On various generalizations of semi-$\mathcal{A}$-Fredholm operators

Abstract Starting from the definition of $\mathcal{A}$-Fredholm and semi-$\mathcal{A}$-Fredholm operator on the standard module over a unital $C^*$ algebra $\mathcal{A}$, introduced in [8] and [4], we construct various generalizations of these operators and obtain several results as an analogue or a generalization of some of the results in [1], [2], [3], [7]. Moreover, we study also non-adjointable semi-$\mathcal{A}$-Fredholm operators as a natural continuation of the work in [6] on non-adjointable $\mathcal{A}$-Fredholm operators and obtain an analogue or a generalization in this setting of the results in [4], [5].

Keywords Generalized $\mathcal{A}$-Fredholm operator, generalized $\mathcal{A}$-Weyl operator, semi-$\mathcal{A}$-$\mathcal{B}$-Fredholm operator, non-adjointable semi-$\mathcal{A}$-Fredholm operator.

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1. Introduction

Various generalizations of Fredholm and Weyl operators have been considered in several papers, such as [1], [2], [3], [7]. In [7] K.W. Yang has introduced the following definition of generalized Fredholm operator from Banach space $X$ into a Banach space $Y$: An operator $T \in B(X,Y)$ is generalized Fredholm if $T(X)$ is closed in $Y$, and $\ker T$ and Coker $T$ are reflexive. Then he has obtained several results concerning these operators such as:

Theorem 1.1. [7, Theorem 5.3] If $S \in B(X,Y)$ and $T \in B(Y,Z)$ are generalized Fredholm and $TS$ has a closed range, then $TS$ is generalized Fredholm.

Theorem 1.2. [7, Theorem 5.4] Suppose $S \in B(X,Y)$ and $T \in B(Y,Z)$ are range closed, and suppose $TS \in B(X,Z)$ is generalized Fredholm. Then, (i) $S$ is generalized Fredholm $\iff$ $T$ is generalized Fredholm; (ii) if $\ker T$ is reflexive, then both $S$ and $T$ are generalized Fredholm; (iii) if Coker $S$ is reflexive, then both $S$ and $T$ are generalized Fredholm.

Theorem 1.3. [7, Theorem 5.5] Let $T \in B(X,Y)$ have a closed range. If there exist $S, S' \in B(Y,X)$ with closed ranges such that $ST$ and $TS'$ are generalized Fredholm, then $T$ is generalized Fredholm.

Theorem 1.4. [7, Theorem 5.6] Let $T \in B(X,Y)$ be range closed. Then, $T$ is generalized Fredholm $\iff$ $T^*$ is generalized Fredholm.
In [3] Djordjevic has considered generalized Weyl operators. The class of these generalized Weyl operators acting from a Hilbert space \( H \) into a Hilbert space \( K \) and denoted by \( \Phi^g_0(H, K) \), is defined as: \( \Phi^g_0(H, K) = \{ T \in L(H, K) : \mathcal{R}(T) \) is closed and \( \dim \mathcal{N}(T) = \dim \mathcal{N}(T^*) \} \), where \( L(H, K) \) denotes the set of all bounded operators from \( H \) into \( K \). If \( T \in \Phi^g_0(H, K) \), then \( \mathcal{N}(T) \) and \( \mathcal{N}(T^*) \), may be mutually isomorphic infinite-dimensional Hilbert spaces.

Then he proves the following theorem.

**Theorem 1.5.** [3] Theorem 1] Let \( H, K \) and \( M \) be arbitrary Hilbert spaces, \( T \in \Phi^g_0(H, K) \), \( S \in \Phi^g_0(H, M) \) and \( \mathcal{R}(ST) \) is closed. Then \( ST \in \Phi^g_0(H, M) \).

In the proof of this theorem he applies well known Kato theorem.

Finally, in [1] and [2]. Berkani has defined \( B \)-Fredholm and semi-\( B \)-Fredholm operators in the following way:

Let \( T \in L(X) \) where \( X \) is a Banach space. Then \( T \) is said to be semi-\( B \)-Fredholm if there exists an \( n \) such that \( ImT^n \) is closed and \( T\big|_{ImT^n} \) is a semi-Fredholm operator viewed as an operator from \( ImT^n \) into \( ImT^n \). If \( T\big|_{ImT^n} \) is Fredholm, then \( T \) is said to be \( B \)-Fredholm.

He proves for instance the following statements regarding these new classes of operators:

**Proposition 1.6.** [1] Proposition 2.1] Let \( T \in L(X) \). If there exists an integer \( n \in \mathbb{N} \) such that \( R(T^n) \) is closed and such that the operator \( T_n \) is an upper semi-Fredholm (resp. a lower semi-Fredholm) operator, then \( R(T^n) \) is closed, \( T_m \) is an upper semi-Fredholm (resp. a lower semi-Fredholm) operator, for each \( m \geq n \). Moreover, if \( T_n \) is a Fredholm operator, then \( T_m \) is a Fredholm operator and \( ind(T_m) = ind(T_n) \) for each \( m \geq n \).

**Proposition 1.7.** [1] Proposition 3.3] Let \( T \in L(X) \) be a \(- \) Fredholm operator and let \( F \) be a finite rank operator. Then \( T + F \) is a \( B \)-Fredholm operator and \( ind(T + F) = ind(T) \).

Now, Hilbert \( C^* \)-modules are natural generalization of Hilbert spaces when the field of scalars is replaced by a \( C^* \)-algebra.

Fredholm theory on Hilbert \( C^* \)-modules as a generalization of Fredholm theory on Hilbert spaces was started by Mishchenko and Fomenko in [8]. They have elaborated the notion of a Fredholm operator on the standard module \( H_A \) and proved the generalization of the Atkinson theorem. Their definition of \( A \)-Fredholm operator on \( H_A \) is the following:

**Definition** A (bounded \( A \) linear) operator \( F : H_A \to H_A \) is called \( A \)-Fredholm if

1) it is adjointable;
2) there exists a decomposition of the domain \( H_A = M_1 \oplus N_1 \), and the range, \( H_A = M_2 \oplus N_2 \), where \( M_1, M_2, N_1, N_2 \) are closed \( A \)-modules and \( N_1, N_2 \) have a finite number of generators, such that \( F \) has the matrix from

\[
\begin{bmatrix}
F_1 & 0 \\
0 & F_4
\end{bmatrix}
\]
with respect to these decompositions and \( F_1 : M_1 \to M_2 \) is an isomorphism.

The notation \( \oplus \) denotes the direct sum of modules without orthogonality, as given in [9].

In [4] we went further in this direction and defined semi-\( \mathcal{A} \)-Fredholm operators on Hilbert \( C^* \)-modules. We investigated then and proved several properties of these new semi Fredholm operators on Hilbert \( C^* \)-modules as an analogue or generalization of the well-known properties of classical semi-Fredholm operators on Hilbert and Banach spaces.

The main idea with this paper was to go further in the direction of [4], [8] and to define generalized \( \mathcal{A} \)-Fredholm operators, generalized \( \mathcal{A} \)-Weyl operators and semi-\( \mathcal{A} \)-\( B \)-Fredholm operators on \( H_\mathcal{A} \) that would be appropriate generalizations of the above mentioned classes of operators on Hilbert and Banach spaces defined by Yang, Djordjevic and Berkani. Moreover the purpose of this paper is to establish in this setting an analogue or a generalization of the above mentioned results concerning generalized Fredholm, generalized Weyl and semi-\( B \)-Fredholm operators on a Hilbert or a Banach space. More precisely, our Proposition 3.2 is an analogue of [3] Theorem 1], our Lemma 3.5 is an analogue of [7] Theorem 5.3], our Proposition 3.6 is an analogue of [7] Theorem 5.4], our Lemma 3.7 is analogue of [7] Theorem 5.5], our Proposition 5.2 is a generalization of [2] Proposition 2.1] and our Theorem 5.5 is a generalization of [2] Proposition 3.3].

Next, in addition to adjointable \( \mathcal{A} \)-Fredholm operator, Mishchenko also considers in [6] non adjointable \( \mathcal{A} \)-Fredholm operators on the standard module \( l_2(\mathcal{A}) \). In this paper, we go further in this direction and consider non adjointable semi-\( \mathcal{A} \)-Fredholm operators on \( l_2(\mathcal{A}) \). We establish some of the basic properties of these operators in terms of inner and external (Noether) decompositions and show that these operators are exactly those that are one sided invertible in \( B(l_2(\mathcal{A}))/K(l_2(\mathcal{A})) \), where \( K(l_2(\mathcal{A})) \) denotes the set of all compact operators on \( l_2(\mathcal{A}) \) in the sense of [6]. Then we prove that an analogue or a modified version of results in [4], [5] hold when one considers these non adjointable semi-\( \mathcal{A} \)-Fredholm operators.

2. Preliminaries

In this section we are going to introduce the notation, and the definitions in [4] that are needed in this paper. Throughout this paper we let \( \mathcal{A} \) be a unital \( C^* \)-algebra, \( H_\mathcal{A} \) be the standard module over \( \mathcal{A} \) and we let \( B^a(H_\mathcal{A}) \) denote the set of all bounded , adjointable operators on \( H_\mathcal{A} \). We also let \( B(l_2(\mathcal{A})) \) denote the set of all \( \mathcal{A} \)-linear, bounded operators on the standard module \( l_2(\mathcal{A}) \), but not necessarily adjointable. According to [8], Definition 1.4.1], we say that a Hilbert \( C^* \)-module \( M \) over \( \mathcal{A} \) is finitely generated if there exists a finite set \( \{x_i\} \subseteq M \) such that \( M \) equals the linear span (over \( C \) and \( \mathcal{A} \) ) of this set.
**Definition 2.1.** [4, Definition 2.1] Let $F \in B^a(H_A)$. We say that $F$ is an upper semi-$\mathcal{A}$-Fredholm operator if there exists a decomposition

$$H_A = M_1 \oplus N_1 \overset{F}{\to} M_2 \oplus N_2 = H_A$$

with respect to which $F$ has the matrix

$$
\begin{bmatrix}
F_1 & 0 \\
0 & F_4
\end{bmatrix},
$$

where $F_1$ is an isomorphism, $M_1, M_2, N_1, N_2$ are closed submodules of $H_A$ and $N_1$ is finitely generated. Similarly, we say that $F$ is a lower semi-$\mathcal{A}$-Fredholm operator if all the above conditions hold except that in this case we assume that $N_2$ (and not $N_1$) is finitely generated.

Set

$$\mathcal{M}\Phi_+(H_A) = \{ F \in B^a(H_A) \mid F \text{ is upper semi-$\mathcal{A}$-Fredholm} \},$$

$$\mathcal{M}\Phi_-(H_A) = \{ F \in B^a(H_A) \mid F \text{ is lower semi-$\mathcal{A}$-Fredholm} \},$$

$$\mathcal{M}\Phi(H_A) = \{ F \in B^a(H_A) \mid F \text{ is $\mathcal{A}$-Fredholm operator on } H_A \}.$$

**Remark 2.2.** [4] Notice that if $M, N$ are two arbitrary Hilbert modules $C^*$-modules, the definition above could be generalized to the classes $\mathcal{M}\Phi_+(M, N)$ and $\mathcal{M}\Phi_-(M, N)$.

Recall that by [9, Definition 2.7.8], originally given in [8], when $F \in \mathcal{M}\Phi(H_A)$ and

$$H_A = M_1 \oplus N_1 \overset{F}{\to} M_2 \oplus N_2 = H_A$$

is an $\mathcal{M}\Phi$ decomposition for $F$, then the index of $F$ is defined by $\text{index } F = [N_1] - [N_2] \in K(\mathcal{A})$ where $[N_1]$ and $[N_2]$ denote the isomorphism classes of $N_1$ and $N_2$ respectively. By [9, Definition 2.7.9], the index is well defined and does not depend on the choice of $\mathcal{M}\Phi$ decomposition for $F$.

**Definition 2.3.** [4, Definition 5.6] Let $F \in \mathcal{M}\Phi_+(H_A)$. We say that $F \in \mathcal{M}\Phi_+(H_A)$ if there exists a decomposition

$$H_A = M_1 \oplus N_1 \overset{F}{\to} M_2 \oplus N_2 = H_A$$

with respect to which

$$F = \begin{bmatrix}
F_1 & 0 \\
0 & F_4
\end{bmatrix},$$

where $F_1$ is an isomorphism, $N_1$ is closed, finitely generated and $N_1 \preceq N_2$. Similarly, we define the class $\mathcal{M}\Phi_+(H_A)$, only in this case $F \in \mathcal{M}\Phi_-(H_A)$, $N_2$ is finitely generated and $N_2 \preceq N_1$.

In [5] we set $\widehat{\mathcal{M}\Phi}_+(H_A)$ to be the space of all $F \in B^a(H_A)$ such that there exists a decomposition

$$H_A = M_1 \oplus N_1 \overset{F}{\to} M_2 \oplus N_2 = H_A,$$

w.r.t. which $F$ has the matrix $\begin{bmatrix}
F_1 & 0 \\
0 & F_4
\end{bmatrix}$, where $F_1$ is an isomorphism, $N_1$ is finitely generated and such that there exist closed submodules $N'_2, N$ where
\[ N_2' \subseteq N_2, N_2' \cong N_1, \ H_A = N_2 \oplus N_1 = N_2' \oplus N_2' \] and the projection onto \( N \) along \( N_2' \) is adjointable.

**Definition 2.4.** [5] Definition 4] We set \( \tilde{M} \Phi_+^c(H_A) \) to be the set of all \( D \in B^a(H_A) \) such that there exists a decomposition

\[
H_A = M_1' \oplus N_1' \xrightarrow{D} M_2' \oplus N_2' = H_A
\]

w.r.t. which \( D \) has the matrix \( \begin{bmatrix} D_1 & 0 \\ 0 & D_4 \end{bmatrix} \), where \( D_1 \) is an isomorphism, \( N_2' \) is finitely generated and such that \( H_A = M_1' \oplus N \oplus N_2' \) for some closed submodule \( N \), where the projection onto \( M_1' \oplus N \) along \( N_2' \) is adjointable.

**Definition 2.5.** [6] Definition 2] A bounded \( \mathcal{A} \)-operator \( l_2(\mathcal{A}) \rightarrow l_2(\mathcal{A}) \) is called a Fredholm \( \mathcal{A} \)-operator if there exists a bounded \( \mathcal{A} \)-operator \( \Phi \) such that

\[
\text{id} - GF \in \mathcal{K}(l_2(\mathcal{A})), \text{id} - GF \in \mathcal{K}(l_2(\mathcal{A})).
\]

**Definition 2.6.** [6] Definition 3] We say that a bounded \( \mathcal{A} \)-operator \( F : l_2'(\mathcal{A}) \rightarrow l_2'(\mathcal{A}) \) admits an inner (Noether) decomposition if there is a decomposition of the preimage and the image \( l_2(\mathcal{A}) = M_1 \oplus N_1, l_2'(\mathcal{A}) = M_2 \oplus N_2 \) where \( C^* \)-modules \( N_1 \) and \( N_2 \) are finitely generated Hilbert \( C^* \)-modules, and if \( F \) has the following matrix from \( F = \begin{bmatrix} F_1 & F_2 \\ 0 & F_4 \end{bmatrix} : M_1 \oplus N_1 \rightarrow M_2 \oplus N_2 \), where \( F_1 : M_1 \rightarrow M_2 \) is an isomorphism.

**Definition 2.7.** [6] Definition 4] We put by definition \( index F = [N_2] - [N_1] \in K(\mathcal{A}) \).

**Definition 2.8.** [6] Definition 5] We say that a bounded \( \mathcal{A} \)-operator \( F : l_2'(\mathcal{A}) \rightarrow l_2''(\mathcal{A}) \) admits an external (Noether) decomposition if there exist finitely generated \( C^* \)-modules \( X_1 \) and \( X_2 \) bounded \( \mathcal{A} \)-operators \( E_2, E_3 \) such that the matrix operator

\[
F_0 = \begin{bmatrix} F & E_2 \\ E_3 & 0 \end{bmatrix} : l_2'(\mathcal{A}) \oplus X_1 \rightarrow l_2''(\mathcal{A}) \oplus X_2, \text{Is an invertible operator.}
\]

**Definition 2.9.** [6] Definition 6] We put by definition \( index F = [X_1] - [X_2] \in K(\mathcal{A}) \).

3. **On Generalized \( \mathcal{A} \)-Fredholm and \( \mathcal{A} \)-Weyl Operators**

**Definition 3.1.** Let \( F \in B^a(H_A) \)

1) We say that \( F \in \mathcal{M} \Phi_0^{gc}(H_A) \) if \( \text{Im} F \) is closed, \( \text{ker} F \) and \( \text{Im} F^\perp \) are self-dual.
2) We say that \( F \in \mathcal{M} \Phi_0^{gc}(H_A) \) is \( \text{Im} F \) is closed and \( \text{ker} F \cong \text{Im} F^\perp \) (here we do not require self-duality of \( \text{ker} F, \text{Im} F^\perp \)).

**Proposition 3.2.** Let \( F, D \in \mathcal{M} \Phi_0^{gc}(H_A) \) and suppose that \( \text{Im} DF \) is closed. Then \( DF \in \mathcal{M} \Phi_0^{gc}(H_A) \).
Proof. Since $ImDF$ is closed, by [9, Theorem 2.3.3] there exists a closed submodule $X$ s.t. $ImD = ImDF \oplus X$. Next, considering the map $D_{|ImF}$ and again using that $ImDF$ is closed, we have that $ker D \cap ImF = ker D_{|ImF}$ is orthogonally complementable in $ImF$ by [9, Theorem 2.3.3], so $ImF = W \oplus (ker D \cap ImF)$ for some closed submodule $W$. Now, since $ker D \cap ImF \oplus W \oplus ImF^\perp = H_A$ and $(ker D \cap ImF) \subseteq ker D$, it follows that $ker D = (ker D \cap ImF) \oplus (ker D \cap (W \oplus ImF^\perp))$. Set $M = ker D \cap (W \oplus ImF^\perp)$, then $ker D = (ker D \cap ImF) \oplus M$. On $ker D^\perp$, $D$ is an isomorphism from $ker D^\perp$ onto $ImD$. Let $S = (D_{|ker D^\perp})^{-1}$. Then $P_{ker D^\perp}$ is an isomorphism from $W$ onto $S(ImDF)$. Indeed, since $D_W$ is injective and $D(W) = ImDF$ is closed, by Banach open mapping theorem $D_{|W}$ is an isomorphism onto $ImDF$. This actually means that $DP_{ker D^\perp}$ is an isomorphism onto $ImDF$, as $D_{|W} = DP_{ker D^\perp}$. Since $D_{|S(ImD)}$ is an isomorphism onto $ImDF$, it follows that $P_{ker D^\perp|W}$ is an isomorphism onto $S(ImDF)$. Hence $\cap_{S(ImDF)}$ denotes the projection onto $S(ImDF)$ along $S$. Therefore we get that $H_A = W \oplus S(X) \oplus ker D$. Thus we have
\[ H_A = W \oplus S(X) \oplus (ker D \cap ImF) \oplus M = W \oplus (ker D \cap ImF) \oplus ImF^\perp. \]

This gives $S(X) \oplus M \cong ImF^\perp$. On the other hand, by classical arguments we have $ker DF = ker F \oplus R$ for some closed submodule $R$ isomorphic to $ker D \cap ImF$. Therefore we get $ker DF \cong (ker F \oplus (ker D \cap ImF)) \cong ImF^\perp \oplus (ker D \cap ImF) \cong S(X) \oplus M \oplus ker D \cap ImF \cong S(X) \oplus ker D \cong X \oplus ImD^\perp \cong ImDF$. (where $\oplus$ denotes now the direct sum in the sense of [9, Example 1.3.3].) \qed

Remark 3.3. This result is a generalization of [8, Theorem 1], however in our proof we do not apply Kato theorem. Indeed, our proof is also valid in the case when $F \in \mathcal{M} \Phi^{ge}_0(M, N), D \in \mathcal{M} \Phi^{ge}_0(N, K)$ where $M, N, K$ are arbitrary Hilbert $C^*$-modules over a unital $C^*$-algebra $A$. Next, by our proof we also obtain easily a generalization of Harte’s ghost theorem:

**Corollary 3.4.** Let $F, D \in B^*(H_A)$ and suppose that $ImF, ImD, ImDF$ are closed. Then $ker F \oplus ker D \oplus ImDF^\perp \cong ImD^\perp \oplus ImF^\perp \oplus ker DF$.

**Proof.** We keep the notation from the previous proof. In that proof we have shown that $ImF^\perp \cong S(X) \oplus M$. Moreover $D = ker D \cap ImF \oplus M$ and $ImDF^\perp = ImD^\perp \oplus X$. This gives
\[ ker F \oplus ker D \oplus ImDF^\perp \cong ker F \oplus ker D \oplus ImD^\perp \oplus X \cong ker F \oplus (ker D \cap ImF) \oplus M \oplus ImD^\perp \oplus X \cong ker DF \oplus M \oplus S(X) \oplus ImD^\perp \cong ker DF \oplus ImF^\perp \oplus ImD^\perp. \]

The next results are inspired by results in [7].

**Lemma 3.5.** Let $F, D \in \mathcal{M} \Phi^{ge}(H_A)$ and suppose that $ImDF$ is closed. Then $DF \in \mathcal{M} \Phi^{ge}(H_A)$. 
Proof. Suppose that $DF \in \mathcal{M}\Phi^\text{gc}(H_A)$. Then $\ker F, \ker D$ are self-dual and $\text{Im} F, \text{Im} D$ are closed. Now, $D_{|\text{Im} F}$ is an operator onto $\text{Im} DF = \text{Im} D_{|\text{Im} F}$ which is closed by assumption and it is adjointable as $D$ is so and $\text{Im} F$ is orthogonally complementable by [9, Theorem 2.3.3]. Hence, again by [9, Theorem 2.3.3] we deduce that $\ker D_{|\text{Im} F} = \ker D \cap \text{Im} F$ is orthogonally complementable in $\text{Im} F$, so $\text{Im} F = (\ker D \cap \text{Im} F) \oplus M$ for some closed submodule $M$. Therefore $H_A = (\ker D \cap \text{Im} F) \oplus M \oplus \text{Im} F^\perp$. It follows that $\ker D = (\ker D \cap \text{Im} F) \oplus M'$ where $M' = \ker D \cap (M \oplus \text{Im} F^\perp)$. On the other hand by classical arguments, one can show that $\ker DF = \ker F \oplus W$ where $W \cong \ker D \cap \text{Im} F$. Since $\ker F$ is self dual, $\ker F$ is therefore an orthogonal direct summand in $\ker DF$ by [9, Proposition 2.5.4], so $\ker DF = \ker F \oplus \tilde{W}$ for some closed submodule $\tilde{W} \cong W \cong \ker D \cap \text{Im} F$. Since $\ker D \cap \text{Im} F$ is self-dual, so is $\tilde{W}$, hence, $\ker DF$ is self-dual being orthogonal direct sum of two self-dual modules.

Next, from the proof of Proposition 3.2 we obtain that $\text{Im} DF^\perp = \text{Im} D^\perp \oplus X$, where $\text{Im} F^\perp \cong X \oplus M$. Since $\text{Im} F^\perp$ is self-dual, so is $X$ being an orthogonal direct summand in a self dual module. Finally since $\text{Im} D^\perp$ is self-dual, it follows that $\text{Im} DF^\perp = \text{Im} D^\perp \oplus X$ is self-dual also.

Proposition 3.6. Let $F, D \in B^a(H_A)$, suppose that $\text{Im} F, \text{Im} D$ are closed and $\text{Im} DF \in \mathcal{M}\Phi^\text{gc}(H_A)$. Then the following statements hold:

a) $D \in \mathcal{M}\Phi^\text{gc}(H_A) \iff F \in \mathcal{M}\Phi^\text{gc}(H_A)$

b) if $\ker D$ is self-dual then $F, D \in \mathcal{M}\Phi^\text{gc}(H_A)$

c) if $\text{Im} F^\perp$ is self-dual, then $F, D \in \mathcal{M}\Phi^\text{gc}(H_A)$.

Proof. Let us prove b) first. If $DF$ is generalized $A$-Fredholm, then $\text{Im} DF$ is closed and $\text{Im} DF^\perp$, $\ker DF$ are self-dual. Now, observe that $\text{Im} DF = \text{Im} D_{|\text{Im} F} = \text{Im} P_{|\text{Im} F} D_{|\text{Im} F}$ where $P_{|\text{Im} D}$ denotes the orthogonal projection onto $\text{Im} D$. Since $P_{|\text{Im} F} D_{|\text{Im} F}$ is adjointable, by [9, Theorem 2.3.3], we have that $\text{Im} DF$ is orthogonally complementable in $\text{Im} D$. Hence $\text{Im} D = \text{Im} DF \oplus N$ for some closed submodule $N$. Therefore $H_A = \text{Im} DF \oplus N \oplus \text{Im} D^\perp$, so $\text{Im} DF^\perp = N \oplus \text{Im} D^\perp$. Since $\text{Im} DF^\perp$ is self-dual, so is $\text{Im} D^\perp$, being an orthogonal direct summand in $\text{Im} DF^\perp$. Next, since $F(\ker DF) = \ker D \cap \text{Im} F$ and $F_{|\ker DF}$ is adjointable, as $F$ is so and $\ker DF$ is orthogonally complementable by [9, Theorem 2.3.3], we deduce that $\ker F = \ker F_{|\ker DF}$ orthogonally complementable in $\ker DF$. Since $\ker DF$ is self-dual, it follows that $\ker F$ is self-dual, being orthogonal direct summand in $\ker DF$. It remains to show that $\text{Im} F^\perp$ is self-dual. But, by earlier arguments, since $\text{Im} DF$ is closed, we have the $\ker D \cap \text{Im} F$ is orthogonally complementable $\text{Im} F$, hence in $H_A$ as $H_A = \text{Im} F \oplus \text{Im} F^\perp$, and therefore in $\ker D$. So $\ker D = (\ker D \cap \text{Im} F) \oplus M'$ for some closed submodule $M'$. Moreover, again by arguments, we have then that $\text{Im} F^\perp \cong N \oplus M'$. Now, $N$ and $M$ are self dual, being orthogonal direct summands in $\text{Im} DF^\perp$ and $\ker D$, respectively, which are self-dual. Hence $M' \oplus N$ is self-dual, thus $\text{Im} F^\perp$ is self-dual. By passing to the adjoints one may obtain c). To deduce a), use b) and c).

Lemma 3.7. Let $F \in B^a(H_A)$ and suppose that $\text{Im} F$ is closed. Moreover, assume that there exist operators $D, D' \in B^a(H_A)$ with closed images such that $D'F, FD \in \mathcal{M}\Phi^\text{gc}(H_A)$. Then $F \in \mathcal{M}\Phi^\text{gc}(H_A)$. 
Proof. By the proof of Proposition \[3.6\] part b), since \(ImF D\) is in \(\mathcal{M}\Phi^g_c(H_A)\) and \(ImF, ImD\) are closed, it follows that \(ImF^\bot\) is self-dual. Now, by passing to the adjoints we obtain that \(F^*(D')^* \in \mathcal{M}\Phi^g_c(H_A)\) as \(D'F \in \mathcal{M}\Phi^g_c(H_A)\). Moreover, by the proof of \[9,\ Theorem 2.3.3\] part ii), \(ImF^*, (ImD')^*\) are closed, as \(ImF, ImD'\) are so (by assumption). Hence, using the previous arguments, we deduce that \(ImF^\bot = \ker F\) is self-dual. \(\square\)

4. REMARKS ON NON-ADJOINTABLE SEMI-FREDHOLM OPERATORS

From [6 Definition 3] it follows as in the proof of [9 Lemma 2.7.10] that \(F\) has the matrix \(\begin{pmatrix} F_1 & 0 \\ 0 & \overline{F}_4 \end{pmatrix}\) w.r.t. the decomposition \(U(M_1) \oplus U(N_1) \xrightarrow{F} M_2 \oplus N_2\). Obviously, such operators are invertible in \(B(l_2(A))_{/K(l_2(A))}\). Now, if only \(N_1\) is finitely generated, we say that \(F\) has upper inner (Noether) decomposition, whereas if only \(N_2\) is finitely generated, we say that \(F\) has lower inner (Noether) decomposition. Based on [6 Definition 4] we give now the following definition.

Definition 4.1. We say that \(F\) has upper external (Noether) decomposition if there exist closed \(C^*\)-modules \(X_1, X_2\) where \(X_2\) finitely generated, s.t. the operator \(F_0\) defined as

\[F_0 = \begin{pmatrix} F & E_2 \\ E_3 & 0 \end{pmatrix} = l_2'(A) \oplus X_1 \rightarrow l_2''(A) \oplus X_1\]

is invertible and s.t. \(ImE_2\) is complementable in \(l_2''(A)\). Similarly, we say that \(F\) has lower external (Noether) decomposition if the above decomposition exists, only in this case we assume that \(X_1\) is finitely generated and that \(\ker E_3\) is complementable in \(l_2''(A)\).

Proposition 4.2. A bounded \(A\)-operator \(F = l_2'(A) \rightarrow l_2''(A)\) admits an upper external (Noether) decomposition iff it admits an upper inner (Noether) decomposition. Similarly, \(F\) admits a lower external (Noether) decomposition iff \(F\) admits a lower inner (Noether) decomposition.

Proof. As in the proof of [6 Theorem 3], we may let, when \(F\) has an inner decomposition, the operator \(F_0\) to be defined as

\[F_0 = \begin{pmatrix} F_1 & F_2 & 0 \\ 0 & id & 0 \end{pmatrix} : M_1 \oplus N_1 \oplus N_2 \rightarrow M_2 \oplus N_2 \oplus N_1.\]

Then \(F_0\) is invertible. Moreover, the operator \(E_2 : X_1 = N_2 \rightarrow l_2''(A) = M_2 \oplus N_2\) is just the inclusion, hence \(ImE_2 = N_2\) is complementable in \(M_2 \oplus N_2 = l_2''(A)\). Also, the operator \(E_3 : l_2'(A) = M_1 \oplus N_1 \rightarrow X_2 = N_1\) is simply the projection onto \(N_1\) along \(M_1\), so \(\ker E_3 = M_1\) is complementable in \(l_2'(A)\). To prove the other direction, when \(F\) has an external decomposition, we may proceed in exactly the same way as in the proof of [6 Theorem 3]. Indeed, to obtain (29) and (34), we use the assumptions in the definition of external decomposition that \(ImE_2\) and \(\ker E_3\) are complementable in \(l_2''(A)\) and \(l_2'(A)\) respectively. \(\square\)
Clearly, any upper semi-Fredholm operator in the sense of our definition is also left invertible in \( B(l_2(A))/K(l_2(A)) \), whereas any lower semi-Fredholm operator is right invertible \( B(l_2(A))/K(l_2(A)) \) (by upper and lower semi-Fredholm we mean here that \( F \) admits upper and lower inner decomposition resp.). The converse also holds:

**Proposition 4.3.** If \( F \) is left invertible in \( B(l_2(A))/K(l_2(A)) \), then \( F \) admits upper inner decomposition. If \( F \) is right invertible in \( B(l_2(A))/K(l_2(A)) \), then it admits lower inner decomposition.

**Proof.** If \( GF = id + K'' \) for some \( G : l''_2(A) \rightarrow l'_2(A), K'' \in K(l_2(A)) \), then by following the proof of [6, Theorem 5] we reach to (45) in [6]. Moreover, by this part of the proof of [6, Theorem 5], we also obtain that \( F \) has the matrix \( \begin{pmatrix} G_1 & G_2 \\ 0 & G_4 \end{pmatrix} \) w.r.t. the decomposition \( l''_2(A) = M_1 \oplus N_1 \xrightarrow{G} M_2 \oplus N_2 = l'_2(A) \) where \( G_1 \) is an isomorphism. Indeed, by (45) in [6] \( M_3 = ImP = ImFK^{-1}_1p_2G \). It follows that \( M_3 = F(M_1) \). Since \( GF_{M_1} \) is an isomorphism onto \( M_2 \), it follows that \( G_{1_{F_{M_1}}} \) is an isomorphism onto \( M_2 \). Then, considering the operator \( G \) and applying the arguments above, one deduces the second statement in the proposition. \( \square \)

The next lemma is again a corollary of [6, Theorem 5]:

**Lemma 4.4.** Let \( F, G \) be bounded \( A \)-operators and suppose that \( GF \) is Fredholm. Then there exist decompositions

\[
l'_2(A) = M_1 \oplus N_1 \xrightarrow{F} l''_2(A) = M_2 \oplus N_2 \xrightarrow{G} l'_2(A) = M_2 \oplus N_2
\]

w.r.t. which \( F, G \) have matrices \( \begin{pmatrix} F_1 & 0 \\ 0 & F_4 \end{pmatrix} \), \( \begin{pmatrix} G_1 & G_2 \\ 0 & G_4 \end{pmatrix} \), respectively, where \( F_1, G_1 \) are isomorphisms, \( N_1, N_2 \) are finitely generated.

From now on, throughout this section we will let \( \mathcal{M}_+(l_2(A)) \) denote the set of all operators left invertible in \( B(l_2(A))/K(l_2(A)) \), whereas \( \mathcal{M}_-(l_2(A)) \) will denote the set of all operators right invertible in \( B(l_2(A))/K(l_2(A)) \). Then we set \( \mathcal{M}_+(l_2(A)) = \mathcal{M}_+(l_2(A)) \cap \mathcal{M}_-(l_2(A)) \) Although the notation here coincides with notation in [4] we do not assume the adjointability of operators here in this section.

Most of the results from [4], [5] are also valid when we consider the non-adjointable semi-Fredholm operators and the same proofs can be applied. Here we are going slightly different formulations and proofs of some of the results from [4], [5] which can not be transfered directly to the non-adjointable case.

**Lemma 4.5.** Let \( V \) be a finitely generated Hilbert submodule of \( l_2(A) \), \( F \in B(l_2(A)) \) and suppose that \( P_{V \perp} F \in \mathcal{M}_+(l_2(A)), V \perp (l_2(A), V \perp) \) where \( P_{V \perp} \) is the orthogonal projection onto \( V \perp \) along \( V \). Then \( F \in \mathcal{M}_-(l_2(A)) \).

**Proof.** Since \( V \) is finitely generated, by [5] Lemma 2.3.7, \( V \) is an orthogonal direct summand in \( l_2(A) \), so \( l_2(A) = V \oplus V \perp \). Consider the decomposition

\[
l_2(A) = M_1 \oplus N_1 \xrightarrow{P_{V \perp} F} M_2 \oplus N_2 = V \perp
\]
where \( N_1, N_2 \) are finitely generated and \((P_{V \perp} F)_1\) is isomorphism. Since \((P_{V \perp} F)_1 = P_{M_2}^v P_{V \perp} F|_{M_1}\) where \( P_{M_2}^v \) is the projection of \( V \perp \) onto \( M_2 \) along \( N_2 \), it follows that \( P_{M_2}^v P_{V \perp} F|_{M_1} \), is an isomorphism of \( M_1 \) onto \( M_2 \). But \( l_2(A) = M_2 \oplus N_2 \oplus V \) and \( P_{M_2}^v P_{V \perp} = P_{M_2} \) where \( P_{M_2} \) is the projection of \( l_2(A) \) onto \( M_2 \) along \( N_2 \oplus V \). Hence \( F \) has the matrix
\[
\begin{bmatrix}
F_1 & F_2 \\
F_3 & F_4
\end{bmatrix}
\]

w.r.t. the decomposition
\[
l_2(A) = M_1 \oplus N_1 \oplus M_2 \oplus (N_2 \oplus V) = l_2(A)
\]

where \( F_1 = P_{M_2} F|_{M_1} \) an isomorphism. Then w.r.t. the decomposition
\[
l_2(A) = U_1(M_1) \oplus U_1(N_1) \oplus U_2^{-1}(M_2) \oplus U_2^{-1}(N_2 \oplus V) = l_2(A)
\]

\( F \) has the matrix
\[
\begin{bmatrix}
\bar{F}_1 & 0 \\
0 & \bar{F}_4
\end{bmatrix}
\]

where
\[
U_1 = \begin{bmatrix}
1 & -F_1^{-1}F_2 \\
0 & 1
\end{bmatrix},
\]
\[
U_2 = \begin{bmatrix}
1 & 0 \\
-F_3F_1^{-1} & 1
\end{bmatrix},
\]

and \( \bar{F}_1 \) are isomorphisms. Now, \( N_2 \oplus V \) is finitely generated, hence \( U_2^{-1}(N_2 \oplus V) \) is finitely generated also. \( \square \)

**Lemma 4.6.** Let \( G, F \in B(l_2(A)) \), suppose that \( \text{Im} G \) is closed and that \( \ker G \) and \( \text{Im} G \) are complementable in \( l_2(A) \). If \( GF \in \mathcal{MF}_-(l_2(A)) \) then \( \cap F \in \mathcal{MF}_-(l_2(A)) \), \( N \) where \( \ker G \oplus N = l_2(A) \) and \( \cap \) denotes the projection onto \( N \) along \( \ker G \).

**Proof.** By the arguments from the proof of Lemma 4.4, since \( GF \in \mathcal{MF}_-(l_2(A)) \), there exists a chain of decompositions
\[
l_2(A) = M_1 \oplus M_2 \xrightarrow{F} R_1 \oplus R_2 \xrightarrow{G} N_1 \oplus N_2
\]

w.r.t. which \( F \) and \( G \) have matrices \( \begin{pmatrix} F_1 & 0 \\ 0 & F_4 \end{pmatrix} \), \( \begin{pmatrix} G_1 & G_2 \\ 0 & G_4 \end{pmatrix} \) whence \( F_1, G_1 \) are isomorphisms and \( N_2 \) is finitely generated. Indeed, considering the \( \mathcal{MF}_- \) decomposition \( M_1 \oplus M_2 \xrightarrow{GF} N_1 \oplus N_2 \), the arguments of the proof of until (45) in \[9\] applies also in the case when \( N_1 \) on \( N_2 \) are not finitely generated. Hence \( G \) has the matrix \( \begin{pmatrix} G_1 & 0 \\ 0 & G_4 \end{pmatrix} \) w.r.t. the decomposition \( R_1 \oplus U(R_2) \xrightarrow{G} N_1 \oplus N_2 \).
where \( U \) is an isomorphism. It is not hard to see that \( \ker G \subseteq U(R_2) \). Since 
\[ \ker G \oplus N = l_2(A) \quad \text{and} \quad \ker G \subseteq U(R_2), \]
we get that \( U(R_2) = \ker G \oplus (U(R_2) \cap N) \). As \( \text{Im} G \) is closed, \( G_{|_{\text{Im} G}} \) is an isomorphism onto \( \text{Im} G \) by open mapping theorem. Hence \( G_{|_{(U(R_2) \cap N)}} \) is an isomorphism. Thus \( \text{Im} G = N_1 \oplus G(U(R_2) \cap N) \). As \( \text{Im} G \) is complementable in \( l_2(A) \), we have that \( G(U(R_2) \cap N) \) is also complementable in \( l_2(A) \). Since \( G(U(R_2) \cap N) \subseteq N_2 \), it follows that \( G(U(R_2) \cap N) \) is complementable in \( N_2 \) also. But \( N_2 \) is finitely generated, hence \( G(U(R_2) \cap N) \) must be finitely generated being a direct summand in \( N_2 \). Hence \( U(R_2) \cap N \) is finitely generated being isomorphic to \( G(U(R_2) \cap N) \). W.r.t. the decomposition 
\[ M_1 \oplus M_2 \xrightarrow{F} R_1 \oplus U(R_2), \]
\( F \) has the matrix \( \begin{pmatrix} F_1 & 0 \\ 0 & \tilde{F}_4 \end{pmatrix} \), w.r.t. the decomposition \( M_1 \oplus \tilde{U}(M_2) \xrightarrow{\tilde{F}} R_1 \oplus U(R_2) \) where \( \tilde{U} \) is an isomorphism. Moreover, since \( l_2(A) = R_1 \oplus (U(R_1) \cap N) \oplus \ker G \), it follows that \( \cap_{|R_1} \) is an isomorphism (recall that \( \cap \) is the projection onto \( N \) along \( \ker G \)). It is then easy to see that \( \cap F \) has the matrix \( \begin{pmatrix} (\cap F)_1 & 0 \\ 0 & (\cap F)_4 \end{pmatrix} \), w.r.t. the decomposition 
\[ M_1 \oplus \tilde{U}(M_2) \xrightarrow{\cap F} \cap (R_1) \oplus (U(R_1) \cap N) \]
where \( (\cap F)_1 \) is an isomorphism. Now, \( U(R_1) \cap N \) is finitely generated. \( \square \)

Recall now the definition of classes \( \mathcal{M} \Phi_+^-(l_2(A)) \), \( \mathcal{M} \Phi_+^+(l_2(A)) \), from \( \[4\] \). Again we are going to use the same notation, but we are not going to assume adjointability.

**Lemma 4.7.** \( F \in B((l_2(A))) \) admits upper external (Noether) decomposition with the property that \( X_2 \preceq X_1 \) iff \( F \in \mathcal{M} \Phi_+^-(l_2(A)) \). Similarly \( F \) admits lower external (Noether) decomposition with the property that \( X_1 \preceq X_2 \) iff \( F \in \mathcal{M} \Phi_+^+(l_2(A)) \).

**Proof.** Statements can be shown in a similar way as in the proof of Proposition \( \[4.2\] \). \( \square \)

**Lemma 4.8.** Let \( F \in \mathcal{M} \Phi_+^+(l_2(A)) \). Then \( F + K \in \mathcal{M} \Phi_+^+(l_2(A)) \) for all \( K \in K(l_2(A)) \).

**Proof.** Let \( l_2(A) = M_1 \oplus M_2 \xrightarrow{F} N_1 \oplus N_2 = l_2(A) \) be an \( \mathcal{M} \Phi_+^+ \) decomposition for \( F \). Then \( N_2 \) is finitely generated and \( N_2 \preceq N_1 \). We may assume that \( N_2 \preceq L_n \), \( L_n = N_2 \oplus P \) and \( M_2 = L_n^1 \oplus P \) for some \( n \in \mathbb{N} \) and \( P \) finitely generated. Moreover, we may choose an \( n \) big enough s.t. \( \| q_n K \| < \| F_1^{-1} \|^{-1} \). Then we may proceed as in the proof of \( \[2.7.13\] \) and use that \( N_2 \preceq N_1 \) to deduce the lemma. \( \square \)

As regards \( \[2\] \), we need to slightly reformulate some definitions and results from that paper when we consider the nonadjointable case.

**Definition 4.9.** We set \( \widehat{\mathcal{M} \Phi}_+(H_A) \) to be as the set \( \widehat{\mathcal{M} \Phi}_+(H_A) \) in \( [?] \), but we demand that \( R(PF_{|R(P)}) \) should be complementable in \( R(P) \), instead of the adjointability of \( P \).
Recall from [3] that $P(l_2(A))$ denote the set of projections, not necessarily adjointable, with finitely generated kernel. Put

$$\sigma_{e_0}^{A}(F) = \{ \alpha \in Z(A) \mid (F - \alpha I) \notin \overline{\hat{\mathcal{M}}\Phi_+}(l_2(A)) \}.$$ 

Then we have the following non adjointability version of [?, Theorem 2]:

**Theorem 4.10.** For $F \in B(l_2(A))$ we have

$$\sigma_{e_0}^{A}(F) = \cap \{ \sigma_{e_0}^{A}(P F |_{R(P)}) \mid P \in P(l_2(A)) \}$$

where $\sigma_{e_0}^{A}(P F |_{R(P)}) = \{ \alpha \in Z(A) \mid (P F - \alpha I) |_{R(P)} \}$ is bounded below on $R(P)$ or that $R(PF - \alpha P)$ is complementable in $R(P)$. 

**Proof.** If $\alpha \notin \sigma_{e_0}^{A}(P F |_{R(P)})$ for some $P \in P(l_2(A))$, then $(P F - \alpha I) |_{R(P)}$ is bounded below and $R(PF - \alpha P)$ is complementable in $R(P)$. Hence we may proceed as in the proof of the [3, Theorem 10], to deduce that $F - \alpha I \notin \overline{\hat{\mathcal{M}}\Phi_+}(l_2(A))$. Conversely, if $\alpha \in Z(A) \setminus \sigma_{e_0}^{A}(F)$, then by the proof of [3, Theorem 10] we obtain a decomposition

$$l_2(A) = V^{-1}(M_2) \oplus N_2 = V^{-1}(M_2) \oplus N'_2 \oplus N'_2 = N \oplus N'_2$$

and $N'_2 \cong N_1$, $N_2 = N'_2 \oplus N'_2$, $U, V$ are isomorphism, $N_1$ is finitely generated and $(F - \alpha I) |_{N}$ maps $N$ isomorphically onto $V^{-1}(M_2)$. If we let, as in that proof, $P$ be the projection ont $N$ along $N'_2$, then $R_{V^{-1}(M_2) \oplus N'_2}$ is an isomorphism onto $N$. Set $\tilde{N} = P(V^{-1}(M_2))$, $\tilde{N} = P(N'_2)$. We have then $N = \tilde{N} = \tilde{N}$. Hence $P(F - \alpha I) |_{N}$ is an isomorphism onto $\tilde{N}$ which is complementable in $\tilde{N} = R(P)$, so $\alpha \notin \sigma_{e_0}^{A}(P F |_{R(P)})$. \hfill \Box

**Remark:** It can be shown that $\overline{\hat{\mathcal{M}}\Phi_+}(l_2(A))$ is open.

Set now $\overline{\hat{\mathcal{M}}\Phi_+}(l_2(A))$ to be the set as $\overline{\hat{\mathcal{M}}\Phi_+}(H_A)$ in [3], only we do not demand the adjointability of the projection $P$ onto $M'_2 \oplus N$ along $N'_2$, but we require that $R(P)$ splits into $R(P) = \tilde{N} \oplus \tilde{N}$ s.t. $PG |_{\tilde{N}_2}$ is an isomorphism from $\tilde{N}$ onto $R(P)$. Then we put

$$\sigma_{e_0}^{A}(G) = \{ \alpha \in Z(A) \mid (G - \alpha I) \notin \overline{\hat{\mathcal{M}}\Phi_+}(l_2(A)) \}$$

and reach to the following non adjointable analogue of [3, Theorem 11].

**Theorem 4.11.** For $G \in B(l_2(A))$ we have

$$\sigma_{e_0}^{A}(G) = \cap \{ \sigma_{e_0}^{A}(PG |_{R(P)}) \mid P \in P(l_2(A)) \}$$

where $\sigma_{e_0}^{A}(PG |_{R(P)}) = \{ \alpha \in Z(A) \mid R(P) \}$ does not split into the decomposition $R(P) = \tilde{N} \oplus \tilde{N}$ where $PG |_{\tilde{N}_2}$ is an isomorphism onto $R(P)$. 

**Proof.** If $\alpha \notin \sigma_{e_0}^{A}(PG |_{R(P)})$ for some $P \in P(l_2(A))$, then $R(P) = \tilde{N} \oplus \tilde{N}$ for some closed submodules $\tilde{N}, \tilde{N}$ or $R(P)$ s.t. $(PG - \alpha I)$ is an isomorphism onto $R(P)$.
Letting $\tilde{N}$ play the role of $N(PD - \alpha I)$ in the proof of [5, Theorem 11], we may proceed in the same way as in that proof to conclude that $G - \alpha I \in \widehat{M\Phi}_+(l_2(A))$. On the other hand, if $\alpha \in Z(\mathcal{A}) \setminus \sigma_{\text{rad}}(G)$, then $G - \alpha I \in \widehat{M\Phi}_-(l_2(A))$. As in the proof of [5, Theorem 11] (and using the same notation) we may consider the projection $P$ onto $M'_1 \oplus N$ along $N'_2$ and obtain that $P(G - \alpha I)|_{M'_1}$ is an isomorphism onto $M'_1 \oplus V$. 

\textbf{Remark 4.12.} Similarly as for $\widehat{M\Phi}_+(l_2(A))$, one can show that $\widehat{M\Phi}_-(l_2(A))$ is open.

### 5. On semi-$\mathcal{A}$-$\mathcal{B}$-Fredholm Operators

\textbf{Lemma 5.1.} Let $F \in B^n(M)$ where $M$ is a Hilbert $C^*$-module and suppose that $\text{Im} F$ is closed. Then

a) $F \in M\Phi_+(M)$, iff ker $F$ is finitely generated.

b) $F \in M\Phi_-(M)$, iff $\text{Im} F^\perp$ is finitely generated.

\textbf{Proof.} a) Let $M = M_1 \oplus M_2 \xrightarrow{F} M'_1 \oplus M'_2 = M$ be an $M\Phi_+$ decomposition for $F$. By the arguments from the proof of [9, Proposition 3.6.8], it is not hard to see that ker $F \subseteq M_2$. Now, by [9, Theorem 2.3.3], ker $F$ is orthogonally complementable in $M$, hence in $M_2$, as ker $F \subseteq M_2$. Since $M_2$ is finitely, it follows that ker $F$ is finitely generated, being a direct summand in $M_2$. Conversely, if ker $F$ is finitely generated, then

$$H_\mathcal{A} = \ker F^\perp \oplus \ker F \xrightarrow{F} \text{Im} F \oplus \text{Im} F^\perp = H_\mathcal{A}$$

is an $M\Phi_+$ decomposition for $F$. (Here we use that $\text{Im} F$ is closed.)

b) This can be shown by passing to the adjoints and using a). Use that $\text{Im} F^*$ is closed if and only if $\text{Im} F$ is closed by the proof of [9, Theorem 2.3.3] part ii). Moreover, $F \in M\Phi_-(M)$ iff $F^* \in M\Phi_+(M)$ by [4, Corollary 2.11] and $\text{Im} F^\perp = \text{ker} F^*$. 

\textbf{Definition 5.2.} Let $F \in B^n(H_\mathcal{A})$. Then $F$ is said to be upper semi-$\mathcal{A}$-$\mathcal{B}$-Fredholm if the following holds: 1) $\text{Im} F^m$ is closed for all $m$ 2) There exists an $n$ s.t. $F|_{\text{Im} F^n}$ upper semi-$\mathcal{A}$-Fredholm.

Similarly, $F$ is said to be lower semi-$\mathcal{A}$-$\mathcal{B}$-Fredholm if 1) and 2) hold, only in this case we assume in 2) that $F|_{\text{Im} F^n}$ is lower semi-Fredholm. Finally, if $F|_{\text{Im} F^n}$ is $\mathcal{A}$-Fredholm, we say that $F$ is $\mathcal{A}$-$\mathcal{B}$-Fredholm.

\textbf{Proposition 5.3.} If $F$ is upper semi-$\mathcal{A}$-$\mathcal{B}$-Fredholm (respectively lower semi-$\mathcal{A}$-$\mathcal{B}$-Fredholm), then $F|_{\text{Im} F^n}$ is upper semi-$\mathcal{A}$-Fredholm (respectively lower semi-$\mathcal{A}$-Fredholm) for all $m \geq n$. Moreover, if $F$ is $\mathcal{A}$-$\mathcal{B}$-Fredholm and $\text{Im} F^n \cong H_\mathcal{A}$, then $\text{Im} F^m \cong H_\mathcal{A}$, for all $m \geq n$, $F|_{\text{Im} F^n}$ is $\mathcal{A}$ Fredholm for all $m \geq n$ and index $F|_{\text{Im} F^n} = \text{for all } m \geq n$.

\textbf{Proof.} We will prove this by induction. Since $\text{Im} F^{n+1} = \text{Im} F|_{\text{Im} F^n}$ and $\text{Im} F^{n+1}$ is closed by assumption, by [9, Theorem 2.3.3] applied to the operator $F|_{\text{Im} F^n}$, we
deduce that \( \ker F|_{Im F^n} \) and \( Im F^{n+1} \) are orthogonally complementable in \( Im F^n \). Namely, by [9] Theorem 2.3.3 applied to \( F^n \) we have that \( Im F^n \) is orthogonally complementable in \( H_A \), as \( Im F^n \) is closed. Hence \( F|_{Im F^n} \in B^a(Im F^n) \) so we can indeed apply [9] Theorem 2.3.3 on \( F|_{Im F^n} \). If \( F|_{Im F^n} \) is upper semi-\( \mathcal{A} \)-Fredholm operator, by Lemma 5.1 we have that \( \ker F|_{Im F^n} = \ker F \cap Im F^n \) is finitely generated, as \( Im F^n \) is closed. If \( F|_{Im F^n} \) is lower semi-\( \mathcal{A} \)-Fredholm, then again by Lemma 5.1 if we let \( R \) denote the orthogonal complement of \( Im F^{n-1} \) in \( Im F^n \), we get that \( R \) is finitely generated.

Consider now the operator \( F|_{Im F^{n+1}} \). Again, \( Im(F|_{Im F^{n+1}}) = Im F^{n+2} \) is closed by assumption, so by the same arguments as above we may apply [9] Theorem 2.3.3 on \( F|_{Im F^{n+1}} \) to deduce that \( \ker F|_{Im F^{n+1}} = \ker F \cap Im F^{n+1} \) is orthogonally complementable in \( Im F^{n+1} \). Since \( Im F^{n+1} \) is orthogonally complementable in \( H_A \), so is \( \ker F \cap Im F^{n+1} \) as well. Now, since we have \( \ker F \cap Im F^{n+1} \cap Im F^n \), it follows that \( \ker F \cap Im F^{n+1} \oplus M = \ker F \cap Im F^n \), where \( M = (\ker F \cap Im F^n) \cap ((\ker F \cap Im F^{n+1})) \). Since \( \ker F \cap Im F^n \), when \( F|_{Im F^n} \) is upper semi-\( \mathcal{A} \)-Fredholm is finitely generated, it follows that \( \ker F \cap Im F^{n+1} \) is finitely generated being a direct summand in \( \ker F \cap Im F^n \). Thus by Lemma 5.1 \( F|_{Im F^{n+1}} \) is upper semi-\( \mathcal{A} \)-Fredholm, when \( F|_{Im F^n} \) is so. Next, again by the same arguments as earlier we get that \( Im F^{n+2} \oplus X = Im F^{n+1} \) for some closed submodule \( X \) (using that \( Im(F|_{Im F^{n+1}}) = Im F^{n+2} \) is closed). By the proof of Proposition 3.2, replacing by \( F \) and \( D \) by \( F|_{Im F^n} \) we obtain that \( R \cong S(X) \oplus M \) where \( S \) is an isomorphism. (recall that \( Im F^{n+1} \oplus R = Im F^n \))

If \( F|_{Im F^n} \) is lower semi-\( \mathcal{A} \)-Fredholm, then \( R \) is finitely generated, as we have seen. Hence \( X \) must be finitely generated also. Thus \( F|_{Im F^n} \) is lower semi-\( \mathcal{A} \)-Fredholm in this case by Lemma 5.1. Finally, if \( F|_{Im F^n} \) is \( \mathcal{A} \)-Fredholm, then by Lemma 5.1 both \( \ker F|_{Im F^n} = \ker F \cap Im F^n \) and the orthogonal complement of \( Im F^n \) in \( Im F^n \) are finitely generated. Thus \( Im F^n = Im F^{n+1} \oplus R' \) for some finitely generated closed submodule \( R' \). Hence, if \( H_A \cong Im F^n \), by Dupre-Filmore theorem \( Im F^{n+1} \cong H_A \) as well. By the same arguments as above we can deduce that both \( \ker F|_{Im F^{n+1}} \) and the orthogonal complement of \( Im F^{n+2} \) in \( Im F^{n+1} \) are finitely generated, as both \( \ker F|_{Im F^n} \) and \( R' \) are so. Hence \( F|_{Im F^{n+1}} \) is \( \mathcal{A} \)-Fredholm and since \( Im F^{n+1} \cong H_A \), by [9] Theorem 2.7.9] the index of \( F|_{Im F^{n+1}} \) is well-defined. If we let \( X' \) denote the orthogonal complement of \( Im F^{n+2} \) in \( Im F^{n+1} \) and \( M' \) denote the orthogonal complement of \( \ker F \cap Im F^{n+1} \) in \( \ker F \cap Im F^n \), by the same arguments as earlier we get that \( R' \cong X' \oplus M' \). Hence we get \( index F|_{Im F^{n+1}} = [\ker F \cap Im F^{n+1}] - [X'] = [\ker F \cap Im F^{n+1}] + [M'] - [X'] - [M'] = [\ker F \cap Im F^n] - [R'] = index F|_{Im F^n} \). \( \square \)

For an \( \mathcal{A} \)-\( \mathcal{B} \)-Fredholm operator \( F \), we set \( index F = index F|_{Im F^n} \), where \( n \) is as in the Definition 5.2 above.

**Lemma 5.4.** Let \( F \in \mathcal{M}\Phi(H_A) \), let \( P \in B(H_A) \) s.t. \( P \) is the projection and \( N(P) \) is finitely generated. Then \( PF|_{\mathcal{R}(P)} \in \mathcal{M}\Phi(R(P)) \) and \( index PF|_{\mathcal{R}(P)} = index F \).

**Proof.** From [5] Lemma 1], we already know that \( PF|_{\mathcal{R}(P)} \in \mathcal{M}\Phi(R(P)) \). If remains to show that \( index PF|_{\mathcal{R}(P)} = index F \). Now, since \( P \in \mathcal{M}\Phi(H_A) \), by [9] Lemma 2.7.11], \( index PF = index P + index F + index P = index F \), as \( index P = \)
0. By the proof of \[9\] Lemma 1, there exists decompositia \( R(P) = P(M) \oplus \tilde{N} \overset{PF}{\longrightarrow} M' \oplus \tilde{N}' = R(P) \) w.r.t. which \( PF \) has the matrix \[
\begin{pmatrix}
(PF)_1 & (PF)_2 \\
0 & (PF)_4
\end{pmatrix}
\]
where \((PF)_1\) is an isomorphism, \(\tilde{N}, \tilde{N}'\) are finitely generated. In addition \( P \) has the matrix \[
\begin{pmatrix}
P_1 & P_2 \\
0 & P_4
\end{pmatrix}
\], w.r.t. the decomposition

\[
H_A = M \oplus N \longrightarrow P(M) \oplus (\tilde{N} \oplus N(P)) = H_A
\]

where \( P_1 \) is an isomorphism and \( N \) is finitely generated. Moreover,

\[
H_A = M \oplus N \overset{PF}{\longrightarrow} M' \oplus N' = H_A
\]
is an \( \mathcal{M} \Phi_\cdot \) decomposition for \( PF \) and \( N' \cong \tilde{N}' \oplus N(P). \) Since \( \text{index}PF = \text{index}F, \) it follows that \([N] - [N'] = \text{index}F \) in \( K(A). \) Next, it is easily seen, by diagonalizing the matrix \[
\begin{pmatrix}
P_1 & P_2 \\
0 & P_4
\end{pmatrix}
\]
as in the proof of \[9\] Lemma 2.7.10 that \([N] - [\tilde{N}] - [N(P)] = [N] - [\tilde{N} \oplus N(P)] = \text{index}P = 0. \) Similarly, by diagonalizing the matrix \[
\begin{pmatrix}
(PF)_1 & (PF)_2 \\
0 & (PF)_4
\end{pmatrix}
\]
we obtain that \( \text{index}(PF_{|R(P)}) = [\tilde{N}] - [N'] \) and \( [\tilde{N}] + [N(P)] = [N']. \) Combining all this together, we obtain \( \text{index}(PF_{|R(P)}) = [\tilde{N}] - [N'] = [\tilde{N}+N(P)] - [\tilde{N}]-[N(P)] = [\tilde{N}\oplus N(P)] - [\tilde{N}]-[N(P)] = [N] - [N'] = \text{index}F. \)

**Theorem 5.5.** Let \( T \) be an \( A-B\)-Fredholm operator on \( H_A, \) and suppose that \( m \) such that \( T_{|mT^m} \) is \( A\)-Fredholm and \( \text{Im} T^m \) is closed for all \( n \geq m. \) Let \( F \) be a finite rank operator (that is \( \text{Im} F \) is finitely generated) and suppose that \( \text{Im}(T+F)^n \) is closed for all \( n \geq m. \) Finally assume that \( \text{Im} T^m \cong H_A \) and that \( \text{Im}(F), T^m(\text{ker} F), T^m(\text{ker} \tilde{F}^\perp), (T+F)^m(\text{ker} \tilde{F}^\perp) \) are closed, where \( \tilde{F} = (T+F)^m - T^m. \) Then \( T+F \) is an \( A-B\)-Fredholm operator and \( \text{index}T+F = \text{index}T. \)

**Proof.** Observe first that since \( \tilde{F} \in B^m(H_A) \) and \( \text{Im} \tilde{F} \) is closed by assumption, we have that \( \text{ker} \tilde{F} \) is orthogonally complementable in \( H_A \) by \[9\] Theorem 2.3.3. Hence \( T^m_{|_{\ker \tilde{F}}} \) is adjointable. Since \( T^m(\text{ker} \tilde{F}) \) is closed by assumption, again by \[9\] Theorem 2.3.3 we have that \( T^m(\text{ker} \tilde{F}) \) is orthogonally complementable in \( H_A. \) As \( T^m(\text{ker} \tilde{F}) \subseteq \text{Im} T^m \cap \text{Im}(T+F)^m, \) it is easy to see that \( \text{Im} T^m = T^m(\text{ker} \tilde{F}) \oplus N, \text{Im}(T+F)^m = T^m(\text{ker} \tilde{F}) \oplus N' \) for some closed submodules \( N, N'. \) Now, since \( \text{Im} \tilde{F} \) is finitely generated, it follows that \( \ker \tilde{F}^\perp \) is finitely generated also, as \( \tilde{F}_{|_{\ker \tilde{F}^\perp}} \) is an isomorphism onto \( \text{Im} \tilde{F}. \) Moreover, \( \text{Im} T^m = T^m(\text{ker} \tilde{F}) + T^m(\text{ker} \tilde{F}^\perp), \text{Im}(T+F)^m = T^m(\text{ker} \tilde{F}) + (T+F)^m(\text{ker} \tilde{F}^\perp). \)

Let \( Q \) denote the orthogonal projection onto \( T^m(\text{ker} \tilde{F})^\perp. \) It is clear then that \( N = Q(\text{Im} T^m) = Q(T^m(\text{ker} \tilde{F}^\perp)) \) and \( N' = Q(\text{Im}(T+F)^m) = Q((T+F)^m(\text{ker} \tilde{F}^\perp)). \) As \( \ker \tilde{F}^\perp \) is finitely generated, it follows that \( N, N' \) are finitely generated also. Since \( T_{|mT^m} \) is \( A\)-Fredholm, by previous lemma it follows that \( \cap T^m_{|_{\ker \tilde{F}}} \) is \( A\)-Fredholm, where \( \cap \) denotes the orthogonal projection onto \( T^m(\text{ker} \tilde{F})^\perp \) along \( N. \) But, since \( T^m(\text{ker} \tilde{F})^\perp = N \oplus \text{Im} T^m_{|_{\ker \tilde{F}^\perp}}, \) \( (\text{Im} T^m \) is orthogonally complementable
again by [9, Theorem 2.3.3]), if we let $P$ denote the orthogonal projection onto $T^m(\ker \bar{F})$ along $T^m(\ker \bar{F})^\perp$, then $PT_{|T^m(\ker \bar{F})}$ is an $A$-Fredholm operator on $T^m(\ker \bar{F})$, as $PT_{|T^m(\ker \bar{F})} = \cap T_{|T^m(\ker \bar{F})}$. By previous lemma, since $ImT^m \cong H_A$ by assumption, it follows that $indexT = indexT_{|ImT^m} = indexPT_{|T^m(\ker \bar{F})}$. Now since $ImT^m \cong H_A$, $ImT^m = T^m(\ker \bar{F}) \oplus N$ and $N$ is finitely generated, by Dupre Filmore theorem it follows easily that $T^m(\ker \bar{F}) \cong H_A$. Since $PF_{|T^m(\ker \bar{F})} \in K(T^m(\ker \bar{F}))$, it follows from [9, Lemma 2.7.13] that $P(T + F)_{|T^m(\ker \bar{F})}$ is an $A$-Fredholm operator on $T^m(\ker \bar{F})$, and $indexPT_{|T^m(\ker \bar{F})} = indexP(T + F)_{|T^m(\ker \bar{F})}$.

But $Im(T + F)^m = T^m(\ker \bar{F}) \oplus N'$ where $N'$ is finitely generated. Hence $P(T + F)_{|T^m(\ker \bar{F})} = \bar{\cap} T_{|T^m(\ker \bar{F})}$ where $\bar{\cap}$ denotes the orthogonal projection onto $T^m(\ker \bar{F})$ along $N'$, as $(T + F)(T^m(\ker \bar{F})) = (T + F)^m(\ker \bar{F}) \subseteq Im(T + F)^m \subseteq Im(T + F)^m$. In addition, since $N'$ is finitely generated and $T^m(\ker \bar{F}) \cong H_A$, by Kasparov stabilization theorem, it follows that $Im(T + F)^m \cong H_A$. By previous lemma, since $\bar{\cap} T_{|T^m(\ker \bar{F})}$ is an $A$-Fredholm operator on $T^m(\ker \bar{F})$, $Im(T + F)^m \cong H_A$ and $N'$ is finitely generated, it follows that $(T + F)_{|Im(T + F)^m}$ is $A$-Fredholm and $index(T + F) = index(T + F)_{|Im(T + F)^m} = index(\bar{\cap}(T + F))_{|T^m(\ker \bar{F})}$. □

Remark 5.6. Proposition 5.3 hold even if $ImF^n$ is not isomorphic to $H_A$ because $ImF^n$ are countably generated being direct summand in $H_A$ by [9, Theorem 2.3.3] Namely, if $M$ a countably generated Hilbert $C^*$-module, then by Kasparov stabilization theorem, $M \oplus H_A \cong H_A$. Given an operator $F \in B^a(M)$, we may consider the induced operator $F' \in B^a(M \oplus H_A)$ given by the operator matrix

$$
\begin{bmatrix}
F & 0 \\
0 & I
\end{bmatrix}
$$

It is clear then that if $M = M_1 \oplus N_1 \xrightarrow{F} M_2 \oplus N_2 = M$ is a decomposition w.r.t. which $F$ has the matrix $\begin{bmatrix} F_1 & 0 \\
0 & F_4 \end{bmatrix}$ where $F_1$ is an isomorphism, then $F'$ has the matrix $\begin{bmatrix} F'_1 & 0 \\
0 & F'_4 \end{bmatrix}$ w.r.t. the decomposition.

$$
M \oplus H_A = (M_1 \oplus H_A) \oplus (N_1 \oplus \{0\}) \xrightarrow{F'} (M_2 \oplus H_A) \oplus (N_2 \oplus \{0\}) = M \oplus H_A
$$

where $F'$ is an isomorphism. It follows then that any semi-Fredholm decomposition for $F$ gives a rise in a natural way to a semi-Fredholm decomposition of $F'$. Moreover, $F'$ can be viewed as an operator in $B^a(H_A)$ as $M \oplus H_A \cong H_A$. It follows easily then that $indexF$ is well defined as $indexF'$ is so, (when $F \in \mathcal{MF}(M)$) and in this case $indexF = indexF'$. Thus [9, Theorem 2.7.9] holds for $F$. Similarly [9, Lemma 2.7.11], [4, Lemma 2.16], [4, Lemma 2.17] also hold for $F$.

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E-mail address: stefan.iv10@outlook.com