Black Holes and Higher Composition Laws

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ABSTRACT

We describe various relations between Bhargava’s higher composition laws, which generalise Gauss’s original composition law on integral binary quadratic forms, and extremal black hole solutions appearing in string/M-theory and related models. The cornerstone of these correspondences is the identification of the charge cube of the \textit{STU} black hole with Bhargava’s cube of integers, which underpins the related higher composition laws.
1 Introduction

In 1801 Gauss introduced a beautiful composition law on binary quadratic forms with integer coefficients. In the modern parlance, the set of SL(2, Z)-equivalence classes of primitive binary quadratic forms of a given discriminant $D$, denoted $\text{Cl}(\text{Sym}^2(Z^2)^2; D)$, has an inherent group structure, which is isomorphic to the narrow class group $\text{Cl}^+(S(D))$ of the unique quadratic ring $S(D)$ of discriminant $D$. This result is all the more remarkable in light of the fact that groups were yet to be defined! This work was clearly ahead of its time and, in the words of Andrew Wiles, “it came to a stop with Gauss”. The original composition law lay in waiting for a little over 200 years, when Manjul Bhargava made ground breaking progress by introducing a set of 14 (subsuming Gauss’s original) higher composition laws [1–3].

In [4] Moore established a relationship between the arithmetics of supersymmetric black holes in string theory and Gaussian composition. Remarkably, Bhargava’s higher composition laws are also closely related to various classes of black hole solutions appearing in string/M-theory [5–9]. In particular, Bhargava’s higher law on $2 \times 2 \times 2$ hypermatrices, or cubes, of integers is directly related to the extremal black hole solutions of the $STU$ model [10,11]. By identifying $\text{SL}(2, Z)$ with the black hole charge cube of [11] with the integers on the corners of Bhargava’s cube, the $U$-duality equivalence classes of extremal $STU$ black hole charge vectors valued in $\mathbb{Z}^2 \otimes \mathbb{Z}^2 \otimes \mathbb{Z}^2$ with Bekenstein-Hawking entropy $S_{BH} = \pi \sqrt{|\Delta|}$ are in one-to-one correspondence [8] with pairs $(S, (I_1, I_2, I_3))$, where $S$ is the unique quadratic ring with non-zero discriminant $D = -\Delta$ and $(I_1, I_2, I_3)$ is an equivalence class of balanced triples of oriented ideals, $I_1I_2I_3 \subseteq S$ and $N(I_1)N(I_2)N(I_3) = 1$. When restricting to projective $1$- $2 \times 2 \times 2$ cubes Bhargava’s composition law endows the equivalence classes with a group structure isomorphic to $\text{Cl}^+(S) \times \text{Cl}^+(S)$, where $\text{Cl}^+(S)$ denotes the narrow class group of the quadratic ring $S$ with discriminant $D = -\Delta$. Correspondingly, the $U$-duality equivalence classes of projective $STU$ black holes are characterised by $\text{Cl}^+(S) \times \text{Cl}^+(S)$ and the corresponding class numbers count the physically distinct projective black hole configurations [8].

The relationship between Bhargava’s composition law on cubes and the black holes of the $STU$ model is but one example. It maintains a special status as the linch-pin holding together all higher composition laws related to quadratic orders. Equivalently, the $STU$ model is the cornerstone of the various relevant (super)gravity theories. In the present contribution we describe the relations between Bhargava’s higher composition laws and black hole solutions in Einstein-Maxwell-scalar theories. Many of these correspond to supergravities with a string/M-theory origin, but supersymmetry is not essential. A number of the examples of have appeared in the literature before [5–9].

The first six examples of higher composition laws (including Gauss’s original) introduced by Bhargava are obtained by through various “symmetrisations” or embeddings of the $2 \times 2 \times 2$ cube and are hence closely related to the $STU$ model. The new cases derived from the law on cubes are defined on: (1) pairs of binary quadratic forms; (2) binary cubic forms; (3) pairs of quaternary alternating 2-forms; and (4) senary alternating 3-forms. In each case there is a corresponding (super)gravity theory. We describe in the detail these six cases as well as the additional cases of $E_7(7)(\mathbb{Z})$ and $SO(6, 6; Z)$ pertaining to $N = 8$ supergravity and its consistent truncation to $SO(6, 6; Z)$ Maxwell-Einstein-scalar theory. In fact, all 14 of Bhargava’s higher composition laws have a realisation in terms of black $p$-branes in (super)gravity theories, as remarked on in the conclusions. However, we shall return to the complete correspondence in future work. We also explain why such a correspondence between black hole charges and (super)gravity theories should exist at all.

1 Projectivity is the natural generalisation of primitivity for higher composition laws [1].
1. $E_{7(7)}(\mathbb{Z})$ & $\mathfrak{g}(\mathbb{R})^{G_2}$ & $\mathfrak{g}(\mathbb{R})^{G_2}$ & $\mathfrak{g}(\mathbb{R})^{G_2}$ & $N = 8$
2. $\text{SO}(6, 6; \mathbb{Z})$ & $\mathfrak{g}(\mathbb{R})^{G_2}$ & $\mathfrak{g}(\mathbb{R})^{G_2}$ & $\mathfrak{g}(\mathbb{R})^{G_2}$ & $N = 0, 16A, 36\phi$
3. $\text{SL}(6, \mathbb{Z})$ & $\wedge^2 \mathbb{Z}^6 \cong \mathfrak{g}(\mathbb{R})^{G_2}$ & $S(D), M_3$ & $\{\ast\}$ & $N = 0, 10A, 20\phi$
4. $\text{SL}(2, \mathbb{Z}) \times \text{SL}(4, \mathbb{Z})$ & $\mathbb{Z}^2 \otimes \wedge^2 \mathbb{Z}^4$ & $S(D), (I_x, M_2)$ & $\mathbb{C}^+(S(D))$ & $N = 0, 6A, 11\phi$
5. $\text{SL}(2, \mathbb{Z}) \times \text{SL}(2, \mathbb{Z}) \times \text{SL}(2, \mathbb{Z})$ & $\mathbb{Z}^2 \otimes \mathbb{Z}^2 \otimes \mathbb{Z}^2$ & $S(D), (I_x, H, I_T)$ & $\mathbb{C}^+(S(D)) \times \mathbb{C}^+(S(D))$ & $N = 2, STU$
6. $\text{SL}(2, \mathbb{Z}) \times \text{SL}(2, \mathbb{Z})$ & $\mathbb{Z}^2 \otimes \text{Sym}^2(\mathbb{Z}^2)$ & $S(D), (I_x, H, I_T)$ & $\mathbb{C}^+(S(D))$ & $N = 2, ST^2$
7. $\text{SL}(2, \mathbb{Z})$ & $\text{Sym}^3(\mathbb{Z}^2)$ & $S(D), (H_x, H, I_T, \delta)$ & $\mathbb{C}^3(S(D))$ & $N = 2, T^3$
8. $\text{SL}(2, \mathbb{Z})$ & $\text{Sym}^3(\mathbb{Z}^2)^*$ & $S(D), I$ & $\mathbb{C}^+(S(D))$ & $[4]$

**Table 1:** Summary of higher composition laws and black holes charge orbits in $D = 4$.

## 2 Black holes, higher composition laws and quadratic orders

In this section we shall describe the relationship between black holes and higher composition laws associated to ideal classes in quadratic orders. These are summarised in Table 1. Before treating the individual cases, we shall expand upon the notion of generalising Gauss composition in the context of quadratic orders and how/why it should be related to black holes appearing in “gravity theories of type $E_7$”.

### 2.1 Prehomogeneous vector spaces and group of type $E_7$

What does it mean to generalise Gauss’s composition law? One perspective, advocated by Bhargava [12] and also put to good use by Krutelevich [13], is motivated by the expression of Gauss’s composition law as a parametrisation result:

**Theorem 1** There is a canonical bijection between the set of $\text{SL}(2, \mathbb{Z})$-equivalence classes of nondegenerate binary quadratic forms $f(x, y) = ax^2 + bxy + cy^2$, $a, b, c \in \mathbb{Z}$, and the set of isomorphism classes of pairs $(S, I)$, where $S$ is a nondegenerate oriented quadratic ring and $I$ is an oriented ideal class of $S$.

Note, in the above we have not restricted to primitive forms, hence the lack of a group structure on the sets of equivalence classes. To summarise, we have a group $G(\mathbb{Z}) \cong \text{SL}(2, \mathbb{Z})$ with a representation $V(\mathbb{Z}) \cong \text{Sym}^2(\mathbb{Z}^2)^*$, such that the space of orbits $V(\mathbb{Z})/G(\mathbb{Z})$ is parametrised by ideal classes in a quadratic order. The two objects of Theorem 1 are connected by the discriminant. On the one hand, the unique algebraically independent $\text{SL}(2, \mathbb{Z})$-invariant on binary quadratic forms is the discriminant, $D = b^2 - 4ac$, which takes values 0, 1 mod 4. On the other, every quadratic order $S$ is determined up to isomorphism by its discriminant, $D = \text{Disc}(S)$, which also necessarily takes values 0, 1 mod 4. The orbits of forms with discriminant $D$ are one-to-one with ideal classes in $S$ with $D = \text{Disc}(S)$.

So, one way to interpret generalisations would be to seek $G$ and $V$ such that $V(\mathbb{Z})/G(\mathbb{Z})$ is naturally parametrised by some algebraic structures. We will take “naturally” here to mean that the two sides of the picture are connected by some $G$-invariant on the space $V$ that is bijectively mapped to an invariant characterising the corresponding algebraic structure, for example, the discriminant in the case of Theorem 1.

This shifts the question to one of identifying suitable candidate $G, V$. How should one go about this? Let us begin by noting that in the case of Theorem 1, $\text{GL}(2, \mathbb{C})$ has a single Zariski-open orbit\(^2\) on the space of binary quadratic forms over $\mathbb{C}$ with non-vanishing discriminant. In view of Theorem 1, this can be thought of as following from the fact that over $\mathbb{C}$ there is a unique class of pairs $(S, I) \cong \mathbb{C} \oplus \mathbb{C}$, where $S = I \cong \mathbb{C} \oplus \mathbb{C}$. Said another way, the pair $\text{GL}(2, \mathbb{C}), \text{Sym}^3(\mathbb{Z}^2)^*$ forms a prehomogeneous vector space:

**Definition 2** Let $G$ be an algebraic group and $\rho$ a rational representation on a vector space $V$. The triple $(G, \rho, V)$ is said to be a prehomogeneous vector space if the action of $G(\mathbb{C})$ on $V(\mathbb{C})$ has exactly one Zariski-dense orbit.

Seeking generalisations of Theorem 1, which are simple in the sense that the algebraic side of the equation reduces to a single object when taken over $\mathbb{C}$, we should therefore focus on prehomogeneous vectors. Fortunately, the reduced irreducible\(^3\) prehomogeneous vector spaces have been classified, as presented in §7 of [14]. If an irreducible prehomogeneous vector space is regular\(^4\) it has a unique (up to a scalar factor) irreducible homogeneous relative $G$-invariant polynomial $f : V(\mathbb{C}) \to \mathbb{C}$ [14], which is the obvious candidate $G$-invariant underpinning our putative bijection to some algebraic structure. For example, in the case of $(\text{GL}(2, \mathbb{C}), \rho, \text{Sym}^3(\mathbb{C}^2)^*)$ it is, of course, nothing but the discriminant.

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\(^1\)One might prefer to think of the family of $\text{SL}(2, \mathbb{C})$ orbits parametrised by the discriminant.

\(^2\)See [14] for definitions.

\(^3\)A prehomogeneous vector space is said to be regular if there is a relative $G$-invariant polynomial $f(x), x \in V$, with a not identically zero Hessian.
The six higher composition laws related to quadratic orders, constructed by Bhargava, are associated to a special class of prehomogeneous vector spaces for which \( G(\mathbb{C}) \) is of the form \( G(\mathbb{C}) \cong \text{GL}(1, \mathbb{C}) \times G_7(\mathbb{C}) \) and \( G_7 \) is a group of type \( E_7 \) [15].

In fact, the relevant \( G_7 \) appearing here are reduced groups of type \( E_7 \), which implies they are built on an underlying cubic Jordan algebra. In the seminal work of Günyaydin, Sierra and Townsend [16, 17] it was shown that precisely such groups, with specific real forms, elegantly underpin the generic Jordan and magic \( N = 2 \) supergravity theories. It has since been understood, largely through the work of Günyaydin, that these structures appear in many places in string/M-theory and black hole physics. See for example [5–7, 18–45]. In particular, see also [46, 47] for early examples and subsequent generalisations to magic triangles. The relationships between black holes in (super)gravity theories, pre-heterotic supergravities are not themselves directly related to Bhargava’s higher composition laws (see comments in subsection 2.5), the appearance of groups of type \( E_7 \) as the global symmetries in various (super)gravity theories is the first stone in the bridge between black holes and higher composition laws.

Accordingly, let us now briefly introduce groups of type \( E_7 \), which are characterised as the automorphisms of Freudenthal triple systems (FTS) [15]:

**Definition 3** A FTS is a finite dimensional vector space \( \mathfrak{F} \) over a field \( \mathbb{F} \) (not of characteristic 2 or 3), such that:

1. \( \mathfrak{F} \) possesses a non-degenerate antisymmetric bilinear form \( \{ x, y \} \).
2. \( \mathfrak{F} \) possesses a symmetric four-linear form \( q(x, y, z, w) \) which is not identically zero.
3. If the ternary product \( T(x, y, z) \) is defined on \( \mathfrak{F} \) by \( \{ T(x, y, z), w \} = q(x, y, z, w) \), then
   \[
   3\{T(x, x, y), T(y, y, y)\} = \{x, y\}q(x, y, y, y). \tag{2.1}
   \]

For notational convenience, let us introduce

\[
2\Delta(x, y, z, w) \equiv q(x, y, z, w), \tag{2.2a}
\]

\[
\Delta(z) \equiv \Delta(x, x, x), \tag{2.2b}
\]

\[
T(x) \equiv T(x, x, x). \tag{2.2c}
\]

**Definition 4** The automorphism group of an FTS is defined as the set of invertible \( \mathbb{R} \)-linear transformations preserving the quartic and quadratic forms:

\[
\text{Aut}(\mathfrak{F}) := \{ \sigma \in \text{Isom}(\mathfrak{F}) | \{ \sigma x, \sigma y \} = \{ x, y \}, \Delta(\sigma x) = \Delta(x) \}, \tag{2.3}
\]

which implies \( \sigma T(x, y, z) = T(\sigma x, \sigma y, \sigma z) \) and thus the automorphism group of the triple product. This defines groups of type \( E_7 \). The prototypical example is, unsurprisingly, given by \( E_7 \), in which case \( \mathfrak{F} \) is the fundamental 56-dimensional representation and the antisymmetric four-linear form and symmetric four-linear form are the unique symplectic quadratic and totally symmetric invariant in \( \mathfrak{F} \times \mathfrak{F} \) and \( \text{Sym}^4(\mathfrak{F}) \), respectively.

As shown in [15, 48], every simple reduced\(^5\) \( \mathfrak{F} \) is isomorphic to an \( \mathfrak{F}(\mathfrak{J}) \), where

\[
\mathfrak{F}(\mathfrak{J}) := \mathbb{F} \oplus \mathbb{F}^* \oplus \mathfrak{J} \oplus \mathfrak{J}^* \tag{2.4}
\]

and \( \mathfrak{J} \) is the Jordan algebra of an admissible cubic form with base point or the Jordan algebra of a non-degenerate quadratic form. With \( \mathfrak{F} \) regarded as an \( \text{Aut}(\mathfrak{F}) \)-module, this corresponds to the decomposition of \( \text{Aut}(\mathfrak{F}(\mathfrak{J})) \) under \( \text{Str}(\mathfrak{J}) \), the structure group of \( \mathfrak{J} \). The notation \( \mathbb{F}^* \cong \mathbb{F} \) and \( \mathfrak{J}^* \cong \mathfrak{J} \) indicates that they transform in conjugate representations of \( \text{Str}(\mathfrak{J}) \). For example, choosing \( \mathfrak{J}^3_\mathbb{O} \), the Jordan algebra of \( 3 \times 3 \) Hermitian octonionic matrices, we have \( \text{Aut}(\mathfrak{F}(\mathfrak{J}^3_\mathbb{O})) \cong E_7(\mathbb{O}) \), \( \text{Str}(\mathfrak{J}^3_\mathbb{O}) \cong U(1) \times E_6(\mathbb{O}) \) and \( \mathfrak{F}(\mathfrak{J}^3_\mathbb{O}) \) is the unique 56-dimensional \( E_7(\mathbb{O}) \)-module, which decomposes as

\[
\mathbb{F} \rightarrow 1_3 + 1_{-3} + 27_1 + 27_{-1} \text{ under } \text{Str}(\mathfrak{J}^3_\mathbb{O}) \subset \text{Aut}(\mathfrak{F}(\mathfrak{J}^3_\mathbb{O})).
\]

The FTS quadratic form, quartic norm and triple product are then defined in terms of the basic Jordan algebra operations [15].

We shall write elements in the basis (2.4) as

\[
x = (\alpha, \beta, A, B), \tag{2.5}
\]

where \( \alpha, \beta \in \mathbb{F} \) and \( A, B \in \mathfrak{J} \), respectively. Then

\[
\{x, y\} = \alpha \delta - \beta \gamma + \text{Tr}(A, D) - \text{Tr}(B, C), \tag{2.6a}
\]

\(\text{An FTS is simple if and only if } \{x, y\} \text{ is non-degenerate, which we assume. An FTS is said to be reduced if it contains a strictly regular element: }u \in \mathfrak{F} \text{ such that } T(u, u, u) = 0 \text{ and } u \in \text{Range } L_{x,y} \text{ where } L_{x,y} : \mathfrak{F} \to \mathfrak{F}; L_{x,y}(z) := T(x, y, z). \text{ Note that FTS on “degenerate” groups of type } E_7 \text{ (as defined in [35], and Refs. therein) are not reduced and hence cannot be written as } \mathfrak{F}(\mathfrak{J}); \text{ they correspond to theories which cannot be uplifted to } D = 5 \text{ dimensions consistently reflecting the lack of an underlying } \mathfrak{J}.\)
Table 2: The automorphism group $\text{Aut}(\mathfrak{J}(3))$ and the dimension of its representation $\dim \mathfrak{J}(3)$ given by the Freudenthal construction defined over the cubic Jordan algebra $\mathfrak{J}$ over $\mathbb{R}$ (with dimension $\dim \mathfrak{J}$ and reduced structure group $\text{Str}_0(\mathfrak{J})$).

| Jordan algebra $\mathfrak{J}$ | $\text{Str}_0(\mathfrak{J})$ | $\dim \mathfrak{J}$ | $\text{Aut}(\mathfrak{J}(3))$ | $\dim \mathfrak{J}(3)$ |
|-------------------------------|-----------------------------|---------------------|-------------------------------|------------------------|
| $\mathbb{R}$                 | $-$                         | 1                   | $\text{SL}(2, \mathbb{R})$  | 4                      |
| $\mathbb{R} \oplus \mathbb{R}$ | $\text{SO}(1, 1)$          | 2                   | $\text{SL}(2, \mathbb{R}) \times \text{SL}(2, \mathbb{R})$ | 6                      |
| $\mathbb{R} \oplus \mathbb{R} \oplus \mathbb{R}$ | $\text{SO}(1, 1) \times \text{SO}(1, 1)$ | 3 | $\text{SL}(2, \mathbb{R}) \times \text{SL}(2, \mathbb{R}) \times \text{SL}(2, \mathbb{R})$ | 8                      |
| $\mathbb{R} \oplus \Gamma_{r,s}$ | $\text{SO}(1, 1) \times \text{SO}(r, s)$ | $r + s + 1$ | $\text{SL}(2, \mathbb{R}) \times \text{SO}(r + 1, s + 1)$ | $2(r + s + 2)$          |
| $\mathfrak{J}^\mathbb{R}$   | $\text{SL}(3, \mathbb{R})$ | 6                   | $\text{Sp}(6, \mathbb{R})$  | 14                     |
| $\mathfrak{J}^\mathbb{C}$   | $\text{SL}(3, \mathbb{C})$ | 9                   | $\text{SU}(3, 3)$           | 20                     |
| $\mathfrak{J}^\mathbb{R}^*$  | $\text{SL}(3, \mathbb{R}) \times \text{SL}(3, \mathbb{R})$ | 9 | $\text{SL}(6, \mathbb{R})$ | 20                     |
| $\mathfrak{J}^\mathbb{R}$   | $\text{SU}^*(6)$           | 15                  | $\text{SO}^*(12)$          | 32                     |
| $\mathfrak{J}^\mathbb{C}$   | $\text{SL}(6, \mathbb{C})$ | 15                  | $\text{SO}(6, 6)$          | 32                     |
| $\mathfrak{J}^\mathbb{R}$   | $\text{E}_6(-26)$          | 27                  | $E_7(-25)$                  | 56                     |
| $\mathfrak{J}^\mathbb{C}$   | $\text{E}_6(6)$            | 27                  | $E_7(7)$                    | 56                     |

For $x = (\alpha, \beta, A, B)$ and $y = (\gamma, \delta, C, D)$ and

$$\Delta(x) = - (\alpha \beta - \text{Tr}(A, B))^2 - 4[\alpha N(A) + \beta N(B) - \text{Tr}(A^2, B^2)]$$

(2.6b)

$$T(x) = (-\alpha \kappa(x) - N(B), \beta \kappa(x) + N(A), - (\beta B^2 - B \times A^2) + \kappa(x) A, (\alpha A^2 - A \times B^2) - \kappa(x) B)$$

(2.6c)

where $\text{Tr} : \mathfrak{J} \times \mathfrak{J} \to \mathbb{F}, \ N : \mathfrak{J} \to \mathbb{F}$ are the trace form and cubic norm of $\mathfrak{J}$, respectively. For details, see [38] and references therein. In Table 2 we list the relevant Jordan algebras, associated FTS and their automorphism groups.

We can then define an integral $\mathfrak{J}_Z$, as introduced in the important work of [13], based on an integral Jordan algebra $\mathfrak{J}_Z$ [49].

$$\mathfrak{J}(3)_Z \equiv \mathfrak{J}_Z \cong \mathbb{Z} \oplus \mathbb{Z} \oplus \mathfrak{J}_Z \oplus \mathfrak{J}_Z.$$

(2.7)

The integral structure on $\mathfrak{J}_Z$ implies that $N(A) \in \mathbb{Z}$ for all $A \in \mathfrak{J}_Z$. Hence, from (2.6b) we see that the quartic norm is quantised,

$$\Delta(x) = 0, 1 \mod 4, \quad \forall x \in \mathfrak{J}_Z.$$

(2.8)

That is, the allowed values of the quartic norm coincide precisely with those of the discriminant of a quadratic order.

The automorphism group is broken to a discrete subgroup

$$\text{Aut}(\mathfrak{J}(3)_Z) := \{ \sigma \in \text{IsoZ}(\mathfrak{J}(3)_Z) | \{ \sigma x, \sigma y \} = \{ x, y \}, \Delta(\sigma x) = \Delta(x) \}$$

(2.9)

and is a model, in the sense of [50], for $\text{Aut}(\mathfrak{J})$ over $\mathbb{Z}$. In addition to the $\text{Aut}(\mathfrak{J})$-invariant quartic norm, there is a set of discrete invariants

$$d_1(x) = \text{gcd}(x)$$
$$d_2(x) = \text{gcd}(3 T(x, x, y) + \{ x, y \} \forall y)$$
$$d_3(x) = \text{gcd}(T(x))$$
$$d_4(x) = |\Delta(x)|$$
$$d'_4(x) = \text{gcd}(x \wedge T(x)),$$

(2.10)

where $\wedge$ denotes the antisymmetric tensor product. For reducible FTS we also have

$$d'_4(x) = \text{gcd}(B^2 - \alpha A, A^2 - \beta B, R(x))$$

(2.11)

where $R(x) : \mathfrak{J} \to \mathfrak{J}$ is a Jordan algebra endomorphism given by

$$R(x)(C) = 2(\alpha \beta - \text{Tr}(A, B)) C + 2\{A, B, C\}, \quad C \in \mathfrak{J},$$

(2.12)

where $\{A, B, C\}$ is the Jordan triple product.

The integral Freudenthal triple systems, $\mathfrak{J}_Z$, and their corresponding discrete automorphism groups, $\text{Aut}(\mathfrak{J}(3)_Z)$, will provide our $V(\mathbb{Z})$ and $G(\mathbb{Z})$, respectively. By definition, they have the desired property that, when taken over $\mathbb{C}$, $\text{GL}(1, \mathbb{C}) \times G(\mathbb{C})$ has one Zariski-dense orbit on $V(\mathbb{C})$. The quartic norm, $\Delta : \text{Sym}^2 \mathfrak{J}(C) \to \mathbb{C}$, provides the generalisation of the discriminant $D = b^2 - 4ac$; all elements in $V(\mathbb{C})$ with non-vanishing quartic norm belong to the unique Zariski-dense orbit. Since over $\mathbb{Z}$ we have (2.8), the quartic norm can be consistently identified with the discriminant of a quadratic order $S(D)$, providing the link to the corresponding algebraic structures.
2.2 Einstein-Maxwell-Scalar theories of type $E_7$

Following the preceding discussion it is perhaps unsurprising that when connecting to higher composition laws associated to quadratic orders, we consider black hole solutions in “Einstein-Maxwell–scalar theories of type $E_7$” or just “theories of type $E_7$” for short. By theories of type $E_7$ we mean Einstein-Hilbert gravity coupled to Abelian 1-forms $A$ and scalar fields $\phi$ (with no potential) such that: (i) the Gaillard-Zumino electromagnetic duality group $[51]$ is the automorphism group $\text{Aut}(\mathfrak{g})$ of some $\mathfrak{g}$; (ii) the Abelian field strengths together with their duals take values in $\Lambda^2 (M) \otimes \mathfrak{g}$; (iii) and the scalars parametrise the coset $\text{Aut}(\mathfrak{g})/[\text{Aut} (\mathfrak{g})]_{\text{ncs}}$, where $[G]_{\text{ncs}}$ denotes the maximal compact subgroup of $G$. They may or may not admit a (not necessarily unique) supersymmetric completion. Theories of type $E_7$ include all $\mathcal{N}$-extended $D = 1 + 3$ supergravities with $\mathcal{N} > 2$ supersymmetries, as well as all $\mathcal{N} = 2$ theories for which the scalar fields belonging to vector multiplets parametrise a symplectic space. Note, however, for $\mathcal{N} = 3$ supergravity (coupled to an arbitrary number of vector multiplets), as well as the minimally coupled $\mathcal{N} = 2$ supergravities, the corresponding $\mathfrak{g}$ are not reduced and their quartic invariant is degenerate in the sense that it is the square of a quadratic invariant. The ‘degeneration’ of groups of type $E_7$ is discussed in [35]. The best known example of a theory of type $E_7$ is provided the low energy effective field theory limit of type II string theory (M-theory) compactified on a 6-torus (7-torus). That is, $D = 1 + 3, \mathcal{N} = 8$ supergravity, which has electromagnetic duality group $E_{7(7)}(R) \cong \text{Aut}(\mathfrak{g}^0_7)$ that is broken to $E_{7(7)}(Z) \cong \text{Aut}(\mathfrak{g}^0_7)$ by the Dirac-Zwanziger-Schwinger charge quantization condition. Here, $\mathfrak{o}$, denotes the composition algebra of split octonions and $\Delta$, the ring of integral split octonions. Another particularly important example in the context of string/M-theory, is type II string theory on $T^2 \times K3$, which corresponds to example 4 of Table 2 with $r = 5, s = 21$.

The two-derivative Einstein-Maxwell-scalar Lagrangian is uniquely determined by the choice of $\mathfrak{g}$, although $\text{Aut}(\mathfrak{g})$ is only a symmetry of the equations of motion due to electromagnetic duality; there is no conventional manifestly covariant $\text{Aut}(\mathfrak{g})$-invariant action for the field strengths. However, there does exist a manifestly covariant $\text{Aut}(\mathfrak{g})$-invariant Lagrangian if one is willing to accept a twisted-self-duality constraint that must be imposed on the Euler-Lagrange equations (but not on the Lagrangian itself) [37,52]. The twisted-self-duality constraint implies both the Bianchi identities and equations of motion for the field strengths, so this construction would be redundant without the coupling to gravity/scalars always present in theories of type $E_7$. This formalism makes the notation compact and we adopt it here. Let us define the “doubled” Abelian gauge potentials $\mathcal{A} = (A, B)^T \in \Lambda^1 (M) \otimes \mathfrak{g}$ transforming as a symplectic vector of $\text{Aut}(\mathfrak{g})$, such that

$$\mathfrak{F} = d \left( \begin{array}{c} A \\ B \end{array} \right),$$

(2.13)

and introduce the manifestly $\text{Aut}(\mathfrak{g})$-invariant Lagrangian,

$$\mathcal{L} = R \star 1 + \frac{1}{4} \text{tr} ( \star d \mathcal{M}^{-1} \wedge d \mathcal{M} ) - \frac{1}{4} \mathfrak{F} \wedge \mathcal{M} \mathfrak{F},$$

(2.14)

with twisted-self-duality constraint [53],

$$\mathfrak{F} = \ast \Omega \mathcal{M} \mathfrak{F}, \quad \Omega = - \mathbb{I}, \quad \mathcal{M} \Omega \mathcal{M}^{-1} = \Omega,$$

(2.15)

where $\mathcal{M}(\phi)$ is the $\text{Aut}(\mathfrak{g})/[\text{Aut}(\mathfrak{g})]_{\text{ncs}}$ scalar coset representative and $\mathcal{F}^T \Omega \mathcal{F} = \{ \mathcal{F}, \mathcal{G} \}$. The doubled Lagrangian (2.14), where the potential $\mathcal{A}$ is treated as the independent variable, together with the constraint (2.15), is on-shell equivalent to the standard Einstein-Maxwell-scalar Lagrangian [52].

For an Einstein-Maxwell-scalar theory of type $E_7$, the most general extremal, asymptotically flat, spherically symmetric, static, dyonic black hole metric with non-vanishing horizon area is given by (cf. for example [54] and the references therein)

$$ds^2 = -e^{2U} dt^2 + e^{-2U} (dr^2 + r^2 d\Omega_2),$$

(2.16)

where $U = U(H (r))$ and

$$e^{-2U} \sqrt{\Delta (H)}], \quad H (r) = H_{\infty} - \frac{Q}{r}.$$  

(2.17)

Here $H_{\infty}$ and $Q$ belong to $\mathfrak{g}$. The Abelian two-form fields strengths are given by

$$\mathfrak{F} = \frac{e^{2U}}{r^2} \Omega \mathcal{M} Q dt \wedge dr + Q \sin \theta d\theta \wedge d\varphi.$$  

(2.18)

so that

$$\frac{1}{4n} \int_{S^2} \mathfrak{F} = Q.$$  

(2.19)

The electromagnetic charges $Q$ are elements of $\mathfrak{g}$. Physically distinct charge configurations $Q$ lie in distinct $\text{Aut}(\mathfrak{g})$ orbits. The Bekenstein-Hawking entropy ($c = h = G = 1$) is given by

$$S_{\text{BH}} = \frac{A_{\text{hor}}}{4} = \pi \sqrt{|\Delta (Q)|} = \pi \{ Q, \bar{Q} \}.$$  

(2.20)
where in the last equality we have used the Freudenthal dual $\tilde{x} := \text{sgn}(\Delta(x))T(x)/\sqrt{\Delta(x)}$ [6].

The Dirac-Zwanziger-Schwinger charge quantisation conditions relating two black holes with charges $Q$ and $Q'$ are given by [6]

$$\{Q, Q'\} \in \mathbb{Z}.$$  \hspace{1cm} (2.21)

Consequently, the black hole charges belong to $\mathfrak{H}$ as given in (2.7). The non-compact global symmetry group of (2.14) is broken to $\text{Aut}(\mathfrak{H})$, which corresponds to the U-duality group [55] in the context of M-theory.

2.3 Higher composition cube law and STU Black Holes

2.3.1 Gauss composition and type II string theory on $T^2 \times K3$

Before graduating to Bhargava’s higher composition law on cubes and the STU model, it will serve us well to first review the relationship between Gauss composition and black hole solutions in type II string theory on $T^2 \times K3$ appearing in Moore’s treatise on the arithmetic of black hole attractors [4]. The first $K3$ compactification was of $D = 11$ supergravity on $T^3 \times K3$ [56], which yields precisely the massless sector of type II supergravity on $T^2 \times K3$ given by $D = 4$, $N = 4$ supergravity coupled to $22 N = 4$ vector multiplets. From the IIB perspective (since $T^2 \times K3$ is self-mirror we need not distinguish IIA/B) the black hole charges originating from the $D = 10$ self-dual 5-form field strength belong to

$$H^3(T^2 \times K3; \mathbb{Z}) \cong H^1(T^2; \mathbb{Z}) \otimes H^2(K3; \mathbb{Z}) \cong \mathbb{Z}^3 \oplus 2 \mathbb{Z}^3 \oplus 10 \mathbb{Z}^2,$$

where $\mathbb{Z}^3$ is the even unimodular self-dual lattice of signature $(3, 19)$. Similarly, the $D = 10$ RR 2-form potential contributes $2 + 2$ charges. The $D = 10$ graviton yields two $D = 4$ Abelian gauge potentials, originating from the $T^2$ alone since the $K3$ has no isometries, providing $2 + 2$ charges. Finally, the NSNS 2-form gives a further $2 + 2$ charges for a total of $28 + 28$ electromagnetic charges belonging to $I^{18,22} \oplus I^{6,22}$. The corresponding FTS in Table 2 is given by case 4, with $r = 5, s = 21$, taken over $\mathbb{Z}$,

$$\mathfrak{H}_{T^2 \times K3} \cong \mathbb{Z} \oplus \mathbb{Z} \oplus (\mathbb{Z} \oplus \Gamma_{5,21}(\mathbb{Z})) \oplus (\mathbb{Z} \oplus \Gamma_{5,21}(\mathbb{Z})).$$

The Jordan algebra $\mathbb{Z} \oplus \Gamma_{5,21}(\mathbb{Z})$ is the space of electric (magnetic) $D = 5$ black hole (string) charges of type II on $S^1 \times K3$. The U-duality group is given by

$$\text{Aut}(\mathfrak{H}_{T^2 \times K3}) \cong \text{SL}(2, \mathbb{Z}) \times \text{SO}(6, 22; \mathbb{Z}).$$

The black hole charges, $Q \in \mathfrak{H}_{T^2 \times K3}$ transform in the $(2, 28)$ of $\text{Aut}(\mathfrak{H}_{T^2 \times K3})$ and, in this case, can be written as

$$Q = Q^i_{\mu} = (P_{\mu}Q_{\mu}), \quad i = 1, 2, 3, \ldots 28.$$  \hspace{1cm} (2.25)

In this basis the quartic norm is given by

$$\Delta(Q) = 4 \det Q^{ij} = P^2 Q^2 - (P \cdot Q)^2$$  \hspace{1cm} (2.26)

where $Q^{ij} = Q^i \cdot Q^j/2$ transforms as the 3 of $\text{SL}(2, \mathbb{Z})$ and is manifestly $\text{SO}(6, 22; \mathbb{Z})$-invariant. To any $Q^i_{\mu} \in \mathfrak{H}_{T^2 \times K3}$ we can associate a binary quadratic form via $x_i = (x, y)$,

$$f_Q(x, y) = [a, b, c]_Q = x_i Q^i_{\mu} x_j = ax^2 + byx + cy^2,$$

where $a = P^2/2, b = P \cdot Q, c = Q^2/2$. We observe that the discriminant is given by the quartic norm, $D(f_Q) := b^2 - 4ac = -\Delta(Q)$. Hence, we have a bijection between nondegenerate binary quadratic forms $f_Q$ and the triplets $Q^{ij}$ derived from black hole charge configurations. Clearly, $f_Q$ and $f_{Q'}$ are in the same equivalence class if and only if $Q^{ij}$ and $Q'^{ij}$ are $\text{SL}(2, \mathbb{Z})$-related. By Theorem 1 the $\text{SL}(2, \mathbb{Z})$-equivalence classes $[Q^{ij}]$ are in one-to-one correspondence with the isomorphism classes of pairs $(S, I)$, where $S$ is a nondegenerate oriented quadratic ring with discriminant $D = -\Delta(Q)$ and $I$ is an oriented ideal class of $S$. Note, all supersymmetric black hole solutions with non-vanishing entropy have $\Delta(Q) > 0$ and, hence, correspond to definite integral binary quadratic forms.

As observed in [4], if $f_Q$ is primitive (i.e. $\text{gcd}(a, b, c) = 1$) then, by Gauss’s theorem, the set of $\text{SL}(2, \mathbb{Z})$-orbits of the associated primitive binary quadratic forms having discriminant $D = -\Delta(Q)$ naturally possesses the structure of a finite abelian group, the narrow class group $\text{Cl}^+(D)$ of the unique quadratic ring $S(D)$ of discriminant $D$. The number, $n_D$, of $\text{SL}(2, \mathbb{Z})$-orbits of primitive $f_Q$ with fixed entropy $S_{\text{BH}} = \pi \sqrt{\Delta(Q)}$ is then given by $|\text{Cl}^+(D)|$ [4]. Since for $f_Q$ not primitive we can simply factor out $\text{gcd}(f_Q)$, it then follows [4] immediately that for arbitrary (not necessarily primitive) $Q^{ij}$

$$n_D = \sum |\text{Cl}^+(D/s^2)|, \quad \frac{D}{s^2} = 0, 1 \mod 4.$$  \hspace{1cm} (2.28)

\[\text{The converse is not true as there are non-BPS solutions with } \Delta(Q) > 0 \text{ [57]. This is possible because } \text{SL}(2, \mathbb{R}) \times \text{SO}(6, 22) \text{ has two orbits for every } \Delta(Q) > 0 \text{ [38], only one of which supports BPS black hole solutions [57].}\]
Siegel [58] demonstrated that the number, \( n_D \), of SL(2, \( \mathbb{Z} \))-orbits of black holes with growing like \( S_{\text{BH}} = \pi \sqrt{-D}; \forall \varepsilon > 0, 3\varepsilon(\varepsilon) > 0 \), such that \( n_D > c(\varepsilon)(S_{\text{BH}})^{1/2}\).

Of course, if \( Q^i_1 \) and \( Q^i_2 \) are SL(2, \( \mathbb{Z} \)) \times SO(6, 22; \( \mathbb{Z} \)) related, then \( Q^{ij}_1 \) and \( Q^{ij}_2 \) are SL(2, \( \mathbb{Z} \)) related. However, the converse is not necessarily true, contrary to the claims of [4]. For SL(2, \( \mathbb{Z} \)) \times SO(6, 22; \( \mathbb{Z} \)), in addition to (2.10), which exist for any \( \mathfrak{g}_\mathbb{Z} \), there is a further discrete invariant, the torsion [59],

\[
t(Q) := \gcd(e_{ij} Q^i_0, Q^j_0) = \gcd(P_i Q_\nu - P_\nu Q_i).
\]  

It is easy to find examples of pairs, \( Q^i_1 \) and \( Q^i_2 \), such that \( Q^{ij}_1 = Q^{ij}_2 \), but \( t(Q) \neq t(Q') \). To illustrate this, we can restrict to an SL(2, \( \mathbb{Z} \)) \times SO(6, 22; \( \mathbb{Z} \)) \times SO(6, 22; \( \mathbb{Z} \)) subsector, choosing primitive \( Q \) in the basis of (2.25) given by

\[
P = \begin{pmatrix} Q_0 \\ J \\ Q_1 \\ 0 \end{pmatrix}, \quad Q = \begin{pmatrix} 0 \\ n \\ 0 \\ 1 \end{pmatrix}, \quad Q_1|J, Q_5
\]

with SO(2, 2; \( \mathbb{Z} \))-invariant metric

\[
\eta = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.
\]

Here, \( Q_5 \) can be considered as representing an NS 5-brane wrapping charge, \( n \) a fundamental string winding charge, while \( J \) and \( Q_1 \) are units of KK monopole charge associated with the two distinct circles of the \( T^2 \). In the canonical FTS basis (2.4) we have

\[
Q = (-1) J (n, Q_5, Q_1) (0, 0, 0).
\]

Since, \( Q_1|J, Q_5 \), we can write

\[
P = Q_1 \begin{pmatrix} q_5 \\ j \\ 1 \\ 0 \end{pmatrix}, \quad Q = \begin{pmatrix} 0 \\ n \\ 0 \\ 1 \end{pmatrix}.
\]

Now let \( Q_1 = 2m \) and consider a second primitive configuration \( Q' \) given by

\[
P' = m \begin{pmatrix} 4q_5 \\ j \\ 1 \\ 0 \end{pmatrix}, \quad Q' = \begin{pmatrix} 0 \\ n \\ 0 \\ 1 \end{pmatrix}.
\]

Then \( Q^{ij}_1 = Q'^{ij}_1 \), but \( t(Q) = 2t(Q') = 2m \) so \( Q \) and \( Q' \) do not lie in the same U-duality orbit. This is our first indication that higher composition laws are relevant to black holes; as we shall see, the full U-duality orbits are related to the equivalence classes associated to higher composition laws. While the case of SL(2, \( \mathbb{Z} \)) \times SO(6, 22; \( \mathbb{Z} \)) does not correspond to one of Bhargava’s higher composition laws, it is closely related to the higher composition law on “cubes”, which applies directly to the \( STU \) model, as we shall explain. Viewed from this perspective, the connection to Gauss’s original composition law is a consequence of the fact that it is implied by the higher composition law on cubes.

### 2.3.2 Bhargava’s Cube Law and extremal \( STU \) black holes

In following we describe how the orbits of dyonic \( STU \) black hole charges are characterised by Bhargava’s higher composition law on “cubes” and ideal classes in quadratic orders [8]. The \( STU \) model, introduced independently in [10, 11], provides an interesting subsector of string compactifications to four dimensions. This model has a low energy limit which is described by \( \mathcal{N} = 2 \) supergravity coupled to three vector multiplets interacting through the special Kähler manifold \([\text{SL}(2, \mathbb{R})/\text{SO}(2)]^3 \). The three complex scalars are denoted by the letters \( S, T, \) and \( U \), hence the name of the model [11, 60]. The remarkable feature that distinguishes it from generic \( \mathcal{N} = 2 \) supergravities coupled to vectors [61] and, in particular, the \( \mathcal{N} = 2 \) generic Jordan sequence [18] given in case 4 of Table 2 with \( r = 1 \), is its \( S-T-U \) triality [11]. There are three different versions with two of the \( \text{SL}(2, \mathbb{R}) \) perturbative symmetries of the Lagrangian and the third a non-perturbative symmetry of the equations of motion. In a fourth version all three are non-perturbative [11, 60]. All four are on-shell equivalent. If there are in addition four hypermultiplets, the \( STU \) model is self-mirror [62, 63]. As a theory of type \( E_7 \) it is given by case 3 of Table 2 over \( \mathbb{Z} \),

\[
\mathfrak{g}_{\text{STU}} = \mathbb{Z} \oplus \mathbb{Z} \oplus \mathfrak{g}_{\text{STU}} \oplus \mathfrak{g}_{\text{STU}}
\]

where \( \mathfrak{g}_{\text{STU}} = \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z} \). The U-duality group\(^7\) is given by

\[
\text{Aut}(\mathfrak{g}_{\text{STU}}) \cong \text{SL}(2, \mathbb{Z}) \times \text{SL}(2, \mathbb{Z}) \times \text{SL}(2, \mathbb{Z}) \ltimes S_3,
\]

\(^7\)In the version of [10], the discrete \( \text{SL}(2, \mathbb{Z}) \) are replaced by a subgroup denoted \( \Gamma_0(2) \).
where the $S_3$ is the triality permutation group. Equivalently, it is given by case 4 with $r = s = 1$, which coincides with the subsector of type II string theory on $T^2 \times K3$ used in (2.30). More generally, the $STU$ model can be considered as a consistent truncation of all theories of type $E_7$ with non-degenerate reduced FTS, with the exception of the $ST^2$ and $T^3$ models, which are however obtained from the $STU$ model by identifying $T = U$ and $S = T = U$, respectively. In this sense it is the key example. Similarly, Bhargava’s cubes provides the key example of a higher composition law.

The $STU$ black hole solutions have $4 + 4$ electromagnetic charges, which in the canonical FTS basis (2.35) are \( Q = (\alpha, \beta, (A_1, A_2, A_3), (B_1, B_2, B_3)) \). See §V of [6] for details. In the physics literature they are typically split into the $4 + 4$ electric and magnetic charges \( Q = (p^I, q_I), I = 0, \ldots, 3 \), belonging to \( \mathbb{Z}^2 \otimes \mathbb{Z}^4 \). In [11] these charges were arranged into a cube and in [5] this cube was identified with Bhargava’s cube. In [64] it was shown that the charges may also be arranged into rank-three two-component tensor, or hypermatrix,

\[
|Q| = a_{ABC}|ABC| \in \mathbb{Z}^2 \otimes \mathbb{Z}^2 \otimes \mathbb{Z}^2 ,
\]

where \( A, B, C = 0, 1 \), so that the the quartic norm is given by Cayley’s hyperdeterminant [65], making the triality symmetry manifest. That is \( \delta_{STU} \cong \mathbb{Z}^2 \otimes \mathbb{Z}^2 \otimes \mathbb{Z}^2 \). The three cases are trivially related by

\[
(\alpha, \beta, (A_1, A_2, A_3), (B_1, B_2, B_3)) = (-q_0, p^0, (p^1, p^2, p^3), (q_0, q_1, q_2, q_3)) = (-a_{111}, a_{000}, (-4a_{001}, -a_{100}), (a_{110}, a_{101}, a_{011})).
\]

Bhargava also arranged the hypermatrix as a cube

\[
\begin{tabular}{ccc}
   & \( a_{001} \) & \( a_{101} \) \\
\( a_{011} \) & & \\
\( a_{000} \) & \( a_{110} \) & \\
\end{tabular}
\]

so that we speak of a higher composition law on \( SL(2, \mathbb{Z}) \times SL(2, \mathbb{Z}) \times SL(2, \mathbb{Z}) \)-equivalence classes of cubes.

In terms of the cube, the quartic norm is given by Cayley’s hyperdeterminant \( \text{Det} : \text{Sym}^4(\delta_{STU}) \rightarrow \mathbb{Z} \), where

\[
\Delta(Q) = -\text{Det} a = \frac{1}{2} \varepsilon^{A_1A_2} \varepsilon^{B_1B_2} \varepsilon^{C_1C_2} \varepsilon^{A_3A_4} \varepsilon^{B_3B_4} \varepsilon^{C_3C_4} a_{A_1B_1C_1} a_{A_2B_2C_2} a_{A_3B_3C_3} a_{A_4B_4C_4}.
\]

Explicitly

\[
\text{Det} a = \frac{1}{2} \left( a_{000}^2 a_{111}^2 + a_{001}^2 a_{110}^2 + a_{010}^2 a_{101}^2 + a_{100}^2 a_{011}^2 - 2 \left( a_{000} a_{001} a_{110} a_{111} + a_{000} a_{010} a_{101} a_{111} + a_{000} a_{010} a_{101} a_{101} + a_{001} a_{110} a_{101} a_{111} + a_{010} a_{101} a_{101} a_{101} + a_{001} a_{110} a_{101} a_{101} \right) + 4 \left( a_{000} a_{101} a_{111} + a_{000} a_{010} a_{101} a_{111} \right) \right),
\]

Following [69] it is useful to write

\[
\Delta(Q) = 4\det \gamma^S = 4\det \gamma^T = 4\det \gamma^U = -\text{Det} a,
\]

where we have defined the three matrices \( \gamma^S \), \( \gamma^T \), and \( \gamma^U \):

\[
(\gamma^S)_{A_1A_2} = \frac{1}{2} \varepsilon^{B_1B_2} \varepsilon^{C_1C_2} a_{A_1B_1C_1} a_{A_2B_2C_2},
\]

\[
(\gamma^T)_{B_1B_2} = \frac{1}{2} \varepsilon^{C_1C_2} \varepsilon^{A_1A_2} a_{A_1B_1C_1} a_{A_2B_2C_2},
\]

\[
(\gamma^U)_{C_1C_2} = \frac{1}{2} \varepsilon^{A_1A_2} \varepsilon^{B_1B_2} a_{A_1B_1C_1} a_{A_2B_2C_2},
\]

transforming respectively as \( (3, 1, 1), (1, 3, 1), (1, 1, 3) \) under \( SL(2, \mathbb{Z}) \times SL(2, \mathbb{Z}) \times SL(2, \mathbb{Z}) \). Explicitly,

\[
\gamma^S = \frac{1}{2} \begin{pmatrix}
2(a_{003}-a_{112}) & a_{0a7} - a_{1a6} + a_{a3} - a_{5a2} & 2(a_{4a7} - a_{5a6}) \\
2(a_{0a7} - a_{1a6} + a_{a3} - a_{5a2}) & a_{003} - a_{112} & a_{0a7} - a_{1a6} + a_{a3} - a_{5a2} \\
2(a_{4a7} - a_{5a6}) & a_{0a7} - a_{1a6} + a_{a3} - a_{5a2} & a_{003} - a_{112}
\end{pmatrix},
\]

\[
\gamma^T = \frac{1}{2} \begin{pmatrix}
a_{0a7} - a_{1a6} + a_{a3} - a_{5a2} & 2(a_{003}-a_{112}) & a_{0a7} - a_{1a6} + a_{a3} - a_{5a2} \\
2(a_{0a7} - a_{1a6} + a_{a3} - a_{5a2}) & a_{003} - a_{112} & a_{0a7} - a_{1a6} + a_{a3} - a_{5a2} \\
2(a_{4a7} - a_{5a6}) & a_{0a7} - a_{1a6} + a_{a3} - a_{5a2} & a_{003} - a_{112}
\end{pmatrix},
\]

\[
\gamma^U = \frac{1}{2} \begin{pmatrix}
2(a_{0a7} - a_{1a6} + a_{a3} - a_{5a2}) & a_{0a7} - a_{1a6} + a_{a3} - a_{5a2} & 2(a_{4a7} - a_{5a6}) \\
2(a_{0a7} - a_{1a6} + a_{a3} - a_{5a2}) & a_{003} - a_{112} & a_{0a7} - a_{1a6} + a_{a3} - a_{5a2} \\
2(a_{4a7} - a_{5a6}) & a_{0a7} - a_{1a6} + a_{a3} - a_{5a2} & a_{003} - a_{112}
\end{pmatrix},
\]

*This identification initiated what came to be known as the black-hole/qubit correspondence [29, 64, 66-68]. Cayley’s hyperdeterminant and the quartic norm, more generally, also give Nambu-Goto string actions [37, 69, 70].
where we have made a binary/decimal conversion $ABC \mapsto 2^6C + 2^1B + 2^2A$. Bhargava’s Cube Law provides a higher composition law for the $\text{SL}(2,\mathbb{Z}) \times \text{SL}(2,\mathbb{Z}) \times \text{SL}(2,\mathbb{Z})$-equivalence classes of projective integer cubes $a_{ABC}$:

**Definition 5** A cube $a_{ABC}$ is projective if the three binary quadratic forms defined by,

$$
\begin{align*}
 f_S(x, y) &= x^A (\gamma^S_A)_{AA} x^A = (a_0 a_3 - a_1 a_2) x^2 + (a_0 a_7 - a_1 a_6 + a_4 a_3 - a_2 a_9) y x + (a_5 a_7 - a_6 a_6) y^2, \\
 f_T(x, y) &= x^B (\gamma^T_B)_{BB} x^B = (a_0 a_5 - a_2 a_1) x^2 + (a_0 a_7 - a_4 a_3 + a_2 a_5 - a_6 a_1) y x + (a_5 a_6 - a_7 a_3) y^2, \\
 f_U(x, y) &= x^C (\gamma^U_C)_{CC} x^C = (a_0 a_6 - a_2 a_4) x^2 + (a_0 a_7 - a_2 a_5 + a_1 a_6 - a_3 a_4) y x + (a_1 a_7 - a_3 a_5) y^2,
\end{align*}
$$

where $x^A = (x, y)$, are each primitive.

Note from (2.41) that the discriminant of $f_S, f_T, f_U$ is given by the quartic norm $D(f_S) = D(f_T) = D(f_U) = \text{Det} \, a$. As demonstrated by Bhargava, this construction provides an alternative definition of Gauss composition through the following theorem [1]:

**Theorem 6 (Gauss composition from cubes)** The following statements hold and are equivalent to Gauss composition:

1. Given a projective cube $|Q\rangle = a_{ABC} |ABC\rangle$ with quartic norm $\Delta(\mathcal{Q}) = -\text{Det} \, a \neq 0$, the Gauss composition of the associated forms $f_S, f_T, f_U$ is the identity element of $\text{Cl}^+(S(D))$, where $D = \text{Det} \, a$.

2. Given three primitive forms $f_1$, $f_2$, $f_3$ all of discriminant $D$ such that their Gauss composition is the identity element of $\text{Cl}^+(S(D))$, there exists a cube $a_{ABC}$ with hyperdeterminant $\text{Det} \, a = D$ such that $f_1 = f_S, f_2 = f_T, f_3 = f_U$.

But Gauss composition concerns the set of $\text{SL}(2,\mathbb{Z})$-equivalence classes $\text{Cl}(\text{Sym}^2(\mathbb{Z})^2) \cap D$ and, as we have already seen during the preceding discussion of type II strings on $K^3$, it is the $\text{SL}(2,\mathbb{Z}) \times \text{SL}(2,\mathbb{Z}) \times \text{SL}(2,\mathbb{Z})$-equivalence classes that we need for the classification of the $STU$ black hole charge configurations [6–8]. As one might anticipate by now, it is Bhargava’s higher composition law that plays the analogous role. The $\text{SL}(2,\mathbb{Z}) \times \text{SL}(2,\mathbb{Z}) \times \text{SL}(2,\mathbb{Z})$-equivalence classes of projective cubes $a_{ABC}$ with discriminant $D = \text{Det} \, a$ themselves form a group, which in analogy to the case of primitive binary forms Bhargava denoted $\text{Cl}((\mathbb{Z})^2 \otimes \mathbb{Z}^2 \otimes \mathbb{Z}^2) \cap D$. This is the first example of a higher composition law. It follows straightforwardly from Theorem 6. Given two projective cubes $a$ and $a'$ we can define their composition $[a] + [a'] = [a'']$, where $[x]$ denotes the $\text{SL}(2,\mathbb{Z}) \times \text{SL}(2,\mathbb{Z}) \times \text{SL}(2,\mathbb{Z})$-equivalence class of $x$, using the fact that Theorem 6 implies $([f_S] + [f'_S]) + ([f_T] + [f'_T]) + ([f_U] + [f'_U]) = \text{Id}$ and the existence and uniqueness, up to $\text{SL}(2,\mathbb{Z}) \times \text{SL}(2,\mathbb{Z}) \times \text{SL}(2,\mathbb{Z})$-equivalence, of a projective cube $a''$ such that $[f_S] + [f'_S] = [f''_S]$ and similarly for $T, U$. According as $D = 0$ or $1 \mod 4$, the identity element is given by the equivalence class of

$$
|\text{id}, D\rangle = a^\text{id}_{ABC} |ABC\rangle = |001\rangle + |010\rangle + |100\rangle + \frac{D}{4} |111\rangle
$$

or

$$
|\text{id}, D\rangle = a^\text{id}_{ABC} |ABC\rangle = |001\rangle + |010\rangle + |100\rangle + |101\rangle + |011\rangle + \frac{D+3}{4} |111\rangle,
$$

respectively. Note, in the context of three-qubit entanglement these are maximally entangled (unnormalised) GHZ states [29, 66, 71–73]. In terms of the cube these correspond respectively to:

As for binary quadratic forms, using the perspective presented in subsection 2.1, we can rephrase this higher composition law as a special case of the following parametrisation result for generic (not necessarily projective) cubes in terms of ideal classes in quadratic orders [1]:

**Theorem 7** There is a canonical bijection between the set of $\text{SL}(2,\mathbb{Z}) \times \text{SL}(2,\mathbb{Z}) \times \text{SL}(2,\mathbb{Z})$-equivalence classes of nondegenerate ($\text{Det} \, a \neq 0$) cubes $a_{ABC} |ABC\rangle \in \mathbb{Z}^2 \otimes \mathbb{Z}^2 \otimes \mathbb{Z}^2$, and the set of isomorphism classes of pairs $S$ and $[I_1, I_2, I_3]$, where $S$ is a nondegenerate oriented quadratic ring of discriminant $D = \text{Det} \, a$ and $[I_1, I_2, I_3]$ is an equivalence class of a balanced triple of ideals in $S$. 

The bijection is remarkably straightforward to state. To summarise, given a black hole with charges \( a_{ABC} \) the quadratic ring \( S(D) \) is determined by the hyperdeterminant via \( D = \text{Det} a \), and the six bases for the ideal classes \( I_i \) are determined by a system of eight equations involving \( a_{ABC} \). Conversely, given any pair \( S(D) \) and \([I_1, I_2, I_3] \) the corresponding black hole \( a_{ABC} \) is directly obtained from the assumption that \( I_1, I_2, I_3 \) are a balanced triple.

In more detail, given \( S(D) \), with oriented basis \( \{1, \tau, \bar{\tau} \} \) such that \( \tau^2 - \frac{D}{4} = 0 \) or \( \tau^2 - \tau + \frac{1-D}{4} = 0 \) according as \( D = 0 \) or \( 1 \mod 4 \), and a balanced triple \((I_S, I_T, I_U)\) with bases \( \alpha_A, \beta_B, \gamma_C \), respectively, we have by assumption

\[
\alpha_A\beta_B\gamma_C = c_{ABC} + a_{ABC}\tau.
\]  

(2.47)

The claim is that \( a_{ABC} \) are the black hole charges with \( S_{\text{BH}} = \pi \sqrt{\text{Det} a} \) under the bijection. If \((I_S, I_T, I_U)\) is replaced by an equivalent triple, \( a_{ABC} \) is left invariant. An orientation preserving basis change on each ideal induces an \( \text{SL}(2, \mathbb{Z}) \times \text{SL}(2, \mathbb{Z}) \times \text{SL}(2, \mathbb{Z}) \) transformation on \( a_{ABC} \). So, the equivalence classes of pairs \( S(D), (I_S, I_T, I_U) \) are mapped injectively into the \( \text{SL}(2, \mathbb{Z}) \times \text{SL}(2, \mathbb{Z}) \times \text{SL}(2, \mathbb{Z}) \)-orbits in \( \mathbb{Z}^2 \otimes \mathbb{Z}^2 \otimes \mathbb{Z}^2 \). Conversely, \( a_{ABC} \) determines \([S(D), (I_S, I_T, I_U)] \) uniquely. First, the balanced hypothesis implies \( \text{Det} a = -D \) for (2.47), fixing \( S(D) \). Second, the balanced assumption implies that the \( c_{ABC} \) are determined uniquely by \( a_{ABC} \) through

\[
c_{ABC} = \frac{1}{2} (T(a)_{ABC} - (\text{Det} a \mod 4) a_{ABC}),
\]  

(2.48)

which is in \( \mathbb{Z}^2 \otimes \mathbb{Z}^2 \otimes \mathbb{Z}^2 \). Here, \( T \) is the triple product of the associated FTS, although this was not used by Bhargava, which is given by the three identical expressions

\[
\begin{align*}
T_{A_3B_2C_1} &= -e^{A_1A_2}a_{A_1B_1C_1}(\gamma^A)_{A_2A_3} \\
T_{A_1B_2C_1} &= -e^{B_1B_2}a_{A_1B_1C_1}(\gamma^B)_{B_2B_3} \\
T_{A_1B_1C_2} &= -e^{C_1C_2}a_{A_1B_1C_1}(\gamma^C)_{C_2C_3},
\end{align*}
\]  

(2.49)

where explicitly

\[
\begin{align*}
T_0 &= a_0 (a_3a_4 + a_2a_5 + a_1a_6 - a_0a_7) - 2a_1a_2a_4 \\
T_1 &= a_1 (-a_3a_4 - a_2a_5 + a_1a_6 - a_0a_7) + 2a_0a_3a_5 \\
T_2 &= a_2 (-a_3a_4 + a_2a_5 - a_1a_6 - a_0a_7) + 2a_0a_3a_6 \\
T_3 &= a_3 (-a_3a_4 + a_2a_5 + a_1a_6 + a_0a_7) - 2a_1a_2a_7 \\
T_4 &= a_4 (a_3a_4 - a_2a_5 - a_1a_6 + a_0a_7) + 2a_0a_3a_6 \\
T_5 &= a_5 (a_3a_4 - a_2a_5 + a_1a_6 + a_0a_7) - 2a_1a_2a_7 \\
T_6 &= a_6 (a_3a_4 + a_2a_5 - a_1a_6 + a_0a_7) - 2a_1a_2a_7 \\
T_7 &= a_7 (-a_3a_4 - a_2a_5 - a_1a_6 + a_0a_7) + 2a_1a_2a_6.
\end{align*}
\]  

(2.50)

This emphasises the implicit role played by the FTS in the higher composition laws. A balanced triple yielding \( c_{ABC}, a_{ABC} \) exists and is determined uniquely up to equivalence by \( c_{ABC}, a_{ABC} \) establishing the bijection.

The key point is that the physically distinct extremal \( STU \) charge configurations can be characterised entirely in terms of ideal classes in quadratic orders. The higher composition law on cubes of fixed \( D \) is then straightforwardly stated through the product of equivalence classes of balanced triples defined by

\[
[S(D); I_S, I_T, I_U] \circ [S(D); I'_S, I'_T, I'_U] := [S(D); I_SI'_S, I_TI'_T, I_UI'_U]
\]  

(2.51)

If we restrict to the set of projective elements \( \text{Cl}(\mathbb{Z}^2 \otimes \mathbb{Z}^2 \otimes \mathbb{Z}^2; D) \), then under the bijection we restrict to invertible ideals and \( I_U \) is determined by \( I_S, I_T \). Consequently, the set of isomorphisms classes has a group structure given by the product of two copies of the narrow class group

\[
\text{Cl}(\mathbb{Z}^2 \otimes \mathbb{Z}^2 \otimes \mathbb{Z}^2; D) \cong \text{Cl}^+(S(D)) \times \text{Cl}^+(S(D)),
\]  

(2.52)

and Theorem 6 induces a group homomorphism

\[
\text{Cl}(\mathbb{Z}^2 \otimes \mathbb{Z}^2 \otimes \mathbb{Z}^2; D) \rightarrow \text{Cl}(\text{Sym}^2(\mathbb{Z}^2)^*; D).
\]  

(2.53)

Hence, the physically distinct \( STU \) extremal large black holes charge configurations \( Q \) with a fixed Bekenstein-Hawking entropy \( S_{\text{BH}} = \pi \sqrt{|\Delta(Q)|} \) are given by the isomorphism classes of pairs of quadratic orders of discriminant \( D = -\Delta(Q) \) and balanced triples of fractional ideals [8]. If we restrict to projective black holes then the set of physically distinct configurations of a fixed entropy \( S_{\text{BH}} = \pi \sqrt{|\Delta(a)|} \) has a group structure isomorphic to \( \text{Cl}^+(S(D)) \times \text{Cl}^+(S(D)) \), where \( D = -\Delta(Q) \), and number of distinct physically distinct projective configurations

\[
r_{D}^{\text{proj}} := |\{ [Q] \mid |\Delta(Q)| = -D, \text{gcd}(f_S) = \text{gcd}(f_T) = \text{gcd}(f_U) = 1 \}|.
\]  

(2.54)
is given by $|\text{Cl}^+(S(D))|^2$ [8]. Interestingly, the number of U-duality inequivalent projective $STU$ black holes is a square number. Note, projectivity implies primitivity, $\gcd(\mathcal{Q}) = 1$, but the converse is not true. For example, in the FTS basis

$$(1, 0, (1, 2, 2), (0, 0, 0))$$

(2.55)

is primitive, but not projective as $f_S = [-2, 0, -2]$. Consequently, one cannot generically divide through by $\gcd(\mathcal{Q})$ to make $\mathcal{Q}$ projective and the straightforward counting of equivalence classes of generic binary quadratic forms in terms of class numbers for primitive forms does not work for generic cubes.

2.3.3 $STU$ U-duality orbits and quadratic orders: examples

Let us make these ideas more concrete with some simple examples. It is straightforward to show that every black hole charge configuration can be brought into a five parameter canonical form using U-duality [6], which in the FTS basis may be expressed as

$$\mathcal{Q}_{\text{can}} = \alpha(1, j, (a, b, c), (0, 0, 0)).$$

(2.56)

The corresponding forms (for $\alpha = 1$) are given by

$$f_S(\mathcal{Q}_{\text{can}}) = [-ab, -j, c],$$

$$f_T(\mathcal{Q}_{\text{can}}) = [-ca, -j, b],$$

$$f_U(\mathcal{Q}_{\text{can}}) = [-bc, -j, a].$$

(2.57)

The discrete invariants (2.10) and three torsions $t_S, t_T, t_U$ on $\mathcal{Q}_{\text{can}}$ are given by

$$d_1(\mathcal{Q}_{\text{can}}) = \alpha$$

$$d_2(\mathcal{Q}_{\text{can}}) = \alpha^2 \gcd(j, 2a, 2b, 2c)$$

$$t_S(\mathcal{Q}_{\text{can}}) = \alpha^2 \gcd(j, a, b)$$

$$t_T(\mathcal{Q}_{\text{can}}) = \alpha^2 \gcd(j, a, c)$$

$$t_U(\mathcal{Q}_{\text{can}}) = \alpha^2 \gcd(j, b, c)$$

(2.58)

$$d_3(\mathcal{Q}_{\text{can}}) = \alpha^3 \gcd(j, 2bc, 2ac, 2ab)$$

$$d_4(\mathcal{Q}_{\text{can}}) = \alpha^4 |j|^2 + 4abc$$

$$d'_4(\mathcal{Q}_{\text{can}}) = 2\alpha^4 \gcd(j^2 + abc, bc, ac, ab).$$

Projectivity implies $\alpha = 1$ and

$$\gcd(a, j, bc) = 1$$

$$\gcd(b, j, ca) = 1$$

$$\gcd(c, j, ab) = 1$$

(2.59)

Note, projectivity implies primitivity and that torsion in the $S, T$ and $U$ frame is one, i.e. configurations with non-trivial torsion are not projective.

Let us consider some simple examples of projective black holes with $D < 0$, which includes all supersymmetric configurations. For $D < 0$ we have $\text{Cl}^+(S(D)) \cong \text{Cl}(S(D))$, the class group. Hence,

$$n^{\text{proj}}_{D < 0} = |\text{Cl}(S(D))|^2.$$  

(2.60)

Consequently, we see that $n^{\text{proj}}_{D < 0}$ grows like $S_{\text{BH}}^2$ as $D \to \infty$.

As a first simple special case take those black hole configurations of fixed Bekenstein-Hawking entropy on which U-duality acts transitively, that is $\Delta(\mathcal{Q}) = D$ for $D$ such that $|\text{Cl}(S(D))| = 1$. This amounts to the classic Gauss class number problem for class number one, solved by Heegner [74], Baker [75] and Stark [76]. For even $D = -4n, n \in \mathbb{N}$, $|\text{Cl}(S(D))| = 1$ iff $n = 1, 2, 3, 4, 7$ (this answers Gauss’s original question). We can parametrise these orbits by letting $j = 0, \alpha = a = b = 1$ and $c = -n$,

$$\gcd(1, 0, n) = 1,$$

$$\gcd(1, 0, n) = 1,$$

$$\gcd(n, 0, 1) = 1,$$

(2.61)

so that every projective supersymmetric black hole with $\Delta(\mathcal{Q}) > 0$ even is U-duality equivalent to one of five canonical forms,

$$\mathcal{Q}_{\text{BPS, even}} = (1, 0, (1, 1, -n), (0, 0, 0)), \quad n = 1, 2, 3, 4, 7,$$

(2.62)
where $n$ is determined uniquely by the U-duality invariant $\Delta(Q_{\text{BPS, even}}) = 4n$. Regarded as embedded in type II string theory on $T^2 \times K3$, every such black hole is U-duality equivalent to one wrapped NS 5-brane, one wrapped fundamental string and $n$ units of KK monopole charge associated with the second circle of the $T^2$.

The odd discriminants $D < 0$ with class number one, are

$$-3, -7, -11, -19, -27, -43, -67, -163.$$  \hspace{1cm} (2.63)

A projective black hole has $\Delta$ odd iff $j$ is odd (an odd number of KK monopole charge). Let us just turn on just one unit of KK monopole charge $j = 1$ on the first circle of $T^2$, then

$$Q_{\text{BPS, 1, even}} = (1, 1, (1, 1, -n), (0, 0, 0)),$$  \hspace{1cm} (2.64)

yields

$$\Delta(Q_{\text{BPS, even}}) = -(1 - 4n) = 3, 7, 11, 19, 27, 43, 67, 163,$$  \hspace{1cm} (2.65)

for a prime $n = 1, 2, 3, 5, 7, 11, 17, 41$ units of KK charge, respectively, as required (for what it is worth $n$, excluding $n = 1$, is given by the first seven primes which are not the sums of two consecutive non-Fibonacci numbers). This exhausts all cases where U-duality acts transitively on the set of projective black holes of a fixed entropy.

Let us consider some examples from the next simplest case, $\text{Cl}(D) \cong \mathbb{Z}_2$ corresponding to four U-duality orbits. Again, at class number two we have an exhaustive list of possible $D < 0$. The simplest example, is given by $D = -15$ [77]. Since there are four orbits, we are seeking four $Q$ such that $\Delta(Q) = 15$ with distinct U-duality invariants, but are projective and hence torsionless. Representatives of the four classes are given by,

$$Q_1 = (1, -1, (-1, 1, 4), (0, 0, 0)), \quad Q_2 = (1, -1, (-1, 2, 2), (0, 0, 0)), \quad Q_3 = (1, -1, (-1, 4, -1), (0, 0, 0)), \quad Q_4 = (1, -1, (2, -1, 2), (0, 0, 0)).$$  \hspace{1cm} (2.66)

The corresponding forms are given by,

$$|f_S| \quad |f_T| \quad |f_U|$$

| $Q_1$ | $[1, 1, 4]$ | $[4, 1, 1]$ | $[-4, 1, -1]$ |
| $Q_2$ | $[2, 1, 2]$ | $[2, 1, 2]$ | $[-4, 1, -1]$ |
| $Q_3$ | $[-4, 1, -1]$ | $[1, 1, 4]$ | $[4, 1, 1]$ |
| $Q_4$ | $[2, 1, 2]$ | $[-4, 1, -1]$ | $[2, 1, 2]$ |

Using elementary $\text{SL}(2, \mathbb{Z})$ operations we can bring them all into reduced form

$$|f_S| \quad |f_T| \quad |f_U|$$

| $Q_1$ | $[1, 1, 4]$ | $[1, 1, 4]$ | $[-1, -1, -4]$ |
| $Q_2$ | $[2, 1, 2]$ | $[2, 1, 2]$ | $[-1, -1, -4]$ |
| $Q_3$ | $[-1, -1, -4]$ | $[1, 1, 4]$ | $[1, 1, 4]$ |
| $Q_4$ | $[2, 1, 2]$ | $[-1, -1, -4]$ | $[2, 1, 2]$ |

from which we immediately see that none are $\text{SL}(2, \mathbb{Z}) \times \text{SL}(2, \mathbb{Z}) \times \text{SL}(2, \mathbb{Z})$-equivalent, although $Q_1$ ($Q_2$) and $Q_3$ ($Q_4$) are triality related.

### 2.4 Symmetrising the Cube law: further examples

In the following we consider the four remaining examples of Bhargava’s higher composition laws related to quadratic orders. These are related to the composition law of cubes discussed in the previous section by symmetrisation and embeddings. In particular, the symmetrisations yield examples relevant to the $\mathcal{N} = 2$ $T^2$ and $T^3$ supergravity models, which admit a stringy derivation. The embeddings give two Einstein-Maxwell-Scalar theories of type $E_7$ that are $\mathcal{N} = 0$ consistent truncations of $\mathcal{N} = 8$ supergravity. Finally, we reconsider the analysis of $\mathcal{N} = 8$ supergravity given in [7], which although not related to a non-trivial composition law, can be treated on an equal footing following [13].

#### 2.4.1 The $\text{ST}^2$ model

The $\text{ST}^2$ model is $\mathcal{N} = 2$ supergravity coupled to two vector multiplets with scalars belonging to

$$\frac{\text{SL}_S(2, \mathbb{R}) \times \text{SL}_{\mathcal{T}^2}(2, \mathbb{R})}{\text{SO}(2) \times \text{SO}(2)}.$$  \hspace{1cm} (2.69)
As a theory of type $E_7$, it corresponds to case 2 of Table 2 over $\mathbb{Z}$,

$$\mathfrak{F}_{ST^2} \cong \mathbb{Z} \oplus \mathbb{Z} \oplus \mathfrak{F}_{ST^2} \oplus \mathfrak{F}_{ST^2},$$

(2.70)

where $\mathfrak{F}_{ST^2} \cong \mathbb{Z} \oplus \mathbb{Z}$. It can be regarded as a “symmetrisation” of the $STU$ model, where the complex structure and Kähler forms are identified $T = U$, along with the two vector multiplets they sit in. It can be uplifted to pure $N = (1,0)$ minimal chiral supergravity in $D = 6$.

The black hole charges are symmetrised, $Q = a_{A(BB')}(A(BB')) \in \mathfrak{F}_{ST^2} \cong \mathbb{Z} \otimes \text{Sym}^2(\mathbb{Z}^2)$, corresponding to the natural inclusion

$$\iota_{ST^2} : \mathbb{Z}^2 \otimes \text{Sym}^2(\mathbb{Z}^2) \hookrightarrow \mathbb{Z}^2 \otimes \mathbb{Z}^2 \otimes \mathbb{Z}^2 : a_{A(BB')} \mapsto a_{ABB'},$$

where $a_{A(BB')} \in \mathbb{Z}$ and $a_{A(BB')} = a_{A(B'B')}. This corresponds to the doubly-symmetric cube:

\[
\begin{array}{ccc}
& a_{0(01)} & \\
\downarrow & & \downarrow \\
a_{0(10)} & a_{0(11)} & a_{1(01)} \\
\downarrow & & \downarrow \\
a_{0(10)} & a_{0(00)} & a_{1(00)} \\
\downarrow & & \downarrow \\
& a_{1(10)} & \\
\end{array}
\]

Correspondingly, the U-duality group is given by the S-duality $SL_S(2, \mathbb{Z})$ together with the diagonal subgroup $SL_T(2, \mathbb{R}) \subset SL_T(2, \mathbb{R}) \times SL_U(2, \mathbb{R})$.

One can regard $\mathbb{Z}^2 \otimes \text{Sym}^2(\mathbb{Z}^2)$ as the space of doublets of (classical, in the Gauss sense) integral binary quadratic forms,

$$f_A(x,y) = a_{A(BB')}^x B^y = \left( a_{000}x^2 + 2a_{001}xy + a_{011}y^2 \right),$$

(2.72)

This identification gives us a parametrisation of U-duality equivalence classes of black hole charge configurations in terms of ideal classes in quadratic orders:

**Theorem 8 (Parametrisation of equivalence classes of pairs of classical integral binary quadratic forms [1])** There is a canonical bijection between the set of nondegenerate $SL(2, \mathbb{Z}) \times SL(2, \mathbb{Z})$-orbits on the space $\mathbb{Z}^2 \otimes \text{Sym}^2(\mathbb{Z}^2)$, and the set of isomorphism classes of pairs $S(D), (I_S, I_T, I_U)$, where $S(D)$ is a nondegenerate oriented quadratic ring and $(I_S, I_T, I_U)$ is an equivalence class of balanced triples of oriented ideals of $S(D)$ such that $I_T = I_U$.

Note, the discriminant is defined by the quartic norm $D = -\Delta(Q)$. The higher composition law on doubly-symmetrised cubes of fixed $D$ is then straightforwardly stated through the product of equivalence classes of balanced triples defined by,

$$[S(D); I_S, I_T, I_U] \circ [S(D); I_S', I_T', I_U'] := [S(D); I_S' I_S, I_T' I_T, I_U' I_U].$$

(2.73)

This bijection follows the same logic as the equivalent statement for the $STU$ model. We will not repeat it here, other than to note that the solution to the required system of equations

$$a_{ABA'B'B'} = c_{A(BB')} + a_{A(BB')}^x,$$

(2.74)

is again given by the triple product of the associated FTS,

$$c_{A(BB')} = \frac{1}{2}(T(a_{A(BB')}) - (\text{Det } a \mod 4)a_{A(BB')}),$$

(2.75)

An element $Q \in \mathbb{Z}^2 \otimes \text{Sym}^2(\mathbb{Z}^2)$ is defined to be projective if the associated cube $\iota_{ST^2}(Q)$ is projective. Under this inclusion

$$f_A^{id,D}(x,y) = \left( \frac{2xy}{x^2 + \frac{D}{4}y^2} \right), \quad f_A^{id,D}(x,y) = \left( \frac{2xy + y^2}{x^2 + 2xy + \frac{D+3}{2}y^2} \right),$$

(2.76)

map to the identity cubes in $\mathbb{Z}^2 \otimes \mathbb{Z}^2 \otimes \mathbb{Z}^2$ for $D = 0, 1 \mod 4$, respectively. This identification then provides a higher composition law with a group structure on projective pairs of classical binary integral forms:

**Theorem 9 (Group law on projective pairs of classical binary quadratic forms [1])** For all $D = 0, 1 \mod 4$, the set of $SL(2, \mathbb{Z}) \times SL(2, \mathbb{Z})$-equivalence classes of projective doublets of binary quadratic forms is a unique group $\text{Cl}(\mathbb{Z}^2 \otimes \text{Sym}^2(\mathbb{Z}^2); D)$ such that
\[ f^a_x \] is the additive identity

\( b) \) The inclusion \( i_{ST^2} : \mathbb{Z}^2 \otimes \text{Sym}^2(\mathbb{Z}^2) \hookrightarrow \mathbb{Z}^2 \otimes \mathbb{Z}^2 \otimes \mathbb{Z}^2 \) induces a group homomorphism

\[
\tilde{i}_{ST^2} : \text{Cl}(\mathbb{Z}^2 \otimes \text{Sym}^2(\mathbb{Z}^2); D) \rightarrow \text{Cl}(\mathbb{Z}^2 \otimes \mathbb{Z}^2 \otimes \mathbb{Z}^2; D)
\]

\[
[fa] \mapsto \tilde{i}_{ST^2}(fa) := i_{ST^2}(fa)
\]

(2.77)

Computing the binary quadratic forms corresponding to \( i_{ST^2}(Q) \) given in (2.44),

\[
f_S(x, y) = (a_0a_1 - a_1^2)x^2 + (a_0a_7 + a_1a_4 - 2a_5a_1)yx + (a_2a_7 - a_7^2)y^2,
\]

\[
f_T(x, y) = (a_0a_5 - a_1a_4)x^2 + (a_0a_7 - a_1a_3)yx + (a_1a_7 - a_3a_5)y^2,
\]

\[
f_U(x, y) = (a_0a_7 - a_4a_4)x^2 + (a_2a_7 - a_4a_4)yx + (a_1a_7 - a_3a_3)y^2,
\]

we observe that \( f_T = f_U \). If projective, any two of \( f_S, f_T, f_U \) determines the third and \( f_T = f_U \), hence the map taking \( a_{A(BB')} \) to \( f_T \) induces an isomorphism \( \text{Cl}(\mathbb{Z}^2 \otimes \text{Sym}^2(\mathbb{Z}^2); D) \cong \text{Cl}(\text{Sym}^2(\mathbb{Z}^2)^*; D) \cong \text{Cl}^+(S(D)) \). The U-duality equivalence classes of the projective black holes with entropy \( S_{BH} = \pi \sqrt{\Delta(Q)} \) of the \( ST^2 \) model are characterised precisely by the narrow class group \( \text{Cl}^+(S(D)) \), where \( D = -\Delta(Q) \).

### 2.4.2 The \( T^3 \) model

The \( T^3 \) model is \( \mathcal{N} = 2 \) supergravity coupled to a single vector multiplet with complex scalar belonging to

\[
\frac{\text{SL}(2, \mathbb{R})}{\text{SO}(2)}.
\]

(2.79)

It follows from the \( S^1 \) dimensional reduction of pure \( \mathcal{N} = 2 \) supergravity in \( D = 5 \). As a theory of type \( E_7 \), it corresponds to case 1 of \( \text{Table 2} \) over \( \mathbb{Z} \),

\[
\tilde{\mathfrak{g}}_{T^3} \cong \mathbb{Z} \oplus \mathbb{Z} \oplus \tilde{\mathfrak{g}}_{T^3} \oplus \tilde{\mathfrak{g}}_{T^3},
\]

(2.80)

where \( \tilde{\mathfrak{g}}_{T^3} \cong \mathbb{Z} \).

It can be regarded as a further “symmetrisation” of the \( ST^2 \) model, with the axion-dilaton and complex structure identified \( S = T \), along with the two vector multiplets they sit in. Consequently the black hole charges are symmetrised, \( Q = a_{(AA'A')}(AA'A') \in \tilde{\mathfrak{g}}_{T^3} \cong \text{Sym}^3(\mathbb{Z}^2) \), corresponding to the natural inclusion

\[
i_{T^3} : \text{Sym}^3(\mathbb{Z}^2) \hookrightarrow \mathbb{Z}^2 \otimes \mathbb{Z}^2 \otimes \mathbb{Z}^2 : a_{(AA'A')} \mapsto a_{AA'A'}
\]

(2.81)

where \( a_{(AA'A')} \in \mathbb{Z} \) and \( a_{(AA'A')} = a(\sigma(A)\sigma(A')\sigma(A')) \), where \( \sigma \in S_3 \). This corresponds to the triply-symmetric cube:

\[
\begin{array}{ccc}
\begin{array}{ccc}
\begin{array}{c}
(001) \\
(011) \\
(000)
\end{array} & \begin{array}{c}
(101) \\
(111) \\
(100)
\end{array}
\end{array} & \begin{array}{c}
(010) \\
(110)
\end{array}
\end{array}
\]

Correspondingly, the U-duality group is given by the diagonal subgroup \( \text{SL}_{T^3}(2, \mathbb{R}) \subset \text{SL}_S(2, \mathbb{R}) \times \text{SL}_{T^3}(2, \mathbb{R}) \).

One can regard \( \text{Sym}^3(\mathbb{Z}^2) \) as the space of classical integral binary cubic forms,

\[
f^Q(x, y) = a_{(AA'A')}x^3x'A'x'A' = a_{(000)}x^3 + 3a_{(001)}x^2y + 3a_{(011)}xy^2 + a_{(111)}y^3.
\]

(2.82)

As before, the discriminant of \( f \) is related to quartic norm by \( D(f) = -\Delta(Q) \).

**Theorem 10 (Parametrisation of equivalence classes of classical integral binary cubic forms [1])** There is a canonical bijection between the set of nondegenerate \( \text{SL}(2, \mathbb{Z}) \)-orbits on the space \( \text{Sym}^3(\mathbb{Z}^2) \) of binary cubic forms of discriminant \( D \), and the set of equivalence classes of triples \( (S(D), I, \delta) \), where \( S \) is a nondegenerate oriented quadratic ring with discriminant \( D \), \( I \) is an ideal of \( S \), and \( \delta \) is an invertible element of \( S \otimes Q \) such that \( I^3 \subset \delta S \) and \( N(I)^3 = N(\delta) \).
The higher composition law on triply-symmetrised cubes of fixed $D$ is then inherited from the law on cubes,

$$[S(D); I, I, I] \circ [S(D); I', I', I'] := [S(D); II', II', II'] \quad \text{.} \tag{2.83}$$

Of course, this can be mapped to $[S(D); I] \circ [S(D); I'] := [S(D); II']$, which superficially looks like Gauss composition. However, the notation in (2.83) serves to remind us that the ideals $I$ are balanced, $I^3 \subset \delta S$ and $N(I)^3 = N(\delta)$, so they are indeed distinct, albeit closely related, composition laws. Again, the solution to the required system of equations underpinning the bijection

$$\alpha_A \alpha_A' \alpha_A'' = c_{(AA'A'')} + a_{(AA'A'')} \tau \quad \text{.} \tag{2.84}$$

is given by the triple product of the associated FTS,

$$c_{(AA'A'')} = \frac{1}{2} \left( T(a)_{(AA'A'')} - \left( \text{Det } a \mod 4 \right) a_{(AA'A'')} \right) \quad \text{,} \tag{2.85}$$

which is the symmetrisation of (2.48). The cubic $f^3$ is defined to be projective if the associated cubic $\nu_T^3(Q)$ is projective. Note, this is not the same as the cubic itself being primitive. Explicitly, for (2.44) evaluated on $\nu_T^3(Q)$ we have

$$f_S = f_T = f_U = (a_0 a_3 - a_1^2)x^2 + (a_0 a_7 - a_3 a_1)x + (a_1 a_7 - a_3^2),$$

so $f^3$ is projective if $\gcd(a_0 a_3 - a_1^2, a_0 a_7 - a_3 a_1, a_1 a_7 - a_3^2) = 1$. Under this inclusion

$$f^3(x, y) = 3x^2 y + \frac{D}{4} y^3,$$

map to the identity cubes in $\mathbb{Z}^2 \otimes \mathbb{Z}^2 \otimes \mathbb{Z}^2$ for $D = 0, 1 \mod 4$, respectively. This identification then provides a higher composition law with a group structure on projective pairs of classical binary integral forms:

**Theorem 11 (Group law on projective classical binary cubic forms [1])** For all $D = 0, 1 \mod 4$, the set of $\text{SL}(2, \mathbb{Z})$-equivalence classes of projective binary cubic forms of discriminant $D$ forms a unique group $\text{Cl}(\text{Sym}^3(\mathbb{Z}^2); D)$ such that

a) $[f^3]_D$ is the additive identity

b) The inclusion $\nu_T^3 : \text{Sym}^3(\mathbb{Z}^2) \rightarrow \mathbb{Z}^2 \otimes \mathbb{Z}^2 \otimes \mathbb{Z}^2$ induces a group homomorphism

$$\nu_T^3 : \text{Cl}(\text{Sym}^3(\mathbb{Z}^2); D) \rightarrow \text{Cl}(\mathbb{Z}^2 \otimes \mathbb{Z}^2 \otimes \mathbb{Z}^2; D) \quad \text{.} \tag{2.88}$$

This corresponds to the surjective group homomorphism

$$\text{Cl}(\text{Sym}^3(\mathbb{Z}^2); D) \rightarrow \{ g \in \text{Cl}(S(D)) | [g]^3 = \text{id} \} \quad \text{.} \tag{2.89}$$

This yields the curious result that the number of physically distinct projective black hole solutions with fixed entropy $\pi \sqrt{\Delta(S)}$ corresponds to the number of invertible ideal classes in $S(D)$, where $D = -\Delta(S)$, having order three in $\text{Cl}(S(D))$.

### 2.4.3 The $\text{SL}(2, \mathbb{Z}) \times \text{SL}(4, \mathbb{Z})$ Einstein-Maxwell-Scalar theory

The theory of type $E_7$ given by case 4 of Table 2 with $r = 2, s = 2$ has quantised electromagnetic duality group

$$\text{Aut} \left( \mathfrak{g}(\mathbb{R} \oplus \Gamma_{1, s}) \right) \cong \text{SL}(2, \mathbb{Z}) \times \text{SO}(3, 3; \mathbb{Z}) \cong \text{SL}(2, \mathbb{Z}) \times \text{SL}(4, \mathbb{Z}) \quad \text{.} \tag{2.90}$$

It corresponds to Einstein-Hilbert gravity coupled to six Abelian gauge fields, whose field strengths and duals belong to the (2, 6) of $\text{SL}(2, \mathbb{R}) \times \text{SO}(3, 3; \mathbb{R})$, and eleven real scalar fields parametrising the coset,

$$\frac{\text{SL}(2, \mathbb{R}) \times \text{SO}(3, 3)}{\text{SO}(2) \times \text{SO}(3) \times \text{SO}(3)} \quad \text{.} \tag{2.91}$$

This theory has been treated previously in [78,79]. Although it does not admit a supersymmetric completion in $D = 1 + 3$, it can be regarded as a consistent truncation of $\mathcal{N} = 8$ supergravity effected through the branching under

$$E_7(7) \supset \text{SL}(2, \mathbb{R}) \times \text{SO}(6, 6) \supset \text{SL}(2, \mathbb{R}) \times [\text{SO}(1, 1) \times \text{SL}(6, \mathbb{R})]$$

$$\supset \text{SL}(2, \mathbb{R}) \times [\text{SO}(1, 1) \times (\text{SL}(2, \mathbb{R}) \times \text{SL}(4, \mathbb{R}))] \quad \text{.} \tag{2.92}$$

---

9Since all $\mathcal{N} \leq 4$ multiplets have an even number of scalars.
so that the $28 + 28$ vectors and their duals break as
\[ 56 \rightarrow (2, 12) + (1, 32) \rightarrow 20 \rightarrow (2, 6) \] (2.93)
where we retain only the $\text{SL}(2, \mathbb{R})$ and $\text{SO}(1, 1)$ singlets at the first and second branchings, respectively. From the above, we see that the $SU$ may be embedded in the $\text{SL}(2, \mathbb{Z}) \times \text{SL}(4, \mathbb{Z})$ Einstein-Maxwell-scalar theory, further branching under $\text{SO}(1, 1) \times \text{SL}(2, \mathbb{R}) \times \text{SL}(2, \mathbb{R}) \subset \text{SL}(4, \mathbb{R})$ and retaining only the $\text{SO}(1, 1)$ singlets,
\[ (2, 6) \rightarrow (2, 1, 1)_2 + (2, 1, 1)_{-2} + (2, 2, 2)_0 \rightarrow (2, 2, 2). \] (2.94)
An alternative equivalent branching, that is better adapted to the scalar sector is given by
\[ E_{7(7)} \supset \text{SO}(6, 6) \times \text{SL}(2, \mathbb{R}) \] (2.95)
\[ \supset \text{SO}(4, 4) \times \text{SL}(2, \mathbb{R}) \times \text{SL}(2, \mathbb{R})_I \times \text{SL}(2, \mathbb{R})_{II} \] (2.96)
\[ 56 \rightarrow (12, 2) + (32, 1) \] (2.97)
\[ \rightarrow (8, 2, 1, 1) + (1, 2, 2, 2) + (8, 1, 2, 1) + (8, 1, 1, 2) : \] (2.98)
\[ \rightarrow (6, 2, 1, 1)_0 + (1, 2, 2, 2)_0 + (1, 2, 1, 1)_2 + (1, 2, 2, 2)_0 + (4, 1, 2, 1)_1 + (4', 1, 2, 1)_1. \] (2.98)
The sequence of maximal subgroups leading to (2.96) can be interpreted as
\[ \text{Aut} \left( \mathfrak{h}_3^O \right) \supset \text{Aut} \left( \mathfrak{h}_3^O \right) \times \text{SO}(2, \mathbb{R})_I \times \text{SO}(2, \mathbb{R})_{II} \] (2.99)
or equivalently as
\[ \text{Aut} \left( \mathfrak{h}_3^O \right) \supset \text{Aut} \left( \mathfrak{h}_3^H \right) \times \text{SL}(2, \mathbb{R})_I \times \text{SL}(2, \mathbb{R})_{II} \] (2.100)
By retaining only the $\text{SL}(2, \mathbb{R})_I \times \text{SL}(2, \mathbb{R})_{II} \times \text{SO}(1, 1)$ singlets one obtains
\[ 56 \rightarrow (6, 2, 1, 1)_0 \rightarrow \text{SO}(3, 3) \times \text{SL}(2, \mathbb{R}) \times \text{SL}(2, \mathbb{R})_I \times \text{SL}(2, \mathbb{R})_{II} \times \text{SO}(1, 1) \rightarrow (6, 2). \] (2.101)
For what concerns the maximal compact subgroups, we have
\[ \text{SU}_8 \supset \text{SO}_8 \times \text{SO}_8 \times \text{U}_1 \] (2.102)
\[ \supset \text{SU}(2)_I \times \text{SU}(2)_{II} \times \text{SU}(2)_{III} \times \text{SU}(2)_{IV} \times \text{U}(1)_I \times \text{U}(1)_{II} \] (2.103)
where $\text{SU}(2)_{d(I, II)} \subset \text{SU}(2)_I \times \text{SU}(2)_{II}$ and $\text{SU}(2)_{d(III, IV)} \subset \text{SU}(2)_{III} \times \text{SU}(2)_{IV}$ are diagonal embeddings. The branchings of (2.103) go as follows:
\[ 8 \rightarrow (4, 1) + (1, 4)_{-1} \] (2.104)
\[ \rightarrow (2, 1, 1)_1, 1, 0 + (1, 2, 2, 1)_{1, -1, 0} + (1, 1, 2, 1)_{-1, 0, 1} + (1, 1, 1, 2)_{-1, 0, -1}; \] (2.105)
\[ 56 \rightarrow (6, 4)_1 + (4, 6)_{-1} + (1, 7, 1)_{-1} + (1, 3, 3)_{-1} \] (2.106)
\[ \rightarrow (2, 2, 2, 1)_{1, 0, -1} + (1, 1, 2, 1)_{1, 2, 1} + (1, 1, 1, 2)_{1, 2, 1} + (1, 1, 2, 1)_{1, 2, -1} \] (2.106)
\[ + (1, 2, 2, 2)_{1, -1, 1} + (2, 1, 1, 1)_{1, -1, -1} + (2, 2, 1, 1)_{1, -1, -1} + (1, 1, 1, 2)_{-1, 1, -1} \] (2.106)
\[ + (1, 1, 2, 1)_{1, -1, -1} + (1, 1, 1, 2)_{1, -1, -1} + (1, 1, 1, 2)_{1, -1, -1} \] (2.106)
\[ + (1, 2, 1)_{1, 1, 1} + (1, 1, 1, 2)_{1, 3, 0, -1} + (1, 1, 1, 2)_{-3, 0, 1}. \] (2.106)
For what concerns the scalars, one has

\[
\begin{align*}
70 &\rightarrow (6,6)_0 + (4,\mathbf{4})_{-2} + (\mathbf{4},4)_2 + (1,1)_4 + (1,1)_{-4} \\
&\rightarrow (2,2,2,2)_{0,0,0,0} + (2,2,1,1)_{0,0,2} + (2,2,1,1)_{0,0,-2} \\
&+ (1,1,2,2)_{0,2,0} + (1,1,1,1)_{0,2,2} + (1,1,1,1)_{0,2,-2} \\
&+ (1,1,1,1)_{0,-2,0} + (1,1,1,1)_{0,-2,2} + (1,1,1,1)_{0,-2,-2} \\
&+ (2,1,1,2)_{-2,1,-1} + (2,1,1,2)_{-2,1,1} + (1,2,2,1)_{-2,-1,-1} + (1,2,2,1)_{-2,-1,1} \\
&+ (2,1,1,2)_{2,-1,-1} + (2,1,1,2)_{2,-1,1} + (1,2,2,1)_{2,1,1} + (1,2,2,1)_{2,1,-1} \\
&+ (1,1,1,1)_{4,0,0} + (1,1,1,1)_{-4,0,0} \\
&\rightarrow (3,1_1,3,1_3 + 1_a)_{0,0,0} + (3,1,1)_{0,0,2} + (3,1,1)_{0,0,-2} \\
&+ (1,3,1)_{0,2,0} + (1,1,1)_{0,2,2} + (1,1,1)_{0,2,-2} + (1,3,1)_{0,-2,0} + (1,1,1)_{0,-2,2} \\
&+ (2,2)_{-2,1,-1} + (2,2)_{-2,1,1} + (2,2)_{-2,-1,-1} + (2,2)_{-2,-1,1} \\
&+ (2,2)_{2,-1,-1} + (2,2)_{2,-1,1} + (2,2)_{2,1,1} + (2,2)_{2,1,-1} \\
&+ (1,1)_{4,0,0} + (1,1)_{-4,0,0} .
\end{align*}
\]

(2.107)

By retaining only the singlets wrt \( U(1)_I \times U(1)_{II} \), from (2.107) one obtains

\[
70 \rightarrow (3,1_a,3,s + 1_a)_{0} + (1,1)_{4} + (1,1)_{-4} .
\]

(2.108)

Therefore, the branching of the scalar fields of \( \mathcal{N} = 8, D = 4 \) supergravity goes as follows

\[
\begin{align*}
\frac{E_7(7)}{SU(8)} &\rightarrow \frac{\text{SO}(3,3)}{\text{SU}(2)_{(III,IV)} \times \text{SU}(2)_{(III,IV)}} \times \frac{\text{SL}(2,R)}{U(1)} \times \frac{\text{SL}(2,R)_I}{U(1)_I} \times \frac{\text{SL}(2,R)_{II}}{U(1)_{II}} \times \text{SO}(1,1) \\
&\rightarrow \frac{\text{SU}(2)_{(III,IV)} \times \text{SU}(2)_0}{\text{SU}(2)_{(III,IV)} \times \text{SU}(2)_{(III,IV)}} \times \frac{\text{SU}(2)}{U(1)} .
\end{align*}
\]

(2.109)

where, from (2.107), the identifications read

\[
\begin{align*}
T_1 \text{SL}(2,R)_{U(1)} &\sim (1,1)_{4,0,0} + (1,1)_{-4,0,0} + (3,s,3,s)_{0,0,0} ; \\
T_1 \text{SU}(2)_{U(1)_I} &\sim (1,1)_{0,2,2} + (1,1)_{0,-2,0} ; \\
T_1 \text{SU}(2)_{U(1)_{II}} &\sim (1,1)_{0,2,2} + (1,1)_{0,-2,0} ; \\
T_1 \text{SU}(1,1) &\sim (1_a,1_a)_{0,0,0} ,
\end{align*}
\]

(2.110)

(2.111)

(2.112)

(2.113)

where the covariance is \( \text{SU}(2)_{(III,IV)} \times \text{SU}(2)_{(III,IV)} \times U(1) \times U(1)_I \times U(1)_{II} \). Note that \((3,s,1_a)_{0,0,0} + (1_a,3,s)_{0,0,0}\), despite being singlets under \( U(1)_I \times U(1)_{II} \), do not correspond to any (reductive) coset. Since

\[
\text{Aut} (\mathbf{3}) \cong \text{SL}(2,R) \times \text{SO}(3,3),
\]

(2.114)

the coset of the \( \mathcal{N} = 0, D = 4 \) Maxwell-Einstein theory based on \( \mathbb{R} \oplus \mathbf{3}_2^{\mathbb{C}} \) can be identified with (2.109), or equivalently with (2.110).

Apply the charge quantisation condition, we have automorphism group \( \text{SL}(2,Z) \times \text{SL}(4,Z) \), which will be related to linear basis changes of ideals in the higher composition law. The quantised black hole charges \( Q = a_{A[i,j]} A[i,j] \), \( i,j = 1,2,3,4 \) belong to

\[
\mathbf{3}(Z \oplus \Gamma_{1,3}(Z)) \cong Z^2 \otimes \wedge^2 Z^4,
\]

(2.115)

which one can regard as the space of pairs of integral 2-forms,

\[
Q = \begin{pmatrix} a_{[i,j]} \\ a_{[i,j]} \end{pmatrix} .
\]

(2.116)

From the embedding of the \( STU \) model, there is a natural linear map of the black hole charges

\[
\text{id} \otimes \wedge Z^2 : Z^2 \otimes Z^2 \otimes Z^2 \rightarrow Z^2 \otimes \wedge^2 Z^4,
\]

(2.117)

which in terms of the hypermatrix is given by

\[
a_{ABC} \mapsto a_{[ij]} = \begin{pmatrix} 0 & a_{ABC} \\ -a_{ACB} & 0 \end{pmatrix} .
\]

(2.118)
In order to state the bijection between the U-duality equivalence classes of black holes and quadratic ideal classes we need a few more notions from the theory of higher rank ideals. A rank $n$ ideal of $S$ is an $S$-submodule of $K^n$, where $K = S \otimes \mathbb{Q}$, of rank $2n$ as a $Z$-module. The determinant of a rank $n$ ideal $M$ is denoted $\det(M)$ and defined as the ideal in $S$ generated by all $\det([M])$, where $[M] \in M^n$ is regarded as an $n \times n$ matrix. A $k$-tuple of oriented ideals $(M_1, \ldots, M_k)$ of ranks $n_1, \ldots, n_k$ is said to be balanced if $\prod_{i=1}^k \det(M_i) \subseteq S$ and $\prod_{i=1}^k N(M_i) = 1$.

With the notion of balanced $k$-tuples of rank $n$ ideals in hand, we can state the parametrisation of U-duality equivalence classes of black hole solutions in terms of ideal classes in quadratic orders:

**Theorem 12 (Parametrisation of equivalence classes of pairs of alternating forms [1])** There is a canonical bijection between the set of nondegenerate $SL(2, Z) \times SL(4, Z)$-orbits on the space $E^2 \otimes \wedge^2 Z^4$ of quartic norm $\Delta(Q)$, and the set of equivalence classes of pairs $(S, (I, M))$, where $S$ is a nondegenerate oriented quadratic ring with discriminant $\Delta = -\Delta(Q)$, and $(I, M)$ is a balanced doublet of ideals of ranks one and two, respectively.

Under the mapping embedding the STU model into the $SL(2, Z) \times SL(4, Z)$ Einstein-Maxwell-Scalar theory, the equivalence classes of pairs $(S, (I_S, I_T, I_U))$ get mapped to the equivalence classes of pairs of the form $(S, (I_S, I_T \oplus I_U))$; the embedding corresponds to the fusion of rank one ideals $(I_T, I_U)$ into a single rank two ideal $(I_T \oplus I_U)$. Clearly, every $SL(2, Z) \times SL(4, Z)$-equivalence class maps into an $SL(2, Z) \times SL(4, Z)$-equivalence class.

What is less obvious is that every $SL(2, Z) \times SL(4, Z)$-equivalence class has a representative in the image of $id \otimes \wedge_2$. Elements of $Z^2 \otimes \wedge^2 Z^4$ can be "diagonalised" by $SL(2, Z) \times SL(4, Z)$ to lie in a subspace isomorphic to $Z^2 \otimes Z^2 \otimes Z^2$. This follows directly from the fact that any torsion-free module over $Z$ is a direct sum of rank one ideals. Thus, the embedding map $(S, (I_S, I_T, I_U)) \mapsto (S, (I_S, I_T \oplus I_U))$ is surjective at the level of equivalence classes and we conclude that every $Q \in Z^2 \otimes \wedge^2 Z^4$ can be "diagonalised" to an element of the form $id \otimes \wedge_2(a_{ABC})$.

This allows one to use projectivity of elements in $Z^2 \otimes Z^2 \otimes Z^2$ to define projectivity of elements in $Z^2 \otimes \wedge^2 Z^4$: $Q \in Z^2 \otimes \wedge^2 Z^4$ is projective if and only if it is $SL(2, Z) \times SL(4, Z)$-equivalent to $id \otimes \wedge_2(a_{ABC})$ for some projective $a_{ABC}$. As before, restricting to projective elements yields a group law:

**Theorem 13 (Group law on projective integral 2-forms [1])** For all $D = 0, 1 \mod 4$, the set of $SL(2, Z) \times SL(4, Z)$-equivalence classes of projective integral $2$-forms, $Q \in Z^2 \otimes \wedge^2 Z^4$ with fixed quartic norm $D = -\Delta(Q)$ is a unique group $Cl(Z^2 \otimes \wedge^2 Z^4; D)$ such that $[a_{ABC}] \mapsto [id \otimes \wedge_2(a_{ABC})]$ is group homomorphism from $Cl(Z^2 \otimes \wedge^2 Z^2; D)$.

One can also maps to integral binary quadratic forms $Z^2 \otimes \wedge^2 Z^4 \rightarrow \text{Sym}^2(Z^2)^*$ using $f^Q(x, y) = x^A a_{A[i][j]} a_{A'[i'][j']} x_{i'j'} y^{*A'}$.

The discriminant $D$ of $f^Q$ is the given by $D = -\Delta(Q)$. Restricting to projective $Q$ implies primitive $f^Q$. Returning to the $S$-module point of view, projective implies the 2-tuples $(I, M)$ satisfy $I \det(M) = S$, which implies $(S, (I, M)) \cong (S, (I, S \oplus I^{-1}))$. Hence, we have a group isomorphism $Cl(Z^2 \otimes \wedge^2 Z^2; D) \rightarrow Cl(\text{Sym}^2(Z^2)^*; D) \cong Cl^+(S(D))$. The projective black hole solutions are in one-to-one correspondence with the elements of the narrow class group.

### 2.4.4 The $SL(6, Z)$ Einstein-Maxwell-Scalar theory

The final example of Barghava’s higher composition laws associated to quadratic orders corresponds to case 7 of Table 2 over the integral split complexes $\mathfrak{c}_s$, $\mathfrak{F}_{\text{SL}(n)}(Z) := \mathfrak{F}(\mathfrak{c}_s^Z)$, where $\mathfrak{c}_s^Z$ is the set of $3 \times 3$ Hermitian matrices over $\mathfrak{c}_s$. The automorphism group is given by $\text{Aut}(\mathfrak{F}(\mathfrak{c}_s^Z)) \cong SL(6, Z)$.

It corresponds theory of type $E_7$ is given by Einstein-Hilbert gravity coupled to 10 Abelian gauge fields, which together with their duals belong to the $20 \cong \wedge^3(R^6)$ of $SL(6, R)$, and 20 real scalar fields parametrising the coset,

$$\frac{SL(6, R)}{SO(6)}$$

It has been treated previously in [78, 79]. Although it does not admit a supersymmetric completion in $D = 1 + 3$, as well as a theory of type $E_7$ in its own right it can be regarded as a consistent truncation of $N = 8$ supergravity effected through the branching under

$$SL(6, R) \subset SL(2, R) \times [SO(1, 1) \times SL(6, R)] \subset SL(2, R) \times SO(6, 6, R) \subset E_{7(7)}$$

so that the 28 + 28 vectors and their duals break as

$$56 \rightarrow (2, 12) + (1, 32) \rightarrow 6_{-2} + 6_{+2} + 20_0 \rightarrow 20$$

$^{10}N = 4$ is ruled out by the even number of scalars, $N = 3$ is ruled out by $20 \neq 6n$, $N = 2$ is ruled out by the requirement that the scalar manifold be special Kähler. This leaves the final possibility of $N = 1$ supergravity coupled to ten vector multiplets and 10 chiral multiplets, but this is ruled out by the fact that the scalar manifold of chiral multiplets must be Kähler, as well as from the fact that the kinetic vector matrix must be holomorphic.
where we retain only the $\text{SL}(2, \mathbb{R})$ and $\text{SO}(1, 1)$ singlets at the first and second branchings, respectively. From the above we see that the $\text{SL}(2, \mathbb{Z}) \times \text{SL}(4, \mathbb{Z})$ theory may be embedded in the $\text{SL}(6, \mathbb{Z})$ Einstein-Maxwell-scalar theory.

The quantized black hole charges $Q = a_{[abc]}[abc]$, $a, b, c = 1, \ldots, 6$, belong to

$$\mathfrak{I}(\mathfrak{I}_3) \cong \wedge^3 \mathbb{Z}^6.$$

From the embedding of the $\text{STU}$ model, there is a natural linear map of the black hole charges

$$\wedge_{2,2,2} : \mathbb{Z}^2 \otimes \mathbb{Z}^2 \otimes \mathbb{Z}^2 \hookrightarrow \wedge^3 \mathbb{Z}^6,$$

which in terms of the canonical FTS basis is given by

$$\wedge_{2,2,2} : \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z} \hookrightarrow \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}$$

$$(\alpha, \beta, (A_1, A_2, A_3), (B_1, B_2, B_3)) \mapsto (\alpha, \beta, \text{diag}(A_1, A_2, A_3), \text{diag}(B_1, B_2, B_3)).$$

As before, the FTS quartic norm can be regarded as the discriminant, $\Delta(Q) = -D$.

**Theorem 14 (Parametrisation of $\text{SL}(6, \mathbb{R})$-equivalence classes in $\wedge^3(\mathbb{Z}^6)$) [1]** There is a canonical bijection between the set of nondegenerate $\text{SL}(6, \mathbb{Z})$-orbits on the space $\wedge^3 \mathbb{Z}^6$ of quartic norm $\Delta(Q)$, and the set of isomorphism classes of pairs $(S, M)$, where $S$ is a nondegenerate oriented quadratic ring with discriminant $D = -\Delta(Q)$, and $M$ is an equivalence class of balanced of ideals of rank three.

Once again the proof of this statement hinges on the identification of $Q \in \wedge^3 \mathbb{Z}^6$ with $a_{[ijk]}$ in

$$\det(\alpha_i, \alpha_j, \alpha_k) = c_{[ijk]} + a_{[ijk]} \tau,$$

where $\{\alpha_i\}_{i=1}^6$ is an oriented $\mathbb{Z}$-basis for $M$. The rather complicated looking unique solution for the $c_{[ijk]}$ is again quite simply given by the triple product

$$c_{[ijk]} = \frac{1}{2} (T(a)_{[ijk]} - (\text{Det } a \mod 4)a_{[ijk]}),$$

emphasising the role played by the FTS.

Under the mapping embedding the $\text{STU}$ model into the $\text{SL}(6, \mathbb{Z})$ Einstein-Maxwell-Scalar theory, the equivalence classes of pairs $(S, (I_S, I_T, I_U))$ get mapped into the equivalence classes of pairs of the form $(S, (I_S \oplus I_T \oplus I_U))$; the embedding corresponds to the fusion of triples of rank one ideals ($(I_S, I_T, I_U)$ into a single rank one ideal. Every $\text{SL}(2, \mathbb{Z}) \times \text{SL}(2, \mathbb{Z}) \times \text{SL}(2, \mathbb{Z})$-equivalence class maps into an $\text{SL}(6, \mathbb{Z})$-equivalence class. Moreover, every $\text{SL}(6, \mathbb{Z})$-equivalence class has a representative in the image of $\wedge_{2,2,2}$. Elements of $\wedge^3 \mathbb{Z}^6$ can be “diagonalised” by $\text{SL}(6, \mathbb{Z})$ to lie in a subspace isomorphic to $\mathbb{Z}^2 \oplus \mathbb{Z}^2 \oplus \mathbb{Z}^2$. Thus, the embedding map $(S, (I_S, I_T, I_U)) \mapsto (S, (I_S \oplus I_T \oplus I_U))$ is surjective at the level of equivalence classes and we conclude that every $Q \in \wedge^3 \mathbb{Z}^6$ can be “diagonalised” to an element of the form $\wedge_{2,2,2}(a_{ABC})$.

As before, this allows one to use projectivity of elements in $\mathbb{Z}^2 \oplus \mathbb{Z}^2 \oplus \mathbb{Z}^2$ to define projectivity of elements in $\wedge^3 \mathbb{Z}^6$. Restricting to projective elements the set of equivalence classes is endowed with a (trivial) group structure:

**Theorem 15 (Group law on projective integral 3-forms) [1]** For all $D = 0, 1 \mod 4$, the set of $\text{SL}(6, \mathbb{Z})$-equivalence classes of projective integral 3-forms, $Q \in \wedge^3 \mathbb{Z}^6$ with fixed quartic norm $D = -\Delta(Q)$ is a unique group $\text{Cl}(\wedge^3 \mathbb{Z}^6; D)$ such that $[a_{ABC}] \mapsto [\wedge_{2,2,2}(a_{ABC})]$ is group homomorphism from $\text{Cl}(\mathbb{Z}^2 \oplus \mathbb{Z}^2 \oplus \mathbb{Z}^2; D)$ and:

1. $\text{SL}(6, \mathbb{Z})$ is transitive on the set of projective elements with fixed $D = -\Delta(Q)$. Hence, $\text{Cl}(\wedge^3 \mathbb{Z}^6; D)$ is a one-element group.

2. An integer is a fundamental discriminant if it is square-free and either 1 mod 4 or 4m, where $m = 2, 3 \mod 4$. If $D = -\Delta(Q)$ is a fundamental discriminant then $Q$ is projective. Hence, up to $\text{SL}(6, \mathbb{Z})$-equivalence there is a unique $Q \in \wedge^3 \mathbb{Z}^4$ with $D = -\Delta(Q)$ a fundamental discriminant.

We conclude, as observed in [7, 8], that there is up to U-duality equivalence a unique projective extremal black hole solution for fixed Bekenstein-Hawking entropy and a unique extremal black solution for $(\frac{1}{2} S_{\text{BH}})^2$ a fundamental discriminant.

### 2.4.5 $\mathcal{N} = 8$ supergravity and the $\text{SO}(6, 6)$ Einstein-Maxwell-scalar theory

A particularly important example of extremal black hole solutions are those of $\mathcal{N} = 8$ supergravity, the low-energy effective field theory limit of type IIA/B string (M-theory) on a 6-torus (7-torus). In this case the black hole charges are elements of $\mathfrak{I}_{\mathcal{N}=8} = \mathfrak{I}(\mathfrak{I}_3)$. Rather than the $A_n$ type automorphism (electromagnetic duality) groups appearing in all the previous examples, here it is given by the exceptional $E_7(\mathbb{Z})$ [53, 55]. This rather obscures any potential
connection to ideal classes in quadratic orders, where previously an electromagnetic duality transformation corresponded to an arbitrary orientation preserving basis changes of the form $SL(n, \mathbb{Z}) \times SL(n, \mathbb{Z}) \times \cdots$. Bhargava does not provide a composition law for this exceptional case, likely for this very reason; it is not straightforward to identify suitable ideals in a quadratic order corresponding to the 56 of $E_7(\mathbb{Z})$. Nonetheless, the natural inclusions

$$3(\mathbb{Z}) \hookrightarrow 3(\mathbb{Z} \oplus \mathbb{Z}) \hookrightarrow 3(\mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}) \hookrightarrow 3(\mathbb{Z} \oplus \Gamma_{\mathbb{Z}}(\mathbb{Z})) \hookrightarrow 3(\mathbb{O}_3^+) \hookrightarrow 3(\mathbb{O}_3^+)^\perp$$

(2.129)

and their associated groups of type $E_7$.

$$SL(2, \mathbb{Z}) \subset [SL(2, \mathbb{Z})]^2 \subset [SL(2, \mathbb{Z})]^3 \subset SL(2, \mathbb{Z}) \times SL(4, \mathbb{Z}) \subset SL(6, \mathbb{Z}) \subset SO(6; \mathbb{Z}) \subset E_7(\mathbb{Z}),$$

(2.130)

allows for the same definition of projectivity [13] to be used for $3(\mathbb{O}_3^+)$:

**Definition 16** An element in $3(\mathbb{O}_3^+)$ is projective if it is $E_7(\mathbb{Z})$-equivalent to an element lying in the image of projective elements in $\mathbb{Z}^2 \otimes \mathbb{Z}^2 \otimes \mathbb{Z}^2$ under the inclusion map given by

$$t_{E_7}: \mathbb{Z} \oplus \mathbb{Z} \oplus 3_{STU} \oplus 3_{STU} \mapsto \mathbb{Z} \oplus \mathbb{Z} \oplus 3_{O}^+ \oplus 3_{O}^+ \quad \forall \alpha, \beta; (A_1, A_2, A_3), (B_1, B_2, B_3) \mapsto (\alpha, \beta, \text{diag}(A_1, A_2, A_3), \text{diag}(B_1, B_2, B_3)).$$

(2.131)

This is the same inclusion and definition used by Bhargava in the previous example of integral 3-forms. In that case, Bhargava showed that every integral 3-form is $SL(6, \mathbb{Z})$-equivalent to some element in the image of the inclusion (2.125) through the bijection to isomorphism classes of pairs $(\mathcal{S}, M)$. In the absence of an equivalent bijection, Krutelevich explicitly demonstrated that every element in $3(\mathbb{O}_3^+)$ is $E_7(\mathbb{Z})$-equivalent to an element lying in the image of (2.131). Note, Krutelevich’s proof relied crucially on the fact that every $3 \times 3$ Hermitian matrix defined over the split integral octonions can be diagonalised [49]. By contrast, not every $3 \times 3$ Hermitian matrix defined over Coxeter’s ring of integral division octonions can be diagonalised [80]; the existence of zero-divisors was essential in the proof of the former case. This has a bearing on the applicability of Bhargava’s higher composition laws to the magic supergravities proposed in [9], as discussed in more detail in subsection 2.5.

Using this notion of projectivity and the orbits classification of [13] we have the trivial group law directly analogous to Bhargava’s $\lambda^\dagger\mathbb{Z}^\dagger$ example:

**Theorem 17 (Group law on projective $E_7(\mathbb{Z})$-equivalence classes)** For all $D = 0, 1 \mod 4$, the set of $E_7(\mathbb{Z})$-equivalence classes of projective $Q \in 3(\mathbb{O}_3^+)$ with fixed quartic norm $D = -\Delta(Q)$ is a unique group $Cl(3(\mathbb{O}_3^+); D)$ such that $[a_{ABC}] \mapsto [t_{E_7}([a_{ABC}])]$ is group homomorphism from $Cl(\mathbb{Z}^2 \otimes \mathbb{Z}^2 \otimes \mathbb{Z}^2; D)$ and:

1. $E_7(\mathbb{Z})$ is transitive on the set of projective elements with fixed $D = -\Delta(Q)$. Hence, $Cl(3(\mathbb{O}_3^+); D)$ is a one-element group.

2. If $D = -\Delta(Q)$ is a fundamental discriminant then $Q$ is projective. Hence, up to $E_7(\mathbb{Z})$-equivalence there is a unique $Q \in 3(\mathbb{O}_3^+)$ with $D = -\Delta(Q)$ a fundamental discriminant.

All extremal black holes with fixed Bekenstein-Hawking entropy in $\mathcal{N} = 8$ supergravity with projective charge configurations are U-duality related [6, 7]. This is not true for non-projective black holes; the discrete invariants (2.10) can be used to demonstrate that there are charge configurations with the same Bekenstein-Hawking entropy that are not U-duality related [6,7]. Ideally, we would like a parametrization result for generic charge configurations. Note that a precisely analogous analysis applies to the SO(6, 6) Einstein-Maxwell-scalar theory of type $E_7$ given by the FTS $\mathcal{F}(\mathcal{F}_{41})$, which has 16 gauge potentials and 36 scalars parameterising $SO(6, 6)/[SO(6) \times SO(6)]$. In particular, we can use the same inclusion map to define projectivity and we have the same trivial group law on projective equivalence classes. A complete description of this case is given in [7].

### 2.5 Comment on the magic $\mathcal{N} = 2$ supergravity theories

In the pioneering work [16–18] it was shown that $D = 5, \mathcal{N} = 2$ supergravity theories with symmetric scalar manifolds are intimately related to Euclidean cubic Jordan algebras. In particular, if the Abelian gauge fields are to transform irreducibly under the global symmetry group of the Lagrangian, then there are four possibilities completely determined by the choice of cubic Jordan algebra $\mathcal{A}_3$, where $\mathcal{A}$ is one of the normed division algebras $\mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}$. These are typically referred to as the magic supergravities since their global symmetries form a row of the Freudenthal-Rosenfeld-Tits magic square of Lie algebras. There is a further countable family given by the $(1, n)$-spin-factor Jordan algebras $\mathbb{R} \oplus \Gamma_{1,n}$, which are sometimes referred to as the generic Jordan supergravities.

Dimensionally reducing the magic supergravities on a circle one obtains the $D = 4$ magic/generic Jordan supergravities with electromagnetic duality groups given by the automorphism groups of the associated FTS, $\mathcal{F}(\mathcal{A})$, over the corresponding $D = 5$ Jordan algebra $\mathcal{A} = \mathcal{A}_3$ or $\mathbb{R} \oplus \Gamma_{1,n}$. Hence, in all cases they are theories of type $E_7$ with electromagnetic duality groups given in Table 2. Given the relationship between theories of type $E_7$ and higher composition
laws already presented, it would perhaps be natural to expect that the magic and generic Jordan supergravities should correspond to Bhargava’s composition laws. Indeed, it was suggested in [9] that the magic complex theory should be related to the composition law on integral 3-forms discussed in subsubsection 2.4.4. Here we discuss why this and other related examples not directly related to the specific composition laws introduced by Bhargava. That is not to say, however, there is no composition law and/or ideal classes associated to such theories.

Let us begin with the magic complex case suggested in [9]. As observed in [7,8,13] and developed in subsubsection 2.4.4, the space of real 3-forms $\Lambda^3(\mathbb{R}^8)$ can be identified with $\mathcal{H}(\mathbb{C}_3)$ with automorphism group $SL(6,\mathbb{R})$. The corresponding theory of type $E_7$ was the Maxwell-Einstein supergravity of subsubsection 2.4.4, which does not admit a supersymmetric extension. Switching from $\mathbb{C}_3$ to $\mathbb{C}$ gives the magic complex $\mathcal{N} = 2$ supergravity once the fermionic sector is included. The correspondence to the higher composition required that the discrete subgroup, obtained by imposing the Dirac-Zwanziger-Schwinger quantisation, corresponded to linear basis changes of the associated rank-3 ideals, that is $SL(6,\mathbb{Z})$ and not $SU(3,3;\mathbb{Z})$. Already over the reals $SL(6,\mathbb{R})$ and $SU(3,3)$ have distinct orbits structures on the rank four elements of $\mathfrak{g}(\mathbb{C}_3^D)$ and $\mathfrak{g}(\mathbb{C}_3^S)$, so the connection to Bhargava’s composition law on 3-forms is lost.

At this stage one might object that the $\mathcal{N} = 8$ case $E_7(\mathbb{Z})$ also lacked an interpretation as the set of basis changes of some ideal classes. One could however “diagonalize” the non-degenerate elements of $\mathcal{H}(\mathbb{C}_3)$ and so apply the notion of projectivity. Perhaps, then, there is some hope for the octonionic magic $\mathcal{N} = 2$ supergravity corresponding $\mathcal{H}(\mathbb{C}_3)$ with automorphism group $E_7(-25)$, but we immediately ran into a problem. The diagonalisation proof of [13] relied on the diagonalisability of all elements in $\mathfrak{g}(\mathbb{C}_3)$ [49], but it is known that there are elements in $\mathfrak{g}(\mathbb{C}_3)$ that cannot be diagonalised [80] and hence the notion of projectivity cannot be applied, at least not without significant further work.

### 2.6 Comments on the relations to $D = 0 + 3$

A powerful technique in the construction stationary black solutions is the time-like dimensional reduction to $D = 0 + 3$, as pioneered in [81]. On performing the dimensional reduction and dualizing all resulting vectors to scalars, the U-duality is enhanced $G_4 \rightarrow G_3$. The scalars parametrise a non-compact coset $G_3/H_3$, where $H_3$ is a non-compact real form of the maximal compact subgroup of $G_3$. For example, in the case of $D = 1 + 3$ $\mathcal{N} = 8$ supergravity we have

$$E_7(\mathbb{Z}) \rightarrow E_8(\mathbb{Z}), \quad E_7(\mathbb{Z}) \rightarrow E_8(\mathbb{Z}) \rightarrow SO(16).$$

(2.132)

The nilpotent orbits of $G_3$ can then be used to classify the classical stationary black hole solutions in $D = 1 + 3$ [27,82–85]. For the six theories of type $E_7$ related to quadratic higher composition laws we have, under time-like dimensional reduction, the following U-duality groups [81]:

$$\begin{align*}
SL(6,\mathbb{Z}) & \rightarrow E_6(\mathbb{Z}) \\
SL(2,\mathbb{Z}) \times SL(4,\mathbb{Z}) & \rightarrow SO(5,5;\mathbb{Z}) \\
SL(2,\mathbb{Z}) \times SL(2,\mathbb{Z}) \times SL(2,\mathbb{Z}) & \rightarrow SO(4,4;\mathbb{Z}) \\
SL(2,\mathbb{Z}) \times SL(2,\mathbb{Z}) & \rightarrow SO(3,4;\mathbb{Z}) \\
SL(2,\mathbb{Z}) & \rightarrow G_{2(2)}(\mathbb{Z}).
\end{align*}$$

(2.133)

Amazingly, Bhargava independently identified these very groups by considering fusions of the ideals entering the parametrization of the orbits in terms of quadratic orders. Using these principles Bhargava was able to identify cubic composition laws, which also have corresponding black $p$-branes. The dimensional reduction of (super)gravity theories knows about higher composition laws and vice versa! Note, the nilpotent orbits of $G_3$ enter into the scattering amplitudes and BPS instantons of closed superstring theory, see for example [86], and so we should expect a connection to Bhargava there too. We leave this for future work.

### 3 Conclusions

We have reviewed the relationships between black hole charge orbits and the higher composition laws of Bhargava that were introduced in [5–8], including two new examples. We gave a more complete description of how this works and how the black hole charge orbits are parametrised by ideal classes through Bhargava’s work. We have explained why this correspondence should exist; it comes down to the dual role of prehomogenous vector spaces in constructing higher composition laws and theories of type $E_7$, which include all the (super)gravity theories of relevance here. In particular, we see the triple product of groups of type $E_7$ appear in the bijection between black hole charge orbits and ideal classes. Finally, we have noted the relationship between higher composition laws and dimensional reduction. We also claimed that all 14 of Bhargava’s higher composition laws are related to black $p$-branes in (super)gravity theories. For example,

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\[For \Delta > 0\] elements the maximally split real form $SL(6,\mathbb{R})$ has a unique orbit, while $SU(3,3)$ has two orbits.

\[\text{In the case of the \textit{STU} model this can also be applied to classify the entanglement of four-qubit states [67]}\]
using the analysis of [87] we see that the two-centered black hole/black string solutions of the $\mathcal{N} = 2$, $D = 1 + 4$ magic model based on $\mathfrak{g}_3^R$ correspond to case I.(8) of the list of regular prehomogeneous vector spaces given in §7 of [14] and Bhargava’s composition law on the $\text{GL}(2, \mathbb{Z}) \times \text{SL}(3, \mathbb{Z})$-equivalence classes of $\mathbb{Z}^2 \otimes \text{Sym}^2 \mathbb{Z}^3$, which are related to order two ideal classes in cubic rings [2]. We leave the details of this for future work.

**Note added**

During the completion of this manuscript the preprint [88] appeared, which has significant overlap with our discussion of the $STU$ model, but also contains interesting and significant developments regarding the $STU$ charge orbits not contained here.

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**References**

[1] M. Bhargava, “Higher composition laws I: A new view on Gauss composition, and quadratic generalizations.” *Ann. Math.* 159 (2004) no. 1, 217-250.

[2] M. Bhargava, “Higher composition laws ii: On cubic analogues of gauss composition,” *Annals of mathematics* (2004) 865–886.

[3] M. Bhargava, “Higher composition laws iii: The parametrization of quartic rings,” *Annals of mathematics* 159 (2004) no. 3, 1329–1360.

[4] G. W. Moore, “Arithmetic and attractors,” arXiv:hep-th/9807087 [hep-th]. 107pp. harvmac b-mode, 4 figures: minor mistakes, typos corrected. references added:v3: typo fixed, reference added Report-no: YCPT-P17-98.

[5] L. Borsten, “$E(7)(7)$ invariant measures of entanglement,” *Fortsch. Phys.* 56 (2008) 842–848.

[6] L. Borsten, D. Dahanayake, M. J. Duff, and W. Rubens, “Black holes admitting a Freudenthal dual,” *Phys. Rev.* D80 (2009) no. 2, 026003, arXiv:0903.5517 [hep-th].

[7] L. Borsten, D. Dahanayake, M. J. Duff, S. Ferrara, A. Marrani, et al., “Observations on Integral and Continuous U-duality Orbits in N=8 Supergravity,” *Class. Quant. Grav.* 27 (2010) 185003, arXiv:1002.4223 [hep-th].

[8] L. Borsten, *Aspects of M-Theory and Quantum Information*. PhD thesis, Imperial College, 2010. Ph.D. thesis, Imperial College.

[9] M. Gunaydin, S. Kachru, and A. Tripathy, “Black holes and Bhargava’s invariant theory,” arXiv:1903.02323 [hep-th].

[10] A. Sen and C. Vafa, “Dual pairs of type II string compactification,” *Nucl. Phys.* B455 (1995) 165–187, arXiv:hep-th/9508064.

[11] M. J. Duff, J. T. Liu, and J. Rahmfeld, “Four-dimensional string-string-string triality,” *Nucl. Phys.* B459 (1996) 125–159, arXiv:hep-th/9508094.

[12] M. Bhargava, “Higher composition laws and applications,” in *International Congress of Mathematicians*, vol. 2, pp. 271–294. 2006.

[13] S. Krutelevich, “Jordan algebras, exceptional groups, and Bhargava composition,” *J. Algebra* 314 (2007) no. 2, 924–977, arXiv:math/0411104.

[14] M. Sato and T. Kimura, “A classification of irreducible prehomogeneous vector spaces and their relative invariants,” *Nagoya Mathematical Journal* 65 (1977) 1–155.

[15] R. B. Brown, “Groups of type $E_7$,” *J. Reine Angew. Math.* 236 (1969) 79–102.
[16] M. Günyaydin, G. Sierra, and P. K. Townsend, “Exceptional supergravity theories and the magic square,” 
Phys. Lett. B133 (1983) 72.

[17] M. Günyaydin, G. Sierra, and P. K. Townsend, “The geometry of $N = 2$ Maxwell-Einstein supergravity and Jordan 
algebras,” Nucl. Phys. B242 (1984) 244.

[18] M. Günyaydin, G. Sierra, and P. K. Townsend, “Gauging the $d = 5$ Maxwell-Einstein supergravity theories: More 
on Jordan algebras,” Nucl. Phys. B253 (1985) 573.

[19] M. Gunaydin, “Generalized conformal and superconformal group actions and Jordan algebras,” 
Mod.Phys.Lett. A8 (1993) 1407–1416, arXiv:hep-th/9301050 [hep-th].

[20] S. Ferrara and M. Günyaydin, “Orbits of exceptional groups, duality and BPS states in string theory,” 
Int. J. Mod. Phys. A13 (1998) 2075–2088, arXiv:hep-th/9708025.

[21] M. Gunaydin, “Generalized conformal and superconformal group actions and Jordan algebras, ” 
Mod.Phys.Lett. A8 (1993) 1407–1416, arXiv:hep-th/9301050 [hep-th].

[22] S. Ferrara and M. Gunaydin, “Orbits of exceptional groups, duality and BPS states in string theory,” 
Int. J. Mod. Phys. A13 (1998) 2075–2088, arXiv:hep-th/9708025.

[23] M. Gunaydin, K. Koepsell, and H. Nicolai, “Conformal and quasiconformal realizations of exceptional Lie 
groups,” Commun. Math. Phys. 221 (2001) 57–76, arXiv:hep-th/0008063.

[24] M. Gunaydin, A. Neitzke, B. Pioline, and A. Waldron, “BPS black holes, quantum attractor flows and automorphic 
forms,” Phys. Rev. D73 (2006) 084019, arXiv:hep-th/0512296.

[25] S. Bellucci, S. Ferrara, M. Gunaydin, and A. Marrani, “Charge orbits of symmetric special geometries and 
attractors,” Int. J. Mod. Phys. A21 (2006) 5043–5098, arXiv:hep-th/0606209.

[26] L. Borsten, D. Dahanayake, M. J. Duff, H. Ebrahim, and W. Rubens, “Black Holes, Qubits and Octonions, ” 
Phys. Rep. 471 (2009) no. 3–4, 113–219, arXiv:0809.4685 [hep-th].

[27] L. Borsten, D. Dahanayake, M. J. Duff, H. Ebrahim, and W. Rubens, “Black Holes, Qubits and Octonions, ” 
Phys. Rep. 471 (2009) no. 3–4, 113–219, arXiv:0809.4685 [hep-th].

[28] L. Borsten, M. J. Duff, S. Ferrara, and A. Marrani, “Small Orbits, ” 
Phys.Rev. D85 (2012) 086002, arXiv:1108.0908 [math.RA].

[29] L. Borsten, M. J. Duff, S. Ferrara, and A. Marrani, “Small Orbits, ” 
Phys.Rev. D85 (2012) 086002, arXiv:1108.0908 [math.RA].
[39] S. L. Cacciatori, B. L. Cerchiai, and A. Marrani, “Squaring the Magic,”
Adv. Theor. Math. Phys. 19 (2015) 923–954, arXiv:1208.6153 [math-ph].

[40] A. Marrani, “Freudenthal Duality in Gravity: from Groups of Type E7 to Pre-Homogeneous Spaces,”
p Adic Ultra. Anal. Appl. 7 (2015) 322–331, arXiv:1509.01031 [hep-th].

[41] M. Chiodaroli, M. Gunaydin, H. Johansson, and R. Roiban, “Complete construction of magical, symmetric and homogenous N=2 supergravities as double copies of gauge theories,”
Phys. Rev. Lett. 117 (2016) no. 1, 011603, arXiv:1512.09130 [hep-th].

[42] L. Borsten, M. J. Duff, L. J. Hughes, and S. Nagy, “A magic square from Yang-Mills squared,”
Phys. Rev. Lett. 112 (2014) 131601, arXiv:1301.4176 [hep-th].

[43] A. Anastasiou, L. Borsten, M. J. Duff, L. J. Hughes, and S. Nagy, “A magic pyramid of supergravities,”
JHEP 1404 (2014) 178, arXiv:1312.6523 [hep-th].

[44] L. Borsten and A. Marrani, “A Kind of Magic,”
Class. Quant. Grav. 34 (2017) no. 23, 235014, arXiv:1707.02072 [hep-th].

[45] L. Borsten, M. J. Duff, and A. Marrani, “Freudenthal duality and conformal isometries of extremal black holes,”
arXiv:1812.10076 [gr-qc].

[46] B. Julia, “Group disintegrations,” in Superspace and Supergravity, S. Hawking and M. Rocek, eds., vol. C8006162 of Nuffield Gravity Workshop, pp. 331–350. Cambridge University Press, 1980.

[47] E. Cremmer, B. Julia, H. Lu, and C. N. Pope, “Higher dimensional origin of D = 3 coset symmetries,”
arXiv:hep-th/9710119 [hep-th].

[48] C. J. Ferrar, “Strictly Regular Elements in Freudenthal Triple Systems,”
Trans. Amer. Math. Soc. 174 (1972) 313–331.

[49] S. Krutelevich, “On a canonical form of a $3 \times 3$ Hermitian matrix over the ring of integral split octonions,”
J. Algebra 253 (2002) no. 2, 276–295.

[50] B. H. Gross, “Groups over $\mathbb{Z}$,”
Invent. Math. 124 (1996) 263–279.

[51] M. K. Gaillard and B. Zumino, “Duality Rotations for Interacting Fields,”
Nucl. Phys. B193 (1981) 221.

[52] E. Cremmer, B. Julia, H. Lu, and C. Pope, “Dualization of dualities. 1.,”
Nucl. Phys. B523 (1998) 73–144, arXiv:hep-th/9710119 [hep-th].

[53] E. Cremmer and B. Julia, “The $SO(8)$ supergravity,”
Nucl. Phys. B159 (1979) 141.

[54] S. Bellucci, S. Ferrara, A. Marrani, and A. Yeranyan, “$STU$ black holes unveiled,”
Enterpy 10 (2008) no. 4, 507–555, arXiv:0807.3503 [hep-th].

[55] C. M. Hull and P. K. Townsend, “Unity of superstring dualities,”
Nucl. Phys. B438 (1995) 109–137, arXiv:hep-th/9410167.

[56] M. J. Duff, B. E. W. Nilsson, and C. N. Pope, “Compactification of $d = 11$ Supergravity on $K(3) \times U(3)$,”
Phys. Lett. B129 (1983) 39, [50(1983)].

[57] B. L. Cerchiai, S. Ferrara, A. Marrani, and B. Zumino, “Duality, Entropy and ADM Mass in Supergravity,”
Phys. Rev. D79 (2009) 125010, arXiv:0902.3973 [hep-th].

[58] C. L. Siegel, “Uber die klassenzahl quadratischer zahlkörper,”
Acta Arith. 1 (1935) no. 1, 83–86.

[59] S. Banerjee and A. Sen, “S-duality Action on Discrete T-duality Invariants,”
JHEP 04 (2008) 012, arXiv:0801.0149 [hep-th].

[60] K. Behrndt, R. Kallosh, J. Rahmfeld, M. Shmakova, and W. K. Wong, “$STU$ black holes and string triality,”
Phys. Rev. D54 (1996) no. 10, 6293–6301, arXiv:hep-th/9608059.

[61] E. Cremmer, C. Kounnas, A. Van Proeyen, J. Derendinger, S. Ferrara, et al., “Vector Multiplets Coupled to N=2 Supergravity: SuperHiggs Effect, Flat Potentials and Geometric Structure,”
Nucl.Phys. B250 (1985) 385.

[62] S. Ferrara and A. Marrani, “$\mathcal{N} = 8$ non-BPS attractors, fixed scalars and magicvsupergravities,”
Nucl. Phys. B788 (2008) 63–88, arXiv:0705.3866 [hep-th].
[63] M. J. Duff and S. Ferrara, “Generalized mirror symmetry and trace anomalies,”
*Class. Quant. Grav.* **28** (2011) 065005, arXiv:1009.4439 [hep-th].

[64] M. J. Duff, “String triality, black hole entropy and Cayley’s hyperdeterminant,”
*Phys. Rev.* **D76** (2007) 025017, arXiv:hep-th/0601134.

[65] A. Cayley, “On the theory of linear transformations.”
*Camb. Math. J.* **4** (1845) 193–209.

[66] L. Borsten, D. Dahanayake, M. J. Duff, W. Rubens, and H. Ebrahim, “Wrapped branes as qubits,”
*Phys. Rev. Lett.* **100** (2008) no. 25, 251602, arXiv:0802.0840 [hep-th].

[67] L. Borsten, D. Dahanayake, M. J. Duff, A. Marrani, and W. Rubens, “Four-qubit entanglement from string theory,”
*Phys. Rev. Lett.* **105** (2010) 100507, arXiv:1005.4915 [hep-th].

[68] L. Borsten, M. J. Duff, and P. Levy, “The black-hole/qubit correspondence: an up-to-date review,”
*Class. Quant. Grav.* **29** (2012) 224008, arXiv:1206.3166 [hep-th].

[69] M. J. Duff, “Hidden symmetries of the Nambu-Goto action,”
*Phys. Lett.* **B641** (2006) 335–337, arXiv:hep-th/0602160.

[70] H. Nishino and S. Rajpoot, “Green-Schwarz, Nambu-Goto actions, and Cayley’s hyperdeterminant,”
*Phys. Lett.* **B652** (2007) 135–140, arXiv:0709.0973 [hep-th].

[71] L. Borsten, M. J. Duff, W. Rubens, and H. Ebrahim, “Freudenthal triple classification of three-qubit entanglement,”
*Phys. Rev. A80* (2009) 032326, arXiv:0812.3322 [quant-ph].

[72] L. Borsten, M. J. Duff, A. Marrani, and W. Rubens, “On the Black-Hole/Qubit Correspondence,”
*Eur. Phys. J. Plus* **126** (2011) 37, arXiv:1101.3559 [hep-th].

[73] L. Borsten, “Freudenthal ranks: GHZ versus W,”
*J. Phys. A46* (2013) 455303, arXiv:1308.2168 [quant-ph].

[74] K. Heegner, “Diophantische analysis und modulfunktionen.,”
*Mathematische Zeitschrift* **56** (1952) 227–253.
http://eudml.org/doc/169287.

[75] A. Baker, “Linear forms in the logarithms of algebraic numbers,”
*Mathematika* **13** (1966) no. 2, 204–216.

[76] H. M. Stark,
“A complete determination of the complex quadratic fields of class-number one.,”
*Michigan Math. J.* **14** (04, 1967) 1–27.
https://doi.org/10.1307/mmj/1028999653.

[77] K. Ireland and M. Rosen, “A classical introduction to modern number theory. 1990,”
*Grad. Texts in Math* (1990).

[78] A. Marrani, G. Pradisi, F. Riccioni, and L. Romano, “Nonsupersymmetric magic theories and Ehlers truncations,”
*Int. J. Mod. Phys.* **A32** (2017) no. 19n20, 1750120, arXiv:1701.03031 [hep-th].

[79] A. Marrani and L. Romano, “Orbits in Non-Supersymmetric Magic Theories,”
arXiv:1906.05830 [hep-th].

[80] N. Elkies and B. H. Gross, “The exceptional cone and the Leech lattice,”
*Internat. Math. Res. Notices* **14** (1996) 665–698.

[81] P. Breitenlohner, D. Maison, and G. W. Gibbons, “Four-Dimensional Black Holes from Kaluza-Klein Theories,”
*Commun. Math. Phys.* **120** (1988) 295.

[82] E. Bergshoeff, W. Chemissany, A. Ploegh, M. Trigiante, and T. Van Riet, “Generating Geodesic Flows and Supergravity Solutions,”
*Nucl. Phys.* **B812** (2009) 343–401, arXiv:0806.2310 [hep-th].

[83] G. Bossard, H. Nicolai, and K. S. Stelle, “Gravitational multi-NUT solitons, Komar masses and charges,”
*Gen. Rel. Grav.* **41** (2009) 1367–1379, arXiv:0809.5218 [hep-th].

[84] G. Bossard, Y. Michel, and B. Pioline, “Extremal black holes, nilpotent orbits and the true fake superpotential,”
*JHEP* **01** (2010) 038, arXiv:0908.1742 [hep-th].

[85] G. Bossard, H. Nicolai, and K. Stelle, “Universal BPS structure of stationary supergravity solutions,”
*JHEP* **0907** (2009) 003, arXiv:0902.4438 [hep-th].

[86] M. B. Green, S. D. Miller, and P. Vanhove, “Small representations, string instantons, and Fourier modes of Eisenstein series,”
*J. Number Theor.* **146** (2015) 187–309, arXiv:1111.2983 [hep-th].
[87] S. Ferrara, A. Marrani, E. Orazi, R. Stora, and A. Yeranyan, “Two-Center Black Holes Duality-Invariants for stu Model and its lower-rank Descendants,” *J.Math.Phys.* **52** (2011) 062302, arXiv:1011.5864 [hep-th].

[88] N. Banerjee, A. Bhand, S. Dutta, A. Sen, and R. K. Singh, “Bhargava’s Cube and Black Hole Charges,” arXiv:2006.02494 [hep-th].