On the Dirac equation for a quark

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Abstract

It is argued from geometrical, group-theoretical and physical points of view that in the framework of QCD it is not only necessary but also possible to modify the Dirac equation so that correspondence principle holds valid. The Dirac wave equation for a quark is proposed and some consequences are considered. In particular, it is shown that interquark potential expresses the Coulomb law for the quarks and, in fact, coincides with the known Cornell potential.

1 Introduction

According to the modern standpoint [1], spacetime theory is any one that possesses a mathematical representation whose elements are a smooth four-dimensional manifold $M$ and geometrical objects defined on $M$. The system of real local coordinates on $M$ is defined as a topological mapping of an open region $U \subset M$ onto the Euclidean four-dimensional space $E_4$. Thus, the Euclidean four-dimensional space $E_4$ is a fundamental structure element of the mathematical formalism of contemporary physics. However, it can be shown that $E_4$ has an underlying structure that is exhibited in the existence of group of transformation that does not coincide with the $SO(4)$ group. In physical space $E_3$ - and, for the sake of comparison, on physical plane $E_2$- we consider the groups of rotations and dilatations with the generators

\[ D = x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y} + z \frac{\partial}{\partial z}, \quad M_1 = z \frac{\partial}{\partial y} - y \frac{\partial}{\partial z} \]

\[ M_2 = z \frac{\partial}{\partial x} - x \frac{\partial}{\partial z}, \quad M_3 = -y \frac{\partial}{\partial x} + x \frac{\partial}{\partial y} \]

in $E_3$ and with generators

\[ D = x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y}, \quad M = -y \frac{\partial}{\partial x} + x \frac{\partial}{\partial y} \]
in $E_2$. We denote these groups by $D \otimes SO(3)$ and $D \otimes SO(2)$, respectively. It can be shown that an element of first group can be parametrized by the real numbers $a, b, c, d$; it is suitable to consider them as a quaternion $q = ai + bj + ck + d$. For the second group, we have two real parameters combined into a complex number $w = u + iv$. It is easy to verify one-to-one correspondence between the algebras quaternions and complex numbers and the $D \otimes SO(3)$ and $D \otimes SO(2)$ groups. The transformations of the $D \otimes SO(3)$ and $D \otimes SO(2)$ groups in $E_3$ and $E_2$ can be represented as

$$R' = qR\bar{q} \quad r' = wrw,$$

where $R = xi + yj + zk$ in the first case and $r = x + iy$ in the second case. When we consider $D \otimes SO(3)$ and $D \otimes SO(2)$ groups as linear spaces it can easily be seen, with the aid of the well-known algebra, that the four-dimensional space of quaternions $Q_4$ and the two-dimensional space of complex numbers, $Q_2$, give spinor representations of the groups in question. These representations are realized as follows:

$$u' = qu, \quad z' = wz, \quad (1)$$

where

$$u = u_1i + u_2j + u_3k + u_4, \quad z = z_1 + z_2i.$$

A remarkable property of this transformations is that there is only one point $u = 0$, $(z = 0)$ that is a stable under the transformations given by (1). When we fix any other point, the transformations reduce to the identity transformation. Another important feature of this transformations (1) is that the Euclidean scalar products in $Q_4$ and $Q_2$

$$(u, u) = uu = u_1^2 + u_2^2 + u_3^2 + u_4^2, \quad (z, z) = zz = z_1^2 + z_2^2$$

are invariant under transformations (1), provided that $q\bar{q} = 1$ and $w\bar{w} = 1$. But this does not mean that $Q_4$ and $Q_2$ are Euclidean spaces because (1) hold. It should be noted, that there is a simple mapping from $Q_2$ to $E_2$ of the form

$$r = z^2.$$

This mapping is known as the Bolin transformation. It can be shown, however, that there is no mapping from $Q_4$ to four-dimensional Euclidean space. Thus, what we usually call four-dimensional Euclidean by the analogy with a three-dimensional physical space is in fact $Q_4$. That there
is no such mapping follows from the fact that real Dirac matrices $\gamma_i$, possessing the properties

$$\gamma_i\gamma_j + \gamma_j\gamma_i = 2\delta_{ij}, \quad i, j = 1, 2, 3, 4$$

do not exist. However, there is mapping from $Q_4$ to $E_3$ which can be defined as

$$R = ui\bar{u}.$$ 

This mapping is known as Hopf mapping. Therefore, the components of a quaternion $u$ are not observable in a direct way, but only through some expressions constructed from these components. In seance, this situation is similar to that with a wave function in quantum mechanics. It is obvious that a three-dimensional sphere in $Q_4$,

$$u_1^2 + u_2^2 + u_3^2 + u_4^2 = \rho^2$$

inherits the properties of the enveloping space. In view of the unusual properties of four-dimensional space and of, respectively, a three-dimensional sphere $S^3$, we consider the investigations associated with the last object.

From the geometrical point of view a three-dimensional sphere is a space of constant positive curvature. The Kepler-Coulomb problem in this space has a long history and was first investigated by Schrödinger [2]. The symmetry properties of the Schrödinger equation for the Kepler-Coulomb problem in a space of constant positive curvature was analyzed by Higgs [3] and by Leemon [4]. On the other hand $S^3$ is the configuration space for a Top. The quantum-mechanical problem for the free motion of a Top was investigated shortly after the creation of quantum mechanics in its moderns form (see for example [5]). It is obvious that connections between these two directions of investigation are very important. Moreover in the 1930s, it was emphasized by Casimir [6] that, from the physical point of view, the notion of a rigid body is as fundamental as the notion of material point. At last, we would like to emphasize that QCD is conceptually a simple theory and that its structure is determined solely by symmetry principles. However, there is no connection between such important phenomena as confinement and quark-lepton symmetry, on one hand, and the first principles of QCD, on the other hand. Despite prolonged and complicated experiments, free quarks have not yet been observed, but it is commonly believed that quarks are true elementary
particles like electrons. Experimentalists gradually arrived at the conclusion that the matter is not in the details of the experiments but rather in the fundamental properties of matter, which were sought under various assumptions. For instance, it is hypothesized that quark confinement can be explained by topological methods that have recently found still a wider use in physics. Nevertheless, the most natural and reliable approach to the problem of confinement must be sought in the possibility of modifying the original Dirac equation to take into account the unusual properties of the quarks.

Summarizing all the facts considered above, we conjecture that the configuration space of a quark coincides with the configuration space for a Top. Within this conjecture, we will derive the Dirac wave equation for a quark and consider some of its properties.

2 Formulation of the Problem

The choices of underlying spacetime manifold can be reasoning as follows. Consider a conventional quantum-mechanical operator of the 4-momentum with components

\[ P_0 = -i\hbar \frac{\partial}{\partial x^0}, \quad P_1 = -i\hbar \frac{\partial}{\partial x^1}, \quad P_2 = -i\hbar \frac{\partial}{\partial x^2}, \quad P_3 = -i\hbar \frac{\partial}{\partial x^3}, \quad (2) \]

where \( x^0 = ct, \quad x^1 = x, \quad x^2 = y, \quad x^3 = z \). Further, the group of translations of spacetime is a finite continuous Lie group with the generators

\[ X_0 = \frac{\partial}{\partial x^0}, \quad X_1 = \frac{\partial}{\partial x^1}, \quad X_2 = \frac{\partial}{\partial x^2}, \quad X_3 = \frac{\partial}{\partial x^3}. \quad (3) \]

Besides, the linear operators (3) are generators of a simply transitive group of transformations \([7]\). In our case it is an Abelian simply transitive group of transformations because

\[ [X_a, X_b] = 0, \quad a, b = 0, 1, 2, 3. \]

The connection between (2) and (3) is obvious. If a simply transitive group is a group of transformations of a certain spacetime, this spacetime is called homogeneous, as this group can transform any point of the spacetime to any a priori given point. To make further analysis more
transparent, we present the general characteristic of homogeneous spacetime manifolds.

As any vector field with components $V^i$ can be associated with the linear operator $X = V^i \partial/\partial x^i$, the operator $X$ is called the vector field. Then, let the vector fields $X_a$

$$X_a = V^i_a \frac{\partial}{\partial x^i} \quad (a = 0, 1, 2, 3)$$

are generators of a simply transitive group of transformations of a four-dimensional spacetime manifold $M$. Indexes from the beginning of the Latin alphabet numerate vectors. In this case

$$[X_a, X_b] = f^c_{ab} X_c,$$  \hspace{1cm} (5)

where $f^c_{ab}$ are structure constants of the group in question. The vector fields $V^i_a$ uniquely determine the system of covector fields $V^a_i$ such that

$$V^i_a V^a_j = \delta^i_j, \quad V^a_i V^i_b = \delta^a_b.$$ \hspace{1cm} (6)

The simply transitive group induces a natural integrable connection $\Gamma$ on $M$ with Christoffel symbols

$$\Gamma^i_{jk} = V^i_a \partial_j V^a_k$$ \hspace{1cm} (7)

and a natural metrics of the Lorentz signature on $M$

$$g_{ij} = \eta_{ab} V^a_i V^b_j, \quad g^{ij} = \eta^{ab} V^i_a V^j_b,$$ \hspace{1cm} (8)

where $\eta_{ab} = \eta^{ab} = \text{diag}(1, -1, -1, -1)$. From (5) and (7) for the torsion tensor and torsion covector of the connection (7) we obtain

$$T^i_{jk} = \Gamma^i_{jk} - \Gamma^i_{kj} = -f^c_{be} V^a_i V^b_j V^c_k, \quad T_i = T^k_{ki} = -f^a_{ib} V^i_b.$$ \hspace{1cm} (9)

From (5) and (8) we have

$$V^i_a V^b_j; i = (f^c_{ab} - \eta_{ad} f^d_{be} \eta^{ce} - \eta_{bd} f^d_{ae} \eta^{ce}) V^j_c,$$ \hspace{1cm} (9)

where semicolon means the covariant derivative with respect to the Levi-Civita connection of the metrics (8) with the Christoffel symbols

$$\{^i_{jk}\} = \frac{1}{2} g^{il} (\partial_j g_{kl} + \partial_k g_{jl} - \partial_l g_{jk}).$$
Knowing generators of the simply transitive group $X_a$ we can find generators $Y_a$ of the mutual simply transitive group by solving the equation $\nabla_i V^j - T^j_{ik} V^k = 0$, where $\nabla_i$ is the covariant derivative with respect to the connection (7). For $X_a, Y_a$ we have $[X_a, Y_b] = 0, \ a, b = 0, 1, 2, 3$.

The manifold $M$ that admits a simply transitive group of transformations and has the metrics (8) will be called the homogeneous space-time manifold. The Minkowski spacetime is a particular case of homogeneous spacetime manifolds. If a homogeneous spacetime defined by a non-Abelian simply transitive group of transformations has a physical meaning, from (2) and (3) it follows that the operator

$$-i\hbar X_0 = -i\hbar V_0^i \frac{\partial}{\partial x^i}$$

will be analog of the operator $P_0$ in the Minkowski spacetime.

3 The Dirac wave equation for a quark

Here we will define a homogeneous spacetime manifold that differs from the Minkowski spacetime by geometrical and topological properties and show that a spacetime manifold of that kind obeys all the required conditions and is of definite interest for the physics of quarks.

In the five-dimensional Minkowski spacetime $M^5_{1,4}$ with Cartesian coordinates $x^A$ (indices denoted by capital letters run through the five values 0, 1, 2, 3, 4) and metrics

$$ds^2 = \eta_{AB} dx^A dx^B = (dx^0)^2 - (dx^1)^2 - (dx^2)^2 - (dx^3)^2 - (dx^4)^2,$$

we will consider the one sheet hyperboloid $H^4$

$$\eta_{AB} x^A x^B = (x^0)^2 - (x^1)^2 - (x^2)^2 - (x^3)^2 - (x^4)^2 = -a^2,$$

where $a$ is the radius of $H^4$, and prove that it is a homogeneous spacetime manifold.

We will use the scalar product $(X, Y) = \eta_{AB} U^A V^B$ for any vector fields $X = U^A \partial_A$ and $Y = V^A \partial_A$ on $M^5_{1,4}$. The vector fields

$$P_A = \delta^C_A \partial_C, \quad M_{AB} = (x_A \delta^C_B - x_B \delta^C_A) \partial_C,$$
where \( x_A = \eta_{AB} x^B \), are generators of the Poincare group of the five-dimensional Minkowski spacetime. All vectors fields \( M_{AB} \) are orthogonal to the radius-vector \( R = x^C \partial_C \), but this is not the case for the vector fields \( P_A \). Representing \( P_A \) as the sum of the component aligned with the direction of the radius vector \( R \) and the component orthogonal to this direction, we obtain the vector fields

\[
M_A = a P_A + \frac{1}{a} (R, P_A) R = (a \delta_A^C + \frac{1}{a} x_A x^C) \partial_C,
\]

which are tangent to \( H^4 \), because from (10), it follows that \((R, M_A) = 0\) at each point of \( H^4 \). The vector fields \( M_A \) and \( M_{AB} \) are generators of the group of conformal transformations of \( H^4 \) because we have

\[
[M_A, M_B] = -M_{AB}, \quad [M_A, M_{BC}] = \eta_{AB} M_C - \eta_{AC} M_B. \quad (11)
\]

Let us now introduce the vector fields

\[
X_0 = M_0, \quad X_1 = M_{14} + M_{23}, \quad X_2 = M_{24} + M_{31}, \quad X_3 = M_{34} + M_{12} \quad (12)
\]

with the components

\[
X_0 = (a + \frac{x_0^2}{a}, \frac{x_0 x_1}{a}, \frac{x_0 x_2}{a}, \frac{x_0 x_3}{a}, \frac{x_0 x_4}{a}),
X_1 = (0, -x_4, -x_3, x_2, x_1),
X_2 = (0, x_3, -x_4, -x_1, x_2),
X_3 = (0, -x_2, x_1, -x_4, x_3).
\]

It is straightforward to see that the vector fields \( X_0, X_1, X_2, \) and \( X_3 \) are continuous and do not vanish at any point of \( H^4 \). Because \((X_a, X_b) = 0\) for \( a \neq b \), \( a, b = 0, 1, 2, 3 \) and

\[
(X_0, X_0) = -(X_1, X_1) = -(X_2, X_2) = -(X_3, X_3) = a^2 + x_0^2,
\]

the vector fields \( X_0, X_1, X_2, \) and \( X_3 \) are linearly independent at each point of \( H^4 \). From (11), it follows that

\[
[X_0, X_i] = 0, \quad [X_i, X_j] = 2 \epsilon_{ijk} X_k, \quad i, j, k = 1, 2, 3,
\]

where \( \epsilon_{ijk} \) is the completely antisymmetric Levi-Civita symbol specified by the equality \( \epsilon_{123} = 1 \). In this way, we have proven that the one sheet hyperboloid (10) admits a simply transitive group of transformations whose
generators are given by (12) and which has only the following nonzero structure constants:

\[ f_{23}^1 = f_{31}^2 = f_{12}^3 = 2. \]  

(13)

Therefore, we will supply \( H^4 \) with a metrics of the type (8) and thus transform \( H^4 \) into the hyperbolic spacetime \( H_{1,3}^4 \). From (9) and (13) it follows that the vector field \( X_0 \) is absolutely parallel with respect to the Levi-Civita connection on \( H_{1,3}^4 \) induced by the vector fields (12). For comparison we note that the vector field \( X_0 = \partial/\partial x^0 \) defined in (2) is also absolutely parallel. Now it is natural to put forward the idea that in the realm of the strong interactions spacetime geometrically can be represented as a one-sheeted hyperboloid (10) in the five-dimensional Minkowski spacetime. A constant \( a \) can be interpreted geometrically as the radius of the three-dimensional sphere \( S^3 \), which is considered here as a space section \( x^0 = 0 \). From the physical point of view we treat \( a \) as the size of region of quark confinement, since a quark is a pointlike particle in the space of constant positive curvature \( S^3 \). For comparison of leptons and quarks we note that free motion of the electron is represented as a straight line in the Euclidean usual space and free motion of the quark is a circumference on the 3d sphere. This correlation between leptons and quarks will be continued with an example of the Coulomb law for these objects. To do this consider the Dirac equation for a quark.

In accordance with the original Dirac equation, we write the Dirac equation in the homogeneous spacetime in the form

\[ \gamma^c P_c \psi = \mu \psi, \]  

(14)

where

\[ \gamma^a \gamma^b + \gamma^b \gamma^a = -2\eta^{ab}, \]

\[ P_c = X_c + \frac{iqa}{\hbar c} A_c - \frac{1}{2} f_c, \quad f_c = f_{ac}^a. \]

Here \( q \) is the charge of a particle, and \( A_c \) are the components of the vector potential of the electromagnetic field in the basis \( X_a \). In \( H_{1,3}^4 \) we also have

\[ \mu = mca/\hbar. \]
For the time being, we do not specify the values of the structure constants of a simply transitive group of transformations of the spacetime $H^4_{1,3}$. In general, $[X_a, X_b] = f_{ab}^c X_c$; therefore, we have

$$[P_a, P_b] = f_{ab}^c P_c + \frac{iqa}{\hbar c} F_{ab},$$

where

$$F_{ab} = X_a A_b - X_b A_a - f_{ab}^c A_c$$

(15)

are the components of the strength tensor of the electromagnetic field in the basis $X_a$. When the wave equation is established it is not difficult to write the equations of electromagnetic field. The Jacobi identity $[P_a[P_b, P_c]] + [P_b[P_c, P_a]] + [P_c[P_a, P_b]] = 0$ results in the first four Maxwell equations

$$X_a F_{bc} + X_b F_{ca} + X_c F_{ab} + f_{ab}^d F_{cd} + f_{bc}^d F_{ad} + f_{ca}^d F_{bd} = 0$$

(16)

To establish the form of other four Maxwell equations, we set $\tilde{F}^{ab} = \frac{1}{2} e^{abcd} F_{cd}$, where $e^{abcd}$ are components of the antisymmetric Levi-Civita unit tensor in the basis $X_a$. Then we can write equations (16) in the following equivalent form

$$X_a \tilde{F}^{ab} + f_a \tilde{F}^{ab} + \frac{1}{2} f_{ab}^b \tilde{F}^{ad} = 0$$

(17)

By analogy, from (17) it follows that the remaining Maxwell equations are of the form

$$X_a F^{ab} + f_a F^{ab} + \frac{1}{2} f_{ab}^b F^{ad} = \frac{4\pi a}{c} j^b,$$

(18)

where $j^b$ are components of the current vector in the basis $X_a$.

Now we will write the Maxwell equations in the three-dimensional vector form. As usual, we put

$$j^a = (c \rho, \vec{j}), \quad A_a = (\varphi, -\vec{A}),$$

$$E_i = F_{0i}, \quad H_i = \frac{1}{2} e_{ijk} F^{jk}, \quad i, j, k = 1, 2, 3.$$  

Then from (13) and (15) we obtain
\[ \vec{E} = -\nabla_0 \vec{A} - \nabla \varphi, \quad \vec{H} = \text{rot} \vec{A} = \nabla \times \vec{A} - 2\vec{A}. \]  

(19)

where

\[ \nabla = (\nabla_1, \nabla_2, \nabla_3), \quad \nabla_0 = X_0, \quad \nabla_i = X_i, \quad i = 1, 2, 3. \]

Considering that \( \text{div} \vec{A} = \sum_{i=1}^{3} \nabla_i A_i \), we can recast the Maxwell equations (17) and (18) into the familiar vector form

\[ -\nabla_0 \vec{H} = \text{rot} \vec{E}, \quad \text{div} \vec{H} = 0, \quad \text{rot} \vec{H} = \nabla_0 \vec{E} + \frac{4\pi a}{c} j, \quad \text{div} \vec{E} = 4\pi a \rho. \]  

(20)

Making use of the commutation relations \( [\nabla_i, \nabla_j] = 2\epsilon_{ijk} \nabla_k \), \( i,j,k = 1, 2, 3 \), we can easily verify the identities

\[ \text{div} \text{rot} = 0, \quad \text{rot} \text{grad} = 0. \]

In addition, we have

\[ \text{div} \text{grad} = \triangle, \]

where \( \triangle \) is the Laplacian on a three-dimensional sphere. Torsion (that is, the non-Abelian character of a simply transitive group of transformations of \( H^4 \)) manifests itself not only in the definition of the operator rot, (21), but also in the identity

\[ (\text{rot} + 1)^2 = -\triangle + 1 + \text{grad} \text{div}. \]

Since the space section of the \( H^4_{1,3} \) is a three-dimensional sphere, it is interesting to show that the Dirac equation (14) is associated with the Schrödinger equation for a spherical Top. To verify this, we will derive eigenvalues \( E \) of the Dirac Hamiltonian in question when there is no electromagnetic field that is, for \( F_{ab} = 0 \). Squaring equation (14) and using (13), we obtain the following equation for \( E \)

\[ E^2 \psi = m^2 c^4 \psi - \frac{c^2 \hbar^2}{a^2} (\triangle + P) \psi, \]  

(21)

where

\[ P = \Sigma_1 \nabla_1 + \Sigma_2 \nabla_2 + \Sigma_3 \nabla_3 \]
and $\Sigma_i = \frac{1}{2} \epsilon_{ijk} \gamma^j \gamma^k$. The operator $P$ has properties analogous to those of the operator rot. In particular, we have

\[(P + 1)^2 = -\Delta + 1\] (22)

Since $\Delta + P = -P(P + 1)$, then

\[E^2 = m^2 c^4 + p(p + 1) \frac{c^2 \hbar^2}{a^2},\]

where $p$ is an eigenvalue of the operator $P$.

To determine eigenvalues of the operator $P$, we consider Hermitian operators acting in the space of solutions to the Dirac equation (14). Generators of the group mutual to the simply transitive group of transformations of the spacetime $H_{1,3}^4$ are given by

\[Y_0 = X_0, \quad Y_1 = M_{14} - M_{23}, \quad Y_2 = M_{24} - M_{31}, \quad Y_3 = M_{34} - M_{12}.\]

This leads to the three Hermitian operators

\[N_i = -\frac{i}{2} Y_i\]

which are analogous to the momentum operators. The remaining three operators,

\[M_i = -\frac{i}{2} (\nabla_i - \Sigma_i) = -\frac{i}{2} (X_i - \Sigma_i)\] (23)

are analogs of the electron angular momentum operators. From (23), it follows that the spin of a particle in question is $\hbar/2$. We have

\[\vec{M} \times \vec{M} = i\vec{M}, \quad \vec{N} \times \vec{N} = i\vec{N}\]

and, in addition,

\[2(M^2 + N^2) = (P + \frac{3}{2})^2 - \frac{3}{4}, \quad 2(M^2 - N^2) = P + \frac{3}{2}.\] (24)

Hence, we have the operator equation

\[2(M^2 + N^2) + \frac{3}{4} = 4(M^2 - N^2)^2.\]
Since, $M^2 = l(l+1)$, $N^2 = k(k+1)$, we find from the operator equation that $l$ and $k$ satisfy the equation

$$2[l(l+1) + k(k+1)] + \frac{3}{4} = 4[l(l+1) - k(k+1)]^2.$$ 

This equation has two solutions, $l = k + \frac{1}{2}$ and $k = l + \frac{1}{2}$, whence it follows that $p = 2[l(l+1) - k(k+1)] - \frac{3}{2} = -2k - 3$. Since $S^3$ has a metric invariant with respect to the isometrical reflection, $p = 2k + 3$ is an eigenvalue too. For the energy, we then have

$$E^2 = m^2 c^4 + n(n+1)\frac{c^2 \hbar^2}{a^2} = m^2 c^4(1 + \frac{\lambda^2}{a^2}),$$

where $n = 2, 3, ...$ and $\lambda = \hbar/mc$. If formula (25) gives the quantum-mechanical value of the energy of the relativistic spherical Top, we can then conclude that, at large $a$, the moment of inertia $I = ma^2$ is also large, so that the angular velocity is small. Therefore, the nonrelativistic limit can be found from the condition $a \gg \lambda$. In the limit of large $a$ it follows from (25) that

$$E = mc^2 + \frac{L^2}{2I},$$

where $L^2 = n(n+1)\hbar^2$ is the angular momentum of the spherical top and $I$ is its moment of inertia. The last formula is in accordance with the classical formula

$$E = \frac{L^2}{2I}$$

for the energy of a Top. Thus, the formula (25) gives energy of rotation.

Let us now consider the Coulomb law. The Coulomb potential can be derived as a solution of the equations of electrostatics, which are invariant under the group of Euclidean motions, including rotations and translations. In the case being considered, we will seek a Coulomb potential in an analogous manner. From (19) and (20), it follows that for a constant electric field $\text{div}\vec{E} = 4\pi a \rho$, $\vec{E} = -\nabla \varphi$, and consequently, $\varphi$ obeys the equation

$$\triangle \varphi = -4\pi a^2 \rho.$$  (26)
As it is well known, the electron Coulomb potential

\[ \phi_e(r) = \frac{e}{r} \]

is the fundamental solution to the Laplace equation \( \Delta \phi = \text{div \ grad} \phi = 0 \). In accordance with our conjecture the Coulomb potential for quarks can be derived as follows. Consider the stereographic projection \( S^3 \) from point \((0,0,0,-a)\) onto the ball \( x^2 + y^2 + z^2 \leq a^2 \):

\[
x^1 = fx, \quad x^2 = fy, \quad x^3 = fz, \quad x^4 = a(1 - f),
\]

where \( f = 2a^2/(a^2 + r^2) \). Then, it follows that the element of length on the three-dimensional sphere can be represented in the form

\[ ds^2 = f^2(dx^2 + dy^2 + dz^2) \]

and hence the Laplace equation on \( S^3 \) can be written as follows

\[ \Delta \phi = f^{-3}\text{div}(f\text{grad} \phi) = 0. \]

We seek the solution to this equation that is invariant under the transformation of the group \( SO(3) \) with generators

\[
x^1 \frac{\partial}{\partial x^2} - x^2 \frac{\partial}{\partial x^1} = x \frac{\partial}{\partial y} - y \frac{\partial}{\partial x}, \quad \text{etc.}
\]

This subgroup of the \( SO(4) \) group is determined by fixing the point \((0,0,0,-a)\). Let us put

\[ \psi = f \frac{1}{r} \frac{d\phi}{dr}. \]

Since

\[ \Delta \phi = f^{-3}(r \frac{d\psi}{dr} + 3\psi) = r^{-2} f^{-3} \frac{d}{dr}(r^3 \psi), \]

then \( r^3 \psi = c_1 = \text{constant} \). Thus, we have

\[ \frac{d\phi}{dr} = c_1 \frac{a^2 + r^2}{2a^2r^2} = c_1 \left( \frac{1}{2r^2} + \frac{1}{2a^2} \right) \]

and hence

\[ \phi_q = c_1 \left( -\frac{1}{2r} + \frac{r}{2a^2} \right) + c_2. \quad (27) \]
Generally speaking, expression (27), that we have derived, coincides with the well-known Cornell potential \([8],[9]\). If we demand that \(\phi_e(a) = \phi_q(a)\), then \(c_2 = e/a\) and the Coulomb law for quarks has the form

\[
\phi_q(r) = q\left(\frac{1}{2r} - \frac{r}{2a^2}\right) + \frac{e}{a},
\]

(28)

where \(q\) is the quark charge.

From our consideration it follows that the idea of rotating matter can be realized in the framework of relativistic quantum mechanics by the Dirac equation (14).

## 4 Conclusion

As the basic wave equation describing the dynamics of quarks, we have proposed the modified Dirac equation (14), which has been written here in homogeneous coordinates. The conclusion that quarks are described by the wave equation different from the conventional wave equation for electrons is quite natural. In fact, it would be strange if the description of such different particles were based on the same equation.

The physical meaning of the confinement phenomenon is tightly connected with idea of a rotating matter and consists in that quarks possess properties of a quantum-mechanical spherical Top. Among other things, this means that the interquark potential expresses the Coulomb law for quarks and, in fact, coincides with the well-known Cornell potential that was first very successfully used by the Cornell group.

For \(a \to \infty\), the theory of electrons can be derived from the theory of quarks, but the electrons are obviously deconfined because in this case, the region of confinement covers the entire Euclidean space. Thus, the symmetry between quarks and leptons has a natural explanation.

Since the kinematics of quarks differs from the kinematics of electrons, decays of hadrons and nuclei such that the energy is conserved, but the momentum is not, are possible. Much has to be done, but the problem is worth efforts as many interesting applications become possible.
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