

Unconventional Bose–Einstein Condensations from Spin-Orbit Coupling

WU Cong-Jun(吴从军)1**, Ian Mondragon-Shem1,2, ZHOU Xiang-Fa(周祥发)3

1Department of Physics, University of California, San Diego, CA 92093
2Instituto de Física, Universidad de Antioquia, AA 1226, Medellín, Colombia
3Key Laboratory of Quantum Information, University of Science and Technology of China, Hefei 230026

(Received 3 August 2011 and accepted by XIANG Tao)

According to the “no-node” theorem, the many-body ground state wavefunctions of conventional Bose–Einstein condensations (BEC) are positive-definite, thus time-reversal symmetry cannot be spontaneously broken. We find that multi-component bosons with spin-orbit coupling provide an unconventional type of BECs beyond this paradigm. We focus on a subtle case of isotropic Rashba spin-orbit coupling and the spin-independent interaction. In the limit of the weak confining potential, the condensate wavefunctions are frustrated at the Hartree–Fock level due to the degeneracy of the Rashba ring. Quantum zero-point energy selects the spin-spiral type condensate through the “order-from-disorder” mechanism. In a strong harmonic confining trap, the condensate spontaneously generates a half-quantum vortex combined with the skyrmion type of spin texture. In both cases, time-reversal symmetry is spontaneously broken. These phenomena can be realized in both cold atom systems with artificial spin-orbit couplings generated from atom-laser interactions and exciton condensates in semiconductor systems.

PACS: 71.35.–y, 73.50.–h, 03.75.Mn, 03.75.Nt

The conventional many-body ground state wavefunctions of bosons satisfy the celebrated “no-node” theorem in the absence of rotation, as written in Feynman’s textbook,[1] which means that they are positive-definite in the coordinate representation. This theorem implies that time-reversal (TR) symmetry cannot be spontaneously broken. It applies to various ground states of bosons including Bose–Einstein condensates (BEC), Mott-insulating states, density-wave states and supersolid states, thus making it a very general statement.

It would be exciting to search for novel types of quantum ground states of bosons beyond the no-node paradigm. This theorem does not apply to spinful bosons with spin-orbit (SO) coupling, whose linear dependence on momentum invalidates Feynman’s proof. Artificial SO coupling from laser-atom interactions has been investigated.[2–9] Excitons in semiconductors[10,11] exhibit SO coupling in their center-of-mass motion.[12,13] In particular, exciting progress has been made in indirect exciton systems in coupled quantum wells where electrons and holes are spatially separate.[14,15] The extraordinarily long lifetime of indirect excitons provides a wonderful opportunity to investigate exciton condensation with SO coupling.

In this Letter, we show that spin-orbit coupled bosons develop unconventional BECs beyond the no-node theorem. A Rashba SO coupled BEC with spin-independent interaction exhibits frustration at the Hartree–Fock level. Quantum zero-point fluctuations select a coherent condensation in the presence of weak spatial inhomogeneities, which exhibits spin-density waves and spontaneous TR symmetry breaking. In a strong external harmonic trap, the ground state condensate develops orbital angular momentum, which can be viewed as a half-quantum vortex. Moreover, the spin density distribution exhibits a cylindrically symmetric spiral pattern as skyrmions.

We begin with a 3D two-component boson system with Rashba SO coupling in the xy-plane and with the contact spin-independent interaction, described by

$$
H = \int d^3r \psi_\alpha^\dagger \left\{ -\frac{\hbar^2\nabla^2}{2M} - \mu \right\} \psi_\alpha + \hbar \lambda_R \psi_\alpha^\dagger \left\{ -i\nabla_y \sigma_x + i\nabla_z \sigma_y \right\} \psi_\beta + \frac{g}{2} \psi_\alpha^\dagger \psi_\beta^\dagger \psi_\beta \psi_\alpha, \tag{1}
$$

where $\psi_\alpha$ is the boson operator; the pseudospin indices $\alpha = \uparrow, \downarrow$ refer to two different internal components of bosons; $\lambda_R$ is the SO coupling strength, which carries the unit of velocity; $g$ describes the s-wave scattering interaction. Equation (1) possesses a Kramer-type TR symmetry $T = i\sigma_2 C$ satisfying $T^2 = -1$, where $C$ is the complex conjugate operation and $\sigma_2$ operates on the boson pseudospin degree of freedom.

In the homogeneous system, the single particle states are the helicity eigenstates of $\sigma \cdot (k \times \hat{z})$ with a dispersion relation given by $\epsilon_\pm(k) = \frac{\hbar^2}{2M}([k + k_{SO}]^2 + k_z^2)$, where $k_{SO} = \frac{\hbar \lambda_R}{M}$. The energy minima are located on the lower branch along a ring with the radius $k_{SO}$ in the plane of $k_z = 0$. The corresponding two-component wavefunction $\psi_\pm(k)$ with $|k| = k_{SO}$ can be solved as $\psi_\pm(k) = \frac{1}{\sqrt{2}}(e^{-i\phi_k/2}, e^{i\phi_k/2})^T$, where $\phi_k$ is the azimuth angle of the projection of $k$ in the xy-plane. The interaction part in Eq. (1) in the helicity

---

DOI:10.1088/0256-307X/28/9/097102
basis can be represented as

\[ H_{\text{int}} = \frac{g}{2} \sum_{\lambda \mu \nu} \sum_{p_1 p_2 q} (p_1 + q; \lambda|p_1; \rho)(p_2 - q; \mu|p_2; \nu) \times \psi_\lambda^\dagger(p_1 + q)\psi_\mu(p_2 - q)\psi_\nu(p_2)\psi_\rho(p_1), \]  

where the Greek indices \( \lambda, \nu, \mu, \rho \) are the helicity indices; the matrix elements denote the inner product of the spin wavefunctions of two helicity eigenstates at different momenta, e.g., \( (p_1 + q; \lambda|p_1; \rho) = \frac{1}{2}[1 + \lambda\rho e^{i\phi_{n_1} - i\phi_{n_1^+}}] \).

For part I, we define boson operators in the lower branch as \( a_q = \psi_\uparrow(-k_{so} \vec{e}_x + q) \) and \( b_q = \psi_\uparrow(k_{so} \vec{e}_x + q) \), and \( (a_q=0) = \sqrt{N_A} \) and \( (b_q=0) = \sqrt{N_B} \), respectively. The low energy excitations in this region have been calculated in Ref.\[9\]. By defining \( \gamma_1^\dagger(q) = \frac{1}{\sqrt{N_A}}(\sqrt{N_A}a_q^\dagger + \sqrt{N_B}b_q^\dagger) \), \( \gamma_2^\dagger(q) = \frac{1}{\sqrt{N_A}}(\sqrt{N_A}a_q^\dagger - \sqrt{N_B}b_q^\dagger) \), the mean-field Hamiltonian, up to the order of \( q^2 \), is expressed as

\[ H_{MF,1} = \sum_q \left\{ E(q)\gamma_1^\dagger(q)\gamma_1(q) + g_{n_0}\gamma_1^\dagger(q)\gamma_1^\dagger(-q) + \text{H.c.} \right\} + E(q)\gamma_2^\dagger(q)\gamma_2(q), \]  

where \( E(\pm q) \approx h(q_x^2 + q_y^2)/(2M) \) up to the order of \( O(q^4/k_{so}) \). The Bogoliubov modes mixing \( \gamma_1^\dagger(q) \) and \( \gamma_1(-q) \) exhibit the spectrum of \( \hbar\omega(q) = \sqrt{E_q(E_q + 2g_{n_0})} \approx \sqrt{\frac{\hbar g_{n_0}}{\sqrt{2}}}q^2 + \frac{q_x^2 + q_y^2}{2} \). This is the phonon mode describing the overall density fluctuations, which exhibits linear dispersion relation for \( q \) in the \( xx \)-plane and becomes soft for \( q \) along \( \vec{e}_x \). The \( \gamma_2 \) mode represents the relative density fluctuations between two condensates which describes spin wave excitations. This mode remains a free particle spectrum \( E(q) \). Both the \( \gamma_1, \gamma_2 \) modes only depend on the total condensation fraction \( N_6 \). Hence, the contribution from part I does not lift the degeneracy between different partitions of \( (N_A, N_B) \) up to the quadratic order of \( q \).

Next we turn to the Bogoliubov spectra in part II where \( \psi_\uparrow(k) \) is degenerate with \( \psi_-(-k) \) but not with

\[ \left| \sqrt{k_x^2 + k_y^2} - k_{so} \right| < \Lambda, \quad |k_z| < \Lambda. \]  

Within this shell, interaction energy is stronger than the kinetic energy, thus particle and hole states are mixed significantly. We further divide this shell into two parts I and II as depicted in Fig. 1. Part I is inside two cylinders with the radius of \( \Lambda \) centering around points \( A, B \), and part II is outside these two cylinders.
The mean-field Hamiltonian reads

\[ H_{MF,2} = \sum_k \psi_+^\dagger(k) \psi_+(k) \{ \epsilon(k) + \frac{g}{2} (n_0 - \Delta n \cos \phi_k) \}. \]

where \( \Delta n = n_a - n_b \). The Bogoliubov spectra can be solved as \( H_{MF,2} = \sum_k \{ \omega(k)(\gamma_0^2(k)\gamma_0(k) + \frac{1}{2}) \} \) with

\[ \omega(k) = \sqrt{\epsilon_K(e_k + g n_0) + \frac{g^2 n_0^2}{4} f(x) + \frac{g n_0}{2} x \cos \phi_k}, \]

where \( x = \Delta n/\Delta \) and \( f(x) = \sin^2 \phi_k + x^2 \cos^2 \phi_k \). The second term in Eq. (8) averages to zero and thus the total zero-point energy in regime II depends on \( n_a \). The radial density profiles of both spin components are chosen as \( \sigma_\alpha(r) = 5 \) for the single particle ground state forms the Kramer doublet as represented in cylindrical coordinates as

\[ \psi_{\alpha} = \left( \frac{f(r)}{g(r)} e^{i \phi} \right), \quad \psi_{\beta} = \left( -\frac{g(r)}{f(r)} e^{-i \phi} \right). \]

The interaction energy scale is defined as \( E_{int} = g N_0/(\pi^2 L_z) \), where \( L_z \) is the system size along the \( z \)-axis and the dimensionless parameter \( \beta = E_{int}/(\hbar \omega_T) \). The GP equation reads

\[ \{ -\hbar^2 \nabla^2 / 2M + \hbar \lambda_R (-i \nabla_y \sigma_{x,\alpha \beta} + i \nabla_x \sigma_{y,\alpha \beta}) + gn(r, \phi) \]

\[ + \frac{1}{2} \frac{\hbar M \omega_T^2 r^2}{\lambda^2} \} \psi_{\alpha}(r, \phi) = E \psi_{\alpha}(r, \phi), \]

where \( n(r, \phi) \) is the particle density. The parameter values are chosen as \( \alpha = 2 \) and \( \beta = 5 \). We show the radial density profiles of both spin components \( |f(r)|^2 \) and \( |g(r)|^2 \) in Fig. 2(a). All of them oscillate, which originate from the low energy ring structure and, thus, are analogous to the Friedel oscillations. The spin density, defined as \( S(r, \phi) \), exhibits a spin texture configuration. Let us first look at its distribution along the \( z \)-axis where the supercurrent is along the \( y \)-direction and the spin lies in the \( xz \)-plane. We show \( S_x(r) = \frac{1}{\pi^2} (|f(r)|^2 - |g(r)|^2) \) and \( S_y(r) = f(r)g(r) \). The radial oscillations of \( |f(r)|^2 \) and \( |g(r)|^2 \) have an approximate \( \pi \) phase shift, which arises from the different angular symmetries. As a result, \( S \) spirals as plotted in Fig. 2(b). The spin density distribution in the whole space can be obtained through a rotation around the \( z \)-axis, which exhibits the skyrmion configuration.
Kramer doublet $\psi_{\pm \frac{1}{2}}$ in Eq. (12) can rotate into one another, thus are degenerate. Therefore, in real experiment systems, if the initial state is prepared with total angular momentum $j_z = 0$, we will obtain a superposition of $\psi_{\pm \frac{1}{2}}$. In addition, if the initial state is prepared with the average $j_z$ per particle $\pm \frac{1}{2}$, say, by cooling down from the fully polarized spin up or down state, then $\psi_{\pm \frac{1}{2}}$ will be reached. If we go beyond the Hartree–Fock level to include the zero-point motion correction, the extra spin-dependent term of Eq. (12) will also lift the above accidental degeneracy.

![Fig. 2.](image)

**Fig. 2.** (a) The radial density distribution of spin up and down components, and the total density distribution in the unit of $\hbar \nu_0$ at $\alpha = 2$ and $\beta = 5$. (b) The skyrmion type spin texture configuration plotted in the $xz$-plane.

![Fig. 3.](image)

**Fig. 3.** The phase diagram boundary of $\beta$ vs $\alpha$ between (I) the half-quantum vortex condensate and (II) spin-spiral condensate. The transition from (I) to (II) breaks rotational symmetry.

So far we have presented two different types of condensations. The half-quantum vortex condensate preserves rotational symmetry, which is stable for weak interaction strengths. Instead, the spin-spiral condensate breaks rotational symmetry, which is favored by interactions. With fixing $\alpha$, a transition between them occurs by increasing the interaction energy scale $\beta$. We have performed the numerical study for the critical line in Fig. 3. We calculate the expectation value of $\langle G | j_z^2 | G \rangle$ of the condensate wavefunction. In regime I, the condensate is chosen as the eigenstate with $j_z = \frac{3}{2}(-\frac{1}{2})$ and thus $\langle G | j_z^2 | G \rangle = \frac{3}{4}$. In regime II, $\langle G | j_z^2 | G \rangle$ deviates from $\frac{3}{4}$. The condensate starts to involve high angular momentum components and thus breaks rotational symmetry. It is qualitatively in the same phase of spin-spiral condensate with cylindrical boundary condition.

The recent research focus of the “synthetic gauge fields” in cold atom systems provides a promising method to observe the above exotic BECs. Other systems are the indirect excitons in 2D coupled double quantum wells. The real space spin configurations of exciton condensations can be detected through photoluminescence from electro-hole recombination. The recent experiment has observed spin-textures of the coherent exciton systems which arise from SO coupling and exhibit a similar pattern shown in Fig. 2.

In summary, we find that bosons with spin-orbit (SO) coupling exhibit complex-valued condensations beyond Feynman’s no-node paradigm. The coherent spin-spiral BEC is realized when interaction energy is dominant, while the half-quantum vortex BEC is stable when the trapping potential is strong. The half-quantum vortex condensate exhibits the skyrmion type spin-texture configuration.

C. W. thanks helpful discussions with L. Butov, L. M. Duan, M. Fogler, J. Hirsch, T. L. Ho, L. Sham, S. C. Zhang and F. Zhou.

*Note added:* After the third version of this paper was posted on arXiv, there appeared two experimental works\cite{20,21} and several theoretical investigations\cite{22–27}.

**References**

1. Feynman R P 1972 Statistical Mechanics, A Set of Lectures (Berlin: Addison-Wesley)
2. Juzeliunas G et al 2008 Phys. Rev. Lett. 100 200405
3. Vaishnav J Y and Clark C W 2008 Phys. Rev. Lett. 100 153002
4. Stanescu T D et al 2007 Phys. Rev. Lett. 99 110403
5. Lin Y J et al 2009 Phys. Rev. Lett. 102 130401
6. Lin Y J et al 2009 Nature 462 628
7. Spielman I B 2009 Phys. Rev. A 79 063613
8. Lin Y J et al 2011 Nature 471 83
9. Stanescu T et al 2008 Phys. Rev. A 78 023616
10. Snoke D W et al 1996 Phys. Rev. B 41 11171
11. Butov L V 2007 J. Phys.: Cond. Matter 19 295202
12. Hakioglu T and Sahin M 2007 Phys. Rev. Lett. 98 166405
13. Yao W and Niu Q 2008 Phys. Rev. Lett. 101 106401
14. Butov L V et al 1994 Phys. Rev. Lett. 73 304
15. Butov L V et al 2002 Nature 418 751
16. Wu C et al 2010 Int. J. Mod. Phys. B 24 311
17. Zhou F 2003 Int. J. Mod. Phys. B 17 2643
18. Larson J and Sjöqvist E 2009 Phys. Rev. A 79 043627
19. Leggett A J 2001 Rev. Mod. Phys. 73 307
20. Madison K W et al 2000 Phys. Rev. Lett. 84 806
21. High A A et al 2011 arXiv:1103.0321
22. Ho T L and Zhang S 2010 arXiv:1007.0650
23. Wang C et al 2010 Phys. Rev. Lett. 105 160403
24. Yip S K 2010 arXiv:1008.2263
25. Zhang Y, Mao L and Zhang C 2011 arXiv:1102.4045
26. Xu Z F et al 2011 Phys. Rev. A 83 053602
27. Kawakami T et al 2011 arXiv:1104.4179