Injectivity and Projectivity Properties of The Category of Representation Modules of Rings

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Abstract. Let \( R, S \) be two rings with unity, \( M \) an \( S \)-module, and \( f : R \to S \) a ring homomorphism. If the map \( M \to M \), \( m \mapsto f(r)m \) is \( S \)-linear for any \( r \in R \), then \( M \) is a representation module of ring \( R \). This condition will be true if \( sf(r) - f(r)s \in \text{Ann}(M) \) for all \( r \in R \) and \( s \in S \). The class of \( S \)-modules \( M \), where \( sf(r) - f(r)s \in \text{Ann}(M) \) for all \( r \in R \) and \( s \in S \), forms a category with its morphisms are all module homomorphisms. This class is denoted by \( \mathcal{S} \). The purpose of this paper is to prove that the category \( \mathcal{S} \) is an abelian category which is under sufficient conditions enabling the category \( \mathcal{S} \) has enough injective objects and enough projective objects. First, we prove the category \( \mathcal{S} \) is stable under kernel and image of module homomorphisms, and a finite direct sum of objects of \( \mathcal{S} \) is also the object of \( \mathcal{S} \). By using this two properties, we prove that \( \mathcal{S} \) is the abelian category. Next, we determine the properties of the abelian category \( \mathcal{S} \), such that it has enough injective objects and enough projective objects. We obtain that, if \( S \) as \( R \)-module is an element of \( \mathcal{S} \), then the category \( \mathcal{S} \) has enough projective objects and enough injective objects.

1. Introduction

Let \( f : R \to S \) be a ring homomorphism, where \( R \) is a ring with unity and \( S \) is a commutative ring with unity. The \( f \)-representation of \( R \) on an \( S \)-module \( M \) is a ring homomorphism \( \mu \) from \( R \) to a ring of all \( S \)-module endomorphisms of \( M(\text{End}_S(M)) \), where for every \( r \in R \), \( \mu(r) := \mu_r \in \text{End}_S(M) \) is defined as

\[
\mu_r : M \to M, m \mapsto f(r)m
\]

Furthermore, the \( S \)-module \( M \) is called an \( f \)-representation module of \( R \)[1]. The \( f \)-representation of rings on modules is generalization of the representation of rings on vector space. The representation of a ring \( R \) on a vector space \( V \) is a ring homomorphism from \( R \) to the ring of all linear transformations of \( V \)[2].
An \( R \)-algebra \( A \) is a ring (not necessary commutative) with a ring homomorphism \( g : R \to Z(A) \), where \( Z(A) \) is a center of \( A \). The class of \( S \)-modules \( M \), where \( S \) is an \( R \)-algebra, is an abelian category. It also has enough projective objects and enough injective objects, and this class satisfy the Krull-Schmidt Theorem [3]. Since \( S \) is commutative, \( Z(S) = S \). Hence \( S \) is an \( R \)-algebra and the class of \( S \)-modules \( M \), which is representation modules of \( R \), is an abelian category and has enough projective objects and enough injective objects.

In [4] and [5], Auslander introduce a category mod \( C \), i.e. the category of all functors from \( C^{op} \) to \( Ab \), where \( C \) is a skeletally small category and \( Ab \) is category of all abelian groups. He also give some properties that a bijection between representation finite Artin algebras and algebras satisfying \( gl.\dim \Gamma \leq 2 \leq \text{dom.\dim} \Gamma \) (is called Auslander algebra) and If \( A \) is the finite-dimension algebra such that \( \text{add} \ M = \text{mod} \ A \), then \( \text{End}_{A}(M) \) is Auslander algebra[6]. This property is very useful in representation theory, since to study representation theory, we must study the properties of \( \text{End}_{A}(M) \) where \( M \) is \( A \)-algebra. There are many mathematicians who examine the representation dimension of Artin algebra, such as Iyama, in [6], proved that the representation dimension of an Artin algebra is always finite; Rouquier proved in [7] that the representation dimension of Artin algebras can be arbitrary large. Based on this result, Yin and Zhang, in [8], proved that the representation dimension of triangular matrix algebras \( T_{2}(A) \) is at most three if \( A \) is Dynkin type and is at most four if \( A \) is not Dynkin type. Another result is given by Assem et al. in [9] and [10].

If the ring \( S \) is not commutative, then for all \( r \in R \), \( \mu_{r} : M \to M, m \mapsto f(r)m \) is not guaranteed to be in \( \text{End}_{A}(M) \). We give an example to show this statement to be true. Let \( R \) be a ring of integer \( Z \), \( S \) a ring of all \( 2 \times 2 \) matrices with entries in \( Z \), and \( M = \{(a,b) \mid a,b \in Z\} \) a module over \( S \) with scalar multiplication defined as matrix multiplication. Consider the ring homomorphism

\[
 f : R \to S, \ a \mapsto \begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix} 
\]

For \( 1 \in R \), \( \mu : M \to M, m \mapsto f(1)m \) is not an endomorphism of \( M \), since there is \( s = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \in S \), such that for \( 0 \neq m \in M \) we have \( \mu_{1}(sm) \neq s\mu_{1}(m) \).

The following proposition is the condition for the map \( \mu_{r} : M \to M, m \mapsto f(r)m \) to be an \( S \)-module endomorphism of \( M \).

**Proposition 1.1.** Let \( R, S \) be rings with unity, \( f : R \to S \) be a ring homomorphism, and \( M \) be an \( S \)-module. For every \( r \in R \), the map \( \mu_{r} : M \to M, m \mapsto f(r)m \) is an \( S \)-linear if and only if \( sf(r) - f(r)s \in \text{Ann}(M) \).

**Proof.** Consider \( \mu_{r} \) is an \( S \)-linear. Then for any \( m \in M \) and \( s \in S \) we have

\[
 s\mu_{r}(m) = \mu_{r}(sm) \iff sf(r)m = f(r)sm \iff (sf(r) - f(r)s)m = 0. 
\]

Hence \( sf(r) - f(r)s \in \text{Ann}(M) \). Conversely, if \( sf(r) - f(r)s \in \text{Ann}(M) \), then \( sf(r)m = f(r)sm \) for all \( m \in M \). So for any \( s \in S \) and \( m \in M \)

\[
 \mu_{r}(sm) = f(r)sm = sf(r)m = sf(r)m = s\mu_{r}(m). 
\]

Thus \( \mu_{r} \) is an \( S \)-linear. \( \blacksquare \). Base on this, the \( S \)-module \( M \) (a ring \( S \) is not necessary commutative) is a representation module of ring \( R \) if \( sf(r) - f(r)s \in \text{Ann}(M) \). The class of \( S \)-modules \( \mathcal{M} \) which is a representation module of \( R \), we denoted as \( \mathcal{Z} \). The class \( \mathcal{Z} \) is more general then the class module over algebra, since the
annihilator of module not always equal to zero. In this paper, we show that the class $\mathcal{C}$ is an abelian category and investigate when $\mathcal{C}$ has enough projective and enough injective objects.

2. Main Result
Recall $\mathcal{C} = \{ M \mid M \text{ is an } S\text{-module and } sf(r) = f(r)s \in \text{Ann}(M) \}$. We define a morphism on $\mathcal{C}$ as an $S$-module homomorphism and we denote $\text{Hom}_S(M_1, M_2)$ as the set of the morphisms from $M_1$ to $M_2$. From [11], we have $\beta \alpha \in \text{Hom}_S(M_1, M_3)$ for any $\alpha \in \text{Hom}_S(M_1, M_2)$ and $\beta \in \text{Hom}_S(M_2, M_3)$, and the identity $1_M \in \text{Hom}_S(M, M)$ is equal with identity $1_M \in \text{Hom}_{S\text{-Mod}}(M, M)$. Hence by Definition 7.2 [12] the class $\mathcal{C}$ is full subcategory of $S\text{-Mod}$.

Let $\alpha : M_1 \to M_2$ be an arbitrary $S$-module homomorphism. The kernel and image of $\alpha$ are consecutively defined by $\ker \alpha = \{ a \in M_1 \mid \alpha(a) = 0 \}$ and $\text{Im} \alpha = \{ a \in M_2 \mid a \in M_1 \}$. We know $\ker \alpha$ is a submodule of $M_1$ and $\text{Im} \alpha$ is a submodule of $M_2$[13]. The following lemma shows that if $M_1$ and $M_2$ are objects of $\mathcal{C}$, then $\ker \alpha$ and $\text{Im} \alpha$ are also objects of $\mathcal{C}$.

**Lemma 2.1.** The class $\mathcal{C}$ is stable under kernel and image of homomorphism

**Proof.** Let $\alpha : M_1 \to M_2$ be an arbitrary $S$-module homomorphism, for $M_1, M_2 \in \mathcal{C}$. Suppose that $a \in \ker \alpha$ and $b \in \text{Im} \alpha$. Then $\alpha(a) = 0$ and there is $c \in M_1$ such that $\alpha(c) = b$. For any $r \in R$, and $s \in S$ we have

$$\alpha((sf(r) - f(r)s)a) = (sf(r) - f(r)s)\alpha(a) = 0,$$

(5)

and

$$(sf(r) - f(r)s)b = (sf(r) - f(r)s)\alpha(c) = \alpha((sf(r) - f(r)s)c) = \alpha(0) = 0.$$  

Hence $sf(r) - f(r)s$ is an element of $\text{Ann} (\ker \alpha)$ and is also an element of $\text{Ann} (\text{Im} \alpha)$. Thus $\ker \alpha$ and $\text{Im} \alpha$ are objects in the category $\mathcal{C}$, in another words the category $\mathcal{C}$ is stable under kernel and image of homomorphism.

From Lemma 2.1, we have that $\text{Im} \alpha$ is an object of category $\mathcal{C}$ for any morphism $\alpha : M_1 \to M_2$ in category $\mathcal{C}$. Since for any $m \in M_2$, $r \in R$, and $s \in S$,

$$(sf(r) - f(r))(m + \text{Im} \alpha) = (sf(r) - f(r)s)m + \text{Im} \alpha = \text{Im} \alpha,$$

(7)
a quotient $S$-module $M'/\text{Im} \alpha$ is also an object category $\mathcal{C}$.

Suppose that $M_1, M_2, \ldots, M_k$ are $S$-modules. Then a finite direct sum

$$\bigoplus_{i=1}^k M_i = \{(m_1, m_2, \ldots, m_k) \mid m_i \in M_i \}$$

is an $S$-module with a scalar multiplication over $S$ defined as

$$sm_1, m_2, \ldots, m_k = (sm_1, sm_2, \ldots, sm_k)$$

for all $s \in S$ and $(m_1, m_2, \ldots, m_k) \in \bigoplus_{i=1}^k M_i[13]$. Analog with kernel and image of homomorphism in the category $\mathcal{C}$, a finite direct sum of objects of $\mathcal{C}$ is also an object of $\mathcal{C}$. It shown in the following lemma.

**Lemma 2.2.** The finite direct sum of objects of $\mathcal{C}$ is object of $\mathcal{C}$.

**Proof.** Let $M_1, M_2, \ldots, M_k$ be objects of $\mathcal{C}$. For any $r \in R$, $s \in S$ and $(m_1, m_2, \ldots, m_k) \in \bigoplus_{i=1}^k M_i$, we have

$$(sf(r) - f(r)s)(m_1, m_2, \ldots, m_k) = ((sf(r) - f(r)s)m_1, (sf(r) - f(r)s)m_2, \ldots, (sf(r) - f(r)s)m_k)$$
\[ = (0,0,\ldots,0). \]  

Hence \( sf(r) - f(r)s \in \text{Ann}(\bigoplus_{i=1}^{k} M_i) \), such that \( \bigoplus_{i=1}^{k} M_i \) is an object of \( \mathcal{A} \). \( \blacksquare \)

Consider \( \mathcal{A} \) is the full subcategory of \( S\text{-Mod} \). Then for every \( \gamma \in \text{Hom}_S(M_0,M_1) \), \( \alpha_1, \alpha_2 \in \text{Hom}_S(M_1,M_2) \), and \( \beta \in \text{Hom}_S(M_2,M_3) \), we have that

\[
\beta(\alpha_1 + \alpha_2) = \beta \alpha_1 + \beta \alpha_2
\]

(9)

\[
(\alpha_1 + \alpha_2)\gamma = \alpha_1 \gamma + \alpha_2 \gamma,
\]

(10)

and a zero object of \( S\text{-Mod} \) is a zero object of \( \mathcal{A} \). Hence \( \mathcal{A} \) is an additive category. Furthermore, by using Lemma 2.1 and Lemma 2.2, we show that \( \mathcal{A} \) is the abelian category.

**Proposition 2.3.** The additive category \( \mathcal{A} \) is an abelian category.

**Proof.** Let \( \theta : M \to N \) be any morphism in \( \mathcal{A} \). The kernel and cokernel of \( g \) in \( S\text{-Mod} \) are the kernel and cokernel of \( \theta \) in \( \mathcal{A} \). Furthermore, we have the direct sum of two objects of \( \mathcal{A} \) is also in \( \mathcal{A} \) (by Lemma 2.2). Hence based on Proposition 5.92 [14], the additive category \( \mathcal{A} \) is the abelian category. \( \square \)

An abelian category is called having enough injective (projective) object if each object of it is contained in its injective (projective) object[12]. The category \( S\text{-Mod} \) for any ring \( S \) is an abelian category that has enough injective objects and enough projective objects. However, it does not necessarily apply to every full subcategory of \( S\text{-Mod} \), although the full subcategory of the \( S\text{-Mod} \) is an abelian category[14]. The abelian category \( \mathcal{A} \) has enough injective (projective) objects, if it satisfies the following proposition.

**Proposition 2.4.** If the ring \( S \) is an object of \( \mathcal{A} \) then \( \mathcal{A} \) has enough injective (projective) objects.

**Proof.**

(i) Let \( M \) be any object of \( \mathcal{A} \). Then By Theorem 3.38 [14], \( M \) can imbedded as a submodule of an injective \( S \)-module, i.e. the \( S \)-module of all \( Z \)-module homomorphism from \( S \) to \( Q(\text{Hom}_Z(S,Q)) \), where \( Q \) is an injective \( Z \)-module containing \( M \) as \( Z \)-module. Next we prove that \( \text{Hom}_Z(S,Q) \) is an object of \( \mathcal{A} \).

Let \( h \in \text{Hom}_Z(S,Q) \) be any element. For any \( r \in R \), \( s,t \in S \),

\[
(sf(r) - f(r)s)h(t) = h((sf(r) - f(r)s)t)
\]

(11)

Since \( S \) is an object of \( \mathcal{A} \),

\[
h((sf(r) - f(r)s)x) = h(0) = 0.
\]

(12)

Hence the \( S \)-module \( \text{Hom}_Z(S,Q) \) is an object of \( \mathcal{A} \). Thus \( \mathcal{A} \) has enough injective objects.

(ii) Let \( M \) be any object of \( \mathcal{A} \). Recall Theorem 2.35[14], \( M \) is a quotient of free \( S \)-module \( F(X) \), i.e. \( M \cong F(X)/\ker h \), where \( h : F(X) \to M \) is an \( S \)-module epimorphism and \( X \) is a generating set of \( M \).

Since every free \( S \)-module is projective, a free \( S \)-module \( F(X) \) is projective. Next we prove \( F(X) \) is an object of \( \mathcal{A} \). Let \( \sum a_i x_i \in F(X) \) be any element. For any \( r \in R \), and \( s \in S \),

\[
(sf(r) - f(r)s)\sum a_i x_i = \sum (sf(r) - f(r)s)a_i x_i
\]

(13)

Since \( S \) is an object of \( \mathcal{A} \),

\[
\sum (sf(r) - f(r)s)a_i x_i = \sum 0 x_i = 0
\]

(14)

Thus the free \( S \)-module is an object of \( \mathcal{A} \). Therefore the category \( \mathcal{A} \) has enough projective objects. \( \square \)
From Proposition 2.4, it is known that the sufficient condition of category \( \mathcal{A} \) has enough injective(projective) objects if the ring \( S \) is an object of \( \mathcal{A} \). As a result, the category \( \mathcal{A} \) is equivalent to the category of the modules over an \( R \)-algebra \( S \), because if \( S \) is an object of \( \mathcal{A} \), then \( S \) is an \( R \)-algebra with a ring homomorphism \( g : R \rightarrow S, r \rightarrow f(r)1_S \).

3. Conclusion

The kernel and image of homomorphism in the category \( \mathcal{A} \) and the finite direct sums of objects of \( \mathcal{A} \) are objects of \( \mathcal{A} \) such that \( \mathcal{A} \) is an abelian category. Furthermore, if the ring \( S \) is an object of \( \mathcal{A} \), then the abelian category \( \mathcal{A} \) has enough injective objects and enough projective objects.

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