1. Introduction and main results

The positivity properties of direct images are at the heart of many recent developments in complex geometry. In the present article our main goal is to explore further some of these topics. Actually the main results we obtain here concern the subharmonicity properties of direct images. As a consequence we answer partially to a conjecture formulated by Pereira-Touzet, cf. [PeTou] and the references therein. We start with the application as follows.

Let $X$ be a compact Kähler manifold whose canonical bundle $K_X$ is pseudo-effective. Let $F \subset T_X$ be a holomorphic foliation such that its first Chern class $c_1(F)$ is trivial. Then Pereira-Touzet show (cf. [PeTou]) that $F$ is non-singular, and they conjecture that if moreover $c_2(F) \neq 0$ and $F$ is stable with respect to some Kähler metric $\omega_X$ on $X$, then the leaves of $F$ are algebraic.

In this article we obtain the following particular case of their conjecture.

**Theorem 1.1.** Let $(X, \omega_X)$ be a smooth projective manifold. Let $F \subset T_X$ be a holomorphic foliation such that the following hold.

(i) The first Chern class of $F$ is zero, i.e. $c_1(F) = 0$ and $c_2(F) \neq 0$.

(ii) The sheaf $\hat{\text{Sym}}^k F^*$ is $\omega_X$-stable for some $k \geq r$, where $r$ is the rank of $F$.

Then the leaves of $F$ are algebraic.

We refer to [HP19], [Dr18], [LPT] and the references therein for other particular cases of the aforementioned conjecture and its relevance in the context of singular Calabi-Yau manifolds.

Now the main new ingredient in the proof of Theorem 1.1 is Theorem 1.2 below that we next discuss. We recall that if $F$ is any torsion free, coherent sheaf on $X$ then there exists a modification $\pi: \hat{X} \to X$ such that the inverse image $\pi^*(F)$ modulo torsion is a vector bundle which we denote by $E$. Let $\mathcal{O}(1) \to \mathbb{P}(E)$ be the (dual of the) tautological bundle over $\mathbb{P}(E)$ (our convention here is that a point of $\mathbb{P}(E)$ corresponds to a line of the fiber of $E$ at some point of $X$).

Then we have the following statement.

**Theorem 1.2.** Let $(X, \omega_X)$ be a compact Kähler manifold, and let $F$ be a reflexive sheaf such that the following hold.

(i) The first Chern class of $F$ is zero, i.e. $c_1(F) = 0$.

(ii) The sheaf $\hat{\text{Sym}}^k F^*$ is $\omega_X$-stable for some $k \geq r + 1$, where $r$ is the rank of $F$. 

Then the bundle $\mathcal{O}(1)$ on $\mathbb{P}(E)$ is not pseudo-effective, or $\mathcal{F}$ is Hermitian flat. Moreover, if $X$ is projective then we can derive the same conclusion provided that $\hat{\text{Sym}}^k \mathcal{F}^*$ is $\omega_X$-stable for some $k \geq r$ instead of (ii).

Unlike the other articles dedicated to these topics, our methods here are relying on new positivity results for direct images that we are now introducing.

Let $p : Y \to X$ be a holomorphic proper map between two Kähler manifolds, and let $(L, h_L)$ be a Hermitian line bundle on the total space $Y$. The metric $h_L$ could be singular, but we assume that

\[ \Theta_{h_L}(L) \geq 0 \]

in the sense of currents on $Y$. In this context it is then established that the direct image sheaf

\[ \mathcal{E} := p_* \left( (K_{Y/X} + L) \otimes \mathcal{I}(h_L) \right) \]

is semi-positively curved in the sense of Griffiths when endowed with the natural $L^2$ metric denoted by $h_{Y/X}$ cf. [HPS], [PT].

The next result we are presenting here concerns the positivity of $\det \mathcal{E}$ as follows.

**Theorem 1.3.** Let $p : Y \to X$ be a holomorphic surjective and proper map, where $X$ and $Y$ are Kähler manifolds. Moreover, we assume that $p$ is locally projective. Let $(L, h_L)$ be a Hermitian line bundle over $Y$ such that (1) holds true, and such that

\[ \Theta_{h_L}(L) \wedge p^* \omega_X^{n-1} \geq \varepsilon_0 p^* \omega_X^n \]

where $\omega_X$ is a Hermitian metric on $X$, $n = \dim X$, and $\varepsilon_0 > 0$ is a positive real number. Then we have

\[ \Theta_{\det h_{Y/X}} (\det(\mathcal{E})) \wedge \omega_X^{n-1} \geq r \varepsilon_0 \omega_X^n \]

where $r$ is the rank of the direct image $\mathcal{E}$.

**Remark 1.4.** If we replace the hypothesis (3) with the following

\[ \Theta_{h_L}(L) \geq \varepsilon_0 p^* \omega_X, \]

then it is well-known that the curvature of $(\mathcal{E}, h_{Y/X})$ is greater than $\varepsilon_0 \omega_X \otimes \text{Id}_{\text{End}(\mathcal{E})}$. Thus our statement [1.3] can be seen as a stronger version of this result.

Very roughly, the proof for Theorem 1.3 goes as follows: by techniques due to [Ber06], [BP08], we show that it would be enough to show the inequality (4) in the particular case of map $p : U \times V \to U$ which is simply the projection on the first factor, where $U$ is an open ball in some Euclidean space and $V$ is a Stein manifold. In this case the statement to prove looks quite different from (4) and it will be established by a (long and) direct computation.

An immediate consequence of this statement is the following.

**Theorem 1.5.** Let $p : Y \to X$ be a holomorphic surjective and proper map, where $X$ and $Y$ are Kähler manifolds. Moreover, we assume that $p$ is locally projective. Let $(L, h_L)$ be a Hermitian holomorphic line bundle over $Y$ such that

\[ \Theta_{h_L}(L) \geq -C p^* \omega_X, \quad \Theta_{h_L}(L) \wedge p^* \omega_X^{n-1} \geq 0 \]
where $\omega_X$ is a Hermitian metric on $X$, $n = \dim X$, and $C > 0$ is a positive real number. Then we have

\begin{equation}
\Theta_{\det h_{Y/X}}(\det(E)) \wedge \omega_X^{n-1} \geq 0,
\end{equation}

where $r$ is the rank of the direct image $E$.

A last result we mention here is the following statement, cf. Theorem 5.4, section 5.

Theorem 1.6. Let $p : Y \to X$ be a holomorphic surjective and proper map, where $X$ and $Y$ are Kähler manifolds. Moreover, we assume that $p$ is locally projective. Let $L$ be a line bundle over the total space $Y$ endowed with a possible singular metric $h_L$ such that $\Theta h_L(L) \geq 0$, and

\begin{equation}
\Theta h_L(L) \wedge p^* \omega_X^{n-1} \geq \epsilon_0 p^* \omega_X^n
\end{equation}

where $\omega_X$ is a Hermitian metric on $X$, $n = \dim X$, and $\epsilon_0 > 0$ is a positive real number. We assume that the space of fiberwise $L_2^m$ sections (with respect to $h_L$) of $p_*(mK_{Y/X} + L)$ is non zero. Then there exists a metric $h$ on the bundle $mK_{Y/X} + L$ such that we have

\begin{equation}
\Theta h(mK_{Y/X} + L) \wedge p^* \omega_X^{n-1} \geq \epsilon_0 : p^* \omega_X^n
\end{equation}

in the sense of currents on $Y$.

Remark 1.7. We see that Theorem 1.6 has the same flavor as the usual results concerning direct images: the positivity of the curvature of $(L, h_L)$ induces similar properties for the twisted pluricanonical bundle $mK_{Y/X} + L$. More precisely, if $\Theta h_L(L) \geq 0$ then we already know that the bundle $mK_{Y/X} + L$ is pseudo-effective. We prove here that the additional positivity requirement (8) is inherited by the twisted pluricanonical bundle of the map $p$.

Organization of the paper. The remaining part of this article will unfold as follows. In the first section we recall a few technical statements which are playing an important role in our arguments. Then as a warm-up, we prove Theorem 1.2 in the projective case; the main ideas are as follows. Assume that $O(1)$ is pseudo-effective. Then it admits a singular metric $e^{-\varphi}$ whose curvature is semi-positive. The stability condition (ii) implies that the multiplier ideal

\[ I(e^{-(r+1-\epsilon)\varphi}_{|p(E)}) \]

is trivial for any $\epsilon > 0$, and for all $x \in X$ generic. In order to prove this we are using the positivity of direct images, combined with the restriction theorem of Mehta-Ramanathan. Now if the multiplier ideal above is trivial, then we can construct a singular Hermitian metric on $F^*$ with positive curvature in the sense of Griffiths. Together with the fact that $c_1(F) = 0$, this implies that $F$ is Hermitian flat by a result in [CPT17].

The general case of Theorem 1.2 is much more subtle, since one has to find an alternative argument in order to compensate the absence of Mehta-Ramanathan theorem. It is at this point that Theorem 1.5 comes into the picture. Thanks to this result we can still analyze the singularities of the metric on $O(1)$ as discussed above.
On the other hand, Theorem 1.3 is interesting in its own right: it can be seen as very precise analysis of the positivity properties of the determinant of direct images. Further results and applications of this statement (and its proof) are discussed in the last section of this article. We can interpret them as a first step towards the analysis of the positivity of direct images on currents of bi-dimension \((1, 1)\).

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2. Stable reflexive sheaves and singular Hermitian metrics

In this preliminary section we collect a few definitions and properties concerning the stability of reflexive sheaves on compact Kähler manifolds, as well as the notion of singular Hermitian metric on a vector bundle.

2.1. Stable reflexive sheaves. We will mostly follow the article by Bando-Siu, cf. [BS94], combined with important clarifications communicated to us by M. Toma, cf. [T19].

Let \(X\) be a compact Kähler manifold, and let \(F\) be a reflexive subsheaf. Then there exists a sequence of blow-up maps of non-singular centers whose composition is denoted by the map

\[
\pi : \hat{X} \to X
\]

such that the inverse image \(\pi^*(F)\) modulo torsion becomes a vector bundle denoted by \(E \to \hat{X}\) in [BS94].

Let \(\omega\) be a Kähler form on \(X\). Then the inverse image \(\hat{\omega} := \pi^*(\omega)\) is not Kähler in general, but nevertheless the class corresponding to \(\hat{\omega}^{n-1}\) is movable. In this context, we have the following result cf. e.g. [GKP1].

**Lemma 2.1.** The sheaf \(F\) is stable with respect to \(\omega\) if and only if \(E\) is stable with respect to \(\hat{\omega}\).

For the basic facts concerning the stability with respect to a movable class we refer to [GKP1].

The important result of Bando-Siu states as follows.

**Theorem 2.2.** [BS94] Let \(F\) be a reflexive sheaf on a compact Kähler manifold, which moreover is stable with respect to a Kähler metric \(\omega\). Then \(F\) has an admissible Hermite-Einstein metric \(h_F\).

We do not recall here the notion of admissible metric of a vector bundle because we do not need it; we will rather work with its regularization defined as follows. Let

\[
K_{\hat{X}/X} = \sum_{i=1}^{N} e_i E_i
\]

be the relative canonical divisor of the map \(\pi\). Then there exist a set of non-singular representatives \(\theta_i \in c_1(E_i)\) such that the form

\[
\hat{\omega}_0 := \hat{\omega} - \varepsilon_0 \sum_{i=1}^{N} \varepsilon_i \theta_i
\]
is Kähler for any positive and small enough $0 < \varepsilon_0 \ll 1$, provided that the coefficients $\varepsilon_i$ are carefully chosen.

By Lemma 2.1 the vector bundle $E$ is stable with respect to $\hat{\omega}$. It is stated as a remark in [BS94] that $E$ is stable with respect to the perturbation $\hat{\omega}_0$ of $\hat{\omega}$. As soon as the $\varepsilon_0$ in (12) is small enough. We were not able to find a reference/proof for this assertion (which is most likely true) so we will rather use the following result established in the recent article [T19].

**Theorem 2.3.** [T19] The bundle $E$ is stable with respect to the degree function induced by the form

\begin{equation}
(1 - \varepsilon)\hat{\omega}_0^{n-1} + \varepsilon\hat{\omega}^{n-1}
\end{equation}

for any $1 \geq \varepsilon \geq 0$, provided that $\varepsilon_0 \ll 1$.

It then follows that there exists a metric $h_{E,\varepsilon}$ on $E$ which verifies the Hermite-Einstein condition with respect to $g_\varepsilon^{n-1}$, where $g_\varepsilon$ is the unique Gauduchon metric on $\hat{X}$ such that

\begin{equation}
g_\varepsilon^{n-1} = (1 - \varepsilon)\hat{\omega}_0^{n-1} + \varepsilon\hat{\omega}^{n-1}.
\end{equation}

Let us fix a coordinate system $(z_j)$ centered at some point $x_0$ of $\hat{X}$. We denote by $G_\varepsilon$ the matrix corresponding to $g_\varepsilon$ and we define

\begin{equation}
(1 - \varepsilon)\hat{\omega}_0^{n-1} + \varepsilon\hat{\omega}^{n-1}|_U = (n - 1)!(-1)^{\frac{n-1}{2}}\sum_{\alpha,\beta}(-1)^{\alpha+\beta}\Omega_{\alpha,\beta,\varepsilon} d\bar{z}_\alpha \wedge dz_\beta.
\end{equation}

We have the formula $\Omega^t_\varepsilon G_\varepsilon = (\det \Omega_\varepsilon)^{\frac{1}{n+1}}$ which implies that we have the equality $\det G_\varepsilon = (\det \Omega_\varepsilon)^{\frac{1}{n+1}}$. We therefore obtain

\begin{equation}
\int_{\hat{X}} dV_{g_\varepsilon} \leq C
\end{equation}

for some constant $C > 0$ independent of $\varepsilon$.

In conclusion, a reflexive sheaf $F$ which is stable with respect to a Kähler metric $\omega$ has an admissible Hermite-Einstein metric $h_F$ which is limit of non-singular Hermite-Einstein metrics on a modification $\hat{X}$ of $X$ with respect to a Gauduchon metric $g_\varepsilon$.

### 2.2. Singular Hermitian Metrics and Positivity of Direct Images.

For the convenience of the reader, we collect here a few basic notions and results concerning the positivity of push-forward of relative canonical bundles, cf. [PT] [HPS], [Pa15] and the references therein. As we will next see, the proof of the projective case of Theorem 2.2 is relying heavily on them.

The following notion appeared naturally in [BP08], and it was subsequently studied in [Rau], [PT], [HPS]. Let $E \to X$ be a holomorphic vector bundle of rank $r$ on a complex manifold $X$. We denote by

\begin{equation}
H_r := \{ A = (a_{ij}) \}
\end{equation}

the set of $r \times r$, semi-positive definite Hermitian matrices. The manifold $X$ is endowed with the Lebesgue measure. We recall next the following notion.
Definition 2.4. A singular Hermitian metric $h$ on $E$ is given locally by a measurable map with values in $H_r$ such that
\begin{equation}
0 < \det h < +\infty
\end{equation}
almost everywhere.

Let $(E, h)$ be a vector bundle endowed with a singular Hermitian metric $h$. Given a local section $v$ of $E$, i.e., an element $v \in H^0(U, E)$ defined on some open subset $U \subset X$, the function $|v|^2_h : U \to \mathbb{R}_{\geq 0}$ is measurable, given by
\begin{equation}
|v|^2_h = tr(hv \overline{v} = \sum h_{ij} v_i \overline{v_j}
\end{equation}
where $v = (v^1, \ldots, v^r)$ is a column vector.

Following [BP08, p. 357], we recall next the notion of positivity/negativity of a singular Hermitian vector bundle as follows.

Definition 2.5. Let $h$ be a singular Hermitian metric on $E$.
\begin{enumerate}
\item The metric $h$ is negatively curved if the function
\begin{equation}
x \to \log |v|^2_{h,x}
\end{equation}
is psh for any local section $v$ of $E$.
\item The metric $h$ is positively curved if the dual singular Hermitian metric $h^* := h^{-1}$ on the dual vector bundle $E^*$ is negatively curved.
\end{enumerate}

We recall the following important property of the notion of positivity.

Proposition 2.6. [CP17, Cor 2.8] Let $(E, h)$ be a vector bundle endowed with a positively curved singular Hermitian metric $h$ on a compact manifold $X$. If the first Chern class $c_1(E) = 0$, then the metric $h$ is non-singular and the curvature of $(E, h)$ is equal to zero.

If $E$ is replaced by a coherent, torsion-free sheaf $E$ then the conclusion of the statement still holds provided that we restrict to the open subset on which $E$ is a vector bundle.

The notion of positively curved singular Hermitian metric is the natural property of push-forwards of relative canonical bundles as we see from the next results.

Theorem 2.7. [PT, Pa15, HPS]. Let $p : Y \to X$ be an algebraic fiber space, and let $(L, h_L)$ be a Hermitian line bundle. The metric $h_L$ could be singular and the corresponding curvature current $\Theta(L, h_L)$ is semi-positive. Then the singular Hermitian metric on the torsion-free push-forward sheaf
\begin{equation}
p_* ((K_Y/X + L) \otimes I(h_L))
\end{equation}
is positively curved in the sense of Griffiths. If moreover $I(h_L|_{Y_x}) = O_{Y_x}$ for a generic fiber $Y_x$, then $h_{Y/X}$ extends naturally as a singular Hermitian metric $p_* ((K_Y/X + L))$, which is also positively curved in the sense of Griffiths.

We also recall the following variant of Theorem 2.7.

Corollary 2.8. [CP17, Lemma 5.25] Let $p : Y \to X$ be an algebraic fiber space, and let $(L, h_L)$ be a Hermitian line bundle. The metric $h_L$ could be singular and the corresponding curvature current $\Theta(L, h_L) + C p^* \omega_X$ is semi-positive for some $C > 0$. 

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Let ξ be a local holomorphic section of the dual of $p_*( (K_{Y/X} + L) \otimes \mathcal{I}(h_L))$. Then we have

\[
\sqrt{-1} \partial \bar{\partial} \log |\xi|^2_{h_{Y/X}} \geq -C \omega_X. 
\]

The property (22) will be formally denoted by

\[
\Theta_{h_{Y/X}}(p_*( (K_{Y/X} + L) \otimes \mathcal{I}(h_L))) \geq -C \omega_X \otimes \text{Id}. 
\]

We will equally need the following generalization of Proposition 2.6 proved in [CH19, CM19].

**Theorem 2.9.** [CH19, CM19] Let $X$ be a projective manifold and $\mathcal{F}$ be a reflexive sheaf on $X$. Let $X_0$ be the locally free locus of $\mathcal{F}$. If $c_1(\mathcal{F}) = 0$ and for every $\varepsilon > 0$, there exists a possible singular hermitian metric $h_\varepsilon$ on $\mathcal{F}|_{X_0}$ such that

\[
\Theta_{h_\varepsilon}(\mathcal{F}) \geq -\varepsilon \omega_X \otimes \text{Id} 
\]
on $X_0$, then $\mathcal{F}$ is a numerically flat vector bundle on $X$, namely $\mathcal{F}$ is nef and $c_1(\mathcal{F}) = 0$.

**Remark 2.10.** The theorem 2.9 is proved in [CH19] for projective surfaces and it was very recently generalized to arbitrary dimension in [CM19]. We recall briefly the idea of the proof and refer to [CM19] for a complete treatment.

We fix a polarisation $H$, and let $0 \to G_1 \to G_2 \to \cdots \to G_m = \mathcal{F}$ be a $H$-stable filtration. By an argument which parallels [DPS94, Thm 1.18], we show that every quotient $(G_{i+1}/G_i)^{**}$ satisfies the same conditions like $\mathcal{F}$, namely

\[
c_1((G_{i+1}/G_i)^{**}) = 0 \quad \text{and} \quad \Theta_{h_{i,\varepsilon}}((G_{i+1}/G_i)^{**}) \geq -\varepsilon \omega_X \otimes \text{Id}, \]

where $h_{i,\varepsilon}$ is a natural metric induced by $h_\varepsilon$. Let $S$ be a surface defined by complete intersection of $H_1 \cap H_2 \cdots \cap H_{n-2}$, where $H_i$ is a generic hypersurface in the class of $|H|$. By applying [CH19, Cor 2.12] to $(G_{i+1}/G_i)^{**}|_S$, we know that $(G_{i+1}/G_i)^{**}|_S$ is hermitian flat. In particular, $c_2((G_{i+1}/G_i)^{**}) \wedge H^{n-2} = 0$. Then [BS94, Cor 3] implies that $(G_{i+1}/G_i)^{**}$ is a hermitian flat vector bundle on $X$. As a consequence, we can prove that $\mathcal{F}$ is a successive extension of hermitian flat vector bundles on $X$. It is thus numerically flat by [DPS94, Thm 1.18].

3. **Proof of the main result: the projective case**

In this section we will prove our main result under the assumption that $X$ is projective. Indeed, in this case all the necessary tools we will be using in our arguments are already available: we show that Theorem 2.2 follows from the positivity of direct images, combined with the existence of Hermite-Einstein metric on stable vector bundles.

To make things precise, the result we establish in this first section is the following.

**Theorem 3.1.** Let $(X, H)$ be a smooth polarized projective manifold. We consider a reflexive sheaf $\mathcal{F}$ with the following properties.

(i) The first Chern class of $\mathcal{F}$ is zero, i.e. $c_1(\mathcal{F}) = 0$.

(ii) The sheaf $\text{Sym}^k \mathcal{F}^*$ is $H$-stable for some $k \geq r$, where $r$ is the rank of $\mathcal{F}$.

Then the bundle $\mathcal{O}(1)$ on $\mathbb{P}(E)$ is not pseudo-effective, or $\mathcal{F}$ is Hermitian flat.
The bundle $E$ in (ii) is $\pi^*(\mathcal{F})$ modulo torsion, cf. section 1.

An immediate consequence of Theorem 3.1 is the following statement.

**Corollary 3.2.** Let $(X, H)$ be a smooth polarized projective manifold. We consider a saturated coherent subsheaf $\mathcal{F} \subset T_X$ satisfying the properties in Theorem 3.1 and moreover we assume that $\mathcal{F}$ is closed under the Lie bracket. Then $\mathcal{F}$ is either Hermitian flat or all of its leaves are algebraic.

We assume that the tautological bundle $\mathcal{O}(1) \to \mathbb{P}(E)$ is pseudo-effective. Then we obtain a singular metric $h = e^{-\varphi}$ on $\mathcal{O}(1)$ whose curvature current $\Theta$ is positive, $\Theta \geq 0$.

The main point in our arguments is the following statement concerning the singularities of the metric $h$.

**Lemma 3.3.** We consider the map $p : \mathbb{P}(E) \to X$ obtained by composing the natural projection $\mathbb{P}(E) \to \tilde{X}$ with $\pi$. If $\text{Sym}^k \mathcal{F}^*$ is $H$-stable for some $k \in \mathbb{N}$, then the multiplier sheaf corresponding to the restriction

$$e^{-(k+1)\varphi|_{\mathbb{P}(E)_x}}$$

is trivial for any $\varepsilon > 0$. Here $x$ is any point in the complement of a countable union of proper analytic subsets of $X$, where we denote by $\mathbb{P}(E)_x$ the $p$-inverse image of the point $x \in X$.

**Remark 3.4.** In this lemma it is not necessary to require $k \geq r$. Ideally, we want to consider the case $k = 1$. The bundle $\mathcal{O}(r+1)$ is endowed with the metric

$$e^{-2\varepsilon \varphi - (r-1+\varepsilon)\varphi_{HE}}$$

where we assume that we are in the projective case and we restrict $\mathcal{F}$ to a complete intersection curve $C$. We denote by $e^{-\varphi_{HE}}$ the metric on $\mathcal{O}(1)$ induced by the flat Hermitian-Einstein metric on $\mathcal{F}|_C$. Of course, the direct image is just $\mathcal{F}|_C$, but it is endowed with a metric depending on $-\varphi_{HE}$. Since the determinant is cohomologically trivial, the hope is that this should have important consequences on the Hessian of $\varphi$ “in the base directions”, cf. calculations section 4.

The rest of this section is divided into three parts. We will first show that Theorem 3.1 follows from Lemma 3.3. Then we prove the lemma, and finally we give a quick argument for Corollary 3.2.

**3.1. Lemma 3.3 implies Theorem 3.1.** Now we can prove Theorem 3.1 since $k \geq r$, thanks to the Lemma 3.3, we know that $I(e^{-(r+1-\varepsilon)\varphi}|_{\mathbb{P}(E)_x}) = \mathcal{O}_{\mathbb{P}(E)_x}$ for a generic $x \in X$. The inclusion

$$p_* \left( K_{\mathbb{P}(E)_X} \otimes \mathcal{O}(r+1) \otimes I(e^{-(r+1-\varepsilon)\varphi}) \right) \subset p_* \left( K_{\mathbb{P}(E)_X} \otimes \mathcal{O}(r+1) \right) = \mathcal{F}^*$$

is thus generically isomorphic.

As $\mathcal{O}(1)$ is relatively ample, there exists a smooth metric $h_0$ on $\mathcal{O}(1)$ such that $\Theta_{h_0} \mathcal{O}(1) \geq -p^* \omega_X$ for some Kähler metric $\omega_X$ on $X$. Set $h_\varepsilon := e^{-(r+1-\varepsilon)\varphi} \cdot h_0^\varepsilon$. It is a metric on $\mathcal{O}(r+1)$ with $\Theta_{h_\varepsilon} \mathcal{O}(r+1) \geq -\varepsilon p^* \omega_X$. By Corollary 3.3 $h_\varepsilon$ induces a metric $h_{\varepsilon, \mathcal{F}^*}$ on $\mathcal{F}^*$ such that

$$i\Theta_{h_{\varepsilon, \mathcal{F}^*}}(\mathcal{F}^* \otimes \mathcal{O}(r+1)) \geq -\varepsilon \omega_X \otimes \text{Id}.$$
Together with the fact that $c_1(F^\ast) = 0$, it follows that $F^\ast$ is numerically flat by Theorem 2.9. Note that the stability of $\text{Sym}^k F^\ast$ implies that $F^\ast$ is stable. $F^\ast$ is thus hermitian flat by using [DPS94, Thm 1.18].

3.2. Proof of Lemma 3.3. By restriction theorem of Mehta-Ramanathan, the bundles

$$\text{Sym}^k F^\ast|_C$$

are still stable provided that $C$ is a generic complete intersection of $n - 1$ divisors in a large enough multiple of $H$.

The curves “$C$” for which the restriction (27) is stable cover a non-empty Zariski open subset of $X$. We assume that our point $x$ belongs to the union of these curves, so $x \in C$ for some complete intersection curve which we fix for the rest of the proof.

Let $h_C$ be the Hermite-Einstein metric on the bundle $F|_C$. It is of course smooth, and thanks to the first hypothesis (i) the curvature tensor $\Theta(F|_C, h_C)$ is identically equal to zero. In particular, $h_C$ induces a smooth metric on $O(1)$ whose curvature is semi-positive. Roughly speaking, this is so because the curvature of $O(1)$ at a point $(y, [v])$ has two components: the Fubini-Study form in the directions of the fibers of the map

$$\pi : \mathbb{P}(F|_C) \to C$$

plus the curvature $\Theta(F|_C, h_C)$ evaluated in the direction $v$.

We assume that $x$ is such that the multiplier ideal $I(e^{-(k+1-\varepsilon)\varphi}|_{\mathbb{P}(F_x)})$ coincides with $I(e^{-(k+1-\varepsilon)\varphi})|_{\mathbb{P}(F_x)}$. Again, this is a genericness requirement which holds true in the complement of a countable union of points of $C$.

Consider the bundle

$$K_{\mathbb{P}(F)/C} + L.$$

We endow $L := O(k + r)$ with a the metric whose local weights are

$$\psi_L := (k + 1 - \varepsilon)\varphi + (r - 1 + \varepsilon)\varphi_C,$$

the corresponding curvature current is semi-positive. By standard $L^2$ estimates cf. Lemma 4.4 the vector space

$$H^0 \left( \mathbb{P}(F_x), O(K_{\mathbb{P}(F)/C} + L) \otimes I(e^{-\psi_L}) \right) = H^0 \left( \mathbb{P}(F_x), O(k) \otimes I(e^{-(k+1-\varepsilon)\varphi}|_{\mathbb{P}(F_x)}) \right)$$

is non-trivial. Moreover, it does not coincides with the space of global sections of $O(k)|_{\mathbb{P}(F_x)}$ if the multiplier ideal sheaf of the metric $e^{-\psi_L}$ is non-trivial.

By Theorem 2.7 we infer that the direct image

$$\mathcal{G} := \pi_* \left( (K_{\mathbb{P}(F)/C} + L) \otimes I(e^{-\psi_L}) \right)$$

is semi-positively curved. In particular, its degree with respect to $C$ is semi-positive. On the other hand, we have

$$\mathcal{G} \subset \pi_* \left( K_{\mathbb{P}(F)/C} + L \right) = \text{Sym}^k F^\ast$$

(modulo a topologically trivial line bundle) and as explained before, the inclusion is strict. Hence the degree of $\mathcal{G}$ is strictly smaller than the degree of $\text{Sym}^k F^\ast$, which is zero. The lemma is proved in the projective case.
Question 3.5. We see that the stability condition allow us to deduce regularity properties of the metric $e^{-\varphi}$ in case $c_1(F) = 0$. Can one formulate (and eventually prove) similar results if $c_1(F)$ contains a negative representative?

3.3. Proof of Corollary 3.2 The statement follows immediately as combination of Theorem 3.1 and the following algebraicity criteria which is a direct consequence of [CP19] and for which we refer to cf. [Dr18].

Theorem 3.6. Let $X$ be a projective manifold and let $F \subset T_X$ be a holomorphic foliation. If $F$ is not algebraic, then the tautological line bundle $O(1)$ over $\mathbb{P}(E)$ is pseudo-effective. Here we denote by $E$ any desingularization of the sheaf $F$.

Indeed, this result does not appears explicitly in [CP19] but it is quickly deduced from the proof of [CP19] Thm 1.1]. More precisely this is proved in section 4.1, pages 14-18 of loc.cit.

4. Proof of the main result: the general case

We establish in the next subsection Theorem 1.3. This statement can be seen as a subharmonicity property of direct images and it represents the new technical result in the present article. Once this is done we show that the Kähler version of Lemma 3.3 follows. Further results based on Theorem 1.3 will be given in the next section.

A good point to start with is the remark that if $h_L$ is non-singular and if $p$ is a submersion, then follows directly from the curvature formula [Ber09 (4.8)]. Indeed, let $x \in X$ be an arbitrary point and let $u \in H^0(Y_x, KY_x + L|_{Y_x})$ be a holomorphic section. Since $h_L$ and $p$ are smooth, we can find a section

$$\hat{u} \in H^0(U, p_*(KY/X + L))$$

extending $u$ as in loc. cit., [Ber09 Prop 4.2]. Here we denote by $U$ a small coordinate set centered at $x$.

The equality [Ber09 (4.8)] implies that we have

$$\left\langle \Theta_{h/Y/X} \left( p_*(KY/X + L) \right) u, u \right\rangle \wedge \omega^n_X \geq c_n \int_{Y_x} \hat{u} \wedge p_*(\omega^n_X) e^{-\varphi_L}.$$ 

Since by hypothesis we have,

$$\Theta_{h_L} \wedge p^*\omega^n_X \geq \varepsilon_0 p^*\omega^n_X$$

it follows that

$$\left\langle \Theta_{h/Y/X} \left( p_*(KY/X + L) \right) u, u \right\rangle \wedge \omega^n_X \geq \varepsilon_0 \|u\|_0^2,$$

at $x$. We obtain thus if $h_L$ is smooth by taking the trace.

4.1. A subharmonicity property of direct images. In general case the metric $h_L$ is not necessary smooth, but nevertheless the function $\log |\sigma|^2_{h_{Y/X}}$ is still psh for any holomorphic section $\sigma$ of $E^*$ defined on some open subset $U$ of $X$. This is the content of Theorem 2.7 we have recalled in Section 2.

The next step is to show that under the assumptions of Theorem 1.3 the following holds.
Claim 4.1. Let $\sigma$ be a local holomorphic section of the dual bundle $E^\ast$. Then we have

\[(34) \quad dd^c \log |\sigma|^2_{h_{Y/X}^\ast} \wedge \omega_X^{n-1} \geq \varepsilon_0 \omega_X^n\]

in the sense of currents on $X$.

Prior to explaining the proof of this claim, we see here that it implies the inequality (4). To this end, we are using a trick from the article [HPS], which consists in considering the relative adjoint bundle

\[(35) \quad K_{\mathbb{P}(E^\ast)/X_0} + O(r)\]

on $\mathbb{P}(E^\ast)$, where $X_0 \subset X$ is the maximal subset of $X$ such that the restriction $E|_{X_0}$ is a vector bundle. The bundle $O(1)$ is endowed with the metric $h$ induced from $h_{Y/X}$. Then the relation (34) implies that we have

\[(36) \quad \Theta_h(O(1)) \wedge \pi^* \omega_X^{n-1} \geq \varepsilon_0 \pi^* \omega_X^n.\]

Now we apply the Claim one more time, for the direct image sheaf

\[(37) \quad \pi_* \left(K_{\mathbb{P}(E^\ast)/X} + O(r)\right)\]

which coincides with $\det E|_{X_0}$. Since $h$ is induced by the metric $h_{Y/X}$, it follows by the definition of a singular metric the corresponding multiplier sheaf is trivial over a large open subset of $\mathbb{P}(E^\ast)$. Then we have

\[(38) \quad \chi_{X_0} \Theta_{\det h_{Y/X}}(\det E) \wedge \omega_X^{n-1} \geq r\varepsilon_0 \omega_X^n.\]

in the sense of currents on $X_0$ as consequence of (34), where $\chi_{X_0}$ is the characteristic function of the set $X_0 \subset X$. Indeed this follows from the fact that

\[(39) \quad T := \Theta_{\det h_{Y/X}}(\det E) \wedge \omega_X^{n-1}\]

is a closed positive current on $X$, and we clearly have

\[(40) \quad T \geq \lim_{k \to \infty} v_k \Theta_{\det h_{Y/X}}(\det E) \wedge \omega_X^{n-1}\]

for a sequence $(v_k)$ converging to $\chi_{X_0}$, cf. [Dem1]. For each $k$ we have

\[(41) \quad v_k \Theta_{\det h_{Y/X}}(\det E) \wedge \omega_X^{n-1} \geq v_k \varepsilon_0 \omega_X^n\]

as currents of bi-degree $(n,n)$ on $X$. Summing up, we have

\[(42) \quad T \geq \lim_{k \to \infty} v_k \varepsilon_0 \omega_X^n\]

and this is equivalent to the inequality (4).

Remark 4.2. It is absolutely crucial that the lower bound (34) is the same as the lower bound of the trace of the curvature of $(L,h_L)$ we have by hypothesis in Theorem 1.5.

We detail now the proof of Claim 4.1. An important point in our argument here is that the function $\log |\sigma|^2_{h_{Y/X}^\ast}$ is already known to be psh. It would be therefore enough to show that the inequality (34) holds true in the complement of an analytic subset of $X$. In particular, we can assume that $p$ is smooth, and that $E$ is a trivial vector bundle of rank $r$. 

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4.1.1. Reduction to a local statement. We invoke here the regularization arguments in [BP08] in order to reduce our claim to a subharmonicity property of fiberwise Bergman kernels for Stein submanifolds in $\mathbb{C}^n$.

Let $s_1, s_2, \cdots, s_r$ be a set of sections of the fibration $p: p^{-1}(U) \to U$ and let $u_i$ be local sections of $\Lambda^n T_{Y/X} \otimes L^{-1}$ defined locally near the image of $s_i$. We define the Bergman kernel-type function on $U$ as follows

\begin{equation}
B(a, a) := \sum_{j,k} a_j \overline{a}_k B(s_j, s_k),
\end{equation}

where

\begin{equation}
B(s_j(y), s_k(y)) = \sum_i u_i(s_j(y)) \otimes \overline{u}_i(s_k(y)),
\end{equation}

and $\{u_1, \ldots, u_r\}$ is an orthonormal basis of $H^0 \left( X_y, (K_{X/Y} + L) \otimes I(h_{\phi_L}) \right)$. Also, we are using the same notation $a_i$ for the section $a_i$ evaluated at $s_i$. Therefore the expression (43) is indeed a function on $U$.

The link between (43) and our problem is as follows. We consider the expression

\begin{equation}
\xi_y := \sum a_j(s_j(y)) \text{ev}_{s_j(y)}
\end{equation}

which is a local holomorphic section of the dual bundle $\mathcal{E}^*$. Conversely, as observed in [BP08] any local section of $\mathcal{E}^*$ can be obtained in this manner. Moreover, the norm of the section (45) with respect to the metric $h_{\phi_L}$ is precisely $B(a, a)$. This is very important, because it shows that $B(a, a)$ is an extremal function.

It would be therefore sufficient to show that for every such $a$ we have

\begin{equation}
dd^c \log B(a, a) \wedge \omega_X^{n-1} \geq \varepsilon_0 \omega_X^n
\end{equation}

on $U$. Actually, thanks to a standard trick which we recall in Lemma 4.3 below it is enough to show the equivalent inequality

\begin{equation}
dd^c B(a, a) \wedge \omega_X^{n-1} \geq \varepsilon_0 B(a, a) \omega_X^n
\end{equation}

It is at this point that we need our map $p$ to be locally projective. In this case we can use the regularization procedure in [Ber06], [BP08] and reduce our problem (i.e. the inequality (47) above) to a local situation in which we can solve it by a direct computation. We will therefore recall the main steps of the relevant part of [BP08] and explain the way in which this is implemented in our case.

- According to the hypothesis, given any point $t_0 \in X$ there exists an open subset $U \subset X$ containing $x_0$ such that

\begin{equation}
D := p^{-1}(U) \setminus H
\end{equation}

is a Stein manifold, where $H$ is a hypersurface. We can therefore replace our initial map with a Stein fibration

\begin{equation}
p : D \to U.
\end{equation}

The corresponding Bergman kernel $B(a, a)$ is defined as in (43) except that we replace the finite dimensional basis $u_1, \ldots, u_r$ with a Hilbert basis of the space of holomorphic $L^2$ top forms on $D_t := D \cap p^{-1}(t)$ with respect to the $L^2$ norm

\begin{equation}
\|u\|_t^2 := \int_{D_t} c_n u \wedge \overline{u} \omega^n.
\end{equation}
Let \( \rho \) be a psh exhaustion function for the Stein manifold \( D \). For \( c \) large enough, the image of the sections \( s_i \) will be contained in \( D^c := (\rho < c) \). Let \( B_c(a,a) \) be the function \( \text{(45)} \) associated to the domain \( D^c \). By the extremal characterization of \( B(a,a) \) and \( B_c(a,a) \) respectively, we see that it would be enough to show that we have
\[
(51) \quad \ddc B_c(a,a) \wedge \omega_X^{n-1} \geq \varepsilon_0 B_c(a,a) \omega_X^n
\]
in the sense of currents on \( D^c \). The inequality \( \text{(17)} \) would follow as \( c \to \infty \).

Let \( c \) be a regular value for the function \( \rho \). There exists a biholomorphism
\[
\Phi : D^{c+1/2} / \{ t \} \to \Omega
\]
such that
\[
D^c_t \subset \Phi \left( D^{c+1/2} / \{ t \} \right) \subset D^{c+1}_t.
\]
This map is obtained as usual, by considering the flow associated to the holomorphic lifting of vector fields of \( U \) (we might be forced to shrink \( U \), but this is fine since the estimates we have to prove are independent of the size of this set).

We can assume that the bundle \( L \) is trivial when restricted to \( D \), and that the weight of the metric \( h_L \) is given by the psh function \( \varphi \). Via the map \( \text{(52)} \), the curvature hypothesis \( \text{(3)} \) becomes
\[
(54) \quad \ddc \phi \wedge \pi^* \omega^{n-1} \geq \varepsilon_0 \pi^* \omega^n
\]
where
\[
(55) \quad \phi := \varphi \circ \Phi, \quad \omega := \omega_X |_U.
\]
and \( \pi \) is the projection on the second factor of \( D^{c+1/2} / \{ t \} \).

Let \( \left( B(x_i, \varepsilon) \right)_{i=1,\ldots,N_\varepsilon} \) be a covering of the base \( X \) with balls of radius \( \varepsilon \). On each such ball we certainly have
\[
(56) \quad (1 - \delta_\varepsilon) \omega_i \leq \omega_X \leq (1 + \delta_\varepsilon) \omega_i
\]
where \( \omega_i \) is a flat metric on \( B(x_i, \varepsilon) \), and \( \delta_\varepsilon \to 0 \) as \( \varepsilon \to 0 \).

Then we have
\[
(57) \quad \ddc \phi \wedge \pi^* \omega_i^{n-1} \geq \varepsilon_0 \frac{(1 - \delta_\varepsilon)^{n-1}}{(1 + \delta_\varepsilon)^n} \pi^* \omega_i^n
\]
for each index \( i \) such that \( x_i \in U \) (so we assume that \( \Phi \) is defined on a domain slightly bigger, which is again harmless).

Now the point is that the inequality \( \text{(57)} \) behaves well with respect to convolutions on the inverse image of \( B(x_i, \varepsilon) \) –since \( \omega_i \) is flat– in the sense that we have
\[
(58) \quad \ddc \phi_\delta \wedge \pi^* \omega_i^{n-1} \geq \varepsilon_0 \frac{(1 - \delta_\varepsilon)^{n-1}}{(1 + \delta_\varepsilon)^n} \pi^* \omega_i^n
\]
for a monotonic sequence of smooth psh functions \( (\phi_\delta) \) converging to \( \phi \). This can be seen e.g. as follows. We first use the fact that the Stein manifold \( D^{c+1/2} / \{ t \} \) can be embedded in an Euclidean space \( \mathbb{C}^N \) (in our case this is much simpler, since it is a domain in a projective manifold). The image of the embedding
\[
(59) \quad D^{c+1/2} / \{ t \} \to \mathbb{C}^N
\]
has a Stein neighborhood $W$ by a result of Siu [], and moreover we have a holomorphic retract $\mu : W \to D_{t_0}^{c+1/2}$. We can regularize the function

$$\tilde{\phi} : U \times W \to \mathbb{R} \cup \{-\infty\}$$

by the usual convolution kernel, and $\phi_\delta$ is the restriction of the convolution to $U \times D_{t_0}^{c+1/2}$. Given this explicit definition of $\phi_\delta$, the inequality (60) becomes obvious.

- It is proved in [BP08] that there exists a sequence of smooth psh functions $\psi_j$ such that if we denote by $B_{j,\delta}(a, a)$ the Bergman kernel-type function induced fiberwise by $e^{-\psi_\delta - \psi_j}$, then the limit

$$\lim_{j} B_{j,\delta}(a, a) \mid_{D_{t_0}^{c+1/2} \times \{t\}}$$

is equal to the Bergman kernel on the fiber $D_t$ of our initial map $p$ with respect to the weight $\phi_\delta \circ \Phi^{-1}$. The role of the sequence $(\psi_j)$ is thus to "erase" the difference between the domains in (53). This is first proved for points $t$ in the complement of measure zero set, and then in general, cf. pages 355-356 in [BP08]. A last observation at this point is that we also have

$$dd^c(\phi_\delta + \psi_j) \wedge \pi^*\omega_t^{-1} \geq \varepsilon_0 \frac{(1 - \delta_\varepsilon)^{n-1}}{(1 + \delta_\varepsilon)^n} \pi^*\omega_t^n$$

for every $j$, since $\psi_j$ is psh.

- We show in the next subsection that we have

$$dd^c B_{j,\delta}(a, a) \wedge \omega_t^{-1} \geq \varepsilon_0 \frac{(1 - \delta_\varepsilon)^{2n-2}}{(1 + \delta_\varepsilon)^{2n}} B_{j,\delta}(a, a) \omega_t^n$$

on $B(x_i, \varepsilon)$. By using (60) and the fact that $B(a, a)$ is psh, we infer that we have

$$dd^c B(a, a) \wedge \omega_t^{-1} \geq \varepsilon_0 \frac{(1 - \delta_\varepsilon)^{2n-2}}{(1 + \delta_\varepsilon)^{2n}} B(a, a) \omega_t^n$$

and we are done, provided that we are able to establish (63). This will be done in the next subsection.

4.1.2. The local computation. As consequence of the discussion in the previous subsection, it is enough to consider the following set-up:

1. The map $p : U \times V \to U$ is simply the projection on the 1st factor. Here $U$ is the unit ball in $\mathbb{C}^n$ and $V$ is a Stein manifold.
2. We denote by $t_1, \ldots, t_n$ the coordinate functions on $U$, and we use the notation $z_1, \ldots, z_d$ for coordinates on $V$ at some point.
3. We have a smooth psh function $\varphi$ on $U \times V$ such that $dd^c \varphi \wedge p^*\omega_t^{-1} \geq \varepsilon_0 p^*\omega_t^n$
   point-wise on $U \times V$.
4. For each $t \in U$ we denote by $B_t$ the Bergman kernel on

$$\{t\} \times V, e^{-\varphi(t, \cdot)}$$

corresponding to $(d, 0)$ holomorphic forms.
(5) Let $s_i : U \to V$ be a set of holomorphic functions (i.e. sections of the projection map $p$), where $i = 1, \ldots, r$. We also consider the holomorphic sections $a_i$ of $s_i^* \Lambda^d T_V$ and we define

$$B_t(a, a) := \sum_{k,p=1}^r a_k(t) \bar{a}_p(t) B_t(s_k(t), s_p(t))$$

Now we will show that we have

$$dd^c B_t(a, a) \wedge \omega^{n-1} \geq \varepsilon_0 B_t(a, a) \omega^n$$

at each point $t \in U$. This would be indeed sufficient, thanks to the following standard fact.

**Lemma 4.3.** Assume that (66) holds true. Then we have

$$dd^c \log B_t(a, a) \wedge \omega^{n-1} \geq \varepsilon_0 \omega^n$$

**Proof.** The argument is well-known in the case of positive psh functions, cf. e.g. [LG], Lemma 3.46. Strictly the same proof goes through in our situation, as we will see in what follows.

The main observation is that if we replace the function $\varphi$ with

$$\varphi_\lambda := \varphi + \Re \left( \sum_{i=1}^n \alpha_i t_i \right)$$

then the hypothesis (3) above is still verified. The corresponding fiberwise Bergman kernel becomes

$$e^{\Re \left( \sum_{i=1}^n \alpha_i t_i \right)} B_t(a, a).$$

Now let $t_0 \in U$ be an arbitrary point. We assume that the coordinates $(t_i)$ are such that $\omega_{t_0}$ is the flat metric. Since the inequality (66) is true for the Bergman kernel (69), we infer that we have

$$\sum_i \frac{\partial^2 B}{\partial t_i \partial \bar{t}_i} + \frac{1}{2} \left( \alpha_i \frac{\partial B}{\partial t_i} + \bar{\alpha}_i \frac{\partial B}{\partial \bar{t}_i} \right) + \frac{1}{4} |\alpha_i|^2 B \geq n \varepsilon_0 B$$

where we denote by $B := B_t(a, a)$, and the quantities above are evaluated at $t = t_0$. The inequality (70) holds true for any choice of the coefficients $\alpha_i$. We then choose

$$\alpha_i := -\frac{2}{B(t_0)} \frac{\partial B}{\partial t_i}(t_0)$$

and the lemma follows. \qed

The inequality (67) will be established by a direct computation detailed along the next lines. The case $n = 1$ was treated by [Ber06], and our arguments below represent a generalization of his approach.

By the reproducing property of Bergman kernels, we have

$$B_t(a, a) = \sum_{i,j} a_i(t) \bar{a}_j(t) \int_V B_t(\xi, s_i(t)) B_t(\xi, s_j(t)) e^{-\varphi(t, \xi)}$$

We first take the anti-holomorphic derivative with respect to $t_\alpha$. 

\[\text{Page 15}\]
\[ \frac{\partial}{\partial t_\alpha} B_t(a, a) = \sum_{i,j} a_i(t) \frac{\partial a_j(t)}{\partial t_\alpha} \int_V B_t(\xi, s_i(t)) \overline{B_t(\xi, s_j(t))} e^{-\varphi(t, \xi)} \]

\[ + \sum_{i,j} a_i(t) a_j(t) \int_V \frac{\partial B_t}{\partial t_\alpha}(\xi, s_i(t)) \overline{B_t(\xi, s_j(t))} e^{-\varphi(t, \xi)} \]

\[ + \sum_{i,j} a_i(t) a_j(t) \int_V \frac{\partial B_t}{\partial t_\alpha}(\xi, s_i(t)) \overline{\frac{\partial s_j}{\partial t_\alpha}(\xi, s_j(t))} e^{-\varphi(t, \xi)} \]

\[ + \sum_{i,j} a_i(t) a_j(t) \int_V B_t(\xi, s_i(t)) \frac{\partial s_j}{\partial t_\alpha}(\xi, s_j(t)) e^{-\varphi(t, \xi)}. \]

\[ =: I_1 + I_2 + I_3 + I_4. \]

Given any holomorphic top form \( h \) which is \( L^2 \) on \( V \) we have

\[ h(s_j(t)) = \int_V h(\xi) \overline{B_t(\xi, s_j(t))} e^{-\varphi(t, \xi)}. \]

We apply the operator \( \frac{\partial}{\partial t_\alpha} \) to (74) and it follows that we have

\[ 0 = \int_V h(\xi) \frac{\partial B_t}{\partial t_\alpha}(\xi, s_j(t)) e^{-\varphi(t, \xi)}. \]

Therefore the last term \( I_4 \) of (73) is equal to zero.

We evaluate next the quantity

\[ \sum_{\alpha, \beta} \omega^{\alpha \beta} \frac{\partial^2}{\partial t_\beta \partial t_\alpha} B_t(a, a) \]

where \( (\omega^{\alpha \beta}) \) be the inverse of the coefficients of the metric \( \omega \) at some point \( t_0 \in U \).

The \( \frac{\partial}{\partial t_\beta} \) of the term \( I_1 \) gives

\[ \frac{\partial I_1}{\partial t_\beta} = \sum_{i,j} \frac{\partial a_i(t)}{\partial t_\beta} \frac{\partial a_j(t)}{\partial t_\alpha} \int_V B_t(\xi, s_i(t)) \overline{B_t(\xi, s_j(t))} e^{-\varphi(t, \xi)} \]

\[ + \sum_{i,j} a_i(t) \frac{\partial a_j(t)}{\partial t_\alpha} \int_V \frac{\partial B_t}{\partial t_\beta}(\xi, s_i(t)) \overline{B_t(\xi, s_j(t))} e^{-\varphi(t, \xi)} \]

\[ + \sum_{i,j} a_i(t) \frac{\partial a_j(t)}{\partial t_\alpha} \int_V \frac{\partial B_t}{\partial t_\beta}(\xi, s_i(t)) \overline{\frac{\partial s_j}{\partial t_\alpha}(\xi, s_j(t))} e^{-\varphi(t, \xi)} \]

\[ + \sum_{i,j} a_i(t) \frac{\partial a_j(t)}{\partial t_\alpha} \int_V B_t(\xi, s_i(t)) \frac{\partial B_t}{\partial t_\beta}(\xi, s_j(t)) \overline{\frac{\partial s_j}{\partial t_\alpha}(\xi, s_j(t))} e^{-\varphi(t, \xi)}. \]
We compute similar derivative for the other terms:

\[
(78) \quad \frac{\partial I_2}{\partial t_\beta} = \sum_{i,j} \frac{\partial a_i(t)}{\partial t_\beta} \int_V \frac{\partial B_i}{\partial t_\alpha}(\xi, s_i(t)) \frac{\partial s_i \gamma}{\partial t_\alpha}(\xi, s_j(t)) e^{-\varphi(t, \xi)}
\]

\[
+ \sum_{i,j} a_i(t) a_j(t) \int_V \frac{\partial^2 B_i}{\partial t_\beta \partial t_\alpha}(\xi, s_i(t)) \frac{\partial B_i}{\partial t_\alpha}(\xi, s_j(t)) e^{-\varphi(t, \xi)}
\]

\[
+ \sum_{i,j} a_i(t) a_j(t) \int_V \frac{\partial B_i}{\partial t_\alpha}(\xi, s_i(t)) \frac{\partial B_i}{\partial t_\beta}(\xi, s_j(t)) e^{-\varphi(t, \xi)}
\]

\[
+ \sum_{i,j} a_i(t) a_j(t) \int_V \frac{\partial B_i}{\partial t_\alpha}(\xi, s_i(t)) \frac{\partial s_i \gamma}{\partial t_\gamma}(\xi, s_j(t)) e^{-\varphi(t, \xi)}
\]

and

\[
(79) \quad \frac{\partial I_3}{\partial t_\beta} = \sum_{i,j} \frac{\partial a_i(t)}{\partial t_\beta} \int_V \frac{\partial B_i}{\partial t_\alpha}(\xi, s_i(t)) \frac{\partial s_i \gamma}{\partial t_\alpha}(\xi, s_j(t)) e^{-\varphi(t, \xi)}
\]

\[
+ \sum_{i,j} a_i(t) a_j(t) \int_V \frac{\partial^2 B_i}{\partial t_\beta \partial t_\alpha}(\xi, s_i(t)) \frac{\partial s_i \gamma}{\partial t_\alpha}(\xi, s_j(t)) e^{-\varphi(t, \xi)}
\]

\[
+ \sum_{i,j} a_i(t) a_j(t) \int_V \frac{\partial B_i}{\partial t_\alpha}(\xi, s_i(t)) \frac{\partial B_i}{\partial t_\beta}(\xi, s_j(t)) e^{-\varphi(t, \xi)}
\]

\[
+ \sum_{i,j} a_i(t) a_j(t) \int_V \frac{\partial B_i}{\partial t_\alpha}(\xi, s_i(t)) \frac{\partial s_i \gamma}{\partial t_\gamma}(\xi, s_j(t)) e^{-\varphi(t, \xi)}
\]

In order to arrange a bit the terms above we make the following observations

(i) The second term in the rhs of (77) equals zero, by the formula (75).

(ii) We introduce the notation

\[
\Xi_{\beta\gamma}(t, \xi) := \frac{\partial a_i(t)}{\partial t_\alpha} + a_i(t) \left( \frac{\partial B_i}{\partial t_\alpha}(\xi, s_i(t)) + \frac{\partial B_i}{\partial t_\beta}(\xi, s_i(t)) \frac{\partial s_i \gamma}{\partial t_\alpha} \right).
\]

Then we have

\[
(80) \quad \frac{\partial^2 B_i(a, a)}{\partial t_\beta \partial t_\gamma} = \sum_{i,j} \int_V \Xi_{\beta\gamma}(t, \xi) \Xi_{\alpha\beta}(t, \xi) e^{\varphi}\n\]

\[
+ \sum_{i,j} a_i(t) a_j(t) \int_V \frac{\partial^2 B_i}{\partial t_\beta \partial t_\alpha}(\xi, s_i(t)) B_i(\xi, s_j(t)) e^{-\varphi(t, \xi)}
\]

\[
+ \sum_{i,j} a_i(t) a_j(t) \int_V \frac{\partial B_i}{\partial t_\beta} \frac{\partial s_i \gamma}{\partial t_\gamma}(\xi, s_j(t)) B_i(\xi, s_j(t)) e^{-\varphi(t, \xi)}
\]
By the equation (75) we infer that

$$\sum_{i,j,\alpha,\beta} \omega^{\alpha \beta} \int_V \Xi_{\alpha \beta}(t, \xi) \Xi_{\alpha \beta}^*(t, \xi) e^\varphi \geq 0$$

for each \( t \in U \) we only have to deal with the last two terms of the previous equality in order to obtain a lower bound of (76).

(iii) The last two terms of (80) are obtained by applying the operator \( \partial^\varphi / \partial t_\beta \circ \partial / \partial t_\alpha \) to the function \( t \to B_t(\xi, s_i(t)) \). The commutation formula reads as

$$\left[ \frac{\partial^\varphi}{\partial t_\beta}, \frac{\partial}{\partial t_\alpha} \right] = \frac{\partial^2 \varphi}{\partial t_\beta \partial t_\alpha}$$

so all in all we infer that we have

$$\sum_{i,j} \omega^{\alpha \beta} \frac{\partial^2 B_t(a, a)}{\partial t_\beta \partial t_\alpha} \geq \sum_{i,j} a_i(t) a_j(t) \omega^{\alpha \beta} \int_V \frac{\partial^2 \varphi}{\partial t_\beta \partial t_\alpha} B_t(\xi, s_j(t)) B_t(\xi, s_j(t)) e^{-\varphi(t, \xi)}$$

$$+ \sum_{i,j} a_i(t) a_j(t) \omega^{\alpha \beta} \int_V \frac{\partial}{\partial t_\alpha} \left( \frac{\partial \varphi}{\partial t_\beta} \right)(\xi, s_j(t)) B_t(\xi, s_j(t)) e^{-\varphi(t, \xi)}$$

By the equation (76) we infer that

$$\int_V \frac{\partial}{\partial t_\alpha} \left( \frac{\partial \varphi}{\partial t_\beta} \right)(\xi, s_i(t)) B_t(\xi, s_j(t)) e^{-\varphi(t, \xi)} = - \int_V \frac{\partial^2 \varphi}{\partial t_\beta \partial t_\alpha} B_t(\xi, s_j(t)) e^{-\varphi(t, \xi)}$$

The computations above were done with respect to an arbitrary coordinate system, so we can as well assume that \( \omega^{\alpha \beta} = \delta_{\alpha \beta} \) at some point \( t_0 \in U \). We therefore get

$$\sum_{i,j} \frac{\partial^2 B_t(a, a)}{\partial t_\alpha \partial t_\beta} (t_0) \geq \sum_{i,j} a_i(t_0) a_j(t_0) \int_V \frac{\partial^2 \varphi}{\partial t_\alpha \partial t_\beta} B_{t_0}(\xi, s_j(t_0)) B_{t_0}(\xi, s_j(t_0)) e^{-\varphi(t_0, \xi)}$$

$$- \sum_{i,j} a_i(t_0) a_j(t_0) \int_V \frac{\partial \varphi}{\partial t_\beta} B_{t_0}(\xi, s_j(t_0)) \frac{\partial^2 \varphi}{\partial t_\alpha \partial t_\beta} B_{t_0}(\xi, s_j(t_0)) e^{-\varphi(t_0, \xi)}$$

$$= \sum_i \int_V \left( \frac{\partial^2 \varphi}{\partial t_\alpha \partial t_\beta} (t_0, \xi) ||\Gamma(\xi)||^2 - |\Lambda_\alpha(\xi)|^2 \right) e^{-\varphi(t_0, \xi)},$$

where the notations used are as follows

(82) \( \Gamma(\xi) := \sum_i a_i(t_0) B_{t_0}(\xi, s_i(t_0)) \)

and

(83) \( \Lambda_\alpha(\xi) := \sum_i a_i(t_0) \frac{\partial \varphi}{\partial t_\alpha} B_{t_0}(\xi, s_i(t_0)) \)

In order to evaluate the norm of \( \Lambda_\alpha \) we compute its \( \bar{\partial} \), and obtain

(84) \( \bar{\partial} \Lambda_\alpha(\xi) = - \sum_{i,k} a_i(t_0) B_{t_0}(\xi, s_i(t_0)) \frac{\partial^2 \varphi}{\partial t_\alpha \partial t_\beta} \delta_{\alpha \beta} \)

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and we see that this can be rewritten as

\[ (85) \bar{\partial} \Lambda_\alpha(\xi) = -\Gamma(\xi) \sum_k \frac{\partial^2 \varphi}{\partial t_\alpha \partial \xi_k} \xi_k. \]

Moreover, \( \Lambda_\alpha \) is perpendicular on the space of holomorphic \( L^2 \) functions – by the property (75) –, and Hörmander estimates show that we much have

\[ (86) \int_V |\Lambda_\alpha(\xi)|^2 e^{-\varphi(t_0,\xi)} \leq \int_V |\Gamma|^2 \frac{\partial^2 \varphi}{\partial t_\alpha \partial \xi_k} \frac{\partial^2 \varphi}{\partial t_\alpha \partial \xi_m} \varphi_m e^{-\varphi(t_0,\xi)}. \]

We thus get the inequality

\[ (87) \sum_\alpha \frac{\partial^2 B_1(a, a)}{\partial t_\alpha \partial \bar{t}_\alpha}(t_0) \geq \sum_\alpha \int_V |\Gamma|^2 \left( \frac{\partial^2 \varphi}{\partial t_\alpha \partial \xi_k} - \frac{\partial^2 \varphi}{\partial t_\alpha \partial \xi_k} \frac{\partial^2 \varphi}{\partial t_\alpha \partial \xi_m} \varphi_m \right) e^{-\varphi(t_0,\xi)}. \]

It turns out that the quantity

\[ (88) \frac{\partial^2 \varphi}{\partial t_\alpha \partial \bar{t}_\alpha} - \frac{\partial^2 \varphi}{\partial t_\alpha \partial \xi_k} \frac{\partial^2 \varphi}{\partial t_\alpha \partial \xi_m} \varphi_m \]

has an intrinsic interpretation: it is equal to

\[ (89) \frac{n}{d+1} (dd^c \varphi)^{d+1} \wedge p^* \omega^{n-1} \]

evaluated at the point \( t_0 \). By hypothesis, \( \varphi \) is psh, which combined with the lower bound for the trace of \( dd^c \varphi \) with respect to \( p^* \omega \) shows that

\[ (90) \frac{n}{d+1} (dd^c \varphi)^{d+1} \wedge p^* \omega^{n-1} \geq n \varepsilon_0. \]

The inequality (90) is easy to justify in our situation, since \( \varphi \) is strictly psh, and we can choose the coordinates in such a way that (90) becomes trivial. It holds however true in more general circumstances, as we will now see: it is enough to assume that \( \varphi \) to be strictly psh on the fibers of \( p \).

To this end, let \( (z_i) \) be local coordinates centered at the origin, such that

\[ (91) dd^c \varphi = \sqrt{-1} \left( \sum_{\alpha, \beta} \varphi_{\alpha \beta} dt_\alpha \wedge d\bar{t}_\beta + \sum_{\alpha, a} \varphi_{\alpha a} dt_\alpha \wedge dz_a + \sum_{a, \beta} \varphi_{a \beta} dz_a \wedge d\bar{t}_\alpha \right) + \sqrt{-1} \sum_a dz_a \wedge d\bar{z}_a. \]

By hypothesis we have

\[ n \varepsilon_0 \prod_\alpha \sqrt{-1} dt_\alpha \bar{d}t_\alpha \leq \sum_\alpha \varphi_{\alpha \alpha} \prod_\alpha \sqrt{-1} dt_\alpha \bar{d}t_\alpha \]

\[ + \sqrt{-1} \sum_{\alpha, a} \varphi_{\alpha a} dt_\alpha \wedge dz_a \prod_\alpha d\bar{t}_\alpha \wedge d\bar{z}_a \]

\[ + \sqrt{-1} \sum_{a, \alpha} \varphi_{a \alpha} dz_a \wedge d\bar{t}_\alpha \prod_\alpha dt_\alpha \wedge \bar{d}t_\alpha \]

\[ + \sqrt{-1} \left( \sum_a dz_a \wedge d\bar{z}_a \right) \prod_\alpha dt_\alpha \wedge d\bar{t}_\alpha, \]
and therefore for any choice of complex numbers \((\lambda_{\alpha\pi})\) we have
\[
\sum_{\alpha} \varphi_{\alpha\pi} - n\varepsilon_0 - 2\Re(\sum_{\alpha,a} \varphi_{\alpha\pi\lambda_{\alpha\pi}}) + \sum |\lambda_{\alpha\pi}|^2 \geq 0.
\]
This now implies that
\[
\sum_{\alpha} \varphi_{\alpha\pi} - \sum_{\alpha,a} |\varphi_{\alpha\pi}|^2 \geq n\varepsilon_0
\]
which is what we wanted to prove.

The quantity we have on the LHS of (88) reads as
\[
\frac{n dd^c B_t\langle a, a \rangle \wedge \omega^{n-1}}{\omega^n}
\]
and we get
\[
dd^c B_t\langle a, a \rangle \wedge \omega^{n-1} \geq \varepsilon_0 B_t\langle a, a \rangle \omega^n.
\]
In conclusion, the local version of our result holds true and Theorem 1.3 is proved.

4.2. Proof of Theorem 1.5. We first assume that \(\omega\) is Kähler. Then Theorem 1.5 is easily reduced to Theorem 1.3 as follows.

Let \(U \subset X\) be an open subset of \(X\), such that
\[
\omega_X|_U = dd^c \varphi_U
\]
for some function smooth \(\varphi_U\) defined on \(U\). We then consider the restriction of \(L\) on the \(p\)-inverse image of \(U\) denoted by \(Y_U\) and we endow it with the metric
\[
h_1 := e^{-C \cdot \phi} h_L
\]
where \(\phi := \varphi_U \circ p\). The hypothesis of Theorem 1.3 are fulfilled. The constant \(\varepsilon_0\) is in this case equal to \(C\). Then we have
\[
\Theta_{\det h_1, Y_U/X} (\det E) \wedge \omega_X^{n-1} \geq rC \omega_X^n,
\]
which is equivalent to
\[
\Theta_{\det h_{Y/X}} (\det E) \wedge \omega_X^{n-1} \geq 0,
\]
by using the relation \(h_{Y,U}/U = h_{Y/X} \cdot e^{-C \cdot \varphi_U}\).

Now for the case of an arbitrary metric \(\omega\) we argue as in the proof of Theorem 1.3. Let \(B(x_i, \varepsilon)\) be a covering of \(X\) with balls of radius \(\varepsilon\), such that the condition (56) is satisfied. Then on each ball we consider the metric
\[
h_{i,L} := e^{-C(1+\delta_2)|t_i|^2} h_L
\]
where \((t_i)\) are coordinates on the ball \(B(x_i, \varepsilon)\). By the Kähler case already discussed, we obtain
\[
\Theta_{\det h_{Y/X}} (\det E) + rC(1 + \delta_2) \omega_i \wedge \omega_i^{n-1} \geq rC(1 + O(\delta_2)) \omega_i^n
\]
on each ball \(B(x_i, \varepsilon)\). This inequality is basically unchanged if we replace \(\omega_i\) by the global metric \(\omega\), again thanks to (56). We therefore obtain
\[
\Theta_{\det h_{Y/X}} (\det E) + rC(1 + \delta_2) \omega \wedge \omega^{n-1} \geq rC(1 + O(\delta_2)) \omega^n
\]
which is therefore established by letting \(\varepsilon \to 0\).
4.3. Kähler version of Lemma 3.3. We will establish here the Kähler version of Lemma 3.3. The statement is absolutely the same, except that we only assume the manifold $X$ to be compact Kähler. Also, the arguments are similar to those used in the projective case but we apply Theorem 1.5 instead of Theorem 2.7. There is however an additional slight complication due to the singularities of $F$, so we will provide a complete treatment in what follows.

Recall that we have the bundle $E \to \hat{X}$ which admits a sequence of smooth metrics $h_{E,\varepsilon}$ such that

\[(103) \quad \Theta(E, h_{E,\varepsilon}) \wedge g_{\varepsilon}^{n-1} = \delta_{\varepsilon} g_{\varepsilon}^n \otimes \text{Id}_{\text{End}(E)}\]

for some constant $\delta_{\varepsilon}$ such that $\lim_{\varepsilon \to 0} \delta_{\varepsilon} = 0$. Let $h$ be a metric on $\mathcal{O}(1)$ which could be singular, but such that

\[(104) \quad \Theta(\mathcal{O}(1), h) \geq 0\]

in the sense of currents on $\mathbb{P}(E)$.

Assume by contradiction that the multiplier sheaf $I((k+1-\delta_0)h|_{\mathbb{P}(E)_x})$ is non-trivial on the generic fiber $\mathbb{P}(E)_x$ for some $\delta_0 > 0$. Then we consider the relative adjoint bundle

\[(105) \quad K_{\mathbb{P}(E)/\hat{X}} + \mathcal{O}(r+k).\]

By a standard $L^2$-estimate (cf. Lemma 4.4), the direct image

\[(106) \quad \mathcal{G} := p^*(K_{\mathbb{P}(E)/\hat{X}} + \mathcal{O}(r+k)) \otimes I((k+1-\delta_0)h)\]

is non zero. We define the metric

\[(107) \quad h_{\varepsilon} := h_{E,\varepsilon}^{(r-1+\delta_0)} \otimes h^{(k+1-\delta_0)}\]

on the bundle $\mathcal{O}(r+k)$ —here we denote with the same symbol $h_{E,\varepsilon}$ the metric on $E$ and the metric induced by it on $\mathcal{O}(1)$.

Then a direct computation shows that we have

\[(108) \quad \Theta(\mathcal{O}(r+k), h_{\varepsilon}) \wedge \pi^* g_{\varepsilon}^{n-1} \geq (r-1) \delta_{\varepsilon} \pi^* g_{\varepsilon}^n.\]

Here we are using (104) combined with the Hermite-Einstein identity (103) and the explicit expression of the curvature of $\mathcal{O}(1)$.

We therefore get a proper subsheaf $\mathcal{G}$ of $\text{Sym}^k E^* \otimes \det E^*$ for which we have

\[(109) \quad \int_{\hat{X}} c_1(\mathcal{G}) \wedge (\pi^* g_{\varepsilon})^{n-1} \geq (r-1) \delta_{\varepsilon} \int_{\hat{X}} (\pi^* g_{\varepsilon})^n\]

by Theorem 1.5. Moreover, $\mathcal{G}$ is independent of $\varepsilon$. As $\varepsilon \to 0$ we obtain

\[(110) \quad \int_{\hat{X}} c_1(\mathcal{G}) \wedge \pi^* g_{\varepsilon}^{n-1} \geq 0\]

since the volume of $\hat{X}$ with respect to $g_{\varepsilon}$ is uniformly bounded.

As in the projective case, the stability hypothesis forbids the existence of such subsheaf $\mathcal{G}$. In conclusion, we have

\[(111) \quad I((k+1-\varepsilon)h|_{\mathbb{P}(E)_x}) = \mathcal{O}_{\mathbb{P}(E)_x},\]

for any $\varepsilon > 0$, where $x$ is a generic point.
Finally, we prove Theorem 1.2 in the case when $X$ is compact Kähler. Thanks to the stability condition $(ii)$ of Theorem 1.2 implies that

$$I(h ^{(r+1)}|_{\mathcal{P}(E)_x}) = \mathcal{O}_{\mathcal{P}(E)_x}.$$ 

By applying Theorem 2.7 to $p^*(K_{\mathcal{P}(E)/X} + \mathcal{O}(r+1) \otimes I(h ^{(r+1)})) = F^*$, $F^*$ is positively curved. Together with $c_1(F^*) = 0$, Proposition 2.6 implies that $F^*$ is hermitian flat.

In our previous arguments we have used the following statement, which is a consequence of $L^2$-estimates.

Lemma 4.4. Let $\mathbb{P}^n$ be the $n$-dimensional projective space, and let $m \in \mathbb{N}^\star$. Let $h$ be a singular metric on $\mathcal{O}_{\mathbb{P}^n}(m)$ such that $i \Theta_h(\mathcal{O}_{\mathbb{P}^n}(m)) \geq 0$ on $X$. Then the space $H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(m-1) \otimes I((1-\varepsilon)h))$ is non zero for every $\varepsilon > 0$.

Proof. Set $L := \mathcal{O}_{\mathbb{P}^n}(m+n)$. Then we have

$$\mathcal{O}_{\mathbb{P}^n}(m-1) = K_{\mathbb{P}^n} + L.$$ 

Let $x \in \mathbb{P}^n \setminus V(I((1-\varepsilon)h))$ and let $\{s_1, s_2, \ldots, s_n\}$ be a basis of $H^0(\mathbb{P}^n, \mathcal{O}(1))$ vanishing on $x$. The local weight

$$\varphi := n \log(\sum_{i} |s_i|^2)$$

defines a metric on $\mathcal{O}(n)$ of isolated singularity at $x$ with Lelong number $n$. We define the metric

$$h_L := h_\varphi + (1-\varepsilon)h + \varepsilon h_{FS}$$
on $L$, where $h_{FS}$ is the Fubini-Study metric. Then the $L^2$-estimates (cf. [Dem2, Cor 5.12]) implies that

$$H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(K_{\mathbb{P}^n} + L) \otimes I((1-\varepsilon)h)) \to (\mathcal{O}_{\mathbb{P}^n}(K_{\mathbb{P}^n} + L))_x$$
is surjective. The lemma is proved. \hfill \Box

5. A few comments about the general conjecture and other results

We recall that the conjecture of Pereira-Touzet cf. [PeTou] states that if $\mathcal{F} \subset T_X$ is a holomorphic foliation such that $c_1(\mathcal{F}) = 0$, $c_2(\mathcal{F}) \neq 0$ and $\mathcal{F}$ is $\omega_X$-stable and $K_X$ is pseudo-effective, then $\mathcal{F}$ is algebraic.

As we have already mentioned, under the conditions above $\mathcal{F}$ is a sub-bundle of $T_X$. If we assume (by contradiction) that $\mathcal{F}$ is not algebraic, then the bundle $\mathcal{O}(1) \to \mathcal{P}(\mathcal{F})$ is pseudo-effective. We can therefore find a singular metric $e^{-\varphi}$ on it such that the curvature $\Theta = dd^c\varphi \geq 0$ in the sense currents. The Siu decomposition of this current writes as

$$(112) \quad \Theta = \sum a_i [E_i] + T,$$

where $\sum a_i [E_i]$ is the divisorial part and $T$ is a positive current with the property that $\operatorname{codim}_X E_c(T) \geq 0$ for every $c > 0$. Here $E_c(T)$ is the locus where the Lelong number of $T$ is at least $c$.

We show now the following result concerning the divisorial part of the current $\Theta$. 

Theorem 5.1. Let $X$ be a smooth projective manifold, such that $K_X$ is pseudo-effective. We assume that $\mathcal{F}$ is a holomorphic foliation on $X$ such that $c_1(\mathcal{F}) = 0$. Assume moreover that $\mathcal{F}$ is $H$–stable, where $H$ is an ample divisor on $X$. If $\mathcal{O}(1)$ is pseudo-effective and let $E$ be one component of the divisorial part of $(112)$, then the map

\begin{equation}
(113) \quad p_E : E \to X
\end{equation}

induced by the restriction of $p$ to $E$ is a locally trivial fibration.

Proof. We first remark that for each $j$ we have

\begin{equation}
(114) \quad E_j \equiv m_j \mathcal{O}(1) + p^*(L_j)
\end{equation}

where $m_j$ is a positive integer, $L_j$ is a line bundle on $X$ and $p : \mathbb{P}(T\mathcal{F}) \to X$ is the projection map. We equally have

\begin{equation}
(115) \quad T \equiv m_T \mathcal{O}(1) + p^*(L_T)
\end{equation}

where $m_T \geq 0$ is a real number and $L_T$ is a $\mathbb{R}$-bundle.

The hypothesis $c_1(\mathcal{F}) = 0$ together with the stability condition shows that for each $j$ we have

\begin{equation}
(116) \quad L_j \cdot H^{n-1} \geq 0
\end{equation}

as soon as $m_j \geq 1$, i.e. if $E_j$ is horizontal with respect to $p$. Now if $E_j$ projects into a proper subvariety of $X$, then we also have $L_j \cdot H^{n-1} \geq 0$ since it means that the corresponding $m_j$ equals zero. Moreover, $T$ is a closed positive current, so a quick approximation argument shows that $L_T \cdot H^{n-1} \geq 0$ as well. On the other hand, the equality $(112)$ shows that numerically we have

\begin{equation}
(117) \quad \sum \nu_j L_j + L_T \equiv 0.
\end{equation}

Hence we infer that $L_j \cdot H^{n-1} = L_T \cdot H^{n-1} = 0$ for every $j$. Therefore, we obtain that $m_j \geq 1$ for every $j$, namely, every component $E_j$ is horizontal with respect to $p$.

Consider the exact sequence

\begin{equation}
(118) \quad 0 \to L_j^{-1} \to \text{Sym}^{m_j} \mathcal{F}^* \to Q_j \to 0
\end{equation}

induced by the section $E_j$. By taking the determinants and using the previous considerations we obtain $\det(Q_j) \wedge \omega^{n-1} = 0$. On the other hand, the foliation $\mathcal{F}$ is smooth so it follows that $\det Q_j$ is pseudo-effective, cf. [CP19]. In conclusion, $c_1(Q_j) = 0$, which in turn implies that $c_1(L_j) = 0$.

After these preliminary considerations, we are ready to prove that $p_E$ is locally trivial. Let

$S := H_1 \cap H_2 \cap \cdots \cap H_{n-1}$

be the complete intersection of some smooth hypersurfaces $H_i \in |kH|$ for $k$ large enough. To prove that $p_E$ is locally trivial, it is sufficient to prove that

$\quad p_E \mid_{p_E^{-1}(S)} : p^{-1}(S) \cap E \to S$

is locally trivial.

Thanks to [Lam04, Thm 5.2], $\mathcal{F}|_S$ is stable if $k$ is large enough (note that we don’t need ask that $H_i$ is a generic hypersurface here). Since $c_1(\mathcal{F}|_S) = 0$, it follows
that \( O_{\mathbb{P}(\mathcal{F})}(1)|_S \) admits a smooth metric \( h \) of semi-positive curvature. As we proved that \( L_j \cdot H^{n-1} = 0 \) for every \( j \), we get
\[
c_1(E) = c_1(O(m)) \in H^{1,1}(\mathbb{P}(\mathcal{F}), \mathbb{R})
\]
for some \( m \in \mathbb{N} \). Let \( a \geq m \) be some number large enough. \( O(a)|_S \) be can equipped with the metric
\[
h_a := h^{a-m} \cdot e^{-\log |s_E|^2},
\]
whose curvature is semipositive. We consider the direct image of \( p : p^{-1}(S) \to S \)
\[
p_*(K_{p^{-1}(S)/S} + O_{\mathbb{P}(\mathcal{F})}(a)) \otimes \mathcal{I}(h_a)) \subset p_*(K_{p^{-1}(S)/S} + O_{\mathbb{P}(\mathcal{F})}(a))
\]
on \( S \). Then RHS is just \( \text{Sym}^{a-2} \mathcal{F}^*|_S \), which is hermitian flat. The LHS is Griffiths-semi-positive by [PT]. Therefore LHS is also hermitian flat, and the flat connection is compatible with the flat connection on \( \mathcal{F}^*|_S \). Note also that the germs of \( p_*(K_{p^{-1}(S)/S} + O_{\mathbb{P}(\mathcal{F})}(a)) \otimes \mathcal{I}(h_a))_x \) is just the degree \( (a-r) \)-polynomials vanishing on \( E \). For \( a \) large enough, the flatness \( p_*(K_{p^{-1}(S)/S} + O_{\mathbb{P}(\mathcal{F})}(a)) \otimes \mathcal{I}(h_a)) \) implies that the common zero locus of these polynomials (which is \( E \)) is also invariant by the flat connection on \( \mathcal{F}|_S \). It means that \( E|_S \) is invariant by the parallel transport of the flat connection on \( \mathcal{F}|_S \). Then \( p_E^{-1}(S) \to S \) is locally trivial. As \( S \) is the complete intersection of any smooth hypersurfaces in \( |kH| \), \( p_E \) is thus locally trivial. 
\[\square\]

If the rank of \( \mathcal{F} \) is equal to two we obtain the following result, which is already known cf. [Dr17], but our arguments here are different.

**Corollary 5.2.** Let \( X \) be a projective manifold such that \( K_X \) is pseudo-effective and let \( \mathcal{F} \subset \mathcal{T}_X \) be a holomorphic subsheaf of rank 2 such that the following hold.

(i) The first Chern class of \( \mathcal{F} \) is zero, i.e. \( c_1(\mathcal{F}) = 0 \).

(ii) The sheaf \( \mathcal{F}^* \) is \( \omega_X \)-strongly stable, meaning that for any finite étale cover \( \pi : X' \to X \), \( \pi^* \mathcal{F}^* \) is \( \pi^* \omega_X \)-stable.

Then the bundle \( O(1) \) on \( \mathbb{P}(\mathcal{F}) \) is not pseudo-effective, or \( \mathcal{F} \) is Hermitian flat.

**Proof.** If the bundle \( O(1) \) on \( \mathbb{P}(\mathcal{F}) \) is pseudo-effective, we consider Siu’s decomposition (112). If there is no divisorial part, namely \( \sum a_i [E_i] = 0 \), then the Lelong number of \( T \) vanishes over the generic fiber on the projection \( p \) (since the fibers are of dimension 1). We have already explained that in this case it follows that \( \mathcal{F} \) is hermitian flat.

If we assume moreover that \( \mathcal{F} \) is not hermitian flat, we have to have a component \( E \) of the divisor-like part of the current \( \Theta \). Thanks to Theorem 5.1, the map
\[
p_E : E \to X
\]
is an étale cover. We prove that \( \pi_E^* \mathcal{F}^* \) is not stable and this contradicts with our strongly stable condition. Indeed, we have the arrows
\[
(119) \quad 0 \to O(-1)|_E \to \pi_E^*(\mathcal{F}^*)
\]
and a quick computation shows that we have
\[
(120) \quad \int_E c_1(O(-1)) \wedge \pi_E^*(\omega_H^{n-1}) = 0,
\]
because \( c_1(\mathcal{F}) = 0 \). This is the end of the proof. \[\square\]
Remark 5.3. Actually Claim 4.1 (on page 7) concerns the positivity of the curvature form of $E$ on the current $\omega_X^{n-1}$. It would be nice to extend this as initiated by Berndtsson-Sibony in [BeSib] i.e. replace $\omega_X^{n-1}$ with more general currents of bidimension $(1, 1)$ on $X$. The following two results are pointing in this direction.

Theorem 5.4. Let $p : Y \to X$ be a holomorphic surjective and proper map, where $X$ and $Y$ are Kähler manifolds. Moreover, we assume that $p$ is locally projective. Let $L$ be a line bundle over $Y$ with a possible singular metric $h_L$ such that $\Theta_{h_L}(L) \geq 0$, and

$$\Theta_{h_L}(L) \wedge p^*\omega_X^{n-1} \geq \varepsilon_0 p^*\omega_X^n$$

where $\omega_X$ is a Hermitian metric on $X$, $n = \dim X$, and $\varepsilon_0 > 0$ is a positive real number. We assume that the space of fiberwise $L^2$ sections (with respect to $h_L$) of $p_*(mK_{Y/X} + L)$ is non zero. Then there exists a metric $h$ on the bundle $mK_{Y/X} + L$ such that we have

$$\Theta_h(mK_{Y/X} + L) \wedge p^*\omega_X^{n-1} \geq \varepsilon_0 \cdot p^*\omega_X^n$$

in the sense of currents on $X$.

Proof. We first show that this is true for $m = 1$. In this case the metric $h$ in our statement 5.4 is precisely the fiberwise Bergman metric, as we will now see.

Actually, this a consequence of the proof of Theorem 1.5. We follow the approximation process for the map $p$ restricted to one of the balls $B(x, \varepsilon)$, on which the distortion between $\omega_X$ and the flat metric in coordinates $(t_i)$ is $1 + O(\delta)$. The inequality (61) applied for the sections $s_1 = s_2 = \cdots = s_r = s$ shows that the regularized version of (122) holds true. Indeed (61) holds true when restricted to the image of any section $s$, hence point-wise by [Dem2, III Criteria 1.6].

In conclusion, the inequality

$$\Theta_{h_1}(K_{Y/X} + L) \wedge p^*\omega_X^{n-1} \geq \varepsilon_0 p^*\omega_X^n$$

follows by taking the limit of several parameters involved.

If $m \geq 2$ we argue as follows. Let $h_B$ be the relative $m$-Bergman kernel metric on $mK_{Y/X} + L$. By our hypothesis, this is not identically $+\infty$. We define the bundle $F := (m-1)K_{Y/X} + L$. Then the local weights

$$\varphi_1 := (1 - \frac{1}{m})\varphi_B + \frac{1}{m}\varphi_L$$

define a metric on $F$ satisfying

$$\Theta_{h_1}(F) \geq 0, \quad \Theta_{h_1}(F) \wedge p^*\omega_X^{n-1} \geq \frac{\varepsilon_0}{m} p^*\omega_X^n.$$

By applying the above $m = 1$ case to $(K_{Y/X} + F, h_1)$, we obtain a Bergman type metric $h_{B,1}$ on $K_{Y/X} + F$ such that

$$\Theta_{h_{B,1}}(K_{Y/X} + F) \geq 0, \quad \Theta_{h_{B,1}}(K_{Y/X} + F) \wedge p^*\omega_X^{n-1} \geq \frac{\varepsilon_0}{m} p^*\omega_X^n.$$

We now iterate this process, namely we define the metric

$$\varphi_2 := (1 - \frac{1}{m})\varphi_{B,1} + \frac{1}{m}\varphi_L$$
on $F$, and let $h_{B,2}$ be the relative Bergman kernel metric corresponding to the data $(K_{Y/X} + F, h_2)$. Note that we have

\[(128) \quad \Theta_{h_2}(F) \wedge p^* \omega_X^{n-1} \geq \left( \frac{1}{m} + \frac{1}{m} \left( 1 - \frac{1}{m} \right) \right) \epsilon_0 p^* \omega_X^n,\]

we get thus $\Theta_{h_{B,2}}(K_{Y/X} + F) \geq 0$ and

\[(129) \quad \Theta_{h_{B,2}}(K_{Y/X} + F) \wedge p^* \omega_X^{n-1} \geq \left( \frac{1}{m} + \frac{1}{m} \left( 1 - \frac{1}{m} \right) \right) \epsilon_0 p^* \omega_X^n.\]

Then for every $k \in \mathbb{N}$, we obtain a metric $h_{B,k}$ on $K_{Y/X} + F$ such that $\Theta_{h_{B,k}}(K_{Y/X} + F) \geq 0$ together with

\[(130) \quad \Theta_{h_{B,k}}(K_{Y/X} + F) \wedge p^* \omega_X^{n-1} \geq \frac{1}{m} \left( \sum_{i=0}^{k-1} (1 - \frac{1}{m})^i \right) \epsilon_0 p^* \omega_X^n.\]

By normalization (and eventually taking a subsequence), $h_{B,k}$ converges to a metric denoted by $h$ on $K_{Y/X} + F = mK_{Y/X} + L$ which satisfies (122). \[\square\]

We remark that although the limit metric $h$ does not exactly the $m$-Bergman kernel metric, but nevertheless it has the following important property. Let $D$ be some divisor such that $p^*(D) = \sum a_i E_i$. By the construction, for every $k \in \mathbb{N}$, we have

\[(131) \quad \Theta_{h_{B,k}}(mK_{Y/X} + L) \geq \sum_i m(a_i - 1)^+[E_i]\]

Then the limit metric $h$ satisfies a similar inequality

\[(132) \quad \Theta_h(mK_{Y/X} + L) \geq \sum_i m(a_i - 1)^+[E_i]\]

as well as

\[(133) \quad (\Theta_h(mK_{Y/X} + L) - \sum_i m(a_i - 1)^+[E_i]) \wedge p^* \omega_X^{n-1} \geq \epsilon_0 \cdot p^* \omega_X^n.\]

We also have the following version of Theorem 5.4.

**Corollary 5.5.** Let $p : Y \to X$ be a holomorphic surjective and proper map, where $X$ and $Y$ are Kähler manifolds. Moreover, we assume that $p$ is locally projective. Let $L$ be a line bundle over $Y$ with a possible singular metric $h_L$ such that we have

\[(134) \quad \Theta_{h_L}(L) \geq -Cp^* \omega_X, \quad \Theta_{h_L}(L) \wedge p^* \omega_X^{n-1} \geq 0\]

where $\omega_X$ is a Hermitian metric on $X$, $n = \dim X$, and $C > 0$ is a positive real number. We assume that the space of fiberwise $L^m$ sections (with respect to $h_L$) of $p^*(mK_{Y/X} + L)$ is non zero. Then there exists a metric $h$ on the bundle $mK_{Y/X} + L$ such that we have

\[(135) \quad \Theta_h(mK_{Y/X} + L) \wedge p^* \omega_X^{n-1} \geq 0\]

in the sense of currents on $X$. \[\square\]
The proof of this statement follows in the same way we have obtained Theorem 1.5 as consequence of Theorem 1.3 so we provide no further explanations about it.

We note that as a direct consequence of the arguments we use in the proof of 5.4 we obtain the existence of a sequence of $m$-Bergman metrics on $mK_Y/X + L$ such that

$$
\Theta_{h_{\rho,k}}(K_Y/X + F) \wedge p^* \omega_X^{n-1} \geq -\delta_k p^* \omega_X^n
$$

where the sequence $(\delta_k)$ converges to zero.

Thanks to Corollary 5.6, we can prove the following variant of [CP17 Thm 3.4] (which is using the fundamental contributions of Viehweg, Tsuji... [V95, T10]).

**Corollary 5.6.** Let $p : Y \to X$ be a holomorphic surjective map between two Kähler manifolds. We assume that $p$ is locally projective. Let $\Sigma \subset X$ be the singular locus of $p$ and we assume that both $\Sigma$ and $p^{-1}(\Sigma)$ are normal crossing. Let $(L, h_L)$ be a hermitian line bundle over $X$ such that

$$
\Theta_{h_L}(L) \geq -Cp^* \omega_X, \quad \Theta_{h_L}(L) \wedge p^* \omega_X^{n-1} \geq 0
$$

where $\omega_X$ is a Kähler metric on $X$, $n = \dim X$, and $C > 0$ is a positive real number. If $I(h_L, Y_x) = \mathcal{O}_{Y_x}$ for a generic $x \in X$ and $E := p_*(K_Y/X + L)$ is non zero, then there exists a divisor $F$ in $Y$ satisfying $\dim_X p_*(F) \geq 2$, such that

$$
\int_Y c_1\left(K_Y/X + L + F - \delta_0 p^*(\det E)\right) \wedge p^* \omega_X^{n-1} \geq 0.
$$

for some positive $\delta_0$.

**Proof.** The proof is a linear combination of [CP17 Thm 3.4] with the arguments above. We sketch the proof here for the convenience of readers.

By hypothesis, $p^* \Sigma$ is normal crossing, and it can be written as

$$
p^* \Sigma = \sum W_i + \sum a_i V_i,
$$

where $\sum W_i + \sum V_i$ is snc and $a_i \geq 2$. Let $r$ be the rank of $E$. Let $Y^r$ be the fiberwise product of $p$, and let $p_i : Y^r \to Y$ be the $i$-directional projection. Let $\varphi : Y^{(r)} \to Y^r$ be a desingularisation, and $p^{(r)} : Y^{(r)} \to X$ be the natural morphism. We set $L^{(r)} := \varphi^* (\sum_i pr_i^* L)$. We have the canonical morphism $\det E \to \otimes i \det E$ over the locally free locus of $p$. This induces a section $s \in H^0(Y^{(r)}, K_{Y^{(r)}/Y} + L^{(r)} - (p^{(r)})^* \det E + E_1 + E_2)$, where $E_1$ and $E_2$ are effective divisors such that $\dim_Y p^{(r)}_*(E_2) \geq 2$ and

$$
E_1 \leq C \sum_{i,k}(\varphi \circ pr_i)^* V_k
$$

for some constant $C$. Let $m \in \mathbb{N}$ sufficiently large. We consider the line bundle

$$
F := mL^{(r)} + (K_{Y^{(r)}/X} + L^{(r)} - (p^{(r)})^* \det E + E_1 + E_2)
$$

with the metric $h_F := e^{-\log |s|^2} h_{L^{(r)}}$. Let $h_k$ be the $m$-relative Bergman kernel metric on $mK_{Y^{(r)}/Y} + F$ constructed in [135] with respect to $h_F$. They satisfy the inequality

$$
\Theta_{h_k}(mK_{Y^{(r)}/Y} + F) \wedge (p^{(r)})^* \omega_X^{n-1} \geq -\delta_k (p^{(r)})^* \omega_X^n
$$

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on $Y^{(r)}$.

Let $\pi : Y \to Y^{(r)}$ be the fiberwise diagonal embedding. Note that if $I(h_L) = \mathcal{O}_X$ and if $m$ large enough with respect to $\text{Div}(s)$, we have

$$I(h^+_F|_{Y^{(r)}}) = \mathcal{O}_Y^{(r)}$$

for a generic $x \in X$. As a consequence, $\pi^* h_k$ is not identically $+\infty$. Therefore $\pi^* \Theta_{h_k} (mK_{Y^{(r)}/X} + F)$ is well defined (thus quasi-psh) on $Y$. We have

$$\pi^* (\Theta_{h_k}) \wedge p^* \omega^{n-1}_X = \pi^* (\Theta_{h_k} \wedge (p^r)^* \omega^{n-1}_X) \geq 0 \quad \text{on} \quad Y,$$

as a consequence of $\text{[B39]}$. Note that $\pi^* (\Theta_{h_k})$ belongs to the class of

$$\pi^* (c_1 (mK_{Y^{(r)}/X} + F)) = c_1 ((mr + r)(K_{Y/X} + L) - \pi^* \det \mathcal{E} + E_3 + F),$$

where $E_3$ is supported $\sum_i V_i$ and $\text{codim}_X p_*(F) \geq 2$. Therefore we have

$$\int_Y c_1 ((mr + r)(K_{Y/X} + L) - \pi^* \det \mathcal{E} + E_3 + F) \wedge p^* \omega^{n-1}_X \geq -\delta_k p^* \omega^n_X.$$

Finally, thanks to $\text{[B39]}$, we know that

$$\sum_x \int_Y c_1 (V_i) \wedge p^* \omega^{n-1}_X \leq \int_Y c_1 (K_{Y/X} + L) \wedge p^* \omega^{n-1}_X$$

Therefore $\text{[B39]}$ is proved as $k \to \infty$. \hfill $\Box$

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