RECURRENCEx EXTRACTION AND DEnOTATIONAL SEMANTICS
WITH RECURSIVE FUNCTION DEFINITIONS

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Abstract. With one exception, our previous work on recurrence extraction and denotational semantics has focused on a source language that supports inductive types and structural recursion. The exception handles general recursion via an initial translation into call-by-push-value. In this note we give an extraction function from a language with general recursive function definitions and recursive types directly to a PCF-like recurrence language. We prove the main soundness result (that the syntactic recurrences in fact bound the operational cost) without the use of a logical relation, thereby significantly simplifying the proof compared to our previous work (at the cost of placing more demands on the models of the recurrence language). We then define two models of the recurrence language, one for analyzing merge sort, and another for analyzing quick sort, as case studies to understand model definitions for justifying the extracted recurrences.

We will not go over the fundamentals of the project of which this note is part, and direct the reader to Danner and Licata [2022] for a thorough discussion. The key point is that we use denotational semantics to give a foundation for the process most of us teach for analyzing program cost: extracting a recurrence that describes the cost of the program in terms of the size of the input. We factor this into two phases: a syntactic extraction from the program source language into a language of recurrences, and then an interpretation of the extracted recurrence into a model. Most of our previous work [Danner et al., 2015, Cutler et al., 2020, Danner and Licata, 2022] uses a source language with inductive types and a structural recursion operator in order to focus on the main ideas. Supporting general recursive function definitions raises (at least) two issues. The first is that the result that connects the bound described by the extracted syntactic recurrence must take into account non-termination, and a necessary condition is that if the extracted recurrence describes a finite bound on the cost, then the original program must terminate. The second is that since the recurrence language is now some flavor of PCF, its models must be defined with two orders simultaneously, one of which satisfies the size order axioms, and the other of which corresponds to an information order, and those two orders must interact. Kavvos et al. [2020] provide one approach to handling these issues. On the syntactic side, the source language from which recurrences are extracted is actually call-by-push-value (CBPV) [Levy, 2003]. Standard translations of call-by-value and call-by-name into CBPV yield recurrence extraction functions for the intended languages. However, the current author finds some of the details of the analysis of the syntactic recurrence language somewhat unsatisfying. For example, there is a precise operational semantics that must be respected in order to discuss non-termination, and that seems somewhat counter to viewing the language as a substitute for ordinary mathematical reasoning. On the semantic side, the models are not investigated in detail, and it behooves us to ensure that models such as those in Danner and Licata [2022] carry over into a framework that supports general recursion.
The main point of this note is to ensure that general recursion can be handled smoothly, and that we can in fact define models that justify informally extracted recurrences, something we have asserted can be done in our prior papers. This is not a deep dive, and there are certainly details that have exposed lacunae in my background knowledge that need to be filled so that they can be addressed. I only occasionally point out relevant results from the literature, and certainly haven’t developed a thorough review of related work.

1. Source and recurrence languages

1.1. The source language. The source language is given in Figures 1-4. Type variables $\alpha$ are only used for recursive types $\mu \alpha. \sigma$; typing only assigns closed types to terms. In the presence of recursive types, recursive functions can be defined without an explicit term former (see, e.g., Pierce [2002, Ch. 20.1]). We include one anyway, because we will ultimately be concerned with models (of the recurrence language). To keep the models somewhat simple, we will restrict our attention to recursive types in which the body is a polynomial over $\alpha$, and such types are not sufficient to implement a fixpoint combinator. However, it turns out that proving the main bounding theorem is not difficult in the presence of recursive types, so we include them because we can.

The evaluation judgment $e \downarrow^c v$ is annotated with a cost $c$ that indicates resource usage, and which is incremented by a programmer-supplied tick operator $\check{e}$. Following Hoffmann et al. [2012], we define both a standard big-step operational semantics along with an incomplete operational semantics that derives judgments of the form $e \downarrow^c$, indicating that $e$ has an incomplete or intermediate evaluation with cost $c$. Ignoring the annotations, using incomplete evaluation is a way of incorporating some of the value of small-step semantics into a big-step setting. It lets us talk about “reducing” an expression without requiring that we end up with a value. This is useful if we want to be able to refer to some part of the evaluation, but we do not care about the result when the evaluation has not yet resulted in a value. That is the setting we are in for this work, because if we have not gotten a value yet, then we only care about the cost of the evaluation so far. Once we get a value, we care about what it is.

We first make some expected observations about evaluation.

Proposition 1.1. For any value $v$, $v \downarrow^0 v$. 

Figure 1: Source language grammar.

\[
\sigma, \tau ::= \alpha \mid \text{unit} \mid \sigma + \sigma \mid \sigma \times \sigma \mid \sigma \rightarrow \sigma \mid \mu \alpha. \sigma
\]

\[
e ::= x \mid () \mid \text{in}_0 e \mid \text{in}_1 e \mid \text{case } e \text{ of } \text{in}_0 x \Rightarrow e \mid \text{in}_1 x \Rightarrow e \mid (e, e) \mid \text{let } (x, y) = e \Rightarrow e \mid \text{in}_0 x \Rightarrow e \mid \text{in}_1 x \Rightarrow e \mid (e, e) \mid \text{let } (x, y) = e \Rightarrow e \mid \text{fun } f x = e \mid \text{fold } e \mid \text{unfold } e
\]

\[
v ::= () \mid \text{in}_i v \mid (v, v) \mid \text{fun } f x = e \mid \text{fold } v
\]
Figure 2: Source language typing.

\[
\begin{array}{ll}
\gamma, x: \sigma \vdash x: \sigma & \gamma \vdash \text{unit} \\
\gamma \vdash e: \sigma_1 & \gamma \vdash \text{case}\ e \text{ of } in_0 x \Rightarrow e_0 \mid \text{in}_1 x \Rightarrow e_1: \sigma \\
\gamma \vdash \text{in}\ e: \sigma_0 + \sigma_1 & \gamma \vdash \text{let}(x_0, x_1) = e'\ \text{in} e: \sigma \\
\gamma \vdash \text{fun}\ f x = e: \sigma \to \tau & \gamma \vdash e_0: \sigma \to \tau & \gamma \vdash e_1: \tau \\
\gamma \vdash e: \sigma \times \sigma & \gamma \vdash \text{let}(x_0, x_1) = e'\ \text{in} e: \sigma \\
\gamma \vdash e: [\mu \alpha. \sigma/\alpha] & \gamma \vdash e: [\mu \alpha. \sigma/\alpha] \\
\gamma \vdash \text{fold} e: \mu \alpha. \sigma & \gamma \vdash \text{unfold} e: \sigma [\mu \alpha. \sigma/\alpha] \\
\end{array}
\]

Figure 3: Source language evaluation.
\[
\begin{align*}
\text{\textbf{Proposition 1.2.}} \quad & \text{For every closed } e, \text{ there is at most one } c \text{ and } v \text{ such that } e \downarrow^c v. \\
\text{\textbf{Proposition 1.3.}} \quad & \text{If } e \downarrow^c v \text{ and } e \downarrow^{c'} \text{, then } c' \leq c.
\end{align*}
\]

1.2. The recurrence language. The recurrence language is the same as the source language, except that \texttt{fun } f x = e \text{ is replaced with } \texttt{fix } x.e, \text{ a cost type } \mathbb{C} \text{ is added, and a monadic type constructor } \mathbb{C}. \text{ is used for complexity expressions. The grammar is}

The monadic type constructor for complexities is a change from previous work, where a complexity type was (transparently) \( \mathbb{C} \times \cdot \). I would like a nicer name for the bind construct for the monad, but I haven’t been able to think of one. In fact, I’m not really even sure \texttt{val} is the right name for the return.
\( \sigma, \tau \ ::= \alpha \mid \mathbf{C} \mid \text{unit} \mid \sigma + \sigma \mid \sigma \times \sigma \mid \sigma \to \sigma \mid \mu \alpha. \sigma \mid \mathbf{C} \sigma \)

e \ ::= \ x \mid 0 \mid 1 \mid e + e
\mid \ () \mid \text{in}_0 \ e \mid \text{in}_1 \ e \mid \text{case} \ e \ \text{of} \ \text{in}_0 \ x \Rightarrow e \mid \text{in}_1 \ x \Rightarrow e \mid (e, e) \mid \text{let} \ ((x, y)) = e \text{ in} \ e
\mid \lambda x. e \mid e \ e \mid \text{fix} \ x. e \mid \text{fold} \ e \mid \text{unfold} \ e
\mid \text{val} \ e \mid \text{bind} \ (x_0, \ldots, x_{n-1}) \leftarrow (e_0, \ldots, e_{n-1}) \text{ in} \ e \mid \text{incr} \ e \mid e_c \mid e_p

Figure 5: Recurrence language grammar.

given in Figure 5 and typing rules in Figure 6. As expected, \( \mathbf{C} \cdot \) has an incr operation corresponding to increasing cost by 1. Informally a complexity (value of type \( \mathbf{C} \sigma \)) consists of a cost and potential, and so we have projections \( (\cdot)_c \) and \( (\cdot)_p \) from \( \mathbf{C} \sigma \) to the complexity type \( \mathbf{C} \times \sigma \). This makes \( \mathbf{C} \sigma \) look a lot like \( \mathbf{C} \times \sigma \), but because \( \mathbf{C} \sigma \) can hide additional “state” about the cost that is not reflected in the cost projection, the first projection on \( \mathbf{C} \times \sigma \) cannot be uniquely factored through it. For example, \( \mathbf{C} \sigma \) could be interpreted as \( (\mathbb{N} \times \mathbb{N}) \times \sigma \) with \( (a, b), x_c = a + b \). The first projection on \( \mathbf{C} \times \sigma \) could then be factored as either \( (a, x) \mapsto ((0, 0), x) \mapsto a \) or \( (a, x) \mapsto ((0, a), x) \mapsto a \). val is the return operation of the monad, and its intended interpretation of \( \text{val} \ e \) is a 0-cost complexity from \( e \). The binding operation binds a tuple of complexities and provides their potentials to the body; the intended semantics is that the cost of the complexities is added onto that of the body. When the tuple has length 1, we drop the parentheses.

Using the monadic type primarily just buys us some cleaner notation for the recurrence extraction function. It also lets us carve out exactly where we want the features of negative products in terms of being able to project exact cost and potential from complexity expressions while interpreting products positively, which gives us finer-grained control of the semantics (see the discussion of quick sort in Section 4).

The size (pre)order is defined in Figure 7. Formally the inference rules presuppose a type judgment, so that, for example, \( (\text{refl}) \) and \( (\text{trans}) \) are really

\[
\frac{\gamma \vdash e : \sigma}{\gamma \vdash e \leq e : \sigma}
\quad \frac{\gamma \vdash e \leq e' : \sigma}{\gamma \vdash e \leq e'' : \sigma}
\]

and so forth. \( \gamma \] ranges over contexts for which we need to ensure monotonicity of \( \leq \). We define \( e = e' \) to mean \( e \leq e' \) and \( e' \leq e \) and observe that \( = \) also satisfies \( (\text{refl}) \), \( (\text{trans}) \), and \( (\text{mon}) \).

1.3. **Recurrence extraction and soundness.** Recurrence extraction is defined in Figure 8. We define an extraction for both values and arbitrary expressions in order to prove an appropriate soundness theorem.

**Lemma 1.4** (Value extraction). For all values \( v \), \( \| v \| = \text{val} \langle\langle v\rangle\rangle \).

**Proof.** The proof is by induction on \( v \); we’ll just do a few cases.

**Case:** \( () \). \( : \| () \| = \text{val} () = \text{val} \langle\langle ()\rangle\rangle \).
\begin{align*}
\gamma, x : \sigma & \vdash x : \sigma & \gamma \vdash () : \text{unit} \\
\gamma \vdash 0 : C & \quad \gamma \vdash 1 : C & \gamma \vdash e_0 : C & \quad \gamma \vdash e_1 : C & \gamma \vdash e_0 + e_1 : C \\
\gamma \vdash e : \sigma_i & \quad \gamma \vdash e : \sigma_0 + \sigma_1 & \gamma \vdash \text{in}_i e : \sigma_0 + \sigma_1 & \quad \gamma \vdash \{ \gamma, x : \sigma_i \vdash e_i : \sigma \}_{i=0,1} & \gamma \vdash \text{case } e \text{ of } \text{in}_0 \text{ x } \Rightarrow e_0 \mid \text{in}_1 \text{ x } \Rightarrow e_1 : \sigma \\
\gamma \vdash (e_0, e_1) : \sigma_0 \times \sigma_1 & \quad \gamma \vdash (e_0, e_1) : \sigma_0 \times \sigma_1 & \gamma \vdash e' : \sigma_0 \times \sigma_1 & \quad \gamma, x_0 : \sigma_0, x_1 : \sigma_1 \vdash e : \sigma & \gamma \vdash \text{let } (x_0, x_1) = e' \text{ in } e : \sigma \\
\gamma, x : \sigma \vdash e : \tau & \quad \gamma \vdash e_0 : \sigma \rightarrow \tau & \quad \gamma \vdash e_1 : \sigma & \quad \gamma \vdash e_0 \, e_1 : \tau \\
\gamma \vdash \mu a. \sigma & \quad \gamma \vdash \text{fold } e : \mu a. \sigma & \gamma \vdash e : \mu a. \sigma & \gamma \vdash \text{unfold } e : \sigma[ju.a.\sigma/o] \\
\gamma, x : \sigma \vdash e : \sigma & \gamma \vdash e : \sigma & \gamma \vdash e : \sigma & \gamma \vdash e_0 : C \sigma & \gamma \vdash \text{val } e : C \sigma \\
\gamma \vdash e_0 : C \tau_0 \ldots \gamma \vdash e_{n-1} : C \tau_{n-1} & \quad \gamma, x_0 : \tau_0, \ldots, x_{n-1} : \tau_{n-1} \vdash e : C \sigma & \gamma \vdash \text{bind } (x_0, \ldots, x_{n-1}) \leftarrow (e_0, \ldots, e_{n-1}) \text{ in } e : C \sigma & \gamma \vdash e : C \sigma & \gamma \vdash \text{in } e : C \sigma \\
\gamma \vdash e : C \sigma & \quad \gamma \vdash e : C \sigma & \quad \gamma \vdash e : C \sigma & \gamma \vdash e : C \sigma & \gamma \vdash e_p : \sigma
\end{align*}

Figure 6: Recurrence language typing.

CASE: \((v_0, v_1)\).
\[
\| (v_0, v_1) \| = \text{bind } p_0, p_1 \leftarrow \| v_0 \|, \| v_1 \| \text{ in } \text{val } (p_0, p_1) \\
= \text{bind } p_0, p_1 \leftarrow \text{val } \langle v_0 \rangle, \text{val } \langle v_1 \rangle \text{ in } \text{val } (p_0, p_1) \\
= \text{val } (\langle v_0 \rangle, \langle v_1 \rangle) \\
= \text{val } \langle \langle v_0 \rangle, \langle v_1 \rangle \rangle.
\]
\[
\begin{align*}
C[] &::= [] | C[] + e | e + C[] | (C[])_c | (C[])_p
| \text{in}_i C[] | \text{case} C[] \text{of} \{ \text{in}_i x_i \Rightarrow e_i \}_{i=0, 1} | (C[], e) | (e, C[]) | \text{let} (x_0, x_1) = C[] \text{ in } e \\
&| C[] e | e C[] | \text{fold} C[] | \text{unfold} C[]
\end{align*}
\]

\[
\begin{align*}
e \leq e & \quad \text{(refl)} & e \leq e' & \leq e'' & \quad \text{(trans)} & e \leq e' & \quad \text{(mon)} \\
0 \leq e & \quad \text{(zero)} & 0 + e = e & \quad \text{(+idl)} & e + 0 = e & \quad \text{(+idr)} & (e + e') + e'' = e + (e' + e'') & \quad \text{(+assoc)}
\end{align*}
\]

- \text{bind} p \leftarrow \text{val} e' \text{ in } e = e'[/p]
- \text{(bind-val)}
- \text{(val-cost)} \quad \text{(val-pot)}

- \text{bind} (x_0, \ldots, x_{n-1}) \leftarrow (e_0, \ldots, e_{n-1}) \text{ in } e_c
- \quad \text{(bind-cost)}

- \text{let} (x_0, x_1) = (e_0, e_1) \text{ in } e
- \quad \text{(bind-pot)}

- e_c + 1 = (\text{incr } e)_c
- \quad \text{(incr-cost)}

- e_p = (\text{incr } e)_p
- \quad \text{(incr-pot)}

- e_i[e/x] \leq \text{case in}_i e \text{ of } \text{in}_0 x \Rightarrow e_0 | \text{in}_1 x \Rightarrow e_1
- \quad \text{(\(\beta_+\))}

- e[e_0, e_1/x_0, x_1] \leq \text{let} (x_0, x_1) = (e_0, e_1) \text{ in } e
- \quad \text{(\(\beta_x\))}

- e[e_1/x] \leq (\lambda x.e) e_1
- \quad \text{(\(\beta\rightarrow\))}

- \quad e \leq \text{unfold } (\text{fold } e)
- \quad \text{(\(\beta\text{fold}\))}

- e[\text{fix } x.e/x] \leq \text{fix } x.e
- \quad \text{(\(\beta\text{fix}\))}

Figure 7: The size (pre)order. Also see Section 1.5 and Figure 9.

\textbf{Case: fold } v.

\[
\|\text{fold } v\| = \text{bind } p \leftarrow \|v\| \text{ in } \text{fold } p
= \text{bind } p \leftarrow \text{val } \langle v \rangle \text{ in } \text{fold } p
= \text{fold } \langle v \rangle
= \langle \text{fold } v \rangle.
\]
\[ \|\sigma\| = C\langle\sigma\rangle \]
\[ \langle\text{unit}\rangle = \text{unit} \]
\[ \langle\sigma_0 + \sigma_1\rangle = \langle\sigma_0\rangle \times \langle\sigma_1\rangle \]
\[ \langle\sigma_0 \times \sigma_1\rangle = \langle\sigma_0\rangle \times \langle\sigma_1\rangle \]
\[ \langle\sigma \rightarrow \tau\rangle = \langle\sigma\rangle \rightarrow \|\tau\| \]
\[ \langle\mu\alpha.\sigma\rangle = \mu\alpha.\langle\sigma\rangle \]
\[ \|x\| = \text{val}\,x \]
\[ \|()\| = \text{val}(\) \]
\[ \|\text{in}_i.e\| = \text{bind}\,p \leftarrow \|e\|\in\text{val}(\text{in}_i\,p) \]
\[ \|\text{case}\,e\,\text{of}\,\text{in}_0\,x \Rightarrow e_0 \mid \text{in}_1\,x \Rightarrow e_1\| = \text{bind}\,p \leftarrow \|e\|\text{in case}\,p\,\text{of}\,\text{in}_0\,x \Rightarrow \|e_0\| \mid \text{in}_1\,x \Rightarrow \|e_1\| \]
\[ \|\text{let}(x_0, x_1) = e'\,\text{in}\,e\| = \text{bind}\,p \leftarrow \|e'\|\text{in let}(x_0, x_1) = p\in\|e\| \]
\[ \|\text{fun}\,f\,x = e\| = \text{val}(\text{fix}\,f.(\lambda x.\|e\|)) \]
\[ \|\text{fold}\,e\| = \text{bind}\,p \leftarrow \|e\|\in\text{val}(\text{fold}\,p) \]
\[ \|\text{unfold}\,e\| = \text{bind}\,p \leftarrow \|e\|\in\text{val}(\text{unfold}\,p) \]
\[ \|\cdot\,e\| = \text{incr}\,\|e\| \]

\[ \langle()\rangle = () \]
\[ \langle\text{in}_i\,v\rangle = \text{in}_i\langle v\rangle \]
\[ \langle(v_0, v_1)\rangle = (\langle v_0\rangle, \langle v_1\rangle) \]
\[ \langle\text{fun}\,f\,x = e\rangle = \text{fix}\,f.(\lambda x.\|e\|) \]
\[ \langle\text{fold}\,v\rangle = \text{fold}\,\langle v\rangle \]

Figure 8: Recurrence extraction.

**Lemma 1.5 (Substitution).** For all expressions \(e\) and values \(\vec{v}\), \(\|e[\vec{v}/\vec{x}]\| = \|e[\langle\vec{v}\rangle]/\vec{x}\|\), where we write \(\langle v_0, \ldots, v_{n-1}\rangle\) for \(\langle v_0\rangle, \ldots, \langle v_{n-1}\rangle\).

**Proof.** The proof is by induction on \(e\); we’ll just do a few cases.

**Case:** \(x_i\).
\[ \|x_i[\vec{v}/\vec{x}]\| = \|v_i\| = \text{val}\,\langle v_i\rangle = (\text{val}\,x_i)[\langle\vec{v}\rangle]/\vec{x} = \|x_i\|[\langle\vec{v}\rangle]/\vec{x}\].
Case: let (y, z) = e' in e.
\[ \| \text{let} (y, z) = e' \in e \| = \| \text{let} (y, z) = e' [\vec{v}/\vec{x}] \in e [\vec{v}/\vec{x}] \| \]
\[ = \text{bind } p \leftarrow \| e' [\vec{v}/\vec{x}] \| \text{ in let } (y, z) = p \in \| e [\vec{v}/\vec{x}] \| \]
\[ = \text{bind } p \leftarrow \| e' \| \text{ in let } (y, z) = p \in \| e [\langle \vec{v} \rangle/\vec{x}] \| \]
\[ = (\text{bind } p \leftarrow \| e' \| \text{ in let } (y, z) = p \in \| e [\langle \vec{v} \rangle/\vec{x}] \| \]
\[ = \| \text{let} (y, z) = e' \| \langle \vec{v} \rangle/\vec{x} \| \]
\[ = (\text{bind } p \leftarrow \| e' \| \text{ in let } (y, z) = p \in \| e [\langle \vec{v} \rangle/\vec{x}] \| \]
\[ = \| \text{let} (y, z) = e' \| \langle \vec{v} \rangle/\vec{x} \|. \]

As one might expect, the Soundness theorem tells us that the cost projection of an extracted recurrence bounds the actual cost, and the potential projection bounds the value when there is one. In the past we’ve used a logical relation between source and recurrence language programs, but Dan Licata observed that Van Stone [2003] proved a comparable result in her thesis without one, so we should be able to do the same.

**Theorem 1.6 (Soundness).** For all e:

1. If \( e \Downarrow^c \), then \( c \leq \| e \|_c \).
2. If \( e \Downarrow^c \) \( v \), then \( c \leq \| e \|_c \) and \( \langle \langle v \rangle \rangle \leq \| e \|_p \).

**Proof.** We prove (1) and (2) simultaneously by induction on the height of the (incomplete) evaluation derivation, breaking into cases according to the last line of the derivation.

Case: \( (\Downarrow^0) \). \( 0 = 0 \) and \( \langle \langle (\) \rangle \rangle = (\) = \( (\text{val} (\) \rangle \rangle_p = \| (\) \|_p \).

Case: \( e \Downarrow^c \) \( \in_i e \Downarrow^c \in_i v \).

By the induction hypothesis, \( c \leq \| e \|_c \) and \( \langle \langle v \rangle \rangle \leq \| e \|_p \). Thus:

\[ \| \in_i e \|_c = (\text{bind } p \leftarrow \| e \| \text{ in val} (\text{in}_i p))_c \]
\[ = \| e \|_c + (\text{val} (\text{in}_i \| e \|_p))_c \]
\[ \geq c + 0 \]
\[ = c \]

and

\[ \| \text{in}_i e \|_p = (\text{bind } p \leftarrow \| e \| \text{ in val} (\text{in}_i p))_p \]
\[ = (\text{val} (\text{in}_i \| e \|_p))_p \]
\[ = \text{in}_i \| e \|_p \]
\[ \geq \text{in}_i \langle \langle v \rangle \rangle \]

Monotonicity uses the contexts \( C[] + 0 \) and \( \text{in}_i C[] \).
Case: \( e \downarrow^c \). By the induction hypothesis, \( c \leq \|e\|_c \). The reasoning is the same as in the cost analysis of the previous case.

Case: \( e \downarrow^c \text{ in}_i v \quad e_i[v/x] \downarrow^{c_i} v_i \). By the induction hypothesis, \( c \leq \|e\|_c \), \( \text{in}_i \langle \langle \text{in}_i v \rangle \rangle \leq \|e\|_p \), \( c_i \leq \|e_i[v/x]\|_c \), and \( \langle \langle v_i \rangle \rangle \leq \|e_i[v/x]\|_p \). Thus:

\[
\|\text{case } e \text{ of } \text{in}_0 x \Rightarrow e_0 \mid \text{in}_1 x \Rightarrow e_1\|_p \\
= (\text{bind } p \leftarrow \|e\| \text{ in case } p \text{ of } \text{in}_0 x \Rightarrow \|e_0\| \mid \text{in}_1 x \Rightarrow \|e_1\|)_p \\
= \|e\|_c + (\text{case } \|e\|_p \text{ of } \text{in}_0 x \Rightarrow \|e_0\| \mid \text{in}_1 x \Rightarrow \|e_1\|)_p \\
\geq c + (\text{case } \text{in}_i \langle \langle \langle v \rangle \rangle \text{ of } \text{in}_0 x \Rightarrow \|e_0\| \mid \text{in}_1 x \Rightarrow \|e_1\|)_p \\
\geq c + (\|e_i\| \langle \langle \langle v \rangle \rangle /x \rangle)_c \\
= c + \|e_i[v/x]\|_c \\
\geq c + c_i \quad \text{IH, (mon)}
\]

and

\[
\|\text{case } e \text{ of } \text{in}_0 x \Rightarrow e_0 \mid \text{in}_1 x \Rightarrow e_1\|_p \\
= (\text{bind } p \leftarrow \|e\| \text{ in case } p \text{ of } \text{in}_0 x \Rightarrow \|e_0\| \mid \text{in}_1 x \Rightarrow \|e_1\|)_p \\
= (\text{case } \|e\|_p \text{ of } \text{in}_0 x \Rightarrow \|e_0\| \mid \text{in}_1 x \Rightarrow \|e_1\|)_p \\
\geq (\text{case } \text{in}_i \langle \langle \langle v \rangle \rangle \text{ of } \text{in}_0 x \Rightarrow \|e_0\| \mid \text{in}_1 x \Rightarrow \|e_1\|)_p \\
\geq (\|e_i\| \langle \langle \langle v \rangle \rangle /x \rangle)_p \\
= \|e_i[v/x]\|_p \quad \text{Lemma 1.5} \\
\geq \langle \langle v \rangle \rangle \quad \text{IH.}
\]

Monotonicity uses the contexts case \( C[\] of \( \text{in}_0 x \Rightarrow \|e_0\| \mid \text{in}_1 x \Rightarrow \|e_1\| \), \( (\text{C[}])_c \), \( (\text{C[}])_p \), and \( c + C[\] 

Case: \( e \downarrow^c \text{ in}_i v \). By the induction hypothesis, \( c \leq \|e\|_c \). Thus:

\[
\|\text{case } e \text{ of } \text{in}_0 x \Rightarrow e_0 \mid \text{in}_1 x \Rightarrow e_1\|_c \\
= (\text{bind } p \leftarrow \|e\| \text{ in case } p \text{ of } \text{in}_0 x \Rightarrow \|e_0\| \mid \text{in}_1 x \Rightarrow \|e_1\|)_c \\
= \|e\|_c + (\text{case } \|e\|_p \text{ of } \text{in}_0 x \Rightarrow \|e_0\| \mid \text{in}_1 x \Rightarrow \|e_1\|)_c \\
\geq \|e\|_c + 0 \\
\geq \|e\|_c.
\]

Case: \( e \downarrow^c \text{ in}_i v \quad e_i[v/x] \downarrow^{c_i} \). The argument is the same as the cost argument for the complete evaluation.
Case: \( \frac{\{e_i \downarrow^c v_i\}_{i=0}^1}{(e_0, e_1) \downarrow^{c_0+c_1} (v_0, v_1)} \). By the induction hypothesis, \( c_i \leq \|e_i\|_c \) and \( \langle\langle v_i \rangle\rangle \leq \|e_i\|_p \).

Thus:

\[ \| (e_0, e_1) \|_c = (\text{bind } p_0, p_1 \leftarrow \|e_0\|, \|e_1\| \text{ in } \text{val } (p_0, p_1) ) \]
\[ = \|e_0\|_c + \|e_1\|_c + (\text{val } (\|e_0\|_p, \|e_1\|_p))_c \]
\[ \geq c_0 + c_1 + 0 \]
\[ = c_0 + c_1 \]

and

\[ \| (e_0, e_1) \|_p = (\text{bind } p_0, p_1 \leftarrow \|e_0\|, \|e_1\| \text{ in } \text{val } (p_0, p_1) )_p \]
\[ = (\text{val } (\|e_0\|_p, \|e_1\|_p))_p \]
\[ = (\|e_0\|_p, \|e_1\|_p) \]
\[ \geq (\langle\langle v_0 \rangle\rangle, \langle\langle v_1 \rangle\rangle) \]
\[ = \langle\langle (v_0, v_1) \rangle\rangle. \]

Case: \( \frac{e_0 \downarrow^{c_0} e_1 \downarrow^{c_1}}{(e_0, e_1) \downarrow^{c_0+c_1}} \). By the induction hypothesis, \( c_0 \leq \|e_0\|_c \). Thus:

\[ \| (e_0, e_1) \|_c = (\text{bind } p_0, p_1 \leftarrow \|e_0\|, \|e_1\| \text{ in } \text{val } (p_0, p_1) ) \]
\[ = \|e_0\|_c + \|e_1\|_c + (\text{val } (\|e_0\|_p, \|e_1\|_p))_c \]
\[ \geq c_0 + 0 + 0 \]
\[ = c_0. \]

Case: \( \frac{e_0 \downarrow^{c_0} v_0 \quad e_1 \downarrow^{c_1}}{(e_0, e_1) \downarrow^{c_0+c_1}} \). By the induction hypothesis, \( c_0 \leq \|e_0\|_c \) and \( c_1 \leq \|e_1\|_c \).

Thus:

\[ \| (e_0, e_1) \|_c = (\text{bind } p_0, p_1 \leftarrow \|e_0\|, \|e_1\| \text{ in } \text{val } (p_0, p_1) ) \]
\[ = \|e_0\|_c + \|e_1\|_c + (\text{val } (\|e_0\|_p, \|e_1\|_p))_c \]
\[ \geq c_0 + c_1 + 0 \]
\[ = c_0 + c_1. \]

Case: \( \frac{e' \downarrow^{c'} (v_0, v_1) \quad e[v_0, v_1/x_0, x_1] \downarrow^{c} v}{\text{let } (x_0, x_1) = e' \text{ in } e \downarrow^{c'+c} v} \). By the induction hypothesis \( c' \leq \|e'\|_c \), \( (\langle\langle v_0 \rangle\rangle, \langle\langle v_1 \rangle\rangle) = \langle\langle (v_0, v_1) \rangle\rangle \leq \|e'\|_p, c \leq \|e[v_0, v_1/x_0, x_1]\|_c \), and \( \langle\langle v \rangle\rangle \leq \|e[v_0, v_1/x_0, x_1]\|_p \). Thus:

\[ \|\text{let } (x_0, x_1) = e' \text{ in } e \|_c = (\text{bind } p \leftarrow \|e'\| \text{ in } \text{let } (x_0, x_1) = p \text{ in } \|e\|) \]
\[ = \|e'\|_c + (\text{let } (x_0, x_1) = \|e'\|_p \text{ in } \|e\|)_c \]
\[ \geq c' + (\|e[v_0, v_1/x_0, x_1]\|)_c \]
\[ \geq c' + (\|e[v_0, v_1/x_0, x_1]\|)_{\langle\langle v_0 \rangle\rangle, \langle\langle v_1 \rangle\rangle} \text{ in } \|e\|}_c \]
\[ = c' + (\|e[v_0, v_1/x_0, x_1]\|)_c \]
\[ \geq c' + c \]

Lemma 1.5

IH, (mon)
Thus:

\[
\| \text{let} (x_0, x_1) = e' \text{ in } e \|_p = (\text{bind } p \leftarrow \| e' \| \text{ in } \text{let} (x_0, x_1) = p \text{ in } \| e \|)_p
\]

\[
= (\text{let} (x_0, x_1) = \| e' \|_p \text{ in } \| e \|)_p
\]

\[
\geq (\text{let} (x_0, x_1) = (\langle \langle v_0 \rangle \rangle, \langle \langle v_1 \rangle \rangle) \text{ in } \| e \|)_p
\]

\[
\geq (\| e \|[[\langle \langle v_0 \rangle \rangle, \langle \langle v_1 \rangle \rangle/x_0, x_1]])_p
\]

\[
= (\| e[v_0, v_1/x_0, x_1] \|)_p
\]

\[
\geq \langle \langle v \rangle \rangle
\]

Monotonicity uses the contexts $C[] + e$, $e + C[]$, $(C[])_c$, $(C[])_p$, and $\text{let} (x_0, x_1) = C[] \text{ in } e$.

**CASE:** \( e_0 \downarrow^0 \text{fun } f x = e \downarrow^0 \text{fun } f x = e \)

The cost bound is immediate, and

\[
\| \text{fun } f x = e \|_p = (\text{val}(\text{fix } f.\lambda x.\| e \|))_p
\]

\[
= \text{fix } f.\lambda x.\| e \|
\]

\[
= \langle \langle \text{fun } f x = e \rangle \rangle.
\]

**CASE:** \( e_0 \downarrow^0 \text{fun } f x = e_0' \quad e_1 \downarrow^0 v_1 \quad e_0'[\text{fun } f x = e_0', v_1/f, x] \downarrow^c v \)

By the induction hypothesis,

\[
\begin{align*}
    e_0 & \leq \| e_0 \|_c \\
    e_1 & \leq \| e_1 \|_c
\end{align*}
\]

\[
\text{fix } f.\lambda x.\| e_0' \| = \langle \langle \text{fun } f x = e_0' \rangle \rangle \leq \| e_0' \|_p
\]

\[
\langle \langle v \rangle \rangle \leq \| e_0' [\text{fun } f x = e_0', v_1/f, x] \|_p = \| e_0' \|_p [\text{fix } f.\lambda x.\| e_0' \|, \langle \langle v_1 \rangle \rangle/f, x] \]
\]

Thus:

\[
\| e_0 e_1 \|_c = (\text{bind } p_0, p_1 \leftarrow \| e_0 \|, \| e_1 \| \text{ in } p_0 p_1)_c
\]

\[
= \| e_0 \|_c + \| e_1 \|_c + (\| e_0 \|_p \| e_1 \|_p)_c
\]

\[
\geq c_0 + c_1 + \left( (\text{fix } f.\lambda x.\| e_0' \|) \langle \langle v_1 \rangle \rangle \right)_c
\]

\[
\geq c_0 + c_1 + \left( \| e_0' \| [\text{fix } f.\lambda x.\| e_0' \|, \langle \langle v_1 \rangle \rangle/f, x] \right)_c
\]

\[
\geq c_0 + c_1 + c
\]

and

\[
\| e_0 e_1 \|_p = (\text{bind } p_0, p_1 \leftarrow \| e_0 \|, \| e_1 \| \text{ in } p_0 p_1)_p
\]

\[
= (\| e_0 \|_p \| e_1 \|_p)_p
\]

\[
\geq \left( (\text{fix } f.\lambda x.\| e_0' \|) \langle \langle v_1 \rangle \rangle \right)_p
\]

\[
\geq \left( \| e_0' \| [\text{fix } f.\lambda x.\| e_0' \|, \langle \langle v_1 \rangle \rangle/f, x] \right)_p
\]

\[
\geq \langle \langle v \rangle \rangle.
\]
Case: $e_0 \downarrow^{c_0} e_0 e_1 \downarrow^{c_0}$. By the induction hypothesis, $c_0 \leq \|e_0\|_c$. Thus:

$$\|e_0 e_1\|_c = (\text{bind } p_0, p_1 \leftarrow \|e_0\|, \|e_1\| \text{ in } p_0 p_1)_c$$

$$= \|e_0\|_c + \|e_1\|_c + (\|e_0\|_p \|e_1\|_p)_c$$

$$\geq c_0 + 0 + 0$$

$$\geq c_0.$$

Case: $e_0 \downarrow^{c_0} \text{fun } f x = e_0 \downarrow^{c_0} e_0 e_1 \downarrow^{c_0 + c_1}$. By the induction hypothesis, $c_0 \leq \|e_0\|_c$ and $c_1 \leq \|e_1\|_c$. Thus:

$$\|e_0 e_1\|_c = (\text{bind } p_0, p_1 \leftarrow \|e_0\|, \|e_1\| \text{ in } p_0 p_1)_c$$

$$= \|e_0\|_c + \|e_1\|_c + (\|e_0\|_p \|e_1\|_p)_c$$

$$\geq c_0 + c_1 + 0$$

$$\geq c_0 + c_1.$$

Case: $e_0 \downarrow^{c_0} \text{fun } f x = e_0 \downarrow^{c_0} e_0 e_1 \downarrow^{c_0 + c_1} [\text{fun } f x = e_0', v_1 \text{ }/f, x] \downarrow^{c}$. The argument is the same as the cost argument for the complete evaluation.

Case: $e \downarrow^{c} v$. By the induction hypothesis, $c \leq \|e\|_c$ and $\langle\langle v \rangle\rangle \leq \|e\|_p$. Thus:

$$\|\text{fold } e\|_c = (\text{bind } p \leftarrow \|e\| \text{ in } \text{fold } p)_c$$

$$= \|e\|_c + (\text{val}(\text{fold } \|e\|_p))_c$$

$$\geq c + 0$$

$$= c$$

Case: $e \downarrow^{c} \text{fold } e \downarrow^{c}$. The argument is the same as the cost argument for the complete evaluation.

Case: $e \downarrow^{c} \text{fold } v$. By the induction hypothesis, $c \leq \|e\|_c$ and $\text{fold } \langle\langle v \rangle\rangle = \langle\langle \text{fold } v \rangle\rangle \leq \|e\|_p$. Thus:

$$\|\text{unfold } e\|_c = (\text{bind } p \leftarrow \|e\| \text{ in } \text{val}(\text{unfold } p))_c$$

$$\geq \|e\|_c + (\text{val}(\text{unfold } \|e\|_p))_c$$

$$\geq c + 0$$

$$= c$$
and

\[ \| \text{unfold } e \|_p = (\text{bind } p \leftarrow \| e \| \text{ in } \text{val}(\text{unfold } p))_p \]
\[ \geq (\text{val}(\text{unfold } \| e \|_p))_p \]
\[ = \text{unfold } \| e \|_p \]
\[ \geq \text{unfold } (\text{fold } \langle \langle v \rangle \rangle) \]
\[ \geq \langle \langle v \rangle \rangle. \]

Case: \[ \frac{e \not\downarrow c}{\text{unfold } e \not\downarrow c}. \] The argument is the same as for the cost argument for the complete evaluation.

Case: \[ \frac{e \not\downarrow c v}{\sqrt{e} \not\downarrow c+1 v}. \] By the induction hypothesis, \( c \leq \| e \|_c \) and \( \langle \langle v \rangle \rangle \leq \| e \|_p \). Thus:

\[ \| e \|_c = (\text{incr } \| e \|)_c \]
\[ \geq c + 1 \]
\[ = \| e \|_p \]
\[ \geq \langle \langle v \rangle \rangle. \]

Case: \[ \frac{e \not\downarrow c}{\sqrt{e} \not\downarrow c+1}. \] The argument is the same as for the cost argument for the complete evaluation.

1.4. Bounding relations. Our previous work establishes a Soundness-like result using a logical relation between the source and recurrence languages. We can recover that sort of relation as follows.

Definition 1.7 (Bounding relations).

1. Define \( e \leq E \) to mean that if \( e \not\downarrow c \), then \( c \leq E_c \) and if \( e \not\downarrow c, v \), then \( c \leq E_c \) and \( v \leq^\text{val} E_p \).

2. Define \( v \leq^\text{val} E \) to mean \( \langle \langle v \rangle \rangle \leq E \).

So the Soundness Theorem can be restated as saying that for all \( e, e \leq \| e \| \). The reason we care about the bounding relations is that they tell us that Soundness “propagates down by application.”

Proposition 1.8. If \( \text{fun } f \ x = e \leq^\text{val} E \) and \( v' \leq^\text{val} E' \), then \( e[\text{fun } f \ x = e, v'/f, x] \leq E \cdot E' \).

1.5. A strengthening of the recurrence language. Figure 7 provides the minimum constraints on \( \leq \) that are needed in order to prove Theorem 1.6. As a reminder, these constraints state that \( \leq \) is a preorder, the equivalence relation = induced by \( e \leq e' \leq e \) behaves nicely with respect to the monadic operations, and that the main term constructors are the source of size abstractions. But in practice our models interpret \( \leq \) as a partial order and thus = is ordinary equality. So that we can reduce clutter in our exposition, we add corresponding inference rules to the recurrence language in Figure 9. In that figure, \( T \) ranges over all term formers in Figure 5.
\[
\begin{align*}
\frac{e = e'}{(\text{eq-leq})} & \quad \frac{e \leq e' \leq e}{(\text{antisym})} & \quad \frac{e_0 = e_0'}{e_1 = e_1'} & \cdots & (\text{eq}) \\
\end{align*}
\]

Figure 9: More size order inference rules.

1.6. Lists. For our examples we will work with the standard inductive types of lists along with the standard syntactic sugar for constructors:

\[
\begin{align*}
\sigma \text{list} &= \mu\alpha.\text{unit} + \sigma \times \alpha \\
\text{nil} &= (\text{fold}_{\sigma\text{list}} \circ \text{in}_0)(()) \\
\text{cons} &= \text{fold}_{\sigma\text{list}} \circ \text{in}_1 \\
\end{align*}
\]

We also have comparable notation in the recurrence language and observe that

\[
\begin{align*}
\|\text{nil}\| &= \text{bind} p \leftarrow \|\text{in}_0(())\| \text{in} \text{val}(\text{fold } p) \\
&= \text{bind} p \leftarrow \text{bind} p' \leftarrow \|()\| \text{in} \text{val}(\text{in}_0 p') \text{in} \text{val}(\text{fold } p) \\
&= \text{bind} p \leftarrow \text{bind} p' \leftarrow \text{val } () \text{in} \text{val}(\text{in}_0 p') \text{in} \text{val}(\text{fold } p) \\
&= \text{val}(\text{fold } (\text{in}_0 ())) \\
&= \text{val } \text{nil}.
\end{align*}
\]

Unfortunately, the corresponding computation for the extraction of \text{cons} expressions is not quite so straightforward, because we need to perform simplifications in \text{bind} expressions, and we can only perform non-trivial simplifications underneath a cost- or potential projection. Thus we must perform two separate computations:

\[
\begin{align*}
\|\text{cons}(e, e')\|_c &= (\text{bind } p \leftarrow \text{bind} p' \leftarrow \text{bind } (q, q') \leftarrow \|e\|, \|e'\|) \text{in} \text{val } (q, q') \text{in} \text{val}(\text{in}_1 p') \text{in} \text{val}(\text{fold } p))_c \\
&= (\text{bind } q' \leftarrow \text{bind } (q, q') \leftarrow \|e\|, \|e'\|) \text{in} \text{val } (q, q') \text{in} \text{val}(\text{in}_1 p')_c \\
&= (\text{bind } (q, q') \leftarrow \|e\|, \|e'\|) \text{in} \text{val } (q, q')_c \\
&= \|e\|_c + \|e'\|_c
\end{align*}
\]

and

\[
\begin{align*}
\|\text{cons}(e, e')\|_p &= (\text{bind } p \leftarrow \text{bind} p' \leftarrow \text{bind } (q, q') \leftarrow \|e\|, \|e'\|) \text{in} \text{val } (q, q') \text{in} \text{val}(\text{in}_1 p') \text{in} \text{val}(\text{fold } p))_p \\
&= \text{fold } (\text{bind } p' \leftarrow \text{bind } (q, q') \leftarrow \|e\|, \|e'\|) \text{in} \text{val } (q, q') \text{in} \text{val}(\text{in}_1 p')_p \\
&= \text{fold } (\text{in}_1 (\text{bind } (q, q') \leftarrow \|e\|, \|e'\|) \text{in} \text{val } (q, q'))_p \\
&= \text{fold } (\text{in}_1 (\|e\|_P, \|e'\|_P)) \\
&= \text{cons}(\|e\|_P, \|e'\|_P).
\end{align*}
\]

We define the expression

\[
\text{case } e \text{ of } \text{nil } \Rightarrow e_{\text{nil}} \mid \text{cons}(x, xs) \Rightarrow e_{\text{cons}}
\]

to be syntactic sugar for

\[
\text{case unfold}_{\sigma\text{list}} e \text{ of } \text{in}_0 \Rightarrow e_{\text{nil}} \mid \text{in}_1 z \Rightarrow \text{let } (x, xs) = z \text{ in } e_{\text{cons}}
\]
and have similar sugar in the recurrence language. As expected, the recurrence extracted from one is written in terms of the other, but as with \texttt{cons} expressions, only underneath cost and potential projections.

\[
\begin{align*}
\| \text{case } e \text{ of nil } &\Rightarrow e_{n1} \mid \text{cons}(x, xs) \Rightarrow e_{\text{cons}} \|_c \\
&= \| \text{case unfold}_c \text{1let } e \text{ of } \text{in}_0 \_ \Rightarrow e_{n1} \mid \text{in}_1 z \Rightarrow \text{let}(x, xs) = z \text{ in } e_{\text{cons}} \|_c \\
&= (\text{bind } p \leftarrow \| \text{unfold } e \text{ in case } p \text{ of } \text{in}_0 \_ \Rightarrow \| e_{n1} \| \\
&\quad \mid \text{in}_1 z \Rightarrow \| \text{let}(x, xs) = z \text{ in } e_{\text{cons}} \|)_c \\
&= (\text{bind } p' \leftarrow \| e \| \text{ in val}(\text{unfold } p'))_c + \\
&\quad (\text{case } (\text{bind } p' \leftarrow \| e \| \text{ in val}(\text{unfold } p'))_p \text{ of } \text{in}_0 \_ \Rightarrow \| e_{n1} \| \\
&\quad \mid \text{in}_1 z \Rightarrow \| \text{let}(x, xs) = z \text{ in } e_{\text{cons}} \|)_c \\
&= \| e \|_c + (\text{val}(\text{unfold } \| e \|_p))_c + \\
&\quad (\text{case } (\text{val}(\text{unfold } \| e \|_p))_p \text{ of } \text{in}_0 \_ \Rightarrow \| e_{n1} \| \\
&\quad \mid \text{in}_1 z \Rightarrow \| \text{let}(x, xs) = z \text{ in } e_{\text{cons}} \|)_c \\
&= \| e \|_c + (\text{case unfold } \| e \|_p \text{ of } \text{in}_0 \_ \Rightarrow \| e_{n1} \| \\
&\quad \mid \text{in}_1 z \Rightarrow \| \text{let}(x, xs) = z \text{ in } e_{\text{cons}} \|)_c \\
&= \| e \|_c + (\text{case unfold } \| e \|_p \text{ of } \text{in}_0 \_ \Rightarrow \| e_{n1} \| \\
&\quad \mid \text{in}_1 z \Rightarrow \text{bind } q \leftarrow \text{val } z \text{ in } \text{let}(x, xs) = q \text{ in } \| e_{\text{cons}} \|)_c \\
&= \| e \|_c + (\text{case } \| e \|_p \text{ of } \text{nil } \Rightarrow \| e_{n1} \| \mid \text{cons}(x, xs) \Rightarrow \| e_{\text{cons}} \|)_c.
\end{align*}
\]

and a similar calculation shows that

\[
\begin{align*}
\| \text{case } e \text{ of nil } &\Rightarrow e_{n1} \mid \text{cons}(x, xs) \Rightarrow e_{\text{cons}} \|_p \\
&= (\text{case } \| e \|_p \text{ of } \text{nil } \Rightarrow \| e_{n1} \| \mid \text{cons}(x, xs) \Rightarrow \| e_{\text{cons}} \|)_p.
\end{align*}
\]

2. Models

It is the interpretation of syntactic recurrences that correspond to informally-extracted recurrences and provide actual bounds on the cost of source language programs, so let us define the notion of model for the recurrence language and see how models arise in cost analysis. For this discussion, syntax refers to the recurrence language unless explicitly stated otherwise.

A \textit{type frame} is a set \( \mathbf{A} \) of preordered sets along with a denotation function \([\cdot] : \text{Type} \to \mathbf{A} \). For a type frame \( \mathbf{A} \) and a type context \( \gamma \), a \( \gamma \)-\textit{environment} is a function \( \eta : \text{Var} \to \bigcup \mathbf{A} \) such that for all \( x \in \text{dom } \gamma \), \( \eta(x) \in [\!(\gamma(x))\!] \). We write \( \text{Env}_\gamma \) for the set
of $\gamma$-environments and $\text{Term}_\gamma$ for the set of type derivations with type context $\gamma$. A type frame is an interpretation if there is a denotation function $\llbracket \cdot \rrbracket : \text{Term}_\gamma \rightarrow \text{Env}_\gamma \rightarrow \bigcup A$ such that if $\gamma \vdash e : \sigma$ and $\eta \in \text{Env}_\gamma$, then $\llbracket e \rrbracket \eta \in \llbracket \sigma \rrbracket$. An interpretation is a model if it satisfies the size order rules of Figure 7. That is, writing the preorder on $\llbracket \sigma \rrbracket$ as $\leq^{\llbracket \sigma \rrbracket}$, if $\gamma \vdash e \leq e' : \sigma$ and $\eta \in \text{Env}_\gamma$, then $\llbracket e \rrbracket \eta \leq^{\llbracket \sigma \rrbracket} \llbracket e' \rrbracket \eta$.

Our interest in models stems from a kind of adequacy-like result for costs: we want to show that if $\llbracket \parallel e \parallel \rrbracket \leq n$ in some model, then $e \downarrow c v$ for some $c \leq n$—i.e., if the semantic recurrence yields a finite cost, then the program evaluates with no greater cost. We might be able to prove this unconditionally, but that seems to be complicated by the fact that evaluation cost is determined by placement of the $\checkmark$ operator. For now let us consider a relatively simple proof that relies on doing so in a “reasonable” way, which in turn lets us take into account the actual size of the evaluation derivation.

**Definition 2.1.** A derivation $\Pi$ of $e \downarrow^c$ is size-maximal if it is the maximum size of any derivation of $e \downarrow^{c'}$ (for any $c'$).

**Lemma 2.2.** Suppose $\Pi$ is a size-maximal derivation of $e \downarrow^c$. Then there is a derivation of $e \downarrow^c v$ for some $v$ that is no smaller than $\Pi$.

**Proof.** The proof is by induction on $\Pi$, breaking into cases according to the last rule.

Suppose $\Pi$ is $\llbracket e \rrbracket 0, 0$. If $e$ is a value, then the claim follows from Prop. 1.1. If $e$ is not a value, then there is another derivation of $e \downarrow^c$ using one of the other rules, contradicting size-maximality of $\Pi$.

The proofs for the other rules all follow the same pattern; we do the pairing rules as an example.

$\Pi_0$

**CASE:** $e_0 \downarrow^0 v_0 \quad (e_0, e_1) \downarrow^c v_0 \quad (e_0, e_1) \downarrow^c v_0$

Size-maximality of $\Pi$ implies the same of the hypothesis, so by the induction hypothesis there is a derivation $\Pi'$ of $e_0 \downarrow^c v_0$ that is no smaller than $\Pi_0$. But that means that there is a larger derivation of $(e_0, e_1) \downarrow^c v_0$ of the form

$\Pi'$

$e_0 \downarrow^0 v_0 \quad e_1 \downarrow^0 \quad (e_0, e_1) \downarrow^c v_0$

contradicting size-maximality of $\Pi$.

$\Pi_1$

**CASE:** $e_0 \downarrow^c v_0 \quad e_1 \downarrow^{c_1} \quad (e_0, e_1) \downarrow^{c_0 + c_1}$

As before, size-maximality of $\Pi$ implies that by the III there is a derivation $\Pi'$ of $e_1 \downarrow^{c_1} v_1$ that is no smaller than $\Pi_1$, and hence

Is this an appropriate use of the term “adequacy?” It is saying something like denotational behavior implies operational behavior. But it isn’t saying that denotational equivalence implies syntactic equivalence (of some kind). Note that Kavvos et al. [2020] prove this sort of result via actual adequacy for the recurrence language.
is the desired derivation.

**Proposition 2.3.** For all closed \(e\), either (1) there are \(v\) and \(c\) such that \(e \downarrow^c v\), or (2) for all \(k\) there is some \(c\) and a derivation of \(e \downarrow^c\) of size at least \(k\).

Our use of the ticking mechanism to count cost means that we cannot conclude anything about the cost of the incomplete evaluations when \(e\) does not evaluate to a value, since \(e\) could be a non-terminating computation with no ticked subexpressions at all. Of course, that is not a particularly well-ticked program.

**Definition 2.4.** Define \(e\) to be **sensibly ticked** if there are constants \(a > 0\) and \(b > 0\) such that whenever there is a size \(k\) derivation of \(e \downarrow^c\), \(ak + b \leq c\).

**Definition 2.5.** For a model \(A\), we write \(n\) for \[
\begin{array}{c}
\underbrace{1 + \cdots + 1} \\
\text{n times}
\end{array}
\]. A model \(A\) is **cost-standard** if \(\leq_{\text{C}}\) is a partial order and \(0 < 1 < 2 < \cdots\).

**Proposition 2.6** (Adequacy for costs). If \(e\) is closed and sensibly ticked and \(A\) is a cost-standard model and there is some \(n\) such that \([\|e\|_{\text{C}}]\) \(\leq n\), then there are \(c \leq n\) and \(v\) such that \(e \downarrow^c v\).

**Proof.** If there is no complete evaluation of \(e\), then by Prop. 2.3, for every \(k\) there is \(c_k\) such that there is a derivation of \(e \downarrow^{c_k}\) of size \(\geq k\). By sensible ticking there are fixed \(a > 0\) and \(b > 0\) such that \(ak + b \leq c_k\) for all \(k\), and hence there are arbitrarily large \(c\) such that \(e \downarrow^c\). So by the Soundness Theorem, there are arbitrarily large \(c\) such that \(c \leq \|e\|_{\text{C}} \leq n\), which contradicts the assumption that \(A\) is cost-standard. So by Prop. 2.3 again, there is a derivation of \(e \downarrow^c v\) for some \(c\), and hence by the Soundness Theorem, \(c \leq \|e\|_{\text{C}} \leq n\). 

So establishing a cost bound on a source language program \(e\) consists of defining a model \(A\) and showing that the semantic recurrence \([\|e\|]\) extracted from \(e\) is bounded in \(A\). When a collection of models all agrees on \([C]\), the models differ in how they yield an interpretation of size/potential. Analyzing a program, then, typically involves understanding the appropriate notion of size. As we will see, understanding “appropriate notion of size” is necessary not only for values of inductive type, but also for other types such as sums and products.

On its own, this adequacy result is probably less useful than it appears, because in an arbitrary model it is probably difficult to prove that there is \(n\) such that \([\|e\|_{\text{C}}] \leq n\). The issue is that \(\|e\|_{\text{C}}\) is a complex expression, and so computing \([\|e\|_{\text{C}}]\) probably relies
on reducing subexpressions along the lines of \( \|e\|_c = E_0 \to E_1 \to E_2 \to \cdots \to E \) and computing \( [E] \). But the size order tells us that \( [E_i] \geq [E_{i+1}] \), so even if \( [E] = m \), that only tells us that \( \|e\|_c \geq m \), not \( \|e\|_c \leq n \) for some \( n \). But the models we define in the sequel satisfy most of the \( \beta \) axioms in a very strong way: each such axiom that asserts that \( e \leq e' \) is valid because \( e \) and \( e' \) are in fact equal. We use that to simplify the semantic recurrences. As long as our simplifications only use those \( \beta \) axioms, we know that the simplified recurrences are equal to the originally extracted recurrences, and thus Proposition 2.6 gives us useful information.

### 3. A Constructor-Counting Model for Merge Sort

Our first goal is to analyze the standard merge sort algorithm on lists, counting item comparisons. The usual analysis is in terms of the length of the argument list, and hence we need a model in which the denotation of a value of type \( \sigma \) list is its length. To do so, we define a model that counts the number of “main non-nul- lary constructors” in each inductive value. That is, we interpret a recursive value \( v \) of type \( \delta \) as the number of non-nul- lary \texttt{fold} constructors in \( v \) used to construct type \( \delta \) subvalues. Thus the potential (size) of a source language list would be the number of \texttt{cons} constructors, the potential of a source language tree would be the number of \texttt{node} constructors, etc. To simplify the description, we restrict ourselves to the setting in which recursive types in the source language have the form \( \mu \alpha. F \), where \( F \) is given by the grammar

\[
F ::= \alpha \mid \sigma \mid F_0 + F_1 \mid F_0 \times F_1
\]

and the \( \sigma \) production is restricted to closed \( \sigma \). We restrict recursive types in the recurrence language similarly and observe that the translation functions in Figure 8 map the restricted source language to the restricted recurrence language. As mentioned earlier, in this restricted setting, we require \texttt{fun} and \texttt{fix} for recursive function definitions.
3.1. The constructor counting model. The interpretation of types is given in Figure 10, with the size order shown as well (we will discuss the information order needed to interpret recursive function definitions momentarily). We assume that $\eta$ is a map from type variables to semantic types, $\mathbb{N}\infty = \mathbb{N} \cup \{\infty\}$, where $\mathbb{N}$ is the natural numbers with the usual order, and $\mathcal{P}(\cdot)$ is the powerset operator. We will frequently write $[[\sigma]]$ instead of $[[\sigma]]\eta$ when $\eta$ is irrelevant. A key feature of this type interpretation is that $[[\sigma]]$ is always a complete lattice under $\leq_{[[\sigma]]}$, a fact that is straightforward to verify. We interpret products and arrows in the usual way, writing $fst$ and $snd$ for left- and right-projection, but sums are a bit more complex. For ordered sets $X_0$ and $X_1$, define the disjoint union $X_0 \sqcup X_1$ as usual, with injection functions $in_i : X_i \rightarrow X_0 \sqcup X_1$, and define $\leq_{X_0 \sqcup X_1}$ by $in_i x \leq_{X_0 \sqcup X_1} in_i x'$ if and only if $x \leq_{X_i} x'$. Of course, $X_0 \sqcup X_1$ is not a complete lattice under this order. One solution is to add a top element, but this ends up resulting in very weak solutions to extracted recurrences; see Danner and Licata [2022] for a discussion. This same issue arises in static program analysis, and the solution is the same: use (a subset of) $\mathcal{P}(X_0 \sqcup X_1)$ instead. Previously we have used order ideals (i.e., downward-closed sets) with the inclusion ordering. Here we will use all subsets, ordered by $X \preceq Y$ if every element of $X$ is bounded above by some element of $Y$. The reason for this more complex ordering is that for arbitrary subsets, desired monotonicity properties are lost in the inclusion order. For example, the implication $a \leq a' \Rightarrow \{in_0 a\} \subseteq \{in_0 a'\}$ fails, where $in_0$ is injection into the left summand. We will have occasion to interpret products similarly, but because calculating the semantic recurrences is messier with these interpretations, we avoid them unless they are needed.

As already mentioned, for every $\sigma$, $[[\sigma]]$ is a complete upper semilattice with respect to $\leq$ (i.e., the least upper bound $\bigvee X$ exists for every $X \subseteq [[\sigma]]$). It is a standard fact that a complete upper semilattice is a complete lattice, with $\bigwedge X = \{a \mid \forall x \in X.a \leq x\}$. We observe that $\bigvee_{[\sigma_0 + \sigma_1]} X' = \bigcup X'$, but in general $\bigwedge_{[\sigma_0 + \sigma_1]} X' \neq \bigcap X'$ (cf. the complete upper semilattice of open sets of a topological space). In order to interpret fix, we also need an information order $\sqsubseteq$ on each type such that $([[[\sigma]]], \sqsubseteq_{[[\sigma]]})$ is chain-complete. We set $\sqsubseteq = \geq$ for all types, so every type is also a complete lattice with respect to $\sqsubseteq$, and hence trivially chain-complete. However, we only require that arrow types be interpreted by monotone functions, not $\sqsubseteq$-continuous functions. Continuity is not really necessary, because in a complete lattice, monotone functions still have a least fixpoint. More precisely, if $(D, \sqsubseteq)$ is a complete lattice (which necessarily has a bottom element $\bot_{\sqsubseteq}$) and $f : D \rightarrow D$ a $\sqsubseteq$-monotone function, define $x_\alpha$ for ordinals $\alpha$ by

$$x_0 = \bot \quad x_{\alpha+1} = f(x_\alpha) \quad x_\lambda = \bigvee_{\sqsubseteq} \{x_\gamma \mid \gamma < \lambda\}$$

Then there is some (in fact, exactly one) $\alpha$ such that $x_{\alpha+1} = x_\alpha$; $x_\alpha$ is the least fixpoint of $f$ with respect to $\sqsubseteq$, and that will be the value of $fix_D(f)$. In practice we will see that for the functions $f$ that we consider, $x_\omega$ is a fixpoint, and hence the least fixpoint. In our setting with $\sqsubseteq = \geq$, $\bot_{\sqsubseteq}$ is the top element with respect to $\preceq$—that is, the least defined semantic value is the semantic value that provides the least useful size bound. Similarly, $\bigvee_{\sqsubseteq}$ is $\bigwedge_{\leq}$, and in particular an $\sqsubseteq$-increasing chain is a $\preceq$-decreasing chain—that is is, a sequence of more defined semantic values is a sequence of better bounds.

The interpretation of terms is given in Figure 11. It is straightforward to prove that for all $e$, $\eta$, and $x$, $[[e]]\eta[x \mapsto \cdot]$ is monotone with respect to $\leq$ (and hence $\geq$), and so the preceding discussion justifies the definition of $[[fix.x.e]]\eta$. What is left is the interpretations

\[\text{That is because so far, I’ve been unable to prove that the term denotation function is } \geq\text{-continuous.}\]
\begin{align*}
[x] \eta = \eta(x) \\
[()] \eta = * \\
[0] \eta = 0 \\
[1] \eta = 1 \\
e_0 + e_1 \eta = [e_0] \eta + [e_1] \eta \\
in_i e \eta = \{ in_i([e] \eta) \} \\
[\text{case } e \text{ of } \text{in}_0 x \Rightarrow e_0 | \text{in}_1 x \Rightarrow e_1] \eta = \sqrt{\{ [e_0] \eta\{x \mapsto a\} \mid \text{in}_0(a) \in [e] \eta \} \cup \{ [e_1] \eta\{x \mapsto a\} \mid \text{in}_1(a) \in [e] \eta \}} \\
((e_0, e_1]) \eta = ([e_0] \eta, [e_1] \eta) \\
[\text{let } (x_0, x_1) = e' \text{ in } e] \eta = [e] \eta\{x_0, x_1 \mapsto \text{fst}([e] \eta), \text{snd}([e] \eta)\} \\
[\lambda x.e] \eta = \lambda a. [e] \eta\{x \mapsto a\} \\
e_0 e_1 \eta = ([e_0] \eta \{[e_1] \eta\}) \\
[\text{fix } x.e] \eta = \text{fix}_e (\lambda a. [e] \eta\{x \mapsto a\}) \\
[\text{fold}_{\mu} F e] \eta = \text{succ}(\text{csize}_F([e] \eta)) \\
[\text{unfold}_{\mu} F e] \eta = \sqrt{\{a \mid \text{succ}(\text{csize}_F(a)) \leq [e] \eta \}} \\
[\text{val } e] \eta = (0, [e] \eta) \\
[\text{bind } x \leftarrow e' \text{ in } e] \eta = (\text{fst}([e] \eta\{x \mapsto p'\}) + [e'] \eta, \text{snd}([e] \eta\{x \mapsto p'\})) \\
[\text{incr } e] \eta = (\text{fst}([e] \eta) + 1, \text{snd}([e] \eta)) \\
[e_c] \eta = \text{fst}([e] \eta) \\
[e_p] \eta = \text{snd}([e] \eta)
\end{align*}

Figure 11: The interpretation of terms in the constructor size model. The semantic functions \text{succ} and \text{csize}_F are defined in Figure 12.

of \text{fold} and \text{unfold}. Remember that our goal is to interpret a value of type \( \delta = \mu \alpha.F \) as the number of \text{fold} applications in the value whose arguments do not contain any subvalues of type \( \delta \). To do so we define an auxiliary function \( \text{csize}_F \) that maps \( F[N^\infty] \to \{\bot\} \cup N^\infty \), where \( \bot \leq y \) for all \( y \in N^\infty \) and \( x + \bot = \bot + x = x \) for all \( x \in N^\infty \). The idea is that to compute \( [\text{fold}_\delta e] \), we inductively compute the size of \([e]\) and add 1. But we want to ensure that if \( e \) has no type-\( \delta \) subterms, then the result has total size 0. Thus we “shift the count down by one” (that is the role of \( \bot \) and “shift back up” after counting (that is the role of \( \text{succ} \)). The definition can be adapted to yield height (e.g., for trees) by replacing the sum in the \( F_0 \times F_1 \) case with maximum.

Since we interpret product by ordinary cartesian product and \( C \) as \( C \times \cdot \), the following fact is trivial to verify this in this model:

**Proposition 3.1.** For all \( z \in [\mathbb{C} \sigma] \), \( z = (\text{fst } z, \text{snd } z) \). In particular, if \( E : \mathbb{C} \sigma \), then \([E] \eta = ([E_c] \eta, [E_p] \eta)\).
\[
\text{csize}_F : F[N^\infty] \rightarrow \{\bot\} \cup N^\infty
\]
\[
csize_\alpha(n) = n
\]
\[
csize_\sigma(a) = \bot
\]
\[
csize_{F_0 + F_1}(X_0 \sqcup X_1) = \bigvee \left( \{\text{csize}_{F_0}(a) \mid a \in X_0\} \cup \{\text{csize}_{F_1}(a) \mid a \in X_1\} \right)
\]
\[
csize_{F_0 \times F_1}(a_0, a_1) = \text{csize}_{F_0}(a_0) + \text{csize}_{F_1}(a_1)
\]
\[
succ : \{\bot\} \cup N^\infty \rightarrow N^\infty
\]
\[
succ(\bot) = 0
\]
\[
succ(n) = n + 1
\]
\[
succ(\infty) = \infty
\]

Figure 12: The semantic constructor size function \(\text{csize}\).

\[
e \text{ withcost } c = (c, e)
\]
\[
\text{add } c \text{ to } e = (c + e_\text{c}, e_\text{p})
\]

Figure 13: Notation for describing costs in the constructor-counting model.

Since a complexity in this model is a pair consisting of a cost and a potential, we introduce the notation in Figure 13; though wordier than writing pairs, we find it easier to read. We will use this notation extensively in the recurrence language syntax, and for the remainder of this section, when we write syntactic expressions, we are really referring to their denotations in the constructor counting model.

Let us make some observations about this model:

1. The axioms \((\beta_+), (\beta_\times), (\beta_\rightarrow), \text{ and } (\beta_\text{fix})\) are all satisfied because in each case, if the axiom asserts \(e \leq e'\), then in fact \([e] = [e]'\).
2. The intent behind this model is that a list is interpreted as the number of \textit{cons} constructors in it. To verify this, we first compute the denotations of \textit{nil} and \textit{cons} expressions:

\[
[[\text{nil}]] = [[\text{fold} \ (\text{in}_0 \ ())] = \text{succ}(\text{csize}_{\text{unit} + \sigma \times \alpha}([[\text{in}_0 \ ()]]) = \text{succ}(\text{csize}_{\text{unit} + \sigma \times \alpha}(\{\text{in}_0 \ast\})) = \text{succ}(\bot) = 0
\]
and
\[
\llbracket \text{cons}(e, es) \rrbracket = \llbracket \text{fold} \ (\text{in} \_1 (\llbracket e \rrbracket, \llbracket es \rrbracket)) \rrbracket \\
= \text{succ}(\text{csize}_{\text{unit}+\sigma \times \alpha}(\llbracket \text{in} \_1 (\llbracket e \rrbracket, \llbracket es \rrbracket)) \\
= \text{succ}(\text{csize}_{\text{unit}+\sigma \times \alpha}(\llbracket e \rrbracket, \llbracket es \rrbracket)) \\
= \text{succ}(\text{csize}_{\sigma}(\llbracket e \rrbracket) + \text{csize}_{\alpha}(\llbracket es \rrbracket)) \\
= \text{succ}(\perp + \llbracket es \rrbracket) \\
= \text{succ}(\perp) \\
= 1 + \llbracket es \rrbracket \quad \text{(because } \llbracket e' \rrbracket \neq \perp \text{ for any } e')
\]

We can now use this to simplify the extraction of \text{nil} and \text{cons} expressions. We use Proposition 3.1 to simplify the denotation of many extracted recurrences by writing them as cost-potential pairs and using the calculations in Section 1.6. Thus:
\[
\llbracket \| \text{nil} \| \rrbracket = \llbracket \text{val} \text{ nil} \rrbracket \\
= \llbracket \text{nil} \rrbracket \text{ withcost } 0 \\
= 0 \text{ withcost } 0
\]

and
\[
\llbracket \| \text{cons}(e, es) \| \rrbracket = \llbracket \| \text{cons}(e, es) \|_p \rrbracket \text{ withcost } \llbracket \| \text{cons}(e, es) \|_c \rrbracket \\
= \llbracket \| e \|_p, \| es \|_p \rrbracket \text{ withcost } \llbracket \| e \|_c + \| es \|_c \rrbracket \\
= 1 + \llbracket \| es \|_p \rrbracket \text{ withcost } \llbracket \| e \|_c + \| es \|_c \rrbracket.
\]

(3) We next compute the values of \text{unfold} expressions, where we will be a little informal and write elements of \(\mathbb{N}^\infty\) as though they are syntax:
\[
\llbracket \text{unfold } 0 \rrbracket = \{\ast\} \sqcup \emptyset \quad \llbracket \text{unfold } n \rrbracket = \{\ast\} \sqcup \{j \mid j < n\} \quad \llbracket \text{unfold } \infty \rrbracket = \{\ast\} \sqcup \mathbb{N}^\infty
\]

(4) Now we can compute the denotations of \text{case}_{\sigma \text{\_\_\_\_}} \text{\_\_\_\_} \text{\_\_\_\_} expressions.
\[
\llbracket \text{case } 0 \text{ of } \text{nil } \Rightarrow e_{\text{nil}} \mid \text{cons}(x, xs) \Rightarrow e_{\text{cons}} \rrbracket \eta \\
= \llbracket \text{case } \text{unfold } 0 \text{ of } \text{in}_0 \_ \Rightarrow e_{\text{nil}} \mid \text{in}_1 z \Rightarrow \text{let } (x, xs) = z \text{ in } e_{\text{cons}} \rrbracket \eta \\
= \bigvee \left( \{ \llbracket e_{\text{nil}} \rrbracket \eta \mid \_ \in \{\ast\} \} \cup \{ \llbracket \text{let } (x, xs) = z \text{ in } e_{\text{cons}} \rrbracket \eta \{z \mapsto (a, j)\} \mid a \in \llbracket \sigma \rrbracket, j < 0 \} \right) \\
= \llbracket e_{\text{nil}} \rrbracket \eta
\]
and for \( n > 0 \),
\[
\llbracket \text{case } n \text{ of } \text{nil} \Rightarrow e_{\text{nil}} \mid \text{cons}(x, xs) \Rightarrow e_{\text{cons}} \rrbracket \eta \\
= \llbracket \text{case unfold } n \text{ of } \text{in}_0 \_ \Rightarrow e_{\text{nil}} \mid \text{in}_1 \_ z \Rightarrow \text{let} (x, xs) = z \text{ in } e_{\text{cons}} \rrbracket \eta \\
= \bigvee \left( \{ \llbracket e_{\text{nil}} \rrbracket \eta \mid _{\text{nil}} \in \{ \ast \} \} \cup \right. \\
\left. \{ \llbracket \text{let} (x, xs) = z \text{ in } e_{\text{cons}} \rrbracket \eta \{ z \mapsto (a, j) \} \mid a \in \llbracket \sigma \rrbracket, j < n \} \right) \\
= \bigvee \left( \{ \llbracket e_{\text{nil}} \rrbracket \eta \cup \{ \llbracket \text{let} (x, xs) = z \text{ in } e_{\text{cons}} \rrbracket \eta \{ z \mapsto (a, j) \} \mid a \in \llbracket \sigma \rrbracket, j < n \} \right) \\
= \bigvee \left( \{ \llbracket e_{\text{nil}} \rrbracket \eta \cup \{ \llbracket e_{\text{cons}} \rrbracket \eta \{ x, xs \mapsto \infty, n - 1 \} \} \right) \\
= \llbracket e_{\text{nil}} \rrbracket \eta \lor \llbracket e_{\text{cons}} \rrbracket \eta \{ x, xs \mapsto \infty, n - 1 \}
\]

Similarly,
\[
\llbracket \text{case } \infty \text{ of } \text{nil} \Rightarrow e_{\text{nil}} \mid \text{cons}(x, xs) \Rightarrow e_{\text{cons}} \rrbracket \eta = \llbracket e_{\text{nil}} \rrbracket \eta \lor \llbracket e_{\text{cons}} \rrbracket \eta \{ x, xs \mapsto \infty, \infty \}.
\]

(5) We also observe that (\text{mon}) holds for arbitrary contexts in this model.

**Proposition 3.2.** The following inference rule is valid in this model for any term context \( C[] \):

\[
\frac{e \leq e'}{C[e] \leq C[e']}
\]

Thanks to these observations, we shall use the equations in Figure 14 when reasoning about complexity language expressions to be interpreted in this model.

### 3.2. The analysis of merge sort

Merge sort over \texttt{int list} is defined in Figure 15. We assume \texttt{int} is defined as a large enumerated type with \( \leq \) defined by table lookup, and we only count \texttt{int} comparisons. The syntactic recurrences are given in Figure 16. These are recurrences that have been simplified making use of equations valid in the model. The derivations are given in Appendix A.1. To reduce clutter, these simplifications make use of equations valid in the constructor counting model—that is, the reader should pretend as though there are denotation brackets around every term in that figure.

Let us now write \texttt{split} for \( \llbracket \text{fix } \text{split} \cdots \rrbracket \), \( \text{split}_c = \text{fst} \circ \text{split} \) and \( \text{split}_p = \text{snd} \circ \text{split} \). It is the last two functions we care most about. Since our interest is in showing that our method results in the same recurrences as those derived informally, we shall use the final expression in Figure 20 to write down the recurrence relations satisfied by \( \text{split}_c \) and \( \text{split}_p \). To that end, let \( F \) be the functional defined by that expression so that \( \text{split} = \text{fix}(F) \). Then, again imagining denotation brackets wherever appropriate, we see that

\[
\text{split}_c = \text{fst} \circ \text{split} = \text{fst} \circ \llbracket \text{fun } \text{split \_ xs} = \cdots \rrbracket_p = \text{fst} \circ \text{fix}(F) = \text{fst} \circ F(\text{fix}(F)) = \text{fst} \circ (F \text{ split}) = \text{fst} \circ \llbracket \lambda xs. \cdots \rrbracket \{ \text{split} \mapsto \text{split} \}.
\]

So when interpreting the term defining \( F \),
\[
\llbracket (\text{split zs})_c \rrbracket = (\text{fst} \circ \text{split}) \llbracket \text{zs} \rrbracket = \text{split}_c(\llbracket \text{zs} \rrbracket)
\]
\[
e_i[e/x] = \text{case } i_1 e \text{ of } i_0 x \Rightarrow e_0 \mid i_1 x \Rightarrow e_1
\]
\[
e[e_0, e_1/x_0, x_1] = \text{let } (x_0, x_1) = (e_0, e_1) \text{ in } e
\]
\[
e[e_1/x] = (\lambda x.e) e_1
\]
\[
e[\text{fix } x/e/x] = \text{fix } x.e
\]

\[
\text{bind } p \leftarrow e' \text{ in } e = \text{add } e'_i \text{ to } e[e'_p/p]
\]
\[
\text{bind } p \leftarrow e' \text{ withcost } c \text{ in } e = \text{add } c \text{ to } e[e'/p]
\]
\[
\text{bind } p_0, \ldots, p_{n-1} \leftarrow e_0, \ldots, e_{n-1} \text{ in } e = e[e'_0, \ldots, e'_{n-1}/p_0, \ldots, p_{n-1}]
\]
\[
(e_i = e'_i \text{ withcost } 0 \text{ or } e_i = \text{val } e'_i)
\]
\[
\text{add } 0 \text{ to } e = e
\]

\[
\| \text{cons}(e, es) \| = 1 + \| es \|_p \text{ withcost } \| e \|_c + \| es \|_c
\]
\[
\| \text{case } e \text{ of } \text{nil} \Rightarrow e_{n11} \mid \text{cons}(y, ys) \Rightarrow e_{cons} \| = \text{add } \| e \|_c
\]
\[
\text{to } (\text{case } \| e \|_p \text{ of } \text{nil} \Rightarrow \| e_{n11} \| \mid \text{cons}(y, ys) \Rightarrow \| e_{cons} \|)
\]

---

**Figure 14:** Equations valid in the constructor-counting model.

```plaintext
fun split xs =
  case xs of
  nil ⇒ (nil, nil)
  | cons(y, ys) ⇒ case ys of
    nil ⇒ ([y], nil)
    | cons(z, zs) ⇒ let (as, bs) = split zs
      in (cons(y, as), cons(z, bs))

fun merge xsys =
  let (xs, ys) = xsys
  in case xs of
    nil ⇒ ys
    | cons(x', xs') ⇒ case ys of
      nil ⇒ xs
      | cons(y', ys') ⇒ if \( x' \leq y' \) then
        cons(x', merge (xs', ys'))
      else
        cons(y', merge (xs, ys'))

fun msort xs =
  case xs of
    nil ⇒ nil
    | cons(y, ys) ⇒ case ys of
      nil ⇒ [y]
      | cons(_, _) ⇒ let (ws, zs) = split xs
        in merge (msort ws, msort zs)
```

---

**Figure 15:** Merge sort
and similarly \( \langle \text{split } z s \rangle \_p = \text{split}_p (\llbracket z s \rrbracket) \). This lets us conclude that

\[
\text{split}_c(0) = 0 \\
\text{split}_c(1) = 0 \lor 0 = 0 \\
\text{split}_c(n) = 0 \lor 0 \lor \text{split}_c(n - 2) + 0 = 0 \\
\text{split}_c(\infty) = 0 \lor 0 \lor \text{split}_c(\infty) + 0 = \text{split}_c(\infty)
\]
and
\[
\begin{align*}
split_p(0) &= (0, 0) \\
split_p(1) &= (0, 0) \lor (1, 0) \\
split_p(n) &= (0, 0) \lor (1, 0) \lor (\text{fst}(\text{snd}(\text{split}(n - 2)))) + 1, \text{snd}(\text{snd}(\text{split}(n - 2))) + 1) \\
split_p(\infty) &= (\text{fst}(\text{split}(\infty)) + 1, \text{snd}(\text{split}(\infty)) + 1)
\end{align*}
\]

Of course, this is the same solution that we would get by finding a least upper bound with respect to the usual information order (partial function containment) for finite \(n\). Presumably that has something to do with the fact that \(F\) is also monotone in the information order and the least upper bound with respect to the converse size order or ordinary information order occurs at \(f_\omega\).

So how do we go about solving these recurrences? Formally the solution is the least upper bound of the iterates of the functional defined by the recurrence. But we should keep in mind that when we say “least upper bound,” that is in the order \(\geq\) (i.e., the converse of the size order). Let’s do that for \(split_p\). Let \(F\) be defined by
\[
\begin{align*}
F f 0 &= (0, 0) \\
F f 1 &= (1, 0) \\
F f n &= (\text{fst}(f(n - 2)) + 1, \text{snd}(f(n - 2)) + 1) \\
F f \infty &= (\text{fst}(f \infty), \text{snd}(f \infty))
\end{align*}
\]

and let \(f_k\) be the iterates of \(F\):
\[
\begin{align*}
\quad & f_0(x) = \infty \\
\quad & f_{k+1}(x) = F f_k
\end{align*}
\]

By induction on \(k\) we have that
\[
\begin{align*}
f_k(x) &= \begin{cases} 
  ([x/2], [x/2]), & x < k \\
  (\infty, \infty), & x \geq k.
\end{cases}
\end{align*}
\]

We then observe that \(split_p(x) = ([x/2], [x/2])\) is an upper bound of all the \(f_k\) (with respect to \(\geq\)), and that if \(f\) is an upper bound of all the \(f_k\) with respect to \(\geq\), then \(f(x) \leq split_p(x)\) and hence \(split_p\) is the least upper bound. Thus \(split_p\) is in fact the solution to the recurrence.

Figure 21 shows the syntactic recurrence extracted from \(merge\), again simplifying using equations that are valid in the current model. Using notation analogous to that used for \(split\), we have that
\[
\begin{align*}
merge_p(0, l) &= l \\
merge_p(k, 0) &= 0 \lor k \\
merge_p(k, l) &= l \lor k \lor (merge_p(k - 1, l) + 1 \lor merge_p(k, l - 1) + 1)
\end{align*}
\]
and

\[ m_sort_c(0, l) = 0 \]
\[ m_sort_c(k, 0) = 0 \lor 0 \]
\[ = 0 \]
\[ m_sort_c(k, l) = 0 \lor 0 \lor (m_sort_c(k - 1, l) + 1 \lor m_sort_c(k, l - 1) + 1) \]

where we define \( \infty - 1 \) to be \( \infty \). Using an argument similar to the one we used for \( split_p \), we see that \( m_sort_p(k, l) = k + l \) and \( m_sort_c(k, l) = k + l \). Here we are making use of the fact that \( \text{int} = \text{unit} + \cdots + \text{unit} \) for some large \( K \) and \( \text{bool} = \text{unit} + \text{unit} \) for interpreting if expressions. We must compute

\[
\left[ \text{if } x' \leq y' \text{ then } e_0 \text{ else } e_1 \right] \{ x, y \mapsto \infty \}_{\text{int}} \}
\]

where if \( x' \leq y' \) then \( e_0 \) else \( e_1 \) is notation for case \( x' \leq y' \) of \( \text{in}_0 \) \( \Rightarrow e_0 \) | \( \text{in}_1 \) \( \Rightarrow e_1 \). Keeping in mind how we interpret sums, \( \infty_{\text{int}} = \{ \text{unit} \} \cup \cdots \cup \{ \text{unit} \} \). Furthermore, \( x' \leq y' \) is itself a case expression with \( K^2 \) branches. Since we take the maximum (union, in this case) over all the branches corresponding to values in \( \infty_{\text{int}} \), we end up taking that maximum over \( \{ \text{in}_0*, \text{in}_1* \} \), and this in turn tells us that

\[
\left[ \text{if } x' \leq y' \text{ then } e_0 \text{ else } e_1 \right] \{ x, y \mapsto \infty_{\text{int}} \} = \left[ e_0 \right] \lor \left[ e_1 \right]
\]

(which in this case does not depend on \( x' \) or \( y' \) because \( \left[ \text{cons}_e \right] = 1 + \left[ e \right] \).

Finally, the syntactic recurrence for \( m_sort \) is given in Figure 15. Keep in mind that in Figure 15, the “identifiers” \( split \) and \( merge \) in the definition of \( m_sort \) really stand for \( \text{fun} split xs = \cdots \) and \( \text{fun} merge xsys = \cdots \), respectively. Thus the use of \( split \) and \( merge \) in Figure 22 really stand for \( \text{fix} split \cdots \) and \( \text{fix} merge \cdots \) from Figures 15 and 20, respectively. The key point is that the denotation of, e.g., \( (split zs)= \) is again \( split_c([zs]) \), as we would expect.

The denotation of \( m_sort_p \) is

\[ m_sort_p(0) = 0 \]
\[ m_sort_p(1) = 0 \lor 1 = 1 \]
\[ m_sort_p(n) = 0 \lor 1 \lor m_sort_p(m_sort_p(k), m_sort_p(l)) \]
\[ = 0 \lor 1 \lor m_sort_p(m_sort_p([n/2]), m_sort_p([n/2])) \]
\[ = 1 \lor (m_sort_p([n/2]) + m_sort_p([n/2])) \]

The denotation of \( m_sort_c \) is

\[ m_sort_c(0) = 0 \]
\[ m_sort_c(1) = 0 \lor 0 = 0 \]
\[ m_sort_c(n) = 0 \lor 0 \lor split_c(n) + m_sort_c(k, l) + m_sort_c(k) + m_sort_c(l) \]
\[ = m_sort_c([n/2], [n/2]) + m_sort_c([n/2]) + m_sort_c([n/2]) \]
\[ = [n/2] + [n/2] + m_sort_c([n/2]) + m_sort_c([n/2]) \]
\[ = n + m_sort_c([n/2]) + m_sort_c([n/2]) \]

and the standard proof tells us that \( m_sort_c(n) \in O(n \lg n) \).
fun part \( x, xs \) =
\[
\text{case } xs \text{ of } \text{nil} \Rightarrow (\text{nil}, \text{nil}) \\
| \text{cons}(y, ys) \Rightarrow \text{let } (ws, zs) = \text{part}(x, ys) \\
\quad \text{in if } (x \leq y) \text{ then } (ws, \text{cons}(y, zs)) \text{ else } (\text{cons}(y, ws), zs)
\]

fun app \((xs, ys)\) = case xs of nil ⇒ ys | cons\((x', xs')\) ⇒ cons\((x', \text{app}(xs', ys))\)

fun qsort \(xs\) =
\[
\text{case } xs \text{ of } \text{nil} \Rightarrow \text{nil} \\
| \text{cons}(y, ys) \Rightarrow \text{let } (ws, zs) = \text{part}(y, ys) \\
\quad \text{in let } (ws', zs') = (\text{qsort ws, qsort zs}) \\
\quad \text{in app}(ws', \text{cons}(y, zs'))
\]

Figure 17: Quick sort.

4. Adapting the constructor counting model for quick sort

It would seem that the constructor counting model of the previous section would be appropriate for analyzing any list algorithm in terms of the length of the argument list. However, that is not the case; let us see what the problem is when analyzing the worst-case cost of deterministic quick sort as given in Figure 17. The trouble arises in the analysis of \(\text{part}\), for which the syntactic recurrence is given in Figure 18.

As usual, let \(\text{part}_p(X, x)\) be the semantic recurrence \(\llbracket \text{fun part } xs = \cdots \rrbracket_p\). Then the usual analysis tells us that \(\text{part}_p(X, x) = (x, x)\). That is, the extracted recurrence tells us that if \(xs\) is a list of length \(n\), \(\text{part}(x, xs)\) partitions \(xs\) into two lists, each of which has length \(\leq n\). Though a correct statement (necessarily, since the model satisfies the size order rules), it is too weak, because the semantic recurrences extracted for \(\text{qsort}\) will then tell us only that

\[
\text{qsort}_c(n) = \text{part}_c(\llbracket \text{int} \rrbracket, n) + 2\text{qsort}_c(n - 1),
\]

from which we will conclude that the cost of quick sort is \(o(2^n)\).

The proximate problem is that the conditional in the definition of \(\text{part}\) is interpreted as the maximum of its branches, and the maximum in a product is interpreted componentwise; in this case, the maximum potential of the branches is \((1 + ws, 1 + zs)\). One perspective of the underlying problem is that our interpretation of products forces us to come up with a pair that simultaneously bounds both branches. While there is such a bound, it obscures any non-trivial relationship between the components of the pairs being bounded. This is similar to the problem we discussed with interpreting sums as the ordinary disjoint sum with an additional top element; when we happen to have elements on both sides of the sum, our only choice is to bound them by the top element.

The problem is analogous to the one for sums, and the solution is likewise similar: we adapt the model by interpreting products as (a subset of) the powerset rather than the ordinary cartesian product. That is, we replace the clauses for the interpretation of products from Figures 10, 11, and 12 with those in Figure 19. Although the proofs are more painful, the equations of Figure 14 are still valid in the adapted model. The additional pain in many cases arises from the one-step unfolding of \(\sigma \text{ list}\). We now have that \(\llbracket \text{unit } + \sigma \times \alpha \rrbracket \{\alpha \mapsto \)}
\[ \| \text{fun } \text{part} (x, xs) = \cdots \|_p = \]
\[ \text{fix part.} \]
\[ \lambda x . \text{let } (x, xs) = xs \]
\[ \text{in case } xs \text{ of } \]
\[ \text{nil } \Rightarrow (\text{nil, nil}) \text{ withcost 0} \]
\[ \mid \text{cons}(y, ys) \Rightarrow \text{add } (\text{part}(x, ys))_c \text{ to let } (ws, zs) = (\text{part}(x, ys))_p \]
\[ \text{in add 1} \]
\[ \text{to if } x \leq y \text{ then} \]
\[ (ws, 1 + zs) \text{ withcost 0} \]
\[ \text{else} \]
\[ (1 + ws, zs) \text{ withcost 0} \]
\[ \| \text{fun } \text{app xsys} = \cdots \|_p = \]
\[ \text{fix app.} \]
\[ \lambda x . \text{let } (x, ys) = xsys \]
\[ \text{in case } xs \text{ of } \]
\[ \text{nil } \Rightarrow ys \text{ withcost 0} \mid \text{cons}(x', xs') \Rightarrow 1 + \text{app}(xs', ys) \text{ withcost 0} \]
\[ \| \text{fun } \text{qsort xs} = \cdots \|_p = \]
\[ \text{fix qsort.} \]
\[ \lambda x . \text{case } xs \text{ of } \]
\[ \text{nil } \Rightarrow \text{nil withcost 0} \]
\[ \mid \text{cons}(y, ys) \Rightarrow \text{add } (\text{part}(x, ys))_c \]
\[ \text{to let } (ws, zs) = (\text{part}(x, ys))_p \]
\[ \text{in add } (\text{qsort ws})_c + (\text{qsort zs})_c \]
\[ \text{to let } (ws', zs') = ((\text{qsort ws})_p, (\text{qsort zs})_p) \]
\[ \text{in app}(ws', 1 + zs') \text{ withcost 0} \]

Figure 18: The syntactic recurrences for quick sort.

\[ [\sigma_0 \times \sigma_1] = \mathcal{P}([\sigma_0] \times [\sigma_1]) \]
\[ X \leq Y \iff \forall (x, x') \in X \exists (y, y') \in Y . x \leq y \land x' \leq y' \]
\[ [\{e_0, e_1\}] \eta = \{([e_0] \eta, [e_1] \eta)\} \]
\[ [\text{let } (x_0, x_1) = e' \text{ in } e] \eta = \bigvee \{[e] \eta\{x_0, x_1 \mapsto a_0, a_1\} \mid (a_0, a_1) \in [e'] \eta\} \]
\[ \text{csize}_{F_0 \times F_1}(X) = \bigvee \{\text{csize}_{F_0}(a_0) + \text{csize}_{F_1}(a_1) \mid (a_0, a_1) \in X\} \]

Figure 19: Interpreting products as powersets.
The reader should be aware that the musings in this section have not been carefully thought through. They might be meaningless or trivial on their face, and certainly expose lacunae in the author’s knowledge that he wishes did not exist.

Some closing thoughts

The reader should be aware that the musings in this section have not been carefully thought through. They might be meaningless or trivial on their face, and certainly expose lacunae in the author’s knowledge that he wishes did not exist.
5.1. Different axioms. The recurrence language of Kavvos et al. [2020] does not include the axiom \( 0 \leq e \), which we have relied on here (in the argument that for an incomplete derivation of the form \( e \downarrow 0 \), \( 0 \leq \|e\|_c \)). Does this matter? We lose the exact cost model of Danner and Licata [2022], in which \( \leq \) is interpreted as the equality relation. And it seems strange to not have an exact cost model. But what about the standard full type structure in which \( C \) is interpreted as \( \mathbb{N}^\infty \) with the usual order, and \( \leq_\sigma \) is the identity for all other \( \sigma \)? It seems like this sort of structure should be a model, and I suspect you still get exact costs, because same size means same (semantic) value. But then in order to do the “standard thing,” and ensure that types are interpreted by domains (at least with respect to the information order) so that we can use the standard interpretation for inductive types, either the information order is the standard one, in which case it is not quite \( \geq \), or the size order isn’t quite the identity. Either way, I think there is a little bit of work to be done here. And unfortunately what I’ve learned thinking about it is that I don’t actually understand the standard interpretation of inductive types as well as I thought I did.

Kavvos et al. [2020] also includes a fixpoint induction type of axiom and axioms that say that partial approximants of a fixpoint go down in the size order. It seems like the latter ought to correspond roughly to the \( \beta_{fix} \) axiom that we have here, though maybe only in models. And we might only be able to usefully compute bounds from extracted recurrences in the semantics provided the model actually satisfies some form of fixpoint induction. The reason is that the recurrence extracted from a recursive function definition is itself a fixpoint, and hence its interpretation is defined as a limit of the finite approximants. That sequence of approximants goes down in the size order. To conclude useful information about the fixpoint itself, we need to know that a lower bound on all the approximants is also a lower bound on the limit. So even if we don’t need fixpoint induction in the syntax, we do seem to need it in the semantics. This doesn’t seem to be an issue in the constructor counting model here, but that might be because \( (\beta_{fix}) \) is actually an equality in this model. So are there interesting models in which it isn’t? Is this really even the operative issue?

5.2. Different cost models. We keep talking about how this extraction technique should extend easily to other cost models. It is time to make good on that promise. Raymond [2016] approaches parallel complexity. There, the cost annotations in the operational semantics describe work/span cost graphs [Blelloch and Greiner, 1995], and because the cost type in the recurrence language is transparent, this is directly carried over into the recurrence language as well. Is there a way to have a more generic description of cost in the source language so that the operational semantics doesn’t have to be changed for each cost model? And what do we have to do to the now-opaque cost type in the recurrence language so that it can be usefully interpreted as work/span cost graphs? I also think it would be especially nice if there were models that resulted in probabilistic analyses. What would such a model be? I’m thinking about the probabilistic analysis of deterministic quick sort: if all lists of a given length are equally likely, then the expected cost is \( O(n \lg n) \). That sounds like [\( \sigma \text{list} \)] should be a probability distribution for each \( n \), in which case \( qsort_c \) maps probability distributions to expected costs, so \( C \) is interpreted as some kind of expectation. Or perhaps we really need to go with the idea that \( C\sigma \) is really an algebra, and the cost component potentially differs for different \( \sigma \).
5.3. Abstract types. Something I’ve had a bee in my bonnet about for awhile is abstract types. Consider heap sort, in which the elements of a list \( xs \) are entered into a priority queue, then the priority queue is emptied out. Assuming the cost of the priority queue operations are \( O(\lg n) \), the cost of the sorting algorithm is \( O(n \lg n) \). The cost analysis needs to know nothing more about the implementation of the priority queue beyond the cost of the operations, any more than proving correctness needs to know anything more about the implementation beyond its correctness. So how can we bring this sort of reasoning into this recurrence extraction setting? Presumably we want to use existential types to model abstract types, though for a first pass I tend to lean toward the abstract type declarations of Mitchell and Plotkin [1985] for concreteness. I think the idea is to have both existential/abstract types in both the source and recurrence language. Of course, it is the semantics of the abstract type declarations that is interesting. I think what happens is that the abstract type can be interpreted one way, with the concrete implementation interpreted another. All that should be necessary is that there be the standard Galois connection corresponding to the \( \beta \) axioms in the recurrence language. For example, we might implement priority queues with leftist heaps. The size of a priority queue (i.e., the denotation of the abstract type) would just be its size. The denotation of the concrete type (leftist heaps) would have to include rank (length of the rightmost branch), because that is the quantity that the cost recurrences are defined in terms of. On the side, one proves that the rank is logarithmic in the size, so in fact the denotation of the concrete type is probably includes size and rank—i.e., \( \mathbb{N}^\infty \times \mathbb{N}^\infty \), or possibly something like the pairs \( (n, r) \) with \( r \leq \lg n \) (I forget the exact relation). So what is the relation between the abstract and concrete sizes? I would guess that in fact we interpret concrete sizes by \( \mathcal{P}(\mathbb{N}^\infty \times \mathbb{N}^\infty) \), the abstract size \( n \) is mapped to \( \{ (n, r) \mid r \leq \lg n \} \), and the concrete size \( X \) is mapped to \( \bigvee \{ n \mid (n, r) \in X \} \). The key point is that the cost of the concrete leftist heap operations are linear in (concrete) rank. But that can then be translated to (concrete) size, which in turn tells us that the cost of the abstract priority queue operations are logarithmic in (abstract) size. And from there, we can analyze the heap sort algorithm itself.

In fact, it should be that we can go further. It ought to be possible to include some form of cost information in the abstract/existential type in the source language (and hence in the recurrence language). Niu et al. [2022] tells us that Acar and Blelloch [2019] define a notion of cost signature to go along with functional signatures for abstract types, but a quick skim of the latter didn’t turn it up. Nonetheless, it seems to consist of annotation functions with cost information. That cost information could presumably be used by the \( \acute{\phi} \) operation as well as in recurrence extraction. It would probably also be carried over into the recurrence language. A model of the recurrence language would then have to validate those cost annotations.

5.4. Things I don’t know enough about. We’ve proved soundness for recursive types, but haven’t used them at all. Part of that is because I don’t understand models for arbitrary recursive types. But maybe that’s a non-issue here; after all, the point is that we would probably model recursive types by a simple notion of size. Another part is that I don’t really know much about recursive types, and in particular don’t know much about typical algorithms that use them and how we talk about their cost.

It seems like Proposition 2.6 should be provable without the sensible ticking hypothesis, just as in Kavvos et al. [2020]. Is that true? And is it interesting? That is, do we really care about the cost of a program that is not sensibly ticked? On the other hand, we haven’t
\[
\| \text{fun split } xs = \ldots \|_p \\
= \text{fix split.}
\]
\[
\lambda xs. \text{case } xs \text{ of } \\
\quad \text{nil } \Rightarrow \| (\text{nil}, \text{nil}) \| \\
\quad \text{cons}(y, ys) \Rightarrow \| \text{case } ys \text{ of } \\
\text{nil } \Rightarrow \quad \| \text{cons}(y, \text{nil}) \|, \| \text{nil} \| \\
\text{in } \text{val } (p_0, p_1) \\
\quad \text{cons}(y, ys) \Rightarrow \| \text{cons}(y, \text{nil}) \|, \| \text{nil} \| \\
\text{in } \text{val } (p_0, p_1) \\
\quad \text{cons}(z, zs) \Rightarrow \quad \| \text{split } zs \| \\
\text{in let } (as, bs) = p \\
\quad \| (\text{cons}(y, as), \text{cons}(z, bs)) \| \\
= \text{fix split.}
\]
\[
\lambda xs. \text{case } xs \text{ of } \\
\quad \text{nil } \Rightarrow \quad \| \text{nil} \| \\
\quad \text{cons}(y, ys) \Rightarrow \| \text{case } ys \text{ of } \\
\text{nil } \Rightarrow \quad \| \text{cons}(y, \text{nil}) \|, \| \text{nil} \| \\
\text{in } \text{val } (p_0, p_1) \\
\quad \text{cons}(y, ys) \Rightarrow \| \text{cons}(y, \text{nil}) \|, \| \text{nil} \| \\
\text{in } \text{val } (p_0, p_1) \\
\quad \text{cons}(z, zs) \Rightarrow \quad \| \text{split } zs \| \\
\text{in let } (as, bs) = p \\
\quad \| (\text{cons}(y, as), \text{cons}(z, bs)) \| \\
= \text{fix split.}
\]
\[
\lambda xs. \text{case } xs \text{ of } \\
\quad \text{nil } \Rightarrow \quad (0, 0) \text{ withcost } 0 \\
\quad \text{cons}(y, ys) \Rightarrow \| \text{case } ys \text{ of } \\
\text{nil } \Rightarrow \quad (1, 0) \text{ withcost } 0 \\
\text{in } \text{let } (as, bs) = p \\
\quad \text{cons}(z, zs) \Rightarrow \quad \text{add } (\text{split } zs)\_c \\
\text{to let } (as, bs) = (\text{split } zs)\_p \\
\text{in } (1 + as, 1 + bs) \text{ withcost } 0
\]

Figure 20: The syntactic recurrence for split.

actually proved that our ticking of merge sort and quick sort is sensible. So this still takes some cleaning up.

APPENDIX A. SYNTACTIC RECURRENCE SIMPLIFICATIONS

A.1. Merge sort. Simplifications of the syntactic recurrences for the functions used to define merge sort are given in Figures 20, 21, and 22.
∥fun merge xsys = ···∥_p

= fix merge.

λxsys.bind p ← val xsys

in let (xs, ys) = p

in case xs of

| cons(x', xs') ⇒ case ys of
  nil ⇒ val xs

| cons(y', ys') ⇒ ||if ··· then ··· else ···||

= fix merge.

λxsys.let (xs, ys) = xsys

in case xs of

| nil ⇒ ys withcost 0

| cons(x', xs') ⇒ case ys of
  nil ⇒ xs withcost 0

| cons(y', ys') ⇒ bind p ← ||x' ≤ y'||

  in if p then
    ||cons(x', merge (xs', ys))||
  else
    ||cons(y', merge (xs, ys'))||

= fix merge.

λxsys.let (xs, ys) = xsys

in case xs of

| nil ⇒ ys withcost 0

| cons(x', xs') ⇒ case ys of
  nil ⇒ xs withcost 0

| cons(y', ys') ⇒ add 1

  to if x' ≤ y' then
    let χ = merge (xs', ys)
    in 1 + χ_p withcost χ_c
  else
    let χ = merge (xs, ys')
    in 1 + χ_p withcost χ_c

Figure 21: The syntactic recurrence for merge. We write let χ = e' in e for e[e'/χ].

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\[ \parallel \text{fun } msort \; xs = \cdots \parallel_p \]
\[ = \text{fix} msort. \]
\[ \lambda xs. \text{case } xs \text{ of} \]
\[ \quad \text{nil } \Rightarrow \text{val} \; \text{nil} \]
\[ \quad \text{cons}(y, ys) \Rightarrow \text{case } ys \text{ of} \]
\[ \quad \quad \text{nil } \Rightarrow \text{cons}(y, \text{nil}) \text{ withcost } 0 \]
\[ \quad \quad \text{cons(\_\_)} \Rightarrow \text{bind } p \leftarrow \parallel (\text{fun } \text{split } x) = \cdots \parallel_p (xs) \]
\[ \quad \quad \text{in let } (ws, zs) = p \]
\[ \quad \quad \text{in } \parallel (\text{fun } \text{merge } xs) = \cdots \parallel (msort \; ws, msort \; zs) \parallel_p \]
\[ = \text{fix} msort. \]
\[ \lambda xs. \text{case } xs \text{ of} \]
\[ \quad \text{nil } \Rightarrow 0 \text{ withcost } 0 \]
\[ \quad \text{cons}(y, ys) \Rightarrow \text{case } ys \text{ of} \]
\[ \quad \quad \text{nil } \Rightarrow 1 \text{ withcost } 0 \]
\[ \quad \quad \text{cons(\_\_)} \Rightarrow \text{bind } p \leftarrow \parallel (\text{fun } \text{split } x) = \cdots \parallel_p (xs) \parallel_p \]
\[ \quad \quad \text{in let } (ws, zs) = p \]
\[ \quad \quad \text{in } \parallel (\text{fun } \text{merge } xs) = \cdots \parallel_p (msort \; ws, msort \; zs) \parallel_p \]
\[ = \text{fix} msort. \]
\[ \lambda xs. \text{case } xs \text{ of} \]
\[ \quad \text{nil } \Rightarrow 0 \text{ withcost } 0 \]
\[ \quad \text{cons}(y, ys) \Rightarrow \text{case } ys \text{ of} \]
\[ \quad \quad \text{nil } \Rightarrow 1 \text{ withcost } 0 \]
\[ \quad \quad \text{cons(\_\_)} \Rightarrow \text{bind } p \leftarrow \text{split } x \]
\[ \quad \quad \text{in let } (ws, zs) = p \]
\[ \quad \quad \text{in } \parallel (\text{msort } ws)_c + (\text{msort } zs)_c \parallel_p \]
\[ \quad \quad \text{to } \parallel (\text{msort } ws)_p, (\text{msort } zs)_p \parallel_p \]
\[ = \text{fix} msort. \]
\[ \lambda xs. \text{case } xs \text{ of} \]
\[ \quad \text{nil } \Rightarrow 0 \text{ withcost } 0 \]
\[ \quad \text{cons}(y, ys) \Rightarrow \text{case } ys \text{ of} \]
\[ \quad \quad \text{nil } \Rightarrow 1 \text{ withcost } 0 \]
\[ \quad \quad \text{cons(\_\_)} \Rightarrow \text{add } (\text{split } x)_c \]
\[ \quad \quad \text{to let } (ws, zs) = (\text{split } x)_p \]
\[ \quad \quad \text{in } \parallel (\text{msort } ws)_c + (\text{msort } zs)_c \parallel_p \]
\[ \quad \quad \text{to } \parallel (\text{msort } ws)_p, (\text{msort } zs)_p \parallel_p \]

Figure 22: The syntactic recurrence for \textit{msort}. 
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