THE MORPHISM INDUCED BY FROBENIUS PUSH-FORWARDS

LINGGUANG LI

Abstract. Let $X$ be a smooth projective curve of genus $g(X) \geq 1$ over an algebraically closed field $k$ of characteristic $p > 0$ and $F_{X/k}: X \to X^{(1)}$ be the relative Frobenius morphism. Let $\mathcal{M}_{X}^{ss}(r, d)$ (resp. $\mathcal{M}_{X}^{ss}(r \cdot p, d + r(p - 1)(g - 1))$) be the moduli space of (semi)-stable vector bundles of rank $r$ and degree $d$ (resp. $r \cdot p$) and degree $d$ (resp. $d + r(p - 1)(g - 1)$) on $X$ (resp. $X^{(1)}$). We show that the set-theoretic map $S_{\text{Frob}}: \mathcal{M}_{X}(r, d) \to \mathcal{M}_{X}^{ss}(r \cdot p, d + r(p - 1)(g - 1))$ induced by $[\mathcal{E}] \mapsto [F_{X/k}(\mathcal{E})]$ is a proper morphism. Moreover, if $g(X) \geq 2$, the induced morphism $S_{\text{Frob}}: \mathcal{M}_{X}(r, d) \to \mathcal{M}_{X}^{ss}(r \cdot p, d + r(p - 1)(g - 1))$ is a closed immersion. As an application, we obtain that the locus of moduli space $\mathcal{M}_{X}(r, d)$ consists of stable vector bundles whose Frobenius pull back have maximal Harder-Narasimhan Polygon is isomorphic to Jacobian variety $\text{Jac}_X$ of $X$.

1. Introduction

Let $k$ be an algebraically closed field of characteristic $p > 0$, $X$ a smooth projective curve of genus $g$ over $k$. The absolute Frobenius morphism $F_{X}: X \to X$ is induced by $\mathcal{O}_{X} \to \mathcal{O}_{X}$, $f \mapsto f^p$. Let $F_{X/k}: X \to X^{(1)} := X \times_k k$ denote the relative Frobenius morphism of $X$ over $k$. Let $\mathcal{M}_{X}^{ss}(r, d)$ be the moduli space of (semi)-stable vector bundles of rank $r$ and degree $d$ on $X$.

It is well known that the (semi)-stability of vector bundles is not preserved by Frobenius pull back $F_{X}^{*}$. Therefore, the set-theoretic map

$$V_{r,d}: \mathcal{M}_{X}^{ss}(r, d) \to \mathcal{M}_{X}^{ss}(r, pd)$$

is not well-defined on the whole moduli space $\mathcal{M}_{X}^{ss}(r, d)$. Denote

$$U_{X}^{ss}(r, d) := \{[\mathcal{E}] \in \mathcal{M}_{X}^{ss}(r, d) \mid F_{X}^{*}(\mathcal{E}) \text{ is a semi-stable vector bundle}\}.$$

Then $U_{X}^{ss}(r, d) \subseteq \mathcal{M}_{X}^{ss}(r, d)$ is an open sub-variety, and $V_{r,d}|_{U_{X}^{ss}(r,d)}: U_{X}^{ss}(r, d) \to \mathcal{M}_{X}^{ss}(r, pd)$ is a well defined morphism (See the proof of [7] Proposition 9). The rational map $V_{r,d}$ is called generalized Verschiebung rational map.

On the other hand, the (semi)-stability of Frobenius direct image has been study by many mathematicians.

(i) H. Lange and C. Pauly [3] Proposition 1.2] showed that if $g \geq 2$, $F_{X/k_{s}}(\mathcal{L})$ is stable on $X^{(1)}$ for any line bundle $\mathcal{L}$ on $X$;

(ii) V. Mehta and C. Pauly [6] Theorem 1.1] proved that if $g \geq 2$, then for any semi-stable bundle $\mathcal{E}$ on $X$, $F_{X/k_{s}}(\mathcal{E})$ is also semistable.

(iii) X. Sun [9] Theorem 2.2] showed that if $g \geq 1$, then $F_{X/k_{s}}(\mathcal{E})$ is semi-stable whenever $\mathcal{E}$ is semi-stable on $X$. Moreover, if $g \geq 2$, then $F_{X/k_{s}}(\mathcal{E})$ is stable whenever $\mathcal{E}$ is stable.
In section 3, we show that the set-theoretic map
\[ S^s_{\text{Frob}} : \mathcal{M}^s_X(r, d) \to \mathcal{M}^s_{X^{(1)}}(r \cdot p, d + r(p - 1)(g - 1)) \]
\[ \mathcal{E} \mapsto [F_{X/k_s}(\mathcal{E})] \]
is a proper morphism. Moreover, if \( g(X) \geq 2 \), the induced morphism
\[ S^s_{\text{Frob}} : \mathcal{M}^s_X(r, d) \to \mathcal{M}^s_{X^{(1)}}(r \cdot p, d + r(p - 1)(g - 1)) \]
is a closed immersion (Theorem 3.5).

Let \( \mathcal{E} \) a vector bundle on \( X \), we consider the Harder-Narasimhan filtration of \( \mathcal{E} \)
\[ \mathcal{E}^{\text{HN}} : 0 = \mathcal{E}_m \subset \mathcal{E}_{m-1} \subset \cdots \subset \mathcal{E}_1 \subset \mathcal{E}_0 = \mathcal{E} \]
For any subbundle \( \mathcal{E}_i \), we may associate to it the point \((\text{rk}(\mathcal{E}_i), \deg(\mathcal{E}_i))\) in the plane \( \mathbb{R}^2 \), and we connect point \((\text{rk}(\mathcal{E}_i), \deg(\mathcal{E}_i))\) to point \((\text{rk}(\mathcal{E}_{i-1}), \deg(\mathcal{E}_{i-1}))\) successively by line segments for \( 0 \leq i \leq m \). Then we get a convex polygon in the plane \( \mathbb{R}^2 \) which we call the Harder-Narasimhan Polygon of \( \mathcal{E} \), denote by \( \text{HNP}(\mathcal{E}) \).

Let \( \text{ConPg}(r, d) \) be the category of convex polygons with starting point at \((0, 0)\) and terminal point \((r, d)\), there is a natural partial order structure, denote by “\( \geq \)”, on \( \text{ConPg}(r, d) \). Let \( \mathcal{P}_1, \mathcal{P}_2 \in \text{ConPg}(r, d) \), we say \( \mathcal{P}_1 \geq \mathcal{P}_2 \) if and only if \( \mathcal{P}_1 \) lies on or above \( \mathcal{P}_2 \).

Consider the natural map
\[ \mathcal{M}^*_{X^{(1)}}(r, d) \to \text{ConPg}(r, pd) \]
\[ \mathcal{E} \mapsto \text{HNP}(F_{X/k}^*(\mathcal{E})) \]
There is a canonical stratification (Frobenius stratification) on \( \mathcal{M}^*_X(r, d) \) by Harder-Narasimhan polygons [3]. By a theorem of S. S. Shatz [8, Theorem 3], the subset
\[ S_{\mathcal{P}} := \{ \mathcal{E} \in \mathcal{M}^*_X(r, d) \mid \text{HNP}(F_{X/k}^*(\mathcal{E})) \supseteq \mathcal{P} \} \]
is a closed subvariety of \( \mathcal{M}^*_X(r, d) \) for any \( \mathcal{P} \in \text{ConPg}(r, pd) \).

The fundamental question of the Frobenius stratification of moduli space of (semi)-stable vector bundles in positive characteristic is: what is the geometric properties, such as non-emptiness, irreducibility, smoothness, dimension and so on, of each stratum of Frobenius stratification. However, very little is known about the strata of Frobenius stratification. Some results are only known in special cases or for small values of \( p, g, r \) and \( d \). For example, K. Joshi, S. Ramanan, E. Z. Xia and J.-K. Yu [3] given a completely understand of moduli space \( \mathcal{M}^*_X(1, 2) \) when \( p = 2 \) and \( g \geq 2 \). They proved the irreducibility of each non-empty stratum of Frobenius stratification and obtained their respective dimensions. This is the only case understood completely.

In section 4, we show that a stable bundles \( \mathcal{E} \) of rank \( p \) and degree \( d \) on \( X^{(1)} \) whose Frobenius pull back \( F_{X/k}^*(\mathcal{E}) \) have maximal Harder-Narasimhan Polygon if and only if \( \mathcal{E} \cong F_{X/k}^*(\mathcal{Z}) \) for some line bundle \( \mathcal{Z} \) on \( X \). It follows that the locus
\[ W = \{ \mathcal{E} \in \mathcal{M}^*_X(p, d) \mid \text{HNP}(F_{X/k}^*(\mathcal{E})) \supseteq \text{HNP}(F_{X/k}^*(\mathcal{F})) \} \]
of moduli space \( \mathcal{M}^*_X(1, d - (p - 1)(g - 1)) \) is precisely the image of the morphism
\[ S^s_{\text{Frob}} : \mathcal{M}^*_X(1, d - (p - 1)(g - 1)) \to \mathcal{M}^*_X(p, d) \]
Hence, \( W \) is a closed sub-variety of \( \mathcal{M}^*_X(p, d) \) which is isomorphic to Jacobian variety \( \text{Jac}_X \) of \( X \) (Corollary 4.4). This result generalize the partial result of [8, Theorem 4.6.4] via different method.
2. Canonical Connection and Canonical Filtration

Let $k$ be an algebraically closed field of characteristic $p > 0$, and $X$ a smooth projective variety over $k$. Consider the commutative diagram

For any coherent sheaf $\mathcal{F} \in \text{Cooh}(X^{(1)})$, there exists a canonical connection on the coherent sheaf $F^*_{X/k}(\mathcal{F})$, denote by $(F^*_{X/k}(\mathcal{F}), \nabla_{\text{can}})$:

$$\nabla_{\text{can}} : F^*_{X/k}(\mathcal{F}) \to F^*_{X/k}(\mathcal{F}) \otimes_{\mathcal{O}_X} \Omega^1_{X/k}$$

locally defined by $f \otimes m \mapsto m \otimes d(f)$, where $m \in \mathcal{F}$, $f \in \mathcal{O}_X$, $d : \mathcal{O}_X \to \Omega^1_{X/k}$ is the canonical exterior differentiation.

**Definition 2.1.** Let $k$ be an algebraically closed field of characteristic $p > 0$, and $X$ a smooth projective variety over $k$. For any coherent sheaf $\mathcal{E}$ on $X$, let

$$\nabla_{\text{can}} : F^*_{X/k}(\mathcal{E}) \to F^*_{X/k}(\mathcal{E}) \otimes_{\mathcal{O}_X} \Omega^1_{X/k}$$

be the canonical connection on $F^*_{X/k}(\mathcal{E})$. Set

$$V_0 := F^*_{X/k}(\mathcal{E}),$$
$$V_1 := \ker(F^*_{X/k}(\mathcal{E}) \to \mathcal{E}),$$
$$V_{l+1} := \ker\{V_l \to \bigoplus F^*_{X/k}(\mathcal{E}) \otimes_{\mathcal{O}_X} \Omega^1_{X/k} \to (F^*_{X/k}(\mathcal{E})/V_l) \otimes_{\mathcal{O}_X} \Omega^1_{X/k}\}$$

The filtration

$$F^*_{X/k}(\mathcal{E}) = V_0 \supset V_1 \supset V_2 \supset \cdots$$

is called the canonical filtration of $F^*_{X/k}(\mathcal{E})$.

X. Sun proved the following theorem in [9].

**Theorem 2.2.** [9 Theorem 3.7] Let $k$ be an algebraically closed field of characteristic $p > 0$, and $X$ a smooth projective variety of dimension $n$ over $k$. Let $\mathcal{E}$ be a vector bundle. Then the canonical filtration of $F^*_{X/k}(\mathcal{E})$ is

$$0 = V_0 \supset V_1 \supset \cdots \supset V_{n(p-1)} \supset V_{n(p-1)+1} = F^*_{X/k}(\mathcal{E})$$

with $\nabla^l : V_l/V_{l+1} \cong \mathcal{E} \otimes_{\mathcal{O}_X} T^l(\Omega^1_{X/k})$, $0 \leq l \leq n(p-1)$.

Let $\mathcal{E}$ be a vector bundle of rank $n$ on a variety, then $T^l(\mathcal{E}) \subset \mathcal{E}^{\otimes l}$ is defined to be the associated vector bundle of the frame bundle of $\mathcal{E}$ (principal $\text{GL}_n(k)$-bundle) through the representation $T^l(V)$ (See [9 Definition 3.4]).
3. The Morphism Induced by Frobenius Push-Forwards

In this section, we will study the natural morphism between moduli spaces of (semi)-stable bundles on curves induced by Frobenius push-forwards.

**Proposition 3.1.** Let $k$ be an algebraically closed field of characteristic $p > 0$, and $X$ a smooth projective curve of genus $g \geq 1$ over $k$. Then

1. If $g \geq 1$, then the set-theoretic map
   \[
   S_{\text{Frob}}^{ss} : \mathcal{M}_X^{ss}(r, d) \to \mathcal{M}_X^{ss}(1)(r \cdot p, d + r(p - 1)(g - 1))
   \]
   is a proper morphism.

2. If $g \geq 2$, then the morphism $S_{\text{Frob}}^{ss}$ restrict to sub-variety $\mathcal{M}_X^{ss}(r, d)$ induces a proper morphism $S_{\text{Frob}}^{ss} : \mathcal{M}_X^{ss}(r, d) \to \mathcal{M}_X^{ss}(1)(r \cdot p, d + r(p - 1)(g - 1))$.

**Proof.** Let $T$ be an algebraic variety over $k$, and $E \in \mathcal{Coh}(T \times X)$ a flat family of semi-stable bundles on $X$ of rank $r$ and degree $d$ parameterized by $T$. Consider the morphism $1_T \times F_{X/k} : T \times X \to T \times X^{(1)}$. Then $1_T \times F_{X/k}(s)\mathcal{E}(E)$ is flat over $T$, and $(1_T \times F_{X/k}(s)\mathcal{E}(E)) \subseteq F_{X/k}(s)\mathcal{E}(E)$ for any $t \in T$. If $g \geq 1$, then $F_{X/k}(s)\mathcal{E}(E)$ are semi-stable vector bundles of rank $r \cdot p$ and degree $d + r(p - 1)(g - 1)$ by [9] Theorem 2.2, Lemma 4.2. Thus, $1_T \times F_{X/k}(s)\mathcal{E}(E)$ is a flat family of semi-stable bundles of rank $r \cdot p$ and degree $d + r(p - 1)(g - 1)$ parameterized by $T$. Hence, by the universal property of $\mathcal{M}_X^{ss}(1)(r \cdot p, d + r(p - 1)(g - 1))$, the set-theoretic map
   \[
   V_{\mathcal{E}} : T \to \mathcal{M}_X^{ss}(1)(r \cdot p, d + r(p - 1)(g - 1))
   \]
   \[
   t \mapsto [F_{X/k}(s)\mathcal{E}(E)]
   \]
   is a morphism.

By the arbitrariness of $T$ and $E$ and the universal property of moduli space $\mathcal{M}_X^{ss}(r, d)$, the set-theoretic map
   \[
   S_{\text{Frob}}^{ss} : \mathcal{M}_X^{ss}(r, d) \to \mathcal{M}_X^{ss}(1)(r \cdot p, d + r(p - 1)(g - 1))
   \]
   \[
   [E'] \mapsto [F_{X/k}(s)\mathcal{E}(E)]
   \]
   is a morphism. Since $\mathcal{M}_X^{ss}(r, d)$ and $\mathcal{M}_X^{ss}(1)(r \cdot p, d + r(p - 1)(g - 1))$ are projective varieties, it follows that $S_{\text{Frob}}^{ss}$ is a proper morphism.

(2). If $g \geq 2$, then by [9] Theorem 2.2] we have
   \[
   S_{\text{Frob}}^{ss}(\mathcal{M}_X^{ss}(r, d)) \subseteq \mathcal{M}_X^{ss}(1)(r \cdot p, d + r(p - 1)(g - 1)).
   \]

On the other hand, we claim that for any vector bundle $\mathcal{E} \in \mathcal{Coh}(X)$, the (semi)-stability of $F_{X/k}(s)\mathcal{E}$ implies the (semi)-stability of $\mathcal{E}$. In fact, let $\mathcal{F} \subseteq \mathcal{E}$ be a coherent sub-sheaf with $0 < \text{rk}(\mathcal{F}) < \text{rk}(\mathcal{E})$, then $F_{X/k}(s)\mathcal{F} \subseteq F_{X/k}(s)\mathcal{E}$ and $0 < \text{rk}(F_{X/k}(s)\mathcal{F}) < \text{rk}(F_{X/k}(s)\mathcal{E}))$, since $F_{X/k}$ is a left exact functor. As (cf Lemma 4.2 in [3])
   \[
   \mu(F_{X/k}(s)\mathcal{E}) = \frac{1}{p} \mu(F_{X/k}(s)\mathcal{E}) = \frac{(p - 1)(2 \cdot g - 2)}{2p} + \frac{\mu(\mathcal{E})}{p}.
   \]
   Thus $\mu(F_{X}(\mathcal{F})) < \mu(F_{X}(\mathcal{E}))$ implies $\mu(\mathcal{F}) < \mu(\mathcal{E})$. Hence,
   \[
   \text{Frob}(\mathcal{M}_X^{ss}(1)(r \cdot p, d + r(p - 1)(g - 1))) = \mathcal{M}_X^{ss}(r, d).
   \]

Thus the properness of morphism $S_{\text{Frob}}^{ss} : \mathcal{M}_X^{ss}(r, d) \to \mathcal{M}_X^{ss}(1)(r \cdot p, d + r(p - 1)(g - 1))$ follows from the properness of $S_{\text{Frob}}^{ss}$. □
The following lemma asserts that push-forwards preserve the determinant of vector bundles on curves.

**Lemma 3.2.** Let $k$ be an algebraically closed field, $f : X \to Y$ a finite morphism of smooth projective curves over $k$. Let $\mathcal{E}$ be a vector bundle, $\det(\mathcal{E}) = \mathcal{O}_X(\sum n_i P_i)$, where $P_i \in X$, $n_i \in \mathbb{Z}$. Then

$$\det(f_*(\mathcal{E})) \cong (\det(f_*\mathcal{O}_X))^{\otimes \text{rk}(\mathcal{E})} \otimes_{\mathcal{O}_Y} (\sum n_i f(P_i)).$$

In particular, let $\mathcal{E}_1$ and $\mathcal{E}_2$ be vector bundles such that $\text{rk}(\mathcal{E}_1) = \text{rk}(\mathcal{E}_2)$ and $\det(\mathcal{E}_1) = \det(\mathcal{E}_2)$. Then $\det(f_*(\mathcal{E}_1)) = \det(f_*(\mathcal{E}_2))$.

**Proof.** We use induction on the rank of $\mathcal{E}$. Suppose $\text{rk}(\mathcal{E}) = r$. In the case $r = 1$, by [H] Excise IV.2.6 we have

$$\det(f_*(\mathcal{E})) \cong (\det(f_*\mathcal{O}_X)) \otimes_{\mathcal{O}_Y} (\sum n_i f(P_i)).$$

The lemma is true. Suppose the lemma is true for any vector bundles of rank less than $r$. Choose any sub-line bundle $\mathcal{L} \subset \mathcal{E}$ such that $\mathcal{E}/\mathcal{L}$ is also a vector bundle. Consider the exact sequence of $\mathcal{O}_Y$-modules

$$0 \to f_*(\mathcal{L}) \to f_*(\mathcal{E}) \to f_*(\mathcal{E}/\mathcal{L}) \to 0,$$

then we have $\det(f_*(\mathcal{E})) \cong \det(f_*(\mathcal{L})) \otimes_{\mathcal{O}_Y} \det(f_*(\mathcal{E}/\mathcal{L}))$. On the other hand, since $\det(\mathcal{E}) \cong \det(\mathcal{L}) \otimes_{\mathcal{O}_Y} \det(\mathcal{E}/\mathcal{L})$, by induction hypothesis we have

$$\det(f_*(\mathcal{E})) \cong (\det(f_*\mathcal{O}_X))^{\otimes \text{rk}(\mathcal{E})} \otimes_{\mathcal{O}_Y} (\sum n_i f(P_i)),$$

where $\det(\mathcal{E}) = \mathcal{O}_X(\sum n_i P_i)$, $P_i \in X$, $n_i \in \mathbb{Z}$. □

**Corollary 3.3.** Let $k$ be an algebraically closed field of characteristic $p > 0$, and $X$ a smooth projective curve of genus $g \geq 1$ over $k$, $\mathcal{L} \in \text{Pic}(X)$. Then the set-theoretic map

$$S^\mathcal{L}_{\text{Frob}} : \mathcal{M}_X^{ss}(r, \mathcal{L}) \to \mathcal{M}_X^{ss}(r \cdot p, \det(\mathcal{F}_X/k, (\mathcal{E})), (\forall \mathcal{E} \in \mathcal{M}_X^{ss}(r, \mathcal{L}))$$

$$[\mathcal{E}] \mapsto [\mathcal{F}_X/k, (\mathcal{E})]$$

is a proper morphism.

**Proof.** For any $[\mathcal{E}] \in \mathcal{M}_X^{ss}(r, \mathcal{L})$. Since $\mathcal{M}_X^{ss}(r, \mathcal{L})$ and $\mathcal{M}_X^{ss}(r \cdot p, \det(\mathcal{F}_X/k, (\mathcal{E})))$ are closed sub-varieties of $\mathcal{M}_X^{ss}(r, \mathcal{L})$ and $\mathcal{M}_X^{ss}(r \cdot p, d+r(p-1)(g-1))$ respectively. Then the corollary follows from Proposition [3.1] and Lemma [3.2] □

We now use the functoriality of canonical filtration and the uniqueness of Harder-Narasimhan filtration to prove the following theorem.

**Theorem 3.4.** Let $k$ be an algebraically closed field of characteristic $p > 0$, and $X$ a smooth projective variety with a fixed ample divisor $H$. Let $\mathcal{E}_1$, $\mathcal{E}_2$ be slope semi-stable vector bundles such that $\mathcal{E}_i \otimes \mathcal{O}_X T^i(\Omega^1_{X/k})$ are slope semi-stable for any integer $0 \leq l \leq n(p-1)$, $(i = 1, 2)$. If $\mu(\Omega^1_{X/k}) > 0$. Then

1. The canonical filtration of $F^*_X/k, (\mathcal{E}_1)$ is precisely the Harder-Narasimhan filtration of $F^*_X/k, (\mathcal{E}_1)$, $(i = 1, 2)$. In particular, $F_X/k, (\mathcal{E}_1) \cong F_X/k, (\mathcal{E}_2)$ implies $\mathcal{E}_1 \cong \mathcal{E}_2$. 
(2). If $\mu(\mathcal{E}_1) = \mu(\mathcal{E}_2)$, Then the natural $k$-linear homomorphism

$$\Phi : \text{Ext}^1_X(\mathcal{E}_1, \mathcal{E}_2) \rightarrow \text{Ext}^1_X(F_{X/k*}(\mathcal{E}_1), F_{X/k*}(\mathcal{E}_2))$$

$$[0 \rightarrow \mathcal{E}_1 \rightarrow \mathcal{F} \rightarrow \mathcal{E}_2 \rightarrow 0] \rightarrow [0 \rightarrow F_{X/k*}(\mathcal{E}_1) \rightarrow F_{X/k*}(\mathcal{F}) \rightarrow F_{X/k*}(\mathcal{E}_2) \rightarrow 0]$$

is an injective homomorphism.

**Proof.** (1). Consider the canonical filtration of $F_{X/k*}(\mathcal{E}_i)$:

$$\mathcal{F}^\text{can}_{\mathcal{E}_i} : 0 = V^0_{X/k*} \subset V^1_{X/k*} \subset \cdots \subset V^e_{X/k*}$$

Since $V^e_{X/k*} / V^i_{X/k*} \simeq \mathcal{O}_X \otimes \mathcal{O}_X T^i(\Omega^1_{X/k})(0 \leq i \leq n(p - 1))$ are slope semi-stable vector bundles, and for any integer $0 \leq s < t \leq n(p - 1)$,

$$\mu(\mathcal{E}_i \otimes \mathcal{O}_X T^s(\Omega^1_{X/k})) = \mu(\mathcal{E}_i) + s \cdot \mu(\Omega^1_{X/k}) < \mu(\mathcal{E}_i) + t \cdot \mu(\Omega^1_{X/k}) = \mu(\mathcal{E}_i \otimes \mathcal{O}_X T^t(\Omega^1_{X/k})).$$

Then by the uniqueness of Harder-Narasimhan filtration we know that canonical filtration $\mathcal{F}^\text{can}_{\mathcal{E}_i}$ is precisely the Harder-Narasimhan filtration of $F_{X/k*}(\mathcal{E}_i)$. It is easy to see that $F_{X/k*}(\mathcal{E}_1) \simeq F_{X/k*}(\mathcal{E}_2)$ implies $\mathcal{E}_1 \simeq \mathcal{E}_2$ by the uniqueness of Harder-Narasimhan filtration.

Similarly.

(2). Let $e_i := [0 \rightarrow \mathcal{E}_1 \rightarrow \mathcal{F}_i \rightarrow \mathcal{E}_2 \rightarrow 0] \in \text{Ext}^1_X(\mathcal{E}_1, \mathcal{E}_2)$, $i = 1, 2$, such that $\Phi(e_1) = \Phi(e_2)$, i.e. there exists isomorphism of $\mathcal{O}_{X(1)}$-modules $\phi : F_{X/k*}(\mathcal{F}_1) \simeq F_{X/k*}(\mathcal{F}_2)$ such that the following diagram

$$0 \rightarrow F_{X/k*}(\mathcal{E}_1) \rightarrow F_{X/k*}(\mathcal{F}_1) \rightarrow F_{X/k*}(\mathcal{E}_2) \rightarrow 0$$

is commutative. Taking $F_{X/k*}$ to the above exact sequence, we get the following commutative diagram of $\mathcal{O}_X$-modules

$$0 \rightarrow F_{X/k*}F_{X/k*}(\mathcal{E}_1) \rightarrow F_{X/k*}F_{X/k*}(\mathcal{F}_1) \rightarrow F_{X/k*}F_{X/k*}(\mathcal{E}_2) \rightarrow 0$$

On the other hand, for any integer $0 \leq l \leq n(p - 1)$, consider the exact sequence of $\mathcal{O}_X$-modules $0 \rightarrow \mathcal{E}_j \otimes \mathcal{O}_X T^l(\Omega^1_{X/k}) \rightarrow \mathcal{F}_j \otimes \mathcal{O}_X T^l(\Omega^1_{X/k}) \rightarrow \mathcal{E}_2 \otimes \mathcal{O}_X T^l(\Omega^1_{X/k}) \rightarrow 0$. Because $\mathcal{E}_j \otimes \mathcal{O}_X T^l(\Omega^1_{X/k}) (j = 1, 2)$ are slope semi-stable bundles and $\mu(\mathcal{E}_j) = \mu(\mathcal{E}_2)$, then $\mathcal{F}_j \otimes \mathcal{O}_X T^l(\Omega^1_{X/k}) (j = 1, 2)$ are also slope semi-stables. It follows from (1) that the canonical filtration $\mathcal{F}^\text{can}_{\mathcal{F}_i}$ of $F_{X/k*}F_{X/k*}(\mathcal{F}_i)$ is also coincided with the Harder-Narasimhan filtration of $F_{X/k*}(\mathcal{F}_i)$.
Then we have the following commutative diagram

\[
\begin{array}{c}
\begin{array}{ccc}
0 & \longrightarrow & \mathcal{F}_X/k_s(\mathcal{E}_1) \\
\downarrow & & \downarrow \\
0 & \longrightarrow & \mathcal{F}_X/k_s(\mathcal{E}_2) \\
\end{array}
\end{array}
\]

Restricting the isomorphism \( F^*_X/k(\phi) : F^*_X/k\mathcal{F}_X/k_s(\mathcal{F}_1) \to F^*_X/k\mathcal{F}_X/k_s(\mathcal{F}_2) \) to \( V_1^{\mathcal{F}_1} \), we get an isomorphism \( F^*_X(k)(\phi)|_{V_1^{\mathcal{F}_1}} : V_1^{\mathcal{F}_1} \to V_1^{\mathcal{F}_2} \), which induces a natural isomorphism \( \psi : \mathcal{F}_1 \cong \mathcal{F}_2 \) with following commutative diagram

\[
\begin{array}{c}
\begin{array}{ccc}
0 & \longrightarrow & \mathcal{E}_1 \\
\downarrow & & \downarrow \\
0 & \longrightarrow & \mathcal{E}_2 \\
\end{array}
\end{array}
\]

Hence \( \Phi : \text{Ext}^1_X(\mathcal{E}_1, \mathcal{E}_2) \to \text{Ext}^1_X(\mathcal{F}_X/k_s(\mathcal{E}_1), \mathcal{F}_X/k_s(\mathcal{E}_2)) \) is an injection. \( \square \)

**Theorem 3.5.** Let \( k \) be an algebraically closed field of characteristic \( p > 0 \), and \( X \) a smooth projective curve of genus \( g \geq 2 \) over \( k \). Then the morphism

\[
S^*_\text{Frob} : \mathcal{M}^s_X(r, d) \to \mathcal{M}^s_{X(1)}(r \cdot p, d + r(p - 1)(g - 1))
\]

is a closed immersion. In particular, for any \( \mathcal{L} \in \text{Pic}(X) \), the morphism

\[
S^*_{\text{Frob}} : \mathcal{M}^s_X(r, \mathcal{L}) \to \mathcal{M}^s_{X(1)}(r \cdot p, \det(\mathcal{F}_X/k_s(\mathcal{E}))), (\forall \mathcal{E} \in \mathcal{M}^s_X(r, \mathcal{L}))
\]

is also a closed immersion.

**Proof.** Since \( \dim X = 1 \), \( T^r(\Omega^1_X/k) \cong (\Omega^1_X/k)^{\oplus l} \) for any integer \( 0 \leq l \leq n(p - 1) \). Then for any stable bundle \( \mathcal{E} \) and any integer \( 0 \leq l \leq n(p - 1) \), \( \mathcal{E} \otimes X T^r(\Omega^1_X/k) \) are stable. Therefore, by Proposition \([3, 1]\) and Theorem \([3, 3]\) the morphism

\[
S^*_{\text{Frob}} : \mathcal{M}^s_X(r, d) \to \mathcal{M}^s_{X(1)}(r \cdot p, d + r(p - 1)(g - 1))
\]

is an injective proper morphism. Since for any \( \mathcal{E} \in \mathcal{M}^s_X(r, d) \), \( F_X/k_s(\mathcal{E}) \) is also stable by \([3, \text{Theorem } 2.2]\). Then the tangent space of \( \mathcal{M}^s_X(r, d) \) at \( \mathcal{E} \) is

\[
T_{[\mathcal{E}]} \mathcal{M}^s_X(r, d) = \text{Ext}^1_X(\mathcal{E}, \mathcal{E}),
\]

and tangent space of \( \mathcal{M}^s_{X(1)}(r \cdot p, d + r(p - 1)(g - 1)) \) at \( [F_X/k_s(\mathcal{E})] \) is

\[
T_{[F_X/k_s(\mathcal{E})]} \mathcal{M}^s_{X(1)}(r \cdot p, d + r(p - 1)(g - 1)) = \text{Ext}^1_X(F_X/k_s(\mathcal{E}), F_X/k_s(\mathcal{E})).
\]

Moreover, tangent map of \( S^*_{\text{Frob}} \) at \( \mathcal{E} \) is precisely the homomorphism

\[
T_{S^*_{\text{Frob}}, \mathcal{E}} : \text{Ext}^1_X(\mathcal{E}, \mathcal{E}) \to \text{Ext}^1_X(F_X/k_s(\mathcal{E}), F_X/k_s(\mathcal{E}))
\]

\([0 \to \mathcal{E} \to \mathcal{F} \to \mathcal{E} \to 0] \mapsto [0 \to F_X/k_s(\mathcal{E}) \to F_X/k_s(\mathcal{F}) \to F_X/k_s(\mathcal{E}) \to 0]
\]

Then by Theorem \([3, 3]\) we have the tangent map \( T_{S^*_{\text{Frob}}, \mathcal{E}} \) is an injective homomorphism. Therefore the morphism \( S^*_\text{Frob} \) is a closed immersion by the well known criterion of closed immersion. \( \square \)
4. The Stratum of Frobenius Stratification with Maximal 
Harder-Narasimhan Polygon

**Definition 4.1.** Let $k$ be an algebraically closed field, $X$ a smooth projective curve 
over $k$. A local system on $X$ is pair $(\mathcal{E}, \nabla)$ consists of a vector bundle $\mathcal{E}$ on $X$ and a 
connection $\nabla$ on $X$. A local system $(\mathcal{E}, \nabla)$ is called semi-stable (resp. stable) if an 
any proper $\nabla$-invariant subbundle $\mathcal{F} \subset \mathcal{E}$ satisfies $\mu(\mathcal{F}) \leq \mu(\mathcal{E})$ (resp. $\mu(\mathcal{F}) < \mu(\mathcal{E})$).

An oper on $X$ is a local system $(\mathcal{E}, \nabla)$ together with a filtration of subbundles $\mathcal{E}_\bullet$:

$0 = \mathcal{E}_m \subset \mathcal{E}_{m-1} \subset \cdots \subset \mathcal{E}_1 \subset \mathcal{E}_0 = \mathcal{E}$, such that

(i). $\nabla(\mathcal{E}_i) \subseteq \mathcal{E}_{i-1} \otimes_{\mathcal{O}_X} \Omega_X$ for any $1 \leq i \leq m - 1$;

(ii). $\mathcal{E}_i/\mathcal{E}_{i+1} \otimes_{\mathcal{O}_X} \Omega_X$ is an isomorphism for any $1 \leq i \leq m - 1$.

Let $r \in \mathbb{Z}_{>0}$ and $d \in \mathbb{Z}$ such that $r|d$. We introduce the oper-polygon for pair $(r, d)$ in the plane $\mathbb{R}^2$

$\mathcal{P}_{oper}^{r, d}$ : with vertices $(i, \frac{d}{r} + i(r - i)(g - 1))$ for $0 \leq i \leq r$.

$\mathcal{P}_{oper}^{r, 0}$ is simply denoted by $\mathcal{P}_{oper}$ (See also section 5.3 of [2]).

**Remark 4.2.** If $g \geq 2$, then the Harder-Narasimhan polygon of any oper $(\mathcal{E}, \nabla, \mathcal{E}_\bullet)$ of rank $r$, degree $d$ and type $1$ (i.e. $\text{rk}(\mathcal{E}_i/\mathcal{E}_{i+1}) = 1$ for $0 \leq i \leq r - 1$) is $\mathcal{P}_{oper}^{r, d}$.

Since in this case the Harder-Narasimhan filtration of $\mathcal{E}$ is just $\mathcal{E}_\bullet$.

**Lemma 4.3.** ([2] Lemma 5.1.1) Let $k$ be an algebraically closed field, $X$ a smooth projective curve of genus $g$ over $k$. Let $(\mathcal{E}, \nabla)$ be a semi-stable local system with 
Harder-Narasimhan filtration $0 = \mathcal{E}_m \subset \mathcal{E}_{m-1} \subset \cdots \subset \mathcal{E}_1 \subset \mathcal{E}_0 = \mathcal{E}$. Then for any 
$1 \leq i \leq m - 1$, $\mu(\mathcal{E}_i/\mathcal{E}_{i-1}) - \mu(\mathcal{E}_{i+1}/\mathcal{E}_i) \leq 2g - 2$.

**Proposition 4.4.** Let $k$ be an algebraically closed field, $X$ a smooth projective curve of genus $g$ over $k$, $r \in \mathbb{Z}_{>0}$, $d \in \mathbb{Z}$ such that $r|d$. Let $(\mathcal{E}, \nabla)$ be a semi-stable local system $(\mathcal{E}, \nabla)$ of rank $r$ and degree $d$ on $X$. Then

1. $\mathcal{P}_{oper}^{r, d} \supset \text{HNP}(\mathcal{E})$,

2. $\mathcal{P}_{oper}^{r, d} = \text{HNP}(\mathcal{E})$ if and only if $(\mathcal{E}, \nabla, \mathcal{E}_{HN})$ is an oper of type $1$, where 
$\mathcal{E}_{HN}$ is the Harder-Narasimhan filtration of $\mathcal{E}$.

**Proof:** (1). Let $0 = \mathcal{E}_m \subset \mathcal{E}_{m-1} \subset \cdots \subset \mathcal{E}_1 \subset \mathcal{E}_0 = \mathcal{E}$ be the Harder-Narasimhan filtration of $\mathcal{E}$. Suppose that $\mathcal{P}_{oper}^{r, d} \not\supset \text{HNP}(\mathcal{E})$. Then there exists some $0 \leq j \leq m - 1$ such that the point $(\text{rk}(\mathcal{E}_{j+1}), \text{deg}(\mathcal{E}_{j+1}))$ lies on or below the $\mathcal{P}_{oper}^{r, d}$, and the point $(\text{rk}(\mathcal{E}_j), \text{deg}(\mathcal{E}_j))$ lies above the $\mathcal{P}_{oper}^{r, d}$, i.e. there exist $0 \leq j \leq m - 1$ and $1 \leq i \leq r$ with the properties $\mu(\mathcal{E}_j/\mathcal{E}_{j+1}) > \frac{d}{r} + (r - 2i + 1)(g - 1)$ and $\text{deg}(\mathcal{E}_j) > i\frac{d}{r} + i(r - i)(g - 1)$. Then by Lemma 4.3 we have

$$
\text{deg}(\mathcal{E}) = \text{deg}(\mathcal{E}_j) + \sum_{l=0}^{j-1} \text{rk}(\mathcal{E}_l/\mathcal{E}_{l+1}) \cdot \mu(\mathcal{E}_l/\mathcal{E}_{l+1}) \\
\geq \text{deg}(\mathcal{E}_j) + \sum_{l=0}^{j-1} \text{rk}(\mathcal{E}_l/\mathcal{E}_{l+1}) \cdot [\mu(\mathcal{E}_j/\mathcal{E}_{j+1}) - (j - l)(g - 1)] \\
\geq \text{deg}(\mathcal{E}_j) + \sum_{l=0}^{r-i} [\mu(\mathcal{E}_j/\mathcal{E}_{j+1}) - (r - i - l + 1)(g - 1)] \\
> d.
$$
This contradicts to the fact \( \deg(\mathcal{E}) = d \). Hence \( \mathcal{H}^{oper} \ni \text{HNP}(\mathcal{E}) \).

(2). Note that the part two of [2] Theorem 5.3.1 make the assumption that \( d = 0 \). In fact, their proof is also valid for any \( d \) such that \( r | d \).

\[\text{Theorem 4.5.} \] Let \( k \) be an algebraically closed field of characteristic \( p > 0 \), and \( X \) a smooth projective curve of genus \( g \geq 2 \) over \( k \). Let \( \mathcal{E} \) be a stable vector bundle on \( X \) of rank \( p \) and degree \( d \). Then the following conditions are equivalent

(i). \( \text{HNP}(F_{X/k}^* (\mathcal{E})) = \mathcal{H}^{oper} \).

(ii). \( \mu_{\max}(F_{X/k}^*(\mathcal{E})) - \mu_{\min}(F_{X/k}^*(\mathcal{E})) = (p - 1)(2g - 2) \).

(iii). \( \mathcal{E} \cong F_{X/k}^*(\mathcal{L}) \) for some line bundle \( \mathcal{L} \) on \( X \).

(iv). \( \text{HNP}(F_{X/k}^*(\mathcal{E})) \succ \text{HNP}(F_{X/k}^*(\mathcal{F})) \) for any \( \mathcal{F} \in \mathfrak{M}_{X(1)}^s(p,d) \).

\[\text{Proof.} \] (i) \( \Rightarrow \) (ii) is obvious by definition of \( \mathcal{H}^{oper} \).

(ii) \( \Leftrightarrow \) (iii) follows from [3] Proposition 1.4.

(iii) \( \Rightarrow \) (iv). By [3] Theorem 5.3, we have \( (F_{X/k}^*(\mathcal{E}), \nabla_{\text{can}}) \) is an oper of type 1. It follows from Remark 4.2 that \( \text{HNP}(F_{X/k}^*(\mathcal{L})) = \mathcal{H}^{oper} \). Then \( \text{HNP}(F_{X/k}^*(\mathcal{L})) \succ \text{HNP}(F_{X/k}^*(\mathcal{F})) \) for any \( \mathcal{F} \in \mathfrak{M}_{X(1)}(p,d) \) by Proposition 4.4 since \( (F_{X/k}^*(\mathcal{F}), \nabla_{\text{can}}) \) is a semi-stable local system by Cartier’s theorem ([4] Theorem 5.1]).

(iv) \( \Rightarrow \) (i). For any \( \mathcal{F} \in \mathfrak{M}_{X(1)}^s(p,d) \), \( (F_{X/k}^*(\mathcal{F}), \nabla_{\text{can}}) \) is a semi-stable local system. Then \( \mathcal{H}^{oper} \ni \text{HNP}(F_{X/k}^*(\mathcal{F})) \) by Lemma 4.4. On the other hand, choose any line bundle \( \mathcal{L} \) of degree \( d - (p - 1)(g - 1) \), then \( F_{X/k}^*(\mathcal{L}) \) is a stable vector bundle of rank \( p \) and degree \( d \) and \( \text{HNP}(F_{X/k}^*(\mathcal{L})) = \mathcal{H}^{oper} \). Hence if \( \text{HNP}(F_{X/k}^*(\mathcal{E})) \succ \text{HNP}(F_{X/k}^*(\mathcal{F})) \) for any \( \mathcal{F} \in \mathfrak{M}_{X(1)}^s(p,d) \), then \( \text{HNP}(F_{X/k}^*(\mathcal{E})) = \mathcal{H}^{oper} \).

\[\text{Corollary 4.6.} \] Let \( k \) be an algebraically closed field of characteristic \( p > 0 \), and \( X \) a smooth projective curve of genus \( g \geq 2 \) over \( k \). Then the subset

\[ W = \{ \mathcal{E} \in \mathfrak{M}_{X(1)}^s(p,d) \mid \text{HNP}(F_{X/k}^*(\mathcal{E})) \succ \text{HNP}(F_{X/k}^*(\mathcal{F})) \text{ for any } \mathcal{F} \in \mathfrak{M}_{X(1)}(p,d) \} \]

is a closed sub-variety of \( \mathfrak{M}_{X(1)}^s(p,d) \), which is isomorphic to Jacobian variety \( \text{Jac}_X \) of \( X \). In particular, \( W \) is an irreducible smooth projective variety of dimension \( g \).

\[\text{Proof.} \] By Theorem 4.5 we know that \( W \) is precisely the image of the morphism

\[ S_p^{\text{Frob}} : \mathfrak{M}_X^s(1, d - (p - 1)(g - 1)) \to \mathfrak{M}_{X(1)}(p,d). \]

Then this Corollary follows from Theorem 4.5 and the trivial fact

\[ \mathfrak{M}_X^s(1, -(p - 1)(g - 1)) \cong \text{Jac}_X. \]

In the case of \( (p,r) = (2,2) \), K. Joshi, S. Ramanan, E. Z. Xia and J.-K. Yu [3] Theorem 4.6.4 show that the locus of moduli space \( \mathfrak{M}_{X(1)}^s(2,d) \) consist of stable vector bundles \( \mathcal{E} \) whose Frobenius pull back \( F_{X/k}^*(\mathcal{E}) \) have maximal Harder-Narasimhan Polygon is an irreducible projective variety of dimension \( g \). The Corollary 4.6 generalize this result to more general case via different method.

5. Acknowledgments

I would like to express my hearty thanks to my advisor Professor X. Sun for helpful discussions.
References

[1] R. Hartshorne: Algebraic geometry. Graduate Texts in Mathematics 52, Springer Verlag, New York (1977).

[2] K. Joshi, C. Pauly: Hitchin-Mochizuki morphism, Opers and Frobenius-destabilized vector bundles over curves. arXiv: 0912.3602 (2009).

[3] K. Joshi, S. Ramanan, E. Z. Xia, J.-K. Yu: On vector bundles destabilized by Frobenius pull-back. Compos. Math. 142 (2006), 616-630.

[4] N. M. Katz: Nilpotent connections and the monodromy theorem: Applications of a result of Turrittin. Publ. Math., Inst. Hautes´ etud. Sci. 39 (1970), 175-232.

[5] H. Lange, C. Pauly: On Frobenius-destabilized rank-2 vector bundles over curves. Comment. Math. Helv. 83 (2008), no. 1, 179-209.

[6] V. Mehta, C. Pauly: Semistability of Frobenius direct images over curves. Bull. Soc. Math. France. 135 (2007), no. 1, 105-117.

[7] B. Osserman: The generalized Verschiebung map for curves of genus 2. Math. Ann. 336 (2006), 963-986.

[8] S. S. Shatz, The decomposition and specializations of algebraic families of vector bundles. Comp. math. 35 (1977), 163-187.

[9] X. Sun: Direct images of bundles under Frobenius morphisms. Invent. Math. 173 (2008), no. 2, 427-447.

School of Mathematical Sciences, Fudan University, Shanghai, P. R. China

E-mail address: LG.Lee@amss.ac.cn