FROM \( p_0(n) \) TO \( p_0(n + 2) \)

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Abstract. In this note, we study global existence to the Cauchy problem for the semilinear wave equation with a non–effective scale-invariant damping, namely

\[ v_{tt} - \Delta v + \frac{2}{1 + t} v_t = |u|^p, \quad v(0, x) = v_0(x), \quad v_t(0, x) = v_1(x), \]

where \( p > 1, \ n \geq 2 \). We prove blow-up in finite time in the subcritical range \( p \in (1, p_2(n)) \) and an existence result for \( p > p_2(n) \), \( n = 2, 3 \). Namely, we find the critical exponent for the small data solution to this problem. We see that for \( n \geq 2 \), \( p_2(n) = p_0(n + 2) \) where \( p_0(n) \) is the Strauss exponent for the classical wave equation.

1. Introduction

In this paper, we study small data global solutions to

\[
\begin{aligned} 
  v_{tt} - \Delta v + \frac{2}{1 + t} v_t &= |v|^p, & t \geq 0, & x \in \mathbb{R}^n, \\
  v(0, x) &= v_0(x), & x \in \mathbb{R}^n, \\
  v_t(0, x) &= v_1(x), & x \in \mathbb{R}^n, 
\end{aligned}
\]

in space dimension \( n \geq 2 \). We prove that the critical exponent of (1) is \( p_0(n + 2) \), where \( p_0(n) \) is the critical exponent for the semilinear wave equation i.e. the positive solution to

\[ (n - 1)p^2 - (n + 1)p - 2 = 0. \]

By critical exponent, we mean that for small initial data in a suitable space, global solutions to (1) exist if \( p > p_0(n + 2) \), moreover there exist suitable data such that (1) admits no global solutions if \( p \in (1, p_0(n + 2)) \).

It has been recently shown that the critical exponent for

\[
\begin{aligned} 
  v_{tt} - \Delta v + \frac{\mu}{1 + t} v_t &= |v|^p, & t \geq 0, & x \in \mathbb{R}^n, \\
  v(0, x) &= v_0(x), & x \in \mathbb{R}^n, \\
  v_t(0, x) &= v_1(x), & x \in \mathbb{R}^n, 
\end{aligned}
\]

is \( 1 + 2/n \) for a sufficiently large \( \mu \) (see Section 2 for details). The exponent \( 1 + 2/n \) is the same of the semilinear heat equation, and it is related to the effectiveness of the damping, i.e., the property of the damping term to makes suitable linear estimates for the wave equation similar to the ones for the corresponding heat equation \( \mu v_t - (1 + t)\Delta v = 0 \) (in particular, the \( L^1 - L^p \) low frequencies estimates). We set

\[ p_\infty(n) = 1 + 2/n, \]

where the pedex \( \infty \) means that \( \mu \) is sufficiently large.

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On the contrary, it appears difficult to show that for small positive values of $\mu$, the critical exponent $p_{\mu}(n)$ is strictly larger than $p_{\infty}$. In this latter case, we say that the scale-invariant damping is non--effective for the nonlinear problem. More precisely one would expect that

$$p_{\infty}(n) < p_{\mu}(n) \leq p_0(n).$$

In this paper, we reach this aim by setting $\mu = 2$ in \cite{2}, and showing that

$$p_2(n):= \max\{p_0(n+2), p_{\infty}(n)\} = \begin{cases} 3 & \text{if } n = 1, \\ p_0(n+2) & \text{if } n \geq 2. \end{cases}$$

We also notice that $p_{\infty}(2) = p_0(2 + 2) = 2$. Hence for $n \geq 3$, we have a non--effective critical exponent.

We prove the following.

**Theorem 1.** Assume that $v \in C^2([0,T) \times \mathbb{R}^n)$ is a solution to (1) with initial data $(v_0, v_1) \in C^2([0,T) \times \mathbb{R}^n)$ such that $v_1, v_0 \geq 0$, and $(v_0, v_1) \neq (0,0)$. If $p \in (1, p_2(n)]$, then $T < \infty$.

Being the 1-dimensional existence result already proved in \cite{3}, we prove the existence result in space dimension $n = 2$ and $n = 3$.

**Theorem 2.** Let $n = 2$ and $p > 2$. Let $(\bar{v}_0, \bar{v}_1) \in C^2([0,T) \times \mathbb{R}^2)$. Then there exists $\varepsilon_0 > 0$ such that for any $\varepsilon \in (0, \varepsilon_0)$, if $v_0 = \varepsilon \bar{v}_0$ and $v_1 = \varepsilon \bar{v}_1$, the Cauchy problem (1) admits a unique global solution $v \in C([0, \infty), H^2) \cap C^2([0, \infty), H^1) \cap C^2([0, \infty), L^2)$.

**Theorem 3.** Let $n = 3$ and $p > p_0(5)$. Let $(\bar{v}_0, \bar{v}_1) \in C_c^\infty([0,T) \times \mathbb{R}^3) \times C^1([0,T) \times \mathbb{R}^3)$, radial. Then there exists $\varepsilon_0 > 0$ such that for any $\varepsilon \in (0, \varepsilon_0)$, if $v_0 = \varepsilon \bar{v}_0$ and $v_1 = \varepsilon \bar{v}_1$, the Cauchy problem (1) admits a unique global radial solution $v \in C([0, \infty) \times \mathbb{R}^3) \cap C^2([0, \infty) \times (\mathbb{R}^3 \setminus \{0\})$).

For the sake of brevity, we use the notation

$$\langle y \rangle = 1 + |y|, \quad \text{for any } y \in \mathbb{R}^n.$$ 

To prove our results, we perform the change of variable $u(t, x) = \langle t \rangle v(t, x)$, so that problem (1) becomes

$$
\begin{cases}
  u_{tt} - \Delta u = \langle t \rangle^{-(p-1)}|u|^p, & t \geq 0, \quad x \in \mathbb{R}^n, \\
  u(0, x) = u_0(x), & x \in \mathbb{R}^n, \\
  u_t(0, x) = u_1(x), & x \in \mathbb{R}^n,
\end{cases}
$$

with $u_0 = v_0 + v_1$ and $u_1 = v_1$. This means we are dealing with a semilinear wave equation with a time dependent coefficient in the nonlinearity.

For proving Theorem 1 we will extend to this equation the classical blow-up technique due to R.T. Glassey; for Theorem 2 we use Klainerman vector fields; due to the lack of regularity of the nonlinear term, for $p \in (p_0(5), 2)$, the proof of Theorem 3 requires a different idea. We will establish an appropriate version of the pointwise estimates for the wave equation.

By the aid of these estimates, in this latter case, we will also find a decay behaviour for the solution to (1) which is the same of the $(n+2)$-dimensional wave-equations. For details, see Theorem 4 and Remark 5. The technique of the pointwise estimates could be applied to prove the existence for $p > p_0(n+2)$ with $n \geq 4$.

2. An overview of some existing results

For the semilinear wave equation,

$$u_{tt} - \Delta u = |u|^p, \quad u(0, x) = u_0(x), \quad u_t(0, x) = u_1(x),$$

it is well-known that the critical exponent for the existence of small data global solutions is $p_0(n)$. More precisely if $1 < p \leq p_0(n)$ then solutions to \cite{8} blow-up in finite time, for a suitable choice of initial data (see \cite{8}, \cite{12}, \cite{13}, \cite{19}, \cite{20}, \cite{23}), whereas for $p \in (p_0(n), (n+3)/(n-1)]$ a unique global small data
solution exists (see\cite{7,9,12,21,26}). In space dimension $n = 1$, solutions to \cite{3} blow-up in finite time for any $p > 1$, hence we put $p_0(1) = \infty$ (see \cite{8}).

The known results on the global existence of small data solutions to \cite{2} can be summarized as follows:

- Non-existence of weak solutions for $\mu > 1$ and $p \leq 1 + 2/n$, provided $\int (u_0 + (\mu - 1)^{-1}u_1)\, dx > 0$. See Theorem 1.1 and Example 3.1 in \cite{4}.
- Non-existence of weak solutions for $\mu \in (0, 1]$ and $p \leq 1 + 2/(n - 1 + \mu)$, provided $\int u_1\, dx > 0$. See Theorem 1.4 in \cite{25}.
- According to Theorems 2 and 3 in \cite{3}, global existence for small data if $p > 1 + 2/n$ for energy solutions

$-$ if $n = 1$ and $\mu \geq 5/3$,
$-$ if $n = 2$ and $\mu \geq 3$,
$-$ for any $n \geq 3$ if $\mu \geq n + 2$.

The equation in \cite{2} has many interesting properties. In particular, if $\mu \in \mathbb{R}$, by the change of variable

$$u^\sharp(t, x) = \langle t \rangle^{\mu - 1} u(t, x),$$

one sees that $u$ solves \cite{2} if, and only if, $u^\sharp(t, x)$ solves

$$\begin{cases}
    u^\sharp_{tt} - \triangle u^\sharp + \frac{\mu^\sharp}{\langle t \rangle} u^\sharp_t = \langle t \rangle^{\langle \mu^\sharp - 1 \rangle/2} |u^\sharp|^p, \\
    u^\sharp(0, x) = u_0(x), \\
    u^\sharp_t(0, x) = u_1(x) + (1 - \mu^\sharp)u_0(x)
\end{cases}$$

with $\mu^\sharp = 2 - \mu$.

If $\mu \in (-\infty, 1)$ in \cite{2}, by introducing the change of variable $\tilde{u}(t, x) = u(\Lambda(t) - 1, x)$, where

$$\Lambda(t) = \frac{\langle t \rangle^{\ell + 1}}{\ell + 1}, \quad \text{and} \quad \ell = \frac{\mu}{1 - \mu},$$

the Cauchy problem \cite{2} becomes a Cauchy problem for a semilinear free wave equation with polynomial propagation speed

$$\begin{cases}
    \tilde{u}_{tt} - \langle t \rangle^{2\ell} \triangle \tilde{u} = \langle t \rangle^{2\ell} |\tilde{u}|^p, \\
    \tilde{u}(\tilde{t}, x) = u_0(x), \\
    \tilde{u}_t(\tilde{t}, x) = (1 - \mu)^{-1}u_1(x)
\end{cases}$$

where $\tilde{t} = (1 - \mu)^{-1} - 1$. We notice that:

- $\ell > 0$ if, and only if, $\mu \in (0, 1)$. On the other hand, $\ell \in (-1, 0)$ if $\mu \in (-\infty, 0)$;
- $\ell \in (0, e^{1/2} - 1]$ if $\mu \in (0, 1)$ and $\ell \to 0$ as $\mu \to 0$ and as $\mu \to 1$;
- $\ell \in (-1, 0)$ if $\mu \in (-\infty, 0)$.

Similarly, by virtue of \cite{6,7} and \cite{8}, if $\mu > 1$, the Cauchy Problem \cite{2} becomes

$$\begin{cases}
    \tilde{u}_{tt} + \langle t \rangle^{2\ell} \triangle \tilde{u} = c_\mu (t)^{2\ell - (p-1)} |\tilde{u}|^p, \\
    \tilde{u}^\sharp(t, x) = u_0(x), \\
    \tilde{u}_t^\sharp(t, x) = (\mu - 1)^{-1}(\mu - 1) - 1, \quad \text{and} \quad c_\mu = (\mu - 1)^{1-(p-1)}.
\end{cases}$$

On the other hand, if $\mu = 1$, by setting $\Lambda(t) = e^\ell$, problem \cite{2} becomes

$$\begin{cases}
    \tilde{u}_{tt} - e^{2\ell} \triangle \tilde{u} = e^{2\ell} |\tilde{u}|^p, \\
    \tilde{u}(0, x) = u_0(x), \\
    \tilde{u}_t(0, x) = u_1(x).
\end{cases}$$
By means of these transformations, following the reasoning in Example 4.4 in [4], we can obtain again the non-existence of weak solutions to (2) for $\mu \in (0, 1)$ and

$$p \leq 1 + \frac{2(\ell + 1)}{n(\ell + 1) - 1} = 1 + \frac{2}{n - 1 + \mu},$$

as in [25].

Since in [4, 25] the test function method is employed, the blow-up dynamic remains unknown. However, one can apply an argument similar to those developed in [8] to [9, 10] and [11], obtaining that all the $L^q$ norms of local solutions blow-up in finite time. Indeed, in Example 2a in [22] the author gives sufficient conditions on

$$u_{tt} - a^2(t)\Delta u = m(t)|u|^p,$$

which guarantee that $\lim_{t \to T} \|u(t)\|_q = +\infty$ for any $1 \leq q \leq +\infty$, where $T$ is the maximal existence time for a smooth solution with nonnegative, compactly supported, initial data. See also [4] for the 1-dimensional case. By means of [9, 10] and [11] from these results one can deduce the blow-up in finite time for (2)

- if $\mu \in (0, 1)$ and $p < 1 + \frac{2}{n - 1 + \mu}$,
- if $\mu = 1$ and $p \leq p_{\infty}$,
- if $\mu \in (1, 2]$ and $p < p_{\infty}$.

We notice that blow-up in finite time is proved for the limit case $p = 1 + 2/(n - [1 - \mu]^+)$ only for $\mu = 1$, while non-existence of weak solution for $\mu \in (0, 1) \cup (1, 2]$ is also known for $p = 1 + 2/(n - [1 - \mu]^+)$. Up to our knowledge no other information in literature leads to the existence or non-existence for (2), in particular the blow-up dynamic is not known for $\mu > 2$.

After this discussion, it was natural to ask if the blow-up exponent $p_{\infty}(n) = 1 + 2/n$ could be improved for some $\mu \in [1, 5/3]$ if $n = 1$, for some $\mu \in [1, 3]$ if $n = 2$, or for some $\mu \in [1, n + 2]$ if $n \geq 3$. On the other hand, one may ask if a counterpart result of global existence can be proved. Theorems [1, 2, 6] give a positive answer to these questions. The special case $\mu = 2$ may give precious hints about the general case of small $\mu$.

For the sake of completeness, we remark that the case of wave equation with space-dependent damping

$$\begin{aligned}
v_{tt} - \Delta v + \mu(x)^{-\alpha} v_t &= |v|^p, & t \geq 0, & x \in \mathbb{R}^n, \\
v(0, x) &= v_0(x), & x \in \mathbb{R}^n, \\
v_t(0, x) &= v_1(x), & x \in \mathbb{R}^n,
\end{aligned}$$

where $\mu > 0$ and $\alpha \in (0, 1]$, is also particularly difficult when $\alpha = 1$. On the one hand, in [10], R. Ikeda, G. Todorova and B. Yordanov proved that the critical exponent for the existence of small data global solution is $1 + 2/(n - \alpha)$ if $\alpha \in (0, 1)$. On the other hand, in [11] they proved that the linear estimates for the energy of (12) show a decay rate which depends on $\mu$ for $\mu \leq n$. This property hints to a $\mu$-depending critical exponent for (12), for small $\mu$.

To complete our overview, we mention that the critical exponent for the wave equation with time-dependent damping $\mu(t)^{\kappa} u_t$ is $1 + 2/n$ if $\kappa \in (-1, 1)$ (see [5, 17, 18]), whereas global existence of small data solutions for $p > 1 + 2/(n - \alpha)$ for the wave equation with damping $\mu(x)^{-\alpha}(t)^{-\beta}$, if $\alpha, \beta > 0$ and $\alpha + \beta < 1$ has been derived in [24].

3. Proof of Theorem 1

Let us remind the ODE blow-up dynamic for polynomial nonlinearity, which will play a fundamental role in proving our result.

**Lemma 1.** Let $p > 1$, $q \in \mathbb{R}$. Let $F \in \mathcal{C}^2([0, T])$, positive, satisfying

$$\bar{F}(t) \geq K_1(t + R)^{-q}(F(t))^p \quad \text{for any } t \in [T_1, T),$$

for $p > 1$, $q \in \mathbb{R}$.
for some $K_1, R > 0$, and $T_1 \in [0, T)$. If
\[ F(t) \geq K_0(t + R)^a \quad \text{for any } t \in [T_0, T), \]
for some $a \geq 1$ satisfying $a > (q - 2)/(p - 1)$, and for some $K_0 > 0$, $T_0 \in [0, T)$, then $T < \infty$.

Moreover, let $q \geq p + 1$ in $[13]$. Then there exists $K_0 = K_0(K_1) > 0$ such that, if $[13]$ holds with $a = (q - 2)/(p - 1)$ for some $T_0 \in [0, T)$, then $T < \infty$.

Proof. The case $a > (q - 2)/(p - 1)$ corresponds to Lemma 4 in [20]. Let $a = (q - 2)/(p - 1)$. Following Lemma 2.1 in [23], our problem reduces to find $K_0$ such that the function $G(s) = (T_0 + 1)^{-a}F((T_0 + 1)s + 1)$ blows up. One has
\[ \tilde{G}(s) \geq K_2(s)^{-q}(G(s))^p \]
with $K_2 = (T_0 + 1)^{-a+2}K_1$. It follows that $\tilde{G}$ is positive and $\tilde{G}(s) \geq K_2K_0^{p-1}s^{-2}(G(s))$ so that $G(s) \geq (s)^{K_0^{p-1}K_2}$. For large $K_0$, the exponent $a := K_0^{p-1}K_2$ satisfies $a > (q - 2)/(p - 1)$, and we may conclude the proof. These ideas are contained in [8]. \[ \square \]

Transforming problem [1] into [3], the statement follows as a consequence of the next proposition. Here we follow [23], taking into account of the time-dependence of the nonlinear term.

**Proposition 1.** Let $f \in C^2(\mathbb{R}^n)$ and $g \in C^1(\mathbb{R}^n)$, nonnegative, compactly supported, such that $f + g \not\equiv 0$. Assume that $u \in C^2([0, T) \times \mathbb{R}^n)$ is the maximal solution to
\[ \begin{cases} 
  u_{tt} - \Delta u = (t)^{-(p-1)}|u|^p, \\
  u(0, x) = f(x), \\
  u_t(0, x) = g(x).
\end{cases} \tag{15} \]
If $p \leq p_2(n)$, with $p_2(n)$ as in [3], then $T < \infty$.

In the following, let $R > 0$ be such that supp $f, \text{supp } g \subset B(R)$. Therefore, supp $u(t, \cdot) \subset B(R + t)$. Without loss of generality, we assume $R = 1$.

Let us define
\[ F(t) := \int_{\mathbb{R}^n} u(t, x) \, dx \]
Thanks to the finite speed of propagation of $u$, and by Hölder’s inequality,
\[ \tilde{F}(t) = (t)^{-(p-1)} \int_{\mathbb{R}^n} |u(t, x)|^p \, dx = (t)^{-(p-1)} \int_{B(1+t)} |u(t, x)|^p \, dx \geq (t)^{-(n+1)(p-1)} |F(t)|^p, \tag{16} \]
In order to apply Lemma 1 we need to establish that $F(t)$ is positive.

We consider the functions
\[ \phi_1(x) := \int_{S^{n-1}} e^{\tau \cdot \omega} \, d\omega, \quad \psi_1(t, x) := \phi_1(x) e^{-t} \]
and
\[ F_1(t) := \int_{\mathbb{R}^n} u(t, x) \psi_1(t, x) \, dx. \]
It follows that
\[ \tilde{F}(t) \geq (t)^{-(p-1)} |F_1(t)|^p \left( \int_{B(1+t)} (\psi_1(t, x))^\frac{p}{p-1} \, dx \right)^{(p-1)}. \]
Let us estimate this last integral. Recalling that $\psi_1(t, x) = e^{-t} \phi_1(x)$, we see that
\[ \int_{B(1)} (\psi_1(t, x))^\frac{p}{p-1} \, dx \leq C(K, A, p) (t)^{-A} \]
for any fixed $K < 1 + t$ and $A > 0$. By using
\[
\phi_1 \lesssim |x|^{-\frac{n+1}{2}} e^{|x|} \quad \text{as } |x| \to \infty,
\]
for large $K$, we get
\[
\int_{B(1+t) \setminus B(K)} (\psi_1(t,x)) \frac{dx}{(t^+)^{n+1}} \lesssim \int_K^{t+1} (\rho)^{n-1} e^{\frac{-p}{p-1} \frac{\alpha}{p-1} (\rho-t)} d\rho.
\]
Putting
\[
\alpha := n - 1 - (n - 1)p/(2(p - 1)),
\]
we have
\[
\int_K^{t+1} (\rho)^{\alpha} e^{\frac{-p}{p-1} (\rho-t)} d\rho \lesssim (t)^{\alpha}.
\]
If $\alpha \geq 0$, i.e. $p \geq 2$, we may immediately conclude
\[
\int_K^{t+1} (\rho)^{\alpha} e^{\frac{-p}{p-1} (\rho-t)} d\rho \lesssim (t)^{\alpha}.
\]
The same estimate holds if $\alpha < 0$, i.e. $p \in (1, 2)$, since we may write
\[
\left(1 + \frac{\alpha(p-1)}{p(1+K)}\right) \int_K^{t+1} (\rho)^{\alpha} e^{\frac{-p}{p-1} (\rho-t)} d\rho \lesssim (t)^{\alpha}
\]
and for large $K$ and $t$ we turn to (17). As a conclusion
\[
\hat{F}(t) \gtrsim \langle t \rangle^{-n(p-1) + (n-1)\frac{p}{2}} |F_1(t)|^p.
\]
To estimate $|F_1(t)|^p$, the sign of the nonlinear term comes into play. More precisely the following result holds for any smooth solution to $u_{tt} - \Delta u = G(t,x,u)$ with positive $G$.

**Lemma 2.** [Lemma 2.2 in [23]] There exists $t_0 > 0$ such that, for $t \geq t_0$ it holds
\[
F_1(t) \gtrsim \frac{1}{2} \left(1 - e^{-2t}\right) \int_{\mathbb{R}^n} (f(x) + g(x)) \phi_1(x) dx + e^{-2t} \int_{\mathbb{R}^n} f(x) \phi_1(x) dx.
\]

In particular, due to our assumption on $f$ and $g$ it holds $F_1(t) > c > 0$. Therefore, we proved
\[
\hat{F}(t) \gtrsim \langle t \rangle^{-n(p-1) + (n-1)\frac{p}{2}} = \langle t \rangle^{-\frac{n+1}{2} p + n}.
\]
Integrating twice, we obtain
\[
F(t) \gtrsim \langle t \rangle^{-\frac{n+1}{2} p + n + 2} + t \hat{F}(0) + F(0) \gtrsim \langle t \rangle^{\max\left\{-\frac{n+1}{2} p + n + 2, 0\right\}},
\]

since $\hat{F}(0) \geq 0$ and $F(0) \geq 0$.

**The subcritical case.** Recalling (16), we may apply the first part of Lemma 1 once we have one of the following:
\[
\frac{n+1}{2} p + n + 2 > \frac{(n+1)(p-1) - 2}{p - 1},
\]
\[
1 > \frac{(n+1)(p-1) - 2}{p - 1}.
\]
Condition (21) corresponds to $p < p_0(n+2)$, whereas condition (22) corresponds to $p < p_{\infty}(n)$, hence we derive $p < \max\{p_0(n+2), p_{\infty}(n)\}$.

**Critical 1D case.** First, let $n = 1$ and $p = 3$. By (16), it follows (14) with $q = 4$. Directly using (20) into (16), we derive
\[
\hat{F}(t) \gtrsim \langle t \rangle^{2-p} F(t)^3 \gtrsim \langle t \rangle^{-1},
\]
and, integrating twice, $F(t) \gtrsim \langle t \rangle \ln(t)$. Therefore, for any $K_0 > 0$ there exists $T_0 > 0$ such that (14) holds with $a = 1$. The proof follows from Lemma 1.

**Critical 2D case.** By (16) and (20), we have again $\hat{F}(t) \gtrsim \langle t \rangle^{-1}$, hence the conclusion follows.
Critical case with \( n \geq 3 \). We notice that \( p_2(n) \leq 2 \). We can prove the blow up for the spherical mean of \( u \),

\[
\hat{u}(t, r) = \frac{1}{\omega_n} \int_{|\omega| = 1} u(t, r\omega) dS_\omega ,
\]

which satisfies the differential inequality (see [13])

\[
\hat{u}_{tt} - \Delta \hat{u} \geq (t)^{-(p-1)}|\hat{u}|^p .
\]

We can assume that \( u \) is radial. Following [23], we consider the Radon transform of \( u \) on the hyper-planes orthogonal to a fixed \( \omega \in \mathbb{R}^n \):

\[
Ru(t, \rho) = \int_{x \cdot \omega = \rho} u(t, x) dS_x ,
\]

where \( dS_x \) is the Lebesgue measure of \( \{ x : x \cdot \omega = \rho \} \). One can see that \( Ru \) is independent of \( \omega \) and that

\[
Ru(t, \rho) = c_n \int_{|r|}^\infty u(t, r) (r^2 - \rho^2)^{\frac{1}{2}} r dr .
\]

(23)

We will assume that \( \rho \geq 0 \). Since \( Ru \) satisfies

\[
\partial_t^2 Ru - \partial_\rho^2 Ru = (t)^{-(p-1)} R(\|u\|^p)
\]

and \( f \geq 0, g \geq 0 \), it follows

\[
Ru(t, \rho) \geq \frac{1}{2} \int_0^t \int_{|s|}^{\rho + (t-s)} R(\|u\|^p)(s, \rho_1) d\rho_1 ds .
\]

Since \( \text{supp} \ R(\|u\|^p)(s, \cdot) \subset B(s+1) \), following [23] we may estimate

\[
Ru(t, \rho) \geq \frac{1}{2} \int_0^t \int_\mathbb{R} (s)^{-(p-1)} R(\|u\|^p)(s, \rho_1) d\rho_1 ds
\]

\[
= \frac{1}{2} \int_0^t \int_\mathbb{R}^n |u(s, y)|^p dy ds = \frac{1}{2} \int_0^t \tilde{F}(s) ds .
\]

Recalling [19], we get

\[
Ru(t, \rho) \geq \frac{1}{2} \int_0^{t \wedge \rho - 1} (s)^{-\frac{n+1}{2} p+n} ds .
\]

Since \( p = p_2(n) \leq 2 \), it holds

\[
Ru(t, \rho) \geq (1 + t - \rho)^{-\frac{n+1}{2} p+n+1} .
\]

(24)

Coming back to [23], and recalling that \( \text{supp} \ u(t, \cdot) \subset B(1+t) \), since \( r + \rho \leq 2r \) in the integral, we may estimate

\[
Ru(t, \rho) = c_n \int_{|r|}^{1+t} u(t, r)(r+\rho)^{\frac{n-3}{2}} (r-\rho)^{\frac{n-3}{2}} dr \leq c_n 2^{n-3} \int_{|r|}^{1+t} u(t, r)^{\frac{n-1}{2}} (r-\rho)^{\frac{n-3}{2}} dr .
\]

(25)

The operator \( T : L^p(\mathbb{R}) \to L^p(\mathbb{R}) \) defined by

\[
Tf(\tau) = \frac{1}{|1+t-\tau|^{\frac{n}{2}}} \int_\tau^{1+t} f(r) |r-\tau|^{\frac{n-3}{2}} dr , \quad \text{for any } \tau \in \mathbb{R},
\]

is bounded. Therefore if we put

\[
f(r) = \begin{cases} |u(t, r)|^r & \text{if } r \geq 0, \\ 0 & \text{if } r \leq 0, \end{cases}
\]

so that \( f(r)^p = |u(t, r)|^p r^{n-1} \) for \( r \geq 0 \), we get

\[
\int_0^{1+t} \left( \frac{1}{|1+t-\rho|^{\frac{n}{2}}} \int_{|\rho|}^{1+t} |u(t, r)|^r (r-\rho)^{\frac{n-3}{2}} dr \right)^p d\rho \leq \int_0^\infty |u(t, r)|^p r^{n-1} dr = C \int_\mathbb{R}^n |u(t, x)|^p dx .
\]
Due to \( p \leq 2 \) and \( r \geq \rho \), it holds \( r^{\frac{n-1}{p}} \geq r^{\frac{n-1}{p} - \frac{n+1}{p} + \frac{n+1}{p}} \), so that, by (24), we obtain
\[
\int_0^{1+t} \frac{(Ru(t,p))^p}{|1+t-\rho|^\frac{n-1}{p}} \rho^{n-1-(n-1)\frac{p}{2}} d\rho \lesssim \int_{\mathbb{R}^n} |u(t,x)|^p dx.
\]

Thanks to (24), we obtain
\[
\int_{\mathbb{R}^n} |u(t,x)|^p dx \gtrsim \int_0^{1+t} (1 + t - \rho)^{-\frac{n+1}{p} + \frac{n+1}{p}} \rho^{n-1-(n-1)\frac{p}{2}} d\rho.
\]

Recalling that \( p = p_2(n) \), we may use
\[
\frac{n+1}{2} p^2 - \frac{n+3}{2} p = 1,
\]

obtaining
\[
\int_{\mathbb{R}^n} |u(t,x)|^p dx \gtrsim \int_0^{1+t} (1 + t - \rho)^{-1} \rho^{n-1-(n-1)\frac{p}{2}} d\rho \gtrsim \langle t \rangle^{1+(n-1)\frac{p}{2}} \ln(t).
\]

Thus,
\[
\tilde{F}(t) \gtrsim \langle t \rangle^{-(p-1)} \int_{\mathbb{R}^n} |u|^p dx \gtrsim \langle t \rangle^{(n-1)\frac{p}{2} + 1} \ln(t),
\]

hence,
\[
F(t) \gtrsim \langle t \rangle^{(n-1)\frac{p}{2} + 1} \ln(t).
\]

Similarly to the case \( n = 1 \), the end of the proof follows by Lemma 1.

4. Proof of Theorem 2

Remark 1. In the statement of Theorem 2 we may relax the assumption on the data from \((\bar{v}_0, \bar{v}_1) \in \mathcal{C}_c^2 \times \mathcal{C}_c^1\) to \((\bar{v}_0, \bar{v}_1) \in H^2 \times H^1\), compactly supported.

As in [16], for any \( p, q \in [1, \infty] \), let us define
\[
\|f\|_{(p,q)} := \|f(r\omega) r^{\frac{n-1}{p}}\|_{L_p^q(S^{n-1})}.
\]

It holds \( \|f\|_{(p,p)} = \|f\|_{L^p} \) and Hölder inequality
\[
\|f_1 f_2\|_{(p,q)} \lesssim \|f_1\|_{(p_1,q_1)} \|f_2\|_{(p_2,q_2)},
\]

is valid if
\[
\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2} \leq 1, \quad \frac{1}{q} = \frac{1}{q_1} + \frac{1}{q_2} \leq 1.
\]

Moreover, since \( S^{n-1} \) is compact, it holds
\[
\|f\|_{(p,q_1)} \lesssim \|f\|_{(p,q_2)} \quad \text{for any } q_2 \geq q_1.
\]

Let \( i, j = 1, 2 \) with \( i \neq j \), we put
\[
\Gamma = (D, L_0, L_j, \Omega_{ij}), \quad D = \langle \partial_i, \partial_j \rangle, \quad L_0 = \langle t \rangle \partial_t + x \cdot \nabla,
\]
\[
L_j = \langle t \rangle \partial_j - x_j \partial_t, \quad \Omega_{ij} = x_i \partial_j - x_j \partial_i,
\]

and
\[
\|f\|_{\Gamma, s, (p,q)} = \sum_{|\alpha| \leq s} \|\Gamma^\alpha f\|_{(p,q)},
\]
\[
\|f\|_{\Gamma, s, p} = \|f\|_{\Gamma, s, (p,p)},
\]
\[
\|f\|_{\Gamma, s, \infty} = \sum_{|\alpha| \leq s} \|\Gamma^\alpha f\|_{\infty}.
\]
It holds
\[ [□, \Gamma^\alpha] = \sum_{|\beta| \leq |\alpha| - 1} a_\beta \Gamma^\beta □, \quad (28) \]
\[ [D, \Gamma^\alpha] = \sum_{|\beta| \leq |\alpha| - 1} b_\beta \Gamma^\beta D. \quad (29) \]

By using the argument in \cite{15}, one has the following Sobolev-type inequalities in these generalized spaces:
\[
\begin{align*}
\|w(t, \cdot)\|_\infty & \lesssim (t)^{-\frac{n-1}{2}} \|w(t, \cdot)\|_{\Gamma, s, 2}, \quad \text{if } s > n/2, \\
\|w(t, \cdot)\|_\infty & \lesssim (t)^{1-\frac{n-1}{2}} (\|w(t, \cdot)\|_{\Gamma, s, 2} + \|Dw(t, \cdot)\|_{\Gamma, s, 2}), \quad \text{if } s + 1 > n/2, \\
\|w(t, \cdot)\|_q & \lesssim (t)^{-(n-1)(\frac{q}{2}-\frac{1}{2})} \|w(t, \cdot)\|_{\Gamma, s, 2} \quad \text{if } 2 \leq q < \infty, \quad \frac{1}{q} \geq \frac{1}{2} - \frac{s}{n} \geq 0, \quad (32)
\end{align*}
\]
for any \( t > 0 \) and any \( w(t, \cdot) \) such that right-hand sides are finite. The previous statements can be found in \cite{29}.

The energy estimates in these spaces, for the solution to the Cauchy problem for the inhomogeneous wave equation
\[
\begin{cases}
  u_{tt} - \Delta u = f(t, x), & t \geq 0, \quad x \in \mathbb{R}^n, \\
  u(0, x) = u_0(x), & x \in \mathbb{R}^n, \\
  u_t(0, x) = u_1(x), & x \in \mathbb{R}^n,
\end{cases} \quad (33)
\]
are given by
\[
\|Du\|_{\Gamma, s, 2} \lesssim \|\nabla u_0\|_{\Gamma, s, 2} + \|u_1\|_{\Gamma, s, 2} + \int_0^t \|f(\tau, x)\|_{\Gamma, s, 2} d\tau, \quad s \in \mathbb{N}. \quad (34)
\]
Indeed, we may combine (28), (29) with the classical energy estimate
\[
\|Du\|_2 \lesssim \|\nabla u_0\|_2 + \|u_1\|_{L^2} + \int_0^t \|f(\tau, x)\|_{L^2} d\tau.
\]
It is also necessary to estimate \( \|u\|_{\Gamma, s, 2} \), here the space dimension \( n = 2 \) comes into play.

**Lemma 3.** Let \( n = 2 \), and \( u \) be the solution to (33). Then, for any \( \epsilon > 0 \) there exists \( \delta(\epsilon) > 0 \), satisfying \( \delta(\epsilon) \to 0 \) as \( \epsilon \to 0 \), such that
\[
\|u(t, \cdot)\|_{\Gamma, s, 2} \lesssim \|u_0\|_{\Gamma, s, 2} + t^\delta \|u_1\|_{\Gamma, s, (1+\epsilon, 2)} + \int_0^t (t - \tau)^\delta \|f(\tau, \cdot)\|_{\Gamma, s, (1+\epsilon, 2)} d\tau.
\]

**Proof.** Due to (28), it suffices to consider the case \( s = 0 \). First, let \( f \equiv 0 \). Following \cite{10}, by using the change of variables \( x = ty \), we may estimate
\[
\|u(t, \cdot)\|_2 \lesssim \|u_0\|_2 + t^2\|G\|_{H^{-1}(\mathbb{R}^2)}^2, \quad \text{where } G(y) = u_1(ty).
\]
Recalling that
\[
\|G\|_{H^{-1}} = \sup_{v \in H^1, v \neq 0} \frac{\int_{\mathbb{R}^2} G(y)v(y) dy}{\|v\|_{H^1}},
\]
by virtue of (20)–(27) and Sobolev embeddings, it holds
\[
\|Gv\|_{L^1} \lesssim \|G\|_{(q, 2)} \|v\|_{(q', 2)} \lesssim \|G\|_{(q, 2)} \|v\|_{q'} \lesssim \|G\|_{(q, 2)} \|v\|_{H^1},
\]
where \( q = 1 + \epsilon \), for some \( \epsilon \in (0, 1) \). Since
\[
\|G\|_{(q, 2)} \lesssim t^{-\frac{2}{q}} \|u_1\|_{(q, 2)},
\]
summarizing, we proved that
\[
\|u(t, \cdot)\|_2 \lesssim \|u_0\|_2 + t^{2(1 - \frac{1}{q' + 1})} \|u_1\|_{(1+\epsilon, 2)}.
\]
The case \( f \neq 0 \) follows by Duhamel’s principle. \( \square \)
Since thanks to the finite speed of propagation, i.e.

\[ \bar{\alpha} \]

Let \( \delta \) is given by Lemma \( \ref{lemma} \). For any \( w \in X(T) \), let \( u = S[w] \) be the solution to

\[ u_t - \Delta u = (t)^{-p(1)}|w|^p, \quad u(0, x) = u_0(x), \quad u_t(0, x) = u_1(x). \]

Thanks to Lemma \( \ref{lemma} \) for \( s = 1 \), we may estimate

\[ \|u(t, \cdot)\|_{\Gamma, 1, 2} \leq \|u_0\|_{\Gamma, 1, 2} + t^\delta \|u_1\|_{\Gamma, 1, (1+\epsilon, 2)} + \int_0^t (t - \tau)^\delta \|\langle \tau \rangle^{-(p-1)}|w(\tau, \cdot)|\|^p_{\Gamma, 1, (1+\epsilon, 2)} \, d\tau. \]

Since

\[ [\partial_\tau, \langle \tau \rangle^\alpha] = \alpha \frac{1}{\langle \tau \rangle^\alpha}, \quad [L_0, \langle \tau \rangle^\alpha] = \alpha \langle \tau \rangle^\alpha, \quad [L_j, \langle \tau \rangle^\alpha] = \alpha \frac{x_j}{\langle \tau \rangle} \langle \tau \rangle^\alpha, \]

thanks to the finite speed of propagation, i.e. \( |x| \lesssim \langle t \rangle \) in supp \( u \), we get:

\[ \|u(t, \cdot)\|_{\Gamma, 1, 2} \leq \|u_0\|_{\Gamma, 1, 2} + t^\delta \|u_1\|_{\Gamma, 1, (1+\epsilon, 2)} + \int_0^t (t - \tau)^\delta \|\langle \tau \rangle^{-(p-1)}|w(\tau, \cdot)|\|^p_{\Gamma, 1, (1+\epsilon, 2)} \, d\tau. \]

We may now estimate:

\[ \|w(\tau, \cdot)\|^p_{\Gamma, 1, (1+\epsilon, 2)} \leq \|w\|^{p-1}_{(q, \infty)} \|w(\tau, \cdot)\|_{\Gamma, 1, 2}, \]

where \( q(\epsilon) \in (2, \infty) \) is given by:

\[ \frac{1}{1 + \epsilon} = \frac{1}{2} + \frac{1}{q}. \]

Let \( \gamma(\epsilon) := 2/q = (1 - \epsilon)/(1 + \epsilon) \). Since \( p > 2 \) it holds \( \gamma + 1 < p \). Then

\[ \|w(\tau, \cdot)\|^{p-1}_{(q, \infty)} \lesssim \|w(\tau, \cdot)\|^{p-1}_{(2, \infty)} \|w(\tau, \cdot)\|^{\gamma}_{(2, \infty)}. \]

By Sobolev embeddings on the unit sphere \( S^1 \), we may estimate

\[ \|w(\tau, \cdot)\|_{(2, \infty)} \lesssim \|w(\tau, \cdot)\|_{H^s} \leq \|w(\tau, \cdot)\|_2 + \|Dw(\tau, \cdot)\|_2. \]

Thanks to (54), we have

\[ \|w(\tau, \cdot)\|_{\Gamma, 1, 2} \lesssim \langle \tau \rangle^{\frac{1}{2}} \|w(\tau, \cdot)\|_{\Gamma, 1, 2} + \|Dw(\tau, \cdot)\|_{\Gamma, 1, 2}, \]

therefore, taking into account that \( w \in X(T) \) we conclude

\[ \int_0^t (t - \tau)^\delta \|\langle \tau \rangle^{-(p-1)}|w(\tau, \cdot)|\|^p_{\Gamma, 1, (1+\epsilon, 2)} \, d\tau \lesssim \|w\|^{p}_{X(T)} \int_0^t (t - \tau)^\delta \langle \tau \rangle^{-(p-1)+p\delta + \frac{p-1+\gamma}{2}} \, d\tau. \]

Since \( \delta(\epsilon) \to 0 \) and \( \gamma(\epsilon) \to 1 \) as \( \epsilon \to 0 \), for any \( p > 2 \), one may find a sufficiently small \( \epsilon \) such that

\[ -(p - 1) + p\delta + \frac{p-1+\gamma}{2} < -1. \]

To estimate \( \|Du\|_{\Gamma, 1, 2} \) we apply (54). Now

\[ \|w(\tau, \cdot)\|_{\Gamma, 1, 2} \lesssim \|w(\tau, \cdot)\|_{2+\epsilon_1} \|w(\tau, \cdot)\|_{\Gamma, 1, q_1} \]

for some \( \epsilon_1 > 0 \), where \( q_1(\epsilon_1) \) is such that

\[ \frac{1}{2} = \frac{1}{2 + \epsilon_1} + \frac{1}{q_1}. \]

By Sobolev embeddings,

\[ \|w(\tau, \cdot)\|_{\Gamma, 1, q_1} \lesssim \|w(\tau, \cdot)\|_{\Gamma, 1, 2} + \|Dw(\tau, \cdot)\|_{\Gamma, 1, 2}. \]

On the other hand, since \( p > 2 \), we have

\[ \|w(\tau, \cdot)\|^{p-1}_{2+\epsilon_1} \leq \|w(\tau, \cdot)\|^{p-1}_{(2+\epsilon_1)(p-1)} \lesssim \langle \tau \rangle^{-\frac{1}{2+\epsilon_1}} \|w(\tau, \cdot)\|^{p-1}_{\Gamma, 1, 2} \leq \|w(\tau, \cdot)\|^{p-1}_{\Gamma, 1, 2}. \]
More precisely, we will prove that for any $p > p$ is the required solution.

Summarizing, we proved that $\epsilon < \epsilon$

Clearly, we set $v \in C^2$.

Remark

We extend to negative values of $p$, there exists $\epsilon > 0$ such that if $(v_0, v_1) \in C^2(\mathbb{R}^3) \times C^1(\mathbb{R}^3)$ are radial, namely, $v_0 = v_0(|x|)$, $v_1 = v_1(|x|)$, and

$(r)^{\kappa + 1} (|v_0(r) + v_1(r)| + (r)|v_0'(r) + v_1'(r)| + (r)\kappa \left(|v_0(r)| + (r)|v_0'(r)| + (r)^2 |v_0''(r)|\right)) < \epsilon,$ \hspace{1cm} (35)

for some $\epsilon < \epsilon_0$, then (1) admits a (radial) global, solution $v \in C([0, \infty) \times \mathbb{R}^3) \cap C^2([0, \infty) \times (\mathbb{R}^3 \setminus \{0\})].$

Clearly, we set $r = |x|$ in (35).

Remark 2. We may also replace the nonlinear term $|u|^p$ in (1) by $f(u)$, where $f \in C^1$ is an even function satisfying $|f^{(h)}(u)| \lesssim |u|^{p-h}$, for $h = 0, 1$.

In particular,

$f(0) = 0$, \hspace{1cm} $|f(u) - f(v)| \lesssim |u - v|(|u|^{p-1} + |v|^{p-1}).$ \hspace{1cm} (36)

To fulfill our objective, we apply to (1) the technique introduced by Asakura and developed in different works, in particular in (14). For the sake of simplicity, let $v_0 = 0$ and let $g := u_1 = v_1$. Condition (35) becomes:

$|g^{(h)}(r)| \leq \epsilon (r)^{-h+\kappa} \hspace{1cm} h = 0, 1.$ \hspace{1cm} (37)

We extend $g$ to negative values of $r$, by defining

$g(r) := g(-r), \hspace{1cm}$ for any $r < 0$,

then, by symmetry, we rewrite (1) as

$$
\begin{cases}
 u_{tt} - u_{rr} - \frac{2}{r} u_r = (t)^{-(p-1)} |u|^p, \hspace{1cm} t \geq 0, \hspace{1cm} r \in \mathbb{R}, \\
 u(0, r) = 0, \hspace{1cm} u_t(0, r) = g(r), \hspace{1cm} r \in \mathbb{R}.
\end{cases} \hspace{1cm} (38)
$$

Definition 1. We say that $u(t, |x|) = u(t, r)$ is a radial global solution to (38), if $u \in C([0, \infty) \times \mathbb{R}),$ $r^2 u \in C^2([0, \infty) \times \mathbb{R})$ and

$$
\begin{cases}
r^2 u_{tt} - r^2 u_{rr} + 2r u_r = r^2 (t)^{-(p-1)} |u|^p, \hspace{1cm} t \geq 0, \hspace{1cm} r \in \mathbb{R}, \\
 u(0, r) = 0, \hspace{1cm} u_t(0, r) = g(r), \hspace{1cm} r \in \mathbb{R}.
\end{cases} \hspace{1cm} (38)
$$

Remark 3. Any solution to (38), in the sense of Definition 1 gives a $C([0, \infty) \times \mathbb{R}^3) \cap C^2([0, \infty) \times (\mathbb{R}^3 \setminus \{0\})])$ solution to (1) and in turn of to (1).
5.1. The linear equation.

**Definition 2.** Let us consider

\[
\begin{cases}
u_{tt} - u_{rr} - \frac{2}{r} u_r = 0, & t \geq 0, \quad r \in \mathbb{R}, \\
u(0, r) = 0, \quad u_t(0, r) = g(r), & r \in \mathbb{R}.
\end{cases}
\]

(39)

We say that \( u \in C([0, \infty) \times \mathbb{R}) \) is a solution to (39) if \( r^2 u \in C^2([0, \infty) \times \mathbb{R}) \) and

\[
\begin{cases}
r^2 u_{tt} - (r^2 u_{rr} + 2r u_r) = 0, & t \geq 0, \quad r \in \mathbb{R}, \\
u(0, r) = 0, \quad u_t(0, r) = g(r), & r \in \mathbb{R}.
\end{cases}
\]

(40)

We see that \( r^2 u \in C^2 \) and \( u \in C \) give enough regularity to write the equation (40). Indeed we have \( ru_r = \partial_r (r^2 u) - 2ru \in C \) and \( r^2 u_{rr} + 2ru_r = \partial_{rr} (r^2 u) - 2u - 2ru_r \in C \); clearly also \( r^2 u_{tt} \in C^2 \).

According to Definition 2, the function

\[ u^{\text{lin}}(t, r) = \int_{-1}^{1} H_g(t + r \sigma) \, d\sigma \quad \text{with} \quad H_g(\rho) := \frac{\rho g(\rho)}{2}, \]

is the solution to (39). This result can be found in [2], but we rewrite the computation for completeness. Indeed, for any \( H = H(\rho), H \in C^1 \), we put

\[ v(t, r) := \frac{1}{r} \int_{t-r}^{t+r} H(\rho) \, d\rho = \int_{-1}^{1} H(t + r \sigma) \, d\sigma. \]

(41)

For any \( r \neq 0 \) it holds

\[
\begin{align*}
v_t &= \int_{-1}^{1} H'(t + r \sigma) \, d\sigma = \frac{1}{r} (H(t + r) - H(t - r)), \\
v_{tt} &= \frac{1}{r} (H'(t + r) - H'(t - r)), \\
v_r &= \int_{-1}^{1} \sigma H'(t + r \sigma) \, d\sigma = \frac{1}{r} (H(t + r) + H(t - r)) - \frac{1}{r} v, \\
v_{rr} &= \int_{-1}^{1} \frac{1}{r^2} (H(t + r) + H(t - r)) + \frac{1}{r} (H'(t + r) - H'(t - r)) + \frac{1}{r^2} v - \frac{1}{r} v_r \quad \text{\[44\]}
\end{align*}
\]

\[ = \frac{1}{r} (H'(t + r) - H'(t - r)) - \frac{2}{r} v_r. \]

In particular, \( v \) solves the equation in (44) for any \( r \neq 0 \). Moreover, \( r^2 v \in C^2([0, \infty), \mathbb{R}), \) and \( v \) solves the equation in (44) for any \( r \in \mathbb{R} \), as one may immediately check by multiplying (42) by \( r \) and (43) by \( r^2 \).

We remark that \( v(0, r) = 0 \) if \( H \) is odd. In this latter case, \( rv_t(0, r) = 2H(r) \).

In particular, this proves that \( u^{\text{lin}} \) solves (39).

For our convenience, we also compute

\[
\begin{align*}
\partial_r(rv) &= v + rv_r = H(t + r) + H(t - r), \\
\partial_r^2(r^2 v) &= \partial_r(rv) + r \partial_r^2(rv) = H(t + r) + H(t - r) + rH'(t + r) - H'(t - r).
\end{align*}
\]

(46)

(47)

For any fixed \( \kappa > 1 \), we introduce the Banach space

\[ X_\kappa := \{ u \in C([0, \infty), \mathbb{R}), \text{ even in } r : \quad \partial_r (ru) \in C([0, \infty), \mathbb{R}), \| u \|_{X_\kappa} < \infty \} \]

with norm

\[ \| u \|_{X_\kappa} := \| (t + |r|) \langle t - |r| \rangle^{\kappa - 1} u \|_{L_t^\infty L_r^\infty} + \| (r)^{-1} \langle t + |r| \rangle \langle t - |r| \rangle^{\kappa - 1} \partial_r (ru) \|_{L_t^\infty L_r^\infty}, \]
Theorem 4. Suppose \([37]\) holds for some \(\kappa > 1\). Then
\[
\|u^{\text{lin}}\|_{X_\kappa} \leq C \varepsilon,
\]
for a suitable constant \(C > 0\).

Proof. We notice that
\[
|H^{(h)}_g(\rho)| \leq \varepsilon(\rho)^{-\kappa-h} \quad h = 0, 1.
\]
Thanks to \([40]\), we immediately derive
\[
|\partial_r(ru^{\text{lin}})| = |H_g(t + r) + H_g(t - r)| \lesssim \varepsilon \langle r \rangle^{-\kappa}.
\]
We distinguish two cases. If \(t \geq 2|\rho|\), then \(\langle t \pm |\rho| \rangle \simeq \langle t \rangle\) and we get
\[
|\partial_r(ru^{\text{lin}})| \lesssim \varepsilon \langle t - |\rho| \rangle^{-(\kappa-1)} \langle t + |\rho| \rangle^{-1} \lesssim \varepsilon \langle t - |\rho| \rangle^{-(\kappa-1)} \langle t + |\rho| \rangle^{-1} \langle r \rangle,
\]
where in the last inequality we used the trivial estimate \(1 \leq \langle r \rangle\).

If \(t \leq 2|\rho|\), then \(\langle t + |\rho| \rangle \lesssim 3\langle r \rangle\), therefore
\[
|\partial_r(ru^{\text{lin}})| \lesssim \varepsilon \langle t - |\rho| \rangle^{-\kappa} \langle t + |\rho| \rangle^{-1} \langle r \rangle \lesssim \varepsilon \langle t - |\rho| \rangle^{-(\kappa-1)} \langle t + |\rho| \rangle^{-1} \langle r \rangle,
\]
where in the last inequality we use the trivial estimate \(\langle t - |\rho| \rangle^{-1} \leq 1\).

In order to estimate \(\|\langle t + |\rho| \rangle \langle t - |\rho| \rangle^{-1} u^{\text{lin}}\|_{L^p L^\infty}\), we observe that
\[
|u^{\text{lin}}(t, r)| \lesssim \frac{\varepsilon}{r} \int_{t-r}^{t+r} \langle \rho \rangle^{-\kappa} d\rho = \frac{1}{|r|} C \varepsilon \int_{t-|\rho|}^{t+|\rho|} \langle \rho \rangle^{-\kappa} d\rho.
\]
If \(t \geq 2|\rho|\), then \(\langle t \pm |\rho| \rangle \simeq \langle t \rangle\) and \(\langle t - |\rho| \rangle \lesssim |\rho|\), hence
\[
|u^{\text{lin}}(t, r)| \lesssim \frac{\varepsilon}{|\rho|} \langle t - |\rho| \rangle^{-\kappa+1} \simeq \varepsilon \langle t - |\rho| \rangle^{-(\kappa-1)} \langle t + |\rho| \rangle^{-1}.
\]
If \(t \leq 2|\rho|\), we distinguish two cases. If \(|\rho| \leq 1\), then \(t + |\rho| \langle t - |\rho| \rangle^{-\kappa-1} \simeq 1\) and it is sufficient to estimate
\[
|u^{\text{lin}}(t, r)| \leq \int_{t-1}^{t} H_g(t + r \sigma) d\sigma \leq C.
\]
On the other hand, if \(t \leq 2|\rho|\) and \(|\rho| \geq 1\), then \(\langle t + |\rho| \rangle \leq 3\langle r \rangle\) and \(\langle t \rangle \simeq \langle r \rangle\), therefore
\[
|u^{\text{lin}}(t, r)| \lesssim \frac{1}{|\rho|} \varepsilon \int_{t-|\rho|}^{t+|\rho|} \langle \rho \rangle^{-\kappa} d\rho \lesssim \frac{1}{\langle r \rangle} \varepsilon \langle t - |\rho| \rangle^{-(\kappa-1)}
\]
\[
\lesssim \varepsilon \langle t - |\rho| \rangle^{-1} \langle t + |\rho| \rangle^{-(\kappa-1)}
\]
thanks to \(\kappa > 1\). This concludes the proof that \(\|u^{\text{lin}}\|_{X_\kappa} \leq C \varepsilon\).

\(\square\)

5.2. Duhamel principle and basic nonlinear estimates. For any \(u \in X_\kappa\), let
\[
Lu(t, r) := \int_0^r \langle s \rangle^{-(p-1)} \int_{-1}^{1} H_u[s](t - s + r \sigma) d\sigma ds + \frac{r}{C} \int_0^t \langle s \rangle^{-(p-1)} \int_{t-s-r}^{t-s+r} \langle s \rangle^{-(p-1)} H_u[s](\rho) d\rho ds,
\]
where
\[
H_u[s](\rho) := \frac{\rho f(u(s, \rho))}{2}.
\]
We denote by \(H_u[s]'(\rho)\) the derivative of \(H_u[s](\rho)\) with respect to \(\rho\), considering \(s\) as a parameter.

Let us consider \(f(u(s, \rho))\) and \(\rho \partial_{\rho} f(u(s, \rho))\). If \(u \in X_\kappa\), recalling that \(ru_r = \partial_r(ru) - u\), we may estimate:
\[
|f(u(s, \rho))| \lesssim \|u\|_{X_\kappa} \langle s + |\rho| \rangle^{-p} \langle s - |\rho| \rangle^{-p(\kappa-1)}
\]
and
\[
|\rho^{-1} \rho \partial_{\rho} f(u(s, \rho))| \lesssim \|u\|_{X_\kappa} \langle s + |\rho| \rangle^{-p} \langle s - |\rho| \rangle^{-p(\kappa-1)},
\]
Having in mind \([15]\), it follows, in particular, that
\[
|H_u[s](\rho) + H_u[s]'(\rho)| \lesssim \|u\|^p_{X_\kappa} \langle s + |\rho| \rangle^{-p} \langle s - |\rho| \rangle^{-p(\kappa-1)} (\rho).
\]
**Proposition 2.** Let \( u \in X_\kappa \), even with respect to \( r \). Then \( Lu \in X_\kappa \) and \( r^2 Lu \in C^2([0, \infty) \times \mathbb{R}) \). Moreover, \( Lu \) is even with respect to \( r \) and satisfies

\[
(\partial_t^2 - \partial_r^2) Lu - 2r \partial_r Lu = (t)^{-(\kappa - 1)} r^2 f(u), \quad t \geq 0, \quad r \in \mathbb{R}
\]

with zero initial data, i.e. for \( g = 0 \).

**Proof.** From the continuity of \( H_u[s](\rho) \) (which follows from \( u \in X_\kappa \subset C \)), it follows that \( Lu \in X_\kappa \), i.e. \( Lu, \partial_r(r Lu) \in C \). Being \( u \) even with respect to \( r \), and \( f \) even in \( u \), we get \( H_u[s] \) odd for any \( s \); it follows that \( Lu \) is even. We now notice that if \( \partial_r \) is even, then

\[
\langle \partial_r^2 Lu \rangle = 1
\]

Recalling the definition of the involved norm, for proving (52) it suffices to find

\[
\| \partial_r^2 Lu \|_{X_\kappa} \lesssim \| u \|_{X_\kappa}^{p-1} + \| u \|_{X_\kappa}^{-1}.
\]

In particular we gain \( \partial_r^2 Lu \in C \). Recalling (45), we see that \( Lu \) solves (50) and we get the continuity of the \( r \)-derivatives for \( r^2 Lu \).

In order to prove global existence through contraction mapping principle, we need to prove the following.

**Theorem 5.** Let \( p > p_0(5) \) and let

\[
\frac{3 - p}{p - 1} \leq \kappa \leq 2(p - 1), \quad \text{if } p \in (p_0(5), 2), \quad \text{or } 1 < \kappa \leq 2(p - 1) \quad \text{if } p \geq 2.
\]

If \( u \in X_\kappa \), then

\[
\| Lu \|_{X_\kappa} \lesssim \| u \|_{X_\kappa}^{p-1} + \| u \|_{X_\kappa}^{-1}.
\]

Recalling the definition of the involved norm, for proving (52) it suffices to find

\[
|Lu(t, r)| \lesssim \| u \|_{X_\kappa}^{p-1} |t - |r||^{-(\kappa - 1)} \| u \|_{X_\kappa}^p,
\]

\[
|\partial_r(r Lu)(t, r)| \lesssim \| u \|_{X_\kappa}^{p-1} |t - |r||^{-(\kappa - 1)} \| u \|_{X_\kappa}^p.
\]

Since \( Lu \) is even in \( r \), it suffices to deal with \( r > 0 \). Proceeding as in (49), from (49), we have

\[
|Lu(t, r)| \lesssim \frac{1}{r} \int_0^t \langle s \rangle^{-(p-1)} \int_{t-s-r}^{t-s+r} \| \partial_r(H_u[s](\rho)) \| \, d\rho \, ds
\]

\[
\lesssim \frac{1}{r} \| u \|_{X_\kappa}^p \int_0^t \langle s \rangle^{-(p-1)} \int_{t-s-r}^{t-s+r} \langle s + |\rho| \rangle^{-p} \langle s - |\rho| \rangle^{-(\kappa - 1)} \langle \rho \rangle \, d\rho \, ds.
\]

By using (49), we get

\[
|\partial_r(r Lu)(t, r)| \lesssim \| u \|_{X_\kappa}^p \sum_{\pm} \int_0^t \langle s \rangle^{-(p-1)} \langle s + |t - s \pm r| \rangle^{-p} \langle s - |t - s \pm r| \rangle^{-(\kappa - 1)} \langle t - s \pm r \rangle \, ds.
\]
Hence our aim reduces to estimate the quantities

\[ I_0(t, r) = \int_0^t \langle s \rangle^{-(p-1)} \int_{t-s-r}^{t-s+r} \langle s + |\rho| \rangle^{-p} \langle s - |\rho| \rangle^{-p(\kappa-1)} \langle \rho \rangle \, d\rho \, ds \]

\[ I_{1, \pm}(t, r) = \int_0^t \langle s \rangle^{-(p-1)} \langle s + |t - s \pm r| \rangle^{-p} \langle s - |t - s \pm r| \rangle^{-p(\kappa-1)} \langle t - s \pm r \rangle \, ds \]

Similarly, to prove \ref{53} it suffices to find

\[ |Lu(t, r) - Lv(t, r)| \lesssim (t + |r|)^{-1} (t - |r|)^{-\kappa-1} \|u - v\|_{X_{\kappa}} \left( \|u\|_{X_{\kappa}}^{-1} + \|v\|_{X_{\kappa}}^{-1} \right), \tag{56} \]

\[ |\partial_t (rLu)(t, r) - \partial_t (rLv)(t, r)| \lesssim (t + |r|)^{-1} (t - |r|)^{-\kappa-1} \langle r \rangle \|u - v\|_{X_{\kappa}} \left( \|u\|_{X_{\kappa}}^{-1} + \|v\|_{X_{\kappa}}^{-1} \right). \tag{57} \]

We have

\[ |Lu(t, r) - Lv(t, r)| \lesssim \frac{1}{r} \int_0^t \langle s \rangle^{-(p-1)} \int_{t-s-r}^{t-s+r} |H_u[s](\rho) - H_v[s](\rho)| \, d\rho \, ds. \]

Moreover from \ref{56} and \ref{57}, it follows that

\[ |H_u[s](\rho) - H_v[s](\rho)| \lesssim |\rho| |u(s, \rho) - v(s, \rho)| \left( \|u(s, \rho)\|_{X_{\kappa}}^{-1} + \|v(s, \rho)\|_{X_{\kappa}}^{-1} \right). \]

As a conclusion

\[ |Lu(t, r) - Lv(t, r)| \lesssim \|u - v\|_{X_{\kappa}} \left( \|u\|_{X_{\kappa}}^{-1} + \|v\|_{X_{\kappa}}^{-1} \right) I_0(t, r). \]

Similarly, we get

\[ |\partial_t (rLu(t, r) - rLv(t, r))| \lesssim \|u - v\|_{X_{\kappa}} \left( \|u\|_{X_{\kappa}}^{-1} + \|v\|_{X_{\kappa}}^{-1} \right) \sum \pm I_{1, \pm}(t, r). \]

If \( t \leq r \), we may simplify our approach, thanks to the following.

**Remark 4.** If \( t \leq r \), it holds:

\[ Lu(t, r) = \frac{1}{r} \int_0^t \langle s \rangle^{-(p-1)} \int_{r(t-s)}^{r(t+s)} H_u[s](\rho) \, d\rho \, ds. \]

Indeed,

\[ \int_{r(t-s)}^{r(t+s)} H_u[s](\rho) \, d\rho = 0, \]

being \( H_u[s] \), defined in \ref{58}, odd, thanks to the assumption that \( f(u) \) is even with respect to \( u \), and thanks to the fact that \( u \) is even with respect to \( r \). Therefore, we may replace \( I_0(t, r) \) by

\[ I_0'(t, r) := \int_0^t \langle s \rangle^{-(p-1)} \int_{r(t-s)}^{r(t+s)} \langle s + |\rho| \rangle^{-p} \langle s - |\rho| \rangle^{-p(\kappa-1)} \langle \rho \rangle \, d\rho \, ds. \]

The estimates for \( I_0, I_{1, \pm} \) and \( I_0' \) are based on the following lemma.

**Lemma 4.** Let \( p > p(5) \) and let

\[ \frac{3 - p}{p - 1} \leq \kappa \leq 2(p - 1) \quad \text{if} \quad p \in (p(5), 2), \tag{58} \]

\[ 1 < \kappa \leq 2 \quad \text{if} \quad p = 2, \tag{59} \]

\[ \frac{1}{p - 1} \leq \kappa \leq 2(p - 1) \quad \text{if} \quad p > 2. \tag{60} \]

Then

\[ I(\xi) = \int_0^\xi (\eta + \xi)(\eta - \xi)^{-(p-1)}(\eta)^{-p(\kappa-1)} \, d\eta \lesssim (\xi)^{-(\kappa-p)}. \tag{61} \]
Remark 5. If \( p < 2 \), the interval \([55]\), i.e. \((3 - p)(p - 1)^{-1} \leq \kappa \leq 2(p - 1)\), is nonempty if, and only if, \( p > p_0(5) \). If \( p > 2 \), the interval \([60]\), i.e. \((p - 1)^{-1} \leq \kappa \leq 2(p - 1)\) is nonempty for any \( p > 2 \).

We observe that clearly this latter range contains the range \((1, 2(p - 1)]\) required in the assumption \([51]\).

Proof. We split \( I(\xi) \) in \( I_1(\xi) = \int_{-\xi/2}^{\xi/2} \ldots d\eta \) and \( I_2(\xi) \) the remainder.

Let \( \eta \in [0, \xi/2] \); we have \( \langle \xi \rangle \simeq \langle \eta + \xi \rangle \simeq \langle \xi - \eta \rangle \) hence

\[
I_1(\xi) \lesssim \langle \xi \rangle^{2-p} \int_0^{\xi/2} \langle \eta \rangle^{-p(\kappa-1)} d\eta
\]

We get \( I_1(\xi) \lesssim \langle \xi \rangle^{-(\kappa-p)} \) if:

\[
\kappa < 1 + \frac{1}{p} \quad \text{and} \quad -3 + p\kappa \geq \kappa - p
\]

\[
\kappa = 1 + \frac{1}{p} \quad \text{and} \quad -2 + p > \kappa - p
\]

\[
\kappa > 1 + \frac{1}{p} \quad \text{and} \quad -2 + p \geq \kappa - p.
\]

The first condition corresponds to the interval

\[
\left[\frac{3 - p}{p - 1}, 1 + \frac{1}{p}\right),
\]

which is nonempty for any \( p > p_0(5) \). The second condition holds for any \( p > p_0(5) \), therefore \( \kappa = 1 + 1/p \) is admissible. The third condition corresponds to the interval

\[
\left(1 + \frac{1}{p}, 2(p - 1)\right],
\]

which is nonempty for any \( p > p_0(5) \). Gluing together the three intervals above, we obtain the admissible range in \([58]\), i.e.

\[
\left[\frac{3 - p}{p - 1}, 2(p - 1)\right].
\]

Now, let \( \eta \in [\xi/2, \xi] \). We have \( \langle \eta \rangle \simeq \langle \xi \rangle \simeq \langle \xi + \eta \rangle \). It follows that

\[
I_2(\xi) \simeq \langle \xi \rangle^{1-p(\kappa-1)} \int_{\xi/2}^{\xi} \langle \eta - \xi \rangle^{-(p-1)} d\eta + \langle \xi \rangle^{-(p-1)-p(\kappa-1)} \int_{\xi/2}^{\xi} \langle \eta - \xi \rangle d\eta = I_{2,1}(\xi) + I_{2,2}(\xi).
\]

For any \( p > 1 \) we have

\[
I_{2,2}(\xi) \lesssim \langle \xi \rangle^{-(p-1)-p(\kappa-1)+2},
\]

in particular \( I_{2,2} \leq \langle \xi \rangle^{-(\kappa-p)} \) for any

\[
\kappa \geq \frac{3 - p}{p - 1}.
\] (62)

The estimate of \( I_{2,1} \) depends on the range of \( p \):

\[
I_{2,1}(\xi) \lesssim \langle \xi \rangle^{1-p(\kappa-1)} \quad \text{if} \quad p > 2
\]

\[
I_{2,1}(\xi) \lesssim \langle \xi \rangle^{1-p(\kappa-1)} \ln \langle \xi \rangle \quad \text{if} \quad p = 2
\]

\[
I_{2,1}(\xi) \lesssim \langle \xi \rangle^{1-p(\kappa-1)-(p-1)+1} \quad \text{if} \quad p < 2.
\]

For \( p < 2 \) the assumption \( \kappa \geq \frac{3 - p}{p - 1} \) directly gives \( I_{2,1}(\xi) \leq \langle \xi \rangle^{-(\kappa-p)} \). For \( p = 2 \), we get \( 1-p(\kappa-1) < p-\kappa \) if, and only if, \( \kappa > 1 \). For \( p > 2 \), we get \( 1-p(\kappa-1) \leq p-\kappa \), if, and only if, \( \kappa \geq \frac{1}{p-1} \).

Therefore, combining the lower bound on \( \kappa \) obtained for \( I_{2,1} \) with the upper bound for \( \kappa \) derived for \( I_1 \), we obtain \([59]\) if \( p = 2 \) and \([60]\), if \( p > 2 \).
Proposition 3. Let $p > p_0(5)$, and $\kappa$ be as in (51). It holds

$$I_0(t, r) \lesssim \begin{cases} \frac{r}{(t + r)^{-\kappa}} & \text{if } t \geq 2r, \text{ or } r \leq 1 \\ \frac{1}{(t - r)^{(\kappa - 1)}} & \text{if } r \leq t \leq 2r \text{ and } r \geq 1 \end{cases}$$

Moreover,

$$I_0'(t, r) \lesssim \langle t - r \rangle^{-(\kappa - 1)} \quad \text{if } t \leq r \text{ and } r \geq 1.$$ 

In particular (51) and (50) hold.

Proof. First, let us estimate $I_0$. Being $|t - s - r| < t - s + r$, we have

$$I_0(t, r) \leq 2 \int_0^t \langle s \rangle^{-(p - 1)} \int_{\max\{0, t - s - r\}}^{t + s + r} \langle s + \rho \rangle^{-p} \langle s - \rho \rangle^{-p(\kappa - 1)} \langle \rho \rangle \, d\rho \, ds.$$ 

Now we use the change of variable $\xi = s + \rho, \eta = \rho - s$. Since $\rho \geq 0$, then $|\eta| \leq \xi$. Moreover $\xi = s + \rho \leq s + (t - s - r) = t + r$ and $\xi \geq s + \max\{0, t - s - r\} \geq (t - r)$. Finally we have

$$I_0(t, r) \lesssim \int_{(t - r)_+}^{t + r} \langle \xi \rangle^{-p} \int_{-\xi}^{\xi} \langle \eta + \xi \rangle^{-p} \langle \eta - \xi \rangle^{-(p - 1)} \langle \eta \rangle^{-p(\kappa - 1)} \, d\eta \, d\xi (63)$$

with $I(\xi)$ as in Lemma 4. From Lemma 4 we can conclude that

$$I_0(t, r) \lesssim \int_{(t - r)_+}^{t + r} \langle \xi \rangle^{-\kappa} d\xi.$$ 

(64)

We now use different approaches into different zones of the $(t, r)$ plane.

The zone $t \geq 2r$. Here, in $[(t - r), (t + r)]$ we have $\langle \xi \rangle \approx \langle t + r \rangle$, therefore $I_0(t, r) \lesssim r \langle t + r \rangle^{-\kappa}$.

The zone $r \leq 1$ and $t \leq 2r$. In such zone $\langle t + r \rangle \approx 1$. It is enough to find $I_0(t, r) \lesssim r$, which trivially follows from (64), being $\kappa \geq 0$.

The zone $r \geq 1$ and $r \leq t \leq 2r$. If $r \leq t \leq 2r$ then from (51) we derive

$$I_0(t, r) \lesssim \int_{t - r}^{t + r} \langle \xi \rangle^{-\kappa} d\xi \lesssim \langle t - r \rangle^{-(\kappa - 1)},$$

where we used $\kappa > 1$.

Now, let us estimate $I_0'$ for $r \geq 1$ and $t \leq r$. Applying the same change of variables to

$$I_0'(t, r) = \int_0^t \langle s \rangle^{-(p - 1)} \int_{r - (t - s)}^{r + (t - s)} \langle s + \rho \rangle^{-p} \langle s - \rho \rangle^{-p(\kappa - 1)} \langle \rho \rangle \, d\rho \, ds,$$

we obtain

$$I_0'(t, r) \lesssim \int_{r - t}^{r + t} \langle \xi \rangle^{-p} \int_{r - t}^{r - t} \langle \eta + \xi \rangle^{-p} \langle \eta - \xi \rangle^{-(p - 1)} \langle \eta \rangle^{-p(\kappa - 1)} d\eta \, d\xi.$$ 

Moreover $[(r - t), \xi] \subset [-\xi, \xi]$; from Lemma 4 we have

$$I_0(t, r) \lesssim \int_{r - t}^{r + t} \langle \xi \rangle^{-p} I(\xi) \, d\xi \lesssim \int_{r - t}^{r + t} \langle \xi \rangle^{-\kappa} d\xi \lesssim \langle t - r \rangle^{1 - \kappa},$$

where we used once again $\kappa > 1$.

Finally we prove (54). If $t \geq 2r$ or $r \leq 1$, from $\langle t + r \rangle \geq (t - r)$, we get

$$|Lu(t, r)| \lesssim \langle t + r \rangle^{-\kappa} \|u\|_{X_{p\kappa}} \lesssim \langle t + r \rangle \langle t - r \rangle^{-(\kappa - 1)}.$$ 

For $r \geq 1$ and $t \leq 2r$ we have

$$|Lu(t, r)| \lesssim \langle r \rangle^{-1} \langle t - r \rangle^{-(\kappa - 1)} \|u\|_{X_{p\kappa}} \lesssim \langle t + r \rangle^{-1} \langle t - r \rangle^{-(\kappa - 1)} \|u\|_{X_{p\kappa}}.$$ 

The same arguments leads to (56).
Proposition 4. Let $p > p_0(5)$ and $\kappa$ be as in \eqref{3.7}. One has
\[
I_{1,-}(t,r) \lesssim \begin{cases} 
(t-r)^{-\kappa} & \text{if } t \geq 2r, \\
(t-r)^{-(\kappa-1)} & \text{if } t \leq 2r,
\end{cases}
\]
and $I_{1,+} \lesssim (t+r)^{-\kappa}$. In particular, \eqref{3.30} and \eqref{3.40} hold.

Proof. We start with the estimate of $I_{1,-}$.

**The zone** $t \geq 2r$. Since $t + r \simeq t - r$ so that if $s \in [t-r, t]$, then

$$s + |t - s - r| \simeq t - r.$$ 

Conversely if $s \in [0, t-r]$, then

$$s + |t - s - r| = s + t - s - r = t - r.$$ 

Therefore,

$$I_{1,-} \lesssim \langle t - r \rangle^{-p} \int_0^t \langle s \rangle^{-(p-1)} \langle s - |t - s - r| \rangle^{-p(\kappa-1)} \langle t - s - r \rangle ds = \langle t - r \rangle^{-p}(Q_- + Q_+),$$

where

$$Q_- = \int_0^{t-r} \langle s \rangle^{-(p-1)} \langle 2s - t + r \rangle^{-p(\kappa-1)} \langle t - s - r \rangle ds,$$

$$Q_+ = \langle t - r \rangle^{-p(\kappa-1)} \int_{t-r}^t \langle s \rangle^{-(p-1)} \langle t - s - r \rangle ds.$$ 

We may directly estimate

$$Q_+ \leq \langle t - r \rangle^{-p(\kappa-1)-(p-1)} \int_{t-r}^t \langle s \rangle^{-(p-1)} \langle s - t - r \rangle ds = \langle t - r \rangle^{-p(\kappa-1)-(p-1)} \int_0^{t-r} \langle \rho \rangle d\rho$$

$$\lesssim \langle t - r \rangle^{-p(\kappa-1)-(p-1)+2}.$$ 

Being $\kappa \geq \frac{3-p}{p-1}$ we have the required estimate $Q_+ \lesssim \langle t - r \rangle^{p-\kappa}$.

In order to estimate $Q_-$ we plan to use Lemma \[1\]. By the change of variable $\eta = \frac{t-r}{2} - s$, we have

$$Q_- \lesssim \int_{-\frac{t-r}{2}}^{0} \langle \eta + \frac{t-r}{2} \rangle^{-(p-1)} \langle \eta \rangle^{-p(\kappa-1)} \langle \eta - \frac{t-r}{2} \rangle ds = I \left( \frac{t-r}{2} \right) \lesssim \langle t-r \rangle^{p-\kappa}. \quad \text{(65)}$$

Together with the estimate of $Q_+$, this gives $I_{1,-} \lesssim \langle t - r \rangle^{-\kappa}$.

**The zone** $t \leq 2r$. We write $I_{1,-} = \tilde{Q}_+ + \tilde{Q}_-$, where:

$$\tilde{Q}_- = \int_0^{(t-r)+} \langle s \rangle^{-(p-1)} \langle t - r \rangle^{-p} \langle 2s - t + r \rangle^{-p(\kappa-1)} \langle t - s - r \rangle ds$$

$$= \langle t - r \rangle^{-p} Q_-,$$

$$\tilde{Q}_+ = \int_{(t-r)+}^t \langle s \rangle^{-(p-1)} \langle 2s - t + r \rangle^{-p} \langle t - r \rangle^{-p(\kappa-1)} \langle t - s - r \rangle ds$$

$$= \langle t - r \rangle^{-p(\kappa-1)} \int_{(t-r)+}^t \langle s \rangle^{-(p-1)} \langle 2s - t + r \rangle^{-p} \langle t - s - r \rangle ds = \langle t - r \rangle^{-p(\kappa-1)} Q_+^d.$$ 

Since estimate \eqref{3.30} holds for any $t \geq r$, directly we can conclude $\tilde{Q}_- \lesssim \langle t - r \rangle^{-\kappa}$.

Since $p > 1$, in order to gain $\tilde{Q}_+ \lesssim \langle t - r \rangle^{-(\kappa-1)}$, it suffices to estimate $Q_+^d$ by a constant.
Since \(2s - (t - r) \geq s - (t - r)\), we trivially have
\[
Q_{t}^{+} \lesssim \int_{0}^{\infty} (s)^{-(p-1)}(s - (t - r))^{-(p-1)} ds
\]
\[
\lesssim \int_{0}^{(t-r)/2} (s)^{-(p-1)} ds + \int_{(t-r)/2}^{\infty} (t - s - r)^{-2(p-1)} ds \leq 2 \int_{0}^{\infty} (s)^{-2(p-1)} ds.
\]
This quantity is finite, being \(2(p-1) \geq 2(p_0(5) - 1) > 1\).

The estimate for \(I_{1,+}\) is simpler, indeed

\[
I_{1,+} = \langle t + r \rangle^{-p} \int_{0}^{t} (s)^{-(p-1)}(2s - t - r)^{-p(\kappa-1)}(t - s + r) ds,
\]
due to \(t + r - s \geq 0\). After the change of variable \(\eta = \frac{t + r}{2} - s\), we are in a position to apply Lemma 4 and conclude

\[
I_{1,+} \lesssim \langle t + r \rangle^{-p} \int_{-\frac{t+r}{2}}^{\frac{t+r}{2}} \eta^{-p} (\eta)^{-p(\kappa-1)}(\eta - \frac{t + r}{2}) d\eta = \langle t + r \rangle^{-p} \frac{1}{\kappa} \lesssim \langle t + r \rangle^{-\kappa}.
\]

Now, we can easily gain (56), and similarly (57). If \(t \geq 2r\) then we use \(\langle t + r \rangle \simeq \langle t - r \rangle\) and \(\langle r \rangle \geq 1\) to conclude

\[
|\partial_{r}(rLu)(t,r)| \lesssim \|u\|_{X_{\kappa}} \|I_{1,\pm}\|_{X_{\kappa}} \lesssim \|u\|_{X_{\kappa}} \langle t - r \rangle^{-\kappa} \lesssim \langle t + r \rangle^{-1} \langle t - r \rangle^{-1} \|u\|_{X_{\kappa}}.
\]

If \(t \leq 2r\), then \(\langle r \rangle \simeq \langle t + r \rangle\), hence

\[
|\partial_{r}(rLu)(t,r)| \lesssim \|u\|_{X_{\kappa}} \sum_{\pm} I_{1,\pm} \lesssim \|u\|_{X_{\kappa}} \langle t - r \rangle^{-\kappa} \lesssim \langle t + r \rangle^{-1} \langle t - r \rangle^{-1} \|u\|_{X_{\kappa}}.
\]

\[
\square
\]

5.3. Existence theorem.

**Theorem 6.** Let \(p > p_0(5)\) and \(\kappa\) as in (31). There exists \(\varepsilon_0 > 0\) such that if (37) holds with \(\varepsilon < \varepsilon_0\), then the Cauchy Problem (38) admits a unique global solution \(u(t,r)\) in the sense of Definition 1. In particular \(u \in X_{\kappa}\) and the following decay estimate holds:

\[
|u(t,r)| + |\partial_{r}u(t,r)| \lesssim \langle t + |r| \rangle^{-1} \langle t - |r| \rangle^{-\kappa-1}.
\]

**Proof.** Let us define the sequence

\[
u_{0} = u_{\text{lin}}, \quad u_{n+1} = u_{\text{lin}} + Lu_{n}.
\]

By using Theorem 4 and Theorem 3 we get

\[
\|u_{n+1}\|_{X_{\kappa}} \leq \|u_{\text{lin}}\|_{X_{\kappa}} + C_{1}\|u_{n}\|_{X_{\kappa}}^{p} \leq C_{0}\varepsilon + C_{1}\|u_{n}\|_{X_{\kappa}}^{p},
\]

\[
\|u_{n+1} - u_{n}\|_{X_{\kappa}} \leq C_{2}\|u_{n} - u_{n-1}\|_{X_{\kappa}} \left(\|u_{n}\|_{X_{\kappa}}^{p-1} + \|u_{n-1}\|_{X_{\kappa}}^{p-1}\right),
\]

for suitable constant \(C_{0}, C_{1}, C_{2} > 0\). For \(\varepsilon_0 < (2C_{0}C_{1}^{1/(p-1)})^{-1}\), via induction argument we find

\[
\|u_{n}\|_{X_{\kappa}} \leq 2\|u_{\text{lin}}\|_{X_{\kappa}} \leq 2C_{0}\varepsilon_0.
\]

In turn, for \(\varepsilon_0 < (2^{p+1}C_{2}C_{0}^{p-1})^{-1}\), this gives

\[
\|u_{n+1} - u_{n}\|_{X_{\kappa}} \leq 2^{-n}\|u_{1} - u_{0}\|_{X_{\kappa}}.
\]

We can conclude that \(\{u_{n}\}\) is a Cauchy sequence, it converges in \(X_{\kappa}\) to the solution to \(u = u_{\text{lin}} + Lu\). According to Proposition 2, this solution is the required one. The decay estimates simply follow from the definition of \(X_{\kappa}\).  

\[
\square
\]
Remark 6. From the decay estimate \([66]\), we may derive an estimate for the solution to the scale-invariant damping Cauchy Problem \([1]\). Coming back, by the inverse Liouville transformation, we find

\[
|v(t, |x|)| \leq \langle t \rangle^{-1} (t + |x|)^{-1} (t - |x|)^{-\left(n - 1\right)}
\]

The worst situation is close to the light cone, where we only have

\[
|v(t, |x|)| \leq \langle t \rangle^{-2}.
\]

The decay behavior \(\langle t \rangle^{-2}\) in 3-dimensional case, can be seen as \(\langle t \rangle^{-\frac{n+2}{2}}\): the same decay of the wave equation in dimension \(n + 2\). This motivates the 2-dimension shift of the critical exponent.

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FROM \( p_0(n) \) TO \( p_0(n + 2) \)

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