The exact solution of a three-body problem of Calogero type in one dimension by the SUSY-QM and SGA methods

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Abstract. Analytic expressions of the eigenvalues for a non-relativistic Hamiltonian of a three-body problem of the Calogero type can be derived using both the well known method of supersymmetric quantum mechanics SUSY-QM and an independent group theoretic method, namely the spectrum generating algebra SGA. We demonstrate the equivalence of the two methods by developing an algebraic framework through the shape invariance SI property with a change of parameters which involves nonlinear extensions of Lie Algebra.

1. Introduction
The research of exact solvable problems in quantum mechanics still attract attention as result of their importance both in physics and mathematics [2]. The origin of the exact solvability is linked to some of their intrinsic properties as hidden symmetries, which make all of them related to Lie algebras. It is is shown that they posses a spectrum generating algebra SGA, as it was demonstrated by numerous authors, starting with Pauli and Fock [7], [10]. Among these systems are the one-dimensional many-body problems. The Calogero model and some of its generalisations are the famous examples which have been studied through the early works of Calogero, Sutherland, Moser, Prelomov, Olshanetsky, Marchioro, Wolfes... [1], [4], [5] The attention for these systems is stemming from the introduction of two important concepts, namely supersymmetric quantum mechanics SUSY-QM (Witten, Copper, Freedman...) and shape invariance SI property (Gendanshtein) [2], [8]. In quantum mechanics, SUSY-QM allows to determine both the energy eigenvalues and eigenfunctions with more ease than for the case of operators methods. This takes place through an additional property called “shape invariance”. It is shown by the authors (Cooper and al), that this property is a sufficient condition, but not a necessary one for the exact solvability [2]. A list of one-dimensional shape invariant potentials is available in the literature [3]. The purpose of the present work is to show the equivalence between the SUSY-QM method and the SGA method to determine the eigenvalues of a non-relativistic quantum mechanics problem by using them with a three-body problem of Calogero type, depending on their connection to the shape invariant property. The paper is organized as follows: In Sect 2, we start with a brief review of the mathematical formalism of SUSY-QM. The Sect 3 is devoted to study the 3-body problem of Calogero type, and determining their eigenvalues using the SUSY-QM method. The mathematical formalism of SGA method...
is included in Sect 4. Section 5 is devoted to the determination of the eigenvalues of the same problem but using the SGA method. Conclusions are drawn in Sect 6.

2. The mathematical formalism of the SUSY-QM method

In the formalism of the SUSY-QM, the Hamiltonians of two systems are related to each other by an algebraic transformation of "supersymmetric" kind. Due to this symmetry, the spectrum of the two Hamiltonians are identical except for the ground state. [2]

The SUSY-QM generalizes the P.A.M. Dirac method to all exactly solvable problems of quantum mechanics. It consists of writing the Hamiltonian of a system as product of two first order differential operators $A^+$ and $A^-$, such that [2], [8]:

$$H_\pm = A^\pm A^\mp + E_0^\mp$$

Or: $H_\pm$ in the context of SUSY-QM, they are known as the partners Hamiltonians, and the $E_0^\mp$ are the ground states energies. where:

$$A^\pm = \pm \frac{d}{dx} + W(x, a)$$

$W(x,a)$ is known as the "superpotential", is a real function of $x$ and a parameter $a$

$A^+, A^-$ are hermitian conjugates to each other $A^- = (A^+)^+$

For the choice of $E_0^\mp = 0$ which corresponds to the unbroken SUSY case [2], [8], we can write:

$$H_\pm = -\frac{d}{dx} + V_\pm(x, a)$$

Where $V_\pm$: are "superpartners potentials", they are defined by:

$$V_\pm(x, a) = W^2(x, a) \pm \left( \frac{dW(x, a)}{dx} \right)$$

The partners Hamiltonians are semi-positive Hamiltonians i.e their eigenvalues are positives or nulls($E_n^\pm \geq 0$), and they are also characterized by the iso-spectrality property, which means that they have the same set of energy eigenvalues except possibly for the ground state [2], [8]:

$$E_n^+ = E_{n+1}^- \quad n = 0, 1, ...$$

$$E_0^- = 0$$

(5)

The ground state of the $H_-$ is related to the superpotential $W(x, a)$, as:

$$W(x, a) = -\frac{d[ln(\Psi_0^-)]}{dx}$$

(6)

The eigenfunctions of the $H_-$ and $H_+$ are connected by the operators $A^-$ and $A^+$, as [3]:

$$\Psi_{n+1}^-(x) = (E_n^-)^{-\frac{1}{2}} A^+ \Psi_n^+(x)$$

(7)

$$\Psi_n^+(x) = (E_{n+1}^-)^{-\frac{1}{2}} A^- \Psi_{n+1}^-(x)$$

(8)

**Shape Invariance:** If the superpartners potentials of the problem obeys a further constraint "shape invariance", then for either Hamiltonian we can derive all the eigenvalues and step-by-step construct all the eigen functions, i.e the problem become "Exactly Solvable". The superpotentials are said to be shape invariant potentials SIP if they have the same dependence in the $x$ variables and differ only in parameters, which be expressed by [2], [3], [8]:

$$V_+(x, a_0) = V_-(x, a_1) + R(a_1)$$

(9)
with: \( a_0, a_1 \) are sets of parameters related by the functional as:

\[
a_1 = f(a_0)
\]  

(10)

Or: \( R(a_1) \) independent of variable \( x \).

It has been shown that the eigenvalues of the Hamiltonian are given by:

\[
E_n^- = \sum_{k=1}^{n} R(a_k) \quad n = 1, 2, \ldots \quad E_{n=0}^- = 0
\]  

(11)

\[
a_k = f^k(a_0)
\]  

(12)

The eigenfunctions can also be given algebraically, by:

\[
\Psi_n^-(x, a_0) = N_0 A^+_n(x, a_0) + A_{n+1}(x, a_0) \ldots A^{n-1}(x, a_{n-1}) \Psi_n^-(x, a_n)
\]  

(13)

3. The Exact solution of a three-body problem of Calogero type in 1-dim by SUSY-QM

3.1. Position of the problem:

We consider the three-body of Calogero type in 1-dim Hamiltonian:

\[
H = \sum_{i=1}^{3} \left( -\frac{\partial^2}{\partial x_i^2} - \frac{\alpha}{(x_1^2 + x_2^2 + x_3^2)^{1/2}} \right) + \lambda \sum_{i<j}^{3} \frac{1}{(x_i - x_j)^2} + \mu \sum_{i=1}^{3} x_i^2
\]  

(14)

This Hamiltonian represents a system of three light particles on the line with the same mass (with unit \( \hbar = 2m = 1 \)) confined in a Coulombian field with coupling constant \( \alpha \) and interact pairwise by two-body inverse square potentials, of Calogero type i.e \( \frac{1}{(x_i - x_j)^2} \) [1], with an additional non-translationally invariant three-body potential of type \( \frac{1}{\sum_{i=1}^{3} x_i^2} \), associated with coupling constants \( \lambda, \mu \) respectively. In order to solve the Schrödinger equation corresponding to the above Hamiltonian separately, we need to make two changes of coordinate systems. [5], [6]

The 1st change is from the Cartesian coordinates \( (x_1, x_2, x_3) \) to the center of mass and Jacobi coordinates \( (R, u, v) \), according to:

\[
R = \frac{1}{3}(x_1 + x_2 + x_3) \quad ; u = \frac{1}{\sqrt{2}}(x_1 - x_2) \quad ; v = \frac{1}{\sqrt{6}}(x_1 + x_2 - 2x_3)
\]  

(15)

In this coordinates system, the potential is written as:

\[
V(R, u, v) = -\frac{\alpha}{(3R^2 + u^2 + v^2)^{1/2}} + \frac{9\lambda(u^2 + v^2)}{2(u^3 - 3uv^2)^2} + \frac{\mu}{3R^2 + u^2 + v^2}
\]  

(16)

Since it isn’t separable in \( (R, u, v) \) coordinates system, we need to make a 2nd change of system coordinate from the center of mass and Jacobi coordinates \( (R, u, v) \) to the spherical coordinates \( (r, \theta, \phi) \), according to:

\[
x_1 = r \sin \theta \cos \phi; \quad x_2 = r \sin \theta \sin \phi; \quad x_3 = r \cos \theta
\]  

(17)

\[
0 \leq r \leq \infty; \quad 0 \leq \theta \leq \pi; \quad 0 \leq \phi \leq 2\pi
\]  

(18)

The stationary Schrödinger equation corresponds to the above Hamiltonian, in the spherical coordinates:

\[
\left( -\frac{\partial^2}{\partial r^2} - \frac{2}{r} \frac{\partial}{\partial r} - \frac{1}{r^2} \left( \frac{\partial^2}{\partial \theta^2} + \cot \theta \frac{\partial}{\partial \theta} + \frac{1}{\sin^2 \theta} \frac{\partial^2}{\partial \phi^2} \right) + V(r, \theta, \phi) \right) \psi(r, \theta, \phi) = E \psi(r, \theta, \phi)
\]  

(19)
where \( V(r, \theta, \varphi) \) is the potential in \((r, \theta, \varphi)\) coordinates:

\[
V(r, \theta, \varphi) = -\frac{\alpha}{r} + \frac{\mu}{r^2} + \frac{1}{r^2 \sin^2 \theta} \left( -\frac{9\lambda}{2 \cos^2(3\varphi)} \right).
\]  

The three-body problem in 1-dim described by the Hamiltonian (14) is converted according to the above equation (19) to the problem of one-particle moving in 3-dim space, with a non central potential of the form:

\[
V(r, \theta, \varphi) = f_1(r) + \frac{f_2(\varphi)}{r^2 \sin^2 \theta}
\]  

It is clear that the problem becomes separable in the spherical coordinates system \((r, \theta, \phi)\). In order to solve the problem, we need to factorise the wave function as follows:

\[
\Psi(r, \theta, \varphi) = F(r) \Theta(\theta) \Phi(\varphi)
\]  

The substitution of (22) in (19) allows us to obtain three differential equations, each of them depending on a single variable:

\[
\left( -\frac{d^2}{d\varphi^2} + \frac{9\lambda}{2 \cos^2(3\varphi)} \right) \Phi(\varphi) = B \Phi(\varphi)
\]  

\[
\left( -\frac{d^2}{d\theta^2} + \frac{B - \frac{1}{4}}{\sin^2 \theta} \right) \Theta(\theta) = D \Theta(\theta)
\]  

\[
\left( -\frac{d^2}{dr^2} - \frac{\alpha}{r} + \frac{\mu + D - \frac{1}{4}}{r^2} \right) F(r) = E F(r)
\]

This problem has two types of solutions, the 1\textsuperscript{st} one corresponding \(E > 0\) "Diffusion state", "continued spectrum Energy", the 2\textsuperscript{nd} one corresponds \(E < 0\) "Binding state", "Discrete spectrum Energy". We interest only to the Binding state. Our aim is to use the mathematical formalism of SUSY-QM and SI property to determine the spectrum of each equation and therefore determine the eigenvalues of the Hamiltonian.

3.2. Calcul of the spectrum of the azimuthal angular equation

In the interval \(0 \leq \varphi \leq 2\pi\), the potential involved in equation (23) has a periodicity of \(\frac{\pi}{3}\), and has the singularities at \(\varphi = (2k+1)\frac{\pi}{6}\), \(k = 0, 1, \ldots, 5\). The equation is first solved in the interval \([-\frac{\pi}{6}, \frac{\pi}{6}]\). Assuming the following superpotential [3]:

\[
W(A, \varphi) = A \tan(3\varphi)
\]

The superpartners potential are calculated according to (4):

\[
V_{\pm}(A, \varphi) = \frac{A (A \pm 3)}{\cos(3\varphi)^2} - A^2
\]

They exhibit the shape invariance property (9) as:

\[
V_+(\varphi, A) - V_-(\varphi, A + 3) = (A + 3)^2 - A^2 = R(a_1)
\]
where: \( a_0 = A \) \( a_1 = A + 3 \)

So the spectrum of the superpartner potential \( V^- (\phi, A) \) is given according to (11), by:

\[
E_n^- = 3n(2A + 3n), \quad n = 0, 1, 2, \ldots \quad \text{Or: } E_{n=0}^- = 0.
\]  

(29)

The azimuthal angular potential expressed in (23) can be written using the superpartner potential as:

\[
V(\lambda, \varphi) = V^- (\phi, A) + A^2
\]  

(30)

with:

\[
\frac{9\lambda}{2} = A(A - 3)
\]  

(31)

So the spectrum of the azimuthal angular potential is given by:

\[
B = B_n = E_n^- + A^2 = 3n(2A + 3n) + A^2 = (3n + A)^2, \quad n = 0, 1, 2, \ldots
\]  

(32)

The equation (31) is a second order equation of the variable \( A \), and its solutions are:

\[
A_\pm = \frac{3}{2} \left( 1 \pm \sqrt{1 + 2\lambda} \right) = \frac{3}{2} \pm 3a,
\]  

(33)

\[
a = \frac{1}{2} \sqrt{1 + 2\lambda}, \quad \lambda > -\frac{1}{2}.
\]  

(34)

We take only the positive value of \( A \) in order to ensure the positivity of the \( E_n^- \).

As a result, we obtain the spectrum of the azimuthal angular differential equation:

\[
B_n = 9\left(n + \frac{1}{2} + a\right)^2, \quad n = 0, 1, 2, \ldots
\]  

(35)

3.3. Computation of the spectrum of the polar equation

The equation (24) will be solved using the Dirichlet boundary conditions in the interval \( 0 \leq \theta \leq \pi \).

We follow closely the previous steps, and take as the superpotential [3]:

\[
W(\theta, A) = A \cot \theta
\]  

(36)

According to (4), the superpartners potentials are given by:

\[
V_\pm (\theta, A) = \frac{A(A \mp 1)}{\sin^2 \theta} - A^2
\]  

(37)

They are obeying the shape invariance property (9), since:

\[
V_+ (\theta, A) - V_- (\theta, A - 1) = -A^2 - (A - 1)^2 = R(a_1)
\]  

(38)

where: \( a_0 = A \) \( a_1 = A - 1 \)

So, the spectrum of the superpartner potential \( V_- (\theta, A) \) are given according to (11), by:

\[
E_l^- = l(-2A + l), \quad l = 0, 1, 2, \ldots \quad \text{Or: } E_{l=0}^- = 0.
\]  

(39)

We use the superpartner potential to express the polar angular potential (24), as:

\[
V(b_n, \theta) = V_- (\theta, A) + A^2,
\]  

(40)
with:

$$B_n = A(A+1)$$  \( (41) \)

So, the spectrum of the polar angular potential is given by:

$$D = D_l = E_l^- + A^2 = (l-A)^2. \quad \quad (42)$$

The solution of the second order equation of \( A \) (41) gives:

$$A_{\pm} = \frac{1}{2} \left( -1 \pm 2\sqrt{B_n} \right) = -\frac{1}{2} \pm b_n \quad \quad (43)$$

$$b_n = \sqrt{B_n}, \; B_n > 0. \quad \quad (44)$$

We take only the negative value of the \( A \) to ensure the positivity of \( E_l^- \).

As result, we obtain the spectrum of the polar angular differential equation:

$$D_{n,l} = (l - A^-)^2 = (l + \frac{1}{2} + b_n)^2, \quad n = 0, 1, 2, ..., \quad l = 0, 1, 2, ... . \quad \quad (45)$$

$$b_n = \sqrt{B_n} = 3(n + \frac{1}{2} + a), \quad a = \frac{1}{2} \sqrt{1+2\lambda}, \; \lambda > -\frac{1}{2}. \quad \quad (46)$$

3.4. Computation of the spectrum of the radial equation

The equation (25) will be solved with the Dirichlet boundary conditions in the interval \( 0 \leq r < \infty \).

We follow the previous way and suppose the superpotential to be [3]:

$$W(r,A,C) = \frac{A^2}{2C} - \frac{C}{r}. \quad \quad (47)$$

using (4), the superpartners potentials are:

$$V_{\pm}(r,A,C) = -\frac{A}{r} + \frac{C(C \pm 1)}{r^2} + \frac{A^2}{4C^2} \quad \quad (48)$$

They are shape invariant potentials since we can write them as:

$$V_+(r,A,C) - V_-(r,A,C+1) = \frac{A^2}{4} \left( \frac{1}{C^2} - \frac{1}{(C+1)^2} \right) = R(a_1) \quad \quad (49)$$

where:

$$a_0 = C \quad \quad a_1 = C + 1$$

So, according to (11) the spectrum of the superpartner potential \( V_-(r,A,C) \) is given by:

$$E_k^- = \frac{A^2}{4} \left( \frac{1}{C^2} - \frac{1}{(C+k)^2} \right), \quad k = 0, 1, 2, ... \quad \text{Or:} \quad E_k^- = 0 = 0. \quad \quad (50)$$

In terms of the superpartner potential, the radial potential is written as:

$$V(r,\alpha,\mu+D_{n,l}) = V_-(r,A,C) - \frac{A^2}{4C^2} \quad \quad (51)$$

with:

$$A = \alpha \quad \quad (52)$$

$$C(C-1) = \mu + D_{n,l} - \frac{1}{4} \quad \quad (53)$$
So, the spectrum of the radial potential is given by:

\[ E = E_k^{(-)}(A, C) - \frac{A^2}{4C^2} = -\frac{A^2}{4(C + k)^2}, \quad k = 0, 1, 2, \ldots . \]  

(54)

Where the value of \( A \) is given by (52).

The second order equation of \( C \) (53) has two solutions:

\[ C_{\pm} = \frac{1}{2} \left( 1 \pm 2\sqrt{\mu + D_{n,l}} \right) = \frac{1}{2} \pm c \]  

(55)

\[ c = \sqrt{\mu + D_{n,l}} \]  

(56)

with: \( \mu + D_{n,l} > 0 \).  

(57)

The condition on the values of \( \mu \) (57), with (45) and (46) implies that:

\[ \mu + (l + 3n + 3a + 2)^2 > 0, \quad \forall n = 0, 1, 2, \ldots, \quad \forall l = 0, 1, 2, \ldots \]  

(58)

\[ \mu > -(l + 3n + 3a + 2)^2 \]  

(59)

It gets minimized for: \( n = l = a = 0 \), which is given:

\[ \mu > -4 \]  

(60)

We accept only the positive value of \( C \), to assure the positivity of the spectrum of \( E_k^{(-)} \).

As result, we obtain the spectrum of the radial differential equation which are the eigenvalues of the Hamiltonian of the three-body problem of Calogero type in 1-dim:

\[ E_{n,l,k} = -\frac{\alpha^2}{(1 + 2c + 2k)^2} \]  

(61)

\[ c = \sqrt{\mu + D_{n,l}}, \quad D_{n,l} = \left( l + \frac{1}{2} + b_n \right)^2, \]  

(62)

\[ b_n = 3(n + \frac{1}{2} + a), \quad a = \frac{1}{2} \sqrt{1 + 2\lambda}, \]

\[ \lambda > -\frac{1}{2}, \quad \mu > -4, \quad n = 0, 1, 2, \ldots, \quad l = 0, 1, 2, \ldots \quad k = 0, 1, 2, \ldots \]  

4. The mathematical formalism of the SGA method

The main idea of the SGA method is related to the existence of an algebra behind every shape invariant potential. This idea was very popular in the (1960-1970's), and numerous papers investigated different approaches and potentials [2], [8]. The mathematical formalism of this method consists of two steps [8]:

1\(^{st}\) step: Build an algebra of differential operators corresponding to any shape invariant system.

2\(^{nd}\) step: Using this algebra to obtain the energy spectrum of the Hamiltonian. In this paper, we follow the approach developed in [8], [10], who are constructing the operator algebra by expressing the shape invariance property of a system in terms of the operators \( A^+ \) and \( A^- \):

\[ A^+ (z, a_1) A^- (z, a_1) - A^- (z, a_0) A^+ (z, a_0) = R (a_0) \]  

(63)

with:

\[ a_1 = f (a_0) \]  

(64)
So, we can associate an algebra generated by:

\[ J_+ = e^{(i\phi)} A^+ (z, X (i\partial\phi)) \]
\[ J_- = A^- (z, X (i\partial\phi)) e^{(-i\phi)} \]
\[ J_3 = -\frac{i}{s} \partial\phi \]  

(65)

(66)

(67)

The operators \( A^+ (z, X (i\partial\phi)) \) and \( A^- (z, X (i\partial\phi)) \) are obtained from (2), with the replacement:

\[ a_0 \rightarrow X (i\partial\phi); \quad a_1 \rightarrow X (i\partial\phi) + s \]  

(68)

where \( X \) is an arbitrary real function must satisfy the compatibility equation:

\[ X (i\partial\phi) + s = f (X (i\partial\phi)) \]  

(69)

But the function \( f \) models the change of parameter \( a_1 \) (64), and \( s \) is an arbitrary real constant. \( \phi \) : is an arbitrary variable introduced to obtain a closed algebra.

As a consequence, we obtain a deformed Lie algebra whose generators \( J_3, J_- \) and \( J_+ \) are satisfying the commutation relations:

\[ [J_3, J_+] = J_+; \quad [J_3, J_-] = -J_-; \quad [J_+, J_-] = F (J_3) \]  

(70)

where: \( F (J_3) \) defines the deformation of the Lie algebra.

\[ F (J_3) = -R (X (-s J_3)) \]  

(71)

while \( R \) satisfy the shape invariance property of the system (63).

Depending on the choice of the \( X \) function, we have different reparametrizations corresponding to several models [8], [10] such as translation: \( a_1 = a_0 + s \) \ that \: \( X(z) = z \). Other changes of parameters follow from more complicated choices of the \( X(z) \) function.

Using (1), with the choice \( E_0 = 0 \) and considering the parameters changes (68), we get:

\[ H_- (z, i\partial\phi + s) = A^+ (z, X (i\partial\phi + s)) A^- (z, X (i\partial\phi + s)) = J_+ J_- \]  

(72)

To obtain the spectrum of the Hamiltonian \( H_- \), we should first construct the unitary representation of the deformed Lie algebra defined by (70), which is determined up to an additive function \( G(J_3) \), such that:

\[ F (J_3) = G (J_3) - G (J_3 - 1) \]  

(73)

But the Casimir operator of this algebra is:

\[ C = J_- J_+ + G (J_3) = J_+ J_- + G (J_3 - 1) \]  

(74)

The basis in which \( C \) and \( J_3 \) are diagonal, \( J_+ \), \( J_- \) plays the role of raising and lowering operators, respectively.

Operating on an arbitrary state |\( h \rangle \) as:

\[ J_3 |\( h \rangle \rangle = h |\( h \rangle \rangle \]  

(75)

\[ J_- |\( h \rangle \rangle = a (h) |\( h - 1 \rangle \rangle \]  

(76)

\[ J_+ |\( h \rangle \rangle = a^* (h + 1) |\( h + 1 \rangle \rangle \]  

(77)

\[ H_- |\( h \rangle \rangle = E (h) |\( h \rangle \rangle = J_+ J_- |\( h \rangle \rangle = |a(h)|^2 |\( h \rangle \rangle \]  

(78)
\( J_+ , J_- \) are adjoint to each other, and hence \((J_+ J_-)\) is a semi-positive definite, i.e its eigenvalues must be positive or zero, so:

\[
E = |a(h)|^2 \geq 0
\]  

From the commutator \([J_+, J_-] = F(J_3)\), and the expressions (75), (76), (77), (78), we find that:

\[
|a(h)|^2 - |a(h + 1)|^2 = G(h) - G(h - 1)
\]  

To determine the allowed values of \(h\) and their corresponding values of \(a(h)\), we start from the lowest state in the representation \(h_{\text{min}}\) or \((a(h_{\text{min}}) = 0)\), then increasing in integer steps still the \(h = h_{\text{min}} + n\), after detailed calculations and according to (70) and (80), (more details are shown in [8], [10]), we get a finite representation, choosing their coefficients \(a(h)\) to be real, given by:

\[
|a(h)|^2 = E_n = G(h - n - 1) - G(h - 1)
\]  

5. The exact solution of a three-body problem of Calogero type in 1-dim by SGA method

In the subsection (3.1), we recall that, the problem of the three-body problem of Calogero type in 1-dim is converted through the transformations of coordinate system, to the problem of one-particle in 3-dim space describing by three differential equations treating by SUSY-QM method in the subsections (3.2), (3.3) and (3.4). In this section, our aim is to solve these differential equations within the mathematical formalism of the SGA method.

5.1. Computation of the spectrum of the azimuthal angular equation

We start by changing the parameters included in the shape invariance property (28) of the superpartners potentials, to define the main objects of the corresponding algebra.

- Change of parameters: \(a_1 = A + 3 = f(a_0) = a_0 + 3\) "translational model"

\[
a_0 = A \rightarrow X(i\partial_\phi) = i\partial_\phi
\]  

\[
a_1 = A + 3 \rightarrow X(i\partial_\phi + s) = i\partial_\phi + 3
\]  

- Identify the \(X(i\partial_\phi + s)\) function: Using (69) with the translational model, we get:

\[
X(i\partial_\phi + s) = f(X(i\partial_\phi))
\]  

\[
X(i\partial_\phi + s) = X(i\partial_\phi) + 3
\]  

\[
i\partial_\phi + s = i\partial_\phi + 3
\]  

So:

\[
s = 3
\]  

- Identify the \(F(J_3)\) function: It is given in (71), where the shape invariance property of this problem is found in (28)

\[
F(J_3) = -R(-3J_3) = 18J_3 + 9
\]  

We have already shown that the azimuthal potential is written in terms of the superpartner potential in (30) according to the change of parameters (82), (83):

\[
V(\lambda, \varphi) = V_-(\varphi, i\partial_\phi) + (i\partial_\phi)^2
\]  

\(V_-(\varphi, i\partial_\phi)\) is a shape invariant superpartner potential generated by: \(W(i\partial_\phi, \varphi) = i\partial_\phi \tan(3\varphi)\).

So, we can associate an algebra generated by: \(J_\pm, J_3\)
• Defining \( J_\pm, J_3 \) operators: as prescribed by (65, 66, 67)

\[
J_+ = e^{(i\phi)} A^+ (\varphi, i\partial_\phi) \quad (87)
\]

\[
J_- = A^- (\varphi, i\partial_\phi) e^{-i\phi} \quad (88)
\]

\[
J_3 = -\frac{i}{3} \partial_\phi \quad (89)
\]

which satisfy the commutation relations (70), and is seen to be the SO (2, 1) algebra.

• Identifying the \( G(J_3) \) function: According to (73), we get:

\[
G(J_3) = 9 (J_3)^2 + 18 J_3 \quad (90)
\]

Then we use this function to identify the spectrum \( E_n^- \) of the superpartner potential, according to (81):

\[
E_n^- = 3n [3n + 2 (-3h)] \quad n = 0, 1, 2...
\]

\[
E_0^- = 0
\]

which are the same results as those found by SUSY-QM method in (29) if we put \(-3h = A\). According to (86), the spectrum of the azimuthal angular potential is given by:

\[
B = B_n = E_n^- + (i\partial_\phi)^2
\]

With:

\[
\frac{9\lambda}{2} = i\partial_\phi (i\partial_\phi - 3) \quad (93)
\]

We replace \( J_3 \) by their expression (89) in (75), we get:

\[
i\partial_\phi = -3h
\]

by substitution both in (92), and (93), we find:

\[
B = B_n = E_n^- + 9 (h)^2 = (3n - 3h)^2
\]

and:

\[
\frac{9\lambda}{2} = 3h (3h + 3)
\]

This second order equation of \( h \) gives:

\[
h^\pm = \frac{1}{2} \pm a \quad ; \quad a = \frac{\sqrt{1 + 2\lambda}}{2} \quad \lambda > -\frac{1}{2}
\]

We take only the negative value of \( h \) to ensure the positivity of \( E_n^- \).

As a result, the spectrum of the azimuthal angular potential is given by:

\[
B = B_n = \left[3n - 3 \left(\frac{1}{2} - a \right)\right]^2 = 9 \left(n + a + \frac{1}{2}\right)^2 \quad n = 0, 1, 2...
\]

which are here too exactly the same results as those found by the SUSY-QM method in (35).
5.2. Computation of the spectrum of the polar angular potential

We follow the similar steps of those of the precedent subsection:
The change of parameters are included in (38):
\[ a_1 = A - 1 = f(a_0) = a_0 - 1 \] "translational model"
and using (68) and (69) to identify the \( X(i\partial_{\phi} + s) \) function, we get:
\[ a_0 \rightarrow X(i\partial_{\phi}) = i\partial_{\phi} \] \[ a_1 \rightarrow X(i\partial_{\phi} + s) = i\partial_{\phi} - 1 \]
\[ s = -1 \]

• Identifying the \( F(J_3) \) function: It gives by (71) where the shape invariance property of this problem is obtained from (38)
\[ F(J_3) = -R(J_3) = 2J_3 + 1 \]
We have already shown that the polar potential is written in terms of the superpartner potential in (40), according to the change of parameters (99), (100):
\[ V (B_n, \theta) = V_-(\theta, i\partial_{\phi}) + (i\partial_{\phi})^2 \]
\( V_-(\theta, i\partial_{\phi}) \) is a shape invariant superpartner potential generated by:
\[ W(i\partial_{\phi}, \theta) = i\partial_{\phi} \cot \theta. \]
So, we can associate an algebra, generated by (65, 66 ,66):
\[ J_+ = e^{(-i\phi)} A^+(\theta, i\partial_{\phi}) \]
\[ J_- = A^-(\theta, i\partial_{\phi}) e^{(i\phi)} \]
\[ J_3 = i\partial_{\phi} \]
which satisfy the commutation relations (70), which is the \( \text{SO}(2, 1) \) algebra.

• Identifying the \( G(J_3) \) function: Depending on (73):
\[ G(J_3) = (J_3)^2 + 2J_3 \]
Then we use this function to identify the spectrum \( E_l^- \) of the superpartner potential following (81), we get:
\[ E_l^- = l(l - 2h) \quad l = 0, 1, 2... \]
\[ E_0^- = 0 \]
which are the same results as those found by the SUSY-QM method in (39) if we put: \( h = A \).
Then, according to (103), the spectrum of the polar angular potential, is given by:
\[ D = D_l = E_l^- + (i\partial_{\phi})^2 \]
With: \( B_n = i\partial_{\phi} (i\partial_{\phi} + 1) + \frac{1}{4} \)
We replace \( J_3 \) by its expression (106) in (75), we get:
\[ i\partial_{\phi} = h \]
By substitution both in (109) and in (110), we find.
\[ D = D_l = E_l^- + h^2 = (l - h)^2 \]
\[ and: B_n = h (h + 1) + \frac{1}{4} \]
We take only the negative value of \( h \) derived from the resolution of the second order equation of \( h \) (113) to ensure the positivity of \( E_l^- \).

\[
h_{\pm} = \frac{1}{2} \pm b_n \quad ; \quad b_n = \sqrt{B_n} \quad B_n > 0 \tag{114}\]

As a result, the spectrum of the polar angular potential is given by:

\[
D = D_l = \left( l + \frac{1}{2} + b_n \right)^2 \quad l = 0, 1, 2... \tag{115}\]

which are exactly the results obtained by SUSY-QM method, in the (45).

5.3. Computation of the spectrum of the radial equation

We repeat the same procedure followed in the two previous subsections.

Change of parameters included in (49):

\[
a_1 = C + 1 = f(a_0) = a_0 + 1 \quad "\text{translational model}"
\]

Using (68) and (69) to identify the \( X(i\partial_\phi + s) \) function, we get:

\[
a_0 = C \rightarrow X (i\partial_\phi) = i\partial_\phi \tag{116}
\]

\[
a_1 = C + 1 \rightarrow X (i\partial_\phi + s) = i\partial_\phi + 1 \tag{117}
\]

\[
s = 1 \tag{118}
\]

\( F(J_3) \) function identifies by (71), where the shape invariance property of this problem is gotten from (49)

\[
F(J_3) = \frac{A^2}{4} \left[ \frac{1}{(J_3)^2} - \frac{1}{(J_3 + 1)^2} \right] \tag{119}\]

According to the change of parameters (116),(117) and using (51), the radial potential is found:

\[
V(\alpha, \mu + D_{n,l}, r) = V_- (A, i\partial_\phi, r) - \frac{A^2}{4 (i\partial_\phi)^2} \tag{120}\]

\( V_- (A, i\partial_\phi, r) \) is a S.I superpartner potential generated by:\( W(A, i\partial_\phi, r) = \frac{A}{2 (i\partial_\phi)} - \frac{i\partial_\phi}{r} \). Thus, we can associate an algebra generated by (65, 66 ,67):

\[
J_+ = e^{i\phi} A^+ (r, A, i\partial_\phi) \tag{121}
\]

\[
J_- = A^- (r, A, i\partial_\phi) e^{-i\phi} \tag{122}
\]

\[
J_3 = -i\partial_\phi \tag{123}
\]

which satisfy the commutation relations (70), which is seen to be the SO (2, 1) algebra.

From (73), we get:

\[
G(J_3) = -\frac{A^2}{4} \left[ \frac{1}{(J_3 + 1)^2} \right] \tag{124}\]

Then we use this function to identify \( E^-_k \) of the superpartner potential, according to (81), we get:

\[
E^-_k = \frac{A^2}{4} \left[ \frac{1}{h^2} - \frac{1}{(h - k)^2} \right] \quad k = 0, 1, 2... \tag{125}\]

\[E^-_0 = 0\]
which are the same results as those found by the SUSY-QM method in the (50), if we put: $h = -C$.

Thus following (120), the spectrum of the radial potential is given by:

$$E = E_{k}^- - \frac{A^2}{4(i\partial_\phi)^2}$$

(126)

With:

$$A = \alpha$$

(127)

$$\mu + D_{n,l} - \frac{1}{4} = i\partial_\phi (i\partial_\phi - 1)$$

(128)

We replace $J_3$ by their expression (123) in (75), we get:

$$i\partial_\phi = -h$$

(129)

by substitution of both (126) and (128), we find:

$$E = E_{n,l,k} = E_{k}^- + \frac{A^2}{4(\mu + D_{n,l})^2} = -\frac{A^2}{4(k + h)^2}$$

(130)

and:

$$\mu + D_{n,l} - \frac{1}{4} = -h (-h - 1)$$

(131)

We take only the negative value of $h$ derived from resolution of (131) to ensure the positivity of $E_{k}^-$. 

$$h = -\frac{1}{2} \pm c ; \quad c = \sqrt{\mu + D_{n,l}} \quad \mu + D_{n,l} > 0$$

(132)

By substitution (124) in (127), we get the spectrum of the radial potential, which are the eigenvalues of the Hamiltonian of the three-body problem of Calogero type in 1-dim:

$$E_{n,l,k} = -\frac{\alpha^2}{4(k + \frac{1}{2} + c)^2} \quad l = 0, 1, 2...$$

(133)

which are exactly the results found by SUSY-QM method in the (61).

### 6. Conclusion

In this paper, we studied the connection of the exact solvability and the shape invariance property for a three-body Hamiltonian of the Calogero type, which describes three interacting light particles on the line confined in a Coulombien mean field. It was shown that the two angular and radial potentials derived from the 1-dim Hamiltonian of this system are each of them separately supersymmetric and shape invariant so that the eigenvalues could be obtained using the methods known in the literature in the context of SUSY-QM, where we restrict our study to only the binding state case. On the other hand, the shape invariant property allows the angular and radial potentials of the system to admit an algebraic structure i.e potential algebra.

Using the approach developed in [10], as well the fact that our potentials are translational type change of parameters, we have derived the algebra and therefore the spectrum of the angular and radial potentials of the system. According to the definition of the exact solvability in literature, the Hamiltonian is considered as exactly solvable if its spectrum can be calculated algebraically. The SUSY-QM and the SGA methods are equivalent for treating shape invariant potentials. However, the SUSY-QM method has an advantage compared to the SGA method, since it is able to give the eigenfunctions of the potentials in addition to the eigenvalues.
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