Analytical treatment of critical collapse in 2+1 dimensional AdS spacetime: a toy model

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Abstract

We present an exact collapsing solution to 2+1 gravity with a negative cosmological constant minimally coupled to a massless scalar field, which exhibits physical properties making it a candidate critical solution. We discuss its global causal structure and its symmetries in relation with those of the corresponding continuously self-similar solution derived in the $\Lambda = 0$ case. Linear perturbations on this background lead to approximate black hole solutions. The critical exponent is found to be $\gamma = 2/5$.

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1 Introduction

Since its discovery, the BTZ black hole solution [1] of 2+1 dimensional AdS gravity has attracted much interest because it represents a simplified context in which to study the classical and quantum properties of black holes. A line of approach which has been opened only recently [2, 3, 4, 5] concerns black hole formation through collapse of matter configurations coupled to 2+1 gravity with a negative cosmological constant. As first discovered in four dimensions by Choptuik [6], collapsing configurations which lie at the threshold of black hole formation exhibit properties, such as universality, power-law scaling of the black hole mass, and continuous or discrete self-similarity, which are characteristic of critical phenomena [7]. In the case of a spherically symmetric massless, minimally coupled scalar field, a class of analytical continuously self-similar (CSS) solutions was first given by Roberts [8, 9, 10]. These include critical solutions, lying at the threshold between black holes and naked singularities, and characterized by the presence of null central singularities. Linear perturbations of these solutions [11, 12] lead to approximate black hole solutions with a spacelike central singularity.

Numerical simulations of circularly symmetric scalar field collapse in 2+1 dimensional AdS spacetime were recently performed by Pretorius and Choptuik [4] and Husain and Olivier [3]. Both groups observed critical collapse, which was determined in [4] to be continuously self-similar near r = 0. In [4], Garfinkle has found a one-parameter family of exact CSS solutions of 2+1 gravity without cosmological constant, and argued that one of these solutions should give the behaviour of the full critical solution (Λ ≠ 0) near the singularity.

The purpose of this paper is to present a new CSS solution to the field equations with Λ = 0 which can be extended to a threshold solution of the full Λ ≠ 0 equations. The new Λ = 0 solution is derived in Sect. 3. It presents a null central singularity and, besides being CSS, possesses four Killing vectors. In Sect. 4 we address the extension of this CSS solution to a quasi-CSS solution of the full Λ < 0 problem, and show that the requirement of maximal symmetry selects a unique extension. This inherits the null central singularity of the Λ = 0 solution, and has the correct AdS boundary at spatial infinity. Finally, we perform in Sect. 5 the linear perturbation analysis in this background, find that it does lead to black hole formation, and determine the critical exponent.
2 CSS solutions

The Einstein equations for cosmological gravity coupled to a massless scalar field in (2+1) dimensions are
\[ G_{\mu\nu} - \Lambda g_{\mu\nu} = \kappa T_{\mu\nu}, \quad (2.1) \]
with the stress-energy tensor for the scalar field
\[ T_{\mu\nu} = \partial_\mu \phi \partial_\nu \phi - \frac{1}{2} g_{\mu\nu} \partial^\lambda \phi \partial_\lambda \phi. \quad (2.2) \]
The signature of the metric is (+ - -), and the cosmological constant \( \Lambda \) is negative for AdS spacetime, \( \Lambda = -l^{-2} \). Static solutions of these equations include the BTZ black hole solutions \[1\] with a vanishing scalar field \( \phi = 0 \), and singular solutions when a non-trivial scalar field is coupled with the positive sign for the gravitational constant \( \kappa \) \[13\].

We shall use for radial collapse the convenient parametrisation of the rotationally symmetric line element in terms of null coordinates \((u, v)\):
\[ ds^2 = e^{2\sigma} du dv - r^2 d\theta^2, \quad (2.3) \]
with metric functions \( \sigma(u, v) \) and \( r(u, v) \). The corresponding Einstein equations and scalar field equation are
\[
\begin{align*}
  r_{,uv} &= \frac{\Lambda}{2} re^{2\sigma}, \\
  2\sigma_{,uv} &= \frac{\Lambda}{2} e^{2\sigma} - \kappa \phi_{,u} \phi_{,v}, \\
  2\sigma_{,u} r_{,u} - r_{,uu} &= \kappa r \phi_{,u}^2, \\
  2\sigma_{,v} r_{,v} - r_{,vv} &= \kappa r \phi_{,v}^2, \\
  2r \phi_{,uv} + r_{,u} \phi_{,v} + r_{,v} \phi_{,u} &= 0. 
\end{align*} \quad (2.4)-(2.8)
\]

From the Einstein equations, the Ricci scalar is
\[ R = -6\Lambda + 4\kappa e^{-2\sigma} \phi_{,u} \phi_{,v}. \quad (2.9) \]
It follows from (2.9) and (2.3) that the behavior of the solutions near the singularity is governed by the equations (2.4)-(2.8) with vanishing cosmological constant \( \Lambda = 0 \) (see also \[5\]). Assuming \( \Lambda = 0 \), Garfinkle has found \[4\] the following family of exact CSS solutions to these equations
\[
\begin{align*}
  ds^2 &= -A \left( \frac{(\sqrt{v} + \sqrt{-u})^4}{-uv} \right) \kappa c^2 du dv - \frac{1}{4} (v + u)^2 d\theta^2, \\
  \phi &= -2c \ln(\sqrt{v} + \sqrt{-u}), 
\end{align*} \quad (2.10)
\]
depending on an arbitrary constant \( c \) and a scale \( A > 0 \). In \( (2.10) \), \( u \) is retarded time, and \(-v\) is advanced time. These solutions are continuously self-similar with homothetic vector \((u \partial_u + v \partial_v)\). An equivalent form of these CSS solutions, obtained by making the transformation
\[
-u = (-\bar{u})^{2q}, \quad v = (\bar{v})^{2q} \quad (1/2q = 1 - \kappa c^2)
\]
to the barred null coordinates \((\bar{u}, \bar{v})\), is
\[
\begin{align*}
  ds^2 &= -\bar{A}(\bar{v}^q + (-\bar{u})^q)^{2(2q-1)/q} d\bar{u} d\bar{v} - \frac{1}{4}(\bar{v}^{2q} - (-\bar{u})^{2q})^2 d\theta^2, \\
  \phi &= -2c \ln(\bar{v}^q + (-\bar{u})^q).
\end{align*}
\]
The corresponding Ricci scalar is
\[
R = \frac{4\kappa c^2}{A}(\bar{v}^q + (-\bar{u})^q)^{2(1-3q)/q}(-\bar{u})^{q-1}(\bar{v})^{q-1}.
\]
Garfinkle suggested that the line element \((2.10)\) describes critical collapse in the sector \( r = -(u + v)/2 \geq 0 \), near the future point singularity \( r = 0 \) (where the Ricci scalar behaves, for \( v \propto u \), as \( u^{-2} \)). The corresponding Penrose diagram (Fig. 1) is a triangle bounded by past null infinity \( u \to -\infty \), the other null side \( v = 0 \), and the central regular timelike line \( r = 0 \). For \( \kappa c^2 \geq 1 \ (q < 0) \), the Ricci scalar
\[
R \sim (\bar{v})^{q-1} \sim (v)^{(q-1)/2q}
\]
is regular near \( v = 0 \), which moreover turns out to be at infinite geodesic distance. To show this, we consider the geodesic equation
\[
(e^{2s} \dot{v}) = -2rr_{,u} \dot{\theta}^2 = -2l^2r^{-3}r_{,u}
\]
\((l \text{ constant})\) near \( v = 0 \), \( u \text{ constant} \), which gives \( v \propto (ls)^{4q} \) for \( l \neq 0 \), or \( s^{2q} \) for \( l = 0 \), so that in all cases the affine parameter \( s \to \infty \) for \( v \to 0 \), and the spacetime is geodesically complete. For \( \kappa c^2 < 1 \ (q > 0) \), we see from \((2.13)\) that the null line \( v = 0 \) is a curvature singularity if \( \kappa c^2 < 1/2 \ (q < 1) \). If \( 1/2 \leq \kappa c^2 < 1 \ (q \geq 1) \), the surface \( v = 0 \) is regular. However, as discussed by Garfinkle, the metric \((2.12)\) can be extended through this surface only for \( q = n \), where \( n \) is a positive integer. For \( n \) even, the extended spacetime is made of two symmetrical triangles joined along the null side \( \bar{v} = 0 \), and has two coordinate singularities \( r = 0 \), one timelike \( (\bar{u} - \bar{v} = 0) \) and one spacelike \( (\bar{u} + \bar{v} = 0) \), but no curvature singularity. For \( n \) odd, one of the \( r = 0 \) sides becomes a future spacelike curvature singularity \( (e^{2s} = 0) \), similar to that of
Brady’s supercritical solutions for scalar field collapse in (3+1) dimensions [9], except for the fact that in the present case the singularity is not hidden behind a spacelike apparent horizon (Fig. 2).

Let us point out that, besides the solutions (2.10), the system (2.4)-(2.8) also admits for \( \Lambda = 0 \) another family of CSS solutions

\[
\begin{align*}
\frac{ds^2}{A} &= (\sqrt{v} - \sqrt{-u})^4 du dv - \frac{1}{4} (v + u)^2 d\theta^2, \\
\phi &= \frac{-2c\ln(\sqrt{v} - \sqrt{-u})}{-uv},
\end{align*}
\]

with \( \phi = -2c\ln(\sqrt{v} - \sqrt{-u}) \), and we choose \( A > 0 \) and consider the sector \( 0 \leq v \leq -u \). These solutions have a future spacelike central \( (r = 0) \) curvature singularity at \( (-\bar{u})^q = \bar{v}^q \) (where the Ricci scalar (2.13) diverges) for all \( q < 0 \) or \( q > 0 \) (implying \( q > 1/2 \)). For \( q < 0 \), the Penrose diagram is a triangle bounded by past null infinities \( \bar{u} \to -\infty \) and \( \bar{v} = 0 \) (which is at infinite geodesic distance). For \( q > 0 \), geodesics terminate at \( \bar{v} = 0 \), unless \( q = n \) integer. For \( n \) even, the extended spacetime has two central curvature singularities \( r = 0 \), one spacelike and the other timelike. The extended spacetime for \( n \) odd is more realistic. In this case the extension from \( \bar{v} > 0 \) to \( \bar{v} < 0 \) amounts to replacing (2.16) with \( A > 0 \) by the original Garfinkle solution (2.10) with \( A > 0 \), the resulting Penrose diagram being that of Fig. 2.

3 A new CSS solution for \( \Lambda = 0 \)

Among the one-parameter (\( c \) or \( q \)) family of CSS solutions (2.10), the special solution, corresponding to \( \kappa c^2 = 1 \),

\[
\begin{align*}
\frac{ds^2}{A} &= 4A(\sqrt{u} + \sqrt{-u})^4 du dv - \frac{1}{4} (v + u)^2 d\theta^2, \\
\phi &= -2\ln(\sqrt{u} + \sqrt{-u}),
\end{align*}
\]

is singled out by the fact that the transformation (2.11) breaks down for this value. The transformation appropriate to this case,

\[
- u = 2e^{-U}, \quad v = 2e^{V} = 2e^{U-2T}
\]

(with \( T \geq U \) for \( u + v \leq 0 \)) transforms the solution (3.1) to

\[
\begin{align*}
ds^2 &= e^{-2U}[-4A(1 + e^{U-T})^4 dU dV - (1 - e^{2(U-T)})^2 d\theta^2], \\
\phi &= U - 2\ln(1 + e^{U-T})
\end{align*}
\]
(we use from now on units such that \(\kappa = 1\), and have dropped an irrelevant additive constant from \(\phi\)).

Starting from this special CSS solution of the Garfinkle class, we now derive, by a limiting process, a new CSS solution which, as we shall see, exhibits a null singularity. We translate \(T\) to \(T - T_0\), and take the late-time limit \(T_0 \to -\infty\), leading to the new CSS solution (written for \(A = -1/2\))

\[
ds^2 = e^{-2U}(2dUdV - d\theta^2), \quad \phi = U ,
\]

with a very simple form which is reminiscent of the Hayward critical solution for scalar field collapse in 3+1 dimensions \cite{12},

\[
ds^2 = e^{2\rho}(2d\tau^2 - 2d\rho^2 - d\Omega^2), \quad \phi = \tau .
\]

The transformation

\[
\bar{u} = -e^{-2U}, \quad \bar{v} = V
\]

leads from (3.4) to the even more simple form of this solution

\[
ds^2 = d\bar{u} d\bar{v} + \bar{u} d\theta^2 , \quad \phi = -\frac{1}{2} \ln(-\bar{u}) ,
\]

which is reminiscent of the other form of the Hayward solution

\[
ds^2 = 2 d\bar{u} d\bar{v} + \bar{u} \bar{v} d\Omega^2 , \quad \phi = -\frac{1}{2} \ln(-\bar{u}/\bar{v}) .
\]

The solution (3.4) or (3.7) is continuously self-similar, with homothetic vector

\[
K = \partial_U = -2\bar{u} \partial_{\bar{u}} .
\]

It also has a high degree of symmetry, with 4 Killing vectors

\[
L_1 = \partial_U + 2V \partial_V + \theta \partial_{\theta} , \\
L_2 = \theta \partial_V + U \partial_{\theta} , \\
L_3 = \partial_V , \\
L_4 = \partial_{\theta} ,
\]

generating the solvable Lie algebra

\[
[L_1, L_2] = L_4 - L_2 , \quad [L_2, L_3] = 0 , \\
[L_1, L_3] = -2L_3 , \quad [L_2, L_4] = -L_3 , \\
[L_1, L_4] = -L_4 , \quad [L_3, L_4] = 0 .
\]
The Ricci scalar (2.9) is identically zero for the solution (3.4), for which the sole nonvanishing Ricci tensor component is $R_{UU} = 1$. It follows that this metric is devoid of curvature singularity. However there is an obvious coordinate singularity at $U \to +\infty$, or $\bar{u} = 0$ (where $r = 0$). To determine the nature of this singularity, we study geodesic motion in the spacetime (3.7). The geodesic equations are integrated by

$$
\dot{\bar{u}} = \pi, \quad \bar{u} \dot{\theta} = l, \quad \pi \dot{\bar{v}} + l \dot{\theta} = \varepsilon,
$$

(3.12)

where $\pi$ and $l$ are the constants of the motion associated with the Killing vectors $L_3$ and $L_4$, and the sign of $\varepsilon$ depends on that of $d\bar{s}^2$ along the geodesic. The null line $\bar{u} = 0$ can be reached only by those geodesics with $\pi \neq 0$. Then, the third equation (3.12) integrates to

$$
\bar{v} = \frac{\varepsilon}{\pi^2} \bar{u} - \frac{l}{\pi} \theta + \text{const.} = \frac{\varepsilon}{\pi^2} \bar{u} - \frac{l^2}{\pi^2} \ln(-\bar{u}) + \text{const.}.
$$

(3.13)

It follows that nonradial geodesics ($l \neq 0$) terminate at $\bar{u} = 0, \bar{v} \to +\infty$, while radial geodesics ($l = 0$), which behave as in cylindrical Minkowski space, can be continued through the null line $\bar{u} = 0$ to $\bar{u} \to +\infty$. So in this sense only the endpoint $\bar{v} \to +\infty$ of the null line $\bar{u} = 0$ is singular. However formal analytic continuation of the metric (3.7) from $\bar{u} < 0$ to $\bar{u} > 0$ involves a change of signature from (+ - -) to (+ - +), leading to the appearance of closed timelike curves. So the null line $\bar{u} = 0$ corresponds to a singularity in the causal structure of the spacetime, analogous to the central singularity in the causal structure of the BTZ black holes [1]. The resulting Penrose diagram, reminiscent of that of the Hayward critical solution [12], is a diamond bound by three lines at null infinity ($\bar{v} = -\infty, \bar{u} = -\infty, \bar{v} = +\infty$) and the null singularity $\bar{u} = 0$ (Fig. 3).

## 4 Extending the new solution to $\Lambda \neq 0$

In the preceding section we have found an exact solution for scalar field collapse with $\Lambda = 0$, which presents a central null singularity. This property makes it a candidate threshold solution, lying at the boundary between naked singularities and black holes. However black holes exist only for $\Lambda < 0$, so the solution (3.7) can only represent the behavior of the true threshold solution near the central singularity, where the cosmological constant can be neglected. This hypothetical $\Lambda < 0$ solution cannot be self-similar, essentially because the scale is fixed preferentially by the cosmological constant.
So what we need is to find some other way to extend (3.7) to a solution of the full system of Einstein equations with $\Lambda < 0$.

A first possible approach is to expand this solution in powers of $\Lambda$, with the zeroth order given by the CSS solution (3.7). In the parametrisation (2.3), this zeroth order is (dropping the bars in (3.7))

$$r_0 = (-u)^{1/2}, \quad \sigma_0 = 0, \quad \phi_0 = -\frac{1}{2} \ln |u|. \quad (4.1)$$

We look for an approximate solution to first order in $\Lambda$ of the form

$$r = (-u)^{1/2} + \Lambda r_1, \quad \sigma = \Lambda \sigma_1, \quad \phi = -\frac{1}{2} \ln |u| + \Lambda \phi_1, \quad (4.2)$$

with the boundary condition that the functions $r_1, \sigma_1$ and $\phi_1$ vanish on the central singularity $u = 0$. Eq (2.4) gives

$$r_1 = (-u)^{1/2} \left( \frac{1}{3} uv + f(u) \right), \quad (4.3)$$

with $f(0) = 0$. Then, the linearized Eq. (2.5) gives

$$2r_0^{1/2}(r_0^{1/2} \phi_{1,v})_u = -r_1,v\phi_{0,u} = \frac{1}{6}(-u)^{1/2}, \quad (4.4)$$

which is solved by

$$\phi_1 = \left( \frac{1}{15} uv + g(u) \right). \quad (4.5)$$

The linearized Eq. (2.3)

$$2\sigma_{1,uv} + 1 - \phi_{0,u} \phi_{1,v} = \frac{8}{15} \quad (4.6)$$

then gives

$$\sigma_1 = \frac{4}{15} uv + h(u). \quad (4.7)$$

Finally Eq. (2.5) leads to the relation between the arbitrary functions $f, g, h$

$$u f''(u) + f'(u) = g'(u) + h'(u). \quad (4.8)$$

Not only does this first order solution break the continuous self-similarity generated by (3.9), as expected, but it also breaks the isometry group generated by the Killings (3.10) down to $U(1)$ (generated by $L_4 = \partial_\theta$), except in the special case $f = g = h = 0$, where the Killing subalgebra $(L_1, L_4)$
remains. This suggests looking for an exact $\Lambda < 0$ extension of the $\Lambda = 0$
CSS solution of the form

$$ds^2 = e^{2\sigma(x)} du dv + u \rho^2(x) d\theta^2, \quad \phi = -\frac{1}{2} \ln |u| + \psi(x),$$  (4.9)

with $x = uv$. This will automatically preserve to all orders the Killing
subalgebra $(L_1, L_4)$. Inserting this ansatz into the field equations (2.4)-(2.8)
leads to the system

$$x \rho'' + \frac{3}{2} \rho' = \frac{\Lambda}{2} \rho e^{2\sigma},$$  (4.10)

$$2(x \sigma'' + \sigma') + \psi' (x \psi' - \frac{1}{2}) = \frac{\Lambda}{2} e^{2\sigma},$$  (4.11)

$$x^2 (-\rho'' + 2 \rho' \sigma' - \rho \psi'^2) + x (-\rho' + \rho (\sigma' + \psi')) = 0,$$  (4.12)

$$-\rho'' + 2 \rho' \sigma' - \rho \psi'^2 = 0,$$  (4.13)

$$2x (\rho \psi')' + \frac{5}{2} \rho \psi' = \frac{1}{2} \rho'.$$  (4.14)

$(\prime = d/dx)$. The unique, maximally symmetric extension of the CSS solution (3.7) reducing to (3.7) near $u = 0$ is the solution of the system (4.10)-(4.14)
with the boundary conditions

$$\rho(0) = 1, \quad \sigma(0) = 0, \quad \psi(0) = 0.$$  (4.15)

The comparison of (4.12) and (4.13) yields

$$\rho = e^{\sigma + \psi}.$$  (4.16)

The combination (4.10) + $x$(4.13) then gives, together with (4.16),

$$x(2 \sigma'' + 2 \sigma' \psi' - \psi'^2) + \frac{3}{2} (\sigma' + \psi') = \frac{\Lambda}{2} e^{2\sigma}.$$  (4.17)

The third independent equation is for instance (4.11):

$$2(x \sigma'' + \sigma') + \psi' (x \psi' - \frac{1}{2}) = \frac{\Lambda}{2} e^{2\sigma}.$$  (4.18)

Using these last two equations with the boundary conditions (4.15), one can
in principle write down series expansions for $\sigma(x)$ and $\psi(x)$. Another simple
relation, deriving from (4.13) and (4.16), is

$$\sigma'' + \psi'' - \sigma'^2 + 2 \psi'^2 = 0.$$  (4.19)
We are interested in the behavior of this extended solution in the sector \( u < 0, v > 0, \) i.e. \( x < 0. \) In this sector, Eqs. (4.10), (4.14) and (4.11) can be integrated to

\[
-x^{3/2} \rho' = \frac{\Lambda}{2} \int_x^0 (-x)^{1/2} p e^{2\sigma} dx, \tag{4.20}
\]

\[
-x^{5/4} \psi' = \frac{1}{4} \int_x^0 (-x)^{1/4} \rho' dx, \tag{4.21}
\]

\[
-x \sigma' = \frac{1}{2} \int_x^0 \left( \frac{\Lambda}{2} e^{2\sigma} + \psi' \left( \frac{1}{2} - x \psi' \right) \right) dx. \tag{4.22}
\]

As long as \( \rho > 0, \) Eq. (4.20) (with \( x < 0, \Lambda < 0) \) implies \( \rho' < 0, \) so that \( \rho(x) \) decreases to 1 when \( x \) increases to 0. It then follows from (4.21) that \( \psi' < 0. \) Also, (4.21) can be integrated by parts to

\[
x \psi' = \frac{1}{4} - \frac{1}{16(-x)^{1/4}} \int_x^0 (-x)^{-3/4} \rho dx, \tag{4.23}
\]

showing that \( x \psi' < 1/4. \) It then follows from (4.22) that \( \sigma' < 0. \) So, as \( x \) decreases, the functions \( \rho \) and \( e^{2\sigma} \) increase and possibly go to infinity for a finite value \( x = x_1. \) If this is the case, the behavior of these functions near \( x_1 \) must be

\[
\rho = \rho_1 \left( \frac{1}{\bar{x}} + \frac{1}{4x_1} - \frac{\bar{x} \ln(\bar{x})}{48x_1^2} + \ldots \right)
\]

\[
e^{2\sigma} = \frac{4x_1}{\Lambda \bar{x}^2} \left( 1 + \frac{\bar{x}^2 \ln(\bar{x})}{48x_1^2} + \ldots \right)
\]

\[
\psi = \psi_1 + \frac{\bar{x}}{4x_1} - \frac{\bar{x}^2}{32x_1^2} \ln(\bar{x}) + \ldots \tag{4.24}
\]

\((\bar{x} = x - x_1).\)

These expectations are borne out by the actual numerical solution of the system

\[
x \rho'' + \frac{3}{2} \rho' = -\rho e^{2\sigma},
\]

\[-\rho'' + 4 \rho' \sigma' = \rho^2 + \rho^2 \sigma'^2, \tag{4.25}
\]

(this last equation comes from (4.13) where \( \psi' \) is given by derivation of (4.16) ) where we have set \( \Lambda = -2, \) with the boundary conditions \( \rho(0) = 1, \rho'(0) = -2/3 \) (see eqs. (4.3) and (4.2) ), \( \sigma(0) = 0. \) The plots of the functions
\( \rho(x), \sigma(x) \) and \( \psi'(x) \) are given in Figs. (4,5,6.). The value of \( x_1 \) is found to be approximately \(-1.94\) (i.e. \( \Lambda x_1 = +3.88 \)).

The coordinate transformation

\[
u = \Lambda^{-1} e^{-\bar{U}}, \quad v = e^{\bar{V}} \quad (\bar{U} = \bar{T} - \bar{R}, \quad \bar{V} = \bar{T} + \bar{R}) \tag{4.26}
\]

leads to \( x = \Lambda^{-1} e^{2\bar{R}} \) and, on account of (4.9) and (4.16), to the form of the metric

\[
ds^2 = -\Lambda^{-1} e^{2(\sigma(R) + \bar{R})} (d\bar{U} d\bar{V} - e^{2\psi(R) - \bar{V}} d\theta^2). \tag{4.27}
\]

Near the spacelike boundary \( \bar{R} = \bar{R}_1 \) of the spacetime, the collapsing metric and scalar field behave, from (4.24), as

\[
ds^2 \simeq -\Lambda^{-1} (\bar{R}_1 - \bar{R})^{-2} (dT^2 - d\bar{R}^2 - e^{\bar{T}_1 - \bar{T}} d\theta^2), \quad \phi = \phi_1 + \bar{T}/2 \tag{4.28}
\]

\((\bar{R} - \bar{R}_1 \simeq \bar{x}/2x_1)\). This metric is asymptotically AdS, as may be shown by making the further coordinate transformation,

\[
\bar{R} - \bar{R}_1 = -2/XT, \quad \bar{T} - \bar{T}_1 = 2 \ln(T/2), \tag{4.29}
\]

leading to

\[
ds^2 \simeq -\Lambda^{-1} \left(X^2 dT^2 - \frac{dX^2}{X^2} - X^2 d\theta^2\right), \quad \phi = \phi_1 + \ln(T/2). \tag{4.30}
\]

The next-to-leading terms in the metric containing logarithms, this asymptotic behavior differs from that of BTZ black holes.

It follows from this discussion that the Penrose diagram of the \( \Lambda < 0 \) threshold solution in the sector \( v > 0, u < 0 \) is a triangle bounded by the null line \( v = 0 \), the null causal singularity \( u = 0 \), and the spacelike AdS boundary \( X \to \infty \). The null singularity \( u = 0 \) remains naked, i.e. is not hidden behind a trapping horizon, which would correspond to

\[
\partial_v r = -(\bar{u})^{3/2} \rho'(x) = 0, \tag{4.31}
\]

because \( \rho' < 0 \) (as discussed above) implies that the only solution of this equation is \( u = 0 \).

For the sake of completeness, let us also discuss the behavior of the solution of the system (4.10)-(4.14) in the sector \( x > 0 \). In this case, one can write down integro-differential equations similar to (1.20)-(1.22), from

\[1\] We have taken care that in (4.3) \( u \) has the dimension of a length squared while \( v \) is dimensionless.
which one again derives that \( \rho' < 0, \psi' < 0 \) and \( \sigma' < 0 \). It follows that the metric function \( e^{2\sigma} \) decreases as \( x \) increases, eventually vanishing for a finite value \( x = x_0 \), corresponding to a spacelike curvature singularity (this has been confirmed numerically). The behavior of the solution near this singularity is found to be

\[
\psi \simeq \gamma \ln(x_0 - x), \quad \sigma \simeq \frac{\gamma^2}{2} \ln(x_0 - x), \quad \rho \propto (x_0 - x) \quad (\gamma = \sqrt{3} - 1),
\]

(4.32)

and the coordinate transformation \( u = e^U, v = e^V(x = e^{2T}) \) leads to the form of the metric near the singularity

\[
ds^2 \simeq (T_0 - T)^{\gamma^2} (dT^2 - dR^2) + e^{R_0 - R}(T_0 - T)^2 d\theta^2.
\]

(4.33)

5 Perturbations

To check whether the quasi-CSS solution (4.9) of the full \( \Lambda \neq 0 \) problem determined in the preceding section is indeed a threshold solution, we now study linear perturbations of this solution. Our treatment will follow the analysis of perturbations of critical solutions in the case of scalar field collapse in 3+1 dimensions [11, 12].

The relevant time parameter in critical collapse being the retarded time \( U = -(1/2) \ln(-u) \) (the “scaling variable” of [11]), we expand these perturbations in modes proportional to \( e^{kU} = (-u)^{-k/2} \), with \( k \) a complex constant. We recall that only the modes with \( \text{Re } k > 0 \) grow as \( U \to +\infty (u \to -0) \) and lead to black hole formation, whereas those with \( \text{Re } k < 0 \) decay and are irrelevant. The other relevant variable is the “spatial” coordinate \( x = uv \), and the perturbations are decomposed as

\[
\begin{align*}
 r &= (-u)^{1/2} (\rho(x) + (-u)^{-k/2} \tilde{r}(x)), \\
 \phi &= -\frac{1}{2} \ln |u| + \psi(x) + (-u)^{-k/2} \tilde{\phi}(x), \\
 \sigma &= \sigma(x) + (-u)^{-k/2} \tilde{\sigma}(x).
\end{align*}
\]

(5.1)

Then, the Einstein equations (2.4)-(2.8) are linearized in \( \tilde{r}, \tilde{\phi}, \tilde{\sigma} \), using

\[
\delta \phi_{,u} = -(-u)^{-k/2 - 1}(x \tilde{\phi}' - \frac{k}{2} \tilde{\phi}), \quad \delta \phi_{,v} = -(-u)^{-k/2 + 1} \tilde{\phi}'.
\]

(5.2)

The resulting equations are homogeneous in \( u \), which drops out, and the linearized system reduces to

\[
x \tilde{r}'' + (-k/2 + 3/2) \tilde{r}' = \frac{\Lambda}{2} e^{2\sigma} (\tilde{r} + 2 \tilde{\rho} \tilde{\sigma}),
\]

(5.3)
\begin{align}
2x\tilde{\sigma}'' + (-k + 2)\tilde{\sigma}' &= \Lambda e^{2\alpha} \tilde{\sigma} - (2x\psi' - 1/2)\tilde{\phi}' + (k/2)\psi' \tilde{\phi}, \\
-(-k + 1)x\tilde{r}' + ((-k + 1)x\sigma' - (k^2 - 1)/4)\tilde{r} + \rho x\tilde{\sigma}' - k(x\rho' + \rho/2)\tilde{\sigma} = \\
&-\rho(x\tilde{\phi}' - k(1/2 - x\psi')\tilde{\phi}) + (1/4 - x\psi')\tilde{r}, \\
2(\rho'\tilde{\sigma}' + \sigma'\tilde{r}') - \tilde{r}'' &= \psi'(2\rho\tilde{\phi}' + \psi \tilde{\phi}), \\
2x\rho\tilde{\phi}'' + (2x\rho' + (-k + 5/2)\rho)\tilde{\phi}' - (k/2)\rho' \tilde{\phi} + (2x\psi' - 1/2)\tilde{r}' \\
&+(2x\psi'' + (-k/2 + 5/2)\psi')\tilde{r} = 0.
\end{align}

What is the number of the independent constants for this system? The perturbed Klein-Gordon equation (5.7) is clearly redundant, while Eqs. (5.5) and (5.6) are constraints. So, as in the (3+1)-dimensional case [11, 12], the order of the system is four, and the general solution depends on four integration constants. However, one of these four independent solutions corresponds to a gauge mode and is irrelevant. The parametrisation (4.9) is invariant under infinitesimal coordinate transformations \( v \rightarrow v + f(v) \). For \( f(v) = -\alpha v^{1+k/2} \), these lead to \( x \rightarrow x - \alpha(-u)^{-k/2}(-x)^{1+k/2} \), giving rise to the gauge mode

\begin{align}
\tilde{r}_k(x) &= \alpha(-x)^{1+k/2} \rho'(x), \\
\tilde{\phi}_k(x) &= \alpha(-x)^{1+k/2} \psi'(x), \\
\tilde{\sigma}_k(x) &= \alpha[(-x)^{1+k/2} \sigma'(x) - \frac{k + 2}{4}(-x)^{k/2}],
\end{align}

which solves identically the system (5.3)-(5.7). So, up to gauge transformations, the general solution of this system depends only on three independent constants.

These will be determined, together with the possible values of \( k \) (the eigenfrequencies) by enforcing appropriate and reasonable boundary conditions. We shall use here the “weak boundary conditions” of [12] on the boundaries \( u = 0 \) and \( x = x_1 \) (\( X \rightarrow \infty \))

\[ \lim_{u \rightarrow 0} r^{-1} \neq 0, \quad \lim_{x \rightarrow x_1} r \neq 0, \]

\[ \tilde{r}(0) = 0, \]

which guarantees that the singularity of the perturbed solution starts smoothly from that of the unperturbed one. On the third boundary \( v = 0 \), we shall impose a stronger condition by requiring that the perturbations are analytic in \( v \), in order for the perturbed solution to be extendible beyond \( v = 0 \) to negative values of \( v \) at finite \( u \).
First, we consider the region $x \to 0$ where, according to Eqs. (4.1), (4.3), (4.5) and (4.7),
\begin{align*}
\rho & \simeq 1 + \frac{1}{3} \Lambda x, \\
\rho^{2} & \simeq 1 + \frac{4}{15} \Lambda x, \\
\sigma & \simeq \frac{1}{15} \Lambda x.
\end{align*}
(5.11)

Let us assume a power-law behavior
\begin{equation}
\tilde{r}(x) \sim a(-x)^{p}
\end{equation}
(5.12)
where $p$ is a constant to be determined. Then Eqs. (5.3), (5.4) and (5.6) can be approximated near $x = 0$ as
\begin{align*}
4x \tilde{r}'' + (-k/2 + 3/2) \tilde{r}' & \simeq \Lambda \tilde{\sigma}, \\
4x \tilde{\sigma}'' + (-k/2 + 1) \tilde{\sigma}' & \simeq \frac{1}{4} \tilde{\phi}', \\
2 \rho' \tilde{\sigma}' - \tilde{r}'' & \simeq 2 \rho \psi' \tilde{\phi}'.
\end{align*}
(5.13) (5.14) (5.15)

Eliminating the functions $\tilde{\sigma}$ and $\tilde{\phi}$ between these three equations and using Eq. (5.11), we obtain the fourth-order equation
\begin{equation}
4x^{2} \tilde{r}''' + (-4k + 13) x \tilde{r}'' + (k/2 - 1)(2k - 5) \tilde{r}' \simeq 0,
\end{equation}
(5.16)
which implies the power-law behavior (5.12) with the exponent $p$ constrained by
\begin{equation}
p(p - 1)(p - k/2 - 3/4)(p - k/2 - 1) = 0.
\end{equation}
(5.17)

Obviously the root $p = k/2 + 1$ corresponds to the gauge mode (5.8) and must be discarded as irrelevant. As a consequence the general solution near $x = 0$ can be given in terms of three independent constants as
\begin{align*}
\tilde{r}(x) & \sim A + B(-x) + \Lambda C(-x)^{3/4 + k/2}, \\
\tilde{\sigma}(x) & \sim -\frac{A}{2} + \Lambda^{-1} \frac{(k - 3)B}{2} - \frac{5C}{8} (k + \frac{3}{2})(-x)^{-1/4 + k/2}, \\
\tilde{\phi}(x) & \sim \frac{(1 - k)A}{2} - \Lambda^{-1} \frac{(k - 3)B}{2} + \frac{5C}{8} (k + \frac{3}{2})(-x)^{-1/4 + k/2}.
\end{align*}
(5.18) (5.19) (5.20)

Let us note that this solution remains valid in the limit $\Lambda \to 0$, leading to the limiting solution $\tilde{r} \sim A + B(-x)$ (with $B = 0$ for $k \neq 3$), which could also be obtained directly by solving the equation $\tilde{r}'' = 0$ which results from (5.6) in the limit $\Lambda \to 0$, together with the stronger condition (from Eq. (5.3)) $(k - 3) \tilde{r}' = 0$. 

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Now we enforce the boundary conditions at $x = 0$. For $k > 0$, $\tilde{r}$ is dominated by its first constant term in (5.18), so that the condition (5.10) can only be satisfied for $u \to 0$ if

$$A = 0.$$  \hspace{1cm} (5.21)

Then, for $k > 1/2$, $\tilde{r}$ is dominated by its second term $-Bx$, leading to a perturbation $(-u)^{1/2 - k/2} \tilde{r}(x)$ which blows up as $u \to 0$ and violates (5.9) unless

$$k \leq 3.$$  \hspace{1cm} (5.22)

Then we impose the condition of analyticity in $v$ at fixed $u$. This is satisfied if

$$k = 2n - 3/2,$$  \hspace{1cm} (5.23)

where $n$ is a positive integer. Combining eqs. (5.22) and (5.23) we find that $k$ has only two positive eigenvalues

$$k = 1/2, \quad k = 5/2.$$  \hspace{1cm} (5.24)

However, in the above analysis we have disregarded the fact that $k = 1/2$ is a double root of the secular equation (5.17). For $k = 1/2$ the correct behavior of the general solution near $x = 0$ is

$$\tilde{r}(x) \sim A + B(-x) + \Lambda C(-x) \ln |x|,$$  \hspace{1cm} (5.25)

$$\tilde{\sigma}(x) \sim -\frac{A}{2} - \Lambda^{-1} \frac{5B}{4} - \frac{9C}{4} - \frac{5C}{4} \ln |x|,$$  \hspace{1cm} (5.26)

$$\tilde{\phi}(x) \sim \frac{A}{4} + \Lambda^{-1} \frac{5B}{4} + \frac{9C}{4} + \frac{5C}{4} \ln |x|,$$  \hspace{1cm} (5.27)

which satisfies the condition of analyticity only if $C = 0$.

At the AdS boundary ($x \to x_1$) the leading behaviour of the background is, from Eqs. (4.24),

$$\rho \simeq \frac{\rho_1}{x - x_1}, \quad e^{2\sigma} \simeq \left( \frac{4x_1}{\Lambda} \right) \frac{1}{(x - x_1)^2}, \quad \psi \simeq \psi_1.$$  \hspace{1cm} (5.28)

We again assume a power-law behavior

$$\tilde{\sigma} \sim b\bar{x}^q$$  \hspace{1cm} (5.29)

($\bar{x} = x - x_1$). Then Eq. (5.4), where $\tilde{\phi}$ can be neglected, gives

$$q(q - 1) = 2,$$  \hspace{1cm} (5.30)
i.e. \( q = -1 \) or \( q = 2 \). Then, Eq. (5.3) reduces near \( \bar{x} = 0 \) to

\[
\tilde{\rho}'' - 2\bar{x}^{-2}\tilde{\rho} \approx 4b\rho_1\bar{x}^{q-3}.
\] (5.31)

If \( q = -1 \), the behavior of the solution is governed by the right-hand side, i.e. \( \tilde{\rho} \propto \bar{x}^{-2} \), which violates the boundary condition (5.9) for \( x \to x_1 \). So the behavior \( \tilde{\sigma} \sim b\bar{x}^{-1} \) must be excluded, which fixes another integration constant \( D = 0 \) (where \( D \) is a linear combination of \( B \) and \( C \)). Then, the generic behavior of the solution of Eq. (5.31) with \( q = 2 \) is governed by that for the homogeneous equation, i.e.

\[
\tilde{\rho} \sim \frac{E}{x - x_1}.
\] (5.32)

This is consistent with the boundary condition (5.9), and is an admissible small perturbation if its amplitude is small enough, \( E \ll \rho_1 \).

For \( k = 1/2 \), we have seen that two of the three integration constants in (5.25)-(5.27) are fixed (\( A = C = 0 \)) by condition (5.10) and the analyticity condition, while the weak boundary condition at the AdS boundary fixes a third constant \( D = 0 \). However this is impossible, as the perturbation amplitude must remain as a free parameter. So the mode \( k = 1/2 \) cannot satisfy all our boundary conditions, and we are left with a single eigenmode,

\[
k = 5/2,
\] (5.33)

completely determined up to an arbitrary amplitude by the two conditions \( A = D = 0 \).

The corresponding perturbed metric function \( \rho \) behaves near \( x = 0 \) as

\[
\rho \simeq (-u)^{1/2}\left[1 + \frac{1}{3}\Lambda x - (-u)^{-5/4}Bx\right].
\] (5.34)

For \( B < 0 \), the central singularity \( \rho = 0 \) is approximately given by

\[
(-u)^{1/4} \simeq -Bv.
\] (5.35)

Our boundary conditions guarantee that it starts at \( u = v = 0 \) (as for the unperturbed solution) and then becomes spacelike in the region \( v > 0 \). This singularity is hidden behind a trapping horizon (defined by Eq. (4.31)) which, near \( x = 0 \), is null,

\[
(-u)^{5/4} = \frac{3B}{\Lambda}
\] (5.36)
(a null trapping horizon was also found in [12]). Let us point out the crucial role played by the cosmological constant $\Lambda$ in the formation of this trapping horizon. For $\Lambda = 0$, $\rho(x) = 1$, while, as discussed after Eq. (5.20), the perturbation $\tilde{r}$ with the boundary condition (5.10) vanishes for $\Lambda = 0$, so that the perturbed radial function $r$ is (as in [11]) identical to the CSS one, and the trapping horizon does not exist. Near the AdS boundary $x \to x_1$, it follows from (5.28) and (5.32) that both the central singularity and the trapping horizon are tangent to the null line

$$(-u)^{5/4} = -E\left(\frac{4x_1}{\Lambda}\right)^{-1/2}.$$  

(5.37)

Thus, perturbations of the quasi-CSS solution lead to black hole formation, showing that this solution is indeed a threshold solution, and is a candidate to describe critical collapse. Near-critical collapse is characterized by a critical exponent $\gamma$, defined by the scaling relation $Q \propto |p - p^*|^s\gamma$, for a quantity $Q$ with dimension $s$ depending on a parameter $p$ (with $p = p^*$ for the critical solution). Choosing for $Q$ the radius $r_{AH}$ of the apparent horizon, and identifying $p - p^*$ with the perturbation amplitude $B$, we obtain from (5.36)

$$r_{AH} \simeq \left(\frac{3B}{\Lambda}\right)^{2/5},$$  

(5.38)

leading to the value of the critical exponent $\gamma = 2/5$, in agreement with the renormalization group argument [14] leading to $\gamma = 1/k$.

6 Conclusion

We have discussed in detail the causal structure of the Garfinkle CSS solutions (2.10) to the $\Lambda = 0$ Einstein-scalar field equations. From a special solution of this class, we have derived by a limiting process a new CSS solution, which we have extended to a unique solution of the full $\Lambda < 0$ equations, describing collapse of the scalar field onto a null central singularity. This is not a curvature singularity (all the curvature invariants remain finite), but a singularity in the causal structure similar to that of the BTZ black hole. Finally, we have analyzed linear perturbations of the $\Lambda < 0$ solution, found a single eigenmode $k = 5/2$, checked that this mode does indeed give rise to black holes, and determined the critical exponent $\gamma = 2/5$.

For comparison, Choptuik and Pretorius [2] derived, by analysing the observed scaling behavior of the maximum scalar curvature, the value $1.15 < \gamma < 1.25$ for the critical exponent. This value is different from the value
\( \gamma \sim 0.81 \) obtained in the numerical analysis of Husain and Olivier [3] from the scaling behavior of the apparent horizon radius. Our value \( \gamma = 0.4 \), while significantly smaller than these two conflicting estimates, is of the order of the theoretical value \( \gamma = 1/2 \) derived either from the analysis of dust-ring collapse [15], of black hole formation from point particle collisions [16], or of the \( J = 0 \) to \( J \neq 0 \) transition of the BTZ black hole [17].

It is worth mentioning here that, even though they were obtained for a vanishing cosmological constant and thus solve the \( \Lambda \neq 0 \) equations only near the singularity, the Garfinkle CSS solutions are, for the particular value (chosen in order to better fit the numerical curves) \( c = (7/8)^{1/2} \approx 0.935 \), in good agreement [4] with the numerical results of [2] at an intermediate time. The fact that this value is close to 1 suggests that the \( c = 1 \) CSS solution (3.3) approximately describes near-critical collapse at intermediate times. If this the case, then it would not be surprising if its late-time limit, our new CSS solution Eq. (3.4), gives a good description of exactly critical collapse near the singularity. A fuller understanding of the relationship between the numerically observed near-critical collapse and these various \( \Lambda = 0 \) CSS solutions could be achieved by extending them to \( \Lambda < 0 \), as done in the present work for the special solution (3.7).
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Figure 1: Penrose diagram of the solutions eq. (2.13) for $q < 0$.

Figure 2: Causal structure for $q = n$ odd.
Figure 3: Penrose diagram of our new CSS solution (3.7). The null line $\bar{u} = 0$ is a singularity in the causal structure.

Figure 4: Numerical plot of the function $\rho(x)$ as derived from the system (4.25) with $\rho(0) = 0$ and $\rho'(0) = -2/3$, showing the divergence of $\rho$ for $x \to x_1$ as the AdS boundary is approached (the behaviour is given in the first of Eqs. (4.24)).
Figure 5: Numerical graph of $\sigma(x)$ starting from $\sigma(0) = 0$. In the limit $x \to x_1$ this is well represented in the second of Eqs. (4.24).

Figure 6: Plot of $\psi'(x)$. In particular it is clear that $\psi''(x) \to \infty$ as $x \to x_1$. This feature is reproduced in the third of Eqs. (4.24) (giving $\psi'' \sim \ln(x - x_1)$).