Relative locality in $\kappa$-Poincaré

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Abstract
We show that the $\kappa$-Poincaré Hopf algebra can be interpreted in the framework of curved momentum space leading to relative locality. We study the geometric properties of the momentum space described by $\kappa$-Poincaré and derive the consequences for particle propagation and energy–momentum conservation laws in interaction vertices, obtaining for the first time a coherent and fully workable model of the deformed relativistic kinematics implied by $\kappa$-Poincaré. We describe the action of boost transformations on multi-particle systems, showing that the covariance of the composed momenta requires a dependence of the rapidity parameter on the particle momenta themselves. Finally, we show that this particular form of the boost transformations keeps the validity of the relativity principle, demonstrating the invariance of the equations of motion under boost transformations.

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(Some figures may appear in colour only in the online journal)

1. Introduction
The problem of quantum gravity is one of the most elusive in modern physics. It has been repeatedly suggested that radical new ideas might be needed to tackle it; in particular, we might be approaching the limit of applicability of Riemannian geometry in the description of spacetime. A ‘bottom-up’ approach to this problem might begin by attempting to describe some particular regimes in which the quantum properties of the geometry of spacetime are under control. A very interesting regime is the one in which all the gravitational degrees of freedom are integrated away, leading to an effective field theory for the matter fields. Of course, this cannot be done explicitly for the full quantum theory of gravity, but in [1], it has been
done in 2 + 1 dimensions, where gravity can be quantized as a topological field theory and can be coupled to point particles, represented by topological defects. Interestingly, the effective theory describing the dynamics of the particles after integrating away the gravitational degrees of freedom does not look like the dynamics of particles moving on a background spacetime manifold: such a situation is recovered only in the low-energy limit. The matter degrees of freedom are representations of an algebraic structure known as the κ-Poincaré group \[2–4\] (in the (2 + 1)-dimensional version). \(\kappa\) refers to an energy constant, setting the scale at which the effective description of quantum-gravity effects provided by the model should begin to break down. Its role is analogous to that of the Fermi constant \(G_F\) in particle physics, which represents the only coupling of the effective theory of weak interaction formulated by Fermi, and is today understood as coming from a deeper theory, whose constants \((g, m_W)\) combine to form \(G_F\). κ-Poincaré is a ‘quantum deformation’ of the Poincaré group, making it into a so-called quantum group or Hopf algebra \[5\], a structure mathematicians proposed a few decades ago as the geometrical tool to describe the symmetries of a noncommutative space \[6\]. The analysis of \[1\] cannot be repeated, at this stage, in the more physical (3 + 1)-dimensional case, but the κ-Poincaré group can be easily generalized to arbitrary dimensions, and therefore it becomes an interesting tool that is expected to capture some essential features of the low-energy limit of quantum gravity.

A recent series of works \[7, 8\] proposed a new framework, called ‘relative locality’, in which to understand the physics of a quantum-gravity regime characterized by negligible \(\hbar\) and \(G\). In this regime, both quantum and gravitational effects are small, but the limits \(\hbar \to 0\) and \(G \to 0\) are taken so that their ratio is kept fixed, and we still have an energy scale \(M_p \sim \sqrt{\hbar/G}\) governing modifications of standard physics. In relative locality, the fundamental notion is that of momentum space, which is a (pseudo-)Riemannian manifold, which might be curved and have other nontrivial geometrical properties, such as torsion and nonmetricity. Space and time, on the other hand, loose their geometrical status. In particular, the notion of locality becomes observer-dependent: the fact that two events take place at the same spacetime point is not an absolute concept, and can only be established by observers close to the events themselves (relativity of locality). The relative locality proposal resulted from a deepening in the understanding of the fate of the locality principle in quantum-gravity-motivated generalizations of Special Relativity \[9–12\].

The relation between the ‘relative locality’ regime and the one considered in \[1\] is apparent, and therefore it is natural to explore the relationship between the relative locality framework and κ-Poincaré. Interestingly, several authors over the last decade have suggested the interpretation of the group manifold underlying κ-Poincaré as a curved momentum space (e.g. \[13–15\]), but the physical meaning of that is still unclear.

In this paper, we apply all the machinery developed in \[7\] to the case of κ-Poincaré, identifying the geometrical properties of the momentum space it describes, such as the metric, curvature, torsion and nonmetricity, which reflect into different kinematical and dynamical properties of the motion and interaction of particles. This construction allows us to deduce the physical implications of κ-Poincaré in a simple model whose physical interpretation is clear, which is what has been missing the most since the discovery of this Hopf algebra.

In the next section, we briefly review the physical implications of the geometrical properties of momentum space emerging in the relative locality framework.

In section 3, we show how the translation sector of the κ-Poincaré Hopf algebra can be used to represent the coordinates over a momentum space, establishing a general correspondence between commutative Hopf algebras and the geometric structures introduced in \[7\], in a way that can be applied also to other Hopf algebras.
In section 4, we review the construction of $\kappa$-Poincaré as a momentum space with de Sitter metric and with torsion and nonmetricity. We can then follow the prescriptions given in section 2 to make the connection between the geometrical properties of this momentum space and the physics that it describes. A byproduct of our analysis is the identification of a dispersion relation that is natural from the perspective of the geometry of momentum space (it is the geodesic distance from the origin), and such that the mass corresponds to the particle’s rest energy.

Within this interpretation of $\kappa$-Poincaré, it is possible to show (and we do this in section 5) that Lorentz transformations act nonlinearly on momenta, and, even more curiously, that they have to act differently on different momenta, when they belong to an interaction vertex, in order to keep covariant the total momentum of the vertex. In particular, we find that different particles participating to an interaction vertex transform under the Lorentz transformation with different rapidities, which depend on the momenta of the other particles involved.

In section 6, we show that in this physical framework the relativity principle still holds, in the sense that the equations of motion are invariant under boost transformations. This result is particularly relevant because in the past, the issue of whether $\kappa$-Poincaré implies a breakdown of the relativity principle was subject to debate \cite{16}. Of course, the interest of $\kappa$-Poincaré as an algebra of physical symmetries would be seriously reduced, if it turned out that it implied the breakdown of those symmetries. Our result provides the first explicit example of how the equivalence between inertial observers is realized in the context of relative locality, and it turns out to be realized in a particularly nontrivial way.

In section 7, we identify a structure, related to the tangent space at the origin of momentum space, which reproduces the commutation relations of $\kappa$-Minkowski, a noncommutative spacetime whose symmetries are thought to be described by $\kappa$-Poincaré \cite{4}. This result suggests that such a noncommutative space could emerge upon the quantization of certain (spacetime) degrees of freedom of our model.

2. Preliminaries on the relative locality principle

The relativity of locality is achieved in \cite{7} by assuming the phase space as the fundamental arena where physics takes place, considered as the cotangent bundle to momentum space. Momentum space is assumed to be a pseudo-Riemannian manifold $\Sigma$ which possess a distinguished point $0$ (the origin), a metric $g$ and a connection $\Gamma$, which does not necessarily need to be metric.

Physical observables are related to intrinsic geometric concepts. The mass of a particle is measured by the geodesic distance of the particle’s representing point in momentum space from the origin

$$d^2(p, 0) = m^2. \quad (1)$$

This equation gives the dispersion relation. In this sense, the metric of momentum space is related to the kinematical properties of a single particle\footnote{Note that the dispersion relation depends on the particular choice of coordinate system over the momentum space.}

Dynamics, or the interaction between particles, is related to the connection, since the connection defines the composition law of momenta, $\oplus : \Sigma \times \Sigma \to \Sigma$\footnote{This law is assumed to admit a left and right inverse $\ominus : \Sigma \to \Sigma$, such that $\ominus p \oplus p = p \oplus (\ominus p) = 0$}, through

$$\frac{\partial}{\partial p_\mu} \frac{\partial}{\partial q_\nu} (p \oplus_k q) \bigg|_{p=q=k} = -\Gamma^\mu_{\rho \nu} (k), \quad (2)$$
where $\oplus_k$ is the composition law ‘translated’ at the point $k$:

$$p \oplus_k q = \oplus_k ((\oplus_k p) \oplus (\oplus_k q)).$$  (3)

The antisymmetric part of the connection is the torsion, which measures the noncommutativity of the composition law

$$\left. \frac{\partial}{\partial p_\mu} \frac{\partial}{\partial q_\nu} (p \oplus_k q - q \oplus_k p) \right|_{p=q=k} = -T^{\mu\nu}_\rho (k),$$  (4)

while the curvature measures its nonassociativity

$$\left. \frac{\partial}{\partial p_\mu} \frac{\partial}{\partial q_\nu} \frac{\partial}{\partial r_\rho} ((p \oplus_k q) \oplus_k r - p \oplus_k (q \oplus_k r)) \right|_{p=q=r=k} = R^{\mu\nu\rho}_\sigma (k).$$  (5)

The nonmetricity, defined from the metric and the connection as

$$N^{\mu\nu\rho} = \nabla_\rho g_{\mu\nu} (k),$$  (6)

has been shown [7, 17] to be responsible for the leading order time-delay effect in the arrival of photons from distant sources, which is an effect that is currently under experimental verification [18].

The dynamics of interacting particles is obtained from a variational principle. In the case of a single vertex (interaction among $n$ particles with momenta $p^j, j = 1, \ldots, n$), we need to minimize the following action:

$$S = \sum_j \left[ \pm \int_{\sigma_0}^{\pm \infty} d\sigma \left( -x^\mu_j \dot{p}^j_\mu + N_j (d^2(p^j, Q) - m^2) + z^\mu_j K_\mu (p^1(\sigma_0), \ldots, p^n(\sigma_0)) \right) \right].$$  (7)

The $\pm$ sign is chosen according to whether the $j$th particle is outgoing or incoming. $N_j$ and $z^\mu$ are Lagrange multipliers, but $z^\mu$ also gives the coordinates of the interaction point. $K_\mu (p^1(\sigma_0), \ldots, p^n(\sigma_0))$ may be any combination of all the momenta in the vertex and gives the momentum conservation law, performed with the rules $\oplus$ and $\ominus$. $x^\mu_j$ are the spacetime coordinates of the $j$th particle, and the dot represents the derivative with respect to $\sigma$, a unphysical variable that parametrize the trajectory of the system in phase space. $\sigma_0$ is an arbitrary value of $\sigma$ at which the interaction is assumed to take place.

The constraints given by the variation with respect to the Lagrange multipliers $N$ and $z^\mu$ are

$$d^2(p^j, Q) = m^2,$$  (8)

which is the dispersion relation, and

$$K_\mu (p^1(\sigma_0), \ldots, p^n(\sigma_0)) = 0,$$  (9)

which gives the conservation of energy and momentum in the interaction vertex.

The (bulk) equations of motion resulting from the minimization of the action are

$$\dot{p}^j_\mu = 0, \quad \dot{x}^\mu_j = -N_j \frac{\partial}{\partial p^j_\mu} d^2(p^j, Q).$$  (10)

The first equation expresses the conservation of particle momenta during free propagation. The second one implies that the spacetime worldlines are straight lines, and their speed is $v_k = \frac{\partial}{\partial p^j_\mu} d^2(p^j, Q)/\frac{\partial}{\partial \sigma_0} d^2(p^j, Q)$. In the case of special relativity, the angular coefficient is the relativistic speed $v_k = p_k/\sqrt{p^2 + m^2}$. Note that the Lagrange multiplier $N_j$ simply amounts to a normalization constant for the tangent vector to the trajectory, and has no physical meaning.
The boundary terms give the initial conditions
\[ x^\mu_j(\sigma_0) = z^\mu \frac{\partial}{\partial p^\mu_j} K_\mu(p^1, \ldots, p^n). \] (11)

In the case of special relativity, \( K_\mu(p^1, \ldots, p^n) = \sum_j p^\mu_j \) and all the worldlines simply end up at the interaction point \( z^\mu \). If the nonlinearity of momentum space induces corrections to the composition law of momenta, then the worldlines will have slightly different endpoints. So the interaction does not appear local. Locality is recovered when the observer lies near \( z^\mu \), that is, the interaction takes place near the origin of the coordinate system, so that \( z^\mu \simeq 0 \) and \( x^\mu_j(\sigma_0) \simeq 0 \ \forall \ j \). This expresses the principle of the relativity of locality.

To describe the physical picture perceived by different inertial observers, which are connected by (spacetime) translations and Lorentz transformations, we need the Poisson brackets of dynamical quantities with generators. Then, the transformation law of coordinates is
\[ x^\mu_j = x^\mu_j + a^\nu \{ K_\nu(p^1, \ldots, p^n), x^\mu_j \}, \] (12)

and it is easy to prove that, at the level of the equations of motion, this action effectively corresponds to translating \textit{classically} the interaction point \( z^\mu = z^\mu + a^\mu \).

The translation generator in the case of more than one vertex is not known, and neither is the Lorentz transformation generator, even with a single vertex, if the momentum space is not simply a maximally symmetric space, where isometries are homomorphisms of the composition law:
\[ \Lambda (p \oplus q) = \Lambda (p) \oplus \Lambda (q). \] (13)

From paper [7], it is not clear whether Lorentz transformations are a symmetry of the theory only in this simple case or also in more general cases.

We are going to use the results of this section on the equations of motion to find out how the Lorentz transformations look like in \( \kappa \)-Poincaré, checking that they are indeed symmetries of the dynamics.

3. \( \kappa \)-Poincaré representation on momentum space

Hopf algebras possess in principle a sufficiently powerful structure to specify univocally a manifold with a metric and a flat connection, which does not have to be necessarily the Levi-Civita one, because torsion and nonmetricity are allowed.

The bicrossproduct structure of \( \kappa \)-Poincaré, identified by Majid and Ruegg, allows one to distinguish between the translation sector, whose generators we call \( P_\mu \), from the Lorentz sector, generated by boosts \( N_j \) and rotations \( R_k \). The translation sector can be interpreted as the algebra of functions over a manifold, which can be identified with the momentum space \( \Sigma^1 \), such that the generators \( P_\mu \) assign coordinates to points on the manifold in a certain coordinate system according to
\[ P_\mu(p) = p_\mu. \] (14)

where \( p \) represents a point on the manifold, and \( p_\mu \) its coordinates.

A change of basis in the algebra generated by \( P_\mu \) corresponds to a change of coordinate system on the manifold:
\[ P'_\mu = P_\mu(p) \rightarrow P'_\mu(p) = p'_\mu, \] where \( p'_\mu = p'_\mu(p) \). (15)

The coproduct map is related to the composition rule \( \oplus \) of momentum space points:
\[ \Delta P_\mu(p, q) = (p \oplus q)_\mu. \] (16)
Table 1. Duality between Hopf algebra and momentum space structures.

| Hopf algebra $\mathcal{H}$ | Momentum space $\Sigma$ |
|-----------------------------|--------------------------|
| $\Delta : \mathcal{H} \to \mathcal{H} \times \mathcal{H}$ | $\oplus : \Sigma \times \Sigma \to \Sigma$ |
| $S : \mathcal{H} \to \mathcal{H}$ | $\ominus : \Sigma \to \Sigma$ |
| $\varepsilon : \mathcal{H} \to \mathbb{R}$ | $0 : \mathbb{R} \to \Sigma$ |
| Generators $P_\mu$ | Coordinate system $p_\mu$ |
| Change of basis | Diffeomorphism |

Then, from the coassociativity axiom of Hopf algebras

$$(\Delta \otimes \text{id}) \circ \Delta = (\text{id} \otimes \Delta) \circ \Delta,$$  

(17)

the associativity of the momentum composition rule follows,

$$((p \oplus q) \oplus k) = (p \oplus (q \oplus k)).$$  

(18)

which, in turn, implies the flatness of the connection on the momentum manifold; cf equation (5). The counit can be used to identify the coordinates of the origin of momentum space $0$:

$$P_\mu(0) = \varepsilon(P_\mu),$$  

(19)

in a way that is compatible with the antipode axiom ($\mu$ is the multiplication of the Hopf algebra):

$$\mu \circ (S \otimes \text{id}) \circ \Delta = \mu \circ (\text{id} \otimes S) \circ \Delta = 1\varepsilon,$$  

(20)

if we relate the antipode $S$ with the inversion $\ominus$ in the following way:

$$S(P_\mu)(p) = (\ominus p)_\mu.$$  

(21)

We end up having a neat picture relating the geometric structures on momentum space introduced in [7], and the algebraic structure of the $\kappa$-Poincaré Hopf algebra, which we summarize in table 1. The reason why we were able to obtain this is simple: the momentum composition rule $\oplus$ together with the origin $0$ and the bilateral inverse $\ominus p$ equips the momentum space with an algebra loop structure: a group without the associativity axiom. If $\oplus$ is associative, then we have a group. In particular, we have a Lie group because its elements are points on a manifold. Now, it is well known [5] that Abelian Hopf algebras are dual structures to Lie groups, and they are introduced as algebras of functions over the group. The duality allows us to reconstruct everything about the group from the Hopf algebra and vice versa.

4. Geometric properties of the $\kappa$-Poincaré Hopf algebra

In the previous section, we have shown the relation between the structures of the $\kappa$-Poincaré Hopf algebra and the ones of the associated momentum space. So now we can deduce the physical properties of particles living on this momentum space according to the framework

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7 With Hopf algebras, due to the coassociativity axiom, we are able only to describe momentum spaces with flat connections. If we wanted to find the algebraic structure associated with a momentum space with a non-associative composition law, we would have had to rely on Hopf quasigroups [19].
of relative locality outlined in section 2. To do this, we need to describe in more detail the geometric properties of the momentum space associated with the $\kappa$-Poincaré algebra, specifying the metric, which allows us to deduce the dispersion relation of particles (see equation (1)), and the connection, with its nonmetricity and torsion, which are instead related to particle interactions (see equations (2), (4) and (6)).

For simplicity, we will restrict the calculations to the $(1+1)$-dimensional version of the $\kappa$-Poincaré algebra. The generalization to $3+1$ dimensions is straightforward (and is discussed in section 8). To fix the notation, we report the main properties of $\kappa$-Poincaré in the bicrossproduct basis. The generators satisfy the commutation rules

$$[P, E] = 0, \quad [N, P] = \frac{\kappa}{2} \left(1 - e^{-2E/\kappa}\right) - \frac{1}{2\kappa}P^2, \quad [N, E] = P, \quad (23)$$

where $E$ and $P$ are the translation generators and $N$ is the boost generator. The coalgebra is

$$\Delta E = E \otimes 1 + 1 \otimes E, \quad \Delta P = P \otimes 1 + e^{-E/\kappa} \otimes P, \quad (24)$$

and, finally, the antipodes and counits are

$$S(E) = -E, \quad S(P) = -e^{E/\kappa}P, \quad S(N) = -e^{E/\kappa}N, \quad (25)$$

$$\epsilon(E) = \epsilon(P) = \epsilon(N) = 0. \quad (26)$$

### 4.1. Metric

It was stated several times in the literature that $\kappa$-Poincaré describes a curved momentum manifold [4, 20] and this manifold has been claimed to be a de Sitter space of radius $\kappa$ (see, in particular, [13–15]). Here, we conclusively demonstrate that the metric is indeed that of a de Sitter space$^8$, but in the next sections, we also show that the momentum space is not simply de Sitter because it is endowed with torsion and nonmetricity, which change the connection in such a way that the curvature tensor is zero, unlike what happens in a de Sitter space with the Levi-Civita connection.

To find the metric, we observe that the de Sitter line element in comoving coordinates

$$ds^2 = dE^2 - e^{2E/\kappa} dp^2 \quad (27)$$

is invariant under the action of the $\kappa$-Poincaré boosts (23). In fact, we can exponentialize the action of the boost generators on the momenta, in order to obtain the finite Lorentz transformations, as done in [22] (also see [23]):

$$E' = E + \kappa \log \left[ \left(\cosh \frac{\xi}{2} + \frac{P}{\kappa} \sinh \frac{\xi}{2}\right)^2 - e^{-2E/\kappa} \sinh^2 \frac{\xi}{2}\right],$$

$$p' = \kappa \frac{\left(\cosh \frac{\xi}{2} + \frac{P}{\kappa} \sinh \frac{\xi}{2}\right) \left(\sinh \frac{\xi}{2} + \frac{P}{\kappa} \cosh \frac{\xi}{2}\right) - e^{-2E/\kappa} \cosh \frac{\xi}{2} \sinh \frac{\xi}{2}}{\left(\cosh \frac{\xi}{2} + \frac{P}{\kappa} \sinh \frac{\xi}{2}\right)^2 - e^{-2E/\kappa} \sinh^2 \frac{\xi}{2}}. \quad (28)$$

Then, plugging these expressions into the line element (27), one verifies that it is invariant:

$$ds'^2 = dE'^2 - e^{2E'/\kappa} dp'^2. \quad (29)$$

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$^8$ During the final stages of preparation of this work, we became aware, through a talk given by L Smolin at the meeting Loops11, of an ongoing related project [21] which reached similar conclusions about the metric and connection for the $\kappa$-Poincaré Hopf algebra.
We can also show that the metric is de Sitter in a constructive way, which will also provide a useful coordinate system to do the computations in the following subsection. Consider the change of basis (remember that a change of basis in the algebra corresponds to a change of coordinates on the momentum manifold):

\[ \eta_0 = \kappa \sinh (E/\kappa) + e^{E/\kappa} P^2 / 2\kappa, \]
\[ \eta_1 = e^{E/\kappa} P. \]  
In this basis, the algebra reduces to the Poincaré algebra

\[ [\eta_0, \eta_1] = 0, \quad [N, \eta_0] = \eta_1, \quad [N, \eta_1] = \eta_0, \]  
but the transformation is not one to one, because it can be inverted in two ways:

\[ E_\pm = \kappa \log \left( \frac{\eta_0 \pm \sqrt{\kappa^2 + \eta_0^2 - \eta_1^2}}{\eta_0 \pm \eta_4} \right), \quad P_\pm = \frac{\kappa \eta_1}{\eta_0 \pm \sqrt{\kappa^2 + \eta_0^2 - \eta_1^2}}. \]  
This implies that the coalgebra in the new basis does not close:

\[ \Delta \eta_1 = \eta_1 \otimes e^{E/\kappa} + 1 \otimes \eta_1 \]
\[ \Delta \eta_0 = \eta_0 \otimes e^{E/\kappa} + e^{-E/\kappa} \otimes \eta_0 + \frac{1}{\kappa} e^{-E/\kappa} \eta_1 \otimes \eta_1, \]  
because we are not able to express the \( e^{E/\kappa} \) factor in a unique way as a function of \( \eta_0 \) and \( \eta_1 \).

However, if we introduce now the additional coordinate:

\[ \eta_4 = \kappa \cosh (E/\kappa) - e^{E/\kappa} P^2 / 2\kappa, \]  
we see that, since \( \eta_4 \) can be both positive and negative, it makes the change of basis invertible in a unique way:

\[ E = \kappa \log \left( \frac{\eta_0 + \eta_4}{\kappa} \right), \quad P = \frac{\kappa \eta_1}{\eta_0 + \eta_4}, \]  
and the coalgebra then closes:

\[ \Delta \eta_0 = \frac{1}{\kappa} \eta_0 \otimes (\eta_0 + \eta_4) + \frac{\kappa}{\eta_0 + \eta_4} \otimes \eta_0 + \frac{\eta_1}{\eta_0 + \eta_4} \otimes \eta_1, \]
\[ \Delta \eta_1 = \frac{1}{\kappa} \eta_1 \otimes (\eta_0 + \eta_4) + 1 \otimes \eta_1, \]
\[ \Delta \eta_4 = \frac{1}{\kappa} \eta_4 \otimes (\eta_0 + \eta_4) - \frac{\kappa}{\eta_0 + \eta_4} \otimes \eta_0 - \frac{\eta_1}{\eta_0 + \eta_4} \otimes \eta_1. \]  
We are then able to recognize the \( \eta_a (a = 0, 1, 4) \) generators as the ones associated (from a momentum space perspective) to the embedding coordinates of a two-dimensional de Sitter space of radius \( \kappa \), since they satisfy the constraint:

\[ \eta_0^2 - \eta_1^2 - \eta_4^2 = -\kappa^2. \]  
See figure 1 for a plot of two-dimensional de Sitter spacetime in embedding conformities. It is also possible to show that then \( N \) is the generator of the Lorentz subalgebra \( \text{so}(1, 1) \in \text{so}(2, 1) \) of the isometries of the space:

\[ [N, \eta_0] = \eta_1, \quad [N, \eta_1] = \eta_0, \quad [N, \eta_4] = 0. \]  
If now we induce the metric on the manifold defined by equation (37) from the flat metric in the ambient space, and we change back to the \([E, P]\) coordinates according to (32), we recover the metric (27).

9 The coordinates \([\eta_0, \eta_1, \eta_4]\) were first introduced in [13–15], where a relation between \( \kappa \)-Poincaré and de Sitter space was first conjectured.

8
4.2. Geodesics and particle dispersion relation

Now that we have the metric of the momentum space associated with the κ-Poincaré algebra, we can derive the physical properties of particles living on this momentum space, studying its geodesic equation, the connection, the torsion and the nonmetricity.

In the relative locality framework, the mass of a particle is given by the geodesic distance of the particle’s representing point on momentum space from the origin. So each particle with mass $m$ will live on a curve of constant geodesic distance from the origin, and equation (1) relating mass and geodesic distance gives the particle’s dispersion relation.

The geodesics in a de Sitter space are easily obtained in the embedding coordinates. They are given \[24\] by the intersection of the hyperboloid (37) with the planes passing through the center (in embedding coordinates: \{$\eta_0, \eta_1, \eta_4$\} ≡ \{0, 0, 0\}).

To write the dispersion relation for particles living on this de Sitter momentum space, we need the geodesics that cross the origin O, which in the $\eta_a$ coordinates represent the point \{0, 0, $\kappa$\}\[10\]. So all the geodesics we are interested in are given by the intersections with the planes that contain the $\eta_4$ axis (see figure 2).

The curves with the constant geodesic distance from the origin are obtained through the intersection with the planes that are orthogonal to the $\eta_4$ axis (see figure 3). Their equation in embedding coordinates is

\[\eta_4 = \kappa \cosh (d/\kappa),\]  (39)

where $d$ is the (constant) geodesic distance of the curve. This can be seen by restricting oneself to the plane $\eta_1 = 0$. The equation of the curve obtained by intersecting the de Sitter space with that plane is $\eta_4^2 - \eta_0^2 = \kappa^2$. This curve can be trivially parametrized by the arc length as $(\eta_4, \eta_0) = \kappa (\cosh s, \sinh s)$, and the relationship between the dimensionless arc length and the geodesic distance is $d = s \kappa$, in a de Sitter space of radius $\kappa$.

\[\text{Note how the essential role of the counit is here manifest: we know that the origin in the } \eta_a \text{ coordinate system has these coordinates because } \varepsilon(\eta_0, \eta_1, \eta_4) = \{0, 0, \kappa\}.\]
Figure 2. Geodesics (in red) from the origin of the $\eta_a$ coordinate system in de Sitter space. They are given by the intersections with planes through the $\eta_4$ axis (in blue).

Figure 3. Curves of constant geodesic distance (in blue) from the origin of the $\eta_a$ coordinate system. They are given by the intersections with the planes orthogonal to $\eta_4$ (in red).

Then, in the bicrossproduct coordinates $E, P$, the equation satisfied by constant geodesic distance curves is

\[ d(E, P) = \kappa \text{arccosh} \left( \cosh \left( \frac{E}{\kappa} \right) - e^{E/\kappa} P^2/2\kappa^2 \right). \]  

(40)

So, according to the relative locality construction, this should be taken as the dispersion relation of particles whose momentum space has coordinates and isometries described by the $\kappa$-Poincaré algebra.
Note that the usual proposal for the dispersion relation of $\kappa$-Poincaré is based on the Casimir [2, 25, 26]:

$$\Box \kappa = 4\kappa^2 \sinh^2(E/2\kappa) - e^{E/\kappa} P^2 \equiv 2\kappa(\eta_4 - \kappa),$$

(41)

which is a nonlinear function of our geodesic distance. The difference can be reabsorbed into a nonlinear redefinition of the mass.

An interesting observation following from this analysis is that the geodesic distance naturally selects a definition of mass as the rest energy of a particle. In fact, according to equation (1), the mass satisfies the relation:

$$\cosh(m/\kappa) = \cosh(E/\kappa) - e^{E/\kappa} P^2/2\kappa^2,$$

(42)

so that when $P = 0$, the dispersion relation gives $E = m$, and when $\kappa \to \infty$, the relation reduces to $E^2 - P^2 = m^2$. If instead, as it was customary to do in the literature until now [2, 25, 26], one uses the Casimir (41) as the definition of the dispersion relation, then the rest energy and the mass would be related in a nonlinear way, $4\kappa^2 \sinh^2(E^2/2\kappa) = m^2$.

### 4.3. Connection, torsion, nonmetricity and composition of particle momenta

In section 3, we have derived the properties of momenta composition rules from the properties of $\kappa$-Poincaré translation generators. On the other hand, in section 2 we have stated that in the relative locality framework, momenta composition rules are related to the geometric properties (connection, torsion) of the momentum space. Here, we show explicitly this relation for the $\kappa$-Poincaré momentum space.

From the coassociativity of the coproduct of the $\kappa$-Poincaré generators, which means that the composition rule of momenta is associative (see equation (18)), it follows that the curvature vanishes\footnote{The associativity of the composition law $\oplus$ trivially implies that of the ‘translated’ law $\oplus_k$, so the curvature vanishes everywhere, according to equation (5).}.

The coproduct of the $P$ and $E$ generators, equation (24), can be used to write explicitly the ‘translated’ composition law (3):

$$(p \oplus_k q)_0 = p_0 + q_0 - k_0,$$

$$(p \oplus_k q)_1 = p_1 + e^{(k_0 - p_0)/\kappa} (q_1 - k_1),$$

(43)

which is needed to calculate the connection at an arbitrary point as in equation (2). Then, the expressions of the connection and the torsion are

$$\Gamma^{\mu\nu}_{\rho} = -\frac{\partial}{\partial p_\mu} \frac{\partial}{\partial q_\nu} (p \oplus_k q)_\rho \bigg|_{p=q=k} = \frac{1}{\kappa} \delta^{\mu\nu}_{0\delta^1_{1\delta^1_{\rho}},}$$

(44)

$$T^{\mu\nu}_{\rho}(k) = \frac{1}{\kappa} \delta^{[\mu}_{0\delta^1_{\nu}]_{1\delta^1_{\rho}}}.$$  

(45)

From the connection and the metric, we can derive the nonmetricity:

$$\nabla^\rho g^{\mu\nu} = \partial^\rho g^{\mu\nu} + \Gamma^{\mu\rho}_{\sigma} g^{\sigma\nu} + \Gamma^{\nu\rho}_{\sigma} g^{\mu\sigma} = -\frac{1}{\kappa} \left(2 \delta^\rho_{1\delta^1_{0\delta^1_{\nu}}} + \delta^\mu_{0\delta^1_{\nu}} \delta^\sigma_{1\delta^1_{\rho}} + \delta^\mu_{0\delta^1_{\nu}} \delta^1_{1\delta^1_{\rho}} \right) e^{2E/\kappa}.$$  

(46)

As already noted in section 3, the connection is flat, in the sense that the Riemann tensor vanishes, due to the associativity of the composition law.
5. Lorentz transformations

The translation sector of \( \kappa \)-Poincaré can be interpreted as the algebra of functions over a curved momentum space, while, as we have shown in subsection 4.1, the Lorentz sector generates a subalgebra of isometries on the momentum space.

We have also seen that we can state a correspondence between the de Sitter momentum space defined by \( \kappa \)-Poincaré and the physical properties of particles living on it, but it is still not clear if the isometries on the momentum space represented by the boost generator actually correspond to transformations leaving the dynamics invariant. In particular, the boost transformations need to be covariant also with respect to the composition of momenta.

In this section, we will be actually able to find this covariant action of boosts on composed momenta. A poorly known ‘back-reaction’ of the momenta on the Lorentz sector, found by Majid in [20], is the key to find this action.

Let us define the boost transformations in momentum space (in the bicrossproduct basis) as in equation (28):

\[
\Lambda(x, p) = \left(\begin{array}{c}
p_0 + \kappa \log\left[\cosh\frac{\xi}{2} + \frac{p_1}{\kappa} \sinh\frac{\xi}{2}\right] - e^{-2p_0/\kappa} \sinh^2\frac{\xi}{2} \\
\kappa \left(\frac{\cosh\frac{\xi}{2} + \frac{p_1}{\kappa} \sinh\frac{\xi}{2}}{\cosh\frac{\xi}{2} + \frac{p_1}{\kappa} \sinh\frac{\xi}{2}}\right) - e^{-2p_0/\kappa} \sinh^2\frac{\xi}{2}
\end{array}\right)
\]

where \( \xi \) is the rapidity. Of course, since these transformations preserve the metric (cf subsection 4.1), they also leave invariant the geodesic distance

\[
d(\Lambda(x, p), \Lambda(x, q)) = d(p, q).
\]

Moreover, they close an Abelian group\(^{12}\):

\[
\Lambda(x, \Lambda(y, p)) = \Lambda(x + y, p),
\]

and they reduce to ordinary Lorentz transformations in the limit \( \kappa \rightarrow \infty \):

\[
\Lambda(x, p) = \left(\begin{array}{c}
p_0 \cosh\frac{\xi}{2} + p_1 \sinh\frac{\xi}{2} \\
p_1 \cosh\frac{\xi}{2} + p_0 \sinh\frac{\xi}{2}
\end{array}\right).
\]

We want to find how these transformations act on composed momenta: this allows us to determine how the momenta of various particles that interact in a vertex would appear to a boosted observer. The trivial solution, valid in special relativity, that each momentum transforms independently from the others, of course does not work here, since

\[
\Lambda(x, p \oplus q) \neq \Lambda(x, p) \oplus \Lambda(x, q).
\]

A solution to this problem comes if we exploit this relation found in [20]: momenta on which finite Lorentz transformations act turn out to have a ‘back-reaction’ on them, since they change the rapidity in a momentum-dependent way, which is compatible with the coproduct of momenta, and with the action of Lorentz transformations on momenta themselves. This ‘back-reaction’ is defined as the right action \( \triangleleft \) \( \mathbb{R} \times \Sigma \rightarrow \mathbb{R} \), which in bicrossproduct coordinates reads

\[
\xi \triangleleft p = 2 \arcsinh\left(\frac{e^{-p_0/\kappa} \sinh\frac{\xi}{2}}{\sqrt{\cosh\frac{\xi}{2} + \frac{p_1}{\kappa} \sinh\frac{\xi}{2}} - e^{-2p_0/\kappa} \sinh^2\frac{\xi}{2}}\right).
\]

\(^{12}\)This is the only point in which the \((3 + 1)\)-dimensional case shows some complications with respect to the \((1 + 1)\)-dimensional because the Lorentz group in the \((3 + 1)\)-dimensional case is non-Abelian. However, there are no novelties with respect to special relativity here because the Lorentz subgroup is classical, and \( \kappa \) would not intervene in the composition law for rapidities.
This equation allows us to write the Lorentz transformation of two composed momenta as

$$\Lambda(\xi, q \oplus k) = \Lambda(\xi, q) \oplus \Lambda(\xi \triangleleft q, k).$$

(52)

Then, if we call $q'$ and $k'$ the boosted momenta,

$$(q \oplus k)' = q' \oplus k', \quad q' = \Lambda(\xi, q), \quad k' = \Lambda(\xi \triangleleft q, k),$$

(53)

and this law ensures that both the transformed momenta, $q'$ and $k'$, are still on the mass shell, because $k'$ is just boosted, even if with a $q$-dependent rapidity:

$$d(k', 0) = d(\Lambda(\xi \triangleleft q, k), 0) = d(k, 0).$$

(54)

Equation (52) gives a physical interpretation to the back-reaction, as a peculiar transformation law for the momenta of particles interacting in a vertex. Each particle ends up transforming with a different rapidity, and its rapidity depends on the momenta of the particle with which it interacts.

Interestingly, the transformation law of any number of momenta participating to a vertex is highly asymmetric with respect to the exchange of momenta, and it keeps track of the order in which the momenta enter the vertex.

Considering the Lorentz transformation of three composed momenta, and applying the Lorentz transformation in the two possible orders (thanks to the associativity of $\oplus$, we can forget about the brackets in the 3-momenta sum)

$$\Lambda(\xi, p \oplus q \oplus k) = \Lambda(\xi, p) \oplus \Lambda(\xi \triangleleft p, q) \oplus \Lambda(\xi \triangleleft p \triangleleft q, k)$$

$$= \Lambda(\xi, p) \oplus \Lambda(\xi \triangleleft p, q) \oplus \Lambda(\xi \triangleleft p \triangleleft q, k),$$

(55)

we deduce that the associativity of $\oplus$ implies that the composition law of two consecutive actions of the momenta on the rapidity is

$$\xi \triangleleft p \triangleleft q = \xi \triangleleft p \oplus q,$$

(56)

expressing the covariance of the right action of momenta on rapidities with respect to the momenta composition law.

A few remarks on the boosts $\Lambda(\xi, p)$ and the back-reaction $\xi \triangleleft p$ have been used. As observed in [20], the boost and the back-reaction are defined for every value of the rapidity only if the momentum lies within the upper light cone $d(p, 0) \geq 0$. Otherwise for every other $p$, there exists a finite critical boost $\xi_c$ that makes $p_0 \to -\infty$, and after which the transformation $\Lambda(\xi, p)$ is not defined. Moreover, for every $\xi$, there exists a critical curve in momentum space, which lies outside the upper light cone, on which $\xi \triangleleft p_c \to \pm \infty$, and after which the back-reaction is not defined.

In [20], a physical meaning is attributed to this critical curve. In fact, the commutation relations of the $\kappa$-Poincaré quantum group dual to the algebra are such that the commutator between translations and Lorentz transformations has a singularity on the critical curve. That is interpreted as an infinite uncertainty for certain states of this algebra. The physical meaning of generalizing to quantum operators the parameters of Lorentz transformations or translations connecting different inertial observers has not yet been clarified; therefore, the meaning of these infinite uncertainty states remains mysterious.

The geometric interpretation of the bicrossproduct momentum space suggests that the singularity encountered in [20] might be unphysical. Here, we want to remark that, in the geometric setting provided by relative locality, the critical curve appears to be due only to a coordinate singularity, which is a well-known property of the comoving coordinate system. In fact, these coordinates only cover half of the de Sitter space and the $p$ coordinate diverges over the critical curves, shown in green in figure 4. The de Sitter space is cut in half by the
Figure 4. The comoving coordinate system of de Sitter space. In solid red, there are the $E = \text{const}$ curves, and in dotted red, the $p = \text{const}$ curves. The dispersion curves are in blue. In green, the two critical curves corresponding to the $p = \pm \infty$ coordinate singularity are shown.

two critical curves, and another complementary set of coordinates is needed to cover the other half. This feature of the bicrossproduct basis has already been noted in [13].

So there appears to be nothing special with comoving coordinates, being just a (possibly convenient) choice of coordinate system for a manifold. It would be interesting to compute the dual to the $\kappa$-Poincaré Hopf algebra in the ‘embedding’ basis $\eta_a$, but we leave this for further studies.

An issue with the critical curves is, however, present: the two halves of de Sitter space that are delimited by the two curves are closed under the coproduct. This means that a momentum lying in one of the two halves might have been generated only by the combination of two momenta in the same half. However, the two halves are not closed under Lorentz transformations, and one can move any momentum from one half to another. This has led to speculations regarding a possible breakdown of Lorentz invariance in $\kappa$-Poincaré [27]. We refer the reader to the most recent discussion of the issue, and its possible solution [28].

6. Equivalence between inertial observers

In the previous section, we have seen that when applied to interacting particles, Lorentz transformations act differently on each particle. The rapidity with which they act on each single particle depends on the momenta of the other particles which participate to the vertex and on their order. Here, we show that physics is left invariant by these kinds of Lorentz transformations, showing that the equations of motion (10) and (11) for the particles in the vertex are invariant.

Let us consider a vertex with $n$ interacting particles whose momenta composition law is $K_j = k_1 \oplus \ldots \oplus k_j$, which is boosted with the rapidity parameter $\xi$, and let us call

$$\xi^j \equiv \xi \text{ < } k_1 \oplus \ldots \oplus k_{j-1}.$$
the rapidity with which the $j$th moment boosts, so that
\[ (k^j)' = \Lambda(\xi^j, k^j), \]  
and the conservation law transforms as
\[ K' = \Lambda(\xi_1, k^j) \oplus \ldots \oplus \Lambda(\xi_n, k^n) = \Lambda(\xi, K), \]  
which means that it is invariant ($K = 0 \Rightarrow \Lambda(\xi, K) = 0$). From this and from the invariance of the geodesic distance under boosts, we see that both the constraints (8) and (9) are invariant.

The particle coordinates will transform according to the transformation rule of covectors under diffeomorphisms:
\[ (x^j)' = x^\nu \frac{\partial}{\partial k^\nu} \frac{\partial \Lambda(-\xi^j, (k^j)')}{\partial (k^j)'_\mu}. \]  
The bulk equations of motion (10) are invariant under these transformations:
\[ \dot{k}^j = 0 \Rightarrow (\dot{k}^j)'_\mu = 0, \quad K'_\mu = 0 \Rightarrow K'_\mu = 0, \]  
if the $z$'s transform as
\[ z'_\mu = z^\nu \frac{\partial K^\nu}{\partial K'_\mu}. \]  
Note that $\frac{\partial K^\nu}{\partial K'_\mu} = \Lambda^\nu_{\mu}$, and $\Lambda'_\mu$ is the classical Lorentz transformation of rapidity $-\xi^{13}$.

So, interestingly, it turns out that the vertex coordinates $z^\mu$ transform classically both under translations and under Lorentz transformations. Mathematically, this is a consequence of the fact that both these transformations are identical to the classical ones near the origin of momentum space\(^{14}\), and the fact that $z^\mu$'s transform under diffeomorphisms $p_\mu = f_\mu(p)$ as (see [17])
\[ z'_\mu = z^\nu \left[ \left( \frac{\partial f}{\partial p} \right)^{-1} \right]^{\nu}_{\mu} \bigg|_{p=0}, \]  
where $\left[ \left( \frac{\partial f}{\partial p} \right)^{-1} \right]^{\nu}_{\mu} \bigg|_{p=0}$ is the inverse diffeomorphism calculated at the origin.

A comment on Lorentz transformations between inertial observers. In special relativity, the rapidity of the boost is related to the velocity of one reference frame with respect to the other, irrespective of the particle content of the system under consideration. In our framework, the Lorentz transformations are informed about that content. This does not prevent us to associate a rapidity to the observers: since the back-reaction of the momenta on the rapidity has the group property we have just shown, one can always express the transformation in terms
\[ \frac{\partial \alpha}{\partial \alpha'} \bigg|_{\alpha=0} \]  
which can be easily shown to be equal to a classical Lorentz transform.

\(^{13}\) A fact that we relate to the 'dual equivalence principle' formulated in [7], which states that locally the geometry of momentum space is that of Minkowski.
of the rapidity with which one chosen particle transforms. An inertial system has to be defined in an operational way, e.g. Alice is in the reference system where particle 1 is measured to have momentum $p_1$, and Bob is defined by having measured the momentum of particle 1 to be

$$p'_1 = \Lambda(\xi_1, p_1).$$

Then, one can predict the value of the momenta $p'_2, \ldots, p'_N$ of all the other particles in the process as measured by Bob, knowing their momenta $p_2, \ldots, p_N$ in Alice’s frame. Nothing will be left undetermined. An issue arises, however, if some of the particles in the process cannot be measured. In that case, one is not able to predict some of the $p'_2, \ldots, p'_N$, an issue that is not present in special relativity, where all the momenta transform independently and with the same rapidity. This is an additional difficulty which makes things significantly harder for the experimenter. Interestingly, this might turn out even to be an advantage over ordinary special relativity: in special relativity, one has only limited knowledge of the momenta of the particles that cannot be measured directly, which is the knowledge coming from the kinematical constraints (i.e. missing energy and momentum). In the framework discussed in this paper, one can deduce something more about the unobserved particles by the way the momenta of the observed particles transform under boost.

7. $z_\mu$ coordinates and $\kappa$-Minkowski spacetime

One can use the $\kappa$-Poincaré connection (44) to calculate the parallel transport along a geodesic of an infinitesimal vector $dq$, living in the tangent space to the momentum space at the point $p$, from the point $p$ to the origin [7]:

$$(p \oplus dq)_\mu = p_\mu + dq_\nu (\tau_L)_{\nu \mu}(p), \quad \tau_L(p) = \begin{pmatrix} 1 & 0 \\ 0 & e^{-p_0/\kappa} \end{pmatrix},$$

where $\tau_L$ is the parallel transport matrix, which relates the components of $dq$ at $p$ to its parallely transported components at the origin.

In [7], the authors obtain the coordinates $z^\mu_j$ from the coordinates of the $j$th particle with momentum $p_j$ (therefore living in the tangent space to the momentum space at the point $p$), by parallely transporting them along a geodesic toward the origin of the momentum space:

$$z^\mu_j \equiv x^j_\nu (\tau_L)^{\nu \mu}(p') .$$

so that $z^0 = x^0$ and $z^1 = x^1 e^{-p_0/\kappa}$. These coordinates are not canonical as $x^\mu_j$’s, which close canonical Poisson brackets with $p^\nu_k$:

$$\{x^\mu_j, p^\nu_k\} = \delta^\mu_\nu \delta^j_k.$$

(65)

They instead close a Lie algebra among them

$$\{z^1_j, z^0_k\} = \frac{1}{\kappa} z^1_j \delta_{jk}.$$

(66)

This algebra is the same satisfied by the coordinates of $\kappa$-Minkowski space, which is expected to be the noncommutative spacetime whose symmetries are described by $\kappa$-Poincaré (see [4, 29, 26]).

Equation (67) indicates that relative locality may describe the ‘$\hbar \to 0$ relics’ of the $\kappa$-Minkowski noncommutative spacetime, which should be recovered upon the quantization, transforming the Poisson brackets into commutators and the coordinates $z^\mu_j$ into operators $\hat{z}^\mu_j$:

$$\{z^1_j, z^0_k\} = \frac{1}{\kappa} z^1_j \delta_{jk} \to [\hat{z}^1_j, \hat{z}^0_k] = i \lambda \hat{z}^1_j \delta_{jk},$$

(67)

where $\lambda = \hbar/\kappa$ is a length scale. This provides a hint for the physical interpretation of the $\kappa$-Minkowski algebra, as an algebra of functions over a noncommutative spacetime. One comment on the meaning of this ‘quantization’: since in the relative locality framework
the creation and annihilation of particles are allowed, we expect that a quantization of the model should involve second quantization methods, with the introduction of a Fock space to represent multi-particle states. The operators $\hat{z}_{\mu}^{\nu}$ in (68) still make sense, in this setting, as the particle coordinates of a system of free particles, in which the effects of particle creation and annihilation can be ignored.

8. (3 + 1)-dimensional $\kappa$-Poincaré

Throughout this paper, we have discussed the ($1 + 1$)-dimensional $\kappa$-Poincaré algebra. Here, we show how to generalize to the more physical ($3 + 1$)-dimensional case.

Let us start by noting down the ($3 + 1$)-dimensional $\kappa$-Poincaré algebra ($\mu, \nu = 0, \ldots, 3$, $j, k, l = 1, \ldots, 3$):

\[
\begin{aligned}
[P_{\mu}, P_{\nu}] &= 0, \quad [N_j, P_0] = P_j, \\
[N_j, P_k] &= \delta_{jk} \left( \frac{\kappa}{2} (1 - e^{-2P_0/\kappa}) + \frac{1}{2\kappa} |\vec{P}|^2 \right) - \frac{1}{\kappa} P_j P_k, \\
[N_j, N_k] &= -\epsilon_{jkl} R_l, \quad [R_j, P_0] = 0, \quad [R_j, P_k] = \epsilon_{jkl} P_l, \\
[R_j, N_k] &= \epsilon_{jkl} N_l, \quad [R_j, R_k] = \epsilon_{jkl} R_l;
\end{aligned}
\]

and coalgebra:

\[
\begin{aligned}
\Delta P_j &= P_j \otimes 1 + e^{-P_0/\kappa} \otimes P_j, \quad \Delta P_0 = P_0 \otimes 1 + 1 \otimes P_0, \\
\Delta N_k &= N_k \otimes 1 + e^{-P_0/\kappa} \otimes N_k + \frac{1}{\kappa} \epsilon_{ijkl} P_j \otimes R_l, \\
\Delta R_j &= R_j \otimes 1 + 1 \otimes R_j,
\end{aligned}
\]

and, finally, antipodes and counits

\[
\begin{aligned}
\varepsilon (P_\mu) &= 0, \quad \varepsilon (N_j) = 0, \quad \varepsilon (R_k) = 0, \\
S (P_0) &= -P_0, \quad S (P_j) = -e^{P_0/\kappa} P_j, \\
S (N_j) &= -e^{P_0/\kappa} N_j + \frac{i}{\kappa} \epsilon_{ijkl} e^{P_0} P_j R_l, \quad S (R_k) = -R_k.
\end{aligned}
\]

It is easy to check that the metric

\[
ds^2 = dE^2 - e^{2E/\kappa} |d\vec{p}|^2
\]

is invariant under the infinitesimal Lorentz transformations (first order in the transformation parameters $\xi$ and $\vec{\theta}$ associated, respectively, with the generators $N$ and $\vec{R}$):

\[
\begin{aligned}
E' &= E + \vec{p} \cdot \vec{\xi}, \\
\vec{p}' &= \vec{p} + \vec{\xi} \left( \frac{\kappa}{2} (1 - e^{-2E/\kappa}) + \frac{1}{2\kappa} |\vec{P}|^2 \right) - \frac{\vec{P}}{\kappa} (\vec{\xi} \cdot \vec{p}) + \vec{\theta} \times \vec{p}.
\end{aligned}
\]

The embedding coordinates in the ($3 + 1$)-dimensional case are a trivial generalization of equations (30) and (34):

\[
\begin{aligned}
\eta_0 &= \kappa \sinh (P_0/\kappa) + e^{P_0/\kappa} |\vec{P}|^2 / 2\kappa, \\
\vec{\eta} &= e^{P_0/\kappa} \vec{P}, \\
\eta_4 &= \kappa \cosh (P_0/\kappa) - e^{P_0/\kappa} |\vec{P}|^2 / 2\kappa.
\end{aligned}
\]
Because of the undeformed action of rotations, also the dispersion relation, the expression for the connection,
\[ \Gamma_{\rho}^{\mu \nu} = \frac{1}{\kappa} \sum_{j=1}^{3} \delta_{\rho}^{\mu} \delta_{\nu}^{j} \delta_{\rho}^{j}, \]  
that of the torsion and nonmetricity are the trivial generalizations of the ones reported in section 4, when one assumes invariance under rotations.

What requires more attention when generalizing to 3+1 dimension is the composition of Lorentz transformations and the transformation of a system of particles. The composition of two transformations is modified because in 3+1 dimensions, the Lorentz group is non-Abelian.

The composition of two boosts in 3+1 dimensions is different from a trivial generalization of equation (48) but the modification is the same as the one needed in special relativity when going from 1+1 to 3+1 dimensions (i.e. there is no \( \kappa \)-dependent effect):
\[ \Lambda(\xi_1, \Lambda(\xi_2, p)) = \Lambda(\xi_1 \circ \xi_2, p), \]  
where \( \gamma_{\xi_i} = \frac{1}{\sqrt{1-|\xi_i|^2}} \). A similar formula for two Lorentz transformations in full generality (i.e. including rotations) is not, to our knowledge, available.

Concerning the transformation of a system of particles under the Lorentz sector of \( \kappa \)-Poincaré algebra, we have seen in section 5 that we need to introduce a ‘back-reaction’ of the momenta on the transformation parameters. The reason for this has to be traced back to the nontrivial form of the coproduct of the boost generator. In the following, we discuss what happens in 3+1 dimensions, limiting the discussion to the first order in the transformation parameters \( \xi \) and \( \theta \).

In 3+1 dimensions, the coproduct of boost generators \( N_j \) contains a rotation generator (see the coalgebra reported at the beginning of this section). So we do expect that the back-reaction of momenta on the \( \xi \) transformation parameter also generates a rotation transformation. Indeed, it is possible to show that the Lorentz transformation of a couple of particles with momenta \( p \) and \( q \) is
\[ \Lambda([[\xi, \theta], p \oplus q]) = \Lambda([\xi, \theta], p) \oplus \Lambda([\xi, \theta] \triangleleft p, q), \]  
where
\[ [[\xi, \theta] \triangleleft p = \left\{ e^{-p_0/\kappa} \xi, \theta - \frac{\xi \times \vec{p}}{\kappa} \right\}. \]  
A few comments are in order.

(i) Equation (78) involves a back-reaction of the momenta on the rotations. This could come unexpected to some readers, as the common wisdom regarding \( \kappa \)-Poincaré is that the rotations are undeformed. But equation (78) shows that, in the presence of a boost, the rotation parameter receives some back-reaction by the other momenta entering a vertex. Interestingly, even if the Lorentz transformation acting on the left momentum contains no rotation part (\( \theta = 0 \)), then the back-reaction induces a rotation of the right momentum with the infinitesimal rotation vector \( \frac{1}{\kappa} \xi \times \vec{p} \). This is the physical manifestation of the ‘no-pure-boost’ principle discovered in [26], and clarifies the way it is physically realized in processes.

(ii) Note how the first-order expression of the back-reaction (78) is closely related to the coproduct of the boost and rotation generators (70). The same happens in the 1+1 case, where the back-reaction takes the form \( \xi \triangleleft p = e^{-p_0/\kappa} \xi \) and the second term in the boost
coproduct is $e^{-E/\kappa} \otimes N$. The back-reaction is clearly a consequence of the coproduct of the Lorentz generators. An explicit dictionary translating the Hopf algebra structures of the Lorentz sector into momentum-space structures like the back-reaction, on the lines of table 1 above, is clearly desirable. We would not comment further on the issue, leaving it for future work.

9. Conclusions and outlook

The $\kappa$-Poincaré Hopf algebra has been subject to an intense study since its discovery, almost 20 years ago. It attracted such a large interest because it provides a treatable example of ‘quantum geometry’, described through the language of symmetries. But until now, a coherent picture had not been found, where its physical implications could be unambiguously determined, and a connection with the experiments be made.

In this paper, we established relative locality as the natural paradigm in which one should interpret the implications of $\kappa$-Poincaré for physics. This paradigm comes with a simple and coherent physical model, which allows us to unambiguously determine the new effects that $\kappa$-Poincaré implies, thus finally allowing for the long-sought connection with the experiments.

We determined that $\kappa$-Poincaré determines a relative locality model with a momentum space which metrically is de Sitter, with a radius of curvature $\kappa$. This momentum space has a nonmetric connection with zero curvature, and non-zero torsion and nonmetricity.

These results lead to a fully workable model of point particles interacting with point-like, but ‘relatively-local’, interactions, with a non-symmetric and associative deformed conservation law of momentum.

We showed also the compatibility of the model with the relativity principle and found the form of the Lorentz transformations for every system of particles, while previously this was known only for free particles. This allows us to confront, in real-world scenarios, the observations of different inertial observers, a task that was previously made impossible by the lack of a consistent law of transformation for systems of interacting particles. Interestingly, under boost transformations, on-shell momenta remain on-shell, with the same masses, but when we apply boosts to several interacting particles, the rapidity with which the momentum of each particle is boosted depends on the momenta of the other particles taking part to the interaction.

To achieve our results, we showed the equivalence between the Hopf algebra structures of $\kappa$-Poincaré and the geometric construction that realizes the principle of relative locality. This construction is directly applicable also to other Hopf algebras, and some analyses on the same lines are under development.

An observation. $\kappa$-Poincaré was initially obtained as the Inönü–Wigner contraction of another Hopf algebra, known as q-de Sitter [30–33]. This algebra depends on two constants: $H$ (dimensionful) and $q$ (dimensionless), since it has been introduced as the $q$-deformation of the de Sitter algebra with the radius $H^{-1}$. The contraction to $\kappa$-Poincaré corresponds to the limit $H \to 0, q \to 1$, where $H/\log q \to \kappa$. One can trade $q$ for $\kappa = H/\log q$, which plays the role of an ultraviolet constant, while $H$ is infrared [34]. This suggests an interpretation of q-de Sitter as the symmetry algebra of a system which possesses both curvature in momentum space and in spacetime. An interesting question is whether this situation can be fitted into the relative locality scheme.

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