The thermodynamic free energy $F(\beta)$ is calculated for a gas consisting of the transverse oscillations of a piecewise uniform bosonic string. The string consists of $2N$ parts of equal length, of alternating type I and type II material, and is relativistic in the sense that the velocity of sound everywhere equals the velocity of light. The present paper is a continuation of two earlier papers, one dealing with the Casimir energy of a $2N$–piece string [I. Brevik and R. Sollie (1997)], and another dealing with the thermodynamic properties of a string divided into two (unequal) parts [I. Brevik, A. A. Bytsenko and H. B. Nielsen (1998)]. Making use of the Meinardus theorem we calculate the asymptotics of the level state density, and show that the critical temperatures in the individual parts are equal, for arbitrary spacetime dimension $D$. If $D = 26$, we find $\beta = (2/N)\sqrt{2\pi/T_{II}}$, $T_{II}$ being the tension in part II. Thermodynamic interactions of parts related to high genus $g$ is also considered.

1. Introduction

Whereas the bosonic string of length $L$ in $D$-dimensional spacetime is assumed to be uniform, the composite string is imagined to consist of two or more uniform pieces. In a Casimir context, such a model was introduced in 1990 [1]. The string was assumed to divided into two pieces, of length $L_I$ and $L_{II}$, and the model was relativistic in the sense that the velocity of sound was everywhere required to be equal to the velocity of light. With this constraint imposed on the model, the Casimir energy of the string, i.e., the zero–point energy associated with its discontinuity...
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properties, was easily calculable as a function of the length ratio \( s = L_{II}/L_I \).

Later, various aspects of the relativistic piecewise uniform string model were studied
in which the model finds application in relation to the Green-Schwarz superstring.

The present paper focuses attention on the thermodynamic free energy \( F(\beta) \) at
inverse temperature \( \beta = 1/T \) of a \( 2N \)-piece string, made up of \( 2N \) parts of equal
length, of alternating type I and type II material. The model is relativistic, in the
sense explained above. In an earlier paper \( ^8 \) we developed the Casimir theory for
a string of this type, whereas in another paper \( ^9 \) we considered the free energy for
the case where the string consists of two pieces only, i.e., the model of Ref. \( ^4 \). The
calculation of \( F(\beta) \) for a \( 2N \)-piece string has to our knowledge not been undertaken
before. It turns out, similarly as in Ref. \( ^{11} \), that the Meinardus theorem \( ^{21}, ^{22}, ^1 \)
is powerful, allowing us to find the asymptotics of the level state density. Using this we
find, for a general spacetime dimension \( D \), that the critical (Hagedorn) temperatures
for the two kinds of pieces are the same. When \( D = 26 \), the common spacetime
dimension for a bosonic string, we find \( \beta_c = (2/N)^{\sqrt{2\pi/T_{II}}}, \) \( T_{II} \) being the tension
in region II. This result is derived in Section 6. In Section 7, we comment upon the
thermodynamic properties of the composite string for arbitrary genus \( g \).

2. Resumé of the 2N–Piece String Theory

Assume, as mentioned, that the string of total length \( L \) is divided into \( 2N \)
equally large pieces, of alternating type I and type II material; see Fig. 1. The
string is relativistic, in the sense that the velocity of sound is everywhere equal to
the velocity of light, \( v_s = \sqrt{T_I/\rho_I} = \sqrt{T_{II}/\rho_{II}} = c \), where \( T_I, T_{II} \) are the tensions
and \( \rho_I, \rho_{II} \) the mass densities in the two pieces. We will study the transverse
oscillations \( \psi = \psi(\sigma, \tau) \) of the string; \( \sigma \) denoting as usual the position coordinate
and \( \tau \) the time coordinate of the string. We can thus write in the two regions

\[
\begin{align*}
\psi_I &= \xi_I e^{i\omega(\sigma-\tau)} + \eta_I e^{-i\omega(\sigma+\tau)}, \\
\psi_{II} &= \xi_{II} e^{i\omega(\sigma-\tau)} + \eta_{II} e^{-i\omega(\sigma+\tau)},
\end{align*}
\]

where \( \xi \) and \( \eta \) are constants. The junction conditions are that \( \psi \) itself as well as the
transverse elastic force \( T \partial \psi / \partial \sigma \) are continuous, i.e.,

\[
\psi_I = \psi_{II}, \quad T_I \partial \psi_I / \partial \sigma = T_{II} \partial \psi_{II} / \partial \sigma,
\]

at each of the \( 2N \) junctions. We define \( x \) as the tension ratio, \( x = T_I/T_{II}, \) and
define also the symbols \( p_N \) and \( \alpha \) by \( p_N = \omega L/N, \alpha = (1-x)/(1+x) \). Now
introduce the matrix \( \Lambda \),

\[
\Lambda(\alpha, p_N) = \begin{pmatrix} a & b \\ b^* & a^* \end{pmatrix},
\]

(3)
with
\[ a = e^{-ipN} - \alpha^2, \quad b = \alpha(e^{-ipN} - 1). \] (4)

Then, as shown in Fig. 1, the eigenfrequencies \( \omega \) are determined from the equation
\[ \text{Det} \left[ (1 - \alpha^2)^{-N} \Lambda^N (\alpha, p_N) - 1 \right] = 0. \] (5)

For practical purposes it is convenient to reformulate the condition (5). Let us define two new quantities \( \lambda_{\pm} \):
\[ \lambda_{\pm}(p_N) = \cos p_N - \alpha^2 \pm [\cos p_N - \alpha^2 - (1 - \alpha^2)^2]^{1/2}. \] (6)
Then, we can re-express the condition (5) as
\[ \lambda_+^N + \lambda_-^N = 2(1 - \alpha^2)^N. \] (7)

We can now make use of the following recursion formula for the quantity \( S_N \equiv \lambda_+^N + \lambda_-^N \):
\[ S_N = 2(\cos p - \alpha^2)S_{N-1} - (1 - \alpha^2)^2S_{N-2}, \quad N \geq 2, \] (8)
in which it is assumed that \( \omega L/N \) is constant, at all recursive steps. The initial values of \( S_N \) are \( S_0 = 2, \quad S_1 = \lambda_+ + \lambda_- = 2(\cos p - \alpha^2) \).

Assume now that \( L = \pi \), in conformity with usual practice. Thus \( p_N = \pi \omega/N \). We let \( X^\mu(\sigma, \tau) \), with \( \mu = 0, 1, 2, \ldots, (D - 1) \), specify the coordinates on the world sheet. For each of the eigenvalue branches determined by the dispersion equation (5) we can write \( X^\mu \) on the form

Fig. 1. Sketch of the composite 2\( N \) string, when \( N = 6 \).
where $x^\mu$ is the centre–of–mass position, $p^\mu$ the total momentum of the string, and $T_0 = \frac{1}{2}(T_I + T_{II})$ is the mean tension. Further, $X_I^\mu$ and $X_{II}^\mu$ are decomposed into oscillator coordinates,

\[X_I^\mu = \frac{i}{2} \sum_{n \neq 0} \frac{1}{n} \left[ \alpha_n e^{i\omega(\sigma - \tau)} + \tilde{\alpha}_n e^{-i\omega(\sigma + \tau)} \right],\]

(11)

\[X_{II}^\mu = \frac{i}{2} \sum_{n \neq 0} \frac{1}{n} \left[ \alpha_n e^{i\omega(\sigma - \tau)} + \tilde{\alpha}_n e^{-i\omega(\sigma + \tau)} \right].\]

(12)

Here, $\ell_s$ is the fundamental string length, unspecified so far, and $\alpha_n, \tilde{\alpha}_n$ are oscillator coordinates of the right– and left–moving waves, respectively. A characteristic property of the composite string is that the oscillator coordinates have to be specified for each of the various branches determined by Eq. (5). This makes the handling of the formalism complicated, in general. A significant simplification can be obtained if, following Ref. 9, we limit ourselves to the case of extreme string ratios only. Since $\alpha$ occurs quadratically in Eqs. (6) and (7), the eigenvalue spectrum has to be invariant under the transformation $x \rightarrow \frac{1}{x}$. It is sufficient, therefore, to consider the tension ratio interval $0 < x \leq 1$ only. The case of extreme tensions corresponds to $x \rightarrow 0$. We consider only this case in the following.

3. The Case of Extreme Tensions

We assume that $T_{II}$ has a finite value, so that the limiting case $x \rightarrow 0$ corresponds to $T_I \rightarrow 0$. Thus $T_0 \rightarrow (1/2)T_{II}$. Since now $\alpha \rightarrow 1$ we get from Eq. (6) $\lambda_- = 0$, $\lambda_+ = \cos p_N - 1$, so we obtain from Eq. (7) the remarkable simplification that all the eigenfrequency branches degenerate into one single branch determined by $\cos p_N = 1$. That is, the eigenvalue spectrum becomes

\[\omega_n = 2Nn, \quad n = \pm 1, \pm 2, \pm 3, \ldots\]

(13)

Then, choosing the fundamental length equal to $\ell_s = (\pi T_I)^{-1/2}$, we can write the expansion (11) in region I as (subscript I on the $\alpha_n$’s omitted)

\[X_I^\mu = \frac{i}{2\sqrt{\pi T_I}} \sum_{n \neq 0} \frac{1}{n} \left[ \alpha_n e^{2iNn(\sigma - \tau)} + \tilde{\alpha}_n e^{-2iNn(\sigma + \tau)} \right].\]

(14)

The junction conditions (2) permit all waves to propagate from region I to region II. When $x \rightarrow 0$, they reduce to the equations
\[ \xi_I + \eta_I = 2\xi_{II} = 2\eta_{II}, \] (15)

which show that the right– and left– moving amplitudes \( \xi_I \) and \( \eta_I \) in region I can be chosen freely and that the amplitudes \( \xi_{II}, \eta_{II} \) in region II are thereafter fixed. This means, in oscillator language, that \( \alpha_n^\mu \) and \( \tilde{\alpha}_n^\mu \) can be chosen freely. The expansion in region II can in view of Eq. (15) be written as

\[ X_{II}^\mu = \frac{i}{2\sqrt{\pi T_I}} \sum_{n \neq 0} \frac{1}{n^\mu} e^{-iNn\tau \cos(2Nn\sigma)}, \] (16)

where we have defined \( \gamma_n^\mu \) as

\[ \gamma_n^\mu = \alpha_n^\mu + \tilde{\alpha}_n^\mu, \quad n \neq 0. \] (17)

The oscillations in region II are thus standing waves. This is the same kind of behaviour as that found for the two-piece string \( \Delta \).

The action of the string is

\[ S = -\frac{1}{2} \int d\tau d\sigma T(\sigma) \eta^{\alpha\beta} \partial_\alpha X^\mu \partial_\beta X_\mu, \] (18)

where \( \alpha, \beta = 0, 1 \) and \( T(\sigma) = T_I \) in region I, \( T(\sigma) = T_{II} \) in region II. The momentum conjugate to \( X^\mu \) is \( P^\mu(\sigma) = T(\sigma)\dot{X}^\mu \), and the Hamiltonian is accordingly

\[ H = \int_0^\pi \left[ P_\mu(\sigma)\dot{X}^\mu - \mathcal{L} \right] d\sigma = \frac{1}{2} \int_0^\pi T(\sigma)(\dot{X}^2 + X'^2) d\sigma, \] (19)

where \( \mathcal{L} \) is the Lagrangian.

As for the constraint equation for the string, some care has to be taken. Conventionally, in the classical theory for the uniform string the constraint equation reads \( T_{\alpha\beta} = 0 \), \( T_{\alpha\beta} \) being the energy–momentum tensor. As discussed in Ref. \( \Delta \), the situation is here more complicated, since the junctions restrict the freedom one has to take the variations \( \delta X^\mu \). We thus have to replace the strong condition \( T_{\alpha\beta} = 0 \) by a weaker condition, and the most natural choice, which we will adopt, is to impose that \( H = 0 \) when applied to the physical states.

Let us introduce lightcone coordinates, \( \sigma^- = \tau - \sigma \) and \( \sigma^+ = \tau + \sigma \). The derivatives conjugate to \( \sigma^\pm \) are \( \partial_\pm = \frac{1}{2}(\partial_\tau \mp \partial_\sigma) \). In region I,

\[ \partial_- X^\mu = \frac{N}{\sqrt{\pi T_I}} \sum_{-\infty}^{\infty} \alpha_n^\mu e^{2iNn(\sigma-\tau)}, \]

\[ \partial_+ X^\mu = \frac{N}{\sqrt{\pi T_I}} \sum_{-\infty}^{\infty} \tilde{\alpha}_n^\mu e^{-2iNn(\sigma+\tau)}, \] (20)

and in region II


\[ \partial_\pi X^\mu = \frac{N}{2\sqrt{\pi T_I}} \sum_{-\infty}^{\infty} \gamma_n e^{\pm 2in(\sigma \pi \tau)}, \quad (21) \]

where we have defined
\[ \alpha_0^\mu = \tilde{\alpha}_0^\mu = \frac{p^\mu}{N T_{II} \sqrt{\frac{T_I}{\pi}}}, \quad \gamma_0^\mu = 2\alpha_0^\mu. \quad (22) \]

Inserting these expressions into the Hamiltonian
\[ H = \int_{0}^{\pi} T(\sigma) (\partial_- X \cdot \partial_- X + \partial_+ X \cdot \partial_+ X) d\sigma \]
\[ = NT_1 \int_{0}^{\pi/(2N)} (\partial_- X \cdot \partial_- X + \partial_+ X \cdot \partial_+ X) d\sigma \]
\[ + NT_{II} \int_{\pi/N}^{\pi/(2N)} (\partial_- X \cdot \partial_- X + \partial_+ X \cdot \partial_+ X) d\sigma \quad (23) \]

we get
\[ H = \frac{1}{2} N^2 \sum_{-\infty}^{\infty} (\alpha_{-n} \cdot \alpha_n + \tilde{\alpha}_{-n} \cdot \tilde{\alpha}_n) + \frac{N^2}{4x} \sum_{-\infty}^{\infty} \gamma_{-n} \cdot \gamma_n. \quad (24) \]

Now consider the expression for the square \( M^2 \) of the mass of the string. One must have \( M^2 = -p^\mu p_\mu \), as in the case of a uniform string. We start from the constraint \( H = 0 \) when applied to physical states, making use of Eq. (24) in which we separate out the \( n = 0 \) terms. Using that \( \alpha_0 \cdot \alpha_0 = -M^2 x/\pi N^2 T_{II} \) according to Eq. (22) we obtain in this way, when again observing that \( x \ll 1 \),
\[ M^2 = \pi N^2 T_{II} \sum_{n=1}^{\infty} \left[ \alpha_{-n} \cdot \alpha_n + \tilde{\alpha}_{-n} \cdot \tilde{\alpha}_n + \frac{1}{2x} \gamma_{-n} \cdot \gamma_n \right]. \quad (25) \]

4. Quantization

The momentum conjugate to \( X^\mu \) is at any position on the string equal to \( T(\sigma) \dot{X}^\mu \). We accordingly require the commutation rules in region I to be
\[ T_I [\dot{X}^\mu(\sigma, \tau), X^\nu(\sigma', \tau)] = -i\delta(\sigma - \sigma')\eta^{\mu\nu}, \quad (26) \]
and in region II
\[ T_{II} [\dot{X}^\mu(\sigma, \tau), X^\nu(\sigma', \tau)] = -i\delta(\sigma - \sigma')\eta^{\mu\nu}, \quad (27) \]
\( \eta^{\mu\nu} \) being the \( D \)-dimensional flat metric. The other commutators vanish. The quantities to be promoted to Fock state operators are \( \alpha_{\mp n} \) and \( \gamma_{\mp n} \). We insert the
expansions for $X^\mu$ and $\dot{X}^\mu$ in regions I and II into Eqs. (26) and (27) and make use of the effective relationship
\[
\sum_{n=-\infty}^{\infty} e^{2iNn(\sigma-\sigma')} = 2 \sum_{n=-\infty}^{\infty} \cos 2Nn\sigma \cos 2Nn\sigma' \rightarrow \frac{\pi}{N} \delta(\sigma-\sigma'). \tag{28}
\]
We then get in region I
\[
[a_\mu^n, a_\nu^m] = n\delta_{n+m,0} \eta^{\mu\nu}, \tag{29}
\]
with a similar relation for $\tilde{a}_n$. In region II,
\[
[\gamma_\mu^n, \gamma_\nu^m] = 4nx \delta_{n+m,0} \eta^{\mu\nu}. \tag{30}
\]
We introduce annihilation and creation operators by
\[
\alpha_\mu^n = \sqrt{n} a_\mu^n, \quad \alpha_\mu^+ = \sqrt{n} a_\mu^+,
\gamma_\mu^n = \sqrt{4nx} c_\mu^n, \quad \gamma_\mu^+ = \sqrt{4nx} c_\mu^+,
\tag{31}
\]
and find for $n \geq 1$ the standard form
\[
[a_\mu^n, a_\nu^m] = \delta_{nm} \eta^{\mu\nu},
\]
\[
[c_\mu^n, c_\nu^m] = \delta_{nm} \eta^{\mu\nu}. \tag{32}
\]
These expressions are formally the same as those found for a two–piece string\(^9\).
From Eq. (24) we get, when separating out the $n = 0$ term,
\[
H = -\frac{M^2}{\pi T_{II}} + \frac{1}{2} N \sum_{n=1}^{\infty} \omega_n \left( a_\mu^+ a_\mu + \tilde{a}_n^+ \tilde{a}_n + 2 c_\mu^+ c_\mu \right). \tag{33}
\]
Here $a_\mu^+ a_\mu \equiv a_\mu^+ a_\mu$, and $\omega_n = 2Nn$ as before. From the condition $H = 0$ we now get
\[
M^2 = \frac{1}{2} \pi NT_{II} \sum_{i=1}^{24} \sum_{n=1}^{\infty} \omega_n \left( a_{ni}^+ a_{ni} + \tilde{a}_{ni}^+ \tilde{a}_{ni} + 2 c_{ni}^+ c_{ni} - C \right), \tag{34}
\]
where we have put $D = 26$ and summed over the transverse 24 oscillator operators. Further, we have introduced a constant $C$ in order to account for ordering ambiguities.

5. Quantum Thermodynamics

The constraint for the closed string (fat circles at Fig. 2), expressing the invariance of the theory in the region I under shifts of the origin of the co-ordinate, has the form
The commutation relations for the above operators are given by Eq. (32). The mass of state (obtained by acting on the Fock vacuum $|0\rangle$ with creation operators) can be written as follows: $(\text{mass})^2 \sim a_{n_1}^\dagger \ldots a_{n_i}^\dagger c_{n_1}^\dagger \ldots c_{n_i}^\dagger |0\rangle$.

Let us start with the discussion of the free energy in field theory at non–zero temperature $T = \beta^{-1}$ (we put $k_B = 1$). As usual the physical Hilbert space consists of all Fock space states obeying the condition (35), which can be implemented by means of the integral representation for Kronecker deltas. Thus the free energy of the field content in the "proper time" representation becomes

$$F(\beta) = \mathcal{F}(\beta = \infty) - \pi(2\pi)^{-14} \int_0^{\infty} \frac{d\tau_4}{\tau_4^{14}} \left[ \theta_3 \left( 0, \frac{i\beta^2}{2\pi\tau_4} \right) - 1 \right] \text{Tr} \exp \left\{ -\frac{\tau_2 M^2}{2} \right\} \times \int_{-\pi}^{\pi} \frac{d\tau_1}{2\pi} \text{Tr} \exp \left\{ iN\tau_1 \sum_{i=1}^{24} \sum_{n=1}^{\infty} \omega_n \left[ a_{n_i}^\dagger a_{n_i} - \tilde{a}_{n_i}^\dagger \tilde{a}_{n_i} \right] \right\},$$

where $\mathcal{F}(\beta = \infty)$ is the temperature independent part of $F(\beta)$ (the Casimir energy), while the second term in (36) presents the temperature dependent part (the statistical sum). Once the free energy has been found, the other thermodynamic quantities can readily be calculated. For instance, the energy $U$ and the entropy $S$ of the system are $U = (\partial/\partial \beta)(\beta F(\beta))$, $S = \beta^2 (\partial/\partial \beta)(F(\beta))$.

6. The Critical Temperature
First we consider some mathematical results on the asymptotics of the level
degeneracy which leads to the asymptotics of the level state density. Let
\[
G(z) = \prod_{n=1}^{\infty} [1 - e^{-zn}]^{-a_n} = 1 + \sum_{n=1}^{\infty} \Xi(n)e^{-zn}, \tag{37}
\]
be the generating function, where Re\(z \rangle > 0\) and \(a_n\) are non-negative real numbers.
Let us consider the associated Dirichlet series
\[
D(s) = \sum_{n=1}^{\infty} a_n n^{-s}, \quad s = \sigma + it, \tag{38}
\]
which converges for \(0 < \sigma < p\). We assume that \(D(s)\) can be analytically continued
in the region \(\sigma \geq -C_0\) (\(0 < C_0 < 1\)) and here \(D(s)\) is analytic except for a pole
of order one at \(s = p\) with residue \(A\). Besides we assume that \(D(s) = O(|t|^{C_1})\)
uniformly in \(\sigma \geq -C_0\) as \(|t| \to \infty\), where \(C_1\) is a fixed positive real number. The
following lemma is useful with regard to the asymptotic properties of \(G(z)\)
at \(z = 0\):

\textbf{Lemma 1} If \(G(z)\) and \(D(s)\) satisfy the above assumptions and \(z = y + 2\pi ix\) then
\[
G(z) = \exp \{ A \Gamma(1 + p) \zeta_R(1 + p) z^{-p} - D(0) \log z + \frac{d}{ds} D(s)|_{s=0} + O(y^{C_0}) \} \tag{39}
\]
uniformly in \(x\) as \(y \to 0\), provided \(|\arg z| \leq \pi/2\) and \(|x| \leq 1/2\). Moreover there
exists a positive number \(\varepsilon\) such that
\[
G(z) = O(\exp \{ A \Gamma(1 + p) \zeta_R(1 + p) y^{-p} - C y^{-\varepsilon} \}), \tag{40}
\]
uniformly in \(x\) with \(y^\alpha \leq |x| \leq 1/2\) as \(y \to 0\), \(C\) being a fixed real number and
\(\alpha = 1 + p/2 - \nu/4, 0 < \nu < 2/3\).

The main result below follows from the Lemma and permits one to calculate the
complete asymptotics of \(\Xi(n)\).

\textbf{Theorem 1} (Meinardus) For \(n \to \infty\) one has
\[
\Xi(n) = C_p n^k \exp \left\{ \frac{1+p}{p} [A \Gamma(1 + p) \zeta_R(1 + p)]\frac{1}{1+p} n^\frac{p}{1+p} \right\} (1 + O(n^{-k_1})) , \tag{41}
\]
\[
C_p = [A \Gamma(1 + p) \zeta_R(1 + p)]^{1 - \frac{2p+1}{2(1+p)}} \frac{\exp \left( \frac{d}{ds} D(s)|_{s=0} \right)}{[2\pi(1 + p)]^{1/2}} , \tag{42}
\]
\[
k = \frac{2D(0) - p - 2}{2(1+p)} , \quad k_1 = \frac{p}{1+p} \min \left( \frac{C_0}{p} - \frac{\nu}{4}, \frac{1}{2} - \nu \right) . \tag{43}
\]
Coming back to composite string problem note that the generation function has the form

\[ W(\beta) = \text{Tr} \left[ e^{-\beta M^2} \right]_{\beta=0} = W^{(I)}(\beta)W^{(II)}(\beta) \]

\[ \equiv \prod_{n=1}^{\infty} \left[ 1 - e^{-n \beta Q(N)} \right]^{-48} \prod_{n=1}^{\infty} \left[ 1 - e^{-2n \beta Q(N)} \right]^{-24}, \quad (44) \]

where \( Q(N) = \pi T_{II} N^2 \). Some remarks are in order. Taking into account \( a_n = D-2 \) (or \( a_n = 2(D-2) \) in the case of region II), we have \( p = 1 \). In fact Eq. (38) gives the Riemann zeta function. Therefore, from the Meinardus theorem, Eqs. (41)–(43), it follows that

\[ \Xi^{(I)}(n) = C^{(I)}_1 n^{k^{(I)}} \exp \left\{ \pi \sqrt{\frac{4n \pi(D-2)}{3Q(N)}} \right\} (1 + O(n^{-k})) , \quad (45) \]

\[ \Xi^{(II)}(n) = C^{(II)}_1 n^{k^{(II)}} \exp \left\{ \pi \sqrt{\frac{2n(D-2)}{3Q(N)}} \right\} (1 + O(n^{-k})) . \quad (46) \]

Using the mass formula \( M^2 = n \) (for the sake of simplicity we assume a tension parameter, with dimensions of \((\text{mass})^{p+1}\), equal to 1) we find for the number of bosonic string states of mass \( M \) to \( M + dM \)

\[ \nu(M)dM \simeq 2C_1 M^{\frac{1-D}{2}} \exp(bM)dM , \quad b = \pi \sqrt{\frac{D-2}{3Q(N)}} . \quad (47) \]

One can show that the constant \( b \) is the inverse of the Hagedorn temperature.

For the closed bosonic string in the region I the constraint \( N_0 = \tilde{N}_0 \) should be taking into account, where \( N_0 \) is a number operator related to \( M^2 \). As a result in Eq. (45) the total degeneracy of the level \( n \) is simply the square of \( \Xi^{(I)}(n) \). Therefore the critical temperatures of composite string is given by

\[ \beta^{(I)}_c = \pi \sqrt{\frac{D-2}{3Q(N)}} = \frac{2}{\sqrt{N}} \sqrt{\frac{2\pi}{T_{II}}} , \quad (48) \]

\[ \beta^{(II)}_c = \pi \sqrt{\frac{D-2}{3Q(N)}} = \frac{2}{N} \sqrt{\frac{2\pi}{T_{II}}} , \quad (49) \]

and \( \beta^{(I)}_c = \beta^{(II)}_c = \beta_c \).

7. High Genera
The aim of this section is to consider thermodynamic properties of the composite string to arbitrary genus \( g \) associated with Riemann surface world–sheet \( \Sigma_g \). Such considerations allow us to identify the critical temperature at arbitrary loop order. It is well-known that the genus-\( g \) temperature contribution to the free energy for the bosonic string can be written as

\[
F_g(\beta) = \sum_{\mathbf{m}, \mathbf{n} \in \mathbb{Z}^{2g}/\{0\}} \int (d\tau)_{WP} \langle \det P^+P \rangle^{1/2}(\det \Delta_g)^{-13} e^{-\Delta S(\beta; \mathbf{m}, \mathbf{n})}, \tag{50}
\]

where \((d\tau)_{WP}\) is the Weil–Petersson measure on the Teichmüller space. This measure as well as the factors \(\det(P^+P)\) and \(\det \Delta_g\) are each individually modular invariant. In addition,

\[
I_g(\tau) = (\det P^+P)^{1/2}(\det \Delta_g)^{-13} = e^{(2g-2)} \left( \frac{d}{ds} Z(s) \right)_{s=1}^{-13} Z(2), \tag{51}
\]

where \(Z(s)\) is the Selberg zeta function and \(c\) an absolute constant. Furthermore, the winding–number factor has the form of a metric over the space of windings, namely

\[
\Delta S(\beta; \mathbf{m}, \mathbf{n}) = \pi NT_{II} \beta^2 [m_i \Omega_{ii} - n_i]((\Im \Omega)^{-1})_{ij} [\bar{\Omega}_{jk} m_k - n_j] = g^{\mu \nu}(\Omega) N_\mu N_\nu, \tag{52}
\]

where \(\mu, \nu = 1, 2, \ldots, 2g\). The periodic matrix \(\Omega\), corresponding to the string world–sheet of genus \( g \), is a holomorphic function of the moduli, \(\Omega_{ij} = \Omega_{ji}\) and \(3\Omega > 0\). The matrix \(\Omega\) admits a decomposition into real symmetric \(g \otimes g\) matrices: \(\Omega = \Omega_1 + i \Omega_2\). As a result

\[
g(\Omega_1 + i \Omega_2) = \left( \begin{array}{cc} \Omega_1 \Omega_2^{-1} \Omega_1 + \Omega_2^{-1} \Omega_1 & -\Omega_1 \Omega_2^{-1} \\ -\Omega_2^{-1} \Omega_1 & \Omega_2 \end{array} \right). \tag{53}
\]

Besides, \(g(\Omega) = \hat{\Lambda}(\Lambda(\Omega)) \hat{\Lambda} \), where \(\Lambda\) is an element of the symplectic modular group \(Sp(2g, \mathbb{Z})\) and the associated tranformation of the periodic matrix reads \(\Omega \mapsto \Omega' = \Lambda(\Omega) = (A + B)(C\Omega + D)^{-1}\). As a consequence, the winding factor

\[
\sum_{\mathbf{m}, \mathbf{n}} \exp \left[ -\Delta S(\beta; \mathbf{m}, \mathbf{n}) \right]
\]

is also modular invariant. It can be shown that the 2\(g\) summations present in the expression for \(F_g(\beta)\) can be replaced by a single summation together with a change in the region of integration from the fundamental domain to the analogue of the strip \(S_{a_1}\) related to the cycle \(a_1\), whose choice is entirely arbitrary. Then, one has

\[
F_g(\beta) = \sum_{r=1}^{\infty} \int (d\tau)_{WP} I_g(\tau) \exp \left\{ -\frac{\pi}{2} NT_{II} \beta^2 \tau^2 (\Omega_{1i}((\Im \Omega)^{-1})_{ij} \bar{\Omega}_{j1}) \right\}. \tag{54}
\]
To make use of the Mellin transform the genus–g free energy can be present in the form
\[
F_g(\beta) = \frac{1}{2\pi i} \int_{C, s = s_0} ds \Gamma(s) \zeta(2s) \left( \frac{\pi}{2} NT_{II} \beta^2 \right)^{-s} \times \left\{ \int (d\tau) WP I_g(\tau)[\Omega_1((3\Omega)^{-1})_{ij}\bar{\Omega}_j]^{-s} \right\}_{\text{(Reg)}}.
\]
In order to deal with Eq. (55) the integrals on a suitable variable in \((d\tau)_{WP}\) should be understood as the regularized ones. In this way the order of integration may be interchanged.

The critical behaviours of closed and open strings of the composite model coincide (at least at level \(g = 1\)). Let us consider, for example, the open string genus-g contribution to the free energy. The matrix \(\Omega\) may be chosen as \(\Omega = \text{diag}(\Omega_2, \Omega_2^{-1})\).

In the limit \(\Omega_2 \to 0\), one has
\[
\exp\left\{ -\frac{\pi}{2} NT_{II} \beta^2 (N'\Omega N')^{-s} \right\} \longrightarrow \exp\left\{ -\frac{\pi}{2} NT_{II} \beta^2 \Omega_2^{-1} N'\Omega N' \right\},
\]
and
\[
\left( \sum_{N \in \mathbb{Z}^g/\{0\}} (N'\Omega N')^{-s} \right) \longrightarrow \Omega_2^s \sum_{N \in \mathbb{Z}^g/\{0\}} (N'\Omega N')^{-s} = \Omega_2 Z_{g|0}^0(2s),
\]
where the Epstein zeta function of order \(g\) is defined by
\[
Z_g^{h|0}(s) = \sum_{N \in \mathbb{Z}^g/\{0\}} [(n_1 + b_1)^2 + ... + (n_g + b_g)^2]^{-s/2} \exp[2\pi i(N', h)].
\]
The corresponding contribution is given by
\[
\frac{1}{2\pi i} \int_{C, s = s_0} ds \Gamma(s) \left( \frac{\pi}{2} NT_{II} \beta^2 \right)^{-s} Z_{g|0}^{0}(2s) \left\{ \int (d\tau)_{WP} \Omega_2^s I_g(\tau) \right\}_{\text{(Reg)}}.
\]
Since a tachyon is present in the spectrum, the total free energy will be divergent, for any \(g\). The infrared divergence may be regularized by means of a suitable cutoff parameter. This divergence could be associated with pinching a cycle non homologous at zero. The behavior of the factor \((d\tau)_{WP} I_g(\tau)\) is given by the Belavin–Knizhnik double–pole result and has a universal character for any \(g\). It should also be noticed that this divergence is \(\beta–\)independent and the meromorphic structure is similar to genus–one case. As a consequence, the whole genus dependence of the critical temperature is encoded in the Epstein zeta function \(Z_{g|0}^{0}(2s)\) (see for details Ref. [4]).
For this reason, we mention the asymptotic properties of function $Z_g(0,2s)$. The following result holds:

**Corollary 1** (Ref. [12])

$$
B_g \equiv \lim_{s \rightarrow +\infty} \frac{Z_g(0,b_1,2s+2)}{Z_g(0,2s)} = [(\hat{b}_1 - \eta_1)^2 + \ldots + (\hat{b}_g - \eta_g)^2]^{-1}, \quad (60)
$$

where at least one of the $b_i$ is noninteger, $\hat{b}_i = b_i - [b_i]$ with $[b_i]$ the noninteger (decimal) part of $b_i$ and

$$
\eta_i = \begin{cases} 
0, & 0 \leq \hat{b}_i \leq 1/2, \\
1, & 1/2 \leq \hat{b}_i < 1. 
\end{cases} \quad (61)
$$

Furthermore, if $b = (0,0,\ldots,0)$, then $B_g = 1$.

As a consequence, the interactions of bosonic strings do not modify the critical temperature. However one can consider different linear real bundles over compact Riemann surfaces and spinorial structures on them. The procedure of evaluation of the free energy in terms of the path integral over the metrics does not depend on whatever type of real scalars are considered. This fact leads to new contributions to the genus–g integral (50).

One could investigate the role of these contributions for the torus compactification [17,18,2]. In this case, the sum in Eq. (50) should be taken over the vectors on the lattice on which some space dimensions are compactified. The half–lattice vectors can be labelled by the multiplets $(b_1,\ldots,b_p)$, with $b_i = 1/2$. The critical temperature related to the multiplet $b = (b_1,\ldots,b_p,0,\ldots,0)$ can be easily evaluated by means of Eq. (60), which gives $B_p = 4p^{-1}$. As a result $\beta_{c,p} = (2/\sqrt{p}) \beta_c$. As an example, we note the particular multiplets $(0,\ldots,1/2,\ldots,0)$ and $(0,\ldots,1/2,1/2,\ldots,0)$, where only one $b_i$ and two $b_i$ are different from zero. In this case we have “minimal” critical temperatures given by $\beta_{c,1}^{-1} = \beta_c^{-1}/2$ and $\beta_{c,2}^{-1} = \beta_c^{-1}/\sqrt{2}$ respectively.

8. **Concluding Remarks**

Making use of the Meinardus theorem in Section 6, we found the critical temperatures of the two kinds of pieces in the string (I and II), to be equal, and to be given by Eqs. (48) and (49) for arbitrary spacetime dimension $D$. The calculation generalizes earlier calculations, of the Casimir energy of the $2N$– piece string in Ref. [1], and of thermodynamics of the 2–piece string in Ref. [2]. Interactions of bosonic parts of a piecewise uniform string do not modify the critical temperatures. However, for the sectors of parts having a spinor structure, the critical temperatures, associated with genus $g$, depend on the windings.

9. **Acknowledgements**
The research of A.A. Bytsenko has been supported in part by the Russian Foundation for Basic Research (grant No. 01–02–17157).

10. References

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