On the Existence of Doubly Convex MOTS

Henri Roesch

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Abstract

In [31] the notion of Double Convexity for a foliation of a Null Cone was introduced to give a proof, if satisfied, of the Null Penrose Inequality. In this paper, for a class of strictly stable, Weakly Isolated Horizons we show the existence of a unique foliation by Doubly Convex Marginally Outer Trapped Surfaces (MOTS). Moreover, we show that any sufficiently small metric perturbation continues to support the existence of a Doubly Convex MOTS. Using this, and a lemma of S. Alexakis [1], we verify that the Null Penrose Inequality remains satisfied for a class of physically reasonable metric perturbations off of the standard Schwarzschild Null Cone.

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1 Introduction

In [29, 30], the Penrose Conjecture for asymptotically flat spacetimes posits a geometric interpretation of the assumption that the total mass of an isolated physical system, as measured by an observer ‘at infinity’, should be no smaller than the mass of its black holes. Cases in which the Conjecture have been verified include spherical symmetry [18, 23] and time symmetric hypersurfaces [19, 9, 10]. The setting of this paper involves a more recent approach utilizing null hypersurfaces.

From this perspective, a quasi-local black hole $\Sigma_0$ is given by a spherical space-like surface with vanishing future null expansion, called a Marginally Outer Trapped Surface (or MOTS), connected
to past null infinity $I^-$ by a smooth null hypersurface $\Omega$. From every asymptotically round foliation of $\Omega$ there is associated an abstract observer at infinity that we may think of as having a fixed velocity relative to our isolated system and measuring a total Trautman-Bondi energy $E_{TB}$. The infimum over all these energies giving the total Trautman-Bondi mass $m_{TB}$ of $\Omega$. Since all cross sections of $\Omega$ to the past of $\Sigma_0$ have greater area we have no need to invoke any outermost minimal area enclosure restrictions (as in the general case) and the Penrose Conjecture takes the form,

$$\sqrt{\frac{|\Sigma_0|}{16\pi}} \leq m_{TB}.$$ 

This form of the conjecture is often referred to as the \textit{Null Penrose Inequality} and has been verified when $\Omega$ is shear-free and vacuum by J. Sauter \cite{32}. For small vacuum perturbations of the metric around the Schwarzschild Null Cone it was shown to hold for the weaker upper bound $E_{TB}$ by S. Alexakis \cite{1}, also known as the \textit{Weak Null Penrose Inequality}. Work by M.T. Wang \cite{33} and M. Mars-A. Soria \cite{25} also study the problem for shells in Minkowski, related to the original formulation of Penrose concerning null shells of dust propagating in Minkowski spacetime. An interesting related conjecture in Schwarzschild spacetime has also been shown by S. Brendle-M.T. Wang \cite{11}.

A general proof of the Weak Null Penrose Inequality was claimed by M. Ludvigsen and J.A.G Vickers \cite{22} but G. Bergqvist \cite{8} identified, amongst decay assumptions of the past null expansion, that no guarantee of ‘asymptotic roundness’ for their given foliation had been justified in order to form a comparison with total energy. In \cite{27}, M. Mars - A. Soria was able to show unique existence of a foliation called ‘Geodesic Asymptotically Bondi’ exhibiting the decay in the Ludvigsen-Vickers-Bergqvist approach. Along with the use of a new energy functional the authors were subsequently able to bound the MOTS mass $\sqrt{|\Sigma_0|/16\pi}$ by the asymptotic limit of the Hawking Energy:

\textbf{Definition 1.1.} Given a spacelike 2-sphere $\Sigma$ with mean curvature $\vec{H} = \text{tr}_\Sigma \Pi$, the Hawking Energy is given by

$$E_H(\Sigma) := \sqrt{\frac{|\Sigma|}{16\pi}} \left(1 - \frac{1}{16\pi} \int_{\Sigma} \langle \vec{H}, \vec{H} \rangle dA\right).$$

Unfortunately, the GAB foliation does not necessarily become round at infinity, and therefore the difficulty of relating the resulting limit to the Trautman-Bondi energy persists.

In \cite{31}, the author constructs a new mass functional $m(\Sigma)$ for a 2-sphere $\Sigma$ in spacetime that, given certain convexity conditions (see (3) and (4) below), is non-decreasing along any past null flow, or equivalently, along any foliation of $\Omega$. Interestingly, a consequence of these convexity conditions on a MOTS is that the mass satisfies $m(\Sigma_0) = \sqrt{|\Sigma_0|/16\pi}$. Moreover, along all asymptotically geodesic foliations $\{\Sigma_s\} \subset \Omega$ we find this mass approaches a unique limit independent of the choice of foliation. From this it can be shown, provided one member amongst all asymptotically geodesic foliations of $\Omega$ can be found satisfying the aforementioned convexity, that $\lim_{s \to \infty} m(\Sigma_s) \leq m_{TB}$. The study of this paper concerns these convexity conditions which, as our discussion identifies, is our only obstruction to proving the Null Penrose Conjecture:

$$\sqrt{\frac{|\Sigma_0|}{16\pi}} = m(\Sigma_0) \leq \lim_{s \to \infty} m(\Sigma_s) \leq m_{TB}.$$ 

\subsection{Overview of Main Results}

In Section 2 we show a class of Weakly Isolated Horizon admits a unique foliation by MOTS satisfying our desired convexity:
**Theorem. 2.1** Let $\mathcal{H}$ be a strictly stable, and optically rigid Weakly Isolated Horizon with positive surface gravity $\kappa > 0$. Then it admits a unique foliation of MOTS $\{\Sigma_s\}$ along its associated null generator $l$ satisfying

$$K_s + \nabla \cdot \tau_s = \frac{4\pi}{|\Sigma|}$$

where $K$ is the Gauss curvature, and $\tau$ the connection 1-form (or torsion).

We also show the existence of such MOTS persists under small metric perturbations around our Weakly Isolated Horizon:

**Theorem. 2.4** Let the metric $g_0$ admit a strictly stable MOTS, $\Sigma$, such that $\delta_{L^+} \langle \bar{H}, \bar{H} \rangle = 0 = K + \nabla \cdot \tau - \frac{4\pi}{|\Sigma|}$. Then, for any smooth variation of metrics $g_\lambda$, there exists $\epsilon > 0$ and a corresponding family of smooth 2-spheres, $\Sigma_\lambda$, satisfying

$$K_\lambda + \nabla \cdot \tau_\lambda = \frac{4\pi}{|\Sigma_\lambda|}, \quad \langle \bar{H}, \bar{H} \rangle_\lambda = 0$$

for $0 \leq \lambda \leq \epsilon$.

In section 3, after imposing ‘reasonable’ decay similar to that of S. Alexakis modeling asymptotically flat metric perturbations of the black hole exterior in Schwarzschild spacetime:

**Theorem. 3.5** Let $g_\lambda$ be a smooth family of metrics satisfying the Dominant Energy Condition off of the Schwarzschild metric $g_0$, then there exists $\epsilon > 0$ and a corresponding family of smooth $\Sigma_\lambda$ as in Theorem 2.4. If the past null cones $\Omega_\lambda \supset \Sigma_\lambda$ are smooth and $g_\lambda$ is close to Schwarzschild according to the decay conditions (20)-(23), then we have the Null Penrose Inequality

$$\sqrt{\frac{|\Sigma_\lambda|}{16\pi}} \leq m_{TB}(\lambda).$$

### 1.2 Initial Constructions and Useful Results

A spacetime $(\mathcal{M}, g)$ is defined to be a four dimensional smooth manifold $\mathcal{M}$ equipped with a Lorentzian metric $g(\cdot, \cdot)$ (or $\langle \cdot, \cdot \rangle$). We assume that the spacetime is both orientable and time orientable, i.e. admits a nowhere vanishing timelike vector field, defined to be future-pointing.

From this, for $\Sigma$ a spacelike embedding of a sphere in $\mathcal{M}$ with induced metric $\gamma$, it follows that $\Sigma$ has trivial normal bundle $T^\perp \Sigma$ with induced Lorentzian metric. From any choice of null section $L \in \Gamma(T^\perp \Sigma)$, we have a unique null partner $L \in \Gamma(T^\perp \Sigma)$ according to $\langle L, L \rangle = 2$. Our convention for the second fundamental form $\Pi$ and mean curvature $\bar{H}$ of $\Sigma$ are

$$\Pi(V, W) = D_V W, \quad \bar{H} = \mathrm{tr}_\Sigma \Pi$$

for $V, W \in \Gamma(T\Sigma)$ and $D$ the Levi-Civita connection of the spacetime.

**Definition 1.2.** For the null basis $\{L, L\}$, we define the associated symmetric 2-tensors $\chi, \chi$ and torsion (connection 1-form) $\zeta$ by

$$\chi(V, W) := \langle D_V L, W \rangle = -\langle L, \Pi(V, W) \rangle$$

$$\chi(V, W) := \langle D_V L, W \rangle = -\langle L, \Pi(V, W) \rangle$$

$$\zeta(V) := \frac{1}{2} \langle D_V L, L \rangle = -\frac{1}{2} \langle D_V L, L \rangle$$

where $V, W \in \Gamma(T\Sigma)$. 

| 3 |
1.2.1 Null Inflation Basis, Flux, and Mass

Whenever $L$ satisfies $\text{tr} \chi =: \sigma > 0$ we define a gauge invariant, canonical null basis $\{L^-, L^+\}$ called the Null Inflation Basis by

$$L^- := \frac{L}{\sigma}, \quad L^+ := \sigma L,$$

from this the corresponding data of Definition 1.2 satisfies $\text{tr} \chi^- = 1$, $\text{tr} \chi^+ = \langle \vec{H}, \vec{H} \rangle$, $\tau(V) := \langle D_V L^-, L^+ \rangle = \zeta(V) - V \log \sigma$. We may now define

**Definition 1.3.** For $\Sigma$ admitting a Null Inflation Basis, the geometric flux $\rho$ and mass $m(\Sigma)$ is given by

$$\rho = K - \frac{1}{4} \langle \vec{H}, \vec{H} \rangle + \nabla \cdot \tau,$$

$$m(\Sigma) = \frac{1}{2} \left( \int_{\Sigma} \rho^2 \frac{dA}{4\pi} \right)^{\frac{3}{2}}$$

where $K$ represents the Gauss Curvature of $\Sigma$, and $\nabla$ the induced covariant derivative.

One of the fundamental results in [31], motivating our convexity assumption on $\Sigma$, was the following monotonic result for the mass functional $m(\Sigma)$

**Theorem 1.1.** ([31], Theorem 1.1) Let $\Omega$ be a null hypersurface foliated by spacelike spheres $\{\Sigma_s\}$ expanding along $L = \sigma L^-$ such that $|\rho(s)| > 0$ for each $s$. Then the mass $m(s) := m(\Sigma_s)$ has rate of change

$$\frac{dm}{ds} = \frac{(2m)^{\frac{1}{2}}}{8\pi} \int_{\Sigma_s} \frac{\sigma}{\rho^2} \left( (|\dot{\chi}|^2 + G(L^-, L^-))(\frac{1}{4} \langle \vec{H}, \vec{H} \rangle - \frac{1}{3} \Delta \log |\rho|) + \frac{1}{2} |\nu|^2 + G(L^-, N) \right) dA$$

where

- $\dot{\chi} = \chi - \frac{1}{2} \sigma \gamma$
- $G$ is the Einstein tensor for the ambient metric $g$
- $\nu := \frac{2}{3} \dot{\chi} \cdot d \log |\rho| - \tau$
- $N := \frac{1}{6} |\nabla \log |\rho||^2 L^+ + \frac{1}{3} \nabla \log |\rho| - \frac{1}{2} L^+$

Taking as our convention that the Riemann Curvature tensor is given by

$$R_{XY}Z := D_{[X,Y]}Z - [D_X, D_Y]Z,$$

then a spacetime $(\mathcal{M}, g)$ is said to satisfy the Dominant Energy Condition (DEC) if, for any two future pointing causal vectors $V, W \in \Gamma(T\mathcal{M})$ the Einstein Curvature tensor satisfies $G(V, W) \geq 0$ (where we recall $G := \text{Ric} - \frac{1}{2} \text{R}g$, for $\text{Ric}$ the Ricci tensor and $\text{R}$ the Ricci scalar). The DEC models the physical assumption (by way of the energy-momentum tensor in the Einstein Field Equations) that spacetime exhibits non-negative energy density. It is here that we recognize that when we couple the DEC with Theorem 1.1 a non-decreasing mass $\frac{dm}{dt} \geq 0$ is achieved irrespective of the flow vector $L = \sigma L^-$ whenever the convexity conditions

$$\rho > 0$$

$$\frac{1}{4} \langle \vec{H}, \vec{H} \rangle \geq \frac{1}{3} \Delta \log \rho$$

are satisfied on $\Sigma$. We say $\Sigma$ is Doubly Convex if conditions (3) and (4) are satisfied.
1.2.2 Null Geometry and the Structure Equations

Since we’ll be analyzing null hypersurfaces in \( \mathcal{M} \) we wish to introduce a setup that will be applicable to our various needs, the setup as in \([26]\) which we now describe will suffice.

Suppose \( \mathcal{N} \) is a smooth connected hypersurface embedded in \((\mathcal{M}, \langle \cdot, \cdot \rangle)\). It follows that the induced metric on \( \mathcal{N} \) is degenerate if we’re able to find a smooth, non-vanishing, null vector field \( L \in \Gamma(T\mathcal{N}) \). It’s a well known fact (see, for example, \([13]\)) that the integral curves of \( L \) are geodesic giving \( \kappa \in \mathcal{F}(\mathcal{N}) \) such that \( D_tL = \kappa L \). We assume the existence of an embedded surface \( \Sigma \subset \mathcal{N} \) such that any integral curve of \( L \) intersects \( \Sigma \) precisely once. We will refer to such \( \Sigma \) as cross-sections of \( \mathcal{N} \). This gives rise to a natural submersion \( \pi : \mathcal{N} \to \Sigma \) sending \( p \in \mathcal{N} \) to the intersection with \( \Sigma \) of the integral curve \( \gamma_p^L \) of \( L \) for which \( \gamma_p^L(0) = p \). Given \( L \) and a constant \( s_0 \) we construct a function \( s \in \mathcal{F}(\mathcal{N}) \) from \( L(s) = 1 \) and \( s|_\Sigma = s_0 \). For \( q \in \Sigma \), if \( (s_-(q), s_+(q)) \) represents the range of \( s \) along \( \gamma_q^L \), letting \( S_- = \sup_{\Sigma} s_- \) and \( S_+ = \inf_{\Sigma} s_+ \) we notice that the interval \( (S_-, S_+) \) is non-empty. Given that \( L(s) = 1 \) the Implicit Function Theorem gives for \( t \in (S_-, S_+) \) that \( \Sigma_t := \{ p \in \mathcal{N} | s(p) = t \} \) is diffeomorphic to \( \Sigma \). For \( s < S_- \) or \( s > S_+ \), in the case that \( \Sigma_s \) is non-empty, although smooth it may no longer be connected. We have that the collection \( \{ \Sigma_s \} \) gives a foliation of \( \mathcal{N} \). An adapted null vector field \( L \) to \( \Sigma_s \) is constructed by assigning at every \( p \in \mathcal{N} \) the unique null vector satisfying \( \langle L, L \rangle = 2 \) and \( \langle L, v \rangle = 0 \) for any \( v \in T_p\Sigma_s(p) \). As before each \( \Sigma_s \) is endowed with an induced metric \( \gamma_s \), two null second fundamental forms \( \chi = -\langle \bar{H}, L \rangle \) and \( \zeta = -\langle \bar{H}, L \rangle \) as well as the connection 1-form (or torsion) \( \zeta(V) = \frac{1}{2} \langle D_VL, L \rangle \).

**Proposition 1.2** (Structure Equations). Along the foliation \( \{ \Sigma_s \} \subset \mathcal{N} \) the following holds:

\[
\begin{align*}
L \kappa &= -\text{tr} \chi \kappa - \frac{1}{2} \Delta \text{tr} \chi + \nabla \cdot (\nabla \cdot \hat{\chi}) \\
L_L \chi &= 2\hat{\chi} \\
L_L \chi &= -\alpha + \frac{1}{2} |\hat{\chi}|^2 \gamma + \text{tr} \chi \hat{\chi} + \frac{1}{4} (\text{tr} \chi)^2 \gamma + \kappa \chi \\
L \text{tr} \chi &= -\frac{1}{2} (\text{tr} \chi)^2 - |\hat{\chi}|^2 - G(L, L) + \kappa \text{tr} \chi \\
L_L \chi &= \left( \kappa + \hat{\chi} \cdot \hat{\chi} + \frac{1}{2} G(L, L) \right) \gamma + \frac{1}{2} \text{tr} \chi \hat{\chi} + \frac{1}{2} \text{tr} \chi \hat{\chi} - \hat{G} - 2S(\nabla \zeta) - 2\zeta \otimes \zeta - \kappa \chi \\
L \text{tr} \chi &= G(L, L) + 2\kappa - 2\nabla \cdot \zeta - 2|\zeta|^2 - \langle \bar{H}, \bar{H} \rangle - \kappa \chi \\
L_L \zeta &= G_L - \nabla \cdot \hat{\chi} - \text{tr} \chi \zeta + \frac{1}{2} d \text{tr} \chi + d\kappa
\end{align*}
\]

where \( \alpha \) is the symmetric 2-tensor given by \( \alpha(V, W) = \langle R_{LV}L, W \rangle \), \( S(T) \) represents the symmetric part of a 2-tensor \( T \), \( G_L = G(L, \cdot)|_{\Sigma_s} \) and \( \hat{G} = G|_{\Sigma_s} - \frac{1}{2} (\text{tr} G) \gamma \).

**Proof.** See, for example \([31, 15]\). \( \square \)

Given a cross section \( \Sigma \subset \mathcal{N} \) and \( v \in T_q(\Sigma) \) we may extend \( v \) along the generator \( \gamma_q^L \) according to

\[
\dot{V}(s) = D_V(s) L \\
V(0) = v.
\]

Since \( x \in T_p \mathcal{N} \iff \langle L_p, x \rangle = 0 \) we see from the fact that \( \langle \dot{V}(s), L \rangle = \kappa \langle V(s), L \rangle \), \( \langle V(0), L \rangle = 0 \) that \( \langle V(s), L \rangle = 0 \) for all \( s \). As a result any section \( W \in \Gamma(T\Sigma) \) may be extended to \( \mathcal{N} \) satisfying
\[ [L, W] = 0. \] Along each generator \( 0 = [L, W] = L(Ws) = Ws, \) so that \( Ws|_\Sigma = 0 \) gives \( Ws = 0 \) on \( \mathcal{N}. \) We conclude that \( Ws|_\Sigma = \Gamma(TS_\Sigma) \) and denote by \( E(\Sigma) \subset \Gamma(T\mathcal{N}) \) the set of such extensions off of \( \Sigma \) along \( L. \) We also note that linear independence is preserved along generators by standard uniqueness theorems allowing us to extend basis fields \( \{X_1, X_2\} \subset \Gamma(TS_\Sigma) \) to \( \mathcal{N}. \) Having established a background foliation \( \{\Sigma_s\}, \) the fact that \( \mathcal{N} \) is generated by null geodesics along \( L \) then uniquely characterizes any spacelike cross-section \( S \rightarrow \mathcal{N} \) as a graph over \( \Sigma := \Sigma_{s_0} \) with graph function \( \sigma(\pi|_S)^{-1} = \omega \in F(\Sigma) \) and image \( \Sigma_\omega \subset \mathcal{N}. \) By Lie-dragging \( \omega \) along \( L \) to all of \( \mathcal{N}, \) we have for any \( V \in E(\Sigma), \) that \( (V + V_\omega L)(s - \omega) = 0 \) so that \( \tilde{V} := V + V_\omega L \) restricts to an element of \( \Gamma(T\Sigma_\omega). \) By also defining \( \nabla_\omega \in \Gamma(T\mathcal{N}) \) according to \( \langle L, \nabla_\omega \rangle = 0, \) \( \langle \nabla_\omega, V \rangle = V_\omega \) for any \( V \in \Gamma(T\Sigma_\omega) \) we see that \( \langle L - |\nabla_\omega|^2L - 2\nabla_\omega, \tilde{V} \rangle = 0 \) so that \( \{L_\omega := L - |\nabla_\omega|^2L - 2\nabla_\omega, L\} \) restricts to an adapted null basis for \( \Sigma_\omega. \)

### 1.2.3 The Stability Operator

Next we will need to introduce the notion of a stable MOTS. To this end, assume we have a 2-sphere \( \Sigma \subset \mathcal{M} \) satisfying \( \text{tr} \chi = 0 \) and a differentiable map \( \Phi : \Sigma \times (-\epsilon, \epsilon) \rightarrow \mathcal{M} \) such that \( \Phi(\cdot, t) \) is an immersion and \( \alpha_p(t) := \Phi(p, t), p \in \Sigma, \) is a curve satisfying \( \alpha_p(0) = p \) with initial velocity \( \alpha'(0) = \psi(p)(L_p - \phi(p)L_p) \) for some \( \psi, \phi \in F(\Sigma). \) It follows (see [2], Lemma 3.1) that the linearization satisfies \( \delta_{\psi(L-\phi L)} \text{tr} \chi := \frac{d}{dt}|_{t=0} \text{tr} \chi(\Phi(p, t)) = 2\mathcal{L}(\psi) \) where

**Definition 1.4.** Given a MOTS \( \Sigma \) in \( \mathcal{M}, \)

\[
\mathcal{L}(\psi) = -\Delta \psi - 2\nabla \cdot (\psi \zeta) + \left( K + \nabla \cdot \zeta - |\zeta|^2 + \frac{1}{2}G(U, L) + \phi |\hat{\chi}|^2 \right) \psi
\]

is called the stability operator of \( \Sigma \) along \( U = L - 2\phi L. \) It’s adjoint with respect to the \( L^2 \) inner product on \( \Sigma \) is also given by

\[
\mathcal{L}^*(\psi) = -\Delta \psi + 2\zeta(\nabla \psi) + \left( K + \nabla \cdot \zeta - |\zeta|^2 + \frac{1}{2}G(U, L) + \phi |\hat{\chi}|^2 \right) \psi.
\]

Like any second order elliptic PDE on a compact Riemannian manifold, the stability operator \( \mathcal{L} \) admits a principal real-valued eigenvalue \( \mu \) with one-dimensional eigenspace of the form \( \{c\varphi\} \) where \( 0 < \varphi \in F(\Sigma) \) is smooth, \( c \in \mathbb{R}. \) Moreover, both \( \mathcal{L}, \mathcal{L}^* \) share a common principal eigenvalue (see [2], Lemma 4.1). We say \( \Sigma \) is stable (strictly stable) if \( \mu \geq (>) 0 \) and unstable if \( \mu < 0. \) We leave it to the reader to verify that any scale change \( L \rightarrow aL, L \rightarrow \frac{1}{a}L \) for some smooth function \( a > 0 \) results in the stability operator (along the same direction, i.e. \( \phi \rightarrow a^2\phi \)) changing to

\[
\mathcal{L}^a(\psi) = \frac{1}{a} \mathcal{L}(a\psi).
\]

From this we also conclude that stability is invariant under scale variations since the new principle eigenfunction is given by \( \varphi_a = \frac{1}{a}\varphi > 0. \) For further discussion regarding the notion of stability we refer the reader to [2]. Throughout this paper, \( C^{k, \alpha}(\Sigma) \) refers to the space of \( k \)-times differentiable functions on our 2-sphere \( \Sigma \) with \( k^{th} \) partial derivatives being Hölder continuous with exponent \( 0 < \alpha \leq 1. \) The (irrelevant for our purposes) Hölder seminorm \( | \cdot |_\alpha \) depending on some fixed background metric (i.e. the standard round metric will do).

**Lemma 1.2.1.** ([2], Lemma 4.2) Let \( \mathcal{L} \) be the stability operator of a MOTS \( \Sigma. \) Let \( \mu \) and \( \varphi > 0 \) be the principal eigenvalue and eigenfunction of \( \mathcal{L}, \) respectively, and let \( \psi \in C^{2, \alpha}(\Sigma) \) be a solution of \( \mathcal{L}(\psi) = f \) for some function \( 0 \leq f \in C^{0, \alpha}(\Sigma). \) Then the following holds,
1. If \( \mu = 0 \), then \( f \equiv 0 \) and \( \psi = C \varphi \) for some constant \( C \).

2. If \( \mu > 0 \) and \( f \not\equiv 0 \), then \( \psi > 0 \).

3. If \( \mu > 0 \) and \( f \equiv 0 \), then \( \psi \equiv 0 \).

To conclude this section we observe that both the operators \( L, L^* : C^{2,\alpha}(\Sigma) \to C^{0,\alpha}(\Sigma) \) have smooth coefficients on a compact manifold and by standard results are therefore bounded. As a consequence of the Fredholm Alternative and the Bounded Inverse Theorem, Lemma 1.2.1 therefore ensures that whenever \( \mu > 0 \) both operators have bounded linear inverses.

2 Existence and Stability of the Doubly Convex MOTS

In order for us to have any hope of propagating a non-decreasing mass from a MOTS \( \Sigma_0 \) to null infinity, conditions (3) and (4) must be satisfied on \( \Sigma_0 \). From the fact that \( \langle \vec{H}, \vec{H} \rangle|_{\Sigma_0} = 0 \), the Maximum Principle along with condition (4) dictates that \( \rho \) must be a constant function on \( \Sigma_0 \). As a result, the Gauss-Bonnet and Divergence Theorems coupled with (3) tell us to look for a MOTS \( \Sigma_0 \) satisfying

\[
\rho = \frac{4\pi}{|\Sigma_0|}
\]

which we'll call a Doubly Convex MOTS.

2.1 Weakly Isolated Horizons

From (8) we notice that if \( \text{tr} \bar{\chi} \equiv 0 \) then the DEC coupled with the fact that all cross-sections are Riemannian give \( \bar{\chi} = \hat{\chi} + \frac{1}{2} \text{tr} \chi \gamma = 0 = G(L, L) \). We conclude that \( \langle D_X Y, L \rangle = 0 \) for any \( X, Y \in \Gamma(TN) \), equivalently \( D_X Y \in \Gamma(TN) \), and the connection \( D \) restricts to a connection on \( N \). Whenever \( S_- = -\infty \), \( S_+ = \infty \), this particular null hypersurface is called a Non Expanding Horizon (NEH) which we denote by \( \mathcal{H} \) with null generator \( l \) (in favor of \( L \)). Given a background foliation \( \{\Sigma_s\} \subset \mathcal{H} \) with an adapted null normal field \( k \) (such that \( \langle k, l \rangle = 2 \)), any cross-section \( \Sigma_\omega \subset \mathcal{H} \) therefore has vanishing null expansion \( \theta_\omega l := \text{tr} \chi_\omega = 0 \) and is a MOTS. We’re searching for a Doubly Convex candidate \( \Sigma_\omega \subset \mathcal{H} \) to propagate in the direction \( k^\omega = k - |\nabla_\omega|^2 l - 2\nabla_\omega \) “off of \( \mathcal{H} \)”. We will denote the torsion 1-form of \( \Sigma_\omega \) by \( t_\omega (V) := \frac{1}{2} \langle D_V k^\omega, l \rangle = -\zeta^\omega (V) \).

NEHs have been extensively studied in the literature \([3, 4, 5, 6, 15, 16, 20]\) as a model for black hole horizons in Relativity theory with various refinements arising from specific contexts. Arguably the most prolific example being the class of Killing Horizons whereby \( l = \xi|_\mathcal{H} \) is the restriction of a Killing vector \( \xi \) in \( \mathcal{M} \). Notice, the Killing equation gives directly from the fact that \( \xi|_\mathcal{H} \in \Gamma(T\mathcal{H}) \cap \Gamma(T^\perp \mathcal{H}) \) that \( \chi = 0 \). Killing Horizons are more restrictive (see \([24]\)) than we’ll need so we generalize to the following subclass of NEHs:

**Definition 2.1.**

1. (\([3]\)) We say \( \mathcal{H} \) is a **Weakly Isolated Horizon** (WIH) if the tensor \( ([L_1, D]|_l)(X) = 0 \)

2. We say a WIH, \( \mathcal{H} \), is **optically rigid** if we can find a foliation \( \{\Sigma_s\} \subset \mathcal{H} \) along the null generator \( l \) such that the adapted null normal \( k \in \Gamma(T^\perp \Sigma_s) \), whereby \( \langle k, l \rangle = 2 \), satisfies \( \delta_l \theta_k = 0 \).
Remark 2.1. Regarding the first part of Definition 2.1, given a background foliation \( \{ \Sigma_s \} \subset H \) and the fact that \( G(l,l) = 0 \), we have for any \( \epsilon > 0 \) according to the DEC that

\[
G(l, \frac{1}{4}l - \epsilon \tilde{G}_l - \epsilon^2 |\tilde{G}_l|^2 k) = -\epsilon |\tilde{G}_l|^2 (1 + \epsilon G(l,k)) \geq 0.
\]

This is impossible unless also \( \tilde{G}_l \equiv 0 \), equivalently \( G(l,X) = 0 \) for any \( X \in \Gamma(TH) \). If we extend the definition of \( \zeta \) to \( \tilde{\zeta}(X) := \frac{1}{2} \langle D_X l, k \rangle \) for any \( X \in \Gamma(TH) \), then from the definition of a WIH we have

\[
0 = (\mathcal{L}_l, D)[l](X) = \mathcal{L}_l(D_X l) - D_{[l,X]} l - D_X (\mathcal{L}_l l)
\]

having used (11) and \( l\tilde{\zeta}(l) = \kappa_l \) in the final line. We conclude therefore that the surface gravity \( \kappa_l \) (according to \( D_l l = \kappa_l l \)) of \( H \) in a physical spacetime identifies a WIH by whether it remains constant on all of \( H \).

Remark 2.2. Regarding the second part of Definition 2.1.

Lemma 2.1.1. \((24), \text{Lemma 3}\)

Given a cross-section \( \Sigma_\omega \subset H \) with null normal \( k_\omega \in \Gamma(T^\bot \Sigma_\omega) \) such that \( \langle k_\omega, l \rangle = 2 \). Then the submersion \( \pi_\omega : \Sigma_\omega \to \Sigma_{s_0} \) is an isometry and:

\[
t_\omega = \pi_\omega^* (t - \kappa l d\omega) \quad (12)
\]

\[
\theta^\omega_k = [\theta_k - 2\Delta \omega - 2\kappa l |\nabla \omega|^2 + 4t(\nabla \omega)] \circ \pi_\omega \quad (13)
\]

We notice from Lemma 2.1.1 that optical rigidity implies \( \theta^\omega_k + s = \theta^\omega_k \) for any given constant \( s \). In turn, we conclude that the definition of optical rigidity is independent of the choice of foliation along \( l \) (i.e. \( \omega \)).

Notice, from (5), (6) and Lemma 2.1.1 the stability operator \( \mathcal{L} \) is independent of the leaf \( \Sigma_s \). Infact, one can show that the stability operator on \( \Sigma_\omega \) satisfies \( \mathcal{L}^\omega(\psi \circ \pi_\omega) = e^{\kappa_\omega} \mathcal{L}(e^{-\kappa_\omega} \psi) \circ \pi_\omega \) (see [24], Proposition 3) and therefore all \( \Sigma_\omega \) share the same principle eigenvalue (see [24], Proposition 4). So for a WIH the stability of any cross-section dictates the stability on all of \( H \).

We’re almost ready now to specify conditions on a WIH \( H \) in order to ensure the unique existence of foliation by Doubly Convex MOTS, but first we’ll need two lemmas.

Lemma 2.1.2. For any cross-section \( \Sigma_\omega \) in a WIH \( H \), and \( V, W \in E(\Sigma) \),

1. \( \nabla^\omega_V W = (\nabla_V W + (\nabla_V W \omega) l)|_{\Sigma_\omega} \)

2. \( (\nabla^\omega_V t^\omega)(\tilde{W}) = ((\nabla_V t)(W) - H^{\kappa_\omega}(V,W))|_{\Sigma_\omega} \)

where \( H^{\kappa_\omega} \) is the background Hessian.
Proof. For the first part of the proof it suffices to show that $\langle \nabla_\tilde{V}^{\omega} \tilde{W}, U \rangle = \langle \nabla_V W, U \rangle$ for any $U \in E(\Sigma)$:

$$\langle \nabla_\tilde{V}^{\omega} \tilde{W}, U \rangle = \langle D_\tilde{V} \tilde{W} + \frac{1}{2} \chi(\tilde{V}, \tilde{W}) I, U \rangle = \langle \tilde{V}(W, U) - \langle \tilde{W}, D_\tilde{V} U \rangle = V \langle W, U \rangle - (V \omega \langle W, U \rangle - W \omega \langle l, D_\tilde{V} U \rangle - V \omega \langle W, D_l U \rangle) \rangle = V \langle W, U \rangle - \langle W, D V U \rangle = \langle \nabla_V W, U \rangle$$

where the third and final terms of the penultimate line vanishes due to (6) and $\chi = 0$, and the forth vanishes since $D$ restricts to $\mathcal{H}$.

For the second part of the lemma, we have from Lemma 2.1.1 and the result above:

$$(\nabla^{\omega} t^{\omega})(\tilde{W}) = \tilde{V} t^{\omega}(\tilde{W}) - t^{\omega}(\nabla^{\omega} \tilde{W}) = (V + V \omega l)(t(W)) - t(\nabla_V W) + \nabla_V W(\kappa_l \omega) = (\nabla_V t)(W) - H^{\kappa_l \omega}(V, W)$$

\[\square\]

Lemma 2.1.3. For any cross-section $\Sigma_\omega$ of an optically rigid $\mathcal{H}$, we have

$$\frac{\kappa_l}{2} \theta_k^\omega = e^{-\kappa_l \omega} \mathcal{L}^*(e^{\kappa_l \omega})$$

where $\mathcal{L}$ is the background stability operator.

Proof. From Lemma 2.1.1 with slight abuse of notation, we have

$$\kappa_l \theta_k^\omega = \kappa_l \theta_k - 2\Delta(\kappa_l \omega) - 2|\nabla(\kappa_l \omega)|^2 + 4t(\nabla(\kappa_l \omega)) = (G(l, k) + 2\mathcal{K} + 2\nabla \cdot t - 2|t|^2) + 4t(\nabla(\kappa_l \omega)) = 2e^{-\kappa_l \omega} \left( - \Delta e^{\kappa_l \omega} + 2t(\nabla e^{\kappa_l \omega}) + \left( \mathcal{K} + \nabla \cdot t - |t|^2 + \frac{1}{2} G(l, k) e^{\kappa_l \omega} \right) \right)$$

having used the optical rigidity of $\mathcal{H}$ coupled with (10) to obtain the second line.

\[\square\]

Theorem 2.1. If $\mathcal{H}$ is strictly stable, optically rigid, and $\kappa_l > 0$, then it admits a unique foliation along $l$ satisfying

$$\rho = \frac{4\pi}{|\Sigma|}.$$ 

Proof. Existence:

From standard results for the Laplace-Beltrami operator on compact Riemannian manifolds, the equation,

$$\Delta u = \mathcal{K} + \nabla \cdot t - \frac{4\pi}{|\Sigma|}$$

is solvable since the Gauss-Bonnet and Divergence theorems ensure both sides integrate to zero. From Elliptic Regularity and the Maximum Principle (see [12, 14]) we also know that the solution $u$ is smooth and unique up to an additive constant respectively. Next, from the stability hypothesis on $\mathcal{H}$, the operator $\mathcal{L}^*$ has bounded inverse so that we may solve for $\psi$ in the equation

$$\mathcal{L}^*(\psi) = e^u.$$
Elliptic regularity once again ensures $\psi$ is smooth, and from Lemma 2.2.1 $\rho > 0$. Defining $\omega := \frac{1}{|\kappa|} \log \psi$ we’ve found a cross-section $\Sigma_\omega \hookrightarrow \mathcal{H}$ which by Lemma 2.1.3 satisfies $\frac{1}{|\kappa|} e^{\kappa \omega} \theta_\omega^p = e^u$. We conclude that $\theta_\omega^p > 0$ and therefore Lemma 2.1.2 coupled with optical rigidity gives:

$$
\rho_\omega = K + \nabla \omega \cdot \tau - \Delta_\omega \log \theta_\omega^p \\
= K + \nabla \cdot (t - d(\kappa_\omega)) - \Delta \log \theta_\omega^p \\
= \frac{4\pi}{|\Sigma|} - \Delta \log (e^{-u + \kappa_\omega \theta_\omega^p}) = \frac{4\pi}{|\Sigma|}.
$$

We also observe that $\rho_{\omega+s} = \rho_\omega$ for any constant $s$ by optical rigidity of $\mathcal{H}$.

Uniqueness:
In order for $\Sigma_{\omega'}$ to satisfy $\rho_{\omega'} = \frac{4\pi}{|\Sigma|}$ we must have that $u + C = \log(e^{\kappa_\omega' \theta_\omega' - C})$ for some constant $C$ by the Maximum Principle. From Lemma 2.1.3 we therefore conclude that $e^u = \mathcal{L}^* \left( \frac{1}{|\kappa|} e^{\kappa_\omega' - C} \right)$. Since $\mathcal{L}^*$ has bounded inverse we have $\frac{1}{|\kappa|} e^{\kappa_\omega' - C} = \psi = e^{\kappa_\omega}$ and therefore

$$
\omega' = \omega + \frac{C + \log \kappa}{\kappa}.
$$

We see that $\Sigma_{\omega'}$ is simply a translate of $\Sigma_\omega$ along $l$, moreover, as the constant $C$ runs through all values of $\mathbb{R}$ we recover the foliation of the existence argument. \hfill $\square$

### 2.2 Stability

In this section, we will assume our MOTS $\Sigma_0$ satisfies the necessary conditions allowing the construction of a Null Inflation Basis. Consequently, we will henceforth take $\mathcal{L}$ to be the stability operator along $L^-$ relative to the Null Inflation basis:

$$
\mathcal{L}(\psi) = -\Delta \psi - 2 \nabla \cdot (\psi \tau) + \left( \rho_0 - |\tau|^2 + \frac{1}{2} G(L^+, L^-) \right) \psi
$$

where $\rho_0 = K + \nabla \cdot \tau$.

**Proposition 2.2.** Given a surface $\Sigma$ admitting a Null Inflation basis the following holds:

$$
\delta_{\psi L^-} \mathcal{L} \psi = -\psi \mathcal{L} \psi - \frac{1}{2} \Delta \psi + \nabla \cdot \nabla \cdot (\psi \hat{\chi}^-) \quad (14)
$$

$$
\delta_{\psi L^+} \mathcal{L} \psi = -\psi (\mathcal{H}, \mathcal{H}) \mathcal{L} \psi - \frac{1}{2} \Delta (\psi \langle \mathcal{H}, \mathcal{H} \rangle) + \nabla \cdot \nabla \cdot (\psi \hat{\chi}^+). \quad (15)
$$

$$
\delta_{\psi L^-} \langle \mathcal{H}, \mathcal{H} \rangle = 2 \mathcal{L} (\psi) - \langle \mathcal{H}, \mathcal{H} \rangle \left( \frac{3}{2} |\chi^-|^2 + G(L^-, L^-) \right) \psi \quad (16)
$$

$$
\delta_{\psi L^+} \langle \mathcal{H}, \mathcal{H} \rangle = 2 \langle \mathcal{H}, \mathcal{H} \rangle \mathcal{L}^* (\psi) - \left( \frac{3}{2} \langle \mathcal{H}, \mathcal{H} \rangle + |\chi^+|^2 + G(L^+, L^+) \right) \psi \quad (17)
$$

$$
\delta_{\psi L^-} \tau = -\psi \tau - \nabla \cdot (\psi \hat{\chi}^-) + \psi G_{L^-} + d(\psi |\chi^-|^2 + G(L^-, L^-)) \quad (18)
$$

$$
\delta_{\psi L^+} \tau = -2d \mathcal{L}^* (\psi) - \psi G_{L^+} + \nabla \cdot (\psi \hat{\chi}^+) - \psi \langle \mathcal{H}, \mathcal{H} \rangle \tau + \frac{1}{2} \langle \mathcal{H}, \mathcal{H} \rangle d \psi - \psi d \langle \mathcal{H}, \mathcal{H} \rangle \quad (19)
$$

**Proof.** Firstly, we start by considering any neighborhood where $\psi \neq 0$ and sufficiently small to proceed as if $\Sigma$ is embedded in $\mathcal{M}$. The result follows for (14) directly from the structure equations by setting $L = \psi L^+$, $L = \frac{1}{\psi} L^-$, similarly for (15) by setting $L = \frac{1}{\psi} L^-$, $L = \psi L^+$. Toward showing
(16) and (17) we calculate
\[ L(\bar{H}, \bar{H}) = L(\text{tr} \chi \text{tr} \chi) \]
\[ = \left(-\frac{1}{2} \text{tr} \chi^2 - |\dot{\chi}|^2 - G(L, L) + \kappa \text{tr} \chi \right) \text{tr} \chi \]
\[ + \text{tr} \chi \left(G(L, L) + 2\kappa - 2\nabla \cdot \zeta - 2|\zeta|^2 - \langle \bar{H}, \bar{H} \rangle - \kappa \text{tr} \chi \right) \]
\[ = \frac{3}{2} \text{tr} \chi \langle \bar{H}, \bar{H} \rangle - \text{tr} \chi \left(|\dot{\chi}|^2 + G(L, L) \right) + 2 \text{tr} \chi \left(\frac{1}{2} G(L^+, L^+) + \kappa - \nabla \cdot \zeta - |\zeta|^2 \right). \]

Setting \( L = \psi L^+ \), \( L = \frac{1}{\psi} L^- \), whereby \( \zeta = d \log |\psi| \pm \tau \), we see
\[ \psi \left(\frac{1}{2} G(L^-, L^+) + \kappa - \nabla \cdot \zeta - |\zeta|^2 \right) \]
\[ = \psi \left(\frac{1}{2} G(L^-, L^+) + \rho_0 - \nabla \cdot \tau - \Delta \log |\psi| \mp \nabla \cdot \tau - |\tau|^2 \mp 2\tau(\nabla \log |\psi|) - |\nabla \log |\psi||^2 \right) \]
\[ = \psi \left(\frac{1}{2} G(L^-, L^+) + \rho_0 - |\tau|^2 \right) - \Delta \psi \mp 2\tau(\nabla \psi) + (\mp 1 - 1)\psi \nabla \cdot \tau \]
\[ = \frac{1}{2} (1 \pm 1) \mathcal{L}(\psi) + \frac{1}{2} (1 \mp 1) \mathcal{L}^*(\psi). \]

So we conclude with (16) by setting \( L = \psi L^- \), where \( \text{tr} \chi = \psi \), and (17) by setting \( L = \psi L^+ \), where \( \text{tr} \chi = \psi \langle \bar{H}, \bar{H} \rangle \). To show (18) we calculate
\[ \mathcal{L}_{\bar{L}} \tau = \mathcal{L}_{\bar{L}} \zeta - d \frac{1}{\text{tr} \chi} \left(-\frac{1}{2} \text{tr} \chi^2 - |\dot{\chi}|^2 - G(L, L) + \kappa \text{tr} \chi \right) \]
\[ = G_L - \nabla \cdot \dot{\chi} - \text{tr} \chi \tau + d \left(\frac{1}{\text{tr} \chi} (|\dot{\chi}|^2 + G(L, L)) \right) \]
and the result follows for \( L = \psi L^- \). For (19) we observe, by switching the roles of \( L \) and \( \bar{L} \) in the structure equations, that
\[ \mathcal{L} \zeta = -G_L + \nabla \cdot \dot{\chi} - \text{tr} \chi \zeta - d \text{tr} \chi - d \kappa \]
\[ \bar{L} \log \text{tr} \chi = \frac{1}{\text{tr} \chi} \left(G(L, L) + 2\kappa + 2\nabla \cdot \zeta - 2|\zeta|^2 \right) - \text{tr} \chi - \kappa \]
\[ \mathcal{L}_{\bar{L}} \tau = -G_L + \nabla \cdot \dot{\chi} - \text{tr} \chi \zeta + \frac{1}{2} d \text{tr} \chi - d \frac{1}{\text{tr} \chi} \left(G(L, L) + 2\kappa + 2\nabla \cdot \zeta - 2|\zeta|^2 \right). \]

The result follows by setting \( L = \psi L^+ \), \( L = \frac{1}{\psi} L^- \) whereby \( \zeta = \tau - d \log |\psi| \) and recalling our calculations for (19).

In any neighborhood where \( \psi \) vanishes identically (14)-(19) holds, since by construction, the geometry remains invariant. The remaining possibilities are settled by continuity of both the left and right sides of the equality in (14)-(19).

**Theorem 2.3.** Given a MOTS, \( \Sigma \), such that \( \delta_{L^+} \langle \bar{H}, \bar{H} \rangle = \rho - \frac{4\pi}{|\Sigma|} = 0 \) the following linearization holds:
\[ \delta_{\psi L^+ + \phi L^-} \left(\rho_0 - \frac{4\pi}{|\Sigma|} \right) = \left(\begin{array}{cc} -2\Delta L^* & G \\ 0 & 2L \end{array} \right) \left(\begin{array}{c} \psi \\ \phi \end{array} \right) \]
whereby \( G(\phi) = \frac{4\pi}{|\Sigma|} \left(\phi - \frac{4\pi}{|\Sigma|} \right) + \nabla \cdot (\phi(G(L^- - 2\chi^+ \circ \tau)) + \Delta (\phi(|\dot{\chi}|^2 + G(L^-, L^-) - \frac{1}{2})) \right). \) Moreover, if \( \Sigma \) is strictly stable then the linearization has bounded inverse on \( C^{k,\alpha}(\Sigma) \times C^{l,\beta}(\Sigma) \), where \( f \in C^{k,\alpha}(\Sigma) \subset C^{k,\alpha}(\Sigma) \) indicates \( \int f dA = 0 \).
From the hypotheses on \( \Sigma \) and \((17)\), it follows that \( \chi^+ = 0 \) and \( G(L^+, L^+) = 0 \), and from the Dominant Energy Condition that \( G_{L^+} = 0 \). Therefore, the second row of the matrix representation for the linearization follows from \((16)\) and \((17)\) of Proposition \ref{prop:linearization}. By the first variation of area formula, we have \( \delta_{L^+} |\Sigma| = \int_\Sigma \langle \hat{H}, \psi L^+ \rangle dA = \int_\Sigma \psi \langle \hat{H}, \hat{H} \rangle dA = 0 \). Therefore, using \((15)\) of Proposition \ref{prop:linearization}, the first entry of the row satisfies \( \delta_{L^+} \rho_0 = \delta_{L^+} (\nabla \cdot \tau) \). It’s a standard exercise (see, for example, \cite{[31]} Corollary 3.1.1) to verify that \( \bar{\psi} \) therefore follows from \((16)\) and \((17)\) of Proposition \ref{prop:linearization}. By the first variation of area formula, we have \( \delta_{L^+} |\Sigma| = \int_\Sigma \langle \hat{H}, \phi L^- \rangle dA \). The formula for \( G(\phi) \) therefore follows from the formula for \( L \nabla \cdot \tau \), \((14)\), and \((18)\).

For the second part of our Theorem, since all operators have smooth coefficients on a compact manifold, it is a standard argument using a partition of unity to locally reduce to an operator on \( \mathbb{R}^2 \) (see, for example, \cite{[2]} \cite{[7]}) from which it follows that the linearization is a bounded operator. It suffices therefore, by way of the Bounded Inverse Theorem, to show that the linearization is a bijection. Since the linearization is upper diagonal, this in turn is equivalent to showing the operators along the diagonal of the linearization are bijective in their respective Banach spaces. By the hypothesis that \( \Sigma \) be strictly stable both \( \mathcal{L} \) and \( \mathcal{L}^* \) have bounded inverses so it remains to show \( \Delta \mathcal{L}^* : \dot{C}^{k,\alpha}(\Sigma) \to \dot{C}^{k-2,\alpha}(\Sigma) \) is bijective.

**Injectivity**
Given \( \psi \in \dot{C}^{k+4,\alpha}(\Sigma) \), \( \Delta \mathcal{L}^*(\psi) = 0 \) necessitates that \( \mathcal{L}^*(\psi) = C \) for some constant \( C \). Given the case that \( C = 0 \), \( \psi = 0 \) follows by the existence of a bounded inverse for \( \mathcal{L}^* \). On the other hand, if it happens that \( C \neq 0 \), then \( \mathcal{L}^*(\psi) > 0 \) implies that \( \dot{\psi} > 0 \) by Lemma \ref{lem:linearization} and therefore \( \int_\Sigma \dot{\psi} dA > 0 \). This contradicts the fact that \( \int \psi dA = 0 \), so \( \psi = 0 \) and we conclude that \( \Delta \mathcal{L}^* \) is injective.

**Surjectivity**
It’s a well known fact that \( \Delta : C^{k,\alpha}(\Sigma) \to \dot{C}^{k-2,\alpha}(\Sigma) \) is surjective so it suffices to show, for each \( u \in C^{k,\alpha}(\Sigma) \), the existence of a constant \( C_u \) and \( v \in \dot{C}^{k+2,\alpha}(\Sigma) \) such that \( \mathcal{L}^* v = u + C_u \). Once again using Lemma \ref{lem:linearization} and the fact that \( \mathcal{L}^* \) has bounded inverse, we find unique \( \psi_1 > 0, \psi_u \in C^{k+2,\alpha}(\Sigma) \) such that \( \mathcal{L}^*(\psi_u) = u \) and \( \mathcal{L}^*(\psi_1) = 1 \). The desired function is therefore given by \( v = \psi_u - \frac{\int \psi_u}{\int \psi_1} \psi_1 \) whereby \( C_u = - \frac{\int \psi_u}{\int \psi_1} \).

For the final result of this section we will need to construct a convenient coordinate system in a neighborhood of a 2-sphere. The following result is an adaptation of the more general result found in \cite{[2]} (Lemma 6.1):

**Lemma 2.2.1.** Given an embedded 2-sphere \( \Sigma \hookrightarrow \mathcal{M} \), there exists a spacetime neighborhood \( \mathcal{V} \) of \( \Sigma \), with local coordinates \( (t, r, x^i) \) on \( \mathcal{V} \) and functions \( Z, \vartheta, \eta^i, h_{ij} \) such that the metric takes the form

\[
g = e^Z (dt \otimes dr + dr \otimes dt) + h_{ij}(dx^i - \eta^i dr) \otimes (dx^j - \eta^j dr)
\]

where \( \Sigma \cap \mathcal{V} = \{t = 0, r = 0\} \), \( Z(t = 0, r = 0, x^i) = \log 2 \), \( \eta^i(t = 0, r = 0, x^i) = 0 \), and \( h_{ij} \) is a positive definite 2-matrix.

**Proof.** We start by choosing a null basis \( \{L, L_t\} \subset \Gamma(T^+ \Sigma) \). For sufficiently small \( |t| \) the map \( (p,t) \to \exp(tL_t)|_p \), \( p \in \Sigma \), defines a smooth embedding of a null hypersurface \( \mathcal{N} \hookrightarrow \mathcal{M} \) with corresponding foliation \( \{\Sigma_t\} \subset \mathcal{N} \) whereby \( \Sigma_0 = \Sigma \) and each \( \Sigma_t \) is a spacelike 2-sphere. If we denote the null tangent along \( \mathcal{N} \) also by \( L \) then each \( \Sigma_t \) admits an adapted null normal \( L_t \) such that \( \langle L, L_t \rangle = 2 \), moreover, \( L_0 = L \). By collecting null geodesics along \( L_t \) we fill-in a neighborhood \( \mathcal{V} \) of \( \Sigma \) foliated by smooth null hypersurfaces \( \{S_t\} \) whereby \( \mathcal{N} \cap S_t = \Sigma_t \). By shrinking \( \mathcal{V} \) if necessary,
the parameter $t$ extends to a smooth function whereby $\{t = t_0\}$. Moreover, since each $S_t$ is null we have $\nabla t \in \Gamma(T^+ S_t) \subset \Gamma(T S_t)$. From the identity $D_{\nabla t} \nabla t = \frac{1}{2} \nabla \nabla t^2$ it follows that $\nabla t$ generates null geodesics ruling the leaves of thefoliation $\{S_t\}$ and the vector field $2\nabla t$ extends $L_t$ off of $N$ to all of $\mathcal{V}$.

For sufficiently small $|r|$ the map $(p, r) \rightarrow \exp(2r \nabla t)|_p$, $p \in \Sigma$, induces a foliation $\{\Sigma_r\} \subset S_0$ with adapted null basis $\{L_r, 2\nabla t\}$. Repeating the process above we obtain another smoothfoliation of $\mathcal{V}$ by null hypersurfaces $\{S_r\}$, generated by the null geodesic vector field $\nabla r$. We now simply carry local co-ordinate functions $(x^1, x^2)$ from $\Sigma$ to $S_{t_0} \cap S_{t_0}$ by Lie-dragging $x^i$ along $\nabla t$ from $\Sigma$ to $\Sigma_{t_0}$, and then along $\nabla r$ to $S_{t_0} \cap S_{t_0}$. This construction gives $t$-co-ordinate curves that are null pre-geodesic, $\partial_t \propto \nabla r$, so that

$$g = e^Z (dt \otimes dr + dr \otimes dt) + \vartheta dr \otimes dr + h_{ij}(dx^i - \eta^i dr) \otimes (dx^j - \eta^j dr)$$

with $L = \partial_t$, $L = \partial_r$ on $\Sigma$, also $Z \equiv \log 2$, $\eta \equiv 0$, and $h_{ij}$ positive definite.

Since $\nabla t$ is null, we have $\nabla \nabla t = a \partial_r + a^i \partial_i$ with $a \neq 0$ since $\partial_i := \partial_i \partial_i$ is spacelike. From this we see that $0 = \partial_i(t) = (\partial_i, \nabla t) = -a \eta_i + \alpha$, whereby $\eta_i = h_{ij} \eta^j$ (similarly for $\alpha_i$). Moreover, $0 = \partial_r(t) = (\partial_r, \nabla t) = a \theta + a |\tilde{\eta}|^2 - \tilde{\eta} \cdot \tilde{\alpha} = a \theta$ where $\tilde{\eta} := \eta^i \partial_i$, giving $\vartheta = 0$. $\square$

**Theorem 2.4.** Let the metric $g$ admit a strictly stable MOTS, $\Sigma$, satisfying the conditions of Theorem 2.3. Then, for any smooth variation of metrics $g\lambda$ ($g_0 = g$, $0 \leq \lambda \leq \Lambda$), there exists $\epsilon > 0$ and a corresponding family of smooth Doubly Convex MOTS $\Sigma_\lambda$ for $0 \leq \lambda \leq \epsilon$.

*Proof.* We take $\epsilon > 0$ sufficiently small to ensure the induced metric $h_\lambda = g_\lambda|_\Sigma$ remains positive definite for $0 \leq \lambda \leq \epsilon$. We also choose a smoothly varying null normal $L_\lambda \in \Gamma(T^+ \Sigma_\lambda)$ and shrink $\epsilon$ so that $tr \chi_\lambda = L_\lambda \log \sqrt{\det h_{ij}(\lambda)} > 0$, giving a smoothly varying Null Inflation Basis $\{L^-_\lambda, L^+_\lambda\} \subset \Gamma(T^+ \Sigma_\lambda)$. Since our co-ordinates in Lemma 2.2.1 depend smoothly on the metric and choice of normal null basis, we apply the construction using $\{L^-_\lambda, L^+_\lambda\}$ and conclude that a sufficiently small neighborhood $\mathcal{V}$ exists on which all metrics take the form

$$g_\lambda = e^Z (dt \otimes dr + dr \otimes dt) + h_{ij}(dx^i - \eta^i dr) \otimes (dx^j - \eta^j dr)$$

$$g_\lambda^{-1} = e^{-Z} (\partial_t \otimes (\partial_r + \eta^i \partial_i) + (\partial_r + \eta^i \partial_i) \otimes \partial_i) + h^{ij} \partial_i \otimes \partial_j$$

where $\Sigma \cap \mathcal{V} = \{t = r = 0\}$, $Z(\lambda, t = 0, r = 0, x^i) = \log 2$, $\eta^i(\lambda, t = 0, r = 0, x^i) = 0$, and $h_{ij}(\lambda, t, r, x^i)$ is positive definite.

Now, for sufficiently small $C$ such that $f, g \in C^{k+4,\alpha}(\Sigma)$ satisfying $|f|_{k+4,\alpha, \Sigma}, |g|_{k+4,\alpha, \Sigma} \leq C$ ensures $(t = f(x_1), r = g(x_1), x_1) \in \mathcal{V}$, this defines an embedding $\Phi(f, g)(\Sigma):=\Sigma_{f, g}$ with induced metric $\gamma_{f, g} = (h_{ij} + A_{ij})dx^i \otimes dx^j$ whereby

$$A_{ij} = e^Z (f_j g_i + f_i g_j) + |\eta|^2 g_i g_j - (\eta_i g_j + \eta_j g_i) \in C^{k+3,\alpha}(\Sigma).$$

Defining $tr A := h^{ij} A_{ij}$, $\hat{A} := A - \frac{1}{2}(tr A)h$, we leave it to the reader to verify, by shrinking $C$ to ensure $(1 + \frac{1}{2} tr A) > \frac{1}{\sqrt{2}} |\hat{A}|$ for all $\lambda \leq \epsilon$, we have:

$$\gamma_{f, g}^{-1} = \frac{(1 + \frac{1}{2} tr A) h^{ij} - \hat{A}^{ij}}{(1 + \frac{1}{2} tr A)^2 - \frac{1}{2} |\hat{A}|^2} \partial_i \otimes \partial_j.$$

It follows that $g_\lambda|_{T^+ \Sigma_{f, g}}$ is non-degenerate and $T^+ \Sigma_{f, g}$ is trivial with basis vectors

$$N_1 := \nabla (t - f) = -e^{-Z} \tilde{\eta} \cdot \tilde{\nabla} f \partial_t + e^{-Z} (\partial_i + \tilde{\eta}^i) - \tilde{\nabla} f$$

$$N_2 := \nabla (r - g) = e^{-Z} (1 - \tilde{\eta} \cdot \tilde{\nabla} g) \partial_t - \tilde{\nabla} g$$

13
when restricted to $\Sigma_{f,g}$. From the embedding we have (with a slight abuse of notation) $d\Phi(f,g)(\partial_t) = \partial_t + f\partial_t + g\partial_r$ and we conclude that

$$D_{d\Phi(\partial_t)}d\Phi(\partial_j) = f_{ij}\partial_t + g_{ij}\partial_r + f_jD_\partial\partial_t + f_if_jD_\partial\partial_r + g_ig_jD_\partial\partial_r + D_\partial g_{ij}.$$ 

Since $\langle \partial_t, N_1 \rangle = \partial_t(t - f) = 1$, $\langle \partial_t, N_2 \rangle = \partial_t(r - g) = 0$, $\langle \partial_r, N_1 \rangle = \partial_r(t - f) = 0$, $\langle \partial_r, N_2 \rangle = \partial_r(r - g) = 1$ we have

$$\langle \bar{F}, N_1 \rangle = \gamma^{ij}\langle D\Phi(\partial_j)\Phi(\partial_r), N_1 \rangle = \gamma^{ij}f_{ij} + F_1(\partial f, \partial g, f, g) =: \mathcal{L}_1(f)$$

$$\langle \bar{F}, N_2 \rangle = \gamma^{ij}g_{ij} + F_2(\partial f, \partial g, f, g) =: \mathcal{L}_2(g)$$

with $F_i$ smooth functions. From the non-degeneracy of $g_\lambda|_{T^\perp\Sigma_{f,g}}$ we conclude $D^2 := \langle N_1, N_2 \rangle^2 - \langle N_1, N_1 \rangle\langle N_2, N_2 \rangle > 0$ so that $a, b \in C^{k+3,\alpha}(\Sigma)$ given by

$$a := -\frac{\langle N_1, N_1 \rangle}{D + \langle N_1, N_2 \rangle}, \quad b = \frac{1}{\langle N_1, N_2 \rangle + a\langle N_2, N_2 \rangle}$$

gives null vector fields $L_{f,g} := 2N_1 + 2aN_2$, $L_{f,g} := bN_2 - \frac{1}{2}\langle N_2, N_2 \rangle b^2 L_{f,g} \in \Gamma(T^\perp\Sigma_{f,g})$ satisfying $\langle L, L \rangle = 2$ (since $b(L_{f,g}, N_2) = 2$). It’s easily verified for $f = g = 0$ that $a = 0$, $b = 2$ giving $L_{0,0} = \partial_r = \Lambda_\Sigma$, $L_{0,0} = \partial_t = \Lambda_\Sigma^+$ on $\Sigma$. Moreover, we conclude that the expansions along $L_{f,g}$ are given by

$$\left(\begin{array}{c} \text{tr} \chi \\ \text{tr} \chi \end{array}\right) = -\left(\begin{array}{cc} 2 & 2a \\ -\langle N_2, N_2 \rangle b^2 & b(1 - \langle N_2, N_2 \rangle ab) \end{array}\right) \left(\begin{array}{c} \mathcal{L}_1(f) \\ \mathcal{L}_2(g) \end{array}\right)$$

for each $\lambda$. Since $\text{tr} \chi(\lambda, t = 0, r = 0, x^i) = 1$, we shrink $C > 0$ so that $\text{tr} \chi(\lambda, f(x^i), g(x^i), x^i) > 0$. Thus, for $\lambda \leq \epsilon$, the map $\hat{\chi}^{k+4,\alpha}(\Sigma) \times C^{k+4,\alpha}(\Sigma) \rightarrow \hat{\chi}^{k,\alpha}(\Sigma) \times C^{k+2,\alpha}(\Sigma)$ given by

$$\left(\begin{array}{c} f \\ g \end{array}\right) \rightarrow \left(\begin{array}{c} \hat{\chi}_{\Sigma_{f,g}} + \nabla \cdot \zeta - \Delta \log \text{tr} \chi - \frac{4\pi}{|\Sigma_{f,g}|} \\ \text{tr} \chi \end{array}\right)$$

is well defined. We recognize for $\lambda = 0$ that the linearization of this map at $(f, g) \equiv 0$ is given in Theorem 2.3 having a bounded inverse, therefore satisfying the hypotheses of the Banach Space Implicit Function Theorem. By shrinking $\epsilon > 0$ if necessary, we therefore conclude with the unique existence of some $(f_\lambda, g_\lambda) \in \hat{\chi}^{k+4,\alpha}(\Sigma) \times C^{k+4,\alpha}(\Sigma)$ for each $\lambda \leq \epsilon$ as desired in the statement of our Theorem. From the induced metric on $\Sigma_{f,g}$ and the expressions of $N_i$ we conclude that

$$-\Delta \log \text{tr} \chi = \frac{4\pi}{|\Sigma_{f,g}|} - \hat{\chi}_{\Sigma_{f,g}, \lambda} - \nabla \cdot \zeta_{f_\lambda, g_\lambda} \in C^{k+1,\alpha}(\Sigma).$$

From standard regularity results for second order elliptic PDE (see, for example [14]) we conclude that $\text{tr} \chi \in C^{k+3,\alpha}(\Sigma)$ and therefore

$$\left(\begin{array}{c} \mathcal{L}_1(f_\lambda) \\ \mathcal{L}_2(g_\lambda) \end{array}\right) = \left(\begin{array}{cc} \frac{1}{2}(\langle N_2, N_2 \rangle ab - 1) & \frac{a}{b} \\ -\frac{1}{2}\langle N_2, N_2 \rangle b & -\frac{b}{a} \end{array}\right) \left(\begin{array}{c} \text{tr} \chi \\ 0 \end{array}\right) \in C^{k+3,\alpha}(\Sigma) \times C^{k+3,\alpha}(\Sigma).$$

It follows that $f_\lambda, g_\lambda \in C^{k+5,\alpha}(\Sigma)$, bootstrapping from this we conclude that $f_\lambda, g_\lambda \in C^\infty(\Sigma)$. \qed
3 Stability of the Schwarzschild Null Penrose Inequality

The vacuum spherically symmetric and static model for an isolated black hole is given by the Schwarzschild spacetime $\mathbb{P} \times_r S^2$. In ingoing Eddington-Finkelstein coordinates the Schwarzschild spacetime is given by the metric

$$g_S = -(1 - \frac{2M}{r})dv \otimes dv + (dv \otimes dr + dr \otimes dv) + r^2(d\theta \otimes d\theta + \sin(\theta)^2d\varphi \otimes d\varphi).$$

From the independence of all coefficients in $g_S$ on the coordinate function $v$ it follows that the vector field $\partial_v$ is Killing and restricts to be the null tangent $l$ of the Killing Horizon at $H = \{r = 2M\}$. We also see

$$\kappa_l = \langle D_{\partial_v} \partial_v, \partial_v \rangle = -\frac{1}{2} \partial_v \langle \partial_v, \partial_v \rangle = \frac{1}{2} \partial_t (1 - \frac{2M}{r}) = \frac{1}{4M}.$$

From the Warped Product structure (see, for example, [28] Chapter 7) and spherical symmetry we have $\Sigma_{v_0} := \{v = v_0, r = 2M\} \hookrightarrow H$ satisfying $\tau \equiv t \equiv 0 \equiv \hat{\tau} =: \chi^- = -\frac{1}{2} \gamma$, and $K = \frac{1}{4M^2}$, giving the stability operator

$$L(\psi) = (-\Delta + \frac{1}{4M^2})\psi.$$

So any positive constant function produces the principal eigenvalue $\frac{1}{4M^2}$ where $M$ is the mass of the black hole. We conclude $H$ is strictly stable, moreover, $\rho_v = \frac{1}{4M^2}$ identifies $\{\Sigma_v\}$ as our unique foliation by Doubly Convex MOTS.

Using a lemma of S. Alexakis in [1], we will show that small metric perturbations of Schwarzschild spacetime satisfies the Null Penrose Inequality.

The analysis of ODEs in this section hinges on the following result:

**Lemma 3.0.1.** ([17], Corollary 6.3)

Let $w(t, u)$ be continuous on $a \leq t < b$, $u \geq 0$ with the initial value problem $\dot{u} = w(t, u)$ having a unique solution $u(t) \geq 0$ for $a \leq t < b$. If $f : [a, b) \times \mathbb{R}^n \to \mathbb{R}^n$ is continuous and

$$|f(t, x)| \leq w(t, |x|), \quad a \leq t < b, \quad x \in \mathbb{R}^n,$$

then the solutions of

$$\dot{x} = f(t, x), \quad |x(a)| \leq u(a)$$

exist on $[a, b)$ and $|x(t)| \leq u(t)$.

3.1 Assumptions

Having applied Theorem [24] to a strictly stable, Doubly Convex MOTS, $\Sigma_v$, of Schwarzschild, we assume the existence of some $\epsilon > 0$, so that any $\lambda \leq \epsilon$ ensures the existence of an infinite null hypersurface off of $\Sigma_\lambda$ along $2L^- \in \Gamma(T^+\Sigma_\lambda)$ which we will denote by $\Omega_\lambda$. We denote by $L_\lambda \in \Gamma(T^+\Omega_\lambda) \subset \Gamma(T\Omega_\lambda)$ the extension of $L_\lambda|\Sigma_\lambda = 2L^- \otimes \delta_{L\lambda}L_\lambda = 0$. The following definition follows in spirit from that of [26]:

**Definition 3.1.** Denoting a smooth k-tensor by $T|_{p} : T_p\Omega_\lambda \otimes_1 ... \otimes_{k-1} T_p\Omega_\lambda \to \mathbb{R}$, then for a basis extension $\{X_i\} \subset \mathcal{E}_\lambda$:

1. We say $T$ is **transversal** whenever $T(L, X_{i_1}, ..., X_{i_{k-1}}) = ... = T(X_{i_1}, ..., X_{i_{k-1}}, L) = 0$

\(^1\)for an in depth analysis of the stability operator for the larger class of Non-Evolving Horizons we refer the reader to [24].
2. We say \( T = O_n(s^{-m}) \) whenever 
\[
s^m(\mathcal{L}_{X_1} \cdots \mathcal{L}_{X_j} T(s))(X_1, \ldots, X_k) = O(1), \quad (0 \leq j \leq n)
\]

3. Given \( |X_i(1)|^2 \leq C \) for some fixed constant \( C \) and all \( \lambda \leq \epsilon \), we say \( T(\lambda, s) = O_n^\lambda(s^{-m}) \) whenever \( T = O_n(s^{-m}) \) and 
\[
\limsup_{\lambda \to 0} \left( \sup_{\Omega_\lambda} |s^m(\mathcal{L}_{X_1} \cdots \mathcal{L}_{X_j} T(\lambda, s))(X_1, \ldots, X_k)| \right) = 0, \quad (0 \leq j \leq n).
\]

With this definition in hand, and some \( 0 < \delta < 1 \), we assume the following conditions:

\[
O(X_i, X_j) := (R_{L_\lambda X_i L_\lambda, X_j} = O^\lambda_4(s^{-1-\delta}) \quad (20)
\]
\[
\mathcal{L}_{L_\lambda} \mathcal{G} = O^\lambda(s^{-2-\delta}) \quad (21)
\]
\[
G_{L_\lambda}(X_i) := G(L_\lambda, X_i) = O^\lambda_3(s^{-2-\delta}) \quad (22)
\]
\[
G(L_\lambda, L_\lambda) = O^\lambda_2(s^{-3-\delta}). \quad (23)
\]

### 3.2 Total Energy and Mass

From assumption (20) we will be able to make sense of the notion of total mass for \( \Omega_\lambda \). In order to do so we need the following known result which, to the author’s understanding, is due to S. Alexakis \( \Pi \). We provide a proof for completeness and context regarding later results.

**Proposition 3.1.** \( \Pi \) Lemma 4.1 For sufficiently small \( \epsilon \) we find a function \( \theta = O_4^\lambda(1) \), and transverse 2-tensors \( \hat{\gamma}, \hat{\chi} = O_4^\lambda(1) \) on \( \Omega_\lambda \) such that
\[
\gamma(\lambda, s) = s^2 \gamma(\lambda, 1) + s^2 \hat{\gamma}, \quad \text{tr} \chi = \frac{2}{s} + \frac{\theta}{s^2}, \quad \mathcal{L}_{L_\lambda} \gamma = \hat{\chi} + \frac{1}{2} \text{tr} \chi \gamma
\]

Moreover, for \( T = \theta, \hat{\gamma}, \hat{\chi} \), we have that \( \lim_{s \to \infty} \mathcal{L}_{X_i} \cdots \mathcal{L}_{X_j} T(\lambda, s) \) \( (0 \leq j \leq 4) \) is a continuous tensor when viewed over the 2-sphere \( \Sigma_\lambda \) for each \( \lambda \).

**Proof.** Taking a basis extension \( \{X_i\} \subset E_\lambda \) and defining \( \hat{\gamma} := \frac{1}{s^2} \gamma \), the structure equations give:
\[
\sqrt{\det \hat{\gamma}^{-1}} = -\frac{\sqrt{\det \gamma}}{\det \hat{\gamma}} = -\frac{1}{\det \hat{\gamma}} \left( \frac{2}{s^2} \sqrt{\det \gamma} + \frac{1}{s^2} \text{tr} \chi \sqrt{\det \gamma} \right)
\]
\[
= \frac{\theta}{s^2} \sqrt{\det \hat{\gamma}^{-1}}
\]
\[
\hat{\gamma}_{ij} = -\frac{2}{s^2} \hat{\gamma}_{ij} + \frac{2}{s^2} \hat{\chi}_{ij} + \frac{1}{s^2} \text{tr} \chi \gamma_{ij}
\]
\[
= \frac{\theta}{s^2} \hat{\gamma}_{ij} + \frac{2}{s^2} \hat{\chi}_{ij}
\]
\[
\hat{\theta} = 2s \text{tr} \chi - 4 + s^2 \left( \frac{2}{s^2} - \frac{1}{2} \text{tr} \chi^2 - |\hat{\chi}|^2 - G(L_\lambda, L_\lambda) \right)
\]
\[
= -\frac{1}{2s^2} \left( s^4 \text{tr} \chi^2 - 4s^3 \text{tr} \chi + 4 \right) - s^2 \gamma_{ij} \gamma_{kl} \hat{\chi}_{ik} \hat{\chi}_{jl} - s^2 \gamma_{ij} \theta_{ij}
\]
\[
= -\frac{1}{2s^2} \theta^2 - \frac{1}{s^2} \hat{\gamma}_{ij} \hat{\gamma}_{kl} \hat{\chi}_{ik} \hat{\chi}_{jl} - \hat{\gamma}_{ij} \theta_{ij}
\]
\[
\hat{\chi}_{ij} = -\hat{\theta}_{ij} + 2 \gamma_{kl} \hat{\chi}_{ik} \hat{\chi}_{lj}
\]
\[
= -\hat{\theta}_{ij} + \frac{2}{s^2} \gamma_{kl} \hat{\chi}_{ik} \hat{\chi}_{lj}.
\]
By Lie-dragging $\gamma(\lambda, 1)$ along $L_\lambda$ to the rest of $\Omega_\lambda$, denoted by $\gamma_0$, we may define $\bar{\gamma}_{ij} := \gamma_{ij} - \gamma_{0ij}$, $D := \sqrt{\det \gamma^{-1}} - \sqrt{\det \gamma_0^{-1}}$, and $u^2(\lambda, s) := D^2 + \theta^2 + \sum_{ij} \left( (\bar{\gamma}_{ij})^2 + \dot{\bar{\gamma}}_{ij}^2 \right)$. We therefore have

\[
|\dot{\dot{D}}| \leq \frac{u}{s^2} (u + \sqrt{\det \gamma_0^{-1}}) \leq \frac{u}{s^2} (u + |\gamma_0|) + \frac{2}{s^2} u
\]

\[
|\dot{\theta}| \leq \frac{1}{2s^2} u^2 + \frac{2}{s^2} |\bar{\gamma}^{-1}|^2 |\dot{\bar{\gamma}}|^2 + |\bar{\gamma}^{-1}| \left| \frac{c(\lambda)}{s^{1+\delta}} \right|
\]

\[
= \frac{1}{2s^2} u^2 + \frac{2}{s^2} \det \bar{\gamma}^{-2} |\dot{\bar{\gamma}}|^2 + \det \bar{\gamma}^{-1} |\dot{\bar{\gamma}}| \left| \frac{c(\lambda)}{s^{1+\delta}} \right|
\]

\[
\leq \frac{1}{2s^2} u^2 + (u + \sqrt{\det \gamma_0^{-1}})^2 (u + |\gamma_0|)^2 \left( \frac{2u^2}{s^2} (u + \sqrt{\det \gamma_0^{-1}})^2 (u + |\gamma_0|) + \frac{c(\lambda)}{s^{1+\delta}} \right)
\]

\[
\sum_{ij} \bar{\gamma}_{ij}^2 \leq |\alpha - \frac{1}{2} v^{ij} a_{ij} \dot{\gamma}| + \frac{2}{s^2} \det \bar{\gamma}^{-1} |\dot{\bar{\gamma}}| |\dot{\bar{\gamma}}|^2
\]

\[
\leq \frac{c(\lambda)}{s^{1+\delta}} (1 + \det \bar{\gamma}^{-1} |\dot{\bar{\gamma}}|^2) + \frac{u^2}{s^2} (u + \sqrt{\det \gamma_0^{-1}})^2 (u + |\gamma_0|)
\]

\[
\leq \frac{c(\lambda)}{s^{1+\delta}} + (u + \sqrt{\det \gamma_0^{-1}})^2 (u + |\gamma_0|) \left( \frac{c(\lambda)}{s^{1+\delta}} (u + |\gamma_0|) + \frac{u^2}{s^2} \right)
\]

for some continuous function $c : [0, \epsilon] \to [0, \infty)$ whereby $c(0) = 0$. After a simple modification of $c(\lambda)$ we may therefore conclude that

\[
\sqrt{\dot{D}^2 + \dot{\theta}^2 + \sum_{ij} (\bar{\gamma}_{ij}^2 + \dot{\bar{\gamma}}_{ij}^2)} \leq \frac{uP(u) + c(\lambda)}{s^{1+\delta}}
\]

for some seventh order polynomial $P$ with positive coefficients.

We now spend some time analyzing the solutions to the ODE;

\[
y = \frac{yP(y) + c(\lambda)}{s^{1+\delta}}, \quad y(\lambda, 1) = \sup_{\Sigma_\lambda} |\dot{\bar{\gamma}}|(1).
\]

Immediately we note that $y(\lambda, s)$ is monotone increasing in $s$. For any constant $\alpha > 0$ we also find an $\epsilon(\alpha) > 0$ such that $y(\lambda, \infty) \leq \alpha$ for all $\lambda \leq \epsilon(\alpha)$, otherwise there exists a sequence $\{\lambda_i\}$ such that $\lim_{i \to \infty} \lambda_i = 0$ and $y(\lambda_i, \infty) > \alpha$. The inequality

\[
\int_{\gamma(\lambda, 1)}^{\epsilon} \frac{du}{uP(\alpha) + c(\lambda)} \leq \int_{\gamma(\lambda, 1)}^{y(\lambda_i, \infty)} \frac{du}{uP(u) + c(\lambda)} = \frac{1}{\delta},
\]

provides a contradiction since $\lim_{\lambda \to 0} \sup_{\Sigma_\lambda} |\dot{\bar{\gamma}}|(1) = 2$ causes the lesser integral to blow up.

In-fact, for

\[
F(x, \lambda) := \int_{\sup_{\Sigma_\lambda} |\bar{\gamma}|}^{\epsilon} \frac{du}{uP(u) + c(\lambda)},
\]

we see $F_x(x, \lambda) > 0$ is continuous for all $(x, \lambda) \in (0, \infty) \times (0, \epsilon)$. Therefore, a standard generalization of the Implicit Function Theorem (see, for example, [21] Theorem 9.3) ensures that $y(\lambda, \infty)$, given

\[^2\text{recall } \chi = \frac{1}{4} \text{tr} \gamma_{ij} \text{ on } \Sigma_\omega \text{ in Schwarzschild}\]
Implicitly via \( F(y(\lambda, \infty), \lambda) = \frac{1}{\lambda} \), is both unique and continuous in \( \lambda \). With a similar blow-up argument as above we also conclude that \( \lim_{\lambda \to 0^+} y(\lambda, \infty) = 0 \). From Lemma 3.0.1 it follows that,

\[
\sqrt{D^2 + \theta^2 + \sum_{ij} \left( \tilde{\gamma}_{ij} - \tilde{\xi}_{ij} \right)^2} = u(\lambda, s) \leq y(\lambda, s) = O^\lambda(1).
\]

Denoting \( \tilde{T} = (\mathcal{D}, \tilde{\gamma}_{ij}, \theta, \tilde{\xi}_{ij}) \) an integration of the propagation equations using the bound on \( u(\lambda, s) \) yields

\[
\sup_{\Sigma_\lambda} |\tilde{T}(s_m) - \tilde{T}(s_n)| \leq c(\lambda)(\frac{1}{s_{m}^\delta} - \frac{1}{s_{n}^\delta}).
\]

We conclude, from uniform convergence, the existence of a continuous limit \( \tilde{T}^\infty = \lim_{s \to \infty} \tilde{T}(s) \). By taking a derivative of the propagation equations we can write

\[
\mathcal{L}_{X_i} \tilde{T} = A \mathcal{L}_{X_i} \tilde{T} + \tilde{h}
\]

whereby \( A_{ij}, h_i = \frac{1}{s^\alpha} O^\lambda(1) \). From this we have \( |\mathcal{L}_{X_i} \tilde{T}| \leq \frac{c(\lambda)}{s^\alpha} (|\mathcal{L}_{X_i} \tilde{T}| + 1) \) and it follows by Lemma 3.0.1 that

\[
|\mathcal{L}_{X_i} \tilde{T}(s)| \leq \left( 1 + \sqrt{\sum_{jk} \mathcal{L}_{X_i} \tilde{T}(s, \lambda, 1)^2} \right) e^{c(1-s^\alpha)} - 1 = O^\lambda(1).
\]

From this we once again integrate the linear system of equations to conclude

\[
\sup_{\Sigma_\lambda} |\mathcal{L}_{X_i} \tilde{T}(s_m) - \mathcal{L}_{X_i} \tilde{T}(s_n)| \leq c(\lambda)(\frac{1}{s_{m}^\delta} - \frac{1}{s_{n}^\delta})
\]

and we have uniform convergence to a limit \( \lim_{s \to \infty} \mathcal{L}_{X_i} \tilde{T} = \tilde{T}^\infty_i \). Iterating this procedure up to three additional times gives our result from the established decay on lower derivatives. \( \square \)

We are now in a position to define the total Trautman-Bondi energy and mass of the null cone \( \Omega_\lambda \). By temporarily denoting by \( E^s_\lambda \) the the set of vector field extensions off of \( \Sigma_\lambda \) and tangent along the background \( \{ \Sigma_s \} \), and by \( E^\phi_\lambda \) the set of extensions along \( \{ \Sigma \} \) whereby \( s - 1 = \phi(t - 1) \) \((0 < \phi \in \mathcal{F}(\Sigma_\lambda))\), its an easy exercise to show that \( X \in E^s_\lambda \implies X + X(\log \phi)(s - 1)L \in E^\phi_\lambda \). Therefore, for a basis extension \( \{ X_i \} \subset E^s_\lambda \) we observe from Proposition 3.1 that the sphere ‘at infinity’ inherits the metric \( \gamma^\infty_{ij} := \lim_{s \to \infty} \frac{\gamma_{ij}(\lambda, s)}{s^2} \), \( \lim_{s \to \infty} \frac{1}{s^2} (X_i, X_j) \) along \( \{ \Sigma_s \} \). Along \( \{ \Sigma_t \} \), the sphere at infinity inherits the metric

\[
\lim_{t \to \infty} \frac{1}{t^2} (X_i + X_i(\log \phi)(s(t) - 1)L, X_j + X_j(\log \phi)(s(t) - 1)L) = \lim_{t \to \infty} \frac{\gamma_{ij}(\lambda, s)}{s^2} \frac{s^2}{t^2} = \phi^2 \gamma_{ij}.
\]

By the Uniformization Theorem we may therefore choose \( \tilde{\phi} \) such that \( \tilde{\phi}^2 \gamma^\infty = \tilde{\gamma} \) where \( \tilde{\gamma} \) is a round metric on \( \mathbb{S}^2 \). It also follows that \( \omega_{\tilde{\gamma}}^2 = 1 \), if and only if

\[
\omega_{\vartheta} = \frac{\sqrt{1 - |\tilde{v}|^2}}{1 - \tilde{v} \cdot \tilde{n}(\vartheta, \varphi)}
\]

for some \( \tilde{v} \in B^3(1) \), \( \tilde{n}(\vartheta, \varphi) \in \partial B^3(1) \) where \( B^3(1) \) is the open ball in \( \mathbb{R}^3 \) (see, [24]).
Definition 3.2. The total Trautman-Bondi Energy $E_{TB}(\lambda, \vec{v})$ of $\Omega_\lambda$ is given by

$$E_{TB}(\lambda, \vec{v}) := \lim_{t \to \infty} E_H(\Sigma_t)$$

whereby $s - 1 = (\dot{\phi} \omega)(t - 1)$. The total Trautman-Bondi Mass $m_{TB}(\lambda)$ is given by

$$m_{TB}(\lambda) = \inf_{\vec{v}} E_{TB}(\lambda, \vec{v}).$$

3.3 Stability of the Null Penrose Inequality

For our next result we need the known fact:

**Proposition 3.2.** (\cite{Y91}, Theorem 3.2)

Assume $\{\Sigma_s\}$ expands along the flow vector $\vec{L} = \sigma L$, then

$$\dot{\rho} + \frac{3}{2} \sigma \rho = \frac{\sigma}{2} \left( \frac{1}{2} (\vec{H}, \vec{H}) \left( |\dot{\chi}|^2 + G(L^-, L^-) \right) + |\tau|^2 - \frac{1}{2} G(L^-, L^+) \right) + \Delta \left( \sigma |\dot{\chi}|^2 + G(L^-, L^-) \right) - 2 \nabla \cdot \left( \sigma \dot{\chi} \circ \tau \right) + \nabla \cdot (\sigma G_{L^-}).$$

Including conditions (22)-(23) we have

**Proposition 3.3.** For sufficiently small $\epsilon$ we conclude $\tau = O_3^\lambda(s^{-1})$, and

$$\frac{1}{4} \langle \vec{H}, \vec{H} \rangle - \frac{1}{3} \Delta \log \rho \geq 0.$$

Moreover, we find functions $\bar{\rho}, \vec{H}^2 = O_3^\lambda(1)$ on $\Omega_\lambda$ such that

$$\rho = \frac{1}{s^3} \left( \frac{4\pi}{|\Sigma_\lambda|} + \bar{\rho} \right), \quad \langle \vec{H}, \vec{H} \rangle = \frac{1}{s^2} \left( \frac{16\pi}{|\Sigma_\lambda|} (1 - \frac{1}{s}) + \vec{H}^2 \right),$$

and for $T = \bar{\rho}, \vec{H}^2$, we also have that $\lim_{s \to \infty} \mathcal{L}_{X_{i_1}} \cdots \mathcal{L}_{X_{i_j}} T(\lambda, s) (0 \leq j \leq 2)$ is continuous over the 2-sphere $\Sigma_\lambda$ for each $\lambda$.

**Proof.** From the structure equations and Proposition 8.1 we see that

$$\left( \sqrt{\det \gamma} \tau_i \right) = \sqrt{\det \gamma} \left( - \nabla \cdot \dot{\chi} + G(L_\lambda, X_i) + X_i |\dot{\chi}|^2 + G(L_\lambda, L_\lambda) \right) / \text{tr} \chi = O_3^\lambda(1).$$

From which we conclude that $\tau_i = \frac{\sqrt{\det \gamma}}{\sqrt{\det \gamma}} \tau_i(\lambda, 1) + \int_0^{\infty} O_3^\lambda(1) dt = O_3^\lambda(s^{-1})$. As a result, we also have

\footnote{We direct the reader to \cite{Y91} for further discussion and context}
from Proposition \textbf{3.2} for \( \dot{\rho} := s^3 \rho \), and Proposition \textbf{2.4} for \( \dot{H}^2 := s^2 \langle H, \dot{H} \rangle \)

\[
\dot{\rho} = -\frac{3\theta}{2s^2} \dot{\rho} + \frac{1}{4} \dot{H}^2 \frac{s}{\text{tr } \chi} \left( (|\dot{\chi}|^2 + G(L_\lambda, L_\lambda)) + \frac{1}{2} (s^3 \text{tr } \chi)|\tau|^2 - \frac{1}{4} (s^3 \text{tr } \chi) G(L_\lambda, L_\lambda) \\
+ s^3 \Delta \frac{|\dot{\chi}|^2 + G(L_\lambda, L_\lambda)}{\text{tr } \chi} + s^3 \nabla \cdot G_{L_\lambda} - 2s^3 \nabla \cdot (\dot{\chi} \circ \tau) \right) = -\frac{3\theta}{2s^2} \dot{\rho} + \frac{1}{4} \dot{H}^2 \frac{s}{\text{tr } \chi} \left( (|\dot{\chi}|^2 + G(L_\lambda, L_\lambda)) + O^2_2(s^{-1-\delta}) \right)
\]

\[
\dot{H}^2 = \frac{2}{s} \dot{\rho}^2 - \frac{3}{2} \text{tr } \chi \dot{H}^2 - \dot{H}^2 \frac{|\dot{\chi}|^2 + G(L_\lambda, L_\lambda)}{\text{tr } \chi} + \frac{1}{2} \text{tr } \chi \dot{H}^2 + \frac{2}{s} \text{tr } \chi \dot{\rho} \\
- 2s^2 \Delta \text{tr } \chi - 4s^2 \nabla \cdot (\text{tr } \chi \tau) - 2s^2 \text{tr } \chi |\tau|^2 + (s^2 \text{tr } \chi) G(L_\lambda, L_\lambda)
\]

\[
= \frac{2}{s} \text{tr } \chi \dot{\rho} - \left( \frac{\theta}{s^2} + \frac{|\dot{\chi}|^2 + G(L_\lambda, L_\lambda)}{\text{tr } \chi} \right) \dot{H}^2 + O^2_2(s^{-2}).
\]

Defining \( T^i = (\dot{\rho}, \dot{H}^2) \) we see from the propagation equations that

\[
\dot{T}^i = AT^i + h
\]

where \( A_{ij}, h_i = \frac{1}{4\pi s^{1+\rho}} O(1) \). From this we conclude that \( |\dot{T}| \leq \frac{C}{s^{1+\rho}} (|\dot{T}| + 1) \) and therefore Lemma \textbf{3.0.1} gives

\[
|\dot{T}| \leq (1 + \frac{4\pi}{|\Sigma_\lambda|}) e^{C(1-\frac{1}{s^2})} - 1 = O(1).
\]

With this bound, we return to the propagation equations to find

\[
\dot{\rho} = -\frac{3\theta}{2s^2} \dot{\rho} + O^2(s^{-1-\delta})
\]

from which we deduce, similarly as in Proposition \textbf{3.1} a continuous limit \( \rho_\infty := \lim_{s \to \infty} \dot{\rho} \). Moreover, defining \( \tilde{\rho} := \rho - \frac{4\pi}{|\Sigma_\lambda|} \), we have \( |\tilde{\rho}| \leq \frac{c(\lambda)}{s^{1+\rho}} (|\tilde{\rho}| + 1) \) ensuring that \( |\tilde{\rho}| \leq e^{C(1-\frac{1}{s^2})} - 1 = O^2(1) \). We leave it to the reader to similarly verify that this now allows us to conclude that \( \tilde{H}^2 := \tilde{H}^2 - \frac{16\pi}{|\Sigma_\lambda|} (1 - \frac{1}{s}) = O^2(1) \), with a continuous limit as \( s \to \infty \). Consider now the propagation of \( (\dot{\rho}, \dot{H}^2) \):

\[
\begin{pmatrix} \dot{\rho} \\ \dot{H}^2 \end{pmatrix} = \begin{pmatrix} O^2_2(s^{-2}) \\ O^2_2(s^{-2}) \end{pmatrix} \frac{\theta}{s^2} + \left( \begin{pmatrix} O^2_2(s^{-1-\delta}) \\ O^2_2(s^{-1-\delta}) \end{pmatrix} \right)
\]

\[
= \begin{pmatrix} O^2_2(s^{-2}) \\ O^2_2(s^{-2}) \end{pmatrix} + \left( \begin{pmatrix} O^2_2(s^{-1-\delta}) \\ O^2_2(s^{-1-\delta}) \end{pmatrix} \right)
\]

Taking a derivative results in a linear system of the form \( \mathcal{L}_X \dot{T} = A \mathcal{L}_X \dot{T} + h \) whereby \( A_{ij}, h_i = \frac{1}{4\pi s^{1+\rho}} O(1) \) from which we deduce boundedness of \( |\mathcal{L}_X \dot{T}| \). From this we can bootstrap similarly as before to deduce continuous limits and decay in \( \lambda \) for up to two derivatives.

Finally, returning to the ODE

\[
\dot{\rho} = -\frac{3\theta}{2s^2} \dot{\rho} + O^2(s^{-1-\delta})
\]

we conclude that

\[
\begin{align*}
\dot{\rho} &= \frac{4\pi}{|\Sigma_\lambda|} e^{\int_1^s \frac{\theta}{t^2} dt} + \int_1^s O^2_2(t^{-1-\delta}) dt \\
X_i \dot{\rho} &= \frac{4\pi}{|\Sigma_\lambda|} \left( \int_1^s X_i \theta \frac{\theta}{t^2} dt \right) e^{\int_1^s \frac{\theta}{t^2} dt} + \int_1^s O^2_1(t^{-1-\delta}) dt \\
H^i(X_i, X_j) &= \frac{4\pi}{|\Sigma_\lambda|} \left( \int_1^s X_i \theta \frac{\theta}{t^2} dt \right) + \int_1^s O^2_1(t^{-1-\delta}) dt
\end{align*}
\]
Proof. For any geodesic foliation $\gamma$ where, according to Proposition 3.1, the Hawking Energy along $\gamma$ is the Trautman-Bondi Mass of $\Omega$ for some continuous $c(\lambda)$ such that $c(0) = 0$. We leave to the reader the simple exercise of verifying $\langle \tilde{H}, \tilde{H} \rangle \geq \frac{16\pi}{s^2}|\Sigma_\lambda|(1 - \frac{1}{s}) - \frac{c(\lambda)}{s^2}(1 - \frac{1}{s})$. As a result, for some modified $c(\lambda)$,

$$\frac{1}{4} \langle \tilde{H}, \tilde{H} \rangle - \frac{1}{3} \Delta \log \rho \geq \frac{1}{s^2} \left( \frac{4\pi}{|\Sigma_\lambda|}(1 - \frac{1}{s}) - c(\lambda)(1 - \frac{1}{s^2}) \right)$$

which remains non-negative for sufficiently small $\rho$.

Finally, from condition (21) and the following Proposition we’re ready to prove stability of the Penrose Inequality.

Proposition 3.4. ([31], Theorem 4.1) Given any cross-section $\Sigma \subset \Omega_\lambda$, it follows that $\Sigma = \{s = \omega\}$ for some $\omega \in \mathcal{F}(\Sigma_\lambda)$. Defining $\phi := \rho(\Sigma)$ we have, with respect to the background data,

$$\dot{\phi} = \rho + \frac{|\dot{\chi}|^2 + G(L_\lambda, L_\lambda)(\Delta \omega - 2\dot{\chi} \nabla \omega, \nabla \omega)}{\text{tr} \chi} + 2\nabla \omega^2 \dot{\chi}^2 + G(L_\lambda, L_\lambda) \nabla \omega |^2 + G(L_\lambda, \nabla) - 2\dot{\chi}(\tilde{\nabla}, \nabla \omega).$$

Theorem 3.5. For sufficiently small $\epsilon$, our metric perturbation off of the Schwarzschild spacetime gives rise to the Null Penrose Inequality

$$\sqrt{\frac{|\Sigma_\lambda|}{16\pi}} \leq m_{TB}(\lambda)$$

where $m_{TB}$ is the Trautman-Bondi Mass of $\Omega_\lambda$.

Proof. For any geodesic foliation $\{\Sigma_t\} \subset \Omega_\lambda$ whereby $\omega_t := s|\Sigma_t = \phi(t - 1) + 1$, combining Proposition 3.4, Proposition 3.3, and (21) applied to Proposition 3.4 it follows

$$\dot{\phi}(t) = \frac{\rho|\Sigma_t}{\omega_t^3} + O^3(t^{-3-\delta}) \implies \lim_{t \to \infty} t^3 \dot{\phi}(t) = \frac{\rho_\infty}{\phi_\infty} \geq 0.$$

The Hawking Energy along $\{\Sigma_t\}$ satisfies

$$\lim_{t \to \infty} E_H(\Sigma_t) = \lim_{t \to \infty} \left( \frac{1}{4\pi} \int_{\Sigma_t} t^3 \phi(t) dA \right) = \frac{1}{16\pi} \int_{S_2} \phi^2 dA_{\infty} =: E_H^\infty(\phi)$$

where, according to Proposition 3.1, $\gamma_{ij}^\infty = \lim_{s \to \infty} \frac{1}{s^2} \gamma_{ij}$ is the metric on the 2-sphere ‘at infinity’. From Theorem 3.1, we conclude

$$\sqrt{\frac{|\Sigma_\lambda|}{16\pi}} \leq \lim_{s \to \infty} \frac{1}{s^2} \gamma^\infty_{ij}$$

a simple application of the Hölder inequality giving the final equality above ([31], Lemma 2.2.2). The geodesic foliation $\{\Sigma_t\}$ induces a conformal re-scaling of the metric at infinity according to $\lim_{t \to \infty} \frac{\gamma_{ij}^\infty}{\gamma_{ij}} = \phi^2$. From the Uniformization Theorem we know all metrics are conformally equivalent, in particular, all round metrics are reached along corresponding geodesic foliations of $\Omega_\lambda$. As a result we conclude that $m_{TB}(\lambda) \geq \inf_{\phi \geq 0} E_H^\infty(\phi)$. 

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