Stochastic duality of ASEP with two particle types via symmetry of quantum groups of rank two

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Abstract
We study two generalizations of the asymmetric simple exclusion process (ASEP) with two types of particles, which will be called type $A_2$ ASEP and type $C_2$ ASEP. Particles of type 1 force particles of type 2 to switch places. In the $C_2$ case, particles of type 2 can only influence the jump rates of particles of type 1, and in the $A_2$ case particles of type 2 do not influence particles of type 1 at all. We prove that the processes are self-dual and explicitly write the duality function, which is a generalization of the self-duality function for ASEP. The construction and proofs of duality are accomplished using symmetry of the quantum groups $\mathcal{U}_q(\mathfrak{gl}_3)$ and $\mathcal{U}_q(\mathfrak{sp}_4)$ for the $A_2$ and $C_2$ ASEP respectively.

Keywords: ASEP, quantum groups, duality

1. Introduction
The asymmetric simple exclusion process (ASEP) is a widely studied model in mathematics and physics. Particles occupy a one-dimensional lattice, with at most one particle at each site. The particles jump to neighboring sites asymmetrically, meaning that particles will drift to either the right or the left. If the particle attempts to jump to an occupied site, the jump is blocked.

In two-type ASEP, there are additionally second-class particles. These particles jump according to the same rule as ASEP. However, if a first-class particle attempts to jump to a site occupied by a second-class particle, the particles switch positions. If a second-class particle attempts to jump to a site occupied by a first-class particle, the jump is blocked. Observe that the second-class particles do not affect the first-class particles, or in other words, the projection to the first-class particles is Markov. Second-class particles were introduced in [17] and then further developed in [12, 13], and also occur more recently in [3, 10] and [21].
This paper examines two-type ASEP, and will also introduce ‘semi-second-class’ particles. These are particles which can not jump over the first-class particles; however, their presence can influence the jump rates of adjacent first-class particles. Thus, the projection to first-class particles is not Markov. These particles will also be called ‘type 1’ and ‘type 2’ particles.

Two particular sets of values for the jump rates will be studied in detail. In these two cases, we will show that the processes are self-dual and explicitly write the duality function. The use of algebra symmetries to prove duality has a well-established history (e.g. [4, 15, 19, 20]). The proofs in this work follow the method laid out in [9]. Recent work [7, 11] has also developed proofs for duality using more ‘direct’ (that is, without algebraic) methods. It has also been previously known that ASEP with second-class particles is integrable (e.g. [1]) and satisfies $U_q(gl_1)$ symmetry [2], but the explicit representations had not been constructed. Similar models have also been shown to be integrable (e.g. [8, 12]).

The remainder of this paper is organized as follows: section 2 gives an explicit description of the processes, states the duality results as well as an application, and states the quantum group symmetry. Section 3 reviews the background on quantum groups and their relationships to ASEP, and section 4 constructs a central element necessary for the proof. Section 5 finishes the proofs for the $U_q(gsp_4)$ case, and section 6 finishes the proofs for the $U_q(gl_3)$ case.

During the writing of this paper, another paper [5] was posted to the arXiv with similar results. That paper studies only the process arising from $U_q(gl_2)$ symmetry and finds a duality function similar to the one presented here. The approach is different in that it uses the Perk-Schultz quantum spin chain [18] to construct the representations, rather than explicitly constructing a central element. That paper also explicitly constructs all invariant measures and proves an interesting sum rule for the duality functions, neither of which are addressed here.

2. Overview

2.1. Description

Consider a one-dimensional lattice, which will be viewed as a subset of $\mathbb{Z}$. Each lattice site has three possible states: either empty, occupied by a first-class particle, or occupied by a (semi-)second-class particle. Each particle has two independent exponential clocks, one for left jumps and one for right jumps. The rate of the left clock depends on the state of the site to the left of the particle, and the rate of the right clock depends on the state of the site to the right of the particle. Let $L(i, j)$ denote the rate of the left clock of the $i$th class particle when the site to the left has state $j$, and similarly let $R(i, j)$ denote the rate of the right clock of the $i$th class particle when the site to the right has state $j$. The symbol $\infty$ will be used to denote a hole.

The particles interact according to the following rules: if a first-class particle attempts to jump to a site occupied by a semi-second-class particle, then the particles switch positions. If a semi-second-class particle attempts to jump to a site occupied by a first-class particle, the jump is blocked. This implies that

$$L(2, 1) = R(2, 1) = 0.$$ 

If a first-class particle attempts to jump to a site occupied by another first-class particle, then the jump is blocked. The same holds for the semi-second-class particle. This implies that

$$L(1, 1) = L(2, 2) = R(2, 2) = R(1, 1) = 0.$$ 

This leaves six remaining jump rates, $L(1, \infty), R(1, \infty), L(2, \infty), R(2, \infty), L(1, 2), R(1, 2)$. Observe that if
then the first class particles evolve independently of the semi-second-class particles. In other words, the behavior of the first-class particles is Markov. In this case, the semi-second-class particles have been described in the literature as second-class particles. In general, however, the semi-second-class particles still affect the jump rates of the first-class particles, even if they cannot jump over them. Also observe that if the six jump rates are all multiplied by the same positive constant, then this corresponds to rescaling the time, and hence has no effect on the interaction of the particles.

In this paper, we will consider two particular sets of values for the jump rates. The first is when

\[ L(1, \infty) = L(2, \infty) = L(1, 2) = 1, \quad R(1, \infty) = R(2, \infty) = R(1, 2) = q^{-2}. \]

This is called spin 1/2 type ASEP, or ASEP with second-class particles. Here, the asymmetry parameter is \( q \) for all particles. The second set of values for the jump rates is when

\[
L(1, \infty) = L(2, \infty) = L(1, 2) = a, \\
R(1, \infty) = R(2, \infty) = q^{-2}, \\
R(1, 2) = aq^{-4},
\]

where \( a = q^2(q^2 + q^{-2})^2(q^{-4} + q^6)^{-1} \). This is called spin 1/2 type C ASEP. In other words, the asymmetry parameter is \( q \) for particles of type 1 and 2, and \( q^2 \) when particles of type 1 and type 2 interact. The reasons for the names will become clear later in the paper.

2.2. Duality results

Let us review the definition of duality.

**Definition 2.1.** Suppose that \( X(t) \) and \( Y(t) \) are Markov processes on state spaces \( X \) and \( Y \) respectively. Given a function \( D \) on \( X \times Y \), let \( S_D \subseteq X \times Y \) be the set of all \((x, y)\) such that

\[
\mathbb{E}_x[D(x(t), y)] = \mathbb{E}_y[D(x, y(t))],
\]

where on the left-hand side, the process \( x(t) \) starts at \( x(0) = x \), and on the right-hand side, the process \( y(t) \) starts at \( y(0) = y \). If \( S_D = X \times Y \), then we say that \( X(t) \) and \( Y(t) \) are dual with respect to \( D(x, y) \). If furthermore, \( X(t) \) and \( Y(t) \) are the same process, then we say that \( X(t) \) is self-dual.

In order to write the duality function explicitly, some more notation is necessary. There are two ways of denoting a particle configuration. The first is called occupation variables, in which a particle configuration is written as \( \eta = \{ \eta_i \} \), where \( \eta_i \in \{ \infty, 1, 2 \} \) for each lattice site \( i \), corresponding to an empty site, occupation by a first-class particle, and occupation by a (semi-)second-class particle, respectively. The second is called particle variables, where \( \xi = (n_1, m_2, \ldots, n_r, m_{r+1}, \ldots, m_s) \) means that there are type 1 particles at lattice sites \( n_1, \ldots, n_r \), type 2 particles at lattice sites \( m_2, \ldots, m_{r+1} \), and all other lattice sites are empty. The symbol \( \varnothing \) will be used to indicate that there are no particles of a particular type, so for example \( \xi^{(n_1, \varnothing)} \) denotes the particle configuration with a type 1 particle at site \( n \) and all other sites empty.
With this notation, and with the lattice sites indexed by \( \{1, \ldots, L\} \), define
\[
N^L_i(\eta) = \sum_{j=1}^{i-1} 1\{\eta_j = \infty\}, \quad \tilde{N}^L_i(\eta) = \sum_{j=1}^{i-1} 1\{\eta_j = 1\},
\]
\[
N^R_i(\eta) = \sum_{j=i+1}^L 1\{\eta_j = \infty\}, \quad \tilde{N}^R_i(\eta) = \sum_{j=i+1}^L 1\{\eta_j = 1\}.
\]
In words, \( N^L_i(\eta) \) and \( \tilde{N}^R_i(\eta) \) count the total number of particles to the left and right of lattice site \( i \), respectively, while \( N^L_i(\eta) \) and \( \tilde{N}^R_i(\eta) \) count the number of type 1 particles to the left and right of lattice site \( i \), respectively.

**Theorem 2.2.** In the \( A_2 \) case, the function \( D(\cdot, \cdot, \cdot) \) defined by
\[
D(\eta, \xi^{(n_1, \ldots, n, m_1, \ldots, m)}) = \prod_{s=1}^r 1\{\eta_s = 1\} q^{2N^L_s(\eta) + 2n_s}
\]
\[
\times \prod_{s'=1}^{r'} 1\{\eta_s = \infty\} q^{2N^R_s(\eta) + 2m_{s'}}
\]
is a self-duality function (where \( \eta = \{\eta_s\} \) are occupation variables).

This function is similar to proposition 2 of [15] or 3.12 of [20]. Indeed, if \( \xi \) only contains type 2 particles, one recovers the self-duality function for the projection of type \( A_2 \) ASEP to the number of particles, which is still ASEP. If \( \xi \) only contains type 1 particles, one recovers the self-duality function for the projection of type \( A_2 \) ASEP to the type 1 particles, which is again still ASEP.

In the \( C_2 \) case, we give two duality functions:

**Theorem 2.3.**

1. In the \( C_2 \) case, the function \( D(\eta, \xi) = \prod_{i=1}^L \left( 1\{\xi_i = \eta_i = 1\} q^{2(i-1)} + 1\{\xi_i = \eta_i = 2\} q^{2(i-1)N^L_i(\eta_i) + N^R_i(\xi_i)} \right) \)

is a self-duality function.

2. In the \( C_2 \) case, there is a function \( D \) such that
\[
S_D = \{\infty, 1, 2\}^L \times \{\infty, 1\}^L.
\]

Explicitly, \( D(\eta, \xi) = \prod_{s=1}^r 1\{\eta_s = \infty\} q^{2N^L_s(\eta) + 2n_s} \).

**Remark.** Theorem 2.3 (2) can also be stated as ‘spin 1/2 type \( C_2 \) ASEP is dual to usual ASEP with respect to \( D' \). Also observe that the function \( D(\cdot, \cdot, \cdot) \) only detects the number, not the type, of particles. The projection of type \( A_2 \) and \( C_2 \) ASEP to particle occupation is simply the usual ASEP, and the duality function matches that from [20]. The interest lies in...
that $D(\cdot, \cdot, \cdot)$ can be constructed from the representation theory of $\mathcal{U}_q(\mathfrak{sp}_4)$, which will be seen below.

**Remark.** If one takes $q \to 1$ in theorems 2.2 and 2.3, the resulting function generalizes the duality function for the SSEP in [19].

**Remark.** In type $A_2$ ASEP, one can consider the current of first-class particles through a bond, or the current of total particles through a bond. Theorem 2.2 provides a formula for the mixed $r + r'$ moments of the exponentials of these two types of currents, multiplied by indicator functions. By following the argument in [15], it should be possible to remove the indicator functions by writing in terms of $k$-particle evolution for $k \leq r + r'$. The main difficulty would be establishing commutation relations between the two number operators and the two annihilation operators. However, the number operator counting first-class particles is simply $E_{11}$, which commutes with the second annihilation operator $E_{23}$. Likewise, the number operator counting all particles is $E_{11} + E_{22}$, which commutes with the first annihilation operator $E_{12}$. A full proof is not pursued here.

### 2.3. Construction

In [9], there is a general description of how to construct particle systems from quantum groups, as well as finding self-duality functions for these particle systems. Let us briefly review the idea in several steps; more details and intuition will be given in the next section.

The first step is to consider the quantum group $\mathcal{U}_q(\mathfrak{g})$ for some finite-dimensional simple Lie algebra $\mathfrak{g}$. Find an explicit central element $C \in \mathcal{U}_q(\mathfrak{g}) \mathcal{U}_q(\mathfrak{h}) \mathcal{U}_q(\mathfrak{g})$. Next, consider a finite-dimensional irreducible representation $V$ of $\mathcal{U}_q(\mathfrak{g})$ with a basis $v_1,\ldots,v_d$ consisting of weight space vectors. If $v_1$ denotes the highest weight vector, then compute the value of $a$ for which $Cv_1 v_1 = D$, where $\Delta$ is the co-product on $\mathcal{U}_q(\mathfrak{g})$. Now compute the $d \times d$ matrix of $A : = \Delta(C - a)$ acting on $V \otimes V$ with respect to the basis $\{v_i \otimes v_j, 1 \leq i, j \leq d\}$. Observe that $C - a$ is still central. Assume that this matrix has non-positive diagonal entries, and non-negative off-diagonal entries (this will not always be true).

Now consider the operator on $V^{\otimes L}$ defined by

$$\sum_{i=1}^{L-1} \sum_{j=1}^{L-1} A^{i\otimes j} \otimes A^{L-i-j}.$$ 

Suppose that we have a vector $g \in V^{\otimes L}$ such that $A^{L} g = 0$. It is possible to find such a vector by applying elements of $\mathcal{U}_q(\mathfrak{h})$ to the lowest weight vector $v_1 \otimes \cdots \otimes v_d$ (in physics language, this is applying creation operators to the vacuum state). Write $g$ in terms of the canonical basis

$$\sum_{1 \leq i_1,\ldots,i_L \leq d} g(i_1,\ldots,i_L) v_{i_1} \otimes \cdots \otimes v_{i_L}.$$ 

Assume that $g(i_1,\ldots,i_L)$ is always positive and define $G$ to the diagonal operator on $V^{\otimes L}$ defined by

$$G(v_{i_1} \otimes \cdots \otimes v_{i_L}) = g(i_1,\ldots,i_L) v_{i_1} \otimes \cdots \otimes v_{i_L}.$$ 

By the assumptions on $A$, the matrix $L = G^{-1} A^{L} G$ is the generator of a continuous-time Markov chain on the state space $[1,\ldots,d]^L$. Another way to think of this is a lattice with $L$ sites, and each site can either be empty or be occupied by a single particle, with $d - 1$
different particle types. The boundary conditions will be reflecting boundary conditions, meaning that particles cannot jump to the left of 1 or to the right of $L$.

Finally, if $A^{(L)}$ is self-adjoint on $V^{\otimes L}$ and $S$ is an operator that commutes with $A$, then $D = G^{-1}S^{-1}$ is a self-duality function for the particle system generated by $L$.

This paper will consider the situation in which $g = sp_4$ or $gl_3$ and $V$ is the fundamental representation. It is also possible to take $V$ to be a higher spin representation, but this is not pursued here. The precise statements are as follows:

**Theorem 2.4.** There exists a central element $C \in U_q(gl_3)$ and an operator $G$ on $V^{\otimes L}$ such that for

$$A^{(L)} := \sum_{i=1}^{L-1} V^{i-1} \otimes \Delta(C) \otimes V^{L-i-1},$$

$G^{-1}A^{(L)}G$ is the generator of spin $1/2$ type $A_2$ ASEP on the lattice $\{1, \ldots, L\}$ with reflecting boundary conditions.

The self-duality function $D(\cdot, \cdot, \cdot)$ in theorem 2.2 is of the form $G^{-1}S^{-1}$ for a symmetry $S$ of $A^{(L)}$.

In the following theorem, the notation for $A^{(L)}$ is the same. In this case, however, the matrix of $A$ will have negative diagonal entries, corresponding to a ‘negative probability’ of a site with both a type 1 and a type 2 particle. To circumvent this problem, one conjugates by $G$, so that the ‘negative probability’ is of order $-\epsilon$, and then takes $\epsilon \to 0$.

**Theorem 2.5.** There exists a central element $C \in U_q(sp_4)$ and operators $G_i$ on $V^{\otimes L}$ such that the limit $\lim_{\epsilon \to 0} G_i^{-1}A^{(L)}G_i$ is the generator of spin $1/2$ type $C_2$ ASEP on the lattice $\{1, \ldots, L\}$ with reflecting boundary conditions.

The functions in theorem 2.3 are of the form $G^{-1}S^{-1}$ for a symmetry $S$ of $A^{(L)}$.

### 3. Background on Quantum Groups

Before continuing with the remainder of the paper, let us review some background on quantum groups, especially in its relationship to ASEP. A more thorough introduction to quantum groups can be found in [16].

#### 3.1. $U_q(sl_2)$ and ASEP

Consider the simplest non-trivial example of a Lie algebra, the Lie algebra $sl_2$ of traceless $2 \times 2$ matrices. This has a basis consisting of $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. It has a natural two-dimensional representation $V$, whose action is given by the natural action of $2 \times 2$ matrices on two-dimensional vectors.

The universal enveloping algebra $U(sl_2)$ is the algebra generated by $e, f, h$ with the commutation relations $[e, f] = h$, $[h, e] = 2e$, $[h, f] = -2f$. Observe that if the three matrices $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ are identified with the three generators $e, f, h$, respectively, then they also satisfy the same relations (taking $[a, b] = ab - ba$ with matrix multiplication). With this identification, $V$ is also a natural two-dimensional representation of $U(sl_2)$. In fact, there is a natural one-to-one correspondence between representations of $sl_2$ and representations of $U(sl_2)$. 
The algebra $\mathcal{U}(\mathfrak{sl}_2)$ is related to the SSEP. SSEP is simply ASEP where the asymmetry parameter is 1, i.e., the jump rates to the left and to the right are equal. On the lattice \{1, ..., $L$\}, SSEP has a total of $2^L$ particle configurations. Each particle configuration can be identified with a basis element of the $2^L$-dimensional representation $V^{\otimes L}$, by identifying $v_1 := \begin{pmatrix} 1 \\ 0 \end{pmatrix} \in V$ with a particle and $v_2 := \begin{pmatrix} 0 \\ 1 \end{pmatrix} \in V$ with a hole. For example, $v_1 \otimes v_2 \otimes v_2 \otimes v_1$ corresponds to the state of SSEP on four lattice sites \{1, 2, 3, 4\} with particles at 1 and 4 and empty sites at 2 and 3. Under this identification, the Hamiltonian of SSEP is a map $V^{\otimes L} \rightarrow V^{\otimes L}$, and $e, f, h$ act as creation and annihilation operators on $V^{\otimes L}$. It turns out that $e, f, h$ all commute with the Hamiltonian, and this commutation property is used to prove duality results.

The action of $\mathcal{U}(\mathfrak{sl}_2)$ on $V^{\otimes L}$ is given by an algebra homomorphism $\Delta : \mathcal{U}(\mathfrak{sl}_2) \rightarrow \mathcal{U}(\mathfrak{sl}_2) \otimes \mathcal{U}(\mathfrak{sl}_2)$ called the co-product. Explicitly,

$$\Delta(e) = e \otimes 1 + 1 \otimes e,$$

$$\Delta(f) = f \otimes 1 + 1 \otimes f,$$

$$\Delta(h) = h \otimes 1 + 1 \otimes h.$$ 

This extends to a homomorphism $\mathcal{U}(\mathfrak{sl}_2) \rightarrow \mathcal{U}(\mathfrak{sl}_2)^{\otimes L}$ by co-associativity. For example,

$$e \cdot (w_1 \otimes \cdots \otimes w_L) = \sum_{j=1}^{L} w_1 \otimes \cdots \otimes w_{j-1} \otimes ew_{j} \otimes w_{j+1} \otimes \cdots \otimes w_L.$$ 

In order to introduce asymmetry, one considers a quantization $\mathcal{U}_q(\mathfrak{sl}_2)$. This is done by replacing $h$ with an invertible element $k_q$, which can be thought of as $q^n h$. The commutation relation $[e, f] = h$ is replaced with its quantization

$$[e, f] = \frac{k - k^{-1}}{q - q^{-1}}.$$ 

Furthermore, the co-product is quantized via

$$\Delta(e) = k_q \otimes e + e \otimes 1, \quad \Delta(f) = 1 \otimes f + f \otimes k_q^{-1}.$$ 

The commutation relations $[h, e] = 2e, [h, f] = -2f$ and the co-product formula $\Delta(h) = 1 \otimes h + h \otimes 1$ become\(^1\)

$$ke = q^2 ek, \quad kf = q^{-2}fk, \quad \Delta(k) = k \otimes k.$$ 

Observe that in the $q \rightarrow 1$ limit, one can see from L’Hopital’s rule that $\mathcal{U}_q(\mathfrak{sl}_2)$ is recovered.

Once again, $\mathcal{U}_q(\mathfrak{sl}_2)$ has a $2^L$-dimensional representation $V^{\otimes L}$, whose basis elements are identified with ASEP particle configurations on $L$ lattice sites. As before, $e, f$ act as creation and annihilation operators, and the ASEP Hamiltonian commutes with the quantum group action.

\subsection*{3.2. Higher rank Lie algebras}

For higher rank Lie algebras, more generators are necessary. For example, consider $\mathfrak{g} = \mathfrak{sl}_3$, which is also known as the type $A_2$ Lie algebra. If $E_{ij}$ denotes the matrix with a 1 at the $(i, j)$-entry and 0 elsewhere, and

\(^1\) For example, to see that the relation $ke = q^2 ek$ is equivalent to $[h, e] = 2e$, set $q = \exp(r)$. Then $[h, e] = 2e$ implies $he = e(h + 2)$ and $b^r e = e(h + 2)^r$, so that $q^r e = \sum_{n=0}^{\infty} \frac{(2e)^n}{n!} (h + 2)^r = q^r e q^h$. 

\[7\]
then \( e_1, e_2, e_1 e_2, f_1, f_2, f_1 f_2, h_1, h_2 \) is a basis for \( \mathfrak{sl}_3 \), and hence \( \mathcal{U}_q(\mathfrak{sl}_3) \) is generated by the elements \( e_i, f_i, k_i = q^{h_i} \). Observe that for \( i = 1, 2 \), the generators \( e_i, f_i, h_i \) satisfy the same relations as for \( \mathfrak{sl}_2 \), and hence \( \mathfrak{sl}_3 \) contains two copies of \( \mathfrak{sl}_2 \). These two copies are related to each other by

\[
\begin{bmatrix} h_1, e_2 \end{bmatrix} = -e_2, \quad \begin{bmatrix} h_1, f_2 \end{bmatrix} = f_2, \quad \begin{bmatrix} h_2, e_1 \end{bmatrix} = -e_1, \quad \begin{bmatrix} h_2, f_1 \end{bmatrix} = f_1.
\]

A natural way to encode this information is to consider certain elements of \( \mathfrak{h}^* \), where \( \mathfrak{h} \) is the two-dimensional linear span of \( h_1, h_2 \). If \( \alpha_i \in \mathfrak{h}^* \) are defined by \( \alpha_i(h_1) = 2 \), \( \alpha_i(h_2) = -1 \), and \( \alpha_1(h_1) = -1, \alpha_2(h_2) = 2 \), then the relations can be written succinctly as

\[
\begin{bmatrix} h_i, e_j \end{bmatrix} = \alpha_j(h_i)e_j, \quad \begin{bmatrix} h_i, f_j \end{bmatrix} = -\alpha_j(h_i)f_j.
\]

The definitions of \( \alpha_1, \alpha_2 \) can be encapsulated with a symmetric inner product \( \langle \cdot, \cdot \rangle \) on \( \mathfrak{h}^* \). If \( \langle \alpha_1, \alpha_2 \rangle = \langle \alpha_2, \alpha_1 \rangle = 2 \) and \( \langle \alpha_1, \alpha_2 \rangle = -1 \), then \( \alpha_i(h_j) = (\alpha_j, \alpha_i) \). Another way of expressing the inner product is to identify \( \mathfrak{h}^* \) with the space \( \mathfrak{h} \) itself:

\[
\begin{bmatrix} a, b \end{bmatrix} = \langle a, b \rangle = -1.
\]

We will also use Greek letter subscripts \( \kappa \) to denote the \( \kappa_i \), when it is notationally more convenient to do so, and \( \kappa_i \kappa_j \) denotes \( \kappa_i \kappa_j \). For example, \( \kappa_{1,1} \) denotes \( \kappa_1 \). By appending an additional generator \( \kappa_{1,1,1} \) to \( \mathcal{U}_q(\mathfrak{sl}_3) \), one obtains \( \mathcal{U}_q(\mathfrak{g}_3) \). This element commutes with all of \( \mathcal{U}_q(\mathfrak{sl}_3) \) and satisfies \( \Delta(k_{1,1,1}) = k_{1,1,1} \otimes k_{1,1,1} \) and \( S(k_{1,1,1}) = k_{1,1,1}^{-1} \).
Given a representation $W$ of $\mathfrak{gl}_3$, and $\mu \in \mathfrak{h}^*$, define $W[\mu]$ to be

$$W[\mu] = \{ w \in W : k_i w = q^{\mu(i,j)} w \ \text{for} \ i = 1, 2 \}.$$ 

The element $\mu$ is called a weight of $W$ and $W[\mu]$ is the weight space. Any finite-dimensional representation decomposes as a direct sum of weight spaces, and are preserved in the sense that $e_i W[\mu] \subseteq W[\mu + \alpha_i], f_i W[\mu] \subseteq W[\mu - \alpha_i]$. One can think of weight spaces as generalized eigenspaces, and weights as generalized eigenvalues.

Let us also describe $\mathfrak{sp}_4$, which is also known as the type $C_2$ Lie algebra. Recall that this is the Lie algebra consisting of $4 \times 4$ matrices of the form

$$
\begin{pmatrix}
A & B \\
C & D
\end{pmatrix} : A = -D^T, B = B^T, C = C^T
$$

Again using the same notation for $E_{ij}$, set

$$
e_1 = E_{12} - E_{21}, \quad f_1 = E_{21} - E_{12}, \quad h_1 = E_{11} - E_{22} - E_{33} + E_{44},$$

$$
e_2 = E_{24}, \quad f_2 = E_{42}, \quad h_2 = E_{22} - E_{44}.$$

Then $\mathfrak{sp}_4$ has a basis $e_1, e_2, f_1, f_2, h_1, h_2$. Once again, each $e_i, f_i, h_i$ generate a copy of $\mathfrak{sl}_2$, and the two copies are related by

$$[h_1, e_2] = -2e_2, \quad [h_2, f_2] = 2f_2,$$

where we have written $2h_2$ to preserve the symmetry. So $\mathcal{U}_q(\mathfrak{sp}_4)$ is generated by $e_1, f_1, k_1 = q^{h_1}, e_2, f_2, k_2 = q^{h_2}$. As before, these relations can be summarized by describing the roots of $\mathfrak{sp}_4$. Let $\mathfrak{h}$ denote the linear span of $h_1, h_2$ and identify $\mathfrak{h} \cong \mathbb{R}^2$ via $h_1 \mapsto x_1, 2h_2 \mapsto x_2$. Define $\alpha_1(x_1) = 2, \alpha_1(x_2) = -2$ and $\alpha_2(x_1) = -2, \alpha_2(x_2) = 4$ so that the above relations are

$$[x_i, e_j] = \alpha_j(x_i) e_j \quad [x_i, f_j] = -\alpha_j(x_i) f_j.$$

If $\mathfrak{h}^*$ is identified with $\mathbb{R}^2$ then $\alpha_1 = (1, -1), \alpha_2 = (0, 2)$, so that $\alpha_j(x_i) = (\alpha_i, \alpha_j)$ where $(\cdot, \cdot)$ is the usual Euclidean inner product.

Therefore, $\mathcal{U}_q(\mathfrak{sp}_4)$ is the Hopf algebra generated by $\{e_i, f_i, k_i\}, i = 1, 2$ with the Weyl relations

$$[e_i, f_j] = \delta_{ij} \begin{pmatrix} k_i - k_i^{-1} \\ q_i - q_i^{-1} \end{pmatrix}, \quad [k_i, k_j] = 0,$$

and the Serre relations (again, these relations are not used in this paper)

$$e_i^2 e_j - (q^2 + 1 + q^{-2}) e_i e_j e_i + (q^2 + 1 + q^{-2}) e_i e_j e_i = 0 \quad i \neq j,$$

$$f_i^2 f_j - (q^2 + 1 + q^{-2}) f_i f_j f_i + (q^2 + 1 + q^{-2}) f_i f_j f_i = 0 \quad i \neq j.$$

The co-product is

$$\Delta(e_i) = e_i \otimes 1 + k_i \otimes e_i \quad \Delta(f_j) = 1 \otimes f_j + f_j \otimes k_j^{-1}, \quad \Delta(k_i) = k_i \otimes k_i.$$

Again, this extends to a homomorphism $\mathcal{U}_q(\mathfrak{sp}_4) \rightarrow \mathcal{U}_q(\mathfrak{sp}_4)^{\otimes 2}$ by co-associativity. The antipode (again, not used in this paper) is
As before, we will also use Greek letter subscripts $k_{ij}$ to denote the $k_i$. Once again, any finite-dimensional representation decomposes as a direct sum of weight spaces which are preserved under the action of the quantum group.

4. Central Element

The first step is to find a suitable central element in $U_q(\mathfrak{g})$ for $\mathfrak{g} = \mathfrak{gl}_3, \mathfrak{sp}_4$. (Any central element will yield a Markov process, at least in a neighborhood of $q = 1$, but a proof of this is beyond the scope of this paper). In order to describe the Markov process the central element needs to be explicitly written in terms of the generators of $U_q(\mathfrak{g})$. This will be done with the quantum Harish-Chandra isomorphism. In principle, one could directly check that the resulting element is central using only the commutation relations, but the whole proof is presented here in order to make the construction less mysterious and more applicable for other Lie algebras.

Letting $b_\pm \subseteq \mathfrak{g}$ denote the Borel subalgebras (that is, $b_+$ is generated by $e_i, h_i$ and $b_-$ is generated by $f_i, h_i$), there is a bi-linear pairing (see proposition 6.12 of [16]) on $\mathfrak{g} (\mathfrak{g})^{-1}_{-}$ defined on generators by

$$\langle k_i, k_j \rangle = q^{-\alpha_i \alpha_j}, \quad \langle f_i, e_j \rangle = \frac{-\delta_{ij}}{q_i - q_i^{-1}},$$

$$\langle k_i, e_j \rangle = \langle f_i, k_j \rangle = \langle 1, e_i \rangle = \langle f_i, 1 \rangle = 0, \quad \langle 1, 1 \rangle = 1$$

and extended to all of $U_q(b_-) \times U_q(b_+)$ by

$$\langle y, xx' \rangle = \langle \Delta(y), x' \otimes x \rangle, \quad \langle yy', x \rangle = \langle y \otimes y', \Delta(x) \rangle,$$

$$\langle y \otimes y', x \otimes x' \rangle = \langle y, x \rangle \langle y', x' \rangle,$$

(1) where $(\cdot, \cdot)_q$ is the invariant, non-degenerate invariant symmetric bilinear form on $\mathfrak{h}^*$ from section 3. Furthermore, according to lemma 6.16 of [16],

$$\langle \omega(x), \omega(y) \rangle = \langle y, x \rangle = \langle \tau(y), \tau(x) \rangle,$$

(2)

where $\omega$ is the automorphism and $\tau$ is the antiautomorphism defined by

$$\omega(e_i) = f_i, \quad \omega(f_i) = e_i, \quad \omega(k_i) = k_i^{-1},$$

$$\tau(e_i) = e_i, \quad \tau(f_i) = f_i, \quad \tau(k_i) = k_i^{-1}.$$

Let $V$ be a fundamental representation of $\mathfrak{g}$ and let $\{v_\mu \} \subseteq V$ be a basis of $V$ such that $v_\mu \in V[\mu]$, the $\mu$-weight space of $V$. It is a classical result in representation theory that the set of weights $\mu \in \mathfrak{h}^*$ such that $V[\mu]$ is non-zero for some finite-dimensional representation $V$ is a discrete subset of $\mathfrak{h}^*$. Define an ordering $\geq$ on this discrete set by $\mu \geq \lambda$ if $\mu(f) \geq \lambda(f)$, where $f$ is a fixed element of $\mathfrak{h}$ such that $\mu(f) = 0$ for all relevant weights $\mu$. For any $\mu \geq \lambda$ such that $V[\mu]$ and $V[\lambda]$ are non-zero, let $e_{\mu, \lambda}$ and $f_{\mu, \lambda}$ be elements generated by $\{e_i\}$ and $\{f_i\}$ respectively such that $e_{\mu, \lambda} v_\lambda = v_\mu$ and $f_{\mu, \lambda} v_\mu = v_\lambda$. Let $\rho$ be half the sum of the positive roots of $\mathfrak{g}$, where a root $\alpha$ is positive if $\alpha > 0$. Explicitly, $\rho = (1, 0, -1)$ for $\mathfrak{sl}_3$ and $\rho = (2, 1)$ for $\mathfrak{sp}_4$.

The next lemma constructs a central element of $U_q(\mathfrak{g})$. Recall that the root lattice of $\mathfrak{g}$ is the lattice in $\mathfrak{h}^*$ spanned by the roots of $\mathfrak{g}$. When $\mathfrak{g} = \mathfrak{sp}_4$, the root lattice can be explicitly written as $\{(x_1, x_2) : x_1 + x_2 \in 2\mathbb{Z}\} \subseteq \mathbb{Z}^2$. 

$$S(e_i) = -k_i^{-1} e_i, \quad S(f_i) = -f_i k_i, \quad S(k_i) = k_i^{-1}.$$
Lemma 4.1. If $q$ is not a root of unity and $2\mu$ is in the root lattice of $\mathfrak{g}$ for all weights $\mu$ of $V$, then the element
\[
\sum_{\mu \geq \lambda} q^{(\mu-\lambda,\mu)} q^{-2(\mu,\mu)} e^{*}_{\mu,\lambda} k_{-\lambda-2\mu} f^{*}_{\mu}
\]
(3)
is central in $\mathcal{U}_q(\mathfrak{g})$, where the star $^*$ denotes the dual element under $(\cdot, \cdot)$.

Proof. This will follow from the construction of the Harish-Chandra isomorphism in [16]. The notation here will closely follow the notation in [16]. Lemma 6.23 says that if $\lambda$ is a dominant weight with $2\mu$ in the weight lattice, there exists a central element $z_{\mu}$ such that $u z_{\mu}$ is equal to the trace of $u k_{-\lambda-2\mu}$ acting on $L(\lambda)$ for all $u$. We will use this for the case when $\lambda$ is a fundamental weight. The existence is because if $m_1, \ldots, m_r$ is a basis of $L(\lambda)$ and $f_1, \ldots, f_r$ is the dual basis, then the trace of $u k_{-\lambda-2\mu}$ acting on $L(\lambda)$ is equal to
\[
\sum_{i=1} c_{f_i, k_{-\lambda-2\mu}}(u)
\]
where $c_{f_m}$ is the linear form defined by $c_{f_m}(v) = f(vm)$.

Now, lemma 6.22 says that for all $f, m$, there exists a unique $u$ such that $c_{f_m}(v) = (v, u)$. This is because there are only finitely many pairs of weights $(\mu, \nu)$ such that $c_{f_m}(U^- U^0 U^+_n) = 0$, and a computation shows that for each $x \in U^+_n$, $y \in U^-_m$, and each $\eta$ in the weight lattice,
\[
c_{f_m}(yk_x k_\eta x) = \left( q^{1/2} \right)^{(y, 2\lambda + 2\nu)} f(y k_x x)
\]
where we have assumed that $m \in M_\lambda$. Now equation (1) from section 6.22 of [16] says that we can take $u = \sum_{(\mu, \nu), u_{\nu \mu}} u_{\nu \mu}$ where each $u_{\nu \mu}$ is the element
\[
\sum_{i,j} q^{-2(\mu,\nu)} f(v^\mu_{\nu j} u^\mu_{\nu j} m) v^\nu_{\nu i} k_{-2(\lambda + \nu)} u^\mu_{\nu i}
\]
where $\{u^\mu_{\nu i}\}$ is an arbitrary basis of $U^+_\nu$ and $\{v^\nu_{\nu i}\}$ is a dual basis of $U^-_{\nu}$. Since the sum will be over $f_i, m_i$ corresponding to a basis and dual basis of the fundamental representation, the sum is only non-zero when $\mu = \nu$. Combining these statements, we have
\[
\sum_{\lambda} \sum_{\nu} \sum_{i,j} q^{-2(\mu,\nu)} f(v^\mu_{\nu j} u^\mu_{\nu j}^{-1} m) v^\nu_{\nu i} k_{-2(\lambda + \nu)} u^\mu_{\nu i},
\]
where the sum is over the weights $\lambda$ in the fundamental representation and the weights $\nu$ such that $\nu + \lambda$ is also a weight in the fundamental representation. Since $\nu$ is the difference of fundamental weights, the sum over $i, j$ is only over a single term. Thus, setting $\nu = \mu - \lambda$, one can take $u^\mu_{\nu i} = f_{\mu,\nu}$ and $v^\nu_{\nu i} = e_{\mu,\nu}$. The action of $k_{-\nu}$ on $m \in M_\mu$ yields $q^{-2(\mu,\nu)}$ and the action of $u^\mu_{\nu i} m \in M_\mu$ yields $q^{\mu,\nu}$. Thus, one sees that
\[
\sum_{\mu \geq \lambda} q^{(\nu,\mu)} q^{-2(\mu,\nu)} e^{*}_{\mu,\lambda} k_{-\lambda-2\nu} f^{*}_{\mu}
\]
(3) is central, which simplifies to (3) by directing plugging in $\nu = \mu - \lambda$. \qed
4.1. \( \mathfrak{sp}_4 \)

Let \( V \) be the natural four-dimensional representation of \( \mathfrak{sp}_4 \). It has a basis \( v_1, v_2, v_3, v_4 \) which are in the weight spaces \( (1, 0), (0, 1), (0, -1), (-1, 0) \). It is immediate that the condition of lemma 4.1 holds. In order to write the central element, the dual elements need to be calculated.

**Lemma 4.2.** The dual elements are:

\[
\begin{align*}
(e_1)^* &= -(q - q^{-1})f_1, \\
(e_2)^* &= -(q^2 - q^{-2})f_2, \\
(f_1)^* &= -(q^2 - q^{-2})e_1, \\
(f_2)^* &= -(q - q^{-1})e_2, \\
(e_1e_2)^* &= (q - q^{-1})(q^2 f_2 f_2 - f_2 f_1), \\
(f_1f_2)^* &= (q - q^{-1})(q^2 e_1 e_2 - e_2 e_1), \\
(e_2e_1)^* &= (q - q^{-1})(q^2 f_2 f_2 - f_2 f_1), \\
(f_2f_1)^* &= (q - q^{-1})(q^2 e_1 e_2 - e_2 e_1), \\
(e_1e_2e_1)^* &= (q - q^{-1})(q^2 f_2 f_2 - f_2 f_1) - (q^2 f_2 f_2 - f_2 f_1) f_1, \\
(e_2e_1e_2)^* &= (q - q^{-1})(q^2 f_2 f_2 - f_2 f_1) - (q^2 f_2 f_2 - f_2 f_1) f_1, \\
(f_2f_1f_2)^* &= (q - q^{-1})(q^2 e_1 e_2 - e_2 e_1) e_1 - e_1 (q^2 e_1 e_2 - e_2 e_1). \\
\end{align*}
\]

**Proof.** The first two lines follow immediately from the definition of the pairing. In order to prove the next two lines, it suffices to compute the four pairings \( \langle f_2, e_1 e_2 \rangle, \langle f_2, e_2 e_1 \rangle, \langle f_2, f_1 e_1 \rangle, \langle f_2, e_1 e_2 \rangle \). The definition (1) implies that

\[
\begin{align*}
\langle f_2, e_1 e_2 \rangle &= \langle f_2 \otimes f_2, e_2 \otimes e_1 \rangle \\
&= \langle f_2, e_2 \rangle \langle f_1 \otimes k^{-1}, e_1 \otimes 1 \rangle \\
&= (q^2 - q^{-2})^{-1} (q - q^{-1})^{-1} \\
\langle f_2, e_2 e_1 \rangle &= \langle f_1 \otimes f_2, e_2 e_1 \rangle \\
&= \langle f_1 \otimes k^{-1}, e_1 \otimes k^{-1} \rangle \langle f_2, e_2 \rangle \\
&= q^{-2} (q^2 - q^{-2})^{-1} (q - q^{-1})^{-1}. \\
\end{align*}
\]
By applying $\tau$ and using (2),

$$\langle f_2 f_1, e_2 e_1 \rangle = \left( q^2 - q^{-2} \right)^{-1} \left( q - q^{-1} \right)^{-1},$$

$$\langle f_2 f_1, e_1 e_2 \rangle = q^{-2} \left( q^2 - q^{-2} \right)^{-1} \left( q - q^{-1} \right)^{-1}.$$ 

This proves lines three and four.

Now for the remainder of the lemma. In order to finish the computations, at least $9 = 3^2$ pairings need to be computed. Observe that $\tau((e_1 e_2)^*) = (e_2 e_1)^*$, so using (2) shows that

$$\langle (e_1 e_2)^* f_1, e_1 e_2 e_1 \rangle = \langle f_1, (e_2 e_1)^* e_2 e_1 \rangle$$

$$\langle (e_1 e_2)^* f_1, e_1 e_1 e_2 \rangle = \langle f_1, (e_2 e_1)^* e_2 e_1 \rangle$$

$$\langle f_1, (e_1 e_2)^* f_1, e_2 e_1 e_1 \rangle = \langle (e_2 e_1)^* f_1, e_2 e_1 e_1 \rangle$$

$$\langle (e_1 e_2)^* f_1, e_2 e_1 e_2 \rangle = \langle f_1, (e_2 e_1)^* e_1 e_1 e \rangle$$

which leaves 5 pairings that need to be computed. Then using direct computations, the first three lines are

$$\langle (e_1 e_2)^* f_1, e_1 e_2 e_1 \rangle = \langle (e_1 e_2)^* \otimes f_1, e_1 e_2 k_1 \otimes e_1 \rangle$$

$$= \langle \Delta((e_1 e_2)^*), k_1 \otimes e_1 e_2 \rangle \langle e_1, f_1 \rangle$$

$$= \langle k_1 k_2 \otimes (e_1 e_2)^*, k_1 \otimes e_1 e_2 \rangle \langle e_1, f_1 \rangle$$

$$= \left( q - q^{-1} \right)^{-1}$$

$$\langle (e_1 e_2)^* f_1, e_1 e_1 e_2 \rangle = \langle (e_1 e_2)^* \otimes f_1, e_1 k_1 e_2 \otimes e_1 + k_1 e_1 e_2 \otimes e_1 \rangle$$

$$= (1 + q^{-2}) \langle (e_1 e_2)^* \otimes f_1, k_1 e_1 e_2 \otimes e_1 \rangle$$

$$= \left( 1 + q^{-2} \right) \left( q - q^{-1} \right)^{-1}$$

$$\langle f_1, (e_1 e_2)^* f_1, e_1 e_1 e_2 \rangle = \langle f_1 \otimes (e_1 e_2)^*, e_1 k_1 k_2 \otimes e_1 e_2 + k_1 e_1 k_2 \otimes e_1 e_2 \rangle$$

$$= (1 + q^2) \langle f_1 \otimes (e_1 e_2)^*, e_1 k_1 k_2 \otimes e_1 e_2 \rangle$$

$$= \left( 1 + q^2 \right) \left( q - q^{-1} \right)^{-1}.$$

Furthermore, the fourth line is

$$\langle (e_1 e_2)^* f_1, e_2 e_1 e_1 \rangle = 0.$$ 

because $e_1 e_2$ cannot appear in the left tensor power of $\Delta(e_2 e_1 e_1)$. Finally, the fifth computation is

$$\langle (e_2 e_1)^* f_1, e_1 e_2 e_1 \rangle = \langle (e_2 e_1)^* \otimes f_1, k_1 e_1 e_1 \otimes e_1 \rangle$$

$$= \left( q - q^{-1} \right)^{-1}.$$

Combining these nine pairings finishes the proof of lines five through seven. Lines eight through ten follow from applying (2) to lines five through seven. \qed
Proposition 4.3. If $q$ is not a root of unity, the element

\[
q^{-4}k_{(-2,0)} + q^{-2}k_{(0,-2)} + q^2k_{(0,2)} + q^4k_{(2,0)}
+ q^{-3}(q - q^{-1})^2f_1k_{(-1,-1)} \epsilon_1 + q^{-3}(e_1e_2)^k_{(-1,1)}(f_2f_1)^k + (q^2 + q^{-2})f_2\epsilon_2
+ q^{-2}(e_1e_2)^k(f_2f_1)^k + q^{-1}(e_2\epsilon_1)^k_{(-1,1)}(f_1f_2)^k + q^3(q - q^{-1})^2f_1k_{(1,1)} \epsilon_1
\]

\[
=q^{-4}k_{(-2,0)} + q^{-2}k_{(0,-2)} + q^2k_{(0,2)} + q^4k_{(2,0)}
+ (q - q^{-1})^2g^{-3}f_1k_{(-1,-1)} \epsilon_1
+ (q - q^{-1})^2g^3f_1k_{(1,1)} \epsilon_1 + (q^2 - q^{-2})^2f_2\epsilon_2
+ (q - q^{-1})^2(q^{-1}(qf_2 - q^{-1}f_2)k_{(-1,1)}(qe_1\epsilon_1 - q^{-1}e_1\epsilon_1)
+ q(qf_2 - q^{-1}f_2)k_{(1,1)}(qe_1\epsilon_2 - q^{-1}e_1\epsilon_2))
+ (q - q^{-1})^2(f_1f_2 - (q^2 + q^{-2})f_1f_2)
+ f_2f_1(e_1\epsilon_1\epsilon_2 - (q^{-2} + q^2)e_1\epsilon_2\epsilon_1 + e_2\epsilon_1\epsilon_1)
\]

is central in $U_q(\mathfrak{sp}_2)$.

Proof. Use (3) and the definition of $\rho$. The terms with $\mu = \lambda$ yield

\[
q^{-4}k_{(-2,0)} + q^{-2}k_{(0,-2)} + q^2k_{(0,2)} + q^4k_{(2,0)}
\]

Furthermore, $\mu = (1,0)$, $\lambda = (0,1)$ yields

\[
q^{-3}(q - q^{-1})^2f_1k_{(-1,-1)} \epsilon_1
\]

and $\mu \in \{(1,0), (0,1)\}$, $\lambda = (0,-1)$ yields

\[
q^{-3}(e_1e_2)^k_{(-1,1)}(f_2f_1)^k + (q^2 + q^{-2})f_2\epsilon_2
\]

and $\mu \in \{(1,0), (0,1), (0,-1)\}$, $\lambda = (-1,0)$ yields

\[
q^{-2}(e_1e_2)^k(f_2f_1)^k + q^{-1}(e_2\epsilon_1)^k_{(-1,1)}(f_1f_2)^k + q^3(q - q^{-1})^2f_1k_{(1,1)} \epsilon_1.
\]

Observe that in each monomial in this central element, $e_i$ and $f_i$ occur the same number of times. In fact, this is true of every central element. Probabilistically, this means that the number of particles of each type is preserved under the dynamics.

One can check (with some calculation) that this element acts as $q^6 + q^{-2} + q^2 + q^6$ times the identity on $V$, which is consistent with the Harish-Chandra isomorphism.
4.2. $g|_{\mathfrak{g}_3}$

It was shown in [14] that the following element is central:

$$C := (q - q^{-1})^{-2} q^{-1} \left( - (q^{-2} + 1 + q^2) + q^{-2}k_{(2,0,0)} + k_{(0,2,0)} + q^2k_{(0,0,2)} \right)$$
$$+ \left( q - q^{-1} \right)^2 (q^{-1}k_{(1,1,0)}e_1f_1 + qk_{(0,1,1)}e_2f_2)$$
$$+ qk_{(1,0,1)}(e_1e_2 + q^{-1}e_2e_1)(f_2f_1 - q^{-1}f_1f_2)),$$

(4)

5. Type $C_2$ ASEP

5.1. Notation

Because different authors use slightly different notation, it is necessary to first establish notation for this paper. There are two creation operators $e_1$, $e_2$. In the $A_2$ case, the operator $e_2$ creates a particle of type 2 and the operator $e_1$ replaces a particle of type 2 with a particle of type 1. The annihilation operator $f_1$ replaces a particle of type 1 with a particle of type 2, and $f_2$ annihilates a particle of type 2. In the $C_2$ case the operator $e_1$ creates a particle of type 1 and $e_2$ replaces a particle of type 1 with a particle of type 2, and similarly for $f_1$, $f_2$. In a sense, $e_1$, $f_1$ are more accurately called ’replacement’ operators instead of creation and annihilation operators. In the $C_2$ case, $v_1$ corresponds to a site with both a type 1 and a type 2 particle.

For $i \leq 4$, let $v_i$ denote the vector with a 1 in the $i$th co-ordinate and a 0 elsewhere. The vector $v_1$ corresponds to a completely full site, so that the creation operators $e_1$, $e_2$ act as $e_1v_1 = e_2v_1 = 0$. The vector $v_3$ corresponds to a completely empty site, so that the annihilation operators $f_1$, $f_2$ act as $f_1v_3 = f_2v_3 = 0$. (It is essentially a coincidence that $v_3$ is the lowest weight vector in both the $A_2$, $C_2$ cases.) In the $A_2$ case, this means that $v_3$ is an empty site, $v_2$ is a particle of type 2 and $v_1$ is a particle of type 1. In the $C_2$ case, this means that $v_3$ is an empty site, $v_4$ is a particle of type 1, $v_2$ is a particle of type 2 and $v_1$ is a site occupied by both a particle of type 1 and 2. The vacuum vector $v_L$ corresponds to $L$ lattice sites all completely empty.

Under this identification, the generator $\mathcal{L}$ of a Markov process $X_t$ on the state space $\{\infty, 1, 2\}^L$ can be identified as a linear operator on $V^\otimes L$. In the $C_2$ case, we a priori need to include a state where both a type 1 and a type 2 particle occupy a site; however, in the construction of the dynamics, the probability of jumping to such a state is zero. An initial condition can be expressed as a vector $A_0 \in V^\otimes L$ by

$$A_0 := \sum_v |X_0 = v \rangle \langle v|.$$ 

Here, and below, the summation $\sum_v$ is over tensors of the form $v_i \otimes \cdots \otimes v_k$. A random variable $\mathcal{O}$ on $\{\infty, 1, 2\}^L$ can be identified with a diagonal operator on $V^\otimes L$ via $v \mapsto \mathcal{O}(v) v$. The same letter $\mathcal{O}$ will refer to both the random variable and the operator.

The inner product $\langle \cdot, \cdot \rangle$ on $V^\otimes L$ is defined by

$$\langle v_i \otimes \cdots \otimes v_k, v_i' \otimes \cdots \otimes v_k' \rangle = \delta_{i=i_1, \ldots, i_k=i_k}.$$

This is essentially the usual bra-ket notation. The expectation of a random variable $\mathcal{O}$ at time $t$ of a Markov process with generator $\mathcal{L}$ and initial condition $A_0$ can be computed as
5.2. Construction

Let $C$ be the central element in proposition 4.3 and let $A$ be the operator on $V \otimes V$ defined by

$$
(q^{-4} + q^{6})^{-1}(q - q^{-1})^{-2} \Delta \left( C - (q^{-8} + q^{-2} + q^{2} + q^{8}) \right).
$$

The constant $(q^{-4} + q^{6})^{-1}$ is chosen so that the jump rates $L(1, \infty), L(2, \infty)$ are equal to 1. Note that $V \otimes V$ has the decomposition into nine different weight spaces (where $W[a, b]$ refers to the $(a, b)$ weight space of the representation $W$)

$$
V \otimes V = (V \otimes V)[2, 0] \oplus (V \otimes V)[1, 1] \oplus (V \otimes V)
\times [0, 2] \oplus (V \otimes V)[1, -1] \oplus (V \otimes V)[0, 0]
= (V \otimes V)[-1, 1] \oplus (V \otimes V)[0, -2] \oplus (V \otimes V)[-1, -1] \oplus (V \otimes V)[-2, 0]
$$

which have dimensions 1, 2, 1, 2, 4, 2, 1, 2, 1 respectively. Order the basis elements of $V \otimes V$ as

$$
v_1 \otimes v_1, v_2 \otimes v_1, v_1 \otimes v_2, v_2 \otimes v_2, v_4 \otimes v_1,
v_1 \otimes v_4, v_2 \otimes v_4, v_4 \otimes v_2, v_3 \otimes v_1, v_1 \otimes v_3,
v_3 \otimes v_2, v_2 \otimes v_3, v_4 \otimes v_4, v_3 \otimes v_4, v_4 \otimes v_3, v_3 \otimes v_3.
$$

This ordering preserves the ordering of the weight spaces.

As explained in section 2.3, the operator $A$ needs to be conjugated with a diagonal operator corresponding to an eigenvector of $A$ with eigenvalue 0.

**Lemma 5.1.** The following are linearly independent eigenvectors of $A$ with eigenvalue 0:

$$
v_3 \otimes v_3 \in (V \otimes V)[2, 0],
e_{1}(v_3 \otimes v_3) \in (V \otimes V)[1, 1],
e_{1}^{2}(v_3 \otimes v_3) \in (V \otimes V)[0, 2],
e_{2}e_{1}(v_3 \otimes v_3) \in (V \otimes V)[1, -1],
e_{2}e_{1}^{2}(v_3 \otimes v_3) \in (V \otimes V)[0, 0],
e_{1}e_{2}e_{1}(v_3 \otimes v_3) \in (V \otimes V)[0, 0],
e_{1}^{2}e_{2}e_{1}(v_3 \otimes v_3) \in (V \otimes V)[-1, 1],
(e_{2}e_{1})^{2}(v_3 \otimes v_3) \in (V \otimes V)[0, -2],
e_{1}(e_{2}e_{1})^{2}(v_3 \otimes v_3) \in (V \otimes V)[-1, -1],
e_{1}^{2}(e_{2}e_{1})^{2}(v_3 \otimes v_3) \in (V \otimes V)[-2, 0].
$$

So the 0- eigenspace of $A$ is at least ten-dimensional.

**Proof.** This follows because $C(v_3 \otimes v_3) = (q^{-8} + q^{-2} + q^{2} + q^{8})v_3 \otimes v_3$ and $A$ commutes with $U_{4}(Sp_{4})$. Note that $e_{2}e_{1}^{2}$ and $e_{1}e_{2}e_{1}$ produce linearly independent eigenvectors because the latter has $v_1 \otimes v_3, v_3 \otimes v_1$ terms and the former does not. □
From the explicit form of $C$ and the fact that $eW[\mu] = W[\mu + e_i]$, $f_i W[\mu] = W[\mu - e_i]$, it follows that $A$ must preserve each summand in the weight space decomposition, so $A$ decomposes into a block matrix with nine blocks. By lemma 5.1, for the one-dimensional weight spaces with weights $(2, 0), (0, 2), (-2, 0), (0, -2)$, the corresponding block matrices are $1 \times 1$ zero matrices. Therefore $A$ has five non-zero blocks corresponding to the weights $(1, 1), (1, -1), (0, 0), (-1, 1), (-1, -1)$, with sizes $2, 2, 4, 2, 2$ respectively. Write this decomposition as

$$A = (q^{-4} + q^6)^{-1}(A_{(1,1)} + A_{(1,-1)} + A_{(0,0)} + A_{(-1,1)} + A_{(-1,-1)}).$$

**Lemma 5.2.** As matrices with respect to the ordered basis in (5),

$$A_{(0,0)} = \begin{pmatrix}
-q^{-2}(q^2 + q^{-2})^2 & (q^2 + q^{-2})^2 \\
(q^2 + q^{-2})^2 & -q^{-2}(q^2 + q^{-2})^2 \\
-q^{-3} + q^{-1} + 2q^3 & q^{-5} - q^{-3} - 2q \\
-2q^{-1} - q^3 & 2q^{-3} + q^{-3} \\
-q^{-3} + q^{-1} + 2q^3 & -2q^{-1} - q^{-3} + q^5 \\
q^{-5} - q^{-3} - 2q & 2q^{-3} + q^{-3} \\
-q^{-4} + q^{-2} - 1 - 2q^4 - q^6 & (q^2 + q^{-2})^2 \\
(q^2 + q^{-2})^2 & -q^{-6} - 2q^{-4} - 1 + q^2 - q^4
\end{pmatrix},$$

$$A_{(1,1)} = A_{(1,-1)} = A_{(-1,1)} = A_{(-1,-1)} = \begin{pmatrix}
-q^{-4} + q^6 & (q^{-5} + q^6) \\
(q^{-5} + q^6) & -(q^{-6} + q^4)
\end{pmatrix}.$$ 

**Proof.** From the definition of the co-product, the matrices for the generators can be written explicitly. For $1 \leq i, j \leq 16$, let $E_{ij}$ denote the matrix with a 1 in the $(i, j)$-entry and 0 elsewhere. Then

$$\Delta(e_i) = E_{12} + qE_{13} + q^{-1}E_{24} + E_{34} + qE_{47} + E_{68} + E_{59} + qE_{510} + q^{-1}E_{911},$$

$$+ E_{1011} + E_{712} + q^{-1}E_{812} + qE_{1314} + E_{1315} + E_{1416} + q^{-1}E_{1516},$$

$$\Delta(f_i) = q^{-1}E_{21} + E_{31} + E_{42} + qE_{43} + q^{-1}E_{95} + E_{105} + E_{76} + q^{-1}E_{86} + qE_{127},$$

$$+ E_{128} + E_{119} + qE_{1110} + E_{1413} + q^{-1}E_{1513} + qE_{1614} + E_{1615},$$

$$\Delta(e_2) = E_{25} + E_{36} + q^2E_{48} + E_{84} + E_{68} + q^2E_{1013} + E_{1214} + E_{1115},$$

$$\Delta(f_2) = E_{52} + E_{63} + E_{84} + q^{-2}E_{104} + q^2E_{138} + E_{1310} + E_{1511} + E_{1412},$$

$$\Delta(k_{(a,b)}) = \text{diag}(q^{2a}, q^{a+b}, q^{-a+b}, q^{2b}, q^{a-b}, q^{-a-b}, q^{-2a}).$$

Using proposition 4.3 and explicit multiplication of $16 \times 16$ matrices yields the result.
Define the operator $A^L$ on $V^\otimes L$ by

$$A^L = \sum_{i=1}^{L-1} 1^i \otimes A \otimes 1^{L-1-i}.$$ 

This will be the Hamiltonian of spin 1/2 type \(C_2\) ASEP. As in the usual ASEP case, it commutes with the action of the quantum group.

**Lemma 5.3.** For any $u \in U_q(\mathfrak{sp}_3)$,

$$[A^L, \Delta^L(u)] = 0.$$

**Proof.** It suffices to prove this for $u = e_i, f_i, k_i$. Since $\Delta^L(k_i) = k_i \otimes k_i$ and

$$[1^{i-1}, k_i] = [A, k_i \otimes k_i] = [1^{i-1}, k_i \otimes k_i] = 0,$$

this shows it for $u = k_i$. Now we have that

$$\Delta^L(e) = \sum_{j=1}^{L-1} k^0 \otimes \Delta(e) \otimes 1^{i-1-j}$$

and that

$$\left[ k^0 \otimes \Delta(e) \otimes 1^{i-1-j}, \sum_{i=1}^{L-1} 1^i \otimes A \otimes 1^{L-1-i} \right]$$

$$= \left[ k^0 \otimes \Delta(e) \otimes 1^{i-1-j}, 1^{0-2} \otimes A \otimes 1^{i-1-0+1} + 1^{0} \otimes A \otimes 1^{i-1-(j+1)} \right]$$

because for all other $j$ terms we can apply

$$[1 \otimes 1, k \otimes k] = [\Delta(e), 1 \otimes 1] = [\Delta(e), A] = [k \otimes k, A] = 0.$$

This then equals

$$\left[ k^0 \otimes e \otimes 1^{i-1-j} + k^0 \otimes e \otimes 1^{i-j}, 1^{0-2} \otimes A \otimes 1^{i-1-0+1} + 1^{0} \otimes A \otimes 1^{i-1-(j+1)} \right]$$

$$= k^0 \otimes [e \otimes 1, A] \otimes 1^{i-2-j} + k^0 \otimes [k \otimes A, A] \otimes 1^{i-2-j}.$$ 

Summing over $j$ yields

$$\sum_{j=0}^{L-2} k^0 \otimes [e \otimes 1, A] \otimes 1^{L-2-j} + \sum_{j=0}^{L-2} k^0 \otimes [k \otimes e, A] \otimes 1^{i-2-j}$$

$$= \sum_{j=0}^{L-2} k^0 \otimes [\Delta(e), A] \otimes 1^{L-2-j} = 0.$$

The argument for $f$ is similar. \(\square\)

In lemma 5.2, there is no value of $q$ for which the off-diagonal entries of $A_{10,0}$ are all non-negative, since the second row is $-q^2$ times the first row. This would indicate a ‘negative probability’ of transitioning to a state with both a type 1 and a type 2 particle occupying a site. To get around this issue, we conjugate with a $G_\epsilon$ such that as $\epsilon \to 0$, these ‘negative probabilities’ converge to 0.
Give $V^{\otimes L}$ the standard basis $B := \{ v_i \otimes \cdots \otimes v_i : i_1, \ldots, i_L \in \{1, 2, 3, 4\} \}$. Partition $B$ into $B_1 \cup B_2$, where

$$B_1 := \{ v_i \otimes \cdots \otimes v_i : i_1, \ldots, i_L \in \{2, 3, 4\} \}.$$ 

Note that $|B_1| = 3^L$ and $|B_2| = 4^L - 3^L$.

Define the sets

$$E_1 = \{ e_j^i e_k^i : 1 \leq j \leq k \leq L \},$$

$$E_2 = \{ e_i^j e_k^j : 1 \leq i \leq j \leq k \leq L \}.$$ 

Let $\Omega$ be the vacuum vector $v_3^{\otimes L}$. We then have that

$$e(\Omega) \in \text{span}(B_1) \text{ for all } e \in E_1, \quad e(\Omega) \not\in \text{span}(B_1) \text{ for all } e \in E_2.$$ 

Let $g_e \in V^{\otimes L}$ be a vector in the kernel of $A^{(L)}$, and for $x \in B$, define $g_e(x)$ to be the coefficient of $x$. Suppose it satisfies

$$g_e(x) > 0 \text{ for all } x \in B_1, \quad \lim_{\epsilon \to 0} g_e(y) = 0 \text{ for } y \in B_2, \quad \lim_{\epsilon \to 0} g_e(x) > 0 \text{ for } x \in B_1. \quad (6)$$

Let $G_e$ be the diagonal matrix on $V^{\otimes L}$ with entries $G_e(x, x) = g_e(x)$. Let $L_e$ be

$$L_e = G_e^{-1} A^{(L)} G_e. \quad (7)$$

For a matrix $S$ that commutes with $A^{(L)}$, let $D_e = G_e^{-1} S G_e^{-1}$. In the $\epsilon \to 0$ limit, the subscript will be dropped. The idea for this construction of $D$ comes from Proposition 2.1 of [9].

**Proposition 5.4.** If $x \in \text{span}(B_1)$, then

$$\lim_{\epsilon \to 0} \langle y, L_e D_e(x) \rangle = \lim_{\epsilon \to 0} \langle y, D_e^{(x)}(x) \rangle \text{ for all } y \in \text{span}(B_1)$$

(and this limit is finite).

**Proof.** Since $A^{(L)}$ is symmetric,

$$L_e D_e = G_e^{-1} A^{(L)} S G_e^{-1} = G_e^{-1} S G_e^{-1} G_e A^{(L)} G_e^{-1} = D_e^{(x)}$$

so it remains to check that the limit is finite. But by (6), the limit can only be infinite if $x$ or $y$ is not in the span of $B_1$. \qed

The next step is to find an explicit $g_e$ satisfying (6). Applying an appropriate linear combination of products of $\Delta^{(L)} e_1, \Delta^{(L)} e_2$ to $v_3^{\otimes L}$ will produce $g_e$. It turns out that for a certain choice of coefficients, the expression for $g_e$ will factor over lattice sites, which will make computations and proofs simpler. This is the motivation for defining the $q$-exponential below.

**Definition 5.5.** The $q$-analog of the exponential function is

$$\exp_q(x) := \sum_{n=1}^{\infty} \frac{x^n}{[n]_q!},$$

where $[n]_q! := [1]_q [2]_q \cdots [n]_q$. 

...
where

\[ \{n\}_q \equiv \frac{1 - q^n}{1 - q}. \]

The following is proposition 5.1 from [9].

**Proposition 5.6.** Let \( \{g_i, k_i; 1 \leq i \leq L\} \) be operators such that \( k_i g_i = r g_k k_i \). Define

\[
\begin{align*}
  k^{(i)} & := k_{i-1} \cdots k_1 \quad g^{(L)} = \sum_{i=1}^L k^{(i-1)}_g, \\
  h^{(i)} & := k_{i-1}^{-1} \cdots k_1^{-1}, \\
  \hat{g}^{(L)} & := \sum_{i=1}^L g_i h^{(i+1)}.
\end{align*}
\]

Then

\[
\begin{align*}
  \exp_r(g^{(L)}) & = \exp_r(g_1) \cdot \exp_r(k^{(1)} g_2) \cdot \exp_r(k^{(2)} g_3) \cdot \cdots, \\
  \exp_r(\hat{g}^{(L)}) & = \exp_r(g_1 h^{(2)}) \cdot \exp_r(g_{L-1} h^{(L)}) \exp_r(g_L).
\end{align*}
\]

In this paper, the proposition will be applied with

\[
\begin{align*}
  g_i & = 1^0 \otimes e \otimes 1^{\otimes L-i}, \\
  k_i & = 1^0 \otimes k \otimes 1^{\otimes L-i},
\end{align*}
\]

where \( e, k \) can be either \( e_1, k_1 \) or \( e_2, k_2 \). Note that the \( L \)-fold co-product \( \Delta^{(L)} e \) is of the form \( g^{(L)} \) in the proposition.

Now let

\[
\begin{align*}
  g_r := \left( \exp_q(\Delta^{(1)} e_2) \cdot \exp_q(\Delta^{(1)} e_3) + e \sum_{v \in \mathbb{Z}} \Delta^{(1)} e \right) (v_1 \otimes \cdots \otimes v_3).
\end{align*}
\]

It is immediate from the definitions that (6) holds. The fact that \( g_r \) is in the kernel of \( A^{(L)} \) follows from lemma 5.3. The first statement in theorem 2.5 can now be proved.

**Theorem 5.7.** The restriction of \( \mathcal{L}toB_1 \) is the generator of spin 1/2 type \( C_2 \) ASEP on \( \{1, \ldots, L\} \) with reflecting boundary conditions.

**Proof.** We will use the lemma:

**Lemma 5.8.** The generator of the generalized two particle type ASEP on \( \{1, \ldots, L\} \) with reflecting boundary conditions is of the form

\[
\sum_{l=1}^{L-1} 1^{\otimes l-1} \otimes H \otimes 1^{\otimes L-l-1},
\]

where the matrix of \( H \) with respect to the basis \( (\infty, \infty), (\infty, 1), (1, \infty), (1, 1), (2, 1), (1, 2), (2, 2), (\infty, 2), (2, \infty) \) is
Proof. Since the particles in two particle type ASEP only jump at most one site, and all the jump bond rates are the same, the generator can be written in that form. The matrix entries can be found from the definition of a generator of a Markov process.

To finish the proof of the theorem, it remains to show that $G^{-1} A^{(L)} G$ matches the expression in the lemma. From proposition 5.6, $G$ can be written in the form

$$G(v) = g_1(v) \cdots g_L(v), \text{ for } v = v_h \otimes \cdots \otimes v_h,$$

where $g(v_h \otimes \cdots \otimes v_h)$ only depends on the values of $i_1, \ldots, i_r$ irrespective of order. In other words, $g$ only depends on the cardinalities of the sets $\{k : 1 \leq k \leq j, i_k = r\}$, where $r$ ranges from 1 to 4. Thus,

$$G^{-1} A^{(L)} G(v) = \sum_{i=1}^L g^{i-1} \otimes H \otimes 1^{L-i-1}(v),$$

where $H = B^{-1} A B$ for some diagonal matrix $B$.

The last step is to show that $H$ is of the form in lemma 5.8. Since $H = B^{-1} A B$ for some diagonal matrix $B$, the $(x, y)$ entry of $H$ equals

$$H(x, y) = \frac{A(x, y)B(y, y)}{B(x, x)}.$$  

Recall that by lemma 5.2, the matrix of $A$ consists of several $2 \times 2$ blocks and one $4 \times 4$ block. Because the theorem restricts $\mathcal{L}$ to $B_1$, the $4 \times 4$ block restricts to the upper left $2 \times 2$ block, so each row of $H$ has at most two non-zero entries. Furthermore, since $g$ is in the kernel of $A^{(L)}$, each row of $H$ must sum to 0. Therefore, it suffices to check the diagonal entries of $H$. Conjugating by a diagonal matrix does not change the diagonal entries, so by lemma 5.2 (and with the appropriate normalizing constant in the definition of $A$ in section 5.2), this shows that $H$ has the necessary form.

5.3. Duality

We first prove an equivalent definition of duality.

Lemma 5.9. Suppose that $\mathcal{L}$ is the generator of the Markov process $X(t)$ on state space $X$. Let $D$ be a function on $X \times X$ viewed as an operator in the sense of the formal sum

$$D(y) = \sum_{x \in X} D(x, y)x.$$
If $Z, Y$ are subsets of $X$ such that for all $(z, y) \in Z \times Y$

$$\langle z, LD(y) \rangle = \langle z, DL^*(y) \rangle$$

then $Z \times Y \subseteq S_D$.

**Proof.** By definition

$$e^{t\mathcal{L}} D(y) = \sum_x e^{t\mathcal{L}}(D(x, y)x)$$

$$= \sum_{x,z} D(x, y)e^{t\mathcal{L}}(z, x)z$$

$$= \sum_{x,z} P_t(z \to x)D(x, y)z$$

and

$$De^{t\mathcal{L}^*}(y) = \sum_x D(e^{t\mathcal{L}^*}(y, x)x)$$

$$= \sum_{z,x} D(z, x)e^{t\mathcal{L}^*}(y, x)z$$

$$= \sum_{z,x} P_t(y \to x)D(z, y)z.$$ 

By the assumptions of the lemma this implies that for all $z \in Z$ and $y \in Y$,

$$\sum_x P_t(z \to x)D(x, y) = \sum_x P_t(y \to x)D(z, x)$$

which is equivalent to saying that for all $z \in Z$, $y \in Y$,

$$E_z[D(X(t), y)] = E_z[D(z, X(t))],$$

which means exactly that $(z, y) \in S_D$.  

By proposition 5.4, $D$ can be used to obtain a suitable duality function. The difficulty lies in the simple fact: in the equation (9), ignoring the summation over states $x$ with sites containing both a particle of type 1 and a particle of type 2 will not always leave the sum unchanged. However, certain duality functions will still work.

**Lemma 5.10.** Suppose $y, z$ are such that

$$D(x, y) = D(z, x) = 0 \text{ for all } x \notin \text{span}(B_1).$$

Then $(z, y) \in S_D$.

**Proof.** With the assumptions of the lemma, the summation over $x \in \text{span}(B_1)$ in (9) is 0, as needed.

Now it remains to find proper duality functions $D$ satisfying lemma 5.10. There are two natural choices. The first is to consider

$$S := \exp_q\{\Delta^{(1)}e_2\} \cdot \exp_q\{\Delta^{(1)}e_1\}$$
and set \( D_i = G_i^{-1} S G_i^{-1} \), with \( D = \lim_{\gamma \to 0} D_i \). The idea behind this choice is as follows. In order for lemma 5.10 to hold, the symmetry \( S \) should not create a site with both a type 1 and a type 2 particle. Since \( e_1 \) creates a particle of type 1 and \( e_2 \) replaces a particle of type 1 with a particle of type 2, this holds as long as \( x \) does not contain any particles of type 2.

Below, recall that \( \nu_3 \in V(-1,0), \nu_4 \in V(0,-1), \nu_2 \in V(0,1) \).

**Proposition 5.11.** If \( \xi = \infty \), 1 for all \( i \), then

\[
S(\eta, \xi) = \prod_{i=1}^{L} 1_{\xi_i = \eta_i} \times q^{\left| \{ i, \xi_i = \eta_i = \infty \} \right| + \sum_{j=1}^{L-1} \left| \{ j, \xi_j = \eta_j = \infty \} \right|} \sum_{j=1}^{L-1} \left| \{ j, \xi_j = \eta_j = \infty \} \right|} \sum_{j=1}^{L-1} \left| \{ j, \xi_j = \eta_j = \infty \} \right|}.
\]

**Proof.** Use proposition 5.6. Since \( e_1^2 \) and \( e_2^2 \) act as 0 on \( V \), it is equivalent to consider

\[
(1 + e_2 \otimes 1^{L-1})(1 + k_2 \otimes e_2 \otimes 1^{L-2}) \ldots (1 + (k_2)^{\otimes (L-1)} \otimes e_2)
\]

\[
\times (1 + e_1 \otimes 1^{L-1})(1 + k_1 \otimes e_1 \otimes 1^{L-2}) \ldots (1 + (k_1)^{\otimes (L-1)} \otimes e_1).
\]

First, move the \( e_2 \) terms from left to right to get

\[
(1 + e_2 \otimes 1^{L-1})(1 + e_1 \otimes 1^{L-1})(1 + k_2 \otimes e_2 \otimes 1^{L-2})(1 + k_1 \otimes e_1 \otimes 1^{L-2})
\]

\[
\ldots (1 + (k_2)^{\otimes (L-1)} \otimes e_2)(1 + (k_1)^{\otimes (L-1)} \otimes e_1).
\]

Due to the commutation relation \( k_2 e_1 = q^2 e_1 k_2 \), this produces the term

\[
\prod_{i=1}^{L} (q^{-2})^{\left| \{ \eta_i = \infty \} \right|} \sum_{j=1}^{L-1} \left| \{ j, \xi_j = \eta_j = \infty \} \right|}.
\]

Next, applications of the \( e_1 \) terms to \( \xi \) yield

\[
\prod_{i=1}^{L} q^{\left| \{ \xi_i = \infty \} \right|} \sum_{j=1}^{L-1} \left| \{ j, \xi_j = \eta_j = \infty \} \right|}.
\]

And then the applications of the \( e_2 \) yields

\[
\prod_{i=1}^{L} (q^{-2})^{\left| \{ \xi_i = \infty \} \right|} \sum_{j=1}^{L-1} \left| \{ j, \xi_j = \eta_j = \infty \} \right|}
\]

and combining all three lines gives the result. \( \square \)

Recall

\[
N^R_k(\eta) = \left\{ j > i : \eta_j \neq \infty \right\}
\]

\[
N^L_k(\eta) = \left\{ i < j : \eta_j \neq \infty \right\}.
\]
For \( n_1 < \ldots < n_r \), let \( \xi^{(n_1, \ldots, n_r)} \) be the state, where \( \xi_{n_1} = 1 \) and all other \( \xi_i = \infty \). As before, \( \Omega \) is the vacuum vector. Proposition 5.11 immediately implies:

**Corollary 5.12.** We have

\[
G(\eta, \Omega) = \prod_{i=1}^{L} q^{1}_{\eta_i = \infty} (1 - i(q - 2)^{(1, \ldots, 1)})
\]

and

\[
G(\xi^{(n_1, \ldots, n_r)}) = \prod_{s=1}^{r} q^{1}_{n_s = \infty}
\]

\[
S(\eta, \xi^{(n_1, \ldots, n_r)}) = \prod_{s=0}^{r} L_{\eta_s = \infty} (q - 2)^{(1)(1)} N_{n_s}^{(1)} N_{n_s}^{(1)} \prod_{i=n_s+1}^{n_s} q^{1}_{\eta_i = \infty} (1 - i)(q - 2)^{(1, \ldots, 1)} N_{n_s}^{(1)} N_{n_s}^{(1)}
\]

\[
= \prod_{s=0}^{r} L_{\eta_s = \infty} \prod_{i=n_s+1}^{n_s+1} q^{1}_{\eta_i = \infty} (2s - i + 1)
\]

Theorem 2.3(1) can now be proved. Suppose that \( \eta_i = 2 \) exactly when \( i \in \{m_1, \ldots, m_L \} \) (and possibly 1 elsewhere). Then

\[
S(\eta, \xi^{(n_1, \ldots, n_r)}) = \prod_{s=0}^{r} L_{\eta_s = \infty} \prod_{i=n_s+1}^{n_s+1} q^{2N_{n_s}^{(1)}} \prod_{s=0}^{r} \prod_{i=n_s+1}^{n_s+1} q^{1}_{\eta_i = \infty} (2s - i + 1)
\]

so that

\[
D(\eta, \xi^{(n_1, \ldots, n_r)}) = \frac{1}{G(\eta, \Omega)} \prod_{s=0}^{r} L_{\eta_s = \infty} \prod_{i=n_s+1}^{n_s+1} q^{2N_{n_s}^{(1)}} \prod_{s=0}^{r} \prod_{i=n_s+1}^{n_s+1} q^{1}_{\eta_i = \infty} (2s - i + 1)
\]

\[
= \prod_{s=0}^{r} L_{\eta_s = \infty} \prod_{i=n_s+1}^{n_s+1} q^{1}_{\eta_i = \infty} (i - 1)
\]

\[
= \prod_{s=0}^{r} L_{\eta_s = \infty} \prod_{i=n_s+1}^{n_s+1} q^{1}_{\eta_i = \infty} (2s - i + 1)
\]

\[
= \prod_{s=0}^{r} L_{\eta_s = \infty} \prod_{s=0}^{r} q^{2(n_s - 1)} \prod_{s=0}^{r} q^{1}_{\eta_s = \infty} (2s)
\]

\[
= \prod_{s=0}^{r} L_{\eta_s = \infty} q^{2(n_s - 1)} q^{2N_{n_s}^{(1)} - (r - s)}
\]

\[
= q^{2(n_s - 1)} R_{s=1}^{r} L_{\eta_s = \infty} q^{2n_s + 2N_{n_s}^{(1)}}
\]

which is theorem 2.3(2).

Now consider the case when

\[
S = \exp_{q} \left( \Delta^{(L)} \varepsilon_2 \right)
\]

In this case, any \( \xi \) will work.
Lemma 5.13.

\[
S(\eta, \xi) = \prod_{i=1}^{L} \left( 1_{\xi_i = \eta_i} + 1_{\xi_i = 1, \eta_i = 2} \left( q^2 \right)^{\sum_{j=i}^{i-1} 1_{\xi_i = 2} - 1_{\xi_j = 1}} \right).
\]

**Proof.** The applications of the \( e_2 \) only occur when \( \xi_i = 1, \eta_i = 2 \), and the lemma follows because \( k_{(0,2)} \) maps \( v_3 \) to \( v_3 \), \( v_3 \) to \( q^2 v_3 \) and \( v_2 \) to \( q^2 v_2 \).

Since

\[
G(\eta) = \prod_{i=1}^{L} q^{1_{\eta_i = 1} (1 - i)} \left( q^{-2} \right)^{1_{\eta_i = 2} N^e(\eta)}
\]

we have

\[
D(\eta, \xi) = \prod_{i=1}^{L} \left( 1_{\xi_i = \eta_i = 1} \left( q^2 \right)^{i-1} + 1_{\xi_i = 1, \eta_i = 2} \left( q^2 \right)^{i-1 + N^e(\eta) + N^e(\xi)} + 1_{\xi_i = 1, \eta_i = 2} \left( q^2 \right)^{i-1 + \sum_{j=i}^{i-1} 1_{\xi_j = 2} - 1_{\xi_j = 1}} \right)
\]

which simplifies to the expression in theorem 2.3(1).

6. Type \( A_2 \) ASEP

Let \( C \) be the central element of \( \mathcal{U}_q(\mathfrak{sl}_3) \) from (4).

**Lemma 6.1.** With respect to the basis \( v_1 \otimes v_1, v_2 \otimes v_2, v_3 \otimes v_3, v_2 \otimes v_1, v_1 \otimes v_2, v_3 \otimes v_2, v_2 \otimes v_3, v_3 \otimes v_3 \), the matrix of \( \Delta(C) \) on \( V \otimes V \) is

\[
\begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & q & 0 & 0 \\
0 & 0 & q & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -q^2 & q & 0 & 0 \\
0 & 0 & 0 & 0 & q & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -q^2 & q \\
0 & 0 & 0 & 0 & 0 & 0 & q & -1 \\
\end{pmatrix}.
\]
Proof. By computation:

\[
\begin{align*}
\Delta(e_1) &= q^{-1}E_{42} + E_{52} + E_{14} + qE_{15} + E_{68} + E_{79}, \\
\Delta(f_1) &= q^{-1}E_{41} + E_{51} + E_{24} + qE_{25} + E_{86} + E_{97}, \\
\Delta(k_{a,b,c}) &= \text{diag}(q^{2a}, q^{2b}, q^{2c}, q^{a+b}, q^{a+c}, q^{b+c}, q^{a+b+c}, q^{b+c}), \\
\Delta(e_2) &= q^{-1}E_{83} + E_{93} + E_{46} + E_{57} + E_{28} + qE_{29}, \\
\Delta(f_2) &= q^{-1}E_{82} + E_{92} + E_{64} + E_{75} + E_{38} + qE_{39}.
\end{align*}
\]

The symmetry in this case is

\[
S := \exp_q(\Delta^{L_1} e_2) \cdot \exp_q(\Delta^{L_1} e_1).
\]

Proposition 6.2.

\[
S(\eta, \xi) = \prod_{i=1}^L q^{[\eta=\xi]} \sum_{j=1}^{L-1} (1_{\eta=1} - 1_{\eta=\infty})
\times (q^{-1})^{[\eta=\xi]} \sum_{j=1}^{L-1} (1_{\eta=1} - 1_{\eta=\infty})
\]

implying that

\[
G(\eta) := S(\eta, \Omega) = \prod_{i=1}^L q^{[\eta=\xi]} (1 - q^{-1})^{[\eta=\xi]} \sum_{j=1}^{L-1} 1_{\eta=\infty}.
\]

Proof. The argument is identical to that of proposition 5.11.

Therefore we see that

Theorem 6.3. The operator

\[ L := G^{-1}A^{L_1} G \]

is the generator of spin 1/2 type \( A_2 \) ASEP on \( \{1, \ldots, L\} \) with reflecting boundary conditions. The function

\[ D := G^{-1}SG^{-1} \]

is a self-duality function explicitly given by the expression given in theorem 2.2.

Proof. The first statement follows from an argument similar to that of theorem 5.7. The second statement is a direct computation using proposition 6.2.
By proposition 6.2,
\[ D(n, \xi) = \prod_{i=1}^{L} q^{\sum_{j=1}^{i-1} \left( \frac{1}{q^{\xi+1}} - \frac{1}{q^{\xi}} \right)} \times (q^{-1})^{\sum_{j=1}^{i} \left( \frac{1}{q^{\xi+1}} - \frac{1}{q^{\xi}} \right)} \times q^{\sum_{j=1}^{i} \left( \frac{1}{q^{\xi+1}} - \frac{1}{q^{\xi}} \right)} \]
which equals \( \prod_{i=1}^{L} f(n_i, \xi_i) \) where \( f(\cdot, \cdot) \) equals
\[
q^{\sum_{j=1}^{i-1} \left( \frac{1}{q^{\xi+1}} - \frac{1}{q^{\xi}} \right)} \cdot q^{i-1}, \quad \text{if } \xi = \infty, \eta = \infty,
q^{\sum_{j=1}^{i-1} \left( \frac{1}{q^{\xi+1}} - \frac{1}{q^{\xi}} \right)} \cdot q^{i-1}, \quad \text{if } \xi = \infty, \eta = 2,
q^{\sum_{j=1}^{i-1} \left( \frac{1}{q^{\xi+1}} - \frac{1}{q^{\xi}} \right)} \cdot q^{i-1} - N_i(0), \quad \text{if } \xi = \infty, \eta = 1,
q^{2(i-1)}, \quad \text{if } \xi = 2, \eta = 2,
q^{2(i-1)} + N_i(0), \quad \text{if } \xi = 2, \eta = 1,
q^{2(i-1) + N_i(0) + N_i(0)}, \quad \text{if } \xi = 1, \eta = 1.
\]
If there are \( s_2 \) type 2 particles and \( s \) type 1 particles in \( \xi \) to the left of \( i \), then this becomes
\[
1, \quad \text{if } \xi = \infty, \eta = \infty,
q^{2(i-1) - s_2 - s}, \quad \text{if } \xi = \infty, \eta = 2,
q^{2(i-1) - s_2 - s}, \quad \text{if } \xi = \infty, \eta = 1,
q^{2(i-1)}, \quad \text{if } \xi = 2, \eta = 2,
q^{2(i-1) + s}, \quad \text{if } \xi = 1, \eta = 1.
\]
The \( q^{2s} \) term in the fifth line and \( q^{s+s_2} \) in the sixth line result in a contribution from the configuration of \( \xi \). If \( \xi \) has a total of \( r \) type 1 particles all to the left of \( r \) type 2 particles, then the contribution is \( q^{r(r+2)} \), which is constant with respect to the dynamics of \( \xi \). Each time a type 1 particle jumps to the right of a type 2 particle, the contribution is unchanged, and hence remains a constant.
Let \( \xi \) denote the particle configuration with particles of type 1 at \( n_1, \ldots, n_r \) and particles of type 2 at \( m_1, \ldots, m_r \). The sixth line yields
\[
\prod_{s=1}^{r} q^{N_s(0)}(q) = \prod_{i=1}^{L} q^{\sum_{j=1}^{i-1} \left( \frac{1}{q^{\xi+1}} - \frac{1}{q^{\xi}} \right)} \times \sum_{j=1}^{i} q^{i-1} \prod_{s=0}^{n_i-1} q^{n_i-1 - 1}
\]
This combines with \( q^{r} \) and \( q^{3s} \) in the second and third lines to contribute
\[
\prod_{s=0}^{r} q^{\sum_{j=1}^{i} \left( \frac{1}{q^{\xi+1}} - \frac{1}{q^{\xi}} \right)} q^{r-1} + q^{2s-1} = \text{const} \prod_{s=0}^{r} q^{2s} \left( \frac{1}{q^{\xi+1}} - \frac{1}{q^{\xi}} \right).
\]
Similarly, the $2s_2$ contributes
\[
\prod_{i=1}^{r'} \prod_{i=m_i+1}^{r'} q^{2s_i} l_{i, \eta^i=1} = \text{const} \prod_{i=1}^{r'} q^{2s'_i} \left( N_{s_i+1}^{\eta^i} (\eta) - N_{s_i}^{\eta^i} (\eta) - 1 \right).
\]
Combining the terms yields
\[
D(\eta, \xi) = \text{const} \prod_{i=1}^{r'} \left\{ \eta_{i+1} = 1 \right\} q^{2N_{s_i}^{\eta^i} (\eta) + 2m_i} \prod_{i=1}^{r'} \left\{ \eta_{i+1} = \infty \right\} q^{2N_{s_i}^{\eta^i} (\eta) + 2m'_i},
\]
which proves theorem 2.2.

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