Four-point Amplitudes in $\mathcal{N} = 8$ Supergravity and Wilson Loops

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Abstract

Prompted by recent progress in the study of $\mathcal{N} = 4$ super Yang-Mills amplitudes, and evidence that similar approaches are relevant to $\mathcal{N} = 8$ supergravity, we investigate possible iterative structures and applications of Wilson loop techniques in maximal supergravity. We first consider the two-loop, four-point MHV scattering amplitude in $\mathcal{N} = 8$ supergravity, confirming that the infrared divergent parts exponentiate, and we give the explicit expression which represents the failure for this to occur for the finite part. We observe that each term in the expansion of the one- and two-loop amplitudes in the dimensional regularisation parameter $\epsilon$ has a uniform degree of transcendentality. We then turn to consider Wilson loops in supergravity, showing that a natural definition of the loop, involving the Christoffel connection, fails to reproduce the one-loop amplitude. An alternative expression, which involves the metric explicitly, is shown to have a close relationship with the physical amplitude. We find that in a gauge in which the cusp diagrams vanish, the remaining diagrams for this Wilson loop correctly generate the full one-loop, four-point $\mathcal{N} = 8$ supergravity amplitude.

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1 Introduction

Evidence of recursive structures in the S-matrix of gauge theories has emerged in the past few years. In 2003 Anastasiou, Bern, Dixon and Kosower (ABDK) [1] made the remarkable observation that the planar, four-point MHV scattering amplitude in $\mathcal{N}=4$ supersymmetric Yang-Mills theory at two loops can be written as a polynomial of the one-loop amplitude, plus a kinematic-independent numerical constant. Subsequently, innovations prompted by twistor string theory, and in particular improved generalised unitarity techniques, have made possible the calculation of higher-loop amplitudes both in Yang-Mills and in gravity. Bern, Dixon and Smirnov (BDS) [2] were able to show that the iterative structure uncovered in [1] holds up to three loops, and put forward a conjecture for the all-loop, $n$-point MHV amplitude in $\mathcal{N}=4$ super Yang-Mills at the planar level, in which the all-loop amplitude is obtained by a suitable exponential of the one-loop amplitude multiplied by the cusp anomalous dimension.

Iterative structures were first discovered by analysing the soft and collinear behaviour of amplitudes in gauge theory [3–10]; the remarkable fact uncovered in [1, 2] was that the finite parts of the MHV amplitude also follow the same pattern induced by the expected exponentiation of the infrared divergences. The BDS proposal was checked at three loops in the four-point case in [2], and subsequently in [11] for the two-loop, five-point amplitude. In a very recent paper [12], a discrepancy was found between the form of the amplitude conjectured by BDS and an explicit two-loop calculation of the six-point amplitude. The result at six points shows that the structure is that of a polynomial in the one-loop amplitude, plus a kinematic-dependent finite remainder function.

In a related development [13], Alday and Maldacena proved the correctness of the BDS proposal for the four-point amplitude at strong coupling using the AdS/CFT correspondence. In their calculation, the exponentiation of the one-loop amplitude occurs through a saddle point approximation of the string path integral, which in the AdS case turns out to be exact. Furthermore, they showed that the computation of amplitudes at strong coupling is dual to the problem of finding the area of a string ending on a lightlike polygonal loop embedded in the boundary of AdS space. This, in turn, is equivalent to the method for computing a lightlike polygonal Wilson loop at strong coupling using the AdS/CFT correspondence, where the edges of the polygon are determined by the momenta of the scattered particles. In a subsequent paper [15] the same authors showed that the BDS conjecture should be violated for a sufficiently large number of scattered particles. Further evidence of a breakdown of the BDS conjecture was also found in [16].

\[^1\]For a recent review, see [14].
The work of [13] suggested that the calculation of a Wilson loop with the same polygonal contour could be related to that of the MHV scattering amplitude even at weak coupling. This was proved in [17] for the one-loop four-point $\mathcal{N} = 4$ amplitude, and by three of us in [18] for the infinite sequence of one-loop MHV amplitudes in $\mathcal{N} = 4$ super Yang-Mills. This surprising Wilson loops/amplitudes duality was later confirmed at two loops for the four- [19], five- [20], and six-point case [21, 22]. On the Wilson loop side, exponentiation naturally emerges as a result of the maximal non-Abelian exponentiation theorem [23, 24]. Furthermore, the form of the four- and five-point expression of the Wilson loop is determined (up to a constant) by an anomalous dual conformal Ward identity [20], and found to be of the form predicted by the BDS ansatz. A similar dual conformal symmetry was found for the integral functions appearing in the expression of the multi-loop amplitudes in [25]. Since conformal invariance is not restrictive enough to fully constrain the $n$-side polygonal Wilson loop for $n \geq 6$, it was perhaps not surprising that at precisely six points the BDS conjecture turned out to be incorrect [12]. It is intriguing however that the Wilson loops/amplitudes duality does not seem to break down – indeed, the results of [12] and [22] show numerical agreement between the Wilson loop and the six-point gluon amplitude at two loops.

These iterative structures in gauge theory and string theory have been found at the planar level. Planarity appears to be a key ingredient of the story – for instance, the non-planar parts of the four-point MHV amplitude at two loops do not respect the same iterative structure as the planar part [1]. Planarity would also appear to be an important ingredient in any relation to integrability – the cusp anomalous dimension appearing in the BDS proposal is also determined by an integral equation derived in [27] using integrability. An analytical solution to this equation was recently presented in [28].

It is natural to ask if gravity shares any of these remarkable properties. Gravity is a non-planar theory, hence it is perhaps even more unexpected to find regularities in the higher-loop structure of its S-matrix. However, the mounting evidence of interconnections between the maximally supersymmetric theories of $\mathcal{N} = 4$ Yang-Mills and $\mathcal{N} = 8$ supergravity gives reason to be more optimistic. Perhaps the potentially most impressive similarity between these two theories is the conjecture that the $\mathcal{N} = 8$ theory could be ultraviolet finite [30–36], just like its non-gravitational maximally supersymmetric cousin. Furthermore, gravity is also well understood in the infrared thanks to the results of [37], where it was found that infrared singularities can be resummed to the exponential of the one-loop infrared divergences, in complete similarity to those of QED [38, 39].

With these motivations in mind, in this paper we would like to initiate a twofold

\footnote{The paper [29] reviews the subject up to 2002.}
in investigation in $\mathcal{N} = 8$ supergravity. Our first goal will consist in looking for possible iterative structures and cross-order relations using the known results at one and two loops for the MHV four-point scattering amplitudes. We confirm that the infrared-divergent parts exponentiate, but we observe a failure for this to occur for the finite parts, in contradistinction with the four- and five-point amplitudes in $\mathcal{N} = 4$ Yang-Mills. On the other hand, we find that, similarly to the $\mathcal{N} = 4$ MHV amplitude, each term in the expansion of the one- and two-loop $\mathcal{N} = 8$ MHV amplitudes in the dimensional regularisation parameter $\epsilon$ has a uniform degree of transcendentality (or polylogarithmic weight). This is very intriguing, and leads to the speculation that maximal transcendentality \cite{40} could be yet another common feature of $\mathcal{N} = 4$ super Yang-Mills and $\mathcal{N} = 8$ supergravity.

Our second aim is to investigate possible relationships between gravitational scattering amplitudes and gravitational Wilson loops. This second objective is further motivated by some calculations of gravity amplitudes in the eikonal approximation \cite{41, 42}, and by our belief that there should exist a strong link between the eikonal approximation \cite{43–45} (performed in specific kinematic regions) and the more recent polygonal Wilson loop calculations (performed without reference to any specific kinematic region).

The rest of the paper is organised as follows. In the next section we will describe the known one- and two-loop MHV amplitudes in $\mathcal{N} = 8$ supergravity, and use them to show that the two-loop amplitude, minus one half of the square of the one loop amplitude, is finite, consistently with general arguments concerning the exponentiation of infrared divergences in gravity. We give the explicit expression for this finite term.

In Section 3 we turn to a one-loop Wilson loop calculation. One candidate for the Wilson loop expression, given by an integral of an exponential involving the Christoffel connection, is shown not to give the one-loop supergravity amplitude correctly. A second expression for the gravity Wilson loop is then studied, motivated by its application in the eikonal approximation to gravity. This involves the metric explicitly and is not gauge invariant, however the failure of gauge invariance is restricted to terms localised at the cusps of the Wilson loop.

The individual cusp diagrams and finite diagrams have the structure expected for the $\mathcal{N} = 8$ MHV amplitude (with the tree-level amplitude stripped off); however, after summing over all diagrams, we find an incorrect relative factor of $-2$ between the infrared-singular and the finite terms in comparison to the gravity amplitude. This is presumably related to the lack of gauge invariance of the Wilson loop at the cusps. Motivated by these results, we then turn in Section 4 to consider a gauge where the cusp diagrams vanish, which we call the conformal gauge. We show that in this gauge the Wilson loop diagrams, where the propagator connects two non-adjacent
segments, precisely yield the full four-point $\mathcal{N} = 8$ supergravity amplitude, including finite and divergent terms, to all orders in the dimensional regularisation parameter $\epsilon$. This is in complete analogy to what happens in $\mathcal{N} = 4$ Yang-Mills in a similar gauge, as we show in Appendices A and B.

Note added: After this work was completed, the preprint [75] appeared, which overlaps with Section 2 of this paper.

2 MHV amplitudes in $\mathcal{N} = 8$ supergravity and iterative structures

In this section we start by briefly reviewing the expressions of the four-point MHV amplitude in $\mathcal{N} = 8$ supergravity at one and two loops, and we then move on to study iterative structures at two loops.

2.1 Background

The form of the four-point MHV amplitude at $L$ loops in maximal supergravity is very simple. It is given by the tree-level four-point MHV amplitude $\mathcal{M}_4^{\text{tree}}$, times a scalar (helicity-blind) function,

$$\mathcal{A}_4^{(L)} = \mathcal{M}_4^{\text{tree}} \mathcal{M}_4^{(L)}.$$  \hspace{1cm} (2.1)

This amplitude was first calculated at one loop in [46] from the $\alpha' \to 0$ limit of a string theory calculation, and later rederived in [47] using string-inspired techniques [48], as well as unitarity [49, 50]. The infinite sequence of one-loop MHV amplitudes was obtained in [51]. Recently, the four- and five-point MHV amplitudes were also rederived in [52] using MHV diagrams. The two- and three-loop expressions were derived in [26], [35], respectively.

At one loop, the function $\mathcal{M}_4^{(1)}$ is simply given by a sum of three zero-mass box functions,

$$\mathcal{M}_4^{(1)} = -i s t u \left(\frac{k^2}{2}\right)^2 \left[ \mathcal{I}_4^{(1)}(s,t) + \mathcal{I}_4^{(1)}(s,u) + \mathcal{I}_4^{(1)}(u,t) \right],$$  \hspace{1cm} (2.2)

where

$$\mathcal{I}_4^{(1)}(s,t) := \int \frac{d^D l}{(2\pi)^D} \frac{1}{l^2(l-p_1)^2(l-p_1-p_2)^2(l+p_4)^2}.$$  \hspace{1cm} (2.3)
is a zero-mass box function with external, cyclically ordered null momenta \( p_1, p_2, p_3 \) and \( p_4 \), which sum to zero. We set \( s := (p_1 + p_2)^2, t := (p_2 + p_3)^2, u := (p_1 + p_3)^2 = -s - t \), and \( D = 4 - 2\epsilon \). Explicitly

\[
I_4^{(1)}(s,t) = i \frac{c_\Gamma}{st} \left[ \frac{2}{\epsilon^2} \left[ (-s)^{-\epsilon} + (-t)^{-\epsilon} \right] - \left( \log \frac{s}{t} + \pi^2 \right) \right],
\]

where \( c_\Gamma := (4\pi)^{-2}\Gamma(1+\epsilon)\Gamma^2(1-\epsilon)/\Gamma(1-2\epsilon) \). Using (2.4), we can rewrite (2.2) as

\[
M^{(1)}_4 = \left( \frac{\kappa}{2} \right)^2 c_\Gamma \left[ \frac{2}{\epsilon^2} \left[ (-s)^{1-\epsilon} + (-t)^{1-\epsilon} + (-u)^{1-\epsilon} \right] - u \log \frac{s}{t} - s \log \frac{t}{u} - t \log \frac{u}{s} \right].
\]

The simplicity of (2.1), where the tree-level amplitude factors out leaving a helicity-blind function of the particle momenta is clearly reminiscent of the structure for the infinite sequence of MHV scattering amplitudes in maximally supersymmetric Yang-Mills. This motivates the search for

a. an iterative structure in the higher-loop amplitude similar to that discovered in [1, 2] for the \( \mathcal{N} = 4 \) amplitude, and

b. a derivation of the functions \( \mathcal{M}^{(L)}_4 \) using Wilson loops.

The investigation of possible iterative structures of MHV amplitudes in \( \mathcal{N} = 4 \) Yang-Mills was motivated by the known structure of the infrared divergences. This led BDS to propose in [2] the following conjecture for the all-loop MHV amplitude in \( \mathcal{N} = 4 \) super Yang-Mills:

\[
\mathcal{M}_{n,\text{YM}} := 1 + \sum_{L=1}^{\infty} a^L \mathcal{M}^{(L)}_{n,\text{YM}}(\epsilon) = \exp \left[ \sum_{L=1}^{\infty} a^L \left( f^{(L)}(\epsilon) \mathcal{M}^{(1)}_{n,\text{YM}}(L\epsilon) + C^{(L)} + E^{(L)}(\epsilon) \right) \right],
\]

where \( a =\left[ g^2 N/(8\pi^2)\right] (4\pi e^{-\gamma})\epsilon \). In (2.4), \( f^{(L)}(\epsilon) = f^{(L)}_0 + f^{(L)}_1 \epsilon + f^{(L)}_2 \epsilon^2 \) is a set of functions, one at each loop order, which make their appearance in the exponentiated all-loop expression for the infrared divergences in generic amplitudes in dimensional regularisation [8]. Specifically, \( f^{(L)}_0 = \gamma_K^{(L)}/4 \), where \( \gamma_K \) is the cusp anomalous dimension (related to the anomalous dimension of twist-two operators of large spin). Importantly, the constants \( C^{(L)} \) do not depend on the kinematics or on the number of particles \( n \). The non-iterating contributions \( E^{(L)}_n(\epsilon) \) vanish as \( \epsilon \to 0 \) and depend explicitly on \( n \) and the kinematics.

As we have mentioned in the Introduction, the BDS proposal has been confirmed at two [1] and three loops [2] in the four-point case, at two loops for the five-point
amplitude [11], but the recent work of [12] shows a breakdown at two loops in the six-point case [12]. The infrared-singular part of $\mathcal{M}_{n,\text{YM}}$ is of course correctly reproduced by the infrared-divergent part of the right hand side of (2.4).

In order to check (2.6), one takes the log and expands both sides in perturbation theory; for example, at two loops, one gets

$$\mathcal{M}_{n,\text{YM}}^{(2)}(\epsilon) - \frac{1}{2}\left(\mathcal{M}_{n,\text{YM}}^{(1)}(\epsilon)\right)^2 = f^{(2)}(\epsilon)\mathcal{M}_{n,\text{YM}}^{(1)}(2\epsilon) + C^{(2)} + E^{(2)}. \tag{2.7}$$

We wish to follow the same path here for $\mathcal{N} = 8$ supergravity, starting from the observation that in gravity the one-loop infrared divergences exponentiate [37]. In the four-point case, the leading infrared divergences are expected to resum to

$$\exp\left[c_T\left(\frac{\kappa}{2}\right)^2\frac{2}{\epsilon}\left(s \log(-s) + t \log(-t) + u \log(-u)\right)\right]. \tag{2.8}$$

Notice the appearance of the invariant $u = (p_1 + p_3)^2$, due to the lack of colour ordering. Moreover, in [51] it was shown that the tree-level soft and collinear splitting amplitudes in gravity are exact to all orders in perturbation theory. This is due to the fact that the coupling constant $\kappa$ is dimensionful, and it is always accompanied by a power of a kinematic invariant which vanishes in the limit considered [37, 51].

We write the four-point MHV amplitude in $\mathcal{N} = 8$ supergravity (stripped of the tree-level prefactor) as

$$\mathcal{M}_4 = 1 + \sum_{L=1}^{\infty} \mathcal{M}_4^{(L)} = \exp\left[\sum_{L=1}^{\infty} m_4^{(L)}\right], \tag{2.9}$$

where

$$m_4^{(1)} = \mathcal{M}_4^{(1)}, \tag{2.10}$$

$$m_4^{(2)} = \mathcal{M}_4^{(2)} - \frac{1}{2}\left(\mathcal{M}_4^{(1)}\right)^2, \tag{2.11}$$

and so on. Motivated by (2.6) and, specifically at two loops, by (2.7), we will calculate in the following section the difference appearing on the right hand side of (2.11).

Let us make a final comment before moving on to explore in detail iterative structures at two loops. We observe that, unlike in the $\mathcal{N} = 4$ Yang-Mills case, the simplicity of (2.1) does not extend immediately beyond the four-particle case, as the explicit results for the $n$-point amplitude of [51] show. It was shown in [51], using $\mathcal{N} = 8$ Ward identities, that the ratio $\mathcal{M}^{(L)}(1^+, 2^+, \ldots, i^-, \ldots, j^-, \ldots, n^+)/\langle i j\rangle^8$ is independent of

\footnote{Notice that in (2.9) we absorb the appropriate power of $\kappa$ in the definition of $\mathcal{M}_4^{(L)}$ and $m_4^{(L)}$.}
the positions $i, j$ of the negative-helicity gravitons, i.e. it is helicity blind. This is similar to the Yang-Mills case [53], where $\mathcal{N} = 4$ supersymmetric Ward identities allow one to move the position of the negative-helicity particle, and show that the corresponding ratio in $\mathcal{N} = 4$ Yang-Mills $\mathcal{M}_{\text{YM}}^{(L)}(1^+, 2^+, \ldots, i^-, \ldots, j^-, \ldots, n^+)/\langle ij\rangle^4$ is independent of $i$ and $j$. In gravity however, this helicity-blind function is in general expressed as a sum of terms containing different spinor bracket valued coefficients. Two immediate consequences of this we would like to stress are that, firstly, it is not immediately clear what sort of iterative structures could be realised beyond four points; and, secondly, it is not obvious how a Wilson loop calculation could reproduce such terms (this situation somewhat parallels the problems one would encounter in attempting a derivation of non-MHV amplitudes in $\mathcal{N} = 4$ super Yang-Mills from Wilson loops). For these reasons, in this paper we only concentrate on the four-point MHV scattering amplitudes.

2.2 Iterative structure of the $\mathcal{N} = 8$ MHV amplitude at two loops

The previous discussion shows that, in searching for prospective iterative structures in the $\mathcal{N} = 8$ MHV amplitudes at two loops, it is meaningful to analyse the quantity \eqref{eqn:2.11} in supergravity, corresponding to the two-loop term in the expansion of the logarithm of the amplitude. We will carry out this computation in detail for the four-point MHV gravity amplitude described in the previous subsection. We observe that unlike the Yang-Mills ABDK conjecture [1], but in agreement with Weinberg’s result for gravity amplitudes [37], the one-loop infrared divergent terms of the amplitude exponentiate. More precisely, we will show that

$$\mathcal{M}_4^{(2)} - \frac{1}{2} (\mathcal{M}_4^{(1)})^2 = \text{finite},$$

and calculate the function on the right hand side of \eqref{eqn:2.12}.

The one-loop amplitude $\mathcal{M}_4^{(1)}$ is given in \eqref{eqn:2.2}. The two-loop amplitude was computed in [26], and is

$$\mathcal{M}_4^{(2)} = \left( \frac{\kappa}{2} \right)^4 stu \left[ s^2 I_4^{(2),P}(s,t) + s^2 I_4^{(2),P}(s,u) + s^2 I_4^{(2),NP}(s,t) + s^2 I_4^{(2),NP}(s,u) + \text{cyclic} \right].$$

\eqref{eqn:2.13}
Here $I_4^{(2), P}(s, t)$ and $I_4^{(2), NP}(s, t)$ are the planar and non-planar double box functions:

$$I_4^{(2), P}(s, t) = \int \frac{d^D l}{(2\pi)^D} \frac{d^D k}{(2\pi)^D} \frac{1}{l^2 (l - p_1)^2 (l - p_1 - p_2)^2 (l + k)^2 (k - p_4)^2 (k - p_3 - p_4)^2},$$

$$I_4^{(2), NP}(s, t) = \int \frac{d^D l}{(2\pi)^D} \frac{d^D k}{(2\pi)^D} \frac{1}{l^2 (l - p_2)^2 (l + k)^2 (l + k + p_1)^2 k^2 (k - p_3)^2 (k - p_3 - p_4)^2},$$

(2.14)

and in (2.13) we have to sum over the three cyclic permutations of the momenta $p_2$, $p_3$ and $p_4$ (i.e. over the three cyclic permutations of $s$, $t$ and $u$).

The two-loop planar box function was first evaluated by Smirnov [54] (see also [2]) and the non-planar double-box function was evaluated by Tausk [55]. These expressions need to be evaluated in different analytic regions, due to the permutation of kinematic invariants: we fix $s, t < 0$ but we will then need functions in which $s$ or $t$ are replaced by $u = -s - t > 0$, requiring a rather delicate procedure for analytic continuation. This procedure is outlined in Appendix C.

Smirnov's result for the planar double box integral (we use the form given in [2]) is given in terms of functions $F^{(2), P}(s, t)$ as

$$I_4^{(2), P}(s, t) = \alpha^2 \left[ \frac{s}{s^2 t} \right],$$

(2.15)

where $\alpha := i (4\pi)^{-2} \Gamma(1 + \epsilon)$ and

$$F^{(2), P}(s, t) = -\frac{e^{-2\epsilon\gamma}}{\Gamma^2(1 + \epsilon)} (-s)^{-2\epsilon} \sum_{j=0}^{4} \frac{c_j (-t/s)}{e^j},$$

(2.16)

with the coefficients $c_j$ in (B.5) of [2]. This expression is valid in the region $s, t < 0$ and we must carefully analytically continue into other regions as described in Appendix C.

Tausk's expression [55] for the non-planar double box is given in terms of functions $F^{(2), NP}(s, t)$ as

$$I_4^{(2), NP}(s, t) = \alpha^2 \left[ \frac{F^{(2), NP}(s, t)}{s^2 t} + \frac{F^{(2), NP}(s, u)}{s^2 u} \right].$$

(2.17)

The function $F^{(2), NP}(s, t)$ is given in [55] in all analytic regions (there it is called $F_t$).

Using the above results for the integrals, we arrive at the following expression for
the two-loop amplitude,
\[ \mathcal{M}_4^{(2)} = \left( \frac{\kappa^2 \alpha_s}{4} \right)^2 \left[ s u F^{(2),P}(s, t) + 2 s u F^{(2),NP}(s, t) + su F^{(2),P}(u, t) + 2 su F^{(2),NP}(u, t) \right. \]
\[ \left. + \text{cyclic} \right] . \]  

(2.18)

Notice that the functions \( F^{(2),P}(s, t) \) and \( F^{(2),NP}(s, t) \) always appear together in the combination \( F^{(2),P} + 2 F^{(2),NP} \), although \( F^{(2),P}(s, t) \) corresponds to the planar double box function (2.15), whereas \( F^{(2),NP}(s, t) \) corresponds to one of the two terms in the non-planar double box function (2.17).

The one-loop amplitude (2.2) is expressed as a sum of zero-mass box functions \( \mathcal{I}_4^{(1)} \), where
\[ \mathcal{I}_4^{(1)}(s, t) = \alpha_s \left[ \frac{F^{(1)}(s, t)}{s t} \right] , \]

and
\[ F^{(1)}(s, t) = \frac{e^{-\epsilon \gamma}}{\Gamma(1 + \epsilon)} (-s)^{-\epsilon} \sum_{j=-2}^2 \frac{\bar{c}_j (-t/s)}{\epsilon^j} . \]

The coefficients \( \bar{c}_j \) are given in (B2) of [2]. Again this is valid for \( s, t < 0 \) and we analytically continue to other regions. Together with (2.2), this gives the following expression for the one-loop amplitude,
\[ \mathcal{M}_4^{(1)} = -i \left( \frac{\kappa^2 \alpha_s}{4} \right) \left[ u F^{(1)}(s, t) + t F^{(1)}(s, u) + s F^{(1)}(u, t) \right] . \]

(2.21)

On putting in the functions for all permutations – correctly defined in their respective analytic regions – into the formula for the amplitude (2.18), we find that \( \mathcal{M}_4^{(2)} - \frac{1}{2} (\mathcal{M}_4^{(1)})^2 \) is finite. This finite remainder is explicitly given in (C.6). As described in detail in Appendix C, this function can be considerably simplified to the following expression:\footnote{Notice that (2.22) is somewhat formal, as there is no common region where all the functions appearing are away from their branch cuts. The precise analytic continuations for the case \( s, t < 0 \) are explained in detail in Appendix C, and the explicit, somewhat lengthier expression for the right hand side of (2.22) valid in that region, is given in (C.6).}
\[ \mathcal{M}_4^{(2)} - \frac{1}{2} (\mathcal{M}_4^{(1)})^2 = - \left( \frac{\kappa}{8 \pi} \right)^4 \left[ u^2 [k(y) + k(1/y)] + s^2 [k(1 - y) + k(1/(1 - y))] \right. \]
\[ + t^2 [k(y/(y - 1)) + k(1 - 1/y)] \]
where
\[
k(y) := \frac{L^4}{6} + \frac{\pi^2 L^2}{2} - 4 S_{1,2}(y)L + \frac{1}{6} \log^4(1 - y) + 4 S_{2,2}(y) - \frac{19\pi^4}{90}
+ i \left[ -\frac{2}{3} \pi \log^3(1 - y) - \frac{4}{3} \pi^3 \log(1 - y) - 4L\pi \mathrm{Li}_2(y) + 4\pi \mathrm{Li}_3(y) - 4\pi \zeta(3) \right]
\]
(2.23)

where \( y = -s/t \) and \( L := \log(s/t) \). Generalised polylogarithms, including the Nielsen polylogarithms \( S_{m,n} \) which appear above, are discussed in [56].

After submitting this paper, we have compared our results to those of [75], which contains a different form for the finite remainder (2.22). The two expressions are in fact in complete agreement. Specifically, one can rewrite (2.22) as
\[
\mathcal{M}_4^{(2)} - \frac{1}{2}(\mathcal{M}_4^{(1)})^2 = \left( \frac{\kappa}{8\pi} \right)^4 \left[ \text{st} h\left( \frac{-s}{u} \right) + \text{st} h\left( -\frac{t}{u} \right) + \text{permutations} \right] + \mathcal{O}(\epsilon) ,
\]
(2.24)

where
\[
h(w) := \frac{\log^4(w)}{3} + 8 S_{1,3}(w) + \frac{4\pi^4}{45} + i \left[ \frac{4}{3} \pi \log^3(w) - 8 \pi S_{1,2}(w) + 8 \pi \zeta(3) \right] ,
\]
(2.25)

which after taking into account the different analytic regions considered (here we consider \( s, t < 0 \) whereas the authors of [75] consider \( s, u < 0 \)) is in precise agreement with the result of [75].

An interesting observation is that the functions appearing in the expression for the amplitude have uniform transcendentality. This is somewhat surprising – although the box function and the planar double box function have uniform transcendentality, the non-planar double box does not. Nevertheless, the combination of functions \( F^{(2),NP}(s,t) + F^{(2),NP}(u,t) \), which appears after summing over all permutation, does have uniform transcendentality. We notice that amplitudes in \( \mathcal{N} = 1, 4 \) supergravity do not have this property. This is explicitly shown by the calculations in [47] of the one-loop four-graviton MHV amplitudes, see Eq. (4.6) of that paper. Perhaps unexpectedly, the \( \mathcal{N} = 6 \) MHV amplitude is also maximally transcendental at one loop. It would be interesting to know if this property persists at higher loops in the perturbative expansion of the amplitudes in these theories.

### 3 The one-loop Wilson loop calculation

In this section we describe the one-loop calculation of the four-point MHV amplitude of gravitons from a Wilson loop.
The expression we are going to use is motivated by its application in the eikonal approximation [43–45] to gravity [41, 42], and it reads

$$W[C] := \left< \mathcal{P} \exp \left[ i \kappa \oint_C h_{\mu\nu}(x(\tau)) \dot{x}^\mu(\tau) \dot{x}^\nu(\tau) \right] \right> , \quad (3.1)$$

where $x^\mu(\tau)$ parametrises the loop $\mathcal{C}$. Note that the exponent in (3.1) can be rewritten as

$$\int d^D x \, T^{\mu\nu}(x) h_{\mu\nu}(x) , \quad (3.2)$$

where, in the linearised approximation, the energy-momentum tensor is

$$T^{\mu\nu}(x) := \int d\tau \, \dot{x}^\mu(\tau) \dot{x}^\nu(\tau) \delta^{(D)}(x - x(\tau)) . \quad (3.3)$$

The specific form of the contour $\mathcal{C}$ we choose is dictated by the graviton momenta $p_1, \cdots, p_4$. In gravity there is no colour ordering – the amplitude (2.2) is a sum over the permutations (1234), (1243), (1324) of the four external gravitons. In order to match this from the Wilson loop side, we will therefore include the contribution of three Wilson loops with contours $\mathcal{C}_{1234}$, $\mathcal{C}_{1243}$, $\mathcal{C}_{1324}$, where $\mathcal{C}_{ijkl}$ is a contour made by joining the four graviton momenta $p_i, p_j, p_k, p_l$ in this order. More precisely, the quantity we calculate at one loop will be

$$W := W[\mathcal{C}_{1234}] \, W[\mathcal{C}_{1243}] \, W[\mathcal{C}_{1324}] . \quad (3.4)$$

Writing $W[\mathcal{C}_{ijkl}] := 1 + \sum_{L=1}^{\infty} W^{(L)}[\mathcal{C}_{ijkl}] = \exp \sum_{L=1}^{\infty} w^{(L)}_{ijkl}$, the one-loop term of (3.4) is

$$W^{(1)} = W^{(1)}[\mathcal{C}_{1234}] + W^{(1)}[\mathcal{C}_{1243}] + W^{(1)}[\mathcal{C}_{1324}] . \quad (3.5)$$

Before presenting the one-loop calculation, we would like to make a few preliminary comments.

1. One can check that the expression in (3.1) is not invariant under the gauge transformations

$$h_{\mu\nu} \rightarrow h_{\mu\nu} + \partial_\mu \xi_\nu + \partial_\nu \xi_\mu , \quad (3.6)$$

where $\xi^\mu(x)$ is an arbitrary vector field. Furthermore, it is easy to see that for contours composed of straight line segments joined at cusps such as those considered in this paper, the failure of gauge invariance is restricted to terms localised at the cusps. We think it is therefore not completely surprising that the infrared divergent parts of the Wilson loop will come out with an incorrect numerical prefactor from our calculation, compared to the finite parts, as we shall see below.

5The same expression for the gravity Wilson loop has recently been used in [57].

6In this section we set $D = 4 - 2\epsilon_{UV}$.
2. The expression \((3.1)\) is not explicitly reparametrisation invariant, but it can be seen to arise from a reparametrisation invariant expression involving an einbein \(e\), by writing the action of a free, massless particle as

\[
S \sim \int \frac{d\tau}{e(\tau)} \dot{x}^{\mu} \dot{x}^{\nu} g_{\mu\nu} .
\]

The energy momentum tensor resulting from this action is the one we use in our definition of the Wilson line in \((3.1)\), after gauge fixing \(e = 1\). The equation of motion for the einbein just imposes the condition that the path of the particle is null. The contour of the Wilson loop we use is piecewise null so that no problems can arise from reparameterisation invariance away from the cusps.

3. We note that the three contours appearing in \((3.4)\) are obtained by permuting the external momenta, not the vertices. Due to the inherently non-planar character of gravity, one cannot consistently associate T-dual momenta to the external graviton momenta. For this reason, it is therefore unlikely that a version of dual conformal invariance might constrain the form of the amplitude here.

4. A different expression for a gravity Wilson loop has been considered by Modanese \([58, 59]\), where the right hand side of \((3.1)\) is replaced by

\[
\langle \text{Tr} \mathcal{U}(\mathcal{C}) \rangle ,
\]

where

\[
\mathcal{U}_\beta^\alpha(\mathcal{C}) := \mathcal{P} \exp \left[ i\kappa \oint_{\mathcal{C}} dy^\mu \Gamma^\alpha_{\mu\beta}(y) \right] ,
\]

and \(\Gamma^\alpha_{\mu\beta}\) is the Christoffel connection. The quantity \(\text{Tr} \mathcal{U}(\mathcal{C})\) has the advantage of being manifestly invariant under coordinate transformations \([59]\). The calculation of the one-loop correction to \(\text{Tr} \mathcal{U}(\mathcal{C})\) for a closed loop has been considered already in \([59]\), and the result is proportional to

\[
\kappa^2 \oint_{\mathcal{C}} dx^{\mu_1} dy^{\mu_2} \langle \Gamma^\alpha_{\mu_1\beta}(x) \Gamma^\beta_{\mu_2\alpha}(y) \rangle .
\]

We refer the reader to Appendix \([D]\) for the details of the evaluation of \((3.9)\) in the linearised gravity approximation. The result is, dropping boundary terms,

\[
\kappa^2 \oint_{\mathcal{C}} dx dy \langle \Gamma^\alpha_{\mu\beta}(x) \Gamma^\beta_{\nu\alpha}(y) \rangle = c(D) \oint_{\mathcal{C}} dx^\mu dy^\nu \delta^{(D)}(x - y) ,
\]

where \(c(D)\) is a numerical constant which is finite as \(D \to 4\). Parameterising the contour as \(x = x(\sigma)\), we can rewrite the right hand side of \((3.10)\) as

\[
c(D) \int d\tau \int d\sigma \ \dot{x}_\mu(\tau) \dot{x}^{\mu}(\sigma) \delta^{(D)}(x(\tau) - x(\sigma)) .
\]
Some readers may notice that the divergent expression in (3.11) already appears in the lowest-order expansion of the Makeenko-Migdal loop equation [60,61] in Yang-Mills. An evaluation of (3.11) has been carried out in $\mathcal{N} = 4$ super Yang-Mills in [62] for a cusped contour, by using a regularisation of the Dirac delta function which employs a cutoff of width $a$. Interestingly, the right hand side of (3.11) is then found to be proportional to $1/a^2$ times the one-loop cusp anomalous dimension. We observe that, because of the delta function appearing in it, the expression in (3.11) receives contribution only from cusps and self-intersections present in the contour. The main point we would like to make here is that (3.11) does not reproduce (parts or all of) the $\mathcal{N} = 8$ supergravity amplitude (2.2). Therefore, in the following we will work with the Wilson loop defined for a polygonal contour as in (3.1).

We now proceed to describe the calculation. We work in the de Donder gauge, where the propagator is given by

$$\langle h_{\mu_1 \nu_1}(x) h_{\nu_1 \nu_2}(0) \rangle = \frac{1}{2} \left( \eta_{\mu_1 \nu_1} \eta_{\mu_2 \nu_2} + \eta_{\mu_1 \nu_2} \eta_{\mu_2 \nu_1} - \frac{2}{D-2} \eta_{\mu_1 \mu_2} \eta_{\nu_1 \nu_2} \right) \Delta(x), \quad (3.12)$$

where

$$\Delta(x) := -\frac{\pi^2 - \frac{D}{2}}{4\pi^2} \Gamma \left( \frac{D}{2} - 1 \right) \left( \frac{1}{-x^2 + i\epsilon} \right)^{\frac{D}{2} - 1} \quad (3.13)$$

The gravity calculation is very similar to the one-loop calculation performed in [17,18] for the one-loop Wilson loop in maximally supersymmetric Yang-Mills theory. As in that case, three different classes of diagrams contribute at one loop. In the first one, a graviton stretches between points belonging to the same segment. As in the Yang-Mills calculation, these diagrams give a vanishing contribution since the momenta of the gravitons are null. In the second class of diagrams, a graviton stretches between two adjacent segments meeting at a cusp. In the Yang-Mills case, such diagrams lead to ultraviolet divergences [64–71]. As in the Yang-Mills Wilson loop case [17], these divergences are associated with infrared divergences of the amplitude by identifying $\epsilon_{UV} = -\epsilon$.

We will now see how in our gravity calculation, these divergences are still present but will be softened (from $1/\epsilon_{UV}^2$ to $1/\epsilon_{UV}$) after taking into account the sum over the contributions of the three Wilson loops.

A typical diagram in the second class is pictured in Figure 1. There one has

7In [63] this result was extended to two loops, and conjectured to hold to any loop order in perturbation theory.

8For a Wilson loop bounded by gravitons, only gravitons can be exchanged to one-loop order.
Figure 1: A one-loop correction to the Wilson loop bounded by momenta $p_1, \cdots, p_4$, where a graviton is exchanged between two lightlike momenta meeting at a cusp. Diagrams in this class generate infrared-divergent contributions to the four-point amplitude which, after summing over the appropriate permutations give rise to (3.16).

\[ x_1(\tau_1) - x_2(\tau_2) = p_1(1 - \tau_1) + p_2\tau_2. \]

The cusp diagram gives

\[ -(i\kappa \tilde{\mu}^{\epsilon_{\text{UV}}})^2 \frac{\Gamma(1 - \epsilon_{\text{UV}})}{4\pi^2 - \epsilon_{\text{UV}}} \int_0^1 d\tau_1 d\tau_2 \left( \frac{(p_1 p_2)^2}{[-(p_1 \tau_1 + p_2 \tau_2)^2]^{1 - \epsilon_{\text{UV}}}} \right). \]

(3.14)

Notice that we need to choose $\epsilon_{\text{UV}} > 0$ in order to regulate the divergence in (3.14). Furthermore the scale used in the Wilson loop calculation is related to the scale used to regulate the amplitudes $\mu$ as $\tilde{\mu} = (c\mu)^{-1}$ (the precise coefficient $c$ in front of $\mu$ can be reabsorbed into an appropriate redefinition of the coupling constant).

Summing this over the four cusps of the first Wilson loop, one gets\(^9\)

\[ \frac{c(\epsilon_{\text{UV}})}{2\epsilon_{\text{UV}}^2} \left[ (-s)^{1+\epsilon_{\text{UV}}} + (-t)^{1+\epsilon_{\text{UV}}} \right]. \]

(3.15)

Adding the contributions of the two other Wilson loops, we get

\[ \frac{c(\epsilon_{\text{UV}})}{\epsilon_{\text{UV}}^2} \left[ (-s)^{1+\epsilon_{\text{UV}}} + (-t)^{1+\epsilon_{\text{UV}}} + (-u)^{1+\epsilon_{\text{UV}}} \right]. \]

(3.16)

Upon expanding this expression in $\epsilon_{\text{UV}}$, the cancellation of the $1/\epsilon_{\text{UV}}^2$ pole becomes manifest (after using $s + t + u = 0$), and (3.16) becomes, up to terms vanishing as

\(^9\)We set $c(\epsilon_{\text{UV}}) = (\kappa \tilde{\mu}^{\epsilon_{\text{UV}}})^2 \Gamma(1 - \epsilon_{\text{UV}})/(4\pi^2 - \epsilon_{\text{UV}})$. 

14
\[ \epsilon_{UV} \to 0, \]
\[ -c(\epsilon_{UV}) \left[ \frac{1}{\epsilon_{UV}} \left( s \log(-s) + t \log(-t) + u \log(-u) \right) + \frac{1}{2} \left( s \log^2(-s) + t \log^2(-t) + u \log^2(-u) \right) \right]. \]  

(3.17)

We recognise that this expression is the infrared-divergent part of the four-point MHV gravity amplitude (2.2). We notice however that, after summing over the appropriate permutations as in (3.4), one finds that these infrared-divergent terms have an extra factor of \(-2\) compared to the finite parts, to be calculated below. We believe this mismatch is not unexpected, given that the failure of gauge invariance of (3.1) occurs at the cusps.\(^{10}\)

![Figure 2: Diagrams in this class, where a graviton stretches between two non-adjacent edges of the loop, are finite, and give in the four-point case a contribution equal to the finite part of the zero-mass box function \( F^{(1)}(s,t) \) multiplied by \( u \).](image)

We now move on to the last class of diagrams, where a graviton is exchanged between two non-adjacent edges with momenta \( p \) and \( q \); one such example is depicted in Figure 2. In the Yang-Mills case these diagrams were found to be in one-to-one correspondence with the finite part of the two-mass easy box functions with massless legs \( p \) and \( q \). We will show now that (3.1) leads exactly to the same kind of correspondence with the finite part of the one-loop four-graviton amplitude.

Indeed, the one-loop diagram in Figure 2 is equal to

\[ c(\epsilon_{UV}) \int_0^1 d\tau_1 d\tau_2 \frac{(p_1 p_3)^2}{\left[ -\left( p_1(1 - \tau_1) + p_2 + p_3 \tau_2 \right) \right]^{1-\epsilon_{UV}}}. \]  

(3.18)

\(^{10}\)A factor of 2 could be explained because we are effectively double-counting the cusps in summing over the permutations, however at the moment we are unable to explain the relative minus sign.
This integral is finite in four dimensions, and gives
\[
c(\epsilon_{UV}) \frac{u}{2} \frac{1}{4} \left[ \log^2 \left( \frac{s}{t} \right) + \pi^2 \right].
\] (3.19)

Summing over the two possible pairs of non-adjacent segments and including the contributions of the two other Wilson loop configurations, we get exactly the finite part of the one-loop MHV amplitude in \( \mathcal{N} = 8 \) supergravity (2.5) up to the tree-level amplitude.\(^{11}\)

## 4 Calculation in the conformal gauge

The gravity Wilson loop defined above, unlike the Yang-Mills Wilson loop, is gauge dependent. It turns out that one can define a gauge in both cases in which the cusp diagrams vanish completely. We call these “conformal” gauges.\(^ {12}\) In the Yang-Mills Wilson loop one obtains the same answer in either gauge, but in the gravity Wilson loop the conformal gauge appears to be the unique gauge which gives the amplitude, both infrared-divergent and finite pieces correctly, to all orders in \( \epsilon \).

### 4.1 Gravity propagator in general gauges

We first need to define a general class of gauges in the gravity case. To do this, we consider the free Lagrangian of linearised gravity:
\[
\mathcal{L} = -\frac{1}{2} (\partial_\mu h_{\nu \rho})^2 + (\partial_\nu h^\nu_\mu)^2 + \frac{1}{2} (\partial_\mu h^\lambda_\lambda)^2 + h^\lambda_\lambda \partial_\mu \partial_\nu h_{\mu \nu},
\] (4.1)

which can be easily checked to be invariant with respect to the gauge transformation \( \delta h_{\mu \nu} = 2 \partial_\mu \xi_\nu \). We then add a gauge fixing term of the following form:
\[
\mathcal{L}^{(gf)} = \frac{\alpha}{2} \left( \partial_\nu h^\nu_\mu - \frac{1}{2} \partial_\mu h_\alpha^\alpha \right)^2,
\] (4.2)

\(^{11}\)A Wilson loop calculation clearly cannot produce any dependence on helicities and/or spinor brackets. Incidentally, we also observe that in Yang-Mills, a Wilson loop calculation cannot produce any parity-odd terms such as those appearing in the five- and six-point two-loop MHV amplitudes.

\(^{12}\)This name is motivated by the fact that, in the Yang-Mills case, the \( D \)-dimensional propagator turns out to be proportional to the inversion tensor \( J_{\mu \nu}(x) := \eta_{\mu \nu} - 2 x_\mu x_\nu / x^2 \). The Yang-Mills conformal propagator is described in Appendix [A](#), where we show that it can be obtained from a Feynman-'t Hooft gauge-fixing term with a specific coefficient. In Appendix [B](#) we perform the calculation of the \( n \)-point polygonal Wilson loop. The outcome of this calculation is that cusp diagrams in the conformal gauge vanish, and the \( \mathcal{N} = 4 \) amplitude is obtained from summing over diagrams where a gluon connects non-adjacent edges. In this case, each such diagram is in one-to-one correspondence with a complete two-mass easy box function.
which is de Donder-like, but with an arbitrary free parameter $\alpha$. We will call this the $\alpha$-gauge.

In momentum space, the corresponding gauge-fixed Lagrangian has the form

$$\frac{1}{2} h^{\mu \nu} K_{\mu \nu', \mu'} h_{\mu' \nu'}$$

where

$$K_{\mu \nu, \mu', \nu'}(k) = k^2 \eta_{\mu}(\mu \eta_{\nu})(\nu' k_{\mu'}) - 2 k_{(\mu} \eta_{\nu)}(\nu' k_{\mu'}) - k^2 \eta_{\mu \nu} \eta_{\mu' \nu'} + \eta_{\mu \nu} k_{\mu'} k_{\nu'} + \eta_{\mu' \nu'} k_{\mu} k_{\nu}
- \alpha \left[ k_{(\mu} \eta_{\nu)}(\nu' k_{\mu'}) - \frac{1}{2}(\eta_{\mu \nu} k_{\mu'} k_{\nu'} + \eta_{\mu' \nu'} k_{\mu} k_{\nu}) + \frac{1}{4} k^2 \eta_{\mu \nu} \eta_{\mu' \nu'} \right]. \quad (4.3)$$

Now we define the propagator $D_{\mu \nu, \mu', \nu'}$ to be the inverse of $K_{\mu \nu, \mu', \nu'}$, i.e.

$$K_{\mu \nu, \mu', \nu'} D_{\mu' \nu', \mu \nu} = \delta^m_{\mu} \delta^{\nu}_{\nu'}. \quad (4.4)$$

By writing down the most general Lorentz covariant terms which have the correct index symmetries and have mass dimension equal to -2, we see that $D_{\mu \nu, \mu', \nu'}$ must take the form

$$D_{\mu \nu, \mu', \nu'}(k) = \frac{1}{k^2} \eta_{\mu}(\mu \eta_{\nu})(\nu' k_{\mu'}) + \frac{a}{k^4} k_{(\mu} \eta_{\nu)}(\nu' k_{\mu'}) + \frac{b}{k^2} \eta_{\mu \nu} \eta_{\mu' \nu'}
+ \frac{c}{k^4} (\eta_{\mu \nu} k_{\mu'} k_{\nu'} + \eta_{\mu' \nu'} k_{\mu} k_{\nu}) + \frac{d}{k^6} k_{\mu} k_{\nu} k_{\mu'} k_{\nu'}. \quad (4.5)$$

Then (4.4) gives a set of equations for the free parameters which have the unique solution (for $D \neq 2$), $a = -(4 + 2\alpha)/\alpha$, $b = -1/(D - 2)$, $c = d = 0$. Thus, the propagator corresponding to the $\alpha$-gauge defined by the gauge-fixing term (4.2) is given by

$$D_{\mu \nu, \mu', \nu'}(k) = \frac{1}{k^2} \left( \eta_{\mu}(\mu \eta_{\nu})(\nu' k_{\mu'}) - \frac{1}{D - 2} \eta_{\mu \nu} \eta_{\mu' \nu'} \right) - \frac{4 + 2\alpha}{\alpha} \frac{1}{k^4} k_{(\mu} \eta_{\nu)}(\nu' k_{\mu'}) \quad (4.6)$$

Notice that (4.6) reproduces the standard de Donder propagator for $\alpha = -2$.

### 4.2 Propagator in position space in the $\alpha$-gauge

We now perform the Fourier transform to position space. The Fourier transform of $1/k^{2\lambda}$ has the form

$$\mathcal{F}[1/k^{2\lambda}] = c(D, \lambda) (-x^2)^{\lambda - D/2}, \quad (4.7)$$

where

$$c(D, \lambda) = -4^{-\lambda} \pi^{-D/2} \frac{\Gamma(2 - D/2) \Gamma(D/2 - 1)}{\Gamma(\lambda + 1 - D/2) \Gamma(\lambda)}. \quad (4.8)$$

\[\text{More details can be found in Appendix A.}\]
By differentiating twice with respect to \( x \) and setting \( \lambda = 2 \) we find that the Fourier transform of \( k_\mu k_\nu / k^4 \) is
\[
2c(D, 2)\epsilon_{UV} \left[ \frac{\eta_{\mu\nu}}{(-x^2)^{1-\epsilon_{UV}}} + \frac{2x_\mu x_\nu}{(-x^2)^{2-\epsilon_{UV}}} (1 - \epsilon_{UV}) \right]. \tag{4.9}
\]
Using this we take the Fourier transform of (4.6), and obtain the propagator in position space:
\[
D_{\mu\nu,\mu'\nu'}(x) = A \frac{\eta_{\mu'\nu'(\mu\nu)}}{(-x^2)^{1-\epsilon_{UV}}} \frac{c(D, 1)}{D-2} \frac{1}{(-x^2)^{1-\epsilon_{UV}}} \eta_{\mu\nu} \eta_{\mu'\nu'} + B \frac{1}{(-x^2)^{2-\epsilon_{UV}}} x(\mu\nu)(\nu'\mu'), \tag{4.10}
\]
where
\[
A = c(D, 1) + 2a \epsilon_{UV} c(D, 2) \quad B = 4a \epsilon_{UV}(1 - \epsilon_{UV}) c(D, 2), \tag{4.11}
\]
and
\[
a = -\frac{4 + 2\alpha}{\alpha}. \tag{4.12}
\]

4.3 The conformal gauge

By direct analogy with the Yang-Mills case, discussed in Appendix B, where we show that in the “conformal” gauge the cusp diagrams vanish, we define the gravity conformal gauge to be the gauge in which the cusp diagrams vanish. We show in this section that this particular gauge can be obtained from an \( \alpha \)-gauge fixing term as defined in the previous section for an appropriate value of the parameter \( \alpha \).

To begin with, consider the cusp defined by momenta \( p, q \) and then let \( x = p\sigma + q\tau \). Then the term appearing in the cusp at one loop is
\[
p^\mu p'^\nu D_{\mu\nu,\mu'\nu'}(x)q^\mu q'^\nu = (-x^2)^{3\epsilon_{UV}-2}(pq)^3\sigma\tau (B - 2A). \tag{4.13}
\]
Therefore, the cusp diagrams vanish for \( B = 2A \). One can quickly check that this implies \( a = -c(D, 1)/(2\epsilon_{UV}^2 c(D, 2)) = 4/(D - 4) \). The result is the propagator in the conformal gauge:
\[
D_{\mu\nu,\mu'\nu'}(x) = c(D, 1) \frac{\epsilon_{UV} - 1}{\epsilon_{UV}} \left[ \frac{1}{(-x^2)^{1-\epsilon_{UV}}} \left( \eta_{\mu'\nu'(\mu\nu)} + \frac{\epsilon_{UV}}{2(\epsilon_{UV} - 1)^2} \eta_{\mu\nu} \eta_{\mu'\nu'} \right) \right.
+ \frac{1}{(-x^2)^{2-\epsilon_{UV}}} x(\mu\nu)(\nu'\mu'), \tag{4.14}
\]
which requires
\[
\alpha = -2(D - 4)/(D - 2). \tag{4.15}
\]
4.4 Gravity Wilson loop in the conformal gauge

We now proceed to calculate the gravity Wilson loop in this conformal gauge. We have shown that the cusp diagrams are equal to zero in this gauge, therefore we need only calculate the “finite” diagrams (which are now no longer finite). Consider the Wilson loop with edges $p_1, p_2, p_3, p_4$ (in that order) and the graviton stretching between sides 1 and 3. Then we have $x = \sigma p_1 + \tau p_3 + p_2$ and $x^2 = s\sigma + t\tau + u\sigma\tau$. The contribution of this diagram is then

$$\int_0^1 d\sigma d\tau \, p_1^{\mu'} p_1^\nu D_{\mu\nu,\mu'^\nu}(x)p_3^{\mu'} p_3^{\nu'}$$

(4.16)

$$= c(D, 1) \frac{\epsilon_{\text{UV}} - 1}{\epsilon_{\text{UV}}} \frac{u}{4} \int_0^1 d\sigma d\tau \frac{st}{(-s\sigma + t\tau + u\sigma\tau)^{2-\epsilon_{\text{UV}}}}$$

$$= c(D, 1) \frac{1}{\epsilon_{\text{UV}}^2} \frac{u}{4} \left[ -(-s)^{\epsilon_{\text{UV}}} F_1(1, \epsilon_{\text{UV}}, 1 + \epsilon_{\text{UV}}, 1 + \frac{s}{t}) - (-t)^{\epsilon_{\text{UV}}} F_1(1, \epsilon_{\text{UV}}, 1 + \epsilon_{\text{UV}}, 1 + \frac{t}{s}) \right] .$$

We see that we obtain the complete (infrared-divergent as well as finite pieces) two-mass easy box function to all orders in $\epsilon_{\text{UV}}$. Adding the other diagram (which gives the same result) and then summing over the remaining permutations as described above, gives the correct one loop $N = 8$ supergravity amplitude (2.21).

Despite this encouraging result, we should remember that our starting expression for the Wilson loop (3.1) was not gauge invariant. It would be important to remedy this gauge non-invariance, which is localised at the positions of the cusps, by an appropriate subtraction procedure. Furthermore, it would be interesting to study infrared divergences, as well as the derivation of finite parts of gravity amplitudes at higher loops using the Wilson loop proposed in (3.4).

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A The conformal propagator in Yang-Mills

In this section we briefly outline the construction of the conformal propagator. It is defined to be proportional to the inversion tensor

\[ J_{\mu \nu}(x) := \eta_{\mu \nu} - 2\frac{x_\mu x_\nu}{x^2} \]  

(A.1)

By using

\[ \int \frac{d^D p}{(2\pi)^D} e^{i p x} \frac{1}{p^2} = \frac{-\pi^{D/2}}{4} \Gamma \left( \frac{D}{2} - 1 \right) \frac{1}{(-x^2 + i\varepsilon)^{(D/2)-1}} \]  

(A.2)

\[ \int \frac{d^D p}{(2\pi)^D} e^{i p x} \frac{p_\mu p_\nu}{p^4} = \frac{-\pi^{D/2}}{8} \Gamma \left( \frac{D}{2} - 1 \right) \frac{\eta_{\mu \nu} - (D - 2)x_\mu x_\nu/x^2}{(-x^2 + i\varepsilon)^{(D/2)-1}} \]

(A.2)

it is easy to see that the following combination has the desired property:

\[ \int \frac{d^D p}{(2\pi)^D} e^{i p x} \frac{\eta_{\mu \nu}}{p^2} + \frac{4}{D - 4} \int \frac{d^D p}{(2\pi)^D} e^{i p x} \frac{p_\mu p_\nu}{p^4} = \Delta_{\mu \nu}^{\text{conf}}(x) \]  

(A.3)

where we define the conformal propagator

\[ \Delta_{\mu \nu}^{\text{conf}}(x) := -\frac{D - 2}{D - 4} \frac{\pi^{D/2}}{4} \Gamma \left( \frac{D}{2} - 1 \right) \left[ \eta_{\mu \nu} - 2\frac{x_\mu x_\nu}{x^2} \right] \]

(A.4)

Thus, the expression (A.4) is obtained by choosing a Feynman–t’Hooft gauge-fixing term \((\alpha/2) \int d^D x (\partial_\mu A^\mu)^2\) for the particular choice of \(\alpha = (D - 4)/D\). The vanishing of this gauge-fixing term in \(D = 4\) dimensions is reflected in the presence of a factor of \(1/(D - 4)\) in (A.4), which makes this propagator not well defined in four dimensions.

B The Yang-Mills Wilson loop with the conformal propagator

As a simple but illuminating application of the above conformal propagator, we would like to outline the calculation of the Yang-Mills Wilson loop with a contour made of \(n\) lightlike segments performed in [18]. Of course, the usual expression of the Wilson loop in Yang-Mills is gauge invariant, hence evaluating it in any gauge leads to the same result. The use of this gauge leads however to a recombination of terms, where the cusp diagrams vanish. Consider for instance the cusped contour depicted in...
Figure 3: A one-loop correction for a cusped contour. We show in the text that, when evaluated in the conformal gauge, the result of this diagram vanishes.

Figure 3 Using the conformal propagator, and $x_{p_1}(\tau_1) - x_{p_2}(\tau_2) = p_1(1 - \tau_1) + p_2\tau_2$, we see that the one-loop correction to the cusp is given by an expression proportional to

\[
\int d\tau_1 d\tau_2 \frac{\eta^{\mu\nu} - 2[p_1(1-\tau_1)+p_2\tau_2]^{\mu}[p_1(1-\tau_1)+p_2\tau_2]^{\nu}}{[-2(p_1p_2)(1-\tau_1)\tau_2]^{D/2-1}},
\]

(B.1)

which vanishes.

We now move on to consider diagrams where a gluon is exchanged between non-adjacent segments, such as that in Figure 4. In [18] it was shown that this diagram is equal to the finite part of a two-mass easy box function. In the conformal gauge, a simple calculation shows that it is equal to

\[
f_{\epsilon} \cdot \frac{1}{2} (st - P^2Q^2) \int_0^1 d\tau_1 d\tau_2 \frac{d\tau_1 d\tau_2}{[-D(\tau_1, \tau_2)]^{2+\epsilon}},
\]

(B.2)

where

\[
D(\tau_1, \tau_2) := (x_p(\tau_1) - x_q(\tau_2))^2
\]

\[= P^2 + (s - P^2)(1 - \tau_1) - (t - P^2)\tau_2 - u(1 - \tau_1)\tau_2,
\]

(B.3)

where we used $2(pP) = s - P^2, 2(qP) = t - P^2$, and $s + t + u = P^2 + Q^2$. We have also introduced

\[
f_{\epsilon} := \frac{1 + \epsilon \Gamma(1 + \epsilon)}{\epsilon \pi^{2+\epsilon}}.
\]

(B.4)

14The usual infrared-divergent terms are produced by diagrams which, in the Feynman gauge calculation of [18], were finite.

15In the following we set $\epsilon = -\epsilon_{UV}$. 21
Figure 4: A one-loop diagram where a gluon connects two non-adjacent segments. In the Feynman gauge employed in [18], the result of this diagram is equal to the finite part of a two-mass easy box function $F^{2me}(p, q, P, Q)$, where $p$ and $q$ are the massless legs of the two-mass easy box, and correspond to the segments which are connected by the gluon. In the conformal gauge, this diagram is equal to the full box function. The diagram depends on the other gluon momenta only through the combinations $P$ and $Q$. In this example, $P = p_3 + p_4$, $Q = p_6 + p_7 + p_1$.

In [18] it was found that

$$\int_0^1 d\tau_1 d\tau_2 \frac{\mathcal{F}_{\epsilon+1}}{P^2 + Q^2 - s - t},$$

where

$$\mathcal{F}_{\epsilon} = -\frac{1}{\epsilon^2}$$

and

$$\mathcal{F}_{\epsilon} = \frac{1}{\epsilon^2} \left[ \left( \frac{a}{1 - aP^2} \right)^{\epsilon} \binom{2}{1} F_1 \left( \epsilon, \epsilon + 1, \frac{1}{1 - aP^2} \right) + \left( \frac{a}{1 - aQ^2} \right)^{\epsilon} \binom{2}{1} F_1 \left( \epsilon, \epsilon + 1, \frac{1}{1 - aQ^2} \right) \right] - \left( \frac{a}{1 - as} \right)^{\epsilon} \binom{2}{1} F_1 \left( \epsilon, \epsilon + 1, \frac{1}{1 - as} \right) - \left( \frac{a}{1 - at} \right)^{\epsilon} \binom{2}{1} F_1 \left( \epsilon, \epsilon + 1, \frac{1}{1 - at} \right),$$

where we have introduced

$$a := \frac{P^2 + Q^2 - s - t}{P^2 Q^2 - st}.$$ 

Notice that in (B.5) the function $\mathcal{F}$ appears with argument $\epsilon + 1$. After a moderate use of hypergeometric identities, we find that the one-loop correction in (B.2) is equal to

$$\frac{1}{2} \frac{\Gamma(1 + \epsilon)}{4\pi^{2+\epsilon}} F^{2me}(s, t, P^2, Q^2),$$

(B.8)
where $F_{\text{2me}}(s, t, P^2, Q^2)$ is the all-orders in $\epsilon$ expression of the two-mass easy box function derived in [72].

\[
F_{\text{2me}}(s, t, P^2, Q^2) = -\frac{1}{\epsilon^2} \left[ \left( \frac{-s}{\mu^2} \right)^{-\epsilon} 2F_1(1, -\epsilon, 1 - \epsilon, as) + \left( \frac{-t}{\mu^2} \right)^{-\epsilon} 2F_1(1, -\epsilon, 1 - \epsilon, at) 
- \left( \frac{-P^2}{\mu^2} \right)^{-\epsilon} 2F_1(1, -\epsilon, 1 - \epsilon, aP^2) - \left( \frac{-Q^2}{\mu^2} \right)^{-\epsilon} 2F_1(1, -\epsilon, 1 - \epsilon, aQ^2) \right] \quad (B.9)
\]

Summing over all possible gluon contractions in the Wilson loop, one finds complete agreement with the result derived in [18] for the same Wilson loop, as anticipated.

C Analytic continuation of two-loop box functions

In Section 2 the one and two loop amplitudes are given in terms of functions $F^{(2),P}(s, t)$, $F^{(2),NP}(s, t)$ and $F^{(1)}(s, t)$. In Yang-Mills, colour ordering means that we need to define the functions explicitly in only one analytic regime. In gravity however, we must sum over permutations of the kinematic invariants. Even if we fix the kinematic regime to be $s, t < 0$ we must also consider for example $F(s, u)$, and the second argument of this function will be greater than zero (recall that $u = -s - t$). There will be three different kinematic regimes of interest and, following Tausk [55], we label them in the following way:

\[
F(s, t) = \begin{cases} 
F_1(s, t) & t, u < 0 \\
F_2(s, t) & s, u < 0 \\
F_3(s, t) & s, t < 0 
\end{cases} 
\quad (C.1)
\]

Tausk gives explicit formulae for the non-planar box function in all three regions, but it is nevertheless useful to know how to obtain the function in any region from its manifestation in a particular region. The Mathematica package HPL [74] is very useful for this.

We will sketch the procedure below. Let us begin by considering the analytic continuation from region 1 to 2. In general, functions in this region take the following form:

\[
F_1(s, t) = f(\log(s), \log(-t), \log(-u), H_{\tilde{a},1}(-t/s)) . \quad (C.2)
\]

Here $H_{\tilde{a},1}(z)$ is a harmonic polylogarithm where $\tilde{a}$ represents a string of zeros or ones. Note that at two loops we need not use harmonic polylogarithms as they can all be

\[\text{Omitting a factor of } c_\Gamma = \Gamma(1+\epsilon)\Gamma^2(1-\epsilon)/(4\pi)^2 = \epsilon \] compared to [72].
re-expressed in terms of Nielsen polylogarithms. On the other hand, at higher loops harmonic polylogarithms will appear which cannot be so expressed; it is nevertheless useful to use harmonic polylogarithms even here (see \[56, 74\] for more details on harmonic polylogarithms). Such a harmonic polylogarithm is analytic everywhere on the complex plane except for a branch cut on the real axis for \(z > 1\). Note that the arguments of all the (poly)logarithm functions in (C.2) lie away from the branch cut.

Now the function continued to region 2 takes the following form:

\[
F_2(s, t) = f(\log(-s) + i\pi, \log(t) - i\pi, \log(-u), H_{\vec{a},1}(-t/s)) .
\]  

We have analytically continued the logs appropriately, however the argument of the HPL functions now lies on the branch cut in region 2 \((-t/s = 1 + u/s > 1)\). We use the HPL package to transform away from the branch cut. Specifically putting \(-t/s = 1/y\) the command ‘HPLConvertToSimplerArgument’ will rewrite this in terms of HPLs with the argument \(y = -s/t\) which lies off the branch cut (one must also use the command ‘HPLReduceToMinimalSet’ to write the functions in a standard form).

If we wish to obtain the formula in region 3 from that in region 1 we immediately have a problem. The argument of our HPL functions is \(-t/s\) which is not on a branch cut for either region. However, close examination shows that as we pass smoothly from region 1 to region 3, we must first pass along the branch cut – for example we must pass through the point \(s = 0\), i.e. \(-t/s = \infty\). The HPL programme will not take this into account and the naive analytic continuation gives the wrong result. So it is better to first perform a transformation \(y \to 1 - y\) on the HPLs in \(F_1(s, t)\) to find a new expression for \(F_1(s, t)\) in terms of HPLs with argument \(1 + t/s = -u/s\), i.e.

\[
F_1(s, t) = g(\log(s), \log(-t), \log(-u), H_{\vec{a},1}(-u/s)) .
\]  

Then in region 3 we find \(-u/s > 0\), and hence we are on the branch cut and we can proceed as before. We analytically continue as follows,

\[
F_3(s, t) = g(\log(-s) + i\pi, \log(-t), \log(-u) - i\pi, H_{\vec{a},1}(-u/s)) .
\]  

Now use the HPL programme to transform back off the cut using the transformation \(y \to 1/y\) yielding HPLs with argument \(-s/u\).

Now we have found the functions in all three analytic regions, and we can transform the arguments to obtain all the different permutations entering in the two-loop amplitude (2.13). For example \(F(s, t) = F_3(s, t)\) since we are in the region \(s, t < 0\), but \(F(u, t) = F_1(u, t)\) since the first argument is positive etc.

At this point, after summing all contributions, the two-loop amplitude will be a linear combination of harmonic polylogarithms with different arguments. We therefore use the HPL programme again to transform them all to the same argument,
ensuring that we never land on a branch cut in so doing. For example, for harmonic
polylogarithms of the form \( H_{a,1}(x) \) (i.e. where the defining string of numbers ends in
a ‘1’) we restrict ourselves to transformations of the form \( y \rightarrow 1 - y \) and \( y \rightarrow y/(y-1) \)
which the HPL program performs assuming we are away from the branch cut.

Using the above techniques we obtain the following form for the two-loop finite
remainder \( \mathcal{M}_{4}^{(2)} - \frac{1}{2}(\mathcal{M}_{4}^{(1)})^{2} \):

\[
\mathcal{M}_{4}^{(2)} - \frac{1}{2}(\mathcal{M}_{4}^{(1)})^{2} = \left( \frac{\alpha_{s}}{4 \pi} \right)^{2} \left[ s^{2} f^{(s)}(y) + t^{2} f^{(t)}(y) + u^{2} f^{(u)}(y) \right],
\] (C.6)

where

\[
f^{(s)}(y) = \frac{L^{4}}{3} - \frac{2}{3} \log(1 - y)L^{3} - \log^{2}(1 - y)L^{2} + \pi^{2}L^{2} + \frac{2}{3} \log^{3}(1 - y)L

- 4\pi^{2} \log(1 - y)L + 8S_{1,2}(y)L - 4\pi^{2} \text{Li}_{2}(y) + 8S_{1,3}(y) - 8S_{2,2}(y) - \frac{7\pi^{4}}{30}

+ i \left[ \frac{2\pi L^{3}}{3} + 2\pi \log(1 - y)L^{2} - 2\pi \log^{2}(1 - y)L + 8\pi \text{Li}_{2}(y)L

+ \frac{4\pi^{3}L}{3} - 8\pi \text{Li}_{3}(y) + 8\pi S_{1,2}(y) \right],
\] (C.7)

\[
f^{(t)}(y) = \frac{2}{3} \log(1 - y)L^{3} + \log^{2}(1 - y)L^{2} + 4\text{Li}_{2}(y)L^{2} - \pi^{2}L^{2} - \frac{2}{3} \log^{3}(1 - y)L

+ 4\pi^{2} \log(1 - y)L - 8\text{Li}_{3}(y)L + 4\pi^{2} \text{Li}_{2}(y) + 8\text{Li}_{4}(y) - 8S_{1,3}(y) + \frac{\pi^{4}}{2}

+ i \left[ \frac{2\pi L^{3}}{3} - 2\pi \log(1 - y)L^{2} + 2\pi \log^{2}(1 - y)L

- \frac{4\pi^{3}L}{3} - 8\pi S_{1,2}(y) + 8\pi \zeta(3) \right],
\] (C.8)

\[
f^{(u)}(y) = \frac{1}{3} \log^{4}(1 - y) - \frac{2}{3} L \log^{3}(1 - y) + L^{2} \log^{2}(1 - y) - \frac{2}{3} L^{3} \log(1 - y) - 4L^{2} \text{Li}_{2}(y)

+ 8L \text{Li}_{3}(y) - 8\text{Li}_{4}(y) - 8LS_{1,2}(y) + 8S_{2,2}(y) - \frac{\pi^{4}}{2} + L^{2} \pi^{2}

+ i \left[ - \frac{2\pi L^{3}}{3} - 2\pi \log(1 - y)L^{2} + 2\pi \log^{2}(1 - y)L - 8\pi \text{Li}_{2}(y)L

+ \frac{4\pi^{3}L}{3} - \frac{4}{3} \pi \log^{3}(1 - y) - \frac{8}{3} \pi^{3} \log(1 - y) + 8\pi \text{Li}_{3}(y) - 8\pi \zeta(3) \right],
\] (C.9)

where \( y = -s/t \) and \( L := \log(-y) \).

Since the amplitude is invariant under crossing symmetry (arbitrary permutations
of the momenta or equivalently arbitrary permutations of \( s, t, u \)) we must have

\[
f^{(u)}(y) = f^{(u)}(1/y) = f^{(s)}(1 - y) = f^{(t)}(y/(y-1)),
\] (C.10)
which one can indeed verify as long as one takes suitable care over the analytic continuation in the manner outlined above.

Simplifying slightly \( f^{(u)}(y) \) by writing it as \( k(y) + k(1/y) \) we obtain the form of the amplitude given in (2.22).

### D Derivation of (3.10)

In this Appendix we derive (3.10) from (3.9) in the linearised gravity approximation. Upon expanding the metric about flat space, \( g_{\mu\nu}(x) = \eta_{\mu\nu} + \kappa h_{\mu\nu}(x) \), one finds that (3.9) is equal to

\[
\kappa^2 \oint_C dx^\mu dx^\nu \langle \Gamma^\alpha_{\mu\beta}(x) \Gamma^\beta_{\nu\alpha}(y) \rangle \quad (D.1)
\]

\[
= \frac{1}{2} \oint_C dx^\mu dx^\nu \left[ -\partial^\alpha_x \partial^\beta_y \langle h^\alpha_{\mu}(x) h^\beta_{\nu}(y) \rangle + \Box_x \langle h_{\mu\beta}(x) h^\beta_{\nu}(y) \rangle \right].
\]

To perform the calculation in (D.1) we choose the de Donder gauge, where the propagator in \( D = 4 - 2\epsilon_{\text{UV}} \) dimensions is given by (3.12). Boundary terms can be dropped as the contour is a closed loop. Doing this, one easily finds that\(^{17}\)

\[
\kappa^2 \oint_C dx^\mu dy^\nu \langle \Gamma^\alpha_{\mu\beta}(x) \Gamma^\beta_{\nu\alpha}(y) \rangle = c(D) \oint_C dx^\mu dy^\mu \Box_x \Delta(x - y)
\]

\[
= c(D) \oint_C dx^\mu dy^\mu \delta^{(D)}(x - y),
\]

where \( c(D) \) is a numerical constant, finite as \( D \to 4 \). This is the result quoted in (3.10).

\(^{17}\)In [59], terms such as those appearing on the right hand side of (D.2) are referred to as “ultra-local”.
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