Effects in Gauge Theories and a Harmonic Function on $E_{10}$

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Abstract

In a previous paper we conjectured that the structure of various gauge theories as well as M-theory on $T^8$ is encoded in a unique function $\Xi$ on the coset $E_{10}(\mathbb{Z}) \backslash E_{10}(\mathbb{R})/K$ and that this function is harmonic with respect to the $E_{10}(\mathbb{R})$ invariant metric. In this paper we elaborate on the conjecture. We discuss various mass deformations of the D-instanton integral and their realizations in $\Xi$. We then present a conjectured prescription for extracting partition functions of the twisted little-string theory out of $\Xi$. We also study various effects of combinations of branes such as D0-branes near D4-branes with 2-form flux, D-instantons near Taub-NUT metrics, and more, in terms of harmonic functions on $E_d(\mathbb{R})/K$. We propose tests of the conjecture that are related to BPS states of global symmetries in gauge theories.
1 Introduction

In a previous paper [1], we conjectured that there exists a unique function Ξ that encodes the partition functions of a large class of gauge theories. The function Ξ was defined on the coset space \( \tilde{M} = E_{10}(\mathbb{Z})/E_{10}(\mathbb{R})/\mathcal{K} \) of the group \( E_{10}(\mathbb{R}) \) (the exponentiation of the Lie algebra \( E_{10} \)) by the maximal compact subgroup \( \mathcal{K} \) on the left and a discrete subgroup \( E_{10}(\mathbb{Z}) \) on the right. We proposed that various field-theoretic partition functions can be extracted from \( \Xi \) by appropriate Fourier transforms with respect to periodic variables of \( \tilde{M} \). Furthermore, we proposed that a partition function of M-theory on \( T^9 \) might be well-defined if we include generic transverse \( SO(2) \)-twists and that this partition function is also encoded in \( \Xi \). (See [1] for references on previous works related to \( E_{10} \).)

The purpose of the present paper is to refine the conjecture. The basic idea is the connection between Euclidean branes that wrap cycles in M-theory on \( T^d \) and positive roots...
of \( E_d \) \((d = 1 \ldots 8)\). In \( \tilde{\mathcal{M}} \), and of course also in the moduli space, \( E_{d(d)}(\mathbb{Z})\backslash E_{d(d)}(\mathbb{R})/\mathcal{K}_d \), of M-theory on \( T^d \), a positive root is related to a periodic variable \( \phi \). An instanton made of \( N \) wrapped Euclidean branes comes with a characteristic factor of \( e^{iN\phi} \). Thus, by extracting the terms in \( \Xi \) that behave as \( e^{iN\phi} \) we can extract information about the gauge (or other) theory associated with \( N \) branes at low-energy. This is, in general, a theory with 16 supersymmetries and noncompact moduli whose partition function is not well-defined. For example, the D-instanton action is an integral with 10\( N \) non-compact modes. To get a well-defined function we have to augment the partition function with mass-terms that break the supersymmetry and get rid of the moduli. For example, in the case of the D-instanton action, the mass-terms are quadratic in the variables. How do we interpret this augmentation in terms of variables of \( \tilde{\mathcal{M}} \)?

We will argue that the general procedure is as follows. We have to find another periodic variable, \( \psi \) in \( \tilde{\mathcal{M}} \), that will be mapped to the coefficient of the term in the deformed action (in the D-instanton example this would be proportional to the mass). However, as we will see, setting \( \psi \) to the desired value is not enough, by itself. We have to find a third periodic variable, \( \chi \), that will “connect” \( \psi \) to the action. The prescription will then be to extract out of \( \Xi \) the term that behaves as \( e^{iN\phi+i\chi} \) and study it as a function of \( \psi \). More generically, we have to identify pairs of variables, \( \psi_j \) and \( \chi_j \), \((j = 1, \ldots)\) and isolate out of \( \Xi \) the term that behaves as \( e^{iN\phi+i\sum \chi_j} \). We then have to study it as a function of the deformations \((\psi_1,\psi_2,\ldots)\). As will be reviewed below, the variables \( \psi_j \) and \( \chi_j \) correspond to positive roots, \( \beta_j \) and \( \gamma_j \), of \( E_{10} \). We will call \( \beta_j \) the “hook” and \( \gamma_j \) will be its corresponding “bait”.

The motivation for this procedure is that deformations of the theories with 16 supersymmetries that describe the dynamics of the branes \([4,5,6]\) can be realized by inserting other objects near the branes \([7,8]\). The most important case will be when the “bait” \( \gamma \) is orthogonal, in the root lattice of \( E_{10} \), to the root \( \alpha \) which corresponds to \( \phi \) and the “hook” \( \beta \) is at 60° to \( \alpha \) and 120° to \( \gamma \). (In other words, \( \langle \alpha, \gamma \rangle = 0 \) and \( \langle \alpha, \beta \rangle = -\langle \gamma, \beta \rangle = 1 \) where \( \langle \cdot, \cdot \rangle \) is the inner product in the weight lattice of \( E_{10} \).) This system corresponds to a deformation that preserves half the supersymmetry and has various manifestations. Among them are the compactifications with R-symmetry twists \([9]\), the elliptic models of \([10]\), a D-instanton inside a D3-brane \([11]\) with background NS-NS 2-form flux and more. Another interesting case is when \( \langle \alpha, \gamma \rangle = -2 \). This system corresponds to D0-branes near D8-branes \([12,13,14]\).
and some other cases as well.

In [1] we suggested the above procedure for extracting partition functions of gauge theories out of $\Xi$ but we did not specify exactly which \textit{hooks} and \textit{baits} need to be chosen for a particular theory. One of the goals of the present paper is to identify them more precisely. We use as a case-study the D-instanton action that is an integral over $10N \times N$ matrices. We will suggest a set of hooks and baits that correspond to mass deformations of the D-instanton action that break supersymmetry completely and lift all the flat directions. The result of the integration should therefore be a nontrivial function of the deformation parameters and we propose that it is encoded in a Fourier transform of $\Xi$.

In this paper we will adhere to the interpretation of [1] and consider only the deformed theories with well-defined partition functions (i.e. with all the flat directions lifted by mass deformations). However, our discussion is also relevant to the study of instanton effects in M-theory on $T^8$ and lower dimensional tori (see [15, 16, 17] and refs. therein). The $R^4$ term, which is also related to 16-fermion, $\lambda^{16}$, terms by supersymmetry [18, 19, 20], can be calculated from single-weight instantons (i.e. terms made from a single BPS-brane wrapped $N$ times). The instantons corresponding to Euclidean $\frac{1}{4}$-BPS states give rise to terms of the form $H^{4g-4}R^4$ [21] where $H$ is an RR field strength. They are likely to be related by supersymmetry to terms of the form $e^{iN\phi + ik\chi}\lambda^{24}$ where $\lambda^{24}$ is a shorthand for a 24-fermion term and $\phi$ and $\chi$ are two periodic phases in the moduli space. They correspond to roots $\alpha$ and $\gamma$ that satisfy $\langle \alpha, \gamma \rangle = 0$. The power of $H$ in $H^{4g-4}R^4$ is related to $k$ and $N$ via $g = Nk$. Similarly, instanton configurations that preserve $2^r$ supersymmetries are likely to contribute to terms of the form $e^{iN\phi + \sum_{j} k_j \chi_j \lambda^{24}}$.

The paper is organized as follows. In section (2) we review the relation between instantons and positive roots of $E_d$ and between harmonic functions on $E_{d(d)}/K$ and the action of the instantons. (See also the comprehensive reviews in [15, 16, 17].) In section (3) we discuss the relation between mass-like terms in the instanton actions and hooks and baits in $\Xi$. We discuss various U-dual systems that demonstrate this principle. We also conjecture that the generic action of a BPS instanton in M-theory on $T^8$ (preserving $\frac{1}{16}$ of the supersymmetry) is a harmonic function on $E_{8(8)}/SO(16)$. In section (4) we restrict to the case of the D-instanton integral and we proceed to study mass deformations that preserve $\frac{1}{4}$ or less of the
supersymmetry. They are realized by two or more hooks (and their corresponding baits). In section (5) we return to systems made up of a pair of BPS instantons that correspond to roots \( \alpha \) and \( \beta \) with inner product \( \langle \alpha, \beta \rangle = -2 \). Whereas the pairs corresponding to roots \( \alpha \) and \( \beta \) with \( \langle \alpha, \beta \rangle = 0 \) have been extensively studied the pairs with \( \langle \alpha, \beta \rangle = -2 \), that also preserve half the supersymmetry, have been studied less. We briefly discuss a particular case of a D(-1)-brane near a D7-brane and suggest that other U-dual systems might be interesting to study. In section (6) we study the effect of baits with \( \langle \alpha, \beta \rangle = -2 \) and its relation to supersymmetry breaking. In section (7) we suggest various models for extracting the partition function of higher dimensional theories. We briefly discuss an example where some of the BPS particle spectrum can be manifested. In section (8) we study the Laplacian on \( E_{10} \) in conjunction with the conjecture that \( \Xi \) is harmonic and given the proposed procedure for mass deformations.

2 Instantons, Branes, and Positive Roots

The moduli space of M-theory on \( T^d \) is given by \( E_{d(d)}(\mathbb{Z}) \backslash E_{d(d)}(\mathbb{R})/K \) where \( K \) is a maximal compact subgroup. An element \( g \in E_{d(d)}(\mathbb{R})/K \) can be decomposed as \( g = n \circ a \) where \( a \in (\mathbb{R}^+)^d \) is an element in a maximal abelian subgroup and \( n \in \mathcal{N} \) is an element in a nilpotent subgroup \( \mathcal{N} \). For example, for \( d = 8 \) we have \( K = SO(16) \) and if \( T^8 \) is of the form \( (S^1)^8 \) with no fluxes of the 3-form or dual 6-form and no VEVs to the 2+1D duals of the vectors then \( a \) can be taken as the vector \( (R_1, \ldots, R_8) \) of the 8 radii of \( S^1 \). The elements of \( n \) contain all the other moduli, i.e. fluxes, Dehn twists and duals of vectors. These become periodic phases after modding out by \( \mathcal{N} \cap E_{8(8)}(\mathbb{Z}) \).

2.1 Single instantons

Various terms in the low-energy effective action of M-theory on \( T^8 \) receive contributions from 2+1D space-time instantons. The simplest of these instantons can be described by taking a BPS particle of M-theory on \( T^7 \) with a Euclidean world-line along the remaining cycle of \( T^8 \). These instanton terms have a characteristic coefficient of the form, \( e^{-2\pi T + 2\pi i \phi} \) where \( T \) is the action of the instanton and \( \phi \) is the phase that couples to it. Restricting to \( T^8 \)'s of
the form \((S^1)^8\) we find 4 kinds of BPS instantons:

- KK states with Euclidean world lines with \(T = R_i R_j^{-1} \) (\(i \neq j\)).
- Wrapped membranes with \(T = \prod_{k=1}^{3} R_{ik}\).
- Wrapped fivebranes with \(T = \prod_{k=1}^{6} R_{ik}\).
- KK monopoles with \(T = R_{i8}^2 \prod_{k=1}^{7} R_{ik}\).

For each of those instantons, the phase \(\phi\) is one periodic variable in \(\mathcal{N}\) and hence corresponds to a positive root \(\alpha\) in the root lattice \(\Delta\) of \(E_8\). The tension \(T\) can then be calculated as follows. Identify,

\[ \lambda = (\log R_1, \ldots, \log R_8) \]

as a vector in the coroot space \(\mathcal{H}\) of \(E_8\). Let \(\langle \cdot, \cdot \rangle\) be the inner product (using the Cartan matrix). The tension is then given by,

\[ T_\alpha = e^{\langle \alpha, \lambda \rangle}. \]

It is interesting to note that the factor \(e^{-2\pi T_\alpha}\) can also be determined by looking for a harmonic function on the moduli space that behaves as \(e^{2\pi i \phi_\alpha}\). Up to prefactors, the function \(e^{-2\pi T_\alpha + 2\pi i \phi_\alpha}\) is harmonic!

### 2.2 Pairs of instantons

Let \(\alpha, \beta \in \Delta_+\) be positive roots (here \(\Delta_+\) is the set of positive roots). There are certain terms in the low-energy effective action that receive contributions from BPS instantons and behave as \(e^{-2\pi T_\alpha + 2\pi i \phi_\alpha}\) and \(e^{-2\pi T_\beta + 2\pi i \phi_\beta}\). We will now discuss terms that behave as

\[ e^{-2\pi T + 2\pi i k \phi_\alpha + 2\pi i m \phi_\beta}, \]

where \(T\) is a real function of the moduli. We will not be very specific about whether these are 16-fermion terms or something else. More important for us will be the behavior of \(T\). Given \(T_\alpha\) and \(T_\beta\), the behavior is determined by the product \(\langle \alpha, \beta \rangle\) of the roots in the weight lattice.

Before we proceed let us present two formulas for calculating \(\langle \alpha, \beta \rangle\). Since each positive root corresponds to an instanton, it is convenient to characterize \(\alpha\) by the vector of integers
\((n_1, \ldots, n_d)\) such that the action of the instanton on \((S^1)^d\) is given by,

\[ T_\alpha = \prod_{i=1}^{d} R_{i}^{n_i}. \]

If \(\beta\) is similarly characterized by \((m_1, \ldots, m_d)\) then,

\[ \langle \alpha, \beta \rangle = \sum_{i=1}^{d} n_i m_i - \frac{1}{9} \left( \sum_{i} n_i \right) \left( \sum_{j} m_j \right). \]

We will in general use \(d = 8\) but later on it will be necessary to extend this to \(d = 10\) that formally corresponds to M-theory.

Sometimes it will be more convenient to express the roots in the type-II language. Suppose we compactify type-IIA (or type-IIB) on \((S^1)^{d-1}\) with radii of lengths \(l_1, \ldots, l_{d-1}\) in string units and a string coupling constant \(\lambda\). A positive root \(\alpha\) can be characterized by the numbers \((p, s_1, \ldots, s_{d-1})\) such that,

\[ T_\alpha = \lambda^{-p} \prod_{i=1}^{d-1} l_i^{s_i}. \]

If we take another root \(\alpha'\) with,

\[ T_{\alpha'} = \lambda^{-p'} \prod_{i=1}^{d-1} l_i^{s'_i}, \]

then product is then given by,

\[ \langle \alpha, \alpha' \rangle = 2pp' - \frac{1}{2} p \sum s'_i - \frac{1}{2} p' \sum s_i + \sum s_is'_i. \]

Note that T-duality on the \(k^{th}\) direction acts as:

\[ s_k \rightarrow p - s_k, \]

leaving \(p\) and all the other \(s_i\)'s intact. S-duality of type-IIB, on the other hand, keeps all the \(s_i\)'s intact but changes:

\[ p \rightarrow \frac{1}{2} \sum s_i - p. \]

It is amusing to note that in higher dimensions there are exotic “branes” that correspond to roots \(|\alpha|^2 = 2\) (see [22]). Some of them are invariant under S-duality. For example, in addition to the D3-brane and KK-monopole that are invariant, we have the formal object with action \(\frac{1}{\lambda} l_1 l_2 l_3 l_4 l_5 l_6 l_7 l_8 l_9\).
Before we discuss the various combinations of two instantons let us mention one more mathematical detail. Suppose $\alpha, \beta \in \Delta_+$ are such that $\alpha + \beta \in \Delta_+$ is also a root. The three periodic variables $e^{2\pi i \phi_\alpha}, e^{2\pi i \phi_\beta}$ and $e^{2\pi i \phi_{\alpha + \beta}}$ do not parameterize $T^3$ but rather $e^{2\pi i \phi_{\alpha + \beta}}$ is a section of an $S^1$ bundle of first Chern class $c_1 = 1$ over the $T^2$ parameterized by $e^{2\pi i \phi_\alpha}$ and $e^{2\pi i \phi_\beta}$ (see [1] and refs therein).

Given an instanton contribution of the form

$$e^{-2\pi T + 2\pi i N \phi_\alpha + 2\pi i K \phi_\beta},$$  \hspace{1cm} (1)

we would like to ask how $T$ behaves as a function of $T_\alpha$ and $T_\beta$. To be rigorous, we have to be more specific about the other phases. In general, if $\alpha - \beta$ (or $\beta - \alpha$) is a positive root we have to set $\phi_{\alpha - \beta} = 0$ because $e^{2\pi i \phi_\alpha}$ is a section of a nontrivial line bundle, as we explained above. As for the other $\phi_\gamma$’s, we can assume that the expression is independent of them. There are various cases according to the value of $\langle \alpha, \beta \rangle$.

- If $\langle \alpha, \beta \rangle = 1$ then $T = \sqrt{N^2 T_\alpha^2 + K^2 T_\beta^2}$ and (1) is the contribution of a BPS instanton. For example, $\alpha$ might correspond to a Euclidean D1-brane wrapped on the 1st and 2nd directions and $\beta$ might correspond to a D1-brane wrapped on the 1st and 3rd directions. Then there exists a single BPS D1-brane wrapped on the 1st direction and the diagonal of the torus made from the 2nd and 3rd directions. For another example, $\alpha$ might correspond to a D(-1)-brane and $\beta$ might correspond to a D1-brane which combine to a D1-brane with electric flux (T-dual to a D0-brane and a D2-brane). It is interesting to note that the functional behavior $T = \sqrt{N^2 T_\alpha^2 + K^2 T_\beta^2}$ also comes out from the leading order behavior of a harmonic function on the moduli space that behaves as (1).

- If $\langle \alpha, \beta \rangle = 0$ then $T = NT_\alpha + KT_\beta$ and (1) is the contribution of a $\frac{1}{4}$BPS instanton. For example, $\alpha$ might correspond to a D(-1)-brane (with action $\frac{1}{2} \lambda$) and $\beta$ to a D3-brane (with action $\frac{1}{2} \lambda l_1 \cdots l_4$). For another example, $\alpha$ might correspond to an M5-brane (with action $R_1 \cdots R_6$) and $\beta$ to a KK-monopole (with action $R_1 \cdots R_7 R_8^2$) that engulfs the M5-brane. A third example is a D4-brane (with action $\frac{1}{2} \lambda l_1 \cdots l_5$) and an NS5-brane (with action $\frac{1}{2\lambda} l_1 \cdots l_4 l_6 l_7$) that intersect along a 4-dimensional hyper-plane. A fourth, U-dual example is furnished by $N$ M2-branes (with action $R_1 R_2 R_3$) intersecting $K$ M2-branes (with action $R_1 R_4 R_5$). Once again, the behavior $T = NT_\alpha + KT_\beta$ also
comes out from the leading order behavior of a harmonic function on the moduli space that behaves as (1). These combinations of instantons contribute to terms of the form $H^{4g-4}R^4$ in the low-energy description of M-theory on $T^d$, where $H$ is an appropriate field-strength of a low-energy field and $g = kN$ [21].

• If $\langle \alpha, \beta \rangle = -1$ then the instanton again preserves $\frac{1}{2}$ of the supersymmetry and has action $\sqrt{N^2T_\alpha^2 + K^2T_\beta^2}$. One example is furnished by a D(-1)-brane and a D5-brane. This is T-dual to the system of a D0-brane and a D6-brane studied in [23]. Supersymmetry is broken when the D0-brane and D6-brane are far from each other. However, type-IIA on $T^6$ actually has a BPS particle that has the same charge of a D0-brane and a D6-brane. To see this, recall that type-IIA on $T^6$ has an $SL(2, \mathbb{Z})$ duality group (a subgroup of the full $E_7(\mathbb{Z})$ U-duality) that acts on $\tau = \frac{V}{\lambda^2} + \chi$. Here $V$ is the volume of $T^6$ (in string units) and $\lambda$ is the 10D string coupling constant. The periodic modulus $\chi$ is the axion (dual to the NSNS 2-form). The S-duality $\tau \to -1/\tau$ transforms a D0-brane into its dual, the wrapped D6-brane. The transformation $\tau \to \tau + 1$ transforms a D6-brane into an object with the charges of a D0-brane and a D6-brane together. Another way of obtaining this “dyonic” object is by starting with a wrapped D6-brane and quantizing the collective coordinate corresponding to rotations of the 11th (M-theory) direction. There is no contradiction between these statements and the results of [23] because there the systems had more charges. The behavior $T = \sqrt{N^2T_\alpha^2 + K^2T_\beta^2}$ can also be deduced from a harmonic function, since the Laplacian is defined to be U-duality invariant. However, as a part of a harmonic function $T$ would also depend on $\phi_{\alpha+\beta}$ and the expression $\sqrt{N^2T_\alpha^2 + K^2T_\beta^2}$ is obtained only when we set $\phi_{\alpha+\beta}$ to zero.

• If $\langle \alpha, \beta \rangle = -2$ the instanton is again $\frac{1}{4}$BPS and $T = NT_\alpha + KT_\beta$. Unlike the previous BPS cases, it is not obvious that this relation does not seem to be directly related to a harmonic function. However, this case is more complicated for various reasons. First note that $|\alpha + \beta|^2 = 0$. This means that the Cartan matrix is either semidefinite or indefinite. Thus, we must have $d \geq 9$ which means that we are dealing with (the abstract) compactification to 1+1D or less. The characterization of the root lattice of $E_9$ and $E_{10}$ (see [24]) then implies that $\alpha + \beta$ is also a positive root. These systems will be discussed section (5).
2.3 Harmonic functions and BPS actions

As we have seen in the examples, the exponentials of the actions of $\frac{1}{2}$-BPS and $\frac{1}{4}$-BPS instantons are harmonic functions on the moduli space. The most generic statement of this sort would be that the exponential of the action of a generic BPS instanton on $T^8$ (that preserves only $\frac{1}{16}$ of the supersymmetry) is given by a harmonic function on $E_{8(8)}(Z)\backslash E_{8(8)}(R)/SO(16)$. I will not attempt to prove this statement here. However, let us outline a possible direction. The idea is to find 60 complex linear differential operators $\mathcal{L}_i$ that annihilate the action of an instanton and such that the Laplacian on the moduli space can be written as $\nabla = \sum \mathcal{L}_i^\dagger \mathcal{L}_i$.

For each set of instanton charges the set of $\mathcal{L}_i$’s could be different. These operators are generalizations of the statement that, for example, the instanton action of 4D Yang-Mills theory is holomorphic in $\tau = \frac{8\pi i}{g^2} + \frac{\theta}{2\pi i}$. Given the two supersymmetry generators that preserve the instanton charges, we can construct 1-forms of the form $A_i(\phi) d\phi^i$ (with $i = 1 \ldots 120$) on the moduli space that should remain constant in an instanton configuration (i.e. $A_i(\phi) \partial_\mu \phi^i = 0$).

The operators $\mathcal{L}_i$ can be constructed as a basis for the orthogonal space to these 1-forms.

If the conjecture that the generic BPS instanton of M-theory on $T^8$ is described by a harmonic action is true then a lot of information about BPS states in gauge theories can be extracted from it. For example, in section (7) we will construct instantons that are described by mass deformed $\mathcal{N} = 4$ SYM. We will argue that the mass of the adjoint scalar can be extracted from the instanton action by considering an extra charge. This resulting instanton can be embedded in M-theory on $T^8$ and, according to the conjecture, could be described by a harmonic function.

3 “Catching” deformations of gauge theories

Let $\alpha_0$ be the root corresponding to the D(-1)-brane. The “action” is $T_{\alpha_0} = \frac{1}{\lambda}$ and the phase is $\phi_{\alpha_0} = \chi$, the RR partner of the dilaton. The contribution of $N$ D(-1)-branes to low-energy processes is of the form,

$$Ze^{-2\pi NT_{\alpha_0} + 2\pi i N\chi}.$$
In appropriate asymptotic regions of the moduli space, the prefactor $Z$ can be calculated from the action,

$$I_0 = -\frac{1}{4} \operatorname{tr} \{ [X_I, X_J][X^I, X^J] \} + \Gamma_{\alpha\beta}^I \operatorname{tr} \{ \psi^\alpha [X_I, \psi^\beta] \}, \quad I, J = 1 \ldots 10, \quad \alpha, \beta = 1 \ldots 64.$$  \hfill (2)

Here $X$ are measured in Einstein units, $\Gamma_{\alpha\beta}^I$ are Dirac matrices of $SO(10)$, $X_I$ are $N \times N$ hermitian matrices in the vector representation of $SO(10)$ and $\psi^\alpha$ are $N \times N$ hermitian matrices with anti-commuting elements in the spinor representation of $SO(10)$. To get a specific quantity one must insert certain couplings to the background fields and integrate over the $X$’s and $\psi$’s (as in [18]).

Now consider the modified action,

$$I = I_0 + (M^2)_{IJ} X^I X^J + m_{\alpha\beta} \psi^\alpha \psi^\beta.$$  \hfill (3)

We would like to realize such deformations in terms of instantons in M-theory on $T^d$.

### 3.1 Hooks and baits

The general idea is to map a mass term $m$ to a phase $\phi_\beta$ for an appropriate positive root $\beta \in \Delta_+$ such that $m = c \phi_\beta$ in the limit $\phi_\beta \to 0$ and $c \to \infty$ is a function of the radii $R_1 \ldots R_d$. We will see that in order to execute the plan we need another root $\gamma \in \Delta_+$ and then we have to consider terms that behave as,

$$(\ldots) e^{2\pi i N \phi_\alpha + 2\pi i \phi_\beta}.$$  

For appropriately chosen $\beta, \gamma \in \Delta_+$, and in appropriate asymptotic regions of the moduli space, the prefactor $(\ldots)$ will be calculated from the massive D-instanton integral. We will call the root $\gamma$ the “bait” and the root $\beta$ will be called the “hook”. We will study several examples below. The examples will be:

- A D-brane inside a KK-monopole with a twist.
- The elliptic brane configurations of [10].
- A D-instanton near a KK-monopole with a B-field turned on.
- A D-instanton near a D3-brane with a B-field turned on.
In these examples it will turn out that we need the roots to satisfy

\[ \langle \alpha_0, \gamma \rangle = 0, \quad \langle \alpha_0, \beta \rangle = 1, \quad \langle \beta, \gamma \rangle = -1. \tag{4} \]

The low-energy description of each example is different. However the examples are U-dual to each other and the main point is that once \( \phi _\beta \neq 0 \), there is a “bound” configuration with action \( T_{\alpha_0} + T_\gamma \) whereas if the instantons are separate, their action is

\[ T_\gamma + T_{\alpha_0} \sqrt{1 + \left( \frac{\phi_\beta}{T_\beta} \right)^2} > T_\gamma + T_{\alpha_0}. \tag{5} \]

This forces the instanton with action \( T_{\alpha_0} \) to be at the center of the instanton with action \( T_\gamma \) and creates an effective mass term for the separation mode. (See also [25, 26]).

The formula \( T_1 \equiv T_{\alpha_0} + T_\gamma \) for the action of the bound state is actually only valid in a certain region of the moduli space. In another region, when the competing

\[ T_2 \equiv T_{\alpha_0} \sqrt{1 + \left( \frac{\phi_\beta}{T_\beta} \right)^2} + \frac{T_\gamma}{\sqrt{1 + \left( \frac{\phi_\beta}{T_\beta} \right)^2}} \tag{6} \]

becomes smaller, there is a “phase-transition” to the other action \( T_2 \). This phenomenon is well-known for \((p,q)\)-string networks on a slanted \( T^2 \) [27]. In this case we can take \( T_{\alpha_0} = \frac{1}{\lambda} l_2 l_1 \) and \( T_\gamma = R_3 R_1 \). Here \( R_2 R_3^{-1} = \text{Im}\tau \) and we take \( \phi_\beta = \text{Re}\tau \). When \( \lambda \) is small, \( T_1 \) is the correct formula for the action of the string network. When \( \lambda \) is large \( T_2 \) is the correct formula.

### 3.2 The twisted instanton actions

The first example is a modification of [9]. Take a Euclidean D0-brane that corresponds to a root \( \alpha_0 \) with action \( \lambda^{-1} l_1 \). Now embed it inside a KK-monopole with action \( \lambda^{-2} l_2^2 l_1 \cdots l_6 \).

This will be the bait-root \( \gamma \). The hook-root \( \beta \) will be the Dehn twist of the circle in the \( 7^{th} \) direction as we go around the \( 1^{st} \) direction. It corresponds to an action \( l_1 l_7^{-1} \). These roots satisfy (11). When the Dehn twist is small it acts as an effective mass term to 4 out of the 9 zero modes of the fields of the D0-brane just as in [9]. \( M^2 \) in (3) has eigenvalues proportional to:

\[ M^2 \sim 4 \{ \phi^2_\beta \}, 6 \{ 0 \}. \]
Let us discuss in what regions of moduli space the approximation of a massive 0D integral is valid. Let $\zeta$ be the vector $(\log R_1, \ldots, \log R_d)$. The D0-brane action is a good approximation when the string coupling constant $\lambda \to 0$. Let $X$ be a generic variable in the D-instanton integral, measured in string units, such that the D-instanton action is proportional to $\frac{1}{\lambda}$. If $X$ is measured in string units, the D0-brane action is, schematically, $\frac{1}{\lambda} \int (\dot{X}^2 + X^4)$. Corrections in $\alpha'$ behave as $\frac{1}{\lambda} X^k$ where $k \geq 6$. String loop corrections give even smaller contributions.

Let us first suppress the time dependence. The order of magnitude of the zero mode $X_0$ is $X_0 \sim \lambda^{1/4} l_1^{-1/4}$. The compactified time interval is $l_1$. The Fourier modes of $X$ along $l_1$ have a quadratic term of the form $\frac{n^2}{l_1} X_n^2$ which implies $X_n \sim \lambda^{1/2} l_1^{1/2}$. These fluctuations can be ignored if $\lambda^{1/2} l_1^{1/2} \ll \lambda^{1/4} l_1^{-1/4}$ so we require $l_1 \ll \lambda^{-1/3}$. For the KK-monopole, we want the fluctuations in $X$ to be small compared to $l_7$. Thus, $\lambda^{1/4} l_1^{-1/4} \ll l_7$.

To summarize we have,

$$\lambda^{1/4} l_1^{-1/4} \ll l_7, \quad \lambda \ll 1, \quad l_1 \ll \lambda^{-1/3}.$$  

We also need to require $\lambda^{1/4} l_1^{-1/4} \ll l_i$ for all $i \neq j$.

### 3.3 Elliptic brane configuration

Another U-dual example is the elliptic model of [10]. Take a Euclidean D0-brane with action $\frac{1}{\lambda} l_1$ (corresponding to the root $\alpha_0$) and an NS5-brane with action $\frac{1}{\lambda^2} l_2 \cdots l_7$ (corresponding to $\gamma$). Now add a Dehn twist that corresponds to the root $l_2^{-1} l_1$ (corresponding to $\beta$). According to the arguments of [10], at low-energies the action induced on the D0-brane, dimensionally reduced to 0D, is of the form (3). $M^2$ has eigenvalues proportional to:

$$M^2 \sim 4 \left\{ \frac{\phi_\beta^2}{l_2^2}, 6 \{0\} \right\}.$$  

Note that in this example (3) is satisfied as follows. If we separate the Euclidean D0-brane from the NS5-brane, with the Dehn twist $\phi_\beta$ turned on, the length of the world-line of the D0-brane will be $\sqrt{l_1^2 + l_2^2 \phi_\beta^2}$ which is bigger than $l_1$. 

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3.4 D-instanton inside a D3-brane

Let us recall the system of a D(-1)-brane near a D3-brane. Let us first take the D3-brane to be the primary root $\alpha_0$ and the “hook” root will be an NSNS 2-form flux.

$$T_{\alpha_0} = \frac{1}{\lambda} l_1 l_2 l_3 l_4, \quad T_\gamma = \frac{1}{\lambda}, \quad T_\beta = l_1 l_2.$$  \hspace{1cm} (7)

The phase is $\phi_\beta = B_{12} l_1 l_2$. Now take $l_1, l_2, l_3, l_4 \to 0$. This yields the construction of $[28, 29]$ of Yang-Mills theories on a noncommutative torus. The D3-brane becomes an instanton $[11]$ of the noncommutative theory. Because of the $(F - B)^2$ term in the action of the D3-brane, the action of the unbound system is bigger by:

$$\frac{1}{2\lambda} B_{12}^2 l_1 l_2 l_3 l_4 = T_{\alpha_0} \left( \frac{\phi_\beta}{T_\beta} \right)^2,$$

to lowest order in $B_{12}$.

On the other hand we can take the D(-1)-brane to be the primary root, $\alpha_0$, and the D3-brane to be the hook $\gamma$. In the limit of $[30]$ this system would become an instanton of $U(1)$ SYM with the noncommutativity set by $B_{12}$. In this case, we must use formula (3). The D(-1)-brane has an action of $\frac{1}{\lambda}$ outside the D3-brane. Inside the D3-brane it has an effective coupling constant of (see eqn (2.44) of $[31]$):

$$\lambda \left( \frac{\det(G + 2\pi \alpha B)}{\det G} \right)^{1/2}$$

where $G$ is the metric. Thus the action of the D(-1)-brane is smaller in the bound state, in accord with (3).

At weak coupling and for large $B$ field, the system is described by an integral on the moduli space of noncommutative instantons. For small $B$ fields, the system is described by a matrix-model with fundamental hyper-multiplets and a Fayet-Illiopoulos term proportional to the $B$ field (see $[30]$ and references therein).

3.5 A graviton trapped in a string

Let us describe yet another example of the same kind. This example is likely to contribute to 24-fermion terms in the low-energy effective action of type-II string theory on $T^2$ in
8-dimensions. The actions are as follows:

\[ T_{\alpha_0} = l_1 l_3^{-1}, \quad T_\gamma = \frac{1}{\lambda} l_1, \quad T_\beta = l_2 l_3^{-1}. \]  

(8)

This system describes the bound state of a string and graviton. The string is stretched on one of the cycles of a slanted \( T^2 \). If \( \tau = \tau_1 + i \tau_2 \) is the complex structure of the \( T^2 \) then \( \phi_\beta = \tau_1 \). The bound state describes a string carrying momentum in one direction. The bound state has an energy gap because the graviton will have energy proportional to \( \tau_2^{-1} \) outside the string, but only \( |\tau|^{-1} \) inside the string.

### 3.6 An instanton near a KK monopole

Applying T-duality to the primary root and the hook of (3.2) we get a D-instanton inside a KK-monopole. The primary root \( \alpha_0 \) is a D-instanton with action \( \frac{1}{\lambda} \). The bait, \( \gamma \) is a KK-monopole with action \( \frac{1}{\lambda} l_1 \cdots l_6 l_7^2 \). The hook \( \beta \) corresponds to a B-field along the 7\textsuperscript{th} and 1\textsuperscript{st} directions, i.e. a string with action \( l_1 l_7 \). Although this system differs from that studied in (3.2) by the sign of \( \langle \alpha_0, \beta \rangle \) and \( \langle \gamma, \beta \rangle \), it is likely to have similar features. The B-field modifies the D-instanton action and forces the instanton to sit at the origin. Let \( y \) be a coordinate along the 7\textsuperscript{th} circle, and let us take the 8, 9, 10 directions to be noncompact with coordinates \( x_8, x_9, x_{10} \) and choose spherical coordinates with \( r \) being the distance to the origin and \( \Omega \) being a coordinate on the \( S^2 \). The metric of a KK-monopole is the Taub-NUT metric:

\[
ds^2 = l_7^2 U (dy - A_i dx^i)^2 + U^{-1} (d\bar{x})^2,
\]

where,

\[ U = \left( 1 + \frac{l_7}{2|x|} \right)^{-1}, \]

and \( A_i \) is the gauge field of a monopole centered at the origin.

The 2-form \( B_{\mu \nu} dx^\mu \wedge dx^\nu \) has to be proportional to \( dy \wedge dx_1 \). However, in the presence of the Taub-NUT metric, \( dy \) is not globally defined over the sphere \( S^2 \). Instead, \( dy - A_i dx^i \) is the well-defined angular-form. However, \( B = (dy - A_i dx^i) \wedge dx_1 \) has a non-vanishing field strength \( H = dB = F \wedge dx_1 \), where \( F \) is the 2-form field-strength of a monopole on \( S^2 \). The presence of the nonzero \( H \) will modify the equations of motion both for the metric as well
as the dilaton. The dilaton will now have a maximum as \( r \to 0 \). This will make the action \( \frac{1}{\lambda} \) smaller at the origin. The difference in the instanton action \( \frac{1}{\lambda} \) at infinity and in the core of the instanton should be given exactly by eqn (6). The second derivative of the function \( \frac{1}{\lambda} \) at the origin will create an effective quadratic term for the the fields \( X^I \) in (2). Although the field-strength \( H = dB \) could be large at the origin, we will assume that it has no effect on the D(-1)-brane action. The exact solution will be explored further in [31].

Let us also note in passing that we can similarly study an M2-brane near a KK-monopole in M-theory. This time the metric along the directions of the M2-brane will probably be smaller at the origin which will cause the M2-brane to be attracted to the center.

4 Deformations with two-hooks and more

So far we have considered examples that deform the integral in (2) by a mass term (3) that preserves half the supersymmetry and gives mass to 4 out of the 10 \( X^I \)'s. Our final goal is to give mass to all the fields and also break supersymmetry completely. As a first step we will add a mass term that preserves only \( \frac{1}{4} \) of the SUSY. We will find it easy to use a model similar to the elliptic model of [10]. (For somewhat related constructions see [32, 33]).

4.1 Two NS5-branes: variant I

We start with a D1-brane with Euclidean world-sheet stretching along directions 1, 2. we add an NS5-brane along directions 2, 3, 4, 5, 6, 7 and add a Dehn twist such that as we go around the 1\(^{st}\) direction we translate along the 3\(^{rd}\). We add a second NS5-brane along directions 1, 3, 4, 5, 6, 8. We also add a Dehn twist such that as we go around the 2\(^{nd}\) direction we translate along the 4\(^{th}\) direction. Note that both NS5-branes include the directions 3, 4. Each NS5-brane creates a mass in the directions orthogonal to it.

The configuration preserves \( \frac{1}{4} \) supersymmetry. Let us calculate the intersection matrix. We define the roots corresponding to the branes as follows:

\[
T_{\alpha_0} = \frac{1}{\lambda} l_1 l_2, \\
T_{\beta_1} = l_1 l_3^{-1}, 
\]
They give the corresponding products:

|   | $\gamma_1$ | $\beta_1$ | $\gamma_2$ | $\beta_2$ |
|---|---|---|---|---|
| $\gamma_1$ | 2 | -1 | 0 | 0 |
| $\beta_1$ | -1 | 2 | 0 | 0 |
| $\gamma_2$ | 0 | 0 | 2 | -1 |
| $\beta_2$ | 0 | 0 | -1 | 2 |

The eigenvalues of the mass term in (3) are proportional to (according to the rules for “brane-boxes” [10, 33]):

$$M^2 \sim 2 \left\{ \left( \frac{\phi_{\beta_1}}{l_3} \right)^2 \right\}, 2 \left\{ \left( \frac{\phi_{\beta_2}}{l_4} \right)^2 \right\}, 2 \left\{ \left( \frac{\phi_{\gamma_1}}{l_3} \right)^2 + \left( \frac{\phi_{\gamma_2}}{l_4} \right)^2 \right\}, 4 \{0\}. \quad (10)$$

### 4.2 An NS5-brane and a KK-monopole: variant-I

Analyzing the constructions with two NS5-branes involves some guesswork because the dynamics of strings joining the open ends of the D-branes (and necessarily passing through NS5-branes) is strongly coupled.

Instead, we will present a U-dual construction which, we believe, is simpler to analyze. We start with the elliptic brane configuration of a D2-brane ending on an NS5-brane. This system realizes the dimensional reduction of a system with $\mathcal{N} = 2$ in 3+1D and a massive adjoint hypermultiplet. We can now immerse that construction inside a KK-monopole that will realize an R-symmetry twist as we go along the other direction of the D2-brane. The corresponding actions are:

$$T_{\alpha_0} = \frac{1}{\lambda} l_1 l_2,$$

$$T_{\beta_1} = l_1 l_3^{-1},$$

$$T_{\gamma_0} = \frac{1}{\lambda^2} l_2 l_3 l_4 l_5 l_6 l_7,$$

$$T_{\beta_2} = l_2 l_6^{-1},$$

$$T_{\gamma_2} = \frac{1}{\lambda^2} l_1 l_2 l_3 l_4 l_5 l_6^2 l_8,$$
Note that the NS5-brane is wrapped on a 2-manifold inside the 4D space transverse to the KK-monopole (the Taub-NUT space). This 2-manifold includes the Taub-NUT direction and is smooth. Because there are no new singularities other than those already present in the elliptic models of [10], we can argue that at low energies the construction gives a term in (3) with $M^2$ having the same form as (10).

$$M^2 \sim 2 \left\{ \left( \frac{\phi_{\beta_1}}{l_3} \right)^2, 2 \left\{ \phi_{\beta_2}^2 \right\}, 2 \left\{ \left( \frac{\phi_{\beta_1}}{l_1} \right)^2 + \phi_{\beta_2}^2 \right\}, 4 \{0\} \right\}.$$ 

After T-duality to obtain the previous example, it is easily seen that this agrees with the rules for "brane-boxes" [13]. The intersection matrix is as before:

|   | $\gamma_1$ | $\beta_1$ | $\gamma_2$ | $\beta_2$ |
|---|-------------|-------------|-------------|-------------|
| $\gamma_1$ | 2          | -1          | 0           | 0           |
| $\beta_1$   | -1         | 2           | 0           | 0           |
| $\gamma_2$  | 0          | 0           | 2           | -1          |
| $\beta_2$   | 0          | 0           | -1          | 2           |

### 4.3 Two NS5-branes: variant II

We start with a D1-brane with Euclidean world-sheet stretching along directions 1, 2. We add an NS5-brane along directions 2, 3, 4, 5, 6, 7 and add a Dehn twist such that as we go around the 1$^{st}$ direction we translate space along the 3$^{rd}$ direction. We add a second NS5-brane along directions 1, 3, 4, 5, 6, 8. We also add a Dehn twist such that as we go around the 2$^{nd}$ direction we translate along the same 3$^{rd}$ direction. Note that both NS5-branes include the direction 3 as they should. Each NS5-brane creates a mass in the directions orthogonal to it. We define the roots corresponding to the branes as follows:

$$T_{\alpha_0} = \frac{1}{\lambda} l_1 l_2,$$

$$T_{\beta_1} = l_1 l_3^{-1},$$

$$T_{\gamma_1} = \frac{1}{\lambda^2} l_2 l_3 l_4 l_5 l_6 l_7,$$

$$T_{\beta_2} = l_2 l_3^{-1},$$

$$T_{\gamma_2} = \frac{1}{\lambda^2} l_1 l_3 l_4 l_5 l_6 l_8.$$ 

They give the corresponding products:
This time the masses in (3) are:

\[
M^2 \sim 2 \left\{ \left( \frac{\phi_{\beta_1}}{l_3} \right)^2 \right\}, \; 2 \left\{ \left( \frac{\phi_{\beta_2}}{l_4} \right)^2 \right\}, \; \left\{ \left( \frac{\phi_{\gamma_1}}{l_3} + \frac{\phi_{\gamma_2}}{l_4} \right)^2 \right\}, \; \left\{ \left( \frac{\phi_{\beta_1}}{l_3} - \frac{\phi_{\beta_2}}{l_4} \right)^2 \right\}, \; 4 \{0\}.
\]  (11)

We can replace \( T_{\alpha_0} \) with \( l_1 l_2 l_4 l_5 l_6 \) to obtain a 3D theory and therefore the instanton integral can be the dimensional reduction of a supersymmetric 3D theory. It cannot be the dimensional reduction of a supersymmetric 4D theory because one of the masses is not doubled.

It is also not completely clear to me if new Yukawa couplings are generated or not (unlike the case of variant-I where the KK-monopole derivation was safe, at least for \( l_i \gg 1 \) and \( \lambda \ll 1 \)).

### 4.4 Two KK-monopoles

The above construction is U-dual to the following:

\[
\begin{align*}
T_{\alpha_0} &= \frac{1}{\lambda} l_1 l_2 l_3 l_4, \\
T_{\beta_1} &= l_1 l_7^{-1}, \\
T_{\gamma_1} &= \frac{1}{\lambda^2} l_1 l_2 l_3 l_4 l_5 l_6 l_7^2, \\
T_{\beta_2} &= l_1 l_6^{-1}, \\
T_{\gamma_2} &= \frac{1}{\lambda^2} l_1 l_2 l_3 l_4 l_7 l_6^2.
\end{align*}
\]

This system seems to describe the dimensional reduction of \( \mathcal{N} = 4 \) SYM compactified on \( S^1 \) (the 1st direction) with an \( SO(6) \) R-symmetry twist along that direction. Each pair of hook and bait \((\beta_j, \gamma_j)\) \((j = 1, 2)\) on its own creates a twist with \( SU(4) \sim SO(6) \) eigenvalues:

\[
(e^{i\phi_{\beta_j}}, e^{-i\phi_{\beta_j}}, 0, 0).
\]
However, the masses in (11) cannot be obtained from the limit of a small twist in $SU(4)$. If we compactify $\mathcal{N} = 4$ SYM on $S^1$ with an R-symmetry twist with $SU(4)$ eigenvalues:

$$(e^{i\alpha_1}, e^{i\alpha_2}, e^{i\alpha_3}, e^{-i(\alpha_1+\alpha_2+\alpha_3)}),$$

Then the bare 3D masses of the scalars are going to be proportional to:

$$M^2 \sim 2 \left\{ |\alpha_1 + \alpha_2|^2 \right\}, 2 \left\{ |\alpha_1 + \alpha_3|^2 \right\}, 2 \left\{ |\alpha_2 + \alpha_3|^2 \right\}.$$ 

This does not agree with (11). There is no immediate contradiction, though, because the configuration of two KK-monopoles cannot be realized geometrically. Once we compactify the transverse space to one monopole, we cannot find a solution any more.

### 4.5 A deformation with three hooks

Let us consider a combination of a D2-brane and 3 NS5-branes as follows:

$$T_{\alpha_0} = \frac{1}{\lambda} l_1 l_2 l_3,$$

$$T_{\beta_1} = l_4^{-1},$$

$$T_{\gamma_1} = \frac{1}{\lambda^2} l_2 l_4 l_5 l_6 l_7,$$

$$T_{\beta_2} = l_5^{-1},$$

$$T_{\gamma_2} = \frac{1}{\lambda^2} l_1 l_4 l_5 l_6 l_8,$$

$$T_{\beta_3} = l_6^{-1},$$

$$T_{\gamma_3} = \frac{1}{\lambda^2} l_1 l_2 l_4 l_5 l_6 l_9.$$ 

This is chosen so that the configuration preserves $\frac{1}{16}$ of the supersymmetry. The corresponding deformation in (8) has $M^2$ with eigenvalues proportional to:

$$M^2 \sim \left( \frac{\phi_{\beta_1}}{l_4} \right)^2, \left( \frac{\phi_{\beta_2}}{l_5} \right)^2, \left( \frac{\phi_{\beta_3}}{l_6} \right)^2,$$

$$\left( \frac{\phi_{\beta_1}}{l_4} \right)^2 + \left( \frac{\phi_{\beta_2}}{l_5} \right)^2 + \left( \frac{\phi_{\beta_3}}{l_6} \right)^2,$$

$$\left( \frac{\phi_{\beta_1}}{l_4} \right)^2 + \left( \frac{\phi_{\beta_2}}{l_5} \right)^2 + \left( \frac{\phi_{\beta_3}}{l_6} \right)^2, 3 \{0\}. \quad (12)$$
5 Instanton pairs with $\langle \alpha, \beta \rangle = -2$

Our goal is to add hooks and baits such that the induced D-instanton integral will have no supersymmetry at all and also will have no flat directions. As long as the primary root $\alpha_0$ and all the hooks and baits can be realized as particles in M-theory on $T^7$ with a Euclidean world-line around an extra $S^1$, it is obvious that some supersymmetry will be preserved. This is because for any configuration of charges in M-theory on $T^7$, one can find the maximal eigenvalue of the central charge and get a corresponding BPS state. It is also likely that inside M-theory on $T^8$ we cannot completely break supersymmetry with a combination of instantons corresponding to positive roots.

In order to break supersymmetry completely, it is very likely that we need to go beyond M-theory on $T^8$ and therefore go beyond the finite group $E_8$. One of the new features that the infinite groups $E_9$ and $E_{10}$ have is pairs of roots with $\langle \alpha, \beta \rangle = -2$. We will see in section (2) that adding two baits $\gamma_1$ and $\gamma_2$ that satisfy $\langle \gamma_1, \gamma_2 \rangle = -2$ has, on the face of it, the potential to break supersymmetry, in certain cases.

In this section we will study in more detail the cases in which the main root $\alpha_0$ and the bait $\beta$ satisfy

$$\langle \alpha_0, \beta \rangle = -2, \quad \alpha^2_0 = \beta^2 = 2.$$

Various U-dual examples are:

- A D(-1)-brane (with action $\frac{1}{\lambda}$) near a D7-brane (with action $\frac{1}{\lambda} l_1 \cdots l_8$) in type-IIB on $T^8$.
- A D0-brane near a D8-brane in type-IA. This system was studied in [12, 13, 14].
- A KK-monopole with respect to the 8th direction and with action $R_1 \cdots R_7 R_8^2$ and a KK-monopole with respect to the 9th direction with action $R_1 \cdots R_7 R_9^2$ in M-theory on $T^9$.
- A KK-monopole with respect to the 9th direction (with action $\frac{1}{\lambda} l_1 \cdots l_6 l_9^2$) intersecting a D7-brane (with action $\frac{1}{\lambda} l_1 \cdots l_8$) in type-IIB on $T^9$ (formally).
- An NS5-brane (with action $\frac{1}{\lambda^2} l_1 \cdots l_6$) submerged inside a D8-brane (with action $\frac{1}{\lambda} l_1 \cdots l_9$) in type-IIA on $T^9$ (formally).
• A D7-brane (with action $\frac{1}{\chi l_1 \cdots l_8}$) intersecting a D3-brane (with action $\frac{1}{\chi l_1 l_2 l_9 l_{10}}$) along a 2-dimensional plane.

5.1 D(-1)-brane near a D7-brane

In this case the $\alpha_0$ root corresponds to a D-instanton with action $\frac{1}{\lambda}$ and the bait is $\beta$, a D7-brane with action $\frac{1}{\chi l_1 \cdots l_8}$. On their own, the $N$ D-instantons will be described by the action $I_0$ of (2). We wish to know the effect of the D7-brane. The D7-brane changes the value of the complex dilaton in the space around it such that the D(-1)-brane action, $e^{-\frac{i}{\lambda} + i\chi}$, becomes $e^{\log(\frac{z}{\Lambda})} = z/\Lambda$. Here, $\Lambda$ is a cutoff which in the usual case of F-theory signifies the presence of another $(p, q)$ 7-brane at that distance. $|z|$ is the distance to the origin where the D7-brane is located. On top of that, there are fermionic variables that come from quantizing the open strings with one end on the D(-1)-brane and the other on the D7-brane. These variables have mass $|z|$. With a single D(-1)-brane, they produce a factor of $z$ when integrated (see [34]). Together they produce a prefactor of $|z|^2$. This is just as well, since the phase of the coordinate $z$ is arbitrary and depends on our choice of coordinates.

In the case of $N$ D-instantons, the natural generalization seems to be $|\det(X_9 + iX_{10})|^2$ where $X_9$ and $X_{10}$ are $N \times N$ matrices. The argument for this is that we get a factor of $\det(X_9 + iX_{10})$ from integrating the fermionic variables. To cancel the phase, we expect to get the complex conjugate from the heuristic $\prod e^{-\frac{i}{\lambda} + i\chi}$ where $\lambda_i$ and $\chi_i$ are the dilaton values at the positions of the D-instantons – which makes sense only when they are far apart. We propose that the modification to the action due to the D7-brane is a term

$$I_d = 2 \log |\det(X_9 + iX_{10})|.$$ 

5.2 NS5-brane near a D8-brane

Another system that falls into the category of the present discussion is an NS5-brane near a D8-brane. We realize this system by considering an NS5-brane in type-IA on $S^1/Z_2$ [35]. We can T-dualize along the segment to obtain type-I on $S^1$ as in [35] and the NS5-brane would become a KK-monopole with respect to $S^1$. The position of the D8-branes in the original type-IA system is related to the $SO(32)$ Wilson line along $S^1$ and the position of
the NS5-brane is related to a 2-form flux in the Taub-NUT solution corresponding to the
KK-monopole. Now take $N$ such KK-monopoles. There are low-energy fields which
classically come from the $A_{N-1}$ singularity at the core of the solution. After S-duality the
question becomes what lives on an $A_{N-1}$ singularity in the heterotic string. This question
was recently studied in \cite{36, 37}. We will not discuss it further here.

In \cite{1}, we suggested that a partition function for M-theory on a space $X$ built as an $\mathbb{R}^2$
fibration over $T^9$ might exist. We proposed that the $SO(2)$ twists in the $MR2$, along the $i^{th}$
direction of $T^9$ might be captured by the following hook and bait:

$$T_\beta = R_i R_{i_0}^{-1}, \quad T_\gamma = R_1 R_2 R_3 R_4 R_5 R_6 R_7 R_8 R_9 R_{10}^3.$$ 

Formally, $\gamma$ corresponds to a D8-brane if we pick the 10th direction for the M-theory/type-IIA
reduction. If we now insert an M5-brane with:

$$T_{\alpha_0} = R_1 R_2 R_3 R_4 R_5 R_6,$$

we conjecture that for an appropriate limit of all the $R_j$'s, the phase $\phi_\beta$ will be related to a
nonsupersymmetric twist in the partition function of the M5-brane. If we formally take the
transverse directions of the M5-brane to be $7 \ldots 11$ then the twist will be in the $SO(2)$ that
rotates directions $10, 11$ (because it will have to preserve directions $7, 8, 9$. The intersection
matrix of the relevant roots is:

|     | $\alpha_0$ | $\beta$ | $\gamma$ |
|-----|------|------|------|
| $\alpha_0$ | 2    | 1    | -2   |
| $\beta$    | 1    | 2    | -2   |
| $\gamma$   | -2   | -2   | 2    |

If the conjecture is true, it seems that the $-2$ products of roots play a crucial role in
supersymmetry breaking.

### 6 Pairs of baits with $\langle \alpha, \beta \rangle = -2$

We will now study the effect of having two baits $\gamma_1$ and $\gamma_2$ with $\langle \gamma_1, \gamma_2 \rangle = -2$. The first
elementary preserves $1/2$ of the supersymmetry but is useful for getting rid of many flat directions.
The rest of the examples in this section seem to break supersymmetry completely.

\footnote{I am grateful to S. Sethi for discussions on this system.}
6.1 A D-instanton near two KK-monopoles

We have conjectured in subsection (3.6) that $N$ D-instantons near a KK-monopole with an appropriate NSNS B-field flux at infinity are pinned to the center of the Taub-NUT space and are described by a mass deformation (3) that breaks $\frac{1}{2}$ supersymmetry. What happens if we insert another KK-monopole? Let us take the following actions (written formally for $T^{10}$ to indicate the directions):

\[
\begin{align*}
T_{\alpha_0} &= \frac{1}{\lambda}, \\
T_{\beta_1} &= l_1 l_7, \\
T_{\gamma_1} &= \frac{1}{\lambda^2} l_1 l_2 l_3 l_4 l_5 l_6 l_7, \\
T_{\beta_2} &= l_2 l_6, \\
T_{\gamma_2} &= \frac{1}{\lambda^2} l_1 l_2 l_4 l_7 l_8 l_9 l_{10}.
\end{align*}
\]

The configuration of the two KK-monopoles is certainly a solution and can even be realized as a decompactification limit of type-II on $K_3 \times K_3$. We conjecture that together this configuration gives mass to $X_3 \ldots X_{10}$ and leaves 4 supersymmetries in the D-instanton action. The action is a dimensional reduction of a 2D theory with $N = (4,0)$. The 2D theory is just a $U(N)$ gauge theory with two mass terms.

| $\gamma_1$ | $\beta_1$ | $\gamma_2$ | $\beta_2$ |
|------------|------------|------------|------------|
| $\gamma_1$ | 2          | -1         | -2         | 0          |
| $\beta_1$  | -1         | 2          | 0          | 0          |
| $\gamma_2$ | -2         | 0          | 2          | -1         |
| $\beta_2$  | 0          | 0          | -1         | 2          |

The intersection matrix is:

The corresponding mass term in (3) has eigenvalues proportional to:

\[
M^2 \sim 4 \left\{ \phi_{\beta_1}^2 \right\}, \ 4 \left\{ \phi_{\beta_2}^2 \right\}, \ 2 \left\{ 0 \right\}. \quad (13)
\]

I do not know if there is a solution of $\alpha_0, \beta_1, \beta_2, \gamma_1, \gamma_2$ inside $E_{10}$ or one actually has to consider $E_{11}$ to realize the roots as branes.

We can combine this construction with that of (4.1) to obtain a system that removes all flat directions:

\[
M^2 \sim 2 \left\{ \left( \frac{\phi_{\beta_1}}{l_3} \right)^2 \right\}, \ 2 \left\{ \left( \frac{\phi_{\beta_2}}{l_4} \right)^2 \right\}, \ 2 \left\{ \left( \frac{\phi_{\beta_1}}{l_3} \right)^2 + \left( \frac{\phi_{\beta_2}}{l_4} \right)^2 \right\}, \ 4 \left\{ \phi_{\beta_3}^2 \right\}. \quad (14)
\]
and we need to find roots with intersection matrix:

|   | $\alpha_0$ | $\gamma_1$ | $\beta_1$ | $\gamma_2$ | $\beta_2$ | $\gamma_3$ | $\beta_3$ |
|---|---|---|---|---|---|---|---|
| $\alpha_0$ | 2 | 0 | -1 | 0 | -1 | 0 | -1 |
| $\gamma_1$ | 0 | 2 | -1 | 0 | 0 | -2 | 0 |
| $\beta_1$ | -1 | -1 | 2 | 0 | 0 | 0 | 0 |
| $\gamma_2$ | 0 | 0 | 0 | 2 | -1 | -2 | 0 |
| $\beta_2$ | -1 | 0 | 0 | -1 | 2 | 0 | 0 |
| $\gamma_3$ | 0 | -2 | 0 | -2 | 0 | 2 | -1 |
| $\beta_3$ | -1 | 0 | 0 | 0 | 0 | -1 | 2 |

6.2 Supersymmetry breaking twists

Instead of getting rid of the noncompact moduli by mass terms, as we did above, one can also get rid of some of the noncompact moduli by R-symmetry twists. For example, we can ask what is the partition function of the the D0-brane with generic

$$U(1)_L \times U(1)_R \subset SU(2)_L \times SU(2)_R \sim SO(4) \subset SO(4) \times SO(6) \subset SO(10)$$

R-symmetry twists along $S^1$ (as in [38]). We conjecture that to realize it we have to take:

$$T_{\alpha_0} = \frac{1}{\lambda} l_1,$$
$$T_{\beta} = l_1 l_7^{-1},$$
$$T_{\gamma} = \frac{1}{\lambda^2} l_1 l_2 l_3 l_4 l_5 l_6 l_7^2,$$
$$T_{\beta'} = l_1 l_8^{-1},$$
$$T_{\gamma'} = \frac{1}{\lambda^2} l_1 l_2 l_3 l_4 l_5 l_6 l_8^2,$$

Here, $\beta$ and $\gamma$ correspond to the $U(1)_L$ twist and $\beta'$ and $\gamma'$ correspond to the $U(1)_R$ twist.

The intersection matrix is:

|   | $\alpha_0$ | $\gamma$ | $\beta$ | $\gamma'$ | $\beta'$ |
|---|---|---|---|---|---|
| $\alpha_0$ | 2 | 0 | 1 | 0 | 1 |
| $\gamma$ | 0 | 2 | -1 | -2 | 1 |
| $\beta$ | 1 | -1 | 2 | 1 | 1 |
| $\gamma'$ | 0 | -2 | 1 | 2 | -1 |
| $\beta'$ | 1 | 1 | 1 | -1 | 2 |
Formally, once the 7th direction is compact there is no Taub-NUT solution with respect to the 8th circle. As we mentioned before, we treat the Taub-NUT solution only as a motivation for an abstract procedure inside $E_9$ or $E_{10}$. However, we have to add a caveat. We have seen in subsection 4.4 that treating to intersecting KK-monopoles formally does not always give the expected intuitive results. Although I do not have a convincing argument that $\gamma'$ indeed does the trick, let us describe another system with two baits with $\langle \gamma_1, \gamma_2 \rangle = -2$ and the following property. If the $U(1)_L$ twists $\phi_{\beta_1} = \cdots = \phi_{\beta_6} = 0$ or the $U(1)_R$ twists $\phi_{\beta'_1} = \cdots \phi_{\beta'_6} = 0$, then $\frac{1}{8}$ of the supersymmetry is preserved. However, if both twists are turned on then supersymmetry is broken.

### 6.3 Interfering hooks: D-instanton and two D3-branes

This system is made of a D-instanton in the presence of two transverse Euclidean D3-branes:

\[
T_{\alpha_0} = \frac{1}{\lambda}, \\
T_{\gamma} = \frac{1}{\lambda} l_1 l_2 l_3 l_4, \\
T_{\gamma'} = \frac{1}{\lambda} l_5 l_6 l_7 l_8,
\]

We will also exhibit two hooks as NSNS B-field fluxes:

\[
T_\beta = l_1 l_2, \\
T_{\beta'} = l_5 l_6.
\]

The intersection matrix is:

\[
\begin{array}{|c|cccc|}
\hline
& \alpha_0 & \gamma & \beta & \gamma' & \beta' \\
\hline
\alpha_0 & 2 & 0 & -1 & 0 & -1 \\
\gamma & 0 & 2 & 1 & -2 & -1 \\
\beta & -1 & 1 & 2 & -1 & 0 \\
\gamma' & 0 & -2 & -1 & 2 & 1 \\
\beta' & -1 & -1 & 0 & 1 & 2 \\
\hline
\end{array}
\]

Note that $\langle \alpha_0, \beta \rangle = -1$ and not +1 as before. Nevertheless, we can see the “interference” of the two hooks as follows. In the presence of the fluxes the D-instanton becomes a noncommutative Yang-Mills instanton inside the D3-brane. However, it can only be a large
noncommutative instanton in one of the D3-branes but not the other. In the presence of fluxes on both D3-branes, supersymmetry has to be broken. Note that $\langle \gamma, \gamma' \rangle = -2$.

### 6.4 Interfering hooks: surfaces in $T^8$

The following system has a related, though somewhat different behavior. Here, if any of the hooks is nonzero supersymmetry seems to be broken, while if both are zero, supersymmetry is preserved.

Take 3 Euclidean D3-branes inside $T^8$ as follows:

\[
\begin{align*}
T_{\alpha_0} &= \frac{1}{\lambda} l_1 l_3 l_5 l_7, \\
T_{\gamma_1} &= \frac{1}{\lambda} l_2 l_3 l_6 l_7, \\
T_{\gamma_2} &= \frac{1}{\lambda} l_1 l_4 l_5 l_8, \\
T_{\beta_1} &= l_1 l_2^{-1}, \\
T_{\beta_2} &= l_3 l_4^{-1}.
\end{align*}
\]

The intersection matrix is:

\[
\begin{array}{c|cccc}
\alpha_0 & \gamma_1 & \beta_1 & \gamma_2 & \beta_2 \\
\hline
\alpha_0 & 2 & 0 & 1 & 0 & 1 \\
\gamma_1 & 0 & 2 & -1 & -2 & 1 \\
\beta_1 & 1 & -1 & 2 & 1 & 0 \\
\gamma_2 & 0 & -2 & 1 & 2 & -1 \\
\beta_2 & 1 & 1 & 0 & -1 & 2
\end{array}
\]

This system describes a (2-complex dimensional) surface inside a product of two slanted $T^4$’s. Let us denote the first $T^4$ by $X$ and the second by $Y$. The hooks $\beta_1$ and $\beta_2$ specify two Dehn twists in $X$ and $Y$ respectively. We would like to argue that with any of the two Dehn twists turned on, there is no complex structure on $T^8$ such that the sum of the cohomology classes of the three D3-branes is analytic (i.e. a 4-form of type $(2, 2)$ in Dolbeaux cohomology).

To begin, let us recall some facts about abelian tori (see [33]). We can regard $T^{2n}$ as $\mathbb{C}^n/\Lambda$ where $\Lambda$ is a lattice and we can pick a basis for the lattice $\hat{e}_1, \ldots, \hat{e}_{2n} \in \mathbb{C}^n$. We can also pick a basis of $\mathbb{C}^n$ such that the first $n$ vectors $\hat{e}_1, \ldots, \hat{e}_n$ will be unit vectors in $\mathbb{C}^n$. The
remaining vectors $\hat{e}_{n+1}, \ldots, \hat{e}_{2n}$ form an $n \times n$ matrix $Z$. An abelian variety is a torus that can be embedded inside some $\mathbb{CP}^k$ (for large enough $k$). It can be shown [33] that $T^{2n}$ is an abelian variety if and only if one can choose $Z$ to be symmetric and such that $\text{Im} Z$ is positive definite.

Let us now take the example of $T^4$ constructed as a $T^2$ fibration over a base $T^2$ with a Dehn twist turned on. We can pick a coordinate $z_1$ for the base and $z_2$ for the fiber and we have the identifications:

\[(z_1, z_2) \sim (z_1 + 1, z_2) \sim (z_1 + \tau', z_2 + \lambda') \sim (z_1, z_2 + 1) \sim (z_1, z_2 + \sigma') .\]

Here $\tau' = \tau'_1 + i \tau'_2$ and $\sigma' = \sigma'_1 + i \sigma'_2$ are the complex structures of the base and fiber and $\lambda'$ is the Dehn twist. Let us change coordinates to:

\[ w_1 = z_1 - \frac{1}{2} b(z_1 - \overline{z}_1) - \frac{i}{2} a(z_2 - \overline{z}_2), \]
\[ w_2 = z_2 - \frac{1}{2} c(z_2 - \overline{z}_2) - \frac{i}{2} a(z_1 - \overline{z}_1), \]

where $a, b, c$ are real. This preserves the Kähler class

\[ dw_1 \wedge \overline{dw}_1 + dw_2 \wedge \overline{dw}_2 = dz_1 \wedge d\overline{z}_1 + dz_2 \wedge d\overline{z}_2 .\]

The matrix $Z$ takes the form:

\[ \begin{pmatrix} \tau + b \tau'_2 + a \lambda'_2 & a \sigma'_2 \\ \lambda' + c \lambda'_2 + a \tau'_2 & \sigma' + c \sigma'_2 \end{pmatrix} \]

Let us denote the 6 cycles in the integer homology group $H_2(T^4, \mathbb{Z})$ as $[\hat{e}_i \hat{e}_j]$ ($1 \leq i < j \leq 4$). Let us look for a $(1, 1)$ form, $\omega$, such that:

\[ \int_{[\hat{e}_i \hat{e}_j]} \omega = n, \quad \int_{[\hat{e}_2 \hat{e}_4]} \omega = k, \]

and all the other $\int_{[\hat{e}_i \hat{e}_j]} \omega = 0$. It is easy to check that such an $\omega$ exists if and only if $Z_{12} = \frac{k}{n} Z_{21}$.

Now we return to $T^8$ and take $Z$ to be of the form:

\[ Z = \begin{pmatrix} \tau & \lambda & 0 & 0 \\ \lambda & \sigma & 0 & 0 \\ 0 & 0 & \overline{\tau} & \overline{\lambda} \\ 0 & 0 & \overline{\lambda} & \overline{\sigma} \end{pmatrix} \]
We wish to find a $(2,2)$ form, $\omega$ on $T^8 = T^4 \times T^4$ such that

\[
\int_{[\hat{e}_1 \hat{e}_2 \hat{e}_5 \hat{e}_7]} \omega = n, \quad \int_{[\hat{e}_1 \hat{e}_4 \hat{e}_5 \hat{e}_8]} \omega = \int_{[\hat{e}_2 \hat{e}_3 \hat{e}_6 \hat{e}_7]} \omega = 1,
\]

and all the other $\int_{[\hat{e}_i \hat{e}_j \hat{e}_k \hat{e}_l]} \omega = 0$. There are 70 4-cycles $[\hat{e}_i \hat{e}_j \hat{e}_k \hat{e}_l]$. We get 70 equations in 70 variables. The coefficients are the $4 \times 4$ minors of the matrix:

\[
C = \begin{pmatrix}
1 & 0 & 0 & 0 & \tau & \lambda & 0 & 0 \\
0 & 1 & 0 & 0 & \lambda & \sigma & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & \bar{\tau} & \bar{\lambda} & 0 \\
0 & 0 & 0 & 1 & 0 & \bar{\lambda} & \bar{\sigma} & 0 \\
1 & 0 & 0 & 0 & \bar{\tau} & \bar{\lambda} & 0 & 0 \\
0 & 1 & 0 & 0 & \bar{\lambda} & \bar{\sigma} & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & \bar{\tau} & \bar{\lambda} & 0 \\
0 & 0 & 0 & 1 & 0 & \bar{\lambda} & \bar{\sigma} & 0
\end{pmatrix}.
\]

The coefficient matrix is $C^{(4)}$ and the inverse matrix is proportional to the complementary minors. Up to a constant, we find the following 4-form that is Poincaré dual to the homology class:

\[
\omega = \frac{1}{\Delta} (n\omega_1 + \omega_2 + \omega_3),
\]

\[
\Delta \equiv \frac{1}{4} \det C = 4(\lambda_2^2 - \tau_2 \sigma)(\bar{\lambda}_2^2 - \bar{\tau}_2 \bar{\sigma}_2),
\]

\[
\omega_1 = -4\tau_2 \bar{\tau}_2 w_1 \wedge w_3 \wedge \bar{w}_1 \wedge \bar{w}_3 \\
-2i\tau_2 \bar{\lambda} w_1 \wedge w_3 \wedge \bar{w}_1 \wedge \bar{w}_4 - 2i\bar{\tau}_2 \lambda w_1 \wedge w_3 \wedge \bar{w}_2 \wedge \bar{w}_3 \\
+2i\tau_2 \bar{\lambda} w_1 \wedge w_4 \wedge \bar{w}_1 \wedge \bar{w}_3 + 2i\bar{\tau}_2 \lambda w_2 \wedge w_3 \wedge w_1 \wedge \bar{w}_3 \\
+2i\tau_2 \lambda w_1 \wedge w_2 \wedge w_3 \wedge \bar{w}_3 + 2i\tau_2 \bar{\lambda} w_1 \wedge w_3 \wedge w_4 \wedge w_1 \\
+2i\tau_2 \bar{\lambda} w_1 \wedge \bar{w}_1 \wedge w_3 \wedge \bar{w}_4 + 2i\bar{\tau}_2 \lambda w_3 \wedge w_1 \wedge w_2 \wedge w_3 + O(\lambda)^2
\]

\[
\omega_2 = -4\sigma_2 \bar{\sigma}_2 w_1 \wedge w_4 \wedge \bar{w}_1 \wedge \bar{w}_4 \\
+2i\tau_2 \bar{\lambda} w_1 \wedge w_3 \wedge \bar{w}_1 \wedge \bar{w}_4 - 2i\tau_2 \bar{\lambda} w_1 \wedge w_4 \wedge \bar{w}_1 \wedge \bar{w}_3 \\
-2i\bar{\sigma}_2 \lambda w_1 \wedge w_4 \wedge \bar{w}_2 \wedge \bar{w}_4 + 2i\bar{\sigma}_2 \lambda w_2 \wedge w_4 \wedge \bar{w}_1 \wedge \bar{w}_4 \\
+2i\bar{\sigma}_2 \lambda w_1 \wedge w_2 \wedge w_4 \wedge \bar{w}_4 - 2i\tau_2 \bar{\lambda} w_1 \wedge w_3 \wedge w_4 \wedge w_1
\]
\[-2i\tau_{2}\bar{\lambda}w_{1}\wedge\bar{w}_{1}\wedge\bar{w}_{3}\wedge\bar{w}_{4} + 2i\bar{\sigma}_{2}\bar{\lambda}w_{4}\wedge\bar{w}_{1}\wedge\bar{w}_{2}\wedge\bar{w}_{4} + O(\lambda)^{2}\]

\[\omega_{3} = -4\sigma_{2}\bar{\tau}_{2}w_{2}\wedge w_{3}\wedge\bar{w}_{2}\wedge\bar{w}_{3} + 2i\bar{\tau}_{2}\bar{\lambda}w_{1}\wedge w_{3}\wedge\bar{w}_{3} - 2i\bar{\tau}_{2}\bar{\lambda}w_{2}\wedge w_{3}\wedge\bar{w}_{1}\wedge\bar{w}_{3} - 2i\bar{\sigma}_{2}\bar{\lambda}w_{2}\wedge w_{3}\wedge\bar{w}_{2}\wedge\bar{w}_{3} + O(\lambda)^{2}\]

It is easy to check that there is no $SO(8)$ matrix that when acting on $w_{1},\ldots,w_{4},\bar{w}_{1},\ldots,\bar{w}_{4}$ (preserving the metric) brings $\omega$ to a $(4,4)$ form (at least to first order in $\lambda$ and $\bar{\lambda}$).

## 7 Higher dimensional gauge theories

So far we focussed on the 0D D-instanton actions. In this section we will describe various deformations of higher dimensional gauge theories and the corresponding hooks and baits that realize them.

### 7.1 Mass deformed $\mathcal{N} = 4$ SYM

In subsection (3.3) we described a deformation that corresponds to a mass term in (3) that preserves half the supersymmetry. The same roots $\alpha_{0},\beta$ and $\gamma$, in another limiting region of the parameters $\lambda,l_{1},\ldots,l_{7}$ can describe a mass deformation of $\mathcal{N} = 4$ SYM exactly as in [10]. We can take:

\[T_{\alpha_{0}} = \frac{1}{\lambda}l_{1}l_{2}l_{3}l_{4}l_{5}, \quad T_{\gamma} = \frac{1}{\lambda^{2}}l_{1}l_{2}l_{3}l_{4}l_{6}l_{7}, \quad T_{\beta} = l_{3}l_{6}^{-1}.\]

Similarly, we can get the twisted $(2,0)$ and little-string theories as in [9]:

\[T_{\alpha_{0}} = \frac{1}{\lambda}l_{1}l_{2}l_{3}l_{4}l_{5}l_{6}, \quad T_{\gamma} = \frac{1}{\lambda^{2}}l_{1}l_{2}l_{3}l_{4}l_{5}l_{6}l_{7}, \quad T_{\beta} = l_{6}l_{7}^{-1}.\]
These constructions preserve $\frac{1}{2}$ of the supersymmetry and we conjecture that to break supersymmetry we need to add more hooks and baits. For example, one can add more R-symmetry twists. We can ask what is the partition function of the little-string theory with generic $U(1)_L \times U(1)_R \subset SU(2)_L \times SU(2)_R$ R-symmetry twists along $T^6$. This question was raised in [1]. To get the answer out of $\Xi$ we take

$$T_{\alpha 0} = \frac{1}{\lambda^2} l_1 l_2 l_3 l_4 l_5 l_6.$$ 

We propose to add hooks

$$T_{\beta i} = l_i l_7^{-1}$$

corresponding to the $U(1)_L$ twists and a corresponding KK-monopole bait:

$$T_\gamma = \frac{1}{\lambda^2} l_1 l_2 l_3 l_4 l_5 l_6 l_7^2.$$ 

In order to trap the other $U(1)_R$ twists we conjecture that we need to add hooks:

$$T_{\beta 7} = l_7 l_8^{-1}$$

and a second bait:

$$T_{\gamma 8} = \frac{1}{\lambda^2} l_1 l_2 l_3 l_4 l_5 l_6 l_8^2.$$ 

As discussed in subsection (6.2), once the $7^{th}$ direction is compact there is no Taub-NUT solution with respect to the $8^{th}$ circle. However, as we mentioned before, we treat the Taub-NUT solution only as a motivation for an abstract procedure inside $E_{10}$.

### 7.2 The particle spectrum

In principle we can also “see” (at least part of) the spectrum by Fourier transforming with respect to appropriate phases. Let us look again at the elliptic brane configuration of [10] in 3+1D. We have:

$$T_{\alpha 0} = \frac{1}{\lambda} l_1 l_2 l_3 l_4 l_5,$$

$$T_\beta = l_5 l_6^{-1},$$

$$T_\gamma = \frac{1}{\lambda^2} l_1 l_2 l_3 l_4 l_6 l_7,$$

31
Suppose we managed to somehow break supersymmetry and get rid of the remaining moduli by adding more, unspecified, hooks and baits but let us assume that they are small. Part of the spectrum of the model has an adjoint hypermultiplet of bare mass proportional to \( m = l_6 \phi_\beta \). Let us think of \( l_1 \) as the time direction. How do we see that the action indeed has contributions of the form \( e^{-ml_1} \)? We can trap the contribution with a given (net) number of such particles by counting the string winding-number in direction \( l_5 \). Note that as far as M-theory is concerned, the string winding number along the 5th direction is an integer. Even though the massive hypermultiplets come from strings that seem to have fractional winding number along the 6th direction (a fraction of \( \phi_\beta \)) the endpoints of the strings of the D4-brane are a pair of oppositely charged points and the electric flux emanating from them along the 5th direction effectively “closes” the open string. Thus, it follows that we need to add a root with corresponding action:

\[
T_\delta = l_1 l_5,
\]

and look for terms proportional to \( e^{2\pi i k \phi_\delta} \) in \( \Xi \) in order to extract the contribution with \( k \) massive hypermultiplets. If we do not Fourier-transform with respect to \( \phi_\delta \), we have to set \( \phi_\delta = 0 \) effectively summing over all \( k \). Let us also add a fifth root, \( \eta \), that corresponds to momentum, say, around the 4th direction. We take: \( T_\eta = l_1 l_{\eta}^{-1} \). Now we expect the behavior:

\[
e^{-2\pi(T_0+kT)+2\pi i (\phi_\alpha_0+\phi_\gamma+\phi_\delta+\phi_\eta)}, \quad T \equiv \sqrt{l^2 T_\eta^2 + k^2 \left( \frac{T_\delta}{T_\beta} \right)^2 \phi_\beta^2}.
\]

Similarly, in the example of the previous subsection which involves the twisted little string theory, we can take \( \delta \) such that \( T_\delta = l_1 l_{\eta}^{-1} \) where \( l_1 \) is taken as the “time” direction. This corresponds to momentum along the 7th circle which measures R-symmetry charge. In any case, the relevant roots have intersection matrix:

|   | \( \alpha_0 \) | \( \beta \) | \( \gamma \) | \( \delta \) | \( \eta \) |
|---|---|---|---|---|---|
| \( \alpha_0 \) | 2 | 1 | 0 | 1 | 0 |
| \( \beta \) | 1 | 2 | -1 | 1 | 0 |
| \( \gamma \) | 0 | -1 | 2 | -1 | 0 |
| \( \delta \) | 1 | 1 | -1 | 2 | 1 |
| \( \eta \) | 0 | 0 | 0 | 1 | 2 |

It would be interesting to see whether the expected behavior (13) is originating from a harmonic function.
Let us note that one can realize the same intersection matrix of five roots in M-theory on $T^6$ as follows:

$$
T_{\alpha_0} = R_1 R_{-1}^{5}, \\
T_{\beta} = R_1 R_{-1}^{4}, \\
T_{\gamma} = R_2 R_3 R_4, \\
T_{\delta} = R_1 R_{-1}^{3}, \\
T_{\eta} = R_2 R_{-1}^{3}.
$$

This configuration is a $\frac{1}{8}$-BPS instanton and is likely to contribute to $\lambda^{28}$-terms in the 6D low-energy effective action of M-theory on $T^5$.

### 7.3 The bait for gravity

In [1] we discussed M-theory on $T^7$ with generic $Spin(4)$ twists of the transverse $R^4$. The twists mean that the space is an $R^4$ fibration over $T^7$ and as we go around 1-cycles of $T^7$ we have to rotate the transverse $R^7$ by an appropriate element of (the spin cover of) $SO(4)$. We then generalized this construction to include U-duals of twists but we will not discuss that here. We proposed that for generic twists there exists a well-defined partition function of M-theory on this space. This partition function should be encoded in $\Xi$.

We therefore search for the corresponding hooks and baits. The natural guess is that for one of the $SU(2)$ factors we take the bait $\gamma$ with action $R_1 \cdots R_7 R_{8}^{-1}$ and hook $\beta_i$ with action $R_i R_{8}^{-1}$ ($i = 1 \ldots 7$) and for the other $SU(2)$ factor we take the bait $\gamma'$ with action $R_1 \cdots R_7 R_{9}^{-1}$ and hook $\beta'_i$ with action $R_i R_{9}^{-1}$.

|   | $\gamma$ | $\beta_j$ | $\gamma'$ | $\beta'_j$ |
|---|---|---|---|---|
| $\gamma$ | 2 | -1 | -2 | 1 |
| $\beta_i$ | -1 | 2 | $-\delta_{ij}$ | $\delta_{ij}$ |
| $\gamma'$ | -2 | -2 | 2 | -1 |
| $\beta'_i$ | 1 | $\delta_{ij}$ | -1 | $2 - \delta_{ij}$ |

The system with the two KK-monopoles is hard to analyze because a semi-classical description of the system is not known. If we start with a KK-monopole with respect to the $8^{th}$ direction that fills directions $1 \ldots 7$ and compactify another transverse direction, say the
9th, it is not clear how to construct the solution.

Note that in [1], we suggested a different prescription for calculating the twisted M-theory action on $T^7$ by starting with the twisted M-theory action on $T^9$. The present prescription seems to be different but more symmetrical. I do not know if the two prescriptions agree or not. Both prescriptions are, of course, conjectures.

8 Harmonic functions on $E_{10}(Z) \setminus E_{10}(R)/K$

In this section we will write down an equation for the Laplacian in terms of the group elements and study some of its properties.

8.1 $\mathcal{N}\mathcal{A}\mathcal{K}$ decomposition

We will deal with maximally split Lie group $G$ that can be decomposed into a product of a nilpotent ($\mathcal{N}$), an abelian ($\mathcal{A}$) and a compact ($\mathcal{K}$) subgroups. We take the $\mathcal{N}\mathcal{A}\mathcal{K}$ decomposition to be as follows. For $\lambda \in \Delta$ (the root lattice), Let $V_\lambda$ be the space of elements in the Lie algebra with weight $\lambda$. $V_0$ is the Cartan subalgebra.

$$\mathcal{N} = e^{\sum \phi_i \tau_i}, \quad \mathcal{A} = e^{\sum \lambda_i \tau_i}, \quad \mathcal{K} = e^{\sum c_u (\tau^u - \omega(\tau^u))}. \quad (17)$$

Here $\tau^i \in V_0$ and $\tau^u \in V_{\alpha(u)}$ with $\alpha(u) \in \Delta_+$ a positive root. $\omega$ is the Chevalley involution and in particular $\omega(\tau^u) \in V_{-\alpha(u)}$ (i.e. $\omega(\tau^u)$ corresponds to a negative root).

We also use the Clebsch-Gordan coefficients:

$$[\tau^u, \tau^v] = \sum_w C^w_{uv} \tau^w.$$

Here $C^w_{uv}$ are integers. They are zero unless $\alpha(w) = \alpha(u) + \alpha(v)$.

8.2 The Laplacian

The Laplacian is defined to be the quadratic $G$-invariant operator of the form:

$$\nabla = \frac{1}{2} \sum h_{ij} \frac{\partial^2}{\partial \lambda_i \partial \lambda_j} - \sum_i \frac{\partial}{\partial \lambda_i} + \sum_{uv} W_{uv} e^{\langle \lambda, \alpha(u) \rangle + \langle \lambda, \alpha(v) \rangle} \frac{\partial^2}{\partial \phi_u \partial \phi_v} + \sum W_a e^{\langle \lambda, \alpha(u) \rangle} \frac{\partial}{\partial \phi_a}. \quad (18)$$
Here \( \lambda_i \) correspond to the simple roots \( \alpha_i \in \Delta_+ \). \( h_{ij} \) is the Cartan matrix. The term \( \sum \frac{\partial}{\partial \lambda_i} \) can be written as \( \langle \delta, \partial/\partial \lambda \rangle \) where \( \delta \) is half the sum of all positive roots. Although \( \delta \) itself is infinite, the functional \( \langle \delta, \cdot \rangle \) is finite and is given by the formula above. The functions \( W_u \) and \( W_{uv} \) are determined by invariance under the group action and by the requirement that when all the phases \( \phi_w \) are set to zero:

\[
W_{uv} \{ \phi_w = 0 \} = \delta_{uv}, \quad W_{u} \{ \phi_w = 0 \} = 0.
\]

The result is as follows. \( W_{uv} \) and \( W_u \) are functions only of

\[
\xi_w \equiv e^{-\langle \alpha(w), \lambda \rangle} \phi_w.
\]

They satisfy

\[
0 = \frac{\partial W_{uv}}{\partial \xi_w} + \sum_{k=1}^{\infty} \sum_{u_1, ..., u_k} \sum_{v_1, ..., v_k} b_k \xi_{u_1} \cdots \xi_{u_n} C_{u_{n,v_n}}^{v_1} \cdots C_{u_{2,v_2}}^{v_3} C_{u_{1,v_1}}^{v_2} \frac{\partial W_{uv}}{\partial \xi_x}
\]

\[
- \sum_{k=1}^{\infty} \sum_{u_1, ..., u_k} \sum_{v_1, ..., v_k} b_k \xi_{u_1} \cdots \hat{\xi}_{u_j} \cdots \xi_{u_n} C_{u_{n,v_n}}^{v_1} \cdots C_{u_{2,v_2}}^{v_3} C_{u_{1,v_1}}^{v_2}
\]

\[
0 = \frac{\partial W_u}{\partial \xi_w} + \sum_{k=1}^{\infty} \sum_{u_1, ..., u_k} \sum_{v_1, ..., v_k} b_k \xi_{u_1} \cdots \xi_{u_n} C_{u_{n,v_n}}^{v_1} \cdots C_{u_{2,v_2}}^{v_3} C_{u_{1,v_1}}^{v_2} \frac{\partial W_u}{\partial \xi_x}
\]

\[
- \sum_{k=1}^{\infty} \sum_{u_1, ..., u_k} \sum_{v_1, ..., v_k} b_k \xi_{u_1} \cdots \hat{\xi}_{u_j} \cdots \xi_{u_n} C_{u_{n,v_n}}^{v_1} \cdots C_{u_{2,v_2}}^{v_3} C_{u_{1,v_1}}^{v_2}
\]

\[
- \sum_{k=1}^{\infty} \sum_{u_1, ..., u_k} \sum_{v_1, ..., v_k} b_k \xi_{u_1} \cdots \hat{\xi}_{u_j} \cdots \xi_{u_n} C_{u_{n,v_n}}^{v_1} \cdots C_{u_{2,v_2}}^{v_3} C_{u_{1,v_1}}^{v_2}
\]

(19)

(20)

where \( b_k \) are the coefficients of

\[
\frac{x}{1 - e^{-x}} = \sum_{k=0}^{\infty} b_k x^k = 1 + \frac{1}{2} x + \frac{1}{12} x^2 - \frac{1}{720} x^4 + \frac{1}{30240} x^6 + \cdots
\]

and \( \hat{\xi}_{u_j} \) means that the term \( \xi_{u_j} \) should be excluded from the monomial. One can solve \((19,20)\) as a power series in \( \xi \). We can start with:

\[
W_{uv} = \delta_{uv} + O(\xi), \quad W_u = O(\xi).
\]

It is easy to see that \( W_u \) and \( W_{uv} \) will depend only on those \( \xi_w \)’s that satisfy either \( \alpha(w) < \alpha(u) \) or \( \alpha(w) < \alpha(v) \). Since the number of positive roots that are smaller than any given
root is finite, it also follows from the iterative procedure and (19-20) that \(W_{uv}\) and \(W_u\) are polynomials in the \(\xi_w\’s.

Now we can pick a positive root \(u_0\) and look for solutions of \(\nabla \Phi = 0\) of the form:

\[
\Phi \equiv \Phi(\{\lambda_i\}, \{\phi_u\}_{\alpha(u) \leq \alpha(u_0)}).
\]

Since there is only a finite number of \(u\’s\) such that \(\alpha(u) < \alpha(u_0)\), and since \(W_u\) and \(W_{uv}\) are independent of \(\xi_w\’s\) that do not satisfy \(\alpha(w) < \alpha(u_0)\), the equation \(\nabla \Phi = 0\) will reduce to a differential equation in a finite number of variables. This might be a good approximation for \(\Xi\) in regions of the \(\{\lambda_i\}\) parameter space that satisfy:

\[
1 \ll \langle \lambda, \beta \rangle, \quad \text{if } \beta \not\leq \alpha(u_0).
\]

### 8.3 First order iterative solution

To first order we find:

\[
W_{uv} = \delta_{uv} + \frac{1}{2} \sum_w (C^w_{uw} + C^w_{vw}) \xi_w
\]

\[
-\frac{1}{24} \sum_{x,y,w} (C^y_{ux}C^w_{xy} + C^y_{uw}C^w_{xy} + C^y_{vx}C^w_{yw} + C^y_{vw}C^w_{xy}) \xi_x \xi_w
\]

\[
+ \frac{1}{8} \sum_{x,y,w} (C^u_{xy}C^w_{xw} + C^u_{wy}C^w_{xw}) \xi_x \xi_w + O(\xi)^2
\]

\[
W_u = \sum_{x,y} C^u_{xy}C^y_{xw} \xi_w + O(\xi)^2
\]

The linear term in \(W_u\) is nonzero only if \(2\alpha(x) + \alpha(w) = \alpha(u)\). The linear term in \(W_{uv}\) is zero unless \(\alpha(w) = \pm(\alpha(u) - \alpha(v))\). The quadratic term with the \(\frac{1}{24}\) perefactor is zero unless \(\alpha(x) + \alpha(w) = \pm(\alpha(u) - \alpha(v))\). The quadratic term with the \(\frac{1}{8}\) perefactor is zero unless

\[
\alpha(w) = \alpha(v) - \beta, \quad \alpha(x) = \alpha(u) - \beta, \quad \beta = \alpha(y) > 0,
\]

or:

\[
\alpha(x) = \alpha(v) - \beta, \quad \alpha(w) = \alpha(u) - \beta, \quad \beta = \alpha(y) > 0.
\]

We see that for given \(u, v\) there are only a finite number of terms in the sum.
8.4 Harmonic functions

We can now check the statements made in previous sections about the relation between harmonic functions and actions of branes.

We will start with the examples in section (3). First let us take a single BPS instanton. To make things simple, let us assume that it corresponds to a simple root $\alpha_i$. From the discussion above it follows that we can look for a harmonic function $\Phi$ that depends only on $\phi \equiv \phi_{\alpha_k}$ and no other phases. Laplace’s equation becomes:

$$\frac{1}{2} \sum h_{ij} \frac{\partial^2 \Phi}{\partial \lambda_i \partial \lambda_j} - \sum \frac{\partial \Phi}{\partial \lambda_i} + e^{2\lambda_k} \frac{\partial^2 \Phi}{\partial \phi^2} = 0.$$ 

We are looking for a solution of the form: $\Phi = e^{-2\pi n T(\lambda) + 2\pi i n \phi}$. The function $T$ satisfies:

$$\pi n \sum h_{ij} \frac{\partial^2 T}{\partial \lambda_i \partial \lambda_j} + 2\pi^2 n^2 \sum h_{ij} \frac{\partial T}{\partial \lambda_i} \frac{\partial T}{\partial \lambda_j} - 2\pi n \sum \frac{\partial T}{\partial \lambda_i} = 4\pi^2 n^2 e^{2\lambda_k}.$$ 

We see that $T = e^{\lambda_k}$ is a good solution.

Similarly, for pairs of distinct simple roots $\alpha_k$ and $\alpha_l$ with $\langle \alpha_k, \alpha_l \rangle = 0$, one can separate variables and see that $e^{-2\pi (n e^{\lambda_k} + m e^{\lambda_l}) + 2\pi i (n \phi_k + m \phi_l)}$, is also a solution.

Now let us take the case $\langle \alpha, \beta \rangle = 1$. In this case $\alpha - \beta$ is also a root and we cannot take both $\alpha$ and $\beta$ to be simple roots. We can take $\alpha = \alpha_k$ and $\beta = \alpha_k + \alpha_l$ such that $\langle \alpha_k, \alpha_l \rangle = -1$. Now the solution must depend on $\phi_\alpha \equiv \phi_{\alpha_k}$ and $\phi_\beta$ but also on $\phi_l \equiv \phi_{\alpha_l}$. If we take $u$ to be the generator such that $\alpha(u) = \alpha_k$, $v$ the generator such that $\alpha(v) = \alpha_l$ and $w$ the generator such that $\alpha(w) = \alpha_k + \alpha_l$, we find from (19-20) the equation:

$$0 = \frac{1}{2} \sum h_{ij} \frac{\partial^2 \Phi}{\partial \lambda_i \partial \lambda_j} - \sum \frac{\partial \Phi}{\partial \lambda_i} + e^{2\lambda_k} \frac{\partial^2 \Phi}{\partial \phi_\alpha^2} + e^{2(\lambda_k + \lambda_l)} \frac{\partial^2 \Phi}{\partial \phi_\beta^2} + e^{2\lambda_l} \frac{\partial^2 \Phi}{\partial \phi_l^2} - e^{2\lambda_l} \phi_\alpha \frac{\partial^2 \Phi}{\partial \phi_l \partial \phi_\beta} + e^{2\lambda_k} \phi_l \frac{\partial^2 \Phi}{\partial \phi_\alpha \partial \phi_\beta} + \frac{1}{4} (e^{2\lambda_l} \phi_\alpha^2 + e^{2\lambda_k} \phi_l^2) \frac{\partial^2 \Phi}{\partial \phi_\beta}.$$ 

Let us look for a solution that behaves like:

$$\Phi = e^{-T(\lambda, \phi) + 2\pi i n \phi_\alpha + 2\pi i m (\phi_\beta + \frac{1}{2} \phi_\alpha \phi_l)}.$$
We find the equation:

\[
0 = -\frac{\partial^2 T}{\partial \lambda^2_k} + \frac{\partial^2 T}{\partial \lambda_k \partial \lambda_l} - \frac{\partial^2 T}{\partial \lambda^2_l}
+ \left( \frac{\partial T}{\partial \lambda_k} \right)^2 + \left( \frac{\partial T}{\partial \lambda_l} \right)^2 - \left( \frac{\partial T}{\partial \lambda_k} \right) \left( \frac{\partial T}{\partial \lambda_l} \right) + \frac{\partial T}{\partial \lambda_k} + \frac{\partial T}{\partial \lambda_l}
- 4\pi^2 (n + m\phi_l)^2 e^{2\lambda_k} - e^{2\lambda_l} \frac{\partial^2 T}{\partial \phi^2_l} + e^{2\lambda_l} \left( \frac{\partial T}{\partial \phi_l} \right)^2
\]

One solution is, as expected from U-duality:

\[
T = 2\pi e^{2\lambda_k} \sqrt{m^2 e^{2\lambda_l} + (n + m\phi_l)^2}.
\]

Finally, we would like to recall the case of subsection (7.2). This case is particularly interesting because it gives us a glimpse of the particle spectrum.

The intersection matrix is:

|     | $\alpha_0$ | $\beta$ | $\gamma$ | $\delta$ | $\eta$ |
|-----|-------------|---------|-----------|----------|-------|
| $\alpha_0$ | 2 | 1 | 0 | 1 | 0 |
| $\beta$ | 1 | 2 | -1 | 1 | 0 |
| $\gamma$ | 0 | -1 | 2 | -1 | 0 |
| $\delta$ | 1 | 1 | -1 | 2 | 1 |
| $\eta$ | 0 | 0 | 0 | 1 | 2 |

It would be interesting to check that the behavior suggested in (15) is related to a harmonic function. We will check this in another work [31], but we will make a few comments. For the check, it seems imperative to find a realization of $\alpha_0 \ldots \eta$ such that the set of roots that are smaller than at least one of $\alpha_0 \ldots \eta$ has the smallest number of elements as possible. We can then search for a harmonic function $\Phi$ that depends only on $\phi_{\alpha_0}, \ldots, \phi_{\eta}$ and the phases that correspond to these extra roots, because the extra roots will enter (19-20). We should also make sure that relations among roots such as $\alpha_0 > \beta$ should be preserved.

Let us ignore the extra root $\eta$ and check the simplest version of (15) with $l = 0$. If we choose the simple roots of $E_8$ to be $\rho_1, \ldots, \rho_8$ with

\[
T_{\rho_1} = R_1 R_2^{-1}, T_{\rho_2} = R_2 R_3^{-1}, \ldots, T_{\rho_7} = R_7 R_8^{-1}, T_{\rho_8} = R_6 R_7 R_8,
\]

then a minimal choice for $\alpha_0, \ldots, \delta$ can be taken as:

\[
\alpha_0 = \rho_5 + \rho_6 + \rho_8, \quad T_{\alpha_0} = R_5 R_6 R_8,
\]
\[ \beta = \rho_5 + \rho_8, \quad T_\beta = R_5 R_7 R_8, \]
\[ \gamma = \rho_4 + \rho_5 + \rho_6, \quad T_\gamma = R_4 R_6^{-1}, \]
\[ \delta = \rho_8, \quad T_\delta = R_6 R_7 R_8, \]

In addition to these roots, the Laplacian depends on the following roots:

\[ \chi_1 = \rho_4, \quad T_{\chi_1} = R_4 R_5^{-1}, \]
\[ \chi_2 = \rho_5, \quad T_{\chi_2} = R_5 R_6^{-1}, \]
\[ \chi_3 = \rho_6, \quad T_{\chi_3} = R_6 R_7^{-1}, \]
\[ \chi_4 = \rho_4 + \rho_5, \quad T_{\chi_4} = R_4 R_6^{-1}, \]
\[ \chi_5 = \rho_5 + \rho_6, \quad T_{\chi_5} = R_3 R_7^{-1}, \]

Physically, this means that the equation will depend on five more angles \((\phi_{\chi_1}, \ldots, \phi_{\chi_5})\). Note that all this roots can be embedded inside an \(SO(4, 4)\) subgroup. We will not pursue this direction here, but we note that if the conjecture at the end of section (2) is correct, then since this instanton can be embedded as an instanton in M-theory on \(T^8\) its action has to be harmonic and the expectation (15) would be met.

### 9 Discussion

There are two established facts that seem fascinating and were part of the motivation for the conjectures presented above. The first fact is that the actions of wrapped branes are encoded in \textit{exact} harmonic functions on \(E_{d(d)}(Z) \setminus E_{d(d)}(R)/K_d\). Precisely which brane we are asking about is encoded in the dependence of the function on the periodic variables (the “phases”) in the moduli space. It is not only the action of single wrapped BPS branes that is the exponent of a harmonic function but, as we have seen in section (2), combinations of several branes are also encoded in harmonic functions. This leads one to suspect that Laplace’s equation on the moduli space is analogous to a second-quantized equation of motion rather than just a first-quantized equation. If harmonic functions with a given behavior as a function of phases encode the action of the BPS instantons, then the natural question is what would harmonic functions with more complicated behavior, as a function of phases, encode.
The second fascinating fact is that one can realize mass-like deformations of gauge theories (such as giving mass to the adjoint hypermultiplet in $N = 4$ SYM) by studying the behavior of the corresponding branes in the presence of another, “spectator” brane. If this construction is embedded inside $T^d$ then each type of brane is coupled to its own periodic phase in the moduli space and the mass parameter corresponds to a different periodic phase.

We might also be able to realize certain nonsupersymmetric deformations of gauge theories in this manner but one is forced to work with M-theory on tori $T^d$ with $d \geq 9$. We can now define a harmonic function on the generalization of the “moduli-space”, $E_{d(d)}(Z)/E_{d(d)}/K_d$, and we can extract the piece of it that has the desired behavior as a function of the phases. The question stands: what would this mode of the harmonic function describe? The natural conjecture is that it will correspond to the partition function of the gauge theory (multiplied by the contribution of the tensions of the branes to the action).

If that is true then it follows that there is a single harmonic function $\Xi$ on $E_d(R)/E_d(R)/\mathcal{K}$ that encodes all of the separate partition functions discussed above. The different partition functions can be obtained from $\Xi$ by extracting particular Fourier modes of the function with respect to appropriate periodic phases. Here, $d$ should be large enough to accommodate the nonsupersymmetric constructions and should probably be $d = 10$, or perhaps higher!

In [1] we also conjectured that certain types of “gravity” partition functions can be defined and that they are also encoded in $\Xi$. The conjectured partition functions were defined to be the partition function of M-theory on a space that is constructed as an $R^4$ fibration over $T^7$ and as $R^2$ fibrations over $T^9$. In [1] we argued that the former is a special case of the latter and we presented a conjectured prescription for extracting the latter out of $\Xi$. In section (7) we suggested a different prescription for extracting out of $\Xi$ the partition function of M-theory on an $R^4$ fibration over $T^7$.

I do not see how this agrees with the prescription in [1] (in the special case of an $R^4$ fibration over $T^7$). The present prescription treats both $SU(2)_L$ and $SU(2)_R$ factors of the fibration group $SO(4)$ symmetrically. Perhaps the conjectured prescription of [1] for an $R^2$ fibration over $T^9$ is wrong, or perhaps it somehow reduces to the present one in a nontrivial fashion (or perhaps both are wrong!).

In this paper we showed how certain deformations of the D-instanton actions can be
realized using the phases of $E_{d(d)}(Z)/E_{d(d)}/\mathcal{K}_d$. We were only able to realize certain mass deformations and we haven’t discussed deformations that are not quadratic in the fields (with the exception of section (4)).

It would be interesting to understand what other deformations of the D-instanton integral are possible using hooks and baits. In [40], the effect of RR fields on the D0-brane action was studied. It was found that certain fields induce quartic and higher terms. It would also be interesting to understand the behavior of the D0-brane or D-instanton in nonzero RR field strengths. Such configurations can arise when the D-instanton “probes” other objects. In a related paper [41], the coupling of closed string states to the D0-brane action was studied and many augmentations of the D0-brane action can arise this way. Although the closed string fields are not directly related to moduli and therefore variables of $E_8$, perhaps one can turn them on by adding more hooks and baits. One would probably have to utilize the other roots of $E_{10}$ (perhaps even the “imaginary” roots that satisfy $\langle \alpha, \alpha \rangle \leq 0$).

We have argued in section (8) that in many cases one can separate from the infinite dimensional $E_{10}$ a subset of a finite number of variables that are relevant for the problem and ignore the rest. We have suggested that the conjecture can therefore be tested in certain cases for which we know the existence of BPS states in the spectrum. Thus, one can test the conjecture that (15) is a limit of a harmonic function. Similarly, the mass formulas in sections (4) might be tested along similar lines.

The fact that one can separate from the infinite dimensional $E_{10}$ a subset of a finite number of variables seems suspicious at first, because we do not expect a differential equation in a finite number of variables to encode the partition functions of complicated gauge theories. However, first of all, I do not see an immediate contradiction. Moreover, given any partition function $Z(R_1, R_2, R_3, R_4)$, one can always add even just one single variable say $\lambda$ to the existing list of variables and find a harmonic function $Z(R_1, R_2, R_3, R_4, \lambda)$ that reduces to $Z(R_1, R_2, R_3, R_4)$ in the limit $\lambda \to \infty$. In the context of $E_{10}$, it would seem that the heart of the matter is the boundary conditions imposed on $\Xi$ by the U-duality group $E_{10}(Z)$.

We have suggested that, if one adds more variables, the partition function of the D-instanton matrix integrals can be read off from a harmonic equation. It is actually a well known fact that if one adds all the deformations to the matrix integral one can write down
a set of second-order differential equations (see [42] for the case with many matrices). We
start with:

\[ Z(\{\sigma_{\mu_1,\ldots,\mu_r}\}) = \sum_N \int \prod_{\mu=1}^{10} dX_{\mu} \exp \text{Tr} \left\{ - \sum \sigma_{\mu_1,\ldots,\mu_r} X_{\mu_1} \cdots X_{\mu_r} \right\}. \]

In particular, \( \sigma \) (with no indices) is the coefficient of \( N = \text{Tr}1 \). To get a second-order
equation we insert:

\[ \sum_N \int \prod_{\mu=1}^{10} dX_{\mu} \ \text{Tr} \left[ X_{\nu_1} \cdots X_{\nu_s} \frac{\partial}{\partial X_{\mu}} \right] \exp \text{Tr} \left\{ - \sum \sigma_{\mu_1,\ldots,\mu_r} X_{\mu_1} \cdots X_{\mu_r} \right\} \quad (21) \]

and integrate by parts. We find:

\[ \left[ \sum_j \delta_{\mu \nu_1} \cdots \delta_{\mu \nu_s} \frac{\partial}{\partial \sigma_{\nu_1,\ldots,\nu_s}} + \sum_j \delta_{\mu \nu_1} \cdots \delta_{\mu \nu_{j-1}} \frac{\partial^2}{\partial \sigma_{\nu_{j+1},\ldots,\nu_s}} \right] Z = 0. \quad (22) \]

This is an infinite set of equations. Perhaps they are somehow related to the harmonic
equation on \( \Xi \) by expanding in certain regions of moduli space where a certain subset of
variables decouple. Expanding the equation in these variables might lead to an infinite set
of second order equations.

One aspect of the conjecture about \( \Xi \) is that the U-duality \( E_{10}(Z) \) plays an important
role. It is not hard to find harmonic functions that reduce to any function that we want
once we take a limit of a certain variable. After all, we can solve Laplace’s equation with
any given initial condition. What is nontrivial, is to find harmonic functions which satisfy
certain boundary conditions. In our case the boundary conditions are set by \( E_{10}(Z) \) that
relate various regions of the 10 noncompact parameters of \( E_{10} \). It is reasonable to expect
that \( \Xi \) should vanish in the limits that correspond to decompactification of enough (probably
3 or more) directions and all their U-dual limits. However, not all limits of the \( E_{10} \) moduli
space can be obtained this way [43]. It is not clear to me what the behavior of \( \Xi \) should be
in those other regions.

Another, perhaps interesting, direction for research would be to further explore the com-
bination of instantons that correspond to roots \( \alpha \) and \( \beta \) with \( \langle \beta, \alpha \rangle = -2 \). This seems to
be the next interesting case (in the sense that supersymmetry is preserved) after the case of
\( \langle \beta, \alpha \rangle = 0 \). Among the examples presented in section [3] are various exotic combinations of
KK monopoles and branes that may lead to interesting physical effects.
There are also interesting systems that are U-dual to the well studied cases. One is the behavior of D-branes, M2-branes and M5-branes near the core of KK-monopoles with NSNS or 3-form fluxes turned on. This was discussed briefly in subsection (3.6) and will hopefully be explored further somewhere else [31]. Another system that will be explored further somewhere else is the system of surfaces in $T^8$ and its U-dual manifestations that was discussed briefly in (6.4).

We have seen that the usual questions about partition functions of gauge theories and gravity are only a part of the answers that $\Xi$ is conjectured to encode. $\Xi$ probably contains other information about questions that we cannot formulate in terms of space and time.

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