The general pattern of Kač Moody extensions in supergravity and the issue of cosmic billiards†

Pietro Fré*, Floriana Gargiulo#, Ksenya Rulik† and Mario Trigiante#

* Dipartimento di Fisica Teorica, Università di Torino, & INFN - Sezione di Torino
via P. Giuria 1, I-10125 Torino, Italy
# Dipartimento di Fisica Politecnico di Torino, C.so Duca degli Abruzzi, 24, I-10129 Torino, Italy

Abstract

In this paper we study the systematics of the affine extension of supergravity duality algebras when one steps down from $D = 4$ to $D = 2$ which is instrumental for the study of cosmic billiards. For all $D = 4$ supergravities (with $N \geq 3$) there is a universal field theoretical mechanism promoting the extension, which relies on the coexistence of two non locally related lagrangian descriptions of the corresponding $D = 2$ degrees of freedom: the Ehlers lagrangian and the Matzner–Misner one. This and the existence of a generalized Kramer–Neugebauer non local transformation relating the two models, provide a Chevalley-Serre presentation of the affine Kač–Moody algebra which follows a universal pattern for all supergravities. This is an extension of the mechanism considered by Nicolai for pure $N=1$ supergravity, but has general distinctive features in extended theories ($N \geq 3$) related to the presence of vector fields and to their symplectic description. Moreover the novelty is that in the general case the Matzner-Misner lagrangian is structurally different from the Ehlers one, since half of the scalars are replaced by gauge 0–forms subject to SO(2n, 2n) electric–magnetic duality rotations representing in $D = 2$ the Sp(2n, $\mathbb{R}$) rotations of $D = 4$. The role played by the symplectic bundle of vectors in this context, suggests that the mechanism of the affine extension can be studied also for $N = 2$ supergravity, where one deals with geometries rather than with algebras, the scalar manifold being not necessarily a homogeneous manifold $U/H$. We also show that the mechanism of the affine extension commutes with the Tits Satake projection of the relevant duality algebras. This is very important for the issue of cosmic billiards, as we show in a separate paper. Finally we also comment on the general field theoretical mechanism of the further hyperbolic extension obtained in $D = 1$, which we plan to analyze in detail in a forthcoming paper. The possible uses of our results and their relation to outstanding problems are pointed out.

†This work is supported in part by the European Union network contract MRTN-CT-2004-005104
1 Introduction

The discovery that General Relativity hides dynamical symmetries associated with infinite Lie algebras of the Kač–Moody type is an old one and dates back to the work of Geroch [1]. Considering metrics that admit two commuting, space-like Killing vectors, Geroch effectively reduced Einstein Gravity to $D = 2$ and showed that the space of such solutions is mapped into itself by an infinite group of symmetries, whose Lie algebra, in modern parlance, is $A_1^\Lambda$, namely the affine Kač–Moody extension of $A_1$, the Lie algebra of the three-parameter group $\text{SL}(2, \mathbb{R})$. These quite intriguing symmetries were extensively analyzed at the end of the seventies and in the beginning of the eighties by several groups (for a comprehensive review see [2]) and a formalism was developed to cope with them in a mathematical rigorous and effective way that is centered on the notion of linear system. To summarize a complicated story in a few words we just recall that Kač–Moody algebras [3] are the Lie algebras of centrally extended loop groups, namely the groups of analytic maps from a circle into an ordinary simple Lie group $G$:

$$G^\infty \ni \gamma : \mathbb{S}^1 \mapsto G$$

(1.1)

Calling $z = \exp [i \theta]$ a local coordinate on $\mathbb{S}^1$, a group element $\hat{g} \in G^\Lambda$ is represented by a pair

$$\hat{g} = (g(z), e^a)$$

(1.2)

where $g(z) \in G^\infty$ is a group element of the simple Lie group which depends analytically on $z$ and $e^a$ is an exponent of the central element. In the set up of linear systems all possible solutions of Einstein gravity reduced to $D = 2$ are associated with all possible maps from the $D = 2$ space-time $\mathcal{M}_{D=2}$ to the infinite-dimensional coset $\text{SL}(2, \mathbb{R})^\Lambda / H^\Lambda$, where $\text{SL}(2, \mathbb{R})^\Lambda$ is the centrally extended loop group of $\text{SL}(2, \mathbb{R})$ and $H^\Lambda$ is its maximal compact subgroup. In other words, reduced to $D = 2$ Einstein Gravity becomes a sort of $\sigma$–model:

$$\mathcal{M}_{D=2} \mapsto \frac{\text{SL}(2, \mathbb{R})^\Lambda}{H^\Lambda}$$

(1.3)

where the target space space is the infinite dimensional affine generalization of the maximally split, non compact coset $\text{SL}(2, \mathbb{R}) / O(2)$. As we are going to stress in the present paper by recalling and generalizing results which were obtained times ago by Nicolai [5], naive dimensional reduction of the Einstein lagrangian produces a $2D$–gravity coupled $\sigma$-model, where the target manifold is indeed $\text{SL}(2, \mathbb{R}) / O(2)$:

$$\mathcal{M}_{D=2} \mapsto \frac{\text{SL}(2, \mathbb{R})}{O(2)}$$

(1.4)

The replacement of equation (1.4) with eq.(1.3) occurs because there are two different ways of obtaining (1.4) via dimensional reductions which are non locally related to each other. Covering the two pictures at the same time is equivalent to the affine extension, namely to eq.(1.3). The main point of the present article is that this mechanism of generation of the infinite symmetries is actually completely general and follows a regular algebraic pattern for all supergravity theories which we plan to describe hereby, emphasizing its role in the discussion of cosmic billiard dynamics. This, however, is just an anticipation of our subsequent discussions.
For the time being it suffices to note that an explicit map of the type advocated in eq. (1.3) is given by a coset representative

\[
\hat{L} = \left( L \left( z(x^\pm), x^\pm \right), e^{-d(x)} \right)
\]  

(1.5)

where \( L \left( z(x^\pm), x^\pm \right) \) is an element of \( \text{SL}(2, \mathbb{R}) \) depending on the 2D coordinates \( x^\pm \) both explicitly and implicitly through the dependence of \( z \) on \( x^\pm \), and \( d(x) \) is a logarithm of the conformal factor of two-dimensional metric. This is just the mathematical transcription of the formal map defined in eq.(1.3). In the context of such a discussion, the coordinate \( z \) on the circle is named the **spectral parameter**. Once the algorithm that to each coset representative \( \hat{L} \) associates a solution of gravity and vice versa to each solution associates a coset representative has been established:

\[
\mathcal{A} : \hat{L} \leftrightarrow \text{solution of Einstein gravity}
\]  

(1.6)

then the action of the affine group on the space of solutions is easily defined by the left action of \( \hat{g}(z) \) on \( \hat{L} \).

After this clarification occurred in the context of pure gravity the interest in infinite symmetries was stimulated by the advent of supergravity. By dimensionally reducing \( D = 10 \) or \( D = 11 \) supergravity to lower dimensions one discovers the so-called hidden symmetries, that act as isometries of the kinetic metric on the scalars and as generalized electric/magnetic duality rotations on the vector fields or \( p \)-forms. Hidden symmetries were discovered specially through the work of Cremmer and Julia who found \( E_{7(7)} \) in \( \mathcal{N} = 8, D = 4 \) supergravity and later clarified that \( E_{11-D,(11-D)} \) is the duality symmetry for maximal supergravity in \( D \)-dimensions, obtained by compactification of M-theory on a \( T^{11-D} \) torus. The continuation of the series to \( D < 3 \) leads to algebras that are no longer finite dimensional, rather they are Kač–Moody algebras. Indeed it was Julia [7] who already long time ago noted this phenomenon, pointing out that \( E_{9(9)} \) is just the affine extension of \( E_{8(8)} \), while \( E_{10(10)} \) is its double hyperbolic extension. Later, extensive work on \( E_{10} \) and in general on the infinite symmetries of lower dimensional supergravities was performed by Nicolai and collaborators in a large set of papers [8]. Notwithstanding the interest and the potential relevance of all these results it must be said that for a long time they did not find a convincing arena of applications in the context of superstring and brane theory, remaining an unexploited truth. This is even more surprising if one considers that precisely the finite dimensional relatives of these symmetries, namely the hidden symmetries in \( D \geq 4 \) played a fundamental role in the so-called second string revolution, namely in establishing the non-perturbative dualities among the various string models. Indeed, as it is well known, those dualities that unify the various perturbative quantum strings into a unique M-theory are elements of a unified group \( U(\mathbb{Z}) \) which is the suitable restriction to integers of a corresponding Lie group \( U(\mathbb{R}) \) encoded in compactified supergravity and given, for toroidal compactifications, by the earlier mentioned \( E_{11-D,(11-D)} \) series.

This situation changed significantly, after 2001, with the renewed interest in Kač–Moody symmetries generated by the discovery of the **cosmic billiard phenomenon**. This phenomenon encodes a profound link between the features of time evolution of the cosmological scale factors and the structure of the hidden symmetry algebra \( U(\mathbb{R}) \). Let us name \( N_Q \) the number of supersymmetry charges. For \( N_Q > 8 \) the scalar manifold is always a homogeneous space \( U/H \) and what actually happens is that the cosmological scale factors \( a_i(t) \) associated with
The various dimensions of supergravity can be interpreted as exponentials of those scalar fields $h_i(t)$ which lie in the Cartan subalgebra of $\mathbb{U}$, while the other scalar fields in $\mathbb{U}/H$ correspond to positive roots $\alpha > 0$ of the Lie algebra $\mathbb{U}$. In this way the cosmological evolution is described by the motion of a fictitious ball in the CSA of $\mathbb{U}$. This space is actually a billiard table whose walls are the hyperplanes orthogonal to the various roots. The fictitious ball bounces on the billiard walls and this means that there are inversions in the time evolution of the scale factors. Certain dimension that were previously expanding almost suddenly begin to contract and others do the reverse. Such a scenario was introduced by Damour, Henneaux, Julia and Nicolai in [9], and in a series of papers with collaborators [10], [11], [12], which generalize classical results obtained in the context of pure General Relativity [13], [14].

In their approach the quoted authors analyzed the cosmic billiard phenomenon as an asymptotic regime in the neighborhood of space-like singularities and the billiard walls were seen as delta function potentials provided by the various $p$–forms of supergravity localized at sharp instants of time. The main focus of attention was centered on establishing whether and under which conditions there may be a chaotic behaviour in the evolution of the scale factors and the same authors established that this may occur only when the billiard table, identified with the Weyl chamber of the duality algebra $\mathbb{U}$ is hyperbolic, namely when the double Kač–Moody extension of the hidden symmetries is taken in proper account.

With a different standpoint it was observed in [15] that the fundamental mathematical setup underlying the appearance of the billiard phenomenon is the so named Solvable Lie algebra parametrization of supergravity scalar manifolds, pioneered in [16] and later applied to the solution of a large variety of superstring/supergravity problems ([16], [17], [18], [19]). In particular we showed that one can implement the following programme:

1. Reduce the original supergravity in higher dimensions $D \geq 4$ (for instance $D = 10, 11$) to a gravity-coupled $\sigma$–model in $D \leq 3$ where gravity is non–dynamical and where the original higher dimensional bosonic field equations reduce to geodesic equations for a solvable group-manifold, metrically equivalent to a non compact coset manifold $\exp[Solv(\mathbb{U}/H)] \cong \mathbb{U}/H$.

2. Utilize the algebraic structure of the solvable Lie algebra $Solv(\mathbb{U}/H)$ associated with the pair of the algebra $\mathbb{U}$ and its maximal compact subalgebra $\mathbb{H}$ in order to integrate analytically the geodesic equations.

3. Dimensionally oxide the solutions obtained in this way to exact time dependent solutions of $D \geq 4$ supergravity.

Within this approach it was proved in [15] that the cosmic billiard phenomenon is indeed a general feature of exact time dependent solutions of supergravity and has smooth realizations. Calling $h(t)$ the $r$–component vector of Cartan fields (where $r$ is the split rank of $\mathbb{U}$) and $h_\alpha(t) \equiv \alpha \cdot h(t)$ its projection along any positive root $\alpha$, a bounce occurs at those instant of times $t_i$ such that:

$$\exists \alpha \in \Delta_+ \quad \text{and} \quad \dot{h}_\alpha(t) |_{t=t_i} = 0$$

namely when the Cartan field in the direction of some root $\alpha$ inverts its behaviour and begins to shrink if it was growing or viceversa begins to grow if it was shrinking. Since all higher dimensional bosonic fields (off-diagonal components of the metric $g_{\mu\nu}$ or $p$–forms $A^{[p]}$) are,
via the solvable parametrization of $U/H$, in one-to-one correspondence with roots $\phi_{\alpha} \Leftrightarrow \alpha$, it follows that the bounce on a wall (hyperplane orthogonal to the root $\alpha$) is caused by the sudden growing of that particular field $\phi_{\alpha}$. Indeed in the exact smooth solutions which we obtained in \cite{15}, each bounce is associated with a typical bell-shaped behaviour of the root field $\phi_{\alpha}$ and the whole process can be interpreted as a temporary localization of the Universe energy density in a lump on a spatial brane associated with the field $\phi_{\alpha}$.

Although very much encouraging, the analysis of \cite{15} was limited in various respects. One limitation, whose removal is the main motivation of the present paper, consists of the following. The dimensional reduction process which is responsible for making manifest the duality algebra $U$ and hence for creating the whole algebraic machinery utilized in deriving the smooth cosmic billiard solutions was stopped at $D = 3$ namely at the first point where all the bosonic degrees of freedom can be represented by scalars. In $D = 3$, $U$ is still a finite dimensional Lie algebra and the whole richness of the underlying algebraic structure is not yet displayed. As implied by the results of \cite{12}, in order to investigate the most challenging aspects of billiard dynamics, in particular chaos, within the framework of exact solutions, we should derive these latter better in a $D = 2$ or $D = 1$ context where Kač–Moody symmetries become manifest. Although the appearance of Kač–Moody extensions is algebraically well established, their exploitation in deriving solutions is not as clear as the exploitation of ordinary symmetries. Indeed the main issue to clarify is the field theoretical realization of the Kač–Moody extensions which is the prerequisite for their utilization in deriving billiard dynamics.

The present paper aims at providing a clearcut step-forward in this direction. In particular we want to show that there is a general mechanism underlying the affine Kač–Moody extension of the $D=3$ algebra $U_{D=3}$ when we step down to $D = 2$ and that this mechanism follows a general algebraic pattern for all supergravity theories, independently of the number of supercharges $N_Q$. As already mentioned above this mechanism relies on the existence of two different reduction schemes from $D = 4$ to $D = 2$, respectively named the Ehlers reduction and the Matzner–Misner reduction, which are non locally related to each other. Nicolai observed this phenomenon time ago in the case of pure gravity (or better of $N=1$ pure supergravity)\cite{5} and showed that one obtains two identical lagrangians, each displaying an $SL(2,\mathbb{R})$ symmetry. The fields appearing in one lagrangian have a non local relation to those of the other lagrangian and one can put together both $SL(2,\mathbb{R})$ algebras. One algebra generates local transformations on one set of fields the other algebra generates non local ones. Together the six generators of the two $SL(2,\mathbb{R})$ provide a Chevalley basis for the Kač–Moody extension $SL(2,\mathbb{R})^\wedge$ namely for $A_1^\wedge$. Our analysis will be an extension of the argument by Nicolai. For a generic supergravity theory, the two reduction schemes Ehlers and Matzner–Misner lead to two different lagrangians with different local symmetries. The first is a normal $\sigma$–model the second is a twisted $\sigma$–model. We shall discuss in detail the symmetries of both theories. Just as in Nicolai case we can put together the symmetries of both lagrangians and in this way we obtain a Chevalley basis for the Kač–Moody algebra. In this way we can write down a precise field theoretic realization of the affine symmetries setting the basis to exploit them in billiard dynamics. We shall then comment on the further hyperbolic extension occurring in $D = 1$ and on the nature of the billiard chamber.
2 \quad D = 4 \text{ supergravity and its duality symmetries}

Rather than starting from $D = 10$ supergravity or 11–dimensional M-theory we begin our analysis from $D = 4$. How we stepped down from $D = 10, 11$ to $D = 4$ is not necessary to specify at this level. It is implicitly encoded in the number of residual supersymmetries that we consider. If $N_Q = 32$ is maximal it means that we used toroidal compactification. Lower values of $N_Q$ correspond to compactifications on manifolds of restricted holonomy, Calabi Yau three-folds, for instance, or orbifolds. The relevant point is that for $D = 4$ ungauged supergravity the bosonic lagrangian admits a general form which we presently discuss and exploit in our argument. We have:

$$
\mathcal{L}^{(4)} = \sqrt{\det g} \left[ -2R[g] - \frac{1}{6} \partial_\mu \phi^a \partial^\mu \phi^b h_{ab}(\phi) + \text{Im} N_{\Lambda \Sigma} F^\Lambda_{\mu \nu} F^{\Sigma}_{\delta \epsilon} \right] \\
+ \frac{1}{2} \text{Re} N_{\Lambda \Sigma} F^\Lambda_{\mu \nu} F^{\Sigma}_{\delta \epsilon} \epsilon^{\mu \nu \delta \epsilon}
$$

(2.1)

In eq.(2.1) $\phi^a$ denotes the whole set of $n_S$ scalar fields parametrizing the scalar manifold $\mathcal{M}_{\text{scalar}}^{D=4}$, which, for $N_Q > 8$, is necessarily a coset manifold:

$$
\mathcal{M}_{\text{scalar}}^{D=4} = \frac{U_{D=4}}{H}
$$

(2.2)

For $N_Q \leq 8$ eq.(2.2) is not obligatory but it is possible. Particularly in the $N = 2$ case, i.e. for $N_Q = 8$, a large variety of homogeneous special Kähler [21] fall into the set up of the present general discussion. The fields $\phi^a$ have $\sigma$–model interactions dictated by the metric $h_{ab}(\phi)$ of $\mathcal{M}_{\text{scalar}}^{D=4}$.

The theory includes also $n$ vector fields $A^\Lambda_\mu$ for which

$$
\mathcal{F}_{\pm|A}^{\mu \nu} \equiv \frac{1}{2} \left[ F^\Lambda_{\mu \nu} \pm \frac{\sqrt{\det g}}{2} \epsilon^{\mu \nu \delta \epsilon} F_{\delta \epsilon} \right]
$$

(2.3)

denote the self-dual (respectively antiself-dual) parts of the field-strengths. As displayed in eq.(2.1) they are non minimally coupled to the scalars via the symmetric complex matrix

$$
N_{\Lambda \Sigma}(\phi) = i \text{Im} N_{\Lambda \Sigma} + \text{Re} N_{\Lambda \Sigma}
$$

(2.4)

which transforms projectively under $U_{D=4}$. Indeed the field strengths $F^\Lambda_{\mu \nu}$ plus their magnetic duals fill up a $2n$–dimensional symplectic representation of $U_{D=4}$ which we call by the name of $\mathbf{W}$.

Following the notations and the conventions of [20], we rephrase the above statements by asserting that there is always a symplectic embedding of the duality group $U_{D=4}$,

$$
U_{D=4} \mapsto \text{Sp}(2n, \mathbb{R}) \quad ; \quad n = n_V \equiv \# \text{ of vector fields}
$$

(2.5)

so that for each element $\xi \in U_{D=4}$ we have its representation by means of a suitable real symplectic matrix:

$$
\xi \mapsto \Lambda_\xi \equiv \begin{pmatrix} A_\xi & B_\xi \\ C_\xi & D_\xi \end{pmatrix}
$$

(2.6)
satisfying the defining relation:

\[
A^T \begin{pmatrix}
0_{n\times n} & 1_{n\times n} \\
-1_{n\times n} & 0_{n\times n}
\end{pmatrix}
A_\xi = \begin{pmatrix}
0_{n\times n} & 1_{n\times n} \\
-1_{n\times n} & 0_{n\times n}
\end{pmatrix}
\] (2.7)

which implies the following relations on the \(n \times n\) blocks:

\[
A^T C - C^T A = 0 \\
A^T D - C^T B = 1 \\
B^T C - D^T A = -1 \\
B^T D - D^T B = 0
\] (2.8)

Under an element of the duality groups the field strengths transform as follows:

\[
\begin{pmatrix}
F^+ \\
G^+
\end{pmatrix}' = \begin{pmatrix}
A_\xi & B_\xi \\
C_\xi & D_\xi
\end{pmatrix}
\begin{pmatrix}
F^+ \\
G^+
\end{pmatrix}; \quad \begin{pmatrix}
F^- \\
G^-
\end{pmatrix}' = \begin{pmatrix}
A_\xi & B_\xi \\
C_\xi & D_\xi
\end{pmatrix}
\begin{pmatrix}
F^- \\
G^-
\end{pmatrix}
\] (2.9)

where, by their own definitions:

\[
G^+ = \mathcal{N} F^+; \quad G^- = \overline{\mathcal{N}} F^-
\] (2.10)

and the complex symmetric matrix \(\mathcal{N}\) transforms as follows:

\[
\mathcal{N}' = (C + D \mathcal{N}) (A + B \mathcal{N})^{-1}
\] (2.11)

Eq. (2.1) is the lagrangian that we are supposed to dimensionally reduce according to the two available schemes, the Ehlers reduction and the Matzner-Misner reduction respectively. We will perform such reductions in later sections. Prior to that we dwell on an algebraic interlude anticipating the result of the Ehlers reduction and analyzing the algebraic structure of the \(U_{D=3}\) algebra. Such an analysis is very important in order to establish the properties of its affine extension and single out a basis of candidate Chevalley-Serre generators.

### 3 Structure of the duality algebras in D=3

from the Ehlers reduction and their affine extensions

Upon toroidal dimensional reduction from \(D = 4\) to \(D = 3\) and then full–dualization of the vector fields, which is the Ehlers scheme to be described in detail later, we obtain supergravity theories admitting a duality Lie algebra \(U_{D=3}\) whose structure is universal in the following sense. It always contains, as subalgebra, the duality algebra \(U_{D=4}\) of the parent supergravity theory in \(D = 4\) times an \(SL(2, \mathbb{R})_E\) algebra which is produced by the dimensional reduction of pure gravity (see section 4 for the details). Furthermore, with respect to this subalgebra \(U_{D=3}\) admits the following universal decomposition, holding for all \(\mathcal{N}\)-extended supergravities having semisimple duality algebras:

\[
\text{adj}(U_{D=3}) = \text{adj}(U_{D=4}) \oplus \text{adj}(SL(2, \mathbb{R})_E) \oplus W_{(2,\mathbf{w})}
\] (3.1)
where $W$ is the symplectic representation of $U_{D=4}$ discussed in the previous section. Indeed the scalar fields associated with the generators of $W_{(2,W)}$ are just those coming from the vectors in $D = 4$. Denoting the generators of $U_{D=4}$ by $T^A$, the generators of $SL(2,\mathbb{R})_E$ by $L^x$ and denoting by $W^{i\alpha}$ the generators in $W_{(2,W)}$, the commutation relations that correspond to the decomposition (3.1) have the following general form:

$$
\begin{align*}
[T^A, T^B] &= f^{AB}_\ C T^C \\
[L^x, L^y] &= f^{xy}_z L^z, \\
[T^A, W^{i\alpha}] &= (\Lambda^A)^\beta_\alpha W^{i\beta}, \\
[L^x, W^{i\alpha}] &= (\lambda^x)^i_j W^{j\alpha}, \\
[W^{i\alpha}, W^{j\beta}] &= \omega^{ij}_{\ C} (K_A)^{\alpha\beta} T^A + \Omega^{\alpha\beta} k^{ij}_x L^x
\end{align*}
$$

(3.2)

where the matrices $(\lambda^x)^i_j$, which are $2 \times 2$ are the canonical generators of $SL(2,\mathbb{R})$ in the fundamental, defining representation:

$$
\begin{align*}
\lambda_3 &= \left( \begin{array}{cc} 1 & 0 \\ 0 & -1 \end{array} \right) \quad ; \quad \lambda_1 &= \left( \begin{array}{cc} 0 & 1 \\ 1 & 0 \end{array} \right) \quad ; \quad \lambda_2 &= \left( \begin{array}{cc} 0 & \frac{1}{2} \\ -\frac{1}{2} & 0 \end{array} \right)
\end{align*}
$$

(3.3)

while $\Lambda^A$ are the generators of $U_{D=4}$ in the symplectic representation $W$. By

$$
\Omega^{\alpha\beta} \equiv \left( \begin{array}{cc} 0_{n \times n} & 1_{n \times n} \\ -1_{n \times n} & 0_{n \times n} \end{array} \right)
$$

(3.4)

we denote the antisymmetric symplectic metric in $2n$ dimensions, $n = n_V$ being the number of vector fields in $D = 4$, as we have already stressed. The symplectic character of the representation $W$ is asserted by the identity:

$$
\Lambda^A \Omega + \Omega \left( \Lambda^A \right)^T = 0
$$

(3.5)

The fundamental doublet representation of $SL(2,\mathbb{R})$ is also symplectic and we have denoted by $\omega^{ij} = \left( \begin{array}{cc} 0 & 1 \\ -1 & 0 \end{array} \right)$ the 2-dimensional symplectic metric, so that:

$$
\lambda^x \omega + \omega (\lambda^x)^T = 0,
$$

(3.6)

The matrices $(K_A)^{\alpha\beta} = (K_A)^{\beta\alpha}$ and $(k_x)^{ij} = (k_y)^{ji}$ are just symmetric matrices in one-to-one correspondence with the generators of $U_{D=4}$ and $SL(2,\mathbb{R})$, respectively. Implementing Jacobi identities, however, we find the following relations:

$$
K_A \Lambda^C + \Lambda^C K_A = f^{BC}_A K_B, \quad k_x \lambda^y + \lambda^y k_x = f^{yz}_x k_z,
$$

which admit the unique solution:

$$
K_A = \alpha g_{AB} \Lambda^B \Omega, \quad ; \quad k_x = \beta g_{xy} \lambda^y \omega
$$

(3.7)

where $g_{AB}, g_{xy}$ are the Cartan-Killing metrics on the algebras $U_{D=4}$ and $SL(2,\mathbb{R})$, respectively, and $\alpha$ and $\beta$ are two arbitrary constants. These latter can always be reabsorbed into the
normalization of the generators $W^{\alpha}$ and correspondingly set to one. Hence the algebra (3.3) can always be put into the following elegant form:

$$[T^A, T^B] = f^{AB} C T^C$$
$$[L^x, L^y] = f^{xy} L^z,$n
$$[T^A, W^{\alpha}] = (\Lambda^A)^\alpha_\beta W^{\beta},$$
$$[L^x, W^{\alpha}] = (\lambda^x)^{ij} W^{ij},$$
$$[W^{\alpha}, W^{\beta}] = \omega^{ij} (\Lambda_A)^{\alpha\beta} T^A + \Omega^{\alpha\beta} \lambda^{ij}_x L^x,$$

where we have used the convention that symplectic indices $(\alpha, i)$ are raised and lowered with the symplectic metrics $(\Omega$ and $\omega)$, while adjoint representation indices $(A, x)$ are raised and lowered with the euclidean Cartan-Killing metrics.

3.1 Affine Kač–Moody extension of $\mathbb{U}_{D=3}$ from an algebraic viewpoint

Any simple Lie algebra admits an affine Kač–Moody extension. From a purely algebraic point of view let us discuss the affine extension of the $\mathbb{U}_{D=3}$ algebra in the presentation provided by its decomposition with respect to its $\mathbb{U}_{D=4}$–subalgebra, namely in the presentation given by eqs. (3.8) which are well adapted to the Ehlers reduction, as we have already stressed. This will be a preparatory study for the argument we shall develop after presenting the two dimensional reductions. In full generality we can write the ansatz:

$$[T^A_n, T^B_m] = f^{AB}_C T^C_{n+m} + c_1 \delta^{AB} n \delta_{m+n,0},$$
$$[L^x_n, L^y_m] = f^{xy} L^z_{m+n} + c_2 \delta^{xy} n \delta_{m+n,0},$$
$$[T^A_n, W^{\alpha}_m] = (\Lambda^A)^{\alpha\beta}_n W^{\beta}_m,$n
$$[L^x_n, W^{\alpha}_m] = (\lambda^x)^{ij}_n W^{ij}_m,$n
$$[W^{\alpha}_n, W^{\beta}_m] = \omega^{ij} (\Lambda^{AB})^{\alpha\beta}_n T^A_{m+n} + \Omega^{\alpha\beta} \lambda^{ij}_x L^x_{m+n} + c_3 \omega^{ij} \Omega^{\alpha\beta} n \delta_{m+n,0},$$

where $c_{1,2,3}$ are three apparently different central charges. Implementation of the Jacobi identities immediately shows that these charges are actually related and there is just one independent charge $c$. Explicitly we find the relations:

$$c_1 = c_2 = c_3 = c \quad (3.10)$$

In eq. (3.9) we wrote the affine extension of the $\mathbb{U}_{D=3}$ duality algebra in a compact notation that emphasizes the role of those generators which are associated with the dimensional reduction of vector fields. We shall shortly from now see the relevance of this presentation in order to discuss the merging of Ehlers symmetries with those of the Matzner–Misner reduction. For other purposes it is now convenient to fix our notations for affine Kač–Moody algebras in the Weyl-Dynkin basis. In this case the generators will be denoted as $\{H_n^i, E_n^\alpha\}$ and the relevant commutation relations are:

$$[H_n^i, H_m^j] = \frac{1}{2} k \cdot c_g(R) \delta^{ij} \cdot n \delta_{n+m,0},$$

8
\[ [\mathcal{H}_n^i, E_m^\alpha] = \alpha^i \cdot E_{n+m}^\alpha, \]
\[ [E_n^\alpha, E_m^\beta] = N_{\alpha\beta} E_{n+m}^{\alpha+\beta}, \]
\[ [E_n^\alpha, E_m^{-\alpha}] = \alpha^i \cdot \mathcal{H}_{n+m} + \frac{1}{2} k \cdot c_g(R)n\delta_{n+m,0} \]  
(3.11)

where \( c_g(R) \) is the value of the 1st Casimir operator in the adjoint representation.

It is also useful to introduce the extra generator \( d \), which measures the level
\[ [d, \mathcal{H}_n^i] = n \mathcal{H}_n^i ; \quad [d, E_n^\alpha] = n E_n^\alpha \]  
(3.12)

The Cartan subalgebra of the affine Kač–Moody algebra \( \hat{\mathcal{G}} \) consists of the following generators
\( \hat{\mathcal{C}} = \{ \mathcal{H}_0^i, k, d \} \), where \( k \) is the central element. In this way the roots are now vectors with \( r + 2 \) components, \( r \) being the rank of the simple Lie algebra that we extend and they form an infinite set \( \Delta \). The set of positive roots \( \Delta^+ \) is composed of three type of roots
\[ 0 < \hat{\alpha} = \begin{cases} (\alpha, 0, 0) & \alpha > 0 \text{ as a root of } \mathcal{G} \\ (\alpha, 0, n) & n > 0 \text{ for both } \alpha > 0 \text{ and } \alpha < 0 \text{ as roots of } \mathcal{G} \\ (0, 0, n) & n > 0 \end{cases} \]  
(3.13)

which can be expressed as integer non negative linear combinations of a set of \( r + 1 \) simple positive roots. For these latter we take
\[ \hat{\alpha}^i = (\alpha^i, 0, 0) ; \quad \hat{\alpha}^0 = (-\psi, 0, 1) \]  
(3.14)

where \( \alpha^i \) are the simple roots of \( \mathcal{G} \) and \( \psi \) denotes the highest root also of \( \mathcal{G} \). The invariant bilinear form on the CSA and hence on the root space \( \hat{\alpha} = (\alpha, n, m) \) has a Lorentzian signature and it is given by
\[ < \hat{\alpha}_1, \hat{\alpha}_2 > = < \alpha_1, \alpha_2 > + n_1 m_2 + m_1 n_2 \]  
(3.15)

Considering now eq.s (3.9) we see that \( \mathcal{U}_D^\alpha \), namely the affine extension of \( \mathcal{U}_{D=3} \), contains, as a subalgebra, \( A_1^\wedge \), namely the affine extension of \( \text{SL}(2, \mathbb{R})_E \). This is evident from the second of eq.s (3.9) and plays an important role in our argument. To this effect let us focus on an algebra \( A_1^\wedge \) and write it in the Weyl-Dynkin basis as in eq.s (3.11). It takes the form:
\[ [\mathcal{H}_n, \mathcal{H}_m] = \frac{1}{2} c_{A_1} \cdot k \cdot n \delta_{n+m,0} \]
\[ [\mathcal{H}_n, E^\pm_m] = \pm \sqrt{2} E^\pm_{n+m}, \]
\[ [E^+_n, E^-_m] = \sqrt{2} \mathcal{H}_{n+m} + \frac{1}{2} c_{A_1} \cdot k \cdot n \delta_{n+m,0} \]
\[ [E^+_n, E^+_m] = 0 \]  
(3.16)

where \( c_{A_1} \) is the quadratic Casimir of the Lie algebra \( A_1 \) in the adjoint representation. Next let us observe that the infinite dimensional \( A_1^\wedge \) algebra contains not just one but several \( A_1 \equiv \text{SL}(2, \mathbb{R}) \) subalgebras whose standard commutation relation have originated the KM-extension (3.16) and are:
\[ [L_0, L_\pm] = \pm \sqrt{2} L_\pm, \]
\[ [L_+, L_-] = \sqrt{2} L_0 \]  
(3.17)
One $A_1$ subalgebra is the obvious one obtained by taking all level zero generators, namely by setting:

$$L_0 = H_0 \quad ; \quad L_\pm = E_0^\pm$$

(3.18)

yet it must be realized that eq.(3.18) is just one instance in an infinite family of $A_1$ subalgebras obtained by setting:

$$L_0^{[m]} = H_0 - m \frac{1}{2\sqrt{2}} c_A, k \quad ; \quad L_\pm^{[m]} = E_\pm^{\pm m}$$

(3.19)

Secondly let us observe that by using two distinct elements in this infinite family of $A_1$ subalgebras we can list six generators that provide a standard Chevalley-Serre presentation of the entire affine Kač–Moody algebra $A_1^\wedge$. Let us recall the concept of Chevalley-Serre presentation. This is the analogue for Lie algebras of the presentation of discrete groups through generators and relations. Given a simple Lie algebra of rank $r$ defined by its Cartan matrix $C_{ij}$, a Chevalley-Serre basis is given by $r$-triplets of generators:

$$(h_i, e_i, f_i) \quad ; \quad i = 1, \ldots, r$$

(3.20)

such that the following commutation relations are satisfied:

$$[h_i, h_j] = 0$$
$$[h_i, e_j] = C_{ij} e_j$$
$$[h_i, f_j] = -C_{ij} f_j$$
$$[e_i, f_j] = \delta_{ij} h_i$$
$$\text{adj} [e_i]^{(C_{ij}+1)} (e_j) = 0$$
$$\text{adj} [f_i]^{(C_{ij}+1)} (f_j) = 0$$

(3.21)

When such $r$-triplets are given the entire algebra is defined. Indeed all the other generators are constructed by commuting these ones modulo the relations (3.21). For simply-laced finite simple Lie algebras a Chevalley basis is easily constructed in terms of simple roots. Let $\alpha_i$ denote the simple roots, then it suffices to set:

$$(h_i, e_i, f_i) = (H_{\alpha_i}, E^{\alpha_i}, E^{-\alpha_i})$$

(3.22)

where $H_{\alpha_i} \equiv \alpha_i \cdot H$ are the Cartan generator associated with the simple roots and $E^{\pm \alpha_i}$ are the step operators respectively associated with the simple roots and their negative.

The Cartan matrix of the affine algebra $A_1^\wedge$ is:

$$C_{ij}^{A_1^\wedge} = \begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix}$$

(3.23)

and, as noted by Nicolai \[10\] time ago a Chevalley-Serre basis for this algebra is provided by setting:

$$(h_1, e_1, f_1) = \left( \sqrt{2} L_0^{[0]}, L_+^{[0]}, L_-^{[0]} \right)$$
$$(h_0, e_0, f_0) = \left( -\sqrt{2} L_0^{[m]}, L_-^{[m]}, L_+^{[m]} \right) \quad ; \quad \text{for any choice of } 0 \neq m \in \mathbb{Z}$$

(3.24)
This observation has far reaching consequences in field-theory. Suppose that we have two
formally identical lagrangians $L_1, L_2$ each with a locally realized global symmetry $\text{SL}(2, \mathbb{R})$, that we distinguish as $\text{SL}(2, \mathbb{R})_1$ and $\text{SL}(2, \mathbb{R})_2$, respectively. Let us suppose furthermore that the local fields $\phi_{[1]}^i$ appearing in the first Lagrangian have a non local (invertible) relation to the fields $\phi_{[2]}^i$ of the second lagrangian. Schematically we denote the transformation from one set of fields to the others as the action of a non local operator $T$:

$$T : \phi_{[1]}^i(x) \mapsto \phi_{[2]}^i(x)$$

$$T^{-1} : \phi_{[2]}^i(x) \mapsto \phi_{[1]}^i(x) \quad (3.25)$$

Then, by use of $T$ and allowing also non local transformations, we can define on the same set of fields, say $\phi_{[1]}^i(x)$, two sets of $\text{SL}(2, \mathbb{R})$ transformations, the original ones and also those associated with the second Lagrangian. Indeed if $\delta_2 \phi_{[2]}^i(x)$ is a local $\text{SL}(2, \mathbb{R})_2$ transformation of the fields $\phi_{[2]}^i(x)$, we can define:

$$\overline{\delta}_2 \phi_{[1]}^i(x) \equiv T^{-1} \delta_2 T \phi_{[1]}^i(x) \quad (3.26)$$

which will be a non local transformation. In this way we can introduce a set of six generators defined as follows:

$$\left( h_1, e_1, f_1 \right) = \left( \sqrt{2}L_0, L_+, L_- \right)$$

$$\left( h_0, e_0, f_0 \right) = \left( -\sqrt{2}T^{-1}L_0T, T^{-1}L_-T, T^{-1}L_+T \right) \quad (3.27)$$

The names given to the generators anticipate that they might constitute a Chevalley-Serre basis. In order for this to be true, they should close the commutation relations (3.21). This is not obvious a priori but it can be explicitly checked in concrete field-theoretical models. The verification of (3.21) is the necessary and sufficient condition to prove that the system described by the two lagrangians admits the full Kač–Moody algebra as a symmetry. This is what Nicolai did in the case of pure gravity [5] by showing that the two dimensional reduction schemes, Ehlers and Matzner–Misner, lead to two $2D$–gravity coupled $\sigma$–models with target manifold $\text{SL}(2, \mathbb{R})/\text{O}(2)$, which exactly realize the situation described above. He also advocated the same argument to argue that in the case of maximal supersymmetry we have $E^\wedge_{8(8)} \equiv E_{9(9)}$, but a systematic analysis of the mechanism of generation of infinite symmetries was not performed so far for generic supergravities. Our paper aims at filling such a gap, clarifying also the relation with billiard dynamics. As we are going to see, things change somewhat in the case of generic supergravities since the two lagrangians $L^{(E)}$ and $L^{(MM)}$, respectively produced by the Ehlers and by the Matzner–Misner reduction, do not have two identical copies of the same symmetry algebra. The Ehlers lagrangian has symmetry algebra $U_{D=3}$ and it is extended with the $\text{SL}(2\mathbb{R})_{MM}$ part of the Matzner–Misner symmetry (coming from the Einstein gravity), that gives the affine extension $U^\wedge_{D=3}$.

### 3.1.1 Systematics of the affine extension

Let us now consider in more detail the structure of the decomposition (3.1) which is the crucial ingredient for the affine (and the hyperbolic) extension. The various cases corresponding to
the various values of $N_Q$ are listed in table 1 which contains also more information, namely the completely split Tits Satake subalgebra of each duality algebra $U_D$. This is relevant for the discussion of the billiard phenomenon as we are going to discuss extensively in a forthcoming paper [22] and just touch upon later on in the present one.

As we recalled in eq.(3.14), crucial for the affine extension $G^\wedge$ of any simple Lie algebra $G$ is the highest root $\psi$ of this latter, since it is by means of $\psi$ that we write the additional affine root $\alpha_0$ and correspondingly a Chevalley-Serre basis. Indeed, in view, of eq.(3.14) the Chevalley-Serre basis of any $G^\wedge$ which extends that of $G$ displayed in eq.(3.22) is the following one:

\[
(h_i, e_i, f_i) = (H_{\alpha_i}, E^{\alpha_i}, E^{-\alpha_i}) ; \quad (i = 1, \ldots, r)
\]

\[
(h_0, e_0, f_0) = (H_{\alpha_0}, E^{\alpha_0}, E^{-\alpha_0})
\]

\[
= \left( \frac{1}{2} c g k - H_\psi, E^{\psi}, E^{-\psi} \right)
\]

(3.28)

From the algebraic viewpoint a crucial property of the general decomposition in eq.(3.1) is encoded into the following statements which are true for all the cases $^1$:

1. The $A_1$ root-system associated with the $SL(2, \mathbb{R})_E$ algebra in the decomposition (3.1) is made of $\pm \psi$ where $\psi$ is the highest root of $U_{D=3}$.

2. Out of the $r$ simple roots $\alpha_i$ of $U_{D=3}$ there are $r - 1$ that have grading zero with respect to $\psi$ and just one $\alpha_W$ that has grading 1:

\[
(\psi, \alpha_i) = 0 \quad i \neq W
\]

\[
(\psi, \alpha_W) = 1
\]

(3.29)

3. The only simple root $\alpha_W$ that has non vanishing grading with respect $\psi$ is just the highest weight of the symplectic representation $W$ of $U_{D=4}$ to which the vector fields are assigned.

4. The Dynkin diagram of $U_{D=4}$ is obtained from that of $U_{D=3}$ by removing the dot corresponding to the special root $\alpha_W$.

5. Hence we can arrange a basis for the simple roots of the rank $r$ algebra $U_{D=3}$ such that:

\[
\alpha_i = \left\{ \pi_i, 0 \right\} ; \quad i \neq W
\]

\[
\alpha_W = \left\{ \frac{\pi_W}{\sqrt{2}} \right\}
\]

\[
\psi = \left\{ 0, \sqrt{2} \right\}
\]

(3.30)

$^1$An apparent exception is given by the case of N=3 supergravity of which we shall give a short separate discussion. The extra complicacy, there, is that the duality algebra in $D = 3$, namely $U_{D=3}$ has rank $r + 2$, rather than $r + 1$ with respect to the naive algebra $U_{D=4}$. By this latter we mean the isometry algebra of the scalar manifold in $D = 4$ supergravity. Actually in this case there is an extra $U(1)_{Z}$ factor that is active on the vectors, but not on the scalars and which is responsible for the additional complications. Indeed it happens in this case that there are two vector roots, one for the complex representation to which the vectors are assigned and one for its conjugate and the phase group and they have opposite charges under $U(1)_{Z}$.  

12
### Table 1

| $\mathcal{N}$ | $n$ | Lie Algebra | $\mathcal{D}=4$ | $\mathcal{D}=3$ | $\mathcal{D}=2$ | $\mathcal{D}=1$ |
|---------------|-----|-------------|-----------------|-----------------|-----------------|-----------------|
| $\mathcal{N}=8$ | $n=2$ | $E_7(7)$ | $E_8(8)$ | $E_7(7)$ | $E_7(7)$ | $E_7(7)$ |
| $\mathcal{N}=6$ | $n=2$ | $SO(10)$ | $SO(12) \times SO(3)$ | $SO(12) \times SO(3)$ | $SO(12) \times SO(3)$ | $SO(12) \times SO(3)$ |
| $\mathcal{N}=5$ | $n=2$ | $SU(5,1)$ | $SU(5,1)$ | $SU(5,1)$ | $SU(5,1)$ | $SU(5,1)$ |
| $\mathcal{N}=4$ | $n=6$ | $SU(6,6)$ | $SU(6,6)$ | $SU(6,6)$ | $SU(6,6)$ | $SU(6,6)$ |
| $\mathcal{N}=3$ | $n=6$ | $SU(3,3)$ | $SU(3,3)$ | $SU(3,3)$ | $SU(3,3)$ | $SU(3,3)$ |

Table 1: In this table we present the duality algebras $\mathcal{U}_D$ in $D = 4, 3, 2, 1$, for various values of the number of supersymmetry charges. We also mention the corresponding Tits Satake projected algebras (where they are well defined) that are relevant for the discussion of the cosmic billiard dynamics.

where $\overline{\alpha}_i$ are $(r-1)$--component vectors representing a basis of simple roots for the Lie algebra $\mathcal{U}_{D=4}$, $\overline{w}_h$ is also an $(r-1)$--vector representing the highest weight of the representation $\mathcal{W}$.

The above properties imply that the Dynkin diagram of $\mathcal{U}_{D=4}$ is just obtained by attaching $\alpha_0$ with a single line to $\alpha_W$. Furthermore, from the point of view of the Chevalley-Serre construction, any triplet $(h_0, e_0, f_0)$ which, added to the generators of $\text{SL}(2, \mathbb{R}_E)$, promotes this latter to its affine extension, automatically promotes the entire $\mathcal{U}_{D=3}$ to its own affine extension. This is so because the root of $\text{SL}(2, \mathbb{R}_E)$ is the highest root of $\mathcal{U}_{D=3}$. From the field-theory point of view this is just what happens. Indeed, as we prove in next sections, in the Matzner–Misner reduction we obtain $\text{SL}(2, \mathbb{R})_{MM}$ which yields the affine extension of $\text{SL}(2, \mathbb{R}_E)$ and as a consequence of the full $\mathcal{U}_{D=3}$.

Before considering the field theoretic realization, let us conclude this section by discussing the various instances of eq. (3.31) in some detail, by making reference to table (II).

**N=8** This is the case of maximal supersymmetry and it is illustrated by fig. (II).

In this case all the involved Lie algebras are maximally split and we have

$$\text{adj E}_8(8) = \text{adj E}_7(7) \oplus \text{adj SL}(2, \mathbb{R}_E) \oplus (2, 56) \quad (3.31)$$

The highest root of $\text{E}_8(8)$ is

$$\psi = 3\alpha_1 + 4\alpha_2 + 5\alpha_3 + 6\alpha_4 + 3\alpha_5 + 4\alpha_6 + 2\alpha_7 + 2\alpha_8 \quad (3.32)$$
The only simple root which has grading one with respect to the highest root $\psi$ is $\alpha_8$ (painted black). With respect to the algebra $U_{D=4} = E_{7(7)}$ whose Dynkin diagram is obtained by removal of the black circle, $\alpha_8$ is the highest weight of the symplectic representation of the vector fields, namely $W = 56$. The affine extension is originated by attaching the extra root $\alpha_0$ to the root corresponding to the vector fields.

and the unique simple root not orthogonal to $\psi$ is $\alpha_8 = \alpha_W$, according to the labeling of roots as in fig. This root is the highest weight of the fundamental $56$-representation of $E_{7(7)}$. As a consequence of this the affine extension of $E_{8(8)}$ has the same Dynkin diagram as it would have $E_{9(9)}$ formally continuing the $E_r$ series to $r > 8$.

The well adapted basis of simple $E_8$ roots is constructed as follows:

$$
\begin{align*}
\alpha_1 & = \{1, -1, 0, 0, 0, 0, 0, 0\} = \{\overline{\alpha}_1, 0\} \\
\alpha_2 & = \{0, 1, -1, 0, 0, 0, 0, 0\} = \{\overline{\alpha}_2, 0\} \\
\alpha_3 & = \{0, 0, 1, -1, 0, 0, 0, 0\} = \{\overline{\alpha}_3, 0\} \\
\alpha_4 & = \{0, 0, 0, 1, -1, 0, 0, 0\} = \{\overline{\alpha}_4, 0\} \\
\alpha_5 & = \{0, 0, 0, 0, 1, -1, 0, 0\} = \{\overline{\alpha}_5, 0\} \\
\alpha_6 & = \{0, 0, 0, 0, 1, 1, 0, 0\} = \{\overline{\alpha}_6, 0\} \\
\alpha_7 & = \{-\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, \frac{1}{\sqrt{2}}, 0\} = \{\overline{\alpha}_7, 0\} \\
\alpha_8 & = \{-1, 0, 0, 0, 0, 0, -\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\} = \{w_h, \frac{1}{\sqrt{2}}\} \\
\end{align*}
$$

In this basis we recognize that the seven $7$-vectors $\overline{\alpha}_i$ constitute a simple root basis for the $E_7$ root system, while:

$$
w_h = \{-1, 0, 0, 0, 0, 0, -\frac{1}{\sqrt{2}}\}$$

is the highest weight of the fundamental $56$ dimensional representation. Finally in this basis the highest root $\psi$ defined by eq. takes the expected form:

$$
\psi = \{0, 0, 0, 0, 0, 0, \sqrt{2}\}
$$
\[ \alpha_7 - \alpha_6 - \alpha_4 - \alpha_3 - \alpha_2 - \alpha_1 \]

\[ \psi = \alpha_1 + 2\alpha_2 + 3\alpha_3 + 4\alpha_4 + 2\alpha_5 + 3\alpha_6 + 2\alpha_7 \]

\[ \langle \psi, \alpha_i \rangle = 1 \quad ; \quad \langle \psi, \alpha_i \rangle = 0 \quad i \neq 7 \]

Figure 2: The Dynkin diagram of \( E_{7(-5)} \) and of its affine Kač–Moody extension \( E_{7(-5)}^\wedge \). The only simple root which has grading one with respect to the highest root \( \psi \) is \( \alpha_7 \) (painted black). With respect to the algebra \( U_{D=4} = SO^*(12) \) whose Dynkin diagram is obtained by removal of the black circle, \( \alpha_7 \) is the highest weight of the symplectic representation of the vector fields, namely the \( W = 32_s \). The affine extension is originated by attaching the extra root \( \alpha_0 \) to the root corresponding to the vector fields.

\( N=6 \) In this case the \( D = 4 \) duality algebra is \( U_{D=4} = SO^*(12) \), whose maximal compact subgroup is \( H = SU(6) \times U(1) \). The scalar manifold:

\[ SK_{N=6} \equiv \frac{SO^*(12)}{SU(6) \times U(1)} \tag{3.36} \]

is an instance of special Kähler manifold which can also be utilized in an \( N = 2 \) supergravity context. The \( D = 3 \) algebra is just dictated by the c-map of homogeneous special Kähler manifolds \([21]\) which yields quaternionic manifolds. Indeed in \( D = 3 \) we obtain the quaternionic manifold:

\[ Q = \frac{E_{7(-5)}}{SO(12) \times SO(3)} \tag{3.37} \]

and we have \( U_{D=3} = E_{7(-5)} \). The 16 vector fields of \( D = 4, N = 6 \) supergravity with their electric and magnetic field strengths fill the spinor representation \( 32_s \) of \( SO^*(12) \), so that the decomposition (3.1), in this case becomes:

\[ \text{adj } E_{7(-5)} = \text{adj } SO^*(12) \oplus \text{adj } SL(2, \mathbb{R})_E \oplus (2, 32_s) \tag{3.38} \]

The simple root \( \alpha_W \) is \( \alpha_7 \) and the highest root is:

\[ \psi = \alpha_1 + 2\alpha_2 + 3\alpha_3 + 4\alpha_4 + 2\alpha_5 + 3\alpha_6 + 2\alpha_7 \tag{3.39} \]

Correspondingly the affine extension is described by the Dynkin diagrams in fig(2).
A well adapted basis of simple $E_7$ roots can be written as follows:

$$\begin{align*}
\alpha_1 &= \{1, -1, 0, 0, 0, 0, 0\} = \{\alpha_1, 0\} \\
\alpha_2 &= \{0, 1, -1, 0, 0, 0, 0\} = \{\alpha_2, 0\} \\
\alpha_3 &= \{0, 0, 1, -1, 0, 0, 0\} = \{\alpha_3, 0\} \\
\alpha_4 &= \{0, 0, 0, 1, -1, 0, 0\} = \{\alpha_4, 0\} \\
\alpha_5 &= \{0, 0, 0, 0, 1, -1, 0\} = \{\alpha_5, 0\} \\
\alpha_6 &= \{0, 0, 0, 0, 1, 1, 0\} = \{\alpha_6, 0\} \\
\alpha_7 &= \{-\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}\} = \{\mathbf{w}_h, \frac{1}{\sqrt{2}}\}
\end{align*}$$

(3.40)

In this basis we recognize that the six 6-vectors $\alpha_i$ ($i = 1, \ldots, 6$) constitute a simple root basis for the $D_6 \simeq SO^*(12)$ root system, while:

$$w_h = \{-\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}\}$$

(3.41)

is the highest weight of the spinor 32-dimensional representation of $SO^*(12)$. Finally in this basis the highest root $\psi$ defined by eq. (3.39) takes the expected form:

$$\psi = \{0, 0, 0, 0, 0, 0, \sqrt{2}\}$$

(3.42)

As we anticipated, in this case, as in most cases of lower supersymmetry, neither the algebra $U_{D=4}$ nor the algebra $U_{D=3}$ are maximally split. In short this means that the non-compact rank $r_{nc} < r$ is less than the rank of $U$, namely not all the Cartan generators are non-compact. Rigorously $r_{nc}$ is defined as follows:

$$r_{nc} = \text{rank}(U/H) \equiv \dim \mathcal{H}^{n.c.}; \quad \mathcal{H}^{n.c.} \equiv \text{CSA}_{U}(C) \bigcap \mathbb{K}$$

(3.43)

For instance in our case $r_{nc} = 4$. As we extensively discuss in the forthcoming paper [22], when this happens it means that the billiard dynamics is effectively determined by a maximally split subalgebra $U^{TS} \subset U$ named the Tits Satake subalgebra of $U$, whose rank is equal to $r_{nc}$. Effectively determined does not mean that the smooth billiard solutions of the big system $E_7(-5)/SO(12) \times SO(3)$ coincide with those of the smaller system $F_{4(4)}/Usp(6) \times SU(2)$, rather it means that the former can be obtained from the latter by means of rotations of a compact subgroup of the big algebra $G_{\text{paint}} \subset U$ which we name the paint group. We refer for all details to the forthcoming paper [22]. Here we just emphasize a very important fact, relevant for the affine extensions. To this effect we recall that the Tits Satake algebra is obtained from the original algebra via a projection of the root system of $U$ onto the smaller rank root system of $U^{TS}$:

$$\Pi^{TS}; \quad \Delta_U \mapsto \Delta_{U^{TS}}$$

(3.44)

In this projection the essential algebraic features of the affine extension are preserved. In particular we have that the decomposition (3.1) commutes with the projection, namely:

$$\begin{align*}
\text{adj}(U_{D=3}) &= \text{adj}(U_{D=4}) \oplus \text{adj}(SL(2, \mathbb{R})_E) \oplus W_{(2,W)} \\
\downarrow & \downarrow \\
\text{adj}(U^{TS}_{D=3}) &= \text{adj}(U^{TS}_{D=4}) \oplus \text{adj}(SL(2, \mathbb{R})_E) \oplus W_{(2,W^{TS})}
\end{align*}$$

(3.45)
Figure 3: The Dynkin diagram of $F_4(4)$ and and of its affine Kač–Moody extension $F_4^\wedge$. The only root which is not orthogonal to the highest root is $\varpi_V = \varpi_1$ and consequently the Dynkin diagram of $F_4^\wedge$ is that displayed above. In the Tits Satake projection $\Pi^{TS}$ the highest root $\psi$ of $F_4(4)$ is the image of the highest root of $E_7(-5)$ and the root $\varpi_V = \varpi_1 = \Pi^{TS}(\alpha_7)$ is the image of the root associated with the vector fields and yielding the Kač–Moody extension of $E_7(-5)$.

In other words the projection leaves the $A_1$ Ehlers subalgebra untouched and has a non trivial effect only on the duality algebra $U_{D=4}$. Furthermore the image under the projection of the highest root of $\mathbb{U}$ is the highest root of $\mathbb{U}_{TS}$:

$$\Pi^{TS} : \psi \rightarrow \psi^{TS}$$

The explicit form of eq. (3.45) is the following one:

$$\begin{array}{l}
\text{adj}(E_7(-5)) = \text{adj}(SO^*(12)) \oplus \text{adj}(SL(2,\mathbb{R})_E) \oplus (2,32_s) \\
\downarrow \\
\text{adj}(F_4(4)) = \text{adj}(Sp(6,\mathbb{R}) \oplus \text{adj}(SL(2,\mathbb{R})_E) \oplus (2,14)
\end{array}$$

and the affine extension of the Tits Satake algebra $F_4(4)$ is described in fig. 3. The representation $14$ of $Sp(6,\mathbb{R})$ is that of an antisymmetric symplectic traceless tensor:

$$\dim_{Sp(6,\mathbb{R})} = 14$$

$\mathbb{N}=5$ The case of $N = 5$ supergravity is described by fig. 4. From the point of view of the Tits - Satake projection this case has some extra complications since the projected root system is not the root system of a simple Lie algebra. This explains the entry $bc$-system appearing in table 1 and because of that we postpone the discussion of its Tits Satake projection to a next paper. Indeed, as we already stressed, the focus of the present paper is in a different direction.
\[ E_6(-14) \]

\[ \psi = \alpha_1 + 2\alpha_2 + 3\alpha_3 + 2\alpha_4 + 2\alpha_5 + \alpha_6 \]

\[ (\psi, \alpha_i) = 1 \ ; \quad (\psi, \alpha_i) = 0 \quad i \neq 4 \]

\[ E_6^\wedge(-14) \]

Figure 4: The Dynkin diagram of \( E_6(-14) \) and of its affine Kač–Moody extension \( E_6^\wedge(-14) \). The only simple root which has grading one with respect to the highest root \( \psi \) is \( \alpha_7 \) (painted black). With respect to the algebra \( \mathbb{U}_{D=4} = \text{SU}(5,1) \) whose Dynkin diagram is obtained by removal of the black circle, \( \alpha_4 \) is the highest weight of the symplectic representation of the vector fields, namely the \( W = 20 \). The affine extension is originated by attaching the extra root \( \alpha_0 \) to the root corresponding to the vector fields.

In the \( N = 5 \) theory the scalar manifold is a complex coset of rank \( r = 1 \),

\[ \mathcal{M}_{N=5,D=4} = \frac{\text{SU}(1,5)}{\text{SU}(5) \times \text{U}(1)} \]

and there are 10 vector fields whose electric and magnetic field strengths are assigned to the 20-dimensional representation of \( \text{SU}(1,5) \), which is that of an antisymmetric three-index tensor

\[ \text{dim}_{\text{SU}(1,5)} = 20 \]

The decomposition (3.1) takes the explicit form:

\[ \text{adj}(E_6(-14)) = \text{adj}(\text{SU}(1,5)) \oplus \text{adj}((\text{SL}(2,\mathbb{R}))_E) \oplus (2, 20) \]

and we have that the highest root of \( E_6 \), namely

\[ \psi = \alpha_1 + 2\alpha_2 + 3\alpha_3 + 2\alpha_4 + 2\alpha_5 + \alpha_6 \]

has non vanishing scalar product only with the root \( \alpha_4 \) originating the affine extension in the form depicted in fig.4.
Writing a well adapted basis of \( E_6 \) roots is a little bit more laborious but it can be done. We find:

\[
\begin{align*}
\alpha_1 &= \left\{ 0, 0, -\frac{\sqrt{2}}{2}, \frac{1}{2\sqrt{3}}, \sqrt{\frac{2}{5}}, 0 \right\} = \{ \alpha_1, 0 \} \\
\alpha_2 &= \left\{ \frac{1}{\sqrt{2}}, \frac{1}{2\sqrt{3}}, 0, 0, 0 \right\} = \{ \alpha_2, 0 \} \\
\alpha_3 &= \left\{ \sqrt{2}, 0, 0, 0, 0, 0 \right\} = \{ \alpha_3, 0 \} \\
\alpha_4 &= \left\{ \frac{1}{\sqrt{2}}, \frac{1}{2\sqrt{3}}, \frac{1}{\sqrt{5}}, 0, 0, 0 \right\} = \{ \alpha_4, 0 \} \\
\alpha_5 &= \left\{ \frac{1}{\sqrt{2}}, 0, 0, 0, 0, 0 \right\} = \{ \alpha_5, 0 \} \\
\alpha_6 &= \left\{ 0, \frac{3}{2}, -\frac{1}{2\sqrt{3}}, -\frac{\sqrt{2}}{2}, 0, 0 \right\} = \{ \alpha_6, 0 \}
\end{align*}
\]

In this basis we can check that the five 5-vectors \( \bar{\alpha}_i \) \((i = 1, \ldots, 5)\) constitute a simple root basis for the \( A_5 \simeq SU(1,5) \) root system, namely:

\[
\langle \bar{\alpha}_i, \bar{\alpha}_j \rangle = \begin{pmatrix}
2 & -1 & 0 & 0 & 0 \\
-1 & 2 & -1 & 0 & 0 \\
0 & -1 & 2 & -1 & 0 \\
0 & 0 & -1 & 2 & -1 \\
0 & 0 & 0 & -1 & 2 \\
\end{pmatrix} = \text{Cartan matrix of } A_5 \quad (3.54)
\]

while:

\[
\mathbf{w}_h = \left\{ -\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{6}}, \frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{5}}, \frac{1}{\sqrt{10}} \right\} \quad (3.55)
\]

is the highest weight of the spinor 20-dimensional representation of \( SU(1,5) \). Finally in this basis the highest root \( \psi \) defined by eq.(3.52) takes the expected form:

\[
\psi = \{ 0, 0, 0, 0, 0, \sqrt{2} \} \quad (3.56)
\]

\[\textbf{N=4} \quad \text{The case of } N = 4 \text{ supergravity is the first where the scalar manifold is not completely fixed, since we can choose the number } n_m \text{ of vector multiplets that we can couple to the graviton multiplet. In any case, once } n_m \text{ is fixed the scalar manifold is also fixed and we have:}
\]

\[
\mathcal{M}_{N=4,D=4} = \frac{\text{SL}(2,\mathbb{R})_0}{O(2)} \otimes \frac{\text{SO}(6,n_m)}{\text{SO}(6) \times \text{SO}(n_m)} \quad (3.57)
\]

The total number of vectors \( n_V = 6 + n_m \) is also fixed and the symplectic representation \( \mathbf{W} \) of the duality algebra

\[
U_{D=4} = \text{SL}(2,\mathbb{R})_0 \times \text{SO}(6,n_m) \quad (3.58)
\]

to which the vectors are assigned and which determines the embedding:

\[
\text{SL}(2,\mathbb{R})_0 \times \text{SO}(6) \times \text{SO}(n_m) \mapsto \text{Sp}(12 + 2n_m,\mathbb{R}) \quad (3.59)
\]

is also fixed, namely \( \mathbf{W} = (2_0, 6 + n_m) \), \( 2_0 \) being the fundamental representation of \( \text{SL}(2,\mathbb{R})_0 \) and \( 6 + n_m \) the fundamental vector representation of \( \text{SO}(6, n_m) \). The \( D = 3 \) algebra that one
Figure 5: The Dynkin diagram of $D_{4+k+1}$ and of its affine Kac–Moody extension $D^\wedge_{4+k+1}$. The algebra $D_{4+k+1}$ is that of the group SO($8, 2k + 2$) corresponding to the Ehlers reduction of $N = 4$ supergravity coupled to $n_m = 2k$ vector multiplets. The only simple root which has non-vanishing grading with respect to the highest one $\psi$ is $\alpha_2$. Removing it (black circle) we are left with the algebra $D_{4+k-1} \times A_1$ which is indeed the duality algebra in $D = 4$, namely $\text{SO}(6, 2k) \times \text{SL}(2, \mathbb{R})_0$. The black root $\alpha_2$ is the highest weight of the symplectic representation of the vector fields, namely the $W = (2_0, 6 + 2k)$. The affine extension is originated by attaching the extra root $\alpha_0$ to the root $\alpha_2$ which corresponds to vector fields.

obtains in the Ehlers dimensional reduction is, for all number of vector multiplets given by $U_{D=3} = \text{SO}(8, n_m + 2)$, leading to the manifold:

$$\mathcal{M}_{N=4, D=3} = \frac{\text{SO}(8, n_m + 2)}{\text{SO}(8) \times \text{SO}(n_m + 2)} \quad (3.60)$$

Correspondingly the form taken by the general decomposition (3.1) is the following one, for all values of $n_m$:

$$\text{adj}(\text{SO}(8, n_m + 2)) = \text{adj}(\text{SL}(2, \mathbb{R})_0) \oplus \text{adj}(\text{SO}(6, n_m)) \oplus \text{adj}(\text{SL}(2, \mathbb{R})_E) \oplus (2_{E, 2_0, 6 + n_m}) \quad (3.61)$$

where $2_{E, 0}$ are the fundamental representations respectively of $\text{SL}(2, \mathbb{R})_E$ and of $\text{SL}(2, \mathbb{R})_0$.

In order to give a Dynkin Weyl description of these algebras, we are forced to distinguish the case of an odd and even number of vector multiplets. In the first case both $U_{D=3}$ and $U_{D=4}$ are non simply laced algebras of the $B$–type, while in the second case they are both simply laced algebras of the $D$-type

$$n_m = \begin{cases} 2k \\ 2k + 1 \end{cases} \quad \Rightarrow \quad U_{D=4} \simeq D_{k+3} \quad (3.62)$$

Just for simplicity and for shortness we choose to discuss only the even case $n_m = 2k$ which is described by fig 5.
In this case we consider the $\mathbb{U}_{D=3} = \text{SO}(8, 2k + 2)$ Lie algebra whose Dynkin diagram is that of $D_{5+k}$. Naming $\epsilon_i$ the unit vectors in an Euclidean $\ell$-dimensional space where $\ell = 5+k$, a well adapted basis of simple roots for the considered algebra is the following one:

$$
\begin{align*}
\alpha_1 &= \sqrt{2} \epsilon_1 \\
\alpha_2 &= -\frac{1}{\sqrt{2}} \epsilon_1 - \epsilon_2 + \frac{1}{\sqrt{2}} \epsilon_\ell \\
\alpha_3 &= \epsilon_2 - \epsilon_3 \\
\alpha_4 &= \epsilon_3 - \epsilon_4 \\
\vdots &= \vdots \\
\alpha_{\ell-1} &= \epsilon_{\ell-2} - \epsilon_{\ell-1} \\
\alpha_\ell &= \epsilon_{\ell-2} + \epsilon_{\ell-1}
\end{align*}
$$

(3.63)

which is quite different from the usual presentation but yields the correct Cartan matrix. In this basis the highest root of the algebra:

$$
\psi = \alpha_1 + 2 \alpha_2 + 2 \alpha_3 + \ldots + 2 \alpha_{\ell-2} + \alpha_{\ell-1} + \alpha_\ell
$$

(3.64)

takes the desired form:

$$
\psi = \sqrt{2} \epsilon_\ell
$$

(3.65)

In the same basis the $\alpha_W = \alpha_2$ root has also the expect form:

$$
\alpha_W = \left( w, \frac{1}{\sqrt{2}} \right)
$$

(3.66)

where:

$$
w = -\frac{1}{\sqrt{2}} \epsilon_1 - \epsilon_2
$$

(3.67)

is the weight of the symplectic representation $W = (2_0, 6 + 2k)$. Indeed $-\frac{1}{\sqrt{2}} \epsilon_1$ is the fundamental weight for the Lie algebra $\text{SL}(2, \mathbb{R})_0$, whose root is $\alpha_1 = \sqrt{2} \epsilon_1$, while $-\epsilon_2$ is the highest weight for the vector representation of the algebra $\text{SO}(6, 2k)$, whose roots are $\alpha_3, \alpha_4, \ldots, \alpha_\ell$.

Next we briefly comment on the Tits Satake projection. The algebra $\text{SO}(8, n_m + 2)$ is maximally split only for $n_m = 6$ which, from the superstring view point, corresponds to the case of Neveu-Schwarz vector multiplets in a toroidal compactification. For a different number of vector multiplets, in particular for $n_m > 6$ the study of billiard dynamics involves considering the Tits Satake projection, which just yields the universal manifold:

$$
\mathcal{M}_{N=4, D=3}^{TS} = \frac{\text{SO}(8, 8)}{\text{SO}(8) \times \text{SO}(8)}
$$

(3.68)

The detailed study of these aspects is however postponed to future publications as we have already stressed.
\[ A_{\ell=4+n} \]

\[ A_{\ell=4+n}^\wedge \]

Figure 6: The Dynkin diagrams of \( A_{4+n} \) and of its affine extension \( A_{4+n}^\wedge \). \( A_{4+n} \) is the algebra of \( SU(4, n+1) \) generated by the Ehlers reduction of \( N = 3 \) supergravity with \( n_m = n \) vector multiplets. Two simple roots \( \alpha_1, \ell \) have non vanishing grading with respect to the highest root \( \psi \). Removing them (black circles) we are left with the algebra \( A_{2+n} \) of \( SU(3, n) \). The black roots \( \alpha_{1,\ell} \) are, respectively, the highest weight of the representation \( (3+n) \) and of its conjugate \( (3+n)^* \) which together make up the symplectic representation of the vector fields, \( W \). The affine extension is originated by attaching the extra root \( \alpha_0 \) to the vector roots.

\[ N=3 \] The case of \( N=3 \) supergravity \[25\] is similar to that of \( N=4 \) since also here the only free parameter is the number of vector multiplets \( n_m \) leading to the following scalar manifold:

\[ \mathcal{M}_{N=3,D=4} = \frac{SU(3, n_m)}{SU(3) \times SU(n_m)} \]  (3.69)

It might seem that the duality algebra in \( D = 4 \) should be \( \mathbb{U}_{D=4} = SU(3, n_m) \), yet, in this case there is a subtlety. The actual algebra is rather

\[ \mathbb{U}_{D=4} = U(3, n_m) = SU(3, n_m) \times U(1)_Z \]  (3.70)

the overall phase group \( U(1)_Z \) having a vanishing action on the scalars, but not on the vector fields. The symplectic representation \( W \) to which the vectors are assigned is just made out of the fundamental \( 3 + n_m \) of \( U(3, n_m) \) plus its complex conjugate \( \overline{3 + n_m} \) leading to the embedding:

\[ U(3, n_m) \mapsto Sp(6 + 2 n_m, \mathbb{R}) \]  (3.71)

which was explicitly described in \[25\]. In short it goes as follows. Let \( L \) be a \( (3+n) \times (3+n) \) matrix in the fundamental representation of \( SU(3, n) \). We map it into a matrix belonging to
Usp(3 + n, 3 + n) in the following way:

$$SU(3, n) \ni L = \begin{pmatrix} X & Y \\ Z & W \end{pmatrix} \rightarrow \mathbb{L}_{Usp} = \begin{pmatrix} X & 0 & 0 & Y \\ 0 & W^* & Z^* & 0 \\ 0 & Y^* & X^* & 0 \\ Z & 0 & 0 & W \end{pmatrix} \in \text{Usp}(3 + n, 3 + n)$$

(3.72)

Then we use the Cayley isomorphism relating Usp(n, n) and Sp(2n, R) to obtain a real symplectic matrix $\mathbb{L}_{Sp}$ representing the original SU(3, n) element [20]:

$$\mathbb{L}_{Sp} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix} \mathbb{L}_{Usp} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix}$$

(3.73)

This matrix construction is the explicit definition of the representation $\mathbf{W}$ of the vectors.

The subtlety of the $N = 3$ case is that the map (3.72) extends to the full U(3, n) algebra, namely also to matrices \( \begin{pmatrix} X & Y \\ Z & W \end{pmatrix} \) which are not traceless. In particular we see that the $U(1)_Z$ group whose generator corresponds to $Y = Z = 0$, $X = W = i \mathbf{1}$ has a non trivial image in the symplectic group and hence a non trivial action on the vector fields, although its action is zero on the scalars of the coset manifold (3.69). Hence it is correct to say that the true duality algebra in $D = 4$ is not just SU(3, n), rather it is:

$$\mathbb{U}_{D=4} = \text{U}(3, n) \equiv \text{SU}(3, n) \oplus U(1)_Z$$

(3.74)

With this proviso the general decomposition (3.1) is still true also in the $N = 3$ case. Indeed in $D = 3$, via the Ehlers reduction, we obtain $\mathbb{U}_{D=4} = \text{SU}(4, n_m + 1)$ and the form taken by the general decomposition (3.1) is:

$$\text{adjSU}(4, n_m + 1) = \text{adjSU}(3, n_m) \oplus U(1)_Z \oplus \text{adjSL}(2, \mathbb{R})_E \oplus (2_E, 3 + n_m) \oplus (2_E, 3 + n_m)$$

(3.75)

The Weyl Dynkin description of this case is now easily provided and it is summarized in fig[6]

The simple roots of the Lie algebra $A_{\ell=4+n}$ can be written according to a standard presentation as euclidean vectors in $\ell + 1 = 5 + n$ dimensions and we can write:

$$\begin{align*}
\alpha_1 &= \epsilon_{5+n} - \epsilon_1 \\
\alpha_{i+1} &= \epsilon_i - \epsilon_{i+1} \quad (i = 1, \ldots, 2 + n) \\
\alpha_{\ell=4+n} &= \epsilon_{3+n} - \epsilon_{4+n}
\end{align*}$$

(3.76)

The highest root $\psi = \sum_{i=1}^{\ell} \alpha_i$ takes the very simple form:

$$\psi = -\epsilon_{4+n} + \epsilon_{5+n}$$

(3.77)

and has grading one with respect to the roots $\alpha_1$ and $\alpha_{\ell}$. The Dynkin diagram of $A_{2+n}$, namely of the Lie algebra SU(3, n), is obtained by deleting $\alpha_{1,\ell}$ which are indeed the weights of the fundamental and anti-fundamental representation of SU(3, n), out of which we cook up the
symplectic W-representation as we have described above. Indeed calling $\lambda_i$ the fundamental weights of $A_{2+n}$ described by the roots $\alpha_{i+1}, \ (i = 1, \ldots, 2+n), it follows by their own definition that we can rewrite:

$$\alpha_1 = \lambda_1 + \Lambda; \quad \alpha_{4+n} = \lambda_{n+2} + \Lambda$$

(3.78)

where $\Lambda$ is the fundamental weight of $SL(2, \mathbb{R})_E$, i.e. the vector orthogonal to all the other roots and such that $\psi \cdot \Lambda = 1$ This discussion shows that the $N=3$ case is actually no exception to the general discussion we present in this paper. Yet since the W representation is made of one representation plus its conjugate there are actually two conjugate weights and hence two roots participating into the mechanism of affine extension.

$N=2$ In $N=2$ supergravity we have geometries rather than algebras since, from a general point of view, we just know that the scalar manifold is of the special Kähler type. Yet many considerations are still valid since the symplectic embedding is just an integral part of the very definition of special geometry and the so called c-map [21] is another name for the Ehlers dimensional reduction. Hence the affine extension as well as the Tits Satake projection can be considered also at the level of $N = 2$ supergravity. We plan to devote a separate publication to this extensions.

Having concluded the algebraic discussion of the affine extensions, we turn to its field theoretic realization, namely to the Ehlers and Matzner–Misner reduction schemes.

4 The Ehlers reduction

The dimensional reduction à la Ehlers consists of three steps:

1. First one dimensionally reduces the $D = 4$ supergravity lagrangian (2.1) to $D = 3$ in the standard way based on the triangular gauge for the vielbein and for the vector fields.

2. Then one dualizes all the vector fields obtained in the $D = 3$ lagrangian to scalars: namely one dualizes both those vector fields that were already present in $D = 4$ and the new ones generated by the Kaluza Klein mechanism. In this way one obtains 3D–gravity coupled $\sigma$–model in $D = 3$ which is based on a new coset manifold $U_{D=3}/H$ enlarging the original four-dimensional $\sigma$–model with the new scalars.

3. Finally one further reduces the $D = 3$ gravity coupled $\sigma$–model to $D = 2$. In this step nothing new happens to the $\sigma$–model part of the lagrangian. The only novelty comes from the reduction of gravity which just produces the coupling to a dilaton field.

According to the above plan we introduce the reduction ansatz for the $D = 4$ metric in the following form:

$$ds^2_{(4)} = \Delta^{-1}ds^2_{(3)} + \Delta(dx^3 + b_\mu dx^\mu)^2,$$

(4.1)

where the index $\mu = 0,1,2$ corresponds to the $D = 3$ space-time dimensions, $\Delta$ denotes the Kaluza–Klein scalar and $b_\mu$ denotes the Kaluza-Klein vector, for which we fix the following Coulomb gauge $b_0 = 0$. In this frame the dimensional reduction of the Einstein term yields:

$$-2 \sqrt{g^{(4)}} R[g^{(4)}] = -2 \sqrt{g} \left[R[g] - \frac{1}{4} (\partial \ln \Delta)^2 + \Delta^2 G_{\mu\nu} G^{\mu\nu} \right]$$

(4.2)
where
\[ G_{\mu\nu} = \frac{1}{2} (\partial_\mu b_\nu - \partial_\nu b_\mu) \]  
(4.3)
is the field strength of the Kaluza Klein vector field.

Let us now consider the dimensional reduction of the gauge fields. These latter are redefined in the customary way as follows:
\[ A^\Lambda_{[D=4]} = (\bar{A}^\Lambda_\mu + \tau^\Lambda b_\mu) \, dx^\mu + \tau^\Lambda \, dx^3 \]  
(4.4)
where \( \bar{A}^\Lambda_\mu \) are the three-dimensional gauge fields, while
\[ \tau^\Lambda \equiv A^\Lambda_3 \]  
(4.5)
are the scalar fields generated by the internal components of the \( D = 4 \) gauge fields. Hence the field strengths \( \bar{F}^{\Lambda\mu\nu} \) of the \( D = 3 \) vector fields are related to their higher dimensional ancestors \( F^{\Lambda} = dA^{\Lambda}_{[D=4]} \) by the following formula:
\[ \bar{F}^{\Lambda\mu\nu} = F^{\Lambda\mu\nu} - \tau^\Lambda G_{\mu\nu} - \frac{1}{2} [b_\nu \partial_\mu \tau^\Lambda - b_\mu \partial_\nu \tau^\Lambda] \]  
(4.6)
This being set, the vector sector of the \( D = 4 \) lagrangian (2.1) reduces as follows
\[ \frac{1}{2} \text{Re} N_{\Lambda\Sigma} F^{\Lambda}_{\mu\nu} F^{\Sigma}_{\rho\sigma} g^{\mu\nu} g^{\rho\sigma} = \epsilon^{\mu\nu\rho} \text{Re} N_{\Lambda\Sigma} (\bar{F}^{\Lambda}_{\mu\nu} + \tau^\Lambda G_{\mu\nu}) \partial_\rho \tau^\Sigma \]
\[ \sqrt{-\det g_{[4]}} \text{Im} N_{\Lambda\Sigma} F^{\Lambda}_{\mu\nu} F^{\Sigma}_{\rho\sigma} g^{\mu\nu} = \sqrt{-\det g} \text{Im} N_{\Lambda\Sigma} \left[ \Delta (\bar{F}^{\Lambda}_{\mu\nu} + \tau^\Lambda G_{\mu\nu}) (\bar{F}^{\Sigma}_{\mu\nu} + \tau^\Sigma G_{\mu\nu}) + \frac{1}{2} \Delta \partial_\mu \tau^\Lambda \partial_\nu \tau^\Sigma \right] \]  
(4.7)
In this way we have completed step one of the Ehlers reduction procedure. The second step is the dualization of all the vector fields to scalars, which in three dimensions is always possible. Explicitly we can replace the Kaluza Klein vector with an axion \( B \) and the vector fields \( \bar{A}^\Lambda_\mu \) with as many axions \( \sigma^\Lambda \) by means of the following non–local relations:
\[ G_{\mu\nu} = \frac{1}{4\Delta^2} \sqrt{g} \epsilon_{\mu\nu\rho\sigma} g^{\rho\sigma} [\partial_\rho B + \frac{1}{2} \sigma^\Lambda \partial_\rho \tau^\Lambda] \]  
(4.8)
\[ \bar{F}^{\Sigma}_{\mu\nu} = -\frac{\sqrt{g}}{2\Delta^2} \left[ \Delta \text{Im} N^{-1}\Sigma\Lambda (\text{Re} N_{\Lambda\Omega} \partial_\rho \tau^\Omega + \partial_\rho \sigma^\Lambda) + \frac{1}{2} \left( \partial_\rho B + \frac{1}{2} \sigma^\Lambda \partial_\rho \tau^\Lambda \right) \tau^\Sigma \right] \]
where we have introduced the short-hand notation: \( a \leftrightarrow b = a \partial b - b \partial a \).

Upon use of the above construction blocks and collecting our results, the lagrangian (2.11) dimensionally reduced à la Ehlers takes the following general form which holds true in all cases and is given by the sum of three terms:
\[ \mathcal{L}_E = -2\sqrt{g} R[g] + \mathcal{L}_{\sigma\text{-model}} + \mathcal{L}_{\text{vec+sl}} \]  
(4.9)
the first term being the \( D = 3 \) Einstein action, the second the \( \sigma \)–model with target manifold \( \frac{\text{Up}_{D=4}}{\text{H}} \), directly inherited from higher dimension
\[ \mathcal{L}_{\sigma\text{-model}} = -\frac{1}{6} \sqrt{g} h_{ab} \partial_\mu \phi^a \partial^\mu \phi^b \]  
(4.10)
while the third term, coming from the reduction of gravity and vectors, has the following structure:

\[
L_{\text{vec}+\text{sl}_2} = -\frac{\sqrt{g}}{4\Delta^2} \left( (\partial \Delta)^2 + \left[ \partial B + \frac{1}{2} \sigma_\Lambda \partial^\Lambda \tau^\Lambda \right]^2 \right) \\
+ \frac{\sqrt{g}}{2\Delta} (\partial \tau \partial \sigma) \mathbf{M}(\phi) \begin{pmatrix} \partial \tau \\ \partial \sigma \end{pmatrix}
\]

(4.11)

The \(2n \times 2n\) matrix \(\mathbf{M}(\phi)\) has the following form

\[
\mathbf{M}(\phi) = \begin{pmatrix}
\text{Im} \mathcal{N} + \text{Re} \mathcal{N} \text{Im} \mathcal{N}^{-1} \text{Re} \mathcal{N} & \text{Re} \mathcal{N} \text{Im} \mathcal{N}^{-1} \\
\text{Im} \mathcal{N}^{-1} \text{Re} \mathcal{N} & \text{Im} \mathcal{N}^{-1}
\end{pmatrix}
\]

(4.12)

and works as kinetic metric of the axionic scalar fields \(\{\tau^\Lambda, \sigma_\Sigma\}\): it depends only on the original scalar fields \(\phi\) of the \(D = 4\) lagrangian. This particular matrix is not a newcomer. It already appeared in the discussion of the geodesic potential for supersymmetric black-holes (for a review see for instance [23]). This latter, which is \(U_{D=4}\) invariant is given in terms of the black-hole electric \(q_\Pi\) and magnetic charges \(p_{\Lambda}\) by:

\[
V^{\text{geo}}(\phi) = (p, q) \mathbf{M}(\phi) \begin{pmatrix} p \\ q \end{pmatrix}
\]

(4.13)

The \(U_{D=4}\) invariance is guaranteed by the fact that under a symplectic transformation \(\Lambda_\xi\) generated by an isometry \(\xi\) of the \(D = 4\) \(\sigma\)-model (see eq. (2.6)), the matrix \(\mathbf{M}(\phi)\) transforms as follows:

\[
\mathbf{M}(\xi \phi) \mapsto \Omega \Lambda_\xi \mathbf{M}(\phi) \Lambda_\xi^T \Omega
\]

(4.14)

as a consequence of the linear fractional transformation (2.11) of the vector kinetic matrix \(\mathcal{N}\).

We can now group all the scalar fields into a single set:

\[
\Phi^I = \left\{ \phi^a_\text{m}, \Delta, B, \tau^\Lambda_{\text{2n}}, \sigma_{\Sigma_{\text{2n}}} \right\}
\]

\[
m = \dim \frac{U_{D=4}}{\text{H}_{D=4}} \ ; \ 2 = \dim \frac{\text{SL}(2,\mathbb{R})}{\text{O}(2)} \ ; \ n = \dim \mathbf{W}
\]

(4.15)

and we can regard the sum \(L_{\sigma\text{-model}} + L_{\text{vec}+\text{sl}_2}\) as the definition of a new metric \(\tilde{h}_{IJ}(\Phi)\) and a new \(\sigma\)-model lagrangian on the set of \(m + 2 + 2n\) fields \(\Phi^I\):

\[
L_{\sigma\text{-model}} + L_{\text{vec}+\text{sl}_2} \equiv -\sqrt{g} \tilde{h}_{IJ}(\Phi) \partial_\mu \Phi^I \partial^\mu \Phi^J
\]

(4.16)

Eq. (4.15) is the field theoretical counterpart of the general algebraic decomposition (3.31) leading to the presentation (3.38) of the Lie algebra \(U_{D=3}\). This latter is the isometry algebra of the new metric \(\tilde{h}_{IJ}(\Phi)\) defined in eq. (4.16). The invariance of \(\tilde{h}_{IJ}(\Phi)\) under the original \(D = 4\) algebra \(U_{D=4}\) is guaranteed by the symplectic embedding (2.5) and by the transformation properties (4.14) of the matrix \(\mathbf{M}(\phi)\). Indeed it suffices to assign the scalar fields \(\{\tau^\Lambda, \sigma_\Sigma\}\) to
the $\mathbf{W}$ representation of $U_{D=4}$ realized by the matrices $\Lambda_\xi$ and the invariance is proved. To this effect, note also that the term $\tau^A \dot{\partial}_\sigma \sigma_A$ can be more effectively written as:

$$
\tau^A \dot{\partial}_\sigma \sigma_A = (\sigma, \sigma)^T \Omega \left( \frac{\partial \tau}{\partial \sigma} \right)
$$

(4.17)

which shows its symplectic invariance.

Let us also note that, if we set $\tau^A = \sigma_\Sigma = 0$, namely if we suppress all the scalars coming from the vector fields in $D = 4$, the Ehlers lagrangian reduces to:

$$
\mathcal{L} = \sqrt{g} \left[ -2R[g] - \frac{1}{4 \Delta^2} (\partial \Delta^2 + \partial B^2) \right]
$$

(4.18)

namely to an $\text{SL}(2, \mathbb{R})_E/\text{O}(2)$ $\sigma$–model coupled to 2D–gravity. Indeed,

$$
ds^2_{\text{Poin}} = \frac{1}{\Delta^2} (d\Delta^2 + dB^2)
$$

(4.19)

is the standard Poincaré metric on the upper complex plane parametrized by the complex variable $z = B + i \Delta$. The Ehlers $\text{SL}(2, \mathbb{R})_E$ generates isometries of the reduced lagrangian (4.18) but also of the complete one (4.16). This is just the statement that the full isometry algebra $U_{D=3}$ always includes the $\text{SL}(2, \mathbb{R})$ subalgebra as asserted by eq.(3.1). Indeed the further invariance of the metric $\tilde{h}_{IJ}(\Phi)$ under the transformations $L^x$ and $W^{i\alpha}$ that close the algebra (3.8) together with the generators $T^A$ of $U_{D=4}$ will be discussed in the next section 4.1.

We can now conclude the Ehlers programme by performing the last step, namely the further reduction from $D = 3$ to $D = 2$. In the $\sigma$–model part of the lagrangian there is nothing to do apart from restricting the dependence to the first two coordinates $t, x_1$. The only novelty comes from the reduction of the Einstein term. We choose the following reduction ansatz:

$$
\begin{align*}
ds^2_{(3)} &= ds^2_{(2)} + \rho^2 dx_2^2 \\
ds^2_{(2)} &= -N^2 dt^2 + \lambda^2 dx_1^2
\end{align*}
$$

(4.20)

which can be motivated as follows. In the first line there is no off-diagonal term (internal-space-time), namely there is no Kaluza-Klein vector. This is no restriction in the case of reduction from $D = 3$ to $D = 2$ since in $D = 2$ vector fields carry no degrees of freedom. In the second line of eq.(4.20) the diagonal form assigned to the $D = 2$ metric is again an always available choice in two–dimensional space–times, due to conformal invariance. With this choices we obtain:

$$
-2 \sqrt{g^{[3]}} R[g^{[3]}] = -2 \rho \sqrt{g^{[2]}} R[g^{[2]}]
$$

(4.21)

and the final form of the Ehlers lagrangian is:

$$
L_{\text{Ehlers}}^{D=2} = -2 \rho \sqrt{g} R[g] - \rho \sqrt{g} \tilde{h}_{IJ}(\Phi) \partial_\mu \Phi^I \partial^\mu \Phi^J
$$

(4.22)

which is just a standard 2D $\sigma$–model with target manifold the following coset

$$
\mathcal{M}_\text{target}^E = \frac{U_{D=3}}{H_{D=3}}
$$

(4.23)

and further coupled to the dilaton $\rho$. 
4.1 Field theoretic realization of the \( \mathbb{U}_{D=3} \) Lie algebra in the Ehlers reduction

As just announced, in this section we write the explicit form of the local transformations of the Ehlers fields under the action of the duality algebra \( \mathbb{U}_{D=3} \). As explained above, the Ehlers lagrangian is that of a standard \( \sigma \)–model, with \([4.23]\) as target manifold. Hence the action of the Lie algebra generators on the fields is obtained by standard techniques, once a parametrization of the coset representative is given. Let us call \( T_A \) the generators of \( \mathbb{U}_{D=3} \), namely:

\[
T_A = \{ T^A, L^x, W^{i\alpha} \}
\]

and let \( L(\Phi) \in \mathbb{U}_{D=3} \) be the coset representative, depending on the set of all \( \sigma \)–model fields. Following general prescriptions we have:

\[
\xi^A T_A L(\Phi) = \xi^A (L(\Phi + \delta_A \Phi) - L(\Phi) W_A(\Phi))
\]

where \( W_A(\Phi) \in \mathbb{H}_{D=3} \) is a suitable compact subalgebra compensator and \( \xi^A \) are generic parameters identifying an element of the Lie algebra \( \mathbb{U}_{D=4} \). With the definition \([4.25]\) the variations \( \delta_A \Phi \) fulfill the commutation relations of the generators with identical structure constants:

\[
[T_A, T_B] = f_{AB}^C T_C \\
\delta_A \delta_B \Phi - \delta_B \delta_A \Phi = -f_{AB}^C \delta_C \Phi
\]

and it is our programme to work them out explicitly.

To this effect we consider the algebra \([3.8]\) and we introduce a new basis for the generators of the \( \text{SL}(2, \mathbb{R}) \) subalgebra, and for those associated with the \( \mathbf{W} \) representation. First we recall that, by definition, the symplectic representation \( \mathbf{W} \) is even dimensional, namely \( \dim \mathbf{W} = 2n_V \) where \( n_V \) is the number of vector fields in \( D = 4 \). Hence the index \( \alpha \) runs on a set of \( 2n_V \) values that can be split into two subsets of \( n_V \) elements each, respectively corresponding to the positive and negative weights of the representation, from the algebraic viewpoint, and to the electric and magnetic field strengths from the physical viewpoint. Hence we write:

\[
W^{i\alpha} \equiv \{ W^{i\Lambda}, W^{i\Sigma} \}
\]

where the index \( \Lambda \) is that which enumerates the vector field strengths in the lagrangian \([2.1]\). Secondly we introduce the further notations:

\[
L_0 = \sqrt{2} L^3; \quad L_{\pm} = L^1 \pm L^2 \\
W^\alpha \equiv W^{i\alpha} = \{ W^\Lambda, W_\Sigma \} \\
\tilde{W}^\alpha \equiv W^{2\alpha} = \{ \tilde{W}^\Lambda, \tilde{W}_\Sigma \}
\]

In terms of these objects, the commutation relations \([3.8]\) become:

\[
\left[ W^\Lambda, \tilde{W}_\Sigma \right] = (\Lambda_\alpha)_{\Lambda}^{\Lambda} T^A + \frac{1}{2\sqrt{2}} \delta_\Sigma^\Lambda L_0, \\
\left[ W_\Lambda, \tilde{W}^\Sigma \right] = (\Lambda_\alpha)_{\Lambda}^{\Sigma} T^A - \frac{1}{2\sqrt{2}} \delta^\Lambda_\Sigma L_0,
\]
\[ [L_+, W^\alpha] = [L_-, \hat{W}^\alpha] = 0, \]
\[ [L_+, \hat{W}^\alpha] = W^\alpha, \quad [L_-, W^\alpha] = \hat{W}^\alpha, \]
\[ [L_0, W^\alpha] = \frac{1}{\sqrt{2}} W^\alpha, \quad [L_0, \hat{W}^\alpha] = -\frac{1}{\sqrt{2}} \hat{W}^\alpha, \]
\[ [L_0, L_\pm] = \pm \sqrt{2} L_\pm; \quad [L_+, L_-] = \sqrt{2} L_0 \]  \tag{4.29}

Let \( T_a \) be a basis of generators for the solvable Lie algebra \( \text{Solv}(U_{D=4}/H_{D=4}) \) of the scalar coset appearing in \( D = 4 \) supergravity. The full set of \( D = 3 \) scalar fields and a basis for the solvable Lie algebra \( \text{Solv}(U_{D=3}/H_{D=3}) \) can be paired in the following way:

\[ \sqrt{2} \log \Delta \Leftrightarrow L_0 \]
\[ B \Leftrightarrow L_+ \]
\[ \varpi^\alpha \Leftrightarrow W^\alpha \quad \alpha = 1, \ldots, \frac{1}{2} \dim W = n_V \]
\[ \phi^a \Leftrightarrow T_a \quad a = 1, \ldots, \dim (U_{D=4}/H_{D=4}) \]  \tag{4.30}

Correspondingly we write the coset representative as follows

\[ \mathbb{L}(\Phi) = \exp (\text{Solv}) = \exp [BL_+] \exp [\varpi^\alpha W^\alpha] \mathbb{L}(\phi_a) \exp \left[ \sqrt{2} \log \Delta L_0 \right] \]  \tag{4.31}

where \( \mathbb{L}(\phi_a) \in G_{D=4}/H_{D=4} \) is the coset representative for the scalar \( \sigma \)-model of \( D = 4 \) supergravity.

Under a generic element \( U_{D=3} \ni T \equiv \xi^A T_A \) of the Lie algebra of parameters \( \xi^A \), the Ehlers fields transform as follows:

- **generator** \( \gamma = \xi^+ L_+ \), that is the step–up operator of \( \text{SL}(2, \mathbb{R})_E \)
  \[ \delta_{\gamma} \varpi = \delta_{\gamma} \Delta = \delta_{\gamma} \phi_a = 0 ; \quad \delta B = \xi^+ \]  \tag{4.32}

- **generator** \( \gamma \equiv \xi^- L_- \) that is the step–down operator of \( \text{SL}(2, \mathbb{R})_E \):
  \[ \delta_{\gamma} B = \left[ \Delta^2 - B^2 - \frac{a}{2 \cdot 4!} \varpi_{\alpha_1} \cdots \varpi_{\alpha_4} \Lambda_A^{(a_1 a_2)} \Lambda^{a_3 a_4} \right] \xi^- \]  \tag{4.33}
\[ \delta_{\gamma} \varpi = \left[ (\Delta - B) \varpi - \frac{1}{3!} \varpi_{\alpha_1} \varpi_{\alpha_2} \varpi_{\alpha_3} (\Lambda_A)^{(a_1)} (\Lambda^{A})^{a_2 a_3} \right] \xi^- \]  \tag{4.34}
\[ \delta_{\gamma} \Delta = -2 B \Delta \xi_-; \quad \delta_{\gamma} \phi^a = -\frac{\varpi^a \varpi^\beta}{2} \Lambda_{A B} \Lambda^{a_3 a_4} \xi^-, \]  \tag{4.35}

where \( k_B^a(\phi) \) are the Killing vectors of \( U_{D=4} \).

- **generator** \( \gamma = \xi^0 L_0 \), that is the Cartan of \( \text{SL}(2, \mathbb{R})_E \)
  \[ \delta_{\gamma} B = B \xi^0 ; \quad \delta_{\gamma} \varpi = \frac{1}{2} \varpi \xi^0 ; \quad \delta \Delta = \xi^0 \Delta ; \quad \delta \phi^a = 0 \]  \tag{4.36}

- **generators** \( \gamma = \xi^A T_A \) of the \( U_{D=4} \) Lie algebra
  \[ \delta_{\gamma} B = \delta_{\gamma} \Delta = 0 ; \quad \delta_{\gamma} \varpi = \xi^A (\Lambda_A)^{a_3 a_4} \varpi_{a_3 a_4} ; \quad \delta_{\gamma} \phi^a = \xi^A k_B^a(\phi) \]  \tag{4.37}

29
generators $\xi_\alpha W^\alpha$ associated with weights of $W$

$$\delta_\gamma \omega_\alpha = \xi_\alpha \quad ; \quad \delta_\gamma B = \frac{1}{4} \omega_\alpha \Omega^{\alpha\beta} \xi_\beta \quad ; \quad \delta_\gamma \Delta = \delta_\gamma \phi_a = 0 \quad (4.38)$$

• generators $\gamma = \hat{\xi}_\alpha \hat{W}^\alpha$ associated with the dual weights of $W$

$$\delta_\gamma B = -\frac{1}{24} \hat{\xi}_\alpha \omega_{\beta_1} \omega_{\beta_2} \omega_{\beta_3} (\Lambda_A)^{\beta_3\alpha} (\Lambda_A)^{\beta_1\beta_2} + \frac{1}{2} \Omega^{\alpha\beta} \hat{\xi}_\alpha \omega_\beta (B + \Delta) \quad (4.39)$$

$$\delta_\gamma \omega_\beta = -\frac{1}{2} \left[ \hat{\xi}_\alpha \omega_{\beta_1} \omega_{\beta_2} \Lambda^{\beta_3\alpha}_A (\Lambda_A)^{\beta_3\beta_2} + \frac{1}{4} \hat{\xi}_\alpha \omega_\beta \omega_\gamma \Omega^{\gamma\alpha} \right] + (\Delta - B) \hat{\xi}_\beta \quad (4.40)$$

$$\delta_\gamma \Delta = -\frac{1}{2} \hat{\xi}_\alpha \omega_\beta \Omega^{\alpha\beta} \Delta \quad ; \quad \delta_\gamma \phi^a = -\hat{\xi}_\alpha \omega_\beta (\Lambda_A)^{\beta_3\alpha} k_A^a(\phi) \quad (4.41)$$

This concludes the analysis of the Ehlers lagrangian and of its symmetries. In the next section we turn our attention to the Matzner Misner dimensional reduction scheme.

5 The Sp$(2n, \mathbb{R}) \hookrightarrow$ SO$(2n, 2n)$ embedding and the general Matzner–Misner lagrangian

The key point in the Maztner Misner reduction of a general supergravity lagrangian from $D = 4$ to $D = 2$ is provided by the following pseudorthogonal embedding:

$$\text{Sp}(2n, \mathbb{R}) \hookrightarrow \text{SO}(2n, 2n) \quad (5.1)$$

which we presently discuss. In $D = 4$ the duality group $G_{D=4}$ is simultaneously realized as an isometry group of the scalar manifold metric $h_{ab}(\phi)$ and as a group of electric-magnetic duality transformations on the vector field strengths $F_{\mu\nu}^A$ as we have already emphasized. The general form of the lagrangian was given in eq.(2.1). Furthermore, as already stressed there is always a symplectic embedding of the duality group $G_{D=4}$ as mentioned in eq.(2.5) where $n = n_V$ is the total number of vector fields in the theory. Let us now consider the general form of a lagrangian in $D = 2$. Here we have two type of scalars, namely the scalars-scalars $\phi^a$ and the twisted scalars or scalar-forms $\pi^\alpha$. This distinction is important. The scalars-scalars appear in the lagrangian under the form of a usual $\sigma$-model while the scalar-forms appear only covered by derivatives and in two terms, one symmetric, one antisymmetric. The coefficients of these two terms are matrices depending on the scalars-scalars. Explicitly the lagrangian has the form (see [20] for a general review):

$$L_{(D=2)} = \int d^2x \sqrt{-\det g} \left\{ -2 R[g] - \frac{1}{8} h_{ab}(\phi) \partial_\mu \phi^a \partial_\mu \phi^b + \frac{1}{2} \kappa \left[ -\partial_\mu \pi^\alpha \gamma_{\alpha\beta}(\phi) \partial_\mu \pi^\beta + \partial_\mu \pi^\alpha \theta_{\alpha\beta}(\phi) \partial_\mu \pi^\beta \epsilon^{\mu\nu} \right] \right\} \quad (5.2)$$

where $\kappa$ is a normalization parameter that can always be reabsorbed into the definition of the 0-forms $\pi^\alpha$ and where, according to the general theory for the dimensions $D = 4\nu + 2$ (see
section 2.4 of [20] if $G_{D=2}^{iso}$ is the isometry group of the $\sigma$-model metric $h_{ab}(\phi)$, then there is a pseudo-orthogonal embedding:

$$G_{D=2}^{iso} \hookrightarrow SO(m, m) \quad (5.3)$$

where $m$ is the total number of the scalar-forms $\sigma^\alpha$. Hence for each element $\xi \in G_{D=2}^{iso}$ we have its representation by means of a suitable pseudorthogonal matrix:

$$\xi \mapsto \Sigma \xi \equiv \begin{pmatrix} A \xi & B \xi \\ C \xi & D \xi \end{pmatrix} \quad (5.4)$$

which satisfies the defining equation:

$$\Sigma^T \begin{pmatrix} 0_{m \times m} & 1_{m \times m} \\ 1_{m \times m} & 0_{m \times m} \end{pmatrix} \Sigma \xi = \begin{pmatrix} 0_{m \times m} & 1_{m \times m} \\ 1_{m \times m} & 0_{m \times m} \end{pmatrix} \quad (5.5)$$

implying the following relations on the $m \times m$ blocks:

$$A^T C + C^T A = 0$$
$$A^T D + C^T B = 1$$
$$B^T C + D^T A = 1$$
$$B^T D + D^T B = 0 \quad (5.6)$$

Defining the $m \times m$ matrix

$$\mathcal{M} \equiv \theta + \gamma \quad (5.7)$$

under the group $G_{D=2}^{iso}$ it transforms as follows:

$$\mathcal{M}' = (C + D \mathcal{M}) (A + B \mathcal{M})^{-1}$$

$$-\mathcal{M}' = (C - D \mathcal{M}) (A - B \mathcal{M})^{-1} \quad (5.9)$$

We can now link the $D = 4$ supergravity lagrangian (2.1) to the $D = 2$ lagrangian that will emerge from the Matzner-Misner reduction, which is of the type (5.2).

Consider the transformation rule of the matrix $\mathcal{N}$ and multiply it by $-i$, we obtain:

$$(-i\mathcal{N})' = [-iC + D (-i\mathcal{N})] [A + iB (-i\mathcal{N})]^{-1} \quad (5.10)$$

Next let us represent the imaginary unit by a $2 \times 2$ matrix $\varepsilon$ such that

$$\varepsilon^2 = -1_{2 \times 2} \quad (5.11)$$

In this way we can write:

$$\mathcal{M} \equiv -i\mathcal{N} = \text{Im} \mathcal{N} \otimes 1_{2 \times 2} - \text{Re} \mathcal{N} \otimes \varepsilon_{2 \times 2}$$
$$A \equiv A = A \otimes 1_{2 \times 2}$$
$$B \equiv iB = B \otimes \varepsilon_{2 \times 2}$$
$$C \equiv -iC = -C \otimes \varepsilon_{2 \times 2}$$
$$D \equiv D = D \otimes 1_{2 \times 2} \quad (5.12)$$
and the transformation (5.10) becomes the transformation (5.8). Furthermore the \(2n \times 2n\) blocks \(A, B, C, D\), defined by equation (5.12) satisfy the relations (5.6) as a consequence of (2.8) and (5.11). This provides the required embedding (5.1) in the form:

\[
\text{Sp}(2n, \mathbb{R}) \ni \left( \begin{array}{cc} A_\xi & B_\xi \\ C_\xi & D_\xi \end{array} \right) \mapsto \left( \begin{array}{cc|cc} A & 0 & 0 & B \\ 0 & A & -B & 0 \\ \hline 0 & -C & D & 0 \\ C & 0 & 0 & D \end{array} \right) \in \text{SO}(2n, 2n) \tag{5.13}
\]

The matrix \(M\) transforms correctly under the pseudorthogonal embedded group \(G_{\text{iso}}^{D=2}\) as a consequence of the same symplectic embedding of \(G_{\text{iso}}^{D=4}\). From its definition in the first of equations (5.12), we derive the symmetric and antisymmetric parts (\(A, B = 1, 2\)):

\[
\gamma_{\alpha\beta} = \text{Im}N \otimes 1_{2 \times 2} = \text{Im}N_{\Lambda\Sigma} \delta_{AB} \\
\theta_{\alpha\beta} = -\text{Re}N \otimes 1_{2 \times 2} = -\text{Re}N_{\Lambda\Sigma} \varepsilon_{AB} \tag{5.14}
\]

It follows that if the reduced lagrangian takes the \(D = 2\) form:

\[
L_{(D=2)}^{MM} = \int d^2x \sqrt{-\det g} \left\{ -2R[g] - \frac{1}{6} h_{ab}(\phi) \partial_\mu \phi^a \partial_\mu \phi^b \\
+ \frac{1}{2} \left[ -\nabla_\mu \pi^A \text{Im}N_{\Lambda\Sigma}(\phi) \delta_{AB} \nabla_\nu \pi^{\Sigma B} + \nabla_\mu \pi^A \text{Re}N_{\Lambda\Sigma} \varepsilon_{AB} \nabla_\nu \pi^{\Sigma B} \epsilon^{\mu\nu} \right] + \text{more} \right\} \tag{5.15}
\]

then the symmetry group \(U_{D=4}\) is realized in \(D = 2\) just as in \(D = 4\), namely as an isometry group on the scalars-scalars and as a group of generalized duality transformations on the scalars-forms or twisted scalars. An enlargement of symmetries can arise, in the Matzner Misner reduction only by the more that we mentioned in eq.(5.15). What is this more? It is just the contribution of pure gravity, that in the Matzner–Misner reduction yields another \(\text{SL}(2, \mathbb{R})_{\text{MM}}/\text{O}(2)\) sigma-model as we have already anticipated in the introduction. Hence, taking into account also the constant shifts of the scalar forms \(\pi^a\), the symmetry of Matzner–Misner lagrangian will eventually be:

\[
U_{D=2}^{MM} = U_{D=4} \times \text{SL}(2, \mathbb{R})_{\text{MM}} \times \mathbb{R}^m \tag{5.16}
\]

opposed to the more extended \(U_{D=3}\) symmetry displayed by the Ehlers lagrangian.

Let us now see the details of the Matzner–Misner reduction and let us prove that the reduced lagrangian does indeed take the form (5.15).

### 5.1 The Matzner–Misner reduction

The reduction à la Matzner–Misner is just the straightforward dimensional reduction of the \(D = 4\) lagrangian (2.1) on a 2-dimensional torus \(T^2\), without any dualization of the vector fields. To this effect we split the space-time indices in the following way: \(\mu, \nu = 0, 1\) and \(i, j = 2, 3\). Then the Matzner–Misner reduction consists of two steps:
1. dimensional reduction of the $D = 4$ supergravity lagrangian in eq. (2.1) on a 2–torus.

This gives a system that is a mixture of an ordinary $\sigma$–model and a twisted $\sigma$–model, namely the coupling of 0-forms with duality symmetries as described in the previous section. Both sectors are coupled to 2–dimensional dilaton gravity and we have a dilaton kinetic term.

2. a rescaling of the 2–dimensional metric appropriate to cancel the kinetic term of the 2–dimensional dilaton, thus putting the lagrangian into a standard form.

The reduction ansatz for the $D = 4$ metric is the following one:

$$ds^2_{(4)} = ds^2_{(2)} + g_{ij} dx^i dx^j,$$

where the indices $i, j = 2, 3$ correspond to the dimensions of the internal torus. More explicitly, we parametrize the internal metric $g_{ij}$ as it follows:

$$g_{ij} = \rho \left( \frac{\tilde{\Delta} + \frac{\bar{B}^2}{\tilde{\Delta}}} \frac{\bar{B}}{\tilde{\Delta}} \frac{1}{\bar{B}} \right) \quad (5.18)$$

where $\tilde{\Delta}$ denotes the Kaluza–Klein scalar, $\rho = \sqrt{-\det(g_{ij})}$ is the two-dimensional dilaton and $\bar{B} = b_2 = g_{23} \tilde{\Delta} / \rho$ denotes the internal component of the Kaluza–Klein vector. In this frame the dimensional reduction of the Einstein term yields:

$$- 2\sqrt{-\det g^{(4)} R[g^{(4)}]} = -2\sqrt{-\det g_{(2)} \rho} \left[ R[g_{(2)}] + \frac{1}{8} (g^{ik} g^{jl} - g^{il} g^{jk}) \partial g_{ik} \partial g_{jl} \right] \quad (5.19)$$

Inserting the explicit form of the internal metric (5.18), we obtain the kinetic term for an SL$(2, \mathbb{R})$ $\sigma$–model plus the kinetic term for the 2–dimensional dilaton $\rho$

$$L_{\text{grav}} = -2\sqrt{-\det g_{(2)} \rho} \left( R_2[g_{(2)}] + \frac{1}{4} \frac{(\partial \rho)^2}{\rho^2} - \frac{1}{4 \tilde{\Delta}^2} \left[ (\partial \tilde{\Delta})^2 + (\partial \bar{B})^2 \right] \right) \quad (5.20)$$

Now we perform the second step, absorbing the dilaton kinetic term in the gravity term by the rescaling $g_{\mu\nu}^{(2)} = \bar{g}_{\mu\nu}^{(2)} \rho^{-\frac{1}{2}}$.2

Eventually the $D = 4$ gravity term reduced à la Matzner–Misner gives a pure SL$(2, \mathbb{R})$ $\sigma$–model coupled to 2D dilaton–gravity

$$L_{\text{grav}}^{MM} = -2\sqrt{-\det \bar{g}_{(2)} \rho} \left( R_2[\bar{g}_{(2)}] - \frac{1}{4 \tilde{\Delta}^2} \left[ (\partial \tilde{\Delta})^2 + (\partial \bar{B})^2 \right] \right) \quad (5.21)$$

The lagrangian (5.21) is formally identical to the reduced lagrangian (4.18) of the Ehlers case and as such admits an SL$(2, \mathbb{R})_{MM}$ group of isometries. Yet the fields $\Delta, B$ are different from the fields $\tilde{\Delta}, \bar{B}$ a non local relation existing between the two. As noted years ago by Nicolai (4) the coexistence of the two formally identical lagrangians (4.18) and (5.21), together with the non-local map between the fields $\Delta, B$ and the fields $\tilde{\Delta}, \bar{B}$ is the mechanism which

---

2Any two dimensional metric is conformally flat, so there will be no overall rescaling of the gravity lagrangian, but there is a contribution from the boundary term, that exactly cancels $\frac{1}{4} (\partial \rho)^2 / \rho^2$. 

---

33
promotes the SL(2, \( \mathbb{R} \)) symmetry to its affine extension in pure gravity. Our present paper aims at generalizing the same mechanism to all cases of supergravity. The further complicacy in the general case is that two lagrangians, Ehlers and Matzner–Misner are not formally identical and have different isometry algebras. Yet they coincide in one sector and it is there that the affine extension mechanism works.

So let us continue by considering the dimensional reduction of the gauge fields. To this effect we restrict the structure of the metric in \( D = 2 \), assuming that there is no off-diagonal term

\[
ds_{(2)}^2 = -\tilde{N}^2 dt^2 + \tilde{\lambda}^2 dx_i^2
\]

(5.22)

Then, in the absence of Kaluza–Klein fields, there is no need to redefine the \( D = 2 \) vectors and they can be parametrized as follows:

\[
A^{\Lambda}_{[D=4]} = A^\Lambda_{(2)\mu} dx^\mu + A^\Lambda_i dx^i
\]

(5.23)

where \( A^\Lambda_{(2)\mu} \) are the two-dimensional gauge fields that do not propagate. In what follows we set the gauge: \( A^\Lambda_{(2)\mu} = 0 \).

This being set, the vector sector of the \( D = 4 \) lagrangian reduces as follows

\[
\frac{1}{2} \text{Re} N_{\Lambda \Sigma} F^\Lambda_{\mu \nu} F^\Sigma_{\mu \nu} e^{i \tilde{\nu} i \tilde{\mu}} = 2 \epsilon^{\mu \nu} \epsilon^{ij} \text{Re} N_{\Lambda \Sigma} \partial_\mu A^\Lambda_i \partial_\nu A^\Sigma_j,
\]

\[
\sqrt{-\text{det} g_{(4)}} \text{Im} N_{\Lambda \Sigma} F^\Lambda_{\mu \nu} F^\Sigma_{\mu \nu} = 2 \sqrt{-\text{det} g_{(2)}} \epsilon^{ij} \text{Im} N_{\Lambda \Sigma} \partial_\mu A^\Lambda_i \partial_\nu A^\Sigma_j
\]

(5.24)

In order to make contact with the action described in the previous section, which implements the \( \text{Sp}(2n, \mathbb{R}) \mapsto \text{SO}(2n, 2n) \) embedding we have to introduce new notations. We define the following doublets of 0-forms with flat internal indices

\[
\pi^{\Lambda|A} = e^{i A} A_i^\Lambda
\]

(5.26)

where \( e^{i A} \) is the internal vielbein defined as usual by:

\[
e^{i A} e^{j B} \delta_{AB} = g^{ij}
\]

(5.27)

The \( 0 \)-forms \( \pi^{\Lambda|A} \) are by definition sections of the \( \text{O}(2) \) vector bundle defined by the coset \( \text{SL}(2, \mathbb{R})_{\text{MM}}/\text{O}(2) \). Introducing the coset representative of this latter

\[
\mathbb{L}_2(\Delta, \bar{B}) = \begin{pmatrix} \sqrt{\Delta} & \bar{B} \\ 0 & \sqrt{\Delta} \end{pmatrix} \equiv \exp \left[ \bar{B} L_+ \right] \exp \left[ \sqrt{2} \log \Delta L_0 \right]
\]

\[
L_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} ; \quad L_0 = \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 \\ 0 & -\frac{1}{\sqrt{2}} \end{pmatrix}
\]

(5.28)

we can identify \( e^{i A} = \mathbb{L}_2^{-1}(\Delta, \bar{B}) \) so that the precise relation between the vector components \( A^\Lambda_i \) and the fields \( \pi^{\Lambda|A} \) reads as follows:

\[
A^\Lambda_3 = \rho^{1/2} \Delta^{-1/2} \pi^{3|A},
\]

\[
A^\Lambda_2 = \rho^{1/2} \Delta^{1/2} \pi^{2|A} + \rho^{1/2} \Delta^{-1/2} \pi^{3|A}.
\]

(5.29)
Furthermore, separating the left invariant one–form \( L^{-1}_2 dL_2 \) into its vielbein and connection parts:

\[
L^{-1}_2 dL_2 = \omega \mathbb{H} + \text{vielbein}
\]

\[
\mathbb{H} \equiv \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}
\]

we can define the O(2)–covariant derivatives of the 0–forms \( \pi^{A|A} \) as follows:

\[
\nabla_\mu \pi^{A|A} = \partial_\mu \pi^{A|A} + \epsilon^{AB} \omega_\mu \pi^{A|B}
\]

In this way the fully reduced and redefined vector lagrangian in \( D = 2 \) takes the following form:

\[
L_{vec}^{MM} = 2 \left[ \sqrt{-\text{det}g(2)} \Im N_{A\Sigma}(\phi) \delta_{AB} \nabla_\mu \pi^{A|A} \partial_\mu \pi^{\Sigma|B} + \Re N_{A\Sigma} \varepsilon_{AB} \nabla_\mu \pi^{A|A} \nabla_\nu \pi^{\Sigma|B} \epsilon^{\mu
u} \right]
\]

The reduction of the \( D = 4 \) scalars is straightforward and gives

\[
L_{\sigma}^{MM} = -\frac{\rho}{6} \sqrt{-\text{det}g(2)} h_{ab} \partial^a \phi^a \partial_b \phi^b
\]

Putting together these results we have completed all the steps of the Matzner–Misner reduction and we have shown that the complete lagrangian

\[
L_{\text{complete}}^{MM} = L_{\text{grav}}^{MM} + L_{\sigma}^{MM} + L_{vec}^{MM}
\]

is, as announced, of the form \((5.13)\) realizing the pseudorthogonal embedding \( \text{Sp}(2n, \mathbb{R}) \rightarrow \text{SO}(2n, 2n) \) and that the \textit{more} is the Matzner–Misner SL(2, \( \mathbb{R} \))\( \text{MM} \)/O(2) sigma model:

\[
\text{more} = -\frac{1}{4\Delta^2} [(\partial \bar{\Delta})^2 + (\partial \bar{B})^2]
\]

### 5.2 Action of the SL(2, \( \mathbb{R} \))\( \text{MM} \) algebra on the Matzner–Misner fields

We denote the generators of the Matzner–Misner SL(2, \( \mathbb{R} \))\( \text{MM} \) group as \((L^{mm}_0, L^{mm}_+, L^{mm}_-)\), with just the same conventions as in the case of the Ehlers SL(2, \( \mathbb{R} \))\( \text{E} \) group. We will then see that, using these three generators, we can cook up the Chevalley-Serre triplet \((h_0, e_0, f_0)\) needed to generate the affine extension of the \( U_{D=3} \) Lie algebra. Yet the way \((L^{mm}_0, L^{mm}_+, L^{mm}_-)\) are associated with \((h_0, e_0, f_0)\) is different in the case of \textit{pure gravity} and in the case of \textit{extended supergravity}, namely when we have also the vector sector of the theory. This is not surprising since the affine Cartan matrix of pure gravity namely \( A^\alpha_1 \) is structurally different from those obtained in the various supergravity models: in pure gravity we add a double line to the Dynkin diagram, while in supergravity we always add a simple line. For this reason we do not immediately commit ourselves with the association:

\[
(L^{mm}_0, L^{mm}_+, L^{mm}_-) \leftrightarrow (h_0, e_0, f_0)
\]
and we just write the transformations of the $SL(2, \mathbb{R})_{\text{MM}}$ generators on the Matzner–Misner fields $\tilde{\Delta}, \tilde{B}$ and $\pi^{|A|^A}$. Applying the general formula (4.25) to the coset representative (5.28)

On the other hand, the scalars $\pi^{|A|^A}$ are, as we have emphasized in a linear doublet representation of the $O(2)$–compensator so that:

This concludes the discussion of the local symmetries of our two lagrangian models. We can now go to the affine extension by considering also non local symmetries. This involves consideration of the non local map $\mathcal{T}$ which relates the two set of fields the Ehlers and the Massner Misner ones which we can trace back reconsidering the steps of our two reduction schemes.

6 The $\mathcal{T}$-map and the affine extension

The two lagrangian models obtained via the two dimensional reductions schemes, respectively à la Ehlers and à la Matzner–Misner, are related to each other by a non–local transformation which is the main token in combining the symmetries of the Ehlers lagrangian with those of the Matzner–Misner one. Let us describe this map in detail. We write

\[
\mathcal{T} : \quad \begin{array}{c}
\text{MM} \\ \tilde{N} \\ \tilde{\lambda} \\ \tilde{\rho} \\ \tilde{\Delta} \\ \tilde{B} \\ \pi^{|A|^A} \\ \tilde{\phi}^a
\end{array} \rightarrow \begin{array}{c}
\text{Ehlers} \\ N \\ \lambda \\ \rho \\ \Delta \\ B \\ \tau^A, \sigma_A \\ \phi^a
\end{array}
\]

(6.1)

which is a generalization of the so named Kramer–Neugebauer transformation, firstly considered in the dimensional reduction of pure $D = 4$ Einstein gravity [24].

Tracing back the rescalings made for the two–dimensional metric and the dilaton, we reconstruct immediately the following map:

\[
\begin{align*}
\tilde{N} &= N \rho^{1/4} \Delta^{-1/2} \\
\tilde{\lambda} &= \lambda \rho^{1/4} \Delta^{-1/2} \\
\tilde{\rho} &= \rho
\end{align*}
\]
The Kaluza–Klein scalars in Ehlers and Matzner–Misner models are connected in the following way

\[ \tilde{\Delta} = \rho \Delta \]  

The scalars coming from the off–diagonal part of the 4–dimensional metric and from the 4–dimensional vectors \((B, \tau^\Lambda, \sigma^\Lambda)\), which were partially dualized in the Ehlers reduction \((\tilde{\Lambda}, \pi^\Lambda|)\), have a non–local relation to the Matzner–Misner fields \((\tilde{B}, \pi^\Lambda)\)

\[ \partial_\mu \tilde{B} = -\frac{1}{2} \frac{N\rho}{\Delta^2} g^{\rho\rho} \epsilon_{\mu\rho} \left[ \partial_\rho B + \frac{1}{2} \sigma^\Lambda \partial_\rho \tau^\Lambda \right], \]  

(6.3)

Finally, the 4–dimensional scalars \(\phi^a\) went through the dimensional reduction untouched and therefore are the same in the Ehlers and in the Matzner–Misner models:

\[ \tilde{\phi}^a = \phi^a \]  

(6.4)

This concludes the description of the map \(T\). By means of this token we can trace the action of the Ehlers symmetry algebra \(U_{D=3}\) on the Matzner–Misner fields and vice versa, trace the action of the Matzner–Misner algebra \(U_{D=4} \times \text{SL}(2, \mathbb{R})_{\text{MM}}\) on the Ehlers fields. The found extension of the Kramer-Neugebauer transformation \(T\) allows to merge these two algebras into a larger one which, as claimed, turns out to be the affine extension of the Ehlers algebra \(U_{D=3}\).

We shall prove this by showing that by using the Matzner–Misner generators \((L_{0m}^{mm}, L_+^{mm}, L_-^{mm})\) we can add to a Chevalley–Serre presentation \((\hbar, e_i, f_i) \ (i = 1, \ldots, r)\) of the Ehlers algebra \(U_{D=3}\) a new Chevalley-Serre triplet \((h_0, e_0, f_0)\), which has, with the generators \((\hbar, e_i, f_i)\), the correct commutation relations corresponding to the Cartan matrix \(C^\Lambda\), if \(C\) was the Cartan matrix of \(U_{D=3}\).

### 6.1 Field Theory identification of the affine Chevalley-Serre triplet

As anticipated the identification of the affine Chevalley-Serre triplet is different in the case of pure gravity and in that of supergravity. Let us begin with the case of pure gravity.

**Pure Gravity** This is the case originally discussed by Nicolai in [5]. Here we just have two copies of the \(\text{SL}(2, \mathbb{R})\) algebra, the Ehlers and the Matzner–Misner realization. The transformations of each algebra on its own fields are:

| \(\text{SL}(2, \mathbb{R})_E\) | \(\text{SL}(2, \mathbb{R})_{\text{MM}}\) |
|-----------------|-----------------|
| \(\delta L_0^B \Delta = \sqrt{2} \Delta\) ; \(\delta L_0^B B = \sqrt{2} B\) | \(\delta L_{0m}^m \tilde{\Delta} = \sqrt{2} \tilde{\Delta}\) ; \(\delta L_{0m}^m \tilde{B} = \sqrt{2} \tilde{B}\) |
| \(\delta L_+^B \Delta = 0\) ; \(\delta L_+^B B = 1,\) | \(\delta L_+^{mm} \tilde{\Delta} = 0\) ; \(\delta L_+^{mm} \tilde{B} = 1,\) |
| \(\delta L_-^B \Delta = -2 \Delta B,\) ; \(\delta L_-^B B = \Delta^2 - B^2\) | \(\delta L_-^{mm} \tilde{\Delta} = -2 \tilde{\Delta} \tilde{B},\) ; \(\delta L_-^{mm} \tilde{B} = \tilde{\Delta}^2 - \tilde{B}^2\) |

(6.5)
and the relation between the two sets is obtained from the general form of the \( \mathcal{T} \)-map (6.2–6.4) by deleting all the scalars coming from vector fields \( \omega^a, n^A \). We have:

\[
\tilde{\rho} = \rho; \quad \tilde{\Delta} = \frac{\rho}{\Delta}
\]

\[
\partial_0 \tilde{B} = -\frac{N\rho}{2\Delta^2 \lambda} \partial_1 B; \quad \partial_1 \tilde{B} = -\frac{\lambda\rho}{2N\Delta^2} \partial_0 B
\]

(6.6)

Using eq.s (6.5) and (6.6) we can combine the two algebras. Inspired by the algebraic discussion following eq.(3.24) we set

\[
(h_1, e_1, f_1) = \left( \sqrt{2} L_0^E, L_+^E, L_-^E \right)
\]

\[
(h_0, e_0, f_0) = \left( \sqrt{2} L_0^{mn}, L_+^{mn}, L_-^{mn} \right)
\]

(6.7)

and we calculate the commutators of one triplet with the other triplet. This defines the Cartan matrix of the extended algebra. In particular we evaluate the commutator \([h_0, e_1]\) that is nonvanishing only on the Ehlers field \( B \) since on the other Ehlers fields \( \delta_{e_1} = 0 \). The commutator of the two transformations is calculated as

\[
[\delta_{h_0}, \delta_{e_1}] = \delta_{h_0} \delta_{e_1} - \delta_{e_1} \delta_{h_0} = -\delta_{[h_0, e_1]}
\]

(6.8)

and we find:

\[
[\delta_{h_0}, \delta_{e_1}] = -2 \delta_{e_1} \delta_{h_0} B = 2 \delta_{e_1} B
\]

(6.9)

The last step in eq.(6.9) is motivated by the following calculation

\[
\delta_{h_0} \partial_0 B = \delta_{h_0} \left( -2 \frac{N}{\lambda} \frac{\rho}{\Delta^2} \partial_1 \tilde{B} \right) = 2 \left( -2 \frac{N}{\lambda} \frac{\rho}{\Delta^2} \partial_1 \tilde{B} \right) = -2 \partial_0 B
\]

(6.10)

which is consistent only with

\[
\delta_{h_0} B = -2 B
\]

(6.11)

Eq.(6.9), on the other hand, implies

\[
[h_0, e_1] = -2 e_1
\]

(6.12)

which is just one of the Serre relations for the affine extension of the \( A_1 \) Lie algebra. Indeed this commutation relation verifies the entry \( \langle \alpha_0, \alpha_1 \rangle \) of the Cartan matrix (3.23). The other necessary relation is:

\[
[h_0, f_1] = 2 f_1
\]

(6.13)

which can also be verified by evaluating the commutators:

\[
\delta_{[h_0, f_1]} \Delta = 2 \delta_{f_1} \Delta
\]

(6.14)

\[
\delta_{[h_0, f_1]} B = 2 \delta_{f_1} B
\]

(6.15)

The results in eq.s (6.14) and (6.15) are obtained with straightforward calculations similar to that in eq.(6.10). This concludes the proof that the symmetry of pure gravity reduced to \( D = 2 \) is indeed \( A_1 \) and that the identification (6.7) is the correct one for the Chevalley Serre pair of triplets occurring in this case. Let us now turn our attention to supergravity.
Supergravity

It is fairly simple, within the algebra (3.8), to identify the Chevalley–Serre triple \(\{h_1, e_1, f_1\}\) which is relevant for the affine extension, once we rewrite the commutation relations in the form (1.29). From the algebraic viewpoint we know that the Chevalley–Serre triple \(\{h_w, e_w, f_w\}\) generating the affine node is associated with the unique black root \(\alpha_W\), which is not orthogonal to the highest root \(\psi\). Namely we can write:

\[
\begin{align*}
  h_w &= \alpha_W \cdot \mathcal{H} \\
  e_w &= E^{\alpha_W} \\
  f_w &= E^{-\alpha_W}
\end{align*}
\]  

having denoted by \(\mathcal{H}\) the CSA of \(U_{D=3}\). The problem, in order to perform field–theoretical calculations is to identify \(\alpha_W \cdot \mathcal{H}\) and \(E^{\pm\alpha_W}\) within the presentation (1.29) of the Lie algebra. To this effect, we just have to recall that the universal subalgebra \(SL(2, \mathbb{R})_E\) coming from the Ehlers reduction of Einstein gravity is nothing else but the \(A_1\) subalgebra associated with the highest root \(\psi\). In other words, with reference to eq.s(4.29), we have the identification:

\[
\begin{align*}
  L_0 &= \psi \cdot \mathcal{H} \quad ; \quad L_+ = E^\psi \quad ; \quad L_- = E^{-\psi}
\end{align*}
\]  

Next we recall that for the simple roots we can always choose a basis of the form (3.30), namely as Cartan generators we can use:

\[
\{ \mathcal{H}_I \mid I = 1, \ldots, r \} \cup \{ \mathcal{T}_i \mid i = 1, \ldots, r-1 \}
\]

where \(\mathcal{T}_i\) is a Cartan basis for the \(U_{D=4}\) Lie algebra. With these considerations we conclude that:

\[
\alpha_W \cdot \mathcal{H} = \mathcal{W}_h \cdot \mathcal{H} + \frac{1}{\sqrt{2}} L_0
\]  

where the term \(\mathcal{W}_h \cdot \mathcal{H}\) is necessarily a linear combination of the generators \(T^A\) of \(U_{D=4}\); which one we still has to determine.

From the commutation relations (1.29) we see that the only generators having non vanishing grading with respect to \(L_0\) are the \(W^A\). Hence we learn that \(W^A \propto E^{\alpha_V}\) where we have collectively denoted by \(\alpha_V\) the \(U_{D=3}\) roots corresponding to weights of the symplectic representation \(W\). For one particular value \(\Lambda_h\) we retrieve the highest weight \(w_h\), namely the black root \(\alpha_W\). At this point we have enough information to fix also the absolute normalizations. Indeed comparing the first of eq.s (1.29) with eq. (6.19) we conclude that:

\[
\begin{align*}
  h_w &= 2 \left( \Lambda_A \right)_{\Lambda_h} T^A + \frac{1}{\sqrt{2}} L_0 \\
  e_w &= \sqrt{2} W^A \\
  f_w &= \sqrt{2} \tilde{W}_A
\end{align*}
\]  

The relevance of this identification is that now via eq.s (1.33–1.32) we know the action of the Chevalley-Serre generators on the Ehlers fields and via eq.s (6.2–6.4) also on the Matzner–Misner fields. Hence we can evaluate commutators of the found triplet \(\{h_w, e_w, f_w\}\) with the
affine triplet that is the same as in eq. (6.7). We illustrate the result on the for the commutator $[h_0, e_w]$ that is nonvanishing only on the Ehlers fields $B$ and $\varpi_\alpha = (\tau^\Lambda, \sigma^\Sigma)$. For instance we calculate $\delta[h_0, e_w] \tau^\Lambda$. First from eq.(6.4) we easily calculate the action of $h_0$ on, for example, $\tau^\Lambda$. Indeed we have:

$$\delta h_0 \tau^\Lambda = \delta h_0 \left(\sqrt{\frac{\rho}{\Delta}} \pi^\Lambda|3\right) = \left(\sqrt{\frac{\rho}{\Delta}} \pi^\Lambda|3\right) = -\tau^\Lambda$$

(6.21)

Then, using eq.s(4.38) and the identification (6.20) we have

$$[\delta h_0, \delta e_w] \tau^\Lambda = \delta e_w \tau^\Lambda \Rightarrow [h_0, e_w] = -e_w$$

(6.22)

Using eqns.(4.39–4.41) it is straightforward to check that $[h_0, f_w] = f_w$.

In order to evaluate the commutators $[h_w, e_0]$ and $[h_w, f_0]$ we note that due to the fact that the generators $L_{m\mu}$ of the SL(2, $\mathbb{R}$)$_{MM}$ commute with the generators $T^A$ of $U_{D=4}$, we have only to calculate the commutators $[L_{m0}, e_0]$ and $[L_{m0}, f_0]$. Then, using (4.29) and tracing the action of the generators $e_0$ and $f_0$ on the Ehlers fields via the map $T$, one finds that $[h_w, f_0] = f_0$ (the action of the generator $e_0$ on the Ehlers fields is trivial).

This allows to reconstruct the Cartan matrix of the extended symmetry algebra that corresponds to the affine extension of $U_{D=3}$. The final check of the Serre relations $ad[e_i]C_{ij}^{\mu+1}e_j = 0$ and $ad[f_i]C_{ij}^{\mu+1}f_j = 0$ should be more involved, but relying on the algebraic arguments presented in the section (3.1.1) it should follow.

So, differently from the case of pure gravity, in all supergravity models the affine extended Dynkin diagram has a new simple line and not a double line as in the case of pure gravity. This is due to the replacement of the eigenvalue $-2$ with the eigenvalue $-1$ shown in the above calculation. This change is due to the fact that the affine node is linked to the vector root $\alpha_W$ and not to the root of SL(2, $\mathbb{R}$)$_E$ as in the pure gravity case. We stress again that the extension is possible due to the coexistence of the Ehlers and Matzner–Misner dimensional reduction schemes that lead to well distinct results in the supergravity case, while they lead to formally identical lagrangians in pure gravity.

7 Conclusions

As emphasized in the introduction, in this paper we have analyzed the field theoretical mechanism that leads to the affine extension of the duality algebra in $D = 2$. We have shown that there is a uniform pattern underlying such mechanism and that this is based on the coexistence of two non locally related dimensional reduction schemes, the Ehlers and the Matzner–Misner scheme, respectively. In this way we have extended an original idea that Nicolai had applied to pure gravity to the more general setup of all $D = 4$ supergravities. In particular we have stressed the structural differences that arise in extended or matter coupled supergravity with respect to pure gravity and which are related to the presence of vector fields as well. This leads to the general form of the Matzner–Misner lagrangian which is different from the Ehlers one and which to our knowledge had not been discussed so far in the literature.
The main motivation for our study is provided by the issue of cosmic billiards. Indeed we plan to use our results in order to discuss how the compensator method to generate solutions of the first order equations which we introduced in [15] and which we recently rediscussed in the context of the Tits Satake projection for non maximally split cosets [22] can be extended to the infinite dimensional compact subalgebras of Kač–Moody algebras.

Alternatively the field theoretical understanding of the affine extensions is relevant in studying wave–solutions of supergravity, as already emphasized by Nicolai [5] and other authors. In this context special attention is to be devoted to the pp-waves and to the Penrose limit. Indeed, as it will be pointed out in a forthcoming paper [26], the Penrose limit in supergravity models can be thoroughly understood within the framework of Lie algebra contractions and the relation between the isometry Lie algebra of a wave solution and the duality (affine) Lie algebra that generates the solution itself is a quite challenging conceptual issue, potentially very important in the quest for a deeper understanding of string theory and brane physics. It should also be stressed that while all purely time dependent solutions (in particular cosmic billiards) break all supersymmetries, wave–like solutions as the pp–waves can preserve several SUSY charges (for a comprehensive review of the vast recent literature on supergravity pp-wave solutions, see for instance [27]). So an obvious research line streaming from our present results is the systematic investigation of wave solutions dynamically generated by the affine Kač–Moody extension of the duality algebra, along the lines already pioneered in pure gravity by [2] and [5], and their classification according to Killing spinors.

Another direction which is to be pursued is the systematic analysis of the double, or hyperbolic, extension of the duality algebra occurring in one–dimension.

It is easy to anticipate that the field theoretical mechanism underlying this is the coexistence of two dimensional reduction schemes, the Ehlers, the analog of the Matzner–Misner one, in which we step directly down from $D = 4$ to $D = 1$, by compactifying on a $T^3$–torus. In this way we obtain a rank two σ–model $\text{SL}(3, \mathbb{R})/\text{O}(3)$ which describes the degrees of freedom of pure gravity. This is also an obvious research line which we plan to carry out in the immediate future.

Let us finally mention that the analysis of cosmic billiards was so far given only in the context of ungauged supergravities. The extension to gauged supergravities and hence to flux compactifications (see, for instance, [28]) is clearly overdue and is in agenda. Here we know that the crucial item governing the classification of gaugings is the so named embedding matrix, originally introduced in [29] and later shown in [30] to be algebraically described by suitable irreducible representations of the duality algebras in $D = 4$ and $D = 3$. It goes without saying that the affine and hyperbolic extension of such an analysis is also a must.

References

[1] R. Geroch, A method for generating solutions of Einstein’s equations, J. Math. Phys. 12 (1971) 918-924; R. Geroch, A method for generating new solutions of Einstein’s equations. II, J. Math. Phys. 13 (1972) 394-404.

[2] P. Breitenlohner and D. Maison, On the Geroch group, Ann. Inst. H. Poincare. Phys. Theor. 46 (1987) 215.
[3] P. Goddard and D. Olive, *Kac-Moody and Virasoro algebras in relation to quantum physics*, Int. J. Mod. Phys. A 1 (1986) 303. Univ. Press (1956).

[4] V.G. Kac, *Infinite dimensional Lie algebras*, 3rd edition. Cambridge University Press 1990.

[5] H. Nicolai, *A Hyperbolic Lie algebra from supergravity*, Phys. Lett. B 276, 333 (1992); H. Nicolai, A. Nagar, *Infinite-dimensional symmetries in gravity*, in "Gravitational Waves", Chapter 14, eds. I. Ciufolini, V. Gorini, U. Moschella and P. Frè, (publisher IOP), (2001).

[6] E. Cremmer and B. Julia, ”The N=8 Supergravity Theory. 1. The Lagrangian,” Phys. Lett. B 80 (1978) 48; E. Cremmer and B. Julia, ”The SO(8) Supergravity,” Nucl. Phys. B 159 (1979) 141; E. Cremmer *Dimensional reduction in field theory and hidden symmetries in extended supergravity*, Lectures given at ICTP Spring School Supergravity, Trieste (1981), Published in Trieste Supergrav.School 1981:0313.

[7] B. Julia, *Infinite Lie Algebras In Physics*, LPTENS-81-14-Invited talk given at Johns Hopkins Workshop on Current Problems in Particle Theory, Baltimore, May 25-27, 1981; B. Julia, *Infinite Lie Algebras in Physics*, invited talk presented at the Johns Hopkins Workshop on Particle Theory (1981); *Application of Supergravity to Gravitation Theory*, based on lectures given at the International School of Cosmology and Gravitation held at Erice (1982); E. Cremmer, B. Julia, H. Lu, C.N. Pope, *Dualisation of Dualities, I*, Nucl.Phys. B523 (1998) 73-144, hep-th/9710119; E. Cremmer, B. Julia, H. Lu, C.N. Pope, *Dualisation of Dualities, II: Twisted self-duality of doubled fields and superdualities*, Nucl.Phys. B535 (1998) 242-292, hep-th/9806106.

[8] H. Nicolai, *Two-dimensional Gravities and Supergravities as Integrable System*, in "Schladming 1991, Proceedings, Recent Aspects of Quantum Fields", eds. H. Mitter and H. Gausterer, Springer (1991) 231; B. Julia, H. Nicolai, *Conformal internal symmetry of 2d σ-models coupled to gravity and a dilaton*, Nucl.Phys. B482 (1996) 431-465, hep-th/9608082; H. Nicolai, H. Samtleben, *Integrability and Canonical Structure of d=2, N=16 Supergravity*, Nucl.Phys. B533 (1998) 210-242, hep-th/9804152.

[9] T. Damour, M. Henneaux, B. Julia, H. Nicolai, *Hyperbolic Kac-Moody Algebras and Chaos in Kaluza-Klein Models*, Phys.Lett. B509 (2001) 323-33, hep-th/0103094.

[10] T. Damour, S. de Buyl, M. Henneaux, C. Schomblond, *Einstein billiards and overextensions of finite-dimensional simple Lie algebras*, JHEP 0208 (2002) 030, hep-th/0206125; T. Damour, M. Henneaux, H. Nicolai, *Cosmological Billiards*, Class.Quant.Grav. 20 (2003) R145-R200, hep-th/0212256.

[11] S. de Buyl, M. Henneaux, B. Julia, L. Paulot, *Cosmological billiards and oxidation*, Fortsch.Phys. 52 (2004) 548-554, hep-th/0312251; J. Brown, O. J. Ganor, C. Helfgott, *M-theory and E10: Billiards, Branes, and Imaginary Roots*, JHEP 0408 (2004) 063, hep-th/0401053; F. Englert, M. Henneaux, L. Houart, *From very-extended to overextended gravity and M-theories*, JHEP 0502 (2005) 070, hep-th/0412184; T. Damour, *Cosmological Singularities, Einstein Billiards and Lorentzian Kac-Moody Algebras*, invited talk at
[12] M. Henneaux, B. Julia, Hyperbolic billiards of pure $D=4$ supergravities, JHEP 0305 (2003) 047, hep-th/0304233

[13] V.A. Belinsky, I.M. Khalatnikov, E.M. Lifshitz, Oscillatory approach to a singular point in the relativistic cosmology, Adv.Phys.19:525-573 (1970); V.A. Belinsky, I.M. Khalatnikov, E.M. Lifshitz, A general solution of the Einstein equations with a time singularity,Adv.Phys.31:639-667, (1982); J. K. Erickson, D. H. Wesley, P. J. Steinhardt, N. Turok, Kasner and Mixmaster behavior in universes with equation of state $w \geq 1$, Phys.Rev. D69 (2004) 063514, hep-th/0312009.

[14] J. Demaret, M. Henneaux, P. Spindel Nonoscillatory behavior in vacuum Kaluza-Klein cosmologies, Phys.Lett.B164:27-30 (1985); J. Demaret, J.L. Hanquin, M. Henneaux, P. Spindel, A. Taormina, The fate of the mixmaster behavior in vacuum inhomogeneous Kaluza-Klein cosmological models, Phys.Lett.B175:129-132 (1986); J. Demaret, Y. De Rop, M. Henneaux, Chaos in nondiagonal spatially homogeneous cosmological models in space-time dimensions $j=10$, Phys.Lett.B211:37-41 (1988).

[15] P. Frè, V. Gili, F. Gargiulo, A. Sorin, K. Rulik, M. Trigiante, Cosmological backgrounds of superstring theory and Solvable Algebras: Oxidation and Branes, Nucl.Phys. B685 (2004) 3-64, hep-th/0309237; P. Frè, K. Rulik, M. Trigiante, Exact solutions for Bianchi type cosmological metrics, Weyl orbits of $E_{8(8)}$ subalgebras and p–branes, Nucl.Phys. B694 (2004) 239-274, hep-th/0312189

[16] L. Andrianopoli, R. D’Auria, S. Ferrara, P. Frè, M. Trigiante, $R–R$ Scalars, $U$–Duality and Solvable Lie Algebras, Nucl.Phys. B496 (1997) 617-629, hep-th/9611014; L. Andrianopoli, R. D’Auria, S. Ferrara, P. Frè, R. Minasian, M. Trigiante, Solvable Lie Algebras in Type IIA, Type IIB and M Theories, Nucl.Phys. B493 (1997) 249-280, hep-th/9612202

[17] L. Andrianopoli, R. D’Auria, S. Ferrara, P. Frè and M. Trigiante, $E(7)(7)$ duality, BPS black hole evolution and fixed scalars, Nucl.Phys.B509:463-518,(1998), hep-th/9707087

[18] P. Frè, U Duality, Solvable Lie Algebras and Extremal Black-Holes, Talk given at the III National Meeting of the Italian Society for General Relativity (SIGRAV) on the occasion of Prof. Bruno Bertotti’s 65th birthday. Rome September 1996. hep-th/9702167; P. Frè, Solvable Lie Algebras, BPS Black Holes and Supergravity Gaugings, Fortsch.Phys. 47 (1999) 173-181, hep-th/9802045, G. Arcioni, A. Ceresole, F. C��rdo, R. D’Auria, P. Frè, L. Gualtieri, M. Trigiante, N=8 BPS Black Holes with 1/2 or 1/4 Supersymmetry and Solvable Lie Algebra Decompositions, Nucl.Phys. B542 (1999) 273-307, hep-th/9807136; M. Bertolini, P. Frè, M. Trigiante, N=8 BPS black holes preserving 1/8 supersymmetry, Class.Quant.Grav. 16 (1999) 1519-1543, hep-th/9811251; M. Bertolini, P. Frè, M. Trigiante, The generating solution of regular $N=8$ BPS black holes, Class.Quant.Grav. 16 (1999) 2987-3004, hep-th/9905143.
[19] L. Andrianopoli, F. Cordaro, P. Fré, L. Gualtieri, Non-Semisimple Gaugings of $D=5$ $N=8$ Supergravity and FDA.s, Class.Quant.Grav. 18 (2001) 395-414, hep-th/0009048. L. Andrianopoli, F. Cordaro, P. Fré, L. Gualtieri, Non-Semisimple Gaugings of $D=5$ $N=8$ Supergravity, Fortsch.Phys. 49 (2001) 511-518, hep-th/0012203.

[20] P. Fré Gaugings and other supergravity tools for p-brane physics, Lectures given at the RTN School Recent Advances in M-theory, Paris February 1-8 IHP, hep-th/0102114.

[21] B. de Wit, A. Van Proeyen, Special geometry, cubic polynomials and homogeneous quaternionic spaces, Commun.Math.Phys.149:307-334 (1992), hep-th/9112027; B. de Wit, A. Van Proeyen, Broken sigma model isometries in very special geometry, Phys.Lett.B293:94-99 (1992), hep-th/9207091; B. de Wit, F. Vanderseypen, A. Van Proeyen, Symmetry structure of special geometries, Nucl.Phys. B400 (1993) 463-524, hep-th/9210068; B. de Wit, A. Van Proeyen, Hidden symmetries, special geometry and quaternionic manifolds, Int.J.Mod.Phys. D3 (1994) 31-48, hep-th/9310067; S. Ferrara, S. Sabharwal, Quaternionic manifolds for type II superstring vacua of Calabi-Yau spaces, Nucl.Phys.B332:317 (1990).

[22] P. Fré, F. Gargiulo, K. Rulik, Cosmic Billiards with Painted Walls in Non Maximal Supergravities: A worked out example, arXiv:hep-th/0507256.

[23] R. D’Auria and P. Fre, BPS black holes in supergravity: Duality groups, p-branes, central charges and the entropy, arXiv:hep-th/9812160.

[24] D. Kramer, G. Neugebauer, Commun. Math. Phys. 10 (1968), 133

[25] L. Castellani, A. Ceresole, S. Ferrara, R. D’Auria, P. Fre and E. Maina, The Complete $N=3$ Matter Coupled Supergravity, Nucl. Phys. B 268 (1986) 317; L. Castellani, A. Ceresole, R. D’Auria, S. Ferrara, P. Fre and E. Maina, Sigma Models, Duality Transformations And Scalar Potentials In Extended Supergravities, Phys. Lett. B 161 (1985) 91.

[26] A. Campoleoni, P. Fré, to appear.

[27] M. Blau, J. Figueroa-O’Farrill, C. Hull and G. Papadopoulos, A new maximally supersymmetric background of IIB superstring theory JHEP 0201 (2002) 047 arXiv:hep-th/0110242; M. Blau, J. Figueroa-O’Farrill, C. Hull and G. Papadopoulos, Penrose limits and maximal supersymmetry, Class. Quant. Grav. 19 (2002) L87 arXiv:hep-th/0201081; M. Blau, J. Figueroa-O’Farrill and G. Papadopoulos, Penrose limits, supergravity and brane dynamics, Class. Quant. Grav. 19 (2002) 4753 arXiv:hep-th/0202111.

[28] For an exhaustive review on flux compactifications see A. R. Frey, Warped strings: Self-dual flux and contemporary compactifications, hep-th/0308156; S. B. Giddings, S. Kachru and J. Polchinski, Hierarchies from fluxes in string compactifications, Phys. Rev. D 66, 106006 (2002); A. R. Frey and J. Polchinski, Phys. Rev. D65, 126009 (2002); S. Kachru, M. Schulz and S. Trivedi, JHEP 0310, 007 (2003); P. K. Tripathy and S. P. Trivedi, JHEP 0303, 028 (2003); R. Blumenhagen, D. Lust and T. R. Taylor, Moduli stabilization in chiral type IIB orientifold models with fluxes, Nucl. Phys. B 663, 319 (2003).
M. Berg, M. Haack and B. Kors, *Nucl. Phys.* **B669**, 3 (2003); T. R. Taylor and C. Vafa, *Phys. Lett.* **B474**, 130 (2000); A. Giryavets, S. Kachru, P. K. Tripathy and S. P. Trivedi, *Flux compactifications on Calabi-Yau threefolds*, JHEP **0404**, 003 (2004) [hep-th/0312104]; M. Grana, T. W. Grimm, H. Jockers and J. Louis, *Soft supersymmetry breaking in Calabi-Yau orientifolds with D-branes and fluxes*, Nucl. Phys. B **690**, 21 (2004) [hep-th/0312232]; T. W. Grimm and J. Louis, *The effective action of N = 1 Calabi-Yau orientifolds*, [hep-th/0403067]; D. Lust, S. Reffert and S. Stieberger, *Flux-induced soft supersymmetry breaking in chiral type IIB orientifolds with D3/D7-branes*, [hep-th/0406092]; D. Lust, P. Mayr, S. Reffert and S. Stieberger, *F-theory flux, destabilization of orientifolds and soft terms on D7-branes*, [hep-th/0501139].

[29] F. Cordaro, P. Frè, L. Gualtieri, P. Termonia, M. Trigiante, *N=8 gaugings revisited: an exhaustive classification*, Nucl. Phys. B **532** (1998) 245-279, [hep-th/9804056].

[30] H. Nicolai and H. Samtleben, *Maximal gauged supergravity in three dimensions*, Phys. Rev. Lett. **86** (2001) 1686 [arXiv:hep-th/0010076]; B. de Wit, H. Samtleben and M. Trigiante, *On Lagrangians and gaugings of maximal supergravities*, Nucl. Phys. B **655** (2003) 93 [arXiv:hep-th/0212239]; B. de Wit, H. Samtleben and M. Trigiante, *The maximal D = 5 supergravities*, Nucl. Phys. B **716** (2005) 215 [arXiv:hep-th/0412173].