ALTERNATINGLY INCREASING PROPERTY AND
BI-GAMMA-POSITIVITY OF POLYNOMIALS

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Abstract. A polynomial \( p(z) \) of degree \( d \) is alternatingly increasing if and only if it can be decomposed into a sum \( p(z) = a(z) + zb(z) \), where \( a(z) \) and \( b(z) \) are symmetric and unimodal polynomials with \( \deg a(z) = d \) and \( \deg b(z) \leq d - 1 \). We say that \( p(z) \) is bi-gamma-positive if \( a(z) \) and \( b(z) \) are both gamma positive. In this paper, we present a unified elementary proof of the bi-\( \gamma \)-positivity of several polynomials that appear often in algebraic, topological and geometric combinatorics, including \( q \)-Eulerian polynomials of types \( A \) and \( B \), descent polynomials of multipermutations and signed multipermutations, Eulerian and derangement polynomials of colored permutations. As an application, we get that these polynomials are all alternatingly increasing.

1. Introduction

1.1. Alternatingly increasing property.

Let \( f(x) = \sum_{i=0}^{n} f_i x^i \) be a polynomial with nonnegative coefficients. We say that \( f(x) \) is unimodal if \( f_0 \leq f_1 \leq \cdots \leq f_k \geq f_{k+1} \geq \cdots \geq f_n \) for some \( k \) and it is called symmetric if \( f_i = f_{n-i} \) for all indices \( 0 \leq i \leq n \). Following [52, Definition 2.9], we say that \( f(x) \) is alternatingly increasing if the coefficients of \( f(x) \) satisfy

\[
0 \leq f_0 \leq f_n \leq f_1 \leq f_{n-1} \leq \cdots f_{\lfloor \frac{n+1}{2} \rfloor}.
\]

Clearly, alternatingly increasing property is a stronger property than unimodality.

Let \( \mathcal{P} \) be a \( d \)-dimensional lattice polytope. The Ehrhart polynomial of \( \mathcal{P} \) is the function \( L_{\mathcal{P}}(t) = |t\mathcal{P} \cap \mathbb{Z}^d| \), where \( t\mathcal{P} := \{(tx_1, tx_2, \ldots, tx_d) : (x_1, \ldots, x_d) \in \mathcal{P}\} \) denotes the dilated polytope for \( \mathcal{P} \) and \( t \) is a nonnegative integer. A classical result of Ehrhart [35] asserts that \( L_{\mathcal{P}}(t) \) is a polynomial in \( t \) of degree \( d \). The generating function containing \( L_{\mathcal{P}}(t) \) as coefficients is called the Ehrhart series of \( \mathcal{P} \):

\[
\text{Ehr}_{\mathcal{P}}(z) = 1 + \sum_{t \geq 1} L_{\mathcal{P}}(t)z^t.
\]

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The $h^*$-polynomial of $P$ can be defined as follows:

$$\text{Ehr}_P(z) = \frac{h_P^*(z)}{(1-z)^{d+1}}.$$  

Stanley’s nonnegativity theorem \[58\] says that if $P$ an integral convex $d$-polytope, then the coefficients of $h_P^*(z)$ are all nonnegative integers.

We say that a lattice polytope $P \in \mathbb{R}^d$ has the integer decomposition property (IDP, for short) if for all integers $t \geq 1$ and every $p \in tP \cap \mathbb{Z}^d$, there are $p_1, \ldots, p_t \in P \cap \mathbb{Z}^d$ such that $p = p_1 + p_2 + \cdots + p_t$. Following \[52, Proposition 2.17\], the $h^*$-polynomial of a closed lattice parallelepiped with at least one interior point is alternatingly increasing. Moreover, Schepers and Langenhoven \[52, Question 2.21\] conjectured that the $h^*$-polynomial of every polytope having the IDP property and an interior lattice point is alternatingly increasing. Since then, there is a large literature devoted to the alternatingly increasing property (see \[2, 3, 4, 13, 57\]). Very recently, Brändén and Solus \[13\] related the alternatingly increasing property to real-rootedness of the symmetric decomposition of a polynomial.

The following is a fundamental result.

**Proposition 1** (\[5, 13\]). Let $p(z)$ be a polynomial of degree $d$. There is a unique symmetric decomposition $p(z) = a(z) + zb(z)$, where $a(z)$ and $b(z)$ are symmetric polynomials satisfying $a(z) = z^d a(\frac{1}{z})$ and $b(z) = z^{d-1} b(\frac{1}{z})$.

It is routine to verify that

$$a(z) = \frac{z^{d+1} p(1/z) - p(z)}{1-z}, \quad b(z) = \frac{z^d p(1/z) - p(z)}{1-z}.$$  

From (1.1), we see that $\deg a(z) = d$ and $\deg b(z) \leq d - 1$. In fact, if $p(z)$ is symmetric, then $b(z) = 0$. We call the ordered pair of polynomials $(a(z), b(z))$ defined by (1.1) the symmetric decomposition of $p(z)$. As pointed out by Brändén and Solus \[13\], $p(z)$ alternatingly increasing if and only if $a(z)$ and $b(z)$ are both unimodal and have nonnegative coefficients. It is well known \[7, Theorem 10.5\] that if $P$ is an integral $d$-polytope that contains an interior lattice point, then there exist a symmetric decomposition of $h_P^*(z)$, and the reader is referred to \[57, Remark 4.1\] and \[7, Section 10.3\] for the combinatorial interpretation of this symmetric decomposition.

1.2. **Gamma-positivity.**

If $f(x)$ is symmetric, then $f(x)$ can be expanded uniquely as

$$f(x) = \sum_{k=0}^{\lfloor n/2 \rfloor} \gamma_k x^k (1+x)^{n-2k},$$

and it is said to be $\gamma$-positive if $\gamma_k \geq 0$ for $0 \leq k \leq \lfloor \frac{n}{2} \rfloor$ (see, e.g., \[12, 36, 41, 66\]). The $\gamma$-positivity of $f(x)$ implies symmetry and unimodality of $f(x)$. Gamma
positive polynomials appear often in combinatorial and geometric contexts. We refer the reader to Athanasiadis’s survey article [3] for details.

**Definition 2.** Let \((a(z), b(z))\) be the symmetric decomposition of the polynomial \(p(z)\). If \(a(z)\) and \(b(z)\) are both \(\gamma\)-positive, then we say that \(p(z)\) is bi-\(\gamma\)-positive.

It is clear that if a polynomial is bi-\(\gamma\)-positive, then it is alternatingly increasing. An alternatingly increasing polynomial does not have to be bi-\(\gamma\)-positive. For example, \(f(x) = 1 + 2x + 3x^2 + x^3\) is alternating increasing, but \(f(x)\) is not bi-\(\gamma\)-positive.

As discussed in Section 5, using the theory of the homology of Rees products of posets, Athanasiadis [2, Theorem 1.3] discovered that colored derangement polynomial is bi-\(\gamma\)-positive. In [4, Section 5.1], Athanasiadis pointed out that it would be interesting to find other classes of bi-\(\gamma\)-positive polynomials. In this paper, we show that bi-\(\gamma\)-positive polynomials appear often in algebraic and geometric context.

### 1.3. Eulerian polynomials.

Let \([n] = \{1, 2, \ldots, n\}\). Let \(\mathfrak{S}_n\) denote the symmetric group of all permutations of \([n]\) and let \(\pi = \pi_1\pi_2 \cdots \pi_n \in \mathfrak{S}_n\). A *descent* of \(\pi\) is an index \(i \in [n-1]\) such that \(\pi_i > \pi_{i+1}\). Let \(\text{des}(\pi)\) denote the number of descents of \(\pi\). The classical *Eulerian polynomial* is defined by

\[
A_n(x) = \sum_{\pi \in \mathfrak{S}_n} x^{\text{des}(\pi)}.
\]

The \(\gamma\)-positivity of \(A_n(x)\) was first shown by Foata and Schützenberger [29]. An explicit combinatorial interpretation of the \(\gamma\)-coefficients of \(A_n(x)\) appeared in the work of Foata and Strehl [30]. An index \(i \in [n]\) is a *double descent* of \(\pi\) if \(\pi_{i-1} > \pi_i > \pi_{i+1}\), where \(\pi_0 = \pi_{n+1} = n+1\). Let \(a(n, k)\) be the number of permutations in \(\mathfrak{S}_n\) with no double descents and \(\text{des}(\pi) = k\). It is well known that

\[
A_n(x) = \sum_{k=0}^{\lfloor (n-1)/2 \rfloor} a(n, k) x^k (1 + x)^{n-1-2k}.
\]

Given a Coxeter system \((W, S)\) and let \(\sigma \in W\). Let \(\ell_W(\sigma)\) be the length of \(\sigma\) in \(W\) with respect \(S\). Let \(D_W(\sigma) = \{ s \in S \mid \ell_W(\sigma s) < \ell_W(\sigma) \}\). The number of \(W\)-*descents* of \(\sigma\) is \(\text{des}_W(\sigma) = \# D_W(\sigma)\). The Eulerian polynomial \(P(W, x)\) of a finite Coxeter group \(W\) is the polynomial

\[
P(W, x) = \sum_{\sigma \in W} x^{\text{des}_W(\sigma)}.
\]

This polynomial is also the \(h\)-polynomial of the Coxeter complex associated to \((W, S)\), see, e.g., [15, 36, 62]. In particular, we have \(P(A_n, x) = A_n(x)\).
There is a large literature on the refinement of $P(W, x)$, see, e.g., [8, 31, 47, 48]. For example, let

$$a_j(n, k) = \# \{ \pi \in \mathfrak{S}_n \mid \pi_n = n + 1 - j, \ \des (\pi) = k \}.$$ 

The corresponding $(A, j)$-Eulerian polynomial is

$$(1.2) \quad A_n^{(j)}(x) = \sum_{k=0}^{n-1} a_j(n, k) x^k.$$ 

Beck, Jochemko and McCullough [8, Lemma 3.5] discovered that $A_n^{(j)}(x)$ is alternating increasing if $\frac{n}{2} < j \leq n$. Motivated by the recent work of Beck et al. [8], in Section 2, we show that a $q$-analog of $A_n(x)$ introduced by Foata and Schützenberger [29] is bi-γ-positive for each $0 < q \leq 1$, and in Section 3, we show that a $q$-analog of $P(B_n, x)$ introduce by Brenti [15] is bi-γ-positive for each $q \geq 1$.

1.4. $s$-inversion sequences and $s$-Eulerian polynomials.

Let $s = \{s_i\}_{i=1}^{\infty}$ be a sequence of positive integers. A geometric interpretation of Eulerian polynomials is obtained by considering the $s$-lecture hall polytope $P_n^{(s)}$, which is defined by

$$P_n^{(s)} = \left\{ (\lambda_1, \lambda_2, \ldots, \lambda_n) \in \mathbb{R}^n \mid 0 \leq \frac{\lambda_1}{s_1} \leq \frac{\lambda_2}{s_2} \leq \cdots \leq \frac{\lambda_n}{s_n} \leq 1 \right\}.$$ 

Let $I_n^{(s)} = \{ e = (e_1, e_2, \ldots, e_n) \in \mathbb{Z}^n \mid 0 \leq e_i < s_i \text{ for } i \in [n] \}$ be the set of $n$-dimensional $s$-inversion sequences. The number of ascents of $e$ is defined by

$$\asc (e) = \# \left\{ i \in \{0, 1, 2, \ldots, n-1\} \mid \frac{e_i}{s_i} < \frac{e_{i+1}}{s_{i+1}} \right\},$$

where we assume that $e_0 = 0$ and $s_0 = 1$. The $s$-Eulerian polynomial is defined as

$$E_n^{(s)}(x) = \sum_{e \in I_n^{(s)}} x^{\asc (e)}.$$ 

It is well known [50] that the $s$-Eulerian polynomial is the $h^*$-polynomial of $P_n^{(s)}$. In particular, according to [50] Lemma 1, we have $E_n^{(1, 2, \ldots, n)}(x) = A_n(x)$. Savage and Visontai [51] showed that for any sequence $s$ of positive integers, the $s$-Eulerian polynomial $E_n^{(s)}(x)$ has only real zeros.

Consider the following sequences

- $s_1 = (1, k + 1, 2k + 1, \ldots, (n-1)k + 1, \ldots),$
- $s_2 = (1, 1, 3, 2, 5, 3, 7, 4, \ldots, 2n - 1, n, \ldots),$
- $s_3 = (1, 4, 3, 8, \ldots, 2n - 1, 4n, \ldots),$
- $s_4 = (r, 2r, \ldots, nr, \ldots).$

The corresponding $s$-Eulerian polynomials have been extensively studied in algebraic, topological and geometric combinatorics (see [8, 13, 40, 57, 64] for instance).
By using the corresponding enumerative polynomials on combinatorial structures, we present a unified elementary proof of the bi-$\gamma$-positivity of these classical $s$-Eulerian polynomials.

1.5. **Context-free grammars.**

The main tool of this paper is context free grammars. Let $V$ be an alphabet whose letters are regarded as independent commutative indeterminates. A *context-free grammar* $G$ over $V$ is a set of substitution rules replacing a variable in $V$ by a Laurent polynomial of variables in $V$, see [18, 21, 27, 45] for details. The formal derivative $D := D_G$ with respect to $G$ is defined as a linear operator acting on Laurent polynomials with variables in $V$ such that each substitution rule is treated as the common differential rule that satisfies the relations: $D(u + v) = D(u) + D(v)$, $D(uv) = D(u)v + uD(v)$. For a constant $c$, we have $D(c) = 0$.

Context-free grammar is an elementary tool for studying labeled structures, including permutations [18, 35, 45, 65] and increasing trees [21, 22, 27]. For example, Chen and Yang [22] found a grammar for the Ramanujan-Shor polynomials. Moreover, context-free grammars can be used to prove the $\gamma$-positivity of $A_n(x)$ and $P(B_n, x)$, see [45]. We now recall two basic definitions.

**Definition 3 ([21]).** A grammatical labeling is an assignment of the underlying elements of a combinatorial structure with variables, which is consistent with the substitution rules of a grammar.

**Definition 4 ([45]).** A change of grammar is a substitution method in which the original grammar is replaced with functions of other grammar.

Consider the grammar

$$G = \{ x \to f_1(x, y, z, \ldots), y \to f_2(x, y, z, \ldots), z \to f_3(x, y, z, \ldots), \ldots \}.$$ 

If $f_1(x, y, z, \ldots) = f_2(y, x, z, \ldots)$, then we say that $G$ is partial symmetric. As illustrated in the proof of Theorem 7 if a structure corresponds to a partial symmetric grammar, then the type of change of grammars may be given as follows:

$$\begin{cases} 
  u = xy, \\
  v = x + y.
\end{cases}$$

1.6. **The organization of the paper.**

In Section 2 we study the bi-$\gamma$-positivity of $q$-Eulerian polynomials and $1/k$-Eulerian polynomials. In Section 3 we consider $q$-Eulerian polynomials of type $B$. In Section 4 we show that the descent polynomials of multipermutations and signed multipermutations are bi-$\gamma$-positive. In Section 5 we discuss the bi-$\gamma$-positivity of colored Eulerian and derangement polynomials.
2. \(q\)-Eulerian polynomials and the \(1/k\)-Eulerian polynomials

2.1. \(q\)-Eulerian polynomials.

Let \(\pi = \pi_1 \pi_2 \cdots \pi_n \in S_n\). An \textit{excedance} of \(\pi\) is an entry \(\pi_i\) such that \(\pi_i > i\). Let \(\text{exc}(\pi)\) (resp. \(\text{cyc}(\pi)\)) be the number of excedances (resp. cycles) of \(\pi\). In [29], Foata and Schützenberger introduced a \(q\)-analog of \(A_n(x)\) defined by

\[
A_n(x; q) = \sum_{\pi \in S_n} x^{\text{exc}(\pi)} q^{\text{cyc}(\pi)}.
\]

The \(q\)-Eulerian polynomials satisfy the recurrence relation

\[
A_{n+1}(x; q) = (nx + q)A_n(x; q) + x(1 - x) \frac{d}{dx} A_n(x; q),
\]

with the initial conditions \(A_0(x; q) = 1\), \(A_1(x; q) = q\) and \(A_2(x; q) = q(x+q)\) (see [16, Proposition 7.2]). By using the theory of homomorphism of symmetric function, Brenti [16] studied \(A_n(x; q)\). In particular, he obtained that

\[
\sum_{n \geq 0} A_n(x, q) \frac{x^n}{n!} = \left( \frac{1 - x}{e^{x(x-1)} - x} \right)^q.
\]

Using multiplier sequences, Brändén [11] proved that if \(q > n + q \leq 0\) or \(q \in \mathbb{Z}\), then \(A_n(x; q)\) has only real zeros.

2.2. \(1/k\)-Eulerian polynomials.

In the rest of this section, we always let \(k\) be a fixed positive integer. The \(q\)-Eulerian polynomial is closely related to \(1/k\)-Eulerian polynomials. Let

\[
s = (1, k + 1, 2k + 1, \ldots, (n-1)k + 1, \ldots).
\]

The \(1/k\)-Eulerian polynomials

\[
N_n,k(x) = E_n^{(1,k+1,2k+1,\ldots,(n-1)k+1)}(x)
\]

was defined and studied in [43, 49, 51]. In particular, Savage and Viswanathan [49, Section 1.5] discovered that

\[
N_n,k(x) = k^n A_n(x; 1/k) = \sum_{\pi \in S_n} x^{\text{exc}(\pi)} k^{n-\text{cyc}(\pi)}.
\]

Note that \(A_n(x; q) = q^n N_{n;1/q}(x)\). Hence

\[
\sum_{n \geq 0} N_{n,k}(x) \frac{x^n}{n!} = \left( \frac{1 - x}{e^{x(x-1)} - x} \right)^{1/k}.
\]

Let \(N_{n,k}(x) = \sum_{j=0}^{n-1} N_{n,j;k} x^j\). It follows from (2.1) that

\[
N_{n+1,j;k} = (1 + kj)N_{n,j;k} + k(n - j + 1)N_{n,j-1;k},
\]

with the initial condition \(N_{1,0,k} = 1\) and \(N_{1,i;k} = 0\) for \(i \neq 0\). The first few \(N_{n,k}(x)\) are \(N_{1,k}(x) = 1\), \(N_{2,k}(x) = 1 + kx\), \(N_{3,k}(x) = 1 + 3kx + k^2 x(1 + x)\).
2.3. \textit{k-Stirling permutations}.

For \( n \geq 1 \), let \([n]_2\) denote the multiset \( \{1, 1, 2, 2, \ldots, n, n\} \), in which we have two copies of each integer \( i \), where \( 1 \leq i \leq n \). Stirling permutations were introduced by Gessel and Stanley \([37]\). A \textit{Stirling permutation} of order \( n \) is a permutation of the multiset \([n]_2\) such that for each \( i, 1 \leq i \leq n \), all entries between the two occurrences of \( i \) are larger than \( i \). Denote by \( Q_n \) the set of \textit{Stirling permutations} of order \( n \).

Let \( \sigma = \sigma_1 \cdots \sigma_{2n} \in Q_n \). For \( 1 \leq i \leq 2n \), we say that an index \( i \) is a \textit{descent} (resp. \textit{ascent}) of \( \sigma \) if \( \sigma_i > \sigma_{i+1} \) or \( i = 2n \) (resp. \( \sigma_i < \sigma_{i+1} \) or \( i = 1 \)). A \textit{plateau} of \( \sigma \) is an index \( i \in [2n-1] \) such that \( \sigma_i = \sigma_{i+1} \). Various statistics on Stirling permutations were repeatedly discovered, see e.g., \([9, 34, 39]\). For example, Haglund and Visontai \([34]\) studied the multivariate refinements of the descent polynomials of generalized Stirling permutations.

Let \( j^i = (j_1, j_2, \ldots, j_i) \) for \( i, j \geq 1 \). We call a permutation of \( \{1^k, 2^k, \ldots, n^k\} \) a \textit{k-Stirling permutation} of order \( n \) if for each \( i, 1 \leq i \leq n \), all entries between the two occurrences of \( i \) are at least \( i \). Denote by \( Q_n(k) \) the set of \textit{k-Stirling permutations} of order \( n \). Let \( \sigma = \sigma_1 \sigma_2 \cdots \sigma_{nk} \in Q_n(k) \). We say that an index \( i \in \{2, 3, \ldots, nk-k+1\} \) is a \textit{longest ascent plateau} if \( \sigma_{i-1} < \sigma_i = \sigma_{i+1} = \sigma_{i+2} = \cdots = \sigma_{i+k-1} \). A \textit{longest left ascent plateau} of \( \sigma \) is a longest ascent plateau of \( \sigma \) endowed with a 0 in the front of \( \sigma \). Let \( \text{ap}(\sigma) \) (resp. \( \text{lap}(\sigma) \)) be the number of longest ascent plateaus (resp. longest left ascent plateaus) of \( \sigma \). If \( \sigma_1 = \sigma_2 = \cdots = \sigma_k \), then \( \text{lap}(\sigma) = \text{ap}(\sigma) + 1 \). Otherwise, \( \text{lap}(\sigma) = \text{ap}(\sigma) \).

Following \([13]\), we have

\begin{equation}
\label{eq:2.2}
N_{n;k}(x) = \sum_{\sigma \in Q_n(k)} x^{\text{ap}(\sigma)} x^n N_{n;k} \left( \frac{1}{x} \right) = \sum_{\sigma \in Q_n(k)} x^{\text{lap}(\sigma)}.
\end{equation}

Note that \( \text{deg} \ N_{n;k}(x) = n-1 \). Let \( (N_{n;k}^{+}(x), N_{n;k}^{-}(x)) \) be the symmetric decomposition of \( N_{n;k}(x) \). Combining (1.1) and (2.2) yields

\[ N_{n;k}^{+}(x) = \frac{\sum_{\sigma \in Q_n(k)} x^{\text{ap}(\sigma)} - \sum_{\sigma \in Q_n(k)} x^{\text{lap}(\sigma)}}{1-x} = \sum_{\sigma \in Q_n(k)} x^{\text{ap}(\sigma)}.
\]

Let \( \overline{Q}_n(k) = \{ \sigma \in Q_n(k) \mid \sigma_j < \sigma_{j+1} \text{ for some } j \in [k-1] \} \). Then we obtain the following result.

\textbf{Proposition 5.} For \( n \geq 1 \), we have

\[ N_{n;k}^{+}(x) = \sum_{\sigma \in Q_n(k)} x^{\text{ap}(\sigma)}, \quad N_{n;k}^{-}(x) = \sum_{\sigma \in Q_n(k)} x^{\text{ap}(\sigma)-1}. \]

In the following subsection, we show the bi-\(\gamma\)-positivity of \( N_{n;k}(x) \).
2.4. Main results.

Lemma 6. Let

\[ G = \{ I \to Iy, \ x \to kxy, \ y \to kxy \}. \]

Then we have

\[ D^n(I) = I \sum_{j=0}^{n-1} N_{n,j,k} x^j y^{n-j}. \]

Proof. Note that \( D(I) = Iy, \ D^2(I) = I(y^2 + kxy) \). Hence the result holds for \( n = 1, 2 \). Now assume that (2.4) holds for some \( n \), where \( n \geq 2 \). Note that

\[ D^{n+1}(I) = D \left( I \sum_{j=0}^{n-1} N_{n,j,k} x^j y^{n-j} \right) \]

\[ = \sum_j N_{n,j,k} I \left( x^j y^{n-j+1} + k j x^j y^{n-j+1} + k(n-j) x^{j+1} y^{n-j} \right). \]

Taking coefficients of \( x^j y^{n-j+1} \) on both sides yields

\[ N_{n+1,j;k} = (1 + k) N_{n,j;k} + k(n-j+1) N_{n,j-1;k}, \]

as desired. So the proof follows by induction.

We can now conclude the first main result of this paper.

Theorem 7. For each \( k \geq 1 \) and \( n \in \mathbb{N} \), the polynomial \( N_{n;k}(x) \) is bi-\( \gamma \)-positive. More precisely, we have \( N_{n,k}(x) = N_{n,k}^+(x) + x N_{n,k}^-(x) \), where

\[ N_{n,k}^+(x) = \sum_{i=0}^{\lfloor (n-1)/2 \rfloor} N_{n,i;k}^+ x^i (1+x)^{n-1-2i}, \]

\[ N_{n,k}^-(x) = \sum_{i=0}^{\lfloor (n-2)/2 \rfloor} N_{n,i;k}^- x^i (1+x)^{n-2-2i}. \]

The \( \gamma \)-coefficients \( N_{n,i;k}^+ \) and \( N_{n,i;k}^- \) satisfy the recurrence system

\[ \begin{align*}
N_{n+1,i;k}^+ &= (1+ki) N_{n,i;k}^+ + 2k(n-2i+1) N_{n,i-1;k}^+ + N_{n,i-1;k}^- \\
N_{n+1,i;k}^- &= k(i+1) N_{n,i;k}^- + 2k(n-2i) N_{n,i-1;k}^- + (k-1) N_{n,i;k}^+,
\end{align*} \]

with the initial conditions \( N_{1,0;k}^+ = 1, \ N_{1,i;k}^- = 0 \) for \( i \neq 0 \) and \( N_{1,i;k}^- = 0 \) for any \( i \). For each \( k \geq 1 \), the \( \gamma \)-coefficients \( N_{n,i;k}^+ \) and \( N_{n,i;k}^- \) are nonnegative integers. Equivalently, the \( q \)-Eulerian polynomial \( A_n(x;q) \) is bi-\( \gamma \)-positive for each \( 0 < q \leq 1 \).

Proof. Consider a change of the grammar (2.3). Note that \( D(I) = Iy, \ D(Iy) = Iy(x+y) + (k-1)Ixy, \ D(x+y) = 2kxy, \ D(xy) = kxy(x+y) \). Setting \( J = Iy, u = \)
Multiplying both sides of (2.5) by \( x \), we obtain \( D(I) = J, D(J) = Ju + (k - 1)Iv, D(u) = 2kv, D(v) = kuv \). Consider the grammar

\[
(2.6) \quad G = \{ I \to J, J \to Ju + (k - 1)Iv, u \to 2kv, v \to kuv \}.
\]

By induction, it is routine to verify that there are nonnegative integers \( N_{n,i,k}^+ \) and \( N_{n,i,k}^- \) such that

\[
(2.7) \quad D^n(I) = J \sum_{i=0}^{\lfloor (n-1)/2 \rfloor} N_{n,i,k}^+ v^i u^{n-1-2i} + Iv \sum_{i=0}^{\lfloor (n-2)/2 \rfloor} N_{n,i,k}^- v^i u^{n-2-2i}.
\]

In particular, we have \( D(I) = J, D^2(I) = Ju + (k - 1)Iv \). Note that

\[
D^{n+1}(I) = D(D^n(I)) = (Ju + (k - 1)Iv) \sum_{i=0}^{\lfloor (n-1)/2 \rfloor} N_{n,i,k}^+ v^i u^{n-1-2i} + Ju \sum_{i=0}^{\lfloor (n-2)/2 \rfloor} N_{n,i,k}^- v^i u^{n-2-2i} + J \sum_{i} N_{n,i,k}^+ (kiu^{n-2i} + 2k(n - 1 - 2i)v^{i+1}u^{n-2-2i}) +
\]

\[
I \sum_{i} N_{n,i,k}^- (k(i + 1)v^{i+1}u^{n-1-2i} + 2k(n - 2 - 2i)v^{i+2}u^{n-3-2i}).
\]

Taking coefficients of \( Jv^i u^{n-2i} \) and \( Iv^{i+1} u^{n-1-2i} \) on both sides and simplifying yields (2.5). When \( k \geq 1 \), it is clear that \( N_{n,i,k}^+ \) and \( N_{n,i,k}^- \) are both nonnegative integers. Comparing (2.4) and (2.7), we see that \( N_{n,k}(x) = N_{n,k}^+(x) + xN_{n,k}^-(x) \),

\[
N_{n,k}^+(x) = \sum_{i=0}^{\lfloor (n-1)/2 \rfloor} N_{n,i,k}^+ x^i (1 + x)^{n-1-2i},
\]

\[
N_{n,k}^-(x) = \sum_{i=0}^{\lfloor (n-2)/2 \rfloor} N_{n,i,k}^- x^i (1 + x)^{n-2-2i}.
\]

This completes the proof. \( \Box \)

Define

\[
\hat{N}_{n,k}^+(x) = \sum_{i=0}^{\lfloor (n-1)/2 \rfloor} N_{n,i,k}^+ x^i, \quad \hat{N}_{n,k}^-(x) = \sum_{i=0}^{\lfloor (n-2)/2 \rfloor} N_{n,i,k}^- x^i.
\]

Then

\[
N_{n,k}^+(x) = (1 + x)^{n-1} \hat{N}_{n,k}^+(x), \quad N_{n,k}^-(x) = (1 + x)^{n-2} \hat{N}_{n,k}^-(x).
\]

Multiplying both sides of (2.5) by \( x^i \) and summing over all \( i \), we obtain that

\[
\begin{align*}
\hat{N}_{n+1,k}^+(x) &= (1 + 2k(n - 1)x)\hat{N}_{n,k}^+(x) + kx(1 - 4x)\frac{d}{dx} \hat{N}_{n,k}^+(x) + x\hat{N}_{n,k}^-(x), \\
\hat{N}_{n-1,k}^-(x) &= k(1 + 2(n - 2)x)\hat{N}_{n,k}^-(x) + kx(1 - 4x)\frac{d}{dx} \hat{N}_{n,k}^-(x) + (k - 1)\hat{N}_{n,k}^+(x).
\end{align*}
\]

Substituting \( x \to x/(1 + x)^2 \) into the above recurrence system and simplifying some terms gives the following result.
Corollary 8. For $n \geq 1$, we have

$$
\begin{align*}
N_{n+1;k}^+(x) &= (1 + x + k(n - 1)x)N_{n;k}^+(x) + kx(1 - x)\frac{d}{dx}N_{n;k}^+(x) + xN_{n;k}^-(x), \\
N_{n+1;k}^-(x) &= k(1 + (n - 1)x)N_{n;k}^-(x) + kx(1 - x)\frac{d}{dx}N_{n;k}^-(x) + (k - 1)N_{n;k}^+(x),
\end{align*}
$$

with the initial conditions $N_{1;k}^+(x) = 1$ and $N_{1;k}^-(x) = 0$.

3. $q$-Eulerian Polynomials of Type $B$

Let $\pm [n] = [n] \cup \{-1, -2, \ldots, -n\}$. Denote by $B_n$ the hyperoctahedral group of rank $n$. Elements of $B_n$ are signed permutations of $\pm [n]$ with the property that $\pi(-i) = -\pi(i)$ for all $i \in [n]$. The number of $B_n$-descents of $\pi$ is defined by

$$
\text{des}_B(\pi) = \# \{i \in \{0, 1, 2, \ldots, n - 1\} \mid \pi(i) > \pi(i + 1)\},
$$

where $\pi(0) = 0$. By considering the type $B$ peak algebra, Petersen [47] showed that the polynomial $P(B_n, x)$ is $\gamma$-positive. By using the theory of hyperplane arrangements, Stembridge [62, Lemma 9.1] proved a connection among the Eulerian polynomials of types $A_n, B_n$ and $D_n$:

$$
P(B_n, x) = n2^{n-1}xP(A_{n-1}, x) + P(D_n, x).
$$

Let $N(\pi)$ denote the number of negative entries of $\pi \in B_n$. In [15], Brenti introduced a $q$-analog of $P(B_n, x)$:

$$
B_n(x, q) = \sum_{\pi \in B_n} x^{\text{des}_B(\pi)} q^{N(\pi)}.
$$

He proved that the polynomials $B_n(x, q)$ satisfy the recurrence relation

$$
B_n(x, q) = (1 + (1 + q)nx - x)B_{n-1}(x, q) + (1 + q)(x - x^2)\frac{\partial}{\partial x}B_{n-1}(x, q),
$$

with the initial condition $B_0(x, q) = 1$, and $B_n(x, q)$ has only simple real zeros for each $q \geq 0$. The polynomial $B_n(x, q)$ can be also defined by the generating function

$$
\sum_{n \geq 0} B_n(x, q) \frac{z^n}{n!} = \frac{(1 - x)e^{z(1-x)}}{1 - xe^{(1-x)(1+q)}}.
$$

In particular, we have $B_1(x, q) = 1 + qx$, $B_2(x, q) = 1 + (1 + 4q + q^2)x + q^2x^2$. The polynomials $B_n(x, q)$ has been further studied by Adin et al. [1], Bränden [11], Savage and Visontai [51], Zhuang [64] and so on. For example, Bränden [11, Corollary 6.9] obtained that the polynomial

$$
\sum_{\pi \in B_n \atop N(\pi) \in S} x^{\text{des}_B(\pi)}
$$

has only real zeros, where $S$ is any subset of $[n]$. 

Very recently, Beck, Jochemko and McCullough [3, Theorem 3.9] discovered that the polynomial
\[ B_n^{(\ell)}(x) = 2^{\ell-1} \sum_{j=0}^{n-\ell} \binom{n-\ell}{j} A_n^{(j+\ell)}(x) \]
is alternating increasing for all \( 1 \leq \ell \leq d \), where \( A_n^{(j)}(x) \) is defined by (1.2) and \( B_n^{(\ell)}(x) \) is so called \( (B, \ell) \)-Eulerian polynomial defined on \( B_n \). Motivated by this result, we shall prove the alternating increasing property of \( B_n(x,q) \).

For a permutation \( \pi \in B_n \), an ascent of \( \pi \) is a position \( i \in \{0,1,\ldots, n-1\} \) such that \( \pi(i) < \pi(i+1) \), where \( \pi(0) = 0 \). Let \( B(\pi) \) be the number of ascents of \( \pi \in B_n \). Following [44], a weight of \( \pi \in B_n \) may be given as follows:
\[ w(\pi) = xy(xz)^{\text{asc} B(\pi)}(yz)^{\text{des} B(\pi)}q^{\lambda B(\pi)} \]
We now recall a grammatical interpretation of a joint distribution on \( B_n \).

Lemma 9 ([44, Theorem 5]). Let \( q \) be a constant. If
\[ (3.1) \quad G = \{x \to qxyu, y \to xyz, z \to yzu, u \to qxzu\}, \]
then we have
\[ (3.2) \quad D^n(xy) = xy \sum_{\pi \in B_n} (xz)^{\text{asc} B(\pi)}(yz)^{\text{des} B(\pi)}q^{\lambda B(\pi)} \]
The second main result of this paper is the following.

Theorem 10. For each \( q \geq 1 \) and \( n \in \mathbb{N} \), the polynomial \( B_n(x,q) \) is bi-\( \gamma \)-positive. More precisely, let \( (B_n^+(x,q), B_n^-(x,q)) \) be the symmetric decomposition of \( B_n(x,q) \). Then we have \( B_n(x,q) = B_n^+(x,q) + xB_n^-(x,q) \), where
\[ B_n^+(x,q) = \sum_{i=0}^{[n/2]} \zeta^+_n(i)(q)x^i(1+x)^{n-2i}, \]
\[ B_n^-(x,q) = \sum_{i=0}^{[n-1/2]} \zeta^-_n(i)(q)x^i(1+x)^{n-1-2i}. \]
The \( \gamma \)-coefficients satisfy the following recurrence system
\[
\begin{align*}
\zeta^+_n(i+1)(q) &= (1 + (1 + q)i)\zeta^+_n(i)(q) + 2(1 + q)(n - 2i + 2)\zeta^+_n(i-1)(q) + 2\zeta^-_n(i-1)(q), \\
\zeta^-_n(i+1)(q) &= (i + q(i + 1))\zeta^-_n(i)(q) + 2(1 + q)(n - 2i + 1)\zeta^-_n(i-1)(q) + (q - 1)\zeta^+_n(i)(q).
\end{align*}
\]
In particular, \( \zeta^+_1(q) = 1 \) and \( \zeta^-_1(q) = q - 1 \). If \( q \geq 1 \), then the \( \gamma \)-coefficients are all nonnegative numbers. Therefore, \( B_n(x,q) \) is alternatingly increasing if \( q \geq 1 \).

Proof. Consider the grammar [43]. Note that \( D(x) = qx(yu), D(y) = y(xz), \)
\[ D(yu) = (1 + q)(xz)(yu), \quad D(xz) = (1 + q)(xz)(yu). \]
Set \( A = xz, B = yu, U = (xz)(yu), V = xz + yu \) and \( L = xy \). Then \( U = AB \) and \( V = A + B \). It follows that

\[
D(U) = (1 + q)UV, \quad D(V) = 2(1 + q)U, \quad D(L) = LA + qLB = LV + (q - 1)LB.
\]

Setting \( LB = H \), we have \( D(H) = 2LU + qHV \). Consider the grammar

\[
G_1 = \{ L \rightarrow LV + (q - 1)H, H \rightarrow 2LU + qHV, U \rightarrow (1 + q)UV, V \rightarrow 2(1 + q)U \}.
\]

For \( n = 1, 2, 3 \), we have

\[
\begin{align*}
D_{G_1}(L) &= VL + (q - 1)H, \quad D^2_{G_1}(L) = (V^2 + 4qU)L + (q^2 - 1)VH, \\
D^3_{G_1}(L) &= ((2 + 12q + 6q^2)UV + V^3)L + (2(q - 1)(1 + 4q + q^2)U + (q^3 - 1)V^2)H.
\end{align*}
\]

By induction, it is routine to verify that

\[
D^n_{G_1}(L) = L \sum_{i=0}^{\left\lfloor \frac{n}{2} \right\rfloor} \zeta^+_n,i(q)U^iV^{n-2i} + H \sum_{i=0}^{\left\lfloor \frac{n-1}{2} \right\rfloor} \zeta^-_n,i(q)U^iV^{n-1-2i}.
\]

Therefore, applying the operator \( D_{G_1} \) to both sides of (3.3) yields

\[
\begin{align*}
D^{n+1}_{G_1}(L) &= (LV + (q - 1)H) \sum_i \zeta^+_n,i(q)U^iV^{n-2i} + \\
&\quad (2LU + qHV) \sum_i \zeta^-_n,i(q)U^iV^{n-1-2i} + \\
&\quad L \sum_i \zeta^+_n,i(q) (1 + q) U^i V^{n-2i+1} + 2(1 + q)(n - 2i) U^{i+1} V^{n-2i-1} + \\
&\quad H \sum_i \zeta^-_n,i(q) (1 + q) U^i V^{n-2i} + 2(1 + q)(n - 1 - 2i) U^{i+1} V^{n-2i-2}.
\end{align*}
\]

Comparing the coefficients of \( LU^iV^{n+1-2i} \) and \( HU^iV^{n-2i} \) on both sides and simplifying yields the recurrence system of \( \zeta^+_n,i(q) \) and \( \zeta^-_n,i(q) \). Define

\[
\zeta^+_n(x, q) = \sum_{i=0}^{\left\lfloor \frac{n}{2} \right\rfloor} \zeta^+_n,i(q)x^i, \quad \zeta^-_n(x, q) = \sum_{i=0}^{\left\lfloor \frac{n-1}{2} \right\rfloor} \zeta^-_n,i(q)x^i.
\]

Comparing (3.2) and (3.3), we obtain that

\[
D^n(xy) \big|_{x=y=z=1} = \sum_{i=0}^{\left\lfloor \frac{n}{2} \right\rfloor} \zeta^+_n,i(q)u^i(1 + u)^{n-2i} + u \sum_{i=0}^{\left\lfloor \frac{n-1}{2} \right\rfloor} \zeta^-_n,i(q)u^i(1 + u)^{n-1-2i},
\]

which gives the desired \( \gamma \)-expansions. This completes the proof. \( \square \)
4. Descent polynomials of multipermutations and signed multipermutations

4.1. Descent polynomials of multipermutations.

For any \( n \)-element multiset \( M \), a descent (resp. ascent, plateau) in a multiset permutation \( \sigma \) of \( M \) is an index \( i \) such that \( \sigma_i > \sigma_{i+1} \) (resp. \( \sigma_i < \sigma_{i+1} \), \( \sigma_i = \sigma_{i+1} \)), where \( i \in [n-1] \). Let \( \text{des} (\sigma) \) (resp. \( \text{asc} (\sigma) \), \( \text{plat} (\sigma) \)) denote the number of descents (resp. ascents, plateaus) of \( \sigma \). Let \( \mathcal{C}_n \) denote the set of permutations of \([n]_2\). Set \([\overline{n}]_2 = [n]_2 \cup \{n+1\} \). Let \( \mathcal{D}_n \) denote the set of permutations of \([\overline{n}]_2\). Define

\[
\begin{align*}
P_n(x) &= \sum_{\sigma \in \mathcal{C}_n} x^{\text{des}(\sigma)} = \sum_{k=0}^{2n-2} P_{n,k} x^k, \\
Q_n(x) &= \sum_{\sigma \in \mathcal{D}_n} x^{\text{des}(\sigma)} = \sum_{k=0}^{2n-1} Q_{n,k} x^k.
\end{align*}
\]

The first few \( P_n(x) \) and \( Q_n(x) \) are given as follows:

\[
\begin{align*}
P_1(x) &= 1, & Q_1(x) &= 1 + 2x, & P_2(x) &= 1 + 4x + x^2, \\
Q_2(x) &= 1 + 12x + 15x^2 + 2x^3, & P_3(x) &= 1 + 20x + 48x^2 + 20x^3 + x^4.
\end{align*}
\]

Let \( s = (1,1,3,2,5,\ldots) \), where \( s_{2i} = i, s_{2i-1} = 2i - 1 \). By using the identity

\[
\sum_{k \geq 0} \binom{k+2}{2} x^k = \frac{P_n(x)}{(1-x)^{2n+1}},
\]

Savage and Visontai [51] Theorem 3.23 showed that

\[
P_n(x) = E_{2n}^{(1,1,3,2,5,3,7,\ldots,2n-1,n)}(x).
\]

The polynomial \( xP_n(x) \) has been studied by Carlitz and Hoggatt [17]. They found that \( xP_n(x) \) is a symmetric polynomial. And so \( P_n(x) \) is symmetric. It follows from [17] Eq. (2.8) that the numbers \( P_{n,k} \) satisfy the recurrence relation

\[
(4.1) \quad P_{n+1,k} = \binom{k+2}{2} P_{n,k} + (k+1)(2n-k+1) P_{n,k-1} + \binom{2n-k+2}{2} P_{n,k-2},
\]

with the initial condition \( P_{1,0} = 1 \) and \( P_{1,k} = 0 \) for \( k \neq 0 \).

From [50] Theorem 14], we see that

\[
\sum_{t \geq 0} (t+1)^{n+1} \binom{t+2}{2} x^t = E_{2n+1}^{(1,1,3,2,5,3,7,\ldots,2n-1,n,2n+1)}(x).
\]

MacMahon [16] Vol 2, Chapter IV, p. 211] showed that

\[
\sum_{\pi \in P(\{1^{p_1}, 2^{p_2}, \ldots, n^{p_n}\})} x^{\text{des}(\pi)} = \sum_{t \geq 0} \frac{(t+1)\cdots(t+p_1)\cdots(t+1)\cdots(t+p_n)}{p_1!p_2!\cdots p_n!} x^t.
\]

where \( P(\{1^{p_1}, 2^{p_2}, \ldots, n^{p_n}\}) \) denote the set of multipermutations of \( \{1^{p_1}, 2^{p_2}, \ldots, n^{p_n}\} \).

When \( p_1 = p_2 = \cdots = p_n = 2 \) and \( p_{n+1} = 1 \), we get
$Q_n(x) = E_{2n+1}^{(1,1,3,2,5,3,7,\ldots,2n-1,n,2n+1)}(x)$. 

4.2. Descent polynomials of signed multipermutations.

Let $C_n^\pm$ be the set of all signed permutations of the multiset $[n]_2$. The elements of $C_n^\pm$ are those of the form $\pm\sigma_1 \pm \sigma_2 \cdots \pm \sigma_{2n}$, where $\sigma_1\sigma_2\cdots\sigma_{2n} \in C_n$. A descent of $\sigma \in C_n^\pm$ is an index $i$ such that $\sigma_i > \sigma_{i+1}$, where $i \in \{0, 1, 2, \ldots, 2n-1\}$ and $\sigma_0 = 0$. As usual, we write $-i$ by $\overline{i}$ for each $i \in [n]$. Let $s = (1, 4, 3, 8, \ldots, 2n-1, 4n)$, where $s_{2i} = 4i, s_{2i-1} = 2i - 1$. The following equidistributed result was first conjectured by Savage and Visontai [51, Conjecture 3.25], and then independently proved by Chen et al. [20] and Lin [40]:

$$\sum_{\sigma \in C_n^\pm} x^{\text{des}(\sigma)} = E_{2n}^{(1,1,3,2,5,3,7,\ldots,2n-1,4n)}(x).$$

Let $D_n^\pm$ be the subset of signed permutations of $[\overline{n}]_2$ consisting of signed permutations such that the element $n+1$ carries a positive sign. In other words, any element of $D_n^\pm$ can be generated from one element of $C_n^\pm$ by inserting the entry $n+1$. Hence $\#D_n^\pm = (2n+1)\#C_n^\pm$. For example, $C_1^\pm = \{11, 1\overline{1}, 1\overline{T}, \overline{T}1\}$ and

$$D_1^\pm = \{112, 1\overline{T}2, 1\overline{1}2, 1\overline{T}2, 121, 1\overline{T}2, 2\overline{T}2, 2\overline{1}1, 2\overline{T}1, 2\overline{1}T, 2\overline{T}T\}.$$ 

A descent of $\sigma \in D_n^\pm$ is an index $i$ such that $\sigma_i > \sigma_{i+1}$, where $i \in \{0, 1, 2, \ldots, 2n\}$ and $\sigma_0 = 0$. Define

$$S_n(x) = \sum_{\sigma \in C_n^\pm} x^{\text{des}(\sigma)} = \sum_{k=0}^{2n-1} S_{n,k}x^k,$$

$$T_n(x) = \sum_{\sigma \in D_n^\pm} x^{\text{des}(\sigma)} = \sum_{k=0}^{2n} T_{n,k}x^k.$$

Following [40] Lemma 4], the numbers $S_{n,k}$ satisfy the recurrence relation

(4.2) 

$$S_{n+1,k} = \left(\frac{2i + 2}{2}\right)S_{n,i} + (2i(4n - 2i + 3) + 2n + 1)S_{n,i+1} + \left(\frac{4n - 2i + 5}{2}\right)S_{n,i-2},$$

with the initial conditions $S_{1,0} = 1, S_{1,1} = 3$ and $S_{1,i} = 0$ for $i < 0$ or $i > 1$. The first few $S_n(x)$ and $T_n(x)$ are $S_1(x) = 1 + 3x$, $T_1(x) = 1 + 8x + 3x^2$.

$$S_2(x) = 1 + 31x + 55x^2 + 9x^3, T_2(x) = 1 + 66x + 258x^2 + 146x^3 + 9x^4.$$ 

We now recall another interpretation of $T_n(x)$. Let $V_n$ be subset of signed permutations of $[\overline{n}]_2$ consisting of signed permutations such that the element $n+1$ carries a minus sign. For $\sigma \in V_n$, let $\text{des}_r(\sigma) = \{i \in [2n+1] \mid \sigma_i > \sigma_{i+1}\}$, where
\(\sigma_{2n+2} = 0\). Chen et al. [20, Theorem 3.1] proved that
\[
\sum_{\sigma \in \mathcal{D}_n} x^{\text{des}(\sigma)} = E_{2n+1}^{1,4,3,8,\ldots,2n-1,4n,2n+1}(x).
\]
In fact, via the following bijections
\[
\sigma = \sigma_1\sigma_2\cdots\sigma_n \mapsto \sigma' = \sigma_n\sigma_{n-1}\cdots\sigma_1 \mapsto (-\sigma_n)(-\sigma_{n-1})\cdots(-\sigma_1),
\]
we see that
\[
\sum_{\sigma \in \mathcal{D}_n} x^{\text{des}(\sigma)} = \sum_{\sigma \in \mathcal{D}_n} x^{\text{des}(\sigma)}.
\]
Therefore,
\[
\sum_{\sigma \in \mathcal{D}_n} x^{\text{des}(\sigma)} = E_{2n+1}^{1,4,3,8,\ldots,2n-1,4n,2n+1}(x).
\]

### 4.3. Main results.

The third main result of this paper can be summarized as follows.

**Theorem 11.** For any \(n \geq 1\), the polynomial \(P_n(x)\) is \(\gamma\)-positive, and the polynomials \(Q_n(x), S_n(x)\) and \(T_n(x)\) are all bi-\(\gamma\)-positive.

We divide the proof of Theorem 11 into four lemmas. Recall that \(N(\sigma)\) is the number of negative entries of \(\sigma\). In the following lemma, we assume that signed multipermutations are prepended and appended by 0. Define
\[
\text{des}^*(\sigma) = \text{des}(0\sigma 0), \; \text{asc}^*(\sigma) = \text{asc}(0\sigma 0).
\]

The following lemma is fundamental.

**Lemma 12.** Let \(G_1 = \{x \to w, \; y \to w\} \) and
\(\{x \to (1+q)^2 x^2 y^2, \; y \to (1+q)^2 x^2 y^2, \; w \to xy(q(x+y)+(1+q^2)y)\}\).
Then we have
\[
(D_2D_1)^n(x) = \sum_{\sigma \in \mathcal{C}_{n+1}^\pm} x^{\text{des}^*(\sigma)} y^{\text{asc}^*(\sigma)+\text{plat}(\sigma)} q^{N(\sigma)},
\]
\[
D_1(D_2D_1)^n(x) = w \sum_{\sigma \in \mathcal{D}_n^\pm} x^{\text{des}^*(\sigma)-1} y^{\text{asc}^*(\sigma)+\text{plat}(\sigma)-1} q^{N(\sigma)}.
\]

**Proof.** For any negative entry of a signed permutation, we put a subscript label \(q\).
For \(\sigma \in \mathcal{C}_{n+1}^\pm\), we label a descent of \(\sigma\) by \(x\) and label an ascent or a plateau by \(y\).
Thus the weight of \(\sigma \in \mathcal{C}_{n+1}^\pm\) is given by
\[
W(\sigma) = x^{\text{des}^*(\sigma)} y^{\text{asc}^*(\sigma)+\text{plat}(\sigma)} q^{N(\sigma)}.
\]
For example, \(\mathcal{C}_1^\pm = \{1y1\bar{y}, \bar{y}1\bar{y}, \bar{y}1\bar{y}, \bar{y}1\bar{y}1\bar{y}, \bar{y}1\bar{y}1\bar{y}1\bar{y}, \bar{y}1\bar{y}1\bar{y}1\bar{y}\}\). It is clear that the sum of weights of the elements in \(\mathcal{C}_{n+1}^\pm\) is given by \(D_2D_1(x)\). A grammatical labeling of \(\sigma' \in \mathcal{D}_n^\pm\) is given as follows: If \(i\) is a descent and \(\sigma'_i \neq n+1\), then put a label \(x\) right after \(\sigma'_i\).
Clearly, the action of \( D \) corresponds to one substitution rule in three cases: 

\[ W(\sigma') = w x^{\text{des} (\sigma') - 1} y^{\text{plat} (\sigma') + \text{asc} (\sigma') - 1} q^{N(\sigma)}. \]

For example, \( 2T_q^w 1^x \in D_1^\pm \). Note that \( D_2 D_1(x) = xy(q(x + y) + (1 + q^2)y) \) and 

\[ D_1(D_2 D_1(x)) = w(q(x + y)^2 + (1 + q^2)xy + y(1 + q^2)(x + y)). \]

It is easy to verify that the sum of weights of the elements in \( D_1^\pm \) is given by \( D_1(D_2 D_1(x)) \). Then the result holds for \( n = 1 \). We proceed by induction on \( n \).

Now we insert the entry \( n \) into \( \sigma \in C_{n-1}^\pm \), where \( n \geq 2 \). Note that the insertion of \( n \) corresponds to one substitution rule in \( G_1 \), since we always replace \( x \) by \( y \).

Clearly, the action of \( D_1 \) on elements of \( C_{n-1}^\pm \) generates all the elements in \( D_{n-1}^\pm \).

Let \( \sigma \in D_{n-1}^\pm \). Suppose that \( \sigma_i = n \). Now we insert \( n \) or \( n^\prime \) into \( \sigma \). We distinguish three cases:

\[(L_1) \text{ We can insert } n \text{ or } n^\prime \text{ right after } \sigma_i, \text{ or we can first replace } \sigma_i \text{ by } n^\prime, \text{ and then insert } n \text{ or } n^\prime \text{ right after } n^\prime. \text{ In this case, the insertion corresponds to applying the substitution rule } w \rightarrow xy(q(x + y) + (1 + q^2)y);\]

\[(L_2) \text{ Suppose that } j \text{ is a descent, where } |i - j| \geq 2. \text{ Then we can insert } n \text{ or } n^\prime \text{ right after } \sigma_j, \text{ or we can first replace } \sigma_i \text{ by } n^\prime, \text{ and then insert } n \text{ or } n^\prime \text{ right after } \sigma_j. \text{ It should be noted that we can get the same permutation if the first } n \text{ in the } (j + 1) \text{th position of } \sigma. \text{ In this case, the insertion corresponds to applying the substitution rule } x \rightarrow \frac{(1 + q)^2 x^2 y^2}{2w};\]

\[(L_3) \text{ Suppose that } j \text{ is an ascent or a plateau, where } |i - j| \geq 2. \text{ Then we can insert } n \text{ or } n^\prime \text{ right after } \sigma_j, \text{ or we can first replace } \sigma_i \text{ by } n^\prime, \text{ and then insert } n \text{ or } n^\prime \text{ right after } \sigma_j. \text{ We can get the same permutation if the first } n \text{ in the } (j + 1) \text{th position of } \sigma. \text{ In this case, the insertion corresponds to applying the substitution rule } y \rightarrow \frac{(1 + q)^2 x^2 y^2}{2w}.\]

Then by induction, it is easy to verify that the action of \( D_2 \) on elements of \( D_{n-1}^\pm \) generates all the elements in \( C_{n}^\pm \). This completes the proof. \( \square \)

The \( q = 0 \) case of Lemma 12 says that

\[ (D_2 D_1)^n(x) = x \sum_{\sigma \in C_n} x^{\text{des} (\sigma)} y^{2n - \text{des} (\sigma)}, \]

\[ D_1(D_2 D_1)^n(x) = w \sum_{\sigma \in D_n} x^{\text{des} (\sigma)} y^{2n - \text{des} (\sigma)}. \]

Equivalently, we have

\[ (D_2 D_1)^n(x) = x \sum_{k=0}^{2n-2} P_{n,k} x^k y^{2n-k}, \]

\[(4.4)\]
Proof. Let $p$ with the initial conditions $p(4.6)$.

**Lemma 13.** The polynomial $P_n(x)$ is $\gamma$-positive. For $n \geq 1$, we have

$$P_n(x) = \sum_{k=0}^{n-1} p_{n,k} x^k (1 + x)^{2n-2-2k},$$

where the numbers $p_{n,k}$ satisfy the recurrence relation

$$(4.6) \quad p_{n+1,k} = \left(\frac{k + 2}{2}\right) p_{n,k} + (k + 1)(4n - 4k + 1) p_{n,k-1} + 4 \left(\frac{2n - 2k + 2}{2}\right) p_{n,k-2},$$

with the initial conditions $p_{1,0} = 1$ and $p_{1,k} = 0$ for $k \geq 1$.

**Proof.** Let $G_1 = \{x \rightarrow w, y \rightarrow w\}$ and $G_2 = \{x \rightarrow \frac{x^2 y^2}{2w}, y \rightarrow \frac{x^2 y^2}{2w}, w \rightarrow xy^2\}$. Note that $D_1(x) = w, D_2 D_1(x) = xy^2, D_1(D_2 D_1(x)) = w(xy + y^2 + xy)$. By induction, it is routine to verify that there are nonnegative integers $p_{n,k}$ such that

$$(4.7) \quad (D_2 D_1)^n(x) = xy^2 \sum_{k=0}^{2n-2} p_{n,k}(xy)^k (x + y)^{2n-2-2k},$$

Note that $(D_2 D_1)^{n+1}(x) = I_1 + I_2 + I_3$, where

$$I_1 = xy^2 \sum_k p_{n,k}(x + y)^{2n-2-2k} (x^{k+1} y^{k+1} + (k + 1)(2n - 1 - 2k)x^{k+1} y^{k+1})+$$

$$xy^2 \sum_k p_{n,k} 2(2n - 2 - 2k)(2n - 3 - 2k)x^{k+2} y^{k+2} (x + y)^{2n-4-2k},$$

$$I_2 = xy^2 \sum_k p_{n,k} (2n - 2 - 2k)(2x^{k+1} y^{k+1} x^{2n-3-2k} + x^{k+2} y^{k+1} (x + y)^{2n-3-2k} +$$

$$xy^2 \sum_k p_{n,k} (2n - 2 - 2k)(x + y)^{2n-3-2k} ((k + 1)x^{k+1} y^{k+2} + (k + 2)x^{k+2} y^{k+1},$$

$$I_3 = xy^2 \sum_k p_{n,k} (k + 1)x^{k+1}(x + y)^{2n-1-2k} +$$

$$\frac{1}{2} xy^2 \sum_k p_{n,k} ((k + 1)^2 x^{k+1} y^k (x + y)^{2n-1-2k} + (k + 1)kx^k y^{k+1} (x + y)^{2n-1-2k} +$$

$$\frac{1}{2} xy^2 \sum_k p_{n,k} ((k + 1)x^{k+2} y^k (x + y)^{2n-2-2k} + (k + 1)x^{k+1} y^{k+1} (x + y)^{2n-2-2k}.$$
Combining like terms, we get that
\[ I_1 = xy^2 \sum_k p_n,k(1 + (k + 1)(2n - 1 - 2k))x^{k+1}y^{k+1}(x + y)^{2n - 2 - 2k} + \]
\[ xy^2 \sum_k p_n,k(2n - 2 - 2k)(2n - 3 - 2k)x^{k+2}y^{k+2}(x + y)^{2n - 4 - 2k}, \]
\[ I_2 = xy^2 \sum_k p_n,k(2n - 2 - 2k)(k + 3)x^{k+1}y^{k+1}(x + y)^{2n - 2 - 2k}, \]
\[ I_3 = xy^2 \sum_k p_n,k \left( \frac{k + 2}{2} \right) x^k y^k (x + y)^{2n - 2k}. \]
Comparing the coefficients of \( xy^2(x+y)^{2n-2k} \) and simplifying yields (1.0).\]

**Lemma 14.** We have \( Q_n(x) = R_n(x) + xP_n(x) \), where
\[ (4.8) \quad R_n(x) = \sum_{k=0}^{\lfloor (2n-1)/2 \rfloor} r_{n,k} x^k (1 + x)^{2n-1-2k}. \]
The \( \gamma \)-coefficients \( r_{n,k} = (k + 1)p_{n,k} + 4(n - k)p_{n,k-1} \). Thus \( (R_n(x), P_n(x)) \) is the symmetric decomposition of \( Q_n(x) \) and \( Q_n(x) \) is bi-\( \gamma \)-positive.

**Proof.** It follows from (1.3) that
\[ D_1(D_2D_1)^n(x) = w \sum_{k=0}^{2n-1} p_{n,k} x^ky^{2n-k-1}((k + 1)y + (2n - k - 1)x) + \]
\[ wx \sum_{k=0}^{2n-1} p_{n,k} x^ky^{2n-k-1}, \]
If we take \( R_{n,k} = (k + 1)p_{n,k} + (2n - k)p_{n,k-1} \), then we get
\[ D_1(D_2D_1)^n(x) = w \sum_{k=0}^{2n-1} R_{n,k} x^ky^{2n-k} + wx \sum_{k=0}^{2n-1} p_{n,k} x^ky^{2n-k-1}. \]
Using the symmetry of \( P_n(x) \), we get \( R_{n,k} = R_{n,2n-k-1} \). Let \( R_n(x) = \sum_{k=0}^{2n-1} R_{n,k} x^k \).
It follows from (1.3) that \( Q_n(x) = R_n(x) + xP_n(x) \). It is to verify that
\[ R_n(x) = (1 + (2n - 1)x)P_n(x) + x(1 - x)P'_n(x). \]
It follows from (1.7) that
\[ D_1(D_2D_1)^n(x) \]
\[ = w \sum_k p_{n,k}(x + y)^{2n-3-2k} ((k + 1)x^{k+1}y^{k+1}(x + y)^2 + 2(2n - 2 - 2k)x^{k+1}y^{k+2}) + \]
\[ wx \sum_k p_{n,k}x^{k+1}y^{k+1}(x + y)^{2n-2-2k}. \]
Comparing the coefficient of \( x^ky^{k+1}(x + y)^{2n-1-2k} \) yields
\[ r_{n,k} = (k + 1)p_{n,k} + 4(n - k)p_{n,k-1}. \]
This completes the proof.

Note that

\[ xP_n(x) = \sum_{\sigma \in D_n} x^{\des(\sigma)}. \]

By Lemma 14 we immediately get the following result.

**Proposition 15.** For \( n \geq 1 \), we have

\[ R_n(x) = \sum_{\sigma \in D_n} x^{\des(\sigma)}. \]

In the rest of this subsection, we consider the descent polynomials of signed multipermutations. The \( q = 1 \) case of Lemma 12 says that

\[ \sum_{\sigma \in C_{\pm}^{\mathbb{D}} n} x^{\des^*(\sigma)} y^{\asc^*(\sigma) + \plat(\sigma)} \]

(4.9)

\[ D_1(D_2 D_1)^n(x) = w \sum_{\sigma \in D_{\pm}^{\mathbb{D}}} x^{\des^*(\sigma) - 1} y^{\asc^*(\sigma) + \plat(\sigma) - 1}. \]

(4.10)

Recall that

\[ T_{n,k} = \# \{ \des(\sigma) = k \mid \sigma \in D_{\pm}^n \}. \]

We first give a connection between the numbers \( T_{n,k} \) and \( S_{n,k} \). There are two ways that we can get an element \( \sigma \in D_{\pm}^n \) with \( \des(\sigma) = k \) from a permutation \( \sigma' \in C_{\pm}^{\mathbb{D}} n \) by inserting the entry \( n + 1 \) into \( \sigma' \). If \( \des(\sigma') = k \), then we can insert \( n + 1 \) at the end of \( \sigma' \), or put the entry \( n + 1 \) between two entries that form a descent. This gives \( k + 1 \) choices for the positions of \( n + 1 \). If \( \des(\sigma') = k - 1 \), then we can insert the entry \( n + 1 \) into the other \( 2n - (k - 1) \) positions. Therefore, we have

(4.11)

\[ T_{n,k} = (k + 1) S_{n,k} + (2n - k + 1) S_{n,k-1}. \]

**Lemma 16.** Let \( G_1 = \{ x \to w, y \to w \} \) and

\[ G_2 = \{ x \to \frac{2x^2y^2}{w}, y \to \frac{2x^2y^2}{w}, w \to xy(x + 3y) \}. \]

Then we have

\[ (D_2 D_1)^n(x) = \frac{1}{w} \sum_{k=0}^{2n-1} S_{n,k} x^{2n-1-k} y^k, \]

(4.12)

\[ D_1(D_2 D_1)^n(x) = w \sum_{k=0}^{2n} T_{n,k} x^{2n-k} y^k. \]

Furthermore, we have

\[
\begin{cases}
S_n(x) = \sum_k \eta_{2n,k}^+ x^k (1 + x)^{2n-1-2k} + x \sum_k \eta_{2n,k}^- x^k (1 + x)^{2n-2-2k}, \\
T_n(x) = \sum_k \eta_{2n+1,k}^+ x^k (1 + x)^{2n-2k} + x \sum_k \eta_{2n+1,k}^- (xy)^k (1 + x)^{2n-1-2k},
\end{cases}
\]
where the coefficients of these expansions satisfy following the recurrence system

\[
\begin{align*}
\eta^{+}_{2m+1,k} &= (1 + k)\eta^{+}_{2m,k} + 2(2m - 2k + 1)\eta^{+}_{2m,k-1} + \eta^{-}_{2m,k-1}, \\
\eta^{-}_{2m+1,k} &= (1 + k)\eta^{-}_{2m,k} + 4(m - k)\eta^{-}_{2m,k-1}, \\
\eta^{+}_{2m+2,k} &= (1 + 2k)\eta^{+}_{2m+1,k} + 8(m - k + 1)\eta^{+}_{2m+1,k-1}, \\
\eta^{-}_{2m+2,k} &= (3 + 2k)\eta^{-}_{2m+1,k} + 4(2m - 2k + 1)\eta^{-}_{2m+1,k-1} + 2\eta^{+}_{2m+1,k},
\end{align*}
\]

(4.13)

with the initial conditions \( \eta^{+}_{1,0} = 1, \eta^{-}_{1,0} = 0, \eta^{+}_{2,0} = 1 \) and \( \eta^{-}_{2,0} = 2 \). Since the \( \gamma \)-coefficients are all nonnegative integers, the polynomials \( S_n(x) \) and \( T_n(x) \) are bi-\( \gamma \)-positive.

**Proof.** For \( n = 1 \), we have \( D_1(x) = w, D_2D_1(x) = xy(x + 3y) \). Assume that

\[
(D_2D_1)^n(x) = \sum_{k=0}^{2n-1} \tilde{S}_{n,k}x^{2n-k}y^{k+1}.
\]

Note that \( D_1(D_2D_1)^n(x) = w\sum_k \tilde{S}_{n,k}\left((2n - k)x^{2n-k-1}y^{k+1}+(k+1)x^{2n-k}y^k\right) \). Then we have

\[
D_2(D_1(D_2D_1)^n(x)) = \sum_k \tilde{S}_{n,k}\left((2n - k)(4n - 2k + 1)x^{2n-k}y^{k+3} + \right.
\]

\[
\sum_k \tilde{S}_{n,k}\left((2n - k)(4k + 5) + 3(k + 1)x^{2n-k+1}y^{k+2} + \right)
\]

\[
\sum_k \tilde{S}_{n,k}(1 + k)(1 + 2k)x^{2n-k+2}y^{k+1}.
\]

Comparing the coefficient of \( x^{2n+2-k}y^{k+1} \), we see that \( \tilde{S}_{n,k} \) satisfy the same recurrence relation and initial conditions as \( S_{n,k} \), so they agree. Comparing (4.11) and the following expansion,

\[
D_1(D_2D_1)^n(x) = w\sum_k S(n,k)\left((2n - k)x^{2n-k-1}y^{k+1}+(k+1)x^{2n-k}y^k\right),
\]

we immediately get (4.12). For \( n = 1, 2 \), we have

\[
D_1(x) = w, \quad D_2D_1(x) = xy(x + y) + 2xy^2,
\]

\[
D_1(D_2D_1)(x) = w((x + y)^2 + 4xy) + 2wy(x + y),
\]

\[
(D_2D_1)^2(x) = xy((x + y)^3 + 20xy(x + y)) + xy^2(8(x + y)^2 + 16xy).
\]

Assume that the following expansions hold for \( n = m \), where \( m \geq 1 \).

\[
(D_2D_1)^n(x) = xy\sum_k (xy)^k(x + y)^{2n-2-2k}\left(\eta^{+}_{2n,k}(x + y) + y\sum_k \eta^{-}_{2n,k}\right),
\]

\[
D_1(D_2D_1)^n(x) = w\sum_k (xy)^k(x + y)^{2n-1-2k}\left(\eta^{+}_{2n+1,k}(x + y) + y\sum_k \eta^{-}_{2n+1,k}\right).
\]
We proceed by induction. Note that
\[ D_1(D_2 D_1)^m \]
\[ = w \sum_k \eta_{2m,k}^+(xy)^k(x + y)^{2m-2-k}((k + 1)(x + y)^2 + 2(2m - 1 - 2k)xy) + \]
\[ w \sum_k \eta_{2m,k}^- y(xy)^k(x + y)^{2m-3-2k}((k + 1)(x + y)^2 + 4(m - 1 - k)xy) + \]
\[ w \sum_k \eta_{2m,k}^- (xy)^{k+1}(x + y)^{2m-2-2k}. \]

Comparing the coefficients of \((xy)^k(x + y)^{2n-2k}\) and \((xy)^k(x + y)^{2n-1-2k}\), we obtain the first two recurrence relations in \((4.13)\).

Note that
\[ D_2(D_1(D_2 D_1)^m(x)) \]
\[ = xy \sum_k \eta_{2m+1,k}^+(1 + 2k)x^ky^k(x + y)^{2m+1-2k} + \]
\[ 2xy^2 \sum_k \eta_{2m+1,k}^+ x^k y^k(x + y)^{2m-2k} + \]
\[ 8xy \sum_k \eta_{2m+1,k}^+ (m - k)x^{k+1} y^{k+1}(x + y)^{2m-1-2k} + \]
\[ xy^2 \sum_k \eta_{2m+1,k}^+ (3 + 2k)x^k y^k(x + y)^{2m-2k} + \]
\[ 4xy^2 \sum_k \eta_{2m+1,k}^-(2m - 1 - 2k)x^{k+1} y^{k+1}(x + y)^{2m-2-2k}. \]

Comparing the coefficients of \(xy(xy)^k(x + y)^{2m+1-2k}\) and \(xy^2(xy)^k(x + y)^{2m-2k}\), we obtain the last two recurrence relations in \((4.13)\). This completes the proof. \(\square\)

Define
\[ S_n^+(x) = \sum_k \eta_{2n,k}^+ x^k(1 + x)^{2n-1-2k}, \quad S_n^-(x) = \sum_k \eta_{2n,k}^- x^k(1 + x)^{2n-2-2k}, \]
\[ T_n^+(x) = \sum_k \eta_{2n+1,k}^+ x^k(1 + x)^{2n-2-2k}, \quad T_n^-(x) = \sum_k \eta_{2n+1,k}^- (xy)^k(1 + x)^{2n-1-2k}. \]

By Lemma 16 we see that \((S_n^+(x), S_n^-(x))\) is the symmetric decomposition of \(S_n(x)\) and \((T_n^+(x), T_n^-(x))\) is the symmetric decomposition of \(T_n(x)\). Combining \((4.9), (4.10)\) and Lemma 16 we see that
\[ \#\{\sigma \in C_n^+ | \text{asc}^+(\sigma) + \text{plat} (\sigma) = k + 1\} = \#\{\sigma \in C_n^+ | \text{des} (\sigma) = k\}, \]
\[ \#\{\sigma \in D_n^+ | \text{asc}^+(\sigma) + \text{plat} (\sigma) = k + 1\} = \#\{\sigma \in D_n^+ | \text{des} (\sigma) = k\}, \]

Hence
\[ S_n(x) = \sum_{\sigma \in C_n^+} x^{\text{asc}^+(\sigma) + \text{plat} (\sigma) - 1}, \quad T_n(x) = \sum_{\sigma \in D_n^+} x^{\text{asc}^+(\sigma) + \text{plat} (\sigma) - 1}. \]
Equivalently, we have
\[ x^{2n} S_n \left( \frac{1}{x} \right) = \sum_{\sigma \in C_n^+} x^{\mathrm{des}^*(\sigma)}, \quad x^{2n+1} T_n \left( \frac{1}{x} \right) = \sum_{\sigma \in D_n^+} x^{\mathrm{des}^*(\sigma)}. \]
Therefore, we obtain
\[ S_n^+(x) = \sum_{\sigma \in C_n^+, \sigma_{2n} > 0} x^{\mathrm{des}(\sigma)} - \sum_{\sigma \in C_n^+, \sigma_{2n} < 0} x^{\mathrm{des}^*(\sigma)} = \sum_{\sigma \in C_n^+, \sigma_{2n} > 0} x^{\mathrm{des}(\sigma)}, \]
\[ T_n^+(x) = \sum_{\sigma \in D_n^+, \sigma_{2n} > 0} x^{\mathrm{des}(\sigma)} - \sum_{\sigma \in D_n^+, \sigma_{2n} < 0} x^{\mathrm{des}^*(\sigma)} = \sum_{\sigma \in D_n^+, \sigma_{2n} > 0} x^{\mathrm{des}(\sigma)}. \]

Therefore, we get the following result.

**Corollary 17.** For \( n \geq 1 \), we have
\[ S_n^+(x) = \sum_{\sigma \in C_n^+, \sigma_{2n} > 0} x^{\mathrm{des}(\sigma)}, \quad S_n^-(x) = \sum_{\sigma \in C_n^+, \sigma_{2n} < 0} x^{\mathrm{des}(\sigma)-1}. \]
\[ T_n^+(x) = \sum_{\sigma \in D_n^+, \sigma_{2n} > 0} x^{\mathrm{des}(\sigma)}, \quad T_n^-(x) = \sum_{\sigma \in D_n^+, \sigma_{2n} < 0} x^{\mathrm{des}(\sigma)-1}. \]

5. Colored Eulerian and derangement polynomials

5.1. Colored Eulerian polynomials.

For nonnegative integers \( m \) and \( n \), let \( [m,n] = \{m, m+1, \ldots, n\} \). For integers \( n, r \geq 1 \), an \( r \)-colored permutation can be written as \( \pi^c \), where \( \pi \in \mathcal{S}_n \) and \( c = (c_1, c_2, \ldots, c_n) \in [0, r - 1]^n \). As usual, \( \pi^c \) can be denoted as \( \pi_1^{c_1} \pi_2^{c_2} \cdots \pi_n^{c_n} \), where \( c_i \) can be thought of as the color assigned to \( \pi_i \). Denote by \( \mathbb{Z}_r \wr \mathcal{S}_n \) the set of all \( r \)-colored permutations of order \( n \).

Given an element \( \pi^c \in \mathbb{Z}_r \wr \mathcal{S}_n \). We say that an index \( k \in [n] \) is a descent of \( \pi^c \) if either \( c_k > c_{k+1} \), or \( c_k = c_{k+1} \) and \( \pi_k > \pi_{k+1} \), where \( \pi(n+1) := n+1 \) and \( c_{n+1} = 0 \). Let \( \text{des}(\pi^c) \) be the number of descents of \( \pi^c \). We say that an entry \( \pi_i^{c_i} \) is an excidence of \( \pi^c \) if \( \pi_i > i \) or \( \pi_i = i \) and \( c_i > 0 \). Let \( \text{exc}(\pi^c) \) be the number of excidences of \( \pi^c \). The \( r \)-colored Eulerian polynomial is defined as follows:
\[ A_{n,r}(x) = \sum_{\pi^c \in \mathbb{Z}_r \wr \mathcal{S}_n} x^{\text{exc}(\pi^c)}. \]

In [61], Steingrímsson showed that
\[ A_{n,r}(x) = \sum_{\pi^c \in \mathbb{Z}_r \wr \mathcal{S}_n} x^{\text{exc}(\pi^c)} = \sum_{\pi^c \in \mathbb{Z}_r \wr \mathcal{S}_n} x^{\text{des}(\pi^c)}. \]
The polynomials \( A_{n,r}(x) \) satisfy the following recurrence relation
\[ A_{n,r}(x) = (1 + (rn - 1)x) A_{n-1,r}(x) + rx(1 - x) A'_{n,r}(x). \]
Let $A_{n,r}(x) = \sum_{k=0}^n A_r(n,k)x^k$. It follows from (5.1) that
\[ A_r(n,k) = (rk + 1)A_r(n-1,k) + (r(n - k) + r - 1)A_r(n - 1, k - 1), \]
with $A_r(0,0) = 1$. The first few $A_{n,r}(x)$ are given as follows:
\[ A_{0,r}(x) = 1, \ A_{1,r}(x) = 1 + (r - 1)x, \ A_{2,r}(x) = 1 + (r^2 + 2r - 2)x + (r - 1)^2x^2. \]

Following [25, Theorem 5], we have
\[ A_{n,r}(x) = E_n^{(r, 2r, \ldots, nr)}(x). \]
When $r = 1$ and $r = 2$, the polynomial $A_{n,r}(x)$ reduces to $A_n(x)$ and type $B$ Eulerian polynomial $P(B_n, x)$, respectively.

5.2. Colored derangement polynomials.

A fixed point of $\pi^c \in \mathbb{Z}_r \wr \mathfrak{S}_n$ is an entry $\pi^c_k$ such that $\pi_k = k$ and $c_k = 0$. An element $\pi^c \in \mathbb{Z}_r \wr \mathfrak{S}_n$ is called a derangement if it has no fixed points. Let $D_{n,r}$ be the set of derangements in $\mathbb{Z}_r \wr \mathfrak{S}_n$. The $r$-colored derangement polynomial is defined as follows:
\[ d_{n,r}(x) = \sum_{\pi^c \in D_{n,r}} x^{\text{exc}(\pi^c)}. \]
Following [25], we have
\[ d_{n+1,r}(x) = rnx(d_{n,r}(x) + d_{n-1,r}(x)) + (r - 1)xd_{n,r}(x) + rx(1 - x) \frac{d}{dx} d_{n,r}(x). \]
The first few $d_{n,r}(x)$ are given as follows:
\[ d_{0,r}(x) = 1, \ d_{1,r}(x) = (r - 1)x, \ d_{2,r}(x) = r^2x + (r - 1)^2x^2. \]
By using combinatorial theory of continued fractions, Shin and Zeng [55, Theorem 3] obtained the following expansion:
\[ d_{n,r}(x) = \sum_{1 \leq i + 2j \leq n} \gamma_{n,i,j}x^{i+j}(1 + x)^{n-i-2j}(r - 1)^j r^{n-i}. \]
When $r = 1$ and $r = 2$, the polynomial $d_{n,r}(x)$ reduces to the classical derangement polynomial $d_n(x)$ and type $B$ derangement polynomial. It should be noted that $d_n(x)$ is $\gamma$-positive (see [51]). In recent years, the type $B$ derangement polynomial has been extensively studied, see, e.g., [19, 24, 45, 55].

Using results of Shareshian and Wachs [53] and Linusson, Shareshian and Wachs [42] on the homology of Rees products of posets, Athanasiadis [2, Theorem 1.3] obtained the following result.

**Proposition 18.** We have $d_{n,r}(x) = d_{n,r}^+(x) + d_{n,r}^-(x)$, where
\[ d_{n,r}^+(x) = \sum_{i=0}^{\lfloor n/2 \rfloor} \xi_{n,r,i}^+ x^i(1 + x)^{n-2i}, \]
\[ d_{n,r}^-(x) = \sum_{i=1}^{\lfloor (n+1)/2 \rfloor} \xi_{n,r,i}^- x^i(1 + x)^{n+1-2i}. \]
Set $\text{Asc}(\pi^c) = [n] \setminus \text{Des}(\pi^c)$. Then $\xi^+_{n,r,i}$ is the number of colored permutations $\pi^c \in \mathbb{Z}_r \wr S_n$ for which $\text{Asc}(\pi^c) \subseteq [2,n]$ has exactly $i$ elements, no two consecutive, and contains $n$, $\xi^-_{n,r,i}$ is the number of colored permutations $\pi^c \in \mathbb{Z}_r \wr S_n$ for which $\text{Asc}(\pi^c) \subseteq [2,n-1]$ has exactly $i$ elements, no two consecutive.

Local $h$-polynomials were introduced by Stanley [59] as a fundamental tool in the theory of face enumeration for subdivisions of simplicial complexes. Athanasiadis [2, Theorem 1.3] showed that the polynomial $d_{n,r}^+(x)$ is equal to the local $h$-polynomial of a suitable simplicial subdivision of the $(n-1)$-dimensional simplex. By using the principle of inclusion-exclusion, it is clear that $A_{n,r}(x) = \sum_{k=0}^{n} \binom{n}{k} d_{k,r}(x)$. By this identity, Athanasiadis [2, Eq. (21)] obtained the following expansion:

$$A_{n,r}(x) = \sum_{k=0}^{n} \binom{n}{k} d_{k,r}^+(x) + \sum_{k=0}^{n} \binom{n}{k} d_{k,r}^-(x).$$

In this section, we give an elementary proof of the bi-$\gamma$-positivity of $A_{n,r}(x)$ and $d_{n,r}(x)$. In the following discussion, we always assume that $r \geq 2$.

5.3. Main results.

Lemma 19. If

$$(5.3) \quad G = \{u \to uv^r, v \to u^r v\},$$

then we have

$$(5.4) \quad D^n(u^{r-1}v) = u^{r-1}v \sum_{k=0}^{n} \binom{n}{k} A_r(n,k)u^{(n-k)r}v^{kr}.$$

Proof. Note that $D^0(u^{r-1}v) = u^{r-1}v$ and $D(u^{r-1}v) = u^{r-1}v(u^r + (r-1)v^r)$. Thus the result holds for $n = 0, 1$. Assume that the result holds for $n = m$. Then

$$D^{m+1}(u^{r-1}v)$$

$$= D \left( u^{r-1}v \sum_{k=0}^{m} A_r(m,k)u^{(m-k)r}v^{kr} \right)$$

$$= u^{r-1}v \sum_{k} A_r(m,k)((mr - kr + r - 1)u^{(m-k)r}v^{(k+1)r} + (kr + 1)u^{(m-k+1)r}v^{kr}).$$

So we get that $A_r(m+1,k) = (rk+1)A_r(m,k) + (r(m+1-k) + r-1)A_r(m,k-1)$. Thus the result holds for $n = m + 1$. This completes the proof. \hfill \Box

Theorem 20. For each $r \geq 3$ and $n \in \mathbb{N}$, the polynomial $A_{n,r}(x)$ is bi-$\gamma$-positive. More precisely, we have

$$(5.5) \quad A_{n,r}(x) = \sum_{k=0}^{\lfloor n/2 \rfloor} \alpha^+_{n,k;r} x^k (1+x)^{n-2k} + x \sum_{k=0}^{\lfloor (n-1)/2 \rfloor} \alpha^-_{n,k;r} x^k (1+x)^{n-1-2k},$$
where the numbers $\alpha_{n,k,r}^+$ and $\alpha_{n,k,r}^-$ satisfy the recurrence system

\[
\begin{aligned}
\alpha_{n+1,k,r}^+ &= (1 + rk)\alpha_{n,k,r}^+ + 2r(n - 2k + 2)\alpha_{n,k-1,r}^+ + 2\alpha_{n,k-1,r}^-,
\alpha_{n+1,k,r}^- &= (r - 2)\alpha_{n,k,r}^+ + (r - 1 + rk)\alpha_{n,k,r}^- + 2r(n - 2k + 1)\alpha_{n,k-1,r}^-,
\end{aligned}
\]

with the initial conditions $\alpha_{1,0,r}^+ = 1$, $\alpha_{1,0,r}^- = r - 2$, $\alpha_{1,k,r}^+ = \alpha_{1,k,r}^- = 0$ for $k \neq 0$.

**Proof.** Consider a change of the grammar (5.3). Note that

\[
\begin{aligned}
&\alpha &
g(5.5).
\end{aligned}
\]

We say that an entry $\pi \in Q_r \subseteq G$ is a

\[
\begin{aligned}
&\pi \in \mathbb{Z}_r \subseteq \mathbb{G}_n &\text{ if } \pi \in \mathbb{Z}_r \subseteq \mathbb{G}_n.
\end{aligned}
\]

and $\alpha_{n,k,r}$ such that

\[
\begin{aligned}
&\pi \in \mathbb{Z}_r \subseteq \mathbb{G}_n &\text{ if } \pi \in \mathbb{Z}_r \subseteq \mathbb{G}_n.
\end{aligned}
\]

We proceed to the inductive step. Note that

\[
\begin{aligned}
&D_{G_1}^n(I) = D_{G_1}^n(D_{G_1}^n(I))
\end{aligned}
\]

Taking coefficients of $a^{k}b^{n+1-2k}I$ and $a^{k}b^{n-2k}c$ on both sides and simplifying yields the desired recurrence system for $\gamma$-coefficients. Setting $u^r = 1$ and $v^r = x$, we have $a = x$, $b = 1 + x$ and $c = Ix$. Comparing (5.4) and (5.6), we immediately get (5.5).

We say that an entry $\pi^c$ is an **anti-excedance** of $\pi^c \in \mathbb{Z}_r \subseteq \mathbb{G}_n$ if $\pi^c < i$. Let $\text{aexc}(\pi^c)$ (resp. $\text{fix}(\pi^c)$) be the number of anti-excedances (resp. fixed points) of $\pi^c$. For any $\pi^c \in \mathbb{Z}_r \subseteq \mathbb{G}_n$, it is clear that $\text{exc}(\pi^c) + \text{aexc}(\pi^c) + \text{fix}(\pi^c) = n$. □
Lemma 21. If
\begin{equation}
G = \{ I \rightarrow (r-1)xI + zI, x \rightarrow rxy, y \rightarrow rxy, z \rightarrow rxy \},
\end{equation}
then we have
\begin{equation}
D^n(I) = I \sum_{\pi^c \in \mathbb{Z}_r \wr S_n} x^{\text{exc} (\pi^c)} y^{\text{aexc} (\pi^c)} z^{\text{fix} (\pi^c)}.
\end{equation}

Proof. We now introduce a grammatical labeling of \( \pi^c \in \mathbb{Z}_r \wr S_n \) as follows:

1. If \( \pi^c_i \) is an excedance, then put a subscript label \( x \) right after \( i \);
2. If \( \pi^c_i \) is an anti-excedance, then put a subscript label \( y \) right after \( i \);
3. If \( \pi^c_i \) is a fixed point, then put a subscript label \( z \) right after \( \pi^c_i \);
4. Put a subscript label \( I \) right after \( \pi^c \).

The weight of \( \pi^c \) is the product of its labels. Note that the weight of \( \pi^c \) is given by
\[ w(\pi^c) = I x^{\text{exc} (\pi^c)} y^{\text{aexc} (\pi^c)} z^{\text{fix} (\pi^c)}. \]

For \( n = 1 \), we have \( \mathbb{Z}_r \wr S_1 = \{(1_1)I, (1^1_1)I, (1^2_1)I, \ldots, (1^n_1)I\} \). Note that \( D(I) = (r-1)xI + zI \). Then the sum of weights of the elements in \( \mathbb{Z}_r \wr S_1 \) is given by \( D(I) \).

Hence the result holds for \( n = 1 \). We proceed by induction on \( n \). Suppose we get all labeled permutations in \( \pi^c \in \mathbb{Z}_r \wr S_{n-1} \), where \( n \geq 2 \). Let \( \hat{\pi}^c \) be obtained from \( \pi^c \in \mathbb{Z}_r \wr S_{n-1} \) by inserting \( n^{c_j} \), where \( 0 \leq c_j \leq r-1 \). When the inserted \( n^{c_j} \) forms a new cycle, the insertion corresponds to the substitution rule \( I \rightarrow (r-1)xI + zI \) since we have \( r \) choices for \( c_j \). For the other cases, the changes of labeling are illustrated as follows:
\[ \cdots (\cdots \pi^c_i x^{n^{c_i}} y^{n_i} z^{n_{i+1}} \cdots) \rightarrow \cdots \cdots \rightarrow \cdots (\pi^c_i x^{n^{c_i}} y^{n_i} z^{n_{i+1}} \cdots) \cdots; \]
\[ \cdots (\cdots \pi^c_i y^{n_i} z^{n_{i+1}} \cdots) \rightarrow \cdots \cdots \rightarrow \cdots (\pi^c_i y^{n_i} z^{n_{i+1}} \cdots) \cdots; \]
\[ \cdots (\pi^c_i z^{n_{i+1}} \cdots) \rightarrow \cdots \cdots \rightarrow \cdots (\pi^c_i z^{n_{i+1}} \cdots) \cdots. \]

In each case, the insertion of \( n^{c_j} \) corresponds to one substitution rule in \( G \). By induction, it is routine to check that the action of \( D \) on elements of \( \mathbb{Z}_r \wr S_{n-1} \) generates all elements of \( \mathbb{Z}_r \wr S_n \). This completes the proof. \( \square \)

Setting \( z = 0 \) in \( (5.9) \), we see that
\begin{equation}
D^n(I) \big|_{z=0} = I \sum_{\pi^c \in \mathbb{D}_{n,r}} x^{\text{exc} (\pi^c)} y^{\text{aexc} (\pi^c)}.
\end{equation}

Theorem 22. For each \( r \geq 2 \) and \( n \in \mathbb{N} \), the polynomial \( d_{n,r}(x) \) is bi-\( \gamma \)-positive. More precisely, for \( n \geq 2 \), we have
\[ d_{n,r}(x) = \sum_{k=1}^{\lfloor n/2 \rfloor} f_{n,k} x^k (1 + x)^{n-2k} + x \sum_{k=0}^{\lfloor (n-1)/2 \rfloor} h_{n,k} x^k (1 + x)^{n-1-2k}, \]
where the \( \gamma \)-coefficients satisfy the recurrence system

\[
\begin{aligned}
    h_{n+1,k}^{(r)} &= (rk + r - 1)h_{n,k}^{(r)} + 2r(n - 2k + 1)h_{n,k-1}^{(r)} + (r - 1)f_{n,k}^{(r)} + rnf_{n-1,k-1}^{(r)}, \\
    f_{n+1,k}^{(r)} &= rkf_{n,k}^{(r)} + 2r(n - 2k + 2)f_{n,k-1}^{(r)} + rnf_{n-1,k-1}^{(r)} + h_{n,k-1}^{(r)},
\end{aligned}
\]

with the initial conditions \( h_{2,0}^{(r)} = (r - 1)^2, h_{2,k}^{(r)} = 0 \) for \( k \neq 0 \), \( f_{2,1}^{(r)} = 2r - 1 \) and \( f_{2,k}^{(r)} = 0 \) for \( k \neq 1 \).

**Proof.** Consider a change of the grammar (5.7). Note that

\[
    u = xy, v = x + y, J = xI,
\]

Setting \( u = xy, v = x + y, J = xI \), we obtain

\[
    D(I) = (r - 1)J + zI, \quad D(xI) = (r - 1)JxI(x + y) + xyI + xIz,
\]

\[
    D(xy) = rxy(x + y), \quad D(x + y) = 2rxy, \quad D(z) = rxy.
\]

In the following, we consider the grammar

\[
    G_2 = \{ I \rightarrow (r - 1)J + zI, \quad J \rightarrow (r - 1)Jv + uI + Jz, \quad D(u) = ruv, \quad D(v) = 2ru, \quad D(z) = ru. \}
\]

For \( n = 1, 2, 3 \), we have

\[
    D_{G_2}(I) = (r - 1)J + zI,
\]

\[
    D_{G_2}^2(I) = (r - 1)^2vJ + (2r - 1)uI + 2(r - 1)Jz + Iz^2,
\]

\[
    D_{G_2}^3(I) = ((1 - 3r + 2r^3)u + (r - 1)^3v^2)J + (1 - 3r + 3r^2)uvI + (3(r - 1)^2vJ + 3(2r - 1)uvI)Iz + (3(r - 1)Jv + 3(r - 1)vI)z + 3(r - 1)Jz^2 + Iz^3.
\]

By induction, it is routine to verify that there exist nonnegative integers \( h_{n,k,i}^{(r)} \) and \( f_{n,k,i}^{(r)} \) such that

\[
    D_{G_2}^n(I) = \sum_{i=0}^{n} z^i \left( \sum_{k=0}^{(n-1-i)/2} h_{n,k,i}^{(r)} u^k v^{n-1-2k-i} J + \sum_{k=1}^{(n-i)/2} f_{n,k,i}^{(r)} u^k v^{n-2k-i} I \right).
\]

Set \( D_{G_2}^n(I) = \lambda_1 + \lambda_2 \), where

\[
    \lambda_1 = \sum_{i=0}^{1} z^i \left( \sum_{k=0}^{(n-1-i)/2} h_{n,k,i}^{(r)} u^k v^{n-1-2k-i} J + \sum_{k=1}^{(n-i)/2} f_{n,k,i}^{(r)} u^k v^{n-2k-i} I \right),
\]

\[
    \lambda_2 = \sum_{i=2}^{n} z^i \left( \sum_{k=0}^{(n-1-i)/2} h_{n,k,i}^{(r)} u^k v^{n-1-2k-i} J + \sum_{k=1}^{(n-i)/2} f_{n,k,i}^{(r)} u^k v^{n-2k-i} I \right).
\]
Comparing (5.9) and (5.10), we get the symmetric decomposition of $d$ evidence, we propose the following.

Setting $h_{n,k,0}$ be a sequence of nonnegative integers. Let $N$ be a partial ordering of $[n]$, where $\deg f = d$ and $f(0) > 0$. Let $v = (v_1, v_2, \ldots, v_m)$ be a sequence of nonnegative integers. Let $N_k(v)$ be the number of permutations of the multiset $M_v = \{1^{v_1}, 2^{v_2}, \ldots, m^{v_m}\}$ with exactly $k$ descents. Simion [56] proved that the descent polynomial $\sum_{k\geq 0} N_k(v)x^k$ has only real zeros. Based on empirical evidence, we propose the following.

**Conjecture 23.** Let $T_v(x) = \sum_{k\geq 0} N_k(v)x^k$. Set $T_v(x) = x^\mu f_v(x)$, where $\mu$ is a nonnegative integer and $f_v(0) > 0$. Then $f_v(x)$ is alternatingly monotone.

It is worth mentioning that, as pointed out by Haghud, Ono and Wagner [33], Simion’s result proves a special case of the Neggers-Stanley conjecture. Let $P$ be a partial ordering of $[n]$. The $W$-polynomial $W_P(x)$ is the generating function counting the linear extensions of $P$ by their descent numbers. The Neggers-Stanley conjecture asserts that the descent polynomial of the linear extensions of a partially ordered set on $[n]$ has only real zeros, see, e.g., [32, 63]. Bränden [10] and Stembridge [63] discovered some counterexamples to Neggers-Stanley conjecture. We note that the counterexamples to real-rootedness given in [63] are all alternatingly monotone. Set $W_P(x) = x^\delta Y_P(x)$, where $\delta$ is a nonnegative integer and $Y_P(0) > 0$. It is a challenging problem to explore the alternatingly monotone property of $Y_P(x)$.  

In order to extract the coefficient of $z^0$ in the above expansion, it suffices to consider $\lambda_1$. Note that

$$D_{G_2}(\lambda_1) = D_{G_3} \left( \sum_k h_{n,k,0}^{(r)} u^k v^{n-1-2k} J + \sum_k f_{n,k,0}^{(r)} u^k v^{n-2k} I \right) +$$

$$D_{G_2} z \left( \sum_k h_{n,k,1}^{(r)} u^k v^{n-2-2k} J + \sum_k f_{n,k,1}^{(r)} u^k v^{n-1-2k} I \right).$$

It is easy to verify that

$$f_{n+1,k,0}^{(r)} = h_{n,k-1,0}^{(r)} + rk f_{n,k,0}^{(r)} + 2r(n - 2k + 2) f_{n,k-1,0}^{(r)} + rf_{n,k-1,1}^{(r)},$$

$$h_{n+1,k,0}^{(r)} = (rk + r - 1) h_{n,k-1,0}^{(r)} + 2r(n - 2k + 1) h_{n,k-1,0}^{(r)} + (r - 1) f_{n,k,0}^{(r)} + rh_{n,k-1,1}^{(r)}.$$  

Since $z$ marks the statistic $\fix$, we have $h_{n,k-1,1}^{(r)} = nh_{n,k-1,1}^{(r)}$, $f_{n,k-1,1}^{(r)} = nf_{n,k-1,0}^{(r)}$. Setting $h_{n,k}^{(r)} = h_{n,k,0}^{(r)}$ and $f_{n,k}^{(r)} = f_{n,k,0}^{(r)}$, we obtain the desired recurrence system.

Comparing (5.9) and (5.10), we get the symmetric decomposition of $d_{n,r}(x)$.  

It follows from Proposition 8 and Theorem 22 that

$$\xi_{n,r,i}^+ = f_{n,i}^{(r)}, \quad \xi_{n,r,i+1}^- = h_{n,i}^{(r)}.$$  

6. Concluding remark

We say that a polynomial $f(x)$ is alternatingly monotone if $f(x)$ or $x^d f(\frac{1}{x})$ is alternatingly increasing, where $\deg f = d$ and $f(0) > 0$. Let $v = (v_1, v_2, \ldots, v_m)$ be a sequence of nonnegative integers. Let $N_k(v)$ be the number of permutations of the multiset $M_v = \{1^{v_1}, 2^{v_2}, \ldots, m^{v_m}\}$ with exactly $k$ descents. Simion [56] proved that the descent polynomial $\sum_{k\geq 0} N_k(v)x^k$ has only real zeros. Based on empirical evidence, we propose the following.

**Conjecture 23.** Let $T_v(x) = \sum_{k\geq 0} N_k(v)x^k$. Set $T_v(x) = x^\mu f_v(x)$, where $\mu$ is a nonnegative integer and $f_v(0) > 0$. Then $f_v(x)$ is alternatingly monotone.

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The s-Eulerian polynomials has been extensively studied in recent years, see e.g., [51][64]. It would be interesting to explore geometric significance of symmetric decomposition of the s-Eulerian polynomials considered in this paper.

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