NON-CLOSED MINIMAL HYPERSURFACES OF $S^4(1)$ WITH IDENTICALLY ZERO GAUSS-KRONECKER CURVATURE

TSASA LUSALA

ABSTRACT. We give a partial local description of minimal hypersurfaces $M^3$ with identically zero Gauß-Kronecker curvature function in the unit 4-sphere $S^4(1)$, without assumption on the compactness of $M^3$.

Keywords and phrases: Minimal hypersurfaces in spheres, isoparametric hypersurfaces, identically zero Gauß-Kronecker curvature, nowhere zero second fundamental form.

§1. INTRODUCTION

Let $x: M^3 \rightarrow S^4(1) \subset \mathbb{R}^5$ be a hypersurface immersion of a connected and orientable 3-dimensional manifold $M^3$ of class $C^\infty$ into $S^4(1) \subset \mathbb{R}^5$. Let $\lambda_1$, $\lambda_2$ and $\lambda_3$ be the three principal curvature functions. The normalized elementary symmetric curvature functions of the immersion $x$ are given by:

$$H := \frac{1}{3}(\lambda_1 + \lambda_2 + \lambda_3),$$
$$H_2 := \frac{1}{3}(\lambda_1\lambda_2 + \lambda_1\lambda_3 + \lambda_2\lambda_3),$$
$$K := \lambda_1\lambda_2\lambda_3.$$

S. Almeida and F. Brito [1] suggested to classify closed hypersurface immersions for which two of the three functions $H$, $H_2$, $K$ are constant. The paper [3] gives a survey of results on closed hypersurfaces in $S^4(1)$ with two constant curvature functions. Particularly, the paper [2] investigated closed minimal hypersurfaces with constant Gauß-Kronecker curvature function, corresponding to $H \equiv 0$ and $K \equiv const$. There it is proved that closed minimal hypersurfaces with constant Gauß-Kronecker curvature $K \neq 0$ are isoparametric, therefore closed minimal hypersurfaces with constant Gauß-Kronecker curvature $K \neq 0$ are classified. Brito conjectured that all hypersurfaces in $S^4(1)$ with $K \equiv const \neq 0$ and $H \equiv const$ (or $H_2 \equiv const$) must be isoparametric (personal communication). If $K \equiv 0$ on $M^3$, the following is well known: a closed minimal hypersurface immersion in $S^4(1)$ with nowhere zero second fundamental form is a boundary of a tube

2000 Mathematics Subject Classification. 53 B 25, 53 C 40.

This work was partially supported by the DFG-project Si-163/7-2 and a DFG-NSFC exchange project.
which is built over a non-degenerate minimal 2-dimensional surface immersion in \( S^4(1) \) with geodesic radius \( \frac{\pi}{2} \). This nice result proves the existence of non-isoparametric closed minimal hypersurfaces with \( K \equiv 0 \) in \( S^4(1) \). But so far no explicit non-isoparametric example has been given. In this paper we investigate local descriptions of minimal hypersurfaces (not necessarily closed) in \( S^4(1) \) with identically zero Gauß-Kronecker curvature \( K \), but with nowhere zero second fundamental form. In particular we present the following two explicit non-isoparametric examples:

**Example 1.1.** The mapping

\[
x_1: \mathbb{R}^3 \rightarrow S^4(1) \subset \mathbb{R}^5
\]

\[
x_1(u, v, z) = \frac{1}{\sqrt{1 + z^2}} \left( \cos(\sqrt{2}u)C_1 + \sin(\sqrt{2}u)C_2 + \cos(\sqrt{2}v)C_3 + \sin(\sqrt{2}v)C_4 + zC_5 \right),
\]

where \( C_1, C_2, C_3, C_4, C_5 \in \mathbb{R}^5 \) are constant orthogonal vectors in \( \mathbb{R}^5 \) such that

\[
\frac{1}{2} = < C_1, C_1 > < C_2, C_2 > < C_3, C_3 > < C_4, C_4 > < C_5, C_5 > = 1,
\]

defines a minimal hypersurface immersion with zero Gauß-Kroneker curvature. The principal curvature functions take the values \( \lambda_1(u, v, z) = \sqrt{z^2 + 1} \), \( \lambda_2(z) = -\sqrt{z^2 + 1} \) and \( \lambda_3(z) = 0 \); they depend only on \( z \).

**Example 1.2.** Let \( I \subset \mathbb{R} \) be an open interval, \( 0 < c_1, c_2 \in \mathbb{R} \) such that \( c_2e^{2v} - 1 - c_1^2e^{4v} > 0 \) for all \( v \in I \), and \( g, h: I \rightarrow \mathbb{R} \) two differentiable functions on \( I \) which are linearly independent solutions of the second order differential equation

\[
(c_2e^{2v} - 1 - c_1^2e^{4v})A''_5(v) + (1 - c_1^2e^{4v})A'_5(v) + 2A_5(v) = 0,
\]

and such that \( g^2(v) + h^2(v) = 1 - c_1^2e^{2v} \), for all \( v \in I \). The existence of such functions will be proved below, see Lemma 3.4 and Remark 3.5. Then the mapping

\[
x_2: \mathbb{R} \times I \times \mathbb{R} \rightarrow S^4(1) \subset \mathbb{R}^5,
\]

\[
x_2(u, v, z) = \frac{e^{-v}}{\sqrt{c_2(z^2 + 1)}} \left( \cos(u)C_1 + \sin(u)C_2 \right) + \frac{1}{\sqrt{z^2 + 1}} \left( zC_3 + g(v)C_4 + h(v)C_5 \right),
\]

where \( C_1, C_2, C_3, C_4, C_5 \in \mathbb{R}^5 \) are constant orthonormal vectors, defines a minimal hypersurface immersion in \( S^4(1) \) with identically zero Gauß-Kroneker curvature. The principal curvatures take the values

\[
\lambda_1(v, z) = c_1e^{2v}\sqrt{z^2 + 1} \quad \lambda_2(v, z) = -c_1e^{2v}\sqrt{z^2 + 1} \quad \text{and} \quad \lambda_3(v, z) = 0,
\]

thus they depend only on \( v \) and \( z \).

The principal curvatures of both examples do depend on at most two parameters. We prove the following local classification of such hypersurfaces.

**Main result:** Let \( x: M^3 \rightarrow S^4(1) \subset \mathbb{R}^4 \) be a minimal hypersurface immersion (with nowhere zero second fundamental form) of a connected and orientable \( C^\infty \)-manifold \( M^3 \) in \( S^4(1) \) with identically zero Gauß-Kroneker curvature. If one of the two nowhere zero principal curvature functions is constant along its associated principal curvature line, then there exist local coordinates so that the immersion \( x \) locally can be described by one of the two non-isoparametric hypersurfaces \( x_1 \) and \( x_2 \) (see Example 1.1 and Example 1.2) above, or locally by Cartan’s minimal isoparametric hypersurface with principal curvatures \( \sqrt{3}, -\sqrt{3} \) and 0.
§2. Notations and integrability conditions

Let \( x : M^3 \rightarrow \mathbb{S}^4(1) \subset \mathbb{R}^5 \) be an immersion of a connected, orientable 3-dimensional \( C^\infty \)-manifold \( M^3 \) into the unit 4-sphere \( \mathbb{S}^4(1) \). Denote by \( y \) a unit normal vector field on \( \mathbb{S}^4(1) \) along the immersion \( x \), by \( <, > \) the canonical inner product of the Euclidean structure, and by \( \nabla \) the flat connection of \( \mathbb{R}^5 \). Referring to [4] for details on geometry of submanifolds, recall that as immersion of codimension 2 in \( \mathbb{R}^5 \) the structure equations (Gauß and Weingarten equations) for \( x \) state:

\[
\begin{aligned}
\nabla_u dx(v) &:= dx(\nabla_u v) + \mathcal{I}(u, v)y - \mathcal{I}(v, u)x, \\
\quad dy(u) &:= -dx(Su),
\end{aligned}
\tag{2.1}
\]

for all \( u, v \in TM^3 \).

where \( \mathcal{I} \) denotes the first fundamental form (induced metric) with Levi-Civita connection \( \nabla \), \( \mathcal{I} \) defines the second fundamental form and \( S \) denotes the shape operator.

The structure equations imply the following integrability conditions (Gauß formula and Codazzi equation) for any \( u, v, w \in TM^3 \):

\[
\begin{aligned}
R(u, v)w &= \mathcal{I}(w, v)u - \mathcal{I}(w, u)v + \mathcal{I}(u, v)Su - \mathcal{I}(v, u)Sv, \\
(\nabla_u S)v &= (\nabla_v S)u,
\end{aligned}
\tag{2.2}
\tag{2.3}
\]

where \( R \) denotes the Riemannian curvature tensor for the induced metric \( \mathcal{I} \).

Let \( (e_1, e_2, e_3) \) be a \( \mathcal{I} \)-orthonormal local differentiable frame of principal curvature vector fields on \( M^3 \):

\[
Se_1 = \lambda_1 e_1, \quad Se_2 = \lambda_2 e_2 \quad \text{and} \quad Se_3 = \lambda_3 e_3.
\]

There are 9 functions \( \alpha_1, \cdots, \alpha_9 \) such that

\[
\begin{aligned}
\nabla e_1 e_1 &= \alpha_1 e_2 + \alpha_2 e_3, \quad \nabla e_1 e_2 = -\alpha_1 e_1 + \alpha_3 e_3, \quad \nabla e_1 e_3 = -\alpha_2 e_1 - \alpha_3 e_2; \\
\nabla e_2 e_1 &= -\alpha_4 e_2 + \alpha_6 e_3, \quad \nabla e_2 e_2 = \alpha_4 e_1 + \alpha_5 e_3, \quad \nabla e_2 e_3 = -\alpha_6 e_1 - \alpha_5 e_2; \\
\nabla e_3 e_1 &= \alpha_9 e_2 - \alpha_7 e_3, \quad \nabla e_3 e_2 = -\alpha_9 e_1 - \alpha_8 e_3, \quad \nabla e_3 e_3 = \alpha_7 e_1 + \alpha_8 e_2.
\end{aligned}
\tag{2.4}
\]

Remark 2.1. Consider the situation that the three principal curvature functions \( \lambda_1, \lambda_2, \lambda_3 \) are everywhere distinct; the fact that the frame \( (e_1, e_2, e_3) \) is orthonormal implies that the functions \( \alpha_i, i = 1, \cdots, 9 \) in (2.4) are defined on \( M^3 \) uniquely up to sign.

Applying the Codazzi equation (2.3) to the vector fields \( e_1, e_2, e_3 \) and using (2.4), one gets the following equations:

\[
\begin{aligned}
e_1(\lambda_2) &= \alpha_4(\lambda_2 - \lambda_1), \\
e_1(\lambda_3) &= \alpha_7(\lambda_3 - \lambda_1), \\
e_2(\lambda_1) &= \alpha_1(\lambda_1 - \lambda_2), \\
e_2(\lambda_3) &= \alpha_8(\lambda_3 - \lambda_2), \\
e_3(\lambda_1) &= \alpha_2(\lambda_1 - \lambda_3), \\
e_3(\lambda_2) &= \alpha_5(\lambda_2 - \lambda_3), \\
\alpha_9(\lambda_1 - \lambda_2) &= \alpha_3(\lambda_2 - \lambda_3) = \alpha_6(\lambda_1 - \lambda_3).
\end{aligned}
\tag{2.5}
\]
From now we assume that the immersion $x$ (with nowhere zero second fundamental form) is minimal and has identically zero Gauß-Kronecker curvature ($K \equiv 0$). There exists a positive non-zero function $\lambda$ such that the principal curvature functions associated to the immersion $x$ are $\lambda_1 = \lambda$, $\lambda_2 = -\lambda$ and $\lambda_3 = 0$. From the equations (2.5), one gets

$$
\begin{align*}
(2.6) & \quad e_1(\lambda) = 2\alpha_4\lambda, \quad e_2(\lambda) = 2\alpha_1\lambda, \quad e_3(\lambda) = \alpha_2\lambda; \\
(2.7) & \quad \alpha_5 = \alpha_2, \quad 2\alpha_9 = -\alpha_3 = \alpha_6, \quad \alpha_7 = 0 = \alpha_8.
\end{align*}
$$

Applying the Gauß formula (2.2) to the vector fields $e_1, e_2, e_3$ and using the equations (2.4) and (2.7), one gets

$$
\begin{align*}
(2.8) \quad & \begin{cases}
  e_1(\alpha_4) + e_2(\alpha_1) = 1 - \lambda^2 + \alpha_1^2 + \alpha_2^2 + 2\alpha_3^2 + \alpha_4^2 \\
  e_3(\alpha_1) + \frac{1}{2} e_1(\alpha_3) = \alpha_1\alpha_2 - \frac{1}{2}\alpha_3\alpha_4 \\
  e_3(\alpha_4) - \frac{1}{2} e_2(\alpha_3) = \alpha_2\alpha_4 + \frac{1}{2}\alpha_1\alpha_3, \\
  e_3(\alpha_2) = 1 + \alpha_2^2 - \alpha_3^2, \\
  e_3(\alpha_3) = 2\alpha_2\alpha_3, \\
  e_1(\alpha_2) = e_2(\alpha_3), \\
  e_1(\alpha_3) = -e_2(\alpha_2).
\end{cases}
\end{align*}
$$

Note that the Lie brackets with respect to the vector fields $e_1, e_2$ and $e_3$ are given by:

$$
[e_1, e_2] = -\alpha_1 e_1 + \alpha_4 e_2 + 2\alpha_3 e_3, \quad [e_1, e_3] = -\alpha_2 e_1 - \frac{1}{2}\alpha_3 e_2, \quad [e_2, e_3] = \frac{1}{2}\alpha_3 e_1 - \alpha_2 e_2.
$$

The fundamental equations (2.1) applied to the vector fields $e_1, e_2, e_3$ give rise to the following (partial) differential equations:

$$
\begin{align*}
(2.9) \quad & \begin{cases}
  \nabla_{e_1} dx(e_1) = \alpha_1 dx(e_2) + \alpha_2 dx(e_3) + \lambda y - x \\
  \nabla_{e_1} dx(e_2) = -\alpha_1 dx(e_1) + \alpha_3 dx(e_3) \\
  \nabla_{e_1} dx(e_3) = -\alpha_2 dx(e_1) - \alpha_3 dx(e_2) \\
  \nabla_{e_2} dx(e_1) = -\alpha_4 dx(e_2) - \alpha_3 dx(e_3) \\
  \nabla_{e_2} dx(e_2) = \alpha_4 dx(e_1) + \alpha_2 dx(e_3) - \lambda y - x \\
  \nabla_{e_2} dx(e_3) = \alpha_3 dx(e_2) - \alpha_2 dx(e_2) \\
  \nabla_{e_3} dx(e_1) = -\frac{1}{2}\alpha_3 dx(e_2) \\
  \nabla_{e_3} dx(e_2) = \frac{1}{2}\alpha_3 dx(e_1) \\
  \nabla_{e_3} dx(e_3) = -x \\
  dy(e_1) = -\lambda dx(e_1), \\
  dy(e_2) = \lambda dx(e_2), \\
  dy(e_3) = 0.
\end{cases}
\end{align*}
$$
§3. Proof of the main result

To describe locally hypersurface immersions in $\mathbb{S}^4(1)$ with $K \equiv 0$, one has to find local coordinates to solve the structure equations (2.9) using the integrability conditions (2.6) and (2.8). It seems to be very difficult to solve this problem in full generality.

In this section, we consider natural additional assumptions on the functions $\alpha_1$, $\alpha_2$, $\alpha_3$ and $\alpha_4$ to solve the structure equations (fundamental equations) for minimal hypersurface immersions in $\mathbb{S}^4(1)$ with $K \equiv 0$; namely we assume that the function $\lambda$ is constant along the $e_1$-direction. This additional assumption is suggested by the Examples 1.1 and 1.2.

**Proposition 3.1.** Let $x: M^3 \longrightarrow \mathbb{S}^4(1)$ be a minimal hypersurface immersion with identically zero Gauß-Kronecker curvature function and nowhere zero second fundamental form. Let $\alpha_1, \ldots, \alpha_9$ be the functions as defined in (2.4). If the function $\alpha_4$ vanishes identically on an open subset $U$ of $M^3$, i.e., the function $\lambda$ is constant along the curvature line of the vector field $e_1$, then also the function $\alpha_3$ vanishes identically on $U$, or the immersion $x$ is a minimal Cartan isoparametric hypersurface on $U$.

**Proof.** Using the equations (2.6) and
\[
e_1e_2(\lambda) - e_2e_1(\lambda) - [e_1, e_2](\lambda) = 0 = e_1e_3(\lambda) - e_2e_1(\lambda) - [e_1, e_3](\lambda),
\]
one gets:
\[
e_1(\alpha_1) = \alpha_2\alpha_3 \quad \text{and} \quad e_1(\alpha_2) = -\alpha_1\alpha_3.
\]
Similarly, using (2.8) and
\[
e_2e_3(\alpha_3) - e_3e_2(\alpha_3) - [e_2, e_3](\alpha_3) = 0,
\]
one has:
\[
\alpha_3e_2(\alpha_2) = 0.
\]
Assume now that $\alpha_3 \neq 0$ everywhere. This implies that $e_2(\alpha_2)$ vanishes identically. Inserting again the equations (2.6) into
\[
e_1e_2(\alpha_3) - e_2e_1(\alpha_3) - [e_1, e_2](\alpha_3) = 0,
\]
one gets:
\[
\alpha_3^2\alpha_2 = 0.
\]
Therefore $\alpha_2 = 0 = \alpha_1$ and $\alpha_3 = \pm 1$. Consequently, $\lambda^2 = 3$. Thus the immersion is isoparametric with principal curvatures $\lambda_1 = \sqrt{3}$, $\lambda_2 = -\sqrt{3}$ and $\lambda_3 = 0$, i.e. the immersion is a Cartan’s minimal isoparametric hypersurface in $\mathbb{S}^4(1)$.

**Corollary 3.2.** Let $x: M^3 \longrightarrow \mathbb{S}^4(1) \subset \mathbb{R}^5$ be a closed minimal hypersurface immersion of a connected and orientable manifold $M^3$ into $\mathbb{S}^4(1) \subset \mathbb{R}^5$ with nowhere zero second fundamental form and $K \equiv 0$. Assume that one of the functions $\alpha_1$ and $\alpha_4$ vanish identically on $M^3$. Then $x(M)$ is a Cartan’s minimal isoparametric hypersurface of $\mathbb{S}^4(1)$, i.e., the boundary of the tube $\text{Tube}(V^2, \pi)$ with radius $\pi$ around the Veronese surface $V^2 \subset \mathbb{S}^4(1)$. 

Proof. From the proposition above we have two possibilities:

(i) \( \alpha_3 \) vanishes identically on \( M^3 \);
(ii) or \( \alpha_1 = \alpha_2 = \alpha_4 \equiv 0 \) and \( \alpha_3^2 = 1 \).

Assuming that the hypersurface \( M^3 \) is closed, the case (i) above cannot happen because the function \( e_3(\alpha_2) = 1 + \alpha_2^2 - \alpha_3^2 \) should be zero at the minimum and maximum points of the function \( \alpha_2 \); but with \( \alpha_3 \equiv 0 \), \( e_3(\alpha_2) = 1 + \alpha_2^2 \) has no zeros. □

Proposition 3.3. Let \( x: M^3 \rightarrow S^4(1) \subset \mathbb{R}^5 \) be a minimal hypersurface immersion of a connected and orientable manifold \( M^3 \) into \( S^4(1) \subset \mathbb{R}^5 \) with nowhere zero second fundamental form and \( K \equiv 0 \). Assume that the immersion is non-isoparametric and the functions \( \alpha_1 \) and \( \alpha_4 \) vanish identically on \( M^3 \). Then there are local coordinates so that the immersion \( x \) can be locally described by the parametrization of the hypersurface given in Example 1.1.

Proof. From Proposition 3.1 we may assume that the function \( \alpha_3 \) vanishes identically on \( M^3 \). Then the following equations hold:

\[
\begin{align*}
e_2(\alpha_2) &= 0 = e_1(\alpha_2) = e_1(\lambda) = e_2(\lambda), \\
e_3(\alpha_2) &= \alpha_2^2 + 1, \\
e_3(\lambda) &= \lambda \alpha_2, \\
\lambda^2 &= \alpha_2^2 + 1.
\end{align*}
\]

The vector fields \( \frac{1}{\lambda} e_1 \), \( \frac{1}{\lambda} e_2 \) and \( \frac{1}{\alpha_2^2 + 1} e_3 \) satisfy:

\[
0 = \left[ \frac{1}{\lambda} e_1, \frac{1}{\lambda} e_2 \right] = \left[ \frac{1}{\lambda} e_1, \frac{1}{\alpha_2^2 + 1} e_3 \right] = \left[ \frac{1}{\lambda} e_2, \frac{1}{\alpha_2^2 + 1} e_3 \right].
\]

Therefore there are local coordinates \((u,v,z)\) on \( M^3 \) such that

\[
\frac{1}{\lambda} e_1 = \frac{\partial}{\partial u}, \quad \frac{1}{\lambda} e_2 = \frac{\partial}{\partial v}, \quad \frac{1}{\alpha_2^2 + 1} e_3 = \frac{\partial}{\partial z}.
\]

The foregoing equations give

\[
\alpha_2 = z \quad \text{and} \quad \lambda = \sqrt{z^2 + 1}.
\]

With respect to the frame \((\frac{\partial}{\partial u}, \frac{\partial}{\partial v}, \frac{\partial}{\partial z})\) the structure equations (Gauß and Weingarten equations) are given by the following system of second order partial differential equations:
\(\lambda^2 x_{uu} = z(z^2 + 1)x_z + \lambda y - x,\)  
\(x_{vv} = 0 = x_{uv},\)
\(x_{zu} = -z\) 
\(x_{z2} + 1\) 
\(x_u = x_{uz},\)  
\(\lambda^2 x_{vv} = z(z^2 + 1)x_z - \lambda y - x,\)
\(x_{vz} = -z\) 
\(x_{z2} + 1\) 
\(x_v = x_{vz},\)  
\((z^2 + 1)^2 x_{zz} = -2z(z^2 + 1)x_z - x,\)
\(y_u = -\lambda x_u,\)
\(y_v = \lambda x_v,\)
\(y_z = 0.\)

Differentiating (3.10) with respect to \(u\) and using (3.12) and (3.16), one gets
\[x_{uuu} = -2x_u.\]

There are two vector valued functions \(A_1 \equiv A_1(v, z)\) and \(A_2 \equiv A_2(v, z)\) in \(\mathbb{R}^5\) depending only on \(v\) and \(z\) such that
\[x_u = \sqrt{2}\left(-\sin(\sqrt{2}u)A_1 + \cos(\sqrt{2}u)A_2\right).\]

One has
\[\frac{1}{\lambda^2} = 1(x_u, x_u)\]
\[= 2\left(< A_1, A_1 > \sin^2(\sqrt{2}u) - < A_1, A_2 > \sin(2\sqrt{2}u) + < A_2, A_2 > \cos^2(\sqrt{2}u)\right)\]
\[< A_1, A_1 > + < A_2, A_2 > + ( < A_2, A_2 > - < A_1, A_1 >) \cos(2\sqrt{2}u)\]
\[- 2 < A_1, A_2 > \sin(2\sqrt{2}u).\]

The linear independence of the functions 1, \(\sin(2\sqrt{2}u)\) and \(\cos(2\sqrt{2}u)\) implies
\[< A_1, A_1 > = \frac{1}{2\lambda^2} = < A_2, A_2 > \quad \text{and} \quad < A_1, A_2 > = 0.\]

Furthermore there is a vector valued function \(A_3 \equiv A_3(v, z)\) depending only on \(v\) and \(z\) such that
\[x(u, v, z) = \cos(\sqrt{2}u)A_1(v, z) + \sin(\sqrt{2}u)A_2(v, z) + A_3(v, z).\]

One has
\[1 = < x, x >\]
\[= \frac{1}{2\lambda^2} + 2 < A_1, A_3 > \cos(\sqrt{2}u) + 2 < A_2, A_3 > \sin(\sqrt{2}u) + < A_3, A_3 > .\]
From the linear independence of the functions 1, \sin(\sqrt{2}u) and \cos(\sqrt{2}u), one gets
\begin{equation}
1 = \frac{1}{2 \lambda^2} + < A_3, A_3 > \quad \text{and} \quad < A_1, A_3 > = 0 = < A_2, A_3 >.
\end{equation}

Differentiating (3.19) with respect to \(z\) (and with respect to \(v\), resp.) and using the equation (3.12) (the equation (3.11), resp.) and the linear independence of the functions 1, \sin(\sqrt{2}u) and \cos(\sqrt{2}u), one gets the following first order partial differential equations for the vector valued functions \(A_1(v, z)\) and \(A_2(v, z)\):
\[ (z^2 + 1) \frac{\partial A_1}{\partial z} = -zA_1, \quad (z^2 + 1) \frac{\partial A_2}{\partial z} = -zA_2 \quad \text{and} \quad \frac{\partial A_1}{\partial v} = 0 = \frac{\partial A_2}{\partial v}. \]

There are constant vectors \(C_1\) and \(C_2\) in \(\mathbb{R}^5\) such that
\[ A_1 = \frac{C_1}{\sqrt{z^2 + 1}} \quad \text{and} \quad A_2 = \frac{C_2}{\sqrt{z^2 + 1}}. \]

Because of (3.20), one has
\[ < C_1, C_1 > = \frac{1}{2} = < C_2, C_2 > \quad \text{and} \quad < C_1, C_2 >= 0. \]

The immersion \(x\) takes the form
\[ x(u, v, z) = \frac{1}{\sqrt{z^2 + 1}} \left( \cos(\sqrt{2}u)C_1 + \sin(\sqrt{2}u)C_2 \right) + A_3(v, z). \]

After inserting the above expression for \(x\) into the equation (3.15), one gets the following second order partial differential equation for the vector valued function \(A_3(v, z)\):
\[ (z^2 + 1)^2 \frac{\partial^2 A_3}{\partial z^2} + 2z(z^2 + 1) \frac{\partial A_3}{\partial z} + A_3 = 0. \]

There are two vector valued functions \(A_4 \equiv A_4(v)\) and \(A_5 \equiv A_5(v)\) in \(\mathbb{R}^5\) such that
\[ A_3(v, z) = \frac{A_4(v)}{\sqrt{z^2 + 1}} + \frac{zA_5(v)}{\sqrt{z^2 + 1}}. \]

The equation (3.14) implies that the vector valued function \(A_5(v)\) is constant:
\[ A_5(v) = C_5 \equiv \text{const.} \]

Because of (3.21), one has
\[ 0 = < C_5, A_4(v) > = < C_1, A_4(v) > = < C_2, A_4(v) > = < C_1, C_5 > = < C_2, C_5 >, \]
\[ < A_4(v), A_4(v) > = \frac{1}{2} \quad \text{and} \quad < C_5, C_5 > = 1. \]
Now eliminating $y$ from the equations (3.10) and (3.13), one has that the vector valued function $A_4(v)$ is a solution of the following second order differential equation:

$$A_4''(v) = -2A_4(v).$$

Therefore there are constant vectors $C_3, C_4 \in \mathbb{R}^3$ such that

$$A_4(v) = \cos(\sqrt{2}v)C_3 + \sin(\sqrt{2}v)C_4.$$

The constant vectors $C_3, C_4$ are orthogonal to $C_1, C_2, C_5$ and satisfy

$$< C_3, C_3 > = \frac{1}{2} = < C_4, C_4 > \quad \text{and} \quad < C_3, C_4 > = 0.$$

Finally, the local description of the immersion $x$ is given by

$$x(u, v, z) = \frac{1}{\sqrt{z^2 + 1}} \left( \cos(\sqrt{2}u)C_1 + \sin(\sqrt{2}u)C_2 + \cos(\sqrt{2}v)C_3 + \sin(\sqrt{2}v)C_4 + zC_5 \right).$$

Note that the vector field $y$ defined by

$$y(u, v, z) = -\cos(\sqrt{2}u)C_1 - \sin(\sqrt{2}u)C_2 + \cos(\sqrt{2}v)C_3 + \sin(\sqrt{2}v)C_4$$

is unit and normal to $x$. It is then easy to check that the mapping (3.22) defines a minimal hypersurface immersion with $K \equiv 0$ in $S^4(1)$. Clearly this hypersurface immersion is non-isoparametric. \hfill \Box

Now we want to characterize Example 1.2. In order to succeed, we need to prove the following lemma:

**Lemma 3.4.** Let $I \subset \mathbb{R}$ be an open interval and $0 < c_1, c_2 \in \mathbb{R}$ be two constant positive real numbers such that $c_2e^{2v} - 1 - c_1^2e^{4v} > 0$ for every $v \in I$. Consider the following second order differential equation for some function $A$ on $I$:

$$\begin{align*}
(c_2e^{2v} - 1 - c_1^2e^{4v})A''(v) + (1 - c_1^2e^{4v})A'(v) + 2A(v) &= 0.
\end{align*}$$

Then there exists a function $\phi: I \rightarrow \mathbb{R}$ such that the general solution $A(v)$ for the equation (3.23) takes the form

$$A(v) = a\sqrt{c_2 - e^{-2v}}\cos(\phi(v)) + b\sqrt{c_2 - e^{-2v}}\sin(\phi(v)),$$

where $a, b \in \mathbb{R}$ are constants.
**Proof.** Let \( B : I \rightarrow \mathbb{R} \) be the function defined by
\[
A(v) = \sqrt{c_2 - e^{-2v}B(v)}.
\]
Inserting this expression of \( A(v) \) into the equation (3.23), one gets the following second order differential equation in \( B(v) \):
\[
(3.24) \quad B''(v) - \left( \frac{3 + c_1^2e^{4v} + c_2c_1^2e^{6v} - 3c_2e^{2v}}{(c_2e^{2v} - 1)(c_2e^{2v} - 1 - c_1^2e^{4v})} \right)B'(v) + \frac{c_2c_1^2e^{6v}B(v)}{(c_2e^{2v} - 1)^2(c_2e^{2v} - 1 - c_1^2e^{4v})} = 0.
\]
Now take \( \phi \) to be the function on \( I \) such that
\[
\phi'(v) = \sqrt{\frac{c_2c_1^2e^{6v}}{(c_2e^{2v} - 1)^2(c_2e^{2v} - 1 - c_1^2e^{4v})}}.
\]
It follows that
\[
\frac{\phi''(v)}{\phi'(v)} = \left( \frac{3 + c_1^2e^{4v} + c_2c_1^2e^{6v} - 3c_2e^{2v}}{(c_2e^{2v} - 1)(c_2e^{2v} - 1 - c_1^2e^{4v})} \right).
\]
Thus the equation (3.24) becomes
\[
B''(v) - \frac{\phi''(v)}{\phi'(v)}B'(v) + \phi'^2(v)B(v) = 0.
\]
Therefore, there are constants \( a, b \in \mathbb{R} \) such that
\[
B(v) = a \cos(\phi(v)) + b \sin(\phi(v)).
\]

**Remark 3.5.** The particular solutions \( g(v) \) and \( h(v) \) of the equation (3.23) given by
\[
g(v) = \frac{1}{\sqrt{c_2}} \sqrt{c_2 - e^{-2v}} \cos(\phi(v)) \quad \text{and} \quad g(v) = \frac{1}{\sqrt{c_2}} \sqrt{c_2 - e^{-2v}} \sin(\phi(v))
\]
are linearly independent and satisfy \( g^2(v) + h^2(v) = 1 - \frac{e^{-2v}}{c_2} \).

**Proposition 3.6.** Let \( x : M^3 \rightarrow S^4(1) \subset \mathbb{R}^5 \) be a minimal hypersurface immersion of a connected and orientable manifold \( M^3 \) into \( S^4(1) \subset \mathbb{R}^5 \) with identically zero Gauß-Kronecker curvature, but with nowhere zero second fundamental form. Assume that the function \( \alpha_4 \) vanishes identically and \( \alpha_1 \) is nowhere zero on \( M^3 \). Then there are local coordinates \((u, v, z)\) such that the immersion \( x \) can be locally described by the parametrization of the hypersurface given in Example 1.2.
**Proof.** The vector fields $\frac{1}{\alpha_1} e_2$ and $\frac{1}{\alpha_2^2 + 1} e_3$ satisfy

$$\left[ \frac{1}{\alpha_1} e_2, \frac{1}{\alpha_2^2 + 1} e_3 \right] = 0.$$ 

Define the function $f$ on $M^3$ by

$$f := \frac{1}{\sqrt{1 + \lambda^2 + \alpha_1^2 + \alpha_2^2}}.$$ 

The function $f$ satisfies the following equations:

$$e_1(f) = 0, \quad e_2(f) = -\alpha_1 f \quad \text{and} \quad e_3(f) = -\alpha_2 f.$$ 

Consequently,

$$\left[ fe_1, \frac{1}{\alpha_1} e_2 \right] = 0 = \left[ fe_1, \frac{1}{\alpha_2^2 + 1} e_3 \right].$$

Therefore there are local coordinates $(u, v, z)$ on $M^3$ such that

$$\frac{\partial}{\partial z} = \frac{1}{\alpha_2^2 + 1} e_3, \quad \frac{\partial}{\partial v} = \frac{1}{\alpha_1} e_2, \quad \frac{\partial}{\partial u} = fe_1.$$ 

From above we get the following equations for $\alpha_2$, $\lambda$ and $\alpha_1$.

Equations for $\alpha_2$:

$$\frac{\partial \alpha_2}{\partial u} = fe_1(\alpha_2) = 0, \quad \frac{\partial \alpha_2}{\partial v} = \frac{1}{\alpha_1} e_2(\alpha_2) = 0, \quad \frac{\partial \alpha_2}{\partial z} = \frac{1}{1 + \alpha_2^2} e_3(\alpha_2) = 1.$$ 

So

$$(3.26) \quad \alpha_2 = z.$$ 

Equations for $\lambda$:

$$\frac{\partial \lambda}{\partial u} = 0, \quad \frac{\partial \lambda}{\partial v} = 2\lambda, \quad \text{and} \quad \frac{\partial \lambda}{\partial z} = \frac{z \lambda}{z^2 + 1}.$$ 

Therefore

$$(3.27) \quad \lambda \equiv \lambda(v, z) = c_1 e^{2v} \sqrt{z^2 + 1},$$ 

where $0 < c_1 \in \mathbb{R}$.

Equations for $\alpha_1$:

$$\frac{\partial \alpha_1}{\partial z} = \frac{z \alpha_1}{z^2 + 1}, \quad \frac{\partial \alpha_1}{\partial v} = \frac{\alpha_1^2 + z^2 + 1 - \lambda^2}{\alpha_1}, \quad \text{and} \quad \frac{\partial \alpha_1}{\partial u} = 0;$$
thus

\[(3.28) \quad \alpha_1^2 \equiv \alpha_1^2(v, z) = (z^2 + 1)(c_2e^{2v} - 1 - c_1^2e^{4v}),\]

where \(v \in I \subset \mathbb{R}\) an open interval and \(0 < c_2 \in \mathbb{R}\) is a constant such that \(c_2e^{2v} - 1 - c_1^2e^{4v} > 0\) for all \(v \in I\). Inserting the expressions (3.26), (3.27) and (3.28) (of \(\alpha_2\), \(\lambda\) and \(\alpha_1\), respectively) into (3.25), we see that the function \(f\) satisfies

\[(1 + z^2)f^2(v) = \frac{e^{-2v}}{c_2}.\]

With respect to the frame \((\frac{\partial}{\partial u}, \frac{\partial}{\partial v}, \frac{\partial}{\partial z})\) the structure equations are given by the following system of second order partial differential equations:

\[(3.29) \quad x_{uz} = -\frac{\alpha_2}{\alpha_2^2 + 1}x_v = x_{zv},\]

\[(3.30) \quad x_{zu} = -\frac{\alpha_2}{\alpha_2^2 + 1}x_u = x_{uz},\]

\[(3.31) \quad x_{vu} = -x_u = x_{uv},\]

\[(3.32) \quad x_{uu} = f^2(\alpha_1^2x_v + \alpha_2(\alpha_2^2 + 1)x_z + \lambda y - x),\]

\[(3.33) \quad x_{zz} = -\frac{2\alpha_2}{\alpha_2^2 + 1}x_z - \frac{1}{(\alpha_2^2 + 1)^2}x,\]

\[(3.34) \quad x_{vv} = \frac{1}{\alpha_1^2}\left(- (\alpha_1^2 + \alpha_2^2 + 1 - \lambda^2)x_v + \alpha_2(\alpha_2^2 + 1)x_z - \lambda y - x\right),\]

\[(3.35) \quad y_u = -\lambda x_u,\]

\[(3.36) \quad y_v = \lambda x_v,\]

\[(3.37) \quad y_z = 0.\]

Differentiating the equation (3.32) with respect to \(u\), one has:

\[(3.38) \quad x_{uuu} = f^2(-\alpha_1^2 - \alpha_2^2 - \lambda^2 - 1)x_u = -x_u.\]

There are vector valued functions \(A_1 \equiv A_1(v, z)\) and \(A_2 \equiv A_2(v, z)\) in \(\mathbb{R}^5\) depending only on \(v\) and \(z\) such that

\[(3.39) \quad x_u = -\sin(u)A_1 + \cos(u)A_2.\]

One has:

\[
\begin{align*}
    f^2 &= I(x_u, x_u) \\
    &= <A_1, A_1> \sin^2(u) - <A_1, A_2> \sin(2u) + <A_2, A_2> \cos^2(u) \\
    &= \frac{<A_2, A_2> + <A_1, A_1>}{2} + \frac{<A_2, A_2> - <A_1, A_1>}{2} \cos(2u) \\
    &\quad - <A_1, A_2> \sin(2u).
\end{align*}
\]
Using the linear independence of the functions 1, \( \cos(2u) \) and \( \sin(2u) \), from the equation above we deduce

\[
< A_1, A_1 > = f^2 = < A_2, A_2 > \quad \text{and} \quad < A_1, A_2 > = 0.
\]

Furthermore, there is a vector valued function \( A_3 \equiv A_3(v, z) \) depending only on \( v \) and \( z \) such that

\[
x(u, v, z) = \cos(u)A_1(v, z) + \sin(u)A_2(v, z) + A_3(v, z).
\]

One has

\[
1 = < x, x > = f^2 + < A_3, A_3 > + 2 < A_1, A_3 > \cos(u) + 2 < A_2, A_3 > \sin(u).
\]

Using the linear independence of the functions 1, \( \cos(u) \) and \( \sin(u) \), we get

\[
1 = f^2 + < A_3, A_3 > \quad \text{and} \quad < A_1, A_3 > = 0 = < A_2, A_3 >.
\]

We differentiate the equation (3.39) with respect to \( z \) and \( v \) and use the equations (3.30) and (3.31); then the linear independence of the functions \( \cos(u) \) and \( \sin(u) \) implies the following first order partial differential equations for the vector valued functions \( A_1(v, z) \) and \( A_2(v, z) \):

\[
\begin{align*}
\frac{\partial A_1}{\partial z} &= -\frac{z}{z^2 + 1}A_1, \\
\frac{\partial A_2}{\partial z} &= -\frac{z}{z^2 + 1}A_2,
\end{align*}
\]

Therefore,

\[
A_1(v, z) = f(v, z)C_1 \quad \text{and} \quad A_2(v, z) = f(v, z)C_2,
\]

where \( C_1, C_2 \in \mathbb{R}^5 \) are constant vectors; they are orthonormal because of (3.40), and from (3.42) they are orthogonal to \( A_3 \).

Differentiating (3.41) with respect to \( z \) we have:

\[
x_z = \frac{\partial f}{\partial z} \left( \cos(u)C_1 + \sin(u)C_2 \right) + \frac{\partial A_3}{\partial z},
\]

\[
x_{zz} = \frac{\partial^2 f}{\partial z^2} \left( \cos(u)C_1 + \sin(u)C_2 \right) + \frac{\partial^2 A_3}{\partial z^2}.
\]

Using the equation (3.33), we get the following partial differential equation for \( A_3 \), depending only on \( z \):

\[
\frac{\partial^2 A_3}{\partial z^2} = -\frac{2z}{z^2 + 1} \frac{\partial A_3}{\partial z} - \frac{1}{(z^2 + 1)^2} A_3.
\]
Therefore there are vector valued functions $A_4 \equiv A_4(v)$ and $A_5 \equiv A_5(v)$ depending only on $v$ such that:

\[(3.46) \quad A_3(v, z) = \frac{z}{\sqrt{z^2 + 1}} A_4(v) + \frac{1}{\sqrt{z^2 + 1}} A_5(v).\]

The equation (3.41) becomes

\[x(u, v, z) = f(v, z) \cdot \left( \cos(u)C_1 + \sin(u)C_2 \right) + \frac{z}{\sqrt{z^2 + 1}} A_4(v) + \frac{1}{\sqrt{z^2 + 1}} A_5(v).\]

Differentiating the equation above and using (3.29), we have:

\[0 = x_{vz} + \frac{z}{z^2 + 1} x_v = \frac{1}{\sqrt{z^2 + 1}} A_4'(v).\]

Therefore the vector valued function $A_4(v)$ is constant: $A_4(v) \equiv C_3 \in \mathbb{R}^5$.

From the equation (3.42), one has

\[1 = f^2 + \frac{1}{z^2 + 1} \left( z^2 < C_3, C_3 > + 2z < C_3, A_5(v) > + < A_5(v), A_5(v) > \right)\]

and thus

\[< C_3, C_3 >= 1, \quad < C_3, A_5(v) >= 0,\]

and

\[(3.47) \quad < A_5(v), A_5(v) >= 1 - (1 + z^2)f^2 = 1 - \frac{e^{-2v}}{c_2}.\]

Eliminating $y$ from the equations (3.32) and (3.34), we get

\[0 = \alpha_1^2 x_{vv} + f^{-2} x_{uu} + (z^2 + 1 - \lambda^2)x_v - 2z(z^2 + 1)x_z + 2x\]

\[= \sqrt{z^2 + 1} \left( (c_2 e^{2v} - 1 - c_1^2 e^{4v}) A_5''(v) + (1 - c_1^2 e^{4v}) A_5'(v) + 2A_5(v) \right).\]

Therefore the vector valued function $A_5(v)$ satisfies the following linear ordinary differential equation of second order:

\[(3.48) \quad (c_2 e^{2v} - 1 - c_1^2 e^{4v}) A_5''(v) + (1 - c_1^2 e^{4v}) A_5'(v) + 2A_5(v) = 0.\]

By the Lemma 3.4, one can conclude that the general solution of the equation (3.48) is

\[A_5(v) = g(v)C_4 + h(v)C_5,\]

where $C_4, C_5 \in \mathbb{R}^5$ are constant vectors, and $g$ and $h$ are the functions given in Remark 3.5.
But since from (3.47) the vector valued function $A_5$ satisfies

$$< A_5(v), A_5(v) > = 1 - \frac{e^{-2v}}{c_2},$$

we have that $C_4$ and $C_5$ are orthonormal. They constitute together with $C_1, C_2, C_3$ an orthonormal basis of $\mathbb{R}^5$. This proves Proposition 3.6.

Our classification theorem summarizes the results from Propositions 3.1-3.6.

Acknowledgements The author would like to express his special thanks to Professor An-Min Li and Professor Guosong Zhao for their hospitality during his scientific visit in August 2001 at the Sichuan University in Chengdu (China).

References

[1] De Almeida S.C. and Brito F., Closed hypersurfaces of $S^4(1)$ with two constant symmetric curvatures, Ann. Fac. Sci. Toulouse, VI. Ser., Math. 6 (1997), 187–202.

[2] De Almeida S.C. and Brito F., Minimal hypersurfaces of $S^4$ with constant Gauß-Kronecker curvature, Math. Z. 195 (1987), 99–107.

[3] Brito F., Liu H.L., Simon U. and Wang C.P., Geometry and topology of submanifolds, vol. IX, (Defever F. et al., eds.), World Scientific Singapore, 1999, pp. 48–63.

[4] Do Carmo M.P., Riemannian Geometry, Birkhäuser Boston, 1992.