Kernels and point processes associated with Whittaker functions

Gordon Blower* and Yang Chen

Department of Mathematics and Statistics, Lancaster University, LA1 4YF, England, UK, and
Department of Mathematics, University of Macau, Avenda da Universidade, Taipa, Macau, China

* Corresponding author’s email: g.blower@lancaster.ac.uk

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Abstract. This article considers Whittaker’s function \( W_{\kappa,\mu} \) where \( \kappa \) is real and \( \mu \) is real or purely imaginary. Then \( \varphi(x) = x^{-\mu-1/2}W_{\kappa,\mu}(x) \) arises as the scattering function of a continuous time linear system with state space \( L^2(1/2, \infty) \) and input and output spaces \( \mathbb{C} \). The Hankel operator \( \Gamma_{\varphi} \) on \( L^2(0, \infty) \) is expressed as a matrix with respect to the Laguerre basis and gives the Hankel matrix of moments of a Jacobi weight \( w \). The operation of translating \( \varphi \) is equivalent to multiplying \( w \) by an exponential factor to give \( w_\varepsilon \). The determinant of the Hankel matrix of moments of \( w_\varepsilon \) satisfies the \( \sigma \) form of Painlevé’s transcendental differential equation \( PV \). It is shown that \( \Gamma_{\varphi} \) gives rise to the Whittaker kernel from random matrix theory, as studied by Borodin and Olshanski (Comm. Math. Phys. 211 (2000), 335–358).

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1. Introduction

The Whittaker function \( W_{\kappa,\mu} \) is the solution of the second order linear differential equation

\[
y'' + \left( \frac{-1}{4} + \frac{\kappa}{x} + \frac{1/4 - \mu^2}{x^2} \right)y = 0
\]

(1.1)

that is asymptotic to \( x^{\kappa}e^{-x/2} \) as \( x \to \infty \) through real values, and possibly has a logarithmic singularity at \( x = 0 \). The general solution of (1.1) is given by linear combinations of the Whittaker functions \( M_{\kappa,\pm\mu} \). See [14, 15, 36 p. 343] for basic definitions and properties, such as \( W_{\kappa,\mu} = W_{\kappa,-\mu} \). We consider the case in which \( \kappa \) is real, and \( \mu \) is either real or purely imaginary; hence \( W_{\kappa,\mu}(x) \) is real for all \( x > 0 \). In random matrix theory [4, 5, 6, 23], kernels such as

\[
K(x, y) = ((\kappa - 1/2)^2 - \mu^2) \sqrt{xy} \frac{W_{\kappa-1,\mu}(x)W_{\kappa,\mu}(y) - W_{\kappa,\mu}(x)W_{\kappa-1,\mu}(y)}{x-y},
\]

(1.2)
provide integral operators on $L^2((0, \infty); \mathbb{C})$, and are associated with determinantal random point fields. The purpose of this note is to provide some transparent proofs of some basic properties of these kernels and their associated determinantal random point fields. We show that the Whittaker kernels are closely related to systems of orthogonal polynomials for a Pollaczek–Jacobi type weight.

The classical Jacobi weight function [30] can be translated onto $[0, 1]$ to become

$$w(x) = x^b(1 - x)^a \quad (0 < x < 1),$$

where $a, b > -1$. We will consider deformations of this weight which arise from multiplying by a positive factor, for instance $e^{-t/x}$ where $t$ is the deformation parameter.

**Definition** With $a, b > -1$ and $t \geq 0$, we introduce

$$D_N(t; a, b) = \det \left[ \int_0^1 x^{j+k} x^b(1 - x)^a e^{-t/x} \, dx \right]_{j,k=0}^{N-1}$$

or equivalently

$$D_N(t; a, b) = \frac{1}{N!} \int_{[0,1]^N} \prod_{1 \leq j < k \leq N} (x_j - x_k)^2 \prod_{\ell=1}^N x_{\ell}^b(1 - x_{\ell})^a e^{-t/x_{\ell}} \, dx_{\ell}. \quad (1.5)$$

In section two, we introduce the notion of a linear system and scattering function. In section three, we show how the determinants $D_N(t; a, b)$ are related to Whittaker functions and state the Painlevé transcendental differential equations [16, 20, 24] that they satisfy. In section four, we introduce the Whittaker kernels and obtain factorization theorems in the style of Tracy and Widom [32, 33, 34], which express Fredholm determinants of the kernels as determinants of Hankel integral operators on $L^2(0, \infty)$. These determinants do not involve $N$ directly. The case $\kappa = a + 1/2$ and $\mu = a$ is of particular interest, as in [25], and we consider this in sections 5 and 6.

A stochastic point process is a probability measure on the space of point configurations. See section 5.4 of [11] for the general definition of finite point processes. The process is said to be determinantal when the correlation functions are given by Fredholm determinants, as follows.

**Definition** (Determinantal point process) Let $S$ be a continuous kernel on $(0, \infty)$ such that

(i) $S(x, y) = S(y, x)$ for all $x, y \in (0, \infty)$;

(ii) the integral operator with kernel $S$ satisfies the operator inequality $0 \leq S \leq I$ as self-adjoint operators on $L^2(0, \infty)$;
We define the scattering function \( \varphi \) as follows: When \( \int S(x,y)I_{[u,v]} \) is trace class, where \( I_{[u,v]} \) denotes the indicator function of \([u,v]\). Then \( S \) gives rise to a determinantal point process on \((u,v)\) in Soshnikov’s sense [7, 29]. Let \( T \) satisfy \( I + T = (I - S)^{-1} \) as operators on \( L^2(u,v) \). Then the probability that there are exactly \( N \) points in the realization, one in each subset \( dx_j \) for \( j = 1, \ldots, N \) and none elsewhere is equal to

\[
\det(I + T)^{-1} \det[T(x_j,x_k)]_{j,k=1}^{N} dx_1 \ldots dx_N,
\]

where the \( x_j \in [0,1] \).

2. Linear systems for the Whittaker functions

Definition (Linear systems) Let \( H \) be a complex separable Hilbert space known as the state space, and \( H_0 \) a finite dimensional complex Hilbert space which serves as the input and output space. Let \( \mathcal{L}(H) \) be the space of bounded linear operators on \( H \) with the operator norm. A linear system \((-A,B,C)\) consists of:

(i) \(-A\), the generator of a \( C_0 \) (strongly continuous) semigroup \( \{e^{-tA} \}_{t \geq 0} \) of bounded linear operators on \( H \) such that \( \|e^{-tA}\| \leq Me^{-\omega_0 t} \) for all \( t \geq 0 \) and some \( M, \omega_0 \geq 0 \);

(ii) \( B : H_0 \to H \) a bounded linear operator;

(iii) \( C : H \to H_0 \) a bounded linear operator.

We define the scattering function \( \varphi \in C((0,\infty); \mathcal{L}(H_0)) \) by \( \varphi(t) = Ce^{-tA}B \), and the Hankel operator with scattering function \( \varphi \) by

\[
\Gamma_{\varphi} f(x) = \int_{0}^{\infty} \varphi(x+y)f(y) \, dy \quad (f \in L^2((0,\infty); H_0)),
\]

as in [26]. Note that if the integral \( \int_{0}^{\infty} t\|\varphi(t)\|^2_{\mathcal{L}(H_0)} \, dt \) converges, then \( \Gamma_{\varphi} \) defines a Hilbert–Schmidt operator. In particular, this holds if \( \omega_0 > 0 \).

We also introduce \( R_\varepsilon : H \to H \) by

\[
R_\varepsilon = \int_{0}^{\infty} e^{-tA}B_\varepsilon C_\varepsilon e^{-tA} \, dt,
\]

and this integral plainly converges whenever \( \omega_0 > 0 \). See [3] for some related results.

For Whittaker functions, the basic linear system is the following. Let \( \varepsilon > 0 \), \( H_0 = C \) and \( H = L^2((1/2, \infty); C) \) and \( \mathcal{D}(A) = \{ f(s) \in H : sf(s) \in H \} \). Then

\[
A : f(s) \mapsto sf(s), \quad (f \in \mathcal{D}(A));
\]

\[
B_\varepsilon : b \mapsto e^{-\varepsilon s}(s + 1/2)^{(\kappa+\mu-1/2)/2}(s - 1/2)^{(-\kappa+\mu-1/2)/2}b, \quad (b \in C);
\]

\[
C_\varepsilon : f(s) \mapsto \int_{1/2}^{\infty} e^{-\varepsilon s}(s + 1/2)^{(\kappa+\mu-1/2)/2}(s - 1/2)^{(-\kappa+\mu-1/2)/2} \frac{f(s)}{\Gamma(\mu - \kappa + 1/2)} \, ds, \quad (f \in \mathcal{D}(A)).
\]
2.1 Lemma (i) Then the scattering function is

\[ \varphi(\varepsilon)(x) = \frac{W_{\kappa,\mu}(x + 2\varepsilon)}{(x + 2\varepsilon)^{\mu + 1/2}}. \]  

(ii) Suppose that \( \kappa, \mu \in \mathbb{R} \) and \(-\kappa + \mu + 1/2 > 0\). Then \( R_\varepsilon \) is self-adjoint and nonnegative.

(iii) Suppose that \( \Re \mu > -\kappa - 1/2 \), and either \( \varepsilon > 0 \), or \( \Re \mu < 1/2 \) and \( \varepsilon = 0 \). Then \( R_\varepsilon \) is trace class and

\[ \det(I - \lambda R_\varepsilon) = \det(I - \lambda \Gamma_{\varphi(\varepsilon)}) \quad (\lambda \in \mathbb{C}). \]  

\[ \varphi(\varepsilon)(x) = \frac{W_{\kappa,\mu}(x + 2\varepsilon)}{(x + 2\varepsilon)^{\mu + 1/2}}. \]  

Proof. (i) This is by direct computation, in which one uses a representation formula

\[ \frac{W_{\kappa,\mu}(x)}{x^{\mu + 1/2}} = \int_{1/2}^{\infty} e^{-sx}(s + 1/2)^{\kappa + 1/2}(s - 1/2)^{-\kappa + 1/2} \frac{ds}{\Gamma(-\kappa + \mu + 1/2)} \]  

for the Whittaker function from [13, 15].

(ii) The operator \( R_\varepsilon \) on \( L^2(0, \infty) \) is represented by the kernel

\[ \frac{e^{-\varepsilon(s + 1/2)^{\kappa + 1/2}}}{(s - 1/2)^{\kappa + 1/2}} \frac{e^{-\varepsilon(t + 1/2)^{\kappa + 1/2}}}{(t - 1/2)^{\kappa + 1/2}} \frac{1}{(s + t)^{\kappa + 1/2}}. \]  

The first two factors are multiplication operators, while the final factor in Carleman’s operator \( \Gamma \) on \( L^2(0, \infty) \) with kernel \( 1/(x + y) \), as discussed in [26, p.440]. Now \( \Gamma \) is non negative as an operator, hence \( R_\varepsilon \) is also non negative by (2.7). See [17] for more analysis of operators of this form.

(iii) One can introduce Hilbert–Schmidt operators \( \Xi, \Theta : L^2(0, \infty) \to L^2(1/2, \infty) \) by \( \Theta f = \int_0^\infty e^{-tA} C^\dagger f(s) \, ds \) and \( \Xi f = \int_0^\infty e^{-tA} B_{\varphi(\varepsilon)} f(s) \, ds \) such that \( \Gamma_{\varphi(\varepsilon)} = \Theta^\dagger \Xi \) and \( R_\varepsilon = \Xi \Theta^\dagger \); see [3] for more details. The kernel of \( \Xi \) is

\[ e^{-\varepsilon s - st}(s + 1/2)^{\kappa + 1/2}(s - 1/2)^{-\kappa + 1/2} \]  

which is Hilbert–Schmidt, hence \( R_\varepsilon \) and likewise \( \Gamma_{\varphi(\varepsilon)} \) are trace class with

\[ \det(I - \lambda R_\varepsilon) = \det(I - \lambda \Xi \Theta^\dagger) = \det(I - \lambda \Theta^\dagger \Xi) = \det(I - \lambda \Gamma_{\varphi(\varepsilon)}). \]  

For \( \alpha > 0 \), let \( L_n^{(\alpha)}(x) \) be the monic generalized Laguerre polynomial of degree \( n \), defined by

\[ L_n^{(\alpha)}(x) = (-1)^n \frac{e^x}{x^\alpha} \frac{d^n}{dx^n} (x^{n+\alpha} e^{-x}); \]  

then the \( n \)th generalized Laguerre function is

\[ \phi_n^{(\alpha)}(x) = e^{-x/2} x^\alpha L_n^{(\alpha)}(x)/\Gamma(n + 1). \]
2.2 Lemma Let $\varphi$ be the scattering function $\varphi(x) = W_{\kappa, \mu}(x)x^{-\mu-1/2}$, and let $w$ be the translated Jacobi weight

$$w(\xi) = \xi^{\mu-\kappa-1/2}(1-\xi)^{2\alpha-2\mu+1} \quad (0 < \xi < 1).$$

Then the operation of $\Gamma_{\varphi}$ on the generalized Laguerre basis is represented by a matrix of moments for the weight $w$, so

$$\langle \Gamma_{\varphi} \phi_n^{(\alpha)} , \phi_\ell^{(\alpha)} \rangle_{L^2(0,\infty)} = \frac{\Gamma(n+1+\alpha)}{\Gamma(n+1)} \frac{\Gamma(\ell+1+\alpha)}{\Gamma(\ell+1)} \int_0^1 \xi^{\ell+n} w(\xi) d\xi. \quad (2.12)$$

Proof. This is suggested by [27]. The Laplace transform of $\phi_n^{(\alpha)}$ satisfies

$$\hat{\phi}_n^{(\alpha)}(s) = \frac{1}{\Gamma(\alpha+n)} (s-1/2)^n \frac{(s+1/2)^{n+1+\alpha}}{\Gamma(s+1)},$$

as one checks by repeatedly integrating by parts. Using the representation formula for $\varphi$ as in (2.6), one can express $\langle \Gamma_{\varphi} \phi_n^{(\alpha)} , \phi_\ell^{(\alpha)} \rangle$ as an integral with respect to $s$ over $(1/2, \infty)$. By changing variables to $\xi = (s-1/2)/(s+1/2)$, one obtains the integral of moments with respect to the weight $w$.

We now show how the leading minors of the Hankel operator $\Gamma_{\varphi}$ are related to the Jacobi unitary ensemble. The joint probability density function of the Jacobi unitary ensemble on $[0,1]^N$ as in [23, 34] is

$$\frac{1}{N! \Gamma(\kappa + \mu + 1)^N} \prod_{j=0}^{N-1} \frac{\Gamma(j+1+\alpha)}{\Gamma(j+1)} \prod_{0 \leq j < k \leq N-1} (x_j-x_k)^2 \prod_{j=0}^{N-1} w(x_j). \quad (2.14)$$

Let $\Delta_N(t)$ be the multiple integral

$$\Delta_N(t) = \frac{1}{N! \Gamma(\kappa + \mu + 1)^N} \prod_{j=0}^{N-1} \frac{\Gamma(j+1+\alpha)}{\Gamma(j+1)} \int_{[t,1]^N} \prod_{0 \leq j < k \leq N-1} (x_j-x_k)^2 \prod_{j=0}^{N-1} w(x_j) dx_j,$$

as in Chen and Zhang [10].

2.3 Proposition (i) The leading minors of the determinant of $\Gamma_{\varphi}$ satisfy

$$\det \left[ \langle \Gamma_{\varphi} \phi_n^{(\alpha)} , \phi_\ell^{(\alpha)} \rangle \right]_{\ell,n=0}^{N-1} = \Delta_N(0). \quad (2.16)$$

(ii) Let $x_0, \ldots, x_{N-1}$ be a sample of $N$ points from the Jacobi unitary ensemble. Then the probability of the event $[x_j \geq t \text{ for all } j]$ equals $\Delta_N(t)/\Delta_N(0)$. 

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Proof. (i) This identity follows from Lemma 2.2 and the Heine–André identity from [30].

(ii) Let \( P_n^{(a,b)}(x) \) be the monic Jacobi polynomial of degree \( n \) for the weight \( w(x) = x^b(1-x)^a \) on \([0,1]\), where we choose \( b = \mu + \kappa - 1/2 \) and \( a = 2\alpha - 2\mu + 1 \). See [30, page 58]. Then with the constants \( \gamma_j = \int P_j^{(a,b)}(x)^2 w(x) dx \), the kernel

\[
J_N(x, y) = \sum_{j=0}^{N-1} \frac{P_j^{(a,b)}(x)P_j^{(a,b)}(y)}{\gamma_j}
\]

(2.17)
defines a self-adjoint operator on \( L^2(w, [0,1]) \) such that \( 0 \leq J_N \leq I \). Hence

\[
\frac{\Delta_N(t)}{\Delta_N(0)} = \det(I - J_N(I_{[0,t]})).
\]

(2.18)

2.4 Proposition Let

\[
H_N(t) = t(1-t)\frac{d}{dt} \log \Delta_N(t).
\]

(2.19)

Then \( \sigma(t) = H_N(t) - d_1 - td_2 \) satisfies the \( \sigma \) form of Painlevé’s transcendental differential equation PVI, so

\[
\sigma'[t(t-1)\sigma'']^2 + [2\sigma'(t\sigma'-\sigma')-(\sigma')^2-\nu_1\nu_2\nu_3\nu_4]^2 = (\sigma'+\nu_1^2)(\sigma'+\nu_2^2)(\sigma'+\nu_3^2)(\sigma'+\nu_4^2)
\]

(2.20)

where \( \nu_1 = (a+b)/2, \nu_2 = (b-a)/2, \nu_3 = \nu_4 = (2N + a + b)/2 \) with initial conditions \( \sigma(0) = d_2 \) and \( \sigma'(0) = d_1 \).

Proof. See [10], and [20].

3. Determinant formulas for the Pollaczek–Jacobi type weight

In this section, we consider the Pollaczek–Jacobi type weights, and show that translating the scattering function \( \varphi \) has the same effect as deforming the weights through multiplication by \( e^{-2t/x} \). The \( m^{th} \) moment of the Pollaczek–Jacobi weight is defined to be

\[
\mu_m(t; a, b) = \int_0^1 x^m x^b(1-x)^a e^{-t/x} \, dx \quad (m = 0, 1, \ldots).
\]

(3.1)

3.1 Proposition The moments satisfy

\[
\mu_m(t; a, b) = \Gamma(a+1)e^{-t/2}t^{(b+m)/2}W_{-(2a+b+m+2)/2, -(b+m+1)/2}(t).
\]

(3.2)

Proof. Making the change of variable \( 1/x = s + 1/2 \) in (2.6) we have

\[
\mu_m(t; a, b) = e^{-t/2} \int_{1/2}^{\infty} e^{-st}(s - 1/2)^{-\kappa + \mu - 1/2}(s + 1/2)^{\kappa + \mu - 1/2} \, ds
\]

(3.3)
with \( \kappa = -(2a + b + m + 2)/2 \) and \( \mu = -(b + m + 1)/2 \).

See [15, 3.4712, page 367]. With the change of integration variable \( x \mapsto 1/\xi \),

\[
\mu_m(t; a, b) = \int_1^\infty \xi^{-2-a-b-m}(\xi - 1)^a e^{-t\xi} d\xi.
\]

The translation operation \( \varphi(t) \mapsto \varphi(\varepsilon)(t) = \varphi(t + 2\varepsilon) \) replaces \( w \) by

\[
w_\varepsilon(x) = w(x) \exp \left( -2\varepsilon \left( \frac{1}{x} - \frac{1}{2} \right) \right).
\]

For this problem, we shall be concerned with

\[
D_N(\varepsilon) = C_N(\varepsilon) D_N(2\varepsilon; a, b)
\]

where \( D_N(t; a, b) \) is defined in (1.4) and

\[
C_N(\varepsilon) = e^{\varepsilon N}[\Gamma(\kappa - \mu + 1/2)]^{-N} \prod_{j=0}^{N-1} \frac{\Gamma(j + \alpha + 1)}{\Gamma(j + 1)}.
\]

As in Lemma 2.2 and Proposition 2.3, we have

\[
D_N(\varepsilon) = \det \left[ \langle \Gamma(\varphi(\varepsilon) \phi_n^{(\alpha)}, \phi_\ell^{(\alpha)} \rangle \right]_{\ell, n=0}^{N-1}.
\]

Let the quantity \( \tilde{H}_N(t) \) be defined as follows

\[
\tilde{H}_N(t) = t \frac{d}{dt} \log D_N(t) - N(N + b + a).
\]

**3.2 Theorem** Then \( \tilde{H}_N(t) \) satisfies the Jimbo–Miwa–Okamoto \( \sigma \) form of Painlevé’s PV for a special choice of parameters, so

\[
(t\tilde{H}_N')^2 = -4t(\tilde{H}_N')^3 + (\tilde{H}_N')^2(4\tilde{H}_N + (b + 2a + t)^2 + 4N(N + a + b) - 4a(b + a)) + 2\tilde{H}_N'(-(b + 2a + t)\tilde{H}_N - 2Na(N + b + a)) + (\tilde{H}_N')^2.
\]
Proof. This was found in Chen and Dai [9, p. 2161].

3.3 Remarks (i) Chen and Dai [9, Theorem 6.1] also show that \((\tilde{H}_N)_{N=1}^\infty\) satisfies a second order nonlinear difference equation.

(ii) \(D_N(t; a, b)\) is the Wronskian determinant of

\[
\{\mu_{2N-1}(t; a, b), \mu'_{2N-1}(t; a, b), \ldots, \mu_{2N-1}^{(N-1)}(t; a, b)\};
\]  

(3.11)

thus \(\mu_{2N-1}(t; a, b)\) determines \(D_N(t; a, b)\).

(iii) The Painlevé V equation has previously appeared in various ensembles in random matrix theory. Tracy and Widom obtained \(PV\) in [34] from the Laguerre ensemble, and also from the Bessel ensemble from [32] which is associated with hard edge distributions. In [34], they also considered the ensembles \(U_N\) of \(N \times N\) complex unitary matrices \(U\) with Haar measure, and obtained \(PV\) from the distribution of trace(\(U\)). Remarkably, this is related to the uniform measure on the symmetric group \(S_N\) of permutations as \(N \to \infty\). Borodin and Olshanski considered the pseudo-Jacobi ensemble and obtained \(PV\) from a Fredholm determinant associated with the Whittaker functions \(M_{\kappa,\mu}(1/x)\) with \(\kappa\) and \(\mu\) real as in [6]. Their results have applications to the infinite dimensional unitary group \(U_\infty\). Lisovyy considered how the determinant for the hypergeometric kernel degenerates to the determinant for the Whittaker kernel, and realised \(PV\) as a limiting case of \(PVI\); see [22, section 10].

The modified Bessel function as in (6.1) satisfies \(K_\mu(z) = (2z/\pi)^{-1/2}W_{0,\mu}(2z)\) as in [28]; so in view of these results, it is fitting that \(PV\) should emerge from the Whittaker kernel.

4. The matrix Whittaker kernel

Borodin and Olshanski [5, 6] have considered kernels of the form (1.2). We can factorize the kernel in terms of products of Hankel operators, by analogy to the results of [31, 34]. In section 5, we see that the case \(\kappa = a + 1/2\) and \(\mu = a\) is of particular interest.

4.1 Proposition The kernel satisfies

\[
\sqrt{zw}W_{\kappa,\mu}(z)W_{\kappa-1,\mu}(w) - W_{\kappa-1,\mu}(z)W_{\kappa,\mu}(w)
\]

\[
= \int_1^\infty \left(\sqrt{zw}W_{\kappa,\mu}(sz)\sqrt{w}W_{\kappa-1,\mu}(sw) + \sqrt{zw}W_{\kappa-1,\mu}(sz)\sqrt{w}W_{\kappa,\mu}(sw)\right)\frac{ds}{2s}. \tag{4.1}
\]

Proof. Note that the left-hand side is a continuously differentiable function of \((z, w) \in (0, \infty)^2\) and that the left-hand side converges to zero as \(z \to \infty\) or \(w \to \infty\).
In the following proof we use that matrices

\[ I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad J = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \quad \tilde{J} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}. \] (4.2)

Combining the differential equation (1.1) with the recurrence relation

\[
z \frac{d}{dz} W_{\kappa,\mu}(z) = (\kappa - z/2) W_{\kappa,\mu}(z) - (\mu^2 - (\kappa - 1/2)^2) W_{\kappa-1,\mu}(z),
\]
we obtain the matrix differential equation

\[ z \frac{d}{dz} W = ((1/2)I + \Omega(z)) W, \quad (4.4) \]

\[ W(z) = \begin{bmatrix} W_{\kappa,\mu}(z) \\ W_{\kappa-1,\mu}(z) \end{bmatrix}, \quad \Omega(z) = \begin{bmatrix} \kappa - 1/2 - z/2 & (\kappa - 1/2)^2 - \mu^2 \\ -1 & 1/2 - \kappa + z/2 \end{bmatrix} \] (4.5)

in which \( \text{trace}(\Omega) = 0 \). The eigenvalues of \((1/2)I + \Omega(0)\) are \((1/2) \pm \mu\), so the eigenvalues differ by an integer if and only if \(2\mu\) is an integer; this characterizes the exceptional case for Birkhoff factorization into canonical form [35].

The system (4.4) resembles the system of matrix differential equation considered by Tracy and Widom [34], although the trace of the coefficient matrix is non-zero, so we use a variant on their methods as in [1]. We compute

\[
\left( z \frac{d}{dz} + w \frac{d}{dw} \right) \frac{\langle JW(z), W(w) \rangle}{z - w} = \frac{\langle JzW'(z), W(w) \rangle}{z - w} + \frac{\langle JW(z), WW'(w) \rangle}{z - w}
\]

\[
- \frac{\langle JW(z), W(w) \rangle}{z - w}
\]

\[
= \frac{\langle J \Omega(z) W(z), W(w) \rangle}{z - w} + \frac{\langle JW(z), \Omega(w) W(w) \rangle}{z - w},
\]

(4.6)

where \( J \Omega(z) + \Omega(w)^t J = -(1/2)(z - w) \tilde{J} \), so

\[
\left( z \frac{d}{dz} + w \frac{d}{dw} \right) \frac{\langle JW(z), W(w) \rangle}{z - w} = -\frac{1}{2} \langle \tilde{J} W(z), W(w) \rangle.
\]

(4.7)

For comparison, we have

\[
\left( z \frac{d}{dz} + w \frac{d}{dw} \right) \frac{1}{2} \int_1^\infty \frac{\langle J W(sz), W(sw) \rangle}{s} \frac{ds}{s}
\]

\[
= \frac{1}{2} \int_1^\infty \left( \langle \tilde{J} z W'(sz), W(sw) \rangle + \langle JW(sz), WW'(sw) \rangle \right) ds
\]

\[
= \frac{1}{2} \int_1^\infty \frac{d}{ds} \langle \tilde{J} W(sz), W(sw) \rangle ds
\]

\[
= -\frac{1}{2} \langle \tilde{J} W(z), W(w) \rangle.
\]

(4.8)
Hence the sum
\[
\langle JW(z), W(w) \rangle \frac{1}{z - w} - \frac{1}{2} \int_1^\infty \langle JW(sz), W(sw) \rangle \frac{ds}{s}
\]
(4.9)
is a function of \(z/w\), which converges to zero as \(z \to \infty\) or \(w \to 0\) in any way whatever, so this sum is zero. To obtain the stated result, we multiply by \(\sqrt{zw}\) and rearrange the terms.

Let \(K\) be the integral operator on \(L^2((a_1, a_2); dx)\) with kernel
\[
K(x, y) = \frac{\langle JW(x), W(y) \rangle}{x - y}.
\]
(4.10)

For suitable \(a_1\) and \(a_2\), we can suppose that \(\|K\| < 1\) as an operator on \(L^2((a_1, a_2); dx)\) and let \(S = K(I - K)^{-1}\); so that \((I + S)(I - K) = I\) and \(S : L^2((a_1, a_2); dx) \to L^2((a_1, a_2); dx)\) has a kernel
\[
S(x, y) = \frac{Q(x)P(y) - Q(y)P(x)}{x - y},
\]
(4.11)
and we take \(S(x, x) = Q'(x)P(x) - Q(x)P'(x)\) on the diagonal.

Let \(D(a_1, a_2; K)\) be the Fredholm determinant of \(K\), regarded as a function of the endpoints \(a_1\) and \(a_2\).

4.2 Proposition (i) The restrictions of the kernels of the operators \(S\) and \(S^2\) to the diagonal satisfy
\[
x \frac{d}{dx} S(x, x) = -P(x)Q(x) + (S^2)(x, x).
\]
(4.12)

(ii) The exterior derivative with respect to the endpoints satisfies
\[
d \log D(a_1, a_2; K) = S(a_1, a_1)da_1 - S(a_2, a_2)da_2.
\]
(4.13)

Proof. (i) Let \(M\) be the operator of multiplication by the independent variable \(x\) and \(D\) the operator of differentiation with respect to \(x\). For a integral operator \(T\) let \(\delta T = [MD, T] - T\), so that \(\delta T\) has kernel \((x\partial/\partial x + y\partial/\partial y)T(x, y)\); thus \(\delta\) is a pointwise derivation on the kernels, while \(T \mapsto [MD, T]\) is a derivation on operator composition. We use \(\doteq\) to mean that an operator corresponds to a certain kernel. Then
\[
[M, K] \doteq W_{\kappa, \mu}(x)W_{\kappa-1, \mu}(y) - W_{\kappa, \mu}(y)W_{\kappa-1, \mu}(x)
\]
(4.14)
and (4.7) shows that
\[
\delta K \doteq -2^{-1} \langle JW(x), W(y) \rangle
\].
Then \([M, S][I - K] = (I + S)[M, K]\), so the kernels have
\[
[M, S] = Q(x)P(y) - Q(y)P(x) \quad (4.15)
\]
where \(Q = (I - K)^{-1}W_{\kappa, \mu}\) and \(P = (I - K)^{-1}W_{\kappa - 1, \mu}\) are differentiable functions. Hence \(S\) is also an integral operator with kernel of the form (4.11). Also, by some straightforward manipulations, we have
\[
\delta S = (I - K)^{-1}(\delta K)(I - K)^{-1} + S^2, \quad (4.16)
\]
where
\[
(I - K)^{-1}(\delta K)(I - K)^{-1} = -(1/2)(P(x)Q(y) + P(y)Q(x)) \quad (4.17)
\]
and a short calculation shows that \((\delta S)(x, x) = x(d/dx)S(x, x)\). Hence the result.

(ii) This is a standard consequence of the results from [19].

Let \(\phi_{\kappa - 1, \mu}(t) = W_{\kappa - 1, \mu}(e^t)\) and \(\phi_{\kappa - 1, \mu}(t) = W_{\kappa, \mu}(e^t)\) and form the matrix
\[
\Phi(t) = \begin{bmatrix}
0 & 0 & \phi_{\kappa, \mu}(t) & 0 \\
0 & \phi_{\kappa - 1, \mu}(t) & 0 & 0 \\
\phi_{\kappa - 1, \mu}(t) & \phi_{\kappa, \mu}(t) & 0 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix} \quad (4.18)
\]
and the integral operator on \(L^2((0, \infty); \mathbb{C})\) with kernel
\[
T = 2\frac{\phi_{\kappa, \mu}(t)\phi_{\kappa - 1, \mu}(u) - \phi_{\kappa, \mu}(u)\phi_{\kappa - 1, \mu}(t)}{e^t - e^u}. \quad (4.19)
\]

4.3 Corollary The Hankel operator with scattering function \(\Phi\) operates on \(L^2((0, \infty); \mathbb{C}^4)\) and satisfies
\[
det(I - T) = det(I + \Gamma_\Phi). \quad (4.20)
\]

Proof. By Proposition 4.1, with the change of variable \(s = e^u\), we see that the integral operator satisfies \(T = \Gamma_{\phi_{\kappa - 1, \mu}}\Gamma_{\phi_{\kappa, \mu}} + \Gamma_{\phi_{\kappa, \mu}}\Gamma_{\phi_{\kappa - 1, \mu}}\) where \(\Gamma_\phi\) is in the standard form of a Hankel operator on \(L^2(0, \infty)\) as in (2.1). By some elementary determinant manipulations, the Hankel with matrix valued scattering function satisfies \(det(I + \Gamma_\Phi) = det(I - \Gamma_{\phi_{\kappa - 1, \mu}}\Gamma_{\phi_{\kappa, \mu}} - \Gamma_{\phi_{\kappa, \mu}}\Gamma_{\phi_{\kappa - 1, \mu}})\), which produces the stated result.

4.4 Remark By [15, 9.237], Whittaker’s differential equation generalizes the differential equation that the associated Laguerre functions satisfy. The column vector \(Y(z) = \sqrt{z}W(z)\) satisfies \(z(d/dz)Y(z) = \Omega(z)Y(z)\), which resembles the differential equation for generalized Laguerre functions on [33, page 60]. In Remark 5.2 of [2] we obtained a
factorization theorem for certain Whittaker kernels which expressly excluded the case of
generalized Laguerre functions.

5. A special case of the Whittaker kernel

For $-1/2 < a < 1/2$, let $R_a$ be the integral operator on $L^2(0, \infty)$ that has kernel

$$R_a \doteq \frac{(y/x)^a e^{-(x+y)/2}}{x+y}$$  \hspace{1cm} (5.1)

and $R$ be the operator on $L^2((0, \infty); \mathbb{C}^2)$ given by the block matrix

$$R = \begin{bmatrix} 0 & R_a \\ -R_a & 0 \end{bmatrix}$$  \hspace{1cm} (5.2)

and let $\Psi$ be the matrix-valued scattering function

$$\Psi(t) = \begin{bmatrix} 0 & \Gamma(1-2a)/(1+t)^{1-2a} \\ -\Gamma(1+2a)/(1+t)^{1+2a} & 0 \end{bmatrix} \quad (t > 0).$$  \hspace{1cm} (5.3)

5.1 Proposition (i) The operators $R_{\pm a}$ are bounded on $L^2(0, \infty)$ for $|a| < 1/2$.

(ii) For all $0 < a_1 < b_1 < \infty$, the operators $R$ and $\Gamma_\Psi$ are trace class on $L^2((a_1, b_1); \mathbb{C}^2)$ and satisfy

$$\det(I - \lambda R) = \det(I - \lambda \Gamma_\Psi) \quad (\lambda \in \mathbb{C}).$$  \hspace{1cm} (5.4)

(iii) The operator $R$ is skew self-adjoint, and $I + R$ is invertible with block matrix form

$$R(I + R)^{-1} = \begin{bmatrix} R_a R_{-a} (I + R_a R_{-a})^{-1} & -R_a (I + R_a R_{-a})^{-1} \\ -R_{-a} (I + R_a R_{-a})^{-1} & R_{-a} R_a (I + R_a R_{-a})^{-1} \end{bmatrix}.$$  \hspace{1cm} (5.5)

(iv) The integral operator $R_a R_{-a}$ is represented by the kernel

$$\Gamma(1-2a) \frac{W_{a+1/2,a}(x)W_{a-1/2,a}(y) - W_{a-1/2,a}(x)W_{a+1/2,a}(y)}{(x-y)\sqrt{xy}}.$$  \hspace{1cm} (5.6)

Proof. (i) This was proved by Olshanski [25] using a Fourier argument. For completeness
we give an equivalent proof involving Mellin transforms. First we check that the operators
$R_{\pm a}$ are bounded on $L^2(0, \infty)$. The expression

$$\int_0^\infty \frac{(x/y)^a}{(x/y) + 1} \frac{f(y)dy}{y} = \frac{1}{\sqrt{x}} \int_0^\infty \frac{(x/y)^a}{\sqrt{(x/y) + \sqrt{(y/x)^2}}} \sqrt{y} f(y) dy$$  \hspace{1cm} (5.7)

is a Mellin convolution as in [28], and one shows by a standard contour integration argument that

$$\int_0^\infty \frac{z^{a+i\sigma-1/2}dz}{z+1} = \pi \text{sech} \pi(a + i\sigma)$$  \hspace{1cm} (5.8)
is bounded for all $\sigma \in \mathbb{R}$ for all $|a| < 1/2$.

(ii) This follows as in Lemma 2.1. Letting $M_2$ be the $2 \times 2$ complex matrices with Hilbert–Schmidt norm, we introduce the state space $H = L^2((\omega_0, \infty); M_2)$ and the input and output space $H_0 = \mathbb{C}^2$, then

\[
A: \mathcal{D}(A) \to H: \quad f(x) \mapsto x f(x);
\]

\[
B_{\omega_0}: H_0 \to H: \quad b \mapsto \begin{bmatrix} 0 & e^{-x/2} x^{-a} \\ -e^{-x/2} x^a & 0 \end{bmatrix} b;
\]

\[
C_{\omega_0}: H \to H_0: \quad g(y) \mapsto \int_0^\infty \begin{bmatrix} e^{-y/2} y^{-a} & 0 \\ 0 & e^{-y/2} y^a \end{bmatrix} g(y) dy.
\] (5.9)

Then $\|e^{-tA}\| \leq e^{-t\omega_0}$ for all $t > 0$, and the scattering function is

\[
C_{\omega_0} e^{-tA} B_{\omega_0} = \int_0^\infty \begin{bmatrix} 0 & e^{-t y} y^{-2a} \\ -e^{-t y} y^2 a & 0 \end{bmatrix} dy;
\] (5.10)

and we note that

\[
C_{\omega_0} e^{-tA} B_{\omega_0} \to \Psi(t)
\] (5.11)
as $\omega_0 \to 0+$, the corresponding $R$ operator has kernel

\[
R = \int_0^\infty e^{-tA} B_{\omega_0} C_{\omega_0} e^{-tA} dt;
\] (5.12)

so as $\omega_0 \to 0$, we obtain

\[
R \doteq \int_0^\infty \begin{bmatrix} 0 & e^{-t(x+y)-(x+y)/2} (y/x)^a \\ -(x/y)^a e^{-t(x+y)-(x+y)/2} & 0 \end{bmatrix} dt
\]

\[
\doteq \begin{bmatrix} 0 & (y/x)^a e^{-(x+y)/2} / (x + y) \\ -(x/y)^a e^{-(x+y)/2} / (x + y) & 0 \end{bmatrix}.
\] (5.13)

(iii) Note that $R^*_a = R_{-a}$ hence $R^* = -R$ and $I + R_{-a} R_a$ is invertible. By elementary row operations one checks that

\[
\begin{bmatrix} I & R_a \\ -R_{-a} & I \end{bmatrix}^{-1} = \begin{bmatrix} (I + R_a R_{-a})^{-1} & -R_a (I + R_{-a} R_a)^{-1} \\ R_{-a} (I + R_{-a} R_a)^{-1} & (I + R_{-a} R_a)^{-1} \end{bmatrix},
\] (5.14)
in which we observe that the indices $a$ and $-a$ alternate.

(iv) We have

\[
R_a R_{-a} \doteq \int_0^\infty \begin{bmatrix} (x/y)^a e^{-(x+y)/2} (y/z)^a e^{-(y+z)/2} \\ x + y & (y/z)^a e^{-(y+z)/2} \\
\end{bmatrix} \frac{dy}{x + y} \frac{dz}{y + z}
\]

\[
\doteq \frac{(xz)^a e^{-(x+z)/2}}{z - x} \int_0^\infty \left( \frac{1}{x + y} - \frac{1}{y + z} \right) y^{-2a} e^{-y} dy
\] (5.15)
where [14, 14.2 (17), and page 431] gives the final integral in terms of an incomplete Gamma function which reduces to Whittaker’s function
\[ \int_0^\infty \frac{y^{-2a}e^{-y}}{x+y} \, dy = \Gamma(1 - 2a)x^{-a-1/2}e^{x/2}W_{a-1/2, a}(x). \] (5.16)

We also have the identity from [14, p 432]
\[ x^a e^{-x/2} = x^{-1/2}W_{a+1/2, a}(x). \] (5.17)

The result follows on substituting (5.16) and (5.17) into (5.15).

6. Diagonalizing the Whittaker kernel

The Whittaker’s functions are related to Bessel’s functions and Laguerre’s functions. Let \( K_\nu(x) \) be the modified Bessel function of the second kind given by
\[ K_\nu(x) = \int_0^\infty \cosh(\nu t)e^{-x\cosh t} \, dt, \] (6.1)
which is holomorphic on the open right half plane \( \{ x : \Re x > 0 \} \) and decays rapidly as \( x \to \infty \) through real values. When \( \nu = im \) is purely imaginary, one refers to McDonald’s function [28].

Let \( L_a \) be the differential operator
\[ L_a f = - \frac{d}{dx}\left( x^2 \frac{df}{dx} \right) + (x/2 + a)^2 f(x). \] (6.2)

Then \( L_\kappa \) has eigenfunctions
\[ f_{\kappa,m}(x) = x^{-1}W_{\kappa,im}(x) \quad (m \geq 0), \] (6.3)
so that
\[ L_\kappa f_{\kappa,m}(x) = (1/4 + m^2 + \kappa^2)f_{\kappa,m}(x). \] (6.4)

6.1 Proposition (i) The differential operator \( L_{-a} \) commutes with \( R_a R_{-a} \) on \( C_c^\infty((0, \infty); \mathbb{C}) \), and
\[ L_{-a}R_a = R_aL_a. \] (6.5)

(ii) \( R \) can be expressed as a diagonal operator with respect to \( (f_{\pm a,m})_{m \geq 0} \).

Proof. (i) Suppose that \( f \) is smooth and has compact support inside \( (0, \infty) \). Then we can repeatedly integrate by parts the integral
\[ \int_0^\infty \frac{(y/x)^a e^{-(x+y)/2}}{(x+y)} \left( -\frac{d}{dy}\left( y^2 \frac{df}{dy} \right) + (y/2 + a)^2 f(y) \right) dy \] (6.6)
without introducing boundary terms, to obtain
\[
\int_0^\infty \frac{(x/y)^a e^{-(x+y)/2}}{x+y} \left( \frac{y^2}{4} + a^2 + \frac{y^2}{(x+y)^2} + ay \right.
\]
\[
+ \frac{y^2}{x+y} + \frac{2ay}{x+y} - a - y + \frac{y^2}{(x+y)^2} - \frac{2y}{x+y} \bigg) f(y) \, dy. \tag{6.7}
\]

After a little reduction, one shows that this coincides with
\[
\left(-x^2 \frac{d^2}{dx^2} - 2x \frac{d}{dx} + (x/2 - a)^2\right) \int_0^\infty \frac{(y/x)^a e^{-(x+y)/2}}{x+y} f(y) \, dy. \tag{6.8}
\]

Likewise, we have \( L_a R_{-a} = R_{-a} L_a \); so \( L_{-a} \) commutes with \( R_a R_{-a} \).

(ii) Erdelyi [14, 14.3 (53)] quotes the following formula for the Stieltjes transform
\[
\int_0^\infty \frac{y^{-1-a} e^{-y/2}}{x+y} W_{-a,im}(y) \, dy = \Gamma((1/2) - a + im) \Gamma((1/2) - a - im)x^{-a-1} e^{x/2} W_{a,im}(x),
\]
\[
\tag{6.9}
\]
so \( f_{-a,m}(x) = x^{-1} W_{-a,im}(x) \) satisfies
\[
R_a f_{-a,m}(x) = \Gamma((1/2) - a + im) \Gamma((1/2) - a - im)f_{a,m}(x). \tag{6.10}
\]

It follows that \( f_{-a,m} \) is an eigenvector for \( R_{-a} R_a \). Wimp [37] considered a Fourier–
Plancherel formula which decomposes \( L^2((0, \infty); \mathbb{C}) \) as a direct integral with respect to 
\( f_{a,m}(x) \), where \( f_{a,m}(x) \) are eigenfunctions of \( L_{-a} \), so there is a transform pair
\[
g(m) = \Gamma(1/2 - \kappa - im) \Gamma(1/2 - \kappa + im) \int_0^\infty f(t) f_{\kappa,m}(t) \, dt,
\]
\[
f(x) = \frac{1}{\pi^2} \int_0^\infty t \sinh(2\pi t) f_{\kappa,t}(x) g(t) \, dt. \tag{6.11}
\]

Hence we can take \((f_{a,m})_{m \geq 0}\) to be a basis of \( L^2((0, \infty); \mathbb{C}) \), and \((f_{-a,m})_{m \geq 0}\) to be a basis of another copy of \( L^2((0, \infty); \mathbb{C}) \) and diagonalize \( R \) with respect to the combined basis \((f_{\pm a,m})_{m \geq 0}\) of \( L^2((0, \infty); \mathbb{C}^2) \).

6.2 Corollary For \(-1/2 < a < 1/2\), the kernel
\[
K \doteq \frac{\Gamma(1-2a) \cos^2 \pi a}{\pi^2} \frac{W_{a+1/2,a}(x) W_{a-1/2,a}(y) - W_{a-1/2,a}(x) W_{a+1/2,a}(y)}{\sqrt{x/y}(x-y)} \tag{6.12}
\]
gives a determinantal point process on \([0, \infty)\) such that random points \( \lambda_j \), ordered by 
\( 0 \leq \lambda_1 \leq \lambda_2 \leq \ldots \), satisfy
\[
\Pr[\lambda_1 \geq s] = \det(I - K I_{[0,s]}). \tag{6.13}
\]
Proof. Note that
\[ \Gamma(a+1/2+im)\Gamma(a+1/2-im)\Gamma(-a+1/2+im)\Gamma(-a+1/2-im) = 2\pi^2/(\cos 2\pi a + \cosh 2\pi m) \]
so
\[ \|R_a R_a\|_{L(0,\infty)} = \pi^2 \sec^2 \pi a. \quad (6.14) \]
Hence we have \(0 \leq K \leq I\) as operators on \(L^2(0,\infty)\), and \(K(x,y)\) has a continuous kernel. Hence by Mercer's theorem, \(K\) restricts to a trace class operator on \(L^2(0,b)\) for all \(0 < b < \infty\) with
\[ \text{trace}(K) = \int_0^b K(x,x) \, dx. \quad (6.15) \]
Note that the right-hand side is finite for all \(b > 0\), but diverges to \(\infty\) as \(b \to \infty\), hence \(\det(I - K I[0,b]) \to 0\) as \(b \to \infty\).

By Soshnikov's theorem [29], there is a point process on \([0,b]\) with \(K\) as the generating kernel. The point process distributes random \(x\) in \([0,b]\) such that only finitely many \(x\) can lie in each Borel subset of \([0,b]\), and the joint distribution of the points is specified as follows. Let \(B_j\) \((j = 1, \ldots, m)\) be disjoint Borel subsets of \([0,b]\) and \(n_j = \sharp\{x \in B_j\}\) the number of random points that lie in \(B_j\). Then the joint probability generating function of the random variables \(n_j\) is
\[ \mathbb{E} \prod_{j=1}^m z_j^{n_j} = \det\left(I + \sum_{j=1}^m (z_j - 1)K I_{B_j}\right) \quad (z_j \in \mathbb{C}). \quad (6.16) \]
Equivalently, as in [7, p. 597] we compress \(K\) to \(I_{[0,b]} K I_{[0,b]}\) on \(L^2([0,b])\) and introduce \(T_{[0,b]} = I_{[0,b]} K I_{[0,b]} (I - I_{[0,b]} K I_{[0,b]})^{-1}\) so that
\[ \det(I + T_{[0,b]}) = (\det(I - I_{[0,b]} K I_{[0,b]}))^{-1}. \quad (6.17) \]
For each integer \(\ell \geq 0\), the \(\ell\)-point correlation function is defined to be the positive symmetric function
\[ \rho_\ell(x_1, \ldots, x_\ell) = \det(I + T_{[0,b]}^{-1}) \det[T_B(x_j, x_k)]_{j,k=1}^\ell \quad (x_j \in [0,b]), \quad (6.18) \]
such that
\[ \Pr(\text{there are exactly } \ell \text{ particles in } B) = \frac{1}{\ell!} \int_B \rho_\ell(x_1, \ldots, x_\ell) \, dx_1 \ldots dx_\ell \quad (6.20) \]
for all Borel subsets \(B\) of \((0,b)\).
Remarks 6.3 (i) In particular [28, (7.4.25)] we can write

\[ f_{0,m}(x) = \frac{1}{\sqrt{\pi x}} K_{im}(x/2) = \frac{1}{\sqrt{2}} \int_1^\infty P_{-1/2+im}(s)e^{-sx/2}ds \]  

(6.21)

in terms of the MacDonald and associated Legendre functions. The function \( W_{0,im} \) also occurs in the spectral decomposition of the Laplace operator over the fundamental domain that arises from the action of \( SL(2,\mathbb{Z}) \) on the upper half plane; see [21, p 318] for a discussion of Maass cusp forms. In [2], we considered the Hankel operators that commute with second order differential operators, and found the case \( L_0 \) and \( R_0 \) as Q(vii).

(ii) Another case of Q(vii) from [2] is \( L_{-\kappa} \) commuting with the Hankel operator with scattering function \( x^{-1}W_{\kappa,1/2}(x) \) on \( C_\infty^c(0, \infty) \). If \( a \neq 0 \), then \( R_a \) is not a Hankel operator but an operator of Howland’s type as in [17], and \( L_a \) does not commute with \( R_{-a} \).

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