Curve implicitization in the bivariate tensor-product Bernstein basis

Ana Marco ¹, José-Javier Martínez ²

Departamento de Matemáticas, Universidad de Alcalá,
Campus Universitario, 28871-Alcalá de Henares (Madrid), Spain

Abstract

The approach to curve implicitization through Sylvester and Bézout resultant matrices and bivariate interpolation in the usual power basis is extended to the case of Bernstein-Bezoutian matrices constructed when the polynomials are given in the Bernstein basis. The coefficients of the implicit equation are also computed in the bivariate tensor-product Bernstein basis, and their computation involves the bidiagonal factorization of the inverses of certain totally positive matrices.

Key words: Curve; Implicitization; Interpolation; Bernstein basis; Total positivity

1 Introduction

When studying rational plane algebraic curves, there are two standard ways of representation, the implicit equations and the parametric equations. The intersection of two curves is more easily computed when we have the implicit equation of one curve and the parametric equations of the other, and hence it is very important to be able to change from one representation to another.

We will concentrate on the implicitization problem, that is to say, on finding an implicit representation starting from a given rational parametrization of the curve.

In [12] we have presented an approach to the implicitization problem based on interpolation using the usual power basis for the corresponding space of

¹ E-mail: ana.marco@uah.es
² Corresponding author. E-mail: jjavier.martinez@uah.es
bivariate polynomials. However, very recent work [2], related to polynomials expressed in the Bernstein basis has showed the importance of evaluating resultants from Bernstein basis resultant matrices directly, avoiding a basis transformation. In this sense, in [2] it is indicated that for numerical computations involving polynomials in Bernstein form it is essential to consider algorithms which express all intermediate results using this form only.

Although those papers study univariate polynomials, it must be observed that the construction of the resultant matrices can be extended to the case in which the entries of the resultant matrix are polynomials. It must also be taken into account that the Bernstein basis has also important advantages in the context of tracing implicit algebraic curves [13].

So our aim is to use bivariate interpolation for obtaining in the bivariate tensor-product Bernstein basis the implicit equation of a plane algebraic curve given by its parametric equations in Bernstein form (which is the usual situation in the case of Bézier curves). Although we present all the details with an example in exact rational arithmetic, it must be taken into account that the process can also be carried out in (high) finite precision arithmetic. In that situation some important results of numerical linear algebra we use will have a major importance. More precisely the total positivity of certain matrices will be an important issue, as it happens in several instances of computer aided geometric design (see, for example, the recent work [18] and references therein).

The rest of the paper is organized as follows. In Section 2 several basic results will be presented. In Section 3 we introduce the interpolation algorithm for computing the implicit equation as a factor of the determinant of the resultant matrix, while in Section 4 we consider some results related to total positivity which will be relevant for the solving of the linear system associated with the interpolation problem. Finally, in Section 5 we briefly examine the computational complexity of the whole algorithm.

2 Preliminaries

Let $P(t) = (x(t), y(t))$ be a proper parametrization of a rational plane algebraic curve $C$, where $x(t) = \frac{u_1(t)}{v_1(t)}$ and $y(t) = \frac{u_2(t)}{v_2(t)}$ and $gcd(u_1, v_1) = gcd(u_2, v_2) = 1$. A parametrization $P(t) = (x(t), y(t))$ of a curve $C$ is said to be proper if every point on $C$ except a finite number of exceptional points is generated by exactly one value of the parameter $t$. It is well known that every rational curve has a proper parametrization, so we can assume that the parametrization is proper. Several recent results on the properness of curve
parametrizations can be seen in [17].

In connection with the implicitization problem, the following theorem [17] holds:

**Theorem 1.** Let \( P = (x(t) = \frac{u_1(t)}{v_1(t)}, y(t) = \frac{u_2(t)}{v_2(t)}) \) be a proper rational parametrization of an irreducible curve \( C \), with \( \gcd(u_1, v_1) = \gcd(u_2, v_2) = 1 \). Then the polynomial defining \( C \) is \( \text{Res}_{t}(u_1(t) - xv_1(t), u_2(t) - yv_2(t)) \) (the resultant with respect to \( t \) of the polynomials \( u_1(t) - xv_1(t) \) and \( u_2(t) - yv_2(t) \)).

Our aim is to compute the implicit equation \( F(x, y) = 0 \) of the curve \( C \) by means of polynomial interpolation, which taking into account Theorem 1 is equivalent to compute \( \text{Res}_{t}(u_1(t) - xv_1(t), u_2(t) - yv_2(t)) \).

First of all, we remark that the concept of interpolation space will be essential. The following result, also in [17], shows which is in our case the most suitable interpolation space:

**Theorem 2.** Let \( P = (x(t) = \frac{u_1(t)}{v_1(t)}, y(t) = \frac{u_2(t)}{v_2(t)}) \) be a proper rational parametrization of the irreducible curve \( C \) defined by \( F(x, y) \), and let \( \gcd(u_1, v_1) = \gcd(u_2, v_2) = 1 \). Then \( \deg_{y}(F) = \max\{\deg_{t}(u_1), \deg_{t}(v_1)\} \) and \( \deg_{x}(F) = \max\{\deg_{t}(u_2), \deg_{t}(v_2)\} \).

Theorem 2 tells us that the polynomial \( F(x, y) \) defining the implicit equation of the curve \( C \) belongs to the polynomial space \( \Pi_{n,m}(x, y) \), where \( n = \max\{\deg_{t}(u_2), \deg_{t}(v_2)\} \) and \( m = \max\{\deg_{t}(u_1), \deg_{t}(v_1)\} \). The dimension of \( \Pi_{n,m}(x, y) \) is \( (n + 1)(m + 1) \), and a basis is given by \( \{x^iy^j|i = 0, \cdots, n; j = 0, \cdots, m\} \). Moreover \( \deg_{y}(F(x, y)) = n \) and \( \deg_{y}(F(x, y)) = m \), and therefore there is no interpolation space \( \Pi_{r,s}(x, y) \) with \( r < n \) or \( s < m \) such that \( F(x, y) \) belongs to \( \Pi_{r,s}(x, y) \).

Let us note that these theorems refer to the degree of polynomials in the power basis, so since now we will be using the Bernstein basis some care will be needed. For the sake of clarity we will illustrate all our results with a small example. Let

\[
\{\beta_0^{(4)}(t), \beta_1^{(4)}(t), \beta_2^{(4)}(t), \beta_3^{(4)}(t), \beta_4^{(4)}(t)\}
\]

be the (univariate) Bernstein basis of the space of polynomials of degree less than or equal to 4, where the Bernstein polynomials are defined as follows,

\[
\beta_i^{(n)}(t) = \binom{n}{i}(1-t)^{n-i}t^i, \quad i = 0, \ldots, n.
\]
and let us consider the algebraic curve given by the parametric equation

\[ x(t) = \frac{4\beta_0^{(4)}(t) + 4\beta_1^{(4)}(t) + 3\beta_2^{(4)}(t) + 3\beta_3^{(4)}(t) + 7\beta_4^{(4)}(t)}{\beta_0^{(4)}(t) + \beta_1^{(4)}(t) + \beta_2^{(4)}(t) + \beta_3^{(4)}(t) + 3\beta_4^{(4)}(t)} \]

\[ y(t) = 2\beta_0^{(4)}(t) + 3\beta_1^{(4)}(t) + 3\beta_2^{(4)}(t) + 3\beta_3^{(4)}(t) + 4\beta_4^{(4)}(t). \]

If we call \( p(t) = u_1(t) - xv_1(t) \) and \( q(t) = u_2(t) - yv_2(t) \), their coefficients in the Bernstein basis are given by

\[ p_0 = 4 - x, \quad p_1 = 4 - x, \quad p_2 = 3 - x, \quad p_3 = 3 - x, \quad p_4 = 7 - 3x, \]

and

\[ q_0 = 2 - y, \quad q_1 = 3 - y, \quad q_2 = 3 - y, \quad q_3 = 3 - y, \quad q_4 = 4 - y. \]

However, let us observe that

\[ p(t) = 4 - x - 6t^2 + 8t^3 + (-2x + 1)t^4 \]

(a polynomial of degree 4 in \( t \)), while

\[ q(t) = 2 - y + 4t - 6t^2 + 4t^3, \]

a polynomial of degree 3 in \( t \).

Therefore, the polynomial defining the implicit equation will be a polynomial belonging to the space \( \Pi_{n,m}(x,y) \) with \( n = 3 \) and \( m = 4 \). We will use for that space the tensor-product bivariate Bernstein basis given by

\[ \{B_{ij}^{(n,m)}\} = \{\beta_i^{(n)}(x)\beta_j^{(m)}(y), i = 0, \ldots, n; j = 0, \ldots, m\}. \]

Finally we will recall, following [2], the algorithm for constructing Bernstein-Bézout matrix of \( p(t) \) and \( q(t) \). Although in [2] the coefficients of the polynomials are always numbers, in our application we will construct the symbolic (i.e. with the entries being polynomials in \( x, y \)) Bernstein-Bézout matrix of \( p(t) \) and \( q(t) \) which we denote by \( BS \). For the reader’s convenience, we present the algorithm written in Maple language:

```maple
for i from 1 to n do
    BS[i,1] := (n/i)*(p[i]*q[0]-p[0]*q[i]);
od;

for j from 1 to n-1 do
```

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Let us observe that if \( m = n \), the resultant is the determinant of the Bernstein-Bézout matrix, while -as a consequence of the corresponding result for the Bézout resultant [16]- if \( m > n \), that determinant is equal to the resultant multiplied by the factor \((\tilde{p}_m(t))^{m-n}\), where \( \tilde{p}_m(t) \) is the leading coefficient of \( p(t) \) in the power basis.

So in our example, the determinant of \( BS \) will be the implicit equation we are looking for multiplied by the factor \((-2x + 1)\), since the degree of \( p \) is 4 and the degree of \( q \) is 3 and the coefficient of \( t^3 \) in \( p \) is \((-2x + 1)\). In the following section we will show how to compute the coefficients in the bivariate tensor-product Bernstein basis of the implicit equation (which will be a scalar multiple of the resultant computed by using the approach of [12], where the equation is obtained in the usual power basis).

### 3 The interpolation process

Since the expansion of the symbolic determinant is very time and space consuming, our aim is to compute the polynomial defining the implicit equation by means of Lagrange bivariate interpolation, but using the Bernstein basis instead of the power basis. A good introduction to the theory of interpolation can be seen in [5].

If we consider the interpolation nodes \((x_i, y_j) (i = 0, \cdots, n; j = 0, \cdots, m)\) and the interpolation space \( \Pi_{n,m}(x, y) \), the interpolation problem is stated as follows:

Given \((n + 1)(m + 1)\) values

\[ f_{ij} \in K \quad (i = 0, \cdots, n; j = 0, \cdots, m) \]
(the interpolation data), find a polynomial
\[ F(x, y) = \sum_{(i,j) \in I} c_{ij} \beta_i^{(n)}(x) \beta_j^{(m)}(y) \in \Pi_{n,m}(x, y) \]
(where \( I \) is the index set \( I = \{(i, j) | i = 0, \ldots, n; j = 0, \ldots, m \} \)) such that
\[ F(x_i, y_j) = f_{ij} \quad \forall \ (i, j) \in I. \]

If we consider for the interpolation space \( \Pi_{n,m}(x, y) \) the basis
\[ \{ B_{ij}^{(n,m)}, i = 0, \ldots, n; j = 0, \ldots, m \} = \]
\[ \{ \beta_i^{(n)}(x) \beta_j^{(m)}(y), i = 0, \ldots, n; j = 0, \ldots, m \} = \]
\[ \{ B_0^{(n,m)}, B_0^{(n,m)}, \ldots, B_0^{(n,m)}, B_0^{(n,m)} \ldots, B_1^{(n,m)}, \ldots, B_m^{(n,m)}, \ldots, B_m^{(n,m)} \ldots, B_m^{(n,m)} \ldots, B_{mn}^{(n,m)} \ldots, B_{nm}^{(n,m)} \} \]
with that precise ordering, and the interpolation nodes
\[ \{(x_i, y_j) | i = 0, \ldots, n; j = 0, \ldots, m \} = \]
\[ \{(x_0, y_0), (x_0, y_1), \ldots, (x_0, y_m), \]
\[ (x_1, y_0), (x_1, y_1), \ldots, (x_1, y_m), \ldots, (x_n, y_0), \ldots, (x_n, y_m) \} \]
then the \((n+1)(m+1)\) interpolation conditions \( F(x_i, y_j) = f_{ij} \) can be written as a linear system
\[ Ac = f, \]
where the coefficient matrix \( A \) is given by a Kronecker product
\[ B_x \otimes B_y, \]
with
\[ B_x = ((\beta_j^{(n)}(x_i)), i = 0, \ldots, n; j = 0, \ldots, n, \]
\[ B_y = ((\beta_j^{(m)}(y_i)), i = 0, \ldots, m; j = 0, \ldots, m, \]
\[ c = (c_{00}, \ldots, c_{0m}, c_{10}, \ldots, c_{1m}, \ldots, c_{n0}, \ldots, c_{nm})^T, \]
and
\[ f = (f_{00}, \ldots, f_{0m}, f_{10}, \ldots, f_{1m}, \ldots, f_{n0}, \ldots, f_{nm})^T. \]
The Kronecker product \( D \otimes E \) is defined by blocks as \((d_{kl}E)\), with \( D = (d_{kl}) \).

For reasons which will be explained in Section 4 we will select as interpolation nodes \((x_i, y_j) = (\frac{i+1}{n+2}, \frac{j+1}{m+2}) \quad (i = 0, \ldots, n; j = 0, \ldots, m)\). In the general case we must avoid the value of \( x_i \) for which the leading coefficient of \( p(t) \) in the power basis evaluates to 0, and the value \( y_j \) for which the leading coefficient of \( q(t) \) in the power basis evaluates to 0.

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In our example we have \( n = 3 \) and \( m = 4 \), and consequently \( B_x \) will be the matrix

\[
B_x = \begin{pmatrix}
64 & 48 & 12 & 1 \\
27 & 54 & 36 & 8 \\
8 & 36 & 54 & 27 \\
1 & 12 & 48 & 64
\end{pmatrix},
\]

and \( B_y \) will be the matrix

\[
B_y = \begin{pmatrix}
625 & 125 & 25 & 5 & 1 \\
16 & 32 & 8 & 8 & 1 \\
1 & 4 & 3 & 4 & 16 \\
1 & 8 & 8 & 32 & 16 \\
1 & 25 & 125 & 625
\end{pmatrix}.
\]

As it is well known, since \( B_x \) and \( B_y \) are nonsingular matrices the Kronecker product \( B_x \otimes B_y \) will also be nonsingular.

As for the generation of the interpolation data, let us remark that they can be obtained without constructing the symbolic Bernstein-Bézout matrix \( BS \). That is to say, we can obtain each interpolation datum by means of the evaluation of \( p(t) \) and \( q(t) \) followed by the computation of the determinant of the corresponding numerical Bernstein-Bézout matrix \( B \) making use of the Bini-Gemignani algorithm which constructs (in \( O(n^2) \) arithmetic operations) the Bernstein-Bézout matrix for the evaluated polynomials.

In addition, we must divide the value of the determinant by \(-2x_i + 1\).

An algorithm for solving linear systems with a Kronecker product coefficient matrix is derived in a self-contained way (in a more general setting) in [14]. For the case of the power basis considered in [12], taking into account that every linear system to be solved was a Vandermonde linear system, it was convenient to use the Björck-Pereyra algorithm [3, 9] to solve those linear systems. For the Bernstein basis being used here, an appropriate algorithm which takes advantage of the special properties of the coefficient matrices \( B_x \) and \( B_y \) will be presented in Section 4.

In the general case, we must solve \( n+1 \) linear systems with the same matrix \( B_y \) and \( m+1 \) linear systems with the same matrix \( B_x \).
4 Total positivity of $B_x$ and $B_y$

From [4] we know that the Bernstein basis of the space of polynomials of degree less than or equal to $n$ is a *strictly totally positive basis* on the open interval $(0, 1)$, which implies that all the collocation matrices

$$M = (\beta_j^{(n)}(t_i)), i, j = 0, \ldots, n$$

with $t_0 < t_1 < \ldots < t_n$ in $(0, 1)$ are *strictly totally positive*, i.e. all their minors are strictly positive. In particular, due to our choice of the interpolation nodes the matrices $B_x$ and $B_y$ are strictly totally positive matrices.

Making use of the results of [7, 8], we know that performing the complete Neville elimination on a strictly totally positive matrix $A$ a *bidiagonal factorization* of its inverse $A^{-1}$ can be obtained, that is to say, we have

$$A^{-1} = G_1 G_2 \ldots G_{n-1} D^{-1} F_{n-1} F_{n-2} \ldots F_1,$$

where $D^{-1}$ is a diagonal matrix and $F_i$ and $G_i$ are bidiagonal matrices.

So, after having obtained that factorization (with a computational cost of $O(n^3)$ arithmetic operations), all the systems $Az = b$ with coefficient matrix $A$ can be solved (with a cost of $O(n^2)$ arithmetic operations) by performing the product

$$G_1 G_2 \ldots G_{n-1} D^{-1} F_{n-1} F_{n-2} \ldots F_1 b.$$  

An early application of these ideas to solve structured linear systems can be seen in [15], and a recent extension has been presented in [6].

A detailed error analysis of Neville elimination, which shows the advantages of this type of elimination for the class of totally positive matrices, has been carried out in [1], and related work for the case of Vandermonde linear systems can be seen in Chapter 22 of [10].

In our situation we must notice that the bidiagonal factorization can be done in exact arithmetic, and the results of the factorization can then be rounded if the subsequent computations must be carried out in finite precision arithmetic.

After having obtained the bidiagonal factorization of the inverse of $B_y$, the solution of the linear system $B_y z = b$ can be obtained in $O(n^2)$ arithmetic operations by computing the product
\[ G_1G_2 \ldots G_{n-1}D^{-1}F_{n-1}F_{n-2} \ldots F_1b, \]
and analogously for the linear systems with coefficient matrix \( B_x \) [6].

In our example, the coefficients of the desired implicit equation in the tensor-product bivariate Bernstein basis (using the lexicographical ordering we are considering) are:

\[
(25264/27, 66256/81, 167852/243, 45652/81, 36137/81, 15728/27, 125312/243, \\
320120/729, 29164/81, 69421/243, 29164/81, 79024/81, 36137/81, 15728/27, 9391/81).
\]

5 Computational complexity

In this section we will briefly examine the computational complexity of our algorithm in terms of arithmetic operations. In view of the algorithm, we must solve \( n + 1 \) systems of order \( m + 1 \) with the same matrix \( B_y \) and \( m + 1 \) systems of order \( n + 1 \) with the same matrix \( B_x \).

The factorization of the inverse of a matrix of order \( n \) by means of complete Neville elimination takes \( O(n^3) \) operations, but that factorization is used for solving all the systems with the same matrix, so each of the remaining systems can be solved with \( O(n^2) \) operations.

For the sake of clarity in the comparison, we will consider here the case \( m = n \). Then, the interpolation part of the algorithm has computational complexity \( O(n^3) \). Let us observe that in this situation, if we solve the linear system \( Ac = f \) of order \( (n + 1)^2 \) by means of Gaussian elimination, without taking into account the special structure of the matrix, we have computational complexity \( O(n^6) \). Moreover, using the approach we are describing, there is no need of constructing the matrix \( A \), which implies an additional saving in computational cost.

Let us remark that, since the construction of the numerical Bernstein-Bézout matrix requires \( O(n^2) \) arithmetic operations and the complexity of the computation of each determinant is \( O(n^3) \), the generation of the interpolation data has a computational complexity of \( O(n^5) \). Therefore with our approach, which exploits the Kronecker product structure, the whole process has complexity \( O(n^5) \), while using Gaussian elimination it would be \( O(n^6) \).
It is worth noting that the main cost of the process corresponds to the generation of the interpolation data, and not to the computation of the coefficients of the interpolating polynomial. So, the main effort to reduce the computational cost must be focused on that stage. In this sense, an interesting issue would be to take advantage of the displacement structure of the Bernstein-Bézout matrices [2, 11] to develop an algorithm with complexity $O(n^2)$ for computing each determinant.

Remark. Finally, let us observe that all the linear systems with matrix $B_y$ can be solved simultaneously, and the same can be said of the systems with matrix $B_x$. Therefore the algorithm exhibits a high degree of intrinsic parallelism. This parallelism is also present in the computation of the interpolation data since we can compute simultaneously the determinants involved in this process.

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