MODULI OF SHEAVES SUPPORTED ON QUARTIC SPACE CURVES

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Abstract. As a generalization of the work of Freiermuth and Trautmann, we study the geometry of the moduli space of stable sheaves in \( \mathbb{P}^3 \) with Hilbert polynomial \( 4m + 1 \). The moduli space has three irreducible components whose elements are respectively sheaves supported on rational quartic curves, on elliptic quartic curves, or on planar quartic curves. The main idea of proof is to relate the moduli space with the Hilbert scheme of curves by wall crossing. We also present the list of possible free resolutions of the stable sheaves on \( \mathbb{P}^3 \).

1. Introduction

For the fixed polynomial \( P(m) \) with rational coefficient, it is well-known that the space parameterizing Gieseker-Mumford stable sheaves on a smooth projective variety \( X \) with Hilbert polynomial \( P(m) \) is a projective scheme. In this paper, we are specifically interested in the case when \( P(m) = rm + \chi \) is a linear polynomial as the moduli space is a compactification of the moduli space of curves. When \( X \) is the projective plane \( \mathbb{P}^2 \), Le Potier studied the geometry the moduli space \([14]\). He showed that the moduli space is an irreducible projective variety of dimension \( r^2 + 1 \) and that it is locally factorial variety whose Picard group is generated by two explicitly constructed divisors. When the leading coefficient \( r \) is small, the moduli spaces have been studied by many authors. Stratifications in terms of the free resolution of sheaves \([7, 17, 18]\), topological invariants such as Poincaré polynomials \([6, 13, 15, 24]\), stable base locus decompositions \([9]\) are known.

In this paper, we study the case when \( X \) is \( \mathbb{P}^3 \). Let \( M(P(m)) \) denote the moduli space of stable sheaves on \( \mathbb{P}^3 \) with Hilbert polynomial \( P(m) \). Freiermuth and Trautmann \([8]\) studied \( M(3m + 1) \). They gave a complete classification of sheaves in \( M(3m + 1) \) and proved \( M(3m + 1) \) has two irreducible components and studied their intersection. The first component parameterizes the structure sheaves of twisted cubic curves and the second parametrizes the sheaves of the form \( \mathcal{O}_C(p) \) where \( C \) is a plane cubic curve and \( p \in C \).

In this paper, we investigate \( M(4m + 1) \). Our main result is that the moduli space \( M(4m + 1) \) consists of three irreducible components. One of components of \( M(4m + 1) \) is the compactification of the space of rational quartic curves which is predicted as a birational flip model of the space of maps. So our project can be regarded as the starting point of understanding the various compactified moduli space of curves in projective space.

Theorem 1. Let \( M(4m + 1) \) be the moduli space of stable sheaves in \( \mathbb{P}^3 \) with Hilbert polynomial \( 4m + 1 \). Then the space \( M(4m + 1) \) consists of three irreducible components whose general points parameterizes

1. the structure sheaves of the rational quartic curves;
2. the dual of the ideal sheaves \( I_{p,C}^D \) of elliptic quartic curves \( C \) with a point \( p \);
3. the planar sheaves.

Remark 2. • The dual is defined by \( F^D := \text{Ext}^2(F, \omega_{\mathbb{P}^3}) \).
• We call a sheaf \( F \) is planar if \( F \cong F|_H \) for some plane \( H \subset \mathbb{P}^3 \), or equivalently, the scheme theoretic support of \( F \) is contained in \( H \).
We denote three components by $R$, $E$, $P$.

The main idea to relate $M(4m + 1)$ with the Hilbert scheme of degree 4 curves by using wall crossing which we will review in §2. It is known that the Hilbert scheme consists of four irreducible components. One of four components must be dropped by wall crossing as its elements correspond to unstable sheaves. The other three components correspond to three irreducible components. We study how elements in the Hilbert scheme are modified by wall crossing and show that we have three corresponding irreducible components. We also study the intersections of irreducible components $R$, $E$, $P$ (See §5). In the last section, we present the possible free resolution of the stable sheaves in $M(4m + 1)$ for completeness. This can be used in the local analysis of each component in a forthcoming work.

2. Review of the wall crossing

2.1. General framework. In this section, we review the wall crossing technique we will use in the paper. Motivated by the conjectural Donaldson-Thomas/Pandharipande-Thomas correspondence, Stoppa and Thomas [22] study a GIT wall crossing between the Hilbert scheme of curves and the moduli space of stable pairs. The stable pair here is a pair of a one-dimensional sheaf and its section which generate the sheaf way from only finitely many points. The moduli space of stable pairs was used to defined the Pandharipande-Thom as invariants for threefolds. They realized both space by GIT quotient of certain space of pairs and by altering the linearization they showed that these two moduli space are related by GIT wall crossing.

One can go further. In [3], the authors study the wall-crossing for the moduli space $M^\alpha(P(m))$ of $\alpha$-stable pairs, where $\alpha$ is a positive rational number. The pair here is also a pair of a sheaf and a nonzero section, but the stability condition is different. A pair $(s,F)$ is called $\alpha$-semistable if $F$ is pure and for any proper nonzero subsheaves $F' \subset F$, the inequality

$$\frac{\chi(F'(m)) + \delta \cdot \alpha}{\tau(F')} \leq \frac{\chi(F(m)) + \alpha}{\tau(F)}$$

holds for $m \gg 0$. Here $\tau(F)$ is the leading coefficient of the Hilbert polynomial $\chi(F(m))$ and $\delta = 1$ if the section $s$ factors through $F'$ and $\delta = 0$ otherwise. When the strict inequality holds, $(s,F)$ is called $\alpha$-stable.

By the definition of $\alpha$-stability, when $\alpha$ goes to infinity (denoted by $\alpha = \infty$), the cokernel of the pair $s : \mathcal{O} \to F$ should be supported on zero dimensional scheme (possibly empty set). In other words, we get the moduli space of Pandharipande-Thomas stable pairs. On the other hand, when $\alpha$ gets sufficiently small (denoted by $\alpha = 0^+$), the $\alpha$-stability is now equivalent to the Gieseker stability of the sheaf. That is, by wall crossing, conditions on the section is changed into conditions on the sheaf. Now since there is no condition on sections, by forgetting the section, we get a map to our moduli space $M(P(m))$ of sheaves.

When $P(m) = 4m + 1$, by an easy calculation as in [3], we see there is only one wall at $\alpha = 3$. We denote by $(1,F)$ the pair of sheaf $F$ with a nonzero section and by $(0,F)$ the pair of sheaf $F$ with zero section. When a pair is given by the extension

$$0 \to (0, \mathcal{O}_L) \to (1,F) \to (1,G) \to 0,$$

where $L$ is a line and $G$ is a sheaf with Hilbert polynomial $3m$, it is unstable when $\alpha < 0$ because $\frac{1}{\alpha} > \frac{1}{\alpha^+}$. After crossing the wall, this pair is modified into a pair given by “flipped” extension

$$0 \to (1,G) \to (1,F) \to (0, \mathcal{O}_L) \to 0.$$
Hence the whole wall crossing picture is as follows.

\[
\begin{array}{ccc}
\text{Hilb} & \rightarrow & M^{\infty}(4m+1) \\
\downarrow & & \downarrow \\
\text{Chow} & \rightarrow & M^{\alpha=3}(4m+1) \\
\downarrow & & \downarrow \\
M^\infty & \rightarrow & M[4m+1].
\end{array}
\]

Here the Chow is defined by

\[
\text{Chow} := \bigsqcup [CM^{1m+1-g} \times S^g(\mathbb{P}^3)]
\]

where \( CM^{4m+1-g} \) is the space of CM curves with Hilbert polynomial \( 4m + 1 - g \) and \( S^g(\mathbb{P}^3) \) is the \( g \)-fold symmetric product.

**Lemma 3.**

1. A pair \((s,F)\) is a \( \infty \)-stable if and only if the section \( s \) is nonzero and the cokernel of \( s \) is supported on a zero-dimensional subscheme of the support of \( F \).
2. Let \(|d,\chi| = 1\). Then \((s,F)\) is a \( 0^+ \)-stable if and only if \( F \) is a stable sheaf.

**Remark 4.**

1. By a simple calculation, one can see that there is no wall for \( d = 1, 2 \).
   Hence \( M^\alpha(d,1) \) are all isomorphic to each other.
2. As a first non-trivial case, when the Hilbert polynomial \( P(m) = 3m + 1 \), Freiermuth and Trautmann prove that \( M^{\infty}(3m+1) \) consists of two smooth irreducible components \( R \cup E \) which intersects transversally, where \( R \) parameterizes the structure sheaves of the twisted cubic curves and \( E \) parameterizes the universal cubic plane curves.
   This can be explained as follow by wall-crossing. By the result of Piene and Schlessinger [23], \( \text{Hilb}^{3m+1}(\mathbb{P}^3) \) consists of two irreducible components: rational cubic curves and elliptic curves with a point. After wall crossing, \( M^{\infty}(3m+1) \) consists of two irreducible components \( R \cup E \) as above. Note that the locus of elliptic curves with a free point is excluded by the wall-crossing. There is no wall-crossing for pairs and all sheaves \( F \) has \( h^0(F) = 1 \). So, we have \( M^{\infty}(3m+1) \cong M^{0^+}(3m+1) \cong M(3m+1) \). Freiermuth and Trautmann show that these two components meet transversely by analyzing the deformation space of each sheaf in the intersection.

2.2. **Geometry of the Hilbert scheme.** The irreducible components of the Hilbert scheme \( \text{Hilb}^{4m+1}(\mathbb{P}^3) \) have been studied in [2].

**Proposition 5.** The Hilbert scheme of curves with Hilbert polynomial \( 4m + 1 \) in \( \mathbb{P}^3 \) consists of four irreducible components:

1. The locus of the quartic rational curves
2. The locus of the unions of a line and plane cubic
3. The locus of the quartic elliptic curves with an embedded point (of unique embedded structure)
4. The locus of the planar quartic curves with three embedded points.

Moreover, the space of quartic elliptic curves which are supporting curves in (3) has been studied in [1, 2].

**Proposition 6.**

- The CM-curves component in \( \text{Hilb}^{4m}(\mathbb{P}^3) \) is obtained from the Grassmannian \( \text{Gr}(2, 10) \) by blowing up twice where the blow-up loci are degenerated loci of determinental.
- Every CM-curve with Hilbert polynomial \( 4m \) is ACM and also the ideal sheaf of \( C \) has the resolution:

\[
0 \to \mathcal{O}(-4) \oplus \mathcal{O}(-3) \to \mathcal{O}(-3) \oplus \mathcal{O}(-2) \oplus 2 \to I_C \to 0.
\]
The possible types (up to projective equivalence) of defining equation of $C$ are

1. $I_C = \langle q_1, q_2 \rangle$.
2. $I_C = \langle xy, xz, yq_1 + zq_2 \rangle$, that is, the curve $C$ is the union of the plane cubic and line meeting at a point.
3. $I_C = \langle x^2, xy, xq_1 + yq_2 \rangle$, that is, the curve $C$ is the union of plane conics and the double lines with genus $-2$.

Here, $q_1$ and $q_2$ are quadric polynomials.

3. The moduli space $M^\infty(4m + 1)$

In this section we will describe all stable pairs in $M^\infty_P(4m + 1)$. Considering the Hilbert-Chow morphism and the Hilbert scheme described in §2.2, we classify all stable pairs according to their supports.

Lemma 7. Let $(s, F) \in M^\infty_P(4, 1)$ be a stable pair. Then,

1. $s : O_C \cong F$ where $\chi(O_C(m)) = 4m + 1$ such that $C$ is a rational quartic curve or a union of a line and a plane cubic curve not intersecting each other.
2. $s : O_C \subset F$ such that the cokernel $\ker(\cdot) = C_p$ and $\chi(O_C(m)) = 4m$, $p \in C$;
3. $s : O_C \subset F$ such that the cokernel $\ker(\cdot) = K$ and $\chi(O_C(m)) = 4m - 2$, $l(K) = 3$.

Proof. By stability, the cokernel of $s$ must be supported on zero dimensional subscheme of the support of the sheaf. By using the classification of CM-curve given in §2.2 and comparing the Hilbert polynomial, one can check the above list are complete. □

Remark 8. In case (1), the case where $C$ is a rational quartic curve is nothing but the Hilbert scheme of connected CM curve with Hilbert polynomial $4m + 1$, which is well-known to be irreducible. We denote this components by $R^\infty$. When the $C$ is a union of a line and a plane cubic, we denote by $X^\infty$. This component will be dropped from the moduli space after wall crossing.

In case (3), the sheaf $F$ must be a planar sheaf. Let $F$ be the pure sheaf fitting into the short exact sequence

$$0 \to O_C \to F \to K \to 0$$

with $l(K) = 3$. By degree-genus formular, the curve $C$ should be degree 4 contained in a plane $H \subset \mathbb{P}^3$. Let us consider the natural restriction map

$$\phi : F \to F|_H.$$

Since $F|_H$ has the subsheaf $O_C$, the kernel of the map $\phi$ should be a torsion sheaf. This contradicts the pureness of $F$. Hence $F \cong F|_H$, that is, $F$ is a planar sheaf. Such sheaves form an irreducible components because it is isomorphic to the relative moduli space $M^\infty(U, 4m + 1)$ where $U$ is a universal bundle over $\text{Gr}(3, 4)$. We denote this component by $P^\infty$.

In case (2), the sheaf $F$ is given by the extension

$$0 \to O_C \to F \to C_p \to 0,$$

where $C$ is as in Proposition 6. However unlike in the degree 3 case studied by Freiermuth and Trautmann, the sheaf $F$ is not uniquely determined by a pair $(p, C)$, that is, the extension group $\text{dim Ext}^1(C_p, O_C)$ can have dimension more than one. In fact, $\text{dim Ext}^1(C_p, O_C) \leq 2$ (See Lemma 9). We denote loci of such pairs by $E^\infty = E^\infty_1 \cup E^\infty_2$, where the lower index is $\text{dim Ext}^1(C_p, O_C)$. 
Lemma 9. Let \( \{p, C\} \in \mathcal{C} \). Then

\[
1 \leq \dim(\text{Ext}^1(C_p, \mathcal{O}_C)) \leq 2.
\]

Moreover, \( \text{Ext}^1(C_p, \mathcal{O}_C) \cong \mathbb{C}^2 \) precisely when (up to projective equivalence)

1. \( I_C = \langle xy, xz, yq_1 + zq_2 \rangle \) and \( q_1(p) = q_2(p) = 0 \).
2. \( I_C = \langle x^2, xy, xq_1 + yq_2 \rangle \) and \( q_1(p) = q_2(p) = 0 \).

Proof. By the Serre duality, we have an isomorphism

\[
\text{Ext}^1(C_p, \mathcal{O}_C) \cong \text{Ext}^2(\mathcal{O}_C, C_p)^* \cong \text{Ext}^1(I_C, C_p)^*
\]

where the second one comes from the short exact sequence \( 0 \to I_C \to O_{P^3} \to O_C \to 0 \). From the resolution of \( I_C \) in Proposition\[6\] we have

\[
0 \to \text{Ext}^0(I_C, C_p) \to \text{Ext}^0(O(-3) \oplus O(-2), C_p) \to \text{Ext}^0(O(-4) \oplus O(-3), C_p) \to \text{Ext}^1(I_C, C_p) \to 0.
\]

Now we calculate the dimension of the map \( \delta \) for each case described in Proposition\[6\]. With out loss of the generality we may assume that \( p = \{0:0:0:1\} \).

1. If \( C \) is general (\( I_C = \langle q_1, q_2 \rangle \)), the map \( \delta \) is given as \[
\begin{bmatrix}
0 & -q_2 \\
0 & q_1 \\
1 & 0
\end{bmatrix}
\]
and then \( \delta \) is the rank one matrix when it is evaluated at \( C_p \). Hence \( \text{Ext}^1(C_p, \mathcal{O}_C) \cong \mathbb{C} \).

2. Let \( C \) is the union of the line \( L \) and cubic \( C_0 \) (\( I_C = \langle xy, xz, yq_1 + zq_2 \rangle \), \( I_L = \langle y, z \rangle \), and \( I_{C_0} = \langle x, y, q_1 + zq_2 \rangle \)). Then the matrix is given by

\[
\begin{bmatrix}
-q_1 & 0 & z \\
-q_2 & -y & 0 \\
x & 0 & 0
\end{bmatrix}
\]

The rank of the map \( \delta \) varies depending the position of the point \( p \). That is,

(a) The rank of \( \delta \) is zero if and only if the point \( p \) lies on the singular point of \( C_0 \) (i.e., \( q_1(p) = q_2(p) = 0 \)) and lies on \( L \). In this case, \( \text{Ext}^1(C_p, \mathcal{O}_C) \cong \mathbb{C} \).

(b) Otherwise, the rank of \( \delta \) is one. In this case, \( \text{Ext}^1(C_p, \mathcal{O}_C) \cong \mathbb{C} \).

3. If \( C \) is the union of the conic and the double line of genus \(-2\) (\( I_C = \langle x^2, xy, xp_1 + yp_2 \rangle \), \( I_L = \langle x, y \rangle \), and \( I_Q = \langle x, q_2 \rangle \)), then the matrix is given by

\[
\begin{bmatrix}
-q_1 & -y \\
-q_2 & x \\
x & 0
\end{bmatrix}
\]

As we did before, we see that \( \text{Ext}^1(C_p, \mathcal{O}_C) \cong \mathbb{C} \) if and only if \( q_1(p) = q_2(p) = 0 \). Geometrically this is when the point \( p \) lies on the intersection \( L \cap Q \) and \( q_1(p) = 0 \).

\[ \square \]

For a planar curve \( C \), we denote by \( \langle C \rangle \) the plane containing \( C \).

Proposition 10. The moduli space \( M^\infty(4,1) \) are decomposed into the disjoint union

\[
\mathbb{R}^\infty \cup \mathbb{E}^\infty \cup \mathbb{P}^\infty \cup \mathbb{X}^\infty.
\]

Furthermore, \( \mathbb{E}^\infty \) are decomposed into

1. \( \mathbb{E}_1^\infty = \)
   (a) \( I_C = \langle q_1, q_2 \rangle \) with a point \( p \).
   (b) \( I_C = I_{L \cup C_0} \) with a point \( p \) such that \( L \not\in \langle C_0 \rangle \), \( p \not\in \text{Sing}(C_0) \) for some line \( L \) and a cubic curve \( C_0 \).
   (c) \( I_C = I_{L \cup Q} \) with a point \( p \) such that \( L(p) \neq 0 \) or \( p_1(p) \neq 0 \) or \( p_2(p) \neq 0 \) for some line \( L \) and the conic \( Q \) where \( L \subset \langle Q \rangle \).
2. \( \mathbb{E}_2^\infty = \)
Lemma 14. Let \( E \subset \text{Hilb}^{4m}(\mathbb{P}^3) \) be the locus of non-planar locally CM-curves with Hilbert polynomial \( 4m \). Let \( C = \{ (p, C) \mid C \subset E, p \in C \} \) be the universal family of \( E \). Then \( C \) is an irreducible variety of dimension 17.

**Proof.** Let \( \pi : C \subset E \times \mathbb{P}^3 \to E \) be the projection map. Then \( \pi \) is a flat map. Also, by [1], we know that \( E \) is an irreducible variety. Now we apply [9, III. Proposition 9.6]. For each \( e \in E \), every irreducible component of the fiber \( \pi^{-1}(e) \) has dimension one. So every irreducible component of \( C \) has dimension \( 16 + 1 = 17 \). By Proposition [9] \( E \) is a blown up space of a Grassmannian. So the inverse image of \( \pi \) away from the exceptional locus in \( E \) is irreducible. But the inverse image of the exceptional locus in \( E \) has dimension 16. Hence it does not form a new irreducible component of \( C \). \( \square \)

4. PROOF OF THE MAIN THEOREM

We use the wall crossing of \( \alpha \)-stable pairs to relate \( \text{M}^{\infty}(4m + 1) \) with \( \text{M}^{\infty^+}(4m + 1) \). Note that since a sheaf in \( \text{M}(4m + 1) \) has at least one nonzero section, the forgetting map from \( \text{M}^{\infty^+}(4m + 1) \) to \( \text{M}(4m + 1) \) is surjective. Moreover, we have the following.

**Lemma 12.** For \( F \in \text{M}(4m + 1) \), we have \( 1 \leq h^0(F) \leq 2 \). Moreover if \( h^0(F) = 2 \), then \( F \) must be planar.

**Proof.** This is clear from the possible resolution of \( F \) we will study in [6]. \( \square \)

Hence if \( F \) is nonplanar, by choosing the unique nonzero section, we will often regard a sheaf as a pair. The locus of planar sheaves was studied in [3].

We will use the following well-known lemma frequently.

**Lemma 13.** Let \( X \) be a projective scheme and \( Y \) a subscheme of \( X \). Let \( F \) be a coherent \( \mathcal{O}_X \)-module and let \( G \) be a coherent \( \mathcal{O}_Y \)-module. Then there is an exact sequence of vector spaces

\[
0 \to \text{Ext}^1_{\mathcal{O}_Y}(F \otimes \mathcal{O}_X, G) \to \text{Ext}^1_{\mathcal{O}_X}(F, G) \to \text{Hom}_{\mathcal{O}_Y}(\mathcal{O}_X \otimes_{\mathcal{O}_X} F, G)
\]

\[
\quad \to \text{Ext}^2_{\mathcal{O}_Y}(F \otimes \mathcal{O}_X, G) \to \text{Ext}^2_{\mathcal{O}_X}(F, G)
\]

In particular, if \( F \) is an \( \mathcal{O}_Y \)-module, then there is an exact sequence

\[
0 \to \text{Ext}^1_{\mathcal{O}_Y}(F \otimes \mathcal{O}_X, G) \to \text{Ext}^1_{\mathcal{O}_X}(F, G) \to \text{Hom}_{\mathcal{O}_Y}(F \otimes \mathcal{O}_X \mathcal{O}_Y, G)
\]

\[
\quad \to \text{Ext}^2_{\mathcal{O}_Y}(F \otimes \mathcal{O}_X, G) \to \text{Ext}^2_{\mathcal{O}_X}(F, G).
\]

The numerical type of wall of the spaces \( \text{M}^\alpha(4,1) \) is given by

**Lemma 14.** The wall crossing on \( \text{M}^\alpha(4,1) \) occurs at \( \alpha = 3 \) with the \( JH \)-filtration

\[
(\mathcal{O}_L) \oplus (1, \mathcal{O}_C)
\]

where \( C \) is a plane cubic curve.
**Proof.** By numerical computation, the possible type of wall is given by

\[(0, F_{m+1}) \oplus (1, F_{3m}),\]

where the subscripts indicate the Hilbert polynomials of the sheaves. Obviously, \(F_{m+1} \cong O_L\).
The sheaf \(F_{3m}\) is a planar sheaf because it should have a section. Then \(F_{3m}\) is isomorphic to the
structure sheaf of a plane cubic curve.

Therefore by wall crossing, the pairs of the form

\[0 \rightarrow (0, O_L) \rightarrow (1, F) \rightarrow (1, O_C) \rightarrow 0\]

in \(M^{\infty}(4, 1)\) is modified into the pairs of the form

\[0 \rightarrow (0, O_C) \rightarrow (1, F) \rightarrow (1, O_L) \rightarrow 0\]

in \(M^{\infty}+ (4, 1)\). We call them the wall crossing loci.

Away from the wall crossing loci, the moduli space is unchanged. We denote by \(R^+, E^+_1, E^+_2\)
and \(P^+\) respectively the loci corresponding to \(R^{\infty}, E^{\infty}_1, E^{\infty}_2\) and \(P^{\infty}\). The immediate
consequence of the above lemma is that the locus \(X^{\infty}\) is dropped after the wall crossing.

**Lemma 15.**
(1) The locus \(R^{\infty}\) does not intersect with the wall crossing locus. So \(R^{\infty} \simeq R^+\).

(2) The locus \(P^+\) is the relative \(O^+\)-stable pairs space \(M^{\infty}+ (P, 4m+1)\) such that \(P\) is
the universal rank 3 bundle over the Grassmannian \(Gr(3, 4)\). In particular, \(P^+\) is irreducible.

**Proof.** A rational quartic curve cannot be a union of a line and a cubic meeting with each other.
Hence (1) follows. For (2), we can apply the wall crossing of [2]. Note that by the result of [2],
\(M^{\infty}+ (P^2, 4m+1)\) is a blow up of \(M(P^2, 4m+1)\) and hence it is irreducible. So, it follows that
\(P^+\) is irreducible.

Hence, we have two irreducible components \(R^+\) and \(P^+\) of \(M^{\infty}+ (4, 1)\). We denote non-planar
wall crossing loci by \(W^{\infty}\) and \(W^+\) respectively. Now we show all the other locus is contained
in the closure of \(E^+_1 \setminus W^+\), which is irreducible by Lemma 11. Let \(E^+\) denote \(E^+_1 \setminus W^+\).

In Proposition 10, we divide \(E^{\infty}_2\) into two cases: support of the sheaf is either a union of
a line and a cubic or a union of a double line and a conic. After wall crossing, we also have
corresponding two loci. Let us denote these by \(E^{\infty}_{2a}\) and \(E^{\infty}_{2b}\).

**Proposition 16.** The locus \(E^+_2 \setminus W^+\) consists of

(1) \(E^{\infty}_{2a} : P(\text{Ext}^1(C, O_C)) - P(\text{Ext}^1(C, O_L(-1))) = (P^1 - pt)\)-bundle over the \(C\) such
that \(I_C = I_{L\cap C_0}\) with a point \(p\) such that \(L \not\subseteq \langle C_0 \rangle\), \(p \subseteq C \cap L \subset \text{Sing}(C_0)\) for some
line \(L\) and a cubic curve \(C_0\).

(2) \(E^{\infty}_{2b} : P(\text{Ext}^1(C, O_C)) - P(\text{Ext}^1(C, O_L(-1))) = (P^1 - pt)\)-bundle over the \(C\) such
that \(I_C = I_{L\cap Q}\) with a point \(p\) such that \(L(p) = \text{pt}\) \(p = p_1(p) = p_2(p) = 0\) for some line \(L\) and
the conic \(Q\) where \(L \subset \langle Q \rangle\).

**Proof.** Consider the exact sequence

\[0 = \text{Ext}^0(C, O_{C_0}) \rightarrow \text{Ext}^1(C, O_L(-1)) \cong C \rightarrow \text{Ext}^1(C, O_C) \rightarrow \text{Ext}^1(C, O_{C_0}) \cong C.\]

we know that, when \(F \in P(\text{Ext}^1(C, O_C))\) such that \(\delta(F) = G \neq 0\), then \(F\) fits exact sequence

\[0 \rightarrow O_L(-1) \rightarrow F \rightarrow G \rightarrow 0.\]

Then one can easily check that \(F\) is a stable sheaf and thus it does not contain the subsheaf
\(O_L\).
Lemma 17. \( W^+ \cup E^+_{2a} \) is an irreducible variety of dimension 15 and is contained in \( E^+ \).

Proof. Let \( F \) fit into the extension class
\[
(4) \quad 0 \to \mathcal{O}_C \to F \to \mathcal{O}_L \to 0.
\]

Denote \( \{p\} = L \cap C \). Let \( H \) be the plane containing \( C \). Tensoring (3) with \( \mathcal{O}_H \) we get the exact sequence
\[
0 = \text{Tor}^1_{\mathcal{O}_H}(\mathcal{O}_L, \mathcal{O}_H) \to \mathcal{O}_C \to F|_H \to C_p \to 0
\]
from which we see that \( F|_H \simeq \mathcal{O}_C(p) \). We obtain the extension
\[
0 \to \mathcal{O}_L(-1) \to F \to \mathcal{O}_C(p) \to 0.
\]

Conversely, a sheaf of this form is stable.

Let
\[
T := \{(p, C, L) \mid \{p\} = L \cap C \text{ and } L \not\subseteq H\}
\]
where \( C \) is a plane cubic curve and \( H \) is a plane containing \( C \). Also, we denote by \( T_{\text{sing}} \subseteq T \) as the sublocus of the pairs \((p, C, L)\) such that \( p \in \text{Sing}(C) \). Obviously, \( T \) is a \((\mathbb{P}^2 - \mathbb{P}^1)\)-bundle over the universal plane cubic curves where \( \mathbb{P}^2 - \mathbb{P}^1 \) parameterizes the choice of the line \( L \). Also, the space \( T_{\text{sing}} \) has the same bundle structure over the singular cubic curves. Hence \( T \) and \( T_{\text{sing}} \) are smooth of dimension 15, 13 respectively. Let
\[
\phi : W^+ \cup E^+_{2a} \to T
\]
be the natural morphism corresponding \( F \) with
\[
0 \to \mathcal{O}_L(-1) \to F \to \mathcal{O}_L(p) \to 0 \quad \text{to} \quad (p, C, L).
\]

We claim that \( \phi \) is a blow-up morphism \( T \) along \( T_{\text{sing}} \). This shows that \( W^+ \cup E^+_{2a} \) is irreducible. By duality, \( \text{Ext}^1_{\mathcal{O}_L}(\mathcal{O}_C(p), \mathcal{O}_L(-1)) \cong \text{Ext}^1_{\mathcal{O}_L}(\mathcal{O}_C(p), (\mathcal{O}_L(p))^{\mathbb{D}}) \). Note that \( \mathcal{O}_C(p)^{\mathbb{D}} \cong I_{p,C} \). From the exact sequence (2), we know that
\[
0 \to \text{Ext}^1_H(\mathcal{O}_L(-1)|_H, I_{p,C}) \cong \text{Ext}^1_{\mathcal{O}_L}(\mathcal{O}_C(p), \mathcal{O}_L(-1), I_{p,C}) \to \text{Ext}^0_H(I_{p,C}, I_{p,C}) = 0.
\]

But
\[
\text{Ext}^1_H(\mathcal{O}_L(-1)|_H, I_{p,C})^* \cong \text{Ext}^1_H(I_{p,C}, \mathcal{O}_C(p)) \cong \text{Hom}_C(I_{p,C}, \mathcal{O}_C(p)) \cong \text{Ext}^1_C(\mathcal{O}_C(p), I_{p,C}) \subset \text{Ext}^1_H(C_p, C_p)
\]
where the first isomorphism is given by the Serre duality. The second isomorphism is given by the exact sequence (2). Note that, since \( C \) is a curve on a projective plane and thus the dualizing sheaf \( \omega_C \) is a line bundle so that one can apply the Serre duality. The space \( \text{Ext}^1_C(C_p, C_p) \) can be naturally identified with the tangent space of \( C \) at \( p \). Hence it is naturally embedded into the tangent space \( \text{Ext}^1_H(C_p, C_p) \) of \( H \) at \( p \). More precisely, we have \( \text{Ext}^1_C(C_p, C_p) \cong \mathbb{C} \) if \( p \) is a smooth point of \( C \) and \( \mathbb{C}^2 \) otherwise. Hence the inverse image of \( \phi \) over the locus \( T \setminus T_{\text{sing}} \) is a single point. Hence \( \phi \) is a birational map. Furthermore by [12, §3.1.1], one can identify the normal space of \( T_{\text{sing}} \) in \( T \) with the extension group \( \text{Ext}^1_C(C_p, C_p) \) which is classifying the fiber of \( \phi \) (up to scalar). This implies that the morphism \( \phi \) is a blow-up \( T \) along \( T_{\text{sing}} \).

As general points in \( W^+ \cup E^+_{2a} \) are contained in \( E^+ \) and since it is irreducible, \( W^+ \cup E^+_{2a} \) itself is contained in \( E^+ \).

It remains to show \( E^+_{2b} \setminus W^\infty = E^+_{2b} \setminus W^+ \) is contained in \( E^+ \).

To this end, let \( Z \) be the locus of the extensions
\[
0 \to \mathcal{O}_L(-1) \to F \to \mathcal{O}_{LQ}(p) \to 0
\]
such that \( p \in Q \). Then clearly \( E^+_{2b} \) is contained in \( Z \). We will show \( Z \) is irreducible. Since general elements in \( Z \) are contained in \( E^+ \), this proves
Lemma 18. The locally closed subset $S \subset \text{Hilb}_{p^2}(m + 1) \times M_{p^2}(3m + 1)$ of pairs $(L, |O_C(p)|)$ for which $C = L \cup Q$ and $p \in Q$ for a conic curve $Q \subset \mathbb{P}^2$ is irreducible.

Proof. Let $M_{p^2}^\infty(2m + 2)$ be the moduli space of pairs with Hilbert polynomial $2m + 2$. Then one can easily see that it is isomorphic to the universal conic curves space which is an $\mathbb{P}^4$-bundle $\mathbb{P}^2$. Specially, it is irreducible. Let us construct a morphism

$$\text{Hilb}_{p^2}(m + 1) \times M_{p^2}^\infty(2m + 2) \to S$$

by associates $(L, p, Q) \mapsto (L, |O_{Q,L}(p)|)$. Then, it is well-defined and surjective. Thus we obtain the result. \hfill \square

Proposition 19. The locus $Z$ is irreducible of dimension 14.

Proof. By Lemma 18 we know that the image of the morphism $Z \to S$ is irreducible. Hence it suffices to show that this morphism is surjective and that its fibers are irreducible of the same dimension. We prove that Ext$_{O_C}$($O_C(p)$, $O_L(-1)$) $\simeq \mathbb{C}^4$ for all $(L, |O_C(p)|) \in S$. From (2) we have the exact sequence

$$0 \to \text{Ext}_{O_L}^1(O_C(p)|_L, O_L(-1)) \to \text{Ext}_{O_C}^1(O_C(p), O_L(-1)) \to \text{Hom}(\text{Tor}_1^{O_C}O_C(p), O_L(-1)) \to \text{Ext}_{O_L}^2(O_C(p)|_L, O_L(-1)) = 0$$

Assume firstly that $p \notin L$. The long exact sequence of torsion sheaves associated to the short exact sequence

$$0 \to O_C \to O_C(p) \to \mathbb{C}_p \to 0$$

yields the isomorphisms

$$O_C(p)|_L \simeq O_C|_L \simeq O_L,$$  
$$\text{Tor}_1^{O_C}O_C(p), O_L)$ \simeq \text{Tor}_1^{O_C}O_C, O_L) \simeq O_L(-1) \oplus O_L(-3)$$

We obtain the isomorphisms

$$\text{Ext}_{O_L}^1(O_C(p)|_L, O_L(-1)) = 0, \quad \text{Hom}(\text{Tor}_1^{O_C}O_C(p), O_L), O_L(-1)) \simeq \mathbb{C}^4$$

Assume now that $p \in L$. We have the exact sequence

$$0 \to 2O(-3) \to 3O(-2) \oplus O(-1) \to O(-1) \oplus O \to O_C(p) \to 0$$

$$\delta = \begin{bmatrix} u & 0 \\ 0 & u \\ -u_1 & -u_2 \\ 0 & -g_2 \end{bmatrix}, \quad \gamma = \begin{bmatrix} u_1 & u_2 & u & 0 \\ 0 & g_2 & 0 & u \end{bmatrix}$$

where $p$ is given by the equations $u = 0$, $u_1 = 0$, $u_2 = 0$, $L$ is given by the equations $u = 0$, $u_1 = 0$, $u_2 = 0$, and $Q$ is given by the equations $u = 0$, $g_2 = 0$. Tensoring with $O_L$, shows that $O_C(p)|_L$ is isomorphic to the cokernel of the morphism

$$O_L(-2) \xrightarrow{\delta_1} O_L(-1) \oplus O_L$$

and that $\text{Tor}_1^{O_C}O_C(p), O_L$) is isomorphic to the middle cohomology of the sequence

$$2O_L(-3) \xrightarrow{\delta_1} 3O_L(-2) \oplus O_L(-1) \xrightarrow{\gamma_1} O_L(-1) \oplus O_L$$
Theorem 22. Let
\[
\delta_L = \begin{bmatrix}
0 & 0 \\
0 & 0 \\
0 & -u_{2|L} \\
0 & -g_{2|L}
\end{bmatrix}, \quad \gamma_L = \begin{bmatrix}
0 & u_{2|L} & 0 & 0 \\
0 & g_{2|L} & 0 & 0
\end{bmatrix}
\]
which is isomorphic to the cokernel of the morphism
\[
\mathcal{O}_L(-3) \begin{bmatrix}
\delta \\
u_{2|L} \\
g_{2|L}
\end{bmatrix} 2\mathcal{O}_L(-2) \oplus \mathcal{O}_L(-1)
\]
By hypothesis \(p\) is a point on \(Q\), hence \(u_{2|L}\) divides \(g_{2|L}\). It now becomes clear that
\[
\mathcal{O}_C(p)|_L \simeq \mathbb{C}_p \oplus \mathcal{O}_L \quad \text{and} \quad \text{Tor}_1^{\mathcal{O}_L}(\mathcal{O}_C(p), \mathcal{O}_L) \simeq \mathbb{C}_p \oplus \mathcal{O}_L(-2) \oplus \mathcal{O}_L(-1)
\]
We obtain the isomorphisms
\[
\text{Ext}^1_{\mathcal{O}_L}(\mathcal{O}_C(p), \mathcal{O}_L(-1)) \simeq \mathbb{C}, \quad \text{Hom}(\text{Tor}_1^{\mathcal{O}_L}(\mathcal{O}_C(p), \mathcal{O}_L), \mathcal{O}_L(-1)) \simeq \mathbb{C}^3
\]

By summarizing above discussion, we obtain the following.

**Proposition 20.** The \(M^{m^+}(4m+1)\) consists of three irreducible components \(R^+, E^+\) and \(P^+\).

By forgetting the section, we have the following.

**Theorem 21.** The moduli space \(M_{p^+}(4m+1)\) consists of three irreducible components \(R, E, P\).

## 5. Description of the Intersections

**Theorem 22.** Let \(H \subset \mathbb{P}^3\) be a plane and let \(C \subset H\) be an irreducible quartic curve having three distinct nodal singular points \(P_1, P_2, P_3\). Then the unique extension of \(\mathbb{C}_P_1 \oplus \mathbb{C}_P_2 \oplus \mathbb{C}_P_3\) by \(\mathcal{O}_C\), denoted \(\mathcal{O}_C(P_1 + P_2 + P_3)\), gives a point in \(\mathbb{R} \cap \mathbb{P}\).

We denote by \(\mathbb{R} \cap \mathbb{P}\) the set of such sheaves. The intersection \(\mathbb{R} \cap \mathbb{P}\) is irreducible and is the closure of \((\mathbb{R} \cap \mathbb{P})_0\).

**Proof.** Either by [1, Proposition 3.3.2] or by the discussion in previous sections, \(\mathcal{O}_C(P_1 + P_2 + P_3)\) gives a point in \(M_{H}(4m+1)\), hence in \(\mathbb{P}\). We will show that \(\mathcal{O}_C(P_1 + P_2 + P_3)\) also gives a point in \(\mathbb{R}\). We fix a point \(O\) in \(\mathbb{P}^3 \setminus H\), blow up \(\mathbb{P}^3\) at \(O\) and we denote by \(\mathbb{O}_P_1, \mathbb{O}_P_2, \mathbb{O}_P_3\) the strict transforms of the lines \(OP_1, OP_2, OP_3\). We blow up a second time along \(\mathbb{O}_P_1, \mathbb{O}_P_2, \mathbb{O}_P_3\) and we denote the resulting variety by \(\mathbb{P}^3\) and the blowing down map by \(\beta: \mathbb{P}^3 \to \mathbb{P}^3\). Let \(S \subset \mathbb{P}^3\) be the cone with vertex \(O\) and base \(C\) and let \(\tilde{S} \subset \mathbb{P}^3\) denote its strict transform. The strict transform \(\tilde{C}\) of \(C\) is a desingularization of \(C\), so it is isomorphic to \(\mathbb{P}^1\). Clearly, \(S\) is a \(\mathbb{P}^1\)-bundle with base \(\tilde{C}\), so it is a Hirzebruch surface. In point of fact, \(\tilde{S} \simeq \Sigma_4\), which can be checked as follows. Let \(H' \subset \mathbb{P}^3\) be a plane that does not contain \(O, P_1, P_2, P_3\) and such that the line \(H'\cap H\) meets \(C\) at four distinct points. Let \(C' = H'\cap S\) and let \(\tilde{C}'\) be the strict transform of \(C'\). Then \(\tilde{C}'\) is a section of \(S \to C\), so it is homologous to \(C\) inside \(\tilde{S}\). Moreover, \(\tilde{C}' \cdot \tilde{C} = 4\), hence \(\tilde{C} \cdot \tilde{C} = 4\). Let \(\tilde{L}\) be a fiber of \(\tilde{S} \to \tilde{C}\), let \(\tilde{E} \subset \mathbb{P}^3\) be the total transform of \(O\), and let \(\mathbb{E} = \tilde{E} \cap \tilde{S}\). From the relations \(\mathbb{E} \cdot \mathbb{L} = 1\) and \(\mathbb{E} \cdot \tilde{C} = 0\) we get the relation \(\mathbb{E} = \tilde{C} - 4\mathbb{L}\). Thus \(\mathbb{E} \cdot \mathbb{E} = -4\), which proves that \(\tilde{S} \simeq \Sigma_4\). We fix an isomorphism \(\tilde{S} \to \mathbb{P}(\mathcal{O}_P + \mathcal{O}_P(4))\) identifying \(\tilde{C}\) with the section \((1, 0)\). The open subset \(\Gamma(\mathbb{P}^1, \mathbb{O}_P(4)) \subset \Gamma(\mathbb{P}^1, \mathbb{O}_P(4))\) of sections \(s\), such that the curve \(R_s \subset \tilde{S}\) given by the section \((1, s)\) maps to a smooth rational quartic curve \(R_s \subset \mathbb{P}^3\), is non-empty. Indeed, let \(P'_1, P''_1 \in \tilde{S}\) be the preimages of \(P_1\). We can find \(s\) such that \(R_s\) is smooth,
We claim that the family \( \{ h_s \} \) is ample on \( \mathbb{R} \), hence \( \beta \) is an isomorphism. According to [11, Proposition 2.1.10], there is a resolution of \( \mathcal{O}_{\mathbb{R}} \) of degree \( 0 \) such that \( \mathcal{I} \) is an irreducible quasi-projective curve containing \( 0 \) such that \( \mathcal{I} \) is flat relative to \( \mathbb{R} \). The sheaf \( \beta_*(\mathcal{O}_{\mathbb{R}}) \sim \mathcal{O}_{\mathbb{R}} \) gives a point in \( \mathbb{R} \) for \( s \in \mathcal{I} \). Moreover, \( \beta_*(\mathcal{O}_{\mathbb{R}}) \sim \mathcal{O}_C(1 + P_1 + P_2 + P_3) \).

To prove the claim, let \( \mathcal{R} \) be a coherent sheaf on \( \mathbb{G} \times \mathbb{P}^3 \) that is flat relative to \( \mathbb{G} \) and that restricts to \( \mathcal{O}_{\mathbb{R}} \) on each fiber \( s \times \mathbb{P}^3 \). We will prove that \( (\text{id} \times \beta)_*((\mathcal{R})) \) is flat relative to \( \mathbb{G} \), and that it parametrizes the family \( \{ \beta_*(\mathcal{O}_{\mathbb{R}}) \} \). Let \( D_1, D_2, D_3 \subset \mathbb{P}^3 \) be the total transforms of \( \mathcal{O}P_1, \mathcal{O}P_2, \mathcal{O}P_3 \), and let \( D = D_1 + D_2 + D_3 \). It is easy to check that for an integer \( 1 \) sufficiently large the line bundle

\[
L = (1 + 3)\beta^*(\mathcal{O}_{\mathbb{P}^3}(1)) - \mathcal{O}(\tilde{E}) - 2\mathcal{O}(\tilde{D})
\]

is ample on \( \mathbb{P}^3 \). Note that

\[
L_{\mathcal{I}} = (1 + 3)\mathcal{O}(\tilde{C}) - \mathcal{O}(\tilde{E}) - 2\mathcal{O}(\tilde{L}) = (1 + 3)\mathcal{O}(\tilde{E} + 4\tilde{L}) - \mathcal{O}(\tilde{E}) - 2\mathcal{O}(\tilde{L}) = \mathcal{O}(\tilde{C}) + 2\mathcal{O}(\tilde{E})
\]

The support of \( \mathcal{R} \) does not meet \( \mathbb{G} \), hence \( \mathcal{R} \otimes \mathcal{O}(\tilde{E}) \simeq \mathcal{R} \), and hence, for any integer \( m \),

\[
\mathcal{R} \otimes \mathcal{L}^m \simeq \mathcal{R} \otimes \beta^*(\mathcal{O}_{\mathbb{P}^3}(1m))
\]

Let \( \bar{\beta} : \mathbb{G} \times \mathbb{P}^3 \rightarrow \Gamma \) and \( \beta : \mathbb{G} \times \mathbb{P}^3 \rightarrow \Gamma \) be the projection maps onto the first component. By [11, Proposition 2.1.2], the property that \( \mathcal{R} \) be flat relative to \( \mathbb{G} \) is equivalent to the property that \( \beta_*(\mathcal{R} \otimes \mathcal{L}^m) \) be locally free for \( m \gg 0 \). By the projection formula

\[
(\text{id} \times \beta)_*(\mathcal{R} \otimes \beta^*(\mathcal{O}_{\mathbb{P}^3}(1m)) \simeq (\text{id} \times \beta)_*(\mathcal{R}) \otimes \mathcal{O}_{\mathbb{P}^3}(1m)
\]

hence

\[
(\text{id} \times \beta)_*(\mathcal{R} \otimes \beta^*(\mathcal{O}_{\mathbb{P}^3}(1m)) \simeq \beta_*(\mathcal{R} \otimes \mathcal{L}^m)
\]

Since the right-hand-side is locally free on \( \Gamma \) for large \( m \), we deduce, again from [11, Proposition 2.1.2], that \( (\text{id} \times \beta)_*(\mathcal{R}) \) is \( \Gamma \)-flat.

Let \( s \in \mathbb{G} \) be an arbitrary closed point. Consider the cartesian diagram

\[
\begin{array}{ccc}
(s) \times \mathbb{P}^3 & \xrightarrow{j} & \mathbb{G} \times \mathbb{P}^3 \\
\downarrow{\beta} & & \downarrow{\text{id} \times \beta} \\
(s) \times \mathbb{P}^3 & \xrightarrow{i} & \mathbb{G} \times \mathbb{P}^3
\end{array}
\]

We need to show that the base-change homomorphism

\[
i^*(\text{id} \times \beta)_*(\mathcal{R}) \rightarrow \beta_*j^*(\mathcal{R})
\]

is an isomorphism. According to [11, Proposition 2.1.10], there is a resolution

\[
0 \rightarrow A_3 \otimes B_3 \rightarrow A_2 \otimes B_2 \rightarrow A_1 \otimes B_1 \rightarrow A_0 \otimes B_0 \rightarrow \mathcal{R} \rightarrow 0
\]

where \( A_k \) is locally free on \( \Gamma \) and \( B_k \) is a tensor power of \( L \). Choose an integer \( m \) so large that \( R^n \beta_*(A_k \otimes L^m) = 0 \) for all \( n > 0 \) and \( k \). We tensor the above exact sequence with \( L^m \) and then we apply \( j^* \). We obtain an exact sequence again, because \( \mathcal{R} \otimes L^m \) is \( \Gamma \)-flat. Applying \( \beta_* \) we get the sequence

\[
0 \rightarrow \beta_*j^*(A_3 \otimes (B_3 \otimes L^m)) \rightarrow \cdots \rightarrow \beta_*j^*(A_0 \otimes (B_0 \otimes L^m)) \rightarrow \beta_*j^*(\mathcal{R} \otimes L^m) \rightarrow 0
\]

which is exact, because for all \( n > 0 \) and \( k \)

\[
R^n \beta_*(A_k \otimes (B_k \otimes L^m)) \simeq A_k \otimes R^n \beta_*(B_k \otimes L^m) = 0
\]
On the other hand, the sequence

\[ 0 \to i^*(\mathrm{id} \times \beta)_*(A_3 \boxtimes (B_3 \otimes L^m)) \to \cdots \to i^*(\mathrm{id} \times \beta)_*(A_0 \boxtimes (B_0 \otimes L^m)) \to i^*(\mathrm{id} \times \beta)_*(\mathcal{R} \boxtimes L^m) \to 0 \]

is exact because

\[ R^n(\mathrm{id} \times \beta)_*(A_k \boxtimes (B_k \otimes L^m)) = 0 \]

for all \( n > 0 \) and \( k \) and, also, because, as shown in the previous paragraph, \((\mathrm{id} \times \beta)_*(\mathcal{R} \boxtimes L^m)\) is flat over \( \Gamma \). Trivially, the base-change homomorphisms

\[ i^*(\mathrm{id} \times \beta)_*(A_k \boxtimes (B_k \otimes L^m)) \to \beta_*j^*(A_k \boxtimes (B_k \otimes L^m)) \]

are isomorphisms for \( k = 0, 1, 2, 3 \). From the five-lemma we deduce that the base-change homomorphism

\[ i^*(\mathrm{id} \times \beta)_*(\mathcal{R} \boxtimes L^m) \to \beta_*j^*(\mathcal{R} \boxtimes L^m) \]

is an isomorphism. Using the projection formula we get the isomorphisms

\[ i^*(\mathrm{id} \times \beta)_*(\mathcal{R} \boxtimes L^m) \simeq i^*(\mathrm{id} \times \beta)_*(\mathcal{R} \boxtimes \beta^*O_{p^3}(1m)) \simeq i^*((\mathrm{id} \times \beta)_*(\mathcal{R}) \boxtimes O_{p^3}(1m)) \simeq i^*(\mathrm{id} \times \beta)_*(\mathcal{R}) \boxtimes O_{p^3}(1m) \]

and

\[ \beta_*j^*(\mathcal{R} \boxtimes L^m) \simeq \beta_*j^*(\mathcal{R} \boxtimes \beta^*O_{p^3}(1m)) \simeq \beta_*j^*(\mathcal{R} \boxtimes O_{p^3}(1m)) \simeq (\beta_*j^*(\mathcal{R}) \boxtimes O_{p^3}(1m)) \]

This finishes the proof of the claim and, also, of the first part of the theorem.

Let \( R \subset \mathbb{P}^3 \) be a smooth rational quartic curve, let \( H \subset \mathbb{P}^3 \) be a plane, let \( O \) be an arbitrary point in \( \mathbb{P}^3 \setminus (R \cup H) \), and let \( \pi_O: \mathbb{P}^3 \setminus \{O\} \to H \) be the projection with center \( O \). Denote by \((\mathbb{P}^3 \setminus (R \cup H))_0\) the subset of points \( O \in \mathbb{P}^3 \setminus (R \cup H) \) satisfying the following properties: no line passing through \( O \) meets \( R \) at three points or is tangent to \( R \), and for any two distinct points \( A, B \) on \( R \) that are colinear with \( O \) the tangent vectors to \( R \) at \( A \) and \( B \) are not coplanar. For \( O \in (\mathbb{P}^3 \setminus (R \cup H))_0 \) the sheaf \((\pi_O)_*(O_R)\) is supported on an irreducible quartic curve \( C \subset H \) whose singularities are nodal. We have a canonical injective morphism \( O_C \to (\pi_O)_*(O_R) \) whose cokernel is the structure sheaf of the set of singular points of \( C \). Since \((\pi_O)_*(O_R)\) has Hilbert polynomial \( 4m + 1 \) and \( P_{O,R}(m) = 4m - 2 \), we see that \( C \) has precisely three singular points \( P_1, P_2, P_3 \), and that \((\pi_O)_*(O_R)\) is the unique extension of \( C_{P_1} \oplus C_{P_2} \oplus C_{P_3} \) by \( O_C \). Consider now a point \( O' \in \mathbb{P}^3 \setminus (R \cup H) \) and let \( \Gamma \subset \mathbb{P}^3 \) be an irreducible quasi-projective curve containing \( O' \) such that \( \Gamma \cap \{O'\} \subset \mathbb{P}^3 \setminus (R \cup H) \). The family \((\pi_O)_*(O_R)|_{O \in \Gamma} \) on \( H \) is flat relative to \( \Gamma \). Each member of the family has irreducible support and has no zero-dimensional torsion, hence it is semi-stable, and hence it gives a point in \( M_H(4m + 1) \). In point of fact, \([([\pi_O]_*(O_R)) \in \Gamma \) belongs to \((R \cap P)_0 \) for \( O \neq O' \). We conclude that \((\pi_O)_*(O_R)\) gives a point in the closure of \((R \cap P)_0 \).

Assume that \( F \) gives a point in \( R \cap P \), where \( \text{supp}(F) \subset H \). Let \( \Gamma \subset \mathbb{R} \) be an irreducible quasi-projective curve with associate flat family \( \{F_s\}_{s \in \Gamma} \). Assume that \( F \) belongs to \( \Gamma \) and \( \Gamma \setminus \{[F]\} \subset R_0 \). Choose a point \( O \in \mathbb{P}^3 \setminus H \) that is not on the support of \( F_s \) for all \( s \in \Gamma \). Let \( \pi: \mathbb{P}^3 \setminus \{O\} \to H \) be the projection with center \( O \). We claim that the family \( \{\pi_*F_s\}_{s \in \Gamma} \) on \( H \) is flat relative to \( \Gamma \). Indeed, let \( \mathcal{R} \) be a coherent \( \Gamma \)-flat sheaf on \( \Gamma \times \mathbb{P}^3 \) that restricts to \( F_s \) on each fiber \( \{s\} \times \mathbb{P}^3 \). Arguing as before, we can show that the sheaf \( (\mathrm{id} \times \pi)_*(\mathcal{R}) \) on \( \Gamma \times H \) is flat relative to \( \Gamma \) and parametrizes the family \( \{\pi_*F_s\}_{s \in \Gamma} \). We saw above that \([\pi_*F_s]\in \Gamma \) lies in the closure of \((R \cap P)_0 \) if \( s \neq [F] \). Since \( \pi_*F_s \simeq F \), we conclude that \([F] \in \Gamma \) belongs to the closure of \((R \cap P)_0 \), as well. This proves that \( R \cap P \) is the closure of \((R \cap P)_0 \).

To finish the proof of the theorem we will show that \((R \cap P)_0 \) is irreducible. Since this set fibers over the dual of \( \mathbb{P}^3 \), it is enough to show that the subset of sheaves that have support in a fixed plane \( H \) is irreducible. Take two such sheaves \( O_{C}(P_1 + P_2 + P_3) \) and \( O_{C}(P_1' + P_2' + P_3') \). Choose a point \( O \in \mathbb{P}^3 \setminus H \) and denote by \( \pi: \mathbb{P}^3 \setminus \{O\} \to H \) the projection with center \( O \). We
saw above that there are smooth rational quartic curves \( R, R' \subset \mathbb{P}^3 \setminus \{ O \} \) such that \( \pi_s(\mathcal{O}_R) \simeq \mathcal{O}_C(P_1 + P_2 + P_3) \) and \( \pi_s(\mathcal{O}_R') \simeq \mathcal{O}_C(P_1' + P_2' + P_3') \). Let \( \Gamma \subset R_0 \) be an irreducible quasi-projective curve containing \( \mathcal{O}_R \) and \( \mathcal{O}_R' \) with associated flat family \( \{ \mathcal{O}_{R,s} \}_{s \in \Gamma} \). We may assume that \( O \not\in R_s \) for all \( s \in \Gamma \). As before, the family \( \{ \pi_s(\mathcal{O}_{R,s}) \}_{s \in \Gamma} \) is flat over \( \Gamma \), so it corresponds to a map \( f: \Gamma \to \mathbb{R} \cap M_{14}(4m + 1) \). Thus both \( \mathcal{O}_C(P_1 + P_2 + P_3) \) and \( \mathcal{O}_C(P_1' + P_2' + P_3') \) belong to \( f(\Gamma) \), which is irreducible, so they belong to the same irreducible component of \( R \cap M_{14}(4m + 1) \). But \( R \cap M_{14}(4m + 1) \) is the closure of \( \{ R, P \}_{0} \cap M_{14} \). We conclude that this set is irreducible.

**Theorem 23.** Let \( H \subset \mathbb{P}^3 \) be a plane and let \( C \subset H \) be an irreducible quartic curve having two distinct nodal singular points \( P_1 \) and \( P_2 \) and no other singularities. Let \( \mathcal{P} \) be a regular point of \( C \). Then \( \mathcal{O}_C(P_1 + P_2 + P) \), gives a point in \( \mathcal{E} \cap \mathcal{P} \). We denote by \( \{ \mathcal{E} \cap \mathcal{P} \}_{0} \) the set of such sheaves. The intersection \( \mathcal{E} \cap \mathcal{P} \) is irreducible and is the closure of \( \{ \mathcal{E} \cap \mathcal{P} \}_{0} \).

**Proof.** The argument is analogous to the argument at Theorem 22 with the following modifications. This time \( \mathbb{P}^3 \) is obtained by blowing up \( \mathbb{P}^3 \) at a point \( O \in \mathbb{P}^3 \setminus H \) and then at the strict transforms of the lines \( OP_1 \) and \( OP_2 \). The strict transform \( \tilde{C} \) of \( C \) is a smooth elliptic curve and the strict transform \( \tilde{S} \) of the cone with vertex \( O \) and base \( C \) is isomorphic to \( \mathbb{P}(\mathcal{O}_{C} \oplus \mathcal{L}) \), where \( \mathcal{L} = \beta^*(\mathcal{O}_{\mathbb{P}^3}(1)) \) is a line bundle of degree 4 on \( C \). Let \( P' \in \tilde{C} \) be the preimage of \( P \). Let \( \Gamma \subset \tilde{C} \) be an irreducible quasi-projective curve containing \( \mathcal{O}_C(P_1 + P_2 + P) \), and maps to a smooth elliptic curve \( E_s \subset \overline{\mathbb{P}^3} \). The family \( \{ \mathcal{O}_{E_s} \}_{s \in \Gamma} \) on \( \overline{\mathbb{P}^3} \) is flat relative to \( \Gamma \). Let \( S \) be the sheaf on \( \Gamma \times \overline{\mathbb{P}^3} \) parametrizing this family. Let \( \mathcal{C} \) be the structure sheaf of the subscheme \( \Gamma \times \{ P' \} \subset \Gamma \times \overline{\mathbb{P}^3} \). We have a canonical surjective morphism \( S \to \mathcal{C} \) whose kernel is denoted by \( T \). Since both \( S \) and \( C \) are \( \Gamma \)-flat, also \( T \) is \( \Gamma \)-flat. In point of fact, \( T \) parametrizes the family \( \{ \mathcal{O}_{E_s}(-P') \}_{s \in \Gamma} \). According to [16], Lemma 12, the dual of \( T \) is \( \Gamma \)-flat and it parametrizes the dual family \( \{ (\mathcal{O}_{E_s}(-P'))^0 \}_{s \in \Gamma} \). But \( \mathcal{O}_{E_s} \) is self-dual, being the structure sheaf of an elliptic curve, hence \( (\mathcal{O}_{E_s}(-P'))^0 \simeq \mathcal{O}_{E_s}(P') \). We deduce that the family \( \{ \mathcal{O}_{E_s}(P') \}_{s \in \Gamma} \) on \( \overline{\mathbb{P}^3} \) is flat relative to \( \Gamma \). Applying \( \pi_s \), we obtain a flat family on \( \mathbb{P}^3 \) whose generic element \( \mathcal{O}_{E,s}(P) \) lies in \( E_0 \). Thus \( \{ \mathcal{O}_{C}(P_1 + P_2 + P) \} \) lies in \( \mathcal{E}_0 \).

The rest of the proof mimics the proof of Theorem 22. A generic projection \( \pi \) onto a plane maps a smooth elliptic quartic curve \( E \) to a quartic curve \( C \) with precisely two singular points \( P_1, P_2 \), which are nodal. Thus \( \pi_s(\mathcal{O}_E(P)) = \mathcal{O}_C(P_1 + P_2 + P) \) gives a point in \( \{ \mathcal{E} \cap \mathcal{P} \}_{0} \) and any sheaf in \( \mathcal{E} \cap \mathcal{P} \) can be approximated by such sheaves.
6. The resolutions of the stable sheaves $F \in M(4m + 1)$

Notations.

$V$ = vector space of dimension 4 over $\mathbb{C}$,
$\mathbb{P}^3 = \mathbb{P}(V)$,
$\{X, Y, Z, W\} =$ basis of $V^*$,
$F =$ coherent sheaf on $\mathbb{P}^3$ with support of dimension 1,
$M_{\mathbb{F}^n}(rm + \chi) =$ moduli space of semi-stable sheaves on $\mathbb{P}^n$ with Hilbert polynomial $rm + \chi$,
$M = M_{\mathbb{P}^3}(4m + 1)$,
$L =$ line in $\mathbb{P}^3$,
$H =$ plane in $\mathbb{P}^3$,
$\mathcal{T}^0(F) =$ the zero-dimensional torsion of a sheaf $F$.

Beilinson spectral sequence for $F$. The relevant part of the $E^1$-term:

$H^1(F(-1)) \otimes O(-3) \xrightarrow{\varphi_7} H^1(F \otimes \Omega^2(2)) \otimes O(-2) \xrightarrow{\varphi_5} H^1(F \otimes \Omega^1(1)) \otimes O(-1) \xrightarrow{\varphi_6} H^1(F) \otimes O$

$H^0(F(-1)) \otimes O(-3) \xrightarrow{\varphi_4} H^0(F \otimes \Omega^2(2)) \otimes O(-2) \xrightarrow{\varphi_3} H^0(F \otimes \Omega^1(1)) \otimes O(-1) \xrightarrow{\varphi_2} H^0(F) \otimes O$

The $E^2$-term:

$\begin{array}{cccc}
\text{Ker}(\varphi_1) & \text{Ker}(\varphi_2)/\text{Im}(\varphi_1) & \text{Ker}(\varphi_3)/\text{Im}(\varphi_2) & \text{Coker}(\varphi_3) \\
\varphi_7 & \varphi_4 & \varphi_5 & \\
\text{Ker}(\varphi_4) & \text{Ker}(\varphi_5)/\text{Im}(\varphi_4) & \text{Ker}(\varphi_6)/\text{Im}(\varphi_5) & \text{Coker}(\varphi_6) \\
\end{array}$

The $E^3 = E^\infty$-term:

$\begin{array}{cccc}
\text{Ker}(\varphi_7) & \text{Ker}(\varphi_8) & \text{Ker}(\varphi_3)/\text{Im}(\varphi_2) & \text{Coker}(\varphi_3) \\
\end{array}$

Thus $\varphi_7$ is an isomorphism, $\varphi_3$ is surjective, $\varphi_4$ is injective, $\text{Ker}(\varphi_3) = \text{Im}(\varphi_4)$ and we have the exact sequence

(5) $0 \rightarrow \text{Ker}(\varphi_2)/\text{Im}(\varphi_1) \xrightarrow{\varphi_4} \text{Coker}(\varphi_6) \rightarrow F \rightarrow \text{Ker}(\varphi_3)/\text{Im}(\varphi_2) \rightarrow 0$

Denote $p = h^0(F \otimes \Omega^1(1)), q = h^0(F \otimes \Omega^2(2))$. Assume that $F$ is semi-stable and has Hilbert polynomial $P_F(m) = 4m + 1$. According to [11] we have the relations

$h^0(F(-1)) = 0, \quad h^1(F) = 0$ or 1.

The Beilinson monad with middle cohomology $F$ yields an exact sequence

(6) $0 \rightarrow 3O(-3) \oplus qO(-2) \xrightarrow{\psi_1} (q + 5)O(-2) \oplus pO(-1) \xrightarrow{\psi_2} \text{Ker}(\varphi_3) \oplus H^0(F) \otimes O \rightarrow F \rightarrow 0$

in which $\psi_{12} = 0$ and $\psi_{12} = 0$. We recall two well-known facts. The sheaves $E$ on $\mathbb{P}^3$ having no zero-dimensional torsion and such that $P_E(m) = m + 1$ are precisely the structure sheaves of lines, so they have resolution

(7) $0 \rightarrow O(-2) \rightarrow 2O(-1) \rightarrow O \rightarrow E \rightarrow 0$
The sheaves \( E \) giving points in \( M_{3p}(2m + 1) \) are precisely the structure sheaves of conic curves, so they have resolution
\[
0 \rightarrow \mathcal{O}(-3) \rightarrow \mathcal{O}(-2) \oplus \mathcal{O}(-1) \rightarrow \mathcal{O} \rightarrow E \rightarrow 0
\]

**Theorem 24.** Let \( F \) give a point in \( M_{3p}(4m + 1) \). Then precisely one of the following is true:

(i) \( h^0(F \otimes \Omega^2(2)) = 0, h^0(F \otimes \Omega^3(1)) = 0, h^0(F) = 1; \)
(ii) \( h^0(F \otimes \Omega^2(2)) = 0, h^0(F \otimes \Omega^3(1)) = 1, h^0(F) = 1; \)
(iii) \( h^0(F \otimes \Omega^2(2)) = 1, h^0(F \otimes \Omega^3(1)) = 3, h^0(F) = 2. \)

The sheaves satisfying conditions (i) are precisely the sheaves having a resolution of the form
\[
0 \rightarrow 3\mathcal{O}(-3) \rightarrow 5\mathcal{O}(-2) \rightarrow \mathcal{O}(-1) \oplus \mathcal{O} \rightarrow F \rightarrow 0
\]

where \( \dim(\text{span}(l_1, l_2, l_3, l_4, l_5)) \geq 3. \)

The sheaves satisfying conditions (ii) are precisely the sheaves having a resolution of the form
\[
0 \rightarrow 3\mathcal{O}(-3) \rightarrow 5\mathcal{O}(-2) \oplus \mathcal{O}(-1) \rightarrow 2\mathcal{O}(-1) \oplus \mathcal{O} \rightarrow F \rightarrow 0
\]

where \( \varphi_{12} = 0 \) and \( \varphi_{11} : 5\mathcal{O}(-2) \rightarrow 2\mathcal{O}(-1) \) is not equivalent to a morphism of the form

\[
\begin{bmatrix}
* & * & * & 0 & 0 \\
* & * & * & * & 0
\end{bmatrix}
\]

or

\[
\begin{bmatrix}
* & * & * & * & 0 \\
* & * & * & * & 0
\end{bmatrix}
\]

The sheaves satisfying conditions (iii) are precisely the sheaves having a resolution of the form
\[
0 \rightarrow \mathcal{O}(-4) \oplus \mathcal{O}(-2) \rightarrow \mathcal{O}(-3) \oplus 3\mathcal{O}(-1) \rightarrow 2\mathcal{O} \rightarrow F \rightarrow 0
\]

where \( l_1, l_1, l_2 \) are linearly independent one-forms. If \( H \subset \mathbb{P}^3 \) is the plane given by the equation \( l = 0 \), then \( F \) has resolution
\[
0 \rightarrow \mathcal{O}_H(-3) \oplus \mathcal{O}_H(-1) \rightarrow 2\mathcal{O}_H \rightarrow F \rightarrow 0
\]

where \( \tilde{f}_1, \tilde{f}_2, \tilde{l}_1, \tilde{l}_2 \) denote classes modulo \( l \).

**Proof.** (i) Assume first that \( h^0(F) = 1 \). The exact sequence becomes
\[
0 \rightarrow 3\mathcal{O}(-3) \oplus \mathcal{O}(-2) \rightarrow (q + 5)\mathcal{O}(-2) \oplus p\mathcal{O}(-1) \rightarrow (p + 1)\mathcal{O}(-1) \oplus \mathcal{O} \rightarrow F \rightarrow 0
\]

We claim that \( p = 0 \) or \( 1 \). Indeed, if \( p = 2 \), then we would get a commutative diagram

\[
\begin{CD}
p\mathcal{O}(-1) @>>> \mathcal{O} \oplus \mathcal{O}_l \rightarrow 0 \\
@VV\varphi_{2,2}V @VV\varphi_0V @VV\varphi_1V @VVFV \\
(q + 5)\mathcal{O}(-2) \oplus p\mathcal{O}(-1) @>>> (p + 1)\mathcal{O}(-1) \oplus \mathcal{O} \rightarrow F \rightarrow 0
\end{CD}
\]

Both \( \mathcal{O}_l \) and \( F \) are stable and \( p(\mathcal{O}_l) = 1 > p(F) \), hence \( \text{Hom}(\mathcal{O}_l, F) = 0 \). Thus \( \mathcal{O} \rightarrow F \) is the zero morphism. On the other hand \( H^0(\mathcal{O}) \rightarrow H^0(F) \) is injective because \( H^0(\text{Coker}(\varphi)) = 0. \)
We have obtained a contradiction. If \( p = 3 \) or \( p \geq 4 \), then \( \text{Coker}(\varphi_{22}) \) would be the structure sheaf of a point, respectively, it would be zero. Both cases would yield contradictions as above.

Assume that \( p = 0 \). Then \( q = 0 \) because \( \varphi_5 \) is injective. We obtain the resolution

\[
0 \rightarrow 3\mathcal{O}(-3) \rightarrow 5\mathcal{O}(-2) \stackrel{\varphi}{\rightarrow} \mathcal{O}(-1) \oplus \mathcal{O} \rightarrow F \rightarrow 0
\]

\[
\varphi = \begin{bmatrix}
l_1 & l_2 & l_3 & l_4 & l_5 \\
q_1 & q_2 & q_3 & q_4 & q_5
\end{bmatrix}
\]

If \( \dim(\text{span}(l_1, l_2, l_3, l_4, l_5)) = 1 \), then we may assume that \( l_1 \neq 0 \) and that \( l_2, l_3, l_4, l_5 \) are zero. We would get a commutative diagram

\[
\begin{array}{ccc}
5\mathcal{O}(-2) & \xrightarrow{\varphi} & \mathcal{O}(-1) \oplus \mathcal{O} \\
\downarrow & & \downarrow \\
\mathcal{O}(-2) & \xrightarrow{l_1} & \mathcal{O}(-1) \rightarrow \mathcal{O}_H(-1) \rightarrow 0
\end{array}
\]

showing that \( F \) maps surjectively to \( \mathcal{O}_H(-1) \). This is absurd, because \( \dim(\text{supp}(F)) = 1 \).

Likewise, if \( \dim(\text{span}(l_1, l_2, l_3, l_4, l_5)) = 2 \), then \( F \) would have a quotient sheaf of the form \( \mathcal{O}_L(-1) \), in violation of semi-stability. We conclude that \( F \) has resolution \( \mathfrak{A} \).

Conversely, we assume that \( F \) has resolution \( \mathfrak{A} \) and we must show that \( F \) is semi-stable. At every point \( P \in \mathbb{P}^3 \) we have \( \dim_{\mathcal{O}_P}(F) \leq 2 \), hence \( \dim_{\mathcal{O}_P}(F) \geq 1 \). From Grothendieck’s Criterion we deduce that \( \mathcal{H}^0_{\mathcal{O}_P}(F) = 0 \), that is, \( F \) has no sections supported on \( \{P\} \). Thus \( F \) has no zero-dimensional torsion. Assume that \( F \) had a destabilizing subsheaf \( E \). We may assume that \( E \) is semi-stable. Since \( h^0(E) \leq h^0(F) = 1 \) the Hilbert polynomial of \( E \) may be one of the following: \( m + 1, 2m + 1, 3m + 1 \). In the first case \( E \simeq \mathcal{O}_L \) and resolution \( \mathfrak{A} \) fits into a commutative diagram

\[
\begin{array}{ccc}
0 & \xrightarrow{\gamma} & 2\mathcal{O}(-1) \xrightarrow{\beta} \mathcal{O} \xrightarrow{\alpha} E \rightarrow 0 \\
0 & \xrightarrow{\gamma} & 3\mathcal{O}(-3) \xrightarrow{\beta} 5\mathcal{O}(-2) \xrightarrow{\alpha} F \rightarrow 0
\end{array}
\]

Since \( \alpha \neq 0 \), we have \( \text{Ker}(\alpha) = 0 \), hence \( \text{Ker}(\gamma) \simeq \text{Ker}(\beta) = 2\mathcal{O}(-1) \). This is absurd. In the second case we have resolution \( \mathfrak{A} \), which fits into a commutative diagram

\[
\begin{array}{ccc}
0 & \xrightarrow{\gamma} & \mathcal{O}(-3) \xrightarrow{\beta} \mathcal{O}(-2) + \mathcal{O}(-1) \xrightarrow{\alpha} \mathcal{O} \xrightarrow{\beta} E \rightarrow 0 \\
0 & \xrightarrow{\gamma} & 3\mathcal{O}(-3) \xrightarrow{\beta} 5\mathcal{O}(-2) \xrightarrow{\alpha} \mathcal{O}(-1) + \mathcal{O} \xrightarrow{\alpha} F \rightarrow 0
\end{array}
\]

Since \( \alpha \neq 0 \), we have \( \text{Ker}(\alpha) \simeq \text{Ker}(\beta) \). It follows that \( \mathcal{O}(-1) \) is a subsheaf of \( \text{Ker}(\gamma) \). This is absurd. Finally, assume that \( P_\mathcal{E}(m) = 3m + 1 \). The quotient \( G = F/E \) has no zero-dimensional torsion and \( P_\mathcal{E}(m) = m \). It follows that \( G \simeq \mathcal{O}_L(-1) \). We have a commutative diagram

\[
\begin{array}{ccc}
0 & \xrightarrow{\gamma} & 3\mathcal{O}(-3) \xrightarrow{\beta} 5\mathcal{O}(-2) \xrightarrow{\alpha} \mathcal{O}(-1) + \mathcal{O} \xrightarrow{\alpha} F \rightarrow 0 \\
0 & \xrightarrow{\beta} & \mathcal{O}(-3) \xrightarrow{\beta} 2\mathcal{O}(-2) \xrightarrow{\alpha} \mathcal{O}(-1) \xrightarrow{\alpha} G \rightarrow 0
\end{array}
\]

From the commutativity of the middle square we see that

\[
\varphi \sim \begin{bmatrix}
\ast & \ast & 0 & 0 & 0 \\
\ast & \ast & \ast & \ast & \ast
\end{bmatrix}
\]
This contradicts our hypothesis. We conclude that there are no destabilising subsheaves $E \subset F$.

(ii) We next examine the case when $h^0(F \otimes \Omega^1(1)) = 1$ and $h^0(F) = 1$. Since $\varphi_5$ is injective we see that $q = 0$ or 1. If $q = 1$, then $\varphi_6$ would be generically zero, hence $\varphi_6 = 0$, hence $\mathcal{Ker}(\varphi_6)/\mathcal{Im}(\varphi_5) \approx \mathcal{O}_H(-1)$. Recall that $\varphi_7: \mathcal{Ker}(\varphi_1) \rightarrow \mathcal{Ker}(\varphi_6)/\mathcal{Im}(\varphi_5)$ is an isomorphism. It would follow that $\mathcal{O}_H(-1)$ is a subsheaf of $3\mathcal{O}(-3)$. This is absurd. Thus $q = 0$ and we have a resolution

$$0 \rightarrow 3\mathcal{O}(-3) \rightarrow 5\mathcal{O}(-2) \oplus \mathcal{O}(-1) \overset{\varphi_1}{\rightarrow} 2\mathcal{O}(-1) \oplus \mathcal{O} \rightarrow F \rightarrow 0$$

If $\varphi_{11}$ were equivalent to a morphism of the form

$$\left[\begin{array}{ccccc}
* & * & 0 & 0 & 0 \\
* & * & * & * & *
\end{array}\right]$$

then $F$ would have a quotient sheaf of the form $\mathcal{O}_H(-1)$, or of the form $\mathcal{O}_t(-1)$. This, we saw above, yields a contradiction. If $\varphi$ were equivalent to a morphism of the form

$$\left[\begin{array}{ccccc}
* & * & * & 0 & 0 \\
* & * & * & * & 0 \\
* & * & * & q & 1
\end{array}\right]$$

then we would have a commutative diagram

$$\begin{align*}
\mathcal{O}(-2) \oplus \mathcal{O}(-1) & \xrightarrow{\begin{bmatrix} q & 1 \end{bmatrix}} \mathcal{O} \\
5\mathcal{O}(-2) \oplus \mathcal{O}(-1) & \xrightarrow{\varphi} 2\mathcal{O}(-1) \oplus \mathcal{O} \\
& \xrightarrow{F} 0
\end{align*}$$

in which $C$ is the conic curve given by the equations $q = 0$, $l = 0$. Both $\mathcal{O}_C$ and $F$ are stable with $p(\mathcal{O}_C) = 1/2 > p(F)$, hence $\text{Hom}(\mathcal{O}_C, F) = 0$. Thus, the map $\mathcal{O} \rightarrow F$ is zero. This, as we saw above, yields a contradiction. We conclude that $F$ has resolution (10).

Conversely, if $F$ has resolution (10), then, by arguments analogous to the arguments in the case of resolution (9), we can show that $F$ is semi-stable.

(iii) Finally, we consider the case when $h^0(F) = 2$. Then $p \geq 3$ and resolution (10) takes the form

$$0 \rightarrow 3\mathcal{O}(-3) \oplus q\mathcal{O}(-2) \overset{\psi}{\rightarrow} (q + 5)\mathcal{O}(-2) \oplus p\mathcal{O}(-1) \overset{\varphi}{\rightarrow} \Omega^1 \oplus (p - 3)\mathcal{O}(-1) \oplus 2\mathcal{O} \rightarrow F \rightarrow 0$$

The morphism $\varphi_{32}: p\mathcal{O}(-1) \rightarrow 2\mathcal{O}$ cannot be equivalent to a morphism represented by a matrix of the form

$$\left[\begin{array}{ccc}
* & \cdots & * \\
* & \cdots & l_1 \ l_2
\end{array}\right]$$

otherwise we would have a commutative diagram

$$\begin{align*}
2\mathcal{O}(-1) & \xrightarrow{\begin{bmatrix} l_1 & l_2 \end{bmatrix}} \mathcal{O} \\
(q + 5)\mathcal{O}(-2) \oplus p\mathcal{O}(-1) & \xrightarrow{\begin{bmatrix} q & 1 \end{bmatrix}} \Omega^1 \oplus (p - 3)\mathcal{O}(-1) \oplus 2\mathcal{O} \\
& \xrightarrow{F} 0
\end{align*}$$

But $\text{Hom}(\mathcal{O}_t, F) = 0$, hence the morphism $2\mathcal{O} \rightarrow F$ is not injective. On the other hand $H^0(2\mathcal{O}) \rightarrow H^0(F)$ is injective because $H^0(\text{Coker}(\psi)) = 0$. This yields a contradiction. Thus
p \leq 5. Denote \( E = \text{Coker}(\varphi_{32}) \). Assume first that \( p = 5 \). We may write

\[
\varphi_{32} = \begin{bmatrix}
X & Y & Z & W & 0 \\
0 & l_1 & l_2 & l_3 & l_4
\end{bmatrix}
\]

If \( X \) and \( l_4 \) are linearly independent, then \( E \) is supported on the line \( L \) given by the equations \( X = 0, l_4 = 0 \). Thus

\[
E/T^0(E) \simeq \mathcal{O}_L(d_1) \oplus \cdots \oplus \mathcal{O}_L(d_n)
\]

Since \( F \) is stable, \( \text{Hom}(\mathcal{O}_L(d), F) = 0 \) if \( d \geq 0 \). Thus \( H^0(E) \to H^0(F) \) is the zero morphism, hence \( H^0(2\mathcal{O}) \to H^0(F) \) is also the zero morphism. This is a contradiction. We have reduced to the case when

\[
\varphi_{32} = \begin{bmatrix}
X & Y & Z & W & 0 \\
0 & l_1 & l_2 & l_3 & X
\end{bmatrix}
\]

where \( l_1, l_2, l_3 \) are linearly independent one-forms in the variables \( Y, Z, W \). Note that

\[
\begin{bmatrix}
Y & Z & W \\
l_1 & l_2 & l_3
\end{bmatrix} \sim \begin{bmatrix}
0 & * & * \\
* & * & *
\end{bmatrix}
\]

hence the maximal minors of this matrix

\[
q_1 = \begin{bmatrix}
Z & W \\
l_2 & l_3
\end{bmatrix}, \quad q_2 = \begin{bmatrix}
Y & W \\
l_1 & l_3
\end{bmatrix}, \quad q_3 = \begin{bmatrix}
Y & Z \\
l_1 & l_2
\end{bmatrix}
\]

are linearly independent and have no common factor. It follows easily that there is an exact sequence

\[
0 \to \mathcal{O}(-4) \xrightarrow{\beta} \mathcal{O}(-3) \oplus 3\mathcal{O}(-2) \xrightarrow{\alpha} 5\mathcal{O}(-1) \xrightarrow{\varphi_{32}} 2\mathcal{O} \to E \to 0
\]

where

\[
\alpha = \begin{bmatrix}
0 & -Y & -Z & -W \\
-q_1 & X & 0 & 0 \\
-q_2 & 0 & X & 0 \\
q_3 & 0 & 0 & X \\
0 & -l_1 & -l_2 & -l_3
\end{bmatrix}, \quad \beta = \begin{bmatrix}
-X & q_1 & -q_2 & q_3
\end{bmatrix}
\]

From this we get \( P_E = 3 \), hence \( \text{Hom}(E, F) = 0 \), hence \( 2\mathcal{O}_L \to F \) is the zero morphism. This, as we saw above, yields a contradiction.

Assume now that \( p = 4 \). We examine first the case when

\[
\varphi_{32} = \begin{bmatrix}
X & Y & Z & 0 \\
0 & l_1 & l_2 & l_3
\end{bmatrix}
\]

If \( X \) and \( l_3 \) are linearly independent, then \( E/T^0(E) \) is supported on a line and we get a contradiction as above. Thus, we may write

\[
\varphi_{32} = \begin{bmatrix}
X & Y & Z & 0 \\
0 & l_1 & l_2 & X
\end{bmatrix}
\]

where \( l_1 \) and \( l_2 \) are linear forms in the variables \( Y, Z, W \). It is easy to see that there is an exact sequence

\[
0 \to 2\mathcal{O}(-2) \xrightarrow{\alpha} 4\mathcal{O}(-1) \xrightarrow{\varphi_{32}} 2\mathcal{O} \to E \to 0
\]

where

\[
\alpha = \begin{bmatrix}
-Y & -Z \\
X & 0 \\
0 & X \\
-l_1 & -l_2
\end{bmatrix}
\]
We have $P_E(m) = 2m + 2$ and $E$ has no zero-dimensional torsion. From the semi-stability of $F$ we see that the morphism $E \to F$ is zero or it factors through a subsheaf $F' \subset F$ with $P_{F'}(m) = m - k, k \geq 0$. Thus $H^0(F') = 0$, so, at any rate, $H^0(E) \to H^0(F)$ is the zero map. It follows that the map $H^0(2O) \to H^0(F)$ is zero, which yields a contradiction.

Assume next that

$$
\varphi_{32} = \begin{bmatrix}
  l_{11} & l_{12} & l_{13} & l_{14} \\
  l_{21} & l_{22} & l_{23} & l_{24}
\end{bmatrix} \sim \begin{bmatrix}
  * & * & 0 \\
  0 & * & *
\end{bmatrix}
$$

Then we may assume that

$$
\varphi' = \begin{bmatrix}
  l_{11} & l_{12} & l_{13} \\
  l_{21} & l_{22} & l_{23}
\end{bmatrix} \sim \begin{bmatrix}
  0 & * \\
  * & *
\end{bmatrix}
$$

Note that the maximal minors of $\varphi'$ are linearly independent and have no common factor. According to [5] and [10], the sheaf $E' = \text{Coker}(\varphi')$ gives a point in $M_{p^3}(3m + 2)$. Note that $E$ is a quotient sheaf of $E'$. Since $\text{Hom}(E', F) = 0$, it follows that $\text{Hom}(E, F) = 0$, hence $2O \to F$ is the zero morphism. We have reached again a contradiction.

Thus far we have proved that $p = 3$. Resolution (10) takes the form

$$
0 \to 3O(-3) \oplus qO(-2) \xrightarrow{\psi} (q + 5)O(-2) \oplus 3O(-1) \xrightarrow{\varphi} \Omega^1 \oplus 2O \to F \to 0
$$

The morphism $\varphi_{32}: 3O(-1) \to 2O$ has linearly independent maximal minors. We claim that these maximal minors have a common linear factor. If this were not the case, then, as mentioned above, $\text{Coker}(\varphi_{32})$ would give a point in $M_{p^3}(3m + 2)$ and we would reach the contradictory conclusion that $2O \to F$ is the zero morphism. It is clear now that $\text{Ker}(\varphi_{32}) \sim O(-2)$. The isomorphism $\varphi_7: \text{Ker}(\varphi_1) \to \text{Ker}(\varphi_{32})/\text{Im}(\varphi_3)$ shows that $q = 1$. Resolving $\Omega^1$ in the above sequence gives the resolution

$$
0 \to O(-4) \oplus 3O(-3) \oplus O(-2) \xrightarrow{\psi} 4O(-3) \oplus 6O(-2) \oplus 3O(-1) \xrightarrow{\varphi} 6O(-2) \oplus 2O \to F \to 0
$$

where

$$
\varphi = \begin{bmatrix}
  \varphi_{11} & \varphi_{12} & 0 \\
  0 & \varphi_{22} & 0 \\
  0 & \varphi_{32} & \varphi_{33}
\end{bmatrix}, \quad \psi = \begin{bmatrix}
  \psi_{11} & \psi_{12} & 0 \\
  0 & \psi_{22} & 0 \\
  0 & \psi_{32} & \psi_{33}
\end{bmatrix}
$$

$$
\psi_{11} = \begin{bmatrix}
  x \\
  y \\
  z \\
  w
\end{bmatrix}, \quad \psi_{33} = \begin{bmatrix}
  l_1 \\
  -l_2 \\
  l_3 \\
  l_{21} & l_{22} & l_{23}
\end{bmatrix}, \quad \varphi_{23} = \begin{bmatrix}
  l_{11} & l_{12} & l_{13} \\
  l_{21} & l_{22} & l_{23}
\end{bmatrix}
$$

for some $u \in V^*$. We claim that $\text{rank}(\psi_{12}) = 3$. To see this we dualise the above exact sequence. According to [10] Lemma 3, we have the exact sequence

$$
0 \to 2O(-4) \oplus 6O(-2) \xrightarrow{\psi^T} 3O(-3) \oplus 6O(-2) \oplus 4O(-1) \xrightarrow{\psi^T} \Omega(-2) \oplus 3O(-1) \oplus O \to F^0 \to 0
$$

According to [10], $F^0$ gives a point in $M_{p^4}(4m - 1)$. If $\psi_{12}$ were zero, then we would get a commutative diagram
showing that the morphism $\mathcal{O} \rightarrow F^0$ is zero. But the map $H^0(\mathcal{O}) \rightarrow H^0(F^0)$ is injective because $H^0(\mathrm{Coker}(\varphi)) = 0$. If rank$(\psi_{12}) = 1$, then the map $\mathcal{O} \rightarrow F^0$ would factor through the structure sheaf of a point, so it would be zero. If rank$(\psi_{12}) = 2$, then the map $\mathcal{O} \rightarrow F^0$ would factor through the structure sheaf $\mathcal{O}_L$ of a line, so it would be zero, because $\text{Hom}(\mathcal{O}_L, F^0) = 0$. We reach again contradictions. This proves the claim.

Canceling $3\mathcal{O}(-3)$ we get the resolution

$$0 \longrightarrow \mathcal{O}(-4) \oplus \mathcal{O}(-2) \longrightarrow \mathcal{O}(-3) \oplus 6\mathcal{O}(-2) \oplus 3\mathcal{O}(-1) \xrightarrow{\varphi} 6\mathcal{O}(-2) \oplus 2\mathcal{O} \longrightarrow F \longrightarrow 0$$

We have rank$(\psi_{12}) = 6$, otherwise $F$ would map surjectively to $\mathcal{O}_H(-2)$ for a plane $H \subset \mathbb{P}^3$. This is clearly impossible. Canceling $6\mathcal{O}(-2)$ we finally get the resolution

$$0 \longrightarrow \mathcal{O}(-4) \oplus \mathcal{O}(-2) \xrightarrow{\psi} \mathcal{O}(-3) \oplus 3\mathcal{O}(-1) \xrightarrow{\varphi} 2\mathcal{O} \longrightarrow F \longrightarrow 0$$

with

$$\psi = \begin{bmatrix} 1 & 0 \\ f_1 & l_1 \\ f_2 & -l_2 \\ f_3 & l_3 \end{bmatrix}, \quad \varphi = \begin{bmatrix} g_1 & l_{11} & l_{12} & l_{13} \\ g_2 & l_{21} & l_{22} & l_{23} \end{bmatrix}$$

The sheaf $E = \mathrm{Coker}(\psi_{12} : 3\mathcal{O}(-1) \rightarrow 2\mathcal{O})$ is supported on $H \cup \{P\}$, where $H$ is the plane given by the equation $u = 0$ and $P$ is the point given by the ideal $(l_1, l_2, l_3)$. Since $F$ is a quotient sheaf of $E$ and since $F$ has no zero-dimensional torsion, we see that $\text{supp}(F) \subset H$. Applying the snake lemma to the commutative diagram in which the middle row is the dual of the above exact sequence

$$\begin{array}{cccccccc}
0 & 
\mathcal{O}(-1) & \longrightarrow & \mathcal{O} & \longrightarrow & \mathcal{O}_H' & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & \\
0 & 
2\mathcal{O}(-4) & \longrightarrow & 3\mathcal{O}(-3) \oplus \mathcal{O}(-1) & \longrightarrow & \mathcal{O}(-2) \oplus \mathcal{O} & \longrightarrow & F^0 & \longrightarrow 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & \\
3\mathcal{O}(-3) & \longrightarrow & \mathcal{O}(-2) & \longrightarrow & \mathbb{C}_P & \longrightarrow & 0 & \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & \\
0 & & 0 & & 0 & & 0 & \\
\end{array}$$

we get the exact sequence

$$\mathcal{O}_{H'} \longrightarrow F^0 \longrightarrow \mathbb{C}_P \longrightarrow 0$$

As $F^0$ has no zero-dimensional torsion, we see that $P \in H'$, that is, $l \in \text{span}\{l_1, l_2, l_3\}$. Moreover, $\text{supp}(F^0) \subset H'$, hence also $\text{supp}(F) \subset H'$. It follows that $H = H'$, otherwise $F$ would be supported on a line, yet a vector bundle of rank greater than one on $\mathbb{P}^1$ is not stable. Thus, we may assume that $u = 1$ and we may write

$$\psi = \begin{bmatrix} 1 & 0 \\ f_1 & 1 \\ f_2 & -l_2 \\ f_3 & -l_3 \end{bmatrix}, \quad \varphi = \begin{bmatrix} g_1 & l_2 & l & 0 \\ g_2 & l_3 & 0 & 1 \end{bmatrix}$$
From the relations
\[ g_1 l + l_2 f_1 + lf_2 = 0, \]
\[ g_2 l + l_3 f_1 + lf_3 = 0, \]
we see that \( f_1 \) is divisible by \( l \). Performing column operations on \( \psi \), we may assume that \( f_1 = 0 \).

Thus \( g_1 = -f_2 \), \( g_2 = -f_3 \). We have obtained resolution (11).

Conversely, given resolution (11), we also have resolution (12), hence, by [7, Theorem 3.2.1], \( F \) is semi-stable.

\[ \square \]

**Remark 25.** The general sheaves in \( R, E \), and \( P \) have the same resolution of the form (9). The sheaves in the wall-crossing has the resolution in (10). The stable sheaves \( F \) with \( h^0(F) = 2 \) have the resolution (11).

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