On multiple solutions to a family of nonlinear elliptic systems in divergence form coupled with an incompressibility constraint

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ON MULTIPLE SOLUTIONS TO A FAMILY OF NONLINEAR ELLIPTIC SYSTEMS IN DIVERGENCE FORM COUPLED WITH AN INCOMPRESSIBILITY CONSTRAINT

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Abstract. The aim of this paper is to prove the existence of multiple solutions for a family of nonlinear elliptic systems in divergence form coupled with a pointwise gradient constraint:

\[
\begin{cases}
\text{div} \{ A(|x|, |u|^2, |\nabla u|^2)\nabla u \} + B(|x|, |u|^2, |\nabla u|^2)u = \text{div} \{ \mathcal{P}(x) [\text{cof} \ \nabla u] \} & \text{in } \Omega, \\
\det \nabla u = 1 & \text{in } \Omega, \\
u = \varphi & \text{on } \partial \Omega,
\end{cases}
\]

where \( \Omega \subset \mathbb{R}^n \) (\( n \geq 2 \)) is a bounded domain, \( u = (u_1, \ldots, u_n) \) is a vector-map and \( \varphi \) is a prescribed boundary condition. Moreover \( \mathcal{P} \) is a hydrostatic pressure associated with the constraint \( \det \nabla u \equiv 1 \) and \( A = A(|x|, |u|^2, |\nabla u|^2), B = B(|x|, |u|^2, |\nabla u|^2) \) are sufficiently regular scalar-valued functions satisfying suitable growths at infinity. The system arises in diverse areas, e.g., in continuum mechanics and nonlinear elasticity, as well as geometric function theory to name a few and a clear understanding of the form and structure of the solutions set is of great significance. The geometric type of solutions constructed here draws upon intimate links with the Lie group \( \text{SO}(n) \), its Lie exponential and the multi-dimensional curl operator acting on certain vector fields. Most notably a discriminant type quantity \( \Delta = \Delta(A, B) \), prompting from the PDE, will be shown to have a decisive role on the structure and multiplicity of these solutions.

1. Introduction

This paper is motivated by questions on the existence and multiplicity of solutions to the following family of nonlinear elliptic systems in divergence form coupled with a pointwise gradient constraint:

\[
\begin{cases}
\text{div} A(x, u, \nabla u) + B(x, u, \nabla u) = \text{div} \{ \mathcal{P}(x) [\text{cof} \ \nabla u] \} & \text{in } \Omega, \\
\det \nabla u = 1 & \text{in } \Omega, \\
u = \varphi & \text{on } \partial \Omega.
\end{cases}
\]  

(1.1)

Here \( \Omega \subset \mathbb{R}^n \) (with \( n \geq 2 \)) is a bounded domain having a sufficiently smooth boundary \( \partial \Omega \), \( A = A(x, u, \nabla u) \) and \( B = B(x, u, \nabla u) \) are sufficiently regular \( n \times n \) and \( n \times 1 \) matrix fields respectively, \( u = (u_1, \ldots, u_n) \) is an unknown vector-map defined on \( \Omega \) with \( \nabla u = [\partial u_i/\partial x_j : 1 \leq i, j \leq n] \) its gradient field, that is required to satisfy the pointwise incompressibility constraint \( \det \nabla u \equiv 1 \), and \( \text{cof} \ \nabla u \) denotes the cofactor matrix of \( \nabla u \). In (1.1) \( \mathcal{P} \) is an a priori unknown scalar function (technically the Lagrange multiplier but also known as the hydrostatic pressure). Furthermore \( \varphi \) is a prescribed boundary

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condition and the divergence operator acts on the rows of the two matrix fields on the left and right respectively.

The presence of the term $\text{div}\{\mathcal{P}[\text{cof} \nabla u]\}$ is linked to the incompressibility constraint which makes the problem much harder compared to the unconstrained case where one typically either has no condition or an orientation preserving (pointwise positivity) condition on the Jacobian determinant. Note that in the unconstrained case the expression on the right in the first line is zero (or alternatively constant $\mathcal{P}$).

This system arises in various fields ranging from nonlinear elasticity and continuum mechanics to geometric function theory (see [1, 3, 12, 16, 24] and the references therein) and the fundamental problem here is to establish the existence of solution pairs $(u, \mathcal{P})$ subject to prescribed Dirichlet (or what is often called in the elasticity context as pure displacement) boundary conditions $u \equiv \varphi$ on $\partial \Omega$. For background and motivation see [1, 3, 6, 12, 26, 36, 43, 44, 45] and for further studies and works in this and closely related directions see [2, 4, 7, 9, 11, 13, 18, 24, 25, 28, 29, 30, 31, 32, 37].

For the sake of clarity, by a solution to the system (1.1) in this paper, we mean a pair $(u, \mathcal{P})$ where the vector-map $u = (u_1, \ldots, u_n)$ is of class $C^2(\Omega, \mathbb{R}^n) \cap C(\Omega, \mathbb{R}^n)$, $\mathcal{P}$ is of class $C^1(\Omega) \cap C(\Omega)$ and the pair satisfy the system (1.1) in the pointwise (classical) sense. If the choice of $\mathcal{P}$ is clear from the context, or when no explicit reference is needed, we often abbreviate by saying that $u$ is a solution.

In nonlinear elasticity where a form of (1.1) is encountered, the system represents the equilibrium equations of an incompressible material occupying the region $\Omega \subset \mathbb{R}^n$ ($n = 2$ or 3) in its reference (undeformed) configuration. Solutions here are incompressible deformations for which the body under the action of the external body forces and the prescribed displacement boundary conditions is at rest (or equilibrium) and are thus of great physical significance. Note that in the hyperelastic case, the system additionally represent the Euler-Lagrange equations associated with the constrained total elastic energy and solutions in this context are equilibria as well as energy extremisers. (See below for more and Section 7. See also [3, 4, 11, 12, 14, 24, 25].)

Now in order to further motivate (1.1) and discuss the above in more detail consider a twice continuously differentiable stored energy density $W = W(x, u, \zeta)$ with $x \in \Omega$, $u \in \mathbb{R}^n$ and $\zeta \in \mathbb{R}^{n \times n}$ [or $\zeta$ in some fixed neighbourhood of the linear group $\text{SL}(n)$]. For any incompressible deformation $u$ of $\Omega$, i.e., any weakly differentiable map $u = (u_1, \ldots, u_n)$ satisfying $\det \nabla u \equiv 1$ a.e. in $\Omega$, let its total elastic energy be given by the integral

$$
E[u] = \int_{\Omega} W(x, u(x), \nabla u(x)) \, dx.
$$

Incorporating boundary conditions and the growth of $W$ prompts one to introduce the space $\mathcal{A}^p(\Omega) = \{u \in \mathcal{W}^{1,p}(\Omega, \mathbb{R}^n)| \det \nabla u = 1 \text{ a.e.} \text{ in } \Omega, \text{ with } u = \varphi \text{ on } \partial \Omega \}$ for suitable choice of $1 \leq p < \infty$. Here boundary values are interpreted in the sense of traces of Sobolev functions. We hereafter refer to $\mathcal{A}^p(\Omega)$ as the space of admissible incompressible maps or deformations.

The Euler-Lagrange equation associated with the energy integral (1.2) over the space $\mathcal{A}^p(\Omega)$ can be formally derived (see Section 7 at the end) and seen to be given by the nonlinear system,

$$
- \text{div}\{W_\zeta(x, u, \nabla u) - \mathcal{P}[\text{cof} \nabla u]\} + W_u(x, u, \nabla u) = 0,
$$

(1.3)
where \( W_\zeta = [\partial W/\partial \zeta_{ij} : 1 \leq i, j \leq n] \) and \( W_u = [\partial W/\partial u_i : 1 \leq i \leq n] \). This system is evidently in the form \((1.1)\) with \( A = W_\zeta \) and \( B = -W_u \). Note however that \((1.1)\) is more general than \((1.3)\) in that there need not be any inherent relations between \( A \) and \( B \) in \((1.1)\) whereas in the variational case leading to \((1.3)\) we have \( A_u = -B_\zeta \) (specifically, \( \partial A_{ij}/\partial u_k = -\partial B_k/\partial \zeta_{ij} \) with \( 1 \leq i, j, k \leq n \)). In passing let us also note that by using the Piola identity (see, e.g., \([3, 12]\)) and recalling the assumed regularity of solution pairs \((u, \mathcal{P})\) we can write \((1.3)\) as

\[
- \text{div}\{W_\zeta(x, u, \nabla u)\} + W_u(x, u, \nabla u) + [\text{cof} \nabla u] \nabla \mathcal{P} = 0. \tag{1.4}
\]

The system \((1.3)\) can be independently derived using the Lagrange multiplier method in the context of infinite dimensional differentiable manifold of incompressible maps (cf. \([24]\) for details). An easy inspection here then shows that \((1.3)\) is also the Euler-Lagrange equation associated with the unconstrained energy integral \( \mathcal{E}_{\mathcal{P}} \) incorporating the Lagrange multiplier and the constraint (which we leave the formal verification to the reader) given by

\[
\mathcal{E}_{\mathcal{P}}[u] = \int_\Omega \{W(x, u(x), \nabla u(x)) - \mathcal{P}(x)[\det \nabla u(x) - 1]\} \, dx. \tag{1.5}
\]

Here evidently for any \( u \) in \( \mathcal{A}^p(\Omega) \) we have \( \mathcal{E}_{\mathcal{P}}[u] = \mathcal{E}[u] \). Let us point out that due to the \( a \text{ priori} \) unknown regularity of the pressure field \( \mathcal{P} \), and integrability of the Jacobian determinant \( \det \nabla u \) the unconstrained energy integral \( \mathcal{E}_{\mathcal{P}} \) in \((1.5)\) need not be everywhere well-defined, let alone, continuously Frechet differentiable on Sobolev spaces \( \mathcal{W}^{1,p}(\Omega, \mathbb{R}^n) \). As a result standard tools from critical point theory do not carry over immediately to this setting (cf. \([33, 34, 35]\)) and so for the construction of energy extremisers (or critical points) other approaches and ideas are needed (cf. \([23, 40]\)).

For the sake of this paper we focus on the case where the nonlinearities take the forms

\[
A = A(|x|, |u|^2, |\nabla u|^2) \nabla u \quad \text{and} \quad B = B(|x|, |u|^2, |\nabla u|^2)u \quad \text{with} \quad A = A(r, s, \xi) \quad \text{and} \quad B = B(r, s, \xi)
\]

being sufficiently regular scalar-valued functions. This is called the \textit{isotropic} case. [Note that in the setting of \((1.2)\) the latter amount to \( W_\zeta = A(|x|, |u|^2, |\nabla u|^2) \nabla u \) and \( W_u = -B(|x|, |u|^2, |\nabla u|^2)u \) where by writing \( W(x, u, \nabla u) = F(|x|, |u|^2, |\nabla u|^2) \), it follows that \( A = F_\xi \) and \( B = -F_s \) (and so \( A_s + B_\xi \equiv 0 \)). In the general case however there are no assumptions or relations linking \( A \) and \( B \) and apart from standard regularity and growth (see below) the choices of \( A \) and \( B \) are independent and arbitrary.]

Now in view of the structure assumptions on the nonlinearities in place, the assumed regularity of solution pairs \((u, \mathcal{P})\), and an application of Piola identity, the system in \((1.1)\) can be re-written in the form,

\[
\begin{cases}
\text{div}[A(|x|, |u|^2, |\nabla u|^2) \nabla u] + B(|x|, |u|^2, |\nabla u|^2)u = [\text{cof} \nabla u] \nabla \mathcal{P} & \text{in } \Omega, \\
\det \nabla u = 1 & \text{in } \Omega, \\
u = \varphi & \text{on } \partial \Omega.
\end{cases} \tag{1.6}
\]

Since \( \det \nabla u \equiv 1 \) we have \( \det \text{cof} \nabla u \equiv 1 \) and \( [\text{cof} \nabla u]^{-1} = [\nabla u]^t \) and so we can write the \textit{constrained} system \((1.6)\) in the more tractable gradient form

\[
\Sigma[(u, \mathcal{P}); (A, B)] = \begin{cases}
[\nabla u]^t \{\text{div}[A \nabla u] + Bu\} = \nabla \mathcal{P} & \text{in } \Omega, \\
\det \nabla u = 1 & \text{in } \Omega, \\
u = \varphi & \text{on } \partial \Omega.
\end{cases} \tag{1.7}
\]
where we have written \( A = A(|x|, |u|^2, |\nabla u|^2) \) and \( B = B(|x|, |u|^2, |\nabla u|^2) \) for brevity. It is also convenient to abbreviate the PDE in the first line of (1.7) to \( \mathcal{L}[u; A, B] = \nabla P \) by introducing the differential operator

\[
\mathcal{L}[u; A, B] = [\nabla u]^t \{ \text{div}[A \nabla u] + Bu \} = [\nabla u]^t [\nabla u] A + A[\nabla u]^t \Delta u + B[\nabla u]^t u. \tag{1.8}
\]

Our aim here is to establish the existence of multiple solutions to the nonlinear system (1.7). We confine to the geometric setting where the domain is an \( n \)-annulus, for definiteness, \( \Omega = \{ a < |x| < b \} \) with \( b > a > 0 \) and \( \varphi \) is the identity map. In this context a class of incompressible smooth maps with geometric significance are introduced and shown to lead to an infinitude of solutions. For related works on non-uniqueness in the incompressible setting see [8, 9, 27, 28, 29, 31, 32] and for results on uniqueness see [22, 38] (see also [1, 12, 24, 45]).

The study of multiple solutions to the constrained system \( \Sigma[(u, P); A, B] \), by way of construction, links here to a closely related unconstrained system for a vector-map \( f = (f_1, \ldots, f_d) \), in a set of new variables \( y = (y_1, \ldots, y_N) \), and on a new region \( \mathbb{A}_n \subset \mathbb{R}^N \), given by (see Section 4 for details)

\[
\begin{aligned}
\text{div} \left[ \mathcal{A}_i(y, \nabla f) \nabla f_i \right] &= 0 \quad \text{in} \ \mathbb{A}_n, \\
f &\equiv \mathbf{g} \quad \text{on} \ \partial \mathbb{A}_n \text{D}, \\
\mathcal{A}_i(y, \nabla f) \partial_{\nu} f_i &= 0 \quad \text{on} \ \partial \mathbb{A}_n \text{N}. 
\end{aligned}
\tag{1.9}
\]

Here \( \mathbf{g} = (g_1, \ldots, g_d) \) is a map defined on the so-called Dirichlet part of the boundary \( \partial \mathbb{A}_n \text{D} \) (see below) describing the boundary values of the vector-map \( f \) itself whilst on the Neumann part \( \partial \mathbb{A}_n \text{N} = \partial \mathbb{A}_n \setminus \partial \mathbb{A}_n \text{D} \) (the remainder of \( \partial \mathbb{A}_n \)) \( f \) is free. Additionally

\[
\mathcal{A}_i(y, \nabla f) = y_i^2 \mathcal{A} \left( z, z^2, n + \sum_{j=1}^{d} y_j^d \nabla f_j |^2 \right) \mathcal{J}(y), \quad 1 \leq i \leq d, \tag{1.10}
\]

with \( \mathcal{J}(y) = y_1 \cdots y_d \) and \( z = ||y|| \) denoting the Euclidean 2-norm of the \( N \)-vector \( y \). The existence and multiplicity of solutions to this unconstrained system is discussed in Sections 4-5 and the crucial connection between the two systems proved in Proposition 4.2 and its two corollaries Propositions 4.3 and 4.4. The main existence and multiplicity results of the paper are then presented in Theorem 5.2 and Theorem 6.2. As is apparent from the analysis in Section 5, a discriminant like object \( \Delta = \Delta(u; A, B) \), plays a crucial role in the structure and dimensional parity of solutions. Let us end this introduction by formalising the assumptions on \( A, B \) and fixing some key notation and terminology.

**Assumptions on \( A, B \).** We assume \( A = A(r, s, \xi) \), \( B = B(r, s, \xi) \) to be of class \( \mathcal{C}^1(U) \), where \( U = U[a, b] = [a, b] \times [0, \infty] \times [0, \infty] \) with \( A > 0 \), \( A_\xi \geq 0 \) for all \( (r, s, \xi) \in U \) and that for every compact set \( K \subset [0, \infty] \times [0, \infty] \) there are constants \( c_1 = c_1(K), c_2 = c_2(K) > 0 \) such that \( c_1 |\xi|^p \leq A(r, s, \xi) |\xi| \leq c_2 |\xi|^{p-1} \) for all \( (r, s, \xi) \in U \), \( s \in K \) and \( p > 1 \).

**Notation.** Throughout the paper we write \( |x| = r \) and \( \Theta = \nabla |x| = x|x|^{-1} \). By \( I = I_n \) we denote the \( n \times n \) identity matrix. We write \( J = \sqrt{-I} \) for the \( 2 \times 2 \) skew-symmetric matrix with \( J_{12} = -1 \) and write \( \text{R}[\alpha] = \exp{\alpha J} \) for the \( \text{SO}(2) \) matrix of rotation by angle \( \alpha \in \mathbb{R} \) (in particular \( J = \text{R}[\pi/2] \)). We write \( y = (y_1, \ldots, y_N) \) for the vector of 2-plane radial variables associated with \( x = (x_1, \ldots, x_n) \) defined as follows: when \( n = 2N \) we set \( y_\ell = (x_{2\ell-1}^2 + x_{2\ell}^2)^{1/2} \) for \( 1 \leq \ell \leq N \) and when \( n = 2N - 1 \) we set \( y_\ell \) as before for
1 \leq \ell \leq N-1 \text{ and } y_N = x_n. \text{ For } b > a > 0 \text{ we write } X_n = \{ x \in \mathbb{R}^n : a < |x| < b \} \text{ and set } 
abla A_n = \{ y \in \mathbb{R}_+^n : a \leq \| y \| \leq b \text{ when } n = 2N \text{ and } A_n = \{ y \in \mathbb{R}_+^{n-1} \times \mathbb{R} : a \leq \| y \| \leq b \} \text{ when } n = 2N - 1. \text{ Here } |x| = (x_1^2 + \cdots + x_n^2)^{1/2} \text{ and } \|y\| = (y_1^2 + \cdots + y_N^2)^{1/2} \text{ denote the } 2\text{-norms of the } n\text{-vector } x \text{ and } N\text{-vector } y \text{ respectively. In either case we have } X_n \subset \mathbb{R}^n \text{ and } A_n \subset \mathbb{R}^N. \text{ Finally we write } (\partial A_n)_D = \{ y \in \partial A_n : \| y \| = a \} \cup \{ y \in \partial A_n : \| y \| = b \} \text{ and } (\partial A_n)_N = \partial A_n \setminus (\partial A_n)_D. \text{ Thus here } \partial A_n = (\partial A_n)_D \cup (\partial A_n)_N. \text{ It is often convenient to write } z = \| y \| \text{ and } 1 = 1_D = (1, \ldots, 1) \text{ for the } d\text{-vector whose components are all } 1. \text{ Vector inner product is denoted by } \langle u, v \rangle \text{ and matrix inner product by } E : F = \text{tr}(E^t F). \text{ Finally we use the standard notation for Sobolev spaces } W^{1,p} \text{ (as, e.g., [43]).}

2. The action \mathcal{L}[u; A, B] \text{ and the radial and spherical parts of } u

Given a nowhere vanishing \( u \in W^{1,p}(\Omega, \mathbb{R}^n) \) (i.e., \( u \) non-zero a.e. in \( \Omega \)) we decompose \( u \) into a radial part \( R_u \) and a spherical part \( S_u \) by writing \( R_u = |u| \) and \( S_u = u|u|^{-1} \) respectively. A straightforward calculation then gives the gradients

\[
\nabla R_u = \nabla |u| = \frac{[\nabla u]^t u}{|u|}, \quad \nabla S_u = \nabla (u|u|^{-1}) = \left( I_n - \frac{u}{|u|} \otimes \frac{u}{|u|} \right) \frac{\nabla u}{|u|},
\]

with \( I_n \) the \( n \times n \) identity matrix. Of particular interest below are the two symmetric matrix fields relating to the left and right Cauchy-Green tensors (\([1, 3, 12]\)):

\[
X[u] = [\nabla u]^t [\nabla u] - I_n, \quad Y[u] = [\nabla u]^t [\nabla u]^t - I_n.
\]

Clearly these fields vanish iff \( \nabla u \) is an orthogonal matrix a.e. in \( \Omega \) and so as such serve as a measure of the closeness of \( \nabla u \) to the group \( O(n) \). Note also that \( |S_u| = 1 \) and so \( [\nabla S_u]^t S_u = 0 \) whilst \( R_u^2 |\nabla R_u|^2 = \langle Y[u], u \rangle + R_u^2 \). Further conclusions are as below.

**Lemma 2.1.** Suppose \( R_u, S_u \) are as in \((2.1)\) and \( X[u], Y[u] \) are as in \((2.2)\). Then the following relations hold:

(i) \( \nabla u = R_u \nabla S_u + S_u \otimes \nabla R_u, \)

(ii) \( [\nabla u]^t u = R_u([\nabla S_u]^t [\nabla S_u] + \nabla R_u \otimes S_u]) S_u = R_u \nabla R_u, \)

(iii) \( X[u] = R_u^2 [\nabla S_u]^t [\nabla S_u] + \nabla R_u \otimes \nabla R_u - I_n, \)

(iv) \( Y[u] = R_u^2 [\nabla S_u]^t [\nabla S_u] + R_u \nabla S_u \nabla R_u \otimes S_u + R_u S_u \otimes \nabla S_u \nabla R_u + |\nabla R_u|^2 S_u \otimes \nabla S_u - I_n, \)

(v) \( |\nabla u|^2 = tr\{[\nabla u]^t [\nabla u]\} = tr\{[\nabla u]^t [\nabla u]^t\} = R_u^2 |\nabla S_u|^2 + |\nabla R_u|^2, \)

(vi) \( X[u] |\nabla u|^2 = 2 R_u (R_u^2 [\nabla S_u]^t [\nabla S_u] |\nabla R_u|^2 + |\nabla R_u|^2 \nabla R_u - \nabla R_u). \)

**Lemma 2.2.** If \( u \) is second order differentiable then with \( R_u, S_u \) as in \((2.1)\) we have:

(i) \( \Delta u = R_u \Delta S_u + 2 \nabla S_u \nabla R_u + R_u \Delta R_u S_u, \)

(ii) \( [\nabla u]^t \Delta u = R_u^2 [\nabla S_u]^t \Delta S_u + R_u [2 \nabla S_u]^t [\nabla S_u] + (S_u, \Delta S_u) I_n |\nabla R_u| + \Delta R_u \nabla R_u, \)

(iii) \( \nabla (|\nabla u|^2) = 2 R_u |\nabla S_u|^2 \nabla R_u + 2 R_u^2 \nabla^2 S_u \nabla S_u + 2 R_u^2 \nabla R_u \nabla R_u. \)

**Proof.** These are all consequences of direct differentiation and routine calculations. \( \Box \)
Lemma 2.3. Suppose $u$ is second order differentiable and $\mathcal{L}[u; A, B]$ is as in (1.8). Then the following relation holds:

$$
\mathcal{L}[u; A, B] - \nabla A = X[u] \nabla A + A[\nabla u^i] \Delta u + B[\nabla u^i] u \tag{2.3}
$$

Here $A_r = A_r(r, s, \xi)$, $A_s = A_s(r, s, \xi)$ and $A_\xi = A_\xi(r, s, \xi)$ denote the respective partial derivatives of $A$ whilst $B = B(r, s, \xi)$. All arguments are at $(r, s, \xi) = (|x|, |u|^2, |\nabla u|^2)$.

Proof. This follows from (1.8) after substituting for $X[u]$ from (2.2) and then rearranging terms.

Generalities on maps with $\mathcal{R}_u = |x|$, $\mathcal{J}_u = Q\Theta$: The class of maps we are interested in here are those whose radial and spherical parts are $\mathcal{R}_u(x) = |x|$, $\mathcal{J}_u(x) = Q(x)\Theta$ respectively. Here $Q$ is an $SO(n)$-valued matrix field whose dependence on the spatial variables $x = (x_1, \ldots, x_n)$ is through the 2-plane radial variables $y = (y_1, \ldots, y_N)$ described earlier. Thus with a slight abuse of notation we hereafter write and think of $Q = Q(y)$ with $y = y(x)$ (see [28, 29]).

We next define the set of $2N$ orthogonal $n$-vectors: $w^i = (0, \ldots, 0, x_{2i-1}, x_{2i}, 0, \ldots, 0)$, $[w^i] = (0, \ldots, 0, -x_{2i}, x_{2i-1}, 0, \ldots, 0)$ for $1 \leq i \leq d$; when $n = 2d$ is even this completes the picture but when $n = 2d + 1$ is odd we set $w^N = (0, \ldots, 0, x_n)$, $[w^N] = (0, \ldots, 0)$. Hence $x = w^1 + \cdots + w^N$, $\langle w^i, w^j \rangle = 0$, $\langle [w^i], [w^j] \rangle = 0$ for $1 \leq i \neq j \leq N$ and $\langle w^i, [w^j] \rangle = 0$ for all $1 \leq i, j \leq N$. Furthermore in relation to the variables $y_1, \ldots, y_N$ introduced earlier we have $y_{\ell} = |w^\ell| = ||[w^\ell]||$ when $1 \leq \ell \leq d$ noting that when $n = 2d + 1$ we have $w^N = (0, \ldots, 0, y_N)$ and $|y_N| = |w^N| = |x_n|$.

Lemma 2.4. For the 2-plane radial variables $y = (y_1, \ldots, y_N)$ we have: $\nabla y_{\ell} = w^\ell / y_{\ell}$, $\langle \nabla y_{\ell}, \nabla y_k \rangle = \delta_{\ell k}$ and $\Delta y_{\ell} = 1 / y_{\ell}$ except for $n = 2d + 1$ where $\Delta y_N = 0$. Here $\nabla, \Delta$ are taken with respect to the $x = (x_1, \ldots, x_n)$ variables.

Proof. These follow by straightforward differentiation and considering the cases corresponding to even and odd $n$ separately.

Lemma 2.5. Assume $u = Q(y_1, \ldots, y_N)x$ with the matrix field $Q$ being of class $C^1$. Then $\nabla \mathcal{J}_u = x/|x|$, $\Delta \mathcal{J}_u = (n - 1)/|x|$ and

$$
\nabla \mathcal{J}_u = \frac{1}{|y|} Q(1_n - \Theta \otimes \Theta) + \sum_{\ell=1}^N \partial_\ell Q \Theta \otimes \nabla y_{\ell}, \quad \nabla \mathcal{J}_u \Delta \mathcal{J}_u = \sum_{\ell=1}^N \langle \nabla y_{\ell}, \Theta \rangle \partial_\ell Q \Theta. \tag{2.4}
$$

Moreover if $Q$ is of class $C^2$ then $u$ is second order differentiable and

$$
\Delta \mathcal{J}_u = \frac{1 - n}{|y|^2} Q \Theta + \frac{1}{|y|} \sum_{\ell=1}^N \left\{ 2[\partial_\ell Q \nabla y_{\ell} - 2(\nabla y_{\ell}, \Theta) \partial_\ell Q \Theta] + \partial_\ell^2 Q \Theta + \Delta y_{\ell} \partial_\ell Q \Theta \right\}. \tag{2.5}
$$

We next prove further identities associated with such maps in line with Lemmas 2.1 and 2.2. Note that $\partial_\ell$ stands for partial differentiation with respect to $y_{\ell}$ whilst $\nabla$ and $\Delta$ as applied to the variables $y = (y_1, \ldots, y_N)$ are all with respect to $x = (x_1, \ldots, x_n)$. 

Lemma 2.6. Assume \( u = Q(y_1, \ldots, y_N)x \) with the matrix field \( Q \) being of class \( \mathcal{C}^1 \). Then the following identities hold:

\[
(i) \quad \nabla u = Q + \sum_{\ell=1}^{N} \partial_\ell Qx \otimes \nabla y_\ell,
\]

\[
(ii) \quad [\nabla u]^i u = \left[ Q^i + \sum_{\ell=1}^{N} \nabla y_\ell \otimes \partial_\ell Qx \right] Qx = x,
\]

\[
(iii) \quad |\nabla u|^2 = \text{tr}\{[\nabla u]^{i}[\nabla u]\} = \text{tr}\{[\nabla u][\nabla u]^i\} = n + \sum_{\ell=1}^{N} \left[ 2\langle Q^i \partial_\ell Qx, \nabla y_\ell \rangle + |\partial_\ell Qx|^2 \right].
\]

Lemma 2.7. Under the assumptions of the previous lemma on \( u \) and with \( X[u], Y[u] \) denoting the matrix fields in (2.2) the following identities hold:

\[
(i) \quad X[u] = \sum_{\ell=1}^{N} \left[ Q^i \partial_\ell Qx \otimes \nabla y_\ell + \nabla y_\ell \otimes Q^i \partial_\ell Qx \right] + \sum_{\ell=1}^{N} \sum_{k=1}^{N} \langle \partial_\ell Qx, \partial_k Qx \rangle \nabla y_\ell \otimes \nabla y_k,
\]

\[
(ii) \quad Y[u] = \sum_{\ell=1}^{N} \left[ Q \nabla y_\ell \otimes \partial_\ell Qx + \partial_\ell Qx \otimes Q \nabla y_\ell + \partial_\ell Qx \otimes \partial_\ell Qx \right],
\]

\[
(iii) \quad X[u] \nabla (|u|^2) = 2 \left[ \sum_{\ell=1}^{N} \langle \nabla y_\ell, x \rangle Q^i \partial_\ell Qx + \sum_{\ell=1}^{N} \sum_{k=1}^{N} \langle \partial_\ell Qx, \partial_k Qx \rangle \langle \nabla y_k, x \rangle \nabla y_\ell \right].
\]

Proof. The identities (i) and (ii) above follow from (iii) and (iv) in Lemma 2.1 and the identities in Lemma 2.5. To conclude (ii) we use the relation \( \langle Q^i \partial_\ell Qx, x \rangle = 0 \) resulting from skew-symmetry. The third identity follows at once by noting \( |u|^2 = R_u = r^2 \).

Lemma 2.8. Assume \( u = Q(y_1, \ldots, y_N)x \) with the matrix field \( Q \) being of class \( \mathcal{C}^2 \). Then \( u \) is second order differentiable and the following identities hold:

\[
(i) \quad \Delta u = \sum_{\ell=1}^{N} \left[ \partial_\ell^2 Qx + \Delta y_\ell \partial_\ell Qx + 2\partial_\ell Q \nabla y_\ell \right],
\]

\[
(ii) \quad [\nabla u]^i \Delta u = \sum_{\ell=1}^{N} \left[ Q^i \partial_\ell^2 Qx + \Delta y_\ell Q^i \partial_\ell Qx + 2Q^i \partial_\ell Q \nabla y_\ell \right]
\]

\[
+ \sum_{\ell=1}^{N} \sum_{k=1}^{N} \left[ \langle \partial_\ell Qx, \partial_k^2 Qx \rangle + \Delta y_k \langle \partial_\ell Qx, \partial_k Qx \rangle + 2\langle \partial_\ell Qx, \partial_k Q \nabla y_k \rangle \right] \nabla y_\ell,
\]

\[
(iii) \quad \nabla (|\nabla u|^2) = 2 \sum_{\ell=1}^{N} \left[ \partial_\ell Q^i Q \nabla y_\ell + \nabla^2 y_\ell Q^i \partial_\ell Qx + \partial_\ell Q^i \partial_\ell Qx \right]
\]

\[
+ \sum_{\ell=1}^{N} \sum_{k=1}^{N} 2 \left[ \langle \partial_\ell Q^i \partial_\ell Q + Q^i \partial_\ell Qx, \nabla y_\ell \rangle + \langle \partial_\ell^2 Qx, \partial_\ell Qx \rangle \right] \nabla y_k.
\]

Proof. For (i) we use the identities in Lemma 2.5 together with the description of the Laplacian given in identity (i) in Lemma 2.2. We then obtain (ii) by pre-multiplying
this with \( \nabla u \) using the description of \( \nabla u \) given by (i) in Lemma 2.6. For (iii) by invoking (iii) in Lemma 2.6 and expanding the gradient directly on each term we have
\[
\nabla \langle Q^t \partial_t Q, \nabla y_t \rangle = \partial_t Q^t Q \nabla y_t + \nabla^2 y_t Q^t \partial_t Q + \sum_{k=1}^N \nabla y_k \otimes \nabla y_t (\partial_k Q^t \partial_t Q + Q^t \partial_t k Q)x, \tag{2.6}
\]
and likewise
\[
\nabla |\partial_t Qx|^2 = \nabla \langle \partial_t Qx, \partial_t Qx \rangle = 2 \partial_t Q^t \partial_t Qx + \sum_{k=1}^N 2 \nabla y_k \otimes \partial^2_{t_k} Qx \partial_t Qx. \tag{2.7}
\]
Putting these together and rearranging terms gives at once the desired conclusion. \( \square \)

3. WHIRLS, MAXIMAL TORI, AND THE BLOCK DIAGONAL \( \text{SO}(n) \)-VALUED MATRIX FIELDS \( Q[f] \)

Returning to the decomposition of \( u \) into its radial and spherical parts, and prompted by symmetry considerations, we now specialise to the class of maps \( u \) whose \( \text{SO}(n) \)-valued matrix field \( Q \) in the spherical part \( S^u \) takes values on a fixed maximal torus \( T \) of \( \text{SO}(n) \) (cf. [29] for more on this). As any two maximal tori on a compact Lie group are conjugate to one another, for our purposes, and without loss of generality, we take the canonical maximal torus \( T=\{\text{diag}(R_1, \ldots, R_d)\} \) for \( n=2d \) and \( T=\{\text{diag}(R_1, \ldots, R_d, 1)\} \) for \( n=2d+1 \). Here \( R_j = R[\alpha_j] \in \text{SO}(2) \) \( (1 \leq j \leq d) \) with \( (\alpha_1, \ldots, \alpha_d) \in \mathbb{R}^d \) (see [21, 46] for background and more on Lie groups and representations).

The implication of this is that we will express \( Q \) as a similar block diagonal matrix with each block described by a suitable angle of rotation function \( f = f_\ell(y) \) \( (1 \leq \ell \leq d) \). Specifically, this leads to the explicit descriptions of the \( \text{SO}(n) \)-valued matrix fields
\[
Q[f](y) = \begin{pmatrix}
R[f_1(y)] & 0 & \ldots & 0 & 0 \\
0 & R[f_2(y)] & \ldots & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \ldots & R[f_{d-1}(y)] & 0 \\
0 & 0 & \ldots & 0 & R[f_d(y)]
\end{pmatrix}, \tag{3.1}
\]
for when \( n=2d \) and
\[
Q[f](y) = \begin{pmatrix}
R[f_1(y)] & 0 & \ldots & 0 & 0 & 0 \\
0 & R[f_2(y)] & \ldots & 0 & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
0 & 0 & \ldots & R[f_{d-1}(y)] & 0 & 0 \\
0 & 0 & \ldots & 0 & R[f_d(y)] & 0 \\
0 & 0 & \ldots & 0 & 0 & 1
\end{pmatrix}, \tag{3.2}
\]
for when \( n=2d+1 \). Hereafter we write \( f = f(y) \) for the vector-map \( f = (f_1, \ldots, f_d) \) and we refer to the resulting map \( u = Q[f](y)x \) as a whirl map or a whirl for simplicity.
It can be seen by direct verification that

\[ Q^t \partial_t Q \nabla y_k = y_k^{-1} \partial_t f_k [w^k] \perp, \quad Q^t \partial_t Q x = \sum_{i=1}^{d} \partial_t f_i [w^i] \perp, \quad (3.3) \]

and likewise

\[ \partial_t Q^t \partial_t Q x = \sum_{i=1}^{d} (\partial_t f_i)^2 w^i, \quad Q^t \partial^2_{\ell k} Q x = \sum_{i=1}^{d} [\partial^2_{\ell k} f_i [w^i] \perp - \partial_t f_i \partial_k f_i w^i], \quad (3.4) \]

(with \(1 \leq \ell \leq N\) and \(1 \leq k \leq N\)). Next using the first identity in (3.4) we have

\[ \langle \nabla y_\ell, \partial_\ell Q^t \partial_\ell Q x \rangle = \left\langle \nabla y_\ell, \sum_{i=1}^{d} (\partial_\ell f_i)^2 w^i \right\rangle, \]

which then upon making note of the inner product relation \(\langle \nabla y_\ell, w^j \rangle = y_\ell \delta_{i\ell}\) leads to

\[ \langle \nabla y_\ell, \partial_\ell Q^t \partial_\ell Q x \rangle = \begin{cases} y_\ell (\partial_\ell f_\ell)^2 & \text{if } 1 \leq \ell \leq d, \\ 0 & \ell = N, \ n \text{ odd} \end{cases}. \quad (3.5) \]

**Lemma 3.1.** Let \( u \) be a whirl as defined above with matrix field \( Q \) of class \( \mathcal{C}^2 \). Then

\[ X[u] \nabla (|\nabla u|^2) = 2 \sum_{k=1}^{N} \sum_{\ell=1}^{N} \left[ \langle \nabla y_k, \partial_\ell Q^t \partial_\ell Q x \rangle + \langle \partial^2_{\ell k} Q x, \partial_\ell Q x \rangle \right] \times \]

\[ \left[ Q^t \partial_\ell Q x + \sum_{i=1}^{N} \langle \partial_\ell Q x, \partial_i Q x \rangle \nabla y_i \right]. \quad (3.6) \]

**Proof.** Since \( \langle Q^t \partial_\ell Q x, \nabla y_\ell \rangle = 0 \) we have \( \nabla (|\nabla u|^2) = \sum_{\ell=1}^{N} \nabla |\partial_\ell Q x|^2 \). Utilising (2.7), pre-multiplying by \( X[u] \) using (i) in Lemma 2.7 and (3.3), (3.4) give the result. \( \square \)

We now turn to formulating the action of the differential operator \( \mathcal{L}[u; A, B] \) on a whirl \( u \) whose radial and spherical parts have the forms \( \mathcal{R}_u(x) = |x|, \mathcal{S}_u(x) = Q[f](y)\Theta. \)
**Proposition 3.1.** Suppose \( u \) is a whirl with matrix field \( Q \) of class \( \mathcal{C}^2 \). The action of the differential operator \( \mathcal{L} \) on \( u \) can be reformulated as

\[
\mathcal{L}[u; A, B] = \nabla A + B x + 2A_t \sum_{k=1}^{N} \sum_{\ell=1}^{N} \left[ (\nabla y_k, \partial_\ell Q^i \partial_\ell Qx) + \langle \partial_\ell^2 Qx, \partial_\ell Qx \rangle \right] \times \tag{3.7}
\]

\[
\times \left[ Q^i \partial_\ell Qx + \sum_{i=1}^{N} \langle \partial_\ell Qx, \partial_i Qx \rangle \nabla y_i \right] + \left[ 2A_s + r^{-1} A_r \right] \sum_{\ell=1}^{N} \left\{ (\nabla y_\ell, x) Q^i \partial_\ell Qx + \sum_{k=1}^{N} \langle \partial_\ell Qx, \partial_k Qx \rangle \langle \nabla y_k, x \rangle \nabla y_\ell \right\}
\]

\[
+ A \sum_{\ell=1}^{N} \left\{ [Q^i \partial_\ell^2 Qx + \Delta y_\ell Q^i \partial_\ell Qx + 2Q^i \partial_\ell Q \nabla y_\ell] + \sum_{k=1}^{N} \left[ (\partial_\ell Qx, \partial_k Qx) + \Delta y_\ell \langle \partial_\ell Qx, \partial_k Qx \rangle + 2\langle \partial_\ell Qx, \partial_k Q \nabla y_k \rangle \right] \nabla y_\ell \right\}.
\]

The arguments of \( A = A(r, s, \xi) \), \( B = B(r, s, \xi) \) and all subsequent derivatives in (3.7) are \( (r, s, \xi) = (|x|, |u|^2, \nabla u^2) = (r^2, n + \sum_{\ell=1}^{N} |\partial_\ell Qx|^2) \).

**Proof.** This follows by referring to (2.3). Firstly, the coefficient of \( A_t \) is \( X[u] \nabla(|u|^2) \), given by identity (3.6) in Lemma 3.1. The coefficient of \( A_s \) is \( X[u] \nabla(|u|^2) \), given by identity (iii) in Lemma 2.7 and with \( \nabla x = \Theta \), the coefficient of \( A_s \) is appropriately described above. Similarly, the coefficient of \( A \) is \( [\nabla u]^2 \Delta u \), described by identity (ii) in Lemma 2.8 and by noting (ii) in Lemma 2.6 we recover \( B x \) in (3.7). \( \square \)

**Remark 3.2.** Using (i) in Lemma 2.6 a whirl is seen to satisfy the incompressibility constraint. Indeed \( \det \nabla u = \det[Q + \sum_{i=1}^{N} \partial_i Qx \otimes \nabla y_i] = \det[I_n + \sum_{i=1}^{N} Q^i \partial_i Qx \otimes \nabla y_i] \). Now since \( p_i = Q^i \partial_i Qx, q_j = \nabla y_j \) we have \( \langle p_i, q_j \rangle = 0 \) for all \( 1 \leq i, j \leq N \) it follows that \( \det[I_n + \sum_{i=1}^{N} p_i \otimes q_i] = 1 \) (cf. Lemma 3.1 in [28]) and so \( \det \nabla u = 1 \) as claimed.

**Remark 3.3.** The boundary condition \( u \equiv x \) on \( \partial \mathcal{X}_n \) (equivalently \( Q[f] \equiv I_n \) on \( \partial \mathcal{X}_n \)) translates to \( f \equiv 0 \) on \( \{ |x| = a \} \) and \( f \equiv 2m\pi \) at \( \{ |x| = b \} \) with \( m = (m_1, \ldots, m_d) \in \mathbb{Z}^d \). This follows by observing that \( x \in \partial \mathcal{X}_n \iff y = y(x) \in (\partial \mathcal{A}_n)_D \) with the segments \( \{ |x| = a \} \) and \( \{ |x| = b \} \) of \( \partial \mathcal{X}_n \) corresponding to the segments \( \{ z = a \} \) and \( \{ z = b \} \) of \( (\partial \mathcal{A}_n)_D \) respectively whilst \( R[x] = I_2 \iff \alpha = 2m\pi \).

4. **An auxiliary system and the interrelation of two differential operators**

We begin the section by introducing a nonlinear unconstrained system in divergence form

\[
\begin{cases}
\text{div} \left[ \mathcal{A}(y, \nabla f) \nabla f \right] = 0 & \text{in } \mathcal{A}_n, \\
\mathcal{A}(y, \nabla f) \partial_i f = 0 & \text{on } (\partial \mathcal{A}_n)_D, \quad 1 \leq i \leq d, \\
f \equiv g & \text{on } (\partial \mathcal{A}_n)_N.
\end{cases}
\]

\[ (4.1) \]

Here \( f = (f_1, \ldots, f_d) \) is the unknown vector with \( \nabla f = [\partial k f_\ell : 1 \leq \ell \leq d, 1 \leq k \leq N] \), the divergence is taken with respect to the \((y_1, \ldots, y_d)\) variables and the nonlinearity
(coefficients) in the PDE are given by

\[ \mathcal{A}_i(y, \nabla f) := A \left( z, z^2, n + \sum_{\ell=1}^{d} y_\ell^2 |\nabla f_\ell|^2 \right) y_i^2 \mathcal{J} (y), \quad z^2 = \|y\|^2 = \sum_{j=1}^{N} y_j^2, \]  

where \( \mathcal{J} (y) = y_1 \cdots y_d \). Recall that \( (\partial \mathbb{A}_n)_D = \{ y \in \partial \mathbb{A}_n : z = a \} \cup \{ y \in \partial \mathbb{A}_n : z = b \} \), \( g = g(y, m) \) is the piecewise constant map defined by \( g|_{z=a} \equiv 0 \) and \( g|_{z=b} \equiv 2m\pi \) with \( m \in \mathbb{Z}^d \) fixed whilst \( (\partial \mathbb{A}_n)_N = \mathbb{A}_n \setminus (\partial \mathbb{A}_n)_D \) and \( \partial_r f_i = \nabla f_i \cdot \nu \) with \( \nu \) the unit outward normal field on \( (\partial \mathbb{A}_n)_N \). The motivation for studying this system by way of its relation to \( \mathcal{L}[u; A, B] \) and the system (1.7) will become clear later on. First we establish the uniqueness of solutions to (4.1). We set \( \mathcal{B}_p[\mathbb{A}_n; g] = \{ f = (f_1, \ldots, f_d) \in W^{1,p}(\mathbb{A}_n, \mathbb{R}^d) : f \equiv g \text{ on } (\partial \mathbb{A}_n)_D \} \) and note that the unconstrained system (4.1) here is strictly elliptic but not uniformly elliptic as a result of \( \mathcal{J} (y) > 0 \) in \( \mathbb{A}_n \) but \( \mathcal{J} (y) \equiv 0 \) on \( (\partial \mathbb{A}_n)_N \) [see (4.2)]. Thus interestingly even the existence of solution falls outside standard theory.

**Proposition 4.1.** Given \( m = (m_1, \ldots, m_d) \in \mathbb{Z}^d \) the solution \( f = (f_1, \ldots, f_d) \in \mathscr{C}^2(\mathbb{A}_n, \mathbb{R}^d) \) to the system (4.1) is unique. \(^1\)

**Proof.** Let \( f^1, f^2 \) be two solutions to (4.1) in \( \mathcal{B}_p[\mathbb{A}_n; m] \) and put \( h = f^2 - f^1 \). Then \( h \equiv 0 \) on \( (\partial \mathbb{A}_n)_D \). Now using the monotonicity inequality \( |A(z, z^2; \zeta_2) - A(z, z^2; \zeta_1)| \geq 0 \) for \( \zeta_1, \zeta_2 \in \mathbb{R} \) with \( \zeta_1 = n + \sum_j y_j^2 |\nabla f_j|^2 \), \( \zeta_2 = n + \sum_j y_j^2 |\nabla f_j|^2 \) (the sums over \( 1 \leq j \leq d \)) it follows after multiplying by \( \mathcal{J} \geq 0 \) and substitution using (4.2) that

\[
0 \leq \sum_{\ell=1}^{d} y_\ell^2 \left[ A \left( z, z^2, n + \sum_{j=1}^{d} y_j^2 |\nabla f_j|^2 \right) - A \left( z, z^2, n + \sum_{j=1}^{d} y_j^2 |\nabla f_j^1|^2 \right) \right] \times \mathcal{J} (|\nabla f_\ell|^2 - |\nabla f_\ell^1|^2) = \sum_{\ell=1}^{d} \left[ \mathcal{A}_\ell(y, \nabla f^2) - \mathcal{A}_\ell(y, \nabla f^1) \right] (|\nabla f_\ell|^2 - |\nabla f_\ell^1|^2) \quad \text{(4.3)}
\]

\[
\leq \sum_{\ell=1}^{d} 2(\mathcal{A}_\ell(y, \nabla f^2) - \mathcal{A}_\ell(y, \nabla f^1)) |\nabla f_\ell| |\nabla h_\ell| - \sum_{\ell=1}^{d} \left[ \mathcal{A}_\ell(y, \nabla f^2) + \mathcal{A}_\ell(y, \nabla f^1) \right] |\nabla h_\ell|^2.
\]

Now integrating the above and taking advantage of \( f^1, f^2 \) being solutions to (4.1) it follows after an application of the integration by parts formula and noting the vanishing of the integral of the expression on the second line above that

\[
\int_{\mathbb{A}_n} \sum_{\ell=1}^{d} -[\mathcal{A}_\ell(y, \nabla f^1) + \mathcal{A}_\ell(y, \nabla f^2)] |\nabla h_\ell|^2 \, dy \geq 0. \quad \text{(4.4)}
\]

As \( \mathcal{A}_\ell > 0 \) inside \( \mathbb{A}_n \) it follows by taking into account the connectedness of \( \mathbb{A}_n \) and the Dirichlet boundary condition on \( h \) that \( h \equiv 0 \). Thus \( f^1 \equiv f^2 \) as required. \( \Box \)

We now aim to make the link between the unconstrained system (4.1) and the PDE \( \mathcal{L}[u; A, B] = \nabla \mathcal{P} \) in the original system (1.7) more transparent. Towards this end, we

\(^1\)For some explicit examples of solutions to this system with the required degree of regularity see Sections 5 and 6.
begin by expanding the divergence in (4.1) thus obtaining the formulation
\[
\frac{1}{\mathcal{J} y_i^2} \text{div}[\mathcal{A}_i(y, \nabla f) \nabla f_i] = \frac{1}{\mathcal{J} y_i^2} \sum_{k=1}^{N} \partial_k \left[ A \left( z, z^2, n + \sum_{j=1}^{d} y_j^2 |\nabla f_j|^2 \right) y_i^2 y_1 \cdots y_d \partial_k f_i \right]
\]
\[
= \sum_{k=1}^{d} \sum_{\ell=1}^{N} 2A_\xi y_k (\partial_\ell f_k)^2 \partial_\ell f_i + \sum_{k=1}^{d} y_k^{-1} A \partial_k f_i + 2y_i^{-1} A \partial_i f_i
\]
\[
+ \sum_{k=1}^{N} \left[ \sum_{j=1}^{d} \sum_{\ell=1}^{N} 2A_\xi y_j^2 \partial_\ell^2 f_j \partial_\ell f_k \partial_\ell f_i + y_k [2A_s + |x|^{-1} A_\tau] \partial_k f_i + A \partial^2 f_i \right].
\]
This then relates to the operator \( \mathcal{L}[u; A, B] \) by way of the following result.

**Proposition 4.2.** Suppose \( u \) is a whirl associated with the matrix field \( Q = Q[f] \) with \( f = (f_1, \ldots, f_d) \) of class \( C^2 \) [see (3.1)-(3.2)]. Then
\[
\mathcal{L}[u; A, B] = \nabla A - A \sum_{i=1}^{d} |\nabla f_i|^2 w_i + \sum_{i=1}^{d} \frac{1}{\mathcal{J} y_i^2} \text{div}[\mathcal{A}_i(y, \nabla f) \nabla f_i][w_i]^\perp
\]
\[
+ \sum_{k=1}^{N} \sum_{i=1}^{d} \frac{\partial_i f_i}{\mathcal{J} y_i} \text{div}[\mathcal{A}_i(y, \nabla f) \nabla f_i] w_i + Bx.
\]

The arguments of \( A, B \) are \((r, s, \xi) = (||y||, ||y||^2, n + \sum_{j=1}^{d} y_j^2 |\nabla f_j|^2) \) and the coefficients \( \mathcal{A}_i = \mathcal{A}_i(y, \nabla f) \) (with 1 \( \leq i \leq d \)) are as in (4.2).

**Proof.** Starting from (3.7) and making use of (3.3), (3.4) we can rewrite \( \mathcal{L}[u; A, B] \) in terms of the components of the vector-map \( f \) as
\[
\mathcal{L}[u; A, B] = \nabla A + Bx + 2A_\xi \times
\]
\[
= \left\{ \sum_{k=1}^{N} \sum_{\ell=1}^{N} \left( y_k (\partial_\ell f_k)^2 + \sum_{i=1}^{d} y_i^2 \partial^2_{\ell k} f_j \partial_\ell f_i \right) \right\} \left[ \sum_{j=1}^{d} \left( \partial_k f_j [w_j]^\perp + \sum_{i=1}^{N} y_j^2 \partial_i f_j \partial_\ell f_i \frac{w_i}{y_i} \right) \right]
\]
\[
+ [2A_s + |x|^{-1} A_\tau] \sum_{\ell=1}^{N} \left\{ \sum_{i=1}^{d} y_\ell \partial_\ell f_i [w_i]^\perp + \sum_{k=1}^{N} \sum_{j=1}^{N} y_j^2 y_k \partial_i f_j \partial_\ell f_k \frac{w_\ell}{y_\ell} \right\}
\]
\[
+ A \left\{ \sum_{\ell=1}^{N} \sum_{i=1}^{d} \left[ (\partial^2 f_i + \Delta y_\ell \partial_\ell f_i) [w_i]^\perp - (\partial_\ell f_i)^2 w_i \right] + \sum_{\ell=1}^{d} \frac{2}{y_\ell} \partial_\ell f_i [w_\ell]^\perp \right\}
\]
\[
+ \sum_{\ell=1}^{N} \left\{ \sum_{k=1}^{N} \sum_{j=1}^{d} y_j^2 \partial_\ell f_j \left( \partial^2 f_j + \Delta y_k \partial_\ell f_j \right) + \sum_{k=1}^{d} 2y_k \partial_\ell f_k \partial_\ell f_k \right\} \frac{w_\ell}{y_\ell}. \]

Now referring to the expansion of the divergence operator prior to the proposition [see (4.5)] after a rearrangement of terms and a tedious but routine set of calculations we arrive at the required conclusion. □

The above result leads to two main consequences. The first underlines the role of the unconstrained system (4.1) in relation to the solvability of the original system (1.7) and the second describes a stark simplification of the vector field \( \mathcal{L}'[u; A, B] \) given that \( u \) satisfies the restricted system (4.1).
Proposition 4.3. If a whirl \( u \) associated with the matrix field \( \mathbf{Q} = \mathbf{Q}[f] \) of class \( \mathcal{C}^2 \) is a solution to (1.7) then the vector-map \( f = (f_1, \ldots, f_d) \) is a solution to (4.1).

Proof. Fixing \( 1 \leq j \leq d \) and taking the inner product of \( \mathcal{L}[u; A, B] \) with \( [w^j]^\perp \) by using the formulation in (4.6) and utilising the various orthogonality relations it is seen that

\[
\langle \mathcal{L}[u; A, B], [w^j]^\perp \rangle = \langle \nabla \mathcal{A}, [w^j]^\perp \rangle + \langle B x, [w^j]^\perp \rangle - \langle A \sum_{i=1}^{d} \vert \nabla f_i \vert^2 w^i, [w^j]^\perp \rangle
\]

\[
+ \frac{1}{\mathcal{J}} \sum_{i=1}^{d} \frac{1}{y_i} \langle \text{div} \mathcal{A}(y, \nabla f_i) \nabla f_i, [w^j]^\perp \rangle
\]

\[
+ \sum_{i=1}^{d} \sum_{\ell=1}^{N} \frac{\partial f_i}{\partial y_{\ell}} \text{div} \mathcal{A}(y, \nabla f_i) \nabla f_i w^\ell, [w^j]^\perp \rangle
\]

\[
= \langle \nabla \mathcal{A}, [w^j]^\perp \rangle + \frac{1}{\mathcal{J}} \text{div} \mathcal{A}(y, \nabla f) \nabla f_j. \tag{4.8}
\]

Now since \( \mathbf{A} \) here is a function of \( y = (y_1, \ldots, y_N) \) an easy differentiation shows that its gradient \( \nabla \mathcal{A} \) is a linear combination of the vectors \( w^1, \ldots, w^N \) and so \( \langle \nabla \mathcal{A}, [w^j]^\perp \rangle \equiv 0 \). As a result (4.8) simplifies further to

\[
\langle \mathcal{L}[u; A, B], [w^j]^\perp \rangle = 1/\mathcal{J} \text{div} \mathcal{A}(y, \nabla f) \nabla f_j.
\]

Let \( x = (x_1, \ldots, x_n) \) be associated with the 2-plane radial variables \( y = (y_1, \ldots, y_N) \). Consider the circle of radius \( y_j \) given by \( \gamma(t) = w^1 + \cdots + w^j(t) + \cdots + w^N \) \( 0 \leq t \leq 2\pi \). Here \( w^j(t) = y_j(0, 0, \cos t, \sin t, 0, \ldots, 0) \) and except for \( w^j \) all the other coordinates \( w^\ell \) are independent of \( t \). Then firstly all the points \( x = \gamma(t) \) \( 0 \leq t \leq 2\pi \) are associated with the same \( y \) and secondly at the point \( x = \gamma(t) \) we have \( \dot{\gamma}(t) = d\gamma/dt(t) = [w^j(t)]^\perp \). Therefore by (4.8) and the PDE we have \( \langle \nabla \mathcal{P}, [w^j]^\perp \rangle = 1/\mathcal{J} \text{div} \mathcal{A}(y, \nabla f) \nabla f_j \) and hence substituting \( x = \gamma(t) \), noting that the right-hand side is independent of \( t \) and integrating over \( 0 \leq t \leq 2\pi \) we arrive at

\[
\frac{2\pi}{\mathcal{J}(y)} \text{div} \mathcal{A}(y, \nabla f) \nabla f_j = \int_{0}^{2\pi} \langle \nabla \mathcal{P}(\gamma(t)), [w^j(t)]^\perp \rangle \, dt \tag{4.9}
\]

\[
= \int_{0}^{2\pi} \langle \nabla \mathcal{P}(\gamma(t)), \dot{\gamma}(t) \rangle \, dt = \int_{0}^{2\pi} \frac{d}{dt} \mathcal{P}(\gamma(t)) \, dt = 0,
\]

as required where the last identity follows from the closedness of \( \gamma \). The proof is thus complete. \( \square \)

Proposition 4.4. Assume \( f = (f_1, \ldots, f_d) \) of class \( \mathcal{C}^2 \) is a solution to the system (4.1). Then denoting by \( u \) the whirl associated with the matrix field \( \mathbf{Q} = \mathbf{Q}[f] \) we have

\[
\mathcal{L}[u; A, B] = \nabla \mathcal{A} + B x - \sum_{i=1}^{d} A \vert \nabla f_i \vert^2 w^i. \tag{4.10}
\]

Proof. This follows from (4.6) upon noting that \( \text{div} \mathcal{A}(y, \nabla f) \nabla f_i = 0 \) by assumption for all \( 1 \leq i \leq d \). \( \square \)

Proceeding forward recall that the overarching goal is to resolve the PDE \( \mathcal{L}[u; A, B] = \nabla \mathcal{P} \). Towards this end we consider the following general result before scrutinising the curl of the vector field in (4.10).
**Lemma 4.1.** Consider the vector field \( U(x) = \sum_{k=1}^{N} A_k(y)w^k \) for \( A_k \in \mathcal{C}^1(\mathbb{R}^n) \) and \( k = 1, \ldots, N \). Then writing \( \nabla w^k = \left[ \frac{\partial w^k}{\partial x_j} : 1 \leq i, j \leq n \right] \) we have
\[
\text{curl } U = \sum_{k=1}^{N} \sum_{\ell=1}^{N} \frac{1}{n} \left[ \partial_\ell A_k w^k \otimes w^\ell - w^\ell \otimes \partial_\ell A_k w^k \right] + \sum_{k=1}^{N} \left( A_k \nabla w^k - \nabla^t A_k \right). \tag{4.11}
\]

**Proof.** By linearity it suffices to consider only the case \( U = A_k w^k \). This then follows by a summation over \( 1 \leq k \leq N \). Towards this end by directly evaluating the curl and employing the product rule we have
\[
[\text{curl } A_k w^k]_{ij} = [A_k w^k]_{i,j} - [A_k w^k]_{j,i} = \sum_{\ell=1}^{N} \left[ \partial_\ell A_k w^k \frac{w^\ell}{y_\ell} + [A_k \nabla w^k]_{i,j} - [A_k \nabla w^k]_{j,i} \right].
\]
This upon shifting to tensor notation immediately leads to the desired conclusion. \( \square \)

**Remark 4.2.** An easy inspection shows that \( \nabla w^k = \text{diag}(0, \ldots, 0, 1, 0, \ldots, 0) \) with \( 1 \) as the \( k \)th block except for when \( n = 2d + 1 \) and \( k = N \) in which case \( \nabla w^N = \text{diag}(0, \ldots, 0, 1) \).

**Remark 4.3.** In the case \( A_k(y) = \Gamma_k(y)I_n \) with \( U = \sum_{k=1}^{N} \Gamma_k(y)w^k \) and \( \Gamma_k = \Gamma_k(y) \) suitable scalar functions, by using Lemma 4.1 and \( A_k \nabla w^k - \nabla^t A_k = 0 \), we have
\[
\text{curl } U = \sum_{k=1}^{N} \sum_{\ell=1}^{N} \frac{\partial_\ell \Gamma_k}{y_\ell} \left[ w^k \otimes w^\ell - w^\ell \otimes w^k \right]
= \sum_{1 \leq k < \ell \leq N} \left( \frac{\partial_\ell \Gamma_k}{y_\ell} - \frac{\partial_k \Gamma_\ell}{y_k} \right) \left[ w^k \otimes w^\ell - w^\ell \otimes w^k \right]. \tag{4.12}
\]

By virtue of the independence of the skew-symmetric tensors \( [w^k \otimes w^\ell - w^\ell \otimes w^k] \) it follows by a continuity argument that \( \text{curl } U \equiv 0 \iff \partial_\ell \Gamma_k / y_\ell - \partial_k \Gamma_\ell / y_k \equiv 0 \) for all \( 1 \leq k < \ell \leq N \).

Returning now to (4.10) and by subtracting the gradient term \( \nabla A \) from both sides (and for the sake of uniformity in notation, extending the vector-map \( f \) in the case \( n = 2d + 1 \) to an \( N \)-vector by setting \( f_N \equiv 0 \)) we consider the vector field \( U = \mathcal{L}[u; A, B] - \nabla A \). This corresponds to the case in Remark 4.3 with \( \Gamma_k(y) = B - A|\nabla f_k|^2 \) (\( 1 \leq k \leq N \)) clearly of class \( \mathcal{C}^1 \). The ongoing analysis leading to (4.12) then gives
\[
\text{curl } U = \sum_{1 \leq k < \ell \leq N} \left( \frac{\partial_\ell B}{y_\ell} - \frac{\partial_k B}{y_k} + \frac{\partial_k [A|\nabla f_k|^2]}{y_k} - \frac{\partial_\ell [A|\nabla f_k|^2]}{y_\ell} \right) \left[ w^k \otimes w^\ell - w^\ell \otimes w^k \right].
\]

**Corollary 4.4.** The \( \mathcal{C}^1 \) vector field \( U = \mathcal{L}[u; A, B] - \nabla A \) satisfies \( \text{curl } U \equiv 0 \) iff
\[
y_k \partial_\ell B - y_\ell \partial_k B + y_\ell \partial_k [A|\nabla f_\ell|^2] - y_k \partial_\ell [A|\nabla f_k|^2] \equiv 0, \tag{4.13}
\]
for all \( 1 \leq k < \ell \leq N \).
In odd dimensions the operator (1.8) and the associated PDE take the form

\[
\mathcal{L}[u; H, B] = \nabla \cdot [\nabla^2 u + 2(\nabla u \nabla H(\|u\|)) + H(\|u\|)|\nabla u|^2 u] + B(\|u\|)|\nabla u|^2 u = \nabla \mathcal{P}.
\]

(5.1)

In line with the preceding analysis we also consider the unconstrained system (4.1), for the vector function \( f = (f_1, \ldots, f_d) \), that in this context takes the form

\[
\begin{aligned}
\text{div} \left[ \mathcal{A}(y) \nabla f_\ell \right] &= 0 \quad \text{in } A_n, \\
\mathcal{A}(y) \partial_{\nu} f_\ell &= 0 \quad \text{on } (\partial A_n)_D, \\
\mathcal{A}(y) \partial_\nu f_\ell &= 0 \quad \text{on } (\partial A_n)_N.
\end{aligned}
\]

(5.2)

Here \( \mathcal{A}(y) = y^2 H(z, z^2) \mathcal{J}(y) \), \( g_\ell = 0 \) at \( z = a \) and \( g_\ell = 2m \pi \) at \( z = b \) with \( m \in \mathbb{Z} \). Note that since \( A(r, s, \xi) = H(r, s) \) has no explicit \( \xi \)-dependence, unlike the original system (5.1), here, the unconstrained system decouples and the \( \ell \)th PDE depends solely on the component \( f_\ell \) rather than the full vector-map \( f = (f_1, \ldots, f_d) \). This allows us to explicitly solve (5.1) in all dimensions which then leads to interesting consequences.

**Theorem 5.1.** Given \( m \in \mathbb{Z}^d \) the system (5.2) has the unique solution \( f = (f_1, \ldots, f_d) \) given by

\[
f = f(y, m) = 2m \pi \frac{H(||y||)}{H(b)}, \quad H(r) = \int_a^b \frac{dz}{z^{n+1} H(z, z^2)}.
\]

(5.3)

**Proof.** The Dirichlet boundary condition \( f = g \) on \((\partial A_n)_D\) is easily seen to be satisfied as a result of the normalisation and end-point conditions on \( H \). The Neumann boundary conditions \( \mathcal{A}(y) \partial_\nu f_\ell = 0 \) on \((\partial A_n)_N\) follow suit as a result of the quantity \( \mathcal{J} \equiv 0 \) on \((\partial A_n)_N\). Now referring to (5.3) a direct verification gives with \( z = ||y|| \),

\[
\nabla f = \left[ \frac{\partial f_i}{\partial y_j} = 2m \pi \frac{H(z) y_j}{H(b)} : 1 \leq i \leq d, 1 \leq j \leq N \right] = \frac{2\pi}{H(b)} m \otimes \frac{y}{z^{n+2} H(z, z^2)}.
\]

(5.4)

In even dimensions with \( n = 2d \) and \( N = d \), we see by a direct calculation that,

\[
\text{div} \left[ \mathcal{A}(y) \nabla f_\ell \right] = \sum_{j=1}^N \frac{\partial}{\partial y_j} \frac{2m \pi}{H(b)} \left[ \frac{y_j y_i}{||y||^{n+2}} \mathcal{J}(y) \right] = \frac{\mathcal{J}(y)}{||y||^{n+2}} \left[ d\ell - (2d + 2)y_\ell + 2y_\ell + dy_\ell \right] = 0.
\]

(5.5)

In odd dimensions with \( n = 2d + 1 \) and \( N = d + 1 \), proceeding similarly and separating the first \( y_1, \ldots, y_d \) variables from the last variable \( y_N \) in calculating the divergence, we
see by a straightforward differentiation that,

\[
\text{div} \left[ \mathcal{A}(y) \nabla f_i \right] = \sum_{j=1}^{d} \frac{2m_{ij} \pi}{H(b)} \frac{\partial}{\partial y_j} \left[ \frac{y_j^2}{||y||^{n+2}} \mathcal{J}(y) \right] + \frac{2m_{ij} \pi}{H(b)} \frac{\partial}{\partial y_N} \left[ \frac{y_N y_j^2}{||y||^{n+2}} \mathcal{J}(y) \right] \\
= \mathcal{J}(y) \frac{2m_{ij} \pi}{||y||^{n+2} H(b)} y_i \left[ dy_{\ell} - \frac{(2d + 3)}{||y||^2} \sum_{i=1}^{d} y_i^2 + 2y_\ell + dy_\ell \right] \\
+ \mathcal{J}(y) \frac{2m_{ij} \pi}{||y||^{n+2} H(b)} y_i \left[ y_i - \frac{(2d + 3)}{||y||^2} y_\ell y_N^2 \right] = 0. 
\]

Having verified the solution to the PDE in (5.2) in both even and odd dimensions the assertion is justified and the proof is thus complete. \(\square\)

Henceforth we shall write \(\mathcal{H}(r) = H(r)/H(b)\). It is then evident that \(\mathcal{H}\) is a solution to the linear ODE \(d/dr[\ell^2 H(r, r^2) \mathcal{H}^2] = 0\) on \(a < r < b\). We write \(Q = Q[f](y, m)\) for the matrix field associated with the solution \(f = f(y, m)\) from Theorem 5.1 [see (3.1) and (3.2)]. Thus it is plain that \(Q[f](y, m) = \text{diag}(R[2m_1 \pi \mathcal{H}^2(||y||)], \ldots, R[2m_d \pi \mathcal{H}^2(||y||)])\) when \(n = 2d\) and \(Q[f](y, m) = \text{diag}(R[2m_1 \pi \mathcal{H}^2(||y||)], \ldots, R[2m_d \pi \mathcal{H}^2(||y||)], 1)\) when \(n = 2d + 1\). With these assumptions in place the action of the differential operator \(\mathcal{L}\) on the map \(u = Q[f](y) x\) after the subtraction of \(\nabla H\) [cf. (4.10)] can be written

\[
U = \mathcal{L}[u; H, B] - \nabla H = -H \sum_{i=1}^{d} |\nabla f_i|^2 w_i + B \left( z, z^2, n + \sum_{j=1}^{d} y_j^2 |\nabla f_j|^2 \right) x. 
\]

We now turn to the curl of the vector field (5.7) in anticipation of solving the PDE \(\mathcal{L}[u; H, B] = \nabla \mathcal{P}\). Here we use Lemma 4.1 and Remark 4.3 with the choice of functions

\[
\Gamma_k(y) = -4m_k^2 \pi^2 \mathcal{H}^2 H + B \left( z, z^2, n + \sum_{j=1}^{d} 4m_j^2 \pi^2 y_j^2 \mathcal{H}^2 \right), \quad 1 \leq k \leq N, 
\]

noting that \(|\nabla f_j|^2 = 4m_j^2 \pi^2 \mathcal{H}^2\). Remark 4.3 and Corollary 4.4 direct us to compute the expressions \(\partial_i \Gamma_k/y_{\ell} - \partial_k \Gamma_i/y_{\ell}\). Towards this end we first observe that

\[
\partial_i \Gamma_k(y) = 8\pi^2 B_{\ell} \left[ \mathcal{H}^2 H \frac{y_{\ell}}{||y||} \sum_{j=1}^{d} m_j^2 y_j^2 + \mathcal{H}^2 m_k^2 y_k \right] + \left( 2||y||B_s + B_r \right) \frac{y_{\ell}}{||y||} \\
- 4m_k^2 \pi^2 \left[ 2\mathcal{H}^2 H + \mathcal{H}^2 \frac{dH}{dr} \right] \frac{y_{\ell}}{||y||}. 
\]

(5.9)
where we have abbreviated the arguments of $B_r, B_s$ and $B_\ell$. Therefore it follows that

$$
\frac{\partial \Gamma_k}{\partial y} - \frac{\partial \Gamma_\ell}{\partial y_k} = -\frac{1}{y_k} \left\{ 8\pi^2 B_\ell \left[ \tilde{\mathcal{H}} \tilde{\mathcal{H}} \frac{y_k}{|y|} \sum_{j=1}^d m_j^2 y_j^2 + \tilde{\mathcal{H}} \tilde{\mathcal{H}}^2 m_k^2 y_k \right] + \left( 2 \frac{|y| |B_s + B_r|}{|y|} \right) \frac{y_k}{|y|} 
- 4m^2 \pi^2 \left[ 2 \tilde{\mathcal{H}} \mathcal{H} + \tilde{\mathcal{H}} \tilde{\mathcal{H}}^2 \frac{dH}{dr} \right] \frac{y_k}{|y|} \right\} 
+ \frac{1}{y_k} \left\{ 8\pi^2 B_\ell \left[ \tilde{\mathcal{H}} \tilde{\mathcal{H}} \frac{y_\ell}{|y|} \sum_{j=1}^d m_j^2 y_j^2 + \tilde{\mathcal{H}} \tilde{\mathcal{H}}^2 m_\ell^2 y_\ell \right] + \left( 2 \frac{|y| |B_s + B_r|}{|y|} \right) \frac{y_\ell}{|y|} 
- 4m^2 \pi^2 \left[ 2 \tilde{\mathcal{H}} \mathcal{H} + \tilde{\mathcal{H}} \tilde{\mathcal{H}}^2 \frac{dH}{dr} \right] \frac{y_\ell}{|y|} \right\} 
= 4(m^2 - m^2) \frac{\tilde{\mathcal{H}} \tilde{\mathcal{H}}^2}{|y|^2} \Delta(H, B).
$$

(5.10)

Here we have introduced the discriminant $\Delta(H, B) := 2(n + 1)H + rH_r + 2r^2[H_s - B_\ell]$ associated with $(H, B)$ and the putative map $u$. Now (4.12) and Corollary 4.4 give

$$
curl [\mathcal{L}(u; H, B) - \nabla H] = \frac{4\pi^2 \tilde{\mathcal{H}}^2}{|y|^2} \Delta(H, B) \sum_{1 \leq k < \ell \leq N} (m_k^2 - m_\ell^2) [w^k \otimes w^\ell - w^\ell \otimes w^k].
$$

(5.11)

The following theorem gives a complete characterisation of all whirl solutions to the system (1.7)-(1.8) [with $\mathcal{L}$ as in (5.1)] pointing at an interesting dimensional parity.

**Theorem 5.2.** A whirl $u$ associated with the matrix field $Q = Q[f] \in C^2(\mathbb{R}_n, SO(n))$ and satisfying the boundary condition $Q \equiv I$ on $(\partial A_n)_D$ is a solution to the system $\Sigma[(u, \mathcal{P}); H, B]$ in (1.7) with $\mathcal{L}[u; H, B]$ as in (5.1) if and only if the following hold.

- If $\Delta(H, B) \neq 0$ then depending on the dimension being even or odd, we have:
  - (i) $n = 2d$: $Q = Q[f](y) = \text{diag}(R[2m_1 \pi \mathcal{H}(|y|)], \ldots, R[2m_d \pi \mathcal{H}(|y|)])$ with $m = (m_1, \ldots, m_d) \in \mathbb{Z}^d$ satisfying $|m_1| = \cdots = |m_d|$.
  - (ii) $n = 2d + 1$: $Q = Q[f] \equiv I_n$, corresponding to $m_1 = \cdots = m_d = 0$.

- If $\Delta(H, B) \equiv 0$ then depending on the dimension being even or odd, we have:
  - (i) $n = 2d$: $Q = Q[f] = \text{diag}(R[2m_1 \pi \mathcal{H}(|y|)], \ldots, R[2m_d \pi \mathcal{H}(|y|)])$.
  - (ii) $n = 2d + 1$: $Q = Q[f] = \text{diag}(R[2m_1 \pi \mathcal{H}(|y|)], \ldots, R[2m_d \pi \mathcal{H}(|y|)], 1)$.

In either case $m = (m_1, \ldots, m_d) \in \mathbb{Z}^d$ and there is no further restrictions needed on $m_1, \ldots, m_d$.

**Proof.** We shall split the proof into two parts justifying the necessity and the sufficiency arguments separately.

**Necessity.** By Proposition 4.3 if a whirl associated with the matrix field $Q = Q[f]$ is a solution to the system $\Sigma[(u, \mathcal{P}); H, B]$ then the vector-map $f$ must be a solution to (4.1), or more specifically, here, to (5.2). Therefore $f$ must be exactly as described by Theorem 5.1. It suffices now to use Proposition 4.4, the curl analysis in Section 4 and the calculation leading to (5.11) to get a complete characterisation of $f$ and to do so we proceed by considering the cases $\Delta(H, B) \neq 0$ and $\Delta(H, B) \equiv 0$ separately.
If \( \Delta(H, B) \neq 0 \) then by virtue of the independence of the tensors \( w^k \otimes w^l - w^l \otimes w^k \) \((1 \leq \ell < k \leq N)\) and the fact these tensos vanish at most on coordinate hyperplanes, (5.11) and a basic continuity argument gives \( \text{curl} \ U \equiv 0 \iff m_1^2 = \cdots = m_N^2 \).

In the case \( n = 2d \) this gives the conclusion in (i) and in the case \( n = 2d + 1 \) this gives \( m_1 = \cdots = m_N = 0 \) and so \( Q \equiv I_n \) as stated in (ii). If \( \Delta(h, B) \equiv 0 \) then again by (5.11) \( \text{curl} \ U \equiv 0 \) irrespective of the choice see there can be no restriction on the integers \( m_\ell \).

(Sufficiency.) We shall do this only for the case \( \Delta(H, B) \equiv 0 \) as the case \( \Delta(H, B) \neq 0 \) is straightforward. Towards this end we assume hereafter that \( \Delta(H, B) \equiv 0 \) and show that \( \mathcal{L}[u; H, B] = \nabla \mathcal{R} \). We claim that \( U = \mathcal{L}[u; H, B] - \nabla H = \nabla \mathcal{R}(|x|, |Hx|^2) \) for a suitable choice of \( \mathcal{R} = \mathcal{R}(r, z) \) of class \( C^2 \). Here \( \mathbf{H} \) stands for the \( n \times n \) skew-symmetric matrix \( \mathbf{H} = \text{diag}(2m_1\pi\mathbf{J}, \ldots, 2m_d\pi\mathbf{J}) \) or \( \mathbf{H} = \text{diag}(2m_1\pi\mathbf{J}, \ldots, 2m_d\pi\mathbf{J}, 0) \) depending as to whether \( n = 2d \) or \( n = 2d + 1 \). Indeed assuming the claim to be true a direct calculation and comparison with \( U \) leads to

\[
\nabla \mathcal{R}(|x|, |Hx|^2) = \mathcal{R}_r(r, |Hx|^2)\Theta - 2r\mathcal{R}_z(r, |Hx|^2)\mathbf{H}^2 \Theta = rB(r, r^2, n + \mathbf{H}^2 z) + rH(\mathbf{H}^2 + \mathbf{H}^2)^2 \Theta = U,
\]

where \( \mathcal{R}_r, \mathcal{R}_z \) denote the derivatives of \( \mathcal{R} \) in the first and second arguments respectively. Thus the second equality in (5.12) would be valid [cf. (5.7)] provided that \( \mathcal{R}_r(r, z) = rB(r, r^2, n + \mathbf{H}^2 z) \) and \( \mathcal{R}_z(r, z) = -H(\mathbf{H}^2 + \mathbf{H}^2)^2/2 \). Let us thus turn on to constructing \( \mathcal{R} \). To this end let \( \mathcal{B} = \{(r, z) : r = |x|, z = |Hx|^2 \text{ with } x \in \mathbb{R}^n \} \). Then \( \mathcal{B} \subset [a, b] \times \mathbb{R} \) is seen to be simply-connected; as a matter of fact, denoting by \( m, \bar{m} \geq 0 \) the minimum and maximum eigenvalues of the diagonal matrix \( \mathbf{H}'\mathbf{H} \) respectively it is easily seen that \( \mathcal{B} = \{(r, z) : a < r < b, 0 \leq \frac{mr^2}{\bar{m}} \leq z \leq \frac{m}{\bar{m}}r^2 \} \). Since \( \Delta(H, B) \equiv 0 \) it is not difficult to see that \( \partial_z \mathcal{R}_r(r, z) - \partial_r \mathcal{R}_z(r, z) = \partial_z[rB(r, r^2, n + \mathbf{H}^2 z)] + \partial_r[H(\mathbf{H}^2 + \mathbf{H}^2)^2/2] \equiv 0 \) in \( \mathcal{B} \). As a result the 1-form \( \omega = rB(r, r^2, n + \mathbf{H}^2 z) dr - H(r, r^2, \mathbf{H}^2 + \mathbf{H}^2)^2/2 dz \) is closed in \( \mathcal{B} \) and hence exact in view of \( \mathcal{B} \) being simply-connected. Thus \( \omega = d\mathcal{R} \) for a function (a 0-form) \( \mathcal{R} = \mathcal{R}(r, z) \) of class \( C^2 \). To describe \( \mathcal{R} \) more specifically pick a base point \( (r^*, z^*) \) in \( \mathcal{B} \) and let \( \gamma \) be any piecewise continuously differentiable Jordan curve in \( \mathcal{B} \) connecting \( (r^*, z^*) \) to \( (r, z) \) and set

\[
\mathcal{R}(r, z) = \int_\gamma \omega = \int_\gamma rB(r, r^2, n + \mathbf{H}^2 z) dr - H(r, r^2, \mathbf{H}^2 + \mathbf{H}^2)^2/2 dz, \quad (r, z) \in \mathcal{B}.
\]

The integral is seen to be independent of the choice of \( \gamma \) and hence well-defined. The function \( \mathcal{R} \) is of class \( C^2 \) in the interior of \( \mathcal{B} \) with continuously differentiable tangential gradients on the upper and lower boundary curves of \( \mathcal{B} \). One can thus verify that (5.12) holds (both for \( (r, z) = (|x|, |Hx|^2) \) in the interior of \( \mathcal{B} \) and the upper and lower boundary curves). Thus \( U = \nabla \mathcal{R}(|x|, |Hx|^2) \) and the proof is complete.

6. Infinitely many whirl solutions to \( \Sigma[(u, \mathcal{P}); A, B] \) in even dimensions

In this last section we prove the existence of an infinitude of solutions to the original system \( \Sigma[(u, \mathcal{P}), A, B] \) in even dimensions. In terms of our earlier notation here \( n = 2d \) with \( d = N \) and the Dirichlet boundary condition will be chosen \( g = 2m\pi 1_{\chi_{z=b}} \) with \( m \in \mathbb{Z} \). Here \( \chi_{z=b} \) is the characteristic function of the set \( \{z = b\} \).
Theorem 6.1. Let \( n = 2d \) and \( m = m1 \in \mathbb{Z}^d \). Then (4.1) admits the unique solution \( f(y;m) = (f_1, \ldots, f_d) = g(||y||; m)1 \) where \( g = \mathcal{G}(r;m) \in \mathcal{C}^2[a,b] \) is the unique solution to the two point boundary-value problem \(^2\)

\[
\begin{aligned}
\frac{d}{dr} \left[ r^{n+1} A(r, r^2, n + r^2 \mathcal{G}^2) \mathcal{G} \right] = 0, \quad a < r < b, \\
\mathcal{G}(a) = 0, \\
\mathcal{G}(b) = 2m\pi.
\end{aligned}
\]

Proof. The boundary conditions on \((\partial \mathcal{A})_D\) in (4.1) follow from the imposed end-point conditions on \( \mathcal{G} \) in (6.1). Now in order to verify the PDE in (4.1) we first observe that,

\[
\nabla f = \left[ \frac{\partial f_i}{\partial y_j} : 1 \leq i \leq d, 1 \leq j \leq N \right] = \mathcal{G} \frac{1}{||y||} \implies \sum_{i=1}^{N} y_i^2 |\nabla f_i|^2 = \sum_{i=1}^{N} y_i^2 \mathcal{G}^2 = ||y||^2 \mathcal{G}^2.
\]

Now upon noting that if \( n = 2d \) we have \( y = (y_1, \ldots, y_d) \) and \( \text{div} = \sum_{j=1}^{d} \partial / \partial y_j \), we can proceed directly and write

\[
\text{div}[\mathcal{A}(y, \nabla f) \nabla f_i] = \sum_{j=1}^{N} \frac{\partial}{\partial y_j} \left[ \mathcal{A}(y, \nabla f) \frac{\partial f_i}{\partial y_j} \right] = \sum_{j=1}^{d} \frac{\partial}{\partial y_j} \left[ y_i^2 \mathcal{A} \mathcal{J}(y) \mathcal{G} \frac{y_j}{||y||} \right] = \frac{y_i^2 \mathcal{J}(y)}{||y||} \left\{ r \mathcal{G} \frac{d}{dr} A + r A \mathcal{G} + (2d + 1) A \mathcal{G} \right\} = \frac{y_i^2 \mathcal{J}(y)}{||y||} \frac{d}{dr} \left[ r^{n+1} A(r, r^2, n + r^2 \mathcal{G}^2) \mathcal{G} \right],
\]

where in the first two lines we have written \( A = A(r, r^2, n + r^2 \mathcal{G}^2) \). It is now plain that if \( \mathcal{G} \) is a solution to the ODE in (6.1) then the vector \( f \) satisfies the above PDE. This therefore completes the proof.

We turn now to the system (1.7) and prove the multiplicity result announced at the start of the section. Indeed we prove that for each \( m \in \mathbb{Z} \) the whirl map \( u = u(x;m) \) with \( \mathcal{R}_u(x) = |x|, \mathcal{J}_u(x) = Q[f](y,m) \Theta \) where \( Q = Q[f](y,m) = \exp\{\mathcal{G}(||y||; m)H\} = \text{diag}(R[\mathcal{G}(||y||; m)], \ldots, R[\mathcal{G}(||y||; m)]) \) and \( \mathcal{G} \) is as in Theorem 6.1 serves as a solution to (1.7). Here we write \( H \) for the so(n) matrix \( H = \text{diag}(J, \ldots, J) \).

Theorem 6.2. For \( n \geq 2 \) even and \( m \in \mathbb{Z} \) let \( u = r \exp\{\mathcal{G}(r;m)H\} \Theta \) where \( \mathcal{G} \in \mathcal{C}^2[a,b] \) is the solution to (6.1) and \( H = \text{diag}(J, \ldots, J) \). Then \( \mathcal{L}[u;A,B] = \nabla \mathcal{P} \) where the pressure field takes the form \( \mathcal{P} = A + \mathcal{G} \). Here the radial scalar-valued function \( \mathcal{G} = G(r) \) is chosen such that \( \nabla G = r[B(r,r^2,n+r^2\mathcal{G}^2) - A(r,r^2,n+r^2\mathcal{G}^2)\mathcal{G}^2] \Theta \). As a result the system (1.7) has an infinitesimal of \( \mathcal{C}^2 \) solutions.

Proof. The boundary conditions in (1.7) follow immediately from those of \( \mathcal{G} \) in (6.1) and as seen earlier \( \det \nabla u = 1 \). It thus remains to prove that \( \mathcal{L}[u;A,B] \) is a gradient field. Taking \( f \) as in Theorem 6.1 it follows after an application of Proposition 4.4 that \( \mathcal{L}[u;A,B] = \nabla A + r[B(r,r^2,n+r^2\mathcal{G}^2) - A(r,r^2,n+r^2\mathcal{G}^2)\mathcal{G}^2] \Theta \). From this the description of \( \mathcal{P} = \mathcal{P}(x) \) as in the statement of the theorem follows and the proof is complete. \( \square \)

\(^2\)The existence of such solutions \( \mathcal{G} \) with the required \( \mathcal{C}^2 \)-regularity can be established as in [32].
7. Appendix

In this appendix we give a short derivation of the equations of first variation for the total elastic energy integral (1.2) subject to the incompressibility constraint (that is, the system (1.1)-(1.3)). The argument is known among the experts and is given here for the sake convenience of the reader. Towards this end we pick a map $u = (u_1, \ldots, u_n)$. As is standard we derive the equations under the assumption of sufficient smoothness of $u$. Moreover $\det \nabla u \equiv 1$ and we assume that $u$ is injective on $\Omega$. Setting $U = u(\Omega) \subset \mathbb{R}^n$ it follows from the invariance of domain that $U$ is open. We denote the inverse of $u$ by $u^{-1}$.

Pick a smooth compactly supported vector field $v \in C^\infty_0(U, \mathbb{R}^n)$ and assume that $\text{div } v = \text{tr } \nabla v = 0$ in $U$. Consider the integral curves of the vector field $v$ in $U$, i.e., for every $y \in U$ and $t \in \mathbb{R}$ let $\Upsilon = \Upsilon(y, t)$ denote the solution to the initial value problem

$$\begin{cases}
\frac{d}{dt} \Upsilon(y, t) = v(\Upsilon(y, t)), & t \in \mathbb{R}, \\
\Upsilon(y, 0) = y.
\end{cases} \quad (7.1)$$

Now recalling the relations $d(det P)/dP = \text{cof } P$ and $P[\text{cof } P]^t = [\text{cof } P]^t P = (\det P)I_n$, a straightforward differentiation gives

$$\frac{d}{dt} \det \nabla \Upsilon = \sum_{ij} [\text{cof } \nabla \Upsilon]_{ij} \frac{d}{dt} [\nabla \Upsilon]_{ij} = \sum_{ij} [\text{cof } \nabla \Upsilon]_{ij} [\nabla v(\Upsilon)]_{ij} = \sum_{ijk} [\text{cof } \nabla \Upsilon]_{ij} [\nabla v]_{ik} [\nabla \Upsilon]_{kj} = [\nabla v][\text{cof } \nabla \Upsilon]^t : [\nabla v]^t = (\det \nabla \Upsilon) \text{ div } v. \quad (7.2)$$

Thus in particular as the vector field $v$ is chosen to be divergence free we have

$$\frac{d}{dt} \det \nabla \Upsilon = (\det \nabla \Upsilon) \text{ div } v = 0. \quad (7.3)$$

Next as by (7.1) we have $\det \nabla \Upsilon(y, 0) = 1$ it follows from (7.3) that $\det \nabla \Upsilon(y, t) = 1$ for all $y \in U$, $t \in \mathbb{R}$. Let us now set $u_t(x) = \Upsilon(u(x), t)$ for $x$ in $\Omega$ and $t \in \mathbb{R}$. A basic calculation then gives

$$\det \{\nabla_x u_t(x)\} = \det \{\nabla_x [\Upsilon(u(x), t)]\} = \det \{[\nabla \Upsilon(u(x), t)][\nabla_x u(x)]\} = 1.$$

Furthermore by an easy inspection $u_t = u$ near the boundary $\partial \Omega$ whilst $u_0 = u$. As a result $u_t$ constitutes a one parameter family of incompressible deformations in $\mathcal{A}^p_0(\Omega)$ passing through $u$ at $t = 0$. Hence by referring to the energy integral (1.2), for $u$ to be an energy extremiser, we must have

$$\frac{d}{dt} \mathbb{E}[u_t] \bigg|_{t=0} = \frac{d}{dt} \int_\Omega W(x, u_t(x), \nabla u_t(x)) \, dx \bigg|_{t=0} = 0. \quad (7.4)$$

\footnote{Note that the whirl solutions $u = Q[f](y)x$ constructed in the paper have both the required degree of smoothness and are injective.}
A direct calculation and making use of (7.1) now gives
\[
\frac{d}{dt} \mathbb{E}[u_t] \bigg|_{t=0} = \int_\Omega \left( \left\langle W_u(x, u(x), \nabla u(x)), \frac{du}{dt} \right\rangle + W_\xi(x, u(x), \nabla u(x)) : \frac{d}{dt} \nabla u_t \bigg|_{t=0} \right) \, dx
\]
\[
= \int_\Omega \langle W_u(x, u(x), \nabla u(x)), [v \circ u](x) \rangle + W_\xi(x, u(x), \nabla u(x)) : \nabla [v \circ u](x) \rangle \, dx. \tag{7.5}
\]

Let us set \(W_u(x) = W_u(x, u(x), \nabla u(x))\) and \(W_\xi(x) = W_\xi(x, u(x), \nabla u(x))\) for brevity. Then from (7.5) and after a change of variables (using the invertibility of \(U\)) we have
\[
\frac{d}{dt} \mathbb{E}[u_t] \bigg|_{t=0} = \int_U \langle (W_u(u^{-1}(y)), v(y)) + W_\xi(u^{-1}(y)) : [\nabla y v](y)[\nabla x u](u^{-1}(y)) \rangle \, dy
\]
\[
= \int_U \langle (W_u(u^{-1}(y)), v(y)) + W_\xi(u^{-1}(y))\{[\nabla x u](u^{-1}(y)) : [\nabla y v](y) \rangle \, dy. \tag{7.6}
\]

Next let us denote the integral on the right in (7.6) by \(L(v)\), that is, let
\[
L(v) = \int_U \langle (W_u(u^{-1}(y)), v(y)) + W_\xi(u^{-1}(y))\{[\nabla x u](u^{-1}(y)) : [\nabla y v](y) \rangle \, dy. \tag{7.7}
\]

Then it is easily seen that there exists \(c > 0\) (depending on \(u, W\) but independent of \(v\)) such that for all vector fields \(v \in \mathcal{C}_0^1(U, \mathbb{R}^n)\) we have
\[
|L(v)| \leq c \|v\|_{L^\infty(U, \mathbb{R}^n)} + \|\nabla y v\|_{L^\infty(U, \mathbb{R}^{n \times n})}. \tag{7.8}
\]

Thus \(L\) is a bounded linear functional on \(\mathcal{C}_0^1(U, \mathbb{R}^n)\). As from (7.4) and (7.6) we have \(L(v) = 0\) for when \(\text{div} \ v = 0\) it then follows (see Section 1.4 and Proposition 1.1 in [44]) that there exists \(p \in \mathcal{D}'(U)\) such that \(L = -\nabla p\). In particular we can write
\[
L(v) = -(\nabla p, v) = (p, \text{div} v) = (p, \text{tr} [\nabla v]). \tag{7.9}
\]

Now a reference to (7.7) and an application of the integration by parts formula on the second term in the integral together with (7.9) gives
\[
\int_U \langle (W_u(u^{-1}(y)) - \text{div} \{W_\xi(u^{-1}(y))\{[\nabla x u](u^{-1}(y))], v(y) \rangle \, dy = -(\nabla p, v). \tag{7.10}
\]

This in particular implies that \(\nabla p\) can be represented by an integrable function (in fact continuous) on \(U\). Now let us take \(\phi \in \mathcal{C}_0^\infty(\Omega; \mathbb{R}^n)\) and set \(v = \phi \circ u^{-1}\). Then a straightforward differentiation results in
\[
\nabla_y v(y) = [\nabla_x \phi](u^{-1}(y))[\nabla_y u^{-1}](y) = [\nabla_x \phi](u^{-1}(y))[\nabla_x u]^{-1}(u^{-1}(y)). \tag{7.11}
\]

Hence substitution in the integral on the right (7.7) and changing variables by transforming back to \(\Omega\) gives
\[
L(v) = \int_U \langle (W_u(u^{-1}(y)), \phi(u^{-1}(y))) + W_\xi(u^{-1}(y)) : [\nabla x \phi](u^{-1}(y)) \rangle \, dy
\]
\[
= \int_\Omega \langle (W_u(x), \phi(x)) + W_\xi(x) : [\nabla x \phi](x) \rangle \, dx. \tag{7.12}
\]
Likewise substitution in (7.9) and a similar argument after setting $\mathcal{P} = p \circ u$ results in
\[
L(v) = -(\nabla p, v) = \int_U p(y) \text{div} v(y) dy = \int_U p(y) \text{tr}\{[\nabla_x \phi](u^{-1}(y)[\nabla u]^{-1}(u^{-1}(y))}\} dy
\]
\[
= \int_{\Omega} (p \circ u)(x) \text{tr}\{[\nabla_x \phi][\text{cof} \nabla_x u]'\} dx
\]
\[
= \int_{\Omega} \mathcal{P}(x) [\text{cof} \nabla_x u] : [\nabla_x \phi] dx. \quad (7.13)
\]
Finally equating (7.12) and (7.13) leads to
\[
\int_{\Omega} ((W_u(x), \phi) + (W_\xi(x) - \mathcal{P}(x) [\text{cof} \nabla_x u] : [\nabla_x \phi]) dx = 0, \quad (7.14)
\]
which after taking into account the arbitrariness of $\phi \in C_0^\infty(\Omega, \mathbb{R}^n)$ and standard arguments formally results in (1.3).

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