Existence of Solutions to Mean Field Equations on Graphs

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Abstract

In this paper, we prove two existence results of solutions to mean field equations

$$\Delta u + e^u = \rho \delta_0$$

and

$$\Delta u = \lambda e^u (e^u - 1) + 4\pi \sum_{j=1}^M \delta_{p_j}$$

on an arbitrary connected finite graph, where $\rho > 0$ and $\lambda > 0$ are constants, $M$ is a positive integer, and $p_1, ..., p_M$ are arbitrarily chosen vertices on the graph.

1 Introduction

The mean field equation

$$\Delta u + e^u = \rho \delta_0$$ (1.1)

has its origin in the prescribed curvature problem in geometry. Closely related is the Kardan-Warner equation [9]

$$\Delta u + he^u = c.$$ (1.2)

The name of the equation (1.1) comes from statistical physics as the mean field limits of the Euler flow [1]. It has also been shown to be related to the Chern-Simons-Higgs model. The existence of solutions to equation (1.1) has been studied in [3], [4], [10], [11] on Euclidean spaces and on the two dimensional flat tori. For example, on the two dimensional flat tori, when $\rho \neq 8m\pi$ for any $m \in \mathbb{Z}$, equation (1.1) always has solutions, see [3], [4]. When $\rho = 8\pi$, it was shown in [10] that equation (1.1) has solutions if and only if the Green’s function on the two dimensional flat tori has critical points other than the three half period points.

In [5], Grigorigan, Lin and Yang have obtained a few sufficient conditions when equation (1.2) has a solution on a finite graph. There are several further results regarding the solutions of (1.2) on graphs in [6], [7], [8].

In this paper, we study equation (1.1) and also the following mean field equation on graphs:
\[ \Delta u = \lambda e^u(e^u - 1) + 4\pi \sum_{j=1}^{M} \delta_{p_j}, \quad (1.3) \]

where \( \lambda > 0 \), \( M \) is any fixed positive integer, and \( p_1, \ldots, p_M \) are arbitrarily chosen vertices on the graph.

Caffarelli and Yang in [2] proved an existence result of solutions to equation (1.3) on doubly periodic regions in \( \mathbb{R}^2 \) (the 2-tori), depending on the value of the parameter \( \lambda \).

In this paper, we show that equation (1.1) always has a solution on any connected finite graph (Theorem 2.1), in contrast to the continuous case. We shall also prove an existence result for equation (1.3) on a connected finite graph (Theorem 2.2), depending on the value of the parameter \( \lambda \), which is in line with the result of Caffarelli and Yang on the 2-tori.

We obtain these results by a mostly straightforward adaption of existing treatments from the continuous case [9], [2], [5]. Once we have the setup, some analysis tend to simplify on finite graphs since there is only a finite number of degrees of freedom. Theorem 2.1 on the other hand shows that the existence of solutions for (1.1) on the discrete two dimensional tori graph given as the quotient of the two dimensional infinite lattice graph by a rank 2 sublattice, differs from that on the continuous limit – the two dimensional flat tori, when the parameter \( \rho \) takes on certain special values such as \( 8\pi \).

**Remark 1.** As a side remark, it appears interesting to study the Green’s function on the 2-tori by studying the corresponding discrete Green’s function on the 2-tori graph stated above. For example, when the torus parameter \( \tau = \frac{1}{2} + i \), there exist two additional critical points of the Green’s function besides the half periods by [10]. A computer study aided by this discrete Green’s function indicates that the slope of the line through these two additional critical points of the Green’s function is equal to \( \frac{25}{64} \).

### 2 Settings and main results

Let \( G = (V, E) \) be a connected finite graph, where \( V \) is the set of vertices and \( E \) is the set of edges. Denote \( N = |V| \). We allow positive symmetric weights \( w_{xy} = \omega_{yx} \) on edges \( xy \in E \). Let \( \mu : V \to \mathbb{R}^+ \) be a finite measure. For any function \( u : V \to \mathbb{R} \), the Laplace operator acting on \( u \) is defined by

\[ \Delta u(x) = \frac{1}{\mu(x)} \sum_{y \sim x} w_{xy}(u(y) - u(x)), \]

where \( y \sim x \) means \( xy \in E \). The gradient form of \( u \) is by definition

\[ \Gamma(u) = \frac{1}{2} \int_V |\nabla u|^2 := \sum_{x \in V} \frac{1}{2\mu(x)} \sum_{y \sim x} w_{xy}(u(y) - u(x))^2. \]

We use the notation \( \int_V f(x)d\mu(x) = \sum_{x \in V} f(x)\mu(x) \). As in [5], we define a Sobolev Space and a norm by

\[ W^{1,2}(V) = \{ u : V \to \mathbb{R} : \int_V (|\nabla u|^2 + u^2)d\mu < +\infty \} \]
respectively. Since \( V \) is a finite graph, \( W^{1,2}(V) \) is \( R \), the finite dimensional vector space of all real functions on \( V \). We have the following Sobolev embedding (Lemma 5 in [5]):

**Lemma 2.1.** Let \( G = (V, E) \) be a finite graph. The Sobolev Space \( W^{1,2}(V) \) is precompact. Namely, if \( \{u_j\} \) is bounded in \( W^{1,2}(V) \), then there exits some \( u \in W^{1,2}(V) \) such that up to a subsequence, \( u_j \to u \) in \( W^{1,2}(V) \).

**Remark 2.** For finite graphs, Lemma 2.1 can be avoided for the purpose of the present paper. But we include it for potential generalizations to infinite graphs.

By using the variational principle (see the similar approach in [9] and [5]), we prove the following

**Theorem 2.1.** Equation (1.1) has a solution on \( G \).

Using an iteration method, we next prove the following

**Theorem 2.2.** There is a critical value \( \lambda_c \) depending on \( G \) satisfying

\[
\lambda_c \geq \frac{16\pi M}{|V|},
\]

such that when \( \lambda > \lambda_c \), the equation (1.3) has a solution on \( G \), and when \( \lambda < \lambda_c \), the equation (1.3) has no solution.

3 The proof of Theorem 2.1

**Proof.** For \( u \in W^{1,2}(V) \), we consider the functional

\[
J(u) = \frac{1}{2} \int_V |\nabla u|^2 + \int_V \rho \cdot \delta_0 \cdot u.
\]

Let the set

\[
B = \{ u \in W^{1,2}(V) : \int_V e^u = \int_V \rho \cdot \delta_0 = \rho \}.
\]

First we verify \( B \neq \emptyset \): let \( l > 0 \), define

\[
u_l(x) = \begin{cases} e^l, & x = x_0 \\ 0, & \text{otherwise} \end{cases}
\]

and

\[
\tilde{u}_l(x) = \begin{cases} e^l, & x = x_0 \\ 0, & \text{otherwise} \end{cases}
\]
\[ \int_V e^{u_l} = e^l \to +\infty \ (l \to +\infty), \]

and

\[ \int_V e^{\tilde{u}_l} = e^{-l} \to 0 \ (l \to +\infty), \]

Let \( \Phi(t) = \int_V e^{u_l+(1-t)\tilde{u}_l} \), then for sufficiently big \( l \),

\[ \Phi(0) < \rho < \Phi(1), \]

so there exists \( t \in (0, 1) \) such that \( \Phi(t) = \rho \), therefore \( B \neq \emptyset \).

For any \( u \in B \), \( \int_V e^{u_l} = \rho \), choose \( x_D \in V \) such that

\[ e^{u(x_D)} = \min_{x \in V} \{e^{u(x)}\}, \]

then

\[ Ne^{u(x_D)} \leq \rho, \]

\[ u(x_D) \leq \log \frac{\rho}{N}. \]

Choose a shortest path on \( G \) from \( x_0 \) to \( x_D \) (therefore non-backtracking): \( x_0 \sim x_1 \sim \ldots \sim x_D \), fix any \( 0 < \epsilon < 1 \),

\[ \frac{1}{2} \int_V |\nabla u|^2 \geq \sum_{x \in V} \frac{1}{2\mu(x)} \sum_{y \sim x} \omega_{xy}(u(y) - u(x))^2 \]

\[ \geq D_{eg} \left[ (u(x_1) - u(x_0))^2 + (u(x_2) - u(x_1))^2 + \ldots + (u(x_D) - u(x_{D-1}))^2 \right] \]

\[ \geq D_{eg} \frac{(u(x_0) - u(x_D))^2}{D} \]

\[ \geq \frac{D_{eg}}{D} \cdot (u(x_0) - \log \frac{\rho}{N})^2 \]

where \( D_{eg} = \min_{x \in V, y \sim x} \omega_{xy} \frac{\omega_{xy}}{2\mu(x)} \).

So there exists \( c > 0 \) depending on \( \epsilon \), such that when \( u(x_0) \geq c, \frac{1}{2} \int_V |\nabla u|^2 \geq M_1 \cdot |u(x_0)| \)

for some \( M_1 \geq \frac{\epsilon}{\epsilon} \). Therefore we have in this case

\[ J(u) \geq (1 - \epsilon) \cdot \frac{1}{2} \int_V |\nabla u|^2. \quad (3.1) \]

When \( |u(x_0)| < c \),

\[ J(u) > \frac{1}{2} \int_V |\nabla u|^2 - \rho c. \quad (3.2) \]

Therefore \( J(u) \) has a lower bound on \( B \). So we can choose

\[ u_k(x) \in B, J(u_k(x)) \to b \quad (k \to \infty), \]
where \( b = \inf_{u \in B} J(u) \).

From (3.1) and (3.2), for all \( k \),
\[
\int_V |\nabla u_k|^2 \leq c_1
\]
for some constant \( c_1 \), since \( |J(u_k)| \leq c_2 \) for some constant \( c_2 \). As
\[
J(u_k) = \frac{1}{2} \int_V |\nabla u_k|^2 + \rho \cdot u_k(x_0),
\]
there exists a constant \( c' \), such that \( |u_k(x_0)| \leq c' \) for all \( k \). For any \( x \in V \), choose a shortest path on \( G \) from \( x_0 \) to \( x \):
\[
x_0 \sim x_1 \sim \ldots \sim x_{D'-1} \sim x,
\]
\[
|u_k(x)| \leq |u_k(x) - u_k(x_{D'-1})| + |u_k(x_{D'-1}) - u_k(x_{D'-2})| + \ldots + |u_k(x_1) - u_k(x_0)| + |u_k(x_0)|
\]
\[
\leq D \cdot \left( |u_k(x) - u_k(x_{D'-1})|^2 + \ldots + |u_k(x_1) - u_k(x_0)|^2 \right)^{1/2} + c'
\]
\[
\leq \frac{D'}{D_{eg}} \int_V |\nabla u_k|^2 + c'
\]

As \( D' \leq N \), the \( L^\infty \) norm of \( u_k(x) \) is uniformly bounded, and therefore \( u_k(x) \) are uniformly bounded in \( W^{1,2}(V) \). From the Sobolev embedding (Lemma 2.1), there exits a subsequence \( u_{k_1}(x) \rightarrow u_\infty(x) \in W^{1,2}(V) \) in \( W^{1,2}(V) \), and
\[
\int_V e^{u_\infty} = \lim_{k_1 \rightarrow \infty} \int_V e^{u_{k_1}} = \rho.
\]

Finally we prove that \( u_\infty \) is the solution of equation (1.1). This is based on the method of Lagrange multiplies. Let
\[
L(t, \lambda) = \frac{1}{2} \int_V |\nabla (u_\infty + t\varphi)|^2 + \int_V \rho \cdot \delta_0(u_\infty + t\varphi) + \lambda(- \int_V e^{u_\infty + t\varphi} + \rho),
\]
where \( \varphi \in W^{1,2}(V) \). So we have
\[
\frac{\partial L}{\partial \lambda} |_{t=0} = - \int_V e^{u_\infty} + \rho = 0,
\]
since \( u_\infty \in B \). And
\[
0 = \frac{\partial L}{\partial t} |_{t=0} = - \int \Delta u_\infty \cdot \varphi + \int \rho \cdot \delta_0 \cdot \varphi - \lambda \int e^{u_\infty} \cdot \varphi = 0.
\]

Therefore by the variational principle,
\[
-\Delta u_\infty + \rho \cdot \delta_0 - \lambda \cdot e^{u_\infty} = 0.
\]
Since \( \int_V \Delta u_\infty = 0 \), we have
\[
\lambda \int_V e^{u_\infty} = \int_V \rho \cdot \delta_0 = \rho.
\]

So \( \lambda = 1 \), and
\[
\Delta u_\infty + e^{u_\infty} = \rho \cdot \delta_0.
\]

This finishes the proof of Theorem 2.1.
\[\Box\]
The proof of Theorem 2.2

We use the method of upper and lower solutions to prove Theorem 2.2, adapting methods from [9], [2] and [5] to the graph setting.

Lemma 4.1. (Maximum principle) Let \(G = (V, E)\), where \(V\) is a finite set, and \(K \geq 0\) is a constant. Suppose a real function \(u(x) : V \to \mathbb{R}\) satisfies
\[
(\Delta - K)u(x) \geq 0 \quad \text{for all} \ x \in V,
\]
then \(u(x) \leq 0\) for all \(x \in V\).

Proof. Let \(u(x_0) = \max_{x \in V} \{u(x)\}\), we only need to show that \(u(x_0) \leq 0\). Suppose this is not the case. Since
\[
(\Delta - K)u(x_0) \geq 0,
\]
we have
\[
\sum_{y \sim x} u(y) \geq (d_{x_0} + K)u(x_0) \geq d_{x_0}u(x_0),
\]
where we have used the assumption that \(u(x_0) > 0\), and that \(K \geq 0\) in the last inequality. This implies that for any \(y \sim x, u(y) \geq u(x_0)\). Since \(G\) is a connected graph, by induction, for any \(xy \in E, u(y) = u(x_0)\). From
\[
K \int_V u(x) \leq \int \Delta u(x) = 0
\]
and \(K \geq 0\) we get that \(u(x_0) \leq 0\). This is a contradiction.  

Let \(u_0\) be a solution of the Poisson equation
\[
\Delta u_0 = -\frac{4\pi M}{|V|} + 4\pi \sum_{j=1}^{M} \delta_{p_j}. \tag{4.1}
\]
The solution of (4.1) always exists, as the integral of the right side is equal to 0. Inserting \(u = u_0 + v\) into equation (1.3), we get
\[
\Delta v = \lambda e^{u_0+v}(e^{u_0+v} - 1) + \frac{4\pi M}{|V|}. \tag{4.2}
\]
Sum the two sides of the about equation, we get
\[
\lambda(e^{u_0+v} - \frac{1}{2})^2 = \lambda \frac{4\pi M}{|V|},
\]
which implies that
\[ \lambda \geq \frac{16\pi M}{|V|}. \]  
(4.3)

We call a function \( v_+ \) an upper solution of (4.2) if for any \( x \in V \), it satisfies
\[ \Delta v_+(x) \geq \lambda e^{u_0(x)+v_+(x)}(e^{u_0(x)+v_+(x)} - 1) + \frac{4\pi M}{|V|}. \]  
(4.4)

Let \( v_0 = -u_0 \), we define a sequence \( \{v_n\} \) by iterating for a constant \( K \geq 2\lambda \),
\[ (\Delta - K)v_n = \lambda e^{u_0 + v_{n-1}}(e^{u_0 + v_{n-1}} - 1) - Kv_{n-1} + \frac{4\pi M}{|V|}. \]  
(4.5)

We next prove that \( \{v_n\} \) is a monotone sequence and it converges to a solution of equation (4.2).

**Lemma 4.2.** Let \( \{v_n\} \) be a sequence defined by (4.5). Then
\[ v_0 \geq v_1 \geq v_2 \geq \ldots \geq v_n \geq \ldots \geq v_+ \]
for any upper solution \( v_+ \) of (4.2).

**Proof.** We prove the Lemma by induction. As \( v_0 = -u_0 \), for \( v_1 \) we have by (4.5),
\[ (\Delta - K)v_1 = Ku_0 + \frac{4\pi M}{|V|}. \]
Together with (4.1), we obtain
\[ (\Delta - K)(v_1 - v_0)(x) = 4\pi \sum_{j=1}^{M} \delta_{p_j}(x) \geq 0 \]
for any \( x \in V \), and
\[ K \int_V (v_1 - v_0) = -4\pi M < 0. \]
Therefore \( v_1 - v_0 \leq 0 \) by Lemma 4.1. Suppose that \( v_0 \geq v_1 \geq \ldots \geq v_k \) for \( k \geq 1 \). From (4.5) and \( K \geq 2\lambda \), we get
\[ (\Delta - K)(v_{k+1} - v_k) = \lambda e^{2u_0+2v_k} - \lambda e^{u_0+v_k} - K v_k - \lambda e^{2u_0+2v_{k-1}} + \lambda e^{u_0+v_{k-1}} + K v_k \]
\[ = \lambda e^{2u_0}(e^{2v_k} - e^{2v_{k-1}}) - \lambda e^{u_0}(e^{v_k} - e^{v_{k-1}}) - K(v_k - v_{k-1}) \]
\[ \geq \lambda e^{2u_0}(e^{2v_k} - e^{2v_{k-1}}) - K(v_k - v_{k-1}) \]
\[ = 2\lambda e^{2u_0+2v_0}(v_k - v_{k-1}) - K(v_k - v_{k-1}) \]
\[ \geq K(e^{2u_0+2v_0} - 1)(v_k - v_{k-1}) \]
\[ \geq 0. \]
Where \( v_k \leq v^* \leq v_{k-1} \leq v_0 \). Lemma 4.1 then implies that \( v_{k+1} - v_k \leq 0 \) on \( V \).
Next we prove that $v_k \geq v_+$ for any $k$. First consider the case $k = 0$. From (4.1) and (4.4),

\[
\Delta (v_+ - v_0) \geq \lambda e^{u_0 + v_+} (e^{u_0 + v_+ - 1}) + 4\pi \sum_{j=1}^{M} \delta_{p_j} \\
\geq \lambda e^{u_0 + v_+} (e^{u_0 + v_+ - 1}) \\
= \lambda e^{v_+ - v_0} (e^{v_+ - v_0} - 1).
\]

(4.6)

Let $v_+(x_0) - v_0(x_0) = \max_{x \in V} \{v_+(x_0) - v_0(x_0)\}$. We only need to prove that $v_+(x_0) - v_0(x_0) \leq 0$. Suppose not, then from (4.6) we have

\[
\Delta (v_+ - v_0)(x_0) > 0.
\]

which contradicts with the assumption that $x_0$ is a point where $v_+ - v_0$ attains maximum in $V$. Hence $v_+ - v_0 \leq 0$ in $V$. Now suppose that $v_+ \leq u_k$ for $k \geq 0$. From (4.3) and (4.5), we have

\[
(\Delta - K)(v_+ - v_{k+1}) = \lambda e^{2u_0}(e^{2v_+} - e^{2v_k}) - K(v_+ - v_k) - \lambda e^{u_0}(e^{v_+} - e^{v_k}) \\
\geq \lambda e^{2u_0}(e^{2v_+} - e^{2v_k}) - K(v_+ - v_k) \\
= 2\lambda e^{2u_0+2v^*}(v_+ - v_k) - K(v_+ - v_k) \\
\geq K(e^{2u_0+2v_0} - 1)(v_+ - v_k) \\
= 0,
\]

where $v_+ \leq v^* \leq v_k \leq v_0$. So Lemma 4.1 implies that $v_{k+1} \geq v_+$.

This finishes the proof of Lemma 4.2.

\[\square\]

**Lemma 4.3.** The equation (1.3) has a solution on $G$, when $\lambda$ is sufficiently big.

**Proof.** We only need to prove that equation (4.2) has an upper solution $v_+$. Suppose $u_0$ is a solution of (4.1). Choose $v_+ = -c'' < 0$ to be a constant function, where $-c''$ is sufficiently small such that $u_0 + v_+ < 0$ in $V$. Then $e^{u_0+v_+} - 1 < 0$. So we can choose $\lambda > 0$ big enough such that

\[
\lambda e^{u_0+v_+} (e^{u_0+v_+} - 1) + \frac{4\pi M}{|V|} < 0.
\]

Therefore

\[
0 = \Delta v_+ > \lambda e^{u_0+v_+} (e^{u_0+v_+} - 1) + \frac{4\pi M}{|V|}.
\]

So $v_+ \equiv -c$ is an upper solution of (4.2).

\[\square\]

**Lemma 4.4.** If $u$ is a solution of equation (1.3) on $G$, then $u < 0$ on $G$. 8
Proof. Let $u(x_0) = \max_{x \in V} \{u(x)\}$, we only need to show that $u(x_0) < 0$. Suppose $u(x_0) \geq 0$. Then $e^{u(x_0)} - 1 \geq 0$. From equation (1.3) we get that
\[
\Delta u(x_0) \geq 0,
\]
that is
\[
\sum_{y \sim x} u(y) \geq d_{x_0} u(x_0).
\]
This implies that for any $y \sim x, u(y) \geq u(x_0)$.
Since $G$ is a connected finite graph, by iterating the above process, we get that for any $y \in V, u(y) = u(x_0)$.

So the left side of equation (1.3) is 0 and the right side is positive on $p_j \in V$, which is a contradiction.

Now we prove Theorem 2.2, which is similar to the proof of Lemma 4 in [2].

Proof. Denote
\[
\Lambda = \{\lambda > 0 | \lambda \text{ is such that equation (1.3) has a solution}\}.
\]
We will show that $\Lambda$ is an interval. Suppose that $\lambda' \in \Lambda$. We need to prove that
\[
[\lambda', +\infty) \in \Lambda.
\]
In fact, let $u' = u_0 + v'$ is the solution of equation (1.3) at $\lambda = \lambda'$, where $v'$ is the corresponding solution of equation (4.2). Since
\[
u' = u_0 + v' < 0,
\]
we see that $v'$ is an upper solution of equation (4.2) for any $\lambda \geq \lambda'$. By Lemma 4.2, we obtain that $\lambda \in \Lambda$ as desired.

Set $\lambda_c = \inf \{\lambda | \lambda \in \Lambda\}$. Then $\lambda \geq \frac{16\pi M}{|V|}$ for any $\lambda > \lambda_c$ by (1.3) and that $\Lambda$ is an interval. Taking the limit, we get that
\[
\lambda_c \geq \frac{16\pi N}{|V|}.
\]

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