Topologically non–trivial chiral transformations and their representations in a finite model space†

R. Alkofer, H. Reinhardt, J. Schlienz and H. Weigel‡
Institute for Theoretical Physics
Tübingen University
Auf der Morgenstelle 14
D-72076 Tübingen, FR Germany

Abstract

The role of chiral transformations in effective theories modeling Quantum Chromodynamics is reviewed. In the context of the Nambu–Jona–Lasinio model the hidden gauge and massive Yang–Mills approaches to vector mesons are demonstrated to be linked by a special chiral transformation which removes the chiral field from the scalar–pseudoscalar sector. The role of this transformation in the presence of a topologically non–trivial chiral field is illuminated. The fermion determinant for such a field configuration is evaluated by summing the discretized eigenvalues of the Dirac Hamiltonian. This discretization is accomplished by demanding certain boundary conditions on the quark fields leaving a finite model space. The properties of two sets of boundary conditions are compared. When the topologically non-trivial chiral transformation is applied to the meson fields the associated transformation of the boundary conditions is shown to be indispensable. A constructive procedure for transforming the boundary conditions is developed.

† Supported by the Deutsche Forschungsgemeinschaft (DFG) under contract Re 856/2-1.
‡ Supported by a Habilitanden–scholarship of the DFG
1 Introduction

Since a solution to Quantum Chromo Dynamics (QCD) is not yet available one has to recede on models in order to explore processes described by the strong interaction. These models are usually constructed under the requirement that the symmetries of the underlying theory, i.e. QCD are maintained. In this context chiral symmetry and its spontaneous breaking play a key role. In this article we will explore a special chiral transformation when topologically non-trivial meson field configurations like solitons are involved. To begin with, let us briefly review the relevance of chiral symmetry on the one side and solitonic field configurations on the other in the context of strong interactions.

QCD can be extended from SU(3) to SU(\(N_C\)) where \(N_C\) denotes the number of color degrees of freedom. It was observed by 't Hooft\[1\] that in the limit \(N_C \rightarrow \infty\) QCD is equivalent to an effective theory of weakly interacting mesons. Subsequently Witten\[2\] conjectured that baryons emerge as solitons of the meson fields within this effective theory. Stimulated by Witten’s conjecture much interest has been devoted to the description of baryons as chiral solitons during the past decade\[3, 4\]. In the soliton description of baryons the chiral field and in particular its topological structure play a key role. The topological character of the chiral field especially endows the soliton with baryonic properties like baryon charge and spin\[5\]. This comes about via the chiral anomaly\[6\] which is a unique feature of all quantum field theories where fermions live in a gauge group and couple to a chiral field. The fermionic part of such a theory has the generic structure

\[
Z_F[\Phi] = \int D\Psi D\bar{\Psi} \exp \left[ i \int d^4x \bar{\Psi} (i\partial - \hat{m}_0) \Psi \right] = \det (i\partial - \hat{m}_0) \tag{1}
\]

where \(\hat{m}_0\) denotes the current quark mass matrix which will be ignored in the ongoing discussions. Furthermore

\[
i\slashed{\partial} = i\partial - \Phi = i(\partial + \Gamma) - MP_R - M^\dagger P_L \tag{2}
\]

represents the Dirac–operator of the fermions in the external Bose–field \(\Phi\). \(\Phi\) in general contains vector \(V_\mu\), axial vector \(A_\mu\) fields as well as scalar \(S\) and pseudo-scalar fields \(P\)

\[
\Gamma_\mu = V_\mu + A_\mu \gamma_\mu, \quad M = S + iP = \xi_L^\dagger \Sigma \xi_R. \tag{3}
\]

Here \(P_{R/L} = \frac{1}{2}(1 \pm \gamma_5)\) are the chiral projectors. Accordingly one defines left(L)– and right(R)–handed quark fields: \(\Psi_{L/R} = P_{L/R} \Psi\). The chiral field \(U\) is defined via the polar decomposition of the meson fields

\[
U = \xi_L^\dagger \xi_R = \exp(i\Theta) \tag{4}
\]

The chiral anomaly arises because there is no regularization scheme which simultaneously preserves local vector and axial vector (chiral) symmetries. In renormalizable theories the chiral anomaly can be calculated in closed form and is given by the Wess-Zumino action\[4, 5, 8\]

\[
\mathcal{A} = S_{WZ} = \frac{iN_C}{240\pi^2} \int_{M^5} ( UdU^\dagger)^5 \tag{5}
\]

\[1\]In our notation \(V_\mu\) and \(A_\mu\) are anti-hermitian.
where the notation of alternating differential forms has been used. \( M^5 \) denotes a five dimensional manifold whose boundary is Minkowski space. Obviously the chiral anomaly is tightly related to the chiral field since the Wess-Zumino action vanishes when the chiral field disappears \((U = 1)\). The chiral anomaly is, however, not merely a technical artifact but has well established physical consequences. In the meson sector it gives rise to the so-called “anomalous decay processes” like e.g. \( \pi \to 2\gamma \) and \( \omega \to 3\pi \). In the soliton sector the chiral anomaly requires for \( N_C = 3 \) the soliton to be quantized as a fermion and endows the soliton with half integer spin and integer baryon number \( \frac{1}{2} \).

For many purposes it is convenient to perform a chiral rotation of the fermions\(^2\)

\[
\Psi = \Omega \Psi \quad \text{with} \quad \Omega = P_L \xi_L + P_R \xi_R, \quad \text{i.e.} \quad \tilde{\Psi}_{L,R} = \xi_{L,R} \psi_{L,R}.
\]

This transformation defines a chirally rotated Dirac-operator

\[
\bar{\Psi} iD \Psi = \bar{\tilde{\Psi}} iD \tilde{\Psi}
\]

which acquires the form

\[
i\bar{\tilde{D}} = \Omega^\dagger iD \Omega = i\gamma_\mu \left( \partial^\mu + \vec{V}^\mu + \vec{A}^\mu \gamma_5 \right) - \Sigma.
\]

The chiral rotation has removed the chiral field from the scalar pseudo-scalar sector of the rotated Dirac operator \( i\bar{\tilde{D}} \). As a consequence the vector and axial vector fields become now chirally rotated

\[
\begin{align*}
V_\mu + A_\mu &= \xi_R (\partial_\mu + V_\mu + A_\mu) \xi_R^\dagger, \\
\bar{V}_\mu + \bar{A}_\mu &= \xi_L (\partial_\mu + V_\mu - A_\mu) \xi_L^\dagger.
\end{align*}
\]

Even in the absence of vector and axial vector fields in the original Dirac operator \((V_\mu = A_\mu = 0)\) the chiral rotation induces vector and axial vector fields

\[
\begin{align*}
V_\mu (V_\mu = A_\mu = 0) &= v_\mu = \frac{1}{2} \left( \xi_R \partial_\mu \xi_R^\dagger + \xi_L \partial_\mu \xi_L^\dagger \right), \\
\bar{A}_\mu (V_\mu = A_\mu = 0) &= a_\mu = \frac{1}{2} \left( \xi_R \partial_\mu \xi_R^\dagger - \xi_L \partial_\mu \xi_L^\dagger \right).
\end{align*}
\]

For the soliton description of baryons the chiral field is usually assumed to be of the hedgehog type

\[
U = \exp (i\Theta(r) \tau \cdot \hat{r})
\]

The non–trivial topological structure of this configuration is then exhibited by the boundary conditions \( \Theta(0) = -n\pi \) and \( \Theta(\infty) = 0 \). The chiral field thus represents a mapping from the compactified coordinate space (all points at spatial infinity are identified) to SU(2) flavor space, i.e. \( S^3 \to S^3 \). The associated homotopy group, \( \Pi_3(S^3) \), is isomorphic to \( Z \), the group of integer numbers. The isomorphism is given by the winding number \( (\Theta(0) - \Theta(\infty))/\pi = -n \). Assuming the unitary gauge \( (\xi_L^\dagger = \xi_R) \) the induced vector field is of the Wu–Yang form\(^\[11\]

\[
\begin{align*}
v_0 &= 0, \quad v_i = i v_i^a \tau^a, \quad v_i^a = \epsilon^{iak} r_k \frac{G(r)}{r},
\end{align*}
\]

\(^2\)For this proof it is mandatory to consider flavor SU(3).
with the profile function $G(r)$ given by the chiral angle $\Theta(r)$

$$G(r) = -2 \sin^2 \frac{\Theta(r)}{2}. \quad (13)$$

For odd $n$ the topological non–trivial character of the chiral rotation is also reflected by a non–vanishing value of the induced vector field at $r = 0$. The induced axial vector field $a_i = i a_i^a \tau^a / 2$ becomes

$$a_i^a = \hat{r}_i \hat{r}_a \left( \Theta'(r) - \frac{\sin \Theta(r)}{r} \right) + \delta_{ia} \sin \Theta(r) r \quad (14)$$

where the prime indicates the derivative with respect to the argument.

The use of the chirally rotated fermions is advantageous since the rotated Dirac-operator does no longer contain a chiral field and its determinant is hence anomaly free. The chiral anomaly, however, has not been lost by the chiral rotation but is hidden in the integration measure over the fermion fields. In fact, as observed by Fujikawa [12], a chiral rotation of the fermion fields gives rise to a non–trivial Jacobian of the integration measure

$$D\Psi D\bar{\Psi} = J(U) D\tilde{\Psi} D\tilde{\bar{\Psi}} \quad (15)$$

which is precisely given by the chiral anomaly

$$J(U) = \exp (iA) \quad (16)$$

Thus we have the relation

$$-i \text{Tr} \log i D = -i \text{Tr} \log i \tilde{D} + A \quad (17)$$

In many cases it is convenient to work with the chirally rotated fermion fields because of the absence of the anomaly from the fermion determinant.

As already mentioned above the chiral anomaly or equivalently the non–trivial Jacobian in the fermionic integration measure arise due to the need for regularization, which introduces a finite cut–off. In the regularized theory the chiral anomaly can be evaluated in a gradient expansion. In leading order the anomaly is then given by the Wess-Zumino action (3). There are, however, higher order terms which are suppressed by inverse powers of the cut–off. In renormalizable theories where the cut–off goes to infinity, these higher order terms disappear and the chiral anomaly is known in closed form. In non–renormalizable effective theories, however, the cut–off of the regularization scheme has to be kept finite and acquires a physical meaning, indicating the range of validity of the effective theory. In this case the higher order terms of the gradient expansion do not disappear but contribute to the anomaly which is then no longer available in closed form.

Furthermore, when the soliton sector of such effective mesonic theories is studied it is not sufficient to only consider the leading and sub–leading contributions from the gradient expansion and one has to perform a full non–perturbative evaluation of the fermion determinant [13]. The non–perturbative calculations have to be performed numerically [14], with the continuous space being discretized. Also in the non–perturbative studies of the soliton sector of the effective theory the use of the chiral rotation is in many cases advantageous [11]. As noticed above for the soliton description of baryons the topological
nature of the chiral field is crucial. Actually, topology is a property of continuous spaces (manifolds) and it is a priori not clear whether the chiral rotation with a topologically non–trivial chiral field can be represented in a finite dimensional and discretized model space used in the numerical calculations. In this respect let us recall that a single point defect in a manifold changes its topological properties already drastically.

The present paper is devoted to a study of the chiral rotation in non–perturbative soliton calculations where the fermion determinant has to be numerically evaluated in the background of a topologically non–trivial chiral field. For definiteness we shall use the Nambu-Jona-Lasinio (NJL) model as a microscopic fermion theory which shares all the relevant properties of chiral dynamics with QCD. Its bosonized version gives a quite satisfactory description of mesons and also of baryons when the soliton picture is assumed. The organization of the paper is as follows: After these introductory remarks we review the importance of the local chiral rotation for the extraction of meson properties from the NJL action. In section 3 we discuss the soliton solution to the NJL model of pseudoscalar fields with special emphasis on the choice of boundary conditions in the finite model space. Section 4 finally is devoted to the study of the local chiral rotation for topologically non–trivial field configurations and its influence on the soliton solution. Concluding remarks are given in section 5.

2 Properties of local chiral transformations

As pointed out in the introduction the chirally rotated formulation of the NJL model is suited to investigate properties of (axial–) vector mesons. In the present section we will therefore briefly review the results and illuminate the connection with the hidden gauge symmetry (HGS) and massive Yang–Mills (MYM) approaches for the description of (axial–) vector mesons. Most of the results reported in this section are taken from earlier works. Nevertheless we repeat these results here in order to put our work into perspective and have the paper self–contained.

In the chirally rotated formulation the bosonized version of the NJL model action reads

\[ A_{NJL} = -i \text{Tr} \log i \tilde{D} + A - \frac{1}{4G_1} \text{tr} \left( \Sigma^2 - m^2 \right) - \frac{1}{4G_2} \text{tr} \left[ \left( \tilde{V}_\mu - v_\mu \right)^2 + \left( \tilde{A}_\mu - a_\mu \right)^2 \right] \]  

(18)

with the constituent quark mass \( m \) being the vacuum expectation value of the scalar field \( \Sigma \) i.e. \( \langle \Sigma \rangle = m \). Again we have discarded terms proportional to the current quark mass matrix. The chirally transformed (axial–) vector fields are defined in eqns (9) and (10). Next we have to face the fact that the functional trace of the logarithm in (18) is ultra–violet divergent and thus needs regularization. This is achieved by first continuing to Euclidean space \( (x_0 = -ix_4) \) and then representing the real part of the Euclidean action by a parameter integral

\[ \frac{1}{2} \text{Tr} \log \left( \tilde{\mathcal{D}}_E \tilde{\mathcal{D}}_E \right) \longrightarrow -\frac{1}{2} \int_1^{\infty} \frac{ds}{s} \exp \left( -s \tilde{\mathcal{D}}_E \tilde{\mathcal{D}}_E \right) \]  

(19)

which introduces the cut–off \( \Lambda \). This substitution is an identity up to an irrelevant constant for \( \Lambda \to \infty \). The Euclidean Dirac operator \( \tilde{\mathcal{D}}_E \) is obtained by analytical continuation of \( \tilde{\mathcal{D}} \) to Euclidean space. The prescription (19) is known as the proper–time regularization. For the purpose of the present paper it is sufficient to only consider
the normal parts of the action. We may therefore neglect the imaginary part as well as the anomaly $\mathcal{A}$. Thus the actual starting point of our considerations is represented by

$$
\mathcal{A}_{\text{NJL}} = -\frac{1}{2} \int_{1/\Lambda^2}^{\infty} \frac{ds}{s} \text{Tr} \exp \left( -s \bar{\psi}^i \gamma^0 \psi^i \right) - \frac{1}{4G_1} \text{tr} \left( \Sigma^2 - m^2 \right) - \frac{1}{4G_2} \text{tr} \left[ (\tilde{V}_\mu - v_\mu)^2 + (\tilde{A}_\mu - a_\mu)^2 \right].
$$

(20)

In order to extract information about the properties of the (axial–) vector mesons commonly a (covariant) derivative expansion of the fermion determinant is performed. We choose to consider a covariant derivative expansion since, in contrast to an on–shell determination of the parameters \cite{20}, it preserves gauge invariance and does not lead to artificial mass terms. Furthermore, this procedure leaves the extraction of the axial–vector meson mass unique. Continuing back to Minkowski space and substituting the scalar field $\Sigma$ by its vacuum expectation value yields the leading terms \cite{17, 11}

$$
\mathcal{L}_{\text{NJL}} = \frac{1}{2g_V^2} \text{tr} \left( \tilde{V}_{\mu\nu} + \tilde{A}_{\mu\nu}^2 \right) - \frac{6m^2}{g_V^2} \text{tr} \tilde{A}_{\mu}^2 - \frac{1}{4G_2} \text{tr} \left[ (\tilde{V}_\mu - v_\mu)^2 + (\tilde{A}_\mu - a_\mu)^2 \right] + \ldots.
$$

(21)

Here $\tilde{V}_{\mu\nu}$ and $\tilde{A}_{\mu\nu}$ denote the field strength tensors

$$
\begin{align*}
\tilde{V}_{\mu\nu} &= \partial_\mu \tilde{V}_\nu - \partial_\nu \tilde{V}_\mu + [\tilde{V}_\mu, \tilde{V}_\nu] + [\tilde{A}_\mu, \tilde{A}_\nu], \\
\tilde{A}_{\mu\nu} &= \partial_\mu \tilde{A}_\nu - \partial_\nu \tilde{A}_\mu + [\tilde{V}_\mu, \tilde{A}_\nu] + [\tilde{A}_\mu, \tilde{V}_\nu]
\end{align*}
$$

(22)

of vector and axial–vector fields, respectively. Obviously in our convention these fields contain the coupling constant $g_V$ which in the proper–time regularization is given by

$$
g_V = 4\pi \left[ \frac{2N_C}{3} \Gamma \left( 0, \left( \frac{m}{\Lambda} \right)^2 \right) \right]^{-\frac{1}{2}}.
$$

(23)

For the description of the pion fields we adopt the unitary gauge for the chiral fields: $\xi = \xi_L^\dagger = \xi_R$. The pions come into the game by the non–linear realization $\xi = \exp \left( i \tau \cdot \pi / f \right)$. Then the last term in eqn (21) contains the axial–vector pion mixing which is eliminated by a corresponding shift in the axial field: $\tilde{A}_\mu \rightarrow \tilde{A}_\mu' = \tilde{A}_\mu + (ig_V m^2 / 12fG_2) \partial_\mu \tau \cdot \pi$. This shift obviously provides an additional kinetic term for the pions and thus effects the pion decay constant

$$
f_\pi^2 = \frac{M_A^2 - M_V^2}{4M_A^2G_2}
$$

(24)

with the (axial–) vector meson masses

$$
M_V^2 = \frac{g_V^2}{4G_2} \quad \text{and} \quad M_A^2 = M_V^2 + 6m^2.
$$

(25)

This brief summary of known results has demonstrated the usefulness of the chiral rotation (6) especially in the context of the derivative expansion since it eliminates the derivative of the chiral field from the fermion determinant (8).

The Lagrangian of the hidden gauge approach can be obtained from (21) by the following approximation. One neglects the kinetic parts for the axial–vector field $\tilde{A}_\mu$...
which leaves this field only as an auxiliary field. This allows to employ the corresponding equation of motion to eliminate $\tilde{A}'_{\mu}$ resulting in

$$L \sim \frac{1}{2g_{V}^{2}} \text{tr} \tilde{V}_{\mu\nu}^{2} - af_{\pi}^{2} \text{tr} \left( \tilde{V}_{\mu} - v_{\mu} \right)^{2} - \frac{1}{4f_{\pi}^{2}} \text{tr} a^{2}_{\mu}. \quad (26)$$

In the work of Bando et al.\[21\] $a$ was left as an undetermined parameter. Here it is fixed in terms of physical quantities

$$a = \frac{M_{A}^{2}}{M_{A}^{2} - M_{V}^{2}}. \quad (27)$$

Assuming the constituent quark mass $m = M_{V}/\sqrt{6}$ not only yields the Weinberg relation $M_{A} = \sqrt{2}M_{V}$[22] but also the KSRF relation $a = 2$[23].

Alternatively one might apply the same manipulations to the formulation in terms of the unrotated fields (3). Then the chiral field still appears in the fermion determinant and one has to deal with the covariant derivative

$$D_{\mu}U = \partial_{\mu}U + [V_{\mu}, U] - \{A_{\mu}, U\}. \quad (28)$$

The leading terms in the Lagrangian can readily be obtained[10]

$$L \sim \frac{3m^{2}}{2g_{V}^{2}} \text{tr} \left( D_{\mu}U D^{\mu}U^{\dagger} \right) + \frac{1}{2g_{V}^{2}} \text{tr} \left( V_{\mu\nu}^{2} + A_{\mu\nu}^{2} \right) - \frac{1}{4G_{2}} \text{tr} \left( V_{\mu}^{2} + A_{\mu}^{2} \right) \quad (29)$$

which exactly represent the massive Yang–Mills Lagrangian[8, 24]. Transforming the (axial–) vector fields according to (9) and noting that\[17]

$$\text{tr} \tilde{A}_{\mu}^{2} = -\frac{1}{4} \text{tr} \left( D_{\mu}U D^{\mu}U^{\dagger} \right) \quad (30)$$

one immediately observes that (29) and (21) describe the same physics. In the language of the NJL model the identity of the HGS and MYM approaches stems from the invariance of the module of the fermion determinant under the special chiral rotation (6).

Let us next explore the behavior of the fields under flavor rotations $g_{L}$, $g_{R}$. These are defined for the unrotated left– and right–handed quark fields

$$\Psi_{L} \rightarrow g_{L}\Psi_{L} \quad \text{and} \quad \Psi_{R} \rightarrow g_{R}\Psi_{R}. \quad (31)$$

The term which describes the coupling of the quarks to the scalar and pseudoscalar mesons is left invariant by demanding

$$\xi_{L}^{\dagger}\Sigma\xi_{R} \rightarrow g_{L}\xi_{L}^{\dagger}\Sigma\xi_{R}g_{R}^{\dagger} \quad (32)$$

which introduces the hidden gauge transformation $h[10]

$$\xi_{L} \rightarrow h^{\dagger}\xi_{L}g_{L}^{\dagger}, \quad \xi_{R} \rightarrow h^{\dagger}\xi_{R}g_{R}^{\dagger} \quad \text{and} \quad \Sigma \rightarrow h^{\dagger}\Sigma h. \quad (33)$$

Obviously the scalar fields transform homogeneously under the hidden gauge transformation. In this context it is important to note that $h$ may not be chosen independently
but rather depends on the gauge adopted for the chiral fields. Consider e.g. the unitary gauge $\xi_L^\dagger = \xi_R = \xi$. This requires the transformation property

$$\xi \rightarrow g_L \xi h = h^\dagger \xi g_R^\dagger. \quad (34)$$

For vector type transformations $g_L = g_R = g_V$ this equation is obviously solved by $h = g_V^\dagger$. Contrary, for axial type transformations $g_L = g_R^\dagger = g_A h$ is obtained as the solution to $g_A \xi h = h^\dagger \xi g_A$ which depends on the field configuration $\xi$. Thus even for global flavor transformations $g_{A,V}$ the hidden gauge transformation $h$ may be coordinate–dependent for coordinate dependent $\xi(x)$. The unrotated (axial–) vector fields transform inhomogeneously under the flavor rotations

$$V_\mu + A_\mu \rightarrow g_R (\partial_\mu + V_\mu + A_\mu) g_R^\dagger \quad \text{and} \quad V_\mu - A_\mu \rightarrow g_L (\partial_\mu + V_\mu - A_\mu) g_L^\dagger. \quad (35)$$

It is then straightforward to verify that the flavor transformation of the rotated fields only involves the hidden symmetry transformation $h$.

$$\tilde{\Psi}_{L,R} \rightarrow h^\dagger \tilde{\Psi}_{L,R}, \quad \tilde{V}_\mu \rightarrow h^\dagger (\partial_\mu + \tilde{V}_\mu) h \quad \text{and} \quad \tilde{A}_\mu \rightarrow h^\dagger \tilde{A}_\mu h. \quad (36)$$

The fact that $\tilde{A}_\mu$ transforms homogeneously has the important consequence that one can build models by setting $\tilde{A}_\mu \equiv 0$ without breaking chiral symmetry. These models then do no longer contain axial–vector degrees of freedom. Such models have frequently been utilized in the investigation of $\rho$ and $\omega$ meson physics. They have also been successfully applied to describe baryons as solitons.

### 3 The NJL soliton

In the baryon number one sector the NJL model has the celebrated feature to possess localized static solutions with finite energy, i.e. solitons. Here we wish to briefly review this solution and discuss different boundary conditions for the quark wavefunctions.

For static field configurations it is convenient to introduce a Dirac Hamiltonian

$$\mathcal{H} = \mathbf{\alpha} \cdot \mathbf{p} + \beta \left( P_R \xi \langle \Sigma \rangle \xi + P_L \xi^\dagger \langle \Sigma \rangle \xi^\dagger \right) \quad (37)$$

where we have assumed the unitary gauge (i.e. $\xi_L^\dagger = \xi_R = \xi$). This Hamiltonian enters the Euclidean Dirac operator via

$$i\beta \partial_E = -\partial_\tau - \mathcal{H}. \quad (38)$$

For static mesonic background fields the fermion determinant can conveniently be expressed in terms of the eigenvalues $\epsilon_\mu$ of the Dirac Hamiltonian

$$\mathcal{H} \Psi_\mu = \epsilon_\mu \Psi_\mu. \quad (39)$$

These eigenvalues obviously are functionals of the mesonic background fields. Depending of the specific boundary condition (which fixes the quantum reference state) to the
fermion fields in the functional integral ([1]), the fermion determinant ([1]) contains in
general besides a vacuum part \( A^0 \) also a valence quark part \( A^{\text{val}} \)
\[ \mathcal{A} = A^0 + A^{\text{val}}. \] (40)

The valence quark part arising from the explicit occupation of the valence quark levels
is given by
\[ A^{\text{val}} = -E^{\text{val}}[\xi]T, \quad E^{\text{val}}[\xi] = N_C \sum_{\mu} \eta_{\mu} |\epsilon_{\mu}|. \] (41)

Here \( \eta_{\mu} = 0, 1 \) denote the occupation numbers of the valence (anti-) quark states. The
vacuum part is conveniently evaluated for infinite Euclidean times (\( T \to \infty \)) which fixes
the vacuum state as the quantum reference state (no valence quark orbit occupied). For
the present considerations it will be sufficient to evaluate the real vacuum part
\[ A^0_R = \frac{1}{2} \text{Tr} \log \mathcal{D}_E \mathcal{D}_E^\dagger. \] (42)

Since for static configurations one has \([\partial_\tau, \mathcal{H}] = 0\) and thus \( \mathcal{D}_E \mathcal{D}_E^\dagger = -\partial_\tau^2 + \mathcal{H}^2 \). Then it
is straightforward to evaluate the real part of the fermion determinant in proper–time
regularization\(3\)
\[ A^0_R = -T N_C \int_1^{\infty} \frac{dz}{2\pi} \sum_{\mu} \int_{1/\Lambda^2}^{\infty} \frac{ds}{s} \exp \left( -s \left( z^2 + \epsilon_{\mu}^2 \right) \right). \] (43)

The temporal part of the trace has become the \( z \) integration. As this integral is Gaussian
it can readily be carried out yielding
\[ A^0_R = -T N_C \int_{1/\Lambda^2}^{\infty} \frac{ds}{\sqrt{4\pi s^3}} \sum_{\mu} \exp \left( -s \epsilon_{\mu}^2 \right). \] (44)

This expression allows to read off the static energy functional \( E[\xi] \) since \( A^0_R \to -T E^0[\xi] \)
as \( T \to \infty \)
\[ E^0[\xi] = N_C \int_{1/\Lambda^2}^{\infty} \frac{ds}{\sqrt{4\pi s^3}} \sum_{\mu} \exp \left( -s \epsilon_{\mu}^2 \right). \] (45)

The occupation numbers \( \eta_{\mu} \) of the valence quark orbits have to adjusted such the baryon
number
\[ B = \sum_{\mu} \left( \eta_{\mu} - \frac{1}{2} \text{sign}(\epsilon_{\mu}) \right) \] (46)
equals unity. The total energy functional is finally given by
\[ E[\xi] = E^{\text{val}}[\xi] + E^{\text{vac}}[\xi] - E^0[\xi] = 1 \] (47)

\[ \text{The imaginary part does not contribute for the field configurations under consideration.} \]
which is normalized to the energy of the vacuum configuration $\xi = 1$. In the chiral limit $(m_\pi = 0)$, which we have adopted here, the meson part of the action does not contribute to the soliton energy. The chiral soliton is the $\xi$ configuration which minimizes $E[\xi]$ and the minimal $E[\xi]$ is then identified as the soliton mass.

To be specific we employ the hedgehog ansatz for the chiral field
\[ \xi(r) = \exp \left( \frac{i}{2} \mathbf{r} \cdot \mathbf{\hat{r}} \Theta(r) \right) \] (48)
while the scalar fields are constrained to the chiral circle, i.e. $\langle \Sigma \rangle = m$. Substituting this ansatz into the Dirac Hamiltonian (37) yields
\[ H = \alpha \cdot p + \beta m(\cos\Theta(r) + i\gamma_5 \mathbf{r} \cdot \mathbf{\hat{r}} \sin\Theta(r)). \] (49)

The stationary condition $\delta E[\xi]/\delta \xi = 0$ is made explicit by functionally differentiating the energy–eigenvalues $\epsilon_\mu$ with respect to $\Theta$
\[ \frac{\delta \epsilon_\mu}{\delta \Theta(r)} = m \int d\Omega \Psi_\mu^\dagger(r) \beta (-\sin\Theta(r) + i\gamma_5 \mathbf{r} \cdot \mathbf{\hat{r}} \cos\Theta(r)) \Psi_\mu(r). \] (50)

This leads to the equation of motion
\[ \cos\Theta(r) \text{ tr} \int d\Omega \rho_S(r,r)i\gamma_5 \mathbf{r} \cdot \mathbf{\hat{r}} = \sin\Theta(r) \text{ tr} \int d\Omega \rho_S(r,r) \] (51)
where the traces are over flavor and Dirac indices only. According to the sum (47) the scalar quark density matrix $\rho_S(x,y) = \langle q(x)\bar{q}(y) \rangle$ is decomposed into valence quark and Dirac sea parts:
\[ \rho_S(x,y) = \rho_S^{\text{val}}(x,y) + \rho_S^{\text{vac}}(x,y) \]
\[ \rho_S^{\text{val}}(x,y) = \sum_\mu \Psi_\mu(x)\eta_\mu \bar{\Psi}_\mu(y)\text{sign}(\epsilon_\mu) \]
\[ \rho_S^{\text{vac}}(x,y) = -\frac{1}{2} \sum_\mu \Psi_\mu(x)\text{erfc} \left( \frac{\epsilon_\mu}{\Lambda} \right) \bar{\Psi}_\mu(y)\text{sign}(\epsilon_\mu). \] (52)

Technically the discretized eigenvalues $\epsilon_\mu$ of the Dirac Hamiltonian $\mathcal{H}$ (37) are obtained by restricting the space $R_3$ to a spherical cavity of radius $D$ and demanding certain boundary conditions at $r = D$. Eventually the continuum limit $D \to \infty$ has to be considered. In order to discuss pertinent boundary conditions it is necessary to describe the structure of the eigenstates of $\mathcal{H}$. Due to the special form of the hedgehog ansatz the Dirac Hamiltonian commutes with the grand spin operator
\[ G = J + \frac{\mathbf{\tau}}{2} = l + \frac{\mathbf{\sigma}}{2} + \frac{\mathbf{\tau}}{2} \] (53)
where $J$ labels the total spin and $l$ the orbital angular momentum. $\mathbf{\tau}/2$ and $\mathbf{\sigma}/2$ denote the isospin and spin operators, respectively. The eigenstates of $\mathcal{H}$ are then as well eigenstates of $G$. The latter are constructed by first coupling spin and orbital angular momentum to the total spin which is subsequently coupled with the isospin to the grand
spin[31]. The resulting states are denoted by $|ljGM\rangle$ with $M$ being the projection of $\mathbf{G}$. These states obey the selection rules

$$j = \begin{cases} G + 1/2, & l = \{ G + 1 \} \\ G - 1/2, & l = \{ G - 1 \} \end{cases}$$

(54)

The Dirac Hamiltonian furthermore commutes with the parity operator. Thus the eigenstates of $\mathcal{H}$ with different parity and/or grand spin decouple. The coordinate space representation of the eigenstates $|\mu\rangle$ is finally given by

$$\Psi^{(G, +)}_\mu = \begin{pmatrix} ig^{(G, +)}(r)|GG + \frac{1}{2}GM\rangle \\ f^{(G, +)}(r)|G + 1G + \frac{1}{2}GM\rangle \end{pmatrix} + \begin{pmatrix} ig^{(G, +)}(r)|GG - \frac{1}{2}GM\rangle \\ f^{(G, +)}(r)|G - 1G - \frac{1}{2}GM\rangle \end{pmatrix}$$

(55)

$$\Psi^{(G, -)}_\mu = \begin{pmatrix} ig^{(G, -)}(r)|G + 1G + \frac{1}{2}GM\rangle \\ -f^{(G, -)}(r)|GG + \frac{1}{2}GM\rangle \end{pmatrix} + \begin{pmatrix} ig^{(G, -)}(r)|G - 1G - \frac{1}{2}GM\rangle \\ f^{(G, -)}(r)|GG - \frac{1}{2}GM\rangle \end{pmatrix}.$$  

(56)

The second superscript labels the intrinsic parity $\Pi_{\text{intr}}$ which enters the parity eigenvalue via $\Pi = (-1)^G \times \Pi_{\text{intr}}$. In the absence of the soliton (i.e. $\Theta = 0$) the radial functions $g^{(G, +)}_{\mu}(r)$, $f^{(G, +)}_{\mu}(r)$, etc. are given by spherical Bessel functions. E.g. 

$$g^{(G, +)}_{\mu}(r) = N_k \sqrt{1 + m/E} \ j_G(kr), \quad f^{(G, +)}_{\mu}(r) = N_k \text{sign}(E) \sqrt{1 - m/E} \ j_{G + 1}(kr)$$ 

(57)

and all other radial functions vanishing represents a solution to $\mathcal{H}(\Theta = 0)$ with the energy eigenvalues $E = \pm \sqrt{k^2 + m^2}$ and parity $(-1)^G$. $N_k$ is a normalization constant.

Two distinct sets of boundary conditions have been considered in the literature. Originally Kahana and Ripka[31] proposed to discretize the momenta by enforcing those components of the Dirac spinors to vanish at the boundary which possess identical grand spin and orbital angular momentum, i.e.

$$g^{(G, +)}_{\mu}(D) = g^{(G, +)}_{\mu}(D) = f^{(G, +)}_{\mu}(D) = f^{(G, +)}_{\mu}(D) = 0.$$ 

(58)

This boundary condition has the advantage that for a given grand spin channel $G$ only one set of basis momenta $\{k_{nG}\}$ is involved. These $k_{nG}$ make the $G^{\text{th}}$ Bessel function vanish at the boundary ($j_G(k_{nG}D) = 0$). However, this boundary condition has (among others) the disadvantage that the matrix elements of flavor generators, like $\tau_3$ are not diagonal in momentum space. If the matrix elements of the flavor generators are not diagonal in the momenta a finite moment of inertia will result even in the absence of a chiral field[33]. Stated otherwise, in this case the boundary conditions violate the flavor symmetry. This problem can be cured[32] by changing the boundary conditions for the states with $\Pi_{\text{intr}} = -1$

$$g^{(G, +)}_{\mu}(D) = g^{(G, +)}_{\mu}(D) = g^{(G, -)}_{\mu}(D) = g^{(G, -)}_{\mu}(D) = 0$$ 

(59)

i.e. the upper components of the Dirac spinors always vanish at the boundary. The diagonalization of the Dirac Hamiltonian[37] with the condition (58) is technically less feasible since it involves three sets of basis momenta $\{k_{nG - 1}\}$, $\{k_{nG + 1}\}$ and $\{k_{nG}\}$ for a given grand spin channel. In table 1 we compare some properties of the two boundary conditions (58) and (59) in the case when no soliton is present. The first four quantities
Table 1. Properties of the two boundary conditions (58) and (59) in the baryon number zero sector. $f(r)$ represents an arbitrary radial function.

| Quantity | Condition (58) | Condition (59) |
|----------|----------------|----------------|
| $\sum_\mu \Psi^\dagger_\mu \beta_5 r^\tau \cdot \hat{r} \Psi_\mu \text{erfc} \left( \frac{\epsilon_\mu}{\Lambda} \right) \text{sign}(\epsilon_\mu) = 0$ | yes | yes |
| $\sum_\mu \Psi^\dagger_\mu \alpha \cdot (\tau \times \hat{r}) \Psi_\mu \text{erfc} \left( \frac{\epsilon_\mu}{\Lambda} \right) \text{sign}(\epsilon_\mu) = 0$ | no | yes |
| $\sum_\mu \Psi^\dagger_\mu \alpha \cdot \tau \Psi_\mu \text{erfc} \left( \frac{\epsilon_\mu}{\Lambda} \right) \text{sign}(\epsilon_\mu) = 0$ | yes | yes |
| $\sum_\mu \Psi^\dagger_\mu \alpha \cdot \hat{r} \tau \cdot \hat{r} \Psi_\mu \text{erfc} \left( \frac{\epsilon_\mu}{\Lambda} \right) \text{sign}(\epsilon_\mu) = 0$ | yes | yes |
| $\text{tr} \left( \beta f(r) \right) = 0$ | yes | no |
| $\text{tr} \left( \gamma \cdot \tau f(r) \right) = 0$ | yes | yes |
| $\langle \mu | \tau_i | \nu \rangle = 0$ for $k_\mu \neq k_\nu$ | no | yes |

appearing in that table show up in various equations of motion when e.g. also (axial–)
vector mesons are included[33]. In case such a quantity is non–zero the vacuum gives a
spurious contribution to the associated equation of motion. For an iterative solution to
the equations of motion this spurious contribution has to be subtracted. It should be
noted, however, that the relations listed in table 1 are all satisfied for both boundary
conditions in the continuum limit $D \to \infty$.

So far the discussion of the boundary conditions has only effected the point $r = D$.
In the context of the local chiral rotation it is equally important to consider the wave–
functions at $r = 0$. As already mentioned the solutions to the Dirac equation (49) are
given by spherical Bessel functions in the free case, $\Theta = 0$. Except of $j_0$ these vanish at
the origin. In the case $\Theta \neq 0$ we adopt the boundary conditions $\Theta(0) = -n\pi$ thus no
singularity appears in the Dirac Hamiltonian (49) at $r = 0$. Therefore the radial parts
of the quark wave–functions may be expressed as linear combinations of the solutions to
the free Dirac Hamiltonian. E.g.

$$g^{(G,+:1)}_\mu (r) = \sum_k V_{\mu k}[\Theta] N_k \sqrt{1 + m/E_{kG} j_G(k_{kG}r)},$$

$$f^{(G,+:1)}_\mu (r) = \sum_k V_{\mu k}[\Theta] N_k \text{sign}(E_{kG}) \sqrt{1 - m/E_{kG} j_{G+1}(k_{kG}r)}$$

(60)

where the eigenvectors $V_{\mu k}[\Theta]$ are obtained by diagonalizing the Dirac Hamiltonian in the
free basis. It should be stressed that the use of the free basis is only applicable because
the point singularity hidden in $\tau \cdot \hat{r}$ has disappeared. If singularities show up for certain
field configurations the basis for diagonalizing $\mathcal{H}$ has to be altered. This will be the
central issue of the next section.

4 The chirally rotated fermion determinant

In the previous section we have demonstrated that the normalizable solutions to the
free Dirac equation with spherical boundary conditions represent a pertinent basis for
the diagonalization of the Dirac Hamiltonian with the soliton present. This property is
based on the fact that the Hamiltonian (43) is free of singularities. In this section we will
explain how singularities appearing in a Dirac Hamiltonian influence the choice of basis
states. Let us for this purpose consider the Hamiltonian for the chirally rotated quark fields $\tilde{\Psi} = \Omega(\Theta)\Psi$ (cf. eqns (12)–(14) and ref. [11]):

$$H_R = \Omega(\Theta)H\Omega^\dagger(\Theta) = \alpha \cdot \mathbf{p} + \beta m - \frac{1}{2}(\sigma \cdot \mathbf{f})(\tau \cdot \mathbf{f}) \left(\Theta'(r) - \frac{1}{r} \sin \Theta(r)\right)$$

$$- \frac{1}{2r}(\sigma \cdot \tau) \sin \Theta(r) - \frac{1}{r} \alpha \cdot (\mathbf{f} \times \tau) \sin^2 \left(\frac{\Theta(r)}{2}\right)$$

(61)

since in unitary gauge $\Omega(\Theta) = \cos(\Theta/2) + i\gamma_5 \tau \cdot \mathbf{f} \sin(\Theta/2)$. Obviously the $\Theta$–dependence in the Hamiltonian has been transferred to induced (axial–) vector meson fields. As expected the rotated Hamiltonian, $H_R$, contains an explicit singularity in the $1/r$ term at $r = 0$. Additionally there are “coordinate singularities” in the expressions involving $\mathbf{f}$. All these singularities appear because the “coordinate singularity” in $\Omega(\Theta)$ is not compensated by corresponding values of the chiral angle $\Theta$. Stated otherwise: the transformation $\Omega(\Theta)$ with $\Theta(0) - \Theta(\infty) = -n\pi$ is topologically distinct from the unit transformation. Although $\Omega(\Theta)$ represents a unitary transformation it is then not astonishing that a numerical diagonalization

$$H_R \tilde{\Psi}_\mu = \tilde{\epsilon}_\mu \Psi_\mu$$

(62)

in the basis of the free Hamiltonian does not render the eigenvalues of the unrotated Hamiltonian, $H$, (i.e. $\tilde{\epsilon}_\mu \neq \epsilon_\mu$) despite the relevant matrix elements being finite. This finiteness is merely due to the $r^2$ factor in the volume element. One might suspect that the Hamiltonian $H_{2R} = \Omega(2\Theta)H\Omega^\dagger(2\Theta)$ obtained by a $2\Theta$ rotation has the same spectrum as $H$, since $H_{2R}$ is free of singularities. Although this behavior is exhibited by the numerical solution for the low–lying energy eigenvalues, the topological character of the transformation has drastic consequences for the states at the lower and upper ends of the spectrum in momentum space. Adopting the same basis states for diagonalizing $H$ and $H_{2R}$ one observes that the eigenvalues of $H_{2R}$ are shifted against those of $H$, i.e. the most negative energy eigenvalue is missing while an additional one has popped up at the upper end of the spectrum. Up to numerical uncertainties the eigenvalues in the intermediate region agree for both $H$ and $H_{2R}$. This behavior is sketched in figure 1 and repeats itself for $H_{4R} = \Omega(4\Theta)H\Omega^\dagger(4\Theta)$. Thus the chiral rotation represents another example of the so–called “infinite hotel story”[34] which is an interesting feature reflecting the topological character of this transformation.

Let us now return to the problem of diagonalizing $H_R$ and restrict ourselves for the moment to the channel $G = 0^+$. At $r = 0$ the chiral rotation

$$\Omega(r = 0) = -i(\tau \cdot \mathbf{f})\gamma_5$$

(63)

obviously exchanges upper and lower components of Dirac spinors. The corresponding wave–functions are given by

$$\Omega(r = 0)\Psi_{\mu}^{(0,+)}(r = 0) = \begin{pmatrix} -i f_{\mu}^{(0,+;1)}(r = 0)|0\vec{1}00\rangle \\ g_{\mu}^{(0,+;1)}(r = 0)|1\vec{1}00\rangle \end{pmatrix}$$

(64)

which obviously cannot be represented by the free basis since $g_{\mu}^{(0,+;1)}(r = 0) \neq 0$, in general. However, the lower components of the eigenstates of the free Hamiltonian in the

In the standard representation $\gamma_5 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. 13
Fig. 1. A schematic plot of the spectrum of the rotated Hamiltonian $H_{nR} = \Omega(n\Theta)H\Omega^\dagger(n\Theta)$.

The $\Pi^0 = 0^+$ channel have this property (cf. eqns [55,57]). We therefore apply the rotation (63) to the eigenstates of the free Hamiltonian as well, resulting in a new basis given by

$$
N_k \left( -i \text{sign}(E)\sqrt{1 - E/m} \ j_1(kr)|0\rangle + m \right),
$$

$$
\sqrt{1 + E/m} \ j_0(kr)|1\rangle - m
$$

This is, of course, no longer a solution to the free Dirac equation. Moreover, a single component of this spinor does not even solve the free Klein Gordan equation. At $r = D$ the chiral rotation equals unity. Thus we demand the discretization condition $j_1(kD) = 0$ according to eqn (58) i.e. the momenta are taken from the set $\{k_{n1}\}$. With this basis we have succeeded in eliminating the singularities at $r = 0$ and keeping track of the boundary conditions at $r = D$. We are thus enabled to numerically diagonalize $H_R$. When comparing with the eigenvalues of $\mathcal{H}$ we again encounter a form of the “infinite hotel story”: one state is missing in the negative part of the spectrum while an additional shows up in the positive part. The missing state turns out to be the one at the upper end of the negative Dirac sea, i.e. $E \approx -m$. Then it is important to note that in addition to the states with finite $k$ the basis (63) together with the boundary condition $j_1(kD) = 0$ also allows for the “constant state” with $k = 0$. In the continuum limit ($D \to \infty$) this state is absent. Including, however, this state for finite $D$ in the process of diagonalizing $H_R$ finally renders the missing state. This is not surprising since application of the inverse chiral rotation $\Omega^\dagger(r = 0)$ onto this “constant state” leads to an eigenstate of the free Dirac Hamiltonian with the eigenvalue $-m$. It should be remarked that for the free unrotated problem a “constant state” with eigenvalue $-m$ is only allowed in the $G^\Pi = 1^-$ channel. Although $\Omega(\Theta)$ does commute with the grand spin operator its topological character mixes various grand spin channels via the boundary conditions.
Accordingly the diagonalization of $H_R$ in the $G^\Pi = 0^-$ channel demands the basis states

$$N_k \left( i \text{sign}(E) \sqrt{1 - E/m} j_0(kr)|1\overline{00}\rangle \right)$$

(66)

with the boundary condition $j_1(kD) = 0$ in order to be compatible with the Kahana–Ripka\[31] diagonalization of $H$. The additional “constant state” needed here corresponds to a state with eigenvalue $+m$ of the free unrotated Hamiltonian.\[31]

We have finally been able to diagonalize the chirally rotated Hamiltonian $H_R$ in the $G = 0$ sectors by very tricky means. It should also be kept in mind that there are now additional states at the upper (from $0^+$) and lower (from $0^-$) ends of the spectrum which do not possess “counterstates” in the $G = 0$ part of the spectrum in the unrotated problem. For the dynamics of the problem they are of no importance because their contribution to physical quantities is damped by the regularization. However, their existence reflects the topological character of the chiral rotation.

In the other channels, i.e. $G \geq 1$ we have unfortunately not been able to construct a set of basis states which rendered the eigenvalues of the unrotated Hamiltonian along the approach described above. In the $G = 0$ sector we have already seen that a “global rotation” $-i(\tau \cdot \hat{r})\gamma_5$ is needed for the basis states in order to accommodate the boundary conditions at $r = 0$. Furthermore a mixture of different grand spin channels appears since this “global rotation” deviates from unity at $r = D$. This problem can be avoided by defining a basis in the topologically non–trivial sector via

$$\tilde{\Psi}_{\mu0} = \Omega^\dagger(\phi)\Psi_{\mu0}$$

(67)

with $\Psi_{\mu0}$ being the solutions to the free unrotated Hamiltonian. $\phi$ represents an auxiliary radial function satisfying the boundary conditions $\phi(0) = -\pi$ and $\phi(D) = 0$. E.g. we may take

$$\phi(r) = -\pi \left( 1 - \frac{r}{D} \right) \exp \left( -tmr \right)$$

(68)

with $t$ being a free parameter. The diagonalization of $H_R$ in the basis $\tilde{\Psi}_0$ is equivalent to diagonalizing

$$\alpha \cdot p + m\beta \left( \cos \phi(r) + i\gamma_5(\tau \cdot \hat{r}) \sin \phi(r) \right)$$

$$+ \frac{1}{2} \left[ \phi'(r) - \Theta'(r) + \frac{1}{r} \sin(\Theta(r) - \phi(r)) \right] (\sigma \cdot \hat{r})(\tau \cdot \hat{r})$$

$$+ \frac{1}{2r} \sin(\phi(r) - \Theta(r))(\sigma \cdot \tau) - \frac{1}{2r} [1 - \cos(\Theta(r) - \phi(r))] \alpha \cdot (\hat{r} \times \tau)$$

(69)

in the standard basis $\{\Psi_{\mu0}\}$. At this point it should be stressed again that $\phi$ is not a dynamical field but merely an auxiliary field which transforms the basis such as to eliminate the singularities from the Dirac Hamiltonian. This property is completely determined by the boundary values of $\phi$. Thus the results ought to be independent of the parameter $t$.

For the numerical investigation we have first considered the energy eigenvalues of $H_R$ employing the rotated basis (67) for the diagonalization and compared to the eigenvalues

---

\[31\]The additional “constant states” thus do not alter the trace of the Hamiltonian.
of $H$. In a wide range $t \approx 0.1 \ldots 1.4$ the difference for the eigenvalues which are contained in the interval $(-5m, 5m)$ is negligible, i.e. less than tenth of a percent. For the higher grand spin channels the window for $t$ which yields reasonable agreement lies somewhat higher $t \approx 0.8 \ldots 1.6$. Summing up the energy eigenvalues according to eqn (45) in order to obtain the vacuum part of the soliton energy the deviation compared to the unrotated formulation is about 0.5MeV. This is, of course, negligibly small since the inherited mass scale is given by the constituent quark mass, $m$, which is of the order of several hundred MeV. By using the locally transformed basis (57) the wave–functions corresponding to the eigenstates of $H_R$ agree reasonably well with the rotated wave–functions $\Omega(\Theta)\Psi_\mu$ of the original Dirac Hamiltonian $H$. It should be noted that a large number of momentum states is required to numerically gain this result. This is not surprising since in order to represent the functional unity an infinite number of momentum states is needed.

Before turning to the detailed discussion of the equation of motion in the rotated system we would like to mention that the self–consistent profile being obtained from the unrotated problem also minimizes the soliton mass in the chirally rotated frame. Stated otherwise: each change in this profile function leads to an increase of the energy obtained from the eigenvalues of $H_R$.

As in the formulation with in the unrotated frame the equation of motion is gained by extremizing the energy functional (17). The energy eigenvalues in the rotated frame, however, exhibit a different functional dependence on the chiral field. The functional derivative of these eigenvalues with respect to the chiral angle reads

$$\frac{\delta \epsilon_\mu}{\delta \Theta(r)} = \int d\Omega \left\{ \frac{1}{2} \left( \frac{\partial}{\partial r} \cos \Theta(r) \right) r^2 \tilde{\Psi}_\mu^\dagger(r)(\sigma \cdot \hat{r})(\tau \cdot \hat{r})\tilde{\Psi}_\mu(r) - \frac{r}{2} \cos \Theta(r) \tilde{\Psi}_\mu^\dagger(r)\sigma \cdot \tau \tilde{\Psi}_\mu(r) - \frac{r}{2} \sin \Theta(r) \tilde{\Psi}_\mu^\dagger(r)\alpha \cdot (\hat{r} \times \tau)\tilde{\Psi}_\mu(r) \right\}$$

(70)

with $\tilde{\Psi}_\mu(r)$ being the eigenstates of the rotated Dirac Hamiltonian (61). The derivatives (70) enter the stationary condition for the energy functional resulting in the equation of motion

$$A_L(r) + A_T(r)\cos \Theta(r) - V(r)\sin \Theta(r) = 0. \quad (71)$$

In order to display the radial functions $A_L$, $A_T$ and $V$ it is convenient to introduce the charge density $\tilde{\rho}_C = \langle \rho(x)\rho(y) \rangle = \tilde{\rho}_C^{val} + \tilde{\rho}_C^{vac}$ involving the eigenstates $\tilde{\Psi}_\mu$ of $H_R$ (cf. eqn (52)) [14]:

$$\tilde{\rho}_C^{val}(x,y) = \sum_\mu \tilde{\Psi}_\mu(x)\eta_\mu \tilde{\Psi}_\mu^\dagger(y)\text{sign}(\epsilon_\mu)$$

$$\tilde{\rho}_C^{vac}(x,y) = -\frac{1}{2} \sum_\mu \tilde{\Psi}_\mu(x)\text{erfc} \left( \frac{\epsilon_\mu}{\Lambda} \right) \tilde{\Psi}_\mu^\dagger(y)\text{sign}(\epsilon_\mu). \quad (72)$$

The radial functions $A_L$, $A_T$ and $V$ are of axial–$(A_{L,T})$ and vector($V$) character

$$A_L(r) = \frac{1}{\rho(r)\partial r} \text{tr} \int d\Omega \ r^2 \tilde{\rho}_C(r,r)(\sigma \cdot \hat{r})(\tau \cdot \hat{r}) \quad (73)$$

$$A_T(r) = \text{tr} \int d\Omega \tilde{\rho}_C(r,r)[(\sigma \cdot \hat{r})(\tau \cdot \hat{r}) - (\sigma \cdot \tau)] \quad (74)$$

$$V(r) = \text{tr} \int d\Omega \tilde{\rho}_C(r,r)\alpha \cdot (\hat{r} \times \tau). \quad (75)$$

16
It should be noted that \( A_L \) does not depend on the auxiliary field \( \phi \). It is then straightforward to verify that for any meson configuration \( A_L \) and \( A_T \) satisfy the relation

\[
A_L(r = 0) = (-1)^k A_T(r = 0)
\]

(76)

where \( k \) is defined by the value of the auxiliary field \( \phi \) at the origin \( \phi(r = 0) = k\pi \). The vector type radial function \( V \) vanishes at the origin. Thus the equation of motion (71) together with the relation (76) yield the boundary condition \( \Theta(r = 0) = (2n + 1)\pi \) for \( k = 1 \). This is stronger than the boundary condition derived from the original equation of motion (71) which also allows for even multiples of \( \pi \) for \( \Theta(r = 0) \) since \( \int d\Omega \rho_S(r = 0, r = 0) i\gamma_5 \tau \cdot \hat{r} = 0 \). Assuming the Kahana–Ripka[31] boundary conditions for the unrotated basis states \( \Psi_{\mu 0} \) similar considerations for \( r = D \) show that \( \Theta(r = D) = 2l\pi \) since \( A_L(r = D) = -A_T(r = D) \) as long as \( \phi(D) = 0 \). Obviously the topological charge associated with the chiral field in the hedgehog ansatz \( (\Theta(r = 0) - \Theta(r = D))/\pi \) can assume odd values only when \( k \) is odd. Especially \( \phi \equiv 0 \) is prohibited in the case of unit baryon number. Thus the study of the boundary conditions in the chirally transformed system corroborates the conclusion drawn from investigating the eigenvalues and –states of \( \mathcal{H}_R \) that it is mandatory to also transform the basis spinors and in particular the boundary conditions to the topological non–trivial sector.

Before turning to the discussion of the numerical treatment of eqn (71) we would like to make the remark that substituting the transformation \( \Psi_{\mu}(r) = \Omega^\dagger(\Theta) \tilde{\Psi}_{\mu}(r) \) into the original equation of motion (51) does not result in the relation (70) but rather yields the constraint

\[
0 = \sum_{\mu} \left( \eta_{\mu} \text{sign}(\epsilon_{\mu}) - \frac{1}{2} \text{erfc} \left( \frac{\epsilon_{\mu}}{\Lambda} \right) \right) \tilde{\Psi}_{\mu}(r) \gamma_5 \tau \cdot \hat{r} \tilde{\Psi}_{\mu}(r).
\]

(77)

Thus the equation of motion (71) cannot be obtained by transforming the states \( \Psi_{\mu} \) but only by employing the Dirac equation in the rotated frame (72) to extremize the energy functional. The constraint (77) does not represent an over–determination of the the system since infinitely many states are involved.

Due the transcendent character of the equation of motion in the rotated frame (71) solutions cannot be obtained for arbitrary values of the radial functions \( A_L, A_T \) and \( V \). E.g. for large distances \( \Theta \to 0 \) requires \( |A_L| \geq |A_T| \) in order to find a solution to eqn (71). Thus the treatment of the NJL soliton in the chirally rotated frame is not very well suited for an iterative procedure to find the self–consistent solution. The reason is that a small deviation of the radial functions \( A_L, A_T \) and \( V \) from those corresponding to this solution can render eqn (71) indissoluble for \( \Theta(r) \). Then it is not unexpected that at large distances the solution to the rotated equation of motion (71) becomes unstable and the original profile function cannot be reproduced for \( r \geq 2\text{fm} \). For smaller values of \( r \) the original profile function is well reproduced. In figure 2 this behavior is displayed. The self–consistent solution to eqn (71) serves as ingredient to evaluate the radial functions \( A_L, A_T \) and \( V \). The solution to eqn (71) is then constructed and compared to the original profile function.
5 Conclusions

We have investigated the role of chiral transformations for the evaluation of fermion determinants. When these transformations are topologically trivial they provide a useful tool to evaluate the chiral anomaly. Furthermore they can be used to demonstrate the equivalence between the hidden gauge and massive Yang–Mills approaches to vector mesons. A generalization to the case when chiral fields have a topological charge different from zero is not straightforward. Even though the special transformation we have been considering is unitary its topological character prevents the eigenvalues and -vectors of the original Dirac Hamiltonian to be regained from the rotated Hamiltonian unless the boundary conditions are transformed to the topologically non–trivial sector accordingly. Furthermore we have observed that the stationary conditions to the static energy functional in the topologically distinct sectors are not related by the transformation of the equation of motion. The boundary conditions for the chiral field obtained from the stationary condition have been found being invariant under the chiral rotation only when the basis quark fields are taken from the topological sector associated with the chiral transformation. Diagonalization of the rotated Dirac Hamiltonian in this basis can be reformulated into a problem where the induced vector fields (10) belong to the topologically trivial sector (cf. eqn (69)). In order to diagonalize the resulting operator (69) the standard basis [31, 32] may be employed.

These explorations on chiral transformation properties of the fermion determinant when the Dirac Hamiltonian contains topologically non–trivial (axial–) vector mesons may prove to be very helpful when considering the chiral invariant elimination of the

Fig. 2. Comparison of the self–consistent profile in the unrotated formulation (dashed line) and the solution to eqn (71) (solid line).
axial–vector fields as described at the end of section 2[26]. In that case the chirally
rotated Dirac Hamiltonian becomes as simple as
\[ H_R = \alpha \cdot p + \beta m + \frac{G(r)}{r} \alpha \cdot (\hat{r} \times \tau). \] (78)

Here \( G(r) \) refers to a dynamical field and should not be confused with the induced vector
field discussed in eqn (13). The chiral field then only appears in the purely mesonic part
of the energy functional
\[ E_m = \frac{2\pi}{G_2} \int dr \left\{ (G(r) + 1 - \cos \Theta(r))^2 + \frac{1}{2} r^2 \Theta'(r) + \sin^2 \Theta(r) \right\}. \] (79)

\( \Theta(r = 0) = -\pi \) then implies that \( G(r = 0) = -2 \)[28, 29]. One thus has to deal with a
Dirac Hamiltonian which contains a topologically non–trivial vector meson field. Thus
(78) cannot be treated using the standard basis[31, 32] but rather by employing techniques
which are analogous to those developed in section 4. Investigations in this direction are
in progress.

Finally we would also like to remark that this kind of singularities does not only
appear when the Dirac Hamiltonian is considered. Such topological defects have also
caused problems when fluctuations off vector meson solitons were investigated[30]. In
that case the boundary conditions for the vector meson fluctuations had to undergo a
special gauge transformation which corresponds to the transformation of the basis quark
spinors described here (67).

Acknowledgement

We would like to thank U. Zückert for helpful contributions in the early stages of this
work.

References

[1] G. ’t Hooft, Nucl. Phys. B72 (1974) 461.
[2] E. Witten, Nucl. Phys. B160 (1979) 57.
[3] G. S. Adkins, C. R. Nappi and E. Witten, Nucl. Phys. B228 (1983) 552.
[4] For an extensive compilation of articles see, e.g. “Baryons as Skyrme Solitons”, G.
   Holzwarth (Ed.), World Scientific Publ. Comp., Singapore 1993.
[5] E. Witten, Nucl. Phys. B223 (1983) 422; 433;
[6] W. Bardeen, Phys. Rev. 184 (1969) 1848.
[7] J. Wess and B. Zumino, Phys. Lett. 37B (1971) 95.
[8] Ö. Kaymakcalan, S. Rajeev and J. Schechter, Phys. Rev. D30 (1984) 594.
[9] A. Manohar, Nucl. Phys. B278 (1984) 19;
   A. P. Balachandran, A. Barducci, F. Lizzi, V. Rodgers and A. Stern, Nucl. Phys. B256 (1985) 525.

[10] D. Ebert and H. Reinhardt, Nucl. Phys. B271 (1986) 188.

[11] H. Reinhardt and B. V. Dang, Nucl. Phys. A500 (1989) 563.

[12] K. Fujikawa, Phys. Rev. D21 (1980) 2848.

[13] H. Reinhardt, Nucl. Phys. A503 (1989) 825.

[14] H. Reinhardt and R. Wünsch, Phys. Lett. B215 (1988) 577; B230 (1989) 93;
   T. Meissner, F. Grümmer and K. Goeke, Phys. Lett. B227 (1989) 296;
   R. Alkofer, Phys. Lett. B236 (1990) 310.

[15] Y. Nambu and G. Jona-Lasinio, Phys. Rev. 122 (1961) 345; 124 (1961) 246.

[16] A status report and a list of references may e.g. be found in the article by R. Alkofer in ref. [4].

[17] M. Wakamatsu and W. Weise, Z. Phys. A331 (1988) 183.

[18] M. Wakamatsu, Ann. Phys. (N. Y.) 193 (1989) 287.

[19] J. Schwinger, Phys. Rev. 82 (1951) 664.

[20] M. Jaminon, G. Mendez Galain, G. Ripka and P. Stassart, Nucl. Phys. A537 (1992) 418.

[21] M. Bando, T. Kugo and K. Yamawaki, Phys. Rep. 164 (1987) 217, and references therein.

[22] S. Weinberg, Phys. Rev. Lett. 18 (1967) 507.

[23] K. Kawarbayashi and M. Suzuki, Phys. Rev. Lett. 16 (1966) 255;
   Riazuddin and Fayazuddin, Phys. Rev. 147 (1966) 1071.

[24] H. Gomm, Ö. Kaymakcalan and J. Schechter, Phys. Rev. D30 (1984) 2345.

[25] S. Callan, S. Coleman, J. Wess and B. Zumino, Phys. Rev. 177 (1969) 2247.

[26] Ö. Kaymakcalan and J. Schechter, Phys. Rev. D31 (1984) 1109.

[27] J. Schechter, A. Subbaraman and H. Weigel, Phys. Rev. D48 (1993) 339 (1993).

[28] P. Jain, R. Johnson, U.-G. Meißner, N. W. Park and J. Schechter, Phys. Rev. D37 (1988) 3252;
   M. Chemtob, Nucl. Phys. A487 (1988) 509;
   The SU(3) version is discussed in: N. W. Park and H. Weigel, Nucl. Phys. A541 (1992) 453.

[29] U.-G. Meißner, Phys. Rep. 161 (1988) 213.

[30] B. Schwesinger, H. Weigel, G. Holzwarth and A. Hayashi, Phys. Rep. 173 (1989) 173.
[31] S. Kahana and G. Ripka, Nucl. Phys. A429 (1984) 462.

[32] H. Weigel, R. Alkofer and H. Reinhardt, Nucl. Phys. B387 (1992) 638.

[33] C. Schüren, F. Döring, E. Ruiz Arriola and K. Goeke, Nucl. Phys. A565 (1993) 687; U. Zückert, R. Alkofer, H. Reinhardt and H. Weigel, “The Chiral Soliton of the Nambu-Jona-Lasinio Model with Vector and Axial Vector Mesons”, hep-ph/9303271, Nucl. Phys. A to be published.

[34] H.B. Nielsen and M. Ninomiya, Phys. Lett. 130B (1983) 389; S. Nadkarin and H. B. Nielsen, Nucl. Phys. B263 (1986) 1.