Extended Formulations for Stable Set Polytopes of Graphs Without Two Disjoint Odd Cycles

Michele Conforti\textsuperscript{1}, Samuel Fiorini\textsuperscript{2}, Tony Huynh\textsuperscript{3}, and Stefan Weltge\textsuperscript{4}

\textsuperscript{1} Dipartimento di Matematica, Università degli Studi di Padova, Padova, Italy  
conforti@math.unipd.it  
\textsuperscript{2} Département de Mathématique, Université libre de Bruxelles, Brussels, Belgium  
sfiorini@ulb.ac.be  
\textsuperscript{3} School of Mathematics, Monash University, Melbourne, Australia  
tony.bourbaki@gmail.com  
\textsuperscript{4} Fakultät für Mathematik, Technische Universität München, Munich, Germany  
weltge@tum.de

Abstract. Let $G$ be an $n$-node graph without two disjoint odd cycles. The algorithm of Artmann, Weismantel and Zenklusen (STOC’17) for bimodular integer programs can be used to find a maximum weight stable set in $G$ in strongly polynomial time. Building on structural results characterizing sufficiently connected graphs without two disjoint odd cycles, we construct a size-$O(n^2)$ extended formulation for the stable set polytope of $G$.

1 Introduction

It is a classic result that integer programs with a totally unimodular constraint matrix $A$ are solvable in strongly polynomial time. Very recently, Artmann, Weismantel and Zenklusen \cite{Artmann17} generalized this to bimodular matrices $A$. These include all matrices with all subdeterminants in $\{-2, -1, 0, 1, 2\}$. As noted in \cite{Artmann17}, this has consequences for the maximum weight stable set problem in graphs as follows.

Let $\text{STAB}(G)$ be the stable set polytope of a graph $G$ and note that

$$\text{STAB}(G) = \text{conv}\{x \in \{0, 1\}^{|V(G)|} \mid Mx \leq 1\},$$

where $M \in \{0, 1\}^{E(G) \times V(G)}$ is the edge-node incidence matrix of $G$. It is well-known that the maximum absolute value of a subdeterminant of $M$ is equal to $2^{\text{ocp}(G)}$, where $\text{ocp}(G)$ is the maximum number of (node-)disjoint odd cycles of $G$ (see \cite{Artmann17}). Therefore, the bimodular algorithm of \cite{Artmann17} can be used to efficiently compute a maximum weight stable set in a graph without two disjoint odd cycles.

Although the bimodular algorithm is extremely powerful, it provides limited insight on which properties of graphs with $\text{ocp}(G) \leq 1$ are relevant to derive efficient algorithms for graphs with higher odd cycle packing number. Indeed, in light of recent work linking the complexity and structural properties of integer programs to the magnitude of its subdeterminants \cite{Artmann17, Fazio18, Fazio19, Fazio20}, it is tempting to believe that integer programs with bounded subdeterminants can be solved in polynomial time. This would imply in particular that the stable set problem on graphs with $\text{ocp}(G) \leq k$ is polynomial for every
fixed $k$. Conforti, Fiorini, Huynh, Joret, and Weltge [7] recently proved this is true under the additional assumption that $G$ has bounded (Euler) genus.

Furthermore, by itself the bimodular algorithm does not imply any linear description of the stable set polytope of graphs $G$ with $\text{ocp}(G) = 1$. It turns out that for such graphs, $\text{STAB}(G)$ may have many facets with high coefficients that do not seem to allow a “nice” combinatorial description in the original space. While stable set polytopes have been studied for several classes of graphs, very little is known about $\text{STAB}(G)$ when $\text{ocp}(G) = 1$. Our main result is to show that every such stable set polytope admits a compact description in an “extended” space. To this end, we say that an extended formulation of a polyhedron $P$ is a description of the form $P = \{x \mid \exists y : Ax + By \leq b\}$ whose size is the number of inequalities in $Ax + By \leq b$. The extension complexity of $P$, denoted $\text{xc}(P)$, is the minimum size of an extended formulation of $P$. Our main result is the following.

**Theorem 1.** For every $n$-node graph $G$ with $\text{ocp}(G) \leq 1$, $\text{STAB}(G)$ admits a size-$O(n^2)$ extended formulation. Moreover, this extended formulation can be constructed in polynomial time.

Note that this does not follow from the main result of [1]. As noted in [3, Thm. 5.4], integer hulls of bimodular integer programs can have exponential extension complexity. Moreover, Theorem 1 does also not follow from [7] since here we are dealing with arbitrary graphs $G$ with $\text{ocp}(G) \leq 1$.

On the one hand, our proof uses a characterization of graphs with $\text{ocp}(G) \leq 1$ due to Lovász (see Seymour [15]). Kawarabayashi and Ozeki [12] later gave a short, purely graph-theoretical proof of the same result. Before stating Lovász’ theorem, we need a few more definitions. The odd cycle transversal number of a graph $G$, denoted $\text{oct}(G)$, is the minimum size of a set of nodes $X$ such that $G - X$ is bipartite. The projective plane is the surface obtained from a closed disk by identifying antipodal points on its boundary. An embedding of a graph $G$ in a surface is an even-face embedding if every face of $G$ is an open disk bounded by an even cycle of $G$.

**Theorem 2 (Lovász, cited in [15]).** Let $G$ be a 4-connected graph with $\text{ocp}(G) \leq 1$. Then

(i) $\text{oct}(G) \leq 3$, or
(ii) $G$ has an even-face embedding in the projective plane.

Note that if a graph $G$ satisfies (i) of Theorem 2 then $\text{STAB}(G)$ has a compact extended formulation since it is the convex hull of the union of at most eight polytopes described by nonnegativity and edge constraints. As a special case of [7, Theorem 3], $\text{STAB}(G)$ also has a compact extended formulation if $G$ satisfies (ii) of Theorem 2.

However, the decomposition portion of our proof is non-trivial since Theorem 2 is stated for 4-connected graphs. Hence, we have to deal with the polyhedral aspects of performing 2- and 3-sums, using the properties of graphs without two disjoint odd cycles. In general

---

The Euler genus of graph $G$ is the minimum of $|E(G)| - |V(G)| - |F(G)| - 2$, taken over all embeddings of $G$ in a (orientable or non-orientable) surface, where $F(G)$ denotes the set of faces of $G$ with respect to the embedding.
graphs, performing multiple $k$-sums does not preserve small extended formulations for the respective stable set polytopes, even for $k = 2$.

On the other hand, our polyhedral analysis crucially relies on new insights about the structure of facets of stable set polytopes (see Lemma 17) and a transformation of stable set polytopes into the edge space (see Sections 3 and 5). We believe that this perspective can be equally beneficial for other future investigations of (general) stable set polytopes.

Finally, we remark that our proof also can be turned into a direct, purely graph-theoretic strongly polynomial time algorithm for the stable set problem in graphs $G$ with $\text{ocp}(G) < 1$.

Outline In Section 2 we build on Theorem 2 and its signed version due to Slilaty [16] to describe the structure of graphs without two disjoint odd cycles. Roughly, we prove that each such graph $G$ either has $\text{oct}(G) \leq 3$ or can be obtained from a graph $H_0$ having an even-face embedding in the projective plane by gluing internally disjoint bipartite graphs $T_1, \ldots, T_\ell$ “around” $H_0$ in a certain way. Section 3 gives a short account of the known compact extended formulation for STAB($G$) for graphs $G$ admitting an even-face embedding in the projective plane, see [17]. This is our base case. The general case is treated in Sections 4 and 5 by a delicate argument using certain gadgets $H_1, \ldots, H_\ell$ “simulating” the bipartite graphs $T_1, \ldots, T_\ell$.

2 The structure of graphs without two disjoint odd cycles

In this section we show that every graph without two disjoint odd cycles either has a small odd cycle transversal or has a structure that we will exploit later. For this purpose we use the notion of separations. A $k$-separation of a graph $G$ is an ordered pair $(G_0, G_1)$ of edge-disjoint subgraphs of $G$ with $G = G_0 \cup G_1$, $|V(G_0) \cap V(G_1)| = k$, and $E(G_0), E(G_1), V(G_1) \setminus V(G_0), V(G_0) \setminus V(G_1)$ all non-empty. We say that a $k$-separation is linked if for every two distinct nodes of $V(G_0) \cap V(G_1)$ there exists a $u$–$v$ path in $G_1$ whose internal nodes are disjoint from $G_0$. 

**Definition 3.** A comb structure of a graph $G$ are subgraphs $H_0, T_1, \ldots, T_\ell$ of $G$ such that for all $i \in [\ell]$, $T_i$ is bipartite, $(H_0 \cup \bigcup_{j \neq i} T_j, T_i)$ is a linked $k$-separation of $G$ with $k \leq 3$, and $V(T_i) \cap V(T_j) \subseteq V(H_0)$ for all $j \neq i$.

For our structural result we will also use the notion of signed graphs. A signed graph is a pair $(G, \Sigma)$ where $G$ is a graph and $\Sigma \subseteq E(G)$. A subgraph of $G$ is said to be $\Sigma$-odd if it contains an odd number of edges in $\Sigma$, and is $\Sigma$-even otherwise. The odd cycle packing number of a signed graph $(G, \Sigma)$ is the maximum number of disjoint $\Sigma$-odd cycles in $(G, \Sigma)$, and is denoted by $\text{ocp}(G, \Sigma)$. A signed graph $(G, \Sigma)$ is balanced if $\text{ocp}(G, \Sigma) = 0$. The odd cycle transversal number of $(G, \Sigma)$ is the minimum number of nodes in $(G, \Sigma)$ intersecting every $\Sigma$-odd cycle in $(G, \Sigma)$, and is denoted by $\text{oct}(G, \Sigma)$. An embedding of a signed graph $(G, \Sigma)$ in a surface is an even-face embedding if every face of $(G, \Sigma)$ is an open disk bounded by a $\Sigma$-even cycle of $(G, \Sigma)$. Graphs in this section may have parallel edges.

In the definition below, $\cup$ is used to denote the edge-disjoint union of graphs.

**Definition 4.** Let $G$ be a graph with comb structure $H_0, T_1, \ldots, T_\ell$. For each $i \in [\ell]$, let $S_i = V(H_0) \cap V(T_i)$ and note that there is a signed clique $(K_i, \Sigma_i)$ with $V(K_i) = S_i$.
such that \((K_i \cup T_i, \Sigma_i \cup E(T_i))\) is balanced. The signed graph \((H^+, \Sigma)\) is then defined via \(H^+ := H_0 \cup K_1 \cup \cdots \cup K_\ell\) and \(\Sigma := E(H_0) \cup \Sigma_1 \cup \cdots \cup \Sigma_\ell\).

The structural result is the following.

**Theorem 5.** Let \(G\) be a graph with \(\text{ocp}(G) = 1\) and \(\text{oct}(G) \geq 4\). Then \(G\) admits a comb structure \(H_0, T_1, \ldots, T_\ell\), such that \(S_1, \ldots, S_\ell\) and \((H^+, \Sigma)\) from Definition 4 have the following properties:

1. \(S_i\) is not a subset of \(S_j\) for all distinct \(i, j \in [\ell]\).
2. \((H^+, \Sigma)\) has an even-face embedding in the projective plane, and
3. the nodes of each \(S_i\) are on the boundary of some face of the embedding.

In order to obtain the above statement, we will use a finer version of Theorem 2 that is suited for signed graphs, due to Slilaty [16]. The latter result was previously known by Gerards, Lovász, and others, but [16] is the first time it appears in print.

**Theorem 6 (Slilaty [16]).** Let \((G, \Sigma)\) be a 4-connected signed graph with \(\text{ocp}(G, \Sigma) \leq 1\). Then

1. \(\text{oct}(G, \Sigma) \leq 3\) or
2. \((G, \Sigma)\) has an even-face embedding in the projective plane.

**Lemma 7.** Let \(G\) be a graph with comb structure \(H_0, T_1, \ldots, T_\ell\) and let \((H^+, \Sigma)\) be as in Definition 4. Then \(\text{ocp}(H^+, \Sigma) \leq \text{ocp}(G)\) and \(\text{oct}(H^+, \Sigma) \geq \text{oct}(G)\).

**Proof.** Let \((K_1, \Sigma_1), \ldots, (K_\ell, \Sigma_\ell)\) be as in Definition 4. Let \(C_1, \ldots, C_k\) be disjoint \(\Sigma\)-odd cycles in \((H^+, \Sigma)\). Observe that for each \(i \in [\ell], j \in [k]\), \(C_j\) contains at most two edges from \(K_i\). Otherwise, \(C_j = K_i\) since \(|V(K_i)| \leq 3\), which contradicts that \((K_i, \Sigma_i)\) is balanced. Now, for each \(j \in [k]\) we will replace \(C_j\) by an odd cycle \(C_j'\) in \(G\) as follows.

If \(C_j\) uses two edges of some \(K_i\), say \(uv\) and \(vw\), we replace them by a \(u-v\) path \(P\) in \(T_i\). Note that \(P\) exists since the separation \((H_0 \cup_{j \neq i} T_j, T_i)\) is linked. Furthermore, since \((K_i \cup T_i, \Sigma_i)\) is balanced and \(|E(P) \cap \Sigma| = 0\), \(|\{uv, vw\} \cap \Sigma|\) is also even. Therefore, \(C_j'\) is odd. If \(C_j\) uses only one edge \(uv\) of some \(K_i\), we replace it by a \(u-v\) path \(P\) in \(T_i\). Again, \(C_j'\) will be odd. If \(C_j\) is edge-disjoint from each \(K_i\), we let \(C_j' = C_j\). Since \(E(H_0) \subseteq \Sigma\), \(C_j'\) is odd in this case as well. Finally, the cycles \(C_1', \ldots, C_k'\) are still disjoint since for each

![Fig. 1. A comb structure.](image-url)
\(i \in [\ell]\) there is at most one cycle \(C_j\) that contains an edge from \(K_i\) (due to \(|V(K_i)| \leq 3\)). Thus, \(\text{ocp}(H^*, \Sigma) \leq \text{ocp}(G)\).

For the second assertion, consider an arbitrary odd cycle \(C'\) in \(G\). By reversing the construction from the previous paragraph, there exists a \(\Sigma\)-odd cycle \(C \in (H^*, \Sigma)\) with \(V(C) \subseteq V(C')\). It follows that \(\text{oct}(H^*, \Sigma) \geq \text{oct}(G)\). \(\square\)

**Proof of Theorem 2.** Let \(G\) be a graph with \(\text{ocp}(G) = 1\) and \(\text{oct}(G) \geq 4\). Let \(H_0, T_1, \ldots, T_\ell\) be a comb structure with \([(V(H_0), \ell)\) lexicographically minimal. Note that such a comb structure exists since \(G\) is a comb structure of itself.

Suppose there exist distinct \(i, j \in [\ell]\) such that \(S_j \subseteq S_i\). Since \(|S_i| \leq 3\) and \(\text{ocp}(G) \geq 4\), \(G - S_i\) contains an odd cycle \(C\). Note that \(C\) is not a subgraph of \(T_i \cup T_j\) because \(T_i\) and \(T_j\) are both bipartite and hence every odd cycle of \(T_i \cup T_j\) must intersect \(S_i\). Since \(\text{ocp}(G) \leq 1\) this implies that \(T_i \cup T_j\) is bipartite, a contradiction to the minimality of \(\ell\).

Suppose \((H^*, \Sigma)\) is not 4-connected. Let \(((H_1, T_1), (H_2, T_2))\) be a separation of \((H^*, \Sigma)\) with \(X := V(H_1) \cap V(H_2)\) and \(|X| \leq 3\). By Lemma 7 \(\text{ocp}(H^*, \Sigma) \leq 1\) and \(\text{oct}(H^*, \Sigma) \geq 4\). Therefore, exactly one of \((H_1, T_1) - X\) or \((H_2, T_2) - X\) is balanced. By symmetry, we may assume that \((H_2, T_2) - X\) is balanced, and by taking \([V(H_2)]\) to be minimal we may assume that \(((H_1, T_1), (H_2, T_2))\) is linked. Recall that \((H^*, \Sigma)\) arises from \(H_0\) by adding (balanced) signed cliques \((K_1, \Sigma_1), \ldots, (K_\ell, \Sigma_\ell)\) corresponding to the bipartite graphs \(T_1, \ldots, T_\ell\). Replacing each \((K_i, \Sigma_i)\) by the bipartite graph \(T_i\), we see that \(G\) admits a comb structure \(H'_0, T'_1, \ldots, T'_\ell\) where \(V(H'_0) = V(H_0)\), a contradiction to the minimality of \(|V(H_0)|\). Since \(\text{ocp}(H^*, \Sigma) \leq 1\) and \(\text{oct}(H^*, \Sigma) \geq 4\), Theorem 6 implies that \((H^*, \Sigma)\) has an even-face embedding in the projective plane.

Suppose \(S_i\) is not contained on the boundary of a face of the embedding for some \(i \in [\ell]\). Since all nodes in \(S_i\) are adjacent in \((H^*, \Sigma)\), this implies \(|S_i| = 3\). But now, \(S_i\) is a cutset of \((H^*, \Sigma)\), contradicting that \((H^*, \Sigma)\) is 4-connected. \(\square\)

### 3 Review of the projective planar case

In this section, we briefly review the compact extended formulation from [7] for \(\text{STAB}(G)\), when \(k\) and \(g\) are fixed constants, where \(k = \text{ocp}(G)\) and \(g\) denotes the Euler genus of \(G\). Since here we are only interested in the case \(k = g = 1\), the extended formulation is much easier to describe. Our starting point is the unbounded polyhedron

\[
P(G) := \text{conv}\{x \in \mathbb{Z}^{V(G)} \mid Mx \leq 1\},
\]

where \(M\) is the edge-node incidence matrix of \(G\). Its relationship to \(\text{STAB}(G)\) is as follows.

**Lemma 8 (7, Prop. 49).** For every graph \(G\), \(\text{STAB}(G) = P(G) \cap [0, 1]^{V(G)}\).

Thus, it suffices to study \(P(G)\) instead of \(\text{STAB}(G)\). To this end, it is convenient to switch from the node space of \(G\) to the edge space of \(G\) by considering the affine map \(\sigma: \mathbb{R}^{V(G)} \to \mathbb{R}^{E(G)}\) defined via

\[
\sigma(x) := 1 - Mx.
\]

Under \(\sigma\), a vector \(x \in \mathbb{R}^{V(G)}\) is mapped to \(y = \sigma(x) \in \mathbb{R}^{E(G)}\) where \(y_{vw} = 1 - x_v - x_w\) for every edge \(vw \in E(G)\). Since \(\sigma\) is invertible if and only if \(G\) has no bipartite component, we can focus on \(Q(G) := \sigma(P(G))\).
We provide an extended formulation for $Q(G)$, assuming that $G$ is even-face embedded in the projective plane. Let $G^*$ be the dual graph of $G$. An orientation $D$ of the edges of $G^*$ is called alternating if in the local cyclic ordering of the edges incident to each dual node $f$, the edges alternatively leave and enter $f$. We say that a graph $G$ satisfies the standard assumptions if it is non-bipartite, 2-connected, and even-face embedded in the projective plane.

**Lemma 9 ([7, Lem. 17]).** Let $G$ be a graph satisfying the standard assumptions. Then the dual graph $G^*$ of $G$ has an alternating orientation.

Let $G$ be even-face embedded in the projective plane and $D$ be an alternating orientation of $G^*$. Note that there is a bijection between the edges of $G$ and the arcs of $D$. Therefore, we may regard a vector $y \in \mathbb{R}^{E(D)}$ as a vector in $\mathbb{R}^{A(D)}$, and vice versa. With this identification, $Q(G)$ turns out to be the convex hull of all non-negative integer circulations of $D$ that satisfy one additional constraint.

**Lemma 10 ([7, Lem. 18]).** Let $G$ be a graph satisfying the standard assumptions, $D$ be an alternating orientation of $G^*$, and $C$ be an arbitrary odd cycle in $G$. Then

$$Q(G) = \text{conv}\{y \in \mathbb{Z}^{E(G)}_{\geq 0} \mid y \text{ is a circulation in } D \text{ and } y(E(C)) \text{ is odd}\}.$$  

Motivated by Lemma 10 we now introduce an auxiliary directed graph to design an extended formulation for $Q(G)$. Let $G$ be a graph satisfying the standard assumptions, $D$ be an alternating orientation of $G^*$, and $C$ be an odd cycle in $G$. The cover graph of $D$ is the directed graph $\overline{D}$ with node set $\{(f,p) \mid f \in V(D), \ p \in \mathbb{Z}/2\mathbb{Z}\}$ and an arc from $(f_1,p_1)$ to $(f_2,p_2)$ if and only if $(f_1,f_2) \in A(D)$ and $p_1 + p_2 \equiv \chi^C_e \mod 2$, where $e$ is the edge of $G$ corresponding to the arc $(f_1,f_2)$. For each node $f \in V(D)$ we let $Q_f$ be the polyhedron of all (uncapacitated) unit flows from $(f,0)$ to $(f,1)$ in $\overline{D}$. Finally, we let $\overline{Q}(G)$ denote the convex hull of the union of all polyhedra $Q_f$ for $f \in V(D)$.

By [7, Sec. 12.3], $\overline{Q}(G)$ is an extension of $Q(G)$. Moreover, each $Q_f$ has $O(|A(\overline{D})|) = O(|E(G)|) = O(|V(G)|)$ facets. Finally, by applying Balas’ theorem [2], we obtain a quadratic size extended formulation for $Q(G)$, and thus for $\text{STAB}(G)$ in case $G$ satisfies the standard assumptions. By [7, Sec. 12.1], this result extends to all graphs that are even-face embedded in the projective plane.

**Theorem 11.** Let $G$ be an $n$-node graph that is even-face embedded in the projective plane. Then $\text{STAB}(G)$ has a size-$O(n^3)$ extended formulation.

## 4 The general case

In this section, we describe how Theorem 1 can be proven using Theorems 5 and 11. Let $G$ be a graph with $\text{ocp}(G) = 1$. If $\text{oct}(G) \leq 3$, then $\text{STAB}(G)$ has a linear-size extended formulation by Balas’ theorem [2]. Otherwise, $\text{oct}(G) \geq 4$ and $G$ can be decomposed as in Theorem 5. In particular, $G$ is the union of graphs $H_0, T_1, \ldots, T_t$ where $H_0$ has an even-face embedding in the projective plane and $T_1, \ldots, T_t$ are bipartite. Although the stable set polytopes of $H_0, T_1, \ldots, T_t$ admit small extended formulations and each $T_i$ intersects $H_0 \cup_{j \neq i} T_j$ in at most three nodes, it is not obvious how to obtain a small extended formulation for $\text{STAB}(G)$. However, in some cases it is possible to use linear descriptions...
of the stable set polytopes of graphs $G_1, G_2$ to obtain a description of $\text{STAB}(G_1 \cup G_2)$, provided that $G_1 \cap G_2$ has a specific structure, see [6,8,3].

With this idea in mind, recall that not only $H_0$ but also the signed graph $(H^+, \Sigma)$ has an even-face embedding in the projective plane. We will replace each signed clique used to define $(H^+, \Sigma)$ by a constant size gadget $H_i$ corresponding to each $T_i$ in a way that the resulting graph $G^{(\ell)} := H_0 \cup H_1 \cup \cdots \cup H_\ell$ (the “core”) still has an even-face embedding in the projective plane. Moreover, each $T'_i := T_i \cup H_i$ will still be bipartite. In this way $G$ is obtained from $G^{(\ell)}$ by iteratively performing $k$-sums with $T'_1, \ldots, T'_\ell$ along $H_1, \ldots, H_\ell$. In each such operation, the specific choice of the gadget will allow us to relate the extension complexities of the stable set polytopes of the participating graphs in a controlled way. Let us start with describing the gadgets that will be used.

**Definition 12.** A gadget is a graph isomorphic to $P_3, P_4, S_{2,2,2}$ or $S_{2,3,3}$, see Figure 2. Let $G$ be a graph with a linked $k$-separation $(G_0, G_1)$ such that $k \in \{2,3\}$ and $G_1$ is bipartite. We say that a gadget $H$ is attachable to $G_1$ (with respect to separation $(G_0, G_1)$) if its set of leaf nodes equals $V(G_0) \cap V(G_1)$, its set of non-leaf nodes is disjoint from $V(G)$, and $G'_i := G_1 \cup H$ is bipartite.

Note that if $G$ is a graph with a linked $k$-separation $(G_0, G_1)$ such that $k \in \{2,3\}$ and $G_1$ is bipartite, then there is a unique gadget $H \in \{P_3, P_4, S_{2,2,2}, S_{2,3,3}\}$ that is attachable to $G_1$.

![Fig. 2. Gadgets and their names.](image)

Next, let us formally describe how the signed cliques used to define $(H^+, \Sigma)$ are replaced by gadgets in order to obtain the core.

**Definition 13.** Let $G$ be a 2-connected graph with comb structure $H_0, T_1, \ldots, T_\ell$. For each $i \in [\ell]$, pick a gadget $H_i$ that is attachable to $T_i$ with respect to the separation $(H_0 \cup \bigcup_{j \neq i} T_j, T_i)$. (We always assume that the set of non-leaf nodes of the gadgets $H_i$, $i \in [\ell]$ are mutually disjoint.) We call the graph $H_0 \cup H_1 \cup \cdots \cup H_\ell$ the core.

**Lemma 14.** Every 2-connected graph $G$ with $\text{oCP}(G) = 1$ and $\text{oct}(G) \geq 4$ admits a comb structure whose core has an even-face embedding in the projective plane.

**Proof.** The proof is immediate by choosing a comb structure that satisfies Theorem 5.

The remaining ingredient for our proof of Theorem 1 will be the following result. To this end, let $(G_0, G_1)$ be a separation of graph $G$. Below, for $i \in \{0,1\}$, we call a vertex internal if it belongs to $V(G_i) \cup V(G_{1-i})$ and an edge of $G_i$ internal if at least one of its endnodes is not in $G_{1-i}$.
Proof of Theorem 15. Let $G$ be a 2-connected, non-bipartite graph. Assume that $G$ has a $k$-separation $(G_0, G_1)$ such that $G_1$ is bipartite, and $k \in \{2, 3\}$. Let $\mu_1$ denote the number of internal vertices and edges of $G_1$. Let $H$ be a gadget that is attachable to $G_1$, and let $G_0' := G_0 \cup H$. Then

$$xc(STAB(G)) \leq xc(STAB(G_0')) + O(\mu_1).$$

Before we continue with the proof of Theorem 15 in the next section, let us see how this yields a proof of our main result.

Proof of Theorem 1 By induction on the number of nodes $n$, we may assume that $G$ is 2-connected. Indeed, suppose that $G$ has a $k$-separation $(G_0, G_1)$ with $k \in \{0, 1\}$. For $i \in \{0, 1\}$, let $n_i := |V(G_i)|$. Thus $n = n_0 + n_1 - k$. If $c$ is any constant such that $xc(STAB(G_i)) \leq c \cdot n_i^2$ for $i \in \{0, 1\}$, we get

$$xc(STAB(G)) \leq xc(STAB(G_0)) + xc(STAB(G_1)) \leq c \cdot n_0^2 + c \cdot n_1^2 \leq c \cdot n^2.$$

As observed above, if $oc(G) \leq 3$ then $STAB(G)$ trivially has a size-$O(n^2)$ extended formulation. Now assume that $ocp(G) = 1$ and $oc(G) \geq 4$. Let $H_0, T_1, \ldots, T_\ell$ be a comb structure of $G$ as in Lemma 13. Since $G$ is 2-connected, each separation $(H_0 \cup \cup_j T_j, T_i)$ is either a 2- or a 3-separation. For each $i \in [\ell]$, we consider the graph

$$G^{(i)} := H_0 \cup H_1 \cup \cdots \cup H_i \cup \cup_{i+1} T_i \cup \cdots \cup T_\ell.$$

where $H_i$ denotes a gadget attachable to $T_i$. For $i \in [\ell]$, let $\mu_i$ denote the number of internal vertices and edges of $T_i$. Notice that $G^{(i)}$ is the core, and thus by Lemma 13 has an even-face embedding in the projective plane. By Theorem 15

$$xc(STAB(G^{(i-1)})) \leq xc(STAB(G^{(i)})) + O(\mu_i).$$

Since $|V(G^{(i)})| = O(n)$, Theorem 11 implies $xc(STAB(G^{(i)})) = O(n^2)$. Since moreover $\sum_{i=1}^\ell \mu_i \leq |V(G)| + |E(G)| = O(n^2)$, we have

$$xc(STAB(G)) = xc(STAB(G^{(0)})) \leq xc(STAB(G^{(\ell)})) + O\left(\sum_{i=1}^\ell \mu_i\right) = O(n^2). \quad \Box$$

5 Extended formulation for small separations

In this section we describe an extended formulation that yields the bound claimed in Theorem 15. Given a stable set $S$ in a graph $G$, we say that an edge is slack if none of its endnodes is in $S$. We denote by $\sigma(S)$ the set of slack edges, or $\sigma_G(S)$ should the graph not be clear from the context. An edge is said to be tight if it is not slack.

Lemma 16. Let $G, G_0, G_1$ and $H$ be as in Theorem 15. Letting $\overline{STAB(G_1')}$ denote the convex hull of characteristic vectors of stable sets $S$ in $G_1'$ having at most one slack edge in $H$, we have

$$STAB(G) = \{(x^0, x^1, x^{01}) \in \mathbb{R}^{V(G)} | \exists x^H : (x^0, x^{01}, x^H) \in STAB(G_0'), \quad (x^1, x^{01}, x^H) \in \overline{STAB(G_1')}\},$$

where $x^0 \in \mathbb{R}^{V(G_0) \setminus V(G_1)}$, $x^1 \in \mathbb{R}^{V(G_1) \setminus V(G_0)}$, $x^{01} \in \mathbb{R}^{V(G_0) \cap V(G_1)}$ and $x^H \in \mathbb{R}^{V(H) \setminus V(G)}$. 

Let us first verify that Lemma 16 indeed implies Theorem 15.

Proof of Theorem 15. By Lemma 16 we have
\[ \text{xc}(\text{STAB}(G)) \leq \text{xc}(\text{STAB}(G_0)) + \text{xc}(\overline{\text{STAB}}(G_1)). \]
Since gadget $H$ has constant size, $\overline{\text{STAB}}(G_1)$ is the convex hull of the union of a constant number of faces of $\text{STAB}(G_1)$ in which the coordinates of the nodes in $H$ are fixed. Hence by Balas’ union of polytopes [2], we obtain $\text{xc}(\text{STAB}(G_1)) = O(\text{xc}(\text{STAB}(G_1))) = O(|V(G_1)| + |E(G_1)|)$. Since $|V(G_1)| + |E(G_1)| - \mu_1 \leq 6$ and $\mu_1 \geq 1$, we conclude
\[ \text{xc}(\overline{\text{STAB}}(G_1)) = O(\mu_1). \]
This proves the claim. □

In the proof of Lemma 16 we will exploit that the facets of stable set polytopes have a special structure, which we describe next.

5.1 Reducing to edge-induced weights

We call a weight function $w : V(G) \to \mathbb{R}$ on the nodes of $G$ *edge-induced* if there is a nonnegative cost function $c : E(G) \to \mathbb{R}_{\geq 0}$ such that $w(v) = c(\delta(v))$ for all $v \in V(G)$. For a given node-weighted graph $(G, w)$ we let $\alpha(G, w)$ denote the maximum weight of a stable set.

Lemma 17. Let $G = (V, E)$ be a graph without isolated nodes and let $w : V \to \mathbb{R}$ be a weight function. There exists an edge-induced weight function $w' : V \to \mathbb{R}$ such that $w(v) \leq w'(v)$ for all nodes $v$ and $\alpha(G, w) = \alpha(G, w')$. In particular, the node weights of every non-trivial facet-defining inequality of $\text{STAB}(G)$ are edge-induced.

Proof. Let $x^*$ denote an optimal solution of the LP $\max \{ \sum_{v \in V} w(v)x_v \mid x_v + x_{w} \leq 1 \ \forall vw \in E, \ x \geq 0 \}$ and $y^*$ be an optimal solution of its dual $\min \{ \sum_{e \in E} y_e \mid y(\delta(v)) \geq w(v) \ \forall v \in V, \ y \geq 0 \}$.

Consider the weight function $w'$ such that $w'(v) := y^*(\delta(v))$. Clearly, $w'(v) \geq w(v)$ for all nodes $v$ and $w'$ is edge-induced. Consider the above LPs where $w'$ replaces $w$. Then $x^*$ and $y^*$ remain optimal solutions as they are feasible and satisfy complementary slackness. Moreover the values of the new LPs remain unchanged, as the objective function of the dual is not changed.

Let $V_0 := \{ v \in V \mid x^*_v = 0 \}$. Since $w(v) = y^*(\delta(v))$ for all $v \in V \setminus V_0$ by complementary slackness, $w(v) = w'(v)$ for all $v \in V \setminus V_0$. We have
\[ \alpha(G, w) \leq \alpha(G, w') = \alpha(G - V_0, w') = \alpha(G - V_0, w) \leq \alpha(G, w). \]
Above, the first inequality follows from $w \leq w'$. The first equality follows from a result of Nemhauser and Trotter [13]. Their result implies that $(G, w')$ has a maximum weight stable set disjoint from $V_0$. The second equality follows from the fact that $w(v) = w'(v)$ for all $v \in V \setminus V_0$. Hence, equality holds throughout and $\alpha(G, w) = \alpha(G, w')$.

Finally, if $\sum_{v \in V} w(v)x_v \leq \alpha(G, w)$ induces a non-trivial facet of $\text{STAB}(G)$, there cannot exist $w' \neq w$ such that $w' \geq w$ and $\alpha(G, w') = \alpha(G, w)$. Hence the above argument shows that the node weights of every non-trivial facet-defining inequality of $\text{STAB}(G)$ are edge-induced. □
For $c : E(G) \to \mathbb{R}_{\geq 0}$, we let

$$
\beta(G, c) := \min \left\{ \sum_{e \in E(G)} c(e)y_e \mid y \in \sigma(\text{STAB}(G)) \right\}
$$

(2)

In fact, in [1] Propositions 11 and 14] it is shown that one can optimize over $Q(G) = \sigma(P(G))$ instead of $\sigma(\text{STAB}(G))$ without changing the optimum. However, we will not need this here. Our last lemma follows from [1 Observation 13].

**Lemma 18.** Let $G = (V, E)$ be a graph. If $w : V(G) \to \mathbb{R}$ is induced by $c : E(G) \to \mathbb{R}_{\geq 0}$, then $\alpha(G, w) = c(E(G)) - \beta(G, c)$.

### 5.2 Correctness of the extended formulation

In this section we prove Lemma 16. To this end, let $R(G)$ denote the right-hand side of (1). Notice that for each stable set $S$ of $G$, there exists a stable set $S'$ of $G'$ such that $S' \cap V(G) = S$ and moreover at most one edge of $H$ is slack with respect to $S'$. The inclusion $\text{STAB}(G) \subseteq R(G)$ follows directly from this.

In order to prove the reverse inclusion $R(G) \subseteq \text{STAB}(G)$, first observe that $R(G) \subseteq \mathbb{R}_{\geq 0}$. Thus, by Lemma 17 it suffices to show that, for all edge-induced node weights $w : V(G) \to \mathbb{R}$, the inequality

$$
\sum_{v \in V(G)} w(v)x_v \leq \alpha(G, w)
$$

(3)

is valid for all $x \in R(G)$. As in Section 3 it will be convenient to work in the edge space instead of the node space. To this end, let $c : E(G) \to \mathbb{R}_{\geq 0}$ be non-negative edge costs, and let $w(v) := c(\delta(v))$ for every node $v$. By Lemma 18 we see that (3) is valid for $R(G)$ if and only if

$$
\sum_{e \in E(G)} c(e)y_e \geq \beta(G, c)
$$

(4)

is satisfied by all points $y \in \sigma(R(G))$. Our proof strategy to obtain (4) is to seek additional costs $c^H : E(H) \to \mathbb{R}_{\geq 0}$ such that

$$
\sum_{e \in E(G_0)} c(e)y^0_e + \sum_{e \in E(H)} c^H(e)y^H_e \geq \beta(G, c)
$$

(5)

is valid for all $(y^0, y^H) \in \sigma(\text{STAB}(G'_0))$ and

$$
\sum_{e \in E(G_1)} c(e)y^1_e - \sum_{e \in E(H)} c^H(e)y^H_e \geq 0
$$

(6)

is valid for all $(y^1, y^H) \in \sigma(\text{STAB}(G'_1))$.

We claim that this will yield (4). Indeed, for every vector $y = (y^0, y^1) \in \sigma(R(G))$ there exists a vector $y^H$ (the image of $(x^0, x^H)$ under $\sigma_H$) with $(y^0, y^H) \in \sigma(\text{STAB}(G'_0))$ and $(y^1, y^H) \in \sigma(\text{STAB}(G'_1))$. This implies that the inequalities in (5) and (6) are satisfied. Now (4) follows since it is the sum of these two inequalities.

Let us first focus on Inequality (5). Independently of how the edge costs $c^H$ are defined, in order to prove that it holds for all $(y^0, y^H) \in \sigma(\text{STAB}(G'_0))$, we may assume
that \( y^H \) is a 0/1-vector with at most one nonzero entry. The general case follows by convexity. Since the case \( y^H = 0 \) is trivial, assume that \( y^H = \chi(f) \) for some \( f \in E(H) \). Hence (6) can be rewritten as

\[
\sum_{e \in E(G_1)} c(e)y_e^1 \geq c^H(f).
\]

This suggests the following definition of \( c^H \). For \( F \subseteq E(H) \), we let

\[
\gamma(F) := \min \{ c(\sigma(S) \cap E(G_1)) \mid S \text{ stable set of } G_1', \sigma(S) \cap E(H) = F \} \in \mathbb{R}_{\geq 0} \cup \{ \infty \}.
\]

We say that \( F \) is feasible if \( \gamma(F) \) is finite, that is, there exists a stable set \( S \) of \( G_1' \) such that \( \sigma(S) \cap E(H) = F \). Notice that \( F := \{ f \} \) is feasible for all \( f \in E(H) \). By setting \( c^H(f) := \gamma(\{ f \}) \in \mathbb{R}_{\geq 0} \) for each \( f \in E(H) \) we clearly satisfy (7), and hence (6) is valid for all \( (y^0, y^H) \in \sigma(\text{STAB}(G_1')) \) for this choice of \( c^H \).

It remains to prove that with this choice of \( c^H \) the inequality in (6) is valid for all \( (y^0, y^H) \in \sigma(\text{STAB}(G_1')) \). To this end, we need the following two observations.

**Lemma 19.** Let \( G_1 \) and \( H \) be as in Theorem 15 and let \( G_1' := G_1 \cup H \). Hence, \( G_1' \) is bipartite. Let \( c : E(G_1) \to \mathbb{R}_{\geq 0} \) be nonnegative edge costs. Assume that \( F \subseteq E(H) \) is feasible. Letting \( x \) and \( y = (y^1, y^H) \) denote arbitrary points in \( \mathbb{R}^{V(G_1')} \) and \( \mathbb{R}^{E(G_1')} \) respectively, and letting \( M \) denote the incidence matrix of \( G_1' \), consider the following LPs:

\[
\text{LP}_1(F) := \min \left\{ \sum_{e \in E(G_1)} c(e)y_e^1 \mid Mx + y = 1, \ y \geq 0, \ y^H = \chi^F, \ x \geq 0 \right\}
\]

and

\[
\text{LP}_2(F) := \min \left\{ \sum_{e \in E(G_1)} c(e)y_e^1 \mid Mx + y = 1, \ y \geq 0, \ y^H = \chi^F \right\}.
\]

Then \( \gamma(F) = \text{LP}_1(F) = \text{LP}_2(F) \).

**Proof.** That \( \gamma(F) = \text{LP}_1(F) \) follows directly from the fact that \( G_1' \) is bipartite. Furthermore, it is clear that \( \text{LP}_1(F) \geq \text{LP}_2(F) \). If \( F \) is empty, then \( \text{LP}_1(F) = \text{LP}_2(F) = 0 \) since \( y := 0 \) is optimal for both LPs. From now on, assume that \( F \) is nonempty, and let \( v_0 \in V(H) \) be any node that is incident to some edge of \( F \).

Now consider the LP obtained from \( \text{LP}_3(F) \) by adding the constraint \( x_{v_0} = 0 \):

\[
\text{LP}_3(F) := \min \left\{ \sum_{e \in E(G_1)} c(e)y_e^1 \mid Mx + y = 1, \ y \geq 0, \ y^H = \chi^F, \ x_{v_0} = 0 \right\}.
\]

Since \( G_1' \) is bipartite, \( \text{LP}_2(F) = \text{LP}_3(F) \) since adding the extra constraint does not change the set of feasible \( y \) vectors. Thanks to the extra constraint, the feasible region of \( \text{LP}_3(F) \) is pointed.

Consider an extreme optimal solution \((\bar{x}, \bar{y})\) of \( \text{LP}_3(F) \). Since \( M \) is TU, we may assume that both \( \bar{x} \) and \( \bar{y} \) are integral. Since \( F \) is feasible, \( \bar{x}_v \in \{0,1\} \) for all \( v \in V(H) \). We claim that \((\bar{x}, \bar{y})\) is feasible for \( \text{LP}_1(F) \). Observe that the claim implies \( \text{LP}_1(F) \leq \text{LP}_3(F) = \text{LP}_2(F) \) and thus \( \text{LP}_1(F) = \text{LP}_2(F) \).
If $\bar{x}$ is nonnegative, we are done. Otherwise, we can find disjoint sets $V_\alpha$ and $V_{1-\alpha}$ for some $\alpha \in \mathbb{Z}_{\geq 0}$ such that $\bar{x}_v = \alpha$ for all $v \in V_\alpha$, $\bar{x}_v = 1-\alpha$ for all $v \in V_{1-\alpha}$ and no edge $e$ with $y_e = 0$ has exactly one end in $V_\alpha \cup V_{1-\alpha}$. Since $\alpha < 0$ and $1-\alpha > 1$, we see that both $V_\alpha$ and $V_{1-\alpha}$ are disjoint from $V(H)$. Let $\bar{x}' := \bar{x} + \chi_{V_\alpha} - \chi_{V_{1-\alpha}}$, $\bar{y}' := 1 - M \bar{x}'$, $\bar{x}'' := \bar{x} - \chi_{V_\alpha} + \chi_{V_{1-\alpha}}$ and $\bar{y}'' := 1 - M \bar{x}''$. Both $(\bar{x}', \bar{y}')$ and $(\bar{x}'', \bar{y}'')$ are feasible for LP$_3(F)$, contradicting the extremality of $(\bar{x}, \bar{y})$.

Lemma 20. If $F \subseteq E(H)$ is feasible and the disjoint union of $A$ and $B$, then $\gamma(F) \leq \gamma(A) + \gamma(B)$.

Proof. We may assume that $A$ and $B$ are both feasible, otherwise there is nothing to prove. Let $(x, y)$ and $(z, t)$ be optimal solutions of LP$_2(A)$ and LP$_2(B)$ respectively (see Lemma 19). If we let $u := x + z - \frac{1}{2}1$ and $v := y + t$, then $(u, v)$ is feasible for LP$_2(F)$ since $M u + v = M x + M z - \frac{1}{2}M 1 + v = (1 - y) + (1 - t) - 1 + v = 1$, $v \geq 0$ and $\chi_A + \chi_B = \chi_F$. By Lemma 19 this shows that $\gamma(F) \leq \gamma(A) + \gamma(B)$. 

To prove that the inequality in 5 is valid for all $(y^0, y^H) \in \sigma(\text{STAB}(G'_0))$, it suffices to consider any vertex $(y^0, y^H)$ of $\sigma(\text{STAB}(G'_0))$ minimizing the left-hand size of 5. We may even assume that $(y^0, y^H)$ minimizes $\|y^H\|_1$ among all such vertices.

Let $S^0$ denote the stable set of $G'_0$ corresponding to $(y^0, y^H)$ and let $F := \sigma(S^0) \cap E(H)$. Note that $y^H = \chi_F$. Observe that $S^0$ is not properly contained in another stable set, since this would contradict the minimality of $y$. Moreover, we claim that $F$ has at most one edge. In order to prove the claim, we consider only the case where $H = S_{2,2,2}$, see Figure 2. The other cases are easier or similar, and we leave the details to the reader.

Let us assume that $F$ contains at least two edges, that is, $\|y^H\|_1 \geq 2$. We will replace $y^H$ by a new vector $\tilde{y}^H \in \{0,1\}^{E(H)}$ such that $(y^0, \tilde{y}^H) \in \sigma(\text{STAB}(G'_0))$ with smaller $\ell_1$-norm in such a way that the cost of $(y^0, \tilde{y}^H)$ is not higher than that of $(y^0, y^H)$, arriving at a contradiction. In order to prove that $(y^0, \tilde{y}^H) \in \sigma(\text{STAB}(G'_0))$ we will explain how to obtain the corresponding stable set $S^0$ from stable set $S^0$ in each case. To guarantee that the cost of $(y^0, \tilde{y}^H)$ does not exceed that of $(y^0, y^H)$, we will mainly rely on Lemma 29.

To distinguish the different cases, let $v_1$, $v_2$ and $v_3$ denote the leaves of $H$ and $v_0$ denote its degree-3 node. For $i, j \in \{0, 1, 2, 3\}$ we let $P_{ij}$ denote the $v_i$–$v_j$ path in $H$. For $i \in [3]$, let $v_{0i}$ denote the middle vertex of $P_{ij}$ and let $e_i$ and $f_i$ denote the edges of the path $P_{0i}$ incident to $v_i$ and $v_0$ respectively. The relevant cases and the replacements are listed in Figure 3. We treat each of them below. Notice that the case $|S^0 \cap \{v_1, v_2, v_3\}| = 3$ cannot arise since this would contradict the maximality of $S^0$.

Case 1: $|S^0 \cap \{v_1, v_2, v_3\}| = 0$. In this case we set $\tilde{y}^H := 0$, which corresponds to letting $S^0 := (S^0 \cup \{v_{01}, v_{02}, v_{03}\}) \setminus \{v_0\}$. In this case it is clear that the cost of $(y^0, \tilde{y}^H)$ is at most the cost of $(y^0, y^H)$.

Case 2: $|S^0 \cap \{v_1, v_2, v_3\}| = 1$. We may assume that $S^0 \cap \{v_1, v_2, v_3\} = \{v_3\}$. Since $|F| \geq 2$ and $S^0$ is maximal, we must have $S^0 \cap V(H) = \{v_0, v_3\}$ and hence $y^H = \chi_{\{e_1, e_2\}}$.

We let $\tilde{y}^H := \chi_{\{f_3\}}$, which corresponds to letting $S^0 := S^0 \cup \{v_0, v_{03}\}$. The cost of $(y^0, \tilde{y}^H)$ equals the cost of $(y^0, y^H)$ minus $\gamma(\{e_1\}) + \gamma(\{e_2\}) - \gamma(\{f_3\}) = \gamma(\{e_1\}) + \gamma(\{e_2\}) - \gamma(\{e_1\}) - \gamma(\{e_2\}) \geq 0$. The equality follows from the fact that stable sets $S$ of $G'_1$ such
that $\sigma(S) \cap E(H) = \{f_3\}$ and stable sets $S$ of $G'_1$ such that $\sigma(S) \cap E(H) = \{e_1, e_2\}$ have
the same intersection with the leaves of $H$. The inequality follows from Lemma 20.

Case 3: $|S^0 \cap \{v_1, v_2, v_3\}| = 2$. We may assume that $S^0 \cap \{v_1, v_2, v_3\} = \{v_1, v_2\}$. Again, since
$|F| \geq 2$ and $S^0$ is maximal, we must have $S^0 \cap V(H) = \{v_1, v_2\}$ and hence $y^H = \chi_{\{f_1, f_2\}}$.
We let $\bar{y}^H = \chi_{\{v_3\}}$, which corresponds to letting $S^0 := S^0 \setminus \{v_{03}\} \cup \{v_{01}, v_{02}\}$. Similar to
the previous case, we obtain that the cost of $(y^0, \bar{y}^H)$ equals the cost of $(y^0, y^H)$ minus
$\gamma(\{f_1\}) + \gamma(\{f_2\}) - \gamma(\{v_3\}) = \gamma(\{f_1\}) + \gamma(\{f_2\}) - \gamma(\{f_1, f_2\}) \geq 0$.

Thus, $F$ has indeed at most one edge. There exists a stable set $S^1$ of $G'_1$ that is
a minimizer for $\gamma(F)$ such that $S^1 \cap V(G) \cap V(H) = S^0 \cap V(G) \cap V(H)$. Hence, $S := S^1 \cup S^0$ is a stable set of $G$. Let $(y^0, y^1)$ denote the characteristic vector of $\sigma(S)$, so that
$(y^0, y^1) \in \sigma(\text{STAB}(G))$. We get
$$
\sum_{e \in E(G_0)} c(e)y_e^0 + \sum_{e \in E(H)} c^H(e)y_e^H = \sum_{e \in E(G_0)} c(e)y_e^0 + \gamma(F)
= \sum_{e \in E(G_0)} c(e)y_e^0 + \sum_{e \in E(G_1)} c(e)y_e^1 \geq \beta(G, c).
$$

Above, the first equality comes from the fact that $F$ has at most one edge, the definition
of $c^H(f)$ for $f \in E(H)$ and $\gamma(\emptyset) = 0$. The second equality follows from the hypothesis
that $S^1$ is a minimizer for $\gamma(F)$. Finally, the inequality is due to the validity of (4) for
$\gamma(\text{STAB}(G))$. This shows that (5) is indeed valid for $(y^0, y^H) \in \sigma(\text{STAB}(G'_0))$, which
concludes the proof of Lemma 16.

| Case 1 | Case 2 | Case 3 |
|--------|--------|--------|
| $y^H$ and $S^0 \cap V(H)$ | $y^H$ and $S^0 \cap V(H)$ | $y^H$ and $S^0 \cap V(H)$ |
| Fig. 3. Replacements in the proof of Lemma 16 (top row: before, bottom row: after). Red thick edges are slack. Blue thick, dotted edges are tight. Red nodes are in the stable set, blue nodes are not. |

Acknowledgements

We would like to mention that the total unimodularity test of Truemper & Walter 19 provided valuable insights in the early stages of our project, especially for gaining a better understanding of the algorithm in 11.
References

1. Artmann, S., Weismantel, R., Zenklusen, R.: A strongly polynomial algorithm for bimodular integer linear programming. In: STOC'17—Proceedings of the 49th Annual ACM SIGACT Symposium on Theory of Computing, pp. 1206–1219. ACM, New York (2017)
2. Balas, E.: Disjunctive programming. Ann. Discrete Math. 5, 3–51 (1979), discrete optimization (Proc. Adv. Res. Inst. Discrete Optimization and Systems Appl., Banff, Alta., 1977), II
3. Barahona, F., Mahjoub, A.R.: Compositions of graphs and polyhedra ii: stable sets. SIAM Journal on Discrete Mathematics 7(3), 359–371 (1994)
4. Bonifas, N., Di Summa, M., Eisenbrand, F., Hähnle, N., Niemeier, M.: On sub-determinants and the diameter of polyhedra. Discrete & Computational Geometry 52(1), 102–115 (2014)
5. Cevallos, A., Weltge, S., Zenklusen, R.: Lifting linear extension complexity bounds to the mixed-integer setting. In: Proceedings of the Twenty-Ninth Annual ACM-SIAM Symposium on Discrete Algorithms, pp. 788–807. SODA ’18, Society for Industrial and Applied Mathematics, Philadelphia, PA, USA (2018), http://dl.acm.org/citation.cfm?id=3174304.3175321
6. Chvátal, V.: On certain polytopes associated with graphs. Journal of Combinatorial Theory, Series B 18(2), 138–154 (1975)
7. Conforti, M., Fiorini, S., Huynh, T., Joret, G., Weltge, S.: The stable set problem in graphs with bounded genus and bounded odd cycle packing number. https://arxiv.org/abs/1908.06300 to appear in SODA ’20
8. Conforti, M., Gerards, B., Pashkovich, K.: Stable sets and graphs with no even holes. Mathematical Programming 153(1), 13–39 (2015)
9. Dyer, M., Frieze, A.: Random walks, totally unimodular matrices, and a randomised dual simplex algorithm. Mathematical Programming 164(1-2), 325–339 (2017)
10. Eisenbrand, F., Vempala, S.: Geometric random edge. Mathematical Programming 164(1-2), 325–339 (2017)
11. Grossman, J.W., Kulkarni, D.M., Schochetman, I.E.: On the minors of an incidence matrix and its smith normal form. Linear Algebra and its Applications 218, 213–224 (1995)
12. Kawarabayashi, K.i., Ozeki, K.: A simpler proof for the two disjoint odd cycles theorem. J. Combin. Theory Ser. B 103(3), 313–319 (2013), https://doi.org/10.1016/j.jctb.2012.11.004
13. Nemhauser, G.L., Trotter, J.L.E.: Properties of vertex packing and independence system polyhedra. Math. Programming 6, 48—61 (1974)
14. Paat, J., Schlöter, M., Weismantel, R.: Most IPs with bounded determinants can be solved in polynomial time. arXiv:1904.06874 (2019)
15. Seymour, P.D.: Matroid minors. In: Handbook of combinatorics, Vol. 1, 2, pp. 527–550. Elsevier Sci. B. V., Amsterdam (1995)
16. Sllaty, D.: Projective-planar signed graphs and tangle signed graphs. J. Combin. Theory Ser. B 97(5), 693–717 (2007), https://doi.org/10.1016/j.jctb.2006.10.002
17. Tardos, E.: A strongly polynomial algorithm to solve combinatorial linear programs. Operations Research 34(2), 250–256 (1986)
18. Veselov, S.I., Chirkov, A.J.: Integer program with bimodular matrix. Discrete Optimization 6(2), 220–222 (2009)
19. Walter, M., Truemper, K.: Implementation of a unimodularity test. Math. Program. Ser. C 5(1), 57–73 (2013). https://doi.org/10.1007/s12532-012-0048-x
http://dx.doi.org/10.1007/s12532-012-0048-x