A Finite Temperature Treatment of Ultracold Atoms in a 1-D Optical Lattice

B. G. Wild, P. B. Blakie and D. A. W. Hutchinson

Department of Physics, University of Otago, Dunedin, New Zealand

Abstract

We consider the effects of temperature upon the superfluid phase of ultracold, weakly interacting bosons in a one dimensional optical lattice. We use a finite temperature treatment of the Bose-Hubbard model based upon the Hartree-Fock-Bogoliubov formalism, considering both a translationally invariant lattice and one with additional harmonic confinement. In both cases we observe an upward shift in the critical temperature for Bose condensation. For the case with additional harmonic confinement, this is in contrast with results for the uniform gas.
I. INTRODUCTION

Ultracold atoms confined within optical lattice potentials are of great current interest, both theoretically \[1, 2, 3, 4, 5\] and experimentally \[6, 7\]. Of particular interest are the connections to solid state regimes where optical lattices can be manufactured with extraordinary control so as to simulate more complicated, less perfect and, often, less manipulable systems, initially of interest in a condensed matter sphere (e.g. Heisenberg or Ising Hamiltonians) \[8, 9, 10\]. Ultracold atoms in optical lattices have also been proposed as candidates for quantum information processing \[11\], which has applications in quantum cryptography and quantum computing. Various methods of loading Bose-Einstein condensates into optical lattices have been proposed \[12, 13, 14, 15\], and this is now routinely performed \[14\], as is the manipulation and control of the atoms in such a lattice \[16\].

The focus of this paper is on the microscopic treatment of Bose-Einstein condensates in a one dimensional optical lattice. We extend previous treatments at zero temperature \[2, 3\] to finite temperature using the Hartree-Fock-Bogoliubov mean-field treatment as applied to a discrete Bose-Hubbard model yielding modified Gross-Pitaevskii and Bogoliubov-de Gennes equations. This set of equations is solved for both the case of a translationally invariant lattice (no external trapping potential) and an inhomogeneous lattice (optical lattice in an external harmonic trapping potential). The use of the mean field based treatment means we are only considering the superfluid phase for the atoms in the optical lattice. The model is not valid in the Mott Insulator regime. We use the model to estimate the superfluid to normal phase transition temperature in each case. Thus one is able to obtain a phase diagram for the superfluid and normal gas states in the low effective interaction strength limit.

II. FORMALISM

We consider a one-dimensional optical lattice with \(I\) lattice sites. We begin from the Bose-Hubbard Hamiltonian for atoms in a one-dimensional optical lattice \[1, 2, 3\]

\[
\hat{H} = \sum_{i=1}^{I} \hat{n}_i \epsilon_i - J \sum_{i=1}^{I} \left( \hat{a}_{i+1}^\dagger \hat{a}_i + \hat{a}_i^\dagger \hat{a}_{i+1} \right) + \frac{V}{2} \sum_{i=1}^{I} \hat{n}_i (\hat{n}_i - 1),
\]

(1)
where $J$ represents the coupling strength between adjacent lattice sites, $V$ is the interaction potential acting between atoms on the same site, and $\hat{a}_i$ is the Bose field operator for the $i^{th}$ lattice site, and $\hat{n}_i = \hat{a}_i^\dagger \hat{a}_i$. $\epsilon_i$ is the energy on each lattice site $i$ due to the trapping potential. The usual commutation relations apply for the Bose field operator $\hat{a}_i$. Assuming a macroscopic occupation of the ground state, we express the Bose annihilation approximation such that the Gross-Pitaevskii equation to be discussed below, and take the self-consistent mean-field potential acting between atoms on the same site, and a trapping potential. The usual commutation relations apply for the Bose field operator $\hat{a}_i$. The resulting Hamiltonian can now be diagonalised using the Bogoliubov (canonical) transformation

$$\hat{\delta}_i = \sum_q \left[ u_q^q \hat{\alpha}_q e^{-i\omega_q t} - v_q^q \hat{\alpha}_q^\dagger e^{i\omega_q t} \right]$$

and

$$\hat{\delta}_i^\dagger = \sum_q \left[ u_q^q \hat{\alpha}_q^\dagger e^{i\omega_q t} - v_q^q \hat{\alpha}_q e^{-i\omega_q t} \right],$$

where $u_q^q$ and $v_q^q$ are the quasiparticle amplitudes, $\omega_q$ are the quasiparticle excitation frequencies, and $\hat{\alpha}_q^\dagger (\hat{\alpha}_q)$ is the quasiparticle creation (annihilation) operator. This yields a set of coupled equations comprising of a modified Gross-Pitaevskii equation

$$\mu' z_i = \epsilon'_i z_i - (z_{i+1} + z_{i-1}) + V_{\text{eff}} (n_{c_i} z_i + 2\bar{n}_i z_i + \bar{m}_i z_i)$$

and the Bogoliubov-de Gennes equations

$$\hbar \omega_q u_q^q + c_q' z_i = [2V_{\text{eff}} (n_{c_i} + \bar{n}_i) - \mu' + \omega'_q] u_q^q - [u_{q+1}^q + u_{q-1}^q] - V_{\text{eff}} z_i^2 v_q^q,$n

$$-\hbar \omega_q v_q^q - c_q' z_i = [2V_{\text{eff}} (n_{c_i} + \bar{n}_i) - \mu' + \epsilon'_q] v_q^q - [v_{q+1}^q + v_{q-1}^q] - V_{\text{eff}} z_i^2 u_q^q,$$

with

$$\bar{n}_i = \langle \hat{\delta}_i^\dagger \hat{\delta}_i \rangle = \sum_q \left[ |v_q^q|^2 + (|u_q^q|^2 + |v_q^q|^2) \right] N_{BE} (\hbar \omega_q')$$

where

$$N_{BE} (\hbar \omega_q') = \frac{1}{Z^{-1} e^{\hbar \omega_q'} - 1}$$
is the usual Bose distribution, the primed quantities are measured in units of $J$ (which depends on the depth of the optical lattice), and where we have assumed without loss of generality the condensate amplitudes $z_i$ to be real. $\mu$ is the energy eigenvalue for the modified Gross-Pitaevskii equation (and approximates the chemical potential closely for values of temperature well below the transition temperature) and $\hbar \omega_q$ is the energy eigenvalue of the Bogoliubov-de Gennes equations. The $c^q$'s are necessary in order to ensure the orthogonality of the condensate with the excited states [3, 18]. $Z$ is a fugacity term resulting from the difference between the true chemical potential $\mu_T$, and the chemical potential as estimated using the eigenvalue $\mu$ corresponding to the ground state of the modified Gross-Pitaevskii equation. Thus $Z = \exp (\beta (\mu_T - \mu))$.

In the case of a homogeneous gas with a large number of particles in the ground state (ie. $n_c \gg 1$), the fugacity may be approximated by [18]

$$Z = 1 + 1/n_c$$

(8)

In the lattice, this is not always a good approximation, since the condition $n_c \gg 1$ does not always hold, and we will take $Z = 1$. This will lead to increased values for the excited atom population, and will therefore result in under-estimated values of the transition temperature.

Superfluid flow of the condensate occurs when there is a phase gradient. A phase gradient of the condensate modifies the hopping term of the Hamiltonian by the introduction of Peierls phase factors. The resulting energy shift may be estimated using second order perturbation theory. Since this energy shift is due entirely to the kinetic energy associated with the superfluid flow, and hence due to the superfluid fraction, it follows that the superfluid fraction $f_s$ may be calculated in terms of this energy shift [2]

$$f_s = \frac{1}{N} E_\phi - E_0 = -\frac{1}{2NJ} \langle \psi_0 \mid \hat{T} \mid \psi_0 \rangle - \frac{1}{NJ} \sum_{\nu \neq 0} \frac{\langle \psi_\nu \mid \hat{J} \mid \psi_\nu \rangle^2}{E_\nu - E_0}$$

(9)

where $\hat{J} = iJ \sum_{i=1}^l \left( \hat{a}_{i+1}^+ \hat{a}_i - \hat{a}_i^+ \hat{a}_{i+1} \right)$ and $\hat{T} = J \sum_{i=1}^l \left( \hat{a}_{i+1}^+ \hat{a}_i + \hat{a}_i^+ \hat{a}_{i+1} \right)$. The superfluid fraction can then be expressed in terms of the condensate and quasiparticle amplitudes

$$f_s = f_s^{(1)} - f_s^{(2)}$$

(10)

where

$$f_s^{(1)} = \frac{1}{2N} \sum_{i=1}^l \left( z_{i+1}^* z_i + z_{i+1}^* z_i \right) + \sum_q \left( v_i^q v_{i+1}^{q^*} + v_i^{q^*} v_{i+1}^q \right)$$

(11)
and

\[ f_s^{(2)} = \frac{J}{N} \sum_{q,q'} \left[ \sum_i \left( \frac{u_{i+1}^q v_i^{q'} - u_i^q v_{i+1}^{q'}}{\hbar \omega_q + \hbar \omega_{q'}'} \right)^2 + \delta_{qq'} \sum_i \left( \frac{u_{i+1}^q v_i^q - u_i^q v_{i+1}^{q}}{2\hbar \omega_q} \right)^2 \right]. \]  (12)

A. Translationally Invariant Lattice

In the case of a translationally invariant lattice, periodic boundary conditions apply, and the quasi-particle amplitudes are given by

\[ u_j^q = \frac{u_{q}^{j(q)}}{\sqrt{I}} , \quad v_j^q = \frac{v_{q}^{j(q)}}{\sqrt{I}} , \quad 1 \leq j \leq I - 1 \]  (13)

Furthermore the condensate amplitudes are equal for each site \( j \) and, since there is no trapping potential, \( \epsilon_i = \epsilon = 0 \). Solving for the quasiparticle amplitudes we obtain

\[ |u^q|^2 = \frac{V z^2 + 4 J \sin^2 \left( \frac{q a}{2} \right) + \hbar \omega_q}{2 \hbar \omega_q} \]  (14)

\[ |v^q|^2 = \frac{V z^2 + 4 J \sin^2 \left( \frac{q a}{2} \right) - \hbar \omega_q}{2 \hbar \omega_q} \]  (15)

and

\[ \hbar \omega_q = \sqrt{4 J \sin^2 \left( \frac{q a}{2} \right) \left[ 2 n_c V + 4 J \sin^2 \left( \frac{q a}{2} \right) \right]} \]  (16)

Thus from equations (14), (15) and (16), for the translationally invariant lattice, the superfluid fraction is given by

\[ f_s = \frac{1}{N} \left( I |z|^2 + \sum_{j=1}^{I-1} |v^q|^2 \cos \left( \frac{2 \pi j}{I} \right) \right) \]  (17)

since \( q = 2\pi j / I a \) for \( 1 \leq j \leq I - 1 \). The condensate fraction for a given lattice site is given by \( f_c = n_c / n_0 \) where \( n_0 = N / I \) is the number of atoms per site.

To calculate the condensate and superfluid fraction, we first determine the condensate amplitude. Then defining

\[ g_{j(n)} = |z|_{(n-1)}^2 V_{\text{eff}} + 4 \sin^2 \left( \frac{\pi j}{I} \right) \]  (18)

\[ e_{j(n)} = 2 \sin \left( \frac{\pi j}{I} \right) \sqrt{2 |z|_{(n-1)}^2 V_{\text{eff}} + 4 \sin^2 \left( \frac{\pi j}{I} \right)} \]  (19)
and

$$N_{BE(n)} = \frac{1}{\exp(\beta' e_{j(n)}(n-1)) - 1},$$  \hspace{1cm} (20)

where $\beta' = J/k_B T$, $k_B$ is Boltzman's constant and where the subscript $(n)$ refers to the variable in question at the $n^{th}$ iteration, one can solve for $|z|^2$ using an iterative scheme

$$|z|^2(n) = \frac{1}{I} \left( N - \sum_{j=1}^{I-1} \left[ \frac{g_j(n) - e_j(n)}{2e_j(n)} + \frac{g_j(n)}{e_j(n)} N_{BE_j(n)} \right] \right).$$  \hspace{1cm} (21)

As an initial guess, the value $|z|^2(n) = \frac{N}{I}$ is used, and the calculations (18), (19), (20) and (21) repeated until convergence is attained (ie. $|z|^2(n) - |z|^2(n-1) < \text{Error Tolerance}$), or the maximum number of iterations is exceeded (divergent solution).

One first calculates $g_j(n)$, $e_j(n)$ and $N_{BE_j(n)}$ using equations (18), (19) and (20). The quasiparticle amplitudes (given by equations (14) and (15)) may be calculated using the equations

$$|u^{(j)}|^2(n) = \frac{g_j(n) + e_j(n)}{2e_j(n)}$$  \hspace{1cm} (22)

and

$$|v^{(j)}|^2(n) = \frac{g_j(n) - e_j(n)}{2e_j(n)}$$  \hspace{1cm} (23)

The condensate and superfluid fractions are then readily determined.

**B. Inhomogeneous Lattice**

The condensate amplitudes $z_i$ are found by solving equation (4), where the trapping potential is given by $\epsilon_i = \Omega(i - (I + 1)/2)^2$ for site $i$, with $\Omega = \frac{1}{2}m\omega^2a^2$, and $a$ is the inter-lattice spacing. The quasi-particle amplitudes for site $i$, $u_i^q$ and $v_i^q$, are found by solving equations (5).

This set of equations can again be solved iteratively. In performing the calculation, we actually set the $c^q$'s to zero when solving the Bogoliubov-de Gennes equations, but do so in the Hartree-Fock basis, thus ensuring orthogonality of the ground state and the excited states [18]. The Hartree-Fock basis is given by the normalised solutions to the eigenvalue problem given by equation (4), but where the ground state (zero energy solution) is excluded.

First, let us rewrite the Bogoliubov-de Gennes equations (5) in matrix form

$$\hat{h}\omega_q \begin{bmatrix} u^q \\ v^q \end{bmatrix} = \begin{bmatrix} \hat{L} & M \\ -M & -\hat{L} \end{bmatrix} \begin{bmatrix} u^q \\ v^q \end{bmatrix},$$  \hspace{1cm} (24)
where

\[
\begin{align*}
\mathbf{u}^q &= \begin{bmatrix} u_1^q \\ \vdots \\ u_{q-1}^q \\ u_q^q \end{bmatrix}, & \mathbf{v}^q &= \begin{bmatrix} v_1^q \\ \vdots \\ v_{q-1}^q \\ v_q^q \end{bmatrix},
\end{align*}
\]

(25)

\[
\hat{\mathcal{L}} = 2V_{\text{eff}}(n_{c_i} + \bar{n}_i) - \mu + \Omega(i - (I + 1)/2)^2 - \hat{J}
\]

(26)

and

\[
M = -V_{\text{eff}} z_i^2
\]

(27)

Here \(\hat{J}\) is defined as the operator acting on \(z_i, u_i^q, \) and \(v_i^q\) as follows:

\[
\hat{J} u_i^q = u_{i+1}^q + u_{i-1}^q.
\]

(28)

Now, let \(\{z_i^q\}\) constitute the eigenstates of equation (4) with eigenvalues \(\mu^q\). We order the eigenvalues into ascending order, and order the normalised eigenstates accordingly, call these \(\{\xi_i^q\}\). The state \(\xi_0^q\) corresponds to the Goldstone mode, we must exclude this in order to obtain the Hartree-Fock-Bogoliubov basis. Let us define the matrix

\[
\mathbf{U} = \begin{bmatrix} \xi_1 & \cdots & \xi^q & \cdots & \xi^{q-1} \end{bmatrix},
\]

(29)

where

\[
\xi^q = \begin{bmatrix} \xi_1^q \\ \vdots \\ \xi_i^q \\ \vdots \\ \xi_{q-1}^q \end{bmatrix}.
\]

(30)

Let \(\{\mathbf{u}^q_{HFB}, \mathbf{v}^q_{HFB}\}\) be the solution to the Bogoliubov-de Gennes equations in the Hartree-Fock-Bogoliubov basis, then the solution to the matrix equation

\[
\hbar \omega_q \begin{bmatrix} \mathbf{u}^q_{HFB} \\ \mathbf{v}^q_{HFB} \end{bmatrix} = \begin{bmatrix} U^T \hat{\mathcal{L}} U & U^T M U \\ -U^T M U & -U^T \hat{\mathcal{L}} U \end{bmatrix} \begin{bmatrix} \mathbf{u}^q_{HFB} \\ \mathbf{v}^q_{HFB} \end{bmatrix}
\]

(31)

gives the Bogoliubov quasiparticle amplitudes in the Hartree-Fock-Bogoliubov basis. To obtain the Bogoliubov quasiparticle amplitudes \(\{u_i^q, v_i^q\}\) one transforms back using

\[
\begin{align*}
\mathbf{u}^q &= \mathbf{U} \mathbf{u}^q_{HFB}, \\
\mathbf{v}^q &= \mathbf{U} \mathbf{v}^q_{HFB}.
\end{align*}
\]

(32)
The iterative solution of these equations is then obtained numerically.

III. RESULTS

We present results for a lattice of depth of approximately $16.8 E_R$, where the recoil energy is defined as $E_R = \frac{\hbar^2}{8ma^2}$ with $a$ the inter-site spacing. This is defined for the optical lattice parameters defined in reference 20, and using the band structure calculations of 21.

A. Translationally Invariant Lattice

In figures 1 and 2 we present results for a translationally invariant lattice with one atom and ten atoms per site respectively for various values of the effective interaction potential $V_{\text{eff}}$. In both cases ten lattice sites, with periodic boundary conditions, were used. One observes, a decrease in both the condensate and superfluid fractions with temperature and we interpret the point at which these densities approach zero as indicative of the critical temperature $T_c$ for the superfluid to normal gas phase transition. This interpretation is supported by an examination of the low lying excitation spectrum which, for the case $V_{\text{eff}} = 20$, is shown in panel (b) of figure 2. The “softening” of the modes, indicative of the phase transition, is clearly seen and coincides with the transition temperature obtained by noting the temperature at which the condensate and superfluid densities approach zero.

We obtain such a transition temperature for each value of the effective interaction potential $V_{\text{eff}}$, producing a phase diagram as shown in figures 1(c) and 2(c), where the superfluid phase and the normal phase are as indicated, the superfluid lying to the left of the curve. The transition to the Mott insulator phase cannot be determined by this analysis, but one would expect the transition from the superfluid phase to the Mott insulator phase to occur when the site coupling strength $J$ is decreased below a certain point, depending on the on-site interaction strength $V_{\text{eff}}$. In practice the Mott insulator phase transition will occur when $V_{\text{eff}} = V/J$ exceeds some critical value. Thus a transition to the Mott insulator phase is also possible by increasing the on-site interaction strength $V$ for a given coupling strength $J$. We note here that, in all instances, the transition temperature increases with the effective interaction potential, as is indeed the case with a homogeneous Bose gas (ie. a Bose gas in the absence of an optical lattice and of a confining potential) 22, 23. This reentrant
FIG. 1: Translationally invariant lattice with ten atoms and ten sites. Overall condensate (a) and superfluid (b) fractions as a function of temperature. The corresponding phase diagram is shown in panel (c).

behaviour has also been predicted for the translationally invariant lattice by Kleinert et al. [24]. Our results are consistent with their conclusions. In addition, they predict a reduction of the critical temperature as the effective interaction strength is increased further. We, however, are unable to explore this regime as it extends beyond the validity of our model.
FIG. 2: Translationally invariant lattice with ten atoms per site. Condensate Fraction (a), and excitation spectrum as a function of temperature (b) with the corresponding phase diagram (c).

B. Inhomogeneous Lattice

In this section we present results for the case of an optical lattice in the presence of a harmonic trapping potential. The condensate fraction $f_c$ and the superfluid fraction $f_s$ are evaluated as a function of the effective potential $V_{\text{eff}}$ and the temperature.

Figure 3 shows plots of the number of condensate atoms, excited atoms and of the total number of atoms for each lattice site for the case of an inhomogeneous optical lattice consisting of forty one lattice sites (odd case) with ten atoms in total, for $V_{\text{eff}} = 1$ at various temperatures ranging from $T = 0$ nK to $T = 1.6$ nK. At zero temperature, the condensate
FIG. 3: Inhomogeneous lattice with forty one lattice sites and ten atoms in total for $V_{\text{eff}} = 1$, at (a) $T = 0$ nK, (b) $T = 1$ nK (c) $T = 1.6$ nK. Circles represent the number of condensate atoms, squares the number of excited atoms and crosses the total number of atoms

atom distribution is bell-shaped, peaked at the central lattice site. There is a small quantum depletion even at zero temperature, and the distribution of excited atoms is shaped as a bimodal distribution, centred about the central lattice site. As the temperature increases, the condensate population decreases, but the distribution still remains bell-shaped, and the excited population increases. In figure 4 we present the overall condensate and superfluid fractions and the corresponding phase diagram for optical lattices in a harmonic potential consisting of forty lattice sites (even case). Panel (a) corresponds to the condensate fraction, (b) the superfluid fraction, and (c) the phase diagram. We note that for higher tempera-
FIG. 4: Overall condensate and superfluid fractions as a function of temperature, and the corresponding phase diagram for an optical lattice in a harmonic potential with forty lattice sites (even case) and ten atoms. (a) corresponds to the condensate fraction, (b) to the superfluid fraction and (c) to the phase diagram.

Tures, calculations performed using forty one lattice sites begin to show a marked difference. This is indicative of the fact that we are pushing the bounds of validity of our model. In particular we are seeing significant finite size effects, which affect the value of the chemical potential and, ultimately, a failure of the mean field approximation. The dotted continuation of the lines in figure [4] indicate where our calculations become unreliable, but are included for completeness. We are still able to use our model to obtain an estimate of the critical tem-
perature and hence the trend in its dependence upon the effective interaction strength. We therefore conclude from figure 4 that the superfluid to normal phase transition temperature increases with increasing $V_{\text{eff}}$. It is clear, then, that the shift in critical temperature with effective interaction potential is positive definite for a Bose gas in a one-dimensional optical lattice, regardless of whether the system is confined in a (harmonic) trapping potential or not. This is in contrast to the case of a three-dimensional Bose gas, where $\Delta T_c$ changes sign for the trapped gas.

IV. CONCLUSIONS

We have applied the descretized Hartree-Fock-Bogoliubov formulation to the Bose-Hubbard model in order to calculate the dependence of the condensate and superfluid fractions on the temperature. We have used this to estimate the critical temperature for the superfluid to normal phase transition for both a translationally invariant optical lattice (no external trap present), and an inhomogeneous optical lattice (contained within an external harmonic trap). This has enabled us to investigate the phase diagram for both cases and we observe that the transition temperature increases with increasing effective interaction potential $V_{\text{eff}}$. Unlike the homogeneous case with no optical lattice, this positive shift in the critical temperature with interaction strength is present in both the translationally invariant case and when a parabolic confining potential is imposed. In the homogeneous gas the shift in the critical temperature is only positive in the absence of a confining potential. These conclusions are consistent with previous work for the translationally invariant case\cite{b}, extending this result to include parabolic confinement.

V. ACKNOWLEDGEMENTS

We would like to thank the Marsden Fund of the Royal Society of New Zealand and the University of Otago for financial support.

\[1\] D. Jaksch, C. Bruder, J. I. Cirac, C. W. Gardiner, and P. Zoller, Phys. Rev. Lett. 81, 3108 (1998).
[2] K. Burnett, M. Edwards, C. W. Clark, and M. Shotter, J. Phys. B: At. Mol. Opt. Phys. 35, 1671 (2002).
[3] A. M. Rey, K. Burnett, R. Roth, M. Edwards, C. J. Williams, and C. W. Clark, J. Phys. B: At. Mol. Opt. Phys. 36, 825 (2003).
[4] D. van Oosten, P. van der Straten, and H. T. C. Stoof, Phys. Rev. A 63, 053601 (2001).
[5] D. B. M. Dickersheid, D. Van Oosten, P. J. H. Denteneer, and H. T. C. Stoof Phys. Rev. A 68, 043623 (2003).
[6] S. Friebel, C. D'Andrea, J. Walz, M. Weitz, and T. W. M. Hansch, Phys. Rev. A 57, R20 (1998).
[7] L. Guidoni and P. Verkerk, Phys. Rev. A 57, R1501 (1998).
[8] J. J. Garcia-Ripoll and J. I. Cirac, New Journal of Physics 5, 76.1 (2003).
[9] J. J. Garcia-Ripoll, M. A. Martin-Delgado, and J. I. Cirac, Phys. Rev. Lett. 93, 250405 (2004).
[10] V. W. Liu, F. Wilczek, and P. Zoller, Phys. Rev. A 70, 033603 (2004).
[11] T. Calarco, H. J. Briegel, D. Jaksch, J. I. Cirac, and P. Zoller, J. Mod. Optics 47, 2137 (2000).
[12] J. Plata, Phys. Rev. A 69, 033604 (2004).
[13] P. B. Blakie and J. V. Porto, Phys. Rev. A 69, 013603 (2004).
[14] A. S. Mellish, G. Duffy, C. McKenzie, R. Guerson, and A. C. Wilson, Phys. Rev. A 68, 051601 (2003).
[15] S. Peil, J. V. Porto, T. B. Laburthe, J. M. Olbrecht, B. E. King, M. Subbotin, S. L. Rolston, and W. D. Phillips, Phys. Rev. A 67, 051603(R) (2003).
[16] H. L. Haroutyunyan and G. Nienhuis, Phys. Rev. A 64, 033424 (2001).
[17] A. Griffin, Phys. Rev. B 53, 9341 (1996).
[18] S. A. Morgan, J. Phys. B: At. Mol. Opt. Phys. 33, 3847 (2000).
[19] D. A. W. Hutchinson, K. Burnett, R. J. Dodd, S. A. Morgan, M. Rusch, E. Zaremba, N. P. Proukakis, M. Edwards, and C. W. Clark, J. Phys. B: At. Mol. Opt. Phys. 33, 3825 (2000).
[20] M. Greiner, O. Mandl, T. Eslinger, T. W. Hansch, and I. Bloch, Nature 415, 39 (2002).
[21] P. B. Blakie and C. W. Clark, J. Phys. B: At. Mol. Opt. Phys. 37, 1391 (2004).
[22] P. Arnold, Phys. Rev. Lett. 87, 120401 (2001).
[23] B. Kastening, Phys. Rev. A 69, 043613 (2004).
[24] H. Kleinert, S. Schmidt, and A. Pelser, Phys. Rev. Lett. 93, 160402 (2004).