GLOBAL UNIQUENESS OF STEADY TRANSONIC SHOCKS IN TWO-DIMENSIONAL COMPRESSIBLE EULER FLOWS

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ABSTRACT. We prove that for the two-dimensional steady complete compressible Euler system, with given uniform upcoming supersonic flows, the following three fundamental flow patterns (special solutions) in gas dynamics involving transonic shocks are all unique in the class of piecewise $C^1$ smooth functions, under appropriate conditions on the downstream subsonic flows: (i) the normal transonic shocks in a straight duct with finite or infinite length, after fixing a point the shock-front passing through; (ii) the oblique transonic shocks attached to an infinite wedge; (iii) a flat Mach configuration containing one supersonic shock, two transonic shocks, and a contact discontinuity, after fixing the point the four discontinuities intersect. These special solutions are constructed traditionally under the assumption that they are piecewise constant, and they have played important roles in the studies of mathematical gas dynamics. Our results show that the assumption of piecewise constant can be replaced by some more weaker assumptions on the downstream subsonic flows, which are sufficient to uniquely determine these special solutions.

Mathematically, these are uniqueness results on solutions of free boundary problems of a quasi-linear system of elliptic-hyperbolic composite-mixed type in bounded or unbounded planar domains, without any assumptions on smallness. The proof relies on an elliptic system of pressure $p$ and the tangent of the flow angle $w = v/u$ obtained by decomposition of the Euler system in Lagrangian coordinates, and a newly developed method for the $L^\infty$ estimate that is independent of the free boundaries, by combining the maximum principles of elliptic equations, and careful analysis of shock polar applied on the (maybe curved) shock-fronts.

1. INTRODUCTION

In discussing steady flows which are supersonic in the entrance section of a duct and then become subsonic, R. Courant and K. O. Friedrichs wrote in their classical monograph *Supersonic Flow and Shock Waves* [15, p. 372] that

"\ldots\ldots\ We know in general that this assumption (the flow is continuous) is not tenable and that we must consider flows involving shocks. It is a question of great importance to know under what circumstances a steady flow involving shocks is uniquely determined by the boundary conditions at the entrance, and when further conditions at the exit are appropriate."
This paper is exactly devoted to showing that, for the two-dimensional steady complete compressible Euler system, with given uniform upstream supersonic flows, under certain reasonable conditions of the downstream of the subsonic flows, in the class of piecewise $C^1$ functions, the uniqueness of the following three important flow patterns in gas dynamics:

(i) The transonic normal shock in a two–dimensional straight duct with finite or infinite length, after fixing a point the shock-front passing through (see Figure 1);

(ii) The transonic oblique shocks attached to an infinite wedge against the uniform supersonic flow (see Figure 2);

(iii) A flat Mach configuration involving three shocks and a contact discontinuity, where one shock is supersonic, and another two are transonic, with the contact line separating two subsonic regions, after fixing a point $O$ where the four discontinuities intersect (see Figure 3).

These flow patterns have been constructed in [15, p.311, pp.332-333] by using shock polar, under the assumptions that the shock-fronts (contact line) are straight lines, and the subsonic flows behind the shock-fronts are also uniform — i.e., the solution is piecewise constant (see §2 below). The uniqueness results we obtain in this work indicate that under certain conditions of the downstream subsonic flows, these assumptions may be relaxed to that the flow fields are only piecewise continuously differentiable — then for given uniform upcoming supersonic flows, the transonic shock-fronts (and the contact line appeared in the Mach configuration) must be straight lines and the subsonic flows behind them must be uniform. From mathematical point of view, since the steady Euler system for subsonic flows is of elliptic-hyperbolic composite-mixed type, these are all results on uniqueness of solutions of free boundary value problems of a nonlinear system of mixed type, with the transonic shock-fronts and contact line being the free boundaries.

![Figure 1. Normal transonic shock in a straight duct.](image)
Figure 2. Oblique transonic shocks attached to a slim infinite symmetric wedge with open angle $2\theta_W$ and zero attacking angle in uniform supersonic flows.

Figure 3. A flat “direct” Mach configuration. Here $S_1$ is a supersonic shock lying in the quadrant II, $S_2, S_3$ are transonic shocks lying in quadrant IV, I respectively, and $D$ is a contact line. They are all straight lines. The flow field is piecewise constant.

established the stability of transonic shocks in finitely long or infinitely long duct with square sections or arbitrary sections, under various conditions at the exit of the duct. Z. Xin and H. Yin studied similar problems but employed different methods [26]. For the Euler system, H. Yuan showed in [29] various stability and
instability results by using Lagrangian coordinates introduced by S. Chen in [11] and characteristic decomposition technique [10]. G.-Q. Chen, J. Chen and M. Feldman also introduced the method of stream function in Lagrangian coordinates by which the Euler system can be written as a single second order equation to study the stability, local uniqueness and asymptotic behavior of transonic shocks in infinite ducts for the Euler system. Z. Xin, H. Yin and their collaborators also studied the problem of stability of transonic shock in nozzles for the steady Euler system, see [25, 27].

For the study of stability of oblique transonic shocks attached to a wedge in supersonic flow, there are papers of S. Chen, B. Fang [13, 17], E. H. Kim [20], and H. Yin, C. Zhou [28]. Both the potential flow equation and the complete Euler system were used as models and the authors showed stability under suitable downstream conditions.

For the Mach configurations, one has to use the Euler system since there is a contact discontinuity. S. Chen firstly studied the stability of a Mach configuration under perturbations of upcoming supersonic flow, for the steady Euler flow and pseudo-steady Euler flow [11, 12]. Later on he and B. Fang also showed the stability of a wave pattern of regular reflection-refraction of shocks upon an interface [13], which is similar to a Mach configuration.

Comparing to stability, the progress on the study of uniqueness is rather slow. For potential flows, G.-Q. Chen and H. Yuan [7] proved the transonic shock in a two-dimensional or three-dimensional straight duct is unique modulo a translation. L. Liu showed uniqueness of subsonic potential flow in various domains [22], see also [23]. By Proposition 3.1 in [9], in two-dimensional case, many of the uniqueness results on subsonic flows also hold for the Euler system, since under appropriate boundary conditions, the flow is in itself irrotational, i.e., governed by the potential flow equation. However, to our knowledge, there is no any result on global uniqueness of transonic shocks for the complete Euler system before.

In this paper, we are going to establish the global uniqueness, for the complete Euler system, of the three flow patterns we previously mentioned. It turns out that the method of maximum principles employed in [7] for potential flow also works for Euler flow, but the point is that one should consider now the extreme values of pressure $p$ and slope of the flow angle $w = v/u$, with $u, v$ the velocity component of the flow along $x, y$ axis, respectively. In [29], H. Yuan has shown that $p$ and $w$ satisfy a first order elliptic system for subsonic flow. Then as in [7], one needs to show if there are nontrivial extremes of $p$ (or $w$) on the shock-front $S$, there would be contradictions by a Hopf-type boundary point lemma. It is here we need to utilize some nice properties of the Rankine-Hugoniot conditions — particularly, a rather simple curve called $p − w$ shock polar (see Figure 4), which shows the relation between $p$ and $w$ behind the shock-front. We recommend [15, 24] for detailed computations of these relations.

This paper is organized as follows. In §2, we formulate the above mentioned physical problems (flow patterns) and state rigorously our results, Theorem 2.1–Theorem 2.6. We also introduce briefly the $p − w$ shock polar for the comfort of those readers not familiar with this subject. Then we prove in §3 the uniqueness of transonic shocks in straight ducts (Theorem 2.1–Theorem 2.2), and in §4, the
The Mach number of the flow is defined by \( M \) for subsonic flow. As in \cite{29}, we introduce the system \( (2.1)-(2.4) \) is symmetric hyperbolic for supersonic flow, while of Theorem 2.6, on the uniqueness of the Mach configuration studied by S. Chen \cite{11}, is established in §5.

2. Shock Polar, Special Solutions and Main Results

2.1. The Euler System and \( p-w \) Shock Polar. The complete compressible Euler system governing the steady motion of polytropic gas flow in two-dimensional space \( (x,y) \in \mathbb{R}^2 \) expresses the conservation of mass, momentum and total energy (cf. \cite{15} pp.14–23):

\[
\begin{align*}
\partial_x (\rho u) + \partial_y (\rho v) &= 0, \\
\partial_x (\rho u^2 + p) + \partial_y (\rho uv) &= 0, \\
\partial_x (\rho uv) + \partial_y (\rho c^2 + p) &= 0, \\
\frac{1}{2}(u^2 + v^2) + \frac{c^2}{\gamma - 1} &= b_0.
\end{align*}
\]

The last equation is also called as Bernoulli law, with \( b_0 \) the Bernoulli constant which is invariant on each flow trajectory, even across a shock-front. Here \( p, \rho, c, u, v \) are the pressure, mass density, sonic speed, velocity component along x and y axis respectively, of the flow, and \( \gamma > 1 \) is the adiabatic exponent. For polytropic gas, the state function is \( p = A(s) \rho^\gamma \), with \( s \) the entropy, we then have \( c = \sqrt{\partial p/\partial \rho} = \gamma p/\rho \).

The Mach number of the flow is defined by \( M = \sqrt{(u^2 + v^2)/c} \). When \( M < 1 \) (resp. \( M > 1 \)), the flow is called subsonic flow (resp. supersonic flow). It is well known that the system \( (2.1)-(2.4) \) is symmetric hyperbolic for supersonic flow, while of elliptic-hyperbolic composite-mixed type for subsonic flow. As in \cite{29}, we introduce \( w = v/u \) to denote the slope of the flow angle \( \theta \) in case \( u \neq 0 \), which, together with \( p \), plays a key role in our analysis later.

Let \( S = \{(x, y) : x = f(y)\} \) be a shock-front — that is, a curve where the state of the gas is discontinuous and physical entropy condition holds (cf. Definition 2.1 in §2.2 below). It is well known that the following Rankine-Hugoniot conditions should hold across \( S \) \cite{11} p.11:

\[
\begin{align*}
[pu] - f'(y)[\rho v] &= 0, \\
[pu^2 + p] - f'(y)[\rho uv] &= 0, \\
[puv] - f'(y)[\rho c^2 + p] &= 0.
\end{align*}
\]

Here \([\cdot]\) denotes the jump of a quantity across \( S \). For example, if the pressure on the left hand side of \( S \) (upstream) is \( p_0 \), on the right hand side of \( S \) (downstream) is \( p \), then we set \([p] = p - p_0 \). Note that \( b_0 \) in \( (2.1) \) is constant along a flow trajectory even across the shock-front, which is the Rankine-Hugoniot condition corresponding to conservation of energy.

For a fixed point \( Q = (f(y), y) \) on the shock-front, supposing the uniform state \((p_0 > 0, u_0 > 0, v_0 \equiv 0 \equiv v_0, p_0 > 0)\) of the gas ahead of the shock-front is given\(^1\), we consider the four algebraic equations \( (2.3)-(2.7) \), with five independent variables \( p(Q), u(Q), w(Q), \rho(Q) \), which are the state of the flow on the right hand side of \( S \), and \( f'(y) \). In \cite{15} pp.306–309, it has been shown that, the flow angle \( \theta = \arctan w \), horizontal velocity \( u(Q) \), and density \( \rho(Q) \), as well as the slope of the

\(^1\)The flow angle \( \theta \) is the angle between the velocity \((u, v)\) and the x axis. Hence \( w = \tan \theta \).

\(^2\)Therefore \( b_0 \) in \( (2.4) \) is \( u_0^2/2 + \gamma p_0/(\gamma - 1)\rho_0 \).
Figure 4. A $p - w$ shock polar. $C = (0, p_0)$ represents the upcoming supersonic flow; $A = (0, p^+)$ represents the strongest normal transonic shock; $w$ reaches its maximum at $B = (w_*, p_*)$, and $S = (w_{\text{sonic}}, p_{\text{sonic}})$ represents a shock that the state behind the shock-front is exactly sonic. $B' = (-w_*, p_*)$ and $S' = (-w_{\text{sonic}}, p_{\text{sonic}})$ are reflections of $B, S$ with respect to the $p$ axis. The open arc $\overline{SAS'}$ represents all possible transonic shocks. The open arc $\overline{CS}$ and $\overline{CS'}$ represent all possible supersonic shocks. The dashed line below $C$ corresponds to those states that do not satisfy physical entropy condition. $E = (w, p_{\text{weak}})$ represents the weak shock; $F = (w, p_{\text{strong}})$ represents the strong shock.

shock-front $f'(y)$, can all be expressed by $p(Q)$. Particularly we have the following important relation on $p$ and $w$ \cite[p. 347]{15}:

$$w = \pm \frac{p - p_0}{\gamma M_0^2 - (\frac{p}{p_0} - 1)} \sqrt{\frac{\frac{2\gamma}{\gamma + 1}(M_0^2 - 1) - \frac{p}{p_0}}{\frac{p}{p_0} + \frac{2\gamma}{\gamma + 1}}}.$$  \hspace{1cm} (2.8)

Here $M_0 = u_0/c_0$ is the Mach number of the flow ahead of $S$, and $c_0 = \sqrt{\gamma p_0/p_0}$. Figure 4 shows the curve represented by this expression in $w - p$ plane, which is traditionally called a $p - w$ shock polar. It is easy to see that this curve is symmetric with respect to the line $w = 0$ due to “$\pm$” in (2.8).

There are several critical points on the $p - w$ shock polar. $A$ is the point where $p$ reaches its maximum (denoted by $p^+$); $B$ is the point where $w$ reaches its maximum $w_\ast$. The point $S = (w_{\text{sonic}}, p_{\text{sonic}})$ has the special property that the state of the gas behind the shock-front is exactly sonic, i.e., $M = 1$. For all points on the shock polar with $p > p_{\text{sonic}}$ (i.e., on the open arc $\overline{SAS'}$), the state behind the shock-front they represent are all subsonic. $B', S'$ are the mirror images of $B, S$ with respect to the line $w = 0$. 
For convenience of later reference, we also write out the following relations on the state of the flow behind the shock-front [24, p.44, p.48]:

\[ u = u_0 - \frac{p - p_0}{\rho_0 u_0}, \quad (2.9) \]

\[ \rho = \rho_0 \frac{(\gamma + 1)p + (\gamma - 1)p_0}{(\gamma - 1)p + (\gamma + 1)p_0}, \quad (2.10) \]

\[ \sin \alpha = \frac{1}{M_0} \sqrt{\frac{\gamma + 1}{2\gamma} \left( \frac{p}{\rho_0} + \frac{\gamma - 1}{\gamma + 1} \right)}, \quad \cot \alpha = f'(y). \quad (2.11) \]

Here \( \alpha \) is the angle between the tangent of the shock-front and the \( x \) axis.

2.2. The Physical Problems and Main Results. Now we formulate the physical problems we are concerned with, construct the special solutions (flow patterns) by using the shock polar, and then state the results on their uniqueness.

2.2.1. Problem A: Transonic Shocks in a Finite Duct. Let \( D = \{(x, y) \in \mathbb{R}^2 : -1 < x < 1, 0 < y < 1\} \) be a straight duct with finite length. We set \( \Sigma_{-1} = \{(-1, y) : 0 \leq y \leq 1\} \) and \( \Sigma_1 = \{(1, y) : 0 \leq y \leq 1\} \) as the entry and exit of the duct, \( \Gamma_0 = \{(x, 0) : -1 < x < 1\} \) and \( \Gamma_1 = \{(x, 1) : -1 < x < 1\} \) as the lower and upper wall of the duct, respectively.

On the walls, it is natural to impose the impenetrability or slip condition \( v = 0 \). On the entry, we assume the flow is supersonic and moves in \( D \), so the Euler system \( (2.1) - (2.4) \) is hyperbolic in the positive \( x \) direction. Therefore we impose the initial condition \( p = p_0 > 0, u = u_0 > 0, v = v_0 = 0, \rho = \rho_0 > 0 \), with \( p_0, u_0, \rho_0 \) all being constants and satisfying \( M_0 > 1 \). On the exit, one Dirichlet condition on \( p \) is given:

\[ p = p_1, \quad p_1 \text{ is a constant.} \quad (2.12) \]

It is indicated in [15, p.373, p.385] that this restriction on pressure is more physically interesting for nozzle flows. Therefore we get a boundary value problem of \( (2.1) - (2.4) \) in \( D \). We call this Problem A. We are interested in those solutions of problems like this with the following structure, i.e., transonic shock solutions:

**Definition 2.1.** Suppose \( D \) is a domain in \( \mathbb{R}^2 \). For a \( C^2 \) function \( x = f(y) \) defined on an interval \( I \) of \( \mathbb{R} \), let

\[ S = \{(f(y), y) \in D : y \in I\}, \]

\[ D^- = \{(x, y) \in D : x < f(y)\}, \]

\[ D^+ = \{(x, y) \in D : x > f(y)\}. \]

Then \( p, u, v, \rho \in L^\infty_{loc}(D) \cap C^1(D^- \cup S) \cap C^1(D^+ \cup S) \) is a transonic shock solution of \( (2.1) - (2.4) \) if it is supersonic in \( D^- \) and subsonic in \( D^+ \), satisfies equations \( (2.1) - (2.4) \) in \( D^- \cup D^+ \) and the boundary conditions pointwise in the classical sense, the Rankine-Hugoniot jump conditions \( (2.5) - (2.7) \), as well as the physical entropy condition across \( S \):

\[ [p] > 0. \]

**Remark 2.1.** By \( (2.4) \) and entropy condition, there holds \( f'(y) = pv/(\rho u^2 + |p|) \) and the denominator never vanishes. So if \( p, u, v, \rho \in L^\infty_{loc}(D) \cap C^1(D^- \cup S) \cap C^1(D^+ \cup S), \) we naturally have \( f \in C^2 \) in Definition 2.1.
2.2. Problem

U
sonic shocks in an infinitely long straight duct

Remark

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exit at infinity.

Definition 2.1 are reasonable. Hence to prove uniqueness, our assumptions on regularity in

§ Example shocks attached to a wedge, in our proof we do not require the solution

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subsonic behind the shock-front, i.e.,

M < M

in the flow at infinity, under the rather strong assumptions that the flow is uniformly

perpendicularly. So as explained in [29, p.1348], at the corner points where they intersect, the solution will in general be singular. However, for uniform upcoming supersonic flow and straight duct, it can be shown that the first order compatibility condition holds (cf. Lemma 3.3 in [25, p.19]), that guarantees the solution \(u, v, p, \rho\) to be \(C^1\) at the corners, and hence \(C^1\) up to boundary in \(D\). For oblique shocks attached to a wedge, in our proof we do not require the solution \(u, v, p, \rho\) to be \(C^1\) up to the vertex, where the shock-front meet the wedge (continuous is enough, see Remark 4.1). Hence to prove uniqueness, our assumptions on regularity in Definition 2.1 are reasonable.

As a simple application of the shock polar, we show existence of special normal transonic shock solutions to Problem A. For given uniform upcoming supersonic flow \((p_0 > 0, u_0 > 0, v_0 = 0, \rho_0 > 0)\), a \(p-w\) shock polar is determined (see Figure 4), and for \(w = 0\), we have a unique \(p^+ > p_0\) (i.e., the point \(A\) in Figure 4) representing a subsonic state, and hence \(u^+, \rho^+, \text{ and } f' \equiv 0\) may be calculated by \((2.9)\) \((2.10)\) \((2.11)\) with \(p\) there replaced by \(p^+\). Suppose the shock-front passes the point \((t, 0)\) with \(-1 < t < 1\), then we may take the shock-front as \(x \equiv t\), and the state behind it is \(U^+ = (p^+, u^+, w^+ = 0, \rho^+)\), which is uniform and subsonic. It is easy to see that what we get in this way is a transonic shock solution to Problem A, with \(p_1 = p^+\) in \((2.12)\). We denote this solution by \(U^+_b\).

The following theorem is one of the main results of this paper.

**Theorem 2.1.** For given supersonic initial data \(U_0 = (p_0 > 0, u_0 > 0, v_0 = 0, \rho_0 > 0)\) on \(\Sigma_{-1}\), supposing the flow behind the shock-front satisfies \(u > 0\), then Problem A has a transonic shock solution (in the sense of Definition 2.1) if and only if \(p_1 = p^+\) in \((2.12)\). If this holds, let \(U\) be a transonic shock solution to Problem A with shock-front passing through the point \((t, 0)\), then \(U\) must be the special solution \(U_b\) constructed above.

2.2.2. Problem A': Transonic Shocks in an Infinite Duct. This is to consider transonic shocks in an infinitely long straight duct \(D' = \{(x, y) \in \mathbb{R}^2 : -1 < x < \infty, 0 < y < 1\}\). The boundary conditions are the same as in Problem A, except that \((2.12)\) is dropped. We call this boundary value problem of Euler system in the unbounded domain \(D'\) as Problem A'. Of course \(U_b\) is a solution to this problem. Results like Theorem 2.1 also hold:

**Theorem 2.2.** For given supersonic initial data \(U_0 = (p_0 > 0, u_0 > 0, v_0 = 0, \rho_0 > 0)\) on \(\Sigma_{-1}\), supposing the flow behind the shock-front satisfies \(u > u_*\) and \(M < M^*\) with fixed numbers \(u_* > 0\) and \(M^* < 1\), then any transonic shock solution to Problem A' must be the special solution \(U_b\), provided that the shock-front passes the point \((t, 0)\) \(\in \partial D'\).

Like Theorem 1.3 in [7, p.568], here we do not need any asymptotic condition of the flow at infinity, under the rather strong assumptions that the flow is uniformly subsonic behind the shock-front, i.e., \(M < M_* < 1\), and moves uniformly to the exit at infinity.

We also note that Theorem 2.1 and Theorem 2.2 imply particularly the instability of the special normal transonic shocks under any perturbations of the downstream pressure, as have been shown in [7, 29].
2.2.3. Problem B: Transonic Shocks Attached to an Infinite Wedge. Next we consider the transonic oblique shocks attached to an infinite wedge against a uniform supersonic flow. By symmetry with respect to the x axis (see Figure 2), we may consider only the “half-wedge” (a ramp) with an open angle $\theta_W > 0$ and zero attacking angle against the uniform supersonic flow. Let $D = \{(x,y) \in \mathbb{R}^2 : x > 0, y > x \tan \theta_W\}$, and set $\Sigma_0 = \{(x,y) : x = 0, y \geq 0\}$ be the “entry” where the uniform supersonic flow $(p_0 > 0, u_0 > 0, w_0 = 0, \rho_0 > 0)$ flows from left to right to the ramp $W = \{(x,y) \in \mathbb{R}^2 : x > 0, 0 < y < x \tan \theta_W\}$ with wall $\Gamma = \{(x,y) : x > 0, y = x \tan \theta_W\}$. On the wall we have the slip condition $w = \tan \theta_W$. We would like to find transonic shock in the flow field. We call this Problem B.

Assuming the transonic shock solution is piecewise constant — that is, the shock-front is a straight line issuing from the origin $O$, which is the vertex of the ramp, and the subsonic flow behind the shock-front is also uniform, we may obtain special transonic shock solution to Problem B by the shock polar. Indeed, for $\tan \theta_W \in (0, w_{\text{sonic}})$ (see Figure 4), the line $w = \tan \theta_W$ meets the $p-w$ shock polar at three different points $D, E, F$. The point $D$ represents a state that does not satisfy the physical entropy condition, so we ignore it. The point $E$ corresponds to a weaker supersonic shock, that is, the flow behind the shock-front is also supersonic. This supersonic shock has been intensively studied (see, for example, [6, 8, 21, 30] and references therein), and is not of our concern. The point $F$ represents a transonic shock which we are interested in, and we may obtain the slope of the shock-front, the density and velocity behind the shock-front by (2.9)–(2.11). We call this solution the strong transonic shock $U_s$. It always corresponds to the point $(w = \tan \theta_W \in (0, w_s), p = p_{\text{strong}})$ on the upper part of the shock polar (i.e., on the closed arc $\overline{AB}$). For $w = \tan \theta_W \in (w_{\text{sonic}}, w_s)$, both the weak shock and strong shock are transonic shocks, and for $w = w_s$, they coincide. We will denote the weak transonic shock by $U_w$: it corresponds to the point $(w = \tan \theta_W \in (w_{\text{sonic}}, w_s), p = p_{\text{weak}} < p_{\text{strong}})$ on the lower part of the shock polar (i.e., on the open arc $\overline{BS}$). For $w > w_s$, detached bow shock will appear, and for $w = w_{\text{sonic}}$, the weak shock is sonic. We will not consider these two cumbersome cases in this paper.

For the special strong transonic shock, we have the following uniqueness result.

**Theorem 2.3.** For given supersonic initial data $U_0 = (p_0 > 0, u_0 > 0, w_0 = 0, \rho_0 > 0)$ on $\Sigma_0$, supposing the flow behind the shock-front satisfies $u > 0$ and the following asymptotic condition:

$$w \to \tan \theta_W \quad \text{as} \quad (x,y) \to \infty,$$

(2.13)

here $\tan \theta_W \in (0, w_s)$, then any transonic shock solution to Problem B with the restriction at the vertex of the ramp

$$w = \tan \theta_W, \quad p = p_{\text{strong}} \quad \text{at} \quad O$$

(2.14)

must be the strong transonic shock solution $U_s$.

Under a further rather strong asymptotic condition on the shock-front at infinity, namely,

$$p < p_{\infty} \quad \text{as} \quad (x,y) \to \infty \quad \text{on} \quad S$$

(2.15)

Note that this is the initial data to the Euler system on $\Sigma_0$. 

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The text references various equations and theorems, which are not repeated here for brevity.
with $p_\infty$ a constant less than $p_*$, we also have uniqueness of the weak transonic shock solution.

**Theorem 2.4.** For given supersonic initial data $U_0 = (p_0 > 0, u_0 > 0, w_0 = 0, \rho_0 > 0)$ on $\Sigma_0$, supposing the flow behind the shock-front satisfies $u > 0$, and the asymptotic conditions (2.13), (2.15), then any transonic shock solution to Problem B with the restriction at the vertex of the ramp

$$w = \tan \theta_W, \quad p = p_{\text{weak}} \quad \text{at} \quad O$$

(2.16)

must be the weak transonic shock solution $U_w$. Here $\tan \theta_W \in (w_{\text{sonic}}, w_*)$.

Here we note that by shock polar, one can always determine that at $O$ either (2.14) or (2.16) holds. Therefore our requirements in the Theorems are natural.

We may replace (2.13) in Theorem 2.3 by

$$p \to p_{\text{strong}} \quad \text{as} \quad (x, y) \to \infty$$

(2.17)

to guarantee uniqueness of $U_s$, and replace (2.13) in Theorem 2.4 by

$$p \to p_{\text{weak}} \quad \text{as} \quad (x, y) \to \infty,$$

(2.18)

as well as dropping (2.15) to show uniqueness of $U_w$.

**Theorem 2.5.** For given supersonic initial data $U_0 = (p_0 > 0, u_0 > 0, w_0 = 0, \rho_0 > 0)$ on $\Sigma_0$, supposing the flow behind the shock-front satisfies $u > 0$, and the asymptotic condition (2.17) (resp. (2.18)), then any transonic shock solution to Problem B with the restriction (2.13) (resp. (2.10)) at the vertex of the ramp must be the strong (resp. weak) transonic shock solution $U_s$ (resp. $U_w$).

We recall that, in discussing the shocks attached to a wedge, R. Courant and K. O. Friedrichs had written [15, pp.317-318]: “Quite aside from the question of stability, the problem of determining which of the possible shocks occurs cannot be formulated and answered without taking the boundary conditions at infinity into account. · · · · · · If the pressure prescribed there is below an appropriate limit, the weak shock occurs in the corner. If, however, the pressure at the downstream end is sufficiently high, a strong shock may be needed for adjustment. Under appropriate circumstances this strong shock may begin just in the corner and thus, of the two possibilities mentioned, the one giving a strong shock may actually occur. · · · · · · All statements made here are conjectures so far. While there is little doubt that they are in general correct, they should be supported, if possible, by detailed theoretical investigation.” Our results above, may be considered as a part of such investigation.

2.2.4. Problem C: Uniqueness of a Mach Configuration. Mach configuration is one of the fundamental flow patterns in gas dynamics which is closely connected to shock reflection-diffraction phenomena. As explained by R. Courant and K. O. Friedrichs in [15, pp.332-333]: “Three shocks separating three zones of different continuous states are impossible. · · · · · · Configurations involving three shock fronts must therefore involve an additional discontinuity. The simplest assumption, in agreement with many observations, is that there occurs in addition to the three shock fronts a single contact discontinuity line.” Such a wave structure is called a Mach configuration (cf. [11, p.2]). Figure 3 shows a possible Mach configuration, where $S_1$ is an incident shock separating the two supersonic regions $\Omega_0$ and $\Omega_1$; $S_3$ is called the reflected shock from the contact discontinuity line $D$; $S_2$ is called a Mach front. The flow in $\Omega_2$ and $\Omega_3$ are both subsonic. So $S_1$ is a supersonic shock
and $S_2, S_3$ are transonic shocks. Following terminologies in [11], we call $\Omega_0 \cup S_1 \cup \Omega_1$ the supersonic part of the Mach configuration, and $S_3 \cup \Omega_3 \cup D \cup \Omega_2 \cup S_2$ its subsonic part.

For the case that all the discontinuities are straight lines and the flow field is piece-wise constant, the existence of such a “flat” Mach configuration showed in Figure 3 can be proved by using shock polar. In Figure 5, the point $C$ represents the upstream supersonic flow in $\Omega_0$. $I_1$ corresponds to the flow state in $\Omega_1$. By the shock polar $R$ roots at $I_1$, we can determine all the possible states in $\Omega_3$; and by the polar roots at $C$, we may determine all the possible states in $\Omega_2$. Since $w$ and $p$ should be continuous across the contact line $D$, the point

$$I_{2,3} = (w_m, p_m)$$

(2.19)

where the two loops intersect represents the $w$ and $p$ in $\Omega_2, \Omega_3$. Hence by (2.9)–(2.11), the slope of the lines $S_1, S_2, S_3, D$ and $u, v$, as well as $\rho$, in $\Omega_1, \Omega_2, \Omega_3$, can all be uniquely determined. We recommend [15, pp.346-350] or [1, pp.22-24] for a detailed discussion.

**Figure 5.** Existence of a flat Mach configuration by using the $p - w$ shock polar.

It arises naturally the question of uniqueness: under what conditions, can we show the flat Mach configuration constructed above is the only possible one in the class of piecewise $C^1$ functions? We have the follows.

**Theorem 2.6.** Suppose the supersonic part, i.e., the uniform upcoming supersonic flow in $\Omega_0$ and a straight supersonic shock-front $S_1$, are given (see Figure 6), and $u > 0$, as well as the following asymptotic condition in the subsonic region $\Omega_2, \Omega_3$:

$$p \to p_m \quad \text{as} \quad (x, y) \to \infty,$$

(2.20)

where $p_m$ is determined in (2.19). For definiteness, we also fix the point $O \in S_1$ where it meets the possibly curved discontinuities $S_2, S_3, D$. Then any weak entropy solution $(p, w, u, \rho) \in L^\infty_{\text{loc}}(\mathbb{R}^2) \cap C^1(\Omega_0) \cap C^1(\Omega_1) \cap C^1_{\text{loc}}(\Omega_2 \setminus \{O\}) \cap C^1_{\text{loc}}(\Omega_3 \setminus \{O\})$ of
the Euler system must be the flat Mach configuration. Here $C^1_{\text{loc}}(\overline{\Omega}\{O\})$ is the set of those functions continuous in $\overline{\Omega}$ and continuously differentiable in any compact subset of $\overline{\Omega}\{O\}$. (2.20) may be replaced by

$$w \to p_m \quad \text{as} \quad \Omega_2 \cup \Omega_3 \ni (x,y) \to \infty,$$

and the same conclusion holds.

For the four Problems stated above, by the theory of classical solutions of hyperbolic systems [21], it is easy to show that the supersonic flow ahead of the transonic shock-fronts must be uniform (cf. [2, p.216] for a sketch). Particularly, we may prescribe supersonic initial data on $\{x = 0\}$ in Problem C (see Figure 6). To prove these uniqueness results, one then only needs to show the state behind the transonic shock must be uniform, and hence by shock polar, the shock-front must be straight. We will show how these hold and prove Theorem 2.1, Theorem 2.2 in §3, Theorem 2.3–Theorem 2.5 in §4, and Theorem 2.6 in §5, by using a rather unified and newly developed approach that combining various maximum principles and analysis of the shock polar.

3. Uniqueness of Transonic Shocks in Ducts

3.1. Reduction of the Euler System. Supposing

$$\rho u > 0 \quad \text{and} \quad 0 < M < 1,$$

the Euler system (2.1)–(2.4) may be written in the Lagrangian coordinates $(\xi, \eta)$ as (see [29] pp.1351–1356)

$$\partial_\xi p + \lambda_R \partial_\eta p - \beta_1 \partial_\eta w = 0,$$

$$\partial_\xi w + \beta_2 \partial_\eta p + \lambda_R \partial_\eta w = 0,$$

Figure 6. The possible curved Mach configuration in Euler coordinates $(x, y)$.
together with the invariance of entropy along flow trajectories
\[ \partial_\xi \left( \frac{p}{\rho} \right) = 0, \]  
and the Bernoulli law
\[ \frac{1}{2} u^2 (1 + w^2) + \frac{c^2}{\gamma - 1} = b_0. \]  
Here \( \xi = x \), and \( \eta \) measures the mass flux along two flow trajectories. The flow trajectory in Lagrangian coordinates is the line \( \eta = \text{constant} \). The coefficients in (3.2) and (3.3) are
\[
\begin{cases}
\lambda_R = \frac{\rho c^2 u w}{u^2 - c^2}, \\
\beta_1 = \frac{-\rho^2 c^2 u^3}{u^2 - c^2} > 0, \\
\beta_2 = \frac{(M^2 - 1)c^2}{u(u^2 - c^2)} > 0.
\end{cases}
\]  
These may be obtained from (4.8)(4.19) in [29, pp. 1355–1356], by taking \( \psi = \psi' = 0 \) there. Note our assumption \( 0 < M < 1 \) in (3.1) implies \( u \neq c \), which is required in (4.21) in [29, p. 1356]. By (3.11) or (3.12) in [29, p. 1352] and our assumptions in Definition 2.1, the transformation \( (x,y) \mapsto (\xi,\eta) \) is \( C^2 \) and invertible, provided that \( \rho u > 0 \). It is easy to check that for subsonic flows, (3.2) and (3.3) constitute an elliptic system, which is the elliptic part of the Euler system. Equations (3.4)(3.5) are the hyperbolic part of the Euler system.

The equations (3.2) (3.3) may also be expressed as
\[ Dw = WDp, \]  
with \( Dw = (\partial_\xi w, \partial_\eta w)^T \) the gradient of a function \( w \), and \( W \) the matrix
\[ W = \frac{-1}{\beta_1} \begin{pmatrix} \lambda_R & \beta_1 \beta_2 + \lambda_R^2 \\ -1 & -\lambda_R \end{pmatrix}. \]  
Note that \( \det W = \beta_2 / \beta_1 = (1 - M^2) / (\rho^2 u^4) > 0 \), so \( W \) is invertible. By acting \( (\partial_\eta, -\partial_\xi) \) from left to (3.7), we get a second order elliptic equation of \( p \) in divergence form:
\[ \partial_\xi \left( \frac{\partial_\xi p}{\beta_1} + \frac{\lambda_R}{\beta_1} \partial_\eta p \right) + \partial_\eta \left( \frac{\lambda_R}{\beta_1} \partial_\xi p + (\beta_2 + \frac{\lambda_R^2}{\beta_1}) \partial_\eta p \right) = 0. \]  
By our assumptions in Definition 2.1 the coefficients\(^4\) are all \( C^1 \) functions\(^5\), and the equation is uniformly elliptic in any fixed compact domain. Therefore the strong maximum principle for weak solutions of elliptic equations in divergence form is available, see Theorem 8.19 in [19, p. 189]. We also need the following boundary point lemma due to R. Finn and D. Gilbarg (see Lemma 7 in [18, p. 31]).\(^6\)

**Lemma 3.1.** Let the coefficients of the system
\[
- v_y = au_x + b'u_y, \quad v_x = b''u_x + cu_y, \\
\Delta = 4ac - (b' + b'')^2 \geq \delta > 0
\]  
\(^4\)They are \( 1/\beta_1, \lambda_R/\beta_1, \beta_2 + \lambda_R^2/\beta_1 \).
\(^5\)Hence bounded in any fixed compact domain.
\(^6\)Although the formulation here is a little different from that in [18], this lemma is obviously a direct consequence of that.
be Hölder continuous in the closure of a simply connected domain $\Omega \subset \mathbb{R}^2$ with $C^{1,\alpha}$ boundary $(0 < \alpha < 1)$, and $u, v \in C^1(\Omega)$ is a solution to (3.10). Suppose there is a $Q \in \partial \Omega$ satisfying $u(x) > u(Q)$ ($u(x) < u(Q)$) for all $x \in \Omega$, then there holds
\begin{align}
\frac{\partial u}{\partial n}(Q) > 0, \\
\frac{\partial v}{\partial n}(Q) < 0,
\end{align}
where $n$ is the inward drawn normal of the curve $\partial \Omega$ at $Q$.

It is clear that the inner normal $n$ in (3.12) may be replaced by any vector $l$ at $Q$ with $l \cdot n > 0$, since along the tangent $\tau$ of $\partial \Omega$ we have $\frac{\partial u}{\partial \tau}(Q) = 0$. In this paper we always use “$\cdot$” to denote the scalar product of vectors.

### 3.2. Finitely Long Duct: Proof of Theorem 2.1

Under the assumptions in Definition 2.1 and Theorem 2.1, we show (3.1) holds. We directly have $u > 0$ and $M < 1$. By (2.8), (2.10) and (3.4), the entropy $A(s) = p/\rho$ has the estimate
\begin{align}
0 < s_* < A(s) < s^* \quad \text{in } D^+
\end{align}
with two numbers $s_*, s^*$ depending only on the upcoming supersonic flow $p_0, u_0, \rho_0$ and the adiabatic exponent $\gamma > 1$. Hence from $u > 0$ and $M < 1$ we get $0 < u^2 < c^2 = \gamma A(s) \rho^{-1}$, therefore $\rho > 0$ in $D^+$. One then checks that for (3.7), $\Delta = 4\beta_2/\beta_1 = 4(1 - M^2)/(\rho^2 u^4)$ is bounded away from zero in a compact subset of $D^+$. So we may employ the results in §3.1 for the following analysis.

The duct, in the Lagrangian coordinates, is now the rectangle $\{ (\xi, \eta) : -1 < \xi < 1, 0 < \eta < \eta_0 \}$, with $\eta_0 = \rho_0 u_0 > 0$ (see Figure 7). At the exit $\Sigma_1 = \{ (1, \eta) : 0 \leq \eta \leq \eta_0 \}$ we still have the Dirichlet condition for pressure:
\begin{align}
p = p_1,
\end{align}
with $p_1$ a constant (cf. (2.12)). On the upper and lower walls $\Gamma_1 = \{ (\xi, \eta_0) : -1 < \xi < 1 \}, \Gamma_0 = \{ (\xi, 0) : -1 < \xi < 1 \}$, there should hold $w = 0$, hence $\partial_{\xi} w = 0$, and by (3.7), this is a Neumann condition of $p$:
\begin{align}
\partial_{\eta} p = 0,
\end{align}
since now we have $\lambda_R = 0$ and $\beta_1 \beta_2 > 0$.

![Figure 7](image-url)

**Figure 7.** The duct and transonic shock in $(\xi, \eta)$ coordinates.

Without loss of generality, we suppose the shock-front $S$, with the equation $\xi = \psi(\eta) \in C^2[0, \eta_0]$ in Lagrangian coordinates, passing through the origin $O$ (i.e., $\psi(0) = 0$ or $f(0) = 0$). We do not need write out the Rankine-Hugoniot
conditions in Lagrangian coordinates here because the relations (2.8)–(2.11) do not depend on coordinate system (these Rankine-Hugoniot conditions are available in [29, p.1358]). Since \( w = 0 \) on the walls, the shock-front always meets the walls perpendicularly (i.e., \( \psi'(0) = 0 = \psi'(\eta_0) \)), even in the (\( \xi, \eta \)) coordinates.

Let \( \Omega \) be the domain bounded by the shock-front, walls and exit, where the flow is subsonic. Now if \( p \) is a constant in \( \Omega \), that is, \( p = p_1 \) according to (3.14), then by (3.7), \( w \) is a constant, and by boundary conditions on walls, we have \( w = 0 \). So from the shock polar, we can infer that \( p_1 = p^+ \). If \( p_1 \neq p^+ \), then we get a contradiction and either the solution is not constant or \( C^1(\Omega) \) solution does not exist. We further may solve on \( S \) that \( p = p^+, u = u^+, \rho = p^+ \) and \( f' = 0 \) by (2.8)–(2.11), and by the hyperbolic part (3.4)–(3.5), obtain that \( \rho = \rho^+, u = u^+ \) in \( \Omega \). Hence if \( p \) is a constant in \( \Omega \), then \( p_1 = p^+ \) and the solution is exactly the special solution \( U_b \). The uniqueness is therefore proved.

Now we show that if \( p \) is not a constant in \( \Omega \), then there will occur contradictions by using maximum principles. In the following, we always assume that \( p \) is not a constant in \( \Omega \).

**Step 1.** First, by (3.9) and the strong maximum principle (Theorem 8.19 in [19, p.189]), \( p_{\text{max}}, p_{\text{min}} \), the maximum and minimum of \( p \) in \( \overline{\Omega} \), respectively, can only be attained on the boundary \( \partial \Omega = \Sigma_1 \cup (\Gamma_0 \cap \overline{\Omega}) \cup (\Gamma_1 \cap \overline{\Omega}) \cup S \). Since \( p_1 \) is a constant, \( p_{\text{max}} \) and \( p_{\text{min}} \) can not be attained simultaneously on \( \Sigma_1 \). By Lemma 3.1 and (3.15), none of them can be attained on the walls. \(^7\) So at least one of them should be attained on \( S \).

**Step 2.** Now we suppose that \( p_{\text{max}} \) is attained on \( S \). By shock polar, we can see the maximum is \( p^+ \) (corresponding to \( w = 0 \), that is the point \( A \) in Figure 4) and is certainly attained at \( O = (0, 0) \) and \( H = (\psi(\eta_0), \eta_0) \). Let \( \tau = (\psi'(\eta), 1) \) and \( n = (1, -\psi'(\eta)) \) be respectively the tangential and inner normal vector of \( S \) at \( (\psi(\eta), \eta) \), where \( p \) attains its maximum. By (3.17), we have

\[
\frac{\partial w}{\partial \tau} = \tau Dw = (\tau W) Dp.
\]

But

\[
(\tau W) \cdot n = \tau W n^T = \frac{1}{\beta_1}((\lambda_1 \psi' - 1)^2 + \beta_1 \beta_2 \psi'^2) > 0,
\]

by Lemma 3.1 we get

\[
(\tau W) Dp < 0.
\]

Note that at the corner points \( O, H \), we cannot employ Lemma 3.1 directly. However, since both the corner angles are \( \pi/2 \), we may use a reflection argument to overcome this difficulty. Taking \( O \) as an example. We first reflect \( S, \Omega \) with respect to \( \eta = 0 \) and obtain its image \( S', \Omega' \). Since the corner angle is \( \pi/2 \), the curve \( S \cup S' \) is still \( C^2 \). Let \( B_r(O) \) be an open disk centered at \( O \) with a very small radius \( r \), we set \( \mathcal{O} = B_r(O) \cap (\Omega \cup \Omega' \cup \Gamma_0) \). Then we use even reflection to extend \( p \) (both the solution \( p \) and those appeared in the coefficients through (3.7)) to \( \mathcal{O} \), and by Neumann condition (3.15), \( p \in C^1(\mathcal{O}) \). Similarly we use even reflection to extend \( \rho, u, v \) odd reflection to extend \( v, w \) to \( \mathcal{O} \). Hence in \( \mathcal{O} \) equation (3.9) still holds and

\(^7\) See (6.5) in [29, p.1358], that is, \( \psi'(\eta) = \frac{|uv|}{|p|} \).

\(^8\) Here the walls are the line segments \( \Gamma_i \cap \overline{\Omega} \) (\( i = 0, 1 \)) excluding their extreme points, hence they are open. Note that, however, by our definition, \( \Sigma_1 \) and \( S \) are closed.
the coefficients are at least Lipschitz continuous. We then use Lemma 3.1 at \(O \in O\) to obtain (3.18).

Hence whenever \(p\) reaches its maximum, we have \(\frac{\partial w}{\partial \tau} < 0\) by (3.16) and (3.18). Thus from \(w(O) = 0\), we may infer that \(w\) is always negative on \(S \setminus \{O\}\). This is impossible since \(w(H) = 0\).

**Step 3.** Next we consider the case that only the minimum of \(p\) is attained on \(S\). If \(p_{\text{min}} = p^+\), then by shock polar, we see that on \(S\) there should always hold \(p = p^+\) and \(w = 0\), hence \(\psi' = 0\). We then get a contradiction from (3.16) and (3.17): the left hand side of (3.16) is zero, while the right hand side should be positive due to Lemma 3.1.

**Step 4.** We then consider the case that \(p_{\text{min}} < p^+\). First we show \(p_{\text{min}} \neq p_s\). Indeed, if \(p_{\text{min}} = p_s\), by shock polar there are only two possibilities: the points corresponding to \(B\) or \(B'\) in Figure 4, where \(p = p_s\), and \(w\) attains its maximum \(w_s\) (at \(B\)), or minimum \(-w_s\) (at \(B'\)). By (3.17), we still see the left hand side of (3.16) is zero, while the right hand side is positive, a contradiction!

**Step 5.** Then we conclude that \(p_{\text{min}} \in (p_{\text{sonic}}, p_s) \cup (p_s, p^+)\). We remark that \(p_{\text{min}}\) should be larger than \(p_{\text{sonic}}\) by Definition 2.4 since for those points on shock polar below \(S\) and \(S'\), the flow is supersonic, and sonic at \(S, S'\).

Considering \(p\) as a function of \(w\) for \(p \in (p_{\text{sonic}}, p_s)\) and \(p \in (p_s, p^+)\) respectively, we see that \(p'(w) \neq 0\). Now suppose \(p\) attains its minimum \(p_{\text{min}}\) at a point \(P\) on \(S\), then at \(P\) we have

\[
0 = \tau Dw = \tau(w'Dw) = p'(w)\tau Dw = p'(w)\tau WDP
\]

due to (3.7). Therefore \(\tau WDP = 0\), a contradiction to Lemma 3.1 by (3.17), which yields \(\tau WDP > 0\). This finishes the proof of Theorem 2.1.

**Remark 3.1.** We may replace (2.12) or (3.14), the Dirichlet condition of \(p\), by \(w = 0\) on the exit \(\Sigma_1\). The proof is similar to those presented below in §4.1, and we omit the details.

### 3.3. Infinitely Long Duct: Proof of Theorem 2.2

For infinitely long duct, we set \(\Omega'\) as the subsonic region behind the shock-front \(S\):

\[
\Omega' = \{(\xi, \eta) : \xi > \psi(\eta), 0 < \eta < \eta_0\}.
\]

We first show that under the assumptions in Theorem 2.2 there holds

\[
\|p, \rho, u, v\|_{C(\Omega')} \leq C_0,
\]

with \(C_0\) depending only on \(p_0, u_0, \rho_0\) and \(\gamma\). Indeed, by Bernoulli law, the bounds of \(u, v\) and \(c\) is max\{\(\sqrt{\gamma}, \sqrt{\gamma - 1}\)\} \(\sqrt{\rho_0}\). But \(p = (c^2/(\gamma A(s)))^{1/(\gamma - 1)}\), by (3.13), \(\rho\) is bounded. Hence due to \(p = c^2\rho/\gamma\), \(p\) is also bounded.

We claim the coefficients in equation (3.9) is bounded, with a bounds determined by \(p_0, \rho_0, u_0\) and \(\gamma, u_*, M^*\). To prove this, from (3.13), we have a lower bound on \(\rho\):

\[
\rho = \left(\frac{c^2}{\gamma A(s)}\right)^{\frac{1}{\gamma - 1}} > \rho_* := \left(\frac{u_0^2}{\gamma s^*}\right)^{\frac{1}{\gamma - 1}}.
\]
Then by (3.6), it is easy to check that
\[|\lambda_R| \leq \frac{\rho|v|}{1-M^2} \leq \frac{C_0^2}{1-M^2},\]
\[\rho_*^2 u_*^3 < \beta_1 \leq \frac{C_0^3}{1-M^2},\]
\[\frac{1}{C_0} \leq \beta_2 \leq \frac{1}{u_* (1-M^2)},\]
and the claim is proved.

Finally, since \(M < M^* < 1\), it is straightforward to check that the equation (3.9) is uniformly elliptic in \(\Omega'\), and (3.11) holds in \(\Omega'\).

We now state an extreme principle of bounded solutions to general linear elliptic equations in the unbounded domain \(\Omega'\) with Neumann conditions on the lateral boundaries.

**Lemma 3.2.** Let \(Lu = \partial_i (a^{ij} \partial_j u) + \partial_n (a^{ij} \partial_j u) = 0\) be a linear uniformly elliptic equation with bounded coefficients \(a^{ij} \in C^{1}_{\text{loc}}(\overline{\Omega})\). Suppose \(u \in C(\overline{\Omega}) \cap C^{1}_{\text{loc}}(\overline{\Omega})\) is a bounded weak solution \(\Phi\) to this equation that satisfies the homogeneous Neumann conditions on the lateral boundaries \(\Gamma_0 = \{ (\xi, \eta) : \xi > \psi(0), \eta = 0 \}\) and \(\Gamma_1 = \{ (\xi, \eta) : \xi > \psi(\eta_0), \eta = \eta_0 \}\). Then either \(u_{\inf} = \inf_{\overline{\Omega}} u\) or \(u_{\sup} = \sup_{\overline{\Omega}} u\) is attained on \(\mathcal{S}\).

By this lemma, we see at least one of the extremes, \(p_{\inf}\) and \(p_{\sup}\), which are respectively the infimum and supremum of \(p\) in \(\overline{\Omega}\), is attained on \(\mathcal{S}\). The analysis in §3.2 then shows the uniqueness of the transonic shock solution, hence Theorem 2.2 is proved. \(^1\)

To prove Lemma 3.2, we need an up to boundary Harnack inequality for elliptic equations with Neumann boundary conditions:

**Lemma 3.3.** Let \(\Omega \subset \mathbb{R}^n\) be a domain with part of its boundary \(T \subset \partial \Omega\) lying in a plane, and \(\sum_{i,j=1}^{n} \partial_i (a^{ij} \partial_j u) = 0\) a linear equation with coefficients \(a^{ij}\) satisfying
\[\sum_{i,j=1}^{n} a^{ij} \xi_i \xi_j \geq \lambda |\xi|^2\quad \text{for any } (\xi_1, \cdots, \xi_n) \in \mathbb{R}^n,\]
(3.22)
\[\sum_{i,j=1}^{n} |a^{ij}| \leq \Lambda \quad \text{in } \Omega\]
(3.23)
for two fixed positive constants \(\lambda, \Lambda\). Suppose \(u \in W^{1,2}_{\text{loc}}(\Omega)\) is a continuous nonnegative weak solution satisfying the homogeneous Neumann condition on \(T\), that is, for any \(\varphi \in C^\infty_c(\mathbb{R}^n)\) whose support \(\text{supp } \varphi\) is disjoint with \(\partial \Omega \setminus T\), there holds
\[\int_{\text{supp } \varphi \cap (\Omega \setminus T)} \sum_{i,j=1}^{n} a^{ij} \partial_j u \partial_i \varphi \, dx = 0.\]
(3.24)
\(^9\)\(C^{1}_{\text{loc}}(\overline{\Omega})\) is the set of those functions which are \(C^1\) in any compact subset of \(\overline{\Omega}\).
\(^1\)A \(W^{1,2}_{\text{loc}}(\Omega)\) function \(u\) is a weak solution to a mixed boundary value problem if it satisfies the Dirichlet condition on \(\mathcal{D}\) in the sense of trace, the Neumann condition on \(\Gamma_0, \Gamma_1\) as stated in Lemma 3.2 below.
\(^1\)It is worth to note that we need not assume \(u, v, p \in C^1(\overline{\Omega})\) in the proof of Theorem 2.2. The assumption \(u, v, p \in C^{1}_{\text{loc}}(\overline{\Omega})\) on regularity is enough.
Then for any compact subset $K$ of $\Omega \cup T$, there holds
\[ \max_K u \leq C \min_K u, \] (3.25)
with the positive constant $C$ relying only on $K, \Omega, n, \Lambda/\lambda$.

**Proof.** Let $B_r(Q)$ be the open ball centered at $Q \in \mathbb{R}^n$ with radius $r$. For $K = \overline{B_r(Q)} \subset \subset \Omega$, (3.25) is the classical interior Harnack inequality (Theorem 8.20 in [19, p. 177]).

We then consider the case that $Q \in T$. Without loss of generality, we may suppose $T$ lying in the hyperplane $R_0^i := \{x = (x_1, \ldots, x_n) \in \mathbb{R}^n : x_n = 0\}$. By even reflections of $a^j(1 \leq i, j \leq n - 1)$ and $u$ with respect to $R_0^i$, and odd reflections of $a^j(i = n$ or $j = n$ but $i \neq j$) with respect to $R_n^i$, they are all extended from $B_{2r}^{-}(Q) := B_{2r}(Q) \cap \{x = (x_1, \ldots, x_n) \in \mathbb{R}^n : x_n > 0\} \subset \Omega$ to the whole ball $B_{2r}(Q)$ for a small $r$. We also get an elliptic equation satisfying (3.22) in $B_{2r}(Q)$. It is straightforward to check that $u$ is still a nonnegative weak solution (in the sense of (8.2) in [19, p. 177]) to this equation. Hence by the interior Harnack inequality we also have $\max_{B_{2r}(Q)} u \leq C \min_{B_{2r}(Q)} u$, and (3.25) follows with $K = B_{2r}^{-}(Q)$.

For any compact subset $\Omega \cup T$, we may cover it by finite balls (or half balls with center on $T$), and (3.25) holds by an argument as that in the proof of Theorem 2.5 in [19, p. 16].

**Proof of Lemma 3.2.** If $u$ is a constant, then obviously Lemma 3.2 holds. In the following we always assume that $u$ is not a constant. We prove by contradiction, that is, we assume that
\[ u_{\text{inf}} < \min_{\bar{S}} u, \quad u_{\text{sup}} > \max_{\bar{S}} u. \] (3.26)

The argument below is similar to that of proof of Theorem 1.2 in [7, p. 570].

Set $\Omega'_L = \Omega \cap \{\xi < L\}$ and $\Sigma_L = \Omega' \cap \{\xi = L\}$. By (3.26), and applying Lemma 3.1 as well as the strong maximum principle in $\Omega'_L$, we infer that there is an increasing sequence $\{L_k\}$ tends to infinity, and there are points $P_k, Q_k \in \Sigma_{L_k}$, such that
\[ u(P_k) = \max_{\Omega'_{L_k}} u < u_{\text{sup}}, \] (3.27)
\[ u(Q_k) = \min_{\Omega'_{L_k}} u > u_{\text{inf}} \] (3.28)
as $k \to \infty$.

We may choose $L_k$ rather large. Then define $U_k = \{(\xi, \eta) \in \Omega' : L_k - 2 < \xi < L_k + 2, 0 \leq \eta \leq \eta_0\}$, and $V_k = \{(\xi, \eta) \in \Omega' : L_k - 1 \leq \xi \leq L_k + 1, 0 \leq \eta \leq \eta_0\}$. By translation along $\xi$:
\[ \xi = \xi' + L_k, \quad \eta = \eta', \]
we may translate $U_k$ to $U_0 = \{(\xi', \eta') : -2 < \xi' < 2, 0 \leq \eta' \leq \eta_0\}$, and $V_k$ to $V_0 = \{(\xi', \eta') : -1 \leq \xi' \leq 1, 0 \leq \eta' \leq \eta_0\}$, and the equation $L u = 0$ in $U_k$ are transformed to $L' u_k = 0$ in $U_0$, with $u_k(\xi', \eta') = u(\xi' + L_k, \eta')$.

Now let $v_k = u_{\text{sup}} - u_{\text{inf}}$, which is positive and also a weak solution of $L' v = 0$ in $U_0$. Applying Lemma 3.2 to $v_k$, by the strong maximum principle, we have
\[ u_{\text{sup}} - u(Q_k) < \max_{V_0} v_k \leq C \min_{V_0} v_k < C (u_{\text{sup}} - u(P_k)). \] (3.29)
It is crucial to note that by our assumptions in Lemma 3.2, the constant $C$ here actually does not depend on $k$. Then taking $k \to \infty$, we have
\begin{equation}
0 < u_{sup} - u_{inf} < C \times 0 = 0,
\end{equation}
a contradiction desired. This finishes the proof of Lemma 3.2.

4. **Uniqueness of Transonic Shocks Attached to a Wedge**

4.1. **Uniqueness of Strong Transonic Shock: Proof of Theorem 2.3.** Now we consider the oblique transonic shocks attached to an infinite wedge against uniform supersonic flow (see Figure 2).

![Figure 8. The oblique transonic shock attached to a wedge in Lagrangian coordinates.](image)

In Lagrangian coordinates, the wall of the wedge is $\Gamma = \{(\xi, \eta) : \xi > 0, \eta = 0\}$, see Figure 8. On $\Gamma$ the slip condition is still
\begin{equation}
w = \tan \theta W. \tag{4.1}
\end{equation}

As in §3.2, let the equation of the shock-front $S$ be $\xi = \psi(\eta)$ with $\eta \geq 0$ and $\psi(0) = 0$, and $\Omega = \{(\xi, \eta) : \eta > 0, \psi(\eta) < \xi < \infty\}$. Condition (2.14) is now
\begin{equation}
w = \tan \theta W, \quad p = p_{strong} \tag{4.2}
\end{equation}
which holds at the origin $O$, and (2.13) is replaced by
\begin{equation}
w \to \tan \theta W \quad \text{as} \quad (\xi, \eta) \to \infty. \tag{4.3}
\end{equation}

By the latter requirement and shock polar, we infer that on $S$, there holds
\begin{equation}
p \to p_{strong} \quad \text{or} \quad p \to p_{weak} \quad \text{as} \quad (\xi, \eta) \to \infty. \tag{4.4}
\end{equation}

We derive a second order elliptic equation of $w$. Indeed, by (3.7), we have
\begin{equation}
Dp = W^{-1} Dw, \quad W^{-1} = -\frac{\beta_1}{\beta_2} W, \tag{4.5}
\end{equation}
then $w$ satisfies
\begin{equation}
\partial_\xi \left( \frac{\partial w}{\beta_2} + \frac{\lambda R}{\beta_2} \partial_\eta w \right) + \partial_\eta \left( \frac{\lambda R}{\beta_2} \partial_\xi w + (\beta_1 + \frac{\lambda R}{\beta_2}) \partial_\eta w \right) = 0, \tag{4.6}
\end{equation}
and \((3.17)\) is now
\[
\tau W^{-1} n^T = -\frac{\beta_1}{\beta_2} \tau W n^T < 0.
\]
\((4.7)\)

One may check that Lemma \(3.1\) holds for \((4.5)\), and the strong maximum principle holds for \((4.6)\).

If \(w\) is a constant, i.e., \(w = \tan \theta_W\) in \(\Omega\), then by \((4.2)\) and shock polar, we infer that there should hold \(p = p_{\text{strong}}\) on \(S\) and \(S\) is the special strong transonic shock-front by \((2.11)\). From \((2.9)\) and \((2.10)\), \(u\) and \(\rho\) can also be calculated on \(S\).

Then using the hyperbolic part \((3.4)\) \((3.5)\), we show the solution must be \(U_s\), and Theorem \(2.3\) is proved.

In the following we show there are contradictions if \(w\) is not a constant. Suppose now that \(w\) is not identical to \(\tan \theta_W\) in \(\Omega\). Then by Lemma \(3.1\) and the strong maximum principle, either the supremum of \(w\) in \(\Omega\), \(w_{\sup}\), or the infimum of \(w\) in \(\Omega\), \(w_{\inf}\), must be attained on \(S\). (By \((4.1)\) and \((4.3)\), \(w_{\sup}\) is indeed the maximum of \(w\), while \(w_{\inf}\) is the minimum of \(w\).)

We consider firstly the case that \(\tan \theta_W \in (0, w_s)\).

**Step 1.** Suppose \(w_{\sup}\) is attained at a point \(P\) on \(S\). If \(\tan \theta_W < w_{\sup} < w_s\), then \(P \neq O\). Note that by shock polar, we may regard \(w\) as a function of \(p\), \(w = w(p)\), which shares \(w'(p) \neq 0\) for \(w \neq \pm w_s\). Now considering the directional derivative of \(w\) along \(S\) at \(P\), \(\frac{\partial w}{\partial \nu}(P)\), which vanishes, we have
\[
0 = \frac{\partial w}{\partial \tau}(P) = \tau D w(P) = w'(p) \tau D p(P) = w'(p) \tau W^{-1} D w(P).
\]
\((4.8)\)

Hence \(\tau W^{-1} D w(P) = 0\). However, by \((4.7)\) and Lemma \(3.1\), one gets \(\tau W^{-1} D w(P) > 0\), a contradiction!

**Step 2.** Secondly consider the case \(w_{\sup} = w_s\), whence \(p(P) = p_s, P \neq O\). Then by \((4.5)\) and Lemma \(3.1\) we have
\[
\tau D p = \tau W^{-1} D w > 0.
\]
\((4.9)\)

This means \(p\) is strictly increasing once it attains \(p_s\). However, let \(P\) be the first point on \(S\) where \(w\) attains its maximum as \(p\) varies from 0 to \(\infty\). Note that \(p_{\text{strong}} > p_s\), so \(p\) decreases near the left side of \(P\), we should have \(\tau D p(P) \leq 0\), a contradiction to the assumption of \(p \in C^1(S)\).

**Step 3.** We then conclude that \(w_{\sup} = \tan \theta_W\), which is attained at \(O\). So \(w_{\inf} < \tan \theta_W\), and definitely \(w_{\inf}\) is attained on \(S\{O\}\). If \(w_{\inf} \neq -w_s\) and \(w_{\inf} \neq 0\), then as in **Step 1** above we can show a contradiction since \(w'(p) \neq 0\).

**Step 4.** Now for \(w_{\inf} = -w_s\), as that of \((4.9)\), we may get \(\tau D p = \tau W^{-1} D w < 0\). Therefore, whenever \(w\) attains its minimum \(-w_s\), \(p\) is strictly decreasing (hence less than \(p_s\)) and \(w\) should run into \((-w_s, -w_{\text{sonic}}\) (the arc \(B^9 S^1\) of shock polar), this contradicts to our asymptotic condition \((4.3)\): \(w \to \tan \theta_W > 0\).

**Step 5.** For the case \(w_{\inf} = 0\), where \(p\) attained its maximum on \(S\{O\}\), we get \(0 = \tau D p = \tau W^{-1} D w < 0\), a contradiction.

Secondly, for the case that \(\tan \theta_W = w_s\), we have \(w(O) = w_{\sup} = w_s\) by \((4.2)\) and \((4.3)\). This fact, together with \((4.1)\), \((4.3)\) and strong maximum principle, implies \(w_{\inf} < w_s\) should be attained on \(S\{O\}\). We then repeat **Step 3 – Step 5** above to reach contradictions.

This finishes proof of Theorem \(2.3\).
4.2. Uniqueness of Weak Transonic Shock: Proof of Theorem 2.4

Next we consider the uniqueness of weak transonic shock. Now at the vertex of the ramp $O$, we have

$$w = \tan \theta W, \quad p = p_{\text{weak}}$$

(4.10)

by (2.16). The condition at infinity is (4.3) plus

$$p < p_\infty \quad \text{as } \eta \to \infty \text{ on } S$$

(4.11)

with $p_\infty$ a constant less than $p_\ast$.

As in §4.1, to prove uniqueness, we only need to show that there are contradictions if $w$ is not a constant, and then either its supremum $w_{\text{sup}}$ or infimum $w_{\text{inf}}$ is attained on $S$.

**Step 1.** Now consider the case $w_{\text{sup}}$ is attained on $S$. If $\tan \theta W < w_{\text{sup}} < w_\ast$, the proof is the same as in Step 1 in §4.1. For $w_{\text{sup}} = w_\ast$, by (4.9), once $w$ reaches its maximum, $p$ should increase (larger than $p_\ast$). This would be a contradiction to our assumption (4.11) and (4.4).

**Step 2.** For the case that $w_{\text{inf}} < \tan \theta W$ is attained on $S$, if $w_{\text{inf}}$ lies on the arc $\hat{BAS}$, by continuity, $w$ should first attain its maximum $w_\ast$ (note now $w$ runs counterclockwise on the shock polar from $(\tan \theta W, p_{\text{weak}})$ to $B'$ on the arc $\hat{BAS}$).

There is a contradiction as we showed above in Step 1; if $w_{\text{inf}}$ lies on the arc $\hat{SB}$, just repeat Step 1 in §4.1 for a contradiction.

Theorem 2.4 is also proved.

4.3. Proof of Theorem 2.5

Finally, we show Theorem 2.5 holds. The arguments are very similar to that in §3.2, so we just outline the main ideas.

By (3.7) and (4.1), one gets an oblique derivative condition of $p$ on $W$:

$$\lambda R \partial_\xi p + (\beta_1 \beta_2 + \lambda R^2) \partial_\eta p = 0.$$ 

(4.12)

The asymptotic conditions (2.17) (2.18) are respectively

$$p \to p_{\text{strong}} \quad \text{as } (\xi, \eta) \to \infty,$$

(4.13)

$$p \to p_{\text{weak}} \quad \text{as } (\xi, \eta) \to \infty.$$ 

(4.14)

To show uniqueness, as before, it is sufficient to show contradictions if $p$ is not constant. In the following we always assume $p$ is not identical to $p_{\text{strong}}$ (resp. $p_{\text{weak}}$). Then by (4.12)–(4.14) and strong maximum principles applied to equation (3.9), either $p_{\text{sup}} = \sup_{\Omega} p$ or $p_{\text{sup}} = \inf_{\Omega} p$ is attained on $S$.

4.3.1. Uniqueness of Strong Transonic Shock. **Step 1.** Consider the case that $p_{\text{sup}} > p_{\text{strong}}$ is attained at a point $(\psi(\eta_*), \eta_*) \in S$. If $p_{\text{sup}} = p^+$, then as of Step 2 in §3.2, $w$ should be negative on $S$ for $\eta > \eta_*$. Note that (4.13) implies particularly $w \to \pm \tan \theta W$. So we get $w \to - \tan \theta W < 0$. Hence $\psi'(\eta) = uw/[p]$ has a negative upper bound for large $\eta$, which means $S$ will eventually intersect with the boundary $\{ (\xi, \eta) : \xi = 0, \eta > 0 \}$ where hyperbolic initial data is prescribed, that is not allowed.

**Step 2.** If $p^+ > p_{\text{sup}} > p_{\text{strong}}$, then $p_{\text{sup}}$ lies on the arc $\hat{AB}$ of $p - w$ shock polar, and by (3.19) there is a contradiction. So $p_{\text{sup}} = p_{\text{strong}}$. Since $p$ is not constant, we infer $p_{\text{inf}} < p_{\text{strong}}$ and is attained on $S\{O\}$. Just repeat **Step

---

Footnote 5. Note that $[p] > 0$ and $u > 0$ are both bounded away from zero on the shock-front which can be checked by using shock polar.
4–Step 5 in §3.2 for contradictions. This finishes the proof of Theorem 2.5 for the case of strong transonic shocks. Note that the arguments here also work for the case tan $\theta_W = w_*$. 

4.3.2. Uniqueness of Weak Transonic Shock. Consider as well the case that $p_{\text{sup}} > p_{\text{weak}}$ is attained at a point $(\psi(\eta_*), \eta_*) \in S$. If $p_{\text{sup}} = p^+$, then the argument is the same as in Step 1 in §4.3.1. If $p_{\text{sup}} = p_*$, refer Step 4 in §3.2 for a contradiction. If $p_{\text{sup}} \in (p_{\text{weak}}, p_*) \cup (p_*, p^+)$, refer Step 5 there. Hence we conclude that $p_{\text{sup}} = p_{\text{weak}}$, and $p_{\text{inf}} < p_{\text{weak}}$ is attained on $S \setminus \{O\}$. Therefore $p_{\text{inf}} \in (p_{\text{sonic}}, p_*)$ and refer also Step 5 in §3.2 for a contradiction. This finishes proof of Theorem 2.5 for the case of weak transonic shock.

Remark 4.1. In the above proofs we do not need to take care of the corner points $O$, so we may only require that $p, \rho, w, u \in C^1_{\text{loc}}(\Omega \setminus \{O\}) \cap C_{\text{loc}}(\Omega)$. 

5. Uniqueness of a Mach Configuration

Since the contact line $D$ is also a trajectory of the flow, by virtue of Lagrangian coordinates, it may be transformed to the $\xi$ axis, see Figure 9. In the following we use superscript $(i)$, such as $u^{(i)}, p^{(i)}, w^{(i)}, \rho^{(i)}$ to denote quantities in $\Omega_i$, with $i = 2, 3$. $D$ is a contact line means

$$w^{(2)} = w^{(3)}, \quad p^{(2)} = p^{(3)} \quad \text{on} \quad \eta = 0. \quad (5.1)$$

That is, $w$ and $p$ are continuous in $\Omega_2 \cup \Omega_3$, although others, such as, $\rho, u, v, s$, are not. The supersonic initial data is prescribed on the $\eta$ axis.

![Figure 9](image_url)

**Figure 9.** The possible curved Mach configuration in Lagrangian coordinates $(\xi, \eta)$.

We note that by the analysis of shock polar in §2.2.4, we can uniquely solve $p^{(2)}(O), w^{(2)}(O), u^{(2)}(O), \rho^{(2)}(O), p^{(3)}(O), w^{(3)}(O), u^{(3)}(O), \rho^{(3)}(O)$, as well as the $^{13}C^1_{\text{loc}}(\overline{\Omega})$ is the set of those functions that are continuous in any compact subset of $\overline{\Omega}$.
slope of the shock-fronts $S_2, S_3$ at $O$. Particularly, we have (cf. (2.19))

$$p^{(2)}(O) = p^{(3)}(O) = p_m, \quad w^{(2)}(O) = w^{(3)}(O) = w_m.$$  \hfill (5.2)

In the subsonic region $\Omega_i, i = 2, 3$, the Euler system reads

$$\partial_t p^{(i)} + \lambda^{(i)}_R \partial_t p^{(i)} - \beta_1^{(i)} \partial_t w^{(i)} = 0, \hfill (5.3)$$

$$\partial_t w^{(i)} + \beta_2^{(i)} \partial_t p^{(i)} + \lambda^{(i)}_R \partial_t w^{(i)} = 0, \hfill (5.4)$$

$$\partial_t \xi \left( \frac{p^{(i)}}{\rho^{(i)}} \right) = 0, \hfill (5.5)$$

$$\frac{4}{7} (u^{(i)})^2 (1 + (w^{(i)})^2) + \frac{(c^{(i)})^2}{2} = b_0. \hfill (5.6)$$

Here the coefficients are

$$\begin{align*}
\lambda^{(i)}_R &= \frac{\rho^{(i)} (c^{(i)})^2 u^{(i)} w^{(i)}}{(u^{(i)})^2 - (c^{(i)})^2}, \\
\beta_1^{(i)} &= -\frac{(\rho^{(i)})^2 (c^{(i)})^2 (u^{(i)})^3}{(u^{(i)})^2 - (c^{(i)})^2} > 0, \\
\beta_2^{(i)} &= \frac{(M^{(i)})^2 - 1 (c^{(i)})^2}{u^{(i)}((u^{(i)})^2 - (c^{(i)})^2)} > 0.
\end{align*} \hfill (5.7)$$

Now we begin to prove Theorem 2.6. It is divided into several steps. We consider only the case of the condition (2.20). The case of (2.21) may be proved similarly by adopting arguments in §4.1 and then combining the following reasoning, so we omit it.

**step 1.** If both $p^{(2)}$ and $p^{(3)}$ are constants in $\Omega_2, \Omega_3$ respectively, then we see the subsonic flow in $\Omega_2, \Omega_3$ are uniform, and by shock polar, the transonic shock-fronts are straight lines, and then according to analysis in §2.2.4, the Mach configuration must be the flat one constructed there. Theorem 2.6 is then proved.

**Step 2.** Now we show there will be contradictions if either $p^{(2)}$ or $p^{(3)}$ is not constant. There are two cases: I. One is constant while the other is not; II. Both are not constant.

**Step 3. Case I.** Without loss of generality, we assume $p^{(2)}$ is a constant, but $p^{(3)}$ is not. Then $p^{(2)} = p_m$ in $\overline{\Omega_2}$, hence $p^{(3)} = p_m$ on $\eta = 0$.

Let $p^{(3)}_{sup}$ and $p^{(3)}_{inf}$ be respectively the supremum and infimum of $p^{(3)}$ in $\overline{\Omega_3}$. If $p^{(3)}_{sup} > p_m$ or $p^{(3)}_{inf} < p_m$, by the strong maximum principle, they can only be attained on the shock-front $S_3 \setminus \{O\}$. As shown in §4.3, there will be contradictions.

**Step 4. Case II.** Denote $p^{(2)}_{sup}$ and $p^{(2)}_{inf}$ be respectively the supremum and infimum of $p^{(2)}$ in $\overline{\Omega_2}$.

**Step 4.1.** We firstly show $p^{(2)}_{sup} > p_m$ implies contradictions. Suppose now $p^{(2)}_{sup} > p_m$. Then by the strong maximum principle, the extreme $p^{(2)}_{sup}$ can only be attained on the shock-front $S_2 \setminus \{O\}$ or on the contact line $\eta = 0$. As shown in §4.3, there will be contradictions if this happens on $S_2$. So it is only possible to take place on $\eta = 0$. By (5.7), we infer $p^{(3)}_{sup} > p_m$. As before, $p^{(3)}_{sup}$ also can only be attained on $\eta = 0$. Hence we get

$$p^{(3)}_{sup} = p^{(2)}_{sup} \hfill (5.8)$$
and they are attained at the same point $Q$ ($Q \neq O$) on the contact line. Therefore, by Lemma 5.3 we have

$$\partial_\eta p^{(3)}(Q) < 0,$$
$$\partial_\eta p^{(2)}(Q) > 0.$$  

Now let us look at the equations (5.3)(5.4). We may solve (cf. (4.5))

$$\partial_\eta p^{(i)} = -\frac{1}{\beta_2^{(i)}}(\partial_\xi w^{(i)} + \lambda_R^{(i)} \partial_\eta w^{(i)}),$$  

Then by (5.11), we have

$$\partial_\xi p^{(i)} = \frac{1}{\beta_2^{(i)}}(\lambda_R^{(i)} \partial_\xi w^{(i)} + (\beta_1^{(i)} \beta_2^{(i)} + (\lambda_R^{(i)})^2) \partial_\eta w^{(i)}).$$  

Then by (5.3)(5.4) and (5.5)(5.10), we may get, at $Q$,

$$\partial_\eta w^{(3)} \cdot (1, \lambda_R^{(3)}) > 0,$$
$$\partial_\eta w^{(2)} \cdot (1, \lambda_R^{(2)}) < 0.$$

On the other hand, note that we have $\partial_\xi p^{(2)}(Q) = \partial_\xi p^{(3)}(Q) = 0$ by (5.8), therefore at $Q$,

$$(\partial_\xi w^{(i)}, \partial_\eta w^{(i)}) \cdot (\lambda_R^{(i)}, \beta_1^{(i)} \beta_2^{(i)} + (\lambda_R^{(i)})^2) = 0.$$  

We then get

$$\partial_\xi w^{(i)}(Q) = (\partial_\xi w^{(i)}, \partial_\eta w^{(i)}) \cdot (1, 0)$$
$$= -\frac{\lambda_R^{(i)}}{\beta_1^{(i)} \beta_2^{(i)}}(\partial_\xi w^{(i)}, \partial_\eta w^{(i)}) \cdot (\lambda_R^{(i)}, \beta_1^{(i)} \beta_2^{(i)} + (\lambda_R^{(i)})^2)$$
$$+ \frac{\beta_1^{(i)} \beta_2^{(i)} + (\lambda_R^{(i)})^2}{\beta_1^{(i)} \beta_2^{(i)}}(\partial_\xi w^{(i)}, \partial_\eta w^{(i)}) \cdot (1, \lambda_R^{(i)})$$
$$= \frac{\beta_1^{(i)} \beta_2^{(i)} + (\lambda_R^{(i)})^2}{\beta_1^{(i)} \beta_2^{(i)}}(\partial_\xi w^{(i)}, \partial_\eta w^{(i)}) \cdot (1, \lambda_R^{(i)}) \begin{cases} < 0 & i = 2, \\ > 0 & i = 3. \end{cases}$$  

However, by (5.4), we should have $\partial_\xi w^{(2)}(Q) = \partial_\xi w^{(3)}(Q)$, a contradiction!

**Step 4.2.** We have now $p_{m}^{(2)} = p_{m}$. Since $p^{(2)}$ is not constant, one has $p_{m}^{(2)} < p_{m}$. Arguments as in **Step 4.1** show that we must have $p_{m}^{(2)} = p_{m}^{(3)}$ and they are attained at the same point $Q \neq O$ on the contact line, and a contradiction similar to (5.10) then follows.

This finishes the proof of Theorem 2.6.

**Acknowledgments.** B. Fang is supported in part by NNSF of China under Grant No. 10801096. L. Liu is supported in part by NNSF of China under Grant No. 10971134. H. Yuan is supported in part by NNSF of China under Grants No. 10901052, No. 10871071, and Chenguang Program (09CG20) sponsored by Shanghai Municipal Education Commission and Shanghai Educational Development Foundation.
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