On O’Grady’s Generalized Franchetta Conjecture

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We study relative zero cycles on the universal polarized $K3$ surface $X \to \mathcal{F}_g$ of degree $2g-2$. It was asked by O’Grady if the restriction of any class in $\text{CH}^2(X)$ to a closed fiber $X_s$ is a multiple of the Beauville–Voisin canonical class $c_{X_s} \in \text{CH}_0(X_s)$. Using Mukai models, we give an affirmative answer to this question for $g \leq 10$ and $g = 12, 13, 16, 18, 20$.

1 Introduction

Throughout, we work over the complex numbers. Let $S$ be a projective $K3$ surface. In [2], Beauville and Voisin studied the Chow ring $\text{CH}^*(S)$ of $S$. They showed that there is a canonical class $c_S \in \text{CH}_0(S)$ represented by a point on a rational curve in $S$, which satisfies the following properties:

(i) The intersection of two divisor classes on $S$ always lies in $\mathbb{Z} c_S \subset \text{CH}_0(S)$.
(ii) The second Chern class $c_2(T_S)$ equals $24c_S \in \text{CH}_0(S)$.

This result is rather surprising since the Chow group $\text{CH}_0(S)$ is infinite-dimensional by Mumford’s theorem [11].
Let $\mathcal{F}_g$ denote the moduli space of (primitively) polarized $K3$ surfaces of degree $2g - 2$. For $g \geq 3$, let $\mathcal{F}_g^0 \subset \mathcal{F}_g$ be the open dense subset parametrizing polarized $K3$ surfaces with trivial automorphism groups, which carries a universal family $X \to \mathcal{F}_g^0$. Motivated by Franchetta’s conjecture on the moduli spaces of curves (see [1]), O’Grady asked the following question in [12], referred to as the generalized Franchetta conjecture.

**Question 1.1** (Generalized Franchetta conjecture). Given a class $\alpha \in \text{CH}^2(X)$ and a closed point $s \in \mathcal{F}_g^0$, is it true that $\alpha|_{X_s} \in \mathbb{Z}c_{X_s}$?

The goal of this article is to give an affirmative answer to Question 1.1 for a list of small values of $g$. By the work of Mukai [7–10], for these $g$ a general polarized $K3$ surface can be realized in a variety with “small” Chow groups as a complete intersection with respect to a vector bundle.

**Theorem 1.2.** The generalized Franchetta conjecture holds for $g \leq 10$ and $g = 12, 13, 16, 18, 20$.

This article is organized as follows. In Section 2 we review Mukai’s constructions and make some comments about Question 1.1. In Section 3 we prove Theorem 1.2 for all cases except $g = 13, 16$. Two independent proofs are presented, one using Voisin’s result [17], the other via a direct calculation. The cases $g = 13, 16$ have a different flavor and are treated in Section 4.

This work is inspired by a recent preprint of Pedrini [13]. However, contrary to what was claimed there, it does not suffice to show that $\text{CH}^2(X)_Q$ is finite-dimensional. Our proof relies deeply on the result of Beauville–Voisin [2].

## 2 Mukai Models and the Basic Setting

In this section we review Mukai’s work [7–10] on the projective models of general polarized $K3$ surfaces of small degrees. Using Mukai’s models, we set up the framework for the proof of Theorem 1.2.

The following table summarizes the ambient varieties $G_g$ and vector bundles $\mathcal{U}_g$ involved in the constructions. It is also accompanied by a glossary.
| $g$ | $g$ | $\mathcal{U}_g$ | $g$ | $g$ | $\mathcal{U}_g$ |
|-----|-----|---------------|-----|-----|---------------|
| 2   | $\mathbb{P}(1,1,1,2)$ | $\mathcal{O}(6)$ | 9   | $\mathbb{G}(3,6)$ | $\mathcal{O}(1)^{\oplus 4} \oplus \wedge^2 \mathcal{Q}$ |
| 3   | $\mathbb{P}^3$     | $\mathcal{O}(4)$ | 10  | $\mathbb{G}(2,7)$ | $\mathcal{O}(1)^{\oplus 3} \oplus \wedge^4 \mathcal{Q}$ |
| 4   | $\mathbb{P}^4$     | $\mathcal{O}(2) \oplus \mathcal{O}(3)$ | 12  | $\mathbb{G}(3,7)$ | $\mathcal{O}(1) \oplus \left(\wedge^2 \mathcal{E}^\vee\right)^{\oplus 3}$ |
| 5   | $\mathbb{P}^5$     | $\mathcal{O}(2)^{\oplus 3}$ | 13  | $\mathbb{G}(3,7)$ | $\left(\wedge^2 \mathcal{E}^\vee\right)^{\oplus 2} \oplus \wedge^3 \mathcal{Q}$ |
| 6   | $\mathbb{G}(2,5)$  | $\mathcal{O}(1)^{\oplus 3} \oplus \mathcal{O}(2)$ | 16  | $\mathbb{G}(2,3,4)$ | $\mathcal{V}_{16}^{\oplus 2} \oplus \tilde{\mathcal{V}}_{16}^{\oplus 2}$ |
| 7   | $\mathcal{O}\mathbb{G}(5,10)$ | $\mathcal{V}_{7}^{\oplus 8}$ | 18  | $\mathcal{O}\mathbb{G}(3,9)$ | $\mathcal{V}_{18}^{\oplus 5}$ |
| 8   | $\mathbb{G}(2,6)$  | $\mathcal{O}(1)^{\oplus 6}$ | 20  | $\mathbb{G}(4,9)$ | $\left(\wedge^2 \mathcal{E}^\vee\right)^{\oplus 3}$ |

$\mathbb{P}(1,1,1,2)$: 3-dimensional weighted projective space with weights $(1,1,1,2)$

$\mathbb{G}(r,n)$: Grassmannian parametrizing $r$-dimensional subspaces of a fixed $n$-dimensional vector space

$\mathcal{O}(i)$: Line bundle on $\mathbb{G}(r,n)$ with respect to the Plücker embedding

$\mathcal{O}\mathbb{G}(r,n)$: Orthogonal Grassmannian parametrizing $r$-dimensional isotropic subspaces of a fixed $n$-dimensional vector space equipped with a nondegenerate symmetric 2-form

$\mathcal{V}_7$: Line bundle on $\mathcal{O}\mathbb{G}(5,10)$ corresponding to a spin representation

$\mathcal{Q}$: Universal quotient bundle on $\mathbb{G}(r,n)$

$\mathcal{E}$: Universal subbundle on $\mathbb{G}(r,n)$

$\mathbb{G}(2,3,4)$: Ellingsrud–Pień–Strømme moduli space of twisted cubic curves, constructed as the GIT quotient of $\mathbb{C}^2 \otimes \mathbb{C}^3 \otimes \mathbb{C}^4$ by the action of $\text{GL}_2 \times \text{GL}_3$ (see [3])

$\mathcal{V}_{16}$: Rank 3 tautological vector bundle on $\mathbb{G}(2,3,4)$

$\tilde{\mathcal{V}}_{16}$: Rank 2 tautological vector bundle on $\mathbb{G}(2,3,4)$

$\mathcal{V}_{18}$: Rank 2 vector bundle on $\mathcal{O}\mathbb{G}(3,9)$ corresponding to a spin representation

For all $g$ listed above, Mukai showed that a general K3 surface over $\mathcal{F}_g$ is given as the zero locus of a general global section of $\mathcal{U}_g$ (the cases $g \leq 5$ are classical).

Let

$$\mathbb{P}_g = \mathbb{P}\mathcal{H}^0(\mathbb{G}_g, \mathcal{U}_g)$$
be the projectivization of the space of global sections of $\mathcal{U}_g$, and let

$$Y = \{(s, x) \in \mathbb{P}_g \times \mathbb{C}_g \mid s(x) = 0\}$$

be the incidence scheme. We have a diagram

$$
\begin{array}{ccc}
Y & \xrightarrow{\iota} & G_g \\
\downarrow{\pi} & & \\
\mathbb{P}_g & & 
\end{array}
$$

where $\pi, \iota$ are the two projections.

The discussion above shows that a general fiber of $\pi : Y \to \mathbb{P}_g$ is a polarized K3 surface of degree $2g - 2$ and that $\mathbb{P}_g$ rationally dominates the moduli space $\mathcal{F}_g$. Moreover, since $\mathcal{U}_g$ is globally generated, we know that $\iota : Y \to G_g$ is a projective bundle. Its fiber over a point $x \in G_g$ is given by

$$
\mathbb{P}H^0(G_g, \mathcal{U}_g \otimes I_x),
$$

where $I_x$ is the ideal sheaf of $x$. We have the following lemma regarding the Chow group $\text{CH}^2(Y)$ and its restriction to a general fiber of $\pi$.

**Lemma 2.1.** Given a closed point $s \in \mathbb{P}_g$ with K3 fiber $Y_s$, let $\phi_s : Y_s \hookrightarrow Y$ and $\iota_s : Y_s \hookrightarrow G_g$ be the natural embeddings. Then we have

$$
\text{Im}(\phi_s^* : \text{CH}^2(Y)_\mathbb{Q} \to \text{CH}^0(Y_s)_\mathbb{Q}) = \text{Im}(\iota_s^* : \text{CH}^2(G_g)_\mathbb{Q} \to \text{CH}^0(Y_s)_\mathbb{Q}). \quad \square
$$

**Proof.** Let $\xi \in \text{CH}^1(Y)$ be the relative hyperplane class of $\iota : Y \to G_g$. By the projective bundle formula, we have for $k \geq 0$,

$$
\text{CH}^k(Y) = \xi^k \cdot i^*\text{CH}^0(G_g) \oplus \xi^{k-1} \cdot i^*\text{CH}^1(G_g) \oplus \cdots \oplus i^*\text{CH}^k(G_g).
$$

(2.1)

Let $h \in \text{CH}^1(\mathbb{P}_g)$ be the hyperplane class. Then we have

$$
\pi^* h = a \cdot \xi + \iota^* \beta
$$

for some $a \in \mathbb{Z}$ and $\beta \in \text{CH}^1(G_g)$. We claim that $a \neq 0$, otherwise

$$
\pi^* (h^{\dim \mathbb{P}_g}) = \iota^* (\beta^{\dim \mathbb{P}_g}).
$$
Since $\dim P_g > \dim G_g$, the right-hand side vanishes, but the left-hand side is the pullback of a point class and is nonzero. Contradiction. Hence

$$\xi = \frac{1}{a}(\pi^*h - \iota^*\beta) \in \text{CH}^1(Y)_\mathbb{Q}.$$ 

The lemma then follows from (2.1) for $k = 2$ and the fact that $\phi_s^*\pi^*h = 0$. □

We end this section by a few remarks on the generalized Franchetta conjecture.

(i) By a standard “spreading out” argument (see [16, Chapter 1]), it is equivalent to answer Question 1.1 for general (in fact, very general) fibers $X_s$ over $\mathcal{F}_g^0$. Moreover, classes in $\text{CH}^2(X)$ supported over a proper closed subset of $\mathcal{F}_g^0$ vanish when restricted to a fiber $X_s$. Hence one may work with a family $Y \to B$ such that a general fiber $Y_s$ is a polarized $K3$ surface of degree $2g - 2$ and that $B$ rationally dominates $\mathcal{F}_g$ via the natural rational map $B \to \mathcal{F}_g$. It then suffices to answer (the analog of) Question 1.1 for classes in $\text{CH}^2(Y)$ and $K3$ fibers $Y_s$. See Section 4 for an even more precise statement.

One may also formulate Question 1.1 in terms of the Chow group $\text{CH}_0(X_\eta)$ of the generic fiber $X_\eta$, but we omit this point of view.

(ii) By Roîtman’s theorem [14], the Chow group $\text{CH}_0(S)$ of a complex $K3$ surface $S$ is torsion-free. Hence in Question 1.1 it is equivalent to work with $\mathbb{Q}$-coefficients. This also means that under Lemma 2.1, we have

$$\text{Im}(\phi_s^* : \text{CH}^2(Y) \to \text{CH}_0(Y_s)) \subset \mathbb{Z}c_{Y_s}$$

if and only if

$$\text{Im}(\iota_s^* : \text{CH}^2(G_g) \to \text{CH}_0(Y_s)) \subset \mathbb{Z}c_{Y_s}.$$ 

(iii) Instead of restricting to $\mathcal{F}_g^0$, one may work with the moduli stack and the universal family over it. Question 1.1 can then be formulated using the Chow groups of smooth Deligne–Mumford stacks with $\mathbb{Q}$-coefficients. This notably covers the case $g = 2$, where a general $K3$ surface over $\mathcal{F}_2$ carries an involution. Our proof in Section 3 works in this case without change.
3 Polarized K3 Surfaces as Unique Complete Intersections

In this section we deal with the cases \( g \leq 10 \) and \( g = 12, 18, 20 \). For these \( g \), the Mukai model embeds a general polarized K3 surface of degree \( 2g - 2 \) in \( G_g \) as a complete intersection with respect to \( U_g \), and the embedding is unique up to automorphisms of \( G_g \) and \( U_g \). Moreover, the variety \( G_g \) is a Grassmannian or an orthogonal Grassmannian.

Since \( \mathbb{P}_g \) rationally dominates the moduli space \( F_g \), to prove Theorem 1.2 it suffices to show that the restriction of any class in \( CH^2(Y) \) to a K3 fiber \( Y_s \) lies in \( \mathbb{Z}c_{Y_s} \). By Lemma 2.1, it is equivalent to show that

\[
\text{Im}(\iota_s^*: CH^2(G_g) \to CH^2(Y_s)) \subset \mathbb{Z}c_{Y_s}.
\]

This allows us to work with a single K3 surface \( S \) with an embedding

\[ i: S \hookrightarrow G_g. \]

If \( g \leq 5 \), the variety \( G_g \) is a projective space and its Chow ring is generated by the hyperplane class. Thus Theorem 1.2 follows from property (i) of \( c_S \) in Section 1.

Now assume that \( G_g \) is not a projective space. It is well known that the Chow group \( CH^2(G(r, n)) \) of the Grassmannian is generated by the Chern classes \( c_1(Q)^2 \) and \( c_2(Q) \), where \( Q \) is the universal quotient bundle. For the orthogonal Grassmannians, we have instead

\[
CH^2(\mathbb{O}G(5, 10)) = \mathbb{Z}\left(\frac{1}{2}c_2(Q)\right) \oplus \mathbb{Z}\left(\frac{1}{4}c_1(Q)^2\right)
\]

and

\[
CH^2(\mathbb{O}G(3, 9)) = \mathbb{Z}\left(\frac{1}{2}c_2(Q)\right) \oplus \mathbb{Z}c_1(Q)^2,
\]

where \( Q \) is the corresponding universal quotient bundle (see [15]). Hence in all cases, a class \( \alpha \in CH^2(G_g) \) can be uniquely expressed as

\[
\alpha = a \cdot c_2(Q) + b \cdot c_1(Q)^2,
\]

with \( a \in \mathbb{Z} \) if \( G_g \) is a Grassmannian, or \( a \in \frac{1}{2}\mathbb{Z} \) if \( G_g \) is an orthogonal Grassmannian. For convenience we define the index \( I(\alpha) \) of \( \alpha \in CH^2(G_g) \) to be the coefficient \( a \).

By property (i) of \( c_S \) in Section 1, we have \( i^*(c_1(Q)^2) \in \mathbb{Z}c_S \). Hence the following proposition implies Theorem 1.2 for \( g = 6, 7, 8, 9, 10, 12, 18, 20 \).
Proposition 3.1. With the notation as above, we have \( i^*c_2(Q) \in \mathbb{Z}_{cS} \).
\[ \square \]

We give two independent proofs of the proposition.

**First proof.** Mukai showed in [7, 8] that the restriction of either \( Q \) or \( E^\vee \) to a general \( S \) is simple and rigid, where \( E \) is the universal subbundle. In fact, the rigidity ensures that the embedding of \( S \) in \( G_g \) is unique. The proposition follows from a strong result of Voisin [17, Corollary 1.10] that the second Chern class of any simple rigid vector bundle on a \( K3 \) surface \( S \) lies in \( \mathbb{Z}_{cS} \), which was conjectured by Huybrechts earlier in [5].

Since part of the original motivation of the generalized Franchetta conjecture was to make Huybrechts’ conjecture as its consequence (see [12, Section 5]), we give a direct proof of Proposition 3.1 without using Voisin’s result.

**Second proof.** We first consider the cases where \( G_g \) is a Grassmannian. The standard exact sequence of normal bundles

\[
0 \to T_S \to i^*T_{Gg} \to i^*U_g \to 0
\]

yields the following relation in \( CH_0(S) \):

\[
i^*c_2(T_{Gg}) = c_2(T_S) + i^*c_2(U_g).
\]

(3.1)

Here \( T_{Gg} \) and \( T_S \) are the corresponding tangent bundles. Using the index of classes in \( CH^2(G_g) \), the relation (3.1) can be written as

\[
\left( I(c_2(T_{Gg})) - I(c_2(U_g)) \right) \cdot i^*c_2(Q) = c_2(T_S) + \gamma,
\]

(3.2)

where \( \gamma \) can be expressed in terms of divisor classes on \( S \). By properties (i) and (ii) of \( c_S \) in Section 1, both \( c_2(T_S) \) and \( \gamma \) lie in \( \mathbb{Z}_{cS} \). Hence it suffices to verify that

\[
I(c_2(T_{Gg})) - I(c_2(U_g)) \neq 0.
\]

(3.3)

The tangent bundle \( T_{G(r,n)} \) of the Grassmannian is \( \mathcal{H}om(E, Q) \), where \( E \) is the universal subbundle. By computing the Chern character

\[
ch(\mathcal{H}om(E, Q)) = ch(E^\vee \otimes Q) = ch(E^\vee) \cdot ch(Q)
\]
and the standard relation $c(E) \cdot c(Q) = 1$ between the total Chern classes, we have

$$I(c_2(T_{G(r,n)})) = 2r - n.$$ 

The following is a case-by-case study:

$g = 6, 8$ Here $U_g$ is a direct sum of line bundles. Hence $I(c_2(U_g)) = 0$ and

$$I(c_2(T_{G_g})) - I(c_2(U_g)) = 2r - n \neq 0.$$ 

$g = 9$ We have $I(c_2(T_{G_9})) = 0$ and $I(c_2(U_9)) = I(c_2(\wedge^2 Q)) = 1$. Hence

$$I(c_2(T_{G_9})) - I(c_2(U_9)) = -1 \neq 0.$$ 

$g = 10$ We have $I(c_2(T_{G_{10}})) = -3$ and $I(c_2(U_{10})) = I(c_2(\wedge^4 Q)) = 1$. Hence

$$I(c_2(T_{G_{10}})) - I(c_2(U_{10})) = -4 \neq 0.$$ 

$g = 12, 20$ We have $I(c_2(T_{G_g})) = -1$ and $I(c_2(U_g)) = 3I(c_2(\wedge^2 E))$. Hence

$$I(c_2(T_{G_g})) - I(c_2(U_g)) = -1 - 3I(c_2(\wedge^2 E)) \neq 0.$$ 

The orthogonal Grassmannian cases ($g = 7, 18$) are similar. The relation (3.2) still holds, and it suffices to show (3.3). Here the left-hand side of (3.3) may be a half integer.

The natural embedding $j : \mathcal{OG}(r, n) \hookrightarrow \mathcal{G}(r, n)$ can be realized as the zero locus of a smooth section of the vector bundle $\mathcal{W}$, which is given by the cohomology group $H^0(\mathbb{P}^{r-1}, \mathcal{O}(2))$ over every closed point

$$[\mathbb{P}^{r-1} \subset \mathbb{P}^{n-1}] \in \mathcal{G}(r, n).$$ 

Hence we have

$$I(c_2(T_{\mathcal{OG}(r,n)})) = I(j^*c_2(T_{\mathcal{G}(r,n)})) - I(j^*c_2(\mathcal{W})).$$ 

The term $I(j^*c_2(T_{\mathcal{G}(r,n)}))$ was already calculated, and the term $I(j^*c_2(\mathcal{W}))$ can be determined by the following Grothendieck–Riemann–Roch calculation.

We consider $p : \mathbb{P}(E) \to \mathcal{G}(r, n)$ the projective bundle on $\mathcal{G}(r, n)$ associated to the universal subbundle $\mathcal{E}$. Let $L = \mathcal{O}_{\mathbb{P}(E)}(1)$ and let $\xi$ be the relative hyperplane class $c_1(L)$.
We have \( R^k p_* L = 0 \) for \( k > 0 \). Hence by the Grothendieck–Riemann–Roch theorem, we have
\[
ch(W) = ch(R p_* L^{\otimes 2}) = p_*(\exp(2\xi) \cdot td(T_p)).
\]
Together with the exact sequence
\[
0 \to O_{\mathbb{P}(E)} \to p^* E \otimes L \to T_p \to 0,
\]
we obtain for \( r = 3, 5 \),
\[
I(c_2(W)) = -(r + 2).
\]
We finish the proof of Proposition 3.1:

\[ g = 7 \quad \text{Here } U_7 \text{ is a direct sum of line bundles. Hence } I(c_2(U_7)) = 0 \text{ and}
\]
\[
I(c_2(G_7)) - I(c_2(U_7)) = 0 - (-7) = 7 \neq 0.
\]
\[ g = 18 \quad \text{We have } I(c_2(G_{18})) = -3 - (-5) \text{ and } I(c_2(U_{18})) = 5I(c_2(V_{18})). \text{ Hence}
\]
\[
I(c_2(G_{18})) - I(c_2(U_{18})) = 2 - 5I(c_2(V_{18})) \neq 0. \quad \square
\]

4 Polarized K3 Surfaces as Nonunique Complete Intersections

In this section we treat the remaining cases \( g = 13, 16 \). In both cases, the embedding of a polarized K3 surface \( S \) of degree \( 2g - 2 \) in \( G_g \) is not unique and the restriction of the tautological bundles to \( S \) may not be rigid. Hence the methods in Section 3 break down.

We keep the notation of Section 2 and write \( \Phi : \mathbb{P}_g \dashrightarrow \mathcal{F}_g \) for the dominant rational map. Let \( t \in \mathcal{F}_g^0 \) be a closed point outside the indeterminacy locus of \( \Phi \) in \( \mathcal{F}_g \). Given two closed points \( s_1, s_2 \in \mathbb{P}_g \) with \( \Phi(s_1) = \Phi(s_2) = t \), there are canonical isomorphisms
\[
Y_{s_1} \cong Y_{s_2} \cong X_t.
\]
We identify \( CH_0(Y_{s_1}), CH_0(Y_{s_2}) \) with \( CH_0(X_t) \), and define
\[
CH^2(Y)_{\text{inv}} = \{ \alpha \in CH^2(Y) \mid \phi_{s_1}^* \alpha = \phi_{s_2}^* \alpha \text{ for all } s_1, s_2 \in \mathbb{P}_g \text{ above} \}.
\]
Recall that \( \phi_s : Y_s \hookrightarrow Y \) is the natural embedding for \( s \in \mathbb{P}_g \).
Again by the “spreading out” argument and the fact that classes supported over a proper closed subset of $F_g^0$ do not contribute, to prove Theorem 1.2 it suffices show that

$$\text{Im}(\phi_s^*: \text{CH}^2(Y)_{\text{inv}} \to \text{CH}_0(Y_s)) \subset \mathbb{Z}c_{Y_s}$$

for all (or general, or very general) K3 fibers $Y_s$.

First we consider the case $g = 13$. As described by the Mukai model, let

$$i: S \hookrightarrow \mathbb{G}(3,7)$$

be the embedding of a K3 surface $S$ in $\mathbb{G}(3,7)$. The restriction of $E^\vee$ (dual of the universal subbundle) to $S$ is semi-rigid, which carries a 2-dimensional deformation. Let $M_S$ be the moduli space of stable vector bundles on $S$ with Mukai vector $(3,H,4)$, where $H$ is the polarization class. A general point of $M_S$ is represented by $i^*E^\vee$ for some $i$; see [9] for details. Note that $M_S$ is also a polarized K3 surface with $g = 13$.

Let $s \in \mathbb{P}_{13}$ be a closed point with K3 fiber $Y_s$, and let $i_s: Y_s \hookrightarrow \mathbb{G}(3,7)$ be as in Section 2. By Lemma 2.1, the restriction $\phi_s^*\alpha$ of a class $\alpha \in \text{CH}^2(Y)_{\text{inv}}$ can be expressed as

$$\phi_s^*\alpha = a \cdot i_s^*c_2(Q) + b \cdot i_s^*(c_1(Q)^2), \quad (4.1)$$

where $Q$ is the universal quotient bundle and $a, b \in \mathbb{Q}$ are constants independent of $s \in \mathbb{P}_{13}$. By property (i) of $c_{Y_s}$ in Section 1, we have $i_s^*(c_1(Q)^2) \in \mathbb{Z}c_{Y_s}$.

Theorem 1.2 for $g = 13$ is a direct consequence of the following lemma.

**Lemma 4.1.** In the expression (4.1), the coefficient $a$ is zero. \hfill $\square$

**Proof.** We choose closed points $s_1, s_2 \in \mathbb{P}_{13}$ with $\Phi(s_1) = \Phi(s_2) = t \in F_{13}^0$, such that the vector bundles $i_{s_1}^*E^\vee, i_{s_2}^*E^\vee$ represent different point classes in $\text{CH}_0(M_{X_t})$. This is possible by [9, Theorem 2], which shows that $\mathbb{P}_{13}$ rationally dominates the moduli space of triples $(S,H,E)$, where $S$ is a K3 surface, $H$ is a polarization with $H^2 = 24$, and $E$ is a stable vector bundle with Mukai vector $(3,H,4)$.

Since $\alpha \in \text{CH}^2(Y)_{\text{inv}}$, we have by definition $\phi_{s_1}^*\alpha = \phi_{s_2}^*\alpha$ and hence

$$a \cdot i_{s_1}^*c_2(Q) = a \cdot i_{s_2}^*c_2(Q), \quad (4.2)$$

viewed as an equality in $\text{CH}_0(X_t)_Q$. 

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On the other hand, let $F$ be a universal sheaf over $M_{X_t} \times X_t$ (which exists by the numerics of the Mukai vector; see [6, Corollary 4.6.7]). The correspondence

$$\text{ch}(F) \cdot \sqrt{\text{td}(TM_{X_t} \times X_t)} \in \text{CH}^*(M_{X_t} \times X_t)$$

induces an isomorphism of (ungraded) Chow groups

$$\theta : \text{CH}^*(M_{X_t}) \xrightarrow{\sim} \text{CH}^*(X_t).$$

Here for $[E] \in M_{X_t}$, we have

$$\theta([E]) = 3[X_t] + H + 15c_{X_t} - c_2(E) \in \text{CH}^*(X_t).$$

According to our choice of $s_1, s_2 \in \mathbb{P}_1$, the vector bundles $t_{s_1}^*E^\vee, t_{s_2}^*E^\vee$ represent different classes in $\text{CH}_0(M_{X_t})$. By applying $\theta$, we find

$$t_{s_1}^*c_2(E^\vee) \neq t_{s_2}^*c_2(E^\vee)$$

in $\text{CH}_0(X_t)$, and together with (4.2) we obtain $a = 0$. 

Finally we consider the case $g = 16$. The variety $G_{16} = G(2, 3, 4)$ is realized as a GIT quotient of $\mathbb{C}^2 \otimes \mathbb{C}^3 \otimes \mathbb{C}^4$ by the obvious action of $\text{GL}_2 \times \text{GL}_3$ on the first two factors. As described in [3] (see also [10]), there are two tautological vector bundles $V_{16}$ and $\tilde{V}_{16}$ of ranks 3 and 2, respectively, as well as a morphism

$$V_{16} \otimes (\mathbb{C}^4)^\vee \to \tilde{V}_{16}.$$ 

Further, it was shown in [3, Proposition 2] that the Chow ring $\text{CH}^*(G(2, 3, 4))$ is generated by the Chern classes of $V_{16}, \tilde{V}_{16}$. To prove Theorem 1.2 we have to take care of the second Chern classes of both tautological bundles.

Let $i : S \hookrightarrow G(2, 3, 4)$ be the embedding of a K3 surface $S$ in $G(2, 3, 4)$ as in the Mukai model. By the same reasoning as in Section 3, we have the following relation in $\text{CH}_0(S)$:

$$i^*c_2(T_{G(2,3,4)}) = c_2(T_S) + i^*c_2(U_{16}). \quad (4.3)$$

Here $U_{16} = V_{16}^{\text{op}} \oplus \tilde{V}_{16}^{\text{op}}$. Using the exact sequence (see [4, (44)])

$$0 \to \mathcal{O}_{G(2,3,4)} \to \mathcal{E}nd(V_{16}) \oplus \mathcal{E}nd(\tilde{V}_{16}) \to \mathcal{H}om(V_{16} \otimes (\mathbb{C}^4)^\vee, \tilde{V}_{16}) \to T_{G(2,3,4)} \to 0,$$
the relation (4.3) can be written as

\[ 6c_2(i^*\tilde{V}_{16}) = c_2(T_S) + \gamma, \]

where \(\gamma\) can be expressed in terms of divisor classes on \(S\). By properties (i) and (ii) of \(c_S\) in Section 1, this verifies that \(i^*c_2(\tilde{V}_{16}) \in \mathbb{Z}c_S\).

Alternatively, by Mukai’s results [10, Propositions 1.3 and 2.2], for a general \(S\) the vector bundle \(i^*\tilde{V}_{16}\) is simple and rigid. The statement \(i^*c_2(\tilde{V}_{16}) \in \mathbb{Z}c_S\) also follows from Voisin’s result [17, Corollary 1.10].

Let \(s \in \mathbb{P}_{16}\) be a closed point with K3 fiber \(Y_s\), and let \(\iota_s : Y_s \hookrightarrow \mathbb{G}(2, 3, 4)\) be as before. Again by Lemma 2.1 and property (i) of \(c_{Y_s}\) in Section 1, the restriction \(\phi_s^*\alpha\) of a class \(\alpha \in \text{CH}^2(Y)_{\text{inv}}\) can be expressed as

\[ \phi_s^*\alpha = a \cdot \iota_s^*c_2(\tilde{V}_{16}) + \tilde{a} \cdot \iota_s^*c_2(\tilde{V}_{16}) + b \cdot c_{Y_s}, \]

where \(a, \tilde{a}, b \in \mathbb{Q}\) are constants independent of \(s \in \mathbb{P}_{16}\). Moreover, the fact that \(\iota_s^*c_2(\tilde{V}_{16}) \in \mathbb{Z}c_{Y_s}\) implies

\[ \phi_s^*\alpha = a \cdot \iota_s^*c_2(\tilde{V}_{16}) + b' \cdot c_{Y_s} \tag{4.4} \]

for some \(a, b' \in \mathbb{Q}\) independent of \(s \in \mathbb{P}_{16}\). Since \(\iota_s^*\tilde{V}_{16}\) is semi-rigid with Mukai vector \((3, H, 5)\) by [10, Proposition 2.2], an identical argument as in the proof of Lemma 4.1 yields \(a = 0\) in the expression (4.4).

This finishes the proof of Theorem 1.2 for \(g = 16\).

**Funding**

This work was supported by the grant ERC-2012-AdG-320368-MCSK [to J.S. and Q.Y.].

**Acknowledgments**

We are grateful to Rahul Pandharipande for his constant support and his enthusiasm in this project. We also thank Kieran O’Grady for his careful reading of a preliminary version of this paper.

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