HIGHER ORDER KIRILLOV–RESHETIKHIN MODULES FOR $U_q(A_n^{(1)})$, IMAGINARY MODULES AND MONOIDAL CATEGORIFICATION

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Abstract. We study the family of irreducible modules for quantum affine $\mathfrak{sl}_{n+1}$ whose Drinfeld polynomials are supported on just one node of the Dynkin diagram. We identify all the prime modules in this family and prove a unique factorization theorem. The Drinfeld polynomials of the prime modules encode information coming from the points of reducibility of tensor products of the fundamental modules associated to $A_m$ with $m \leq n$. These prime modules are a special class of the snake modules studied by Mukhin and Young. We relate our modules to the work of Hernandez and Leclerc and define generalizations of the category $\mathcal{O}$. This leads naturally to the notion of an inflation of the corresponding Grothendieck ring. In the last section we show that the tensor product of a (higher order) Kirillov–Reshetikhin module with its dual always contains an imaginary module in its Jordan–Holder series and give an explicit formula for its Drinfeld polynomial. Together with the results of [20] this gives examples of a product of cluster variables which are not in the span of cluster monomials. We also discuss the connection of our work with the examples arising from the work of [25]. Finally, we use our methods to give a family of imaginary modules in type $D_4$ which do not arise from an embedding of $A_r$ with $r \leq 3$ in $D_4$.

Introduction

In [24], Kirillov and Reshetikhin introduced a family of irreducible finite–dimensional modules for the Yangian of a simple Lie algebra and conjectured a character formula for these representations. Subsequently, these modules were also defined for the quantum loop algebra associated to a simple Lie algebra and these are now called the Kirillov–Reshetikhin (KR) modules. These modules have many nice properties and there is an extensive literature on the subject, which includes their connections with integrable systems, the combinatorics of crystal bases, the fermionic formula [11, 12, 17, 23, 29, 31, 32], and more recently [19, 20, 21] they have been shown to be connected to cluster algebras via the notion of monoidal categorification.

In this paper we are interested in higher order versions of KR–modules for a quantum loop algebra associated to a simple Lie algebra of type $A_n$. To explain this, we recall that the KR–modules are indexed by three parameters $(i, a, r)$ where $i \in [1, n]$, $a$ is a complex number and $r$ a positive integer and the corresponding module is denoted $W_{i,a}^{(r)}$. Equivalently, one can think of these modules as being indexed by $i \in [1, n]$ and the $q$–segment $(a, aq^2, \ldots, aq^{2r-2})$.

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We show that the tensor product $W_{1,a}^{(1)} \otimes W_{1,b}^{(1)}$ is reducible if and only if $b = aq^{\pm 2}$ and also that the module $W_{1,a}^{(r)}$ is the simple socle of the tensor product $W_{1,a}^{(1)} \otimes W_{1,aq^{2}}^{(1)} \otimes \cdots \otimes W_{1,aq^{2r-2}}^{(1)}$. In the higher rank case, it remains true that $W_{i,a}^{(1)} \otimes W_{i,b}^{(1)}$ is reducible if $b = aq^{\pm 2}$ but there are many more points of reducibility. For instance in the case of $A_3$ the module $W_{2,a}^{(1)} \otimes W_{2,aq^{2}+4}^{(1)}$ is also reducible. A complete list of the points of reducibility simple Lie algebras of classical type can be found in [4] (see also [1]).

By a higher order KR–module for the quantum loop algebra of type $A_n$ we mean a module which is indexed by a pair $(i, a)$ where $a$ is an increasing $r$–tuple of integers $(a_1, \ldots, a_r)$, $r \geq 2$ such that $W_{i,a}^{(1)} \otimes W_{i,a}^{(1)}$ is reducible and at least one of $a_j - a_{j-1} \neq 2$. We call the family which includes both the KR–modules and higher order KR–modules at node $i$ as KR–modules associated to $(i, n)$–segments. In fact these modules are very special examples of the prime snake modules which appear in the work of [27, 28]. The results of those two papers are used extensively in our work.

We describe the results of this paper. Following [6] we introduce the notion of two $(i, n)$–segments being in general or special position. We prove that the associated tensor product is irreducible if and only if the segments are in pairwise general position. (In particular, we recover a special case of the result of [28], namely that the modules associated to an $(i, n)$–segments are prime, i.e., cannot be written as a tensor product of nontrivial representations).

This allows us to prove a classification result and a unique factorization result. Namely, suppose that we have an irreducible module for quantum affine $A_n$ whose associated Drinfeld polynomial (see [7]) is trivial at all nodes different from $i$. Then it is prime if and only if the Drinfeld polynomial defines an $(i, n)$–segment. Otherwise it can be written uniquely as a tensor product of modules associated with $(i, n)$–segments in general position.

In Section 3 we study the relation between the higher order KR–modules and the work of Hernandez and Leclerc. In particular given a triple of integers $(i, i, n)$ with $n + 1 = i(i+1)$ we define a full subcategory $\mathcal{C}_{i,n}$ of finite–dimensional representations of the quantum loop algebra of type $A_n$. In the case when $i = n$ this is precisely the category $\mathcal{C}^{-}$ defined in [20]. We show that $\mathcal{C}_{i,n}$ is a monoidal tensor category whose Grothendieck ring is isomorphic to the Grothendieck ring $\mathcal{C}_{i,\bar{i}}$ (equivalently the Grothendieck ring of $\mathcal{C}^{-}$ for the quantum loop algebra associated to $A_i$). The isomorphism is defined by requiring that the class of the module $W_{i,a}^{(1)}$ for $A_i$ maps to the class of $W_{ij,ia}^{(1)}$ for $A_n$. We conjecture that in general the isomorphism maps the class of an irreducible object of $\mathcal{C}_{i,\bar{i}}$ to the class of an irreducible object of $\mathcal{C}_{i,n}$. We prove that the conjecture holds for any KR–module associated to a $(j, \bar{i})$–segment. We also establish the conjecture for the category $\mathcal{C}_{1}$ defined in [19]. For this we define the analogous category $\mathcal{C}_{i,n}^{1}$ and show that it is a tensor category and is the image of $\mathcal{C}_{i,\bar{i}}^{1}$. It now follows from [21] that $K_{0}(\mathcal{C}_{i,n}^{1})$ is a monoidal categorification of the cluster algebra of type $A_i$ with the frozen variables being higher order KR–modules.
Recall that a finite–dimensional irreducible module is said to be imaginary if its tensor square is reducible. The first example of such a module was given by Leclerc in \cite{26} in types $A_3$, $B_3$, $C_2$ and $G_2$. Using embeddings of the associated quantum loop algebra it was immediate that they existed in all simple Lie algebras of higher rank. This was the only example in the literature of an imaginary modules until the work \cite{25} of Lapid and Minguez which gives examples in type $A_n$. Their work, as well as the original example, come from affine Hecke algebras and affine Schur–Weyl duality.

In Section 4 of this paper we use the results of the earlier sections to give a systematic way to construct imaginary modules. We work with tensor products of dual KR–modules associated to $(i, n)$–segments. In particular, we prove (see Theorem 3 for the general statement) that if $r \geq 2$ the tensor product $W_{i,a}^{(r)} \otimes W_{n+1-i,a-n-1}^{(r)}$ has an imaginary module in its Jordan–Holder series. The example given by Leclerc in type $A_4$ can be viewed as coming from our result in type $A_3$ and the tensor product $W_{2,a}^{(2)} \otimes W_{2,aq}^{(2)}$. Our examples do not satisfy the conditions for imaginary modules imposed in the work of \cite{25} except for small values of $r$. In type $D_4$ we find an imaginary module in the tensor product of $W_{1,a}^{(1)}$ and its dual for all $r \geq 3$. This example is not obtained by an embedding of any $A_r$–subalgebra in $D_4$, in fact the associated Drinfeld polynomial would define a real module for any $A_3$ type subalgebra.

We conclude the introduction with a brief discussion on higher order KR–modules in other types. They clearly exist, for instance in type $D_4$ one could just look at the top irreducible constituent of the tensor product of a $W_{i,a}^{(1)}$ and its dual. This representation is obviously prime. However the problem arises in these higher ranks because the points of reducibility of $W_{i,a}^{(1)} \otimes W_{i,b}^{(1)}$ have “gaps”. In $D_n$ and $i = 1$ the reducibility is when $a = bq^{\pm 2}$ and $a = bq^{2n-2}$. (In type $A_n$ in contrast the points of reducibility are of the form $\{aq^{\pm 2s} : 1 \leq s \leq p\}$ for a suitable choice of $p$.) The existence of the gaps makes it difficult to define a good analog of $(i, n)$–segments; it would appear that one has to “glue” two different kinds of segments together multiple times. The difficulty is analogous in some sense to the difficulty in type $A_2$ of gluing segments associated to nodes 1 and 2 to give a prime representation. We hope to return to these ideas in the future.

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1. Preliminaries

In this section we set up the notation to be used in the paper and recall several results which will play a crucial role in the later sections.

We assume throughout that $q$ is a non–zero complex number and not a root of unity. As usual $\mathbb{C}$ (resp. $\mathbb{C}^\times$, $\mathbb{Z}$, $\mathbb{Z}_+$, $\mathbb{N}$) will denote the set of complex numbers (resp. non-zero complex numbers and integers, non-negative integers and natural numbers).
numbers, integers, non-negative integers, positive integers). Given $i, j \in \mathbb{N}$ with $i \leq j$ we let $[i, j]$ be the set of integers $\{i, i + 1, \ldots, j\}$.

1.1. The algebra $\hat{U}_n$. Let $\hat{U}_n$ be the quantum loop algebra over $\mathbb{C}$ associated to $\mathfrak{sl}_{n+1}$; we refer the reader to [7] for precise definitions. For our purposes, it is enough to recall that $\hat{U}_n$ is a Hopf algebra and is generated as an algebra by elements $x_{i,s}^\pm, \phi_{i,s}^\pm, i \in [1, n]$ and $s \in \mathbb{Z}$. The algebra generated by the elements $\phi_{i,s}^\pm, i \in [1, n], s \in \mathbb{Z}$, is denoted by $\hat{U}_n^0$ and is a commutative subalgebra of $\hat{U}_n$.

Given $a \in \mathbb{C}^\times$ let $\tau_a : \hat{U}_n \to \hat{U}_n$ be the Hopf algebra homomorphism given by

$$x_{i,s}^\pm \to a^s x_{i,s}^\pm, \quad \tau_a(\phi_{i,s}^\pm) = a^s \phi_{i,s}^\pm, \quad i \in [1, n], \quad s \in \mathbb{Z}. $$

The quantum affine analog of the Cartan involution of $A_n$ is the unique algebra involution $\Omega : \hat{U}_n \to \hat{U}_n$ given by

$$\Omega(x_{i,s}^\pm) = -x_{i,-s}^\pm, \quad \Omega(\phi_{i,s}^\pm) = \phi_{i,-s}^\pm, \quad i \in [1, n], \quad s \in \mathbb{Z}. $$

Given $J = [i, j] \subset [1, n]$ let $\hat{U}_{n,J}$ be the subalgebra of $\hat{U}_n$ generated by the elements $x_{p,s}^\pm, \phi_{p,s}^\pm, s \in \mathbb{Z}, p \in J$. Then we have an algebra (but not a Hopf algebra) homomorphism $\hat{U}_{j-i+1} \to \hat{U}_{n,J}$.

1.2. The $\ell$–weight and $\ell$–root lattice. Let $\mathcal{P}_n$ (resp. $\mathcal{P}_n^+$) be the (multiplicative) free abelian group (resp. monoid) generated by elements $\{\omega_{i,a} : i \in [1, n], a \in \mathbb{Z}\}$ and denote by $1$ the identity element. It will be convenient to set $\omega_{k,a} = 1$ for all $a \in \mathbb{Z}$ if $k \notin [1, n]$. The elements of $\mathcal{P}_n$ are called $\ell$–weights and those of $\mathcal{P}_n^+$ the dominant $\ell$–weights. Let $P_n$ be the free (additive) abelian group on generators $\{\omega_i : i \in [1, n]\}$ and $P_n^+$ the corresponding monoid. Define a morphism of groups by extending the assignment

$$\text{wt} : \mathcal{P}_n \to P_n, \quad \text{wt} \omega_{i,a} = \omega_i, \quad i \in [1, n], \quad a \in \mathbb{Z}. $$

For $i \in [1, n]$ and $a \in \mathbb{Z}$ define $\alpha_{i,a} \in \mathcal{P}_n$ and $\alpha_i \in P_n$ by

$$\alpha_{i,a} = \omega_{i-1,a}^{-1} \omega_{i,a-1} \omega_{i,a+1} \omega_{i+1,a}^{-1}, \quad \alpha_i = 2 \omega_i - \omega_{i-1} - \omega_{i+1} $$

and let $Q_n$ (resp. $Q_n^+$) be the subgroup (resp. submonoid) of $\mathcal{P}_n$ generated by these elements. The subgroup (resp. submonoid) of $P_n$ spanned by the elements $\alpha_i, i \in [1, n]$, is denoted $Q_n$ (resp. $Q_n^+$).

Given $J = [i, j] \subset [1, n]$ let $\mathcal{P}_{n,J}$ be the subgroup of $\mathcal{P}_n$ generated by the elements $\omega_{j,c}$ with $j \in J$ and $c \in \mathbb{Z}$. Clearly $\mathcal{P}_{n,J} \cong \mathcal{P}_{j-i+1}$ and we shall use this freely without mention. Define $P_{n,J}^+$ in the obvious way and identify it with the corresponding monoid in $\hat{U}_{j-i+1}$.

Define a homomorphism $\mathcal{P}_n \to P_{j-i+1}$ sending $\omega \mapsto \omega_J$ by extending the assignment

$$\omega_{i,c} \mapsto \omega_{i,c}, \quad i \in J, \quad \omega_{i,c} \mapsto 1, \quad i \notin J. $$

Clearly the homomorphism maps $Q_n$ to $Q_{j-i+1}$.
1.3. The category $\mathcal{F}_n$. Denote by $\mathcal{F}_n$ the category of type 1 finite-dimensional representations of $\hat{U}_n$. In particular if $V$ is any object of $\mathcal{F}_n$ we can write

$$V = \bigoplus_{\mu \in P} V_\mu, \quad V_\mu = \{v \in V : \phi_i^\pm v = q_\mu^\pm v\}, \quad \mu = \sum_{i=1}^n \mu_i \omega_i, \quad \mu_i \in \mathbb{Z}, \quad i \in [1, n],$$

and we set $\text{wt} \ V = \{\mu \in P : V_\mu \neq 0\}$. Given any object $V$ of $\mathcal{F}_n$ we can regard it as a module for the commutative subalgebra $\hat{U}_n^0$. It follows that we can write $V$ as a direct sum of generalized eigenspaces for the action of this subalgebra. The generalized eigenspaces are called $\ell$-weight spaces.

The Hopf algebra structure on $\hat{U}_n$ is chosen so that the comultiplication on the generators $x_{i,0}^+$ is given by $\Delta(x_{i,0}^+) = x_{i,0}^+ \otimes 1 + \phi_i^+ \otimes x_{i,0}^+$ (see [3] for details). Note that the Hopf algebra structure ensured that $\mathcal{F}_n$ contains the trivial representation, and is closed under tensor products and duals.

**Definition.** We say that an object $V$ of $\mathcal{F}_n$ is prime if it is not isomorphic to $U \otimes U'$ where $U$ and $U'$ are non-trivial objects of $\mathcal{F}_n$.

Let $K_0(\mathcal{F}_n)$ be the corresponding Grothendieck ring of $\mathcal{F}_n$ and denote by $[V]$ the isomorphism class of an object $V$ of $\mathcal{F}_n$.

Any object $V$ of $\mathcal{F}_n$ has a right and a left dual denoted by $V^*$ and $^*V$ respectively, and we have $\hat{U}_n$-maps

$$\mathbb{C} \hookrightarrow V^* \otimes V, \quad ^*V \otimes V \rightarrow \mathbb{C} \rightarrow 0.$$

We shall freely use properties of duals, in particular, the isomorphisms

$$(W \otimes V)^* \cong V^* \otimes W^*, \quad ^*(W \otimes V) \cong V^* \otimes W,$$

$$\text{Hom}_{\hat{U}_n}(V \otimes U, W) \cong \text{Hom}_{\hat{U}_n}(U, V^* \otimes W), \quad \text{Hom}_{\hat{U}_n}(U \otimes V, W) \cong \text{Hom}_{\hat{U}_n}(U, W \otimes ^*V).$$

For $a \in \mathbb{C}^\times$ and $V \in \mathcal{F}_n$, let $\tau_a V$ and $\Omega(V)$ be the corresponding objects of $\mathcal{F}_n$ obtained by pulling $V$ back by the automorphisms $\tau_a$ and $\Omega$ respectively. Then

$$\tau_a(V \otimes W) \cong \tau_a V \otimes \tau_a W, \quad \Omega(V \otimes W) \cong \Omega(W) \otimes \Omega(V).$$

1.3.1. The modules $W(\omega)$ and $V(\omega)$. It is convenient to identify $\mathcal{P}_n^+$ with the monoid consisting of $n$-tuple of polynomials by extending the assignment $\omega_{i,a} \mapsto (1 - \delta_{i,j} q^a u)_{j \in [1, n]}$ to a multiplicative homomorphism. For $\omega \in \mathcal{P}_n^+$ we let $W(\omega)$ be the $\hat{U}_n$-module generated by an element $v_\omega$ satisfying the relations

$$x_{i,s}^+ v_\omega = 0 = (x_{i,0}^-)^{\deg \pi_i(u)+1} v_\omega, \quad \phi_i^\pm v_\omega = \gamma_{i,s}^\pm v_\omega, \quad i \in [1, n], \quad s \in \mathbb{Z},$$

where $\gamma_{i,s}^\pm \in \mathbb{C}$ are defined by

$$\sum_{s=0}^\infty \gamma_{i,s}^\pm u^{\pm s} = q^{\deg \pi_i} \frac{\pi_i(q^{-1} u)}{\pi_i(q u)}, \quad \omega = (\pi_i(u))_{i \in I}.$$
Moreover
\[
\dim W(\omega)_{\text{wt} \omega} = 1, \quad \text{and for } \mu \in P, \quad \dim W(\omega)_{\mu} \neq 0 \implies \text{wt} \omega - \mu \in Q^+.
\]

Any quotient of \(W(\omega)\) is called an \(\ell\)-highest weight module with highest \(\ell\)-weight \(\omega\) and we continue to denote by \(v_\omega\) the image of the generator of \(W(\omega)\) in any quotient. It follows from [9] that the module \(W(\omega)\) is finite-dimensional and has a unique irreducible quotient which we denote as \(V(\omega)\). Finally, any irreducible module in \(\mathcal{F}_n\) is isomorphic to a tensor product of objects of the form \(\tau_b V(\omega)\) for some \(b \in \mathbb{C}^\times\) and \(\omega \in P_n^+\).

Setting
\[
\omega_{i,a} = \omega_{n+1-i,a+n+1}, \quad \omega_{i,a} = \omega_{n+1-i,a-n-1}, \quad \Omega(\omega_{i,a}) = \omega_{n+1-i,-a}
\]
we get corresponding isomorphisms \(\omega \mapsto \omega^*, \omega \mapsto \omega^*\) and \(\omega \mapsto \Omega(\omega)\) of \(P_n\) and
\[
V(\omega)^* \cong V(\omega^*), \quad *V(\omega) \cong V(\omega^*), \quad \Omega(V(\omega)) \cong V(\Omega(\omega)).
\]

### 1.3.2. Tensor products

Part (i) of the next proposition was proved in [7, 8]. Part (ii) was proved independently in [11] and [12] and part (iii) in [13]. Given \(m, r \in [1, n]\) set
\[
S_{m,r,n} = \{2p+2-m-r : \max\{m, r\} \leq p \leq \min\{m+r-1, n\}\}. \tag{1.1}
\]

**Proposition.** Suppose that \(\omega = \omega_{i_1,a_1} \cdots \omega_{i_k,a_k} \in P_n^+\) with \(a_1 \leq \cdots \leq a_k\).

(i) Let \(\omega' \in P_n^+\). The module \(V(\omega \omega')\) occurs with multiplicity one in the Jordan–Holder series of \(V(\omega) \otimes V(\omega')\). Moreover, \(V(\omega \omega')\) is isomorphic to \(V(\omega) \otimes V(\omega')\) if and only if \(V(\omega) \otimes V(\omega')\) and its left (or right) dual are \(\ell\)-highest weight modules.

(ii) We have,
\[
W(\omega) \cong V(\omega_{i_k,a_k}) \otimes \cdots \otimes V(\omega_{i_1,a_1})
\]
and hence for all \(\omega, \omega' \in P_n^+\) the following holds in \(K_0(\mathcal{F}_n)\):
\[
[W(\omega \omega')] = [W(\omega)][W(\omega')].
\]

(iii) The module \(V(\omega_{i_1,a_1}) \otimes V(\omega_{i_2,a_2}) \otimes \cdots \otimes V(\omega_{i_k,a_k})\) is \(\ell\)-highest weight if and only if \(\omega_2 - \omega_k \in S_{i_2,i_1-1,n}, \, 2 \leq j \leq k\). If \(\omega_2 - \omega_k\) is also not in this set for all \(2 \leq j \leq k\) then the module is irreducible.

### 1.3.3. \(\ell\)-lowest weight modules

An \(\ell\)-lowest weight module is defined in the obvious way; it is generated by an element \(v\) which is an eigenvector for the elements \(\phi_{i,s}^+\) and \(x_{i,s}^- v = 0\) for all \(i \in [1, n], \, s \in \mathbb{Z}\).

**Proposition.** (i) Any \(\ell\)-highest weight module with \(\ell\)-highest weight \(\omega\) in \(\mathcal{F}_n\) is also a lowest \(\ell\)-weight module with lowest weight \((\omega^*)^{-1}\).

(ii) Let \(V, V'\) be \(\ell\)-highest weight modules with \(\ell\)-highest weight \(\omega, \omega' \in P_n^+\), respectively. Let \(v^-\) and \(v^+\) be non-zero lowest and highest \(\ell\)-weights of \(V\) and \(V'\). Then \(v^- \otimes v^+\) is an \(\ell\)-weight vector with \(\ell\)-weight \((\omega^*)^{-1}\omega'\) and
\[
V \otimes V' = \mathcal{U}_n(v^- \otimes v^+).
\]

In particular if \(U\) is a proper quotient of \(V \otimes V'\) then \(\dim U(\omega^*)^{-1}\omega' \neq 0\).
Proof. We sketch a proof. Let \( \lambda = \text{wt} \, \omega \). Since \( V \) is an \( \ell \)-highest weight module we have \( \text{wt} \, V \subset \lambda - Q^+ \) and \( \dim V_\lambda = 1 \). Since \( V \) is a finite-dimensional module for \( \hat{U}_n \) and hence also for \( \hat{U}_n \) (the subalgebra generated by the elements \( x_{i,0}^\pm, \phi_{i,0}^\pm, \) \( i \in [1,n] \)) it follows that \( \dim V_{w_o} \lambda = 1 \) where \( w_o \) is the longest element of the Weyl group \( S_{n+1} \) of \( A_n \); in particular any non-zero element of \( V_{w_o} \lambda \) is an \( \ell \)-weight vector. It was shown in [15] that if \( V = V(\omega) \) then \( V_{w_o} \lambda \) was an \( \ell \)-weight space with \( \ell \)-weight \( (\omega^*)^{-1} \). Since \( V(\omega) \) is a quotient of any \( \ell \)-highest weight module with \( \ell \)-weight \( \omega \), part (i) follows.

Part (ii) is immediate from the formulae for the comultiplication [10] (see also [4]). \( \square \)

1.3.4. We shall use the following consequence of Proposition 1.3.3.

Lemma. Suppose that \( \pi, \pi_1, \pi_2 \in \mathcal{P}_n^+ \). Then

\[
\text{Hom}_{\hat{U}_n}(W(\pi), V(\pi_1) \otimes V(\pi_2)) \neq 0 \implies \pi \pi_1^{-1} \in \text{wt}_\ell V(\pi_2) \quad \text{and} \quad (\pi^*)^{-1} \pi_2^* \in \text{wt}_\ell V(\pi_1).
\]

Proof. Using duals we see that \( \text{Hom}_{\hat{U}_n}(W(\pi), V(\pi_1) \otimes V(\pi_2)) \neq 0 \) implies

\[
\text{Hom}_{\hat{U}_n}(V(\pi_1)^* \otimes W(\pi), V(\pi_2)) \neq 0, \quad \text{Hom}_{\hat{U}_n}(W(\pi) \otimes V(\pi_2)^*, V(\pi_1)) \neq 0.
\]

Since \( (\pi_1)^* = \pi_1 \) it follows from Proposition 1.3.3 that

\[
\pi_1^{-1} \pi \in \text{wt}_\ell V(\pi_2), \quad (\pi^*)^{-1} \pi_2^* \in \text{wt}_\ell V(\pi_1).
\]

\( \square \)

1.4. The \( \ell \)-weight space decomposition and \( q \)-characters. Let \( \mathcal{F}_{n,\mathbb{Z}} \) be the full subcategory of \( \mathcal{F}_n \) whose Jordan–Holder constituents are of the form \( V(\omega), \omega \in \mathcal{P}_n^+ \). It is well–known that \( \mathcal{F}_{n,\mathbb{Z}} \) is a rigid tensor subcategory of \( \mathcal{F}_n \) and we let \( \mathcal{K}_0(\mathcal{F}_{n,\mathbb{Z}}) \) be the corresponding Grothendieck ring. It was proved in [15] that if \( V \) is an object of \( \mathcal{F}_{n,\mathbb{Z}} \) then one can write \( V \) as a direct sum of generalized eigenspaces for \( \hat{U}_n^0 \) and that the eigenvalues are indexed by elements of \( \mathcal{P}_n \),

\[
V = \bigoplus_{\omega \in \mathcal{P}_n} V_\omega, \quad \text{wt}_\ell V = \{\omega \in \mathcal{P}_n : V_\omega \neq 0\}, \quad \text{wt}_\ell^+ V = \text{wt}_\ell V \cap \mathcal{P}_n^+.
\]

Given a subgroup \( \mathcal{G} \) of \( \mathcal{P}_n \) we define \( \chi^\mathcal{G}(V) \) to be the element of the group ring of \( \mathcal{G} \) given by,

\[
\chi^\mathcal{G}(V) = \sum_{\omega \in \mathcal{G}} \dim V_\omega c(\omega) \in \mathbb{Z}[\mathcal{G}]. \tag{1.2}
\]

The \( q \)-character of \( V \) is the element \( \chi^{\mathcal{P}_n}(V) \).

1.4.1. The following was proved in [15].

Theorem. The assignment \( [V] \mapsto \chi^{\mathcal{P}_n}(V) \) gives an injective homomorphism \( \chi^{\mathcal{P}_n} : \mathcal{K}_0(\mathcal{F}_{n,\mathbb{Z}}) \rightarrow \mathbb{Z}[\mathcal{P}_n] \) of rings. Moreover, \( \mathcal{K}_0(\mathcal{F}_{n,\mathbb{Z}}) \) is a polynomial ring in the generators \( [V(\omega_{i,a})] \) with \( i \in [1,n] \) and \( a \in \mathbb{Z} \). In particular if \( V \) and \( V' \) are objects of \( \mathcal{F}_{n,\mathbb{Z}} \) we have,

\[
\text{wt}_\ell(V \otimes V') = \text{wt}_\ell V \text{wt}_\ell V'.
\]

\( \square \)
1.4.2. We shall use the following result which proof can be found in [5], for instance.

**Proposition.** Let $V$ be an $\ell$–highest weight module with $\ell$–highest $\omega$. Then
\[ \text{wt}_\ell V \subset \omega(Q^+)^{-1}. \]

\[ \square \]

1.5. **Restrictions to $\hat{U}_{n,J}$**. Given $J = [i, j] \subset [1, n]$ and $\omega \in P_n^+$ we have an isomorphism of $\hat{U}_{n,J}$–modules $V(\omega_J) \cong \hat{U}_{n,J} v_\omega \subset V(\omega)$ and
\[ V(\omega_J) \otimes V(\omega_J') \cong \hat{U}_{n,J} v_\omega \otimes \hat{U}_{n,J} v_{\omega'}. \]
In particular if $v \in V(\omega_J) \otimes V(\omega_J')$ is an $\ell$–highest weight vector with $\ell$–highest $\omega, \omega' \alpha_j^{-1}$ where $\alpha \in Q^+, J$, then $V(\omega) \otimes V(\omega')$ has an $\ell$–highest weight vector with $\ell$–highest $\omega' \alpha^{-1}$.

1.6. **Some results of Mukhin and Young**. We recall some results of Mukhin and Young which were established in [27, 28] and which will play an important role in the subsequent sections.

1.6.1. **The set $P_{i,a}$**. For $i \in [1, n]$ and $a \in \mathbb{Z}$, let $P_{i,a}$ be the set of all functions $p : [0, n+1] \to \mathbb{Z}$ satisfying the following:
\[ p(0) = i + a, \quad p(r + 1) - p(r) \in \{-1, 1\}, \quad 0 \leq r \leq n, \quad p(n + 1) = n + 1 - i + a. \]

For $p \in P_{i,a}$ set
\[ c^+_p = \{ r \in [1, n] : p(r - 1) = p(r) \pm 1 = p(r + 1) \}, \]
\[ \omega(p) = \prod_{r \in c^+_p} \omega_{r, p(r)} \prod_{r \in c^-_p} \omega_{r, p(r)}^{-1} \in P_n. \]
In particular $\omega(p)$ is in the subgroup of $P_n$ generated by the elements $\{ \omega_{j,c} : j \in [1, n], \ a \leq c \leq a + n + 1 \}$.

Let $p_{i,a}$ and $p_{i,a}^*$ be the elements of $P_{i,a}$ given as follows:
\[ p_{i,a}(j) = \begin{cases} i - j + a, & 0 \leq j \leq i, \\ j - i + a, & i < j \leq n + 1 \end{cases}, \quad p_{i,a}^*(j) = \begin{cases} a + i + j, & 0 \leq j \leq n + 1 - i, \\ a + 2n + 2 - i - j, & n + 2 - i \leq j \leq n + 1. \end{cases} \]
Then
\[ \omega(p_{i,a}) = \omega_{i,a}, \quad \omega(p_{i,a}^*) = \omega_{n+1-i,a+n+1}^{-1}. \]
The following is a simple calculation.

**Lemma.** Let $a, b, c$ be integers with $b - a = 2m_1$ and $c - b = 2m_2$ for some $m_1, m_2 \in \mathbb{N}$. Then, for all $p \in P_{i,b}$ and $j \in [0, n+1]$ we have
\[ p_{i,a}(j) < p(j) < p_{i,c}^*(j), \quad p \in P_{i,b}. \]
\[ \square \]
1.6.2. More generally, given \( j \in [1, n] \), \( m \in [0, \min\{j, n + 1 - j\}] \) and \( a \in \mathbb{Z} \) it is not hard to see that there exist elements \( g^m_{j,a} \in \mathbb{P}_{j,a} \) satisfying,

\[
\omega(g^m_{j,a}) = \left\{
\begin{array}{ll}
\omega_{j,a}, & m = 0, \\
\omega_{j-m,a+m,j}^{-1}, & 0 < m \leq \min\{j, n + 1 - j\}.
\end{array}
\right.
\]

Notice that

\[
m = \min\{j, n + 1 - j\} \implies \omega(g^m_{j,a}) = \omega_{n+1-j, a+n+1}^{-1} = p^a_{j,a}.
\]

It is easily checked that

\[
\{g^m_{j,a} : 1 \leq m \leq \min\{j, n + 1 - j\}\} = \{g \in \mathbb{P}_{j,a} : c^{-1}_g = \{n + 1 - j\}\}. \quad (1.3)
\]

Similarly there exist elements \( p^m_{j,a} \in \mathbb{P}_{j,a} \) such that

\[
\omega(p^m_{j,a}) = \left\{
\begin{array}{ll}
\omega_{j,a}, & m = 0, \\
\omega_{j-m,a+m,j}^{-1}, & 0 < m \leq \min\{j, n + 1 - j\}.
\end{array}
\right.
\]

and

\[
\{p^m_{j,a} : 1 \leq m \leq \min\{j, n + 1 - j\}\} = \{p \in \mathbb{P}_{j,a} : c^{-1}_p = \{j\}\}. \quad (1.4)
\]

1.6.3. The set \( \mathbb{P}_\omega \). Given \( \omega = \omega_{i_1,a_1} \cdots \omega_{i_k,a_k} \in \mathbb{P}^+_n \) with \( a_1 \leq a_2 \leq \cdots \leq a_k \), define

\[
\mathbb{P}_\omega \subset \mathbb{P}_{i_1,a_1} \times \cdots \times \mathbb{P}_{i_k,a_k}
\]

to consist of \( k \)-tuples \((p_1, \cdots, p_k)\) satisfying:

\[
p_j(k) < p_s(k) \text{ for all } k \in [0, n + 1] \text{ and all } 1 \leq j < s \leq r. \quad (1.5)
\]

Given \( \mathbb{P} \in \mathbb{P}_\omega \), set

\[
\omega(\mathbb{P}) = \omega(p_1) \cdots \omega(p_r), \quad \mathbb{P} = (p_1, \cdots, p_r). \quad (1.6)
\]

The restriction in (1.5) guarantees that the expression on the right hand side of (1.6) is a reduced word in \( \mathbb{P}_\omega \). Here we emphasize that this is equivalent to saying that there are no cancellations between the \( \omega(p_j) \). As an example, consider \( n = 3 \) and \( \omega = \omega_{2,0} \omega_{2,4} \). Then \((p^a_{2,0}, p_{2,4}) \notin \mathbb{P}_\omega \), since \( p^a_{2,0}(2) = 4 = p_{2,4}(2) \). Moreover

\[
\omega(p^a_{2,0}) \omega(p_{2,4}) = \omega_{2,4}^{-1} \omega_{2,4} = 1 \notin \text{wt}_V(\omega).
\]

For \( i \in [1, n] \) and integers \( a_1 \leq \cdots \leq a_k \), we shall write

\[
\omega_{i,a} = \omega_{i,a_1} \cdots \omega_{i,a_k}, \quad \mathbb{P}_{i,a} = \mathbb{P}_{\omega_{i,a}}. \quad (1.7)
\]

Following [28] we shall say that \( \omega \in \mathbb{P}_n^+ \) is a prime snake if we can write

\[
\omega = \omega_{i_1,a_1} \cdots \omega_{i_k,a_k} \quad \text{with} \quad a_p - a_{p-1} \in S_{i_p,i_{p-1},n}, \quad p \in [2, r].
\]

The next result was proved in [27, 28].

Proposition. Suppose that

\[
\omega = \omega_{i_1,a_1} \cdots \omega_{i_k,a_k}, \quad \omega' = \omega_{j_1,b_1} \cdots \omega_{j_m,b_m}
\]

satisfy (1.8), i.e. are prime snakes.
\section*{Proposition} \hfill 

(i) If \(b_1 - a_k \in S_{j_1,i_k,n} \), then the \( \hat{U}_n \)-module \( V(\omega) \otimes V(\omega') \) is reducible.

(ii) We have
\[
\text{wt}_\ell V(\omega) = \{ \omega(p) : p \in P_\omega \}, \quad \dim V(\omega)_\pi = 1 \quad \text{if} \quad \pi \in \text{wt}_\ell V(\omega),
\]
\[
\text{wt}_\ell^+ V(\omega) = \{ \omega \}.
\]

(iii) Suppose that \( k = m \) and \( j_s = i_{s+1} \) and \( b_s = a_{s+1} \) for \( s \in [2, k-1] \). The following equality holds in \( K_0(\mathcal{P}_n, \mathbb{Z}) \):
\[
[V(\omega) \otimes V(\omega')] = [V(\omega w_{j_k,b_k})][V(\omega^{-1}_j)] \oplus [V(\omega^+)][V(\omega^-)],
\]
where
\[
\omega^\pm = \prod_{p=1}^k \omega_\pm^{(i_p + i_{p+1} \pm (a_p - a_{p+1})), \pm (a_p + a_{p+1} \pm (i_p - i_{p+1}))},
\]
and we understand \( i_{k+1} = j_k \) and \( a_{k+1} = b_k \). Moreover,
\[
\omega^+ \omega^- \not\in \text{wt}_\ell(V(\omega w_{j_k,b_k}) \otimes V(\omega^{-1}_j)).
\]

(iv) For \( p, p' \in P_\omega \) we have
\[
\omega(p) \in \omega(p') Q^+ \iff p'(k) \geq p(k) \quad \text{for all} \quad k \in [0, n+1]. \tag*{\square}
\]

1.6.4. We note some consequences of Proposition 1.6.3 for later use.

\textbf{Proposition.} Let \( i \in [1, n] \) and \( a = (a_1, \ldots, a_r) \) with \( a_1 \leq \cdots \leq a_r \) for some \( r \geq 1 \) be such that \( \omega_{i,a} \) is a prime snake. For \( j, s \in [1, r] \), with \( j \leq s \), set \( a_{j,s} = (a_j, \ldots, a_s) \) and \( i_{j,s} = (i_j, \ldots, i_s) \).

(i) We have \( (p_1, \ldots, p_r) \in P_{i,a} \implies (p_{j_1}, \ldots, p_{s_1}) \in P_{i_{j,s}}. \)

(ii) Conversely \( (p_{j_1}, \ldots, p_{s_1}) \in P_{i_{j,s}} \implies (p_{i_1,a_1}, \ldots, p_{j_{s-1},a_{j-1}}, p_{j_1}, \ldots, p_{s_1}, p_{j_{s+1},a_{j+1}}, \ldots, p_{i_r,a_r}) \in P_{i,a}. \)

1.7. A result of Kang, Kashiwara, Kim and Oh. We shall make crucial use of the main result of \cite{22} which we now recall.

\textbf{Theorem.} \cite{22} Theorem 3.12] Suppose that \( \pi \in \mathcal{P}^+_n \) is such \( V(\pi) \) is real. Then for all \( \pi' \in \mathcal{P}^+_n \) the module \( V(\pi) \otimes V(\pi') \) has simple head and simple socle. Moreover the socle of \( V(\pi) \otimes V(\pi') \) is the head of \( V(\pi) \otimes V(\pi'). \) \tag*{\square}

\section*{2. Higher order KR–modules and a prime factorization theorem}

In this section we prove the analog of the main result of \cite{3} for elements of \( \mathcal{P}^+_n \) concentrated at a node \( i \in [1, n] \). We reformulate the notion of a prime snake module \( (i, a) \) in terms of \((i, n)-segments\) and define a notion of two \((i, n)-segments\) being in general position. We show that if \( b \in \mathbb{Z}^r \) for some \( r \geq 1 \) then \( V(\omega_{i,b}) \) can be written uniquely (up to an overall permutation) as a tensor product of modules associated to \((i, n)-segments\) in general position.

\subsection*{2.1. The set \( S_{i,n} \) and \((i, n)-segments\)} For \( i \in [1, n] \), let
\[
S_{i,n} = \{ 2j : 1 \leq j \leq \min(i, n+1-i) \} = S_{n+1-i,n}. \tag{2.1}
\]
Notice that this is precisely the set \( S_{i,i,n} \) defined in Section 1.3.2.
2.1.1. Segments.

**Definition.** Say that an element \( \mathbf{a} = (a_1, \cdots, a_r) \in \mathbb{Z}^r \) is an \((i, n)\)-segment of length \( r \) if \( a_p - a_{p-1} \in S_{i,n} \) for all \( 2 \leq p \leq r \). Equivalently we say that \( \mathbf{a} \) is an \((i, n)\)-segment if and only if \((i, \mathbf{a})\) is a prime snake of type \( n \).

Set
\[
\mathbf{a}^* = (a_1 + n + 1, \cdots, a_k + n + 1), \quad \mathbf{a}^* = (a_1 - n - 1, \cdots, a_k - n - 1),
\]
and notice that they are \((n+1-i, n)\)-segments. \( \Box \)

Since \( 0 \notin S_{i,n} \) the entries of \( \mathbf{a} \) are all distinct and so in what follows we will also think of segments as sets.

**Example.** If \( n = 3 \) we have
\[ S_{1,3} = \{2\} = S_{3,3}, \quad S_{2,3} = \{2, 4\}. \]
The element \( \mathbf{a} = (0, 4, 6, 10) \) is the union of the three \((1, 3)\)-segments (and also \((3, 3)\)-segments) namely: \((0)\), \((4, 6)\) and \((10)\). However \( \mathbf{a} \) is a \((2, 3)\)-segment of length 4.

2.1.2. General and special position of segments.

**Definition.** Say that two \((i, n)\)-segments \( \mathbf{a} = (a_1, \cdots, a_r) \) and \( \mathbf{b} = (b_1, \cdots, b_s) \) are in general position if their union does not contain an \((i, n)\)-segment of length greater than \( \max\{r, s\} \). Otherwise we say that they are in special position. \( \Box \)

**Examples.**

(i) An \((i, n)\)-segment is in general position with itself.

(ii) Consider the \((2, 3)\)-segments \( \mathbf{a} = (0, 2, 6, 10) \), \( \mathbf{b} = (4) \), \( \mathbf{c} = (16, 18) \). Then \( \mathbf{a} \) and \( \mathbf{b} \) are in special position since their union contains the \((2, 3)\)-segment \((0, 2, 4, 6, 10)\) while \( \mathbf{a}, \mathbf{c} \) (and also \( \mathbf{b}, \mathbf{c} \)) are in general position.

2.2. The KR–modules of type \((i, n)\).

Recall that for \( i \in [1, n] \) and \( \mathbf{a} \in \mathbb{Z}^r \) we set
\[ \omega_{i, \mathbf{a}} = \omega_{i,a_1} \cdots \omega_{i,a_r} \in \mathcal{P}_n^+. \]

**Definition.** Given \( \omega \in \mathcal{P}_n^+ \) we say that \( V(\omega) \) is a KR–module of type \((i, n)\) if there exists an \((i, n)\)-segment \( \mathbf{a} \) such that \( \omega = \omega_{i, \mathbf{a}} \). \( \Box \)

We note that the usual KR–module for \( \hat{\mathbb{U}}_n \) is of the form \( \omega_{i, \mathbf{a}} \) where \( \mathbf{a} = (a, a+2, \cdots a+2r-2) \) for some \( a \in \mathbb{Z} \) and \( r \geq 1 \). We refer to the KR–modules associated with more general segments as the higher order KR–modules since they encode the reducibility data of \( V(\omega_{i, \mathbf{a}}) \otimes V(\omega_{i, \mathbf{b}}) \) in higher rank.

**Remark.** The KR–modules of type \((i, n)\) are a special family of the prime snake modules \([27, 28]\).
2.3. A prime factorization result. We state our first main theorem, which generalizes the result of \([g]\) in the rank one case.

**Theorem 1.** Let \(i \in [1, n]\) and \(a \in \mathbb{Z}^r\). There exists a unique integer \(k \geq 1\) and unique (up to a permutation) \((i, n)\)-segments \(a_1, \ldots, a_k\) which are in pairwise general position such that
\[
V(\omega_{i,a}) \cong V(\omega_{i,a_1}) \otimes \cdots \otimes V(\omega_{i,a_k}).
\]

In particular, \(V(\omega_{i,a}) \otimes V(\omega_{i,a})\) is irreducible for all \(a \in \mathbb{Z}^r\). In particular, \(V(\omega_{i,a})\) is prime if and only if \(a\) is an \((i, n)\)-segment.

The proof of the theorem occupies the rest of the section.

2.4. Combinatorics of \((i, n)\)-segments. We give a more explicit formulation for a pair of segments to be in special or general position. We use this to prove that an element \(\omega_{i,a}, a \in \mathbb{Z}^r\) can be written uniquely (up to a renumbering) as a product of elements associated to \((i, n)\)-segments in general position.

2.4.1.

**Proposition.** Let \(r \geq m\) and assume that \(a = (a_1, \ldots, a_r)\) and \(b = (b_1, \ldots, b_m)\) are \((i, n)\)-segments.

(i) The segments \(a\) and \(b\) are in general position if and only if one of the following holds:
- \(b_1 - a_r > 2\min\{i, n + 1 - i\}\) or \(b_1 - b_m > 2\min\{i, n + 1 - i\}\) or \(b_1 - a_1 \notin 2\mathbb{Z}\),
- \(\{b_1, \ldots, b_m\} \subset \{a_1, \ldots, a_r\}\).

(ii) The segments \(a\) and \(b\) are in special position if and only if there exists \(1 \leq j \leq m\) such that one of the following hold:
- \(b_j - a_r \in S_{i,n}\) or \(a_1 - b_j \in S_{i,n}\),
- \(b_1 - a_1 \in 2\mathbb{Z}\) and there exists \(1 \leq k < r\) such that \(a_k < b_j < a_{k+1}\).

**Proof.** It is clear that if one of conditions in (i)(a) or (i)(b) holds then \(a\) and \(b\) are in general position. For the converse we suppose that none of the conditions in (a) and (b) are satisfied and show that \(a \cup b\) contains a segment of length \(r+1\). If \(b_1 > a_r\) (resp. \(b_m < a_1\)) then \(b_1 - a_r \in S_{i,n}\) (resp. \(a_1 - b_m \in S_{i,n}\)) which means that \((a_1, \ldots, a_r, b_1)\) (resp. \((b_m, a_1, \ldots, a_r)\)) is an \((i, n)\)-segment of length \(r + 1\) as needed.

Hence to complete the proof of (i) we must consider the case when all of the following hold:
\(b_1 \leq a_r\), \(b_m \geq a_1\), and \(b_1 - a_1 \in 2\mathbb{Z}\). If \(a_r < b_k\) (resp. \(b_k < a_1\)) for some \(2 \leq k \leq m\) (resp. \(1 \leq k \leq m - 1\)) and \(k\) is minimal (resp. maximal) with this property, then \(b_{k-1} \leq a_r < b_k\) (resp. \(b_k < a_1 \leq b_{k+1}\)) and so \((a_1, \ldots, a_r, b_k)\) (resp. \((b_k, a_1, \ldots, a_r)\)) is an \((i, n)\)-segment. Otherwise, we have \(a_1 \leq b_1 < b_2 < \cdots < b_m \leq a_r\). Since condition (b) does not hold, it follows that \(b_p \notin \{a_1, \ldots, a_r\}\) for some \(1 \leq p \leq m\); in other words, there exists \(2 \leq k \leq r\) such that \(a_{k-1} < b_p < a_k\). It follows that \((a_1, \ldots, a_{k-1}, b_p, a_k, \ldots, a_r)\) is an \((i, n)\)-segment. The proof of part (i) is complete.

If either of the conditions in part (ii) hold, then \(\{a_1, \ldots, a_r, b_j\}\) is an \((i, n)\)-segment after applying a suitable permutation and hence \(a\) and \(b\) are in special position. Suppose that \(a\) and \(b\) are in special position and assume that there does not exist \(1 \leq j \leq m\) and \(1 \leq k < r\) with \(a_k < b_j < a_{k+1}\). By part (i) we see that we cannot have \(\{b_1, \ldots, b_m\} \subset \{a_1, \ldots, a_r\}\).
Hence either there exists $j$ maximal with $b_j < a_1$ or $j$ minimal with $b_j > a_r$. In the first (resp. second) case either $j = m$ or $j < m$ and $b_j < a_1 \leq b_{j+1}$ (resp. $j = 1$ or $j > 1$ and $b_{j-1} \leq a_r < b_j$). If $j = m$ (resp. $j = 1$) then the segments can be in special position only if $a_1 - b_m \in S_{i,n}$ (resp. $b_1 - a_r \in S_{i,n}$). If $j < m$ (resp. $j > 1$) then $b_{j+1} - b_j \in S_{i,n}$ (resp. $b_{j-1} - b_j \in S_{i,n}$) and so $a_1 - b_j \in S_{i,n}$ (resp. $b_j - a_r \in S_{i,n}$) and the proof is complete. \hfill $\square$

**Corollary.** Suppose that $a = (a_1, \ldots, a_r)$ and $b = (b_1, \ldots, b_m)$ are $(i,n)$-segments in general position with $r \geq m$. Then,

$$\{a_1, \ldots, a_r\} \cap \{b_1, \ldots, b_m\} \neq \emptyset \implies \{b_1, \ldots, b_m\} \subset \{a_1, \ldots, a_r\}.$$ 

\hfill $\square$

### 2.4.2.

Given $a = (a_1, \ldots, a_r) \in \mathbb{Z}^r$ and $b = (b_1, \ldots, b_n) \in \mathbb{Z}^s$ set

$$a \lor b = (a_1, \ldots, a_r, b_1, \ldots, b_n).$$

**Proposition.** Suppose that $a = (a_1, \ldots, a_r) \in \mathbb{Z}^r$. Then $\omega_{i,a}$ can be written uniquely (up to a permutation) as a product $\omega_{i,a} = \omega_{i,a_1} \cdot \cdots \cdot \omega_{i,a_k}$ where $a_1, \ldots, a_k$ are $(i,n)$-segments in pairwise general position.

Before proving the proposition we give an example.

**Example.** Let $a = (0, 6, 4, 2, 10, 16, 10) \in \mathbb{Z}^7$, then the associated $(2,3)$-segments are

$$a_1 = (0, 2, 4, 6, 10), \quad a_2 = (10), \quad a_3 = (16),$$

or any permutation of these by an element of $S_3$.

**Proof.** Clearly it suffices to prove that there exists a unique integer $k \geq 1$ and $(i,n)$-segments $a_1, \ldots, a_k$ (which are unique up to a renumbering) in pairwise in general position and a permutation $\sigma \in S_r$ such that

$$(a_{\sigma(1)}, \ldots, a_{\sigma(r)}) = a_1 \lor a_2 \lor \cdots \lor a_k.$$ 

We proceed by induction on $r$ with induction beginning trivially at $r = 1$. After applying an element of $S_r$ if needed, we may assume without loss of generality that: $a_1 \leq a_s$ for all $1 \leq s \leq r$ and that $r_1 \in [1, r]$ is maximal so that $a_1 = (a_1, a_2, \ldots, a_{r_1})$ is an $(i,n)$-segment. If $r_1 = r$ we are done and otherwise we let $b = (a_{r_1+1}, \ldots, a_r)$. The inductive hypothesis applies to $b$ and we let $a_2, \ldots, a_k$ be the $(i,n)$-segments associated to $b$.

We prove that $a_1$ is in general position with $a_s$ for $s \in [2, k]$. Suppose that $a_s = (b_1, \ldots, b_m)$ and recall that $a_1 \leq b_1$. Assume for a contradiction that $a_s$ and $a_1$ are in special position. If $m \leq r_1$, then by Proposition 2.4.1(ii) there exists $1 \leq p \leq m$ such that either $a_{k-1} < b_p < a_k$ for some $2 \leq k \leq r_1$ or $a_{r_1} < b_p$ with $b_p - a_{r_1} \in S_{i,n}$. In either case after applying a suitable permutation if needed, we see that $(a_1, \ldots, a_{r_1}, b_p)$ defines an $(i,n)$-segment contradicting our choice of $r_1$. If $m > r_1$ then the maximality of $r_1$ implies that $b_1 > a_1$ and also that $b_1 - a_{r_1} \notin S_{i,n}$. Since by assumption $a_1$ and $a_s$ are in special position, this forces $b_1 \leq a_{r_1}$. Choose $p \in [1, r_1]$ minimal so that $a_p < b_1 \leq a_{p+1}$. Then, $(a_1, \ldots, a_p, b_1, \ldots, b_m)$ is a $(i,n)$-segment and again we have a contradiction to the maximality of $r_1$. This proves that $a_1$ and $a_s$ are in general position.
It remains to prove that \( k \) is unique and that the segments are unique up to an element of \( S_k \). For this, suppose that \( c_1, \ldots, c_\ell \) is another set of \((i,n)\)–segments in pairwise general position and assume that \((a_{\sigma(1)}, \ldots, a_{\sigma(r)}) = c_1 \lor \cdots \lor c_\ell \) for some \( \sigma \in S_\ell \). Since \( a_1 \) is minimal it must occur as the first term in \( c_p \) for some \( 1 \leq p \leq \ell \) and assume without loss of generality that \( p = 1 \) and also that \( c_1 \) has maximal length say \( s_1 \) amongst those \( c_s \) with first term \( a_1 \). Since \( r_1 \) is the maximum length of an \((i,n)\)-segment starting at \( a_1 \) we have \( s_1 \leq r_1 \). We claim that \( c_1 = a_1 \). Otherwise, there exists \( 1 < p \leq r_1 \) minimal such that \( a_p \) does not occur in \( c_1 \). All the other segments \( c_s \) whose initial term is \( a_1 \) have length at most \( s_1 \) and hence by Corollary \[2.4.3\] must be contained in \( c_1 \). Hence none of these segments contain \( a_p \) and so there must exist an \((i,n)\)-segment \( c_j \) of length \( s_j \) containing \( a_p \). The minimal term of \( c_j \) is \( a_m \) for some \( m \). This means that \( a_m \in a_1 \) and so \( a_m - a_m - 1 \in S_{k,n} \). Consider \( c_1 \cup c_j \). If \( s_1 \geq s_j \) then since \( a_{p-1} \in c_1 \) it follows that \( c_1 \cup \{a_p\} \) is a longer \((i,n)\)-segment in the union. If \( s_j > s_1 \) the preceding discussion shows that \( \{a_m\} \cup c_j \) is a longer segment in the union. In any case we have a contradiction to the fact that \( c_1 \) and \( c_j \) are in general position. It follows that \( a_1 = c_1 \) and that \( b = c_2 \lor \cdots \lor c_\ell \). The uniqueness is now immediate by the inductive hypothesis.

\[ \square \]

2.4.3. We need a further equivalent formulation on segments in special position. We remark that in the language of \[25\] the next proposition proves that any pair of \((i,n)\)-segments in special position contains a pair of segments which form an extended \( T \)-system. In the following, given any \((i,n)\)-segment \( c = (c_1, \ldots, c_k) \) we adopt the convention that \( c_m = -\infty \) if \( m < 1 \) and \( c_m = \infty \) if \( m > k \).

**Lemma.** Let \( a = (a_1, \ldots, a_r) \) and \( b = (b_1, \ldots, b_s) \) be \((i,n)\)-segments with \( r \geq s \). Then \( a \) and \( b \) are in special position if and only if \( b_1 - a_1 \in 2\mathbb{Z} \) and there exists \( p \in \mathbb{Z}_{\geq 0} \), \( a_1 = (a_j, \ldots, a_{j+p}) \subset a \) and \( b_1 = (b_m, \ldots, b_{m+p}) \subset b \) such that

\[
\max\{a_{j-1}, b_{m-1}\} \prec \min\{a_j, b_m\}, \quad \min\{a_{j+p+1}, b_{m+p+1}\} > \max\{a_{j+p}, b_{m+p}\},
\]

(2.2)

and either

\[
a_{j+k} = b_{m+k-1} \quad \text{for all } k \in [1,p] \quad \text{or} \quad a_{j+k-1} = b_{m+k} \quad \text{for all } k \in [1,p].
\]

(2.3)

Before proving the Lemma we give an example.

**Example.** Consider the \((2,3)\)-segments

\[
a = (-4, 0, 4, 8, 10, 14), \quad b = (0, 2, 4, 8, 12, 14).
\]

They are in special position since their union contains the \((2,3)\)-segment \((-4, 0, 2, 4, 8, 10, 12, 14)\). In the notation of the lemma, taking \( p = 2 \), \( j = 3 \) and \( m = 2 \) we have \( a_1 = (4, 8, 10) \), \( b_1 = (2, 4, 8) \), which clearly satisfy

\[
\max\{a_2, b_1\} = 0 < 2 = \min\{a_3, b_2\}, \quad \min\{a_6, b_5\} = 12 > 10 = \max\{a_5, b_4\}
\]

and (2.3) as well.

**Proof.** Suppose that \( a \) and \( b \) satisfy (2.2) and that \( a_{j+k} = b_{m+k-1} \) for \( k \in [1,p] \) (resp. \( a_{j+k-1} = b_{m+p} \)) for all \( k \in [1,p] \). Then \( \{a_1, \ldots, a_{j+p}, b_{m+p}, a_{j+p+1}, \ldots, a_r\} \) is an \((i,n)\)-segment of length \( r + 1 \) which proves the converse direction.
For the forward direction, we proceed by induction on $s$. To see that induction begins at $s = 1$ notice that if $r = 1$ there is nothing to prove. If $r > 1$ then by Proposition 2.4.1 we have $a_j < b_1 < a_{j+1}$ for some $0 \leq j \leq r$. Clearly (2.2) and (2.3) hold by taking $p = 0$, $a_1 = (a_{j+\delta_j, 0})$ and $b_1 = (b_1)$. For the inductive step assume that the result holds for all $(i, n)$-segments in special position for all $s' < s$ and all $r \geq s'$. We consider two main cases.

For the first case we assume that $b_1 \leq a_1$. If $b_2 < a_1$ then observe that $a$ and $b' = (b_2, \cdots, b_s)$ are in special position. Hence the inductive hypothesis applies to these pairs and so we can choose $a_1$ and $b_1$ so that they satisfy (2.2) and (2.3). If $b_1 = a_1$ or $b_1 < a_1$ and $a_k < b_1 < a_{k+1}$ for some $j \in [2, s]$ and $k \in [1, r]$ then the pairs $a' = (a_2, \cdots, a_r)$ and $b' = (b_2, \cdots, b_s)$ are in special position. The induction hypothesis applies to these pairs, and so we can choose $a_1$ and $b_1$ satisfying (2.2) and (2.3). To complete the first case we must consider the situation when $b_1 < a_1 \leq b_2$ and there does not exist $j \geq 2$ with $a_k < b_j < a_{k+1}$. Since $r \geq s$ there exists $p \in [2, s + 1]$ maximal so that $a_{m-1} = b_m$ if $2 \leq m \leq p - 1$ and $a_p = b_p$. In particular, this means that $a_{p-1} < b_p < b_p$ and so the elements $b_1 = (b_1, \cdots, b_{p-1})$ and $a_1 = (a_1, \cdots, a_{p-1})$ satisfy the conditions of the proposition and give the inductive step.

For the second case we assume that $b_1 > a_1$. Working in a similar fashion from the other side we see that we can further reduce to the case when $b_s < a_r$ as well. Choose $p$ maximal and $p'$ minimal so that $a_p < b_1 < b_s < a_{p'}$. Since $a$ and $b$ are in special position, Proposition 2.4.1 shows that there exists $j \in [1, s]$ minimal such that $a_k < b_j < a_{k+1}$ for some $k \in [p, p']$. Define an integer $m$ as follows: if either $j = 1$ or $j > 1$ and $b_{j-1} < a_k$ we take $m = 0$; otherwise we take $m$ so that $a_{k-m'} = b_{j-m'-1}$ for all $0 \leq m' < m$ and $a_{k-m} > b_{j-m-1}$. Notice that $m$ must exist since $a_p < b_1$ and $j$ was chosen minimal. This time we take

$$a_1 = (a_{k-m}, \cdots, a_k), \quad b_1 = (b_{j-m}, \cdots, b_j).$$

In particular we have

$$a_{k-m-1}, b_{k-m-1} < a_{k-m} < b_{k-m}, \quad a_k < b_j < \min\{a_{k+1}, b_{k+1}\},$$

and the proof of the inductive step is complete.

\[\square\]

2.5. Proof of Theorem 2.1. Reducibility. Assume that $a = (a_1, \cdots, a_r)$ and $b = (b_1, \cdots, b_s)$ are $(i, n)$-segments in special position with $r \geq s$. We prove that $V(\omega_i, a) \otimes V(\omega_i, b)$ is reducible by showing that

$$\text{wt}_\ell(V(\omega_i, a) \otimes V(\omega_i, b)) \neq \text{wt}_\ell V(\omega_i, a_0 \omega_i, b_0).$$

(2.4)

Choosing $a_1$ and $b_1$ as in Lemma 2.4.3 we use Proposition 1.6.3 to choose elements $\pi$ and $\omega$ in $\mathcal{P}_n$ as follows:

$$\pi \in \text{wt}_\ell(V(\omega_i, a_1) \otimes V(\omega_i, b_1)) \setminus \text{wt}_\ell V(\omega_i, a_1 \omega_i, b_1), \quad \omega = \omega_i a_0 \omega_i a_1 \omega_i a_2 \omega_i \cdots \omega_i \cdots \omega_i a_{m+n-1-i} \omega_i \omega_i^{-1} a_{m+n-1-i} b_{m+n-1-i}.$$  \hspace{1cm} (2.5)

Here $a_0 = (a_1, \cdots, a_{j-1})$, $a_2 = (a_{j+p+1}, \cdots, a_r)$ and $b_0$, $b_2$ are defined similarly, and we understand that these segments can be empty. Using (2.2) we see that $\pi$ is in the subgroup of $\mathcal{P}_n$ generated by elements $\omega_i, \varnothing$ with

$$\min\{a_{j-1}, b_{m-1}\} < c \leq \max\{a_{j+p}, b_{m+p}\} + n + 1 < \min\{a_{j+p+1}, b_{m+p+1}\} + n + 1.$$
It follows that the expression in (2.5) for $\omega$ is reduced (c.f. Section 1.6.3). Writing $\pi = \omega_1 \omega_2$ with $\omega_1 \in \text{wt}_\ell V(\omega_{i,a})$ and $\omega_2 \in V(\omega_{i,b})$ we see by using Proposition 1.6.4 that

$$\omega = (\omega_{i,a_0} \omega_{1-i,a_2}) (\omega_{i,b_0} \omega_{n+1-i,b^2}) \in \text{wt}_\ell V(\omega_{i,a}) \otimes V(\omega_{i,b}).$$

The assertion in (2.4) follows if we prove that

$$\omega \notin \text{wt}_\ell (W(\omega_{i,a_0} \omega_{i,b_0}) \otimes V(\omega_{i,a_1} \omega_{i,b_1}) \otimes W(\omega_{i,a_2} \omega_{i,b_2})).$$

This is because the module $V(\omega_{i,a} \omega_{i,b})$ occurs in the Jordan–Holder series of the triple tensor product. Suppose for a contradiction that $\omega = \omega_{i,a_0} \omega_{i,b_0}$ where $\pi_1$, $\pi_2$, $\pi_3$ are $\ell$–weights of the corresponding modules in the tensor product. This gives,

$$\pi_1 \pi_2 \pi_3 = \omega = \omega_{i,a_0} \omega_{i,b_0} \pi \omega_{n+1-i,a_2} \omega_{n+1-i,b^2}^{-1}.$$

If $\pi_1 \neq \omega_{i,a_0} \omega_{i,b_0}$ then some $\omega_{i,c}$ with $c \in a_0 \cup b_0$ survives in a reduced expression of the left hand side. On the other hand (2.2) shows that the right hand side is in the subgroup of $P^+_n$ generated by $\omega_{j,d}$ with $d > \max \{a_j-1, b_{m-1}\}$ and hence cannot have $\omega_{i,c}$ in a reduced expression and we have a contradiction. It follows that $\pi_1 = \omega_{i,a_0} \omega_{i,b_0}$. Similarly as before write

$$\pi_3 \omega_{n+1-i,a_2} \omega_{n+1-i,b^2}^{-1} = \pi_2 \pi_{-1}.$$

If $\pi_3 \neq \omega_{n+1-i,a_2} \omega_{n+1-i,b^2}^{-1}$ there exists $c \in a_2 \cup b_2^\ast$ such that $\omega_{i,c}$ survives in a reduced expression of the left hand side. Another application of (2.2) shows that $\pi_2 \pi_3$ is in the subgroup of $P^+_n$ generated by $\omega_{j,d}$,

$$d \leq \max \{a_j+p, b_{m+p}\} + n + 1 < \min \{a_j+p+1, b_{m+p+1}\} + n + 1 \leq c,$$

and hence we have a contradiction. Then we must also have $\pi_3 = \omega_{n+1-i,a_2} \omega_{n+1-i,b^2}$ and thus $\pi = \pi_2$. But this again contradicts our choice of $\pi$ and so the assertion in (2.6) holds and the proof of reducibility is complete.

### 2.6. Proof of Theorem 11 Irreducibility

We prove irreducibility by induction on $k$. Let $k = 2$ and $a$ and $b$ be $(i, n)$–segments in general position. The irreducibility of $V(\omega_{i,a}) \otimes V(\omega_{i,b})$ follows from the fact that $\dim(V(\omega_{i,a}) \otimes V(\omega_{i,b})) = 1$ and the following claim:

$$\text{soc}(V(\omega_{i,a}) \otimes V(\omega_{i,b})) \cong V(\omega_{i,a} \omega_{i,b}) \cong \text{hd}(V(\omega_{i,a}) \otimes V(\omega_{i,b})).$$

(2.7)

For the first isomorphism assume that $V(\pi)$ is in the socle of $V(\omega_{i,a}) \otimes V(\omega_{i,b})$ for some $\pi \in P^+_n$. Using Lemma 1.3.4 write

$$\pi = \omega_{i,a} \omega_p, \quad (\pi^*)^{-1} \omega_{n+1-i,b^*} = \omega_{g}, \quad p \in P_{i,b}, \quad g \in P_{i,a}.$$

(2.8)

Since $\pi \in P^+_n$ it follows from (1.4) of Section 1.6.2 that there exist integers $0 \leq m_1, \cdots, m_s \leq \min \{i, n+1-i\}$ so that $g = (p_{i,b_1}^{m_{i,b_1}} \cdots, P_{i,b_s}^{m_{i,b_s}})$.

We use the equivalent formulation given in Proposition 2.4.1 for a pair of $(i, n)$–segments to be in general position. Without loss of generality we can assume that either $a_1 - b_s > 2 \min \{i, n+1-i\}$ or that $a_1 - b_1 \notin 2\mathbb{Z}$ or $\{b_1, \cdots, b_s\} \subset \{a_1, \cdots, a_r\}$.
Suppose that $a_1 - b_s > 2 \min\{i, n + 1 - i\}$ or that $a_1 - b_1 \notin 2\mathbb{Z}$. Then $b_j + 2m_j \notin \{a_1, \ldots, a_r\}$ and since $\pi \in \mathcal{P}^+_n$ we must have $m_j = 0$ for all $j \in [1, s]$ proving $\pi = \omega_{i,a}\omega_{i,b}$.

Consider the case when $\{b_1, \ldots, b_s\} \subset \{a_1, \ldots, a_r\}$. Suppose for a contradiction that there exists $k$ minimal with $m_k \neq 0$ and choose $j \in [1, r]$ with $b_k = a_j$. The first equality in (2.8) can be written as:

$$\pi = \omega_{i,a_1}^\epsilon \cdots \omega_{i,a_{j-1}}^\epsilon \omega_{i,a_j} \pi_1, \quad \pi_1 = \omega_{i,a_{j+1}} \cdots \omega_{i,a_r} \omega(p_k) \cdots \omega(p_r) \in \mathcal{P}^+_n,$$

where $\epsilon_m = 1$ if $a_m \notin \{b_1, \ldots, b_k-1\}$ and $\epsilon_m = 2$ otherwise. Substituting in the second equality in (2.8), we get

$$\omega_{n+1-i,a_1+n+1}^{-1} \cdots \omega_{n+1-i,a_{j-1}+n+1}^{-1} \pi_1^{-1} \omega_{n+1-i,b_{k+1}+n+1} \cdots \omega_{n+1-i,b_s} = \omega(q).$$

Notice that $\pi_1$ is in the submonoid of $\mathcal{P}^+_n$ generated by elements $\omega_{p,c}$ with $c > a_j$. Hence $\omega_{n+1-i,a_1+n+1}$ does not occur on the left hand side of the preceding equation and so cannot occur on the right. This means that $\omega(q_j)$ has an expression of the form $\omega_{p,c}$ with $c < a_j + n + 1$ which does not occur on the left hand side of the previous equality. This gives the desired contradiction and so $\omega(p) = \omega_{i,b}$ and the first isomorphism of (2.7) is proved.

If $a = b$ then the second isomorphism follows by duality. In particular this proves that the module $V(\omega_{i,a})$ is real. If $a \neq b$ then using Theorem 1.7 the second isomorphism follows if we prove that

$$\text{soc}(V(\omega_{i,b}) \otimes V(\omega_{i,a})) \cong V(\omega_{i,a}\omega_{i,b}).$$

Assume that $V(\pi)$ is in the socle and write

$$\pi = \omega_{i,b}\omega(p), \quad p = (p_1, \ldots, p_r) \in \mathcal{P}_{i,a}.$$  

By (1.4) there exist integers $m_j \in [0, \min\{i, n + 1 - i\}]$ such that $p_j = \frac{m_j}{p_{i,a_j}}$. If $a_1 - b_s > 2 \min\{i, n + 1 - i\}$ or if $b_1 - a_1 \notin 2\mathbb{Z}$ then $a_j + 2m_j \notin \{b_1, \ldots, b_s\}$ for $m_j > 0$. Hence the restriction that $\pi \in \mathcal{P}^+_n$ forces $m_j = 0$ for all $j \in [1, r]$ and $\pi = \omega_{i,b}\omega_{i,a}$.

Suppose that $\{b_1, \ldots, b_s\} = \{a_{i_1}, \ldots, a_{i_s}\}$ and that there exists $j \in [1, r]$ maximal with the property that $m_j > 0$. Since $g_j(i) = a_j + 2m_j$ the condition that $\pi \in \mathcal{P}^+_n$ forces $a_j + 2m_j = a_{i_p}$ for some $p \in [1, s]$ with $i_p > j$. But the maximality of $j$ implies that $g_{i_p}(i) = a_{i_p}$ and we get a contradiction to $g \in \mathcal{P}_{i,a}$. Hence we have proved the second isomorphism in (2.8) and the irreducibility is established when $k = 2$.

For the inductive step, assume that $a_1, \ldots, a_k$ are $(i, n)$–segments in general position of length $r_1, \ldots, r_k$ and that the irreducibility is known for all $k' < k$. To prove it for $k$, we proceed by a further induction on $N = r_1 + \cdots + r_k$. This induction clearly begins at $N = k$ by Proposition 1.3.2 (iii). Let $a$ be the maximal element of the set $a_1 \cup \cdots \cup a_k$ and let $r$ be the number of segments with $a = \max a_s$ and assume without loss of generality that $a_1, \ldots, a_s$ are the segments whose maximal value is $a$. Let $a'_s = a_s \setminus \{a\}, 1 \leq s \leq k$. Then $a_1, \ldots, a_k'$ are $(i, n)$–segments in general position and the sum of their lengths $N - r$. The inductive hypothesis implies that $V(\omega_{i,a'_1}) \otimes \cdots \otimes V(\omega_{i,a'_k})$ is irreducible and hence a quotient
of \(W(\omega_{i,a_1} \cdots \omega_{i,a_k})\). By Proposition 1.3.2(iii) we have
\[
V(\omega_{i,a})^\otimes_r \otimes W(\omega_{i,a_1} \cdots \omega_{i,a_k}) \cong W(\omega_{i,a_1} \cdots \omega_{i,a_k}).
\]
It follows that the quotient \(V(\omega_{i,a})^\otimes_r \otimes V(\omega_{i,a_1} \cdots \omega_{i,a_k})\) is \(\ell\)-highest weight and hence so is its quotient \(V(\omega_{i,a})^\otimes_r(V(\omega_{i,a_1} \otimes V(\omega_{i,a_2}) \otimes \cdots \otimes V(\omega_{i,a_k})\). Again using the fact that we have shown that \(V(\omega_{i,a}) \otimes V(\omega_{i,a_1})\) is irreducible it follows that \(V(\omega_{i,a_1}) \otimes V(\omega_{i,a_2}) \otimes \cdots \otimes V(\omega_{i,a_k})\) is \(\ell\)-highest weight. An iteration of this argument proves that \(V(\omega_{i,a_1}) \otimes \cdots \otimes V(\omega_{i,a_k})\) is \(\ell\)-highest weight. Working with the duals as usual establishes irreducibility.

3. Inflation and Monoidal Categorification

Throughout this section we work with a triple of positive integers \((i, \bar{i}, n)\) which are related by requiring
\[
n + 1 = i(\bar{i} + 1). \tag{3.1}
\]
Equivalently if we fix \(n\) then \(i\) can vary over all divisors of \(n+1\) and \(\bar{i}\) is defined by the preceding equation. We study the relationship between higher order Kirillov–Reshetikhin modules for \(U_i\) and \(\mathbf{U}_n\). We also study connections of our theory with the work of [19] [21] on monoidal categorification of cluster algebras.

3.1. The homomorphism \(\Phi_{i,n}\) and the category \(\mathcal{C}_{i,n}\). Let \(\Phi_{i,n} : \mathcal{P}_i \to \mathcal{P}_n\) be the group homomorphism defined by extending,
\[
\Phi_{i,n}(\omega_{j,a}) = \omega_{ij,ia}, \quad j \in [1, \bar{i}], \quad a \in \mathbb{Z}.
\]
We shall continue to denote by \(\Phi_{i,n}\) the induced homomorphism \(\mathbb{Z}[\mathcal{P}_i] \to \mathbb{Z}[\mathcal{P}_n]\). Notice that for \(j_1, j_2 \in [1, \bar{i}]\) and \(a, b \in \mathbb{Z}\) we have,
\[
a - b \in S_{j_1,j_2,\bar{i}} \implies i(a - b) \in S_{ij_1,ij_2,n}.
\]
It follows that if \(\omega \in \mathcal{P}_i^+\) is a prime snake then \(\Phi_{i,n}(\omega)\) is a prime snake in \(\mathcal{P}_n^+\).

Let \(\mathcal{H}_{i,n}\) (resp. \(\mathcal{H}_{i,n}^+\)) be the subgroup (resp. submonoid) of \(\mathcal{P}_n\) generated by the elements of the set
\[
\{\omega_{ij,ia} : j \in [1, \bar{i}], \quad j - a \in 2\mathbb{Z}, \quad a \in (-\infty, 0]\}.
\]
Let \(\mathcal{C}_{i,n}\) be the full subcategory of \(\mathcal{P}_n\) consisting of objects whose Jordan–Holder components are of the form \(V(\omega), \quad \omega \in \mathcal{H}_{i,n}^+\). Notice that \(\mathcal{C}_{n,n}\) is precisely the category \(\mathcal{C}^-\) studied in [21]. Let \(\mathcal{C}_{i,n}^1\) be the full subcategory consisting of objects in \(\mathcal{C}_{i,n}\) whose objects have Jordan–Holder constituents \(V(\pi)\) where \(\pi\) in the subgroup \(\mathcal{H}_{i,n}^+\) generated by \(\omega_{ij,ic} \in \mathcal{H}_{i,n}\), with \(c \geq -3\). This is the analog of the category \(\mathcal{C}_1\) studied in [19].
3.2. Recall from equation (1.2) the definition of the truncation of the \( q \)-character of an object of \( \mathcal{F}_n \) at a subgroup \( \mathcal{G} \) of \( \mathcal{P}_n \). Given \( \mathbf{a} = (a_1, \ldots, a_r) \in \mathbb{Z}^r \) it will be convenient to set
\[
i a = (ia_1, \ldots, ia_r).
\]
The following is the main result of this section.

**Theorem 2.** (i) The category \( \mathcal{C}_{\bar{i},n} \) is a monoidal category and the Grothendieck ring \( K_0(\mathcal{C}_{\bar{i},n}) \)

is a polynomial algebra generated by the elements \( \{ [V(\omega_{ij},ia)] : \omega_{ij,ia} \in H_{i,n} \} \).

(ii) The assignment
\[
\chi^{\mathcal{H}_{i,n}} : K_0(\mathcal{C}_{\bar{i},n}) \to \mathbb{Z}[H_{i,n}],
\]
is an injective homomorphism and
\[
\Phi_{i,n} \circ \chi^{\mathcal{H}_{i,n}}([V(\omega_{j,a})]) = \chi^{\mathcal{H}_{i,n}}([V(\Phi_{i,n}(\omega_{j,a}))]), \quad \omega_{j,a} \in H_{i,i}^+.
\]

(iii) We have an isomorphism \( \Psi_{i,n} : K_0(\mathcal{C}_{\bar{i},i}) \to K_0(\mathcal{C}_{\bar{i},n}) \)

such that
\[
\Psi_{i,n}([V(\omega)]) = [V(\Phi_{i,n}(\omega))],
\]

where \( \omega \in \mathcal{H}_{i,i}^+ \) is of snake type or \( \omega = \omega_{i,a} \) for some \( a \in \mathbb{Z}^r \).

(iv) The restriction of \( \Psi_{i,n} \) to \( K_0(\mathcal{C}_{1,i}) \) satisfies
\[
\Psi_{i,n}([V(\omega)]) = [V(\Phi_{i,n}(\omega))], \quad \omega \in H_{i,i}^1.
\]

In particular \( \mathcal{C}_{1,i} \) is a monoidal tensor category and \( K_0(\mathcal{C}_{1,i}) \) is a monoidal categorification of a cluster algebra of type \( A_i \).

**Remark.** Parts (i) and (ii) of the theorem were proved for \( \mathcal{C}_{i,i} \) in [21]. For a more detailed description of the relationship with cluster algebras we refer the reader to [19, Section 4.3] and [2].

**Conjecture.** We conjecture that
\[
\Psi_{i,n}([V(\omega)]) = [V(\Phi_{i,n}(\omega))], \quad \omega \in \mathcal{H}_{i,i}^1.
\]

Clearly parts (iii) and (iv) of the theorem establish special cases of the conjecture.

The proof of the theorem is in several steps.

*From now on for ease of notation we shall set \( \Phi_{i,n} = \Phi \) and \( \Psi_{i,n} = \Psi \).*

3.3. Properties of \( \Phi \).

3.3.1. Let \( \phi : P_i \to P_n \) be the group homomorphism defined by extending the assignment
\[
\phi(\omega_j) = \omega_{ij}, \quad j \in [1,\bar{i}].
\]

Clearly \( wt \circ \Phi = \phi \circ wt \).

**Lemma.** We have \( \phi(Q_{i}^+) \subset Q_{n}^+ \). Moreover for \( \pi, \omega \in \mathcal{P}_i \),
\[
wt \omega - wt \pi \in Q_{i}^+ \iff \phi(wt \omega) - \phi(wt \pi) \in Q_{n}^+.
\]
Proof. Writing $\alpha_j = 2\omega_j - \omega_{j-1} - \omega_{j+1}$, it is easily checked that

$$\phi(\alpha_j) = \sum_{p=1}^{i} \sum_{s=p-i}^{i-p} \alpha_{ij+s}, \quad j \in [1, i].$$

The forward direction of the second assertion of the lemma is now immediate. For the converse assume that $\text{wt } \omega - \text{wt } \pi = \sum_{j=1}^{i} s_j \alpha_j$ with $s_m < 0$ for some $m \in [1, i]$. The assertion follows once we notice that the coefficient of $\alpha_{im}$ in $(\phi(\text{wt } \omega) - \phi(\text{wt } \pi))$ is $is_m$. \hfill \Box

Remark. More generally, one computes that $\Phi(\mathbb{I}_{i,n}(\alpha_{j,a}), j \in [1, i], a \in \mathbb{Z}$ is equal to

$$\left( \prod_{k=1}^{i-1} \prod_{p=1}^{k} \alpha_{i(j-1)+k, i(a+1)-k+2p-2} \right) \prod_{j=1}^{i} \alpha_{ij, i(a+1)-i+2p-2}.$$

3.3.2. The set $\Phi(\text{wt}_\ell V(\omega_{k,a}))$. Given $a \in \mathbb{Z}$ and $k \in [1, i]$ we would like to show that $\Phi(\omega) \in \text{wt}_\ell V(\omega_{k,a})$ for each $\omega \in \text{wt}_\ell V(\omega_{k,a})$. Since $\omega = \omega(p)$ for some $p \in \mathbb{P}_{k,a}$, it becomes natural in view of the results of Section 1.6 to define an element $\Phi(p) \in \mathbb{P}_{k,a}$ such that $\omega(\Phi(p)) = \Phi(\omega(p))$.

For $p \in \mathbb{P}_{k,a}$ define $\Phi(p) : [0, n+1] \rightarrow \mathbb{Z}$ as follows: for $j \in [0, i] + 1$ and $j' \in [0, i - 1]$ with $ij + j' \in [0, n + 1]$,

$$\Phi(p)(ij + j') = \begin{cases} ip(j) + j', & p(j+1) - p(j) = 1, \\ ip(j) - j', & p(j+1) - p(j) = -1. \end{cases}$$

A straightforward checking shows that

$$\Phi(p) \in \mathbb{P}_{ik, ja} \quad \text{and } \quad \omega(\Phi(p)) = \Phi(\omega(p)).$$

Conversely, suppose that $g \in \mathbb{P}_{ik, ja}$ is such that $\omega(g) \in \Phi(\mathbb{P}_{i})$; equivalently:

$$m \in c_{g}^- \cup c_{g}^+ \cup \{0, n+1\} \implies m, \ g(m) \in i\mathbb{Z}.$$

Let $r \in [0, i] + 1$ and choose $m \in c_{g}^- \cup c_{g}^+ \cup \{0, n+1\}$ with

$$m \leq ir \quad \text{and} \quad g(ir) = g(m) \pm (ir - m).$$

It is immediate that $g(ir)$ is also in $i\mathbb{Z}$ and that $g(ir + j) = g(ir) \pm j$ for all $1 \leq j \leq i$. Define

$$\Phi^{-1}(g) : [0, i + 1] \rightarrow \mathbb{Z}, \quad \Phi^{-1}(g)(r) = g(ir)/i, \quad r \in [0, i] + 1.$$

It is straightforward to see that $\Phi^{-1}(g) \in \mathbb{P}_{k,a}$ and that

$$\omega(g) = \Phi(\omega(\Phi^{-1}(g))).$$

The following is trivial.

Lemma. Suppose that $s \in \{1, 2\}$ and $p_s \in \mathbb{P}_{k,s}$, for some $p_s \in [1, i]$ and $a_s \in \mathbb{Z}$. Then

$$p_1(j) < p_2(j) \quad \text{for all } j \in [0, i] + 1 \implies \Phi(p_1)(j) < \Phi(p_2)(j) \quad \text{for all } j \in [0, n+1].$$

A similar assertion holds for $s = 1, 2$ and $g_s \in \mathbb{P}_{ik, ia}$ satisfying $\omega(g_s) \in \Phi(\mathbb{P}_{i}).$ \hfill \Box
3.3.3. The set \( \text{wt}_\ell V(\Phi(\omega)) \). The following proposition is crucial in the proof of Theorem 2.

**Proposition.** Suppose that \( \omega \in \mathcal{P}_i^+ \) is a prime snake. For \( \pi \in \mathcal{P}_i \) we have
\[
\pi \in \text{wt}_\ell V(\omega) \iff \Phi(\pi) \in \text{wt}_\ell V(\Phi(\omega)).
\]

**Proof.** Writing \( \pi = \omega(p_1) \cdots \omega(p_r) \) with \( (p_1, \ldots, p_r) \in \mathbb{P}_\omega \), it is immediate from Lemma 3.3.2 that \( (\Phi(p_1), \ldots, \Phi(p_r)) \in \mathbb{P}_\Phi(\omega) \) proving that \( \Phi(\pi) \in \text{wt}_\ell V(\Phi(\omega)) \). For the converse let \( (g_1, \ldots, g_r) \in \mathbb{P}_{\Phi(\omega)} \) be such that \( \omega(g_1) \cdots \omega(g_r) \in \Phi(\mathcal{P}_i) \). Since the expression is reduced it follows that each \( \omega(g_s) \in \Phi(\mathcal{P}_i) \). By Lemma 3.3.2 we have \( (\Phi^{-1}(g_1), \ldots, \Phi^{-1}(g_r)) \in \mathbb{P}_\omega \) and the converse follows.

The next corollary is immediate using Proposition 3.3.2.

**Corollary.** For all \( \omega \in \mathcal{P}_i^+ \), we have
\[
\pi \in \text{wt}_\ell W(\omega) \implies \Phi(\pi) \in \text{wt}_\ell W(\Phi(\omega)).
\]

3.3.4. The set \( \text{wt}_\ell V(\Phi(\omega_{k,0})) \setminus \mathcal{H}_{i,n} \).

**Lemma.** Suppose that \( k \in [1, \hat{i}] \) and \( a \in \mathbb{Z} \) are such that
\[
\omega_{ik,ia} \in \mathcal{H}_{i,n}, \quad p \in \mathbb{P}_{ik,ia}, \quad \omega(p) \in \text{wt}_\ell V(\omega_{ik,ia}) \setminus \mathcal{H}_{i,n}.
\]

There exists \( s \in \mathbb{C}_p^- \) satisfying,
\[
\omega_{s,p}(s) \notin \mathcal{H}_{i,n} \quad \text{and} \quad p(s) > p(j) \quad \text{if} \quad j \in \mathbb{C}_p^+ \quad \text{with} \quad \omega_{j,p(j)} \notin \mathcal{H}_{i,n}.
\]

**Proof.** Let \( 1 \leq j_1 < \cdots < j_r \leq n+1 \) be an enumeration of \( \mathbb{C}_p^- \cup \mathbb{C}_p^+ \) and for \( m \in [1, r] \) let \( \epsilon_m = \pm 1 \) if \( j_m \in \mathbb{C}_p^+ \). Write \( \omega(p) = \omega_{j_1,p(j_1)} \cdots \omega_{j_r,p(j_r)} \). Since \( \omega(p) \notin \mathcal{H}_{i,n} \) it follows that
\[
r \geq 2, \quad \mathbb{C}_p^+ \neq \emptyset, \quad \text{and} \quad p(j_m) - p(j_{m-1}) = \epsilon_m(j_m - j_{m-1}), \quad m \in [2, r].
\]

Suppose that \( \omega_{j_m,p(j_m)} \notin \mathcal{H}_{i,n} \) and \( j_m \in \mathbb{C}_p^+ \). Then \( j_{m-1} \in \mathbb{C}_p^- \) if \( m > 1 \) and \( j_{m+1} \in \mathbb{C}_p^- \) if \( m < r \). Setting \( (j_0, e_0) = (0, 0) \) and \( (j_{r+1}, e_{r+1}) = (n+1, 0) \) we have
\[
p(j_{m-1}) - p(j_m) = j_m - j_{m-1}, \quad p(j_{m+1}) - p(j_m) = j_{m+1} - j_m.
\]
If \( \omega_{j_{m+1},p(j_{m+1})} \in \mathcal{H}_{i,n} \) the first equation gives \( p(j_m) + j_m = 0 \mod 2i \) while the second equation gives \( p(j_m) - j_m = 0 \mod 2i \) which contradicts \( \omega_{j_m,p(j_m)} \notin \mathcal{H}_{i,n} \). Hence at least one of \( \omega_{j_{m+1},p(j_{m+1})} \) does not belong to \( \mathcal{H}_{i,n} \). Since \( p(j_{m+1}) > p(j_m) \) it follows that \( p(j_m) \) is not maximal and the Lemma is proved.

3.3.5. The next proposition gives a partial converse to Corollary 3.3.3.

**Proposition.** Suppose that \( \omega = \omega_{j_1,a_1} \cdots \omega_{j_k,a_k} \in \mathcal{H}_{i,i}^+ \) and \( \pi = \omega(p_1) \cdots \omega(p_k) \in \text{wt}_\ell W(\Phi(\omega)) \) for some \( p_s \in \mathbb{P}_{ij,ia_s}, \quad s \in [1, k] \). Then
\[
\pi \in \mathcal{H}_{i,n} \cup \mathcal{P}_n^+ \implies \omega(p_s) = \Phi(\omega(g_s)), \quad g_s \in \mathbb{P}_{js,sa_s}, \quad 1 \leq s \leq k.
\]

**Proof.** In view of the Proposition 3.3.3 it suffices to prove that \( \omega(p_s) \in \mathcal{H}_{i,n} \) for all \( s \in [1, k] \). Suppose for a contradiction that \( \omega(p_m) \notin \mathcal{H}_{i,n} \) and let \( m \) be minimal with this property. Let \( s \in \mathbb{C}_{pm}^- \) be as in Lemma 3.3.3. Since \( \pi \in \mathcal{H}_{i,n} \cup \mathcal{P}_n^+ \) there exists \( m_1 \in [1, k] \) with \( s \in \mathbb{C}_{pm_1}^+ \) and \( p_{m_1}(s) = p_m(s) \). In particular this means that \( \omega(p_{m_1}) \notin \mathcal{H}_{i,n} \) and so \( m_1 > m \). Clearly this process can never stop which is absurd and the proposition follows.
Corollary. Suppose that $\omega_s \in \mathcal{H}_{i,n}^+$ and $\pi_s \in \text{wt}_\ell(V(\Phi(\omega_s)))$ for $s = 1, 2$. Then
\[ \pi_1 \pi_2 \in \mathcal{H}_{i,n} \cup \mathcal{P}_n^+ \implies \pi_s \in \mathcal{H}_{i,n}, \ s = 1, 2. \]

Proof. Recall from Section 1.3.2 that $V(\omega)$ is the unique irreducible quotient of $W(\omega)$ and that
\[ \text{wt}_\ell V(\omega) \subseteq \text{wt}_\ell W(\omega), \ \text{wt}_\ell W(\omega_1) \text{wt}_\ell W(\omega_2) = \text{wt}_\ell W(\omega_1 \omega_2). \]
Hence $\pi_1 \pi_2 \in \text{wt}_\ell V(\Phi(\omega_1) \Phi(\omega_2))$ and the corollary is now an immediate consequence of the proposition.

3.4. Proof of Theorem 2(i)–(iii). To prove that $\mathcal{C}_{i,n}$ is a monoidal tensor category it suffices to prove that the Jordan–Holder constituents of $V(\omega_1) \otimes V(\omega_2)$ with $\omega_s \in \mathcal{H}_{i,n}^+$, $s = 1, 2$ are of the form $V(\pi)$ with $\pi \in \mathcal{H}_{i,n}^+$. Writing $\pi = \pi_1 \pi_2$ with $\pi_s \in \text{wt}_\ell V(\omega_s)$, $s = 1, 2$ it follows from Corollary 3.3.5 that $\pi_s \in \mathcal{H}_{i,n}$ for $s = 1, 2$ and hence $\pi \in \mathcal{H}_{i,n}^+$.

Let $\mathcal{A}$ be the subalgebra of $K_0(\mathcal{C}_{i,n})$ generated by elements of the form $[V(\omega_{ij,ia})]$ with $\omega_{ij,ia} \in \mathcal{H}_{i,n}$. Since $K_0(\mathcal{F}_{n,\mathbb{Z}})$ is a polynomial algebra in the (infinitely many) algebraically independent generators $[V(\omega_{j,a})], \omega_{j,a} \in \mathcal{P}_n^+$ (see Theorem 1.4.1) it follows that $\mathcal{A}$ is a polynomial algebra.

To prove that $\mathcal{A} = K_0(\mathcal{C}_{i,n})$ we show that
\[ \omega \in \mathcal{H}_{i,n} \implies [V(\omega)] \in \mathcal{A} \]
by using induction on the partial order on $P$, $\lambda \leq \mu \iff \mu - \lambda \in \mathcal{Q}_n^+$. The minimal dominant elements of the partial order are 0 and $\{\omega_j : 1 \leq j \leq n\}$ and induction obviously begins by definition of $\mathcal{A}$. For the inductive step we observe that Proposition 1.3.2 gives
\[ \omega \in \mathcal{H}_{i,n} \implies [W(\omega)] \in \mathcal{A}. \]
On the other hand we can write
\[ [W(\omega)] = \sum_{\omega' \in \mathcal{P}_n^+} a_{\omega'} [V(\omega')] \in K_0(\mathcal{F}_{n,\mathbb{Z}}), \ a_{\omega'} \in \mathbb{Z}. \]
Noting that $W(\omega)_{\mu} \neq 0$ only if $\mu \leq \text{wt} \omega$ and that $\dim W(\omega)_{\text{wt} \omega} = 1$ and using Corollary 3.3.5 get
\[ a_{\omega} = 1, \ a_{\omega'} \neq 0 \implies \omega' \in \mathcal{H}_{i,n} \text{ and } \omega' \neq \omega \implies \text{wt} \omega' < \text{wt} \omega. \]
Hence the inductive hypothesis applies to all the terms with $\omega' \neq \omega$ and $a_{\omega'} \neq 0$. It follows that $[V(\omega)] \in \mathcal{A}$ and part (i) of the theorem is proved.

For part (ii), note that Corollary 3.3.5 gives
\[ \omega \in \mathcal{H}_{i,n}, \ \dim(V(\omega_1) \otimes V(\omega_2))_{\omega} \neq 0 \implies \omega = \pi_1 \pi_2, \ \pi_s \in \mathcal{H}_{i,n} \cap \text{wt}_\ell V(\omega_s), \ s = 1, 2. \]
Together with the fact that $\chi_{\mathcal{P}_n}$ is a homomorphism it follows that for $\omega_1, \omega_2 \in \mathcal{H}_{i,n}$ and $\omega \in \mathcal{H}_{i,n}$ we have
\[ \dim (V(\omega_1) \otimes V(\omega_2))_{\omega} = \sum_{\pi_1 \pi_2 = \omega} \dim (V(\omega_1)_{\pi_1} \otimes V(\omega_2)_{\pi_2}), \]
proving that $\chi_{H_{i,n}}$ is a homomorphism. To prove that it is injective, recall from [14] Lemma 6.17 that if $V_1$ and $V_2$ are two objects of $\mathcal{F}_{n,Z}$ such that $\dim(V_1)_\pi = \dim(V_2)_\pi$, for all $\pi \in \mathcal{P}_n^+$, then $[V_1] = [V_2]$ in $K_0(\mathcal{F}_{n,Z})$. Since Corollary 3.3 implies that $\ell$–dominant weight of an object of $\mathcal{C}_{i,n}$ is in $H_{i,n}^\ell$, the injectivity is now clear. The final statement of (ii) is immediate from (3.4).

For part (iii) the existence of the isomorphism $\Psi_{i,n}$ sending $[V(\omega_{j,a})] \mapsto [V(\omega_{ij,ia})]$, $j \in [1, \bar{i}]$, $a \in \mathbb{Z}$ is clear since both algebras are polynomial on these algebraically independent generators. Using part (ii) of the theorem for $V$ an object of $\mathcal{C}_{i,i}$ we have,

$$\chi_{H_{i,n}}(\Psi([V])) = \Phi \circ \chi_{H_{i,i}}([V]) = \sum_{\omega \in H_{i,i}} \dim V_{\omega} e(\Phi(\omega)).$$

Suppose that $\omega \in \mathcal{P}_i^+$ is a prime snake. By Proposition 1.6.3 we have

$$\dim V(\omega)_{\omega'} \in \{0, 1\}, \quad \omega' \in \mathcal{P}_i.$$

By Proposition 3.3.3 we have

$$\dim V(\omega)_{\omega'} = 1 \iff \dim V(\Phi(\omega))_{\Phi(\omega')} = 1.$$

Hence,

$$\chi_{H_{i,n}}([V(\Phi(\omega))]) = \sum_{\pi \in H_{i,n}} \dim V(\Phi(\omega))_{\pi} e(\pi) = \sum_{\pi \in H_{i,i}} \dim V(\omega)_{\pi} e(\Phi(\pi))$$

$$= \chi_{H_{i,n}}(\Psi([V(\omega)])),$$

Since $\chi_{H_{i,n}}$ is injective it follows that $\Psi([V(\omega)]) = [V(\omega)]$. If $\omega$ is an arbitrary snake, then by [27] we can write $V(\omega)$ as a tensor product of prime snakes. If $\mathbf{a} \in \mathbb{Z}^r$ is arbitrary with $\omega_{j,a} \in H_{i,i}$, then we use Theorem 1 to write

$$V(\omega_{j,a}) \cong V(\omega_{j,a_1}) \otimes \cdots \otimes V(\omega_{j,a_k}),$$

where $\mathbf{a}_1, \ldots, \mathbf{a}_k$ are $(j, \bar{i})$–segments in general position. Clearly $\omega_{j,a_s} \in H_{i,i}$ for all $s \in [1, k]$. Using Proposition 2.4.4 we see that the $(ij, n)$–segments $i\mathbf{a}_1, \ldots, i\mathbf{a}_k$ are also in general position (since $a_j - a_{j-1} \in 2\mathbb{Z}$ by our hypothesis). Since $\Psi$ is a ring homomorphism we get

$$\Psi([V(\omega_{j,a})]) = \Psi([V(\omega_{j,a_1})]) \cdots \Psi([V(\omega_{j,a_k})]) = [V(\omega_{ij,ia_1})] \cdots [V(\omega_{ij,ia_k})],$$

and part (iii) is proved.

### 3.5. Monoidal Categorification and Proof of Theorem 2(iv)

The notion of a monoidal categorification of a cluster algebra was introduced in [19] by Hernandez and Leclerc. They showed that the cluster algebra of type $A_1$ is isomorphic to $K_0(\mathcal{C}^1_{i,i})$ via an isomorphism which maps a cluster monomial to the class of an irreducible representation.

In terms of representation theory (see [2]) this amounts to solving the following problems in $K_0(\mathcal{C}^1_{i,i})$: classify the prime representations in the category $\mathcal{C}^1_{i,i}$, give a necessary and sufficient condition for a tensor product of prime representations to be irreducible and show that the exchange relations hold for a pair of irreducible representations whose tensor product is reducible. In [2] we also describe the Jordan–Holder series of a reducible tensor product of prime representation. Hence the equality in (3.3) in Theorem 2(iv) implies that that we
also know the Jordan–Holder series of the image under $\Psi$. It is then immediate that $C_{i,n}^1$ is a monoidal tensor category and hence also that $K_0(C_{i,n}^1)$ is a monoidal categorification of type $A_i$.

From now on our focus is to prove that $\Psi([V(\omega)]) = [V(\Phi(\omega))]$ for all $\omega \in \mathcal{H}_{i,n}^1$ and the proof occupies the rest of the section.

3.5.1. Set $I_{i,n} = \{i, 2i, \cdots, ii\} \subset [1, n]$ and let

$$\epsilon_j = 0, \quad j \in 2i\mathbb{Z}, \quad \epsilon_j = -i, \quad j \in i(2\mathbb{Z} - 1).$$

For $j, k \in I_{i,n}$ define the following elements of $\mathcal{H}_{i,n}$,

$$f_j = \omega_{j,3e_j}\omega_{j,-\epsilon_j-2i}, \quad \omega(j, k) = \omega_{j,3\epsilon_j}\omega_{j+i,3\epsilon_j+i}\cdots\omega_{k,3\epsilon_k}, \quad j \leq k, \quad \omega(j, k) = 1, \quad j > k.$$

Set

$$Pr_{i,n} = \{f_j : j \in I_{i,n}\} \cup \{\omega(j, k) : j, k \in I_{i,n}, \quad j \leq k\} \cup \{\omega_{m,-2i-\epsilon_m} : m \in I_{i,n}\}.$$

Clearly

$$Pr_{i,n} = \Phi(Pr_{i,n}).$$

Given $\omega = \omega_{j_1,a_1}\cdots\omega_{j_k,a_k} \in \mathcal{P}_n^+$ we set $ht \omega = k$.

3.5.2. The next proposition was proved in the special case when $i = 1$ in [19] (see also [2]).

Proposition. Let $\pi_1, \pi_2 \in Pr_{i,n}$. Then

$$[V(\pi_1)][V(\pi_2)] = [V(\pi_1\pi_2)]$$

if one of the following hold:

(i) $\pi_1 = f_j$ for some $j \in I_{i,n}$,

(ii) $\pi_1 = \omega(j, k)$ and $\pi_2 = \omega(m, r)$ for $j, k, m, r \in I_{i,n}$ with $j \leq k$, $m \leq r$ and $m \neq k + i$ and one of the following hold:

\begin{itemize}
  \item $j = m, \quad k \leq m, \quad k = r, \quad j < m < k < r$ and $\epsilon_{k-m} = 0, \quad j < m < r < k$ and $\epsilon_{r-m} = -i$.
  \item $\pi_1 = \omega_{j,-\epsilon_j-2i}$ and $\pi_2 = \omega_{k,-\epsilon_k-2i}$ or $\pi_2 = \omega(m, r)$ with $j \notin [m, r]$.
\end{itemize}

Proof. The proof of the proposition follows the lines of the proof of the corresponding result for $C_{i,n}^1$ given in [2] Theorem 3. The basic idea is to proceed by induction on $ht \pi_1\pi_2$. We shall prove that induction begins; the proof of the inductive step is identical (and easier) to the one given in [2] Section 3.7.

To see that induction begins for part (i) we must prove that $V := V(\omega_{k,a}) \otimes V(f_j)$ is irreducible for all $j, k \in I_{i,n}$ and $a \in \{3\epsilon_j, -2i - \epsilon_k\}$. If $k = j$ the result is a particular case of Theorem [1]. If $|j - k| > i$ it is easy to check that for $b \in \{3\epsilon_j, -2i - \epsilon_j\}$ we have

$$\pm(a - b) \notin \{2p + 2 - j - k : \max\{j, k\} \leq p < \min\{j + k - 1, n\}\}.$$  \hfill (3.5)

It follows from [1] that $V$ is irreducible. If $|j - k| = i$ there are four cases to consider corresponding to $\{\epsilon_k, \epsilon_j\} = \{0, -i\}$ and $a \in \{3\epsilon_j, -2i - \epsilon_k\}$. Assume first that $\epsilon_k = 0$. If $a = -2i$ then (3.5) again holds and the irreducibility follows from [1] as before. If $a = 0$ then it suffices to prove that,

$$wt_\ell(V(\omega_{k,0}) \otimes V(f_j)) \cap \mathcal{P}_n^+ = \{\omega_{k,0}f_j\}.$$

To prove this we assume that \( \omega \) is an element of the intersection. Proposition 3.3.3 and Corollary 3.3.5 imply that we can write \( \omega = \Phi(\omega_1) \Phi(\omega_2) \) with \( \omega_1, \omega_2 \in P_i \). \( \omega_1 \in \text{wt}_i \omega_1 \) and \( \omega_2 \in V(f_{j/i}) \). Hence \( \omega_1 \omega_2 \in P_i^+ \cap \text{wt}_i \omega_1 V(\omega_{k/i,0}) \) and it follows from [21 Proposition 4.3(a)] that \( \omega_1 \omega_2 = \omega_{k/i,0} \omega_{j/i-1} \omega_{j/i-3} \).

Suppose that \( \epsilon_k = -i \). Then
\[
\Omega(V(\omega_{k,a}) \otimes V(f_j))^* = V(\omega_{k,-a+n+1}) \otimes V(\omega_{j,-3\epsilon_j+n+1} \omega_{j,2i+\epsilon_j+n+1}).
\]

It follows from the previous case, after a suitable shift of parameters, that the module on the right is irreducible and hence, so is \( V(\omega_{k,a}) \otimes V(f_j) \).

For parts (ii)-(iv) to see that induction begins we have to establish the cases when \( h \pi_1 + h \pi_2 \leq 4 \). Suppose that \( h \pi_1 + h \pi_2 = 2 \). In this case, if we write \( \pi_1 \pi_2 = \omega_{j,a} \omega_{k,b} \) it follows that \( \pi_1 \pi_2 \notin \text{Pr}_{i,n} \). Then \( \pm(a-b) \notin \{2p + 2 - j - k : p < \min\{j + k - 1, n\}\} \) and it follows from [3] that \( V(\omega_{j,a}) \otimes V(\omega_{k,b}) \) is irreducible.

Suppose that \( h \pi_1 + h \pi_2 = 3 \). In this case we have to prove that if \( \omega(j, j+i) \omega_{k,b} \notin \text{Pr}_{i,n} \cup k \text{Pr}_{i,n} \) then \( V(\omega(j, j+i)) \otimes V(\omega_{k,b}) \) is irreducible. Note that the hypothesis implies that
\[
\omega_{j,3\epsilon_j} \omega_{k,b} \notin \text{Pr}_{i,n}, \quad \text{and} \quad \omega_{j+i,3\epsilon_j+i} \omega_{k,b} \notin \text{Pr}_{i,n}.
\]

If \( \epsilon_j = -i \) and hence \( \epsilon_{j+i} = 0 \) it follows from Proposition 1.3.2(iii) and the preceding case that
\[
V(\omega_{j+i,0}) \otimes V(\omega_{j,-3i}) \otimes V(\omega_{k,b}), \quad V(\omega_{k,b}) \otimes V(\omega_{j+i,0}) \otimes V(\omega_{j,-3i})
\]
are both \( \ell \)-highest weight modules. Hence \( V(\omega(j, j+i)) \otimes V(\omega_{k,b}) \) and its dual are \( \ell \)-highest weight and thus irreducible by Proposition 1.3.2(i). The case \( \epsilon_j = 0 \) is completely analogous and we omit the details.

If \( h \pi_1 + h \pi_2 = 4 \) then we have to prove that the following modules are irreducible:
\[
V(\omega(j, j+i)) \otimes V(\omega(j, j+i)), \quad V(\omega(j, j+i)) \otimes V(\omega(j+i, j+2i)).
\]
The first assertion is immediate from [13] since \( \omega(j, j+i) \) is a prime snake. For the second, if \( \epsilon_j = 0 \) then
\[
\omega(j, j+i) = \omega(j,0) \omega_{j+i,-3i}, \quad \omega(j+i, j+2i) = \omega_{j+i,-3i} \omega_{j+2i,0}.
\]
Using Proposition 1.3.2(iii) and the previous case it follows that the modules
\[
V(\omega_{j,0}) \otimes V(\omega_{j+i,-3i}) \otimes V(\omega(j+i, j+2i)), \quad V(\omega_{j+2i,0}) \otimes V(\omega_{j+i,-3i}) \otimes V(\omega(j, j+i))
\]
are \( \ell \)-highest weight and the irreducibility of \( V(\omega(j, j+i)) \otimes V(\omega(j+i, j+2i)) \) follows as before. The proof when \( \epsilon_j = -i \) is entirely analogous so we omit details. \( \square \)

**Proposition.** For \( p, j, k \in I_{i,n} \) with \( j < k \), the following identities hold in \( K_0(\mathcal{C}_i) \).
\[
(i) \quad [V(\omega_{p,3p})(V(\omega_{p,-2i-\epsilon_p})) = [V(f_p)] + [V(\omega_{p-i,\epsilon_p-i})][V(\omega_{p-i,\epsilon_p-i})],
\]
\[
(ii) \quad [V(\omega_{j,-2i-\epsilon_j})][V(\omega(j, k))] = [V(f_j)][V(\omega(j+i, k))] + [V(\omega_{j-i,\epsilon_j-i})][V(f_{j+i}(\omega(j+2i, k))],
\]
\[(iii) \quad [V(\omega, j, 3\epsilon_j)] [V(\omega, j + i, k)] = [V(\omega, j, k)] + [V(\pi)],\]

where

\[ [V(\pi)] = [V(\omega, j - i, \epsilon_j - i)] [V(\omega, j + 2i, \epsilon_j + i)] [V(f_{j + 2i}, \omega, j + 3i, k)]^{-\delta_{j + i, k}}, \]

**Proof.** For \( j, k \in I_{i,n} \) with \( k \geq j \), set

\[
M(j, k) = V(\omega, j, 3\epsilon_j) \otimes V(\omega, j + i, k) \otimes V(\omega, j - 2i, \epsilon_j)
\]

\[
U(j, k) = V(\omega, j, k) \otimes V(\omega, j - 2i, \epsilon_j), \quad W(j, k) = V(f_j) \otimes V(\omega, j + i, k),
\]

Then parts (i) and (ii) the proposition can be formulated as:

\[
[M(j, j)] = [U(j, j)] + [V(\omega, j, j - i)][V(\omega, j - i, \epsilon_j - i)]
\]

\[
[U(j, k)] = [W(j, k)] + [V(\omega, j - i, \epsilon_j - i)][W(j + i, k)], \quad k > j.
\]

Since the Grothendieck ring has no zero divisors we see that the following equations are equivalent to part (iii) of the proposition:

\[
[M(j, j + i)] = [U(j, j + i)] + [V(\omega, j - i, \epsilon_j - i)][V(\omega, j + 2i, \epsilon_j + i)][V(\omega, j - 2i, -\epsilon_j)],
\]

\[
[M(j, k)] = [U(j, k)] + [V(\omega, j - i, \epsilon_j - i)][V(\omega, j - 2i, \epsilon_j)][W(j + 2i, k)], \quad k \geq j + 2i.
\]

The equalities in (3.6) and (3.8) are immediate from Proposition 1.6.3. Since the module \( V(\omega, j, k) \) occurs in the Jordan–Holder series of the reducible tensor product \( V(\omega, j, k) \otimes V(\omega, j, 3\epsilon_j) \) we can write

\[
[M(j, k)] = [U(j, k)] + [K_1(j, k)], \quad k \geq j
\]

where \([K_1(j, j)] = 0\) and \([K_1(j, k)]\) is the class of an object of \( \mathcal{F}_{n,Z} \). Note that by (3.8) we have

\[
[K_1(j, j + i)] = [V(\omega, j - i, \epsilon_j - i)][V(\omega, j + 2i, \epsilon_j + i)][V(\omega, j - 2i, -\epsilon_j)].
\]

The module \( W(j, k) \) is irreducible by Proposition 3.5.2 and hence occurs in the Jordan–Holder series of both \( M(j, k) \) and \( U(j, k) \). In particular we can write

\[
[M(j, k)] = [W(j, k)] + [K_2(j, k)], \quad [U(j, k)] = [W(j, k)] + [K_3(j, k)], \quad k \geq j
\]

where \( K_s(j, k), \quad s = 2, 3 \) is the class of an object of \( \mathcal{F}_{n,Z} \). Combining the expressions for \([M(j, k)]\) and \([U(j, k)]\) we get

\[
[K_1(j, k)] + [K_3(j, k)] = [K_2(j, k)].
\]

Note that

\[
[K_2(j, j)] = [K_3(j, j)] = [V(\omega, j - i, \epsilon_j - i)][V(\omega, j + i, \epsilon_j - i)].
\]

Proposition 1.6.3 gives

\[
[K_2(j, k)] = [V(\omega, j - i, \epsilon_j - i)][V(\omega, j + i, \epsilon_j - i)][V(\omega, j + i, k)] = [V(\omega, j - i, \epsilon_j - i)][U(j + i, k)],
\]

where the last equality holds if \( k > j \). Hence we have

\[
[K_3(j, j + i)] + [K_1(j, j + i)] = [V(\omega, j - i, \epsilon_j - i)][(V(f_{j + i})] + [V(\omega, j - 2i, -\epsilon_j)][V(\omega, j + 2i, \epsilon_j + i)]]
\]

\[
[K_3(j, k)] + [K_1(j, k)] = [V(\omega, j - i, \epsilon_j - i)][K_3(j + i, k)] + [W(j + i, k)], \quad k \geq j + 2i.
\]

The equalities in (3.7) and (3.9) hold if we prove that

\[
[K_1(j, k)] = [V(\omega, j - i, \epsilon_j - i)][K_3(j + i, k)],
\]
[K_3(j,j+i)] = [V(\omega_{j+1-i})]_{i}^{j+i}, \quad [K_3(j,k)] = [V(\omega_{j-i,\epsilon_j-i})]_{i}^{j+k}, \quad k \geq j + 2i.

We proceed by an induction on \(k\). To see that induction begins at \(k = j + i\), we observe that (3.11) and (3.8) give \([K_3(j,j+i)] = [V(\omega_{j-i,\epsilon_j-i})][W(j+i,k)]\) as needed.

Suppose that \(k \geq j + 2i\). Note that the \(\ell\)-weight \(\omega_{j-i,\epsilon_j-i}\), \(f_{j+1}^i\omega(j+2i,k)\) of \(V(\omega_{j-i,\epsilon_j-i}) \otimes W(j+i,k)\) occurs in both \(M(j,k)\) and \(U(j,k)\) with multiplicity one so it cannot occur in \(K_1(j,k)\). By Proposition 3.5.2 each of the terms on the right hand side of (3.12) is the class of an irreducible representation. It follows that

\[ [K_1(j,k)] = [V(\omega_{j-i,\epsilon_j-i})][K_3(j,i,k)], \quad [K_3(j,k)] = [V(\omega_{j-i,\epsilon_j-i})][W(j+i,k)]. \]

Using the inductive hypothesis we get the desired formulae. □

3.5.4. **Proof of Theorem 2(iii).** Notice that by Theorem 2(iii) we know part (iv) for the elements \(\omega \in \Pr_{\tilde{\ell}}\) such that \(\text{ht} \omega \leq 2\). A straightforward induction on \(\text{ht} \omega(j,k)\) along with Proposition 3.5.3(iii) show that part (iv) holds for all \(\omega(j,k) \in \Pr_{\tilde{\ell}}\).

To complete the proof of part (iv) it suffices to show that if \(\omega, \omega' \in \Pr_{\tilde{\ell}}\) are such that \(V(\omega) \otimes V(\omega')\) is irreducible then so is \(V(\Phi(\omega)) \otimes V(\Phi(\omega'))\). For this we recall that in [2] we actually proved that in the category \(\mathcal{C}^1_{\tilde{\ell}}\), the conditions for irreducibility given in Proposition 3.5.2 are also necessary. Observing that if \(\omega, \omega' \in \Pr_{\tilde{\ell}}\) satisfy the conditions in Proposition 3.5.2 then \(\Phi(\omega), \Phi(\omega') \in \Pr_{\tilde{\ell}}\) satisfy the conditions in Proposition 3.5.2 and hence \(V(\Phi(\omega)) \otimes V(\Phi(\omega'))\) is irreducible and the proof is complete. □

**Remark.** In [19] Theorem 7.8 it is proved that for each \(\omega \in \Pr_{\tilde{\ell}}\) we have

\[ \chi^{H_{i,i}}V(\omega) = \omega F_\omega(\alpha_{i,\epsilon_j-1}^{-1}, \ldots, \alpha_{i,\epsilon_j-1}^{-1}) \]

where \(F_\omega(y_1, \ldots, y_\ell)\) is the \(F\)-polynomial of the cluster variable associated to \([V(\omega)]\). Therefore, using (3.2) we have that

\[ \chi^{H_{i,a}}V(\Phi(\omega)) = \Phi \circ \chi^{H_{i,i}}V(\omega) = \Phi(\omega)F_\omega(\Phi(\alpha_{1,\epsilon_j-1}^{-1}), \ldots, \Phi(\alpha_{\ell,\epsilon_j-1}^{-1})) \]

(see Remark 3.3.1). In this sense, \(F_\omega\) can be identified with the evaluation of the \(F\)-polynomial of \([V(\Phi(\omega))]\), however the evaluation is different from the one in [19].

4. **Imaginary Modules**

Recall that an irreducible \(\tilde{U}_n\)-module is said to be real if its tensor square is irreducible and otherwise the module is said to be imaginary. It has been known since 1991 (see [6] that imaginary modules do not exist when \(n = 1\). The first example of an imaginary module was given in 2006 by Leclerc in [26], where he proved that if \(n = 5\), the module \(V(\omega)\) with \(\omega = \omega_{2,6,\omega_{1,3,3}}\omega_{2,0,0} \in \Pr_{5}\) is imaginary. Subsequently further examples of imaginary modules were given in [25]. In both cases, the proof that the modules are imaginary is indirect; examples are first constructed for the affine Hecke algebra and then affine Schur–Weyl duality gives that the corresponding module for a quantum affine algebra \(\tilde{U}_N\) for some \(N\) sufficiently large is imaginary. Note that the reducibility of a tensor product in rank \(N\) does not necessarily imply reducibility for all \(r < N\). For instance, as we shall see in this section, the example given by Leclerc, is actually imaginary for \(\tilde{U}_3\) but this cannot be deduced by using
Since and Leclerc speculated imaginary simple objects existed in $C$ and reminded the reader that Hernandez sor products in (4.1). In particular any submodule of $V$ mention.

Setting $an (i, n)$ be the composition $\id \circ \otimes \hat{U}$ with tensor products of $\hat{U}$.

In this section we shall give a systematic way to construct imaginary modules by working with tensor products of $\hat{U}$-modules associated to $(i, n)$-segments with $n \geq 2i$ and $i \geq 2$. In Section 4.5 we give a family of examples of imaginary modules for the quantum affine algebra $D$.

In the case of type $A$ which do not arise from an $A_3$-example. In fact we shall see that the corresponding $A_3$-module is real.

4.1. The following is the main result of this section.

**Theorem 3.** Let $n \geq 3$ and $i \in [2, n - 1]$ with $2i \leq n + 1$ and $r \geq 2$. Let $b = (b_1, \cdots, b_r)$ be an $(i, n)$-segment with $b_j - b_{j-1} < 2i$ for all $j \in [2, r]$ and let

$$a = (a_1, \cdots, a_r), \quad a_k = b_k - n - 1, \quad k \in [1, r], \quad s_j = \frac{1}{2}(b_{j-1} - b_j + 2i), \quad j \in [2, r].$$

Setting

$$\omega = \omega_{i, b_r} \omega_{i-s_r, b_{r-1}-s_r, \omega_{n+1-i+s_r, a_r+s_r}} \cdots \omega_{i-s_2, b_1-s_2, \omega_{n+i-s_2, a_2+s_2}} \omega_{n-i+1, a_1},$$

the module $V(\omega)$ is imaginary.

The proof of the theorem is given in Sections 4.2–4.4.

4.2. The key step. The proof of the following proposition can be found in Section 4.4.

**Proposition.** Retain the notation established so far and let $\pi \in \wt^+ W(\omega)$. Then

$$\Hom_{\hat{U}}(W(\pi), V(\omega_{i, b}) \otimes V(\omega_{n+1-i, a})) \neq 0 \iff \pi \in \{1, \omega\}.$$

4.3. Proof of Theorem 3. Assuming Proposition 4.2 we complete the proof of the theorem.

4.3.1. The map $\Phi$. Noting that $\omega_{n+1-i, a} = \omega_{i, b}$ we have $\hat{U}$-maps,

$$\C \hookrightarrow V(\omega_{i, b}) \otimes V(\omega_{n+1-i, a}), \quad \langle : V(\omega_{n+1-i, a}) \otimes V(\omega_{i, b}) \to \C \to 0. \quad (4.1)$$

Let

$$\Phi : V(\omega_{i, b}) \otimes V(\omega_{n+1-i, a}) \otimes V(\omega_{i, b}) \otimes V(\omega_{n+1-i, a}) \to V(\omega_{i, b}) \otimes V(\omega_{n+1-i, a}) \to 0,$$

be the composition $\id \circ \langle \circ \id$.

Since $V(\omega_{i, b})$ is real by Theorem 4.1 it follows that $\C$ is the socle (resp. head) of the tensor products in (4.1). In particular any submodule of $V(\omega_{i, b}) \otimes V(\omega_{n+1-i, a})$ must contain the trivial representation. In what follows, we shall make use of this remark without further mention.
4.3.2. We need the following result on the set of $\ell$–weights of a tensor product of a prime snake module and its dual.

**Lemma.** Let $\pi = \omega_{i_1,c_1} \cdots \omega_{i_k,c_k}$ with $i_1, \ldots, i_k \in [1,n]$ and $c_1 < \cdots < c_k$ be a prime snake and set $V := V(\pi) \otimes V(\pi^*)$. Then

$$\dim V_1 = 1 = \dim(V \otimes V)_1.$$  

Moreover if $\pi_1 \in P_n$ has a reduced expression containing $\omega_{n+1-i_1,c_1-n-1}^{s}$ for some $s \geq 2$ then $\pi_1 \notin \text{wt}_\ell V.$

**Proof.** Set $\ell \geq 1$. Proposition 4.3.4. by Lemma 4.3.2 it follows that $V_{\ell}$ has the highest weight vector with $\pi$ of Proposition 4.2 and proves that $\ell \geq 1$. Moreover if $\pi_1 \in P_n$ has a reduced expression containing $\omega_{n+1-i_1,c_1-n-1}^{s}$ for some $s \geq 2$ then $\pi_1 \notin \text{wt}_\ell V.$

**Proof.** Suppose that $\pi_1 = 1$ for some $\pi \in \text{wt}_\ell V(\pi)$ and $\pi' \in \text{wt}_\ell V(\pi^*)$ and write

$$\pi' = \omega(g_1') \cdots \omega(g_k'), \quad \pi = \omega(g_1) \cdots \omega(g_k), \quad (g_1', \ldots, g_k') \in \mathbb{P}^* \pi, (g_1, \ldots, g_k) \in \mathbb{P} \pi.$$  

Since $\text{wt}_\ell V(\pi)$ is in the subgroup of $P$ generated by elements $\omega_{j,c}$ with $c \geq c_1$ we see that $\pi'$ must also be in this subgroup. It follows that $\omega(g_1') = \omega_{i_1,c_1}^{-1}$. Since $\omega_{i_1,c_1}$ cannot occur in $\omega(g_s)$ for any $s > 1$ we get $\omega(g_1') = \omega_{i_1,c_1}^{-1}$. An obvious iteration proves that $\pi' = \omega_{i_1,b}^{-1}$ and hence $\dim V_1 = 1$. The proof that $\dim(V \otimes V)_1 = 1$ is identical. The second assertion is immediate from Proposition 1.6.3 applied to $\pi$ since $\omega_{n+1-i_1,c_1-n-1}$ cannot occur in any $\ell$–weight of $V(\pi)$.

4.3.3. **Lemma.** There exists an $\ell$–highest submodule $M(\omega)$ of $V(\omega_{i,b}) \otimes V(\omega_{n+1-i,a})$ and a non–split short exact sequence

$$0 \to \mathbb{C} \to M(\omega) \to V(\omega) \to 0.$$  

**Proof.** By Proposition 4.2 there exists a non–zero map of $\mathbb{U}_n$–modules

$$f : W(\omega) \to V(\omega_{i,b}) \otimes V(\omega_{n+1-i,a}), \quad f(v_\omega) \neq 0.$$  

Then $M(\omega) = \mathbb{U}_nf(v_\omega)$ is an $\ell$–highest weight module and hence so is any quotient of it. Moreover, $V(\omega)$ is its unique irreducible quotient and since $\mathbb{C} \hookrightarrow M(\omega)$ it follows that $M(\omega)/\mathbb{C}$ is $\ell$–highest weight. Suppose that $M(\omega)/\mathbb{C}$ is reducible and let $N$ be the maximal submodule of $M(\omega)/\mathbb{C}$. Taking $N$ to be the preimage in $M(\omega)$ of $N$ we see that $N$ must contain an $\ell$–highest weight vector with $\ell$–weight $\pi \notin \{1, \omega\}$. But this is impossible by the forward direction of Proposition 4.2 and proves that $M(\omega)/\mathbb{C}$ is irreducible. The fact that the short exact is non–split is immediate since $M(\omega)$ is $\ell$–highest weight and the lemma is proved.

4.3.4. **Theorem.** The next proposition and its corollary establishes Theorem 3.

**Proposition.** The restriction of $\Phi$ to $M(\omega) \otimes M(\omega)$ is non–zero and $\Phi(M(\omega) \otimes M(\omega))$ has an irreducible quotient $V(\pi)$ with $\pi \in P_n^+ \setminus \{1, \omega, \omega^2\}$.

**Proof.** Set $V := V(\omega_{i,b}) \otimes V(\omega_{n+1-i,a})$. Using Lemma 4.3.2 and Lemma 4.3.3 we have

$$\dim(M(\omega) \otimes M(\omega))_1 = \dim(V \otimes V)_1 = \dim V_1 = 1.$$  

Since $\Phi : V \otimes V \to V$ is surjective it follows that $\Phi(M(\omega) \otimes M(\omega)) \neq 0$. Since $\omega^2 \notin \text{wt}_\ell V$ by Lemma 4.3.2 it follows that $V(\omega^2)$ is not a quotient of $\Phi(M(\omega) \otimes M(\omega))$. We claim that
\( \mathbb{C} \) is also not a quotient of \( M(\omega) \otimes M(\omega) \). Otherwise we would have a non–zero element, say \( f \in \text{Hom}_{U_\omega}(M(\omega), \star M(\omega)) \). By Lemma 1.3.3 we have
\[
0 \to V(\star \omega) \to \star M(\omega) \to \mathbb{C} \to 0.
\]
Since \( M(\omega) \) is \( \ell \)-highest weight it follows that the image of \( f \) must be isomorphic to \( V(\star \omega) \). But this is absurd since \( \omega \neq \star \omega \). Hence \( \mathbb{C} \) is also not a quotient of \( \Phi(M(\omega) \otimes M(\omega)) \). To prove that \( V(\omega) \) is not a quotient as well, it suffices to show (since \( M(\omega) \) is an \( \ell \)-highest weight module) that
\[
\text{Hom}_{U_\omega}(W(\omega) \otimes W(\omega), V(\omega)) = 0.
\]
This follows from Lemma 1.3.3 if we prove that \( (\omega^\star)^{-1} \omega \not\in \text{wt}_\ell W(\omega) \). Write,
\[
\omega = \omega_1 \omega_2, \quad \omega_1 = \omega_{n+1-i,a_1} \prod_{j=2}^{m} \omega_{i-s_j,b_1-s_j} \prod_{j=2}^{p} \omega_{n+1-i+s_j,a_j+s_j},
\]
where \( m = \max\{j \in [2, r] : b_j - 1 - s_j < b_1\} \) and \( p = \max\{j \in [2, r] : a_j + s_j < b_1\} \). Notice that \( \omega_1 \omega_1^{-1} \in \{1, \omega_{b_1, b_1}\} \) occurs in a reduced expression for \( (\omega^\star)^{-1} \omega \). We have \( \text{wt}_\ell W(\omega) = \text{wt}_\ell W(\omega_1) \cdot \text{wt}_\ell W(\omega_2) \) and also that \( \text{wt}_\ell W(\omega_2) \) is contained in the submonoid generated by \( \omega_{k,c}^+ \) with \( c > b_1 \) and \( \omega_{k,b_1} \) for \( k \in [1, n] \). This means that \( \omega_1 \omega_1^{-1} \omega \) must be in \( \text{wt}_\ell W(\omega_1) \) which is clearly impossible.

Corollary. There exists \( \pi \in \mathcal{P}_n^+ \) with \( \pi \neq \omega^2 \) and a non–zero map \( V(\omega) \otimes V(\omega) \to V(\pi) \). In particular, \( V(\omega) \) is imaginary.

Proof. The proposition implies that there exists
\[
f : \Phi(M(\omega) \otimes M(\omega)) \to V(\pi) \to 0, \quad \pi \in \mathcal{P}_n^+ \setminus \{1, \omega, \omega^2\}.
\]
Noting that
\[
V(\omega) \otimes V(\omega) \cong (M(\omega) \otimes M(\omega))/[(M(\omega) \otimes \mathbb{C} + \mathbb{C} \otimes M(\omega))],
\]
\[
\pi \notin \{1, \omega\} \implies f(\Phi(M(\omega) \otimes \mathbb{C} + \mathbb{C} \otimes M(\omega))) = 0,
\]
we have an induced map \( V(\omega) \otimes V(\omega) \to V(\pi) \to 0 \). Since \( \pi \neq \omega^2 \) this map cannot be an isomorphism and the corollary follows. □

4.4. Proof of Proposition 1.4.2

4.4.1. A simple calculation shows that
\[
\omega = \omega_{i,0}^{m}(g), \quad g = (g_{n+1-i,0}^{m_1}, \cdots, g_{n+1,0}^{m_r}) \in \mathbb{P}_{n+1-i,a}.
\]
Assume that \( \text{Hom}_{U_{\omega}}(W(\pi), V(\omega_{i,0}) \otimes V(\omega_{n+1-i,0})) \neq 0 \) for some \( \pi \in \mathcal{P}_n^+ \). Using Lemma 1.3.3 and Section 1.6.2 we write
\[
\pi = \omega_{i,0}^{m}(g), \quad g = (g_{n+1-i,0}^{m_1}, \cdots, g_{n+1-i,0}^{m_r}) \in \mathbb{P}_{n+1-i,a}, \quad m_j \in [0, i], \quad j \in [1, r],
\]
where \( m_j > 0 \implies b_j - 2i + 2m_j \in \{b_1, \cdots, b_j\} \). (4.2)

If in addition \( \pi \in \text{wt}_\ell W(\omega) \) then \( \omega(p) \in \omega(g) \mathbb{Q}^+ \), and an application of Proposition 1.6.3(iv) now gives \( m_j \geq s_j > 0 \) for all \( j \in [2, r] \). Since
\[
i \geq m_j \geq s_j \implies b_{j-1} = b_j + 2(s_j - i) \leq b_j + 2(m_j - i) \leq b_j,
\]
we see that \(4.2\) forces \(b_j - 2i + 2m_j = b_j\), i.e. \(m_j \in \{s_j, i\}\) for all \(j \in [2, r]\). If \(j = 1\) then again the restriction that \(\pi \in P^+_1\) implies \(m_1 \in \{0, i\}\). Finally note that these arguments prove that \(m_j = i\) implies \(m_{j+1} = i\). Summarizing, we have proved that if \(\pi \notin \{1, \omega\}\) then

\[
\pi = \omega_{i,b} \omega_{b_1, b_1 - s_j} \omega_{n + 1 - i, s_j, a, + s_j} \cdots \omega_{i-s_2, b_1 - s_2, \omega_{n + 1 - i, s_2, a_2 + s_2}} \omega_{n + 1 - i, a_1}
\]

for some \(j \in [1, r - 1]\). The second condition in Lemma \[1.3.4\] gives

\[
(\pi^*)^{-1} \omega_{i,b} = (g_1) \cdots (g_r), \quad (g_1, \cdots, g_r) \in P_{i,b}.
\]

A reduced expression of the left hand side of the preceding equality always contains \(\omega_{i,b}\) for all \(k \in [2, r]\) and does not contain \(\omega_{n + 1 - i, b_k + n + 1}^{-1}\) for any \(k \neq j\). Hence a standard argument using Proposition \[1.6.3\] shows that \(\omega(g_p) = \omega(i,b)\) for all \(p \in [2, r]\) with \(p \neq j\) and \(\omega(g_j) \in \{\omega_{i,b}, \omega_{i,b + n + 1}\}\). But this means that

\[
\omega(g_1) \cdots \omega(g_r) = \omega_{i,b} \cdots \omega_{i,b}^{-1} g_{n+1-i-b_n+1} \cdots \omega_{i,b} \omega(g_1), \quad \epsilon = 0, 1
\]

or equivalently that

\[
(\pi^*)^{-1} \omega_{i,b} \omega_{i,b} = \omega_{i,b}^{-1} g_{n+1-i-b_n+1} \omega(g_1).
\]

Substituting for \(\pi\) we are forced to have \(\epsilon = 0\). i.e. \(\omega(g_j) = \omega_{n + 1 - i, b_j + n + 1}^{-1}\). But this means that \(g_j(n + 1 - i) = b_j + n + 1 > g_{j+1}(n + 1 - i)\) and hence we have a contradiction to \((g_1, \cdots, g_r) \in P_{i,b}\).

**4.4.2.** The following proposition is needed for the converse direction.

**Proposition.** Let \(J = \{j + 1, \cdots, k - 1\}\) be a subset of \(I\) and assume that \(\pi\) (resp. \(\pi_1, \pi_2\)) are in the submonoid of \(P^+_n\) generated by \(\omega_{j,c}\) with \(j \in J\) (resp. \(I \setminus J\)). Setting

\[
\pi = \omega_{j_1, c_1} \cdots \omega_{j_s, c_s},
\]

\[
\pi' = \omega_{j_1, c_1 + k - j} \cdots \omega_{j_s, c_s + k - j}, \quad I = \omega_{k-j, c_1 - k + j} \cdots \omega_{k-j, c_s - k + j}.
\]

we have non–zero maps

\[
W(\pi_1 \pi_2) \rightarrow V(\pi_1 \pi') \otimes V(\pi_2), \quad W(\pi_1 \pi - \pi_2) \rightarrow V(\pi_1 \pi) \otimes V(\pi_2),
\]

where, \(\pi^* = (\omega_{j_1, c_1 + k - j_1} \cdots \omega_{j_s, c_s + k - j_s})\).

**Proof.** Let \(v_1 \in V(\pi_1 \pi')\) and \(v_2 \in V(\pi_2)\) be the highest \(\ell\)-weight vectors. By Section \[1.5\] we have the following maps of \(U_{n,J}\)-modules

\[
V_f(\pi) \hookrightarrow V(\pi s_n), \quad s = 1, 2, \quad U_{n,J} v_2 \cong V_f(\pi), \quad V_f(\pi)^* \cong U_{n,J} v_1 \cong V_f(\pi').
\]

Hence we have non–zero maps of \(U_{n,J}\)-modules,

\[
C \hookrightarrow V_f(\pi') \otimes V_f(\pi) \hookrightarrow V(\pi_1 \pi') \otimes V(\pi_2),
\]

and we let \(v \in V(\pi_1 \pi') \otimes V(\pi_2)\) be the image of \(1 \in C\). It is straightforward to compute that,

\[
\pi' \pi = \alpha_j, \quad \alpha = \prod_{p=1}^{k-1} \prod_{r = j_p}^{j_p-j-1} \prod_{t=0}^{\alpha_{r-t, c_p + t + 1}}.
\]

Section \[1.5\] implies that \(v\) is an \(\ell\)-highest weight vector with \(\ell\)-weight \(\pi_1 \pi_2 \pi' \pi \alpha^{-1}\) and a further simple computation shows that \(\pi' \pi \alpha^{-1} = \pi^+\) which proves the existence of the first
map. The existence of the second map in the statement of the proposition is proved in an identical fashion and we omit the details.

4.4.3. Set,

\[ \omega_0 = \omega_{n+1-s_2, a_2-i+s_2} \cdots \omega_{n+1-s_r, a_r-i+s_r}, \]

and write \( \omega = \omega_1 \omega_2 \), where

\[ \omega_1 = \omega_{i, b} \omega_{i-s_r, b_r-1-s_r} \cdots \omega_{i-s_2, b_2-1-s_2}, \quad \omega_2 = \omega_{n+1-i+s_r, a_r+s_r} \cdots \omega_{n+1-i+s_2, a_2+s_2} \omega_{n+1-i, a_1}. \]

Taking \( J = \{ n+2-i, \cdots, n \} \), \( \pi = \omega_0 \), \( \pi_1 = \omega_{n+1-i, a_r} \), \( \pi_2 = 1 \),

we see that Proposition \[4.4.2\] gives,

\[ \text{Hom}_{\hat{U}_n}(W(\omega_{n+1-i, a}), V(\omega_2) \otimes V(\omega_0)) \neq 0. \]

Similarly, taking \( J = \{ i, \cdots, i-1 \} \), \( \pi = \omega_0^* \), \( \pi_2 = \omega_{i, b}^* \), \( \pi_1 = 1 \),

we get

\[ \text{Hom}_{\hat{U}_n}(W(\omega_{i, b}), V(\omega_0^*) \otimes V(\omega_1)) \neq 0. \]

We claim that in fact we have,

\[ \text{Hom}_{\hat{U}_n}(V(\omega_{n+1-i, a}), V(\omega_2) \otimes V(\omega_0)) \neq 0, \quad \text{Hom}_{\hat{U}_n}(V(\omega_{i, b}), V(\omega_0^*) \otimes V(\omega_1)) \neq 0. \]

Assuming the claim we complete the proof of the reverse direction. The claim implies that

\[ \text{hd}(V(\omega_2) \otimes V(\omega_{n+1-i, a})) \cong V(\omega_0) \cong \text{soc}(V(\omega_1) \otimes V(\omega_{n+1-i, a})), \]

Using [22, Theorem 3.12] this is equivalent to

\[ \text{soc}(V(\omega_{n+1-i, a}) \otimes V(\omega_2)) \cong V(\omega_0) \cong \text{hd}(V(\omega_{n+1-i, a}) \otimes V(\omega_1)). \]

This means that we have a non-zero map,

\[ V(\omega_{n+1-i, a}) \otimes V(\omega_1) \rightarrow V(\omega_{n+1-i, a}) \otimes V(\omega_2), \]

and hence a non-zero map,

\[ f : V(\omega_1) \otimes V(\omega_2) \rightarrow V(\omega_{i, b}) \otimes V(\omega_{n+1-i, a}). \]

Since \( V(\omega_1) \) and \( V(\omega_2) \) are not dual representations it follows that \( \text{Im} f \neq \mathbb{C} \). Hence \( \text{Im} f \) must contain an \( \ell \)-highest weight vector with \( \ell \)-highest weight \( \pi \) satisfying,

\[ \pi \in \text{wt}_\ell(V(\omega_1) \otimes V(\omega_2)) \subset \text{wt}_\ell(W(\omega)), \quad \text{Hom}_{\hat{U}_n}(W(\pi), V(\omega_{i, b}) \otimes V(\omega_{n+1-i, a})) \neq 0. \]

But now the forward direction of the proposition implies that \( \pi = \omega \) which gives the reverse direction of the proposition.

It remains to prove the claim. Let \( f : W(\omega_{n+1-i, a}) \rightarrow V(\omega_2) \otimes V(\omega_0) \) be a non-zero map and set \( v_0 = f(w_{n+1-i, a}) \). Then \( \hat{U}_n v_0 \) is the image of \( f \) and it suffices to prove that \( \hat{U}_n v_0 \) is irreducible. For this, it suffices to show that

\[ \pi \in \text{wt}_\ell^+(W(\omega_{n+1-i, a})), \quad \text{Hom}_{\hat{U}_n}(W(\pi), V(\omega_2) \otimes V(\omega_0)) \neq 0 \implies \pi = \omega_{n+1-i, a}. \]

Setting \( s_{r+1} = 0 \) and \( s_1 = 2i \) we have

\[ b_j - s_{j+1} - b_{j-1} - s_j = 2i - s_{j+1} - s_j \in S_{i-s_{j+1}-i-s_j, n}. \]
Hence \( V(\omega_s), s = 0, 1, 2 \) are prime snake modules. By Lemma [1.3.4] and Proposition [1.6.3] we can write

\[
\omega_{n+1-i,a} = \omega_2 \omega(g), \quad \pi = \omega_2 \omega(g'), \quad g' = (g'_2, \ldots, g'_r) \in P_{\omega_0}, \quad g = (g_2, \ldots, g_r) \in P_{\omega_0},
\]

where

\[
\omega(g_p) = \omega_{n+1-i+s_p,a_p+s_p}^{-1} \omega_{n+1-i,a_p}, \quad 2 \leq p \leq r.
\]

Since \( \omega_{i,a} \in \pi Q^+ \), it follows from Proposition [1.6.3(iv)] that

\[
g'(k) \geq g(k), \quad \text{for all } \quad k \in [0,n+1].
\]

Hence if \( g' \neq g \) there exists \( 2 \leq p \leq r \) minimal such that \( g_p(n + 1 - i) > a_p \). The definition of \( P_{n+1-s_p,a_p-i+s_p} \) gives

\[
a_p \leq g'_p(n + 1 - i) \leq a_p + 2s_p, \quad g'_p(0) = a_p + n + 1 - i.
\]

If \( c_{g_p}^- \subset [n + 2 - i, n] \), since \( g'_p(n + 1 - i) > a_p \) we would have

\[
g'_p(n + 1 - i) = g'_p(0) + n + 1 - i = a_p + 2(n + 1 - i) > a_p + 2s_p,
\]

and hence a contradiction. It follows that there exists \( j \leq n + 1 - i \) with \( j \in c_{g_p}^- \) and \( g'_p(j) \geq a_p + 2 \). But this contradicts \( \omega_2 \omega(g') \in P^+_n \). Hence we have proved that \( \pi = \omega_{n+1-i,a} \) as needed. The proof of the second assertion of the claim is identical and we omit the details.

### 4.5. Imaginary modules in \( A_2 \)

In the case of the quantum affine algebra associated to \( A_2 \), the tensor product of a KR–module and its dual has Jordan–Holder series of length two. Hence the methods of the previous section do not yield an imaginary module. However, the methods do work for a different family of modules and in fact they work in all ranks.

The evaluation modules are a particular family of snake modules, which are indexed by elements of \( P^+ \). In the case of \( A_2 \) they correspond to the following elements (and their duals) of \( P^+_2 \): for \( r_1, r_2 \in \mathbb{N} \) and \( b_1 \in \mathbb{Z} \), we have

\[
\pi = \omega_{1,b_1}, \omega_{2,b_2}, \quad b_j = (b_j, b_j + 2, \ldots, b_j + 2r_j - 2), \quad b_1 - 3 = b_2 + 2r_2 - 2. \quad (4.4)
\]

We shall call these elements snakes of evaluation type. Write

\[
*\pi = \omega_{1,a_1}, \omega_{2,a_2}, \quad a_1 = (b_2 - 3, \ldots, b_2 + 2r_2 - 5), \quad a_2 = (b_1 - 3, \ldots, b_1 + 2r_1 - 5).
\]

**Proposition.** Suppose that \( \pi \) is a snake of evaluation type in \( A_2 \) and assume that \( r_1, r_2 \geq 2 \). Then the module \( V(*\pi \pi) \) is imaginary.

**Proof.** Let \( M \) be the submodule of \( V(\pi) \otimes V(*\pi) \) generated by the tensor product \( v \) of the highest weight vectors. Consider the quantum affine subalgebra associated to \( J = \{2\} \). It was proved in [9] that \( \hat{U}_J v \) is a proper submodule of \( V_J(\omega_{2,b_2}) \otimes V_J(\omega_{2,a_2}) \). It follows that \( M \) must be a proper submodule of \( V(\pi) \otimes V(*\pi) \). Next we prove that there exists a non–split short exact sequence

\[
0 \to \mathbb{C} \to M \to V(*\pi \pi) \to 0. \quad (4.5)
\]

For this along of Lemma [4.3.2] it is enough to show that

\[
\text{Hom}_{\hat{U}_n}(W(\pi), V(\pi) \otimes V(*\pi)) \neq 0 \iff \pi = \{1, *\pi \pi\}.
\]
Notice that the reverse direction is obvious. For the forward direction we use Lemma 1.3.1 to deduce that $\bar{\pi}$ must satisfy,
\[
\bar{\pi} = \pi \omega(g), \quad \text{and} \quad (*\bar{\pi})^{-1} \pi = \omega(p), \quad g \in P_{\pi}, \quad p \in P_{\bar{\pi}}.
\] (4.6)

Writing
\[
g = (g_1, g_2), \quad g_1 \in P_{1,a_1} \quad g_2 = (g_1, \cdots, g_r) \in P_{2,a_2},
\]
we see by using Proposition 1.6.3 that
\[
\omega(g_1) = \omega_{2,b_1+2r_1-5} \implies \omega(g_2) = \omega_{2,a_2}.
\]
We claim that this further implies that $\omega(g_1) = \omega_{1, a_1}$. Otherwise we would have an element of the form $\omega_{1, b_1-4}$ or $\omega_{2, b_1-3}$ in a reduced expression for $\omega(g_2)$. The first would contradict $\bar{\pi} \in P^+$ and the second is impossible by Proposition 1.6.3 since $\omega_{2, b_1-3}$ occurs in $\omega(g_2)$. This proves the claim and hence we have proved that $\bar{\pi} = *\pi \pi$.

If $\omega(g_1) \neq \omega_{2, b_1+2r_1-5}$, then the condition that $\bar{\pi} \in P^+$ forces $\omega(g_1) = \omega_{1, b_1+2r_1-2}$ and so we get
\[
\tilde{\pi} = \omega_{1, b_1}\{b_1+2r_1-2\} \omega_{1}(g_1) \omega(g'_2), \quad g'_2 = (g_1, \cdots, g_r-1) \in P_{2,a_2}\{b_1+2r_1-5\}.
\]

Now we observe that the preceding assertion implies that $\omega_{1, b_1+2r_1}$ does not occur on the left hand side of the second equality in (4.6). Writing $\bar{p} = (p_2, \bar{p}_1)$ with $p_s \in P_{s,b_s}, \ s = 1, 2, \bar{p}_1 = (p_1, \cdots, p_r)$ we see that we must have $\omega(p_1) = \omega_{1, b_1+2r_1-2}$. But this again forces $\omega(p_2) = \omega_{1, b_2}$. If $\omega(p_2) \neq \omega_{1, b_2}$ then an expression of the form $\omega_{1, b_1-2}$ or $\omega_{1, b_1-3}$ must occur in $\omega(p_2)$. The first is impossible since such a term does not occur on the left hand side of the second equation in (4.6) while the second would contradict the fact that $\bar{p} \in P_{\bar{\pi}}$. Hence we have proved that $\omega(p_2) = \omega_{1, b_2}$ which now forces $\bar{\pi} = 1$.

Let $\Phi : V(\pi) \otimes V(*\pi) \otimes V(\pi) \otimes V(*\pi) \to V(\pi) \otimes V(*\pi)$ be defined as in Section 4.3.1 Again Lemma 1.3.1 shows that $\Phi(M \otimes M)$ is a non–zero submodule of $V$. By Proposition 1.3.3 we have
\[
\Phi(M_{\pi \pi^*}) \otimes M_{\pi \pi} \neq 0.
\]
A standard argument (as in the previous examples) using Proposition 1.6.3 shows that $\dim(V(\pi) \otimes V(*\pi))_{\pi(\pi^*)^{-1}} = 1$ and so we get,
\[
\Phi(M_{\pi \pi^*}^{-1} \otimes M_{\pi \pi}) = (V(\pi) \otimes V(*\pi))_{\pi(\pi^*)^{-1}}.
\]

Let $\bar{\pi} \in P^+$ be the head $V(\pi) \otimes V(*\pi)$ and note that $\bar{\pi} \neq 1, *\pi \pi, (*\pi^2)$. Lemma 1.3.3 now gives that the image of $(V(\pi) \otimes V(*\pi))_{\pi(\pi^*)^{-1}}$ in $V(\pi)$ is non–zero. The discussion so far shows that the composite map:
\[
M \otimes M \to V(\pi) \otimes V(*\pi) \to V(\pi)
\]
is surjective. Using the short exact sequence in (4.5) we see that the composite map obviously factors through to a map from $V(*\pi \pi) \otimes V(*\pi \pi)$ and the proof of the proposition is complete. \qed
4.6. $D_4$. We give the first family of examples of imaginary modules in type $D_4$ which does not arise from an embedding of $A_3$. Assume that $2$ is the trivalent node in $D_4$. We shall prove that if
\[ \omega = \omega_{1,2r} \omega_{1,2r-2} \omega_{2,2r-5} \cdots \omega_{2,5} \omega_{1,2r-5} \]
then the module $V(\omega)$ is imaginary $r \geq 5$. The proof follows the same strategy as in the case of $A_n$.

**Remark.** Note that regarded as module for the quantum affine $A_3$ subalgebras given by $\{j, 2, k\}$ with $j \neq k \in \{1, 3, 4\}$ the module $V(\omega)$ is an example of a snake module $\mathfrak{S}$ and hence by [13] real for the subalgebra. \[ \square \]

It follows from $\mathfrak{S}$ that $V(\omega_{j,a+6}) \cong V(\omega_{j,a+6})$. We take
\[ b = (6, 8, \ldots, 2r), \quad a = (0, 2, \ldots, 2r - 6), \]
\[ V = V(\omega_{1,b}) \otimes V(\omega_{1,a}). \]
We have a map of $\hat{U}(D_4)$-modules $\mathbb{C} \leftarrow V$ and a projection $< , > : V(\omega_{1,a}) \otimes V(\omega_{1,b}) \rightarrow \mathbb{C}$ and we let $\Phi : V \otimes V \rightarrow V$ be the composition $\circ < , > \circ \text{id}$.

It is helpful to recall that the elements of $\text{wt}_\ell V(\omega_{1,0})$ are the following:
\[ \omega_{1,0}, \omega_{2,1\omega_{1,2}^{-1}}, \omega_{2,3\omega_{1,2}\omega_{3,4}}, \omega_{3,4\omega_{1,2}}, \omega_{4,4\omega_{3,2}}^{-1}, \omega_{2,3\omega_{3,4}\omega_{4,2}}, \omega_{1,4\omega_{2,5}}, \omega_{1,6} \cdot \omega_{1,5}. \]
Notice that
\[ \omega = \omega_{i,b}\omega', \quad \omega' = \omega_{i,a}\alpha_{1,2r-5} \cdots \alpha_{1,5} \in \text{wt}_\ell V(\omega_{i,a}). \]

4.6.1. The following result will be used.

**Lemma.** Suppose that $\pi \in \text{wt}_\ell V(\omega_{i,a})$ is such that $\pi = \omega_{1,c_1}^{-1} \cdots \omega_{1,c_r}^{-1} \pi'$ for some $c_1, \cdots, c_k$ and $\pi' \in \mathcal{P}^+$. There exists $1 \leq s \leq p \leq r + 1$ such that
\[ \pi = \omega_{1,a_0+b}^{-1} \cdots \omega_{1,a_p+b}^{-1}(\omega_{1,a_0+b+2}^{-1} \cdots \omega_{1,a_p+b+1}^{-1}) \cdots (\omega_{1,a_s+2}^{-1} \omega_{1,a_{s+1}} \cdots \omega_{1,a_1}). \]

**Proof.** Since $\pi \in \text{wt}_\ell V(\omega_{i,a})$ and $\pi \neq \omega_{i,a}$ it follows that there exists $1 \leq s \leq r$ minimal such that
\[ \pi \leq \omega_{i,a} \alpha_{1}^{-1}, \quad \alpha_1 = \alpha_{1,a_r+1} \cdots \alpha_{1,a_s+1}. \]
Hence either the Lemma is proved or the inequality is strict. In the latter case, there must exist $p \geq s$ minimal with
\[ \pi \leq \omega_{i,a}(\alpha_1 \alpha_2)^{-1}, \quad \alpha_2 = \alpha_{2,a_{r+2}} \cdots \alpha_{2,a_{p+2}}. \]
The assumptions on $\pi$ now means that this inequality must be strict. Now working with $\alpha_j$ with $j = 3, 4$ we see in a similar fashion that
\[ \pi \leq \omega_{i,a}(\alpha_1 \alpha_2 \alpha_3 \alpha_4)^{-1}, \quad \alpha_3 = \alpha_{3,a_{r+3}} \cdots \alpha_{a_{p+3}}, \quad j = 3, 4. \]
Again the assumption on $\pi$ mean that the inequality must be strict and so we get
\[ \pi \leq \omega_{i,a}(\alpha_1 \alpha_2 \alpha_3 \alpha_4 \alpha_5)^{-1}, \quad \alpha_5 = \alpha_{5,a_{r+4}} \cdots \alpha_{2,a_{p+4}}. \]
A final application of the property of $\pi$ gives
\[ \pi \leq \omega_{i,a}(\alpha_{1,2} \alpha_3 \alpha_4 \alpha_5 \alpha_1)^{-1}, \quad \alpha_1 = \alpha_{1,a_{r+5}} \cdots \alpha_{1,a_{p+5}}, \]
or equivalently
\[ \pi \leq \omega_{1,a}^{-1} \cdots \omega_{1,a_r}^{-1}(\omega_{1,a_{p-1}+2}^{-1} \omega_{2,a_p+1}^{-1}) \cdots (\omega_{1,a_s+2} \omega_{2,a_s+1}) \omega_{1,a_s-1} \cdots \omega_{1,a_1}. \]
The minimality of the choice of \( p, s \) now proves this must be an equality as needed. \( \square \)

4.6.2.

Lemma. Let \( \varpi \in \text{wt}^+_{\ell} V(\omega). \) Then
\[ \text{Hom}_U(W(\varpi), V) \neq 0 \iff \varpi \in \{1, \omega\}. \]

Proof. The converse is clear by using the result for quantum affine \( \mathfrak{sl}_2. \) For the forward direction we use Lemma 1.3.4 to write \( \varpi = \omega_{1,b} \pi \) with \( \pi \in \text{wt} V(\omega_{1,a}). \) Since \( \varpi \in \mathcal{P}^+ \) we see that \( \pi \) must satisfy the conditions of Lemma 1.6.1 and let \( s, p \) be as in that Lemma.

Since \( \varpi \leq \omega \) we are forced to take \( s \in \{1, 2, 3\}. \) If \( s = 1, 2 \) and if \( p - 1 \geq s \) then the term \( \omega_{1,a_2+2} \) occurs in a reduced expression for \( \varpi \in \mathcal{P}^+ \) which is absurd. Hence we must have \( p - 1 < s \) which forces \( p = s. \) This means that \( \varpi \in \{1, \omega_{1,b} \omega_{1,a_1}\}. \) Using the second assertion of Lemma 1.3.4 we see that
\[ \varpi = \omega_{1,b_1} \omega_{1,a_1} \implies \omega_{1,b_1+6}^{-1} \omega_{1,b_1} \varpi \in \text{wt} V(\omega_{1,b}). \]

If \( r \geq 6 \) we get a contradiction since \( \omega_{1,b_1+6}^{-1} \omega_{1,b_1} \varpi \in \mathcal{P}^+ \) and \( \text{wt} V(\omega_{1,b}) = \{\omega_{1,b}\}, \) by 30 (see also [18 Theorem 2.12]). If \( r = 5 \) this means that
\[ \omega_{1,b_2} \omega_{1,b_3} \omega_{1,b_4}^6 \in \text{wt} V(\omega_{1,b}). \]
It follows from [18] Lemma 5.8 that
\[ \omega_{1,b_2} \omega_{1,b_3} \omega_{1,b_4} \leq (\omega_{1,b_1+2} \omega_{1,b_2+2} \omega_{1,b_3+2})^{-1} \omega_{2,b_1+1} \omega_{2,b_2+1} \omega_{2,b_3+2}(\mathcal{Q}^+)^{-1}, \]
which is absurd since \( \omega_{1,b_2} \) can never occur in the right hand side.

Suppose next that \( s = 3 \) and let \( 3 \leq p \leq r + 1. \) If \( p = r + 1 \) then \( \varpi = \omega \) and there is nothing to prove. If \( p \leq r \) then,
\[ \varpi = \omega_{1,a_1} \omega_{1,a_2} \omega_{2,a_3+1} \cdots \omega_{2,a_{p-1}+1} \omega_{1,b_{p-2}} \omega_{1,b_p}. \]
Using Lemma 1.3.4 we have
\[ (\omega_{1,b_3} \omega_{2,b_3+1}) \cdots (\omega_{1,b_{p-1}} \omega_{2,b_{p-1}+1})(\omega_{1,b_{p-2}+6} \omega_{1,b_{p-1}+6})^{-1} \omega_{1,b_{p-1}} \cdots \omega_{1,b_p} \in \text{wt} V(\omega_{1,b}). \]
In particular we can write the above element as
\[ \omega_{1} \cdots \omega_{p-1} \omega' \in \text{wt} V(\omega_{1,b_1}) \otimes \cdots \otimes V(\omega_{1,b_{p-1}}) \otimes V(\omega_{1,b'}), \quad \omega' \in \text{wt} V(\omega_{1,b'}), \]
where \( b' = (b_p, \cdots, b_r). \) Comparing the possible weights we are forced to have
\[ \omega_s = \omega_{1,b_s+2} \omega_{2,b_s+2+1}, \quad 1 \leq s \leq p - 3, \]
and hence
\[ (\omega_{1,b_{p-2}+6} \omega_{1,b_{p-1}+6})^{-1} \omega_{1,b_{p-2}} \cdots \omega_{1,b_{p}} \in \text{wt} V(\omega_{1,b'}). \]
If \( p \leq r - 2 \) the corresponding \( \ell \)-weight is dominant and again we have a contradiction since \( \text{wt} V(\omega_{1,b'}) = \{\omega_{1,b'}\}. \) If \( p = r - 1 \) it means that
\[ \omega_{1,b_{p}} \cdots \omega_{1,b_{r-1}} \omega_{1,b_r+2}^{-1} \in \text{wt} V(\omega_{1,b'}) \implies \omega_{1,b_{r+2}}^{-1} \in \text{wt} V(\omega_{1,b_r}). \]
Similarly, if $p = r$ we have
\[
\omega_{1,b_p} \cdots \omega_{1,b_r} \omega^{-1}_{1,b_r+2} \omega^{-1}_{1,b_r+4} \in \text{wt}_\ell V(\omega_{1,b}).
\]
In either case we have a contradiction. \hfill \Box

### 4.6.3. The proof that $V(\omega)$ is imaginary is now completed as in the case of $A_n$. Let $f : W(\omega) \to V$ be a non–zero map of $\hat{U}$–modules and let $v = f(\nu)$ and set $M(\omega) = \hat{U}v$. The preceding Lemma and [22] show that we have a non–split short exact sequence
\[
0 \to \mathbb{C} \to M(\omega) \to V(\omega) \to 0.
\]
Suppose that $\omega_1 = \omega_2^{-1}$ with $\omega_1 \in \text{wt}_\ell V(\omega_{1,b})$ and $\omega_2 \in \text{wt}_\ell V(\omega_{1,a})$. Using [18] again we see that if $\omega_1 \neq \omega_{1,b}$ then $\omega_1$ must involve a term $\omega_{j,c}$ with $c > b_r$. Since $\omega_{j,c}^{-1}$ cannot occur in an element of $\text{wt}_\ell V(\omega_{1,a})$ we see that $\omega_1 = \omega_{1,b}$ and so $\dim V_1 = 1$. Similarly if $\omega_1 \omega_2 = 1$ with $\omega_1 \in \text{wt}_\ell V(\omega_{1,b}) \otimes \omega_2 \in \text{wt}_\ell V(\omega_{1,a}) \otimes \omega_2$ one proves that $\omega_1 = \omega_{1,b}^2$ and hence $\dim(V \otimes V)_1 = 1$. It follows that the restriction of $\Phi$ to $M(\omega) \otimes M(\omega)$ is non–zero.

Next we note that $\omega^2 \notin \text{wt}_\ell V$. This proves that $V(\omega^2)$ cannot be in the head of $\Phi(M(\omega) \otimes M(\omega))$.

Finally, we prove that $(\omega^s)^{-1} \omega \notin \text{wt}_\ell W(\omega)$. Suppose that $(\omega^s)^{-1} = \omega_1 \cdots \omega_{2r+2}$ where $\omega_s \in \text{wt}_\ell W(\omega_{1,2s})$, $s = r - 1, r$, $\omega_s \in \text{wt}_\ell W(\omega_{1,2s})$, $s = 1, 2$, $\omega_s \in \text{wt}_\ell W(\omega_{1,2s})$ otherwise. Then we must have $\omega_s = \omega_{1,2s}$ if $s = 1, 2$. But then $\omega_{1,2s}^{-1}$ does not occur as part of an $\ell$–weight of the other modules and we are done. Hence Proposition [18,3] shows that $V(\omega)$ cannot occur in the head of $\Phi(M(\omega) \otimes M(\omega))$. Since $\omega$ is not self dual it is now clear that $\Phi(M(\omega) \otimes M(\omega))$ has an irreducible module $V(\pi)$ in its head with $\pi \neq 1, \omega, \omega^2$ and this map factors through to give a non-zero map $V(\omega) \otimes V(\omega) \to V(\pi)$ proving that $V(\omega)$ is imaginary.

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