PROTORI AND TORSION-FREE ABELIAN GROUPS

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ABSTRACT. The Resolution Theorem for Compact Abelian Groups is applied to show that the profinite subgroups of a finite-dimensional compact connected abelian group (protorus) which induce tori quotients comprise a lattice under intersection (meet) and + (join), facilitating a proof of the existence of a universal resolution. A finite rank torsion-free abelian group X is algebraically isomorphic to a canonical dense subgroup X_0 of its Pontryagin dual G. A morphism between protori lifts to a product morphism between the universal covers, so morphisms in the category can be studied as pairs of maps: homomorphisms between finitely generated profinite abelian groups and linear maps between finite-dimensional real vector spaces. A concept of non-Archimedean dimension is introduced which acts as a useful invariant for classifying protori.

1. INTRODUCTION

Finite rank torsion-free abelian groups, isomorphic to additive subgroups of \( \mathbb{Q}^n \), \( 0 \leq n \in \mathbb{Z} \), have been an active area of research for more than a century. This paper is the first, as far as we are aware, to intrinsically approach the study from within the dual category of finite-dimensional protori. The nature of compact abelian groups manifests an approach which would not emerge simply by dualizing results from the discrete torsion-free category.

The results are organized into four sections: Background (Section 2), Profinite Theory (Section 3), Structure of Protori (Section 4), and Morphisms of Protori (Section 5).

Section 4 establishes the structural properties unique to protori in the category of compact abelian groups. The Resolution Theorem for Compact Abelian Groups [4, Theorem 8.20] provides a starting point for our investigation. In Lemma 4.2, a profinite subgroup of a torus-free protorus G inducing a torus quotient is shown to intersect the path component of the identity to form a dense free abelian subgroup of the profinite subgroup. An isogeny class for a finitely generated profinite abelian group has a representative that is a profinite algebra. Proposition 4.3 shows the collection of profinite subgroups of a torus-free protorus inducing tori quotients comprise a countable lattice. Proposition 4.8 establishes the existence of a protorus with a prescribed profinite subgroup inducing a torus quotient. Theorem 4.11 establishes a structure theorem for protori in terms of the subgroup \( \Delta_G \), generated by the profinite subgroups, and \( \exp \Sigma(G) \), the path component of the identity.

Section 5 presents an analysis of morphisms between protori. Lemma 5.5 gives that \( \Delta_G \), \( \exp \Sigma(G) \), and their countable intersection, are fully invariant subgroups under continuous endomorphisms. Proposition 5.6 gives that a morphism of protori \( G \to H \) lifts to a product morphism \( \Delta_G \times \Sigma(G) \to \Delta_H \times \Sigma(H) \). Theorem 5.7 completes the paper, giving that a morphism of protori \( G \to H \) lifts to a morphism \( \Delta_G \times \exp G \Sigma(G) \to \Delta_H \times \exp H \Sigma(H) \).

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2. Background

A protorus is a compact connected abelian group. The name *protorus* derives from the formulation of its definition as an inverse limit of finite-dimensional tori [4, Corollary 8.18, Proposition 1.33], analogous to a profinite group as an inverse limit of finite groups. A morphism between topological groups is a continuous homomorphism. A topological isomorphism is an open bijective morphism between topological groups, which we denote by \( \cong \). Set \( T = \mathbb{R}/\mathbb{Z} \) with the quotient topology induced from the Euclidean topology on \( \mathbb{R} \) (note that \( \mathbb{Z} \) is discrete in the subspace topology). A torus is a topological group topologically isomorphic to \( T^n \) for some positive integer \( n \). A protorus is *torus-free* if it contains no subgroups topologically isomorphic to a torus.

Pontryagin duality defines a one-to-one correspondence between locally compact abelian groups given by \( G^\vee = \text{Hom}(G, \mathbb{T}) \) under the topology of compact convergence, satisfying \( G^\vee \cong \mathbb{T} \), and restricts to an equivalence between the categories of discrete abelian groups and compact abelian groups [4, Theorem 7.63] wherein compact abelian groups are connected if and only if they are divisible [4, Proposition 7.5(i)], [4, (24.3)], [3, (23.17)], [4, (24.25)]. Some locally compact abelian groups, such as finite cyclic groups \( \mathbb{Z}(n) \), the real numbers \( \mathbb{R} \), the \( p \)-adic numbers \( \mathbb{Q}_p \), and the adeles \( \mathbb{A} \) are fixed points of the contravariant Pontryagin duality functor \( \text{Hom}(\mathbb{Z}, \mathbb{T}) \).

All groups in this paper are abelian and all topological groups are Hausdorff. All torsion-free abelian groups have finite rank and all protori are finite-dimensional. All finite-dimensional real topological vector spaces are topologically isomorphic to a real Euclidean vector space of the same dimension [4, Proposition 7.24.(iii)]. All rings are commutative with 1. Finitely generated in the context of profinite groups will always mean topologically finitely generated.

Note: Some authors use the term solenoid to describe finite-dimensional protori; for us, a solenoid is a 1-dimensional protorus. Some authors use the term solenoidal group to describe a finite-dimensional protorus. Also, some authors prefer the spelling pro-torus as having the Greek root proto-. After much reflection, we decided to use protorus both for the spelling and to connote compact connected abelian group. While a protorus does not have to be finite-dimensional, all protori herein are finite-dimensional. These usage decisions were motivated by the strong parallel between protori and profinite abelian groups and the frequency of the term solenoid in the literature as it applies to 1-dimensional compact connected abelian groups.

For a compact abelian group \( G \), the Lie algebra \( \mathfrak{L}(G) \cong \text{Hom}(\mathbb{R}, G) \), consisting of the set of continuous homomorphisms under the topology of compact convergence, is a real topological vector space [4, Proposition 7.36]. The exponential function \( \exp : \mathbb{L}(G) \to G \) given by \( \exp(r) = r(1) \), is a morphism of topological groups, and \( \exp \) is injective when \( G \) is torus-free [4, Corollary 8.47]. Let \( G_0 \) denote the connected component of the identity and let \( G_0 \) denote the path component of the identity in \( G \); then \( G_0 = \exp(\mathbb{L}(G)) \) by [4, Theorem 8.30].

The dimension of a compact abelian group \( G \) is \( \text{dim}G \equiv \dim_{\mathbb{R}} \mathbb{L}(G) \). When \( G \) is a finite-dimensional compact abelian group \( \mathbb{L}(G) \cong \mathbb{R}^{\text{dim}G} \) as topological vector spaces [4, Proposition 7.24]. For a compact abelian group of positive dimension, \( \text{dim}G = \text{dim}(\mathbb{Q} \otimes G^\vee) \) [4, Theorem 8.22]. A sequence of compact abelian groups \( G_1 \overset{\phi}{\to} G_2 \overset{\psi}{\to} G_3 \) is exact if \( \phi \) and \( \psi \) are, respectively, injective and surjective morphisms of topological groups and \( \text{Ker}\psi = \text{Im}\phi \); note that automatically \( \phi \) is open onto its image and \( \psi \) is open [3, Theorem 5.29]. For a morphism \( \rho : G \to H \) of locally compact abelian groups, the adjoint of \( \rho \) is the morphism \( \rho^\vee : H^\vee \to G^\vee \) given by \( \rho^\vee(\chi) = \chi \circ \rho \) [4, Theorem 24.38]. A sequence of compact abelian groups \( G_1 \overset{\phi}{\to} G_2 \overset{\psi}{\to} G_3 \) is exact if and only if \( G_2^\vee \overset{\phi^\vee}{\to} G_3^\vee \) is an exact of discrete abelian groups [4, Theorem 2.1]. A compact abelian group \( G \) is totally disconnected \( \iff \text{dim}G = 0 \iff G^\vee \) is torsion \( \iff \text{dim}(\mathbb{Q} \otimes G^\vee) = 0 \) [4, Corollary 8.5].

Finite rank torsion-free abelian groups \( A \) and \( B \) are quasi-isomorphic if there is \( f : A \to B \), \( g : B \to A \), and \( 0 \neq n \in \mathbb{Z} \) such that \( fg = n \cdot 1_B \) and \( gf = n \cdot 1_A \). By [1, Corollary 7.7], \( A \) and \( B \) are quasi-isomorphic.
and $B$ are quasi-isomorphic if and only if there is a monomorphism $h: A \to B$ such that $A/f(B)$ is finite. It follows by Pontryagin duality that $A$ and $B$ are quasi-isomorphic if and only if there is a surjective morphism $h^\vee: B^\vee \to A^\vee$ with finite kernel. This is exactly the definition of isogeny between finite-dimensional protori: $G$ and $H$ are isogenous if there is a surjective morphism $G \to H$ with finite kernel. As is evident from the definition, quasi-isomorphism of torsion-free groups is an equivalence relation, whence isogeny of protori is an equivalence relation.

For reasons we do not delve into here, the definition of isogeny between profinite abelian groups is slightly different from that of isogeny between protori. Profinite abelian groups $D$ and $E$ are isogenous if there are morphisms $f: D \to E$ and $g: E \to D$ such that $E/f(D)$ and $D/g(E)$ are bounded torsion groups. In the setting of finite-dimensional protori, the profinite abelian groups that emerge are always finitely generated, so this definition is equivalent to the stipulation that $E/f(D)$ and $D/g(E)$ are finite for morphisms $f$ and $g$ [8, Lemma 4.3.7]. It is evident from the symmetry of the definition that isogeny between profinite abelian groups is an equivalence relation. Proceeding strictly according to Pontryagin duality, one would conclude that torsion abelian groups $A$ and $B$ be defined as quasi-isomorphic if there are morphisms $h: A \to B$ and $k: B \to A$ such that $B/h(A)$ and $A/k(B)$ are bounded torsion groups; this is, in fact, the definition for quasi-isomorphism between torsion abelian groups: see, for example, [1] Proposition 1.8.

3. Profinite Theory

The development of a structure theory for protori is very much dependent on the theory of profinite abelian groups. The profinite theory developed in this section is derived in large part from the standard reference for profinite theory, namely Ribes and Zaleskii [8]. The content comprises a separate section because of the unique nature of the requisite theory.

We begin by showing that the additivity of dimension for vector spaces also holds for compact abelian groups.

**Lemma 3.1.** If $0 \to G_1 \to G_2 \to G_3 \to 0$ is an exact sequence of finite-dimensional compact abelian groups, then $\dim G_2 = \dim G_1 + \dim G_3$.

**Proof.** The exactness of $0 \to G_1 \to G_2 \to G_3 \to 0$ implies the exactness of $0 \to G_1^\vee \to G_2^\vee \to G_3^\vee \to 0$ and this implies the exactness of $0 \to \mathbb{Q} \otimes G_1^\vee \to \mathbb{Q} \otimes G_2^\vee \to \mathbb{Q} \otimes G_3^\vee \to 0$ because $\mathbb{Q}$ is torsion-free [2, Theorem 8.3.5]. But this is an exact sequence of $\mathbb{Q}$-vector spaces and hence $\dim_{\mathbb{Q}}(\mathbb{Q} \otimes G_2) = \dim_{\mathbb{Q}}(\mathbb{Q} \otimes G_1) + \dim_{\mathbb{Q}}(\mathbb{Q} \otimes G_3)$. This establishes the claim because, in general $\dim G = \dim_{\mathbb{Q}} \mathbb{Q} \otimes G^\vee$ by [4, Theorem 8.22] for $\dim G \geq 1$ and $\dim G = 0 \iff \dim(\mathbb{Q} \otimes G^\vee) = 0$.  

Fix $n \in \mathbb{Z}$. Denote by $\mu_n$ the multiplication-by-$n$ map $A \to A$ for an abelian group $A$, given by $\mu_n(a) = na$ for $a \in A$.

**Lemma 3.2.** $\mu_n: G \to G, \ 0 \neq n \in \mathbb{Z}$, is an isogeny for a finite-dimensional protorus $G$.

**Proof.** $\mu_n$ is a surjective morphism because $G$ is a divisible abelian topological group, so the adjoint $\mu_n^\vee: G^\vee \to G^\vee$ is injective, whence $[G^\vee: \mu_n^\vee(G^\vee)]$ is finite by [1] Proposition 6.1.(a). It follows that $\ker \mu_n$ is finite and $\mu_n$ is an isogeny.  

A **profinite group** $\Delta$ is a compact totally disconnected group or, equivalently, an inverse limit of finite groups [4, Theorem 1.34]. A profinite group is either finite or uncountable [8, Proposition 2.3.1]. A profinite group is finitely generated if it is the topological closure of a finitely generated subgroup. The **profinite integers** $\hat{\mathbb{Z}}$ is defined as the inverse limit of cyclic groups of order $n$; $\hat{\mathbb{Z}}$ is topologically isomorphic to $\prod_{p \in \mathbb{P}} \hat{\mathbb{Z}}_p$, where $\hat{\mathbb{Z}}_p$ denotes the $p$-adic integers and $\mathbb{P}$ is the set of prime numbers [8, Example 2.3.11]; also, $\hat{\mathbb{Z}}$ is topologically isomorphic to the profinite completion of $\mathbb{Z}$ [8, Example 2.1.6.(2)]; so $\hat{\mathbb{Z}}^m$ is topologically finitely generated, $0 \leq m \in \mathbb{Z}$.

For a finite-dimensional protorus $G$, the **Resolution Theorem for Compact Abelian Groups** states that $G$ contains a profinite subgroup $\Delta$ such that $G \cong _1 \Delta \times \mathbb{Z}$ where $\Gamma$ is a discrete
Proposition 3.5. A finitely generated profinite abelian group is isomorphic to $\prod_{p \in \mathbb{P}} \hat{\mathbb{Z}}(p^\infty)$.

Proof. For each $p$ in an alternative representation, the associated group will be isomorphic to the one given.

Lemma 3.3. The algebraic structure of a finitely generated profinite abelian group uniquely determines its topological structure.

Proof. A profinite group has a neighborhood basis at $0$ consisting of open (whence closed) subgroups $[4]$ Theorem 1.34]. A subgroup of a finitely generated profinite abelian group is open if and only if it has finite index $[8]$ Lemma 2.1.2, Proposition 4.2.5]. It follows that finitely generated profinite abelian groups are topologically isomorphic if and only if they are isomorphic as abelian groups. □

As a result of Lemma 3.3, we usually write $\cong$ in place of $\cong_t$ when working with finitely generated profinite abelian groups.

Set $\mathbb{Z}(p^\alpha) \cong \hat{\mathbb{Z}}(p^\infty)$ for $0 \leq r \in \mathbb{Z}$. We introduce the notation $\hat{\mathbb{Z}}(p^\infty)$ if $n < \infty$ and $\hat{\mathbb{Z}}(p^\infty) \cong \hat{\mathbb{Z}}(p^\infty)$ for $p \in \mathbb{P}$. With the conventions $p^\infty \hat{\mathbb{Z}}_p \cong 0$ and $p^\infty \hat{\mathbb{Z}}_p \cong \prod_{p \neq q} \hat{\mathbb{Z}}_q$, we see that $p^n\hat{\mathbb{Z}}_p = p^n\hat{\mathbb{Z}}_p \times \prod_{p \neq q} \hat{\mathbb{Z}}_q$ and $\hat{\mathbb{Z}}(p^n) \cong \hat{\mathbb{Z}}(p^n)$ for $p \in \mathbb{P}$ and $0 \leq n \in \mathbb{Z} \cup \{\infty\}$.

Lemma 3.4. A finitely generated profinite abelian group is isomorphic to $\prod_{j=1}^m \prod_{p \in \mathbb{P}} \hat{\mathbb{Z}}(p^{r_j})$ for some $0 \leq r_j(j) \in \mathbb{Z} \cup \{\infty\}$, $p \in \mathbb{P}$, $1 \leq j \leq m$.

Proof. $[8]$ Theorem 4.3.5, 4.3.6]. □

Proposition 3.5. A finitely generated profinite abelian group is isomorphic to

$$\Delta = \prod_{j=1}^m \hat{\mathbb{Z}}(p^{r_j})$$

for some $0 \leq r_j(j) \in \mathbb{Z} \cup \{\infty\}$, $p \in \mathbb{P}$, $1 \leq j \leq m$, where $r_j(j) \geq r_j(k) \iff j \leq k$, and $r_q(m) > 0$ for some $q \in \mathbb{P}$.

Proof. Fix a representation as in Lemma 3.3. The representation is indexed by $\{1, \ldots, m\} \times \mathbb{P}$. With regard to uniqueness up to isomorphism, there is no significance to the order of the factors $\hat{\mathbb{Z}}(p^{r_j})$ appearing. As long as the exact same aggregate list of $r_j(j)$ appears in an alternative representation, the associated group will be isomorphic to the one given.

For each $p \in \mathbb{P}$ we rearrange the $m$ exponents $r_p(1), \ldots, r_p(m)$ into descending order and relabel the ordered exponents $s_p(1), \ldots, s_p(m)$: $\{r_p(1), \ldots, r_p(m)\} = \{s_p(1), \ldots, s_p(m)\}$ and $s_p(1) \geq s_p(2) \geq \cdots \geq s_p(m)$. If, after applying this ordering for each $p \in \mathbb{P}$, we get $r_p(m) = 0$ for all $p \in \mathbb{P}$, then we remove all $\hat{\mathbb{Z}}(p^{r_p(1)})$ for $p \in \mathbb{P}$, and reduce the value of $m$ accordingly. We repeat this weaning process right-to-left, so it terminates in a finite number of steps because $1 \leq m \in \mathbb{Z}$. In this way we see that, without loss of generality, $m$ is minimal for a representation with the given characteristics. □

Define the standard representation of a finitely generated profinite abelian group to be the $\Delta$ of Proposition 3.5 to which it is isomorphic. We introduce the notation $\Delta_j = \prod_{p \in \mathbb{P}} \hat{\mathbb{Z}}(p^{r_j})$, $1 \leq j \leq m$, and $\Delta_p = \prod_{j=1}^m \hat{\mathbb{Z}}(p^{r_j})$, $p \in \mathbb{P}$.

Let $D$ be a finitely generated profinite abelian group with standard representation $\Delta$ as in Proposition 3.5. Define the non-Archimedean width of $D$ to be $width_D \Delta \equiv m$. Define the non-Archimedean dimension of $D$ to be $dim_{\Delta} D \equiv \sum_{j=1}^m |\{j \in \{1, \ldots, m\} : \Delta_j \text{ is infinite}\}|$. 
Corollary 3.6. Non-Archimedean dimension of finitely generated profinite abelian groups is well-defined.

Proof. Isomorphic finitely generated profinite abelian groups have the same standard representation given by Proposition 3.5.

Corollary 3.7. Set $\Delta = m \prod_{j=1}^{m} \hat{\mathbb{Z}}(p^{r_p(j)})$. Let $k = p_1^{\alpha_1} \cdots p_{\ell}^{\alpha_{\ell}} \in \mathbb{P}$ where $p_1, \ldots, p_{\ell} \in \mathbb{P}$ are distinct and $0 < \ell, \alpha_1, \ldots, \alpha_{\ell} \in \mathbb{Z}$. Then

$$\frac{\Delta}{k \Delta} \cong \hat{\mathbb{Z}}(p_1^{\min\{r_p(j)\alpha_1\}} \times \cdots \times \hat{\mathbb{Z}}(p_{\ell}^{\min\{r_p(j)\alpha_{\ell}\}})).$$

In particular, $\frac{\hat{\mathbb{Z}}_n}{p^k \hat{\mathbb{Z}}_n} \cong \mathbb{Z}(p^n)$ for $0 \leq n \in \mathbb{Z}$.

Proof. Scalar multiplication $\mathbb{Z} \times \Delta \rightarrow \Delta$ is componentwise: if $x = (x_1, \ldots, x_m) \in \Delta$, where $x_j = (x_{j,p}) \in \mathbb{P}$, then $kx = (kx_1, \ldots, kx_m)$ where the scalar multiplication in each coordinate is given by $kx_j = (kx_{j,p}) \in \mathbb{P}$, applying the usual scalar multiplications for $\hat{\mathbb{Z}}_p$ and $\mathbb{Z}(p^k)$.

A profinite abelian group has a unique $p$-Sylow subgroup, $p \in \mathbb{P}$, and is the product of its $p$-Sylow subgroups [3] Proposition 2.3.8. Let $0 \leq n \in \mathbb{Z}$ and fix $p \in \mathbb{P}$. If $q$ is relatively prime to $p$, then $q^p \hat{\mathbb{Z}}_p = \hat{\mathbb{Z}}_p$ and $q^p \mathbb{Z}(p^k) = \mathbb{Z}(p^k)$, so the only nonzero $p$-Sylow factors of the profinite group $\frac{\Delta}{k \Delta}$ correspond to primes $p|k$.

If $r_p(j) < \infty$, then $p^p \hat{\mathbb{Z}}(p^{r_p(j)}) = \hat{\mathbb{Z}}(p^{\max\{r_p(j)\alpha_1\}})$ so $\frac{\hat{\mathbb{Z}}(p^{r_p(j)})}{p^k \hat{\mathbb{Z}}(p^{r_p(j)})} \cong \hat{\mathbb{Z}}(p^{\min\{r_p(j)\alpha_1\}})$. If $r_p(j) = \infty$, then $p^p \hat{\mathbb{Z}}(p^{r_p(j)}) = p^p \hat{\mathbb{Z}}_p$ so $\frac{\hat{\mathbb{Z}}(p^{r_p(j)})}{p^k \hat{\mathbb{Z}}(p^{r_p(j)})} \cong \hat{\mathbb{Z}}(p^n)$.

A supernatural number is a formal product $\mathfrak{n} = \prod_{p \in \mathbb{P}} p^{n_p}$ where $0 \leq n_p \in \mathbb{Z}$ or $n_p = \infty$ for $p \in \mathbb{P}$ [8] Section 2.3. Let $S$ denote the set of all supernatural numbers. A supernatural vector is any $\tilde{\mathfrak{n}} = (n_1, \ldots, n_m) \in \mathbb{S}_m$, $0 \leq m \in \mathbb{Z}$. Set $1 \triangleq \prod_{p \in \mathbb{P}} p^0 \in S$ and $\tilde{1} \triangleq (1, 1, \ldots, 1) \in \mathbb{S}_m$. Fix a finitely generated profinite abelian group $\Delta = m \prod_{j=1}^{m} \hat{\mathbb{Z}}(p^{r_p(j)})$ as in Proposition 3.5. We write $\Delta(\mathfrak{n}) \triangleq \prod_{p \in \mathbb{P}} \hat{\mathbb{Z}}(p^{r_p})$ for $\mathfrak{n} \in S$ and $\Delta(\tilde{\mathfrak{n}}) \triangleq \prod_{j=1}^{m} \hat{\mathbb{Z}}(p^{r_p(j)})$ for $\tilde{\mathfrak{n}} \in \mathbb{S}_m$. Similarly, we introduce the notation $K(\mathfrak{n}) \triangleq \prod_{p \in \mathbb{P}} p^{n_p} \hat{\mathbb{Z}}_p$ for $\mathfrak{n} \in S$ and $K(\tilde{\mathfrak{n}}) \triangleq \prod_{j=1}^{m} p^{r_p(j)} \hat{\mathbb{Z}}_p$ for $\tilde{\mathfrak{n}} \in \mathbb{S}_m$.

Set $r \cdot n \triangleq \prod_{p \in \mathbb{P}} p^{r_p+n_p}$ for $r, n \in S$, where $k + \infty = \infty + k = \infty$ for $0 \leq k \leq \infty$. Set $r \cdot \tilde{\mathfrak{n}} \triangleq (r \cdot n_1, \ldots, r \cdot n_m)$ and set $r \cdot K(\tilde{\mathfrak{n}}) \triangleq K(r \cdot \tilde{\mathfrak{n}})$ for $r \in S, \tilde{\mathfrak{n}} \in \mathbb{S}_m$. Write $r < \infty$ if $r_p < \infty$ for $p \in \mathbb{P}$.

Corollary 3.8. A finitely generated profinite abelian group $\Delta$ is isomorphic to $\Delta(\tilde{\mathfrak{n}})$ for some $\tilde{\mathfrak{n}} \in \mathbb{S}_m$, $m = \text{width}_{\mathfrak{n}_{\mathbb{A}}} \Delta$. If $r < \infty$, $r \in S$, then the following sequence is exact:

$$r \cdot K(\tilde{\mathfrak{n}}) \rightarrow r \cdot \hat{\mathbb{Z}}_m \rightarrow \Delta(\tilde{\mathfrak{n}}).$$

Proof. Proposition 3.5 gives that a finitely generated profinite abelian group is isomorphic to $\Delta(\tilde{\mathfrak{n}})$ for some $\tilde{\mathfrak{n}} \in \mathbb{S}_m$ with $m$ minimal. For each $p \in \mathbb{P}$, $K(\tilde{\mathfrak{n}}) \triangleq \prod_{j=1}^{m} \hat{\mathbb{Z}}(p^{r_p(j)})$ has $p$-Sylow subgroup isomorphic to a product of $m$ or less copies of the $p$-adic integers, so $K(\tilde{\mathfrak{n}})$ is torsion free [3] Theorem 25.8. By Corollary 3.7 $\frac{\hat{\mathbb{Z}}_n}{p^k \hat{\mathbb{Z}}_n} \cong \hat{\mathbb{Z}}(p^n)$ for $0 \leq n \in \mathbb{Z} \cup \{\infty\}$.

Thus, $\frac{\hat{\mathbb{Z}}_m}{K(\tilde{\mathfrak{n}})} = \frac{\hat{\mathbb{Z}}_m}{\prod_{j=1}^{m} \hat{\mathbb{Z}}(p^{r_p(j)})} \cong \prod_{j=1}^{m} \frac{\hat{\mathbb{Z}}_p}{\hat{\mathbb{Z}}(p^{r_p(j)})} \cong \prod_{j=1}^{m} \frac{\hat{\mathbb{Z}}_p}{p^{r_p(j)} \hat{\mathbb{Z}}_p} = \Delta(\tilde{\mathfrak{n}})$. With $\hat{\mathbb{Z}}_m = K(\tilde{1})$ and $p^r \hat{\mathbb{Z}}_p \cong \hat{\mathbb{Z}}_p$ for $0 \leq r \in \mathbb{Z}$, the result follows.
4. Structure of Protori

For a torus-free protorus $G$ with profinite subgroup $\Delta$ inducing a torus quotient, we have by [4, Corollary 8.47] an accompanying injective morphism $\exp_G : \mathcal{L}(G) \to G$ given by $\exp_G(r) = r(1)$. Set

- $Z_\Delta = \Delta \cap \exp \mathcal{L}(G)$,
- $\Gamma_\Delta = \{(\alpha, -\exp_G^{-1}(\alpha)) : \alpha \in Z_\Delta\}$,
- $\pi_\Delta : \Delta \times \mathcal{L}(G) \to \Delta$, the projection map onto $\Delta$,
- $\pi_\Gamma : \Delta \times \mathcal{L}(G) \to \mathcal{L}(G)$, the projection map onto $\mathcal{L}(G)$.

Then $\pi_\Gamma(\Gamma_\Delta) = Z_\Delta$ and $\pi_\Delta(\Gamma_\Delta) = \exp^{-1} Z_\Delta$ by the Resolution Theorem for Compact Abelian Groups [4, Theorem 8.20].

**Lemma 4.1.** If $\Delta$ is a profinite subgroup of a torus-free finite-dimensional protorus $G$ such that $G/\Delta \cong \mathbb{T}^\dim G$, then $\varphi_\Delta : \Delta \times \mathcal{L}(G) \to G$, given by $\varphi_\Delta(\alpha, r) = \alpha + \exp r$, satisfies $\ker \varphi_\Delta = \mathbb{T}^\dim G$.

**Proof.** By [4, Theorem 8.20], $\ker \varphi_\Delta = \Gamma_\Delta$ and the projection $\pi_\Gamma : \Delta \times \mathcal{L}(G) \to \mathcal{L}(G)$ restricts to a topological isomorphism $\pi_\Gamma|_{\Gamma_\Delta} : \Gamma_\Delta \to \exp_G^{-1}(\Delta) = \exp^{-1} Z_\Delta$, where $\exp_G$ is injective because $G$ is torus-free [4, Corollary 8.47]. Also, $\mathcal{L}(G) \cong \mathbb{R}^\dim G$ by [4, Theorem 8.22 (5) $\Rightarrow$ (6)]. By [4, Theorem 8.22 (7)], $\Gamma_\Delta$ is discrete, so $\Gamma_\Delta \cong \exp^{-1} Z_\Delta \cong \mathbb{Z}^\dim G$ for some $0 \leq k \leq \dim G$ [4, Theorem A1.12]. But $[\Delta \times \mathcal{L}(G)]/\Gamma_\Delta \cong G$ is compact, so it follows $k = \dim G$. Since $\varphi_\Delta$ is a morphism, $\Gamma_\Delta$ is closed. Thus, $\ker \varphi_\Delta = \Gamma_\Delta \cong \mathbb{T}^\dim G$ as discrete groups. □

The next lemma identifies a simultaneously set-theoretic, topological, and algebraic property unique to profinite subgroups in a protorus which induce tori quotients.

**Lemma 4.2.** If $\Delta$ is a profinite subgroup of a torus-free finite-dimensional protorus $G$ such that $G/\Delta \cong \mathbb{T}^\dim G$ then $\mathbb{T}^\dim G = \Delta$ and $\mathbb{T}^\dim G$ is closed in the subspace $\exp \mathcal{L}(G)$.

**Proof.** By [4, Theorem 8.20] a profinite subgroup $\Delta$ such that $G/\Delta \cong \mathbb{T}^\dim G$ always exists and for such a $\Delta$ we have $G \cong \varprojlim G_\alpha = \Delta \cap \exp \mathcal{L}(G)$, where $\Gamma_\Delta = \{(\exp r, -r) : r \in \mathcal{L}(G), \exp r \in \Delta\}$ is a free abelian group and $\text{rank } \Gamma_\Delta = \dim G = \text{rank } [\Delta \cap \exp \mathcal{L}(G)]$ by Lemma 4.1 and the fact that $\exp$ is injective when $G$ is torus-free [4, Corollary 8.47]. We have $\pi_\alpha(\Gamma_\Delta) = Z_\Delta \subseteq \Delta' \cong \mathbb{Z}^\dim G$, so $\Delta_\alpha$ is a subgroup of $\Delta' \times \mathcal{L}(G)$. Because $\Delta \to \Delta' \times \mathcal{L}(G)$ is a topological isomorphism onto its image, $\Delta' \to \Delta' \times \mathcal{L}(G) \subseteq G_\alpha$ as well. Since $\Delta$ is discrete in $\Delta \times \mathcal{L}(G)$ [4, Theorem 8.20], it is discrete in $\Delta' \times \mathcal{L}(G)$, so $\Delta'_\alpha \equiv [\Delta' \times \mathcal{L}(G)]/\Gamma_\Delta$ is a Hausdorff subgroup of $G_\alpha$. But $[\Delta' \times \mathcal{L}(G)]/\Gamma_\Delta$ is open in $\Delta' \times \mathcal{L}(G)$ and the quotient map $q_\Delta : \Delta \times \mathcal{L}(G) \to G_\alpha$ is an open map. By Pontryagin duality, $\Delta' \times \mathcal{L}(G)$ embeds in the torsion-free group $G_\alpha^\vee$, whence $\Delta' \times \mathcal{L}(G)$ is torsion. □

A lattice is a partially ordered set in which any two elements have a a greatest lower bound, or meet, and least upper bound, or join. It follows that a lattice is directed upward and directed downward as a poset.

Define $L(G) = \{\Delta \subset G : 0 \neq \Delta \text{ a profinite subgroup such that } G/\Delta \text{ is a torus}\}$ for a protorus $G$. If $\Delta_1, \Delta_2 \in L(G)$, then $\Delta_1 \cap \Delta_2$ is the greatest lower bound and $\Delta_1 + \Delta_2$ is the least upper bound. We next prove a number of closure properties for $L(G)$; in particular we show that $L(G)$ is closed under $\cap$ and $+$, so that $L(G)$ is a lattice.
**Remark 4.3.** Going forward, we will apply the following facts without further mention.
(i) By [14] Theorem 8.46.(iii), a path-connected protorus is a torus. So, if a protorus \( G \) has a closed subgroup \( D \) and \( G/D \) is the continuous image of a (path-connected) torus, then automatically \( D \in L(G) \).
(ii) By (i) and Lemma 3.1 a profinite subgroup of a finite-dimensional torus is finite.

**Proposition 4.4.** For a torus-free protorus \( G \), \( L(G) \) is a countable lattice under \( \cap \) for meet and + for join. \( L(G) \) is closed under:

1. Preimages via \( \mu_n \), \( 0 \neq n \in \mathbb{Z} \),
2. Finite extensions,
3. Scalar multiplication by nonzero integers,
4. Join (+), and
5. Meet (\( \cap \)).

Given any \( \Delta, \Delta' \in L(G) \) there exists \( 0 < k \in \mathbb{Z} \) such that \( k\Delta \subseteq \Delta' \). If \( \Delta' \subseteq \Delta \), then \( [\Delta : \Delta'] < \infty \).

**Proof.** Each \( \Delta \in L(G) \) corresponds via Pontryagin duality to a unique-up-to-isomorphism torsion abelian quotient of \( X = G^{\vee} \) by a free abelian subgroup \( Z_\Delta \) with \( \text{rk}Z_\Delta = \text{rk}X \). Because \( X \) is countable and there are countably many finite subsets of a countable set (corresponding to bases of \( Z_\Delta \)'s, counting one basis per \( Z_\Delta \)), it follows that \( L(G) \) is countable.

(1): \( \mu_n: G \rightarrow G \) has finite kernel by Lemma 3.2 so its restriction \( \mu_n^{-1}[\Delta] \rightarrow \Delta \) has finite kernel for \( \Delta \in L(G) \). Since \( \ker\mu_n \) and \( \Delta \in L(G) \) are 0-dimensional compact abelian groups, it follows from Lemma 3.1 that the compact Hausdorff subgroup \( \mu_n^{-1}[\Delta] \) is 0-dimensional, whence profinite. Because the natural map \( G/\Delta \rightarrow G/\mu_n^{-1}[\Delta] \) is surjective and \( G/\Delta \) is a torus, it follows that \( G/\mu_n^{-1}[\Delta] \) is path-connected, whence \( G/\mu_n^{-1}[\Delta] \) is a torus [4]. Theorem 8.46.(iii) and \( \mu_n^{-1}[\Delta] \in L(G) \).

(2) If \( \Delta \in L(G) \) has index \( 1 \leq m \in \mathbb{Z} \) in a subgroup \( D \) of \( G \), then \( D \) is the sum of finitely many copies of \( \Delta \), so \( D \) is compact. Thus, \( \Delta \subseteq D \subseteq \mu_m^{-1}[\Delta] \in L(G) \) by (1), \( D \) is profinite. The natural morphism \( G/\Delta \rightarrow G/D \) is surjective and \( G/\Delta \) is a torus, so \( D \in L(G) \).

(3) \( \mu_j|_{\Delta}: \Delta \rightarrow j\Delta \) is surjective with finite kernel by Lemma 3.2 so \( j\Delta \) is profinite. \( G \) is divisible so \( \mu_j: G \rightarrow G \) is surjective, thus inducing a surjective morphism \( \frac{G}{\Delta} \rightarrow \frac{G}{j\Delta} \) which follows that \( j\Delta \in L(G) \).

(4) Addition defines a surjective morphism \( \Delta \times \Delta' \rightarrow \Delta + \Delta' \). By Lemma 3.1 it follows that \( \Delta + \Delta' \), whence \( \Delta + \Delta' \), is profinite. Because the natural map \( \frac{G}{\Delta} \rightarrow \frac{G}{\Delta + \Delta'} \) is surjective, \( \Delta + \Delta' \in L(G) \).

(5) The kernel of \( \frac{G}{\Delta} \rightarrow \frac{G}{\Delta + \Delta'} \) is \( \Delta + \Delta' \), a 0-dimensional subgroup of \( \frac{G}{\Delta} \) by Lemma 3.1. As a 0-dimensional subgroup of a torus, \( \frac{\Delta + \Delta'}{\Delta} \approx \frac{\Delta'}{\Delta} \) is finite, so there is a nonzero integer \( l \) such that \( l\Delta' \subseteq \Delta \). Lemma 3.1 gives that \( \Delta \cap \Delta' \) is 0-dimensional, whence profinite. We know that \( l\Delta' \in L(G) \), so the natural map \( \frac{G}{\Delta} \rightarrow \frac{G}{\Delta + \Delta'} \) is a surjective morphism, whence \( \Delta \cap \Delta' \in L(G) \).

It follows from (4) and (5) that \( L(G) \) is a lattice. It remains to show that if \( \Delta' \subseteq \Delta \), then \( [\Delta : \Delta'] < \infty \). Arguing as in (5), there exists \( 0 < k \in \mathbb{Z} \) such that \( k\Delta \subseteq \Delta' \). And \( \Delta \) is a finitely generated profinite abelian group, so \( [\Delta : \Delta'] \leq [\Delta : k\Delta] < \infty \) by Corollary 3.7.

**Corollary 4.5.** Elements of \( L(G) \) are mutually isogenous in a torus-free protorus \( G \).

**Proof.** Suppose that \( \Delta_1, \Delta_2 \in L(G) \). We proved in Proposition 4.4 that there exist nonzero integers \( k \) and \( l \) such that \( k\Delta_1 \subseteq \Delta_2 \), \( l\Delta_2 \subseteq \Delta_1 \), \( [\Delta_2 : k\Delta_1] < \infty \), and \( [\Delta_1 : l\Delta_2] < \infty \). The multiplication-by-\( k \) and multiplication-by-\( l \) morphisms thus exhibit an isogeny between \( \Delta_1 \) and \( \Delta_2 \). Hence, all elements of \( L(G) \) are mutually isogenous.

**Lemma 4.6.** Non-Archimedean dimension of finitely generated profinite abelian groups is invariant under isogeny.

**Proof.** If two such groups, say \( D \) and \( D' \) are isogenous, then so are their standard representations, say \( D_{\Delta} \) and \( D_{\Delta'} \) as in Proposition 3.5. Multiplying both groups by the same sufficiently large integer, say \( N \), produces isogenous groups \( ND \) and \( ND' \) with standard representations, say \( \Delta(\mathbf{n}) \) and \( \Delta(\mathbf{n}') \) for some supernumerary vectors \( \mathbf{n} \) and \( \mathbf{n}' \). If \( \Delta(\mathbf{n}) \) and
\( \Delta(\tilde{n}') \) have distinct non-Archimedean dimensions, then one will have an extra coordinate for infinitely many primes (fixed) and/or with one or more copies of \( \mathbb{Z}_p \) (distinct \( p \)), and it is impossible that \( \Delta(\tilde{n}) \) and \( \Delta(\tilde{n}') \) are isogenous. Thus, the definition of non-Archimedean dimension and its preservation under multiplication by \( N \) give that
dim_{\mathbb{A}} D = dim_{\mathbb{A}} (ND) = dim_{\mathbb{A}} \Delta(\tilde{n}) = dim_{\mathbb{A}} \Delta(\tilde{n}') = dim_{\mathbb{A}} D'. \]

Define the non-Archimedean dimension of a protorus \( G \) to be \( dim_{\mathbb{A}} G \equiv dim_{\mathbb{A}} (\Delta) \) for a profinite subgroup \( \Delta \) of \( G \) for which \( G/\Delta \) is a torus.

**Corollary 4.7.** Non-Archimedean dimension of protori is well-defined.

**Proof.** Profinite subgroups of a protorus \( G \) which induce tori quotients are isogenous by Corollary 4.5, so the result follows by Lemma 4.6.

A protorus \( G \) is **factorable** if there exist non-trivial protori \( G_1 \) and \( G_2 \), such that \( G \cong G_1 \times G_2 \), and \( G \) is completely factorable if \( G \equiv \prod_{i=1}^m G_i \) where \( \dim G_i = 1, 1 \leq i \leq m \). A result by Mader and Schultz [6] has the surprising implication that the classification of finite-dimensional protori up to topological isomorphism reduces to that of finite-dimensional protori with no 1-dimensional factors.

**Proposition 4.8.** If \( D \) is a finitely generated profinite abelian group, then there is a completely factorable protorus \( G \) containing a closed subgroup \( \Delta \cong D \) such that \( G/\Delta \) is a torus.

**Proof.** First note that the finite cyclic group \( \mathbb{Z}(r), 0 < r \in \mathbb{Z} \), is isomorphic to the closed subgroup \( \mathbb{Z}(r) / \mathbb{Z} \) of the torus \( \mathbb{R}^n / \mathbb{Z}^n \), so it follows that \( \mathbb{Z}(r) / \mathbb{Z} \) is a profinite subgroup of \( \mathbb{R}^n / \mathbb{Z}^n \), inducing a torus quotient. Next, by Proposition 4.7, there is no loss of generality in assuming \( D = \Delta(\tilde{n}) \) for some \( \tilde{n} \in \mathbb{S}^n \) where \( \Delta(\tilde{n}) \neq 0 \). If \( \Delta(\tilde{n}) \) is finite then it must be isomorphic to \( \mathbb{Z}(r_j) \) for some \( 0 < r_j \in \mathbb{Z} \), and \( \Delta(\tilde{n}) \) is a torus. If \( \Delta(\tilde{n}) \) is not finite, then \( G \equiv [\Delta(\tilde{n}) \times \mathbb{R}] / \mathbb{Z}(1,1) \) is a solenoid (1-dimensional protorus) containing a closed subgroup \( \Delta \equiv \Delta(\tilde{n}) \) satisfying \( G/\Delta \cong \mathbb{T}^m \). It follows that \( G \cong G_1 \times \cdots \times G_m \) is a finite-dimensional protorus containing the closed subgroup \( \Delta \equiv \Delta(\tilde{n}) \times \cdots \times \Delta(\tilde{n}) \) and satisfying \( G/\Delta \cong \mathbb{T}^m \).

**Corollary 4.9.** If \( D \) is a finitely generated profinite abelian group with \( \text{width}_{\mathbb{A}} D = dim_{\mathbb{A}} D \), then there is a completely factorable torus-free protorus \( G \) containing a closed subgroup \( \Delta \cong D \) such that \( G/\Delta \) is a torus.

**Proof.** As in the case above, \( \Delta \) is a factor of \( G \) and is finite cyclic in the proof of Proposition 4.8.

A torsion-free abelian group is **coreduced** if it has no free summands; equivalently, its dual has no torus factors. Next we show a protorus splits into three factors, each factor unique up to topological isomorphism — a torsion-free factor (its dual is a rational vector space), a maximal torus, and a protorus whose dual is both reduced and coreduced.

**Lemma 4.10.** A protorus \( K \) is topologically isomorphic to \( K_{\mathbb{Q}} \times K_{\mathbb{T}} \times G \) where \( K_{\mathbb{Q}} \) is a rational vector space, \( K_{\mathbb{T}} \) is a torus, \( G \) is a reduced and coreduced protorus, and each factor of the decomposition is unique up to topological isomorphism.

**Proof.** [2] Theorem 4.2.5 \( K \equiv K_{\mathbb{Q}} \oplus K_{\mathbb{R}} \) where \( K_{\mathbb{Q}} \) is a rational vector space, \( K_{\mathbb{R}} \) is reduced, and each summand is unique up to isomorphism. By [2] Corollary 3.8.3], \( K_{\mathbb{R}} = \mathbb{Z} \oplus R \) where \( \mathbb{Z} \) is free abelian, \( R \) is both reduced and coreduced, and each summand is unique up to isomorphism. It follows that \( K \equiv K_{\mathbb{Q}} \times K_{\mathbb{T}} \times G \) where \( K_{\mathbb{T}} \) is a torus and \( G \) is a protorus for which \( G \) is both reduced and coreduced.

For a torsion-free protorus \( G \), set
- \( \Lambda_G \equiv \sum_{\Delta \in L(G)} \Delta \subset G \),
- \( X_G \equiv \sum_{\Delta \in L(G)} \mathbb{Z} \Delta \),
- \( \Gamma_G \equiv \{ (\alpha, -\exp_1^{-1} \alpha : \alpha \in X_G) \}. \)
The next result establishes a *universal resolution* for a finite-dimensional torus-free protorus $G$, and in the process exhibits a canonical dense subgroup which is algebraically isomorphic to the finite rank torsion-free dual of $G$. Thus, a coreduced finite rank torsion-free abelian group is isomorphic to a canonical dense subgroup of its Pontryagin dual.

**Theorem 4.11.** (Structure Theorem for Protors) A finite-dimensional protorus factors as $K_Q \times K_T \times G$ where each factor is unique up to topological isomorphism, $K_Q$ is a maximal torsion-free protorus, $K_T$ is a maximal torus, $G$ is a torsor-free protorus with no torsion-free protorus factors, $m \equiv \dim G \geq \dim_{mG}$, and $G$ has the following structure:

1. $L(G) = \{ \Delta \subseteq G : 0 \neq \Delta \text{ a profinite subgroup and } G/\Delta \text{ a torus} \}$ is a countable lattice,
2. $\exp \mathcal{L}(G) \cong \mathbb{R}^m$, the path component of $0$, is a dense divisible subgroup of $G$,
3. $\tilde{\Delta}_G = \bigcup_{\Delta \in L(G)} \Delta \cong \prod_{p \in \mathbb{P}} \left( \mathbb{Z}/p^d \right)^{\dim_{mG} G - r_p} [\text{for some } 0 \leq r_p \leq \dim_{mG}, \ p \in \mathbb{P}]$,
4. $\text{tor}(G) \cong \bigoplus_{p \in \mathbb{P}} \mathbb{Z}/p^{\dim_{mG} G - r_p}$ is a dense subgroup of $G$ contained in $\tilde{\Delta}_G$,
5. $Z_{\Delta} = \Delta \cap \exp \mathcal{L}(G)$ is a dense rank-$m$ free abelian subgroup of $\Delta$ for $\Delta \in L(G)$,
6. $X_G = \bigcup_{\Delta \in L(G)} Z_{\Delta} = \tilde{\Delta}_G \cap \exp \mathcal{L}(G)$ is a countable dense subgroup of $G$,
7. $G \cong \prod_{\Delta \in L(G)} \mathbb{Z} / \tilde{\Delta}_G$, where $\Gamma_G = \{(\alpha, -\exp_{G}^{\Delta} \alpha) : \alpha \in X_G \} \cong X_G$,
8. $G \cong \lim_{\Delta \in L(G)} (G/\Delta)$,
9. $X_G \cong \lim_{\Delta \in L(G)} \dim \{ G/\Delta \} \cong G^\vee$,
10. $\tilde{\Delta}_G$ and $\exp \mathcal{L}(G)$ are incomplete metric subgroups of $G$, and
11. $\dim_{mG} G > 0$ if and only if $G \neq 0$.

**Proof.** The first assertion is a reformulation of Lemma 4.10. $K_Q$ and $K_T$ are uniquely determined up to topological isomorphism by their dimensions.

1. $L(G)$ is a countable lattice by Proposition 4.4 and $L(G) = \{ \Delta : 0 \leq i \in \mathbb{Z} \}$.
2. $\mathcal{L}(G) \cong \mathbb{R}^m$ by [4, Proposition 7.24]. Since $G^\vee$ is reduced, $G$ is torus-free. Hence, $\exp : \mathcal{L}(G) \to G$ is injective by [4, Corollary 8.47], whence $\exp \mathcal{L}(G) \cong \mathbb{R}^m$. The path component of the identity in $G$ is $\exp \mathcal{L}(G)$ by [4, Theorem 8.30] and $\exp \mathcal{L}(G)$ is dense by [4, Theorem 8.20]. Divisibility of $\exp \mathcal{L}(G)$ follows from that of $\mathcal{L}(G)$.

3. Clearly, $\bigcup_{\Delta \in L(G)} \Delta \subseteq \mathbb{Z}$ $\Delta = \tilde{\Delta}_G$. Conversely, if $x \in \tilde{\Delta}_G$, then $x$ is in a finite sum of elements of $L(G)$, whence $x$ lies in a single element of $L(G)$, and thus $x \in \bigcup_{\Delta \in L(G)} \Delta$. Next, $\tilde{\Delta}_G$ is divisible. Let $g \in \tilde{\Delta}_G$ and $p \in \mathbb{P}$. Then $g \in \Delta$ for some $\Delta \in L(G)$. Also, $\mu_p^{-1} \Delta \in L(G)$ by Proposition 4.4. $G$ is divisible, so $p^n g = g$ for some $y \in G$, whence $y \in \mu_p^{-1} \Delta \subseteq \tilde{\Delta}_G$. Since $g$ and $p$ were arbitrary, it follows that $\tilde{\Delta}_G$ is divisible. The displayed algebraic structure of $\tilde{\Delta}_G$ then follows, where $r_p$ is the $p$-adic rank of $G$, $p \in \mathbb{P}$.

4. It suffices to show that $G/\tilde{\Delta}_G$ is torsion-free. Suppose that $g \in G$ and $p g \in \tilde{\Delta}_G$ for some prime $p$. Then $p g \in \Delta$ for some $\Delta \in L(G)$, whence $g \in \mu_p^{-1} \Delta \subseteq \tilde{\Delta}_G$, as desired. $G^\vee$ is reduced $\text{tor}(G)$ is dense in $G$ by [4, Corollary 8.9 (ii)].

5. $\tilde{\Delta}_G = \bigcup_{\Delta \in L(G)} \{ \Delta \subseteq G : 0 \neq \Delta \text{ a dense divisible subgroup of } G \}$.

6. $X_G = \bigcup_{\Delta \in L(G)} Z_{\Delta} \cong Z_{\Delta} = \Delta$ for each $\Delta \in L(G)$ by Lemma 4.4. Define a partial order $\prec$ on $M(G) \cong \{ [\Delta : \Delta \in L(G)] : \exists \Delta_1 < \Delta_2, \Delta_2 \subseteq \Delta_1 \}$. Set $\bigcap \Delta_1 \cap \bigcap \Delta_2 \subseteq \Delta_1 \cap \bigcap \Delta_2$. Then $\bigcap \Delta_1 \cap \bigcap \Delta_2 = \{ \Delta_1 \cap \Delta_2 \} \cap \exp \mathcal{L}(G) = \{ \Delta_1 \cap \exp \mathcal{L}(G) \} \cap \{ \Delta_2 \cap \exp \mathcal{L}(G) \} = \Delta_1 \cap \Delta_2$. Set $\bigcup \Delta_1 \cup \Delta_2 \subseteq \Delta_1 \cup \Delta_2$. If $\bigcap \Delta_1 \cap \bigcap \Delta_2 \subseteq \bigcup \Delta_1 \cup \Delta_2$, then $\bigcup \Delta_1 \cup \Delta_2 \subseteq \bigcap \Delta_1 \cap \bigcap \Delta_2$. Thus, $M(G)$ is well-defined. It follows that $M(G)$ is a lattice. In particular, $\Delta_G = \bigcup_{\Delta \in L(G)} \Delta$ implies that $X_G = \bigcup_{\Delta \in L(G)} Z_{\Delta}$. Next, $X_G = \tilde{\Delta}_G \cap \exp \mathcal{L}(G)$: if $z \in X_G$, then $z \in \Delta \cap \exp \mathcal{L}(G)$ for some $\Delta \in L(G)$, so $z \in \tilde{\Delta}_G \cap \exp \mathcal{L}(G)$; conversely, if $z \in \tilde{\Delta}_G \cap \exp \mathcal{L}(G)$, then $z \in \Delta$ for some $\Delta \in L(G)$, whence $z \in \Delta \cap \exp \mathcal{L}(G) = Z_{\Delta} \subseteq X_G$. 


Lastly, \( \tilde{\Delta} = \bigcup_{\alpha \in \Delta} \Delta \) is dense in \( G \) by (4) and \( X_G \) is the union over the countable set \( L(G) \) of free abelian subgroups \( \mathbb{Z}_\Delta \) with \( \bigcup_{\alpha \in \Delta} \mathbb{Z}_\Delta = \Delta \), so \( X_G \) is a countable dense subgroup of \( G \).

(7) For each \( \Delta \in L(G) \), there is an exact sequence \( \mathbb{Z}_\Delta \xrightarrow{\eta_\Delta} \Delta \times \mathbb{L}(G) \xrightarrow{\phi_\Delta} G \) where \( \eta_\Delta(\alpha) = (\alpha, -exp^{-1} \alpha) \) and \( \phi_\Delta(r, r) = \alpha + exp\ r \). In particular, \( \eta(\mathbb{Z}_\Delta) = \Gamma_\Delta \cong \ker \phi_\Delta \) is a discrete subgroup of \( \Delta \times \mathbb{L}(G) \), although \( \mathbb{Z}_\Delta \) is not closed in \( G \). One checks that a subset \( U \) is open in the subspace \( \tilde{\Delta} \), and only if \( U \cap \Delta \) is open in \( \Delta \) for all \( \Delta \in L(G) \), and that \( V \) is open in the subspace \( X_G \) if and only if \( V \cap \mathbb{Z}_\Delta \) is open in \( \mathbb{Z}_\Delta \) for all \( \Delta \in M(G) \); in other words, the subspace topology on \( \tilde{\Delta} \) is the final topology coherent with \( L(G) \), and the subspace topology on \( X_G \) is the final topology coherent with \( M(G) \). The elements of \( L(G) \) are directed upward because \( L(G) \) is a lattice, so the products \( \Delta \times \mathbb{L}(G) \) are directed upward as well. Thus,

\[
\{ \mathbb{Z}_\Delta : \Delta \in L(G) \} \xrightarrow{\eta_\Delta : \Delta \in L(G)} \{ \Delta \times \mathbb{L}(G) : \Delta \in L(G) \} \xrightarrow{\phi_\Delta : \Delta \in L(G)} \{ G : \Delta \in L(G) \}
\]

is a short exact sequence of direct systems of abelian groups. By [2, Theorem 2.4.6],

and the realization of the subspace topologies on \( \tilde{\Delta} \) and \( X_G \) as final topologies, each consistent with the respective topology on the direct limit in the category of topological spaces, imply that \( X_G \xrightarrow{\eta_\Delta} \tilde{\Delta} \times \mathbb{L}(G) \xrightarrow{\phi_\Delta} G \) is an exact sequence of topological groups (though we will soon see that \( \tilde{\Delta} \) is not closed in \( G \)).

(8) As a lattice, the elements of \( L(G) \) are directed downward. For each \( i \geq 0 \) the sequence \( \Delta_i \xrightarrow{\phi} G \) is exact. Inclusions induce surjective bonding maps \( f_i : G/\Delta_i \to G/\Delta_j \), where \( i \leq j \) if and only if \( \Delta_i \leq \Delta_j \). Because \( \bigcap_{\Delta \in L(G)} \Delta = 0 \) [4, Corollary 8.18], we conclude that \( \lim_{\Delta \in L(G)} (G/\Delta) \cong G \), and the limit maps \( f_i : G \to G/\Delta_i \), satisfying \( f_i = f_{ij} \circ f_j \) when \( \Delta_j \leq \Delta_i \), are the quotient maps \( q_\Delta : G \to G/\Delta_i \), \( 0 \leq i \in \mathbb{Z} \) [4, Proposition 1.33(ii)].

(9) The exact sequence of inverse systems of compact abelian groups

\[
\{ \Delta : \Delta \in L(G) \} \xrightarrow{\{ q_\Delta : \Delta \in L(G) \}} \{ G/\Delta \} \xrightarrow{\{ \phi_\Delta : \Delta \in L(G) \}} \{ G : \Delta \in L(G) \}
\]

effects the exact sequence \( 0 \to \lim_{\Delta \in L(G)} \Delta \to \lim_{\Delta \in L(G)} G \to \lim_{\Delta \in L(G)} G/\Delta \to 0 \) because the inverse limit functor is left exact. Computing limits, we get \( 0 \to 0 \to G \to G \) is exact. Because \( f_i = q_\Delta \), \( i \geq 0 \), it follows by definition of the inverse limit that the morphism \( G \to G \) on the right is surjective. Thus, in fact, \( \lim_{\Delta \in L(G)} \Delta \to \lim_{\Delta \in L(G)} G \to \lim_{\Delta \in L(G)} G/\Delta \) is an exact sequence of inverse limits. Dualizing, we get an exact sequence \( (\lim_{\Delta \in L(G)} G/\Delta) \to (\lim_{\Delta \in L(G)} G) \to (\lim_{\Delta \in L(G)} \Delta) \), or equivalently, \( (\lim_{\Delta \in L(G)} G/\Delta) \to G \to 0 \). The dual of an inverse limit of compact abelian groups is the direct limit of the duals by [5, Chapter II, (20.8)], so we get the exact sequence \( \lim_{\Delta \in L(G)} (G/\Delta)^{\vee} \to G^{\vee} \to 0 \). The correspondences \( \mathbb{Z}_\Delta \leftrightarrow \Delta \leftrightarrow (\mathbb{Z}_\Delta)^{\vee} \) define bijections between the partial orders \( M(G) \), \( \langle L(G), \subseteq \rangle \), \( \langle L(G), \supseteq \rangle \), and, via Pontryagin duality, a countable collection of discrete free abelian groups \( (G/\Delta)^{\vee} \) with rank equal to \( \dim G \). Because the latter two bijective correspondences compose to form a single order isomorphism, we conclude that \( X_G \cong \lim_{\Delta \in L(G)} \mathbb{Z}_\Delta \cong \lim_{\Delta \in L(G)} (G/\Delta)^{\vee} \cong G^{\vee} \), as desired.

(10) The result is vacuously true when \( G = 0 \). Assume \( G \neq 0 \). The path component of 0, namely \( \exp \mathbb{L}(G) \), is a proper subgroup because \( G \) is torus-free. Thus, \( \exp \mathbb{L}(G) \) is dense, but not closed, in \( G \), and hence is an incomplete metric subgroup with completion \( G \).
To show that $\hat{\Delta}_G$ is an incomplete metric subgroup with completion $G$, it suffices by (4) to show that $\hat{\Delta}_G$ is not closed. Suppose on the contrary that $\hat{\Delta}_G$ is closed. Then the second isomorphism theorem applies [3] Theorem 5.33; $\frac{G}{\Delta_G} = \frac{\Delta_G + \exp \Sigma(G)}{\Delta_G} \cong_1 \frac{\exp \Sigma(G)}{\Delta_G - \exp \Sigma(G)} = \frac{\exp \Sigma(G)}{X_G}$ so $X_G$ is closed in $\exp \Sigma(G)$, whence (9) implies that $\exp^{-1} X_G \cong X_G \cong G^v$ is closed in $\Sigma(G)$. By [3] Theorem A1.12, $\exp^{-1} X_G$ is the direct sum of a free abelian group and a real vector space. But $G^v$ has no free summands, so $X_G$ is isomorphic to the additive subgroup of a nontrivial real vector space, contradicting the fact that $X_G$ is countable.

(11) Finally, in an exact sequence $\Delta \rightarrow H \rightarrow \mathbb{T}^m$ where $H$ is an $m$-dimensional protorus with profinite subgroup $\Delta$ inducing a torus quotient, $0 = \dim_{\mathbb{A}} H = \dim_{\mathbb{A}} \Delta \Rightarrow \Delta$ is finite $\Leftrightarrow H$ is isogenous to a torus $\Leftrightarrow H$ is a torus. Because $G$ has no torus factors, it follows that $\dim_{\mathbb{A}} G > 0 \Leftrightarrow G \neq 0$. □

Define the universal resolution of a torus-free finite-dimensional protorus $G$ to be $\hat{\Delta}_G \times \Sigma(G) \rightarrow L_G$. The factors of the product $\hat{\Delta}_G \times \Sigma(G)$ are neither locally compact nor complete; however, the canonical nature of the exact sequence $X_G \rightarrow \hat{\Delta}_G \times \Sigma(G) \rightarrow G$ suggests that $\hat{\Delta}_G \times \Sigma(G)$ is a natural candidate for a universal covering group of $G$.

5. MORPHISMS OF PROTORI

Promotion structure in place, several results dealing with morphisms of protori follow.

Lemma 5.1. A morphism $f_\Delta: \Delta_G \rightarrow \Delta_H$ with $f(\mathbb{Z}_{\Delta_G}) = \mathbb{Z}_{\Delta_H}$ for some torus-free protori $G, H$ and $\Delta_G \in L(G)$, $\Delta_H \in L(H)$ extends to an epimorphism $f: G \rightarrow H$.

Proof. The morphism $\varphi_G: \Delta_G \times \Sigma(G) \rightarrow G$ of the Resolution Theorem is an open map and $\mathbb{Z}_{\Delta_G} \cong_1 \exp_1 [\Delta] \cong_1 \ker \varphi_G$. Let $V \cong_k \mathbb{R}^k$, $0 \leq k \in \mathbb{Z}$, denote a real vector space satisfying $\Sigma(G) = \text{span}_\mathbb{R} (\exp_1 [\Delta]) \oplus V$. Then $G \cong \varphi_G (\Delta_G \times V) \cong_1 \Delta_G \times V$. The compactness of $G$ implies $k = 0$, so $\exp_1 [\mathbb{Z}_G] = \exp_1 [\Delta] \cong \Sigma(G)$.

Continuity of $f_\Delta$ with $f(\mathbb{Z}_{\Delta_G}) = \mathbb{Z}_{\Delta_H}$ ensures $f_\Delta$ is surjective and $\dim_{\mathbb{A}} \Sigma(G) = \text{rank} \mathbb{Z}_{\Delta_G} \geq \text{rank} \mathbb{Z}_{\Delta_H} = \dim_{\mathbb{A}} \Sigma(H)$. Define $f_R: \Sigma(G) \rightarrow \Sigma(H)$ by setting $f_R (\exp_1 (z)) = \exp_1 (f(z))$ for $z \in \mathbb{Z}_{\Delta_G}$ and extending $\mathbb{R}$-linearly. Then $f_\Delta \times f_R: \Delta_G \times \Sigma(G) \rightarrow \Delta_H \times \Sigma(H)$ is an epimorphism with $(f_\Delta \times f_R)(\Gamma_G) = \Gamma_H$, so $f_\Delta \times f_R$ induces an epimorphism $\tilde{f}: \hat{\Delta}_G \times \Sigma(G) \rightarrow \hat{\Delta}_H \times \Sigma(H)$ and $\tilde{f}$ in turn induces an epimorphism of protori $f: G \rightarrow H$ with $f|_{\Delta_G} = f_\Delta$. □

A projective resolution of a protorus $G = G_0$ is an exact sequence $K \rightarrow P \rightarrow G$ where $P$ is a torsion-free protorus and $K$ is a torsion-free profinite group: [4] Definitions 8.80.

Corollary 5.2. A protorus has a projective resolution.

Proof. Let $G$ be a protorus and set $r = \dim G$. By the Resolution Theorem, $G$ has a profinite subgroup inducing a torus quotient, which we can take without loss of generality to be $\Delta(\mathbf{n})$ for some $\mathbf{n} \in \mathbb{S}^m$, $m = \text{width}_{\mathbb{A}} \Delta(\mathbf{n})$. Identifying $\mathbb{Z}^r$ in the natural way as a subgroup of $\hat{\mathbb{Z}}^r$, an isomorphism of free abelian groups $\mathbb{Z}^r \rightarrow \mathbb{Z}_{\Delta(\mathbf{n})}$ extends by continuity to an isomorphism $f_\Delta: \hat{\mathbb{Z}}^r \rightarrow \Delta(\mathbf{n})$, thus inducing an exact sequence $K \rightarrow \hat{\mathbb{Z}}^r \rightarrow \Delta(\mathbf{n})$ where $K$ is torsion-free profinite. We have $\text{span}(\mathbb{Z}^r)/\text{span}(\mathbb{Z}^r) \cong \mathbb{A}(\mathbb{Q} \otimes \mathbb{G}^v)^\vee$. By Lemma 5.1, $f_\Delta$ induces a projective resolution $K \rightarrow \mathbb{Z}^r \rightarrow \mathbb{Z}(P(G))/\Gamma P(G) \cong [\Delta(\mathbf{n}) \times \Sigma(G)]/\Gamma G$. □

A completely decomposable group is a torsion-free abelian group isomorphic to the dual of a completely factorable protorus. An almost completely decomposable (ACD) group is a torsion-free abelian group quasi-isomorphic to a completely decomposable group.

Corollary 5.3. If $G$ is a protorus with $\dim G = \dim_{\mathbb{A}} G$, then $G^v$ is an ACD group.

Proof. Let $\Delta_G \in L(G)$. Multiplying $\Delta_G$ by sufficiently large $N \in \mathbb{Z}$ effects a sufficient generation that $\text{width}_{\mathbb{A}} \Delta_G = \dim_{\mathbb{A}} N \Delta_G$. Since $NG = G$, we can assume without loss of generality that $\text{width}_{\mathbb{A}} \Delta_G = \dim_{\mathbb{A}} \Delta_G = \dim G$. Let $\Delta_H$ denote the standard representation $\Delta_G$ and $\psi: \Delta_G \rightarrow \Delta_H$ an isomorphism with $\psi(\mathbb{Z}_{\Delta_G}) = \mathbb{Z}_{\Delta_H} \cong \mathbb{Z}_e \oplus \cdots \oplus \mathbb{Z}_{\dim G}$ where $\{e_1, \ldots, e_{\dim G}\}$ is the standard basis of $\Delta_H$ as a $\mathbb{Z}$-module. By Corollary 4.9 there is a completely factorable protorus $H$ with $\dim H = \dim G$ and $\Delta_H \in L(H)$. By Lemma 5.1 there is an epimorphism...
Lemma 3.1. The natural map \( \Delta \) is a continuous epimorphism, so \( \Delta + D \) is surjective, so \( \Delta + D \subseteq \Delta_G \). We conclude that \( \Delta_G = \bigcup \{ D : D \) a profinite subgroup of \( G \} \); similarly for \( \Delta_H \). In particular, \( \Delta_G \) contains all profinite subgroups of \( G \); similarly for \( \Delta_H \).

Let \( f \) denote a morphism \( G \to H \). If \( \Delta \in L(G) \), then \( \Delta + D \) is profinite because \( \Delta/K \cong \dim \Delta \) is profinite. Thus, \( \dim \Delta \) is a lifting of \( \Delta \). It follows from Proposition 5.6 because \( \Delta \subseteq \Delta_G \). Hence, \( \Delta \subseteq \Delta_H \). Consequently, \( \Delta \subseteq \Delta_H \).

**Remark 5.4.** \( \mathcal{L} \) is a functor between the categories of topological abelian groups and real topological vector spaces [4, Corollary 7.37]: for a morphism \( f : G \to H \) of topological abelian groups, the map \( \mathcal{L}(f) : \mathcal{L}(G) \to \mathcal{L}(H) \) given by \( \mathcal{L}(f)(r) = f \circ \exp_r \) is a morphism of real topological vector spaces satisfying \( \exp_H \circ \mathcal{L}(f) = f \circ \exp_G \).

**Proposition 5.5.** A morphism \( G \to H \) between torus-free protors restricts to maps between subgroups \( \Delta_G \to \Delta_H \), \( \exp_G \mathcal{L}(G) \to \exp_H \mathcal{L}(H) \), and \( X_G \to X_H \).

**Proof.** Let \( D \) be a profinite subgroup of \( G \). If \( \Delta \in L(G) \), then \( \Delta + D \) is profinite because the intersection of any two elements of \( L(G) \) is an element of \( L(H) \) with finite index. Hence, \( \Delta + D \subseteq \Delta_G \). Similarly, \( \Delta_H \). Therefore, \( \Delta \subseteq \Delta_G \). Consequently, \( \Delta \subseteq \Delta_H \).

**Proposition 5.6.** For a morphism \( f : G \to H \) of torus-free protors there exist \( \Delta_G \in L(G) \), \( \Delta_H \in L(H) \) such that \( f \) lifts to a product map \( f|_{\Delta_G} \times \mathcal{L}(f) : \Delta_G \times \mathcal{L}(G) \to \Delta_H \times \mathcal{L}(H) \).

**Proof.** Let \( \Delta_G \in L(G) \). By Proposition 5.5, \( f(\Delta_G) \subseteq \Delta_H \). By Theorem 4.1, \( \Delta_H = \bigcup_{\Delta \in L(H)} \Delta \).

Each \( \Delta \in L(H) \) is open in \( \Delta_H \) because the intersection of any two elements of \( L(H) \) is an element of \( L(H) \) with finite index. Hence, \( \Delta \subseteq \Delta_H \). Consequently, \( \Delta \subseteq \Delta_H \).

**Theorem 5.7.** (Structure Theorem for Morphisms) A morphism \( f : G \to H \) of torus-free protors restricts to a product map \( f|_{\Delta_G} \times \exp_G \mathcal{L}(f) : \Delta_G \times \exp_G \mathcal{L}(G) \to \Delta_H \times \exp_H \mathcal{L}(H) \).

**Proof.** This follows from Proposition 5.6 because \( \Delta_G = \bigcup_{\Delta \in L(G)} \Delta \).

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