Finite $XXZ$ critical chain
with double boundaries

Takeo Kojima

Department of Mathematics, College of Science and Technology,
Nihon University, Chiyoda-ku, Tokyo 101-0062, Japan

November 19, 2018

Abstract

Finite $XXZ$ chain with double boundaries is considered at critical regime $-1 < \Delta < 1$. We construct the eigenvectors of finite Hamiltonian by means of the vertex operators and the quasi-boundary states. Using the free field realizations of the vertex operators and the quasi-boundary states, integral representations for the correlation functions are derived.

1 Introduction

The $XXZ$ chain is a fundamental model in understanding of the integrable systems. Many attentions have been paid to the $XXZ$ integrable systems [1, 2]. The purpose of this paper is to derive correlation functions for finite $XXZ$ chain with double boundaries at critical regime $-1 < \Delta < 1$, by means of the free field approach.

In the earlier works [3, 4] the $XXZ$ chain with a boundary was considered at massive regime $\Delta < -1$, in the framework of the free field approach. The integral representations of the correlation functions were derived. It was shown that boundary quantum
Knizhnik-Zamolodchikov equations with the certain shift, governed the correlation functions. The $U_q(\widehat{sl}_n)$-generalization of the papers on the XXZ chain [3, 4] was given in [5, 6]. In the paper [7] results for finite XXZ chain at massive regime $\Delta < -1$ were extended to critical regime $-1 < \Delta < 1$, using bosonizations of vertex operators [8].

Y.Fujii and M.Wadati [9] noticed that solutions of boundary quantum Knizhnik-Zamolodchikov equations without shift became eigenstates of finite XXZ chain with double boundaries. They constructed eigenstates of finite XXZ chain with double boundaries at massive regime $\Delta < -1$, by means of the free field approach. In the present paper we shall study finite XXZ chain with double boundaries at critical regime $-1 < \Delta < 1$. We shall derive the correlation functions as integrals of meromorphic functions involving Multi-Gamma functions.

Now a few words about organization of this paper. In section 2 we formulate the problem. In section 3 we construct the realizations of eigenstates. In section 4 we derive integral representations for the correlation functions. In Appendix A we summarized the bosonizations of the vertex operators [8]. In Appendix B we summarized the Multi-Gamma functions.

## 2 Boundary quantum KZ-equation

In 1984 I.V.Cherdnik [10] proposed the following systems of difference equations, now called boundary quantum Knizhnik-Zamolodchikov equations.

$$ F(\beta_1, \ldots, \beta_j + i\lambda, \ldots, \beta_N) = T_j(\beta_1, \ldots, \beta_N|\lambda)F(\beta_1, \ldots, \beta_j, \ldots, \beta_N), \ (j = 1, \ldots, N), \quad (2.1) $$

where the shift operator $T_j(\beta_1 \cdots \beta_N|\lambda)$ is given by

$$ T_j(\beta_1, \ldots, \beta_N|\lambda) = R_{j,j-1}(\beta_j - \beta_{j-1} + i\lambda) \cdots R_{j,1}(\beta_j - \beta_1 + i\lambda) \bar{K}_j(\beta_j) $n

$$ \times R_{1,j}(\beta_1 + \beta_j) \cdots R_{j-1,j}(\beta_{j-1} + \beta_j)R_{j,j+1}(\beta_{j+1} + \beta_j) \cdots R_{N,j}(\beta_N + \beta_j) $$

$$ \times K_j(\beta_j)R_{j,N}(\beta_j - \beta_N) \cdots R_{j,j+1}(\beta_j - \beta_{j+1}). \quad (2.2) $$

The $R$-matrix $R(\beta)$ and the boundary $K$-matrix $K(\beta), \bar{K}(\beta)$ are specified later. The solutions of the boundary quantum KZ equations represent various physical quantities. For the case of the shift parameter $\lambda = 2\pi$, the certain solutions of the quantum KZ
equations represents $N$-point correlation functions for the massless $XXZ$ chain with a boundary, which is described by the following Hamiltonian:

$$\mathcal{H} = -\frac{1}{2} \sum_{n=1}^{\infty} \left( \sigma_{n+1}^x \sigma_n^x + \sigma_{n+1}^y \sigma_n^y + \Delta \sigma_{n+1}^z \sigma_n^z \right) + h \sigma_1^z,$$

(2.3)

where we set a parameter $-1 < \Delta < 1$ and a parameter $h$ represents the boundary external field. The $\sigma_n^x, \sigma_n^y$ and $\sigma_n^z$ stand for the Pauli matrices acting on the $n$-th site of the half Infinite spin chain: $\cdots \otimes \mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2$. The author [7] derived the integrable representations of the correlation functions for the above model.

In the present paper we shall consider the case of the shift parameter $\lambda = 0$. In this case the solution of the boundary quantum KZ equation (2.1) represents the eigenvector of finite $XXZ$ chain with double boundaries at critical regime $-1 < \Delta < 1$. The Hamiltonian $\mathcal{H}_F$ of our considering model is given by

$$\mathcal{H}_F = -\frac{1}{2} \sum_{n=1}^{N-1} \left( \sigma_{n+1}^x \sigma_n^x + \sigma_{n+1}^y \sigma_n^y + \Delta \sigma_{n+1}^z \sigma_n^z \right) + h_1 \sigma_1^z + h_N \sigma_N^z,$$

(2.4)

where we set a parameter $-1 < \Delta < 1$. Parameters $h_1, h_N$ represent the boundary external fields. The $\sigma_n^x, \sigma_n^y$ and $\sigma_n^z$ stand for the Pauli matrices acting on the $n$-th site of the Finite spin chain: $(\mathbb{C}^2)^{\otimes N}$.

Let us set the $R$-matrix as

$$R(\beta) = r(\beta) \begin{pmatrix} 1 & b(\beta) & c(\beta) \\ b(\beta) & c(\beta) & b(\beta) \\ c(\beta) & b(\beta) & 1 \end{pmatrix},$$

(2.5)

where we set the components as

$$b(\beta) = -\frac{\text{sh} \left( \frac{\beta}{\xi + 1} \right)}{\text{sh} \left( \frac{\beta + \pi i}{\xi + 1} \right)}, \quad c(\beta) = \frac{\pi i}{\xi + 1}.$$

(2.6)

Here we set

$$r(\beta) = -\frac{S_2(i\beta, \pi, \pi(\xi + 1)) S_2(-i\beta + \pi, \pi, \pi(\xi + 1))}{S_2(-i\beta, \pi, \pi(\xi + 1)) S_2(i\beta, \pi, \pi(\xi + 1))}.$$

(2.7)
where $S_2(\beta | \omega_1 \omega_2)$ is the double sine function defined in Appendix B. Let $\{v_+, v_-\}$ denote the natural basis of $V = \mathbb{C}^2$. When viewed as an operator on $V \otimes V$, the matrix elements of $R(\beta)$ are defined by

$$R(\beta) v_{k_1} \otimes v_{k_2} = \sum_{j_1, j_2 = \pm} v_{j_1} \otimes v_{j_2} R(\beta)^{k_1 k_2}_{j_1 j_2}.$$  

(2.8)

The $R$-matrix satisfies the Yang-Baxter equation:

$$R_{12}(\beta_1 - \beta_2) R_{13}(\beta_1 - \beta_3) R_{23}(\beta_2 - \beta_3) = R_{23}(\beta_2 - \beta_3) R_{13}(\beta_1 - \beta_3) R_{12}(\beta_1 - \beta_2).$$  

(2.9)

The normalization factor $r_0(\beta)$ is so chosen that the unitarity and crossing relations are

$$R_{12}(\beta) R_{21}(-\beta) = \text{id},$$

(2.10)

$$R(-\beta)^{k_1 k_2}_{j_1 j_2} = R(\beta - \pi i)^{-j_2 k_1}_{-k_2 j_1}.$$  

(2.11)

Let us set the boundary $K$-matrix $K(\beta)$ by

$$K(\beta) = k(\beta)\begin{pmatrix} 1 & 0 \\ 0 & \frac{\sh(\nu + \beta)}{\sh(\xi + 1)} \\ 0 & \frac{\sh(\nu - \beta)}{\sh(\xi + 1)} \end{pmatrix},$$  

(2.12)

where the normalization factor is given by

$$k(\beta) = k_0(\beta) k_1(\beta),$$  

(2.13)

where

$$k_0(\beta) = \frac{S_2(-2i \beta + 4\pi|4\pi, \pi(\xi + 1)) S_2(2i \beta + 3\pi|4\pi, \pi(\xi + 1))}{S_2(2i \beta + 4\pi|4\pi, \pi(\xi + 1)) S_2(-2i \beta + 3\pi|4\pi, \pi(\xi + 1))},$$

(2.14)

$$k_1(\beta) = \frac{S_2(-i \beta + iv + \pi|2\pi, \pi(\xi + 1)) S_2(i \beta + iv + 2\pi|2\pi, \pi(\xi + 1))}{S_2(i \beta + iv + \pi|2\pi, \pi(\xi + 1)) S_2(-i \beta + iv + 2\pi|2\pi, \pi(\xi + 1))}.$$  

(2.15)

The matrix elements $K(\beta)^{j}_j$ are defined by

$$K(\beta) v_k = \sum_{j = \pm} v_j K(\beta)^{j}_j.$$  

(2.16)

The $R$-matrix and the $K$-matrix satisfy the Boundary Yang-Baxter equation:

$$K_{22}(\beta_2) R_{21}(\beta_1 + \beta_2) K_{11}(\beta_1) R_{12}(\beta_1 - \beta_2) = R_{21}(\beta_1 - \beta_2) K_{11}(\beta_1) R_{12}(\beta_1 + \beta_2) K_{22}(\beta_2).$$  

(2.17)
The normalization factor $k(\beta)$ is so chosen that the boundary unitarity and the boundary crossing relations are

$$K(\beta) K(-\beta) = id, \quad (2.18)$$

$$K\left(\beta + \frac{\pi i}{2}\right)^j_j = \sum_{k=\pm} R(2\beta)_{k,-k}^j K\left(-\beta + \frac{\pi i}{2}\right)^k_k. \quad (2.19)$$

Let us set the boundary $K$-matrix $\bar{K}(\beta)$ by

$$\bar{K}(\beta) = K(\beta)|_{\mu \leftrightarrow \nu}. \quad (2.20)$$

**Note.** For the another shift parameter $\lambda = 2\pi$ case, we take another choice of the $K$-matrix $\bar{K}(\beta)$. See the reference [3].

The derivatives of $R$-matrix and $K$-matrix are given by

$$\frac{\partial}{\partial \beta} R_{j,j+1}(\beta) P_{j,j+1}\bigg|_{\beta=0} = \frac{-1}{2(\xi + 1)} \times \frac{1}{\text{sh} \left( \frac{\pi i}{\xi + 1} \right)} \left( \sigma_j^x \sigma_{j+1}^x + \sigma_j^y \sigma_{j+1}^y + \Delta \sigma_j^z \sigma_{j+1}^z \right) + \text{const.} \quad (2.21)$$

$$\frac{\partial}{\partial \beta} K_j(\beta)\bigg|_{\beta=0} = \frac{-1}{\xi + 1} \times \frac{\text{ch} \left( \frac{\nu}{\xi + 1} \right)}{\text{sh} \left( \frac{\nu}{\xi + 1} \right)} \sigma_j^z + \text{const.} \quad (2.22)$$

Here the anisotropic parameter $\Delta = -\cos \left( \frac{\pi}{\xi + 1} \right)$.

Therefore the shift operator $T_j(\beta_1 \cdots \beta_N|0)$ is related to the Hamiltonian $H_F$ [2.4] as following.

$$\text{sh} \left( \frac{\pi i}{\xi + 1} \right) \times \frac{\xi + 1}{2} \times \left( \frac{\partial}{\partial \beta_j} T_j \right)(0, \cdots, 0|0) = H_F + \text{const.} \quad (2.23)$$

Here the boundary magnetic fields $h_1, h_N$ are related with parameters $\mu, \nu$,

$$h_1 = -\frac{1}{2} \times \text{sh} \left( \frac{\pi i}{\xi + 1} \right) \frac{\text{ch} \left( \frac{\nu}{\xi + 1} \right)}{\text{sh} \left( \frac{\nu}{\xi + 1} \right)}, \quad h_N = -\frac{1}{2} \times \text{sh} \left( \frac{\pi i}{\xi + 1} \right) \frac{\text{ch} \left( \frac{\mu}{\xi + 1} \right)}{\text{sh} \left( \frac{\mu}{\xi + 1} \right)}. \quad (2.24)$$
We have

\[ T_j(0, \cdots, 0|0) = id. \]  (2.25)

Let us set the eigenvector \(|\beta_1 \cdots \beta_N\rangle\) by

\[ T_j(\beta_1, \cdots, \beta_N|0|\beta_1, \cdots, \beta_N) = |\beta_1, \cdots, \beta_N\rangle. \]  (2.26)

Let us set the dual eigenvector by

\[ \langle \beta_1, \cdots, \beta_N|T_j(\beta_1, \cdots, \beta_N|0) = \langle \beta_1, \cdots, \beta_N|. \]  (2.27)

The above eigenvectors satisfy the followings.

\[ \mathcal{H}_F|0, \cdots, 0\rangle = Const.|0, \cdots, 0\rangle, \quad \langle 0, \cdots, 0|\mathcal{H}_F = Const.|0, \cdots, 0|. \]  (2.28)

In the next section we shall construct the eigenvector \(|\beta_1 \cdots \beta_N\rangle\) explicitly.

## 3 Eigenvectors

In this section we solve following eigenvector problem.

\[ T_j(\beta_1, \cdots, \beta_N|0|\beta_1, \cdots, \beta_N) = |\beta_1, \cdots, \beta_N\rangle. \]  (3.1)

The eigenvector is realized by using the vertex operators \(\Phi_j(\beta)\).

\[ |\beta_1, \cdots, \beta_N\rangle = \frac{1}{\langle G|G \rangle} \sum_{\epsilon_1 \cdots \epsilon_N = \pm} \langle G|\Phi_{\epsilon_1}(\beta_1) \cdots \Phi_{\epsilon_N}(\beta_N)|G\rangle(v_{\epsilon_1} \otimes \cdots \otimes v_{\epsilon_N}). \]  (3.2)

Here the vector \(|G\rangle\) and the dual vector \(\langle G|\) are characterized by the following relations.

\[ \langle G|\Phi_j(\beta) = \bar{K}(\beta)^{j}_{j'}\langle G|\Phi_{j'}(-\beta), \quad (j = \pm), \]  (3.3)

\[ \Phi_j(-\beta)|G\rangle = K(\beta)^{j}_{j'}\Phi_j(\beta)|G\rangle, \quad (j = \pm), \]  (3.4)

We call the auxiliary states \(\langle G|\) and \(|G\rangle\) “quasi-boundary state”. Using the following commutation relation of the vertex operator and the characterizing relations (3.3, 3.4) of the quasi-boundary state, we have the equation (3.1).

\[ \Phi_{j_1}(\beta_1)\Phi_{j_2}(\beta_2) = \sum_{k_1, k_2 = \pm} R(\beta_1 - \beta_2)^{k_1,k_2}_{j_1,j_2}\Phi_{k_2}(\beta_2)\Phi_{k_1}(\beta_1). \]  (3.5)
We give the free field realization of the quasi-boundary state.

Let us introduce the free bosons \( b(t), (t \in \mathbb{R}) \) by

\[
[b(t), b(t')] = \frac{\text{sh} \left( \frac{\pi t}{2} \text{sh} (\pi t) \text{sh} \left( \frac{\pi t \xi}{2} \right) \right)}{t \text{sh} \left( \frac{\pi t (\xi + 1)}{2} \right)} \delta (t + t').
\] (3.6)

Let us set the Fock space \( \mathcal{H} \) generated by the vacuum vector \( \langle \text{vac} | \) which satisfies

\[
\langle \text{vac} | b(-t) = 0, \text{ if } t > 0.
\] (3.7)

The quasi-boundary state \( \langle G | \) is realized as followings.

\[
\langle G | = \langle \text{vac} | e^{G},
\] (3.8)

Here we have set

\[
G = \frac{1}{2} \int_0^\infty \frac{G_2(t|\mu)}{[b(t), b(-t)]} b(t)^2 dt + \int_0^\infty \frac{G_1(t|\mu)}{[b(t), b(-t)]} b(t) dt,
\] (3.9)

where

\[
G_2(t|\mu) = -1,
\] (3.10)

\[
G_1(t|\mu) = \frac{1}{t} \frac{\text{sh} \left( \frac{\pi t}{2} \right) \text{sh} \left( \frac{(i\mu + \frac{\pi}{2} \xi) t}{2} \right)}{\text{sh} \left( \frac{\pi t (\xi + 1)}{2} \right)} + \frac{1}{t} \frac{\text{sh} \left( \frac{\pi}{4} t \right) \text{ch} \left( \frac{\pi}{4} \xi t \right)}{\text{sh} \left( \frac{\pi}{4} (\xi + 1) t \right)}.
\] (3.11)

Let us prove the relation (3.3). In what follows we use the abbreviations.

\[
U_+ (\beta) = \exp \left( - \int_0^\infty \frac{b(t)}{\text{sh} \pi t} e^{i\beta t} dt \right),
\]

\[
U_- (\beta) = \exp \left( \int_0^\infty \frac{b(-t)}{\text{sh} \pi t} e^{-i\beta t} dt \right),
\] (3.12)

\[
\bar{U}_+ (\alpha) = \exp \left( \int_0^\infty \frac{b(t)}{\text{sh} \frac{\pi}{2} t} e^{i\alpha t} dt \right),
\]

\[
\bar{U}_- (\alpha) = \exp \left( - \int_0^\infty \frac{b(-t)}{\text{sh} \frac{\pi}{2} t} e^{-i\alpha t} dt \right).
\] (3.13)

In what follows we omit non-essential constant factors.

At first we explain the formulas of the form

\[
X(\beta_1) Y(\beta_2) = C_{XY} (\beta_1 - \beta_2) : X(\beta_1) X(\beta_2) :,
\] (3.14)

where \( X, Y = U_j \), and \( C_{XY} (\beta) \) is a meromorphic function on \( \mathbb{C} \). These formulae follow from the commutation relation of the free bosons. When we compute the contraction of the basic operators, we often encounter an integral

\[
\int_0^\infty F(t) dt,
\] (3.15)
which is divergent at $t = 0$. Here we adopt the following prescription for regularization: it should be understood as the contour integral,

$$
\int_{C} F(t) \frac{\log(-t)}{2\pi i} dt, \quad (3.16)
$$

where the contour $C$ is given by

![Contour C](image)

**Contour C**

The actions of the basic operators on quasi-boundary state $\langle G \rangle$ are evaluated as followings.

$$
\langle G \mid U_-(\beta) = Const.m(\beta)\langle G \mid U_+(-\beta), \quad (3.17)
$$

$$
\langle G \mid \bar{U}_-(\alpha) = Const.J(\alpha)\langle G \mid \bar{U}_+(-\alpha). \quad (3.18)
$$

Here we have set

$$
m(\beta) = \frac{\Gamma_2(2i\beta + 4\pi|2\omega_1\omega_2)\Gamma_2(2i\beta + \pi(\xi + 1)|2\omega_1\omega_2)}{\Gamma_2(2i\beta + 3\pi|2\omega_1\omega_2)\Gamma_2(2i\beta + \pi(\xi + 2)|2\omega_1\omega_2)} \times \frac{\Gamma_2(i\beta + i\mu + \pi|\omega_1\omega_2)\Gamma_2(i\beta - i\mu + \pi(\xi + 2)|\omega_1\omega_2)}{\Gamma_2(i\beta + i\mu + 2\pi|\omega_1\omega_2)\Gamma_2(i\beta - i\mu + \pi(\xi + 1)|\omega_1\omega_2)}, \quad (3.19)
$$

$$
J(\alpha) = \alpha \times \frac{\Gamma\left(\frac{-i\mu+i\alpha}{\pi(\xi+1)} + 1 - \frac{1}{2(\xi+1)}\right)}{\Gamma\left(\frac{i\mu+i\alpha}{\pi(\xi+1)} + \frac{1}{2(\xi+1)}\right)} \quad (3.20)
$$

We have

$$
\langle G \mid \Phi_+(\beta) = m(\beta)\langle G \mid U_+(\beta)U_+(-\beta). \quad (3.21)
$$

Because the function $m(\beta)$ satisfies

$$
\bar{K}(\beta)^+ = \frac{m(\beta)}{m(-\beta)}, \quad (3.22)
$$

we have proved the "$+$"-part of the characterizing relation (3.3).

We will prove the "$-$"-part of the equation (3.3). Using the actions formulae of the basic
operators on the quasi-boundary state, we have the following.

\[ \langle G | \Phi - (\beta) = \text{Const}.m(\beta) \int_{-\infty}^{\infty} d\alpha \times \alpha \times \prod_{\epsilon_1,\epsilon_2=\pm} \Gamma \left( \frac{i(\epsilon_1\alpha + \epsilon_2\beta)}{\pi(\xi + 1)} + \frac{1}{2(\xi + 1)} \right) \]

\[ \times \text{sh} \left( \frac{\alpha + \beta}{\xi + 1} + \frac{\pi i}{2(\xi + 1)} \right) \frac{\Gamma \left( \frac{-i\mu + i\alpha}{\pi(\xi + 1)} + 1 - \frac{1}{2(\xi + 1)} \right)}{\Gamma \left( \frac{i\mu + i\alpha}{\pi(\xi + 1)} + \frac{1}{2(\xi + 1)} \right)} \]

\[ \times \langle G | U_+ (\beta) U_+(-\beta) \bar{U}_+ (\alpha) \bar{U}_+(-\alpha) \rangle. \] \hspace{1cm} (3.23)

Note that the operator part of the above equation: \( \langle G | U_-(\beta) U_-(\beta) \bar{U}_-(\alpha) \bar{U}_-(\alpha) \rangle \) is invariant under the changes of variables: \( \alpha \leftrightarrow -\alpha, \beta \leftrightarrow -\beta \).

We have

\[ m(\beta)^{-1} \text{sh} \left( \frac{\mu - \beta}{\xi + 1} \right) \langle G | \Phi - (\beta) - m(-\beta)^{-1} \text{sh} \left( \frac{\mu + \beta}{\xi + 1} \right) \langle G | \Phi - (-\beta) \right) \]

\[ = \text{Const.} \times \text{sh} \left( \frac{2\beta}{\xi + 1} \right) \int_{-\infty}^{\infty} d\alpha \prod_{\epsilon_1,\epsilon_2=\pm} \Gamma \left( \frac{i(\epsilon_1\alpha + \epsilon_2\beta)}{\pi(\xi + 1)} + \frac{1}{2(\xi + 1)} \right) \]

\[ \times \prod_{\epsilon=\pm} \Gamma \left( \frac{-i\mu + \epsilon\alpha}{\pi(\xi + 1)} + 1 - \frac{1}{2(\xi + 1)} \right) \times \alpha \prod_{\epsilon=\pm} \text{sh} \left( \frac{\mu + \epsilon\alpha}{\xi + 1} - \frac{\pi i}{2(\xi + 1)} \right) \]

\[ \times \langle G | U_-(\beta) U_-(\beta) \bar{U}_-(\alpha) \bar{U}_-(\alpha) \rangle. \] \hspace{1cm} (3.24)

The integrand of (RHS) is anti-symmetric to a change of integral variable \( \alpha \leftrightarrow -\alpha \). It means the left-hand side becomes zero after taking integral. Therefore we get

\[ m(\beta) \text{sh} \left( \frac{\mu - \beta}{\xi + 1} \right) \langle G | \Phi - (\beta) = m(\beta) \text{sh} \left( \frac{\mu + \beta}{\xi + 1} \right) \langle G | \Phi - (-\beta). \] \hspace{1cm} (3.25)

We have proved the "−"-part of the characterizing relation (3.3).

The quasi-boundary state \( |G\rangle \) is given by the following.

\[ |G\rangle = e^{G^*} |\text{vac}\rangle, \] \hspace{1cm} (3.26)

where

\[ G^* = \frac{1}{2} \int_0^{\infty} \frac{G_2^*(t|\mu)}{[b(t), b(-t)]} b(-t)^2 dt + \int_0^{\infty} \frac{G_1^*(t|\mu)}{[b(t), b(-t)]} b(-t) dt, \] \hspace{1cm} (3.27)

where

\[ G_2^*(t|\mu) = G_2(t|\mu), \ G_1^*(t|\mu) = -G_1(t|\mu). \] \hspace{1cm} (3.28)

As the same manner as the above we can prove the characterizing equations (3.4).
Note. When we consider the massless XXZ chain with a boundary \[7\]. We introduce the boundary state \(|B\rangle\) and the dual boundary state \(\langle B|\). Left quasi-boundary state \(\langle G|\) differs from the dual boundary state \(\langle B|\). Right quasi-boundary state \(|G\rangle\) coincides with the boundary state \(|B\rangle\). Physically the boundary state \(|B\rangle\) of the paper \[7\] corresponds to the vacuum expectation value \(\langle G|\Phi(\beta_1)\cdots\Phi(\beta_N)|G\rangle\) of the present paper. Both quantities \(|B\rangle\) and \(\langle G|\Phi(\beta_1)\cdots\Phi(\beta_N)|G\rangle\) represent an eigenvector of the Hamiltonian for each model.

Let us construct the dual stationary state,

\[
\langle \beta_1, \cdots \beta_N | T_j (\beta_1, \cdots, \beta_N) | 0 \rangle = \langle \beta_1, \cdots \beta_N \rangle. \tag{3.29}
\]

The dual stationary state is realized by using the dual vertex operators \(\Phi_j^*(\beta)\).

\[
\langle \beta_1, \cdots \beta_N \rangle = \frac{1}{\langle F|F \rangle} \sum_{\epsilon_1, \cdots, \epsilon_N = \pm} \langle F|\Phi_{\epsilon_1}^*(\beta_1) \cdots \Phi_{\epsilon_N}^*(\beta_N)|F \rangle (v_{\epsilon_1}^* \otimes \cdots \otimes v_{\epsilon_N}^*). \tag{3.30}
\]

The dual vertex operators are related to the vertex operators,

\[
\Phi_j^*(\beta) = \Phi_{-j}(\beta + \pi i), \quad (j = \pm). \tag{3.31}
\]

The quasi-boundary state \(|F\rangle\) and \(|F\rangle\) are characterized by

\[
\langle F|\Phi_j^*(\beta) = K(\beta)\langle F|\Phi_j^*(-\beta), \quad (j = \pm), \tag{3.32}
\]

\[
\Phi_j^*(-\beta)|F\rangle = K(\beta)\Phi_j^*(\beta)|F\rangle, \quad (j = \pm). \tag{3.33}
\]

The quasi-boundary states \(|F\rangle\) and \(|F\rangle\) are realized as followings.

\[
\langle F\rangle = \langle \text{vac}|e^F, \quad |F\rangle = e^{F^*}|\text{vac}\rangle. \tag{3.34}
\]

Here we have set

\[
F = \frac{1}{2} \int_0^\infty \frac{F_2(t|\mu)}{[b(t), b(-t)]} b(t)^2 dt + \int_0^\infty \frac{F_1(t|\mu)}{[b(t), b(-t)]} b(t) dt, \tag{3.35}
\]

\[
F^* = \frac{1}{2} \int_0^\infty \frac{F_2^*(t|\nu)}{[b(t), b(-t)]} b(-t)^2 dt + \int_0^\infty \frac{F_1^*(t|\nu)}{[b(t), b(-t)]} b(-t) dt, \tag{3.36}
\]

where

\[
F_2(t|\mu) = -e^{-2\pi t}, \tag{3.37}
\]

\[
F_1(t|\mu) = \frac{e^{-\pi t} \text{sh} \left( \frac{\pi t}{2} \right)}{t} \text{sh} \left( \frac{\pi (\xi + 1)t}{2} \right) + \frac{e^{-\pi t} \text{sh} \left( \frac{\pi t}{4} \right)}{t} \text{ch} \left( \frac{\pi \xi t}{4} \right) \text{sh} \left( \frac{\pi (\xi + 1)t}{4} \right). \tag{3.38}
\]
Formally other eigenvectors are constructed by inserting the type-II vertex operators. The characterizing relations are proved as the same manner. Here we omit details.

\[
\sum_{\epsilon_1, \ldots, \epsilon_N = \pm} \langle G| \Phi_{\epsilon_1} (\beta_1) \cdots \Phi_{\epsilon_N} (\beta_N) \Psi^*_{j_1} (\xi_1) \cdots \Psi^*_{j_M} (\xi_M) | G \rangle (\nu_{\epsilon_1} \otimes \cdots \otimes \nu_{\epsilon_N}) \tag{3.41}
\]

4 Correlation Functions

In this section we calculate the vacuum expectation values of the type-I vertex operators, and obtain them as integrals of meromorphic functions involving Multi-Gamma functions. We compute the following 2\(N\)-point function,

\[
P_{\epsilon_1, \ldots, \epsilon_N; \epsilon_N; \ldots, \epsilon_1} (\{ \beta^*_j \}; \{ \beta_j \}) = \frac{\langle F_\eta | \Phi_{\epsilon_1} (\beta_1^*) \cdots \Phi_{\epsilon_N} (\beta_N^*) | F_\eta \rangle}{\langle F_\eta | F_\eta \rangle} \times \frac{\langle G_\eta | \Phi_{\epsilon_N} (\beta_N) \cdots \Phi_{\epsilon_1} (\beta_1) | G_\eta \rangle}{\langle G_\eta | G_\eta \rangle} \tag{4.1}
\]

Here we set the state \( \langle G_\eta |, | G_\eta \rangle \) and \( \langle F_\eta |, | F_\eta \rangle \) by

\[
\langle G_\eta | = \langle \text{vac} | e^{G_\eta}, \quad | G_\eta \rangle = e^{G^*_\eta} | \text{vac} \rangle, \tag{4.2}
\]

\[
\langle F_\eta | = \langle \text{vac} | e^{F_\eta}, \quad | F_\eta \rangle = e^{F^*_\eta} | \text{vac} \rangle. \tag{4.3}
\]

where

\[
G_\eta = \frac{1}{2} \int_0^\infty \frac{e^{-\eta t} G_2 (t | \mu) \mu}{[b(t), b(-t)]} b(t)^2 dt + \int_0^\infty \frac{G_1 (t | \mu) \mu}{[b(t), b(-t)]} b(t) dt, \tag{4.4}
\]

\[
G^*_\eta = \frac{1}{2} \int_0^\infty \frac{e^{-\eta t} G_2^* (t | \mu) \mu}{[b(t), b(-t)]} b(-t)^2 dt + \int_0^\infty \frac{G_1^* (t | \mu) \mu}{[b(t), b(-t)]} b(-t) dt, \tag{4.5}
\]

\[
F_\eta = \frac{1}{2} \int_0^\infty \frac{e^{-\eta t} F_2 (t | \nu) \nu}{[b(t), b(-t)]} b(t)^2 dt + \int_0^\infty \frac{F_1 (t | \nu) \nu}{[b(t), b(-t)]} b(t) dt, \tag{4.6}
\]

\[
F^*_\eta = \frac{1}{2} \int_0^\infty \frac{e^{-\eta t} F_2^* (t | \nu) \nu}{[b(t), b(-t)]} b(-t)^2 dt + \int_0^\infty \frac{F_1^* (t | \nu) \nu}{[b(t), b(-t)]} b(-t) dt. \tag{4.7}
\]

We have

\[
\lim_{\eta \to 0} \langle G_\eta | = \langle G |, \quad \lim_{\eta \to 0} | G_\eta \rangle = | G \rangle, \quad \lim_{\eta \to 0} \langle F_\eta | = \langle F |, \quad \lim_{\eta \to 0} | F_\eta \rangle = | F \rangle. \tag{4.8}
\]

First we compute the normal-ordering of the vertex operators.

Let us denote by \( A \) the index set

\[
A = \{ a | \epsilon_a = -, 1 \leq a \leq N \} \tag{4.9}
\]
We have the following expressions.

\[
\frac{\langle G_\eta | \Phi_e (\beta_N) \cdots \Phi_{\epsilon_1} (\beta_1) \rangle | G_\eta \rangle}{\langle G_\eta | G_\eta \rangle} = \prod_{1 \leq b_1 < b_2 \leq N} \frac{\Gamma (i(\beta_{b_2} - \beta_{b_1}) + 2\pi |\omega_1 \omega_2|)}{\Gamma (i(\beta_{b_2} - \beta_{b_1}) + \pi |\omega_1 \omega_2|)} \frac{\Gamma (i(\beta_{b_2} - \beta_{b_1}) + \pi |\omega_1 \omega_2|)}{\Gamma (i(\beta_{b_2} - \beta_{b_1}) + (\xi + 2) |\omega_1 \omega_2|)} \\
\times \prod_{a \in A} \int_{-\infty}^{\infty} d\alpha_a \prod_{a \in A} \Gamma \left( \frac{i(\alpha_a - \beta_a)}{\pi (\xi + 1)} + \frac{1}{2(\xi + 1)} \right) \frac{\Gamma \left( \frac{i(\beta_a - \alpha_a)}{\pi (\xi + 1)} + \frac{1}{2(\xi + 1)} \right)}{\Gamma \left( \frac{i(\alpha_a - \beta_a)}{\pi (\xi + 1)} + \frac{1}{2(\xi + 1)} \right)} \frac{\Gamma \left( \frac{i(\beta_a - \alpha_a)}{\pi (\xi + 1)} + \frac{1}{2(\xi + 1)} \right)}{\Gamma \left( \frac{i(\alpha_a - \beta_a)}{\pi (\xi + 1)} + \frac{1}{2(\xi + 1)} \right)} \\
\times \prod_{a > b} \frac{\Gamma \left( \frac{i(\beta_a - \alpha_a)}{\pi (\xi + 1)} + \frac{1}{2(\xi + 1)} \right)}{\Gamma \left( \frac{i(\beta_a - \alpha_a)}{\pi (\xi + 1)} + \frac{1}{2(\xi + 1)} \right)} \frac{\Gamma \left( \frac{i(\beta_a - \alpha_a)}{\pi (\xi + 1)} + \frac{1}{2(\xi + 1)} \right)}{\Gamma \left( \frac{i(\beta_a - \alpha_a)}{\pi (\xi + 1)} + \frac{1}{2(\xi + 1)} \right)} \\
\times I'' \left( \{ \beta^*_a \} | \{ \alpha_a \} \right).
\]

Let us denote by \( A^* \) the index set

\[
A^* = \{ a | \epsilon_a = +, 1 \leq a \leq N \}
\]

We have the following expressions.

\[
\frac{\langle F_\eta | \Phi_{\epsilon_1}^* (\beta_1^* - \pi i) \cdots \Phi_{\epsilon_N}^* (\beta_N^* - \pi i) \rangle | F_\eta \rangle}{\langle F_\eta | F_\eta \rangle} = \prod_{1 \leq b_1 < b_2 \leq N} \frac{\Gamma (i(\beta_{b_2} - \beta_{b_1}) + 2\pi |\omega_1 \omega_2|)}{\Gamma (i(\beta_{b_2} - \beta_{b_1}) + \pi |\omega_1 \omega_2|)} \frac{\Gamma (i(\beta_{b_2} - \beta_{b_1}) + \pi |\omega_1 \omega_2|)}{\Gamma (i(\beta_{b_2} - \beta_{b_1}) + (\xi + 2) |\omega_1 \omega_2|)} \\
\times \prod_{a \in A^*} \int_{-\infty}^{\infty} d\alpha_a \prod_{a \in A^*} \Gamma \left( \frac{i(\alpha_a - \beta_a)}{\pi (\xi + 1)} + \frac{1}{2(\xi + 1)} \right) \frac{\Gamma \left( \frac{i(\beta_a - \alpha_a)}{\pi (\xi + 1)} + \frac{1}{2(\xi + 1)} \right)}{\Gamma \left( \frac{i(\alpha_a - \beta_a)}{\pi (\xi + 1)} + \frac{1}{2(\xi + 1)} \right)} \frac{\Gamma \left( \frac{i(\beta_a - \alpha_a)}{\pi (\xi + 1)} + \frac{1}{2(\xi + 1)} \right)}{\Gamma \left( \frac{i(\alpha_a - \beta_a)}{\pi (\xi + 1)} + \frac{1}{2(\xi + 1)} \right)} \\
\times \prod_{a > b} \frac{\Gamma \left( \frac{i(\beta_a - \alpha_a)}{\pi (\xi + 1)} + \frac{1}{2(\xi + 1)} \right)}{\Gamma \left( \frac{i(\beta_a - \alpha_a)}{\pi (\xi + 1)} + \frac{1}{2(\xi + 1)} \right)} \frac{\Gamma \left( \frac{i(\beta_a - \alpha_a)}{\pi (\xi + 1)} + \frac{1}{2(\xi + 1)} \right)}{\Gamma \left( \frac{i(\beta_a - \alpha_a)}{\pi (\xi + 1)} + \frac{1}{2(\xi + 1)} \right)} \\
\times I'' \left( \{ \beta^*_a - \pi i \} | \{ \alpha_a \} \right).
\]
Here we have set

\[ I_\eta(\{\beta_b\}|\{\alpha_a\}) = \frac{\langle G_\eta| \exp \left( \int_0^\infty X_A(t)b(-t)dt \right) \exp \left( \int_0^\infty Y_A(t)b(t)dt \right) |G_\eta \rangle}{\langle G_\eta|G_\eta \rangle}, \]  

(4.13)

\[ I_\eta^\ast(\{\beta_b^\ast - \pi i\}|\{\alpha_a\}) = \frac{\langle F_\eta| \exp \left( \int_0^\infty X_A^\ast(t)b(-t)dt \right) \exp \left( \int_0^\infty Y_A^\ast(t)b(t)dt \right) |F_\eta \rangle}{\langle F_\eta|F_\eta \rangle}, \]  

(4.14)

with

\[ X_A(t) = \sum_{b=1}^N \frac{e^{-i\beta_b t}}{\tsh(\pi t)} - \sum_{a \in \Lambda} \frac{e^{-i\alpha_a t}}{\tsh \left( \frac{\pi t}{2} \right)} \]  

(4.15)

\[ Y_A(t) = -\sum_{b=1}^N \frac{e^{i\beta_b t}}{\tsh(\pi t)} + \sum_{a \in \Lambda} \frac{e^{i\alpha_a t}}{\tsh \left( \frac{\pi t}{2} \right)}. \]  

(4.16)

We evaluate the quantities \( I_\eta(\{\beta_b\}|\{\alpha_a\}), I_\eta^\ast(\{\beta_b^\ast - \pi i\}|\{\alpha_a\}) \). Using the completeness relation of the coherent state \([3, 7]\), and performing the integral calculations, we have

\[ I_\eta(\{\beta_b\}|\{\alpha_a\}) = \exp \left( \int_0^\infty \frac{1}{1 - e^{-2\eta t}} \frac{\tsh \left( \frac{\pi t}{2} \right)}{\tsh \left( \frac{\pi t}{2} \right)} \right) \times \left( \frac{1}{2} e^{-\eta t} X_A(t)^2 + e^{-2\eta t} X_A(t)Y_A(t) - \frac{1}{2} e^{-\eta t} Y_A(t)^2 \right) \]  

(4.17)

\[ + \int_0^\infty \frac{1}{1 - e^{-2\eta t}} \left\{ (G_1(t)|\mu) - e^{-\eta t} G_1^\ast(t)|\nu) \right\} X_A(t) + (G_1^\ast(t)|\nu) - e^{-\eta t} G_1(t)|\mu) \right\} Y_A(t) \right\} dt \right). \]

and

\[ I_\eta^\ast(\{\beta_b^\ast - \pi i\}|\{\alpha_a\}) = \exp \left( \int_0^\infty \frac{1}{1 - e^{-2\eta t}} \frac{\tsh \left( \frac{\pi t}{2} \right)}{\tsh \left( \frac{\pi t}{2} \right)} \right) \times \left( \frac{1}{2} e^{-\eta t - 2\pi t} X_A^\ast(t)^2 + e^{-2\eta t} X_A(t)Y_A^\ast(t) - \frac{1}{2} e^{-\eta t + 2\pi t} Y_A(t)^2 \right) \]  

(4.18)

\[ + \int_0^\infty \frac{1}{1 - e^{-2\eta t}} \left\{ (F_1(t)|\mu) - e^{-\eta t - 2\pi t} F_1^\ast(t)|\nu) \right\} X_A^\ast(t) + (F_1^\ast(t)|\nu) - e^{-\eta t + 2\pi t} F_1(t)|\mu) \right\} Y_A^\ast(t) \right\} dt \right). \]

In what follows we use the abbreviations:

\[ \omega_1 = 2\pi, \omega_2 = \pi(\xi + 1), \omega_3 = 2\eta, \mu_+ = \mu, \mu_- = \nu, \]  

(4.19)

The vacuum expectation value is evaluated as following.

\[ I_\eta(\{\beta_b\}|\{\alpha_a\}) = I_\eta^\beta(\{\beta_b\}) I_\eta^\alpha(\{\beta_b\}) I_\eta^\alpha(\{\alpha_a\}). \]  

(4.20)
Here we set

\[ I_\eta^n(\{\beta_b\}) = \prod_{b=1}^N \prod_{\epsilon = \pm} \sqrt{\frac{S_3(2i\epsilon \beta_b + \pi + \eta|\omega_1 \omega_2 \omega_3)}{S_3(2i\epsilon \beta_b + 2\pi + \eta|\omega_1 \omega_2 \omega_3)}} \prod_{b_1 < b_2} \prod_{\epsilon_1 < \epsilon_2 = \pm} \frac{\Gamma_2(2i\epsilon \beta_{b_1} - \beta_{b_2}) + \pi|\omega_1 \omega_2 \omega_3)}{\Gamma_2(2i\epsilon \beta_{b_1} - \beta_{b_2}) + 2\pi|\omega_1 \omega_2 \omega_3)}
\]

\[ \times \prod_{b_1 < b_2} \prod_{\epsilon_1 < \epsilon_2 = \pm} \frac{S_3(i\epsilon(\beta_{b_1} + \beta_{b_2}) + \pi + \eta|\omega_1 \omega_2 \omega_3)S_3(i\epsilon(\beta_{b_1} - \beta_{b_2}) + \pi|\omega_1 \omega_2 \omega_3)}{S_3(i\epsilon(\beta_{b_1} + \beta_{b_2}) + 2\pi + \eta|\omega_1 \omega_2 \omega_3)S_3(i\epsilon(\beta_{b_1} - \beta_{b_2}) + 2\pi|\omega_1 \omega_2 \omega_3)}
\]

\[ \times \prod_{b=1}^N \prod_{\epsilon = \pm} \frac{\Gamma_3(i\epsilon \beta_b + i\mu_e + \pi|\omega_1 \omega_2 \omega_3)\Gamma_3(i\epsilon \beta_b - i\mu_e + \pi \xi + 2\pi|\omega_1 \omega_2 \omega_3)}{\Gamma_3(i\epsilon \beta_b - i\mu_e + \pi \xi + 2\pi|\omega_1 \omega_2 \omega_3)\Gamma_3(i\epsilon \beta_b + i\mu_e + 2\pi|\omega_1 \omega_2 \omega_3)}
\]

\[ \times \prod_{b=1}^N \prod_{\epsilon = \pm} \frac{\Gamma_3(i\epsilon \beta_b - \eta + i\mu_e + \pi \omega_1 \omega_2 \omega_3)\Gamma_3(i\epsilon \beta_b + \eta - i\mu_e + \pi \omega_1 \omega_2 \omega_3)\Gamma_3(i\epsilon \beta_b + \eta + i\mu_e + 2\pi|\omega_1 \omega_2 \omega_3)}{\Gamma_3(i\epsilon \beta_b + \eta - i\mu_e + \pi \omega_1 \omega_2 \omega_3)\Gamma_3(i\epsilon \beta_b - \eta + i\mu_e + 2\pi|\omega_1 \omega_2 \omega_3)}
\]

\[ \times \prod_{b=1}^N \prod_{\epsilon = \pm} \frac{\Gamma_3(i\epsilon \beta_b + \pi \xi + 2\pi \omega_1 \omega_2 \omega_3)\Gamma_3(i\epsilon \beta_b + \pi \xi + 4\pi \omega_1 \omega_2 \omega_3)}{\Gamma_3(i\epsilon \beta_b + \pi \xi + 3\pi \omega_1 \omega_2 \omega_3)\Gamma_3(i\epsilon \beta_b + 2\pi \omega_1 \omega_2 \omega_3)}
\]

\[ \times \prod_{b=1}^N \prod_{\epsilon = \pm} \frac{\Gamma_3(i\epsilon \beta_b + \pi \xi + \pi \omega_1 \omega_2 \omega_3)\Gamma_3(i\epsilon \beta_b + \pi \xi + 4\pi \omega_1 \omega_2 \omega_3)}{\Gamma_3(i\epsilon \beta_b + \pi \xi + 3\pi \omega_1 \omega_2 \omega_3)\Gamma_3(i\epsilon \beta_b + 2\pi \omega_1 \omega_2 \omega_3)}
\]

\[ \times \prod_{b=1}^N \prod_{\epsilon = \pm} \frac{\Gamma_3(i\epsilon \beta_b + 2\eta + \pi \omega_1 \omega_2 \omega_3)\Gamma_3(i\epsilon \beta_b + 2\eta + \pi \xi + 4\pi \omega_1 \omega_2 \omega_3)}{\Gamma_3(i\epsilon \beta_b + 2\eta + \pi \xi + 3\pi \omega_1 \omega_2 \omega_3)\Gamma_3(i\epsilon \beta_b + 2\pi \omega_1 \omega_2 \omega_3)}
\]

\[ I_\eta^a(\{\alpha_a\}) = \prod_{a \in A} \prod_{\epsilon = \pm} \sqrt{\frac{S_2(i\epsilon \alpha_a + \pi + \eta|\omega_2 \omega_3)}{S_2(i\epsilon \alpha_a + \pi + \eta|\omega_2 \omega_3)}} \prod_{a_1 < a_2} \prod_{\epsilon_1 < \epsilon_2 = \pm} \frac{\Gamma(\frac{i(\alpha_{a_1} - \alpha_{a_2})}{\pi(\xi + 1)} + 1)}{\Gamma(\frac{i(\alpha_{a_1} - \alpha_{a_2})}{\pi(\xi + 1)} + 1)}
\]

\[ \times \prod_{a_1 < a_2} \prod_{\epsilon_1 < \epsilon_2 = \pm} \frac{S_2(i\epsilon(\alpha_{a_1} + \alpha_{a_2}) + \eta|\omega_2 \omega_3)f_2(i\epsilon(\alpha_{a_1} - \alpha_{a_2}) + \pi|\omega_2 \omega_3)}{S_2(i\epsilon(\alpha_{a_1} + \alpha_{a_2}) + \pi \xi + \eta|\omega_2 \omega_3)f_2(i\epsilon(\alpha_{a_1} - \alpha_{a_2}) + \pi(\xi + 1)|\omega_2 \omega_3)}
\]

\[ \times \prod_{a \in A} \prod_{\epsilon = \pm} \frac{\Gamma_2(i\epsilon \alpha_a + \pi \xi + \frac{\pi}{2} - i\mu_e|\omega_2 \omega_3)\Gamma_2(-i\epsilon \alpha_a + \eta + \pi \xi + \frac{\pi}{2} - i\mu_e|\omega_2 \omega_3)}{\Gamma_2(i\epsilon \alpha_a + \pi \xi + \frac{\pi}{2} + i\mu_e|\omega_2 \omega_3)\Gamma_2(-i\epsilon \alpha_a + \eta + \pi \xi + \frac{\pi}{2} + i\mu_e|\omega_2 \omega_3)}
\]

\[ \times \prod_{a \in A} \prod_{\epsilon = \pm} \sqrt{\frac{\Gamma_2(2i\epsilon \alpha_a + \pi |\omega_2 \omega_3)\Gamma_2(2i\epsilon \alpha_a + \pi (\xi + 1)|\omega_2 \omega_3)}{\Gamma_2(2i\epsilon \alpha_a |\omega_2 \omega_3)\Gamma_2(2i\epsilon \alpha_a + \pi \xi |\omega_2 \omega_3)}}
\]

\[ \times \prod_{a \in A} \prod_{\epsilon = \pm} \sqrt{\frac{\Gamma_2(2i\epsilon \alpha_a + 2\eta + \pi |\omega_2 \omega_3)\Gamma_2(2i\epsilon \alpha_a + 2\eta + \pi (\xi + 1)|\omega_2 \omega_3)}{\Gamma_2(2i\epsilon \alpha_a + 2\eta |\omega_2 \omega_3)\Gamma_2(2i\epsilon \alpha_a + 2\eta + \pi \xi |\omega_2 \omega_3)}}
\]
The vacuum expectation value is evaluated as following.

\[
I^\beta\alpha_\eta(\{\beta_b\}|\{\alpha_a\}) = \prod_{a \in \mathcal{A}} \prod_{b=1}^N \left\{ \Gamma \left( \frac{ie(\alpha_a - \beta_b)}{\pi(\xi + 1)} + \frac{1}{2(\xi + 1)} \right) S_2 \left( ie(\alpha_a - \beta_b) + \frac{\pi}{2} |\omega_2\omega_3| \right) \right\}^{-1}.
\]

The vacuum expectation value is evaluated as following.

\[
I^\beta\alpha_\eta(\{\beta_b^* - \pi i\}|\{\alpha_a\}) = I^\beta\alpha_\eta(\{\beta_b^*\}) I^\beta\alpha_\eta(\{\beta_b^*\}) I^\alpha\alpha_\eta(\{\alpha_a\}).
\]

Here we set

\[
I^\beta\alpha_\eta(\{\beta_b^*\}) = \prod_{b=1}^N \prod_{\epsilon = \pm} \sqrt{S_3(2ie\beta_b^* + \pi + 2\pi \epsilon + \eta |\omega_1\omega_2\omega_3|) \prod_{b_1 < b_2} \frac{\Gamma_2(ie(\beta_b^* - \beta_{b_1}^*) + \pi |\omega_1\omega_2|)}{\Gamma_2(ie(\beta_{b_1}^* - \beta_{b_2}^*) + 2\pi |\omega_1\omega_2|)} \prod_{b_1 < b_2} \frac{\Gamma_3(ie(\epsilon_i \beta_b^* + i\mu_{\epsilon_i} + \pi \epsilon + \pi |\omega_1\omega_2\omega_3|)}{\Gamma_3(ie(\epsilon_i \beta_b^* - i\mu_{\epsilon_i} + \pi \epsilon + \pi |\omega_1\omega_2\omega_3|}}}
\]
The magnetization on a site which is free from a difficulty of divergence. However it’s physical meaning is not clear.

Finite XXZ chain with double boundaries at massive regime,

Note. In paper [9] the authors considered the following vacuum expectation value for

\[
\langle \text{vac}|e^F \Phi_{\xi_1}(\zeta_1^*) \cdots \Phi_{\xi_N}(\zeta_N)e^{E*}e^G \Phi_{\xi_N}(\zeta_N) \cdots \Phi_{\xi_1}(\zeta_1)e^{G*}|\text{vac}\rangle
\]

\[
\langle \text{vac}|e^F e^{E*} e^G|\text{vac}\rangle
\]

which is free from a difficulty of divergence. However it’s physical meaning is not clear.
Acknowledgements  This work was partly supported by Grant-in-Aid for Encouragement for Young Scientists (A) from Japan Society for the Promotion of Science (11740099).

References

[1] M.Jimbo and T. Miwa : *Algebraic Analysis of Solvable Lattice Models*. CBMS Regional Conference Series in Mathematics vol 85, AMS, 1994, and references therein.

[2] R.Baxter : *Exactly Solved Models in Statistical Mechanics*. Academic Press, London, 1982, and references therein.

[3] M.Jimbo,R.Kedem,T.Kojima,H.Konno and T.Miwa : XXZ chain with a boundary, *Nucl.Phys.* **B441**[FS], 437-470, (1995).

[4] M.Jimbo,R.Kedem,H.Konno,T.Miwa and R.Weston : Difference Equations in Spin Chains with a Boundary, *Nucl.Phys.* **B448**, 429-456, (1995).

[5] H.Furutsu and T.Kojima : The \( U_q(\hat{sl}_n) \)-analogue of the XXZ chain with a boundary, *J.Math.Phys.* **41**, No.7, 4413-4436, (2000).

[6] T.Kojima and Y.H..Quano : Difference equations for the higher rank XXZ model with a boundary, [nlin.SI/0001038], (2000), to appear in *Int.J.Mod.Phys.A*

[7] T.Kojima : The massless XXZ chain with a boundary, [nlin.SI/0006026], (2000), to appear in *Int.J.Mod.Phys.A*.

[8] M.Jimbo, H.Konno and T.Miwa : Massless XXZ model and degeneration of the elliptic algebra \( A_{q,p}(\hat{sl}_2) \), *Deformation theory and Symplectic Geometry*, Eds. D.Sternheimer, J.Rawnsley and G.Gutt, *Math.Phys.Studies*, Kluwere, 20, 117-138, 1997.

[9] Y.Fujii and M.Wadati : Correlation functions of finite XXZ model with boundaries, *Chaos,Solitons and Fractals* **11**, 565-579, (2000).
Here we summarize the bosonizations of the vertex operators $\Phi$. Let us set free bosons $b(t)$ ($t \in \mathbb{R}$) which satisfy

$$[b(t), b(t')] = \frac{\text{sh}\left(\frac{\pi t}{2}\right) \text{sh}(\pi t) \text{sh}\left(\frac{\pi t \xi}{2}\right)}{t \text{sh}\left(\frac{\pi t (\xi + 1)}{2}\right)} \delta(t + t').$$  \hspace{1cm} (A.1)

Let us set $a(t)$ by

$$b(t) \text{sh}\left(\frac{\pi t (\xi + 1)}{2}\right) = a(t) \text{sh}\left(\frac{\pi t \xi}{2}\right).$$ \hspace{1cm} (A.2)

The bosonization of the type-I vertex operators is given by

$$\Phi_+ (\beta) = U(\beta),$$

$$\Phi_- (\beta) = \int_{C_I} d\alpha : U(\beta) \bar{U}(\alpha) : \times \Gamma\left(\frac{i(\alpha - \beta)}{\pi (\xi + 1)} - \frac{1}{2\xi}\right) \Gamma\left(\frac{-i(\alpha - \beta)}{\pi (\xi + 1)} + \frac{1}{2\xi}\right),$$ \hspace{1cm} (A.3)

where we have set

$$U(\alpha) =: \exp\left(-\int_{-\infty}^{\infty} b(t) \frac{e^{i\alpha t}}{\text{sh}\pi t} dt\right), \quad \bar{U}(\alpha) =: \exp\left(\int_{-\infty}^{\infty} b(t) \frac{e^{i\alpha t}}{\text{sh}\frac{\pi t}{2}} dt\right).$$ \hspace{1cm} (A.4)

The bosonization of the type-II vertex operators is given by

$$\Psi_+ (\beta) = V(\beta),$$

$$\Psi_- (\beta) = \int_{C_{II}} d\alpha : V(\beta) \bar{V}(\alpha) : \times \Gamma\left(\frac{i(\alpha - \beta)}{\pi \xi} - \frac{1}{2\xi}\right) \Gamma\left(\frac{-i(\alpha - \beta)}{\pi \xi} + \frac{1}{2\xi}\right),$$ \hspace{1cm} (A.5)
where we have set
\[ V(\alpha) =: \exp \left( -\int_{-\infty}^{\infty} \frac{a(t)}{\operatorname{sh} \pi t} e^{i\alpha t} dt \right), \quad \bar{V}(\alpha) =: \exp \left( -\int_{-\infty}^{\infty} \frac{a(t)}{\operatorname{sh} \frac{\pi}{2} t} e^{i\alpha t} dt \right). \]

(A.8)

Here the integration contours are chosen as follows. The contour \( C_I \) is \((-\infty, \infty)\). The poles
\[ \alpha - \beta = \frac{\pi i}{2} + n\pi (\xi + 1)i, \quad (n \in \mathbb{N}) \]  
(A.9)
of \( \Gamma \left( \frac{i(\alpha - \beta)}{\pi (\xi + 1)} + \frac{1}{2(\xi + 1)} \right) \) are above \( C_I \) and the poles
\[ \alpha - \beta = -\frac{\pi i}{2} - n\pi (\xi + 1)i, \quad (n \in \mathbb{N}) \]  
(A.10)
of \( \Gamma \left( -\frac{i(\alpha - \beta)}{\pi (\xi + 1)} + \frac{1}{2(\xi + 1)} \right) \) are below \( C_I \). The contour \( C_{II} \) is \((-\infty, \infty)\) except that the poles
\[ \alpha - \beta = -\frac{\pi i}{2} + n\pi \xi i, \quad (n \in \mathbb{N}) \]  
(A.11)
of \( \Gamma \left( \frac{i(\alpha - \beta)}{\pi \xi} - \frac{1}{2\xi} \right) \) are above \( C_{II} \) and the poles
\[ \alpha - \beta = \frac{\pi i}{2} - n\pi \xi i, \quad (n \in \mathbb{N}) \]  
(A.12)
of \( \Gamma \left( -\frac{i(\alpha - \beta)}{\pi \xi} - \frac{1}{2\xi} \right) \) are below \( C_{II} \).

B Multi Gamma functions

Here we summarize the multiple gamma and the multiple sine functions.

Let us set the functions \( \Gamma_1(x|\omega), \Gamma_2(x|\omega_1, \omega_2) \) and \( \Gamma_3(x|\omega_1, \omega_2, \omega_3) \) by

\[ \log \Gamma_1(x|\omega) + \gamma B_{11}(x|\omega) = \int_C \frac{dt}{2\pi i t} e^{-xt} \frac{\log(-t)}{1 - e^{-\omega t}}, \]  
(B.1)

\[ \log \Gamma_2(x|\omega_1, \omega_2) - \frac{\gamma}{2} B_{22}(x|\omega_1, \omega_2) = \int_C \frac{dt}{2\pi i t} e^{-xt} \frac{\log(-t)}{(1 - e^{-\omega_1 t})(1 - e^{-\omega_2 t})}, \]  
(B.2)

\[ \log \Gamma_3(x|\omega_1, \omega_2, \omega_3) + \frac{\gamma}{3!} B_{33}(x|\omega_1, \omega_2, \omega_3) = \int_C \frac{dt}{2\pi i t} e^{-xt} \frac{\log(-t)}{(1 - e^{-\omega_1 t})(1 - e^{-\omega_2 t})(1 - e^{-\omega_3 t})}, \]  
(B.3)

where the functions \( B_{jj}(x) \) are the multiple Bernoulli polynomials defined by

\[ \frac{t^r e^{xt}}{\prod_{j=1}^{r} (e^{\omega_j t} - 1)} = \sum_{n=0}^{\infty} \frac{t^n}{n!} B_{r,n}(x|\omega_1 \cdots \omega_r), \]  
(B.4)
more explicitly

\[ B_{11}(x|\omega) = \frac{x}{\omega} - \frac{1}{2}, \]  \hspace{1cm} (B.5)

\[ B_{22}(x|\omega) = \frac{x^2}{\omega_1\omega_2} - \left( \frac{1}{\omega_1} + \frac{1}{\omega_2} \right)x + \frac{1}{2} + \frac{1}{6} \left( \frac{\omega_1}{\omega_2} + \frac{\omega_2}{\omega_1} \right). \]  \hspace{1cm} (B.6)

Here \( \gamma \) is Euler’s constant, \( \gamma = \lim_{n \to \infty} \left( 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} - \log n \right) \).
Here the contour of integral is given by

\[ \text{Contour } C \]

Let us set

\[ S_1(x|\omega) = \frac{1}{\Gamma_1(\omega - x|\omega)\Gamma_1(x|\omega)}, \]  \hspace{1cm} (B.7)

\[ S_2(x|\omega_1, \omega_2) = \frac{\Gamma_2(\omega_1 + \omega_2 - x|\omega_1, \omega_2)}{\Gamma_2(x|\omega_1, \omega_2)}, \]  \hspace{1cm} (B.8)

\[ S_3(x|\omega_1, \omega_2, \omega_3) = \frac{1}{\Gamma_3(\omega_1 + \omega_2 + \omega_3 - x|\omega_1, \omega_2, \omega_3)\Gamma_3(x|\omega_1, \omega_2, \omega_3)}. \]  \hspace{1cm} (B.9)

We have

\[ \Gamma_1(x|\omega) = e^{(\frac{d}{dx} - \frac{x}{2})\log \Gamma(x/\omega)} \frac{\Gamma(x/\omega)}{\sqrt{2\pi}}, \quad S_1(x|\omega) = 2\sin(\pi x/\omega), \]  \hspace{1cm} (B.10)

\[ \frac{\Gamma_2(x + \omega_1|\omega_1, \omega_2)}{\Gamma_2(x|\omega_1, \omega_2)} = \frac{1}{\Gamma_1(x|\omega_2)}, \quad \frac{S_2(x + \omega_1|\omega_1, \omega_2)}{S_2(x|\omega_1, \omega_2)} = \frac{1}{S_1(x|\omega_2)}, \quad \frac{\Gamma_1(x + \omega|\omega)}{\Gamma_1(x|\omega)} = x. \]  \hspace{1cm} (B.11)

\[ \frac{\Gamma_3(x + \omega_1|\omega_1, \omega_2, \omega_3)}{\Gamma_3(x|\omega_1, \omega_2, \omega_3)} = \frac{1}{\Gamma_2(x|\omega_2, \omega_3)}, \quad \frac{S_3(x + \omega_1|\omega_1, \omega_2, \omega_3)}{S_3(x|\omega_1, \omega_2, \omega_3)} = \frac{1}{S_2(x|\omega_2, \omega_3)}. \]  \hspace{1cm} (B.12)
\[
\log S_2(x|\omega_1\omega_2) = \int_C \frac{\text{sh}(x - \frac{\omega_1 + \omega_2}{2})t}{2\text{sh}\frac{\omega_1 t}{2}\text{sh}\frac{\omega_2 t}{2}} \log(-t) \frac{dt}{2\pi it}, \quad (0 < \text{Re} x < \omega_1 + \omega_2). \quad (B.13)
\]

\[
S_2(x|\omega_1\omega_2) = \frac{2\pi}{\sqrt{\omega_1\omega_2}} x + O(x^2), \quad (x \to 0). \quad (B.14)
\]

\[
S_2(x|\omega_1\omega_2)S_2(-x|\omega_1\omega_2) = -4\sin\frac{\pi x}{\omega_1}\sin\frac{\pi x}{\omega_2} \quad (B.15)
\]