Problems with Pencils: Lensing Covariance of Supernova Distance Measurements

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While luminosity distances from Type Ia supernovae (SNe) provide a powerful probe of cosmological parameters, the accuracy with which these distances can be measured is limited by cosmic magnification due to gravitational lensing by the intervening large-scale structure. Spatial clustering of foreground mass fluctuations leads to correlated errors in distance estimates from SNe. By including the full covariance matrix of supernova distance measurements, we show that a future survey covering more than a few square degrees on the sky, and assuming a total of ~2000 SNe, will be largely unaffected by covariance noise. “Pencil beam” surveys with small fields of view, however, will be prone to the lensing covariance, leading to potentially significant degradations in cosmological parameter estimates. For a survey with 30 arcmin mean separation between SNe, lensing covariance leads to a ~45% increase in the expected errors in dark energy parameters compared to fully neglecting lensing, and a ~20% increase compared to including just the lensing variance. Given that the lensing covariance is cosmology dependent and cannot be mapped out sufficiently accurately with direct weak lensing observations, surveys with small mean SN separation must incorporate the effects of lensing covariance, including its dependence on the cosmological parameters.

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Introduction.—Type Ia supernovae (SNe) have proven to be powerful probes of the expansion history of the universe [1], contributing to the discovery that this expansion is accelerating. A mysterious dark energy component that comprises ~70% of the energy density of the universe is presumed to be responsible for this acceleration. While the presence of dark energy is by now well established, its properties and provenance remain a complete mystery. As the precise nature of the dark energy has profound implications for both cosmology and particle physics, the elucidation of its properties is one of the foremost observational and theoretical challenges. It is hoped that more accurate cosmological measurements will constrain parameters describing dark energy, and eventually shed light on the underlying physical mechanism [2]. Several ongoing programs, including the Supernova Legacy Survey1, Carnegie Supernova Project2, Essence3, Sloan Supernova Survey, and Supernova Factory4, are underway to observe large samples of low, intermediate, and high-redshift SNe and thereby obtain ~10% constraints on the equation of state parameter of dark energy. Future attempts to measure crucial properties of the dark energy, such as its time evolution, include a dedicated space-based instrument as part of the NASA/DOE Joint Dark Energy Mission (JDEM).

It is well known that gravitational lensing provides a limit to the accuracy with which the true luminosity distance can be determined for an individual SN [3]. The effect comes from the slight modification of the observed SN flux due to lensing by the intervening large-scale structure. In fact, the total error budget for SNe at redshifts higher than z ~ 1 will have statistical errors due to lensing comparable to the intrinsic luminosity distance dispersion [3]. These lensing effects may have already been detected in the current supernova sample [4], although the evidence is still inconclusive [5]. Assuming that lensing contributes to the variance of the observed SN luminosity distribution (i.e. affects each SN observation individually) and using the expected distribution function for the cosmic magnification [6], it has been suggested that the intrinsic power of SNe Ia observation can be restored in the presence of lensing provided the SN sample is increased by a factor of 2–3 [2].

In addition to the increased variance of SN distance measurements due to lensing, spatial fluctuations in the foreground mass structures will lead to correlation of distance estimates of SNe. Even SNe that are widely separated in the radial direction will be lensed by common (sufficiently large-scale) modes of the foreground mass distribution.

In principle, one can use fluctuations of the mean intrinsic luminosity to measure magnification statistics [7]. While such measurements are useful in the context of weak lensing studies, lensing correlations provide a significant challenge for precision measurement of dark energy properties. The additional covariance due to lensing can lead to significant degradation of cosmological parameter estimates for future small-field SN searches. It is to be emphasized that our results apply to any standard-candle approach (e.g., gravitational-wave standard sirens [10]).

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1 http://www.cfht.hawaii.edu/SNLS/
2 http://csp1.lco.cl/~csuserl1/CSP.html
3 http://www.ctio.noao.edu/~wsne/
4 http://snfactory.lbl.gov
\[ \mu = [(1 - \kappa)^2 - |\gamma|^2]^{-1} \approx 1 + 2\kappa + 3\kappa^2 + |\gamma|^2 + \ldots, \]  
(1)

where \( \kappa \ll 1 \) is the lensing convergence and \( |\gamma| = \sqrt{\gamma_1^2 + \gamma_2^2} \) is the total lensing shear. Since \( f_{\text{obs}} \propto \delta_L^{-2}(z) \), where \( \delta_L(z) \) luminosity distance to a source at a redshift of \( z \), fluctuations in \( \mu \) lead to fluctuations in inferred distance so that \( \delta d_L / \delta_L = -\delta \mu / 2 \). Ignoring higher order terms (which are suppressed by an order of magnitude in the lensing variance \( \delta_L^2 \)), one can take \( \mu \approx 1 + 2\kappa \) and relate SN distance fluctuations due to lensing to the convergence along the line-of-sight. Thus, the full covariance matrix of fractional distance estimates for a sample of supernovae is

\[ \text{Cov}_{ij} \approx \sigma_{\text{int}}^2 \delta_{ij} + C^\kappa (z_i, z_j, \theta_{ij}), \]  
(2)

where \( \sigma_{\text{int}} \) is the intrinsic error that affects each SN distance measurement.

**Calculational Method.**—In order to quantify these statements, we first summarize the lensing magnification of background SNe due to the foreground mass distribution and estimate the full covariance matrix associated with lensing. Lensing modifies the true SN flux by a magnification factor \( \mu \), so that the observed flux is given by \( f_{\text{obs}}(n, z) = \mu(n, z) f_{\text{true}}(z) \), where \( n \) represents the direction of the SN on the sky. In the weak lensing limit, this magnification can be related to other well-known quantities through \[11\]

\[ \mu = [(1 - \kappa)^2 - |\gamma|^2]^{-1} \approx 1 + 2\kappa + 3\kappa^2 + |\gamma|^2 + \ldots, \]  
(1)

where \( \kappa \ll 1 \) is the lensing convergence and \( |\gamma| = \sqrt{\gamma_1^2 + \gamma_2^2} \) is the total lensing shear. Since \( f_{\text{obs}} \propto \delta_L^{-2}(z) \), where \( \delta_L(z) \) luminosity distance to a source at a redshift of \( z \), fluctuations in \( \mu \) lead to fluctuations in inferred distance so that \( \delta d_L / \delta_L = -\delta \mu / 2 \). Ignoring higher order terms (which are suppressed by an order of magnitude in the lensing variance \( \delta_L^2 \)), one can take \( \mu \approx 1 + 2\kappa \) and relate SN distance fluctuations due to lensing to the convergence along the line-of-sight. Thus, the full covariance matrix of fractional distance estimates for a sample of supernovae is

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Using the angular cross power spectrum of convergence between two different redshifts, computed under the Limber approximation \[13\]

\[ C^\kappa (z_i, z_j) = \int_0^{\min(r_i, r_j)} dr W(r, r_i) W(r, r_j) \frac{P_{\text{dim}}}{d_A^2} \left( \frac{k}{d_A(r)} \right)^2, \]  
(3)

the lensing contribution to the covariance is

\[ C^\kappa (z_i, z_j, \theta_{ij}) = \int \frac{d^2}{(2\pi)^2} C^\kappa_{ij} (z_i, z_j) J_0(\theta_{ij}). \]  
(4)

Here \( J_0 \) is the 0th order Bessel function of the first kind. In Eq. (3), \( r_i \) and \( r_j \) are comoving distances corresponding to SNe at redshifts \( z_i \) and \( z_j \) respectively, \( d_A \) is the angular diameter distance, and \( P_{\text{dim}}(k, r) \) is the three-dimensional power spectrum of dark matter evaluated at the distance \( r \); we calculate it using the halo model of the large-scale structure mass distribution \[14\].

Eq. (2) defines the full covariance matrix due to lensing for supernovae at redshifts \( z_i \) and \( z_j \) with projected angular separation of \( \theta_{ij} \) on the sky. For reference, the previously considered excess variance due to lensing corresponds to diagonal elements of \( \text{Cov}_{ij} \) with \( z_i = z_j \) and \( \theta_{ij} = 0 \). In this limit, \( J_0(\theta_{ij}) \rightarrow 1 \) in Eq. (4) and one recovers the variance, \( \sigma^2(z) = \int d\ell C^\kappa_{ij}(z) / 2\pi. \)
FIG. 2: Left panel: Expected errors on $w = \text{const}$ (bottom plot) or $w_a$ (with a prior on $\Omega_M$ of 0.01; top plot) as a function of the side length of the observed field. The two dashed curves show errors in corresponding parameters when lensing is completely ignored, and when only the lensing variance is considered. It is apparent that the lensing covariance contributes to the error budget appreciably when the size of the field is $\lesssim 1\text{deg}$. We show results for two values of $\sigma_8$ that roughly span the currently favored values of the amplitude of mass fluctuations and hence the SN lensing covariance. Right panel: The full expected constraints projected into the $\Omega_M - w$ plane (bottom plot; assuming $w = \text{const}$) and $w_0 - w_a$ plane (top plot; with a prior on $\Omega_M$ of 0.01) when $\sigma_8 = 0.95$ and for the cases of no lensing, lensing variance only, and a few selected survey sizes. We have assumed a fixed total number of SNe ($N = 1700$) throughout, regardless of the parameter set and survey sky coverage.

In the left panel of Figure 1 we show the covariance $C_{\kappa}(z_i, z_j, \theta)$ as a function of $\theta \equiv \theta_{ij}$ (which is assumed fixed for the moment) for several values of $z_i = z_j$, while in the right panel we show the covariance as a function of $z_1$ with the other redshift fixed at $z_2 = 1.7$. For reference we also plot the variance as a function of redshift $z$ and compare it to the intrinsic SN magnitude errors of 0.10 and 0.15 mag, roughly spanning the error expected in upcoming surveys. To estimate the resulting effect on cosmological parameter estimates, we compute the Fisher information matrix

$$F_{\alpha\beta} = \sum_{ij} \frac{\partial d_L(z_i)}{\partial p_\alpha} (\text{Cov}^{-1})_{ij} \frac{\partial d_L(z_j)}{\partial p_\beta}. \quad (5)$$

If the variance of SN distance measurements alone is considered, the Fisher matrix reduces to the familiar form, with the factors $N(z_i)/(\sigma_{\text{int}}^2 + \sigma_{\text{lens}}^2)$ representing the inverse covariance terms; here $N(z_i)$ is the number of SNe in the redshift bin centered at $z_i$ and $\sigma_{\text{lens}}^2$ is the variance due to lensing. With the full covariance matrix considered, this simple form no longer holds. Moreover, a full $N_{\text{tot}} \times N_{\text{tot}}$ Fisher matrix (and not the redshift-binned smaller version) is now required in order to obtain the cosmological parameter accuracy estimates; however, this is not a novel problem since a correct treatment of SN calibration uncertainties similarly requires the full $N_{\text{tot}} \times N_{\text{tot}}$ (or even larger) covariance matrix [15]. Here we implicitly neglect information from the cosmological parameter dependence of the covariance matrix; there would only be significant information in the covariance if the off-diagonal terms were comparable to the diagonal ones.

Discussion.—To estimate cosmological parameter measurement errors, we assume a survey with 1700 SNe distributed uniformly in redshift out to $z = 1.7$ (roughly following Ref. [3]). To speed up the calculation of the $1700 \times 1700$ covariance matrix, we compute it in discrete redshift bins, stepping by 0.1 in both $z_i$ and $z_j$. The covariance also depends on the angular separation of SNe, and we distribute the SNe randomly in a square field whose side (or total area) we are free to change. The histogram of the angular separations is a smooth bell curve that peaks at roughly half the field size.

Figure 2 summarizes the effect of lensing covariance on dark energy measurements from the assumed future SN survey. We model the evolution of the dark energy equa-
tion of state with redshift as \( w(a) = w_0 + (1-a)w_a \) where \( a \) is the scale factor, and consider measurements of four parameters: the matter energy density relative to critical, \( \Omega_M \), \( w_0 \), \( w_a \), and the nuisance parameter \( M \) that combines the Hubble constant and absolute SN magnitude information. Our fiducial model is standard \( \Lambda \)CDM with \( \Omega_M = 0.3 \), \( w_0 = -1 \), and \( w_a = 0 \). Figure 2 (left panel) shows the expected errors on \( w = \text{const} \) (bottom plot) or \( w_a \) (with a prior on \( \Omega_M \) of 0.01; top plot) as a function of the size of the observed field. The two dotted curves show errors in corresponding parameters when lensing is completely ignored, and when solely the lensing variance is considered. It is apparent that the lensing covariance contributes to the error budget appreciably when the size of the field is \( \lesssim 1 \text{ deg}^2 \). Furthermore, the effects of lensing covariance depend on the fiducial convergence power \( C^\kappa(z_1,z_2,\theta_{ij}) \), which in turn is sensitive to the amplitude of mass fluctuations \( \sigma_8 \) (and, to a lesser extent, other cosmological parameters). Since \( \sigma_8 \) is somewhat uncertain at present, we show results for two values, \( \sigma_8 = 0.8 \) and \( \sigma_8 = 0.95 \), that roughly span the currently favored values of the amplitude of mass fluctuations in the universe.

Figure 2 (right panel) shows the full expected constraints projected into the \( \Omega_M - w \) plane (bottom plot; assuming \( w = \text{const} \) and \( w_0 - w_a \) plane (top plot; with a prior on \( \Omega_M \) of 0.01) with \( \sigma_8 = 0.95 \) and for the cases of no lensing, lensing variance only, and a few selected survey sizes. Again we see that surveys of less than about one square degree will suffer from considerable error due to lensing covariance. As the mean separation between supernovae is increased, off-diagonal terms in the covariance matrix decrease, and the resulting effect on cosmological parameters is reduced.

Our results can be understood simply in the limit of equal off-diagonal covariance terms. In this case, the Fisher matrix estimate of error in parameter \( p_\alpha \) is increased by a factor \( \sqrt{1+(N-1)/2} \) relative to the case with no off-diagonal terms, where \( r = C^\kappa(z_1,z_2,\theta)/\sqrt{C^\kappa(z_1,z_2,\theta)C^\kappa(z_2,z_1,\theta)} \) and \( N \) is the total number of SNe in the sample. With \( N \sim 2000 \) or more in upcoming searches, parameter errors will increase by a factor of \( \sqrt{2} \) when \( r \sim 1/\sqrt{N} \sim 0.02 \). Furthermore, Figure 1 reveals that when \( z_2 \geq 1 \) correlations are at the percent level when SNe are separated by \( \theta \sim 10 \text{ arcmin} \). Note that in order to accurately estimate the errors on dark energy parameters one will need to allow for the dependence of the covariance matrix on imprecisely known cosmological parameters that determine the weak lensing convergence power spectrum. Accounting for the uncertainty in the covariance is even more important since galaxy shear maps are not useful to correct for lensing-modified SN fluxes [16].

Conclusions.—We have discussed gravitational lensing covariance as an additional source of error for cosmological surveys utilizing standard candles. Future supernova surveys that plan \( \sim 10-20 \text{ deg}^2 \) coverage with \( \sim 2000 \) SNe will be largely unaffected by lensing covariance. Lensing variance remains an issue, but is reduced through increased numbers of SNe (about 50 SNe per redshift bin of width 0.1 are necessary to reduce the lensing variance so that it is negligible compared to the systematic floor [9]). The cosmological parameter accuracies for a survey with a rectangular field that is wide in one direction and narrow in another may be compromised, since the histogram of the angular distribution of SNe now has a peak at an angle of order the narrow side of the survey (albeit with a very pronounced tail). We find that true “pencil beam” surveys with a sky coverage of \( \lesssim 1 \text{ deg}^2 \) in a single field are subject to significant degradation in cosmological parameter accuracies due to lensing covariance. The consideration of lensing covariance thus argues against pencils, and in favor of wider-field surveys.

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