WELL-POSEDNESS AND ILL-POSEDNESS
FOR THE CUBIC FRACTIONAL SCHröDINGER EQUATIONS

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Abstract. We study the low regularity well-posedness of the 1-dimensional cubic nonlinear fractional Schrödinger equations with Lévy indices $1 < \alpha < 2$. We consider both non-periodic and periodic cases, and prove that the Cauchy problems are locally well-posed in $H^s$ for $s \geq \frac{2-\alpha}{4}$. This is shown via a trilinear estimate in Bourgain's $X^{s,b}$ space. We also show that non-periodic equations are ill-posed in $H^s$ for $\frac{2-3\alpha}{4(\alpha+1)} < s < \frac{2-\alpha}{4}$ in the sense that the flow map is not locally uniformly continuous.

1. Introduction. We consider the Cauchy problem for the one dimensional fractional Schrödinger equations with cubic nonlinearity in periodic and non periodic settings:

\begin{equation}
\begin{cases}
    i\partial_t u + (-\Delta)^{\alpha/2} u = \gamma|u|^2 u,
    \\
    u(0, \cdot) = \phi \in H^s(\hat{Z}),
\end{cases}
\end{equation}

where $\hat{Z} = \mathbb{R}$ or $\mathbb{T}$, $\alpha \in (1, 2)$ is the Lévy index, $\gamma \in \mathbb{R} \setminus \{0\}$ and $s \in \mathbb{R}$. In this paper we are concerned with well-posedness of the Cauchy problem in low regularity Sobolev spaces. As the linear part generalizes the usual second-order Schrödinger equation, our main interest is to investigate how the weaker dispersion...
affects dynamics and well-posedness. The fractional Schrödinger equations were introduced in the theory of the fractional quantum mechanics where the Feynmann path integral approach is generalized to the α-stable Lévy process [15]. Also this type of equation also appears in the water wave models (for example, see [13] and references therein).

In what follows $Z$ denotes $\mathbb{R}$ (non-periodic) or $\mathbb{Z}$ (periodic). Accordingly, the Sobolev space $H^s(Z)$ is defined by

$$H^s(Z) = \{ f \in \mathcal{S}': \| f \|_{H^s(Z)} := \| (1 + |\xi|^2)^{s/2} \mathcal{F} f \|_{L^2(Z)} < \infty \},$$

where $L^2(Z)$ denotes $L^2(\mathbb{R})$ or $\ell^2(\mathbb{Z})$ and $\mathcal{F} f$ is the Fourier transform or the Fourier coefficient of $f$ given by $\mathcal{F} f(\xi) = \int_Z e^{-ix\xi} f dx$ for $\xi \in Z$.

We define the linear propagator $U(t)$ by setting

$$U(t) \phi = e^{\ii (-\Delta)^{\alpha/2}} \phi = \mathcal{F}^{-1} e^{\ii |\xi|^\alpha} \mathcal{F} \phi,$$

where $\mathcal{F}^{-1}$ denotes the inverse Fourier transform. Then, by the Duhamel formula the equation $(1)$ is written as an integral equation

$$u = U(t) \phi - i\gamma \int_0^t U(t - t')(|u|^2 u(t')) dt'.$$

Well-posedness. If $s > 1/2$, by the Sobolev embedding and the energy method one can easily show the local well-posedness in $H^s$ for $0 < \alpha < 2$ for both periodic and non periodic cases. The equation $(1)$ also has mass and energy conservation:

$$M(u) = \int |u|^2, \quad E(u) = \frac{1}{2} \int |\nabla|\alpha/2 u|^2 + \gamma \frac{1}{4} \int |u|^4.$$

Thus, for $s \geq \alpha/2$ and $s > 1/2$, the global well-posedness in $H^s$ follows from the conservation laws. (For instance see [5, 6].)

For the less regular initial data, i.e. $s \leq 1/2$, particularly in the non periodic case, a plausible approach may be to use the Strichartz estimate for $U(t)$. In fact, it is known that the estimate

$$\| \nabla^{-\alpha} U(t) \phi \|_{L^4_t L^4_x(\mathbb{R} \times \mathbb{R})} \leq \| \phi \|_{L^2}$$

(3)

holds for $2/q + 1/r = 1/2$, $2 \leq q, r \leq \infty$ (see [10]). However, due to weak dispersion the estimate accompanies a derivative loss of order $2 - \alpha$ unless one imposes additional assumptions on $\phi$ ([7, 8, 9]). This makes difficult for general data to use the usual iteration argument which relies on (3).

To get around shortcoming of Strichartz estimates we use Bourgain’s $X^{s,b}_p$ space, which has been widely used in the studies of dispersive equations for both non periodic and periodic settings. For the fractional Schrödinger equation, $X^{s,b}_p$ is defined by

$$X^{s,b}_p = \{ \varphi \in \mathcal{S}': \| \varphi \|_{X^{s,b}_p} := \| \langle \xi \rangle^s (\tau - |\xi|^\alpha)^{b} \hat{\varphi}(\tau, \xi) \|_{L^p_x(\mathbb{R} \times \mathbb{Z})} < \infty \},$$

where $\hat{\varphi}(\tau, \xi)$ is the Fourier transform of $\varphi$ with respect to the space variables. Here $\langle \cdot \rangle$ denotes $1 + |\cdot|$. In the standard iteration argument, the main step is to show the trilinear estimate in terms of $X^{s,b}$ spaces:

$$\| uvw \|_{X^{s,-1,b}_p} \lesssim \| u \|_{X^{s,b}_p} \| v \|_{X^{s,b}_p} \| w \|_{X^{s,b}_p}.$$

(4)

We obtain this estimate by adapting the dyadic method in Tao [17] where multilinear estimates in weighted $L^2$ spaces are systematically studied. The argument similarly applies to both non periodic and periodic cases.
The following is our local well-posedness result.

**Theorem 1.1.** For $1 < \alpha < 2$, the Cauchy problem (1) is locally well-posed in $H^s(\mathbb{Z})$, if $s \geq \frac{2-\alpha}{4}$.

Recently, for the periodic case, Demirbas, Erdoğan and Tzirakis [11] showed that the equation (1) is locally well-posed for $s > 2 - \frac{\alpha}{4}$ and globally well-posed for $s > 5\alpha + \frac{1}{2}$. Our result gives local well-posedness at the missing endpoint $s = 2 - \frac{\alpha}{4}$.

The regularity threshold $s = 2 - \frac{\alpha}{4}$ is optimal in that if $s < 2 - \frac{\alpha}{2}$ we do not expect to solve (1) via the contraction mapping principle. Firstly, the estimate (4) fails for $s < 2 - \frac{\alpha}{4}$ due to the resonant interaction of high–high–high to high (frequencies). Compared to the usual Schrödinger equation, the curvature of the characteristic curve is smaller ($(\text{frequency})^{\alpha - 2}$). So, the stronger such resonant interactions make the threshold regularity higher. See the counter-example in Section 4. In [12], the authors claimed that (1) is globally well posed if $\phi \in L^2$. But Theorem 1.2 below shows that their result is incorrect. Their proof is based on a trilinear estimate, namely (13) with $s = 0$ ([12, Theorem 3.2]), which is not true.

**Ill-posedness.** Now we consider ill-posedness in the non periodic setting. Following Christ, Colliander, and Tao [3], we approximate solutions of the fractional equations with those of the cubic NLS, at $(N, N^\alpha)$ in the Fourier space by Taylor expansion of the phase function. This allows to transfer an ill-posedness result of NLS to (1). A similar trick was also used in the fifth-order modified KdV equation [14]. The following is our second result.

**Theorem 1.2.** Let $2 - \frac{3\alpha}{4(\alpha+1)} < s < 2 - \frac{\alpha}{4}$. Then the solution map of the initial value problem (1) fails to be locally uniformly continuous on $C_T H^s(\mathbb{R})$ for any $T > 0$. More precisely, for $0 < \delta \ll \varepsilon \ll 1$, there are two solutions $u_1, u_2$ to (1) with initial data $\phi_1, \phi_2$ such that

\[
||\phi_1||_{H^s}, ||\phi_2||_{H^s} \lesssim \varepsilon, \quad (5)
\]

\[
||\phi_1 - \phi_2||_{H^s} \lesssim \delta, \quad (6)
\]

\[
\sup_{0 \leq t \leq T} ||u_1(t) - u_2(t)||_{H^s} \gtrsim \varepsilon. \quad (7)
\]

In view of the counter-example of the trilinear estimate (4) it seems natural to expect the similar ill-posedness result for the periodic equations. However, it doesn’t seem simple to set up a counter-example because the frequency supports should be distributed in a wide region of length $N^{\frac{2-\alpha}{4}}$. Currently we are not able to prove ill-posedness\(^1\).

**Organization of the paper.** The paper is organized as follows. In section 2, we introduce notations and recall previously known estimates which we need in the subsequent sections. In section 3, bilinear estimates in $X^{s,b}_z$ space are established. Finally, we prove Theorem 1.1 in section 4 and Theorem 1.2 in section 5.

**2. Notations and preliminaries.** We use the same notations as in [17]. Let us invoke that $Z$ denotes $\mathbb{R}$ for the non-periodic case and $Z$ for the periodic case. For an integer $k \geq 2$, let $\Gamma_k(\mathbb{R} \times Z)$ denote the hyperplane

\[
\Gamma_k(\mathbb{R} \times Z) := \{ \zeta = (\zeta_1, \cdots, \zeta_k) \in (\mathbb{R} \times Z)^k : \zeta_1 + \cdots + \zeta_k = 0 \},
\]

\(^1\) The same counter-example as the cubic NLS in [1] gives the ill-posedness for $s < 0$. 

and define
\[ \int_{\Gamma_k(\mathbb{R} \times Z)} f := \int_{(\mathbb{R} \times Z)^k-1} f(\zeta_1, \cdots, \zeta_{k-1}, -\zeta_1 - \cdots - \zeta_{k-1}) d\zeta_1 \cdots d\zeta_{k-1}, \]
where \( d\zeta_j \) is the product of Lebesgue and the counting measure for the periodic case, and the Lebesgue measure on \( \mathbb{R}^2 \) for the non-periodic case. Note that the integral is symmetric under permutations of \( \zeta_j \).

Let us define a \([k; \mathbb{R} \times Z]\)-multiplier to be any function \( m : \Gamma_k(\mathbb{R} \times Z) \to \mathbb{C} \). When \( m \) is a \([k; \mathbb{R} \times Z]\)-multiplier, the norm \( \|m\|_{[k; \mathbb{R} \times Z]} \) is defined to be the best constant so that the inequality
\[ \left| \int_{\Gamma_k(\mathbb{R} \times Z)} m(\zeta) \prod_{j=1}^{k} f_j(\zeta_j) \right| \leq \|m\|_{[k; \mathbb{R} \times Z]} \prod_{j=1}^{k} \|f_j\|_{L^2(\mathbb{R} \times Z)} \]
holds for all test functions \( f_j \) on \( \mathbb{R} \times Z \). Here we recall some of the results about \([k; \mathbb{R} \times Z]\)-multiplier from [17].

**Lemma 2.1** (Lemma 3.1 in [17]). If \( m \) and \( M \) are \([k; \mathbb{R} \times Z]\)-multipliers, and \( |m(\zeta)| \leq M(\zeta) \) for all \( \zeta \in \Gamma_k(\mathbb{R} \times Z) \), then \( \|m\|_{[k; \mathbb{R} \times Z]} \leq \|M\|_{[k; \mathbb{R} \times Z]} \). Also, if \( m \) is a \([k; \mathbb{R} \times Z]\)-multiplier, and \( g_1, \cdots, g_k \) are functions from \( \mathbb{R} \times Z \) to \( \mathbb{R} \), then
\[ \|m(\zeta) \prod_{j=1}^{k} g_j(\zeta_j) \|_{[k; \mathbb{R} \times Z]} \leq \|m\|_{[k; \mathbb{R} \times Z]} \prod_{j=1}^{k} \|g_j\|_{\infty}. \]

**Lemma 2.2** (Lemma 3.4 in [17]). For \( \zeta_0 \in \Gamma_k(\mathbb{R} \times Z) \) and a \([k; \mathbb{R} \times Z]\)-multiplier \( m \), we have
\[ \|m(\zeta)\|_{[k; \mathbb{R} \times Z]} = \|m(\zeta + \zeta_0)\|_{[k; \mathbb{R} \times Z]}. \]
From this and Minkowski’s inequality, we thus have the averaging estimate, for any finite measure \( \mu \) on \( \Gamma_k(\mathbb{R} \times Z) \),
\[ \|m * \mu\|_{[k; \mathbb{R} \times Z]} \leq \|m\|_{[k; \mathbb{R} \times Z]} \|\mu\|_{L^1(\Gamma_k(\mathbb{R} \times Z))}. \]

**Lemma 2.3** (Lemma 3.7 in [17]). Let \( k_1, k_2 \geq 1 \), and \( m_1, m_2 \) be functions defined on \((\mathbb{R} \times Z)^{k_1}, (\mathbb{R} \times Z)^{k_2}\), respectively. Then
\[ \|m_1(\zeta_1, \cdots, \zeta_{k_1})m_2(\zeta_{k_1+1}, \cdots, \zeta_{k_1+k_2})\|_{[k_1+k_2; \mathbb{R} \times Z]} \leq \|m_1\|_{[k_1+1; \mathbb{R} \times Z]} \|m_2\|_{[k_2+1; \mathbb{R} \times Z]}. \]
As a special case, we have the \( TT^* \) identity, for all functions \( m : (\mathbb{R} \times Z)^k \to \mathbb{R} \),
\[ \|m(\zeta_1, \cdots, \zeta_k)m(-\zeta_k, \cdots, -\zeta_1)\|_{[2k; \mathbb{R} \times Z]} \leq \|m(\zeta_1, \cdots, \zeta_k)\|_{[k+1; \mathbb{R} \times Z]}^2. \]

Let \( m \) be a \([k; \mathbb{R} \times Z]\) multipliers. For \( 1 \leq j \leq k \) we define the \( j \)-support \( \text{supp}_j(m) \subset \mathbb{R} \) of \( m \) to be the set
\[ \text{supp}_j(m) := \{ \eta_j \in \mathbb{R} : \Gamma_k(\mathbb{R} \times Z; \zeta_j = \eta_j) \cap \text{supp}(m) \neq \emptyset \}, \]
where \( \Gamma_k(\mathbb{R} \times Z; \zeta_j = \eta_j) = \{ (\zeta_1, \cdots, \zeta_k) \in \Gamma_k(\mathbb{R} \times Z ; \zeta_j = \eta_j) \} \). And if \( J \) is a non-empty subset of \( \{1, \cdots, k\} \), we define the set \( \text{supp}_J(m) \subset \mathbb{R}^J \) by
\[ \text{supp}_J(m) := \prod_{j \in J} \text{supp}_j(m). \]
Lemma 2.4 (Lemma 3.11 in [17]). Let $J_1, J_2$ be disjoint non-empty subsets of \{1, \cdots, k\} and $A_1, A_2 > 0$. Suppose that $(m_a)_{a \in I}$ is a collection of $[k; \mathbb{R} \times \mathbb{Z}]$ multipliers such that
\[
\# \{a \in I : \xi \in \text{supp}_I^J(m_a) \} \leq A_i
\]
for all $\xi \in \mathbb{R}^k$ and $i = 1, 2$. Then
\[
\| \sum_{a \in I} m_a \|_{[k; \mathbb{R} \times \mathbb{Z}]} \leq (A_1 A_2)^{\frac{1}{2}} \sup_{a \in I} \| m_a \|_{[k; \mathbb{R} \times \mathbb{Z}]}
\]
In particular, if $m_a$ is non-negative and $A_1, A_2 \sim 1$, then we have
\[
\| \sum_{a \in I} m_a \|_{[k; \mathbb{R} \times \mathbb{Z}]} \sim \sup_{a \in I} \| m_a \|_{[k; \mathbb{R} \times \mathbb{Z}]}
\]
We set, for $j = 1, 2, 3$,
\[
h_j = \pm|\xi_j|^{\alpha_j}, \quad \xi_j = (\tau_j, x_j), \quad \lambda_j = \tau_j - h_j(\xi_j).
\]
For the $X^b_{\sigma} \{3; \mathbb{R} \times \mathbb{Z}\}$ space estimates, we need to consider the $[3; \mathbb{R} \times \mathbb{Z}]$-multiplier
\[
m(\xi_1, \xi_2, \xi_3) = \frac{\tilde{m}(\xi_1, \xi_2, \xi_3)}{\prod_{j=1}^3 (\lambda_j)^{b_j}}
\]
for a function $\tilde{m}$ on $\mathbb{R}^3$ which will be specified later. By averaging over unit time scale (Lemmas 2.1 and 2.2), one may restrict the multiplier to the region $|\lambda_j| \geq 1$. And we define the function $h : \Gamma^3_{\text{med}}(\mathbb{R} \times \mathbb{Z}) \to \mathbb{R}$ by setting
\[
h(\xi_1, \xi_2, \xi_3) := h_1(\xi_1) + h_2(\xi_2) + h_3(\xi_3) = -\lambda_1 - \lambda_2 - \lambda_3,
\]
which plays an important role in what follows.

Let $N_j, L_j, H$ ($j = 1, 2, 3$) be dyadic numbers. By dyadic decomposition along the variables $\xi_j, \lambda_j$, as well as the function $h(\xi_1, \xi_2, \xi_3)$, we have
\[
\| m \|_{[3; \mathbb{R} \times \mathbb{Z}]} \lesssim \left\| \sum_{N_{\text{max}} \geq 1} \sum_{L_1, L_2, L_3 \geq 1} \frac{m(N_1, N_2, N_3)}{L_1^{b_1} L_2^{b_2} L_3^{b_3}} X_{N_1, N_2, N_3; H; L_1, L_2, L_3} \right\|_{[3; \mathbb{R} \times \mathbb{Z}]}^3
\]
where $X_{N_1, N_2, N_3; H; L_1, L_2, L_3}$ is the multiplier given by
\[
X_{N_1, N_2, N_3; H; L_1, L_2, L_3}(\tau_1, \tau_2, \tau_3, \xi_1, \xi_2, \xi_3) := X(|h(\xi_1, \xi_2, \xi_3)| \sim H) \prod_{j=1}^3 X(|\xi_j| \sim N_j) X(|\lambda_j| \sim L_j)
\]
and
\[
m(N_1, N_2, N_3) := \sup_{|\xi_j| \sim N_j, j=1,2,3} |\tilde{m}(\xi_1, \xi_2, \xi_3)|.
\]
From the identities $\xi_1 + \xi_2 + \xi_3 = 0$ and $\lambda_1 + \lambda_2 + \lambda_3 + h(\xi_1, \xi_2, \xi_3) = 0$ on the support of the multiplier, we see that $X_{N_1, N_2, N_3; H; L_1, L_2, L_3}$ vanishes unless $N_{\text{max}} \sim N_{\text{med}}$ and $L_{\text{max}} \sim \max(H, L_{\text{med}})$.

Suppose for the moment that $N_1 \geq N_2 \geq N_3$. Then we have $N_1 \sim N_2 \sim 1$. As $N_1$ ranges over the dyadic numbers, the symbols in the summation in (8) are supported on essentially disjoint regions of $\xi_1$ and $\xi_2$ spaces. This is true for any permutation of $\{1, 2, 3\}$. Thus, by Lemma 2.4 we have
Lemma 2.5

\[ \|m\|_{[3;\mathbb{R}\times Z]} \lesssim \sup_{N \geq 1} \left( \sum_{N_{\text{max}} \sim N_{\text{med}}} \sum_{H \sim \max(H, L_{\text{med}})} m(N_1, N_2, N_3) \right) \]

in the low modulation case \( H \sim L_{\text{max}} \) and the high modulation case \( L_{\text{max}} \sim L_{\text{med}} \gg H \). The following two lemmas give estimates for (9) in each case.

Lemma 2.5 ((37) in [17]). If \( L_{\text{max}} \sim L_{\text{med}} \gg H \), then

\[ (9) \lesssim L_{\min}^{\frac{3}{2}} \left\| \chi h(\xi) \right\|_{[3;\mathbb{R}^{1+\delta}]} \lesssim L_{\min}^{\frac{3}{2}} \left\| \langle \xi \rangle \right\|_{[3;\mathbb{R}^{1+\delta}]} \lesssim L_{\min}^{\frac{3}{2}} \left\| \langle \xi \rangle \right\|_{[3;\mathbb{R}^{1+\delta}]} \]

Let \( |E| \) denote the Lebesgue measure or counting measure of any measurable subset \( E \) of \( Z \).

Lemma 2.6 (Corollary 4.2 in [17]). Let \( N_1, N_2, N_3 > 0, L_1 \geq L_2 \geq L_3 \). Suppose that \( H \sim L_{\text{max}} \) and \( \xi_1^0, \xi_2^0, \xi_3^0 \) satisfy that

\[ |\xi_j^0| \sim N_j \text{ for } j = 1, 2, 3 \text{ and } |\xi_1 + \xi_2 + \xi_3| \ll N_{\text{min}}. \]

Then we have

\[ (9) \lesssim L_3^\frac{3}{2} \left\| \chi h(\xi) \right\|_{[3;\mathbb{R}^{1+\delta}]} \lesssim \left\| \langle \xi \rangle \right\|_{[3;\mathbb{R}^{1+\delta}]} \lesssim \left\| \langle \xi \rangle \right\|_{[3;\mathbb{R}^{1+\delta}]} \]

for some \( \tau \in \mathbb{R} \) and \( \xi \in Z \) with \( |\xi + \xi_0^0| \ll N_{\text{min}} \). The same statement holds with the roles of the indices 1, 2, 3 permuted.

3. Bilinear estimates. In order to prove well-posedness for (1), we show the bilinear estimates (Proposition 3 below). For this purpose, we first prove a bilinear estimate for \( \|uv\|_{L_2(\mathbb{R}\times Z)} \), which automatically gives the estimate for \( \|uv\|_{L_2(\mathbb{R}\times Z)} \).

Since the resonance function is \( h(\xi_1, \xi_2, \xi_3) = |\xi_1|^\alpha - |\xi_2|^\alpha + |\xi_3|^\alpha \), we have \( |\xi_{\text{max}}|^{\alpha - 1} \lesssim |\xi_{\text{min}}| \lesssim |h(\xi)| \lesssim |\xi_{\text{max}}|^\alpha \).

To begin with, we establish an estimate for (9). Here \( \langle \cdot \rangle_\mathbb{Z} \) denotes \( | \cdot | \) for non-periodic case and \( 1 + | \cdot | \) for periodic case. So, \( |\{\xi \in Z : a \leq \xi \leq b\}| = O((b-a)_{\mathbb{Z}}) \).

Proposition 1. Let \( H, N_1, N_2, N_3, L_1, L_2, L_3 \) be dyadic and \( h(\xi) = |\xi_1|^\alpha - |\xi_2|^\alpha + |\xi_3|^\alpha \). Then we have the following.

- If \( H \sim L_{\text{max}} \sim L_1 \) and \( N_1 \sim N_{\text{max}}, \)
  \[ (9) \lesssim L_{\min}^\frac{3}{2} \langle \min(N_{\text{med}}^{\frac{1}{2}}, N_{\text{max}}^{\frac{1}{2}} \max L_{\text{med}}) \rangle Z. \]

- If \( H \sim L_{\text{max}} \sim L_1 \) and \( N_2 \sim N_3 \gg N_1, \)
  \[ (9) \lesssim L_{\min}^\frac{3}{2} \langle \min(N_{\text{med}}^{\frac{1}{2}}, N_{\text{max}}^{\frac{1}{2}} \max L_{\text{med}}) \rangle Z. \]

- If \( H \sim L_{\text{max}} \sim L_2 \) and \( N_{\text{max}} \sim N_{\text{med}}, \)
  \[ (9) \lesssim L_{\min}^\frac{3}{2} \langle \min(N_{\text{med}}^{\frac{1}{2}}, N_{\text{max}}^{\frac{1}{2}} \max L_{\text{med}}) \rangle Z. \]

- If \( H \sim L_{\text{max}} \sim L_2 \) and \( N_{\text{med}} \gg N_{\text{min}}, \)
  \[ (9) \lesssim L_{\min}^\frac{3}{2} \langle \min(N_{\text{med}}^{\frac{1}{2}}, N_{\text{max}}^{\frac{1}{2}} \max L_{\text{med}}) \rangle Z. \]
(9) $\geq L_{\text{min}}^\frac{1}{2} L_{\text{med}}^{\frac{1}{2}} (10)$

for some $\tau \in \mathbb{R}$ and $\xi \in Z$ with $|\xi + \xi_\tau| \ll N_{\text{min}}$. We observe that the derivative of $|\xi_\tau - \xi_\tau|^\alpha$ is equal to $\alpha (|\xi_\tau|^\alpha - |\xi_\tau|^\alpha - |\xi_\tau|^\alpha (\xi_\tau - \xi))$.

If $N_1 \sim N_{\text{max}}$, then $0 < |\xi_\tau^2 - C| |\xi|$ for some constant $C > 1$. This means $\xi_\tau$ is equal to $\xi_\tau^2$ for some $0 < |\xi_\tau - \xi| < C$ and thus $||\xi_\tau|^\alpha - |\xi_\tau - \xi|^{\alpha-2} (\xi_\tau - \xi) - |\xi_\tau - \xi|^{\alpha-2} (\xi_\tau - \xi)|$ is greater than or equal to $|C|^{\alpha-1} - C^{\alpha-1}|\xi_\tau - \xi|^{\alpha-1}$. So, $\xi_\tau$ is contained in an interval of length $O(N_{\text{max}}^\frac{2}{3} L_{\text{med}})$. Hence, by (10) we get the desired estimate for the first case.

If $N_2 \sim N_3 \gg N_1$, then

$$(10) \geq \int_{\xi_\tau - \xi}^{\xi_\tau} \min(|\xi_\tau|^\alpha - |\xi_\tau - \xi|^{\alpha-2} (\xi_\tau - \xi)).$$

So, $\xi_\tau$ variable is contained in an interval of length $O(N_{\text{max}}^\frac{2}{3} N_{\text{min}}^{-1} L_{\text{med}})$. This and (10) give the estimate for the second case.

We now consider the case $L_2 \sim L_{\text{max}}$. If $N_1 \sim N_2 \sim N_3$, we see that $|\xi_\tau|^{\alpha-2} \xi_\tau + |\xi_\tau - \xi|^{\alpha-2} (\xi_\tau - \xi) \geq 1$ by the Taylor expansion. This means that $\xi_\tau$ is contained in an interval of length $O(N_{\text{max}}^\frac{2}{3} L_{\text{med}})$ by the mean value theorem and the estimate for the third case follows from (10).

If $N_{\text{max}} \sim N_{\text{med}} \gg N_{\text{min}}$, then we have $||\xi_\tau|^\alpha - |\xi_\tau - \xi|^{\alpha-2} (\xi_\tau - \xi) \sim |\xi_\tau - \xi|^{\alpha-1} N_{\text{max}}^{-1}$ and thus (10) and the mean value theorem shows that $\xi_\tau$ is contained in an interval of length $O(N_{\text{max}}^{-\alpha} L_{\text{med}})$. Since $\xi_\tau$ is also contained in an interval of length $\ll N_{\text{min}}$, Proposition 1 follows from (10).

We now show some bilinear estimates for the periodic and non periodic cases.

**Proposition 2.** Let $s \geq \frac{2-\alpha}{4}$ and $0 < \varepsilon \ll 1$. Then, for $u \in X_{\frac{1}{2}}^s$ and $v \in X_{\frac{1}{2}}^s$, we have

$$\|uv\|_{L^2(R \times \hat{Z})} = \|u\|_{L^2(R \times \hat{Z})} \lesssim \|u\|_{X_{\frac{1}{2}}^s} \|v\|_{X_{\frac{1}{2}}^s}.$$
Proof of Lemma 3.1. We observe

\[ \|u\overline{\tau}\|_{L^2(\mathbb{R}^2)} \leq \|(u - \widehat{u}(0)) (\overline{\tau} - \overline{\tau}(0))\|_{L^2(\mathbb{R}^2)} + \|u\overline{\tau}(0)\|_{L^2(\mathbb{R}^2)} \]
\[+ \|\widehat{u}(0)\overline{\tau}\|_{L^2(\mathbb{R}^2)} + \|\widehat{u}(0)\overline{\tau}(0)\|_{L^2(\mathbb{R}^2)} \leq \|(u - \widehat{u}(0)) (\overline{\tau} - \overline{\tau}(0))\|_{L^2(\mathbb{R}^2)} + \|u\overline{\tau}(0)\|_{L^2(\mathbb{R}^2)} \overline{\tau}(0)\|_{L^2(\mathbb{R}^2)} \]
\[+ \|\widehat{u}(0)\|_{L^2(\mathbb{R}^2)} \|\overline{\tau}\|_{L^2(\mathbb{R}^2)} + \|\widehat{u}(0)\|_{L^2(\mathbb{R}^2)} \|\overline{\tau}(0)\|_{L^2(\mathbb{R}^2)} \overline{\tau}(0)\|_{L^2(\mathbb{R}^2)} \]

By Sobolev embedding \(X^{0, \frac{1}{2} + \varepsilon}_2 \hookrightarrow C(\mathbb{R}^2; L^2(\mathbb{T}))\) we have

\[\|\overline{\tau}(0)\|_{L^2(\mathbb{R}^2)} \leq \sqrt{2\pi}\|\overline{\tau}\|_{L^2(\mathbb{R}^2)} \approx \|\overline{\tau}\|_{X^{0, \frac{1}{2} + \varepsilon}_2} \]
and \(\|\widehat{u}(0)\|_{L^2(\mathbb{R}^2)} \leq \sqrt{2\pi}\|u\|_{L^2(\mathbb{R}^2)} \approx \|u\|_{X^{0, \frac{1}{2} - \varepsilon}_2} \). This gives the desired estimate.

Proof of Proposition 2. For the proof it suffices to show that

\[\frac{1}{\langle \xi_1 \rangle^{2} (\tau_1 - |\xi_1|^\alpha)^{\frac{1}{2} + \epsilon} (\tau_2 + |\xi_2|^\alpha)^{\frac{1}{2} - \epsilon} \|\overline{\tau}\|_{3; \mathbb{R}^2} Z} \approx 1.\]

The left hand side is bounded by the sum of

\[\sum_{N_{\text{max}} \sim N_{\text{med}} \sim N} \sum_{L_1, L_2, L_3 \geq 1} \sum_{H \sim \max} \frac{1}{(N_1)^{\alpha} L_1^{\frac{1}{2} + \varepsilon} L_2^{\frac{1}{2} - \varepsilon}} \|X_{N_1, N_2, N_3; H; L_1, L_2, L_3}\|_{3; \mathbb{R}^2} Z,\]

(11)

and

\[\sum_{N_{\text{max}} \sim N_{\text{med}} \sim N} \sum_{\max \sim \min} \frac{1}{\langle N_1 \rangle^{\alpha} L_1^{\frac{1}{2} + \varepsilon} L_2^{\frac{1}{2} - \varepsilon}} \|X_{N_1, N_2, N_3; H; L_1, L_2, L_3}\|_{3; \mathbb{R}^2} Z.\]

(12)

From the Lemma 3.1 we may assume that \(\widehat{u}(0) = \widehat{v}(0) = 0\) and thus we may also assume that \(N_{\text{min}} \geq 1\) when \(Z = Z\).

Using Proposition 1, we have

\[(12) \lesssim \sum_{N_{\text{max}} \sim N_{\text{med}} \sim N} \sum_{L_{\text{max}} \gtrsim N_{\text{med}}} \frac{1}{(N_1)^{\alpha} L_1^{\frac{1}{2} + \varepsilon} L_2^{\frac{1}{2} - \varepsilon}} \frac{L_{\min}^{\frac{1}{2}} (N_{\text{max}}^{\frac{1}{2}} Z)}{N_{\text{min}}^{\frac{1}{2}}}.
\]

Since \(L_1^{\frac{1}{2} + \varepsilon} L_2^{\frac{1}{2} - \varepsilon} \approx (N_{\text{max}}^{\frac{1}{2}} Z)\), we get

\[(12) \lesssim \sum_{N_{\text{max}} \sim N_{\text{med}} \sim N} \frac{1}{\langle N_1 \rangle^{\alpha} N_{\text{max}}^{\frac{1}{2} - \alpha}} N_{\text{min}}^{\frac{1}{2}} \approx 1 + N_{\text{min}}^{\frac{1}{2} - \frac{1}{2} + \alpha} \approx 1.\]

Now we turn to (11). Firstly we consider the case \(L_1 = L_{\text{max}}\) and \(N_{\text{max}} = N_1\) (the estimate for the case \(L_3 = L_{\text{max}}\) and \(N_{\text{max}} = N_3\) follow by symmetry). Proposition 1 gives

\[(11) \lesssim \sum_{N_{\text{max}} \sim N_{\text{med}} \sim N} \sum_{L_1, L_2, L_3 \geq 1} \sum_{H \sim L_1} \frac{L_{\min}^{\frac{1}{2}} (\min(N_{\text{min}}^{\frac{1}{2}}, N_{\text{max}}^{\frac{1}{2}} L_{\text{med}}^{\frac{1}{2}})) Z}{(N_1)^{\alpha} L_1^{\frac{1}{2} + \varepsilon} L_2^{\frac{1}{2} - \varepsilon}}\]

\[\lesssim \sum_{N_{\text{max}} \sim N_{\text{med}} \sim N} \sum_{L_1, L_2, L_3 \geq 1} \frac{L_{\min}^{\frac{1}{2}} (\min(N_{\text{min}}^{\frac{1}{2}}, N_{\text{max}}^{\frac{1}{2}} L_{\text{med}}^{\frac{1}{2}})) Z}{(N_1)^{\alpha} L_1^{\frac{1}{2} + \varepsilon} L_2^{\frac{1}{2} - \varepsilon}}.\]
Here $H$-sum is bounded by an absolute constant. By summing in $L_{\text{min}}$ and then $L_1$, we get

\[
(11) \lesssim \sum_{N_{\text{max}} \sim N_{\text{med}}} \sum_{N_{\text{max}} \geq L_{\text{med}} \geq 1} \frac{\langle \min(N_{\text{min}}^{\frac{1}{2}}, N_{\text{max}}^{\frac{1}{2}} L_{\text{med}}^{\frac{1}{2}}) \rangle}{\langle N_{\text{min}} \rangle^s L_{\text{max}}^{\frac{1}{2} + \varepsilon}}.
\]

If $Z = \mathbb{R}$, then we separate $N_{\text{min}}$ sum as follows:

\[
(11)
\]

\[
\lesssim \left( \sum_{0 < N_{\text{min}} < N^{1 - \alpha}} + \sum_{N^{1 - \alpha} \leq N_{\text{min}} \leq N} \right) \sum_{L_{\text{med}} \geq 1} \frac{\min(N_{\text{min}}^{\frac{1}{2}} L_{\text{med}}^{\frac{1}{2}}, N^{\frac{1 - \alpha}{2}})}{\langle N_{\text{min}} \rangle^s}.
\]

\[
\lesssim \sum_{N_{\text{min}} < N^{1 - \alpha}} \sum_{L_{\text{med}} \geq 1} \frac{N_{\text{min}}^{\frac{1}{2}} L_{\text{med}}^{\frac{1}{2}}}{\langle N_{\text{min}} \rangle^s} + \sum_{N_{\text{min}} = N^{1 - \alpha}} \sum_{L_{\text{med}} \geq 1} \frac{\min(N_{\text{min}}^{\frac{1}{2}} L_{\text{med}}^{\frac{1}{2}}, N^{\frac{1 - \alpha}{2}})}{\langle N_{\text{min}} \rangle^s}.
\]

\[
\lesssim N^{\frac{1 - \alpha}{2}} + \sum_{N_{\text{min}} = N^{1 - \alpha}} \left( \sum_{1 \leq L_{\text{med}} < N_{\text{min}} N^{\alpha - 1}} N^{\frac{1 - \alpha}{2}} + \sum_{L_{\text{med}} \geq N_{\text{min}} N^{\alpha - 1}} N^{\frac{1 - \alpha}{2}} L_{\text{med}}^{-\varepsilon} \right) \lesssim 1.
\]

If $Z = \mathbb{Z}$, then we have

\[
(11)
\]

\[
\lesssim \sum_{N_{\text{min}} = 1}^N \left( \sum_{N^{\alpha - 1} N_{\text{min}} \leq L_{\text{med}} \leq L_{\text{max}}} + \sum_{L_{\text{med}} \leq N^{\alpha - 1} N_{\text{min}}} \right) \frac{\langle 1 \min(N_{\text{min}}^{\frac{1}{2}}, N^{\frac{1 - \alpha}{2}} L_{\text{med}}^{\frac{1}{2}}) \rangle}{\langle N_{\text{min}} \rangle^s L_{\text{max}}^{\frac{1}{2} + \varepsilon}}.
\]

\[
\lesssim \sum_{N_{\text{min}} = 1}^N \left( \sum_{N^{\alpha - 1} N_{\text{min}} \leq L_{\text{med}} \leq L_{\text{max}}} \frac{N_{\text{min}}^{\frac{1}{2}} L_{\text{med}}^{\frac{1}{2}}}{\langle N_{\text{min}} \rangle^s L_{\text{max}}^{\frac{1}{2} + \varepsilon}} + \sum_{L_{\text{med}} \leq N^{\alpha - 1} N_{\text{min}}} \frac{1 + \min(N_{\text{min}}^{\frac{1}{2}}, N^{\frac{1 - \alpha}{2}} L_{\text{med}}^{\frac{1}{2}})}{\langle N_{\text{min}} \rangle^s L_{\text{max}}^{\frac{1}{2} + \varepsilon}} \right).
\]

\[
\lesssim N^{\frac{1 - \alpha}{2}} + \sum_{N_{\text{min}} = 1}^N \sum_{L_{\text{med}} \leq N^{\alpha - 1} N_{\text{min}}} \frac{(1 + N^{\frac{1 - \alpha}{2}} L_{\text{med}}^{\frac{1}{2}})}{\langle N_{\text{min}} \rangle^s L_{\text{max}}^{\frac{1}{2} + \varepsilon}}.
\]

\[
\lesssim N^{\frac{1 - \alpha}{2}} + \sum_{N_{\text{min}} = 1}^N \left( \sum_{1 \leq L_{\text{med}} < N^{\alpha - 1} N_{\text{min}}} \frac{L_{\text{med}}^{\frac{1}{2} + \varepsilon}}{\langle N_{\text{min}} \rangle^s L_{\text{max}}^{\frac{1}{2} + \varepsilon}} + \sum_{N^{\alpha - 1} N_{\text{min}} \leq L_{\text{med}} \leq N^{\alpha - 1} N_{\text{min}}} \frac{N^{\frac{1 - \alpha}{2}} L_{\text{med}}^{\frac{1}{2} + \varepsilon}}{\langle N_{\text{min}} \rangle^s L_{\text{max}}^{\frac{1}{2} + \varepsilon}} \right).
\]

\[
\lesssim N^{\frac{1 - \alpha}{2}} + 1 + \sum_{N_{\text{min}} = 1}^N \sum_{N^{\alpha - 1} N_{\text{min}} \leq L_{\text{med}} \leq N^{\alpha - 1} N_{\text{min}}} N_{\text{min}}^{\frac{1 - \alpha}{2}} \lesssim 1 + N^{\frac{1 - \alpha}{2}} \log N \lesssim 1.
\]

Secondly, we deal with the case $L_2 = L_{\text{max}}$ and $N_{\text{max}} \sim N_{\text{min}}$. Using Proposition 1, we have

\[
(12) \lesssim \sum_{N_{\text{max}} \sim N_{\text{min}}} \sum_{N_{\text{max}} \geq L_{\text{med}} \geq 1} \frac{1}{\langle N_{\text{min}} \rangle^s L_{\text{max}}^{\frac{1}{2} + \varepsilon}} L_{\text{min}}^{\frac{1}{2}} \min(N_{\text{min}}^{\frac{1}{2}}, N_{\text{max}}^{\frac{2 - \alpha}{2}} L_{\text{med}}^{\frac{1}{2}}) Z.
\]
Since $s \geq \frac{2-\alpha}{4}$, we have

$$
\text{(12) } \lesssim \sum_{N_{\text{max}} \sim N_{\text{min}} \sim N_{L_{\text{med}}} \geq 1} \frac{1}{(N)^{s}L_{\text{med}}^{\frac{1}{2}-\varepsilon}} \langle N^{\frac{2}{3}-\alpha}L_{\text{med}}^{\frac{1}{2}} \rangle Z \lesssim 1.
$$

We now handle the remaining three cases: $L_{1} = L_{\text{max}}$ and $N_{2} \sim N_{3} \gg N_{1}$; $L_{2} = L_{\text{max}}$ and $N_{3} \sim N_{1} \gg N_{2}$; $L_{3} = L_{\text{max}}$ and $N_{1} \sim N_{2} \gg N_{3}$.

**Case** $L_{1} = L_{\text{max}}$ and $N_{2} \sim N_{3} \gg N_{1}$. Since $N_{2} \sim N_{3} \gg N_{1}$ and $\xi_{1} + \xi_{2} + \xi_{3} = 0$, one can observe that $H \sim |h(\xi_{1}, \xi_{2}, \xi_{3})| \sim |\xi_{1}||\xi_{2}|^{\alpha-1} \sim N_{\text{min}}N_{\text{max}}^{-1}$. Thus we have $1 \lesssim L_{1} \sim H \sim N_{\text{max}}^{-1}N_{\text{min}}^{-1}$, which means that $N_{\text{min}} \gtrsim N_{\text{med}}^{-1}$. Using Proposition 1 and performing $L_{\text{min}}$, we have

$$
\text{(11) } \lesssim \sum_{N_{\text{max}} \sim N_{\text{med}} \sim N_{L_{\text{max}}} \sim N_{\text{max}}^{-\alpha}N_{\text{min}} \sim 1} \sum_{L_{\text{med}} \geq 1} \frac{L_{\text{med}}^{\frac{1}{2}} \min(N_{\text{max}}^{\frac{1}{2}}, N_{\text{max}}^{\frac{2}{3}-\alpha}N_{\text{med}}^{-\frac{1}{2}}L_{\text{med}}^{-\frac{1}{2}})}{(N_{\text{min}})^{s}(N_{\text{max}}^{-1}N_{\text{min}})^{\frac{1}{2}+\varepsilon}}.
$$

When $Z = \mathbb{R}$, by separating $N_{\text{min}}$ sum into the cases $N_{\text{min}} \leq \frac{N_{\text{med}}}{N_{\text{max}}}$ and $N_{\text{min}} \gtrsim \frac{N_{\text{med}}}{N_{\text{max}}}$, we have

$$
\text{(11) } \lesssim \sum_{N_{\text{max}} \sim N_{\text{med}} \sim N_{L_{\text{max}}} \sim N_{\text{max}}^{-\alpha}N_{\text{min}} \sim 1} \sum_{L_{\text{med}} \geq 1} \frac{N_{\text{med}}^{\frac{1}{2}} \min(N_{\text{max}}^{\frac{1}{2}}, N_{\text{max}}^{\frac{2}{3}-\alpha}N_{\text{med}}^{-\frac{1}{2}}L_{\text{med}}^{-\frac{1}{2}})(N_{\text{min}})^{s}(N_{\text{max}}^{-1}N_{\text{min}})^{\frac{1}{2}+\varepsilon}}{N_{\text{med}}^{\frac{1}{2}} \min(L_{\text{med}}^{\frac{1}{2}})}.
$$

Otherwise ($Z = \mathbb{Z}$), since $N_{\text{min}} \geq 1$, we have

$$
\text{(11) } \lesssim \sum_{1 \leq N_{\text{min}} \lesssim N_{\text{max}}^{-\alpha}N_{\text{med}} \geq 1} \frac{L_{\text{med}}^{\frac{1}{2}} \min(N_{\text{max}}^{\frac{1}{2}}, N_{\text{max}}^{\frac{2}{3}-\alpha}N_{\text{med}}^{-\frac{1}{2}}L_{\text{med}}^{-\frac{1}{2}})}{(N_{\text{min}})^{s}(N_{\text{max}}^{-1}N_{\text{min}})^{\frac{1}{2}+\varepsilon}}.
$$
\[ \sum 1 + N^{\frac{1-\alpha}{2}} \log N \lesssim 1. \]

**Case** \( L_2 = L_{\text{max}} \) and \( N_3 \sim N_1 \gg N_2 \). In this case we have \( L_2 \sim H \sim N^\alpha \). From Proposition 1, summation in \( L_{\text{min}} \) and the assumption \( N_{\text{min}} \geq 1 \) for \( Z = Z \), we have

\[
\begin{align*}
(11) \lesssim & \sum_{N_{\text{max}} \sim N_{\text{med}} \sim N} \sum_{L_{\text{max}} \sim N^\alpha} \sum_{L_{\text{med}} \geq L_{\text{min}} \geq 1} \frac{L_{\text{min}}^{\frac{1}{r}} \langle \min(N_{\text{min}}^{\frac{1}{r}}, N_{\text{max}}^{\frac{1-\alpha}{2}}, L_{\text{med}}^{\frac{1}{r}}) \rangle Z}{(N_1)^s L_1^{\frac{1}{r} + \frac{1}{r} - \varepsilon} L_{\text{max}}^{\frac{1}{r} - \varepsilon}} \\
\lesssim & \sum_{N_{\text{max}} \sim N_{\text{med}} \sim N} \sum_{1 \leq L_{\text{med}} \leq N^\alpha} \frac{\langle \min(N_{\text{min}}^{\frac{1}{r}}, N_{\text{max}}^{\frac{1-\alpha}{2}}, L_{\text{med}}^{\frac{1}{r}}) \rangle Z}{N^{s + \alpha(\frac{1}{r} - \varepsilon)}} \\
\lesssim & \sum_{N_{\text{max}} \sim N_{\text{med}} \sim N} \sum_{N_{\text{min}} \geq N^{1-\alpha}} \sum_{L_{\text{med}} \geq L_{\text{min}} \geq 1} \frac{\langle \min(N_{\text{min}}^{\frac{1}{r}}, N_{\text{max}}^{\frac{2-\alpha}{2}}, L_{\text{med}}^{\frac{1}{r}}) \rangle Z}{N^{s + \alpha(\frac{1}{r} - \varepsilon)} L_{\text{med}}^{\frac{1}{r} - \varepsilon}}.
\end{align*}
\]

Since \( N^{\alpha-1} N_{\text{min}} \sim L_{\text{max}} \gtrsim 1 \) implies \( N_{\text{min}} \gtrsim N^{1-\alpha} \), by breaking \( N_{\text{min}} \)-sum into two parts, we have:

\[
(11) \lesssim \sum_{N_{\text{min}} \geq N^{1-\alpha}} \frac{\langle N_{\text{min}}^{\frac{1}{r}} \rangle Z}{N^s} \\
+ \sum_{N_{\text{min}} \geq N^{1-\alpha}} \sum_{L_{\text{max}} \sim N^\alpha} \sum_{L_{\text{med}} \geq L_{\text{min}} \geq 1} \frac{\langle \min(N_{\text{min}}^{\frac{1}{r}}, N_{\text{max}}^{\frac{2-\alpha}{2}}, N_{\text{med}}^{\frac{1}{r}}) \rangle Z}{N^{s + \alpha(\frac{1}{r} - \varepsilon)} L_{\text{med}}^{\frac{1}{r} - \varepsilon}}
\]

For the second inequality we use \( \sum_{N_{\text{min}} \geq N^{1-\alpha}} \sum_{L_{\text{max}} \sim N^\alpha} \langle (1 + N_{\text{min}}^2) N^{-s} \rangle \lesssim N^{\frac{2-\alpha}{s}} + N^{-s} \log N \lesssim N^{\frac{2-\alpha}{s}} \). Now by dividing \( L_{\text{med}} \)-sum into \( \sum_{1 \leq L_{\text{med}} \leq N^{\alpha-2} N_{\text{min}}^2} \)
2.3 and bilinear estimates Propositions 2.

\[ \sum_{N^{a/2}N_{\min}^{2} \ll L_{\text{med}}} \sum_{N^{a/2} < N_{\min} \leq N} \sum_{L_{\text{med}} \geq 1} \min(N_{\min}^{\frac{3}{2}}, N^{\frac{2a}{2} - \frac{1}{2}}, L_{\text{med}}^{\frac{1}{2}}) \]

\[ \ll N^{\frac{2a}{2} - s} + \sum_{N^{a/2} < N_{\min} \leq N} \sum_{L_{\text{med}} \geq 1} N^{\frac{2a}{2} - \frac{1}{2}} N^{sL_{\text{med}}^{\frac{1}{2} - \epsilon}} \]

Since \( s \geq \frac{2a}{4} \), we get the desired result. \( \square \)

4. Proof of Theorem 1.1. For the proof Theorem 1.1, we need the trilinear estimate

\[ \|u_1 \overline{u}_2 \overline{u}_3\|_{X^{s,-\frac{1}{2} + \epsilon}} \lesssim \prod_{j=1}^{3} \|u_j\|_{X^{s,-\frac{1}{2} + \epsilon} (\mathbb{R} \times \mathbb{R})}. \] (13)

Remark 1 (Failure of (13) for \( s < \frac{2a}{4} \)). It is easy to see that the trilinear estimate fails when \( s < \frac{2a}{4} \). The counter-example is a resonant high-high-high to high interaction. For \( N \gg 1 \), let

\[ \widetilde{u}_1, \widetilde{u}_2 = \chi_{A_N}, \quad A_N = \{ (\xi, \tau) : N \leq \xi \leq N + N^{\frac{2a}{2}}, \ |\tau - |\xi|^\alpha| \leq 1 \}, \]

\[ \widetilde{u}_3 = \chi_{A_N}, \quad A_N = \{ (\xi, \tau) : -N \leq \xi \leq -N + N^{\frac{2a}{2}}, \ |\tau + |\xi|^\alpha| \leq 1 \}. \]

Here, the number \( N^{\frac{2a}{2}} \) is chosen so that the parallelogram \( A_N \) is fit in a width 1 strip of \( \tau = |\xi|^\alpha \). Then, it follows that

\[ \|\widetilde{u}_1 * \overline{u}_2 * \overline{u}_3\|_{X^{s,-1}} \sim N^{\frac{2a}{2}} N^{\frac{2a}{2} - s} N^s N^{\frac{2a}{4}}, \text{ and } \|u_j\|_{X^{s,b}} \sim N^s N^{\frac{2a}{4}}. \]

This and letting \( N \to \infty \) give the necessary condition \( s \geq \frac{2a}{4} \) for (13).

Proposition 3. Let \( s \geq \frac{2a}{4} \) and \( 0 < \epsilon \ll 1 \). For any \( u_1, u_2, \) and \( u_3 \in X^{s,\frac{1}{2} + \epsilon}_2 \), we have

\[ \|u_1 \overline{u}_2 \overline{u}_3\|_{X^{s,-\frac{1}{2} + \epsilon}} \lesssim \prod_{j=1}^{3} \|u_j\|_{X^{s,-\frac{1}{2} + \epsilon}}. \]

Proof. By duality and Plancherel’s theorem it suffices to show that

\[ \|\langle \xi_1 \rangle^s \langle \xi_2 \rangle^s \langle \xi_3 \rangle^s (\tau_1 - |\xi_1|^\alpha)^{\frac{1}{2} + \epsilon} (\tau_2 + |\xi_2|^\alpha)^{\frac{1}{2} + \epsilon} (\tau_3 - |\xi_3|^\alpha)^{\frac{1}{2} - \epsilon} (\tau_4 + |\xi_4|^\alpha)^{\frac{1}{2} - \epsilon} \|_{L^2(\mathbb{R}^4 \times \mathbb{R}^4)} \lesssim 1. \]

Since \( \langle \tau_2 + |\xi_2|^\alpha \rangle^{\frac{1}{2} + \epsilon} \gtrsim \langle \tau_2 + |\xi_2|^\alpha \rangle^{\frac{1}{2} - \epsilon} \), the desired estimate follows from Lemma 2.3 and bilinear estimates Propositions 2.

Proof of Theorem 1.1. We define a nonlinear functional \( N \) by

\[ N(u) = \psi(t) U(t) \phi - i \gamma \psi(t/T) \int_0^T U(t - t')|u|^2 u(t')dt', \]

where \( \psi \) is a fixed smooth cut-off function such that \( \psi(t) = 1 \) if \( |t| < 1 \) and \( \psi(t) = 0 \) if \( |t| > 2 \), and \( 0 < T \leq 1 \) is fixed. For \( s, b \in \mathbb{R} \) we define the norm \( X^{s,b}_t \) for on the time interval \( J_T = [0, T] \) by

\[ \|u\|_{X^{s,b}(J_T)} := \inf \{ \|v\|_{X^{s,b}} : v|_{J_T} = u \}. \]
Theorem 5.1. Let cubic NLS respectively, satisfying (5) and $T > -H$ ill-posed in $\mathbb{R}$ for $j = 1$. For this purpose we recall ill-posedness result for the Schrödinger equation (16) which is $\gamma = 1$. Our strategy is to approximate the solution by the solutions of (16) which is CUBIC FRACTIONAL SCHRÖDINGER EQUATION 2875. Choosing $\varepsilon'$ sufficiently small, from Proposition 3 we see

$$
\|\mathcal{N}(u)\|_{X^{s, \frac{1}{2}+\varepsilon}(J_T)} \lesssim \|\phi\|_{H^s} + T^{\varepsilon'-\varepsilon}\|u\|_{X^{s, \frac{1}{2}+\varepsilon}(J_T)} \lesssim \|\phi\|_{L^2} + T^{\varepsilon'-\varepsilon}\rho^3.
$$

Choosing $\rho$ and $T$ small enough so that $\rho \geq 2C\|\phi\|_{H^s}$ and $CT^{\varepsilon'-\varepsilon}\rho^3 \leq \rho/2$ for some constant $C$, we see that the functional $\mathcal{N}$ is a map from $B_{T, \rho}$ to itself. Similarly one can show that $N(u)$ is a contraction. Therefore there is a unique $u \in X^{s, \frac{1}{2}+\varepsilon}(J_T)$ satisfying (2).

5. Ill-posedness. In this section, we prove that the equation (1) in the non-periodic case is ill-posed for $\frac{2\alpha-3\alpha}{4\alpha+1} < s < \frac{2\alpha}{4\alpha+1}$. For convenience we assume that $\gamma = 1$. Our strategy is to approximate the solution by the solutions of (16) which is ill-posed in $H^s, s < 0$ (see [3] for the non-periodic case and [4, 16] for the periodic one). For this purpose we recall ill-posedness result for the Schrödinger equation

$$
\begin{cases}
i\partial_t u - \Delta u = |v|^2 v, \\
v(0, \cdot) = \phi \in H^s.
\end{cases}
$$

Theorem 5.1. Let $s < 0$. The solution map of the initial value problem of the cubic NLS (16) fails to be uniformly continuous. More precisely, for $0 < \delta \ll \varepsilon \ll 1$ and $T > 0$ arbitrary, there are two solutions $v_1, v_2$ to (16) with initial data $\phi_1, \phi_2$, respectively, satisfying (5), (6) and (7). Moreover we can find solutions to satisfy

$$
\sup_{0 \leq t \leq \infty} \|v_j(t)\|_{H^s} \lesssim \varepsilon,
$$

for $j = 1, 2$.

Let $N \gg 1$ be a large parameter to be chosen later. Let $v(s, y)$ be a solution of the cubic NLS equation (16) and

$$
(s, y) := \left(t, \frac{x + \alpha N^{\alpha-1}t}{(\alpha - \frac{1}{2}) N^{\alpha-2}}\right).
$$

We shall construct approximate solutions which is given by

$$
V(t, x) := e^{inx} e^{in^2 t} v(s, y).
$$
It is easy to see that
\[
(i\partial_t + (-\Delta)^{\frac{\alpha}{2}})V
\]
\[
e^{iN_x e^{iN^\alpha t}}(-N^\alpha v(s, y) + i\partial_x v(s, y) + i\frac{\alpha N^\alpha - 1}{(\frac{\alpha-1}{2}) N^{\alpha-2}} \partial_y v(s, y)) + e^{iN_x e^{iN^\alpha t}}(N^\alpha v(s, y) - i\frac{\alpha N^\alpha - 1}{(\frac{\alpha-1}{2}) N^{\alpha-2}} \partial_y v(s, y) - \partial_y v(s, y) + R(-i\partial_y)v(s, y)),
\]
where
\[
R(\xi) = \left|\frac{\xi}{(\frac{\alpha-1}{2}) N^{\alpha-2}}\right|^\alpha - N^\alpha - \frac{\alpha N^\alpha - 1}{(\frac{\alpha-1}{2}) N^{\alpha-2}} \xi - \xi^2. \tag{19}
\]
Since \(v(s, y)\) is a solution of (16), we have
\[
iV_t + (-\Delta)^{\frac{\alpha}{2}} V - |V|^2 V = E,
\]
where \(E = e^{iN_x e^{iN^\alpha t}} R(-i\partial_y)v(s, y)\). We need to bound the error. First we show the following perturbation result relying on the local well-posedness.

**Lemma 5.2.** Let \(u\) be a smooth solution to (1) and \(V\) be a smooth solution to the equation
\[
iV_t + (-\Delta)^{\frac{\alpha}{2}} V - |V|^2 V = \mathcal{E}
\]
for some error function \(\mathcal{E}\). Let \(e\) be the solution to the inhomogeneous problem \(ie_t + (-\Delta)^{\frac{\alpha}{2}} e = \mathcal{E}, e(0) = 0\) and let \(\eta(t)\) be a compactly supported smooth time cut-off function such that \(\eta = 1\) on \(J = [0, 1]\). Suppose that \(\|u(0)\|_{H^{\frac{\alpha+4}{2}}} \leq \|V(0)\|_{H^{\frac{\alpha+4}{2}}} \),
\[
\|\eta(t)e\|_{X^\alpha(R^2)} \leq \varepsilon. \tag{20}
\]
Then, if \(\varepsilon\) is sufficiently small, we have
\[
\|u(t) - V(t)\|_{H^{\frac{\alpha+4}{2}}(J)} \leq \|u(0) - V(0)\|_{H^{\frac{\alpha+4}{2}}} + \|\eta(t)e\|_{X^\alpha(R^2)} \leq \varepsilon.
\]

**Proof.** Writing the equation for \(V\) in integral form, we have
\[
V(t) = U(t)V(0) + e(t) - i \int_0^t U(t-t')(|V|^2 V)(t')dt'.
\]
By taking \(X^\alpha(R^2)\) norm on both sides and applying (15), we get
\[
\|V\|_{X^\alpha(R^2)} \leq \|V(0)\|_{H^{\frac{\alpha+4}{2}}} + \|\eta(t)e\|_{X^\alpha(R^2)} \leq \|V(0)\|_{H^{\frac{\alpha+4}{2}}} + \|\eta(t)e\|_{X^\alpha(R^2)} \leq \varepsilon.
\]
By continuity argument with sufficiently small \(\varepsilon\), we obtain
\[
\|\eta(t)e\|_{X^\alpha(R^2)} \leq \varepsilon.
\]
Let \(w := u - V\). Then \(w\) satisfies the equation
\[
iw_t + (-\Delta)^{\alpha/2}w = |w|^2 w + 2w^2 w + 2w|w|^2 + w^2 \bar{v} + \bar{w}v^2 - E, \quad w(0) = u(0) - V(0),
\]
which is written in integral form as
\[
w(t) = U(t)w(0) - e(t) - i \int_0^t U(t-t')(|w|^2 w + 2w^2 w + 2w|w|^2 + w^2 \bar{v} + \bar{w}v^2)(t')dt'.
\]
Again taking $X^\frac{2s}{m} - \frac{1}{2} + (J)$ norms on both sides and applying (15), we have

$$
\|w\|_{X^\frac{2s}{m} - \frac{1}{2} + (J)} \lesssim \|u(0) - V(0)\|_{H^\frac{2s}{m} - \frac{1}{2} + (J)} + \|\eta(t)e\|_{X^\frac{2s}{m} - \frac{1}{2} + (J)} + \|\eta(t)e\|_{X^\frac{2s}{m} - \frac{1}{2} + (J)} + \|\eta(t)e\|_{X^\frac{2s}{m} - \frac{1}{2} + (J)} + \|\|\|_{X^\frac{2s}{m} - \frac{1}{2} + (J)} + \|\|_{X^\frac{2s}{m} - \frac{1}{2} + (J)}^2.
$$

If $\varepsilon$ is sufficiently small, the continuity argument with respect to time gives the desired bound.

**Lemma 5.3.** Let $e$ be a solution to the initial value problem $ie_t + (-\Delta)^\frac{s}{2} e = E$, $e(0) = 0$, and let $\eta$ be the smooth time cut-off function given in Lemma 5.2. Then

$$
\|\eta(t)e\|_{X^\frac{2s}{m} - \frac{1}{2} + (J)} \lesssim \varepsilon N^{-\alpha/2}.
$$

For the proof of this lemma, we make use of the following which is in [3].

**Lemma 5.4 (Lemma 2.1 [3]).** Let $-\frac{1}{2} < s, \sigma > 0$, and $w \in H^s(\mathbb{R})$. For $M > 1, \tau > 0, x_0 \in \mathbb{R}$ and $A > 0$ let

$$
\bar{w}(x) = Ae^{iMx}w\left(\frac{\tau - x_0}{\tau}\right).
$$

1. Suppose that $s \geq 0$. Then there exists a constant $C_1 < \infty$, depending only on $s$, such that

$$
\|\bar{w}\|_{H^s} \leq C_1 |A|\tau^{1/2}M^s \|w\|_{H^s}
$$

for all $w, A, x_0$ whenever $M \cdot \tau \geq 1$.

2. Suppose that $s < 0$ and that $\sigma \geq |s|$. Then there exists a constant $C_1 < \infty$, depending only on $s$ and $\sigma$, such that

$$
\|\bar{w}\|_{H^s} \leq C_1 |A|\tau^{1/2}M^s \|w\|_{H^s}
$$

for all $w, A, x_0$ whenever $1 \leq \tau \cdot M^{1+(s/\sigma)}$.

3. There exists $c_1 > 0$ such that for each $w$ there exists $C_w < \infty$ such that

$$
\|\bar{w}\|_{H^s} \geq c_1 |A|\tau^{1/2}M^s \|w\|_{L^2}
$$

whenever $\tau \cdot M \geq C_w$.

**Proof of Lemma 5.3.** Using (15) and Plancherel’s theorem, we have

$$
\|\eta(t)e\|_{X^\frac{2s}{m} - \frac{1}{2} + (J)} \lesssim \|\eta(t)E\|_{X^\frac{2s}{m} - \frac{1}{2} + (J)} = \|\langle \xi \rangle^{\frac{2s}{m}} \langle \tau - |\xi|^2 \rangle^{-\frac{1}{2} + \eta(t)E}\|_{L^2_{\tau, \xi}} \\
\lesssim \|\langle \xi \rangle^{\frac{2s}{m}} \eta(t)E\|_{L^2_{\tau, \xi}} = \|\eta(t)\langle \xi \rangle^{\frac{2s}{m}} FE\|_{L^2_{\tau, \xi}} \\
\lesssim \|\langle \xi \rangle^{\frac{2s}{m}} FE\|_{L^\infty_{\tau} L^2_{\xi}[0,2] \times \mathbb{R}}.
$$

It suffices to show

$$
\sup_{0 \leq t \leq 2} \|E\|_{H^\frac{2s}{m}} \lesssim \varepsilon N^{-\alpha/2}.
$$

Since $E = e^{iN^\alpha t} e^{i\alpha^* t} R(-i\partial_y) v(s, y)$, by Lemma 5.4 with $M = N, \tau = N^2 \tau^{-1}$ we see that $\|E\|_{H^\frac{2s}{m}} \lesssim \|R(-i\partial_y) v\|_{H^\frac{2s}{m}}$. Recalling that $R(\xi)$ is given by (19), it suffices
to show \( \|R(-i\partial_y)v\|_H^{2+\alpha} \lesssim \varepsilon N^{-\alpha/2} \). Since \( \|v\|_H^s \lesssim \varepsilon \) by Theorem 5.1, we need only to show that
\[
|R(\xi)| \leq cN^{-\alpha/2}|\xi|^3 \quad \text{for all } N \gg 1. \tag{20}
\]
Let \( c_1 = \max\left(8\alpha(\frac{\alpha-1}{2})^2, \frac{2\alpha}{6}(2-\alpha)(\frac{\alpha-1}{2})^2\right) \),
\( c_2 = \max\left(8\alpha(\frac{\alpha-1}{2}) \frac{1}{2}, \frac{3\alpha}{\alpha}(\frac{\alpha-1}{2})^2\right) \) and let
\[
f(\xi) = |(\frac{\alpha-1}{2})N^{-2-\frac{1}{2}}\xi + N|^\alpha, \quad P(\xi) = N\alpha + \frac{\alpha N^\alpha - 1}{\frac{\alpha-1}{2}N^{\alpha-2}}\frac{\xi}{\xi} + \xi^2,
\]
so that \( R(\xi) = f(\xi) - P(\xi) \). We also denote \( g(\xi) = -c_1 N^{-\alpha/2} N^3 + P(\xi), \) \( h(\xi) = c_2 N^{-\frac{1}{2}} N^3 + P(\xi) \). Then it suffices to show \( |R(\xi)| \leq c_1 N^{-\alpha/2}|\xi|^3 \) on \( \xi > \xi_1 \) and \( f \leq g, h \leq f \) on \( \xi \leq \xi_1 \) for some \( \xi_1 < 0 \). The following are easy to check:
\[
f'(\xi) = \alpha \left(\frac{\alpha-1}{2}\right)N^{-2-\frac{1}{2}}\frac{\xi}{\xi} + N|\frac{\alpha-1}{2}|\left(\frac{\alpha-1}{2}\right)N^{-2-\frac{1}{2}}\frac{\xi}{\xi} + N \right),
\]
\[
f''(\xi) = 2\left(\frac{\alpha-1}{2}\right)N^{-2-\frac{1}{2}}\frac{\xi}{\xi} + N|\frac{\alpha-1}{2}|\left(\frac{\alpha-1}{2}\right)N^{-2-\frac{1}{2}}\frac{\xi}{\xi} + N \right),
\]
\[
f'''(\xi) = 2(\alpha - 2)\left(\frac{\alpha-1}{2}\right)N^{-2-\frac{1}{2}}\frac{\xi}{\xi} + N|\frac{\alpha-1}{2}|\left(\frac{\alpha-1}{2}\right)N^{-2-\frac{1}{2}}\frac{\xi}{\xi} + N \right),
\]
\[
g'(\xi) = -3c_1 N^{-\frac{1}{2}} N^2 + 2\xi + \frac{\alpha N^{\alpha-1}}{\frac{\alpha-1}{2}N^{\alpha-2}}\frac{\xi}{\xi}, \quad g''(\xi) = -6c_1 N^{-\frac{1}{2}} N^2 + 2,
\]
\[
h'(\xi) = 3c_2 N^{-\frac{1}{2}} N^2 + 2\xi + \frac{\alpha N^{\alpha-1}}{\frac{\alpha-1}{2}N^{\alpha-2}}\frac{\xi}{\xi} \geq \frac{2}{3} \frac{\alpha N^{\alpha-1}}{\frac{\alpha-1}{2}N^{\alpha-2}}\frac{\xi}{\xi},
\]
provided that the derivatives exist.

Let us set \( \xi_1 = -\frac{1}{2} \left(\frac{\alpha-1}{2}\right)N^{\alpha-2}\right)^{\frac{1}{2}} N \) and \( \xi_2 = -2(\frac{\alpha-1}{2})N^{\alpha-2}\right)^{\frac{1}{2}} N \). Then we consider separately three cases \( \xi \geq \xi_1; \xi_2 \leq \xi < \xi_1; \xi < \xi_2 \). If \( \xi \geq \xi_1, f \) is three times differentiable and
\[
|f'''(\xi)| \leq |f'''(\xi_1)| = (2 - \alpha)\left(\frac{\alpha-1}{2}\right)^{-\frac{1}{2}}2^{1-\alpha}N^{-\alpha/2}.
\]
Hence by Taylor’s theorem, we get (20). We need only to handle the remaining two cases.

For both cases it is easy to show \( h(\xi) \leq f(\xi) \). In fact, observe that \( h(\xi_1) \leq \left(\frac{\alpha-1}{2}\right) N^{\alpha} \leq \left(\frac{1}{2}\right) \alpha N^{\alpha} = f(\xi_1) \). Since \( f' \) is increasing, \( f'(\xi) \leq f'(\xi_1) = \left(\frac{\alpha-1}{2}\right) N^{\alpha} \leq \left(\frac{1}{2}\right) \alpha N^{\alpha} \leq f(\xi_1) \). Hence, \( h(\xi) \leq f(\xi_1) \) if \( \xi \leq \xi_1 \).

To show that \( f(\xi) \leq g(\xi) \) for \( \xi_2 \leq \xi < \xi_1 \), observe that \( f(\xi_1) = \left(\frac{1}{2}\right) \alpha N^{\alpha} \leq \left(\frac{1}{2} + 2(\frac{\alpha-1}{2})\right) N^{\alpha} \leq g(\xi_1) \). Hence, it suffices to show \( f'(\xi) \geq g'(\xi) \) for \( \xi_2 \leq \xi < \xi_1 \). Since \( f' \) is increasing, \( f'(\xi) \geq f'(\xi_2) = -\alpha(\frac{\alpha-1}{2})^{-\frac{1}{2}} N^{\frac{3}{2}} \). Since \( g' \) is increasing, \( g'(\xi) \leq g'(\xi_1) = -\alpha(\frac{\alpha-1}{2})^{-\frac{1}{2}} N^{\frac{3}{2}} \). Hence, \( f'(\xi) \geq g'(\xi) \) for \( \xi_2 \leq \xi < \xi_1 \).

Finally, we show \( f(\xi) \leq g(\xi) \) for \( \xi < \xi_2 \). We note that \( f''(\xi) \leq g''(\xi) \) and
\[
f(\xi_2) = N^{\alpha} \leq (64\alpha + 2\alpha(\alpha - 1) - 2\alpha + 1)N^{\alpha} \leq g(\xi_2).
\]
Since $f'(\xi_2) = -\alpha (\frac{\alpha - 1}{2}) N^{\frac{\alpha}{2}} \geq (-96 + 2\alpha) N^{\frac{\alpha}{2}} \geq g'(\xi_2)$, $f'(\xi) \geq g'(\xi)$. This together with $f(\xi_2) \leq g(\xi_2)$ gives $f(\xi) \leq g(\xi)$. □

Now we are ready to prove Theorem 1.2.

**Proof of Theorem 1.2.** Let $0 < \delta \ll \varepsilon \ll 1$ and $T > 0$ be given. From Theorem 5.1 we have two global solution $v_1, v_2$ with initial data $\phi_1, \phi_2$, respectively, such that

$$\|\phi_1\|_{H^s}, \|\phi_2\|_{H^s} \lesssim \varepsilon,$$  \hspace{1cm} (21)

$$\|\phi_1 - \phi_2\|_{H^s} \lesssim \delta,$$  \hspace{1cm} (22)

$$\sup_{0 \leq t \leq T} \|v_1(t) - v_2(t)\|_{H^s} \gtrsim \varepsilon,$$  \hspace{1cm} (23)

$$\sup_{0 \leq t \leq \infty} \|v_1(t)\|_{H^s}, \|v_2(t)\|_{H^s} \lesssim \varepsilon.$$  \hspace{1cm} (24)

Define $V_1, V_2$ by

$$V_j(t, x) := e^{iN\lambda t} e^{iN^\alpha t} v_j(s, y), \quad j = 1, 2,$$  \hspace{1cm} (25)

where $(s, y)$ is given by (18). And let $u_1, u_2$ be smooth global solutions of (1) with initial data $V_1(0, x), V_2(0, x)$, respectively.

Now we rescale these solutions to have the conditions (5), (6) and (7) satisfied. Let $\lambda \gg 1$ be a large parameter to be chosen later. For $j = 1, 2$, set

$$u_j := \lambda u_j(\lambda^\alpha t, \lambda x), \quad V_j := \lambda V_j(\lambda^\alpha t, \lambda x).$$

Thus we have

$$u_j(0, x) = V_j(0, x) = \lambda e^{iN\lambda x} e^{iN^\alpha t} v_j(0, \frac{\lambda x + \alpha N^\alpha t}{\alpha(N^\alpha - 1)}).$$

Lemma 5.4 with $M = N\lambda, \tau = N^{\frac{\alpha}{2}} \lambda^{-1}$ implies that if $s \geq 0$,

$$\|u_j^\lambda(0)\|_{H^s} \lesssim \lambda^{s+1/2} N^{s-(2\alpha)/4} \|v_j(0)\|_{H^s}.\$$

If $\frac{2-3\alpha}{4(\alpha+1)} < s < 0$,

$$\|u_j^\lambda(0)\|_{H^s} \lesssim \lambda^{s+1/2} N^{s-(2\alpha)/4} \|v_j(0)\|_{H^s}.\$$

We choose $\lambda = N^{(2-\alpha)/4-s}/(s+1/2)$. By (21) and (22) we have

$$\|u_j^\lambda(0)\|_{H^s} \lesssim \varepsilon, \|u_1(0) - u_2(0)\|_{H^s} \lesssim \delta.$$

Now we show (7). Rescaling gives

$$\|u_j^\lambda(t) - V_j^\lambda(t)\|_{H^s} \lesssim \lambda^{\max(s,0)+1/2} \|u_j(\lambda^\alpha t) - V_j(\lambda^\alpha t)\|_{H^s} \lesssim \lambda^{\max(s,0)+1/2} \|u_j(\lambda^\alpha t) - V_j(\lambda^\alpha t)\|_{H(2-\alpha)/4}.$$\hspace{1cm} (26)

Lemma 5.2 and induction argument on time interval up to $\log N/\lambda^\alpha$ yield

$$\|u_j(\lambda^\alpha t) - V_j(\lambda^\alpha t)\|_{H(2-\alpha)/4} \lesssim \varepsilon N^{-\alpha/2+\eta},$$

whenever $0 < t \ll \log N/\lambda^\alpha$. Hence we have

$$\|u_j^\lambda(t) - V_j^\lambda(t)\|_{H^s} \lesssim \lambda^{\max(s,0)+1/2} \varepsilon N^{-\alpha/2+\eta}.$$\hspace{1cm} (27)

From the hypothesis $\frac{2-3\alpha}{4(\alpha+1)} < s$ it follows that, for a sufficiently small $\eta > 0$,

$$\|u_j^\lambda(t) - V_j^\lambda(t)\|_{H^s} \ll \varepsilon.$$
Applying Lemma 5.4 with $M = N\lambda, \tau = N^{\frac{\alpha}{2} - \frac{3}{2}}\lambda^{-1}$, we have

$$\|u_{\lambda j}(t)\|_{H^s} \leq \|u_{\lambda j}(t) - V_{\lambda j}(t)\|_{H^s} + \|V_{\lambda j}(t)\|_{H^s} \lesssim \varepsilon + \|v_j(\lambda^\alpha t)\|_{H^s} \lesssim \varepsilon.$$  

From (23), we can find a time $t_0 > 0$ such that $\|v_1(t_0) - v(t_0)\|_{L^2} \gtrsim \varepsilon$. Fixing $t_0$, we may choose $N$ so large that $t_0 \ll \log N$. From (24) and Lemma 5.4, we get

$$\|V_1(t_0/\lambda^\alpha) - V_2(t_0/\lambda^\alpha)\|_{H^s} \sim \varepsilon.$$  

Choosing $N$ large enough, we can make $t_0/\lambda^\alpha < T$. Therefore (7) follows.

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