Classical Hamiltonian Systems with balanced loss and gain

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Abstract

Classical Hamiltonian systems with balanced loss and gain are considered in this review. A generic Hamiltonian formulation for systems with space-dependent balanced loss and gain is discussed. It is shown that the loss-gain terms may be removed completely through appropriate co-ordinate transformations with its effect manifested in modifying the strength of the velocity-mediated coupling. The effect of the Lorentz interaction in improving the stability of classical solutions as well as allowing a possibility of defining the corresponding quantum problem consistently on the real line, instead of within Stokes wedges, is also discussed. Several exactly solvable models based on translational and rotational symmetry are discussed which include coupled cubic oscillators, Landau Hamiltonian etc. The role of $PT$-symmetry on the existence of periodic solution in systems with balanced loss and gain is critically analyzed. A few non-$PT$-symmetric Hamiltonian as well as non-Hamiltonian systems with balanced loss and gain, which include mechanical as well as extended system, are shown to admit periodic solutions. An example of Hamiltonian chaos within the framework of a non-$PT$-symmetric system of coupled Duffing oscillator with balanced loss-gain and/or positional non-conservative forces is discussed. It is conjectured that a non-$PT$-symmetric system with balanced loss-gain and without any velocity mediated interaction may admit periodic solution if the linear part of the equations of motion is necessarily $PT$ symmetric — the nonlinear interaction may or may not be $PT$-symmetric. Further, systems with velocity mediated interaction need not be $PT$-symmetric at all in order to admit periodic solutions. Results related to nonlinear Schrödinger and Dirac equations with balanced loss and gain are mentioned briefly. A class of solvable models of oligomers with balanced loss and gain is presented for the first time along with the previously known results.

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1 Introduction

Dissipation is an ubiquitous phenomenon in nature. There are different approaches to understand dissipation, instabilities induced by it and controlling its effect in physical systems\[1, 2\]. One of the earlier attempts in this direction was Hamiltonian formulation of system with dissipation. This may seem counter intuitive, since a Hamiltonian system is conservative, while dissipation induces the loss of energy to the environment. The apparent conflict can be resolved if the system plus environment is considered as a larger system. The Hamiltonian formulation of Bateman oscillator is based on this approach and describes a system comprising of a damped oscillator plus an auxiliary system of an anti-damped oscillator with the same dissipation/anti-dissipation coefficient —the dimension of the ambient space is twice that of the target space\[3\]. The flow in the position-velocity state-space preserves the volume, allowing a Hamiltonian of the system, since the rate of energy-loss of the damped mode is equal to the rate of gain in energy by the anti-damped oscillator. The Bateman oscillator is an example of a system with balanced loss and gain —the flow is preserved in the position-velocity state-space although individual degrees of freedom are subjected to gain/loss.

The Bateman oscillator has been studied extensively over the last ninety years from different perspectives\[4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14\]. One of the undesirable features of the Hamiltonian formulation of Bateman oscillator is the introduction of the auxiliary degree of freedom. With the growing interest and relevance of $\mathcal{PT}$-symmetric theory\[15\], the concept of auxiliary system is abandoned in the interpretation of generalized versions of Bateman oscillator, where damped and anti-damped modes are coupled and exchange energies. It should be noted that there is no exchange of energies between the damped and anti-damped modes of the standard Bateman oscillator and a clear distinction between the two modes exists. Further, there are descriptions of quantum dissipation\[16\], where the energy is deposited from the system to the bath consisting of infinitely many harmonic oscillators so that the energy can not be transferred back to the system. The sole objective of adding interaction in the generalized models of Bateman oscillator is allowing a bi-directional exchange of energies between the damped and the anti-damped modes such that an equilibrium may be achieved —the energy is reverted back to the damped oscillator from the anti-damped oscillator at the same rate as it is deposited to the anti-damped oscillator from the damped oscillator. The existence of equilibrium state in a $\mathcal{PT}$-symmetric system with balanced loss and gain is worth comparing to a similar situation arising in the context of non-equilibrium thermodynamics, where the invariance of Schwinger-Keldysh action under time-reversal symmetry plus time-translation corresponds to thermodynamic equilibrium\[17\]. The Hamiltonian is the generator of the time-translation, while invariance under time-reversal symmetry conforms to
principle of detailed balance for the quantum system. It should be noted here that \( \mathcal{PT} \)-symmetry may be substituted with a non-standard time-reversal symmetry within the realm of quantum mechanics\[18\]. Thus, the conditions on the Schwinger-Keldysh action for thermodynamic equilibrium and the equilibrium condition for systems with balanced loss and gain may be identified as similar.

The basic model of a generalized Bateman oscillator may be described by considering two linearly coupled identical harmonic oscillators, one of which is subjected to damping and the other to anti-damping with the same strength\[19\]. This describes an experimentally realized system involving coupled whispering galleries\[20\]. The system consists of two degrees of freedom and there is no concept of auxiliary system—the target and the ambient spaces are the same unlike in the case of the standard Bateman oscillator. The abandoning of auxiliary degrees of freedom, while considering models of generalized Bateman oscillator, is indeed a paradigm shift from the traditional treatment of the standard Bateman oscillator. The coupling allows an equilibrium state manifested as periodic solutions within some regions in the parameter-space characterizing the unbroken \( \mathcal{PT} \)-phase. The existence of equilibrium state is one of the novel features of a system with balanced loss and gain. The phase-transitions between broken and unbroken \( \mathcal{PT} \)-phases, existence of quantum bound states in an unbroken \( \mathcal{PT} \)-phase, role of exceptional points etc. are some of the salient aspects of a system with balanced loss and gain\[15\].

The central focus of this article is to review known results on classical mechanical systems with balanced loss and gain for arbitrary number of particles and physically interesting potentials. The criteria for a mechanical system to be identified as a system with balanced loss and gain is discussed in Sec. 2. The Hamiltonian formulation for a generic system with balanced loss and gain is discussed in Sec. 3 along with a representation of the matrices appearing in the Hamiltonian. The loss-gain coefficient may be taken to be constant or space-dependent. It is shown that the loss-gain terms may be removed with its effect manifested in modifying the strength of the velocity-mediated coupling through appropriate co-ordinate transformations. This result is independent of any particular representation of the matrices. The effect of the Lorentz interaction in improving the classical stability as well as defining the quantum problem on the real line is discussed. In Sec. 4 different exactly solvable models based on translational and rotational symmetry are discussed. It is also shown by constructing \( m + 1 \) integrals of motion for a system with \( N = 2m \) particles that the system is at least partially integrable for \( N > 2 \) and completely integrable for \( N = 2 \). The developments in the field of \( \mathcal{PT} \)-symmetry played a significant role in interpreting a Hamiltonian of generalized Bateman oscillator as a system without any auxiliary degrees of freedom. The role of \( \mathcal{PT} \)-symmetry on the existence of periodic solution is critically analyzed in Sec. 5. A non-\( \mathcal{PT} \)-symmetric Hamiltonian system of coupled Duffing oscillators with balanced loss-gain and positional non-conservative forces is shown to admit regular as well as chaotic dynamics. The periodic solutions along with bifurcation diagrams and Lyapunov exponents are presented. Further, a non-\( \mathcal{PT} \)-symmetric non-Hamiltonian system of coupled Duffing oscillators with balanced loss and gain is also shown to admit periodic solution. Other examples of non-\( \mathcal{PT} \)-symmetric Hamiltonian system with balanced loss and gain which admit periodic solution include Landau Hamiltonian, a dimer model and nonlinear Schrödinger equation. It is conjectured that a system with balanced loss-gain and without any velocity mediated interaction may admit periodic solution if the linear part of the equations of motion is necessarily \( \mathcal{PT} \)-symmetric —the nonlinear interaction may or may not be \( \mathcal{PT} \)-symmetric. Further, systems with velocity mediated coupling among different degrees of freedom need not be \( \mathcal{PT} \) symmetric at all in order to admit periodic solutions. The subject of system with balanced loss and gain has many facets which are not included in this review and a few important omitted topics are mentioned in Sec. 6. The nonlinear Schrödinger and Dirac equations with balanced loss and gain are reviewed in this section. Further, a class of solvable models of oligomers with balanced loss and gain is presented for the first time along with the previously known results. Finally, in Sec. 7 the results are summarized along with discussions.
2 Overviews on Systems with balanced loss and gain

The Bateman oscillator is described by the following set of equations:

\[
\begin{align*}
\ddot{x} + 2\gamma \dot{x} + \omega^2 x &= 0 \\
\ddot{y} - 2\gamma \dot{y} + \omega^2 y &= 0
\end{align*}
\]

where \( \dot{x} = \frac{dx}{dt} \) and \( \ddot{x} = \frac{d^2x}{dt^2} \). The \( x \) and \( y \) degrees of freedom are subjected to loss and gain, respectively for \( \gamma > 0 \). The Bateman oscillator was originally introduced as a Hamiltonian system for a dissipative simple harmonic oscillator. The basic idea is to embed the original system described by the \( x \) degree of freedom in a larger system with two degrees of freedom, where the additional \( y \) degree of freedom defines an auxiliary system. The ambient space is defined by taking both the \( x \) and \( y \) degrees of freedom together, while \( x \) degree of freedom alone describes the target space. The anti-damped oscillator being the time-reversed version of the damped oscillator and the vice verse, a Hamiltonian formulation is allowed in the ambient space. The Bateman oscillator has been studied for almost ninety years from different perspectives with the central theme being a Hamiltonian description for a dissipative oscillator. A major unwanted feature of Bateman oscillator is the presence of auxiliary degree of freedom. A few other alternative approaches to study the same problem with a non-standard Hamiltonian description and without any auxiliary degree of freedom have also been proposed — (i) time-dependent Hamiltonian [19], (ii) a complex Lagrangian [20], and (iii) different Hamiltonians for different parameter regimes by using modified Prelle-Singer method [21]. All these methods have their own merits and demerits with a common goal — Hamiltonian description of a dissipative system. However, abandoning the idea of auxiliary system for the case of Bateman oscillator and reinterpreting it as conservative system with balanced loss and gain admitting periodic solutions may be reproduced for \( G \) regions of the parameters space [22, 23, 24, 25, 26, 27, 28, 29, 30, 31].

However, abandoning the idea of auxiliary system for the case of Bateman oscillator and reinterpreting it as conservative system with balanced loss and gain for which the ambient and the target spaces are the same with two degrees of freedom has not been explored until recently.

The Bateman oscillator is one of the simplest examples of a system with balanced loss and gain. The flow in the phase space preserves its volume due to the balanced loss and gain — any imbalance leads to either growing or decaying volume. The flow in the phase-space preserves its volume even if interaction terms are added by coupling the two degrees of freedom:

\[
\begin{align*}
\ddot{x} + 2\gamma \dot{x} + \omega^2 x + G_1(x, y) &= 0 \\
\ddot{y} - 2\gamma \dot{y} + \omega^2 y + G_2(x, y) &= 0,
\end{align*}
\]

where \( G_1(x, y) \) and \( G_2(x, y) \) are arbitrary functions. A different viewpoint may be considered where the mechanical model described by Eq. (2) with two degrees of freedom is taken as a whole system without any auxiliary degree of freedom. In other words, the target and the ambient spaces are taken to be the same. This is a paradigm shift in the interpretation of models with balanced loss and gain and leads to a new type of interacting system with important physical consequences. The modified system defined by Eq. (2) may admit periodic solutions for specific choices of \( G_{1,2}(x, y) \) which is not possible for the Bateman oscillator, i.e. \( G_1(x, y) = G_2(x, y) = 0 \). For example, Eq. (2) with \( G_1 = cy, G_2 = cx, c \in \mathbb{R} \), represents a mathematical model [19] for an experimentally realized \( PT \) symmetric coupled resonators and admits periodic solutions within suitable ranges of the parameters [22]. Similarly, a two particle Calogero-type model with balanced loss and gain admitting periodic solutions may be reproduced for \( G_1 = \alpha(x-y)^{-3}, G_2 = -\alpha(x-y)^{-3}, \alpha \in \mathbb{R} \) [23]. There are many examples of systems with balanced loss and gain admitting periodic solutions in certain regions of the parameters space [24, 25, 26, 27, 28, 29, 30, 31].

The concept of a system with balanced loss and gain may be generalized for arbitrary \( N \) degrees of freedom. The system is governed by the equations of motion,

\[
\ddot{X} - 2D \dot{X} + G(x_1, x_2, \ldots, x_N) = 0,
\]

where the co-ordinates of the \( N \) particles are denoted as \( x_1, x_2, \ldots, x_N \) which are elements of the column matrix \( X \). In particular, \( X^T \equiv (x_1, x_2, \ldots, x_N) \), where \( X^T \) denotes the transpose of \( X \). The term linear in
\(X\) contains information on the nature of loss-gain as well as velocity mediated coupling among the particles depending on the explicit form of the \(N \times N\) matrix \(D\). In general, \(D\) may be decomposed as,

\[
D = D + D_{SO} + D_A
\]

where \(D\) is a diagonal matrix, \(D_{SO}\) is a symmetric matrix with vanishing diagonal elements and \(D_A\) is an anti-symmetric matrix. The symmetric matrix \(D_S = D + D_{SO}\) is decomposed in terms of its diagonal and off-diagonal parts, \(D\) and \(D_{SO}\), respectively. The Lorentz interaction is described by \(D_A\), while velocity mediated non-Lorentzian interaction appears in \(D_{SO}\).

The loss and gain in the system are introduced via the matrix \(D\). In general, \(D\) may be space-dependent and thereby, allowing space-dependent loss-gain terms in the system. The space-mediated coupling among different degrees of freedom is encoded through the \(N\)-component field \(G(x_1, x_2, \ldots, x_N)\) with its components \(G_i \equiv G_i(x_1, x_2, \ldots, x_N), i = 1, 2, \ldots, N\).

The criteria for the system described by Eq. (3) to be non-dissipative may be determined by studying the time-evolution of the flow in the \(2N\) dimensional position-velocity state space. The flow preserves the volume in the position-velocity state-space for a non-dissipative system, while the volume either grows or decays depending on whether the system is anti-damped or damped, respectively. Introducing a \(2N\)-component vector \(\xi\) and a field \(\eta\),

\[
\xi \equiv \begin{pmatrix} X \\ \dot{X} \end{pmatrix}, \eta_i \equiv \xi_{N+i}, \; \eta_{N+i} = -G_i(\xi_i, \ldots, \xi_N) + \sum_{k=1}^{N} D_{ik}(\xi_i, \ldots, \xi_N)\xi_{N+k}
\]

Eq. (5) may be rewritten in terms of \(2N\) first-order coupled differential equations as \(\dot{\xi} = \eta\). The flow preserves the volume \(V = \prod_{i=1}^{2N} d\xi_i\) in the position velocity state space provided \(\eta\) is solenoidal, i.e.

\[
\sum_{i=1}^{2N} \frac{\partial \eta_i}{\partial \xi_i} = 0 \Rightarrow Tr(D) = 0
\]

where \(Tr\) denotes trace of a matrix. The two dimensional examples in Eqs. (1) and (2), when cast into the form of Eq. (3), contain \(D = D = \gamma \text{diag}(-1, 1)\) which is indeed traceless. For a system with \(N > 2\) degrees of freedom, the traceless condition may be implemented in several ways depending on the actual physical scenario. The vanishing trace of \(D\) is the criteria for a model governed by Eq. (3) to be identified as a system with balanced loss and gain for which the flow in the position-velocity state space preserves the volume in spite of the fact that individual degrees of freedom are subjected to loss or gain.

3 Hamiltonian system

The Hamiltonian formulation of a system with balanced loss and gain is a non-trivial problem. For example, the Hamiltonian of two linearly coupled identical Duffing oscillators, one of which is subjected to gain and the other with an equal amount of loss is unknown. The same system with additional nonlinear interaction may be shown to be Hamiltonian. It should be mentioned here that there are well known methods to construct Lagrangian and Hamiltonian from a given set of equations of motion. The inverse variational problem was initiated by Helmholtz. However, a successful implementation of the scheme becomes nontrivial for many-particle systems with nonlinear interaction. Further, a system may not admit a Lagrangian-Hamiltonian formulation at all. Within this background, the Hamiltonian formulation for a class of systems with balanced loss and gain is described below.

The Hamiltonian of the Bateman oscillator has the expression,

\[
H_B = P_x P_y + \gamma (y P_y - x P_x) + (\omega^2 - \gamma^2) xy
\]
\[ P_x = \dot{y} - \gamma y, \quad P_y = \dot{x} + \gamma x, \]  
(7)

where the canonical conjugate momenta corresponding to \(x\) and \(y\) are denoted as \(P_x\) and \(P_y\), respectively. The Hamilton’s equation of motion reproduces Eq. (1). The Hamiltonian \(H_B\) can be rewritten in terms of the generalized momenta \(\Pi_x\) and \(\Pi_y\) as,

\[ H_B = \Pi_x \Pi_y + \omega^2 x y, \quad \Pi_x = P_x + \gamma y, \quad \Pi_y = P_y - \gamma x. \]  
(8)

The following points may be noted:

- The expressions for \(\Pi_{x,y}\) are similar to that of the generalized momenta for a particle in an uniform external magnetic field with magnitude \(\gamma\) and along the direction perpendicular to the \(x-y\) plane. There is no magnetic field in the system. So, \(A_x := \gamma y\), \(A_y := -\gamma x\) may be interpreted as ‘fictitious gauge potentials’ leading to the uniform ‘fictitious magnetic field’ with magnitude \(\gamma\).

- The quadratic term involving the generalized momenta is not positive-definite, \(\Pi_x \Pi_y = \Pi^2_x - \Pi^2_y\), \(\Pi_{\pm} = \frac{1}{2} (\Pi_x \pm \Pi_y)\) and the system may be interpreted as defined in the background of a pseudo-Euclidean metric with signature (1, -1).

These two features are present for known Hamiltonian systems with balanced loss and gain and will be used as essential inputs for constructing Hamiltonian for a generic many-particle system with balanced loss and gain.

The Hamiltonian for a general system with balanced loss and gain is taken to be of the form,

\[ H = \Pi^T \mathcal{M} \Pi + V(x_1, x_2, \ldots, x_N), \]  
(9)

where \(\mathcal{M}\) is an \(N \times N\) real symmetric matrix and \(\Pi = (\pi_1, \pi_2, \ldots, \pi_N)^T\) denotes \(N\) component generalized momenta. The matrix \(\mathcal{M}\) is non-singular so that \(\mathcal{M}^{-1}\) exists and is not necessarily semi-positive definite. The matrix \(\mathcal{M}\) may be interpreted as a constant background metric as in the case of Bateman oscillator. A semi-positive definite \(\mathcal{M}\), if exists, may be interpreted either as a metric or a mass-matrix. The generalized momenta \(\Pi\) has the expression,

\[ \Pi = P + A F(X), \]  
(10)

where \(P = (p_1, p_2, \ldots, p_N)^T\) is the conjugate momentum corresponding to the coordinate \(X\), \(F(X) = (F_1, F_2, \ldots, F_N)^T\) is \(N\) dimensional column matrix whose entries are functions of coordinates and \(A\) is an \(N \times N\) anti-symmetric matrix. It may be noted that \(a_i = \sum_{k=1}^{N} A_{ik} F_k\) can be interpreted as gauge potentials. In general, \(a_i\) may be decomposed as \(a_i = a_i^B + a_i^F\) in terms of realistic gauge potentials \(a_i^B\) and fictitious gauge potentials \(a_i^F\). The realistic gauge potential describes external magnetic field in the system, while the fictitious gauge potential leads to loss-gain terms as in the case of Bateman oscillator. With suitable choices of \(F\), space-dependent gain-loss terms are allowed in the system. The Hamiltonian \(H\) in Eq. (9) reduces to the Hamiltonian of Ref. [19] describing coupled harmonic oscillators with balanced loss-gain for the following choices of \(\mathcal{M}, A, F, V\) and identification \(x_1 = x, x_2 = y, p_1 = P_x, p_2 = P_y\):

\[ \mathcal{M} = \frac{1}{2} \sigma_1, \quad A = i \gamma \sigma_2, \quad F_1 = x, \quad F_2 = y, \quad V(x,y) = \omega^2 x y + \frac{\beta}{2} (x^2 + y^2) \]  
(11)

where \(\sigma_i, i = 1, 2, 3\) are the Pauli matrices. The Hamiltonian \(H_B\) of the Bateman oscillator is reproduced for \(\beta = 0\). The Lagrangian corresponding to the Hamiltonian \(H\) in Eq. (9) may be derived as follows:

\[ \mathcal{L} = \frac{1}{4} \dot{X}^T \mathcal{M}^{-1} \dot{X} - \frac{1}{2} (\dot{X}^T A F + (F^T A^T \dot{X}) - V(x_1, x_2, \ldots, x_N). \]  
(12)

The kinetic energy term in the Lagrangian is not necessarily positive-definite. The presence of terms linear in velocity is essential for incorporating loss-gain in the system. These are general characteristics of Bateman oscillator and its generalized versions.
The equations of motion following from the Hamiltonian \( \mathcal{H} \) or the Lagrangian \( \mathcal{L} \) reads,

\[
\dot{X} - 2MR\dot{X} + 2\mathcal{M} \frac{\partial V}{\partial X} = 0,
\]

where the anti-symmetric matrix \( R \), the Jacobian \( J \) and \( \frac{\partial V}{\partial X} \) are defined as follows:

\[
R \equiv AJ - (AJ)^T, \quad [J]_{ij} = \frac{\partial F_i}{\partial x_j}, \quad \frac{\partial V}{\partial X} = \left( \frac{\partial V}{\partial x_1}, \frac{\partial V}{\partial x_2}, \ldots, \frac{\partial V}{\partial x_N} \right)^T.
\]

The matrix \( R \) is anti-symmetric by definition, while \( J \) has no specified symmetry. However, \( J \) can be made to be symmetric or anti-symmetric with suitable choices of the \( N \)-component field \( F \). The equations of motion \( (13) \) reduces to Eq. \( (3) \) with the following identifications:

\[
\mathcal{D} = \mathcal{MR}, \quad G = 2\mathcal{M} \frac{\partial V}{\partial X}.
\]

The criteria \( \text{Tr}(\mathcal{D}) = 0 \) for a balanced loss and gain system is automatically satisfied, since \( \mathcal{M} \) is symmetric and \( R \) is anti-symmetric. The construction of Hamiltonian for a given Eq. \( (3) \) now reduces to finding solutions for the equations in Eq. \( (13) \). It is known \( [30] \) that a matrix \( \mathcal{D} \) that is similar to \(-\mathcal{D}\) can always be decomposed as a product of a symmetric and an anti-symmetric matrices. However, finding expressions for \( A \) and \( J \) from a known \( R \) is a non-trivial problem, particularly for the case of space-dependent loss-gain terms. Further, the problem of finding \( V \) for a given \( G \) involves solving \( N \) coupled first order differential equations \( \frac{\partial V}{\partial x} = \frac{1}{2} \mathcal{M}^{-1} G \) which, in general, may elude a closed form expression for \( V \). Nevertheless, the Hamiltonian of a large number of physical system with balanced loss and gain may be constructed via specific representation of matrices \( \mathcal{M}, R \) and \( \mathcal{D} \) and closed form expression for \( V \) \( [18, 29, 30] \).

The Hamiltonian for systems with space-independent loss and gain terms for \( N = 2 \) has been considered earlier \( [3, 19] \). The general formalism described above may be used to construct Hamiltonian system with space-dependent loss-gain terms. For example, a coupled Van der Pol-Duffing oscillator model with balanced loss and gain may be obtained by choosing,

\[
\mathcal{M} = \sigma_1 + \alpha^2 I_2, \quad A = -\frac{i\gamma}{2} \sigma_2, \quad F_i = a_i x_i + b_i x_i^3, \quad V(x_1, x_2) = \frac{\omega^2}{2} x_1 x_2 = \frac{3}{4} \left( x_1^2 + x_2^2 \right) + g x_1^2 x_2,
\]

where \( I_2 \) is the 2×2 identity matrix and \( \omega, \alpha, \beta, \gamma, a_i, b_i, g \) are real constants. The choice of the matrix \( A \) and the field \( F(x_1, x_2) \) uniquely fixes the generalized momenta \( \Pi \), while that of the matrix \( \mathcal{M} \) and the potential \( V(x_1, x_2) \) completely specifies the Hamiltonian. The expressions for the matrices \( R(x_1, x_2) \) and \( \mathcal{D}(x_1, x_2) \) can be computed by using Eqs. \( (13) \) and \( (15) \), respectively,

\[
R = -\frac{i\gamma}{2} \sigma_2 Q(x_1, x_2), \quad D = \frac{\gamma}{2} Q(x_1, x_2) \left( \sigma_3 - i\alpha^2 \sigma_2 \right), \quad Q(x_1, x_2) \equiv a_1 + a_2 + 3 \left( b_1 x_1^2 + b_2 x_2^2 \right)
\]

Eq. \( (12) \) describes a coupled Van der Pol-Duffing oscillator \( [38] \),

\[
\ddot{x}_1 - \gamma Q(x_1, x_2) \left( \dot{x}_1 - \alpha^2 \dot{x}_2 \right) + \omega^2 x_1 + \beta x_2 + g x_1^3 = 0
\]

\[
\ddot{x}_2 + \gamma Q(x_1, x_2) \left( \dot{x}_2 - \alpha^2 \dot{x}_1 \right) + \omega^2 x_2 + \beta x_1 + 3g x_1^2 x_2 = 0
\]

where \( \gamma Q \) is the space-dependent loss-gain coefficient, \( \gamma \alpha^2 Q \) is the external magnetic field, \( \omega \) is the angular frequency of the harmonic oscillator. The strengths of the space-mediated linear and non-linear couplings are \( \beta \) and \( g \), respectively. The above equation may be reduced to a few known models with appropriate choices of the parameters. For example, the Hamiltonian of coupled oscillators with constant balanced loss and gain \( [19] \) is obtained for \( a_1 = a_2 = 1, b_1 = b_2 = \alpha = g = 0 \). The Bateman oscillator is obtained if the linear coupling is switched off additionally by taking \( \beta = 0 \). The Hamiltonian formulation of the above
system for \( \beta = \alpha = g = 0, a_1 = a_2 = \frac{1}{2} \) and \( b_1 \to -\frac{b_1}{\sqrt{2}}, b_2 \to -\frac{b_2}{\sqrt{2}} \) has been considered earlier [37] which describes coupled Van der Pol oscillators. The \( x_1 \) degree of freedom describes a Van der Pol-Duffing oscillator for \( b_2 = \beta = \alpha = 0, a_1 = a_2 = \frac{1}{2}, b_1 \to -\frac{b_1}{\sqrt{2}} \) and the \( x_2 \) degree of freedom is unidirectionally coupled to it. This also provides a Hamiltonian formulation for Van der Pol-Duffing oscillator in an ambient space of two dimensions, much akin to the case of Bateman oscillator. The Hamiltonian of Ref. [31] is obtained for \( a_1 = a_2 = 1, b_1 = b_2 = \alpha = 0 \). The system has rich dynamical properties for the generic values of the parameters [38].

Generalizations to \( N = 3 \) may be achieved in several ways depending on the particular physical contexts. For example, the non-vanishing elements of the \( 3 \times 3 \) matrices \( M \) and \( A \) may be chosen as \( M_{12} = M_{21} = M_{33} = 1, M_{11} = M_{22} = \alpha^2, A_{12} = -A_{21} = -\frac{\gamma}{\sqrt{2}} \) along with \( F_3 = 0 \) and \( F_1, F_2 \) as given in Eq. (10). The \( x_3 \) degree of freedom is neither subjected to gain/loss nor it is coupled to \( x_1 \) and \( x_2 \) degrees of freedom via velocity mediated coupling, since all elements of \( D \) are zero except for \( D_{11} = -D_{22} = \frac{\gamma}{\sqrt{2}} Q(x_1, x_2) \). The coupling among \( x_1, x_2 \) and \( x_3 \) degrees of freedom can be incorporated via the potential \( V \equiv V(x_1, x_2, x_3) \). A particular choice of \( V \) and the resulting equation of motions are described below,

\[
V(x_1, x_2, x_3) = \frac{\omega^2}{4} (2x_1 x_2 + x_3^2) + \frac{\beta}{4} (x_1^2 + x_2^2 + 2x_1 x_3 + 2x_2 x_3) + \alpha x_1^3 x_2 + \frac{\delta}{8} x_3^4
\]

\[
\ddot{x}_1 - \gamma Q(x_1, x_2) (\dot{x}_1 - \alpha^2 \dot{x}_2) + \omega^2 x_1 + \beta (x_2 + x_3) + \alpha x_3^2 = 0
\]

\[
\ddot{x}_2 + \gamma Q(x_1, x_2) (\dot{x}_2 - \alpha^2 \dot{x}_1) + \omega^2 x_2 + \beta (x_1 + x_3) + 3 \alpha x_1^2 x_2 = 0
\]

\[
\ddot{x}_3 + \omega^2 x_3 + \beta (x_1 + x_2) + \delta x_3^2 = 0
\]

where the undamped Duffing oscillator described by \( x_3 \) degree of freedom linearly couples with the \( x_1, x_2 \) degrees of freedom. Generalizations to \( N > 3 \) may be continued in a similar way and is not discussed. A particular representation of matrices for arbitrary \( N \) is presented in Sec. 3.2 and several examples are discussed in Refs. [18, 24, 28, 29, 30, 31].

### 3.1 Hiding the loss-gain terms

The Bateman oscillator can be rewritten as,

\[
\ddot{z}_+ + 2\gamma \dot{z}_+ + \omega^2 z_+ = 0, \quad \ddot{z}_- + 2\gamma \dot{z}_- + \omega^2 z_- = 0,
\]

in a rotated co-ordinate system \( z_\pm = \frac{1}{\sqrt{2}} (x_\pm \pm y) \). The loss-gain terms are absent in Eq. (20), since the equation of motion for \( z_+ \) does not contain a term \( \dot{z}_+ \) and similarly the equation of motion for \( z_- \) does not contain a term \( \dot{z}_- \). Further, the system may be interpreted as defined in the background of a pseudo-Euclidean metric with signature \((1, -1)\) and the particle is subjected to external magnetic field proportional to \( \gamma \). This is a generic feature of Hamiltonian systems with balanced loss and gain. The loss-gain terms may always be hidden in a specific coordinate system. The real symmetric matrix \( M \) can be diagonalized by an orthogonal matrix \( \hat{O}, i.e. M_D = \hat{O}^T M \hat{O} \). Defining a rotated co-ordinate system \( \hat{X} = \hat{O}^T X \) and \( \hat{P} = \hat{O}^T P, \) eq. (3) can be rewritten as,

\[
\ddot{\hat{X}} - 2 (M_D \hat{R}) \dot{\hat{X}} + 2 M_D \frac{\partial V}{\partial \hat{X}} = 0, \quad \hat{R} = \hat{O}^T R \hat{O}
\]

The matrix \( \hat{R} \) is anti-symmetric and each diagonal element of the matrix \( M_D \hat{R} \) is zero, i.e. \([M_D \hat{R}])_{ii} = 0 \forall i \). The loss-gain terms are hidden in the co-ordinate system \( \hat{X} \) with the effect manifested by modifying the velocity mediated coupling terms. The general formalism may be exemplified with the Van der Pol-Duffing oscillator model described by Eqs. (10,17,18) for which the matrices \( \hat{O}, M_D \) and \( \hat{R} \) have the expressions,

\[
\hat{O} = \frac{1}{\sqrt{2}} (\sigma_1 + \sigma_3), \quad M_D = \alpha^2 I_2 + \sigma_3, \quad \hat{R} = \frac{i\gamma}{2} Q \sigma_2, \Rightarrow M_D \hat{R} = \frac{\gamma Q}{2} (\sigma_1 + i\alpha^2 \sigma_2)
\]
It is clearly seen that the diagonal elements of the matrix $M_D \tilde{R}$ are zero, i.e., $[M_D \tilde{R}]_{11} = [M_D \tilde{R}]_{22} = 0$. Further, the off-diagonal elements are $[M_D \tilde{R}]_{12} = \frac{2\eta}{\alpha^2} (1 + \alpha^2)$, $[M_D \tilde{R}]_{21} = \frac{2\eta}{\alpha^2} (1 - \alpha^2)$ which can not be interpreted as describing pure Lorentz force, since $[M_D \tilde{R}]_{12} \neq -[M_D \tilde{R}]_{21}$.

The system may be defined in the background of a pseudo-Euclidean metric through a canonical scale transformation. The matrix $M_D$ and two other matrices $S$ and $\eta^a$ are defined as follows:

$$[M_D]_{ij} = \delta_{ij} \text{sgn}(|\lambda_i|)|\lambda_i|, \quad [S]_{ij} = \delta_{ij} \sqrt{|\lambda_i|}, \quad [\eta^a]_{ij} = \delta_{ij} \text{sgn}(|\lambda_i|), \quad a = 1, 2, \ldots, N + 1$$

(23)

where $\lambda_i$’s are the eigenvalues of the matrix $\mathcal{M}$ and \text{sgn}(x) is the sign function. The parameter space of the system may be characterized in terms of at most $N + 1$ distinct regions depending on the number of negative eigenvalues of the matrix $\mathcal{M}$. The superscript $a$ in $\eta^{(a)}$ identifies the Region-$a$ in the parameter space corresponding to $a - 1$ negative eigenvalues of $\mathcal{M}$. The matrix $\eta^{(a)}$ is to be interpreted as the background metric for an effective description of the system defined by the Hamiltonian $H$ and equations of motion following from it in Eqs. (9) and (13), respectively. The canonical scale transformation is defined as follows:

$$\mathcal{X} = S^{-1} \tilde{X}, \quad \mathcal{P} = S \tilde{P},$$

(24)

where $\mathcal{X} \equiv (X_1, X_2, \ldots, X_N)^T$ and $\mathcal{P} \equiv (P_1, P_2, \ldots, P_N)^T$. The purpose of the scale transformation is to normalize the eigenvalues of $\mathcal{M}$ to $\pm 1$ i.e., $S^{-1}\tilde{\mathcal{M}}S^{-1} = \eta^{(a)}$. The Hamiltonian and the equations of motion resulting from the transformation in different regions have the forms:

$$H^{(a)} = \tilde{\Pi}^T \eta^{(a)} \tilde{\Pi} + \mathcal{V}(X_1, X_2, \ldots, X_N),$$

$$\dot{\mathcal{X}} - 2\eta^{(a)} R \mathcal{X} + 2\eta^{(a)} \left( \frac{\partial \mathcal{V}}{\partial \mathcal{X}} \right) = 0, \quad \mathcal{V}(X_1, X_2, \ldots, X_N) = V(x_1, x_2, \ldots, x_N),$$

(25)

where $R = S \tilde{R} S$ and the transformed generalized momenta $\Pi$ after rotation and the scale transformation is denoted as, $\tilde{\Pi} \equiv S \tilde{O}^T \Pi = \mathcal{P} + \frac{\delta}{\alpha} \mathcal{X}$. The quadratic term in momenta for Hamiltonians $H^{(1)}$ and $H^{(N+1)}$ are semi-positive definite, while $H^{(a)}$, $2 \leq a \leq N$ are not definite. The loss-gain terms of Eq. (18) are absent in Eq. (25). It may be noted that Eq. (25) contains velocity mediated Lorentzian and non-Lorentzian interaction in Region-$a$, $2 \leq a \leq N$, while only Lorentzian interaction is present in Region-1 and Region-N+1.

The formalism for an effective description of the system in the background of a pseudo-Euclidean metric may be exemplified with the Van der Pol-Duffing oscillator model described by Eqs. (16) and (17). The matrix $\mathcal{M}$ has eigenvalues $\lambda_1 = \alpha^2 + 1$ and $\lambda_2 = \alpha^2 - 1$. The parameter space may be divided into two regions: (i) Region-I: $\alpha^2 > 1$ with an Euclidean metric $\eta^{(1)} = I_2$, (ii) Region-II: $\alpha^2 < 1$ with a pseudo-Euclidean metric $\eta^{(2)} = \sigma_3$. The new co-ordinates $\mathcal{X}$, the matrix $\mathcal{R}$, the space-dependent co-efficient $Q(X_1, X_2)$ and the potential $\mathcal{V}(X_1, X_2)$ have the expressions:

$$\mathcal{X}_1 = \frac{1}{\sqrt{2|\lambda_1|}} (x_1 + x_2), \quad \mathcal{X}_2 = \frac{1}{\sqrt{2|\lambda_2|}} (x_1 - x_2), \quad \mathcal{R} = \frac{i\sigma_2}{2} Q \lambda \sigma_2$$

$$Q(X_1, X_2) = a_1 + a_2 + \frac{3(b_1 + b_2)}{2} (|\lambda_1| X_1^2 + |\lambda_2| X_2^2) + 3(b_1 - b_2) \lambda X_1 X_2$$

$$\mathcal{V}(X_1, X_2) = \frac{\Omega^+}{4} X_1^2 + \frac{g \lambda^2}{4} X_1^4 - \frac{\Omega^-}{4} X_2^2 - \frac{g \lambda^2}{4} X_2^4 + \frac{g \lambda}{2} X_1 X_2 (|\lambda_1| X_1^2 - |\lambda_2| X_2^2)$$

(26)

where $\lambda \equiv \sqrt{|\lambda_1||\lambda_2|}$, $\Omega_+ \equiv (\omega^2 + \beta)|\lambda_1|$ and $\Omega_- \equiv (\omega^2 - \beta)|\lambda_2|$.

- Region-I: The matrix $\mathcal{M}$ is positive-definite. The effective description of the system is in the background of a two dimensional Euclidean metric. The particle is subjected to Lorentz interaction and there is no other velocity-mediated interaction.

- Region-II: The matrix $\mathcal{M}$ is indefinite. The effective description of the system is in the background of a two dimensional pseudo-Euclidean metric with the signature $(1 - 1)$. The particle is subjected to non-Lorentzian velocity-mediated interaction.
Another simple example with two degrees of freedom is given in Sec. 3.2.1 which explains the general idea presented above.

3.2 Representation of matrices
Several representations of the matrices for a vanishing Lorentz interaction and constant loss-gain terms are presented in Ref. [18]. A few representations for space-dependent loss-gain terms are included in Refs. [29] and [30] for vanishing and non-vanishing Lorentz interaction, respectively. A particular representation for $N = 2m, m \in \mathbb{Z}^+$ with pair-wise balancing of space-dependent loss-gain terms is discussed in this article. The matrix $\mathcal{M}$ is chosen as,

$$\mathcal{M} = M + \alpha^2 I_{2m}, \alpha \in \mathbb{R},$$

(27)

where $M$ is a traceless $2m \times 2m$ symmetric matrix that anti-commutes with $R$, i.e. $\{M, R\} = 0$ and $I_{2m}$ is the $2m \times 2m$ identity matrix. The substitution of $\mathcal{M}$ in eq. (27) to the expression $\mathcal{D} = \mathcal{M} R$ gives,

$$\mathcal{D} = \frac{MR + \alpha^2 R}{\partial S}$$

(28)

The relations $M^T = M, R^T = -R, \{M, R\} = 0$ ensure that $\mathcal{D}_S = MR$ is a symmetric matrix. The parameter $\alpha$ controls the strength of the Lorentz interaction in the system and $\alpha = 0$ corresponds to vanishing Lorentz interaction.

The $m$ functions $Q_i \equiv Q_i(x_{2i-1}, x_{2i})$ are introduced as,

$$Q_i(x_{2a-1}, x_{2a}) = Tr(V_a^{(2)}), \quad V_a^{(2)} = \frac{\partial F_{2a-1}}{\partial x} \frac{\partial F_{2a-1}}{\partial x}.$$

(29)

The representation of the matrices is specified as follows:

$$M = I_m \otimes \sigma_1, \quad A = -\frac{i}{2} \gamma I_m \otimes \sigma_2, \quad D_S = \gamma \chi_m \otimes \sigma_3, \quad [\chi_m]_{ij} = \frac{1}{2} \delta_{ij} Q_i(x_{2i-1}, x_{2i}),$$

(30)

where $I_m$ is $m \times m$ identity matrix. The matrix $R$ may be determined by noting that the matrix $J$ has a block-diagonal form for the choices of $F_i \equiv F_i(x_{2i-1}, x_{2i})$:

$$R = \frac{\gamma}{2} \sum_{i=1}^{m} U_i^{(m)} \otimes \left( \begin{array}{cc} 0 & -Q_i(x_{2i-1}, x_{2i}) \\ Q_i(x_{2i-1}, x_{2i}) & 0 \end{array} \right), \quad [U_i^{(m)}]_{ij} = \delta_{ia} \delta_{ja}.$$

(31)

This completely specifies the representation of the matrices for pair-wise balancing of loss-gain terms and vanishing non-Lorentzian velocity mediated coupling, i.e. $\mathcal{D}_S = D, D_{SO} = 0$. A representation for the case $D_{SO} \neq 0$ may be found in Ref. [30].

3.2.1 Effect of Lorentz interaction:
The Lorentz interaction in the system vanishes for $\alpha = 0$ for which $\mathcal{M} = M$ and $\mathcal{D} = D_S$. It is known [18] that the matrices $M, R$ and $D_S$ anti-commute with each other and, hence,

$$\{\mathcal{M}, R\} = 0, \quad \{\mathcal{M}, D_S\} = 0, \quad \{D_S, R\} = 0.$$

(32)

An immediate consequence is that the matrix $\mathcal{M} = M$ is indefinite and corresponding to each of its $m$ positive eigenvalues $\lambda_i$, there exists an eigenvalue $-\lambda_i$. The term $\Pi^T \mathcal{M} \Pi$ in the Hamiltonian is not semi-positive definite and this has important consequences for the classical as well as quantum systems.
The Hamiltonian is not bounded from below even for a $V$ with a well-defined lower bound. Consequently, the question of stability of the system is much more involved compared to separable Hamiltonian. In general, the Lagrange-Dirichlet theorem\cite{39, 1} for Hamiltonian system does not give any conclusive results on the stability of the solutions.

The quantum problem is not well-defined on the real line and a well-defined ground state does not exist. However, if the quantum Hamiltonian is defined in appropriate Stokes wedges, the system may admit well defined bound states\cite{18, 19, 21, 28, 29, 31}. This is consistent from the viewpoint of axiomatic foundations of quantum mechanics. However, no experimentally realizable system of this type has been found so far.

The Hamiltonian of the Bateman oscillator is unbounded from below. However, for specific choices of $V$, the Hamiltonian may be bounded from below and admit periodic solutions. The Hamiltonian of Ref. \cite{19} and examples considered in Refs. \cite{18, 21, 28, 29, 30, 31} are bounded from below/above within some regions in the parameter-space. These examples admit stable periodic solutions at the classical level and quantum bound states.

The Lorentz interaction is switched on for $\alpha \neq 0$ and introduces external magnetic field in the system. On the other hand, the loss-gain terms may be interpreted as ‘fictitious magnetic field’ arising due to a ‘fictitious gauge potential’ containing in the generalized momenta. The eigenvalues of $M$ are $\alpha^2 \pm \lambda_i, i = 1, 2, \ldots, m$ semi-positive definite for $\alpha^2 \geq \max_i \lambda_i$. The stability properties of a system with balanced loss and gain is improved in this region. The famous Landau problem\cite{32} with balanced loss and gain has been studied in Ref. \cite{30} with various interesting results. In particular, the representation of the matrices $M, R, D$ may be considered as follows:

$$
M = \frac{1}{2} \begin{pmatrix} B + C & \gamma \\ \gamma & B - C \end{pmatrix},
R = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},
D = \frac{1}{2} \begin{pmatrix} -\gamma & B + C \\ -(B - C) & \gamma \end{pmatrix}
$$

for the Hamiltonian $H$ in Eq. (9) with $N = 2$ and $V = 0$. The system is described by the equations of motion,

$$
\ddot{x}_1 + \gamma \dot{x}_1 - B \dot{x}_2 - C \ddot{x}_2 = 0
\ddot{x}_2 - \gamma \dot{x}_2 + B \dot{x}_1 - C \ddot{x}_1 = 0
$$

where $B$ is the external uniform magnetic field, $\gamma$ is the loss-gain parameter and $C$ is the strength of non-Lorentzian velocity mediated coupling which appears in the study of synchronization of coupled oscillators\cite{33}. The eigenvalues of the matrix $M$ are,

$$
\lambda_{\pm} = \frac{1}{2} (B \pm \Delta), \quad \Delta \equiv \sqrt{C^2 + \gamma^2}.
$$

The condition for a positive-definite $M$ for $C = 0$ may be interpreted as the situation in which the magnitude of the external magnetic field $B$ supersedes the magnitude of the ‘fictitious magnetic field’ $\gamma$, i.e. $|B| > |\gamma|$. The matrix $M$ is singular for $B = \Delta$ and the parameter-space may be divided into three disjoint sectors:

(i) Region-I ($B > \Delta$): The Hamiltonian is bounded from below and the system admits periodic solutions with a reduced cyclotron frequency $\omega = \sqrt{B^2 - \Delta^2}$ compared to the standard Landau system with $\omega_L = B$.

(ii) Region-II ($-\Delta < B < \Delta$): The solutions are unbounded.

(iii) Region-III ($B < -\Delta$): The Hamiltonian is bounded from above and in fact, it is identical with the Hamiltonian in Region-I except for an overall multiplication factor of $-1$. The system admits periodic solutions.

The system does not admit any periodic solution for a vanishing Lorentz interaction, while periodic solutions are allowed for $B \neq 0$. The quantum Hamiltonian admits bound states in Region-I and III. Moreover, the quantum bound states are well defined on the real line without the need of defining the problem on any suitable Stokes wedges. The same feature persists for other Hamiltonian system with balanced loss and gain.
The details of classical and quantum solutions of the Landau Hamiltonian with balanced loss and gain along with Hall effect and underlying supersymmetry is described in Ref. [30].

A few comments are in order before the end of the section.

- The Bateman oscillator with a modified kinetic energy term has been considered in Ref. [12] in a different context. The system, when considered on a commutative space and cast into the notation of the present article, describes the model of Ref. [19] with Lorentz interaction. It can be shown that classical periodic solutions are obtained for an extended region of parameter-space compared to that of Ref. [19]. Further, the quantum problem is well defined on the real line.

- The gyroscopic force, which has the same form as Lorentz force, has been used for long to control dissipative induce instabilities in classical systems and there are plenty of important theorems and results[1]. The relevant results as appropriate to a system with balanced loss and gain may be used successfully.

The inclusion of Lorentz interaction in order to stabilize a classical system with balanced loss and gain as well as to define the quantum problem on the real line may be utilized for practical applications.

4 Solvable Models

Several solvable models with balanced loss and gain have been considered for $N \geq 2$ degrees of freedom in Refs. [19, 24, 25, 18, 28, 29, 30]. A few solvable models are discussed below to highlight the general features of these solvable models. The case of uniform external magnetic field and constant loss-gain terms has been discussed along with its physical relevance in Sec. 3.2.1. The space-dependent loss-gain terms make the equations highly nonlinear and in general, solvable models are rare. Nevertheless, a few solvable models with space-dependent loss-gain terms are known[29] which will not be discussed in this article. A few solvable models with vanishing Lorentz interaction and constant loss-gain terms are presented below, the details of which may be found in Ref. [18]. This corresponds to $\alpha = 0$ and $F_i = x_i$ in the representation of matrices in Sec. 3.2. The matrices have the following expressions,

$$M = I_m \otimes \sigma_1, \quad R = -i\gamma I_m \otimes \sigma_2, \quad D_S = \gamma I_m \otimes \sigma_3$$

and $\hat{O} = \frac{1}{\sqrt{2}}[I_m \otimes (\sigma_1 + \sigma_3)]$ diagonalizes $M$ to $M_D = I_m \otimes \sigma_3$. The orthogonal transformation generates the following set of new co-ordinates:

$$z_i^- = \frac{1}{\sqrt{2}}(x_{2i-1} - x_{2i}), \quad z_i^+ = \frac{1}{\sqrt{2}}(x_{2i-1} + x_{2i}), \quad i = 1, 2, \ldots m$$

The Eqs. of motion reads,

$$\ddot{z}_i^+ - 2\gamma \dot{z}_i^- + 2\frac{\partial V}{\partial z_i^+} = 0, \quad \ddot{z}_i^- - 2\gamma \dot{z}_i^+ - 2\frac{\partial V}{\partial z_i^-} = 0, \quad i = 1, 2, \ldots m.$$ (38)

In the subsequent discussions $2V$ is replaced by $V$. The Eqs. (38) is a set of $2m$ coupled differential equations which take simple form for (i) translational and (ii) rotational invariance of the system and solvable for specific choices of potential.

4.1 Translational invariance

The choice of the potential $V \equiv V(z_i^-)$ or $V \equiv V(z_i^+)$ allows a decoupling of Eq. (38). The potential $V(z_i^-)$ remains invariant under the translations $x_{2i-1} \to x_{2i-1} + \eta_i, x_{2i} \to x_{2i} + \eta_i$, where $\eta_i$’s are $m$ independent
parameters. The form of the potential is special in the sense that it allows \(m\) independent parameters \(\eta_i\) instead of a single one. The translational invariance leads to \(m\) integrals of motion \(\Pi_i\) which are in involution:

\[
\Pi_i = 2P_z^+ - \gamma z_i^-, \quad \{\Pi_i, \Pi_j\}_{PB} = 0, \quad \{H, \Pi_i\}_{PB} = 0, \quad (39)
\]

where \(\{,\}_{PB}\) denotes the Poisson bracket and \(P_z^+\) is the conjugate momenta corresponding to \(z_i^-\). The Hamiltonian along with \(\Pi_i\)'s constitute \(m + 1\) integrals of motion for a system with \(N = 2m\) particles, implying that the system is at least partially integrable for \(N > 2\) and completely integrable for \(N = 2\). A similar analysis can be performed for \(V(z_i^-)\). The discussion in this article is restricted to the case \(V = V(z_i^-)\).

With the introduction of new co-ordinates \(z_i = z_i^+ + \frac{\Pi_i}{\gamma}\), Eq. (38) can be rewritten as,

\[
\dot{z}_i^+ - 4\gamma^2 z_i^+ - \frac{\partial V(z_i)}{\partial z_i} = 0, \quad z_i^+(t) = 2\gamma \int z_i(t)\,dt + C_i, \quad i = 1, 2, \ldots, m, \quad (40)
\]

where \(C_i\) are \(m\) integration constants. The potential depends on the gain-loss parameter via \(z_i\), which may be avoided by choosing the constants of motion \(\Pi_i = 0 \quad \forall \ i\) leading to the initial conditions \(\dot{z}_i^+(0) = 2\gamma z_i^-(0) \quad \forall \ i\), where \(\dot{z}_i^+(0)\) and \(z_i^-(0)\) can be chosen depending on the physical requirements. The problem now lies to find \(V\) for which the first equation of Eq. \((40)\) is exactly solvable. Several examples involving coupled chain of cubic oscillators, potential solely dependent on the radial variable \(z\) in the sub-system defined by the co-ordinates \((z_1, z_2, \ldots, z_m)\), Calogero-type inverse square interaction, and Henon-Heils potential are considered in Ref. [18]. The case of cubic oscillator is discussed below.

The choice of \(V\) for the simplest case of \(N = 2, m = 1\) which produces a solvable system is the following:

\[
V(z_1) = -2\omega_0^2 z_1^2 - \frac{\alpha}{4} z_1^4, \quad \omega, \alpha \in \mathbb{R}, \quad \ddot{z}_1 + \omega^2 z_1 + \alpha z_1^3 = 0, \quad \omega^2 \equiv 4(\omega_0^2 - \gamma^2). \quad (41)
\]

There are three distinct regions in the parameter-space for which non-singular stable solutions can be obtained analytically in terms of Jacobi Elliptic functions — Region-I: \(\omega^2 > 0, \alpha > 0\), Region-II: \(\omega^2 > 0, \alpha < 0\) and Region-III: \(\omega^2 < 0, \alpha > 0\). The stability requires \(-\omega_0 < \gamma < \omega_0\) for region-I and Region-II, while \(-\gamma < \omega_0 < \gamma\) in region-III. The nonlinear interaction allows \(\gamma > \omega_0\) which has not been seen for system with linear interaction.

There is an additional constraint in each region involving the amplitude and frequency of the solution, and the nonlinear coupling \(\alpha\) for the existence of non-singular stable solution. For example, the solution in Region-I,

\[
z_1(t) = A \operatorname{cn}(\Omega t, k), \quad z_1^+(t) = \frac{2\gamma}{\Omega} \frac{\cos^{-1}\{dn(\Omega t, k)\}sn(\Omega t, k)}{\sqrt{1 - dn^2(\Omega t, k)}}
\]

\[
\Omega = \sqrt{\omega^2 + \alpha A^2}, \quad k^2 = \frac{\alpha A^2}{2\Omega^2}. \quad (42)
\]

is non-singular and stable for \(-\omega_0 < \gamma < \omega_0\) and \(0 < k < 1\). The solutions in other regions are given in Ref. [18]. The solvable models for higher \(N\) can be constructed by using the known results of coupled cubic oscillators[40]. The case of \(N = 4, m = 2\) along with its exact, non-singular and stable solutions are also discussed in Ref. [18].

### 4.2 Rotational Invariance

The Bateman oscillator admits a constant of motion in addition to the Hamiltonian \(H_B\),

\[
L_B = xy - y\dot{x} + 2\gamma xy \quad (43)
\]
and exactly solvable models may be constructed for suitable choices of $V$ of intricate issues\textsuperscript{4, 5, 6, 7, 8, 9, 10, 11, 12}, the quantization of the quartic oscillator corresponding to the allows stable solutions. Different quantization schemes as applied to Bateman oscillator have revealed a host of nonlinear interaction couples all the modes and the corresponding Hamiltonian is not separable like Bateman where the two modes are decoupled and there is no nonlinear interaction. However, inclusion of coupling among the two modes is always preserved. The same behaviour has been seen for the Bateman oscillator, where the variables of the decay and growth of the solutions are of the form $x_i(t)$ and $z_i(t)$ which are in involution, $\{\hat{L}_i, \hat{L}_j\}_{PB} = 0$, $\{H, \hat{L}_i\}_{PB} = 0$, implying that the system is at least partially integrable for $N > 2$ and integrable for $N = 2$ for a system of $N = 2m$ particles. The values of all the constants of motion are zero, i.e. $\hat{L}_i = 0 \forall i$ and $\hat{r}^2 = \sum_{i=1}^m q_i^2 \equiv q^2$ for the following parameterizations of the co-ordinates,

$$z_i^+(t) = q_i(t) \cosh(\gamma t), \quad z_i^-(t) = q_i(t) \sinh(\gamma t)$$

(48)

The equations of motion can be expressed solely in terms of $q_i$,

$$\ddot{q}_i - \gamma^2 q_i + \frac{1}{q} \frac{\partial V(q)}{\partial q} q_i = 0, \quad i = 1, 2, \ldots, m,$$

(49)

and exactly solvable models may be constructed for suitable choices of $V(q)$. For example, $V(q) = \frac{1}{4} \omega^2 q^2 + \frac{1}{4} \alpha q^4$ leads to an exactly solvable equation\textsuperscript{10},

$$\ddot{q}_i + \Omega^2 q_i + \alpha q^2 q_i = 0, \quad \Omega^2 = \omega^2 - \gamma^2.$$  

(50)

The Bateman oscillator is reproduced for $\alpha = 0, N = 2$ for which the two modes are not coupled. The nonlinear interaction couples all the modes and the corresponding Hamiltonian is not separable like Bateman oscillator. Although the above equation\textsuperscript{50} admits non-singular stable solutions, when expressed in terms of the original variables $x_i$, there are growing as well as decaying modes $-x_i$ are always decaying in time, while that of $x_{2i-1}$ grows with time. The situation can not be saved by allowing $\Omega$ to be complex, i.e $|\gamma| > |\omega|$, since the decay and growth of the solutions are of the form $e^{\pm \gamma t}$ which is independent of $\Omega$. The volume of the flow in the phase-space is always preserved. The same behaviour has been seen for the Bateman oscillator, where the two modes are decoupled and there is no nonlinear interaction. However, inclusion of coupling among different co-ordinates via nonlinear interaction does not change the situation for $\hat{L}_i = 0$. The solutions for $\hat{L}_i \neq 0$ may give stable solutions. The addition of a term that breaks the rotational invariance may also allow stable solutions. Different quantization schemes as applied to Bateman oscillator have revealed a host of intricate issues\textsuperscript{11, 12}, the quantization of the quartic oscillator corresponding to the $V(q)$ deserves further attention.
5 Role of $\mathcal{PT}$-symmetry

The discrete symmetries like parity($\mathcal{P}$), time-reversal($\mathcal{T}$) and charge-conjugation($\mathcal{C}$) play an important role in physics. The $\mathcal{PT}$-symmetric quantum mechanics is one such area where non-relativistic non-hermitian Hamiltonian with $\mathcal{PT}$ symmetry admit entirely real spectra in the unbroken $\mathcal{PT}$-regime[14]. In general, the eigen-states are not orthogonal and the unitary time-evolution is not possible. However, with a modified $\mathcal{CPT}$ norm, a consistent quantum description with unitary time-evolution is allowed[15]. There is no concept of anti-particles in non-relativistic quantum mechanics. The charge conjugation operator $\mathcal{C}$ is introduced with the expectation that at a more fundamental level it will be related to the $\mathcal{CPT}$ invariance of any Lorentz invariant local quantum field theory. It may be noted that the $\mathcal{CPT}$ theorem[12], which was originally derived for a hermitian Hamiltonian, has now been extended to the case of non-hermitian Hamiltonian[13]. There is no signature of violation of Lorentz invariance in nature so far and the $\mathcal{CPT}$ is considered to be a fundamental symmetry of nature.

The studies on $\mathcal{PT}$-symmetric systems have been diversified to many areas, including the obvious choice of $\mathcal{PT}$-symmetric classical systems. If the operator $\mathcal{C}$ of $\mathcal{CPT}$-norm of the $\mathcal{PT}$-symmetric quantum mechanics is expected to be related to the charge-conjugation operator of Lorentz invariant local quantum field theory, then the parity and the time-reversal operation for the classical non-relativistic $\mathcal{PT}$-symmetric system should necessarily be described by a linear transformation, since the Lorentz transformation itself is linear. Within this background, the time-reversal transformation $\mathcal{T}$ and the most general parity transformation $\mathcal{P}$ in two space dimensions may be defined as,

$$\mathcal{T}: t \rightarrow -t, \quad \hat{P}_x \rightarrow -\hat{P}_x, \quad \hat{P}_y \rightarrow -\hat{P}_y$$

$$\mathcal{P}: \begin{pmatrix} x \\ y \end{pmatrix} \rightarrow \begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} x \cos \theta + y \sin \theta \\ x \sin \theta - y \cos \theta \end{pmatrix}, \quad \begin{pmatrix} \hat{P}_x \\ \hat{P}_y \end{pmatrix} \rightarrow \begin{pmatrix} \hat{P}_x' \\ \hat{P}_y' \end{pmatrix} = \begin{pmatrix} \hat{P}_x \cos \theta + \hat{P}_y \sin \theta \\ \hat{P}_x \sin \theta - \hat{P}_y \cos \theta \end{pmatrix}$$

(51)

where $\theta \in (0, 2\pi)$. The Bateman oscillator in Eq. [1] is invariant under $\mathcal{PT}$ symmetry for two values of $\theta$, namely $\theta = \frac{\pi}{2}, \frac{3\pi}{2}$. However, the classical solutions are not invariant under $\mathcal{PT}$ symmetry, i.e. $\mathcal{PT}X \neq \pm X$ in any region of the parameter-space. Thus, there is no $\mathcal{PT}$-symmetric phase for the Bateman oscillator. Consider Eq. [2] with $G_1(x, y) = cy, G_2(x, y) = cx$ which describes the model considered in [19]. The system is $\mathcal{PT}$-symmetric and admits periodic solutions in certain regions in the parameter-space. There are both $\mathcal{PT}$-broken and $\mathcal{PT}$-unbroken regions for this model. The existence of periodic solutions in the $\mathcal{PT}$-unbroken phase lead further investigations on $\mathcal{PT}$-symmetric system with balanced loss and gain in various directions[24, 25, 26, 27, 18, 28, 29, 30] by including nonlinear interaction[24, 26, 27, 18, 29], many-particle system[25, 18, 28], Lorentz interaction[30], etc. The general result common to all these systems is that periodic solutions are obtained in the unbroken $\mathcal{PT}$ phase of a $\mathcal{PT}$-symmetric system.

It has been shown recently that non-$\mathcal{PT}$ symmetric classical system may also admit periodic solutions in a coupled Duffing-oscillator with balanced loss and gain[31]. This is a new result with far reaching consequences. Many non-$\mathcal{PT}$ symmetric system with balanced loss and gain may be included in the mainstream investigations which may exhibit novel features including existence of periodic solutions. It should be mentioned here that there exist many other non-$\mathcal{PT}$-symmetric systems with balanced loss and gain which admit periodic solutions in certain regions of the parameter space[38]. The examples include mechanical systems with finite degrees of freedom, models of dimer, non-relativistic field theory etc., which can be broadly classified into Hamiltonian and non-Hamiltonian systems. A few examples are described below.

5.1 Hamiltonian System

5.1.1 A coupled Duffing Oscillator model:

The equations of motion of the system are,

$$\ddot{x} + 2\gamma \dot{x} + \omega^2 x + \beta_1 y + gx^3 = 0,$$
\[ \ddot{y} - 2\gamma \dot{y} + \omega^2 y + \beta_2 x + 3g x^2 y = 0 \]  

(52)

The linear coupling between the two modes are asymmetric for \( \beta_1 \neq \beta_2 \). The system reduces to the one considered in Ref. [19] for \( g = 0, \beta_1 = \beta_2 \). The equation of motion for \( x \) degree of freedom decouples for \( \beta_1 = 0 \) and describes a damped Duffing oscillator, while the \( y \) degree of freedom is unidirectionally coupled to \( x \). There is no explicit forcing term. However, the term \( \beta_1 y \) in the first equation of (52) acts as a driving term for the damped Duffing oscillator in a non-trivial way. The system also admits a Hamiltonian,

\[ H_D = P_x P_y + \gamma (y P_y - x P_x) + (\omega^2 - \gamma^2) xy + \frac{1}{2} (\beta_2 x^2 + \beta_1 y^2) + gx^3 y, \]  

(53)

where \( P_x = \dot{y} - \gamma y \) and \( P_y = \dot{x} + \gamma x \) are canonical momenta. In general, the system is non-\( \mathcal{PT} \)-symmetric for the \( \mathcal{P} \) and \( \mathcal{T} \) defined in Eq. (51). The first, second and the third terms in \( H_D \) are invariant under \( \mathcal{PT} \) only for \( \theta = \frac{\pi}{2}, \frac{3\pi}{2} \). The fourth and the fifth terms representing linear asymmetric coupling and nonlinear interaction, respectively, are not invariant under \( \mathcal{PT} \) symmetry. It can been shown that for vanishing nonlinear coupling, i.e. \( g = 0 \), the corresponding non-\( \mathcal{PT} \)-symmetric system with asymmetric linear coupling does not admit any periodic solutions. This is consistent with the general folklore that a system with balanced loss and gain admits periodic solution only for a \( \mathcal{PT} \)-symmetric system. However, although the system is non-\( \mathcal{PT} \) symmetric for \( g \neq 0, \beta_1 = \beta_2 \), it admits periodic solutions [31] in some regions in the parameter-space.

The independent scales in the system may be fixed by employing the following transformations,

\[ t \rightarrow \omega^{-1} t, \quad x \rightarrow |\beta_2|^{-\frac{1}{2}} x, \quad y \rightarrow |\beta_1|^{-\frac{1}{2}} y, \quad \beta_1 \neq 0, \beta_2 \neq 0, \]  

(54)

which allows a reduction in total number of independent parameters which is convenient for analyzing the system. The model can be described in terms of three independent parameters \( \Gamma, \beta \) and \( \alpha \) defined as,

\[ \Gamma = \frac{\gamma}{\omega^2}, \quad \beta = \frac{1}{\beta_2 \omega^2}, \quad \alpha = \frac{g}{|\beta_2| \omega^2}, \]  

(55)

and the equations of motion have the following expressions:

\[ \ddot{x} + 2\Gamma \dot{x} + x + \text{sgn}(\beta_1) \beta y + \alpha x^3 = 0, \]

\[ \ddot{y} - 2\Gamma \dot{y} + y + \text{sgn}(\beta_2) \beta x + 3\alpha x^2 y = 0. \]  

(56)

The limit to the linear system \( g \rightarrow 0 \) now corresponds to \( \alpha \rightarrow 0 \). The system admits regular periodic as well as chaotic solutions for the case \( \text{sgn}(\beta_1) \text{sgn}(\beta_2) = 1 \) and the discussion is restricted to \( \text{sgn}(\beta_i) = 1, i = 1, 2 \). The results for \( \text{sgn}(\beta_i) = -1 \) can be obtained from that of results for \( \text{sgn}(\beta_i) = 1 \) by simply allowing \( \beta \rightarrow -\beta \).

The system admits five equilibrium points \( P_0, P_1^\pm, P_2^\pm \) in the phase-space \( (x, y, \dot{x}, \dot{y}) \) of the system, where \( \dot{x} = \dot{y} - \Gamma y, \quad \dot{y} = \dot{x} + \Gamma x \). The equilibrium points are,

\[ P_0 = (0, 0, 0, 0), P_1^\pm = (\pm \delta_+, \pm \eta_+, \mp \Gamma \eta_+, \pm \Gamma \delta_+), P_2^\pm = (\pm \delta_-, \pm \eta_-, \mp \Gamma \eta_-, \pm \Gamma \delta_-), \]  

(57)

where \( \delta_\pm \) and \( \eta_\pm \) are defined as follows:

\[ \delta_\pm = \frac{1}{\sqrt{3\alpha}} \left[ -2 \pm \sqrt{1 + 3\beta^2} \right]^\frac{1}{2}, \quad \eta_\pm = -\frac{\delta_\pm}{3\beta} \left[ 1 \mp \sqrt{1 + 3\beta^2} \right]. \]  

(58)

The application of Dirichlet theorem [39] is inconclusive. The linear stability analysis shows that the points \( P_0 \) and \( P_1^\pm \) are stable in the following regions of the parameter space:

\[ P_0 : \quad \frac{1}{\sqrt{2}} < \Gamma < \frac{1}{\sqrt{2}}, \quad 4\Gamma^2 (1 - \Gamma^2) < \beta^2 < 1 \]

\[ P_1^\pm : \quad \beta^2 > 1, \quad \Gamma^2 \leq \frac{\sqrt{2} - 1}{2}. \]  

(59)
The points $P_2^\pm$ are not stable anywhere in the parameter-space. These results are confirmed by perturbative and numerical analysis\cite{ref31} and the system admits periodic solutions around the equilibrium points. Regular solutions of Eq. (56) in the vicinity of the point $P_0$ and $P_1$ are plotted in Figs. 1 and 2 respectively for $\alpha = 1, \beta = .5, \Gamma = .2$.

Figure 1: (Color online) Regular solutions of Eq. (56) in the vicinity of the point $P_0$ with the initial conditions $x(0) = .1, y(0) = .2, \dot{x}(0) = .03, \dot{y}(0) = .04$ and $\alpha = 1, \beta = .5, \Gamma = .2$. (Reproduced from Ref. \cite{ref31})

The bifurcation diagram is plotted in Fig. 3 for varying $\beta$ and $\Gamma = 0.01, \alpha = .5$. It can be seen that the chaotic regime begins for $\beta > \beta_c \equiv 1.05$. In general, the chaotic regime starts beyond $|\beta| > 1$ for a range of values of $\alpha$ and $\Gamma$. The existence of chaotic dynamics is confirmed numerically through various means —sensitivity of time-series to the initial conditions, Poincaré section, power-spectra, auto-correlation functions and computation of Lyapunov exponents. The details of the numerical investigations are given in Ref.\cite{ref31}, only the plots of the Lyapunov exponents for the initial condition $x(0) = .01, y(0) = .02, \dot{x}(0) = .03, \dot{y}(0) = .04$ are shown in the left panel of the Fig. 4 which have the values (.13248, .0015691, -.0016145, -.13244). The standard result that the sum of the Lyapunov exponents are zero for a Hamiltonian system may be verified within the numerical approximations by taking the values of the Lyapunov exponents up to the third decimal places with an error of the order of $10^{-4}$.

### 5.1.2 Non-$\mathcal{PT}$-symmetric Positional Non-conservative force

The Hamiltonian $H_D$ shows chaotic behaviour even for $\Gamma = 0$, i.e. the system without the vanishing loss-gain terms, thereby providing an example of Hamiltonian chaos for two undamped Duffing oscillators coupled to each other in a specific way. In particular, the Hamiltonian $H_D$ can be rewritten for $\Gamma = 0$ as,

$$H_D = \left(\frac{1}{2} P_u^2 + \frac{\Omega_u}{2} u^2 + \frac{\alpha}{4} u^4\right) - \left(\frac{1}{2} P_v^2 + \frac{\Omega_v}{2} v^2 + \frac{\alpha}{4} v^4\right) + \frac{\alpha}{2} \frac{u v}{2} \left(u^2 - v^2\right), \Omega_{\pm} = 1 \pm \beta,$$

where the new co-ordinates and momenta are defined as,

$$u = \frac{x + y}{\sqrt{2}}, \quad v = \frac{x - y}{\sqrt{2}}, \quad P_u = \frac{\dot{x} + \dot{y}}{\sqrt{2}}, \quad P_v = \frac{\dot{x} - \dot{y}}{\sqrt{2}}.$$

This particular representation of $H_D$ describes two undamped Duffing oscillators, with different angular frequencies corresponding to the harmonic terms and identical nonlinear terms, coupled to each other through specified nonlinear interaction. The Lyapunov exponents are given in the the right panel of the Fig. 4 and
Figure 2: (Color online) Regular solutions of Eq. (56) in the vicinity of the point \( P_1^+ \) with the initial conditions \( x(0) = 0.2, y(0) = -0.1, \dot{x}(0) = .02, \dot{y}(0) = .03 \) and \( \alpha = 1, \beta = 1.01, \Gamma = .3 \). (Reproduced from Ref. [31])

Figure 3: (Color online) Bifurcation diagrams for \( \beta \) with \( \Gamma = 0.01 \) and \( \alpha = .5 \) with the initial conditions \( x(0) = 0.01, y(0) = .02, \dot{x}(0) = .03, \dot{y}(0) = .04 \). (Reproduced from Ref. [31])

have the values \((0.22685, 0.00431, -0.00431, -0.22685)\). It may be noted that the highest Lyapunov exponent for \( \Gamma = 0 \) is greater than the highest Lyapunov exponent for \( \Gamma = .01 \) with all other conditions remaining the same. The numerical results are described in detail in Ref. [31].

The standard damped Duffing oscillator with a driving term is known to show chaotic behaviour and a Hamiltonian formulation of the system is not known. However, for \( H_D \), there is neither any explicit damping term nor any driving term, yet the coupling between the modes allows a chaotic regime. This may be explained as follows by analyzing the equations of motion. In particular, the nonlinear terms in Eq. (56) with \( \Gamma = 0 \) can not be generated as the gradient of a potential function \( V \), i.e. there is no solution to the coupled partial differential equations,

\[
\frac{\partial V}{\partial x} = -gx^3, \quad \frac{\partial V}{\partial y} = -3gx^2y \tag{62}
\]

This implies that the system contains positional non-conservative forces\[1\] or curl-forces\[41\] which are known to admit chaotic behaviour. The driving is provided by the coupling term \( \beta_1 y \) in a non-standard way. This
Figure 4: (Color online) Lyapunov exponents for $\beta = 1.5, \alpha = .5$ and (a) $\Gamma = .01$ (left panel), (b) $\Gamma = 0$ (right panel). The initial conditions for both the cases are same, $x(0) = .01, y(0) = .02, \dot{x}(0) = .03, \dot{y}(0) = .04$. (Reproduced from Ref. [31])

model may play a significant role in the context of investigations on curl-forces admitting a Hamiltonian formalism and quantum chaos, and further investigations in this direction are required.

5.1.3 Landau Hamiltonian with balanced loss and gain:
The coupled Duffing oscillator model in Eq. (52) does not contain any velocity mediated coupling. The observation on a possible relation between $\mathcal{PT}$ symmetry and existence of the equilibrium state changes for system with velocity mediated coupling which may be explained by using Eq. (34) which describes Landau system with balanced loss and gain. The system is not $\mathcal{PT}$-symmetric. The second and the third terms in each equation of (34) breaks $\mathcal{PT}$ symmetry. However, the system admits periodic solutions in Region-I. The observation is that if the linear part of equations of motion of a system contains velocity mediated interaction and is non-$\mathcal{PT}$-symmetric, the system may admit periodic solutions. It may noted that the system of equations is invariant under the transformation:

$$x_1 \rightarrow x_2, x_2 \rightarrow -x_1, t \rightarrow -t, B \rightarrow -B.$$  

(63)

The transformation on the spatial co-ordinates correspond to a rotation by $\frac{\pi}{2}$ around an axis perpendicular the $'x_1 - x_2'$-plane, which can not be identified as discrete $\mathcal{P}$ transformation. The transformation involving $t$ and $B$ correspond to time-reversal symmetry.

5.1.4 Dimer & Non-linear Schrödinger Equation with balanced loss and gain:
The following dimer model describes time-evolution of amplitudes of $x$ and $y$ in the leading order of multiple time-scale analysis of Eq. (52) for $\Gamma \ll 1, \beta \ll 1$.

$$i \frac{\partial \Psi}{\partial t} + (i \Gamma_0 \sigma_3 + \beta \sigma_1) \Psi + \alpha \left( \frac{|\psi_1|^2 \psi_1}{2|\psi_1|^2 \psi_2 + \psi_1^* \psi_2} \right) = 0, \quad \Psi \equiv \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}$$  

(64)

The time-reversal transformation is given by $t \rightarrow -t, i \rightarrow -i$, while $\mathcal{P} : \Psi \rightarrow \sigma_1 \Psi$. The linear part of the equation, i.e. $\alpha = 0$, is invariant under $\mathcal{PT}$ transformation. However, the system is non-$\mathcal{PT}$-symmetric for $\alpha \neq 0$, yet it admits periodic solutions in some region in the parameter-space[30]. In particular, the Stokes variables,

$$Z_a = \frac{1}{2} \Psi^\dagger \sigma_a \Psi, \quad R = \frac{1}{2} \Psi^\dagger \Psi = \sqrt{Z_1^2 + Z_2^2 + Z_3^2}, \quad a = 1, 2, 3$$  

(65)

1Substitute $A \rightarrow \frac{1}{2} \Psi, T_2 \rightarrow t, \beta_0 \rightarrow 2\beta, \alpha \rightarrow \frac{\alpha}{8}$ in Eq. (43) of [31] to get this particular form of the equation.
satisfy a solvable linear equations from which the time-periodic solutions $\Psi$ may be constructed\cite{31}. Adding a dispersion term to Eq. (64) leads to a coupled non-linear Schrödinger equation with balanced loss and gain,

$$\frac{\partial \Psi}{\partial t} + (i\Gamma_0 \sigma_3 + \beta \sigma_1) \Psi + \frac{\partial^2 \Psi}{\partial x^2} + \alpha \left( |\psi_1|^2 \psi_1 + 2|\psi_1|^2 \psi_2 + \psi_1^2 \psi_1^* \right) = 0,$$  \hspace{1cm} (66)

the linear part of which is $\mathcal{P}\mathcal{T}$-symmetric, while the the nonlinear term explicitly breaks $\mathcal{P}\mathcal{T}$-symmetry. The moments $Z_a = \int dx \ Z_a, \ R = \int R \ dx$ satisfy the same time-evolution Eq. as satisfied by the Stokes variables given in Eq. (65) for well behaved fields $\Psi$ vanishing at asymptotic infinity. Thus, the moments are time-periodic for the same region for which the dimer model admits periodic solutions. The spatial degree of freedom has been integrated out and the analytic expression for $\Psi(x,t)$ can not be obtained from the time-dependence of the moments. However, the existence of periodic solutions in some region of the parameter-space is ensured.

5.2 Non-$\mathcal{P}\mathcal{T}$-symmetric Non-Hamiltonian System

The central focus of this review is on Hamiltonian system. However, taking a detour from the main line of discussions, an example of a non-$\mathcal{P}\mathcal{T}$-symmetric non-Hamiltonian System with balanced loss and gain that admits periodic solutions is presented in this section. The system is described by the equations of motion,

$$\ddot{x} + 2\gamma \dot{x} + \omega^2 x + \beta y + \alpha_1 x^3 = 0,$$

$$\ddot{y} - 2\gamma \dot{y} + \omega^2 y + \beta x + \alpha_2 y^3 = 0,$$  \hspace{1cm} (67)

which describe a damped Duffing oscillator linearly coupled to an anti-damped Duffing oscillator with different nonlinear strengths. The gain and loss terms are equally balanced and the flow in the position-velocity state space preserves its volume. However, no Hamiltonian for the system is known for $\alpha_1 \neq 0$ and/or $\alpha_2 \neq 0$. The system is Hamiltonian for $\alpha_1 = \alpha_2 = 0$ as well as $\mathcal{P}\mathcal{T}$-symmetric\cite{19}. The coupled Duffing oscillator model described by Eq. (67) is $\mathcal{P}\mathcal{T}$-symmetric for $\alpha_1 = \alpha_2$ and has been studied in detail in Ref. \cite{44}. The system is non-Hamiltonian and admits periodic solutions in the unbroken $\mathcal{P}\mathcal{T}$-regime. The nonlinear term breaks $\mathcal{P}\mathcal{T}$-symmetry for $\alpha_1 \neq \alpha_2$. It appears that the solutions of Eq. (67) have not been investigated for $\alpha_1 \neq \alpha_2$ for which the system is non-$\mathcal{P}\mathcal{T}$-symmetric. The linear stability analysis predicts periodic solutions around the equilibrium point $(0, 0, 0, 0)$ in the position-velocity state space for

$$-\frac{1}{\sqrt{2}} < \Gamma < \frac{1}{\sqrt{2}}.$$  \hspace{1cm} (68)

This is confirmed by numerical analysis and time-series of $x$ and $y$ for $\alpha_1 = 0.5, \alpha_2 = 1.0, \beta = 0.5, \Gamma = 0.1$ and are shown in the first row of Fig. 5. Plots of $\dot{x}$ vs $x$ and $\dot{y}$ vs. $y$ for the same values of the parameters are shown in the second row of Fig. 5. Numerical investigations show periodic solutions in a large region in the parameter space. The periodic solution of non-Hamiltonian non-$\mathcal{P}\mathcal{T}$-symmetric coupled Duffing oscillator model is presented in this review for the first time and has not been discussed previously.

5.3 A Conjecture

One important observation related to these investigations is that linear part of the equations for systems without any velocity mediated coupling is necessarily $\mathcal{P}\mathcal{T}$-symmetric in order to have periodic solutions, while the nonlinear part may or may not be $\mathcal{P}\mathcal{T}$-symmetric. It may be conjectured at this point that a non-$\mathcal{P}\mathcal{T}$ symmetric system with balanced loss-gain and without any velocity mediated coupling may admit periodic solution if the linear part of the equations of motion is necessarily $\mathcal{P}\mathcal{T}$ symmetric —the nonlinear interaction may or may not be $\mathcal{P}\mathcal{T}$-symmetric. Further, systems with velocity mediated coupling among different degrees of freedom need not be $\mathcal{PT}$ symmetric at all in order to admit periodic solutions. The conjecture has no contradiction with the known results on $\mathcal{P}\mathcal{T}$-symmetric systems. There is no general result in a model
Figure 5: (Color online) Eq. \ref{eq:67} is solved with the initial conditions $x(0) = 0.01, y(0) = 0.02, \dot{x}(0) = 0.03, \dot{y}(0) = 0.04$ for $\alpha_1 = 0.5, \alpha_2 = 1.0, \beta = 0.5$ and $\Gamma = 0.1$. Time-series of $x$ and $y$ are plotted in the first row. Plots of $\dot{x}$ vs. $x$ and $\dot{y}$ vs. $y$ are given in the second row.
independent way to suggest that $\mathcal{PT}$-symmetry of a system is necessary in order to have periodic solutions. On the contrary, it is known that non-Hamiltonian dimer model without any $\mathcal{PT}$ symmetry due to imbalanced loss and gain admits stable nonlinear supermodes[45]. Similarly, within the mean field description of Bose-Einstein condensate, stationary ground state is obtained for non-$\mathcal{PT}$-symmetric confining potential[46]. The results stated above and described in detail in Refs. [31, 38] are related to mechanical systems with finite degrees of freedom, dimers and nonlinear Schrödinger equation. Thus, the conjecture is supported by known results from diverse areas of classical physics.

The conjecture is also consistent with $\mathcal{PT}$-symmetric quantum system due to the following reasons:

- Different realizations of the linear operator $\mathcal{P}$ and the anti-linear operator $\mathcal{T}$ are allowed[18, 47] in quantum mechanics as long as $|\langle \tilde{\phi}_i | \psi_i \rangle| = |\langle \phi | \psi \rangle|$, where $\phi = \mathcal{P}\phi$, $\psi = \mathcal{P}\psi$, $\phi = \mathcal{T}\phi$, $\psi = \mathcal{T}\psi$ and $\phi, \psi, \phi_i, \psi_i$ are state vectors in the relevant Hilbert space. There may exist non-trivial realizations of $\mathcal{P}$ and $\mathcal{T}$ operators in the corresponding quantum system such that it is $\mathcal{PT}$ invariant. An example of this non-standard $\mathcal{T}$-symmetry within the context of $\mathcal{PT}$-symmetric theory is discussed in Ref. [18].

- The standard $\mathcal{PT}$-symmetry may be substituted with an appropriate anti-linear symmetry for which consistent quantum description of a non-hermitian Hamiltonian admitting entirely real spectra and unitary time-evolution is allowed. A pseudo-hermitian system[48] admits anti-linear symmetry. Thus, a non-$\mathcal{PT}$-symmetric classical system with balanced loss and gain and admitting periodic solutions may, upon quantization, become a pseudo-hermitian system.

There is no analogue of anti-linear symmetry and pseudo-hermiticity in the classical physics. This leads to fixing criterion for the existence of periodic solution of a system with balanced loss and gain solely in terms of $\mathcal{PT}$ symmetry. However, any criterion based on $\mathcal{PT}$ symmetry is inadequate and incomplete. A possible resolution of the problem may be to identify appropriate $\mathcal{PT}$-symmetry/anti-linear symmetry/pseudo-hermiticity of the corresponding quantized non-hermitian Hamiltonian which may explain the entirely real spectra with unitary time-evolution. The second step is to take the classical limit of the relevant anti-linear operator, which may not necessarily reduce to the standard $\mathcal{P}$ and $\mathcal{T}$ transformation, and fix the criteria for the existence of periodic solutions for the corresponding classical Hamiltonian. It should be mentioned here that an implementation of the scheme is tricky and nontrivial, since there may be more than one quantum system for a given classical Hamiltonian based on the quantization condition. An intelligent resolution of the problem is desirable.

6 Omitted topics

The discussion so far is restricted to mechanical system with finite degrees of freedom. The discrete and continuum models of non-linear Schrödinger equation with balanced loss and gain or its generalized versions appear in diverse contexts and have been studied extensively in the literature with interesting results[49]. The nonlinear Dirac equation with balanced loss-gain has also been investigated[50, 51]. A brief description on these topics is presented below.

6.1 Non-linear Schrödinger Equation with balanced loss and gain

The non-linear Schrödinger equation in one dimension is an integrable system with exact soliton solutions[52, 53, 54, 55]. Several generalizations of non-linear Schrödinger equation have been considered with the growing interest and relevance on $\mathcal{PT}$ symmetric theory. The $\mathcal{PT}$ symmetry motivated generalizations of non-linear Schrödinger equation may be broadly classified into three major sub-areas: (i) non-linear Schrödinger equation with $\mathcal{PT}$-symmetric confining complex potential[56], (ii) non-local non-linear Schrödinger equation[57, 58, 59] and (iii) non-linear Schrödinger Equation with balanced loss and gain[49, 60, 61, 62, 63, 64, 65, 66]. The topic non-linear Schrödinger equation with balanced loss and gain is relevant for the present review for which the literature is vast and the major results till 2016 are described in Ref. [49]. A recent result on a class of exactly solvable non-linear Schrödinger equation with balanced loss and gain is described below in brief.
The generic form of the non-linear Schrödinger equation with balanced loss and gain is given by,

$$i \left( I \frac{\partial}{\partial t} + iA \right) \Psi + \frac{\partial^2 \Psi}{\partial x^2} + V(x)\Psi + G(\Psi, \Psi^\dagger)\Psi = 0, \quad A := B + iC,$$

(69)

where $I$ is the $N \times N$ identity matrix, $\Psi$ is an $N$-component complex scalar field and $G$ is an $N \times N$ hermitian matrix depending on the field $\Psi$. The hermitian matrix $B$ describes linear coupling among different fields, while the traceless hermitian matrix $C$ describes balanced loss and gain. The time-dependent gain-loss and linear coupling terms may be considered by taking time-dependent matrices $C$ and $B$, respectively. The matrix $G$ encodes nonlinear coupling among different fields and may be chosen depending on the physical system. The external potential $V$ is generally taken as complex and $\mathcal{PT}$-symmetric. However, the discussion in this article is restricted to real $V$. Various soliton solutions of Eq. (69) for $V = 0$ and specific forms of $G$ have been obtained under certain reductions[49, 60, 61, 62, 63, 64, 65]. Recently, exactly solvable models of non-linear Schrödinger equation with balanced loss and gain have been constructed in Ref. [66] for $G = I \Psi^\dagger M \Psi$, where $M$ is a hermitian matrix independent of fields. The space-time modulation of the cubic nonlinearity may be introduced via the matrix $M$.

The continuity equation contains source and sink terms proportional to $\Psi^\dagger C \Psi$:

$$\frac{\partial \tilde{\rho}}{\partial t} + \frac{\partial j}{\partial x} = 2\Psi^\dagger C \Psi, \quad \tilde{\rho} = \Psi^\dagger \Psi, \quad j = -i \left[ \Psi^\dagger \eta \frac{\partial \Psi}{\partial x} - \eta \frac{\partial \Psi^\dagger}{\partial x} \Psi \right].$$

(70)

In general, the total density $\tilde{Q} = \int dx \tilde{\rho}$ is not conserved even for well-behaved $\Psi$ vanishing at asymptotic infinity, $\frac{\partial \tilde{Q}}{\partial t} = \int dx \Psi^\dagger C \Psi$. There may be specific field configurations with additional properties, like $\Psi^\dagger C \Psi$ being an odd function of $x$, for which $\tilde{Q}$ is conserved. It is important to note that a non-standard continuity equation with the associated conserved quantity can be derived for the special case of an $\eta$-pseudo-hermitian[48] $A$, i.e. $A^\dagger = \eta A \eta^{-1}$,

$$\frac{\partial \rho}{\partial t} + \frac{\partial J}{\partial x} = 0, \quad \rho = \Psi^\dagger \eta \Psi, \quad J = -i \left[ \Psi^\dagger \eta \frac{\partial \Psi}{\partial x} - \eta \frac{\partial \Psi^\dagger}{\partial x} \Psi \right].$$

(71)

The quantity $Q = \int dx \rho$ is a conserved quantity for the well-behaved fields $\Psi$ vanishing at infinity. The continuity equation holds irrespective of whether the matrix $\eta$ is positive-definite or indefinite. A positive-definite $\eta$ ensures a positive-definite $Q$. Further, it is known[48] that a pseudo-hermitian matrix $A$ with respect to a positive-definite $\eta$ admits entirely real eigenvalues which is required for the existence time-periodic solution of $\Psi$. It may be noted that Eq. (69) may or may not depend on $\eta$ and if it depends on $\eta$ at all, it should be through the matrix $G$.

The system described by Eq. (69) admits a Lagrangian $L = \int dx \mathcal{L}$,

$$\mathcal{L}_F = \frac{i}{2} \left[ \Psi^\dagger M (D_0 \Psi) - (D_0 \Psi)^\dagger M \Psi \right] - \frac{\partial \Psi^\dagger}{\partial x} M \frac{\partial \Psi}{\partial x} + V(x) \Psi^\dagger M \Psi + W(\Psi, \Psi^\dagger) + \Psi^\dagger F_1 \Psi,$$

(72)

where the matrix $G(\Psi, \Psi^\dagger)$ and the potential $W(\Psi, \Psi^\dagger)$ are related by the equation,

$$G = M^{-1} \frac{\partial W}{\partial \Psi^\dagger}.$$

(73)

and the hermitian matrix $M$ is independent of field $\Psi$. The operator $D_0 := I \frac{\partial}{\partial t} + iA$ has formal resemblance with the temporal component of covariant derivative with non-hermitian gauge potential $A$ and the anti-hermitian matrix $F_1 := \frac{i}{2} (A^\dagger M - MA)$. There may be alternative Lagrangian formulations of Eq. (69) for specific $G$. This particular formulation is chosen due to its conceptual similarity with $\mathcal{L}$ of Eq. (12) describing mechanical system with finite degrees of freedom —presence of fictitious gauge potential and a metric is common to both the formulations. The conjugate momenta corresponding to $\Psi$ and $\Psi^\dagger$ are
\( \Pi_\Psi = \frac{i}{2} \psi^\dagger M \) and \( \Pi_{\psi^\dagger} = -\frac{i}{2} M \psi \), respectively. The Hamiltonian density \( \mathcal{H}_F \) corresponding to \( \mathcal{L}_F \) has the form,

\[
\mathcal{H}_F = \frac{\partial \psi^\dagger}{\partial x} M \frac{\partial \psi}{\partial x} - V(x) \psi^\dagger M \psi - W(\psi, \psi^\dagger) + \psi^\dagger M A \psi
\]

(74)

In general, the Hamiltonian \( \mathcal{H}_F \) is complex-valued and the quantized Hamiltonian is expected to be non-hermitian. However, for the case of an \( M \)-pseudo-hermitian \( A \), \( \mathcal{H}_F \) is real-valued and the corresponding quantum Hamiltonian is hermitian with suitable quantization condition. An \( M \)-pseudo-hermitian matrix \( A \) may be used to define an \( M \)-pseudo-unitary matrix \( U := e^{-iA} \),

\[
U^\dagger M U = M \Leftrightarrow A^\dagger = MA^{-1} M
\]

(75)

The Hamiltonian \( \mathcal{H}_F \) is invariant under pseudo-unitary transformation \( \psi \rightarrow U \psi \) provided \( W \) is invariant under this transformation. For example, the Hamiltonian is invariant under pseudo-unitary transformation for the choice \( W \equiv W(\psi^\dagger M \psi) \). It should be noted that two systems connected by a unitary transformation can be considered as gauge equivalent. However, the same is not true for systems connected by pseudo-unitary transformations—the physical observable like density, square of the width of the wave-packet and its speed of growth have different expressions. The pseudo-unitary transformation can be used to construct solvable models.

6.1.1 \( V = 0 \) and \( W = \frac{i}{2} \left( \psi^\dagger M \psi \right)^2 \):

The non-linear Schrödinger equation in Eq. (69) takes the form,

\[
i \left( \frac{\partial}{\partial t} + iA \right) \psi + \frac{\partial^2 \psi}{\partial x^2} + \delta \left( \psi^\dagger M \psi \right) \psi = 0,
\]

(76)

Defining \( \psi = e^{-iA} \phi \) and using Eq. (69), Eq. (69) is mapped to the equation,

\[
i \phi_t = -\phi_{xx} - \delta \left( \psi^\dagger M \psi \right) \phi.
\]

(77)

Eq. (77) is exactly solvable and with appropriate unitary transformation followed by a scaling, it can be brought to the canonical form of Manakov-Zakharov-Schulman system. The pseudo-unitary mapping can be used to find exact solutions of Eq. (69) with \( V = 0 \) and \( G = \psi^\dagger M \psi \). Exactly solvable models with analytic expression for power-oscillation have been constructed in Ref. by using this mapping. If the non-hermitian matrix \( A \) is not pseudo-hermitian, then the mapping can be used to construct exact solutions for a non-autonomous system. Further, the mapping can be used to construct a variety of solvable models with interesting physical properties like time-dependent gain-loss terms, space-time modulated nonlinear interaction and external confining potential.

6.2 Oligomer with balanced loss and gain

The discrete non-linear Schrödinger equation with finite number of lattice points \( N \) or similar equation with generalized interaction term and self-trapping phenomena is popularly known as oligomer and the \( N = 2 \) is termed dimer. Such equations arise in the study of \( \mathcal{PT} \)-symmetric lattices, nonlinear dynamics of molecules, optics, Landau-Lifshitz equation etc.. The time-evolution of amplitude of waves arising in a mechanical and/or extended system under various approximation schemes may also be modelled as oligomer. For example, a multiple time-scale analysis of Eq. (52) leads to the dimer model describing the time-evolution of amplitudes in the leading order of the perturbation. The same recipe for studying the time-evolution of amplitudes of nonlinear oscillators has been used earlier in Refs. 64, 26, 27.

The equation satisfied by an oligomer has the general form,

\[
i \left( I \partial_t + iA \right) \psi + G(\psi, \psi^\dagger) \psi = 0
\]

(78)
It may be noted that Eq. (69) for vanishing external potential \((V = 0)\) and no dispersion term \(\frac{\partial^2 \psi}{\partial x^2}\) reduces to the oligomer model in Eq. (78). The matrices \(B, C\) and \(G(\Psi, \Psi^*)\) have the same interpretation as in the case of dimer model. The discrete non-linear Schrödinger equation with balanced loss and gain is reproduced for a tridiagonal \(B\) with \([G]_{ij} = \alpha \delta_{ij} |\Psi|^2\), where the real parameter \(\alpha\) denotes the strength of the nonlinear interaction. There are specific forms of \(B, C\) and \(G\) for which a Hamiltonian description is possible and the system may even be integrable \([26, 27, 73, 74]\). The literature on dimer with balanced loss and gain is vast and results with physical settings are nicely reviewed in Ref. [49]. A new class of exactly solvable dimer models which has not been discussed earlier is presented below.

A generic Lagrangian-Hamiltonian formulation of Eq. (78) may be presented by continuing with the general development for mechanical system with finite degrees of freedom and for the extended system. In particular, the Lagrangian and Hamiltonian are,

\[
\mathcal{L}_O = \frac{i}{2} \left[ \Psi^\dagger M \left( D_0 \Psi \right) - \left( D_0 \Psi^\dagger \right) M \Psi \right] + W(\Psi, \Psi^\dagger) + \Psi^\dagger F_1 \Psi,
\]

\[
\mathcal{H}_O = -W(\Psi, \Psi^\dagger) + \Psi^\dagger M A \Psi
\]

where \(G\) and \(W\) are related via Eq. (73). The conjugate momenta corresponding to \(\Psi\) and \(\Psi^\dagger\) are \(\Pi_\Psi = \frac{i}{2} \Psi^\dagger M\) and \(\Pi_{\Psi^\dagger} = -\frac{i}{2} M \Psi\), respectively. A new class of solvable models are obtained for the choice of \(W = \frac{\delta}{n+1} (\Psi^\dagger M \psi)^n\) for which Eq. (78) takes the form,

\[
i \left( I \partial_t + iA \right) \Psi + \delta \left( \Psi^\dagger M \Psi \right)^n \Psi = 0
\]

and is exactly solvable for a \(M\)-pseudo-hermitian \(A\). In particular, substituting \(\Psi = e^{-iAt} \Phi\) in eq. (79), \(\Phi\) satisfies the equation,

\[
i \partial_t \Phi + \delta (\Phi^\dagger M \Phi)^n \Phi = 0.
\]

It immediately follows that \(\Phi^\dagger M \Phi\) is a conserved quantity and denoting its value at \(t = 0\) as the real constant \(C\), i.e. \(\Phi^\dagger(0)M\Phi(0) = C\), \(\Phi\) is solved as,

\[
\Phi(t) = W e^{iCn t},
\]

where \(W\) is an arbitrary \(N\)-component constant complex vector. For the simplest case of a dimer, i.e. \(N = 2\), an example of \(M\)-pseudo-hermitian \(A\) may be presented as,

\[
A = \beta^* \sigma_+ + \beta \sigma_- + i \Gamma \sigma_3, \quad M = \alpha_0 I_2 + \alpha^* \sigma_+ + \alpha \sigma_-
\]

where \(I_2\) is the \(2 \times 2\) identity matrix and \(\sigma_\pm = \frac{1}{2} (\sigma_1 \pm \sigma_2)\). The condition for a \(M\)-pseudo-hermitian \(A\) with positive definite \(M\) has been derived in Ref. [66] in terms of the complex parameters \(\beta = |\beta| e^{i \beta_3}, \alpha = |\alpha| e^{i \alpha_3}\) and the real parameters \(\Gamma, \alpha_0\) as,

\[
\frac{\alpha_0}{|\alpha|} = \frac{|\beta|}{\Gamma} \sin(\theta_\alpha - \theta_\beta) > 0.
\]

Using the expression for \(U := e^{-iAt}\) in Ref. [66], the solutions for \(\Psi\) are obtained as,

\[
\Psi_1 = e^{iCn t} \left[ W_1 \left( \cos(\theta t) + \frac{\Gamma}{\theta} \sin(\theta t) \right) - i W_2 \beta^* \frac{\Gamma}{\theta} \sin(\theta t) \right]
\]

\[
\Psi_2 = e^{iCn t} \left[ -i W_1 \beta \frac{\Gamma}{\theta} \sin(\theta t) + W_2 \left( \cos(\theta t) - \frac{\Gamma}{\theta} \sin(\theta t) \right) \right]
\]

where \(\theta = \sqrt{|\beta|^2 - \Gamma^2}\). The solutions are periodic for \(|\beta| > \Gamma\) and unbounded for \(|\beta| \leq \Gamma\). The matrix \(U\) is unitary for \(\Gamma = 0\) and \(\Psi^\dagger \Psi\) is independent of time. However, for \(\Gamma \neq 0\), \(\Psi^\dagger \Psi\) is periodic in time. Results for higher values of \(N\) may be obtained in a similar way.
6.3 Nonlinear Dirac Equation with balanced loss and gain

The nonlinear Dirac equation appears in diverse branches of modern science[75, 76, 77, 78, 79], although it has not been studied as much as its non-relativistic counterpart, namely non-linear Schrödinger equation. With the advent of PT-symmetric theory, several nonlinear Dirac equations with balanced loss and gain have been considered in the literature[50, 51]. It may be recalled that the balanced loss and gain terms for the case of non-linear Schrödinger equation is introduced via a non-hermitian mass term in the Hamiltonian. The same approach is taken for the construction of nonlinear Dirac equation with balanced loss and gain. The Lagrangian density for the Dirac equation has the generic form

$$L_D = \overline{\Psi} (i\gamma^\mu \partial_\mu - m_1 - m_2 \gamma_5) \psi - F(\bar{\psi}, \psi)$$

(85)

with the non-hermitian mass term $\Psi (m_1 + m_2 \gamma_5) \psi$[80] and Lorentz invariant nonlinear interaction term $F$. The interaction term $F$ may be chosen depending on the physical settings. For example, $F_1 = F_1 (\bar{\psi} \psi)$ reproduces Soler model[75] for $m_2 = 0$ and in 1+1 space-time it is also known as Gross-Neveu model. The nonlinear interaction $F_2 = F_2 (\bar{\psi} \gamma_5 \psi)$ depends on pseudo-scalar $\bar{\psi} \gamma_5 \psi$. Similarly, the choice $F = J_\mu J^\mu$ with the current $J_\mu = \bar{\psi} \gamma_\mu \psi$ introduces vector-type self-interaction and the system describes massive Thirring model[77] for $m_2 = 0$. The interaction term $F$ need not be Lorentz invariant for non-relativistic systems with equations having formal resemblance with the Dirac equation. The Hamiltonian is complex-valued for $m_2 \neq 0$,

$$H_D = \bar{\Psi} (i\gamma^\mu \partial_\mu + m_1 + m_2 \gamma_5) \psi + F(\bar{\psi}, \psi),$$

(86)

and the quantum Hamiltonian is non-hermitian irrespective of whether $F$ is hermitian or not.

The equation of motion following from the Lagrangian reads,

$$(i\gamma^\mu \partial_\mu - m_1 - m_2 \gamma_5) \psi - \frac{\partial F}{\partial \bar{\psi}} = 0.$$  

(87)

The balanced loss-gain terms can be seen explicitly with the following representation of the $\gamma$ matrices,

$$\gamma^0 = \sigma_1 \otimes I_2, \quad \gamma^j = i\sigma_3 \otimes \sigma_j, \quad \gamma_5 := i\gamma^0 \gamma^1 \gamma^2 \gamma^3 = \sigma_2 \otimes I_2,$$

(88)

for which Eq. (87) reads,

$$i \left( D_{\theta} + i \mathcal{M} \right) \psi = -i\sigma_2 \otimes \sigma^j \bar{\Psi} \gamma^j \frac{\partial F}{\partial \bar{\psi}}, \quad \mathcal{M} := (m_1 \sigma_1 + im_2 \sigma_3) \otimes I_2.$$

(89)

The term appearing in the matrix $\mathcal{M}$ with the co-efficient $m_1$ describes linear coupling among different components of the spinor $\Psi$, while the term with the co-efficient $m_2$ describes balanced loss and gain. The loss-gain terms may be hidden by using a unitary equivalent representations of the $\gamma$ matrices. For example, the loss-gain terms are hidden in the Dirac-Pauli representation. The mass-matrix $\mathcal{M}$ is non-hermitian with doubly degenerate eigenvalues $\lambda_\pm = \pm \sqrt{m_1^2 - m_2^2}$ which are real provided $|m_2| < |m_1|$. The mass-gap $2\sqrt{m_1^2 - m_2^2}$ vanishes for $m_2 = \pm m_1$, signalling the existence of zero mode for the linear problem, i.e. $F = 0$.

The ansatz $\Psi = We^{i(\vec{r} - \omega t)}$ with $F = 0$ gives the dispersion $\omega^2 = k^2 + m_1^2 - m_2^2$, where $W$ is a four-component constant complex vector. The frequency $\omega$ is real for real mass, i.e. $|m_2| < |m_1|$ and $\Psi$ is periodic in time.

The charge $Q = \int d^3x \bar{\Psi} \psi$ is not a conserved quantity due to the non-hermitian mass term. However, a non-standard conserved charge $Q = \int d^3x \bar{\Psi} \psi$ can be introduced if a hermitian matrix $\eta$ exists satisfying the following conditions:

$$[\eta, \gamma^0 \gamma^j] = 0 \quad \forall \quad i, \quad \mathcal{M}_i^\dagger = \eta \mathcal{M} \eta^{-1}, \quad W^\dagger = W, \quad W \equiv \bar{\Psi} \psi \gamma^0 \frac{\partial F}{\partial \bar{\psi}}$$

(90)

The ansatz $\eta = \delta_1 I_1 + \delta_2 \sigma_2 \otimes I_2$ solves the first equation for arbitrary real constants $\delta_1$ and $\delta_2$, since $\gamma^0 \gamma^j = \sigma_2 \otimes \sigma_j$. The second equation demands that the mass-matrix $\mathcal{M}$ is $\eta$-pseudo-hermitian, which fixes
$\delta_1 = 1, \delta_2 = \frac{m_2}{m_1}$ so that $\eta = I_4 + \frac{m_2}{m_1} \gamma_5$. It may be noted that $\eta$ is positive-definite for $|m_1| > |m_2|$ and is singular for $m_2 = \pm m_1$. The conserved current $J^\mu$ has the expression $J^\mu = \bar{\Psi} \gamma^\mu \eta \Psi$ leading to the conservation of $Q$ for $F = 0$. This relation for the linear Dirac equation has been obtained earlier. The probability density $J^0$ receives contribution only from right- or left-handed degrees of freedom for $m_2 = m_1$ and $m_2 = -m_1$, respectively. The hermiticity of $W$ has to be ensured such that $J^\mu$ can also be taken as the conserved current for the nonlinear Dirac equation, i.e. $F \neq 0$. There are choices of $F$ for which $W$ is hermitian. For example, $W$ is hermitian for $F = F(\Psi M \Psi)$ provided the constant matrix $M$ satisfies the condition,

$$[M, \gamma^0] + \frac{m_2}{m_1} \{M, \gamma^0 \gamma_5\} = 0. \quad (91)$$

One important solution of the above equation is $M_1 = \eta$. The nonlinear interaction $F$ contains both Lorentz scalar and pseudo-scalar interactions, since $\bar{\Psi} \eta \Psi = \bar{\Psi} \Psi + \frac{m_2}{m_1} \bar{\Psi} \gamma_5 \Psi$. A more general solution can be constructed by taking $M_2 = \sigma_1 \otimes m$, where $m$ is an arbitrary $2 \times 2$ hermitian matrix. It should be emphasized here that the existence of a conserved charge with its interpretation as the probability density is not necessary in the study of nonlinear Dirac equation, since one is dealing with relativistic field theory instead of quantum mechanics. The existence of a conserved charge corresponds to an internal symmetry of the system. However, an appropriate number operator should be defined in the corresponding quantum field theory. The investigations on nonlinear Dirac equation with balanced loss and gain are mainly restricted to its classical solutions and stability in $1 + 1$ dimensions. Several choices of $F$ maintaining Lorentz invariance and $\mathcal{PT}$-symmetry have been considered in Refs. [50, 51] which admit stable periodic as well as soliton solutions.

7 Summary & Discussions

Classical Hamiltonian systems with balanced loss and gain have been reviewed in this article. The emphasis is on mechanical system with finite degrees of freedom. The criteria for a mechanical system to be identified as a system with balanced loss and gain, irrespective of whether a Lagrangian-Hamiltonian formulation is admissible or not, has been presented in terms of the volume conservation of flow in the position-velocity state-space. The Hamiltonian formulation for systems with space-dependent balanced loss and gain has been introduced for arbitrary number of particles and generic potential. It has been shown that the loss-gain terms may be removed completely through appropriate co-ordinate transformations with its effect manifested in modifying the strength of the velocity-mediated coupling. This mapping is inherent to the generic Hamiltonian system with balanced loss and gain and does not depend on any specific form of the potential or the number of particles. In general, the quadratic term in momenta in the Hamiltonian is not positive-definite leading to instabilities. The effect of the Lorentz interaction in improving the stability of classical solutions as well as allowing a possibility of defining the corresponding quantum problem consistently on the real line, instead of within Stokes wedges, has also been discussed.

The system with $N = 2m$ particles admits at least $m+1$ integrals of motion for a potential having specified type of translational or rotational symmetry, thereby implying that the system is at least partially integrable for $N > 2$ and completely integrable for $N = 2$. Several exactly solvable models based on translational and rotational symmetry and specific form of potentials have been discussed which include coupled cubic oscillators, Landau Hamiltonian etc. The Lorentz interaction appear naturally in the Landau Hamiltonian and there are regions in the parameter space where the quadratic term in momenta in the Hamiltonian is positive-definite. This is also the region for which stable classical solutions are obtained. The solution is same as the standard Landau problem with a reduced cyclotron frequency due to the loss-gain term.

An example of Hamiltonian chaos within the framework of a model of coupled Duffing oscillator with balanced loss and gain has been discussed. The dynamical properties of the system are rich —three out of five equilibrium points are stable and admits periodic solutions around these equilibrium points. The damped undriven Duffing oscillator is linearly coupled to another anti-damped oscillator with variable angular frequency depending on the degree of freedom of the Duffing oscillator. This coupling acts effectively as a
driving term, albeit in a nontrivial way. The chaotic behaviour is seen beyond a critical value of this coupling strength. The chaotic behaviour persists even for vanishing gain-loss terms, thereby providing an example of Hamiltonian chaos for coupled Duffing oscillator without any explicit damping and driving terms. It has been argued that the system contains positional non-conservative force or the curl-force, thereby compensating the effect of damping even if $\gamma = 0$. The coupling to the oscillator with variable frequency provides the effect of driving term. The corresponding quantum system may provide some insight into quantum chaos.

The role of $\mathcal{PT}$-symmetry on the existence of periodic solution in systems with balanced loss and gain has been critically analyzed. Examples from many-particle mechanical systems, dimer models and nonlinear Schrödinger equations without any $\mathcal{PT}$-symmetry is analyzed with the understanding that $\mathcal{P}$ corresponds to linear transformation only. This is because the Lorentz transformation is linear and no signature of its violation has been seen in nature. Moreover, $\mathcal{PT}$-symmetric quantum mechanics dwells on $\mathcal{CPT}$ norm, which at a more fundamental level is expected to correspond to the $\mathcal{CPT}$ invariance of a local hermitian Lorentz invariant theory. Based on the observations, it has been conjectured that non-$\mathcal{PT}$-symmetric system with balanced loss-gain and without any velocity mediated interaction may admit periodic solution if the linear part of the equations is necessarily $\mathcal{PT}$ symmetric — the nonlinear interaction may or may not be $\mathcal{PT}$-symmetric. Further, systems with velocity mediated coupling among different degrees of freedom need not be $\mathcal{PT}$ symmetric at all in order to admit periodic solutions. This conjecture has no contradiction with the formulation of $\mathcal{PT}$-symmetric quantum mechanics — the corresponding quantum system may be pseudo-hermitian or invariant under generalized time-reversal symmetry which has no analog in classical mechanics. The criteria for the existence of periodic solutions in terms of standard $\mathcal{PT}$-symmetry alone is not sufficient. Investigations in this direction is desirable so that a large number of non-$\mathcal{PT}$ symmetric system may be included in the mainstream of investigations with possible technological applications.

The central focus of this review is on mechanical system with finite degrees of freedom. However, there is significant advancement in the fields of oligomers, nonlinear Schrödinger and Dirac equations with balanced loss and gain. The developments in the context of oligomers and nonlinear Schrödinger equations are summarized recently in Ref. and excluded for extensive discussions in this review. A very recent result on non-linear Schrödinger equation with balanced loss and gain related to the construction of exactly solvable models via non-unitary transformation has been discussed. This mapping can be used to construct a variety of solvable models with interesting physical properties. The same technique is used to construct an exactly solvable dimer with balanced loss and gain which has not appeared earlier in the literature. Results related to nonlinear Dirac equations with balanced loss and gain are mentioned briefly.

8 Acknowledgments

This work is supported by a grant (SERB Ref. No. MTR/2018/001036) from the Science & Engineering Research Board(SERB), Department of Science & Technology, Govt. of India under the MATRICS scheme. The Author would like to thank Debdeep Sinha, Puspendu Roy and Supriyo Ghosh for discussions and collaboration on the topic.

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