Global embedding of $D$-dimensional black holes with a cosmological constant in Minkowskian spacetimes: Matching between Hawking temperature and Unruh temperature

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We study the matching between the Hawking temperature of a large class of static $D$-dimensional black holes and the Unruh temperature of the corresponding higher dimensional Rindler spacetime. In order to accomplish this task we find the global embedding of the $D$-dimensional black holes into a higher dimensional Minkowskian spacetime, called the global embedding Minkowskian spacetime procedure (GEMS procedure). These global embedding transformations are important on their own, since they provide a powerful tool that simplifies the study of black hole physics by working instead, but equivalently, in an accelerated Rindler frame in a flat background geometry. We discuss neutral and charged Tangherlini black holes with and without cosmological constant, and in the negative cosmological constant case, we consider the three allowed topologies for the horizons (spherical, cylindrical/toroidal and hyperbolic).

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I. INTRODUCTION

In [1] Hawking has shown that black holes are not black at all: they emit radiation, through quantum effects, with a characteristic thermal spectrum of a black-body with temperature $T_H$ proportional to the horizon surface gravity $k_h$. More concretely, an observer at rest at a constant radial coordinate close to the event horizon measures a Hawking temperature given by (we use units in which the light velocity, the Boltzmann constant and the Planck constant are set equal to one)

$$T_H = \frac{1}{2\pi k_h \sqrt{-g_{00}}},$$

(1)

where $g_{00}$ is the time-time component of the gravitational metric. Later on, Unruh [2] has remarked that the detection of thermal radiation is not restricted to an observer in the vicinity of a black hole horizon. Indeed, even in flat spacetime, an observer that moves with a constant acceleration $a$, a Rindler observer, will encounter an acceleration horizon and will also detect thermal radiation. The associated Unruh temperature $T_U$ is given by

$$T_U = \frac{a}{2\pi}.$$  

(2)

Once the Hawking effect is known to exist, one can argue that the Unruh effect follows from the equivalence principle. Indeed, according to this principle, the effects measured by an observer that is at rest in the vicinity of a black hole horizon are the same as the effects measured by an uniformly accelerated observer in a flat spacetime. Moreover, the equivalence principle also suggests that the connection (defined by Eq.(1)) between the temperature and the surface gravity also holds for Rindler motions, with $k_h$ being now the surface gravity of the acceleration horizon [3].

The strong connection between the Hawking and the Unruh temperatures appear in a different context, when one embeds a lower dimensional curved spacetime containing a black hole into higher dimensional flat spacetime with one or more timelike coordinates.

That such an embedding can be done for any 4-dimensional black hole geometry was shown in [4]. His-
torically, the first embedding of the Schwarzschild black hole in a 6-dimensional flat spacetime was performed by Kasner [5] (he himself noted that an embedding in 5 dimensions is impossible [6]). A vast list of embedding transformations for other 4-dimensional black hole geometries are presented in [7]. All these embeddings have the drawback that they are incomplete, since they cannot be extended past the event horizon, for radius less than the black hole radius. The complete, or global, embedding of the Schwarzschild black hole in a 6-dimensional flat spacetime was presented in [8], for a review see [4]. Applying this procedure appropriately, one can then map the black hole horizon to its acceleration horizon counterpart in the higher dimensional flat spacetime. The global embedding of a black hole geometry into a higher dimensional Minkowskian spacetime is generically called global embedding Minkowskian spacetime procedure, or GEMS procedure, for short. Note that any flat spacetime with one or more time coordinates is been called here a Minkowski spacetime. The GEMS procedure is an important procedure on its own, since it provides a powerful tool that simplifies the study of black hole physics by working instead, but equivalently, in a flat background geometry.

Through the GEMS procedure one can match the Hawking temperature associated with the black hole horizon with the Unruh temperature associated with the acceleration horizon of the higher dimensional Rindler spacetime. This matching has been first confirmed in the de Sitter (dS) geometry [9, 10], and in the anti-de Sitter (AdS) geometry [11], in 4-dimensional spacetimes. It was in [11] that the name GEMS appeared. Soon after, this unified computation of the temperatures has been verified for the Schwarzschild black hole [12], for the Reissner-Nordström black hole [13], and for the Schwarzschild and Reissner-Nordström black holes in a asymptotically dS and AdS 4-dimensional spacetimes [13, 14], as well as for the 3-dimensional Bañados-Teitelboim-Zanelli black hole [13]. When dealing with temperature issues, it is essential that one works with a global embedding as an incomplete embedding leads to observers for whom there is no horizon and no temperature [13].

In this paper we will study the connection between the Hawking effect and the Unruh effect for the static black hole solutions in the higher dimensional Einstein-Maxwell theory, generalizing thus the 4-dimensional results of Deser and Levin [13]. The higher dimensional counterparts of the Schwarzschild and of the Reissner-Nordström black holes – the Tangherlini black holes – have been found and discussed in [15]. The higher dimensional Schwarzschild and Reissner-Nordström black holes in an asymptotically de Sitter (dS) spacetime and in an asymptotically anti-de Sitter (AdS) spacetime have also been discussed in [15]. Now, in an asymptotically AdS 4-dimensional background, besides the black holes with spherical topology, there are also solutions with planar, cylindrical or toroidal topology found and discussed in [16], and black holes with hyperbolic topology analyzed in [17]. The higher dimensional extensions of these non-spherical AdS black holes are already known. Namely, the D-dimensional AdS black holes with planar, cylindrical or toroidal topology were discussed in [18], and the D-dimensional AdS black holes with hyperbolic topology were analyzed in [18, 19]. We will find the global embedding transformations that describe all the above D-dimensional black holes in a higher dimensional flat background, and then we use them to study the matching between the Hawking and the Unruh temperatures.

In section II we outline the GEMS procedure. Then, the black holes in an asymptotically AdS backgrounds will be analyzed in Section III A and the dS background in III B, while the black holes in an asymptotically flat background will be discussed in Section III C.

II. BLACK HOLE GEOMETRIES. OUTLINE OF THE GEMS PROCEDURE

A. Static black holes in higher dimensions

In a higher dimensional background with a cosmological constant $\Lambda$, the Einstein-Maxwell equation is

$$ R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} + \frac{(D-1)(D-2)}{6} \Lambda g_{\mu\nu} = 8\pi T_{\mu\nu} \quad (3) $$

where $R_{\mu\nu}$ is the Ricci tensor and $T_{\mu\nu}$ is the electromagnetic energy-momentum tensor (for the corresponding action see [20]). This equation allows a three-family of static black hole solutions, parameterized by the constant $k$ which can take the values 1, 0, −1, and whose gravitational field is described by

$$ ds^2 = -f(r) dt^2 + f(r)^{-1} dr^2 + r^2 (d\Omega_D^{k-2})^2, \quad (4) $$

where

$$ f(r) = k - \frac{\Lambda}{3} r^2 - \frac{M}{r^{D-3}} + \frac{Q^2}{r^{D-4}}, \quad (5) $$

$D \geq 4$, and the mass parameter $M$ and the charge parameter $Q$ are proportional to the Arnowitt-Deser-Misner mass and electric charge, respectively [21]. In Eq. (3) the coefficient of the $\Lambda$ term was choosen such that $f(r) = k - (\Lambda/3)r^2$ when $M = 0$ and $Q = 0$, as occurs in $D = 4$. For $k = 1$, $k = 0$ and $k = -1$ one has, respectively,

$$ (d\Omega_D^{k-2})^2 = d\theta_1^2 + \sin^2 \theta_1 d\theta_2^2 + \cdots + d\theta_{D-2}^2, $$

$$ (d\Omega_D^{k-2})^2 = d\theta_1^2 + \sin^2 \theta_1 d\theta_2^2 + \cdots + d\theta_{D-2}^2, $$

$$ (d\Omega_D^{k-2})^2 = d\theta_1^2 + \sinh^2 \theta_1 d\theta_2^2 + \cdots + d\theta_{D-2}^2. \quad (6) $$
The family with \( k = 1 \) describes the Tangherlini black holes with spherical topology [15]. These black holes have support in asymptotically AdS (\( \Lambda < 0 \)), dS (\( \Lambda > 0 \)), or flat (\( \Lambda = 0 \)) backgrounds. Black hole solutions with non-spherical topology (i.e., with \( k = 0 \) and \( k = -1 \)) live only in an AdS background and do not have black hole counterparts in a \( \Lambda > 0 \) or in a \( \Lambda = 0 \) background. The family with \( k = 0 \) yields AdS black holes with planar, cylindrical or toroidal (with genus \( g = 1 \)) topology [16, 18]. Finally, the family with \( k = -1 \) yields AdS black holes with hyperbolic, cylindrical, or toroidal topology with genus \( g \geq 2 \) [17–19]). The radial electromagnetic field produced by an electric charge proportional to \( Q \) with genus \( \alpha \) \((\alpha = 1, 2, \ldots, N - 1)\) live in the black hole geometry into an Unruh detector in the flat background.

The acceleration horizon \( x = 0 \) is mapped into the plane \( \pi \) by \( k_{h}^{2} = -\frac{1}{2}(\nabla^{\mu} \xi^{\nu})(\nabla_{\mu} \xi_{\nu}) \bigg|_{r_{h}} \), where \( \xi^{\nu} \) is the timelike Killing vector \( \xi^{\nu} = (1, 0, \ldots, 0) \). Using the metric (4)-(5) one finds

\[
\kappa_{h} = -\frac{1}{2} \frac{df(r)}{dr} \bigg|_{r_{h}}. 
\]  

It is sometimes also useful to define the cosmological length \( \ell \) as

\[
\ell^{2} = -\frac{3}{\Lambda}. 
\]  

For a detailed description of the properties of these black holes see, e.g., [20] and references therein.

### B. Brief description of the GEMS procedure

In the next sections we will find the transformations \( z^{a}(t, r, \theta_{1}, \ldots, \theta_{D-2}) \) that perform a global embedding of the above \( D \)-dimensional black hole geometries into a \( N \)-dimensional Minkowskian spacetime (with \( N \geq D \)) described by

\[
ds^{2} = \eta_{\alpha\beta} dz^{\alpha} dz^{\beta}, \tag{10}
\]

where \( \eta_{\alpha\beta} \) is the flat metric in \( N \)-dimensions with one or more timelike coordinates, and \( \alpha, \beta = 0, 1, 2, \ldots, N - 1 \).

The GEMS procedure consists of these embedding transformations. These transformations will map a Hawking detector, that moves according to

\[
r = \text{constant} , \quad \theta_{i} = \text{constant}, \tag{11}
\]

in the black hole geometry into an Unruh detector in the \( N \)-dimensional flat spacetime that follows a hyperbolic trajectory or Rindler motion described by

\[
(z^{1})^{2} - (z^{0})^{2} = a^{-2}, \tag{12}
\]

where \( a \) is the constant acceleration of the Unruh detector in the \( N \)-dimensional flat spacetime. With this GEMS procedure we can compare the temperature (defined by Eq.(1)) measured by the Hawking detector in the black hole spacetime with the temperature (defined by Eq.(2)) measured by the Unruh detector in the flat background.

The idea of the GEMS transformations is the following: First, the line element containing the \((t, r)\) pair of coordinates is transformed via a Rindler type transformation into a Minkowski line element comprising the \((z^{0}, z^{1})\) pair of coordinates. In the process some terms containing \( dr^{2} \) are left behind. Second, one has to transform the angular part of the line element into a flat space line element, through the usual procedure. In the process some other terms containing \( dr^{2} \) are again left behind. Third, one now has to put these leftover \( dr^{2} \) terms into a flat form. Depending on the particular case, this can take zero, one or two additional extra flat dimensions.

We first illustrate this procedure, with the simplest, trivial, example: the \( D \)-dimensional Rindler geometry. In this case there are no leftover \( dr^{2} \) terms and thus this is the case where there are zero additional extra flat dimensions. The metric is

\[
ds^{2} = -x^{2} \left( \frac{dt}{2} \right)^{2} + dx^{2} + \sum_{i=2}^{D-1} (dx^{i})^{2}, \tag{13}
\]

which has an acceleration horizon at \( x = 0 \), (we use \( t/2 \) instead of \( t \) for convenience and consistency with what follows; see Sec. III C). This Rindler geometry describes the metric seen by an uniformly accelerated observer in a flat background: an observer at constant \( x \) and \( x^{i} \)'s suffers an acceleration \( a = x^{-1} \). It is just an wedge of Minkowski spacetime written in accelerated coordinates. Consider now the timelike Killing vector \( \xi^{\nu} = (1, 0, \ldots, 0) \). The surface gravity at the horizon \( x = 0 \) is by definition, \( k_{h}^{2} = -\frac{1}{2}(\nabla^{\mu} \xi^{\nu})(\nabla_{\mu} \xi_{\nu}) \bigg|_{x=0} \), which in the present case gives \( k_{h} = 1 \). The Hawking temperature measured by a detector at constant \( x \) and \( x^{i} \)'s is then given by Eq.(1):

\[
T_{H} = \frac{x^{-1}}{2\pi}. \tag{14}
\]

Now, we can define the embedding of this \( D \)-dimensional Rindler geometry in the \( N \)-dimensional Minkowski spacetime given by Eq.(10). This is a trivial embedding since \( N = D \):

\[
\begin{align*}
z^{0} &= x \sinh \left( \frac{t}{2} \right), \\
z^{1} &= x \cosh \left( \frac{t}{2} \right), \\
z^{i} &= x^{i}, \quad \text{for } 2 \leq i \leq D - 1. \tag{15}
\end{align*}
\]

The acceleration horizon \( x = 0 \) is mapped into the plane \( z^{0} = z^{1} \), and transformations given by Eq.(15) constitute a global embedding since they extend the original spacetime behind the horizon and, in particular, they
are analytic at the horizon. Moreover, a Hawking detector that in the original Rindler geometry, described by Eq.(13), moved with constant $x$ and $x$'s is mapped through Eq.(15) into an Unruh detector that moves along the hyperbolic trajectory of Eq.(12), with uniform acceleration $a = x^{-1}$. This Unruh detector measures a temperature given by Eq.(2):

$$T_U = \frac{x^{-1}}{2\pi}.$$  \hspace{1cm} (16)

Thus, the Hawking temperature and the Unruh temperature match, $T_H = T_U$.

In the next sections we will find the GEMS transformations for the black holes presented in Sec. II A. These transformations will be analytic at the horizon whose temperature we will be computing.

III. GLOBAL EMBEDDING OF BLACK HOLES IN A GENERIC COSMOLOGICAL CONSTANT A BACKGROUND

In Sec. III A we give the global embeddings and the matching of the Hawking and Unruh temperatures for higher dimensional black holes in an asymptotically AdS background. Then in sections III B and III C we study the higher dimensional black holes in asymptotically dS and flat backgrounds, respectively.

A. GEMS for the asymptotically AdS black holes

\[\Lambda < 0\]

The Einstein-Maxwell equations with a negative cosmological constant ($\Lambda < 0$) allow a family of static black hole solutions, parameterized by a constant $k$, which can take the values $k = 1$ (horizons with spherical topology [15]), $k = 0$ (planar, cylindrical or toroidal topology [16, 18]), and $k = -1$ (hyperbolic, cylindrical or toroidal topology [17–19]).

In the cases $k = 1, -1$ the $D$-dimensional black hole can be globally embedded into a Minkowski background with $N = D + 3$ dimensions, and in the case $k = 0$ in a Minkowski background with $N = 2 \times D$ dimensions. Note that the global embedding of $D = 4$ black holes with spherical topology ($k = 1$) has been found in [13, 14]. The global embedding of black holes with non-spherical topology ($k = 0$, and $k = -1$) has not been discussed at all, even for $D = 4$.

For all three cases ($k = 1, 0, -1$) the first two dimensions $z^0$ and $z^1$ are given by the following GEMS transformations

\[z^0 = \kappa_h^{-1}\sqrt{f(r)} \sin (k_h t),\]
\[z^1 = \kappa_h^{-1}\sqrt{f(r)} \cosh (k_h t),\]  \hspace{1cm} (17)

where $\kappa_h$ is defined in (8).

For the $z^i$ coordinates related to the angular part of the original line element, one has to separate the embeddings for the $k = 1, -1$ cases from the embedding for $k = 0$. For $k = 1, -1$ one has

$$z^2 = r S(\theta_1) \prod_{i=2}^{D-2} \sin(\theta_i),$$
$$z^j = r S(\theta_1) \prod_{i=2}^{D-j} \sin(\theta_i) \cos(\theta_{D+1-j}),$$
$$z^D = r C(\theta_1).$$  \hspace{1cm} (18)

where

$$S(\theta_1) = \begin{cases} \sin (\theta_1), & \text{for } k = 1 \\ \sinh (\theta_1), & \text{for } k = -1 \end{cases}$$
$$C(\theta_1) = \begin{cases} \cos (\theta_1), & \text{for } k = 1 \\ \cosh (\theta_1), & \text{for } k = -1 \end{cases}$$  \hspace{1cm} (19)

For $k = 0$ one has

$$z^{2j} = r \sin \theta_j, \quad \text{for } 1 \leq j \leq D - 2,$$
$$z^{2j+1} = r \cos \theta_j, \quad \text{for } 1 \leq j \leq D - 2.$$  \hspace{1cm} (20)

Note that the number of $z^i$ coordinates describing the angular part in the $k = 0$ cases is twice the number of angular coordinates $\theta_1$ the black hole has.

The additional extra flat dimensions coming from the leftover $dr^2$ terms can now be dealt with. The gravitational metric of the higher dimensional charged AdS black holes is given by Eq.(4) and Eq.(5) with $\Lambda < 0$. For the three allowed values of the topological parameter $k$, the corresponding black holes have two horizons, $r = r_-$ and $r = r_h$ (say) with $r_- \leq r_h$, and $t$ is a timelike coordinate for $r > r_h$ [20]. This is then the region where a physical Hawking detector might be located, and it is aware only of the $r = r_h$ horizon. In particular, for our purposes, the global embedding of this black hole into a flat background will only have to cover the range $r > r_-$, that includes a vicinity of $r = r_h$. It is useful to swap the mass parameter $M$ for the horizon radius $r_h$. This is achieved through the zero of (5), i.e.,

$$M = k_h^{D-3} + \frac{Q^2}{r_h^{D-3}} + \frac{r_h^{D-1}}{l^2}.$$  \hspace{1cm} (21)

In turn this allows us to write the function $f(r)$ in Eq.(5) as

$$f(r) = \frac{\ell^2 (k r_h^{D-3} r^{D-3} - Q^2) (r^{D-3} - r_h^{D-3})}{\ell^2 r_h^{D-3} r^{2(D-3)}} + \frac{r_h^{D-3} r^{D-3} (r^{D-1} - r_h^{D-1})}{\ell^2 r_h^{D-3} r^{2(D-3)}}.$$  \hspace{1cm} (22)
The surface gravity associated with \( r = r_h \) is then
\[
\kappa_h = \frac{(D - 3) \ell^2 (k^2 r_h^{2(D-3)} - Q^2) + (D - 1) r_h^{2(D-2)}}{2 \ell^2 r_h^{2D-5}}.
\]
(23)

We are now ready to give the remaining GEMS coordinate transformations that yields the additional extra flat dimensions. The GEMS transformations for the case \( k = 1, -1 \) are then

\[
z^m = \int dr \left[ \frac{W_{(D-3)}}{G(r)r^{D-3}} + \frac{(D - 3)^2 W_{(D-1)}}{4 \ell^2 k_h^2 G(r)r_h^{3(D-3)+2} r^{2(D-2)}} \left[ \frac{(k^2 r_h^{2(D-3)} - Q^2) + r_h^{2(D-2)}}{r^{2(D-3)}} \right]^2 + \frac{Q^2 (D - 3)^2 W_{(D-1)}}{\ell^2 k_h^2 G(r)r_h^{3(D-3)} r^{2(D-2)}} + H(k) \right]^\frac{1}{2},
\]
\[
z^n = \int dr \left[ \frac{W_{(D-1)} r^{D-3}}{\ell^2 k_h^2 G(r)r_h^{D-3} r^{2(D-3)}} + \frac{Q^2 (D - 3)^2}{k_h^2 r^{2(D-2)}} + T(k) \right]^\frac{1}{2},
\]
(24)

where \( m_k \) and \( n_k \) are defined by
\[
\text{for } k = 1, -1 \quad \left\{ \begin{array}{l} m_k = D + 1 \\ n_k = D + 2 \end{array} \right. \\
\text{for } k = 0 \quad \left\{ \begin{array}{l} m_k = 2D - 2 \\ n_k = 2D - 1 \end{array} \right.
\]
(25)

and the functions \( H(k) \) and \( T(k) \) are
\[
H(k) \equiv \delta_{-1k} \quad \text{(26)}
\]
\[
T(k) \equiv \delta_{1k} + \delta_{0k}(D - 2), \quad \text{(27)}
\]

where \( \delta_{\mu\nu} \) is the Kronecker symbol. In addition we have defined \( W_{(n)} \) as
\[
W_{(n)} = \prod_{i=1}^{n} r_h^{-1} r^{n-i}.
\]
(28)

and \( G(r) \) as
\[
G(r) = f(r)/(r - r_h)
\]
\[
= \frac{\ell^2 (k r_h^{D-3} r^{D-3} - Q^2) W_{(D-3)} + r_h^{D-3} W_{(D-1)} r^{D-3}}{\ell^2 r_h^{D-3} r^{2(D-3)}}.
\]
(29)

A careful analysis of the arguments of the square roots in Eq.(24) shows that they are always positive and finite for \( r > r_- \) and, in particular, these GEMS transformations are analytical at \( r = r_h \).

Thus, a spherical \((k = 1)\) \( D \) dimensional charged black hole in an AdS background can be embedded in a \( D + 3 \) dimensional Minkowski spacetime with two times, with metric \( ds^2 = -(dz^0)^2 + \sum_{i=1}^{D+1} (dz^i)^2 - (dz^{D+2})^2 \). A hyperbolic \((k = -1)\) \( D \) dimensional charged black hole in an AdS background can be embedded in a \( D + 3 \) dimensional Minkowski spacetime with three times, with metric \( ds^2 = -(dz^0)^2 + \sum_{i=1}^{D-1} (dz^i)^2 - (dz^{D+1})^2 + (dz^{D+2})^2 \).

Finally, a planar \((k = 0)\) \( D \) dimensional charged black hole in an AdS background can be embedded in a \( 2 \times D \) dimensional Minkowski spacetime with two times, with metric \( ds^2 = -(dz^0)^2 + \sum_{i=1}^{D-2} (dz^i)^2 - (dz^{D+1})^2 - (dz^{D+2})^2 \).

\[\text{These GEMS transformations map the horizon at } r = r_h \text{ into the plane } z^1 = z^0, \text{ and the original Hawking detector is mapped into the Unruh detector that follows the hyperbolic motion described by Eq.(12), with constant acceleration } a = r_h/\sqrt{f(r)}. \text{ This Unruh detector then measures a temperature given by Eq.(2) which matches with the temperature given by Eq.(1) measured by its Hawking partner in the original black hole. As a consistency check we remark that the GEMS transformations given in Eqs.(17)-(29) for the Reissner-Nordström } (D = 4 \text{ and } k = 1) \text{ black hole reduce to the transformations found in [14].}

The gravitational metric of the higher dimensional \( k = 1, 0, -1 \) neutral AdS black holes is given by Eq.(4) and Eq.(5) with \( Q = 0 \) and \( \Lambda < 0 \). This case is in everything similar to the charged black hole. To find the GEMS one just has to make the limit \( Q = 0 \) in the Eqs.(17)-(29). As a consistency check we remark that the GEMS transformations for the neutral Schwarzschild \((D = 4 \text{ and } k = 1)\) black hole reduce to the transformations found in [13]. And when we further set \( M = 0 \) we recover the embedding of the pure AdS spacetime found in [13].

Since extremal black holes, \((r_h = r_-)\), have zero temperature, there is no interest here in discussing their GEMS transformations. Indeed, the GEMS transformations showed above do not even apply to extremal cases.

\[\text{B. GEMS for the asymptotically dS black holes} \quad \Lambda > 0\]

When \( \Lambda > 0 \), the Einstein-Maxwell equations allow black hole solutions only for \( k = 1 \), i.e., whose horizons have a spherical topology, see Eqs.(4)-(7).

In the charged case, \( Q \neq 0 \), the solution has three horizons, the Cauchy horizon \( r = r_- \), the event horizon \( r = r_h^+ \), and the cosmological horizon \( r = r_h^- \), with
r_- \leq r_{h+} \leq r_h. A physical detector can be in between r_{h-} and r_{h+}, and it is aware only of r_{h+} and r_{h-}. Use of Eqs.(17)-(29) with \(\ell^2 \to -\ell^2\) and \(r_h \equiv r_{h+}\) yields an embedding that covers the region \(r_- < r < r_{h+}\). This embedding allows us to verify the matching between the Hawking temperature and the Unruh temperature of the embedding to verify that \(T_H = T_U\) also for the cosmological horizon. We can then use this embedding to verify that \(T_H = T_U\) also for the cosmological horizon.

In the neutral case, \(Q = 0\), the black hole has no Cauchy horizon, and the discussion of the last paragraph follows by replacing \(r_-\) by \(r_- = 0\).

C. GEMS for the asymptotically flat black holes \(\Lambda = 0\)

The charged Tangherlini black hole, which is the higher dimensional Reissner-Nordström black hole, has been found in [15]. Its gravitational field is described by Eq.(4) and Eq.(5) with \(k = 1\) and \(\Lambda = 0\), and its Maxwell field is given by Eq.(7). It has two horizons, the event horizon at \(r_{h}^{D-3} = M/2 + \sqrt{M^2/4 - Q^2}\), and the Cauchy horizon at \(r_{Cauchy}^{D-3} = M/2 - \sqrt{M^2/4 - Q^2}\) with \(r_- \leq r_h\). In particular, for our purposes, the global embedding of this black hole into a flat background will only have to cover the region \(r > r_-\), that includes a vicinity of both sides of \(r = r_+\).

The GEMS transformations are given by Eqs.(17)-(29) with \(k = 1\) and \(\Lambda = 0\). In order to make touch with the results of [13] we will give explicitly for this case the transformations of the \(z^{D+1}, z^{D+2}\) coordinates in terms of the horizon radial coordinates \(r_h\) and \(r_-\). From Eq.(24) one finds

\[
\begin{align*}
  z^{D+1} &= \int dr \left( \frac{[W_{(D-1)} - (r_h^{D-3} r + r_h^{D-2})]}{r^{2(D-3) - r_h^{D-3} W_{(D-1)}}} \right)^{\frac{1}{D-3}}, \\
  z^{D+2} &= \int dr \left( \frac{4r_h^{3D-7} r_{Cauchy}^{D-3}}{(r_h^{D-3} - r_{Cauchy}^{D-3})^2 r^{2(D-2)}} \right)^{\frac{1}{D-3}}.
\end{align*}
\]

When \(D = 4\) these transformations reduce to those found in [13]. These transformations map the horizon at \(r = r_h\) into the plane \(z^1 = z^0\), and are due complete to the analyticity of \(z^{D+1}\) and \(z^{D+2}\) in the range \(r > r_-\). Note also that the surface gravity associated with the \(r = r_h\) horizon is \(k_h = \frac{D}{2} \frac{D-3}{D-2} (r_h^{D-3} - r_{Cauchy}^{D-3})/r_h^{D-2}\) and its temperature measured by a Hawking detector at constant \(r\) and \(\theta_i\)’s, is given by Eq.(1), which matches the Unruh temperature (2).

The higher dimensional Schwarzschild black hole, the Tangherlini black hole, is found by setting \(Q = 0\) in the higher dimensional Reissner-Nordström black hole. Its gravitational field is described by Eq.(4) and Eq.(5) with \(k = 1\), \(Q = 0\) and \(\Lambda = 0\). In the \(Q = 0\) limit, one has \(r_- = 0\) and \(z^{D+2} = 0\) in Eq.(30), and this limit yields the Schwarzschild GEMS. Note that in the limit \(M \to \infty\), the near horizon geometry is the Rindler metric and one easily recovers Eqs. (13)-(16).

IV. CONCLUSION

We have discussed the thermal properties of higher dimensional black holes. We have verified that the Hawking temperature matches the Unruh temperature, by mapping the curved space observers into the Rindler-accelerated observers of a higher dimensional spacetime, a formulation introduced by Deser and Levin [13] in the \(D = 4\) case. In order to accomplish this task we have found the global embedding Minkowskian spacetime (GEMS) transformations from \(D\)-dimensional black holes into a higher dimensional Minkowskian spacetime. The global embedding of AdS black holes with non-spherical topology \((k = 0, \text{ and } k = -1)\) had not been discussed previously, even for \(D = 4\). The GEMS transformations are important on their own, since they provide a powerful tool that simplifies the study of black hole physics by working instead, but equivalently, in a flat background geometry.

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