Infinite Tri–Symmetric Group, Multiplication of Double Cosets, and Checker Topological Field Theories

Yury Neretin

Vienna, Preprint ESI 2180 (2009)  
September 23, 2009

Supported by the Austrian Federal Ministry of Education, Science and Culture
Available via http://www.esi.ac.at
Infinite tri-symmetric group, multiplication of double cosets, and checker topological field theories

YURY NERETIN

We consider a product of three copies of infinite symmetric group and its representations spherical with respect to the diagonal subgroup. We show that such representations generate functors from a certain category of simplicial two-dimensional surfaces to the category of Hilbert spaces and bounded linear operators.

1 Introduction

1.1. Infinite symmetric group. Denote by $S_\infty$ the group of all permutations of the set $\mathbb{N} = \{1, 2, 3, \ldots\}$. By $S_\infty(\alpha)$ we denote the stabilizer of points $1, 2, \ldots, \alpha \in \mathbb{N}$. We also assume $S_\infty(0) = S_\infty$.

A topology on the group $S_\infty$ is defined by the condition: subgroups $S_\infty(\alpha)$ are open and cosets $gS_\infty(\alpha) \subset S_\infty$ form a basis of topology. The group $S_\infty$ is a totally disconnected topological group.

Classification of irreducible unitary representations of $S_\infty$ is rather simple, all representations are induced from trivial representations of Young subgroups $S_{m_1} \times \cdots \times S_{m_k} \times S_\infty - \sum_{j} m_j$, see Lieberman [3], see also Olshanski [11], and exposition in [5], VIII.1-2.

By $S_{f_\infty} \subset S_\infty$ we denote the group of finite permutations of $\mathbb{N}$, this group is an inductive limit of finite symmetric groups, $S_{f_\infty} = \bigcup_{n=1}^{\infty} S_n$.

The group $S_{f_\infty}$ is a wild (not type I, see e.g., [1]) discrete group, and description of its representations seems to be a non-reasonable problem.

1.2. n-symmetric groups. Consider the product

$G^{[n]} := S_\infty \times \cdots \times S_\infty$

of $n$ copies of $S_\infty$. We write elements $g \in G^{[n]}$ as collections $g = (g_1, \ldots, g_n)$, where $g_j \in S_\infty$. Denote by $K$ the diagonal subgroup in $G^{[n]}$, i.e., the subgroup consisting of collections $(g, g, \ldots, g)$, where $g \in S_\infty$.

We define the $n$-symmetric group $G = G^{[n]}$ as the subgroup of $G^{[n]}$ consisting of collections $(g_1, \ldots, g_n)$ such that

$$g_ig_j^{-1} \in S_{f_\infty} \quad \text{for all } i, j \leq n.$$

In other words, $G^{[n]}$ consists of all collections

$$(gh_1, gh_2, \ldots, gh_n) \quad \text{such that } g \in S_\infty, h_j \in S_{f_\infty}.$$
Denote by $K(\alpha)$ the image of $S_\infty(\alpha)$ under the diagonal embedding $K \to G^{[n]}$. We define the topology on $G^{[n]}$ from the condition: the subgroups $K(\alpha)$ are open. In other words, the topology of $K \simeq S_\infty$ is the same as above, the quotient-space $G^{[n]} / K$ is countable and equipped with the discrete topology.

1.3. Bisymmetric group. The existing representation theory of infinite symmetric groups is mainly the representation theory of the bisymmetric group $G$, see Thoma [14], Vershik, Kerov [15], Olshanski [13], Okounkov [9], Kerov, Olshanski, Vershik [2]. The situation was explained by Olshanski in [13]. We refer the reader to this paper.

The group $G^{[n]}$ is outside Olshanski’s approach to infinite-dimensional groups, based on imitation of symmetric pairs, see Olshanski [12], [13], and also [5]. However, $G^{[n]}$ is a $(G, K)$-pair in the sense of [5], VIII.5. My another standpoint was a strange Nessonov’s theorem, [7], [8] about spherical functions on certain infinite-dimensional groups with respect to certain small (non-symmetric) subgroups.

A discussion of bisymmetric groups is contained in Addendum to this paper.

1.4. Content of the paper. The tri-symmetric group. In Sections 2–4 we discuss the tri-symmetric group $G := G^{[3]}$.

Section 2. We describe of double cosets\textsuperscript{2}. The space $H \setminus G / K (\alpha) \setminus G / K (\beta)$ is closed in $G^{[n]}$ with respect to the $G / K (\gamma)$-topology. More precisely, we identify a double coset with a triangulated compact two-dimensional surface equipped with certain additional data (‘checker-boards’).

Next, we construct a natural multiplication $(a, b) \mapsto a \circ b$ of double cosets,

$$K(\alpha) \setminus G / K(\beta) \times K(\beta) \setminus G / K(\gamma) \rightarrow K(\alpha) \setminus G / K(\beta)$$

for each $\alpha, \beta, \gamma \in \mathbb{Z}_+$. Thus we get a category $\mathbb{S} = G^{[3]}$, whose objects are 0, 1, 2, . . . and morphisms $\beta \rightarrow \alpha$ are double cosets $K(\alpha) \setminus G / K(\beta)$.

Multiplications of double cosets is a usual phenomenon for infinite-dimensional groups (Ismagilov–Olshanski multiplicativity, see numerous constructions in [5], see also some additional examples in Russian translation of [5], Addendum E, and in [6]).

In this case, we get a construction similar to so-called ‘topological field theories’\textsuperscript{3}.

Section 3. Let $\rho$ be a unitary representation of $G$ in a Hilbert space $H$. Denote by $H(\alpha)$ the space of $K(\alpha)$-fixed vectors. Then each $a \in K_\alpha \setminus G / K_\beta$ determines a well-defined operator $\rho(a) : H(\beta) \rightarrow H(\alpha)$ and

$$\rho(a) \rho(b) = \rho(a \circ b) \quad \text{for } a \in K(\alpha) \setminus G / K(\beta), \ b \in K(\beta) \setminus G / K(\gamma),$$

\textsuperscript{2}Let $G$ be a group, $H, K$ its subgroups. A double coset is a subset in $G$ of the form $HgK$, where $g \in G$. The space $H \setminus G / K$ is the set, whose points are double cosets. Denote by $\mathcal{F}(H \setminus G / K)$ the space of functions on $H \setminus G / K$ If a group $G$ is finite, then there is a well-defined convolution $\mathcal{F}(H_1 \setminus G / H_2) \times \mathcal{F}(H_2 \setminus G / H_3) \rightarrow \mathcal{F}(H_1 \setminus G / H_3)$. But a product of individual double cosets is not well-defined.

\textsuperscript{3}Topological field theories are imitations of conformal field theories on topological level. Usually, this term is used for functors from categories of bordisms to the category of linear spaces and linear operators. Bordisms of simplicial complexes were considered, e.g., by Natanzon [4].
i.e., we get a representation of the category $S$.

Section 4. We construct a family of representations of $G$, write expressions for their spherical functions and for matrix elements of representations of the category $S$.

1.5. $n$-symmetric groups. General $n$-symmetric groups $G[n]$ are similar to the tri-symmetric group. Generally, checker-boards are tiled by $n$-gons. Also, there is another (nonequivalent) construction of a two-dimensional polygonal complex from a double coset. We discuss this in Section 5.

1.6. Abstract theorems. Sections 1–5 form the main part of the paper. In supplementary Section 6, we obtain some abstract theorems about unitary representations of $n$-symmetric groups: uniqueness of a $K$-spherical vector, rough separation of the set of representations, etc.

Acknowledgements. I am grateful to G. I. Olshanski and N. I. Nessonov for discussions of this topic.

2 Multiplication of checker-boards

Here we consider the tri-symmetric group $G := G[3]$.

2.1. Construction of simplicial complexes. We have 3 copies of the set $\mathbb{N}$, say red, yellow, and blue. An element of $G$ is a triple of permutations of $\mathbb{N}$, denote it by $(g_{\text{red}}, g_{\text{yellow}}, g_{\text{blue}})$.

We draw a collection of disjoint oriented black triangles $A_j$, where $j$ ranges in $\mathbb{N}$, and paint their sides in red, yellow, and blue anti-clockwise. We also draw a collection of oriented white triangles $B_j$ and paint sides in red, yellow, and blue clockwise.

Next, we glue a simplicial complex from these triangles. If $g_{\text{red}}$ sends $i$ to $j$, then we identify the red side of the black triangle $A_i$ with the red side of the white triangle $B_j$. We repeat the same operation for $g_{\text{yellow}}$ and $g_{\text{blue}}$. In this way, we get a disjoint countable union of 2-dimensional compact closed triangulated surfaces.

All components except finite number consist of two triangles (black and white, glued along the corresponding sides). We call such components chebureks⁴, see Figure 2.

Remark. Elements of the subgroup $K \subset G$ correspond to countable collections of disjoint chebureks.

2.2. Checker-boards. Now we define a checker-board (see Figure 1) as a countable disjoint union of triangulated closed surfaces equipped with following data:

a) Triangles are painted in black and white, neighbors of black triangles are white and vise versa.

⁴A street fast-food in some countries.
b) Edges are painted in red, yellow, and blue. The boundary of any black (respectively white) triangles is composed of red, yellow, and blue edges situated anti-clockwise (resp., clockwise).

c) Black (respectively, white) triangles are enumerated by natural numbers.

d) Almost all components are chebureks, i.e., unions of two triangles.

**Proposition 2.1** There is a canonical one-to-one correspondence between \( G \) and the set of all checker-boards.

One-side construction was given above. Conversely, take a checker-board. Let a red (yellow, blue) edge be common for \( i \)-th black triangle and \( j \)-th white triangle. Then \( g_{red} \) sends \( i \) to \( j \), see Figure 1.

2.3. **Vertices of checker-boards.** There are 3 types of vertices according colors of edges at a vertex: red–blue, red–yellow, yellow–blue.
Proposition 2.2 The set of red-blue vertices is in a natural one-to-one correspondence with the set of independent cycles of \( g_{\text{red}}^{-1}g_{\text{blue}} \). Moreover, the number of triangles meeting at a vertex is the double length of the corresponding cycle.

This is obvious, see Figure 1.

2.4. Double cosets and checker-boards. We say that an \((\alpha, \beta)\)-board is a compact (generally, disconnected) oriented triangulated surface equipped with the following data:

a) triangles are painted in white and black, edges are painted in red, yellow, blue as above,

b) we assign labels 1, 2, \ldots, \beta to some black triangles and 1, 2, \ldots, \alpha to some white triangles.

c) There is no chebureks without labels.

Proposition 2.3 There is a canonical one-to-one correspondence between the set \( K(\alpha) \setminus G/K(\beta) \) and the set of all \((\alpha, \beta)\)-boards.

Indeed, take a double coset, choose its representative \( \in G \) and compose the checker-board for \( g \). Next, we remove numbers > \beta from black triangles and numbers > \alpha from white triangles. Now almost all components are chebureks without numbers (we call them empty chebureks), and we remove them.

Thus we identify double cosets and checker-boards.

2.5. Multiplication of checker-boards. Consider two checker-boards, \( a \in K(\alpha) \setminus G/K(\beta) \), \( b \in K(\beta) \setminus G/K(\gamma) \).

For each \( j = 1, 2, \ldots, \beta \) we cut off the black triangle of \( a \) with label \( j \) and the white triangle of \( b \) with label \( j \) and identify edges of the triangles according colours and orientations. We obtain an \((\alpha, \gamma)\)-board.

Denote this operation by \( a \circ b \).

Thus we get a category, whose objects are 0, 1, 2, \ldots, and morphisms \( \beta \to \alpha \) are \((\alpha, \beta)\)-boards or, equivalently, double cosets.

The real reason for this definition is Theorem 3.1 about representations. Now we wish to interpret the definition in terms of the group \( G \).

2.6. Weak zeros. Let us write elements of \( K \simeq S_\infty \) as 0-1 matrices. We say that a sequence \( h_j \) in \( K(\alpha) \) is a weak zero if \( h_j \) converges element-wise to the \((\alpha + \infty) \times (\alpha + \infty)\)-matrix \( \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \).

2.7. Another definition of the multiplication of double cosets.

Theorem 2.4 Given \( a \in K(\alpha) \setminus G/K(\beta) \), \( b \in K(\beta) \setminus G/K(\gamma) \).

choose \( p \in a \), \( q \in b \), and a weak zero \( h_j \) in \( K(\beta) \). Then

\( c_j := K(\alpha)p h_j q K(\gamma) \)

equals \( c = a \circ b \) for sufficiently large \( j \).
Figure 3: Gluing of checker-boards. On the resulting surface, the interior part of the fat triangle is removed.
Proof. First, consider two elements \( p, q \in G \) and the corresponding checker-boards \( \mathcal{P}, \mathcal{Q} \). Then the product \( pq \) in the group \( G \) corresponds to the following operation:

- we remove black triangles from \( \mathcal{P} \) and white triangles from \( \mathcal{Q} \);
- we glue perimeters of corresponding 'holes' in \( \mathcal{P} \) and \( \mathcal{Q} \) (according colors and orientations of edges).

Second, let \( h \in K \subset G \). Then the product \( phq \) corresponds to the following operation: for each \( j \) we glue \( j \)-th white triangle of \( \mathcal{Q} \) with \( h \cdot j \)-th black triangle of \( \mathcal{P} \).

Let \( p \in G \). Let draw the corresponding checker-board \( p \). Let us remove all chebureks, whose black side has label \( > \beta \) and white side has label \( > \alpha \), denote by \( \mathcal{P} \) the corresponding reduced checker board. We say that the right (resp. left) \((\alpha, \beta)\)-support of \( p \) is the set of all labels on black (resp. white) triangles of \( \mathcal{P} \).

Equivalently, \( j \) is not in the right support iff

\[
  j > \beta \quad \text{and} \quad p_{\text{red}}j = p_{\text{yellow}}j = p_{\text{blue}} > \beta.
\]

Now, let \( h_j \) be a weak zero in \( K(\beta) \). For sufficiently large \( j \), the permutation \( h_j \in K(\beta) \) sends the left support of \( q \) to the complement of the right support of \( p \).

Now let \( p, q \in G \), let \( \mathcal{P}, \mathcal{Q} \) be the corresponding checker-boards. We evaluate \( ph_jq \) by the rule described above. Then all black triangles of \( \mathcal{Q} \) with labels \( > \beta \) are glued with chebureks, whose white labels are big. A gluing of a cheburek does not change a surface. Big label will be forgotten. The same holds for white triangles of \( \mathcal{P} \) with labels \( > \beta \). Thus, we get the operation of gluing of \((\alpha, \beta)\)-board and \((\beta, \gamma)\)-board described above. \( \square \)

2.8. Involution in the category \( S \). The map \( p \mapsto p^{-1} \) induces a map

\[
  K(\alpha) \setminus G / K(\beta) \to K(\beta) \setminus G / K(\alpha).
\]

For an \((\alpha, \beta)\)-board \( a \) we make a \((\beta, \alpha)\)-board \( a^\square \) by the follow rule:

a) we change 'black' and 'white';

b) we change the orientation of the surface.

Evidently,

\[
  (a \circ b)^\square = b^\square \circ a^\square.
\]

3 Representations. Abstract theorem

3.1. Operators \( \rho(a) \). Let \( \rho \) be a unitary representation of \( G = G[3] \) in a Hilbert space \( H \). Denote by \( H(\alpha) \) the space of \( K(\alpha) \)-fixed vectors. Denote by \( P(\alpha) \) the operator of orthogonal projection to the space \( H(\alpha) \).

Fix \( \alpha, \beta \). For \( p \in G \), consider the operator

\[
  H(\beta) \to H(\alpha)
\]
given by
\[ \overline{\rho}(g) := P(\alpha)\rho(p) = P(\alpha)\rho(p)P(\beta). \]

For \( r_1 \in K(\alpha), \ r_2 \in K(\beta) \) we have
\[ \overline{\rho}(r_1 gr_2) = \overline{\rho}(g). \]

Thus \( \overline{\rho} \) is a function on double cosets \( K(\alpha) \setminus G/K(\beta) \).

**Theorem 3.1** For any unitary representation \( \rho \) of \( G \) for each \( \alpha, \beta, \gamma \in \mathbb{Z}_+ \), for each
\[ a \in K(\alpha) \setminus G/K(\beta), \quad b \in K(\beta) \setminus G/K(\gamma), \]
we have
\[ \overline{\rho}(a)\overline{\rho}(b) = \overline{\rho}(a \circ b). \]

Thus we get a representation of the category \( S \) of checker-boards. Precisely, for each \( \alpha \) we assign the Hilbert space \( H(\alpha) \) and for each \((\alpha, \beta)\)-board \( a \) we assign the operator \( \rho(a) : H(\beta) \to H(\alpha) \).

Evidently, \( \overline{\rho} \) is a \( \ast \)-representation, i.e.,
\[ \overline{\rho}(a \boxdot) = \rho(a)^\ast. \]

**3.2. Lemma on representations of \( S_\infty \).** For representations of \( S_\infty \), see [3], [10], [13], [5]. For proof of the following Proposition 3.2, see, e.g., [5], Corollary 8.1.5.

For a unitary representation of \( S_\infty \) in a Hilbert space \( H(\alpha) \subset H \) as above.

**Proposition 3.2** Let \( h_j \) be a weak zero\(^5\) in \( S_\infty(\alpha) \). Then \( \rho(h_j) \) converges to \( P(\alpha) \) in the weak operator topology.

**3.3. Proof of Theorem 3.1.** Let \( a, b \) be double cosets as above (3.2), let \( p, q \in G \) be their representatives. Let \( u_j \) be a weak zero in \( K(\beta) \). Then
\[ \overline{\rho}(a)\overline{\rho}(b) = P(\alpha)\rho(p)P(\beta)\rho(q)P(\gamma) = \lim_{j \to \infty} P(\alpha)\rho(p)\rho(u_j)\rho(q)P(\gamma) = P(\alpha) \lim_{j \to \infty} \rho(pu_jq)P(\gamma) \]
where \( \lim \) denotes the weak limit.

For sufficiently large \( j \), a product \( pu_jq \) is contained in the double coset \( a \circ b \). Therefore, we get \( \overline{\rho}(a \circ b). \)

\(^5\)See Subsection 2.6
4 Representations. Constructions

In this section we construct some representations of the tri-symmetric group and of the category $\mathcal{S}$.

4.1. Tensor product construction. Here we imitate the construction of [15], [13]. Let $U, V, W$ be (finite dimensional or infinite dimensional) Hilbert spaces. Let $u_i, v_j, w_k$ be their orthonormal bases. Consider the tensor product

$$H := U \otimes V \otimes W.$$ 

Fix a unit vector $h \in H$,

$$h = \sum_{i,j,k} h_{ijk} u_i \otimes v_j \otimes w_k \in U \otimes V \otimes W, \quad \|h\| = 1.$$ 

Consider the tensor product

$$H := (H, h) \otimes (H, h) \otimes \ldots = (U \otimes V \otimes W, h) \otimes (U \otimes V \otimes W, h) \otimes \ldots$$

of infinite number of copies of Hilbert spaces $H$ with the distinguished vector $h$ (see the von Neumann definition of infinite tensor products, [16]). Denote

$$\xi := h \otimes h \otimes \ldots \in H.$$ 

Now we wish to define a certain representation $\nu_h$ of the group $G$ in the space $H$. Let $p = (p_1, p_2, p_3) \in G$. Then $p_1$ acts as a permutation of factors $U$, $p_2$ as permutation of factors $V$, and $p_3$ as a permutation of factors $W$.

Let formulate this more carefully. The operators of permutations are well defined

— for $p_1, p_2, p_3 \in S_\infty^{\text{fin}}$;
— for elements $(r, r, r) \in K$, they act by simultaneous permutations of factors $(U \otimes V \otimes W)$ and preserve the distinguished vector $\xi$.

These operators generate a unitary representation of $G$.

REMARK. The group $S_\infty \times S_\infty \times S_\infty$ does not act in this tensor product. □

REMARK. Let $A, B, C$ be unitary operators in $U, V, W$ respectively. Then

$$\nu_h \simeq \nu(A \otimes B \otimes C)h.$$ 

However, we cannot simplify an expression for $h$ by such transformations. □

4.2. Spherical functions. Our representations are $K$-spherical in the following sense:

**Proposition 4.1** The vector $\xi$ is a unique $K$-fixed vector in $H$.

**Proof.** Indeed, choose an orthonormal basis $e_1, e_2, \ldots \in U \otimes V \otimes W$ with $e_1 = h$. Expand a $K$-fixed vector $\eta \in H$ in Fourier series,

$$\eta = \sum_{i_1, i_2, \ldots} c_{i_1 i_2 \ldots} e_{i_1} \otimes e_{i_2} \otimes \ldots$$

where $i_N = 1$ for large $N$.
Coefficients $c$ must be invariant with respect to all permutations of indices. Let some $i_k \neq 1$ and $c_{i_1,i_2,...} = \varepsilon \neq 0$. Then there is a countable number of pairwise different permutations of the sequence $i_1, i_2, \ldots$ and therefore a countable number of Fourier coefficients $= \varepsilon$. Therefore $\|\eta\| = \infty$. \hfill \Box

Next, we write the spherical function, i.e.,

$$\Phi_h(g) = \langle \eta(h(g)) \xi \rangle.$$

By (3.1, $\Phi_h$ is a function on $K(0) \setminus \mathbb{G}/K(0)$, i.e., on the set of non-labeled checker-boards.

Let us write natural numbers $i \leq \dim U$ on red edges of the simplicial complex, $j \leq \dim V$ on yellow edges, and $k \leq \dim W$ on blue edges. For each triangle $T$, we get 3 numbers on its sides, say $i(T), j(T), k(T)$.

**Proposition 4.2**

$$\Phi_h(a) = \sum_{\text{all arrangements of } i, j, k \text{ on edges}} \prod_{\text{black triangles } T} h_{i(T)j(T)k(T)} \prod_{\text{white triangles } S} \eta_{i(S)j(S)k(S)}$$

\begin{equation}
\tag{4.1}
\end{equation}

**Proof.** We choose a representative $g \in \mathbb{G}$ of the coset $a$ such that $g_1, g_2, g_3$ are finite, let they are contained in $S_N$. Hence we can work in the space $H \otimes^N$,

$$h^{\otimes N} = \sum_{i_1, i_2, \ldots; j_1, j_2, \ldots; k_1, k_2, \ldots} \left( \prod_{m \leq N} h_{i_m,j_m,k_m} \right) (u_{i_1} \otimes v_{j_1} \otimes w_{k_1}) \otimes \cdots \otimes (u_{i_N} \otimes v_{j_N} \otimes w_{k_N}) \tag{4.2}$$

Thus summands are enumerated by collections of numbers written on sides of black triangles.

Next,

$$\rho(g)h^{\otimes N} = \sum_{i_1, i_2, \ldots; j_1, j_2, \ldots; k_1, k_2, \ldots} \left( \prod_{m \leq N} h_{i_m,j_m,k_m} \right) (u_{g_1(i_1)} \otimes v_{g_2(j_1)} \otimes w_{g_3(k_1)}) \otimes \cdots \otimes (u_{g_1(i_N)} \otimes v_{g_2(j_N)} \otimes w_{g_3(k_N)}) \tag{4.3}$$

We change the order of summation,

$$i_m^{\text{new}} = g_1^{-1}(i_m), \quad j_m^{\text{new}} = g_2^{-1}(j_m), \quad k_m^{\text{new}} = g_3^{-1}(k_m)$$

10
and get
\[
\sum_{1, i_2, ..., j_1, j_2, ..., k_1, k_2, ...} \left( \prod_{m \leq N} h_{g_1^{-1}(i_m), g_2^{-1}(j_m), g_3^{-1}(k_m)} \right) (u_{i_1} \otimes v_{j_1} \otimes w_{k_1}) \otimes \cdots \otimes (u_{i_N} \otimes v_{j_N} \otimes w_{k_N})
\]

Evaluating the inner product of (4.2) and (4.3), we come to (4.1). □

Remark. a) The multiplication in \( K(0) \setminus G/K(0) \) is the disjoint union. Therefore, for \( a, b \in K(0) \setminus G/K(0) \) we have
\[
\Phi_h(a \circ b) = \Phi_h(a) \Phi_h(b).
\]

Factors corresponding to chebureks are 1.

b) The function \( a \mapsto \Phi_h(a) \Phi_{h'}(a) \) has the form \( \Phi_K(a) \). Indeed, having spaces
\[
H = (U \otimes V \otimes W, h) \quad \text{and} \quad H' = (U' \otimes V' \otimes W', h'),
\]
we take the space
\[
H \otimes H' = \left( (U \otimes U') \otimes (V \otimes V') \otimes (W \otimes W'), h \otimes h' \right).
\]

4.3. Representations of the category \( \mathcal{S} \). Now we wish to construct the corresponding representations of the category of checker-boards.

**Proposition 4.3** The spaces of \( K(\alpha) \)-fixed vectors are
\[
H(\alpha) = H \otimes \cdots \otimes H \otimes h \otimes h \otimes \ldots
\]

This a rephrasing of Proposition 4.1.

In particular, \( H(\alpha) \) are finite dimensional if \( U, V, W \) are finite dimensional.

Now let us write matrix elements for a checker-board \( a \in K(\alpha) \setminus G/K(\beta) \). Consider basis vectors
\[
\xi := (u_{i_1} \otimes v_{j_1} \otimes w_{k_1}) \otimes \cdots \otimes (u_{i_\beta} \otimes v_{j_\beta} \otimes w_{k_\beta}) \otimes h \otimes h \otimes \ldots \in H(\beta),
\]
\[
\xi' := (u_{i'_1} \otimes v_{j'_1} \otimes w_{k'_1}) \otimes \cdots \otimes (u_{i'_\beta} \otimes v_{j'_\beta} \otimes w_{k'_\beta}) \otimes h \otimes h \otimes \ldots \in H(\alpha).
\]

Then we get the same formula (4.1), but a set of summation changes. Now we are obliged to write \( (i_s, j_s, k_s) \) (according colours) on edges of the white triangle with labels \( s \leq \beta \) and \( (i'_\sigma, j'_\sigma, k'_\sigma) \) on the black triangle with label \( \sigma \leq \gamma \). The products are given over non-labeled triangles.

4.4. Super-tensor products. The previous construction admits an extension. Let each Hilbert space \( U, V, W \) be \( \mathbb{Z}_2 \)-graded,
\[
U = U_\Omega \oplus U_\Gamma, \quad V = V_\Omega \oplus V_\Gamma, \quad W = W_\Omega \oplus W_\Gamma,
\]
sums are orthogonal.

Take bases $u_i \in U$, $v_j \in V$, $W_k \in W$ compatible with gradations (i.e., $u_i \in U_\sigma$ or $U_{-\sigma}$). Take an even vector

$$h \in H =: U \otimes V \otimes W,$$

i.e.,

$$h \in (U_\sigma \otimes V_\sigma \otimes W_\sigma) \oplus (U_{-\sigma} \otimes V_{-\sigma} \otimes W_{-\sigma}) \oplus (U_\sigma \otimes V_{-\sigma} \otimes W_{-\sigma}) \oplus (U_{-\sigma} \otimes V_\sigma \otimes W_{-\sigma})$$

Next, we consider the tensor product

$$H := (H, h) \otimes (H, h) \otimes \ldots \quad (4.4)$$

For $\mathbb{Z}_2$-graded spaces $Y = Y_0 \oplus Y_1$, $Z = Z_0 \oplus Z_1$, the operator of transposition of factors

$$Y \otimes Z \rightarrow Z \otimes Y$$

is given by

$$y_{\sigma} \otimes z_{\sigma} \rightarrow z_{\sigma} \otimes y_{\sigma}, \quad y_{-\sigma} \otimes z_{-\sigma} \rightarrow z_{-\sigma} \otimes y_{-\sigma}.$$ 

This determines the action of symmetric group in tensor products of $\mathbb{Z}_2$-graded spaces.

Taking the action of $G$ in the infinite tensor product (4.4, we get a unitary representation of $G$.

4.5. Spherical functions in super-case. We get the following expression similar to (4.1):

$$\Phi_h(a) = \sum (-1)^\sigma \prod_{\text{black triangles } T} h_{i(T) j(T) k(T)} \prod_{\text{white triangles } S} h_{i(S) j(S) k(S)} \quad (4.5)$$

We must describe a set of summation and a choose of signs $(-1)^\sigma$.

We write basic vectors on edges of the checker-board according to the following rules:

1) $u_i$ on red edges, $v_j$ on yellow edges, $w_j$ on blue edges.

2) The perimeter of a triangle contains even number of odd basis elements, i.e., 0 or 2.

Thus, consider a triangle $T$ having 2 odd basis elements on its perimeter (we say 'odd edges'). Then its neighbors through odd edges also have 2 odd edges. Consider their neighbors through odd edges etc. In this way, we get a collection of closed chains of triangles. Denote by $2l_t$ their lengths (these numbers are even). Then

$$\sigma = \sum l_t - 1.$$
5 \( n \)-symmetric group

Here the situation is similar to the 3-symmetric group. All consideration given above (checker-boards, the multiplication of double cosets, Theorem 3.1, and constructions of representations) survive for general \( n \geq 2 \).

However, there is an alternative (below 'quasidual') description of spaces of double cosets

\[ K(\alpha) \setminus \mathbb{G}[n] / K(\beta). \]

For \( n > 3 \) the two descriptions are essentially different.

5.1. Polygonal checker-boards. We write an element \( p \) of \( \mathbb{G}_n \) as \( p = (p_1, \ldots, p_n) \).

Consider a countable collection \( A_1, A_2, \ldots \) of oriented black \( n \)-gons. On sides of each \( A_k \) we write antclockwise numbers 1, 2, \ldots, \( n \). Consider another countable collection of oriented white \( n \)-gons \( B_1, B_2, \ldots \). On sides of \( B_1, B_2, \ldots \) we write clockwise numbers 1, 2, \ldots, \( n \).

For each \( i \leq n \) for each \( m \in \mathbb{N} \) we glue \( i \)-th side of \( A_m \) with \( i \)-th side of \( B_{p_i(m)} \). In this way we get an \( n \)-gonal complex, which is a union of countable collection of compact closed two-dimensional surfaces.

Again, almost all components are chebureks, i.e., obtained by gluing of a black and white \( n \)-gons along their perimeters.

Now we can repeat one-to-one all considerations of Sections 2–4.

5.2. The quasi-dual polygonal complex. We draw a countable collection of black vertices \( a_1, a_2, \ldots \) and a countable collection of white vertices \( b_1, b_2, \ldots \). For each \( i \leq n \) for each \( m \in \mathbb{N} \) we draw an edge [\( a_i, b_{p_i(m)} \)] and draw the label \( i \) on this edge. In this way, we get a graph\(^6\).

Next, for any \( i, j \leq n \), we consider the permutation \( p_i^{-1}p_j \). Decompose it in independent cycles. We get chains of the form

\[ s_1 \overset{p_i}{\rightarrow} t_1 \overset{p_i^{-1}}{\rightarrow} s_2 \overset{p_i}{\rightarrow} t_2 \overset{p_i^{-1}}{\rightarrow} s_3 \overset{p_i}{\rightarrow} \cdots \overset{p_i}{\rightarrow} t_k \overset{p_i^{-1}}{\rightarrow} s_1 \]

\(^6\)Which is dual to the 1-skeleton of the checker-board.
Figure 5: A cheburek and a quasidual cheburek for $n = 4$.

a) In checker-board model a cheburek is obtained by gluing of two quadrangles along their perimeters.

number of faces = 2, number of vertices = 4, number of edges = 4.

b) In the quasidual model, we glue 6 biangles:

number of faces = 6, number of vertices = 2, number of edges = 4.

Figure 6: $n = 3$. A checker-board and the dual complex.
Figure 7: Gluing of quasidual polygonal complexes. Picture at a vertex.
Then $s_1 t_1 s_2 t_2 \ldots t_k s_1$ is a closed way on our graph. We glue a 2l-gon to this way. Thus we get a polygonal complex.

We get $n(n - 1)/2$ polygons meeting in each vertex. Each pair of polygons has a common edge. Therefore, for $n > 3$ our complex is not a 2-dimensional surface. See Figure 5.

Remark. For $n = 3$ these constructions are dual in the usual sense, see Figure 6.

To pass to double cosets $K(\alpha) \setminus G[n]/K(\beta)$, we remove labels $> \beta$ from black vertices and labels $> \alpha$ from white vertices.

It remains to describe the multiplication

$$K(\alpha) \setminus G[n]/K(\beta) \times K(\beta) \setminus G[n]/K(\gamma) \rightarrow K(\alpha) \setminus G[n]/K(\gamma)$$

For each $i \leq \beta$ we take the link of the black vertex $b_i \in K(\beta) \setminus G[n]/K(\gamma)$ and the link the white vertex $a_i \in K(\alpha) \setminus G[n]/K(\beta)$. We remove neighborhoods of both the vertices and get collection of half-edges in each complex. So we glue halves of edges with the same labels together. We must describe two-dimensional faces of the new complex. We glue the corresponding polygons as it is shown on Figure 7.

6 A priori theorems about representations of $n$-symmetric groups

Here we use the checker-board model for double cosets.

6.1. Representations of products of group. Let $G_1$, $G_2$ be finite groups. Then any irreducible representation of $G_1 \times G_2$ is a tensor product $\tau_1 \times \tau_2$ of irreducible representations of $G_1$ and $G_2$. A similar theorem for unitary representations holds for compact groups, semisimple groups (over $\mathbb{R}$ or $p$-adics), nilpotent Lie groups. However it is not the case for unitary representations of arbitrary groups. Apparently, a minimal necessary condition (see, e.g., [1]) is:

— the group $G_1$ has type I.

We can restrict an irreducible representation of $G[n]$ to the dense subgroup $S_{\text{fin}}^i \times \cdots \times S_{\text{fin}}^i$. But it is not a tensor product of representations of $S_{\text{fin}}^i$. Moreover, consider two classes of representation of $S_{\text{fin}}^i \times \cdots \times S_{\text{fin}}^i$:

— the set of tensor products $\tau_1 \otimes \cdots \otimes \tau_i$;
— the set of representations admitting a continuous extension to the group $G[n]$.

The intersection of these two huge classes is almost trivial. Precisely:

Proposition 6.1 a) Each irreducible representation $\tau_1 \otimes \cdots \otimes \tau_i$ of $S_{\text{fin}}^i \times \cdots \times S_{\text{fin}}^i$ having a non-zero $K$-fixed vector is one-dimensional.
b) Let $\rho_j$ be irreducible representations of $S_{\infty}^{\text{fin}}$. The representation $\tau_1 \otimes \cdots \otimes \tau_n$ admits a continuation to $\mathbb{G}^{[n]}$ if and only if each $\tau_j$ is continuous in the topology of $S_{\infty}$.

Proof is given in Subsection 6.3.

6.2. Reformulations of continuity.

Theorem 6.2

a) Let an irreducible unitary representation $\rho$ of $S_{\infty}^{\text{fin}} \times \cdots \times S_{\infty}^{\text{fin}}$ have a non-zero vector invariant with respect to the diagonal subgroup $K$. Then $\rho$ is continuous in the topology of $\mathbb{G}^{[n]}$.

b) Let an irreducible unitary representation $\rho$ of $S_{\infty}^{\text{fin}} \times \cdots \times S_{\infty}^{\text{fin}}$ have a non-zero vector invariant with respect to some subgroup $K(\alpha) \subset K$. Then $\rho$ is continuous in the topology of $\mathbb{G}^{[n]}$.

c) Let $\tau$ be a unitary (reducible) representation of $S_{\infty}^{\text{fin}} \times \cdots \times S_{\infty}^{\text{fin}}$ in a Hilbert space $H$. Denote by $H(\alpha)$ the space of all $K(\alpha)$-fixed vectors in $H$. Then the following conditions are equivalent:

- $\tau$ is continuous in the topology of $\mathbb{G}^{[n]}$.
- the space $\bigcup H(\alpha)$ is dense in $H$.

Olshanski’s [13] proof for $n = 2$ survives for any finite $n$.

We formulate separately this criterion for the group $\mathbb{G}^{[1]} = S_{\infty}$.

Theorem 6.3 Let $\tau$ be a unitary (reducible) representation of $S_{\infty}^{\text{fin}}$ in a Hilbert space $H$. Then the following conditions are equivalent:

- $\tau$ is continuous in the topology of $S_{\infty}$.
- the space $\bigcup H(\alpha)$ is dense in $H$.

If $\tau$ is irreducible, then these conditions are equivalent to

- the space $\bigcup H(\alpha)$ is non-empty.

6.3. Proof of Proposition 6.1.

Lemma 6.4 Let $\tau_1, \tau_2$ be unitary (generally, reducible) representations of a group $G$. Assume that $\tau_2$ has not finite-dimensional $G$-invariant subspaces. Then the representation $\tau_1 \otimes \tau_2$ also has not finite-dimensional $G$-invariant subspaces.

Proof. Denote by $V_1, V_2$ the spaces of the representations $\tau_1, \tau_2$. Assume that $V_1 \otimes V_2$ admits a finite-dimensional invariant subspace $W$, denote by $\psi$ the subrepresentation in $W$.

Then the tensor product $\psi^* \otimes \tau_1 \otimes \tau_2$ admits a $G$-invariant vector $\eta$. Indeed,

$$W^* \otimes (V_1 \otimes V_2) \supset W^* \otimes W.$$ 

We can regard $W^* \otimes W$ as the space of operators $W \to W$, the identical operator commutes with the action of $G$. 17
Next, we identify $W^* \otimes V_1 \otimes V_2$ with the space of Hilbert–Schmidt operators

$$W \otimes V_1^* \rightarrow V_2.$$ 

An invariant vector $\eta$ corresponds to an intertwining operator $W \otimes V_1^* \rightarrow V_2$. We choose bases in the initial space $W \otimes V_1^*$ and in the target space $V_2$ such that the matrix of $A$ is diagonal. Equivalently, we expand $A$ in a series

$$A = \sum_j \lambda_j R_j,$$

where $\lambda_j$ are the singular numbers of $A$, and $R_j$ are partial isometries,

$$\text{im } R_i \perp \text{im } R_j, \quad \text{im } R_i^* \perp \text{im } R_j^*.$$ 

Then the operators $R_i$ are intertwining, hence $\text{im } R_j$ is an invariant subspace. However $\text{rk}(R_j)$ is the multiplicity of the singular number $\lambda_j$, it is finite. Therefore $\text{im } R_i = 0$ and $R_i = 0$.

**Proof of Proposition 6.1.** The group $S_{\text{fin}}^{\infty}$ has two one-dimensional characters, the trivial one and the signature. They are the only irreducible finite dimensional representations of $S_{\text{fin}}^{\infty}$. Note also that the signature does not admit an extension to the complete group $S_{\infty}$.

Let a representation $\tau_1 \otimes \cdots \otimes \tau_n$ of $S_{\text{fin}}^{\infty} \times \cdots \times S_{\text{fin}}^{\infty}$ has a $K(\alpha) \simeq S_{\text{fin}}^{\infty}(\alpha)$-invariant vector. By Lemma 6.4 each $\tau_j$ has an $S_{\text{fin}}^{\infty}(\alpha)$-eigenvector $v_j$, 

$$\tau_j(g)v_j = \chi_j(g)v_j, \quad g \in S_{\text{fin}}^{\infty}(\alpha), \quad \chi_j \text{ is a character of } S_{\text{fin}}^{\infty}.$$ 

Therefore all representations $\tilde{\tau} \simeq \chi_j\tau_j$ have $S_{\text{fin}}^{\infty}(\alpha)$-invariant vectors and are continuous in the topology of $S_{\infty}$.

Thus, our representation of $G^{[n]}$ has the form

$$(g_1, \ldots, g_n) \mapsto \left( \prod \chi_j(g_j) \right) \tilde{\tau}_1(g_1) \otimes \cdots \otimes \tilde{\tau}_n(g_n).$$

Since the representations

$$(g_1, \ldots, g_n) \mapsto \tilde{\tau}(g_j)$$

are continuous in the topology of $G^{[n]}$, it follows that the character $\prod \chi_j(g_j)$ also is continuous. Therefore it is trivial. 

**6.4. Analogy with $p$-adic groups.** The criterion of continuity (Theorem 6.2) is an imitation of admissibility for $p$-adic groups. For definiteness, consider $G = \text{GL}(n, \mathbb{Q}_p)$. Denote by $\text{GL}^\alpha$ the subgroup consisting of matrices $1 + p^\alpha T$, where $T$ has integer coefficients. Then a representation $\rho$ of $G$ in a space $H$ is *admissible* if each vector is fixed by some subgroup $\text{GL}^\alpha$. A unitary representation of $G$ in a Hilbert space $H$ is admissible, more precisely it is admissible on a dense subspace.

The reason of this parallel, see [5], Proposition VIII.1.3.
6.5. Correspondence between representations of the group $G[n]$ and representations of the category $S[n]$. In Section 3 we obtained the canonical map

$$\left\{\text{Unitary representations of } G[n] \right\} \to \left\{\ast\text{-representations of } S[n] \right\}.$$ 

**Theorem 6.5** This map is a bijection.

Below we prove a stronger version of this statement.

6.6. The completion of the category $S[n]$. Let us define a 'new' category $\overline{S}[n]$ obtained by an adding of an infinite object. Its objects are $0, 1, 2, \ldots, \infty$.

A morphism $\beta \to \alpha$ is the following collection of data:

- a finite or countable collection of checker-boards colored in the usual way;
- almost all components are chebureks;
- we fix an injective map from the set $\{1, 2, \ldots, \beta\}$ to the set of black faces and an injective map $\{1, 2, \ldots, \alpha\}$ to the set of white faces;
- there is no empty chebureks.

The product of morphisms is given in the usual way.

We use the common notation $\text{Mor}(\beta, \alpha)$ for sets of morphisms $\beta \to \alpha$, $\text{End}(\alpha)$ for semigroups of endomorphism, and $\text{Aut}(\alpha)$ for group of automorphisms. Evidently,

$$\text{Aut}(\alpha) = S_\alpha \times \cdots \times S_\alpha, \text{ where } \alpha < \infty, \quad \text{Aut}(\infty) = G[n].$$

At the end of this subsection we define the natural topology on the semigroup $\text{End}(\infty)$ (representations must be continuous\textsuperscript{7}) and discuss some properties of the category $\overline{S}[n]$. Now we formulate the main statement of the subsection.

**Theorem 6.6** Any $\ast$-representation of the category $S[n]$ admits a unique continuous extension to the $\ast$-representation of the category $\overline{S}[n]$.

We need the following distinguished morphisms

$$\lambda_{\beta, \alpha} : \beta \to \alpha, \quad \mu_{\alpha, \beta} : \alpha \to \beta, \quad \theta_\beta^\alpha : \alpha \to \alpha$$

defined for all $\alpha > \beta$.

The morphism $\lambda_{\beta, \alpha} : \beta \to \alpha$ is defined as a disjoint sum of chebureks with the following labels on black/white sides:

$$(j/j), \text{ where } j \leq \beta,$$

$$(p/\text{empty}), \text{ where } \beta < p \leq \alpha.$$  

We set

$$\mu_{\alpha, \beta} := \lambda_{\beta, \alpha}, \quad \theta_\beta^\alpha := \lambda_{\beta, \alpha} \mu_{\alpha, \beta}$$

\textsuperscript{7}With respect to the weak topology on the space of operators in a Hilbert space.
Thus $\mu_{\alpha,\beta}$ is determined by the following collection of chebureks:

\[
(j/j), \text{ where } j \leq \beta,
\]
\[
(\text{empty}/p), \text{ where } \beta < p \leq \alpha.
\]

and $\theta_\beta^\alpha$ by

\[
(j/j), \text{ where } j \leq \beta,
\]
\[
(p/\text{empty}), (\text{empty}/p), \text{ where } \beta < p \leq \alpha.
\]

We need some simple fact concerning these morphisms.

First,

\[
(\theta_\beta^\alpha)^2 = \theta_\beta^\alpha, \quad \theta_\beta^\alpha = (\theta_\beta^\alpha)^\Box
\]

\[\lambda_{\beta,\alpha} \lambda_{\beta,\alpha} = 1\]  \hspace{1cm} (6.4)

Second, the map

\[a \mapsto \lambda_{\beta,\alpha} a \mu_{\alpha,\beta}\]  \hspace{1cm} (6.5)

is an injective homomorphism of the semigroup $\text{End}(\beta)$ to the semigroup $\text{End}(\alpha)$. In fact, we add chebureks (6.2) to the checker-board $a$.

Third, we need the map $\text{End}(\alpha) \to \text{End}(\beta)$ given by

\[b \mapsto \mu_{\alpha,\beta} b \lambda_{\beta,\alpha}\]

This is equivalent to removing of labels $> \beta$ from black and white triangles.

Now we are ready to define the topology on $\text{End}(\infty)$. We say that a sequence $\epsilon_j \in \text{End}(\infty)$ converges to $\epsilon$ if for each $\beta < \infty$

\[\mu_{\infty,\beta} \epsilon_j \lambda_{\beta,\infty} = \mu_{\infty,\beta} \epsilon \lambda_{\beta,\infty}, \quad \text{for sufficiently large } j.\]

**Example.** A weak zero $h_j K(\alpha)$, see Proposition 2.6, converges to $\theta_\infty^\alpha$. \qed

**Proposition 6.7**

a) *The multiplication in $\text{End}(\infty)$ is separately continuous.*

b) *The group $\text{Aut}(\infty)$ is dense in the semigroup $\text{End}(\infty)$.*

**Proof of b.** Let $\epsilon \in \text{End}(\infty)$ have non-labeled white and non-labeled black faces. For each $\beta$ let us make $\delta_\beta \in \text{End}(\infty)$, where all black faces are labeled, in the following way.

We preserve all white labels on $\epsilon$, preserve all black label $\leq \beta$, remove black labels with numbers $> \beta$, and enumerate all initially non-labeled black faces and all faces liberated from labels in arbitrary way. Then $\delta_\beta$ converges to $\epsilon$.

Next, for each $\delta_\beta$ we construct a sequence $\epsilon_{\beta\alpha}$ by repeating the same actions for white faces. Then $\epsilon_{\beta\alpha}$ converges to $\delta_\beta$.

On the other hand, $\epsilon_{\beta\alpha} \in \text{End}(\infty)$. \qed

**6.7. A formal proof of Theorem 6.6.** It is a special case of Theorem VIII.1.10 from [5]. All necessary structures were defined in the previous subsection. We must only verify:
Lemma 6.8 For any $\star$-representation $\overline{\rho}$ of the category $S^{[n]}$ for any morphism $a$,

$$\|\overline{\rho}(a)\| \leq 1.$$  

**Proof.** By (6.3)–(6.4), $\overline{\rho}(\theta^3_\beta)$ is an orthogonal projection and $\overline{\rho}(\lambda_{\beta,\alpha})$ is an isometric embedding. For $\gamma \in \text{Aut}(\gamma)$, the operator $\overline{\rho}(\gamma)$ is unitary.

We can represent any $a \in \text{Mor}(\beta, \alpha)$ as

$$a = \mu_{\gamma,\alpha} \nu_{\beta,\gamma}, \quad \gamma \in \text{Aut}(\gamma)$$

for some $\gamma$. We simply write labels on all empty triangles. This completes the proof. □ □

6.8. Reconstruction of a representation of the group $G^{[n]}$ from a representation of the category. We will not repeat the proof of Theorem VIII.1.10 from [5]. Here we briefly describe the limit passed necessary for Theorem 6.6.

Let $\rho$ be a $\star$-representation of $S^{[n]}$. Let $H(\alpha)$ be the spaces of representation. For $\beta < \alpha$, the operator $\rho(\lambda_{\beta,\alpha})$ is an isometric embedding $H(\beta) \to H(\alpha)$. The semigroup $\text{End}(\beta)$ acts in both spaces, see (6.5); the operator $\rho(\lambda_{\beta,\alpha})$ intertwines these actions.

Therefore, we can assume that $H(\beta) \subset H(\alpha)$ and take the inductive limit $H(\infty)$ of the chain

$$\cdots \to H(\beta) \to H(\beta + 1) \to \cdots$$

of Hilbert spaces (i.e., we take the completion of $\bigcup H(\beta)$). The inductive limit $\bigcup_{\beta < \infty} \text{End}(\beta)$ of the chain of semigroups

$$\cdots \to \text{End}(\beta) \to \text{End}(\beta + 1) \to \cdots$$

acts in $H(\infty)$.

However, the semigroup $\bigcup_{\beta < \infty} \text{End}(\beta)$ is a small subset in $\text{End}(\infty)$ ($\bigcup_{\beta < \infty} \text{End}(\beta)$ is countable and contains no invertible elements at all). To define $\overline{\rho}(\gamma)$ for an arbitrary $\gamma \in \text{End}(\infty)$, we write $\overline{\rho}(\gamma)$ as a weak limit

$$\overline{\rho}(\gamma) := \lim_{\beta \to \infty} \overline{\rho}(\theta^\infty_\beta \gamma \theta^\infty_\beta).$$  \hspace{1cm} (6.6)

6.9. The composition of functors. Thus we get a chain of 3 functors:

$$\begin{align*}
\{ \text{Unitary representations of } G^{[n]} \} & \xrightarrow{T_1} \{ \text{*-Representations of } S^{[n]} \} & \xrightarrow{T_2} \\
\{ \text{*-Representations of } G^{[n]} \} & \xrightarrow{T_3} \{ \text{Unitary representations of } G^{[n]} \}
\end{align*}$$  \hspace{1cm} (6.7)

The last map is the restriction of a representation of $\text{End}(\infty)$ to $\text{Aut}(\infty) \simeq G^{[n]}$.

We must show that this chain is closed,

**Proposition 6.9** The composition $T_3 T_2 T_1$ is the identical map.
Proof. Let \( \rho \) be a unitary representation of \( G^{[n]} \). Let \( b \in \text{End}(\beta) \), where \( \beta < \infty \). In Section 3, we defined the operator \( \overline{\rho}(b) \) as follows. Take \( g_\beta \in G^{[n]} \) such that \( b = \theta_\beta \alpha g_\beta \theta_\beta \). (6.8)

Then

\[ \overline{\rho}(b) := P(\beta) \rho(g_\beta) P(\beta). \] (6.9)

Take \( g \in \text{Aut}(\infty) \cong G^{[n]} \). Keeping in the mind (6.6)–(6.9), we get

\[ \rho(g) = \lim_{\beta \to \infty} P(\beta) \rho(g) P(\beta) \quad (6.10) \]

(the limit is a weak limit). By Theorem 6.2, \( P(\beta) \) strongly tends to the identical operator. Therefore the weak limit (6.10) is \( \rho(g) \).

Thus, \( \overline{\rho}(g) = \rho(g) \) for \( g \in G^{[n]} \). □

Next, we prove a stronger form of Proposition 6.9.

Theorem 6.10 All the maps \( T_1, T_2, T_3 \) are bijections. The composition \( T_3T_2T_1, T_2T_1T_3, T_1T_3T_2 \) are identical.

Proof. The injectivity of \( T_1 \). This follows from the identity

\[ \lim_{\beta \to \infty} P(\beta) \rho(g) P(\beta) = \rho(g), \]

see the previous proof.

The bijectivity of \( T_2 \) is evident, the map \( T_2^{-1} \) is the restriction of a representation of \( S^{[n]} \) to \( G^{[n]} \).

The injectivity of \( T_3 \). The group \( \text{Aut}(\infty) \) is dense in the semigroup \( \text{End}(\infty) \). Therefore a representation of \( \text{Aut}(\infty) \) remembers a representation of \( \text{End}(\infty) \).

On the other hand, all the semigroups \( \text{End}(\alpha) \) can be regarded as subsemigroups in \( \text{End}(\infty) \). □

6.10. Uniqueness of a spherical vector.

Theorem 6.11 Let \( \rho \) be a unitary irreducible representation of \( G^{[n]} \), let \( H(0) \) be the space of \( K \)-fixed vectors. Then

\[ \dim H(0) \leq 1. \]

Proof. Our functors (6.7) send direct sums to direct sums and irreducible representations to irreducible representations. In particular, the representation of \( \text{End}(0) \cong K(0) \setminus G^{[n]}/K(0) \) must be irreducible (see, e.g., [5], Lemma II.8.1.). The semigroup \( K(0) \setminus G^{[n]}/K(0) \) is a commutative semigroup with involution. Its irreducible \( \ast \)-representations are one-dimensional. □

If \( \dim H(0) = 1 \), we say that the representation is \( K \)-spherical.

6.11. Spherical characters. For \( \alpha = 0, 1, \ldots, \infty \), denote by \( \Delta(\alpha) \) the center of \( \text{End}(\infty) \).

\*For categories, see definitions in [5], II.8.
Proposition 6.12 All elements of $\Delta(\alpha)$ have the form: the collection of labeled chebureks

$$\{j/j\}, \text{ where } j \leq \alpha,$$

and a finite disjoint union of compact non-labeled surfaces.

The statement is obvious.

Thus, all semigroups $\Delta(\alpha)$ are isomorphic to

$$\Delta := \Delta(0) = K(0) \setminus \mathbb{G}^n / K(0).$$

For $\alpha > \beta$, the isomorphism $\Delta(\alpha) \to \Delta(\beta)$ can be written as follows:

$$z \mapsto \mu_{\alpha,\beta} z \lambda_{\beta,\alpha}.$$  \hspace{1cm} (6.11)

The center of $\text{End}(\infty)$ acts in irreducible representations by scalar operator.

Observation 6.13 For any irreducible representation $\rho$ of $G[\mathbb{G}^n]$, we get a homomorphism $\chi_\rho$ of $\Delta$ to the multiplicative semigroup of $\mathbb{C}$.

We say that $\chi_\rho$ is the spherical character of the representation $\rho$.

Observation 6.14 For any irreducible spherical representation of $G[\mathbb{G}^n]$ the spherical character coincide with spherical function.

Lemma 6.15 For any irreducible representation $\rho$ of $G[\mathbb{G}^n]$ for each $\alpha$ the semigroup $\Delta(\alpha)$ acts in $H(\alpha)$ as $\chi_\rho$.

Proof. We apply the map (6.11). \hfill $\square$

6.12. Self-similarity and spherical characters. Consider the subgroup $G[\mathbb{G}^n](\alpha) \subset G[\mathbb{G}^n]$ that fixes points of the initial segment 1, . . . , $\alpha$ in all $n$ copies of $\mathbb{N}$. Then

$$G[\mathbb{G}^n](\alpha) \simeq G[\mathbb{G}^n].$$

Restricting a representation of $G[\mathbb{G}^n]$ to $G[\mathbb{G}^n](\alpha)$, we get a collection of (reducible) representations of $G[\mathbb{G}^n]$.

Evidently, the embedding

$$G[\mathbb{G}^n](\alpha) \to G[\mathbb{G}^n]$$

induces an isomorphism of centers of two copies of $\text{End}(\infty)$. This implies the following statement:

Theorem 6.16 Let $\rho$ be an irreducible unitary representation of $G[\mathbb{G}^n]$. Consider the restriction

$$\rho \bigg|_{G[\mathbb{G}^n](\alpha)}$$

and a spherical subrepresentation $\tau$ of the restriction\(^9\). Then the spherical function of $\tau$ coincides with the spherical character of $\rho$.

\(^9\)For sufficiently large $\alpha$ such subrepresentations exist.
Example. A pair of substitutions

\[
\begin{pmatrix}
1 & 2 & 4 & 3 & 5 & 8 & 7 & 6 & 9 & 10 \\
4 & 10 & 5 & 9 & 3 & 6 & 2 & 8 & 1 & 7 \\
\end{pmatrix}, \quad \begin{pmatrix}
10 & 5 & 9 & 3 & 6 & 2 & 8 & 4 & 7 & 1 \\
1 & 2 & 4 & 3 & 5 & 8 & 7 & 6 & 9 & 10 \\
\end{pmatrix}
\]

Notation for colors:
- black
- red
- white
- blue

Example. A representation of $G[n]$ is continuous in the topology of $S_\infty \times \cdots \times S_\infty$ (see Subsection 6.1) if and only if its spherical character = 1 (this is a rephrasing of Proposition 6.1).

Let $\chi$ be a spherical character. For each connected non-labeled checker-board $c$ we have a number $\chi(c)$. For a disjoint sum $c_1 \cup \cdots \cup c_m$ we have

\[
\chi(c_1 \cup \cdots \cup c_m) = \prod_j \chi(c_j).
\]

Thus a spherical character is determined by numbers $\chi(c)$, where $c$ ranges in set of connected checker-boards.

However, numbers $\chi(c)$ are not arbitrary.

6.13. Status of problem of classification of unitary representations of the group $G[n]$. It is not clear, is this problem reasonable or not (there are different arguments 'pro and con').

However, the problem looks like two-step:

a) to describe all possible spherical characters\(^{10}\);

b) to classify all irreducible representations with a given spherical character.

There is also (apparently, simple) problem:

c) to prove that $G[n]$ is a type I group.

For the bisymmetric group $G^{[2]}$, the description of spherical characters is equivalent to the famous Thoma Theorem (1964), see Theorem A.1. The problem b) was solved by Olshanski [13] and Okounkov [9].

\(^{10}\)The construction of Section 4 does not exhaust all spherical characters. Even for $G^{[2]}$ there is an additional parameter.
Figure 9: The analog of a cheburek

Figure 10: The production of a chip from an one-dimensional simplicial complex.
We take an element of $\mathcal{K}(4) \setminus \mathcal{G}^2 / \mathcal{K}(3)$:

Cut labeled edges:

We get a collection of arcs and draw this collections as a chip:
A. Addendum. Bisymmetric group

A.1. Olshanski chips. Let \( n = 2 \). Consider an element \( g = (g_{\text{red}}, g_{\text{blue}}) \in \mathbb{G}^{[2]} \). Take a countable collection of black segments \( A_j \) with red and blue ends. Take a countable collection of white segments with red and blue ends. If \( g_{\text{red}}i = j \), then we glue the red end of \( A_i \) with the red end of \( B_j \). If \( g_{\text{blue}}k = l \), then we glue the blue end of \( A_k \) with the blue end of \( B_l \). In this way, we get an one-dimensional complex. One-dimensional strata of the complex (segments) are colored in black and white, zero-dimensional strata in red and blue. There are labels \( \in \mathbb{N} \) on black segments and labels \( \in \mathbb{N} \) on white segments.

The pair \( g = (g_{\text{red}}, g_{\text{blue}}) \) can be easily reconstructed from these data, see Figure 8.

To realize a double coset

\[ \in K(\alpha) \setminus \mathbb{G}^{[2]}/K(\beta), \]

we remove labels \( \beta \) from black segments and labels \( \alpha \) from white segments. The analog of an empty cheburek is a circle consisting of two non-labeled segments. It is reasonable to throw out all empty chebureks (see Figure 9).

Thus we get a finite disjoint union of closed chains of segments, black and white segments interlace, red and blue ends also interlace. There are pairwise different labels \( 1, \ldots, \beta \) on some black segments and pairwise different labels \( 1, \ldots, \alpha \) on some white segments.

Next, it is reasonable to do the following operation:

1. We cut all labeled segments at midpoints and assign the corresponding labels to the ends.

2. After this we get a finite collection of cycles having even lengths \( 2l_p \) and a finite collection of non-closed chains. If both ends of a chain are white (resp. black), then the chain contains an odd number \( 2m_q + 1 \) of complete segments. If one end is black and another is white, then the chain contains an even number \( 2k_r \) of complete segments. See Figure 10.

Now an element of \( K(\alpha) \setminus \mathbb{G}^{[2]}/K(\beta) \) corresponds to diagram (a chip) of the following form (see Figure 10):

a) There are \( \beta + \beta \) entries. On Figure we show them by \( \beta \) labeled fat points and \( \beta \) labeled square points. There are \( \alpha + \alpha \) exits. On Figure 10 we show them by numbered \( \alpha \) fat points and numbered \( \alpha \) square points.

b) There are arcs of 3 types (horizontal, vertical, and cyclic). For each arc we assign a non-negative integer (length). The rules of designing are:

- a horizontal arc connects a square point and a fat point in the entrance or a square and a fat point in the exit. It has odd length.

- a vertical arc connects a fat point in the entrance with a fat point in the exit or a square point in the entrance with a square point in the exit. It has even length.

- cycles have even length.
Having two chips $a : \gamma \rightarrow \beta$ and $b : \beta \rightarrow \alpha$. Then we connect exits of $a$ with corresponding entrances of $b$ and evaluate lengths of composite arcs.

Remark. Certainly, we can apply the general construction of checker-board for $\mathbb{G}^2$. In this case 2-dimensional complexes are spheres divided into biangles, see Figure 11.

A.2. Characters of $S_\infty$ and the bisymmetric group. In 1964 E. Thoma obtained the classification of all characters of $S^\text{fin}_\infty$. According his definition, a character is an extreme point of the set of all functions $F$ on $S^\text{fin}_\infty$ satisfying the conditions:
- $F$ is central, i.e., $F(hgh^{-1}) = F(g)$;
- $F$ is positive definite;
- $F(1) = 1$.

Theorem A.1 (Thoma, [14]) All characters of $S^\text{fin}_\infty$ have the form

$$
\chi(g) = \prod_{k \geq 2} \left( \sum_j \alpha_j^k - \sum_j (-\beta_j)^k \right)^{r_k(g)}, 
$$

where $r_k(g)$ is the number of cycles of length $k$ in $g$, and the parameters $\alpha$, $\beta$ satisfy

$$
\alpha_1 \geq \alpha_2 \geq \ldots \geq 0, \quad \beta_1 \geq \beta_2 \geq \ldots \geq 0, \quad \sum \alpha_j + \sum \beta_j \leq 1.
$$

Having a positive definite function $\chi$ on the group $S^\text{fin}_\infty$, we define a Hilbert space and a unitary representation of the group in the usual way, see, e.g., [1]. This representation is a $II_1$-factor representation (in particular, it is strongly reducible).

Theorem A.2 ([13]). a) There is a canonical one-to-one correspondence between Thoma characters and spherical representations of the bisymmetric group.
b) Moreover, for a character $\chi(g)$ the corresponding spherical function is given by
$$\Phi(g_1, g_2) = \chi(g_1 g_2^{-1}).$$

Formula (A.1) is simpler than our formulas (4.1) and (4.5). In fact a vector
$$h \in U \otimes V,$$
where $U, V$ are Hilbert spaces,
can be reduced to the form
$$h = \sum \alpha_j^{1/2} u_j \otimes v_j$$
by a change of bases in $U$ and $V$. For a triple tensor product such reduction is impossible.

A.3. Symmetric pairs. Olshanski researches in infinite-dimensional groups (see [12], [13]) partially are an imitation of symmetric pairs and the theory of spherical functions. In Neretin [5], VIII.5 it was observed that some tricks related to infinite dimensional symmetric pairs can be applied in a wider generality.

References

[1] Dixmier, J. Les $C^*$-algebres et leurs representations. Gauthier Villars, 1964.

[2] Kerov, S., Olshanski, G., Vershik, A. Harmonic analysis on the infinite symmetric group. Invent. Math. 158, No. 3, 551-642.

[3] Lieberman, A. The structure of certain unitary representations of infinite symmetric groups. Trans. Am. Math. Soc. 164, 189-198 (1972).

[4] Natanzon, S. M. Cyclic Foam Topological Field Theories. Preprint, arXiv:0712.3557 (2007)

[5] Neretin, Yu. A. Categories of symmetries and infinite-dimensional groups. Oxford University Press, New York, 1996; Russian transl.: URSS, 1998.

[6] Neretin, Yu. A. Spreading maps (polymorphisms), symmetries of Poisson processes, and matching summation. Zap. Nauchn. Sem. S.-Peterburg. Otdel. Mat. Inst. Steklov. (POMI) 292 (2002), Teor. Predst. Din. Sist. Komb. i Algoritm. Metody. 7, 62–91, 178–179. English transl. in J. Math. Sci (New York).

[7] Nessonov, N. I., Factor-representation of the group $GL(\infty)$ and admissible representations $GL(\infty)^X$, J. Math. Phys., Analysis, Geometry, 2003, N4, 167-187.

[8] Nessonov, N.I. Factor-representation of the group $GL(\infty)$ and admissible representations of $GL(\infty)^X$. II. (Russian. English summary) Mat. Fiz. Anal. Geom. 10, No. 4, 524-556 (2003).
[9] Okounkov, A. *Thoma’s theorem and representations of the infinite bisymmetric group*. Funct. Anal. Appl. 28, No.2, 100-107 (1994);

[10] Ol’shanskij (Olshanski, G.I.) *New “large” groups of type I*. (Russian) Itogi Nauki Tekh., Ser. Sovrem. Probl. Mat. 16, 31-52 (1980). English transl.: J. Sov. Math. 18, 22-39 (1982).

[11] Ol’shanskij, G.I. (Olshanski, G.I.) *Unitary representations of the infinite symmetric group: a semigroup approach*. in *Representations of Lie groups and Lie algebras*, Proc. Summer Sch., Budapest 1971, Pt. 2, 181-197 (1985).

[12] Ol’shanskij, G.I. (Olshanski, G.I.) *Unitary representations of infinite dimensional pairs (G,K) and the formalism of R. Howe*. in *Representation of Lie groups and related topics*, Adv. Stud. Contemp. Math. 7, 269-463 (1990).

[13] Ol’shanskij, G.I. (Olshanski, G.I.) *Unitary representations of (G,K)-pairs connected with the infinite symmetric group S(∞)*. Leningr. Math. J. 1, No.4, 983-1014 (1990); translation from Algebra Anal. 1, No.4, 178-209 (1989).

[14] Thoma, E. *Die unzerlegbaren, positiv-definiten Klassenfunktionen der abzählbar unendlichen, symmetrischen Gruppe*. (German) Math. Z. 85, 40-61 (1964).

[15] Vershik, A. M., Kerov, S. V. *Characters and factor-representations of infinite symmetric group*, Dokl. Akad. Nauk SSSR, 257 (1981), 1037–1040, English transl. in Sov. math Dokl, 23.

[16] von Neumann, J. *On infinite direct products*. Compos. Math. 6, 1-77 (1938). Reprinted in von Neumann *Collected works*,

Math.Dept., University of Vienna,
Nordbergstrasse, 15, Vienna, Austria

&

Institute for Theoretical and Experimental Physics,
Bolshaya Cheremushkinskaya, 25, Moscow 117259, Russia

&

Mech.Math. Dept., Moscow State University, Vorob’evy Gory, Moscow
e-mail: neretin(at) mccme.ru
URL: www.mat.univie.ac.at/~neretin
wwwth.itep.ru/~neretin

29