ON A RESULT OF JESSICA LIN

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Abstract. In Theorem 1.5 of [5] an estimate from below is given for nonnegative supersolutions of parabolic equations in terms of the measure of the set where the right-hand side is less than $-1$. The given proof contains many interesting details which are certainly useful in many other situations. However, as long as the statement of Theorem 1.5 of [5] is concerned, it can be proved in a much shorter way, which we present here.

1. Introduction

The author of [5] presents a parabolic version of the lower bound for solutions established by Caffarelli, Souganidis, and Wang in [2], which was used in the error estimates for stochastic homogenization of uniformly elliptic equations in random media [1]. Although the general approach in [5] follows [2], it was necessary to develop a number of new arguments to handle the parabolic structure of the problem. One of the main technical tools in [5] is Theorem 1.5, proved there by invoking many facts which are very useful in many situations. However, as long as the statement of Theorem 1.5 of [5] is concerned, it can be proved in a much shorter way, which we present here expanding a too short to understand argument in Remark 3.4 of [5] attributed to the author. We first consider the case of linear equations and then the general case.

2. Linear case

Let $\mathbb{R}^d$ be a Euclidean space of points $x = (x_1, \ldots, x_d)$, $a(t, x) = (a^{ij}(t, x))$ a $d \times d$ matrix-valued function defined on $\mathbb{R}^{d+1} = \mathbb{R} \times \mathbb{R}^d$. Take a constant $\delta \in (0, 1)$ and assume that for all values of arguments and $\xi \in \mathbb{R}^d$

$$\delta |\xi|^2 \leq a^{ij}\xi^i \xi^j \leq \delta^{-1}|\xi|^2.$$  

Introduce

$$D_i = \frac{\partial}{\partial x^i}, \quad D_{ij} = D_i D_j, \quad \partial_t = \frac{\partial}{\partial t}, \quad L = a^{ij} D_{ij} - \partial_t.$$

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Also set \( B_\alpha = \{ x \in \mathbb{R}^d : |x| < \alpha \} \), \( Q_{\alpha,\beta} = (0, \beta] \times B_\alpha \) and take a function \( u \in W^{1,2}_{d+1,loc}(Q_{1,1}) \cap C(\bar{Q}_{1,1}) \), which is nonnegative on the parabolic boundary of \( Q_{1,1} \). Finally, for a Borel subset \( \Gamma \) of \( \mathbb{R}^{d+1} \) denote by \( |\Gamma| \) its Lebesgue measure.

**Theorem 2.1.** Assume that \(|\{ Lu \leq -1 \} \cap Q_{1,1}| \geq m|Q_{1,1}|\), where \( m \in (0, 1) \). Then there exists a constant \( N, \rho, \beta \in (0, \infty) \) depending only on \( \delta \) and \( d \) and for any \( \kappa \in (0, 1) \) there exists a constant \( c \in (0, \infty) \) depending only on \( \delta, d, \kappa \) such that for \( |x| \leq 1 - \sqrt{1 - \kappa}, 1 \geq t \geq 1 - m\kappa \) we have

\[
|u(t, x)| \geq cm^\rho e^{-\beta/(\gamma m)} - N\|Lu\|_{L_{d+1}(Q_{1,1})},
\]

where \( \gamma = 1 - \kappa \).

Proof. Observe that if the result is true for \( \kappa = 1/2 \), then it is obviously also true for \( \kappa \in (1, 1/2) \) and one can even keep \( c \) the same as for \( \kappa = 1/2 \). Therefore we concentrate on

\[
\kappa \in [1/2, 1).
\]

Then, one can slightly shrink \( Q_{1,1} \) and add to \( u \) a small constant in order to preserve the positivity of the new function on the parabolic boundary of the new domain. After that passing to the limit in the corresponding version of (2.1) shows that without losing generality we may and will assume that \( u \in W^{1,2}_{d+1}(Q_{1,1}) \cap C(\bar{Q}_{1,1}) \). This fact, for obvious reasons, allows us to also assume that \( u \) and the coefficients of \( L \) are infinitely differentiable in \( \mathbb{R}^{d+1} \). In that case, by adding to \( u \) the solution of the equation \( Lu = -(Lu)_+ \) in \( Q_{1,1} \) with boundary data \( v = -u (\leq 0) \) on the parabolic boundary of \( Q_{1,1} \) and using the parabolic Alexandrov estimate, we reduce the situation to the one in which \( Lu \leq 0 \) in \( Q_{1,1} \) and \( u = 0 \) on the parabolic boundary of \( Q_{1,1} \).

The last operation may yield a new function \( u \) which is no longer infinitely differentiable in \( Q_{1,1} \), but yet it will be continuous in \( Q_1 \) and belong to \( W^{1,2}_p(Q_{1,1}) \) for any \( p < \infty \).

After that introduce \( \Gamma = \{(t, x) \in Q_{1,1} : Lu(t, x) \leq -1\} \) and denote by \( w \) the solution of class \( W^{1,2}_{d+1}(Q_{1,2}) \cap C(\bar{Q}_{1,2}) \) of the problem

\[
Lw(t, x) = -I_{\Gamma}(t, x)I_{B_1}(x)I_{(0,T)}(t)
\]

with zero data on the parabolic boundary of \( Q_{1,2} \), where

\[
T = (1 - m)(1 + \gamma^2 m).
\]

As is easy to see

\[
|\{(t, x) \in Q_{1,1} : Lu(t, x) \leq -1\}| \geq m(1 - m)|Q_{1,1}|\gamma^2 \geq m|Q_{1,1}|\gamma^2/2.
\]

It follows by Theorem 4.1 of \([6]\) that \( w(2, 0) \geq c_0 m^\rho \), where \( c_0 = c_0(\delta, d, \kappa) > 0 \), \( \rho = \rho(\delta, d) < \infty \).

Furthermore, observe that if a point \( t \geq 1 - \kappa m \), then its distance to \((0, T)\) is greater than (recall that \( \gamma \leq 1/2 \))

\[
1 - \kappa m - T = \gamma m[1 - \gamma(1 - m)] \geq \gamma m/2.
\]
Also if $|x| < 1 - \sqrt{\gamma}$, then the distance of $x$ to $\partial B_1$ is greater that $\sqrt{\gamma} \geq \sqrt{\gamma m/2}$. By combining this with the fact that for $t \geq T$ the function $w$ satisfies a homogeneous equation by Theorem 3.6 of [4], for $|x| < \kappa$ and $t \geq 1 - \kappa m$, we get that

$$w(2, 0) \leq c(\gamma/2)w(x, t),$$

where $c(\mu)$ is a certain function which depends only on $\mu, \delta,$ and $d$. Since by the maximum principle $w \leq u$ in $Q_{1,1}$ we conclude

$$c(\gamma/2)w(x, t) \geq c_0 u^\rho.$$  

It only remains to figure out how $c(\mu)$ depends on $\mu > 0$. It follows from the three-line proof of Theorem 3.6 in [4] that $c(\mu)$ can be taken as a constant depending only on $\delta$ and $d$ times the Harnack constant, namely, the constant $d(\mu)$ which is such that for any $(x, t), (y, s) \in Q_{1,1}$ for which $t \geq s + \mu \geq 2\mu$, and $|x|, |y| \leq 1 - \mu^{1/2}$, where $\nu = \nu(\delta, d) > 0$, we have

$$w(y, s) \leq d(\mu)w(x, t)$$

(2.2)

for any $w \geq 0$ satisfying the homogeneous equation in $Q_{1,1}$.

It turns out that it suffices to find $d(\mu)$ such that (2.2) holds as long as $t \geq s + \mu \geq 2\mu$ and $|x|, |y| \leq 1 - \mu^{1/2}$. Indeed, if $\nu \geq 1$, then this is obvious. However if $\nu < 1$, then by replacing the condition $t \geq s + \mu \geq 2\mu$, with $t \geq s + \mu^{1/2} \geq 2\mu^{1/2}$, we again would get a stronger result.

Thus, take $(x, t), (y, s) \in Q_{1,1}$ such that $t \geq s + \mu \geq 2\mu$ and $|x|, |y| \leq 1 - \mu^{1/2}$, consider the straight segment joining them, and define $n$ as the smallest integer such that $|x - y|/n \leq \sqrt{\mu}/\sqrt{\gamma}$. Obviously, $n \leq 5/\mu$ for small $\mu$. Then we split the segment into $n$ parts of equal length, so that for any two neighboring points $(x_i, t_i)$ and $(x_{i+1}, t_{i+1})$ we have $t_{i+1} - t_i = (t - s)/n \geq \mu/n)$ and $|x_{i+1} - x_i| = |x - y|/n \leq \sqrt{\mu}/\sqrt{\gamma}$. Now each couple of points $(x_i, t_i)$ and $(x_{i+1}, t_{i+1})$ is in a standard-shape cylinder to which Harnack’s inequality is applicable, so that

$$w(x_i, t_i) \leq d_0 w(x_{i+1}, t_{i+1}),$$

where $d_0 = d_0(\delta, d)$, implying that (2.2) holds with $d(\mu) = d_0^\rho \leq \exp(\beta/\mu)$. This proves the theorem.

3. NONLINEAR CASE

Let $A$ be a closed set of $d \times d$ symmetric matrices with eigenvalues in $[\delta, \delta^{-1}]$. Define

$$F[v](t, x) = \max_{a \in A} a^{ij} D_{ij} v(t, x) - \partial_t v(t, x).$$

(3.1)

We are going to deal with $L_\rho$-viscosity supersolutions the definition of which and their numerous properties can be found in [3]. Here is a somewhat more detailed version of Theorem 1.5 of [3].
Theorem 3.1. Let \( f \in L_{d+2}(Q_{1,1}) \) and \( u \in C(\bar{Q}_{1,1}) \) be such that \( u, f \geq 0 \) and \( u \) is an \( L_{d+2} \)-viscosity supersolution of the equation \( F[v] = -f \) in \( Q_{1,1} \). Assume that \( \{|f \geq 1\} \cap Q_{1,1} \geq m|Q_{1,1}| \), where \( m \in (0,1) \). Then for \( \kappa \in (0,1), |x| \leq \kappa, 1 \geq t \geq 1 - mk \) we have

\[
    u(t, x) \geq cm^\rho e^{-\beta/(\gamma m)},
\]

(3.2)

where \( c, \rho, \beta, \gamma \) are taken from Theorem 2.1.

Proof. By Theorem 8.4 of [3] the equation \( F[v] = -f \) in \( Q_{1,1} \) with zero condition on the parabolic boundary of \( Q_{1,1} \) has a unique solution \( v \in W^{1,2}_{d+2,loc}(Q_{1,1}) \cap C(\bar{Q}_{1,1}) \). It also follows from [3] that \( v \leq u \) in \( \bar{Q}_{1,1} \). Furthermore, observe that since \( A \) is closed, the maximum in (3.1) is attained for each \( (t, x) \), and there exists an operator \( L \) as in Section 2 such that \( Lv = -f \) in \( Q_{1,1} \). After that it only remains to apply Theorem 2.1. The theorem is proved.

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