DISCRETE GAP PROBABILITIES AND
DISCRETE PAINLEVÉ EQUATIONS

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Abstract. We prove that Fredholm determinants of the form det(1 − K_s), where
K_s is the restriction of either the discrete Bessel kernel or the discrete 2F_1 kernel
to {s, s + 1, . . .}, can be expressed through solutions of discrete Painlevé II and V
equations, respectively.

These Fredholm determinants can also be viewed as distribution functions of the
first part of the random partitions distributed according to a poissonized Plancherel
measure and a z-measure, or as normalized Toeplitz determinants with symbols
ε^{(ζ+ζ^{-1})} and (1 + √ξζ)^z(1 + √ξ/ζ)^z'.

The proofs are based on a general formalism involving discrete integrable operators
and discrete Riemann–Hilbert problem. A continuous version of the formalism has
been worked out in [BD].

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Introduction

In recent years we have witnessed a discovery and an intensive study of a class
of discrete probabilistic models which in appropriate limits converge, in one way or
another, to well–known models of Random Matrix Theory (RMT, for short). The
sources of new models are quite diverse, they include Combinatorics, Representation
Theory, Percolation Theory, Growth Processes, tiling problems and others.

One quantity of interest in RMT is the gap probability — the probability of
having no particles–eigenvalues in a given interval. In particular, the level spac-
ing distribution and the distribution of the most right or left particle are easily
expressible in terms of gap probabilities. Naturally, gap probabilities also arise as
limits of relevant quantities of the discrete models mentioned above. In many cases these quantities can be viewed as gap probabilities for certain discrete random point processes.

It is well known that in random matrix models gap probabilities can often be expressed in terms of a solution of a 2nd order nonlinear ordinary differential equation which, quite remarkably, happens to be one of the six Painlevé equations, see, e.g., [TW2].

In this paper we show that the counterparts of gap probabilities in two important discrete models can be expressed through solutions of discrete analogs of Painlevé II and Painlevé V equations. We also develop a general formalism which allows us to handle these 2 examples, and which we expect to work in other models as well.

Let us describe our results.

Let $S_n$ be the symmetric group of degree $n$. Denote by $l_n(\sigma)$ the length of the longest increasing subsequence of a permutation $\sigma \in S_n$. Set

$$p_k^n = \frac{1}{n!} \text{Card}\{\sigma \in S_n \mid l_n(\sigma) \leq k\}, \quad p_k^{(\eta)} = e^{-\eta^2} \sum_{n=0}^{\infty} \frac{\eta^{2n}}{n!} p_k^n. \quad (0.1)$$

Here $\eta$ is a complex parameter.

There are many other ways to define $p_k^{(\eta)}$. For example, thanks to a result of [G], it can be defined using a Toeplitz determinant

$$p_k^{(\eta)} = e^{-\eta^2} \det[f_{i-j}]_{i,j=1}^k, \quad \sum_{m=-\infty}^{+\infty} f_m \zeta^m = e^{\eta(\zeta + \zeta^{-1})}. \quad (0.2)$$

A representation theoretic definition (which can be easily obtained using the Robinson–Schensted correspondence) has the form

$$p_k^{(\eta)} = e^{-\eta^2} \sum_{\lambda_1 \leq k} \left( \frac{\dim \lambda}{|\lambda|!} \eta^{\ell(\lambda)} \right)^2 \quad (0.3)$$

where the summation is taken over all partitions $\lambda = (\lambda_1 \geq \cdots \geq \lambda_l > 0)$ such that $\lambda_1 \leq k$, $|\lambda| = \lambda_1 + \cdots + \lambda_l$ is the size of the partition, and $\dim \lambda$ is the dimension of the irreducible representation of $S_{|\lambda|}$ corresponding to $\lambda$.

A Fredholm determinant representation of $p_k^{(\eta)}$ relevant for us will be given in §3 below.

**Theorem 1.** Let $\{x_n\}_{n=-1}^{\infty}$ be the sequence defined by $x_0 = -1$, $x_1 = f_1/f_0$ with $f_i$’s as in (0.2), and

$$x_{n+1} + x_{n-1} = \frac{n x_n}{\eta(x_n^2 - 1)}, \quad n \geq 0. \quad (0.4)$$

Then for any $k \geq 1$ and generic $\eta$ we have (dropping the superscript $(\eta)$)

$$\frac{p_{k+1} P_k - 1}{P_k^2} = 1 - x_k^2.$$

This result was also proved independently by J. Baik [Bai]. One more proof based on [AvM] was found by M. Adler and P. van Moerbeke. The same equation for a related quantity was derived by C. Tracy and H. Widom in [TW3].
The relation (0.4) is a special case of the discrete PII equation, see, e.g., [GRN].

Theorem 1 is a highly efficient tool for computing \( p_k^{(\eta)} \) numerically. Indeed, the Toeplitz determinant interpretation gives the initial conditions \( p_0 = e^{-\eta^2} \), \( p_1 = e^{-\eta^2} f_0 \), and then Theorem 1 implies \( p_{k+1} = (1 - x_k^2) p_k^2 / p_{k-1} \) for \( k \geq 1 \). Needless to say, this computational scheme is much faster than computing the Toeplitz determinants.

The celebrated result of [BDJ1] claims that if we assume \( \eta > 0 \) and let \( \eta \to +\infty \), then \( p_{2\eta+t\eta^{1/6}} \) converges to a smooth function \( F_2(t) \) (also known as Tracy–Widom distribution in RMT [TW1]) which can be expressed through a solution of the continuous PII equation. Given the existence of the limit, Theorem 1 implies that \((\ln F_2(t))'' = -y^2(t)\) where \( y(t) \) solves \( y'' = ty + 2y^3 \), which also follows from [BDJ1].

To state our second result we introduce the quantity \( q_k^{(z,z',\xi)} \) via the Toeplitz determinant

\[
q_k^{(z,z',\xi)} = (1 - \xi)^{zz'} \det [g_{i-j}]_{i,j=1}^k, \quad \sum_{m=-\infty}^{+\infty} g_m \zeta^m = (1 + \sqrt{\xi} \zeta)^z (1 + \sqrt{\xi} / \zeta)^{z'}.
\]  

(0.5)

It is not hard to show that for any \( k = 1, 2, \ldots, q_k^{(z,z',\xi)} \) extends to an analytic function in \((z, z', \xi) \in \C \times \C \times (\C \setminus [1, +\infty))\).

A representation theoretic definition of \( q_k^{(z,z',\xi)} \) says that if \( z' = \bar{z} \) and \( \xi \in (0, 1) \) then \( q_k^{(z,z',\xi)} \) is the distribution function of the first row of the random Young diagram distributed according to a \( z \)-measure, see [BO2], [BO3]. \( z \)-measures are closely related to the generalized representations of the infinite symmetric group [KOV]. This interpretation of \( q_k^{(z,z',\xi)} \) leads to the identity, cf. (0.3),

\[
q_k^{(z,z',\xi)} = (1 - \xi)^{zz'} \sum_{\lambda_1 \leq k} \prod_{(i,j) \in D(\lambda)} (j - i + z)(j - i + z') \left( \frac{\dim \lambda}{|\lambda|!} \right)^2 \xi^{|\lambda|}.
\]  

(0.6)

Note that both (0.5) and (0.6) imply that

\[
q_k^{(z,z',\xi)} \to p_k^{(\eta)} \quad \text{as} \quad \xi \to 0, \ z, z' \to \infty, \ \xi zz' \to \eta^2.
\]

For positive integral values of \( z, z' \), \( q_k^{(z,z',\xi)} \) also admits a longest increasing subsequence interpretation [BO3, §2]. It can also be viewed as the first passage time in an oriented percolation model, see [J1] and also [Bai]. For integral \( z, z' \) of different signs, \( q_k^{(z,z',\xi)} \) gives a height distribution in a growth model called digital boiling, see [GTW].

A Fredholm determinant representation of \( q_k^{(z,z',\xi)} \) which will be important for us, is given in §6 below.

In what follows we denote by \( F(a, b; c; u) \) the Gauss hypergeometric function.

**Theorem 2.** Let \( \{x_n\}_{n=0}^\infty \) and \( \{y_n\}_{n=0}^\infty \) be the sequences defined by the initial conditions

\[
x_0 = - \frac{F(-z + 1, -z'; 1; \xi)}{z' \xi F(-z + 1, -z' + 1; 2; \xi)}, \quad y_0 = \frac{z'F(-z, -z' - 1; 1; \xi) F(-z + 1, -z' + 1; 2; \xi)}{F(-z, -z'; 1; \xi) F(-z + 1, -z'; 2; \xi)}.
\]
and the recurrence relations
\[
\begin{align*}
x_{n+1} &= \frac{(y_n - (z + z' + n + 1)) (y_n - (z' + n + 1))}{\xi x_n y_n (y_n - z')} , \\
y_{n+1} &= -y_n + \frac{z + n + 1}{1 - x_{n+1}} + \frac{z' + n + 2}{1 - \xi x_{n+1}} + z'.
\end{align*}
\]
(0.7)

Then for any \(k \geq 0\) and generic \((z, z', \xi)\) we have (dropping the superscript \((z, z', \xi)\))
\[
\left(\frac{q_{k+1}}{q_k} - \frac{q_{k+2}}{q_{k+1}}\right) \left(\frac{q_{k+2}}{q_{k+1}} - \frac{q_{k+3}}{q_{k+2}}\right)^{-1} = \frac{((1 - \xi x_k)(y_k - z') - (z + k + 1))}{((1 - \xi x_{k+1})(y_{k+1} - z') - (z + k + 2))} \\
\times \frac{(z + k + 2)(z' + k + 2)(y_k - (z' + k + 1))}{(1 - \xi x_{k+1})(1 - \xi x_k) x_k y_k (y_k - z')^2}.
\]
(0.8)

Similar to Theorem 1, this theorem can be used for numerical evaluation of \(q_k^{(z, z', \xi)}\). Indeed, the initial conditions \(q_0, q_1, q_2\) can be read off the Toeplitz determinant representation, and then (0.8) provides a recurrence for computing \(q_k\) for \(k \geq 3\). The plot of the ‘probability density’ \(q_{k+1}^{(z, z', \xi)} - q_k^{(z, z', \xi)}\) for certain specific values of \((z, z', \xi)\), computed using Theorem 2, can be found on the last page of the paper.

It is known that if \(z' = \bar{z}\), \(\xi \in (0, 1)\), and \(\xi \to 1\), then \(q_{t/(1-\xi)}^{(z, z', \xi)}\) converges to a smooth function \(G(z)(t)\) which is the \(\tau\)-function of the Painlevé V equation, see [BO2], [BD, §8]. If we assume that \(x_t/(1-\xi)\) has a smooth limit \(x(t)\) then (0.7) implies
\[
x''(t) = \left(\frac{1}{2x(t)} + \frac{1}{x(t) - 1}\right) (x'(t))^2 - \frac{x'(t)}{t} + \frac{(z - z' - 1)x(t)}{t} \\
+ \frac{(x(t) - 1)^2}{2t} ((z')^2 x(t) - z^2 / x(t)) - \frac{1}{2} x(t)(x(t) + 1) x(t) - 1 ,
\]
(0.9)
which is a special case of the PV equation. Then the limit of (0.8) provides an algebraic expression for \((\ln\ln G(z)(t))''\) in terms of \(x(t)\) and \(x'(t)\). In fact, the results of [BD] imply that already \((\ln(G(z))'\) can be algebraically expressed in terms of \(x\) and \(x'\) for a certain solution \(x(t)\) of (0.9). It remains unknown whether there exists a discrete analog of this result.

At present there exist several approaches to discretizing the Painlevé equations, see, e.g., [GNR], [NY], [JS], [Sak]. The equations (0.7) turn out to be a special case of the dPV equation of [Sak]. In this paper the discrete analogs of Painlevé equation are derived from a purely algebraic geometric construction. We refer the reader to [Sak] for a further discussion of the subject.

Discrete Painlevé I and II equations have appeared in the physics literature earlier, see e.g. [BK], [FIK], [PS]. One of the main points of the present paper is that dPV also arises in a concrete mathematical/physical model.

Let us also point out that the results of [BD] in the continuous situation suggest that \(q_k^{(z, z', \xi)}\) is a natural candidate for the \(\tau\)-function of (0.7). However, no general definition of a \(\tau\)-function in the geometric setting of [Sak] is available at this moment.
The proofs of both theorems are based on the Riemann–Hilbert approach to discrete integrable operators developed in [Bor2]. Both $p_k^{(q)}$ and $q_k^{(z,z',\xi)}$ are represented as det$(1 - K|_{k,k+1,...})$, where $K$ is either the discrete Bessel kernel of [BOO], [J2], or the hypergeometric (discrete $2F_1$) kernel of [BO2].

Using the results of [Bor2], we reduce the computation of the Fredholm determinant to solving a certain discrete Riemann–Hilbert problem (DRHP, for short). The jump matrices of the DRHPs thus obtained have particularly simple form. This allows us to derive a Lax pair of difference equations for the solutions $m_k(\zeta)$ of these DRHPs which have the form

$$m_{k+1}(\zeta) = A(\zeta)m_k(\zeta), \quad m_k(\zeta - 1) = B(\zeta)m_k(\zeta)C(\zeta)$$

with some unknown rational matrices $A(\zeta)$ and $B(\zeta)$ and a known rational diagonal matrix $C(\zeta)$. The consistency relations for these two equations lead to discrete Painlevé equations on the matrix elements of $A$ and $B$. Additional arguments are needed to express the Fredholm determinants through these matrix elements.

The continuous variant of the same scheme has been worked out in [BD]. There the corresponding matrix $m_t(\zeta)$ solves a RHP with a jump matrix which can be conjugated to a piecewise constant one. This leads to the equations

$$\frac{\partial m_t(\zeta)}{\partial t} = A(\zeta)m_t(\zeta), \quad \frac{\partial m_t(\zeta)}{\partial \zeta} = B(\zeta)m_t(\zeta) + m_t(\zeta)C(\zeta)$$

with known $C$ and unknown rational $A$ and $B$. They form a Lax pair for an appropriate Painlevé equation, and the Fredholm determinant in question is the $\tau$-function of the isomonodromy deformation associated with the RHP. Similar ideas in the continuous setting were also used in [Pal], [HI], [DIZ], [KH].

We expect that our approach can also be applied to a variety of other discrete integrable kernels, in particular, to the Christoffel–Darboux kernels for discrete orthogonal polynomials of the Askey–Wilson scheme. The case of Charlier polynomials, also related to the longest increasing subsequences in random words, will be worked out in the subsequent paper [BB]. It leads to the dPIV equation of [Sak].

I am very grateful to Percy Deift, David Kazhdan, and Grigori Olshanski for interesting and helpful discussions. I would also like to thank Craig Tracy for his letter [T] which initiated my interest in the subject, Jinho Baik and Pierre van Moerbeke for keeping me informed about their work, Masatoshi Noumi for referring me to [Sak], and Hidetaka Sakai for useful comments about his paper.

This research was partially conducted during the period the author served as a Clay Mathematics Institute Long-Term Prize Fellow. This work was also partially supported by the NSF grant DMS-9729992.

1. Integrable operators and discrete Riemann–Hilbert problems

In this section we give a summary of results proved in [Bor2, §4].

Let $\mathcal{Y}$ be a discrete locally finite subset of $\mathbb{C}$. We call an operator $K$ acting in $\ell^2(\mathcal{Y})$ integrable if its matrix has the form

$$K(x,y) = \begin{cases} \sum_{j=1}^{N} F_j(x)G_j(y) \frac{x}{x-y}, & x \neq y, \\ k(x), & x = y, \end{cases}$$

where $F_j, G_j : \mathcal{Y} \to \mathbb{C}$ are functions.
for some complex–valued functions $F_j$, $G_j$, $j = 1, \ldots, N$, and $k$ on $\mathfrak{Y}$.

Integrable operators as a distinguished class were first singled out in the continuous setting in [IIKS], see also [D1]. The definition in the discrete setting was given in [Bor2].

We will assume that

- $F_j, G_j \in \ell^2(\mathfrak{Y})$ for all $j = 1, \ldots, N$, and

$$
\sum_{j=1}^{N} F_j(x)G_j(x) = 0, \quad x \in \mathfrak{Y}. 
$$

- $k$ is a bounded function on $\mathfrak{Y}$ which is also bounded away from 1. That is,

$$
\inf_{x \in \mathfrak{Y}} |k(x) - 1| > 0.
$$

- The operator

$$
(Th)(x) = \sum_{x' \in \mathfrak{Y}, x' \neq x} \frac{h(x')}{x - x'}
$$

is a bounded operator in $\ell^2(\mathfrak{Y})$. This always holds if, for example, $\mathfrak{Y}$ is a subset of a one–dimensional lattice in $\mathbb{C}$.

It is not hard to show that under these assumptions, $K$ is a bounded operator in $\ell^2(\mathfrak{Y})$.

Our goal is to explain how the operator $(1 - K)^{-1}$ (if it exists) can be expressed through a solution of a complex analytic problem which we call the discrete Riemann–Hilbert problem (DRHP, for short).

Let $w$ be a map from $\mathfrak{Y}$ to $\text{Mat}(n, \mathbb{C})$, where $n$ is a fixed integer.

We say that a matrix–valued function $m : \mathbb{C} \setminus \mathfrak{Y} \to \text{Mat}(n, \mathbb{C})$ with simple poles at the points $x \in \mathfrak{Y}$ is a solution of the DRHP $(\mathfrak{Y}, w)$ if the following conditions are satisfied

- $m(\zeta)$ is analytic in $\mathbb{C} \setminus \mathfrak{Y}$,
- $\text{Res } m(\zeta) = \lim_{\zeta \to x} (m(\zeta)w(x))$, $x \in \mathfrak{Y}$.

By analogy with continuous Riemann–Hilbert problems, we call $w(x)$ the jump matrix.

We say that $m$ satisfies the normalized DRHP $(\mathfrak{Y}, w)$ if, in addition to the conditions above, $m(\zeta) \to I$ as $\zeta \to \infty$. Here $I$ is the $n \times n$ identity matrix.

If the set $\mathfrak{Y}$ is infinite, the last condition must be made more precise. Indeed, a function with poles accumulating at infinity cannot have a limit at infinity. One way to make the condition precise is to require uniform asymptotics on a sequence of expanding contours, for example, on a sequence of circles $|\zeta| = a_k$, $a_k \to +\infty$.

Whenever below we give an asymptotic behavior of a function with poles in $\mathfrak{Y}$ at infinity, we mean that there exists a sequence of expanding contours such that the distance from these contours to the set $\mathfrak{Y}$ is bounded away from zero, and the function has uniform asymptotics on these contours.

Let us introduce column–vectors

$$
F = (F_1, \ldots, F_N)^t, \quad G = (G_1, \ldots, G_N)^t.
$$

Then (1.2) can be rewritten as $F^t(x)G(x) = G^t(x)F(x) = 0$. 


Theorem 1.1. Let $K$ be an integrable operator as defined above, and assume that the operator $(1 - K)$ is invertible. Then

(i) There exists a unique solution $m_{\mathfrak{Y}}$ of the normalized DRHP $(\mathfrak{Y}, w)$ where

$$w(x) = (1 - k(x))^{-1} F(x) G^t(x) \in \text{Mat}(N, \mathbb{C}).$$

Moreover, $\det m_{\mathfrak{Y}} \equiv 1$.

(ii) The matrix of the operator $R = (1 - K)^{-1} - 1 = K(1 - K)^{-1}$ has the form

$$R(x, y) = \begin{cases} 
\frac{\sum_{j=1}^N F_j(x) G_j(y)}{x - y}, & x \neq y, \\
\frac{k(x) + r(x)}{1 - k(x)}, & x = y,
\end{cases}$$

(1.3)

where, with the notation $F = (F_1, \ldots, F_N)^t$, $G = (G_1, \ldots, G_N)^t$,

$$F(x) = (1 - k(x))^{-1} \lim_{\zeta \to x} m_{\mathfrak{Y}}(\zeta) F(x),$$

$$G(x) = (1 - k(x))^{-1} \lim_{\zeta \to x} m_{\mathfrak{Y}}(\zeta) G(x),$$

$$r(x) = G^t(x) \lim_{\zeta \to x} (m_{\mathfrak{Y}}(\zeta) F(x)) .$$

Here $m_{\mathfrak{Y}}(\zeta) = dm_{\mathfrak{Y}}(\zeta)/d\zeta$.

Proof. Follows from Proposition 4.3 and Remark 4.2 in [Bor2]. Note the difference in notation: we used $\mathfrak{X}$, $L$, $f$, $g$, and $K = L(1 + L)^{-1}$ in [Bor2] instead of $\mathfrak{Y}$, $K$, $F$, $G$ and $R = K(1 - K)^{-1}$ here. The reason for switching the notation will become clear in the next section. □

2. Simpler DRHP for a class of integrable operators

It turns out that for many integrable operators of interest, the DRHP from Theorem 1.1(i) is rather complicated. In this section we show how to reduce the computation of the inverse operator $(1 - K)^{-1}$ and $\det(1 - K)$ to a much simpler DRHP for a certain subclass of the class of integrable operators.

Our main assumption is that there exists a locally finite set $\mathfrak{X}$, $\mathfrak{X} \supset \mathfrak{Y}$, vector-valued functions

$$f = (f_1, \ldots, f_N)^t, \quad g = (g_1, \ldots, g_N)^t$$
on $\mathfrak{X}$, and a matrix-valued function $m_{\mathfrak{X}} : \mathbb{C} \setminus \mathfrak{X} \to \mathbb{C}$ such that

(1) $f^t(x) g(x) \equiv 0$ on $\mathfrak{X}$;
(2) $m_{\mathfrak{X}}$ solves the DRHP $(\mathfrak{X}, -fg^t)$, not necessarily normalized;
(3) $\det m_{\mathfrak{X}} \equiv 1$;
(4) for any $x \in \mathfrak{Y}$

$$F(x) = \lim_{\zeta \to x} m_{\mathfrak{X}}(\zeta) f(x), \quad G(x) = \lim_{\zeta \to x} m_{\mathfrak{X}}^{-t}(\zeta) g(x)$$

$$k(x) = G^t(x) \lim_{\zeta \to x} (m_{\mathfrak{X}}(\zeta) f(x)).$$

Note that we do not impose any boundedness/decay conditions on $f_j$ and $g_j$.

There are at least two general situations when this assumption is satisfied.
Situation 2.1. There exists an integrable operator $L$ on $\mathfrak{X}$ with the matrix

$$L(x, y) = \begin{cases} \frac{\sum_{j=1}^{N} f_j(x) g_j(y)}{x - y}, & x \neq y, \\ 0, & x = y. \end{cases}$$

Then if we set $K = L(1 + L)^{-1}$, assuming that $-1$ is not in the spectrum of $L$, we will have the main assumption above satisfied by Theorem 1.1 (or Proposition 4.3 in [Bor2]). Note that $m_\mathfrak{X}$ will then satisfy the normalized DRHP.

Situation 2.2. The matrix $K$ has the form

$$K(x, y) = \begin{cases} \frac{\alpha(x)\beta(y)\phi(x)\psi(y) - \psi(x)\phi(y)}{x - y}, & x \neq y, \\ \alpha(x)\beta(x)\phi'(x)\psi(x) - \psi'(x)\phi(x), & x = y, \end{cases}$$

where $\alpha, \beta$ are some functions defined on $\mathcal{Y}$, and $\phi, \psi$ are entire functions. Note that the expression for $K(x, x)$ is obtained from that for $K(x, y)$ by formal limit transition $y \to x$.

Assume that there exist meromorphic functions $\hat{\phi}$ and $\hat{\psi}$ with simple poles, such that for any $x \in \mathcal{Y}$ we have

$$\text{Res}_{\zeta=x} \hat{\phi}(\zeta) = \alpha(x)\beta(x)\phi(x), \quad \text{Res}_{\zeta=x} \hat{\psi}(\zeta) = \alpha(x)\beta(x)\psi(x),$$

and $\phi \hat{\psi} - \psi \hat{\phi} \equiv 1$ on $\mathbb{C}$.

Let us denote by $\mathfrak{X}$ the union of $\mathcal{Y}$ and the set of poles of the functions $\hat{\phi}$ and $\hat{\psi}$. Assume that we can extend the functions $\alpha$ and $\beta$ to $\mathfrak{X}$ in such a way that the residue conditions above hold for $x \in \mathfrak{X}$.

Then we can satisfy the main assumption (1)–(4) by setting

$$f = (\alpha, 0)^t, \quad g = (0, -\beta)^t, \quad F = (\alpha\phi, \alpha\psi)^t, \quad G = (\beta\psi, -\beta\phi)^t,$$

$$m_\mathfrak{X} = \begin{bmatrix} \phi & \hat{\phi} \\ \psi & \hat{\psi} \end{bmatrix}, \quad m_\mathfrak{X}^{-1} = \begin{bmatrix} \hat{\psi} & -\psi \\ \phi & \hat{\phi} \end{bmatrix}.$$

Situation 2.1 comes up when $K$ is the correlation kernel for a determinantal point process defined as an $L$–ensemble, see [BO4, §5], and also [BO2], [BOO], and [Bor2] for concrete examples.

Situation 2.2 suits (restrictions of) Christoffel–Darboux kernels for classical discrete orthogonal polynomials and some limits of such kernels. The functions $\hat{\phi}$ and $\hat{\psi}$ are then the so-called “functions of the second kind”, $\mathfrak{X}$ is the orthogonality set, and the product $\alpha \beta$ is equal to the weight function.

Denote $\mathfrak{Z} = \mathfrak{X} \setminus \mathcal{Y}$.

Theorem 2.3. Under the main assumption (1)–(4) above

(i) There exists a unique solution $m_\mathfrak{Z}$ of the DRHP $(\mathfrak{Z}, -fg^t)$ satisfying the condition $m_\mathfrak{Z}m_\mathfrak{X}^{-1} \to I$ as $\zeta \to \infty$. Moreover, $\det m_\mathfrak{Z} \equiv 1$.

(ii) The matrix of the operator $R = (1 - K)^{-1} - 1 = K(1 - K)^{-1}$ has the form

$$R(x, y) = \begin{cases} \frac{\sum_{j=1}^{N} F_j(x)G_j(y)}{x - y}, & x \neq y, \\ g^t(x)m_\mathfrak{Z}^{-1}(x)m_\mathfrak{Z}(x)f(x), & x = y, \end{cases}$$

(2.1)
where 
\[ F(x) = m_3(x)f(x), \quad G(x) = m_3^{-t}(x)g(x). \] (2.2)

Comments. 1. Since \( \det m_X \equiv \det m_3 \equiv 1 \), the inverse matrices \( m_X^{-1} \) and \( m_3^{-1} \) are well-defined outside \( X \) and \( 3 \) respectively.

2. The formulas (2.1) and (2.2) hold for \( x, y \in \mathcal{Y} \), and \( m_3 \) is analytic around such points because \( \mathcal{Y} \cap 3 = \emptyset \).

3. The part of Theorem 2.3 which will be employed later on, is the formula for the diagonal values \( R(x,x) \). The reason is the relation (assume that \( K \) is a trace class operator)

\[ 1 + R(x,x) = \frac{\det(1-K_x)}{\det(1-K)}, \]

where \( K_x \) is the restriction of the matrix \( K \) to \( \mathcal{Y} \setminus x \).

4. The DRHP of Theorem 2.3 in a number of interesting examples turns out to be much simpler than that of Theorem 1.1. The reason is that the functions \( \{f_j,g_j\} \) are often elementary (or even constant), while the functions \( \{F_j,G_j\} \) are commonly expressed through classical special functions.

Before proceeding to the proof of Theorem 2.3, let us prove the following lemma.

**Lemma 2.4.** Let \( X, Y, \) and \( Z \) be locally finite subsets of \( \mathbb{C} \) such that \( X = Y \sqcup Z \). Let \( w_X \) be an arbitrary matrix-valued function on \( X \) such that for any \( x \in X \), \( w_X(x) \) is nilpotent of rank 1. Let \( m_X \) be a solution of the DRHP \((X, w_X)\) such that \( \det m_X \equiv 1 \).

For any \( x \in X \), denote by \( B(x) \) the constant term in the Laurent expansion of \( m_X \) near \( x \). Then \( \det B(x) \neq 0 \). Furthermore, if \( m_Z \) is a solution of the DRHP \((Z, w_X)\), then \( m_Y \equiv m_Zm_X^{-1} \) solves the DRHP \((Y, w_Y)\), where

\[ w_Y(x) = -B(x)w_X(x)B^{-1}(x). \] (2.3)

Conversely, if \( m_Y \) is a solution of the DRHP \((Y, w_Y)\), then \( m_Z \equiv m_Ym_X \) solves the DRHP \((Z, w_X)\).

**Proof.** Take any \( x \in X \). The residue condition implies that in the neighborhood of \( x \) we have

\[ m_X(\zeta) = B(x) \left( I + \frac{w_X(x)}{\zeta-x} + (\zeta-x)C(\zeta) \right) \] (2.4)

where \( C(\zeta) \) is analytic in the neighborhood of \( x \), cf. [Bor2, Lemma 4.4]. Then \( \det m_X \equiv 1 \) implies \( \det B(x) \neq 0 \).

Since \( w_X \) is of rank 1, there exist column-vectors \( u \) and \( v \) such that \( w_X = uv^t \). The condition \( w_X^2 = 0 \) implies \( u^tv = v^tu = 0 \). Then

\[ \det \left( I + \frac{w_X(x)}{\zeta-x} + (\zeta-x)C(\zeta) \right) = 1 - v^t(x)C(x)u(x) + O(\zeta-x). \]

Hence, \( 1 - v^t(x)C(x)u(x) = (\det B(x))^{-1} \neq 0 \).

Denote \( A(\zeta) = I + (\zeta-x)C(\zeta) \). Clearly, \( A(\zeta) \) is invertible when \( |\zeta-x| \) is small enough, and \( A^{-1}(\zeta) = I - (\zeta-x)C(\zeta) + O((\zeta-x)^2) \). Hence

\[ \zeta - x + v^t(x)A^{-1}(\zeta)u(x) = (\zeta-x) \left( 1 - v^t(x)C(x)u(x) \right) + O((\zeta-x)^2) \]
is not equal to 0 if \( \zeta \neq x \) and \( |\zeta - x| \) is small.

In the neighborhood of \( x \) we have

\[
\left( I + \frac{w_X(x)}{\zeta - x} + (\zeta - x)C(\zeta) \right)^{-1} = \left( A(\zeta) + \frac{u(x)v^t(x)}{\zeta - x} \right)^{-1} \equiv \left( I - \frac{A^{-1}(\zeta)u(x)v^t(x)}{\zeta - x + v^t(x)A^{-1}(\zeta)u(x)} \right) A^{-1}(\zeta)
\]

\[= \left( I - \frac{u(x)v^t(x)}{(\zeta - x)(1 - v^t(x)C(x)v(x))} + O(1) \right) A^{-1}(\zeta) = \left( I - \frac{\det B(x)u(x)v^t(x)}{\zeta - x} \right) A^{-1}(\zeta) + O(1).
\]

Thus, using (2.4) we see that for any \( x \in \mathfrak{X} \)

\[
\text{Res}_{\zeta=x} m_X^{-1}(\zeta) = - \det B(x) \cdot u(x)v^t(x)B^{-1}(x).
\]

On the other hand, relations (2.4), (2.5) imply

\[
m_X^{-1}(\zeta)B(x)u(x)v^t(x)B^{-1}(x) = \frac{(\zeta - x)A^{-1}(\zeta)}{\zeta - x + v^t(x)A^{-1}(\zeta)u(x)} u(x)v^t(x)B^{-1}(x)
\]

\[= \det B(x) \cdot u(x)v^t(x)B^{-1}(x) + O(\zeta - x).
\]

Hence, for \( x \in X \),

\[
\text{Res}_{\zeta=x} m_X^{-1}(\zeta) = - \lim_{\zeta \to x} \left( m_X^{-1}(\zeta) (B(x)w_X(x)B^{-1}(x)) \right).
\]

Now, if \( x \in Y \) then \( m_Z \) is analytic near \( x \), and for \( m_Y = m_Z m_X^{-1} \) we have

\[
\text{Res}_{\zeta=x} m_Y(\zeta) = - \lim_{\zeta \to x} \left( m_Y(\zeta) (B(x)w_X(x)B^{-1}(x)) \right)
\]

as required. For \( x \in Z \), both \( m_X \) and \( m_Z \) satisfy the same residue condition at \( x \), which implies that \( m_Y \) is analytic near \( x \), cf. [Bor2, Lemma 4.5]. This concludes the proof of the first statement of the lemma.

To prove the converse statement it suffices to verify that \( m_Y m_X \) is analytic near any \( x \in Y \). But we have just proved that \( m_X^{-1} \) satisfies the same residue condition as \( m_Y \) does. Thus, their ratio is analytic. \( \Box \)

Proof of Theorem 2.3. Let \( m_2 \) be as in Theorem 1.1(i). Set \( m_3 = m_2 m_X \). Clearly, \( \det m_3 \equiv 1 \). We are going to show that for \( x \in \mathfrak{Y} \)

\[
(1 - k(x))^{-1}F(x)G^t(x) = B_X(x)f(x)g^t(x)B_X(x)^{-1},
\]

where \( B_X(x) \) is the constant term in the Laurent expansion of \( m_X(\zeta) \) near \( x \). Then Lemma 2.4 will imply that \( m_3 \) satisfies the DRHP \( (3, -fg^t) \).\(^1\)

\(^1\)The match with Lemma 2.4 is established by \( X = \mathfrak{X}, Y = \mathfrak{Y}, Z = \mathfrak{Z}, w_X = -fg^t, w_Y = (1 - k)^{-1}FG^t, u = f, v = -g. \)
Condition (2) of the main assumption implies that near \( x \in \mathcal{X} \), \( m_\mathcal{X}(\zeta) \) has the form

\[
m_\mathcal{X}(\zeta) = B_\mathcal{X}(x) \left( I - \frac{f(x)g^t(x)}{\zeta - x} + (\zeta - x)C_\mathcal{X}(\zeta) \right),
\]

where \( C_\mathcal{X}(\zeta) \) is analytic in the neighborhood of \( x \), cf. (2.4). Then condition (1) of the main assumption implies

\[
F(x) = \lim_{\zeta \to x} m_\mathcal{X}(\zeta) f(x) = B_\mathcal{X}(x) f(x). \tag{2.7}
\]

Similarly to the proof of Lemma 2.4, setting \( A_\mathcal{X}(\zeta) = I + (\zeta - x)C_\mathcal{X}(\zeta) \), we have

\[
m_\mathcal{X}^{-t}(\zeta) g(x) = B_\mathcal{X}^{-t}(x)A_\mathcal{X}^{-t}(\zeta) \left( g(x) + \frac{g(x)f^t(x)A_\mathcal{X}^{-t}(\zeta)g(x)}{\zeta - x - f^t(x)A_\mathcal{X}^{-1}(\zeta)f(x)} \right)
\]

\[
= \frac{(\zeta - x)B_\mathcal{X}^{-t}(x)A_\mathcal{X}^{-t}(\zeta)g(x)}{\zeta - x - f^t(x)A_\mathcal{X}^{-1}(\zeta)f(x)} = \det B_\mathcal{X}(x) \cdot B_\mathcal{X}^{-t}(x)g(x) + O(1),
\]
as \( \zeta \to x \). That is,

\[
G(x) = \lim_{\zeta \to x} m_\mathcal{X}^{-t}(\zeta) g(x) = \det B_\mathcal{X}(x) \cdot B_\mathcal{X}^{-t}(x)g(x). \tag{2.8}
\]

Finally,

\[
k(x) = G^t(x) \lim_{\zeta \to x} (m_\mathcal{X}^t(\zeta) f(x)) = G^t(x)B_\mathcal{X}(x)C_\mathcal{X}(x) f(x)
\]

\[
= \det B_\mathcal{X}(x)g^t(x)C_\mathcal{X}(x) f(x) = \det B_\mathcal{X}(x) \left( (\det B_\mathcal{X}(x))^{-1} - 1 \right) = 1 - \det B_\mathcal{X}(x). \tag{2.9}
\]

Then (2.6) follows from (2.7)–(2.9).

The uniqueness of \( m_3 \) follows from the following general argument. If \( n_3 \) is another solution of \((3, -fg^t)\) with the same asymptotics at infinity then \( n_3m_3^{-1} \) has no singularities and tends to \( I \) at infinity. By Liouville’s theorem, \( n_3 \equiv m_3 \). This concludes the proof of (i).

Let us proceed to (ii). Thanks to (2.6), locally near any \( x \in Y \) we can write \( m_\mathcal{Y} \) in the form

\[
m_\mathcal{Y}(\zeta) = B_\mathcal{Y}(x)B_\mathcal{X}(x) \left( I + \frac{f(x)g^t(x)}{\zeta - x} + (\zeta - x)C_\mathcal{Y}(\zeta) \right)B_\mathcal{X}^{-1}(x),
\]

where \( B_\mathcal{Y}(x) \) is a nondegenerate matrix, and \( C_\mathcal{Y}(\zeta) \) is analytic near \( x \). Then, similarly to (2.7), (2.8), using (2.7)–(2.9), we get

\[
\mathcal{F}(x) = (1 - k(x))^{-1} \cdot \lim_{\zeta \to x} m_\mathcal{Y}(\zeta) F(x) = (\det B_\mathcal{X}(x))^{-1} \cdot B_\mathcal{Y}(x)B_\mathcal{X}(x) f(x),
\]

\[
\mathcal{G}(x) = (1 - k(x))^{-1} \cdot \lim_{\zeta \to x} m_\mathcal{Y}^{-t} G(x) = \det (B_\mathcal{Y}(x)) \cdot (B_\mathcal{Y}B_\mathcal{X})^{-t}(x)g(x). \tag{2.10}
\]

Further, near \( x \in Y \) we have

\[
m_3(\zeta) = m_\mathcal{Y}(\zeta)m_\mathcal{X}(\zeta) = B_\mathcal{Y}(x)B_\mathcal{X}(x)
\]

\[
\times \left( I + f(x)g^t(x)C_\mathcal{X}(\zeta) - C_\mathcal{Y}(\zeta)f(x)g^t(x) + (C_\mathcal{X}(\zeta) + C_\mathcal{Y}(\zeta))(\zeta - x) \right). \tag{2.11}
\]
Thus,

\[ m_3(x)f(x) = (1 + g^t(x)C_\chi(x)f(x)) \cdot B_{\gamma}(x)B_{\chi}(x)f(x) = \mathcal{F}(x). \quad (2.12) \]

It is easy to verify the following equality (remember that \(1 + g^tC_\chi f = (\det B_\chi)^{-1}\) and \(1 - g^tC_\gamma f = (\det B_\gamma)^{-1}\))

\[
m_3^{-1}B_\gamma B_\chi = (I + fg^tC_\chi - C_\gamma fg^t))^{-1}
= I - \det B_\chi \cdot fg^tC_\chi + \det B_\gamma \cdot C_\gamma fg^t - \det(B_\chi B_\gamma) \cdot fg^tC_\chi C_\gamma fg^t,
\]

where we have omitted the argument \(x \in Y\) of all the functions above. Hence,

\[
m_3^{-t}(x)g(x) = (B_\gamma B_\chi)^{-t}(x)(1 + \det B_\gamma(x)(g^tC_\gamma f)(x)) \cdot g(x)
= \det B_\gamma(x) \cdot (B_\gamma B_\chi)^{-t}(x)g(x) = \mathcal{G}(x). \quad (2.13)
\]

The formula (2.1) for the nondiagonal entries of \(R\) follows from (1.3), (2.12) and (2.13).

To prove (2.1) for diagonal entries we need to evaluate \(r(x)\) introduced in (1.4). We have

\[
\lim_{\zeta \to x} (m_3'(\zeta)F(x)) = B_\gamma(x)B_\chi(x)C_\gamma(x)f(x).
\]

Thus, with the help of (2.10),

\[
r(x) = \mathcal{G}(x) \lim_{\zeta \to x} (m_3'(\zeta)F(x)) = \det B_\gamma(x) \cdot (g^tC_\gamma f)(x) = \det B_\gamma(x) - 1.
\]

Then

\[
R(x, x) = k(x) + r(x) \quad \frac{1}{1 - k(x)} = \frac{\det B_\gamma(x)}{\det B_\chi(x)} - 1.
\]

On the other hand, (2.11) and (2.13) imply

\[
g^t m_3^{-1}m_3'f = \det B_\gamma \cdot g^t(C_\chi + C_\gamma)f
= \det B_\gamma \cdot ((\det B_\chi^{-1} - 1) + (1 - \det B_\gamma^{-1})) = \frac{\det B_\gamma(x)}{\det B_\chi(x)} - 1. \quad \Box
\]

3. Discrete Bessel kernel and dPII

In this section we will apply the general formalism of §2 to derive the discrete Painlevé II equation (dPII, for short) for the Fredholm determinant of the discrete Bessel kernel.

The discrete Bessel kernel plays an important role in the asymptotic analysis of the Plancherel measures on the symmetric groups. It was derived independently in [BOO] and [J2], see also [Bor2]. We define it as follows.

Let \(Z' = Z + \frac{1}{2} = \{\pm \frac{1}{2}, \pm \frac{3}{2}, \ldots\}\). For \(x, y \in Z'\) set

\[
K(x, y) = \eta \frac{J_{x+\frac{1}{2}}(2\eta)J_{y+\frac{1}{2}}(2\eta) - J_{x-\frac{1}{2}}(2\eta)J_{y-\frac{1}{2}}(2\eta)}{x - y}, \quad (3.1')
\]
where \( \eta > 0 \) is a parameter, \( J_\nu(\cdot) \) is the \( J \)-Bessel function, and \( K(x, x) \) is defined by the L'Hospital rule:

\[
K(x, x) = \eta \left( \frac{\partial J_{x-\frac{1}{2}}(2\eta)}{\partial x} J_{x+\frac{1}{2}}(2\eta) - J_{x-\frac{1}{2}}(2\eta) \frac{\partial J_{x+\frac{1}{2}}(2\eta)}{\partial x} \right). \tag{3.1''}
\]

Note that only the Bessel functions with \emph{integral} indices enter the formula.

For any \( s \in \mathbb{Z}' \), denote by \( K_s \) the operator in \( \ell^2(\{s, s+1, \ldots\}) \) defined by the restriction of \( K \) to \( \{s, s+1, \ldots\} \times \{s, s+1, \ldots\} \). It can be shown that \( K_s \) is a positive trace class operator.\(^2\)

We will be interested in the Fredholm determinants

\[
D_s = \det(1 - K_s).
\]

From the probabilistic interpretation of the kernel \( K \) it immediately follows that \( D_s = 0 \) if \( s < 0 \).\(^3\) Thus, from now on we assume that \( s > 0 \). Then \( D_s > 0 \).

It is worth noting that \( D_s \) can also be interpreted as a Toeplitz determinant, see [BOk, §4]. Namely,

\[
D_s = \exp(-\eta^2) \cdot \det[I_{i-j}(2\eta)]_{i,j=1,\ldots,s-\frac{1}{2}}. \tag{3.2}
\]

Here \( I_\nu(\cdot) \) is the \( I \)-Bessel function. This formula also follows from [G], [BOO], [J2]. The symbol of the Toeplitz determinants above is equal to

\[
\sum_{k \in \mathbb{Z}} I_k(2\eta)\zeta^k = \exp(\eta(\zeta + \zeta^{-1})).
\]

The identity (3.2) can be generalized to Toeplitz determinants with an arbitrary symbol [BOk], and even to block Toeplitz determinants [BW], see [Bot] for a simple proof.

Note that in the Introduction we used the notation \( p^{(\eta)}_{s-\frac{1}{2}} \) for \( D_s \).

The discrete Bessel kernel fits into both situations described in the beginning of §2. We will use Situation 2.1, but let us first show how Situation 2.2 works in this case. Naturally, set \( \mathcal{Y} = \{s, s+1, \ldots\} \).

\textbf{Example 3.1 (a realization of Situation 2.2).} We set \( X = \mathbb{Z}' \), and

\[
\alpha \equiv 1, \quad \beta \equiv 1, \quad \phi(\zeta) = J_{\zeta-\frac{1}{2}}(2\eta), \quad \psi(\zeta) = \eta J_{\zeta+\frac{1}{2}}(2\eta),
\]

\[
\hat{\phi}(\zeta) = -\frac{\pi}{\cos \pi \zeta} J_{-\zeta+\frac{1}{2}}(2\eta), \quad \hat{\psi}(\zeta) = \frac{\pi \eta}{\cos \pi \zeta} J_{-\zeta-\frac{1}{2}}(2\eta).
\]

Then the residue conditions follow from the relation \( J_{-n} = (-1)^n J_n, n \in \mathbb{Z} \). The equality \( \phi \hat{\psi} - \psi \hat{\phi} \equiv 1 \) is the well-known identity

\[
J_{\zeta-\frac{1}{2}}(2\eta)J_{-\zeta-\frac{1}{2}}(2\eta) + J_{\zeta+\frac{1}{2}}(2\eta)J_{-\zeta+\frac{1}{2}}(2\eta) = \frac{\cos \pi \zeta}{\pi \eta}.
\]

\(^2\)This immediately follows from the fact that \( K_s \) is a symmetric correlation kernel for a determinantal point process which has finitely many particles almost surely, see [So, Theorem 4] for a general theorem and [BOO] for the description of the point process.

\(^3\)Vanishing of \( D_s \) is equivalent to the statement that for any partition \( \lambda = (\lambda_1, \lambda_2, \ldots) \), the set \( \{\lambda_i - i + \frac{1}{2}\} \cap \{s, s+1, \ldots\} \) is nonempty, see [BOO].
The jump matrix $m_{\mathcal{X}}(\zeta)$ satisfies the equation

$$m_{\mathcal{X}}(\zeta - 1) = \frac{1}{\eta} \begin{bmatrix} \zeta - \frac{1}{2} & -1 \\ \eta & 0 \end{bmatrix} m_{\mathcal{X}}(\zeta).$$

The only disadvantage of this realization is that $m_{\mathcal{X}}(\zeta)$ does not tend to $I$ as $\zeta \to \infty$. However, one can very well use it to analyze the inverse operators $(1 - K_s)^{-1}$ and the Fredholm determinants $D_s$.

All the details of how Situation 2.1 is applied to the discrete Bessel kernel are explained in [Bor2]. Here we just state the results that we need.

We set $\mathcal{X} = \mathbb{Z}'$. The matrix $m_{\mathcal{X}}$ has the form

$$m_{\mathcal{X}}(\zeta) = \sqrt{\eta} \begin{bmatrix} J_{\zeta - \frac{1}{2}}(2\eta) & J_{-\zeta + \frac{1}{2}}(2\eta) \\ -J_{\zeta + \frac{1}{2}}(2\eta) & J_{\zeta - \frac{1}{2}}(2\eta) \end{bmatrix} \begin{bmatrix} \eta^{-\zeta}\Gamma(\zeta + \frac{1}{2}) & 0 \\ 0 & \eta^\zeta\Gamma(-\zeta + \frac{1}{2}) \end{bmatrix}.$$

It solves the normalized DRHP $(\mathcal{X}, -fg^t)$, where $f = (f_1, f_2)^t$, $g = (g_1, g_2)^t$,

$$f_1(x) = \begin{cases} \frac{\eta^x}{\Gamma(x + \frac{1}{2})}, & x \in \mathbb{Z}_+ \\ 0, & x \in \mathbb{Z}_- \end{cases}, \quad f_2(x) = \begin{cases} 0, & x \in \mathbb{Z}_+ \\ \frac{\eta^{-x}}{\Gamma(-x + \frac{1}{2})}, & x \in \mathbb{Z}_- \end{cases},$$

$$g_1(x) = \begin{cases} 0, & x \in \mathbb{Z}_+ \\ \frac{\eta^{-x}}{\Gamma(-x + \frac{1}{2})}, & x \in \mathbb{Z}_- \end{cases}, \quad g_2(x) = \begin{cases} 0, & x \in \mathbb{Z}_+ \\ \frac{\eta^x}{\Gamma(x + \frac{1}{2})}, & x \in \mathbb{Z}_- \end{cases}.$$

The jump matrix $w_{\mathcal{X}}(x) = -f(x)g(x)^t$ has the form

$$w_{\mathcal{X}}(x) = \begin{cases} \begin{bmatrix} 0 & -\frac{\eta^{2x}}{\Gamma^2(x + \frac{1}{2})} \\ 0 & 0 \end{bmatrix}, & x \in \mathbb{Z}_+ \\ \begin{bmatrix} 0 & 0 \\ -\frac{\eta^{2x}}{\Gamma^2(-x + \frac{1}{2})} & 0 \end{bmatrix}, & x \in \mathbb{Z}_- \end{cases} \quad \text{(3.3)}$$

We proceed to examine the unique solution $m_3$ of the normalized DRHP $(3, w_{\mathcal{X}})$, where

$$3 = \mathcal{X} \setminus \mathcal{Y} = \mathbb{Z} \setminus \{s, s + 1, \ldots\} = \{\ldots, s - 2, s - 1\},$$

see Theorem 2.3(i). It is more convenient to redenote $m_3$ by $m_s$, because we will be working with solutions corresponding to different values of $s$. We also denote $3_s = \{\ldots, s - 2, s - 1\}$. Recall that $\det m_s \equiv 1$.

We aim at proving the following

**Proposition 3.2 (Lax pair).** For any $s \in \mathbb{Z}_+$ there exist a constant nilpotent matrix $A_s$,

$$A_s = \begin{bmatrix} p_s & q_s \\ r_s & -p_s \end{bmatrix}, \quad p_s^2 = -r_s q_s,$$
and constant $a_s, b_s, a_s b_s = 1$, such that
\begin{equation}
m_{s+1}(\zeta) = \left(I + \frac{A_s}{\zeta - s}\right)m_s(\zeta), \tag{3.4}\end{equation}
\begin{equation}
m_s(\zeta - 1) = \begin{bmatrix} \eta^{-1}(\zeta - \frac{1}{2} - ps) & a_s \\ -b_s & 0 \end{bmatrix} m_s(\zeta) \begin{bmatrix} \eta(\zeta - \frac{1}{2})^{-1} & 0 \\ 0 & \eta^{-1}(\zeta - \frac{1}{2}) \end{bmatrix} m_{s+1}(\zeta). \tag{3.5}\end{equation}

**Proof.** The first equation is almost obvious. Indeed, since $w_\chi$ does not depend on $s$, we see that $m_s(\zeta)$ and $m_{s+1}(\zeta)$ satisfy the same residue condition on $3_s$. However, $m_{s+1}$ has an extra pole at the point $\{s\} = 3_{s+1} \setminus 3_s$. Hence, the ratio $m_{s+1}m_s^{-1}$ has only one pole at the point $\zeta = s$. Denoting the residue at this pole by $A_s$, we conclude that the function
\[
m_{s+1}(\zeta)m_s^{-1}(\zeta) - \frac{A_s}{\zeta - s}
\]
is entire. Evaluating the asymptotics at $\zeta = \infty$ we see that, by Liouville’s theorem, this function is identically equal to $I$, which proves the first equation. Furthermore, since $\det m_s \equiv \det m_{s+1} \equiv 1$, we see that $\det(I + A_s/(\zeta - s)) \equiv 1$. This implies that $A_s$ is nilpotent.

To prove the second equation, let us first verify that
\begin{equation}
m_s(\zeta - 1) \begin{bmatrix} \eta^{-1}(\zeta - \frac{1}{2}) & 0 \\ 0 & \eta(\zeta - \frac{1}{2})^{-1} \end{bmatrix} m_{s+1}^{-1}(\zeta) \tag{3.6}\end{equation}
is an entire function. It is easy to see that all the factors above are analytic outside $3_{s+1}$. Let $x \in 3_{s+1}$. Then we have
\[
m_s(\zeta - 1) = H_1(\zeta) \left(I + \frac{w_\chi(x-1)}{\zeta - x}\right),
\]
\[
m_{s+1}^{-1}(\zeta) = \left(I - \frac{w_\chi(x)}{\zeta - x}\right)H_2(\zeta),
\]
where $H_1, H_2$ are analytic and invertible near $x$. Hence, we need to prove that
\[
\left(I + \frac{w_\chi(x-1)}{\zeta - x}\right) \begin{bmatrix} \eta^{-1}(\zeta - \frac{1}{2}) & 0 \\ 0 & \eta(\zeta - \frac{1}{2})^{-1} \end{bmatrix} \left(I - \frac{w_\chi(x)}{\zeta - x}\right)
\]
is analytic. For $x \neq \frac{1}{2}$ this is obvious from (3.3). For $x = \frac{1}{2}$ we have
\[
\begin{bmatrix} 1 & 0 \\ -\eta(\zeta - \frac{1}{2})^{-1} & 1 \end{bmatrix} \begin{bmatrix} \eta^{-1}(\zeta - \frac{1}{2}) & 0 \\ 0 & \eta(\zeta - \frac{1}{2})^{-1} \end{bmatrix} \begin{bmatrix} 1 & \eta(\zeta - \frac{1}{2})^{-1} \\ 0 & 1 \end{bmatrix}
\]
\[
= \begin{bmatrix} \eta^{-1}(\zeta - \frac{1}{2}) & 0 \\ -1 & \eta(\zeta - \frac{1}{2})^{-1} \end{bmatrix} \begin{bmatrix} 1 & \eta(\zeta - \frac{1}{2})^{-1} \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} \eta^{-1}(\zeta - \frac{1}{2}) & 1 \\ -1 & 0 \end{bmatrix}
\]
which is analytic. Thus, (3.6) is entire.\footnote{This is no longer true if $s \leq -\frac{1}{2}$. Then (3.6) has a pole at $\zeta = \frac{1}{2}$.}
The next step is to compute the asymptotics of (3.6) at infinity. From the general formula [Bor2, (4.9)] it follows that

\[ m_s(\zeta) = I + \begin{bmatrix} \alpha_s & \beta_s \\ \gamma_s & \delta_s \end{bmatrix} \zeta^{-1} + O(\zeta^{-2}), \]

\[ m_{s+1}(\zeta) = I + \begin{bmatrix} \alpha_{s+1} & \beta_{s+1} \\ \gamma_{s+1} & \delta_{s+1} \end{bmatrix} \zeta^{-1} + O(\zeta^{-2}), \]

as \( \zeta \to \infty \), with some constants \( \alpha_s, \alpha_{s+1}, \ldots, \delta_s, \delta_{s+1} \). Hence, the asymptotics of (3.6) has the form

\[ \left( I + \begin{bmatrix} \alpha_s & \beta_s \\ \gamma_s & \delta_s \end{bmatrix} \zeta^{-1} \right) \left[ \eta^{-1}(\zeta - \frac{1}{2}) \begin{bmatrix} 0 & 0 \\ \gamma & \delta \end{bmatrix} \right] \left( I - \begin{bmatrix} \alpha_{s+1} & \beta_{s+1} \\ \gamma_{s+1} & \delta_{s+1} \end{bmatrix} \zeta^{-1} \right) + O(\zeta^{-1}) \]


\[ = \eta^{-1} \begin{bmatrix} \zeta - \frac{1}{2} + \alpha_s - \alpha_{s+1} & -\beta_{s+1} \\ \gamma_s & 0 \end{bmatrix} + O(\zeta^{-1}). \]

Denote \( a_s = -\eta^{-1}\beta_{s+1} \), \( b_s = -\eta^{-1}\gamma_s \), \( c_s = \alpha_{s+1} - \alpha_s \). Then Liouville's theorem implies that (3.6) is equal to

\[ \begin{bmatrix} \eta^{-1}(\zeta - \frac{1}{2} - c_s) & a_s \\ -b_s & 0 \end{bmatrix}. \]

It remains to show that \( c_s = p_s \) and \( a_s b_s = 1 \). The second equality follows from the fact that the determinant of (3.6) is equal to 1. To prove that \( c_s = p_s \), let us now substitute (3.4) into what we have just proved. We obtain

\[ m_s(\zeta - 1) \begin{bmatrix} \eta^{-1}(\zeta - \frac{1}{2}) & 0 \\ 0 & \eta(\zeta - \frac{1}{2})^{-1} \end{bmatrix} \]

\[ = \begin{bmatrix} \eta^{-1}(\zeta - \frac{1}{2} - c_s) & a_s \\ -b_s & 0 \end{bmatrix} \left( I + (\zeta - s)^{-1} \begin{bmatrix} p_s & q_s \\ r_s & -p_s \end{bmatrix} \right) m_s(\zeta). \]

Comparing the asymptotics of the (1,1) entry of both sides, we conclude that \( c_s = p_s \). □

**Corollary 3.3 (Compatibility conditions).** For any \( s \in \mathbb{Z}_+ \), we have

\[ a_s = a_{s+1} + \eta^{-1}q_{s+1}, \quad b_s = b_{s+1} + \eta^{-1}r_s, \quad \] (3.7)

\[ a_s r_s = -b_{s+1}q_{s+1}. \quad \] (3.8)

Denote \( a_s r_s \) by \( w_s \). Then

\[ (p_s + p_{s+1})(w_s - \eta) = \left( s + \frac{1}{2} \right) w_s, \quad \] (3.9)

\[ p_{s+1}^2 = w_s w_{s+1}. \quad \] (3.10)

**Proof.** Shifting \( \zeta \) by 1 in (3.5) and substituting the right–hand side of (3.5) into the right–hand side of (3.4) yields

\[ m_{s+1}(\zeta) = \left( I + \frac{A_s}{\zeta - s} \right) \begin{bmatrix} \eta^{-1}(\zeta + \frac{1}{2} - p_s) & a_s \\ -b_s & 0 \end{bmatrix} \]

\[ \times m_s(\zeta + 1) \begin{bmatrix} \eta(\zeta + \frac{1}{2})^{-1} & 0 \\ 0 & \eta^{-1}(\zeta + \frac{1}{2}) \end{bmatrix}. \]
On the other hand, shifting $s$ and $\zeta$ by 1 in (3.4) and (3.5), and substituting the right-hand side of (3.4) into the right-hand side of (3.5) gives

$$m_{s+1}(\zeta) = \left[ \eta^{-1}(\zeta + \frac{1}{2} - p_{s+1}) \quad a_{s+1} \quad 0 \right] \left( I + \frac{A_{s+1}}{\zeta - s} \right) \left( I + \frac{A_{s+1}}{\zeta - s} \right)^{-1} \eta^{-1}(\zeta + \frac{1}{2}).$$

Comparing these two relations, we conclude that

$$\left( I + \frac{A_{s}}{\zeta - s} \right) \left[ \eta^{-1}(\zeta + \frac{1}{2} - p_{s}) \quad a_{s} \quad 0 \right] = \left[ \eta^{-1}(\zeta + \frac{1}{2} - p_{s+1}) \quad a_{s+1} \quad 0 \right] \left( I + \frac{A_{s+1}}{\zeta - s} \right).$$

This is the consistency or compatibility relation for the Lax pair (3.4), (3.5).

If we were in a continuous situation, the consistency relation would have been obtained by cross-differentiation of the Lax pair equations.

Computing the asymptotics of the (1,2) and (2,1) elements of (3.11) at $\zeta = \infty$ yields (3.7). Let us compute the residue of both sides of (3.11) at $\zeta = s$. This gives

$$\left[ \begin{array}{cc} p & q \\ r & -p \end{array} \right] \left[ \eta^{-1}(\zeta + \frac{1}{2} - p_{s}) \quad a_{s} \quad 0 \right] = \left[ \eta^{-1}(\zeta + \frac{1}{2} - p_{s+1}) \quad a_{s+1} \quad 0 \right] \left[ \begin{array}{cc} p_{s+1} & q_{s+1} \\ r_{s+1} & -p_{s+1} \end{array} \right].$$

The (2,2) element of this equality is (3.8). The (1,2) element gives

$$a_{s}p_{s} = \eta^{-1}(\zeta + \frac{1}{2} - p_{s+1})q_{s+1} - a_{s+1}p_{s+1}. \quad \Box$$

Multiplying both sides by $b_{s+1}$, we obtain (recall that $a_{s+1}b_{s+1} = 1$)

$$b_{s+1}a_{s}p_{s} = -\eta^{-1}(\zeta + \frac{1}{2} - p_{s+1})w_{s} - p_{s+1}. \quad (3.12)$$

Multiplying the first relation of (3.7) by $b_{s+1}$ we see that $a_{s}b_{s+1} = 1 - \eta^{-1}w_{s}$. Substituting this into (3.12) yields (3.9).

As for (3.10), we have

$$p_{s+1}^{2} = -q_{s+1}r_{s+1} = (-b_{s+1}q_{s+1})(a_{s+1}r_{s+1}) = w_{s}w_{s+1}. \quad \Box$$

**Proposition 3.4 (dPII).** Assume that $w_{s} \neq 0$ for $s = \frac{1}{2}, \frac{3}{2}, \ldots, S$. Pick $v_{\frac{1}{2}}$ so that $v_{\frac{1}{2}}^{2} = \eta^{-1}w_{\frac{1}{2}}$, and define

$$v_{s} = \frac{p_{s}}{\eta v_{s-1}}, \quad s = \frac{3}{2}, \ldots, S + 1.$$

Then for $s = \frac{3}{2}, \ldots, S$ we have

$$v_{s-1} + v_{s+1} = \frac{(s + \frac{1}{2})v_{s}}{\eta(v_{s}^{2} - 1)}. \quad (3.13)$$

Moreover, $w_{s} = \eta v_{s}^{2}$ for all $s = \frac{1}{2}, \ldots, S + 1$.

The relation (3.13) is called the discrete Painlevé II equation, see e.g. [GNR].

**Proof.** The relation $w_{s} = \eta v_{s}^{2}$ follows from (3.10) and the definition of $v_{s}$. The equation (3.13) follows from (3.9), after we substitute $\eta v_{s}^{2}$ for $w_{s}$, $\eta v_{s-1}v_{s}$ for $p_{s}$, and $\eta v_{s}v_{s+1}$ for $p_{s+1}. \quad \Box$
4. Fredholm determinant and dPII

In this section we show how the Fredholm determinant $D_s$ is related to the solution $v_s$ of the dPII introduced in Proposition 3.4.

We will work under the assumption that $w_s \neq 0$ for all $s \in \mathbb{Z}_+$. As we will see in the next section, this assumption holds for generic values of $\eta$, see Proposition 5.4. Note that the nonvanishing of all $w_s$ implies that $w_s \neq \eta$, see (3.9), and $p_s \neq 0$, see (3.10). Then $q_s r_s = -p_s^2 \neq 0$, and $a_s \neq a_{s+1}$, $b_s \neq b_{s+1}$ by (3.7).

Set $R_s = K_s(1 - K_s)^{-1}$, where $K_s$ is the operator in $\ell^2(\{s, s+1, \ldots\})$ defined by the discrete Bessel kernel restricted to $\{s, s+1, \ldots\}$. Then

$$1 + R_s(s, s) = \frac{\det(1 - K_{s+1})}{\det(1 - K_s)} = \frac{D_{s+1}}{D_s}. \quad (4.1)$$

**Proposition 4.1.** Assume that $w_s \neq 0$ for $s \in \mathbb{Z}_+$, and define $v_s$ as in Proposition 3.4. Then for any $s \in \mathbb{Z}_+$ we have

$$\frac{D_{s+2}}{D_{s+1}} - \frac{D_{s+3}}{D_{s+2}} = \frac{(1 - v_s^2)v_{s+1}^2}{v_s^2} \left( \frac{D_{s+1}}{D_s} - \frac{D_{s+2}}{D_{s+1}} \right). \quad (4.2)$$

**Proof.** By Theorem 2.3(ii) we have

$$R_s(s, s) = g^t(s)m_s^{-1}(s)m_s'(s)f(s), \quad (4.3)$$

$$R_{s+1}(s+1, s+1) = g^t(s+1)m_{s+1}^{-1}(s+1)m'_{s+1}(s+1)f(s+1), \quad (4.4)$$

where

$$f(x) = \left( \frac{\eta^x}{1(x+\frac{1}{2})}, 0 \right)^t, \quad g(x) = \left( 0, \frac{\eta^x}{1(x+\frac{1}{2})} \right)^t.$$

Let us evaluate (4.3) using (3.5). We have

$$m_s^{-1}(s) = \begin{bmatrix} \eta^{-1}(s + \frac{1}{2}) & 0 \\ 0 & \eta(s + \frac{1}{2}) \end{bmatrix} m_{s+1}^{-1}(s+1) \begin{bmatrix} 0 & -a_s \\ b_s & \eta^{-1}(s + 1 - p_s) \end{bmatrix},$$

$$m_s'(s) = \frac{d}{d\zeta} \left( \begin{bmatrix} \eta^{-1}(\zeta - \frac{1}{2} - p_s) & a_s \\ -b_s & 0 \end{bmatrix} m_{s+1}(\zeta) \begin{bmatrix} \eta(\zeta - \frac{1}{2})^{-1} & 0 \\ 0 & \eta^{-1}(\zeta - \frac{1}{2}) \end{bmatrix} \right)_{\zeta=s+1}. $$

It is immediately verified that if the derivative in the last formula falls on the third (diagonal) factor then the corresponding contribution to (4.3) vanishes. If the derivative falls on the second factor, the contribution to (4.3) equals $R_{s+1}(s+1, s+1)$ because of (4.4) and the equalities

$$g^t(s) \begin{bmatrix} \eta^{-1}(s + \frac{1}{2}) & 0 \\ 0 & \eta(s + \frac{1}{2}) \end{bmatrix} = g^t(s+1),$$

$$\begin{bmatrix} \eta(\zeta - \frac{1}{2})^{-1} & 0 \\ 0 & \eta^{-1}(\zeta - \frac{1}{2}) \end{bmatrix} f(s) = f(s+1).$$

---

5 The existence of $(1 - K_s)^{-1}$ follows, for example, from the fact that $D_s = \det(1 - K_s) \neq 0$. 

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Finally, when the derivative falls on the first factor, using the relations above, we see that the contribution to (4.3) is equal to

$$g'(s + 1)m_{s+1}^{-1}(s + 1) \begin{bmatrix} 0 & 0 \\ \eta^{-1}b_s & 0 \end{bmatrix} m_{s+1}(s + 1)f(s + 1). \tag{4.5}$$

For $x \in \mathbb{Z}_+^t$ denote

$$m_x(x) = \begin{bmatrix} m_{x1}^{21} & m_{x2}^{12} \\ m_{x1}^{11} & m_{x2}^{22} \end{bmatrix}.$$ 

Since det $m_x \equiv 1$, we have

$$m_x(x) = \begin{bmatrix} m_{x1}^{22} & -m_{x1}^{12} \\ -m_{x2} & m_{x1}^{11} \end{bmatrix},$$

and (4.5) turns into

$$g'(s + 1) \begin{bmatrix} m_{s+1}^{22} & -m_{s+1}^{12} \\ -m_{s+1} & m_{s+1}^{11} \end{bmatrix} \begin{bmatrix} 0 & 0 \\ \eta^{-1}b_s & 0 \end{bmatrix} \begin{bmatrix} m_{s+1}^{11} & m_{s+1}^{12} \\ m_{s+1} & m_{s+1}^{22} \end{bmatrix} f(s + 1)$$

$$= \frac{\eta^{2s+1}}{\Gamma^2 (s + \frac{3}{2})} b_s (m_{s+1}^{11})^2.$$

Denote $R_s(s, s) - R_{s+1}(s + 1, s + 1)$ by $\delta_s$. We have just proved that

$$\delta_s = \frac{\eta^{2s+1}}{\Gamma^2 (s + \frac{3}{2})} b_s (m_{s+1}^{11})^2. \tag{4.6}$$

The (2,1) element of the equation (3.5) at the point $\zeta = s + 1$ gives

$$m_{s1}^{21} = -b_s m_{s+1}^{11} \eta^{-1}(s + \frac{1}{2})^{-1}.$$

Hence, $m_{s+1}^{11} = -b^{-1}_s m_{s+1}^{21} \eta^{-1}(s + \frac{1}{2})$. Substituting this into the right-hand side of (4.6) and using $a_s = b_s^{-1}$, we obtain

$$\delta_s = \frac{\eta^{2s-1}}{\Gamma^2 (s + \frac{1}{2})} a_s (m_s^{21})^2. \tag{4.7}$$

Look at the residue of $m_{s+1}(\zeta)$ at the point $\zeta = s$. Since the jump matrix $-f(s)g'(s)$ has zero first column, the residue itself also has zero first column. On the other hand, (3.4) implies that this residue equals $A_s m_s(s)$. Equating the (1,1) element of this matrix to zero we obtain

$$p_s m_s^{11} = -q_s m_s^{21}.$$

By our assumption $q_s \neq 0$, hence, $m_s^{21} = -q_s^{-1} p_s m_s^{11}$. Then (4.7) turns into

$$\delta_s = \frac{\eta^{2s-1}}{\Gamma^2 (s + \frac{1}{2})} a_s p_s^2 (m_s^{11})^2 = -\frac{\eta^{2s-1}}{\Gamma^2 (s + \frac{1}{2})} \frac{w_s}{q_s} (m_s^{11})^2,$$

where we used $p_s^2 = -q_s r_s$ and $a_s r_s = w_s$. Comparing the last equality with (4.6) we conclude that

$$\delta_{s+1} = \frac{b_{s+1} w_{s+1}}{b_s w_s} \delta_s. \tag{4.8}$$

The relation (3.7) implies that $b_{s+1} b_s^{-1} = 1 - \eta^{-1} w_s = 1 - v_s^2$, and (4.2) follows. \qed
Corollary 4.2. Under the assumption of Proposition 4.1, there exist constants \( \kappa \) and \( \nu \) such that
\[
\frac{D_{s+1}}{D_s} = \nu + \kappa \cdot \eta b_s
\]
for all \( s \in \mathbb{Z}'_+ \).

Proof. By (4.8) we have \( \delta_s = \kappa b_s w_s = \kappa r_s \) for some constant \( \kappa \). Then the second formula of (3.7) implies
\[
\delta_s = \frac{D_{s+1}}{D_s} - \frac{D_{s+2}}{D_{s+1}} = \kappa \cdot \eta (b_s - b_{s+1}),
\]
and (4.9) follows. \( \square \)

5. Initial conditions for dPII

The goal of this section is to provide initial conditions for the dPII in Proposition 3.4 and to find the constants \( \kappa \) and \( \nu \) from Corollary 4.2. We will also prove that the assumption of Propositions 3.4 and 4.1 holds for generic \( \eta \).

Lemma 5.1. The function \( m_{\frac{1}{2}}(\zeta) \) has the form
\[
m_{\frac{1}{2}}(\zeta) = \begin{bmatrix} 1 & 0 \\ -\sum_{x \in \mathbb{Z}'_+ \cup (-x+\frac{1}{2})}^{1} & 1 \\ \frac{\eta^{-2x}}{\Gamma^2(-x+\frac{1}{2})} & \zeta - x \end{bmatrix}.
\]

Proof. A direct computation shows that the right-hand side of (5.1) solves the normalized DRHP \( (\mathbb{Z}'_-, -fg^t) \). The uniqueness statement in Theorem 2.3(i) concludes the proof. \( \square \)

Proposition 5.2. We have
\[
A_{\frac{1}{2}} = \begin{bmatrix} p_{\frac{1}{2}} & q_{\frac{1}{2}} \\ r_{\frac{1}{2}} & -p_{\frac{1}{2}} \end{bmatrix} = \frac{\eta}{I_0(2\eta)} \begin{bmatrix} -I_1(2\eta) & -1 \\ I_1^2(2\eta) & I_1(2\eta) \end{bmatrix},
\]
\[
a_{\frac{1}{2}} = I_0^{-1}(2\eta), \quad b_{\frac{1}{2}} = I_0(2\eta),
\]
\[
w_{\frac{1}{2}} = a_{\frac{1}{2}} r_{\frac{1}{2}} = \frac{\eta I_1^2(2\eta)}{I_0^2(2\eta)},
\]
where \( I_\nu(\cdot) \) is the \( I \)-Bessel function.

Proof. By (3.4) we have
\[
m_{\frac{1}{2}}(\zeta) = \left( I + \frac{A_{\frac{1}{2}}}{\zeta - \frac{1}{2}} \right) m_{\frac{1}{2}}(\zeta).
\]
Using (5.1) and (3.3) we see that the residue condition for \( m_{\frac{1}{2}}(\zeta) \) at the point \( \zeta = \frac{1}{2} \) looks as follows
\[
A_{\frac{1}{2}} \begin{bmatrix} 1 & 0 \\ -\sum_{x \in \mathbb{Z}'_+ \cup (-x+\frac{1}{2})}^{1} & 1 \\ \frac{\eta^{-2x}}{\Gamma^2(-x+\frac{1}{2})} & \zeta - x \end{bmatrix} = \lim_{\zeta \to \frac{1}{2}} \left( I + \frac{A_{\frac{1}{2}}}{\zeta - \frac{1}{2}} \right) \begin{bmatrix} 1 & 0 \\ -\sum_{x \in \mathbb{Z}'_+ \cup (-x+\frac{1}{2})}^{1} & 1 \\ \frac{\eta^{-2x}}{\Gamma^2(-x+\frac{1}{2})} & \zeta - x \end{bmatrix} = \begin{bmatrix} 0 & -\eta \\ 0 & 0 \end{bmatrix}.
\]
Recall that
\[ I_0(2\eta) = \sum_{k \geq 0} \frac{\eta^{2k}}{\Gamma^2(k+1)}, \quad I_1(2\eta) = \sum_{k \geq 0} \frac{\eta^{2k+1}}{(k+1)\Gamma^2(k+1)}. \]

Since the (1,1) element of the right-hand side of (5.3) vanishes, we obtain
\[ p_{\frac{1}{2}} = I_1(2\eta) \cdot q_{\frac{1}{2}}. \]

Since the matrix \( A_{\frac{1}{2}} \) is nilpotent, this implies that
\[ A_{\frac{1}{2}} = c \cdot \begin{bmatrix} -I_1(2\eta) & -1 \\ I_1^2(2\eta) & I_1(2\eta) \end{bmatrix} \]
with some constant \( c \). Then the (1,2) element of (5.3) gives
\[ -c = -\eta + c \cdot \sum_{x \in \mathbb{Z}'} \frac{\eta^{-2x+1}}{(x+\frac{1}{2})^2 \Gamma^2(x+\frac{1}{2})}. \]

Hence, \( c = \eta/I_0(2\eta) \), which proves the first line of (5.2).

Next, (3.5) implies that \( -b_{\frac{1}{2}} \) is the limit value of the (2,1) element of the matrix
\[ m_{\frac{1}{2}}(\zeta-1) \begin{bmatrix} \eta^{-1}(\zeta-\frac{1}{2}) & 0 \\ 0 & \eta(\zeta-\frac{1}{2})^{-1} \end{bmatrix} \]
as \( \zeta \to \infty \). (Indeed, \( m_{\frac{1}{2}} \to I \) as \( \zeta \to \infty \).) Since the (2,1) element of \( m_{\frac{1}{2}}(\zeta) \), see (5.1), has the form
\[ -\eta I_0(2\eta) \cdot \zeta^{-1} + O(\zeta^{-2}), \quad \zeta \to \infty, \]
we conclude that \( b_{\frac{1}{2}} = I_0(2\eta) \). This means that \( a_{\frac{1}{2}} = b_{\frac{1}{2}}^{-1} = I_0^{-1}(2\eta) \), and the proof of Proposition 5.2 is complete. \( \square \)

**Corollary 5.3 (Initial conditions for dPII).** Assume that \( w_s \neq 0 \) for all \( s \in \mathbb{Z}'_+ \). Then the sequence \( v_s \) from Proposition 3.4 can be defined by the initial conditions
\[ v_{\frac{1}{2}} = \pm 1, \quad v_{\frac{3}{2}} = \pm \frac{I_1(2\eta)}{I_0(2\eta)}, \quad (5.4) \]
and the dPII equation (3.13) with \( s \geq \frac{1}{2} \).

**Proof.** We will omit the argument \( 2\eta \) in the Bessel functions below. From (5.2) and (3.9) we obtain
\[ p_{\frac{1}{2}} \frac{w_{\frac{1}{2}}}{w_{\frac{1}{2}} - \eta} - p_{\frac{3}{2}} \frac{I_1^2}{I_1^2 - I_0^2} + \frac{\eta I_1}{I_0}, \]
and by definition of \( v_s \)
\[ v_{\frac{1}{2}} = \pm \frac{I_1}{I_0}, \quad v_{\frac{3}{2}} = \frac{p_{\frac{1}{2}}}{\eta v_{\frac{1}{2}}} = \pm \frac{I_0 I_1}{\eta (I_1^2 - I_0^2)} \pm 1. \]

This is exactly the same value of \( p_{\frac{1}{2}} \) as we would obtain by substituting (5.4) into (3.13) with \( s = \frac{1}{2} \). \( \square \)
Proposition 5.4. The nonvanishing $w_s \neq 0$ for all $s \in \mathbb{Z}'_+$ holds for all but countably many $\eta$.

Proof. By Proposition 3.4, it is enough to show that for any fixed $s \in \mathbb{Z}'_+$, $v_s$ defined via (5.4) and (3.13) does not vanish for all but countably many values of $\eta$. Clearly, $v_s$ is a meromorphic function in $\eta$. Thus, it suffices to prove that $v_s$ does not vanish identically.

It is easy to verify that the formulas

$$v_{2k-\frac{1}{s}} = (-1)^k + \frac{kv}{s^2-1} \eta^{-1} + O(\eta^{-2}),$$

$$v_{2k+\frac{1}{s}} = (-1)^k v(-1)^k + O(\eta^{-1}),$$

for $k = 0, 1, \ldots$, define an asymptotic solution of (3.13) as $\eta \to \infty$. Here $v$ is an arbitrary constant not equal to 0 or 1, and for a fixed $k$ the asymptotics is uniform in $v$ varying in any compact subset of $\mathbb{C} \setminus \{0,1\}$.

Since $I_1(2\eta)/I_0(2\eta)$ has zeros and poles with arbitrarily large $|\eta|$, see e.g. [Er, 7.9], we can find a sequence $\{\eta_m\}_{m=1}^\infty$ such that $|\eta_m| > m$ and $|I_1(2\eta_m)/I_0(2\eta_m)| \in [\frac{1}{3}, \frac{2}{3}]$. Denote by $\{v_s^{(m)}\}_{s \in \mathbb{Z}'_+}$ the sequence defined by (3.13) and (5.4) with $\eta = \eta_m$.

Then by (5.5), for any fixed $s \in \mathbb{Z}'_+$, $v_s^{(m)}$ is equal to 1, $-1$, $v$, or $-v^{-1}$, where $v = I_1(2\eta_m)/I_0(2\eta_m)$, up to a correction term with goes to zero as $m \to \infty$. Hence, for large enough $m$ we have $v_s^{(m)} \neq 0$. □

Corollary 5.5 (cf. Corollary 4.2). For all but countably many values of $\eta$, we have $D_{s+1}/D_s = b_s$ for all $s \in \mathbb{Z}'_+$.

Proof. Corollary 4.2 implies that $\delta_s = z_1 \gamma \frac{1}{2}$. The interpretation of $D_s$ as a Toeplitz determinant, see (3.2), easily implies that $\delta_s = I_1^2(2\eta)/I_0(2\eta)$. Comparing this with the value of $r_1$ from (5.2) we see that $\tau = \eta^{-1}$. Furthermore, (3.2) implies that $D_s^2/D_s = I_0(2\eta)$. Since $b_s = I_0(2\eta)$, we conclude that $\nu$ in (4.9) vanishes. □

We can now state our final result.

Theorem 5.6. Let $s \in \mathbb{Z}'_+$, and let $D_s$ be the Fredholm determinant of the discrete Bessel kernel as defined in the beginning of §3. Define a sequence $\{v_s\}_{s \in \mathbb{Z}'_+}$ by initial conditions (5.4) and the recurrence equation (3.13). Then for all but countably many values of $\eta$ we have

$$v_s^2 = 1 - \frac{D_s D_{s+2}}{D_{s+1}^2}.$$ (5.6)

Proof. Follows from Corollary 5.5, the definition of $\{v_s\}$ and the relation $\eta^{-1} w_s = 1 - b_{s+1}/b_s$, see (3.7). □

Remark 5.7. If we extend $D_s$ to an entire function in $\eta$ using the Toeplitz determinant interpretation, see (3.2), then (5.6) can be viewed as an equality of meromorphic functions.

6. Discrete $2F_1$ kernel and dPV

In this section we obtain results similar to those of §3, but for a more complicated kernel. In particular, we will see that the Fredholm determinant can be expressed through a solution of a dPV equation.
The discrete \(2F_1\) kernel plays a key role in harmonic analysis on the infinite symmetric group. We refer the reader to [BO2] and [BO3] for a detailed description.

Set

\[
m(\zeta) = \begin{bmatrix}
F\left(-z, -z'; \zeta + \frac{1}{2}; \frac{\xi}{\xi-1}\right) & \frac{\sqrt{z z'}}{1-\xi} F\left(1 + z, 1 + z'; -\zeta + \frac{3}{2}; \frac{\xi}{\xi-1}\right) \\
-\sqrt{z z'} \frac{F\left(1 - z, 1 - z'; \zeta + \frac{3}{2}; \frac{\xi}{\xi-1}\right)}{\zeta + \frac{1}{2}} & F\left(z, z'; -\zeta + \frac{1}{2}; \frac{\xi}{\xi-1}\right)
\end{bmatrix}
\]  

(6.1)

\[
h_+(x) = \frac{(z')^{\frac{1}{2}} \xi^\frac{1}{2} \left(1 - \xi\right)^{-\frac{1}{2} + z'}}{\Gamma(x + \frac{1}{2})} \sqrt{(z + 1)_{x-\frac{1}{2}} (z' + 1)_{x-\frac{1}{2}}},
\]

\[
h_-(x) = \frac{(z')^{\frac{1}{2}} \xi^{-\frac{1}{2}} \left(1 - \xi\right)^{-\frac{1}{2} + z'}}{\Gamma(-x + \frac{1}{2})} \sqrt{(-z + 1)_{-x-\frac{1}{2}} (-z' + 1)_{-x-\frac{1}{2}}}.
\]

Here \(F(a; b; c; u)\) is the Gauss hypergeometric function, \((a)_k = \Gamma(a + k)/\Gamma(a)\) is the Pochhammer symbol, \((z, z')\) are two complex parameters such that \((z + k)(z' + k) > 0\) for all \(k \in \mathbb{Z}\) (for instance, \(z' = \bar{z} \in \mathbb{C} \setminus \mathbb{Z}\)), \(\xi \in (0, 1)\) is also a parameter. It is also convenient to assume that \(z \neq z'\).

The discrete \(2F_1\) kernel is a kernel on \(\mathbb{Z}' \times \mathbb{Z}'\). For \(x, y \in \mathbb{Z}'_+\) it is defined by the formula

\[
K(x, y) = \begin{cases}
h_+(x)h_+(y) \frac{m^{21}(x)m^{11}(y) - m^{11}(x)m^{21}(y)}{x - y}, & x \neq y \\
h_+^2(x) \left(\frac{dm^{21}(x)}{dx}m^{11}(x) - \frac{dm^{11}(x)}{dx}m^{21}(x)\right), & x = y.
\end{cases}
\]

The general definition for \(x\) and \(y\) not necessarily positive can be found in [BO2]. Note that when one of the parameters \(z, z'\) tends to an integer, the kernel (or rather the part of the kernel written above) turns into a Christoffel–Darboux kernel for Meixner orthogonal polynomials, see [BO2, §4].

As in §3, for any \(s \in \mathbb{Z}'_+\), denote by \(K_s\) the operator in \(l^2(\{s, s+1, \ldots\})\) defined by the restriction of \(K\) to \(\{s, s+1, \ldots\} \times \{s, s+1, \ldots\}\). Then \(K_s\) is a positive trace class operator. We will be interested in the Fredholm determinants

\[
D_s = \det(1 - K_s).
\]

These Fredholm determinants can be also expressed as Toeplitz determinants, see [BOk, §4] and also [BW], [Bot]:

\[
D_s = (1 - \xi)^{zz'} \cdot \det[t_{i-j}]_{i,j=1,\ldots,s-\frac{1}{2}},
\]

where

\[
t_k = \begin{cases}
\xi^{k/2} \Gamma(-z + k) \Gamma(-z) \Gamma(k + 1) \ F(-z + k, -z'; k + 1; \xi), & k \geq 0, \\
\xi^{-k/2} \Gamma(-z' - k) \Gamma(-z') \Gamma(k + 1) \ F(-z' - k, -z; -k + 1; \xi), & k < 0.
\end{cases}
\]

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The symbol of the Toeplitz determinants is equal to
\[
\sum_{k \in \mathbb{Z}} t_k \zeta^k = (1 + \sqrt{\xi} \zeta)^{z} (1 + \sqrt{\xi} \zeta^{-1})^{z'}.
\]

Note that in the Introduction we denoted \( D_s \) by \( q_{s-\frac{1}{2}} \).

The discrete \(_2F_1\) kernel fits in Situation 2.1 of \( \S 2 \): There exists an integrable kernel \( L(x, y) \) on \( \mathbb{Z}' \times \mathbb{Z}' \) such that it defines a bounded operator in \( \ell^2(\mathbb{Z}') \), and \( K = L(1 + L)^{-1} \), see [BO2]. Specifically, with respect to the splitting \( \mathbb{Z}' = \mathbb{Z}'_+ \cup \mathbb{Z}'_- \) the kernel \( L \) has the form
\[
L(x, y) = \begin{bmatrix}
0 & \frac{h_+(x) h_-(y)}{x-y} \\
\frac{h_-(x) h_+(y)}{x-y} & 0
\end{bmatrix}
\]

with \( h_\pm \) as above. In the notation of \( \S 2 \) we have \( \mathfrak{X} = \mathbb{Z}' \) and the matrix \( m_\mathfrak{X}(\zeta) \) coincides with \( m(\zeta) \) of (6.1). Furthermore, \( f_1(x) = g_2(x) = \begin{cases} h_+(x), & x \in \mathbb{Z}'_+, \\ 0, & x \in \mathbb{Z}'_- \end{cases} \) and \( f_2(x) = g_1(x) = \begin{cases} 0, & x \in \mathbb{Z}'_+, \\ h_-(x), & x \in \mathbb{Z}'_- \end{cases} \), and the jump matrix \( w_\mathfrak{X}(x) = -f(x)g(x)^t \) has the form
\[
w_\mathfrak{X}(x) = \begin{cases}
\begin{bmatrix} 0 & -h_+^2(x) \\ 0 & 0 \end{bmatrix}, & x \in \mathbb{Z}'_+, \\
\begin{bmatrix} 0 & 0 \\ -h_-^2(x) & 0 \end{bmatrix}, & x \in \mathbb{Z}'_-.
\end{cases}
\]

See [Bor2] for details.

Our next goal is to study the unique solution \( m_3 \) of the normalized DRHP \((3, w_\mathfrak{X})\), where
\[
3 = \mathfrak{X} \setminus \mathfrak{Y} = \mathbb{Z} \setminus \{s, s+1, \ldots\} = \{\ldots, s-2, s-1\},
\]
see Theorem 2.3(i). As in \( \S 3 \), we redenote \( m_3 \) by \( m_s \). We also denote \( 3_s = \{\ldots, s-2, s-1\} \). Recall that \( \det m_s \equiv 1 \). Set
\[
\Xi = \begin{bmatrix} \xi^\frac{1}{2} & 0 \\ 0 & \xi^{-\frac{1}{2}} \end{bmatrix}.
\]

The following statement is an analog of Proposition 3.2.

**Proposition 6.1 (Lax pair).** For any \( s \in \mathbb{Z}'_+ \) there exist constant \( 2 \times 2 \) matrices \( A_s \) and \( B_s \) such that
\[
m_{s+1}(\zeta) = \left( I + \frac{A_s}{\zeta - s} \right) m_s(\zeta), \quad (6.2)
\]
\[
m_s(\zeta - 1) = \Xi^{-1} \left( I + \frac{B_s}{\zeta + z - \frac{1}{2}} \right) m_{s+1}(\zeta) \begin{bmatrix} \zeta^\frac{1}{2} & 0 \\ 0 & \zeta^{-\frac{1}{2}} \end{bmatrix} \Xi. \quad (6.3)
\]
Furthermore,

$$\text{Tr } A_s = \text{Tr } B_s = 0, \quad \det A_s = 0, \quad \det B_s = z' - z,$$

$$A_s + B_s = \begin{bmatrix} -z & \ast \\ \ast & z' \end{bmatrix}. \quad (6.4)$$

**Proof.** The proofs of (6.2) and of the fact that $A_s$ is nilpotent are very similar to the proofs of analogous statements in Proposition 3.2.

Similarly to the proof of Proposition 3.2, it is easy to show that

$$m_s(\zeta - 1) \Xi^{-1} \begin{bmatrix} \frac{\zeta + z - \frac{1}{2}}{\zeta - s} & 0 \\ 0 & \frac{\zeta + z' - \frac{1}{2}}{\zeta - s} \end{bmatrix} m_{s+1}(\zeta) \Xi^{-1} m_{s+1}(\zeta) \Xi \quad (6.5)$$

has no singularities in $\mathbb{Z}'$. Thus, the only possible singularity of (6.5) is a simple pole at $\zeta = -z + \frac{1}{2}$. (Note that by our assumption on the parameters $z, z'$, they are both nonintegral, and $-z + \frac{1}{2} \notin \mathbb{Z}'$.) Denote the residue by $C_s$. Since both $m_s(\zeta)$ and $m_{s+1}(\zeta)$ tend to $I$ as $\zeta \to \infty$, we see that (6.5) tends to $\Xi^{-1}$ as $\zeta \to \infty$, and by Liouville’s theorem we obtain (6.3) with $B_s = \Xi C_s$.

Taking the determinants of both sides of (6.3) we see that

$$\det \left( I + \frac{B_s}{\zeta + z - \frac{1}{2}} \right) = 1 + \frac{z' - z}{\zeta + z - \frac{1}{2}}.$$  
This implies that $\text{Tr } B_s = z' - z$ and $\det B_s = 0$.

Finally, let us substitute (6.2) into the right-hand side of (6.3). We obtain

$$m_s(\zeta - 1) \Xi^{-1} \left( I + \frac{B_s}{\zeta + z - \frac{1}{2}} \right) \left( I + \frac{A_s}{\zeta - s} \right) m_{s+1}(\zeta) \Xi \quad (6.6)$$

We know that $m_s(\zeta) = I + m_s^{(1)} \zeta^{-1} + O(\zeta^{-2})$ as $\zeta \to \infty$, with a constant matrix $m_s^{(1)}$. (This follows from the general formula [Bor2, (4.9)].) Substituting this asymptotic relation into both sides of (6.6) and comparing the asymptotics of the diagonal elements of both sides at $\zeta = \infty$ we obtain

$$A_s^{11} + B_s^{11} = -z, \quad A_s^{22} + B_s^{22} = z'. \quad \square$$

For any $2 \times 2$ matrix $M$ we will denote by $M^\Xi$ the matrix $\Xi M \Xi^{-1}$.

**Corollary 6.2 (Consistency relations).** For any $s \in \mathbb{Z}'_+$ we have

$$A_{s+1} - A_s^\Xi = B_s - B_{s+1} = \frac{A_s B_s - B_{s+1} A_{s+1}}{s + z + \frac{1}{2}}. \quad (6.7)$$

**Proof.** Similarly to the proof of Corollary 3.3, we compute $m_{s+1}(\zeta)$ in two different ways. Using (6.2) and then (6.3) we get

$$m_{s+1}(\zeta) = \left( I + \frac{A_s}{\zeta - s} \right) \Xi^{-1} \left( I + \frac{B_s}{\zeta + z - \frac{1}{2}} \right) m_{s+1}(\zeta + 1) \Xi.$$
On the other hand, using (6.3) first and then (6.2) we get

\[ m_{s+1}(\zeta) = \Xi^{-1} \left( I + \frac{B_{s+1}}{\zeta + z + \frac{1}{2}} \right) \left( I + \frac{A_{s+1}}{\zeta - s} \right) m_{s+1}(\zeta + 1) \left[ \begin{array}{cc} \frac{\zeta + s + \frac{1}{2}}{\zeta + \frac{1}{2}} & 0 \\ \frac{\zeta + s + \frac{1}{2}}{\zeta + s + \frac{3}{2}} & 0 \end{array} \right] \Xi. \]

Comparing the results we see that

\[ \left( I + \frac{A_s}{\zeta - s} \right) \Xi^{-1} \left( I + \frac{B_s}{\zeta + z + \frac{1}{2}} \right) = \Xi^{-1} \left( I + \frac{B_{s+1}}{\zeta + z + \frac{1}{2}} \right) \left( I + \frac{A_{s+1}}{\zeta - s} \right). \]

Equating the residues of both sides at \( \zeta = s \) and \( \zeta = -z - \frac{1}{2} \) yields (6.7). \( \square \)

In order to proceed to deriving the dPV equation we need to impose certain non-degeneracy conditions on \( A_s, B_s \) similar to the condition \( w_s \neq 0 \) used in Proposition 3.4.

We will say that matrices \( A_s \) and \( B_s \) are generic if we can uniquely parameterize them by

\[ A_s = (z + b_s) \begin{bmatrix} -1 & -\alpha_s \beta_s \\ 1/\alpha_s \beta_s & 1 \end{bmatrix}, \quad B_s = \begin{bmatrix} b_s & b_s \beta_s \\ (z' - z - b_s)/\beta_s & z' - z - b_s \end{bmatrix} \] (6.8)

with some \( b_s, \alpha_s \neq 0, \beta_s \neq 0 \). This is certainly true if the off–diagonal entries of \( A_s \) and \( B_s \) are nonzero:

\[ A_{s1}^2, A_{s1}^{21}, B_{s1}^{12}, B_{s1}^{21} \neq 0. \]

It is also true if

\[ A_{s1}^{12}, A_{s1}^{21}, B_{s1}^{21}, B_{s1}^{22} \neq 0, \quad B_{s1}^{11} = B_{s1}^{12} = 0. \]

Then \( b_s = 0, \beta_s = (z' - z)/B_{s1}^{21}, \alpha_s = -A_{s1}^{12}/(z\beta_s) \). Recall that \( z \neq z' \) by the assumption. Similarly, if

\[ A_{s1}^{12}, A_{s1}^{21}, B_{s1}^{11}, B_{s1}^{12} \neq 0, \quad B_{s1}^{21} = B_{s1}^{22} = 0. \]

Then \( b_s = z' - z, \beta_s = B_{s1}^{12}/(z' - z), \alpha_s = -A_{s1}^{12}/(z\beta_s) \). These three cases exhaust all possibilities.

**Proposition 6.3 (dPV).** Fix \( s \in \mathbb{Z}_+ \). Assume that the matrices \( A_s \) and \( B_s \) have nonzero off–diagonal entries, and in the notation (6.8)

\[ \xi \alpha_s \neq 1, \quad b_s + \frac{z' + s + \frac{1}{2}}{1 - \xi \alpha_s} \notin \{-z, z', z' - z, z' + s + \frac{1}{2}, z' - z + s + \frac{1}{2}\}. \] (6.9)

Then the matrices \( A_{s+1} \) and \( B_{s+1} \) are generic, and

\[ \alpha_{s+1} = \frac{\left( b_s + \frac{z' + s + \frac{1}{2}}{1 - \xi \alpha_s} - (z' + s + \frac{1}{2}) \right) \left( b_s + \frac{z' + s + \frac{1}{2}}{1 - \xi \alpha_s} - (z' - z + s + \frac{1}{2}) \right)}{\xi \alpha_s \left( b_s + \frac{z' + s + \frac{1}{2}}{1 - \xi \alpha_s} + \frac{z'}{\xi \alpha_s} \right) \left( b_s + \frac{z' + s + \frac{1}{2}}{1 - \xi \alpha_s} + \frac{z}{\xi \alpha_s} - z' \right)} \] (6.10)

\[ b_{s+1} = -b_s - \frac{z' + s + \frac{1}{2}}{1 - \xi \alpha_s} + \frac{z + s + \frac{1}{2}}{1 - \alpha_{s+1}} - 2z + z', \] (6.11)

\[ \beta_{s+1} = \frac{\xi \alpha_s \left( b_s + \frac{z' + s + \frac{1}{2}}{1 - \xi \alpha_s} + z' \right)}{b_s + \frac{z' + s + \frac{1}{2}}{1 - \xi \alpha_s} - (z' - z + s + \frac{1}{2})} \cdot \beta_s. \] (6.12)
Remarks 6.4. 1. The conditions (6.9) mean that four factors in the right-hand side of (6.10) as well as two factors in the right-hand side of (6.12) and the denominator \((1 - \xi \alpha_s)\) in (6.11) do not vanish. As we will see in the proof, (6.9) also implies that \(\alpha_{s+1} \neq 1\), hence, the right-hand side of (6.11) is also well-defined.

2. If we define \((\alpha_{s+1}, \beta_{s+1}, b_{s+1})\) via (6.10)–(6.12) and define the matrices \(A_{s+1}, B_{s+1}\) using (6.8) (with \(s\) replaced by \(s + 1\)), then the consistency relations (6.7) will hold.

3. The relations (6.10), (6.11) define a map \((\alpha_s, b_s) \mapsto (\alpha_{s+1}, b_{s+1})\). The inverse map has a similar form. Let us assume that \(A_{s+1}\) and \(B_{s+1}\) have nonzero off-diagonal entries, and

\[
\alpha_{s+1} \neq 1, \quad b_{s+1} - \frac{z + s + \frac{1}{2}}{1 - \alpha_{s+1}} \notin \{-z, z' - z, -(z + s + \frac{1}{2}), -(2z + s + \frac{1}{2})\}.
\]

Then the matrices \(A_s\) and \(B_s\) are generic and

\[
\alpha_s = \frac{(b_{s+1} - \frac{z + s + \frac{1}{2}}{1 - \alpha_{s+1}} + z + s + \frac{1}{2})}{\xi \alpha_{s+1} (b_{s+1} - \frac{z + s + \frac{1}{2}}{1 - \alpha_{s+1}} + z)} \left( b_{s+1} - \frac{z + s + \frac{1}{2}}{1 - \alpha_{s+1}} + 2z + s + \frac{1}{2} \right) + 1 - \xi \alpha_s,
\]

\[
b_s = -b_{s+1} + \frac{z + s + \frac{1}{2} - z' + s + \frac{1}{2}}{1 - \alpha_{s+1}} - \frac{z' + s + \frac{1}{2} - 2z + z'}{1 - \xi \alpha_s},
\]

(6.13)

\[
\beta_s = \frac{\alpha_{s+1} (b_{s+1} - \frac{z + s + \frac{1}{2}}{1 - \alpha_{s+1}} + z)}{b_{s+1} - \frac{z + s + \frac{1}{2}}{1 - \alpha_{s+1}} + 2z + s + \frac{1}{2}} \cdot \beta_{s+1}.
\]

4. Let us introduce a new variable

\[
c_s = b_s + \frac{z' + s + \frac{1}{2}}{1 - \xi \alpha_s} + z
\]

and rewrite (6.9)–(6.12) in using \((c_s, c_{s+1})\) instead of \((b_s, b_{s+1})\). Then the formulas become slightly simpler:

\[
\xi \alpha_s \neq 1, \quad c_s \notin \{0, z', z' + s + \frac{1}{2}, z + z' + s + \frac{1}{2}\},
\]

\[
\alpha_s \alpha_{s+1} = \frac{(c_s - (z + z' + s + \frac{1}{2})) (c_s - (z' + s + \frac{1}{2}))}{\xi c_s (c_s - z')},
\]

(6.14)

\[
c_s + c_{s+1} = \frac{z + s + \frac{1}{2} + z' + s + \frac{3}{2}}{1 - \alpha_{s+1}} + z',
\]

\[
\frac{\beta_{s+1}}{\beta_s} = \frac{\xi \alpha_s c_s}{c_s - (z' + s + \frac{1}{2})}.
\]

Similarly, if we set

\[
d_s = b_s - \frac{z + s - \frac{1}{2}}{1 - \alpha_s} + z
\]
then the nondegeneracy condition and (6.13) take the form

\[
\alpha_{s+1} \neq 1, \quad d_{s+1} \notin \{0, z', -(s + \frac{1}{2}), -(z + s + \frac{1}{2})\}, \\
\alpha_s\alpha_{s+1} = \frac{(d_{s+1} + s + \frac{1}{2})(d_{s+1} + z + s + \frac{1}{2})}{\xi d_{s+1}(d_{s+1} - z')}, \\
d_s + d_{s+1} = -\frac{z' + s + \frac{1}{2}}{1 - \xi} = \frac{z + s - \frac{1}{2}}{1 - \alpha_s} + z', \\
\beta_s = \frac{\alpha_{s+1} d_{s+1}}{d_{s+1} + z + s + \frac{1}{2}}.
\] (6.15)

The relations between \((\alpha_s, d_s)\) and \((\alpha_{s+1}, d_{s+1})\) form a special case of the difference Painlevé V equation of [Sak, §7]. The parameters \(a_0, \ldots, a_4\) in our case are as follows:

\[
a_0 = -(z' + s + \frac{1}{2}), \quad a_1 = -(z + s + \frac{1}{2}), \quad a_2 = s + \frac{3}{2}, \quad a_3 = z', \quad a_4 = z, \quad \lambda = a_0 + a_1 + 2a_2 + a_3 + a_4 = 1.
\]

**Proof of Proposition 6.3.**

**Step 1.** Let us first show that the matrices \(A_{s+1}\) and \(B_{s+1}\) are generic. By (6.4) we can write these matrices in the form

\[
A_{s+1} = \begin{bmatrix} -z - b & A^{12} \\ A^{21} & z + b \end{bmatrix}, \quad B_{s+1} = \begin{bmatrix} b & B^{12} \\ B^{21} & z' - z - b \end{bmatrix}
\]

with some constants \(b, A^{12}, A^{21}, B^{12}, B^{21}\) such that

\[
\det A_{s+1} = -(z + b)^2 - A^{12}A^{21} = 0, \quad \det B_{s+1} = b(z' - z - b) - B^{12}B^{21} = 0. \quad (6.16)
\]

Let us prove that \(A^{12}, A^{21}\) are nonzero.

Assume that \(A^{21} = 0\). Then \(b = -z\) from (6.16). The \((1,1)\)–element of the second equality in (6.7) gives

\[
b_s - b = \frac{-(z + b)s - \xi\alpha_s(z + b)s(z' - z - b_s) + b(z + b) - B^{12}A^{21}}{z + s + \frac{1}{2}}. \quad (6.17)
\]

Hence,

\[
(z + b)\left(1 + \frac{b_s + \xi\alpha_s(z' - z - b_s)}{z + s + \frac{1}{2}}\right) = 0.
\]

Since \(\det A_s = 0\) and \(A_s\) has nonzero off–diagonal entries, its diagonal elements are nonzero, and \(z + b_s \neq 0\). Thus, the second factor in the last equality vanishes and

\[
b_s + \frac{z' + s + \frac{1}{2}}{1 - \xi\alpha_s} = z' - z
\]

which contradicts (6.9).

Assume that \(A^{12} = 0\). Then, again, \(b = -z\), and the \((2,2)\)–element of the second equality in (6.7) gives

\[
b - b_s = \frac{(z + b_s)(z' - z - b_s) + b_s(z + b_s)(\xi\alpha_s)^{-1} - (z + b)(z' - z - b) - B^{21}A^{12}}{z + s + \frac{1}{2}}. \quad (6.18)
\]
As above, this leads to

\[ b_s + \frac{z' + s + \frac{1}{2}}{1 - \xi \alpha_s} = z' + s + \frac{1}{2} \]

which again contradicts (6.9).

Now let us proceed to \( B_{s+1} \). Assume \( B_{12}^s = 0 \). Then (6.16) implies that either \( b = 0 \) or \( b = z' - z \). If \( b = z' - z \) then similarly to the above (6.17) yields

\[ b_s + \frac{z' + s + \frac{1}{2}}{1 - \xi \alpha_s} = -z \]

which cannot happen by (6.9). (In this reduction one needs to know that \( b_s + z - z' \neq 0 \). This follows from the fact that \( \det B_s = 0 \) and the hypothesis that the off–diagonal entries of \( B_s \) are nonzero. It follows that the diagonal entries, including \( b_s + z - z' \) are also nonzero.) If \( b = 0 \) then the only situation when \( B_{s+1} \) is not generic is when \( B_{21}^s = 0 \), see below.

Finally, assume that \( B_{21}^s = 0 \). Then by (6.16) either \( b = 0 \) or \( b = z' - z \). If \( b = z' - z \) then similarly to the above (6.17) yields

\[ b_s + \frac{z' + s + \frac{1}{2}}{1 - \xi \alpha_s} = z' - z + s + \frac{1}{2} \]

Once again, this contradicts (6.9).

Thus, we have proved that \( A_{s+1} \) and \( B_{s+1} \) are generic.

**Step 2.** From now on we will use the form (6.8) for \( A_{s+1} \) and \( B_{s+1} \). Since \( A_{s+1}, B_{s+1} \) are generic, \( \alpha_{s+1}, b_{s+1}, \beta_{s+1} \) are well–defined. In this part of the proof we derive (6.10)–(6.12) under the additional assumption \( b_{s+1} \neq b_s \).

Relations (6.17) and (6.18) rewritten in the notation (6.8) give

\[ b_s - b_{s+1} = \frac{-(z + b_s)(b_s + \xi \alpha_s(z' - z - b_s)) + (z + b_{s+1})b_{s+1}(1 - 1/\alpha_{s+1})}{z + s + \frac{1}{2}} \]  
(6.19)

\[ b_{s+1} - b_s = \frac{(z + b_s)((z' - z - b_s) + b_s/(\xi \alpha_s)) - (z + b_{s+1})(z' - z - b_{s+1})(1 - \alpha_{s+1})}{z + s + \frac{1}{2}} \]  
(6.20)

Add (6.19) multiplied by \( \alpha_{s+1} \) and (6.20) multiplied by \( \xi \alpha_s \). We obtain

\[ (b_{s+1} - b_s) \left( \xi \alpha_s - \alpha_{s+1} \right) + \frac{(1 - \alpha_{s+1})(1 - \xi \alpha_s)(b_{s+1} + b_s + 2z - z') + z - z'}{s + z + \frac{1}{2}} = 0. \]

We assumed that \( b_{s+1} - b_s \neq 0 \), and vanishing of the second factor is easily seen to be equivalent to (6.11). (Note that \( \alpha_{s+1} \) cannot equal 1. Indeed, in that case vanishing of the second factor would imply \( \xi \alpha_s = 1 \), which contradicts our hypothesis.) Substituting \( b_{s+1} \) from (6.11) into either (6.19) or (6.20) and solving for \( \alpha_{s+1} \) yields (6.10).
To prove (6.12) we look at the (1,2) and (2,1)–elements of the first equality $A_s^{\Xi} + B_s = A_{s+1} + B_{s+1}$ in (6.7):

\begin{align*}
\beta_s (\xi \alpha_s (z + b_s) + b_s) &= \beta_{s+1} (-\alpha_s (z + b_{s+1}) + b_{s+1}), \\
\beta^{-1}_s ((z + b_s)/ (\xi \alpha_s) + z' - z - b_s) &= \beta_{s+1}^{-1} ((z + b_{s+1})/ \alpha_s + z' - z - b_{s+1}).
\end{align*}

(6.21)

Substituting $\alpha_{s+1}$ and $b_{s+1}$ from (6.10), (6.11) we obtain (after some tedious work)

\begin{align*}
(b_s - \xi \alpha_s (z + b_s)) \left( \beta_s - \frac{b_s + \frac{z' + s + \frac{1}{2}}{1 - \xi \alpha_s} - (z' - z + s + \frac{1}{2}) \beta_{s+1}}{\xi \alpha_s \left( b_s + \frac{z' + s + \frac{1}{2}}{1 - \xi \alpha_s} + z \right)} \right) &= 0, \\
\left( \frac{z + b_s}{\xi \alpha_s} + z' - z - b_s \right) \left( \beta_s^{-1} - \frac{\xi \alpha_s \left( b_s + \frac{z' + s + \frac{1}{2}}{1 - \xi \alpha_s} + z \right)}{b_s + \frac{z' + s + \frac{1}{2}}{1 - \xi \alpha_s} - (z' - z + s + \frac{1}{2}) \beta_{s+1}^{-1}} \right) &= 0.
\end{align*}

This yields (6.12) if at least one of the prefactors

\begin{align*}
b_s - \xi \alpha_s (z + b_s) \quad \text{and} \quad \frac{z + b_s}{\xi \alpha_s} + z' - z - b_s
\end{align*}

(6.22)

is nonzero. But if they are both zero then we obtain $\alpha_s = -z/(\xi z')$, $b_s = -z^2/(z + z')$. The evaluation of $b_{s+1}$ through (6.11) gives $b_{s+1} = b_s$, which contradicts our assumption.

Thus, we have proved (6.10)–(6.12) under the assumption $b_{s+1} \neq b_s$.

**Step 3.** Let us assume that $b_{s+1} = b_s$. Recall that $b_s$, $b_s + z$, $b_s + z - z'$ are all nonzero because otherwise $A_s$ or $B_s$ would have zero off–diagonal entries which is impossible by the hypothesis. Equation (6.19) implies

\begin{align*}
\alpha_{s+1} &= \frac{b_s}{\xi \alpha_s (b_s + z - z')}.
\end{align*}

(6.23)

Let us show that $\alpha_{s+1} \neq 1$. Using the relation $b_{s+1} = b_s$, (6.23), and $\alpha_{s+1} = 1$ we obtain

\begin{align*}
(A_{s+1} + B_{s+1} - A_s^{\Xi} - B_s)^{12} &= \frac{z' b_s \beta_s - z (b_s + z - z') \beta_{s+1}}{b_s + z - z'}, \\
\left( A_{s+1} - A_s^{\Xi} - \frac{A_s^{\Xi} B_s - B_{s+1} A_{s+1}}{s + z + \frac{1}{2}} \right)^{12} &= \frac{(z + b_s) (b_s \beta_s - (b_s + z - z') \beta_{s+1})}{b_s + z - z'}.
\end{align*}

Since $z \neq z'$, both these expressions cannot vanish simultaneously, which contradicts (6.7). Thus, $\alpha_{s+1} \neq 1$.

In order to proceed we assume that both expressions (6.22) are nonzero. The case when one of them vanishes will be considered in Step 4 below.

The first relation (6.21) implies

\begin{align*}
\beta_{s+1} = \frac{b_s - \xi \alpha_s (z + b_s)}{b_{s+1} - \alpha_{s+1} (z + b_{s+1})} \beta_s.
\end{align*}

(6.24)
Let us denote by \( \alpha_{s+1}^\circ \) and \( b_{s+1}^\circ \) the differences of the left and right-hand sides of (6.10) and (6.11). Then with the substitutions \( b_{s+1} = b_s \), (6.23), (6.24) one computes that

\[
\beta_s(b_s + z)(b_s + z + z')(1 - \xi \alpha_s)^2 \left( b_s + \frac{z' + s + \frac{1}{2}}{1 - \xi \alpha_s} + z \right) \left( b_s + \frac{z' + s + \frac{1}{2}}{1 - \xi \alpha_s} + z' \right) \cdot \alpha_{s+1}^\circ
\]

\[
(z + s + \frac{1}{2})(z' + s + \frac{1}{2}) \left( \frac{z + b_s}{\xi \alpha_s} + z' - z - b_s \right)
= \left( A_{s+1} - A_s - \frac{A_s B_s - B_{s+1} A_{s+1}}{s + z + \frac{1}{2}} \right)^{12}
\]

\[
= \frac{(1 - \xi \alpha_s) \beta_s(z + b_s)(b_s - \xi \alpha_s(b_s + z - z'))}{\xi \alpha_s \left( \frac{z + b_s}{\xi \alpha_s} + z' - z - b_s \right)} \cdot b_{s+1}^\circ.
\]

Thanks to all the assumptions, none of the prefactors of \( \alpha_{s+1}^\circ \) and \( b_{s+1}^\circ \) vanishes. Thus, (6.7) implies (6.10), (6.11). Then (6.12) follows in the same way as in Step 2 above.

**Step 4.** Finally, assume that \( b_{s+1} = b_s \) and one of the expressions (6.22) vanishes. Let us show that the other one also vanishes. Indeed, if \( b_s - \xi \alpha_s(z + b_s) = 0 \) then the first equation (6.21) implies that \( b_{s+1} - \alpha_{s+1}(z + b_{s+1}) = 0 \). Substituting \( b_{s+1} = b_s \) and (6.23) we get

\[
b_{s+1} - \alpha_{s+1}(z + b_{s+1}) = \frac{b_s((z + b_s)/((\xi \alpha_s) + z' - z - b_s)}{b_s + z - z' = 0,
\]

hence, \( (z + b_s)/((\xi \alpha_s) + z' - z - b_s) = 0 \). Conversely, if \( (z + b_s)/((\xi \alpha_s) + z' - z - b_s) = 0 \) then the second equation of (6.21) implies \( (z + b_{s+1})/\alpha_{s+1} + z' - z - b_{s+1} = 0 \). With the same substitutions we obtain

\[
(z + b_{s+1})/\alpha_{s+1} + z' - z - b_{s+1} = (\xi \alpha_s(z + b_s) - b_s)/(\xi \alpha_s) = 0,
\]

hence, \( b_s - \xi \alpha_s(z + b_s) = 0 \).

On the other hand, if both expressions (6.22) vanish, we obtain

\[
\alpha_s = -\frac{z}{\xi z'}, \quad b_s = -\frac{z^2}{z + z'}.
\]  

(6.25)

In particular, this means that \( z + z' \neq 0 \). Hence, using (6.23), we get

\[
\alpha_{s+1} = -\frac{z}{z'}, \quad b_{s+1} = b_s = -\frac{z^2}{z + z'}.
\]  

(6.26)

But (6.26) exactly is exactly what one gets by substituting (6.25) into (6.10), (6.11). To prove (6.12) we compute, using (6.25),

\[
\left( A_{s+1} - A_s - \frac{A_s B_s - B_{s+1} A_{s+1}}{s + z + \frac{1}{2}} \right)^{12} = \frac{z^2(\beta_{s+1}(s + \frac{1}{2}) - \beta_s(z + z' + s + \frac{1}{2}))}{(z + z')(z + s + \frac{1}{2})}.
\]

By (6.7) this implies that \( \beta_{s+1} = \beta_s \cdot (z + z' + s + \frac{1}{2})/(s + \frac{1}{2}) \) which coincides with (6.12) under the conditions (6.25). The proof of Proposition 6.3 is complete.  

□
7. Fredholm determinant and dPV

Recall that in the beginning of the previous section we introduced $D_s$ as the Fredholm determinant of $\left(1 - K_s\right)$, where $K_s$ is the hypergeometric kernel restricted to $\{s, s + 1, \ldots\}$. In this section we relate $D_s$ to the sequences $\{b_s\}, \{\alpha_s\}, \{\beta_s\}$. Our goal is to prove the following

**Proposition 7.1.** Fix $s \in \mathbb{Z}_+'$. Under the assumptions of Proposition 6.3, we have

\[
\left(\frac{D_{s+1}}{D_s} - \frac{D_{s+2}}{D_{s+1}}\right) \left((1 - \xi \alpha_{s+1})(b_{s+1} + z - z') - z + z'\right) \quad (7.1)
\]

\[
= \left(\frac{D_{s+2}}{D_{s+1}} - \frac{D_{s+3}}{D_{s+2}}\right) \left((1 - \xi \alpha_s)(b_s + z - z') - z + z'\right)
\]

\[
(1 - \xi \alpha_{s+1})(1 - \xi \alpha_s) \alpha_s \left(b_s + \frac{s + z' + \frac{1}{2}}{1 - \xi \alpha_s} + z\right) \left(b_s + \frac{s + z' + \frac{1}{2}}{1 - \xi \alpha_s} + z - z'\right)^2.
\]

**Remark 7.2.** Using the variables $c_s$ of Remark 6.4(4) we can rewrite (7.1) as

\[
\left(\frac{D_{s+1}}{D_s} - \frac{D_{s+2}}{D_{s+1}}\right) \left((1 - \xi \alpha_{s+1})(c_{s+1} - z') - (z + s + \frac{3}{2})\right)
\]

\[
= \left(\frac{D_{s+2}}{D_{s+1}} - \frac{D_{s+3}}{D_{s+2}}\right) \left((1 - \xi \alpha_s)(c_s - z') - (z + s + \frac{1}{2})\right)
\]

\[
(1 - \xi \alpha_{s+1})(1 - \xi \alpha_s) \alpha_s c_s (c_s - z')^2.
\]

To prove Proposition 7.1 it suffices to prove the following two lemmas.

**Lemma 7.3.** Assume that the matrices $A_s$ and $B_s$ have nonzero off–diagonal entries. Then

\[
\frac{D_{s+1}}{D_s} - \frac{D_{s+2}}{D_{s+1}} = \frac{(zz')^{\frac{1}{2}}(1 - \xi)(z + s + \frac{1}{2})\Gamma(z' + s + \frac{1}{2})}{(z + s + \frac{1}{2})(z' + s + \frac{1}{2})^{2}\Gamma(z' + s + \frac{1}{2})}
\]

\[
\times (1 - \xi \alpha_s) ((1 - \xi \alpha_s)(b_s + z - z') - z + z') \cdot \beta_s \gamma_s^2
\]

where $\gamma_s$ is the $(2,1)$–entry of the matrix $m_s(s)$.

**Lemma 7.4.** Under the assumptions of Proposition 6.3 we have

\[
\gamma_{s+1} = \frac{(s + \frac{1}{2})(1 - \xi \alpha_s) \left(b_s + \frac{s + z' + \frac{1}{2}}{1 - \xi \alpha_s} + z - z'\right)}{\xi (z + s + \frac{1}{2})(z' + s + \frac{1}{2})} \cdot \gamma_s
\]

(7.3)

where $\gamma_s = m_s^{21}(s)$, $\gamma_{s+1} = m_{s+1}^{21}(s + 1)$.

The relation (7.1) is a direct consequence of (7.2), (7.3), and (6.12).
Proof of Lemma 7.3. The proof reminds that of Proposition 4.1 but is a little more technically involved. Theorem 2.3(ii) gives

\[ R_s(s, s) = g^t(s) m_s^{-1}(s) m'_s(s) f(s), \quad (7.4) \]
\[ R_{s+1}(s+1, s+1) = g^t(s+1) m_{s+1}^{-1}(s+1) m'_{s+1}(s+1) f(s+1), \quad (7.5) \]

where (see §6 for the definition of \( h_+ \))

\[ f(x) = (h_+(x), 0)^t, \quad g(x) = (0, h_+(x))^t. \]

(Recall that \( R_s = K_s(1 - K_s)^{-1} \) and \( R_s(s, s) = D_{s+1}/D_s - 1. \))

Let us plug the expression for \( m_{s+1} \) from (6.3) into (7.5). We have

\[ R_{s+1}(s+1, s+1) = g^t(s+1) \begin{bmatrix} s+z+\frac{1}{2} \quad 0 \\ s+\frac{1}{2} \quad 0 \end{bmatrix} \Xi m_s^{-1}(s) \Xi^{-1} \left( I + \frac{B_s}{s + z + \frac{1}{2}} \right) \]
\[ \times \frac{d}{d\zeta} \left( \left( I + \frac{B_s}{\zeta + z - \frac{1}{2}} \right)^{-1} \Xi m_s(\zeta - 1) \Xi^{-1} \begin{bmatrix} \frac{s+\frac{1}{2}}{s+z+\frac{1}{2}} \quad 0 \\ 0 \quad \frac{s+z+\frac{1}{2}}{s+\frac{1}{2}} \end{bmatrix} \right) \bigg|_{\zeta=s+1} \]

(7.6)

It is immediately seen that if the derivative falls on the last (diagonal) factor then the corresponding term vanishes. If the derivative falls on \( m_s \) we obtain

\[ g^t(s+1) \begin{bmatrix} s+z+\frac{1}{2} \quad 0 \\ s+\frac{1}{2} \quad 0 \end{bmatrix} \Xi m_s^{-1}(s) m'_s(s) \Xi^{-1} \begin{bmatrix} s+\frac{1}{2} \quad 0 \\ 0 \quad s+z+\frac{1}{2} \end{bmatrix} f(s+1) \]

which coincides with the right-hand side of (7.4) because

\[ \Xi^{-1} \begin{bmatrix} \frac{s+\frac{1}{2}}{s+z+\frac{1}{2}} \quad 0 \\ 0 \quad \frac{s+z+\frac{1}{2}}{s+\frac{1}{2}} \end{bmatrix} f(s+1) g^t(s+1) \begin{bmatrix} s+z+\frac{1}{2} \quad 0 \\ s+\frac{1}{2} \quad 0 \end{bmatrix} \Xi = f(s) g^t(s). \]

Finally, if the derivative falls on \( (I + B_s/((s + z + \frac{1}{2}))^{-1}) \), we compute

\[ \frac{d}{d\zeta} \left( I + \frac{B_s}{\zeta + z - \frac{1}{2}} \right)^{-1} = \left( I + \frac{B_s}{\zeta + z - \frac{1}{2}} \right)^{-1} \frac{B_s}{(\zeta + z - \frac{1}{2})^2} \left( I + \frac{B_s}{\zeta + z - \frac{1}{2}} \right)^{-1} \]
\[ = \frac{1}{\zeta + z - \frac{1}{2}} \left( I + \frac{B_s}{\zeta + z - \frac{1}{2}} \right)^{-1} \left( I - \left( I + \frac{B_s}{\zeta + z - \frac{1}{2}} \right)^{-1} \right). \]

Substituting into (7.6) we obtain

\[ R_{s+1}(s+1, s+1) = R_s(s, s) - \frac{1}{s + z + \frac{1}{2}} g^t(s) m_s^{-1}(s) \Xi^{-1} \left( I + \frac{B_s}{s + z + \frac{1}{2}} \right)^{-1} \Xi m_s(s) f(s). \quad (7.7) \]
Taking determinants of both sides of (6.3) we obtain \( \det (I + B_s/(s + z + \frac{1}{2})) = (s + z' + \frac{1}{2})(s + z + \frac{1}{2}) \), and hence

\[
\left( I + \frac{B_s}{\zeta + z - \frac{1}{2}} \right)^{-1} = \frac{s + z + \frac{1}{2}}{s + z' + \frac{1}{2}} \left[ \begin{array}{cc} 1 + \frac{B_{22}^{s}}{s + z' + \frac{1}{2}} & -\frac{B_{12}^{s}}{s + z' + \frac{1}{2}} \\ -\frac{B_{21}^{s}}{s + z' + \frac{1}{2}} & 1 + \frac{B_{11}^{s}}{s + z' + \frac{1}{2}} \end{array} \right]
\]

Substituting into (7.7) we obtain

\[
R_{s+1}(s + 1, s + 1) = R_s(s, s) - \frac{1}{(s + z + \frac{1}{2})(s + z' + \frac{1}{2})} \times g^t(s) m_{s}^{-1}(s) \Xi^{-1} \left[ \begin{array}{cc} s + z' + \frac{1}{2} - b_s & -\beta_s b_s \\ (b_s + z - z')/\beta_s & s + z + \frac{1}{2} + b_s \end{array} \right] \Xi m_s(s) f(s). \quad (7.8)
\]

For \( x \in \mathbb{Z}_+ \) denote

\[
m_x(x) = \left[ \begin{array}{cc} m_{x1}^{11} & m_{x2}^{12} \\ m_{x1}^{21} & m_{x2}^{22} \end{array} \right].
\]

Since \( \det m_x \equiv 1 \) we have

\[
m_x^{-1}(x) = \left[ \begin{array}{cc} m_{x2}^{22} & -m_{x2}^{12} \\ -m_{x1}^{21} & m_{x1}^{11} \end{array} \right].
\]

Substituting into (7.8) we obtain

\[
R_{s+1}(s + 1, s + 1) = R_s(s, s) - \frac{h_x^2(s)}{(s + z + \frac{1}{2})(s + z' + \frac{1}{2})} \times \left( \frac{\xi (b_s + z - z')}{\beta_s} (m_{s}^{11})^2 + \frac{b_s \beta_s}{\xi} (m_{s}^{21})^2 + (2b_s + z - z') m_s^{11} m_s^{21} \right). \quad (7.9)
\]

Similarly to the proof of Proposition 4.1, we now look at the residue of \( m_{s+1}(\zeta) \) at the point \( \zeta = s \). Since the jump matrix \(-f(s)g^t(s)\) has zero first column, the residue itself also has zero first column. On the other hand, (6.2) implies that this residue equals \( A_s m_{s}(s) \). Equating the (1,1) element of this matrix to zero and using the fact that \( z + b_s \neq 0 \) by the hypothesis of Proposition 6.3, we obtain

\[
m_s^{11} = -\alpha_s \beta_s m_s^{21}. \quad (7.10)
\]

Substituting this relation into (7.9) we arrive at (7.2). \( \square \)

**Proof of Lemma 7.4.** Let us look at the (1,1)–element of (6.3) with \( \zeta = s + 1 \). We get

\[
\frac{s + \frac{1}{2}}{s + z + \frac{1}{2}} m_s^{11} = \left( 1 + \frac{b_s}{s + z + \frac{1}{2}} \right) m_s^{11} + \frac{b_s \beta_s}{s + z + \frac{1}{2}} m_s^{21}.
\]

Using (7.10) for \( m_s^{11} \) and \( m_s^{11} \) and simplifying we obtain

\[
(s + \frac{1}{2}) \alpha_s \beta_s \cdot \gamma_s = (b_s \beta_s - (b_s + s + \frac{1}{2}) \alpha_{s+1} \beta_{s+1}) \cdot \gamma_{s+1}.
\]

Substituting \( \alpha_{s+1} \) and \( \beta_{s+1} \) from (6.10) and (6.12) and simplifying further we arrive at (7.3). \( \square \)
In this section we compute the initial conditions for the recurrences (6.10-12). We will also show that the assumptions of Proposition 6.3 hold for generic values of parameters \((z, z', \xi)\).

**Proposition 8.1.** We have

\[
\alpha_{\frac{1}{2}} = -\frac{F(-z + 1, -z'; 1; \xi)}{z' \xi F(-z + 1, -z' + 1; 2; \xi)}, \tag{8.1}
\]

\[
b_{\frac{1}{2}} = -\frac{z F(-z + 1, -z'; 1; \xi)}{F(-z, -z'; 1; \xi)}, \tag{8.2}
\]

\[
\beta_{\frac{1}{2}} = -\frac{(zz'\xi)^{\frac{1}{2}}(1 - \xi)^{z+z'}}{z F(-z + 1, -z'; 1; \xi)}. \tag{8.3}
\]

Here \(F(a, b; c; u)\) is the Gauss hypergeometric function.

**Remark 8.2.** Using adjacency relations for the Gauss hypergeometric function, it is not hard to deduce from (8.1), (8.2) the formula

\[
c_{\frac{1}{2}} = b_{\frac{1}{2}} + \frac{z' + 1}{1 - \xi \alpha_{\frac{1}{2}}} + z = \frac{z' F(-z, -z' - 1; 1; \xi) F(-z + 1, -z' + 1; 2; \xi)}{F(-z, -z'; 1; \xi) F(-z + 1, -z'; 2; \xi)}. \tag{8.4}
\]

(See Remark 6.4(4) for the definition of \(\{c_s\}\).)

**Proof of Proposition 8.1.** Similarly to Lemma 5.1, we have

\[
m_{\frac{1}{2}}(\zeta) = \left[ -\sum_{x \in \mathbb{Z}_{-}'} \frac{h^2(x)}{\zeta - x} \right]_{0}^{1}.
\]

Similarly to the computation of \(A_{\frac{1}{2}}\) in Proposition 5.2 we obtain

\[
A_{\frac{1}{2}} = \frac{h^2_{\frac{1}{2}}(\frac{1}{2})}{1 + h^2_{\frac{1}{2}}(\frac{1}{2}) \sum_{x \in \mathbb{Z}_{-}'} h^2(x)} \left[ \sum_{x \in \mathbb{Z}_{-}'} \frac{h^2(x)}{\frac{1}{2} - x} \right] \left[ \sum_{x \in \mathbb{Z}_{-}'} \frac{h^2(x)}{\frac{1}{2} - x} \right]^{-2} - \sum_{x \in \mathbb{Z}_{-}'} \frac{h^2(x)}{\frac{1}{2} - x} \right]. \tag{8.4}
\]

By (6.2) we have

\[
m_{\frac{1}{2}}(\zeta) = \left( I + \frac{A_{\frac{1}{2}}}{\zeta - \frac{1}{2}} \right) m_{\frac{1}{2}}(\zeta) = \left( I + \frac{A_{\frac{1}{2}}}{\zeta - \frac{1}{2}} \right) \left[ -\sum_{x \in \mathbb{Z}_{-}'} \frac{h^2(x)}{\zeta - x} \right]_{0}^{1}.
\]

Since \(A_{\frac{1}{2}}\) is nilpotent we also have

\[
m_{\frac{1}{2}}^{-1}(\zeta) = \left[ \sum_{x \in \mathbb{Z}_{-}'} \frac{1}{\zeta - x} \right]_{0}^{1} \left( I - \frac{A_{\frac{1}{2}}}{\zeta - \frac{1}{2}} \right).
\]

Then (6.3) gives

\[
\left[ -\frac{1}{\xi} \sum_{x \in \mathbb{Z}_{-}'} \frac{h^2(x)}{\zeta - x - \frac{1}{2}} \right]_{0}^{1} \left[ \begin{array}{cc}
\frac{\zeta - \frac{1}{2}}{\zeta + z - \frac{1}{2}} & 0 \\
0 & \frac{\zeta + z - \frac{1}{2}}{\zeta - \frac{1}{2}}
\end{array} \right] \left[ \sum_{x \in \mathbb{Z}_{-}'} \frac{1}{\zeta - x} \right]_{0}^{1} \left( I - \frac{A_{\frac{1}{2}}}{\zeta - \frac{1}{2}} \right) = I + \frac{B_{\frac{1}{2}}}{\zeta + z - \frac{1}{2}}.
\]
Taking the residue of both sides at $\zeta = -z + \frac{1}{2}$ we obtain

$$B_{\frac{4}{z}} = - \left[ \frac{1}{4} \sum_{x \in \mathbb{Z}_-} \frac{h^2(x)}{z + x + \frac{1}{2}} \right] \left( z \cdot I + A_{\frac{1}{z}} \right).$$

Thus,

$$b_{\frac{4}{z}} = B_{\frac{4}{z}}^{11} = -z - A_{\frac{4}{z}}^{11} = -z + \frac{h^2_{\frac{4}{z}}(\frac{1}{2})}{1 + h^2_{\frac{4}{z}}(\frac{1}{2})} \sum_{x \in \mathbb{Z}_-} \frac{h^2(x)}{\frac{3}{2} - x},$$

$$b_{\frac{4}{z}} \beta_{\frac{4}{z}} = B_{\frac{4}{z}}^{12} = -A_{\frac{4}{z}}^{12} = \frac{h^2_{\frac{4}{z}}(\frac{1}{2})}{1 + h^2_{\frac{4}{z}}(\frac{1}{2})} \sum_{x \in \mathbb{Z}_-} \frac{h^2(x)}{\frac{3}{2} - x}.$$

Since $(z + b_{\frac{4}{z}})\alpha_{\frac{4}{z}} = -A_{\frac{4}{z}}^{12}$ is equal to $b_{\frac{4}{z}} \beta_{\frac{4}{z}}$, we also obtain $\alpha_{\frac{4}{z}} = b_{\frac{4}{z}} / (z + b_{\frac{4}{z}})$.

Now we recall the definition of $h_{\pm}$, see the beginning of §6. We have

$$h^2_{\frac{4}{z}}(\frac{1}{2}) = (zz')(1 - \xi)^{\frac{1}{2}}(1 - \xi)^{z + z'},$$

$$\sum_{x \in \mathbb{Z}_-} \frac{h^2(x)}{\frac{1}{2} - x} = (zz') \frac{1}{2}(1 - \xi)^{-z - z'} \sum_{l = 0}^{\infty} \frac{(-z + 1)_l(-z' + 1)_l \xi^l}{l!^2 (l + 1)}$$

$$= (zz') \frac{1}{2}(1 - \xi)^{-z - z'} F(-z + 1, -z' + 1; 2; \xi),$$

$$1 + h^2_{\frac{4}{z}}(\frac{1}{2}) \sum_{x \in \mathbb{Z}_-} \frac{h^2(x)}{\frac{1}{2} - x} = 1 + zz' \sum_{l = 0}^{\infty} \frac{(-z + 1)_l(-z' + 1)_l \xi^{l+1}}{(l + 1)!^2}$$

$$= F(-z, -z'; 1; \xi).$$

Hence,

$$b_{\frac{4}{z}} = -z + \frac{zz' \xi F(-z + 1, -z' + 1; 2; \xi)}{F(-z, -z'; 1; \xi)} = -\frac{z F(-z + 1, -z'; 1; \xi)}{F(-z, -z'; 1; \xi)},$$

$$\alpha_{\frac{4}{z}} = \frac{b_{\frac{4}{z}}}{b_{\frac{4}{z}} + z} = -\frac{F(-z + 1, -z'; 1; \xi)}{z' \xi F(-z + 1, -z' + 1; 2; \xi)},$$

$$\beta_{\frac{4}{z}} = \frac{B_{\frac{4}{z}}^{12}}{b_{\frac{4}{z}}} = -\frac{(zz') \frac{1}{2}(1 - \xi)^{z + z'}}{z F(-z + 1, -z'; 1; \xi)}.$$

The formulas of Propositions 6.3 and 8.1 allow us to extend the definition of the sequences $\{\alpha_s\}$, $\{b_s\}$ to arbitrary parameters $(z, z', \xi) \in \mathbb{C} \times \mathbb{C} \times (\mathbb{C} \setminus [1, +\infty))$ such that the denominators in (6.10), (6.11), (8.1), (8.2) do not vanish. Now we will show that none of these denominators vanishes identically.

According to Proposition 5.4, we can choose $\eta > 0$ such that $w_s \neq 0$ for all $s \in \mathbb{Z}_+$, where $w_s$ was defined in Corollary 3.3. Let us fix such an $\eta$ for the rest of this section, and let us also recall that for the sequence $\{v_s\}$ defined in Proposition 3.4, $v_s \neq 0$, $\pm 1$, for all $s \in \mathbb{Z}_+$. 

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Proposition 8.3. Recurrence relations (6.10), (6.11), (6.12) with initial conditions (8.1), (8.2), (8.3) admit an asymptotic solution of the form
\[ |z| \to \infty, \quad |z'| \to \infty, \quad \xi \to 0, \quad z, z' \in \mathbb{C}, \quad \xi \in (0, 1), \]
\[ z \xi^{\frac{1}{2}} = \eta + o(1), \quad z' \xi^{\frac{1}{2}} = \eta + o(1), \]
\[ \alpha_s = \xi^{-\frac{1}{2}} \left( \frac{v_{s-1}}{v_s} + o(1) \right), \quad b_s = -z - \eta v_{s-1} v_s + o(1), \quad s = \frac{1}{2}, \frac{3}{2}, \ldots, \]
\[ \beta_{\frac{1}{2}} = -\xi^{\frac{1}{2}} (I_0^{-1}(2\eta) + o(1)), \quad \beta_{s+1} = ((1 - v_s^2)^{-1} + o(1)) \cdot \beta_s, \quad s = \frac{1}{2}, \frac{3}{2}, \ldots. \] (8.4)

Remark 8.4. It is not hard to see that in the limit (8.4) the Lax pair (6.2), (6.3) for dPV degenerates to the Lax pair (3.4), (3.5) for dPII. This explains why solutions of dPII provide asymptotic solutions for dPV.

Proof of Proposition 8.3. Using (8.4) we obtain
\[ b_s + \frac{z' + s + \frac{1}{2}}{1 - \xi \alpha_s} - \left( z' + s + \frac{1}{2} \right) = -z + O(1), \]
\[ b_s + \frac{z' + s + \frac{1}{2}}{1 - \xi \alpha_s} - \left( z' - z + s + \frac{1}{2} \right) = \frac{\eta v_{s-1}(1 - v_s^2)}{v_s} + o(1), \]
\[ b_s + \frac{z' + s + \frac{1}{2}}{1 - \xi \alpha_s} + z = z' + O(1), \]
\[ b_s + \frac{z' + s + \frac{1}{2}}{1 - \xi \alpha_s} + z - z' = \frac{\eta v_{s-1}(1 - v_s^2)}{v_s} + s + \frac{1}{2} + o(1). \]

First three expressions are obviously nonzero while the last one is nonzero because (3.13) implies that
\[ \frac{\eta v_{s-1}(1 - v_s^2)}{v_s} + s + \frac{1}{2} = -\eta v_{s+1}(1 - v_s^2), \]
and the sequence \( \{v_s\} \) does not take values 0, \pm 1.

Then (6.10) turns into
\[ \frac{v_s}{v_{s+1}} = -\frac{\eta v_{s-1}(1 - v_s^2)}{\eta v_{s-1}(1 - v_s^2) + (s + \frac{1}{2})v_s} + o(1) \]
which holds by (3.13). Similarly, (6.11) turns into
\[ -\eta v_{s+1} = \eta v_{s-1} v_s - \frac{\eta v_{s-1}}{v_s} - s - \frac{1}{2} - \frac{\eta v_{s+1}}{v_s} + o(1) \]
which again holds by (3.13), and (6.12) turns into the last relation in (8.4).

The asymptotics for initial conditions is also immediate:
\[ \xi^{\frac{1}{2}} \alpha_{\frac{1}{2}} = -\frac{F(-z + 1, -z'; 1; \xi)}{z' \xi^{\frac{1}{2}} F(-z + 1, -z' + 1; 2; \xi)} = -\frac{I_0(2\eta)}{I_1(2\eta)} + o(1) = \frac{v_{-\frac{1}{2}}}{v_s^{\frac{1}{2}}} + o(1) \]
\[ z + b_s = z (F(-z, -z'; 1; \xi) - F(-z + 1, -z'; 1; \xi)) \]
\[ = \frac{zz' \xi F(-z + 1, -z' + 1; 2; \xi)}{F(-z, -z'; 1; \xi)} = \frac{\eta I_1(2\eta)}{I_0(2\eta)} + o(1) = -\eta v_{-\frac{1}{2}} v_s^{\frac{1}{2}} + o(1), \]
\[ \xi^{-\frac{1}{2}} \beta_{\frac{1}{2}} = -\frac{(zz' \xi)^{\frac{1}{2}} (1 - \xi)^{s + z'}}{\xi^{\frac{1}{2}} z F(-z + 1, -z'; 1; \xi)} = -\frac{1}{I_0(2\eta)} + o(1). \quad \square \]
Recall that if we define $D_s$ as a normalized Toeplitz determinant then it is an analytic functions in $(z, z', \xi) \in \mathbb{C} \times \mathbb{C} \times (\mathbb{C} \setminus [1, +\infty))$. Thus, now it makes sense to ask whether the relations (7.1) are, in fact, equalities of analytic functions. The answer is positive.

**Theorem 8.5.** Let $s \in \mathbb{Z}^+_+$ and $D_s$ be the normalized Toeplitz determinant defined in §6. Define $\alpha_s$ and $\beta_s$ by the initial conditions (8.1), (8.2) and recurrence relations (6.10), (6.11). Then for any $(z, z', \xi) \in \mathbb{C} \times \mathbb{C} \times (\mathbb{C} \setminus [1, +\infty))$ in the complement of the set of zeros of a nontrivial analytic function, the equality (7.1) holds.

**Proof.** Since both sides of (7.1) are ratios of analytic functions, it suffices to prove that (7.1) holds on some open set.

Let us assume that $z' = \bar{\varepsilon} \in \mathbb{C} \setminus \mathbb{Z}$ and $\xi \in (0, 1)$. Fix $s \in \mathbb{Z}^+_+$. Clearly, both sides of (7.1) are ratios of analytic functions in $\Re z = (z + z')/2$, $\Im z = (z - z')/2$, $\xi$. According to Proposition 8.3, we can find some $\eta > 0$ and small enough $\epsilon > 0$ such that for

$$\xi \in (0, \epsilon), \quad \Re z \in (\eta \xi^{-\frac{1}{2}}, \eta \xi^{-\frac{1}{2}} + 1), \quad \Im z \in (1, 2),$$

the asymptotics (8.4) ensures that neither numerators nor denominators in the formulas (6.10), (6.11), (6.12) for the indices from $\frac{1}{2}$ to $s$ vanish. Thus, $\alpha_t, \beta_t$ for $t = \frac{1}{2}, \ldots, s$ do not vanish as well. Then Propositions 6.3 and 7.1 prove (7.1) on this set of parameters. Complexifying $(\Re z, \Im z, \xi)$ proves (7.1) on an open subset of $\mathbb{C} \times \mathbb{C} \times (\mathbb{C} \setminus [1, +\infty))$. □

9. Degeneration to continuous PII and PV

**Discrete PII to continuous PII.** Although it does seem to be possible to degenerate the Lax pair (3.4), (3.5) to a continuous limit, it is possible to find a scaling limit of the equation (3.13) leading to the Painlevé II (ordinary differential) equation, see e.g. [ORGT]. Let us introduce a new real variable $t$ by

$$s = 2\eta + \eta^\frac{3}{4} t, \quad t = (s - 2\eta) \eta^{-\frac{3}{4}}.$$

Now let $\eta$ go to infinity, and assume that $v_s \approx (-1)^s \eta^{-\frac{3}{4}} v(t)$ as $\eta \to \infty$, with a smooth function $v(\cdot)$, and $s$ and $t$ related as above. Then we have

$$v_{s+1} = (-1)^{s+1} \eta^{-\frac{3}{4}} \left(v(t) \pm \eta^{-\frac{3}{4}} v'(t) + \eta^{-\frac{3}{4}} v''(t) + O(\eta^{-1})\right),$$

$$\frac{(s + \frac{1}{2}) v_s}{\eta(v_s^2 - 1)} = (-1)^{s+1} \left(2 + \eta^{-\frac{3}{4}} t + \frac{1}{2} \eta^{-1}\right) \eta^{-\frac{3}{4}} v(t) \left(1 + \eta^{-\frac{3}{4}} v^2(t) + O(\eta^{-\frac{3}{4}})\right)$$

Substituting into (3.13) and taking the limit $\eta \to \infty$ we get

$$v''(t) = tv(t) + 2v^3(t) \quad (9.1)$$

which is a special case of Painlevé II equation. Since we also know, see (5.6), that $D_s D_{s+2}/D_{s+1}^2 = 1 - v_s^2$, it is natural to assume that $D_s \approx D(t)$ for some smooth function $D(\cdot)$, and then we obtain

$$(\ln D(t))'' = -v^2(t). \quad (9.2)$$
As a matter of fact, the last formula is correct in the sense that there exists a solution \( v(t) \) of (9.1) such that \( D_s = D(t) + o(1) \) as \( \eta \to \infty \), and (9.2) holds. This is a deep fact and it is the main result of [BDJ1].\(^6\) For the history of this result, other proofs, generalizations, etc., we also refer to [AD], [BDJ2], [BDR], [BOO], [BO3], [D2], [J2], [Ok], [W] and references therein.

**Discrete PV to continuous PV.** The limit procedure considered in this section has a representation theoretic origin, see [BO2, §5], [BO3], and also [Bor2, §8] for more details.

We assume that \( \xi \to 1 \) and introduce a new complex variable \( \omega \) and a new real variable \( t \) by

\[
\omega = (1 - \xi) \zeta, \quad t = (1 - \xi) s.
\]

We will also redenote \( m_s(\zeta) \) as \( m_t(\omega) \) and \( A_s, B_s \) as \( A(t), B(t) \) for \( (\zeta, s) \) related to \( (\omega, t) \) as above. Let us assume that \( m_t(\omega), A(t), B(t) \) all have smooth limits as \( \xi \to 1, \zeta \to \infty, s \to +\infty \) in such a way that \( \omega \) and \( t \) converge to finite limits. Then the Lax pair equations (6.2) and (6.3) (in (6.3) we use the right–hand side of (6.2) instead of \( m_{s+1}(\zeta) \)) tend to

\[
\begin{align*}
\frac{\partial m_t(\omega)}{\partial t} &= \frac{A(t)}{\omega - t} \cdot m_t(\omega), \\
\frac{\partial m_t(\omega)}{\partial \omega} &= -\left( \frac{\sigma_3}{2} + \frac{B(t)}{\omega} + \frac{A(t)}{\omega - t} \right) m_t(\omega) + m_t(\omega) \left( \frac{1}{\omega} \begin{bmatrix} -z & 0 & \zeta' \\ 0 & \zeta & \frac{\sigma_3}{2} \end{bmatrix} + \frac{\sigma_3}{2} \right),
\end{align*}
\]

(9.3)

where \( \sigma_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \). Set

\[
n_t(\omega) = m_t(\omega) \begin{bmatrix} \omega^z e^{-\frac{\zeta'}{\omega}} & 0 \\ 0 & \omega^{-z} e^{\frac{\zeta'}{\omega}} \end{bmatrix}.
\]

Then (9.3) can be written in the form

\[
\begin{align*}
\frac{\partial n_t(\omega)}{\partial t} &= \frac{A(t)}{\omega - t} \cdot n_t(\omega), \\
\frac{\partial n_t(\omega)}{\partial \omega} &= -\left( \frac{\sigma_3}{2} + \frac{B(t)}{\omega} + \frac{A(t)}{\omega - t} \right) n_t(\omega),
\end{align*}
\]

(9.4)

which is a Lax pair for the Painlevé V equation, see [JM, Appendix C]. The consistency relations (6.7) tend to

\[
A'(t) = \frac{[A(t), \sigma_3]}{2} + \frac{[A(t), B(t)]}{t}, \quad B'(t) = \frac{[B(t), A(t)]}{t},
\]

(9.5)

which are, quite naturally, the consistency relations for (9.4). The relations (9.5) are usually called the Schlesinger equations. Now, if we parameterize the matrices \( A(t) \) and \( B(t) \) as

\[
A = (z + b) \begin{bmatrix} -1 & -\alpha \beta \\ \alpha \beta^{-1} & 1 \end{bmatrix}, \quad B = \begin{bmatrix} b & b \beta \\ (z' - z - b)/\beta & z' - z - b \end{bmatrix},
\]

\(^6\)The limit function \( D(t) \) is known as the Tracy–Widom distribution in Random Matrix Theory, and it was obtained for the first time in [TW1].
cf. (6.8), then the diagonal elements of (9.5) give
\[ b'(t) = \frac{(z + b(t)) (b(t)(1/\alpha(t) - \alpha(t)) + (z' - z)\alpha(t))}{t}. \] (9.6)

This is the limit of both (6.19) and (6.20). Note that in the discrete case we derived the recurrence (6.10), (6.11) using just the relations (6.19), (6.20). In the continuous limit (6.19) becomes equivalent to (6.20), and we need additional arguments.

Instead of deriving Painlevé V in the usual way by a more careful examination of (9.5), we will take a shortcut and use the limit of the relation (6.11). (Note that the limit of (6.10) in the first order approximation is trivial.) We obtain
\[ 2b(t) = \frac{t(\alpha'(t) + \alpha(t))}{(1 - \alpha(t))^2} + \frac{z - z'}{1 - \alpha(t)} - 2z + z'. \] (9.7)

Substituting into (9.6) yields
\[
\alpha''(t) = \left( \frac{1}{2\alpha(t)} + \frac{1}{\alpha(t) - 1} \right) (\alpha'(t))^2 - \frac{\alpha'(t)}{t} + \frac{(z' - z - 1)\alpha(t)}{t} + \frac{(\alpha(t) - 1)^2}{2t} \left( (z')^2 \alpha(t) - z^2/\alpha(t) \right) - \frac{1}{2} \frac{\alpha(t)(\alpha(t) + 1)}{\alpha(t) - 1}.
\] (9.8)

which is the Painlevé V equation in the standard form.

Now if we assume that \( D_s \approx D(t) \) with a smooth \( D(\cdot) \), then the relation (7.1) becomes trivial in the first order approximation, and the second asymptotic term gives a rather cumbersome expression for \( (\ln D(t))'''/(\ln D(t))'' \) in terms of \( \alpha(t), b(t), \) and their first derivatives.\(^7\)

In fact, it is known that under the limit transition described above, the Fredholm determinant \( D_s \) of the hypergeometric kernel tends to the Fredholm determinant \( D(t) = \det(1 - K_t) \) where \( K_t \) is the Whittaker kernel restricted to \( (t, +\infty) \), see [BO1], [Bor1] for the definition of the Whittaker kernel, and [BO2], [BO3] for the limit transition. Moreover, as was shown in [BD, §8], \( D(t) \) is the isomonodromy \( \tau \)-function of the Schlesinger equations (9.5). This means that there exist solutions \( \alpha(t), b(t) \) of (9.7), (9.8) such that \( (\ln D(t))' \) is expressed in terms of \( \alpha(t) \) and \( \beta(t) \) in the following simple way:
\[ t(\ln D(t))' = \text{Tr}(AB) + \frac{4}{2} \text{Tr}(A\sigma_3) = (b + z)(z' - z + (b + z - z')\alpha + b/\alpha - t). \] (9.9)

Using (9.5) one also computes
\[ (t(\ln D(t))')' = \frac{1}{2} \text{Tr}(A\sigma_3) = -(b + z). \]

It remains unclear whether there exist a discrete analog of either of these simple formulas. It is also worth noting that the function \( \sigma(t) = t(\ln D(t))' \) itself satisfies a second order differential equation
\[ (t\sigma'')^2 = (2\sigma')^2 - t\sigma' + \sigma + (z + z')\sigma'\sigma' - 4(\sigma')^2(\sigma' + z)\sigma' + z'). \] (9.10)

This result was first proved by C. Tracy [T] using the method of [TW2]; it also follows from (9.7)–(9.9). The equation (9.10) is the so-called \( \sigma \)-form of the Painlevé V equation. It is also not clear if there exists a discrete analog of (9.10).

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\(^7\)We use the fact that for any smooth function \( f(x) \),
\[
\left( \frac{f(x + \epsilon) - f(x + 2\epsilon)}{f(x + \epsilon)} \right) / \left( \frac{f(x + 2\epsilon) - f(x + 3\epsilon)}{f(x + 2\epsilon)} \right) = 1 - \frac{(\ln f(x))''}{(\ln f(x))''} \epsilon + O(\epsilon^2).
\]
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The plot of $q_{x+1}^{(z,z',\xi)} - q_{x}^{(z,z',\xi)}$ for $z = z' = 2.5$, $\xi = 0.85$, see Introduction.