THE K-ENERGY ON SMALL DEFORMATIONS OF 
CONSTANT SCALAR CURVATURE KÄHLER MANIFOLDS

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Dedicated to Professor S.-T. Yau on the occasion of his 60th birthday.

ABSTRACT. We give a simplified proof of a recent result of X.X. Chen, which together with work of G. Székelyhidi implies that on a sufficiently small deformation of a polarized constant scalar curvature Kähler manifold the K-energy has a lower bound.

1. Introduction

The study of canonical metrics in Kähler geometry was initiated by Yau [21], and has developed into a very large and active field, see Phong and Sturm [16] for a survey. A fundamental result in this area says that if a compact Kähler manifold \((X,\omega)\) admits a constant scalar curvature Kähler (cscK) metric cohomologous to \(\omega\), then the Mabuchi K-energy \(K_\omega(\varphi)\) of any Kähler potential \(\varphi\) for \(\omega\) is bounded below uniformly. In fact, if the cscK metric is \(\omega + \sqrt{-1} \partial \bar{\partial} \varphi_{\text{cscK}}\) then one has \(K_\omega(\varphi) \geq K_\omega(\varphi_{\text{cscK}})\), with equality holding if and only if the metric \(\omega + \sqrt{-1} \partial \bar{\partial} \varphi\) is also cscK.

This was first proved for Kähler-Einstein metrics by Bando and Mabuchi [2]. In the groundbreaking papers [8, 10] Donaldson extended this result to cscK metrics on polarized manifolds (i.e. \([\omega] = c_1(L)\) for some holomorphic line bundle \(L\)) without nonzero holomorphic vector fields. This was further extended by Chen and Tian [7] to cscK metrics on any compact Kähler manifold. Later a simpler proof of the lower boundedness of the K-energy for polarized cscK manifolds was provided by Chen and Sun [6], and more recently an even shorter proof was found by Li [15].

The main theorem that we want to prove is the following.

Theorem 1.1. Let \((X, J', L')\) be a polarized complex manifold that admits a constant scalar curvature Kähler metric in the class \(c_1(L')\). If \((X, J, L)\) is a sufficiently small polarized deformation of \((X, J', L')\), then the K-energy in the class \(c_1(L)\) on \((X, J)\) is bounded below.

In this situation it follows from Székelyhidi’s deformation result [18] that there exists a smooth test configuration (see section 2 for definitions) with generic fiber \((X, J, L)\) and central fiber with admits a cscK metric in \(c_1(L)\) (for a proof, see Proposition 6 in [19]). Then Theorem 1.1 follows immediately from the following:
Theorem 1.2 (X.X. Chen [5]). If $\mathcal{L} \to \mathcal{X} \to \mathcal{C}$ is a smooth test configuration with central fiber that admits a cscK metric in $c_1(\mathcal{L})$, then on the generic fiber the K-energy in the class $c_1(\mathcal{L})$ is bounded below.

In fact, we can compute the infimum of the K-energy in $c_1(L)$ by looking at certain smooth paths of Kähler potentials $\varphi_t$ that converge modulo diffeomorphisms to a cscK metric on the central fiber. The infimum of the K-energy on the generic fiber is then equal to the limit of the K-energy along the path $\varphi_t$ when $t$ goes to infinity.

Notice that if the central fiber is not biholomorphic to the generic fiber, then the generic fiber is not K-stable [9] (since the central fiber has vanishing Futaki invariant [14]), and hence it does not admit cscK metrics (at least if it does not have nonzero holomorphic vector fields [11, 17]).

There is an explicit example of such a test configuration, where the central fiber is the Mukai-Umemura threefold [12], and the generic fiber is Tian’s unstable deformation of it [20, 13]. This was the first example of a Kähler manifold that has a Kähler class with K-energy bounded below but without cscK metrics.

The proof of Theorem 1.2 follows closely the arguments of Theorem 1.7 in [5], except that we avoid using a result of Arezzo and Tian [1] and therefore we do not have to explicitly use weak geodesics in the space of Kähler potentials (these are still needed to prove (3.3) below). The key to this simplification is (3.4), which holds for all the paths we consider and not just for geodesics.

This note is organized as follows: in section 2 we will provide the basic setup concerning smooth test configurations, and in section 3 we will prove Theorem 1.2.

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2. Setup

A smooth test configuration, as defined by Donaldson in [9], is the following data.

- A holomorphic proper submersion $\pi: \mathcal{X} \to \mathcal{C}$ with $\mathcal{X}$ a complex manifold of dimension $n + 1$, with a line bundle $\mathcal{L} \to \mathcal{X}$ ample on all fibers of $\pi$.
- An embedding $\mathcal{X} \subset \mathbb{P}^N \times \mathcal{C}$ with $\mathcal{L}$ equal to the pullback of the hyperplane bundle and so that this embedding composed with the second projection equals $\pi$. 

An action of $\mathbb{C}^*$ on $\mathbb{P}^N$ via a 1-parameter subgroup $\rho : \mathbb{C}^* \to GL(N + 1, \mathbb{C})$, which is extended to an action on $\mathbb{P}^N \times \mathbb{C}$ by acting on the second factor using the product in $\mathbb{C}$, and so that $X$ is $\mathbb{C}^*$-invariant and all the maps are $\mathbb{C}^*$-equivariant.

In this case, the complex $n$-manifolds $X_\lambda = \pi^{-1}(\lambda)$ with polarization $L_\lambda = \mathcal{L}|_{X_\lambda}$ are all biholomorphic to a fixed polarized manifold $(X, J, L)$ when $\lambda \neq 0$, while the central fiber $(X_0, J_0, L_0)$ is a complex manifold diffeomorphic to $X$ but usually not biholomorphic to it. Since $c_1(L_0) = c_1(L_\lambda)$ we will identify all these bundles (as complex line bundles), and just call them $L$.

Since we assume that $\pi$ is a submersion, by Ehresmann’s theorem we conclude that the family $X$ is differentiably trivial, so that there is a diffeomorphism

$$F : X \times \mathbb{C} \to X,$$

such that $\pi(F(z, \lambda)) = \lambda$. We can think of $F$ as a family of maps

$$F_\lambda : X \to X, \ \lambda \in \mathbb{C},$$

which are diffeomorphisms with the image $X_\lambda$. Moreover, from the construction of $F$ in Ehresmann’s theorem, we can assume that $F$ extends a given diffeomorphism of the central fiber, and so we may assume that $F_0$ is a biholomorphism between $X$ with the complex structure $J_0$ and its image inside $X$.

We can also define a different trivialization of $X$ over $\mathbb{C}^*$ using the 1-parameter subgroup $\rho$. This acts on $X \subset \mathbb{P}^N \times \mathbb{C}$ as

$$\rho(\lambda) \cdot (z, \lambda') = (\rho(\lambda)(z), \lambda \lambda'), \ \lambda \in \mathbb{C}^*.$$

In particular, if $z$ is in $X_1$ (the fiber over 1) then $\rho(\lambda)(z)$ is in $X_\lambda$, and $\rho(\lambda)$ gives a biholomorphism between $X_1$ and $X_\lambda$ that preserves $L$.

We get a holomorphic map (which is biholomorphic with its image)

$$\rho : X \times \mathbb{C}^* \to X,$$

by sending $(z, \lambda)$ to $\rho(\lambda)(z)$, which is a holomorphic trivialization of the family over $\mathbb{C}^*$ and satisfies $\pi(\rho(z, \lambda)) = \lambda$.

Comparing the two trivializations $F$ and $\rho$, we see that there exists a diffeomorphism

$$f : X \times \mathbb{C}^* \to X \times \mathbb{C}^*$$

such that $F = \rho \circ f$ on $X \times \mathbb{C}^*$. To say this differently, the map $f$ is of the form

$$(z, \lambda) \mapsto (f_\lambda(z), \lambda),$$

where $f_\lambda : X \to X$ is a family of diffeomorphisms. Notice that when $\lambda$ approaches zero, the maps $f_\lambda$ and $\rho_\lambda$ are badly behaved, but their composition

$$F_\lambda = \rho_\lambda \circ f_\lambda$$

has a perfectly nice limit $F_0$.

From now on, we will consider the $S^1$ action on $X$ given by restricting $\rho$ to the circle. Suppose that we have an $S^1$-invariant Kähler metric $\Omega$ on $X$. 

with cohomology class $c_1(\mathcal{L})$, and so that the $S^1$-action is Hamiltonian with moment map $H : \mathcal{X} \to \mathbb{R}$. Recall that this means that if $V$ is the (smooth) vector field on $\mathcal{X}$ that generates the $S^1$-action, then $\iota_V \Omega = dH$.

If we denote by $V_C$ the holomorphic vector field on $\mathcal{X}$ generating the $C^*$-action, then we have that $V = \text{Im} V_C = \frac{1}{2}(V_C - \overline{V_C})$. If we denote by $J$ the complex structure of $\mathcal{X}$ then we have $J V = \text{Re} V_C = \frac{1}{2}(V_C + \overline{V_C})$.

Moreover, from the definition of test configuration, the pushforward $\pi_*(\mathcal{C})$ is equal to the vector field generating the standard action of $C^*$ on $\mathcal{X}$ by multiplication, i.e. $\pi_*(V_C) = z \frac{\partial}{\partial z}$. If we consider its real part $\pi_*(J V)$, then we can explicitly compute that its flow on $C$ is given by $z(t) = e^t z(0)$. It follows that the flow of the vector field $-J V$ on $\mathcal{X}$ is simply given by $\rho_{e^{-t}}$.

If we now let

$$\omega_t = \rho_{e^{-t}}^* \Omega, \ 0 \leq t < \infty,$$

then since $\rho_{e^{-t}}$ is holomorphic we see that $\omega_t$ are Kähler metrics on $(X, J)$ cohomologous to $c_1(L)$. Moreover, if we modify them by the diffeomorphisms $f_{e^{-t}}$ we get Riemannian metrics on $X$

$$f_{e^{-t}}^* \omega_t = f_{e^{-t}}^* \rho_{e^{-t}}^* \Omega = F_{e^{-t}}^* \Omega,$$

which satisfy

$$(2.1) \quad \| f_{e^{-t}}^* \omega_t - F_0^* \Omega \|_{C^k(g)} = \| F_{e^{-t}}^* \Omega - F_0^* \Omega \|_{C^k(g)} < C_k e^{-t},$$

for any $k, t$ (here $g$ is any fixed reference Riemannian metric on $X$ and $C_k$ are constants that depend only on $k$ and on the geometry of $\mathcal{X}, \Omega, \pi$). This is because we are pulling back the fixed Kähler metric $\Omega$ on the ambient space $\mathcal{X}$ via the maps $F_{e^{-t}}$ that converge smoothly exponentially fast to $F_0$.

There is another interesting observation to make. From the definition of Lie derivative we see that

$$\frac{\partial}{\partial t} \omega_t = \rho_{e^{-t}}^* L_{-J V} \Omega = -\rho_{e^{-t}}^* d(\iota_{J V} \Omega) = \rho_{e^{-t}}^* d(\iota_V \Omega) = \rho_{e^{-t}}^* d(J \Omega) = \sqrt{-1} \rho_{e^{-t}}^* \partial \bar{\partial} \Omega = \sqrt{-1} \partial \bar{\partial} \rho_{e^{-t}}^* \Omega.$$

On the other hand, if we fix a reference Kähler metric $\omega$ on $X$ in $c_1(L)$, then we can write $\omega_t = \omega + \sqrt{-1} \partial \bar{\partial} \varphi_t$ for some potentials $\varphi_t$, which are defined only up to addition of a time-dependent constant. Then we see that

$$\frac{\partial}{\partial t} \omega_t = \sqrt{-1} \partial \bar{\partial} \varphi_t,$$

and so we must have that

$$\varphi_t = \rho_{e^{-t}}^* H + c_t,$$

where $c_t$ is a time-dependent constant, that we absorb in $\varphi_t$ by changing the normalization of $\varphi_t$. We can then assume that $c_t = 0$ and pulling this back via the diffeomorphisms $f_{e^{-t}}$ we get

$$f_{e^{-t}}^* \varphi_t = f_{e^{-t}}^* \rho_{e^{-t}}^* H = F_{e^{-t}}^* H.$$
Since \( F_{e^{-t}H} \) approaches \( F_0^*H \) exponentially fast we see that
\[
|f_{e^{-t}H}^* - F_0^*| < Ce^{-t}.
\]
Recall now that on any compact Kähler manifold \((X,\omega)\) the Calabi energy of a Kähler potential \(\varphi\) is defined by
\[
Ca(\varphi) = \int_X (R(\omega_\varphi) - \overline{R})^2 \omega_\varphi^n,
\]
where \(R(\omega_\varphi)\) is the scalar curvature of \(\omega_\varphi = \omega + \sqrt{-1}\partial\bar{\partial}\varphi\) and \(\overline{R}\) is its average (using the volume form \(\omega_\varphi^n\)), while the K-energy of \(\varphi\) is defined by
\[
K_\omega(\varphi) = \int_0^1 \int_X \dot{\varphi}(\overline{R} - R(\omega_t))\omega_t^n,
\]
where \(\varphi_t, 0 \leq t \leq 1\), is any smooth path of Kähler potentials with \(\varphi_0 = 0\) and \(\varphi_1 = \varphi\).

We can now study the behavior of the Calabi energy and of the K-energy along our path \(\omega_t\). For the Calabi energy, note that
\[
Ca(\varphi_t) = \int_X (R(\omega_t) - \overline{R})^2 \omega_t^n = \int_X (R(f_{e^{-t}H}^*\omega_t) - \overline{R})^2 dV_{f_{e^{-t}H}^*\omega_t},
\]
which thanks to (2.1) converges exponentially fast to
\[
\int_X (R(F_0^*\Omega) - \overline{R})^2 (F_0^*\Omega)^n,
\]
which is the Calabi energy of the Kähler metric \(F_0^*\Omega\) on \((X,J_0)\) (here \(\overline{R}\) denotes the average of the scalar curvature, and \(R(f_{e^{-t}H}^*\omega_t)\) is the scalar curvature of the Riemannian metric \(f_{e^{-t}H}^*\omega_t\) and \(dV_{f_{e^{-t}H}^*\omega_t}\) its volume form).

As for the K-energy, its derivative satisfies
\[
\frac{d}{dt}K_\omega(\varphi_t) = \int_X \dot{\varphi}(\overline{R} - R(\omega_t))\omega_t^n = \int_X (f_{e^{-t}H}^*\dot{\varphi})(\overline{R} - R(f_{e^{-t}H}^*\omega_t))dV_{f_{e^{-t}H}^*\omega_t},
\]
which thanks to (2.1) and (2.2) converges exponentially fast to
\[
\int_X (F_0^*H)(\overline{R} - R(F_0^*\Omega))(F_0^*\Omega)^n.
\]
But this is just the Futaki invariant of the vector field \(V\) on the central fiber \((X,J_0)\), which is zero because \((X,J_0)\) admits a cscK metric [14]. So the derivative of the K-energy decays to zero exponentially fast.

3. Proof of Theorem 1.2

As in Theorem 1.2 we assume that the central fiber admits a cscK metric in \(c_1(L)\). The first step of the proof is to construct a Kähler metric \(\Omega\) on \(X\) as in the previous section. First of all we claim that it will be sufficient to construct \(\Omega\) only on a small neighborhood of the central fiber, since the only difference that this will make is that the family \(\omega_t\) constructed above will only be defined for \(t\) sufficiently large (which is enough for all the arguments).
So first we consider the Kähler metric on $X$

$$\Omega_1 = (\omega_{FS} + \sqrt{-1}dt \wedge d\bar{t})|_X,$$

where $\omega_{FS}$ is a Fubini-Study metric on $\mathbb{P}^N$ and $\sqrt{-1}dt \wedge d\bar{t}$ is the flat metric on $\mathbb{C}$. Notice that $\Omega_1$ is clearly $S^1$-invariant and moreover that the $S^1$-action is Hamiltonian (since this is true for $\omega_{FS}$ and trivially also for the flat metric). The cohomology class of $\Omega_1$ is $c_1(\mathcal{L})$.

Since the central fiber admits a cscK metric in $c_1(L)$, it follows that there is a Kähler potential $\psi$ on $(X,J_0)$ so that $F_0^*\Omega_1 + \sqrt{-1}\partial\bar{\partial}\psi$ is cscK. Then we just extend $\psi$ to a smooth $S^1$-invariant function $\tilde{\psi}$ on a neighborhood of the central fiber, and by choosing the neighborhood small enough we can ensure that $\Omega_1 + \sqrt{-1}\partial\bar{\partial}\tilde{\psi}$ is Kähler when restricted to nearby fibers. We then let

$$\Omega = \Omega_1 + \sqrt{-1}\partial\bar{\partial}\tilde{\psi} + C\sqrt{-1}dt \wedge d\bar{t},$$

for some large constant $C$, so that $\Omega$ is Kähler in a small neighborhood of the central fiber. By construction $\Omega$ is also $S^1$-invariant, the action is Hamiltonian, and the cohomology class of $\Omega$ is $c_1(\mathcal{L})$.

We also have that $F_0^*\Omega$ has constant scalar curvature. If we let $\omega_t = \rho_{e^{-t}}^*\Omega$ as before (for $t$ sufficiently large), then it follows that both the Calabi energy and the derivative of the K-energy of $\omega_t$ decay to zero exponentially fast when $t$ goes to infinity.

At this point we need the following inequality of X.X. Chen [4], which in the case of polarized manifolds has a simpler proof due to Chen and Sun [6] (see also Berndtsson [3]). It says that for any two Kähler potentials $\varphi, \psi$ for a Kähler metric $\omega$, connected by a piecewise smooth path $\varphi_t$ of potentials with $0 \leq t \leq T$, $\varphi_0 = \varphi$, $\varphi_T = \psi$, we have

$$(3.3) \quad K_\omega(\psi) - K_\omega(\varphi) \leq \sqrt{C_{a(\psi)}} \int_0^T \int_X \varphi_{\varphi_t}^2 \omega_{\varphi_t}^n dt,$$

where we are using the obvious notation for piecewise smooth (but not smooth) paths.

We now have all the ingredients to complete the proof of Theorem 1.2. As before $\omega$ is a reference Kähler metric on $(X,J)$ cohomologous to $c_1(L)$, and let $\varphi$ be any Kähler potential for $\omega$. We wish to prove a uniform lower bound for $K_\omega(\varphi)$, independent of $\varphi$. Take the family of metrics $\omega_t$ constructed above, with $\omega_t = \omega + \sqrt{-1}\partial\bar{\partial}\varphi_t$, $t \geq t_0$. Notice that this family does not depend on $\varphi$. We connect the potentials $\varphi$ and $\varphi_{t_0}$ with a smooth path $\varphi_t$ with $0 \leq t \leq t_0$ with $\varphi_0 = \varphi$. Concatenating these two paths we get a piecewise smooth path $\varphi_t$ with $t \geq 0$ and we can apply (3.3) to get

$$K_\omega(\varphi) \geq K_\omega(\varphi_t) - \sqrt{C_{a(\varphi_t)}} \int_0^t \int_X \varphi_{\varphi_s}^2 \omega_{\varphi_s}^n ds,$$
for \( t \geq t_0 \), say. First of all, since the derivative of \( K_\omega(\varphi_t) \) decays exponentially fast, we see that

\[
K_\omega(\varphi_t) = K_\omega(\varphi_{t_0}) + \int_{t_0}^{t} \frac{d}{ds} K_\omega(\varphi_s) ds \geq -C - C \int_{t_0}^{t} e^{-s} ds \geq -C_0,
\]

for a uniform constant \( C_0 \) independent of \( \varphi \) and \( t \).

Secondly, we split

\[
\int_{0}^{t} \sqrt{\int_{X} \dot{\varphi}_s^2 \omega^n_{\varphi_s}} ds = \int_{0}^{t_0} \sqrt{\int_{X} \dot{\varphi}_s^2 \omega^n_{\varphi_s}} ds + \int_{t_0}^{t} \sqrt{\int_{X} \dot{\varphi}_s^2 \omega^n_{\varphi_s}} ds,
\]

and we can bound the second term by using

\[
\int_{X} \dot{\varphi}_s^2 \omega^n_{\varphi_s} = \int_{X} (f_{e^{-s}\varphi_s})^2 dV_{f_{e^{-s}\varphi_s}} \leq C,
\]

for some constant \( C \) independent of \( s \) and \( \varphi \), because of (2.2). It follows that

\[
(3.4) \quad \int_{t_0}^{t} \sqrt{\int_{X} \dot{\varphi}_s^2 \omega^n_{\varphi_s}} ds \leq Ct.
\]

The first term \( \int_{0}^{t_0} \sqrt{\int_{X} \dot{\varphi}_s^2 \omega^n_{\varphi_s}} ds \) depends on the initial potential \( \varphi \), but is a fixed number independent of \( t \). On the other hand the term \( \sqrt{Ca(\varphi_t)} \) decays to zero exponentially fast, and so we get

\[
K_\omega(\varphi) \geq -C_0 - \left( Ct + \int_{0}^{t_0} \sqrt{\int_{X} \dot{\varphi}_s^2 \omega^n_{\varphi_s}} ds \right) C e^{-t/2},
\]

for all \( t \geq t_0 \). But since the term \( \int_{0}^{t_0} \sqrt{\int_{X} \dot{\varphi}_s^2 \omega^n_{\varphi_s}} ds \) and the LHS of the inequality are independent of \( t \), we can let \( t \) go to infinity and get

\[
K_\omega(\varphi) \geq -C_0,
\]

which is what we want.

Finally, we can compute the infimum of \( K_\omega(\varphi) \) over all \( \text{Kähler potentials} \ \varphi \) as follows. We take \( \varphi_t, t \geq t_0 \), to be the path constructed above. Notice that since \( \frac{d}{dt} K_\omega(\varphi_t) \) decays exponentially fast, the limit \( K = \lim_{t \to \infty} K_\omega(\varphi_t) \) exists and is finite. The proof of Theorem 1.2 that we have just finished, replacing \( -C_0 \) by \( K \), shows that for any \( \text{Kähler potential} \ \varphi \) for \( \omega \) we have

\[
K_\omega(\varphi) \geq K,
\]

and picking \( \varphi = \varphi_t \) we immediately see that

\[
\inf_{\varphi} K_\omega(\varphi) = K.
\]

It follows then that if we use another path \( \varphi_t \) (still constructed as above) we get the same number \( K \), even if in the construction of \( \varphi_t \) we use different cscK metrics on the central fiber.
It would be interesting to see if one gets the same number $K$ for any path $\varphi_t$ such that the metrics $\omega + \sqrt{-1} \partial \bar{\partial} \varphi_t$ converge modulo diffeomorphisms to some cscK metric.

REFERENCES

[1] Arezzo, C., Tian, G. Infinite geodesic rays in the space of Kähler potentials, Ann. Sc. Norm. Super. Pisa Cl. Sci. (5) 2 (2003), no. 4, 617–630.
[2] Bando, S., Mabuchi, T. Uniqueness of Einstein Kähler metrics modulo connected group actions in Algebraic geometry, Sendai, 1985, 11–40, Adv. Stud. Pure Math., 10, North-Holland, Amsterdam, 1987.
[3] Berndtsson, B. Probability measures related to geodesics in the space of Kähler metrics, arXiv:0907.1806
[4] Chen, X.X. Space of Kähler metrics. III. On the lower bound of the Calabi energy and geodesic distance, Invent. Math. 175 (2009), no. 3, 453–503.
[5] Chen, X.X. Space of Kähler metrics (IV)–On the lower bound of the K-energy, arXiv:0809.4081
[6] Chen, X.X., Sun, S. Space of Kähler metrics (V)–Kähler quantization, arXiv:0902.4149
[7] Chen, X.X., Tian, G. Geometry of Kähler metrics and foliations by holomorphic discs, Publ. Math. Inst. Hautes Études Sci. 107 (2008), 1–107.
[8] Donaldson, S.K. Scalar curvature and projective embeddings, I, J. Differential Geom. 59 (2001), no. 3, 479–522.
[9] Donaldson, S.K. Scalar curvature and stability of toric varieties, J. Differential Geom. 62 (2002), no. 2, 289–349.
[10] Donaldson, S.K. Scalar curvature and projective embeddings, II, Q. J. Math. 56 (2005), no. 3, 345–356.
[11] Donaldson, S.K. Lower bounds on the Calabi functional, J. Differential Geom. 70 (2005), no. 3, 453–472.
[12] Donaldson, S.K. A note on the $\alpha$-invariant of the Mukai-Umemura 3-fold, arXiv:0711.4357
[13] Donaldson, S.K. Kähler geometry on toric manifolds, and some other manifolds with large symmetry, in Handbook of geometric analysis. No. 1, 29–75, Adv. Lect. Math. (ALM), 7, Int. Press, Somerville, MA, 2008.
[14] Futaki, A. On compact Kähler manifolds of constant scalar curvatures, Proc. Japan Acad. Ser. A Math. Sci. 59 (1983), no. 8, 401–402.
[15] Li, C. Constant scalar curvature Kähler metric and K-energy, arXiv:0910.0421
[16] Phong, D.H., Sturm, J. Lectures on stability and constant scalar curvature, in Current developments in mathematics, 2007, 101–176, Int. Press, Somerville, MA, 2009.
[17] Stoppa, J. K-stability of constant scalar curvature Kähler manifolds, Adv. Math. 221 (2009), no. 4, 1397–1408.
[18] Székelyhidi, G. The Kähler-Ricci flow and K-stability, Amer. J. Math. 132 (2010), 1077–1090.
[19] Székelyhidi, G. Greatest lower bounds on the Ricci curvature of Fano manifolds, arXiv:0903.5504 to appear in Compositio Math.
[20] Tian, G. Kähler-Einstein metrics with positive scalar curvature, Invent. Math. 130 (1997), no. 1, 1–37.
[21] Yau, S.-T. On the Ricci curvature of a compact Kähler manifold and the complex Monge-Ampère equation, I, Comm. Pure Appl. Math. 31 (1978), no.3, 339–411.

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