Plausibility measures on test spaces

Tobias Fritz
Perimeter Institute for Theoretical Physics, Waterloo ON, Canada
tfritz@perimeterinstitute.ca

Matthew S. Leifer
Perimeter Institute for Theoretical Physics, Waterloo ON, Canada
matt@mattleifer.info

Plausibility measures are structures for reasoning in the face of uncertainty that generalize probabilities, unifying them with weaker structures like possibility measures and comparative probability relations. So far, the theory of plausibility measures has only been developed for classical sample spaces. In this paper, we generalize the theory to test spaces, so that they can be applied to general operational theories, and to quantum theory in particular. Our main results are two theorems on when a plausibility measure agrees with a probability measure, i.e. when its comparative relations coincide with those of a probability measure. For strictly finite test spaces we obtain a precise analogue of the classical result that the Archimedean condition is necessary and sufficient for agreement between a plausibility and a probability measure. In the locally finite case, we prove a slightly weaker result that the Archimedean condition implies almost agreement.

1 Introduction

It is often stated that a physical theory must (at least) supply probabilities for the outcomes of experiments, but are probabilities strictly necessary, or can we sometimes get away with a weaker predictive structure? Recent years have seen a growth of interest in theories that are variously called possibilistic [12, 19], modal [24, 25, 26], or relational [1], in which the continuum of probability assignments is replaced by a two valued assessment of whether or not an outcome is possible.

If we are serious about using such weaker structures in fundamental physics, then we need to employ a theory that is capable of reproducing probabilistic predictions where they are known to work well, i.e. in most ordinary experiments in quantum theory and statistical mechanics, but allows for weaker predictions in general. In the classical case, the theory of plausibility measures [10, 11, 15, 14] can play this role. Both possibility and probability measures are examples of plausibility measures, but in general plausibility measures allow for comparative assessments, i.e. we may be able to say that A is more likely than B, without specifying a precise numerical probability for either. Possibility measures are obtained when there are only two plausibility values and, as we know from the foundations of subjective probability theory [7], a probability measure can be derived when the ordering relation over events is rich enough. Plausibility measures are therefore a good framework for unifying and generalizing the various predictive structures that could be applied to physical theories, but, so far, they have been limited to classical theories.

In this paper, we generalize the theory of plausibility measures to test spaces, which are capable of representing nonclassical theories such as quantum theory. The fundamental question we address is when a plausibility measure agrees with a probabilistic state on the test space, i.e. when the comparative relations are rich enough to be faithfully represented by probabilities. For finite test spaces, we prove a precise analogue of the classical result that the Archimedean condition is necessary and sufficient for agreement.
For locally finite test spaces, we prove that the Archimedean condition implies a weaker notion called almost agreement. This is an important first step in the program of generalizing physics beyond probabilities, as it enables us to identify the limits in which probability theory applies.

Before embarking on the study of more general predictive structures, it is worth mentioning at least one of the motivations. It has been argued that argued that, along with spacetime, probabilities will inevitably become fuzzy in future theories of quantum gravity \cite{2,3,20}. One proposal for implementing this is to discretize the state space of quantum systems so that, for example, a system with a finite dimensional Hilbert space, such as a spin-1/2 particle, would only be able to occupy a finite number of states \cite{2}. This violates the superposition principle and raises problems such as how the system decides which state to “snap to” when an observable is measured for which the eigenvectors are not allowed states.

However, from a test space perspective, the state space of a theory is derived from the attempt to consistently assign probability measures to the outcomes of measurements, such that outcomes of measurements that are physically identified are always assigned the same probability. For example, in quantum theory, Gleason’s theorem \cite{13} says that any such assignment can be represented by a density operator on Hilbert space. If we no longer have the continuum of probability assignments, then we no longer have the right to assert that the state space of a quantum system must be a Hilbert space, let alone a discrete collection of vectors within one. In our view, if we are interested in employing fuzzified probabilities in physics, it is better to start again from scratch with a well-defined discrete predictive structure, such as a possibility measure or a comparative plausibility measure, and let the theory tell us what the state space must be. This is likely to lead to a more consistent theory, without the need to introduce arbitrary rules for how systems are updated upon measurement. The present work can be viewed as an attempt to take the first few steps along this road.

The remainder of this paper is structured as follows. \S2 introduces the definitions of test spaces and plausibility measures, and gives examples. \S3 defines various notions of agreement between plausibility measures and probability measures and states the main results. \S4 develops a Hahn-Banach theorem for order unit spaces, which we then use in \S5 to prove the main results. \S6 discusses related work and \S7 concludes with a discussion of possible future directions.

2 Plausibility measures on test spaces

Intuitively speaking, a test space is a generalization of a measurable space, designed to allow for incompatible events that cannot be resolved in a single experiment. It has its origins in the work of Foulis and Randall \cite{9,23} (see \cite{29} for a review).

**Definition 2.1.** A test space \((X, \Sigma)\) consists of a set \(X\) together with a set of subsets \(\Sigma \subseteq 2^X\) such that the members of \(\Sigma\) cover \(X\), i.e. \(\bigcup_{T \in \Sigma} T = X\).

The elements of \(X\) are called outcomes, while the elements of \(\Sigma\) are the tests. Each test represents the set of possible outcomes of a measurement that can be performed on the system. A subset \(A \subseteq X\) is called an event if it is a subset of some test. The assumption that \(\Sigma\) covers \(X\) can be rephrased as stating that every singleton set \(\{x\}\) is an event. We write \(E(X, \Sigma)\) for the set of all events. Under subset inclusion, \(E(X, \Sigma)\) is a partially ordered set with \(\emptyset\) as the least element and all tests as maximal elements.

**Example 2.2.** For a classical system with a finite sample space \(X\), the test space is \((X, \{X\})\), i.e. there is only one test that contains all possible outcomes. \(X\) consists of all the possible configurations of the system and its elements can, in principle, be resolved by a single measurement. The events are \(E(X, \{X\}) = 2^X\), as they would be in classical probability theory with a finite sample space.
Example 2.3. For a quantum system with finite dimensional Hilbert space $\mathcal{H}$, the test space is $(P(\mathcal{H}), b(\mathcal{H}))$, where $P(\mathcal{H})$ is the projective Hilbert space of $\mathcal{H}$, or equivalently the set of unit vectors in $\mathcal{H}$ with vectors differing by a global phase identified, and $b(\mathcal{H})$ is the set of rank-1 projector valued measures, or equivalently the set of orthonormal bases of $\mathcal{H}$ with basis vectors identified if they differ by a global phase. Each test represents a measurement on the system that is as fine-grained as possible. Events correspond to projection operators that may be of higher rank.

Definition 2.4. A test space $(X, \Sigma)$ is locally finite if every test $T \in \Sigma$ is finite.

In what follows, all test spaces are assumed locally finite. This includes finite classical systems and quantum systems with finite dimensional Hilbert spaces. Note that the outcome set $X$ can be infinite in a locally finite test space if there are an infinite number of tests, as there are in the quantum case.

Definition 2.5. A probability measure on a test space $(X, \Sigma)$ is a function $\mu : X \to \mathbb{R}_{\geq 0}$ such that $\sum_{x \in T} \mu(x) = 1$ for every test $T \in \Sigma$.

A probability measure can be extended to a function $E(X, \Sigma) \to \mathbb{R}_{\geq 0}$, which we also denote $\mu$, via $\mu(A) = \sum_{x \in A} \mu(x)$.

Note that the same outcome may appear in more than one test in a test space, e.g. $\{\{x, y, z\}, \{\{x, y\}, \{y, z\}, \{z, x\}\}$). This means that, for whatever reason, the outcome $x$ is thought to have the same physical meaning regardless of whether it appears in test $\{x, y\}$ or $\{z, x\}$. Since $\mu$ is a function of $X$, the probability assigned to an outcome does not depend on which test is being measured. This feature is often called noncontextuality of probability assignments.

Example 2.6. A probability measure on a classical test space $(X, \{X\})$ with finite $X$ is just a probability measure on $(X, 2^X)$ in the sense of classical probability theory.

Example 2.7. By Gleason’s theorem [13], for a Hilbert space $\mathcal{H}$ with dim$(\mathcal{H}) \geq 3$, a probability measure on the quantum test space $(P(\mathcal{H}), b(\mathcal{H}))$ is of the form $\mu(\Pi) = \text{tr}(\Pi \rho)$ where $\Pi$ is a projection operator and $\rho$ is a density operator (semi-positive operator with tr$(\rho) = 1$).

Note that a probability measure is usually called a state in the test space literature since, in the quantum case, it corresponds to the usual notion of a quantum state. However, plausibility measures have equal claim to be thought of as physical states, so we do not use this terminology here.

Definition 2.8. A plausibility measure on a test space $(X, \Sigma)$ is a function $\text{Pl} : E(X, \Sigma) \to D$, where $(D, \preceq)$ is a partially ordered set of plausibility values, satisfying the following conditions:

(a) For all $T, R \in \Sigma$, $\text{Pl}(T) = \text{Pl}(R)$.

(b) If $A$ and $B$ are events with $A \subseteq B$, then $\text{Pl}(A) \preceq \text{Pl}(B)$.

(c) For any $T \in \Sigma$, $\text{Pl}(\emptyset) \prec \text{Pl}(T)$.

Here, we write “$\text{Pl}(\emptyset) \prec \text{Pl}(T)$” as shorthand for “$\text{Pl}(\emptyset) \preceq \text{Pl}(T)$ and $\text{Pl}(T) \not\preceq \text{Pl}(\emptyset)$”.

This definition is the natural generalization of a classical plausibility measure [10] [11] [15] to test spaces, and we recover the classical case for the test space $(X, \{X\})$. Axiom [c] is not always imposed, but it prevents the trivial case where every event has the same plausibility.

Under these axioms, the image of $\text{Pl}$ is a bounded poset with minimal element $\text{Pl}(\emptyset)$ and maximal element $\text{Pl}(T)$, where $T \in \Sigma$ is any test. Because of this, without loss of generality, we may assume that $D$ is a bounded poset, with minimal element 0 and maximal element 1, and demand that $\text{Pl}(\emptyset) = 0$ and $\text{Pl}(T) = 1$ for any test $T \in \Sigma$. Axiom [c] then becomes $0 \neq 1$.

Example 2.9. If $D = \{0, 1\}$ with $0 \prec 1$, then $\text{Pl}$ is a possibility measure. $0$ is interpreted as impossibility and 1 as possibility. Note that, given any plausibility measure $\text{Pl}$, we may derive a possibility measure $\text{Po}$ from it by setting $\text{Po}(A) = 0$ if $\text{Pl}(A) = 0$ and $\text{Po}(A) = 1$ otherwise.
Example 2.10. Any probability measure $\mu$ on a test space is also a plausibility measure, where the plausibility values are the unit interval $[0, 1]$, which is totally ordered.

Example 2.11. One way in which plausibility measures can arise is if we have a set $\{\mu_i\}$ of possible probability measures on a test space, but no prior probability distribution over them. One can then set $\text{Pl}(A) \leq \text{Pl}(B)$ iff $\mu_i(A) \leq \mu_i(B)$ for every measure in the set.

3 Agreement

We are interested in determining the conditions under which a plausibility measure can be faithfully represented by a probability measure. In the classical case, this question has been studied extensively [7], from which we adapt the following definitions.

Definition 3.1. A plausibility measure $\text{Pl}$ on a test space $(X, \Sigma)$ agrees with a probability measure $\mu$ if

$$\text{Pl}(A) \leq \text{Pl}(B) \iff \mu(A) \leq \mu(B).$$  \hfill (1)

$\text{Pl}$ almost agrees with $\mu$ if

$$\text{Pl}(A) \leq \text{Pl}(B) \implies \mu(A) \leq \mu(B) \quad \text{and} \quad \mu(x) + \mu(y) = 0 \implies \mu(x) = \mu(y) = 0.$$

Note that eq. (1) is equivalent to

$$\text{Pl}(A) < \text{Pl}(B) \iff \mu(A) < \mu(B).$$ \hfill (3)

Agreement implies that the image of $\text{Pl}$ is totally ordered. In contrast, almost agreement is quite weak as, not only does it not imply total ordering, but it also allows $\mu(A) = 0$ when $\text{Pl}(A) > 0$.

Example 3.2. Finding a $\mu$ that agrees with $\text{Pl}$ is not always possible. For instance, the test space $\{(x, y, z), \{(x, y), (y, z), (z, x)\}\}$ has a possibility measure defined by $\text{Pl}(x) = 0$, $\text{Pl}(y) = 1$ and $\text{Pl}(z) = 1$, with the assignments to other events entailed by axiom [8]. However, there is no probability measure with $\mu(x) = 0$ and both $\mu(y) > 0$ and $\mu(z) > 0$. This is because we must have $\mu(x) + \mu(y) = 1$ and $\mu(y) + \mu(z) = 1$, but these assignments would imply $\mu(x) + \mu(y) = 0 + 1 - \mu(z) < 1$.

Example 3.3. Even in the classical case, agreement is not always possible, as was shown by Kraft, Pratt and Seidenberg [18]. For the test space $\{(1, 2, 3, 4, 5), \{(1, 2, 3, 4, 5)\}\}$, consider the following relations between plausibility values

$$\text{Pl}(\{1, 3\}) < \text{Pl}(\{4\}), \quad \text{Pl}(\{1, 4\}) < \text{Pl}(\{2, 3\}), \quad \text{Pl}(\{3, 4\}) < \text{Pl}(\{1, 5\}), \quad \text{Pl}(\{2, 5\}) < \text{Pl}(\{1, 3, 4\}).$$

If there exists a probability measure $\mu$ that agrees with these assignments then, denoting $\mu(j)$ by $\mu_j$, because of additivity we must have

$$\mu_j + \mu_k < \mu_4, \quad \mu_j + \mu_4 < \mu_2 + \mu_3, \quad \mu_3 + \mu_4 < \mu_1 + \mu_5, \quad \mu_2 + \mu_5 < \mu_1 + \mu_3 + \mu_4,$$

but summing these inequalities and cancelling terms leads to $0 < 0$, so no such probability measure exists.
Clearly then, the existence of an agreeing probability measure requires additional constraints to be imposed on plausibility measures. One such condition is as follows. Let \((A_1, \ldots, A_n)\) and \((B_1, \ldots, B_n)\) be families of events such that every outcome in \(X\) occurs the same number of times in both, then, for a probability measure
\[
\mu(A_1) \leq \mu(B_1), \ldots, \mu(A_{n-1}) \leq \mu(B_{n-1}) \implies \mu(A_n) \geq \mu(B_n).
\]
Consequently, in order for a plausibility measure to agree with \(\mu\), it must satisfy the analogous condition:
\[
\Pi(A_1) \leq \Pi(B_1), \ldots, \Pi(A_{n-1}) \leq \Pi(B_{n-1}) \implies \Pi(A_n) \geq \Pi(B_n).
\]

Definition 3.4. A plausibility measure on a test space \((X, \Sigma)\) is Archimedean\(^1\) if it satisfies eq. \((12)\) for every pair \((A_1, \ldots, A_n)\) and \((B_1, \ldots, B_n)\) of families of events such that every outcome in \(X\) occurs the same number of times in both.

Example 3.5. The plausibility measure in Example 3.2 is non-Archimedean. Consider the families of events \((\{y, z\}, \{x, z\}, \{x\})\) and \((\{x, y\}, \{x, z\}, \{z\})\) in which \(x, y\) and \(z\) appear the same number of times. The possibility measure with \(\Pi(x) = 0, \Pi(y) = 1\) and \(\Pi(z) = 1\) satisfies
\[
\Pi(\{y, z\}) \leq \Pi(\{x, y\}) \quad \Pi(\{x, z\}) \leq \Pi(\{x, z\}),
\]
but \(\Pi(x) \prec \Pi(z)\).

Example 3.6. For Example 3.3, the families \((\{1, 3\}, \{1, 4\}, \{3, 4\}, \{2, 5\})\) and \((\{4\}, \{2, 3\}, \{1, 5\}, \{1, 3, 4\})\) fail the Archimedean condition.

In our terminology, a classic theorem in comparative probability [18][27][7] states Theorem 3.7. A plausibility measure \(\Pi\) on a finite classical test space \((X, \{X\})\) agrees with some probability measure \(\mu\) iff the image of \(\Pi\) is totally ordered and \(\Pi\) is Archimedean.

Our task is to generalize this to locally finite test spaces. This works perfectly for test spaces with finite \(X\), but so far we have only been able to prove a weaker result in the locally finite case.

Theorem 3.8. A plausibility measure \(\Pi\) on a locally finite test space \((X, \Sigma)\) almost agrees with some probability measure \(\mu\) if \(\Pi\) is Archimedean.

Theorem 3.9. A plausibility measure \(\Pi\) on a test space \((X, \Sigma)\) with \(X\) finite agrees with some probability measure \(\mu\) if the image of \(\Pi\) is totally ordered and \(\Pi\) is Archimedean.

Unfortunately, we do not know whether the stronger result still holds for general locally finite test spaces, but it seems likely that additional topological assumptions may be required.

Our proof strategy is to translate the problem into the language of order unit spaces over the rational field and make use of the associated Hahn-Banach theorems. Before proceeding, we therefore review some of this theory.

4 Hahn-Banach theorems for order unit spaces

Here we redevelop two standard Hahn-Banach theorems for order unit spaces (see e.g. [21]) with minor modifications. Let \(V\) be a vector space over an ordered field \(\mathbb{F}\). A subset \(C \subseteq V\) is a convex cone if it is closed under addition and positive scalar multiplication,
\[
a \in C, \ b \in C \implies a + b \in C \quad \quad \lambda \in \mathbb{F}_{\geq 0}, \ a \in C \implies \lambda a \in C.
\]
If the convex cone \(C\) is clear from the context, then we also write \(a \geq b\) as shorthand for \(a - b \in C\). It is straightforward to see that ‘\(\geq\)’ defines a partial ordering on \(V\) with \(a \geq 0\) if and only if \(a \in C\).

\(^1\)This is sometimes also called strong additivity in the literature.
Definition 4.1. An order unit space is a triple \((V, C, u)\) such that \(C \subseteq V\) is a convex cone and \(u \geq 0\) is a distinguished element called the order unit such that

(a) \(-u \not\geq 0,\)

(b) For any \(a \in V\), there is \(\lambda \in F\) such that \(\lambda u + a \geq 0.\)

Axiom \([b]\) states exactly that every \(a \in V\) can be lower bounded by a scalar multiple of \(u.\)

Definition 4.2. A probability measure \(\rho\) on an order unit space \((V, C, u)\) is a linear functional \(\rho : V \to \mathbb{R}\) with \(\rho(a) \geq 0\) for \(a \geq 0\) and \(\rho(u) = 1.\)

Note that the standard terminology for such a functional is “state”, but we prefer “probability measure” for the sake of consistency with our test space terminology.

Theorem 4.3 (Hahn-Banach extension theorem). Let \((V, C, u)\) be an order unit space. If \(U \subseteq V\) is a subspace with \(u \in U\), then \((U, C \cap U, u)\) is again an order unit space. Any probability measure \(\sigma\) on \(U\) can be extended to a probability measure \(\rho\) on \(V\), i.e. there is a probability measure \(\rho : V \to \mathbb{R}\) such that \(\rho | U = \sigma.\)

Proof. The claim that \((U, C \cap U, u)\) is again an order unit space is straightforward to check; the non-trivial part is the second statement. In the following, we present the standard argument adapted to our particular situation.

Consider the collection of all pairs \((W, \rho)\) where \(W \subseteq V\) is a subspace with \(u \in W\) and \(\rho\) is a probability measure on \(W\). This set is partially ordered by defining \((W, \rho) \leq (W', \rho')\) to mean that \(W \subseteq W'\) and \(\rho|_W = \rho\). Since the hypothesis of Zorn’s lemma trivially holds, it follows that this partially ordered set has a maximal element. Likewise, if we consider only all the elements “above” the given \((U, \sigma)\), then Zorn’s lemma still applies and we obtain that there exists a maximal element which is greater than or equal to \((U, \sigma)\). Let \((W, \rho)\) be such a maximal element. Then we have \(\rho|_U = \sigma\) by construction.

Our goal is to show that maximality implies \(W = V\). For if \(W = V\) did not hold, then we could find \(a \in V\) with \(a \not\in W\), consider \(W' := W + Fa\), and extend \(\rho\) to \(\rho' : W' \to \mathbb{R}\) by choosing

\[ \rho'(a) = \sup_{b \in W \text{ s.t. } b + a \geq 0} \left( -\rho(b), \inf_{b \in W \text{ s.t. } b - a \geq 0} \rho(b) \right). \]  

If the interval is non-empty, then we obtain \(\rho' : W' \to \mathbb{R}\) by linear extension. So it remains to show that the interval is non-empty, and that the resulting \(\rho'\) is indeed a probability measure, since then we conclude \((W, \rho) < (W', \rho')\) in contradiction with the maximality assumption. We start with the first. The supremum is not \(+\infty\) since \(u \in W\) and there is \(\lambda \in F\) with \(\lambda u + a \geq 0\); likewise, the infimum is not \(-\infty\). It remains to be shown that its lower end is less than or equal to its upper end by showing that \(-\rho(b_1) \leq \rho(b_2)\) for \(b_1 + a \geq 0\) and \(b_2 - a \geq 0\). But the latter two conditions imply that \(b_1 + b_2 \geq 0\), and hence \(\rho(b_1) + \rho(b_2) \geq 0\) by the assumptions on \(\rho\).

Finally, we still need to show that \(\rho'\) is indeed a probability measure, i.e. that it is positive. But this is easy, since \([14]\) was engineered to guarantee precisely this: any element \(c \in W'\) with \(c \geq 0\) is a unique linear combination \(c = \lambda a + b\) with \(\lambda \in F\) and \(b \in W\). For \(\lambda = 0\), we have \(\rho'(c) = \rho(c) \geq 0\), since \(\rho\) is a probability measure. Otherwise, we can assume \(\lambda = \pm 1\) after rescaling. If \(\lambda = +1\), then we have \(c = b + a \geq 0\), and the claim follows from \(\rho'(a) \geq -\rho(b)\). If \(\lambda = -1\), then we have \(c = b - a\), and the claim follows from \(\rho'(a) \leq \rho(b)\). \(\Box\)

Since there is a unique probability measure on the one-dimensional subspace \(U := Fu\), we obtain immediately:
Corollary 4.4. There is a probability measure $\rho : V \to \mathbb{R}$.

Our goal in the following is to derive a refined version of this statement.

Theorem 4.5 (Hahn-Banach separation theorem). Let $(V, C, u)$ be an order unit space and $a \in V$ an element for which there exists $n \in \mathbb{N}$ with $a + \frac{1}{n}u \not\geq 0$. Then there is a probability measure $\rho : V \to \mathbb{R}$ with $\rho(a) < 0$.

Proof. If $a$ is a scalar multiple of $u$, then it must be a negative scalar multiple, and the claim follows from Corollary 4.4. Otherwise, the subspace $U := \mathbb{F}u + \mathbb{F}a$ is two-dimensional, and this is the case that we consider from now on.

By Theorem 4.3 it is sufficient to define $\rho$ on $U$ only. Obtaining such a $\rho$ can be done as in the proof of Theorem 4.3: there is a unique state on the one-dimensional subspace $\mathbb{F}u$, and we extend this state to $U$ by finding a feasible value for $\rho(a)$ and extending linearly. The interval of feasible values (14) now becomes

$$\rho(a) \in \left[ \sup_{\lambda \in F \text{ s.t. } \lambda a + \lambda u \geq 0} (\lambda), \inf_{\lambda \in F \text{ s.t. } \lambda a - \lambda u \geq 0} \lambda \right],$$

and we have already shown above that this is a non-empty interval. However, we also would like to achieve $\rho(a) < 0$; in order for this to be possible, we need the lower end of the interval to be strictly negative, or equivalently

$$\inf_{\lambda \in F \text{ s.t. } \lambda a + \lambda u \geq 0} \lambda > 0.$$

But this is equivalent to the assumption of existence of an $n \in \mathbb{N}$ such that $\frac{1}{n}u + a \not\geq 0$. $\square$

5 Proof of main results

Theorem 3.8. A plausibility measure $\text{Pl}$ on a locally finite test space $(X, \Sigma)$ almost agrees with some probability measure $\mu$ if $\text{Pl}$ is Archimedean.

Proof. Our strategy is to apply Corollary 4.4 to a suitably constructed order unit space.

Let $V$ be the rational vector space with basis $X$, i.e. the elements of $V$ are finite $\mathbb{Q}$-linear combinations of the outcomes. We write $e_x$ for the basis vector associated to an outcome $x \in X$. If $A$ is an event, then we also write $e_A$ as shorthand for $\sum_{x \in A} e_x$. In particular, we have $e_\emptyset = 0$. We define a convex cone $C$ in this space as the set of all finite non-negative linear combinations of vectors of the form

$$e_A - e_B$$

for all pairs of events $A$ and $B$ with $\text{Pl}(A) \geq \text{Pl}(B)$. The idea here is that if we can find a vector $u$ such that $(V, C, u)$ is an order unit space, then, by Corollary 4.4, there exists a probability measure $\rho : V \to \mathbb{R}$. This induces a probability measure $\mu$ on the test space via $\mu(A) = \rho(e_A)$.

In more detail, this works as follows: we fix a test $T \in \Sigma$ and consider the distinguished element $u := e_T$, claiming that this $u$ is an order unit. In order to show that it indeed is, we need to prove that any other vector $\sum_x \alpha_x e_x$ can be lower bounded by a scalar multiple of $u$. If two vectors can be lower bounded by a multiple of $u$, then so can any positive linear combination of these vectors; therefore, it is sufficient to consider the case of a single basis vector $e_x$ or $-e_x$. The first is trivial: with $A = \{x\}$ and $B = \emptyset$, the inequality $e_x \geq 0$ is itself an instance of (15). For the second, choose a test $S \in \Sigma$ with $x \in S$. Then $-e_x \geq -e_S$ is again an instance of (15), since $\text{Pl}(S) \geq \text{Pl}(\{x\})$. Moreover, since all tests have plausibility 1, we also have $-e_S \geq -e_T = -u$ as an instance of (15). In conclusion, we obtain $-e_x \geq -u$.
In order to complete the verification of the order unit space axioms, we still need to make sure that $-u \not\geq 0$, i.e. that $-e_T$ is not a positive linear combination of vectors of the form $i\mathbb{1}$. This is where the assumption that $\text{Pl}$ is Archimedean comes in. In fact, we will prove something more general: if $A$ and $B$ are events with $\text{Pl}(A) \prec \text{Pl}(B)$, then $e_A - e_B$ is not in our convex cone; the statement $-u \not\geq 0$ is then the special case with $A = \emptyset$ and $B = T$. So assume that $e_A - e_B$ is in our convex cone. This means that there are events $(A_1, \ldots, A_n)$ and $(B_1, \ldots, B_n)$ such that

$$ e_A - e_B = \sum_i \lambda_i (e_{A_i} - e_{B_i}), \quad (16) $$

where $\lambda_i \in \mathbb{Q}$ and $\text{Pl}(A_i) \succeq \text{Pl}(B_i)$. Writing each $\lambda_i$ in lowest terms as $\lambda_i = a_i / b_i$ and multiplying eq. (16) by $N$, where $N$ the least common multiple of the $b_i$'s, gives

$$ N e_A - N e_B = \sum_i r_i (e_{A_i} - e_{B_i}), \quad (17) $$

where each $r_i$ is a positive integer. Defining the list $(A_1', \ldots, A_m')$ so that the first $r_1$ events are $A_1$, the next $r_2$ events are $A_2$, etc., and similarly for $(B_1', \ldots, B_m')$, we obtain the decomposition

$$ N e_A - N e_B = \sum_i (e_{A_i'} - e_{B_i'}), \quad (18) $$

or equivalently

$$ \sum_i e_{B_i'} + N e_A = \sum_i e_{A_i'} + N e_B. \quad (19) $$

If we construct the lists $(A_1', \ldots, A_m', B, \ldots, B)$ and $(B_1', \ldots, B_m', A, \ldots, A)$ by appending $N$ copies of $B$ and $A$ to the end of the lists $(A_1', \ldots, A_m')$ and $(B_1', \ldots, B_m')$ respectively, then eq. (19) says exactly that each $x \in X$ occurs the same number of times in these two lists. By construction we have $\text{Pl}(A_i') \succeq \text{Pl}(B_i')$ and $\text{Pl}(B) \prec \text{Pl}(A)$. But applying the Archimedean property of eq. (12) then gives $\text{Pl}(B) \preceq \text{Pl}(A)$, which is a contradiction.

**Theorem 3.9.** A plausibility measure $\text{Pl}$ on a test space $(X, \Sigma)$ with $X$ finite agrees with some probability measure $\mu$ iff the image of $\text{Pl}$ is totally ordered and $\text{Pl}$ is Archimedean.

**Proof.** The only if part follows from the properties of probability measures, so we focus on the if part.

For finite $X$, the convex cone constructed in the proof of Theorem 3.8 is finite-dimensional and polyhedral. Hence by Theorem 4.5 (or simply Farkas’ lemma), any point outside of the cone can be separated from the cone by a probability measure. In particular, this applies to the vector $e_A - e_B$ for any pair of events with $\text{Pl}(A) \prec \text{Pl}(B)$, which we previously proved to be outside of the cone. So let $\rho_{A,B}$ be such a probability measure with $\rho_{A,B}(e_A - e_B) < 0$. We then consider the new probability measure

$$ \rho' := \frac{1}{N} \sum_{A, B \text{ s.t. } \text{Pl}(A) \prec \text{Pl}(B)} \rho_{A,B}, $$

where $N$ is the appropriate normalization factor equal to the number of terms in the sum. We claim that $\rho'(e_A - e_B) < 0$ for any $A$ and $B$ with $\text{Pl}(A) \prec \text{Pl}(B)$. For any two pairs of events $A, B$ and $A', B'$ with $\text{Pl}(A) \prec \text{Pl}(B)$ and $\text{Pl}(A') \prec \text{Pl}(B')$, we have $\rho_{A,B'}(e_B - e_A) \geq 0$ since $e_B - e_A \geq 0$ and $\rho_{A',B'}$ is a positive functional. By linearity, we then have $\rho_{A',B'}(e_A - e_B) = -\rho_{A',B'}(e_B - e_A) \leq 0$. But since $\rho_{A,B}(e_A - e_B) < 0$
by definition, we obtain the claim. Setting \( \mu(A) = \rho(e_A) \) results in a probability measure on the test space which satisfies

\[
Pl(A) \leq Pl(B) \implies \mu(A) \leq \mu(B),
\]

\[
Pl(A) < Pl(B) \implies \mu(A) < \mu(B).
\]

To see that \( \mu \) also satisfies the converse implication, we make use of the assumption of total ordering. We know that, for every pair of events \( A \) and \( B \), either \( Pl(A) \leq Pl(B) \) or \( Pl(B) \leq Pl(B) \), and that this is reflected in the constructed probability measure \( \mu \). Therefore, whenever \( \mu(A) \leq \mu(B) \), this necessitates \( Pl(A) \leq Pl(B) \).

6 Related work

The use of ordering relations rather than precise numerical probabilities has a long history in the foundations of probability theory, particularly in the intuitive probability of Koopman [17] and in subjective Bayesian probability [6, 22]. The notions of agreement and almost agreement used in this work are derived from the literature on axiomatizing subjective probability in these terms (see [7] for a review). Most of this literature assumes a total ordering, but see [16] for early arguments that a partial ordering should be used instead.

The notion of a plausibility measure is due to Friedman and Halpern [10, 11, 15]. Their work was aimed at providing a unifying framework for various mathematical structures that had been proposed for reasoning in the face of uncertainty in artificial intelligence, such as probabilities and Dempster-Shafer belief functions [4, 28]. In doing so, they effectively reinvented earlier theories of comparative probability theory [5], except with partial rather than total ordering, and in a far more elegant formalism. We have followed their notations here.

On the quantum side, Foulis, Randall and Piron investigated the notion of supports on test spaces [8], which in our terminology are just possibility measures on test spaces. A plausibility measure on a test space is a generalization of this, and can be viewed as a way of unifying it with the usual notion of a probability measure, or state, on a test space.

More recently, there has been much interest in applying possibilistic measures in the foundations of quantum theory. One of the authors of the present paper has investigated possibilistic hidden variable theories [12]. This was followed by work of Abramsky [11], investigating the logical structure of non-locality and contextuality using possibility measures. As both possibility and probability are special cases of plausibility, plausibility measures have the potential to unify Abramsky’s approach with the conventional account of these phenomena in terms of probabilities.

Finally, Schumacher and Westmoreland have developed a version of quantum theory on vector spaces over finite fields, which they call modal quantum theory [24, 25, 26]. They get around the problem of having no inner product on such spaces by basing their theory on possibilistic assignments rather than probabilities. They have noted that not all of the possibility measures in their theory can be represented by probabilities [25, 26] (i.e. that they don’t agree with any probability measure in our terminology) and raised the question of how such cases could be identified. The Archimedean condition can certainly be used for this as, in particular, Example 3.2 occurs in their theory for a modal quantum bit. However, this does not yield an efficient algorithm for checking agreement, as one potentially has to consider all possible families of events. Employing the order unit space construction directly may be preferable.
7 Conclusion

In this paper, we have taken the first steps in developing the theory of plausibility measures for general operational theories, which include quantum theory as a special case. We have shown that the Archimedean property, which successfully identifies when classical plausibility measures agree with classical probability measures, is still useful in general theories. In particular, for theories based on strictly finite test spaces, such as Schumacher-Westmoreland modal quantum theory, it provides a necessary and sufficient condition for agreement. For locally finite test spaces, we have shown that it is sufficient for almost agreement, but it is possible that this might be improved.

There are many potential applications of plausibility measures, some of which will be developed in future work. In particular, they can be used to unify possibilistic and probabilistic approaches to phenomena in quantum foundations, such as nonlocality and contextuality. Additionally, in quantum information science, there are many applications in which reasoning with qualitative comparisons rather than precise numerical probabilities may be beneficial. For example, perhaps quantum cryptography protocols can be proved secure using only a few comparative assessments, or perhaps algorithms for numerically simulating quantum systems can be rendered more efficient by only tracking qualitative information, such as whether or not the system is close to being in an eigenstate of some set of observables. The latter would be analogous to the classical artificial intelligence applications for which Friedman and Halpern originally invented plausibility measures [14].

More speculatively, plausibility measures offer the possibility of developing future theories of physics using a weaker predictive structure than probability, whilst still allowing for precise probabilities in an appropriate limit. This might be necessary in future theories of quantum gravity.

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