Multiplying decomposition of stress/strain, constitutive/compliance relations, and strain energy

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Abstract

To account for phenomenological theories and a set of invariants, stress and strain are usually decomposed into a pair of pressure and deviatoric stress and a pair of volumetric strain and deviatoric strain. However, the conventional decomposition method only focuses on individual stress and strain, so that cannot be directly applied to either formulation in Finite Element Method (FEM) or Boundary Element Method (BEM). In this paper, a simpler, more general, and widely applicable decomposition is suggested. A new decomposition method adopts multiplying decomposition tensors or matrices to not only stress and strain but also constitutive and compliance relation. With this, we also show its practical usage on FEM and BEM in terms of tensors and matrices.

Keywords: Pressure, hydrostatic pressure, Deviatoric stress, Volumetric strain, Mean strain, Deviatoric strain, Stress decomposition, Strain decomposition, Constitutive decomposition, Compliance decomposition, Multiplication decomposition, Decomposition multiplier

1. Introduction

In many references (see Fung (1965); Fung and Tong (2001); Gurtin (1981); Richards Jr (2000); Bower (2009)), stress and strain decomposition are usually used for a set of invariants. From a hydrostatic stress tensor (or volumetric stress tensor) \( p \), the 1st stress invariant \( I_1 \) is given by

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\[ I_1 = 3 \, p \]  \hspace{1cm} (1.1)

Also, from a deviatoric stress tensor \( s_{ij} \), the second and third deviatoric stress invariants are given by

\[
J_2 = \frac{1}{2} s_{ij} s_{ij} \quad (1.2)
\]
\[
J_3 = \frac{1}{3} s_{ij} s_{jk} s_{ki} \quad (1.3)
\]

In 1.1-1.3, a hydrostatic stress tensor \( p \) and a deviatoric stress tensor \( s_{ij} \) result from the decomposition of stress tensor \( \sigma_{ij} \):

\[
\sigma_{ij} = s_{ij} + p \, \delta_{ij} \quad (1.4)
\]
\[
p = \frac{\sigma_{kk}}{3} \quad (1.5)
\]

Similarly, from a volumetric strain tensor \( \epsilon_M \) \( (\epsilon_M = \epsilon_{kk}/3) \) and a deviatoric strain tensor \( \epsilon'_{ij} \) \( (\epsilon'_{ij} = \epsilon_{ij} - \epsilon_M \, \delta_{ij}) \), we have strain invariants \( I_1^e, J_2^e, \) and \( J_3^e \) as

\[
I_1^e = 3 \, \epsilon_M \quad (1.6)
\]
\[
J_2^e = \frac{1}{2} \epsilon'_{ij} \epsilon'_{ji} \quad (1.7)
\]
\[
J_3^e = \frac{1}{3} \epsilon'_{ij} \epsilon'_{jk} \epsilon'_{ki} \quad (1.8)
\]

The other general use of stress and strain decomposition can be found in phenomenological theories (e.g., Houlsby and Puzrin (2002); Perzyna (1966); Dunne and Petrinic (2005); Simo and Hughes (1998); Lubliner (1990); Leigh (1968)). For example, von Mises yield function \( f \) that describes elastoplasticity is written as

\[
f = J_2 - k^2 \quad (1.9)
\]

where, \( k^2 = \frac{1}{3} \sigma_Y^2 \). Also, the flow potential \( \varphi \) in Perzyna formulation that describes rate-dependent plasticity is written as
\[ \varphi = \frac{1}{2\eta} \left( \sqrt{J_2} - k \right)^2 \]  

(1.10)

where, \( \eta \) is a viscosity and \( \langle \cdot \rangle \) represents the Macaulay bracket (ramp function).

From 1.9-1.10, the flow rules for plasticity and viscoplasticity yield

\[ \dot{\varepsilon}_{ij}^p = \dot{\lambda} \frac{\partial f}{\partial \sigma_{ij}} \]

\[ = \dot{\lambda} s_{ij} \]  

(1.11)

and

\[ \dot{\varepsilon}_{ij}^{vp} = \frac{\partial \varphi}{\partial s_{ij}} \]

\[ = \frac{\partial \varphi}{\partial J_2} \frac{\partial J_2}{\partial s_{ij}} \]

\[ = \frac{1}{2\eta} \langle 1 - \frac{k}{\sqrt{J_2}} \rangle s_{ij} \]  

(1.12)

In 1.11-1.12, \( \dot{\varepsilon}_{ij}^p \), \( \dot{\lambda} \), and \( \dot{\varepsilon}_{ij}^{vp} \) represent the plastic strain rate, plastic multiplier, and viscoplastic strain rate, respectively.

Despite such fundamental uses of decomposition in mechanics, the conventional decomposition method is not directly adopted to the formulation in FEM and BEM due to the lack of the generalized decomposition method including decomposition of constitutive or compliance relation. As we shall see, through the multiplication decomposition that developed here, one can directly decompose not only stress and strain, but also constitutive relation and compliance relation. The method is so simple and general that just a decomposition multiplier is applied to all cases mentioned here. And as an example, we will also show its applications to the formulation in FEM and BEM for elastostatics.

2. Decomposition in tensor forms

In this Section, we show the multiplying decomposition method for stress, strain, constitutive relation, and compliance relation in tensor forms. For canonical example, we also show its application to linear isotropic elasticity.
2.1 Decomposition of stress and constitutive relation in tensor form

Conventional stress decompositions such as 1.4-1.5 can be written in tensor form as

\[ s_{ij} = \sigma_{ij} - p\delta_{ij} \]
\[ = \sigma_{ij} - \frac{1}{3}\sigma_{kk}\delta_{ij} \]
\[ = \left( \delta_{ki}\delta_{lj} - \frac{1}{3}\delta_{lk}\delta_{ij} \right)\sigma_{kl} \]  
(2.1)

and

\[ p = \frac{1}{3}\sigma_{kk} \]
\[ = \left( \frac{1}{3}\delta_{lk} \right)\sigma_{kl} \]  
(2.2)

In 2.1-2.2, if we let

\[ M^d_{ijkl} = \delta_{ki}\delta_{lj} - \frac{1}{3}\delta_{lk}\delta_{ij} \]  
(2.3)

and

\[ M^v_{ijkl} = \frac{1}{3}\delta_{lk}\delta_{ij} \]  
(2.4)

then, 2.1 and 2.2 can be written as:

\[ s_{ij} = M^d_{ijkl}\sigma_{kl} \]  
(2.5)

and

\[ p_{ij} = M^v_{ijkl}\sigma_{kl} \]  
(2.6)

In 2.6, \( p_{ij} \) represents

\[ p_{ij} = p\delta_{ij} \]  
(2.7)

Such decompositions as 2.5 and 2.6 can be easily checked through
2.1 Decomposition of stress and constitutive relation in tensor form

\[
\sigma_{ij} = s_{ij} + p_{ij}
\]

\[
= M_{ijkl}^d \sigma_{kl} + M_{ijkl}^v \sigma_{kl}
\]

\[
= \left( \delta_{ki} \delta_{lj} - \frac{1}{3} \delta_{lk} \delta_{ij} \right) \sigma_{kl} + \left( \frac{1}{3} \delta_{lk} \delta_{ij} \right) \sigma_{kl}
\]

\[
= (\delta_{ki} \delta_{lj}) \sigma_{kl}
\]

\[
= \sigma_{ij} \tag{2.8}
\]

and, by them, we can also decompose the constitutive relation \( C_{ijkl} \) in

\[
\sigma_{ij} = C_{ijkl} \epsilon_{kl} \tag{2.9}
\]

That is, by 2.9 and 2.5, the deviatoric stress \( s_{ij} \) can be written by

\[
s_{ij} = M_{ijkl}^d \sigma_{kl}
\]

\[
= M_{ijkl}^d \left( C_{klmn} \epsilon_{mn} \right)
\]

\[
= \left( M_{ijkl}^d C_{klmn} \right) \epsilon_{mn} \tag{2.10}
\]

and, by 2.9 and 2.6, the pressure \( p_{ij} \) can be written by

\[
p_{ij} = M_{ijkl}^v \sigma_{kl}
\]

\[
= M_{ijkl}^v \left( C_{klmn} \epsilon_{mn} \right)
\]

\[
= \left( M_{ijkl}^v C_{klmn} \right) \epsilon_{mn} \tag{2.11}
\]

Then, noting \( M_{ijkl}^d C_{klmn} \) and \( M_{ijkl}^v C_{klmn} \) as

\[
C_{ijmn}^d = M_{ijkl}^d C_{klmn} \tag{2.12}
\]

and

\[
C_{ijmn}^v = M_{ijkl}^v C_{klmn} \tag{2.13}
\]

yields stress decompositions

\[
s_{ij} = C_{ijmn}^d \epsilon_{mn} \tag{2.14}
\]

\[
p_{ij} = C_{ijmn}^v \epsilon_{mn} \tag{2.15}
\]
2.2 Decomposition of strain and compliance relation in tensor form

in terms of decomposed constitutive tensors $C_{ijmn}^d$ and $C_{ijmn}^v$.

Such constitutive decompositions as 2.12 and 2.13 are also checked through

$$
C_{ijmn}^d + C_{ijmn}^v = \left( M_{ijkl}^d + M_{ijkl}^v \right) C_{klmn}
$$

$$
= \delta_{kl}\delta_{ij} C_{klmn}
$$

$$
= C_{ijmn}
$$

(2.16)

2.2. Decomposition of strain and compliance relation in tensor form

Conventional strain decompositions are given by

$$
\epsilon_{ij}' = \epsilon_{ij} - \frac{1}{3} \epsilon_{kk} \delta_{ij}
$$

(2.17)

$$
\epsilon_{ij}^M = \frac{1}{3} \epsilon_{kk} \delta_{ij}
$$

(2.18)

By following similar decomposition procedures to stress, the multiplying decomposition can define strain decompositions as

$$
\epsilon_{ij}' = M_{ijkl}^v \epsilon_{kl}
$$

(2.19)

$$
\epsilon_{ij}^M = M_{ijkl}^d \epsilon_{kl}
$$

(2.20)

and, by them, we also can decompose the compliance relation $D_{ijkl}$ in

$$
\epsilon_{ij} = D_{ijkl} \sigma_{kl}
$$

(2.21)

into

$$
D_{ijmn}^d = M_{ijkl}^d D_{klmn}
$$

(2.22)

and

$$
D_{ijmn}^v = M_{ijkl}^v D_{klmn}
$$

(2.23)

In 2.20, $\epsilon_{ij}^M$ represents

$$
\epsilon_{ij}^M = \epsilon_M \delta_{ij}
$$

(2.24)
2.3 Example for linear isotropic elasticity

and equations 2.19-2.20 can also be written by

\[
\begin{align*}
\epsilon'_{ij} &= D_{ijmn}^d \sigma_{mn} \quad (2.25) \\
\epsilon^M_{ij} &= D_{ijmn}^v \sigma_{mn} \quad (2.26)
\end{align*}
\]

in terms of the decomposed compliance tensors \(D_{ijmn}^d\) and \(D_{ijmn}^v\).

Such compliance decompositions as 2.22 and 2.23 are also checked through

\[
D_{ijmn}^d + D_{ijmn}^v = (M_{ijkl}^d + M_{ijkl}^v) D_{klmn}
\]

\[
= \delta_{ki} \delta_{lj} D_{klmn}
\]

\[
= D_{ijmn} \quad (2.27)
\]

2.3. Example for linear isotropic elasticity

For linear isotropic elasticity, the constitutive relation \(C_{ijkl}\) is given by

\[
C_{ijkl} = \lambda \delta_{kl} \delta_{ij} + 2 \mu \delta_{ki} \delta_{lj} \quad (2.28)
\]

By the suggested decomposition method, the constitutive decomposition tensors \(C_{ijmn}^d\) and \(C_{ijmn}^v\) are expressed as

\[
C_{ijmn}^d = M_{ijkl}^d C_{klmn}
\]

\[
= \left( \delta_{ki} \delta_{lj} - \frac{1}{3} \delta_{lk} \delta_{ij} \right) \left( \lambda \delta_{mn} \delta_{kl} + 2 \mu \delta_{mk} \delta_{nl} \right)
\]

\[
= \frac{2}{3} \mu \left( 3 \delta_{mi} \delta_{nj} - \delta_{ij} \delta_{mn} \right) \quad (2.29)
\]

and

\[
C_{ijmn}^v = M_{ijkl}^v C_{klmn}
\]

\[
= \left( \frac{1}{3} \delta_{lk} \delta_{ij} \right) \left( \lambda \delta_{mn} \delta_{kl} + 2 \mu \delta_{mk} \delta_{nl} \right)
\]

\[
= \frac{1}{3} (3\lambda + 2\mu) \delta_{ij} \delta_{mn}
\]

\[
= \left( \lambda + \frac{2}{3} \mu \right) \delta_{ij} \delta_{mn}
\]

\[
= K \delta_{ij} \delta_{mn} \quad (2.30)
\]
2.3 Example for linear isotropic elasticity

where \( \mu \) and \( K \) are shear modulus and bulk modulus, respectively.

Equations 2.29-2.30 can also be checked through the conventional decomposition method. That is, from the elastic constitutive relation 2.28, we have

\[
\sigma_{ij} = C_{ijmn} \epsilon_{mn} = (\lambda \delta_{mn} \delta_{ij} + 2\mu \delta_{mi} \delta_{nj}) \epsilon_{mn} \tag{2.31}
\]

Letting \( ij \to kk \) in 2.31 yields

\[
\sigma_{kk} = (3\lambda + 2\mu)\epsilon_{kk} = 3K\epsilon_{kk} \tag{2.32}
\]

Then, \( p_{ij} \) that given by 2.7 is expressed as

\[
p_{ij} = \frac{\sigma_{kk}}{3} \delta_{ij} = K\epsilon_{kk} \delta_{ij} \tag{2.33}
\]

Also, \( s_{ij} \) given by 2.1 is expressed as

\[
s_{ij} = \sigma_{ij} - p_{ij} = \sigma_{ij} - \frac{\sigma_{kk}}{3} \delta_{ij} = (\lambda \delta_{kl} \delta_{ij} + 2\mu \delta_{ki} \delta_{lj}) \epsilon_{kl} - \frac{1}{3} ((3\lambda + 2\mu)\epsilon_{kk}) \delta_{ij} = \frac{2}{3} \mu (3\epsilon_{ij} - \delta_{ij} \epsilon_{kk}) \tag{2.34}
\]

Thus, from 2.33 and 2.34, we can verify the constitutive relations such as

\[
p_{ij} = C_{ijmn}^v \epsilon_{mn} \tag{2.35}
\]

and

\[
s_{ij} = C_{ijmn}^d \epsilon_{mn} \tag{2.36}
\]

Similarly, the compliance tensor \( D_{ijkl} \) for linear isotropic elasticity, which is given by
\[ D_{ijkl} = \frac{1}{E} ((1 + \nu)\delta_{ki}\delta_{lj} - \nu\delta_{kl}\delta_{ij}) \]  

(2.37)

can be decomposed into

\[
D_{ijmn}^d = M_{ijkl}^d D_{klmn}
\]

\[
= \frac{1}{2\mu} \left( \delta_{im}\delta_{jn} - \frac{1}{3}\delta_{ij}\delta_{mn} \right)
\]

(2.38)

and

\[
D_{ijmn}^v = M_{ijkl}^v D_{klmn}
\]

\[
= \frac{1}{3K}\delta_{ij}\delta_{mn}
\]

(2.39)

Such equations as 2.38 and 2.39 can also be checked through the conventional decomposition method.

3. Decomposition in matrix form

In this Section, we show the multiplying decomposition method for stress, strain, constitutive relation, and compliance relation in matrix forms.

3.1. Decomposition of stress and constitutive relation in matrix form

With Voigt notation, the decomposition multiplier 2.3 can be expressed in a matrix form as

\[
M^d = \frac{1}{3}
\begin{bmatrix}
2 & -1 & -1 & 0 & 0 & 0 \\
-1 & 2 & -1 & 0 & 0 & 0 \\
-1 & -1 & 2 & 0 & 0 & 0 \\
0 & 0 & 0 & 3 & 0 & 0 \\
0 & 0 & 0 & 0 & 3 & 0 \\
0 & 0 & 0 & 0 & 0 & 3 \\
\end{bmatrix}
\]  

(3.1)

Also, the decomposition multiplier 2.4 can be expressed in a matrix form as
3.1 Decomposition of stress and constitutive relation in matrix form

\[ M^v = \frac{1}{3} \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \] (3.2)

Then, we can express the deviatoric stress in 2.5 and the pressure in 2.6 in a vector form \( s \) and \( p \) as

\[ s = M^d \sigma \] (3.3)

and

\[ p = M^v \sigma \] (3.4)

where, the Cauchy stress \( \sigma \) can also be expressed in terms of the constitutive relation matrix \( C \) as

\[ \sigma = C \epsilon \] (3.5)

Substituting 3.5 into 3.3 and 3.4 yields

\[ s = C^d \epsilon \] (3.6)

\[ p = C^v \epsilon \] (3.7)

where, the constitutive relation \( C \) is decomposed into the matrices \( C^d \) and \( C^v \):

\[ C^d = M^d C \] (3.8)

\[ C^v = M^v C \] (3.9)

Matrix decompositions 3.3-3.5 can be checked through

\[ \sigma = s + p = M^d \sigma + M^v \sigma = (M^d + M^v) \sigma = I \sigma = \sigma \] (3.10)
3.2 Decomposition of strain and compliance relation in matrix form

where, $I$ in 3.10 is the identity matrix of size 6.

The matrix decompositions 3.6-3.9 can be also checked through

$$\sigma = s + p$$
$$= C^d \epsilon + C^v \epsilon$$
$$= (M^d C + M^v C) \epsilon$$
$$= (M^d + M^v) C \epsilon$$
$$= C \epsilon$$  

(3.11)

3.2. Decomposition of strain and compliance relation in matrix form

Similarly, we have the matrix decompositions for strain and compliance relation as

$$\epsilon^d = M^d \epsilon$$  
$$\epsilon^v = M^v \epsilon$$  

(3.12)

$$\epsilon^d = D^d \sigma$$  
$$\epsilon^v = D^v \sigma$$  

(3.13)

(3.14)

(3.15)

and

$$D^d = M^d D$$  
$$D^v = M^v D$$  

(3.16)

(3.17)

In 3.12-3.17, $\epsilon^d$, $\epsilon^v$, $D^d$, and $D^v$ represent the deviatoric strain, the volumetric strain, the deviatoric compliance relation, and the volumetric compliance relation in a vector and matrix form, respectively.

4. Properties of multiplying decomposition method

So far, we have shown the multiplying decomposition method for stress, strain, constitutive relation, and compliance relation in both tensor and matrix forms. The method just takes the multiplication decomposers $M^d_{ijkl}$ (or $M^d$) and $M^v_{ijkl}$ (or $M^v$) and we will see their properties in this Section. As we shall see, the multiplication decomposition is also valid to decompose strain energy density and compatible with physical meaning.
4.1 Decomposition of strain energy

With the multiplying decomposition method, strain energy density \( u \) can be written as

\[
\begin{align*}
\frac{\partial}{\partial x_i} = & \frac{1}{2} C_{ijkl} \epsilon_{ij} \epsilon_{kl} \\
= & \frac{1}{2} (C_{ijkl}^d + C_{ijkl}^d) \epsilon_{ij} \epsilon_{kl} \\
= & \frac{1}{2} (s_{kl} + p_{kl}) (\epsilon'_{kl} + \epsilon^M_{kl}) \\
= & \frac{1}{2} s_{kl} \epsilon'_{kl} + \frac{1}{2} p_{kl} \epsilon^M_{kl}
\end{align*}
\]

(4.1)

While deriving 4.1, we use the relations

\[
\begin{align*}
s_{kl} \epsilon^M_{kl} & = 0 \\
p_{kl} \epsilon_{kl} & = 0
\end{align*}
\]

(4.2)

(4.3)

since we have

\[
\begin{align*}
s_{kl} \epsilon^M_{kl} & = s_{kl} \epsilon_M \delta_{kl} \\
& = \epsilon_M s_{kk} \\
& = 0
\end{align*}
\]

(4.4)

and

\[
\begin{align*}
p_{kl} \epsilon'_{kl} & = p \delta_{kl} (\epsilon_{kl} - \epsilon_M \delta_{kl}) \\
& = p (\epsilon_{kk} - \epsilon_M \delta_{kk}) \\
& = 0
\end{align*}
\]

(4.5)

In 4.1, the multiplying decomposition method also decomposes the strain energy density \( u \) into deviatoric strain energy density \( \frac{1}{2} s_{kl} \epsilon'_{kl} \) and volumetric strain energy density \( \frac{1}{2} p_{kl} \epsilon^M_{kl} \). Such strain energy density decomposition can also be derived from \( u = \frac{1}{2} D_{ijkl} \sigma_{ij} \sigma_{kl} \).
4.2 Compatibility with physical meaning

With the multiplying decomposition method, one may consider this can be also used for

- the evaluation of total stress from either deviatoric stress (strain) or pressure (volumetric strain)
- the evaluation of total strain from either deviatoric strain (stress) or volumetric strain (pressure)

However, this is not possible because both the decomposition matrices $M_d$ and $M_v$ are singular. The eigenvalues of $M_d$ are $\lambda_1 = 0$ and $\lambda_2 = 1$ with multiplicity 5 while the eigenvalues of $M_v$ are $\lambda_1 = 0$ with multiplicity 5 and $\lambda_2 = 1$. In fact, the decomposition matrices $M_d$ and $M_v$ are projection matrices. And having the eigenvalue of multiplicity 5 in $M_d$ and $M_v$ are natural since $M_d$ leaves only five deviatoric stress (or deviatoric strain) components and $M_v$ leaves only one pressure (or volumetric strain) component when multiplied to stress (or strain). Thus, we cannot have $[M_d]^{-1}$ and $[M_v]^{-1}$ in 3.3-3.4 and 3.12-3.13. Consequently, we also cannot have $[C_d]^{-1}, [C_v]^{-1}, [D_d]^{-1}$, and $[D_v]^{-1}$ in 3.8-3.9 and 3.16-3.17.

In physical viewpoint, the properties of multiplying decomposition can be interpreted as:

- Total stress results in both deviatoric and volumetric strain but not vice versa
- Total strain results in both deviatoric stress and pressure but not vice versa

5. Applications

The multiplying decomposition method can be directly used to formulate both FEM and BEM. In this Section, we show its canonical application to elastostatics.

5.1. Finite element formulation

In FEM for elastostatics (see Strang and Fix (1973); Cook et al. (2002); Bathe (1996); Hughes (2000); Braess (2001)), the element stiffness matrix $K_e$ is determined by
5.1 Finite element formulation

\[ K_e = \int_{\Omega_e} B^T C B \, d\Omega \]  

(5.1)

where, \( B \), \( C \), and \( \Omega_e \) are the strain-displacement matrix, the constitutive relation matrix, and the domain of the element, respectively.

After assembling all the stiffness matrices of each element and accounting for boundary conditions to describe the given structure, the global system of equations is given by

\[ K \, u = f \]  

(5.2)

In 5.2, \( K \), \( u \) and \( f \) represent the global stiffness matrix, nodal displacements and nodal forces in a vector form, respectively.

With the adoption of the multiplying decomposition, we can reformulate the element stiffness matrix \( K_e \) in 5.1 as

\[ K_e = \int_{\Omega_e} B^T (C^d + C^v) B \, d\Omega \]

\[ = \int_{\Omega_e} B^T C^d B \, d\Omega + \int_{\Omega_e} B^T C^v B \, d\Omega \]  

(5.3)

\[ = K_e^d + K_e^v \]

where \( K_e^d \) and \( K_e^v \) are decomposed element deviatoric stiffness matrix and element volumetric stiffness matrix.

Also, the global system of equations can be written as

\[(K^d + K^v) \, u = f \]  

(5.4)

\[ K = K^d + K^v \]  

(5.5)

As in 5.5, with the multiplying decomposition, the global stiffness matrix \( K \) is decomposed into the global deviatoric stiffness matrix \( K^d \) and the global volumetric stiffness matrix \( K^v \), where \( K^d \) results from assembling all the \( K_e^d \) and \( K^v \) results from assembling all the \( K_e^v \).

Separate evaluation of \( K_e^d \) and \( K_e^v \) by the multiplying decomposition allows “selective integration technique” in a direct way and some useful applications are listed below.
5.2 Boundary element formulation

- Using the reduced integration for volumetric part and full integration for deviatoric part: in \textit{(nearly) incompressible cases, selectively reduced-integration technique (SRI)} is used for 1st order linear isoparametric elements (4-node elements in two dimensions and 8-node elements in three dimensions), to prevent mesh locking and get accurate solution for \textit{(nearly) incompressible cases} (see Malkus and Hughes (1978); Hughes (2005); Doll et al. (2000); Liu et al. (1994, 1998)).

- Using the reduced integration for deviatoric part and full integration for volumetric part: \textit{shear locking} can be also prevented by \textit{SRI}. (See Bathe (1996); Cook et al. (2002); Braess (2001); Hughes (2000))

- Formulation of plasticity: when von Mises \textit{plasticity} occurs (see 1.9, 1.11), \( K^d \) is evaluated \textit{iteratively} while \( K^v \) remains constant since \( \epsilon_{kk} = 0 \) (See Dunne and Petrinic (2005); Lubliner (1990)). With the adoption of the \textit{multiplying decomposition}, we can separately evaluate \( K^d \) without evaluating \( K^v \) during plastic evolution (computation can be reduced).

5.2 Boundary element formulation

In BEM, an integral equation for interior stress in elasticity can be written as (see Banerjee (1994); Beer et al. (2010); Ang (2007); Wrobel and Aliabadi (2002))

\[
\sigma_{ij}(\xi) = \int_S \left( G^\sigma_{kij}(x, \xi) t_k(x) - F^\sigma_{kij}(x, \xi) u_k(x) \right) dS(x)
\]  

(5.6)

where \( G^\sigma_{kij} \) and \( F^\sigma_{kij} \) are traction and displacement kernel function for stress respectively, and \( S \) means the boundary of a given problem domain.

With the \textit{multiplying decomposition}, deviatoric stress can be derived as

\[
s_{mn}(\xi) = M^d_{ijmn} \sigma_{ij}(\xi)
\]

\[ = M^d_{ijmn} \int_S \left( G^\sigma_{kij}(x, \xi) t_k(x) - F^\sigma_{kij}(x, \xi) u_k(x) \right) dS(x)
\]

(5.7)

Similarly, pressure can be derived as
\[ p_{mn}(\xi) = M_{ijmn}^v \int_S \left( G_{kij}^\sigma(x, \xi)t_k(x) - F_{kij}^\sigma(x, \xi)u_k(x) \right) dS(x) \]  (5.8)

Similarly, decomposed strains are as follows

\[ \epsilon'_{mn}(\xi) = M_{ijmn}^d \int_S \left( G_{kij}^\prime(x, \xi)t_k(x) - F_{kij}^\prime(x, \xi)u_k(x) \right) dS(x) \]  (5.9)

\[ \epsilon^M_{mn}(\xi) = M_{ijmn}^\nu \int_S \left( G_{kij}^\nu(x, \xi)t_k(x) - F_{kij}^\nu(x, \xi)u_k(x) \right) dS(x) \]  (5.10)

where \( G_{kij}^\sigma \) and \( F_{kij}^\sigma \) are traction and displacement kernel function for strain respectively.

6. Conclusions

A simple, clear, and widely applicable way to decompose stress/strain and constitutive/compliance relations is suggested in both tensor and matrix forms: multiplication decomposition. The method is also applicable to decompose strain energy density along with proper physical meaning.

We consider here the application of multiplying decomposition to elastostatics in FEM and BEM formulation, which illustrates the elegance of this approach. Clearly, however, the multiplying decomposition is quite general and can be applied readily to elastoplasticity, viscoplasticity, fluid mechanics and more broadly throughout mechanics. In addition, we anticipate that the multiplying decomposition method developed here will provide an interesting foundation for the development of novel analytic and computational methods.

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