Triangularisation of Singularly Perturbed Logarithmic Differential Systems of Rank 2

Nikita Nikolaev

School of Mathematics and Statistics, University of Sheffield, United Kingdom
Section of Mathematics, University of Geneva, Switzerland

17 December 2021

Abstract

We study singularly perturbed linear systems of rank two of ordinary differential equations of the form $\hbar x \partial_x \psi(x, \hbar) + A(x, \hbar) \psi(x, \hbar) = 0$, with a regular singularity at $x = 0$, and with a fixed asymptotic regularity in the perturbation parameter $\hbar$ of Gevrey type in a fixed sector. We show that such systems can be put into an upper-triangular form by means of holomorphic gauge transformations which are also Gevrey in the perturbation parameter $\hbar$ in the same sector. We use this result to construct a family in $\hbar$ of Levelt filtrations which specialise to the usual Levelt filtration for every fixed nonzero value of $\hbar$; this family of filtrations recovers in the $\hbar \to 0$ limit the eigen-decomposition for the $\hbar$-leading-order of the matrix $A(x, \hbar)$, and also recovers in the $x \to 0$ limit the eigen-decomposition of the residue matrix $A(0, \hbar)$.

Keywords: singular perturbation theory, exact perturbation theory, Borel summation, singular differential systems, ordinary differential equations, Levelt filtration, asymptotic analysis, Gevrey asymptotics, resurgence

2020 MSC: 34M60 (primary); 34E10, 34E20, 40G10,

Contents

1 Introduction 2
  1.1 Definitions and Conventions 4

2 Singularly Perturbed Systems 5
  2.1 Spectral Data and Formal Normal Forms 6
  2.2 Triangularisation 10
  2.3 Singularly Perturbed Levelt Filtrations 11
  2.4 Another Point of View on the Levelt Filtration 15

First appeared: 9 September 2019
Previously, this paper was titled “Triangularisation of Singularly Perturbed Linear Systems of Ordinary Differential Equations: Rank Two at Regular Singular Points”.
Contact: n.nikolaev@sheffield.ac.uk
§ 1. Introduction

We revisit the classical problem in singular perturbation theory of studying systems of linear ordinary differential equations of the form

\[ h x \partial_x \psi + A(x, h) \psi = 0 \]  

where \( x \) is a complex independent variable, \( h \) is a complex parameter, \( \psi = \psi(x, h) \) is a vector function, and \( A(x, h) \) is a matrix of functions which are holomorphic near \( x = 0 \) and admit a uniform asymptotic expansion as \( h \to 0 \) in a sector in the \( h \)-plane. Such systems not only have a regular singular point at the origin \( x = 0 \), but are also singularly perturbed in \( h \). The most important class of examples matrices \( A(x, h) \) which are polynomial in \( h \).

If the perturbation parameter \( h \) were held constant and nonzero, then this system specialises to a usual linear system of ordinary differential equations with a regular singularity. Then standard theory (see e.g., [Was76]) tells us that the (finite-dimensional) vector space of solutions \( V_h \) is naturally filtered as

\[ V_h^\bullet := (0 \subset V_h^1 \subset V_h^2 \subset \cdots \subset V_h^n = V) \]

by increasing growth rate as \( x \to 0 \). Namely, the steps in this filtration are weighted by numbers \( \nu_i \) (the growth rate), and the subspace \( V_h^i \) consists of solutions \( \psi_h(x) \) which grow like \( x^{-\nu_i} \) as \( x \to 0 \). To be precise, \( V_h^i \) is defined to be subspace of \( V_h \) consisting of solutions \( \psi_h(x) \) which satisfy the following growth bound: for all \( \delta > 0 \),

\[ \lim_{x \to 0} x^{\nu_i + \delta} \psi_h(x) = 0 \]

Such filtrations are often called Levelt filtrations [Lev61, Zol06, Boa11].

The main problem we wish to address in this paper is the construction Levelt filtrations \( V_h \) as an asymptotic family. That is, we wish to construct a filtration \( V^\bullet \) on the space of solutions of the singularly perturbed system (1) such that its specialisation to any fixed nonzero \( h \) is the Levelt filtration \( V_h \) above. The main challenge is to construct \( V^\bullet \) in such a way that we maintain a very tight asymptotic control as \( h \to 0 \). The reason this is interesting is that the system (1) in the singular perturbation limit \( h \to 0 \) degenerates (as is very typical in singular perturbation theory) from a differential system to a problem in linear algebra: \( A_0(x) \psi = 0 \), where \( A_0 \) is the limit of \( A \) as \( h \to 0 \). The tight asymptotic control on \( V^\bullet \) that we are able to achieve allows us to make a direct identification of the asymptotic limit of Levelt filtrations.

1.1. Main results. Let us briefly outline the main results in this paper. We focus on the simplest case where \( A \) is a 2 \times 2-matrix (i.e., the rank of the system is 2) of functions which are defined and holomorphic on a domain of the form \( D \times S \) where \( D \) is a disc centred at the origin in the \( x \)-plane and \( S \) is a sector in the based at the origin in the \( h \)-plane with opening angle at least \( \pi \). We also assume that the constant matrix \( A_{00} \), obtained from \( A \) in the limit as \( x \to 0 \) and \( h \to 0 \), has eigenvalues \( m_1, m_2 \) whose real parts satisfy \( \text{Re}(m_1/h) < \text{Re}(m_2/h) \) for all \( h \in S \). We make a further crucial assumption on the asymptotic regularity of \( A \) as \( h \to 0 \): we insist that \( A \) has an asymptotic expansion of class Gevrey. This assumption allows us
to capture strict control of the asymptotics in the sense that our main constructions remain within the same regularity class. For this, we use the powerful machinery of Borel resummation.

Singularly perturbed linear systems and linear ordinary differential equations have been recently studied using Borel resummation techniques (e.g., [BK02, BMF02, BK03, KT05, CDMFS07, KT13]), but to the best of our knowledge the question of constructing the Levelt filtration in the strictly-controlled sense above has not been addressed.

The first main result in this paper is the existence of the Levelt filtration in singular perturbation families.

1.2. Theorem (Levelt filtration for singularly perturbed systems (Theorem 2.13)). Given such a singularly perturbed system (1), the 2-dimensional vector space $V$ of solutions of $A$ has a natural 1-dimensional subspace $L \subset V$ such that for any nonzero $h \in S$, the filtration $L_h \subset V_h$ is the Levelt filtration for the system $A_h$.

The Levelt filtration on $V$ induces a natural filtration on the vector space $C^2$ on which the differential system $A$ is defined in the first place. This is the filtration whose asymptotics can be identified with linear-algebraic data as follows (this is part of Theorem 2.20).

1.3. Theorem. Given such a singularly perturbed system (1), there is a natural 1-dimensional subspace $L = L(x, h) \subset C^2$, which depends on $(x, h)$, defined over a subdomain $D_0 \times S_0 \subset D \times S$ (where $D_0 \subset D$ is a concentric subdisc and $S_0 \subset S$ is a subsector which (crucially) has the same opening angle), and such that $L$ and the quotient space $L' = L'(x, h) := C^2 / L$ have the following properties:

1. The limit vector space $\lim_{x \to 0} L \oplus L'$ is canonically isomorphic to the eigenspace decomposition of the residue matrix $A(0, h)$.

2. The limit vector space $\lim_{h \to 0} L \oplus L'$ is canonically isomorphic to the eigenspace decomposition of the leading order matrix $A_0(x)$.

The main technical tool in proving these theorems is the ability to gauge transform any system $A$ as above to an upper-triangular form in a way to maintains the Gevrey asymptotics. Namely, we prove the following theorem.

1.4. Theorem (Triangularisation Theorem (Theorem 2.12)). Given a singularly perturbed system (1) as above, there is an invertible $2 \times 2$-matrix $G = G(x, h)$ whose entries are holomorphic functions on a subdomain $D_0 \times S_0 \subset D \times S$ (where $D_0 \subset D$ is a concentric subdisc and $S_0 \subset S$ is a subsector which (again crucially) has the same opening angle), with uniform Gevrey asymptotic regularity in $S_0$, which transforms the given system into an upper-triangular system of the form

$$h x \partial_x \varphi + \left( \begin{bmatrix} \lambda_1(x, h) & u(x, h) \\ \lambda_2(x, h) & 0 \end{bmatrix} \right) \varphi = 0 ,$$

where $\lambda_i(x, h)$ is of the form $m_i + x \mu_i(x) + h \kappa_i(h)$ for some holomorphic function $\mu_i(x)$ on $D_0$ and some Gevrey function $\kappa_i(h)$ on $S_0$, and where $u(x, h)$ is a holomorphic function on $D_0 \times S_0$ which is uniformly Gevrey on $S_0$ and which vanishes in the limit $x \to 0$ and in the limit $h \to 0$. 

3
The proof of this theorem is inspired by the argument of Koike-Schäfke on the Borel summability of WKB solutions of Schrödinger equations (see [Tak17, §3.1] for an account). Finally, we remark that these results may be viewed as a case of generalisation to second-order systems of the exact WKB method that is normally applied to second-order scalar equations [Nik21a]. This point will be developed in great detail in a series of forthcoming publications.

1.5. Acknowledgements. The author wishes to thank Marco Gualtieri, Kohei Iwaki, and Shinji Sasaki for very helpful discussions. This work was supported by the NCCR SwissMAP of the SNSF. Several important aspects of this work were initiated during the author’s time as a PhD student at the University of Toronto, and he wishes to thank the Department of Mathematics for the hospitality as well as a very productive and supportive environment.

§ 1.1. Definitions and Conventions

In this paper, we fix, once and for all, complex coordinates \( x \) and \( \hbar \), as well as \( \Theta \) to be either a connected arc \((\theta_-, \theta_+ + \pi/2)\) on the unit circle or a single point \( \theta \). Let

\[
\hat{\Theta} := (\theta_- - \pi/2, \theta_+ + \pi/2) \quad \text{or} \quad \hat{\Theta} := (\theta - \pi/2, \theta + \pi/2).
\]

We refer to points in \( \hat{\Theta} \) as directions. The most typical domain of definition of our objects will be a disc in the \( x \)-plane and a sector in the \( \hbar \)-plane with opening \( \hat{\Theta} \):

\[
D := \{ x \in \mathbb{C} \mid |x| < r_1 \} \subset \mathbb{C}_x, \\
S := \{ \hbar \in \mathbb{C} \mid 0 < |\hbar| < r_2 \quad \text{and} \quad \arg(\hbar) \in \hat{\Theta} \} \subset \mathbb{C}_\hbar,
\]

for some real numbers \( r_1, r_2 > 0 \). The arc \( \hat{\Theta} \) is called the opening of \( S \). Limits as \( \hbar \to 0 \) will always be taken inside the given sector \( S \), so we adopt the following shorthand notation:

\[
\lim_{\hbar \to 0} := \lim_{\hbar \to 0} = \lim_{\hbar \to 0, \arg(\hbar) \in \hat{\Theta}}.
\]

Recall that a holomorphic function \( f = f(\hbar) \) on \( S \) is Gevrey if, for every proper subsector \( S' \subset S \) whose closure lies in \( S \), there are constants \( C, M > 0 \) such that

\[
\sup_{S'} \left| \frac{\partial^k f}{k!} \right| < CM^k k!
\]

for all \( k \in \mathbb{Z}_{\geq 0} \). We will say that \( f \) is strongly Gevrey if the above bound holds for \( S' = S \). If the opening of \( S \) has length exactly \( \pi \) (so that \( \Theta \) is a single direction), we will always assume that \( f \) is strongly Gevrey. Such functions form a ring (in fact a differential algebra) which we will denote by \( R(S) \).

The most typical kind of functions that we will encounter in this paper is those that are holomorphic in \( x \) and Gevrey in \( \hbar \). To be precise, we will say that a holomorphic function \( f = f(x, \hbar) \) on a standard domain \( D \times S \) is a regular function if \( f \) is Gevrey on \( S \) uniformly for all \( x \in D \). Such functions form a ring (again, in fact a differential algebra), which we will denote by \( R(D \times S) \).
Any regular function \( f = f(x, h) \) on \( D \times S \) admits an asymptotic expansion which we will always write as follows:

\[
f(x, h) \sim \hat{f}(x, h) := \sum_{k=0}^{\infty} f_k(x)h^k,
\]

where each \( f_k(x) \) is a holomorphic function on \( D \). The fact that \( f \) is uniformly Gevrey in \( h \) means that the coefficients of the formal power series in \( h \) in (5) satisfy the following uniform bounds: there are constants \( C, M > 0 \) such that for all \( k \geq 0 \),

\[
|f_k(x)| \leq CM^k k!
\]

uniformly for all \( x \in D \). We will always refer to such series \( \hat{f} \) as regular \( h \)-series defined on \( D \). They also form a differential algebra which we denote by \( \hat{R}(D) \).

This paper is concerned only with the local analysis of singularly perturbed systems near the singularities \( x = 0 \) and \( h = 0 \). Therefore, we may as well concentrate our attention on germs of functions. Recall that a germ of a Gevrey function on the arc \( \hat{\Theta} \) is represented by any Gevrey function on a sector \( S \). We will sometimes refer to a germ of a regular function as a regular germ. Regular germs also form a differential algebra which we will denote by \( R \). If the coefficients \( f_k(x) \) of the Gevrey power series \( \hat{f} \) in (5) are germs of holomorphic functions at \( x = 0 \), we will refer to \( \hat{f} \) as simply simply a regular \( h \)-series (without specifying a disc in \( \mathbb{C}_x \) where its coefficients are defined). We denote the ring of germs of regular \( h \)-series by \( \hat{R} \).

§ 2. Singularly Perturbed Systems

In this paper, we study linear systems of differential equations like (1), and we focus on a special class in the following sense.

2.1. Definition. By a singularly perturbed differential system \( A \) (or simply a system from now on) we shall mean a system of linear ordinary differential equations for a 2-dimensional vector function \( \psi = \psi(x, h) \) of the form

\[
A : \quad hx\partial_x \psi + A(x, h)\psi = 0,
\]

where \( A(x, h) \) is a \( 2 \times 2 \)-matrix of germs of regular functions; i.e., \( A \in \mathfrak{gl}(2, \mathbb{R}) \).

Concretely, the entries of \( A \) are regular functions defined on a standard domain \( D \times S \). The most important subclass of systems is when the matrix \( A(x, h) \) is in fact holomorphic at \( h = 0 \), or even altogether independent of \( h \). The most prominent example of this is the local study of the stationary one-dimensional Schrödinger equation

\[
h^2\partial_x^2 \Psi + V(x)\Psi = 0
\]

near a second order pole of the potential \( V(x) \). If we write \( V(x) = x^{-2}Q(x) \), then this equation is equivalent to the system

\[
x\partial_x \Psi + \begin{bmatrix} 0 & -1 \\ Q(x) & 0 \end{bmatrix} \psi = 0.
\]
Note also that any singularly perturbed system $A$ can be specialised at every fixed nonzero $h \in S$ to a usual linear system of ordinary differential equations defined over the disc $D$, which we shall denote by $A_h$:

$$A_h : \quad x\partial_x \psi + h^{-1} A(x, h) \psi = 0 .$$

2.2. Regular and formal equivalence. In this paper, there are two main notions of equivalence of systems. Invertible $2 \times 2$-matrices $G = G(x, h)$ act on systems by gauge transformations:

$$G \bullet A := GAG^{-1} - (\partial_x G)G^{-1} .$$

We will say that $G$ is regular gauge transformation if its entries are germs of regular functions. Regular gauge transformations form a group $G := \text{GL}(2, \mathbb{R})$. If, instead, the entries of $G$ are regular $h$-series, we will call $G$ a formal gauge transformation. They also form a group $G := \text{GL}(2, \mathbb{R})$.

Two systems $A, A'$ are regularly gauge equivalent (and we write $A \sim A'$) if there is a regular gauge transformation $G$ such that $G \bullet A = A'$. We will say that $A, A'$ are formally gauge equivalent (in which case we will write $A \simeq A'$) if there is a formal gauge transformation $\hat{G}$ such that $\hat{G} \bullet A = A'$.

Concretely, if two systems of equations

$$hx\partial_x \psi + A(x, h) \psi = 0 \quad \text{and} \quad hx\partial_x \psi' + A'(x, h) \psi' = 0 ,$$

both defined over a domain $D \times S$ of the form (3), then $A \sim A'$ if there is a subdisc $D_0 \subset D$, a subsector $S_0 \subset S$ with the same opening, and an invertible matrix $G = G(x, h)$ of regular functions defined on $D_0 \times S_0$ such that the transformation $\psi = G\psi'$ carries the system $A$ into $A'$. Likewise, $A \simeq A'$ if there is a subdisc $D_0 \subset D$ and an invertible matrix $\hat{G} = \hat{G}(x, h)$ of regular $h$-series defined on $D_0$ such that the transformation $\psi = \hat{G}\psi'$ carries the system $A$ into $A'$.

2.3. Set of systems. Let Syst denote the set of all systems, and let Syst$(A)$ be the set of all systems formally equivalent to $A$. Obviously, if $A \simeq A'$, then Syst$(A) = \text{Syst}(A')$. We will denote the regular equivalence class of $A$ by $[A]$. One result in this paper is to show that generically a formal equivalence class Syst$(A)$ contains a canonical diagonal system $\Lambda$ which has a very simple and standard form.

§ 2.1. Spectral Data and Formal Normal Forms

2.4. Classical polar data. We will refer to the leading order part of $A$ in both $x$ and $h$ (which is a constant matrix $A_{00}$) as the classical residue of the system $A$:

$$A_{00} := \lim_{\substack{\hbar \to 0 \\ x \to 0}} A(x, \hbar) \in \mathfrak{gl}(2, \mathbb{C}) .$$

The classical residue of a system plays the most central rôle in this paper. Let $m_1, m_2 \in \mathbb{C}$ be the eigenvalues of $A_{00}$. The pair $\{m_1, m_2\}$ is clearly an invariant of the system $A$, which we will call classical polar data. We will say that a classical polar data $\{m_1, m_2\}$ is generic if $m_1 \neq m_2$; we will say it is nonresonant with
2.5. Classical spectral data. The $\hbar$-leading order part of $A$ is matrix of convergent power series $A_0(x) \in \mathfrak{gl}(2, \mathbb{C}\{x\})$, and $A_0(0) = A_{00}$. If the classical polar data $\{m_1, m_2\}$ is generic, then standard theory (e.g., see [Was76, §25.2]) implies that $A_0(x)$ is diagonalisable: there are holomorphic germs $\eta_1, \eta_2 \in \mathbb{C}\{x\}$ such that $\eta_1(0) = m_1$, and an invertible matrix $G = G(x)$ of convergent power series such that

$$GA_0G^{-1} = \begin{bmatrix} \eta_1(x) & 0 \\ 0 & \eta_2(x) \end{bmatrix}.$$ 

We will refer to the set of eigenvalues $\{\eta_1, \eta_2\}$ as the classical spectral data of the system $A$. If the classical polar data is ordered $m_1 \prec m_2$, then we order the classical spectral data accordingly: $\eta_1 \prec \eta_2 :\iff m_1 \prec m_2$. It is easy to see that classical spectral data is also an invariant of the system $A$.

2.6. Polar data. We define the residue of a system $A$ to be the matrix $A(0, \hbar)$ of Gevrey function germs on the arc $\hat{\Theta}$. Its classical limit $\lim_{\hbar \to 0} A(0, \hbar)$ is the classical residue $A_{00}$. Concretely, the entries of $A(0, \hbar)$ are Gevrey functions on a sector $S$ of the form (3). If the classical polar data $\{m_1, m_2\}$ is generic, then the matrix $A(0, \hbar)$ can likewise be diagonalised in a way that retains the asymptotic regularity, thanks to the following proposition, which is a special case of [Nik21b, Theorem 1.1].

2.7. Proposition (diagonalisation in asymptotic families). Let $A = A(\hbar)$ be a $2 \times 2$-matrix of Gevrey functions germs, and assume that its leading-order $A_0$ has distinct eigenvalues $m_1, m_2$. Then there is an invertible Gevrey matrix $G = G(\hbar)$ such that $R := GAG^{-1}$ is a diagonal Gevrey matrix.

Thus, there are Gevrey germs $\rho_1, \rho_2$ on the arc $\hat{\Theta}$ such that $\lim_{\hbar \to 0} \rho_1(\hbar) = m_1$, and an invertible matrix $G = G(\hbar)$ of Gevrey germs on $\hat{\Theta}$ such that

$$GA(0, \hbar)G^{-1} = \begin{bmatrix} \rho_1(\hbar) & 0 \\ 0 & \rho_2(\hbar) \end{bmatrix}.$$ 

We will refer to the pair $\{\rho_1, \rho_2\}$ as the polar data. It is also easy to see that polar data is an invariant of the system $A$.

2.8. Spectral data. If a system $A$ has generic classical polar data $\{m_1, m_2\}$, we will write its classical spectral data $\{\eta_1, \eta_2\}$ and its polar data $\{\rho_1, \rho_2\}$ as follows:

$$\eta_i(x) = m_i + x\overline{\eta}_i(x) \quad \text{and} \quad \rho_i(x) = m_i + \hbar\overline{\rho}_i(\hbar).$$

We will refer to the set $\{\lambda_1, \lambda_2\}$ of germs of regular functions

$$\lambda_i(x, \hbar) := m_i + x\overline{\lambda}_i(x) + \hbar\overline{\lambda}_i(\hbar)$$
as the spectral data of the system $A$. Spectral data is a complete formal invariant in the following sense.

2.9. Definition. Let $\lambda_1, \lambda_2 \in \mathbb{R}$ be any pair of germs of regular functions. We will refer to the diagonal system $\Lambda = \text{diag}(\lambda_1, \lambda_2)$ as the formal normal form corresponding to the spectral data $\{\lambda_1, \lambda_2\}$.

2.10. Theorem. Given a system $A$ with generic classical polar data $\{m_1, m_2\}$, let $\lambda_1, \lambda_2$ be its spectral data. Then $A$ is formally gauge equivalent to the formal normal form

$$\Lambda = \Lambda(x, \hbar) := \begin{bmatrix} \lambda_1(x, \hbar) & 0 \\ 0 & \lambda_2(x, \hbar) \end{bmatrix}.$$  

The proof requires a few preliminary comments.

Concretely, suppose we are given a singularly-perturbed system (6) defined over a domain $D \times S$ of the form (3). Suppose its classical residue matrix $A_{00}$ has distinct eigenvalues $m_1, m_2$. Then there is a subdisc $D_0 \subset D$ and an invertible $2 \times 2$-matrix $\hat{G} = \hat{G}(x, \hbar)$ whose entries are regular formal $\hbar$-series defined over $D_0$ such that the transformation $\psi = \hat{G}(x, \hbar)\varphi$ carries the given system into the diagonal system of the form

$$\Lambda + B = h x \partial_x \varphi + \begin{bmatrix} \lambda_1(x, \hbar) & 0 \\ 0 & \lambda_2(x, \hbar) \end{bmatrix} \varphi = 0 ,$$

for some regular functions $\lambda_i(x, \hbar)$ defined on $D_0 \times S_0$ where $S_0 \subset S$ is a subsector with the same opening.

The first step is to diagonalise the leading-order and the residue of the system $A$, and this can be done using regular gauge transformations. The following lemma is quite evident from the above discussion.

2.11. Lemma (diagonalisation of spectral data). Given a system $A$ with generic classical polar data $\{m_1, m_2\}$, let $\lambda_1, \lambda_2$ be its spectral data. Then $A$ is regularly gauge equivalent to a system of the form $\Lambda(x, \hbar) + B(x, \hbar)$, where $\Lambda$ is the formal normal form corresponding to $\lambda_1, \lambda_2$, and $B(x, \hbar)$ is a system with the following limiting properties:

$$\lim_{x \to 0} B(x, \hbar) = 0 \quad \text{and} \quad \lim_{\hbar \to 0} B(x, \hbar) = 0 .$$

(8)

Concretely, if $A$ is defined over a standard domain $D \times S$, then there is a standard subdomain $D_0 \times S_0$ and an invertible matrix $G = G(x, \hbar)$ of regular functions on $D_0 \times S_0$ such that the transformation $\psi = G\varphi$ carries the given system $A$ to the following system:

$$\Lambda + B = h x \frac{d}{dx} \varphi + \left( \Lambda(x, \hbar) + B(x, \hbar) \right) = 0 ,$$

(9)

where $B(x, \hbar)$ is a $2 \times 2$-matrix of regular functions on $D_0 \times S_0$ satisfying (8).

Proof of Theorem 2.10. For simplicity of notation, assume that the $\hbar$-leading order part $A_0(x)$ and the residue $A(0, \hbar)$ have already been diagonalised to $\Lambda(x, \hbar)$ using
Lemma 2.11. Thus, if we write

\[ A(x, \hbar) = \begin{bmatrix} a_{11}(x, \hbar) & a_{12}(x, \hbar) \\ a_{21}(x, \hbar) & a_{22}(x, \hbar) \end{bmatrix}, \]  

then

\[ A_0(x) = \begin{bmatrix} \eta_1(x) & 0 \\ 0 & \eta_2(x) \end{bmatrix} \quad \text{and} \quad A_{00} = \begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix}. \]  

We search for a gauge transformation \( \hat{G} = \hat{G}(x, \hbar) \) in the following almost form:

\[ G(x, \hbar) := \begin{bmatrix} 1 & g_{12}(x, \hbar) \\ g_{21}(x, \hbar) & 1 \end{bmatrix}, \]  

where \( g_{ij}(x, \hbar) \) are to be solved for. Then \( \hat{G} \) must satisfy the following matrix differential equation:

\[ \hbar x \partial_x \hat{G} = \hat{G} A - \Lambda \hat{G}. \]

It yields four scalar equations:

\[ \eta_1 = a_{11} + a_{21} g_{12}, \quad \eta_2 = a_{22} + a_{12} g_{21}, \]  

\[ x \hbar \partial_x g_{12} = a_{12} + a_{22} g_{12} - \eta_1 g_{12}, \quad x \hbar \partial_x g_{21} = a_{21} + a_{11} g_{21} - \eta_2 g_{21}. \]  

Substituting expressions (13) for \( \eta_i \) into (14), we obtain two uncoupled nonlinear first order differential equations:

\[ x \hbar \partial_x g_{12} = a_{12} + (a_{22} - a_{11}) g_{12} - a_{21} g_{12}^2, \]  

\[ x \hbar \partial_x g_{21} = a_{21} + (a_{11} - a_{22}) g_{21} - a_{12} g_{21}^2. \]  

Thanks to (11), the \( \hbar \)-leading order of the coefficient \((a_{ii} - a_{jj})\) is \( m_i - m_j \), which is nonzero by the assumption that the classical polar data \( \{m_1, m_2\} \) is generic. Furthermore, the leading-order of the coefficients \( a_{12}, a_{21} \) are 0, thanks again to (11). Both differential equations (14) are formal singularly perturbed Riccati equations, so by the Formal Existence and Uniqueness Theorem (see, for example, [Nik20, Theorem 3.8]), we obtain formal \( \hbar \)-power series solutions \( \hat{g}_{ij}(x, \hbar) \) satisfying differential equations (14). Therefore,

\[ \hat{G}(x, \hbar) := \begin{bmatrix} 1 & \hat{g}_{12}(x, \hbar) \\ \hat{g}_{21}(x, \hbar) & 1 \end{bmatrix} \]

is the desired formal gauge transformation. \( \blacksquare \)
§ 2.2. Triangularisation

A given generic and nonresonant system \( A \) can always be formally gauge transformed into its formal normal form, but this usually cannot be done by using regular gauge transformations. However, we can use regular gauge transformations to achieve a simplification of \( A \) which is almost as good.

2.12. Theorem (Triangularisation Theorem). Let \( \Lambda \) be a generic and nonresonant formal normal form. Then any system \( A \in \text{Syst}(\Lambda) \) is regularly gauge equivalent to an upper-triangular system of the form \( \Lambda + U \), where

\[
U = \begin{bmatrix} 0 & u(x, \hbar) \\ 0 & 0 \end{bmatrix}
\]  

(16)

for some regular function germ \( u = u(x, \hbar) \in \mathbb{R} \) which has the following properties:

\[
\lim_{\hbar \to 0} u(x, \hbar) = 0 \quad \text{and} \quad \lim_{x \to 0} u(x, \hbar) = 0 .
\]

uniformly in \( x \) and in \( \hbar \), respectively.

Concretely, suppose we are given a singularly-perturbed system \( (6) \) defined over a domain \( D \times S \) of the form \( (3) \). Suppose its classical residue matrix \( A_0 \) has different nonresonant eigenvalues \( m_1, m_2 \) ordered like \( m_1 < m_2 \). Then there is a subdisc \( D_0 \subset D \), a subsector \( S_0 \subset S \) with the same opening, and an invertible \( 2 \times 2 \)-matrix \( G = G(x, \hbar) \) whose entries are regular functions defined over \( D_0 \times S_0 \) such that the transformation \( \psi = G(x, \hbar)\varphi \) carries the given system \( A \) into an upper-triangular system of the form

\[
\Lambda + U : \quad \hbar x \partial_x \varphi + \left( \Lambda(x) + U(x, \hbar) \right) \varphi = 0 .
\]

(17)

Proof of Theorem 2.12. For simplicity of notation, assume that the \( \hbar \)-leading order part \( A_0(x) \) and the residue \( A(0, \hbar) \) have already been diagonalised to \( \Lambda(x, \hbar) \) using Lemma 2.11. Thus, if we write

\[
A(x, \hbar) = \begin{bmatrix} a_{11}(x, \hbar) & a_{12}(x, \hbar) \\ a_{21}(x, \hbar) & a_{22}(x, \hbar) \end{bmatrix},
\]

(18)

then

\[
A_0(x) = \Lambda(x) = \begin{bmatrix} \lambda_1(x) & 0 \\ 0 & \lambda_2(x) \end{bmatrix} \quad \text{and} \quad A_00 = \begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix}.
\]

(19)

We will first transform our system to a triangular system of the form

\[
\Lambda + V : \quad \hbar x \partial_x \varphi' + \left( \Lambda + V \right) \varphi' = 0 ,
\]

(20)

where

\[
V = V(x, \hbar) := \begin{bmatrix} v_{11}(x, \hbar) & v_{12}(x, \hbar) \\ v_{21}(x, \hbar) & v_{22}(x, \hbar) \end{bmatrix},
\]

for some regular function germs \( v_{ij}(x, \hbar) \in \mathbb{R} \). Then we will apply another transformation to kill the diagonal entries of \( V \) in order to obtain the system (17).

Inspired by techniques in [Sib58, RS66, RS68] (see also [Was76, §11 and §25.3]),

\[
\]
we search for a gauge transformation \( G_1 \) in the following unipotent form:

\[
G_1(x, \hbar) := \begin{bmatrix}
1 \\
\delta(x, \hbar)
\end{bmatrix},
\]

(21)

where \( \delta(x, \hbar) \) is to be solved for. Then matrices \( G_1 \) and \( V \) must satisfy the following matrix differential equation:

\[
\hbar x \partial_x G_1 = G_1 A - \Lambda G_1 - VG_1.
\]

It yields four scalar equations:

\[
\begin{align*}
v_{11} &= a_{11} - \lambda_1 - a_{12}s, & v_{22} &= a_{22} - \lambda_2 + a_{12}s & v_{12} &= a_{12} \\
x \hbar \partial_x s &= a_{21} + (a_{11} - a_{22})s - a_{12}s^2
\end{align*}
\]

(22)

Observe that \( v_{ij} \) are expressed entirely in terms of \( s \) and the known data, so the problem has been reduced to solving the nonlinear differential equation in (22).

To remove the diagonal terms of \( V \), we transform the system (20) into (17) via a diagonal transformation of the form

\[
G_2(x, \hbar) := \begin{bmatrix}
g_{11}(x, \hbar) \\
g_{22}(x, \hbar)
\end{bmatrix},
\]

(23)

where \( g_{ii}(x, \hbar) \in \mathbb{R} \) are to be solved for. The matrices \( G_2 \) and \( U \) must satisfy the following matrix differential equation

\[
\hbar x \frac{d}{dx} G_2 = G_2 A - \Lambda G_2 + \hbar x (G_2 V - U G_2).
\]

It yields three nontrivial scalar equations:

\[
\begin{align*}
\partial_x g_{11} &= v_{11} g_{11}, & \partial_x g_{22} &= v_{22} g_{22}, & u &= v_{12} g_{11} g_{22}^{-1}.
\end{align*}
\]

The first two are easy to solve by integration, and they determine an expression for \( u \). Since \( v_{11}, v_{22} \) are regular germs, so are \( g_{11}, g_{22} \).

\section*{§ 2.3. Singularly Perturbed Levelt Filtrations}

The main application of the Triangularisation Theorem (Theorem 2.12) in this paper is to construct a filtration on the space of solutions which specialises to the Levelt filtration for every fixed nonzero \( \hbar \) and has controlled limits in both \( \hbar \) and \( x \).

2.13. Theorem (The Levelt filtration for singularly perturbed systems). Let \( \Lambda \) be a generic and nonresonant formal normal form, and suppose \( A \in \text{Syst}(\Lambda) \) is a system defined over a standard domain \( D \times S \). Then the 2-dimensional vector space \( V \) of solutions of \( \Lambda \) has a natural 1-dimensional subspace \( L \subset V \) such that for any nonzero \( \hbar \in S \), the filtration \( L_\hbar \subset V_\hbar \) is the Levelt filtration for the system \( A_\hbar \).
The proof of this theorem is to gauge transform $A$ into an upper-triangular system, solve the upper-triangular system explicitly, and use these solutions to construct the desired filtration.

2.14. Solving a triangular system. Any triangular system (17) can be solved directly by integration. To write down an explicit basis of solutions, let $D \times S$ be a domain of the form (3) where the triangular system (17) is defined. We choose any nonzero basepoint $x_\ast \in D$, and introduce the following notation:

$$f_i(x, h) := \exp \left( - \int_{x_\ast}^{x} \lambda_i(t, h) \frac{dt}{ht} \right)$$ and $$f_{ij}(x, h) := \exp \left( - \int_{x_\ast}^{x} \lambda_{ij}(t, h) \frac{dt}{ht} \right),$$

where $\lambda_{ij} := \lambda_i - \lambda_j$. If $(\vec{e}_1, \vec{e}_2)$ is the standard basis of $\mathbb{C}^2$, then using the method of variation of parameters, we obtain a basis of solutions of the system (17):

$$\varphi_1 := f_1 \vec{e}_1 \quad \text{and} \quad \varphi_2 := f_2 (\vec{e}_2 + c_{12} \vec{e}_1),$$

where

$$c_{12} = c_{12}(x, h) := C f_{12}(x, h) - f_{12}(x, h) \int_{x_\ast}^{x} f_{21}(t, h) u(t, h) \, dt,$$ for an integration constant $C$ which is allowed to depend on $h$ and $x_\ast$.

2.15. A vanishing lemma. Our aim is to construct a basis of solutions $\{\varphi_1, \varphi_2\}$ which has $h$-asymptotic behaviour that we can control. For this, we need the following vanishing lemma, whose proof can be found at the end of this subsection.

2.16. Lemma. There is a unique way to choose the integration constant $C = C(h, x_\ast)$ in (25) such that $c_{12}$ is independent of the basepoint $x_\ast$ and satisfies the following bounds:

$$|c_{12}(x, h)| \lesssim |x| \quad \text{as} \quad x \to 0 \text{ uniformly in} \quad h \in S,$$  

$$|c_{12}(x, h)| \lesssim |h| \quad \text{as} \quad h \to 0 \text{ along} \quad I \text{ uniformly in} \quad x \in D.$$

Moreover, $c_{12}$ is holomorphic but possibly multivalued with at most a logarithmic branch singularity at the points $(0, h) \in D \times S$ for $h = \frac{\nu - 2n}{n+1}$ for all $n \in \mathbb{Z}_{\geq 0}$, and admits a uniform Gevrey asymptotic expansion along $\hat{\Theta}$.

For this unique choice of $C$, we write the function $c_{12}$ from (25) as:

$$c_{12}(x, h) := -f_{12}(x, h) \int_{x_\ast}^{x} f_{21}(t, h) u(t, h) \, dt.$$

2.17. Proposition. The vector functions $\varphi_1, \varphi_2$, as defined by (24) with $c_{12}$ given by (28), form an ordered basis of solutions $(\varphi_1, \varphi_2)$ of the triangular system (17) with the following properties:
(1) \( \varphi_i \) has the following leading behaviours:

\[
\varphi_i(x, h) \sim x^{\nu_i/h} e_i(x, h) \quad \text{as } x \to 0, \text{ for all } h \in S; \\
\varphi_i(x, h) \sim f_i(x) e_i(x, h) \quad \text{as } h \to 0 \text{ in } S, \text{ for all } x \in D^*
\]

(2) They satisfy the following dominance relation:

\[
\varphi_1 \prec \varphi_2 \quad \text{as } x \to 0 .
\]

uniformly for all \( h \in S \) with \( \arg(h) \in \Theta \).

Proof. To prove (1), place the basepoint \( x_* \) on the boundary of \( D \). Note that moving the basepoint amounts to multiplication by a constant (depending on \( h \)) which does not affect the Levelt exponent as \( x \to 0 \). We need to show that for any \( \delta > 0 \), the vector functions \( \varphi_1, \varphi_2 \) have the property

\[
x^{\nu_i/h+\delta} \varphi_i \to 0 \quad \text{as } x \to 0 ,
\]

which amounts to showing that

\[
x^{\nu_1/h+\delta} f_1 \to 0, \quad x^{\nu_2/h+\delta} f_2 \to 0, \quad x^{\nu_2/h+\delta} f_2 c_{12} \to 0,
\]

as \( x \to 0 \). Since \( \lambda_i = \nu_i + x \mu_i \), the first two of these claims are obvious. For the third claim, we use the above vanishing lemma (Lemma 2.16), which says that \( |c_{12}| \lesssim |x| \). Since \( (\nu_1 - \nu_2)/h < 0 \) for all \( \arg(h) \in \Theta \), property (2) now follows as well. The second half of (1) is proved similarly.

2.18. Proposition. Let \( \Lambda \) be a generic and nonresonant formal normal form. Then any system \( A \in \text{Syst}(\Lambda) \) has an ordered basis of solutions \( \{ \psi_1, \psi_2 \} \) with the following properties:

(1) \( \psi_i \) has the following leading behaviours:

\[
\psi_i(x, h) \sim x^{\nu_i/h} e_i(x, h) \quad \text{as } x \to 0, \text{ for all } h \in S; \\
\psi_i(x, h) \sim f_i(x) e_i(x, h) \quad \text{as } h \to 0 \text{ in } S, \text{ for all } x \in D^*
\]

where \( e_1(x, h) \) is a regular vector function on \( D \times S \), and \( e_2(x, h) \) is, up to terms involving \( x \log(x) \), is also a regular vector function on \( D^* \times S \), and \( e_1, e_2 \) are linearly independent wherever \( e_2 \) is well-defined.

(2) They satisfy the following dominance relation:

\[
\psi_1 \prec \psi_2 \quad \text{as } x \to 0 .
\]

uniformly for all \( h \in S \) with \( \arg(h) \in \Theta \).

Proof. By the Triangularisation Theorem (Theorem 2.12), \( A \) is regularly gauge equivalent to an upper triangular system \( \Lambda + U \) via a regular gauge transformation \( G \). Let \( (\varphi_1, \varphi_2) \) be an ordered basis of solutions of \( \Lambda + U \) guaranteed by Proposition 2.17. Define \( \psi_i := G \varphi_i \). Since \( G \) is regular, the properties of \( \varphi_1, \varphi_2 \) immediately imply the corresponding properties of \( \psi_1, \psi_2 \).■
**Proof of Theorem 2.13.** By Proposition 2.18, the vector space $V$ of solutions of $A$ has a basis $(\psi_1, \psi_2)$ ordered by dominance as in (30). The 1-dimensional subspace $L \subset V$ is spanned by the vector $\psi_1 = \psi_1(x, h)$. For any fixed nonzero $h \in S$ with $\text{arg}(h) \in \Theta$, the specialisation $\psi_{1,h}(x) := \psi_1(x, h)$ spans the Levelt filtration $L_h \subset V_h$. $\blacksquare$

**Proof of the Vanishing Lemma, Lemma 2.16.** Notice that although the functions $f_{ij}$ depend on the choice of $x_*$, the expression $f_{ij}(x, h) f_{ij}(t, h)$ is independent of it, because
\[
\int_{x_*}^{t} h_{ij}(u, h) \frac{du}{u} + \int_{x_*}^{x} h_{ji}(u, h) \frac{du}{u} = \int_{x_*}^{t} h_{ij}(u, h) \frac{du}{u}.
\]
This makes it clear that it is possible to choose $C$ such that $c_{12}$ is independent of $x_*$, and that this specifies $C$ uniquely.

**Proof of (26).** Write $h_{ij}(x, h) = \nu_{ij}(h) + x \mu_{ij}(x)$ where $\nu_{ij} := \nu_i - \nu_j$ and $\mu_{ij} := \mu_i - \mu_j$, so
\[
f_{ij}(x, h) = \left(\frac{x}{x_*}\right)^{-\nu_{ij}/h} \exp\left(-\int_{x_*}^{x} \mu_{ij}(u) \frac{du}{u}\right).
\]
If we put $g(t, h) := \exp(\int_{x}^{t} \mu_{12}(u) \frac{du}{u}) q(t, h)$ (manifestly holomorphic and independent of the basepoint $x_*$), then (28) becomes
\[
c_{12}(x, h) = -x^{-\nu_{12}/h} \int_{x_*}^{x} t^{\nu_{12}/h} g(t, h) dt = -x^{-\nu_{12}/h} \sum_{n=0}^{\infty} g_n(h) \int_{x_*}^{x} t^{\nu_{12}/h+n} dt,
\]
where we expanded the holomorphic function $g$ as a power series $g(x, h) = \sum g_n(h) x^n$. If $\nu_{12}/h + n \neq 1$ whenever $g_n \neq 0$, then this integrates to
\[
c_{12}(x, h) = -x \sum_{n=0}^{\infty} \frac{g_n(h)}{n + 1 + \nu_{12}/h} x^n.
\]
This expression is holomorphic at $x = 0$ and satisfies the bound $|c_{12}(x, h)| \lesssim |x|$ uniformly in $h \in S_0$ by virtue of the fact that $g(x, h)$ is holomorphic in $U_0 \times S_0$. If, on the other hand, $N \in \mathbb{Z}_{\geq 0}$ is such that $\nu_{12}/h + N = -1$ yet $g_N \neq 0$, then
\[
c_{12}(x, h) = -x \sum_{n=0}^{\infty} \frac{g_n(h)}{n + 1 + \nu_{12}/h} x^n - g_N(h) x \log(x).
\]
This expression has a logarithmic branch singularity at $x = 0$, but it still satisfies the $|c_{12}(x, h)| \lesssim |x|$ uniformly in $h \in S_0$.

**Proof of (27).** Let
\[
\Phi_{ij}(x) := \int_{x_*}^{x} \lambda_{ij}(u) \frac{du}{u},
\]
so (28) becomes
\[
c_{12}(x, h) = -\int_{x_*}^{x} e^{(\Phi_{12}(t, h) - \Phi_{12}(x, h))/h} q(t, h) dt.
\]
Since $q(x, \hbar)$ admits a uniform Gevrey asymptotic expansion along $\Theta$, it follows that $|q(x, \hbar)|$ is uniformly bounded by a constant. Then the obvious inequality
\[ \text{Re} \left( \left( \Phi_{12}(t, \hbar) - \Phi_{12}(x, \hbar) \right) / \hbar \right) \leq \left| \Phi_{12}(t, \hbar) - \Phi_{12}(x, \hbar) \right| / |\hbar| \]
implies
\[ |c_{12}(x, \hbar)| \lesssim \int e^{\left| \Phi_{12}(t, \hbar) - \Phi_{12}(x, \hbar) \right| / |\hbar|} \, dt. \]

We are free to choose the basepoint $x_*$ without affecting $c_{12}$: we place $x_*$ on the boundary of $D_0$. For every $x \in U_0$, there is a phase $\theta = \theta(x)$ such that the trajectory emanating from $x_*$ hits $x$. We integrate along this trajectory. Let
\[ r = r(t) := \left( \Phi_{12}(t, \hbar) - \Phi_{12}(x, \hbar) \right) e^{-i\theta}. \]
Then $r$ is real and positive and satisfies $r = \left| \Phi_{12}(t, \hbar) - \Phi_{12}(x, \hbar) \right|$. Furthermore, $\partial_t \Phi_{12}(t, \hbar)$ is nonvanishing on $D_0$, so $\left| \partial_t \Phi_{12}(t, \hbar) \right|$ is bounded below by a constant. Thus, we find
\[ |c_{12}(x, \hbar)| \lesssim \int_{r=0}^{r=0} e^{r/|\hbar|} \, dr = |\hbar|. \]

§ 2.4. Another Point of View on the Levelt Filtration

We can shift our point of view on the Levelt filtration and consider, instead of the vector space of solutions $V$, the vector space $\mathbb{C}^2$ on which the differential system $A$ is defined in the first place. The problem is that elements of $V$ have no meaning at the pole $x = 0$ (because the solutions are singular at $x = 0$), they have no meaning at $\hbar = 0$ or more precisely in the limit $\hbar \to 0$ (because the solutions are singularly perturbed). The filtration also requires a choice of $\log(x)$ for the vector space $V$ to be well-defined. The advantage of going to the vector space $\mathbb{C}^2$ is that it is obviously well-defined both at $x = 0$ and $\hbar = 0$. The discussion in the previous section and especially the Vanishing Lemma (Lemma 2.16) imply the following result.

2.19. Proposition. Let $\Lambda$ be a generic and nonresonant formal normal form, and suppose $A \in \text{Syst}(\Lambda)$ is a system defined over a standard domain $D \times S$. There exists an ordered pair of linearly independent 2-dimensional vector functions $(e_1, e_2)$, where $e_i = e_i(x, \hbar)$ with the following properties:

1. $e_1$ is a nowhere-vanishing and regular on a standard subdomain $D_0 \times S_0$;
2. $e_2$ is nowhere-vanishing and regular on $D_0 \times S_0$, but possibly with a branch point singularity at $x = 0$ with monodromy that changes it by a multiple of $e_1$.
3. the limits
   \[ e_{i,0*}(\hbar) := \lim_{x \to 0} e_i(x, \hbar) \quad \text{and} \quad e_{i,0*}(x) := \lim_{\hbar \to 0} e_i(x, \hbar) \]
   exist uniformly for $\hbar \in S_0$ and $x \in D_0$ respectively.
4. The vector $e_{i,0*}(\hbar)$ is regular on $S_0$, and it is an eigenvector of the residue matrix $A_0(\hbar)$ with eigenvalue $\rho_i(\hbar)$.
5. The vector $e_{i,0*}(x)$ is holomorphic on $D_0$, and it is an eigenvector of the matrix

\[ ^1 \text{though this issue is far less crucial and can be resolved by considering the local system of solutions; we will not discuss this point of view here.} \]
\[ \lim_{\hbar \to 0} A(x, \hbar) \text{ with eigenvalue } \eta_i(x). \]

(6) The vectors \( e_i \) satisfy the differential equation

\[ h \ell x \partial_x e_i + A e_i = \lambda_i e_i. \]

We conclude by restating this corollary in terms of filtrations on \( \mathbb{C}^2 \).

2.20. Theorem. Let \( \Lambda \) be a generic and nonresonant formal normal form, and suppose \( A \in \text{Syst}(\Lambda) \) is a system defined over a standard domain \( D \times S \). Then there is a natural 1-dimensional subspace \( L = L(x, \hbar) \subset \mathbb{C}^2 \), which depends on \( (x, \hbar) \), defined over a standard subdomain \( D_0 \times S_0 \subset D \times S \), with the following properties:

(1) There is a generator \( e_1 \in L \), which is a regular vector function defined on \( D_0 \times S_0 \), and which satisfies the differential equation

\[ h \ell x \partial_x e_1 + A e_1 = \lambda_1 e_1. \]

(2) The vector space \( \lim_{x \to 0} L(x, \hbar) \) is the \( \rho_1(\hbar) \)-eigenspace of the residue \( A(0, \hbar) \).

(3) The vector space \( \lim_{h \to 0} L(x, \hbar) \) is the \( \eta_1(x) \)-eigenspace of the leading order \( A_0(x) \).

Furthermore, consider the quotient vector space \( L' = L'(x, \hbar) := \mathbb{C}^2 / L \), which also depends on \( (x, \hbar) \) and is defined over the standard domain \( D_0 \times S_0 \). Then:

(1) There is a generator \( e_2 \), which is a regular vector function defined on \( D_0 \times S_0 \), and which satisfies the differential equation

\[ h \ell x \partial_x e_2 + A e_2 = \lambda_2 e_2. \]

(2) The vector space \( \lim_{x \to 0} L'(x, \hbar) \) is canonically isomorphic to the \( \rho_2(\hbar) \)-eigenspace of the residue \( A(0, \hbar) \).

(3) The vector space \( \lim_{h \to 0} L'(x, \hbar) \) is canonically isomorphic to the \( \eta_2(x) \)-eigenspace of the leading order \( A_0(x) \).

References

[BK02] W. Balser and V. Kostov, Singular perturbation of linear systems with a regular singularity, *J. Dynam. Control Systems* 8 no. 3, (2002) 313–322.

[BK03] W. Balser and V. Kostov, Recent progress in the theory of formal solutions for ODE and PDE, *Appl. Math. Comput.* 141 no. 1, (2003) 113–123. Advanced special functions and related topics in differential equations (Melfi, 2001).

[BMF02] W. Balser and J. Mozo-Fernández, Multisummability of formal solutions of singular perturbation problems, *J. Differential Equations* 183 no. 2, (2002) 526–545.

[Boa11] P. P. Boalch, Riemann–Hilbert for tame complex parahoric connections, *Transform. Groups* 16 no. 1, (2011) 27–50, arXiv:1003.3177 [math.DG].

[CDMFS07] M. Canalis-Durand, J. Mozo-Fernández, and R. Schäfke, Monomial summability and doubly singular differential equations, *J. Differential Equations* 233 no. 2, (2007) 485–511.

[KT05] T. Kawai and Y. Takei, Algebraic analysis of singular perturbation theory, vol. 227 of Translations of Mathematical Monographs. American Mathematical Society, Providence, RI, 2005. Translated from the 1998 Japanese original by Goro Kato, Iwanami Series in Modern Mathematics.
[KT13] T. Koike and Y. Takei, Exact WKB analysis of second-order non-homogeneous linear ordinary differential equations, in Recent development of micro-local analysis for the theory of asymptotic analysis, RIMS Kôkyûroku Bessatsu, B40, pp. 293–312. Res. Inst. Math. Sci. (RIMS), Kyoto, 2013.

[Lev61] A. H. M. Levelt, Hypergeometric functions. Doctoral thesis, University of Amsterdam. Drukkerij Holland N. V., Amsterdam, 1961.

[Nik20] N. Nikolaev, Exact Solutions for the Singularly Perturbed Riccati Equation and Exact WKB Analysis, arXiv:2008.06492 [math.CA].

[Nik21a] N. Nikolaev, Existence and Uniqueness of Exact WKB Solutions for Second-Order Linear ODEs, arXiv:2106.10248 [math.AP].

[Nik21b] N. Nikolaev, Gevrey Asymptotic Implicit Function Theorem, arXiv:2112.08792 [math.CV].

[RS66] D. L. Russell and Y. Sibuya, The problem of singular perturbations of linear ordinary differential equations at regular singular points. I, Funkcial. Ekvac. 9 (1966) 207–218.

[RS68] D. L. Russell and Y. Sibuya, The problem of singular perturbations of linear ordinary differential equations at regular singular points. II, Funkcial. Ekvac. 11 (1968) 175–184 (1969).

[Sib58] Y. Sibuya, Sur réduction analytique d'un système d'équations différentielles ordinaires linéaires contentant un paramètre, J. Fac. Sci. Univ. Tokyo. Sect. I 7 (1958) 527–540.

[Tak17] Y. Takei, WKB analysis and stokes geometry of differential equations, in Analytic, Algebraic and Geometric Aspects of Differential Equations, pp. 263–304. Springer, 2017.

[Was76] W. Wasow, Asymptotic expansions for ordinary differential equations. Robert E. Krieger Publishing Co., Huntington, N.Y., 1976. Reprint of the 1965 edition.

[Zol06] H. Zoladek, The Monodromy Group. Monografie Matematyczne. Birkhäuser Basel, 2006.