EXCLUDING WORDS FROM DYCK SHIFTS

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Abstract. We study subshifts that arise by excluding words of length two from Dyck shifts. The words that are to be excluded are taken from a finite set that is not literal-uniform.

1. Introduction

Let $\Sigma$ be a finite alphabet, and let $S$ be the shift on the shift space $\Sigma^\mathbb{Z}$,

$$S((x_i)_{i \in \mathbb{Z}}) = (x_{i+1})_{i \in \mathbb{Z}}, \quad (x_i)_{i \in \mathbb{Z}} \in \Sigma^\mathbb{Z}. $$

An $S$-invariant closed subset $X$ of $\Sigma^\mathbb{Z}$ is called a subshift. For an introduction to the theory of subshifts see [Ki] or [LM]. A word is called admissible for the subshift $X \subset \Sigma^\mathbb{Z}$ if it appears in a point of $X$. We denote the language of admissible words of a the subshift $X \subset \Sigma^\mathbb{Z}$ by $L(X)$. A basic class of subshifts are the subshifts of finite type. A subshift of finite type is constructed from a finite set $F$ of words in the alphabet $\Sigma$ as the subshift that contains the points in $\Sigma^\mathbb{Z}$, in which no word in $F$ appears. More generally, a subshift $X \subset \Sigma^\mathbb{Z}$ and a finite set $F \subset L(X)$ determines a subshift $X(F)$ that contains the points in $X$ in which no word in $F$ appears. We say that the subshift $X(F)$ arises from the subshift $X$ by excluding words.

In this paper we study subshifts that arise from Dyck shifts by excluding words of length two. To recall the construction of the Dyck shifts, let $N > 1$, and let $\alpha^-(n), \alpha^+(n), 0 \leq n < N$, be the generators of the Dyck inverse monoid $\mathcal{D}_N$ with the rules

$$\alpha^-(n)\alpha^+(n') = \begin{cases} 1, & \text{if } n = n', \\ 0, & \text{if } n \neq n'. \end{cases}$$

The Dyck shifts are defined as the subshifts

$$D_N \subset (\{\alpha^-(n) : 0 \leq n < N\} \cup \{\alpha^+(n) : 0 \leq n < N\})^\mathbb{Z}$$

with the admissible words $(\sigma_i)_{1 \leq i \leq I}, I \in \mathbb{N}$, of $D_N, N > 1$, given by the condition

$$\prod_{1 \leq i \leq I} \sigma_i \neq 0.$$  

We denote by $C_N(n)$ the code that contains the words in $L(D_N)$ that have $\alpha^-(n)$ as their first symbol and $\alpha^+(n)$ as their last symbol, and that have no proper prefix with $\alpha^+(n)$ as the last symbol, or, equivalently, that have no proper suffix with $\alpha^-(n)$ as the first symbol, $0 \leq n < N$. The Dyck shift $D_N$ can also be defined as the coded system [BH] of the Dyck code $\cup_{0 \leq n < N} C_N(n)$. In [HI] a necessary and sufficient condition was given for the existence of an embedding of an irreducible subshift of finite type into a Dyck shift. In [HIK] this result was extended to a wider class of target shifts that have presentations that were constructed by using a graph inverse semigroup $S$. These presentations were called $S$-presentations. The Dyck inverse monoids occupy a central place among the graph inverse semigroups. With the semigroup $D_N^+ (D_N^-)$ that is generated by $\{\alpha^-(n) : 0 \leq n < N\} \cup \{\alpha^+(n) : 0 \leq n < N\}$, $D_N$-presentations can be described...
as arising from a finite irreducible directed labelled graph with vertex set \( V \) and edge set \( \Sigma \) and a label map \( \lambda \), such that

\[
\lambda(\sigma) \in \mathcal{D}_N \cup \{1\} \cup \mathcal{D}_N^+.
\]

Extending the label map to paths \( b = (b_i)_{1 \leq i \leq I}, I > 1 \), in the directed graph by setting \( \lambda(b) = \prod_{1 \leq i \leq I} \lambda(b_i) \), the admissible words of the \( \mathcal{D}_N \)-presentations are the paths \( b \) in the directed graph that satisfy the condition \( \lambda(b) \neq 0 \). It is required that one has for \( U, W \in \mathcal{V} \), and for \( \beta \in \mathcal{D}_N \), that in the directed graph there is a path \( b \) from \( U \) to \( W \) such that \( \lambda(b) = \beta \). A periodic point \( p = (p_i)_{i \in \mathbb{Z}} \) of a \( \mathcal{D}_N \)-presentation is said to have non-positive (non-negative) multiplier, if, with \( \Pi(p) \) the period of \( p \) there exists an \( i \in \mathbb{Z} \) such that \( \lambda((p_j)_{1 \leq j < i + \Pi(p)}) \in \mathcal{D}_N \cup \{1\}(\{1\} \cup \mathcal{D}_N^+) \).

Among the invariants, that determine the existence of an embedding of a given irreducible subshift of finite type into a \( \mathcal{D}_N \)-presentation, are periodic point counts and entropies that are associated to periodic points of target shift with non-positive or non-negative multipliers. Examples of \( \mathcal{D}_N \)-presentations can be obtained by excluding finitely many words from Dyck shifts. The examples in Section 4 of \[HIK\] were constructed in this way. This then leads to the problem of determining zeta functions that are associated to subshifts that are obtained by excluding from a Dyck shift \( D_N \) the words in a finite set \( F \) of \( D_N \)-admissible words. In studying this problem one is lead to make a distinction according to the nature of the set \( F \).

In \[IK\], a \( D_N \)-admissible word was called literal-non-positive (literal-non-negative) if all of its symbols are in \( \{\alpha^-(n) : 0 \leq n < N\} \) \( (\{\alpha^+(n) : 0 \leq n < N\}) \), and a set of \( D_N \)-admissible words was called literal uniform if all of its words are literal-non-positive or literal-non-negative. In \[IK\] we considered subshifts that are obtained by excluding from a Dyck shift \( D_N \) the words in a finite literal-uniform set of \( D_N \)-admissible words, and in this paper we consider subshifts that are obtained by excluding from a Dyck shift \( D_N \) the words in a set of \( D_N \)-admissible words of length two that is not literal-uniform (literal non-uniform). More generally, we study subshifts that arise by excluding words from certain subshifts that belong to a class of subshifts, that are constructed from an finite index set \( \Gamma \) and a relation \( \sim \) on \( \Gamma \), with the Dyck shifts as special cases. This wider class of subshifts contains the subshifts \( X(\Gamma, \sim) \) with alphabet

\[
\{\alpha^-(\gamma) : \gamma \in \Gamma\} \cup \{\alpha^+(\gamma) : \gamma \in \Gamma\}
\]

and admissible words \((\sigma_i)_{1 \leq i \leq I} \), that are given, with the rules

\[
\alpha^-(\gamma)\alpha^+(\gamma') = \begin{cases} 1, & \text{if } \gamma \sim \gamma', \\ 0, & \text{if } \gamma \sim \gamma', \end{cases}
\]

by the condition

\[
\prod_{1 \leq i \leq I} \sigma_i \neq 0,
\]

(see \[HK\] Section 4). The notions of multiplier, and of a literal-uniform set and literal-non-uniform set of words, for the subshifts \( X(\Gamma, \sim) \) are analogous to the ones for the Dyck shifts.

In this paper, more specifically, we let \( N > 1 \), choose \( M_n \in \mathbb{N}, 0 \leq n < N \), set \( \Gamma = \bigcup_{0 \leq n < N} \{(n, m) : 1 \leq m \leq M_n\} \), and use the relation \( \sim \), where \( (n, m) \sim (n', m') \) means that \( n = n' \), denoting the resulting subshift \( X(\Gamma, \sim) \) by \( X_N((M_n)_{0 \leq n < N}) \) \( (\text{The subshift } X_N((1)_{0 \leq n < N}) \text{ is } D_N) \). Moreover, we choose sets

\[
\mathcal{A}_{n,n'} \subset [1, M_n], \quad \mathcal{A}_{n,n'}^- \subset [1, M_n], \quad \mathcal{A}_{n,n'}^+ \subset [1, M_n], \quad 0 \leq n, n' < N,
\]

\[
\mathcal{A}_{n,n'} \subset [1, M_n], \quad \mathcal{A}_{n,n'}^- \subset [1, M_n], \quad \mathcal{A}_{n,n'}^+ \subset [1, M_n], \quad 0 \leq n, n' < N,
\]
and set
\[ \mathcal{F}((M_n)_{0 \leq n < N}, (A_{n,m}^-, A_{n,m}^+, A_{n,m}^{+})_{0 \leq n,m < N}) = \bigcup_{0 \leq n,m < N} \bigcup_{1 \leq m \leq M_n} \{ \alpha^+(n,m)\alpha^-(n',m') : m' \notin A_{n,m}^- \} \cup \{ \alpha^+(n,m)\alpha^-(n',m') : m' \notin A_{n,m}^+ \} \cup \{ \alpha^+(n,m)\alpha^-(n',m') : m' \notin A_{n,m}^0 \}. \]

We also set
\[ X((M_n)_{0 \leq n < N}, (A_{n,m}^-, A_{n,m}^+, A_{n,m}^{+})_{0 \leq n,m < N}) = X_N((M_n)_{0 \leq n < N})(\mathcal{F}((M_n)_{0 \leq n < N}, (A_{n,m}^-, A_{n,m}^+, A_{n,m}^{+})_{0 \leq n,m < N})). \]

(There are procedures to decide if these subshifts are empty, or of finite type, or topologically intransitive.) By the use of circular Markov codes and by applying a formula of Keller [Ke], we obtain in Section 2 an expression for the zeta function of the subshifts \( X((M_n)_{0 \leq n < N}, (A_{n,m}^-, A_{n,m}^+, A_{n,m}^{+})_{0 \leq n,m < N}) \).

In section 3 we consider the case \( N = 2 \). The equations for the generating functions of the two essential circular Markov codes of the subshift are in this case of degree at most four, and in the case, that the two generating functions are equal, these equations are of degree at most three. The recursive structure of the code words allows to show that the two generating functions are the same, provided the number of words of length two, four and six, that the codes contain, are the same.

In section 4 we specialize to the case of the Dyck shift \( D_2 \). We determine the literal non-uniform sets \( \mathcal{F} \) of words of length two, such that the subshift, that arise from \( D_2 \) by removing the word in \( \mathcal{F} \), have a 2-block system that yields an \( D_2 \)-presentation, such that the two essential circular Markov codes have the same generating function.

2. Subsystems of \( X((M_n)_{0 \leq n < N}) \)

We denote the generating function of a set \( C \) of words in the symbols of a finite alphabet by \( g(C) \).

We recall from [Ke] the notion of a circular Markov code to the extent that is needed here. We let a Markov code be given by a nonempty set \( C \) of words in the symbols of a finite alphabet \( \Sigma \) together with a finite set \( \mathcal{V} \), a 0-1 transition matrix \( B = (B(U,W))_{U,W \in \mathcal{V}} \) and mappings \( r : C \to \mathcal{V}, s : C \to \mathcal{V} \). To \( (C, r, s) \) there is associated the shift invariant set \( X_C \subset \Sigma^\mathbb{Z} \) of points \( x \in \Sigma^\mathbb{Z} \) such that there are indices \( I_k, k \in \mathbb{Z} \), such that
\[ I_0 \leq 0 < I_1, \quad I_k < I_{k+1}, \quad k \in \mathbb{Z}, \]
and such that
\[ x_{[I_k,I_{k+1})} \in C, \quad k \in \mathbb{Z}, \]
and
\[ B(r(x_{[I_k,I_{k+1})}), s(x_{[I_{k+1},I_{k+2})})) = 1, \quad k \in \mathbb{Z}. \]
\((C, r, s, B)\) is said to be a circular Markov code if for every periodic point \( x \) in \( X_C \) the indices \( I_k, k \in \mathbb{Z} \), such that (2.1) and (2.2) hold, are uniquely determined by \( x \). Given a circular Markov code \((C, r, s, A)\) denote by \( C(U, W) \) the set of words \( c \in C \) such that \( s(c) = U, \ r(c) = W, U, W \in \mathcal{V} \). Introduce the matrix
\[ H^{[C]}(z) = (B(U,W))_{U,W \in \mathcal{V}}(z). \]
Lemma 2.1. For a circular Markov code \((C, s, r, B)\),

\[\zeta_C(z) = \det(I - H^C(z))^{-1}.\]

Proof. This is a variant of a special case of a formula of Keller [Ke]. \qed

We return to the subshifts \(X_N((M_n)_0 \leq n < N, (A_{n, n'}, A_{n, n'}, A_{n, n'}^+)_{0 \leq n, n' < N})\).

Proposition 2.2. Let \(N > 1, \) and \( M_n, n \in \mathbb{N}, 0 \leq n < N, \) and let

\[
A_{n, n'}, A_{n, n'}^+ \subset [1, M_n], \quad A_{n, n'}, A_{n, n'}^+ \subset [1, M_n], \quad A_{n, n'}, A_{n, n'}^+ \subset [1, M_n],
\]

be such that

\[
\text{card } A_{n, n'}^- = \text{card } A_{n, n'}^-, \text{ card } A_{n, n'}^- = \text{card } A_{n, n'}^-, \text{ card } A_{n, n'}^+ = \text{card } A_{n, n'}^+,
\]

\[
0 \leq n, n' < N.
\]

Then the subshifts

\[
X_N((M_n)_0 \leq n < N, (A_{n, n'}, A_{n, n'}, A_{n, n'}^+)_{0 \leq n, n' < N})
\]

and

\[
X_N((M_n)_0 \leq n < N, A_{n, n'}, A_{n, n'}^+ \subset [1, M_n])
\]

are topologically conjugate.

Proof. With permutations \(\Psi_{n, n'}, \Psi_{n, n'}^+, 0 \leq n < N, \) of \([1, M_n]\), \(0 \leq n' < N, \) such that

\[
\Psi^{-}(n, n')(A_{n, n'}^-) = A_{n, n'}^-, \quad \Psi(n, n')(A_{n, n'}^-) = A_{n, n'}^-, \quad \Psi(n, n')(A_{n, n'}^+) = A_{n, n'}^+,
\]

\[
0 \leq n, n' < N,
\]

a topological conjugacy of \(X_N((M_n)_0 \leq n < N, (A_{n, n'}, A_{n, n'}, A_{n, n'}^+)_{0 \leq n, n' < N})\) onto \(X_N((M_n)_0 \leq n < N, A_{n, n'}, A_{n, n'}^+ \subset [1, M_n])\) is given by the mapping that replaces in a point \(X_N((M_n)_0 \leq n < N, (A_{n, n'}, A_{n, n'}, A_{n, n'}^+)_{0 \leq n, n' < N})\) a symbol \(\alpha^{-}(n', m')\), if preceded by the symbol \(\alpha^{-}(n, m)\), in which case \(m' \in A_{n, n'}^-\), by the symbol \(\alpha^{-}(n', \Psi_{n, n'}^-(m'))\), a symbol \(\alpha(n', m')\), if preceded by the symbol \(\alpha(n, m)\), in which case \(m' \in A_{n, n'}^+\), by the symbol \(\alpha(n', \Psi_{n, n'}^-(m'))\), a symbol \(\alpha^+(n', m')\), if preceded by the symbol \(\alpha^+(n, m)\), in which case \(m' \in A_{n, n'}^+\), by the symbol \(\alpha^+(n', \Psi_{n, n'}^+(m'))\), \(\alpha^+(n, m)\), in which case \(m' \in A_{n, n'}^+\), by the symbol \(\alpha^+(n', \Psi_{n, n'}^+(m'))\).

\(\Box\)

Setting

\[
A^{-}(n, n') = \text{card } A_{n, n'}^-, \quad A(n, n') = \text{card } A_{n, n'}, \quad A^+(n, n') = \text{card } A_{n, n'}^+,
\]

\[
0 \leq n, n' < N,
\]

we introduce matrices

\[
A^{-} = (A^{-}(n, n'))_{0 \leq n, n' < N}, \quad A = (A(n, n'))_{0 \leq n, n' < N}, \quad A^+ = (A^+(n, n'))_{0 \leq n, n' < N}.
\]

In view of Proposition (2.2) we will will write \(X_N((M_n)_0 \leq n < N, A^{-}, A, A^+)\) for \(X_N((M_n)_0 \leq n < N, (A_{n, n'}, A_{n, n'}, A_{n, n'}^+)_{0 \leq n, n' < N})\). Alternatively, in case one wants to be more specific, one can let \(X_N((M_n)_0 \leq n < N, A^{-}, A, A^+)\) denote the subshift

\[
X_N((M_n)_0 \leq n < N, ([1, A_{n, n'}^-], [1, A_{n, n'}], [1, A_{n, n'}^+])_{0 \leq n, n' < N}).
\]

Note that this construction carries the restriction

\[
A^{-}(n, n'), A(n, n'), A^+(n, n') \leq M_{n'}, \quad 0 \leq n, n' < N.
\]
Given a subshift $X_N((M_n)_{0 \leq n < N}, A^-, A, A^+)$, we set
\[
\rho(\alpha^-(n, m)) = 1, \quad \rho(\alpha^+(n, m)) = -1, \quad 1 \leq m \leq M_n, \quad 0 \leq n < N,
\]
and we let
\[
C_{(M_n)_{0 \leq n < N}, A^-, A, A^+}(n, m), \quad 1 \leq m \leq M_n, \quad 0 \leq n < N,
\]
denote the circular Markov code that contains the words
\[
c = (c_i)_{1 \leq i \leq 2J} \in \mathcal{L}(X_N((M_n)_{0 \leq n < N}, A^-, A, A^+)), \quad J \in \mathbb{N},
\]
such that
\[
c_1 = \alpha^-(n, m),
\]
and
\[
\sum_{1 \leq i \leq 2J} \rho(c_i) = 0,
\]
\[
\sum_{1 \leq i \leq 2J} \rho(c_i) > 0, \quad 1 < J < I.
\]
Setting
\[
V = \{(n, m) : 0 \leq n < N, 1 \leq m \leq M_n\},
\]
the range and the source map are given here by setting for
\[
c = (c_i)_{1 \leq i \leq 2J} \in \mathcal{C}_{(M_n)_{0 \leq n < N}, A^-, A, A^+}(n, m), \quad I \in \mathbb{N},
\]
if $c_{2J} \in \{\alpha^+(n, m), 1 \leq m \leq M_n\}$, $r(c)$ equal to $(n, 1)$, and if $c_1 = \alpha^-(n, m)$, setting $s(c)$ equal to $(n, m)$, $1 \leq m \leq M_n$, with the transition matrix given by
\[
B((n, m), (n', m')) = \begin{cases} 1, & \text{if } m' \in \mathcal{A}_{n,n'}, \\ 0, & \text{if } m' \notin \mathcal{A}_{n,n'}, \end{cases} \quad 0 \leq n, n' < N.
\]
Note that replacing the first symbol $\alpha^-(n, 1)$ of the words in the code
\[
C_{(M_n)_{0 \leq n < N}, A^-, A, A^+}(n, 1)
\]
by the symbol $\alpha^-(n, m)$ yields the words in the code
\[
C_{(M_n)_{0 \leq n < N}, A^-, A, A^+}(n, m), 0 \leq n < N, 1 < m \leq M.
\]
Writing $g_n((M_n)_{0 \leq n < N}, A^-, A, A^+), 0 \leq n < N$, for
\[
g(C_{(M_n)_{0 \leq n < N}, A^-, A, A^+}(n, m)), 1 \leq m \leq M_n,
\]
we denote by $G((M_n)_{0 \leq n < N}, A^-, A, A^+)$ the diagonal matrix with the diagonal elements
\[
g_n((M_n)_{0 \leq n < N}, A^-, A, A^+), \quad 0 \leq n < N.
\]

Lemma 2.3.
\[
g_n((M_n)_{0 \leq n < N}, A^-, A, A^+) = z^2(M_n 1 + A^-(1 - G((M_n)_{0 \leq n < N}, A^-, A, A^+))^{-1}G((M_n)_{0 \leq n < N}, A^-, A, A^+))_{n,n}, \quad 0 \leq n < N.
\]

Proof. The proof is by the transfer matrix method. \qed

In proving the next proposition we follow [Ke] [I] [KM] [Kr].

Proposition 2.4.
\[
\zeta_{X_N((M_n)_{0 \leq n < N}, A^-, A, A^+)}(z) = \frac{\det(1 - G((M_n)_{0 \leq n < N}, A^-, A, A^+))}{\det(1 - A^- z - G((M_n)_{0 \leq n < N}, A^-, A, A^+) z \det(1 - A^+ z - G((M_n)_{0 \leq n < N}, A^-, A, A^+))}.
\]
Proof. Applying Lemma (2.1), one finds that the zeta function of the neutral periodic points is given by
\[ \det(1 - G_{(M_n)_{0 \le n < N}} A, A+ A)^{-1}, \]
the zeta function of the sets of periodic points with non-positive multiplier by
\[ \det(1 - A^{-} z)^{-1} \det(1 - A^{-} z - G_{(M_n)_{0 \le n < N}} A, A+ A)^{-1}, \]
and the zeta function of the sets of periodic points with non-negative multiplier by
\[ \det(1 - A^{+} z)^{-1} \det(1 - A^{-} z - G_{(M_n)_{0 \le n < N}} A, A+ A)^{-1}. \]
Taking into account that the intersection of the sets of periodic points with non-positive multiplier and non-negative multiplier is the set of neutral periodic points, one obtains the proposition. 

To consider the case of constant assignments, let \( M \in \mathbb{N} \). Let \( M, K, K_{-}, K_{+} \in \mathbb{N}, K_{-}, K, K_{+} \le M, \) and denote the common value of the generating functions
\[ g(C_{(M)})_{1 \le m \le M, (K_{-})_{0 \le n, n' < N}, (K)_{0 \le n, n' < N}, (K_{+})_{0 \le n, n' < N}}, 0 \le n < N, \]
by \( g(M, K_{-}, K, K_{+}) \).

Proposition 2.5.

\[ g(M, K_{-}, K, K_{+})(z) = \frac{1}{2K^2N} [1 + (MK - K^{-}K^{+})Nz^2 - \sqrt{(1 + (MK - K^{-}K^{+})Nz^2)^2 - 4MKNz^2}], \]
\[ \zeta_{X_{M,K_{-},K,K_{+}}}(z) = \frac{1 - KNg(M, K_{-}, K, K_{+})}{(1 - K^{-}Nz - KNg(M, K_{-}, K, K_{+})(z))(1 - K^{+}Nz - KNg_{M,K_{-},K,K_{+}}(z))}. \]

Proof. By Lemma 2.3
\[ g(z) = z^2(M + \frac{K^{-}NK^{+}g(M, K_{-}, K, K_{+})(z)}{1 - KNg(M, K_{-}, K, K_{+})(z)}), \]
Also apply Proposition 2.4. 

3. THE CASE \( N = 2 \)

We consider the case \( N = 2 \).

Theorem 3.1. Let there be given matrices
\[ A^{-} = (A_{\delta, \delta'}_{\delta, \delta' \in \{0, 1\}}, A = (A_{\delta, \delta'}_{\delta, \delta' \in \{0, 1\}}, A^{+} = (A_{\delta, \delta'}^{+})_{\delta, \delta' \in \{0, 1\}}), \]
with entries in a commutative ring, such that
\[ (A^{-}A^{+})_{0,0} = (A^{-}A^{+})_{1,1}, \]
and
\[ (A^{-}A^{+})_{0,0} = (A^{-}A^{+})_{1,1}. \]
Then
\[ (A^{-}A^{+})_{0,0} = (A^{-}A^{k}A^{+})_{1,1}, \quad k \in \mathbb{N}. \]
Proof. The proof is by induction. Assume that (3.1) and (3.2) hold, let \( k > 1 \) and assume that

\[
(A^-A^kA^+)_{0,0} = (A^-A^kA^+)_{1,1}.
\]

Then

\[
(A^-A^{k+1}A^+)_{0,0} = \\
A^-(0,0)A(0,0)A^k(0,0)A^+(0,0) + A^-(0,0)A(0,1)A^k(1,0)A^+(0,0) + \\
A^-(0,0)A(0,1)A^k(0,1)A^+(1,0) + A^-(0,0)A(1,0)A^k(1,1)A^+(1,0) + \\
A^-(0,1)A^k(1,0)A(0,0)A^+(0,0) + A^-(0,1)A^k(1,1)A(1,0)A^+(0,0) = \\
A(0,0)\{A^-(0,0)A^k(0,0)A^+(0,0) + A^-(0,1)A^k(1,0)A^+(0,0)\} + \\
A(0,1)A^k(1,0)\{A^-(0,0)A^+(0,0) + A^-(0,1)A^+(1,0)\} + \\
A^k(1,1)\{A^-(0,0)A(0,1)A^+(1,0) + \\
A^-(0,1)A(1,1)A^+(1,0)\} + \\
A(0,0)\{A^-(1,1)A^k(1,1)A^+(1,1) + \\
A^-(1,1)A^k(1,0)A^+(0,1) + \\
A^-(1,0)A^k(0,1)A^+(1,1) + \\
A^-(1,0)A^k(0,0)A^+(0,1) - \\
A^-(0,1)A^k(1,1)A^+(1,0)\} + \\
A(0,1)A^k(1,0)\{A^-(1,1)A^+(1,1) + A^-(1,0)A^+(0,1)\} + \\
A^k(1,1)\{-A^-(0,0)A(0,0)A^+(0,0) + \\
A^-(1,1)A(1,1)A^+(1,1) + \\
A^-(1,1)A(1,0)A^+(0,1) + \\
A^-(1,0)A(0,1)A^+(1,1) + \\
A^-(0,1)A(0,0)A^+(0,1)\} = \\
A^-(1,1)A^k(1,1)A(1,1)A^+(1,1) + A^-(1,1)A^k(1,0)A(0,1)A^+(1,1) + \\
A^-(1,1)A^k(1,1)A(0,0)A^+(0,1) + A^-(1,1)A^k(1,0)A(0,0)A^+(0,1) + \\
A^-(1,0)A(0,1)A^k(1,0)A^+(0,1) + A^-(1,0)A(0,0)A^k(0,0)A^+(0,1) + \\
A^-(1,0)A(0,1)A^k(1,1)A^+(1,1) + A^-(1,0)A(0,0)A^k(0,1)A^+(1,1) + \\
A(0,0)A^k(1,1)\{A^-(1,1)A^+(1,1) + A^-(1,0)A^+(1,0) - \\
A^-(0,0)A^+(0,0) - A^-(0,1)A^+(0,1)\} = \\
(A^-A^{k+1}A^+)_{1,1}. \quad \square
One checks that (3.3) holds for matrix triples $A^-, A, A^+$ that have one of the following forms $(\star), (\star\star)$ or $(\star\star\star)$:

$(\star) \quad A^- = \begin{pmatrix} A_0 & A_1 \\ A_1 & A_0 \end{pmatrix}, \quad A = \begin{pmatrix} A_0 & A_1 \\ A_1 & A_0 \end{pmatrix}, \quad A^+ = \begin{pmatrix} A_0^+ & A_1^+ \\ A_1^+ & A_0^+ \end{pmatrix},$

$(\star\star) \quad A^- = \begin{pmatrix} A_0 & A_1 \\ A_0 & A_1 \end{pmatrix}, \quad A = \begin{pmatrix} A(0,0) & A(0,1) \\ A(1,0) & A(1,1) \end{pmatrix}, \quad A^+ = \begin{pmatrix} A_0^+ & A_1^+ \\ A_1^+ & A_0^+ \end{pmatrix},$

$(\star\star\star) \quad A^- = \begin{pmatrix} B_0 & B_1 \\ B_2 & B_3 \end{pmatrix}, \quad A = \begin{pmatrix} A_0 & A_1 \\ A_1 & A_2 \end{pmatrix}, \quad A^+ = \begin{pmatrix} B_2 & B_0 \\ B_3 & B_1 \end{pmatrix}.$

Denote the characteristic polynomial of a matrix $A$ by $\chi_A$.

**Theorem 3.2.** Let there be given an $M \in \mathbb{N}$, and matrices

$A^- = (A_{\delta, \delta'})_{\delta, \delta' \in \{0,1\}}, \quad A = (A_{\delta, \delta'})_{\delta, \delta' \in \{0,1\}}, \quad A^+ = (A_{\delta, \delta'})_{\delta, \delta' \in \{0,1\}},$

with entries in $\mathbb{Z}_+$, such that $A^-(\delta, \delta'), A(\delta, \delta'), A^+(\delta, \delta') \leq M, \quad \delta, \delta' \in \{0,1\}$.

Set

$g_0 = \sum_{k \in \mathbb{Z}_+} g_0(k)z^k = g_0((M, M), A^-, A, A^+),$

$g_1 = \sum_{k \in \mathbb{Z}_+} g_1(k)z^k = g_1((M, M), A^-, A, A^+).$

Assume that

$(3.4) \quad g_0(4) = g_1(4),$

and

$(3.5) \quad g_0(6) = g_1(6).$

Then

$(3.6) \quad (A^- A^+)^{0,0} = (A^- A^+)^{1,1},$

and

$(3.7) \quad (A^- AA^+)_{0,0} = (A^- AA^+)_{1,1},$

and also

$(3.8) \quad g_0 = g_1,$

and, denoting the common value of $(A^- A^+)^{0,0}$ and $(A^- A^+)^{1,1}$ by $\eta(4)$ and the common value of $(A^- AA^+)_{0,0}$ and $(A^- AA^+)_{1,1}$ by $\eta(6)$, the common value $g$ of $g_0$ and $g_1$ satisfies the equation

$(3.6) \quad g(z) = z^2[M + \frac{1}{g(z)^2 \chi_A(g(z)^{-1})} \eta(4)(g(z) + (\eta(6) - \eta(4)\text{tr}A)g(z)^2)].$

**Proof.** It is

$g_0(2) = g_1(2) = M.$

From (3.4)

$M(A^- A^+)_{0,0} = g(4) = M(A^- A^+)_{1,1},$

which is (3.6). From (3.5)

$\eta(4)g(4) + M^2(A^- AA^+)_{0,0} = g(6) = \eta(4)g(4) + M^2(A^- AA^+)_{1,1},$

which is (3.7). The proof of (3.8) is now by induction. Let $k \geq 3$, and let

$g_0(2q) = g_1(2q), \quad 3 < q \leq k.$
Denote the common value of $g_0(2q)$ and $g_1(2q)$ by $g(2q)$, $2 \leq q \leq k$. It is by Lemma (3.1)
\[
g_s(2(k+1)) = \sum_{\delta, \delta' \in \{0,1\}, 1 \leq q \leq k} A^- (\delta, \delta) A^q (\delta, \delta') A^+ (\delta', \delta_s)\]
\[
[\sum_{\{s(r)\} : 1 \leq r \leq q \in \mathbb{N} \cap s(r) = k} \prod_{1 \leq r \leq q} g(2s(r))], \quad \delta_s \in \{0, 1\},
\]
which is (3.8).

One has from Lemma 2.3 that
\[
g_s = z^2 (M + \frac{1}{1 - A(0,0) g_0 - A(1,1) g_1 + g_0 g_1 \text{det } A} [\text{A}^- (\delta, \delta) (1 - g_0 A(\delta', \delta')) g_s A^+ (\delta, \delta) + \text{A}^- (\delta, \delta') g_s A(\delta', \delta) g_s A^+ (\delta', \delta) + \text{A}^- (\delta, \delta') (1 - g_s A(\delta, \delta)) g_s A^+ (\delta', \delta)]], \quad (\delta, \delta') = (0, 1), (1, 0).
\]

Apply (3.8). \hfill \Box

Setting
\[
B(z) = - \text{tr } A - [M \text{ det } A + \eta(6) - \eta(4) \text{tr } A] z^2, \\
C(z) = 1 + [M \text{ tr } A - \eta(4)] z^2, \\
D(z) = -M z^2,
\]
one has from equation (3.6) for the case that $\text{det } A = 0$, that
\[
g = \frac{1}{2 \text{B}} (-C + \sqrt{C^2 - 4BD}),
\]
and, also setting
\[
I = B^2 - 3C \text{ det } A, \quad J = 9BC \text{ det } A - 27D \text{ det } A^2 - 2B^3,
\]
we find by the formulas of Cardano and Vieta, from equation (3.1) for the case that $\text{det } A \neq 0$, and $\chi_A$ has real roots, that
\[
g = \frac{1}{3 \text{det } A} (-B + \sqrt{\frac{2}{3}(J + \sqrt{J^2 - 4I^2})} + \sqrt{\frac{2}{3}(J - \sqrt{J^2 - 4I^2})}),
\]
and for the case that $\text{det } A \neq 0$, and $\chi_A$ has complex roots, that
\[
g = \frac{1}{3 \text{det } A} (-B + 2\sqrt{I} \cos(\frac{1}{2}(2\pi + \arccos \frac{J}{2\sqrt{I}}))).
\]

4. Excluding words from $D_2$

We look now more closely at sets $\mathcal{F}$ of $D_2$-admissible words of length two, that we describe by 0-1 matrices
\[
A^- = \begin{pmatrix} A^- (0,0) & A^- (0,1) \\ A^- (1,0) & A^- (1,1) \end{pmatrix}, \\
A = \begin{pmatrix} A (0,0) & A (0,1) \\ A (1,0) & A (1,1) \end{pmatrix}, \\
A^+ = \begin{pmatrix} A^+ (0,0) & A^+ (0,1) \\ A^+ (1,0) & A^+ (1,1) \end{pmatrix},
\]
where
\[
\mathcal{F} = \{ \alpha^- (\delta) \alpha^- (\delta') : \delta, \delta' \in \{0, 1\}, A^- (\delta, \delta') = 0 \} \cup \\
\{ \alpha^+ (\delta) \alpha^- (\delta') : \delta, \delta' \in \{0, 1\}, A (\delta, \delta') = 0 \} \cup \\
\{ \alpha^+ (\delta) \alpha^+ (\delta') : \delta, \delta' \in \{0, 1\}, A^+ (\delta, \delta') = 0 \}.
\]
The subshift that is obtained by removing the words in $\mathcal{F}$ from $D_2$ is identical to the subshift $X(1, 1, A^-_F, A_F, A^+_F)$.

We are interested in the set $\mathcal{T}$ of matrix triplets $(A^-_F, A_F, A^+_F)$ such that $L_2(D_2) \setminus \mathcal{F}$ is the vertex set of a $D_2$-presentation, and such that $\lambda_0(1, 1, A^-_F, A_F, A^+_F)$ is equal to $g_1(1, 1, A^-_F, A_F, A^+_F)$. The set $L_2(D_2) \setminus \mathcal{F}$ is the vertex set of a $D_2$-presentation precisely if each row of $A^-_F$ is non-zero, $A_F$ is irreducible, and each column of $A^+_F$ is non-zero. The edge set of the $D_2$-presentation is then the set of words in $L_3(D_2)$ with a prefix in $L_2(D_2) \setminus \mathcal{F}$, which acts as the initial vertex of the edge, and a suffix in $L_2(D_2) \setminus \mathcal{F}$, which acts as the final vertex of the edge. The label map $\lambda$ is given by

$$\lambda(\beta_0 \beta_1 \beta_2) = \begin{cases} 
\beta_0, & \text{if } \beta_0 \beta_1 \beta_2 \in S^+ \setminus \{1\}, \\
\beta_1, & \text{if } \beta_0 \beta_1 \beta_2 \in S^- \setminus \{1\}, \\
1, & \text{otherwise,}
\end{cases} \quad \beta_0, \beta_1, \beta_2 \in \mathcal{L}(D_2).$$

Dyck shifts have a time reversal, by which is meant a topological conjugacy between the subshift and its inverse. For $D_2$ the time reversal $T$ is given by

$$T(x)_i = \begin{cases} 
\alpha_-(0), & \text{if } x_{i-1} = \alpha_+(0), \\
\alpha_+(1), & \text{if } x_{i-1} = \alpha_-(1), \\
\alpha_-(0), & \text{if } x_{i-1} = \alpha_+(0), \\
\alpha_-(1), & \text{if } x_{i-1} = \alpha_+(1),
\end{cases} \quad x = (x_i)_{i \in \mathbb{Z}} \in D_2.$$

It is

$$T(X(1, 1, A^-_F, A^+_F)) = X(1, 1, (A^-_F)^T, A^-_F, (A^+_F)^T).$$

As an application of theorem (3.2) we list the triplets in the set $\mathcal{T}$ that are neither of the form $(\ast)$ nor of the form $(\ast \ast)$ nor $(\ast \ast \ast)$, choosing a representative out of every set of triplets that can be obtained from one another by exchanging the indices 0 and 1 and/or by time reversal: For

$$A_F = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix},$$

we take

$$A^-_F = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}, \quad A^+_F = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

and for

$$A_F = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad A^-_F = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix},$$

we take

$$A^-_F = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \quad A^+_F = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

$$A^-_F = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad A^+_F = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix},$$

$$A^-_F = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}, \quad A^+_F = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

$$A^-_F = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}, \quad A^+_F = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}.$$

The zeta functions of the periodic points with negative and with positive multipliers, and of the neutral periodic points of the $D_2$-presentations that arise from the triplets in the set $\mathcal{A}$ can be obtained from Proposition 2.5 using the formulas.
(3.7), (3.8) and (3.9). In a number of cases they can also be determined by direct inspection without the use of circular codes that are properly Markov, that is, without recourse to Lemma 2.1.

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