THE ∇ OPERATION OF TROPICAL MATRICES

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ABSTRACT. Although the only invertible tropical matrices are products of diagonal and permutation matrices with invertible determinant, the classic notion \( \frac{1}{\det(A)} \text{adj}(A) \), inherits some classic algebraic properties, and even some surprising new ones. In this paper we study this rather interesting operation, denoted as \( A^{∇} \). We examine its eigenvalues as well as those of tropically similar matrices, establish a connection to the stabilization of powers of matrices in normal form and find an alternative proof to the tropical multiplicativity of the determinant, due to the factorizability of \( A^{∇} \).

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1. Introduction

The tropical max-plus semifield is an ordered group \( \mathcal{G} \) (usually the set of real numbers \( \mathbb{R} \) or the set of rational numbers \( \mathbb{Q} \)), together with \(-∞\), denoted as \( \mathbb{T} = \mathcal{G} \cup \{-∞\} \), equipped with the operations \( a \oplus b = \max\{a, b\} \) and \( a \odot b = a + b \), denoted as \( a + b \) and \( ab \) respectively (see [1] and [12]). The unit element \( 1_{\mathbb{T}} \) is really the element \( 0 \in \mathbb{Q} \). This arithmetic enables one to simplify non-linear questions by answering them in a linear setting (see [10]), which can be applied to discrete mathematics, optimization, algebraic geometry and more, as has been well reviewed in [7], [8], [9], [11] and [22]. In this max-plus language, we may also use notions of linear-algebra to interpret combinatorial problems, such as eigenvectors being used to solve the Longest-Distance problem (see [3]).

In this paper we aspire to use an analog concept of the inverse matrix by passing to a wider structure called the supertropical semiring, equipped with the ghost ideal \( G = \mathcal{G}^{ν} \), as established and studied by Izhakian and Rowen in [15] and [14]. We denote as \( R = T \cup G \cup \{-∞\} \) the “standard” supertropical semiring, where \( T = \mathcal{G} \), which contains the so called tangible elements of the structure and where \( \forall a \in T \) we have \( a^{ν} \in G \) are the ghost elements of the structure, as defined in [15]. So \( G \) inherits the order of \( \mathcal{G} \). This enables us to distinguish between a maximal element \( a \) that is attained only once in a sum, i.e. \( a \in T \) which is invertible, and a maximum that is being attained at least twice, i.e. \( a + a = a^{ν} \in G \), which is not invertible. Note that \( ν \)

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projects the standard supertropical semiring onto $G$, which can be identified with the usual tropical structure.

In this new supertropical sense, we use the following order relation to describe two elements that are equal up to some ghost supplement:

**Definition 1.1.** Let $a, b$ be any two elements in $R$. We say that $a$ **ghost surpasses** $b$, denoted $a \parallel_{gs} b$, if $a = b + \text{ghost}$, i.e. $a \approx_{\nu} b$ or $a \in G$ with $a'' \geq b''$. We say $a$ is $\nu$-equivalent to $b$, denoted by $a \cong_{\nu} b$, if $a'' = b''$. That is, in the tropical structure, $\nu$-equivalence projects to equality.

For matrices $A = (a_{ij}), B = (b_{ij}) \in M_{n \times m}(R)$ (and in particular for vectors) $A \parallel_{gs} B$ means $a_{ij} \parallel_{gs} b_{ij} \quad \forall i = 1, \ldots, n$ and $j = 1, \ldots, m$.

For polynomials $f(x) = \sum_{i=1}^{n} a_{i}x^{i}$, $g(x) = \sum_{i=1}^{n} b_{i}x^{i} \in R[x]$, we say that $f(x) \parallel_{gs} g(x)$ when $a_{i} \parallel_{gs} b_{i} \quad \forall i$.

**Important properties of $\parallel_{gs}$:**

1. $\parallel_{gs}$ is a partial order relation. See [17, Lemma 1.5].

2. If $a \parallel_{gs} b$ then $ac \parallel_{gs} bc$.

Considering this relation, we use the classical notion $\frac{1}{\text{det}(A)} \text{adj}(A)$ (where $\text{det}(A)$ is the permanent and then $\text{adj}(A)$ is defined as usual) to formulate results in the supertropical setting, which are inaccessible in the usual tropical setting. Then we obtain tropical theorems by considering the tangible elements. By Izhakin’s work in [13], this notion satisfies that $\frac{1}{\text{det}(A)} \text{adj}(A) \cdot A$ is equal to the $Id$ matrix on the diagonal, and $\parallel_{gs}$ the $Id$ matrix off the diagonal.

In section 3 we discuss type of matrices with 0 on the diagonal and 0 determinant; these are said to have **normal form**. In section 3.1, we establish that for the set

$\mathcal{N} = \{A^\nabla : A \in M_{n}(R)\}$,

the operation $\nabla$ is of order 2 (see Theorem 3.5), and in section 3.2 we will show that for a matrix $A$ of order $n$ in normal form

$A^\nabla \cong_{\nu} A^{\nabla \nabla} \cong_{\nu} A^* \cong_{\nu} A^k$, $\forall k \geq n - 1$

(see Theorem 3.6 and Proposition 3.7). These results are extended to the supertropical setting from the results obtained over the tropical structure in [28] and [29], regarding these closure operations.

In section 4, we use the factorizability of matrices in $\mathcal{N}$ to give an alternative proof, analogous to the proof in classical linear algebra, of the property

$\text{det}(AB) \parallel_{gs} \text{det}(A)\text{det}(B)$

stated in Theorem 2.10.

In sections 5 and 6 we will prove the supertropical properties

$f_{A^\nabla B} \parallel_{gs} f_{B}$, (see Theorem 5.4)

and
\[ |A| f_{A^v}(x) \models_{gs} x^n f_A(x^{-1}) \] (see Theorem 6.2),
where \( f_M(x) = \text{det}(M+xI) \) denotes the characteristic polynomial of a square matrix \( M \).
As a result, we conclude that
\[ f_{A^v v} = f_A \], whenever \( f_{A^v v} \) is tangible (see Theorem 6.4).

These properties provide a foundation for further research in representation theory and eigenspace decomposition.

2. Preliminaries

2.1. Matrices.

The work in [16], [17] and [18] shows that even though the semiring of matrices over the supertropical semiring lacks negation, it satisfies many of the classical matrix theory properties when using the ghost ideal \( G \). Following [3] and [16] for the tropical and supertropical notations, we give some basic definitions for this theory. One may also find in [3] further combinatorial motivations for the objects discussed.

Definition 2.1. A permutation track, of the permutation \( \pi \in S_n \), is the sequence
\[ a_{1,\pi(1)} a_{2,\pi(2)} \cdots a_{n,\pi(n)} \]
of \( n \) entries of the matrix \( A = (a_{i,j}) \in M_n(R) \).

Definition 2.2. We define the tropical determinant of a tangible matrix \( A = (a_{i,j}) \) to be the usual permanent
\[ \text{det}(A) = \sum_{\sigma \in S_n} a_{1,\sigma(1)} \cdots a_{n,\sigma(n)} \]
(see \([11] \S 5\)). In the special case where \( a_{i,j} \in R, \forall i,j \), we refer to any permutation track yielding the highest value in this sum as a dominant permutation track.

Definition 2.3. We define a matrix \( A \in M_n(T) \) or \( A \in M_n(R) \) to be tropically non-singular if there exist at least two different dominant permutation tracks. Otherwise the matrix is tropically singular.

Consequently a matrix \( A \in M_n(R) \) is supertropically singular if \( \text{det}(A) \in G \cup \{-\infty\} \) and supertropically non-singular if \( \text{det}(A) \in T \). A matrix \( A \) is strictly singular if \( \text{det}(A) = -\infty \).

Notice that over the tropical semifield we cannot determine if the matrix is tropically non-singular from the value of its determinant, which is always invertible over \( T \setminus \{-\infty\} \). Over the supertropical semiring however, a supertropically non-singular matrix has an invertible determinant, while a supertropically singular matrix has a non-invertible determinant.

As the definitions of singularity are identical over the tropical and supertropical structures, we will only indicate ”non-singular” or ”singular” and ”over \( T \)” or ”over \( R \)” (which will effect the value and invertibility of the determinant).
Definition 2.4. Let $\mathbb{T}^n$ be the free module (see [17]) of rank $n$ over the tropical semifield, and $R^n$ be the free module of rank $n$ over the supertropical semiring. We define the **standard base** of $\mathbb{T}^n$, and therefore of $R^n$, to be $e_1, ..., e_n$, where

$$e_i = \begin{cases} 1_T = 1_R, & \text{in the } i^{th} \text{ coordinate} \\ 0_T = 0_R, & \text{otherwise} \end{cases}.$$  

Definition 2.5. The tropical **identity matrix** in the tropical matrix semiring is the $n \times n$ matrix with the standard base for its columns. We denote this matrix as

$$I_T = I_R = I.$$  

Definition 2.6. A matrix $A \in M_n(R)$ is **invertible** if there exists a matrix $B \in M_n(R)$ such that

$$AB = BA = I.$$  

Definition 2.7. A square matrix $P_\pi = (a_{i,j})$ is defined to be a **permutation matrix** if there exists $\pi \in S_n$ such that $a_{i,j} = \begin{cases} 0_R, j \neq \pi(i) \\ 1_R, j = \pi(i) \end{cases}.$  

Remark 2.8. A tropical matrix $A$ is invertible if and only if it is a product of a permutation matrix $P_\pi$ and a diagonal matrix $D$ with an invertible determinant.

Proof. See [16, Proposition 3.9]. These type of products are defined in the literature as **generalized permutation matrices**.  

Definition 2.9. Following the notation in [24], we define three types of tropical elementary matrices, corresponding to the three elementary matrix operations, obtained by applying one such operation to the identity matrix.

An **elementary matrix of type 1** is obtained from the identity matrix by switching two rows (resp. columns). Multiplying a matrix $A$ to the right of such a matrix (resp. to the left) will switch the corresponding rows (resp. columns) in $A$.

An **elementary matrix of type 2** is obtained from the identity matrix where one row (resp. column) has been multiplied by an invertible scalar. Multiplying a matrix $A$ to the right of such a matrix (resp. to the left) will multiply the corresponding row (resp. columns) in $A$ by the same scalar.

An **elementary matrix of type 3** is obtained from the identity matrix where we add one row (resp. column), multiplied by an invertible scalar, to another. Multiplying a matrix $A$ to the right of such a matrix (resp. to the left) will add the corresponding row (resp. column), multiplied by the same scalar, to the corresponding other in $A$.

By Remark 2.8, a product of matrices of type 1, which is a permutation matrix, is invertible. A product of matrices of type 2, which is a diagonal matrix, is invertible. Thus a product of matrices of type 1 and 2, which is a generalized permutation matrix, is invertible. Matrices of type 3 however, are not invertible, and therefore a product including a matrix of type 3 is not invertible.

Theorem 2.10. (The rule of determinants) For $n \times n$ matrices $A,B$ over the supertropical semiring $R$, we have

$$\det(AB) \models gs \det(A)\det(B).$$
Proof. See Theorem 3.5 in [16, §3].

The fact that the determinant of a non-singular matrix $A$ over $R$ is tangible means that the matrix has one dominant permutation track. By using a permutation matrix we can relocate the corresponding permutation to the diagonal, and by using a diagonal matrix we can change the diagonal entries to $1_R$, obtaining a non-singular matrix whose dominant Id-permutation track equals $1_R$. That is, $A = P\bar{A}$ where $P$ is an invertible matrix (See Remark 2.8) such that $\det(P) = \det(A)$ and $\det(\bar{A}) = 1_R$, with $1_R$ on the diagonal. These matrices are referred to in [3] as definite diagonal dominant matrices.

We call $\bar{A}$ the normal form of $A$, and say that $P$ normalizes the dominant permutation track to the diagonal. This is not the same as normal matrices, defined in [2] and [3], which requires the off diagonal entries to be non-positive. Over $T$ the normal form is obtained for not strictly singular matrices, by relocating and normalizing one of the dominant permutation tracks.

Looking at the process of bringing a matrix to normal form, we see that we can normalize a non-singular matrix $A$ using elementary operations of type 1 and 2 either on the rows or on the columns of $A$.

**Definition 2.11.** We denote the matrix of right operations (resp. left) as the right (resp. left) normalizer of $A$, which is invertible. The right (resp. left) normal form corresponds to the right (resp. left) normalizer.

Looking at elementary matrices (including the identity matrix considered as an elementary matrix of type 2) as the "atoms" of matrices, we present the following definitions:

**Definition 2.12.** We define a matrix to be elementarily factorizable if it can be factored into a product of tropical elementary matrices.

Thus, an invertible matrix, is always elementarily factorizable, while a non-invertible matrix is elementarily factorizable if and only if its normal form can be written as a product of tropical elementary matrices of type 3.

Here is a useful combinatorial property of tropical matrices in normal form.

**Lemma 2.13.** For $A = (a_{i,j}) \in M_n(R)$ in normal form, any sequence

\[
a_{i_1,i_2}a_{i_2,i_3} \cdots a_{i_k,i_1}, \text{ where } i_j \in \{1, \ldots, n\} \forall j = 1, \ldots, k
\]

either describes a permutation track, or is dominated by a subsequence which describes a permutation track.

**Remark.** If we write every permutation $\pi$ as a product of disjoint cycles $\sigma_1, \ldots, \sigma_t$, then the permutation track can be decomposed into the cycle tracks $C_1, \ldots, C_t$, where $C_i$ is the cycle track of the cycle $\sigma_i$.

**Proof.** (of Lemma 2.13)

We recall that in normal form the diagonal entries are $1_R$, and the determinant is obtained from the diagonal.

Looking at the left indices, if $i_t \neq i_s \forall t \neq s$, then equation (2.1) is a cycle track, which, when composed with Id disjoint cycle tracks, describes a permutation track.

If there exists a repeating index in (2.1), then it can be factored into cycle tracks. This is done recursively by choosing the repeating indices with no other indices repeating
between them. These subsequences will describe cycle tracks starting and ending at the points of repetition. Then we compose the parts remained by relocating those cycle tracks to the beginning of \((2.1)\) and repeating this process. At the last step we get a string of (not necessarily disjoint) cycle tracks.

For example:

\[
a_{1,2}a_{2,3}a_{3,4}a_{4,2}a_{2,5}a_{5,6}a_{6,7}a_{7,6}a_{8,7}a_{7,6}a_{6,1}.
\]

We can see that the left indices that repeat are 2, 6 and 7 (in bold), but the ones that repeat with no intermediate indices repeating are 2 and 7 (underlined). Therefore

\[
a_{1,2}(a_{2,3}a_{3,4}a_{4,2})a_{2,5}a_{5,6}a_{7,8}a_{8,7}a_{7,6}a_{6,1} = C_1C_2[a_{1,2}a_{2,5}a_{5,6}a_{6,7}a_{7,6}a_{6,1}].
\]

Now the remaining repeating index is 6, and therefore

\[
C_1C_2[a_{1,2}a_{2,5}a_{5,6}(a_{6,7}a_{7,6})a_{6,1}] = C_1C_2C_3[a_{1,2}a_{2,5}a_{5,6}a_{6,1}] = C_1C_2C_3C_4,
\]

where \(C_1 = a_{2,3}a_{3,4}a_{4,2}, C_2 = a_{7,8}a_{8,7}, C_3 = a_{6,7}a_{7,6}\) and \(C_4 = a_{1,2}a_{2,5}a_{5,6}a_{6,1}\) are cycle tracks.

Having factored \((2.1)\) into cycle tracks \(C_1 \cdots C_l\) (not necessarily disjoint of course), we know that for a matrix in normal form each cycle track, when composed with \(\text{Id}\) disjoint cycle tracks, is a permutation track and therefore has value \(\leq 1_R\). Thus,

\[
\prod_{j=1}^l C_j = C_i \cdot \prod_{j \neq i} C_j \leq C_i \forall i,
\]

where \(C_i\) is a cycle track. \(\square\)

For an element \(a \in R\), we denote as \(a^\nu\) the element \(b \in G\) s.t. \(a \cong b\), and as \(\hat{a}\) the element \(b \in T\) s.t. \(a \cong b\).

For a matrix (and in particular for a vector) \(A = (a_{i,j})\) we write

\[
A^\nu = (a^\nu_{i,j}) \text{ and } \hat{A} = (\hat{a}_{i,j}).
\]

For a polynomial \(f(x) = \Sigma a_i x^i\) we write

\[
f^\nu(x) = \Sigma a^\nu_i x^i \text{ and } \hat{f}(x) = \Sigma \hat{a}_i x^i.
\]

**Definition 2.14.** A quasi-zero matrix \(Z_G\) is a matrix equal to \(0_R\) on the diagonal, and whose off-diagonal entries are ghosts or \(0_R\). A quasi-identity matrix \(I_G\) is a nonsingular, multiplicatively idempotent matrix equal to \(I + Z_G\), where \(Z_G\) is a quasi-zero matrix. A ghost quasi-identity matrix is a singular, multiplicatively idempotent matrix equal to \(I^\nu + Z_G\).

**Definition 2.15.** The \(r, c\)-minor \(A_{r,c}\) of a matrix \(A = (a_{i,j})\) is obtained by deleting row \(r\) and column \(c\) of \(A\). The adjoint matrix \(\text{adj}(A)\) of \(A\) is defined as the matrix \((a'_{i,j})\), where \(a'_{i,j} = \text{det}(A_{j,i})\).

The matrix \(A^\nabla\) denotes \(\frac{1}{\text{det}(A)}\text{adj}(A)\), when \(\text{det}(A)\) is invertible, and \(\left(\frac{1}{\text{det}(A)}\right)^\nu\text{adj}(A)\), when \(\text{det}(A)\) is not invertible. Thus over \(R\), \(A^\nabla\) is defined differently for singular and non-singular matrices. Over \(T\), however, \(A^\nabla\) is defined the same for every not strictly singular matrix.
Notice that \( \text{det}(A_{j,i}) \) may be obtained as the sum of all permutation tracks in \( A \) passing through \( a_{j,i} \), with \( a_{j,i} \) removed:

\[
\text{det}(A_{j,i}) = \sum_{\sigma \in S_n : \sigma(j) = i} a_{1,\sigma(1)} \cdots a_{j-1,\sigma(j-1)}a_{j+1,\sigma(j+1)} \cdots a_{n,\sigma(n)}.
\]

When writing such a permutation as the product of disjoint cycles, \( \text{det}(A_{j,i}) \) can be presented as:

\[
\text{det}(A_{j,i}) = \sum_{\sigma \in S_n : \sigma(j) = i} (a_{i,\sigma(i)} \cdots a_{\sigma^{-1}(j),j})C_{\sigma},
\]

where \((a_{i,\sigma(i)} \cdots a_{\sigma^{-1}(j),j})\) is the cycle track missing \( a_{j,i} \), and \( C_{\sigma} \) is the product of the cycle tracks of \( \sigma \) that do not include \( i \) and \( j \).

**Definition 2.16.** We say that \( A^\triangledown \) is the **quasi-inverse** of \( A \) over \( R \), denoting

\[
I_A = AA^\triangledown \text{ and } I'_A = A^\triangledown A,
\]

where \( I_A, I'_A \) are quasi-identities when \( \text{det}(A) \) is invertible, and ghost quasi-identities otherwise (see [17, Theorem 2.8]).

**Theorem 2.17.**

(i) \( \text{det}(A \cdot \text{adj}(A)) = \text{det}(A)^n \).

(ii) \( \text{det}(\text{adj}(A)) = \text{det}(A)^{n-1} \).

**Proof.** [16 Theorem 4.9].

**Remark 2.18.** For \( A \) in normal form, we have \( A^\triangledown = \frac{1}{\text{det}(A)}\text{adj}(A) = \text{adj}(A) \), which is also in normal form.

**Proof.** The diagonal entries in \( \text{adj}(A) \) are sums of cycle tracks of \( A \), and thus the Id summand \( 1_R \) dominates every diagonal entry. Also, by Theorem 2.17(ii), we have

\[
1_R = \text{det}(A^\triangledown),
\]

as required for normal form. \( \square \)

**Proposition 2.19.** \( \text{adj}(AB) \trianglelefteq \text{gs} \text{adj}(B)\text{adj}(A) \).

**Proof.** [16 Proposition 4.8].

Recall the classical Bruhat (LDU) decomposition, whose tropical analog in [20] is called the LDM decomposition.

**Lemma 2.20.**

a. If \( A \) is a not strictly singular triangular matrix over \( T \) (respectively non-singular over \( R \)), then \( A \) is elementarily factorizable.

b. If \( A \) is a not strictly singular matrix over \( T \) (respectively non-singular over \( R \)), then \( A^\triangledown \) (respectively \( A^\triangledown^\triangledown \)) is elementarily factorizable.

**Proof.** See the LDM decomposition in [20], or an alternative proof in [24] Lemma 6.5 and Corollary 6.6]. \( \square \)
One can find a cruder factorization in [2], where sufficient conditions are established for \( \text{trop}(AB) = \text{trop}(A)\text{trop}(B) \), when looking at the tropical structure as the image of a valuation over the field of Puiseux series, using the classical LDU decomposition over this field. In section 3 we will show the connection of normal forms and factorization to the well known tropical closure operation \( * \), studied in [25] and [26].

2.2. Polynomials.

As one can see in [15], the polynomials over the tropical structure are rather straightforward to view geometrically. We notice that the graph of a monomial \( a_i x^i \in T[x] \) is a line, where the power \( i \) represents the slope. Since \( T \) is ordered we may present its elements on an axis, directed rightward, where if \( a < b \) then \( a \) appears left to \( b \) on the \( T \)-axis, for every pair of distinct elements \( a, b \in T \). It is now easy to understand that a tropical polynomial

\[
\sum_{i=0}^{n} a_i x^i \in \mathbb{R}[x]
\]

takes the value of the dominant monomial among \( a_i x^i \) along the \( T \)-axis. That having been said, it is possible that some monomials in the polynomial would not dominate for any \( x \in T \).

**Definition 2.21.** Let

\[
f(x) = \sum_{i=0}^{n} a_i x^i \in \mathbb{R}[x]
\]

be a supertropical polynomial. We call monomials in \( f(x) \) that dominate for some \( x \in \mathbb{R} \) essential, and monomials in \( f(x) \) that do not dominate for any \( x \in \mathbb{R} \) inessential. We write

\[
f^{es} = \sum_{i \in I} a_i x^i \in \mathbb{R}[x],
\]

where \( a_i x^i \) is an essential monomial \( \forall i \in I \), called the essential polynomial of \( f \).

**Remark 2.22. (The Frobenius property)**

a. If \( R \) is commutative then \( \forall n \in \mathbb{N}, r, s \in \mathbb{R} \quad (r + s)^n \models_{gs} r^n + s^n \).

b. If \( R \) is a supertropical algebra then \( (r + s)^n = r^n + s^n \).

**Proof.** See [16] Remark 1.3.

**Definition 2.23.** We say that \( b \) is a \( k \)-root of \( a \), for some \( k \in \mathbb{N} \), denoted as \( b = \sqrt[k]{a} \), if \( b^k = a \).

**Remark 2.24.** If \( a, b \in T \) and \( a^k = b^k \) then \( a^k + b^k = (a+b)^k \in G \), and therefore \( a+b \in G \) and \( a = b \). That is, the \( k \)-root of a tangible element is unique.

**Definition 2.25.** We call an element \( r \in R \) a root of a polynomial \( f(x) \) if

\[
f(r) \models_{gs} 0_R.
\]

We distinguish between two kinds of roots of supertropical polynomials.

**Definition 2.26.** We refer to roots of a polynomial being obtained as an intersection of two leading tangible monomials as corner roots, and to roots that are being obtained from one leading ghost monomial as non-corner roots.
Remark 2.27. Suppose $f(x) = \sum a_i x^i \in R[x]$. We specialize to elements $r \in R$, starting with $r$ small and then increasing.

1. The constant term $a_0$ and the leading monomial $a_n x^n$ dominate first and last, respectively, due to their slopes. Furthermore, they are the only ones that are necessarily essential in every polynomial.

2. The intersection of an essential monomial $a_i x^i$ and the next essential monomial $a_j x^j$ where $j > i$, is the $i^{th}$ root of $f$ (counting multiplicities), denoted as $\alpha_i = \sqrt[k]{\frac{a_i}{a_j}}$, and is of multiplicity $k = j - i$.

Proof: The monomial $a_i x^i$ dominates all monomials between $a_i x^i$ and $a_j x^j$. Therefore, when $r \in (\alpha_i - 1, \alpha_i]$, $f(r) = a_j r^j + a_i r^i = a_j \left( r^j + \frac{a_i}{a_j} r^i \right) \in G \Rightarrow \left( r^j + \frac{a_i}{a_j} r^i \right) = r^i \left( r^{j-i} + \frac{a_i}{a_j} \right) \in G$

which means $(r + \alpha_i)^k \in G$, and therefore $\alpha_i$ is a root of $f$ with multiplicity $k$.

2.3. Supertropical characteristic polynomials and eigenvalues.

We follow the description in [17, §5].

Definition 2.28. $\forall v \in T^n$ and $A \in M_n(R)$ such that $\exists \alpha \in T \cup \{0_R\}$ where $Av \models_{gs} \alpha v$, we say that $v$ is a supertropical eigenvector of $A$ with a supertropical eigenvalue $\alpha$.

The characteristic polynomial of $A$ (also called the maxpolynomial) is defined to be $f_A(x) = \det(x I + A)$. The tangible value of the roots of the characteristic polynomial $f_A$ are the eigenvalues of $A$, as shown in [16, Theorem 7.10]. The coefficient of $x^k$ in this polynomial is a sum of the tropical determinants of all $n - k \times n - k$ minors, obtained by deleting $k$ chosen rows of the matrix, and their corresponding columns. These minors are defined as principal sub-matrices.

The combinatorial motivation for the tropical characteristic polynomial is the Best Principal Submatrix problem, and has been studied by Butkovic in [4] and [5].

Theorem 2.29. (Supertropical Hamilton-Cayley) Any matrix $A$ satisfies its characteristic polynomial.

Proof. See [16, Theorem 5.2].

Proposition 2.30. If $\alpha \in T \cup 0_R$ is a supertropical eigenvalue of a matrix $A \in M_n(T)$, then $\alpha^i$ is a supertropical eigenvalue of $A^i$.

Proof. See [23, Proposition 3.2].

However, we notice that $\{\alpha^i : \alpha$ is an eigenvalue of a matrix $A\}$ need not be the only supertropical eigenvalues of the matrix $A^i$, as shown in the next example.

Example 2.31. Consider the $2 \times 2$ matrix

$$A = \begin{pmatrix} 0 & 0 \\ 1 & 2 \end{pmatrix}.$$
Then \( f_A(x) = x^2 + 2x + 2 \Rightarrow f_A(x) \in G \) when \( x = 0, 2 \).

However,

\[
A^2 = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}.
\]

Thus \( f_{A^2}(x) = x^2 + 4x + 5 \Rightarrow f_{A^2}(x) \in G \) when \( x' \leq 1 \) or \( x = 4 \).

**Theorem 2.32.** Let \( A \) be in \( M_n(R) \). If

\[
f_{A^m}(x) = \sum_{i=0}^{n} \beta_i x^i
\]

is the characteristic polynomial of the \( m \)th power of \( A \), and

\[
f_A(x) = \sum_{i=0}^{n} \alpha_i x^i
\]

is the characteristic polynomial of \( A \), then

\[
f_{A^m}(x^m) \models f_A(x)^m.
\]

**Proof.** See [23, Theorem 3.6]. \( \square \)

**Corollary 2.33.**

a. If \( f_{A^m} \in T[x] \) then equality holds in Theorem 2.32.

b. Every corner root of \( f_{A^m} \) is an \( m \)th power of a corner root of \( f_A \).

**Proof.** See [23, Theorem 3.10]. \( \square \)

### 3. The \( \nabla \) of a Matrix in Normal Form

As shown in [24] and [25], the multiple conditions for a matrix to be in normal form, allow us to approach complicated properties in a very elegant manner. On top of that, matrices of this form enjoy some very interesting properties of their own.

**Remark 3.1.** Since ghost equivalent implies equality in the tropical setting, we will formulate our results in terms of ghost equivalence (i.e. over \( R \)), with the understanding that the corresponding tropical equalities (i.e. over \( T \)) follow automatically.

#### 3.1. Stabilization under the \( \nabla \) operation.

The goal in this section is to show that \( \nabla \) is a closure operation.

**Lemma 3.2.**

(i) \( P^\nabla = P^{-1} \) whenever \( P \) is an invertible matrix.

(ii) \( \det(PA) = \det(P)\det(A) = \det(AP) \), for \( P \) an invertible matrix.

(iii) \( (PA)^\nabla = A^\nabla P^\nabla \) where \( \det(A) \) is invertible and \( P \) is an invertible matrix.

(iv) Let \( \bar{A} \) be the left normal form of the matrix \( A \), i.e. \( A = P\bar{A} \) for some invertible matrix \( P \). Then \( A^\nabla = \bar{A}^\nabla P^{-1} \).
Proof. For (i),(iii) and (iv) see [24 Lemma 5.7].

(ii) If $P = E_{i,j}$ is an elementary matrix of type 1 and $A = (a_{i,j})$, then

$$\det(E_{i,j}A) = \sum_{\sigma} a_{i,\sigma(j)}a_{j,\sigma(i)} \prod_{k \neq j,i} a_{k,\sigma(k)} = \sum_{\rho} a_{i,\rho(i)}a_{j,\rho(j)} \prod_{k \neq j,i} a_{k,\rho(k)} = \det(A) = \det(E_{i,j})\det(A).$$

If $P = E_{\alpha,j}$ is an elementary matrix of type 2 and $A = (a_{i,j})$, then

$$\det(E_{\alpha,j}A) = \sum_{\sigma} \alpha a_{j,\sigma(j)} \prod_{k \neq j} a_{k,\sigma(k)} = \alpha \sum_{\sigma} a_{j,\sigma(j)} \prod_{k \neq j} a_{k,\sigma(k)} = \alpha \det(A) = \det(E_{\alpha,j})\det(A).$$

Inductively the claim holds for every invertible matrix $P$. \hfill \Box

Claim 3.3. If $A$ is in normal form, then $A^\triangledown A \cong_{\nu} A^\triangledown \cong_{\nu} AA^\triangledown$.

Proof. See [24, Claim 6.1]. \hfill \Box

Corollary 3.4. Let $A$ be a matrix with left normal form $\bar{A}$ and right normal form $\bar{A}$, i.e. $A = PA\bar{A} = \bar{Q}A$ for invertible matrices $P$ and $Q$. Then

a. $A^{\triangledown \triangledown} \cong_{\nu} A^{\triangledown} \cong_{\nu} I_{\bar{A}}$, $A^{\triangledown \triangledown} \cong_{\nu} A^{\triangledown} \cong_{\nu} I_{A}$.
b. $A^{\triangledown \triangledown} \cong_{\nu} P A^{\triangledown} P$.

Proof.
a. According to [17, Corollary 4.4] we know that $A^{\triangledown} \cong_{\nu} A^{\triangledown} A^{\triangledown \triangledown} A^{\triangledown}$. By applying Claim 3.3 to $A^{\triangledown}$ we can conclude

$$\bar{A}^{\triangledown} \cong_{\nu} A^{\triangledown} (A^{\triangledown \triangledown} \bar{A}^{\triangledown}) \cong_{\nu} A^{\triangledown} \bar{A}^{\triangledown} \cong_{\nu} \bar{A}^{\triangledown \triangledown}.$$ 

According to Claim 3.3

$$\bar{A}^{\triangledown} \cong_{\nu} A^{\triangledown} \bar{A} = A^{\triangledown} P^{-1} P \bar{A} = A^{\triangledown} A = I_{\bar{A}},$$

and

$$\bar{A}^{\triangledown} \cong_{\nu} \bar{A} \bar{A}^{\triangledown} = \bar{A} QQ^{-1} \bar{A} = AA^{\triangledown} = I_{A}$$

b. By Lemma 3.2 we have

$$A^{\triangledown \triangledown} = (PA)^{\triangledown \triangledown} = P A^{\triangledown} \cong_{\nu} P \bar{A}^{\triangledown} = P A^{\triangledown} P.$$ 

\hfill \Box

Theorem 3.5. We denote the application $k$ times of $\triangledown$ to $A$ as $A^{\triangledown (k)}$. Then

$$A^{\triangledown (k)} \cong_{\nu} A^{\triangledown (k+2)} \quad \forall k \geq 1.$$ 

That is, applying $\triangledown$ to $A^{\triangledown}$ is an operator of order 2, which means that the $\triangledown$ operation on the set $\{A^{\triangledown} : A \in M_n(R)\}$ acts like the inverse operation.

Proof. In the same way as in Corollary 3.4 (b) we see that

$$(A^{\triangledown})^{\triangledown \triangledown} = (A^{\triangledown} P^{-1})^{\triangledown \triangledown} = (PA^{\triangledown})^{\triangledown} \cong_{\nu} (PA^{\triangledown})^{\triangledown} = A^{\triangledown \triangledown} P^{-1} \cong_{\nu} A^{\triangledown} P^{-1} = A^{\triangledown}.$$ 

The general case then follows inductively. \hfill \Box

In summary, $\triangledown \triangledown$ is a sort of closure operation, which yields elementarily factorizable matrices. For example, $A^{\triangledown \triangledown} = A$ for $2 \times 2$ matrices, implying $A$ is elementarily factorizable.
3.2. The closure operation $*$ and power stabilization.

Noticing that $A^\nabla$ arises from supertropical algebraic considerations (as its product with $A$ gives a quasi-identity), we would like to make a connection to the familiar tropical concept of the Kleene star, denoted as $A^*$, which has been widely studied since the 60’s. In the next theorem and proposition we give results relevant to [28] and [25], obtained over the tropical setting, with the understanding that tropical equality implies supertropical ghost-equivalent. Since the definition for $A^*$ requires that $\det(A)$ is bounded by $1_R$, we consider here the special case of normal form. According to Lemma 2.20(a), we can conclude from the LDM factorization of $A^*$ in [21] that $A^*$ is elementarily factorizable.

**Theorem 3.6.** (See [28, Theorem 2]) If $A$ is an $n \times n$ matrix in normal form and $k$ is a natural number, then $A^k \cong_\nu A^{k+1}$, $\forall k \geq n-1$.

**Proposition 3.7.** $A^\nabla \cong_\nu A^* \cong_\nu A^{n-1}$ when $A$ is in normal form.

**Proof.** For $A^\nabla \cong_\nu A^*$ see [28, Theorem 4]. As described in [25], the equivalence to $A^{n-1}$ is immediate from the definition of $A^*$ for $A$ in normal form:

$$A^* = \sum_{i\in \mathbb{N}\cup\{0\}} A^i.$$ 

Indeed, according to [28, Theorem 2] we have that

$$\hat{A} \leq \hat{A}^2 \leq \ldots \leq \hat{A}^{n-1} \cong_\nu A^n \cong_\nu \ldots$$

(due to the normalized diagonal, each position can only increase comparing to the corresponding position in the previous power and the value of each entry will stabilize at power $n-1$ at most).

Combining Corollary 3.4, Theorem 3.6 and Proposition 3.7 we get

$$A^* \cong_\nu A^{n-1} \cong_\nu A^\nabla \cong_\nu A^{\nabla\nabla} \cong_\nu I_A,$$

for $A$ of order $n$ in normal form.

4. An alternative proof to Theorem 2.10: $\det(AB) \vdash \det(A)\det(B)$

The proof of Theorem 2.10 is given by means of a multilinear function. In the classical case there exists an alternative, somewhat easier proof, using the factorization of non-singular matrices into the product of elementary matrices. Due to Lemma 2.20 we can now give an elegant tropical analog of this proof.

If $A$ or $B$ are strictly singular (without lost of generality $B$), then according to [16, Theorem 6.5] the columns of $B$, denoted as $C_1, \ldots, C_n$, are tropically dependent:

$$\sum \alpha_i C_i \in G \cup 0_R, \alpha_i \in T \forall i.$$ 

Denoting the rows of $A$ as $R_1, \ldots, R_n$ and the columns of $AB$ as $c_1, \ldots, c_n$ we get

$$\sum \alpha_i c_i = \left( \begin{array}{c} \sum \alpha_i R_1 C_i \\ \vdots \\ \sum \alpha_i R_n C_i \end{array} \right) \in G_{0_R}.$$ 

Therefore $\det(AB) \in G \cup 0_R$ as required.
In order to obtain the $\nabla$ operation for (not-strictly) singular matrices we work over the tropical semifield and will prove that $\det(AB)$ equals $\det(A)\det(B)$ plus a term that is attained twice.

Using Theorem 2.17 we have that $\det(A^\nabla) = \det(A)^{-1}$, and it suffices to prove the theorem for $A^\nabla, B^\nabla$. By Lemma 3.2(ii), if $A$ or $B$ are elementary matrices of type 1 and 2 then the theorem holds with equality. Due to Lemma 2.20(b) we only have to show the impact of elementary matrices of type 3.

Let $E = E_{\text{row} \ i} + \alpha \cdot \text{row} \ j$ be an elementary matrix of type 3 adding row $j$, multiplied by $\alpha$, to row $i$. We denote the rows of $A$ as $R_1, \ldots, R_n$. Then the rows of $EA$ are the same as $A$’s, except for the $i^{\text{th}}$ row which is $R_i + \alpha R_j$, therefore

$$\det(EA) = \sum_{\sigma \in S_n} a_{1,\sigma(1)} \cdots a_{i,\sigma(i)} \cdots a_{n,\sigma(n)} + \sum_{\sigma \in S_n} a_{1,\sigma(1)} \cdots \alpha a_{j,\sigma(i)} \cdots a_{n,\sigma(n)},$$

where the sum on the left is $\det(A)$. In order to analyze the sum on the right, we note that

$$\forall i, j \exists! \sigma, \rho \in S_n : \sigma(i) = \rho(j), \sigma(j) = \rho(i), \text{ and } \sigma(k) = \rho(k) \forall k \neq i, j,$n

and let $E$ fix $i$ and $j$. Thus

$$\det(EA) = \det(A) + \sum_{\sigma, \rho \in S_n} (a_{1,\sigma(1)} \cdots \alpha a_{j,\sigma(i)} \cdots a_{j,\sigma(j)} \cdots a_{n,\sigma(n)} + a_{1,\sigma(1)} \cdots a_{j,\rho(j)} \cdots \alpha a_{j,\rho(i)} \cdots a_{n,\sigma(n)}),$$

for $\sigma, \rho$ as in (4.2). Therefore the sum is attained twice, as required. If $E$ is multiplied on the right, then the proof is analogous using column operation instead of row operations.

As a result we have

$$|AB| = |(AB)^\nabla|^{-1} = |B^\nabla A^\nabla|^{-1} = |P_1^{-1} B^\nabla A^\nabla P_2^{-1}|^{-1} = |P_1||B^\nabla A^\nabla|^{-1}|P_2| \leq \frac{|P_1||B^\nabla|^{-1}|A^\nabla|^{-1}|P_2|}{|P_1||B^\nabla|^{-1}|A^\nabla|^{-1}|P_2|} = |P_1^{-1} B^\nabla P_2^{-1}||A^\nabla|^{-1} = |B^\nabla|^{-1}|A^\nabla|^{-1} = |A||B|,$n

where $P_1, P_2$ are the right normalizer of $B$ and left normalizer of $A$ respectively, and as shown in Lemma 2.20, $A$ and $B$ are products of elementary matrices of type 3, for which the theorem holds according to (4.3).

5. TROPICALLY CONJUGATE AND SIMILAR MATRICES

Although our factorization results for $A^\nabla$ follow from those for $A^*$, the $\nabla$ operation has additional applications in representation theory.

**Definition 5.1.** A matrix $B$ is **tropically conjugate** to $B'$ if there exists a non-singular matrix $A$ such that

$$A^\nabla B A \equiv B'.$n

If equality holds in (5.1) then we say that $B$ is **tropically similar** to $B'$.

**Remark 5.2.** If $B$ is tropically similar to $B'$, then

a. $\det(B') \equiv \det(B)$, with equality when $B'$ is non-singular.

b. $\text{tr}(B') \equiv \text{tr}(B)$.
These relations can be seen by direct computation, but are also a consequence of Theorem 5.4 below.

Following the next motivating example, we have an analog to the theorem on similar matrices in classical algebra.

**Example 5.3.**

1. Let 
   \[ A = \begin{pmatrix} 2 & 0 \\ 1 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 2 \\ 3 & 1 \end{pmatrix}, \quad B' = \begin{pmatrix} 3 & 1^\nu \\ 5 & 3 \end{pmatrix}. \]
   Therefore \( A \nabla BA = B' \), and \( f_{B'}(x) = x^2 + 3^\nu x + 6^\nu \models f_B(x) = x^2 + 1^\nu x + 5 \). As a result all supertropical eigenvalues of \( B \) are supertropical eigenvalues of \( A \nabla BA \).

2. By taking 
   \[ B = \begin{pmatrix} 0 & 0 \\ 1 & 2 \end{pmatrix}, \quad B' = \begin{pmatrix} 1 & 3 \\ 1 & 2 \end{pmatrix}, \quad A = \begin{pmatrix} 0 & 1^\nu \\ -\infty & 0 \end{pmatrix} \]
   we get 
   \[ A \nabla BA = \begin{pmatrix} 2^\nu & 3^\nu \\ 1 & 2^\nu \end{pmatrix} \models B'. \]
   However, \( |B| = 2 \) and \( |B'| = 4 \) do not ghost-surpass each other, which shows that there is no ghost surpassing relation between the characteristic polynomials of tropically conjugate matrices.

**Theorem 5.4.** If \( B \) is tropically similar to \( B' \), then \( f_{B'} \models f_B \).

*Proof.* Let \( B' = A \nabla BA \), where \( A \) is a non-singular matrix.

Let \( \bar{A} \) be the right normal form of \( A \) and \( E \) be its right normalizer which means 
\[ A = \bar{A}E \text{ and } A^\nabla = E^{-1}\bar{A}^\nabla. \]

Using Lemma 3.2, we notice that \( A^\nabla BA \) and \( \bar{A}^\nabla B\bar{A} \) share the same characteristic polynomial, since 
\[ det(xI + A^\nabla BA) = det(xI + E^{-1}\bar{A}^\nabla B\bar{A}E) \models det(E)^{-1}det(xI + \bar{A}^\nabla B\bar{A})det(E) = \]
\[ det(xI + \bar{A}^\nabla B\bar{A}) = det(xI + EA^\nabla BA^{-1}) \models det(E)det(xI + A^\nabla BA)det(E)^{-1} = \]
\[ det(xI + A^\nabla BA). \]

Therefore it is sufficient to prove this theorem for \( A \) in normal form, which also means 
\[ det(A) = 1_R \text{ and therefore } A^\nabla = \text{adj}(A). \]

We will show that each of the summands in the coefficient of \( x^{n-k} \) in \( f_{B'} \) either has another identical summand in this coefficient, creating a ghost summand, or is dominated by one, or has an identical summand in the corresponding coefficient in \( f_B \).

First, we formulate the relevant terms. The description of the entries in \( B' \) can be quite complicated, as this matrix is obtained as a product of three matrices. Moreover, the entries of one of them, \( A^\nabla = (a_i'_{i,j}) \), includes the sums of monomials of \( n-1 \) entries from \( A = (a_{i,j}) \). Therefore in order to describe the coefficient of \( x^{n-k} \) (which is the sum
of determinants of all $k \times k$ principal sub-matrices, as described in detail in [4], we begin by formulating the entries of $A^\nabla BA$.

The entry in the $(i, \sigma(i))$ position in $B' = A^\nabla BA$ has the form

$$
\sum_{t,s=1}^{n} a'_{i,t} b_{t,s} a_{s,\sigma(i)},
$$

and therefore, the coefficient of $x^{n-k}$ is a sum of monomials of the form

$$(5.2) \prod_{j=1}^{k} a'_{i,j} b_{j,s} a_{s,\sigma(i)}.$$  

for $\sigma \in S_k$, where $t_j, s_j \in \{1, \ldots, n\}$.

We can factor each $\sigma$ into disjoint cycles $(i_j \sigma(i_j) \sigma^2(i_j) \cdots \sigma^{d_j-1}(i_j))$. In order to simplify the indices, we denote $i_j$ as $j$, and powers of $\sigma$ are starting with $\sigma^0(j) = j$, with the understanding that these powers are taken mod$(d_j)$.

$$(5.3) \prod_{cycles\ of\ \sigma} (a'_{j,t_j} b_{t_j,s_j} a_{s_j,\sigma(j)})(a'_{\sigma(j),t_{\sigma(j)}} b_{t_{\sigma(j)},s_{\sigma(j)}} a_{s_{\sigma(j)},\sigma^2(j)}) \cdots (a'_{\sigma^{d_j-1}(j),t_{\sigma^{d_j-1}(j)-1}} b_{t_{\sigma^{d_j-1}(j)-1},s_{\sigma^{d_j-1}(j)-1}} a_{s_{\sigma^{d_j-1}(j)-1},\sigma^{d_j-1}(j)}) =$$

$$\prod_{cycles\ of\ \sigma} \left( \prod_{powers\ of\ \sigma} a'_{j,t_j} b_{t_j,s_j} a_{s_j,\sigma(j)} \right).$$

This term itself is a sum of products, since each $a'_{j,t_j}$ is a sum arising from the adjoint. Therefore, according to Definition 2.15, each $a'_{j,t_j}$ in (5.2) is:

$$a'_{j,t_j} = \sum_{\pi \in S_n} (a_{j,\pi(j)} \cdots a_{\pi^{-1}(t_j),t_j}) C_\pi$$

which is dominated, according to Lemma 2.13, by the summand:

$$(a_{j,\pi(j)} \cdots a_{\pi^{-1}(t_j),t_j}) Id = (a_{j,\pi(j)} \cdots a_{\pi^{-1}(t_j),t_j}).$$

Thus, it is sufficient to refer to (5.3) as

$$\prod_{cycles\ of\ \sigma} \left( \prod_{powers\ of\ \sigma} (a_{j,\pi(j)} \cdots a_{\pi^{-1}(t_j),t_j}) b_{t_j,s_j} a_{s_j,\sigma(j)} \right).$$

Next, we divide our proof to disjoint cases as follows:
Case 1: Assume that $a_{s_j,\sigma(j)}$ is diagonal for every $j$.
Case 1.1: If all $a'_{j,t_j}$ are diagonal entries, then (5.2) is $\prod b_{j,\sigma(j)}$, which is a summand in $f_B$.
Case 1.2: If some $a'_{j,t_j}$ is not diagonal, then (5.3) is
\[
\prod \text{cycles of } \sigma \left( \prod \text{powers of } \sigma a'_{j,t_j} b_{t_j,\sigma(j)} \right) = \prod \text{cycles of } \sigma \left( \prod \text{powers of } \sigma (a_{\pi(j)} \cdots a_{\pi^{-1}(t_j),t_j}) b_{t_j,\sigma(j)} \right)
\]
and we can move the outer brackets one position forward:
\[
\prod \text{cycles of } \sigma \left( \prod \text{powers of } \sigma (a_{\pi(j)} \cdots a_{\pi^{-1}(t_j),t_j}) b_{t_j,\sigma(j)} a_{\sigma(j),\pi(j)} \cdots a_{\pi^{-1}(t_j),t_j}) b_{t_j,\sigma(j)} \right).
\]
We want the new initial indices to describe a permutation, which occurs when all of these indices are different. In this case (5.4) and (5.5) yield a ghost summand. If some index repeats, then according to Lemma 2.13 we can factor (5.5) into (not necessarily disjoint) cycles, and some permutation composed of the Id and one of these cycles will dominate this summand, so (5.5) will be dominated by a ghost summand.
Case 2: If some $a_{s_j,\sigma(j)}$ is not diagonal, then our term can be rewritten by moving each bracket one position backwards:
\[
\prod \text{cycles of } \sigma \left( \prod \text{powers of } \sigma a'_{j,t_j} b_{t_j,s_j} a_{s_j,\sigma(j)} \right) = \prod \text{cycles of } \sigma \left( \prod \text{powers of } \sigma a_{s^{-1}(j),j} a'_{j,t_j} b_{t_j,s_j} \right).
\]
Once again, we want the new initial indices to describe a permutation.
Case 2.1: If all the $s_j$'s are different, this also describes a permutation track, yielding a ghost summand (dual to the previous argument).
Case 2.2: If some index repeats, then by Lemma 2.13 we can factor the term into (not necessarily disjoint) cycles, and there exists a permutation composed of the Id and one of these cycles whose track will dominate this summand.

Moreover, we want each
\[
a_{s^{-1}(j),j} a'_{j,t_j} = a_{s^{-1}(j),j} a_{\sigma(j)} \cdots a_{\pi^{-1}(t_j),t_j}
\]
to describe a summand in an entry of $A^\nabla$, which occurs when the initial index $s_{\sigma^{-1}(j)}$ is different from all of the subsequent ones in (5.6). (Notice that all other indices cannot repeat since they are a part of a permutation.) In this case we get a summand in the $(s_{\sigma^{-1}(j)}, t_j)$ position of $A^\nabla$.

If this index does appear in the following indices in (5.6), then by Lemma 2.13 we can factor this term, including a cycle starting and ending in $s_{\sigma^{-1}(j)}$, dominated by the Id. The assertion then follows. □

**Corollary 5.5.** If $B$ is tropically similar to $B'$, then
1. Every supertropical eigenvalue of $B$ is a supertropical eigenvalue of $B'$.
2. If $f_{B'} \in T[x]$ then $f_{B'} = f_B$.
3. $B$ satisfies $f_{B'}$.

Proof. 1. $B$ is tropically similar to $B'$. If $\lambda$ is an eigenvalue of $B$, then $f_{B'}(\lambda) \equiv f_B(\lambda) \in G$.
2. Follows immediately from the theorem.
3. By Theorem 2.29 we get $f_{B'}(B) \equiv f_B(B) \in G$. □

6. **The characteristic polynomial of $A^\nabla$**

Current research regarding the eigenspaces of a supertropical matrix shows that these spaces may behave in an undesirable fashion, as shown in [17, Example 5.7]. The following theorem states important properties regarding the characteristic polynomial, eigenvalues, determinant and trace of $A^\nabla$. We are hoping this will provide the means of solving the dependency problem of supertropical eigenspaces.

**Example 6.1.** Let $A = \begin{pmatrix} 1 & 0 & - \\ 3 & 4 & - \\ - & - & 1 \end{pmatrix}$ (where $-$ denotes $-\infty$). Then

$$f_A(x) = x^3 + 4x^2 + 5'x + 6$$

and

$$A^\nabla = 6^{-1} \begin{pmatrix} 5 & 1 & - \\ 4 & 2 & - \\ - & - & 5 \end{pmatrix} = \begin{pmatrix} -1 & -5 & - \\ -2 & -4 & - \\ - & - & -1 \end{pmatrix}.$$ 

As a result we get $f_{A^\nabla}(x) = x^3 + (-1)'x^2 + (-2)x - 6$. Multiplying by $\det(A)$ we get

$$6x^3 + 5'x^2 + 4x + 0,$$

which has the same coefficients of $f_A$ but in opposite order. Consequently, the inverse of every supertropical eigenvalue of $A$ is a supertropical eigenvalue of $A^\nabla$, obtained in the opposite order.

In the following theorem we generalize this example and show how the characteristic polynomial of $A^\nabla$ is related to the characteristic polynomial of $A$.

**Theorem 6.2.** If $A = (a_{i,j}) \in M_n(R)$ is a non-singular matrix, then

$$\det(A)f_{A^\nabla}(x) \equiv x^n f_A(x^{-1}).$$
Proof. We denote:
\[ A^\nabla = \left( \frac{a'_{i,j}}{\text{det} A} \right), \] where \( a'_{i,j} \) are the entries of \( \text{adj}(A) \),
\[ |A| = a_{1,\rho(1)} \cdots a_{n,\rho(n)} \] is the dominant permutation track in \( \text{det}(A) \),
\[ f_A(x) = \sum a_i x^i, \]
\[ f_{A^\nabla}(x) = \sum b_i x^i, \]
and \( a'_{i,t} = \sum (a_{i,\pi(i)} a_{\pi(i),\pi^2(i)} \cdots a_{\pi_{\rho(i)}^{-1}(i),t}) C_{\pi_i} \) as defined in 2.15 (\( \pi \) has an index since we are about to have a sequence of entries from \( A^\nabla \)).

We would like to show that \( |A| b_{n-k} = a_k \forall k \).

A typical summand in \( |A| b_{n-k} \) is
\[ (6.1) \quad |A| \prod_{j=1}^{k} \frac{a'_{j,\sigma(i_j)}}{|A|}, \]
where \( \sigma \in S_k \) is applied to \( i_1, \ldots, i_k \). In this term, two types of permutations are in action; \( \sigma \), which is determined by the tracks in the principal \( k \times k \) sub-matrices of \( A^\nabla \),
and \( \pi_{ij} \in S_n, j = 1, \ldots, k \), which are determined in the \( i_j, \sigma(i_j) \) position of \( A^\nabla \). We also consider the dominant permutation track attained by \( \rho \in S_n \) in \( \text{det}(A) \). We simplify the notation by taking \( i_j \) as \( j \).

In the special case where \( \rho(j) = \sigma^{-1}(j) \) and \( \rho = \pi_j \) for every \( j = 1, \ldots, k \) we obtain in (6.1) a summand of \( a_k \):
\[ |A| \prod_{j=1}^{k} \frac{a'_{j,\sigma(i_j)}}{|A|} = |A| \prod_{j=1}^{k} \frac{(a_{j,\pi_j(j)} \cdots a_{\pi_{\rho(j)}^{-1}(j),\sigma(j)}) C_{\pi_j} \cdot a_{\sigma(j),j}}{|A| \cdot a_{\sigma(j),j}} = \]
\[ |A| \prod_{j=1}^{k} \frac{1}{a_{\sigma(j),j}} = \prod_{j=1}^{k} a_{\sigma(j),j} \prod_{j=k+1}^{n} a_{j,\rho(j)} = \prod_{j=k+1}^{n} a_{j,\rho(j)}, \]
where \( \rho \in S_{n-k} \) applied to \( k+1, \ldots, n \) (equivalently to \( i_{k+1}, \ldots, i_n \)).

If \( \rho(j) = \sigma^{-1}(j) \forall j \) but \( \exists j : \rho \neq \pi_j \), then (6.1) will be surpassed by a summand of \( a_k \):
\[ |A| \prod_{j=1}^{k} \frac{a'_{j,\sigma(i_j),j} a_{\sigma(j),j}}{|A| a_{\sigma(j),j}} \leq \frac{|A|}{\prod_{j=1}^{k} a_{\sigma(j),j}} = \prod_{j=k+1}^{n} a_{j,\rho(j)}. \]

If \( \text{det}(A) \) is attained by \( \rho \in S_n \) such that \( \sigma^{-1}(j) \neq \rho(j) \) for some \( j \in \{1, \ldots, k\} \), then (6.1) will be attained twice, creating a ghost summand. We explain this phenomenon by factoring \( \sigma \) into disjoint cycles. As in Theorem 5.4, we consider powers of \( \sigma \) as starting at \( \sigma^0(j) = j \), with the understanding that these powers are considered \( \text{mod}(d_j) \), the order of the cycle of \( j \):
\[ \prod_{j=1}^{k} a'_{j,\sigma(j)} = \prod_{\text{cycles of } \sigma} (a'_{j,\sigma(j)} a_{\sigma(j),\sigma^2(j)} \cdots a_{\sigma^{d_j-1}(j),j}) = \]
\[
\prod_{\text{cycles of } \sigma} \left( \prod_{\text{powers of } \sigma} a'_{j,\sigma(j)} \right) = \prod_{\text{cycles of } \sigma} \left( \prod_{\text{powers of } \sigma} \left( a_{j,\sigma(j)} a_{\pi(j),\sigma(j)}^{2} \cdots a_{\pi^{-1}(\sigma(j)),\sigma(j)}^{k} \right) C_{\pi(j)} \right).
\]

So we are looking at a sequence of brackets; the inner ones, which start with the indices \( j = 1, \ldots, k \), forming the permutation \( \sigma \in S_k \), factored into disjoint cycles, denoted by the outer brackets. We will now shift our attention to the second index in each inner bracket, that is \( \pi_{j}(j) \), and show how it helps us to find an identical term on a different permutation track, using the same technique as in the proof of [23, Theorem 3.6].

If \( \pi_{i}(i) \neq \pi_{j}(j) \forall i \neq j \) then moving each inner bracket one element forward describes the permutation track of \( \pi_{j}(j) \mapsto \pi_{\sigma(j)}(\sigma(j)) \):

\[
\prod_{\text{cycles of } \sigma} \left( \prod_{\text{powers of } \sigma} \left( a_{\pi_{j}(j),\pi_{j}(j)}^{2} \cdots a_{\pi^{-1}(\sigma(j)),\sigma(j)} C_{\pi_{j}} a_{\sigma(j),\pi_{\sigma(j)}(\sigma(j))} \right) \right).
\]

If \( \exists i \neq j : \pi_{i}(i) = \pi_{j}(j) \), then we may use the transposition \( i \mapsto \sigma(j), \ j \mapsto \sigma(i) \), since:

\[
a_{i,\pi_{i}(i)}(a_{\pi_{i}(i),\pi_{j}(i)}^{2} \cdots a_{\pi^{-1}(\sigma(i)),\sigma(i)}) \cdot a_{j,\pi_{j}(j)}(a_{\pi_{j}(j),\pi_{j}(j)}^{2} \cdots a_{\pi^{-1}(\sigma(j)),\sigma(j)}) =
\]

\[
a_{i,\pi_{i}(i)}(a_{\pi_{j}(j),\pi_{j}(j)}^{2} \cdots a_{\pi^{-1}(\sigma(j)),\sigma(j)}) \cdot a_{j,\pi_{j}(j)}(a_{\pi_{i}(i),\pi_{i}(i)}^{2} \cdots a_{\pi^{-1}(\sigma(i)),\sigma(i)}).
\]

Again, as in the proof of [23, Theorem 3.6], this action will cause a decomposition of a cycle with such repeating indices, or a joining of cycles in case the indices repeat in different cycles.

\[\square\]

**Corollary 6.3.**

1. The inverse of every supertropical eigenvalue of \( A \) is a supertropical eigenvalue of \( A^{\triangledown} \).
2. \( |A|tr(A^{\triangledown}) = a_1 \).

**Proof.**

1. If \( \lambda \) is an eigenvalue of \( A \), then \( det(A)f_{A^{\triangledown}}(\lambda^{-1}) \vdash (\lambda^{-1})^n f_{A}(\lambda) \in G \).
2. \( |A|tr(A^{\triangledown}) = |A|b_{n-1} = \sum |A| \frac{a'_{i,i}}{|A|} = \sum a'_{i,i}, \)

which is indeed the sum of all \((n-1) \times (n-1)\) principal sub-matrices in \( A \).

\[\square\]

**Theorem 6.4.**

(i) If \( f_{A^{\triangledown}} \in T[x] \) then \( det(A)f_{A^{\triangledown}}(x) = x^n f_{A}(x^{-1}). \)

(ii) If \( f_{A^{\triangledown^{\triangledown}}} \in T[x] \) then \( f_{A^{\triangledown^{\triangledown}}} = f_{A} \).
Proof.
(i) Follows immediately from Theorem 6.2.

(ii) According to Corollary 5.3, we have
\[ \det(A)^{-1} f_{A^{\nabla \nabla}}(x) \models_{gs} x^n f_A(x^{-1}) \]
and
\[ \det(A) f_{A^{\nabla}}(x) \models_{gs} x^n f_A(x^{-1}). \]

Combining these two equations we get
\[ \det(A) \det(A)^{-1} f_{A^{\nabla \nabla}}(x) \models_{gs} \det(A) x^n f_{A^{\nabla}}(x^{-1}) \models_{gs} x^n x^{-n} f_A(x). \]

Then, if \( f_{A^{\nabla \nabla}} \in T[x] \), we have that \( f_{A^{\nabla \nabla}} = f_A \). \( \Box \)

References

[1] M. Akian, R. Bapat, and S. Gaubert, Max-plus algebra. Hogben L., Brualdi R., Greenbaum A., Mathias R. (eds.), Handbook of Linear Algebra. Chapman and Hall, London, 2006.

[2] A. Buchholz, Tropicalization of linear isomorphisms on plane elliptic curves. PhD dissertation, Mathematical Institute Georg-August, University Göttingen, April 2010.

[3] P. Butkovic, Max-algebra: the linear algebra of combinatorics?. Linear Algebra Appl. 367 (2003), 313-335.

[4] P. Butkovic, On the coefficients of the max-algebraic characteristic polynomial and equation. In proceedings of the workshop on Max-algebra, Symposium of the International Federation of Automatic Control, Prague, 2001.

[5] P. Butkovic and L. Murfitt, Calculating essential terms of a characteristic maxpolynomial. CEJOR 8 (2000) 237-246.

[6] R.A. Cuninghame-Green, Minimax algebra. Lecture notes in Economics and Mathematical Systems, no. 166, Springer, 1979.

[7] M. Fiedler, J. Nedoma, J. Ramik, J. Rohn and K. Zimmermann, Linear optimization problems with inexact data. Springer, New York, 2006.

[8] S. Gaubert and M. Plus, Methods and applications of (max, +) linear algebra, in Lecture Notes in Computer Science. 1200, Springer Verlag, New York, 1997.

[9] J.S. Golan, Semirings and their applications. Kluwer Acad. Publ., Dordrecht, 1999.

[10] M. Gondran, Path algebra and algorithms. In B. Roy, editor, Combinatorial programing: methods and applications, Reidel, Dordrecht, pp. 137-148, 1975.

[11] B. Heidergott, G.J Olsder and J. van der Woude, Max Plus at Work: Modeling and Analysis of Synchronized Systems: A Course on Max-Plus Algebra and Its Applications. Princeton Univ. Press, 2006.

[12] I. Itenberg, G. Mikhalkin and E. Shustin, Tropical Algebraic Geometry. Oberwolfach Seminars, Vol. 35, Birkhauser, Basel e.a., 2007.

[13] Z. Izhakian, Tropical arithmetic and matrix algebra. Comm. in Algebra 37(4):1445-1468, 2009.

[14] Z. Izhakian, M. Knebusch and L. Rowen, Supertropical linear algebra. To appear in Pacific Journal of Mathematics, 2013.

[15] Z. Izhakian and L. Rowen, Supertropical algebra. Adv. Math. 225: 2222-2286, 2010.

[16] Z. Izhakian and L. Rowen, Supertropical matrix algebra. Israel Journal of Mathematics 182(1):383–424, 2011.

[17] Z. Izhakian and L. Rowen, Supertropical matrix algebra II: solving tropical equations. Israel Journal of Mathematics, 186(1):69-97, 2011.

[18] Z. Izhakian and L. Rowen, Supertropical matrix algebra III: Powers of matrices and their supertropical eigenvalues. Journal of Algebra, 341(1):125–149, 2011.

[19] G. L. Litvinov and V. P. Maslov, Idempotent mathematics: correspondence principle and applications. Russian Mathematical Surveys, 51, no. 6 (1996) 1210-1211.
[20] G. L. Litvinov, A. Ya. Rodionov, S.N. Sergeev and A. V. Sobolevski, *Universal algorithms for solving the matrix Bellmann equation over semirings*. Preprint. Moscow, 2012. Submitted to Soft Computing.

[21] G. L. Litvinov and S.N. Sergeev, *Tropical and Idempotent Mathematics*. Contemporary Mathematics, vol 495. AMS, Providence (2009).

[22] G. Mikhalkin, *Tropical geometry and its applications*. In Proceedings of the ICM, Madrid, Spain, vol. II, 2006, pp. 827-852. arXiv: math/0601041v2 [math. AG]

[23] A. Niv, *Characteristic Polynomials of Supertropical Matrices*. Preprint at arXiv:1201.0966v2, 2012. To appear in Communications in Algebra.

[24] A. Niv, *Factorization of tropical matrices*. J. Algebra Appl., vol 13, 1350066 (2014).

[25] S.N. Sergeev, *Multiorder, Kleene stars and cyclic projectors in the geometry of max cones*. In G. L. Litvinov and S. N. Sergeev (Eds.), Contemporary mathematics: Vol. 495. Tropical and idempotent mathematics (pp. 317342). AMS, Providence (2009). Society.

[26] S.N. Sergeev, H. Schneider and P. Butkovic, *On visualization scaling, subeigenvectors and Kleene stars in max algebra*. Linear Algebra and its Applications, 431(12):2395–2406, 2009.

[27] H. Straubing, *A combinatorial proof of the Cayley-Hamilton Theorem*. Discrete Math. 43 (2-3): 273-279, 1983.

[28] M. Yoeli, *A note on a generalization of boolean matrix theory*. Amer. Math. Monthly 68 (1961), 552-557.