Abstract

We obtain the full hamiltonian structure for a parametric coupled KdV system. The coupled system arises from four different real basic lagrangians. The associated hamiltonian functionals and the corresponding Poisson structures follow from the geometry of a constrained phase space by using the Dirac approach for constrained systems. The overall algebraic structure for the system is given in terms of two pencils of Poisson structures with associated hamiltonians depending on the parameter of the Poisson pencils. The algebraic construction we present admits the most general space of observables related to the coupled system.

Keywords: partial differential equations, integrable systems, Lagrangian and Hamiltonian approach.

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1 Introduction

Coupled Korteweg-de Vries (KdV) systems describes several physical interactions of interest. Hirota and Satsuma \[1\] proposed a model that describes interactions of two long waves with different dispersion relations. Gear and Grimshaw \[2\] considered a coupled KdV system to describe linearly stable internal waves in a stratified fluid.
More recently Lou, Tong, Hu and Tang \[3\] proposed models which may be used in the description of atmospheric and oceanic phenomena. Coupled KdV systems were also analyzed in \[4, 5, 6\]. An important area of interest of high energy physics related to coupled systems is provided by the supersymmetric extensions of KdV equations \[7, 8, 9, 10, 11, 12, 13\] and more generally by operator and Clifford valued extensions of KdV equation \[14, 15\].

In this work we consider a parametric coupled KdV system. For some values of the parameter, $\lambda < 0$, the system corresponds to the complexification of KdV equation. For $\lambda = 0$ the system corresponds to one of the Hirota-Satsuma coupled KdV systems, while for $\lambda > 0$ the system is equivalent to two decoupled KdV equations. We analyze the hamiltonian formulation and the associated Poisson bracket structure of the system. Although some properties of the complexification of KdV arise directly from the analogous ones on the solutions of the KdV equation there are new properties, in particular, the full hamiltonian structure, which does not have an analogous on the original real equation. In fact, the complexification approach gives rise only to holomorphic observables on phase space. The full hamiltonian structure of the complex system give rise to self-adjoint hamiltonian functionals, whose hamiltonian flow are the complex KdV equations, and it provides the full structure of observables on phase space, not only the holomorphic ones.

The approach we will follow in our analysis is to construct a family of lagrangians from which the coupled KdV system is obtained by taking independent variations with respect to the fields defining the lagrangian functional. It turns out that these lagrangians are singular ones. This implies that the hamiltonian construction, via a Legendre transformation is formulated on a constrained phase space. In all the cases we will consider the constraints turn out to be primary constraints and of the second class. The unconstrained phase space is equipped with a Poisson bracket structure, however since there are second class constraints we must obtain the Poisson bracket structure on the constrained sub-manifold of the phase space. This Poisson bracket is provided by the Dirac brackets \[16\]. It satisfies all the properties of a Poisson bracket, in particular the Jacobi identity. In this way starting from a lagrangian for the system we can construct a Poisson bracket structure, together with a hamiltonian functional. This approach was followed for the KdV equation in \[17, 18\]. It provides a geometrical picture on phase space of the hamiltonian structure of the integrable system. The other way to proceed is to find a hamiltonian operator together with a hamiltonian functional. Afterwards we may construct a Poisson bracket structure provided the hamiltonian operator satisfies a differential restriction \[19\] ensuring that the Jacobi identity is satisfied. In this approach the set of allowed observables is only a subset of the space of observables of the more general formulation in terms of the constrained phase space approach.

We will obtain a pencil of Poisson bracket structures each of them associated to a hamiltonian functional. In particular this implies compatibility between some of the Poisson structures.
2 The parametric coupled KdV system

We consider a coupled Korteweg-de Vries (KdV) system, formulated in terms of two real differentiable functions \( u(x, t) \) and \( v(x, t) \), given by the following partial differential equations:

\[
\begin{align*}
    u_t + uu_x + u_{xxx} + \lambda vv_x &= 0 \quad (1) \\
    v_t + u_x v + v_x u + v_{xxx} &= 0 \quad (2)
\end{align*}
\]

where \( \lambda \) is a real parameter.

Here and in the sequel \( u \) and \( v \) belong to the real Schwartz space defined by

\[
C^\infty_c = \left\{ w \in C^\infty(\mathbb{R})/ \lim_{x \to \pm\infty} x^p \frac{\partial^q}{\partial x^q} w = 0; p, q \geq 0 \right\}.
\]

By a redefinition of \( v \) given by \( v \to \frac{v}{\sqrt{\lambda}} \) we may reduce the values of \( \lambda > 0 \) to be \( +1 \) and \( \lambda < 0 \) to be \( -1 \). The systems for \( \lambda = +1, \lambda = -1 \) and \( \lambda = 0 \) are not equivalent. The \( \lambda = -1 \) case corresponds to the complexification of KdV equation.

The case \( \lambda = +1 \) corresponds to two decoupled KdV equations.

The system (1), (2) for \( \lambda = -1 \) describes a two-layer liquid model studied in references [2, 3, 20]. It is a very interesting evolution system. It is known to have solutions developing singularities on a finite time [21]. Also, a class of solitonic solutions was reported in [22] via the Hirota approach [23].

The system (1), (2) for \( \lambda = 0 \) corresponds to the ninth Hirota-Satsuma [1] coupled KdV system given in [5] (for the particular value of \( k = 0 \)) (see also [4]) and is also included in the interesting study which relates integrable hierarchies with polynomial Lie algebras [6].

(1), (2) is equivalent to the \( Z_\lambda \)-KdV equation introduced in [24]. It was also considered from a different point of view in [25].

A Bäcklund transformation, the permutability theorem, the Gardner transformation as well as the Gardner equations for the coupled KdV system (1), (2), were obtained in [26]. Also a class of multisolitonic solutions and a class of periodic solutions were found in [26].

3 Poisson structures

In this and in the following section we will show that there exists four basic hamiltonians and four associated basic Poisson structures for the coupled KdV system we are considering. We will use the method of Dirac for constrained systems to deduce them. The hamiltonian as defined in quantum physics must be a selfadjoint operator conjugate to the time, hence our four hamiltonians will be four real functionals in terms of the
real fields \( w(x, t) \) and \( y(x, t) \). We start our construction by considering the lagrangian 
\[
L_1 = \int_0^T dt \int_{-\infty}^{+\infty} dx \mathcal{L}_1, \\
\mathcal{L}_1 = -\frac{1}{2} w_xw_t - \frac{1}{6} w_x^3 + \frac{1}{2} w_{xx}^2 - \frac{\lambda}{2} w_x y_x^2 - \frac{\lambda}{2} y_{xx}^2 + \frac{\lambda}{2} y_{tt}^2 
\]
for \( \lambda \neq 0 \), where
\[
u(x, t) = w_x(x, t) \\
v(x, t) = y_x(x, t).
\]

By taking independent variations of \( L_1 \) with respect to \( w \) and to \( y \) we obtain the field equations
\[
\frac{\delta L_1}{\delta w} = 0, \quad \frac{\delta L_1}{\delta y} = 0,
\]
which are the same as equations (1),(2).

We now introduce a second lagrangian \( L_2 = \int_0^T dt \int_{-\infty}^{+\infty} dx \mathcal{L}_2 \) where
\[
\mathcal{L}_2 = -\frac{1}{2} w_x y_t - \frac{1}{2} w_t y_x - \frac{1}{2} w_x^2 y_x - y_x w_{xxx} - \frac{\lambda}{6} y_x^3
\]
for any \( \lambda \).

By taking independent variations of \( L_2 \) with respect to \( w \) and \( y \) we obtain the same field equations.

We will now construct the hamiltonian structure associated to each of these lagrangians. We start by considering the lagrangian \( L_1 \). We introduce the conjugate momenta associated to \( w \) and \( y \), we denote them \( p \) and \( q \) respectively, we have
\[
p = \frac{\delta \mathcal{L}_1}{\delta w_t} = -\frac{1}{2} w_x, \quad q = \frac{\delta \mathcal{L}_1}{\delta y_t} = -\frac{\lambda}{2} y_x.
\]

We define
\[
\phi_1 \equiv p + \frac{1}{2} w_x, \quad \phi_2 = q + \frac{\lambda}{2} y_x.
\]
\( \phi_1 \) and \( \phi_2 \) do not have any \( w_t \) nor any \( y_t \) dependence, hence \( \phi_1 = \phi_2 = 0 \), they are constraints on the phase space. It turns out that these are the only constraints on the phase space. They are second class contraints.

The hamiltonian may be obtain directly from \( \mathcal{L}_1 \) by performing a Legendre transformation,
\[
\mathcal{H}_1 = pw_t + qy_t - \mathcal{L}_1.
\]
We obtain
\[
\mathcal{H}_1 = \frac{1}{6} w_x^3 - \frac{1}{2} w_{xx}^2 + \frac{\lambda}{2} w_x y_x^2 - \frac{\lambda}{2} y_{xx}^2
\]
and the corresponding Hamiltonian $H_1 = \int_{-\infty}^{\infty} dx \mathcal{H}_1$. We introduce a Poisson structure on the phase space defined by

$$\{w(x), p(\hat{x})\}_{PB} = \delta(x - \hat{x})$$
$$\{y(x), q(\hat{x})\}_{PB} = \delta(x - \hat{x})$$

with all other brackets between these variables being zero.

Since we have a constrained phase space we must introduce the Dirac brackets corresponding to a Lie bracket structure on the constrained submanifold of phase space. The Dirac brackets between two functionals $F$ and $G$ on phase space is defined as

$$\{F, G\}_{DB} = \{F, G\}_{PB} - \langle\langle \{F, \phi_i(x')\}_{PB} \{\phi_j(x''), G\}_{PB} \rangle\rangle_{x', x''}$$

where $\langle\rangle_{x'}$ denotes integration on $x'$ from $-\infty$ to $+\infty$. The indices $i, j = 1, 2$ and the $C_{ij}(x', x'')$ are the components of the inverse of the matrix whose components are $\{\phi_i(x'), \phi_j(x'')\}_{PB}$.

This matrix becomes

$$\begin{bmatrix}
\partial_x \delta(x' - x'') & 0 \\
0 & \lambda \partial_x \delta(x' - x'')
\end{bmatrix}$$

and its inverse, satisfying

$$\langle\left\langle \begin{bmatrix}
\partial_x \delta(x - x'') & 0 \\
0 & -\partial_x \delta(x - x'')
\end{bmatrix} \begin{bmatrix}
C_{11}(x'', \hat{x}) & C_{12}(x'', \hat{x}) \\
C_{21}(x'', \hat{x}) & C_{22}(x'', \hat{x})
\end{bmatrix} \right\rangle_{x''}\rangle = \begin{bmatrix}
\delta(x - \hat{x}) & 0 \\
0 & \delta(x - \hat{x})
\end{bmatrix}$$

is given by

$$[C_{ij}(x', x'')] = \begin{bmatrix}
\int_{x'}^{x''} \delta(s - x'') ds & 0 \\
0 & \frac{1}{\lambda} \int_{x'}^{x''} \delta(s - x'') ds
\end{bmatrix}.$$ 

It turns out, after some calculations, that

$$\{u(x), u(\hat{x})\}_{DB} = -\partial_x \delta(x - \hat{x}), \quad \{v(x), v(\hat{x})\}_{DB} = -\frac{1}{\lambda} \partial_x \delta(x - \hat{x})$$
$$\{u(x), v(\hat{x})\}_{DB} = 0.$$ 

We notice that this Poisson bracket is not well defined for $\lambda = 0$. We have already assumed $\lambda \neq 0$.

From them we obtain the Hamilton equations, which are of course the same as (1),(2):

$$u_t = \{u, H_1\}_{DB} = -uu_x - u_{xxx} - \lambda vv_x$$
$$v_t = \{v, H_1\}_{DB} = -uv_x - v_{xxx}.$$
Moreover, we may obtain directly the Dirac bracket of any two functionals \( F(u, v) \) and \( G(u, v) \) from (3) using the above bracket relations for \( u \) and \( v \). We notice that the observables \( F \) and \( G \) in (3) may be functionals of \( w, y, p \) and \( q \), not only of \( u \) and \( v \). In this sense the phase space approach for singular lagrangians provides the most general space of observables. The same comment will be valid for the phase space construction using lagrangians \( L_2 \) and \( L^M_2 \) in the following sections.

We now consider the lagrangian \( L_2 \) and its associated hamiltonian structure. In this case we denote the conjugate momenta to \( w \) and \( y \) by \( \hat{p} \) and \( \hat{q} \) respectively. We have

\[
\hat{p} = -\frac{1}{2} y_x, \quad \hat{q} = -\frac{1}{2} w_x.
\]

The constraints become in this case

\[
\hat{\phi}_1 = \hat{p} + \frac{1}{2} y_x = 0, \quad \hat{\phi}_2 = \hat{q} + \frac{1}{2} w_x = 0.
\]

The corresponding Poisson brackets between \( \phi_i \) and \( \phi_j, i, j = 1, 2 \), are given by

\[
\{ \hat{\phi}_1(x), \hat{\phi}_1(x') \}_{PB} = 0, \quad \{ \hat{\phi}_2(x), \hat{\phi}_2(x') \}_{PB} = 0,
\]

\[
\{ \hat{\phi}_1(x), \hat{\phi}_2(x') \}_{PB} = \partial_x \delta(x - x').
\]

The corresponding construction of the Dirac brackets yields

\[
\{ u(x), u(\hat{x}) \}_{DB} = 0, \quad \{ v(x), v(\hat{x}) \}_{DB} = 0
\]

\[
\{ u(x), v(\hat{x}) \}_{DB} = -\partial_x \delta(x - \hat{x}).
\]

The hamiltonian \( H_2 = \int_{-\infty}^{+\infty} dx \, \mathcal{H}_2 \) is given by the hamiltonian density

\[
\mathcal{H}_2 = \frac{1}{2} w_x^2 y_x + y_x w_{xxx} + \frac{\lambda}{6} y_x^3.
\]

The Hamilton equations

\[
u_t(x) = \{ u(x), H_2 \}_{DB}, \quad v_t(x) = \{ v(x), H_2 \}_{DB}
\]

now using the corresponding Dirac brackets yields the same fields equations (1),(2) for any \( \lambda \). We have thus constructed two hamiltonian functionals and associated Poisson bracket structures. These two hamiltonian structures arise directly from the basic lagrangians \( L_1 \) and \( L_2 \). We will now construct two additional hamiltonian structures by considering the Miura transformation.

The hamiltonians \( H_1 \) and \( H_2, H^M_1 \) and \( H^M_2 \) in the following section, were presented in [24].
4 The Miura transformation

We consider the Miura transformation

\[ u = \mu_x - \frac{1}{3} \mu^2 - \frac{\lambda}{6} \nu^2 \]
\[ v = \nu_x - \frac{1}{3} \mu \nu. \]  

(5)

The corresponding modified KdV system (MKdVS) is given by

\[ \mu_t + \mu_{xxx} - \frac{1}{6} \mu^2 \mu_x - \frac{\lambda}{3} \mu \nu \nu_x = 0 \]
\[ \nu_t + \nu_{xxx} - \frac{1}{6} \nu^2 \nu_x - \frac{\mu}{3} \mu \nu \mu_x = 0. \]  

(6)

These equations may be obtained from two lagrangians, which we will denote \( L_M^1 = \int_T^0 dt \int_{-\infty}^{+\infty} dx \mathcal{L}_1^M \) and \( L_M^2 = \int_T^0 dt \int_{-\infty}^{+\infty} dx \mathcal{L}_2^M \).

The lagrangian densities \( \mathcal{L}_1^M \), formulated for \( \lambda \neq 0 \), and \( \mathcal{L}_2^M \), formulated for any \( \lambda \), expressed in terms of \( \sigma, \rho \) where \( \mu = \sigma_x, \nu = \rho_x \) are given by

\[ \mathcal{L}_1^M = -\frac{1}{2} \sigma_t \sigma_x - \frac{\lambda}{2} \rho_t \rho_x - \frac{1}{2} \sigma_x \sigma_{xxx} - \frac{\lambda}{2} \rho_x \rho_{xxx} + \frac{1}{72} \sigma_x^4 - \frac{\lambda^2}{72} \rho_x^4 + \frac{\lambda}{12} \rho_x^2 \sigma_x^2 \]  

(7)

and

\[ \mathcal{L}_2^M = -\frac{1}{2} \sigma_t \rho_x - \frac{1}{2} \sigma_x \rho_t - \sigma_{xxx} \rho_x + \frac{1}{18} \sigma_x^3 \rho_x + \frac{\lambda}{18} \rho_x^3 \sigma_x \]  

(8)

respectively.

We will now construct the hamiltonian structure associated to \( \mathcal{L}_1^M \).

We denote by \( \alpha \) and \( \beta \) the conjugate momenta associated to \( \sigma \) and \( \rho \) respectively. We have

\[ \alpha = \frac{\delta \mathcal{L}_1^M}{\delta \sigma_t} = -\frac{1}{2} \sigma_x, \quad \beta = \frac{\delta \mathcal{L}_1^M}{\delta \rho_t} = -\frac{\lambda}{2} \rho_x. \]

These are constraints on the phase space.

The hamiltonian \( H_1^M \) corresponding to this lagrangian density \( \mathcal{L}_1^M \) is given by

\[ H_1^M = v^2 - u^2 \]
\[ H_1^M = \int_{-\infty}^{+\infty} H_1^M dx \]

where \( u \) and \( v \) are given in terms of \( \mu \) and \( \nu \) by the Miura transformation.

The construction of the Dirac brackets follows in the usual way. We end up with the following Poisson structure on the constrained submanifold,

\[ \{ \mu(x), \mu(\hat{x}) \}_{DB} = -\partial_x \delta(x, \hat{x}) \]
\[ \{ \nu(x), \nu(\hat{x}) \}_{DB} = -\frac{1}{\lambda} \partial_x \delta(x, \hat{x}) \]
\[ \{ \mu(x), \nu(\hat{x}) \}_{DB} = 0. \]
From these Poisson bracket structure we obtain for the original $u$ and $v$ fields

\begin{align*}
\{u(x), u(\hat{x})\}_{DB} &= \partial_{xxx} \delta(x, \hat{x}) + \frac{1}{3} u_x \delta(x, \hat{x}) + \frac{2}{3} u \partial_x \delta(x, \hat{x}) \\
\{v(x), v(\hat{x})\}_{DB} &= \frac{1}{\lambda} \partial_{xxx} \delta(x, \hat{x}) + \frac{1}{3\lambda} u_x \delta(x, \hat{x}) + \frac{2}{3\lambda} u \partial_x \delta(x, \hat{x}) \\
\{u(x), v(\hat{x})\}_{DB} &= \frac{1}{3} v_x \delta(x, \hat{x}) + \frac{2}{3} v \partial_x \delta(x, \hat{x})
\end{align*}

which defines the Poisson structure on the original fields inherited from the Poisson structure on the constrained submanifold on the phase space associated to the modified KdV system. This Poisson bracket is not well defined for $\lambda \neq 0$. We have already assumed $\lambda \neq 0$.

From the Dirac brackets of $u$ and $v$ we may obtain directly the hamiltonian field equations

\begin{align*}
u_t = \{u, H^M_1\}_{DB} &= -u u_x - u_{xxx} - \lambda v v_x \\
v_t = \{v, H^M_1\}_{DB} &= -v_{xxx} - (uv)_x
\end{align*} 

(9)

which, as it should be, coincide with system (1),(2).

We have then obtained the Poisson structure associated to the hamiltonian $H^M_1$.

We now proceed to obtain a second Poisson structure starting from the Lagrangian $\mathcal{L}^M_2$.

The hamiltonian obtained via a Legendre transformation is given by $H^M_2 = \int_{-\infty}^{+\infty} (-uv) \, dx$ where $u$ and $v$ are functions of $\mu$ and $\nu$ according to the Miura transformation. We use as before $\mu = \sigma_x, \nu = \rho_x$.

We denote by $\hat{\alpha}$ and $\hat{\beta}$ the conjugate momenta associated to $\sigma$ and $\rho$ respectively.

The constraints on phase space become now

\begin{align*}
\hat{\alpha} &= -\frac{1}{2} \rho_x \\
\hat{\beta} &= -\frac{1}{2} \sigma_x.
\end{align*}

The Dirac brackets are

\begin{align*}
\{\mu(x), \mu(\hat{x})\}_{DB} &= 0 \\
\{\nu(x), \nu(\hat{x})\}_{DB} &= 0 \\
\{\mu(x), \nu(\hat{x})\}_{DB} &= -\partial_x \delta(x, \hat{x}).
\end{align*}

We then obtain, for any $\lambda$,

\begin{align*}
\{u(x), u(\hat{x})\}_{DB} &= \frac{\lambda}{3} v_x \delta(x, \hat{x}) + \frac{2\lambda}{3} v \partial_x \delta(x, \hat{x}) \\
\{v(x), v(\hat{x})\}_{DB} &= \frac{1}{3} v_x \delta(x, \hat{x}) + \frac{2}{3} v \partial_x \delta(x, \hat{x}) \\
\{u(x), v(\hat{x})\}_{DB} &= \partial_{xxx} \delta(x, \hat{x}) + \frac{1}{3} u_x \delta(x, \hat{x}) + \frac{2}{3} u \partial_x \delta(x, \hat{x}).
\end{align*}
This is the Poisson bracket structure inherited from the second Poisson structure on the modified phase space. One may directly verify that the corresponding Hamilton equations exactly coincide with equations (1),(2). We have then constructed four basic lagrangians and associated hamiltonian functionals together with four basic Poisson structures.

5 Two pencils of Poisson structures for the coupled system

We now construct a parametric lagrangian density $L_k$, where $k$ is a real parameter, associated to the two basic lagrangians $L_1$ and $L_2$.

We define the lagrangian density

$$L_k = kL_1 + (1-k)L_2.$$  

The field equations obtained from this lagrangian density are equivalent to (1) and (2) in the following cases: If $\lambda < 0$ for any $k$. If $\lambda = 0$, for $k \neq 1$. If $\lambda > 0$ for $k \neq \frac{1}{1+\sqrt{\lambda}}$ and $k \neq \frac{1}{1-\sqrt{\lambda}}$. From now on we will excluded this particular values of $k$. The corresponding hamiltonian density is given by

$$L_k = pw_t + qy_t - L_k = k\mathcal{H}_1 + (1-k)\mathcal{H}_2$$

and the primary constraints by

$$\phi_1 \equiv \frac{k}{2}w_x + \frac{(1-k)}{2}y_x + p = 0 \quad (10)$$

$$\phi_2 \equiv \frac{\lambda k}{2}y_x + \frac{(1-k)}{2}w_x + q = 0. \quad (11)$$

These are the only constraints on phase space, they are second class ones. The Poisson brackets on the unconstrained phase space are

$$\{\phi_1(x),\phi_1(\hat{x})\}_{PB} = k\partial_x\delta(x,\hat{x})$$

$$\{\phi_2(x),\phi_2(\hat{x})\}_{PB} = \lambda k\partial_x\delta(x,\hat{x})$$

$$\{\phi_1(x),\phi_2(\hat{x})\}_{PB} = (1-k)\partial_x\delta(x,\hat{x}).$$

We will denote by $\{\}_{DB}$ the Dirac bracket corresponding to the parameter $k$. The Dirac brackets are then given by

$$\{u(x),u(\hat{x})\}_{DB}^k = \frac{\lambda k}{-\lambda k^2 + (1-k)^2}\partial_x\delta(x,\hat{x})$$

$$\{v(x),v(\hat{x})\}_{DB}^k = \frac{k}{-\lambda k^2 + (1-k)^2}\partial_x\delta(x,\hat{x})$$

$$\{u(x),v(\hat{x})\}_{DB}^k = \frac{1-k}{-\lambda k^2 + (1-k)^2}(-\partial_x\delta(x,\hat{x})).$$
where the denominator is different from zero for the values of $k$ we are considering. They define the Poisson structure for the hamiltonian $H_k = \int_{-\infty}^{+\infty} \mathcal{H}_k \, dx$.

The associated Hamilton equations coincide with the coupled equations (1),(2). It is interesting to notice that the above Poisson structure is a linear combination of the Dirac brackets introduced associated to hamiltonians $H_1$ and $H_2$. In the present notation $H_2$ corresponds to $k = 0$.

We then have

$$\{F, G\}_k^{DB} = -\frac{\lambda k}{-\lambda k^2 + (1 - k)^2} \{F, G\}_1^{DB} + \frac{1 - k}{-\lambda k^2 + (1 - k)^2} \{F, G\}_0^{DB}$$

where $F, G$ are any functionals of $u$ and $v$. In particular for any $\lambda$ different from one and zero, and $k = \frac{1}{1 - \lambda}$, we obtain

$$\{F, G\}_k^{DB} = \{F, G\}_1^{DB} + \{F, G\}_0^{DB}.$$ 

Consequently, the two basic Poisson brackets for every $\lambda \neq 0, 1$ are then compatible.

We also notice that for any $k$ and $\lambda = -1$, using the above Poisson bracket structure, one gets

$$\{u(x) + i v(x), u(\hat{x}) - i v(\hat{x})\}_k^{DB} = 0,$$

$$\{u(x) + i v(x), u(\hat{x}) + i v(\hat{x})\}_k^{DB} = -\frac{2}{k^2 + (1 - k)^2} \partial_x \delta(x, \hat{x}).$$

We emphasize that only (13) arises from the complexification of the corresponding Poisson structure for real KdV. The relation (12) follows in our approach from first principles. It is not imposed by hand. The existence of a local real hamiltonian $H_k$ for each $k$ is a non-trivial feature of the system (1),(2) and is not an algebraic consequence of the complexification of the real KdV equation.

We may now consider the case $\lambda = 0$. The Poisson bracket for any $k \neq 1$ becomes

$$\{F, G\}_k^{DB} = \frac{k}{2(1 - k)^2} \{F, G\}_1^{DB} + \frac{1 - 2k}{(1 - k)^2} \{F, G\}_0^{DB}$$

in particular for $k = \frac{5}{2}$ the two coefficients are equal, hence the Poisson brackets for $k = \frac{1}{2}$ and $k = 0$ are compatible.

We have thus constructed a pencil of Poisson structures, each of them with an associated local real hamiltonian $H_k = \int_{-\infty}^{+\infty} \mathcal{H}_k$.

We now construct, as we have already done with $\mathcal{L}_1$ and $\mathcal{L}_2$, a parametric lagrangian density $\mathcal{L}_k^M = k \mathcal{L}_1^M + (1 - k) \mathcal{L}_2^M$. The associated hamiltonian density is given by $\mathcal{H}_k^M = k \mathcal{H}_1^M + (1 - k) \mathcal{H}_2^M$ in terms of the other two basic lagrangian densities. The constraints on phase space are given by

$$\phi_1 \equiv \alpha + \frac{k}{2} \sigma_x + \frac{(1 - k)}{2} \rho_x = 0$$

$$\phi_2 \equiv \beta \frac{\lambda k}{2} \rho_x + \frac{(1 - k)}{2} \sigma_x = 0$$

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which are second class constraints and the only constraints on the phase space. The Poisson brackets on the unconstrained phase space are

\[
\{\phi_1(x), \phi_1(\hat{x})\}_{PB} = k \partial_x \delta(x, \hat{x}) \\
\{\phi_2(x), \phi_2(\hat{x})\}_{PB} = \lambda k \partial_x \delta(x, \hat{x}) \\
\{\phi_1(x), \phi_2(\hat{x})\}_{PB} = (1 - k) \partial_x \delta(x, \hat{x}).
\]

and the Dirac brackets are then given by

\[
\{u(x), u(\hat{x})\}_DB^k = -\frac{\lambda k}{-\lambda k^2 + (1 - k)^2} \left( \partial_{xxx} \delta(x, \hat{x}) + \frac{1}{3} u_x \delta(x, \hat{x}) + \frac{2}{3} u \partial_x \delta(x, \hat{x}) \right) - \\
-\frac{1 - k}{-\lambda k^2 + (1 - k)^2} \left( \frac{1}{3} u_x \delta(x, \hat{x}) + \frac{2}{3} v \partial_x \delta(x, \hat{x}) \right)
\]

\[
\{u(x), v(\hat{x})\}_DB^k = -\frac{\lambda k}{-\lambda k^2 + (1 - k)^2} \left( \partial_{xxx} \delta(x, \hat{x}) + \frac{1}{3} u_x \delta(x, \hat{x}) + \frac{2}{3} u \partial_x \delta(x, \hat{x}) \right) + \\
-\frac{1 - k}{-\lambda k^2 + (1 - k)^2} \left( \frac{1}{3} v_x \delta(x, \hat{x}) + \frac{2}{3} v \partial_x \delta(x, \hat{x}) \right)
\]

\[
\{v(x), v(\hat{x})\}_DB^k = -\frac{\lambda k}{-\lambda k^2 + (1 - k)^2} \left( \partial_{xxx} \delta(x, \hat{x}) + \frac{1}{3} u_x \delta(x, \hat{x}) + \frac{2}{3} u \partial_x \delta(x, \hat{x}) \right) + \\
+\frac{1 - k}{-\lambda k^2 + (1 - k)^2} \left( \frac{1}{3} v_x \delta(x, \hat{x}) + \frac{2}{3} v \partial_x \delta(x, \hat{x}) \right).
\]

It follows from the construction that the Hamilton equations in terms of the corresponding Poisson structure,

\[ u_t = \{u(x), H_k^M\}_DB^k, \quad v_t = \{v(x), H_k^M\}_DB^k \]

are equivalent to the coupled KdV system (1),(2).

As in the previous case the pencil of Poisson structures can be rewritten in terms of the basic Poisson structures which corresponds to \(k = 1\) and \(k = 0\) in (14):

\[
\{F, G\}_DB^k = \frac{1 - k}{-\lambda k^2 + (1 - k)^2} \{F, G\}_DB^1 + \frac{1 - k}{-\lambda k^2 + (1 - k)^2} \{F, G\}_DB^0.
\]

We notice that this decomposition is the same as in previous case, however the basic Poisson structure are different.

In particular for \(k = \frac{1}{1 - \lambda}, \lambda \neq 0, 1\), the \(\{., .\}_DB^k\) is the sum of the \(\{., .\}_DB^1\) and \(\{., .\}_DB^0\) basic Poisson structures. For \(\lambda = 0\) and \(k \neq 1\) the same relation (14) holds for the Poisson bracket we are now considering. These are then compatible Poisson structures.

We notice that by construction \(\phi_1\) and \(\phi_2\) as well as any functional of them, in all the cases we have considered, are Casimirs of the Poisson structure defined in terms of the Dirac brackets. In fact,

\[
\{F, \phi_1\}_DB = 0 \\
\{F, \phi_2\}_DB = 0
\]

for any functional \(F\) on phase space. This is a general property of the Dirac bracket.

It is a non-trivial feature that for each real \(k\), the parameter of the pencil of Poisson structures, there are hamiltonians \(H_k\) and \(H_k^M\) which give rise to the coupled KdV system when the corresponding Poisson structure is used.
6 Conclusions

We obtained the full hamiltonian structure for a coupled parametric KdV system. We started from four basic singular lagrangians. The associated hamiltonian formulation on phase space is restricted by second class constraints. The Poisson structure on the constrained variety of phase space was obtained using the Dirac approach. The Dirac brackets on the constrained phase space yields the most general structure of observables. A subset of them are functionals of the original fields \( u(x, t) \), \( v(x, t) \) of the coupled KdV system. We then constructed two pencils of Poisson brackets each of them with an associated parametric hamiltonian in terms of the same parameter of each pencil.

Each pencil of Poisson brackets is obtained from two compatible Poisson brackets of the same dimension. Consequently it is not possible to construct a hierarchy of higher dimensional hamiltonians from them. However the two pencils of Poisson brackets are of different dimensions, hence one may construct a hierarchy of higher order hamiltonians as in the KdV case.
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