Towards Rigorous Derivation of Quantum Kinetic Equations

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Abstract. We develop a rigorous formalism for the description of the evolution of states of quantum many-particle systems in terms of a one-particle density operator. For initial states which are specified in terms of a one-particle density operator the equivalence of the description of the evolution of quantum many-particle states by the Cauchy problem of the quantum BBGKY hierarchy and by the Cauchy problem of the generalized quantum kinetic equation together with a sequence of explicitly defined functionals of a solution of stated kinetic equation is established in the space of trace class operators. The links of the specific quantum kinetic equations with the generalized quantum kinetic equation are discussed.

Key words: quantum kinetic equation; quantum BBGKY hierarchy; cluster expansion; cumulant of scattering operators; quantum many-particle system.

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1 Introduction

It is well known that in certain situations the collective behavior of quantum many-particle systems can be adequately described by the initial-value problem of the kinetic equation, i.e. by the evolution equation for a one-particle (marginal) density operator \cite{1, 2}. To get an understanding of the nature of such phenomenon as the kinetic evolution it is necessary to answer at least two fundamental questions. One is an origin of initial data for such evolution equation or in other words what is the immediate cause that many-particle systems tend to the state described in terms of a one-particle density operator in evolutionary process. If initial data is completely specified by a one-particle density operator then the other fundamental question is the derivation of quantum kinetic equations from microscopic dynamics, i.e. from the von Neumann equation or the quantum BBGKY hierarchy. We note that the main problem herein is whether such intention can be put on a firm mathematical foundation. In the paper we consider the second question, i.e. the problem of rigorous derivation of quantum kinetic equations from underlying many-particle dynamics.

First attempt to justify the kinetic equations was undertaken by N.N. Bogolyubov on basis of the perturbation method of construction of a particular solution of the hierarchy of equations for marginal distribution functions \cite{3} (in quantum case in the paper \cite{4}). Later, drawing an analogy with the equilibrium state expansions, such approach for classical system of particles
was developed in papers of M.S. Green [5], M.S. Green and R.A. Piccirelly [6,7] and in series of E.G.D. Cohen papers, summed up in the review [8] (see also [9]). The current view of this problem consists in the following [10]. Since the evolution of states of infinitely many quantum particles is generally described by a sequence of marginal density operators which is a solution of the initial-value problem of the quantum BBGKY hierarchy, then the evolution of states can be effectively described by a one-particle density operator governed by the kinetic equation only as a result of some approximations or in a suitable scaling limit [11–14]. Recently in the framework of such approach the considerable advance in the rigorous derivation of quantum kinetic equations, namely, the nonlinear Schrödinger equation [15–24] and the quantum Boltzmann equation [25,26], is observed.

In the paper we discuss the problem of potentialities inherent in the description of the evolution of states of many-particle systems in terms of a one-particle density operator. We demonstrate that in fact if initial data is completely defined by a one-particle marginal density operator, then all possible states of infinite-particle systems at arbitrary moment of time can be described within the framework of a one-particle density operator without any approximations.

Now we outline the structure of the paper and the main results. At first in Section 2 we formulate some definitions and preliminary facts about quantum dynamics of finitely many particles. Then the main results related to the origin of kinetic evolution is stated. For initial data specified in terms of trace class operators satisfying a chaos property in case of the Maxwell-Boltzmann statistics we prove that the Cauchy problem of the quantum BBGKY hierarchy can be reformulated as a new Cauchy problem for the certain evolution equation for a one-particle marginal density operator (generalized quantum kinetic equation) and an infinite sequence of explicitly defined functionals of the solution of this evolution equation which characterizes the correlations of particle states. In Section 3 we prove the main results, namely, we develop the method of the kinetic cluster expansions of the cumulants of scattering operators which define the evolution operators of every term of the marginal functional expansions over the products of a one-particle density operator and derive the generalized quantum kinetic equation. In Section 4 a solution of the Cauchy problem of the generalized quantum kinetic equation is constructed and the existence of a strong and a weak solution is proved in the space of trace class operators. Finally in Section 5 we conclude with some observations and perspectives for future research. Among them we discuss the problem of the derivation of the specific quantum kinetic equations such as the nonlinear Schrödinger equation, from the constructed generalized quantum kinetic equation in the appropriate scaling limits. In particular the mean-field scaling limit of a solution of the Cauchy problem of the generalized quantum kinetic equation and the marginal functionals of the state holds up.

2 Origin of kinetic evolution

2.1 The evolution of many-particle systems: the quantum BBGKY hierarchy

Hereinafter we consider a quantum system of a non-fixed (i.e. arbitrary but finite) number of the identical (spinless) particles with unit mass $m = 1$ in the space $\mathbb{R}^\nu$, $\nu \geq 1$. The Hamiltonian $H = \bigoplus_{n=0}^{\infty} H_n$ of such system is a self-adjoint operator ($H_0 = 0$) with the domain $\mathcal{D}(H) =$
\[ \{ \psi = \otimes \psi_n \in \mathcal{F}_H \mid \psi_n \in \mathcal{D}(H_n) \in \mathcal{H}^\otimes n, \sum_n \|H_n \psi_n\|^2 < \infty \} \subset \mathcal{F}_H, \] where \( \mathcal{F}_H = \bigoplus_{n=0}^\infty \mathcal{H}^\otimes n \) is the Fock space over the Hilbert space \( \mathcal{H} \). We adopt the usual convention that \( \mathcal{H}^\otimes 0 = \mathbb{C} \). Assume \( \mathcal{H} = L^2(\mathbb{R}^n) \) (coordinate representation), then an element \( \psi \in \mathcal{F}_H = \bigoplus_{n=0}^\infty L^2(\mathbb{R}^{mn}) \) is a sequence of functions \( \psi = (\psi_0, \psi_1(q_1), \ldots, \psi_n(q_1, \ldots, q_n), \ldots) \) such that \( \|\psi\|^2 = |\psi_0|^2 + \sum_{n=1}^\infty \int dq_1 \ldots dq_n |\psi(q_1, \ldots, q_n)|^2 < +\infty \). On the subspace of infinitely differentiable functions with compact supports \( \psi_n \in L^2_0(\mathbb{R}^{mn}) \subset L^2(\mathbb{R}^{mn}) \) the Hamiltonian \( H_n \) of \( n \geq 1 \) particles acts according to the formula

\[ H_n \psi_n = -\hbar^2 \sum_{i=1}^n \Delta_{q_i} \psi_n + \sum_{i_1 < i_2=1}^n \Phi(q_{i_1}, q_{i_2}) \psi_n, \tag{1} \]

where \( \hbar = 2\pi \hbar \) is a Planck constant, \( \Phi \) is a two-body interaction potential satisfying the Kato conditions \[27] [28].

The states of finitely many quantum particles belong to the space \( \Omega^1(\mathcal{F}_H) = \bigoplus_{n=0}^\infty \Omega^1_0(\mathcal{H}_n) \) of the sequences \( f = (f_0, f_1, \ldots, f_n, \ldots) \) of trace class operators \( f_n \equiv f_n(1, \ldots, n) \in \Omega^1(\mathcal{H}_n) \) and \( f_0 \in \mathbb{C} \), that satisfy the symmetry condition: \( f_n(1, \ldots, n) = f_n(i_1, \ldots, i_n) \) for arbitrary \((i_1, \ldots, i_n) \in (1, \ldots, n)\), equipped with the norm

\[ \|f\|_{\Omega^1(\mathcal{F}_H)} = \sum_{n=0}^\infty \|f_n\|_{\Omega^1(\mathcal{H}_n)} = \sum_{n=0}^\infty \|\text{Tr}_{1,\ldots,n} f_n(1, \ldots, n)\|, \]

where \( \text{Tr}_{1,\ldots,n} \) are partial traces over \( 1, \ldots, n \) particles \[29\]. We denote by \( \Omega^1_0(\mathcal{F}_H) = \bigoplus_{n=0}^\infty \Omega^1_0(\mathcal{H}_n) \) the everywhere dense set of finite sequences of degenerate operators with infinitely differentiable kernels with compact supports \[28] [29].

The evolution of states is described by the sequences \( F(t) = (F_1(t, 1), \ldots, F_s(t, 1, \ldots, s), \ldots) \) of the marginal density operators that satisfy the Cauchy problem of the quantum BBGKY hierarchy

\[ \frac{d}{dt} F_s(t, Y) = -\mathcal{N}_s(Y) F_s(t, Y) + \sum_{i=1}^s \text{Tr}_{s+1}( -\mathcal{N}_{\text{int}}(i, s+1) ) F_{s+1}(t, Y, s+1), \tag{2} \]

\[ F_s(t)_{t=0} = F_s^0, \quad s \geq 1, \]

where \( Y \equiv (1, \ldots, s) \), the operator \( \mathcal{N}_s \) is defined on \( \Omega^1_0(\mathcal{H}_s) \) as follows

\[ \mathcal{N}_s f_s \equiv \frac{i}{\hbar} (f_s H_s - H_s f_s) \tag{3} \]

and correspondingly

\[ \mathcal{N}_{\text{int}}(i, j) f_s \equiv \frac{i}{\hbar} (f_s \Phi(i, j) - \Phi(i, j) f_s). \tag{4} \]

Hereinafter we consider initial data satisfying the factorization property or a "chaos" property \[2\], which means the lack of correlations at initial time. For a system of identical particles, obeying the Maxwell-Boltzmann statistics, we have

\[ F(t)|_{t=0} = F^{(c)} \equiv (F_1^0(1), \ldots, \prod_{i=1}^s F_1^0(i), \ldots). \tag{5} \]
The assumption about initial data is intrinsic for the kinetic description of a gas, because in this case all possible states are characterized only by a one-particle density operator. 

On the space $L^1(\mathcal{H}_n)$ we define the group of operators

$$G_n(-t)f_n = e^{-\frac{it}{\hbar}\mathcal{H}_n}f_n e^{\frac{it}{\hbar}\mathcal{H}_n}.$$ (6)

On the space $L^1(\mathcal{H}_n)$ the mapping: $t \rightarrow G_n(-t)f_n$ is an isometric strongly continuous group which preserves positivity and self-adjointness of operators $[30, 31]$. For $f_n \in L^1_0(\mathcal{H}_n)$ there exists a limit in the sense of a strong convergence by which the infinitesimal generator of the group of evolution operators (6) is determined as follows

$$\lim_{t \to 0} \frac{1}{t}(G_n(-t)f_n - f_n) = -\mathcal{N}_n f_n,$$ (7)

where the operator $(-\mathcal{N}_n)$ is defined by formula (3) and the operator $(-\mathcal{N}_n)f_n$ is defined on the domain $\mathcal{D}(\mathcal{H}_n) \subset \mathcal{H}_n$.

A solution of the quantum BBGKY hierarchy (2) with initial data (5) is represented by the expansion

$$F_s(t, Y) = \sum_{n=0}^{\infty} \frac{1}{n!} \text{Tr}_{s+1,\ldots,s+n} \mathfrak{A}_{1+n}(t, \{Y\}, s+1, \ldots, s+n) \prod_{i=1}^{s+n} F_1^0(i),$$ (8)

where the evolution operator $\mathfrak{A}_{1+n}(t)$, $n \geq 0$, is the $(n+1)$-order cumulant $[31]$ of the groups of operators (6)

$$\mathfrak{A}_{1+n}(t, \{Y\}, X \setminus Y) = \sum_{P: \{Y\}, X \setminus Y = \bigcup_i X_i} (-1)^{|P|-1}(|P| - 1)! \prod_{X_i \in P} G_{|X_i|}(-t, X_i),$$

and the following notation are used: $\{Y\}$ is the set consisting of one element $Y = (1, \ldots, s)$, i.e. $|\{Y\}| = 1$, $\sum_P$ is the sum over all possible partitions of the set $\{\{Y\}, X \setminus Y\} = (\{Y\}, s+1, \ldots, s+n)$ into $|P|$ nonempty mutually disjoint subsets $X_i \subset \{\{Y\}, X \setminus Y\}$. If $\|F_1^0\|_{L^1(\mathcal{H}_1)} < e^{-1}$, series (8) converges in the norm of the space $L^1(\mathcal{H}_s)$ for arbitrary $t \in \mathbb{R}^1$.

Hereinafter in the capacity of a solution expansion of the quantum BBGKY hierarchy we will use its equivalent representation in the space $L^1(\mathcal{F}_H)$, namely expansion (8) with the $(n+1)$-order reduced cumulant $[27, 31]$ of the groups of operators (6)

$$\mathfrak{A}_{1+n}(t, \{Y\}, s+1, \ldots, s+n) = \sum_{k=0}^{n} (-1)^k \frac{n!}{k!(n-k)!} G_{s+n-k}(-t).$$ (9)

If $F_1^0 \in L^1(\mathcal{H}_1)$, in this case series (8) converges in the norm of the space $L^1(\mathcal{H}_s)$ for arbitrary $t \in \mathbb{R}^1$ and the estimate holds

$$\|F_s(t)\|_{L^1(\mathcal{H}_s)} \leq \|F_1^0\|_{L^1(\mathcal{H}_1)}^s \exp \left(2\|F_1^0\|_{L^1(\mathcal{H}_1)}^s\right), \quad s \geq 1.$$ (10)

Thus, in case of initial data (5) the microscopic evolution of states of quantum many-particle systems is described by sequence (8, 9). In the next subsection we formulate the evolution of states in terms of the kinetic theory.
2.2 The kinetic evolution: main results

Since we consider initial data \([5]\) which are completely characterized by the one-particle density operator \(F_1^0\), namely, \(F^{(c)} = (F_1^1(1), \ldots, \prod_{i=1}^n F_1^0(i), \ldots)\), the initial-value problem of the quantum BBGKY hierarchy \([2],[3]\) is not completely well-defined Cauchy problem, because the generic initial data is not independent for every unknown operator \(F_s(t, 1, \ldots, s)\), \(s \geq 1\), in the hierarchy of equations. Thus, it naturally arises the opportunity of reformulating such initial-value problem as a new Cauchy problem for operator \(F_1(t)\) with the independent initial data together with explicitly defined functionals \(F_s(t, 1, \ldots, s \mid F_1(t))\), \(s \geq 2\), of the solution \(F_1(t)\) of this Cauchy problem. We refer to such functionals as the marginal functionals of the state of quantum many-particle systems. At first we define the restated Cauchy problem.

Functionals \(F_s(t, 1, \ldots, s \mid F_1(t))\), \(s \geq 2\), are represented by the following expansions over products of the one-particle density operator \(F_1(t)\)

\[
F_s(t, Y \mid F_1(t)) \doteq \sum_{n=0}^{\infty} \frac{1}{n!} \operatorname{Tr}_{s+1, \ldots, s+n} \mathfrak{V}_{1+n}(t, \{Y\}, s + 1, \ldots, s + n) \prod_{i=1}^{s+n} F_1(t, i),
\]

where the \((n + 1)\)-order evolution operator \(\mathfrak{V}_{1+n}(t)\), \(n \geq 0\), are defined as follows

\[
\mathfrak{V}_{1+n}(t, \{Y\}, X \setminus Y) \doteq n! \sum_{k=0}^{n} (-1)^k \sum_{n_1=1}^{n} \ldots \sum_{n_k=1}^{n-n_1-\ldots-n_k-1} \frac{1}{(n-n_1-\ldots-n_k)!} \times \]

\[
\times \prod_{j=1}^{k} \sum_{D_j : Z_j = \bigcup_{X_{i,j}} X_{i,j}, |D_j| \leq s + n - n_1 - \ldots - n_j} \frac{1}{|D_j|!} \sum_{i_1 \neq \ldots \neq |D_j| = 1} X_{i_j \subseteq D_j, 1} \prod_{|X_{i,j}|!} \hat{A}_{1+|X_{i,j}|}(t, i_j, X_{i,j}),
\]

and \(\sum_{D_j : Z_j = \bigcup_{X_{i,j}} X_{i,j}}\) is the sum over all possible dissections\(^3\) \(D_j\) of the linearly ordered set \(Z_j \equiv (s + n - n_1 - \ldots - n_j + 1, \ldots, s + n - n_1 - \ldots - n_{j-1})\) on no more than \(s + n - n_1 - \ldots - n_j\) linearly ordered subsets. In \((12)\) we denote by \(\hat{A}_{1+n}(t)\) the \((1 + n)\)-order reduced cumulant, i.e.

\[
\hat{A}_{1+n}(t, \{Y\}, s + 1, \ldots, s + n) = \sum_{k=0}^{n} (-1)^k \frac{n!}{k!(n-k)!} \hat{g}_{s+n-k}(t),
\]

of the following groups of scattering operators

\[
\hat{g}_n(t) = g_n(-t, 1, \ldots, n) \prod_{i=1}^{n} g_i(t, i), \quad n \geq 1.
\]

We give below for later use a few examples of the evolution operators \(\mathfrak{V}_n\), \(n \geq 1\), of the

\(^3\)The dissection \(D\) of the linearly ordered set \((1, \ldots, n)\) is its partition on connected subsets, \(|D|\) is the number of subsets of the dissection \(D\). The total number of dissections of an \(n\)-elements set is \(2^{n-1}\). For example, the set \((1, 2, 3)\) has four dissections: \((1,2,3); ((1),(2,3)); ((1,2),(3)); ((1),(2),(3))\).
In terms of groups of scattering operators (13), evolution operators (14) are represented:

\[ \mathfrak{V}_1(t, \{Y\}) = \tilde{\mathfrak{A}}_1(t, \{Y\}), \]  
\[ \mathfrak{V}_2(t, \{Y\}, s + 1) = \tilde{\mathfrak{A}}_2(t, \{Y\}, s + 1) - \tilde{\mathfrak{A}}_1(t, \{Y\}) \sum_{i_1=1}^{s} \tilde{\mathfrak{A}}_2(t, i_1, s + 1), \]  
\[ \mathfrak{V}_3(t, \{Y\}, s + 1, s + 2) = \tilde{\mathfrak{A}}_3(t, \{Y\}, s + 1, s + 2) - 2! \tilde{\mathfrak{A}}_2(t, \{Y\}, s + 1) \times \]  
\[ \sum_{i_1=1}^{s+1} \tilde{\mathfrak{A}}_2(t, i_1, s + 2) - \tilde{\mathfrak{A}}_1(t, \{Y\}) \left( \sum_{i_1=1}^{s} \tilde{\mathfrak{A}}_3(t, i_1, s + 1, s + 2) - \right) \]  
\[ -2! \sum_{i_1=1}^{s} \tilde{\mathfrak{A}}_2(t, i_1, s + 1) \sum_{i_2=1}^{s+1} \tilde{\mathfrak{A}}_2(t, i_2, s + 2) + \sum_{i_1 \neq i_2=1}^{s} \tilde{\mathfrak{A}}_2(t, i_1, s + 1) \tilde{\mathfrak{A}}_2(t, i_2, s + 2) \].

In terms of groups of scattering operators (13), evolution operators (14) are represented:

\[ \mathfrak{V}_1(t, \{Y\}) = \mathfrak{g}_s(t, 1, \ldots, s), \]  
\[ \mathfrak{V}_2(t, \{Y\}, s + 1) = \hat{\mathfrak{g}}_{s+1}(t, 1, \ldots, s + 1) - \hat{\mathfrak{g}}_s(t, 1, \ldots, s) \sum_{i=1}^{s} \hat{\mathfrak{g}}_2(t, i, s + 1) + \]  
\[ (s - 1)\hat{\mathfrak{g}}_s(t, 1, \ldots, s). \]

In what follows it will be clear that functionals (11) characterize the correlations of quantum many-particle states.

The one-particle density operator \( F_1(t) \) is a solution of the following initial-value problem

\[ \frac{d}{dt} F_1(t, 1) = -\mathcal{N}_1(1) F_1(t, 1) + \]  
\[ + \text{Tr}_2 \left( -\mathcal{N}_{\text{int}}(1, 2) \right) \sum_{n=0}^{\infty} \frac{1}{n!} \text{Tr}_{3, \ldots, n+2} \mathfrak{V}_{1+n}(t, \{1, 2\}, 3, \ldots, n + 2) \prod_{i=1}^{n+2} F_1(t, i), \]

\[ F_1(t, 1)|_{t=0} = F_1^0(1), \]  

where the evolution operator \( \mathfrak{V}_{1+n}(t) \) is defined by formula (12). We refer to evolution equation (15) as the generalized quantum kinetic equation. For systems of classical particles such equation was formulated in [2,32,33] and for discrete velocity models in [34].

We observe that the kinetic evolution is described in terms of cumulants of scattering operators (13) in contrast to the evolution of states described by the BBGKY hierarchy (2).

Thus, the principle of equivalence of initial-value problems (15)−(16) and (2), (3) is true.

**Proposition 1.** In the space \( \mathcal{L}^1(F_{\mathcal{H}}) \) under the condition \( \| F_1^0 \|_{\mathcal{L}^1(\mathcal{H})} < e^{-2} \) the initial-value problem of the quantum BBGKY hierarchy (2), (3) is equivalent to the initial-value problem of the generalized quantum kinetic equation (15), (16) together with the sequence of marginal functionals of the state \( F_s(t \mid F_1(t)) \), \( s \geq 2 \), defined by expansions (11).

The proof of the equivalence proposition is the subject of next section.

It should be noted that the possibility for the corresponding initial data to describe the evolution of states only within the framework of a one-particle density operator without any approximations is an inherent property of infinite-particle dynamics.
Remark 1. We illustrate the possibility of reformulating of initial-value problem of a hierarchy of evolution equations in case of depending initial data as a new Cauchy problem for the certain evolution equation together with explicitly defined functionals of a solution of this Cauchy problem by the example of the quantum Vlasov hierarchy \[12\,15\]

\[
\frac{\partial}{\partial t} f_s(t) = \sum_{i=1}^{s} \left( - \mathcal{N}_1(i) \right) f_s(t) + \sum_{i=1}^{s} \text{Tr}_{s+1} \left( - \mathcal{N}_{\text{int}}(i, s + 1) \right) f_{s+1}(t),
\]

(17)

\[
f_s(t, 1, \ldots, s) |_{t=0} = \prod_{j=1}^{s} f_1^0(j), \quad s \geq 1.
\]

(18)

The Cauchy problem (17)-(18) is equivalent to the Cauchy problem of the Vlasov quantum kinetic equation

\[
\frac{\partial}{\partial t} f_1(t, 1) = -\mathcal{N}_1(1) f_1(t, 1) + \text{Tr}_2 \left( - \mathcal{N}_{\text{int}}(1, 2) \right) f_1(t, 1) f_1(t, 2),
\]

(19)

\[
f_1(t) |_{t=0} = f_1^0.
\]

(20)

and a sequence of functionals \( f_s(t, 1, \ldots, s | f_1(t)) \), \( s \geq 2 \), defined by the expressions

\[
f_s(t, 1, \ldots, s | f_1(t)) = \prod_{j=1}^{s} f_1(t, j).
\]

The structure of these functionals is usually interpreted as such that the quantum Vlasov hierarchy (17) preserves chaos property (18) in time for particles obeying Maxwell-Boltzmann statistics.

3 Kinetic evolution of quantum many-particle systems

3.1 Marginal functionals of the state

The straightforward procedure to construct marginal functionals of the state \( \mathcal{H} \) consists in the elimination from expressions of the quantum BBGKY hierarchy solution \( (8),(9) \) for \( s = 1 \) and \( s \geq 2 \) the initial one-particle density operator \( F_1^0 \). With this aim we express the operator \( F_1^0 \) in terms of the operator \( F_1(t) \) from expansion \( (8),(9) \) for \( s = 1 \) applying the contraction mapping principle.

In view of expression \( (8) \) for \( s = 1 \) in the space \( \mathcal{L}^1(\mathcal{H}) \) we have the following equation for the determination of initial one-particle density operator via the operator \( F_1(t) \):

\[
f = \mathcal{A}(f),
\]

where in the space \( \mathcal{L}^1(\mathcal{H}) \) the nonlinear mapping \( \mathcal{A} \) acts according to the formula

\[
(\mathcal{A}(f))(1) = f^0 - \sum_{n=1}^{\infty} \frac{1}{n!} \text{Tr}_{2,\ldots,n+1} \mathcal{G}_1(t, 1) \mathcal{A}_{1+n}(t) \prod_{i=1}^{n+1} f(i),
\]

(21)
and we denote: \( f^0 \equiv G_1(t, 1) F_1(t, 1) \).

Let us find a condition under which nonlinear mapping (21) is a contraction mapping. Let \( f_1 \) and \( f_2 \) are arbitrary elements from the space \( \mathcal{L}^1(\mathcal{H}) \), then according to definition (21) we obtain the estimate

\[
\| A(f_1) - A(f_2) \|_{\mathcal{L}^1(\mathcal{H})} \leq \sum_{n=1}^{\infty} \frac{2^n}{n!} (n+1)(\| f \|_{\mathcal{L}^1(\mathcal{H})})^n \| f_1 - f_2 \|_{\mathcal{L}^1(\mathcal{H})} =
\]

\[
= (e^2 \| f \|_{\mathcal{L}^1(\mathcal{H})} (2 \| f \|_{\mathcal{L}^1(\mathcal{H})} + 1) - 1) \| f_1 - f_2 \|_{\mathcal{L}^1(\mathcal{H})},
\]

where \( \| f \|_{\mathcal{L}^1(\mathcal{H})} = \max(\| f_1 \|_{\mathcal{L}^1(\mathcal{H})}, \| f_2 \|_{\mathcal{L}^1(\mathcal{H})}) \). The mapping \( A \) is contractive under the condition

\[
\| f \|_{\mathcal{L}^1(\mathcal{H})} < x_0,
\]

(22)

where \( x_0 \) is a solution of the equation \( e^{2x}(2x + 1) = 2 \) such that \( x_0 \approx 0, 18742 > e^{-2} \).

Therefore under condition (22) there exists a unique solution of equation (21) in the space \( \mathcal{L}^1(\mathcal{H}) \). This solution is determined as the limit of successive approximations \( f^{(n)} = A(f^{(n-1)}) \) with the first approximation \( f^{(0)} = f^0 \equiv G_1(t, 1) F_1(t, 1) \). This solution expresses initial data \( F_1^0 \) by means of the one-particle density operator \( F_1(t) \). Consequently, assembling the evolution operators before the products of operators \( F_1(t) \), we can represent solution (8) of the quantum BBGKY hierarchy for \( s \geq 2 \) as the marginal functionals with respect to the one-particle density operator \( F_1(t) \).

Thus, if the norm of initial one-particle density operator \( \| F_1^0 \|_{\mathcal{L}^1(\mathcal{H})} \) satisfies established condition, i.e.

\[
\| F_1^0 \|_{\mathcal{L}^1(\mathcal{H})} < e^{-2},
\]

(23)

then in view of estimate (10) there exists a sequence of marginal functionals of the state \( F_s(t \mid F_1(t)) \), \( s \geq 2 \), which are represented by converged series (11). These functionals satisfy the quantum BBGKY hierarchy (2) for \( s \geq 2 \), if the operator \( F_1(t) \) is given by expression (8) for \( s = 1 \). The condition under which the marginal functionals of the state exist was ascertained in [35] and in case of classical systems of particles in the paper [32].

**Remark 2.** Within the framework of the kinetic description of evolution of quantum states we have used the space of trace class operators by reason of the existence of global in time solution of the quantum BBGKY hierarchy [36]. In this space the condition: \( \| F_1^0 \|_{\mathcal{L}^1(\mathcal{H})} < e^{-1} \), guarantees the convergence of series (8) and means that the average number of particles is finite. We can reformulate the convergence condition of series (8) as a condition on the parameter characterizing the density \( \frac{1}{v} \) of a system (the average number of particles in a unit volume). In fact if we consider the quantum BBGKY hierarchy (2) as the evolution equation in the thermodynamic limit, then as a result of the renormalization of initial data \( F_1^0 = \frac{1}{v} \tilde{F}_1^0 \), we obtain expansion (8) over powers of density \( \frac{1}{v} \) which converges under the condition: \( \frac{1}{v} < e^{-1} [36] \) or for arbitrary values of \( \frac{1}{v} \) in case of reduced cumulants [5]. In this case marginal functionals of the state are represented by converged series (11) under the condition: \( \frac{1}{v} < e^{-2} \). We emphasize that intensional spaces for the description of states of infinite-particle systems, that means the description of kinetic evolution or equilibrium states [37], are different from the exploit space [2].
3.2 Kinetic cluster expansions

Now we formulate one more method to define the marginal functionals of the state in the explicit form, namely, we develop the method of kinetic cluster expansions. The following assertion is valid.

**Theorem 1.** Under condition (23) the marginal density operator $F_s(t)$ defined by (8), (9) for $s \geq 2$ and the marginal functional $F_s(t \mid F_1(t))$ defined by (11), (12) are equivalent if and only if the evolution operators $\mathfrak{V}_{1+n}(t), n \geq 0$, satisfy the following recurrence relations

$$\hat{A}_{1+n}(t, \{Y\}, s+1, \ldots, s+n) = \sum_{n_1=0}^{n} \frac{n!}{(n-n_1)!} \mathfrak{V}_{1+n-n_1}(t, \{Y\}, s+1, \ldots, s+n-n_1) \quad (24)$$

$$s + n - n_1 \left( \sum_{D: Z = \bigcup_{l=1}^{s+n} X_l, |D| \leq s+n-n_1} \frac{1}{|D|!} \sum_{i_1 \neq \ldots \neq i_{|D|}=1}^{s+n-n_1} \prod_{X_l \subset D} \frac{1}{|X_l|!} \hat{A}_{1+|X_l|}(t, i_l, X_l), \right.$$

where $\sum_{D: Z = \bigcup_{l=1}^{s+n} X_l, |D| \leq s+n-n_1}$ is the sum over all possible dissections $D$ of the linearly ordered set $Z \equiv (s+n-n_1 + 1, \ldots, s+n)$ on no more than $s+n-n_1$ linearly ordered subsets.

**Proof. Necessity.** To derive recurrence relations (24) we assume that for $s \geq 2$ marginal density operators (8) coincide with the functionals of a one-particle density operator $F_s(t \mid F_1(t)), s \geq 2$. These marginal functionals of the state are represented in the form of series over particle clusters whose evolution is governed by the corresponding order evolution operator acting on products of one-particle density operators defined on Hilbert spaces associated with every particle from the cluster, namely as expansions (11).

Observing that in case of $s = 1$ for a solution of the quantum BBGKY hierarchy defined by expansion (8) the following equality holds

$$\prod_{i=1}^{s+n} F_1(t, i) = \sum_{n_1=0}^{s+n} \sum_{D: Z = \bigcup_{k} X_k, |D| \leq s+n} \frac{1}{|X_k|!} \times$$

$$\times \mathfrak{V}_{1+|X_k|}(t, i_k, X_k) \prod_{l=1}^{s+n} F_1^0(j),$$

where $\sum_{D: Z = \bigcup_{k} X_k, |D| \leq s+n}$ is the sum over all possible dissections $D$ of the linearly ordered set $Z \equiv (s+n+1, \ldots, s+n+n_1)$ on no more than $s+n$ linearly ordered subsets, we transform functionals $F_s(t \mid F_1(t)), s \geq 2$, to the series over products of initial one-particle density operators.

Then, equating term by term both series for $F_s(t), s \geq 2$, and for the transformed functionals $F_s(t \mid F_1(t))$ under the trace signs for the evolution operators acting on the same products of initial data, we obtain the following recurrence relations for the generating evolution operators.
of functionals (11) in terms of cumulants of groups of operators (6)

\[ \mathcal{A}_{1+n}(t, \{Y\}, s+1, \ldots, s+n) = \mathcal{V}_{1+n}(t, \{Y\}, s+1, \ldots, s+n) + \]
\[ + \sum_{n_1=1}^{n} \frac{n!}{(n-n_1)!} \mathcal{V}_{1+n-n_1}(t, \{Y\}, s+1, \ldots, s+n-n_1) \sum_{D: Z = \bigcup_k X_k, |D| \leq s+n-n_1} \frac{1}{|D|!} \times \]
\[ \times \sum_{i_1 \neq \ldots \neq i_{|D|}=1}^{s+n-n_1} \prod_{k \in D} \frac{1}{|X_k|!} \mathcal{A}_{1+|X_k|}(t, i_k, X_k) \prod_{m=1, m \neq i_1, \ldots, i_{|D|}}^{s+n} \mathcal{A}_1(t, m), \]

where the linearly ordered set \( Z = (s+n-n_1+1, \ldots, s+n) \) is dissected on no more than \( s+n-n_1 \) linearly ordered subsets.

Under the trace signs recurrence relations (26) are naturally represented in terms of cumulants of scattering operators as (24). We refer to recurrence relations (24) as the kinetic cluster expansions of reduced cumulants of scattering operators (13).

**Sufficiency.** Using recurrence relations (24), i.e. kinetic cluster expansions of reduced cumulants of scattering operators (13), we construct the expansions of the functionals of a one-particle density operator \( F_s(t \mid F_1(t)) \), \( s \geq 2 \), on basis of solution expansions (8) of the quantum BBGKY hierarchy. Indeed, taking into account relations (26), we represent series over the summation index \( n \) and the sum over the summation index \( n_1 \) as the two-fold series

\[ F_s(t, 1, \ldots, s) = \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{n_1=0}^{\infty} \text{Tr}_{s+1,\ldots,s+n_1} \mathcal{V}_{1+n_1}(t, \{Y\}, s+1, \ldots, s+n_1) \sum_{D: Z = \bigcup_k X_k, |D| \leq s+n} \frac{1}{|D|!} \times \]
\[ \times \sum_{i_1 \neq \ldots \neq i_{|D|}=1}^{s+n} \prod_{k \in D} \frac{1}{|X_k|!} \mathcal{A}_{1+|X_k|}(t, i_k, X_k) \prod_{l=1, l \neq i_1, \ldots, i_{|D|}}^{s+n} \mathcal{A}_1(t, l) \prod_{j=1}^{n+n_1} F^0(j), \]

where \( Z \equiv (s+n+1, \ldots, s+n+n_1) \) is the linearly ordered set and it is used the notations introduced above. The series in the right-hand side converge under condition (23).

In view of formula (26) we identify the series over the summation index \( n_1 \) with the products of one-particle density operators and consequently for \( s \geq 2 \) the following equality holds

\[ F_s(t, 1, \ldots, s) = \sum_{n=0}^{\infty} \frac{1}{n!} \text{Tr}_{s+1,\ldots,s+n} \mathcal{A}_{1+n}(t, \{Y\}, s+1, \ldots, s+n) \prod_{i=1}^{s+n} F^0(i) = \]
\[ = \sum_{n=0}^{\infty} \frac{1}{n!} \text{Tr}_{s+1,\ldots,s+n} \mathcal{V}_{1+n}(t, \{Y\}, s+1, \ldots, s+n) \prod_{i=1}^{s+n} F_1(t, i) = F_s(t \mid F_1(t)), \]

i.e., if kinetic cluster expansions (24) of cumulants of scattering operators (13) hold, then solution expansions (8) for \( s \geq 2 \) can be represented in the form of marginal functionals of the state (11).
We make a few examples of relations (24) of the kinetic cluster expansions:

\[
\hat{A}_1(t, \{Y\}) = \mathcal{V}_1(t, \{Y\}),
\]

\[
\hat{A}_2(t, \{Y\}, s + 1) = \mathcal{V}_2(t, \{Y\}, s + 1) + \mathcal{V}_1(t, \{Y\}) \sum_{i_1=1}^{s} \hat{A}_2(t, i_1, s + 1),
\]

\[
\hat{A}_3(t, \{Y\}, s + 1, s + 2) = \mathcal{V}_3(t, \{Y\}, s + 1, s + 2) +
\]

\[
+ 2! \mathcal{V}_2(t, \{Y\}, s + 1) \sum_{i_1=1}^{s+1} \hat{A}_2(t, i_1, s + 2) +
\]

\[
+ \mathcal{V}_1(t, \{Y\}) \left( \sum_{i_1=1}^{s} \hat{A}_3(t, i_1, s + 1, s + 2) + \sum_{i_1 \neq i_2=1}^{s} \hat{A}_2(t, i_1, s + 1) \hat{A}_2(t, i_2, s + 2) \right).
\]

It is evident that solutions of these relations are given by expressions (14) which the evolution operators (12) of the first, second and third order correspondingly are determined by in the expansions of marginal functionals of the state (11). In general case solutions of recurrence relations (24) are given by expressions (12). This statement is verified as a result of the substitution of expressions (12) into recurrence relations (24).

It should be emphasized that in case under consideration, i.e. the absence of correlations at initial time, the correlations generated by the dynamics of a system are completely governed by evolution operators (12).

Typical properties for the kinetic description of the evolution of constructed marginal functionals of the state (11) are induced by the properties of evolution operators (12). Let us indicate some intrinsic properties of the evolution operators \( \mathcal{V}_{1+n}(t) \), \( n \geq 0 \), representative for cumulants (semi-invariants) of group of operators.

Since in case of a system of non-interacting particles for scattering operators (13) the equality holds: \( \hat{G}_n(t) = I \), where \( I \) is a unit operator, then we have

\[
\mathcal{V}_{1+n}(t) = I \delta_{n,0},
\]

where \( \delta_{n,1} \) is a Kronecker symbol. Similarly, at initial time \( t = 0 \) it is true: \( \mathcal{V}_{1+n}(0) = I \delta_{n,0} \).

The infinitesimal generator of the first-order evolution operator (14) is defined by the following limit in the sense of the norm convergence in the space \( \mathcal{L}^1(\mathcal{H}_n) \)

\[
\lim_{t \to 0} \frac{1}{t} (\mathcal{V}_1(t, \{1, \ldots, n\}) - I) f_n = \sum_{i<j=1}^{n} (-N_{\text{int}}(i,j)) f_n,
\]

where the operator \( (-N_{\text{int}}(i,j)) \) is defined by formula (4) for \( f_n \in \mathcal{L}^1(\mathcal{H}_n) \subset \mathcal{L}^1(\mathcal{H}_n) \).

In general case, i.e. \( n \geq 2 \), in the sense of the norm convergence in the space \( \mathcal{L}^1(\mathcal{H}_n) \) for the \( n \)-order evolution operator (12) it holds

\[
\lim_{t \to 0} \frac{1}{t} \mathcal{V}_n(t, 1, \ldots, n) f_n = 0.
\]

Summarize we observe that in case of initial data (5) which is completely characterized by the one-particle density operator \( F^0_1 \), solution (8) for \( s \geq 2 \) of the quantum BBGKY hierarchy (2) and marginal functionals of the state (11) give two equivalent approaches to the description of states of quantum many-particle systems.
3.3 The derivation of the generalized quantum kinetic equation

Let us construct the evolution equation which satisfies expression (8), (9) for \( s = 1 \).

Taking into account equality (7) and observing the validity of the following equalities for reduced cumulants (9) of groups (6) for \( f \in \mathfrak{L}^1(\mathcal{F}_H) \) in the sense of the norm convergence (for \( n \geq 2 \) it is a consequence that we consider a system of particles interacting by a two-body potential):

\[
\lim_{t \to 0} \frac{1}{t} \text{Tr}_2 \mathfrak{A}_2(t, 1, 2) f_2(t, 1, 2) = \text{Tr}_2 \left( -\mathcal{N}_{\text{int}}(1, 2) \right) f_2(1, 2),
\]

\[
\lim_{t \to 0} \frac{1}{t} \text{Tr}_{2, \ldots, n+1} \mathfrak{A}_{1+n}(t, 1, \ldots, n+1) f_{n+1} = 0, \quad n \geq 2,
\]

we will differentiate over the time variable expression (8), (9) for \( s = 1 \) in the sense of pointwise convergence in the space \( \mathfrak{L}^1(\mathcal{H}_1) \). As result it holds

\[
\frac{d}{dt} F_1(t, 1) = -\mathcal{N}_1(1) F_1(t, 1) + \text{Tr}_2(-\mathcal{N}_{\text{int}}(1, 2)) \sum_{n=0}^{\infty} \frac{1}{n!} \text{Tr}_{3, \ldots, n+2} \mathfrak{A}_{1+n}(t, \{1, 2\}, 3, \ldots, n+2) \prod_{i=1}^{n+2} F_1^0(i).
\]

In second summand in the right-hand side of this equality we expand reduced cumulants (9) of groups (6) into transformed (26) kinetic cluster expansions (24) and represent series over the summation index \( n \) and the sum over the summation index \( n_1 \) as the two-fold series. Then the following equalities take place:

\[
\sum_{n=0}^{\infty} \frac{1}{n!} \text{Tr}_{2, \ldots, n+2}(-\mathcal{N}_{\text{int}}(1, 2)) \mathfrak{A}_{1+n}(t, \{1, 2\}, 3, \ldots, n+2) \prod_{i=1}^{n+2} F_1^0(i) = \\
\sum_{n=0}^{\infty} \frac{1}{n!} \text{Tr}_{2, \ldots, n+2}(-\mathcal{N}_{\text{int}}(1, 2)) \sum_{n_1=0}^{n} \frac{n!}{(n-n_1)!} \mathfrak{A}_{1+n-n_1}(t, \{1, 2\}, 3, \ldots, n+2-n_1) \times \\
\times \sum_{D:Z=\bigcup_{j} X_i} \frac{1}{|D|!} \prod_{i_1 \neq \ldots \neq i_{|D|}=1} \prod_{X_i \subset D} \frac{1}{|X_i|!} \mathfrak{A}_{1+|X_i|}(t, i_1, X_i) \prod_{m=1}^{2+n-n_1} \mathfrak{A}_1(t, m) \prod_{i=1}^{n+2} F_1^0(i) = \\
= \text{Tr}_2(-\mathcal{N}_{\text{int}}(1, 2)) \sum_{n=0}^{\infty} \frac{1}{n!} \text{Tr}_{3, \ldots, n+2} \mathfrak{A}_{1+n}(t, \{1, 2\}, 3, \ldots, n+2) \sum_{n_1=0}^{\infty} \sum_{D:Z=\bigcup_{j} X_i} \frac{1}{|D|!} \times \\
\times \sum_{i_1 \neq \ldots \neq i_{|D|}=1} \prod_{X_i \subset D} \frac{1}{|X_i|!} \mathfrak{A}_{1+|X_i|}(t, i_1, X_i) \prod_{m=1}^{n+2} \mathfrak{A}_1(t, m) \prod_{i=1}^{n+2+n_1} F_1^0(i),
\]

where \( Z \equiv (n+3-n_1, \ldots, n+2) \) and \( Z' \equiv (n+3, \ldots, n+2+n_1) \) are linearly ordered sets and it is used the notations accepted above.
Consequently, applying in case of $s = 2$ formula (25) to the obtained expression, from equality (27) we derive

$$\frac{d}{dt} F_1(t, 1) = -N_1(1) F_1(t, 1) + \text{Tr}_2(-N_{\text{int}}(1, 2)) \sum_{n=0}^{\infty} \frac{1}{n!} \text{Tr}_{3,\ldots,n+2} \mathcal{W}_{1+n}(t, \{1, 2\}, 3, \ldots, n + 2) \prod_{i=1}^{n+2} F_1(t, i).$$

Under condition (23) the series in right-hand side of this equality converges.

The constructed identity (28) for the one-particle (marginal) density operator $F_1(t, 1)$ we will treat as the evolution equation which governs the one-particle states of many-particle quantum systems obeying the Maxwell-Boltzmann statistics.

We remark that one more approach to the derivation of the generalized quantum kinetic equation consists in its construction on the basis of dynamics of correlations [36, 38]. Thus, if initial data is completely defined by a one-particle density operator, then all possible states of infinite-particle systems at arbitrary moment of time can be described within the framework of a one-particle density operator without any approximations. In other words, for mentioned states the evolution of states governed by the quantum BBGKY hierarchy (2) can be completely described by the generalized quantum kinetic equation (15) and therefore Proposition 1 is valid.

3.4 Some properties of marginal functionals of the state

We indicate that expansions (11) of marginal functionals of the state are nonequilibrium analog of the Mayer-Ursell expansions over powers of the density of equilibrium marginal density operators [31, 37].

In case of the description of states in terms of the marginal correlation operators [36, 38]

$$G_s(t, Y) = \sum_{|P|: Y = \bigcup_i X_i} (-1)^{|P|-1}(|P| - 1)! \prod_{X_i \in P} F_{|X_i|}(t, X_i),$$

where $\sum_P$ is the sum over all possible partitions $P$ of the set $Y \equiv \{1, \ldots, s\}$, $s \geq 2$, into $|P|$ nonempty mutually disjoint subsets $X_i \subset Y$ and in particular, $G_1(t) = F_1(t)$, the marginal correlation functionals $G_s(t, Y \mid F_1(t))$, $s \geq 2$, are represented by the expansions similar to (11), namely

$$G_s(t, Y \mid F_1(t)) = \sum_{n=0}^{\infty} \frac{1}{n!} \text{Tr}_{s+1,\ldots,s+n} \mathcal{W}_{1+n}(t, \theta(\{Y\}), s + 1, \ldots, s + n) \prod_{i=1}^{s+n} F_1(t, i).$$

In expansion (29) it is introduced the notion of the declusterization mapping $\theta : \{Y\} \to Y$. This mapping is defined by the formula [39]

$$\theta(\{Y\}) = Y,$$

that it means the declusterization of particle clusters in cumulants of scattering operators, i.e. in contrast to expansion (11) the $n$ term of expansions (29) of marginal correlation functionals
$G_s(t, 1, \ldots, s \mid F_1(t))$ is governed by the $(1 + n)$-order evolution operator (12) of the $(s + n)$-order, $n \geq 0$, cumulants of the scattering operators, for example, as compared to (14) the lower orders evolution operators $\mathfrak{V}_{1+n}(t, \theta(\{Y\}), s + 1, \ldots, s + n), n \geq 0,$ have the form

$$\mathfrak{V}_1(t, \theta(\{Y\})) = \hat{\mathfrak{A}}_s(t, \theta(\{Y\})), \quad (30)$$

$$\mathfrak{V}_2(t, \theta(\{Y\}), s + 1) = \hat{\mathfrak{A}}_{s+1}(t, \theta(\{Y\}), s + 1) - \hat{\mathfrak{A}}_s(t, \theta(\{Y\})) \sum_{i=1}^{s} \hat{\mathfrak{A}}_2(t, i, s + 1),$$

and in case of $s = 2$, it holds

$$\mathfrak{V}_1(t, \theta(\{1, 2\})) = \hat{\mathfrak{G}}_2(t, 1, 2) - I.$$

In the framework of the description of states by marginal functionals of the state (11) the average values, for example, of the additive-type observables $A^{(1)} = (0, a_1, \ldots, \sum_{i=1}^{n} a_1(i), \ldots)$ are given by the functional

$$\langle A^{(1)}(t) \rangle = \text{Tr}_1 a_1(1) F_1(t, 1), \quad (31)$$

i.e. they are defined by a solution of the generalized quantum kinetic equation (15), or in general case of the $s$-particle observables $A^{(s)} = (0, \ldots, 0, a_s(1, \ldots, s), \ldots, \sum_{i_1 < \ldots < i_s=1}^n a_s(i_1, \ldots, i_s), \ldots)$ by the functional

$$\langle A^{(s)}(t) \rangle = \frac{1}{s!} \text{Tr}_{1, \ldots, s} a_s(1, \ldots, s) F_s(t, 1, \ldots, s \mid F_1(t)), \quad s \geq 2.$$

For $A^{(s)} \in \mathcal{L}(\mathcal{F}_H)$ and $F_1(t) \in \mathcal{L}^1(\mathcal{H})$ these functionals exist.

The dispersion of an additive-type observable is defined by a solution of the generalized quantum kinetic equation (15) and marginal correlation functionals (29) as follows

$$\langle (A^{(1)} - \langle A^{(1)}(t) \rangle)^2(t) \rangle =$$

$$\text{Tr}_1 (a_1^2(1) - \langle A^{(1)}(t) \rangle^2) F_1(t, 1) + \text{Tr}_{1,2} a_1(1) a_1(2) G_2(t, 1, 2 \mid F_1(t)),$$

where the functional $\langle A^{(1)}(t) \rangle$ is determined by expression (31). Note that the dispersion of observables is minimal for states characterized by marginal correlation functionals (29) equals to zero, i.e. from macroscopic point of view the evolution of many-particle states with the minimal dispersion is the Markovian kinetic evolution.

In fact functionals (29) or (11) characterize the correlations of states of quantum many-particle systems. We illustrate close links of functionals (29) and (11) in the following way:

$$F_2(t, 1, 2 \mid F_1(t)) = F_1(t, 1) F_1(t, 2) + G_2(t, 1, 2 \mid F_1(t)).$$

Basically this equality gives the classification of all possible currently in use scaling limits [11][13]. In the scaling limits it is assumed that chaos property (5) of initial state preserves in time, i.e. the scaling limit means such limit of dimensionless parameters of a system in which the marginal correlation functional $G_2(t, 1, 2 \mid F_1(t))$ vanishes. According to definition (29), it is possible, if particles of every finite particle cluster move without collisions (30). In conclusions the mean-field scaling limit of functionals (11) and (29) holds up.
Another approach to the derivation of the Markovian kinetic equations was formulated by Bogolyubov [3] (see also [40]) and consists in the construction of marginal functionals of the state \( F_s(t, Y | F_1(t)) \) by the perturbation method.

Let us consider first two terms of expansion (11). If an interaction potential in (11) is a bounded operator and \( f_{s+1} \in \mathcal{L}^1(\mathcal{H}_{s+1}) \), then for the second-order cumulant \( \tilde{A}_2(t, \{ Y \}, s + 1) \) of scattering operators (13) an analog of the Duhamel equation holds

\[
\tilde{A}_2(t, \{ Y \}, s + 1) \big| f_{s+1} = \int_0^t d\tau \, \mathcal{G}_s(-\tau, Y) \mathcal{G}_1(-\tau, s + 1) \sum_{i_1=1}^{s+1} \left( -\mathcal{N}_{\text{int}}(i_1, s + 1) \right) \times \tag{32}
\]

and, consequently, for the second-order evolution operator \( \mathfrak{B}_2(t, \{ Y \}, s + 1) \) we have

\[
\mathfrak{B}_2(t, \{ Y \}, s + 1) \big| f_{s+1} = \left( \tilde{A}_2(t, \{ Y \}, s + 1) - \tilde{A}_1(t, \{ Y \}) \right) \sum_{i_1=1}^{s+1} \tilde{A}_2(t, i_1, s + 1) \big| f_{s+1} = \tag{33}
\]

\[
= \int_0^t d\tau \, \mathcal{G}_s(-\tau, Y) \mathcal{G}_1(-\tau, s + 1) \left( \sum_{i_1=1}^{s} \left( -\mathcal{N}_{\text{int}}(i_1, s + 1) \right) \mathcal{G}_{s+1}(\tau - t, Y, s + 1) - \mathcal{G}_s(\tau - t, Y) \sum_{i_1=1}^{s} \left( -\mathcal{N}_{\text{int}}(i_1, s + 1) \right) \mathcal{G}_2(\tau - t, i_1, s + 1) \right) \prod_{i_2=1}^{s+1} \mathcal{G}_1(\tau, i_2) f_{s+1}.
\]

In the kinetic (macroscopic) scale of the variation of variables [10] groups of operators (6) of finitely many particles depend on microscopic time variable \( \varepsilon^{-1}t \), where \( \varepsilon \geq 0 \) is a scale parameter, and the dimensionless marginal functionals of the state are represented in the form: \( F_s(\varepsilon^{-1}t, Y | F_1(t)) \). Note that on the macroscopic scale the typical length for the kinetic phenomena described, for example, by the quantum Boltzmann equation is the mean free pass. Then according to (33) in the formal Markovian limit \( \varepsilon \to 0 \) the first two terms of the dimensionless marginal functional expansions coincide with corresponding terms constructed by the perturbation method with the use of the weakening of correlation condition in [3] (see also [4, 40])

\[
\lim_{\varepsilon \to 0} F_s(\varepsilon^{-1}t, Y | F_1(t)) = \mathcal{G}_s(\infty, Y) \prod_{i=1}^s F_1(t, i) + \tag{34}
\]

\[
+ \int_0^\infty d\tau \, \mathcal{G}_s(-\tau, Y) \text{Tr}_{s+1} \left( \sum_{i_1=1}^{s} \left( -\mathcal{N}_{\text{int}}(i_1, s + 1) \right) \mathcal{G}_{s+1}(\infty, Y, s + 1) - \mathcal{G}_s(\infty, Y) \sum_{i_1=1}^{s} \left( -\mathcal{N}_{\text{int}}(i_1, s + 1) \right) \mathcal{G}_2(\infty, i_1, s + 1) \right) \prod_{i_2=1}^{s+1} \mathcal{G}_1(\tau, i_2) F_1(t, i_2) + \text{etc.}
\]

Therefore in the kinetic scale the collision integral of the generalized kinetic equation (15) takes the form of Bogolyubov’s collision integral [3, 40] which enables to control correlations of infinite-particle systems. We remark that in the homogeneous case the collision integral of the first approximation in [31] has a more general form than the quantum Boltzmann collision integral.
4 Initial-value problem of generalized kinetic equation

4.1 An existence theorem

Before considering abstract initial-value problem (15)-(16) in the space $\mathfrak{L}^1(\mathcal{H})$ we generalize it for case of $n$-body interaction potential $\Phi^{(n)}$, $n \geq 1$. In this case the Cauchy problem of the generalized quantum kinetic equation has the form

$$\frac{d}{dt} F_1(t, 1) = -\mathcal{N}_1(1) F_1(t, 1) + \sum_{n=1}^{\infty} \sum_{k=1}^{n} \frac{1}{(n-k)! k!} \text{Tr}_{2, \ldots, n+1}(-\mathcal{N}_{\text{int}}^{(k+1)})(1, \ldots, k+1) \mathfrak{V}_{1+n-k}(t, \{1, \ldots, k+1\}, k+2, \ldots, n+1) \prod_{i=1}^{n+1} F_1(t, i),$$

$$F_1(t, 1)|_{t=0} = F_0^1(1),$$

where $\mathfrak{V}_{1+n-k}(t)$, is the $(1+n-k)$-order evolution operator (12) and notations (3),(4) are used, and

$$\mathcal{N}_{\text{int}}^{(n)} f_n = -\frac{i}{\hbar} (f_n \Phi^{(n)} - \Phi^{(n)} f_n).$$

The collision integral in the generalized quantum kinetic equation (35) is defined by the convergent series under condition (23).

For the sake of a comparison of the structure of various collision integral components in (35) we give expressions of the collision integral term describing a two-body interaction and three particle correlations

$$\text{Tr}_{2,3}(-\mathcal{N}_{\text{int}}^{(2)})(1, 2) \mathfrak{V}_2(t, \{1, 2\}, 3) F_1(t, 1) F_1(t, 2) F_1(t, 3),$$

and the collision integral term describing a three-body interaction

$$\frac{1}{2!} \text{Tr}_{2,3}(-\mathcal{N}_{\text{int}}^{(3)})(1, 2, 3) \mathfrak{V}_3(t, \{1, 2, 3\}) F_1(t, 1) F_1(t, 2) F_1(t, 3),$$

where the evolution operators $\mathfrak{V}_2(t, \{1, 2\}, 3)$ and $\mathfrak{V}_3(t, \{1, 2, 3\})$ are defined by formulas (14).

For the Cauchy problem (35)-(36) (and (15)-(16)) in the space $\mathfrak{L}^1(\mathcal{H})$ the following statement is true.

**Theorem 2.** The global in time solution of initial-value problem (35)-(36) is determined by the following expansion

$$F_1(t, 1) = \sum_{n=0}^{\infty} \frac{1}{n!} \text{Tr}_{2, \ldots, 1+n} \mathfrak{A}_{1+n}(t, 1, \ldots, n+1) \prod_{i=1}^{n+1} F_0^1(i),$$

where the reduced cumulants $\mathfrak{A}_{1+n}(t)$, $n \geq 0$, are defined by formula (9). If $\|F_1^0\|_{\mathfrak{L}^1(\mathcal{H})} < e^{-2}$, then for $F_1^0 \in \mathfrak{L}^1_0(\mathcal{H})$ it is a strong (classical) solution and for an arbitrary initial data $F_1^0 \in \mathfrak{L}^1(\mathcal{H})$ it is a weak (generalized) solution.
Proof. Let \( F_1^0 \in \mathfrak{L}_0^1(\mathcal{H}) \). It will be recalled that series (37) converges in the norm of the space \( \mathfrak{L}^1(\mathcal{H}) \) and estimate (10) holds. Series (37) is a strong solution of initial-value problem (35)-(36), if the equality holds

\[
\lim_{\triangle t \to 0} \left\| \frac{1}{\triangle t} (F_1(t + \triangle t, 1) - F_1(t, 1)) - \right. \\
\left. - \left( - \mathcal{N}_1(1) F_1(t, 1) + \sum_{n=1}^{\infty} \sum_{k=1}^{n} \frac{1}{(n - k)! k!} \text{Tr}_{2, \ldots, n+1}(-\mathcal{N}^{(k+1)}_{\text{int}})(1, \ldots, k+1) \times \\
\times \mathfrak{A}_{1+n-k}(t, \{1, \ldots, k+1\}, k + 2, \ldots, n+1) \prod_{i=1}^{n+1} F_1(t, i) \right) \right\|_{\mathfrak{L}^1(\mathcal{H})} = 0, 
\]

where abridged notations are applied: the symbols \( F_1(t, 1) \) and \( \prod_{i=1}^{n+1} F_1(t, i) \) are implied series (37) and for \( s = 1 \) series (25), respectively.

To prove the existence of a strong solution of initial-value problem (35)-(36) we use the result of section 3.4 on the differentiation of expansion (37) over time variable in the sense of pointwise convergence in the space \( \mathfrak{L}^1(\mathcal{H}) \) with a little modification. Taking into account that for \( n \geq 1 \) and \( f_{n+1} \in \mathfrak{L}_0^1(\mathcal{H}_{n+1}) \) the equality is true

\[
\lim_{t \to 0} \left\| \frac{1}{t} \text{Tr}_{2, \ldots, n+1} \mathfrak{A}_{1+n}(t, 1, \ldots, n+1) f_{n+1} - \text{Tr}_{2, \ldots, n+1} (-\mathcal{N}^{(n+1)}_{\text{int}})(1, \ldots, n) f_{n+1} \right\|_{\mathfrak{L}^1(\mathcal{H})} = 0, 
\]

in the sense of the pointwise convergence in the space \( \mathfrak{L}^1(\mathcal{H}) \) we have

\[
\lim_{\triangle t \to 0} \frac{1}{\triangle t} (F_1(t + \triangle t, 1) - F_1(t, 1)) = -\mathcal{N}_1 F_1(t) + \\
+ \sum_{n=1}^{\infty} \frac{1}{n!} \text{Tr}_{2, \ldots, n+1}(-\mathcal{N}^{(n+1)}_{\text{int}})(1, \ldots, n+1) \sum_{k=0}^{\infty} \frac{1}{k!} \text{Tr}_{n+2, \ldots, n+k+1} \mathfrak{A}_{1+k}(t, \\
\{1, \ldots, n+1\}, n + 2, \ldots, n + k + 1) \prod_{i=1}^{n+k+1} F_1^0(i). 
\]

In the second summand in the right-hand side of this equality we expand the reduced cumulants (9) of groups (6) into transformed (20) kinetic cluster expansions (24) and represent series over the summation index \( n \) and the sum over the summation index \( k \) as the two-fold series. Then, applying formula (25) in case of \( s = n + 1 \) to the obtained expression, from equality (39) we derive

\[
\sum_{n=1}^{\infty} \frac{1}{n!} \text{Tr}_{2, \ldots, n+1}(-\mathcal{N}^{(n+1)}_{\text{int}})(1, \ldots, n+1) \sum_{k=0}^{\infty} \frac{1}{k!} \text{Tr}_{n+2, \ldots, n+k+1} \mathfrak{A}_{1+k}(t, \{1, \ldots, n+1\}, n + 2, \ldots, n + k + 1) \prod_{i=1}^{n+k+1} F_1^0(i) = \\
\sum_{n=1}^{\infty} \sum_{k=1}^{n} \frac{1}{(n - k)! k!} \text{Tr}_{2, \ldots, n+1}(-\mathcal{N}^{(k+1)}_{\text{int}})(1, \\
\ldots, k + 1) \mathfrak{A}_{1+n-k}(t, \{1, \ldots, k+1\}, k + 2, \ldots, n+1) \prod_{i=1}^{n+1} F_1(t, i). 
\]
4.2 A weak solution

Let us prove that in case of arbitrary initial data \( F_1^0 \in \mathcal{L}^1(\mathcal{H}) \) expansion (41) is a weak solution of the initial-value problem of the generalized quantum kinetic equation (35). With this purpose we introduce the functional

\[
(f_1, F_1(t)) = \text{Tr}_1 f_1(1) F_1(t, 1),
\]

where \( f_1 \in \mathcal{L}_0(\mathcal{H}) \subset \mathcal{L}(\mathcal{H}) \) is degenerate bounded operator with infinitely times differentiable kernel with compact support and the operator \( F_1(t) \) is defined by expansion (37). According to estimate (10), for \( f_1 \in \mathcal{L}_0(\mathcal{H}) \), functional (41) exists and represents by the convergence series.

Using expansion (37), we transform functional (41) as follows

\[
(f_1, F_1(t)) = \sum_{n=0}^{\infty} \frac{1}{n!} \text{Tr}_{1, \ldots, 1+n} f_1(1) \mathfrak{A}_{1+n}(t, 1, \ldots, n+1) \prod_{i=1}^{n+1} F_1^0(i) = \]

\[
= \sum_{n=0}^{\infty} \frac{1}{n!} \text{Tr}_{1, \ldots, 1+n} \sum_{k=0}^{n} (-1)^k \frac{n!}{k!(n-k)!} \mathcal{G}_{1+n-k}(t) f_1(1) \prod_{i=1}^{n+1} F_1^0(i),
\]

where the group of operators \( \mathcal{G}_{1+n-k}(t) \) is adjoint to the group \( \mathcal{G}_{1+n-k}(-t) \) in the sense of functional (41)). For \( F_1^0 \in \mathcal{L}^1(\mathcal{H}) \) and \( f_1 \in \mathcal{L}_0(\mathcal{H}) \) considering (39) the following equality holds in the sense of the \(*\)-weak convergence (30) of the space \( \mathcal{L}(\mathcal{H}) \)

\[
\lim_{\Delta t \to 0} \sum_{n=0}^{\infty} \frac{1}{n!} \text{Tr}_{1, \ldots, 1+n} \sum_{k=0}^{n} (-1)^k \frac{n!}{k!(n-k)!} \frac{1}{\Delta t} (\mathcal{G}_{1+n-k}(t + \Delta t) - \mathcal{G}_{1+n-k}(t)) f_1(1) \prod_{i=1}^{n+1} F_1^0(i) = \]

\[
= (\mathcal{N}_1 f_1, F_1(t)) + \sum_{n=1}^{\infty} \frac{1}{n!} \text{Tr}_{1, \ldots, n+1} \mathcal{N}_{\text{int}}^{(n+1)}(1, \ldots, n+1) f_1(1) \times \]

\[
\times \sum_{k=0}^{n+k+1} \frac{1}{k!} \text{Tr}_{n+2, \ldots, n+k+1} \mathfrak{A}_{1+k}(t, \{1, \ldots, n+1\}, n+2, \ldots, n+k+1) \prod_{i=1}^{n+k+1} F_1^0(i).
\]

For \( F_1^0 \in \mathcal{L}^1(\mathcal{H}) \) and bounded interaction potentials the limit functionals exist. Using equality (40), we transform the second functional in (42) to the form

\[
\sum_{n=1}^{\infty} \frac{1}{n!} \text{Tr}_{1, \ldots, n+1} \mathcal{N}_{\text{int}}^{(n+1)}(1, \ldots, n+1) f_1(1) \prod_{k=0}^{n+k+1} \frac{1}{k!} \text{Tr}_{n+2, \ldots, n+k+1} \mathfrak{A}_{1+k}(t) \prod_{i=1}^{n+k+1} F_1^0(i) = \]

\[
= \sum_{n=1}^{\infty} \sum_{k=1}^{n} \frac{1}{(n-k)! k!} \text{Tr}_{1, \ldots, n+1} \mathcal{N}_{\text{int}}^{(k+1)}(1, \ldots, k+1) f_1(1) \mathfrak{V}_{1+n-k}(t) \prod_{i=1}^{n+1} F_1(t, i).
\]
Therefore as consequence of equalities (42), (43), for functional (41) we have
\[
\frac{d}{dt}(f_1, F_1(t)) = (\mathcal{N}_1 f_1, F_1(t)) + \\
+ \sum_{n=1}^{\infty} \text{Tr}_{1,\ldots,n+1} \left( \sum_{k=1}^{n} \frac{1}{(n-k)!} \frac{1}{k!} \mathcal{N}^{(k+1)} f_1(1) \Psi_{1+n-k}(t) \prod_{i=1}^{n+1} F_1(t, i) \right).
\]

Equality (44) means that expansion (37) for arbitrary \( F_1^0 \in \mathfrak{L}^1(\mathcal{H}) \) is a weak solution of the Cauchy problem (35)-(36).

For the Cauchy problem (35)-(36) it can be introduced the notion of a weak solution in certain generalized sense. Consider the functional
\[
(f, F(t | F_1(t))) \equiv \sum_{s=0}^{\infty} \frac{1}{s!} \text{Tr}_{1,\ldots,s} f_s F_s(t | F_1(t)),
\]
where \( f = (f_0, f_1, \ldots, f_n, \ldots) \in \mathfrak{L}_0(\mathcal{F}_{\mathcal{H}}) \in \mathfrak{L}(\mathcal{F}_{\mathcal{H}}) \) is a finite sequence of degenerate bounded operators \([48]\) with infinitely times differentiable kernels with compact supports and elements of the sequence \( F(t | F_1(t)) \equiv (F_1(t, 1), F_2(t, 1, 2 | F_1(t)), \ldots, F_s(t, 1, \ldots, s | F_1(t)), \ldots) \) are defined by formulas (37) and (41) for the first and other elements correspondingly. If for functional (45) it is valid the equality
\[
\frac{d}{dt}(f, F(t | F_1(t))) = (\mathcal{B}^+ f, F(t | F_1(t))),
\]
where \( \mathcal{B}^+ \) is the operator dual to the generator of the quantum BBGKY hierarchy \([41]\), i.e.
\[
(\mathcal{B}^+ f)_s(Y) \equiv \mathcal{N}_s(Y) f_s(Y) + \\
+ \sum_{n=1}^{s} \frac{1}{n!} \sum_{k=n+1}^{s} \frac{1}{(k-n)!} \sum_{j_1 \neq \ldots \neq j_k = 1}^{s} \mathcal{N}^{(k)}_{\text{int}}(j_1, \ldots, j_k) f_{s-n}(Y \setminus (j_1, \ldots, j_n)),
\]
we are said to be that expansion (37) is a weak solution of the Cauchy problem (35)-(36) in extended meaning.

To verify this definition we transform functional (45) as follows \([41]\)
\[
(f, F(t | F_1(t))) = \sum_{s=0}^{\infty} \frac{1}{s!} \text{Tr}_{1,\ldots,s} \left( \sum_{n=0}^{s} \frac{1}{(s-n)!} \sum_{j_1 \neq \ldots \neq j_s = 1}^{s} \mathcal{N}^{(s)}_{\text{int}}(j_1, \ldots, j_s) f_{s-n}(1, \ldots, j_s) \right) \prod_{i=1}^{s} F^0_1(i),
\]
where \( \sum_{Z \subset Y \setminus (j_1, \ldots, j_s-n)} \) is a sum over all subsets \( Z \subset Y \setminus (j_1, \ldots, j_s-n) \) of the set \( Y \setminus (j_1, \ldots, j_s-n) \subset (1, \ldots, s) \). For \( F^0_1 \in \mathfrak{L}^1(\mathcal{H}) \) and bounded interaction potentials this functional exists.

Skipping the details, as a result for \( f \in \mathfrak{L}_0(\mathcal{F}_{\mathcal{H}}) \) the derivative of functional (45) over the time variable in the sense of the \(*\)-weak convergence in the space \( \mathfrak{L}(\mathcal{F}_{\mathcal{H}}) \) takes the form \([41]\)
\[
\frac{d}{dt}(f, F(t | F_1(t))) = \sum_{s=0}^{\infty} \frac{1}{s!} \text{Tr}_{1,\ldots,s} (\mathcal{N}_s(Y) f_s(Y) + \sum_{n=1}^{s} \frac{1}{n!} \sum_{k=n+1}^{s} \frac{1}{(k-n)!} \times \\
\times \sum_{j_1 \neq \ldots \neq j_k = 1}^{s} \mathcal{N}^{(k)}_{\text{int}}(j_1, \ldots, j_k) f_{s-n}(Y \setminus (j_1, \ldots, j_n)) F_s(t, Y | F_1(t))).
\]
In the sense of defined notion of a weak solution in extended meaning (46) equality (47) means that for arbitrary initial data $F_0^1 \in \mathcal{L}^1 (\mathcal{H})$ a weak solution of the initial-value problem of the generalized quantum kinetic equation (35) is determined by formula (37).

5 Conclusions

We demonstrate that in fact if initial data is completely defined by a one-particle density operator, then all possible states of infinite-particle systems at arbitrary moment of time can be described within the framework of a one-particle density operator without any approximations and explicitly defined functionals of this one-particle density operator. One of the advantage of such approach is the possibility to construct the kinetic equations in scaling limits if there are correlations of particle states at initial time [2], for instance, correlations characterizing the condensate states [1].

The specific quantum kinetic equations such as the Boltzmann equation and other ones, can be derived from the constructed generalized quantum kinetic equation in the appropriate scaling limits or as a result of certain approximations. For example, in the mean-field limit [13] (the case of scaled interaction potential $\epsilon \Phi$, i.e. $\epsilon \mathcal{N}_{\text{int}}$) we derive the quantum Vlasov equation and for pure states the Hartree equation or the nonlinear Schrödinger equation (in case of a two-body interaction potential with the cubic nonlinear term and for $n$-body interaction potential (35) with the $2n - 1$ power nonlinear term).

Indeed, if there exists the following limit $f_0^1 \in \mathcal{L}^1 (\mathcal{H}_1)$ of initial data (16)

$$\lim_{\epsilon \to 0} \|\epsilon F_0^1 \epsilon f_0^1 \|_{\mathcal{L}^1 (\mathcal{H}_1)} = 0,$$

then for arbitrary finite time interval, there exists the limit of solution (37) of the generalized quantum kinetic equation (15)

$$\lim_{\epsilon \to 0} \|\epsilon F_1 (t) - f_1 (t) \|_{\mathcal{L}^1 (\mathcal{H}_1)} = 0,$$

where $f_1 (t)$ is a strong solution of the Cauchy problem of the quantum Vlasov equation (19)-(20) represented in the form of the following expansion

$$f_1 (t, 1) = \sum_{n=0}^{\infty} \int_{0}^{t} dt_1 \ldots \int_{0}^{t_{n-1}} dt_n \operatorname{Tr}_{s+1, \ldots, s+n} \prod_{j=1}^{s} \mathcal{G}_1 (-t + t_1, j) \times$$

$$\times \sum_{i_1=1}^{s} (-\mathcal{N}_{\text{int}} (i_1, s + 1)) \prod_{j_1=1}^{s+1} \mathcal{G}_1 (-t_1 + t_2, j_1) \ldots \prod_{j_{n-1}=1}^{s+n-1} \mathcal{G}_1 (-t_{n-1} + t_n, j_{n-1}) \times$$

$$\times \sum_{i_n=1}^{s+n} (-\mathcal{N}_{\text{int}} (i_n, s + n)) \prod_{j_n=1}^{s+n} \mathcal{G}_1 (-t_n, j_n) \prod_{i=1}^{s+n} f_0^1 (i),$$

and the operator $\mathcal{N}_{\text{int}}$ is defined by formula (4). For bounded interaction potentials series (50) converges for finite time interval [27].
This statement is a consequence that, if \( f_s \in L^1(\mathcal{H}_s) \), then for arbitrary finite time interval for the strongly continuous group (6) it holds [28]

\[
\lim_{\epsilon \to 0} \left\| G_s(-t) f_s - \prod_{j=1}^{s} G_1(-t, j) f_s \right\|_{L^1(\mathcal{H}_s)} = 0,
\]

and in general case the validity of the following equality

\[
\lim_{\epsilon \to 0} \left\| \frac{1}{\epsilon^n} A_{1+n}(t, \{1, \ldots, s\}, s+1, \ldots, s+n) f_{s+n} - \right.
\]

\[
\int_0^t dt_1 \ldots \int_0^{t_{n-1}} dt_n \prod_{j=1}^{s} G_1(-t + t_1, j) \sum_{i_1=1}^{s} (\mathcal{N}_{\text{int}}(i_1, s+1)) \prod_{j_1=1}^{s+1} G_1(-t_1 + t_2, j_1) \ldots
\]

\[
\ldots \prod_{j_{n-1}=1}^{s+n-1} G_1(-t_{n-1} + t_n, j_{n-1}) \sum_{i_n=1}^{s+n-1} (\mathcal{N}_{\text{int}}(i_n, s+n)) \prod_{j_n=1}^{s+n} G_1(-t_n, j_n) f_{s+n} \left\|_{L^1(\mathcal{H}_{s+n})} = 0.
\]

Then according to definition (12) of the evolution operators \( \mathcal{W}_{1+n}(t, \{1, \ldots, s\}, s+1, \ldots, s+n) \), \( n \geq 0 \), from expansion (11), taking into account an analog of the Duhamel equation for scattering operators

\[
(\tilde{G}_s(t, 1, \ldots, s) - I) f_s = \epsilon \int_0^{t} d\tau \prod_{l=1}^{s} G_1(\tau, l) \left( - \sum_{i<j=1}^{s} \mathcal{N}_{\text{int}}(i, j) \right) G_s(-\tau) f_s,
\]

and (82), we establish formulas of an asymptotic perturbation of evolution operators (12)

\[
\lim_{\epsilon \to 0} \left\| (\mathcal{W}_1(t, \{1, \ldots, s\}) - I) f_s \right\|_{L^1(\mathcal{H}_s)} = 0,
\]

and for \( n \geq 1 \), correspondingly

\[
\lim_{\epsilon \to 0} \left\| \frac{1}{\epsilon^n} \mathcal{W}_{1+n}(t, \{1, \ldots, s\}, s+1, \ldots, s+n) f_{s+n} \right\|_{L^1(\mathcal{H}_{s+n})} = 0.
\]

Since a solution of initial-value problem (15)-(16) of the generalized quantum kinetic equation converges to a solution of initial-value problem (19)-(20) of the Vlasov quantum kinetic equation as (18), (19), for functional (11) for every \( s \geq 2 \) it is true

\[
\lim_{\epsilon \to 0} \left\| \epsilon^s F_s(t, 1, \ldots, s | F_1(t) \right\|_{L^1(\mathcal{H}_s)} = 0,
\]

where \( f_1(t) \) is defined by series (50) which converges for finite time interval, or for marginal correlation functionals (29) it holds

\[
\lim_{\epsilon \to 0} \left\| \epsilon^s G_s(t, 1, \ldots, s | F_1(t) \right\|_{L^1(\mathcal{H}_s)} = 0.
\]

The last equalities mean that in the mean-field scaling limit chaos property (4) preserves in time.
In the case of quantum systems of particles obeying Fermi or Bose statistics \[38\] the generalized quantum kinetic equation \(35\) and functionals \(11\) have different structures. The analysis of these cases will be given in a separate paper.

In the end it should be emphasized that a one-particle marginal density operator which belongs to the space \(\mathcal{L}_1^\alpha(\mathcal{F}_H)\) describes only finitely many particles, i.e. systems for which the average number of particles in a system is finite. In order to describe the evolution of infinitely many particles we have to construct solutions for initial data that belongs to more general Banach spaces than the space of trace class operators \[2\]. For example, it can be the space of sequences of bounded operators containing the equilibrium states \[27, 37\]. In that case every term of the solution expansions of the quantum BBGKY hierarchy \[2\] and correspondingly of the generalized quantum kinetic equation \(15\) as well as marginal functionals of the state \(11\) contains the divergent traces \[2, 8\].

References

[1] M.M. Bogolyubov, Lectures on Quantum Statistics. Problems of Statistical Mechanics of Quantum Systems. Rad. Shkola, 1949 (in Ukrainian).

[2] C. Cercignani, V.I. Gerasimenko and D.Ya. Petrina, Many-Particle Dynamics and Kinetic Equations. Kluwer Acad. Publ., 1997.

[3] N.N. Bogolyubov, Problems of a Dynamical Theory in Statistical Physics. Gostekhizdat, 1946. (In: Studies in Statistical Mechanics, 1, North-Holand Publ., 1962).

[4] N.N. Bogolyubov and K.P. Gurov, Kinetic equations in quantum mechanics. JETP, 17, (1947), 614–628.

[5] M.S. Green, Boltzmann equation from statistical mechanical point of view. J. Chem. Phys. 25, (5), (1956), 836–855.

[6] M.S. Green and R.A. Piccirelly, Basis of the functional assumption in the theory of the Boltzmann equation. Phys. Rev. 132, (3), (1963), 1388–1410.

[7] R.A. Piccirelli, Some properties of the long-time value of the probability densities for moderately dense gases. J. Math. Phys. 7, (1966), 922–934.

[8] E.G.D. Cohen, The Kinetic Theory of Dense Gases. (In: Fundamental Problem in Statistical Mechanics, 2, North-Holand Publishing, 1968), 228–275.

[9] E.G.D. Cohen, Bogolyubov and kinetic theory: the Bogolyubov equations. Ukrainian J. Phys. 54, (2009), 847–861.

[10] C. Cercignani, R. Illner and M. Pulvirenti, The Mathematical Theory of Dilute Gases. Springer-Verlag, 1994.

[11] H. Grad, Principles of the Kinetic Theory of Gases. (In: Handbuch der Physik, 12, Springer, 1958), 205–294.
[12] H. Spohn, *Kinetic equations from Hamiltonian dynamics*. Rev. Mod. Phys. 52, (3),(1980), 569-615.

[13] H. Spohn, *Large Scale Dynamics of Interacting Particles*. Springer-Verlag, 1991.

[14] H. Spohn, *Kinetic equations for quantum many-particle systems*. arXiv:0706.0807v1, (2007).

[15] A. Arnold, *Mathematical properties of quantum evolution equations*. Lect. Notes in Math. 1946, Springer, 2008.

[16] R. Adami, F. Golse and A. Teta, *Rigorous derivation of the cubic NLS in dimension one*. J. Stat. Phys. 127, (6), (2007), 1193–1220.

[17] C. Bardos, F. Golse, A. Gottlieb and N. Mauser, *Accuracy of the time-dependent Hartree-Fock approximation for uncorrelated initial states*. J. Stat. Phys. 115, (2004), 1037–1055. arXiv:quant-ph/0312005.

[18] L. Erdős, B. Schlein and H.-T. Yau, *Derivation of the cubic non-linear Schrödinger equation from quantum dynamics of many-body systems*. Invent. Math. 167, (3), (2007), 515–614.

[19] J. Fröhlich, S. Graffi and S. Schwarz, *Mean-field and classical limit of many-body Schrödinger dynamics for bosons*. Commun. Math. Phys. 271, (2007), 681–697. arXiv:math-ph/0603055.

[20] A. Michelangeli, *Role of scaling limits in the rigorous analysis of Bose-Einstein condensation*. J. Math. Phys., 48, (2007), 102102.

[21] L. Erdős, B. Schlein and H.-T. Yau, *Derivation of the Gross-Pitaevskii equation for the dynamics of Bose-Einstein condensate*. Ann. Math. 172, (2010), 291-370. arXiv:math-ph/0606017.

[22] L. Erdős and B. Schlein, *Quantum dynamics with mean field interactions: a new approach*. J. Stat. Phys. 134, (5),(2009), 859–870. arXiv:0804.3774.

[23] M. Grillakis, M. Machedon, and D. Margetis, *Second-order corrections to mean field evolution of weakly interacting bosons*. Comm. Math. Phys. 294, (2010), 273-301. arXiv:1003.4713.

[24] V.I. Gerasimenko, *Approaches to derivation of quantum kinetic equations*. Ukrainian J. Phys. 54, (8/9), (2009), 834–846. arXiv:0908.2797.

[25] L. Erdős, M. Salmhofer and H.-T. Yau, *On quantum Boltzmann equation*. J. Stat. Phys. 116, (116), (2004), 367–380. arXiv:math-ph/0302034.

[26] D. Benedetto, F. Castella, R. Esposito and M. Pulvirenti, *A short review on the derivation of the nonlinear quantum Boltzmann equations*. Commun. Math. Sci. 5, (2007), 55–71.
[27] D.Ya. Petrina, *Mathematical Foundations of Quantum Statistical Mechanics. Continuous Systems*. Kluwer, 1995.

[28] T. Kato, *Perturbation Theory for Linear Operators*. Springer-Verlag, 1995.

[29] R. Dautray and J.L. Lions, *Mathematical Analysis and Numerical Methods for Science and Technology*. 5, Springer-Verlag, 1992.

[30] O. Bratelli and D.W. Robinson, *Operator Algebras and Quantum Statistical Mechanics*. 1, Springer-Verlag, 1979.

[31] V.I. Gerasimenko, *Groups of operators for evolution equations of quantum many-particle systems*. Operator Theory: Adv. and Appl. 191, (2009), 341–355. arXiv:0804.1153.

[32] V.I. Gerasimenko and D.Ya. Petrina, *On the generalized kinetic equation*. Reports of NAS of Ukraine, 7, (1997), 7–12.

[33] V.I. Gerasimenko and D.Ya. Petrina, *The generalized kinetic equation generated by the BBGKY hierarchy*. Ukrainian J. Phys., 43, (6/7), (1998), 697–702.

[34] G. Borgioli, V.I. Gerasimenko and G. Lauro, *Derivation of a discrete Enskog equation from the dynamics of particles*. Rend. Sem. Mat. Univ. Pol. Torino. 56, (2), (1998), 59–69.

[35] Zh.A. Tsvir, *Cluster expansions in theory of quantum kinetic equations*. Bulletin of Kyiv Nat. Univ., Math. and Mech. 23, (2010), 25–30.

[36] V.I. Gerasimenko and V.O. Shtyk, *Evolution of correlations of quantum many-particle systems*. J. Stat. Mech., (3), (2008), P03007. arXiv:0712.4336.

[37] J. Ginibre, *Some Applications of Functional Integrations in Statistical Mechanics*. (In: Statistical Mechanics and Quantum Field Theory, Gordon and Breach, 1971), 329–427.

[38] V.I. Gerasimenko and D.O. Polishchuk, *Dynamics of correlations of Bose and Fermi particles*. Math. Meth. Apl. Sci., DOI:10.1002/mma.1336, (2010). arXiv:1001.3893.

[39] D.O. Polishchuk, *BBGKY hierarchy and dynamics of correlations*. Ukrainian J. Phys., 55, (5), (2010), 593–598. arXiv:1002.1490.

[40] G.E. Uhlenbeck and G.W. Ford, *Lectures in Statistical Mechanics*. AMS, 1963.

[41] G. Borgioli and V.I. Gerasimenko, *Initial-value problem of quantum dual BBGKY hierarchy*. Nuovo Cimento, 33 C, (1), (2010), 71–78. arXiv:0806.1027.