Dynamic critical exponents of the Ising model with multispin interactions

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Abstract

We revisit the short-time dynamics of 2D Ising model with three spin interactions in one direction and estimate the critical exponents \( z \), \( \theta \), \( \beta \) and \( \nu \). Taking properly into account the symmetry of the Hamiltonian we obtain results completely different from those obtained by Wang et al.. For the dynamic exponent \( z \) our result coincides with that of the 4-state Potts model in two dimensions. In addition, results for the static exponents \( \nu \) and \( \beta \) agree with previous estimates obtained from finite size scaling combined with conformal invariance. Finally, for the new dynamic exponent \( \theta \) we find a negative and close to zero value, a result also expected for the 4-state Potts model according to Okano et al..

Since the work by Janssen et al.¹ and Huse² pointing out the existence of another universal stage at an early time critical dynamic, several statistical models have been investigated to confirm the analytical predictions about the "critical initial slip" and to enlarge the knowledge of critical phenomena ³, ⁴, ⁵, ⁶, ⁷, ⁸. The investigation of the universal behavior in short-time dynamics avoids the critical slowing down effects of the equilibrium and provides an alternate way to calculate the new exponent \( \theta \) which governs the behavior of the magnetization, the dynamic critical exponent \( z \) as well as the static exponents \( \beta \) and \( \nu \).

In this letter we adopt this approach to study the two-dimensional (2D) Ising model with three-spin interactions in one direction and calculate its set of exponents. Motivation came from the fact that in a recent paper by Wang et al.¹⁰ estimates obtained were in complete disagreement with pertinent results.
Here we show that, when the symmetry of the model is taken properly into account, good agreement is obtained with expected results.

The Hamiltonian of the 2D Ising model with three spin interactions \((m = 3)\) in one direction is \[ -\beta H = \sum_{<i,j>} \left\{ K_y S_{i,j} S_{i+1,j} + K_x \prod_{l=0}^{m-1} S_{i,j+l} \right\} \]

where \(S_{i,j} = \pm 1\) is the Ising spin variable. The model is known to be self-dual \([12]\), its critical line being

\[ \sinh 2K_x \sinh 2K_y = 1 \]

for all \(m\). For the particular isotropic case \((K_x = K_y)\) the critical coupling is

\[ K_c = J/k_B T_c = \frac{1}{2} \ln(\sqrt{2} + 1) = 0.4406867, \]

which is the same of the standard 2D Ising model.

Symmetry analysis and ground state degeneracy considerations suggest that the model is in the same universality class as the \(q\)-state Potts model, whenever \(q = 2^{m-1}\) \([12, 13]\). This result is supported by finite-size scaling studies \([14, 12, 17]\), weak and strong coupling expansions \([16]\), conformal invariance \([17, 18, 19]\), standard Monte Carlo simulations \([20, 21, 13]\) and mapping of the \(m = 3\) model in the extreme anisotropic limit of the 4-state Potts model \([22]\). In most of those papers the argument to include the Ising model with three-spin interaction and the 4-state Potts model in the same universality class is based on the value of the exponents \(\nu\) and \(\alpha(\approx 2/3)\) \([23]\). All of them respect the symmetry of the Hamiltonian. Very little is known about the exponents \(\beta\) \([21]\) and \(z\) \([10]\).

The ground state for the general \(m\)-spin interaction is \(2^{m-1}\) degenerated \([14]\), which implies that the 2D Ising model with \(m = 3\) spin interaction is four-fold degenerated. The relevant symmetry of this model is semi-global \([22]\) and the Hamiltonian is symmetric under the reversal of all the spins in any two sublattices, which leads to the existence of three independent interpenetrating sublattices in the system. At \(T = 0\), the possible states consist of repetitions of the patterns \(+ + +, + - -, - + -, - - +\) in the horizontal direction, copied along the lines. According to preceding statement, it is important to take lattice sizes that are multiple of three - in order to respect the symmetry of the Hamiltonian - and to work with the appropriate order parameter - the sublattice magnetization - in order to avoid the effect of staggered magnetization. We suspected that Wang \textit{et al} \([10]\) didn’t consider the \(m\)-spin symmetry in their simulations neither worked with the magnetization of sublattice, since they just presented results for square lattices which do not obey that previous condition (multiple of three). In this sense, it seemed to us relevant to repeat simulations, paying attention
to the above mentioned points, to check the apparent failure of the short-time approach in this case.

We began by repeating the analysis made by Wang et al [10] for the Binder cumulant

\[ U(t, L) = \frac{\langle M^2(t) \rangle}{\langle M(t) \rangle^2} - 1 \tag{4} \]

where \(< >\) means average on samples, \(t\) is the time, \(M(t) = \sum_{i=1}^{L} \sum_{j=1}^{L} S_{i,j}\) is the magnetization at time \(t\) and \(L\) the size of the square lattice. They argue that this expression obeys the power law form:

\[ U(t, L) \propto t^{d/z}, \tag{5} \]

when the dynamical process starts from an ordered state \((m_0 = 1)\), which is a fixed point under renormalization group transformation. Fig. 1 shows explicitly the different results obtained when we use the magnetization of the sublattice. The average is taken over 50000 independent initial configurations and the error bars (smaller than the size of the points) are obtained by repeating five times each simulation. When the simulation is performed without the sublattice considerations the slope of the curve agrees with that presented by Wang et al and confirmed our suspicions. From our point of view [25], though, this cumulant should obey the power law form (5) only when different initial conditions were used in the study of the magnetization and its second moment. Scaling arguments [3] assert that the second moment of magnetization behaves as

\[ \langle M^2(t) \rangle \propto t^{(d-2\beta/\nu)/z}, \tag{6} \]

only when the samples are taken with zero initial magnetization \((m_0 = 0)\). On the other hand, short-time scaling behavior implies

\[ \langle M(t) \rangle \propto t^{-\beta/\nu z} \tag{7} \]

for samples starting from the ordered state \((m_0 = 1)\). Thus, performing two different simulations under those mentioned conditions, we obtain the time evolution of the ratio \(\langle M^2(t) \rangle / \langle M(t) \rangle^2\) which furnishes the exponent \(d/z\) in a log-log plot (Fig. 2). From the slope of that curve we estimate \(z = 2.380 \pm 0.004\).

To confirm our result we used two other approaches: the generalized fourth order Binder's cumulant [3] and the parameters Q and R introduced by de Oliveira [26]. In both cases, the collapse of the curves for different lattice sizes, at critical temperature, are used to determine the dynamical critical exponent \(z\) from short-time simulations. The Binder's cumulant,

\[ U_4(t, L) = 1 - \frac{\langle M^{(4)} \rangle}{3 \langle M^{(2)} \rangle^2} \tag{8} \]
satisfies, at $T = T_c$ the scaling relation

$$U_4(t, L) = U_4(b^{-z}t, b^{-1}L)$$  \hspace{1cm} (9)$$

where $b = L/L'$ since $U_4$ scales as $L^0$. This technique has proved to be useful in determining the exponent $z$ and was applied to the 2D and 3D Ising models [3], [28], the 3-state Potts model [4], the majority vote model [24] and cellular automata [30]. The initial magnetization of samples, in this case, is zero as well as the correlation length. The results are good enough. Error bars, however, are bigger than those obtained by damage spreading technique [31]. In Fig. 3, we show the collapse of the cumulant $U_4$ for lattice pairs $(L, 2L)$ with $z = 2.3$. In fact, the range of $z$ for which the collapse is still observed is given by $z = 2.3 \pm 0.1$. This result supports our previous estimate for $z$ $(2.383 \pm 0.004)$ and can be related to the 4-state Potts model exponent [32]. In order to stress the importance of considering the symmetry of Hamiltonian, we exhibit in Fig. 4 the deformation of the Binder cumulant when there is no sharp preparation of the initial magnetization on the sublattices and the magnetization evolution is calculated without restrictions.

When we use the parameters $Q(t, L) = \langle \text{sign} \left( \frac{1}{L} \sum S_i \right) \rangle$ and $R(t, L) = \langle \text{sign} \left( \frac{1}{L} \sum_{\text{bot}} S_i \right) \langle \text{sign} \left( \frac{1}{L} \sum_{\text{top}} S_i \right) \rangle$ of de Oliveira [27] and scaling relations for $T = T_c$ [33]

$$Q(t, L) = Q(b^{-z}t, b^{-1}L)$$

and

$$R(t, L) = R(b^{-z}t, b^{-1}L)$$

we obtain the collapse among curves (see Fig. 5 and Fig. 6) for different lattices when time is scaled with $z = 2.3$. Samples in these cases were initialized with all spins up ($m_0 = 1$).

In order to calculate the exponent $\nu$, we studied the derivative of the magnetization, which presents the following scaling form:

$$\partial_\tau \ln M(t, \tau)|_{\tau = 0} = t^{1/\nu z} \partial_{\tau'} \ln F(\tau')|_{\tau' = 0}.$$  \hspace{1cm} (10)$$

Fig. 7 shows the power law behavior of the $\partial_\tau \ln M(t, \tau)$ when $\Delta \tau = 0.0002$. The measured slope of the curve gives $1/\nu z = 0.624 \pm 0.005$. Thus, taking $z = 2.383 \pm 0.004$ we find $\nu = 0.67 \pm 0.01$, to be compared with the conjectured value $\nu = 2/3$. 

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Since the values of the exponents $\nu$ and $z$ are already known, the exponent $\beta$ of the magnetization can be obtained from the power law increase of the second moment of magnetization, Eq.\((6)\). Fig. 8 presents, in a double-log scale, the polynomial behavior of $\langle M(2)^2(t) \rangle$. From the slope of these lines we estimate $\beta = 0.11 \pm 0.02$. This result is in agreement with the expected value $\beta = 0.125$ \cite{24,34,35}.

In order to extend the picture of universality we investigate the exponent $\theta$ by a recent technique proposed by Tomé and de Oliveira \cite{36}. In their paper they show that the exponent $\theta$ can also be independently calculated in despite of the sharp preparation of the samples, since the time correlation of the total magnetization in samples with a random initial configuration also exhibits polynomial behavior

$$\langle M(t)M(0) \rangle = \frac{1}{N} \left\langle \sum_i \sum_j S_i(t)S_j(0) \right\rangle \propto t^\theta. \quad (11)$$

This procedure avoids the use of an initial state with a nonzero (but small) magnetization as well as the numerical extrapolation $m_0 \rightarrow 0$ to calculate the dynamic exponent $\theta$. Fig. 9 shows, in a log-log scale, the $\langle M(t)M(0) \rangle$ behavior for different lattice sizes. The slope of those curves give us $\theta = -0.03 \pm 0.01$ which is compatible with the conjecture by Okano et al for the 4-state Potts model \cite{5}.

In summary, we have obtained static and dynamic critical exponents for the Ising model with multispin interactions using short-time Monte Carlo simulations. Our results show that this model and the 4-state Potts one share the same set of critical exponents even at dynamic level. When compared to the paper by Whang et al this letter shows the importance of taking properly into account the symmetry of the Hamiltonian to deal with magnetization and boundary conditions. We stress that the present result for the exponent $\nu$ is better than previous estimates obtained by finite size scaling and Monte Carlo approach \cite{24}.

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Figure 1: Log-log plot of the cumulant $U = \langle M^2 \rangle / \langle M \rangle^2 - 1$ versus time. Our results were obtained using the sublattice magnetization (upper curve), but samples (50000) were originally in the ordered state, as done in the paper by Wang et al [10] (lower curve). The error bars are smaller than the symbols.
Figure 2: Log-log plot of the ratio $U = \langle M^2 \rangle / \langle M \rangle^2$ when different initial conditions are used to measure the magnetization ($m_0 = 1$) and its second moment ($m_0 = 0$). Error bars were calculated over 5 sets of 50000 samples and are smaller than the dot sizes.

$tan\phi = 0.839 \pm 0.004$

$L = 144$
Figure 3: Generalized Binder cumulant for $L=9, 18, 36$ (continuous curves) compared to the rescaled cumulant in time through $t'=t/2^z$ for $L=18, 36, 72$ (dotted curves). The best fit is obtained when $z=2.3$. 

$U = 1 - \langle M^2 \rangle / 3 \langle M \rangle^2$ 

(time)
Figure 4: Deformation of the generalized Binder cumulant ($L=12$) when the sublattices are not considered in the Ising model with three spin interaction.
Figure 5: Time behaviour of $Q(t)$ for lattices $L=18, 36, 48$ (full lines) and the corresponding time rescaled curves for lattices $L=12, 24, 36$ (open circles). The collapse was obtained with $z=2.3$. At $T=0$ all the spins are in the up direction ($m_0=1$). Statistical errors are smaller than the symbols.
Figure 6: Short-time behaviour of $R(t)$ of lattices $L=18, 24, 36$ (full lines) together with the corresponding time rescaled curves for the lattices $L=24, 36, 48$ (open symbols). The initial condition is given by $m_0=1$ and error bars are smaller than the symbol sizes. The best fit is obtained with $z=2.3$. 
Figure 7: Power law behaviour of the derivative of the magnetization (eq. 12) gives the critical exponent $\nu$ of the correlation length.
Figure 8: Log-log plot of the temporal increase of the second moment of the magnetization. The error bars were calculated over 5 sets of 50000 samples and are smaller than the dotsizes. The initial magnetization is given by $m_0=0$ and $\tan \phi = 0.700 \pm 0.001$. 
Time correlation of the total magnetization for samples which have in average $m_0 = 0$. Error bars were calculated over 5 sets of 50000 samples.

$\tan \phi = -0.03 \pm 0.01$

**Figure 9:** Time correlation of the total magnetization for samples which have in average $m_0 = 0$. Error bars were calculated over 5 sets of 50000 samples.