On Fractional and Fractal Formulations of Gradient Linear and Nonlinear Elasticity

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Abstract

In this paper we consider extensions of the gradient elasticity models proposed earlier by the second author to describe materials with fractional non-locality and fractality using the techniques developed recently by the first author. We derive a generalization of three-dimensional continuum gradient elasticity theory, starting from integral relations and assuming a weak non-locality of power-law type that gives constitutive relations with fractional Laplacian terms, by utilizing the fractional Taylor series in wave-vector space. In the sequel we consider non-linear field equations with fractional derivatives of non-integer order to describe nonlinear elastic effects for gradient materials with power-law long-range interactions in the framework of weak non-locality approximation. The special constitutive relationship that we elaborate on, can form the basis for developing a fractional extension of deformation theory of gradient plasticity. Using the perturbation method, we obtain corrections to the constitutive relations of linear fractional gradient elasticity, when the perturbations are caused by weak deviations from linear elasticity or by fractional gradient non-locality. Finally we discuss fractal materials described by continuum models with non-integer dimensional spaces. Using the recently suggested vector calculus for non-integer dimensional spaces, we consider problems of fractal gradient elasticity.

PACS: 45.10.Hj; 62.20.Dc; 81.40.Jj
1 Introduction

Three-dimensional integral elasticity models for strong non-locality are usually defined by the integral linear constitutive relation for the stress $\sigma_{ij}$ in terms of the strain $\varepsilon_{ij}$ in the form [1]–[3]

$$\sigma_{ij}(\mathbf{r}, t) = C_{ijkl} \varepsilon_{kl}(\mathbf{r}, t) + \int_{\mathbb{R}^3} c_{ijkl}(\mathbf{r} - \mathbf{r}') \varepsilon_{kl}(\mathbf{r}', t) \, d\mathbf{r'},$$

(1)

where $C_{ijkl}$ is the local fourth-order elastic stiffness tensor and $c_{ijkl}(\mathbf{r} - \mathbf{r}')$ is the nonlocal elastic stiffness tensor kernel that characterizes nonlocality of materials.

In [4]–[6] it was suggested a generalization of the constitutive relations classical elasticity by a gradient modification that contains the Laplacian $\Delta$. It reads

$$\sigma_{ij} = C_{ijkl} \varepsilon_{kl} - l_s^2 C_{ijkl} \Delta \varepsilon_{kl},$$

(2)

where the newly appearing quantity $l_s$ is an internal length scale parameter. For $l_s = 0$, we have the classical case of Hooke’s law. The Lagrangian term in equation (2) – also emerging from the positive-definite strain energy density and a corresponding variational formulation [7] resulted to stable and well-posed solutions of boundary value problems and enabled the removal of singularities from dislocations and cracks [8]–[9].

To describe complex materials characterized by non-locality of power-law type and fractality, we should further generalize the gradient equation (2), as already suggested by the authors [10, 11].

Derivatives and integrals of non-integer orders [12, 13] have a wide application in mechanics (for example see [14]–[21]). The theory of fractional differential equations is powerful tool to describe materials and media with power-law non-locality, long-range memory and/or fractality. The fractional calculus can, in fact, be used to formulate a generalization of non-local theory of elasticity in both forms: fractional gradient elasticity (weak power-law non-locality) and fractional integral elasticity (strong power-law non-locality).

In this paper, we consider fractional and fractal generalizations of gradient elasticity and plasticity models, including the rather popular GRADELA model, as proposed and utilized by Aifantis and co-workers (see, for example [8]–[9], as well as the references quoted therein), and focus on three cases:

- The elasticity of linear materials with power-law non-locality that can be described by fractional Laplacians of non-integer order of the Riesz type.
- The elasticity of nonlinear materials with power-law nonlocality that can be described by fractional Laplacians of the Riesz and Caputo type.
- The elasticity of materials with fractal structure that can be described in the framework of continuum models by using the recently suggested vector calculus for non-integer dimensional space.
We elaborate, in particular, on the following non-standard generalizations of gradient stress-strain relation.

(a) The fractional gradient elasticity with power-law non-locality
\[ \sigma_{ij} = C_{ijkl} \varepsilon_{kl} - l^2_s(\alpha) C_{ijkl} \left( -R \Delta \right)^{\alpha/2} \varepsilon_{kl} \] where \((-R\Delta)^{\alpha/2}\) is the fractional generalization of the Laplacian in the Riesz form \([13]\).

(b) The gradient elasticity for fractal materials
\[ \sigma_{ij} = C_{ijkl} \varepsilon_{kl} - l^2_F(D, d) C_{ijkl} \Delta^{(D,d)} \varepsilon_{kl} \] where \(\Delta^{(D,d)}\) is the ”fractal-Laplacian” \([22]\) that takes into account the power-law density of states of the fractal medium under consideration.

A new method is proposed for deriving three-dimensional constitutive relation for fractional gradient elasticity starting from the integral relations for a nonlocal elastic continuum and using the fractional Taylor series in wave-vector space. It is well-known that constitutive relations for gradient models can be derived from relations for integral nonlocal models by using the Taylor series approximation in the coordinate space (see, for example, \([6]\)). Specific properties of derivatives of non-integer order do not allow us to use a fractional Taylor series for a direct generalization of this approach. We thus suggest to use a fractional Taylor series expansion in the wave-vector space instead of the coordinate space. This avoids the difficulties arising from the application of the fractional Taylor series in coordinate space. The physical basis of the proposed method is an assumption pertaining to fractional spatial dispersion for a nonlocal elastic continuum.

Fractional nonlinear elasticity is considered, by elaborating on model non-linear field equations with fractional derivatives of non-integer order to describe materials with power-law non-locality and weak non-linearity. A special constitutive equation (involving the second invariants of deviatoric stress and strain) is employed which can be used as a starting point for developing a fractional theory of gradient plasticity. Using the perturbation method, we obtain corrections to linear constitutive relations, where the perturbations are caused by weak deviations from elasticity. Perturbations caused by fractional gradient nonlocality are also discussed.

Fractal media and materials in non-Euclidean space can be described by different methods (see, for example, \([23]\) and references quoted therein). Fractal materials can be considered as special continua with non-integer physical dimensions \([17]\). Continuum models for fractal materials can be formulated using the fractional integration of non-integer orders. The kernels of fractional integrations are interpreted as power-law densities of states, and orders of fractional integrals are equal to the physical dimensions of the material. In these models, the concept of density of states is applied in addition to the notion of distribution functions (such as density of mass), thus allowing to take into account the specific properties of fractal materials. Their interesting feature is that while they use fractional integration of non-integer order, the differential operators that include the density of states are of integer-order. Such models have been considered earlier by the second author \([24]–[26]\), and more recently by other authors \([27]–[29]\).
Fractal materials can also be described by using continuum models with non-integer dimensional spaces (NIDS) [23], where integration and differentiation for NIDS are applied. The integration in non-integer dimensional space is well developed [30]–[31], and it has a wide application in quantum field theory. Differentiation in non-integer dimensional space is considered in [31]-[32], but in these papers only a scalar Laplacian for NIDS was suggested. Recently a generalization of differential operators of first order (gradient, divergence, curl operators) and the vector Laplacian has been proposed in [22, 33]. The suggested vector calculus for NIDS allows us to expand the range of applications of continuum models of isotropic fractal materials. Generalizations of the gradient, divergence, curl operators and the vector Laplace operator for non-integer dimensional and fractional spaces to describe anisotropic fractal materials have been suggested in [34]. Using the vector calculus for non-integer dimensional spaces, which is suggested in [22, 35] to describe different properties of isotropic fractal media, it is possible to consider different problems of fractal gradient elasticity. The corresponding governing equations are differential equations with integer-order derivatives. Therefore the suggested vector calculus can allow us easy to solve the cylindrical and spherical (boundary value) problems for gradient elasticity theory of fractal materials by using tools of integer-order differential equations. The definitions of vector operators for non-integer dimensional spaces, can be realized for two cases: $d = D - 1$ and $d \neq D - 1$, where $D$ is the dimension of the considered fractal material region and $d$ is the dimension of it’s boundary.

The plan of the papers is as follows: In Section 2 we give a rigorous derivation of fractional gradient elasticity, starting from an integral type of constitutive law of nonlocal elasticity and adopting a fractional Taylor series expansion for its kernel in wave-vector space. This expansion generates fractional Laplacians of the Riesz type and the resulting fractional GRADELA model is solved by the Green’s function and Fourier transform techniques. In Section 3, nonlinear elasticity effects are considered and a fractional constitutive equation involving fractional Laplacians of the Riesz and Caputo type is proposed. This nonlinear fractional constitutive equation can be utilized for constructing in the future extended fractional models of the standard deformation theory of plasticity. It is further used here to consider perturbations due to nonlinearity and fractional nonlocality. Finally in Section 4 we consider fractal gradient elasticity based on Laplacians for non-integer dimensional space, generalizing the usual Laplacian in Euclidean space. Various GRADELA models for fractal media are proposed and explicit form of the governing differential equations are derived for problems of radial symmetry. It is interesting that these equations for fractal media and Laplacians for non-integer dimensional space involve derivatives of integer order and, thus, they can be solved by resorting to existing methods.
2 Fractional Gradient Elasticity as an approximation of Nonlocal Elasticity

2.1 Derivation of general constitutive relation for weak nonlocal elasticity

In this section we obtain constitutive relations for fractional gradient elasticity from integral constitutive relations for nonlocal materials by using a fractional Taylor series approximation in wave-vector space. We start from an integral constitutive relation between stress and strain that is a convolution in coordinate space. Then we apply a Fourier transform of the convolution to obtain a multiplication in the dual space (the wave-vector space). Using an assumption of weak power-law (fractional) non-locality, which gives a spatial dispersion, we apply a fractional Taylor series in the dual space to obtain an adequate approximation. Then we realize an inverse Fourier transform that gives a constitutive relation with a fractional Laplacian. As a result, we derive a fractional gradient constitutive relation with a fractional generalization of the Laplacian in the Riesz form in coordinate space. The main idea of the suggested approach is to use a Taylor series in wave-vector space instead of Taylor series in coordinate space that is usually used. It allows us to avoid three problems that can appear if we use the fractional Taylor series in coordinate space: (a) problems with an integration of fractional Taylor series in coordinate space; (b) problems with an exact form of kernels for fractional nonlocality, as derived by different lattice model, in contrast to the suggested phenomenological approach which is free from atomistic details, allows us to work in the framework of the macroscopic approach; (c) problems with derivations arriving at an undesirable sign in front of the Laplacian term, in contrast to the proposed method, which is free from sign constraints in front of the Laplacian.

In the three-dimensional theory of nonlocal elasticity [3], the nonlocal stress tensor $\sigma_{ij}$ is defined by

$$\sigma_{ij}(r,t) = \int_{\mathbb{R}^3} K(|r - r'|) \sigma^0_{ij}(r',t) \, dr',$$

where $K(|r - r'|)$ is the interaction kernel that characterize nonlocality, and $\sigma^0_{ij}$ is the stress tensor of classical (local) elasticity defined as

$$\sigma^0_{ij}(r',t) = C_{ijkl} \varepsilon_{ij}(r',t),$$

where $\varepsilon_{ij}(r',t)$ is the classical strain, and $C_{ijkl}$ is the fourth-order elastic stiffness tensor. For isotropic materials, we have

$$C_{ijkl} = \lambda \delta_{ij}\delta_{kl} + \mu (\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}),$$

where $\lambda$ and $\mu$ are the Lamé constants.

Substitution of (6) into (5) gives the nonlocal linear constitutive relation in the form

$$\sigma_{ij}(r,t) = \int_{\mathbb{R}^3} K(|r - r'|) C_{ijkl} \varepsilon_{kl}(r',t) \, dr'.$$
It is easy to see that equation (8) can be derived from relations (11) with
\[ c_{ijkl}(\mathbf{r} - \mathbf{r}') = C_{ijkl} \left( \mathcal{K}(|\mathbf{r} - \mathbf{r}'|) - \delta^3(\mathbf{r} - \mathbf{r}') \right). \]

Mathematically, equation (8) can be considered as a convolution of the interaction kernel \( \mathcal{K}(|\mathbf{r}|) \) and the strain tensor \( \varepsilon_{kl} \) in the form
\[ \sigma_{ij}(\mathbf{r}, t) = C_{ijkl} \left( \mathcal{K} \ast \varepsilon_{kl} \right)(\mathbf{r}, t). \]

To use the Fourier method, we introduce the Fourier transform \( \mathcal{F} \) – designated by a superimposed bar (\( \sim \)) – as
\[ \tilde{\sigma}_{ij}(k, t) = \int_{\mathbb{R}^3} e^{-i(k, \mathbf{r})} \sigma_{ij}(\mathbf{r}, t) \, d\mathbf{r}, \]
where \( k \) is the wave vector. Similar formulas also hold for the strain tensor and the interaction kernel:
\[ \tilde{\varepsilon}_{ij}(k, t) = \int_{\mathbb{R}^3} e^{-i(k, \mathbf{r})} \varepsilon_{ij}(\mathbf{r}, t) \, d\mathbf{r}, \]
\[ \mathcal{K}(|k|) = \int_{\mathbb{R}^3} e^{-i(k, \mathbf{r})} \mathcal{K}(|\mathbf{r}|) \, d\mathbf{r}. \]

Applying \( \mathcal{F} \) to the convolution equation (10), we obtain
\[ \tilde{\sigma}_{ij}(k, t) = C_{ijkl} \mathcal{K}(|k|) \tilde{\varepsilon}_{kl}(k, t). \]
which can be considered as a general constitutive relation in the wave-vector space for nonlocal elasticity.

The particular dependence of the function \( \mathcal{K}(|k|) \) on the wave-vector \( k = |k| \) defines the type of spatial dispersion and it characterizes the type of material non-locality. For a wide class of nonlocal materials, the wavelength \( \lambda \) satisfies the relation \( kR_0 \sim R_0/\lambda \ll 1 \), where \( R_0 \) denotes the characteristic length of interaction that usually is of the order of the interparticle distance. In this case, the spatial dispersion of the elastic material is weak. To describe materials with such property it is sufficient to know the dependence of the function \( \mathcal{K}(|k|) \) only for small values \( k = |k| \), and then we can replace this function by its Taylor’s polynomial series expansion. For an isotropic continuum, we have
\[ \mathcal{K}(|k|) = \mathcal{K}(0) + a_1 k + a_2 k^2 + o(k^2), \]
where frequency dependent dispersion is neglected for simplicity, i.e., the coefficients \( \mathcal{K}(0), a_1, a_2 \) do not depend on the frequency \( \omega \).

The weak spatial dispersion of materials with fractal power-law type of non-locality cannot be described by the usual Taylor approximation. The fractional Taylor series is very useful for the approximation of non-integer power-law functions [36]. For example, the usual Taylor series for the power-law function
\[ \mathcal{K}(k) = a_0 + a_\alpha k^\alpha \]
contains infinite many terms for non-integer $\alpha$, while the fractional Taylor series of order $\alpha$ has a finite number of terms. For example, we can consider the fractional Taylor series in the Odibat-Shawagfeh form \cite{37} that contains the Caputo fractional derivative $\frac{\mathcal{C}_0}{\mathcal{D}_k^\alpha} K(k)$, where $\alpha$ is the fractional order of differentiation, $0 < \alpha < 1$. This fractional Taylor series has the form

$$\tilde{K}(k) = \tilde{K}(0) + \frac{(\mathcal{C}_0 D_k^\alpha \tilde{K})(0)}{\Gamma(\alpha + 1)} k^\alpha + \frac{(\mathcal{C}_0 D_k^\alpha \frac{\mathcal{C}_0}{\mathcal{D}_k^\alpha} \tilde{K})(0)}{\Gamma(2\alpha + 1)} k^{2\alpha} + o(k^{2\alpha}), \quad (17)$$

where $\frac{\mathcal{C}_0}{\mathcal{D}_k^\alpha}$ is the Caputo fractional derivative \cite{13} of order $\alpha$ with respect to $k = |k|$ that is defined by

$$(\mathcal{C}_0 D_k^\alpha \tilde{K})(k) = \left(\frac{\mathcal{I}_k^{-\alpha}}{\mathcal{D}_k} \right)^n \tilde{K}(k). \quad (18)$$

Here $\mathcal{I}_k^\alpha$ is the left-sided Riemann-Liouville fractional integral of order $\alpha > 0$ with respect to $k$ of the form

$$(\mathcal{I}_k^\alpha \tilde{K})(k) = \frac{1}{\Gamma(\alpha)} \int_0^k \tilde{K}(k') \frac{dk'}{k-k'}^{1-\alpha}, \quad (k > 0). \quad (19)$$

Note that the Caputo fractional derivative of a constant is zero like for the integer order derivative. In general, the third term in (17), which contains repeated fractional derivatives of order $\alpha$, is not the same as the fractional derivative of order $2\alpha$,

$$(\mathcal{C}_0 D_k^\alpha \frac{\mathcal{C}_0}{\mathcal{D}_k^\alpha} \tilde{K})(k) \neq (\mathcal{C}_0 D_k^{2\alpha} \tilde{K})(k).$$

The coefficients of the fractional Taylor series in the Odibat-Shawagfeh form can be found in the usual way by repeated differentiation. Using the equation

$$\mathcal{C}_0 D_k^\beta k^\beta = \frac{\Gamma(\beta + 1)}{\Gamma(\beta - \alpha + 1)} k^{\beta-\alpha}, \quad (k > 0, \alpha > 0, \beta > 0) \quad (20)$$

for the case $\beta = \alpha$, in the form

$$\mathcal{C}_0 D_k^\alpha k^\alpha = \Gamma(\alpha + 1), \quad (n \geq 2), \quad (21)$$

we obtain

$$(\mathcal{C}_0 D_k^\alpha \tilde{K})(0) = \Gamma(\alpha + 1), \quad ((\mathcal{C}_0 D_k^\alpha)^n \tilde{K})(0) = 0, \quad (n \geq 2). \quad (22)$$

It is easy to see that the fractional Taylor series approximation of the function given by equation (16) is exact.

The fractional Taylor series expansion for the function $\tilde{K}(|k|)$, can be written in the form

$$\tilde{K}(|k|) = a_0 + \sum_{j=1}^{N} a_{\alpha_j} |k|^\alpha + o(|k|^\alpha N), \quad (23)$$
where $0 < \alpha_1 < \alpha_2 < \ldots < \alpha_N$; with the small-o notation $o(|k|^s)$ meaning as usual, inclusion of terms with higher powers of $|k|$ than $|k|^s$. The coefficients $a_{\alpha_j}$ in the Odibat-Shawagfeh form of the fractional Taylor series are defined by

$$a_0 = \tilde{K}(0), \quad a_{\alpha_j} = \frac{1}{\Gamma(j\alpha + 1)} \left( (C_0 D_0^{\alpha})^j \tilde{K} \right)(0).$$

As a result, for long wavelengths we can use the approximation

$$\tilde{K}(|k|) \approx a_0 + \sum_{j=1}^{N} a_{\alpha_j} |k|^\alpha_j$$

for materials with power-law type of nonlocality. The order of the fractional Taylor series approximation should be correlated with the type of power-law nonlocality in order that the fractional Taylor approximation of $\tilde{K}(|k|)$ to be valid. In the general case $0 < \alpha_{j+1} - \alpha_j < 1$, we can use the fractional Taylor formula in the Dzherbashyan-Nersesian form [38, 39],

$$\tilde{K}(k) = \sum_{j=0}^{N} a_{\alpha_j} k^{\alpha_j} + R_{N+1}(k), \quad (k > 0),$$

where

$$a_{\alpha_j} = \frac{(D^{(\alpha_j)} \tilde{K})(0)}{\Gamma(\alpha_j + 1)}, \quad R_{N+1}(k) = \frac{1}{\Gamma(\alpha_{N+1} + 1)} \int_{0}^{k} (k - z)^{\alpha_{N+1} - 1} (D^{(\alpha_j)} \tilde{K})(z) \, dz,$$

and $\alpha_j (j = 0, 1, \ldots, m)$ is an increasing sequence of real numbers such that

$$0 < \alpha_j - \alpha_{j-1} \leq 1, \quad \alpha_0 = 0, \quad j = 1, 2, \ldots, N + 1.$$

In equation (27) we use the notation [38, 39] (see also Section 2.8 in [12]) of the form

$$D^{(\alpha_k)} = I_0^{1-(\alpha_j - \alpha_{j-1})} I_0^{1+\alpha_{j-1}}.$$

In general, the fractional derivative $D^{(\alpha_k)}$ differs from the Riemann-Liouville derivative [12, 13] by a finite sum of power functions since $I_0^{\alpha} I_0^{\beta} \neq I_0^{\alpha+\beta}$ (see Eq. 2.68 in [13]).

For the special cases $\alpha_j = j \alpha$, where $\alpha < 1$ and/or $\alpha_j = \alpha + j$, we could use the fractional Taylor formulas in the Riemann formal form [40, 41], in the Riemann-Liouville form (see Chapter 1, Section 2.6 [12]), and the Trujillo-Rivero-Bonilla form [42]. In this connection, it is noted that the fractional Taylor series with Caputo derivatives is physically more meaningful than a series with Riemann-Liouville derivatives, since the Riemann-Liouville derivative of a constant is not equal to zero. If $\alpha_j = j$ for all $j \in \mathbb{N}$, we can use the usual Taylor’s formula.
2.2 Derivation of the fractional GRADELA constitutive relation

Let us obtain a constitutive relation for a special class of fractional gradient elasticity - The fractional GRADELA model:

Substitution of (25) into (14) gives

\[ \tilde{\sigma}_{ij}(k, t) = C_{ijkl} \left( a_0 + \sum_{j=1}^{N} a_{\alpha_j} |k|^\alpha_j \right) \tilde{\varepsilon}_{kl}(k, t). \]  

(30)

where the coefficients \( a_{\alpha_j} \) are defined by the fractional derivatives of \( \tilde{K}(|k|) \) with respect to \( k \) of order \( \alpha_j \) at zero \( k = 0 \). For example, if we use the fractional Taylor series in the Odibat-Shawagfeh form, then we should use (24)

\[ a_{\alpha_j} = \left( \frac{C_0 D^{\alpha_j} \tilde{K}(0)}{\Gamma(j\alpha + 1)} \right). \]  

(31)

The type of the fractional Taylor series should be correlated with the type of non-locality of the under consideration material. In particular the first non-zero coefficient \( a_{\alpha_j} \) is the term, \( \tilde{K}(k) - \tilde{K}(0) \) which is asymptotically equivalent to \( k^{\alpha_j} \) as \( k \to 0 \).

The inverse Fourier transform of (30) gives the constitutive relation for fractional gradient elasticity in the form

\[ \sigma_{ij}(r, t) = C_{ijkl} \left( \tilde{K}(0) + \sum_{j=1}^{N} a_{\alpha_j} (-\Delta)^{\alpha_j/2} \right) \varepsilon_{kl}(r, t), \]  

(32)

where we used the connection between the Riesz fractional Laplacian \( (-\Delta)^{\alpha/2} \) and its Fourier transform [12, 13],

\[ \mathcal{F}[(-\Delta)^{\alpha/2} \varepsilon_{kl}(r, t)](k) = |k|^\alpha \tilde{\varepsilon}_{kl}(k, t) \]  

(33)

in the form

\[ |k|^{\alpha_j} \leftrightarrow (-\Delta)^{\alpha_j/2}. \]  

(34)

In view of the above, we first derive the standard non-fractional constitutive relation for the GRADELA model. We consider the special case \( \alpha_j = j \) for integer \( j \in \mathbb{N} \), and the function \( \tilde{K}(k) \) in the form

\[ \tilde{K}(k) \approx a_0 + a_2 k^2, \]  

(35)

where \( a_2 \neq 0 \), with all other \( a_j = 0 \). Then, the inverse Fourier transform of (14) with (35) gives the constitutive relation

\[ \sigma_{ij}(r, t) = C_{ijkl} \left( a_0 - a_2 \Delta \right) \varepsilon_{kl}(r, t), \]  

(36)

where

\[ a_2 = \left( \frac{\partial^2 \tilde{K}(k)}{\partial k^2} \right)_{k=0}. \]  

(37)
It is also assumed that there is no initial stress, so that
\[ a_0 = \tilde{K}(0) = 1 \]
in consistency with the usual the Hooke’s law
\[ \sigma_{ij}(r, t) = C_{ijkl} \varepsilon_{kl}(r, t). \]

Let us introduce the internal length scale parameter \( l_2^s \) of gradient elasticity given by
\[ l_2^s = |a_2| = \frac{\partial^2 \tilde{K}(k)}{\partial k^2}{|}_{k=0}. \] (38)

Then, the second-gradient term is preceded by a sign that is defined by the sign of \( a_2 \). As a result, relation (36) can be rewritten in the form
\[ \sigma_{ij}(r, t) = C_{ijkl} \left( 1 - l_2^s \Delta \right) \varepsilon_{kl}(r, t), \quad (for \quad a_2 > 0), \] (39)
\[ \sigma_{ij}(r, t) = C_{ijkl} \left( 1 - \text{sgn}(a_2) l_2^s \Delta \right) \varepsilon_{kl}(r, t), \quad (for \quad a_2 < 0). \] (40)

This is the well-known constitutive relation [4]–[9] for gradient elasticity. The suggested approach to obtain constitutive relation can be generalized for the case of the higher order gradient elasticity by using additional integer non-zero values of \( a_j \).

Next, we extend this approach to derive the fractional counterpart of the GRADELA model. We consider the case \( \alpha_j = \alpha \) for some \( j = j_0 \), and \( \alpha_j = 0 \) for all other values of \( j \in \mathbb{N} \). Then the function \( \tilde{K}(k) \) has the form
\[ \tilde{K}(k) \approx 1 + a_\alpha k^\alpha. \] (41)

Substitution (41) into (14), and subsequent application of the inverse Fourier transform gives the constitutive relation
\[ \sigma_{ij}(r, t) = C_{ijkl} \left( 1 + a_\alpha (-\Delta)^{\alpha/2} \right) \varepsilon_{kl}(r, t). \] (42)

Using the new scale parameter \( l_s(\alpha) \), equation (42) can be written in the form
\[ \sigma_{ij}(r, t) = C_{ijkl} \left( 1 + l_2^s(\alpha) (-\Delta)^{\alpha/2} \right) \varepsilon_{kl}(r, t), \quad (for \quad a_\alpha > 0), \] (43)
\[ \sigma_{ij}(r, t) = C_{ijkl} \left( 1 - l_2^s(\alpha) (-\Delta)^{\alpha/2} \right) \varepsilon_{kl}(r, t), \quad (for \quad a_\alpha < 0), \] (44)
where \( l_2^s(\alpha) = |a_\alpha| \) is the scale parameter. For \( \alpha = 2 \), the relation (43) gives (39), and the relation (44) gives (40). If we use the fractional Taylor series in the Odibat-Shawagfeh form (31), then the scale parameter is defined by
\[ l_2^s(\alpha) = |a_\alpha| = \frac{1}{\Gamma(j\alpha + 1)} \left| \left( C_0 D_k^{\alpha/j}\tilde{K}\right)(0) \right|. \] (45)
In general, for the Caputo fractional derivative in equation (45) we have the inequality
\[ \left( \frac{C_0}{0} D^{\alpha/j} \right)^j \neq C_0 D^{\alpha}. \tag{46} \]

For example, for \( j = 2 \) we have
\[
\left( \left( \frac{C_0}{0} D^{\alpha/2} \right)^2 k \right)(0) = \left( C_0 D^{\alpha} k \right)(0) + \frac{(D^{1/0} K)(0)}{\Gamma(1 - \alpha)} k^{1 - \alpha}. \tag{47} \]

Note that \( r, r \) and \( l^{2/5}(\alpha) \) are dimensionless quantities for fractional elasticity. Equation (43) and (44) are the constitutive relations for fractional GRADELA model. One of the advantages of the suggested phenomenological approach is its independence on the details of the underlying micro/nanostructures. As a result, the above fractional constitutive relation can correspond to different lattice models.

The governing equations for the components \( u_i(r, t) \) of the displacement vector can be derived from the linear momentum balance equation for continuous media, i.e.
\[
\rho \ddot{u}_i(r, t) = \sum_j \frac{\partial \sigma_{ij}(r, t)}{\partial x_j} + f_i(r, t), \tag{48} \]

where \( f_i \) denotes body force. The linearized strain tensor is connected to the displacement vector through the usual relation
\[
\varepsilon_{kl}(r, t) = \frac{1}{2} \left( \frac{\partial u_k(r, t)}{\partial x_l} + \frac{\partial u_l(r, t)}{\partial x_k} \right). \tag{49} \]

Substitution of (49) and of the constitutive relation (42) for fractional gradient elasticity into equation (48), results to governing equation for the displacement of the fractional GRADELA model, in the form
\[
\rho \frac{\partial^2 u_i(r, t)}{\partial t^2} = \sum_{j,l,m} C_{ijlm} \frac{\partial^2 u_m(r, t)}{\partial x_j \partial x_l} + \sum_{j,l,m} C_{ijlm} a_\alpha \frac{\partial}{\partial x_j} (-\Delta)^{\alpha/2} \frac{\partial u_m(r, t)}{\partial x_l} + f_i(r, t), \tag{50} \]

Let us now consider the one-dimensional case, with
\[
u_x(r, t) = u(x, t), \quad u_y(r, t) = u_z(r, t) = 0, \tag{51} \]
\[f_x(r, t) = f(x, t), \quad f_y(r, t) = f_z(r, t) = 0. \tag{52} \]

In this case, \( C_{111} = \lambda + 2 \mu \), and equation (50) has the form
\[
\rho \frac{\partial^2 u(x, t)}{\partial t^2} = (\lambda + 2 \mu) \left( \frac{\partial^2 u(x, t)}{\partial x^2} + a_\alpha \frac{\partial}{\partial x} \frac{\partial^\alpha}{\partial |x|^\alpha} \frac{\partial u(x, t)}{\partial x} \right) + f(x, t), \tag{53} \]
where \( \partial^\alpha / \partial |x|^\alpha \) is the Riesz fractional derivative \([12,13]\) with respect to \( x \in \mathbb{R} \). Note that \( \partial^\alpha / \partial |x|^\alpha \) for \( \alpha = 2 \) is a derivative of second order with respect \( x \) with the minus sign

\[
\frac{\partial^2}{\partial |x|^2} = -\frac{\partial^2}{\partial x^2}.
\]  

For the cases of \( \alpha = 1 \) and of others with odd integer values, \( \partial^\alpha / \partial |x|^\alpha \) cannot be considered as a local operator. It is a nonlocal operator; e.g.

\[
\frac{\partial^1}{\partial |x|^1} \neq \frac{\partial}{\partial x}.
\]  

For the static case \((u(x,t) = u(x), f(x,t) = f(x))\), equation (53) can be rewritten in the form

\[
\frac{\partial^2 u(x)}{\partial x^2} + a_\alpha \frac{\partial}{\partial x} \frac{\partial^\alpha}{\partial |x|^\alpha} \frac{\partial u(x)}{\partial x} = -\frac{1}{\lambda + 2\mu} f(x).
\]  

For \( \alpha = 2 \), equation (56) describes the static equation for usual (non-fractional) gradient elasticity in one dimension \([10]\).

### 2.3 Green functions and Fourier method to solve fractional differential equations

Let us consider the fractional partial differential equation in the form

\[
\sum_{j=1}^{m} a_j (-(\Delta)^{\alpha_j/2} \Phi)(r) + a_0 \Phi(r) = f(r),
\]  

where \( \alpha_m > \ldots > \alpha_1 > 0 \), and \( a_j \in \mathbb{R} \ (1 \leq j \leq m) \) are constants. We apply the Fourier method, which is based on the relations

\[
\mathcal{F}[\frac{\partial^{\alpha_j}}{\partial |x_i|^{\alpha_j}} \Phi(r)](k) = |k_i|^{\alpha_j} \hat{\Phi}(k).
\]

\[
\mathcal{F}[-(\Delta)^{\alpha_j/2} \Phi(r)](k) = |k|^{\alpha_j} \hat{\Phi}(k).
\]

for Riesz fractional derivatives and the Riesz Laplacian, valid for the Lizorkin space \([12]\) and infinitely differential functions \( C^\infty(\mathbb{R}^1) \) on \( \mathbb{R}^1 \) with compact support. Applying the Fourier transform \( \mathcal{F} \) to both sides of (57) and using (58), we have

\[
(\mathcal{F} \Phi)(k) = \left( \sum_{j=1}^{m} a_j |k|^{\alpha_j} + a_0 \right)^{-1} (\mathcal{F} f)(k).
\]
The fractional analog of the Green function (see Section 5.5.1. in [13]) is given by
\[ G_\alpha(r) = F^{-1} \left[ \left( \sum_{j=1}^{m} a_j \lambda^{\alpha_j} + a_0 \right)^{-1} \right](r) = \int_{\mathbb{R}^3} \left( \sum_{j=1}^{m} a_j \lambda^{\alpha_j} + a_0 \right)^{-1} e^{i(k \cdot r)} d^3k, \] (60)
where \( \alpha = (\alpha_1, ..., \alpha_m) \). The following relation
\[ \int_{\mathbb{R}^n} e^{i(k \cdot r)} G(|k|) d^n k = \left( \frac{2\pi}{|r|^{(n-2)/2}} \right) \int_0^\infty G(\lambda) \lambda^{n/2} J_{n/2-1}(\lambda |r|) d\lambda, \] (61)
holds (see Lemma 25.1 of [12]) for any suitable function \( G \) such that the integral in the right-hand side of (61) is convergent. Here \( J_\nu \) is the Bessel function of the first kind. As a result, the Fourier transform of a radial function is also a radial function.

On the other hand, using (61), the Green function (60) can be represented (see Theorem 5.22 in [13]) in the form of the one-dimensional integral involving the Bessel function \( J_{1/2} \) of the first kind
\[ G_\alpha(r) = \left| r \right|^{-1/2} \int_0^\infty \left( \sum_{j=1}^{m} a_j \lambda^{\alpha_j} + a_0 \right)^{-1} \lambda^{3/2} J_{1/2}(\lambda |r|) d\lambda, \] (62)
where we use \( n = 3 \) and \( \alpha = (\alpha_1, ..., \alpha_m) \). For the one-dimensional case, we have
\[ J_{-1/2}(z) = \sqrt{\frac{2}{\pi z}} \cos(z), \] (63)
whereas for three-dimensional case, we have
\[ J_{1/2}(z) = \sqrt{\frac{2}{\pi z}} \sin(z). \] (64)

If \( \alpha_m > 1 \) and \( A_m \neq 0, A_0 \neq 0 \), then equation (57) (see, for example, Section 5.5.1. pages 341-344 in [13]) has a particular solution represented in the form of the convolution of the functions \( G(r) \) and \( f(r) \) as follows
\[ \Phi(r) = \int_{\mathbb{R}^3} G_\alpha(r - r') f(r') d^3r', \] (65)
where the Green function \( G_\alpha(z) \) is given by (62). Thus, we can now effectively consider the fractional partial differential equation (57) with \( a_0 = 0 \) and \( a_1 \neq 0 \), when \( m \in \mathbb{N}, m \geq 1 \), as well as the case where \( \alpha_1 < 3, \alpha_m > 1, m \geq 1, a_1 \neq 0, a_m \neq 0, \alpha_m > ... > \alpha_1 > 0 \), i.e. the equation
\[ \sum_{j=1}^{m} a_j ((-\Delta)^{\alpha_j/2}) \Phi(r) = f(r). \] (66)

The above equation has the following particular solution (see Theorem 5.23 in [13])
\[ \Phi(r) = \int_{\mathbb{R}^3} G_\alpha(r - r') f(r') d^3r', \] (67)
with
\[
G_\alpha(r) = \frac{|r|^{-1/2}}{(2\pi)^{3/2}} \int_0^\infty \left( \sum_{j=1}^m a_j |\lambda|^{\alpha_j} \right)^{-1} \lambda^{3/2} J_{1/2}(\lambda |r|) \, d\lambda,
\]
which also describes the electrostatic field in plasma-like media with a spatial dispersion of power-law type.

Let us now apply the multi-dimensional Fourier method to derive particular solutions for the linear fractional differential equations (50) by employing the direct and inverse Fourier transforms, and using the The Fourier transform method for solving the relations
\[
\mathcal{F} \left\{ \frac{\partial u_i(r,t)}{\partial x_l} \right\} (k,t) = i k_l \mathcal{F} \{u_i(r,t)\} (k,t),
\]
\[
\mathcal{F} \{ (-\Delta)^{\alpha/2} u_i(r,t) \} (k,t) = |k|^\alpha \mathcal{F} \{u_i(r,t)\} (k,t).
\]
The Fourier transform of equation (50) has the form
\[
\rho \frac{\partial^2 \bar{u}_i(k,t)}{\partial t^2} = \sum_{j,l,m} C_{ijkl} \left( -k_j k_l - a_\alpha k_j k_l |k|^\alpha \right) \bar{u}_m(k,t) + \tilde{f}_i(k,t),
\]
where \( \tilde{f}_i(k,t) = \mathcal{F} \{f_i(r,t)\} (k,t) \). Using the tensor
\[
\hat{C}_{im} (k) = \sum_{j,l} C_{ijkl} k_j k_l \left( 1 + a_\alpha |k|^\alpha \right),
\]
we can calculate its inverse \( \hat{C}_m^{-1}(k) \) through the identity
\[
\sum_m \hat{C}_m^{-1}(k) \hat{C}_{mi}(k) = \delta_{il}.
\]
For the static case \( \bar{u}_i(k,t) = \tilde{u}_i(k), \tilde{f}_i(k,t) = \tilde{f}_i(k) \), equation (71) has the form
\[
- \sum_m \hat{C}_{im}(k) \bar{u}_m(k) + \tilde{f}_i(k) = 0,
\]
and thus
\[
\bar{u}_m(k) = \sum_i \hat{C}_m^{-1}(k) \tilde{f}_i(k).
\]
Applying the inverse Fourier transform to (75), we obtain a particular solution to equation (50) in the form
\[
u_m(r) = \sum_i \mathcal{F}^{-1} \left\{ \hat{C}_m^{-1}(k) \tilde{f}_i(k) \right\}.
\]
On introducing the fractional analog of the Green function
\[
G'^{mi}(r) = \mathcal{F}^{-1} \left\{ \hat{C}_m^{-1}(k) \right\}.
\]
and applying the convolution property of the Fourier transform, we can then represent the solution \(u_m(r)\) in the form of the convolution of the Green’s function \(G^{mi}(r)\) and the body force \(f_i(r)\) as

\[
u_m(r) = \int_{\mathbb{R}^3} G^{mi}(r - r') f_i(r') \, dr'. \tag{78}
\]

Let us now obtain a solution for the one-dimensional equation \((56)\) in Fourier space, i.e.

\[
(k^2 + a_\alpha k^{\alpha+2}) \tilde{u}(k) = \frac{1}{\lambda + 2 \mu} \tilde{f}(k), \quad (\alpha > 0), \tag{79}
\]

where \(k = k_x\), by utilizing the Green’s function (see Theorem 5.24 in [13]) in the form

\[
G^1_\alpha(x) = \frac{1}{\pi} \int_0^\infty \frac{\cos(\lambda|x|)}{\lambda^2 + a_\alpha \lambda^{\alpha+2}} d\lambda. \tag{80}
\]

For the deformation at positions \(x\), which are large in comparison with the size of the region where the force is applied (point load), we can assume that the force is given by

\[
f(x) = f_0 \delta(x). \tag{81}
\]

Then the displacement, which is a particular solution of equation \((56)\), will be described by the equation

\[
u(x) = \frac{f_0}{\pi(\lambda + 2 \mu)} \int_0^\infty \frac{\cos(\lambda|x|)}{\lambda^2 + a_\alpha \lambda^{\alpha+2}} d\lambda. \tag{82}
\]

The solution \((82)\) for non-integer \(\alpha > 0\) describes a solution for the fractional GRADELA model.

### 2.4 Solution based on Fourier transform

The fractional GADELA displacements for the static case are given by the equation

\[
\mu u_{i,jj} + (\lambda + \mu) u_{j,ji} + l^2_{\alpha} (-\Delta)^{\alpha/2} \left( \mu u_{i,jj} + (\lambda + \mu) u_{j,ji} \right) + f_i = 0, \tag{83}
\]

where \((-\Delta)^{\alpha/2}\) is the fractional Laplacian of order \(\alpha > 0\) of Riesz type. For \(\alpha = 1\), equation \((83)\) gives the usual equations of the standard GRADELA model [7–9]

\[
\mu u_{i,jj} + (\lambda + \mu) u_{j,ji} - l^2 \Delta \left( \mu u_{i,jj} + (\lambda + \mu) u_{j,ji} \right) + f_i = 0. \tag{84}
\]

Using the Fourier transform

\[
\hat{F}(k) = \int_{-\infty}^{+\infty} F(x) e^{-i(k,x)} \, d^3x, \tag{85}
\]

\[
F(x) = \frac{1}{(2\pi)^3} \int_{-\infty}^{+\infty} \hat{F}(k) e^{i(k,x)} \, d^3k, \tag{86}
\]
we obtain
\[ |k|^2 \left( 1 + l_0^2 |k|^\alpha \right) \left( \lambda + 2 \mu \right) k_i^0 k_j^0 + \mu (\delta_{ij} - k_i^0 k_j^0) \right) \hat{u}_i(k) = \hat{f}_i(k), \tag{87} \]
where
\[ |k| = (k_i k_i)^{1/2}, \quad k_i^0 = k_i / |k|. \tag{88} \]
Equation (87) leads us to the expression
\[ \hat{u}_i(k) = \hat{G}_{ij}(k) \hat{f}_j(k), \tag{89} \]
where
\[ \hat{G}_{ij}(k) = \frac{1}{|k|^2 \left( 1 + l_0^2 |k|^\alpha \right)} \left( \frac{1}{\mu} (\delta_{ij} - k_i^0 k_j^0) + \frac{1}{\lambda + 2 \mu} k_i^0 k_j^0 \right). \tag{90} \]
Then displacement \( u_i(x) \) is represented in the form
\[ u_i(x) = \int_{-\infty}^{+\infty} G_{ij}(x - y) d^3y, \tag{91} \]
where
\[ G_{ij}(x) = \frac{1}{(2\pi)^3} \int_{-\infty}^{+\infty} \hat{G}_{ij}(k) e^{i(k \cdot x)} d^3k. \tag{92} \]
To explicitly evaluate \( G_{ij}(x) \), we use spherical coordinates \((k, \theta, \phi)\) with angle \( \theta \) between \( x \) \((\theta = 0)\) and \( k \):
\[ (k, x) = k, x_i = |k| |x| \cos \theta. \tag{93} \]
\[ x = |x|, \quad k = |k|, \quad d^3k = k^2 dk \sin \theta d\theta d\phi. \tag{94} \]
Then we have
\[ G_{ij}(x) = \frac{1}{(2\pi)^3} \int_0^{2\pi} \left( \int_0^{\pi} \left( \int_0^{|x|} \left( \frac{1}{|k|^2 \left( 1 + l_0^2 |k|^\alpha \right)} \left( \frac{1}{\mu} (\delta_{ij} - k_i^0 k_j^0) + \frac{1}{\lambda + 2 \mu} k_i^0 k_j^0 \right) e^{i(k \cdot x) \cos \theta} dk \right) \sin \theta d\theta \right) d\phi = \right. \]
\[ \left. \frac{1}{(2\pi)^3} \int_0^{\pi} \left( \int_0^{2\pi} \left( \left( \frac{1}{\mu} (\delta_{ij} - k_i^0 k_j^0) + \frac{1}{\lambda + 2 \mu} k_i^0 k_j^0 \right) \right) d\phi \right) \cdot \int_0^{|x|} \left( \frac{1}{1 + l_0^2 |k|^\alpha} e^{i(k \cdot x) \cos \theta} dk \right) \sin \theta d\theta. \tag{95} \]
Using that \( \hat{G}_{ij}(-k) = \hat{G}_{ij}(k) \) and \( G_{ij}(-x) = G_{ij}(x) \) along with the identities
\[ \int_0^{2\pi} k_i^0 k_j^0 d\phi = \pi \left( \delta_{ij} \sin^2 \theta - x_i^0 x_j^0 (1 - 3 \cos^2 \theta) \right), \tag{97} \]
where \(x_i^0 = x_i/|x|\), the Fourier transform can be represented as a cosine Fourier transform

\[
\int_0^\infty \frac{1}{1 + l_\alpha^2 |k|^\alpha} e^{i k x \cos \theta} dk = \frac{1}{2} \int_{-\infty}^\infty \frac{1}{1 + l_\alpha^2 |k|^\alpha} e^{i k x \cos \theta} dk = \]

\[
= \int_0^\infty \frac{1}{1 + l_\alpha^2 |k|^\alpha} \cos(k x \cos \theta) dk.
\] (98)

For \(\alpha = 2\), we can use equation (11) of Section 1.2 of [43] in the form

\[
\int_0^\infty \frac{1}{k^2 + \alpha^2} \cos(k x) dk = \frac{\pi}{2\alpha} e^{-\alpha x}\quad (\alpha > 0).
\] (99)

For \(\alpha \neq 2\), the corresponding integrals cannot be evaluated explicitly. For this integrals we introduce the notation

\[
C_n,\alpha(x, l) := \int_0^\infty \frac{\cos(k x)}{k^n (1 + l_\alpha^2 k^\alpha)} \cos(k x \cos \theta) dk
\] (n \in \mathbb{N}).

Note that

\[
C_{0,2}(x, 1) = \frac{\pi e^{-x}}{2},
\] (101)

and the integral \(C_{n,2}(x, 1)\) does not converge for \(n \in \mathbb{N}\). Usually, the integration with respect to \(k\) is realized first, and then the integration with respect to \(\theta\). Here, we will use a reverse sequence of integrations. Using the notation (100), we rewrite equation (96) in the form

\[
G_{ij}(x) = \frac{1}{(2\pi)^3} \int_{-1}^1 \left(\frac{2}{\mu} + \left(\frac{1}{\lambda + 2\mu} - \frac{1}{\mu}\right)(1 - t^2)\right) \delta_{ij} C_{0,\alpha}(x t, l) dt -
\]

\[
- \frac{1}{(2\pi)^3} \int_{-1}^1 \left(\frac{1}{\mu} - \frac{1}{\lambda + 2\mu} - \frac{1}{\mu}\right) x_i^0 x_j^0 (1 - 3t^2) C_{0,\alpha}(x t, l) dt,
\] (102)

where

\[
t = \cos \theta.
\]

To explicitly evaluate this expression (102), we should first calculate the integrals

\[
\int_{-1}^1 C_{0,\alpha}(x t, l) dt = \int_0^\infty \frac{1}{1 + l_\alpha^2 k^\alpha} \left(\int_{-1}^1 \cos(k x t) dt\right) dk.
\] (103)

\[
\int_{-1}^1 C_{0,\alpha}(x t, l) (1 - t^2) dt = \int_0^\infty \frac{1}{1 + l_\alpha^2 k^\alpha} \left(\int_{-1}^1 \cos(k x t)(1 - t^2) dt\right) dk.
\] (104)

\[
\int_{-1}^1 C_{0,\alpha}(x t, l) (1 - 3t^2) dt = \int_0^\infty \frac{1}{1 + l_\alpha^2 k^\alpha} \left(\int_{-1}^1 \cos(k x t)(1 - 3t^2) dt\right) dk.
\] (105)

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The integrals, which appear in the brackets of (103)-(105), can be explicitly represented by elementary functions

\[
\int_{-1}^{1} \cos(kx) \, dt = \frac{2 \sin(kx)}{kx}, \tag{106}
\]

\[
\int_{-1}^{1} \cos(kx) (1 - t^2) \, dt = \frac{4 \sin(kx) - 4kx \cos(kx)}{(xk)^3}, \tag{107}
\]

\[
\int_{-1}^{1} \cos(kx) (1 - 3t^2) \, dt = -\frac{4 ((xk)^2 - 3) \sin(kx) + 12kx \cos(kx)}{(xk)^3}. \tag{108}
\]

Expressions (106)-(108) can be substituted into (103)-(105). This results in new special functions $z$ where $Si(lx)$ and $Ci(lx)$ are sine and cosine integrals, respectively.

Expressions (106)-(108) can be substituted into (103)-(105). This results in new special functions $A_1(\alpha; l; x)$, $A_2(\alpha; l; x)$, and $A_3(\alpha; l; x)$, since the corresponding integrals cannot be represented by elementary and well-known special functions. These new functions are defined by the equations

\[
A_1(\alpha; l; x) := \int_{0}^{\infty} \frac{2 \sin(kx)}{kx (1 + l^2 x^\alpha)} \, dk. \tag{109}
\]

\[
A_2(\alpha; l; x) := \int_{0}^{\infty} \frac{4 \sin(kx) - 4kx \cos(kx)}{(xk)^3 (1 + l^2 k^\alpha)} \, dk. \tag{110}
\]

\[
A_3(\alpha; l^2; x) := -\int_{0}^{\infty} \frac{4 ((xk)^2 - 3) \sin(kx) + 12kx \cos(kx)}{(xk)^3 (1 + l^2 k^\alpha)} \, dk. \tag{111}
\]

Note that for the case $\alpha = 2$, the functions (109)-(111) can be explicitly represented by elementary and well-known special functions. For example, for $l = 1$, we have

\[
A_1(2; 1; x) = x^{-1} \left( \pi - \pi \cosh(x^{3/2}) + I \text{Ci}(-I x^{3/2}) \sinh(x^{-3/2}) - I \text{Ci}(I x^{3/2}) \sinh(x^{3/2}) \right)
\]

\[
A_2(2; 1; x) = -x^{-13/2} \left( 3 \pi \sqrt{x} - 2 \pi \sqrt{x} \cosh(x^{5/2}) + 2 x^3 \pi \sinh(x^{5/2}) \\
+ 2 I \sqrt{x} \text{Ci}(-I x^{5/2}) \sinh(x^{5/2}) + 2 I x^3 \text{Ci}(I x^{5/2}) \cosh(x^{5/2}) \\
- 2 I \sqrt{x} \text{Ci}(I x^{5/2}) \sinh(x^{5/2}) - 2 I x^3 \text{Ci}(-I x^{-5/2}) \cosh(x^{5/2}) \right).
\]

\[
A_3(2; 1; x) = x^{-13/2} \left( -2 \pi x^{11/2} + 2 \pi \cosh(x^{-5/2}) x^{11/2} - 9 \pi \sqrt{x} + 6 \pi \sqrt{x} \cosh(x^{5/2}) \\
- 6 x^3 \pi \sinh(x^{-5/2}) - 6 I x^3 \text{Ci}(I x^{5/2}) \cosh(x^{5/2}) - 6 I \sqrt{x} \text{Ci}(-I x^{-5/2}) \sinh(x^{5/2}) \\
- 2 I \text{Ci}(-I x^{5/2}) \sinh(x^{5/2}) x^{11/2} + 6 I \sqrt{x} \text{Ci}(I x^{-5/2}) \sinh(x^{5/2}) \\
+ 2 I \text{Ci}(I x^{5/2}) \sinh(x^{5/2}) x^{11/2} + 6 I x^3 \text{Ci}(-I x^{-5/2}) \cosh(x^{5/2}) \right),
\]

where $\text{Si}(z)$ and $\text{Ci}(z)$ are sine and cosine integrals, respectively.
Using (109)-(111), equation (102) can be represented in the form

\[
G_{ij}(x) = \frac{1}{(2\pi)^3} \left( \frac{1}{\mu} A_1(\alpha; l; x) \delta_{ij} + \left( \frac{1}{\lambda + 2\mu} - \frac{1}{\mu} \right) A_2(\alpha; l; x) \delta_{ij} - \left( \frac{1}{\lambda + 2\mu} - \frac{1}{\mu} \right) x_i^0 x_j^0 A_3(\alpha; l; x) \right) .
\]  

(112)

As a result, we obtain the equation

\[
G_{ij}(x) = \frac{1}{(2\pi)^3} \left( \psi_\alpha(x) \delta_{ij} - \chi_\alpha(x) x_i^0 x_j^0 \right),
\]  

(113)

where

\[
\psi_\alpha(x) := \frac{1}{\mu} A_1(\alpha; l; x) + \left( \frac{1}{\lambda + 2\mu} - \frac{1}{\mu} \right) A_2(\alpha; l; x),
\]  

(114)

\[
\chi_\alpha(x) := \left( \frac{1}{\lambda + 2\mu} - \frac{1}{\mu} \right) A_3(\alpha; l; x).
\]  

(115)

The functions (109)-(111) can be represented in terms of some sine and cosine Fourier transforms in the form

\[
A_1(\alpha; l; x) := \frac{2}{x} S_{1,\alpha}(x, l),
\]  

(116)

\[
A_2(\alpha; l; x) := \frac{4}{x^3} S_{3,\alpha}(x, l) - \frac{4}{x^2} C_{2,\alpha}(x, l),
\]  

(117)

\[
A_3(\alpha; l^2; x) := -\frac{4}{x} S_{1,\alpha}(x, l) + \frac{12}{x^3} S_{3,\alpha}(x, l) - \frac{12}{x^2} C_{3,\alpha}(x, l),
\]  

(118)

where \( S_{n,\alpha}(x, l) \) and \( C_{n,\alpha}(x, l) \) are the sine and cosine Fourier transforms of corresponding functions in the form

\[
S_{n,\alpha}(x, l) := \int_0^\infty \frac{\sin(k x)}{k^n \left( 1 + l_\alpha^2 k^\alpha \right)} \, dk.
\]  

(119)

\[
C_{n,\alpha}(x, l) := \int_0^\infty \frac{\cos(k x)}{k^n \left( 1 + l_\alpha^2 k^\alpha \right)} \, dk.
\]  

(120)

Note that the integrals \( C_{n,\alpha}(x, l) \) and \( S_{n,\alpha}(x, l) \) do not converge for a wide range of parameters. At the same time, the combinations in the form (116)-(118) converge for some of these cases. In this connection, we give some values of the function \( A_2 \) for \( x = 1 \):

\[
A_2(2.0; 1; 1) = 1.48131675 \quad A_2(2.7; 1; 1) = 1.430697955 \quad A_2(1.2; 1; 1) = 1.550145537
\]  

(121)

\[
A_2(2.0; 0.1; 1) = 3.078796 \quad A_2(2.7; 0.1; 1) = 3.049724200 \quad A_2(1.2; 0.1; 1) = 3.098659886
\]  

(122)

\[
A_2(2.0; 10; 1) = 0.20179058 \quad A_2(2.7; 10; 1) = 0.3025869013 \quad A_2(1.2; 10; 1) = 0.09279446113.
\]  

(123)
3 Fractional Gradient Elasticity with Nonlinear Effects

3.1 Fractional nonlinear constitutive equations

Nonlinearity in the fractional gradient elasticity models can be incorporated in analogy to conventional nonlinear stress-strain relations used for ‘non-gradient’ material behavior. The theories of nonlinear elasticity and plasticity are classical examples which have been well developed and used in a plethora of applications, by also accounting for thermal, diffusion and electromagnetic effects. Such type of multiphysics models for deforming materials lead to systems of differential equations (with derivatives of integer order) which are not only difficult to address analytically or numerically, but also not always able to interpret the observed behavior. The point of view advanced here is that the multiphysics/multivariable approach may be supplemented by a ‘fractional derivatives’ approach utilizing a smaller number of variables and phenomenological parameters. Thus, when a micro/nano structural variable evolves in a complex manner such that non-fractional micro/nano elasticity or plasticity models cannot capture the observed behavior, one may explore the possibility of ignoring the explicit appearance of that variable in the constitutive equations and use non-integer (instead of integer) derivatives to describe deformation with the newly introduced fractional parameter modeling of ‘missing’ phenomenology.

We postpone this general discussion for the future and we focus here on extending the linear fractional gradient elasticity onside various of the previous section to include the nonlinear effects. The proposed nonlinear fractional gradient constitutive equation used involves scalar measures of the stress and strain tensors; i.e. their second invariants, as these quantities enter in both theories of nonlinear elasticity and plasticity. In plasticity theory, in particular, we employ the second invariants of the deviatoric stress and plastic strain tensors ([44, 45]). The effective (equivalent) stress \( \sigma \) is defined by the equation

\[
\sigma = \sqrt{\frac{1}{2}} \sigma'_{ij} \sigma'_{ij},
\]

where \( \sigma_{ij} \) is the stress tensor, and

\[
\sigma'_{ij} = \sigma'_{ij} - \frac{1}{3} \sigma'_{kk} \delta_{ij}.
\]

The effective (equivalent) plastic strain is defined as

\[
\varepsilon = \int dt \sqrt{2 \varepsilon'_{ij} \varepsilon'_{ij}},
\]

where \( \varepsilon'_{ij} \) is the plastic strain rate tensor, which is assumed to be traceless in order to satisfy plastic incompressibility.

Motivated by the above, we propose the following form of nonlinear fractional differential equation for the scalar quantities \( \sigma \) and \( \varepsilon \) which can be used as a basis for a future tensorial formulation of nonlinear elasticity and plasticity theories

\[
\sigma(x) = E \varepsilon(x) + c(\alpha) ((-\Delta)^{\alpha/2} \varepsilon)(x) + \eta K(\varepsilon(x)), \quad (\alpha > 0),
\]
where $K(\varepsilon(\mathbf{x}))$ is a nonlinear function, which describes the usual (homogenous) material’s response beyond linear elasticity or linear hardening plasticity; $c(\alpha)$ is an internal parameter, that measures the nonlocal character of deformation mechanisms; $E$ is a shear-like elastic modulus; $\eta$ is a small parameter of non-linearity; and $(-\Delta)^{\alpha/2}$ is the fractional Laplacian in the Riesz form \cite{13}. As a simple example of the nonlinear function, we can consider

$$K(\varepsilon) = \varepsilon^\beta(\mathbf{x}), \quad (\beta > 0).$$  \hspace{1cm} (127)$$

Equation (126), where $K(\varepsilon)$ is defined by (127) with $\beta = 3$, is the fractional Ginzburg-Landau equation (see, for example, \cite{46}–\cite{48}). For various choices of the parameters $(E, \eta, \beta)$ characterizing the homogeneous material response, different models of elastic and inelastic behavior may result. It is noted, in particular, that (126) may be considered as a fractional generalization of the flow stress expression for the conventional theory of plasticity with $E$ denoting the linear hardening modulus and $(\eta, \beta)$ the Ludwik–Hollomon parameters.

Let us derive a particular solution of equation (126) with $K(\varepsilon) = 0$. To solve the linear fractional differential equation

$$\sigma(\mathbf{x}) = E\varepsilon(\mathbf{x}) + c(\alpha) ((-\Delta)^{\alpha/2}\varepsilon)(\mathbf{x}),$$  \hspace{1cm} (128)$$

which represents the constitutive relation of the fractional gradient elasticity, we apply the Fourier method and the fractional Green function.

Using Theorem 5.22 of \cite{13} for the case $E \neq 0$ and $\alpha > (n-1)/2$, the equation (135) is solvable, and its particular solution is given by the expression

$$\varepsilon(\mathbf{x}) = G_{n,\alpha} * \sigma = \int_{\mathbb{R}^n} G_{n,\alpha}(\mathbf{x} - \mathbf{x}') \sigma(\mathbf{x}') d\mathbf{x'},$$  \hspace{1cm} (129)$$

where the symbol $*$ denotes the convolution operation, and $G_{n,\alpha}(\mathbf{x})$ is defined by (62),

$$G_{n,\alpha}(\mathbf{x}) = \frac{|\mathbf{x}|^{(2-n)/2}}{(2\pi)^{n/2}} \int_0^\infty \frac{\lambda^{n/2} J_{(n-2)/2}(\lambda|\mathbf{x}|)}{c(\alpha) \lambda^{\alpha} + E} d\lambda,$$  \hspace{1cm} (130)$$

where $n = 1, 2, 3$, $\alpha > (n-1)/2$, and $J_{(n-2)/2}$ is the Bessel function of the first kind.

Let us consider an unbounded fractional nonlocal continuum, where the stress is applied to an infinitesimally small region in its interior. In this case, we can assume that the strain $\varepsilon(\mathbf{x})$ is generated by a point stress $\sigma(\mathbf{x})$ at the origin of coordinates, i.e.

$$\sigma(\mathbf{x}) = \sigma_0 \delta(\mathbf{x}).$$  \hspace{1cm} (131)$$

Then, the scalar field $\varepsilon(\mathbf{x})$ is proportional to the Green’s function, and has the form

$$\varepsilon(\mathbf{x}) = \sigma_0 G_{n,\alpha}(\mathbf{x}),$$  \hspace{1cm} (132)$$

i.e.

$$\varepsilon(\mathbf{x}) = \frac{1}{2\pi^2 |\mathbf{x}|} \int_0^\infty \frac{\lambda \sin(\lambda|x|)}{E + c(\alpha) \lambda^{\alpha}} d\lambda.$$  \hspace{1cm} (133)$$

This is the particular solution of the fractional-order differential equation (130) with the point stress for materials distributed in the three-dimensional space.
3.2 Perturbation of gradient elasticity by nonlinearity

Suppose that $\varepsilon(\mathbf{x}) = \varepsilon_0(\mathbf{x})$ is a solution of equation (126) with $\eta = 0$, i.e. $\varepsilon_0(\mathbf{x})$ is a solution of the linear fractional equation

$$\sigma(\mathbf{x}) = E \varepsilon_0(\mathbf{x}) + c(\alpha) \left( (-\Delta)^{\alpha/2} \varepsilon_0(\mathbf{x}) \right).$$

The solution of this equation has the form (129). We will seek a solution of nonlinear equation (126) with $\eta \neq 0$ in the form

$$\varepsilon(\mathbf{x}) = \varepsilon_0(\mathbf{x}) + \eta \varepsilon_1(\mathbf{x}) + \ldots.$$  

This means that we consider perturbations to the strain field $\varepsilon_0(\mathbf{x})$ of the fractional gradient elasticity, which are caused by weak nonlinearity effects.

In this case, equation (134) is an approximation of the zero order. The first order approximation with respect to $\eta$ gives the equation

$$E \varepsilon_1(\mathbf{x}) + c(\alpha) \left( (-\Delta)^{\alpha/2} \varepsilon_1(\mathbf{x}) \right) + K(\varepsilon_0(\mathbf{x})) = 0.$$  

This equation is equivalent to the linear equation

$$\sigma_{eff}(\mathbf{x}) = E \varepsilon_1(\mathbf{x}) + c(\alpha) \left( (-\Delta)^{\alpha/2} \varepsilon_1(\mathbf{x}) \right)$$

where the effective stress $\sigma_{eff}(\mathbf{x})$ is defined by the equation

$$\sigma_{eff}(\mathbf{x}) = -K(\varepsilon_0(\mathbf{x})).$$

Equation (136) can give a particular solution in the form

$$\varepsilon(\mathbf{x}) = \varepsilon_0(\mathbf{x}) + \varepsilon_1(\mathbf{x}) = G_{n,\alpha} * \sigma + \eta G_{n,\alpha} * \sigma_{eff},$$

where the symbol $*$ denotes the convolution operation defined by equation (129). Substitution of (138) into (139), gives

$$\varepsilon(\mathbf{x}) = G_{n,\alpha} * \sigma - \eta G_{n,\alpha} * K(G_{n,\alpha} * \sigma).$$

For point stress (132), equation (140) can written in the form

$$\varepsilon(\mathbf{x}) = \sigma_0 G_{n,\alpha}(\mathbf{x}) - \eta \left( G_{n,\alpha} * K(\sigma_0 G_{n,\alpha}) \right)(\mathbf{x}).$$

For the fractional gradient model that is described by the function $K$ defined by (127), we have

$$\varepsilon(\mathbf{x}) = \sigma_0 G_{n,\alpha}(\mathbf{x}) - \eta \sigma_0^\beta \left( G_{n,\alpha} * (G_{n,\alpha})^\beta \right)(\mathbf{x}),$$

where $\beta > 0.$
3.3 Perturbation by fractional gradient nonlocality

Equilibrium value of \( \varepsilon_0 = \text{const} \) (where \( (-\Delta)^{\alpha/2} \varepsilon_0 = 0 \)) and \( \sigma(x) = \sigma = \text{const} \) is defined by the condition

\[
E \varepsilon_0 + \eta K(\varepsilon_0) = \sigma. \tag{143}
\]

For the case, where the function \( K \) is defined by Eq. (127) with \( \beta = 3 \), we obtain the nonlinear algebraic equation

\[
E \varepsilon_0 + \eta \varepsilon_0^3 = \sigma. \tag{144}
\]

For \( \sigma \neq 0 \), there is no solution \( \varepsilon_0 = 0 \). For \( E > 0 \) and the weak scalar stress field \( \sigma \ll \sigma_c \) with respect to the critical value \( \sigma_c = \sqrt{E/\eta} \), there exists only one solution

\[
\varepsilon_0 \approx \frac{\sigma}{E}. \tag{145}
\]

For negative stiffness materials \( (E < 0) \) and \( \sigma = 0 \), we have three solutions

\[
\varepsilon_0 \approx \pm \sqrt{|E|/\eta}, \quad \varepsilon_0 = 0. \tag{146}
\]

For \( \sigma < (2\sqrt{3}/9)\sigma_c \), also exist three solutions. For \( \sigma \gg \sigma_c \), we can neglect the first term \( (E \approx 0) \),

\[
\eta \varepsilon_0^3 \approx \sigma, \tag{147}
\]

and obtain

\[
\varepsilon_0 \approx \left( \frac{\sigma}{\eta} \right)^{1/3} = \sqrt[3]{\frac{\sigma}{\eta}}. \tag{148}
\]

In any case, the equilibrium values \( \varepsilon_0 \) are solutions of the algebraic equation (143).

Let us consider a deviation \( \varepsilon_1(x) \) of the field \( \varepsilon(x) \) from the equilibrium value \( \varepsilon_0 \). For this purpose we will seek a solution in the form

\[
\varepsilon(x) = \varepsilon_0 + \varepsilon_1(x). \tag{149}
\]

In general, the stress field is not constant, i.e. \( \sigma(x) \neq \sigma \). For the first order approximation, we have the equation

\[
\sigma(x) = c(\alpha) \left( (-\Delta)^{\alpha/2} \varepsilon_1 \right)(x) + \left( E + \eta K'_\varepsilon(\varepsilon_0) \right) \varepsilon_1(x), \tag{150}
\]

where \( K'_\varepsilon = \partial K(\varepsilon)/\partial \varepsilon \). Equation (150) is equivalent to the linear fractional differential equation

\[
\sigma(x) = E_{\text{eff}} \varepsilon_1(x) + c(\alpha) \left( (-\Delta)^{\alpha/2} \varepsilon_1 \right)(x) \tag{151}
\]

with the effective modulus \( E_{\text{eff}} \) defined by

\[
E_{\text{eff}} = E + \eta K'_\varepsilon(\varepsilon_0). \tag{152}
\]

For the case \( K(\varepsilon) = \varepsilon^\beta \), we have

\[
E_{\text{eff}} = E + \beta \eta \varepsilon_0^{\beta-1}. \tag{147}
\]
A particular solution of Eq. (151) can be written in the form (129), where we use $E_{\text{eff}}$ instead of $E$. For the point stress (131–133), Eq. (140) gives
\[
\varepsilon_1(x) = \frac{1}{2\pi^2 |x|} \int_0^\infty \frac{E + E_{\text{eff}} + 2c(\alpha) \lambda^\alpha}{(c(\alpha) \lambda^\alpha + E)(c(\alpha) \lambda^\alpha + E_{\text{eff}})} \sin(\lambda|x|) \, d\lambda. \tag{153}
\]
For the case $\alpha = 2$, the field $\varepsilon_1(x)$ is given by the equation
\[
\varepsilon_1(x) = \frac{\sigma_0}{4\pi c(\alpha) |x|} e^{-|x|/r_c}, \tag{154}
\]
where $r_c$ is defined by
\[
r_c^2 = \frac{c(\alpha)}{E + \eta K_0(\varepsilon_0)}. \tag{155}
\]
It should be noted that on analogous situation exists in classical theory of electric fields. In the electrodynamics the field $\varepsilon_1(x)$ describes the Coulomb potential with the Debye’s screening. For the case $\alpha \neq 2$, we have a power-law type of screening that is described in [49]. The electrostatic potential for media with power-law spatial dispersion differs from the Coulomb’s potential by the factor
\[
C_{\alpha,0}(|x|) = \frac{2}{\pi} \int_0^\infty \frac{\lambda \sin(\lambda|x|)}{E_{\text{eff}} + c(\alpha) \lambda^\alpha} \, d\lambda. \tag{156}
\]
Note that the Debye’s potential differs from the Coulomb’s potential by the exponential factor $C_D(|x|) = \exp(-|x|/r_D)$.

### 3.4 Fractional Laplacian of Caputo type

Fractional gradient models can be based on the Caputo fractional derivatives. Due to reasons concerning the initial and boundary conditions, it is more convenient to use the Caputo fractional derivatives. They allows us to use simpler boundary conditions that contain derivatives of integer order. In fact, the initial and boundary conditions take the same form as for integer-order differential equations. For fractional derivatives of other type (for example, the Riemann-Liouville derivatives) the boundary conditions are represented by integrals and derivatives of non-integer orders [12] [13].

The Caputo fractional derivative is usually denoted as $C_{D_{a+}}^\alpha$, and it is defined by the equation
\[
C_{a+}D_\alpha^\alpha \varepsilon(x) = \frac{1}{\Gamma(n-\alpha)} \int_a^x \frac{dx'}{(x-x')^{1+\alpha-n}} \frac{\partial^n \varepsilon(x')}{\partial x'^n}, \quad n-1 < \alpha < n, \tag{157}
\]
where $n - 1 < \alpha < n$, and $\varepsilon(x)$ is a real-valued function defined on a closed interval $[a, b]$ such that $\varepsilon(x) \in AC^1[a, b]$ or $\varepsilon(x) \in C^1[a, b]$. We define the left-sided Caputo fractional differential operator on $[a, b]$ in the form
\[
C_a D_\alpha^n [x'](\ldots) = \frac{1}{\Gamma(n-\alpha)} \int_a^x \frac{dx'}{(x-x')^{1+\alpha-n}} \frac{\partial^n \varepsilon(x')}{\partial x'^n}(\ldots), \quad n-1 < \alpha < n. \tag{158}
\]
The Caputo operator defined by (158) acts on real-valued functions \( \varepsilon(x) \in AC^n[a, b] \) as \( C^a D^\alpha [x'] \varepsilon(x') \). We note that the Caputo operator can be represented as

\[
C^a D^\alpha x \left[ x' \right] = a I^{n-\alpha}_x [x'] D^n [x'], \quad (n - 1 < \alpha < n).
\]  

(159)

where we use the left-sided Riemann-Liouville fractional integral operator that is defined as

\[
a I^\alpha x \left[ x' \right] \left( \ldots \right) = \frac{1}{\Gamma(\alpha)} \int_a^x \frac{dx'}{(x-x')^{1-\alpha}} \left( \ldots \right), \quad (\alpha > 0).
\]  

(160)

To designate that the operator given by (160) acts on a real-valued function \( \varepsilon(x) \in L_1[a, b] \), we employ the notation \( a I^\alpha x [x'] \varepsilon(x') \).

Let us assume that \( \varepsilon(x) \) is real-valued functions with continuous derivatives up to order \( (m - 1) \) on \( W \subset \mathbb{R}^3 \), such that their \( (m - 1) \) derivatives are absolutely continuous, i.e., \( \varepsilon(x) \in AC^m[W] \). Then, using the notation introduced in (159), we can define the fractional Laplacian of the Caputo type

\[
C^\Delta^\alpha_W = \sum_{l=1}^{3} (C^a D^\alpha [x_l])^2.
\]  

(161)

In the general case, we have the inequality \( (C^\Delta^\alpha_W)^2 \neq C^\Delta^{2\alpha}_W \), since \( (C^a D^\alpha x)^2 \neq C^D^{2\alpha}_x \).

In order to solve the corresponding governing fractional equations, we can use an explicit form of the relationship between the square of the Caputo derivative \( (C^D^a x)^2 \) and the Caputo derivative \( C^D^{2\alpha}_a \) of the form

\[
(C^D^a x)^2 \varepsilon(x) = C^D^{2\alpha} \varepsilon(x) + \frac{\varepsilon'(a)}{\Gamma(1 - 2\alpha)} (x - a)^{1 - 2\alpha}, \quad (0 < \alpha \leq 1),
\]  

(162)

where \( \alpha \neq 1/2 \). Using (162), we can represent the fractional Laplacian of Caputo type as

\[
C^\Delta^\alpha_W \varepsilon(x) = \sum_{k=1}^{3} C^D^{2\alpha}_x \varepsilon(x) + \sum_{k=1}^{3} \frac{(x_k - a_k)^{1 - 2\alpha}}{\Gamma(1 - 2\alpha)} \left( \frac{\partial \varepsilon(x)}{\partial x_k} \right)_{x_k = a_k}. \quad (0 < \alpha \leq 1),
\]  

(163)

Note that the relation given by Eq. (162) cannot be used for \( \alpha > 1 \).

To describe nonlinear effects within the aforementioned fractional gradient framework, we consider the constitutive relation in the form

\[
\sigma(x) = c(\alpha) \left( C^\Delta^\alpha_W \varepsilon(x) + K[x, \varepsilon(x)] \right) \quad (\alpha > 0),
\]  

(164)

where we use the fractional Laplacian of the Caputo type (161) instead of the Laplacian of the Riesz type.

Let us consider the situation for \( \sigma(x) = 0 \) for the one-dimensional case \( x \in \mathbb{R}^1 \). In this case, we have the nonlinear differential equation of order \( \alpha > 0 \) in the form

\[
(C^D^a x)^2 \varepsilon(x) = K[x, \varepsilon(x)] \quad (\alpha > 0, \quad a \leq x \leq b),
\]  

(165)
involving the Caputo fractional derivative $C^{\alpha}D_{a+}$ on a finite interval $[a,b]$ of the real axis $\mathbb{R}$, with the initial conditions

$$\varepsilon^{(k)}(a) = b_k, \quad b_k \in \mathbb{R}, \quad (k = 0, 1, 2, ..., n - 1). \quad (166)$$

In [13], the conditions are given for a unique solution $\varepsilon(x)$ to this problem in the space $C^{\alpha,r}[a,b]$ defined for $0 < \alpha < n$, $r \in \mathbb{N}$ and $0 \leq \gamma < 1$, by $C^{\alpha,r}[a,b] = \{\varepsilon(x) \in C^r[a,b] : C^{\alpha}D_{a+} \varepsilon \in C^\gamma[a,b]\}$.

One possible way of to proceed based on reducing the problem considered to the Volterra integral equation [13]:

$$\varepsilon(x) = \sum_{j=0}^{n-1} \frac{b_j}{j!} (x - a)^j + \frac{1}{\Gamma(\alpha)} \int_a^x \frac{K[z,\varepsilon(z)]dz}{(x - z)^{1-\alpha}}, \quad (a \leq x \leq b). \quad (167)$$

In this connection, noted that such equations with fractional Laplacian of the Caputo type can be numerically solved. In [50], the Cauchy problem of the form

$$(C^{\alpha}D_{0+})\varepsilon(x) = K[x,\varepsilon(x)], \quad (x > 0); \quad \varepsilon^{(k)}(0) = b_k, \quad (k = 0, 1, ..., n - 1) \quad (168)$$

with the Caputo derivative $C^{\alpha}D_{0+}$ of order $n - 1 < \alpha < n$ ($n \in \mathbb{N}$), is numerically solved by using the fixed memory principle described in Chapter 8 of [51]. In [52], an algorithm for the numerical solution of (168), which is a generalization of the classical one-step Adams-Bashforth-Moulton scheme for first-order equations, has been suggested.
4 Towards Fractal Gradient Elasticity

4.1 Laplacian for non-integer dimensional space

Let us begin by giving expressions of fractal–type differential operators for functions $u = u_r(r)e_r$ and $\varphi = \varphi(r)$ in the spherical coordinates in $\mathbb{R}^n$ for arbitrary $n$ (i.e. $n$ with integer or non-integer values $D$).

As a result, we have equations of differential operators in $\mathbb{R}^n$ for continuation from integer $n$ to arbitrary non-integer $D$ in the following forms.

The scalar Laplacian in non-integer dimensional space for the scalar field $\varphi = \varphi(r)$ is

$$S \Delta^D_r \varphi = \text{Div}^D_r \text{Grad}^D_r \varphi = \frac{\partial^2 \varphi}{\partial r^2} + \frac{D - 1}{r} \frac{\partial \varphi}{\partial r}. \quad (169)$$

The vector Laplacian in non-integer dimensional space for the vector field $u = u(r)e_r$ is

$$V \Delta^D_r u = \text{Grad}^D_r \text{Div}^D_r u = \left( \frac{\partial^2 u_r}{\partial r^2} + \frac{D - 1}{r} \frac{\partial u_r}{\partial r} - \frac{D - 1}{r^2} u_r \right) e_r. \quad (170)$$

Let us consider a case of axial symmetry for fractal materials, where the fields $\varphi(r)$ and $u(r) = u_r(r)e_r$ are also axially symmetric. Let the $z$–axis be directed along the axis of symmetry. Therefore it is convenient to use a cylindrical coordinate system. Equations for differential vector operations for cylindrical symmetry case have the following forms.

The scalar Laplacian in non-integer dimensional space for the scalar field $\varphi = \varphi(r)$ is

$$S \Delta^D_r \varphi = \text{Div}^D_r \text{Grad}^D_r \varphi = \frac{\partial^2 \varphi}{\partial r^2} + \frac{D - 2}{r} \frac{\partial \varphi}{\partial r}. \quad (171)$$

The vector Laplacian in non-integer dimensional space for the vector field $u = u(r)e_r$ is

$$V \Delta^D_r u = \text{Grad}^D_r \text{Div}^D_r u = \left( \frac{\partial^2 u_r}{\partial r^2} + \frac{D - 2}{r} \frac{\partial u_r}{\partial r} - \frac{D - 2}{r^2} u_r \right) e_r. \quad (172)$$

Using the analytic continuation of Gaussian integrals the scalar Laplace operator for non-integer dimensional space has already suggested. Specifically for a function $\varphi = \varphi(r, \theta)$ of radial distance $r$ and related angle $\theta$ measured relative to an axis passing through the origin, the Laplacian in non-integer dimensional space proposed by Stillinger [31] is

$$St \Delta^D = \frac{1}{r^{D-1}} \frac{\partial}{\partial r} \left( r^{D-1} \frac{\partial}{\partial r} \right) + \frac{1}{r^2 \sin^{D-2} \theta} \frac{\partial}{\partial \theta} \left( \sin^{D-2} \theta \frac{\partial}{\partial \theta} \right), \quad (173)$$

where $D$ is the dimension of space ($0 < D < 3$), and the variables $r \geq 0$, $0 \leq \theta \leq \pi$. Note that $(St \Delta^D)^2 \neq St \Delta^D$. If the function depends on the radial coordinate $r$ only ($\varphi = \varphi(r)$), then

$$St \Delta^D \varphi(r) = \frac{1}{r^{D-1}} \frac{\partial}{\partial r} \left( r^{D-1} \frac{\partial \varphi(r)}{\partial r} \right) = \frac{\partial^2 \varphi(r)}{\partial r^2} + \frac{D - 1}{r} \frac{\partial \varphi(r)}{\partial r}. \quad (174)$$
It is easy to see that the Stillinger’s form of Laplacian \( St \Delta^D \) for radial scalar functions \( \varphi(r) = \varphi(r) \) coincides with the above suggested scalar Laplacian \( S \Delta^D \) for this function,
\[
St \Delta^D \varphi(r) = S \Delta^D \varphi(r). \tag{175}
\]

The Stillinger’s Laplacian can be applied only for scalar fields and it cannot be used to describe vector fields \( u = u_r(r) e_r \) because Stillinger’s Laplacian for \( D = 3 \) is not equal to the usual vector Laplacian for \( \mathbb{R}^3 \),
\[
St \Delta^3 u(r) \neq \Delta u(r) = \left( \frac{\partial^2 u_r}{\partial r^2} + \frac{2}{r} \frac{\partial u_r}{\partial r} - \frac{2}{r^2} u_r \right) e_r. \tag{176}
\]

For the vector fields \( u = u_r(r) e_r \), we should use the vector Laplace operators (170) and (170).

Note that the gradient, divergence, curl operators and vector Laplacian are not considered in Stillinger’s [31].

In [22], the scalar and vector Laplace operators for the case \( d \neq D - 1 \) and the fields \( \varphi = \varphi(r) \) and \( u = u(r) e_r \), are defined by the equations
\[
S \Delta_r^D \varphi = \text{Div}_r^D \text{Grad}_r^D \varphi, \quad V \Delta_r^D u = \text{Grad}_r^D \text{Div}_r^D u. \tag{177}
\]

Then, the scalar Laplacian for \( d \neq D - 1 \) for the field \( \varphi = \varphi(r) \) is
\[
S \Delta_r^D \varphi = \frac{\Gamma((d + \alpha_r)/2) \Gamma(\alpha_r/2)}{\pi^{\alpha_r-1/2} \Gamma((d + 1)/2)} \left( \frac{1}{r^{2\alpha_r-2}} \frac{\partial^2 \varphi}{\partial r^2} + \frac{d + 1 - \alpha_r}{r^{2\alpha_r-1}} \frac{\partial \varphi}{\partial r} \right), \tag{178}
\]
and the vector Laplacian for \( d \neq D - 1 \) for the field \( u = u(r) e_r \) is
\[
V \Delta_r^D u = \frac{\Gamma((d + \alpha_r)/2) \Gamma(\alpha_r/2)}{\pi^{\alpha_r-1/2} \Gamma((d + 1)/2)} \left( \frac{1}{r^{2\alpha_r-2}} \frac{\partial^2 u_r}{\partial r^2} + \frac{d + 1 - \alpha_r}{r^{2\alpha_r-1}} \frac{\partial u_r}{\partial r} - \frac{d - \alpha_r}{r^{2\alpha_r}} u_r \right) e_r. \tag{179}
\]

The vector differential operators and the Laplacian operator for non-integer dimensional space [22] allow us to describe complex fractal materials with fractal dimensions \( D \) for the interior region and \( d \) for its boundary surface \( (d \neq D - 1) \).

### 4.2 Strain and stress in non-integer dimensional space

Any deformation can be represented as the sum of a pure shear and a hydrostatic component. To do this for fractal materials, we can use the identity
\[
\varepsilon_{kl} = \left( \varepsilon_{kl} - \frac{1}{D} \delta_{kl} \varepsilon_{ii} \right) + \frac{1}{D} \delta_{kl} \varepsilon_{ii}. \tag{180}
\]

The first term on the right is a pure shear, since the sum of diagonal terms is zero. Here we use the equation \( \delta_{ii} = D \) for non-integer dimensional space (for details see Property 4 in Section 4.3 of [30]). The second term is the hydrostatic component. For \( D = 3 \), equation (180) has the well-known form
\[
\varepsilon_{kl} = \left( \varepsilon_{kl} - \frac{1}{3} \delta_{kl} \varepsilon_{ii} \right) + \frac{1}{3} \delta_{kl} \varepsilon_{ii}, \tag{181}
\]
where $\delta_{ii} = 3$ is used.

The stress tensor can then be represented (Hooke’s law of classical elasticity) as

$$\sigma_{kl} = K \varepsilon_{ii} \delta_{kl} + 2\mu \left( \varepsilon_{kl} - \frac{1}{D} \delta_{kl} \varepsilon_{ii} \right),$$  \hspace{1cm} (182)

where the bulk modulus $K$ is related to the Lame coefficients $(\lambda, \mu)$ by

$$K = \lambda + \frac{2\mu}{D}. \hspace{1cm} (183)$$

Under the hydrostatic compression, the stress tensor is

$$\sigma_{kl} = -p \delta_{kl}. \hspace{1cm} (184)$$

Hence we have

$$\sigma_{kk} = -p D. \hspace{1cm} (185)$$

which, in view of (182), gives

$$\sigma_{ii} = (\lambda D + 2\mu) \varepsilon_{ii}. \hspace{1cm} (186)$$

The radial component of the strain tensor is

$$\varepsilon_{rr} = \frac{\partial u_r}{\partial r} = (e_r, \text{Grad}_r u_r).$$  \hspace{1cm} (187)

Using $\text{Div}_r^{D}$ [22], and the trace of the strain tensor

$$e(r) = Tr[\varepsilon_{kl}] = \varepsilon_{kk} = \text{Div}_r^{D} u = \frac{\partial u_r}{\partial r} + \frac{D - 1}{r} u_r, \hspace{1cm} (188)$$

we can consider

$$e(r) - \varepsilon_{rr}(r) = \text{Div}_r^{D} u - (e_r, \text{Grad}_r u_r) = \frac{D - 1}{r} u_r, \hspace{1cm} (189)$$

as a sum of the angular diagonal components in spherical coordinates of the strain tensor. For $D = 3$, we have the well-known result

$$\varepsilon_{\theta\theta} + \varepsilon_{\varphi\varphi} = \frac{2}{r} u_r. \hspace{1cm} (190)$$

When we consider the fractal medium distributed in three-dimensional space we can define the effective value of the diagonal angular components of the strain tensor, as

$$\varepsilon_{\theta\theta}^{\text{eff}} = \varepsilon_{\varphi\varphi}^{\text{eff}} = \frac{D - 1}{2r} u_r. \hspace{1cm} (191)$$

Using (187) and (188), the components of the stress tensor $\sigma_{kl} = \sigma_{kl}(r)$ in spherical coordinates are given below, starting from the radial component $\sigma_{rr}$:

$$\sigma_{rr}(r) = 2\mu \varepsilon_{rr}(r) + \lambda e(r) = (2\mu + \lambda) \frac{\partial u_r}{\partial r} + \lambda \frac{D - 1}{r} u_r. \hspace{1cm} (192)$$
To deduce appropriate expressions for the fractal counterpart of the diagonal angular components, we first note that for \( D = 3 \) in spherical coordinates, they are given by the familiar relations
\[
\sigma_{\theta\theta}(r) = 2\mu \varepsilon_{\theta\theta}(r) + \lambda e(r), \quad \sigma_{\varphi\varphi}(r) = 2\mu \varepsilon_{\varphi\varphi}(r) + \lambda e(r).
\] (193)

For the fractal medium distributed in three-dimensional space we can define the effective value of the diagonal angular components of the stress tensor
\[
\sigma_{\theta\theta}^{\text{eff}}(r) = 2\mu \varepsilon_{\theta\theta}^{\text{eff}}(r) + \lambda e(r),
\] (194)
\[
\sigma_{\varphi\varphi}^{\text{eff}}(r) = 2\mu \varepsilon_{\varphi\varphi}^{\text{eff}}(r) + \lambda e(r).
\] (195)

Using the form for the effective components (191), we obtain
\[
\sigma_{\theta\theta}^{\text{eff}}(r) = \sigma_{\varphi\varphi}^{\text{eff}}(r) = \lambda \frac{\partial u_r}{\partial r} + (\lambda + \mu) \frac{D-1}{r} u_r.
\] (196)

This equation define the diagonal angular components of the stress tensor in spherical coordinates.

### 4.3 Gradient elasticity model for fractal materials

The standard linear elastic constitutive relation for the isotropic case is the well-known Hooke’s law – given by (182) with \( D = 3 \) – written here in the alternative form, as
\[
\sigma_{ij} = \lambda \varepsilon_{kk} \delta_{ij} + 2\mu \varepsilon_{ij},
\] (197)
where \( \lambda \) and \( \mu \) are the Lame constants. The corresponding governing equation for the displacement vector field \( \mathbf{u} = \mathbf{u}(r,t) \), reads
\[
\lambda \text{grad} \text{div} \mathbf{u} + 2\mu \Delta \mathbf{u} + \mathbf{f} = \rho D_t^2 \mathbf{u},
\] (198)
where \( \mathbf{f} = \mathbf{f}(r,t) \) is an external body force field, and \( \rho \) denotes the density of the material.

If the deformation in the material is described by \( \mathbf{u}(r,t) = u(r,t) \mathbf{e}_r \), then equation (198) has the form
\[
(\lambda + 2\mu) \Delta \mathbf{u}(r,t) + \mathbf{f}(r,t) = \rho D_t^2 \mathbf{u}(r,t).
\] (199)

Using the non-integer dimensional vector calculus [22], we can now suggest a fractal generalization of equations (199) for elastic materials in non-integer dimensional space, where the displacement vector \( \mathbf{u} = \mathbf{u}(r,t) \), does not depend on the angular coordinate. It reads
\[
(\lambda + 2\mu) \left(1 \pm \frac{l_1^2(D,d)}{\Delta_\alpha(D,d)}\right) \Delta_\alpha^{D,d} \mathbf{u} + \mathbf{f} = \rho D_t^2 \mathbf{u}.
\] (200)

where \( \Delta_\alpha^{D,d} \mathbf{u} \) is the vector Laplacian for \( d \neq D - 1 \) for the spherically symmetric field \( \mathbf{u} = u(r) \mathbf{e}_r \) that is defined by the equation
\[
\Delta_\alpha^{D,d} \mathbf{u} = \frac{\Gamma((d+\alpha_r)/2) \Gamma(\alpha_r/2)}{\pi^{\alpha_r-1/2} \Gamma((d+1)/2)} \left( \frac{1}{r^{2\alpha_r-2}} \frac{\partial^2 u_r}{\partial r^2} + \frac{d+1-\alpha_r}{r^{2\alpha_r-1}} \frac{\partial u_r}{\partial r} - \frac{d \alpha_r}{r^{2\alpha_r}} u_r \right) \mathbf{e}_r.
\] (201)
with $\alpha_r = D - d$. The vector differential operator (201) allow us to describe complex fractal materials with boundary dimension $d \neq D - 1$. Similarly, we can consider the cylindrical symmetry case for $d \neq D - 1$.

For fractal materials with $d = D - 1$ equation (200) has the form

$$
(\lambda + 2\mu)^{V} \Delta^D_r u(r, t) + f(r, t) = \rho D^2_t u(r, t),
$$

(202)

where $V \Delta^D_r$ is defined by (170). In the next sections we consider the case $d = D - 1$ for simplicity.

### 4.4 Fractal gradient elasticity for $d = D - 1$ and spherical symmetry

Let us assume that the displacement vector $u$ is everywhere radial and it is a function of $r = |r|$ alone ($u_k = u_k(|r|, t)$). Using a continuum model with non-integer dimensional space, a fractal generalization of gradient elasticity equations for this case, where the displacement vector $u = u(r, t)$, does not depend on the angular coordinate, has the form

$$
(\lambda + 2\mu) \left( 1 \pm l_s^2(D) V \Delta^D_r \right) V \Delta^D_r u + f = \rho D^2_t u.
$$

(203)

This is the gradient elasticity equation for homogenous and isotropic fractal materials with the spherical symmetry. Let us consider equation (203) for static case ($D^2_t u = 0$) with a minus in front of Laplacian, i.e. the GRADELA model for fractal materials

$$
(\lambda + 2\mu) \left( 1 - l_s^2(D) V \Delta^D_r \right) V \Delta^D_r u + f = 0.
$$

(204)

We can rewrite this equation as

$$
( V \Delta^D_r )^2 u - l_s^{-2}(D) V \Delta^D_r u - (\lambda + 2\mu)^{-1} l_s^{-2}(D) f = 0.
$$

(205)

For spherical symmetry and $d = D - 1$, the vector Laplacian for non-integer dimensional space has the form [22]

$$
V \Delta^D_r u(r) = \left( \frac{\partial^2 u_r(r)}{\partial r^2} + \frac{D - 1}{r} \frac{\partial u_r(r)}{\partial r} - \frac{D - 1}{r^2} u_r(r) \right) e_r,
$$

(206)

where we assume that the displacement vector is $u(r) = u_r(r)e_r$. For non-fractal materials in the three–dimensional space, we should use equation (206) with $D = 3$. Equation (206) gives

$$
( V \Delta^D_r )^2 u(r) = \left( \frac{\partial^4 u_r(r)}{\partial r^4} + \frac{2(D - 1)}{r} \frac{\partial^3 u_r(r)}{\partial r^3} + \frac{(D - 1)(D - 5)}{r^2} \frac{\partial^2 u_r(r)}{\partial r^2} - 3(D - 1)(D - 3) \frac{\partial u_r(r)}{\partial r} + \frac{3(D - 1)(D - 3)}{r^4} u_r(r) \right) e_r.
$$

(207)

Substitution of expressions (206)−(207) and $f(r) = f(r) e_r$ into equation (205) gives

$$
\frac{\partial^4 u_r(r)}{\partial r^4} + \frac{2(D - 1)}{r} \frac{\partial^3 u_r(r)}{\partial r^3} + \left( \frac{(D - 1)(D - 5)}{r^2} - l_s^{-2}(D) \right) \frac{\partial^2 u_r(r)}{\partial r^2} -
$$
\[- \left( \frac{3(D-1)(D-3)}{r^3} + l_s^{-2}(D) \frac{D-1}{r} \right) \frac{\partial u_r(r)}{\partial r} + \left( \frac{3(D-1)(D-3)}{r^4} + l_s^{-2}(D) \frac{D-1}{r^2} \right) u_r(r) - (\lambda + 2\mu)^{-1} l_s^{-2}(D) f(r) = 0. \]  

(208)

The general solution for the case \( f(r) = 0 \) is

\[ u_r(r) = C_1 r + C_2 r^{1-D} - C_3 I_{l}(D, r) - C_4 I_{K}(D, r), \]  

(209)

where \( I_{l}(D, r) \) and \( I_{K}(D, r) \) are the integrals of Bessel functions, as follows:

\[ I_{l}(D, r) = D r \int dr r^{-D-1} \int dr r^{D/2+1} I_{l}(r/l_s(D)), \]  

(210)

\[ I_{K}(D, r) = D r \int dr r^{-D-1} \int dr r^{D/2+1} K_{l}(r/l_s(D)), \]  

(211)

where \( I_{\alpha}(x) \) and \( K_{\alpha}(x) \) are Bessel functions of the first and second kinds.

### 4.5 Fractal gradient elasticity for \( d = D - 1 \) and cylindrical symmetry

For cylindrical symmetry and \( d = D - 1 \), the vector Laplacian for non-integer dimensional space has the form \[ 22 \]

\[ V \Delta^D_r \mathbf{u}(r) = \left( \frac{\partial^2 u_r(r)}{\partial r^2} + \frac{D-2}{r} \frac{\partial u_r(r)}{\partial r} - \frac{D-2}{r^2} u_r(r) \right) \mathbf{e}_r. \]  

(212)

Equation (212) gives

\[ (V \Delta^D_r)^2 \mathbf{u}(r) = \left( \frac{\partial^4 u_r(r)}{\partial r^4} + \frac{2(D-2)}{r} \frac{\partial^3 u_r(r)}{\partial r^3} + \frac{(D-2)(D-6)}{r^2} \frac{\partial^2 u_r(r)}{\partial r^2} + \frac{3(D-2)(D-4)}{r^3} \frac{\partial u_r(r)}{\partial r} + \frac{3(D-2)(D-4)}{r^4} u_r(r) \right) \mathbf{e}_r. \]  

(213)

Substitution of expressions (212), (213) and \( f(r) = f(r) \mathbf{e}_r \) into equation (205) gives

\[ \frac{\partial^4 u_r(r)}{\partial r^4} + \frac{2(D-2)}{r} \frac{\partial^3 u_r(r)}{\partial r^3} + \left( \frac{(D-2)(D-6)}{r^2} - l_s^{-2}(D) \right) \frac{\partial^2 u_r(r)}{\partial r^2} - \left( \frac{3(D-2)(D-4)}{r^3} + l_s^{-2}(D) \frac{D-2}{r} \right) \frac{\partial u_r(r)}{\partial r} + \left( \frac{3(D-2)(D-4)}{r^4} + l_s^{-2}(D) \frac{D-2}{r^2} \right) u_r(r) - (\lambda + 2\mu)^{-1} l_s^{-2}(D) f(r) = 0. \]  

(214)

The general solution for the case \( f(r) = 0 \) is

\[ u_r(r) = C_1 r + C_2 r^{2-D} - C_3 I_{l}(D-1, r) - C_4 I_{K}(D-1, r), \]  

(215)

where \( I_{l}(D-1, r) \) and \( I_{K}(D-1, r) \) are defined by equations (210) and (211).
5 Conclusions

We proposed (Section 2) a fractional-order generalization of three-dimensional continuum gradient elasticity models by assuming weak non-locality of power-law type that results to constitutive relations with fractional Laplacian terms. A three-dimensional constitutive equation for fractional gradient elasticity is derived from integer-order integral relations for nonlocal elasticity using a fractional Taylor series expansion in the wave-vector space. The suggested fractional constitutive relations can be connected with microscopic atomistic–type models with long-range interactions, formulated based on lattice with long-range interactions, that can be formulated by using fractional-order differential and integral operators on physical lattices, as suggested in [53] – [56].

Fractional generalizations of gradient models with nonlinearity, employing fractional order Laplace operators of the Riesz and Caputo types, are also considered (Section 3) for we consider plasticity of non-local continua with weak nonlocality of power-law type. Using the perturbation method, we obtain corrections to linearized constitutive relations associated with weak deviations from elasticity, as well as with fractional gradient non-locality. We assume that such fractional gradient models can be described by the fractional variational principle suggested in [11] – [57].

Finally, we propose (Section 4) models of gradient elasticity for fractal materials by using vector the calculus on non-integer dimensional spaces recently suggested in [22, 34, 58]. We consider applications for fractal gradient elasticity theory for axially symmetric problems in spherical and cylindrical coordinates. Although the proposed models may not capture all features of underlying material fractality, the suggested vector calculus on non-integer dimensional spaces may enable us to derive concrete results to be checked with experiments commonly employed for characterization of standard (non-fractal) material response. This task is facilitated by the fact that the governing equations to be solved for fractal gradient elastic materials in ‘non-integer’ space, are differential equations with derivatives of ‘integer’ order, as the case of non-fractal media. Solutions to typical problems for both cases \( d = D - 1 \) and \( D \neq d - 1 \) will be given in the future. These will generalize some results already obtained for fractal elasticity [58]. In [59] an operator split method (the Ru–Aifantis theorem) has been used to obtain solutions to gradient elasticity in terms of solutions of corresponding problems in classical elasticity [8, 9, 60]. A generalization of the Ru–Aifantis operator split method can be used to solve boundary value problems for fractal gradient elasticity by using the solution for fractal non-gradient elasticity given in [58]. Such an extension of the Ru–Aifantis operator split method to fractional/fractal elasticity has been illustrated in [11]. Nonlinear elasticity and plasticity effects within a fractal formulation based on non-integer space can also be considered in a similar way.

Acknowledgement

Support of the Ministry of Education and Science of Russian Federation under grant no. 14.Z50.31.0039 is acknowledged.
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