Quantum gravity: a geometrical perspective

Ivano Tavernelli

IBM Research – Zurich, 8803 Rüschlikon, Switzerland

(Dated: August 3, 2018)

We present a theory of quantum gravity that combines a geometrical formulation of quantum field theory in space-time with classical Einstein’s general relativity. This approach is based on the geometrization of quantum mechanics proposed in refs. [1, 2] and combines quantum and gravitational effects into a global curvature of the Finsler’s space associated to the $4N$-dimensional configuration space of a $N$-particle system. In order to make this theory compatible with general relativity, the quantum effects are described in the framework of quantum field theory, where a covariant definition of ‘simultaneity’ for many-body systems is introduced through the definition of a suited foliation of space-time. As for Einstein’s classical gravitation theory, the particles dynamics is finally described by means of a geodesic equation in a curved space-time manifold.

Keywords: Quantum gravity; Gravitation; Covariant quantum field theory; Finsler spaces.

INTRODUCTION

The quest for a theory that conceals quantum theory with general relativity has attracted the interest of many researchers since the early sixties of the past century. However, no single theory has yet emerged as the leading one and, on the contrary, we have witnessed a fruitful proliferation of different approaches that lead to the development of new fields of research in mathematics and theoretical physics. A thorough summary of the different approaches is certainly beyond the purpose of this study and therefore we orient the interested readers towards the more specialized literature.

Following the ‘classification’ proposed by Rovelli [3], the attempts to combine quantum theory with gravitation led to the development of several distinct, but also interconnected, theoretical models. The most promising among these candidates for quantum gravity is probably string theory. From a fundamental prospective string theory is developed from the quantum field theory of one-dimensional objects, the strings, which take the place of the original point particles. One of the vibrational states of the strings corresponds to the graviton, the quantum mechanical particle that carries the gravitational force [4]. Alternatively, other consistent theories of quantum gravity are obtained from the quantization, with different flavours, of the gravitational field and corresponding metric tensor, or of the space-time itself. The most successful among these theories is probably loop quantum gravity [5, 6]. In this case, space-time is quantized producing a granularity of space (quantum of space), which defines a minimum scale distance through which matter can travel, known as Planck length. At this scale, space is conceived as a ‘tissue’ of finite loops also known as spin networks, whose dynamics produces spin foams. Mathematically, these objects correspond to a generalization of the Feynman diagrammatic perturbation theory where instead of a graph, a higher dimensional 2-complex topological space is used [5, 6]. In addition to these two main research lines, there have been several other directions proposed, which include: Euclidean quantum gravity [9] based on Wick-rotation of the Minkowski space; twistor theory [10], which maps Minkowski space to a new geometrical object in a complex coordinate space known as twistor space; and noncommutative quantum field theories that play an important role in M-theory [11]. Despite the beauty of the formalisms, none of these alternative theories has developed into a firm physical model of quantum gravity but they are mainly confined into the realm of mathematical hypotheses with limited experimental evidences.

To these well established theories for quantum gravity we also need to add a series of alternative approaches that aim at deriving a relativistic covariant formulation of quantum theory in the framework of Bohmian mechanics [12–14] and its field theory extension [15]. The first attempt to derive a fully consistent covariant version of Bohmian dynamics is due to DeWitt [16] and Jackiw [17] and was later followed by the extensions developed by Holland [18]. They showed that a consistent covariant formulation of Bohmian field theory is possible for both scalar and spinor fields leading to a new framework for a possible unification of quantum theory with gravitation. More recently, Dürr and coworkers [19] made an additional step in this direction showing how a relativistic space-time Bohmian theory of many-body dynamics can be achieved using a privileged foliation of space-time, the same that we will use in our approach. However, also in this case the purpose was to demonstrate the compatibility of Bohmian dynamics with general relativity and not the one of deriving a consistent theory of quantum gravity.

In this work, we will take a different path and propose a quantum gravity theory where the quantum fields produce a further curvature (in addition to the one induced by the energy-matter tensor) of an extended space-time through the action of the quantum potential, which then guides the time evolution of point particles along geodesic paths.
This new theory of quantum gravity is based on four fundamental pillars: the many-body field theory of relativistic particles, the Bohmian theory of quantum potential extended to field theories, the extension of space-time (pseudo-) Riemann geometry to Finsler’s geometry \[21, 22\] and, ultimately, the generalization of Einstein’s field theory from the 3\(N\)+1 dimensional manifold \(M\) and its tangent space \(TM\) to the Finsler’s space \(TM\) (of dimension \(2 \times 4N\)) and its tangent space \(TTM\), where \(N\) is the number of particles considered. In the first section of this paper, we will present the many-body (Dirac) theory in the relativistic space-time manifold putting particular emphasis on the definition of a covariant concept of ‘simultaneity’ for general relativity \[24, 25\]. This will allow us to define unambiguously the concept of a relativistic invariant many-body quantum potential. The third section will be devoted to the description of Finsler’s geometry in the multi-time extended phase space, where the quantum potential enters in the definition of the metric tensor and the corresponding non-linear Cartan connection. Finally, in section four, we will formulate Einstein’s field theory in the extended Finsler’s space and describe its connection to the original classical theory formulated by Einstein in 1915 \[27\].

**THEORY**

**Relativistic field theory in a space-time manifold**

In field theory, one can formulate a Dirac equation for a many-body system using the multi-time wavefunction \(\Psi(x_1, x_2, \ldots, x_N)\) in the \(N\)-particle spinor space of dimension \((\mathbb{C}^4)^\otimes N\)

\[
i\gamma^\mu \partial_{\mu} \Psi(x_1, x_2, \ldots, x_N) - m_i \Psi(x_1, x_2, \ldots, x_N) = 0 \tag{1}\]

where \(x_i\) belongs to the Minkowski space, \(k = 1, \ldots, N\), \(\gamma^\mu = \mathbb{1} \otimes \cdots \otimes \gamma^\nu \otimes \mathbb{1} \otimes \cdots \otimes \mathbb{1}\) with \(\gamma^\nu\) in position \(k\) (see Appendix A). The system wavefunction \(\Psi(x_1, x_2, \ldots, x_N)\) is derived from the Dirac spinor operator \(\psi(x)\) \[28\]

\[
\Psi(x_1, x_2, \ldots, x_N) = \frac{1}{\sqrt{N!}} \langle 0 | \psi(x_1) \cdots \psi(x_N) | \Psi \rangle \tag{2}\]

where the field operator \(\psi(x)\) for a particle \(i\) fulfils the Dirac field equation

\[
(i\gamma^\mu \partial_{\mu} - m_i)\psi(x) = 0 \tag{3}\]

and \(m_i\) is the particle mass. So far, the theory is formulated for the case of free, non-interacting, Dirac particles. However, we can apply the perturbation theory for field operators to describe particle interactions and then use Eq. (2) to recover the interacting many-particle wavefunction from the interacting fields \(\psi(x_i)\) at different positions \(x_i\).

The use of the many-time formulation with a time variable for each particle introduces the problem of the definition of simultaneity in general relativity. While in special relativity simultaneity acquires a clear physical meaning when an inertial frame is specified, in general relativity there is no global inertial frame and therefore the concept of ‘equal-time events’ remains ambiguous. However, as discussed in ref. \[24\] in general relativity the concept of simultaneity can be replaced by the more general one of ‘events belonging to the same three-dimensional space-like hypersurface’ \[24,25,26\]. In this case, events belonging to the same space-like hypersurface occur at the same global time, \(s\). The time variable, \(x_i^0\), associated to each \(i\)-th particle of the system, \(x_i\), is therefore parametrized by \(s\), which measures the progress of the dynamics of all particles. Further, the families of space-like hypersurfaces corresponding to different global times \(s\) define a slicing or foliation \(F\) of space-time with slices (or leafs) \(\Sigma_s\). More precisely, in general relativity the geometry of space-time is described by the Lorentzian manifold \((M, g(x))\) of dimension \(3 + 1\). At each point \(x\) in space-time the metric \(g(x)\) defines a linear space \((T_x M, g(x))\) which is locally isometric to the Minkowski space-time \(\mathbb{R}^{3+1}\) and defines the null cone \(\mathcal{N}_x = \{X \in T_x \mathbb{R}^{3+1}, g(x)(X, X) = 0\}\). In particular, the corresponding local Minkowski frame defines a spacelike plane of simultaneity that bisects the cone \(\mathcal{N}_x\) and that coincides locally with the selected hypersurface \(\Sigma_s\). It is worth mentioning, that - even though rigorous - the definition of the foliation \(F\) is somehow arbitrary. However, in the next section we will introduce a choice of \(F\), which will generate fully covariant equations of motion for all particles.

The ensemble of the coordinates of all particles in the system is therefore characterized by the extended configuration space vector \(\vec{x}_N^s = (x_1^0, x_2^0, \ldots, x_N^0)\) with \(x_i^0 = (x_i^0(s), x_i^1(s), x_i^2(s), x_i^3(s))\) on the hypersurface \(\Sigma_s\) of the foliation \(F\).
Lorentz invariant Bohmian dynamics in field theory

Central to the definition of dynamics in general relativity is the concept of mass particles. In the Bohmian interpretation of quantum theory, the field introduced in Eq. (3) and the corresponding many-particle wavefunction $\Psi$ (Eq. (2)) are ‘mathematical’ quantities that guide the particle dynamics and are not directly related to any physical observable. In this letter, in order to preserve the particle nature of the relativistic formulation of Bohmian mechanics we will not pursue the Bohmian field theory approach in the so-called Schrödinger representation. Instead, we will use the many-time framework outlined above to formulate a relativistic invariant particle dynamics in which the Bohmian quantum potential is derived from the relativistic quantum fields and the related many-particle wavefunctions defined in Eqs. (2) and (3). For a detailed account on Bohmian field theory we referred to the specialized literature [18, 30, 31].

Following [25], we can define a particle dynamics in the extended configurations space of coordinates $\bar{x}_N^F$ by means of the vector field

$$\dot{x}_i^{\Sigma_s} = K_i v_i^F [\Psi(\bar{x}_N^F), \bar{x}_N^F]$$

(4)

where the 4-vector $v_i^F [\Psi(\bar{x}_N^F), \bar{x}_N^F]$ has components $(\mu_i = 0, 1, 2, 3)$

$$v_i^F \mu_i [\Psi(\bar{x}_N^F), \bar{x}_N^F] = J^{\mu_1 \ldots \mu_N} (\bar{x}_N) n^F_{\mu_1} (x_1) \ldots n^F_{\mu_N} (x_N)$$

(5)

and where $n^F_{\mu_i} (x)$ is the future-directed normal vector of the hyperplane $\Sigma_s$ at $x$, $\ldots$ indicates that the term is omitted, $K_i$ is a constant, and

$$J^{\mu_1 \ldots \mu_N} (x_N) = \langle \Psi | : \frac{1}{N!} \bar{\psi}(x_1) \gamma^{\mu_1} \psi(x_1) \ldots \bar{\psi}(x_N) \gamma^{\mu_N} \psi(x_N) : | \Psi \rangle .$$

(6)

In Eq. (6), $\ldots$ stands for normal ordering. For a covariant choice of the foliation (see [24, 25]), the multi-tensor $J^{\mu_1 \ldots \mu_N} (x_N)$ defines a covariant direction field for all particles, implying that the equation of motion in Eq. (4) is also covariant.

In this work, we propose a unified quantum gravity theory in which both quantum field theory and general relativity act on the curvature of space-time. To this end, as described in refs. [1, 2], we need to define a quantum potential for the Dirac spin wavefunction in Eq. (2). The quantum potential is given by

$$Q_i (x_N) = -\frac{\hbar^2}{2m_i} \frac{\nabla_i^2 A(\bar{x}_N)}{A(\bar{x}_N)}$$

(7)

where $A(\bar{x}_N) = (J^{\mu_1 \ldots \mu_N} (\bar{x}_N) J_{\mu_1 \ldots \mu_N} (\bar{x}_N))^{1/2}$ [18, 32, 33]. Note that $A^2 (\bar{x}_N)$ corresponds to the probability density in the rest frame while the 0-component $J^{0 \ldots 0} (\bar{x}_N) J_{0 \ldots 0} (\bar{x}_N)$ is not Lorentz invariant [18].

Finsler dynamics in relativistic quantum mechanics

Recently, a new geometrical description of quantum dynamics based on Finsler’s geometry was introduced, which defines the metric tensor as a function of the positions and velocities in the cotangent bundle. Within this formulation, matter points evolve along single quantum trajectories described by the geodesics of the curved phase space. This theory is strictly deterministic, as it does not rely on the probabilistic interpretation of the quantum-mechanical wavefunction. In fact, differently from Bohmian dynamics, the particle position is not described by a physical observable. In this letter, in order to preserve the particle nature of the relativistic formulation of Bohmian mechanics we will not pursue the Bohmian field theory approach in the so-called Schrödinger representation. Instead, we will use the many-time framework outlined above to formulate a relativistic invariant particle dynamics in which the Bohmian quantum potential is derived from the relativistic quantum fields and the related many-particle wavefunctions defined in Eqs. (2) and (3). For a detailed account on Bohmian field theory we referred to the specialized literature [18, 30, 31].

In order to guarantee relativistic covariance, it is necessary to move to the multi-time framework already introduced in Eqs. (1)-(3). The dimension of the multi-time phase space is now $2 \times 4N$. The generalization of the non-relativistic geodesic dynamics to the multi-time relativistic case requires the formulation of the equation of motion of each individual particle, separately. In the following, we will use the variable $x = (x_1, \ldots, x_N)$ with $x_i = (x_i^0, x_i^1, x_i^2, x_i^3)$ for the positions and $y = (y_1, \ldots, y_N)$ with $y_i = (\dot{x}_i^0, \dot{x}_i^1, \dot{x}_i^2, \dot{x}_i^3)$ for the corresponding velocities, with $\dot{x}_i^a = \partial x_i^a / \partial s$ ($a \in \{0, 1, 2, 3\}$) where $s$ is the global time variable defined by the choice of the foliation $F$. For a specific particle of
interest (any one of the ensemble) we associate the geodesic curve $s \mapsto x(s) = \zeta(s)$ with components $(\zeta^0(s), \ldots, \zeta^3(s))$ and velocities $s \mapsto y(s) = \dot{\zeta}(s)$ (using Einstein’s summation convention and $a, b, c = 0, \ldots, 3$)

$$\dot{\zeta}^a + N^a_b(\zeta(s), \dot{\zeta}(s))\zeta^b = -g^{ab}\partial V(r)/\partial \zeta_b,$$  

(8)

where

$$N^a_b(\zeta(s), \dot{\zeta}(s)) = \frac{1}{2} \partial_b(\Gamma^a_{cd}(\zeta(s), \dot{\zeta}(s))\dot{\zeta}^c\dot{\zeta}^d)$$  

(9)

is the non-linear Cartan connection (Appendix B) and $\Gamma^a_{bc}(\text{with } g_{ab,c} = \partial_c g_{ab} = \partial g_{ab}/\partial \zeta_c$, and $\dot{\zeta}(s) = \partial_\zeta \zeta(s)$), $g_{ab}$ are the Finsler metric coefficients

$$g_{ab}(\zeta(s), \dot{\zeta}(s)) = \frac{1}{2} \partial_a \partial_b \mathcal{L}_q(\zeta(s), \dot{\zeta}(s))$$  

(10)

where $\partial_\zeta = \partial_{\zeta^i}$ (see Appendix B) and

$$\mathcal{L}_q(\zeta(s), \dot{\zeta}(s)) = T(\dot{\zeta}(s))/\zeta^0 - Q(\zeta(s))\dot{\zeta}^0$$  

(11)

and $T(\dot{\zeta}) = (1/2) \sum_i m_i \sum_{k=1}^3 (\dot{\zeta}^k)^2$ ($m_i$ is the particle mass) is the, still non-relativistic, kinetic energy. In Eq. \[8\], $V(\zeta(s))$ is the classical potential, and $Q(\zeta(s))$ is the quantum potential. $\zeta(s)$ and $\dot{\zeta}(s)$ belong to the hypersurface $\Sigma^s$ and corresponding tangent space defined by the foliation $\mathcal{F}$.

**Deterministic paths in relativistic quantum field theory: the kinematics of Finsler’s geodesics**

In Bohmian mechanics, particles follow deterministic paths driven by the combined action of the classical and quantum potentials. In the quantum field theory extension formulated above, the Bohmian potential is fully determined by the quantum fields, which defines a quantum Lagrangian (Eq. \[11\]) that governs the dynamics of the associated potentials. In the quantum field theory extension formulated above, the Bohmian potential is fully determined and velocities

$$s \mapsto \zeta(s),$$  

for the index over the particles: $i = 1, \ldots, N,$

$$\zeta_i^0 = \sum_i^N \zeta_i^0.$$  

Following the definition in Eq. \[10\], we finally obtained the covariant relativistic Finsler metric

$$F(\zeta, \dot{\zeta}) = \mathcal{L}_C(\zeta, \dot{\zeta})$$  

(14)

where

$$\zeta_0 = \sum_i^N \zeta_i^0.$$  

Note that in Eqs. \[8\]-\[14\], we used $i$ for the index over the particles: $i = 1, \ldots, N,$
and \(a = 0, 1, 2, 3\) represented the 4-coordinates of each particle. With Eq. (15) we introduce a new notation, where \(a, b\) are now global indices for both particle number and coordinates and run between 1 and \(4N\) \((a, b \in \{1, \ldots, \text{dim}(M) = 4N\})\); the tangent bundle (TM) has then dimension \(2 \times 4N\). Within this formulation all quantum effects are absorbed into the evolution of the quantum field operators that obey the well-known dynamical field equations defined by the corresponding quantum field theory Lagrangians (see for instance [35]).

The geodesic equation (Eq. 6) combined with the Finsler metric in Eq. (15) define the trajectory of a relativistic particle associated to a quantum field, which for instance can be visualized by the trace in a bubble chamber between the instants of creation and the one of annihilation. (Note that in this work we only consider non-interacting quantum fields and associated particles, and therefore the particle number is a conserved quantity). In the following section, we derive the simplest possible extension of this theory to general relativity.

**Unified picture: quantum gravity in Finsler’s space**

It is commonly assumed that the gravitation field equations can be derived from the variation of action

\[
\mathcal{A}(g, \psi, A, \ldots) = \int dt \mathcal{L}(g, \psi, A, \ldots) = \mathcal{A}_g(g) + \mathcal{A}_m(\psi, A, \ldots)
\]

where

\[
\mathcal{A}_g(g) = -\frac{1}{16\pi G} \int d^4x R\sqrt{-g},
\]

\(R\) is the Riemannian scalar curvature and the matter-action \(\mathcal{A}_m(\psi, A, \ldots)\) is a functional of the matter field \((\psi)\), the electromagnetic field \((A)\) and all other known interaction fields \((G)\) is the universal gravitational constant, and we use the units \(\hbar = c = 1\). In classical general relativity, by setting \(\delta \mathcal{A}(g, \psi, A, \ldots) = 0\) one derives Einstein’s field equations. The generalization of this procedure to classical (non-quantum) Finsler’s spaces has been already successfully explored in the literature [36]. However, the extension of this approach to the quantum regime is beyond the scope of this investigation. In this explorative study, we will instead proceed by identifying the minimal set of modifications, which enable the derivation of a new set of extended field equations that reproduces Einstein’s theory in the classical limit.

The simplest extension of Einstein’s gravitation theory to the Finsler’s geometry in the cotangent bundle TM is achieved by constructing the TM curvature tensor from the linear Cartan connection together with the corresponding Ricci tensor and Ricci scalar (see Appendix B). The first assumption we make is that all quantum mechanical effects (in particular entanglement) are captured by the quantum potential and therefore the dynamics of the single particle can be resolved in its own sector of the \(N\)-particle tensor space defined in Eq. 14. In the following a, b, c \(\in \{0, 1, 2, 3\}\) label the base manifold \(M\) in the single particle sector of the \(N\)-particle Hilbert space, \(a, b, c \in \{4, 5, 6, 7\}\) the corresponding fibre space, and \(A, B, C \in \{0, 1, \ldots, 7\}\). TM refers now to the single particle tangent bundle.

The field equations for the gravitation metric tensor \(g^T\) are then obtained by combining the extended Einstein tensor in TM to the energy momentum tensor 22, 36

\[
R^c_{\ ab} - \frac{1}{2}(g^{c\ l\ d}R^e_{\ l\ cd} + g^{c\ ab}S^e_{\ l\ cd})g^c_{\ ab} = k_g T_{ab}
\]

\[
S^c_{\ ab} - \frac{1}{2}(g^{c\ l\ d}R^e_{\ l\ cd} + g^{c\ ab}S^e_{\ l\ cd})g^c_{\ ab} = k_g T_{ab}
\]

\[
^1P_{ab} = k_g T_{ab}
\]

\[
^2P_{ab} = -k_g T_{ab}
\]

where simple indices \(a, b\) refer to the base manifold, \(M\), and \(\bar{a}, \bar{b}\) to the fibre space, \(k_g = 8\pi G/c^4\), \(R^c_{\ ab}\), \(S^c_{\ ab}\) are the components of the curvature tensor, \(R\), and \(^1P_{ab}\) and \(^2P_{ab}\) of the corresponding Ricci tensor in the tangent space of TM in the horizontal-vertical split basis (see Appendix B). Considering the minimal possible extension to Einstein’s original formulation, we restrict to the case where \(T_{ab} = \bar{T}_{ab} = T_{\bar{a}\bar{b}} = 0\) and the energy-momentum tensor for the Dirac fields \(\psi(x)\) of Eq. 3 is given by [37]

\[
T_{ab} = i\frac{1}{4} (\bar{\psi}\gamma_a \slashed{D}_b \psi + \bar{\psi}\gamma_b \slashed{D}_a \psi) - i\frac{1}{2} g^\bar{c}_{ab} (g^{\bar{c}\bar{d}} \psi_i \slashed{D}_i \psi),
\]
where $D_a = \partial_a - ieA_a$, and $A_a$ is the QED vector potential (see below). In Eq. (22), $\gamma_a = \gamma_{i,\mu} = \mathbb{1} \otimes \cdots \otimes \gamma_\mu \otimes \mathbb{1} \otimes \cdots \otimes \mathbb{1}$, with $\gamma_\mu$ and $\partial_\mu$ in position $i$, $i \in \{1, \ldots, N\}$, $\mu \in \{0, 1, 2, 3\}$ and $\Psi = \frac{1}{\sqrt{\mathcal{N}}} (\Psi_i(x_N) \cdots \Psi_i(x_1))$ with $\Psi = \psi^4 \gamma_0$.

The classical limit. The generalization of a well established and verified theory is only valid when it can correctly reproduce the corresponding equations in the appropriate limit case. The classical limit of the Lagrangian in Eq. (14) is obtained by setting $Q(\zeta)$ to zero (obtained also by setting $\hbar \to 0$). The Finsler metric for a test particle $i$ becomes

$$g^{LC}_{ab}(x_i; x_N) = \frac{1}{4 \pi} m_i^2 g_{ab}(x_N),$$

which defines a Riemannian metric space (with the metric tensor depending on positions only). In this case, the Finsler non-linear connection coincides with the covariant derivative defined for the based manifold, $M$, and the connection coefficients simplify to $N^a_{\mu b}(x, y) = \Gamma^a_{\mu b}(y^c)$. Here $\Gamma^a_{\mu b}$ are the Christoffel symbols and the curvature obtained from $N^a_{\mu b}(x, y)$ becomes the standard Riemann curvature. Since we are now confined to the base manifold and its tangent space (the tangle bundle TM), the mixed Finsler’s curvature elements in the tangent space of TM, $S^c_{\mathcal{A} \mathcal{B} \mathcal{C}}$, $P^c_{\mathcal{A} \mathcal{B} \mathcal{C}}$ and $W^c_{\mathcal{A} \mathcal{B} \mathcal{C}}$ all vanish. From Eq. (18), we then recover the classical Einstein’s field equations,

$$R_{ab} - \frac{1}{2} R g_{ab} = k \Phi T_{ab},$$

The case where gravity can be neglected and the dynamics is controlled by the quantum potential has been already successfully discussed in refs. [1, 2].

Eqs. (18)-(22) together with the definition of the Finsler space Lagrangian $\mathcal{L}_C$ (Eq. (14)), the corresponding metric $g^{LC}$ (Eq. (15)) and the quantum potential $Q(x_N)$ (Eq. (7)) constitute the main results of this work.

The interacting field contribution. Even though a thorough investigation of the extension of the energy-momentum tensor to the tangent bundle TM goes beyond the scope of this work, it is simple to give a flavour of its implications by looking at the extended electromagnetic field theory. In the case of charged particles, the electromagnetic interaction is mediated through the action of the 4-vector potential $A(x)$. In Finsler space, $A(x)$ acquires an extra dependence on the velocity variables and its components, $A(x, y)$, are $0$-homogeneous in $y$. The corresponding generalized Faraday 2-form is defined as $F = dA$ or, in local coordinates, $F = \frac{1}{4} F_{ab} dx^a \wedge dx^b + F_{ab} dx^a \delta y^b$ with components $F_{ab} = \delta_a A_b - \delta_b A_a$ and $F_{ab} = -\partial_b A_a$. The sources of the electromagnetic field are given by the 4-currents $j^a(x_i^N)$ defined on the spacelike hypersurface $\Sigma_s$ is the usual way, $j(x_i^N) = \sum_i N_i^a (x_i^N - x_i^{\Sigma_s}) \dot{x}_i^N$, where $x_i^{\Sigma_s}$ are the 4-coordinates of the test particle in $\Sigma_s$ at a given value of the global time $s$ (and $x_i^{\Sigma_s}$ is spatial part of the 4-coordinates of particle $i$, $x_i^{\Sigma_s}$). The total current defines the horizontal components $j^a(x_i^N)$ of a vector field in TM. However, the energy-momentum conservation law in Finsler geometry implies the appearance of vertical components of the current, which are not present in pseudo-Riemannian general relativity, and leads to the general form $j = j^a \delta_a + j^b \partial_b$ (see Appendix B for the definitions of $\delta_a$ and $\partial_b$). Note that for $A_a(x)$ function of the x-coordinates only, it follows that $j^a = 0$. The extended energy-momentum tensor in TM is then obtained from the variation of the action $S_F = \frac{1}{16\pi \mathcal{N}} \int F \ast F dx \wedge dy$, with respect to the spacetime metric an the nono-linear connection $N$, where $\ast$ is the Hodge dual [32]. This leads to

$$T = T_{ab} dx^a \otimes dx^b + T_{ab} dx^a \otimes \delta y^b$$

with

$$T_{ab} = \frac{1}{4\pi} (-F_A^B F_{ab} + \frac{1}{4} g_{ab} F_{BC} F^{BC}),$$

where $g_{ab} = 0$ [39].

**Conclusions.**

Most approaches to quantum gravity start from Einstein’s classical field theory and derive a coherent and, possibly, renormalizable quantization of the gravitational field i.e., the metric tensor of a pseudo-Riemannian manifold. In this work, we addressed quantum gravity from the opposite prospective, proposing a theory in which all quantum effects are included into the curvature of a non-Riemannian Finsler’s space. Following the work on the geometrization of
quantum mechanics presented in refs. [1,2], we first generalized the formalism to make the theory relativistic covariant introducing the concept of many-times field theory together with a relativistic invariant definition of 'simultaneity' in general relativity. We then proceeded with the definition of the quantum mechanical potential that contributes to the composition of the metric tensor in the extended $2 \times (4N)$ dimensional Finsler’s space. The new theory is fully covariant, incorporates all quantum mechanical effects and reproduces Einstein’s classical gravitation theory in the limit in which the quantum potential vanishes. Finally, the additional components of the energy-momentum tensor appearing in Eqs (19)-(21) allow for the incorporation of new phenomena that still do not have an explanation within classical general relativity and that are commonly associated with the presence of dark matter and dark energy.

Appendix A

The $N$-particle spinor wavefunction, $\Psi(x_1, x_2, \ldots, x_N) \in (\mathbb{C}^4)^\otimes N$, depends on $N$ electron coordinates $x_i$ in Minkowski space and $N$ spinor indices $s_i \in \{1, 2, 3, 4\}$ [40]. The meaning of $\Psi(x_1, x_2, \ldots, x_N)$ becomes more evident if we use the expansion in a given one-particle basis of length $K \geq N$,

$$\Psi(x_1, \ldots, x_N) \rightarrow \Psi(x_1, s_1, \ldots, x_N, s_N) = \sum_{1 \leq i_1 < \cdots < i_N \leq K} \alpha_{i_1, \ldots, i_N} \langle x_1, s_1, \ldots, x_N, s_N | \phi_{i_1} \cdots \phi_{i_N} \rangle$$

where

$$\langle x_1, s_1, \ldots, x_N, s_N | \phi_{i_1} \cdots \phi_{i_N} \rangle = \frac{1}{\sqrt{N!}} \det(\phi_{ik}(x_i, s_i))_{k,l}$$

with

$$\det(\phi_{ik}(x_i, s_i))_{k,l} = \begin{vmatrix} \phi_1(x_1, s_1) & \phi_2(x_1, s_1) & \cdots & \phi_N(x_1, s_1) \\ \phi_1(x_2, s_2) & \phi_2(x_2, s_2) & \cdots & \phi_N(x_2, s_2) \\ \vdots & \vdots & \ddots & \vdots \\ \phi_1(x_N, s_N) & \phi_2(x_N, s_N) & \cdots & \phi_N(x_N, s_N) \end{vmatrix}$$

and $a \in S$ with

$$S = \{a \in \mathbb{C}^{(K)}_N, ||a||^2 = \sum_{1 \leq i_1 < \cdots < i_N \leq K} |a_{i_1, \ldots, i_N}|^2 = 1 \}.$$

Appendix B

In this appendix, we review the main aspects of Finsler geometry that are related to the subject of this work. For a more detailed account on this subject the interested reader should refer to the specialized literature [20,41]. A point in the tangent bundle TM is represented by the coordinates $(x^1, \ldots, x^n, y^1, \ldots, y^n)$, where $M$ is the base manifold of dimension $n = 4N$, and $N$ is the number of particles. The tangent space of TM is described by the coordinates $(\frac{\partial}{\partial x^1} = \partial_1, \ldots, \frac{\partial}{\partial x^n} = \partial_n, \frac{\partial}{\partial y^1} = \partial_{y_1}, \ldots, \frac{\partial}{\partial y^n} = \partial_{y_n})$.

For a Finsler space $(M,F)$ defined by the configuration space manifold $M$ and the Finsler’s function $F$ the $(0,2)$-d metric tensor field is defined as (using $x$ for $x_N$ and $y$ for $y_N$)

$$g_{ab}(x,y) = \frac{1}{2} \partial_a \partial_b F^2(x,y),$$

while the $(0,3)$-d Cartan tensor is given by the third derivative of $F^2$ with respect to the tangent space coordinates $y$,

$$C_{abc}(x,y) = \frac{1}{4} \partial_a \partial_b \partial_c F^2(x,y).$$

In case $C_{abc}(x,y) = 0$ everywhere in the tangent space, the Finsler’s space becomes a Riemannian metric space with $g_{ab}(x)$ independent on tangent space coordinates $y$. The corresponding non-linear Cartan connection is defined by
the coefficients

\[ N^a_b(x, y) = \Gamma^a_{bc}(x, y)y^c - C^a_{bc}(x, y)\Gamma^c_{pq}(x, y)y^p y^q \]  

(B.3)

(with \( C^a_{bc}(x, y) = g^{ad}(x, y)C_{dbc}(x, y) \)) or, in a more compact form,

\[ N^a_b(x, y) = \frac{1}{2} \partial_y(\Gamma^a_{cd}(x, y)y^c y^d) \]  

(B.4)

where \( g^{ab}(x, y) \) is the inverse of \( g_{ab}(x, y) \) and \( \Gamma^a_{bc} = g^{dq}(\partial_b g_{qc} + \partial_c g_{qb} - \partial_q g_{bc}) \) (removing the dependence on \( x \) and \( y \)). The non-linear curvature derived from \( N^a_b \) is

\[ R^a_{bc} = \delta_a N^a_b - \delta_b N^a_c. \]  

(B.5)

The connection allows us to decompose the tangent space \( T_a TM \) into two subspaces: the vertical space (\( V_a TM \)) tangent to \( T_a M \) (span by \( \partial_a \)) and the horizontal subspace (\( H_a TM \)) tangent to the base space manifold \( M \) as defined by the connection \( N^a_b \). In coordinates, this definition implies the transformation \( \{ \partial_a, \partial_b \} \rightarrow \{ \delta_a = \partial_a - N^b_a \partial_b, \bar{\partial}_b \} \). Regarding the tangent bundle as a manifold in its own, one can associate linear covariant derivatives to this manifold, which are compatible with the structure induced by the non-linear connection and that preserve the horizontal-vertical split of its tangent bundle \( TM \) with basis \( \{ \delta_a, \bar{\partial}_b \} \), causing no mixing. The linear covariant derivative in the horizontal-vertical basis is

\[ \nabla_{\delta_a} \delta_b = \bar{\Gamma}^c_{ab} \delta_c \]  

(B.6)

\[ \nabla_{\delta_a} \bar{\partial}_b = \bar{\Gamma}^c_{ab} \bar{\partial}_c \]  

(B.7)

\[ \nabla_{\bar{\partial}_a} \delta_b = \bar{Z}^c_{ab} \delta_c \]  

(B.8)

\[ \nabla_{\bar{\partial}_a} \bar{\partial}_b = \bar{Z}^c_{ab} \bar{\partial}_c \]  

(B.9)

where \( a, b, c = 1, \ldots, n; \bar{a}, \bar{b} = n + 1, \ldots, 2n; n \) is the dimension of \( M \) and

\[ \bar{\Gamma}^c_{ab} = \frac{1}{2} g^{cq}(\delta_a g_{bq} + \delta_b g_{aq} - \delta_q g_{ab}) \]  

(B.10)

\[ \bar{Z}^c_{ab} = g^{cq}C_{abq}. \]  

(B.11)

In the basis \( \{ \delta_a, \bar{\partial}_b \} \) the linear curvature \((1,3)\)-tensor on the tangent bundle (TM) is given by \((\alpha, \beta, \delta, \gamma, \phi = 1, \ldots, 2 \dim(M))\)

\[ R^\alpha_{\beta \gamma \delta} = X_\delta \Gamma^\alpha_{\beta \gamma} - X_\gamma \Gamma^\alpha_{\beta \delta} + \Gamma^\phi_{\beta \gamma} \Gamma^\alpha_{\phi \delta} - \Gamma^\phi_{\beta \delta} \Gamma^\alpha_{\phi \gamma} + \Gamma^\alpha_{\beta \phi} W^\phi_{\gamma \delta}, \]  

(B.12)

where

\[ X_a = (\delta_a, \bar{\partial}_a), \ W_{\bar{a}}_{bc} = R^a_{bc}, \ W_{\bar{a}}_{bc} = -\partial N^a_{\partial y^b}, \ W_{\bar{a}}_{bc} = \frac{\partial N^a_{\partial y^b}}{\partial y^c} \]  

(B.13)

and zero otherwise. \( R \) decomposes into the following components labelled by different symbols (capital letters) depending on the addressed (horizontal/vertical) sectors

\[ \begin{align*}
R^a_{bcd} &= R^a_{b\bar{c}d} = R^a_{b\bar{c}d} \quad \text{(B.14)}
R^a_{bcd} &= -R^a_{b\bar{c}d} = R^a_{b\bar{c}d} \quad \text{(B.15)}
R^a_{b\bar{c}d} &= R^a_{b\bar{c}d} = S^a_{b\bar{c}d} \quad \text{(B.16)}
\end{align*} \]
while the corresponding Ricci tensor components become

\[ R_{ab} = R_{ab}, \quad (B.17) \]
\[ R_{\bar{a}b} = 1 P_{ab}, \quad (B.18) \]
\[ R_{a\bar{b}} = -2 P_{ab}, \quad (B.19) \]
\[ R_{\bar{a}\bar{b}} = S_{ab}. \quad (B.20) \]

Note that while the Cartan non-linear connection is unique, linear connections are not and therefore alternative definitions are also possible [36]. The horizontal part of the curvature \( R(\delta_a, \delta_b)(\cdot) \) is then given by

\[ ^4R^q_{cab} = \delta_a \tilde{\Gamma}^q_{cb} - \delta_b \tilde{\Gamma}^q_{ca} + \tilde{\Gamma}^q_{ma} \tilde{\Gamma}^m_{cb} - \tilde{\Gamma}^q_{mb} \tilde{\Gamma}^m_{ca} - C^q_{cm} R^m_{ab}, \quad (B.22) \]

while the relation to the non-linear curvature becomes

\[ R^q_{ab} = -^4R^q_{cab} y^c. \quad (B.23) \]

The corresponding geodesic equation for a curve \( s \mapsto \zeta(s) \) is

\[ \dddot{\zeta}^a + N^a_b(\zeta, \dot{\zeta}) \dot{\zeta}^b = -g^{ab} \partial V(\zeta(s))/\partial \zeta^b, \quad (B.24) \]

which corresponds to Eq. [5].

* Electronic address: ita@zurich.ibm.com

[1] I. Tavernelli, Annals of Physics 371, 239 (2016).
[2] I. Tavernelli, arXiv:1710.02583 (2017).
[3] C. Rovelli, Quantum Gravity (Cambridge University Press, 2004).
[4] K. Becker, M. Becker, and J. Schwarz, String Theory and M-Theory: A Modern Introduction (Cambridge University Press, 2007).
[5] C. Rovelli, Living Rev. Rel. 1, 1 (1998).
[6] S. Carlip, Rept.Prog.Phys. 64, 885 (2001).
[7] M. P. Reisenberger and C. Rovelli, Physical Review D 56, 3490 (1997).
[8] J. Engle, E. Livine, R. Pereira, and C. Rovelli, Nuclear Physics B 799, 136 (2008).
[9] B. S. DeWitt, Phys. Rev. D 162, 1195 (1967).
[10] R. Penrose, Journal of Mathematical Physics 8, 345 (1967).
[11] A. Connes, M. R. Douglas, and A. Schwarz, J. High Energy Phys. 2, 35 (1998).
[12] D. Bohm, Phys. Rev. 85, 180 (1952).
[13] D. Bohm, Phys. Rev. 85, 166 (1952).
[14] D. Bohm and B. J. Hiley, The Undivided Universe: An Ontological Interpretation of Quantum Theory (Routledge; Revised edition, 1995).
[15] B. Hatfield, Quantum Field Theory of Point Particles and Strings (Addison Wesley, 1992).
[16] B. S. DeWitt, Supermanifolds (Cambridge University Press, 1984).
[17] R. Folreanini and R. Jackiw, Phys. Rev. D 37, 2206 (1988).
[18] P. R. Holland, The Quantum Theory of Motion - An Account of the de Broglie-Bohm Causal Interpretation of Quantum Mechanics (Cambridge University Press, 1993).
[19] D. Dürr and S. Teufel, Bohmian Mechanics, The Physics and Mathematics of Quantum Theory (Springer-Verlag Berlin Heidelberg, 2009).
[20] H. Rund, The Differential Geometry of Finsler Spaces (Springer-Verlag, 1959).
[21] C. Pfeifer and M. N. R. Wohlfarth, Phys. Rev. D 85, 064009 (2012).
[22] I. Bucataru and R. Miron, Finsler-Lagrange Geometry (Editura Academiei Romane, Bucarest, 2007).
[23] F. Girelli, S. Liberati, and L. Sindoni, Phys. Rev. D 75, 064015 (2007).
[24] D. Dürr, S. Goldstein, K. Munch-Berndl, and N. Zanghi, Phys. Rev. A 60, 2729 (1999).
[25] D. Dürr, S. Goldstein, T. Norsen, W. Struyve, and N. Zanghi, Proc. R. Soc. A 470, 20130699 (2014).
[26] C. Misner, K. S. Thorne, and J. Wheeler, Gravitation (Freeman, W.H. and Company, 1973), chap. Stress-Energy Tensor and Conservation Laws.
[27] A. Einstein, Sitzungsberichte der Preussischen Akademie der Wissenschaften zu Berlin pp. 844–847 (1915).
[28] M. Le Bellac, Quantum and Statistical Field Theory (Oxford Science Publications, 1995).
[29] H. Nicolic, arXiv:1205.4102 (2012).
[30] H. Freistadt, Phys. Rev. 97, 1158-1161 (1955).
[31] D. Bohm and B. J. Hiley, Phys. Rep. 144, 323 (1987).
[32] T. Takabayasi, Suppl. Prog. Theor. Phys. 4, 1 (1957).
[33] P. Holland, Found. Phys. 16, 701-719 (1986).
[34] R. Feynman, F. B. Morinigo, and W. G. Wagner, Feynman Lectures on Gravitation (Addison Wesley, 1995).
[35] M. E. Peskin and D. V. Schroeder, An Introduction to Quantum Field Theory (CRC Press, 1995).
[36] C. Pfeifer, The Finsler spacetime framework: backgrounds for physics beyond metric geometry (Thesis, University of Hamburg, 2013).
[37] M. Forger and H. Roemer, Ann. Phys. 309, 306 (2004).
[38] S. Weinberg, Gravitation and Cosmology (John Wiley and Sons, 1972).
[39] N. Voicu, Prog. Electromagn. Res. 113, 83 (2011).
[40] A. Levitt, Rev. Math. Phys. 26, 1450014 (2014).
[41] C. Pfeifer, The Finsler spacetime framework: backgrounds for physics beyond metric geometry (Thesis, Hamburg University, 2013).