PROJECTIVE DIMENSION AND REGULARITY OF EDGE IDEALS OF SOME VERTEX-WEIGHTED ORIENTED $m$-PARTITE GRAPHS

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Abstract. In this paper we provide some exact formulas for the projective dimension and the regularity of edge ideals associated to three special types of vertex-weighted oriented $m$-partite graphs. These formulas are functions of the weight and number of vertices. We also give some examples to show that these formulas are related to direction selection and the weight of vertices.

1. Introduction

Let $S = k[x_1, \ldots, x_n]$ be a polynomial ring in $n$ variables over a field $k$ and let $I \subseteq S$ be a homogeneous ideal. There are two central invariants associated to $I$, the regularity $\text{reg}(I) := \max\{j - i \mid \beta_{i,j}(I) \neq 0\}$ and the projective dimension $\text{pd}(I) := \max\{i \mid \beta_{i,j}(I) \neq 0 \text{ for some } j\}$, that in a sense, they measure the complexity of computing the graded Betti numbers $\beta_{i,j}(I)$ of $I$. In particular, if $I$ is a monomial ideal, its polarization $I^P$ has the same projective dimension and regularity as $I$ and is squarefree. Thus one can associate $I^P$ to a graph or a hypergraph or a simplicial complex. Many authors have studied the regularity and Betti numbers of edge ideals of graphs, e.g. [1, 2, 4, 6, 7, 15, 18, 20, 25, 26, 27, 28, 29, 30]. Other authors have studied higher degree generalizations using hypergraphs and clutters [6, 7, 14] or simplicial complexes [8, 9].

A directed graph or digraph $D$ consists of a finite set $V(D)$ of vertices, together with a collection $E(D)$ of ordered pairs of distinct points called edges or arrows. A vertex-weighted digraph is a triplet $D = (V(D), E(D), w)$, where $w$ is a weight function $w : V(D) \to \mathbb{N}^+$, where $\mathbb{N}^+ = \{1, 2, \ldots\}$. Some times for short we denote the vertex set $V(D)$ and the edge set $E(D)$ by $V$ and $E$ respectively. The weight of $x_i \in V$ is $w(x_i)$, denoted by $w_i$ or $w_{x_i}$.

The edge ideal of a vertex-weighted digraph was first introduced by Gimenez et al [11]. Let $D = (V, E, w)$ be a vertex-weighted digraph with the vertex set $V = \{x_1, \ldots, x_n\}$. We consider the polynomial ring $S = k[x_1, \ldots, x_n]$ in $n$ variables over a field $k$. The edge ideal of $D$, denoted by $I(D)$, is the ideal of $S$ given by

$$I(D) = (x_i x_{w_j}^{x_j} \mid x_i x_j \in E).$$

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Edge ideals of weighted digraphs arose in the theory of Reed-Muller codes as initial ideals of vanishing ideals of projective spaces over finite fields [21, 22]. If a vertex $x_i$ of $D$ is a source (i.e., has only arrows leaving $x_i$) we shall always assume $w_i = 1$ because in this case the definition of $I(D)$ does not depend on the weight of $x_i$. If $w_j = 1$ for all $j$, then $I(D)$ is the edge ideal of underlying graph $G$ of $D$. It has been studied in the literature [15, 24]. Especially the study of algebraic invariants corresponding to their minimal free resolutions has become popular (see [1, 2, 4, 6, 8, 13, 18, 20, 25, 26, 27, 28, 29, 30]). In [27], the first three authors derive some exact formulas for the projective dimension and regularity of edge ideals of some oriented unicyclic vertex-weighted rooted forests. In [28, 29], they provide some exact formulas for the projective dimension and regularity of edge ideals of vertex-weighted rooted forests and oriented cycles. In [28], they derive some exact formulas for the projective dimension and regularity of edges of some special types of graphs and cyclic graphs with a common vertex or a common edge. To the best of our knowledge, little is known about the projective dimension and the regularity of $I(D)$ for a vertex-weighted oriented graph.

In this article, we are interested in algebraic properties corresponding to the projective dimension and the regularity of the edge ideals of some types of vertex-weighted oriented $m$-partite graphs. By using the approaches of Betti splitting and polarization, we derive some exact formulas for the projective dimension and the regularity of edge ideals of these oriented graphs. The results are as follows:

**Theorem 1.1.** Let $m \geq 2$ be an integer. Assume that $D = (V, E, w)$ is a vertex-weighted oriented $m$-partite graph, its vertex set $V = \bigcup_{i=1}^m V_i$ and its edge set $E = \bigcup_{i=1}^{m-1} E(D_i)$, where $D_i$ is a complete bipartite graph and it is also an induced subgraph of $D$ on $V_i \cup V_{i+1}$ satisfying the starting point of every edge of $E(D_i)$ belongs to $V_i$ and its ending point belongs to $V_{i+1}$ for $1 \leq i \leq m - 1$. If $w(x) \geq 2$ for any $x \in V \setminus (V_1 \cup V_m)$. Then

1. $\text{reg}(I(D)) = \sum_{x \in V(D)} w(x) - |V(D)| + 2$,
2. $\text{pd}(I(D)) = |V(D)| - 2$.

**Theorem 1.2.** Let $m \geq 2$ be an integer, and suppose that $D = (V, E, w)$ is a vertex-weighted oriented $m$-partite graph, its vertex set $V = \bigcup_{i=1}^m V_i$ with $|V_1| \leq |V_2|$, its edge set $E = \bigcup_{i=1}^{m-1} E(D_i)$, where $D_1$ is a bipartite graph with the vertex set $\{x_{11}, \ldots, x_{1,|V_1|}\} \cup \{x_{21}, \ldots, x_{2,|V_2|}\}$, the edge set $\{x_{21}x_{11}, \ldots, x_{2,|V_1|}x_{1,|V_1|}\}$ and $D_i$ is a complete bipartite graph and it is an induced subgraph of $D$ on $V_i \cup V_{i+1}$ satisfying the starting point of every edge in $E(D_i)$ belongs to $V_i$ and its ending point belongs to $V_{i+1}$ for $2 \leq i \leq m - 1$. If $w(x) \geq 2$ for any $x \in V \setminus (V_1 \cup V_m)$. Then

1. $\text{reg}(I(D)) = \sum_{x \in V(D)} w(x) - |V(D) \setminus V_1| + 1$,
\[(2) \quad pd(I(D)) = \begin{cases} 
|V(D \setminus V_1)| - 2, & \text{if } |V_1| < |V_2|, \\
|V(D \setminus V_2)| - 1, & \text{if } |V_1| = |V_2|. 
\end{cases} \]

**Theorem 1.3.** Let \( m \geq 3 \) be an integer. Assume that \( D = (V, E, w) \) is a vertex-weighted oriented \( m \)-partite graph, its vertex set \( V = \bigcup_{i=1}^{m} V_i \) and its edge set \( E = \bigcup_{i=1}^{m} E(D_i) \), where \( D_i \) is a complete bipartite graph and it is also an induced subgraph of \( D \) on \( V_i \cup V_{i+1} \) satisfying the starting point of every edge of \( E(D_i) \) belongs to \( V_i \) and its ending point belongs to \( V_{i+1} \) for \( 1 \leq i \leq m \), where we stipulate \( V_{m+1} = V_1 \). If \( w(x) \geq 2 \) for all \( x \in V \). Then

\[(1) \quad \text{reg}(I(D)) = \sum_{x \in V(D)} w(x) - |V(D)| + 1, \]

\[(2) \quad \text{pd}(I(D)) = |V(D)| - 1. \]

Our paper is organized as follows. In section 2, we recall some definitions and basic facts used in the following sections. From section 3 to section 5, we provide some exact formulas for the projective dimension and the regularity of the edge ideals of three classes of vertex-weighted oriented \( m \)-partite graphs such as Figure 1. We also give some examples to show that formulas for these three types of oriented graphs are related to direction selection and the weight of vertices.
For all unexplained terminology and additional information, we refer to [19] (for the theory of digraphs), [3] (for graph theory), and [16] (for the theory of edge ideals of graphs and monomial ideals). We greatfully acknowledge the use of the computer algebra system CoCoA ([5]) for our experiments.

2. Preliminaries

In this section, we gather together the needed definitions and basic facts, which will be used throughout this paper. However, for more details, we refer the reader to [2] [3] [10] [13] [16] [18] [19] [21] [23] [25] [27].

A directed graph or digraph \( D \) consists of a finite set \( V(D) \) of vertices, together with a collection \( E(D) \) of ordered pairs of distinct points called edges or arrows. If \( \{u, v\} \in E(D) \) is an edge, we write \( uw \) for \( \{u, v\} \), which is denoted to be the directed edge where the direction is from \( u \) to \( v \) and \( u \) (resp. \( v \)) is called the starting point (resp. the ending point). An oriented graph is a directed graph having no bidirected edges (i.e. each pair of vertices is joined by a single edge having a unique direction). In other words, an oriented graph \( D \) is a simple graph \( G \) together with an orientation of its edges. We call \( G \) the underlying graph of \( D \).

Every concept that is valid for graphs automatically applies to digraphs too. A digraph is said to be connected if its underlying graph is connected. A digraph \( H \) is called an induced subgraph of a digraph \( D \) if \( V(H) \subseteq V(D) \), and for any \( x, y \in V(H) \), \( xy \) is an edge of \( H \) if and only if \( xy \) is an edge of \( D \). For \( P \subset V(D) \), we denote \( D \backslash P \) the induced subgraph of \( D \) obtained by removing the vertices in \( P \) and the edges incident to these vertices. If \( P = \{x\} \) consists of a single element, then we write \( D \backslash x \) for \( D \backslash \{x\} \). The induced subgraph of \( D \) over a subset \( W \subset V(G) \) is a graph with the vertex set \( W \) and the edge set \( \{uw \in E(G) \mid u, v \in W\} \). For \( U \subseteq E(D) \), we define \( D \backslash U \) to be the subgraph of \( D \) with all edges in \( U \) deleted (but its vertices remained). When \( U = \{e\} \) consists of a single edge, we write \( D \backslash e \) instead of \( D \backslash \{e\} \). An oriented path or oriented cycle is an orientation of a path or cycle in which each vertex dominates its successor in the sequence. Let \( G = (V, E) \) be a finite simple graph on the vertex set \( \{x_1, \ldots, x_n\} \), the whisker graph \( G^* \) of \( G \) is the graph with the vertex set \( V \cup \{y_1, \ldots, y_\ell\} \) and the edge set \( E(G^*) = E \cup \{x_{ij}y_i \mid 1 \leq j \leq \ell\} \), where \( \ell \leq n \) and these \( x_{ij} \) are different from each other. Let \( m \) be an integer, a graph \( G = (V, E) \) is called \( m \)-partite if if the set of all its vertices can be partitioned into \( m \) subsets \( V_1, \ldots, V_m \), in such a way that any edge of graph \( G \) connects vertices from different subsets. The terms bipartite graph and tripartite graph are used to describe \( m \)-partite graphs for \( m \) equal to 2 and 3, respectively. A \( m \)-partite graph is called complete if any vertex \( v \in V \) is adjacent to all vertices not belonging to the same partition as \( v \). Unless specifically stated, an oriented bipartite graph with vertex set \( V = V_1 \cup V_2 \) in this article is a bipartite graph in which all edges are oriented from the vertex in \( V_1 \) to the vertex in \( V_2 \).

A vertex-weighted oriented graph is a triplet \( D = (V(D), E(D), w) \), where \( V(D) \) is the vertex set, \( E(D) \) is the edge set and \( w \) is a weight function \( w : V(D) \to \mathbb{N}^+ \), where \( \mathbb{N}^+ = \{1, 2, \ldots\} \). Some times for short we denote the vertex set \( V(D) \) and
edge set $E(D)$ by $V$ and $E$ respectively. The weight of $x_i \in V$ is $w(x_i)$, denoted by $w_i$ or $w_{x_i}$. Given a vertex-weighted oriented graph $D = (V, E, w)$ with the vertex set $V = \{x_1, \ldots, x_n\}$, we consider the polynomial ring $S = k[x_1, \ldots, x_n]$ in $n$ variables over a field $k$. The edge ideal of $D$, denoted by $I(D)$, is the ideal of $S$ given by

$$I(D) = (x_i x_j^{w_{ij}} \mid x_i x_j \in E).$$

If a vertex $x_i$ of $D$ is a source (i.e., has only arrows leaving $x_i$) we shall always assume $w_i = 1$ because in this case the definition of $I(D)$ does not depend on the weight of $x_i$.

For any homogeneous ideal $I$ of the polynomial ring $S = k[x_1, \ldots, x_n]$, there exists a graded minimal finite free resolution

$$0 \rightarrow \bigoplus_j S(-j)^{\beta_{p,j}(M)} \rightarrow \bigoplus_j S(-j)^{\beta_{p-1,j}(M)} \rightarrow \cdots \rightarrow \bigoplus_j S(-j)^{\beta_{0,j}(M)} \rightarrow I \rightarrow 0,$$

where the maps are exact, $p \leq n$, and $S(-j)$ is the $S$-module obtained by shifting the degrees of $S$ by $j$. The number $\beta_{i,j}(I)$, the $(i, j)$-th graded Betti number of $I$, is an invariant of $I$ that equals the number of minimal generators of degree $j$ in the $i$th syzygy module of $I$. Of particular interests are the following invariants which measure the size of the minimal graded free resolution of $I$. The projective dimension of $I$, denoted $pd(I)$, is defined to be

$$pd(I) := \max\{i \mid \beta_{i,j}(I) \neq 0\}.$$ 

The regularity of $I$, denoted $reg(I)$, is defined by

$$reg(I) := \max\{j - i \mid \beta_{i,j}(I) \neq 0\}.$$ 

We now derive some formulas for $pd(I)$ and $reg(I)$ in some special cases by using some tools developed in [10].

**Definition 2.1.** Let $I$ be a monomial ideal, and suppose that there exist monomial ideals $J$ and $K$ such that $G(I)$ is the disjoint union of $G(J)$ and $G(K)$, where $G(I)$ denotes the unique minimal set of monomial generators of $I$. Then $I = J + K$ is a Betti splitting if

$$\beta_{i,j}(I) = \beta_{i,j}(J) + \beta_{i,j}(K) + \beta_{i-1,j}(J \cap K) \text{ for all } i, j \geq 0,$$

where $\beta_{i-1,j}(J \cap K) = 0$ if $i = 0$.

In [10], the authors describe some sufficient conditions for an ideal $I$ to have a Betti splitting. We need the following lemma.

**Lemma 2.2.** ([10, Corollary 2.7]). Suppose that $I = J + K$ where $G(J)$ contains all the generators of $I$ divisible by some variable $x_i$ and $G(K)$ is a nonempty set containing the remaining generators of $I$. If $J$ has a linear resolution, then $I = J + K$ is a Betti splitting.

When $I$ is a Betti splitting ideal, Definition 2.1 implies the following results:

**Corollary 2.3.** If $I = J + K$ is a Betti splitting ideal, then

1. $reg(I) = \max\{reg(J), reg(K), reg(J \cap K) - 1\}$,
2. $pd(I) = \max\{pd(J), pd(K), pd(J \cap K) + 1\}$.
The following lemmas is often used in this article.

**Lemma 2.4.** ([12] Lemma 1.3) Let \( R \) be a polynomial ring over a field and let \( I \) be a proper non-zero homogeneous ideal in \( R \). Then

1. \( \text{pd}(I) = \text{pd}(R/I) - 1 \),
2. \( \text{reg}(I) = \text{reg}(R/I) + 1 \).

**Lemma 2.5.** ([13] Lemma 2.2 and Lemma 3.2) Let \( S_1 = k[x_1, \ldots, x_m], S_2 = k[x_{m+1}, \ldots, x_n] \) and \( S = k[x_1, \ldots, x_n] \) be three polynomial rings, \( I \subseteq S_1 \) and \( J \subseteq S_2 \) be two proper non-zero homogeneous ideals. Then

1. \( \text{pd}(S/(I + J)) = \text{pd}(S_1/I) + \text{pd}(S_2/J) \),
2. \( \text{reg}(S/(I + J)) = \text{reg}(S_1/I) + \text{reg}(S_2/J) \).

From Lemma 2.4 and Lemma 2.5 we have

**Lemma 2.6.** ([25] Lemma 3.1) Let \( S_1 = k[x_1, \ldots, x_m] \) and \( S_2 = k[x_{m+1}, \ldots, x_n] \) be two polynomial rings, \( I \subseteq S_1 \) and \( J \subseteq S_2 \) be two non-zero homogeneous ideals. Then

1. \( \text{pd}(I + J) = \text{pd}(I) + \text{pd}(J) + 1 \),
2. \( \text{reg}(I + J) = \text{reg}(I) + \text{reg}(J) - 1 \).

Let \( G(I) \) denote the minimal set of generators of a monomial ideal \( I \subseteq S \) and let \( u \in S \) be a monomial, we set \( \text{supp}(u) = \{ x_i : x_i|u \} \). If \( G(I) = \{ u_1, \ldots, u_m \} \), we set \( \text{supp}(I) = \bigcup_{i=1}^m \text{supp}(u_i) \). The following lemma is well known.

**Lemma 2.7.** Let \( I, J = (u) \) be two monomial ideals such that \( \text{supp}(u) \cap \text{supp}(I) = \emptyset \). If the degree of monomial \( u \) is \( d \). Then

1. \( \text{reg}(J) = d \),
2. \( \text{reg}(JI) = \text{reg}(I) + d \),
3. \( \text{pd}(JI) = \text{pd}(I) \).

**Definition 2.8.** Suppose that \( u = x_1^{a_1} \cdots x_n^{a_n} \) is a monomial in \( S \). We define the polarization of \( u \) to be the squarefree monomial

\[
P(u) = x_{11} x_{12} \cdots x_{1a_1} x_{21} \cdots x_{2a_2} \cdots x_{n1} \cdots x_{na_n}
\]

in the polynomial ring \( S^P = k[x_{ij} \mid 1 \leq i \leq n, 1 \leq j \leq a_i] \). If \( I \subseteq S \) is a monomial ideal with \( G(I) = \{ u_1, \ldots, u_m \} \), the polarization of \( I \), denoted by \( I^P \), is defined as:

\[
I^P = (P(u_1), \ldots, P(u_m)),
\]

which is a squarefree monomial ideal in the polynomial ring \( S^P \).

Here is an example of how polarization works.
Example 2.9. Let \( I(D) = (x_3x_1^2, x_4x_2^2, x_3x_2^2, x_3x_5^2, x_4x_5^2, x_4x_6^2) \) be the edge ideal of a vertex-weighted digraph \( D = (V, E, w) \), where \( V = \bigcup_{j=1}^{3} V_j \) with \( V_1 = \{x_1, x_2\} \), \( V_2 = \{x_3, x_4\} \) and \( V_3 = \{x_5, x_6\} \). Then the polarization \( I^P \) of \( I(D) \) is the ideal 
\[
(x_3x_1^2, x_4x_2^2, x_3x_2^2, x_3x_5^2, x_4x_5^2, x_4x_6^2). 
\]

A monomial ideal \( I \) and its polarization \( I^P \) share many homological and algebraic properties. The following is a very useful property of polarization.

Lemma 2.10. (\cite{16} Corollary 1.6.3) Let \( I \subset S \) be a monomial ideal and \( I^P \subset S^P \) its polarization. Then
\[
\begin{align*}
(1) \quad \beta_{ij}(I) &= \beta_{ij}(I^P) & \text{for all } i \text{ and } j, \\
(2) \quad \reg(I) &= \reg(I^P), \\
(3) \quad \pd(I) &= \pd(I^P).
\end{align*}
\]

The following lemma can be used for computing the projective dimension and the regularity of an ideal.

Lemma 2.11. (\cite{12} Lemma 1.1 and Lemma 1.2) Let \( 0 \to A \to B \to C \to 0 \) be a short exact sequence of finitely generated graded \( S \)-modules. Then
\[
\begin{align*}
(1) \quad \reg(B) &= \reg(C) & \text{if } \reg(A) \leq \reg(C), \\
(2) \quad \pd(B) &= \pd(A) & \text{if } \pd(A) \geq \pd(C).
\end{align*}
\]

3. Projective dimension and regularity of edge ideals of the first class of vertex-weighted oriented \( m \)-partite graphs

In this section, we will provide some exact formulas for the projective dimension and the regularity of the edge ideals of a class of vertex-weighted oriented \( m \)-partite graphs with the vertex set \( V = \bigcup_{i=1}^{m} V_i \) and the edge set \( E = \bigcup_{i=1}^{m-1} E(D_i) \), where \( D_i \) is a complete bipartite graph and it is also an induced subgraph of \( D \) on \( V_i \sqcup V_{i+1} \) satisfying the starting point of every edge of \( E(D_i) \) belongs to \( V_i \) and its ending point belongs to \( V_{i+1} \) for \( 1 \leq i \leq m - 1 \). We also give some examples to show that these formulas are related to direction selection and the weight of vertices. We shall start with the following lemma.

Lemma 3.1. (\cite{30} Theorem 3.2) Let \( D = (V(D), E(D), w) \) be a vertex-weighted oriented complete bipartite graph. Then
\[
\begin{align*}
(1) \quad \reg(I(D)) &= \sum_{x \in V(D)} w(x) - \vert V(D) \vert + 2, \\
(2) \quad \pd(I(D)) &= \vert V(D) \vert - 2.
\end{align*}
\]

Now we are ready to present the main results of this section.
Theorem 3.2. Let $m \geq 2$ be an integer, and assume that $D = (V, E, w)$ is a vertex-weighted oriented $m$-partite graph, its vertex set $V = \bigcup_{i=1}^{m} V_i$ and its edge set $E = \bigcup_{i=1}^{m-1} E(D_i)$, where $D_i$ is a complete bipartite graph and it is also an induced subgraph of $D$ on $V_i \sqcup V_{i+1}$ satisfying the starting point of every edge in $E(D_i)$ belongs to $V_i$ and its ending point belongs to $V_{i+1}$ for $1 \leq i \leq m-1$. If $w(x) \geq 2$ for any $x \in V \setminus (V_1 \cup V_m)$. Then

$$\text{reg}(I(D)) = \sum_{x \in V(D)} w(x) - |V(D)| + 2.$$ 

Proof. Let $V_i = \{x_{i1}, \ldots, x_{it_i}\}$ for $1 \leq i \leq m$, then

$$I(D) = (x_{11}x_{21}, x_{11}x_{22}, \ldots, x_{11}x_{t_1}, x_{12}x_{22}, \ldots, x_{12}x_{t_2}, \ldots, x_{1t_1}x_{21}, \ldots, x_{1t_1}x_{t_2}, x_{21}x_{31}, \ldots, x_{21}x_{t_3}, x_{22}x_{31}, \ldots, x_{22}x_{t_3}, x_{2t_2}x_{31}, \ldots, x_{2t_2}x_{t_3}, x_{m-1,1}x_{m1}, \ldots, x_{m-1,1}x_{mt_m}, x_{m-1,2}x_{m1}, \ldots, x_{m-1,2}x_{mt_m}, x_{m-1,3}x_{m1}, \ldots, x_{m-1,3}x_{mt_m}).$$

We apply induction on $m$. The case $m = 2$ follows from Lemma 3.1 (1).

Now we assume that $m \geq 3$. Consider the following short exact sequences

$$0 \rightarrow \frac{S}{(I(D); x_{m1})(-w_{m1})} \xrightarrow{-x_{m1}} \frac{S}{I(D)} \xrightarrow{I} \frac{S}{J_1} \rightarrow 0$$

$$0 \rightarrow \frac{S}{(J_1; x_{m2})(-w_{m2})} \xrightarrow{-x_{m2}} \frac{S}{J_1} \xrightarrow{I} \frac{S}{J_2} \rightarrow 0 \quad (\dagger)$$

$$\vdots$$

$$0 \rightarrow \frac{S}{(J_{m-1}; x_{w_{mt_m}})(-w_{mt_m})} \xrightarrow{-x_{mt_m}} \frac{S}{J_{m-1}} \xrightarrow{I} \frac{S}{J_m} \rightarrow 0$$

where $J_i = I(D) + (x_{w_{m1}}, \ldots, x_{w_{mi}})$ for $1 \leq i \leq t_m$. We prove this argument in the following two steps.

1. We first prove $\text{reg}(J_{m}) = \sum_{x \in V(D)} w(x) - |V(D)| + 2$.

In fact, $J_{m} = I(D) + (x_{m1}, \ldots, x_{mt_m}) = I(D \setminus V_m) + (x_{m1}, \ldots, x_{mt_m})$, where $I(D \setminus V_m)$ is the edge ideal of the induced subgraph $D \setminus V_m$ of $D$ on the set $V \setminus V_m$ obtained by removing the vertices in $V_m$ and the edges incident to these vertices. Let $K = (x_{m1}^{w_{m1}}, x_{m2}^{w_{m2}}, \ldots, x_{mt_m}^{w_{mt_m}})$, then the variables appearing in $I(D \setminus V_m)$ and $K$
are different, then by induction hypothesis on $m$ and Lemma 2.6 (2), we get

$$\text{reg} (J_{tm}) = \text{reg} (I(D \setminus V_m)) + \text{reg} (K) - 1$$

$$= \left[ \sum_{x \in V(D \setminus V_m)} w(x) - |V(D \setminus V_m)| + 2 \right] + \left[ \sum_{j=1}^{t_m} w_{mj} - (t_m - 1) \right] - 1$$

$$= \left( \sum_{x \in V(D \setminus V_m)} w(x) + \sum_{j=1}^{t_m} w_{mj} \right) - (|V(D \setminus V_m)| + t_m) + 2$$

$$= \sum_{x \in V(D)} w(x) - |V(D)| + 2.$$

(2) Next we will prove $\text{reg} ((J_i : x_{m,i+1}^{w_{m,i+1}})(-w_{m,i+1})) \leq \text{reg} (J_{tm})$ for $0 \leq i \leq t_m - 1$, where $J_0 = I(D)$. Thus the assertion follows from Lemma 2.4 (2) and by repeatedly using Lemma 2.11 (1) on the short exact sequences $(\dagger)$.

In fact, we can write $(J_i : x_{m,i+1}^{w_{m,i+1}})$ as

$$(J_i : x_{m,i+1}^{w_{m,i+1}}) = I(D \setminus (V_{m-1} \cup V_m)) + (x_{m-1,1}, x_{m-1,2}, \ldots, x_{m-1,t_{m-1}})$$

$$+ (x_{m1}^{w_{m1}}, x_{m2}^{w_{m2}}, \ldots, x_{mi}^{w_{mi}}, x_{m,i+1}^{w_{m,i+1}})$$

$$= L_1 + L_2 + L_3^i$$

where $L_1$ is the edge ideal of the induced subgraph $D \setminus (V_{m-1} \cup V_m)$ of $D$ on the set $V \setminus (V_{m-1} \cup V_m)$ obtained by removing the vertices in $V_{m-1} \cup V_m$ and the edges incident to these vertices, $L_2 = (x_{m-1,1}, x_{m-1,2}, \ldots, x_{m-1,t_{m-1}})$, $L_3^0 = (0)$, and $L_3^i = (x_{m1}^{w_{m1}}, x_{m2}^{w_{m2}}, \ldots, x_{mi}^{w_{mi}}, x_{m,i+1}^{w_{m,i+1}})$ for $1 \leq i \leq t_m - 1$. In fact, the variables appearing in $L_1$, $L_2$ and $L_3^i$ are different from each other and $L_3^0 = (0)$ for any $0 \leq i \leq t_m - 1$. We distinguish into the following two cases:

(1) If $m = 3$, then $L_1 = (0)$. Thus, Lemma 2.6 (2), we have

$$\text{reg} ((J_0 : x_{31}^{w_{31}})(-w_{31})) = \text{reg} ((J_0 : x_{31}^{w_{31}})) + w_{31}$$

$$= \sum_{i=1}^{t_2} \text{reg} (x_{2,i}) - (t_2 - 1) + w_{31} = w_{31} + 1$$

$$= (t_1 + \sum_{j=1}^{t_2} w_{2j} + \sum_{j=1}^{t_3} w_{3j}) - (t_1 + \sum_{j=1}^{t_2} w_{2j} + \sum_{j=2}^{t_3} w_{3j}) + 1$$

$$= \left[ \sum_{x \in V(D)} w(x) - |V(D)| + 2 \right] + (t_1 + t_2 + t_3)$$

$$- (1 + t_1 + \sum_{j=1}^{t_2} w_{2j} + \sum_{j=2}^{t_3} w_{3j})$$

$$\leq \text{reg} (J_{tm}).$$
and, for $1 \leq i \leq t_3 - 1$,
\[
\text{reg} \left( (J_i : x_{3,i+1}^{w_3,i+1})(-w_{3,i+1}) \right) = \text{reg} \left( (J_i : x_{3,i+1}^{w_3,i+1}) \right) + w_{3,i+1} = \text{reg} \left( L_2 + L_3^i \right) + w_{3,i+1} \\
= \left[ \text{reg} \left( L_2 \right) + \text{reg} \left( L_3^i \right) - 1 \right] + w_{3,i+1} \\
= 1 + \left( \sum_{j=1}^{i} w_{3j} - (i - 1) \right) - 1 + w_{3,i+1} \\
= \left[ \sum_{x \in V(D)} w(x) - |V(D)| + 2 \right] + (t_1 + t_2 + t_3 - i) - \\
(1 + t_1 + \sum_{j=1}^{t_2} w_{2j} + \sum_{j=i+2}^{t_3} w_{3j}) \\
\leq \text{reg} \left( J_{t_m} \right)
\]
where the above inequalities hold because of $w_{2j} \geq 2$ for $1 \leq j \leq t_2$.

(II) If $m \geq 4$. By induction hypothesis on $m$ and Lemma 2.6 (2) and similarly arguments at above, we have
\[
\text{reg} \left( (J_0 : x_{m1}^{w_{m1}})(-w_{m1}) \right) = \text{reg} \left( L_1 + L_2 \right) + w_{m1} = \left[ \text{reg} \left( L_1 \right) + \text{reg} \left( L_2 \right) - 1 \right] + w_{m1} \\
= \sum_{x \in V(D \setminus (V_{m-1} \cup V_m))} w(x) - |V(D \setminus (V_{m-1} \cup V_m))| + 2 + w_{m1} \\
= \left[ \sum_{x \in V(D)} w(x) - |V(D)| + 2 \right] + t_{m-1} + t_m - \sum_{j=1}^{t_{m-1}} w_{m-1,j} - \sum_{j=2}^{t_m} w_{mj} \\
\leq \text{reg} \left( J_{t_m} \right)
\]
and, for $1 \leq i \leq t_m - 1$,
\[
\text{reg} \left( (J_i : x_{m,i+1}^{w_{m,i+1}})(-w_{m,i+1}) \right) = \text{reg} \left( L_1 \right) + \text{reg} \left( L_2 \right) + \text{reg} \left( L_3^i \right) - 2 + w_{m,i+1} \\
= \sum_{x \in V(D \setminus (V_{m-1} \cup V_m))} w(x) - |V(D \setminus (V_{m-1} \cup V_m))| + 2 \\
+ 1 + \sum_{j=1}^{i} w_{mj} - (i - 1) - 2 + w_{m,i+1} \\
= \left[ \sum_{x \in V(D)} w(x) - |V(D)| + 2 \right] + (t_{m-1} + t_m - i) \\
- \left( \sum_{j=1}^{t_{m-1}} w_{m-1,j} + \sum_{j=i+2}^{t_m} w_{mj} \right) \\
\leq \text{reg} \left( J_{t_m} \right)
\]
where the above inequalities hold because of $w_{2j} \geq 2$ for $1 \leq j \leq t_2$. This completes the proof. \qed
Theorem 3.3. Let $D = (V, E, w)$ be a vertex-weighted oriented graph as Theorem 3.2. Then

$$pd(I(D)) = |V(D)| - 2.$$ 

Proof. Let $V_i = \{x_{i1}, \ldots, x_{im}\}$ for $1 \leq i \leq m$, then

$$I(D) = \langle x_{11}^{w_{21}}, x_{12}^{w_{22}}, \ldots, x_{11}^{w_{2m}}, x_{12}^{w_{2m}}, \ldots, x_{11}^{w_{31}}, x_{12}^{w_{32}}, \ldots, x_{11}^{w_{33}}, x_{12}^{w_{33}}, \ldots, x_{11}^{w_{3m}}, x_{12}^{w_{3m}}, \ldots, x_{11}^{w_{41}}, x_{12}^{w_{42}}, \ldots, x_{11}^{w_{4m}}, x_{12}^{w_{4m}}, \ldots, x_{11}^{w_{51}}, x_{12}^{w_{52}}, \ldots, x_{11}^{w_{5m}}, x_{12}^{w_{5m}}, \ldots, x_{11}^{w_{61}}, x_{12}^{w_{62}}, \ldots, x_{11}^{w_{6m}}, x_{12}^{w_{6m}}, \ldots \rangle.$$ 

We apply induction on $m$. The case $m = 2$ follows from Lemma 3.1 (2). Now we assume that $m \geq 3$. Consider the following short exact sequences

$$0 \rightarrow \frac{S}{(I(D) : x_{m-1,1})}(-1) \xrightarrow{x_{m-1,1}} \frac{S}{I(D)} \xrightarrow{S} J_1 \xrightarrow{S} 0,$$

$$0 \rightarrow \frac{S}{(J_1 : x_{m-1,2})}(-1) \xrightarrow{x_{m-1,2}} \frac{S}{J_1} \xrightarrow{S} J_2 \xrightarrow{S} 0 \quad (\ddagger\ddagger)$$

$$\vdots$$

$$0 \rightarrow \frac{S}{(J_{m-1} : x_{m-1,1})}(-1) \xrightarrow{x_{m-1,1}t_{m-1}} \frac{S}{J_{m-1}} \xrightarrow{S} J_{m-1} \xrightarrow{S} 0$$

where $J_i = I(D) + (x_{m-1,1}, \ldots, x_{m-1,i})$ for $1 \leq i \leq t_{m-1}$. We prove this argument in the following two steps.

1. We first prove $pd(J_i : x_{m-1,i+1}) = |V(D)| - 2$ for $0 \leq i \leq t_{m-1} - 1$, where $J_0 = I(D)$. We write $(J_i : x_{m-1,i+1})$ as follows:

$$(J_i : x_{m-1,i+1}) = L_i^0 + L_i^1 + L_3,$$

where $L_i^0 = (0), L_i^1 = (x_{m-1,1}, x_{m-1,2}, \ldots, x_{m-1,i})$ for $1 \leq i \leq t_{m-1} - 1, L_i^2 = I(D \setminus (V_{m-1} \cup J_i^0)) + (x_{m-2,1}x_{m-1,1}^{w_{m-1}+1}, x_{m-2,1}x_{m-1,1}^{w_{m-1}+2}, \ldots, x_{m-2,1}x_{m-1,1}^{w_{m-1}t_{m-1}}),$ $x_{m-2,1}x_{m-1,1}^{w_{m-1}t_{m-1}+1}, x_{m-2,1}x_{m-1,1}^{w_{m-1}t_{m-1}+2}, \ldots, x_{m-2,1}x_{m-1,1}^{w_{m-1}t_{m-1}+w_{m-1}+1}, x_{m-2,1}x_{m-1,1}^{w_{m-1}t_{m-1}+w_{m-1}+2}, \ldots, x_{m-2,1}x_{m-1,1}^{w_{m-1}t_{m-1}+w_{m-1}+w_{m-1}+1}), \ldots, x_{m-2,1}x_{m-1,1}^{w_{m-1}t_{m-1}+w_{m-1}+w_{m-1}+w_{m-1}+1})$ for $0 \leq i \leq t_{m-1} - 1$. In fact, $L_2$ is an induced subgraph $H_i$ of $D$ on the set $(\bigcup_{j=1}^{i} V_j) \cup \{x_{m-1,1}, \ldots, x_{m-1,t_{m-1}}\}$ with the weight function $w_i : V(H_i) \rightarrow \mathbb{N}^+$ such that $w_i(x_{m-1,i+1}) = w_{m-1,i+1} - 1$ and $w_i(x) = w(x)$ for any other vertex $x \in V(H_i)$. Therefore, by induction hypothesis on $m$, we obtain

$$pd(L_i^2) = |V(D_i)| - 2 = |V(D)| - t_m - i - 2$$

for any $0 \leq i \leq t_{m-1} - 1$.

Next we will compute $pd(J_i : x_{m-1,i+1})$. Note that for $0 \leq i \leq t_{m-1} - 1$

$$(J_i : x_{m-1,i+1}) = L_i^1 + L_2 + L_3.$$
We consider the following two cases:

Corollary 3.4. Let \( m \geq 3 \). The proof is complete.  

2.4 (1) and by repeatedly using Lemma 2.11 (2) on the short exact sequences (\( \dagger \dagger \)).

Example 3.5. First, we notice that \( J_{m-1} = I(D) \setminus (V_{m-1} \cup V_m) \) and \( J_{m-1} = I(D) \setminus (V_{m-1} \cup V_m) \). Thus the assertion follows from Lemma 3.2 (1) and by repeatedly using Lemma 3.1 (2) on the short exact sequences (\( \dagger \dagger \)).

We consider the following two cases:

(I) If \( m = 3 \), then \( I(D \setminus (V_2 \cup V_3)) = (0) \). Hence

\[
\text{pd}(J_{t_2}) = \text{pd}\left( (x_{21}, x_{22}, \ldots, x_{2t_2}) \right) = t_2 - 1 = |V(D)| - t_1 - t_3 - 1 \leq |V(D)| - 3.
\]

(II) If \( m \geq 4 \). Since all the generators of \( I(D \setminus (V_{m-1} \cup V_m)) \) can not divided by variables \( x_{m-1,i} \), where \( 1 \leq i \leq t_{m-1} \), we have

\[
\text{pd}(J_{t_{m-1}}) = \text{pd}(I(D \setminus (V_{m-1} \cup V_m))) + \text{pd}\left( (x_{m-1,1}, x_{m-1,2}, \ldots, x_{m-1, t_{m-1}}) \right) + 1
\]

\[
= [\! [V(D \setminus (V_{m-1} \cup V_m))] - 2] + (t_{m-1} - 1) + 1
\]

\[
= |V(D)| - 2 - t_m
\]

\[
\leq |V(D)| - 3.
\]

The proof is complete.  

An immediate consequence of the above theorem is the following corollary.

**Corollary 3.4.** Let \( D = (V(D), E(D), w) \) be a vertex-weighted oriented graph as Theorem 3.2. Then

\[
\text{depth}(I(D)) = 2.
\]

**Proof.** It follows from Auslander-Buchsbaum formula and the above theorem.  

The following example shows that the projective dimension and the regularity of the edge ideals of vertex-weighted oriented graphs as Theorem 3.2 are related to direction selection.

**Example 3.5.** Let \( I(D) = (x_1 x_2^2, x_2 x_3^2, x_4 x_2^2, x_3 x_4^2, x_3 x_5^2, x_5 x_2^2, x_6 x_7^2, x_6 x_7^2) \) be the edge ideal of vertex-weighted oriented 6-partite graph \( D = (V, E, w) \) with \( w_1 = w_4 = w_6 \) = 1 and \( w_2 = w_3 = w_5 \) = 1 and \( w_7 = 2 \), where \( V = \bigcup_{j=1}^6 V_j \) with \( V_1 = \{x_1\}, V_2 = \{x_2\}, V_3 = \{x_3, x_4\}, V_4 = \{x_5\}, V_5 = \{x_6\} \) and \( V_6 = \{x_7\} \). By using CoCoA, we obtain
\[ \text{reg}(I(D)) = 7 \] and \( \text{pd}(I(D)) = 4. \) But we have \( \text{reg}(I(D)) = \sum_{i=1}^{7} w_i - |V(D)| + 2 = 6 \) by Theorem 3.3 and \( \text{pd}(I(D)) = |V(D)| - 2 = 5 \) by Theorem 3.3.

The following example shows that the assumption that \( w(x) \geq 2 \) if \( x \in V \setminus (V_1 \cup V_m) \) in Theorem 3.3 cannot be dropped.

**Example 3.6.** Let \( I(D) = \langle x_1 x_2^2, x_1 x_3^2, x_2 x_3^2, x_2 x_4^2, x_3 x_5, x_3 x_6, x_4 x_5, x_4 x_6, x_5 x_7, x_6 x_7^2 \rangle \) be the edge ideal of vertex-weighted 4-partite digraph \( D = (V, E, w) \) with \( w_1 = w_2 = w_5 = w_6 = 1 \) and \( w_3 = w_4 = w_7 = 2, \) where \( V = \bigcup_{j=1}^{4} V_j \) with \( V_1 = \{x_1, x_2\}, V_2 = \{x_3, x_4\}, V_3 = \{x_5, x_6\} \) and \( V_4 = \{x_7\}. \) By using CoCoA, we obtain \( \text{reg}(I(D)) = 4 \) and \( \text{pd}(I(D)) = 4. \) But we have \( \text{reg}(I(D)) = \sum_{i=1}^{7} w_i - |V(D)| + 2 = 5 \) by Theorem 3.3 and \( \text{pd}(I(D)) = |V(D)| - 2 = 5 \) by Theorem 3.3.

4. Projective Dimension and Regularity of Edge Ideals of the Second Class of Vertex-weighted Oriented \( m \)-Partite Graphs

In this section, we will provide some exact formulas for the projective dimension and the regularity of the edge ideals of some vertex-weighted oriented \( m \)-partite graphs with whiskers. Such graphs are another class of vertex-weighted oriented \( m \)-partite graphs with vertex set \( V = \bigcup_{i=1}^{m} V_i \) with \( |V_i| \leq |V_2|, \) the edge set \( E = \bigcup_{i=1}^{m-1} E(D_i), \) where \( D_1 \) is a bipartite graph with the vertex set \( \{x_{11}, \ldots, x_{1,|V_1|}\} \uplus \{x_{21}, \ldots, x_{2,|V_1|}\}, \) the edge set \( \{x_{21}x_{11}, \ldots, x_{2,|V_1|}x_{1,|V_1|}\} \) and \( D_i \) is a complete bipartite graph and it is also an induced subgraph of \( D \) on \( V_i \cup V_{i+1} \) satisfying the starting point of every edge of \( E(D_i) \) belongs to \( V_i \) and its ending point belongs to \( V_{i+1} \) for \( 2 \leq i \leq m - 1. \) We also give some examples to show that these formulas are related to direction selection and the weight of vertices.

Now we are ready to present the main theorem of this section.

**Theorem 4.1.** Let \( m \geq 2 \) be an integer, and suppose that \( D = (V, E, w) \) is a vertex-weighted oriented \( m \)-partite graph, its vertex set \( V = \bigcup_{i=1}^{m} V_i \) with \( |V_1| \leq |V_2|, \) its edge set \( E = \bigcup_{i=1}^{m-1} E(D_i), \) where \( D_1 \) is a bipartite graph with the vertex set \( \{x_{11}, \ldots, x_{1,|V_1|}\} \uplus \{x_{21}, \ldots, x_{2,|V_1|}\}, \) the edge set \( \{x_{21}x_{11}, \ldots, x_{2,|V_1|}x_{1,|V_1|}\} \) and \( D_i \) is a complete bipartite graph and it is also an induced subgraph of \( D \) on \( V_i \cup V_{i+1} \) satisfying the starting point of every edge in \( E(D_i) \) belongs to \( V_i \) and its ending point belongs to \( V_{i+1} \) for \( 2 \leq i \leq m - 1. \) If \( w(x) \geq 2 \) for any \( x \in V \setminus (V_1 \cup V_m). \) Then

1. \( \text{reg}(I(D)) = \sum_{x \in V(D)} w(x) - |V(D \setminus V_1)| + 1, \)
We apply induction on $|V_1|$. In this case, by Lemma 2.6, we have

\[ \text{pd}(I(D)) = \begin{cases} |V(D \setminus V_1)| - 2, & \text{if } |V_1| < |V_2|, \\ |V(D \setminus V_2)| - 1, & \text{if } |V_1| = |V_2|. \end{cases} \]

**Proof.** Let $V_i = \{x_{i1}, \ldots, x_{it_i}\}$ for $1 \leq i \leq m$, then

\[
I(D) = (x_{21}x_{11}, x_{22}x_{12}, \ldots, x_{2,t_1}x_{1,t_1}, x_{21}x_{w11}, x_{21}x_{w12}, \ldots, x_{2,t_1}x_{w1,t_1}),
\]

\[
\ldots, x_{2m-1,t_1}x_{w1,m-1,t_1}, \ldots, x_{2m,t_1}x_{w1,m,t_1},
\]

\[
x_{m-1,1}x_{w1,m-1}, \ldots, x_{m-1,1}x_{w1,m}, \ldots, x_{m-1,t}x_{w1,m},
\]

\[
x_{m-1,2}x_{w2,m-1}, \ldots, x_{m-1,2}x_{w2,m}, \ldots, x_{m-1,t}x_{w2,m},
\]

\[
x_{m,3}x_{w3,m-1}, \ldots, x_{m,3}x_{w3,m}, \ldots, x_{m,t}x_{w3,m}. \]

We apply induction on $m$. If $m = 2$, since the underlying graph $G$ of $D$ is simple, it has not isolated vertex. Thus $t_1 = t_2$ and $I(D) = (x_{21}x_{11}, x_{22}x_{12}, \ldots, x_{2,t_1}x_{1,t_1})$. In this case, by Lemma 2.6, we have

\[
\text{reg}(I(D)) = \sum_{j=1}^{t_1} (1 + w_{ij}) - (t_1 - 1) = \sum_{x \in V(D)} w(x) - |V(D \setminus V_1)| + 1,
\]

\[
\text{pd}(I(D)) = \sum_{j=1}^{t_1} \text{pd}((x_{2j}x_{w_{1j}})) + (t_1 - 1) = t_1 - 1 = |V(D \setminus V_2)| - 1.
\]

Now we assume that $m \geq 3$. For $0 \leq i \leq t_1 - 2$, we set $J_0 = I(D)$, $K_{i+1} = (x_{2,i+1}x_{w1,i+1})$, $L_{i+1} = (x_{2,i+2}x_{w1,i+2}, \ldots, x_{2,t_1}x_{w1,t_1})$, $K_{i+1} = (x_{2,t}x_{w1,t})$ and $L_{i+1} = (0)$. Further, we assume that $J_{i+1} = L_{i+1} + I(D \setminus V_1)$ and $J_i = I(D \setminus V_1)$. Thus, for all $0 \leq i \leq t_1 - 1$, we have

\[
J_i = J_{i+1} + K_{i+1} \quad \text{and} \quad J_{i+1} \cap K_{i+1} = K_{i+1}(L_{i+1} + L)
\]

where $L = (x_{w1,31}, x_{w1,32}, \ldots, x_{w1,3,t_1}) + I(D \setminus (V_1 \cup V_2))$.

For any $0 \leq i \leq t_1 - 1$, because the variable $x_{w1,i+1}$ in $K_{i+1}$ can not divided the generators of $J_{i+1}$ and $K_{i+1}$ has a linear resolution, it follows that $J_i = J_{i+1} + K_{i+1}$ is a Betti splitting. By Corollary 2.3, we obtain

\[
\text{reg}(J_i) = \max\{\text{reg}(K_{i+1}), \text{reg}(J_{i+1}), \text{reg}(K_{i+1} \cap J_{i+1}) - 1\},
\]

\[
\text{pd}(J_i) = \max\{\text{pd}(K_{i+1}), \text{pd}(J_{i+1}), \text{pd}(K_{i+1} \cap J_{i+1}) + 1\}.
\]

Note that the variables appearing in $L_{i+1}$, $K_{i+1}$ and $L$ are different from each other. Repeated using the above formulas, Lemmas 2.6 and 2.7 we obtain

\[ \text{reg}(J_0) = \max\{\text{reg}(K_{i+1}), \text{reg}(J_{i+1}), \text{reg}(K_{i+1} \cap J_{i+1}) - 1\}, \text{ for } 0 \leq i \leq t_1 - 1 \]

\[ = \max\{\text{reg}(K_{i+1}), \text{reg}(J_{i+1}), \text{reg}(K_{i+1} + L_{i+1} + L) - 1, \text{ for } 0 \leq i \leq t_1 - 1 \}
\]

\[ = \max\{\text{reg}(K_{i+1}), \text{reg}(K_{i+1} + L), \text{reg}(L_{i+1} + L) - 2, \text{reg}(J_{i+1}), \text{reg}(K_{i+1}) + \text{reg}(L) - 1, \text{ for } 0 \leq i \leq t_1 - 2 \}
\]

and

\[ \text{pd}(J_0) = \max\{\text{pd}(K_{i+1}), \text{pd}(J_{i+1}), \text{pd}(K_{i+1} \cap J_{i+1}) + 1, \text{ for } 0 \leq i \leq t_1 - 1 \}
\]

\[ = \max\{\text{pd}(K_{i+1}), \text{pd}(J_{i+1}), \text{pd}(L_{i+1} + L) + 1, \text{ for } 0 \leq i \leq t_1 - 1 \}
\]

\[ = \max\{\text{pd}(J_{i+1}), \text{pd}(L_{i+1}) + \text{pd}(L) + 2, \text{pd}(L) + 1, \text{ for } 0 \leq i \leq t_1 - 2 \}.
\]
Next, we will prove that $\text{reg} (L) = \sum_{\ell=3}^{m} (\sum_{j=1}^{t_{\ell}} w_{\ell, j}) - \sum_{j=3}^{m} t_{j} + 1$ and $\text{pd} (L) = \sum_{j=3}^{m} t_{j} - 1$.

We consider the following two cases:

(I) If $m = 3$, then $I(D \setminus (V_{1} \cup V_{2})) = (0)$. This implies that $L = (x_{31}^{w_{31}}, x_{32}^{w_{32}}, \ldots, x_{3,t_{3}}^{w_{3,t_{3}}})$. It follows that from Lemma 2.6

$$
\text{reg} (L) = \sum_{j=1}^{t_{3}} w_{3j} - (t_{3} - 1) = \sum_{j=1}^{t_{3}} w_{3j} - t_{3} + 1,
$$

$$
\text{pd} (L) = t_{3} - 1.
$$

(II) If $m \geq 4$, then the polarization $L^T$ of the ideal $L$ can be regarded as the polarization of the edge ideal of a vertex-weighted oriented graph $H$ with whiskers, its vertex set $(\bigcup_{j=3}^{m} V_{j}) \cup \{y_{31}, \ldots, y_{3,t_{3}}\}$, edge set $E(D \setminus (V_{1} \cup V_{2})) \cup \{x_{31}y_{31}, \ldots, x_{3,t_{3}}y_{3,t_{3}}\}$, and its weight function is $w' : V(H) \to \mathbb{N}^+$ with $w'(x_{3j}) = 1$, $w'(y_{3j}) = w_{3j} - 1$ for $1 \leq j \leq t_{3}$ and $w'(x) = w(x)$ for any other vertex $x$. It follows that $w'(x_{3j}) + w'(y_{3j}) = w_{3j}$ for $1 \leq j \leq t_{3}$. Notice that $H$ has only $(m - 1)$-partition. By induction hypothesis and Lemma 2.10 we obtain

$$
\text{reg} (L) = \sum_{x \in V(H)} w'(x) - |V(H \setminus V_{3})| + 1 = \sum_{\ell=3}^{m} (\sum_{j=1}^{t_{\ell}} w_{\ell, j}) - \sum_{j=3}^{m} t_{j} + 1,
$$

$$
\text{pd} (L) = |V(H \setminus V_{3})| - 1 = \sum_{j=3}^{m} t_{j} - 1.
$$

Therefore, from the formulas $(\ast)$ and $(\ast\ast)$, we have

$$
\text{reg} (J_{0}) = \max \{ \text{reg} (K_{i+1}), \text{reg} (K_{i}), \text{reg} (K_{i+1}) + \text{reg} (L_{i+1}) + \text{reg} (L) - 2, \text{reg} (J_{t_{1}}), \text{reg} (K_{t_{1}}) + \text{reg} (L) - 1, \text{ for } 0 \leq i \leq t_{1} - 2 \}
$$

$$
= \max \{ 1 + w_{1,i+1}, 1 + w_{1,t_{1}}, (1 + w_{1,i+1}) + \lfloor \sum_{j=i+2}^{t_{1}} (1 + w_{1}) - (t_{1} - i - 2) \rfloor + \sum_{\ell=3}^{m} (\sum_{j=1}^{t_{\ell}} w_{\ell, j}) - \sum_{j=3}^{m} t_{j} + 1 - 2, \sum_{x \in V(D \setminus V_{1})} w(x) - |V(D \setminus V_{1})| + 2,
$$

$$
(1 + w_{1,t_{1}}) + \sum_{\ell=3}^{m} (\sum_{j=1}^{t_{\ell}} w_{\ell, j}) - \sum_{j=3}^{m} t_{j} + 1 - 1, \text{ for } 0 \leq i \leq t_{1} - 2 \}
$$

$$
= \sum_{x \in V(D)} w(x) - |V(D \setminus V_{1})| + 1
$$

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where this maximal value is obtained when \( i = 0 \), and
\[
pd(J_0) = \max\{\text{pd}(J_1), \text{pd}(L_{i+1}) + \text{pd}(L) + 2, \text{pd}(L) + 1\}, \text{ for } 0 \leq i \leq t_1 - 2
\]
\[
= \max\{|V(D \setminus V_1)| - 2, (t_1 - i - 2) + (\sum_{j=3}^m t_j - 1) + 2, (\sum_{j=3}^m t_j - 1) + 1\}
\]
\[
= \max\{\sum_{j=2}^m t_j - 2, (\sum_{j=2}^m t_j - 2) + (1 + t_1 - t_2)\}
\]
\[
= \begin{cases} |V(D \setminus V_1)| - 2 & \text{if } t_1 < t_2, \\ |V(D \setminus V_2)| - 1 & \text{if } t_1 = t_2. \end{cases}
\]
The proof is complete. \( \Box \)

An immediate consequence of the above theorem is the following corollary.

**Corollary 4.2.** Let \( D = (V(D), E(D), w) \) be a vertex-weighted oriented graph as Theorem 4.1. Then
\[
\text{depth}(I(D)) = \begin{cases} |V_1| - 2 & \text{if } t_1 < t_2, \\ |V_2| - 1 & \text{if } t_1 = t_2. \end{cases}
\]

**Proof.** It follows from Auslander-Buchsbaum formula and the above theorem. \( \Box \)

The following example shows that the projective dimension and the regularity of the edge ideals of vertex-weighted oriented graphs as Theorem 4.1 are related to direction selection.

**Example 4.3.** Let \( I(D) = (x_2x_1, x_2x_3^3, x_2x_4^3, x_3x_5^3, x_4x_5^3, x_6x_7^3, x_6x_7^3) \) be the edge ideal of vertex-weighted oriented 6-partite graph \( D = (V, E, w) \) with \( w_1 = w_2 = w_6 = 1 \) and \( w_3 = w_4 = w_5 = w_7 = 3 \), where \( V = \bigcup_{j=1}^6 V_j \) with \( V_1 = \{x_1\}, V_2 = \{x_2\}, V_3 = \{x_3, x_4\}, V_4 = \{x_5\}, V_5 = \{x_6\} \) and \( V_6 = \{x_7\} \). By using CoCoA, we obtain \( \text{reg}(I(D)) = 11 \) and \( \text{pd}(I(D)) = 4 \). But we have \( \text{reg}(I(D)) = \sum_{i=1}^7 w_i - |V(D \setminus V_1)| + 1 = 10 \) and \( \text{pd}(I(D)) = |V(D \setminus V_2)| - 1 = 5 \) by Theorem 4.1.

The following example shows that the assumption that \( w(x) \geq 2 \) if \( x \in V \setminus (V_1 \cup V_m) \) in Theorem 4.1 cannot be dropped.

**Example 4.4.** Let \( I(D) = (x_3x_1^3, x_4x_2^3, x_3x_5^3, x_4x_5^3, x_6x_7^3, x_6x_7^3, x_5x_7, x_6x_7, x_7x_8^3) \) be the edge ideal of vertex-weighted whisker \( D = (V, E, w) \) with \( w_1 = w_2 = w_5 = w_6 = w_8 = 3 \) and \( w_3 = w_4 = w_7 = 1 \), where \( V = \bigcup_{j=1}^5 V_j \) with \( V_1 = \{x_1, x_2\}, V_2 = \{x_3, x_4\}, V_3 = \{x_5, x_6\}, V_4 = \{x_7\} \) and \( V_5 = \{x_8\} \). By using CoCoA, we obtain \( \text{reg}(I(D)) = 11 \) and \( \text{pd}(I(D)) = 4 \). But we have \( \text{reg}(I(D)) = \sum_{i=1}^8 w_i - |V(D \setminus V_1)| + 1 = 13 \) and \( \text{pd}(I(D)) = |V(D \setminus V_2)| - 1 = 5 \) by Theorem 4.1.
5. Projective Dimension and Regularity of Edge Ideals of the Third Class of Vertex-Weighted Oriented m-Partite Graphs

In this section, we will give some exact formulas for the projective dimension and the regularity of edge ideals of the third class of vertex-weighted oriented m-partite graphs with vertex set \( V = \bigcup_{i=1}^{m} V_i \) and edge set \( E = \bigcup_{i=1}^{m} E(D_i) \), where \( D_i \) is a complete bipartite graph and it is also an induced subgraph of \( D \) on \( V_i \sqcup V_{i+1} \) satisfying the starting point of every edge of \( E(D_i) \) belongs to \( V_i \) and its ending point belongs to \( V_{i+1} \) for \( 1 \leq i \leq m \), where we stipulate \( V_{m+1} = V_1 \). We also give some examples to show that these formulas are related to direction selection and the weight of vertices.

**Theorem 5.1.** Let \( m \geq 3 \) be an integer, and assume that \( D = (V, E, w) \) is a vertex-weighted oriented m-partite graph with vertex set \( V = \bigcup_{i=1}^{m} V_i \) and edge set \( E = \bigcup_{i=1}^{m} E(D_i) \), where \( D_i \) is a complete bipartite graph and it is also an induced subgraph of \( D \) on \( V_i \sqcup V_{i+1} \) satisfying the starting point of every edge in \( E(D_i) \) belongs to \( V_i \) and its ending point belongs to \( V_{i+1} \) for \( 1 \leq i \leq m \), where we stipulate \( V_{m+1} = V_1 \). If \( w(x) \geq 2 \) for all \( x \in V \). Then

\[
\text{reg}(I(D)) = \sum_{x \in V(D)} w(x) - |V(D)| + 1.
\]

**Proof.** Let \( V_i = \{x_{i1}, \ldots, x_{i,t_i}\} \) for \( 1 \leq i \leq m \). Then

\[
I(D) = (x_1x_2^{w_{21}}, x_1x_2^{w_{22}}, \ldots, x_1x_2^{w_{2,t_2}}, x_2x_1^{w_{1,2}}, \ldots, x_2x_1^{w_{1,2}}, x_1x_2^{w_{2,2}}, \ldots, x_1x_2^{w_{2,2}}, x_2x_1^{w_{1,2}}, \ldots, x_2x_1^{w_{1,2}}).
\]

Consider the following short exact sequences

\[
0 \rightarrow \frac{S}{(I(D):x_{m,1}^{w_{m,1}})} \left(-w_{m,1}\right) \xrightarrow{x_{m,1}^{w_{m,1}}} \frac{S}{J(D)} \rightarrow \frac{S}{J_1} \rightarrow 0
\]

\[
0 \rightarrow \frac{S}{(J_1:x_{m,2}^{w_{m,2}})} \left(-w_{m,2}\right) \xrightarrow{x_{m,2}^{w_{m,2}}} \frac{S}{J_1} \rightarrow \frac{S}{J_2} \rightarrow 0
\]

\[
\vdots \rightarrow \vdots \rightarrow \vdots
\]

\[
0 \rightarrow \frac{S}{(J_{m-1}:x_{m,1}^{w_{m,1}}, x_{m,2}^{w_{m,2}})} \left(-w_{m,1}\right) \xrightarrow{x_{m,1}^{w_{m,1}}} \frac{S}{J_{m-1}} \rightarrow \frac{S}{J_m} \rightarrow 0
\]

where \( J_i = I(D) + (x_{m,1}^{w_{m,1}}, \ldots, x_{m,i}^{w_{m,i}}) \) for \( 1 \leq i \leq t_m \). We prove this argument in the following two steps.
\( (1) \) We first prove \( \text{reg} \left( (J_i : x_{m,i+1}^{w_{m,i+1}}) \right) \leq \sum_{x \in V(D)} w(x) - |V(D)| + 1 - w_{m,i+1} \), for \( 0 \leq i \leq t_m - 1 \), where \( J_0 = I(D) \).

In fact, for \( 0 \leq i \leq t_m - 1 \), we can write \( (J_i : x_{m,i+1}^{w_{m,i+1}}) \) as

\[
(J_i : x_{m,i+1}^{w_{m,i+1}}) = L_1 + L_2 + L^i
\]

where \( L_1 = (x_{w_{11}}^{w_{11}} \cdot x_{w_{12}}^{w_{12}} \cdot x_{w_{11},t_1}^{w_{11},t_1}) + I(D \setminus (V_{m-1} \cup V_m)), L_2 = (x_{w_{m-1,1}}^{w_{m-1,1}}, \ldots, x_{w_{m-1,t_{m-1}}}^{w_{m-1,t_{m-1}}}), L^0 = (0) \) and \( L^i = (x_{w_{m1}}^{w_{m1}}, x_{w_{m2}}^{w_{m2}}, \ldots, x_{w_{mi}}^{w_{mi}}) \) for \( 1 \leq i \leq t_m - 1 \). Thus

\[
\text{reg}(L^0) = 0, \quad \text{reg}(L_2) = 1, \quad \text{and} \quad \text{reg}(L^i) = \sum_{j=1}^{i} w_{mj} - (i - 1) \quad \text{for all} \quad 1 \leq i \leq t_m - 1.
\]

Note that the variables appearing in \( L_1, L_2 \) and \( L^i \) are different from each other. Therefore, it is enough to calculate \( \text{reg}(L_1) \) in order to compute \( \text{reg} \left( (J_i : x_{m,i+1}^{w_{m,i+1}}) \right) \) by Lemma 2.6 (2). We distinguish into the following two cases:

(I) If \( m = 3 \), then \( I(D \setminus (V_2 \cup V_3)) = (0) \). In this case, \( L_1 = (x_{w_{11}}^{w_{11}} \cdot x_{w_{12}}^{w_{12}} \cdot x_{w_{11},t_1}^{w_{11},t_1}) \).

Thus

\[
\text{reg}(L_1) = \sum_{j=1}^{t_1} w_{1j} - t_1 + 1.
\]

(II) If \( m \geq 4 \). In this case, \( L_1 = (x_{w_{11}}^{w_{11}} \cdot x_{w_{12}}^{w_{12}} \cdot x_{w_{11},t_1}^{w_{11},t_1}) + I(D \setminus (V_{m-1} \cup V_m)) \). Let \( L^m_1 \) be the polarization of the ideal \( L_1 \), then it can be regarded as the polarization of the edge ideal of a vertex-weighted oriented graph \( H \) with whiskers, its vertex set \( (\bigcup_{j=1}^{m-2} V_j) \cup \{y_{11}, \ldots, y_{1,t_1}\} \), edge set \( E(D \setminus (V_{m-1} \cup V_m)) \cup \{x_{11}y_{11}, \ldots, x_{1,t_1}y_{1,t_1}\} \), and the weight function is \( w' : V(H) \to \mathbb{N}^+ \) with \( w'(x_{1j}) = 1 \), \( w'(y_{1j}) = w_{1j} - 1 \) for \( 1 \leq j \leq t_1 \) and \( w'(x) = w(x) \) for any other vertex \( x \). Thus \( w'(x_{1j}) + w'(y_{1j}) = w_{1j} \) for \( 1 \leq j \leq t_1 \). By Lemma 2.10 and Theorem 4.1, we obtain

\[
\text{reg}(L_1) = \sum_{x \in V(H)} w'(x) - |V(H \setminus W)| + 1 = \sum_{\ell=1}^{t_1} \left( \sum_{j=1}^{t_\ell} w_{\ell,j} \right) - \sum_{j=1}^{m-2} t_j + 1
\]

where \( W = \{y_{11}, \ldots, y_{1,t_1}\} \).

Next we will prove \( \text{reg} \left( (J_i : x_{m,i+1}^{w_{m,i+1}}) \right) \leq \sum_{x \in V(D)} w(x) - |V(D)| + 1 - w_{m,i+1} \), for \( 0 \leq i \leq t_m - 1 \).
Since the variables that appear in $L_1$, $L_2$ and $L^i$ are different from each other for any $0 \leq i \leq t_m - 1$, by Lemma 2.6 (2), we can get

\[
\text{reg} \left( (J_0 : x_{m1}^{w_0}) \right) = \text{reg} \left( L_1 + L_2 \right) = \text{reg} \left( L_1 \right) + \text{reg} \left( L_2 \right) - 1
\]

\[
= \left[ \sum_{\ell=1}^{m-2} \left( \sum_{j=1}^{t_{\ell}} w_{\ell,j} - \sum_{j=1}^{t_{\ell}} t_j + 1 \right) + 1 - 1 \right.
\]

\[
= \sum_{x \in V(D)} w(x) - \sum_{j=1}^{t_{m-1}} w_{m-1,j} - \sum_{j=1}^{t_m} w_{mj} - |V(D)| + t_{m-1} + t_m + 1
\]

\[
\leq \sum_{x \in V(D)} w(x) - |V(D)| + 1 + t_{m-1} + t_m - \sum_{j=1}^{t_{m-1}} w_{m-1,j} - \sum_{j=1}^{t_m} w_{mj}
\]

\[
\leq \sum_{x \in V(D)} w(x) - |V(D)| + 1 + t_{m-1} + t_m - 2t_{m-1} - 2(t_m - 1) - w_{m1}
\]

and, for $1 \leq i \leq t_m - 1$,

\[
\text{reg} \left( (J_i : x_{m,i+1}^{w_{m,i+1}}) \right) = \text{reg} \left( L_1 + L_2 + L^i \right) = \text{reg} \left( L_1 \right) + \text{reg} \left( L_2 \right) + \text{reg} \left( L^i \right) - 2
\]

\[
= \left[ \sum_{\ell=1}^{m-2} \left( \sum_{j=1}^{t_{\ell}} w_{\ell,j} - \sum_{j=1}^{t_{\ell}} t_j + 1 \right) + 1 + \left( \sum_{j=1}^{t_m} w_{mj} - i + 1 \right) - 2 \right.
\]

\[
= \sum_{x \in V(D)} w(x) - |V(D)| + 1 + t_{m-1} + t_m - i - \sum_{j=1}^{t_{m-1}} w_{m-1,j} - \sum_{j=i+1}^{t_m} w_{mj}
\]

\[
\leq \begin{cases} 
\sum_{x \in V(D)} w(x) - |V(D)| + 1 - w_{m,tm} \\
+ t_{m-1} + t_m - i - 2t_{m-1}, & \text{if } i = t_m - 1, \\
\sum_{x \in V(D)} w(x) - |V(D)| + 1 - w_{m,i+1} \\
+ i + 2 - t_{m-1} - t_m, & \text{if } 1 \leq i \leq t_m - 2.
\end{cases}
\]

\[
\leq \sum_{x \in V(D)} w(x) - |V(D)| + 1 - w_{m,i+1}
\]

where the first inequality in the above formulas is due to $w_{m-1,j} \geq 2$ for $1 \leq j \leq t_{m-1}$, and $w_{mj} \geq 2$ for $i + 2 \leq j \leq t_m$.

(2) Next we will prove $\text{reg} \left( J_{t_m} \right) = \sum_{x \in V(D)} w(x) - |V(D)| + 1$, this implies that $\text{reg} \left( (J_i : x_{m,i+1}^{w_{m,i+1}}) \right) + w_{m,i+1} \leq \text{reg} \left( J_{t_m} \right)$ for all $0 \leq i \leq t_m - 1$. Thus the assertion follows from Lemma 2.4 (2) and by repeatedly using Lemma 2.11 (1) on the short exact sequences $(\dagger \dagger \dagger)$.

In fact, we write $J_{t_m}$ as

\[
J_{t_m} = I(D') + L,
\]
where $I(D')$ is the edge ideal of a vertex-weighted oriented subgraph $D'$ of $D$, where $D'$ obtained from $D$ deleting the edges $\{x_{m-1,1}x_{m1}, \ldots, x_{m-1,1}x_{m1,tm}, x_{m-1,2}x_{m2}, \ldots, x_{m-1,2}x_{m2,tm}, \ldots, x_{m-1,tm-1}x_{m1}, \ldots, x_{m-1,tm-1}x_{m1,tm}, x_{m-1,tm}x_{m1}, \ldots, x_{m-1,tm}x_{m1,tm}\}$, and $L = (x_{m1,1}^w, x_{m2,1}^w, \ldots, x_{m,1}^w)$. Then the polarization $J_{m}^P$ of ideal $J_{m}$ can be regarded as the polarization of the edge ideal of a vertex-weighted oriented graph $D''$ with whiskers, its vertex set $(\bigsqcup_{j=1}^{m} V_j)\cup\{y_{m1}, \ldots, y_{m,tm}\}$, edge set $E(D')\cup\{x_{m1}y_{m1}, \ldots, x_{m,tm}y_{m,tm}\}$ and the weight function is $w'' : V(D'') \to \mathbb{N}^+$ with $w''(x_{mj}) = 1$, $w''(y_{mj}) = w_{mj} - 1$ for $1 \leq j \leq tm$ and $w''(x) = w(x)$ for any other vertex $x$. Thus $w''(x_{mj}) + w''(y_{mj}) = w_{mj}$ for $1 \leq j \leq tm$. By Lemma $2.10$ $(2)$ and Theorem $4.1$ we obtain

$$\begin{align*}
\text{reg}(J_{m}) &= \sum_{x \in V(D'')} w''(x) - |V(D'') \setminus V''| + 1 = \sum_{\ell=1}^{m} (\sum_{j=1}^{t_{\ell}} w_{\ell,j}) - \sum_{j=1}^{m} t_{j} + 1 \\
&= \sum_{x \in V(D)} w(x) - |V(D)| + 1
\end{align*}$$

where $V'' = \{y_{m1}, \ldots, y_{m,tm}\}$. This proof is complete. □

**Theorem 5.2.** Let $D = (V(D), E(D), w)$ be a vertex-weighted oriented graph as Theorem $5.1$. Then

$$pd(I(D)) = |V(D)| - 1.$$ 

**Proof.** Let $V_i = \{x_{i,1}, \ldots, x_{i,t_i}\}$ for $1 \leq i \leq m$. Then

$$I(D) = (x_{11}^{w_{11}}, x_{11}^{w_{12}}, \ldots, x_{11}^{w_{12,t_2}}, x_{12}^{w_{21}}, \ldots, x_{12}^{w_{21,t_2}}, \ldots, x_{1,t_1}^{w_{12}}, x_{21}^{w_{13}}, x_{21}^{w_{13,t_3}}, \ldots, x_{21}^{w_{13,t_3}}, \ldots, x_{2,t_2}^{w_{13}}, x_{22}^{w_{23}}, x_{22}^{w_{23,t_3}}, \ldots, x_{22}^{w_{23,t_3}}, \ldots, x_{m-1,t_{m-1}}^{w_{m-1}}, x_{m-1,t_{m-1}}^{w_{m-1,t_{m-1}}}, x_{m-1,t_{m-1}}^{w_{m-1,t_{m-1}},t_{1}}, x_{m-1,t_{m-1}}^{w_{m-1,t_{m-1}},t_{1}}, x_{m-1,t_{m-1}}^{w_{m-1,t_{m-1}},t_{1}}), \ldots, x_{m-1,t_{m-1}}^{w_{m-1}}, x_{m-1,t_{m-1}}^{w_{m-1,t_{m-1}}}, x_{m-1,t_{m-1}}^{w_{m-1,t_{m-1}},t_{1}}, x_{m-1,t_{m-1}}^{w_{m-1,t_{m-1}},t_{1}}, x_{m-1,t_{m-1}}^{w_{m-1,t_{m-1}},t_{1}}), \ldots, x_{m-1,t_{m-1}}^{w_{m-1}}, x_{m-1,t_{m-1}}^{w_{m-1,t_{m-1}}}, x_{m-1,t_{m-1}}^{w_{m-1,t_{m-1}},t_{1}}, x_{m-1,t_{m-1}}^{w_{m-1,t_{m-1}},t_{1}}, x_{m-1,t_{m-1}}^{w_{m-1,t_{m-1}},t_{1}}).$$

Consider the following short exact sequences

$$
\begin{align*}
0 &\rightarrow \frac{S}{(I(D) : x_{m1})}(-1) \xrightarrow{x_{m1}} \frac{S}{I(D)} \xrightarrow{S} \frac{S}{J_i} \xrightarrow{S} 0 \\
0 &\rightarrow \frac{S}{(J_i : x_{m2})}(-1) \xrightarrow{x_{m2}} \frac{S}{J_i} \xrightarrow{S} \frac{S}{J_2} \xrightarrow{S} 0 \quad (\ddots)
\end{align*}
$$

where $J_i = I(D) + (x_{m1}, x_{m2}, \ldots, x_{mi})$ for $1 \leq i \leq tm$. We prove this argument into the following two steps.

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(1) We first prove \( \text{pd} (J_i : x_{m,i+1}) = |V(D)| - 1 \) for all \( 0 \leq i \leq t_m - 1 \), where \( J_0 = I(D) \). We write \((J_i : x_{m,i+1})\) as follows:

\[
(J_i : x_{m,i+1}) = L_1^i + L_2^i
\]

where \( L_1^0 = (0), L_1^i = (x_{m,1}, x_{m,2}, \ldots, x_{m,i}) \) for \( 1 \leq i \leq t_m - 1 \), \( L_2^i = (x_{i,1}, x_{i,2}, \ldots, x_{i,t_i}) + I(D_i) \), and \( D_i \) is an induced subgraph \( D \setminus \{x_{m,1}, x_{m,2}, \ldots, x_{m,i}\} \) of \( D \) on the set \( V \setminus \{x_{m,1}, x_{m,2}, \ldots, x_{m,i}\} \).

Since the variables appearing in \( L_1^i \) and \( L_2^i \) are different and \( \text{pd} (L_1^i) = i - 1 \), we only need to calculate \( \text{pd} (L_2^i) \) in order to compute \( \text{pd} ((J_i : x_{m,i+1})) \) by Lemma 2.6.

For \( 0 \leq i \leq t_m - 1 \), the polarization \( (L_2^i)^P \) of the ideal \( L_2^i \) can be regarded as the polarization of the edge ideal of a vertex-weighted oriented graph \( H_i \) with whiskers, its vertex set \( V(D_i) \cup \{y_{11}, \ldots, y_{1,t_1}\} \), edge set \( E(D_i) \cup \{x_{11}y_{11}, \ldots, x_{1,t_1}y_{1,t_1}\} \) and the weight function is \( w_i : V(H_i) \rightarrow \mathbb{N}^+ \) with \( w_i(x_{1j}) = 1, w_i(y_{1j}) = w_{1j} - 1, w_i(x_{m,i+1}) = w_{m,i+1} - 1 \) for \( 1 \leq j \leq t_1 \) and \( w_i(x) = w(x) \) for any other vertex \( x \). In fact, \( H_i \) is an oriented graph as Theorem 4.1. Then by Lemma 2.10 (3) and Theorem 4.1, we have

\[
\text{pd} (J_0 : x_{m,1}) = |V(D)| - 1,
\]

\[
\text{pd} (J_i : x_{m,i+1}) = \text{pd} (L_1^i + L_2^i) = \text{pd} (L_1^i) + \text{pd} (L_2^i) + 1 = (i - 1) + (|V(D)| - i - 1) + 1 = |V(D)| - 1.
\]

(2) Next we will compute \( \text{pd} (J_{t_m}) \leq |V(D)| - 2 \). Thus we have \( \text{pd} (J_{t_m}) < \text{pd} (J_i : x_{m,i+1}) \) for all \( 0 \leq i \leq t_m - 1 \). Therefore, the assertion follows from Lemma 2.4 (1) and by repeatedly using Lemma 2.11 (2) on the short exact sequences (‡ ‡ ‡‡).

In fact, we notice that

\[
J_{t_m} = L_1 + L_2
\]

where \( L_1 = (x_{m,1}, x_{m,2}, \ldots, x_{m,t_m}) \) and \( L_2 = I(D \setminus V_m) \). Notice that \( L_2 \) is the edge ideal of the induced subgraph \( D \setminus V_m \) of \( D \), it is a vertex-weighted \((m - 1)\)-partite graph with the vertex set \( \bigcup_{i=1}^{m-1} V_i \). Using Theorem 3.3 and Lemma 2.6 (1), we obtain

\[
\text{pd} (J_{t_m}) = \text{pd} (L_1) + \text{pd} (L_2) + 1 = (t_m - 1) + (|V(D \setminus V_m)| - 2) + 1 = t_m + (|V(D)| - t_m) - 2 = |V(D)| - 2.
\]

The proof is complete. \( \square \)

The following theorem generalizes Theorem 5.1 of [27].

**Corollary 5.3.** Let \( D = (V(D), E(D), w) \) be a weighted oriented cycle such that \( w(x) \geq 2 \) for any vertex \( x \). Then

1. \( \text{reg} (I(D)) = \sum_{x \in V(D)} w(x) - |E(D)| + 1 \),
2. \( \text{pd} (I(D)) = |E(D)| - 1 \).

**Proof.** Let \( V(D) = \{x_1, \ldots, x_n\} \). Then \( D \) is an oriented \( n \)-partite graph as Theorem 5.1 with vertex set \( V = \bigcup_{i=1}^{n} V_i \), where \( V_i = \{x_i\} \), and edge set \( E = \bigcup_{i=1}^{m} E(D_i) \), where
E(D_i) = \{x_i x_{i+1}\}. Thus |E(D)| = |V(D)| = n and the assertion follows from two theorems above. □

The following corollaries are immediate consequences of two theorems above.

**Corollary 5.4.** Let \( D = (V(D), E(D), w) \) be a weighted oriented complete tripartite graph such that \( w(x) \geq 2 \) for any vertex \( x \). Then

1. \( \text{reg}(I(D)) = \sum_{x \in V(D)} w(x) - |V(D)| + 1, \)
2. \( \text{pd}(I(D)) = |V(D)| - 1. \)

**Corollary 5.5.** Let \( D = (V(D), E(D), w) \) be a vertex-weighted oriented graph as Theorem 5.1. Then

\[
\text{depth}(I(D)) = 1
\]

*Proof.* It follows from Auslander-Buchsbaum formula and the above theorem. □

The following example shows that the projective dimension and the regularity of the edge ideals of vertex-weighted oriented graphs as Theorem 5.1 are related to direction selection.

**Example 5.6.** Let \( I(D) = (x_1 x_2^3, x_2 x_3^2, x_3 x_4^3, x_4 x_5^2, x_4 x_5 x_1) \) be the edge ideal of weighted oriented 3-partite graph \( D = (V(D), E(D), w) \) with \( w_1 = w_2 = 1 \) and \( w_3 = w_4 = 3 \), where \( V = \bigsqcup_{j=1}^{3} V_j \) with \( V_1 = \{ x_1, x_2 \}, V_2 = \{ x_3 \} \) and \( V_3 = \{ x_4 \} \). By using CoCoA, we obtain \( \text{reg}(I(D)) = 6 \) and \( \text{pd}(I(D)) = 2. \) But we have \( \text{reg}(I(D)) = \sum_{i=1}^{4} w_i - |V(D)| + 1 = 5 \) by Theorem 5.1 and \( \text{pd}(I(D)) = |V(D)| - 1 = 3 \) by Theorem 5.2.

The following example shows that the assumption that \( w(x) \geq 2 \) for any \( x \in V(D) \) in Theorem 5.1 and Theorem 5.2 cannot be dropped.

**Example 5.7.** Let \( I(D) = (x_1 x_2^2, x_1 x_3^2, x_1 x_4^2, x_2 x_3^2, x_3 x_4^2, x_4 x_5^2, x_5 x_1) \) be the edge ideal of vertex-weighted oriented tripartite graph \( D = (V, E, w) \) with \( w_1 = 1, w_2 = w_3 = w_4 = 2 \) and \( w_5 = 3 \), where \( V = \bigsqcup_{j=1}^{3} V_j \) with \( V_1 = \{ x_1 \}, V_2 = \{ x_2, x_3, x_4 \} \) and \( V_3 = \{ x_5 \} \). By using CoCoA, we obtain \( \text{reg}(I(D)) = 5 \) and \( \text{pd}(I(D)) = 3. \) But we have \( \text{reg}(I(D)) = \sum_{i=1}^{5} w_i - |V(D)| + 1 = 6 \) by Theorem 5.1 and \( \text{pd}(I(D)) = |V(D)| - 1 = 4 \) by Theorem 5.2.

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