemma 4.10) that there exist finite \( p = p(\varepsilon) \gg 1 \) and \( q = q(\varepsilon) \gg 1 \) (with \( jq < \frac{1}{\varepsilon} \leq \frac{p}{2} \)) such that

\[
\mathbb{E} \left[ \left| \left\langle f : \Phi_1^j(t) : \varphi \right\rangle_{L^2(\mathbb{R}^2)} \right|^2 \right]
\lesssim \| f \varphi \|^2_{W^{-\varepsilon, p}(B_{2R})} \left\| (\nabla_x)^\varepsilon (\nabla_y)^\varepsilon (H(x, y; t))^j \right\|^2_{L^q_{\varepsilon,y}(B_{2R} \times B_{2R})}
\lesssim \| \varphi \|^2_{W^{\varepsilon, p'}(B_{2R})} \| f \|^2_{L^p_{\varepsilon, p}(B_{2R})} \| (\nabla_x)^\varepsilon (\nabla_y)^\varepsilon H(x, y; t) \|^j_{L^q_{\varepsilon,y}(B_{2R} \times B_{2R})}
\lesssim \| \varphi \|^2_{W^{\varepsilon, p'}(B_{2R})} \| f \|^2_{W^{\varepsilon, p}(\mathbb{R}^2)}.
\]

Hence, by taking an infimum over the extension \( \tilde{f} \) and applying Lemma 2.14 (with \( j \leq k \)), we obtain

\[
\mathbb{E} \left[ \| f : \Phi_1^j(t) : \|_{W^{-2(k+1), \varepsilon, p}(B_R)} \right] \lesssim \| f \|^p_{W^{-\varepsilon, p}(B_R)},
\]

uniformly in \( t \in \mathbb{R}_+ \). Then, by first conditioning on \( \Xi_0 \), we obtain

\[
\mathbb{E} \left[ \frac{\int_0^T \| \Phi_1^j(t) : \Xi_0(t) \|_{W^{-2(k+1), \varepsilon, p}(B_R)} dt}{\int_0^T \| \Xi_0(t) \|_{W^{-\varepsilon, p}(B_R)} dt} \right] \leq A_{j,t} < \infty.
\]
Hence, we conclude that

\[ E[\Vert \Xi_1(u_1, \xi)\Vert_{\mathcal{C}, R}] = E\left[ \max_{0 \leq j, \ell \leq k} \int_0^T \left\| \Phi_j^\ell(t) : \Xi_0(t)\right\|_{W^{-2(k+1),p}(B_R)}^p dt \right] \]

\[ \leq \sum_{j, \ell=0}^k A_{j, \ell} < \infty. \]

This proves (4.123).

(ii) The bound (4.117) follows from the proof of Proposition 4.4. Namely, in view of (4.37), (4.39), and (4.60) with (4.40), (4.56), (4.57), (4.58), and (4.59), it follows from Kolmogorov’s continuity criterion ([5, Theorem 8.2]) together with the exponential decay in time (see Remark 4.5) that

\[ \rho_L\left(\Vert \Xi_0(\phi)\Vert_{L^\infty W^{-2, p}_\nu} > k\right) \lesssim \frac{1}{K^q} \]

for any finite \(q \geq 1\), from which the desired bound (4.117) follows. \(\square\)

4.4. Global well-posedness on the plane. We are now ready to prove global well-posedness of the hyperbolic \(\Phi_2^{k+1}\)-model (4.1) on the plane. We do this by constructing a solution on each cone \(C_R, R > 0\).

Proposition 4.11. Let \(\bar{\rho}_\infty\) be as in Theorem 1.2(i) and Law\((u_0, u_1) = \bar{\rho}_\infty\). Then, given \(R > 0\), the hyperbolic \(\Phi_2^{k+1}\)-model (4.1) is well-posed on the cone \(C_R\). More precisely, there exists a function \(u\) of the form (4.8) such that the following statements holds true \(\bar{\rho}_\infty\)-almost surely:

- \((u, \partial_t u) \in L^\infty([0, R]; H^{-\varepsilon_k}(B_{R-I}))\),
- the remainder term \(v = u - S(t)u_0 - D(t)u_1 - \Psi\) satisfies the equation (4.11) on \(C_R\), and \((v, \partial_t v) \in L^\infty([0, R]; H^{1-\varepsilon_k}(B_{R-I}))\),

where \(\varepsilon_k = 2(k+1)\varepsilon\) is as in (4.19).

For the convergence of the \(L\)-periodic solution \(u_L\, L \in \mathcal{A}\), to the solution \(u\) constructed in Proposition 4.11 see Remark 4.12 below.

Proof. Let \(\mathcal{A} = \{L_j : j \in \mathbb{N}\} \subset \mathbb{N}\) be the index set such that Theorem 1.2(i) and Proposition 4.6 hold. Given a spatial white noise \(\zeta\) on \(\mathbb{R}^2\) and a space-time white noise \(\xi\) on \(\mathbb{R}^2 \times \mathbb{R}_+\), let \(\zeta_L\) be the \(L\)-periodic spatial white noise defined in (3.53) and (3.54), and \(\xi_L\) be the \(L\)-periodic space-time white noise defined in (4.2) and (4.3), \(L \geq 1\), respectively.

In Proposition 4.6 we proved that the \(L_j\)-periodic enhanced Gibbs measure \(\nu_{L_j} = (\Xi_0)_{\#}\rho_{L_j}\) on \(\mathcal{W} = (C(\mathbb{R}_+; W^{-\varepsilon,p}(\mathbb{R}^2)))_{\#}^{\otimes k}\) converges weakly to \(\nu_{\infty} = (\Xi_0)_{\#}\rho_{\infty}\) as \(j \to \infty\). Then, by the Skorokhod representation theorem (Lemma 2.14), there exist a probability space \((\tilde{\Omega}, \tilde{F}, \tilde{P})\), and random variables \(\Xi_j^0, \Xi_0 : \tilde{\Omega} \to \mathcal{W}\) such that

\[ \text{Law}(\Xi_j^0) = \nu_{L_j} \quad \text{and} \quad \text{Law}(\Xi_0) = \nu_{\infty}, \quad (4.124) \]

and \(\Xi_j^0\) converges \(\tilde{P}\)-almost surely to \(\Xi_0\) in \(\mathcal{W}\) as \(j \to \infty\). Given \(j \in \mathbb{N}\), we define a transport plan \(p_{L_j} \in \Pi(\nu_{\infty}, \nu_{L_j})\) by

\[ p_{L_j} = (\Xi_0, \Xi_j^0)_{\#}\tilde{P}. \quad (4.125) \]
Then, by the bounded convergence theorem, we have
\[
\int_{\mathcal{W} \times \mathcal{W}} d_{\mathcal{W}}(\Theta, \Theta') d\mathbf{p}_{L_j}(\Theta, \Theta') = \int_{\Omega} d_{\mathcal{W}}(\Xi_0(\tilde{\omega}), \Xi_0'(\tilde{\omega})) d\tilde{\mathcal{P}}(\tilde{\omega}) = 0,
\]
(4.126)
as \(j \to \infty\), where the metric \(d_{\mathcal{W}}\) is as in (4.14). See Remark 6.11 in [64].

From (4.36) and (4.61) with (4.12), we have
\[
\rho_{L_j} = (\text{eval}_{t=0} \circ \text{proj}_1)'_{\#} \nu_{L_j} \quad \text{and} \quad \rho_{\infty} = (\text{eval}_{t=0} \circ \text{proj}_1)'_{\#} \nu_{\infty},
\]
(4.127)
where \(\text{eval}_{t=0}\) denotes the evaluation map at time \(t = 0\) and \(\text{proj}_1\) is the projection onto the first component. We now define the first components of the initial data by
\[
u_0 = \text{eval}_{t=0} \circ \text{proj}_1(\Xi_0) \quad \text{and} \quad u_0 = \text{eval}_{t=0} \circ \text{proj}_1(\Xi_0).
\]
(4.128)

Finally, we let
\[
\begin{align*}
\bar{u} &= (u, \partial_t u) = \bar{u}(u_0, u_1, \xi) \text{ denote a solution to the hyperbolic } \Phi_{k+1}^2 \text{-model (4.1) on } \mathbb{R}^2 \\
\bar{u}_{L_j} &= (u_{L_j}, \partial_t u_{L_j}) = \bar{u}_{L_j}(u_0, u_1, L_j, \xi_{L_j}) \text{ denote a solution to the } L_j \text{-periodic hyperbolic } \Phi_{k+1}^2 \text{-model (4.2) on } \mathbb{R}^2 \\
\end{align*}
\]
(4.129)
with \(u_0\) as in (4.128) and \(u_1 = \zeta\), and \(u_0, u_1\) lives.

In the following, we fix \(R \in \mathbb{N}\) and work on the cone \(\mathbf{C}_R\). When \(L_j \gg R\), it follows from the finite speed of propagation and (4.6) with the notation above that
\[
\bar{u}_{L_j}(u_0, L_j, u_1, L_j, \xi_{L_j}) = \bar{u}_{L_j}(u_0, L_j, u_1, \xi) \quad \text{on the cone } \mathbf{C}_R.
\]
(4.129)

In particular, on \(\mathbf{C}_R\), \(\bar{u}_{L_j}(u_0, L_j, u_1, \xi)\) satisfies (4.1) with the initial data \((u_0, L_j, u_1)\). We denote by \(\mathbb{P}_{u_1, \xi} = \mathbb{P}_{u_1} \otimes \mathbb{P}_{\xi}\) the probability distribution of the spatial white noise \(u_1\) and the space-time white noise \(\xi\) which are independent from each other and also from the transport plan \(\mathbf{p}_{L_j}, j \in \mathbb{N}\), defined in (4.128). For simplicity of notation, we set \(\mathcal{X} = \mathcal{D}'(\mathbb{R}^2) \times \mathcal{D}'(\mathbb{R}^2 \times \mathbb{R}_+)\), where \((u_1, \xi)\) lives.

Given \(\Theta = (\Theta_1, \cdots, \Theta_k) \in \mathcal{W}\), let \(v = v(\Theta, u_1, \xi)\) be a solution to (4.11), where we replace \(\Phi_{k+1}^m(u_0)\) by \(\Theta_m\). Namely, \(v = v(\Theta, u_1, \xi)\) satisfies
\[
v(t) = -\sum_{\ell=0}^{k} \sum_{m=0}^{\ell} \binom{k}{\ell} \binom{\ell}{m} \mathcal{D}(t - t') \Theta_m(t') : \Phi_{k+1}^{\ell-m}(u_1, \xi)(t') : v^{k-\ell}(t') dt'.
\]
(4.130)

Define \(E_R \subset \mathcal{W} \times \mathcal{X}\) by
\[
E_R = \{ (\Theta, u_1, \xi) \in \mathcal{W} \times \mathcal{X} : \text{there exists a solution } v = v(\Theta, u_1, \xi) \text{ to (4.130)} \}
\]
on the cone \(\mathbf{C}_R\) such that \(\bar{v} = (v, \partial_t v) \in L^\infty([0, R]; \tilde{H}^{1-\varepsilon_k}(B_{R-t}))\)

Our goal is to show
\[
\rho_{\infty} \otimes \mathbb{P}_{u_1, \xi} \left( \bigcap_{R \in \mathbb{N}} \{ (u_0, u_1, \xi) : (\Xi_0(u_0), u_1, \xi) \in E_R \} \right) = 1,
\]
(4.131)
where $\Xi_0(\cdot)$ is the map defined in (4.12).

Fix $R \in \mathbb{N}$. Given $M_0, M_1, M_2 \geq 1$ and $\delta > 0$, define

$$A_{R,M_0,M_2} = \left\{ (\Theta', u_1, \xi) \in \mathcal{W} \times \mathcal{X} : \|\Theta'\|_{\mathcal{W}(R)} \leq M_0, \right.$$ 

$$\left. \|\bar{v}(\Theta', u_1, \xi)\|_{L^\infty([0,R]\cup \bar{H}^{1-\varepsilon_k(B_{R^{-1}})})} \leq M_2 \right\},$$

$$B_{R,M_1} = \left\{ (\Theta, \Theta', u_1, \xi) \in \mathcal{W} \times \mathcal{W} \times \mathcal{X} : \|\Xi_1(u_1, \xi)\|_{\Theta, \Theta', R} \leq M_1 \right\},$$

$$C_{R,\delta} = \left\{ (\Theta, \Theta') \in \mathcal{W} \times \mathcal{W} : d_{\mathcal{W}}(\Theta, \Theta') \leq \delta \right\},$$

where $\Theta' = (\Theta'_1, \cdots, \Theta'_k)$, the $\mathcal{W}(R)$-norm is as in (4.13), and $\| \cdot \|_{\Theta, \Theta', R}$ is as in Definition 4.1(iii). Furthermore, we define $\tilde{E}_R, \tilde{A}_{R,M_0,M_2}$, and $\tilde{C}_{R,\delta}$ by setting

$$\tilde{E}_R = \left\{ (\Theta, \Theta', u_1, \xi) \in \mathcal{W} \times \mathcal{W} \times \mathcal{X} : (\Theta, u_1, \xi) \in E_R \right\},$$

$$\tilde{A}_{R,M_0,M_2} = \left\{ (\Theta, \Theta', u_1, \xi) \in \mathcal{W} \times \mathcal{W} \times \mathcal{X} : (\Theta', u_1, \xi) \in A_{R,M_0,M_2} \right\},$$

$$\tilde{C}_{R,\delta} = \left\{ (\Theta, \Theta', u_1, \xi) \in \mathcal{W} \times \mathcal{W} \times \mathcal{X} : (\Theta', \Theta) \in C_{R,\delta} \right\},$$

such that all the sets live in a common space $\mathcal{W} \times \mathcal{W} \times \mathcal{X}$. Then, it follows from Propositions 4.2 and 4.3 that

$$\tilde{E}_R \supset \tilde{A}_{R,M_0,M_2} \cap B_{R,M_1} \cap \tilde{C}_{R,\delta},$$

for any $M_0, M_1, M_2 \geq 1$ and $0 < \delta \leq \delta^* = \delta^*(\mu, R, M_0, M_1, M_2)$. Here, $\delta^* = \delta^*(\mu, R, M_0, M_1, M_2)$ is chosen such that, in view of (4.14) and (4.15),

$$d_{\mathcal{W}}(f, g) \leq \delta^* \text{ implies } \|f - g\|_{\mathcal{W}(R)} \leq \delta^*, \tag{4.135}$$

Recalling (2.59) that

$$\int_{\Theta \in \mathcal{W}} dp_{L,J}(\Theta, \Theta') = d\nu_\infty(\Theta) \text{ and } \int_{\Theta \in \mathcal{W}} dp_{L,J}(\Theta, \Theta') = d\nu_\infty(\Theta'), \tag{4.136}$$

it follows from (4.136), (4.133), (4.134), and (4.132) that

$$\rho_\infty \otimes p_{u_1,\xi} \left( \left\{ (u_0, u_1, \xi) : (\Xi_0(u_0), u_1, \xi) \in E_R \right\} \right)$$

$$= \int_{\mathcal{X}} \int_{\mathcal{W} \times \mathcal{W}} 1(\Theta, u_1, \xi) \in E_R(\Theta, \Theta', u_1, \xi) dp_{L,J}(\Theta, \Theta') dp_{u_1,\xi}(u_1, \xi)$$

$$= \int_{\mathcal{X}} \int_{\mathcal{W} \times \mathcal{W}} 1_{\Xi_0(\cdot)}(\Theta, \Theta', u_1, \xi) dp_{L,J}(\Theta, \Theta') dp_{u_1,\xi}(u_1, \xi)$$

$$\geq \int_{\mathcal{X}} \int_{\mathcal{W} \times \mathcal{W}} 1_{A_{R,M_0,M_2} \cap B_{R,M_1} \cap \tilde{C}_{R,\delta}'}(\Theta, \Theta', u_1, \xi) dp_{L,J}(\Theta, \Theta') dp_{u_1,\xi}(u_1, \xi)$$

$$\geq 1 - \int_{\mathcal{W} \times \mathcal{W}} 1_{\|\Theta'\|_{\mathcal{W}(R)} > M_0}(\Theta') dp_{L,J}(\Theta, \Theta')$$

$$- \int_{\mathcal{X}} \int_{\mathcal{W} \times \mathcal{W}} 1_{\|\bar{v}(\Theta', u_1, \xi)\|_{L^\infty([0,R]\cup \bar{H}^{1-\varepsilon_k(B_{R^{-1}})})} > M_2}(\Theta', u_1, \xi) dp_{L,J}(\Theta, \Theta') dp_{u_1,\xi}(u_1, \xi)$$

\[\text{for some } C(\mu, R) > 0.\]
\[- \int_{\mathbb{R}^d_{+} \times \mathbb{R}^d} 1_{B_R(t_1)} \left( (\Theta, \Theta', u_1, \xi) \right) dp_{L_j}(\Theta, \Theta') dp_{u_1, \xi}(u_1, \xi) \]

\[- \int_{\mathbb{R}^d_{+} \times \mathbb{R}^d} 1_{C_R(t)} \left( (\Theta, \Theta') \right) dp_{L_j}(\Theta, \Theta') \]

\[= 1 - I - II - III - IV. \tag{4.137} \]

From (4.136), (4.36), (4.127), Markov’s inequality, and Proposition 4.9(ii), we have

\[I \lesssim \frac{1}{M_0} E_{p_{L_j}} \left[ \|u_0(u_0, L_j)\|_{L^\infty \left( \mathbb{R}^d_{+}; W_{\mu, \rho}^{1-\varepsilon} \right)} \right] \lesssim \frac{1}{M_0}, \tag{4.138} \]

uniformly in \( j \in \mathbb{N} \). Similarly, from (4.136), (4.36), (4.127), and Markov’s inequality, we have

\[II < \frac{1}{M_2} E_{p_{L_j} \otimes p_{u_1, \xi}} \left[ \|u_0(u_0, L_j, u_1, \xi) - S(t) u_0, L_j + D(t) u_1 + \Psi(t) \right. \]

on \( \mathcal{C}_R \), where \( u_{L_j} \) is the solution to the \( L_j \)-periodic hyperbolic \( \Phi^{k+1} \)-model (4.2). Thus, we see that \( v(u_0(u_0, L_j, u_1, \xi) = - \int_0^t D(t - t') \left( 1_{C_R} ; u_{L_j}^k(t') \right) dt' \tag{4.140} \]

on \( \mathcal{C}_R \). Compare this with (4.7), (4.8), and (4.9) (with an additional cutoff function \( 1_{C_R} \) as in (4.121)). Then, from (4.140) Minkowski’s integral inequality, and Hölder’s inequality, we have

\[\|v(u_0(u_0, L_j), u_1, \xi)\|_{L^\infty(\mathbb{R}^d_{+}; H^{1-\varepsilon_k}(B_{R-t}))} \leq \left\| \int_0^t D(t - t') \left( 1_{C_R} ; u_{L_j}^k(t') \right) \right\|_{H^{1-\varepsilon_k}(B_{R-t})} \]

\[\lesssim \int_0^t \| u_{L_j}^k(t') \|_{H^{1-\varepsilon_k}(B_{R-t})} dt'. \tag{4.141} \]

Hence, from (4.139), (4.141), the invariance \( \tilde{p}_{L_j} \) under (4.2), and Proposition 4.9(ii), we obtain

\[II \lesssim_{\mu, R} \frac{1}{M_2} E_{p_{L_j} \otimes p_{u_1, \xi}} \left[ \|u_0(u_0, L_j)\|_{L^\infty \left( \mathbb{R}^d_{+}; W_{\mu, \rho}^{1-\varepsilon} \right)} \right] \lesssim \frac{1}{M_2}, \tag{4.142} \]

uniformly in \( j \in \mathbb{N} \).

From (4.132) and Proposition 4.9(i), we have

\[III < \frac{1}{M_1} E_{p_{L_j} \otimes p_{u_1, \xi}} \left[ \|u_1(u_1, \xi)\|_{L^\infty \left( \mathbb{R}^d_{+}; W_{\mu, \rho}^{1-\varepsilon} \right)} \right] \lesssim \frac{1}{M_1}, \tag{4.143} \]

uniformly in \( j \in \mathbb{N} \). Finally, from (4.132) and (4.126), we have, for each fixed \( \delta > 0 \),

\[IV < \frac{1}{\delta} \int_{\mathbb{R}^d_{+} \times \mathbb{R}^d} d_{W}(\Theta, \Theta') dp_{L_j}(\Theta, \Theta') \rightarrow 0, \tag{4.144} \]

as \( j \to \infty \).
Fix small \( \kappa > 0 \). We first choose \( M_0, M_1, M_2 \gg 1 \) such that \( (4.135) \), \( (4.142) \), and \( (4.143) \) imply

\[
I + II + III < \frac{\kappa}{2}.
\]

Then, we choose sufficiently small \( \delta = \delta(R, M_0, M_1, M_2) > 0 \) such that \( 0 < \delta \leq \delta^* \) (and thus \( (4.137) \) holds). Finally, by taking sufficiently large \( j \gg 1 \), we obtain from \( (4.144) \) that

\[
IV < \frac{\kappa}{2}.
\]

Hence, we obtain \( \mathbb{P}(E_R^c) < \kappa \). Since the choice of \( \kappa > 0 \) was arbitrary, we then conclude from \( (4.137) \), \( (4.145) \), and \( (4.146) \) that

\[
\rho_\infty \otimes \mathbb{P}_{u_1, \xi} \left( \left\{ (u_0, u_1, \xi) : (\Xi_0(u_0), u_1, \xi) \in E_R \right\} \right) = 1
\]

for any \( R \in \mathbb{N} \), which in turn implies \( (4.131) \). This concludes the proof of Proposition \( 4.11 \) \( \square \)

**Remark 4.12.** A slight modification of the proof of Proposition \( 4.11 \) shows that, on each cone \( C_R, R > 0 \), the \( L_j \)-periodic solution \( \tilde{u}_{L_j} = (u_{L_j}, \partial_t u_{L_j}) \) to \( (4.2) \) converges in probability to the solution \( \tilde{u} = (u, \partial_t u) \) to \( (4.1) \) constructed in Proposition \( 4.11 \) in the \( L^\infty([0, R]; \tilde{H}^{-\varepsilon_k}(B_{R-\ell})) \)-topology. In particular, there exists a subsequence \( \{u_{L_{j\ell}}\}_{\ell \in \mathbb{N}} \) such that \( u_{L_{j\ell}} \) converges almost surely to \( \tilde{u} \) in \( L^\infty([0, R]; \tilde{H}^{-\varepsilon_k}(B_{R-\ell})) \) as \( \ell \to \infty \). Since \( u_{L_{j\ell}} \in C(\mathbb{R}_+; \tilde{H}^{-\varepsilon_k}(\mathbb{R}^2)) \), we then deduce that \( u \in C([0, \frac{R}{2}]; \tilde{H}^{-\varepsilon_k}(\mathbb{R}^2)) \) almost surely. Since the choice of \( R > 0 \) was arbitrary, we conclude that \( (u, \partial_t u) \in C(\mathbb{R}_+; \tilde{H}^{-\varepsilon_k}(\mathbb{R}^2)) \) almost surely.

Let \( u_0 \) and \( u_{0, L_j} \) be as in \( (4.128) \). Then, define a set \( F_R \subset \hat{\Omega} \times \mathcal{X} \) by

\[
F_R = \bigcap_{M_0, M_1, M_2 = 1}^{\infty} \bigcap_{m \in \mathbb{N}}^{\infty} \left( \bigcap_{\ell = 1}^{\infty} \bigcup_{j = \ell}^{\infty} \left( D_{R, M_0, M_1, M_2}^{L_j} \cap C_{R, \delta}^{L_j} \right) \right),
\]

where \( \delta^*(\mu, R, M_0, M_1, M_2) \) is as in the proof of Proposition \( 4.11 \) (see also \( (4.135) \)). Here, \( D_{R, M_0, M_1, M_2}^{L_j} \) and \( C_{R, \delta}^{L_j} \) are given by

\[
D_{R, M_0, M_1, M_2}^{L_j} = \left\{ (\Xi_0(u_0(\omega)), \Xi_0(u_{0, L_j}(\omega)), u_1, \xi) \in \tilde{A}_{R, M_0, M_2} \cap B_{R, M_1} \right\},
\]

\[
C_{R, \delta}^{L_j} = \left\{ (\Xi_0(u_0(\omega)), \Xi_0(u_{0, L_j}(\omega))) \in \tilde{C}_{R, \delta} \right\},
\]

where \( \tilde{A}_{R, M_0, M_2} \), \( B_{R, M_1} \), and \( \tilde{C}_{R, \delta} \) are as in \( (4.132) \) and \( (4.133) \). As in \( (4.134) \), we then have

\[
F_R \subset \left\{ (\omega, u_1, \xi) \in \hat{\Omega} \times \mathcal{X} : (\Xi_0(u_0(\omega)), u_1, \xi) \in E_R \right\}.
\]

Given small \( \kappa > 0 \), by proceeding as in \( (4.137) \) with the uniform (in \( j \)) bounds \( (4.138) \), \( (4.142) \), and \( (4.143) \), choosing sufficiently large \( M_0, M_1, M_2 \gg 1 \), and then applying Markov's
inequality and (4.120), we obtain
\[
\tilde{P} \otimes \mathbb{P}_{u_1, \xi}(F_R^c) \leq \kappa + \sup_{m \in \mathbb{N}} \liminf_{j \to \infty} \mathbb{P}_{L_j}(d_W(\Theta, \Theta') > \frac{1}{m}) \\
\leq \kappa + \sup_{m \in \mathbb{N}} \liminf_{j \to \infty} m \cdot \liminf_{j \to \infty} \int_{W \times \tilde{W}} d_W(\Theta, \Theta') d\mathbb{P}_{L_j}(\Theta, \Theta') \\
= \kappa.
\]
Since the choice of \( \kappa > 0 \) was arbitrary, we conclude that
\[
\tilde{P} \otimes \mathbb{P}_{u_1, \xi}(F_R) = 1.
\]

With a slight abuse of notation, let us denote by \( \omega' \) an element \((\tilde{\omega}, u_1, \xi) \in \tilde{\Omega} \times X\). Given \( \omega' \in F_R \), it follows from the definition (4.137) of \( F_R \) with (4.134) and Propositions 4.2 and 4.3 that there exists an \( \omega' \)-dependent subsequence \( L_{j_\ell} \subset A \) such that \( \tilde{u}_{L_{j_\ell}}(\omega') \) converges almost surely to \( \tilde{u}(\omega') \) in \( L^\infty([0, R]; \tilde{H}^{1, -\epsilon_k}(B_{R-\ell})) \) as \( \ell \to \infty \). On the other hand, we claim that \( \{u_{L_j}\}_{j \in \mathbb{N}} \) is Cauchy in probability in \( L^\infty([0, R]; \tilde{H}^{1, -\epsilon_k}(B_{R-\ell})) \). Then, from the uniqueness of a limit, we conclude that \( \tilde{u}_{L_j} \) converges in probability to \( \tilde{u} \) in \( L^\infty([0, R]; \tilde{H}^{1, -\epsilon_k}(B_{R-\ell})) \) as \( j \to \infty \).

It remains to show that \( \{\tilde{u}_{L_j}\}_{j \in \mathbb{N}} \) is Cauchy in probability (with respect to \( \tilde{P} \otimes \mathbb{P}_{u_1, \xi} \)) in \( L^\infty([0, R]; \tilde{H}^{1, -\epsilon_k}(B_{R-\ell})) \). From (4.126) with (4.128), we have, for each fixed \( \delta_0 > 0 \),
\[
\tilde{P}\left(d_W(\Xi_0(u_0, L_{j_1}), \Xi_0(u_0, L_{j_2})) > \delta_0 \right) \\
\leq \tilde{P}\left(d_W(\Xi_0(u_0, L_{j_1}), \Xi_0(u_0)) > \frac{\delta_0}{\delta} \right) + \tilde{P}\left(d_W(\Xi_0(u_0), \Xi_0(u_0, L_{j_2})) > \frac{\delta_0}{\delta} \right) \\
< \frac{2}{\delta_0} \int_{\tilde{\Omega}} d_W(\Xi_0^{j_1}(\tilde{\omega}), \Xi_0(\tilde{\omega})) d\tilde{P}(\tilde{\omega}) + \frac{2}{\delta_0} \int_{\tilde{\Omega}} d_W(\Xi_0(\tilde{\omega}), \Xi_0^{j_2}(\tilde{\omega})) d\tilde{P}(\tilde{\omega}) \\
\to 0,
\]
as \( j_1, j_2 \to \infty \). Given \( \kappa > 0 \), from (4.138), (4.142), and (4.143), there exist \( M_0, M_1, M_2 \gg 1 \) such that
\[
\tilde{P} \otimes \mathbb{P}_{u_1, \xi}\left((D_{L_0, M_0, M_1, M_2}^{L_{j_1}})^c \cup (D_{L_0, M_0, M_1, M_2}^{L_{j_2}})^c \right) < \frac{\kappa}{2},
\]
uniformly in \( j \in \mathbb{N} \), where \( D_{L_0, M_0, M_1, M_2}^{L_{j_1}} \) is as in (4.148). Then, it follows from Proposition 4.3 with (4.149) and (4.150) (see also (4.6)) that
\[
\tilde{P} \otimes \mathbb{P}_{u_1, \xi}\left(\| \tilde{u}_{L_{j_1}}(u_0, L_{j_1}(\tilde{\omega}), u_1, \xi) - \tilde{u}_{L_{j_2}}(u_0, L_{j_2}(\tilde{\omega}), u_1, \xi)\|_{L^\infty([0, R]; \tilde{H}^{1, -\epsilon_k}(B_{R-\ell}))} > \delta \right) \\
\leq \tilde{P}\left(d_W(\Xi_0(u_0, L_{j_1}), \Xi_0(u_0, L_{j_2})) > \delta_0 \right) \\
+ \tilde{P} \otimes \mathbb{P}_{u_1, \xi}\left((D_{L_0, M_0, M_1, M_2}^{L_{j_1}})^c \cup (D_{L_0, M_0, M_1, M_2}^{L_{j_2}})^c \right) \\
< \kappa
\]
for any \( j_2 \geq j_1 \gg 1 \). Here, \( \delta_0 = \delta_0(\delta) > 0 \) is a small number such that \( d_W(f, g) \leq \delta_0 \) implies \( \|f - g\|_{W(R)} \leq \frac{1}{2}C_0^{-1}\delta \), where \( C_0 = C_0(R, M_0, M_1, M_2) > 0 \) are as in Proposition 4.3. This shows that \( \{\tilde{u}_{L_j}\}_{j \in \mathbb{N}} \) is Cauchy in probability in \( L^\infty([0, R]; \tilde{H}^{1, -\epsilon_k}(B_{R-\ell})) \).
4.5. Invariance of the Gibbs measure $\tilde{\rho}_\infty$. We conclude this section by briefly discussing invariance of the Gibbs measure $\tilde{\rho}_\infty$ under the dynamics of the hyperbolic $\Phi_2^{k+1}$-model \((4.1)\) on the plane. We only consider the values of $L$ belonging to $A = \{L_j : j \in \mathbb{N}\} \subset \mathbb{N}$, along which we proved Theorem \((1.2)(i)\).

Let $u$ be the solution to the hyperbolic $\Phi_2^{k+1}$-model \((4.1)\) with $\text{Law}(u_0, u_1) = \tilde{\rho}_\infty$. Our goal is to show

$$\text{Law}(\tilde{u}(t)) = \tilde{\rho}_\infty$$

for each $t \in \mathbb{R}_+$. Fix $t > 0$. Given $j \in \mathbb{N}$, let $u_{L_j}$ be the solution to the $L_j$-periodic hyperbolic $\Phi_2^{k+1}$-model \((4.2)\) with $\text{Law}(u_{0, L_j}, u_{1, L_j}) = \tilde{\rho}_{L_j}$ such that $(u_{0, L_j}, u_{1, L_j})$ converges almost surely to $(u_0, u_1)$ in $\tilde{H}_{L_j}^\infty(\mathbb{R}^2)$, as $j \to \infty$. By the invariance of the Gibbs measure $\tilde{\rho}_{L_j}$ under the $L_j$-periodic dynamics, we have $\text{Law}(\tilde{u}_{L_j}(t)) = \tilde{\rho}_{L_j}$ for any $t \in \mathbb{R}_+$. Then, it follows from the weak convergence of $\{\tilde{\rho}_{L_j}\}_{j \in \mathbb{N}}$ to $\tilde{\rho}_\infty$ that

$$\tilde{\rho}_\infty = \text{w-lim}_{j \to \infty} \text{Law}(\tilde{u}_{L_j}(t)), \quad (4.152)$$

where w-lim denotes a weak limit of probability measures.

On the other hand, given $R > t$, let $\varphi = (\varphi_0, \varphi_1) \in (\mathcal{D}(\mathbb{R}^2))^{\otimes 2} = (C_c(\mathbb{R}^2))^{\otimes 2}$ be a pair of test functions with $\text{supp} \varphi_\ell \subset B_R$, $\ell = 1, 2$. Then, it follows from the proof of Proposition \((4.1)\) (see Remark \((4.2)\)) that the solution $\tilde{u}_{L_j}$ to \((4.2)\) converges in probability to the solution $\tilde{u}$ to \((4.1)\) on $C_{R+t}$ as $j \to \infty$. This in particular implies $(\tilde{u}_{L_j}(t), \tilde{\varphi})$ converges in probability to $(\tilde{u}(t), \tilde{\varphi})$ as $j \to \infty$, where $(\cdot, \cdot)$ denotes $(\mathcal{D}'(\mathbb{R}^2))^{\otimes 2}(\mathcal{D}(\mathbb{R}^2))^{\otimes 2}$ duality pairing. In particular, as a $(\mathcal{D}'(\mathbb{R}^2))^{\otimes 2}$-valued random variable $\tilde{u}_{L_j}(t)$ converges in law to $\tilde{u}(t)$ and, therefore, together with \((4.152)\) and the uniqueness of a limit, we obtain \((4.151)\). This concludes the proof of Theorem \((1.2)(ii)\).

**Appendix A. Embeddings between weighted Sobolev and Besov spaces**

In this appendix, we present the proof of Lemma \((2.3)\). Fix $1 \leq p < \infty$. We first prove

$$\|f\|_{W_{s,p}^\mu} \lesssim \|f\|_{B_{p,1}^{s',\mu'}}. \quad (A.1)$$

for $s < s'$ and $\mu \geq c(p)\mu' > 0$. Let us state an auxiliary lemma.

**Lemma A.1.** Let $s \in \mathbb{R}$, finite $p \geq 1$, and $\mu > 0$. Then, given any $\varepsilon > 0$, there exists $c_0 > 0$ such that

$$\|\phi_jQ_kf\|_{W_{s,p}^\mu} \lesssim e^{c_0\varepsilon^2/2}2^{(s+\varepsilon)\varepsilon}e^{j\varepsilon}2^{2j\varepsilon}2^{2j\varepsilon}|Q_kf|_{L_{\mu}^p} \quad (A.2)$$

for any $j, k \in \mathbb{Z}_{\geq 0}$, where $Q_k$ is as in \((2.8)\).

We first prove \((A.1)\) by assuming Lemma \((A.1)\). We present the proof of Lemma \((A.1)\) at the end of this section. From \((2.23)\) and Lemma \((A.1)\), we have

$$\|f\|_{W_{s,p}^\mu} \leq \sum_{k=0}^{\infty} \|Q_kf\|_{W_{s,p}^\mu} \leq \sum_{k=0}^{\infty} \sum_{j=0}^{\infty} e^{-\varepsilon/2j\varepsilon} \|\phi_jQ_kf\|_{W_{s,p}^\mu} \lesssim \sum_{k=0}^{\infty} \sum_{j=0}^{\infty} e^{j\varepsilon/2j\varepsilon} \|Q_kf\|_{L_{\mu}^p} \lesssim \|f\|_{B_{p,1}^{s',\mu'}},$$
provided that \( \mu > c_0 \mu' \) and \( s < s' \). This proves (A.1).

Next, we prove

\[
\|f\|_{B^{s,\mu}_{p,\infty}} \lesssim \|f\|_{W^{s,p}_{\mu'}}
\]

(A.3)

for any \( s \in \mathbb{R} \) and \( 0 < \mu' < c_1 \mu \) for some small \( c_1 > 0 \).

From (2.17) and (2.18), we have

\[
2^sk \|Q_k f\|_{L^p_{\mu}} \leq \sum_{j,j'=0}^{\infty} 2^sk \|\phi_{j'} Q_k (\phi_j f)\|_{L^p_{\mu}}
\]

\[
\leq \sum_{j,j'=0}^{\infty} e^{-c\frac{\mu'}{p} 2^j s} 2^sk \|\phi_{j'} Q_k (\phi_j f)\|_{L^p_{\mu}}.
\]

(A.4)

Note that we have

\[
(\phi_{j'} Q_k (\phi_j f))(x) = \phi_{j'}(x) \int_{\mathbb{R}^d} \eta_k(x-y)(\phi_j f)(y)dy,
\]

(A.5)

where \( \eta_k \) is as in (2.9). In view of (2.16), we have \(|x| \sim 2^j\cdot 2^s\) and \(|y| \sim 2^j\) in the integration above.

We first estimate the contribution to (A.4) from the case \( j' \geq j - 2 \). By first summing over \( j' \) and applying Bernstein’s inequality and (2.23), we have

\[
\sum_{j=0}^{\infty} \sum_{j'=j-2}^{\infty} e^{-c\frac{\mu'}{p} 2^j s} 2^sk \|\phi_{j'} Q_k (\phi_j f)\|_{L^p_{\mu}}
\]

\[
\lesssim \sum_{j=0}^{\infty} e^{-c' 2^j s} 2^sk \|Q_k (\phi_j f)\|_{L^p_{\mu}}
\]

(A.6)

\[
\lesssim \|f\|_{W^{s,p}_{\mu'}}.
\]

uniformly in \( k \in \mathbb{Z}_{\geq 0} \), provided that \( 0 < \mu' < c_1 \mu \).

Next, we consider the case \( j' \leq j - 3 \). In this case, we have \(|x-y| \sim |y| \sim 2^j\) in (A.5). Then, it follows from (2.10) that

\[
|\eta_k(x-y)| \lesssim 2^d e^{-c(2^j|x-y|)^{\frac{1}{70}}} \lesssim 2^d e^{-c(2^j|y|)^{\frac{1}{70}}} e^{-c(2^j|x-y|)^{\frac{1}{70}}}.
\]

(A.7)

In the following, it is understood that, given \( j \in \mathbb{Z}_{\geq 0} \), we have \(|x-y| \sim |y| \sim 2^j\) in (A.5). Hence, by first summing over \( j' \) and applying Bernstein’s inequality followed by Young’s
inequality with (A.7) and (2.23), we obtain
\[
\sum_{j=0}^{\infty} \sum_{j' = 0}^{j-3} e^{-c \frac{\mu}{p} 2^{j} |j'|^2 2^k} \| \phi_{j'} (\eta_k \ast (\phi_j f)) \|_{L^p} \lesssim \sum_{j=0}^{\infty} \| \eta_k \ast (\nabla)^s (\phi_j f) \|_{L^p} \\
\lesssim \sum_{j=0}^{\infty} \sum_{j' = 0}^{j-3} e^{-c' \frac{\mu'}{p} 2^{j} |j'|^2 2^k} \| \phi_{j'} f \|_{W^{s,p}} \lesssim \sum_{j=0}^{\infty} e^{-c' \frac{\mu'}{p} 2^{j} |j'|^2 2^k} \| f \|_{W^{s,p}} \\
\lesssim \| f \|_{W^{s,p}}
\]
for any \( \mu' \in \mathbb{R} \) since we have \( 0 < \delta < \frac{1}{\theta_0} \); see the definition (2.11) of the weight \( w_\mu \).

Putting (A.4), (A.6), and (A.8) together, we obtain (A.3).

We conclude this paper by presenting the proof of Lemma A.1.

**Proof of Lemma A.1.** We first consider the case \( s \leq 0 \). From (2.23), we have
\[
\|\phi_j Q_k f\|_{W^{s,p}} \leq \|\phi_j Q_k f\|_{L^p} \lesssim e^{\frac{\mu}{p} 2^{j} |k|^2} \|Q_k f\|_{L^p} \tag{A.9}
\]
for any \( \mu > 0 \), uniformly in \( k \in \mathbb{Z}_{\geq 0} \). By (2.27) and Bernstein’s inequality, we have
\[
\|\phi_j Q_k f\|_{W^{s,p}} \lesssim \|Q_k f\|_{W^{s,p}} \lesssim 2^{sk} \|Q_k f\|_{L^p}, \tag{A.10}
\]
uniformly in \( k \in \mathbb{Z}_{\geq 0} \). Then, (A.2) follows from interpolating (A.9) and (A.10); see [7, Theorem 5.4.1] for an interpolation of weighted Lebesgue spaces.

Next, we consider the case \( s > 0 \). In this case, (A.2) follows from interpolating
\[
\|\phi_j Q_k f\|_{L^p} \lesssim e^{\frac{\mu}{p} 2^{j} |k|^2} \|Q_k f\|_{L^p}
\]
and (A.10) (with \( s \) replaced by \( s + \epsilon \)). \( \square \)

**Appendix B. Declarations**

**Funding.** T.O. and G.Z. were supported by the European Research Council (grant no. 864138 “SingStochDispDyn”). Y.W. was supported by the EPSRC New Investigator Award (grant no. EP/V003178/1).

**Competing interests.** The authors have no competing interests to declare that are relevant to the content of this article.

**Data availability statement.** This manuscript has no associated data.

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