We derive the gravitational and electrostatic self-energies of a particle at rest in the background of a cosmic dispiration (topological defect), finding that the particle may experience potential steps, well potentials or potential barriers depending on the nature of the interaction and also on certain properties of the defect. The results may turn out to be useful in cosmology and condensed matter physics.

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INTRODUCTION

Spontaneous symmetry breaking could have caused the appearance of topological defects in the very early universe [1]. Spacetimes corresponding to defects such as cosmic strings, domain walls and monopoles represent solutions of the Einstein equations, giving rise to many gravitational and cosmological effects where particle production during phase transitions (when the formation of defects takes place) [2], gravitational lensing [3] and galaxy formation are examples [4, 5, 6].

The formalism of general relativity to deal with spacetime defects has been used in the context of condensed matter physics as a tool to tackle certain problems involving defects in solids [7, 8, 9, 10]. (Ref. [10], for example, has examined analogous aspects shared by a screw dislocation in solids, a cosmic string and a magnetic vortex, discussing also the dislocation theory in solids as a three-dimensional theory of gravity.) Following this program quantum field theory [11] and classical field theory (see Ref. [12] and references therein) in the bulk of solids with linear defects (such as disclinations and dislocations) have recently been considered.

This work examines classical self-force effects on a test particle at rest in a spacetime with a cosmic dispiration (a screw dislocation plus a disclination [13, 14, 15]), which is locally flat [cf. Eq. (1) below]. Self-forces are more surprising in a locally flat background [16, 17, 18, 19, 20] rather than in one with non vanishing local curvature, such as a black hole [21, 22, 23]. In a spacetime with a dispiration, self forces arise due to the non trivial global geometry which distorts the test particle gravitational and electrostatic fields.

In the next section the Green function for the Poisson equation in a background with a dispiration is obtained, and used in the following section to develop an analytical and numerical study of the gravitational and electrostatic self-forces on a test particle at rest. A discussion of rather curious self-force effects is presented in the last section.

THE GREEN FUNCTION

The geometry of a cosmic dispiration is characterized by the Minkowski line element written in cylindrical coordinates,

$$ds^2 = dt^2 - dr^2 - r^2d\varphi^2 - dZ^2,$$

and by the non trivial identification \((t, r, \varphi, Z) \sim (t, r, \varphi + 2\pi \alpha, Z + 2\pi \kappa)\), where \(\alpha > 0\) and \(\kappa \geq 0\) are parameters corresponding to a disclination and a screw dislocation, respectively. By defining new space coordinates \(\theta := \varphi/\alpha\) and \(z := Z - (\kappa/\alpha)\varphi\), the line element of the space section becomes [13, 14]

$$dl^2 = dr^2 + \alpha^2r^2d\theta^2 + (dz + \kappa d\theta)^2,$$

and the usual identification \((r, \theta, z) \sim (r, \theta + 2\pi, z)\) must be observed. Clearly, when \(\alpha = 1\) and \(\kappa = 0\) the space becomes Euclidean.

Following Ref. [17], in order to obtain self-energies in the space corresponding to Eq. (2), one evaluates the Green function \(G(x, x')\), which is solution of

$$\nabla^2 G(x, x') = \frac{4\pi}{\sqrt{g}} \delta(r - r')\delta(\theta - \theta')\delta(z - z'),$$

(3)
with
\[ \nabla^2 = \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial}{\partial r} \right) + \frac{1}{\alpha^2 r^2} \frac{\partial^2}{\partial \theta^2} + \left( 1 + \frac{\kappa^2}{\alpha^2 r^2} \right) \frac{\partial^2}{\partial z^2} - \frac{2\kappa}{\alpha^2 r^2} \frac{\partial^2}{\partial \theta \partial z}, \]
and \( g = \alpha^2 r^2 \). The regular eigenfunctions of the operator \(-\nabla^2\) are given by
\[ \phi_{n,\nu,\mu}(x) = \frac{1}{2\pi \sqrt{\alpha}} J_{\nu}(\mu r)e^{-in\theta}e^{i\nu z}, \]
where \( n \) is an integer, \( \nu \) is a real number,
\[ \mu := \frac{n + \nu \kappa}{\alpha}, \]
\( \mu \) is a positive real number and \( J_\beta \) denotes the Bessel function of the first kind. The corresponding eigenvalues are \( \lambda_{\mu,\nu} = \mu^2 + \nu^2 \). By considering the completeness relation of the eigenfunctions \( \phi_{n,\nu,\mu}(x) \), it follows that \( G(x,x') \) can be expressed as
\[ G(x,x') = 4\pi i \int_0^\infty dt \sum_{n=-\infty}^{\infty} \int_{-\infty}^{\infty} d\nu \int_0^\infty d\mu \ e^{-it\lambda_{\mu,\nu}} \phi_{n,\nu,\mu}(x) \phi^*_{n,\nu,\mu}(x'). \]
In order to extract from Eq. (3) self-force effects one needs to consider \( G(x,x) \) \[\text{(4)}\]. Accordingly, by evaluating the integral over \( \mu \) \[\text{(26)}\] and defining \( T := it \), Eq. (3) yields
\[ G(r) = \frac{1}{2\alpha\pi^2} \int_{-\infty}^{\infty} d\nu \int_0^\infty \frac{dT}{T} e^{-T\nu^2} \sum_{n=-\infty}^{\infty} I_{\nu}(r^2/2T), \]
where \( I_\beta \) denotes the modified Bessel function of the first kind. Using a convenient integral representation for \( I_\beta \) \[\text{(26)}\] it follows that
\[ \sum_{n=-\infty}^{\infty} I_{\nu}(r^2/2T) = \alpha e^{r^2/2T} - \frac{1}{\pi} \int_0^\infty dx \ e^{-(r^2/2T)\cosh x} \sum_{n=-\infty}^{\infty} \sin(|M|\pi) \ e^{-|M|x}, \]
which holds for \( \alpha > 1/2 \) [smaller values of \( \alpha \) can be considered by taking into account terms that were omitted in Eq. (8)], resulting
\[ G(r) = \frac{1}{2\alpha\pi^2} \int_0^\infty \frac{dT}{T} \int_{-\infty}^{\infty} d\nu \ e^{-T\nu^2} \]
\[ -\frac{1}{\alpha\pi^2} \int_0^\infty \frac{dT}{T} \ e^{-r^2/2T} \sum_{n=-\infty}^{\infty} \int_{-\infty}^{\infty} d\nu \ e^{-T\nu^2} \sin(|M|\pi) \int_0^\infty dx \ e^{-(r^2/2T)\cosh x-|M|x}. \]
The first term on the right hand side (r.h.s.) of Eq. (3) is an infinite constant and does not affect the evaluation of self force effects (cf. next section). One therefore simply drops this term, which amounts to renormalize Eq. (3) with respect to the Euclidean contribution [by considering Eq. (3) it should be noticed that the second term on the r.h.s. of Eq. (3) vanishes when \( \alpha = 1 \) and \( \kappa = 0 \)]. By inserting \( \delta(\lambda-M) \) in Eq. (3) and using Poisson’s formula, one is able to perform the integration over \( \nu \) \[\text{(26)}\]. Finally, the integrations over \( \lambda \) and \( T \) are evaluated \[\text{(26)}\], resulting in
\[ G_{\text{ren}}(r) = -\frac{\ln(2)}{2 \pi r} - 2 \sum_{n=1}^{\infty} \int_0^\infty dx \ \frac{x^2 - \pi^2(4\alpha^2 r^2 - 1)}{[\pi^2(2\alpha n + 1)^2 + x^2][\pi^2(2\alpha n - 1)^2 + x^2][\cosh^2(x/2) + (n\pi r)^2]^{1/2}}. \]

**THE SELF-FORCES**

Self-force effects are better appreciated when described in terms of the flat coordinates appearing in Eq. (1). The gravitational and electrostatic self-energies of a point particle of mass \( m \) and charge \( q \) can be expressed in terms of \( G_{\text{ren}}(r) \) as
\[ U_m(r) = -\frac{Gm^2}{2} G_{\text{ren}}(r) \]
(11)
and

\[ U_q(r) = \frac{q^2}{2} G_{\text{ren}}(r), \tag{12} \]

respectively. These expressions arise from the corresponding Poisson equations by taking the test particle as source \[17, 27\]. The self-forces can then be obtained by taking minus the gradient of the self-energies. However, usually the behavior of a test particle under the action of a force are better understood by considering the corresponding potential energy, rather than the force itself. Therefore the analysis below will be based upon Eqs. (10), (11) and (12).

As \( \kappa/r \to 0 \), the summation in Eq. (10) can be evaluated by considering the power series expansion of \( \psi(z) \) (the logarithmic derivative of the gamma function) and its properties, yielding as leading contribution

\[ G_{\text{ren}}(r) = -\frac{1}{2\pi \alpha r} \sin(\pi/\alpha) \int_{0}^{\infty} dx \frac{1}{\cosh(x/2)[\cosh(x/\alpha) - \cos(\pi/\alpha)]}. \tag{13} \]

As \( \kappa/r \to \infty \), on the other hand, the leading contribution is now given by

\[ G_{\text{ren}}(r) = -\frac{\ln(2)}{\pi r}. \tag{14} \]

Apart from these asymptotic cases, the dependence of \( G_{\text{ren}}(r) \) on \( r \) is non-trivially hidden in Eq. (10) and a numerical analysis is required. The plots (where units were omitted) show how \(-G_{\text{ren}}(r)\) varies with \( r \) for various combinations of values of \( \alpha \) and \( b := 2\pi \kappa \).

\[ \text{FIG. 1: } U := -G_{\text{ren}}(r) \times 10^{-3}. \]

**DISCUSSION**

By inserting Eq. (13) in Eqs. (1) and (2), one reproduces the known results in the literature \([16, 17]\). Namely, the gravitational self-force due to a disclination alone, i.e. when \( \kappa = 0 \), is attractive (repulsive) for \( \alpha < 1 \) (\( \alpha > 1 \)). The opposite holds for the electrostatic self-force (cf. Figs. 1 and 2).

For \( \kappa \neq 0 \), Eq. (13) shows that, as \( r \) gets very large, disclination effects become dominant (cf. Figs. 1 and 2). As \( r \) gets very small though, Eq. (14) shows that screw dislocation effects rule in a peculiar way, rendering the self-forces due to a dispiration independent of \( \kappa \) and \( \alpha \), which are the parameters characterizing the defect (similar effects are rather familiar in the context of vacuum polarization \([28]\)). By observing Eqs. (11) and (14), one sees that the dispiration (more precisely, the screw dislocation) induces a gravitational barrier, keeping the test particle away from the defect. From Eqs. (12) and (14) one sees that the reverse is true for the electrostatic self-force, i.e., a charged test particle is electrostatically attracted by the defect.

For arbitrary values of the radial coordinate \( r \), numerical and analytical considerations show that when \( \alpha \geq 1 \) (including therefore the screw dislocation alone for which \( \alpha = 1 \)) the gravitational self-force induced by the dispiration
is repulsive, whereas the electrostatic self-force is attractive (cf. Figs. 2 and 3). When $\alpha < 1$, the repulsive gravitational self-force due to the screw dislocation and the attractive gravitational self-force due to the disclination dispute, giving rise to an well potential (a potential step, in the case of the electrostatic self-force). Quick dimensional considerations reveal that the minimum of $-G_{ren}(r)$ is proportional to $-1/\kappa$ whose corresponding $r$ is proportional to $\kappa$ (cf. Figs. 1 and 4). It should be noticed that when $\alpha < 1$ the behaviors of the self-forces for $\kappa = 0$ and $\kappa \neq 0$ differ radically from each other as $r \to 0$ [cf. Eq. (13), Eq. (14) and Fig. 1].

Before closing, it should be pointed out that Eq. (13) is the leading contribution as $r \to \infty$ only when $\alpha \neq 1$. When $\alpha = 1$, Eq. (13) vanishes and the sub-leading contribution, which is due to the screw dislocation only, takes over. By dimensional considerations, one may be tempted to assume that such a contribution is proportional to $\kappa^2/r^3$. However an analysis seems to show that $-G_{ren}(r)$ falls much quicker than that as $r \to \infty$ (cf. Fig. 1). (The representation of $G_{ren}(r)$ in Eq. (13) is not very handy to determine the sub-leading contribution, and an alternative representation may show to be more useful on this matter.)

It remains to be seen if the results outlined above have applications in the realms of cosmology and condensed matter physics.

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\[ U := -G_{\text{ren}}(r) \times 10^{-3} \]

FIG. 4: \[ U := -G_{\text{ren}}(r) \times 10^{-3} \].

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