Nontrivial low-frequency topological waves at the boundary of a magnetized plasma

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The topological properties of a magnetized cold gaseous plasma have recently been explored and the existence of topologically protected edge states has been established. These studies are limited to a magnetized plasma, where ions are infinitely massive and provide a neutralizing background. When ion motion is included, a new class of low-frequency unidirectional topological waves emerges in the dispersion relation. The group velocity of these waves is in the opposite direction of high-frequency topological electron waves for a given magnetic field direction. The Berry curvature and Chern numbers are calculated to establish nontrivial topological phase. These finding broadens the possible applications and observations of these exotic excitations in space and laboratory plasmas.

Introduction.- Topological edge states have attracted wide attention due to their unidirectional nature and robustness against perturbation. These modes are found within common bulk-band gaps and characterized by an integer invariant, known as the Chern number. The Chern number follows the bulk-boundary correspondence principle which states that a gapless surface mode exists within a common band gap at the interface of two topologically distinct materials and allows to determine the number of unidirectional edge states from the topological properties of bulk modes. This principle has originated in condensed matter physics and later extended to photonics, cold atomic gases, electron gas and classical fluids.

Recently, nontrivial topology of high-frequency edge states (surface plasmon polariton; SPP’s), occurring at the boundary of a cold magnetized plasma, have been established for gyrotropic plasmonic materials and continuum plasma fluids. It has been shown that for certain choices of plasma density, stationary external magnetic field and parallel wavenumber, SPP’s can exist at the plasma-vacuum or plasma-plasma interface, provided both mediums have different topological phase.

Study of nontrivial topology of low-frequency edge states is still in infancy. In plasmonic materials, the low-frequency SPPs ceases to exist due to typical Drude behavior and resulting dissipation. One can however make an artificial material out of very thin metallic wires, which supports low-frequency SPPs and removes dissipation. In continuum fluids, like a plasma, low-frequency electrostatic and electromagnetic surface waves arises naturally due to ion motion and exist well below plasma frequencies. For a cold magnetized plasma, low-frequency electromagnetic waves have been studied, however, nontrivial topology of low-frequency waves has not been discussed. In a recent work, the nontrivial topology of a magnetized fusion plasma in the Alfven continuum has been discussed in the presence of magnetic shear and found unidirectional reversed-shear Alfven eigenmodes. These low-frequency modes are non-dissipative and exist at the neutral layer of magnetic shear.

In this letter, without going into much complexity, we use a simplified cold magnetized plasma slab model to observe the effect of ion motion on nontrivial topological edge states, occurring at the plasma-vacuum interface, and calculate Chern numbers of energy bands for mobile ions. We discuss the limitation on parallel wavenumber for a chosen value of ion mass, numerically obtain the dispersion relation of low-frequency topological edge states, demonstrate their existence and topological protection by exciting an edge state enroute to a rectangular discontinuity using a full 3D particle-in-cell simulation code.

We consider the cold-plasma slab model of a magnetized, stationary plasma, where electron and ion collisions are neglected. The external magnetic field is uniform and is kept in \( \hat{z} \) direction. The linearized set of governing equations for an infinite homogeneous plasma is

\[
\begin{align*}
\frac{\partial \mathbf{v}_e}{\partial t} &= -\omega_{pe}\mathbf{E} - \Omega_i \mathbf{v}_e \times \hat{z}, \\
\frac{\partial \mathbf{v}_i}{\partial t} &= \omega_{pi}\mathbf{E} + \Omega_i \mathbf{v}_i \times \hat{z}, \\
\frac{\partial \mathbf{E}}{\partial t} &= \nabla \times \mathbf{B} + \omega_{pe} \mathbf{v}_e - \omega_{pi} \mathbf{v}_i, \\
\frac{\partial \mathbf{B}}{\partial t} &= -\nabla \times \mathbf{E},
\end{align*}
\]

where \( \mathbf{v}_e \) and \( \mathbf{v}_i \) are the electron and ion fluid velocities, \( \mathbf{E} \) the electric field, \( \mathbf{B}_0 \) the external stationary background magnetic field, \( \mathbf{B} \) the perturbation magnetic field, \( |e| \) the charge on an electron, \( m \) and \( M \) are the mass of electron and ion, \( \alpha = m/M \), \( c \) the speed of light, \( \omega_{pe} = \sqrt{4\pi n_e e^2/m} \) the electron plasma frequency, \( \omega_{pi} = \sqrt{4\pi n_i e^2/M} \) the ion plasma frequency, \( \Omega_i = eB_0/M \) the ion cyclotron frequency and \( \Omega_e = eB_0/m \) the electron cyclotron frequency. We have used renormalized velocities \( \tilde{\mathbf{v}}_e = \mathbf{v}_e/\omega_{pe} \) and \( \tilde{\mathbf{v}}_i = \mathbf{v}_i/\omega_{pi} \), reference electric field \( E_0 \), and reference frequency \( \omega_{pe} \).

The time is normalized to \( \omega_{pe}^{-1} \), frequency to \( \omega_{pe} \), length to \( k_0^{-1} = c/\omega_{pe} \), electric field to \( E_0 \) and magnetic field to \( E_0/c \), electron and ion velocities to \( eE_0/m \) and \( eE_0/M \), respectively.
and to ensure that Chern numbers are integers, we use the same regularization strategy as discussed in \[3, 17, 18\]. We choose \( \omega_{ps} = \omega_{ps}/(1 + k_4^2/k_z^2) \), where \( k_z \) is a very big cutoff wavenumber, and subscript \( s = e, i \) stands for species electron and ion. The positive frequency bands spectrum \((\omega, k_y)\) for parameters \( \Omega_e/\omega_{p0} = 0.5 \), \( \Omega_i/\omega_{p0} = 0.1 \), \( M/m = 5 \) and two different value of parallel wavenumbers \( k_z/\kappa = 0.4, 1 \) are shown in Fig. 4 (a,b), respectively. We have also shown band Chern numbers obtained using \( k_z = 100 \). The Chern number for each positive frequency band is \((C_1, C_2, C_3, C_4, C_5) = (1, -2, 2, 0, 1)\) for parameters \( k_z/\kappa = 0.4, \Omega_e/\omega_{p0} = 0.5, M/m = 5 \) and \((C_1, C_2, C_3, C_4, C_5) = (1, -2, 1, 1, -1)\) for parameters \( k_z/\kappa = 1.0, \Omega_e/\omega_{p0} = 0.5, M/m = 5 \). The gap Chern number for a low frequency band gap is \( C_{1,2} = 1 \) and for a high frequency band gap is \( C_{2,3} = -1 \). Thus multiple bands are topologically non-trivial and can support multiple topological gapless edge states. Note that, as discussed in \[17\], bands 3 and 4 touch at \( k_z^* = \Omega_e/\sqrt{1 + \Omega_e} \). Since band gap closes at \( k_z^* \), we observe different Chern numbers for bands 3 and 4 for different mass of plasma frequencies \( M/m = 5 \) and found that ion mass does not affect the Chern number of any band. Surprisingly, the Chern number for band 1 \((C_1 = 1)\) agrees with the Chern number obtained by earlier study \[16\]. For infinitely massive ions, the Chern numbers for bands \( \omega_{s1} \) become zero (trivial) and for bands \( \omega_{s2} \) the Chern number becomes \( \mp 1 \), respectively, which is exactly same result earlier studies obtained for single mobile specie plasma \[17, 20\].

Bulk-edge correspondence states that an edge mode exists in the common band gaps at the interface between two topologically distinct materials with different gap Chern numbers. We have identified all existing topological nontrivial band gaps in a band spectrum. To prove existence of gapless edge states between band gaps, we use a numerical eigenvalue solver \[19\] to solve a set 1D differential equations \((1) - (4)\) in an inhomogeneous plasma by replacing \( x \)-coordinate dependent electron and ion plasma frequencies \( \omega_{ps} \rightarrow \omega_{pe}(x) \) and \( \omega_{pi} \rightarrow \omega_{pi}(x) \), respectively. The background magnetic field is uniform \((B = B_0 \hat{z})\) and constant in z-direction. The plasma density is nonuniform only in the \( x \)-direction, shown in Fig. 2 (c,d) by a black color line. We choose the density profile \( n(x) = n_0 (1 + tanh((x_0 - x)/\delta l)) \), where \( n_0 \) is a maximum density of plasma, \( x_0 \) the location of the interface and \( \delta l \) is width of the interface. We use artificial conducting boundary conditions at \( x = 0 \) and \( x = l \). The artificial conducting boundary condition yields a nonphysical mode which is uninteresting and not shown here.

It has been already established \[17, 19\] that for a mag-
FIG. 2. (a) The band spectrum $\omega(k_y)$ with two band gaps is shown for parameters $\Omega_e/\omega_{p0} = 0.5$, $k_z/k_p = 1.0$, $M/m = 5$, $k_p\delta l = 1$. The gray color lines show bulk modes. The high (electron dynamics dominated) and low frequency (ion dynamics dominated) topological edges states are shown by blue and red color lines, respectively. (b) Zoom in view of low-frequency modes. (c) The non-zero eigenvectors (left $y$-scale) for a high frequency mode at $k_y/k_p = -1$. (d) The non-zero eigenvectors (left $y$-scale) for a low frequency mode at $k_y/k_p = 1$. The density profile of plasma in $x$-direction is shown in (c,d) by a black color line (right $y$-scale).

FIG. 3. The band spectrum $\omega(k_y)$ for low frequency modes for the parameters $\Omega_e/\omega_{p0} = 0.5$, $k_z/k_p = 0.01$; $M/m = 1836$.

Netized plasma an edge state arises between bands 2 and 3, and can also be confirmed in our results as shown by the blue color line in Fig. 2(a). This high-frequency edge state is dominated by electron dynamics, hence we name it electron gaseous plasmon polariton (EGPP). EGPPs are localized near the center of the density profile which is confirmed by Fig. 2(c), where the centroid of non-zero components of electric field (eigenmodes), obtained using a numerical eigenvalue solver, is localized near the center of the density profile. Nonetheless, we have another band gap exist between band 1 and 2 and the gap Chern number for this gap is $\Delta_2 = (17,0)$ LCP mode and its dispersion relation is given by

$$k_z^{LCP} = \omega \sqrt{1 + \alpha + \frac{1}{(\Omega_i - \omega)(\Omega_e + \omega)}}.$$  

The second band gap remains open for every finite value of $k_z$ and closes only at $k_z = \infty$, however, in the limit $k_z \rightarrow \infty$, band gap becomes infinitesimally small. A finite band gap can be tuned by using equation (5) at an appropriate frequency value $\omega < \Omega_i$. It should be noted that higher the mass of an ion, lower would be the value of $k_z$ to open a finite band gap to obtain unidirectional frequency window. One needs to use a very small value of $k_z$ to open a finite band gap for higher massive ions. For example, to open a unidirectional frequency window of $\Delta\omega = 0.2\Omega_i$ for mass ratios $M/m = 5$ and $M/m = 1836$, one needs $k_z/k_p = 0.75$ and $k_z/k_p = 0.042$, respectively. The dispersion for a realistic mass ratio $M/m = 1836$ is shown in Fig. 3(a) which is plotted for parameters $k_z/k_p = 0.01, \Omega_e/\omega_{p0} = 0.5, M/m = 1836$ where red color line shows a gapless edge mode and gray color lines represent bulk modes. It is worth mentioning that for higher massive ions, edge states become almost like plane waves. The higher ion mass will also yield slower frequency waves.

3D particle-in-cell simulation: We demonstrate existence and verify their topological protection using a 3D particle-in-cell simulation code[33]. The plasma density profile is shown in Fig. 3(c), which varies linearly (ramp) from $n_0$ to 0 within $k_p\delta l = 1.0$ length. A $z$-direction periodic dipole source chain is kept at $(x, y) = (17, 0)k_p^{-1}$, which excites propagating edge
FIG. 4. The simulation is carried out for parameters $k_z/k_p = 1.0, \Omega_i/\omega_m = 0.5, M_i/m = 5, k_p\delta l = 1.0$. (a) A time snapshot of normalized intensity of electric field $E_z$ at $k_yz = \pi$. (b) The $x$–direction variation of normalized electric fields (left scale) $E_x$ (green line) and $E_z$ (violet color) at $k_y = 4\pi$. The density profile as a function $x$–coordinate (black line; left scale). To save computational time, we have considered incompressible plasma fluid, which does not give any nonlinear effects.

modes in $-yz (+yz)$ plane for positive (negative) external magnetic field ($B_0$). The simulation box size is $L_x \times L_y \times L_z = 30k_p^{-1} \times 50k_p^{-1} \times 6.28k_p^{-1}$. The spatial and temporal resolutions are $\Delta x = 0.02k_p^{-1}, \Delta y = 0.2k_p^{-1}, \Delta z = 0.25k_p^{-1}$ and $\Delta t = 0.8\Delta x$, respectively. We have used 4 particles per cell. The simulation box has periodic boundaries in the $z$–direction for both particles and electromagnetic fields, absorbing in $x$– and $y$–directions for electromagnetic fields and reflecting for particles. A rectangular discontinuity is kept at $k_p y = 12$.

The IGPP mode is excited for the wavenumber $k_y/k_p = 1$ as shown in Fig. 4(b) by a black star. The mode frequency and wavenumber $k_y$ obtained from the simulation results agrees with the theoretical dispersion relation. It is clear from Fig. 4(a,b) that edge states are unidirectional, robust and propagate without any defect and scattering around the discontinuity. The centroid of the edge state is localized near the vacuum as shown in Fig. 4(c), which agrees with the theory.

In conclusion, we have presented that, finite ion mass opens a second nontrivial low-frequency band gap. The Chern numbers are calculated for all bands. The Chern numbers for low-frequency bands are found to be independent of both parallel wavenumber $k_z$ and the mass of an ion. An edge state arises within the low-frequency band gap for a few nonzero values of $k_z$. The dispersion relation has been obtained using a numerical solution of governing equations for a magnetized two-fluid cold plasma.

The excited edge state frequency is smaller than ion cyclotron frequency and no naturally occurring resonance continua exist below ion cyclotron frequency, thus we do not expect any collisionless damping. However, a quasi-edge state between ion cyclotron frequency and band 2 can be obtained in the limit $\delta l \ll 1/k_p$, where a continuum of lower-hybrid frequency exist. The edge state would undergo collisionless damping via resonant interaction with lower-hybrid resonance and would become a quasi-edge state. It is also worthwhile to mention that, since quasi-edge states are confined between $\Omega_i$ and $\omega_2$, they can exist at arbitrary values of $k_z$.

On the separate note, we do not observe any Alfven resonance bands from our solver. The Alfven resonance may not exist in two-fluid description of a cold plasma and a detailed explanation can be found in [35] and its subsequent comments.

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