Abstract. We show that a positive Borel measure of positive finite total mass, on compact Hermitian manifolds, admits a Hölder continuous quasi-plurisubharmonic solution to the Monge-Ampère equation if and only if it is dominated locally by Monge-Ampère measures of Hölder continuous plurisubharmonic functions.

1. Introduction

The analogue of the Calabi-Yau theorem on compact Hermitian manifolds was proven in 2010 by Tosatti and Weinkove [22]. Continuous weak solutions for the right hand side in $L^p$, $p > 1$ were obtained later by the authors [17]. Here we continue to study weak solutions for more general measures.

Consider a compact Hermitian manifold $(X, \omega)$ of dimension $n$, and a positiveRadon measure $\mu$ with finite total mass on $X$. An upper semicontinuous function $u$ on $X$ is called $\omega$-psh if $dd^c u + \omega \geq 0$ (as currents). Then we write $u \in \text{PSH}(\omega)$.

Our objective is to show that if the complex Monge-Ampère equation has Hölder continuous solutions for $\mu$ restricted to local charts then it has Hölder continuous solutions globally on $X$. To be precise we introduce first the following definition.

Definition 1.1. We say that $\mu$ admits a global Hölder continuous subsolution if there exists a Hölder continuous $\omega$-psh function $u$ and $C_0 > 0$ such that

$$\mu \leq C_0 (\omega + dd^c u)^n$$

on $X$.

Let us denote by $\mathcal{M}$ the set of all such measures.

To verify the defining condition it is enough to look at $\mu$ locally.

Lemma 1.2. A measure $\mu$ belongs to $\mathcal{M}$ if and only if for every $x \in X$, there exists a neighborhood $D$ of $x$ and a Hölder continuous psh function $v$ on $D$ such that $\mu|_D \leq (dd^c v)^n$.

Proof. The necessary condition is obvious, so we prove the sufficient condition. Using the strict positivity of $\omega$ we can extend a Hölder continuous psh function $v$ defined in a local coordinate chart to the whole space $X$ so that the extension is a Hölder continuous $C^\omega$-psh function for some large $C > 0$. Taking a finite cover

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by coordinate charts and using the partition of unity one easily constructs a global \(\omega\)-psh function \(u\) satisfying (1.1) (see [15] for details of such a construction). □

Our main result can be viewed as a generalization of Demailly et al. [7, Proposition 4.3] from the Kähler to the Hermitian setting.

**Theorem 1.3.** Assume that \(0 < \mu(X) < +\infty\). There exists a Hölder continuous \(\omega\)-psh \(\varphi\) and a constant \(c > 0\) solving

\[
(\omega + dd^c\varphi)^n = c\mu
\]

if and only if \(\mu\) belongs to \(\mathcal{M}\).

Thanks to this theorem the important class of measures having \(L^p\)-density, for \(p > 1\), admits Hölder continuous solutions. This result was proven in [18, Theorem B] under the extra assumption that the right hand side is strictly positive.

**Corollary 1.4.** Let \(f\) be a non-negative function in \(L^p(\omega^n)\) for \(p > 1\). Assume that \(\int_X f\omega^n > 0\). Then there exists a Hölder continuous \(\varphi \in \text{PSH}(\omega)\) and a constant \(c > 0\) such that

\[
(\omega + dd^c\varphi)^n = c f\omega^n.
\]

**Proof.** By [17, Theorem 0.1] there exists \(\varphi \in \text{PSH}(\omega) \cap C^0(X)\) and a constant \(c > 0\) satisfying

\[
(\omega + dd^c\varphi)^n = c f\omega^n.
\]

Consider a local coordinate chart \(B \subset X\) parametrized by the unit ball in \(\mathbb{C}^n\). Let \(\chi\) be a smooth cut-off function such that

\[
0 \leq \chi \leq 1, \quad \chi = 1 \text{ on } B(0,1/2), \quad \text{supp } \chi \subset B.
\]

Find \(w \in \text{PSH}(B)\) the solution of the Dirichlet problem for the Monge-Ampère equation:

\[
(dd^cw)^n = c\chi f\omega^n, \quad w|_{\partial B} = 0.
\]

By the main result of [13] (see also [5]) we get that \(w \in C^{0,\alpha}(\overline{B})\) for some \(\alpha\) positive depending only on \(n,p\). Therefore, on \(B(0,1/2)\) the right hand side \(c f\omega^n\) is dominated by \((dd^cw)^n\). We conclude from Lemma 1.2 and Theorem 1.3 that \(\varphi\) is Hölder continuous. □

**Remark 1.5.** Using the recent result from [21] instead of [13] we also can show that if \(\mu \in \mathcal{M}\) and \(0 \leq f \in L^p(d\mu)\) for \(p > 1\), then \(f d\mu \in \mathcal{M}\). In other words, \(\mathcal{M}\) satisfies the \(L^p\)-property (see [21]) and the above corollary is a special case.

Another consequence of the main result is the convexity of the range of Monge-Ampère operator acting on Hölder continuous functions.

**Corollary 1.6.** The set

\[
\mathcal{A} := \{ c \cdot (\omega + dd^c\varphi)^n : \varphi \in \text{PSH}(\omega), \ \varphi \text{ is Hölder continuous}, \ c > 0 \}
\]

is convex.

**Proof.** For brevity we use the notation \(\omega^n_\varphi := (\omega + dd^c\varphi)^n\). Let \(c_1 \omega^n_\varphi_1, c_2 \omega^n_\varphi_2 \in \mathcal{A}\). It is easy to see that

\[
\mu := \frac{1}{2} \left( c_1 \omega^n_\varphi_1 + c_2 \omega^n_\varphi_2 \right) \leq 2^{n-1} \left( c_1 + c_2 \right) \left( \omega + dd^c\varphi_1 + \frac{\varphi_2}{2} \right)^n.
\]

Apply Theorem 1.3 to get that \(\omega^n_\varphi = c\mu\) for some Hölder continuous \(\omega\)-psh \(\phi\) and some constant \(c > 0\). Therefore, \(\mu \in \mathcal{A}\). □
Dedication. It is our privilege to dedicate this paper to Jean-Pierre Demailly, a great mathematician and a champion for math education.

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2. Preliminaries

Let us recall the definition of the Bedford-Taylor capacity. For a Borel set $E \subset X$ put

$$
\text{cap}_\omega(E) := \sup \left\{ \int_E \omega^v : v \in PSH(\omega), 0 \leq v \leq 1 \right\}.
$$

By [15, p. 52], this capacity is comparable with the local Bedford-Taylor capacity $\text{cap}'_\omega(E)$. Combining this fact with the work of Dinh-Nguyen-Sibony [11] we get the following result.

Lemma 2.1. Let $\mu \in \mathcal{M}$. Then, for every compact set $K \subset X$,

$$
\mu(K) \leq C \exp \left( \frac{-\alpha_1}{[\text{cap}_\omega(K)]^{\frac{1}{n}}} \right),
$$

where $C, \alpha_1 > 0$ depend only on $X$ and the Hölder exponent of the global Hölder subsolution.

Corollary 2.2. Assume that $\mu \in \mathcal{M}$ and fix $\tau > 0$. Then, there exists $C_\tau > 0$ such that for every compact set $K \subset X$

$$
\mu(K) \leq C_\tau [\text{cap}_\omega(K)]^{1+\tau}.
$$

The set of measures satisfying this inequality is denoted by $\mathcal{H}(\tau)$.

The proof of the next statement can be found in [10, Theorem 2.1].

Lemma 2.3. Let $u \in PSH(\omega) \cap C^0(X)$ with $0 < \alpha < 1$. Then there exists a sequence of smooth $\omega$-psh function $\{u_j\}_{j \geq 1}$ such that $u_j \to u$ in $C^{0,\alpha'}(X)$ as $j \to +\infty$, for any $0 < \alpha' < \alpha$.

We need also an estimate which for Kähler manifolds was given in [12].

Proposition 2.4. Suppose $\psi \in PSH(\omega) \cap C^0(X)$ and $\psi \leq 0$. Let $\mu$ satisfy the inequality (2.3) for some $\tau > 0$, i.e. $\mu \in \mathcal{H}(\tau)$. Assume that $\varphi \in PSH(\omega) \cap C^0(X)$ solves

$$
(\omega + dd^c \varphi)^n = \mu.
$$

Then for $\gamma = \frac{1}{1+(n+2)(\alpha+\frac{1}{2})}$ and some positive $C > 0$ depending only on $\tau, \omega$ and $\|\psi\|_{\infty}$ we have

$$
\sup_X (\psi - \varphi) \leq C \|\psi - \varphi\|_{L^1(\omega)} + \|\psi\|_{\infty}.
$$

Proof. Without loss of generality we may assume that $-1 \leq \psi \leq 0$. Put $U(\varepsilon, s) = \{ \varphi < (1-\varepsilon)\psi + \inf_X [\varphi - (1-\varepsilon)\psi] + s \}$, where $0 < \varepsilon < 1$ and $s > 0$. 
Lemma 2.5. For $0 < s \leq \frac{1}{3} \min \{ \varepsilon^n, \frac{3}{16n} \}$, $0 < t \leq \frac{1}{3}(1 - \varepsilon) \min \{ \varepsilon^n, \frac{3}{16n} \}$ we have
\[
t^n \cap_\omega (U(\varepsilon, s)) \leq C [\cap_\omega (U(\varepsilon, s + t))]^{1 + \tau},
\]
where $C$ is a dimensional constant.

Proof of Lemma 2.5. By [17, Lemma 5.4] (2.4)
\[
t^n \cap_\omega (U(\varepsilon, s)) \leq C \int_{U(\varepsilon, s + t)} \omega^n_x,
\]
The lemma now follows from (2.3). \hfill \Box

Lemma 2.6. Fix $0 < \varepsilon < \frac{3}{4}$ and $\varepsilon_B := \frac{1}{3} \min \{ \varepsilon^n, \frac{3}{16n} \}$. Then, there exists a constant $C = C(\tau, \omega)$ such that for $0 < s < \varepsilon$,
\[
t^n \leq C [\cap_\omega (U(\varepsilon, s + t))]^{1 + \tau}.
\]

Proof of Lemma 2.6. Let us use the notation $a(s) := [\cap_\omega (U(\varepsilon, s))]^{\frac{1}{n}}$.

It follows easily from (2.4) that
\[
a(s) \leq C [a(s + t)]^{1 + \tau}.
\]
This is the inequality [19, Eq. (3.6)]. The arguments that follow in that paper complete the proof of the present lemma. \hfill \Box

To finish the proof of the proposition we proceed as in [19, Theorem 3.11]. One needs to estimate
\[-S := \sup_X (\psi - \varphi) > 0\]
in terms of $\| (\psi - \varphi)_+ \|_{L^1(d\mu)}$ as in the Kähler case [14]. Suppose that
\[
(2.5) \quad \| (\psi - \varphi)_+ \|_{L^1(d\mu)} \leq \varepsilon^a
\]
for $0 < \varepsilon << 3/4$ and $a = \frac{1}{\gamma}$. Let
\[
h(s) := \frac{(s/C)}{\tau}
\]
be the inverse function of $C_x s^\tau$. Consider sublevel sets $U(\varepsilon, t) = \{ \varphi < (1 - \varepsilon)\psi + S_x + t \}$, where $S_x := \inf_X [\varphi - (1 - \varepsilon)\psi]$. It is clear that
\[
(2.6) \quad S - \varepsilon \leq S_x \leq S.
\]
Therefore, $U(\varepsilon, 2t) \subset \{ \varphi < \psi + \varphi + 2t \}$. Then, $(\psi - \varphi)_+ \geq |S| - \varepsilon - 2t > 0$ for $0 < t < \varepsilon_B$ and $0 < \varepsilon < |S|/2$ on the latter set (if $|S| \leq 2\varepsilon$ then we are done).

By (2.4) we have
\[
\cap_\omega (U(\varepsilon, t)) \leq \frac{C}{t^n} \int_{U(\varepsilon, 2t)} d\mu \leq \frac{C}{t^n} \int_X \frac{(\psi - \varphi)_+}{(|S| - \varepsilon - 2t)} d\mu \\
\leq \frac{C \| (\psi - \varphi)_+ \|_{L^1(d\mu)}}{t^n (|S| - \varepsilon - 2t)}.
\]
Moreover, by Lemma 2.6
\[
h(t) \leq [\cap_\omega (U(\varepsilon, t))]^{\frac{1}{n}}.
\]
Combining these inequalities, we obtain
\[ |S| - \varepsilon - 2t \leq \frac{C}{t^n} \left( \psi - \varphi \right)_+ \|L^1(d\mu)\| t^n |h(t)|^n. \]

Therefore, using (2.5),
\[ |S| \leq \varepsilon + 2t + \frac{C}{t^n} \left( \psi - \varphi \right)_+ \|L^1(d\mu)\| t^n |h(t)|^n. \]

Recall that \( \varepsilon_B = \frac{1}{3} \min \{ \varepsilon^n, \frac{\varepsilon}{16B} \} \). So, taking \( t = \varepsilon_B / 2 \geq \varepsilon^n + 2 \) we have
\[ h(t) = \left( \frac{t}{C\tau} \right)^{1/\tau} \geq C\varepsilon^{(n+2)/\tau}. \]

With our choice of \( a \)
\[ \frac{\varepsilon_a}{\varepsilon^{n(n+2)} + \varepsilon^{(n+2)}} = \varepsilon. \]

Hence \( |S| \leq C\varepsilon \) with \( C = C(\tau, \omega) \). Thus,
\[ \sup_X (\psi - \varphi) \leq C \left( \psi - \varphi \right)_+ \|L^1(d\mu)\|. \]

This is the desired stability estimate. \( \square \)

Following [6] we consider \( \rho_\delta \varphi \)- the regularization of the \( \omega \)-psh function \( \varphi \) defined by

\begin{equation}
\rho_\delta \varphi(z) = \frac{1}{\delta^{2n}} \int_{\zeta \in T_z X} \varphi(\exp h_\delta(z)) \rho \left( \frac{|\zeta|^2}{\delta^2} \right) dV_\omega(\zeta), \ \delta > 0;
\end{equation}

where \( \zeta \rightarrow \exp h_\delta(z) \) is the (formal) holomorphic part of the Taylor expansion of the exponential map of the Chern connection on the tangent bundle of \( X \) associated to \( \omega \), and the modifier \( \rho : \mathbb{R}_+ \rightarrow \mathbb{R}_+ \) is given by
\[ \rho(t) = \begin{cases} \frac{\eta}{(1 - t)^2} \exp \left( \frac{t}{1 - t} \right) & \text{if } 0 \leq t \leq 1, \\ 0 & \text{if } t > 1 \end{cases} \]
with the constant \( \eta \) chosen so that
\begin{equation}
\int_{\mathbb{C}^n} \rho(\|z\|^2) dV(z) = 1,
\end{equation}

where \( dV \) denotes the Lebesgue measure in \( \mathbb{C}^n \).

The proof of the following variation of [6] Proposition 3.8] and [2] Lemma 1.12] was given in [18].

**Lemma 2.7.** Fix \( \varphi \in PSH(\omega) \cap L^\infty(X) \). Define the Kiselman-Legendre transform with level \( b > 0 \) by

\begin{equation}
\Phi_{\delta,b}(z) = \inf_{t \in [0,b]} \left( \rho_t \varphi(z) + Kt^2 + b \log \frac{t}{\delta} \right),
\end{equation}

Then for some positive constant \( K \) depending on the curvature, the function \( \rho_t \varphi + Kt^2 \) is increasing in \( t \) and the following estimate holds:
\begin{equation}
\omega + dd^c \Phi_{\delta,b} \geq -(Ab + 2K\delta) \omega,
\end{equation}
where $A$ is a lower bound of the negative part of the Chern curvature of $\omega$.

The next lemma is essentially proven in [7, Theorem 4.3] or [9, Lemma 3.3, Proposition 4.4]. The adaptation of those proofs to the case of compact Hermitian manifolds is straightforward.

**Lemma 2.8.** Let $\mu \in M$ and $\varphi \in PSH(\omega) \cap L^\infty(X)$. Then, there exists $0 < \alpha_1 < 1$ such that

$$
\|\rho_\delta \varphi - \varphi\|_{L^1(\mu)} \leq C \delta^{\alpha_1}.
$$

3. Proof of Theorem 1.3

The necessary condition follows easily. It remains to prove the other one. As $\mu \in M$ there exists $u \in PSH(\omega) \cap C^{0,\alpha_0}(X)$ with $0 < \alpha_0 \leq 1$, and $C_0 > 0$ such that

$$
\mu \leq C_0(\omega + dd^c u)^n.
$$

Using Radon-Nikodym’s theorem, we write $\mu = C_0 h \omega^n_u$ for a Borel measurable function $0 \leq h \leq 1$. Let $u_j$ be the smooth approximation of $u$ as in Lemma 2.3 and denote

$$
\mu_j := C_0 h \omega^n_{u_j}.
$$

Then $\mu_j$ converges weakly to $\mu$ as $j \to +\infty$. Using [17, Theorem 0.1] we find $\varphi_j \in PSH(\omega) \cap C^0(X)$ with normalisation $\sup_X \varphi_j = 0$, and $c_j > 0$ satisfying

$$
\omega^n_{\varphi_j} = c_j \mu_j.
$$

The first thing we need to show is the following.

**Claim 3.1.** There is a uniform constant $C_1 > 0$ such that $1/C_1 < c_j < C_1$.

**Proof.** Since $\mu(X) > 0$, it follows that $\int_X h \omega^n_u > 0$. Therefore, $\int_X h^{1/2} \omega^n_u > 0$. By the Bedford-Taylor convergence theorem [1] we know that $\omega^n_{u_j}$ converges weakly to $\omega^n_u$. Since $C^0(X)$ is dense in $L^1(X, \omega^n_u)$, we have

$$
\int_X h^{1/2} \omega^n_{u_j} > C
$$

for some uniform $C > 0$. Applying the mixed forms type inequality (see [15], [20]) one obtains

$$
\omega_{\varphi_j} \land \omega^{n-1}_{u_j} \geq \left[ \frac{\omega^n_{\varphi_j}}{\omega^n_{u_j}} \right]^{1/2} \omega^n_{u_j} = (c_j C_0 h)^{1/2} \omega^n_{u_j}.
$$

On the other hand,

$$
\int_X \omega_{\varphi_j} \land \omega^{n-1}_{u_j} = \int_X \omega \land \omega^{n-1}_{u_j} + \int_X dd^c \varphi_j \land \omega^{n-1}_{u_j}
$$

$$
= \int_X \omega \land \omega^{n-1}_{u_j} + \int_X \varphi_j dd^c (\omega^{n-1}_{u_j})
$$

$$
\leq \int_X \omega \land \omega^{n-1}_{u_j} + B \int_X |\varphi_j| (\omega^2 \land \omega^{n-2}_{u_j} + \omega^3 \land \omega^{n-3}_{u_j}),
$$
where $B$ is a constant depending only on $\omega$ (see e.g. [8] for details). Since $\|u_j\|_\infty < C \text{ and } \sup_X \varphi_j = 0$, it follows from the Chern-Levine-Nirenberg type inequality ([20, Proposition 1.1]) that the right hand side is uniformly bounded. Thus,

$$\int_X \omega_{\varphi_j} \wedge \omega_{u_j}^{n-1} \leq C.$$  

Combining the above inequalities we get

$$c_j < C_1 := \frac{\int_X \omega_{\varphi_j} \wedge \omega_{u_j}^{n-1}}{\int_X (C_0 h) \omega_{u_j}^{n}} < +\infty.$$  

Hence, by [17, Lemma 5.9] we also have

$$c_j > 1/C_1,$$

increasing $C_1$ if necessary. Thus, Claim 3.1 is proven.  

Thanks to Lemma 2.1, Lemma 2.3 and Claim 3.1 measures $\mu_j$ satisfy the volume-capacity inequality (2.2) with a uniform constant. Thus by [17, Corollary 5.6] we have $\|\varphi_j\|_\infty < C_2$. Passing to a subsequence one may assume that $\{\varphi_j\}$ is a Cauchy sequence in $L^1(\omega^n)$, and $\{c_j\}$ converges. Set

$$(3.3) \quad \varphi := (\limsup_j \varphi_j)^* \quad \text{ and } \quad c = \lim_j c_j.$$

Again passing to a subsequence if necessary we can also assume that

$$(3.4) \quad \varphi_j \to \varphi \quad \text{ in } L^1(\omega^n) \quad \text{ as } j \to \infty.$$  

Lemma 3.2. We have

$$\int_X |\varphi_k - \varphi| \omega_{u_j}^{n} \to 0 \quad \text{ as } \min\{j, k\} \to \infty.$$  

Proof. Using the uniform boundedness of $\|\varphi_j\|_\infty$, $\|u_j\|_\infty$ and the argument in Cegrell[14, Lemma 5.2] (it’s a version of Vitali’s convergence theorem) we get that

$$(3.5) \quad \int_X |\varphi_k - \varphi| \omega_{u_j}^{n} \to 0 \quad \text{ as } k \to \infty.$$  

We shall prove the lemma by the contradiction argument. Assume that there exist subsequences, still denoted by $\{\varphi_k\}_{k \geq 1}$, $\{u_j\}_{j \geq 1}$, and $\delta > 0$ such that

$$\int_X |\varphi_k - \varphi| \omega_{u_j}^{n} > \delta.$$  

Let $a > 0$ be small. By Hartogs’ lemma there exists $k_0$ such that

$$\varphi_k \leq \varphi + a \quad \text{ for } k \geq k_0.$$  

If we choose $a$ small enough, then for $k \geq k_0$ and $j \geq 1$,

$$(3.6) \quad \int_X (\varphi - \varphi_k) \omega_{u_j}^{n} \geq \delta/2.$$  

Next, we are going to show that

$$(3.7) \quad E_{jk} := \int_X (\varphi - \varphi_k) \omega_{u_j}^{n} - \int_X (\varphi - \varphi_k) \omega_{u}^{n} \to 0$$
Let us denote by $T_p(j)$ the current $\omega^p_{u_j} \land \omega^{n-1-p}_{u_j}$. Then

$$dd^c [(\varphi - \varphi_k) T_p(j)] = dd^c (\varphi - \varphi_k) \land T_p(j) + d(\varphi - \varphi_k) \land dT_p(j)$$

$$- d^c (\varphi - \varphi_k) \land dT_p(j) + (\varphi - \varphi_k) dd^c T_p(j)$$

$$=: S_1 + S_2 + S_3 + S_4.$$

By integration by parts

$$(3.8) 
E_{jk} = \int_X (u_j - u) dd^c [(\varphi - \varphi_k) T_p(j)] \leq \int_X (u_j - u)(S_1 + S_2 + S_3 + S_4).$$

Now we shall estimate each term in the right hand side. First, since $S_1 = (\omega_{\varphi - \omega_{\varphi_k}}) \land T_p(j)$,

$$(3.9) \left| \int_X (u - u_j) S_1 \right| \leq \|u - u_j\|_\infty \left( \int_X (\omega_{\varphi} + \omega_{\varphi_k}) \land T_p(j) \right) \to 0$$

as $j \to +\infty$.

Next, we estimate $\int_X (u - u_j) S_2$. As $d^c T_p(j) = d^c \omega \land T_p(j)$, where $T_p(j)$ is a sum of terms of the form $C_3 \omega^p_{u_j} \land \omega^n_{u_j}$ (the constant $C_3$ depending only on $n, p$), we apply the Cauchy-Schwarz inequality \cite{20} Proposition 1.4] to get that

$$(3.10) \left| \int_X (u - u_j) d\varphi \land S_2 \right| \leq C \|u - u_j\|_\infty \left( \int_X d\varphi \land d^c \varphi \land \omega \land T_p(j) \right)^{\frac{1}{2}} \left( \int_X \omega^2 \land T_p(j) \right)^{\frac{1}{2}}.$$

Moreover,

$$(3.11) 2 \int_X d\varphi \land d^c \varphi \land T_p(j) = \int_X dd^c \varphi^2 \land T_p(j) - \int_X 2\varphi \omega \land T_p(j)$$

$$+ 2 \int_X \omega \land T_p(j)$$

$$\leq C \left( \int_X \omega^n + \|\varphi\|_\infty \|u_j\|_\infty \right).$$

where in the last inequality we used \cite{20} Proposition 1.5]. Therefore, we conclude the right hand side of the previous inequality tends to 0 as $j \to +\infty$. Similar estimates are also applied to the remaining terms with $S_3, S_4$. Thus we have shown that $E_{jk} \to 0$ as $\min\{j, k\} \to +\infty$.

Combining (3.5), (3.10), and (3.11) we get a contradiction. The lemma thus follows.

Existence of a continuous solution. Notice that

$$\int_X |\varphi_j - \varphi_k| \omega^n_{u_j} \leq \int_X |\varphi_j - \varphi| \omega^n_{u_j} + \int_X |\varphi_k - \varphi| \omega^n_{u_j} \to 0$$
as $\min\{j, k\} \to +\infty$. Therefore, using Lemma 3.2 and the argument in [17] Theorem 5.8 we get that $\{\varphi_j\}_{j \geq 1}$ is a Cauchy sequence in $C^0(X)$. Thus,

$$\varphi = \lim_{j \to \infty} \varphi_j \quad \text{in } C^0(X).$$

We conclude that $\varphi \in PSH(\omega) \cap C^0(X)$ and it solves

$$\omega_\varphi^n = c \mu,$$

where $c$ is defined in (3.3).

**Hölder continuity of the solution.** We shall show that the solution $\varphi$ obtained in (3.12) is Hölder continuous. Fix $\tau > 0$ and set

$$\alpha = \min\left\{ \frac{1}{1 + (n + 2)(n + 1/2)}, \alpha_1 \right\},$$

where $\alpha_1$ is given in Lemma 2.8. By Corollary 2.2 $\mu \in H(\tau)$ and then Proposition 2.4 holds with $\gamma = \alpha$.

Consider the regularization of $\varphi$ as in (2.7). As explained in [16] and [7] the result follows as soon as we show that

$$\rho_t \varphi - \varphi \leq C t^{\alpha_1}$$

for $t$ small enough.

It follows from Lemma 2.7 that

$$\varphi \leq \Phi_{\delta,b} \leq \rho_\delta \varphi + K(\delta + \delta^2),$$

$$\leq \rho_\delta \varphi + 2K\delta.$$

Choose the level $b = (\delta^\alpha - 2K\delta)/A = O(\delta^\alpha)$ so that

$$Ab + 2K\delta = \delta^\alpha.$$

After fixing the level $b$, we write

$$\Phi_{\delta} := (1 - \delta^\alpha)\Phi_{\delta,b}.$$

Then, by Lemma 2.7

$$\omega + dd^c \Phi_{\delta} \geq \delta^{2\alpha} \omega.$$  

Since $-C_4 \leq \varphi \leq 0$ and $\rho_\delta \varphi \leq 0$ one obtains

$$\Phi_{\delta} \leq (1 - \delta^\alpha)(\rho_\delta \varphi + K\delta + K\delta^2) \leq 2K\delta.$$

It follows that

$$\Phi_{\delta} \leq C_4 \delta^\alpha$$

for $\delta \leq \delta_0$ small. Therefore, by (3.10) and (3.17) we have

$$\Phi_{\delta} - \varphi \leq C_4 \delta^\alpha + (1 - \delta^\alpha)(\rho_\delta \varphi + K\delta + K\delta^2 - \varphi).$$

Next, the stability estimate Proposition 2.4 applied for $\Phi_{\delta} - C_4 \delta^\alpha$ and $\varphi$, and $\gamma = \alpha$ give us that

$$\sup_X (\Phi_{\delta} - \varphi) \leq C_5 \max\{\Phi_{\delta} - \varphi - C_4 \delta^\alpha, 0\} \|\varphi\|_{L^1(d\mu)} + C_4 \delta^\alpha$$

$$\leq C_5 \|\rho_\delta \varphi + K\delta + K\delta^2 - \varphi\|_{L^1(d\mu)} + C_4 \delta^\alpha,$$

where we used (3.18) for the second inequality. Hence, using Lemma 2.8 we conclude that

$$\Phi_{\delta} - \varphi \leq C_6 \delta^{\alpha_1}.$$
For a fixed point \( z \), the minimum in the definition of \( \Phi_{\delta,b}(z) \) is realised for some \( t_0 = t_0(z) \). Then, (3.14) and (3.17) imply
\[
(1 - \delta^\alpha)(\rho_{t_0} \varphi + K t_0^2 + b \log \frac{t_0}{\delta} - \varphi) \leq C_6 \delta^\alpha.
\]
Since \( \rho_t \varphi + K t^2 + K t - \varphi \geq 0 \), we have
\[
 b(1 - \delta^\alpha) \log \frac{t_0}{\delta} \geq -C_6 \delta^\alpha.
\]
Combining this with \( b \geq \delta^\alpha / (2A) \), one gets that
\[
 t_0(z) \geq \delta \kappa \quad \text{for} \quad \kappa = \exp \left( -\frac{2AC_6}{(1 - \delta^\alpha)} \right),
\]
where \( \delta_0 \) is fixed, and \( \kappa \) is a uniform constant.

Now, we are ready to conclude the proof. Since \( t_0 = t_0(z) \geq \delta \kappa \) and \( t \mapsto \rho_t \varphi + K t^2 \) is increasing,
\[
 \rho_{\kappa \delta} \varphi(z) + K(\delta \kappa)^2 + K \delta \kappa - \varphi(z) \leq \rho_{\kappa_0} \varphi(z) + K t_0^2 + K t_0 - \varphi(z) = \Phi_{\delta,b}(z) - \varphi(z) = \frac{\delta^\alpha}{1 - \delta^\alpha} \Phi_\delta + (\Phi_\delta - \varphi).
\]
Combining this, (3.17) and (3.19) we get that
\[
 \rho_{\kappa \delta} \varphi(z) - \varphi(z) \leq C_7 \delta^{\alpha_1}.
\]
The desired estimate follows by rescaling \( \delta := \kappa \delta \) and increasing \( C_7 \).

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