THE WEAK BANACH-SAKS PROPERTY OF THE SPACE \((L^p_n)^m\)

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ABSTRACT. In this paper we show the weak Banach-Saks property of the Banach vector space \((L^p_n)^m\) generated by \(m\) \(L^p_n\)-spaces for \(1 \leq p < +\infty\), where \(m\) is any given natural number. When \(m = 1\), this is the famous Banach-Saks-Szlenk theorem. By use of this property, we also present inequalities for integrals of functions that are the composition of nonnegative continuous convex functions on a convex set of a vector space \(\mathbb{R}^m\) and vector-valued functions in a weakly compact subset of the space \((L^p_n)^m\) for \(1 \leq p < +\infty\) and inequalities when these vector-valued functions are in a weakly* compact subset of the product space \((L^\infty_n)^m\) generated by \(m\) \(L^\infty_n\)-spaces.

1. Introduction

We begin with some notations and definitions used throughout this paper. \(m\) and \(n\) are natural numbers, \(\mathbb{R}\) denotes the real number system, \(\mathbb{R}^n\) is the usual vector space of real \(n\)-tuples \(x = (x_1, x_2, \cdots, x_n)\), \(\mu\) is a nonnegative Lebesgue measure of \(\mathbb{R}^n\); \(L^p_\mu(\mathbb{R}^n)\) represents a Banach space of any measurable function \(\hat{u} = \hat{u}(x)\) with its finite norm

\[\|\hat{u}\|_p = \left(\int_{\mathbb{R}^n} |\hat{u}(x)|^p d\mu\right)^{\frac{1}{p}}\]

for any given \(p \in [1, +\infty)\), and \((L^p_\mu(\mathbb{R}^n))^m\) denotes a Banach vector space where each measurable vector-valued function \(u = u(x)\) with \(m\) components \(\hat{u}(j) = \hat{u}^j(x)\) \((j = 1, 2, \cdots, m)\) in \(L^p_\mu(\mathbb{R}^n)\) is given by \(u = (\hat{u}^1, \hat{u}^2, \cdots, \hat{u}^m)\) and its norm is defined by

\[\|u\|_p = \left(\sum_{j=1}^{m} \|\hat{u}^j\|_p^p\right)^{\frac{1}{p}};\]

similarly, \(L^\infty_\mu(\mathbb{R}^n)\) represents a Banach space of any measurable function \(\hat{u} = \hat{u}(x)\) with its finite norm

\[\|\hat{u}\|_\infty = \text{ess sup}_{x \in \mathbb{R}^n} |\hat{u}(x)|\] (or say \(\|\hat{u}\|_\infty = \inf_{\mu(E^c) = 0} \sup_{x \in E} |\hat{u}(x)|\))

where \(E^c\) represents the complement set of \(E\) in \(\mathbb{R}^n\), and \((L^\infty_\mu(\mathbb{R}^n))^m\) denotes a Banach vector space where each measurable vector-valued function \(u = u(x)\) with \(m\) components \(\hat{u}_j = \hat{u}_j(x)\) \((i = 1, 2, \cdots, m)\) in \(L^\infty_\mu(\mathbb{R}^n)\) is denoted by \(u = (\hat{u}^1, \hat{u}^2, \cdots, \hat{u}^m)\) and its norm is defined by

\[\|u\|_\infty = \sum_{j=1}^{m} \|\hat{u}^j\|_\infty\]

If a function \(u\) and a sequence \(\{u_i\}_{i=1}^{+\infty}\) in \((L^p_\mu(\mathbb{R}^n))^m\) are assumed to satisfy the fact that \(\lim_{i \to \infty} \|u_i - u\|_p = 0\), then this sequence \(\{u_i\}_{i=1}^{+\infty}\) is said to be strongly convergent in \((L^p_\mu(\mathbb{R}^n))^m\) to \(u\). Similarly, if a function \(u\) and a sequence \(\{u_i\}_{i=1}^{+\infty}\) in \((L^\infty_\mu(\mathbb{R}^n))^m\) are assumed to have the property that \(\lim_{i \to \infty} \|u_i - u\|_\infty = 0\), then this sequence \(\{u_i\}_{i=1}^{+\infty}\) is said to be strongly convergent in \((L^\infty_\mu(\mathbb{R}^n))^m\) to \(u\).

Besides the convergence given above, we consider a definition of weak convergence of a sequence in \((L^p_\mu(\mathbb{R}^n))^m\). Assume that \(q = p/(p-1)\) as \(p \in (1, +\infty)\) and that \(q = \infty\) as
p = 1. If a function \( \hat{u} \) and a sequence \( \{ \hat{u}_i \}_{i=1}^{\infty} \) in \( L_p^\mu(\mathbb{R}^n) \) have the following relation:

\[
\lim_{i \to +\infty} \int_{\mathbb{R}^n} \hat{u}_i \hat{v} d\mu = \int_{\mathbb{R}^n} \hat{u} \hat{v} d\mu
\]

for all \( \hat{v} \in L^q_\mu(\mathbb{R}^n) \), then the sequence \( \{ \hat{u}_i \}_{i=1}^{\infty} \) is said to be weakly convergent in \( L^p_\mu(\mathbb{R}^n) \) to \( \hat{u} \). If \( \{ \hat{u}_i^{(j)} \}_{i=1}^{\infty} \) is weakly convergent in \( L^p_\mu(\mathbb{R}^n) \) to \( \hat{u}^{(j)} \) for all \( j = 1, 2, \ldots, m \) as \( i \) goes to \( \infty \), then a sequence \( \{ u_i = (\hat{u}_i^{(1)}, \hat{u}_i^{(2)}, \ldots, \hat{u}_i^{(m)}) \}_{i=1}^{\infty} \) is said to be weakly convergent in \( (L^p_\mu(\mathbb{R}^n))^m \) to \( u = (\hat{u}^{(1)}, \hat{u}^{(2)}, \ldots, \hat{u}^{(m)}) \).

Similarly, we introduce a definition of weak* convergence of a sequence in \( L^\infty_\mu(\mathbb{R}^n) \). If a function \( \hat{u} \) and a sequence \( \{ \hat{u}_i \}_{i=1}^{\infty} \) in \( L^\infty_\mu(\mathbb{R}^n) \) satisfy the equality \( (\) \) for all \( \hat{v} \in L^q_\mu(\mathbb{R}^n) \), then the sequence \( \{ \hat{u}_i \}_{i=1}^{\infty} \) is said to be weakly* convergent in \( L^\infty_\mu(\mathbb{R}^n) \) to \( \hat{u} \). If \( \{ \hat{u}_i^{(j)} \}_{i=1}^{\infty} \) is weakly* convergent in \( L^\infty_\mu(\mathbb{R}^n) \) to \( \hat{u}^{(j)} \) for all \( j = 1, 2, \ldots, m \) as \( i \) goes to \( \infty \), then a sequence \( \{ u_i = (\hat{u}_i^{(1)}, \hat{u}_i^{(2)}, \ldots, \hat{u}_i^{(m)}) \}_{i=1}^{\infty} \) is said to be weakly* convergent in \( (L^\infty_\mu(\mathbb{R}^n))^m \) to \( u = (\hat{u}^{(1)}, \hat{u}^{(2)}, \ldots, \hat{u}^{(m)}) \).

Assume that \( Y \) is a Banach space. \( Y \) is said to be of the weak Banach-Saks property if any sequence \( \{ y_i \}_{i=1}^{\infty} \) weakly convergent in \( Y \) to \( y \) contains a subsequence \( \{ y_{n_k} \}_{k=1}^{\infty} \) such that \( \sum_{k=1}^{\infty} y_{n_k} = y \).

Banach and Saks \( [1] \) first proved that \( L_p^\mu(\mathbb{R}^n) \) has the weak Banach-Saks property for the \( 1 < p < +\infty \) case in 1930 and then the similar result for \( L^\infty_\mu(\mathbb{R}^n) \) was showed by Szlenk \( [8] \) in 1965. This result about \( L_p^\mu(\mathbb{R}^n) \) is the famous Banach-Saks-Szlenk theorem. Now there is not yet this result about the Banach vector space \( (L_p^\mu(\mathbb{R}^n))^m \) when \( m \neq 1 \). The aim of this paper is to extend the Banach-Saks-Szlenk theorem to the case of the vector space \( (L_p^\mu(\mathbb{R}^n))^m \) generated by \( m \) \( L_p^\mu(\mathbb{R}^n) \)-spaces for \( 1 \leq p < +\infty \) and show that \( (L_p^\mu(\mathbb{R}^n))^m \) has the weak Banach-Saks property for any fixed natural number \( m \). An application of this property is to show inequalities for integrals of functions that are the composition of nonnegative continuous convex functions on a convex set of a vector space \( \mathbb{R}^m \) and vector-valued functions in a weakly compact subset of a Banach vector space generated by \( m \) \( L_p^\mu \)-spaces for \( 1 \leq p < +\infty \) and inequalities when these vector-valued functions are in a weakly* compact subset of a Banach vector space generated by \( m \) \( L_p^\infty \)-spaces.

2. The Weak Banach-Saks Property

A detail description of the weak Banach-Saks property of \( (L_p^\mu(\mathbb{R}^n))^m \) for any fixed natural number \( m \) is as follows:

**Theorem 1.** Given a real number \( p \) in \([1, +\infty)\). Assume that a sequence \( \{ u_i = u_i(x) \}_{i=1}^{\infty} \) converges weakly in \( (L_p^\mu(\mathbb{R}^n))^m \) to \( u = u(x) \). Then this sequence contains a subsequence \( \{ u_{i_k} \}_{k=1}^{\infty} \) with its arithmetic means \( \frac{1}{k} \sum_{i=1}^{k} u_{i_k} \) strongly convergent in \( (L_p^\mu(\mathbb{R}^n))^m \) to \( u \) as \( k \) goes to infinity.

We can below show Theorem \( [1] \) using the two following techniques with only minor adjustments: one is given by Banach and Saks \( [1] \) for any fixed \( p \in (1, +\infty) \), and another by Szlenk \( [8] \) in the case when \( p = 1 \).

To prove Theorem \( [1] \) for any fixed \( p \in (1, +\infty) \), we have first to introduce the following lemma:

**Lemma 1 (\( [1] \)).** Let \( a \) and \( b \) be any real numbers and \( 1 < p < +\infty \). Then

\[
|a + b|^p \leq |a|^p + p|a|^{p-1}|sgn(a)|b + A|b|^p + B(p, a, b).
\]
Here, $A$ is a positive constant independent of $a$ and $b$; $\text{sgn}(\tau)$ is defined as follows: $\text{sgn}(0) = 0$, $\text{sgn}(\tau) = 1$ as $\tau > 0$ and $\text{sgn}(\tau) = -1$ as $\tau < 0$; $B(p, a, b) = 0$ as $p \in (1, 2]$ and $B(p, a, b) = \sum_{i=2}^{E(p)} |a|^{p-1}|b|^i$ as $p \in (2, +\infty)$, where $E(p)$ is the largest natural number less than $p$.

The proof of this lemma can be found in [1]. Using the inequality [2], we can get a similar result to that given by Banach and Saks [1]. This result is as follows:

**Lemma 2.** Assume that $p > 1$ and that a sequence $\{\hat{u}_i = \hat{u}_i(x)\}_{i=1}^{\infty}$ in $L^p_\mu(\mathbb{R}^n)$ satisfies
\[
\int_{\mathbb{R}^n} |\hat{u}_i(x)|^p d\mu \leq 1
\]for all $i \geq 1$. Put $\hat{s}_k(x) = \sum_{i=1}^k \hat{u}_i(x)$. Then
\[
\int_{\mathbb{R}^n} |\hat{s}_k(x)|^p d\mu \leq C(k) + p \int_{\mathbb{R}^n} |\hat{s}_{k-1}(x)|^{p-1}[\text{sgn}(\hat{s}_{k-1}(x))]|\hat{u}_k(x)| d\mu + \int_{\mathbb{R}^n} |\hat{s}_{k-1}(x)|^p d\mu, \tag{4}
\]
where, $C(k) = A + Bk^{p-2}$, $A$ and $B$ are positive constants independent of $k$ and $\{\hat{u}_i\}_{i=1}^{\infty}$, $\text{sgn}(\tau)$ is defined as in Lemma [1].

**Proof.** Insert $a = \hat{s}_{k-1}(x)$ and $b = \hat{u}_k(x)$ into the inequality [2] and integrate all its terms over the space $\mathbb{R}^n$. Then, by (3), we can know that (4) holds in the $1 < p \leq 2$ case and that
\[
\int_{\mathbb{R}^n} |\hat{s}_k(x)|^p d\mu \leq \int_{\mathbb{R}^n} |\hat{s}_{k-1}(x)|^p d\mu + p \int_{\mathbb{R}^n} |\hat{s}_{k-1}(x)|^{p-1}[\text{sgn}(\hat{s}_{k-1}(x))]|\hat{u}_k(x)| d\mu
\]
\[+ \sum_{i=2}^{E(p)} \int_{\mathbb{R}^n} |\hat{s}_{k-1}(x)|^{p-1}|\hat{u}_k(x)|^i d\mu + A \tag{5}
\]
for all $p > 2$. Notice that $\int_{\mathbb{R}^n} |\hat{s}_{k-1}(x)|^{p-1}|\hat{u}_k(x)|^i d\mu \leq k^{p-2}$ for all $i \leq p$; this can be obtained by first using the Hölder inequality and then the Minkowski one with the help of the condition (3). Take $B = \sum_{i=2}^{E(p)} \left( \frac{p}{i} \right)$. It can be then found that (5) gives (4) for $p > 2$. This hence completes our proof.

We below give the proof of Theorem [11] for any fixed $p \in (1, +\infty)$. To do this, it suffices to consider the case when $m = 2$. Let us first denote all the vector-valued functions $u_i$ of this sequence in $(L^p_\mu(\mathbb{R}^n))^2$ by $u_i = (\hat{u}_i^{(1)}, \hat{u}_i^{(2)})$, where $\hat{u}_i^{(1)} = \hat{u}_i^{(1)}(x)$ and $\hat{u}_i^{(2)} = \hat{u}_i^{(2)}(x)$ represent two functions in $L^p_\mu(\mathbb{R}^n)$ for any natural number $i$. Since any weak convergent sequence in $L^p_\mu(\mathbb{R}^n)$ is bounded, we may first assume without loss of generality that all the functions $u_i$ of the considered sequence satisfy
\[
\|u_i\|^p \leq 1 \tag{6}
\]for all $i \geq 1$. We may also assume without loss of generality that this sequence $\{u_i\}_{i=1}^{+\infty}$ converges weakly in $(L^p_\mu(\mathbb{R}^n))^2$ to zero. Then, by recursion, we can determine a subsequence $\{u_i = (\hat{u}_i^{(1)}, \hat{u}_i^{(2)})\}_{i=1}^{+\infty}$ ($i_1 = 1$). This recursive process can be roughly divided into two steps and they are described as follows. The first step to do is to take $s_k^{(1)}(x) = \sum_{i=1}^k \hat{u}_i^{(1)}(x)$ for $j = 1, 2$ under the assumption that $k$ previous terms $\{u_i = (\hat{u}_i^{(1)}, \hat{u}_i^{(2)})\}_{i=1}^k$ of this subsequence is determined. It can be then known that $s_k^{(j)}(x) \in L^p_\mu(\mathbb{R}^n)$ and that $|s_k^{(j)}(x)|^{p-1}[\text{sgn}(s_k^{(j)}(x))] \in L^p_\mu(\mathbb{R}^n)$. Since these functions $\hat{u}_i^{(j)}$ converge weakly in $L^p_\mu(\mathbb{R}^n)$ to zero for $j = 1, 2$, there exists a natural
number \( i_k \) such that
\[
\int_{\mathbb{R}^n} |\hat{s}_{k}^{(j)}(x)|^{p-1}[sgn(\hat{s}_{k}^{(j)}(x))]\hat{u}_{i_k}^{(j)}(x)\,d\mu \leq 1
\]  
for all \( i > i_k \) and \( j = 1, 2 \); thus the second one is to define the subscript \( i_{k+1} \) of the next term to be one of all the natural numbers \( i \) satisfying the condition given by (7).

Then, by (7), we can know that for all \( k > 1 \) and \( j = 1, 2 \),
\[
\int_{\mathbb{R}^n} |\hat{s}_{k-1}^{(j)}(x)|^{p-1}[sgn(\hat{s}_{k-1}^{(j)}(x))]\hat{u}_{i_k}^{(j)}(x)\,d\mu \leq 1.
\]  
Combining (6) and (8) and using Lemma 2, we can show that for \( j = 1, 2 \),
\[
\int_{\mathbb{R}^n} |\hat{s}_k^{(j)}(x)|^p\,d\mu \leq (A + p)k + Bk^{p-2} + 1,
\]  
thus giving
\[
\lim_{k \to \infty} \int_{\mathbb{R}^n} \left| \frac{\hat{s}_k^{(j)}(x)}{k} \right|^p \,d\mu = 0 \text{ for } j = 1, 2.
\]  
This hence completes our proof of Theorem 1 for any given \( p \in (1, +\infty) \).

Now it remains to prove Theorem 1 when \( p = 1 \). To do this, we first recall a lemma as follows:

**Lemma 3** ([8]). Assume that \( \hat{u}_i \) belongs to the Banach space \( L[0, 1] \) for any natural number \( i \) and converges weakly to zero as \( i \) goes to infinity. Then, for any given \( \varepsilon > 0 \), there exists a sequence of indexes \( i_r \) such that
\[
\lim_{k \to \infty} \sup_{1 \leq k < \cdots < k_m} \frac{1}{k} \| \sum_{r=1}^{k} \hat{u}_{i_r} \|_L \leq \varepsilon \text{ where } \| \cdot \|_L \text{ represents the norm of the Banach space } L[0, 1].
\]

Lemma 3 and its proof were shown by Szlenk [8] in 1965. By using a similar proof to that given by Szlenk, it can be found that Lemma 3 still holds if \( L[0, 1] \) is replaced by the Banach space \( L([0, 1]^n) \). Since \( L(\mathbb{R}^n) \) is isometric to \( L([0, 1]^n) \) (see [9], Page 83), we can easily deduce that

**Lemma 4.** Assume that \( \hat{u}_i \) belongs to the Banach space \( L(\mathbb{R}^n) \) for any natural number \( i \) and converges weakly to zero as \( i \) goes to infinity. Then, for any given \( \varepsilon > 0 \), there exists a sequence of indexes \( i_r \) such that
\[
\lim_{k \to \infty} \sup_{1 \leq k < \cdots < k_m} \frac{1}{k} \| \sum_{r=1}^{k} \hat{u}_{i_r} \|_1 \leq \varepsilon.
\]

Using Lemma 4, we can get the following result:

**Lemma 5.** Assume that \( u_i \) belongs to the Banach space \( (L(\mathbb{R}^n))^m \) for any natural number \( i \) and converges weakly to zero as \( i \) goes to infinity. Then, for any given \( \varepsilon > 0 \), there exists a sequence of indexes \( i_r \) such that
\[
\lim_{k \to \infty} \sup_{1 \leq k < \cdots < k_m} \frac{1}{k} \| \sum_{r=1}^{k} u_{i_r} \|_1 \leq \varepsilon.
\]

We can below prove Theorem 1 when \( p = 1 \). To do this, it suffices to consider the case of \( u = 0 \). By Lemma 3, for any given \( l \geq 1 \), there exists a sequence of indexes \( i_{l,r} \) such that
\[
\lim_{k \to \infty} \sup_{1 \leq k < \cdots < k_m} \frac{1}{k} \| \sum_{r=1}^{k} u_{i_{l,r}} \|_1 \leq 1 \frac{1}{l}.
\]
Assume that the sequence of indexes $i_{i+1,r}$ is a subsequence of the sequence of indexes $i_{i,r}$. Denote by $\{u_{i,r}\}_{r=1}^{+\infty}$ a sequence of indexes $i_r = i_{r,r}$ corresponding to the condition (10). Then we can know that this sequence $\{u_{i,r}\}_{r=1}^{+\infty}$ satisfies
\[
\frac{1}{k} \sum_{r=1}^{k} u_{i,r} \to \infty \leq \frac{1}{k} \sum_{r=1}^{l} u_{i,r} + \frac{1}{k-l} \sum_{r=1}^{k-l} u_{i,r} \to \infty
\]
for all $k > l$. It follows that
\[
\lim_{k \to \infty} \frac{1}{k} \sum_{r=1}^{k} u_{i,r} \leq \lim_{k \to \infty} \frac{1}{k-l} \sum_{r=1}^{k-l} u_{i,r} = \lim_{k \to \infty} \frac{1}{k} \sum_{r=1}^{k} u_{i,r}.
\]
(11)

Since $\{u_{i,r}\}_{r=1}^{+\infty}$ is a subsequence of this sequence $\{u_{i,r}\}_{r=1}^{+\infty}$, we have
\[
\frac{1}{k} \sum_{r=1}^{k} u_{i,r} \leq \sup_{s_1 < \cdots < s_k} \frac{1}{k} \sum_{r=1}^{k} u_{i,r}.
\]
(12)

Combining (10), (11) and (12), we can know that
\[
\lim_{k \to \infty} \frac{1}{k} \sum_{r=1}^{k} u_{i,r} \leq \frac{1}{l}
\]
for $l = 1, 2, \cdots$.

This implies that $\frac{1}{k} \sum_{r=1}^{k} u_{i,r}$ converges strongly in the Banach space $(L(R^p))^m$ to zero as $k$ tends to infinity. Our proof is hence finished.

We can also extend Theorem 1 to a more general case, that is,

**Theorem 2.** Given a measure space $(X, \mathcal{A}, \mu)$ and a real number $p$ in $[1, +\infty)$. Assume that $\{\hat{u}^{(i)}\}_{i=1}^{+\infty}$ converges weakly in $(L^p(X))^m$ to $\hat{u}^{(i)}$ for $j = 1, 2, \cdots, m$. Take $u = (\hat{u}^{(1)}, \hat{u}^{(2)}, \cdots, \hat{u}^{(m)})$ and $u_i = (\hat{u}_i^{(1)}, \hat{u}_i^{(2)}, \cdots, \hat{u}_i^{(m)})$ for any natural number $i$. Then this sequence $\{u_i\}_{i=1}^{+\infty}$ contains a subsequence $\{u_{i,r}\}_{r=1}^{+\infty}$ with its arithmetic means $\frac{1}{k} \sum_{r=1}^{k} u_{i,r}$ strongly convergent in $(L^p(X))^m$ to $u$ as $k$ goes to infinity.

When $m = 1$, this result appears in the book of Benedetto [2] and there is an explanation of its proof, that is, it is the same as given by Banach and Saks for any $p \in (1, +\infty)$ and by Szlenk in the case when $p = 1$. We below give a simple proof of Theorem 2. Notice that the separable space $L^p(X)$ is isometric to $L^p[0, 1]$ for any $p \in [1, +\infty)$ (see [3]). It is then clear that $(L^p(X))^m$ is also isometric to $(L^p[0, 1])^m$ for any fixed natural number $m$. By Theorem 1, Theorem 2 thus follows.

3. **APPLICATION TO INEQUALITIES FOR INTEGRALS**

By use of the weak Banach-Saks property of $(L^p)^m$, we can show inequalities for integrals of functions which are the composition of nonnegative continuous convex functions on a convex set of a vector space $R^m$ and vector-valued functions in a weakly compact subset of a Banach vector space generated by $m \ L^p$-spaces for any given $p \in [1, +\infty)$. That is the following

**Theorem 3.** Suppose that a sequence $\{u_{i+1}^{+\infty}\}_{i=1}^{+\infty}$ weakly converges in $(L^p(R^n))^m$ to $u$ as $i$ goes to infinity, where $p \in [1, +\infty)$ and $m$ and $n$ are two positive integers. Assume that all the values of $u$ and $u_i$ ($i = 1, 2, 3, \cdots$) belong to a convex set $K$ in $R^m$ and that $f(w)$ is a nonnegative continuous convex function from $K$ to $R$. Then
\[
\lim_{i \to +\infty} \int_{\Omega} f(u_i) d\mu \geq \int_{\Omega} f(u) d\mu
\]
for any measurable set $\Omega \subseteq R^n$. (13)
The estimates of integrals of this kind of composite function is interesting and important in many application areas such as the existence of solutions of differential equations (e.g., see [3] and [10]). A similar result was shown by Jiang et. al [4] if $K$ is assumed to be an open convex set of $\mathbb{R}^m$ instead; Egorov’s theorem is used into their proof except for the weak Banach-Saks property of $(L_p^m)^m$. Meanwhile, a simple proof of another similar one was given in [5] when $K$ is set to be $\mathbb{R}^m$; this proof requires the weak Banach-Saks property of $(L_p^m)^m$ but it does not give any proof of this property; it only shows the case when $m = 1$ in Theorem 2. The former device is valid for only an open convex set $K$ while the latter one is suitable to show inequalities for integrals of these composite functions in a more general case, or more precisely speaking, this case is for any convex set $K$ in $\mathbb{R}^m$. Therefore it is still very necessary to show Theorem 3 and its proof.

It is worth mentioning that some properties of convex functions and weakly compact sets can be found in the literature (e.g., see [2], [6], [7], [9] and [11]).

**Proof of Theorem 3.** Put $\alpha_i = \int_{\Omega} f(u_i)d\mu$ ($i = 1, 2, \cdots$) and $\alpha = \lim_{i \to +\infty} \int_{\Omega} f(u_i)d\mu$ for all $\Omega \subseteq \mathbb{R}^n$. Then there exists a subsequence of $\{\alpha_i\}_{i=1}^{+\infty}$ such that this subsequence, denoted without loss of generality by $\{\alpha_i\}_{i=1}^{+\infty}$, converges to $\alpha$ as $i \to +\infty$.

Since $u_i$ converges weakly in $(L_p^m(\mathbb{R}^n))^m$ to $u$ for $1 \leq p < +\infty$, by Theorem 1 it is easy to see that there exists a subsequence $\{u_{i_j} : j = 1, 2, \cdots\}$ such that $\frac{1}{k} \sum_{j=1}^{k} u_{i_j} \to u$ in $(L_p^m(\mathbb{R}^n))^m$ for $1 \leq p < +\infty$ as $k \to +\infty$. Thus there exists a subsequence of $\{\frac{1}{k} \sum_{j=1}^{k} u_{i_j} : k = 1, 2, \cdots\}$ such that this subsequence (also denoted without loss of generality by $\{\frac{1}{k} \sum_{j=1}^{k} u_{i_j} : k = 1, 2, \cdots\}$) satisfies that, as $k \to +\infty$,

$$\frac{1}{k} \sum_{j=1}^{k} u_{i_j} \to u \text{ a.e. in } \mathbb{R}^n.$$  \hfill (14)

On the other hand, since all the values of $\{u_i\}_{i=1}^{+\infty}$ and $u$ belong to the convex set $K$ in $\mathbb{R}^m$ and $f(w)$ is a nonnegative continuous convex function from $K$ to $\mathbb{R}$, we have

$$f\left(\frac{1}{k} \sum_{j=1}^{k} u_{i_j}\right) \leq \frac{1}{k} \sum_{j=1}^{k} f(u_{i_j}).$$  \hfill (15)

By (15) and Fatou’s lemma, it follows that

$$\int_{\Omega} \lim_{k \to +\infty} f\left(\frac{1}{k} \sum_{j=1}^{k} u_{i_j}\right)d\mu \leq \lim_{k \to +\infty} \frac{1}{k} \sum_{j=1}^{k} \int_{\Omega} f(u_{i_j})d\mu.$$  \hfill (16)

Combining (14) and (16), we can know that

$$\int_{\Omega} f(u)d\mu \leq \lim_{k \to +\infty} \frac{1}{k} \sum_{j=1}^{k} \int_{\Omega} f(u_{i_j})d\mu \equiv \lim_{k \to +\infty} \frac{1}{k} \sum_{j=1}^{k} \alpha_{i_j}.$$  \hfill (17)

Finally, by using the property of the convergence of $\alpha_i$ to $\alpha$, (17) gives (13). This completes our proof. \hfill $\Box$

Furthermore, we can give the following similar result for weakly* convergent sequences in $(L_p^\infty(\mathbb{R}^n))^m$:

**Theorem 4.** Assume that a sequence $\{u_i\}_{i=1}^{+\infty}$ weakly* converges in $(L_p^\infty(\mathbb{R}^n))^m$ to $u$ as $i$ goes to infinity, where $m$ and $n$ are two positive integers. Assume that all the values
of $u$ and $u_i$ $(i = 1, 2, 3, \cdots)$ belong to a convex set $K$ of $\mathbb{R}^m$ and that $f(w)$ is a nonnegative continuous convex function from $K$ to $\mathbb{R}$. Then the inequality (13) holds for any measurable set $\Omega \subseteq \mathbb{R}^n$.

**Proof.** Put $\Omega_R = \Omega \cap \{ x : |x| < R, x \in \mathbb{R}^n \}$. Then $\Omega_R$ is a bounded measurable set in $\mathbb{R}^n$ for all the fixed positive real number $R$. Since $u_i \to u$ weakly* in $(L^\infty_\mu(\mathbb{R}^n))^m$, $u_i \to u$ weakly* in $(L^\infty_\mu(\Omega_R))^m$. Hence, by $L^\infty(\Omega_R) \subset L^1(\Omega_R)$, it can be easily known that $u_i \to u$ weakly in $(L^1(\Omega_R))^m$. Then, using the process of the proof of Theorem 3 we can get

$$\lim_{i \to +\infty} \int_{\Omega_R} f(u_i) d\mu \geq \int_{\Omega_R} f(u) d\mu.$$  \hfill (18)

It follows from the nonnegativity of the convex function $f$ that

$$\lim_{i \to +\infty} \int_\Omega f(u_i) d\mu \geq \int_\Omega f(u) d\mu.$$  \hfill (19)

Finally, by Lebesgue monotonous convergence theorem, as $R \to +\infty$, (19) implies (13). Our proof is completed. \hfill \Box

Also, removing the nonnegativity of $f(w)$ and assuming that the convex set $K$ is closed, by Mazur’s lemma (see [9] and [11]), we can deduce

**Theorem 5** ([4]). Assume that a sequence \( \{u_i\}_{i=1}^{+\infty} \) weakly* converges in \((L^\infty_\mu(\mathbb{R}^n))^m\) to $u$ as $i$ goes to infinity, where $m$ and $n$ are two positive integers. Assume that all the values of $u$ and $u_i$ $(i = 1, 2, 3, \cdots)$ belong to a closed convex set $K$ in $\mathbb{R}^m$ and that $f(w)$ is a continuous convex function from $K$ to $\mathbb{R}$. Then the inequality (13) holds for any bounded measurable set $\Omega \subset \mathbb{R}^n$.

Theorem 5 is in fact an extension of a result given by Ying [10] (or see [4] and [5]) and its detailed proof can be found in [4].

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