Conformally invariant formalism for the electromagnetic field with currents in Robertson-Walker spaces

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We show that the Laplace-Beltrami equation \( \Box_j a = j \) in \((\mathbb{R}^6, \eta)\), \( \eta := \text{diag}(+---+) \), leads under very moderate assumptions to both the Maxwell equations and the conformal Eastwood-Singer gauge condition on conformally flat spaces including the spaces with a Robertson-Walker metric. This result is obtained through a geometric formalism which gives, as byproduct, simplified calculations. In particular, we build an atlas for all the conformally flat spaces considered which allows us to fully exploit the Weyl rescaling to Minkowski space.

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I. INTRODUCTION

This paper describes a geometrical framework for the study of conformally invariant fields in conformally flat spaces in four dimensions. Applications to scalar and electromagnetic fields are made. In particular, we show that the equation \( \Box_j a = j \) in \((\mathbb{R}^6, \eta)\), \( \eta := \text{diag}(+---+) \), where \( \Box_j \) is the Laplace-Beltrami operator, leads under very moderate assumptions to both the Maxwell equations and the conformal Eastwood-Singer gauge condition \cite{1} on 4D conformally flat spaces. In addition, our proof of this result allows us to propose a new fiber bundle in which, broadly speaking, the Maxwell equations on a conformally flat space are converted into constrained scalar equations on Minkowski space. This drastically simplifies the practical calculations.

The basic geometrical idea is to build four-dimensional spaces as the intersection in \((\mathbb{R}^6, \eta)\), of a surface and the five-dimensional null cone (invariant under the linear conformal group \( \text{SO}_0(2,4) \)). The metric on such a space is induced from that of \((\mathbb{R}^6, \eta)\). In particular, for any given Robertson-Walker (RW) metric, one can always find a surface such that the induced metric is the RW metric. Now, to each point of a space obtained in this way corresponds a half line of the cone, hence all spaces can be realized as subsets of the set of the half lines: the cone modulo the dilations. In addition, in the intersection of two such subsets the spaces are related through a Weyl rescaling. As a special case, the Minkowski space can be obtained in that scheme. This allows us to build a particular atlas of the cone modulo the dilations. In effect, copies of that Minkowski space can be obtained by displacing the surface intersecting the cone thanks to the action of \( \text{O}(2,4) \). This generates a covering of the set of the half lines. One can go a step further by introducing coordinates of \( \mathbb{R}^6 \) with the specific property that their restriction to a particular Minkowski space of the covering are the usual Minkowskian Cartesian coordinates. Each space of the covering together with its corresponding set of Minkowskian Cartesian coordinates can be turned into a local chart of the cone modulo the dilations, which is thus endowed with a Minkowskian atlas. This atlas is one of the key ingredients to perform simple calculations. In particular, it allows us to use the local Weyl rescaling to handle global problems.

The study of conformal fields, as the electromagnetic field, viewed as restrictions to four dimensional spaces of fields on \( \mathbb{R}^6 \) can be traced back to the seminal paper of Dirac \cite{2}. There, he introduces the “six cone formalism” which sets the conditions, mainly the homogeneity of the fields on \( \mathbb{R}^6 \), to obtain conformal Minkowskian fields. This approach takes into account the \( \text{SO}_0(2,4) \) symmetry of the Minkowskian equations under consideration from the beginning. This formalism has been extended in group theoretical context by Mack and Salam \cite{3} almost 35 years later and in study of conformal generalizations of QED on Minkowski and Anti-de Sitter spaces in the mid 80’s by many Authors (see \cite{4} and references herein). The fact that the conformal group \( \text{SO}_0(2,4) \) includes as subgroups, besides the Poincaré and the anti-de Sitter groups, the de Sitter group was our starting point in the study of the relation between conformal scalar field in the de Sitter and Minkowski space \cite{5,6}. The generalization to the electromagnetic field in a conformal gauge on the de Sitter space was tackled in \cite{7}. There following \cite{4} we used a method of auxiliary fields to obtain a two-point function in a conformal gauge which reduces to the Eastwood-Singer gauge \cite{1}. These works was concerned by free quantum fields, but part of the methods used also applied in the case of classical fields with sources. In that context, the tools previously build, especially the use of Minkowskian charts on the set of half lines, allowed us to reproduce, with simplified calculations, the result obtained by Higuchi and Cheong \cite{8} for the problem of two charges in de Sitter space. The geometrical framework...
we build up in the present work encompass and generalizes the method we developed in these previous studies. It extends the formalism to a class of conformally flat spaces which contains in particular the spaces endowed with a RW metric. It provides an atlas which permits a global use of the Weyl relation to Minkowski space. Last but not least, it also provides a deeper view on the geometrical nature of the objects involved.

This article is organized as follows. The Sec. II is devoted to the geometrical framework. We first gives a global view and motivate the main definitions. Then we turn to the geometrical formulation and consequences on tensors fields of the assumptions of transversality and homogeneity. This allows us to see how the Weyl rescaling emerges in this context. The action of the SO(2, 4) group on tensor fields is then discussed. The construction of the Minkowskian atlas, and the expression of some properties in it follows. This section ends by the proof that all the RW metrics can be obtained in the present formalism. In Sec. III we discuss the conformal scalar fields, we show how both equations and fields on \( \mathbb{R}^6 \) and on a conformally flat space are related. The Sec. IV contains the proof of the proposition that the Maxwell equations and the Eastwood-Singer gauge condition are obtained from homogeneous one form fields satisfying the Laplace-Beltrami in \( \mathbb{R}^6 \) and some transversality requirement. Although the result is purely geometrical, the proof makes use of calculations performed in the Minkowskian atlas. This make apparent the practical calculation method inherited from this geometrical framework. Some generalizations are discussed in Sec. V. Some properties of homogeneous tensors are reminded in Appendix A. Appendix B resembles a few additional comments on some particular coordinate systems for the sake of completeness. Definitions and conventions are collected in Appendix C. For convenience, the conventions for indices are repeated here:

\[
\begin{align*}
\alpha, \beta, \gamma, \delta & = 0, \ldots, 5, \\
\mu, \nu, \rho, \sigma & = 0, \ldots, 3, \\
i, j, k, l & = 1, \ldots, 3, \\
I, J & = c, 0, \ldots, 3, +.
\end{align*}
\]

The space \( \mathbb{R}^6 \) is provided with the metric \( \eta = \text{diag}(+,-,-,-,-,+) \). A point of \( \mathbb{R}^6 \) is denoted by \( x \) or \( y \), the letter \( y \) refers most often to the linear structure of \( \mathbb{R}^6 \).

II. GEOMETRY

A. Overview of the formalism and definitions

This section is intended to give the Reader a global view of the formalism. We motivate the introduction of key structures as the cone modulo the dilations or the bundle \( B_f \). We give their definitions and their main properties, leaving the proofs to the forthcoming sections.

For convenience a reminder of definitions and conventions is given in Appendix C.

1. The general framework

The main geometrical construction is pictured in Fig. 1. A four dimensional manifold \( X_f \) is obtained as the intersection \( P_f \cap C \) of \( P_f \) a submanifold of \( \mathbb{R}^6 \) defined by the equation \( f(x) = 1, x \in \mathbb{R}^6 \) and the five-dimensional null cone \( C \). To each point of \( X_f \) corresponds a unique half-line \([x] \), that is a point of \( C' \) which can be identified with a point of the intersection (bold line) \( X_\sigma \) of \( C \) with the sphere \( S \).

\[
\begin{array}{c}
\mathbb{R}^6 \\
\downarrow \lambda_f \\
X_f \leftarrow X'_f \subset C'
\end{array}
\]

in which

\[
t'_f : X'_f \rightarrow \mathbb{R}^6 \\
[x] \mapsto x_f := [x] \cap X_f,
\]

\[
\text{FIG. 1: The main construction: A space } X_f \text{ (bold line) is obtained as the intersection } P_f \cap C \text{ of } P_f \text{ a submanifold of } \mathbb{R}^6 \text{ defined by the equation } f(x) = 1, x \in \mathbb{R}^6 \text{ and the five-dimensional null cone } C. \text{ To each point of } X_f \text{ corresponds a unique half-line } [x], \text{ that is a point of } C' \text{ which can be identified with a point of the intersection (bold line) } X_\sigma \text{ of } C \text{ with the sphere } S.\]

where \([x]\) is the half line of \(\mathcal{C}\) (that is a point of \(\mathcal{C}'\)) which contains the point \(x\),

\[
l_f : X_f \to \mathbb{R}^6 \quad \text{with} \quad x \mapsto l_f(x) = x,
\]

is the canonical injection, and

\[
\lambda_f : X'_f \to X_f \quad \text{with} \quad [x] \mapsto x_f := [x] \cap X_f,
\]

which is a diffeomorphism. Note that, thanks to the homogeneity of \(f\) one has \(f(x/f(x)) = 1\), and thus

\[
x_f = \frac{x}{f(x)}.
\]

Now, a set of half lines may cross several manifolds \(X_f\), in other words the realizations \(X'_f\) on \(\mathcal{C}'\) of these manifolds have a non-empty intersection, in that case one can define some common coordinate systems (at least locally). This property allows us to drastically simplify many practical calculations, this can be explained as follows. Firstly, as we will prove in Sec. [III] the cone modulo the dilations as a manifold can be covered by a collection of realizations \(X'_f\) of Minkowski spaces, each endowed with a Cartesian system of coordinates. We will call Minkowskian atlas this covering together with these coordinates, and Minkowskian charts the elements of this atlas. Secondly, we will prove in Sec. [IV] that in their intersection two realizations \(X'_f\) are related through a Weyl rescaling. Finally, due to the existence of the above covering all the manifolds \(X_f\) are Weyl related to a Minkowski space, that is are conformally flat. In addition, and (this is the point for the simplifications) a conformal equation reads under its Minkowskian form in a Minkowskian chart.

Amongst the spaces obtained within this scheme the space \(X_\alpha\), which is obtained as the intersection of the cone with the 5-sphere \(S\), provides a realization of the abstract manifold \(\mathcal{C}'\) itself. Indeed, \(S\) is intercepted only once by each half line of \(\mathcal{C}\) (Fig. [IV]). We will use this realization of \(\mathcal{C}'\), which is included in the Einstein space, for drawing conformal diagrams in Sec. [III] and some practical calculations (Sec. [IV]).

2. Geometry and the Maxwell field

Other important structures, more specifically related to the Maxwell field, have to be introduced. In the scheme we use, which is inspired by the Dirac’s “six cone formalism” [2], the Maxwell field \(A^f\), a one form field on \(X_f\), is obtained from a one form field \(a\) on \(\mathbb{R}^6\) homogeneous of degree zero. We remind (see Appendix [A]) that the Cartesian components of \(a\) are then homogeneous of degree -1. In Sec. [IV] we will prove the following property: for \(a\) homogeneous of degree zero, \(j\) homogeneous of degree -2 satisfying

\[
\begin{aligned}
\Box_a a &= j, \\
\mathcal{Z}_a a|_C &= T(C), \\
\mathcal{Z}_a j|_{X_f} &= T(X_f),
\end{aligned}
\]

the field \(A^f := l_f^*(a)\) and the current \(J^f := l_f^*(j)\) defined on \(X_f\) satisfy the Maxwell equations and the Eastwood-Singer gauge condition [I]:

\[
\begin{aligned}
\left(\Box_f \delta_{\nu}^\mu - \partial_\mu \partial^\nu + R^\nu_{\mu} \right) A^f_\nu &= J^f_\mu, \\
\left(\Box_f \partial^\nu - 2 \partial_\mu \left( R^\nu_{\mu} - \frac{1}{3} R g^{\mu\nu} \right) \right) A^f_\nu &= 0,
\end{aligned}
\]

where \(\Box_f\) is the Laplace-Beltrami operator on \(X_f\). Note that for maximally symmetric spaces the Eastwood-Singer gauge condition reduces to the more familiar form

\[
\Box_f + \frac{R}{6} \partial \cdot A^f = 0,
\]

with a constant Ricci scalar \(R\). Note also that the two conditions appearing in [6] mean that the vector fields \(\mathcal{Z}_a a = a^\alpha \partial_\alpha\) and \(\mathcal{Z}_a j = j^\alpha \partial_\alpha\) are tangent respectively to \(\mathcal{C}\) and to \(X_f\).

Beside the property by itself its proof makes use of a new geometrical structure: the fiber bundle

\[
B^*_f := \bigcup_{x \in X_f} T^*_x (\mathbb{R}^6).
\]

It is obtained in restricting the base space of the cotangent bundle \(T^*(\mathbb{R}^6)\) to \(X_f\). This object allows us to solve some problems in a more simple way than by tackling them directly on \(X_f\). It also describes more accurately the transition between the fields on \(\mathbb{R}^6\) and those on \(X_f\). More precisely, although under the assumptions made in [6] the Maxwell field \(A^f\) is directly obtained as the pullback of \(a\). The proof of this result involves an intermediate step in which the equations \(\Box_a a = j\) are, in a sense which will be made precise in Sec. [IV] restricted to the six conformal scalar equations

\[
\Box_{a^f} + \frac{R}{6} a^f_\alpha = j^f_\alpha,
\]

where \(\Box_{a^f}\) is the scalar Laplace-Beltrami operator on \(X_f\) and in which both the \(a^f\)’s and the \(j^f\)’s belongs to the set \(B^*_f\) of the sections of \(B^*_f\). These equations reduce (when the two constraints of [6] are applied) to the Maxwell equations and the Eastwood-Singer condition [7]. However they are more easier to handle. Even more, when quantizing the electromagnetic field, these equations account directly for the gauge fixing: part of the \(\{a^f\}\)’s components leads to the Maxwell field, the others carry constraints [II III].

The field \(a^f\) is obtained from the field \(a\) by restricting the base of \(T^*(\mathbb{R}^6)\) to \(X_f\) and the field \(A^f\) from the field \(a^f\) by further restricting the fiber \(T^*_x (\mathbb{R}^6)\) to \(T^*_x (X_f)\). The
relations between $a, a^f$ and $A^f$ (also true for $j, j^f$ and $J^f$) are collected in the following diagram

$$a \in \Omega_1(\mathbb{R}^6)$$

$$a^f \in B_f^*$$

$$A^f \ni \Omega_1(X_f) \ni A^f$$

in which

$$\bar{I}_f : \Omega^1(\mathbb{R}^6) \rightarrow B_f^*$$

$$\bar{I}_f : \Omega^1(\mathbb{R}^6) \rightarrow B_f^*$$

such that: $\forall x \in X_f, \forall V \in T_x(\mathbb{R}^6), a^f(x)[V] = a(x)[V]$,

$$r_f : B_f^* \rightarrow \Omega^1(X_f)$$

$$a \mapsto a^f := \bar{I}_f(a)$$

such that: $\forall x \in X_f, \forall U \in T_x(X_f), A^f(x)[U] = a(x)[U]$.

B. The transversality condition

This is a condition on tensor fields which will appear very often. Roughly speaking, for a tensor $t$ fulfilling this relation, the map $t \rightarrow T^f$ has a good behavior with respect to group invariance, Weyl relation, equations... This condition reads as follows: A tensor $t \in T_p^0(\mathbb{R}^6)$ fulfills the transversality condition if and only if, for any $x \in \mathcal{C}$ and for any $V_1, \ldots, V_p \in T_x(\mathcal{C})$, $t(x)(V_1, \ldots, V_p) = 0$ as soon as one of the arguments $V_i$ is equal to the dilation field $\xi := y^\alpha \partial_\alpha$, i.e. $V_i \propto O_x$. Note that, when $t \in \Omega^p(\mathbb{R}^6)$, one often finds the little bit stronger condition $i_\xi t = 0$ which, for $p = 1$, reduces to $y^\alpha \partial_\alpha = 0$ (from the name “transversality condition” originates after Dirac’s paper [2], although in the context of mechanics this kind of condition is usually termed “horizontal”) equivalently in an index-free notation: $\varepsilon_{abc} t|_{\mathcal{C}} \in \mathcal{T}(\mathcal{C})$. This condition will eventually be required for any field considered here. For quantum fields, as usual when a gauge condition is present, the implementation of the constraint at the quantum level (in a Gupta-Bleuler quantization scheme) is done after the quantization.

C. Homogeneity and Weyl relations

In this section, we consider two manifolds, $X_f$ and $X_h$ defined, as in Sec. [1, 4] as the intersections of $\mathcal{C}$ and the manifolds $f(y) = 1$, and $h(y) = 1$. Let $t \in T_p^0(\mathbb{R}^6)$ be an homogeneous tensor field of degree $d(t)$, we want to compare the fields $T^f := l^*_f t$ and $T^h := l^*_h t$. For this purpose we realize both fields on $\mathcal{C}'$ and obtain a Weyl relation between $T'^f := l'^*_f t$ and $T'^h := l'^*_h t$. Let us show that if $t$ fulfills the transversality condition then, on $X'_f \cap X'_h$, the following property holds:

$$(T'^f)([x]) = \left( K_v^f([x]) \right)^{d(t)} (T^h)([x]).$$

where we defined

$$K_v^h([x]) := h(x)^{d(t)}/f(t).$$

Thus the function $K_v^h$ appears as the conformal factor. For further references we also define the related one form on $\mathcal{C}'$

$$W^{(f,h)} := d\ln(K_v^h)^2.$$  

Note that, no reference to a metric structure is made in these two definitions.

In order to prove (12), we begin with showing that, for $x \in \mathcal{C}$ and $V' \in T_{[x]}(\mathcal{C}')$ one has

$$f(x)l'_f([x]) [V'] = h(x)l'_h([x]) [V'] + N,$$

where $N$ is a null vector which belongs to a half-line $[x]$. We first remark that

$$l'_h([x]) = \frac{f(x)}{h(x)} l'_f([x]) = K_v^f([x]) l'_f([x]).$$

Then, differentiating the rightmost term, one obtains, for any $x \in \mathcal{C}$ and $V' \in T_{[x]}(\mathcal{C}')$,

$$l'_h([x]) [V'] = K_v^f([x]) [V'] + K_v^h([x]) l'_f([x]) [V'] + N.$$

The first term of the r.h.s. of this expression is proportional to $x$, and as a consequence, belongs to the half-line $[x]$. The expression (15) follows at once.

Now, let us consider $V_1', \ldots, V_p' \in T_{[x]}(\mathcal{C}')$. Then using in succession the definitions (2)-(4), the homogeneity of $t$, the property (15) and finally the transversality condition for $t$ (Sec. [2, 3]) one has

$$T'^f([x]) [V_1', \ldots, V_p'] = \left( l'^*_f t \right)([x]) [V_1', \ldots, V_p']$$

$$= \left( \frac{x}{f(x)} \right)^{d(t)-p} t(x) \left( l'_f V_1', \ldots, l'_f V_p' \right)$$

$$= \left( \frac{1}{f(x)} \right)^{d(t)-p} t(x) \left( f(x) l'_h V_1', \ldots, f(x) l'_h V_p' \right)$$

$$= \left( \frac{1}{f(x)} \right)^{d(t)} t(x) \left( h(x) l'_h V_1', \ldots, h(x) l'_h V_p' \right)$$

$$= \left( \frac{1}{f(x)} \right)^{d(t)} t(x) \left( h(x) l'_h V_1', \ldots, h(x) l'_h V_p' + N_p \right)$$

$$= \left( \frac{1}{f(x)} \right)^{d(t)} t(x) \left( h(x) l'_h V_1', \ldots, h(x) l'_h V_p' \right)$$

$$= \left( \frac{h(x)}{f(x)} \right)^{d(t)} T^h([x]) \left[ V_1', \ldots, V_p' \right],$$

which is the announced result.
Finally, thanks to the isomorphism $\lambda_f$ between the $X_f$'s and the $X'_f$ and using the notation $T^f$ for the tensors on the $X'_f$'s instead of $T^{xf}$'s, one can recast the above relation \[12\] under the more familiar form

$$T^f(x) = \left( K^f_i(x) \right)^{\alpha} T^\alpha(x),$$

in which $T^f(x)$ stands for $T^{xf}(x) = T^f(x_f)$. In particular, between metrics this relation specializes to

$$g^f(x) = \left( K^f_i(x) \right)^2 g^h(x),$$

which makes apparent that the Eq. \[12\] is in fact a Weyl relation in the usual sense.

**D. Action of $\text{SO}_0(2,4)$**

In this section we specify the action of $\text{SO}_0(2,4)$ on the various objects defined above. In particular, we define the action of $\text{SO}_0(2,4)$ on the sections $a^f$ of the fiber bundle $B^r_f$ and show that the transversality condition ensures the $\text{SO}_0(2,4)$ invariance of the construction.

The natural action $x \mapsto L_g x = gx$ of $\text{SO}_0(2,4)$ on $\mathbb{R}^6$ yields an action $L^g_f$ on $X_f$ defined through:

$$X_f \ni x \mapsto L^g_f x = (gx)/f(gx) \in X_f.$$

Setting $\omega^g_f(x) = f(x)/f(gx)$ for any $x \in \mathbb{R}^6$, one obtains the action of the conformal group on the space time:

$$X_f \ni x \mapsto L^g_f x = \omega^g_f(x)gx \in X_f.$$

1. **Action of the group on the scalar fields**

Let us recall that, for a scalar field $\phi$ homogeneous of degree $r$ the operator $l^r_f$ is defined through:

$$l^r_f \phi(x) = \phi(l_f(x)) =: \Phi^f(x).$$

Also, the group acts on $\phi$ through the natural representation $L^g_f \phi(x) = \phi(g^{-1}x)$. The field $\Phi^f$ is defined on $X_f$ and we now define the representation $(L^f)^c$ of the conformal group on it. We just impose that the operator $l^r_f$ intertwines $(L^f)^c$ and the natural representation $L^c$ on $\phi$:

$$(L^f)^c l^r_f = l^r_f (L^g)^c.$$  \[20\]

That is to say, using that $f(g^{-1}x) = \omega^g_f(g^{-1}x)$ for any $x \in X_f$.

$$((L^f)^c \Phi^f)(x) = (L^f)^c l^r_f \phi(x) = \phi(g^{-1}x) = \phi \left( \frac{g^{-1}x}{f(g^{-1}x)} f(g^{-1}x) \right) = (\omega^g_f(g^{-1}x))^r \Phi^f(L^g_{g^{-1}} x),$$

which is the well-known action of the conformal group on conformal pseudo scalar fields of weight $r$.

2. **Action of the group on the one-forms**

The group acts on a 1-form $a$ of $\mathbb{R}^6$ through \[21\]

$$(L^f)^c a(x) = g \cdot a(g^{-1} x),\quad (L^f)^c r^f a^f = r^f \tilde{L}^g a^f\text{ as soon as } i_{\xi} a = 0,$$  \[22\]

which reads

$$\tilde{L}^g a^f|U] = (L^f)^c A^f(x)|U] \forall x \in X_f, \forall U \in T_x(X_f),$$

as soon as $i_{\xi} a = 0$.

We first build the representation $\tilde{L}^f$ on the field $a^f$ in a very similar way as for the scalar field. We just impose that $\tilde{L}^f$ intertwines the representations $\tilde{L}^g_f$ and $(L^g)^c$ and obtain straightforwardly:

$$\tilde{L}^g f a^f(x)[V] = (\omega^g_f(g^{-1}x))^r g \cdot a^f(L^g_{g^{-1}} x)[V],$$

for any $V \in T_x(\mathbb{R}^6)$, which is a fortiori true for any $U \in T_x(X_f)$.

We now define the representation $(L^f)^c$ of the group on the forms $A^f$. As a result, we cannot define a representation on the $A^f$ in the same way as above. As we will see, one must impose a condition on $a$. In place of this construction, we define directly the representation $(L^f)^c$ through

$$(L^f)^c A^f(x)|U] = (\omega^g_f(g^{-1}x))^r A^f(L^g_{g^{-1}} x) - (L^g)^c A^f(x)|U] = (L^g)^c A^f(x)|U],$$

for $x \in X_f$ and $U \in T_x(X_f)$, \[24\]

We begin with calculating $(L^g)^c$ on any $U \in T_x(X_f)$ using the Leibniz rule:

$$(L^g)^c_{\omega^a} (x)[U] = \frac{g \cdot U}{f(gx)} - g(x) \left( \frac{\partial f(gx)[U]}{f(gx)^2} \right).$$

The crucial remark is that the second term of the r.h.s. belongs to a line of the cone. This term will be denoted as $F$ in the following. At this time we can set down the representation $(L^g)^c$, putting $\omega = : \omega^g_f(g^{-1}x)$ for the readability:

$$(L^f)^c A^f(x)[U] = \omega^r A^f(L^g_{g^{-1}} x) - \left( \frac{1}{\omega} g^{-1} \cdot U + F \right)$$

$$= \omega^r a(L^g_{g^{-1}} x) \left( \frac{1}{\omega} g^{-1} \cdot U + F \right)$$

$$= \omega^r a(L^g_{g^{-1}} x) \left( \frac{1}{\omega} g^{-1} \cdot U \right)$$

$$= (\omega^g_f(g^{-1}x))^r g \cdot a^f(L^g_{g^{-1}} x)[U],$$

\[26\]
the before last equality being due to the transversality condition. The result \(22\) follows immediately from \(23\) and \(26\). As a consequence, the map \(l^f\) intertwines the representations \((L^f)^c\) and \(L^e\) as soon as \(a\) fulfills the transversality condition. Note that the identification \(X_f = X_f^\prime\) allows to realize the group action on \(X_f^\prime\) as well.

E. Minkowskian atlas and conformal diagrams

As explained in Sec. II A, all the spaces \(X_f\) are realized as subsets \(X_f^\prime\) of \(C^\prime\) through the one to one map \(\lambda_f\). Then, one can use the Weyl relation \(16\) in order to simplify practical calculations in problems involving conformally invariant equations. As well known, these calculations can be much simpler than those performed on \(X_f\). Unfortunately, one cannot in general recover the space \(X_f\) by replacing \(y^5\) with only one Minkowskian space. However, we will prove that the covering of the whole \(C^\prime\) is possible using four Minkowskian spaces. These can be endowed with coordinates systems in order to form an atlas. In addition, these systems can be chosen to be Cartesian Minkowskian coordinates to make calculations simpler. Now, the formalism we develop use \(\mathbb{R}^6\) as framework, in practical calculations the coordinates basis on which fields can be expanded are chosen on \(\mathbb{R}^6\). As a consequence, the Cartesian Minkowskian coordinates of the atlas have to be related to coordinates in \(\mathbb{R}^6\). In the sequel, we will call Minkowskian systems both the systems in \(\mathbb{R}^6\) and those deduced from them in the atlas of \(C^\prime\). This atlas will also be named Minkowskian as well as charts that compose it. We will prove that one Minkowskian plane \(P_f\) together with one Minkowskian coordinate system, the \(N\)-system, are sufficient to generate the whole Minkowskian atlas. Note that, some general comments in relation with such kind of coordinates are made in Appendix [3]

First let us introduce a useful graphical representation of \(C^\prime\) as conformal diagram. Remind that \(C^\prime\) can be realized as the intersection \(X_\sigma\) of the \(\delta\)-sphere, obtained through \(P_\sigma\), \(\sigma := \sqrt{\alpha^2 y^5 y^6} / 2\), with the cone \(C\) (see Fig. 1). Points of \(X_\sigma\) and \(C^\prime\) are then identified through the map \(\lambda_\sigma\). This space \(X_\sigma\) is naturally endowed with a very convenient global coordinate system \((\alpha, \beta, \theta, \varphi)\) obtained by setting \(r_1 = r_2 = 1\) in the hyper-spherical coordinate system \(\{r_1, r_2, \alpha, \beta, \theta, \varphi\}\) defined through

\[
\begin{align*}
    y^5 &= r_1 \cos \beta \\
    y^0 &= r_1 \sin \beta \\
    y^4 &= r_2 \sin \alpha \omega^4(\theta, \varphi) \\
    y^4 &= r_2 \cos \alpha,
\end{align*}
\]

in which \(\beta \in [-\pi, \pi], \alpha, \theta \in [0, \pi], \varphi \in [0, 2\pi]\) and \(\omega^4(\theta, \varphi)\) correspond to the usual spherical coordinates on \(S^2\). The condition \(r_1 = r_2 = 1\) gives a system on the cone, \(r = 1\) gives a system on \(X_\sigma\). Now, since as a result of the considerations of the Sec. II C a space \(X_f\) is related to \(X_\sigma\) through a Weyl rescaling, the subsets \(X_f^\prime\) can be conformally pictured in the plane \((\alpha, \beta)\). In the sequel we will call “\((\alpha, \beta)\)-diagram” this conformal mapping (see Fig. 2 for an example). One may note that the metric element on \(X_\sigma\) in the \((\alpha, \beta, \theta, \varphi)\) system is

\[
ds^2 = d\beta^2 - d\alpha^2 - \sin^2 \alpha d\varphi^2.
\]

This form shows that \(X_\sigma\) is indeed included \((\beta \in [0, \pi])\) in the static Einstein space of positive curvature.

Now, let us consider the space \(X_{fN}\), \(f_N := (y^5 + y^4)/2 = 1\). As shown in [2] this space is a Minkowski space. The coordinate system \(\{x_N^I\}\), \(I = c, \mu, +\) defined by

\[
\begin{align*}
x_N^c &= \frac{y^a y_0}{4f_N^2(y^4, y^5)} \\
x_N^\mu &= \frac{y_\mu}{f_N(y^4, y^5)} \\
x_N^+ &= f_N(y^4, y^5),
\end{align*}
\]

(where its inverse is given for convenience in Appendix [3] provides on \(X_f^\prime \subset C^\prime\) a chart, called \(N\)-chart in the sequel. The restriction to the cone is obtained for \(x_N^c = 0\) and that for \(P_{fN}\) for \(x_N^+ = 1\). The metric induced from \(\mathbb{R}^6\) on \(X_{fN}\) is, in the \(\{x_N^I\}\)-coordinate basis, \(g_{\mu\nu}\), the usual form of the Minkowski metric.

The points where the above system becomes singular, that is on the subset \(\{y^5 + y^4 = 0\}\), correspond to points at infinity in the Minkowski space \(X_{fN}\). They are mapped to the boundaries of the \(N\)-chart on the conformal \((\alpha, \beta)\)-diagram in Fig. 2). Similar considerations apply to the space \(X_{fS}\), defined by \(f_S := (y^5 - y^4)/2 = 1\) and the coordinate system \(\{x_S^I\}\), \(I = c, \mu, +\) is obtained by replacing \(N\) by \(S\) in \(29\).

A set of Minkowskian charts covering \(C^\prime\) is finally obtained by moving the two surfaces \(P_{fN}\), \(P_{fS}\) with some elements of SO(2,4). More precisely, we obtain new Minkowski spaces through the action of the one parameter subgroup of SO(2,4) generated by \(X_{50} := y_5 \partial_0 - y_0 \partial_5 = \partial_\beta\). The manifolds \(X_{f\betaN}\) (resp. \(X_{f\betaS}\)) defined through

\[
\begin{align*}
f_{\beta N}(y) := f_N(e^{-\beta X_{50}}y) &= \frac{1}{2} (\cos \beta y_5 + \sin \beta y_0 + y_4) = 1, \\
f_{\beta S}(y) := f_S(e^{-\beta X_{50}}y) &= \frac{1}{2} (\cos \beta y_5 + \sin \beta y_0 - y_4) = 1.
\end{align*}
\]

are Minkowski spaces. Corresponding to each case, one can define a coordinate system \(\{x_{\beta N}^I\}\) analogous to \(\{x_N^I\}\), replacing \(f_N\) by \(f_{\beta N}\) in \(29\) and similarly for \(S\) in place of \(N\). As a consequence, we obtain an atlas of \(C^\prime\), in the usual sense, in which the systems of coordinates make the Weyl relations very transparent. Now every \(X_f^\prime\) space is an open subset of \(C^\prime\) and can be endowed with the above atlas.
The space $P$ of variables in $\mathbb{R}$ say through an (inoffensive) Cartesian isometric change from the elements of our Minkowskian atlas can be obtained respectively from the Minkowski planes, the interior of that chart is on the l.h.s. (moved N-plane), the crossed region obtained respectively from the Minkowski charts can be translated along the the coordinates $x$ satisfying (32), thus a RW space. The same result applies in others Minkowskian coordinates.

As a final remark, the space $P_{fs}$ can be obtained from the space $P_{fn}$ through the transformation $y^j \rightarrow -y^j$ which is also an element of $O(2, 4)$. As a consequence, all the elements of our Minkowskian atlas can be obtained from $P_{fn}$ through a transformation of $O(2, 4)$ that is to say through an (inoffensive) Cartesian isometric change of variables in $\mathbb{R}^6$. Thus, the whole atlas is generated from $P_{fn}$ and the $N$-coordinates as announced in the beginning of this section.

### Some explicit conditions in the $N$-charts

We note that, for $x \in X_f$,

$$x^+_N(x) = f_N(x) = f_N \left( \frac{x}{f(x)} \right) = K^f_{fN}(x).$$

(30)

Also, in $N$-coordinates the dilation field reads

$$\xi := y^\alpha \partial_\alpha = x^+_N \partial_{x^+_N}.$$

Now, a one-form $t \in \Omega^0(\mathbb{R}^6)$ homogeneous of degree zero, is expanded in the coordinate basis $\{dx^0_N, dx^\mu_N, dx^+_N\}$ as

$$t^N = t^c_N dx^c_N + t^\mu_N dx^\mu_N + \frac{t^+_N}{x^+_N} dx^+_N.$$

The last term is for convenience divided by $x^+_N$ in order that $t^+_N$ be of the same degree of homogeneity (zero) as the other components. One has

$$t^+_N = g^\alpha t_\alpha = i \xi t.$$

In particular, the transversality condition reads $t^+_N = 0$. The same result applies in others Minkowskian coordinates.

### F. Robertson-Walker metric

Here we prove that for any Robertson-Walker (RW) metric $g^{RW}$ one can find a function $f_{RW}$ such that the metric induced from $\mathbb{R}^6$ on the conformally flat space $X_{f_{RW}} := P_{f_{RW}} \cap C$ be the Robertson-Walker metric $g^{RW}$. In other words, each Robertson Walker space can be realized at least locally as a space $X_{f_{RW}}$.

Following Ibison [11] the set called \{conformal$(r, t)$\} of conformally flat metrics defined through:

$$ds^2 = A^2(x^0, ||x||)dx^2; \ dx^2 = \eta_{\mu\nu}dx^\mu dx^\nu,$$

(32)

$A^2$ being a conformal factor and \{x$^\mu$\} the Cartesian Minkowskian coordinates, includes the RW metrics. The notation $(x^0, ||x||)$ for the arguments of the function $A$ indicates that it depends separately of the time coordinates and of the space coordinates and that the space coordinates appears only through the radius $||x||$.

Let us assume now that $A$ for some RW metric is given in a Minkowskian coordinate system, say the system $\{x^i_N\}$ [29]. The function

$$f_{RW}(x) := \frac{x^+_N}{A(x^0_N, ||x||))},$$

(33)

is invariant under the subgroup $SO_0(3)$ of $SO_0(2, 4)$ generated by $y_i \partial_j - y_j \partial_i$. The equation $f_{RW}(x) = 1$ defines a $SO_0(3)$ invariant space endowed with a metric which satisfies [32], thus a RW space.

### III. SCALAR FIELDS

In this section, we consider scalar fields on $\mathbb{R}^6$, homogeneous of degree $r$. These fields yield scalar fields on $X_f$. We show that these fields fulfill the scalar conformal equation as soon as the original field fulfill the Laplace-Beltrami equation on $\mathbb{R}^6$.

Let $\phi$ and $j$ two scalar fields on $\mathbb{R}^6$ homogeneous of degree -1 and -3 respectively, and suppose that

$$\Box_6 \phi = j.$$

(34)

We claim that

$$(\Box f + \frac{R}{6}) \Phi^f = J^f,$$

(35)
where $\Box_f$ is the Laplace-Beltrami operator and $R$ the scalar curvature for the space $X_f$. We consider $X_\sigma$ the realization of the cone up to the dilation defined Sec. \[\underline{27}\] on which all $X_f$ spaces realize as subsets. Setting $u = r_1/r_2$ and $v = 1/r_2$ in the hyperspherical coordinates and using the homogeneity of $\phi$, the equation (34) reads

$$v^2 \left( (1 - u^2) \partial_u^2 + \frac{1}{u} (1 - u^2) \partial_u + 1 + \frac{1}{u^2} \partial_\beta^2 - \Delta s^3 \right) \phi = j.$$

The restriction to $X_\sigma$ ($u = v = 1$) then gives

$$(\partial_\beta^2 - \Delta s^3 + 1) \Phi = J^\sigma,$$

which is the desired result. We obtain the similar result for the other hyper surfaces $X_f$ by using the Weyl correspondence between the spaces $X_f$. Note that although straightforward the change of variables leading to the equation (36) is rather cumbersome.

IV. THE MAXWELL FIELD

In this section we deal with one-forms $a$ and $j$ of $\mathbb{R}^6$ satisfying $\Box a = j$ and such that $a$ is homogeneous of degree 0 and $j$ homogeneous of degree $-2$ (remind that this implies that $a_\alpha$ and $j_\alpha$ are homogeneous of degree $-1$ and $-3$ respectively). We first show how in the present formalism the equation $\Box a = j$ leads to a set of six copies of the equation of the conformal scalar field. Then, we give the proof that $A^l \in T^* (X_f)$ satisfies the Maxwell equations as soon as $a$ fulfills the transversality condition, and that an additional condition on $j$ allows us to recover a conformal gauge condition.

A. Equation on $E^f_l$

Suppose that $E^f_l$ is endowed with a coordinate system $(x^l, dy^a)$ where $x^l$ is any coordinate system of $X_f$ and $dy^a$ a Cartesian system of coordinates on $T^*_x (\mathbb{R}^6)$. In such a coordinate system the equation on $a^l$ reads

$$(\Box^{(s)} + \frac{R}{6}) a^l_\alpha = j^l_\alpha,$$

where $\Box^{(s)}$ is the scalar Laplace-Beltrami on $X_f$ and $R$ the Ricci scalar. This is a straightforward consequence of the fact that, on $\mathbb{R}^6$ in Cartesian coordinates one has $\Box a_\alpha = \Box^{(s)} a_\alpha$: since in a Cartesian coordinate system $\{y^a\}$ of $\mathbb{R}^6$ the equation $\Box a = j$ reduces to six copies (one per component of $a$ and $j$) of the scalar equation $\Box a_\alpha = j_\alpha$, each components $a_\alpha$ and $j_\alpha$ being homogeneous of degree $d(a_\alpha) = -1$ and $d(j_\alpha) = -3$, one can thus apply the result (35) of the Sec. \[\underline{11}\] to each scalar equation separately.

Two comments are in order concerning equations (38).

In first, they do not present any gauge ambiguity and we proved in \[\underline{9}\] that they allow a quantization of the Maxwell field on de Sitter space using a Gupta-Bleuler scheme where the condition of transversality is translated to a condition on states after quantization. In second, they can be used to solve classical propagation problems for the Maxwell field: For instance, we considered in \[\underline{7}\] the two-charges problem for the de Sitter space.

B. Maxwell equation on $X_f$

In this section, we implicitly identify the spaces $X_f$ and $X_f'$, writing all the objects on the common manifold $C'$. We now prove the statement of Sec. \[\underline{11A}\] for a homogeneous of degree zero, $j$ homogeneous of degree $-2$ satisfying

$$\begin{cases}
\Box a = j \\
\Box(j^l) = T(C)
\end{cases}$$

the field $A^l := l^l_j (a)$ and the current $J^l = l^l_j (j)$ defined on $X_f$ satisfy the Maxwell equations and the Eastwood-Singer gauge condition \[\underline{11}\] :

$$\begin{cases}
\Box f^\mu \delta^\nu - \nabla_\mu \nabla^\nu + R^\mu_\nu \mu \nu A^l = J^l \\
\Box f^\mu \nabla^\nu + (R^\mu_\nu - \frac{1}{3} R g^{\mu \nu}) A^l = 0.
\end{cases}$$

We first prove the result on the Minkowskian chart $X_{f^\infty}$, after which, using the O(2, 4) invariance of the hypothesis and of the conclusion, and the properties of the Minkowskian charts, we obtain this result on all the Minkowskian charts. In the following, for readability, we note $a^N, A^N$, for $A^{\infty}$ and $A^f$. We first consider the equation $\Box a = j$ and apply (35) on the Minkowski space $X_{f^\infty}$ defined by $f_{\infty}$ (used to build the N-chart), this yields the system

$$\Box^N_N a^N_\alpha = j^N_\alpha,$$

where $\Box^N_N$ is the scalar Laplace-Beltrami operator on $X_{f^\infty} \simeq X_{f^\infty}', a^N := l^N_N (a)$ and $j^N := l^N_N (j)$. As a second step we express in the N-chart of $C'$ the above system of equations (41). This leads to the system already obtained in \[\underline{10}\] (with slightly different notations)

$$\begin{cases}
\partial^2 a^N_\mu + \partial_\mu a^N = j^N_\mu \\
\partial^2 a^N_\alpha - 2 \partial_\beta a^N - 2a^N_\beta = j^N_\alpha
\end{cases}$$

Then, we express the two constraints appearing in (39) in the N-chart. The first one is the transversality condition applied to the field $a$, using the formula of (31) one has

$$a^N = 0.$$
The second condition \( \tau_{n\bar{n}}|_{X_f} \in T(X_f) \) implies the transversality of \( j \) since \( T(X_f) \subset T(C) \), thus
\[
j^N_+ = 0. \tag{44}
\]
In addition, \( \tau_{n\bar{n}}|_{X_f} \in T(X_f) \) rewrites \( \tau_{n\bar{n}}|_{f} = 0 \), which in the \( N \)-coordinates on the cone \((x^N = 0)\) reads
\[
j^N_+ \frac{\partial f}{\partial x^N} + 2 \frac{\eta_{\mu N} j^N_{\mu}}{x^N} \frac{\partial f}{\partial x^N} = 0.
\]
Thanks to the homogeneity of \( f \) this equation rewrites
\[
j^N_+ + 2 \eta^{\mu N} j^N_{\mu} \frac{\partial f}{f} = 0.
\]
Then, using the definition \((14)\), it becomes
\[
j^N_+ = j^N \cdot W(f,J^N), \tag{45}
\]
where the dot refer to the Minkowskian metric \( \eta^N \).

Using the constraint \((43, 45)\), the system \((42)\) becomes
\[
\begin{aligned}
\partial^2 a^N_{\mu} - \partial_{\mu} \partial \cdot a^N &= j^N_{\mu} \\
\partial^2 \partial \cdot a^N &= j^N \cdot W(f,J^N) \\
a^N_{\mu} &= \partial \cdot a^N.
\end{aligned}
\]

Owing to the map \( r_{fN} \), the above system leads to
\[
\begin{aligned}
\frac{\partial^2 A^N_{\mu}}{\partial x^N} - \partial_{\mu} \partial \cdot A^N &= J^N_{\mu} \\
\frac{\partial^2 \partial \cdot A^N}{\partial x^N} &= J^N \cdot W(f,J^N)
\end{aligned}
\]
on the Minkowski space \( X_{fN} \). Note that the map \( r_{fN} \) becomes obvious in the \( N \)-coordinates \((29)\): for all \( x \in X_f \) one has \( A^N_{\mu}(x) = a^N_{\mu}(x) \).

For the sake of argument, let us introduce a self explanatory symbolical notation for the two operators appearing in the l.h.s. of the Maxwell equations and the Eastwood-Singer gauge condition. With them the above system reads:
\[
\begin{aligned}
(M_N[A^N])_{\mu} &= J^N_{\mu} \\
ES_N[A^N] &= J^N \cdot W(f,J^N),
\end{aligned}
\]
This result is available for the \( N \)-chart. Nevertheless, the \( O(2,4) \) invariance of hypothesis and conclusion and the fact that any Minkowskian chart can be deduced from the \( N \)-chart by mean of an \( O(2,4) \) transformation, proves that this result is true for any Minkowskian chart of our atlas.

Now, using the same symbolical notation for the two operators appearing in the l.h.s of \((46)\), and following \[11\] to apply a usual (local) Weyl transformation between \( X_{fN} \) and \( X_f \) to each Eqs. of the system \((47)\), one obtains
\[
\begin{aligned}
(M_f[A^f])_{\mu} &= J^f_{\mu} \\
ES_f[A^f] + W(f,J^N) \cdot (M_f[A^f]) &= J^f \cdot W(f,J^N),
\end{aligned}
\]
where the dots now refer to the metric \( g^f \) and where we have taken into account the homogeneity of both the electromagnetic and the current one-form fields. They have respectively a degree of \(-1\) and of \(-3\) which correspond to conformal weights of zero and \(-2\) for the components of the fields: \( A^f_{\mu} = A^N_{\mu}, J^f_{\mu} = (K^f_{J^N})^{-2} J^N_{\mu} \). Finally, the second equation simplifies and we obtain the announced result on the \( N \)-chart. Then, using the above remark on \( O(2,4) \) invariance, we obtain the result on the whole space.

Note that the Maxwell equations are obtained independently of the condition \( \tau_{n\bar{n}}|_{X_f} \in T(X_f) \), which is used only to obtain the Eastwood-Singer gauge condition. Note also that, in absence of source (\( j = 0 \)), the Eastwood-Singer gauge condition is automatically fulfilled as soon as the transversality condition on \( a \) is fulfilled.

It is important to point out that from a physical perspective the initial conditions and currents are given in the space \( X_f \). In particular, the Minkowskian currents appearing in \((47)\) are defined through the equation \( J^f_{\mu} = (K^f_{J^N})^{-2} J^N_{\mu} \) in order to satisfy the Weyl relation. They have in general no physical meaning in the Minkowski space.

\section{Comments on some applications}

Besides the results presented in the previous sections, the formalism depicted in the present paper explicits the geometrical nature of the various objects used in our previous works. This allows us to consider some straightforward generalizations, let us comment briefly about the classical and quantum situations.

The method used for the classical propagation problem considered in [12] extends naturally to the conformally flat spaces \( X_f \). To summarize, given a set of initial conditions and currents in \( X_f \) the problem of finding the \( A^f \) solution of the Maxwell equations in the Eastwood-Singer gauge (or in a gauge contained in it) in \( X_f \) amounts to use in the Minkowskian atlas the propagation formula established in [12], which uses only the Minkowskian scalar Green’s function.

The quantization scheme used in [9] for the free electromagnetic field on de Sitter space can be transposed here for the most part. The definition of an \( SO(2,4) \)-invariant scalar product on the space of the solutions of \((35)\) and the obtention (from the known mode solutions of \((35)\)) of a reproducing kernel for the \( a^f_{\alpha} \)'s with respect to that product can be reproduced almost verbatim. The result is: \( W^f_{\alpha\beta}(x, x') = -\eta_{\alpha\beta} D^f_J(x, x') \), where \( D^f_J(x, x') \) is the scalar two-point function on \( X_f \). The construction of the Fock space, including the determination of the physical subspace is formally identical. The general form of a covariant two-point function, a bi-tensor on a specific \( X_f \), requires more developments (note that, such a function has been proposed recently in Minkowskian
coordinates \[\mathbb{R}^6\]. This could be the object of future investigations.

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Appendix A: Homogeneous tensors

For reference, we recall here some properties of an homogeneous \((0,p)\)-tensors \(T\) of \(\mathbb{R}^6\).

The set \(\mathbb{R}^6\), together with the scalar product \(<y_1,y_2>: = \eta_{\alpha\beta}y^\alpha y^\beta\), is naturally endowed with a structure of linear space. A point in \(\mathbb{R}^6\) is located by a vector \(y\) whose components in Cartesian coordinates are \(\{y^\alpha\}\). The linear structure allows us to identify the tangent spaces at any point through \(\partial_\alpha(y) = \partial_\alpha(0)\). This makes sense for expressions as \(T(y) = T(\lambda y)\) since the tangent space at \(y\) is identified with the tangent space at \(\lambda y\). Let us consider the dilatation map in \(\mathbb{R}^6\)

\[\rho_\lambda : \mathbb{R}^6 \rightarrow \mathbb{R}^6\]

\[y \mapsto \lambda y,\]

\(\lambda\) being a positive real number. Since \(T_\rho(\mathbb{R}^6) \equiv \mathbb{R}^6\) this is also the result of a push-forward on the vector \(y\). In other words \(\rho_\lambda \partial_\alpha = \lambda \partial_\alpha\). For a \((0,p)\)-tensor field \(T\) of the pullback reads \((\rho_\lambda^*T)(y) = \lambda^pT(\lambda y)\). Now, an homogeneous \((0,p)\)-tensor field \(T\) of degree \(d(T) = r\) is defined through the relation

\[(\rho_\lambda^*T)(y) = \lambda^rT(y).\]

Thus, homogeneous \((0,p)\)-tensor field \(T\) of degree \(r\) satisfy

\[T(\lambda y) = \lambda^{(r-p)}T(y).\]  

(A1)

The homogeneity of a tensor field can be related to that of its components in some coordinate basis whose coordinates \(\{x^I(y)\}\) \(I = 0,\ldots,5\) are each one homogeneous functions of degree \(d(I)\). Precisely, let us show that: for \(T \in T_p^0(\mathbb{R}^6)\) a \((0,p)\)-tensor field of \(\mathbb{R}^6\), homogeneous of degree \(d(T)\) one has

\[d \left(T_{i_1^\ldots i_p}\right) = d(T) - \sum_{k=1}^p d(I_k),\]

in particular for \(x(y) = y\)

\[d \left(T_{\alpha_1\ldots\alpha_p}\right) = d(T) - p.\]

This result is obvious in the particular case \(x(y) = y\) since the Eq. [A1] reads in components

\[T(\lambda y)_{\alpha_1\ldots\alpha_p} = \lambda^{(r-p)}T(y)_{\alpha_1\ldots\alpha_p}.\]

Now, moving to the more general coordinate basis \(\{x^I(y)\}\) on has

\[T_{i_1^\ldots i_p}(\lambda y) = J^{i_1}_1(\lambda y) \ldots J^{i_p}_p(\lambda y)T_{\alpha_1\ldots\alpha_p}(\lambda y)\]

where \(J^\alpha_\beta(u) := \left(\frac{\partial y^\alpha}{\partial x^I}\right)_u, u \in \mathbb{R}^6\), is homogeneous of degree \(1 - d(I)\). Taking homogeneity into account in the r.h.s. of the above expression leads to

\[T_{i_1^\ldots i_p}(\lambda y) = \lambda(\sum_{k=1}^p (d(I_k))-d(T)+p)J^{i_1}_1(y) \ldots J^{i_p}_p(y)T_{\alpha_1\ldots\alpha_p}(\lambda y),\]

from which the result follows.

Appendix B: Notes on Minkowskian systems

The system appearing in Sec. [14] has already been used in the literature (see for instance [10]), it is reminiscent of systems called polyspherical systems (see for instance [13]). One may note that they are framed in such a way that the “extremal”coordinates \(\{x^+,x^\pm\}\) are functions of the constraints (defining the cone \(C\) and the space \(P_{\lambda,\kappa}\) respectively) whereas the “central” coordinates \(\{x^\mu\}\) are those on the background space \(X_f\). Cartesian Minkowskian coordinates can be obtained on \(X_f\) setting \(x^+ := f\) instead of \(x^+ := f_N\), with this choice conformally invariant equations will appear on their Minkowskian form on \(X_f\).

For the de Sitter space, one may verify that the \(N\) and \(S\) Minkowski charts correspond to the stereographic projections from the North and South Poles from which their names originates.

Finally, we note for convenience that the inverse of system [10] reads

\[
\begin{align*}
y^5 &= x_N^+(1 + x_N^c - \frac{1}{4} \eta_{\mu\nu}x_N^\mu x_N^\nu) \\
y^4 &= x_N^+(1 - x_N^c + \frac{1}{4} \eta_{\mu\nu}x_N^\mu x_N^\nu) \\
y^\mu &= x_N^\mu.
\end{align*}
\]

Appendix C: Conventions and definitions

We summarize here the conventions, main structures and maps used in this paper. They are discussed in Sec. [15]. Here are the conventions for indices:

\[\alpha, \beta, \gamma, \delta, \ldots = 0, \ldots, 5,\]

\[\mu, \nu, \rho, \sigma, \ldots = 0, \ldots, 3,\]

\[i, j, k, l, \ldots = 1, \ldots, 3,\]

\[I, J, \ldots = c, 0, \ldots, 3, +.\]

The coefficients of the metric diag(\(+,-,-,-,+,+\)) of \(\mathbb{R}^6\) are denoted \(\eta_{\alpha\beta}\). The definitions of spaces and maps reads:
$P_f := \{ y \in \mathbb{R}^6 : f(y) = 1 \}$, in which $f$ is homogeneous of degree 1.

$C := \{ y \in \mathbb{R}^6 : (y^0)^2 - y^2 - (y^4)^2 + (y^5)^2 = 0 \}$, the null cone of $\mathbb{R}^6$.

$C'$ denotes the set of the half lines of the cone $C$, namely the cone modulo the dilations. One has $[y] \in C'$ iff $[y] = \{ u \in C : \exists \lambda > 0, u = \lambda y \}$

$X_f = P_f \cap C$, the physical space.

$X'_f = \{ [x] \in C' : \exists x \in X_f \}$ the realization of $X_f$ in $C'$.

$B^*_f = \bigcup_{x \in X_f} T^*_x(\mathbb{R}^6)$

$B^*_f$ the set of sections of $B^*_f$

$l_f : X_f \to \mathbb{R}^6$

$x \mapsto l_f(x) = x$

$l_f : X_f \to \mathbb{R}^6$

$x \mapsto l_f(x) = x$

$\tilde{l}_f : \Omega_1(\mathbb{R}^6) \to B^*_f$

$a \mapsto a^l := \tilde{l}_f(a)$

such that:

$\forall x \in X_f, \forall V \in T_x(\mathbb{R}^6), a^l(x)[V] = a(x)[V]$.

$\lambda_f : X'_f \to X_f$

$[x] \mapsto x_f := [x] \cap X_f$

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$[x] \mapsto x_f := [x] \cap X_f$

$r_f : B^*_f \to \Omega_1(X_f)$

$a^l \mapsto r_f(a^l) = A^l$

such that:

$\forall x \in X_f, \forall U \in T_x(X_f), A^l(x)[U] = a(x)[U]$, where $A^l := l_f^*(a)$ is the physical field on $X_f$.

The Laplace-Beltrami operators on $(\mathbb{R}^6, \eta)$ and $X_f$ are denoted respectively by $\Box_6$ and $\Box_f$. The operator $\Box_f^{(s)}$ is the scalar Laplace-Beltrami operator on $X_f$: it acts on its argument as if it were a scalar.

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