Linear and nonlinear analysis of Ion-Temperature-Gradient (ITG) driven mode in the asymmetric pair-ion magnetoplasma

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Abstract
We have investigated linear and nonlinear dynamics of ion-temperature-gradient (ITG) driven drift mode for Maxwellian and non Maxwellian pair-ion plasma embedded in an inhomogeneous magnetic field having gradients in the temperature and number density of ions. Linear dispersion relations are derived and analyzed analytically as well as numerically for different cases. It has been found that growth rate of instability increases with increasing $\eta$. By using the transport equations of Braginskii, model, a set of nonlinear equations are derived. In the nonlinear regime, soliton structures are found to exist. Our numerical analysis shows that amplitude of solitary waves increases by increasing ion to electron number density ratio. These solitary structures are also found to be sensitive to asymmetries in pair plasma and non thermal kappa and Cairns distributed electrons. Our present work may contribute a good illustration of the observation of nonlinear solitary waves driven by the ITG mode in magnetically confined pair-ion plasmas and space pair-ion plasmas as the formation of localized structures along drift modes is one of the striking reasons for L-H transition in the region of improved confinements in magnetically confined devices like tokamaks.

1. Background
Curiosity of the scientists to dig deep into the early universe and processes happening at stellar distances has been the first building block of laboratory astrophysics.

Lepton epoch, which in physical cosmology is believed to have happened one second after the big bang is an era where electron positron ($e-p$) plasma [also known as pair plasma] was dominating the Universe. It’s widely believed that the e-p plasma was created by the process of annihilation of hadrons and anti-hadrons and remained in the thermal equilibrium with electromagnetic radiations (photon gas). Pair plasma possesses unique features of equal mass, opposite charge of same strength and time-space parity. Therefore such plasma if produced in the lab can provide excellent opportunity to understand the matter and anti-matter research [1–7]. However, due to phenomena of annihilation of electrons-positrons in e-p plasma, it is very difficult to maintain this plasma for a longer period of time in the laboratory. At the same time, importance of positrons’ physics has also been well recognized in high energy physics in particular to understand the properties of antimatter such as charge, parity and time reversal (CPT) invariance, Bose–Einstein condensates etc Also production of positrons in the laboratory is carried out for the better understanding of blackholes, gamma ray bursts like astrophysical environments.

To mimic the electron positron plasma in a controlled environment, Japanese scientists Hatekayama and Oohara successfully created lighter pair ion ($p-i$) [consisting of positive and negative ions] (hydrogen ion) plasma and high density Fullerene ion ($C_{60}^+$ and $C_{60}^-$) plasma [6, 8–11]. Unlike the electron-positron (e-p)
plasma systems, in pair ion (p-i) plasma, problem of annihilation due to its long enough lifetime does not arise, and therefore collective behavior can be studied conveniently under controlled conditions. At the same time, some results obtained by the experiments need support from theories, in this regard a kinetic theory taking into account the boundary effects [1,2] has been developed. However, still a theory is required which can support experimental observations and basic fluid theory.

Usually e-p plasma is symmetric in nature but both mass and temperature symmetries can also be broken, the first experimental observations where asymmetries in the temperature of electrons ($T_{e}$) and positrons ($T_{p}$) were observed were carried by Chen et al [13]. Authors reported $T_{p} \lesssim 0.5 T_{e}$, where effective temperature of positrons: $T_{p}$ was found to be $\sim 2.8 \pm 0.3$ MeV. On the other hand at Fermi National Accelerator Laboratory, Batavia, experiments from Tevatron collider indicated an unbalanced proportion of matter to antimatter where matter appears to dominate the antimatter beyond the limit of 1% predicted by the Standard Model. These observation clearly indicated asymmetries in both temperature and mass of the e-p plasma. Plasma at electron positron collider created by using slightly different Lorentz factor beams is also asymmetric. Interactions between the charged particles or some naturally occurring nonlinear process can also lead to mass asymmetries whereas it can also assumed as initial condition [14].

Temperature and mass asymmetries in plasma have been responsible for the onset of many instabilities (such as Weibel like instabilities in case of temperature asymmetry) and other nonlinear phenomena however some contradictory studies also exist in literature. For a pair ion (p-i) plasma case, Verheest et al [15] demonstrated that a strict symmetry destroys the stationary nonlinear structures of acoustic nature and showed such nonlinear structures can exist when there is a thermodynamic asymmetry between both constituents. Whereas in context of propagation of electromagnetic waves in the asymmetric e-p plasma, Mahajan et al, [2] assuming $T_{e} = T_{p}$ derived the nonlinear Schrödinger equation (NLSE) in which the nonlinear term vanishes when the temperatures of the electrons and positrons are equal however this was found inconsistent with the earlier results [16–18]. For instance in [16], Shukla et al investigated nonlinear interaction between photons and phonons in e-p plasma $\left( \text{e}^+ + \text{e}^- \right)$ with $T_{p} (m_{p}) = T_{p} (m_{p})$ where source of nonlinearity was variation of relativistic mass of electrons and positrons and low-frequency ponderomotive forces where latter emerged due to interaction of photons with plasma slow motion. There authors derived cubic nonlinear Schrödinger equation to study the modulation instabilities and nonlinear structures where ponderomotive force of photons leads to the depression of density. These results are also consistent with the nonlinear analysis of e-p plasma presented in [17, 18] where nonlinear term appears in the NLSE for the symmetric $\left( \text{e}^+ + \text{e}^- \right)$ plasma.

Razzaq et al, [19] studied the formation of nonlinear structures in ion-temperature-gradient driven drift waves in p-i plasma system by assuming $T_{-} = T_{+}, m_{-} = m_{+}$, where $T_{-} (T_{+})$ and $m_{-} (m_{+})$ represent temperature and mass of negative (positive) ions respectively.

Recently Ehsan et al, [7] reported a new acoustic-like mode which exists only for asymmetric or non isothermal p-i plasma system when $T_{-} \ll T_{+}$. The real and imaginary parts are given as below:

$$\omega_{r} = \frac{k \left( \frac{T_{e}}{m_{e}} \right)^{1/2}}{\left( 1 + k^{2} \lambda_{D+}^{2} \right)^{1/2}}$$

and

$$\omega_{i} = -\frac{8 \pi}{\sqrt{8}} \frac{k \left( \frac{T_{e}}{m_{e}} \right)^{1/2} \left( m_{e} / m_{i} \right)^{1/2}}{\left( 1 + k^{2} \lambda_{D+}^{2} \right)^{1/2}} \left[ 1 + \left( \frac{T_{e}}{T_{+}} \right) \left( m_{e} / m_{i} \right)^{1/2} \exp \left( -\frac{1}{2} \frac{T_{+}}{T_{e}} \left( 1 + k^{2} \lambda_{D+}^{2} \right) \right) \right]^{1/2}$$

where $\omega_{\pm} = \left( \frac{4 \pi \nu_{e,i}}{m_{e}} \right)^{1/2}$ is the plasma frequency. This is a heavy ion and low frequency branch of longitudinal oscillations in the pair-ion plasma system where dynamics of the heavier negative ions play the dominant role. For $|\omega_{i}| \ll \omega_{r}$, the temperature ratio of the plasma components give:

$$\left( \frac{T_{e}}{T_{+}} \right)^{3/2} \exp \left( -\frac{T_{e}/T_{+}}{2(1 + k^{2} \lambda_{D+}^{2})} \right) \ll 1$$

On the other hand, enormous literature on various type of drift wave instabilities to solve plasma confinement problem exists for such instabilities are considered to be responsible for anomalous energy transport and enhanced particle diffusion. For instance in tokamak and space plasmas much attention has been given towards ion (electron) temperature gradient (I/E)TG for their vital role in the ions (electrons) anomalous thermal transport. ITG mode arises due to inhomogeneities in temperature and density owing to the linear coupling of ion sound and drift wave. Needless to mention initially ITG mode instability was investigated for uniform density plasma [20], later on density inhomogeneity and magnetic shear were also incorporated [21].
In this manuscript we plan to see how a new longitudinal mode identified for the pair ion plasma couples with the drift waves and excites nonlinear stationary structures. This will be intriguing to see how inhomogeneity in temperature, density and then asymmetries in mass (temperature) of the negative and positive ions impact ITG instability. Since temperature and mass asymmetries will enhance the complexities in the system and the associated differential equations therefore their significant role on the ITG instability growth rate is also expected.

Here we shall study different situations, first when there are asymmetries in both temperature and mass of the positive and negative ions [7]. Then we shall see how presence of electrons’ population along with asymmetries in positive and ions as indicated by Saleem [22] and Verheest [15] can affect the instability and subsequent nonlinear structures. We will also examine the above cases for Kappa and Cairns distributed plasmas [23, 24].

To the best of the authors’ knowledge, this has not been studied earlier and the results of the present investigation are useful to study nonlinear structures driven by ITG modes.

This paper is organized in the following manner: In section 2, the basic set of equations for the ITG mode in the asymmetric pair ion plasma have been presented. Sections 3 and 4 deal with the derivation of linear dispersion relation for Maxwellian pure asymmetric p-i plasma and with the presence of electrons, respectively. Whereas nonlinear analysis for both the cases has been carried out in the subsections. Sections 4 provides linear and nonlinear analysis of ITG mode for the non-Maxwellian p-i plasma, while in section 5, numerical results are presented and summarized with a brief discussion of main results.

2. Basic set of equations

We consider a system of collisionless plasma which contains s species embedded in an inhomogeneous external magnetic field \( \mathbf{B}_0 = B_0(x) \hat{z} \) (where \( \hat{z} \) is the unit vector along the z-axis). The gradients in temperature \( (\nabla T_{\omega}(x)) \) and density \( (\nabla n_{\omega}(x)) \), are assumed to be along the x-axis.

The momentum equation for the sth species can be written as

\[
m_s n_s \left( \frac{\partial}{\partial t} + \mathbf{v} \cdot \nabla \right) \mathbf{v}_s = -Z_s e n_s \left( \mathbf{E} + \frac{1}{c} \mathbf{v} \times \mathbf{B}_0 \right) - \nabla (n_s T_s)
\]

where \( \mathbf{E} = -\nabla \phi \) (\( \phi \) is the electrostatic wave potential), the subscripts \( s \) stands for species \( (e \) for electrons, \( \pm \) for positive (negative) ions), \( m_s, Z_s, T_s \), and \( n_{\omega} \) are the mass, the charge state, temperature and number density of the sth species, respectively. The perpendicular fluid velocity for low-frequency \( \omega \ll \omega_i \), electrostatic waves (where \( \omega_{is} = Z_i e B_0 / m_i c \) is the standard gyrofrequency) can be written as \( v_{\perp1} \approx v_{\parallel1} B_0 \), where, \( v_{\parallel1} = \frac{m_s}{e} \frac{\mathbf{E} \times \mathbf{B}}{B_0} \times (n_s T_s) \) are the well known \( \mathbf{E} \times \mathbf{B} \), and diamagnetic drift velocities, respectively.

The continuity equation is given as

\[
\frac{\partial n_s}{\partial t} + \nabla \cdot (n_s \mathbf{v}_s) = 0,
\]

the parallel component of ion fluid velocity can be expressed as

\[
m_s n_s \left( \frac{\partial}{\partial t} + \mathbf{v} \cdot \nabla \right) v_{\parallel s} = -Z_s e n_s \frac{\partial}{\partial z} \phi - \frac{\partial}{\partial z} (n_s T_s)
\]

whereas the energy balance equation is given as:

\[
\frac{3}{2} n_s \left( \frac{\partial}{\partial t} + \mathbf{v} \cdot \nabla \right) T_s + n_s T_s (\nabla \cdot \mathbf{v}_s) = -\nabla \cdot \mathbf{q}_s
\]

where \( \mathbf{q}_s \) is the Righi-Leduc heat flux which is defined as \( \left( 5 \pi n_s T_s / 2 Z_i e B_0 \right) \mathbf{E} \times \nabla T_s \).

To close our system of equations, we shall use the Poisson’s equation given by

\[
\nabla^2 \phi = -4\pi Z_s e n_{\omega1}
\]

Now to derive nonlinear set of mode coupling equations for the ITG driven drift mode in this case we write

\[
n_{+1} = n_{+} - n_{+\omega}(x)
\]

\[
T_{+1} = T_{+} - T_{+\omega}(x)
\]

where \( n_{+} (x) \ll n_{-\omega} \) and \( T_{+}(x) \ll T_{-\omega} \) are perturbed (equilibrium) number density and temperature, respectively.

As the singly charged negative, positive ions and electrons are assumed to be in thermal equilibrium, so we can describe their densities by the well-known Boltzmann distribution expression [25]:
\[ n_+ (\phi) = n_{+0} \exp \left( -\frac{e\phi}{k_B T_+} \right) \]  

(8)

for the case when electrons, singly charged negative and positive ions are taken as nonthermal, their number densities for Kappa distributed electrons and singly charged negative and positive ions can be expressed as [23]:

\[ n_+ (\phi) = n_{+0} \left\{ 1 + \left( \kappa - \frac{3}{2} \right) \frac{e\phi}{k_B T_+} \right\}^{-\kappa+1/2} \]  

(9)

and for the case of Cairns distributed electrons and singly charged negative and positive ions densities are given by [23, 24]

\[ n_+ (\phi) = n_{+0} \left\{ 1 + \Gamma \left( \frac{e\phi}{k_B T_+} \right) + \Gamma \left( \frac{e\phi}{k_B T_+} \right)^2 \exp \left( -\frac{e\phi}{k_B T_+} \right) \right\} \]  

(10)

Here, \( \Gamma = 4\alpha/(1 + 3\alpha) \) comprise of Cairns parameter \( \alpha \) that determines the population of nonthermal particles. It is important to mention here that if we consider \( \kappa \rightarrow \infty \) and \( \Gamma \rightarrow 0 \) in equations (9)–(10) respectively, then we can directly replicate the Maxwell-Boltzmann density distribution for electrons, positively and negatively charged ions.

3. Case 1: (negative and positive ions)

First we consider a case when there is a temperature and mass asymmetry in a two component plasma consisting of positive and negative ions, for this it is assumed that \( T_+ (m_+) \) is the temperature (mass) of lighter and heavier ions. In this case, we assume \( T_+ \gg T_- \) and \( m_- \gg m_+ \) so dynamics of heavier negative ions (given by equations (5)–(7)) will be taken into account while lighter ions are assumed to behave as Boltmanian.

Using drift-approximation, which is valid for low-frequency waves (\( \omega \ll \omega_{ci} \)), the negative ion continuity, parallel component of ion momentum and energy balance equation for the \( - \) ion species (equation (5)–(7)) can be expressed in the normalized form as:

\[ (\zeta_t + v_{b-} \cdot \nabla)N_- + Z_- \tau_-(v_{b-} - v_{n-}) \cdot \nabla \Phi + v_{b-} \cdot \nabla T_- + \frac{\partial}{\partial z} v_{z-} = 0 \]  

(11)

and

\[ (\zeta_t + v_{b-} \frac{\partial}{\partial z}) v_{z-} = -v_{z-}^2 \tau_+ \frac{\partial}{\partial z} (Z_+ \Phi + (N_- + T_-)) \]  

(12)

\[ (\zeta_t + \frac{5}{3} v_{b-} \cdot \nabla) T_- = \frac{2}{3} \zeta_t N_- - Z_- \tau_-(\eta_- - \frac{2}{3}) v_{n-} \cdot \nabla \Phi = 0 \]  

(13)

where \( v_{b-} = -\frac{e\text{\vec{e}} \cdot \text{\vec{B}}}{Ze_B} \times \nabla \ln B_\text{in} \) in (11) and (13) represents the inhomogeneity in magnetic field through drift velocity, \( \zeta_t = \partial_t + (v_{b+} + v_{b-}) \cdot \nabla + v_{b+} \partial_\tau, \Phi = e\phi/T_+ \), \( v_{z-}^2 = T_+/m_- \) and \( \tau_- = T_+/m_- \).

3.1. Linear dispersion relation

To obtain a linear dispersion relation, we shall drop all nonlinear terms and assume that the perturbed quantities \( N_- \), \( T_- \), \( v_{z-} \) and \( \Phi \) are proportional to \( \exp[i(k_y y + k_z z - \omega t)] \), where \( \omega \) and \( (k_y, k_z) \) are perturbation frequency and wave vectors in the \( y \) and \( z \) directions, then we obtain from equations (11)–(13)

\[ (\omega - \omega_{b-}) N_- - Z_- \tau_-(\omega_{b-} + \omega_{b+}) \Phi - \omega_{b-} T_- - k_z v_{z-} = 0 \]  

(14)

and

\[ \omega v_{z-} = v_{z-}^2 \tau_+ k_z (Z_- \Phi + (N_- + T_-)) \]  

and

\[ (\omega - \frac{5}{3} \omega_{b-}) T_- - \frac{2}{3} \omega N_- + Z_- \tau_-(\eta_- - \frac{2}{3}) \omega_{b-} \Phi = 0 \]  

(16)

where \( \omega_{b-} = \text{\vec{k}} \cdot \text{\vec{v}}_{b-} \), and \( \omega_{n-} = \text{\vec{k}} \cdot \text{\vec{v}}_{n-} \). Eliminating \( T_- \) and \( v_{z-} \) from equations (15) and (16) and substituting in equation (14), we get

\[ \omega \left( \omega - \frac{5}{3} \omega_{b-} \right) \left( \omega - \omega_{b-} - \frac{k_z^2 v_{z-}^2}{Z_\tau \omega} \right) - \omega \left( \omega - \frac{5}{3} \omega_{b-} \right) Z_- \tau_-(\omega_{n-} + \omega_{b-}) + \omega Z_- \tau_-(\omega_{n-} - \omega_{b-}) \left( 2 - \eta_- \right) - \left( \omega - \frac{5}{3} \omega_{b-} \right) k_z^2 v_{z-}^2 + k_z^2 v_{z-}^2 \left( \eta_- - \frac{2}{3} \right) \omega_{n-} = 0 \]  

(17)
Equation (17) is the third order dispersion relation of ion-temperature-gradient drift mode for a plasma containing only positive and negative ions.

### 3.2. Nonlinear solution

A possible stationary solution of nonlinear set of equations (11)–(13) can be obtained by introducing a new frame \( \xi = y + \alpha z - ut \), where \( \alpha \) and \( u \) are constant (\( \alpha \) the angle between xy plane and wave front normal while \( u \) is speed of nonlinear structure) and we assume that all the normalized perturbed quantities like \( \Phi, N_\perp, v_{-z} \), and \( T_\perp \) are only functions of \( x \) and \( \xi \) variables. In this new frame, the ion energy balance equation (13) yields the following result,

\[
T_\perp = \frac{2}{3} N_\perp - \tau_\perp \left( \eta_\perp - \frac{2}{3} v_{-z} \right) \Phi
\]

where \( \tilde{v}_{-z} = v_{-z} \cdot \hat{\xi} \).

Now, we transform the parallel component of ion fluid velocity equation (12), using the stationary frame coordinate transformation we get

\[
\left( -u \frac{\partial}{\partial \xi} + \frac{e}{B_0} \times \nabla \phi \cdot \nabla \right) v_{-z} + v_{-z} \alpha \frac{\partial}{\partial \xi} v_{-z} = \frac{2}{3} N_\perp \tau_\perp (Z_\perp \Phi + \Phi + T_\perp)
\]

substituting the value of \( T_\perp \)

\[
-\frac{u}{\partial \xi} v_{-z} + \frac{e}{B_0} \left( \frac{\partial \phi}{\partial x} \frac{\partial \phi}{\partial \xi} v_{-z} - \frac{\partial \phi}{\partial \xi} \frac{\partial \phi}{\partial \xi} v_{-z} \right) + \frac{\partial}{\partial \xi} v_{-z} = \frac{2}{3} N_\perp \tau_\perp (Z_\perp \Phi + \Phi + T_\perp)
\]

after simplifying we obtain the following result

\[
v_{-z} = \frac{\alpha v_\perp^2}{u} \left[ \left( 1 - \left( \eta_\perp - \frac{2}{3} \right) \right) \tilde{v}_{-z} - \frac{5}{3} \tilde{v}_{-z} \right] + \frac{\alpha^3 v_\perp^4}{2u^3} \left[ \left( 1 - \left( \eta_\perp - \frac{2}{3} \right) \right)^2 \tilde{v}_{-z} + \frac{5}{3} \tilde{v}_{-z} \right]
\]

similarly equation of continuity takes the following form

\[
\frac{\partial \Phi}{\partial \xi} - \frac{v_{-z}}{u} \frac{\partial \Phi}{\partial \xi} \tilde{v}_{-z} - \frac{\tau_\perp N_\perp}{u} \frac{\partial \Phi}{\partial \xi} \tilde{v}_{-z} - \frac{\tau_\perp N_\perp}{u} \frac{\partial \Phi}{\partial \xi} \tilde{v}_{-z} = 0
\]

substituting the value of \( v_{-z} \) and solving this system of nonlinear equation we obtain

\[
a_1 \frac{\partial \Phi}{\partial \xi} + a_2 \frac{\partial \Phi}{\partial \xi} = 0
\]

which represents shock like drift wave. Here

\[
a_1 = 1 \left( \frac{1}{u} \frac{v_{-z}}{u} - \frac{1}{u} \tilde{v}_{-z} - \frac{1}{u} \tilde{v}_{-z} \right) \left[ \left( 1 - \left( \eta_\perp - \frac{2}{3} \right) \right) \tilde{v}_{-z} + \frac{5}{3} \tilde{v}_{-z} \right] + \frac{\alpha v_\perp^2}{u} \left[ \left( 1 - \left( \eta_\perp - \frac{2}{3} \right) \right) \tilde{v}_{-z} + \frac{5}{3} \tilde{v}_{-z} \right]
\]

\[
a_2 = \frac{\alpha^3 v_\perp^4}{2u^3} \left[ \left( 1 - \left( \eta_\perp - \frac{2}{3} \right) \right)^2 \tilde{v}_{-z} + \frac{5}{3} \tilde{v}_{-z} \right] + \frac{\alpha v_\perp^4}{2u^3} \left[ \left( 1 - \left( \eta_\perp - \frac{2}{3} \right) \right)^2 \tilde{v}_{-z} + \frac{5}{3} \tilde{v}_{-z} \right]
\]

### 4. Case 2 (presence of electrons)

Observed greater value of the ion acoustic mode frequency than what was expected in the experiments performed by Ohara [12] lead the scientists [22] believe presence of electrons which are not fully filtered out from the chamber. Therefore we discuss a case when electrons are also present in the pair plasma. For this \( T_\perp > T_\parallel, T_\perp \parallel ; m_e < m_i ; T_\perp < T_\parallel \).

Using drift-approximation, which is valid for low-frequency waves \( (\omega \ll \omega_{pe}) \), the pair-ion continuity equation, the parallel component of ion momentum equation and energy balance equation for the sth-species of ions can be expressed in the normalized form as:

\[
(\zeta_i + v_{\parallel i} \cdot \nabla)N_i + Z_i n_i (v_{\parallel i} - \bar{v}_m) \cdot \nabla \Phi + v_{\parallel i} \cdot \nabla T_i + \frac{\partial}{\partial z} v_{-z} = 0
\]
\[
\left( \zeta + v_{nz} \frac{\partial}{\partial z} \right) v_{nz} = -c_s^2 \frac{\partial}{\partial z} (Z_i \Phi + \tau^{-1}_{\nu} (N_i + T_i)) \tag{22}
\]
\[
\left( \zeta + \frac{2}{3} v_{Bz} \cdot \nabla \right) T_z = -\frac{2}{3} \zeta N_i - Z_i \tau_i \left( \eta_i - \frac{2}{3} \right) v_{nz} \cdot \nabla \Phi = 0 \tag{23}
\]
\[
\nu_{Bz} = -\frac{c_{De}}{Z_i e B_z} \times \nabla \ln B_z, \quad \nu_{Bz} = -\frac{c_{De}}{Z_i e B_z} \times \nabla \ln n_{\text{eo}}, \text{are the } \nabla B_z \text{ and ion } \nabla n_{\text{eo}} \text{ drifts, respectively. Here,}
\]
\[
\eta_i = \frac{L_{ni}}{T_i}, \text{with } L_{ni} = 1/|\partial / \partial \ln n_{\text{eo}}| \text{ and } L_{T_i} = 1/|\partial / \partial \ln T_{\text{eo}}| \text{ represent the equilibrium ion density and}
\]
\[
\text{temperature gradient scale lengths, respectively. The normalized parameters used in the above set of}
\]
equations (21) to (23) are defined as: \( N_i = n_{i,0}/n_{\text{eo}}, T_i = T_{i,0}/T_{\text{eo}}, \Phi = e \phi / T_{\text{eo}} \) represents the perturbed number density, temperature, electrostatic potential, respectively. Furthermore, \( \tau_i = T_{\nu}/T_{\text{eo}} \) and \( c_{is} = \sqrt{T_i/m_i} \), is the ion-acoustic speed.

### 4.1. Linear analysis

To obtain a linear dispersion relation, we shall drop all nonlinear terms and assume that the perturbed quantities \( N_{i,0}, T_{\nu}, v_{nz} \) and \( \Phi \) are proportional to \( \exp[i(k_y y + k_z z - \omega t)] \), where \( \omega \) and \( (k_y, k_z) \) are perturbation frequency and wave vectors in the \( y \) and \( z \) directions. Assuming plane-wave solution, equations (21)–(23) can be re-written as

\[
(\omega - \omega_{Bz}) N_i - Z_i \tau_i (\omega_{Bz} + \omega_{Bz}) \Phi - \omega_{Bz} T_i - k_z v_{nz} = 0 \tag{24}
\]
\[
\omega_{v_{nz}} = \frac{c_s^2}{Z_i} k_z (Z_i \Phi + \tau_{\nu}^{-1} (N_i + T_i)) \tag{25}
\]
\[
(\omega - \frac{5}{3} \omega_{Bz}) T_z - 2 \frac{3}{3} \omega_{Bz} N_i + Z_i \tau_i \left( \eta_i - \frac{2}{3} \right) \omega_{Bz} \Phi = 0 \tag{26}
\]

Here, we define \( \omega_{Bz} = k \cdot v_{Bz} \) and \( \omega_{v_{nz}} = k \cdot v_{nz} \). Eliminating \( T_i \) and \( v_i \) from equations (25)–(26), we get

\[
\left( \omega - \omega_{Bz} - \frac{k_z^2 c_s^2}{Z_i \tau_i \omega} \right) N_i = \left[ Z_i \tau_i (\omega_{v_{nz}} + \omega_{Bz}) - \frac{Z_i \tau_i \omega_{Bz} \omega_{Bz}}{\omega - \frac{3}{5} \omega_{Bz}} \right] \left( \eta_i - \frac{2}{3} \right) \Phi + \frac{k_z^2 c_s^2}{Z_i \omega} \left( 1 - \frac{Z_i (\eta_i - \frac{2}{3}) \omega_{Bz}}{\omega - \frac{3}{5} \omega_{Bz}} \right)
\]
\[
(1 + k^2 \lambda_{De}^2) \Phi = Z_i N_i \left( \frac{n_{i,0}}{n_{\text{eo}}} \right) - Z_i N_i \left( \frac{n_{e,0}}{n_{\text{eo}}} \right) \tag{27}
\]

where \( \lambda_{De} = \sqrt{v_{\text{De}}/4 \pi n_{\text{eo}} e^2} \) is the electron Debye length. Equation (27) relates the ion number density of the \( i \)-th species with the normalized electrostatic potential. To close the system of equations, we may use the normalized form of Poisson’s equation expressed by (28). Solving equations (27) and (28) for pair-ion plasma, we obtain the following dispersion relation under the assumption that the perturbation wavelength is much larger than the electron Debye shielding length such that \( k \lambda_{De} \ll 1 \), and obtain the following relation,

\[
Z_i n_{i,0} (Z_i n_{i,0}) \Phi = Z_i n_{i,0} \left( \frac{n_{i,0}}{n_{\text{eo}}} \right) - Z_i n_{i,0} \left( \frac{n_{e,0}}{n_{\text{eo}}} \right)
\]

where \( Z_i n_{i,0} (Z_i n_{i,0}) \) satisfies the quasineutrality condition of the plasma such that we can write
\( Z_i n_{i,0} - Z_i n_{e,0} = n_{\text{eo}} \), for pair-ion electron system. Here \( \alpha \Phi = n_{i,0}/n_{\text{eo}} \) and \( \beta = (1 - Z_i \alpha / Z_i) \). Equation (29) represents a sixth order dispersion relation and its analytical solution is extremely difficult to
obtain. We shall therefore discuss numerical results in section 4. We shall first present here, some interesting limiting cases in the following sub-section.

4.1.1. Case-1
First, we consider a uniform density and magnetic field case, for which we can take $\omega_{n+} = \omega_{n-} = 0$; $\omega_{B+} = \omega_{B-} = 0$, and $\eta_1 = \eta_2 = 0$. If we further assume that the phase velocity of wave lies in the rage of ion-acoustic and ion thermal speed such that: $v_0^2 < \omega^2/k^2 < c_0^2$, we obtain the following dispersion relation:

$$\omega^2 = k^2 [\alpha_0 c_{i+}^2 + \beta c_i^2]$$ (30)

where $\alpha_0$ and $\beta$ are defined earlier. The above dispersion relation represents a modified coupled ion-acoustic mode with variable mass, charge state and number density of positive and negative ions.

4.1.2. Case-2
Secondly, we consider a nonuniform pair-ion plasma having equilibrium density and temperature gradients such that $\omega_{n+} = 0$, $\eta_1 = 0$. To investigate coupled ion-temperature-gradient- driven drift mode with ion-acoustic mode for pair-ion plasma, we assume the phase velocity of the wave in the range $c_0^2 > \omega^2/k^2 > v_0^2$ and keeping magnetic field uniform for simplification, we obtain

$$\omega^3 = k^2 \left[ \frac{\alpha_0}{\beta} c_{i+}^2 \omega + \frac{\beta}{\beta} c_i^2 \omega \right] + Z \frac{\alpha_0}{\beta} \left( \omega^2 \tau_s Z c_{i+}^2 \omega + c_{i+}^2 k^2 \omega - \frac{2}{3} \right)$$

$$- Z \frac{\beta}{\beta} \left( \omega^2 \tau_s Z c_i^2 \omega + c_i^2 k^2 \omega - \frac{2}{3} \right)$$

which is third order dispersion relation, essentially representing a coupled ion-acoustic type mode with ion-temperature-gradient driven drift mode. If we ignore the linear stabilizing term in $\omega$ and assume $c_{i+}^2 k^2 \omega \left( \eta_1 - \frac{2}{3} \right) \gg \omega^2 \tau_s Z \omega$ then the above dispersion relation reduces to a cubic type dispersion relation.

$$\omega^3 = + Z \frac{\alpha_0}{\beta} \left( c_{i+}^2 k^2 \omega \left( \eta_1 - \frac{2}{3} \right) - Z \frac{\beta}{\beta} \left( c_i^2 k^2 \omega - \frac{2}{3} \right) \right)$$ (32)

The threshold condition of instability for the ion temperature gradient driven unstable mode would be when $Z \frac{\beta}{\beta} \left( c_{i+}^2 \omega - \frac{2}{3} \right) > Z \frac{\alpha_0}{\beta} \left( c_i^2 \omega + \left( \eta_1 - \frac{2}{3} \right) \right)$ as well as $\omega_{n-} \left( \eta_1 - \frac{2}{3} \right) > \omega_{n+} \left( \eta_1 - \frac{2}{3} \right)$. Under these conditions, the $\eta_1$ mode would be destabilized with a growth rate

$$\gamma = \sqrt{3} \left[ Z \frac{\beta}{\beta} \left( c_{i+}^2 k^2 \omega - \frac{2}{3} \right) - Z \frac{\alpha_0}{\beta} \left( c_i^2 k^2 \omega + \left( \eta_1 - \frac{2}{3} \right) \right) \right]^{1/3}$$ (33)

4.2. Nonlinear analysis
In the following section, we shall present a possible stationary solution of nonlinear set of equations (21)–(23) by introducing a new frame $\xi = y + \alpha x - ut$, where $\alpha$ and $u$ are constant and we assume that all the normalized perturbed quantities like $\Phi$, $N_i$, $v_{iz}$ and $T_i$ are only functions of $x$ and $\xi$ variables. In this new frame, the ion energy balance equation (23) yields the following result,

$$T_i = \frac{2}{3} N_i - \tau_s \left( \eta_1 - \frac{2}{3} \right) \frac{u_{in} \Phi}{u}$$ (34)

where $\vec{V}_{in} = v_{in} \cdot \vec{y}$.

Now, we transform the parallel component of ion fluid velocity equation (23), using the stationary frame coordinate transformation and obtain the following result

$$v_{iz} = \frac{\alpha_0 c_{i+}^2}{u} \left[ 1 - \left( \eta_1 - \frac{2}{3} \right) \frac{\vec{V}_{in}}{u} \right] \Phi + \frac{5}{3} \tau_s N_i$$

$$+ \frac{\alpha_0 c_{i+}^2}{u} \left[ 1 - \left( \eta_1 - \frac{2}{3} \right) \frac{\vec{V}_{in}}{u} \right] \Phi + \frac{5}{3} \tau_s N_i$$

Similarly the ion continuity equation can be written as,

$$\left( \partial_t + \frac{c}{B_0} \times \nabla \phi \cdot \nabla \right) n_{iz} = - \frac{c}{B_0} \times \nabla n_{io} \cdot \nabla \phi + n_{io} \frac{\partial}{\partial_z} v_{iz} = 0$$
For pair-ion plasma, which can also be re-written as

\[
d_i(Z_{n+1} - Z_{n-1}) - \frac{c}{B_p} \nabla (Z_{n+1} - Z_{n-1}) \cdot \nabla \phi + \frac{\partial}{\partial t} (Z_{n+1} - Z_{n-1}) = 0
\]  

(36)

where \(d_i = \partial_i + \frac{c}{B_p} (2 \times \nabla \phi \cdot \nabla)\). Using the Poisson’s equation and Boltzmann distribution for the electrons, the above equation can approximately be written as,

\[
\left[ \frac{\alpha^2}{u^2} \left( Z_i c_{ij}^2 \left( 1 - \left( \eta - \frac{2}{3} \right) \frac{V_{ni}}{u} \right) - Z_{-} \beta \right) \right] + \frac{u_{ne}}{u} - 1 \frac{\partial \Phi}{\partial \xi} = 0
\]

(37)

Since in most of laboratory produced plasmas, the mass of \(m_+\) and \(m_-\) ions is approximately same, i.e., \(m_+ = m_-\). Furthermore, the temperature of pair-ion is slightly different i.e., \(T_+ \approx T_-\). Hence, we are considering a special case in which \(m_- > m_+\), \(T_+ > T_-\), such that \(T_+/m_+ > T_-/m_-\), thus we can write \(T_+/m_+ = (1 + \epsilon)T_- / m_-\), whereas, in most of the experiments \(\epsilon > 0\) but is a small number. To obtain an approximately good result, in the last term of equation (37), we take \(\epsilon\) very small we finally obtain,

\[
a_1 \frac{\partial \Phi}{\partial \xi} + a_2 \frac{\partial \Phi}{\partial \xi} + a_3 \frac{\partial \Phi}{\partial \xi}^3 = 0
\]

where

\[
a_1 = \left[ 1 + \frac{u_{ne}}{u} - \frac{\alpha^2}{u^2} \left( \frac{5}{3} c_{ij}^2 + Z_i c_{ij}^2 \left( 1 - \left( \eta - \frac{2}{3} \right) \frac{V_{ni}}{u} \right) - Z_{-} \beta \right) \right]
\]

and

\[
a_2 = \frac{u_{ne}}{u}
\]

\[
a_3 = \left( \frac{5 \alpha^2}{3 u^2 c_{ij}^2} - 1 \right) \lambda_{D_e}^2
\]

The above equation is the well known Korteweg–de Vries (KdV) type equation which admits a localized pulse type soliton solution such that

\[
\Phi = \Phi_m \text{sech} \left[ \frac{\xi}{\Delta} \right]
\]

(38)

where \(\Phi_m = 3u/a_{21}, \Delta = \sqrt{4a_{31}/u}\) with \(a_{21} = a_2 / a_1\) and \(a_{31} = a_3 / a_1\). The above solution would be valid for \(a_{31} > 0\), which means \(1 + \frac{u_{ne}}{u} - \frac{\alpha^2}{u^2} \left( \frac{5}{3} c_{ij}^2 + Z_i c_{ij}^2 \left( 1 - \left( \eta - \frac{2}{3} \right) \frac{V_{ni}}{u} \right) - Z_{-} \beta \right) \left( 1 - \left( \eta - \frac{2}{3} \frac{V_{ni}}{u} \right) \right) > 0\).

In this paper, we have obtained ITG driven solitons for pair-ion plasma with electrons follow Boltzmann type distribution.

5. Case 3 (non maxwellian pair ion electron plasma)

Solving equations (5)–(7) for pair-ion plasma, we obtain the following dispersion relation under the assumption that the perturbation wavelength is much larger than the electron Debye shielding length such that \(k \lambda_{D_e} \ll 1\), and obtain the following relation,
Solving the system of non Maxwellian pair ion electron plasma, we obtain

\[
\left( \omega - \frac{5}{3} \omega_{B+} \right) \left( \omega - \frac{5}{3} \omega_{B-} \right) \left( Z \tau_{+} \omega^2 - Z \tau_{+} \omega_{B+} \omega - k_{+}^2 c_{+}^2 \right) (Z \tau_{-} \omega^2 - Z \tau_{-} \omega_{B-} \omega - k_{-}^2 c_{-}^2) = \left\{ \begin{array}{l}
\frac{\tau^2}{Z \alpha_o} \left( \omega_{B+} + \omega_{B-} \right) \left( \omega - \frac{5}{3} \omega_{B+} \right) \omega - \frac{\tau^2}{Z \alpha_o} \omega_{B+} \omega_{B+} \left( \eta_+ - \frac{2}{3} \right) \omega^+ \\
\frac{k_{+}^2 c_{+}^2 \tau_{+}}{Z \alpha_o} \left( \omega - \frac{5}{3} \omega_{B+} - Z \tau_{+} \omega_{B+} \left( \eta_+ - \frac{2}{3} \right) \right) \\
(\omega_{B+} - Z \tau_{+} \omega_{B+} \omega - k_{+}^2 c_{+}^2) \left( \omega - \frac{5}{3} \omega_{B+} \right)
\end{array} \right.
\]

\[
\left\{ \begin{array}{l}
\frac{\tau^2}{Z \alpha_o} \left( \omega_{B+} + \omega_{B-} \right) \left( \omega - \frac{5}{3} \omega_{B-} \right) \omega - \frac{\tau^2}{Z \alpha_o} \omega_{B+} \omega_{B-} \left( \eta_+ - \frac{2}{3} \right) \omega^+ \\
\frac{k_{+}^2 c_{+}^2 \tau_{+}}{Z \alpha_o} \left( \omega - \frac{5}{3} \omega_{B-} - Z \tau_{+} \omega_{B-} \left( \eta_+ - \frac{2}{3} \right) \right) \\
(\omega_{B-} - Z \tau_{+} \omega_{B-} \omega - k_{+}^2 c_{+}^2) \left( \omega - \frac{5}{3} \omega_{B-} \right)
\end{array} \right.
\]

where \( Z \alpha_o (Z \alpha_o) \) satisfies the quasineutrality condition of the plasma such that we can write \( Z \alpha_o - Z \alpha_o = n_0 \) for pair-ion electron system. Here \( \alpha_o = n_+/n_0 \) and \( \beta_0 = (1 - Z \alpha_o) \). Equation (39) represents a sixth order dispersion relation and its analytical solution is extremely difficult to obtain. We shall therefore discuss numerical results in later section. We shall first present here, some interesting limiting cases in the following sub-section.

5.1. Limiting cases

5.1.1. Case-1

First, we consider a uniform density and magnetic field case, for which we can take \( \omega_{B+} = \omega_{B-} = 0 \); \( \omega_{B+} = 0 \), and \( \eta_+ = \eta_- = 0 \). If we further assume that the phase velocity of wave lies in the rage of ion-acoustic and ion thermal speed such that: \( v_{th}^2 < \frac{\omega^2}{k^2} < c_o^2 \), we obtain the following dispersion relation:

\[
\omega^2 = k^2 \left[ \frac{\alpha_o}{\beta} c_{+}^2 + \frac{\beta_o}{\beta} c_{-}^2 \right]
\]

where \( \alpha_0 \) and \( \beta_0 \) are defined earlier. The above dispersion relation represents a modified coupled ion-acoustic mode with variable mass, charge state and number density of positive and negative ions.

5.1.2. Case-2

Secondly, we consider a nonuniform pair-ion plasma having equilibrium density and temperature gradients such that \( \omega_{th} = 0 \); \( \eta_+ = 0 \). To investigate coupled ion-temperature-gradient- driven drift mode with ion-acoustic mode for pair-ion plasma, we assume the phase velocity of wave in the range \( c_{+}^2 > \frac{\omega^2}{k^2} > c_{th}^2 \) and keeping magnetic field uniform for simplification, we obtain

\[
\omega^3 = k^2 \left[ \frac{\alpha_o}{\beta} c_{+}^2 \omega + \frac{\beta_o}{\beta} c_{-}^2 \omega \right] + Z \frac{\alpha_o}{\beta} \left( \omega^2 \tau_{+} Z \omega_{th} + c_{+}^2 k_{+}^2 \omega_{th} \left( \eta_+ - \frac{2}{3} \right) \right) \\
- Z \frac{\beta_o}{\beta} \left( \omega^2 \tau_{-} Z \omega_{th} + c_{-}^2 k_{-}^2 \omega_{th} \left( \eta_- - \frac{2}{3} \right) \right)
\]

which is third order dispersion relation, essentially representing a coupled ion-acoustic type mode with ion-temperature-gradient driven drift mode.

5.2. Nonlinear solution

Solving the system of non Maxwellian pair ion electron plasma we finally obtain

\[
a_1 \frac{\partial \Phi}{\partial \xi} + a_2 \frac{\partial \Phi}{\partial \xi^3} + a_3 \frac{\partial \Psi}{\partial \xi^3} = 0
\]

where \( a_1 = \beta \left( \frac{5 \omega_{th}}{3 \omega_{th}^2} - 1 \right) + \frac{u_{th}}{u} - \frac{1}{\mu} Z \omega_{th} \left( \eta_+ - \frac{2}{3} \right) \left( \frac{u_{th}}{u} \right) - \frac{Z \beta u_{th}}{u} \left( \eta_- - \frac{2}{3} \right) \left( \eta_- - \frac{2}{3} \right) - \frac{a_2}{a_3} \mu \) and

\[
a_2 = \frac{5 \omega_{th}}{3 \omega_{th}^2} - 1 \right) \lambda_{th}^2.\text{ The above equation is the well known Korteweg–de Vries (KdV) type equation which admits a localized pulse type soliton solution such that}

\[
\]
where $\Phi_m = 3u/a_{21}$, $\Delta = \sqrt{4a_{31}/u}$ with $a_{31} = a_3/a_1$ and $a_{31} = a_3/a_1$. The above solution would be valid for $a_{31} > 0$, which means

$$\left[ \beta \left( \frac{5\nu T_e}{3\nu m_e} - 1 \right) + \frac{\nu m_e}{u} + \frac{\nu m_e}{u} \right] \left[ Z_+ \alpha_0 c_+^2 \left( 1 - \left( \eta_+ - \frac{2}{3} \frac{\nu m_e}{u} \right) \right) \right] > 0$$

In this paper, we have obtained ITG driven solitons for pair-ion plasma with electrons following Boltzmann type distribution.

6. Numerical results and discussion

Now we present quantitative analysis of the results, for that we choose $n_i = 2 \times 10^7$ cm$^{-3}$, $B_0 = 0.3$ G, $T_e = 1$ eV, $T_+ = 0.3$ eV, $1.5 T_- = T_e$ [8]. We have solved the linear dispersion relation given by equation (39) numerically and the effect of growth rate of ion-temperature-gradient (ITG) instability on pair ion plasma with nonthermal electron distribution has been studied in figures 1–4. Imaginary and real parts of the growth rate of frequency

![Figure 1](image-url)

(a) Plot of imaginary part of frequency versus $k_z$ for different values of $\eta_+ = \eta_- = \eta = 3, 5, 9$, keeping other parameters fixed. (b) Plot of real part of frequency versus $k_z$ for different values of $\eta_+ = \eta_- = \eta = 3, 5, 9$, keeping other parameters fixed.
versus $k_z$ have been given in figures 1(a)–(b) for different values of $\eta$. It is evident that by varying $\eta$, the growth rate of instability increases. Sharp increase in the temperature as well as number density gradient scale lengths, would destabilize the plasma system which is also in good agreement with several experimental results [26] is also valid for H-mode in tokamak plasmas. For the destabilization, some free source of energy must be present, here ion-temperature, density and magnetic field gradients act as a free source of energy, which would eventually destabilize the plasma system.

Figure 2 has been plotted for different temperature ratios, that is $T_+ = \frac{T_-}{1.5}$ and $T_- = T_-$, where in the former case, growth rate of ITG instability for pair-ion plasma system is more slight change in the equilibrium temperature of both ion species would affect the growth rate of instability.

Imaginary part of frequency has been plotted for different values of nonthermal kappa and Cairns distributed electrons in figures 3 and 4. The growth rate of ITG instability increases with the increase of the value of kappa. Since at larger values of $\kappa$, kappa distribution function approaches towards Maxwellian therefore we can say that the growth rate of ion-temperature-gradient instability is less for kappa distribution as compared to

Figure 2. Plot of imaginary part of frequency versus $k_z$ for $T_+ = \frac{T_-}{1.5}$ and $T_- = T_-$, keeping other parameters fixed.

Figure 3. Plot of imaginary part of frequency versus $k_z$ with different values of kappa ($\kappa = 3, 4, 10$).
the Maxwellian distribution. Similarly, it is evident that the growth rate of ITG instability increases as we increase the nonthermality parameter $\Gamma$ in Cairns distribution.

For the same parameters used above, the solitary wave structures (equation (35)) in ITG driven drift mode for the pair-ion plasma system have been plotted in Figures 5–7. Figure 5 has been plotted for normalized electrostatic potential $\Phi$ as a function of $\xi$ in $(x, \xi)$ plane for different ratio of ion to electron number density. The amplitude of solitary wave structure increases as we increase the ratio of ion to electron number density. This indicates that as this ratio increases, shielding would not be perfect and potential of some order would be leaked out showing an increase of amplitude of solitary wave structure. Figures 6–7 depicts the variation of soliton amplitude for different values of nonthermality parameters $\kappa$ and $\Gamma$ where amplitude of soliton increases by increasing the values of $\kappa$ and $\Gamma$.

Figure 4. Plot of imaginary part of frequency versus $k_z$ with different values of gamma ($\Gamma = 0.20, 0.30, 0.40$).

Figure 5. Effect of ratio of ion to electron number densities ($\alpha_o = 0.75, 0.85, 0.95$) on soliton.
Data availability statement

The data that support the findings of this study are available upon reasonable request from the authors.

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