BLOW UP AT FINITE TIME FOR WAVE EQUATION IN VISCOELASTICITY: A NEW KIND FOR ONE SPATIAL VARIABLE EMDEN-FOWLER TYPE

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Abstract. For one spatial variable, a new kind of nonlinear wave equation for Emden-Fowler type is considered with boundary value null and initial values. Under certain conditions on the initial data and the exponent \( p \), we exhibit that the viscoelastic term leads our problem to be dissipative and the global solutions still non-exist in \( L^2 \) at given finite time.

1. Introduction

We consider a new kind of Emden-Fowler type wave equation in viscoelasticity

\[
\begin{cases}
t^2u'' - u_{xx} + \int_0^t \mu(s) u_{xx}(t-s) ds = u^p \\
u(1, x) = u_0(x) \in H^2(r_1, r_2) \cap H_0^1(r_1, r_2), \\
u'(1, x) = u_1(x) \in H_0^1(r_1, r_2)
\end{cases}
\] (1.1)

where \( p > 1 \), \( r_1 \) and \( r_2 \) are real numbers and the scalar function \( \mu \) (so-called relaxation kernel) is assumed to only be \( \mu : \mathbb{R}^+ \to \mathbb{R}^+ \) of \( C^1 \), nonincreasing and satisfying

\[
\mu(0) > 0, 1 - \int_0^\infty e^{s/2} \mu(s) ds = l > 0.
\] (1.2)

The study of the Emden-Fowler equation originated from earlier theories concerning gaseous dynamics in astrophysics around the turn of the 20-th century. The fundamental problem in the study of stellar structure at that time was to study the equilibrium configuration of the mass of spherical clouds of gas. The Emden-Fowler equation has an impact on many astrophysics evolution phenomena. It has been poorly studied by scientists until now, essentially in the qualitative point of view. Under the assumption that the gaseous cloud is under convective equilibrium (first proposed in 1862 by Lord Kelvin [27]), Lane studied the equation

\[
\frac{d}{dt}(t^2 \frac{du}{dt}) + t^2 u^p = 0,
\] (1.3)

for the cases \( p = 1.5 \) and 2.5. Equation (1.3) is commonly referred to as the Lane-Emden equation [4]. Astrophysicists were interested in the behavior of the solutions of (1.3) which satisfy the initial condition: \( u(0) = 1, u'(0) = 0 \). Special cases of (1.3), namely, when \( p = 1 \) the explicit solution to

\[
\frac{d}{dt}(t^2 \frac{du}{dt}) + t^2 u = 0, \quad u(0) = 1, \ u'(0) = 0
\]
is \( u = \sin(t)/t \), and when \( p = 5 \), the explicit solution to
\[
\frac{d}{dt}(t^2 \frac{du}{dt}) + t^2 u^5 = 0, \quad u(0) = 1, \quad u'(0) = 0
\]
is \( u = 1/\sqrt{1 + t^2/3} \).

Many properties of solutions to the Lane-Emden equation were studied by Ritter [24] in a series of 18 papers published during 1878-1889. The publication of Emden’s treatise Gaskugeln [9] marks the end of first epoch in the study of stellar configurations governed by (1.3). The mathematical foundation for the study of such an equation and also of the more general equation
\[
\frac{d}{dt}(t^\sigma \frac{du}{dt}) + t^\sigma u^\gamma = 0, \quad t \geq 0,
\]
was made by Fowler [10, 11, 12, 13] in a series of four papers during 1914-1931. The first serious study on the generalized Emden-Fowler equation
\[
\frac{d^2 u}{dt^2} + a(t)|u|^\gamma \text{sgn} u = 0, \quad t \geq 0,
\]
was made by Atkinson and al.

Recently, M.-R. Li in [19] considered and studied the blow-up phenomena of solutions to the Emden-Fowler type semilinear wave equation
\[
t^2 u_{tt} - u_{xx} = u^p \quad \text{in } [1, T) \times (a, b)).
\]

The present research aims to extend the study of Emden-Fowler type wave equation to the case when the viscoelastic term is injected in domain \([r_1, r_2]\) where there is no result about this topic. Thus, a wider class of phenomena can be modeled.

The main results here are to exhibit the role of the viscoelasticity, which makes our problem (1.1) dissipative, in the blow up of solutions in \(L^2\) at finite time given by
\[
\ln T^*_1, \ s.t.\ T^*_1 = \frac{2}{p-1}T_1^* = \frac{2}{p-1} \left( \int_{r_1}^{r_2} |u_0|^\gamma dx \right)^{-1} \left( \int_{r_1}^{r_2} u_0 u_1 dx \right)^{-1},
\]
for Emden-Fowler type wave equation when the energy is null which will be the main results of subsection 3.1. In the subsection 3.2, we will discuss the blow up in finite time \(\ln T^*_2 < \ln T^*_1\) of problem (1.4) for large class of solution in the case when the associated energy is negative. The questions of local existence and uniqueness will be also considered in the section 2.

2. Preliminaries, local Existence of unique solution

Under some suitable transformations, we can get the local existence of solutions to equation (1.1). Taking the transform
\[
\tau = \ln t, \quad v = u, \quad u_{xx} = v_{xx},
\]
then
\[
u' = t^{-1}v_{\tau}, \quad t^2u'' = -v_{\tau} + v_{\tau\tau},
\]
equation (1.1) takes the form
\[
v_{\tau\tau} - v_{xx} + \int_0^\tau \mu(s)v_{xx}(\tau - s)ds = v_{\tau} + v^p \quad \text{in } [0, \ln T) \times (r_1, r_2),
\]
\[ v(x, 0) = u_0(x), \quad u_\tau(x, 0) = u_1(x), \]
\[ v(r_1, \tau) = v(r_2, \tau) = 0. \tag{2.1} \]

Let
\[ v(\tau, x) = e^{\tau/2}w(\tau, x), \]
\[ v_\tau(\tau, x) = e^{\tau/2}w_\tau(\tau, x) + \frac{1}{2}e^{\tau/2}w(\tau, x), \]
\[ v_{\tau\tau}(\tau, x) = e^{\tau/2}w_{\tau\tau}(\tau, x) + e^{\tau/2}w_\tau(\tau, x) + \frac{1}{4}e^{\tau/2}w(\tau, x), \]
then (2.1) can be rewritten as
\[ e^{\tau/2}w_{\tau\tau} - e^{\tau/2}w_{xx} + \int_0^\tau e^{s/2}\mu(s)w_{xx}(\tau - s)ds, \]
\[ = \frac{1}{4}e^{\tau/2}w + e^{\tau/2}w_\tau, \]
then
\[ w_{\tau\tau} - w_{xx} + e^{-\tau/2}\int_0^\tau e^{s/2}\mu(s)w_{xx}(\tau - s)ds = \frac{1}{4}w + e^{(p-1)\tau/2}w_\tau. \tag{2.2} \]

The following technical Lemma will play an important role.

**Lemma 2.1.** For any \( w \in C^1(0, T, H^1(r_1, r_2)) \) we have for any nonincreasing differentiable function \( \alpha(\tau) > 0 \)
\[ \int_{r_1}^{r_2} \alpha(\tau) \int_0^\tau e^{s/2}\mu(\tau - s)w_{xx}(s)w'(\tau)dsdx \]
\[ = \frac{1}{2} \frac{d}{d\tau} \alpha(\tau) \int_0^\tau e^{s/2}\mu(\tau - s) \int_{r_1}^{r_2} |w_x(\tau) - w_x(s)|^2dxds \]
\[ - \frac{1}{2} \frac{d}{d\tau} \alpha(\tau) \int_0^\tau e^{s/2}\mu(\tau - s)ds \int_{r_1}^{r_2} |w_x(\tau)|^2dx \]
\[ - \frac{1}{2} \alpha(\tau) \int_0^\tau e^{s/2}\mu(\tau - s) \int_{r_1}^{r_2} |w_x(\tau) - w_x(s)|^2dxds \]
\[ + \frac{1}{2} \alpha(\tau) e^{\tau/2}\mu(\tau) \int_{r_1}^{r_2} |w_x(\tau)|^2dx \]
\[ - \frac{1}{2} \alpha'(\tau) \int_0^\tau e^{s/2}\mu(\tau - s) \int_{r_1}^{r_2} |w_x(\tau) - w_x(s)|^2dxds \]
\[ + \frac{1}{2} \alpha'(\tau) \int_0^\tau e^{s/2}\mu(\tau - s)ds \int_{r_1}^{r_2} |w_x(\tau)|^2dx. \]

**Proof.** It’s not hard to see
\[ \int_{r_1}^{r_2} \alpha(\tau) \int_0^\tau e^{s/2}\mu(\tau - s)w_{xx}(s)w'(\tau)dsdx \]
Consequently, which implies, \[ \int_{r_1}^{r_2} \frac{1}{2} \alpha(\tau) \int_{0}^{\tau} e^{s/2} \mu(\tau - s) w_{xx}(s) w'(\tau) ds dx = -\alpha(\tau) \int_{0}^{\tau} e^{s/2} \mu(\tau - s) \int_{r_1}^{r_2} w_x'(\tau) w_x(s) dx ds. \]

This completes the proof. \( \square \)

We introduce the modified energy associated to problem (2.2).

\[ 2E_w(\tau) = \int_{r_1}^{r_2} |w_t|^2 dx + (1 - \int_{0}^{\tau} e^{s/2} \mu(s) ds) \int_{r_1}^{r_2} |w_x|^2 dx dτ \]

\[ + \int_{0}^{\tau} e^{s/2} \mu(\tau - s) \int_{r_1}^{r_2} |w_x(s) - w_x(\tau)|^2 dx dσ \]

\[ - \frac{1}{4} \int_{r_1}^{r_2} |w|^2 dx - \frac{2}{p+1} e^{(p-1)\tau} \int_{r_1}^{r_2} |w|^{p+1} dx. \] (2.3)
Thus, by Lemma 2.1 with \( \alpha \), we have

\[
2E_w(0) = \int_{r_1}^{r_2} (u_1 - \frac{1}{2} u_0)^2 dx + \int_{r_1}^{r_2} |u_0x|^2 dx + \int_{r_1}^{r_2} u_0 u_1 dx - \frac{2}{p+1} \int_{r_1}^{r_2} |u_0|^{p+1} dx.
\]

Direct differentiation, using (1.2), (2.2), leads to

\[
E'_w(\tau) \leq 0.
\]

We now can obtain the next important Lemma.

**Lemma 2.2.** Suppose that \( v \in C^1(0, T, H^1_0(r_1, r_2)) \cap C^2(0, T, L^2(r_1, r_2)) \) is a solution of the semi-linear wave equation (2.2). Then for \( \tau \geq 0 \),

\[
E_w(\tau) \leq E_w(0) - \frac{p-1}{p+1} \int_0^\tau \int_{r_1}^{r_2} |w|^{p+1} dx ds,
\]

(2.4)

**Proof.** Taking the \( L^2 \) product of (2.2) with \( w_\tau \) yields

\[
\int_{r_1}^{r_2} w_{\tau \tau} w_\tau dx - \int_{r_1}^{r_2} \left( w_{xx} + e^{-\tau/2} \int_0^t e^{s/2} \mu(s) w_{xx}(t-s) ds \right) w_\tau dx
\]

\[
= \frac{1}{4} \int_{r_1}^{r_2} w w_\tau dx + \int_{r_1}^{r_2} e^{(p-1)\tau/2} u^p w_\tau dx.
\]

Thus, by Lemma 2.1 \( \mu(\tau) = e^{-\tau/2} \), we have

\[
\frac{1}{2} \frac{d}{dt} \left[ \int_{r_1}^{r_2} |w_\tau|^2 dx + \left( 1 - \int_0^t e^{s/2} \mu(s) ds \right) \int_{r_1}^{r_2} |w_x|^2 dx \right]
\]

\[
+ \frac{1}{2} \frac{d}{dt} \int_0^\tau e^{s/2} \mu(\tau - s) \int_{r_1}^{r_2} |w_x(s) - w_x(\tau)|^2 dx ds
\]

\[
= \frac{1}{8} \frac{d}{dt} \int_{r_1}^{r_2} |w|^2 dx + \frac{1}{p+1} \frac{d}{dt} \int_{r_1}^{r_2} e^{(p-1)\tau/2} u^p w_\tau dx + \frac{2(p-1)}{p+1} \int_{r_1}^{r_2} e^{(p-1)\tau/2} u^p + \frac{1}{2} \alpha(\tau) \int_{r_1}^{r_2} (e^{s/2} \mu(\tau - s))' \int_{r_1}^{r_2} |w_x(s) - w_x(\tau)|^2 dx ds
\]

\[
- \frac{1}{2} \alpha(\tau) \int_{r_1}^{r_2} |w_x(t)|^2 dx
\]

\[
+ \frac{1}{2} \alpha'(\tau) \int_0^\tau e^{s/2} \mu(\tau - s) \int_{r_1}^{r_2} |w_x(s) - w_x(\tau)|^2 dx ds
\]

\[
- \frac{1}{2} \alpha'(\tau) \int_0^\tau e^{s/2} \mu(s) ds \int_{r_1}^{r_2} |w_x(\tau)|^2 dx.
\]

Then, by conditions on \( \mu, \alpha \) and (2.3), the assertions (2.4) is proved. \( \square \)

3. **Blow up result for \( E_u(0) = 0 \)**

Under small amplitude initial data, we prove that \( w \) blows up in \( L^2 \) at finite time \( \ln T^* \) in the following Theorem 3.1
Theorem 3.1. Suppose that $w \in C^1(0, T, H^1_0(r_1, r_2)) \cap C^2(0, T, L^2(r_1, r_2))$ is a weak solution of equation (2.2) with

$$e(0) := \int_{r_1}^{r_2} u_0 u_1(x) dx > 0, \quad E_w(0) = 0$$

and $0 < r_2 - r_1 \leq 1$. Then there exists $T^*_1$ such that

$$\int_{r_1}^{r_2} |w(t, x)|^2 dx \to +\infty \quad \text{as} \ t \to T^*_1,$$

where

$$T^*_1 = \frac{2}{p - 1} \left( \int_{r_1}^{r_2} |u_0| dx \right) \left( \int_{r_1}^{r_2} u_0 u_1 dx \right)^{-1}.$$

We need to state and prove the next intermediate Lemma.

Lemma 3.2. Suppose that $w$ is a weak solution of equation (2.2). Then

$$\int_{r_1}^{r_2} e^{\frac{p - 1}{2} s} w^{p+1}(s, x) dx$$

$$\geq \frac{p + 1}{2} \left( \int_{r_1}^{r_2} |w_s|^2 dx + \left( 1 - \int_0^t e^{s/2} \mu(s) ds \right) \int_{r_1}^{r_2} |w_x|^2 dx - \frac{1}{4} \int_{r_1}^{r_2} w^2 dx \right)$$

$$+ \int_0^\tau e^{s/2} \mu(\tau - s) \int_{r_1}^{r_2} |w_x(s) - w_x(\tau)|^2 dx ds - (p + 1) E_w(0) e^{\frac{p - 1}{2} s}$$

$$+ \frac{p^2 - 1}{2} \int_0^s e^{\frac{p - 1}{2} (s - r)} \int_{r_1}^{r_2} e^{s/2} \mu(\tau - s) \int_{r_1}^{r_2} |w_x(s) - w_x(\tau)|^2 dx ds.$$

Proof. Set

$$L(s) = \frac{1}{p + 1} \int_0^s e^{\frac{p - 1}{2} r} \int_{r_1}^{r_2} |w|^{p+1} dx dr,$$

$$F(s) = \int_{r_1}^{r_2} |w_s|^2 dx + \left( 1 - \int_0^t e^{\tau/2} \mu(\tau) d\tau \right) \int_{r_1}^{r_2} |w_x|^2 dx$$

$$- \frac{1}{4} \int_{r_1}^{r_2} w^2 dx + \int_0^\tau e^{s/2} \mu(\tau - s) \int_{r_1}^{r_2} |w_x(s) - w_x(\tau)|^2 dx ds,$$

By Lemma 2.3 and Lemma 2.2, equation (2.3) can be rewritten as

$$E_w(0) \geq F - 2L' + (p - 1)L,$$

therefore,

$$(e^{\frac{p - 1}{2} s} L)' = e^{\frac{p - 1}{2} s} \left( L' - \frac{p - 1}{2} L \right)$$

$$\geq \frac{1}{2} e^{\frac{p - 1}{2} s} (F - E_w(0)),$$

and

$$e^{\frac{p - 1}{2} s} L \geq \frac{1}{2} \int_0^s e^{\frac{p - 1}{2} r} (F(r) - E_w(0)) dr.$$
Let \( (Of \text{ Theorem 3.1}) \)

This completes the proof. \( \square \)

We are now ready to prove Theorem 3.1

Proof. (Of Theorem 3.1)

Let

\[ A(s) := \int_{r_1}^{r_2} |w|^2 dx, \]

and

\[ L \geq \frac{1}{2} \int_0^s e^{\frac{s-1}{p}} F(r) dr - \frac{E_w(0)}{p-1} (1 - e^{\frac{s-1}{p}}), \]

(this implies)

\[ \frac{1}{p+1} \int_0^s e^{\frac{s-1}{p}} \int_{r_1}^{r_2} |w|^{p+1} dx dr \]

\[ \geq \frac{1}{2} \int_0^s e^{\frac{s-1}{p}} \left[ \int_{r_1}^{r_2} |w_x|^2 dx + (1 - \int_0^t e^{s/2} \mu(s) ds) \int_{r_1}^{r_2} |w_x|^2 dx - \frac{1}{4} \int_{r_1}^{r_2} |w|^2 dx \right] dr \]

\[ - \frac{p+1}{p-1} E_w(0) (e^{\frac{s-1}{p}} - 1) + \frac{p+1}{2} \int_0^s e^{\frac{s-1}{p}} \int_{r_1}^{r_2} e^{s/2} \mu(\tau - s) \int_{r_1}^{r_2} |w_x(s) - w_x(\tau)|^2 dx ds, \]

and

\[ \int_{r_1}^{r_2} e^{\frac{s-1}{p}} w^{p+1}(r, x) dx \]

\[ \geq \frac{p+1}{2} \left[ \int_{r_1}^{r_2} |w_x|^2 dx + (1 - \int_0^t e^{s/2} \mu(s) ds) \int_{r_1}^{r_2} |w_x|^2 dx - \frac{1}{4} \int_{r_1}^{r_2} |w|^2 dx \right] \]

\[ + \int_0^t e^{s/2} \mu(\tau - s) \int_{r_1}^{r_2} |w_x(s) - w_x(\tau)|^2 dx ds - (p+1) E_w(0) e^{\frac{s-1}{p}}, \]

\[ + \frac{p^2 - 1}{2} \int_0^s e^{\frac{s-1}{p}} \left[ \int_{r_1}^{r_2} |w_x|^2 dx + (1 - \int_0^t e^{s/2} \mu(s) ds) \int_{r_1}^{r_2} |w_x|^2 dx - \frac{1}{4} \int_{r_1}^{r_2} |w|^2 dx \right] dr \]

\[ + \frac{p^2 - 1}{2} \int_0^s e^{\frac{s-1}{p}} \int_{r_1}^{r_2} e^{s/2} \mu(\tau - s) \int_{r_1}^{r_2} |w_x(s) - w_x(\tau)|^2 dx ds. \]  \hspace{1cm} (3.2)

This completes the proof. \( \square \)
then we have
\[ A'(s) = 2 \int_{r_1}^{r_2} w w_s(s, x) dx. \]
and
\[
A''(s) = 2 \int_{r_1}^{r_2} w w_s(s, x) dx + 2 \int_{r_1}^{r_2} w_s^2(s, x) dx
\]
\[
= 2 \int_{r_1}^{r_2} (w w_{xx} - w e^{-\tau/2} \int_0^s e^{s/2} \mu(s) w_x(t - s) ds + \frac{1}{4} w^2 + w_s^2 + e^{\frac{p-1}{2} w} w^{p+1}) dx
\]
\[
= 2 \int_{r_1}^{r_2} (-w_x^2 + w_x e^{-\tau/2} \int_0^s e^{s/2} \mu(s) w_x(t - s) ds + \frac{1}{4} w^2 + w_s^2 + e^{\frac{p-1}{2} w} w^{p+1}) dx.
\]
By Lemma 3.1, Lemmas 3.2 and (3.2), then
\[
A''(s) \geq 2 \left( \int_0^t e^{s/2} \mu(s) ds - 1 \right) \int_{r_1}^{r_2} |w_x|^2 dx + \frac{1}{4} \int_{r_1}^{r_2} |w|^2 dx + \int_{r_1}^{r_2} |w_s|^2 dx
\]
\[
- 2 \int_0^\tau e^{s/2} \mu(\tau - s) \int_{r_1}^{r_2} |w_x(s) - w_x(\tau)|^2 dx ds
\]
\[
+ (p + 1)/2 \left( \int_0^t e^{s/2} \mu(s) ds - 1 \right) \int_{r_1}^{r_2} |w_x|^2 dx + \frac{1}{4} \int_{r_1}^{r_2} |w|^2 dx + \int_{r_1}^{r_2} |w_s|^2 dx
\]
\[
- (p + 1) \int_0^\tau e^{s/2} \mu(\tau - s) \int_{r_1}^{r_2} |w_x(s) - w_x(\tau)|^2 dx ds
\]
\[
+ (p^2 - 1) \int_0^t e^{p-1} (s-r) \left( \int_0^t e^{s/2} \mu(s) ds - 1 \right) \int_{r_1}^{r_2} |w_x|^2 dx + \frac{1}{4} \int_{r_1}^{r_2} |w|^2 dx + \int_{r_1}^{r_2} |w_s|^2 dx
\]
\[
- (p^2 - 1) \int_0^\tau e^{p-1} (s-r) \int_0^\tau e^{s/2} \mu(\tau - s) \int_{r_1}^{r_2} |w_x(s) - w_x(\tau)|^2 dx ds
\]
\[
- 2(p + 1) E_w(0) e^{p-1}
\]
\[
\geq [(p + 3) \int_{r_1}^{r_2} |w_s|^2 dx + (p - 1) \left( 1 - \int_0^t e^{s/2} \mu(s) ds \right) \int_{r_1}^{r_2} |w_x|^2 dx - \frac{p-1}{4} \int_{r_1}^{r_2} |w|^2 dx]
\]
\[
- 2(p + 1) E_w(0) e^{p-1} + (p - 1) \int_0^\tau e^{s/2} \mu(\tau - s) \int_{r_1}^{r_2} |w_x(s) - w_x(\tau)|^2 dx ds
\]
\[
+ (p^2 - 1) \int_0^t e^{p-1} (s-r) \left( \int_{r_1}^{r_2} |w_s|^2 dx + (\int_0^t e^{s/2} \mu(s) ds - 1) \int_{r_1}^{r_2} |w_x|^2 dx + \frac{1}{4} \int_{r_1}^{r_2} |w|^2 dx \right) dr
\]
\[
- (p^2 - 1) \int_0^\tau e^{p-1} (s-r) \int_0^\tau e^{s/2} \mu(\tau - s) \int_{r_1}^{r_2} |w_x(s) - w_x(\tau)|^2 dx ds dr.
\]
As in [19], let us setting
\[ J(s) := A(s)^{-k}, \quad k = \frac{p-1}{4} > 0. \]
Then

\[ J'(s) = -kA(s)^{-k-1}A'(s), \]

and

\[ J''(s) = -kA(s)^{-k-2}[A(s)A''(s) - (k + 1)A'(s)^2] \]

\[ \leq -kA(s)^{-k-1}[A''(s) - 4(k + 1) \int_{r_1}^{r_2} w_s^2 dx]. \quad (3.4) \]

Since \( E_u(0) = 0 \), we have

\[
A''(s) - 4(k + 1) \int_{r_1}^{r_2} |w_s|^2 dx \\
\geq [(p + 3) \int_{r_1}^{r_2} |w_s|^2 dx + (p - 1)(1 - \int_0^1 e^{s/2} \mu(s) ds) \int_{r_1}^{r_2} |w_s|^2 dx - \frac{p - 1}{4} \int_{r_1}^{r_2} |w|^2 dx] \\
+(p^2 - 1) \int_{r_1}^{r_2} e^{\frac{p-1}{4}(s-r)} \left( \int_{r_1}^{r_2} |w_s|^2 dx + (1 - \int_0^1 e^{s/2} \mu(s) ds) \int_{r_1}^{r_2} |w_s|^2 dx - \frac{1}{4} \int_{r_1}^{r_2} |w|^2 dx \right) dr \\
-4(k + 1) \int_{r_1}^{r_2} |w_s|^2 dx + (p - 1) \int_0^1 e^{s/2} \mu(\tau - s) \int_{r_1}^{r_2} |w_x(s) - w_x(\tau)|^2 dx ds \\
+(p^2 - 1) \int_0^\tau e^{\frac{p-1}{4}(s-r)} \left( \int_{r_1}^{r_2} |w_s|^2 dx + (1 - \int_0^1 e^{s/2} \mu(s) ds) \int_{r_1}^{r_2} |w_s|^2 dx - \frac{1}{4} \int_{r_1}^{r_2} |w|^2 dx \right) dr \\
+(p^2 - 1) \int_0^\tau e^{\frac{p-1}{4}(s-r)} \left( \int_{r_1}^{r_2} |w_s|^2 dx + (1 - \int_0^1 e^{s/2} \mu(s) ds) \int_{r_1}^{r_2} |w_s|^2 dx \right) dr \\
\int_{r_1}^{r_2} |w_x(s) - w_x(\tau)|^2 dx ds dr \\
\geq (p - 1)(1 - (r_2 - r_1)^2) \left( \int_{r_1}^{r_2} |w_s|^2 dx + \int_0^\tau e^{s/2} \mu(\tau - s) \int_{r_1}^{r_2} |w_x(s) - w_x(\tau)|^2 dx ds \right) \\
+(p + 1) \int_0^\tau e^{\frac{p-1}{4}(s-r)} \left( \int_{r_1}^{r_2} |w_s|^2 dx + (1 - \int_0^1 e^{s/2} \mu(s) ds) \int_{r_1}^{r_2} |w_s|^2 dx \right) dr \\
+(p + 1) \int_0^\tau e^{\frac{p-1}{4}(s-r)} \left( \int_{r_1}^{r_2} |w_s|^2 dx + (1 - \int_0^1 e^{s/2} \mu(s) ds) \int_{r_1}^{r_2} |w_s|^2 dx \right) dr > 0,
\]

where \( r_2 \leq 1 + r_1 \).

Therefore, by (3.4) we obtain that for, \( r_2 - r_1 \leq 1 \), \( J''(s) < 0 \) for all \( s \geq 0 \).
Further, we have
\[ J'(s) \leq J'(0) = -\frac{p-1}{4} A(0) \frac{t^{p+3}}{r^{p+1}} A'(0) \]
and
\[ J(s) \leq J(0) - \frac{p-1}{2} e(0) \int_{r_1}^{r_2} |u_0|^{-(p+3)} ds. \]

Then
\[ J(s) \to 0 \quad \text{as} \quad s \to T^* = \frac{2}{p-1} \int_{r_1}^{r_2} |u_0| dx. \quad \text{(3.5)} \]
Thus \( w \) solution of (2.2) blows up in \( L^2 \) at finite time \( T^* \). \( \square \)

4. Blow up result for \( E_u(0) < 0 \)

In the following theorem we shall state and prove our second blowing up result

**Theorem 4.1.** Suppose that \( w \in C^1(0,T, H^1_0(r_1, r_2)) \cap C^2(0,T, L^2(r_1, r_2)) \) is a weak solution of equation (1.7) with
\[ e(0) = \int_{r_1}^{r_2} u_0 u_1(x) dx > 0, \quad E_u(0) < 0, \]
and \( 0 < r_2 - r_1 \leq 1 \). Then, there exists \( T^*_2 \) such that
\[ \frac{1}{\int_{r_1}^{r_2} |u(t,x)|^2 dx} \to 0 \quad \text{as} \quad t \to \ln T^*_2. \]

Further, we have \( \ln T^*_2 < \ln T^*_1 \), and the estimate
\[ \int_{r_1}^{r_2} w^2 dx \geq \int_{r_1}^{r_2} u_0^2 dx - 2 E_u(0) \frac{p+1}{p-1} \left[ se^{k-s} - \frac{2}{p-1} (e^{k-s} - 1) \right]. \]

**Proof.** By (3.1), Lemma 2.1, \( E_u(0) < 0, e(0) > 0 \) and \( 0 < r_2 - r_1 \leq 1 \), we have
\[ J''(s) \leq -k \left( \int_{r_1}^{r_2} w^2 dx \right)^{-k-1} \left[ A''(s) - (p+3) \int_{r_1}^{r_2} w^2 x(s, x) dx \right] \]
\[ = -k \left( \int_{r_1}^{r_2} w^2 dx \right)^{-k-1} \left[ -2(p+1) E_w(0) e^{\frac{p+1}{2}} \right], \]
\[ + (p-1) \left( 1 - \int_{0}^{t} e^{s/2} \mu(s) ds \right) \int_{r_1}^{r_2} w^2_s dx - \frac{1}{4} \int_{r_1}^{r_2} |w^2|^2 dx \]
\[ + (p-1) \int_{0}^{t} e^{s/2} \mu(\tau - s) \int_{r_1}^{r_2} |w_s(s) - w_x(\tau)|^2 dx ds \]
This completes the proof. □

By the inequality (4.1) and

\[ u = \left( \int_{r_0}^{r} e^{\frac{e^{s^{-1}}}{s^{-1}}(s-r)} \left( \int_{r_1}^{r_2} |w_x|^2 dx + (1 - \int_{0}^{t} e^{s/2} \mu(s) ds) \int_{r_1}^{r_2} |w_x|^2 dx - \frac{1}{4} \int_{r_1}^{r_2} |w|^2 dx \right) dr \right) \]

\[ + \left( p^2 - 1 \right) \int_{r_0}^{r} e^{\frac{e^{s^{-1}}}{s^{-1}}(s-r)} \left( \int_{r_1}^{r_2} |w_x|^2 dx + \int_{0}^{t} e^{s/2} \mu(s) ds \int_{r_1}^{r_2} |w_x|^2 dx - \frac{1}{4} \int_{r_1}^{r_2} |w|^2 dx \right) dr \]

\[ + \left( p^2 - 1 \right) \int_{r_0}^{r} e^{\frac{e^{s^{-1}}}{s^{-1}}(s-r)} e^{s/2} \mu(\tau - s) \int_{r_1}^{r_2} |w_x(s) - w_x(\tau)|^2 dx ds dr \]

\[ \leq 2k(p+1)E_u(0)e^{\frac{e^{s^{-1}}}{s^{-1}}s}J(s)^{1+\frac{1}{2}} < 0, \]

(4.1)

where \( k = (p - 1)/4 \), we can obtain the same conclusions as in Theorem 3.1.

By the inequality (4.1) and \( J' < 0 \) we can estimate \( J \) further,

\[ J''(s) \leq 2k(p+1)E_u(0)e^{\frac{e^{s^{-1}}}{s^{-1}}s}J(s)^{1+\frac{1}{2}} \]

\[ = \frac{1}{2} \left( p^2 - 1 \right) E_u(0)e^{\frac{e^{s^{-1}}}{s^{-1}}s}J(s)^{1+\frac{1}{2}} < 0, \]

and

\[ J'(s) \leq J'(0) + \frac{s}{2} \left( p^2 - 1 \right) E_u(0)e^{\frac{e^{s^{-1}}}{s^{-1}}s}J(s)^{1+\frac{1}{2}} \]

\[ \leq \frac{s}{2} \left( p^2 - 1 \right) E_u(0)e^{\frac{e^{s^{-1}}}{s^{-1}}s}J(s)^{1+\frac{1}{2}}, \]

and

\[ -k(J(s)^{-\frac{1}{2}}) = J(s)^{-1-\frac{1}{2}}J'(s) \]

\[ \leq \frac{E_u(0)}{2} \left( p^2 - 1 \right) e^{\frac{e^{s^{-1}}}{s^{-1}}s}, \]

and

\[ -k(J(s)^{-\frac{1}{2}} - J(0)^{-\frac{1}{2}}) \leq \frac{E_u(0)}{2} \left( p^2 - 1 \right) \left[ \frac{2}{p-1} e^{\frac{e^{s^{-1}}}{s^{-1}}s} - \left( \frac{2}{p-1} \right)^2 (e^{\frac{e^{s^{-1}}}{s^{-1}}s} - 1) \right] \]

\[ = E_u(0) \left( p + 1 \right) \left[ se^{\frac{e^{s^{-1}}}{s^{-1}}s} - \frac{2}{p-1} (e^{\frac{e^{s^{-1}}}{s^{-1}}s} - 1) \right], \]

which implies

\[ \int_{r_0}^{r_2} u^2 dx \geq \int_{r_1}^{r_2} \left( \int_{r_1}^{r_2} \left( 2 \frac{p+1}{p-1} E_u(0) \left[ se^{\frac{e^{s^{-1}}}{s^{-1}}s} - \frac{2}{p-1} (e^{\frac{e^{s^{-1}}}{s^{-1}}s} - 1) \right] \right) \right) \]

Then \( u \) solution of our initial problem (1.1) blows up in \( L^2 \) at finite time \( \ln T^*_2 \).

This completes the proof. □

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