Hilbert-Schmidt quantum coherence in multi-qudit systems

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Using Bloch’s parametrization for qudits (d-level quantum systems), we write the Hilbert-Schmidt distance (HSD) between two generic n-qudit states as an Euclidean distance between two vectors of observables mean values in $\mathbb{R}^{d^2_n-1}$, where $d_s$ is the dimension for qudit $s$. Then, applying the generalized Gell Mann’s matrices to generate $SU(d_s)$, we use that result to obtain the Hilbert-Schmidt quantum coherence (HSC) of n-qudit systems. As examples, we consider in details one-qubit, one-qudit, two-qubit, and two copies of one-qubit states. In this last case, the possibility for controlling local and non-local coherences by tuning local populations is studied and the contrasting behaviors of HSC, $l_1$-norm coherence, and relative entropy of coherence in this regard are noticed. We also investigate the decoherent dynamics of these coherence functions under the action of qutrit dephasing and dissipation channels. At last, we analyze the non-monotonicity of HSD under tensor products and report the first instance of a consequence (for coherence quantification) of this kind of property of a quantum distance measure.

Keywords: quantum coherence, Hilbert-Schmidt distance, qudit states, Gell Mann matrices

I. INTRODUCTION

Coherent superposition states are an important aspect of quantum systems [1, 2], one which is believed to make possible their use for more efficient realization of energy and information manipulation tasks [3–12]. In the last few years, much attention has been given for the development, analysis, and application of a resource theory for quantum coherence (RTC). For a recent review, see Ref. [13]. It is certainly worthwhile recognizing the existence of several levels of incoherent operations (IO) in the RTC [14, 15]. Naturally, depending on its properties, different coherence functions may or may not be a coherence monotone under a specific element of a given set of IO [16–19]. This kind of observation points for the need of further investigations concerning the similarities and differences among the properties enjoyed by varied quantumness functions.

Here we shall pay special attention to the geometric quantum coherence function which is obtained using Hilbert-Schmidt’s distance (HSD). Although the possible non-contractivity under general quantum operations of the HSD [20–22] demands care when of its application for quantumnesses quantification [23, 24], it is a fact that the HSD has been a handy tool for a variety of investigations in quantum information science [25–34]. In addition to that, recently it was shown that the HSD leads to an observable upper bound for the robustness of coherence, which is a full quantum coherence monotone [17]. Besides, because of its friendliness for analytical calculations, the HSD can be a useful tool for starting on quantumnesses quantification, as happened, for instance, with quantum entanglement and quantum discord [35–44].

Our main goal in this article is obtaining the Hilbert-Schmidt coherence (HSC) of multipartite qudit states and to use it to analyze the distribution and manipulation of quantum coherence in this kind of scenario. We shall then deal with two relevant-associated topics. We will look at the possibility of controlling the local and/or non-local coherences [45–47] of a system by manipulating its local populations. And we will investigate the issue of a possible non-monotonic behavior of quantum distance measures under tensor products [48, 49].

The non-monotonicity under tensor products (NMuTP) of a distance measure, investigated recently for the trace distance [49], is the possibility for an inversion in the dissimilarity relation between pairs of states when we consider their copies. That is to say, for four density matrices $\rho, \sigma, \xi, \eta$, it can happen that

$$d(\rho, \sigma) > d(\xi, \eta) \text{ and } d(\rho \otimes \sigma, \xi \otimes \eta) < d(\xi \otimes \xi, \eta \otimes \eta).$$

Once there is no known analytical expression for the trace distance coherence for general states of systems with dimension greater or equal to three [50–52], and with the aim of unveiling the possible implications of the NMuTP in quantum information science, here we shall analyze it using the HSD and look for its possible implications in coherence quantification.

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The remainder of the article is structured as follows. In Sec. II we make use of Bloch’s representation to write the Hilbert-Schmidt distance between two generic \( n \)-qudit states as an Euclidean distance between the associated rescaled Bloch’s vectors. In Sec. III we use this result to obtain an analytical formula for the Hilbert-Schmidt coherence of \( n \)-qudit states. We instantiate this general result by looking at one-qubit (Sec. III A), one-qutrit (Sec. III B), two-qubit (Sec. III C), and two copies of one-qubit (Sec. III D) states. After writing, in Sec. III C, the total HSC of two-qubit states in terms of its local and nonlocal parts, in Sec. III D we study the possibility for controlling these quantities by manipulating the local populations of two copies of one-qubit states. The contrasting behaviors of HSC, \( l_1 \)-norm coherence, and relative entropy of coherence in this regard are noticed. We also investigate, in Sec. IV, the dynamical behavior of quantum coherence functions under the action of qutrit phase damping and amplitude damping channels. Finally, the non-monotonicity of the HSD under tensor products is investigated in Sec. V, where we report the first example of an important implication of this kind of property of a quantum distance function. We summarize our findings and discuss related open questions in Sec. VI.

II. HILBERT-SCHMIDT DISTANCE BETWEEN \( n \)-QUDIT STATES

Let \( \mathbb{I}_{d_s} = \Gamma_0 = \Gamma_0^d \) denote the \( d_s \times d_s \) identity matrix, and \( \{ \Gamma_j, j_s=1 \}^d \) be a basis for \( SU(d_s) \). We can use these operators to form a basis for \( SU(d) = \bigotimes_{s=1}^n SU(d_s) \), with \( d = \prod_{s=1}^n d_s \), and to write any \( n \)-qudit state as [53, 54]:

\[
\rho = \sum_{j_1=0}^{d_1-1} \sum_{j_2=0}^{d_2-1} \cdots \sum_{j_n=0}^{d_n-1} r_{j_1,j_2\cdots j_n} \Gamma_{j_1} \otimes \Gamma_{j_2} \otimes \cdots \otimes \Gamma_{j_n}.
\]  

(2)

If the matrices \( \{ \Gamma_j, j_s=1 \} \) satisfy the algebraic relations

\[
\text{Tr}(\Gamma_j \Gamma_k) = d_s^{\delta_{jk}} 2^1-\delta_{jk} \delta_{j_s k_s},
\]

(3)

then the unit trace of \( \rho \) requires \( r_{00\cdots 0} = 1/d \). Besides, the other parameters \( r_{j_1 j_2 \cdots j_n} \), which are the components of the so called Bloch’s vector, can be obtained as follows:

\[
\text{Tr}(\rho \Gamma_{k_1} \otimes \Gamma_{k_2} \otimes \cdots \otimes \Gamma_{k_n}) = \sum_{j_1=0}^{d_1-1} \sum_{j_2=0}^{d_2-1} \cdots \sum_{j_n=0}^{d_n-1} r_{j_1,j_2\cdots j_n} \text{Tr}(\Gamma_{j_1} \Gamma_{k_1}) \text{Tr}(\Gamma_{j_2} \Gamma_{k_2}) \cdots \text{Tr}(\Gamma_{j_n} \Gamma_{k_n})
\]

\[
= \sum_{j_1=0}^{d_1-1} \sum_{j_2=0}^{d_2-1} \cdots \sum_{j_n=0}^{d_n-1} r_{j_1,j_2\cdots j_n} d_1^{\delta_{j_1 k_1}} 2^1-\delta_{jk_1} \delta_{j_2 k_2} d_2^{\delta_{j_2 k_2}} 2^1-\delta_{jk_2} \delta_{j_3 k_3} \cdots d_n^{\delta_{j_n k_n}} 2^1-\delta_{jk_n} \delta_{j_n k_n}
\]

\[
= r_{k_1 k_2 \cdots k_n} 2^n-\sum_{s=1}^n \delta_{k_s s} \prod_{s=1}^n \frac{n}{d_s^{\delta_{k_s s}}}.
\]

(4)

Now let’s regard the Hilbert-Schmidt distance (HSD) [30],

\[
d_{hs}(\rho, \zeta) = \sqrt{\text{Tr}(\rho - \zeta)^2},
\]

(5)

between two generic \( n \)-qudit states. Using \( \rho - \zeta = \sum_{j_1=0}^{d_1-1} \cdots \sum_{j_n=0}^{d_n-1} (r_{j_1\cdots j_n} - z_{j_1\cdots j_n}) \Gamma_{j_1} \otimes \cdots \otimes \Gamma_{j_n} \), we shall have

\[
\text{Tr}(\rho - \zeta)^2 = \sum_{j_1,k_1=0}^{d_1-1} \cdots \sum_{j_n,k_n=0}^{d_n-1} (r_{j_1\cdots j_n} - z_{j_1\cdots j_n})(r_{k_1\cdots k_n} - z_{k_1\cdots k_n}) \text{Tr}(\Gamma_{j_1} \Gamma_{k_1}) \cdots \text{Tr}(\Gamma_{j_n} \Gamma_{k_n})
\]

\[
= \sum_{j_1,k_1=0}^{d_1-1} \cdots \sum_{j_n,k_n=0}^{d_n-1} (r_{j_1\cdots j_n} - z_{j_1\cdots j_n})(r_{k_1\cdots k_n} - z_{k_1\cdots k_n}) d_1^{\delta_{j_1 k_1}} 2^1-\delta_{j_1 k_1} \delta_{j_2 k_2} \cdots d_n^{\delta_{j_n k_n}} 2^1-\delta_{j_n k_n} \delta_{j_n k_n}
\]

\[
= \sum_{j_1=0}^{d_1-1} \sum_{j_2=0}^{d_2-1} \cdots \sum_{j_n=0}^{d_n-1} (r_{j_1 j_2 \cdots j_n} - z_{j_1 j_2 \cdots j_n}) 2^n-\sum_{s=1}^n \delta_{j_s s} \prod_{s=1}^n \frac{n}{d_s^{\delta_{j_s s}}}.
\]

(6)
Thus, if we define the components of the rescaled Bloch’s vector as

\[ R_{j_1j_2\cdots j_n} = r_{j_1j_2\cdots j_n} \sqrt{2^n-\sum_{s=1}^n \delta_{j_s}} \prod_{s=1}^n d_s^{\delta_{j_s}} = \left(2^n-\sum_{s=1}^n \delta_{j_s} \prod_{s=1}^n d_s^{\delta_{j_s}}\right)^{-1/2} \text{Tr}(\rho \Gamma_{j_1} \otimes \Gamma_{j_2} \otimes \cdots \otimes \Gamma_{j_n}), \]  

(7)

with the analogous holding for \( Z_{j_1j_2\cdots j_n} \), the HSD between \( \rho \) and \( \zeta \) can be written as the Euclidean distance between the \( \mathbb{R}^{d^2-1} \) vectors \( \vec{R} = \{R_{j_1j_2\cdots j_n}\} \) and \( \vec{Z} = \{Z_{j_1j_2\cdots j_n}\} \):

\[ d_{hs}(\rho, \zeta) = \sqrt{\sum_{j_1=0}^{d^2-1} \sum_{j_2=0}^{d^2-1} \cdots \sum_{j_n=0}^{d^2-1} (R_{j_1j_2\cdots j_n} - Z_{j_1j_2\cdots j_n})^2} = ||\vec{R} - \vec{Z}||. \]  

(8)

We observe that if the generators \( \Gamma_{j_s} \) are Hermitian, then the distinguishability of the two states gains a nice physical significance in terms of differences between the corresponding pairs of observables mean values. Notice that for this kind of generator we have, e.g., \( R_{ij} = \frac{1}{\sqrt{n}} \sum_{k=1}^n \delta_{j_k} \rho_{k} \) and \( \rho_{ij} = \sum_{k=1}^n \delta_{j_k} \delta_{i_k} \). That is to say \( n \) being a probability distribution. The Hilbert-Schmidt coherence (HSC) of a state \( \rho \) is defined as its minimum Hilbert-Schmidt distance (HSD) to incoherent states:

\[ C_{hs}(\rho) = \min_{\iota} d_{hs}(\rho, \iota). \]  

(10)

The HSC is nonnegative and is equal to zero if and only if \( \rho \) is an incoherent state. It is also worthwhile mentioning that although the HSC is not a “full coherence monotone” [16], it is a monotone (i.e. \( C_{hs}(\rho) \geq C_{hs}(\Delta(\rho)) \)) under incoherent operations of the dephasing type (in basis \( \{|j\rangle\}_{j=1}^d \)): \( \Delta(\rho) = \sum_{j=1}^d |\langle j|\rho\rangle\rangle \otimes |\langle j|\rangle\rangle \) [30].

For computing the HSC, one convenient choice for the \( SU(d_s) \) generators is the \( d_s \times d_s \) generalized Gell Mann matrices [54]:

\[ \Gamma_{j_s}^d = \sqrt{\frac{2}{j_s(j_s+1)}} \sum_{k_s=1}^{j_s+1} (j_s)_{d_s,j_s+1}^k |k_s\rangle\langle k_s|, \]  

for \( j_s = 1, \ldots, d_s - 1, \) \( 11 \)

\[ \Gamma_{ls}^s(\kappa_s, t_s) = |k_s\rangle\langle l_s| + |l_s\rangle\langle k_s|, \]  

for \( 1 \leq k_s < l_s \leq d_s, \) \( 12 \)

\[ \Gamma_{ls}^l(\kappa_s, t_s) = -i(|k_s\rangle\langle l_s| - |l_s\rangle\langle k_s|), \]  

for \( 1 \leq k_s < l_s \leq d_s. \) \( 13 \)

These three sets of matrices are named the diagonal, symmetric, and antisymmetric sets, respectively. The particular cases with \( d_s = 2 \) and \( d_s = 3 \) give the well known Pauli and Gell Mann matrices, respectively.

If the reference basis is chosen to be the \( n \)-qubit computational basis \( \bigotimes_{s=1}^n |j_s\rangle \), with \( j_s = 1, \cdots, d_s \), then the presence of any non-null component of the Bloch’s vector corresponding to generators from the symmetric and/or antisymmetric sets implies in the presence of off diagonal elements in the density matrix. Thus, the more general \( n \)-qubit incoherent state is obtained utilizing the identity matrix and the generators from the diagonal set:

\[ \iota = \sum_{j_1=0}^{d_1-1} \sum_{j_2=0}^{d_2-1} \cdots \sum_{j_n=0}^{d_n-1} t_{j_1j_2\cdots j_n} \Gamma_{j_1}^d \otimes \Gamma_{j_2}^d \otimes \cdots \otimes \Gamma_{j_n}^d. \]  

(14)

III. HILBERT-Schmidt Quantum Coherence for \( n \)-QUdit STATES

Incoherent states, \( \iota \), are represented by density matrices which are diagonal in some reference, orthonormal, basis \( \{|j\rangle\}_{j=1}^d \). That is to say [16]

\[ \iota = \sum_{j=1}^d t_j |j\rangle\langle j|, \]  

(9)

with \( \{t_j\}_{j=1}^d \) being a probability distribution. The Hilbert-Schmidt coherence (HSC) of a state \( \rho \) is defined as its minimum Hilbert-Schmidt distance (HSD) to incoherent states:

\[ C_{hs}(\rho) = \min_{\iota} d_{hs}(\rho, \iota). \]  

(10)
As seen in Eq. (8), the HSD is given in terms of the sum of positive terms, which are equal to the square of the difference between corresponding components of the rescaled vectors. Therefore, the minimum in Eq. (10) is obtained if we set \( t_{j_1,j_2,...,j_n} = r_{j_1,j_2,...,j_n} \) for all \( j_s = 0, \ldots, d_s - 1 \) and \( s = 1, 2, \ldots, n \). Thus we see that the optimal incoherent state under the HSD is

\[
t_\rho = \sum_{j_1=0}^{d_1-1} \sum_{j_2=0}^{d_2-1} \cdots \sum_{j_n=0}^{d_n-1} r_{j_1,j_2,...,j_n} \Gamma_j^d \otimes \Gamma_{j_2}^d \otimes \cdots \otimes \Gamma_{j_n}^d.
\]

(15)

We shall define the coherence vector, \( \vec{C} \), as the vector obtained using the components of the rescaled Bloch’s vector in Eq. (7) corresponding to one or more generators from the symmetric and/or antisymmetric sets. Hence the HSC of a \( n \)-qutrit state can be cast as the Euclidean norm of this dimension \( d(d-1) \) coherence vector:

\[
C_{hs}(\rho) = ||\vec{C}||.
\]

(16)

In the next sub-sections, we exemplify the application of this general result to some particular cases.

A. One-qubit state

In this example \( n = 1, d_1 = 2, \) and \( \vec{C} = \left( \langle \Gamma_{(1,2)}^s \rangle, \langle \Gamma_{(1,2)}^a \rangle \right) / \sqrt{2} \). So, the one-qubit HSC is given by

\[
C_{hs}(\rho_{1q}) = 2^{-1/2} \sqrt{\langle \Gamma_{(1,2)}^s \rangle^2 + \langle \Gamma_{(1,2)}^a \rangle^2}.
\]

(17)

In this section, and in the remainder of this article, when there is no chance for confusion, we omit the density matrix in the expression for the mean values.

B. One-qutrit state

Here \( n = 1, d_1 = 3, \) and \( \vec{C} = \left( \langle \Gamma_{(1,2)}^s \rangle, \langle \Gamma_{(1,3)}^s \rangle, \langle \Gamma_{(1,2)}^a \rangle, \langle \Gamma_{(1,3)}^a \rangle, \langle \Gamma_{(2,3)}^a \rangle \right) / \sqrt{3} \). So,

\[
C_{hs}(\rho_{1q}) = 2^{-1/2} \sqrt{\langle \Gamma_{(1,2)}^s \rangle^2 + \langle \Gamma_{(1,3)}^s \rangle^2 + \langle \Gamma_{(1,2)}^a \rangle^2 + \langle \Gamma_{(1,3)}^a \rangle^2 + \langle \Gamma_{(2,3)}^a \rangle^2}.
\]

(18)

C. Two-qubit state

Now \( n = 2, d_1 = 2, d_2 = 2 \) and

\[
C_{hs}^2(\rho_{2q}) = 2^{-1} \sum_{j=0,1} \sum_{\kappa,s,a} \left( \langle \Gamma_j^d \otimes \Gamma_{(1,2)}^s \rangle^2 + \langle \Gamma_j^d \otimes \Gamma_{(1,2)}^a \rangle^2 \right) + 2^{-1} \sum_{\kappa,s,a} \langle \Gamma_{(1,2)}^s \otimes \Gamma_{(1,2)}^a \rangle^2
\]

\[
= C_{l}^{hs}(\rho_{2q}) + C_{nl}^{hs}(\rho_{2q}),
\]

(19)

where the local part of the coherence was set to

\[
C_{l}^{hs}(\rho_{2q}) = 2^{-1} \sum_{\kappa=s,a} \left( \langle \Gamma_0^d \otimes \Gamma_{(1,2)}^s \rangle^2 + \langle \Gamma_0^d \otimes \Gamma_{(1,2)}^a \rangle^2 \right) = C_{hs}^0(\rho_{2q}) + C_{hs}^1(\rho_{2q}),
\]

(20)

with \( \rho_{2q}^{(s)} = \text{Tr}_{1}(\rho_{2q}) \) being the one-qubit reduced states (their HSC read as in Eq. (17)). The non-local part of the coherence \[45]\] may be cast as

\[
C_{nl}^{hs}(\rho_{2q}) = C_{hs}^2(\rho_{2q}) - C_{l}^{hs}(\rho_{2q}).
\]

(21)
D. Two copies of one-qubit states: Controlling coherences via the manipulation of populations

As a special case of the class of states regarded in the last sub-section, let us look to two copies of one-qubit states: $\rho_{qb} \otimes \rho_{qb}$. For this kind of state we have $C_{lh}^1(\rho_{qb} \otimes \rho_{qb}) = 2C_{hs}^2(\rho_{qb})$ and

$$C_{nl}^{lh}(\rho_{qb} \otimes \rho_{qb}) = 2^{-1} \sum_{k=s,a} \left( (\Gamma_{1}^{d} \otimes \Gamma_{(1,2)}^{n})^{2}_{\rho_{qb} \otimes \rho_{qb}} + (\Gamma_{1}^{n} \otimes \Gamma_{(1,2)}^{d})^{2}_{\rho_{qb} \otimes \rho_{qb}} \right) + 2^{-1} \sum_{k,s,a} \left( (\Gamma_{1}^{n} \otimes \Gamma_{(1,2)}^{s})^{2}_{\rho_{qb} \otimes \rho_{qb}} \right) = 2C_{hs}^2(\rho_{qb}) \left( (\Gamma_{1}^{d})^{2}_{\rho_{qb}} + C_{hs}^2(\rho_{qb}) \right). \tag{22}$$

Considering that the local populations are equal to

$$p_{j} = \text{Tr}(\rho_{qb}|\langle j \rangle|) = (1 + (-1)^{\delta_{j2}} (\Gamma_{1}^{d})))/2, \tag{23}$$

with $j = 1, 2$, we notice that although we cannot control the local HSC by manipulating $p_{j}$, we can change the non-local HSC by modifying the difference between those populations: $\langle \Gamma_{1}^{d} \rangle = p_{1} - p_{2}$.

The result above is to be compared with the one for the $l_{1}$-norm coherence (LIC), which, in contrast to the HSC, is a coherence monotone under the incoherent operations regarded in Ref. [16]. The LIC is given by the sum of the absolute values of the off-diagonal elements of a density matrix. Although $\langle \Gamma_{1}^{d} \rangle$ is present in the off-diagonal elements of $\rho_{qb} \otimes \rho_{qb}$, it cancels out when we perform that sum, and we get

$$C_{l_{1}}(\rho_{qb} \otimes \rho_{qb}) = (1 + C_{l_{1}}(\rho_{qb}))^{2} - 1, \tag{24}$$

with $C_{l_{1}}(\rho_{qb}) = 2^{1/2}C_{hs}(\rho_{qb})$. So, we cannot control the non-local LIC by tuning local populations.

Because of its operational interpretation [57], the relative entropy of coherence (REC) may be considered to be one of the main coherence quantifiers introduced up today. The REC is defined and given by [16]:

$$C_{re}(\rho) = \min_{\iota} d_{re}(\rho, \iota) = \min_{\iota} (-\text{Tr}(\rho \log_{2} \iota) - S(\rho)) = S(\rho_{\text{diag}}) - S(\rho), \tag{25}$$

with $\rho_{\text{diag}}$ being the density matrix obtained by setting the off-diagonal elements of $\rho$ to zero and $S(x) = -\text{Tr}(x \log_{2} x)$ is von Neumann’s entropy. Using the additivity of von Neumann’s entropy under tensor products, i.e., $S(x \otimes y) = 2S(x)$ and the fact that $(\rho \otimes \rho)_{\text{diag}} = \rho_{\text{diag}} \otimes \rho_{\text{diag}}$, we see that the REC is also additive under tensor products, i.e., $C_{re}(\rho \otimes \rho) = 2C_{re}(\rho)$. So, we shall be able to change the REC of the two copies of $\rho$ only if we can change its own REC. An interesting result we report here in this direction is that, contrary to what happens with the HSC and LIC, we can modify the REC of a qubit by controlling its populations, a quantity usually considered to have a classical character. For a qubit state, $\rho_{qb} = 2^{-1}(\Gamma_{0}^{d} + \langle \Gamma_{1}^{d} \rangle \Gamma_{1}^{d} + \langle \Gamma_{1,2}^{b} \rangle \Gamma_{1,2}^{b} + \langle \Gamma_{1,2}^{a} \rangle \Gamma_{1,2}^{a})$, where $\Gamma_{1,2}^{z} = \Gamma_{1,2}^{d}$. The form of the equation for $C_{re}(\rho_{qb})$ is an indicative that the term $\langle \Gamma_{1}^{d} \rangle$, related to the populations, does not cancel out. We verify that this is indeed the case via an example, which is shown in Fig. 1.

IV. COHERENCE DYNAMICS UNDER QUTRIT DEPHASING AND DISSIPATION

In this section, we study the dynamics of some coherence functions under decoherence. Once for one-qubit states the Hilbert-Schmidt coherence (HSC) and the $l_{1}$-norm coherence (LIC) are proportional, their dynamic behavior shall be equivalent in this physically meaningful kind of process. For instance, for a qubit the freezing conditions [45, 58] for $C_{hs}$ and for $C_{l_{1}}$ are the same. However, even in this simplest context, the dynamic behavior of relative entropy of coherence (REC) is qualitatively disparate from those of HSC and LIC. And this is due to the dependence of the first on the populations. On the other hand, already for qutrits the proportionality between $C_{hs}$ and $C_{l_{1}}$ does not hold in general. Here we have $C_{l_{1}}(\rho_{qb}) = \sum_{k,l} \sqrt{(\langle \Gamma_{k,l}^{s} \rangle)^{2} + (\langle \Gamma_{k,l}^{n} \rangle)^{2}}$ with $1 \leq k \leq l \leq 3$. So, we can warrant that $C_{l_{1}} \propto C_{hs}$ only if one single pair $(\langle \Gamma_{k,l}^{s} \rangle, \langle \Gamma_{k,l}^{n} \rangle)$ is non-null. In the following we shall regard the dynamics induced by qutrit
We illustrate the time dependence of these functions, and of REC, with the specific examples shown in Fig. 2.

Figure 1: (color online) Relative entropy of coherence as a function of the qubit’s population-controlling parameter \( \langle \Gamma_1^d \rangle \) for a fixed value of the \( l_1 \)-norm coherence \( C_{l_1}(\rho_{pd}) = \sqrt{\langle \Gamma_{(1,2)}^d \rangle^2 + \langle \Gamma_{(1,2)}^d \rangle^2} = \sqrt{0 + 1/4} \). If we look at the Bloch’s sphere in the \( \mathbb{R}^3 \) space \((\langle \Gamma_{(1)}^a \rangle, \langle \Gamma_{(1,2)}^a \rangle, \langle \Gamma_{(1,2)}^a \rangle)\), a constant value of \( C_{l_1} \) (or of \( C_{hs} \)) will be associated with points in the lateral surface of a cylinder of physical states within the sphere. In this cylinder, for \( \langle \Gamma_1^d \rangle \) running from \(-\sqrt{1-C_{l_1}^2}\) to \(\sqrt{1-C_{l_1}^2}\), different values of \(|\langle \Gamma_1^a \rangle|\) shall lead to different values of \(C_{re}\). The uppermost and downmost circles of that cylinder surface touch the sphere surface, and in these circles lie pure states for which the Bloch’s vector norm, shown in the upper (red) curve, is equal to one.

For a qutrit prepared in a general state and submitted to the action of the PD or AD channels, the states at parametrized time \( p \) [61], \( \rho_p = \sum_j K_j \rho K_j^\dagger \), are given as follows:

\[
\rho_p^{pd} = \begin{bmatrix}
3^{-1} + \langle (\Gamma_1^a) + (\Gamma_2^a) \rangle/2 & q(\langle \Gamma_{(1,2)}^a \rangle - i\langle \Gamma_{(1,2)}^a \rangle)/2 & q(\langle \Gamma_{(1,3)}^a \rangle - i\langle \Gamma_{(1,3)}^a \rangle)/2 \\
q(\langle \Gamma_{(1,2)}^a \rangle + i\langle \Gamma_{(1,3)}^a \rangle)/2 & 3^{-1} - \langle (\Gamma_1^a) - (\Gamma_2^a) \rangle/2 & q(\langle \Gamma_{(2,3)}^a \rangle - i\langle \Gamma_{(2,3)}^a \rangle)/2 \\
q(\langle \Gamma_{(1,3)}^a \rangle + i\langle \Gamma_{(2,3)}^a \rangle)/2 & q(\langle \Gamma_{(2,3)}^a \rangle + i\langle \Gamma_{(2,3)}^a \rangle)/2 & 3^{-1} - \langle (\Gamma_1^a) \rangle/2
\end{bmatrix}
\]

\[
\rho_p^{ad} = \begin{bmatrix}
(1+2p)/3 + q(\langle \Gamma_1^a \rangle + (\Gamma_2^a))/2 & q(\langle \Gamma_{(1,2)}^a \rangle - i\langle \Gamma_{(1,2)}^a \rangle)/2 & q(\langle \Gamma_{(1,3)}^a \rangle - i\langle \Gamma_{(1,3)}^a \rangle)/2 \\
q(\langle \Gamma_{(1,2)}^a \rangle + i\langle \Gamma_{(1,3)}^a \rangle)/2 & q(3^{-1} - \langle (\Gamma_1^a) - (\Gamma_2^a) \rangle/2 & q(\langle \Gamma_{(2,3)}^a \rangle - i\langle \Gamma_{(2,3)}^a \rangle)/2 \\
q(\langle \Gamma_{(1,3)}^a \rangle + i\langle \Gamma_{(2,3)}^a \rangle)/2 & q(\langle \Gamma_{(2,3)}^a \rangle + i\langle \Gamma_{(2,3)}^a \rangle)/2 & q(3^{-1} - \langle (\Gamma_2^a) \rangle/2
\end{bmatrix}
\]

with \( q = 1-p \). As all off-diagonal elements of \( \rho_p^{pd} \) decay at the same rate, we see that for the PD channel (PDC) the following motion equation holds for the L1C: \( \dot{C}_{l_1}(\rho_{pd}) = (1-p)C_{l_1}(\rho_{pd}) \). Besides, as all Bloch’s vector components from the symmetric and anti-symmetric sets are proportional to \( 1-p \), we get the following evolution equation for HSC: \( \dot{C}_{hs}(\rho_{pd}) = (1-p)C_{hs}(\rho_{pd}) \). Therefore, even if there is no simple-general proportionality relation between the initial values of L1C and HSC, the rate with which these values diminish with time for the PDC is once again observed to be equivalent. On the other hand, for this particular model of dissipation of three level systems, not all off-diagonal elements of \( \rho_p^{ad} \) change equally with time. Here the time dependence of the coherences for the AD channel are:

\[
C_{l_1}(\rho_{pd}) = \sqrt{1-p} \left( \sqrt{\langle \Gamma_{(1,2)}^a \rangle^2 + \langle \Gamma_{(1,2)}^a \rangle^2} + \sqrt{\langle \Gamma_{(1,3)}^a \rangle^2 + \langle \Gamma_{(1,3)}^a \rangle^2} \right) + (1-p) \sqrt{\langle \Gamma_{(2,3)}^a \rangle^2 + \langle \Gamma_{(2,3)}^a \rangle^2},
\]

\[
C_{hs}(\rho_{pd}) = \frac{1}{\sqrt{2}} \sqrt{(1-p) \left( \langle \Gamma_{(1,2)}^a \rangle^2 + \langle \Gamma_{(1,2)}^a \rangle^2 + \langle \Gamma_{(1,3)}^a \rangle^2 + \langle \Gamma_{(1,3)}^a \rangle^2 \right) + (1-p)^2 \left( \langle \Gamma_{(2,3)}^a \rangle^2 + \langle \Gamma_{(2,3)}^a \rangle^2 \right)},
\]
In the last equation we used the fact that, under the HSD, \( \iota \) result to derive an analytical expression for the Hilbert-Schmidt quantum coherence (HSC) of \( n \) and \( L_1C \) is seen to be similar, but quite different from that of the REC (regarding the functions concavities).

**V. NON-MONOTONICITY OF THE HSD UNDER TENSOR PRODUCTS**

One can verify that, similarly to what happens with the trace distance (TD) \[49\], the Hilbert-Schmidt distance (HSD) does not suffer from the non-monotonicity under tensor product (NMuTP) issue for general pure states or for one-qubit collinear states. Actually, one can show that this holds true for all \( p \)-norm distances \[20\]. For general qudit states, the probability for random quartets of states to present the NMuTP behavior of HSD decays with increasing one-qubit collinear states. Actually, one can show that this holds true for all \( p \)-norm distances \[20\]. For general qudit states, the probability for random quartets of states to present the NMuTP behavior of HSD can be revealed on its associated HSC if we manipulate accordingly; what we will do by using an example. We will consider \( (1 - \Gamma_d) \rho_q \) and \( (1 - \Gamma_d') \rho_{q'} \) for general qudit systems. This

**VI. CONCLUDING REMARKS**

In this article, we started obtaining a closed formula for the Hilbert-Schmidt distance (HSD) between two generic \( n \)-qudit states in terms of the Euclidean distance between the corresponding rescaled Bloch’s vectors. We used this result to derive an analytical expression for the Hilbert-Schmidt quantum coherence (HSC) of \( n \)-qudit systems. This
Figure 3: (color online) Percentage of the random quartets of states leading to the non-monotonic behavior under tensor products of the Hilbert-Schmidt distance. Using the standard method from the library described in Ref. [62] (the associated Fortran code can be accessed freely in [63]), we generated one million quartets of states for each dimension $d$.

The formula was then exemplified in simple cases and we applied it to investigate issues related to the distribution and transformation of quantum coherence in some composite systems.

After splitting the total HSC of two-qubit states into its local and non-local parts, we analyzed the possibility of controlling these coherences by tuning the local populations of two copies of a generic one-qubit state. Related to that matter, we observed some interesting contrasting behaviors of the HSC, $l_1$-norm coherence (L1C), and relative entropy of coherence (REC). We noticed that it is not possible to change the local HSC and L1C by tuning local populations. However, contrary to what was seen for the L1C, we can control the HSC by doing that kind of operation. The REC of two copies of a qubit state can be changed only if the local REC is changed, what we showed being possible by the control of local populations. This raises yet another bold difference among these coherence functions. In future works, it could be fruitful to investigate this kind of issue considering its possible physical implications and also another coherence quantifiers.

We also investigated the time evolution of coherence under qutrit dephasing and dissipation environments. The population-dependent character of the REC leads to its quite different dynamical behavior when compared to those of L1C and HSC. These last two coherence functions are seen to be equivalent for qubits and for some qutrit states. Besides, we have shown that even when HSC and L1C are not proportional, their dynamical behavior can be similar.

We have identified the first important implication of the non-monotonicity under tensor product (NMuTP) property of a quantum distance measure. We showed that the NMuTP of HSD, when applied to some states, can indicate that although a configuration $\rho_{qb}$ is more coherent than another state $\xi_{qb}$, two independent copies of the first state possess less HSC than two copies of the last. We emphasize that we succeeded in doing that at this moment mainly because we used the HSD. This unexpected result, obtained for the HSD, calls for further investigations of the NMuTP issue regarding other quantum distance measures and also considering other of its possible consequences.

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Quântica (INCT-IQ), process 2008/57856-6.

[1] R. P. Feynman, R. B. Leighton, and M. Sands, *The Feynman Lectures on Physics*, Volume 3 (Addison-Wesley, Massachusetts, 1965).

[2] L. Mandel and E. Wolf, *Optical Coherence and Quantum Optics* (Cambridge University Press, Cambridge, England, 2008).

[3] A. Streltsov, U. Singh, H. S. Dhar, M. N. Bera, and G. Adesso, Measuring quantum coherence with entanglement, Phys. Rev. Lett. 115, 020403 (2015).

[4] J. Ma, B. Yadin, D. Girolami, V. Vedral, and M. Gu, Converting coherence to quantum correlations, Phys. Rev. Lett. 116, 160407 (2016).

[5] P. Giorda and M. Allegra, Coherence in quantum estimation, arXiv:1611.02519.

[6] A. Chenu and G. D. Scholes, Coherence in energy transfer and photosynthesis, Annu. Rev. Phys. Chem. 66, 69 (2015).

[7] R. Uzdin, Coherence-induced reversibility and collective operation of quantum heat machines via coherence recycling, Phys. Rev. Applied 6, 024004 (2016).

[8] M. Hillery, Coherence as a resource in decision problems: The Deutsch-Jozsa algorithm and a variation, Phys. Rev. A 93, 012111 (2016).

[9] X. Yuan, K. Liu, Y. Xu, W. Wang, Y. Ma, F. Zhang, Z. Yan, R. Vijay, L. Sun, and X. Ma, Experimental quantum randomness processing using superconducting qubits, Phys. Rev. Lett. 117, 010502 (2016).

[10] A. Streltsov, E. Chitambar, S. Rana, M. N. Bera, A. Winter, and M. Lewenstein, Entanglement and coherence in quantum state merging, Phys. Rev. Lett. 116, 240405 (2016).

[11] A. Misra, U. Singh, S. Bhattacharya, and A. K. Pati, Energy cost of creating quantum coherence, Phys. Rev. A 93, 052335 (2016).

[12] H.-L. Shi, S.-Y. Liu, X.-H. Wang, W.-L. Yang, Z.-Y. Yang, and H. Fan, Coherence depletion in the Grover quantum search algorithm, Phys. Rev. A 95, 032307 (2017).

[13] A. Streltsov, G. Adesso, and M. B. Plenio, Quantum coherence as a resource, arXiv:1609.02439.

[14] E. Chitambar and G. Gour, Critical examination of incoherent operations and a physically consistent resource theory of quantum coherence, Phys. Rev. Lett. 117, 030401 (2016).

[15] I. Marvian and R. W. Spekkens, How to quantify coherence: Distinguishing speakable and unspeakable notions, Phys. Rev. A 94, 052324 (2016).

[16] T. Baumgratz, M. Cramer, and M. B. Plenio, Quantifying coherence, Phys. Rev. Lett. 113, 170401 (2014).

[17] C. Napoli, T. R. Bromley, M. Cianciaruso, M. Piani, N. Johnston, and G. Adesso, Robustness of coherence: An operational and observable measure of quantum coherence, Phys. Rev. Lett. 116, 150502 (2016).

[18] M. Piani, M. Cianciaruso, T. R. Bromley, C. Napoli, N. Johnston, and G. Adesso, Robustness of asymmetry and coherence of quantum states, Phys. Rev. A 93, 042107 (2016).

[19] F. Levi and F. Mintert, A quantitative theory of coherent delocalization, New J. Phys. 16, 033007 (2014).

[20] D. Pérez-García, M. M. Wolf, D. Petz, and M. B. Ruskai, Contractivity of positive and trace-preserving maps under Lp norms, J. Math. Phys. 47, 083506 (2006).

[21] X. Wang and S. G. Schirmer, Contractivity of the Hilbert-Schmidt distance under open-system dynamics, Phys. Rev. A 79, 052326 (2009).

[22] J. Dajka, J. Łuczka, and P. Hänggi, Distance between quantum states in the presence of initial qubit-environment correlations: A comparative study, Phys. Rev. A 84, 032120 (2011).

[23] M. Ozawa, Entanglement measures and the Hilbert-Schmidt distance, Phys. Lett. A 268, 158 (2000).

[24] M. Piani, The problem with the geometric discord, Phys. Rev. A 86, 034101 (2012).

[25] B. Dakic, Y. O. Lipp, X. Ma, M. Ringbauer, S. Kropatschek, S. Barz, T. Paterek, V. Vedral, A. Zeilinger, C. Brukner, and P. Walther, Quantum discord as resource for remote state preparation, Nat. Phys. 8, 666 (2012).

[26] X. Wu and T. Zhou, Geometric discord: A resource for increments of quantum key generation through twirling, Sci. Rep. 5, 13365 (2015).

[27] R. A. Bertlmann, H. Narnhofer, and W. Thirring, A geometric picture of entanglement and Bell inequalities, Phys. Rev. A 66, 032319 (2002).

[28] R. A. Bertlmann, K. Durstberger, B. C. Hiesmayr, and P. Krammer, Optimal entanglement witnesses for qubits and qudits, Phys. Rev. A 72, 052331 (2005).

[29] J. Lee, M. S. Kim, and C. Brukner, Operationally invariant measure of the distance between quantum states by complementary measurements, Phys. Rev. Lett. 91, 087902 (2003).

[30] B. Tamir and E. Cohen, A Holevo-type bound for a Hilbert Schmidt distance measure, J. Quant. Inf. Science 05, 127 (2015).

[31] V. V. Dodonov, O. V. Man’ko, V. I. Man’ko, and A. Wünsche, Hilbert-Schmidt distance and non-classicality of states in quantum optics, J. Mod. Opt. 47, 633 (2000).

[32] K. Zyczkowski and H.-J. Sommers, Hilbert-Schmidt volume of the set of mixed quantum states, J. Phys. A: Math. Gen. 36, 10115 (2003).

[33] S. Popescu, A. J. Short, and A. Winter, Entanglement and the foundations of statistical mechanics, Nat. Phys. 2, 754 (2006).
[34] G. Björk, H. de Guise, A. B. Klimov, P. de la Hoz, and L. L. Sánchez-Soto, Classical distinguishability as an operational measure of polarization, Phys. Rev. A 90, 013830 (2014).

[35] C. Witte and M. Trucks, A new entanglement measure induced by the Hilbert-Schmidt norm, Phys. Lett. A 257, 14 (1999).

[36] F. Verstraete, J. Dehaene, and B. De Moor, On the geometry of entangled states, J. Mod. Opt. 49, 1277 (2002).

[37] J. Maziero, Computing partial transposes and related entanglement functions, Braz. J. Phys. 46, 605 (2016).

[38] B. Dakic, V. Vedral, and C. Brukner, Necessary and sufficient condition for non-zero quantum discord, Phys. Rev. Lett. 105, 190502 (2010).

[39] L. Li, Q.-W. Wang, S.-Q. Shen, and M. Li, Geometric measure of quantum discord with weak measurements, Quantum Inf. Process. 15, 291 (2016).

[40] S. Luo and S. Fu, Geometric measure of quantum discord, Phys. Rev. A 82, 034302 (2010).

[41] F. Verstraete, J. Dehaene, and B. De Moor, On the geometry of entangled states, J. Mod. Opt. 49, 1277 (2002).

[42] J. Maziero, Computing coherence vectors and correlation matrices with application to quantum discord quantification, Adv. Math. Phys. 2016, e6892178 (2016).

[43] G. Kimura, The Bloch vector for N-level systems, Phys. Lett. A 314, 339 (2003).

[44] R. A. Bertlmann and P. Krammer, Bloch vectors for qudits, J. Phys. A: Math. Theor. 41, 235303 (2008).

[45] J. Maziero, Computing coherence vectors and correlation matrices with application to quantum discord quantification, Adv. Math. Phys. 2016, e6892178 (2016).

[46] https://github.com/jonasmaziero/LibForQ.

[47] K. M. R. Audenaert, J. Calsamiglia, R. Munóz-Tapia, E. Bagan, Ll. Masanes, A. Acin, F. Verstraete, Discriminating states: The quantum Chernoff bound, Phys. Rev. Lett. 98, 160501 (2007).

[48] J. Maziero, Non-monotonicity of trace distance under tensor products, Braz. J. Phys. 45, 560 (2015).

[49] S. Rana, P. Parashar, and M. Lewenstein, Trace-distance measure of coherence, Phys. Rev. A 93, 012110 (2016).

[50] L.-H. Shao, Z. Xi, H. Fan, and Y. Li, Fidelity and trace-norm distances for quantifying coherence, Phys. Rev. A 91, 042120 (2015).

[51] S. Luo and S. Fu, Measurement-induced nonlocality, Phys. Rev. Lett. 106, 120401 (2011).

[52] J. Maziero, Computer code for generating random probability vectors, unitaries, and quantum states, Front. ICT 3, 4 (2016).

[53] https://github.com/jonasmaziero/LibForr.