Borel resummation with low order perturbations in QCD

Taekoon Lee\textsuperscript{*}

\textsuperscript{a}Department of Physics, Kunsan National University, Kunsan 573-701, Korea

The bilocal expansion of Borel transform provides an efficient way of Borel resummation with low order perturbations in QCD. Its applications to the heavy quark pole mass, static potential, and lattice calculation are reviewed.

1. Introduction

The perturbative expansions in weak coupling constant in quantum field theories are in general divergent, with the coefficients growing in factorial. One way to make a rigorous sense out of the divergent series is to resum the series using Borel transform. When the large order behavior of the expansion is sign-alternating Borel resummation may yield the full amplitude. But in QCD the series are in general of same-sign and not Borel resummable, and must be augmented by nonperturbative effects. However, Borel resummation can still be useful in these cases when a precise definition of the nonperturbative effects are required, or the coupling is large enough that the usual perturbative amplitude from the low order perturbations is not sufficiently accurate for the purpose.

The same-sign large order behavior of weak coupling expansion gives a singularity (renormalon) on the contour of the Borel integral. When the coupling is small the Borel integral receives most of its contribution from the immediate neighborhood of the origin, where the Borel transform can be well-described by the low order perturbations, and the singular behavior of the Borel transform is not of much concern, but as the coupling increases the contribution from the region near the singularity becomes important, and one needs to have an accurate description of the Borel transform in the region that contains the origin as well as the nearest singularity to the origin.

It is obvious that it is a formidable task to rebuild the singular behavior of the Borel transform using the ordinary perturbative expansions about the origin. In fact, the solvable models like the quantum mechanical instanton models suggest that one need many 10s orders of perturbative terms to obtain a good Borel resummation. This approach is obviously not viable in QCD considering the high cost of loop calculations. Thus an alternative approach is required to do any attempt for realistic Borel resummation with the low order perturbations in QCD.

The nature of the renormalon singularity can be determined by renormalization group technique, and the (singular part of) Borel transform near the singularity can be systematically expanded perturbatively, up to the residue of the singularity that determines the overall normalization constant of the large order behavior (For a review see \cite{1}). It was later realized that, once the nature of the singularity is known, the residue too can be calculated perturbatively, using only the usual perturbative expansions of the quantity in concern \cite{2}.

We thus have a systematic expansion of the Borel transform at two points, the origin as well as the singularity. The bilocal expansion is to combine these expansions via an interpolation to obtain an accurate description of the Borel transform not only about the origin but also about the singularity \cite{3}.

The bilocal expansion has been successfully applied to several QCD observables \cite{3,4,5}, and following is a review of those applications.

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2. Bilocal expansion

Let us assume that the Borel transform \( \tilde{A}(b) \) of an amplitude has the perturbative expansion about the origin to order \( M \) as

\[
\tilde{A}(b) = \sum_{i=0}^{M} \frac{a_i}{i!} b^i
\]  
(1)

and expansion about the nearest renormalon singularity to order \( N \) as

\[
\tilde{A}(b) = N \left( 1 - \frac{b}{b_0} \right)^{1+\nu} \left[ 1 + \sum_{j=1}^{N} c_j (1 - b/b_0)^j \right] + \text{Analytic part}
\]  
(2)

where \( b_0 \) is the location of the singularity. The “Analytic part” denotes terms that are analytic on the disk \(| b - b_0 | < b_0 \), and in general is not known.

We can now combine the expansions (1) and (2) as a bilocal expansion using the following interpolation [3]:

\[
\tilde{A}_{M,N}(b) = \sum_{i=0}^{M} \frac{h_i}{i!} b^i + \frac{N}{(1 - b/b_0)^{1+\nu}} \left[ 1 + \sum_{j=1}^{N} c_j (1 - b/b_0)^j \right] + \text{Analytic part}
\]  
(3)

When we compare this expression to (2) we see that the first sum simulates the “Analytic part”. The coefficients \( h_i \) can be determined by demanding that the bilocal expansion, when expanded about the origin, reproduce the perturbative expansion (1). This yields the relation

\[
h_0 = a_0 - N \left( 1 + \sum_{j=0}^{N} c_j \right),
\]

\[
h_1 = a_1 - \frac{N}{b_0} [1 + \nu + \sum_{j=1}^{N} c_j (1 + \nu - j)],
\]

\[
h_2 = a_2 - \frac{N}{b_0^2} [(\nu + 1)(\nu + 2) + \sum_{j=1}^{N} c_j (j(j - 1) - 2(1 + \nu)j + (\nu + 1)(\nu + 2))],
\]
etc.  
(4)

### Table 1

| \( N_f \) | \( N_f = 1 \) | \( N_f = 2 \) | \( N_f = 3 \) | \( N_f = 4 \) |
|---------|-------------|-------------|-------------|-------------|
| LO      | 0.904      | 0.904      | 0.946       | 1.018       |
| NLO     | 0.521      | 0.546      | 0.592       | 0.674       |
| NNLO    | 0.592      | 0.549      | 0.494       | 0.411       |

The exact Borel transform is given by the limit of the bilocal expansion:

\[
\tilde{A}(b) = \lim_{M,N \to \infty} \tilde{A}_{M,N}(b)
\]  
(5)

For the bilocal expansion to be usable it is necessary to compute the residue \( N \). Since the residue determines the normalization constant of the large order behavior, the Borel transform in bilocal expansion (3) allows one to resum the divergent series to all orders once the residue is known.

The residue can be perturbatively computed as follows. Noticing that

\[
R(b) = \tilde{A}(b) (1 - b/b_0)^{1+\nu}
\]  
(7)

one can expand \( R(b) \) about the origin to evaluate \( N \). Notice that the expansion involves only the usual perturbative coefficients \( a_i \). Although \( R(b) \) is not analytic at \( b_0 \), the expansion is nevertheless convergent because \( R(b) \) is bounded at the singularity. Fortunately, for several QCD observables this expansion gives a rather quick convergence. Adler function is such an example (see Table 1).

3. Pole mass

The perturbative coefficients for the pole mass grow rapidly, and the convergence for the bottom quark, for example, is not good, as can be seen in

\[
m_{\text{pole}} = m_{\overline{\text{MS}}} (1 + 0.093 + 0.045 + 0.032) = m_{\overline{\text{MS}}} (1 + 0.22), \quad N_f = 4.
\]  
(8)

The Borel transform for the heavy-quark pole mass has the nearest renormalon singularity at...
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\( b = 1/2 \) and the bilocal expansion can be computed up to the order \( M = 2, N = 2 \). The perturbative calculation of the residue shows an excellent convergence [3,6]

\[ \mathcal{N} = 0.4244 + 0.1224 + 0.0101 = 0.5569. \] (9)

With this residue and \( N = 2 \), and varying \( M \) from 0 to 2 we obtain the Borel resummed bottom quark pole mass \( m_{\text{BR}} \) as

\[ m_{\text{BR}} / m_{\overline{\text{MS}}} = 1 + 0.1577 + 0.0041 - 0.0003. \] (10)

Notice the remarkable convergence. The leading term already contains the bulk of the contribution, which shows that the proper handling of the singularity in the bilocal expansion accelerates the convergence in an enormous way.

As the above calculation shows the Borel resummation allows one to have a precise relation between the pole mass and \( \overline{\text{MS}} \) mass. This means that the pole mass defined through Borel resummation is as physical as the \( \overline{\text{MS}} \) mass and can be used as an alternative to the short distance masses.

4. Static potential

The heavy quark static potential obtained in perturbation theory disagrees badly with the lattice simulation as shown in Fig.1, even in short distances where perturbation is expected to work.

The nature of the nearest renormalon singularity is the same as the pole mass and the Borel resummation can be performed in a similar fashion [3]. The result is given in Fig.2. The resummed potential agrees remarkably well with the lattice computation. This is a concrete example that shows the efficiency of the bilocal expansion.

5. Lattice determination of heavy-quark mass

The matching relation for the heavy-quark mass (b-quark) in lattice heavy quark effective theory reads [7]

\[ m_{\text{b}}^{\text{pole}} = M_B - \bar{\Lambda}, \quad \bar{\Lambda} = \mathcal{E}(a) - \delta m(a), \] (11)

where \( \bar{\Lambda} \) denotes the renormalized binding energy, \( a \) is the lattice spacing, \( M_B \) is the B-meson mass, \( \delta m \) is the mass shift in the static limit, and \( \mathcal{E} \) is the binding energy that is to be computed in lattice simulation.

The matching relation suggests that with high order perturbative calculations of the pole mass and the mass shift an accurate determination of the \( \overline{\text{MS}} \) mass be possible with precision calculation of \( \mathcal{E} \). There is, however, a problem with that the pole mass and mass shift do not converge well perturbatively, due to the renormalon. To overcome this difficulty the matching relation is expressed in terms of the \( \overline{\text{MS}} \) mass only, bypassing the pole mass, as

\[ m_{\text{b}}^{\overline{\text{MS}}} = \Delta(a)(1 + \sum_{n=0}^{\infty} r_n(\mu/m_{\text{b}}^{\overline{\text{MS}}}, \mu a)\alpha_s(\mu^{n+1})), \] (12)

where \( \Delta(a) \equiv M_B - \mathcal{E}(a) \).

Although this expression does not suffer from the renormalon divergence there could be a convergence problem coming from the mixing of the two independent scales, the lattice spacing and the heavy quark mass, through the RG scale \( \mu \). When the two scales are close, as in the bottom quark case, this problem may not be severe but when the two scales are far-separated this may become a serious problem.

The scale mixing problem can be solved with Borel resummation. The matching relation [11]
with Borel resummation reads

\[ m_{b}^{\text{BR}} = M_{B} - \bar{\Lambda}^{\text{BR}}, \]

where \( \bar{\Lambda}^{\text{BR}} \equiv \mathcal{E}(a) - \delta m^{\text{BR}}(a) \). Here the pole mass and mass shift are resummed independently, using the strong coupling at a scale optimal for each quantities. The scales are cleanly isolated, with \( 1/a \) only in \( \delta m^{\text{BR}} \) and \( m_{b} \) only in \( m_{b}^{\text{BR}} \).

Table 2 shows the \( \overline{\text{MS}} \) mass for the bottom quark from the above two approaches in quenched limit. In this case the results from the two approaches agree remarkably well. However, when the lattice spacing and the heavy-quark mass are far-different the scale mixing problem appears.

For an (imaginary) heavy-light meson with mass \( m_{H} = 500 \text{ GeV} \) we obtain, using (13), the BR mass:

\[ m_{Q}^{\text{BR}} = 499.542, 499.542, 499.523 \text{ (GeV)} \]

(14)

at \( 1/a = 2.12, 2.91, 3.85 \text{ (GeV)} \), respectively, and using the relation

\[ m_{Q}^{\text{BR}}/m_{Q}^{\overline{\text{MS}}} = 1 + 0.03638 + 0.00008 - 0.00004 \]

we obtain the corresponding \( \overline{\text{MS}} \) mass

\[ m_{Q}^{\overline{\text{MS}}} = 481.984, 481.984, 481.965 \text{ (GeV)} \].

What is remarkable with this result is that the uncertainties in these numbers are only \( \pm 20 \text{ MeV} \), entirely coming from \( \bar{\Lambda}^{\text{BR}} \) and the conversion to \( m_{Q}^{\overline{\text{MS}}} \) from the BR mass, both of which cause less than \( 20 \text{ MeV} \) uncertainty.

On the other hand, with the conventional method using (12), the \( m_{Q}^{\overline{\text{MS}}} \) obtained shows strong dependence on the RG scale \( \mu \) which can be anywhere in a wide range between \( 1/a \) and \( M_{Q} \), and the uncertainties in the heavy quark mass from this approach are several hundred MeVs, which are an order of magnitude larger than those in the Borel resummation method. For details we refer the readers to [8].

6. Conclusion

With the bilocal expansion Borel resummation for certain QCD observables can be done to a remarkable accuracy with the low order perturbations. The salient feature of the Borel resummation is the automatic isolation of hierarchical scales, which can be useful in quarkonium systems.

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