VERTICAL STRUCTURE AND CORONAL POWER OF ACCRETION DISKS POWERED BY MAGNETOROTATIONAL-INSTABILITY TURBULENCE

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ABSTRACT

In this paper, we consider two outstanding intertwined problems in modern high-energy astrophysics: (1) the vertical-thermal structure of an optically thick accretion disk heated by the dissipation of magnetohydrodynamic turbulence driven by the magnetorotational instability (MRI), and (2) determining the fraction of the accretion power released in the corona above the disk. For simplicity, we consider a gas-pressure-dominated disk and assume constant opacity. We argue that the local turbulent dissipation rate due to the disruption of the MRI channel flows by secondary parasitic instabilities should be uniform across most of the disk, almost up to the disk photosphere. We then obtain a self-consistent analytical solution for the vertical thermal structure of the disk, governed by the balance between the heating by MRI turbulence and the cooling by radiative diffusion. Next, we argue that the coronal power fraction is determined by the competition between the Parker instability, viewed as a parasitic instability feeding off of MRI channel flows, and other parasitic instabilities. We show that the Parker instability inevitably becomes important near the disk surface, leading to a certain lower limit on the coronal power. While most of the analysis in this paper focuses on the case of a disk threaded by an externally imposed vertical magnetic field, we also discuss the zero net flux case, in which the magnetic field is produced by the MRI dynamo itself, and show that most of our arguments and conclusions should be valid in this case as well.

Key words: accretion, accretion disks – magnetic fields – radiative transfer

Online-only material: color figure

1. INTRODUCTION

1.1. Motivation

Angular momentum transport (AMT) in accretion disks is an important topic in modern high-energy astrophysics, with applications to systems as diverse as young stellar objects (YSOs), accreting stellar-mass compact objects (white dwarfs, neutron stars, and black holes, BHs) in Galactic binary systems (including X-ray binaries, XRBs), supermassive BHs in active galactic nuclei (AGNs), and even collapsar central engines of long gamma-ray bursts (GRBs). The leading candidate for explaining the observed relatively high AMT levels is the magnetohydrodynamic (MHD) turbulence excited by the magnetorotational instability (MRI; Velikhov 1959; Chandrasekhar 1961; Balbus & Hawley 1991, 1998; Balbus 2003). Over the past two decades, there have been many analytical and numerical studies of MRI-turbulent disks (including three-dimensional (3D) MHD simulations; see, e.g., Brandenburg et al. 1995, 1996; Stone et al. 1996; Armitage 1998; Miller & Stone 2000; Turner 2004; Sano et al. 2004; Hirose et al. 2006; Blaes et al. 2007; Fromang & Papaloizou 2007; Suzuki & Inutsuka 2009; Fromang 2010; Davis et al. 2010; Shi et al. 2010; Lesur & Longaretti 2011; Beckwith et al. 2009, 2011; Simon et al. 2009, 2011; Guan & Gammie 2011; Bodo et al. 2011; Blaes et al. 2011; Bai & Stone 2013; Bodo et al. 2012; Fromang et al. 2013) resulting in significant progress in our understanding of how these systems work (see, e.g., Balbus & Hawley 1998; Balbus 2003 for a review). However, most of these studies have focused, at least until recently, on the basic dynamical behavior of MRI-driven MHD turbulence in disks, with the primary goal of determining the $r \phi$ component of the turbulent Reynolds and Maxwell stresses that drive accretion. Relatively little attention, in our view, has been paid to the questions of thermodynamics, in particular, to the vertical thermal structure of MRI-powered disks, governed by the balance between the heating due to turbulent dissipation (which is inextricably linked to the AMT) and the cooling by radiative processes. Correspondingly, most accretion disks simulations have focused on the MHD aspects of the problem, while treating the thermodynamics in a relatively simplified fashion, e.g., considering an isothermal disk, or including optically thin radiative cooling. While these approaches have been very helpful in establishing the basic picture of AMT inside disks, they could not address the structure of the upper layers of disks, where the observable radiation is formed, or the transition to the magnetically dominated disk corona. One notable exception is the numerical work by Hirose et al. (2006; see also Turner 2004; Blaes et al. 2007, 2011; Hirose et al. 2009; Shi et al. 2010), which made the first difficult strides toward a more realistic treatment of MRI-powered radiatively cooled accretion disks. One has to appreciate, of course, the tremendous computational challenge of combining high-resolution 3D MHD simulations necessary to get the basic MHD of the problem right with radiative transfer that spans both optically thick and optically thin regimes.

Another outstanding problem in today’s high-energy astrophysics is understanding the formation of a strongly magnetized, hot, tenuous corona above a turbulent accretion disk, in particular, understanding what governs the fraction of the accretion power that is released in the corona above a relatively cold and dense disk. The observational motivation for our interest in accretion disk coronae (ADCs) stems from the fact that they lie at the base of disk-driven outflows and thus link disks to their winds and jets. These objects are also interesting for their role as emitters of high-energy (e.g., X-ray) radiation in many different types of accreting systems. Examples of astrophysical systems where the presence of an ADC has been inferred include: Galactic BH XRBs such as Cyg X-1 (e.g., Bisnovatyi-Kogan & Blinnikov 1976; Liang & Price 1977;
In accretion disks threaded by a relatively weak (below equipartition with the gas pressure) vertical magnetic field with a non-zero-net-flux, gravitational and rotational energy of the accreting matter is first transformed by the MRI into the magnetic and kinetic energy of the so-called channel flows (Hawley & Balbus 1992; Sano & Inutsuka 2001; Sano et al. 2004; Bodo et al. 2008; Suzuki & Inutsuka 2009). As was shown by Goodman & Xu (1994, hereafter GX94) in the shearing-box formulation, these channel flows are exact nonlinear solutions of the MHD equations; they are basically MRI linear eigen-modes that continue to grow exponentially even when they become large (Goodman & Xu 1994). However, as was also shown by GX94, channel flows cannot grow indefinitely, in reality, since they themselves become unstable to various secondary parasitic instabilities—i.e., instabilities that feed off of the velocity and magnetic field structures produced by the primary MRI mode, which in this context plays the role of a slowly evolving equilibrium. The development of the parasitic modes leads to the disruption of MRI channel flows and a fully developed turbulence that dissipates energy locally in the disk. GX94 themselves considered only ideal-MHD parasitic instabilities in the non-stratified case and found two families, one of which is related to the Kelvin–Helmholtz instability. In principle, however, other parasitic modes are possible, such as non-ideal, visco-resistive modes including the tearing mode (Pessah & Goodman 2009; Latter et al. 2009). It is also important to note that magnetic fields generated by the MRI channel flows have a substantial toroidal component and so may themselves be subject to a toroidal-field version of the MRI, which, in a certain sense, may be viewed here as also playing the role of a secondary parasitic instability.

Another limitation of the above-mentioned analytical studies of parasitic modes is that they restricted themselves to considering an unstratified shearing box; this can be thought of as a representation of a small region inside the disk, small compared with the pressure scale height, \( H \sim c_s / \Omega \). Thus, these analyses were effectively local not only in cylindrical radius but also in height above the disk midplane. Since the characteristic scale of the fastest-growing MRI mode, \( \lambda_{\text{MRI}} \), is proportional to the strength of the vertical magnetic field,

\[
\lambda_{\text{MRI}} \equiv \frac{\lambda_{\text{MRI}}}{2\pi} \sim \frac{V_A}{\Omega},
\]

where \( V_A \) is the Alfvén velocity corresponding to the local density and the vertical magnetic field \( B_z \), this approach is justified only if the vertical magnetic field is sufficiently weak, i.e., \( V_A \ll c_s \). The generalization of the channel-mode analysis to the case of stratified disks has been developed recently by Latter et al. (2010).

We note that recently reported numerical evidence from large shearing-box simulations suggesting that channel flows are not important in the overall energetics and AMT in accretion disks (e.g., Longaretti & Lesur 2010) can probably be attributed to a rather restricted definition of channel flows used in these studies. In particular, these studies restricted channel flows as being only the axisymmetric modes in the turbulent Fourier spectrum. Of course, if the computational box is sufficiently large in the toroidal direction, then the contribution from axisymmetric (i.e., spanning the entire toroidal extent of the box) modes should indeed be small. This is because any coherent flow and magnetic field structures, such as channel flows, have only a limited lifetime before they are disrupted by the parasitic modes as discussed above and thus they cannot be correlated over distances.
much larger than about the MRI wavelength. That is, one should not realistically expect channel-mode structures extending (in any direction) to sizes much larger than $\lambda_{\text{MRI}}$. Since here we are interested in a situation where $\lambda_{\text{MRI}} \ll H$, then the contribution of any axisymmetric (or, for that matter, any large scale; $l \sim H$) structures in the overall energy dissipation and AMT should indeed be inevitably small. We believe, however, that this does not disqualify any localized channel-mode structures. Locally, on scales of the order of $\lambda_{\text{MRI}} \ll H$, we can still expect that the picture presented in GX94 basically holds.

1.3. Objectives of This Paper

The present paper has two main goals. First, we are interested in the vertical structure of a gravitationally stratified accretion disk. In particular, we will restrict the present study to the case of a gas pressure-dominated disk, even though it is not directly applicable to inner parts of BH accretion disks in the high-slow state. Developing a generalization of our model for the case of a radiation pressure-dominated disk is left for a future study.

As mentioned above, most of the previous numerical studies of MRI turbulence (with the exception of Hirose et al. 2006, 2009; Blaes et al. 2007, 2011) were done using either the isothermal approximation or a simple prescription for optically thin radiative cooling. Standard geometrically thin accretion disks, however, are optically thick, and so one inevitably has to face the optically thick radiative transfer problem in order to deduce their structure (Shakura & Sunyaev 1973). Thus, the first main goal of the present study is to construct a self-consistent model of the vertical structure of an optically thick accretion disk heated by the local dissipation of MHD turbulence and cooled by radiative diffusion (we shall ignore any external irradiation). To attack this problem, we will adopt, for definitiveness, the point of view that the path to turbulence onset lies mostly in the nonlinear disruption of the MRI channel modes by parasitic instabilities, as described above. This assumption will allow us to obtain a concrete vertical profile (namely, flat, see Section 3.1) of the magnetic energy dissipation. We note, however, that this assumption is probably not critical and a similar picture may be developed without explicitly relying on the concept of parasitic modes. In any case, having a specific physically motivated prediction for turbulent dissipation, supplemented with some extra assumptions about the radiative transport properties of the disk (see Section 3), will allow us to construct a full theory of the steady-state vertical disk structure, along the lines of the classical analysis of Shakura & Sunyaev (1973).

Our second main objective is to find a way to estimate the fraction of the accretion energy released by the MRI that is not dissipated locally in the disk itself, but instead is transported vertically by buoyantly rising magnetic flux tubes and dissipated in the hot overlying corona. In our view, the coronal fraction of the released power is, to a large degree, determined by the competition between the GX94 parasitic instabilities and the Parker instability (Parker 1966), viewed here as another type of parasitic instability feeding off the horizontal magnetic field of the primary MRI mode (Foglizzo & Tagger 1999; Tout & Pringle 1992; Blackman & Pessah 2009). As a first step toward investigating this issue quantitatively, we will compare the growth rates of the GX94 and Parker parasitic instabilities and investigate how their ratio varies with height. As we will show in this paper, the Parker instability is slower than other parasitic instabilities in most of the bulk of the disk, but starts to become competitive in the disk’s upper layers, leading to a certain lower limit on the coronal power.

In most of the paper, we will focus on the case where the main magnetic field responsible for the MRI is an externally imposed, large-scale vertical magnetic field $B_0$, uniform in $z$. This case is applicable to situations where such a field is relatively strong, namely, stronger than the magnetic field that would be produced in its absence by turbulent dynamo action due to the MRI turbulence itself. We note that, in general, the problem of a large-scale dynamo in stratified MRI turbulent disks is still poorly understood and represents a key frontier in accretion disk research (Brandenburg et al. 1995; Stone et al. 1996; Hawley et al. 1996; Balbus & Hawley 1998; Blackman & Tan 2004; Vishniac 2009; Blackman & Pessah 2009; Simon et al. 2012). Nevertheless, as most of the recent zero net flux numerical simulations indicate (Shi et al. 2010; Davis et al. 2010; Guan & Gammie 2011), the MRI dynamo, by itself, leads to a saturation large-scale magnetic field only at a relatively low level, $B_{\text{dyn}} \ll B_{\text{eq}} \equiv (8 \pi P_0)^{1/2}$, where $P_0$ is the midplane gas pressure. Moreover, even the small-scale turbulent magnetic field responsible for the AMT in zero mean flux simulations is also relatively small, of the order of $B_{\text{turb}}/8 \pi \sim 0.01 P_0$ (e.g., Blackman et al. 2008). Thus, if the external mean vertical field $B_0$ is larger than this characteristic dynamo field but still weak compared to $B_{\text{eq}}$, i.e., if $B_{\text{dyn}} \ll B_0 \ll B_{\text{eq}}$, then one may expect the dynamics to be determined mostly by this superimposed vertical field $B_0$. This point of view is also supported by recent numerical studies by Lesur & Longaretti (2011) and by Bodo et al. (2011) and we will adopt it as our starting point in the main part of the paper where we consider the non-zero net flux case. We acknowledge, however, that, in practice, achieving a good, clean scale separation between the magnetic field strengths in the above inequality may be numerically challenging and one has to exercise caution. For example, if the mean vertical field is as high as a few tens of percent of the equipartition field, the corresponding horizontal magnetic field associated with channel flows starts to be dynamically important and affects the vertical pressure balance, perhaps even leading to the disruption of the disk (Miller & Stone 2000). In addition, even a weak net vertical field can affect the structure of the upper layers of the disk; in particular, it can excite MRI turbulence there, which can drive disk winds/outflows (Suzuki & Inutsuka 2009).

We note, however, that the opposite case—where the externally imposed magnetic field is small or absent and where the MRI is driven by the self-generated dynamo magnetic field,—is also, of course, of considerable interest in astrophysics and we will devote Section 7 to discussing it. In particular, we will argue that most of our results for the vertical disk structure obtained in Section 3 for the finite-$B_0$ case can also be applied to the zero net flux case, with $B_0$ replaced by $B_{\text{dyn}}$. Furthermore, we will recover the basic scalings obtained by Shakura & Sunyaev (1973) for this case and, in addition, will calculate the vertical temperature and density profiles that seem to be in good agreement with the results of numerical simulations by Hirose et al. (2006). However, we will not attempt to estimate the coronal power fraction for the zero net flux case, leaving this task for a future study.

This paper is organized as follows. In Section 2, we describe the basic idea and our overall approach to the problem. In Section 3, we present the calculation of the internal vertical thermal structure of the disk threaded by a non-zero net vertical
magnetic flux. Next, in Section 4, we consider the upper layers of the disk and investigate how close to the disk surface various assumptions of our model break down. Then, in Section 5, we estimate the growth rate of the Parker instability and compare it with that of the GX94 parasitic instabilities, as a function of height; this allows us to estimate the coronal fraction of the accretion power as a function of the system’s parameters. Then, in Section 6, we discuss the implications of our model on the accretion rate and on the longer-term evolution of the accretion disk. The zero net flux case is discussed in Section 7. Finally, we present our conclusions in Section 8.

2. THE OVERALL PHYSICAL PICTURE

The overall physical picture we have in mind can be described as follows.

For MRI to be active in the first place, the vertical magnetic field must be relatively weak, with the corresponding magnetic pressure less than the gas pressure in most of the disk, \( B_0^2/8\pi < P \). (As mentioned above, for simplicity, we ignore radiation pressure in the present work.) Furthermore, following previous studies (Goodman & Xu 1994; Pessah & Goodman 2009; Latter et al. 2009), we shall consider for simplicity the case when this field pressure is not only weak, but very weak compared with the gas pressure: \( B_0^2/8\pi \ll P \). If this condition is not satisfied, e.g., if the vertical magnetic field reaches a few tens of percent of the nominal equipartition field \( B_{eq} = (8\pi P_0)\beta^{1/2} \), then the unchecked development of MRI channel flows may effectively disrupt the disk (Miller & Stone 2000). Having the small parameter \( \beta^{-1} \equiv B_0^2/8\pi P \ll 1 \) at our disposal will allow us to make several important simplifications.

First, in the spirit of GX94 and Pessah & Goodman (2009), we assume that the growth of MRI is checked by the development of the parasitic instabilities. This assumption allows us to get on a direct path toward evaluating the MRI turbulent energy dissipation rate. Once the parasitic instabilities take over, fully developed MHD turbulence sets in and destroys the horizontal magnetic field of the primary MRI mode. As a result, the magnetic and kinetic energies of the primary MRI mode are dissipated by a turbulent cascade on a timescale of the order of the dynamical time, i.e., \( \Omega^{-1} \). The typical maximum amplitude to which the MRI channel flows are able to grow can be represented by the characteristic horizontal magnetic field \( B_{hor, sat} = b_{sat}B_0 \). Following GX94, it can be estimated from the condition that the growth rate of the fastest growing parasitic mode is comparable to the growth rate of the fastest growing (and hence the most relevant) primary MRI mode, which is essentially a numerical constant times \( \Omega \), independent of \( \beta \). The maximum growth rate of the GX94 parasitic instabilities is naturally proportional to the amplitude of the primary mode, i.e., \( \gamma_{GX} \sim b\Omega \). Therefore, the two growth rates become equal at a certain finite critical value \( b = b_{sat} \) of the order of 1 (even though the GX94 theory was formally developed under the assumption that \( b \gg 1 \)). Importantly, within the framework of the GX94 model (an unstratified disk with \( \beta \gg 1 \), ideal-MHD incompressible motions, etc.), there are really no additional parameters on which the dimensionless saturation amplitude \( b_{sat} \) could depend. It is then natural to take \( b_{sat} \) to be just a constant number of the order of unity. In particular, it should be the same for all heights in the disk, as long as we are not close enough to the disk surface, where the assumptions of the model break down (see Section 4).

Another reason why the assumption \( B_0^2/8\pi \ll P \) is important is that it justifies the locality (in \( z \)) of our picture of the MRI turbulence. Indeed, the communication speed in the vertical direction is only \( V_A \), and the lifetime of channel modes limited by the disruption by the parasitic instabilities is of order \( \Omega^{-1}\log b_{sat} \). Any causal connection established during this time extends only over a vertical distance of order \( l_{mri} \log b_{sat} \sim l_{mri} \), which is smaller than the gas pressure scale height \( H \) by a factor \( \beta^{1/2} \). This means that, as long as our assumption \( \beta \gg 1 \) holds, channel flows that develop at substantially different heights interact with each other only weakly and hence can be considered separately.

In other words, MRI at different heights in the disk develop independently of what happens at other heights. Therefore, since we are interested in the vertical structure of the disk including the vertical profile of the MRI turbulence and its energy dissipation rate, we can simply regard properties of MRI channel flows and parasitic instabilities as being local in \( z \). One thus may conclude that, under these circumstances, the exercise of calculating global MRI eigen-modes spanning the entire thickness of a stratified disk may be of only academic interest (although still useful for, e.g., providing benchmarks for numerical solutions and evaluating numerical resolution requirements).

The dissipated turbulent energy provides the main heating source of the gas in the disk. Because MRI turbulence is distributed over the disk thickness, the effective heating source is also distributed, \( Q = Q(z) \). In a thermal steady state, this distributed heating is balanced by the sum of turbulent thermal conduction losses (which, as we shall argue below, effectively just modify \( Q \) by a constant factor of order unity) and radiative losses; we assume that the latter take place via optically thick radiative diffusion (e.g., Shakura & Sunyaev 1973). Solving the corresponding vertical radiative transfer problem yields a fully self-consistent vertical structure for the disk (see Section 3.2).

Having this solution at hand will help us obtain a lower estimate for the coronal power fraction (see Section 5). As mentioned in Section 1.3, our approach to this problem is to regard the Parker instability, which leads to the buoyant rise of magnetic flux tubes into the corona, as a parasitic instability that competes for power with the other (GX94) parasitic instabilities. Numerical evidence for this picture was reported by Miller & Stone (2000) and by Blaes et al. (2007). As we shall show in this paper, the Parker instability growth rate deep inside the disk, at heights \( z \leq H \sim \alpha_{cs}/\Omega \), is smaller than the characteristic GX94 growth rate, roughly by a factor of \( \beta^{1/2} \gg 1 \). This means that in the bulk of the disk, the MRI channel flows are destroyed by the GX94 parasitic instabilities well before the Parker instability can develop. Correspondingly, most of the energy density associated with the MRI channel flows (\( \sim B_{sat}^2/8\pi \sim b^2B_0^2 \)) goes into feeding the resulting local MHD turbulence and is dissipated locally in the disk. However, in the upper layers of the disk, as one approaches the disk’s photosphere, the Parker instability starts to compete effectively with the other parasitic
instabilities and hence one can expect a significant fraction of
the MRI energy in this region to be transported up into the
corona.

Whether the assumptions on which the above arguments are
built are actually valid will require investigation by carefully
designed numerical simulations of MRI turbulence in a stratified
accretion disk, including full gas thermodynamics with radiative
transfer. Since the present paper provides specific predictions for
the disk structure and the coronal power, we hope that it will
motivate such studies in the future.

3. VERTICAL STRUCTURE OF A DISK THREADED
BY A NET VERTICAL MAGNETIC FIELD

3.1. Vertical Profile of Energy Dissipation

The first important conclusion we can derive from the
arguments in the preceding section is that, as long as the parasitic
instabilities dominate, the volumetric dissipation (i.e., heating)
rate should be independent of height \( z \) within the disk.

Indeed, the vertical magnetic field is independent of \( z \), but the
density decreases with height because of stratification. This
means that \( V_A \) increases with \( z \), and so does \( l_{\text{MRI}} \). Then, both the
typical vertical and horizontal extent of MRI channel modes,
of the order of \( l_{\text{MRI}} \), increase with height; however, as long as they remain \( \ll H \), the typical life time of the
channel modes, and the local turn-over timescale of the MRI
turbulence remain constant, of order \( \Omega^{-1} \). The typical horizontal
magnetic field component also remains independent of height,
of the order of \( B_{\text{hor}} \sim B_0 b_{\text{sat}} \). The overall local dissipation rate
per unit volume can then be estimated as

\[
Q \sim \gamma_{\text{MRI}} \frac{B_{\text{sat}}^2}{8\pi} \log^{-1} \left[ \frac{B_{\text{sat}}}{\delta B_{\text{hor}}(0)} \right] \sim \Omega \frac{B_0^2}{8\pi} b_{\text{sat}}^2. \tag{2}
\]

Here, the logarithmic factor represents the number of \( e \)-foldings
needed to grow from some initial perturbation \( \delta B_{\text{hor}}(0) \) to the
saturation amplitude \( B_{\text{sat}} \). In a realistic situation, of course, one
does not expect the magnetic field to return exactly to a pure
vertical-field state; therefore, in the following analysis, we shall
assume that the typical initial perturbation amplitude is of the
order of \( B_0 \) itself, and thus will ignore this logarithmic factor.

Thus, we now see that, because the background vertical field
threading the disk, \( B_0 \), is independent of height \( z \) in the
case under consideration, the above volumetric MRI dissipation
rate is also independent of height, basically unaffected by the
pressure stratification! Importantly, it does not have to follow
the gas density or the gas pressure. That is, the volumetric
dissipation rate does not follow mass, as is sometimes assumed,
but is just constant, uniform in \( z \), at least across the main part of
the disk, where the magnetic field is still dynamically weak. This
expectation is in fact consistent with the results of numerical
simulations (Miller & Stone 2000; Hirose et al. 2006) for the
zero net flux case, which show turbulent magnetic dissipation
that is roughly flat over a few gas scale heights (or even slightly
peaked at about \( z = 2H \)).

It is important to note that since the ultimate source of the heating
is the dissipation of the accretion energy, the fact that \( Q \sim B_0^3 \)
implies that the accretion torque is also proportional to
the square of the mean vertical field. This is in clear contradiction
with the results of a number of numerical simulations reporting
a linear scaling of the accretion stress with \( B_0 \) (Hawley et al.
1995; Pessah et al. 2007; note, however, that Sano et al. 2004
report a \( B_0^{3/2} \) scaling). We believe that this discrepancy may be
attributed to the fact that most of these numerical studies were
not in the asymptotic regime of interest here; i.e., they lacked
the required separation of scales between the disk scale height
(or the vertical box size \( L \)), the prevailing MRI scale \( l_{\text{MRI}} \),
and the dissipative scale (or the grid scale). In other words, these
studies either were not sufficiently resolved, in the sense that
their MRI scale corresponding to the net vertical field \( B_0 \)
were not much larger than the resolution scale, or their MRI scale
was not much lower than the scale height (for stratified simulations)
or the box size (for unstratified simulations), which is equivalent
to saying that \( B_0 \) was not sufficiently small compared with the
pressure-equipartition field (and hence the corresponding MRI
wavelength was not much smaller than \( H \)). In fact, a recent, very
careful numerical study by Bodo et al. (2011) demonstrates that
the accretion torque indeed scales as \( B_0^2 \) in the asymptotic regime
\( \delta \ll l_{\text{MRI}} \ll L \); this result supports the point of view advanced
in the present paper (see also Longaretti & Lesur 2010).

Another important point here is that the specific (i.e., per unit
mass) AMT due to MRI turbulence and, hence, the resulting
effective accretion inflow velocity, are not uniform in height; at
larger heights, the accretion inward drift velocity is higher. This
is similar to the results of numerical simulations of Beckwith
et al. (2009), where this property of MRI turbulence in a stratified
disk provided a mechanism for the efficient inward transport of
vertical magnetic flux (see also Rothstein & Lovelace 2008).

3.2. Vertical Structure of an Optically Thick Disk

Once the vertical profile of turbulent dissipation is established,
we can determine the vertical structure of the disk, similar
to the calculation by Shakura & Sunyaev (1973) for accretion
disks and similar to standard stellar structure calculations.
Here, of course, we are interested in the steady-state profiles of the
gas temperature and density established on timescales longer than
the characteristic orbital time and the cooling time, but shorter
than the overall accretion time (the characteristic radial transport
timescale).

We also neglect external irradiation of the disk by, e.g.,
radiation coming from the inner part of the disk or the central
star.

It is widely recognized that in discussing MRI in the presence
of a mean vertical field, the ratio of the magnetic pressure to the
gas pressure, \( \beta^{-1} \equiv B_0^2/8\pi P \), is an important parameter.
However, it is important to realize that the gas pressure profile
in the disk is not just some arbitrarily prescribed function, but
that it needs to be determined self-consistently using physical
laws, namely energy transport and vertical hydrostatic balance.
This point was made by Hirose et al. (2009), and here we adopt
this point of view. That is, we cannot prescribe the temperature
and density profiles of the disk; they are to be determined as part
of the overall problem. Instead, we can prescribe as fixed
only those quantities that evolve only relatively slowly, namely,
due to the radial transport associated with the accretion process
itself; these quantities are then preserved on the timescales of
interest to us here. In the present problem, there are basically
two such quantities: the disk surface density of mass, \( \Sigma \), and
that of the vertical magnetic flux, i.e., the vertical magnetic
field \( B_0 \). In addition, we can prescribe the local disk rotation
rate \( \Omega \) and the parameters describing the radiative transfer, e.g.,
the scattering opacity \( \kappa \), which for simplicity we assume to be
constant; see below. Thus, \( \Sigma, B_0, \Omega, \) and \( \kappa \) are the only input
parameters determining the disk vertical structure in our model.

It is also convenient to define an important dimensionless
parameter—the disk’s optical depth measured from infinity to
the disk midplane:
\[ \tau_{turb} \equiv \frac{1}{2} \Sigma \kappa, \]
(3)

We assume the disk to be optically thick, \( \tau_{turb} \gg 1 \).

Another important dimensionless parameter in our problem is the midplane plasma \( \beta \) parameter, defined with respect to the superimposed vertical field \( B_0 \):
\[ \beta_0 = \beta(0) \equiv \frac{8 \pi P_0}{B_0^2}, \]
(4)
where \( P_0 \equiv P(z = 0) \). This parameter characterizes the dynamical importance of the vertical magnetic field relative to the gas pressure. In our model, we assume that \( \beta_0 \gg 1 \). Note that since we have not yet computed the midplane plasma pressure, we cannot, at this stage, express \( \beta_0 \) in terms of our principal input parameters. This evaluation will be done at the end of this section.

Now let us solve for the vertical structure based on hydrostatic balance and radiative energy transport. This calculation is similar to the classical analysis by Shakura & Sunyaev (1973) and is also similar to the traditional analyses of the radiative stellar structure except that it is performed in plane, rather than spherical, geometry. One important difference between our analysis and that of Shakura & Sunyaev (1973) is that here we are able, within the framework of our model, to obtain explicit analytical expressions for the dependences of the plasma density \( \rho(z) \) and temperature \( T(z) \) on the height \( z \) within a gas-pressure-dominated disk.

We shall start with the condition of hydrostatic pressure balance. Neglecting magnetic and radiation pressure support compared to the thermal gas pressure, and assuming that gravity is from a central point mass, we have (for a fully ionized hydrogen plasma):
\[ \frac{dP}{dz} = \frac{2}{m_p} \frac{d(\rho k_B T)}{dz} = g_z \rho = -\Omega^2 z \rho. \]
(5)

That is,
\[ \frac{4}{m_p} \frac{d(\rho k_B T)}{dz} = -\Omega^2 \rho. \]
(6)

Next, we need to supplement this equation by the vertical heat balance equation that reads, in a steady state,
\[ Q = \frac{d}{dz} (F_{rad} + F_{turb}), \]
(7)
where \( F_{rad} \) is the radiative energy flux and \( F_{turb} \) is the effective vertical heat flux due to the MRI turbulence itself. The latter can be estimated roughly as a diffusive flux \( F_{turb}(z) = -D_{turb} \frac{d(3nk_B T)}{dz} \), with an effective MRI-turbulent diffusion coefficient \( D_{turb} \sim \lambda_{mri} v_A / 3 \). Then, ignoring factors of order unity, we can write: \( F_{turb}(z) \sim -\lambda_{mri} v_A dP/dz \sim -\Omega^2 V_A^2 dP/dz \). Substituting \( dP/dz \) from Equation (5), we obtain:
\[ F_{turb}(z) \sim \Omega V_A^2 \zeta \rho = \Omega \frac{B_0^2}{4\pi} z, \]
(8)
and hence, since both \( \Omega \) and \( B_0 \) are constant in \( z \),
\[ \frac{dF_{turb}}{dz} \sim \frac{\Omega B_0^2}{4\pi} = \text{const}. \]
(9)

Thus, we see that the effective cooling rate due to the vertical heat transport by the MRI-driven turbulence is basically the same as the MRI turbulent heating rate \( Q \) (see Equation (2)), apart from a constant numerical coefficient of the order of unity: \( dF_{turb}/dz = f_{turb} Q \); in particular, importantly, it is constant in \( z \). This allows us to combine both of the effects of the MRI turbulence—the heating by turbulent dissipation and cooling by vertical turbulent transport—into one single term, the reduced heating rate
\[ Q' = Q - \frac{dF_{turb}}{dz} = \eta Q = \text{const}, \]
where \( \eta \) is the effective MRI-turbulent diffusivity
\[ Q' \equiv Q - \frac{dF_{turb}}{dz} = \eta Q = \text{const}, \]
(10)

In the radiation diffusion approximation, the vertical radiative energy flux is
\[ F_{rad} = \frac{ca}{3} \frac{dT^4}{d\tau} = -\frac{ca}{3} \frac{dT^4}{dz} \lambda_{ph} = -\frac{ca}{3} \frac{dT^4}{\rho k_B dz}, \]
(12)
where \( a = 4\sigma_{SB}/c = \pi^2 k_B^4 / 15 h^3 c^3 = 7.566 \times 10^{-15} \text{ erg cm}^{-3} \text{ K}^{-4} \) is the radiation constant, \( \sigma_{SB} \) is the Stefan–Boltzmann constant, and \( \lambda_{ph} = -dz/d\tau = (\rho k_B)^{-1} \) is the photon mean-free path. This allows us to write our energy balance equation as
\[ \frac{2ca}{3} \frac{dT^4}{dz} = -\rho \kappa Q'. \]
(13)

From Equations (13) and (6), we obtain:
\[ \frac{2ca}{3} \frac{dT^4}{dz} = -\kappa Q' \rho d(z^2) = \frac{\kappa Q'}{\Omega^2} \frac{4}{m_p} (\rho k_B T). \]
(14)

In general, the opacity should be a function of the local plasma parameters such as the temperature, but for simplicity and definiteness in this study we shall assume that it is constant, \( \kappa = \text{const} \) (as it would be for the case of electron scattering, for example). Investigation of more realistic and complicated opacities, including the transition to the free–free opacity \( \sigma_{ff} \sim \rho T^{-7/2} \) in the upper, colder layers of the disk, is left for future work.

We can now immediately integrate Equation (14) to obtain an algebraic relationship between density and temperature:
\[ \rho(z) = \frac{ca\Omega^2 m_p}{6\kappa Q' k_B} T^3(z) \equiv A T^3, \]
where we defined
\[ A = \frac{ca\Omega^2 m_p}{6\kappa Q' k_B} = \frac{2\sigma_{SB} \Omega^2 m_p}{3\pi Q' k_B}, \]
and where we neglected the integration constant by assuming that \( T(z) \) and \( \rho(z) \) effectively become small together at the disk’s photosphere. (We assume that the disk is not subject to any substantial external irradiation (such as that coming from the central star for example), which would invalidate the above boundary condition).
At this point, we can check that a disk with this structure is always convectively stable for a gas adiabatic index equal to \( \gamma_{ad} = 5/3 \). Indeed, because \( \rho(z) \sim T^3(z) \), it follows that

\[
\nabla_{\text{rad}} = \left( \frac{d \log T}{d \log P} \right)_{\text{rad}} = \frac{1}{4} < \nabla_{\text{rad}} = \frac{\gamma_{ad} - 1}{\gamma_{ad}} = 2.
\]

This means that the disk is stable to thermal convection and hence our assumptions regarding the nature of vertical heat transfer are justified.

Substituting Equation (15) into Equation (6), we obtain:

\[
d\frac{T^4}{T^3} = -\frac{m_p\Omega^2}{4k_B} dz^2,
\]

and hence

\[
k_B T(z) = k_B T(0) - \frac{m_p\Omega^2}{16} z^2 = k_B T(0) \left( 1 - \frac{z^2}{z_0^2} \right),
\]

where \( T(0) \) is the midplane temperature and

\[
z_0 \equiv 4c_{t,0}/\Omega.
\]

and \( c_{t,0}^2 \equiv k_B T(0)/m_p \).

Next, using Equation (15), we obtain the density profile:

\[
\rho(z) = \rho(0) \left( 1 - \frac{z^2}{z_0^2} \right)^3 = A T^3(0) \left( 1 - \frac{z^2}{z_0^2} \right)^3.
\]

That is, in this model, the disk has a sharp surface (similar to stars) at a finite height \( z = z_0 \) above midplane. In the following, it will sometimes be convenient to normalize \( z \) by \( z_0 \), i.e., to use the dimensionless height variable

\[
\xi \equiv z/z_0.
\]

The midplane values of temperature and density are then determined from the condition that the surface density has to be equal to a prescribed value:

\[
2 \int_0^{\infty} \rho(z) dz = \Sigma.
\]

Thus, we obtain:

\[
\Sigma = 2z_0 \rho(0) \int_0^1 (1 - \xi^2)^3 d\xi \equiv C_1 z_0 \rho(0).
\]

where \( C_1 \equiv 2 \int_0^1 (1 - \xi^2)^3 d\xi = 32/35 \). Of course, our model is too crude to consider any factors of the order of 1 meaningful, but we will keep them here anyway. Thus, the total optical depth from infinity to the midplane of the disk and the central density are related via:

\[
\tau_{\text{tot}} = \frac{1}{2} \Sigma \kappa = \frac{C_1}{2} z_0 \rho(0) \kappa = \frac{16}{35} z_0 \rho(0) \kappa.
\]

Remembering the definition of \( z_0 \) (Equation (20)), we then obtain the central temperature in terms of the primary input parameters:

\[
T_0 \equiv T(0) = \left[ \frac{35}{128} A^{-1} \Omega (m_p/k_B)^{1/2} \right]^{2/7} = \left[ \frac{105}{128} \frac{Q'}{\Omega} (k_B/m_p)^{1/2} \tau_{\text{tot}}^{7/8} \frac{\Sigma \Omega}{\kappa} \right]^{2/7}.
\]

where we used \( Q' = \eta \Omega B_0^2 b_{\text{sat}}^2/8\pi \) as the fiducial reduced characteristic MRI heating rate to get the last expression. It is interesting to note that the above expression for the midplane disk temperature involves only the vertical magnetic field pressure and the total disk optical depth, plus some universal physical constants. In particular, one can see that \( T(0) \) does not depend on the orbital frequency \( \Omega \). This can be understood qualitatively as follows. The total dissipation rate per unit disk area is of the order of \( \zeta_0 Q' \sim (c_{t,0}/\Omega)(\eta \Omega B_0^2 b_{\text{sat}}^2/8\pi) \sim T^{1/2}(0) \eta (B_0^2 b_{\text{sat}}^2/8\pi) \)—independent of \( \Omega \). On the other hand, in this model this dissipated power is emitted from the disk’s photosphere as thermal emission: \( F \sim \sigma_{SB} T^4 \sim \sigma_{SB} T^4(0) \tau_{\text{tot}}^{-1} \).

Comparing these two expressions, we immediately find the above scaling of \( T(0) \) with \( B_0^2 \) and \( \tau_{\text{tot}} \).

Numerical estimate yields:

\[
T(0) \simeq 220 K \tau_{\text{tot}}^{2/7} \left[ \frac{\eta B_0^2 b_{\text{sat}}^2}{8\pi} \right]^{2/7}.
\]

Thus, in the case of Galactic BH XRBs, for example, with a typical \( B_0 \sim 10^7 \) G, we obtain \( T(0) \simeq 10^6 K \tau_{\text{tot}}^{2/7} \), which is not unreasonable. However, in reality we, of course, do not expect our present model to apply in the inner parts of a BH accretion disk because of the predominance of radiation pressure there.

With the above result for the midplane disk temperature in hand, we can estimate other important disk quantities, e.g., the speed of sound

\[
c_{s,0} = \sqrt{\frac{k_B T(0)}{m_p}} \sim \left[ \frac{105}{128} \frac{Q'}{\Omega} \left( \frac{k_B}{m_p} \right)^4 \tau_{\text{tot}}^{1/7} \frac{\Sigma \Omega}{\kappa} \right]^{1/2}.
\]

and the disk half-thickness:

\[
z_0 = 4c_{t,0}/\Omega = 2 \Omega^{-8/7} \left[ 105 \left( \frac{k_B}{m_p} \right)^4 \tau_{\text{tot}}^{1/7} \frac{\Sigma \Omega}{\kappa} \right]^{1/7}.
\]

Next, the central density becomes

\[
\rho_0 \equiv \rho(0) = A T_0^3 \left[ \frac{35}{32} \frac{\tau_{\text{tot}}}{\kappa} \right]^{3/2} \left[ \frac{\Sigma \Omega}{\kappa} \right]^{1/2}.
\]

The gas pressure profile is given by

\[
P(z) = \frac{2 \rho k_B T}{m_p} = \frac{2 A k_B}{m_p} T^4(z) = \frac{2 A k_B}{m_p} T_0^4 \left( 1 - \frac{z^2}{z_0^2} \right) = P_0 \left( 1 - \frac{z^2}{z_0^2} \right)^4.
\]
where the central gas pressure is
\[ P_0 \equiv P(0) = 2 \rho_0 k_B T_0 / m_p \]
\[ = \frac{35}{128} \Omega \Sigma \sqrt{\frac{k_B}{m_p}} \left[ 105 \frac{Q'}{\Omega} \left( \frac{k_B}{\sqrt{m_p \sigma_{SB}}} \right) \tau_{tot} \right]^{1/7} \]
\[ = \frac{35}{128} \Omega \Sigma \sqrt{\frac{k_B}{m_p}} \left[ 8 \frac{\eta B_0^2}{8\pi} \sqrt{\frac{k_B}{\sqrt{m_p \sigma_{SB}}} \tau_{tot}} \right]^{1/7}. \]  

(32)

It is interesting to note a very weak dependence of \( P_0 \) on \( Q' \) and \( B_0 \), as well as on the opacity \( \kappa \), and a relatively strong dependence on \( \Sigma \) and \( \Omega \).

Likewise, the local plasma-\( \beta \) parameter associated with the vertical field \( B_0 \) is
\[ \beta(z) \equiv 8\pi P / B_0^2 = \beta_0 \left[ 1 - \frac{z}{z_0} \right]^4, \]  

(33)

where
\[ \beta_0 \equiv \beta(0) = \frac{P_0}{B_0^2 / 8\pi} \sim \frac{35}{128} \Omega \Sigma \left( \frac{B_0}{8\pi} \right)^{-6/7} \]
\[ \times \left[ \frac{k_B}{m_p} \left( 105 \frac{\eta B_0^2}{8\pi} \sqrt{\frac{k_B}{\sqrt{m_p \sigma_{SB}}} \tau_{tot}} \right]^{1/7} \right. \]  

(34)

is the midplane value; our model assumes \( \beta_0 \gg 1 \).

These equations complete the calculation of the interior vertical structure of an optically thick MRI-heated gas pressure-dominated accretion disk threaded by a relatively weak vertical magnetic field.

4. BREAKDOWN OF THE MODEL NEAR THE DISK PHOTOSPHERE

The above model of the disk interior vertical structure relies on several assumptions that should be well justified deep within the disk but are expected to break down close to the disk surface, as \( z \to z_0 \). This situation is similar to the one in stellar structure calculations as one approaches the stellar photosphere. These assumptions can be cast in terms of a certain ordering of the relevant physical length scales, which can be summarized as follows:
\[ l_{\text{mid}}(z), \lambda_{\text{ph}}(z) \ll H(z) < \Delta z < z_0. \]  

(35)

Here, \( H(z) \equiv -(d \ln P / dz)^{-1} \) is the disk local gas pressure scale height and \( \Delta z \equiv z_0 - z \) is the geometrical depth, i.e., the distance from the disk edge.

In this section, we shall estimate how rapidly each of these scales varies as one approaches the disk photosphere and will thus estimate at what optical depth the individual components of the above ordering break down (see Figure 1).

Here, we are interested in the region near the disk’s surface:
\[ \Delta z \equiv z_0 - z \ll z_0, \]  

(36)

or, equivalently,
\[ \Delta \xi \equiv \Delta z / z_0 \ll 1. \]  

(37)

In what follows, it will often be convenient to use the optical depth coordinate \( \tau(z) = \int_z^{z_0} \rho(z) dz \), along with \( z \) itself; the two quantities are related to each other by Equation (21). In particular, near the disk edge, \( \Delta \xi \ll 1 \), we have:
\[ \tau(z \sim z_0) \simeq 2 \kappa \rho(0) z_0 \Delta \xi^4 \frac{35}{8} \tau_{tot} \Delta \xi^4. \]  

(38)

Since Equation (21) was derived for the optically thick region, the above expression is valid only for \( \tau \gg 1 \), i.e., for \( 1 \gg \Delta \xi \gg 0.7 \tau_{tot}^{-1/4} \).

It is also interesting to note that because gravity is nearly constant near the disk edge, \( g_z(z \sim z_0) = -\Omega^2 z \simeq -\Omega^2 z_0 \) = const, the vertical hydrostatic pressure balance, Equation (5), yields
\[ P(\tau) \simeq \tau |g_z(z_0)| / \kappa \simeq \tau \Omega^2 z_0 / \kappa. \]  

(39)

That is, the variation of the optical depth \( \tau \) with the geometrical depth \( \Delta z \) is the same as that of the gas pressure in this region and hence \( \tau \) can be used as a proxy for the pressure (c.f., Shakura & Sunyaev 1973; Hirose et al. 2006).

Now we are in a position to check the validity of some of the assumptions underlying our model.

1. Gas scale height, \( H \). First, for our model to be valid near the disk surface, we must require that \( \Delta z \equiv z_0 - z > H(z) \). The gas pressure scale height, using Equation (31), is
\[ H(z) = \left( \frac{d P}{P dz} \right)^{-1} \equiv \frac{z_0 - z^2}{8z} = \frac{z_0}{8} \left( 1 - \frac{\Delta z}{z_0} \right). \]  

(40)

Near the disk edge, \( \Delta \xi \ll 1 \), this equation becomes:
\[ H(z) \simeq 1 / 4 \Delta z. \]  

(41)

so that the assumption \( H(z) \ll \Delta z \) is marginally satisfied.

2. Photon mean-free path, \( \lambda_{\text{ph}} \). Similarly, because of the radiative diffusion approximation, our model is valid only as long as the photon mean-free path is sufficiently short, i.e., \( \lambda_{\text{ph}} \ll H, \Delta z \). Using the above expressions, we have:
\[ \frac{\lambda_{\text{ph}}}{\rho \kappa} = \frac{1}{\rho(0) \kappa} \left( 1 - \frac{1}{2} \Delta z^{2} \right)^{-3} \simeq \frac{\lambda_{\text{ph},0}}{8} \Delta \xi^{-3}. \]  

(42)

where \( \lambda_{\text{ph},0} \) is the photon mean-free path at the disk midplane:
\[ \lambda_{\text{ph},0} = \frac{1}{\rho(0) \kappa} = \frac{16}{35} z_0 \tau_{tot}^{-1}. \]  

(43)

From this, we see that near the disk edge:
\[ \frac{\lambda_{\text{ph}}}{H} \simeq \frac{4 \lambda_{\text{ph}}}{\Delta z} \simeq \frac{\lambda_{\text{ph},0}}{2z_0} \Delta \xi^{-4} = \frac{8}{35} \tau_{tot}^{-1} \Delta \xi^{-4} = \frac{1}{\tau(z)}. \]  

(44)

Thus, we see that the condition that \( \lambda_{\text{ph}}(z) \ll H(z) < \Delta z \) is automatically satisfied as long as we are in the optically thick part of the disk, \( \tau(z) \gg 1 \).

3. MRI scale, \( l_{\text{mri}} \). Next, we want to check that \( l_{\text{mri}} \) is less than \( H \) and \( \Delta z \). We have:
\[ l_{\text{mri}} = \frac{V_\Delta}{\Omega} = l_{\text{mri},0} \left( \frac{\rho}{\rho_0} \right)^{-1/2} = l_{\text{mri},0} (1 - \xi^2)^{-3/2}, \]  

(45)

where
\[ \frac{l_{\text{mri},0}}{2} = \frac{z_0}{2 \sqrt{\rho_0}} \]  

(46)
is the MRI scale at \( z = 0 \).
Near the disk edge, we then have

\[
l_{\text{ini}} \simeq \frac{l_{\text{ini},0}}{2\sqrt{2}} \Delta \zeta^{-3/2} = \frac{z_0}{4\sqrt{2}\rho_0} \Delta \zeta^{-3/2},
\]

or

\[
l_{\text{ini}} \simeq \frac{l_{\text{ini},0}}{2\sqrt{2}} \left( \frac{8}{35} \right)^{-3/8} \simeq 0.31 z_0 \rho_0^{-1/2} \zeta^{-3/8},
\]

where \( \zeta(z) \equiv \frac{\tau(z)}{\tau_{\text{tot}}} \).

Thus, we have

\[
\frac{l_{\text{ini}}}{H(z)} \simeq \frac{4l_{\text{ini}}}{\Delta \zeta} \simeq (2\rho_0)^{-1/2} \Delta \zeta^{-5/2} \simeq 1.8 \rho_0^{-1/2} \zeta^{-5/8},
\]

and so the condition \( l_{\text{ini}}(z) \ll H(z) \) is satisfied as long as \( \Delta \zeta \gg (2\rho_0)^{-1/5} \) or, equivalently, \( \bar{\tau} \gg 2.5 \rho_0^{-4/5} \).

Finally, the calculation presented in the preceding section neglects the magnetic contribution to the vertical pressure balance, which is valid only as long as the local plasma-\( \beta \) is greater than 1. According to Equation (33), in the outer layer of the disk, the plasma-\( \beta \) parameter can be written as

\[
\beta(z \to z_0) = \beta_0 (1 - \zeta^2)^4 \simeq 16\beta_0 \Delta \zeta^4 \simeq \frac{128}{35} \beta_0 \bar{\tau}.
\]

Thus, the assumption \( \beta \gg 1 \) is valid only for \( \bar{\tau} \gg (35/128) \beta_0^{-1} \). At optical depths below this, our model for the disk structure becomes invalid. Note, however, that for \( \beta_0 \gg 1 \), the critical optical depth corresponding to \( \beta = 1 \), \( \bar{\tau} = (128/35) \beta_0^{-1} \), is smaller than the critical optical depth \( \bar{\tau} = 2.5 \beta_0^{-4/5} \) at which \( l_{\text{ini}} \) becomes equal to the gas scale height \( H \). This implies the condition \( l_{\text{ini}} < H \) is more restrictive, and hence more important, than the condition \( \beta \gg 1 \).

5. PARKER INSTABILITY AND CORONAL POWER

Let us now consider the Parker instability in the disk, with the ultimate goal of estimating the coronal fraction of the accretion power. We wish to remind the reader that the spirit of our approach is to consider the Parker instability as a secondary parasitic instability feeding on the horizontal magnetic field of the primary MRI mode (e.g., Tout & Pringle 1992; Miller & Stone 2000; Blaes et al. 2007), and competing with the usual GX94 parasitic instabilities. Our approach is to compare the linear growth rates of these two types of parasitic modes and see under what conditions the Parker instability becomes important.

The linear growth rate of a maximally unstable parasitic instability scales as \( \gamma_{\text{GX94}} \sim \Omega \delta b \), where \( b = B_{\text{hot}}/B_0 \) is the MRI channel flow’s horizontal magnetic field normalized by the vertical magnetic field (GX94). The growth rate of the fastest-growing Parker instability can be estimated as

\[
\gamma_{\text{P}}(z) \sim \frac{V_{\text{A,hot}}}{H(z)} = \frac{b V_A}{H} = \frac{\Omega b}{l_{\text{ini}}}. \tag{51}
\]

Thus, the ratio of the growth rates of the two types of parasitic instabilities is simply proportional to the ratio of the MRI scale \( l_{\text{ini}} \) to the disk local pressure scale height \( H(z) \):

\[
\frac{\gamma_{\text{P}}}{\gamma_{\text{GX94}}} \sim \frac{l_{\text{ini}}}{H}. \tag{52}
\]

As we see, this ratio is small in most of the disk’s volume and hence the GX94 parasitic instabilities dominate over the Parker instability. However, as one approaches the disk surface, the above ratio increases because \( H(z) \) starts to decrease due to decreasing temperature and increasing gravity. Simultaneously, the MRI scale \( l_{\text{ini}} \sim B_0/\sqrt{4\pi \rho} \) increases due to decreasing density. In particular, using our estimate (Equation (49)) near the disk surface, we obtain:

\[
\frac{\gamma_{\text{P}}}{\gamma_{\text{GX94}}} \sim \frac{l_{\text{ini}}}{H} \sim \frac{\beta_0^{-1/2}}{\bar{\tau}}^{-5/8}. \tag{53}
\]
Thus, at a characteristic optical depth of the order of
\[ \tau_P = \tau_{\text{tot}} \beta_0^{4/5} \] (54)
the local MRI scale \( l_{\text{MRI}} \) becomes comparable with \( H \) and hence the growth rate of the secondary Parker instability becomes comparable with that of the GX94 parasitic instabilities. We conjecture that, in the region above the critical depth \( z_P \equiv z(\tau_P) \), the Parker instability starts to compete with other parasitic modes as the main mechanism of destroying MRI channel flows. At the same time, however, we acknowledge that, since the condition \( \lambda_{\text{MRI}} \ll H \) is no longer satisfied in this region, the classical incompressible analysis of GX94 is no longer valid, and the effects of stratification (e.g., Latter et al. 2010) need to be taken into account. Nevertheless, it is important to note that since, according to (Equation (50)), the plasma-\( \beta \) parameter at \( \tau = \tau_P \) is still greater than 1 (it is of the order of \( \beta_P \sim \beta_0^{1/5} \gg 1 \)), there is still a sizable region above \( z_P \) where MRI, although modified by stratification, continues to operate and leads to release of the gravitational energy. It then seems reasonable to suggest that a substantial fraction of the accretion power released in the corona should scale simply as:
\[ f \sim \Delta z_P / z_0 \sim \beta_0^{-1/5}. \] (55)
Since the reduced volumetric MRI energy-release rate \( Q' \) is roughly uniform across the disk (see Section 3.1), we come to the conclusion that the fraction \( f \) of the accretion power released in the corona should scale simply as:
\[ f \sim \Delta z_P / z_0 \sim \beta_0^{-1/5}. \] (56)
Using Equation (34), the dependence of the coronal fraction on the primary input parameters of our model can be expressed as
\[ f \propto \Omega^{-1/5} \Sigma^{-1/5} \left( \frac{B_0^2}{8 \pi} \right)^{6/35} \tau_{\text{tot}}^{1/35}. \] (57)
Thus, within the limitations of our theory, the coronal power fraction is rather insensitive to most of the input parameters.

Recall now that one of the assumptions of our model was that the vertical net field \( B_0 \), while small, is still larger than the field that would be generated by the large-scale MHD dynamo in the zero net flux case. This limits the midplane plasma-\( \beta \) parameter to something of order 100, with the corresponding coronal fraction no less than \( f_{\text{min}} \sim \beta_{0,\text{max}}^{-1/5} \sim 0.4 \).

6. MASS ACCRETION RATE

According to Shakura & Sunyaev (1973), the mass accretion rate \( \dot{M} \) is related to the total energy dissipation in a Keplerian disk via:
\[ 2 \int_0^{z_0} Q' \, dz = 2 Q' z_0 = \frac{3}{8 \pi} \dot{M} \Omega^2. \] (58)
This equation allows us to express the mass accretion rate in terms of the local quantities in our model:
\[ \dot{M} = \frac{64 \pi}{3} \frac{B_0^2 \beta_{\text{sat}}^2}{8 \pi \Omega^2} c_{r,0}. \] (59)
Substituting our Equation (28) for \( c_{r,0} \), we obtain a relationship between \( \dot{M} \) and \( B_0 \), \( \Omega \), and \( \Sigma \) (or \( \tau_{\text{tot}} \)):
\[ \dot{M}[B_0(R), \Omega(R), \Sigma(R)] \simeq 65 \left( \frac{\eta B_0^2 \beta_{\text{sat}}^2}{8 \pi} \right)^{8/7} \times \Omega^{-2} \left( \frac{k_B}{m_p} \right)^{4/7} \left( \frac{\tau_{\text{tot}}}{\sigma_{SB}} \right)^{1/7}. \] (60)
As we can see, the mass accretion rate has a strongly relative dependence on the magnetic field and on the rotation rate, but a rather weak dependence on the total disk column density and optical depth. This effective insensitivity of \( \dot{M} \) to \( \tau_{\text{tot}} \) can be exploited. For example, ignoring the \( \tau_{\text{tot}}^{1/7} \) dependence and taking the \( \Omega(R) \) profile to be Keplerian, \( \Omega_K \sim R^{-3/2} \), we obtain a scaling \( \dot{M} \sim B_0^{16/7}(R) R^3 \). Thus, we see that a stationary accretion regime, \( M(R) = \text{const} \) requires a special particular magnetic field profile, \( B_0^{\text{sat}}(R) \sim R^{-22/16} \), which is in agreement with the results of Shakura & Sunyaev (1973).

In general, Equation (60) gives us an equation that governs a longer-term evolution for the gas surface mass density: \( \Sigma(R, t) = d M / 2 \pi R \, dR \), relating the evolution of the disk density to the radial distribution of the vertical magnetic flux through the disk. Determining the latter, however, is itself an outstanding problem in modern accretion disk research and is still far from being solved (e.g., van Ballegooijen 1989; Lubow et al. 1994; Spruit & Uzdensky 2005; Uzdensky & Goodman 2008; Rothstein & Lovelace 2008; Lovelace et al. 2009; Beckwith et al. 2009; Fromang & Stone 2009).

7. VERTICAL STRUCTURE OF THE DISK WITH ZERO NET FLUX

The picture presented in the previous sections was developed under the assumption that the superimposed net vertical magnetic field, \( B_0 \), while weak compared with the gas pressure inside the disk, nevertheless dominates over the field \( B_{\text{dyn}} \) that would be generated by the MRI-driven turbulent dynamo in the absence of \( B_0 \). Since, as numerical simulations demonstrate, \( B_{\text{dyn}} \) is typically indeed relatively weak, corresponding to a central plasma-\( \beta \) of the order of \( \beta_{\text{dyn}} \sim 100 \) (e.g., Hirose et al. 2006; Davis et al. 2010; Shi et al. 2010), there is indeed a sizable range of parameters where the condition \( B_{\text{dyn}}^2 \ll B_0^2 \ll B_{\text{tot}}^2 \) is satisfied and hence where the above picture applies.

However, it is also interesting to consider the zero net flux case where there is no externally imposed vertical magnetic field (or where this field is weak compared with \( B_{\text{dyn}} \)). This case has attracted a lot of attention in the MRI literature, especially in recent years (e.g., Hirose et al. 2006; Blaes et al. 2007; Fromang & Papaloizou 2007; Pessah et al. 2007; Davis et al. 2010; Shi et al. 2010; Guan & Gammie 2011; Simon et al. 2011). In this case, the magnetic field responsible for driving the MRI is a self-generated turbulent field produced by an MHD dynamo associated with the turbulence itself. One expects both small-scale and large-scale dynamos generating the field on a broad range of spatial scales. The problem of computing analytically the overall spectrum of the resulting magnetic field and its vertical distribution in a stratified disk is an outstanding problem in accretion disk theory. This formidable challenge has so far eluded a full solution, although several important theoretical advances have already been made (e.g., Lesur & Ogilvie 2008; Vishniac 2009; Blackman & Pessah 2009; Blackman 2012). It is, however, reasonable to expect
that the characteristic length scale of the average magnetic pressure (averaged over turbulent fluctuations) is much larger than the dominant MRI wavelength $\lambda_{\text{MRI}}$, and, especially if a strong large-scale MRI dynamo is active, may even be larger than the gas pressure scale height $H$. In that case, in line with the arguments presented in Section 3.1, we may expect the volumetric dissipation rate $Q(z) \sim \Omega B^2/8\pi$ to be also roughly uniform with $z$ within the disk. We note that this conclusion differs drastically from the assumption that $Q(z)$ traces the mass density profile, $Q \sim \rho(z)$, adopted by Shakura & Sunyaev (1973). However, our conclusion is in good agreement with the results of numerical simulations by Hirose et al. (2006) and Shi et al. (2010)—the only numerical studies known to us that address MRI turbulence in a gas pressure-dominated disk with optically thick radiative cooling—which report a flat top-hat energy dissipation profile over three scale heights on each side of the disk. (The flat magnetic energy and dissipation profiles are also consistent with findings by Simon et al. (2011) for isothermal stratified disks.) If this is indeed the case, then we can apply the rest of our analysis presented in Section 3.2 to the zero net flux case, except that $B_{\text{dyn}}^2/8\pi$ should be replaced with $B_{\text{dyn}}^2/8\pi \sim \rho(z)^{1/2} P_0$. In particular, we still recover the $z$-profiles of $\rho$ and $T$ as those given by Equations (19) and (21), e.g., a parabolic profile for the temperature $T(z) = T_0 (1 - z^2/z_0^2)$. We find that these profiles agree well with the actual average profiles measured in numerical simulations by Hirose et al. (2006; see their Figures 2 and 3).

It is important to note that in the zero net flux case the characteristic magnetic field strength can no longer be viewed as an external input parameter but rather has to be determined self-consistently along with the other disk quantities. The only input parameters in this case are $\Omega$ and $\Sigma$, plus the opacity $\kappa$ (which we assume to be constant, as is the case of the dominant electron scattering). In lieu of using the magnetic field as an input parameter, we can just use the result $\beta_0 = \beta_{\text{dyn}} \sim 100$ obtained from numerical simulations. Substituting this on the left-hand side of Equation (34) for $\beta_0$, we can rearrange it to express the characteristic magnetic energy density in terms of $\Omega$, $\Sigma$, and $\kappa$, and then substitute the resulting expression for the magnetic energy into Equations (26)–(30) for $T_0$, $z_0$, $\rho_0$, etc. Ignoring factors of order unity, we obtain:

$$B^2 \sim \left( \frac{k_B}{m_p} \right)^{2/3} \left( \frac{\tau_{\text{tot}}}{\sigma_{\text{SB}}} \right)^{1/6} \rho_{\text{dyn}}^{-7/6} \Omega^{1/6} \Sigma^{7/6},$$

(61)

and then

$$T_0 \sim \left( \frac{k_B}{m_p} \right) \left( \frac{\tau_{\text{tot}}}{\sigma_{\text{SB}}} \right) \rho_{\text{dyn}}^{-1/3} \Omega \Sigma^{1/3},$$

(62)

$$c_s,0 \sim \sqrt{\frac{T_0}{m_p}} \sim \left( \frac{k_B}{m_p} \right) \left( \frac{\tau_{\text{tot}}}{\sigma_{\text{SB}}} \right) \rho_{\text{dyn}}^{-1/6} \Omega^{1/6} \Sigma^{1/6},$$

(63)

$$z_0 \sim c_s,0/\Omega \sim \left( \frac{k_B}{m_p} \right) \left( \frac{\tau_{\text{tot}}}{\sigma_{\text{SB}}} \right) \rho_{\text{dyn}}^{-1/6} \Omega^{-5/6} \Sigma^{5/6},$$

(64)

$$\rho_0 \sim \Sigma/z_0 \sim \left( \frac{k_B}{m_p} \right) \left( \frac{\tau_{\text{tot}}}{\sigma_{\text{SB}}} \right) \rho_{\text{dyn}}^{-1/6} \Omega^{-5/6} \Sigma^{-5/6}.$$ 

(65)

We can then also use Equation (60) to express the mass accretion rate in terms of $\Sigma$ and $\Omega$:

$$M \sim \left( \frac{k_B}{m_p} \right)^{4/3} \left( \frac{\tau_{\text{tot}}}{\sigma_{\text{SB}}} \right)^{1/3} \rho_{\text{dyn}}^{-4/3} \Omega^{-2/3} \Sigma^{4/3}.$$ 

(66)

As one can easily check, these scalings are exactly the same as those obtained by Shakura & Sunyaev (1973; see their Equations (2.16)) in the appropriate limit (gas pressure-dominated disk, constant opacity, etc.).

Thus, we believe that much of the analysis developed in the main part of this paper can also be applied to the zero net flux case, resulting in an essentially very similar vertical disk structure (represented by the temperature and density profiles) and enabling one to recover the classical Shakura & Sunyaev (1973) scalings for the key disk parameters. While in this paper we do not want to make any strong claims regarding the coronal fraction $f$ of the accretion power in the zero net flux case, we nevertheless note that a typical value of $f \sim 100$ expected in this case implies, in conjunction with Equation (56), a relatively high (tens of percent, which is consistent with AGN observations) and universal (independent of any system parameters) value of $f$.

8. CONCLUSIONS

In this paper, we considered the problem of the vertical thermal structure of a thin gas pressure-dominated accretion disk heated by the dissipation of MRI turbulence and cooled by optically thick radiative cooling. We also considered the question of the fraction of the overall accretion energy that is transported by buoyantly rising magnetic loops in the tenuous corona lying above the disk.

In the main part of the paper, we focused on the non-zero net vertical flux case, in which the disk is threaded by an externally imposed vertical magnetic field, $B_0$. Because we are interested in an MRI-active disk, we considered the case where this field is relatively weak, $B_0^2/8\pi \ll P_0$, where $P_0$ is the gas pressure in the middle of the disk. At the same time, we assumed that the mean vertical field $B_0$ is stronger than the dynamo-generated magnetic field $B_{\text{dyn}}$, which, according to numerical simulations, is expected to be of order $B_{\text{dyn}}^2/8\pi \sim 10^{-2} P_0$. We argued that, under these assumptions, the volumetric heating rate, $Q$, due to the dissipation of the MRI turbulence (which we tentatively associate with the disruption of MRI channel modes by parasitic instabilities; GX94) should scale as $\Omega B_0^2/8\pi$; in particular, it should be independent of the height $z$ inside most of the disk. It is particularly important that the $z$-profile of $Q$ does not trace the profiles of the gas density or pressure. Making use of this finding and assuming, in addition, that the opacity $\kappa$ is also constant with $z$, we then were able to solve analytically the combined set of equations governing the vertical structure of the disk—the hydrostatic pressure balance, the energy conservation, and the optically thick radiative transfer equation. As a result, we were able to obtain the $z$ profiles of the gas temperature and density inside the disk: $T(z) = T_0 (1 - z^2/z_0^2)$, $\rho(z) = \rho_0 (1 - z^2/z_0^2)^3$, where $T_0$ and $\rho_0$ are the values at the disk midplane, $z = 0$, and $z_0$ is the effective thickness of the disk (see Equations (19) and (21)).

We were also able to determine all the key disk parameters such as the midplane temperature, density, and pressure and the disk thickness $z_0$, in terms of the governing input parameters in this problem: the external vertical magnetic field $B_0$, the surface density $\Sigma$, the disk rotation rate $\Omega$, and the opacity $\kappa$. This enabled us to evaluate the scaling of the mass accretion rate $M$ with these parameters (see Section 6) and formulate the equation...
governing the time evolution of the radial distribution of mass across the disk, \( \Sigma(r, t) \).

We then also examined how and where various assumptions on which our disk model is based—\( \beta(z) \gg 1, l_{\phi} \ll H \), etc.—break down as one approaches the disk surface. This enabled us to address the question of the coronal power fraction. In our view, the coronal power is governed by the competition between various parasitic instabilities disrupting the primary MRI channel modes: the GX94 instabilities leading to fully developed local MHD turbulence whose dissipation heats the disk locally, and the Parker instability that pumps the Poynting flux into the ADC by buoyantly rising magnetic loops. The magnetic energy of these loops is then dissipated by reconnection in the corona. In our model, the resulting corona fraction turns out to be relatively insensitive to most input parameters, scaling as \( f \sim \beta_0^{-1/5} \). The practical consequence of this conclusion is that it is difficult to avoid a sizable (tens of percent) fraction of the accretion power being released in the overlying corona. The part of this dissipated energy that goes to the electrons powers the observed coronal emission, whereas a significant part of the ion coronal energy may actually be transported back to the disk by ions streaming along the closed field lines. This energy then gets deposited in the dense disk through ion–ion collisions. Finally, some of the energy dissipated by reconnection involving open field lines may power outflows (winds and jets) along these open field lines.

Finally, in Section 7, we turned our attention to the case of a disk with a zero net flux. This is the case where the next externally imposed vertical field \( B_0 \) is either absent altogether or small compared to the MRI dynamo-generated field \( B_{\text{dyn}} \). In this case, the entire steady-state vertical structure of a thin gravitationally stratified disk should be determined solely by \( \Sigma, \Omega, \) and \( \kappa \) since there are no other parameters in the problem (ignoring external irradiation). As numerical simulations show, in this case one expects an effective large-scale MRI dynamo to produce a sizable large-scale (comparable to or larger than the disk pressure scale height \( H \)) magnetic field \( B_{\text{dyn}}^2/8\pi \sim \beta_0^{-1} P_0 \) with \( \beta_0 \sim 10^2 \). We argued that since the magnetic energy density is roughly uniform in \( z \) inside the disk (perhaps up to the equipartition height at which \( P(z) = B_{\text{dyn}}^2/8\pi \gg 10^2 P_0 \)), then the volumetric energy dissipation rate of the MHD turbulence should also be roughly uniform, just as it is for the finite net flux case. This picture is in fact supported by numerical simulations of stratified disks (e.g., Miller & Stone 2000; Hirose et al. 2006). We then argued that we can apply the analysis developed in Section 3.2 to the zero net flux case. As a result, we recover the same vertical profiles for the gas temperature and density, i.e., \( T(z) = T_0(1 - z^2/z_0^2) \), and \( \rho(z) = \rho_0(1 - z^2/z_0^2)^3 \) (see Equations (19) and (21)). These profiles seem to be a good agreement with those obtained in full numerical simulations by Hirose et al. (2006). Furthermore, we find our scalings of the main accretion disk parameters (the midplane values of the temperature and density, the disk thickness \( z_0 \), and the mass accretion rate \( M \), etc.) to be the same as those in the classic paper by Shakura & Sunyaev (1973) for the regime under consideration (constant opacity, gas pressure-dominated disk).

While the theoretical model presented in this paper is successful at being able to provide a set of concrete, physically motivated predictions, it still has to be viewed just as an idealized conceptual toy model. It relies on a number of simplifying assumptions—e.g., gas pressure domination, a single constant opacity, no external irradiation—that preclude its direct application to real astrophysical systems. For example, the inner regions of optically thick BH accretion disks are dominated by radiation pressure, which is completely ignored in the present study. In addition, the magnetic field generated by the MRI dynamo is either completely ignored (in the first half of the paper, where we consider a disk threaded by an external magnetic field) or is treated in a simplified fashion (in Section 7). Perhaps for these reasons the present model is not able to address the questions of spectral state transitions in Galactic BH binaries and, in particular, explain the relatively low level of X-ray coronal activity in the high-soft state. Future theoretical studies should develop generalizations of the present work to take into account radiation pressure, more realistic opacities, and, perhaps, external irradiation. One can also envision a more rigorous and, perhaps, more accurate model of MRI saturation and MHD turbulent dissipation: the development of such a model would benefit from detailed direct comparisons with numerical simulations.

In conclusion, we believe that the future of accretion disk studies lies in the incorporation of more realistic radiation and thermal physics—i.e., the more accurate treatment of disk thermodynamics and radiative cooling processes. We hope that the present paper will help stimulate and pave the way for such studies using advanced numerical simulations, along the lines of the studies by Hirose et al. (2006, 2009), Blaes et al. (2007, 2011), and Shi et al. (2010). Eventually, we hope, this line of research will reach the state of maturity at which it is able to provide meaningful predictions for observations and explain important observational facts.

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