Optical properties of Born-Infeld-dilaton-Lifshitz holographic superconductors

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In this paper, we first study the Lifshitz-dilaton holographic superconductors with nonlinear Born-Infeld (BI) gauge field and obtain the critical temperature of the system for different values of Lifshitz dynamical exponent, $z$, and nonlinear parameter $b$. The critical temperature decreases with increasing $z$. This indicates that the increase of anisotropy between space and time prevents the phase transition. Also, for fixed value of $z$, the critical temperature decrease with increasing $b$. Then, we investigate the optical properties of $(2 + 1)$ and $(3 + 1)$-dimensional BI-Lifshitz holographic superconductors in the presence of dilaton field. We explore the refractive index of the system. For $z = 1$ and $(2 + 1)$-dimensional holographic superconductor, we observe negative real part for permittivity $\text{Re}[\varepsilon]$ as frequency $\omega$ decreases. Thus, in low frequency region our superconductor exhibit metamaterial property. This behavior is independent of the nonlinear parameter and can be seen for either linear ($b = 0$) and nonlinear ($b \neq 0$) electrodynamics. Interestingly, for $(3 + 1)$-dimensional Lifshitz-dilaton holographic superconductors, we observe metamaterial behavior neither in the presence of linear nor nonlinear electrodynamics.

I. INTRODUCTION

The correspondence between gauge fields living on the boundary of a spacetime and the gravity in the bulk, called gauge/gravity duality, provides a powerful tool for studying the strongly coupled systems in quantum field theory [1, 2]. According to this dictionary, one can effectively calculate correlation functions in a strongly interacting field theory by using a dual classical gravity description. A new application of this duality, called holographic superconductor, was recently proposed in order to shed light on the understanding the mechanism governing the high-temperature superconductors in condensed-matter physics [3]. In [3–5], by including the Abelian Higgs model within the AdS black hole spacetime, holographic superconductors have been built. Decreasing the Hawking temperature to a critical value, the black hole exhibits unstability against small perturbations and by condensing some field, grows hair to make the system stable. This process can be adjudged as the holographic description of the superconducting phase transition. The (asymptotically) AdS black hole spacetime is taken as the starting point for this kind of building for holographic superconductors. The studies on the holographic superconductor have got a lot of attentions (see for example [6–31] and references therein). When the gauge field is in the form of nonlinear BI electrodynamics, the holographic superconductors have been explored in [32–41]. It was argued that the nonlinear parameter decrease the critical temperature of the superconductor and make the condensation harder, while the critical exponent associated with the condensation near the critical temperature has still the universal value 1/2 of the mean field theory [32–39].

On the other hand, since in condensed matter physics a dynamical exponent appears near the critical point, the studies on Lifshitz spacetimes have attained a lot of interests. It was shown that the near critical point dynamics of such systems can be described by a relativistic conformal field theory or a more subtle scaling theory respecting the Lifshitz symmetry $t \rightarrow \lambda^2 t, \vec{x} \rightarrow \lambda \vec{x}$ [42]. Various aspects of Lifshitz spacetimes have been explored in the literature [43–57]. The holographic superconductors were also extended to the context of Lifshitz black holes [58, 59]. It has been argued that despite quite different behaviors of asymptotically AdS and Lifshitz geometries as a result of existing dynamical exponent, the holographic Lifshitz superconductors basically behave the same qualitatively as AdS ones.
Effects of dynamical exponent \( z \) on the holographic Lifshitz superconductor models have been disclosed both numerically and analytically in [61]. Holographic superconducting phase transitions of a system dual to Yang-Mills field coupled to an axion as probes of black hole with arbitrary Lifshitz scaling have been studied with \( p_x + ip_y \) condensate [62]. The properties of the Weyl holographic superconductor in the Lifshitz black hole background were explored in [63]. It was shown that the critical temperature of the Weyl superconductor decreases with increasing Lifshitz dynamical exponent, \( z \), indicating that condensation becomes difficult [63]. Analytical study on the properties of the \( s \)-wave holographic superconductor in the presence of Exponential nonlinear electrodynamics in the Lifshitz black hole background in four-dimensions has been done in [64].

It is also interesting to investigate the optical properties of holographic superconductors\([65–69]\). It has been argued that no negative refractive index is seen for Lifshitz-dilaton-Maxwell holographic (2 + 1)-dimensional superconductors in probe limit [67]. Here, we re-investigate this case and show that one could find negative index of refraction albeit just for Lifshitz exponent \( z = 1 \). We also extend the study of [67] in probe limit to (3 + 1)-dimensional Lifshitz-dilaton superconductors as well as nonlinear BI electrodynamics. We show that one could observe negative refractive index for non-linearly charged (2 + 1)-dimensional superconductors just with \( z = 1 \) whereas no such effect is occurred for (3 + 1)-dimensional Lifshitz-dilaton superconductors neither in the presence of linear nor nonlinear electrodynamics for any \( z \). The latter result is in agreement with the conclusions of [65] and [66] in AdS case and further reveal that neither Lifshitz scaling nor presence of nonlinear electrodynamics help (3 + 1)-dimensional Lifshitz-dilaton holographic superconductors to exhibit negative refraction phenomenon.

This paper is organized as follows. In section II, we describe the holographic superconductors in Lifshitz-dilaton setup and in the probe limit, when the gauge field is in the form of BI nonlinear electrodynamics. In section III, we investigate the optical properties of BI-dilaton-Lifshitz holographic superconductors. In particular, we shall study the behavior of refractive index for our holographic superconductors. The last section is devoted to summary and conclusions.

II. HOLOGRAPHIC SETUP OF BI-DILATON-LIFSHITZ SUPERCONDUCTORS

In this section, we will present the holographic setup for dilaton-Lifshitz superconductors in the presence of BI electrodynamics. The \( D \)-dimensional metric which is invariant under the dynamical scaling \( t \to \lambda^z t, \, \mathbf{x} \to \lambda \mathbf{x} \) where \( z \) is dynamical critical exponent is given by [42]

\[
\text{ds}^2 = L^2 \left( -r^{2z} \text{d}t^2 + r^2 \text{d}\mathbf{x}^2 + \frac{\text{d}r^2}{r^2} \right),
\]

(1)

where \( \text{d}x^2 = \sum_{i=1}^{d} \text{d}x_i^2 \), \( D = d + 2 \) and \( 0 < r < \infty \). The AdS \( d+2 \) can be recovered by setting \( z = 1 \) in metric (1). (1) is a solution of the action [70]

\[
S = \frac{1}{16\pi G_{d+2}} \int \text{d}^{d+2} \sqrt{-g} \left( R - 2\Lambda - \frac{1}{2} (\partial \varphi)^2 - \frac{1}{4} e^{\lambda \varphi} H_{\mu\nu} H^{\mu\nu} \right),
\]

(2)

where \( R \) is the Ricci scalar, \( \Lambda \) is the cosmological constant, \( \varphi \) is dilaton scalar field and \( H_{\mu\nu} = \partial_{[\mu} B_{\nu]} \) where \( B_{\nu} \) is a gauge potential, provided

\[
H_{rt} = \sqrt{2(z - 1)(z + d)} L r^{z+d-1}, \quad e^{\lambda \varphi} = r^{-2d},
\]

\[
\Lambda = \frac{(z + d - 1)(z + d)}{2L^2}, \quad \lambda = \frac{2d}{z - 1}.
\]

(3)

The finite temperature generalization of metric (1) which is an asymptotic Lifshitz black hole is [70]

\[
\text{ds}^2 = L^2 \left( -r^{2z} f(r) \text{d}t^2 + \frac{1}{r^2 f(r)} \text{d}r^2 + r^2 \text{d}\mathbf{x}^2 \right),
\]

(4)

where

\[
f(r) = 1 - \frac{r_+^{z+d}}{r^{z+d}},
\]

(5)

where \( r_+ \) is the black hole horizon. The Hawking temperature of this black hole is

\[
T = \frac{(z + d) r_+^z}{4\pi}.
\]

(6)
Here, we intend to build a Lifshitz holographic superconductor in the presence of BI electrodynamics in probe limit. Therefore, we consider the following Lagrangian density for electrodynamics and scalar field

\[
\mathcal{L}_m = \mathcal{L}_{BI} - |\nabla_\mu \psi - iq A_\mu \psi|^2 - m^2 \psi^2,
\]

(7)

where \(\psi\) is a charged complex scalar field, \(A_\mu\) is gauge field and \(q\) and \(m\) are the charge and the mass of the scalar field respectively. \(\mathcal{L}_{BI}\) is Lagrangian density of the BI electrodynamics

\[
\mathcal{L}_{BI} = \frac{1}{b^2} \left( 1 - \sqrt{1 + \frac{b^2 F_{\mu\nu} F^{\mu\nu}}{2}} \right),
\]

(8)

where \(F_{\mu\nu} = \partial_{[\mu} A_{\nu]}\) is the electrodynamic tensor and \(b\) is the BI nonlinear parameter. In the limit \(b \to 0\), (8) reduces to the linear Maxwell case. We choose the ansatz [3]

\[
\psi = \psi(r), \quad A_\mu dx^\mu = \phi(r) dt.
\]

(9)

Varying (7), the equations of motion for the complex scalar field \(\psi(r)\) and the electric potential \(\phi(r)\) in the background (4) are

\[
\psi'' + \left( \frac{d + z + 1}{r} + \frac{f'}{f} \right) \psi' + \frac{q^2 \phi^2}{r^{z+2} f^2} \psi - \frac{m^2 L^2}{r^2 f} \psi = 0,
\]

(10)

and

\[
\phi'' + \frac{d - z + 1}{r} \phi' - \frac{2q^2 L^2 \psi^2 \phi}{r^2 f} \sqrt{1 - \frac{b^2 \phi^2}{L^4 r^{2z-2} \left( 1 - \frac{b^2}{L^4 r^{2(z-1)} \phi'^2} \right)}} - \frac{b^2 d}{L^4 r^{2z-2}} \phi'^3 = 0,
\]

(11)

where prime stands for derivative with respect to \(r\). It is difficult to find the analytical solutions to these nonlinear equations. However, they are reachable numerically. In order to solve the above two equations numerically, we specify
are depicted for the condensation and

\[ T_{c} \]

of anisotropy between space and time prevents the phase transition. This can be explained as follows. As it is seen from the table

\[ \frac{T_{c}}{\rho^{\nu}} \]

different values of the parameter \( \nu \). Also

\[ z \]

d as a function of temperature for various

\[ \phi \]

and charge density of dual field theory respectively. We impose

\[ r \]

the boundary conditions for \( \psi \) and \( \phi \) near the horizon of black hole \( r = r_{+} \) and in the spatial infinity \( r \to \infty \). In order to satisfy the regularity condition at the horizon we demand \( \phi(r_{+}) = 0 \). Also, \( \psi(r_{+}) \) should be regular at horizon. At the boundary \( r \to \infty \), the complex scalar field \( \psi \) and the scalar potential \( \phi \) behave as

\[
\psi(r) = \begin{cases}
(\psi^{(2)} + \psi^{(1)} \ln \xi r) r^{-\nu} + \cdots, & \text{for } \nu = \nu_{+} = \nu_{-}, \\
\psi^{(1)} r^{-\nu_{-}} + \psi^{(2)} r^{-\nu_{+}} + \cdots, & \text{other cases,}
\end{cases}
\]

where

\[
\nu_{\pm} = \frac{z + d \pm \sqrt{(z + d)^2 + 4L^2 m^2}}{2},
\]

and

\[
\phi(r) = \begin{cases}
\mu - \rho r^{z-d} + \cdots, & \text{for } z < d, \\
\mu - \rho \ln \xi r + \cdots, & \text{for } z = d,
\end{cases}
\]

where \( \psi^{(1)}, \psi^{(2)}, \mu, \rho \) and \( \xi \) are constants. According to gauge/gravity duality dictionary, \( \psi^{(1)} \) and \( \psi^{(2)} \) correspond to the vacuum expectation value of the dual operator \( O \) dual to the scalar field and \( \mu \) and \( \rho \) are chemical potential and charge density of dual field theory respectively. We impose \( \psi^{(1)} \) equals zero in order to make the superconducting phase transition a spontaneous breaking of symmetry [61]. Explicitly the modified BF bound for the scalar mass is

\[ m^2 \geq -(z + d)^2/4 \]

Throughout this paper when we discuss the black hole backgrounds we assume that \( \rho \) is fixed. Also \( \nu \) denotes \( \nu_{+} \). For our numerical calculations we focus on the cases \( z = 2 \) in \( d = 2 \) and \( z = 2, 3 \) in \( d = 3 \) for different values of the parameter \( b \). In our numerical calculations, we fix the dimension \( \nu \), in order to see the influence of the dynamical critical exponent \( z \) and nonlinear parameter \( b \). Figs. 1, 2 and 3 are depicted for the condensation as a function of temperature for various \( z \) and \( b \). It is seen from the figures that the condensation decreases as the parameter \( z \) increases. In table I, we show the values of \( T_{c}/\rho^{\nu}/d \) for the cases \( z = 2 \) and \( z = 3 \) with fixed \( \nu \)'s. As it is seen from the table I, for each case as we increase \( z \), the critical temperature decreases, showing that the increase of anisotropy between space and time prevents the phase transition. This can be explained as follows. As it is seen from equation (10) as the dynamical critical exponent \( z \) increases, the effective mass of the scalar field increases near the horizon. This causes a lower critical temperature as \( z \) is increased [61].

\[
\frac{<O>^{1/\nu}}{\rho^{1/\nu}}
\]

FIG. 3: The behavior of \( \langle O \rangle^{1/\nu}/\rho^{1/\nu} \) vs. \( T/T_{c} \) for the case \( d = 3 \) and \( z = 3 \) with \( b = 0, 0.15 \) and 0.3.

| \( b = 0 \) | \( b = 0.3 \) | \( b = 0.6 \) |
|---|---|---|
| \( d = z \) | \( T_{c}/\rho^{\nu/3} \) | \( d = z \) | \( T_{c}/\rho^{\nu/3} \) | \( d = z \) | \( T_{c}/\rho^{\nu/3} \) |
| 0.0 | 0.1184 | 0.1023 | 0.0962 |
| 0.1 | 0.0578 | 0.0475 | 0.0418 |
| 0.2 | 0.0236 | 0.0236 | 0.0214 |
| 0.3 | 0.0180 | 0.0134 | 0.0095 |
| 0.4 | 0.0084 | 0.0025 | 0.0008 |
| 0.5 | 0.0452 | 0.0094 | 0.0004 |

TABLE I: The values of critical temperature for different \( z \) and \( b \).
III. OPTICAL PROPERTIES OF BI-DILATON-LIFSHITZ HOLOGRAPHIC SUPERCONDUCTORS

Here, we will investigate the optical properties of BI-dilaton-Lifshitz holographic superconductors. Specially, we will study the behavior of refractive index (focusing on the negativity of it) for our holographic superconductors. In order to do this, we will study the behavior of Depine-Lakhtakia (DL) index \[n_{DL} = |\epsilon|\Re[\mu] + |\mu|\Re[\epsilon],\] (14)

where \(\epsilon\) is the permittivity and \(\mu\) is effective permeability. The negativity of DL index shows that the phase velocity and energy flow are in opposite directions in the medium. It implies that the refractive index of the system is negative. It is clear from Eq. (14) that in order to compute DL index we need to obtain \(\epsilon\) and \(\mu\). To determine \(\epsilon\) and \(\mu\) corresponding to the superconductor, we apply an external electric field by turning on \(A_j = A_j(r) e^{-i\omega t + iK_i x_i}\) where \(j\) and \(k\) stand for two directions which are perpendicular to each other e.g. \(x\) and \(y\), \(\omega\) is the frequency and \(K_k\) is the momentum. From now on, we replace \(A_j\) with \(A\) and \(K_k\) with \(K\) for abbreviation. Since we are in probe limit, the equation of motion for \(A\) is decoupled from other ones and reads

\[A'' + \left(\frac{d + z - 1}{r} + \frac{f'}{f} - \frac{d b^2}{L^4 r^{2z-1} \phi'^2}\right) A' + \left(\frac{\omega^2}{r^{2(z+1)} f^2} - \frac{K^2}{r^2 f}\right) A - \frac{2L^2 q^2 \psi^2}{r^2 f} \sqrt{1 - \frac{b^2 \phi'^2}{L^4 r^{2z-2}}} \left(A - \frac{b^2 \phi \phi'}{L^4 r^{2(z-1)}} A^0\right) = 0.\] (15)

We restrict our study here to \(d = 2\) and \(3\) dimensions. The behavior of \(A\) as \(r\) goes to infinity is then

\[\lim_{r \to \infty} A = \begin{cases} A_0 + A_1 r^{-z} + \cdots, & d = 2, \\ A_0 + A_1 r^{-z-1} + \cdots, & d = 3 (z \neq 1), \\ A_0 \left(1 + \frac{1}{2} \omega^2 r^{-2} \ln \xi r\right) + A_1 r^{-2} + \cdots, & d = 3 (z = 1). \end{cases}\] (16)

Also, \(A\) behaves as \(f^{-\omega/4\pi T}\) near the horizon (by employing the ingoing wave boundary condition near the horizon). According to gauge/gravity duality dictionary, one could holographically interpret \(A_0\) and \(A_1\) as dual source and the expectation value of boundary current. Therefore, the current-current \((J_j - J_j)\) correlator \(G_{jj}\) (the response to \(A\)) has a relation \(G_{jj} = A_1 / A_0\) \([72]\). On the other hand, \(\epsilon\) and \(\mu\) has relation with \(G_{jj}\) so that \([73]\]

\[\epsilon (\omega) = 1 + 4\pi \omega^{-2} C^2 G_{jj}^0 (\omega),\] (17)

\[\mu (\omega) = \left[1 - 4\pi C^2 G_{jj}^0 (\omega)\right]^{-1},\] (18)

where \(C\) is the electromagnetic coupling constant (which we set it to unity in our calculations) and \(G_{jj}^0 (\omega)\) and \(G_{jj}^0 (\omega)\) come from casting the \(K\)-dependent \(G_{jj}\) \([74]\]

\[G_{jj} (\omega, K) = G_{jj}^0 (\omega) + K^2 G_{jj}^0 (\omega) + \cdots.\] (19)

To find the relation between \(G_{jj}^0\) and \(G_{jj}^0\) with \(A_0\) and \(A_1\), we expand \(A\) in powers of \(K\) in the same way as \(G_{jj}\):

\[A (r) = A^0 (r) + K^2 A^2 (r) + \cdots.\] (20)

Then, the corresponding field equations come from Eq. (15) are

\[A''^0 + \left(\frac{d + z - 1}{r} + \frac{f'}{f} - \frac{d b^2}{L^4 r^{2z-1} \phi'^2}\right) A'^0 + \left(\frac{\omega^2}{r^{2(z+1)} f^2} - \frac{K^2}{r^2 f}\right) A^0 - \frac{2L^2 q^2 \psi^2}{r^2 f} \sqrt{1 - \frac{b^2 \phi'^2}{L^4 r^{2z-2}}} \left(A^0 - \frac{b^2 \phi \phi'}{L^4 r^{2(z-1)}} A^0\right) = 0,\] (21)

and

\[A''^2 + \left(\frac{d + z - 1}{r} + \frac{f'}{f} - \frac{d b^2}{L^4 r^{2z-1} \phi'^2}\right) A'^2 + \left(\frac{\omega^2}{r^{2(z+1)} f^2} - \frac{K^2}{r^2 f}\right) A^2 - \frac{2L^2 q^2 \psi^2}{r^2 f} \sqrt{1 - \frac{b^2 \phi'^2}{L^4 r^{2z-2}}} \left(A^2 - \frac{b^2 \phi \phi'}{L^4 r^{2(z-1)}} A^2\right) - \frac{A^0}{r^4 f} = 0.\] (22)
The behaviors of $A^0$ and $A^2$ as $r \to \infty$ are the same as $A$ in Eq. (16). Finally, we have

$$G_{jj}^0 = \begin{cases} \frac{z A^0}{A_0}, & d = 2, \\ \frac{z+1}{A_0}, & d = 3 (z \neq 1), \\ \frac{A^0}{A_0} + c \omega^2, & d = 3 (z = 1), \end{cases} \quad (23)$$

and

$$G_{jj}^2 = \begin{cases} \frac{z A^0}{A_0} \left( \frac{A^2}{A_1} - \frac{A^2}{A_0} \right), & d = 2, \\ \frac{z+1}{A_0} \left( \frac{A^2}{A_1} - \frac{A^2}{A_0} \right), & d = 3 (z \neq 1), \\ \frac{A^0}{A_0} \left( \frac{A^2}{A_1} - \frac{A^2}{A_0} \right) - c, & d = 3 (z = 1), \end{cases} \quad (24)$$

where $c$ could be fixed so that $G_{jj}^0$ vanishes at large frequencies where the variation of external field is too rapid to be responded by the system. Now, we can solve Eqs. (21) and (22) numerically to find $G_{jj}^0$ and $G_{jj}^2$. Then, we compute the permittivity $\epsilon$ and effective permeability $\mu$. Eventually, using $\epsilon$ and $\mu$, we can calculate DL index $n_{DL}$ via Eq. (14).

A. Numerical results

Here, we present the optical properties of $(2+1)$- and $(3+1)$-dimensional Lifshitz-dilaton holographic superconductors numerically. We start with study the behavior of permittivity $\epsilon$ and effective permeability $\mu$ for $(2+1)$-dimensional superconductors with respect to frequency $\omega$. To ensure we are in superconducting phase, we fix the temperature $T$ to $0.7T_c$. As Fig. 4(a) shows, for $z = 1$, we observe negative real part for permittivity Re[$\epsilon$] as $\omega$ decreases. Therefore, in low frequency region our superconductor exhibit metamaterial property. This fact is held both in the presence of linear ($b = 0$) and nonlinear ($b \neq 0$) electrodynamics. As it is clear from Fig. 4(a), the frequency at which $\epsilon$ changes the sign is larger for larger values of $b$. The imaginary part of permittivity Im[$\epsilon$] is positive for all frequencies and has a pole as $\omega \to 0$ (see Fig. 4(b)). The behavior of effective permeability $\mu$ in terms of $\omega$ is depicted in Fig. 5. The real part of it Re[$\mu$] is always positive as Fig. 5(a) shows. However, the imaginary part of permeability Im[$\mu$] exhibits a seemingly odd behavior. As one can see in Fig. 5(b), it is negative for all frequencies. Actually, this happens for all available probe limit analyses in literature. This behavior may be problematic, although $\mu$ is not an observable [66, 68]. However, the issue of sign of Im[$\mu$] is not yet a settled issue in literature [68, 75]. Including backreaction could make Im[$\mu$] positive [69].

Now, we turn to study the behavior of DL index $n_{DL}$ in terms of frequency. This behavior is shown in Figs. 6(a) and 7(a) for $T = 0.7T_c$ and $0.78T_c$, respectively. Before going on further, it is necessary to give a comment on the
validity of our analysis. As we discussed in previous subsection, to calculate $n_{DL}$, we use $G^0_{jj}$ and $G^2_{jj}$, which are the coefficients of expansion of current-current correlator $G_{jj}$ in terms of momentum $K$. Therefore, in order for the series (19) to be restricted to the first two terms, the condition $|K^2 G^2_{jj}/G^0_{jj}| \ll 1$ has to be satisfied. By virtue of the relation $n^2 = K^2/\omega^2 = \epsilon \mu$, one could rewrite the latter condition as $B \omega^2 |n|^2 \ll 1$ where $B = |G^2_{jj}/G^0_{jj}|$. We depict $B \omega^2 |n|^2$ in terms of frequency in Figs. 6(b) and 7(b). Figs. 6(a) and 7(a), in agreement with Fig. 4, show that in low frequency regime our superconductor exhibits metamaterial property namely, $n_{DL} < 0^1$. In low frequency regime, our analysis is valid (although not strictly $^2$) as Figs. 6(b) and 7(b) confirm. As the value of nonlinear parameter $b$ increases, the frequency at which the system goes from positive refraction phase to negative refraction phase increases too. This fact is again in agreement with the result of Fig. 4. Comparing Figs. 6(a) and 7(a), one could find out that by increasing the temperature in superconducting phase, the negative $n_{DL}$ tends more and more to zero and positive values$^3$ and finally the metamaterial behavior disappears. This fact occurs for the cases with smaller $b$ sooner than the larger $b$ cases. Also, for a fixed $b$, the frequency at which $n_{DL}$ changes the sign decreases as temperature increases (see Figs. 6(a) and 7(a)).

It is also interesting to investigate the dissipation effects. These effects could be obtained form the ratio $\text{Re}[n]/\text{Im}[n]$ which shows the propagation to dissipation ratio. The behavior of $\text{Re}[n]/\text{Im}[n]$ in terms of frequency $\omega$ is depicted in Fig. 8. In a part of negative refraction domain, actually near the zero frequency, this ratio tends to zero meaning that the electromagnetic wave is virtually not propagated. This is not a desirable result, however, we would keep in

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$^1$ Since in our numerical studies $\text{Im}[\mu] < 0$ (as all available probe limit studies), there are some possibilities for interpreting $n_{DL} < 0$ case [68]. This issue is not settled [66, 68, 75] and one could interpret this case as metamaterial.

$^2$ This problem occurs in probe limit like the negativity of imaginary part of permeability $\mu$ discussed later. It might be overcome by including backreaction effect. It is an interesting and important issue for future studies.

$^3$ It could be interesting to mention that a qualitatively similar result has been reported in [76], albeit not for a superconductor.
mind that we are in probe approximation. This problem as others mentioned before may be the result of probe limit and like goes away by including the effects of backreaction \[66\]. In addition, unlike the real metamaterials the signs of real and imaginary parts of $n$ are the same. This also could likely be overcome if one works in backreacted regime.

To close this section, we point out here that for $(2 + 1)$-dimensional Lifshitz-dilaton holographic superconductors we observed the metamaterial behavior just for $z = 1$ (the case we discussed above) both for linear and nonlinear electrodynamics. Also, for $(3+1)$-dimensional Lifshitz-dilaton holographic superconductors, we observed metamaterial behavior neither in the presence of linear nor nonlinear electrodynamics. Fig. 9 shows the behavior of $n_{DL}$ in terms of $\omega$ in superconducting phase for $d = 3$ and $z = 2$ as an example. As one could see, $n_{DL}$ is always positive for this case.

**IV. SUMMARY AND CONCLUSIONS**

The studies on the holographic superconductors theory have got a lot of attentions in the past years. This new prescription of superconductor is based on the gauge/gravity duality which provides a powerful tool for investigation the strongly coupled systems in quantum field theory. The motivation for such studying comes from the unsolved problem in condensed-matter physics. Indeed, it was argued that holographic superconductor may shed light on the problem of high temperature superconductors in condensed-matter physics \[3–5\].

In this paper, we have further continued the studies on the holographic superconductors, by considering the framework of Lifshitz-dilaton black holes when the gauge field is in the form of nonlinear BI electrodynamics. We have considered the probe limit in which the gauge and scalar fields do not affect on the background geometry. Due to the complexity of the field equations, we determine the condensation value as well as the critical temperature by using the numerical calculations. In numerical calculations, we studied the cases with the dynamical critical exponent $z = 2$ in $d = 2$ (where $d + 2$ is the spacetime dimensions) and $z = 2, 3$ with $d = 3$ for different values of the nonlinear parameter $b$. We plotted the condensation as a function of the temperature for various $z$ and $b$. We find out that with increasing $z$, the condensation operator decreases. We also investigated the effects of $z$ and $b$ on the behavior of the critical temperature. We observed that for fixed values of $b$, the critical temperature decreases with increasing $z$. 
showing that the increase of anisotropy between space and time prevents the phase transition. This is due to the fact that as the dynamical critical exponent $z$ increases, the effective mass of the scalar field increases near the horizon. Also, for fixed value of $z$, the critical temperature decrease with increasing $b$.

We also numerically investigated the optical properties of (2+1) and (3+1)-dimensional Lifshitz-dilaton holographic superconductors. We observed that in case of (2 + 1)-dimensions and $z = 1$, the real part of permittivity $\text{Re}[\varepsilon]$ is negative as $\omega$ decreases. This implies that in low frequency region, the superconductor exhibit metamaterial property. This behavior is independent of the nonlinear parameter $b$ and can be seen for both linear case ($b = 0$) and nonlinear case ($b \neq 0$). We find that the imaginary part of permittivity $\text{Im}[\varepsilon]$ is positive for all frequencies and has a pole as $\omega \to 0$. Besides, for (3 + 1)-dimensional Lifshitz-dilaton holographic superconductors, we observed metamaterial behavior neither in the presence of linear nor nonlinear electrodynamics.

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[1] J. M. Maldacena, *The large-N limit of superconformal field theories and supergravity*, Adv. Theor. Math. Phys. 2 (1998) 231 [Int. J. Theor. Phys. 38 (1999) 1113] [hep-th/9711200].
[2] E. Witten, *Anti de Sitter space and holography*, Adv. Theor. Math. Phys. 2 (1998) 253 [hep-th/9802150].
[3] S. A. Hartnoll, C. P. Herzog and G. T. Horowitz, *Building an AdS/CFT superconductor*, Phys. Rev. Lett. 101 (2008) 031601 [arXiv:0803.3295].
[4] S. A. Hartnoll, C. P. Herzog and G. T. Horowitz, *Holographic superconductors*, JHEP 12 (2008) 015 [arXiv:0810.1563].
[5] G. T. Horowitz, *Introduction to holographic superconductors*, Lect. Notes Phys. 828 (2011) 313 [arXiv:1002.1722].
[6] D. Musso, *Introductory notes on holographic superconductors*, arXiv:1401.1504.
[7] R. Gregory, S. Kanno and J. Soda, *Holographic superconductors with higher curvature corrections*, JHEP 0910 (2009) 010 [arXiv:0907.3203].
[8] R. G. Cai, L. Li-Fang Li, R. Yang, *Introduction to holographic superconductor models*, Sci. China Phys. Mech. Astron. 58 (2015) 060401 [arXiv:1502.00437].
[9] R. Banerjee, S. Gangopadhyay, D. Roychowdhury, and A. Lala, *Holographic s-wave condensate with nonlinear electrodynamics: A nontrivial boundary value problem*, Phys. Rev. D 87 (2013) 104001 [arXiv:1208.5902].
[10] D. Momeni, M. Raza, and R. Myrzakulov, *More on Superconductors via Gauge/Gravity Duality with Nonlinear Maxwell Field*, Journal of Gravity 2013 (2013) Article ID 782512 [arXiv:1305.3541].
[11] C. M. Chen and M. F. Wu, *An analytic analysis of phase transitions in holographic superconductors*, Prog. Theor. Phys. 126 (2011) 387 [arXiv:1103.5130].
[12] H. B. Zeng, X. Gao, Y. Jiang, and H. S. Zong, *Analytical computation of critical exponents in several holographic superconductors*, JHEP 1105 (2011) 002 [arXiv:1012.5564].
[13] R. G. Cai, Z.Y. Nie, H.Q. Zhang, *Holographic p-wave superconductors from Gauss-Bonnet gravity*, Phys. Rev. D 82 (2010) 066007 [arXiv:1007.3321].
