Minimal assumption derivation of a Bell-type inequality

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John Bell showed that a big class of local hidden-variable models stands in conflict with quantum mechanics and experiment. Recently, there were suggestions that empirical adequate hidden-variable models might exist, which presuppose a weaker notion of local causality. We will show that a Bell-type inequality can be derived also from these weaker assumptions.

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I. INTRODUCTION

The violation of Bell’s inequality by the outcome of an EPR-type spin experiment [1, 2] seems to exclude a local theory with hidden variables. The underlying reductio ad absurdum proof infers on the grounds of the empirical falsification of the derived inequality that at least one of the required assumptions must be false. The force of the argument requires that the derivation be deductive and that all assumptions be explicit. We aim to extract a minimal set of assumptions needed for a deductive derivation of Bell’s inequalities given perfect correlation of outcomes of an EPR-type spin experiment with parallel settings.

One of the assumptions in Bell’s original derivation was determinism. Later, he succeeded in deriving a similar inequality without determinism, placing in its stead an assumption later dubbed local causality. As Bell stressed, the notion of local causality he and others used might be challenged. In [3] it was pointed out, that Reichenbach’s Common Cause Principle indeed suggests a weaker form of local causality. We will prove here, however, that even from this weaker notion Bell’s inequality can still be derived.1

Consider the so-called EPR-Bohm (EPRB) experiment [1, 2]. Two spin- particles in the singlet state

\[ |\Psi\rangle = \frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle) \tag{1} \]

are separated in such a way that one particle moves to a measurement apparatus in the left wing of the experimental setting and the other particle to a measurement apparatus in the right wing (see FIG. 1). The experimenter can choose arbitrarily one of three directions in which the spin is measured with a Stern-Gerlach magnet.

The following terminology follows the reconstruction by Wigner [10] and which van Fraassen [11] has subsequently expanded on. The event type\(^{2}\) that the left (right) measurement apparatus is set to measure the spin in direction \(i \in \{1, 2, 3\}\) is symbolized by \(L_i\) (\(R_i\)). \(L_i^a\) (\(R_i^a\)) symbolizes the event type that the measurement outcome in the left (right) wing of a spin measurement in direction \(i\) is \(a\). There are two possible measurement outcomes spin up \((a=+\rangle\) and spin down \((a=-\rangle\) for each particle in each direction. The letter \(j \in \{1, 2, 3\}\) will be used like \(i\) to symbolize directions and \(b \in \{+, -\}\) like \(a\) to symbolize measurement outcomes. Formulas in which the variables \(i, j, a, b\) appear are meant to hold—if not otherwise stated—for all possible values of the variables. \(p(X)\) denotes the probability of an event type \(X\), which is empirically measurable as the relative frequency of all runs of an EPRB experiment in which the event type \(X\) is instantiated, with respect to all runs. \(p(X \wedge Y)\) is the probability of the event type ‘\(X\) and \(Y\)’, measurable as the relative frequency of all runs in which both \(X\) and \(Y\) are instantiated. \(p(X|Y) = p(X \wedge Y)/p(Y)\) is the conditional probability of the event type \(X\) given the event type \(Y\), measurable as the relative frequency of instantiations of \(X\) with respect to the subensemble of

\[ \begin{array}{c}
\text{screen} \\
\text{magnets} \\
L_i^+ \\
\text{source} \\
L_i \\
\text{magnets} \\
R_j^- \\
\text{screen}
\end{array} \]

FIG. 1: Setup of the EPR-Bohm experiment. Cf. [3, p. 140].

\(^{1}\) Several of the issues we present in this paper are discussed in more detail in [3].

\(^{2}\) We will speak of event types to distinguish them from the token events which instantiate corresponding event types.
all runs in which \( Y \) is instantiated. E.g.
\[
p(L_i^a \land R_j^b | L_i \land R_j)
\]
denotes the probability that the measurement outcome is \( a \) on the left and \( b \) on the right, when measuring in direction \( i \) on the left and in direction \( j \) on the right. These probabilities are predicted by quantum mechanics as
\[
p(L_i^+ \land R_j^+ | L_i \land R_j) = \frac{1}{2} \sin^2 \frac{\varphi_{ij}}{2},
\]
where \( \varphi_{ij} \) denotes the angle between the two measurement directions \( i \) and \( j \). Also, the outcomes on each side are predicted separately to be completely random:
\[
p(L_i^a | L_i \land R_j) = \frac{1}{2},
\]
\[
p(R_j^b | L_i \land R_j) = \frac{1}{2}.
\]

III. LOCAL CAUSALITY

The derivations of Bell-type inequalities known to us which do not presuppose determinism assume instead what John Bell calls local causality \[3, 12\]. That is, the assumption that there is common cause variable \( V \) which takes on values \( q \in I = \{q_1, q_2, \ldots, q_k\} \) such that for event types ‘the variable \( V \) has the value \( q \)’ (\( Vq \)) we have \( \sum_q p(Vq) = 1 \) and
\[
p(L_i^a \land R_j^b | Vq \land L_i \land R_j) = p(L_i^a | Vq \land L_i)
\times p(R_j^b | Vq \land R_j).
\]

Equation (10) says that event types \( Vq \) or the variable \( V \) “screens off” \( L_i^a \) and \( R_j^b \) from each other \[11, 13\]. Van Fraassen \[11\] pointed out that equation (10) can be motivated through Reichenbach’s Principle of Common Cause (PCC) \[5\]. The principle states, that whenever two different event types \( A \) and \( B \) are statistically correlated
\[
p(A \land B) \neq p(A)p(B)
\]
and neither \( A \) is causally relevant for \( B \) nor \( B \) for \( A \), there exists a common cause variable \( V \) with values \( q \in I = \{q_1, q_2, \ldots, q_k\} \) \( \sum_q p(Vq) = 1 \) such that \( A \) and \( B \) given \( Vq \) are uncorrelated:
\[
p(A \land B | Vq) = p(A | Vq)p(B | Vq).
\]

In its original formulation the principle is stated only for a common cause event type \( C \), which is included in our formulation as the special case where \( Vq \) can take only two values: \( Vq_1 = C, Vq_2 = \neg C \) (‘not \( C \)’). The principle was formulated for general common cause variables by Hofer-Szabó et al. \[17\] and Placek \[18\]. Besides the screening-off condition Reichenbach \[5\] and Hofer-Szabó et al. \[17\] stipulate further restrictions on the common cause variable, which are, however, irrelevant for our purposes.

Now, as can be seen from equations (8)-(10), the event type \( L_i^a \) is in general correlated with event type \( R_j^b \). It is
\[
p(L_i^a | L_i \land R_j) = p(R_j^b | L_i \land R_j) = \frac{1}{2},
\]
and therefore
\[
p(L_i^a \land R_j^b | L_i \land R_j) \neq p(L_i^a | L_i \land R_j)p(R_j^b | L_i \land R_j)
\]
except for \( \varphi_{ij} = \frac{\pi}{2} \mod \pi \).

Supposing that \( L_i^a \) is not causally relevant for \( R_j^b \) and vice versa (which is reinforced by the fact that the setup of the experiment can be chosen so that the instantiations of \( L_i^a \) and \( R_j^b \) in each run of the experiment are space-like separated), PCC requires a common cause variable which fulfills equation (10). There are several different correlations; e.g. \( L_i^+ \) is correlated with \( R_j^+ \), and \( L_i^+ \) is correlated with \( R_j^+ \). For each of these correlations PCC enforces the consequence that a common cause variable exists. As stressed in \[3\] nothing in PCC dictates that the common cause variables of the different correlations have to be the same. However, in all the derivations of Bell’s inequality known to us this identification is made nevertheless. It is further shown in \[6\] and \[17\], that for any set of correlations it is mathematically possible to construct common cause variables. The authors concluded in \[6\] that the apparent contradiction between this possibility and the claim that the EPRB correlations do not allow for a common cause variable \[11, 13\], is resolved by pointing out that in the derivation of Bell’s inequality a common common cause variable for all measurements is assumed:

\[\text{For the sake of simplicity, we assume that this partition is discrete and finite. As will become clear in the following, the derivation of Bell’s inequality can also be done without this restriction.}\]
We use here the definition of Assumption 1 (PCORR) or PCORR for short. Without this assumption Bell’s inequality cannot be derived. But there does not seem to be any obvious reason why common causes should also be common common causes, whether of quantum or of any other sort of correlations.” (Italics in the original.)

Showing the mathematical possibility of constructing common cause variables for any set of correlations and in particular for the correlations found in the EPRB experiment is not sufficient for proving the existence of a physically “natural” hidden-variable model for that experiment, however. Besides being common cause variables (thus fulfilling equation (10)), parameter independence should hold, too (equations (11) and (12)). Also, they should not be correlated with the measurement choices. As shown by Szabó [13], it is possible to construct a model which fulfills these requirements for each of the common cause variables separately. However, the conjunctions and other logical combinations of the event types that the common cause variables have certain values correlate in that model with the measurement operations. Whether a model can be constructed without these correlations was posed as an open question by Szabó. This question is answered negatively by the derivation of Bell’s inequality that we present in the remainder of this article.

IV. BELL’S INEQUALITY FROM SEPARATE COMMON CAUSES

A. A weak screening-off principle

Consider an EPRB experiment where the same direction $i$ ($i \in \{1, 2, 3\}$) is chosen in both wings. That is, in each run the event type $L_i \wedge R_j$ is instantiated. With this special setting quantum mechanics predicts (see equations (9)-8, with $\varphi_{ij} = 0$) that the measurement outcomes in each wing are random but that the outcomes in one wing are perfectly correlated with the outcomes in the other wing: if and only if the spin of the left particle is up, then the spin of the right particle is down, and vice versa. We refer to this assumption as perfect correlation, or PCORR for short.

Assumption 1 (PCORR)

$$p_{ij}(R_i^- | L_i^+) = 1 \text{ and } p_{ii}(L_i^+ | R_i^+) = 1.$$  \hfill (17)

We use here the definition

$$p_{ij}(\ldots) \equiv p(\ldots | L_i \wedge R_j).$$  \hfill (18)

Large spatial separation of coinciding events of type $L_i^n$ and $R_j^b$ suggests that the respective instances are indeed distinct events. This excludes an explanation of the correlations by event identity, as is the case, for example, with a tossed coin for the perfect correlation of the event types ‘heads up’ and ‘tails down’. Such a perfect correlation is explained in that every instance of ‘heads up’ is also an instance of ‘tails down’, and vice versa. Since the separation is even space-like, no $L_i^n$ or $R_j^b$ should be causally relevant for the other. We refer to these two assumptions as separability, SEP for short, and locality 1 (LOC1).

Assumption 2 (SEP) The coinciding instances of $L_i^n$ and $R_j^b$ are distinct events.

Assumption 3 (LOC1) No $L_i^n$ or $R_j^b$ is causally relevant for the other.

Rather, there should be a common cause variable; that is, we assume PCC.

Assumption 4 (PCC) If two event types $A$ and $B$ are correlated and the correlation cannot be explained by direct causation nor event identity, then there exists a common cause variable $V_q$, with values $q \in I = \{q_1, q_2, q_3, \ldots, q_k\}$ such that $\sum_q p(V_q) = 1$ and

$$p(A \wedge B|V_q) = p(A|V_q)p(B|V_q), \quad \forall q.$$  

As already mentioned, we omit the other Reichenbachian conditions [7, 17] since they are not necessary for our derivation.

This principle together with the assumptions PCORR, SEP and LOC1 implies that there is for each of the EPRB correlations a (separate) common cause variable $V_{ij}^\pm$ with $q \in I_{ij}^{\pm}$.

Result 1

$$p_{ii}(L_i^+ \wedge R_i^- | V_{ij}^\pm_q) = p_{ii}(L_i^+ | V_{ij}^\pm_q) \times p_{ii}(R_i^- | V_{ij}^\pm_q).$$ \hfill (19)

Note that common cause variables can be different for different correlations.

B. Perfect correlation and “determinism”

We now show that from the fact that a perfect correlation is screened off by some variable it follows that without loss of generality the common cause variable can be assumed to be two-valued and that the having of one of the two values of the variables is necessary and sufficient for the instantiation of the two perfectly correlated event types, cf. [20].

Let $A$ and $B$ be perfectly correlated,

$$p(A|B) = p(B|A) = 1,$$
and screened-off from each other by a common cause variable,
\[ p(A \land B | Vq) = p(A | Vq)p(B | Vq). \]
We can split the set \( I \) of all values \( V \) completely into two
disjoint subsets, namely in the subset \( I^+ \) of those values
of \( V \) for which \( p(A \land Vq) \) is not zero and in the subset \( I^- \)
of those for which it is zero:
\[ I^+ = \{ q \in I : p(A \land Vq) \neq 0 \}, \]
\[ I^- = \{ q \in I : p(A \land Vq) = 0 \}, \]
\[ I = I^- \cup I^+, \quad I^- \cap I^+ = \emptyset. \]
From this definition of \( I^- \) it follows already that
\[ p(A | Vq) = 0, \quad \forall q \in I^-, \quad (20) \]
i.e. that \( Vq \) with \( q \in I^+ \) is necessary for \( A \). Moreover, for
all \( q \in I^+ \) we have by screening off and perfect correlation
\[ p(A | Vq) = p(A | B \land Vq) = 1. \quad (21) \]
That the variable has a value in \( I^+ \) is a necessary and sufficient
condition for \( A \). The following calculation shows that \( Vq \) with \( q \in I^+ \) is also necessary and sufficient for
\( B \).
From perfect correlation it follows that
\[ p(B | A \land Vq) = 1, \quad \forall q \in I^+. \]
That \( Vq \) screens off \( B \) from \( A \) yields
\[ p(B | A \land Vq) = p(B | Vq). \]
Together with the previous equation this implies that \( Vq \)
is sufficient for \( B \) for all \( q \in I^+ \):
\[ p(B | Vq) = 1 \quad \forall q \in I^+. \quad (22) \]
If \( q \in I^- \) we have by definition \( p(A \land Vq) = 0 \), which implies
\[ p(A \land B \land Vq) = 0. \]
By perfect correlation we have therefore also \( p(B \land Vq) = 0 \), which in turn implies that
\[ p(B | Vq) = 0, \quad \forall q \in I^-, \quad (23) \]
which means that \( Vq \) with \( q \in I^+ \) is also necessary for \( B \).
This calculation shows that in the case of a perfect
correlation the set of values of the common cause variable
decomposes into two relevant sets. This means that
whenever there is an (arbitrarily-valued) common cause variable
for a perfect correlation, there is also a two-valued
common cause variable, namely the disjunction
of all event types \( Vq \) for which \( q \in I^+ \) or \( q \in I^- \), respectively.
\[ C = \bigvee_{q \in I^+} Vq, \]
\[ \neg C = \bigvee_{q \in I^-} Vq. \]
We refer to \( C \) as a common cause event type. In the case
of a perfect correlation no generality is achieved by allowing
for a more than two-valued common cause variable; if there is a common cause variable for a perfect correlation,
there is also a common cause event type. Moreover, the
common cause event type is a necessary and sufficient
condition for the event types that are screened off by it
(equations (20), (21), (22) and (23)).

Result 1 thus implies that there is a common cause
event type \( C_{ii}^{+/-} \) such that
\[ p_{ii}(L_i^+ | C_{ii}^{+/-}) = p_{ii}(R_i^- | C_{ii}^{+/-}) = 1, \quad (24) \]
\[ p_{ii}(L_i^+ | -C_{ii}^{+/-}) = p_{ii}(R_i^- | -C_{ii}^{+/-}) = 0. \quad (25) \]
The sub- and superscripts of \( C_{ii}^{+/-} \) refer to \( C_{ii}^{+/-} \) being the
common cause event type of \( L_i \) and \( R_i \).

The outcome of a spin measurement is always either + or − and nothing else. We call this assumption exactly
one of exactly two possible outcomes (EX).

Assumption 5 (EX)
\[ p_{ii}(L_i^+) + p_{ii}(L_i^-) = 1, \quad p_{ii}(L_i^+ \land L_i^-) = 0, \quad (26) \]
\[ p_{ii}(R_i^+) + p_{ii}(R_i^-) = 1, \quad p_{ii}(R_i^+ \land R_i^-) = 0. \quad (27) \]

As stressed by Fine [21], among the actual measurements
there are always runs in which no outcome is registered,
which is normally attributed to the limited efficiency of
the detectors and not taken to the statistics. If one assumes
instead, that part of these non-outcome runs are
caused by the hidden variable, then it is possible to construct
empirically adequate models for the EPRB experiments [22, 23].
We make use of assumption (EX) to explicitly exclude
such models.

With assumption (EX) while \( C_{ii}^{+/-} \) is necessary and sufficient
for \( L_i^+ \) and \( R_i^- \), its complement, i.e. \( -C_{ii}^{+/-} \) is necessary
and sufficient for the opposite outcomes, i.e. \( L_i^- \) and \( R_i^+ \):
\[ p_{ii}(L_i^- | C_{ii}^{+/-}) = p_{ii}(R_i^+ | C_{ii}^{+/-}) = 0, \quad (28) \]
\[ p_{ii}(L_i^- | -C_{ii}^{+/-}) = p_{ii}(R_i^+ | -C_{ii}^{+/-}) = 1. \quad (29) \]

C. A minimal theory for spins

In section (IV.B) it was found that \( C_{ii}^{+/-} \) is sufficient for
\( L_i^+ \) given parallel settings \( (L_i \land R_i) \), see equation (24). I.e. the conjunction
\( C_{ii}^{+/-} \land L_i \land R_i \) is sufficient for \( L_i^+ \).
But because of space-like separation of events of type \( L_i^+ \)
and \( R_i \) that are instanitated in the same run, the latter
types should not be causally relevant for the former.
The measurement choice in one wing should be causally
irrelevant for the outcomes (and the choices) in the other
wing. Therefore we should discard \( R_i \) from the sufficient
conjunction. The part \( C_{ii}^{+/-} \land L_i \) alone is sufficient for
\( L_i^+ \). A similar reasoning can be applied to \( R_j^+, R_j \) and
\( -C_{jj}^{+/-} \), cf. equation (24). This is our assumption locality 2 (LOC2).
**Assumption 6 (LOC2)** If $L_i \land R_i \land X$ is sufficient for $L_i^+$, then $L_i \land X$ alone is sufficient for $L_i^+$; and similarly for $R_i^+$, i.e. if $L_j \land R_j \land Y$ is sufficient for $R_j^+$, then $R_j \land Y$ alone is sufficient for $R_j^+$.

Moreover, the remaining part $C_{ii}^{+-} \land L_i$ is minimally sufficient, in the sense that none of its parts is sufficient on its own. If, for example, $C_{ii}^{+-}$ is instantiated, but we do not choose to measure $L_i$, then $L_i^+$ will not be instantiated. That is to say, we cannot discard yet another conjunct of $L_i \land C_{ii}^{+-}$ as we discarded $R_i$ from $C_{ii}^{+-} \land L_i \land R_i$.

Let us turn to necessary conditions for $L_i^+$. To begin with, $L_i$ is necessary: If there is no Stern-Gerlach magnet properly set up ($L_i$) the particle is not deflected either up- or downwards; similarly for $L_i^-$, $R_i^+$ and $R_i^-$. Roughly speaking, **no outcome without measurement (NOWM).**

**Assumption 7 (NOWM)**

\[
p(L_i^+ \land \neg L_i) = 0, \quad p(L_i^+ \land \neg L_i) = 0, \quad (30)\]

\[
p(R_i^+ \land \neg R_i) = 0, \quad p(R_i^- \land \neg R_i) = 0. \quad (31)\]

Second, we saw in section V B that if parallel settings are chosen and $\neg C_{ii}^{+-}$ is instantiated an event of type $L_i^+$ does never occur. In other words, $\neg C_{ii}^{+-} \land L_i \land R_i$ implies $\neg L_i^+$:

\[
\neg C_{ii}^{+-} \land L_i \land R_i \rightarrow \neg L_i^+. \quad (32)\]

Again we propose a locality condition based on the idea that the measurement choice in one wing should be causally irrelevant for the outcomes (and the choices) in the other wing: If $\neg C_{ii}^{+-} \land L_i \land R_i$ are sufficient for $L_i^+$, then $\neg C_{ii}^{+-} \land L_i$ alone should be sufficient for $\neg L_i^+$. A similar reasoning can be applied to $R_j^+$, $R_j$ and $C_{jj}^{+-}$, cf. equation (30).

**Assumption 8 (LOC3)** If $L_i \land R_i \land X$ is sufficient for $\neg L_i^+$, then $L_i \land X$ alone is sufficient for $\neg L_i^+$; and similarly for $\neg R_j^+$, i.e. if $L_j \land R_j \land Y$ is sufficient for $\neg R_j^+$, then $R_j \land Y$ alone is sufficient for $\neg R_j^+$.

By LOC3 it follows from equation (32) that

\[
\neg C_{ii}^{+-} \land L_i \rightarrow \neg L_i^+. \quad (33)\]

This is equivalent to

\[
L_i^+ \land L_i \rightarrow C_{ii}^{+-}, \quad (34)\]

and also to

\[
L_i^+ \land L_i \rightarrow C_{ii}^{+-} \land L_i. \quad (35)\]

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Footnotes:

4 Minimal sufficient conditions as defined by [24] and [25].

5 The following version of LOC3 is slightly different from an earlier version of the article. We thank Gábor Hofer-Szabó, Miklos Rédei and Iñaki San Pedro for their comments.

6 For details see [24] and [25]. Note in particular that a correct formal notation of a minimal theory uses what Graßhoff et. al. [24, 25] call a double conditional.
From the logical relations \(2(a), 2(b), 2(c)\) and \(2(d)\) the following probabilities can be derived:

\[
\begin{align*}
p(L_1^+ \land R_2^+ | L_1 \land R_2) &= p(L_1 \land C_{11}^{+} \land R_2 \land -C_{22}^{+}), \\
p(L_2^+ \land R_3^+ | L_2 \land R_3) &= p(L_2 \land C_{22}^{+} \land R_3 \land -C_{33}^{+}), \\
p(L_1^+ \land R_3^+ | L_1 \land R_3) &= p(L_1 \land C_{11}^{+} \land R_3 \land -C_{33}^{+}).
\end{align*}
\]

By NOWM (equations \(30\) and \(31\)) \(p(L_1^+ \land R_2^+)\) is the same as \(p(L_1^+ \land R_2^+ \land L_1 \land R_2)\) etc. and the above equations read:

\[
\begin{align*}
p(L_1^+ \land R_2^+ \land L_1 \land R_2) &= p(L_1 \land C_{11}^{+} \land R_2 \land -C_{22}^{+}), \\
p(L_2^+ \land R_3^+ \land L_2 \land R_3) &= p(L_2 \land C_{22}^{+} \land R_3 \land -C_{33}^{+}), \\
p(L_1^+ \land R_3^+ \land L_1 \land R_3) &= p(L_1 \land C_{11}^{+} \land R_3 \land -C_{33}^{+}).
\end{align*}
\]

\[\text{(36)}\]

\[\text{(37)}\]

\[\text{(38)}\]

\[\text{D. No conspiracy}\]

The events of type \(C_{ii}^{+}\) are not supposed to be influenced by the measuring operations \(L_i\) and \(R_j\). One reason for this assumption is that the measurement operations can be chosen arbitrarily before the particles enter the magnetic field of the Stern-Gerlach magnets and that an event of type \(C_{ii}^{+}\) is assumed to happen before the particles arrive at the magnets. Therefore a causal influence of the measurement operations on events of type \(C_{ii}^{+}\) would be tantamount to backward causation. Also an inverse statement is supposed to hold: The event types \(C_{ij}^{+}\) are assumed not to be causally relevant for the measurement operations. This is meant to rule out some kind of “cosmic conspiracy” that whenever an event of type \(C_{ii}^{+}\) is instantiated, the experimenter would be “forced” to use certain measurement operations. This causal independence between \(C_{ii}^{+}\) and the measurement operations is assumed to imply the corresponding statistical independence. The same is assumed to hold also for conjunctions of common cause event types. We refer to this condition as no conspiracy (NO-CONS).

**Assumption 9 (NO-CONS)**

\[
p(C_{ii}^{+} \land -C_{jj}^{+} | L_i \land R_j) = p(C_{ii}^{+} \land -C_{jj}^{+}).\]

By this condition of statistical independence the three probabilities considered above can be transformed. That is, we have, for instance

\[
p(L_1^+ \land R_2^+ | L_1 \land R_2) = \frac{p(L_1^+ \land R_2^+ \land L_1 \land R_2)}{p(L_1 \land R_2)}
\]

\[
\begin{align*}
&= p(L_1 \land C_{11}^{+} \land R_2 \land -C_{22}^{+}) \\
&= \frac{p(C_{11}^{+} \land -C_{22}^{+} \land C_{33}^{+})}{p(L_1 \land R_2)} \\
&= p(C_{11}^{+} \land -C_{22}^{+}) + p(C_{11}^{+} \land -C_{22}^{+} \land -C_{33}^{+}).
\end{align*}
\]

The dotted equations are true by definition of conditional probability. In step (i) equation \(31\) was used. Step (ii) is valid by “no conspiracy” (equation \(39\)), and (iii) by a theorem of probability calculus, according to which \(p(A) = p(A \land B) + p(A \land \neg B)\) for any \(A\) and \(B\). Transforming the other two expressions in a similar way, we arrive at

\[
p(L_1^+ \land R_2^+ | L_1 \land R_2)
\]

\[
= p(C_{11}^{+} \land -C_{22}^{+} \land C_{33}^{+}) \\
+ p(C_{11}^{+} \land -C_{22}^{+} \land -C_{33}^{+}), \quad \text{(40)}
\]

\[
= p(C_{11}^{+} \land C_{22}^{+} \land -C_{33}^{+}) \\
+ p(-C_{11}^{+} \land C_{22}^{+} \land -C_{33}^{+}), \quad \text{(41)}
\]

\[
p(L_1^+ \land R_3^+ | L_1 \land R_3)
\]

\[
= p(C_{11}^{+} \land C_{22}^{+} \land C_{33}^{+}) \\
+ p(C_{11}^{+} \land -C_{22}^{+} \land -C_{33}^{+}), \quad \text{(42)}
\]

Since both terms of the right-hand side of the last equation appear in the sum of the right-hand sides of the first two equations, the following version of the Bell inequality (BELL) follows\(^7\).

**Result 3 (BELL)**

\[
p(L_1^+ \land R_2^+ | L_1 \land R_2) \leq p(L_1^+ \land R_2^+ | L_1 \land R_2)
\]

\[
+ p(L_2^+ \land R_3^+ | L_2 \land R_3). \quad \text{(43)}
\]

This inequality has been empirically falsified, see e.g. \(24\).

The inequality was derived from the following assumptions.

- Perfect correlation (PCORR),
- separability (SEP),

\(^7\) It was first derived in this form by Wigner \([13]\).
• locality 1 (LOC1),
• principle of common cause (PCC),
• exactly one of exactly two possible outcomes (EX),
• locality 2 (LOC2),
• no outcome without measurement (NOMW),
• locality 3 (LOC3),
• no conspiracy (NO-CONS).

This is a version of Bell’s theorem. It says: If these assumptions are true, the Bell inequality is true. The derivation of the Bell inequality presented here is an improvement on the usual Bell-type arguments, such as Ref. 8 and 11, in two respects: First, it does not assume a common cause variable for different correlations. Second, contrary to the usual locality conditions, the ones assumed here do not presuppose a solution to the problems posed by the relation between causal and statistical (in)dependence (see e.g. 27).

V. DISCUSSION

Our claim to have presented a minimal assumption derivation of a Bell-type inequality is relative: our set of assumptions is weaker than any set known to us from which a Bell-type inequality can be derived and that contains the assumption of perfect correlation (PCORR). It was one of the achievements of Clauser and Horne 12 to show that a Bell-type inequality can be derived also if the correlations of outcomes of parallel spin measurements are not assumed to be perfect. Our assumption of correlation is stronger than the one used by Clauser and Horne. However, they assume a common cause variable for all correlations, which is a stronger assumption than our assumption of possibly different common cause variables for each correlation (PCC). We have not been able to derive a Bell-type inequality without assuming perfect correlation and allowing different common cause variables. If PCORR is indeed a necessary assumption for our derivation of the Bell inequality, it should be possible to construct a model in which PCORR does not hold (being violated by an arbitrary small deviation, say). Since the actually measured correlations are never perfect—a fact that is usually attributed to experimental imperfections—it is not obvious how such a model could be refuted.

Our notion of local causality might be challenged as follows. Even though nothing in PCC dictates that in general the common cause variables of different correlations have to be the same, there might be strong grounds for why they are the same in the context of the EPRB experiment. Indeed, Bell argued for his choice of local causality along the following lines.8 Assume that $L_i^a$ and $R_j^b$ are positively correlated. Then

$$p(L_i^a \mid R_j^b \wedge L_i \wedge R_j) > p(L_i^a \mid L_i \wedge R_j).$$

(44)

Since coinciding instances of $L_i^a$ and $R_j^b$ are space-like separated, neither is causally relevant for the other. Rather, the correlation should be explained by exhibiting some common causes in the overlap of the backward light cones of the coinciding instances. An instance of, say, $L_i^a$ raises the probability of an instantiation of one of the common causally relevant factors, and this raises the probability of an instantiation of $R_j^b$. But given the total state of the overlap of the backward light cones of two coinciding instances, the probability of, say, $R_j^b$ is assumed to be the same whether $L_i^a$ is instantiated or not. If the total state of the overlap of the backward light cones is already given, nothing more that could be causally relevant for $R_j^b$ can be inferred from an instance of $L_i^a$.

Along this line of reasoning the total state $V_q$ of the overlap of the backward light cones9 of $L_i^a$ and $R_j^b$ is a common cause variable which screens off the correlation:

$$p(L_i^a \wedge R_j^b \mid L_i \wedge R_j \wedge V_q) = p(L_i^a \mid L_i \wedge R_j \wedge V_q) 	imes p(R_j^b \mid L_i \wedge R_j \wedge V_q).$$

(45)

The common past $V_q$ cannot be altered by choosing one or the other direction for the spin measurement—“facta infecta fieri non possunt” 13, p. 185). Therefore the total state $V_q$ of the common past is indeed a common cause variable for all correlated outcomes, see FIG. 3.

This reasoning can be questioned along the following lines. It is reasonable that not all event types that are instantiated in the overlap of the backward light cones of two coinciding instances of the correlated event types are causally relevant for these latter event types. Therefore

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8 For a very good and more detailed discussion of this, see 12.

9 One might argue that the total state of the union of the backward light cones is a better candidate for a common cause variable 13.

The following discussion carries over also to this case.
conditionalizing on the total state is conditionalizing not only on the relevant factors but also on the irrelevant. Moreover, it is conceivable that which event types of the common past are relevant and which are not differs for different measurements. Claiming that the total state of the common past is a common common cause variable, one is thus committed to assume that

“conditionalizing on all other events [...] in addition to those affecting [the correlated event types], does not disrupt the stochastic independence induced by conditionalizing on the affecting events.”

In particular in the light of Simpson’s paradox, this assumption has been challenged \[29\]. Here, we will not assess arguments in favour of or against the possibility that conditionalizing on irrelevancies yields unexpected statistical dependencies. Our point is that by weakening the assumption in the way we did, our derivation is conclusive whatever may be the answer to this question.

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