Classification of BPS instantons in $\mathcal{N}=4$ $D=4$ supergravity

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Abstract. This talk is based on the recent work in collaboration with M. Azreg-Aïnou and G. Clément [1] devoted to extremal instantons in the one-vector truncation of the Euclidean $\mathcal{N}=4$, $D=4$ theory. Extremal solutions satisfying the no-force condition can be associated with null geodesic curves in the homogeneous target space of the three-dimensional sigma model arising in toroidal reduction of the four-dimensional theory. Here we (preliminarily) discuss the case of two vector fields sufficient to find all relevant metrics in the full $\mathcal{N}=4$, $D=4$ theory. Classification of instanton solutions is given along the following lines.

1. Introduction

Instantons in vacuum Einstein gravity were subject of intense investigations since the late seventies, which culminated in their complete topological classification [2]. Instantons in extended supergravities are non-vacuum and typically involve multiplets of scalar and vector fields in four dimensions and form fields in higher dimensions. Non-vacuum axionic gravitational instantons attracted attention in the late eighties in connection with the idea of multiverse mechanism of fixing the physical constants [3]. These are particular solutions of the present theory whose bosonic sector is frequently termed as the Einstein-Maxwell-dilaton-axion (EMDA) model. All extremal instanton solutions in the one-vector EMDA theory were found recently in [1]. Here we make some steps toward classification of extremal instantons in the full $\mathcal{N}=4$, $D=4$ theory.

Supersymmetric solutions to the Lorentzian $\mathcal{N}=4$ supergravity were classified solving the Killing spinor equations in [4, 5]. An alternative technique to classify BPS solutions relates to classification of null geodesics of the target space of the sigma model obtained via dimensional reduction of the theory along the Killing coordinate. The method was suggested in the context of the five-dimensional Kaluza-Klein (KK) theory by G. Clement [6], building on the interpretation by Neugebauer and Kramer [7] of solutions depending on a single potential as geodesics of the three-dimensional target space. It was further applied in [8] to classify Lorentzian extremal solutions of the EMDA theory and Einstein-Maxwell (EM) theory. In two of these three cases (KK and EMDA) it was found that the matrix generators $B$ of null geodesics split into a nilpotent class ($B^n = 0$ starting with certain integer $n$), in which cases the matrix is degenerate ($\det B = 0$), and a non-degenerate class ($\det B \neq 0$). The solutions belonging to the first class are regular, while the second class solutions, though still satisfy the no-force constraint on asymptotic charges, generically contain singularities. More recently similar approach partially overlapping with the present one was suggested as the method of nilpotent orbits [9]. The latter
starts with some matrix condition following from supersymmetry, which is generically stronger than our condition selecting the null geodesic subspace of the target space. In the minimal $N=2$ theory all null geodesics are nilpotent orbits, corresponding to the Israel-Wilson-Perjés solutions [8], while in the $N=4$ case there are null geodesics whose generators are not nilpotent. These correspond to solutions satisfying the no-force condition, but presumably not supersymmetric.

2. The setup

Bosonic sector of $\mathcal{N}=4$, $D=4$ supergravity contains

$$g_{\mu\nu}, \phi \ [\text{dilaton}], \kappa \ [\text{axion}], \text{six vector fields } A_{\mu}. \quad (2.1)$$

The theory has global symmetries: S-duality $SL(2,R)$ and $O(6)$, rotating the vector fields. The scalar fields parametrize the coset $SL(2,R)/SO(2)$.

2.1. Euclidean action

Correct choice of the Euclidean action for the axion field follows from the positivity requirement and amounts to starting with the three-form field:

$$S_0 = \frac{1}{16\pi} \int_{\mathcal{M}} \left( -R + 1 + 2d\phi \wedge *d\phi + 2e^{-4\phi}H \wedge *H + 2e^{-2\phi}F_n \wedge *F_n \right) - \frac{1}{8\pi} \int_{\partial\mathcal{M}} e^{\psi/2}K \wedge d\Phi, \quad (2.2)$$

where $F_n = dA_n$ are the Maxwell two-forms and the sum over $n$ from one to six is understood. $H$ is the three-form field strength related to the two-form potential $B$ via the relation involving the Chern-Simons term: $H = dB - A_n \wedge F_n$. The boundary $\partial\mathcal{M}$ of $\mathcal{M}$ is described by $\Phi(x^\mu) \equiv 0$, while $e^{\psi/2}$ is a scale factor ensuring $e^{\psi/2}d\Phi$ measures the proper distance in a direction normal to $\partial\mathcal{M}$, and $K$ is the trace of the extrinsic curvature of $\partial\mathcal{M}$ in $\mathcal{M}$.

To pass to the pseudoscalar axion one has to ensure the Bianchi identity for $H$: $ddB = d(H + A \wedge F) = 0$ which is effected adding to the action (2.2) a new term with the Lagrange multiplier $\kappa$

$$S_\kappa = \frac{1}{8\pi} \int_{\mathcal{M}'} \kappa (dH + A_n \wedge F_n) = \frac{1}{8\pi} \int_{\mathcal{M}'} \kappa (dH + F_n \wedge F_n), \quad (2.3)$$

where $\mathcal{M}'$ is $\mathcal{M}$ with the monopole sources of $H$ cut out. Integrating out the three-form $H$ we obtain the bulk action in terms of the pseudoscalar axion

$$S_E = \frac{1}{16\pi} \int_{\mathcal{M}} \left( -R + 1 + 2d\phi \wedge *d\phi - \frac{1}{2} e^{4\phi}d\kappa \wedge *d\kappa + 2e^{-2\phi}F \wedge *F + 2\kappa F \wedge F \right) \quad (2.4)$$

plus the boundary term. Combining the latter with the gravitational boundary term we get

$$4S_b = \frac{1}{16\pi} \int_{\partial\mathcal{M}'} [\kappa e^{4\phi} \wedge *d\kappa] - \frac{1}{8\pi} \int_{\partial\mathcal{M}} e^{\psi/2}[K] \wedge d\Phi, \quad (2.5)$$

where the pull-back of the three-form $*d\kappa$ onto the boundary $\partial\mathcal{M}$ is understood. Square brackets denote the background subtraction which is necessary to make the action finite. Note that the bulk matter action in the form (2.4) is not positive definite in contrast to (2.2): the difference is hidden in the boundary term.
2.2. 3D sigma-model

To develop generating technique for instantons we apply dimensional reduction to three dimensions, where the equations of motion are equivalent to those of the sigma model on the homogeneous space of the three-dimensional U-duality group. The derivation of the sigma model in the case $p = 1$ was first given in [10] and generalized to arbitrary $p$ in [11]. This leads to the homogeneous space of the group $SO(2, p + 2)$. In the particular case $p = 2$ the coset has a simpler representation $G/H = SU(2, 2)/\{SO(1, 3) \times SO(1, 1)\}$ [12] due to isomorphism $SO(2, 4) \sim SU(2, 2)$.

We parametrize the four-dimensional metric as

$$\text{d}s^2 = g_{\mu \nu} \text{d}x^\mu \text{d}x^\nu = f(\text{d}t - \omega_0 \text{d}x^i)^2 + \frac{1}{f} h_{ij} \text{d}x^i \text{d}x^j,$$  \hspace{1cm} (2.6)

where where $t$ is an Euclidean coordinate with period $\beta$, and $f$, $\omega_0$, $h_{ij}$ are functions of $x^i$ ($i = 1, 2, 3$). Occasionally we will also use an exponential parametrization of the scale factor $f = e^{-\chi}$. To be able to compute the on-shell instanton action one has to keep all boundary terms [1] which were neglected in [10, 11, 12].

The Maxwell fields are parameterized by the electric $v_n$ and magnetic $u_n$ potentials partly solving the equations of motion

$$F_{i4} = \frac{1}{\sqrt{2}} \partial_i v, \hspace{1cm} (2.7)$$

$$e^{-2\phi} F^{ij} - \kappa F^{ij} = \frac{f}{\sqrt{2h}} e^{ijkl} \partial_k u, \hspace{1cm} (2.8)$$

where the index $n$ labeling different vector fields is omitted. The rotation one-form $\omega$ in the metric is dualized to the NUT potential $\chi$:

$$\partial_i \chi + v \partial_i u - u \partial_i v = -f^2 h_{ij} \frac{\epsilon^{jkl}}{\sqrt{h}} \partial_k \omega_l \hspace{1cm} (2.9)$$

(we define $\epsilon_{1234} = +1$). The resulting full bulk action is that of the gravity-coupled three-dimensional sigma model

$$S_\sigma = -\frac{\beta}{16\pi} \int \text{d}^3 x \sqrt{h} \left(\mathcal{R} - G_{AB} \partial_i X^A \partial_j X^B \delta^{ij}\right), \hspace{1cm} (2.10)$$

where the target space variables are $X = (f, \phi, v_n, \chi, u_n)$, integration is over the three-space $\mathcal{E}$ and the target space metric $\text{d}l^2 = G_{AB} \text{d}X^A \text{d}X^B$ reads

$$\text{d}l^2 = \frac{1}{2} f^{-2} \text{d}f^2 - \frac{1}{2} f^{-2} (\text{d}f v_n + v_n \text{d}u_n - u_n \text{d}v_n)^2 + f^{-1} e^{-2\phi} \text{d}v_n^2 - f^{-1} e^{2\phi} (\text{d}u_n - \kappa \text{d}v_n)^2 + 2 \text{d}\phi^2 - \frac{1}{2} e^{4\phi} \text{d}\chi^2. \hspace{1cm} (2.11)$$

This space has the isometry group $G = SO(2, p + 2)$, the same as its Lorentzian counterpart [11]. The metric (2.11) is the metric on the coset $G/H$, whose nature can be uncovered from a signature argument. The Killing metric of $so(2, p + 2)$ algebra has the signature $(+2(p + 1), -(p^2 + 3p + 4)/2)$, with plus standing for non-compact and minus for compact generators. Since the signature of the target space is $(+(p + 2), -(p + 2))$, it is clear that the isotropy subalgebra must contain $p + 2$ non-compact and $p(p + 1)/2$ compact generators. Such a subalgebra of $so(2, p + 2)$ is lie $(H) \sim so(1, p + 1) \times so(1, 1)$. We therefore deal with the coset $SO(2, p + 2)/\{SO(1, p + 1) \times SO(1, 1)\}$. In the $p = 2$ case this is isomorphic to $SU(2, 2)/\{SO(1, 1) \times SO(1, 1)\}$. As was argued in [8], the maximal number of independent harmonic functions with unequal charges is equal to the number of independent isotropic directions in the target space. Here the target space has $p + 2$ positive and $p + 2$ negative direction, thus the number of independent null vectors is $p + 2$. 

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In addition to the bulk action we have a number of surface terms resulting from three-dimensional dualizations as well as from dimensional reduction of the four-dimensional Gibbons-Hawking-axion term. Collecting these together, and taking care of the rescaling of the electric and magnetic potentials, we get:

\[ S_{\text{inst}} = \frac{3}{16\pi} \int \frac{1}{\delta} \left( -2 |k| * d\Psi + * d\varphi(\theta) + \frac{1}{16\pi} \left( \left[ \kappa e^{4\phi} * d\kappa \right] + 2\sqrt{2} u_a F_a + (\chi + u_a v_a) \right) \right). \] (2.12)

Note that the on-shell value of the action which we are interested in for instantons is entirely given by the boundary term \(3S_b\) since the bulk sigma-model action vanishes by virtue of the contracted three-dimensional Einstein equations.

Variation of the bulk action (2.10) over \(X^A\) gives the equations of motion

\[ \partial_i \left( \sqrt{\hbar} h^{ij} G_{AB} \partial_j X^B \right) = \frac{1}{2} G_{BC,A} \partial_j X^B \partial_j X^C \hbar^{ij} \sqrt{\hbar}, \] (2.13)

which can be rewritten in a form explicitly covariant both with respect to the three-space metric \(h_{ij}\), and to the target space metric \(G_{AB}\)

\[ \nabla_i J^i_A = 0, \] (2.14)

where \(\nabla_i\) is the total covariant derivative involving Christoffel symbols both of \(h_{ij}\) and \(G_{AB}\). The currents associated with the potentials read

\[ J_A = h^{ij} \partial_j X^B G_{AB}. \] (2.15)

2.3. Geodesic solutions

Neugebauer and Kramer [7], considering Kaluza-Klein theory, noticed that, if the target space coordinates \(X^A\) depend on \(x^i\) through the only scalar function, \(X^A = X^A(\tau(x^i))\), the geodesic curves of the target space

\[ \frac{d^2X^A}{d\tau^2} + \Gamma^A_{BC} \frac{dX^B}{d\tau} \frac{dX^C}{d\tau} = 0, \] (2.16)

where \(\Gamma^A_{BC}\) are Christoffel symbols of the metric \(G_{AB}\). solve the sigma-model equations of motion, provided \(\tau(x^i)\) is a harmonic function in three-space with the metric \(h_{ij}\):

\[ \Delta \tau = \frac{1}{\sqrt{\hbar}} \partial_i \left( \sqrt{\hbar} h^{ij} \partial_j \tau \right) = 0. \] (2.17)

Therefore certain classes of solutions can be associated with geodesics surfaces in the target space. Note that no assumptions were made here about the metric of the three-space, which is generically curved.

3. Matrix representation

In view of the rotational symmetry in the space of vectors, to get all different metrics it is not sufficient to consider only one, but it is enough to consider two vector fields. Indeed, as we will see later, solutions can be labeled by asymptotic electric \(Q_n\) and magnetic \(P_n\) charges, the metric being dependent on the three invariants \(Q^2 = Q_n Q_n\), \(P^2 = P_n P_n\) and \(QP = Q_n P_n\). In one-vector case \(QP = 0\) implies that either \(Q^2 = 0\) or \(P^2 = 0\). To have the third invariant \(QP\) independent of the first two, it is enough, however, to take \(p = 2\); then, e.g., for \(Q_1 \neq 0\), \(Q_2 = 0\), \(P_1 = 0\), \(P_2 \neq 0\) one has \(QP = 0\) but \(Q^2 \neq 0\), \(P^2 \neq 0\). Using rotation in the space of vector fields, one can can always choose this configuration as a representative of a general one. So in what follows we will consider the case \(p = 2\).

To proceed, we have to introduce the matrix representation of the coset \(SU(2,2)/(SO(1,3 + 1) \times SO(1,1))\). In the Lorentz case the corresponding coset is \(SU(2,2)/(SO(2,2) \times SO(2))\), its representation
in terms of the complex matrices $4 \times 4$ was given in [12]. The analogous representation of the Euclidean coset $G/H = SU(2,2)/(SO(1,3) \times SO(1,1))$ is given by the hermitian block matrix

\[ M = \begin{pmatrix} P^{-1} & P^{-1}Q \\ QP^{-1} & -P + QP^{-1} \end{pmatrix}, \]

with $2 \times 2$ hermitian blocks

\[ P = e^{-2\phi} \begin{pmatrix} f e^{2\phi} + v_n^2 & v_1 - iv_2 \\ v_1 + iv_2 & 1 \end{pmatrix}, \quad Q = \begin{pmatrix} v_n w_n - \chi & w_1 - iw_2 \\ w_1 + iw_2 & -\kappa \end{pmatrix}, \]

where $w_n = u_n - \kappa v_n$. In terms of these matrices the target space metric reads

\[ dl^2 = -\frac{1}{4} \text{tr} (dM dM^{-1}) = \frac{1}{2} [(p^{-1}dp)^2 - (p^{-1}dQ)^2]. \]

To read off the target space potentials from the matrix $M$ it is enough to use its following two blocks:

\[ P^{-1} = f^{-1} \begin{pmatrix} 1 \\ -(v_1 + iv_2) \end{pmatrix}, \]

\[ P^{-1}Q = f^{-1} \begin{pmatrix} -\tilde{\chi} \\ \tilde{\chi}(v_1 + iv_2) + fe^{2\phi}(w_1 + iw_2) \end{pmatrix}, \]

where $\tilde{\chi} = \chi + iW$, $W = v_1 u_2 - v_2 u_1$.

### 3.1. Asymptotic conditions

We will be interested by finite action solution with vanishing target space potentials $v_n = u_n = \kappa = 0$ in the asymptotic region (specified as $r \to \infty$), while the NUT potential and the dilaton may be growing there. If the dilaton tends to a constant value at infinity, the asymptotic metrics can be classified as in pure gravity [2]. These are known to be of two types. The first includes asymptotically locally flat (ALF) solutions, with $f(\infty) = 1$, $\chi(\infty) = 0$ and the asymptotic form of the metric

\[ ds^2 = (dt - 2N \cos \theta d\phi)^2 + dr^2 + r^2(d\theta^2 + \sin^2 \theta d\phi^2), \]

where $N$ is the NUT parameter. Our coset matrix $M_{\alpha\alpha}$, corresponding to this solution reads

\[ M_{ALF} = \sigma_0 \equiv \sigma_3 \otimes \sigma_0 = \begin{pmatrix} \sigma_0 & 0 \\ 0 & -\sigma_0 \end{pmatrix}, \]

where (and in what follows) we use the notation $\sigma_{\mu \nu} = \sigma_\mu \otimes \sigma_\nu$, $\mu, \nu = 0, 1, 2, 3$ for the direct product of the matrices $\sigma_\mu = (1, \sigma_i)$, with $\sigma_i$ being the Pauli matrices.

The second class, with growing $f = \chi \sim r$ at infinity, corresponds to asymptotically locally Euclidean (ALE) solutions with the asymptotic metric

\[ ds^2 = d\rho^2 + \rho^2 d\Omega_3^2, \]

where the three-sphere is parametrized as

\[ d\Omega_3^2 = \frac{1}{4}[d\theta^2 + \sin^2 \theta d\phi^2 + (d\eta + \cos \theta d\phi)^2], \]

with the angular coordinate $\eta = t$, and the radial coordinate $\rho = (4r)^{1/2}$. In this case $M_{\alpha\alpha}$ is

\[ M_{ALE} = \frac{1}{2}(\sigma_{30} - \sigma_{33} - \sigma_{10} - \sigma_{13}). \]

Both these asymptotic solutions satisfy source-free Euclidean Einstein equation.

Two additional asymptotic types satisfy source-free Euclidean Einstein equation.

The ALF solutions are related to Lorentzian solutions with the linear dilaton asymptotic [13], while the ALE ones are dilaton-axion dressed Eguchi-Hanson and lens spaces [1].
3.2. Null geodesic solutions

In the matrix terms the sigma-model field equations (2.14) read

\[ \nabla (M^{-1} \nabla M) = 0, \]  

(3.27)

where \( \nabla \) stands for the three–dimensional covariant derivative, and the scalar product with respect to the metric \( h_{ij} \) is understood. The geodesic solutions then obey the equation

\[ \frac{d}{d\tau} \left( M^{-1} \frac{dM}{d\tau} \right) = 0, \]  

(3.28)

which is first integrated by

\[ M^{-1} \frac{dM}{d\tau} = B, \]  

(3.29)

where \( B \) is a constant matrix generator of the coset. A second integration leads to the solution to the geodesic equation in the exponential form

\[ M = M_\infty e^{B\tau}, \]  

(3.30)

where we assume that \( \tau(\infty) = 0 \).

The three–dimensional Einstein equations now read

\[ \mathcal{R}_{ij} = -\frac{1}{4} \text{tr} \left( \nabla_i M \nabla_j M^{-1} \right). \]  

(3.31)

The parametrisation (3.30) (3.31) to

\[ \mathcal{R}_{ij} = \frac{1}{4} (\text{tr}B^2) \nabla_i \tau \nabla_j \tau. \]  

(3.32)

From this expression it is clear that in the particular case

\[ \text{tr}B^2 = 0 \]  

(3.33)

the three–space is Ricci–flat. In three dimensions the Riemann tensor is then also zero, and consequently the three–space \( \mathcal{E} \) is flat. We shall assume in the following that \( \mathcal{E} = \mathbb{R}^3 \). From Eq. (3.20) one can see [6] that the condition (3.33) corresponds to null geodesics of the target space

\[ ds^2 = \frac{1}{4} (\text{tr}B^2) d\tau^2 = 0. \]  

(3.34)

In terms of the above notation for \( 4 \times 4 \) matrices the eight generators of the coset \( G/H = SU(2, 2)/(SO(1, 3 + 1) \times SO(1, 1)) \) can be chosen as

\[ B = \{ \text{lie}(G) \ominus \text{lie}(H) \} = \{ \sigma_3 \mu, i\sigma_2 \mu \}. \]  

(3.35)

3.3. Charge vectors

In the ALF case one assumes the following behavior of the target space variables at spatial infinity:

\[ f \sim 1 - \frac{2M}{r}, \quad \chi \sim \frac{-2N}{r}, \]
\[ \phi \sim \frac{D}{r}, \quad \kappa \sim \frac{2A}{r}, \]
\[ v_n \sim \frac{\sqrt{2}Q_n}{r}, \quad u_n \sim \frac{\sqrt{2}P_n}{r}. \]  

(3.36)
Then comparing (3.18, 3.19) with (3.23, 3.30) and using the basis (3.35), the matrix generator $B$ can be parametrized by two vectors $\mu^\alpha$, $\nu^\alpha$ in the four-dimensional flat space with the $SO(1,3)$ metric $\eta_{\mu \nu}$ of the signature $(-+++)$ as follows:

$$B(\mu, \nu) = \nu^0 \sigma_{00} + i \nu^i \sigma_{2i} + i \mu^0 \sigma_{20} + \mu^i \sigma_{3i}, \quad (3.37)$$

where explicitly

$$\mu^\alpha = \left( N - A, -\sqrt{2}Q_1, -\sqrt{2}Q_2, M - D \right), \quad \nu^\alpha = \left( M + D, \sqrt{2}P_1, \sqrt{2}P_2, N + A \right). \quad (3.38)$$

In the space of charges the $SO(1,3) \times SO(1,1)$ global symmetry is acting. Fixing the corresponding “Lorentz frame” one can simplify matrices describing physically distinct classes of the solutions.

4. Classification of null geodesics

Squaring the matrix (3.37) we obtain

$$B^2 = (\mu^2 - \nu^2) \sigma_{00} + 2 \sigma_{0i} (\nu^0 \mu^i - \mu^0 \nu^i) + 2 \epsilon_{ijk} \mu^i \nu^j \sigma_{1k}, \quad (4.39)$$

where $\mu^2 = \mu^\alpha \mu^\beta \eta_{\alpha \beta}$ etc. Its diagonal part is proportional to the difference of squared charge vectors, while the non-diagonal part is defined through their skew product.

4.1. No-force condition

To ensure the three-space to be flat, which presumably corresponds to BPS solutions, we must impose the vanishing trace condition on $B^2$, which in view of (4.39) reduces to the equality of the norms of two charge vectors $\mu^2 = \nu^2$, indeed

$$\text{tr} B^2 = 4(\mu^2 - \nu^2) = 0. \quad (4.40)$$

Substituting (3.38) we get the relation between the asymptotic charges

$$M^2 + D^2 + Q^2 = N^2 + A^2 + P^2, \quad (4.41)$$

where $Q^2 = Q_i Q_i$, $P^2 = P_i P_i$. This is the no-force condition in the Euclidean case, where the mass, the dilaton charge and the electric charges are attractive, while the NUT charge, the axion charge and the magnetic charge are repulsive.

4.2. Characteristic equation

Imposing the condition (4.41), we get for the third power of $B$

$$\frac{1}{2} B^3 = (\mu^2 \nu^0 - (\mu \nu) \mu^0) \sigma_{30} + ((\mu \nu) \nu^i - \nu^2 \mu^i) \sigma_{3i} - i (\nu^2 \mu^0 - (\mu \nu) \nu^0) \sigma_{20} - i ((\mu \nu) \mu^i - \mu^2 \nu^i) \sigma_{2i}, \quad (4.42)$$

where $(\mu \nu) = \mu^\alpha \nu_\alpha$. The forth power, again with (4.41), is

$$\frac{1}{4} B^4 = ((\mu \nu)^2 - \mu^2 \nu^2) \sigma_{00}. \quad (4.43)$$

It is easy to check that

$$\text{tr} B = 0, \quad \text{tr} B^3 = 0, \quad (4.44)$$

so, together with (4.41), one finds the following characteristic equation for the matrix $B$

$$B^4 + (\det B) I = 0, \quad (4.45)$$
consistently with $B^4$ being proportional to the unit $4 \times 4$ matrix. In view of (4.45), if the matrix $B$ is degenerate, $\det B = 0$, one has $B^4 = 0$, so the expansion of the exponential in (3.30) contains only the terms up to cubic. In the non-degenerate case the series is infinite. It turns out that in the latter case most of the solutions contain singularities and are not supersymmetric, so we do not discuss them here.

The degeneracy condition in terms of the charge vectors (restricted by the no-force condition) according to (4.43) is one of the two

\[ (M \pm N)(D \mp A) = (Q_n \mp P_n)^2, \]  

where the sum over $n$ is understood.

4.3. Strongly degenerate case

The rank of the degenerate $B$ can be either two or four. In the first case $B^2 = 0$ and the coset matrix $M$ is linear in terms of $B$:

\[ M = \eta (I + B\tau). \]  

According to (4.39), vanishing of $B^2$, apart from (4.41) imposes the following conditions on the charge vectors:

\[ \nu^i \mu^j - \mu^i \nu^j = 0, \quad \mu^i \nu^j - \mu^j \nu^i = 0, \]  

which are equivalent to vanishing of the bivector $\mu^\alpha \wedge \nu^\beta$.

This leads to different subclasses of solutions according to whether the charge vectors $\mu^\alpha, \nu^\alpha$ are time-like, space-like or null. Further details of classification of rank two solutions in the case $p = 1$ can be found in [1]. They include solutions of all mentioned above asymptotic types.

4.4. Weakly degenerate case

In the case or rank three all terms in the series expansion of $M$ up to the third are non-zero:

\[ M = \eta (I + B\tau + B^2 \tau^2 / 2 + B^3 \tau^3 / 6), \]  

while $B^4$ and higher terms vanish by virtue of the degeneracy condition $\det B = 0$. Now we have only two conditions on eight (for $p = 2$) charges: (4.41) and one of the two in (4.46). Again this case includes a variety of new ALF, ALE and dilatonic instantons.

4.5. Multiple harmonic functions

The construction (3.30) may be generalized [6, 8] to the case of several truly independent harmonic functions $\tau_a, \Delta \tau_a = 0$, by replacing the exponent in (3.30) by a linear superposition

\[ M = \lambda \exp \left( \sum_a B_a \tau_a \right). \]  

This solves the field equations (3.27) provided that the commutators $[B_a, B_b]$ commute with the $B_c$ (for the proof see [8]):

\[ [ [B_a, B_b], B_c] = 0. \]  

The three-dimensional Einstein equations (3.31) generalize to

\[ R_{ij} = \frac{1}{4} \sum_a \sum_b \text{tr}(B_a B_b) \nabla_i \tau_a \nabla_j \tau_b, \]  

so that the three-space is Ricci flat if the matrices $B_a$ satisfy

\[ \text{tr}(B_a B_b) = 0. \]  

The number of independent harmonic functions on which an extremal solution of the form (4.50) may depend is limited by the number of independent mutually orthogonal null vectors of the target space. In the present case of Euclidean EMDA with two vector fields this number is four. This gives a number of solutions, whose explicit form (in the case $p = 1$) can be found in [1].
5. Concluding remarks
We described general structure of the space of extremal instantons in N=4 D=4 supergravity as null geodesics of the coset \( G/H = SU(2,2)/\langle SO(1,3) \times SO(1,1) \rangle \). A number of particular \( p = 1 \) new solutions was given in [1]. Apart from some simple extremal solutions, which were previously known explicitly in the purely scalar ALE sector [3], new scalar ALF and ALE were found, such as dilaton-axion dressed Taub-NUT, Eguchi-Hanson and lens-space instantons. There are also new types of wormholes interpolating between ALF or ALE and conical ALF spaces. All electrically and magnetically charged solutions are entirely new except for those which were (or could be) found by euclideanization of known Lorentzian black hole and/or IWP-type solutions, which were rederived in the general treatment as well. The new charged ALE solutions include, among others, purely electric solutions, as well as purely magnetic instantons with linear dilaton asymptotics.

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References
[1] Azreg-Aïnou M, Clément G and Gal’tsov D V 2011 “All extremal instantons in Einstein-Maxwell-dilaton-axion theory,” arXiv:1107.5746 [hep-th] to appear in Phys. Rev. D.
[2] Gibbons G W and Hawking S W 1977 Phys. Rev. D 15, 2752; S.W. Hawking S W 1977 Phys. Lett. A 60 81 ; Gibbons G W and Hawking S W 1979 Commun. Math. Phys. 66 291 .
[3] Giddings S B and Strominger A 1988 Nucl. Phys. B 306 890; 1989 Phys. Lett. B 230 46.
[4] Tod K P 1995 Class. Quant. Grav. 12 1801.
[5] Bellorin J and Ortín T 2005 Nucl. Phys. B 726 171; Meessen P, Ortin T, Vaula S 2010 JHEP 1011 072 (2010).
[6] Clément G 1986 Gen. Rel. and Grav. 18 861; Phys. Lett. A 118 11.
[7] Neugebauer G and Kramer D 1969 Ann. der Physik (Leipzig) 24 62.
[8] Clément G and Gal’tsov D V 1996 Phys. Rev. D 54 6136.
[9] Bossard G, Nicolai H, K. Stelle K S 2009 JHEP 0907, 003 (2009) [arXiv:0902.4438 [hep-th]]; Bossard G 2010 Gen. Rel. and Grav. 42, 539.
[10] Gal’tsov D V and Kechkin O V 1994 Phys. Rev. D 50 7394; 1995 Phys. Lett. B 361, 52 (1995); 1996 Phys. Rev. D 54 1656; Gal’tsov D V 1995 Phys. Rev. Lett. 74 2863.
[11] Gal’tsov D V and Letelier P S 1997 Phys. Rev. D 55 3580.
[12] Gal’tsov D V and Sharakin S A 1997 Phys. Lett. B 399 250.
[13] Clément G, Gal’tsov D V and Leygnac C 2003 Phys. Rev. D 67 024012 .