A study on finding a buried obstacle in a layered medium having the influence of the total reflection phenomena via the time domain enclosure method

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Abstract

An inverse obstacle problem for the wave governed by the wave equation in a two layered medium is considered under the framework of the time domain enclosure method. The wave is generated by an initial data supported on a closed ball in the upper half-space, and observed on the same ball over a finite time interval. The unknown obstacle is penetrable and embedded in the lower half-space. It is assumed that the propagation speed of the wave in the upper half-space is greater than that of the wave in the lower half-space, which is excluded in the previous study: Ikehata and Kawashita (2018) to appear, Inverse Problems and Imaging. In the present case, when the reflected waves from the obstacle enter the upper layer, the total reflection phenomena occur, which give singularities to the integral representation of the fundamental solution for the reduced transmission problem in the background medium. This fact makes the problem more complicated. However, it is shown that these waves do not have any influence on the leading profile of the indicator function of the time domain enclosure method.

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1 Introduction and the statement of the result

Continued on [10], we pursue further study on an inverse obstacle problem for the wave governed by a scalar wave equation in a two layered medium under the framework of the time domain enclosure method [5, 6, 7, 8, 9]. It is a mathematical formulation of a typical and important inverse obstacle problem and the solution may give us a hint to treat other inverse obstacle problems using electromagnetic waves, e.g., those coming from application to subsurface radar, ground probing radar [3] and through-wall imaging [1].

In [10] it is assumed that the unknown obstacle is penetrable and embedded in the lower half-space, and that the propagation speed of the wave in the upper half-space is less than that of the wave in the lower half-space. The wave is generated by an initial data supported on an open ball in the upper half-space and observed on the same ball over a finite time interval. It is shown that one can extract the optical distance from the ball to the obstacle and its qualitative property from the leading profile of the indicator function, which can be computed by using the wave observed over a finite time interval.

When the propagation speed of the wave in the upper half-space is greater than that of the wave in the lower half-space, the total reflection phenomena of the reflected wave by the obstacle may occur and complicate the problem more. The purpose of this article is to show that the leading profile of the indicator function is the same as the case treated in [10].

Let $0 < T < \infty$. Given $f \in L^2(\mathbb{R}^3)$ let $u = u(x,t)$ be the solution of the following initial value problem:

$$\begin{cases}
(\partial_t^2 - \nabla \cdot \gamma \nabla) u = 0 & \text{in } (0, T) \times \mathbb{R}^3, \\
u(0, x) = 0, \quad \partial_t u(0, x) = f(x) & \text{on } \mathbb{R}^3,
\end{cases}
$$

(1.1)

where $\gamma = \gamma(x) = (\gamma_{ij}(x))$ satisfies

$\bullet$ for each $i, j = 1, 2, 3 \gamma_{ij}(x) = \gamma_{ji}(x) \in L^{\infty}(\mathbb{R}^3)$;

$\bullet$ there exists a positive constant $C$ such that $\gamma(x) \xi \cdot \xi \geq C|\xi|^2$ for all $\xi \in \mathbb{R}^3$ and a.e. $x \in \mathbb{R}^3$.

As given in [4] (see e.g. Theorem 1 on p. 558 of [4]), for $f \in L^2(\mathbb{R}^3)$, there exists a unique $u \in L^2(0, T; H^1(\mathbb{R}^3))$ with $\partial_t u \in L^2(0, T; H^1(\mathbb{R}^3))$, $\partial_t^2 u \in L^2(0, T; (H^1(\mathbb{R}^3))')$, such that for all $\phi \in H^1(\mathbb{R}^3)$, $u$ satisfies

$$\langle \partial_t^2 u(t, \cdot), \phi \rangle + \int_{\mathbb{R}^3} \gamma(x) \nabla_x u(t, x) \cdot \nabla_x \phi(x) dx = 0 \quad \text{a.e. } t \in (0, T)$$

and $u(0, x) = 0$ and $\partial_t u(0, x) = f(x)$. This function $u$ is called the (weak) solution of $u$ of (1.1).

As a background medium we choose the whole space $\mathbb{R}^3$ and divide the space into two homogeneous and isotropic media:

$$\mathbb{R}^3 = \mathbb{R}^+_3 \cup \mathbb{R}^-_3,$$
where $\mathbb{R}_\pm^3 = \{ x = (x_1, x_2, x_3) \in \mathbb{R}^3 \mid \pm x_3 > 0 \}$. The propagation speed of the wave in $\mathbb{R}_\pm^3$ is given by $\sqrt{\gamma_\pm}$, where $\gamma_\pm > 0$ are constants. We call $\mathbb{R}_+^3$ (resp. $\mathbb{R}_-^3$) the upper (resp. lower) side of the flat transmission boundary $\partial \mathbb{R}_\pm^3$.

Now we specify the form of $\gamma$ in (1.1). Let $D$ be a bounded open set with $C^2$ boundaries satisfying $\overline{D} \subset \mathbb{R}_-^3$. We assume that $\gamma$ takes the form

$$
\gamma(x) = \begin{cases} 
\gamma_0(x) I_3, & \text{if } x \in \mathbb{R}^3 \setminus D, \\
\gamma_0(x) I_3 + h(x), & \text{if } x \in D,
\end{cases}
$$

where $\gamma_0(x) = \gamma_\pm$ for $\pm x_3 > 0$ and $h(x) = (h_{ij}(x)) \in L^\infty(D)$.

Note that $D$ is a mathematical model of a penetrable obstacle (inclusion) embedded in the lower half-space. We introduce a jump condition of $\gamma(x)$ from $\gamma_0(x)I$ on $D$:

$$
(A)_\pm \quad \begin{cases} 
\text{there exists a positive constant } C' \text{ such that} \\
\pm h(x) \xi \cdot \xi \geq C' \abs{\xi}^2 \quad (\xi \in \mathbb{R}^3 \text{ and a.e. } x \in D).
\end{cases}
$$

We consider the following problem:

**Problem.** Fix a large $T$ (to be determined later). Assume that $\gamma_0$ is known, $\gamma_+ \neq \gamma_-$ and that both $D$ and $h$ are unknown. Let $B$ be an open ball with $\overline{B} \subset \mathbb{R}_+^3$. Fix some $f \in L^2(\mathbb{R}^3)$ satisfying $\text{supp } f \subset \overline{B}$ and $\text{ess.inf}_{x \in B} f(x) > 0$ (or $-\text{ess.inf}_{x \in B} f(x) > 0$). Extract information about the location and shape of $D$ from the measured data $u$ on $B$ over the time interval $(0, T)$, where $u$ is the weak solution of (1.1) for the above $f$.

Note that the problem asks us to extract some information about the unknown obstacle from a single observed wave over a finite time interval. The place where the wave
is observed is the same as the generating place of the wave. This is a near field version of the inverse backscattering problem in the time domain and different from the studies in [2, 11, 12] where the time harmonic reduced case in a two layered medium have been treated.

Note also that the case where $\gamma_+ = \gamma_-$ has been considered in [6] and applying an idea in [7] to this case, one can extract the distance of the ball $B$ to the obstacle $D$, that is $\text{dist}(D,B) = \inf_{x \in D, y \in B} |x - y|$. Moreover, a similar inverse obstacle problem for the wave governed by the equation $(\alpha(x) \partial_t^2 - \Delta) u = 0$ has been considered in [8] for a general inhomogeneous background medium. In this case lower and upper estimates of $\text{dist}(D,B)$ are given.

To describe a solution to the present problem we recall the definition of the optical distance between the ball $B$ and obstacle $D$ given by

$$l(D,B) = \inf_{x \in D, y \in B} l(x,y),$$

where

$$l(x,y) = \inf_{z' \in \mathbb{R}^2} l_{x,y}(z'),$$

$$l_{x,y}(z') = \frac{1}{\sqrt{\gamma_-}} |z' - x| + \frac{1}{\sqrt{\gamma_+}} |z' - y| \quad (\tilde{z}' = (z_1, z_2, 0), z' = (z_1, z_2)). \quad (1.2)$$

As is in Lemma 4.1 of [10], for arbitrary $x$ and $y \in \mathbb{R}^3$ with $x_3 < 0$ and $y_3 > 0$, there exists the unique point $z'(x,y) \in \mathbb{R}^2$ satisfying $l(x,y) = l_{x,y}(z'(x,y))$, and the point $z'(x,y)$ is on the line segment $x'y'$ and $C^\infty$ for $x$ and $y \in \mathbb{R}^3$ with $x_3 < 0$ and $y_3 > 0$.

Recall the indicator function given in [10]:

$$I_f(\tau,T) = \int_{\mathbb{R}^3} f(x)(w(x,\tau) - v(x,\tau))dx,$$

where

$$w(x,\tau) = \int_0^T e^{-\tau t} u(t,x)dt \quad (x \in \mathbb{R}^3)$$

and $v \in H^1(\mathbb{R}^3)$ is the weak solution of

$$(\nabla \cdot \gamma_0 \nabla - \tau^2)v(x,\tau) + f(x) = 0 \quad \text{in } \mathbb{R}^3.$$

Our main result is the following theorem:

**Theorem 1.1** Assume that $\gamma_+ > \gamma_-$, then we have;

$$\lim_{\tau \to \infty} e^{\tau T} I_f(\tau,T) = \left\{ \begin{array}{ll} 0, & \text{if } T < 2l(D,B), \\
-\infty, & \text{if } T > 2l(D,B) \text{ and } \gamma \text{ satisfies } (A)_{\pm}. \end{array} \right.$$  

Moreover, if $\gamma$ satisfies $(A)_{\pm}$, then for all $T > 2l(D,B)$

$$\lim_{\tau \to \infty} \frac{1}{\tau} \log |I_f(\tau,T)| = -2l(D,B). \quad (1.4)$$
Note that we have obtained the same result as [10]. Our studies have completely covered the case $\gamma_+ \neq \gamma_-$. In the case $\gamma_+ < \gamma_-$, the incident waves from the lower half-space do not cause the total reflection. On the other hand, in the present case $\gamma_+ > \gamma_-$ those waves cause the total reflection, which makes the problems more complicated than that of [10]. Theorem 1.1 shows that the total reflection phenomena do not have any influence on the leading profile of the indicator function $I_f(\tau, T)$ as $\tau \to \infty$.

Theorem 1.1 says that we need to take $T > 2l(D, B)$ at least if we wish to know information of $D$ from the indicator function. We think this restriction is optimal and consistent with wave phenomena, since we should wait to the signals going and coming back to the points taking measurements. We can also know whether the propagation speed of wave in the inclusion is greater or less than the speed of wave in the background medium by checking the asymptotic behavior of $e^{\tau T}I_f(\tau, T)$ as $\tau \to \infty$. From the formula (1.4), we can compute the value $l(D, B)$. Moreover, as pointed out in [10] we have

$$D \subset E(D; B, \gamma_+, \gamma_-),$$

where

$$E(D; B, \gamma_+, \gamma_-) = \left\{ x \in \mathbb{R}^3_+ \mid l(x, p) > l(D, B) + \frac{\eta}{\sqrt{\gamma_+}} \right\}$$

and $p$ and $\eta$ are the center point and radius of $B$, respectively. Note that the set $E(D; B, \gamma_+, \gamma_-)$ can be determined by the computed value of $l(D, B)$, $\eta$ and $\sqrt{\gamma_+}$. This means that the one shot yields one information about the geometry of $D$.

The proof of Theorem 1.1 proceeds along the same lines as the case $\gamma_+ < \gamma_-$ in [10]. The indicator function has the well known estimates below (see Lemma 1.2 in [10]).

**Lemma 1.2** We have, as $\tau \to \infty$

$$I_f(\tau, T) \geq \int_{\mathbb{R}^3} (\gamma_0 I_3 - \gamma) \nabla v \cdot \nabla vdx + O(\tau^{-1}e^{-\tau T}) \quad (1.5)$$

and

$$I_f(\tau, T) \leq \int_{\mathbb{R}^3} \gamma_0(\gamma_0 I_3 - \gamma)\gamma^{-1/2}\nabla v \cdot \gamma^{-1/2}\nabla vdx + O(\tau^{-1}e^{-\tau T}). \quad (1.6)$$

From (1.5) and (A)$_-$ (resp. (1.6) and (A)$_+$), we see that Theorem 1.1 immediately follows from the following estimates for $v$.

**Theorem 1.3** Assume that $\partial D$ is $C^1$ and that $\gamma_+ > \gamma_-$. Then, there exist positive numbers $C$ and $\tau_0$ such that, for all $\tau \geq \tau_0$ we have

$$C^{-1} \tau^{-4}e^{-2\tau l(D, B)} \leq \int_D |\nabla v(x)|^2 dx \leq C\tau^2 e^{-2\tau l(D, B)}.$$

Note that Theorem 1.3 in which the assumption $\gamma_+ > \gamma_-$ is replaced with $\gamma_+ < \gamma_-$ has been established in [10].
Thus everything is reduced to showing the validity of Theorem 1.3. For the purpose we make use of the expression

$$v(x) = \int_B \Phi_\tau(x, y) f(y) dy,$$

where $\Phi_\tau(x, y)$ is governed by

$$\nabla_x \cdot (\gamma_0(x) \nabla_x \Phi_\tau(x, y)) - \tau^2 \Phi_\tau(x, y) + \delta(x - y) = 0 \quad \text{in } \mathbb{R}^3.$$

Since it follows that

$$\int_D |\nabla_x v(x)|^2 dx = \int_B dy \int_B d\xi f(y) f(\xi) \int_D \nabla_x \Phi_\tau(x, y) \cdot \nabla_x \Phi_\tau(x, \xi) dx,$$

Theorem 1.3 is given by investigating an asymptotic behavior of $\nabla_x \Phi_\tau(x, y)$ as $\tau \to \infty$ for $x = (x', x_3)$ with $x_3 < 0$, $x' \in \mathbb{R}^2$ and $y \in B$.

In section 2, a complex integral representation of the fundamental solution $\Phi_\tau(x, y)$ is recalled, which is given in [10]. As in (2.1) in section 2, $\Phi_\tau(x, y)$ consists of the part corresponding to the incident wave and the refracted part $E^-_j(x, z')$ for $x \in \mathbb{R}^3$ with $x_3 < 0$ and $z' \in \mathbb{R}^2$. To obtain asymptotics for $\Phi_\tau(x, y)$, the steepest descent method is used for the integral representation of the refracted part. If $\gamma_+ < \gamma_-$, the integrand in the representation of the refracted part is holomorphic near the steepest descent curve. Hence, we can perform asymptotic expansion of the refracted part and $\Phi_\tau(x, y)$.

On the other hand, if $\gamma_+ > \gamma_-$, the total reflection phenomena for incident waves from the lower half-space occur, which correspond to the fact that the steepest descent curve should be across singularities of the integrand when the contour is changed. Because of singularities, it seems difficult to get asymptotics of the refracted part corresponding to the total reflection phenomena. Hence, we only obtain some estimates for the refracted part containing the total reflection phenomena, which is the purpose of section 2.

In section 3, we show the following asymptotics of $\nabla_x \Phi_\tau(x, y)$ (and $\Phi_\tau(x, y)$):

**Proposition 1.4** Assume that $\gamma_+ > \gamma_-$. Then for $k = 0, 1$, we have

$$\nabla^k_x \Phi_\tau(x, y) = \frac{e^{-\tau l(x, y)}}{8\pi \gamma_+ \gamma_- \sqrt{\det H(x, y)}} \left( -\frac{\tau}{\sqrt{\gamma_-}} \right)^k \left( \sum_{j=0}^N \tau^{-j} \Phi_j^{(k)}(x, y) + Q_{N, \tau}^{(k)}(x, y) \right), \quad (1.7)$$

where $H(x, y) = \text{Hess}(l_{x,y})(z'(x, y))$ is the Hessian of $l_{x,y}$ given by (1.3) at $z' = z'(x, y)$, $\Phi_j^{(k)}(x, y)$ $(k = 0, 1)$ are $C^\infty$ in $\overline{D} \times \overline{B}$, for any $N \in \mathbb{N} \cup \{0\}$, $Q_{N, \tau}^{(k)}(x, y)$ $(k = 0, 1)$ are continuous in $\overline{D} \times \overline{B}$ with a constant $C_N > 0$ satisfying

$$|Q_{N, \tau}^{(0)}(x, y)| + |Q_{N, \tau}^{(1)}(x, y)| \leq C_N \tau^{-(N+1)} \quad (x \in \overline{D}, y \in \overline{B}, \tau \geq 1).$$

Moreover, $\Phi_j^{(k)}(x, y)$ $(k = 0, 1)$ are given by

$$\Phi_0^{(0)}(x, y) = \frac{E_0(x - z'(x, y))}{|x - z'(x, y)||z'(x, y) - y|},$$
and

\[ \Phi_0^{(1)}(x, y) = \Phi_0^{(0)}(x, y) \frac{x - z'(x, y)}{|x - z'(x, y)|}, \]

where

\[ E_0(x - z') = \frac{4\sqrt{\gamma_{-}|x_3|\sqrt{a_0^2|x - \tilde{z}'|^2 - |x' - z'|^2}}}{|x - z'|(\sqrt{a_0^2|x - \tilde{z}'|^2 - |x' - z'|^2} + a_0^2|x_3|)}. \quad (1.8) \]

Note that Proposition 1.4 is the same as Proposition 1 of [10] except for the condition \( \gamma_{+} > \gamma_{-}. \) That means that the total reflection phenomena make no difference to asymptotics of \( \Phi_{\tau}(x, y) \). We should consider the influence of the total reflection phenomena on the optical distance, which is discussed in section 2. For the usual inner waves, the optical distance between \( z' \in \mathbb{R}^2 = \partial \mathbb{R}^3_{+} \) and \( x \in \mathbb{R}^3 \) is given by \( |x - z'|/\sqrt{\gamma_{-}} \). Hence, the optical distance between \( x \in \mathbb{R}^3 \) and \( y \in \mathbb{R}^3_{+} \) is given by (1.2) and (1.3) if the total reflection phenomena do not occur. In our case, we should pay attention to the fact that optical distance between \( \tilde{z}' \) and \( x \) corresponding to the total reflection phenomena is different from \( |x - \tilde{z}'|/\sqrt{\gamma_{-}} \) (see (3.1) in section 3). Hence, in this case, the time in which the waves travel from \( x \) to \( y \) via \( \tilde{z}' \in \partial \mathbb{R}^3_{+} \) is also different from \( l_{x,y}(z') \) given in (1.3). But we can show that even in this case, the function \( l(x, y) \) gives the optical distance between \( x \in \mathbb{R}^3_{-} \) and \( y \in \mathbb{R}^3_{+} \) (cf. Lemma 3.1). As is in [10], the fact that \( l(x, y) \) gives the optical distance plays an important role to obtain Proposition 1.4. This is the reason why Proposition 1.4 has the same conclusion as in [10]. Once we obtain asymptotics in Proposition 1.4, Theorem 1.3 can be shown by the same argument as in [10]. This is the outline of this article.

## 2 Asymptotics and estimates of the refracted part

Let us recall an integral representation of the fundamental solution \( \Phi_{\tau}(x, y) \) given in [10]. A usual fundamental solution for the case of no transmission boundary (i.e. the case of \( \gamma_{-} = \gamma_{+} \)) is of the form:

\[ E_{\tau}^{\gamma_{+}, 0}(x, y) = \frac{1}{4\pi \gamma_{+}} \frac{e^{-\tau|x-y|/\sqrt{\gamma_{+}}}}{|x-y|} \quad (x \neq y, \tau > 0), \]

which coincides with that of defined by the Fourier integral

\[ E_{\tau}^{\gamma_{+}, 0}(x, y) = \frac{1}{(2\pi)^3} \int_{\mathbb{R}^3} e^{i\xi \cdot (x-y)} \frac{1}{\gamma_{+} \xi^2 + \tau^2} d\xi = \frac{\tau}{(2\pi)^3} \int_{\mathbb{R}^3} e^{i\tau \xi \cdot (x-y)} \frac{1}{\gamma_{+} \xi^2 + 1} d\xi. \]

As in (11) of [10], we introduce

\[ E_{\tau}^{\gamma_{-}}(x, z') = \frac{\tau}{(2\pi)^3} \int_{\mathbb{R}^3} e^{i\tau \xi \cdot (x-z')} \frac{1}{\gamma_{-} \xi^2 + 1} R(\sqrt{\gamma_{-}}|\xi'|) d\xi \quad (x_3 < 0), \]
where \( \tilde{z}' = (z', 0) \) \((z' \in \mathbb{R}^2)\) is the point on the transmission boundary \( \partial R_{\pm}^3 \) and \( R(|\xi'|) \) is a function of \(|\xi'|\) standing for the transmission coefficient given by

\[
R(\rho) = \frac{4\sqrt{\gamma} - a_0^2 + \rho^2 \sqrt{1 + \rho^2}}{\sqrt{a_0^2 + \rho^2 + a_0^2 \sqrt{1 + \rho^2}}} \quad (\rho \geq 0) \quad \text{with} \quad a_0 = \sqrt{\frac{\gamma}{\gamma_+}}.
\]

Using \( E_{\tau}^{\gamma+0}(x, \tilde{z}') \) and \( E_{\tau}^{-}(x, z') \) we can represent the fundamental solution \( \Phi_{\tau}(x, y) \) for \( y \in \mathbb{R}^3_+ \) and \( x \in \mathbb{R}^3_+ \) as

\[
\Phi_{\tau}(x, y) = \frac{\tau}{4\pi \gamma_+} \int_{\mathbb{R}^2} E_{\tau}^{-}(x, z') e^{-r|\tilde{z}' - y|/\sqrt{\gamma_+}} \frac{dz'}{|\tilde{z}' - y|}.
\]

This is just (10) of [10]. In what follows, as in [10], we call \( E_{\tau}^{-}(x, z') \) the refracted part (of the fundamental solution \( \Phi_{\tau}(x, y) \)).

Put \( \Theta_k(x, z') = \frac{x_k - z_k}{|x_k - z_k|} \) \((k = 1, 2)\) and \( \Theta_3(x, z') = \frac{x_3}{|x_3|} \). Note that (26)-(29) of [10] imply that the refracted part \( E_{\tau}^{-}(x, z') \) is expressed by

\[
E_{\tau}^{-}(x, z') = \frac{\tau}{2(2\pi)^{3/2}} \int_{\mathbb{R}} I_{\tilde{\tau},0}(x - \tilde{z}', \zeta_2) d\zeta_2,
\]

\[
\partial_{x_k} E_{\tau}^{-}(x, z') = \frac{\tau^2}{2(2\pi)^{2}\gamma_+^2} \int_{\mathbb{R}} I_{\tilde{\tau},k}(x - \tilde{z}', \zeta_2) d\zeta_2 \Theta_k(x, z') \quad (k = 1, 2, 3),
\]

where for \( x \in \mathbb{R}^3_+, z' \in \mathbb{R}^2 \) and \( k = 0, 1, 2, 3 \), we put \( \tilde{\tau} = \tau / \sqrt{\gamma_+} \),

\[
I_{\tilde{\tau},k}(x - \tilde{z}', \zeta_2) = \int_{\mathbb{R}} e^{-\tilde{\tau}\sqrt{1 + \zeta_2^2}(|x' - z'|/|x_3| + 1)} Q_k(\zeta_1, \zeta_2) \frac{d\zeta_1}{\sqrt{1 + \zeta_1^2}}
\]

and

\[
\begin{align*}
Q_0(\zeta_1, \zeta_2) &= R\left(\sqrt{\zeta_1^2 + \zeta_2^2 + \zeta_1^2 \zeta_2^2}\right), \\
Q_1(\zeta_1, \zeta_2) &= Q_2(\zeta_1, \zeta_2) = i\zeta_1 Q_0(\zeta_1, \zeta_2), \\
Q_3(\zeta_1, \zeta_2) &= -\sqrt{1 + \zeta_1^2} Q_0(\zeta_1, \zeta_2).
\end{align*}
\]

We use the steepest descent method to the integrals \( I_{\tilde{\tau},k}(x - \tilde{z}', \zeta_2) \). Take \( \theta \) satisfying

\[
\sin \theta = \frac{|x' - z'|}{|x - \tilde{z}'|}, \quad \cos \theta = \frac{|x_3|}{|x - \tilde{z}'|} \quad (0 \leq \theta \leq \pi/2),
\]

and put \( r = |x - \tilde{z}'|\sqrt{1 + \zeta_2^2} \) and

\[
\lambda = \lambda(\zeta_1, x, z') = -i \sin \theta \zeta_1 + \cos \theta \sqrt{1 + \zeta_1^2}.
\]

Note that (2.6) is equivalent to \( \zeta_1 = i\lambda \sin \theta \pm \sqrt{\lambda^2 - 1} \cos \theta \), which yields

\[
\zeta_1 = \zeta_1(\rho, x, z') = i\sqrt{1 + \rho^2} \sin \theta + \rho \cos \theta \quad (\rho \in \mathbb{R}, x \in \mathbb{R}^3, z' \in \mathbb{R}^2)
\]

by putting \( \lambda = \sqrt{1 + \rho^2} \) for \( \lambda \geq 1 \) (cf. (33) in [10]).
In the case of $\gamma_+ < \gamma_-$, the function
\[ Q_0(\zeta_1, \zeta_2) = \frac{4\sqrt{\gamma_+ - 1 + \zeta_1^2} \sqrt{1 + \zeta_1^2}}{P(\zeta_1, \zeta_2) + a_0^2 \sqrt{1 + \zeta_1^2}}, \] (2.8)
where
\[ P(\zeta_1, \zeta_2) = \sqrt{\frac{a_0^2 + \zeta_2^2}{1 + \zeta_2^2}} + \zeta_1^2 \]
is holomorphic for $\zeta_1 \in \mathbb{C} \setminus ((-i\infty, -i] \cup [i, i\infty))$. Hence, we can change the contour of integrals (2.4) to the curve $\Gamma_{x,z'}$ defined by (2.6). This implies
\[ I_{\tau,k}(x - z', \zeta_2) = \int_{\Gamma_{x,z'}} e^{-\tau_\lambda} Q_k(\zeta_1, \zeta_2) \frac{d\zeta_1}{\sqrt{1 + \zeta_1^2}}. \] (2.9)
Using this formula, we can obtain asymptotics of $\Phi_\tau(x, y)$ as $\tau \to \infty$. On the contrary, in the case of $\gamma_+ > \gamma_-$, i.e. $a_0 = \sqrt{\gamma_+ - 1} < 1$, the functions $P$ and $Q_0$ are holomorphic for $\zeta_1 \in \mathbb{C} \setminus ((-i\infty, -ib_0(\zeta_2)) \cup [ib_0(\zeta_2), i\infty))$, where
\[ b_0(\zeta_2) = \sqrt{\frac{a_0^2 + \zeta_2^2}{1 + \zeta_2^2}}. \]
Thus, if $\sin \theta < a_0$, we can change the contour to $\Gamma_{x,z'}$, however, if $\sin \theta > b_0(\zeta_2)$ we should make a detour to connect $\Gamma_{x,z'}$ and the branch point $\zeta_1 = ib_0(\zeta_2)$ of $P(\zeta_1, \zeta_2)$. This corresponds to the total reflection phenomena, which makes us additional arguments.

In what follows, for $0 < \delta < a_0^{-1}$ and $x \in \mathbb{R}_+^3$, we put $U_0(x) = \{ z' \in \mathbb{R}^2 \mid |x' - z'| < a_0\delta |x - z'| \}$. Note that $z' \in U_0(x)$ is equivalent to
\[ |x' - z'| < \frac{a_0\delta}{\sqrt{1 - a_0^2\delta^2}} |x_3|. \] (2.10)

Since $U_0(x) = \{ z' \in \mathbb{R}^2 \mid \sin \theta < a_0\delta \}$, it follows that $\inf\{|ia_0 - \zeta_1| \mid \zeta_1 \in \Gamma_{x,z'}\} = a_0(1 - \delta)$ for any $0 < \delta < 1, x \in \mathbb{R}_+^3$ and $z' \in U_0(x)$. Thus, in this case, the argument for getting Proposition 2 in [10] implies the following expansions of the refracted part:

**Lemma 2.1** Assume that $\gamma_+ > \gamma_-$. Then, for any $0 < \delta < 1$, the refracted part $E^{\gamma_-}_\tau(x, z')$ for $x \in \mathbb{R}_+^3$ and $z' \in \overline{U_0(x)}$ is expanded by
\[ E^{\gamma_-}_\tau(x, z') = \frac{e^{-\tau|x - z'|/\sqrt{\gamma_-}}}{4\pi \gamma_- |x - z'|} \left( \sum_{j=0}^{N-1} E_j(x - z') \left( \frac{\sqrt{\gamma_-}}{\tau|x - z'|} \right)^j + E_N(x, z'; \tau) \right), \]
and for $k = 1, 2, 3$,
\[ \partial_{x_k} E^{\gamma_-}_\tau(x, z') = \frac{-\tau e^{-\tau|x - z'|/\sqrt{\gamma_-}}}{4\pi \gamma_-^{3/2} |x - z'|} \left( \sum_{j=0}^{N-1} G_{k,j}(x - z') \left( \frac{\sqrt{\gamma_-}}{\tau|x - z'|} \right)^j + G_{k,N}(x, z'; \tau) \right), \]
where $E_j(x - z')$, $G_{k,j}(x - z')$ ($k = 1, 2, 3$ and $j = 0, 1, 2, \ldots$) are $C^\infty$ functions for $x$ and $z'$ with $z' \in \overline{U_\delta(x)}$. Here, the remainder terms $\tilde{E}_N(x, z'; \tau)$ and $\tilde{G}_{k,N}(x, z'; \tau)$ ($k = 1, 2, 3$) are estimated by

$$|\tilde{E}_N(x, z'; \tau)| + \sum_{k=1}^{3} |\tilde{G}_{k,N}(x, z'; \tau)| \leq C_{N,\delta} \left( \frac{\sqrt{\gamma_-}}{\tau |x - z'|} \right)^N \quad (x \in \mathbb{R}^3_-, z' \in \overline{U}_\delta(x))$$

for some constant $C_{N,\delta} > 0$ depending only on $N \in \mathbb{N}$ and $\delta$. In particular, we have

$$\begin{cases} G_{k,0}(x - z') = E_0(x - z') \frac{x_k - z_k}{|x - z'|} (k = 1, 2) \vspace{1.5mm} \\
G_{3,0}(x - z') = E_0(x - z') \frac{|x - z'|}{x_3} \end{cases}$$

where $E_0(x - z')$ is given in (1.8).

Thus, once $0 < \delta < 1$ is fixed, we can obtain uniform estimates of the refracted part for $x \in \mathbb{R}^3_-$ and $z' \in \overline{U_\delta(x)}$. On the contrary, for $x \in \mathbb{R}^3_-$ and $z' \in \mathbb{R}^2 \setminus U_\delta(x)$, it seems to be hard to get asymptotics of the refracted part by the total reflection waves. Fortunately, for our purpose, we have only to obtain the estimates for the refracted part. The main part of this section is to show these estimates.

If $\theta$ is near $\theta_0$ and $\theta \leq \theta_0$, we have the following expansions:

**Proposition 2.2** Assume that $\gamma_+ > \gamma_-$. Then, for any fixed $\delta$ with $0 < \delta < 1$, the refracted part $E^-_{\gamma_-}(x, z')$ for $x \in \mathbb{R}^3_-$ and $z' \in \overline{U_1(x)} \setminus U_\delta(x)$ is expanded by

$$E^-_{\gamma_-}(x, z') = e^{-\tau |x - z'|} \frac{\sqrt{\gamma_-}}{4\pi \gamma_- |x - z'|} \left( E_0(x - z') + \tilde{E}_{0,0}(x, z'; \tau) \right),$$

$$\partial_{x_k} E^-_{\gamma_-}(x, z') = \frac{-\tau e^{-\tau |x - z'|} \sqrt{\gamma_-}}{4\pi \gamma_-^{3/2} |x - z'|} \left( G_{k,0}(x - z') + \tilde{E}_{k,0}(x, z'; \tau) \right) \quad (k = 1, 2, 3).$$

In the above, $E_0$ and $G_{k,0}$ are the functions given in Lemma 2.1. For the remainder terms $\tilde{E}_{k,0}(x, z'; \tau)$, for any $0 < \delta < 1$, there exists a constant $C_{\delta} > 0$ such that

$$|\tilde{E}_{k,0}(x, z'; \tau)| \leq C_{\delta} \left( \frac{\sqrt{\gamma_-}}{\tau |x - z'|} \right)^{1/4} \quad (x \in \mathbb{R}^3_-, z \in \overline{U_1(x)} \setminus U_\delta(x), k = 0, 1, 2, 3).$$

For the case of $\theta > \theta_0$, we have the following estimates:

**Proposition 2.3** Assume that $\gamma_+ > \gamma_-$. Then, there exists a constant $C > 0$ such that the refracted part $E^-_{\gamma_-}(x, z')$ for $x \in \mathbb{R}^3_-$ and $z' \in \mathbb{R}^2 \setminus U_1(x)$ is estimated by

$$|\nabla_x^k E^-_{\gamma_-}(x, z')| \leq C \tau^k e^{-\tau T_{x,z'}(\theta_0)} \quad (x \in \overline{D}, z' \in \overline{\mathbb{R}^2} \setminus U_1(x), k = 0, 1),$$

where for $x \in \mathbb{R}^3_-$ and $z' \in \mathbb{R}^2$, $T_{x,z'}(\alpha)$ is defined by

$$T_{x,z'}(\alpha) = \frac{1}{\sqrt{\gamma_-}} \left( |x_3| \cos \alpha + |z' - x'| \sin \alpha \right). \quad (2.11)$$
Note that $T_{x,z'}(\alpha)$ is expressed by

$$T_{x,z'}(\alpha) = \frac{|\tilde{z}' - x|}{\sqrt{\gamma_z}} \cos(\theta - \alpha)$$

(2.12)

by using $\theta$ defined by (2.5). In what follows, we only write $T_{x,z'}(\alpha)$ by $T(\alpha)$ shortly.

The rest of this section is devoted to show Propositions 2.2 and 2.3.

Proof of Proposition 2.2. When $z' \in U_1(x) \setminus U_6(x)$, we can change the contour of integrals (2.4) to the curve $\Gamma_{x,z'}$ defined by (2.6) since $\sin \theta \leq \sin \theta_0 = a_0$. For simplicity we write $\sigma_1 = \rho$, $\sigma_2 = \zeta_2$ and $\sigma = (\sigma_1, \sigma_2)$, and we set

$$f(\sigma) = \sqrt{1 + \sigma_1^2} \sqrt{1 + \sigma_2^2}, \quad F_k(\sigma, x, z') = Q_k(\zeta_1(\sigma_1, x, z'), \sigma_2) \frac{1}{\sqrt{1 + \sigma_1^2}}$$

(2.13)

Then, as in the same way as section 3 of [10], by (2.2), (2.3) and (2.9) we obtain

$$E_r^\gamma(x, z') = \frac{\tau}{2(2\pi)^2 \gamma_z^{3/2}} \int_{\mathbb{R}^2} e^{-\tau|x-z'|} f(\sigma) F_0(\sigma, x, z') d\sigma, \quad (2.14)$$

$$\partial_k E_r^\gamma(x, z') = \frac{\tau^2}{2(2\pi)^2 \gamma_z^{5/2}} \int_{\mathbb{R}^2} e^{-\tau|x-z'|} f(\sigma) F_k(\sigma, x, z') d\sigma \Theta_k(x, z') \quad (k = 1, 2, 3). \quad (2.15)$$

We put $\tilde{P}(\sigma, x, z') = P(\zeta_1(\sigma_1, x, z'), \sigma_2)$, then

$$\tilde{P}(\sigma, x, z') = \sqrt{a_0^2 - \sin^2 \theta + \sigma_1^2 \cos 2\theta + \frac{(1 - a_0^2) \sigma_2^2}{1 + \sigma_1^2} + i \sigma_1 \sqrt{1 + \sigma_1^2} \sin 2\theta}.$$

We should note that $F_k(\sigma, x, z')$ is continuous in $\sigma \in \mathbb{R}^2$ and there exists a constant $C_k > 0$ such that

$$|F_k(\sigma, x, z')| \leq C_k (1 + |\sigma|)^3, \quad (\sigma \in \mathbb{R}^2), \quad (2.16)$$

but $F_k(\sigma, x, z')$ is not $C^\infty$ near $\sigma = (0, 0)$ when $z' \in \overline{U_1(x) \setminus U_6(x)}$ because of $\tilde{P}(\sigma, x, z')$. For small $|\sigma|$ we will show the following continuity at $\sigma = 0$:

$$|F_k(\sigma, x, z') - F_k(0, x, z')| \leq C(\sqrt{|\sigma_1| + |\sigma_2|})$$

(\sigma \in \mathbb{R}^2, |\sigma| \leq 2, z' \in \overline{U_1(x) \setminus U_6(x)}). \quad (2.17)$$

To obtain (2.17), it is enough to show

$$|\tilde{P}(0, x, z') - \tilde{P}(\sigma, x, z')| \leq C(\sqrt{|\sigma_1| + |\sigma_2|})$$

(2.18)

$$\frac{1}{\tilde{P}(0, x, z') + a_0^2} - \frac{1}{\tilde{P}(\sigma, x, z') + a_0^2} \sqrt{1 + \sigma_1^2} \leq C(\sqrt{|\sigma_1| + |\sigma_2|})$$

(2.19)

for $|\sigma| \leq 2$ and $z' \in \overline{U_1(x) \setminus U_6(x)}$ because of the definition of $Q_k$. Estimate (2.19) follows from (2.18), since

$$\frac{1}{\tilde{P}(0, x, z') + a_0^2} - \frac{1}{\tilde{P}(\sigma, x, z') + a_0^2} \sqrt{1 + \sigma_1^2}$$
We know that

\[
|\tilde{P}(\sigma, x, z') - \tilde{P}(0, x, z')| + a_0^2|1 - \sqrt{1 + \sigma_1^2}|
\]

\[
\leq \sqrt{\left(\text{Re}[\tilde{P}(\sigma, x, z')] + a_0^2\sqrt{1 + \sigma_1^2}\right)^2 + \left(\text{Re}[\tilde{P}(0, x, z')] + a_0^2\right)^2}
\]

\[
\leq \frac{|\tilde{P}(\sigma, x, z') - P(0, x, z')| + a_0^2|\sigma_1|}{a_0^4} \quad (|\sigma| \leq 2).
\]

Here we used the fact that \(\text{Re}[\tilde{P}(\sigma, x, z')] \geq 0\), which follows from the definition \(\sqrt{X} = |X|^{1/2}e^{i\text{arg}X/2}\) \((|\text{arg}X| < \pi)\). Now we shall show (2.18). Here we consider

\[
\tilde{P}(0, x, z') - \tilde{P}(\sigma, x, z')
\]

\[
= \tilde{P}(0, x, z') - \tilde{P}(0, \sigma_2, x, z') + \tilde{P}(0, \sigma_2, x, z') - \tilde{P}(\sigma, x, z')
\]

\[
= \sigma_2 \int_0^1 \partial_{\sigma_2} \tilde{P}(0, t\sigma_2, x, z') \, dt + \sigma_1 \int_0^1 \partial_{\sigma_1} \tilde{P}(t\sigma_1, \sigma_2, x, z') \, dt. \tag{2.20}
\]

We know that

\[
\partial_{\sigma_1} \tilde{P}(\sigma, x, z') = \frac{1}{2\tilde{P}(\sigma, x, z')} \left\{ 2\sigma_1 \cos 2\theta + i \sin 2\theta \left( \frac{\sigma_1^2}{\sqrt{1 + \sigma_1^2}} + \sqrt{1 + \sigma_1^2} \right) \right\}, \tag{2.21}
\]

\[
\tilde{P}(0, \sigma_2, x, z') = \sqrt{\frac{a_0^2 - 1}{1 + \sigma_2^2} + 1 - \sin^2 \theta},
\]

\[
\partial_{\sigma_2} \tilde{P}(0, \sigma_2, x, z') = \frac{(1 - a_0^2)\sigma_2}{(1 + \sigma_2^2)^{3/2}\sqrt{1 - \sin^2 \theta} \sqrt{\sigma_2^2 - s(\theta)}}, \tag{2.22}
\]

where \(s(\theta) = (\sin^2 \theta - a_0^2)/(1 - \sin^2 \theta)\). To show (2.18) by using (2.20), we consider

\[
|\tilde{P}(\sigma, x, z')|^4 = \left( a_0^2 - \sin^2 \theta + \sigma_2^2 \cos(2\theta) + \frac{(1 - a_0^2)\sigma_2^2}{1 + \sigma_2^2} \right)^4 + \sigma_1^2(1 + \sigma_1^2)^2 \sin^2(2\theta). \tag{2.23}
\]

We know that there exists a \(\epsilon > 0\) such that

\[
|\tilde{P}(\sigma, x, z')|^4 \geq \sigma_1^2 \sin^2(2\theta) \geq \epsilon \sigma_1^2,
\]

since \(0 < 2\sin^{-1}(a_0\delta) \leq 2\theta \leq 2\theta_0 < \pi\). Then, it follows that there exists a constant \(C\) such that

\[
\frac{1}{|\tilde{P}(\sigma, x, z')|} \leq \frac{C}{|\sigma_1|^{1/2}} \quad (|\sigma| \leq 2, z' \in \overline{U_1(x) \setminus U_6(x)}). \tag{2.24}
\]

From (2.21) and (2.24) it follows that

\[
\left| \sigma_1 \int_0^1 \partial_{\sigma_1} \tilde{P}(t\sigma_1, \sigma_2, x, z') \, dt \right| \leq C \int_0^1 \frac{|\sigma_1|^{1/2}}{\sqrt{t}} \, dt \leq 2C|\sigma_1|^{1/2}. \tag{2.25}
\]

From (2.22) and (2.24) it follows that

\[
\left| \sigma_2 \int_0^1 \partial_{\sigma_2} \tilde{P}(0, t\sigma_2, x, z') \, dt \right| = \frac{1 - a_0^2}{\sqrt{1 - \sin^2 \theta}} \int_0^1 \frac{t \sigma_2^2}{(1 + t^2 \sigma_2^2)^{3/2}(t^2 \sigma_2^2 + |s(\theta)|)^{1/2}} \, dt
\]

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Proof of Proposition 2.3. Here we consider the case $|\sigma| \geq 3/2$ and set
\[ \int_{\mathbb{R}^2} e^{-\tau|x-z'|f(\sigma)} F_k(\sigma, x, z')d\sigma = \int_{\mathbb{R}^2} e^{-\tilde{\tau}|x-z'|f(\sigma)} F_k(\sigma, x, z')\psi(\sigma)d\sigma \]
\[ + \int_{\mathbb{R}^2} e^{-\tilde{\tau}|x-z'|f(\sigma)} F_k(\sigma, x, z')(1 - \psi(\sigma))d\sigma, \]  
(2.27)
where $f(\sigma)$ and $F_k$ are defined by (2.13). Since $f(\sigma) \geq 1 + |\sigma|/4$ for $|\sigma| \geq 1$, it follows that $f(\sigma) \geq 9/8 + |\sigma|/8$ for $|\sigma| \geq 1$. From this estimate and (2.16), we have the estimate of the second integral of (2.27) as
\[ \left| \int_{\mathbb{R}^2} e^{-\tilde{\tau}|x-z'|f(\sigma)} F_k(\sigma, x, z')(1 - \psi(\sigma))d\sigma \right| \leq C e^{-9\tilde{\tau}|x-z'|/8} \int_{\mathbb{R}^2} (1 + |\sigma|)^3 e^{-\tilde{\tau}|x-z'|/8|\sigma|}d\sigma \]
\[ \leq \frac{C_N e^{-\tilde{\tau}|x-z'|}}{(\tilde{\tau}|x-z'|)^N}. \]

For the first integral of (2.27) if we use Laplace method and estimate (2.17), we have
\[ \left| \int_{\mathbb{R}^2} e^{-\tilde{\tau}|x-z'|f(\sigma)} F_k(\sigma, x, z')\psi(\sigma)d\sigma - e^{-\tilde{\tau}|x-z'|} \left( \frac{2\pi}{\tilde{\tau}|x-z'|} \right) F_{k,0}(x - z') \right| \]
\[ \leq C(\tilde{\tau}|x-z'|)^{-5/4} e^{-\tilde{\tau}|x-z'|}. \]

Thus we complete the proof of Proposition 2.2.

Proof of Proposition 2.3. Here we consider the case $z' \in \mathbb{R}^2 \setminus U_1(x)$. The integral $I_{\tilde{\tau},k}(x - z', \zeta_2)$ in (2.4) can be written as below:
\[ I_{\tilde{\tau},k}(x - z', \zeta_2) = \int_\mathbb{R} e^{-\tilde{\tau}r\lambda} Q_k(\zeta_1, \zeta_2) \frac{d\zeta_1}{\sqrt{1 + \zeta_1^2}}, \]  
(2.28)
where $r = |x - z'|\sqrt{1 + \zeta_2^2}$, $\lambda = -i(|x' - z'|/|x - z'|)\zeta_1 + (|x_3|/|x - z'|)\sqrt{1 + \zeta_1^2} = -i\sin \theta \zeta_1 + \cos \theta \sqrt{1 + \zeta_1^2}$ and $k = 0, 1, 2, 3$. When we try to change the contour of the integrals $I_{\tilde{\tau},k}(x - z', \zeta_2)$ $(k = 0, 1, 2, 3)$ in the same way as in the case of $U_1(x) \setminus U_3(x)$, we need to count $a_0 < b_0(\zeta_2)$, and $[ib_0(\zeta_2), i\infty)$ is the branch cut of the integrands. Therefore, in case that $b_0(\zeta_2) < \sin \theta$, we consider the following contour for $\varepsilon > 0$ (see figure 2):
\[ \Gamma_{\varepsilon} : \zeta_1 = ib_0(\zeta_2) + \varepsilon e^{i\phi} \quad (\pi \leq \phi \leq 2\pi), \]
\[ \Gamma_{+\varepsilon} : \zeta_1 = e^{i\pi/2}w + \varepsilon \quad (b_0(\zeta_2) \leq w \leq \sin \theta), \]
\[ \Gamma_{-\varepsilon} : \zeta_1 = e^{i\pi/2}w - \varepsilon \quad (\sin \theta \geq w \geq b_0(\zeta_2)). \]

When \( \zeta_1 \in \Gamma_{\pm\varepsilon}, \zeta_1^2 = -w^2 + 2e^{\pi i/2}w\varepsilon + O(\varepsilon^2), 1 + \zeta_1^2 = 1 - w^2 + O(\varepsilon) \) as \( \varepsilon \downarrow 0 \). Thus

![Figure 2: Contour of the integrals](image)

we have

\[ P(\zeta_1, \zeta_2) = \sqrt{\zeta_1^2 + b_0(\zeta_2)^2} \to \sqrt{w^2 - b_0(\zeta_2)^2}e^{\pm i\pi/2} \quad (\varepsilon \downarrow 0) \]

for \( b_0(\zeta_2) \leq w \leq \sin \theta \). If we put \( X_0(\zeta_2) = 4\sqrt{\gamma - \sqrt{1 + \zeta_2^2}} \) and \( X_1(\zeta_2) = 4\sqrt{\gamma - (1 + \zeta_2^2)} \). Then we have

\[
Q_0(\zeta_1, \zeta_2)|_{\zeta_1 \in \Gamma_{\pm\varepsilon}} = \frac{X_0(\zeta_2)\sqrt{1 - w^2}\sqrt{w^2 - b_0(\zeta_2)^2}e^{\pm i\pi/2}}{\sqrt{w^2 - b_0(\zeta_2)^2}e^{\pm i\pi/2} + a_0\sqrt{1 - w^2}} + O(\varepsilon)
\]
\[ =: Q_0^\pm (w, \zeta_2) + O(\varepsilon) \quad (\varepsilon \downarrow 0), \]

\[
Q_k(\zeta_1, \zeta_2)|_{\zeta_1 \in \Gamma_{\pm\varepsilon}} = \frac{-X_1(\zeta_2)w\sqrt{1 - w^2}\sqrt{w^2 - b_0(\zeta_2)^2}e^{\pm i\pi/2}}{\sqrt{w^2 - b_0(\zeta_2)^2}e^{\pm i\pi/2} + a_0^2\sqrt{1 - w^2}} + O(\varepsilon)
\]
\[ =: Q_k^{\pm} (w, \zeta_2) + O(\varepsilon) \quad (\varepsilon \downarrow 0, k = 1, 2), \]

\[
Q_3(\zeta_1, \zeta_2)|_{\zeta_1 \in \Gamma_{\pm\varepsilon}} = \frac{-X_1(\zeta_2)(1 - w^2)\sqrt{w^2 - b_0(\zeta_2)^2}e^{\pm i\pi/2}}{\sqrt{w^2 - b_0(\zeta_2)^2}e^{\pm i\pi/2} + a_0^2\sqrt{1 - w^2}} + O(\varepsilon)
\]
\[ =: Q_3^{\pm} (w, \zeta_2) + O(\varepsilon) \quad (\varepsilon \downarrow 0), \]

\[
\lambda|_{\zeta_1 \in \Gamma_{\pm\varepsilon}} = -i(\sin \theta)\zeta_1 + (\cos \theta)\sqrt{1 + \zeta_1^2} |_{\zeta_1 \in \Gamma_{\pm\varepsilon}}
\]
\[ = (\sin \theta)w + (\cos \theta)\sqrt{1 - w^2} + O(\varepsilon)
\]
\[ =: \lambda_0(w) + O(\varepsilon) \quad (\varepsilon \downarrow 0). \]

We define the following integrals \( I_{\tau,k}^x(x - z', \zeta_2) \) and \( I_{\tau,k}^m(x - z', \zeta_2) \) for \( k = 0, 1, 2, 3 \) as below:

\[
I_{\tau,k}^x(x - z', \zeta_2) = \int_{\Gamma_{x,z'}} e^{-\tau x} Q_k(\zeta_1, \zeta_2) \frac{d\zeta_1}{\sqrt{1 + \zeta_1^2}}. \quad (2.29)
\]
\[ I^m_{\tau,k}(x - z', \zeta_2) = \lim_{\varepsilon \to 0} \int_{\Gamma_{\varepsilon} \cup \Gamma_{-\varepsilon} \cup \Gamma_{0}} \frac{Q_k(\zeta_1, \zeta_2)}{\sqrt{1 + \zeta_1^2}} e^{-\tau \rho(w)} d\zeta_1 \]
\[ = i \int^{\sin \theta}_{\rho_0(\zeta_1)} \frac{Q_k^+(w, \zeta_2) - Q_k^-(w, \zeta_2)}{\sqrt{1 - w^2}} e^{-\tau \rho_0(w)} dw. \] (2.30)

Then we can change the contour of (2.28) as
\[ I_{\tau,k}(x - z', \zeta_2) = I^s_{\tau,k}(x - z', \zeta_2) + I^m_{\tau,k}(x - z', \zeta_2). \]

Thus (2.2) and (2.3) are reduced to
\[ E^m_{\tau,k}(x, z') = E^s_{\tau,0}(x, z') + E^m_{\tau,0}(x, z'), \]
\[ \partial_{z_k} E^m_{\tau,k}(x, z') = E^s_{\tau,k}(x, z') + E^m_{\tau,k}(x, z') \quad (k = 1, 2, 3), \]
where
\[
E^s_{\tau,k}(x, z') = \frac{\tau}{2(2\pi)^2 \gamma^2} \int_{\mathbb{R}} I^s_{\tau,0}(x - z', \zeta_2) d\zeta_2 \quad (\alpha = s, m), \]
\[
E^m_{\tau,k}(x, z') = \frac{\tau^2}{2(2\pi)^2 \gamma^2} \int_{\mathbb{R}} I^m_{\tau,k}(x - z', \zeta_2) d\zeta_2 \quad (\alpha = s, m, k = 1, 2, 3). \]

At first, we shall consider \( E^s_{\tau,k}(x, z') \). From (2.29), they are reduced to similar forms to (2.14) and (2.15). Hence, if we prove estimates corresponding to (2.18) for \( z' \in \mathbb{R}^2 \setminus \mathcal{U}_1(x) \), the same argument as for (2.14) and (2.15) works. Thus we should show
\[ \frac{1}{|P(\sigma, x, z')|^2} \leq \frac{C}{|\sigma_1|^{1/2}} \quad (|\sigma| \leq 2, z' \in \mathbb{R}^2 \setminus \mathcal{U}_1(x)). \] (2.33)

If we set \( Y = a_0^2 - \sin^2 \theta + \sigma_1^2 \cos(2\theta) + \frac{(1-a_0^2)\sigma_2^2}{1+\sigma_2^2} \) and \( Y_0 = \sigma_1^2 + \frac{1-a_0^2}{1+\sigma_2^2} \), then \( Y = -Y_0 + \cos^2 \theta(1 + 2\sigma_1^2) \) and \( 0 < (1-a_0^2)/5 \leq Y_0 \leq 5 \) as \( |\sigma| \leq 2 \). From
\[ |\tilde{P}(x, z')|^4 \geq Y_0^2 - 2Y_0 \cos^2 \theta(1 + 2\sigma_1^2) \geq Y_0(Y_0 - 18 \cos^2 \theta), \]
it follows that \( |\tilde{P}(x, z')|^4 \geq (1-a_0^2)^2/50 \) for \( \cos^2 \theta \leq (1-a_0^2)/180 \). When \( \cos^2 \theta > (1-a_0^2)/180 \), there exists an \( \epsilon > 0 \) such that \( |\tilde{P}(x, z')|^4 \geq \epsilon \sigma_1^4 \), since we know that \( |\tilde{P}(x, z')|^4 \geq \sigma_2^6 \sin^2 (2\theta) \) from (2.23). Thus, we obtain (2.33).

By using (2.20), we shall estimate \( \tilde{P}(0, x, z') - \tilde{P}(x, x, z') \). Then, the first term is
\[
\sigma_2 \int_{0}^{1} \partial_{\sigma_2} \tilde{P}(0, t \sigma_2, x, z') dt = \int_{0}^{1} \frac{(1-a_0^2)\sigma_2^2}{(1+t^2\sigma_2^2)^{3/2}} \sqrt{1 - \sin^2 \theta} \sqrt{t^2 \sigma_2^2 - s(\theta)} dt
\[ = \frac{1-a_0^2}{2\sqrt{1 - \sin^2 \theta}} \int_{0}^{\sigma_2^2} \frac{d\tau}{(1+\tau)^{3/2}(\tau - s(\theta))^{1/2}}. \]

Thus we have
\[
|\sigma_2 \int_{0}^{1} \partial_{\sigma_2} \tilde{P}(0, t \sigma_2, x, z') dt| \leq \frac{1-a_0^2}{2\sqrt{1 - \sin^2 \theta}} \int_{0}^{\sigma_2^2} |\tau - s(\theta)|^{-1/2} d\tau.
\]
When \( s(\theta) < \sigma_2^2 \), we have
\[
\left| \sigma_2 \int_0^1 \partial_{\sigma_2} \tilde{P}(0, t\sigma_2, x, z') \ dt \right| \\
\leq \frac{1 - a_0^2}{2\sqrt{1 - \sin^2 \theta}} \left( \int_0^{s(\theta)} (s(\theta) - \tau)^{-1/2} \ d\tau + \int_{s(\theta)}^{\sigma_2^2} (\tau - s(\theta))^{-1/2} \ d\tau \right) \\
= \frac{1 - a_0^2}{2\sqrt{1 - \sin^2 \theta}} \left( 2s(\theta)^{1/2} + 2(\sigma_2^2 - s(\theta))^{1/2} \right) \\
\leq C(s(\theta)^{1/2} + |\sigma_2|) \\
\leq C|\sigma_2|.
\]

When \( s(\theta) \geq \sigma_2^2 \), we have
\[
\left| \sigma_2 \int_0^1 \partial_{\sigma_2} \tilde{P}(0, t\sigma_2, x, z') \ dt \right| \leq \frac{1 - a_0^2}{2\sqrt{1 - \sin^2 \theta}} \left( \int_0^{s(\theta)} (s(\theta) - \tau)^{-1/2} \ d\tau \right) \\
\leq \frac{1 - a_0^2}{\sqrt{1 - \sin^2 \theta}} \left\{ \sqrt{s(\theta)} - \sqrt{s(\theta) - \sigma_2^2} \right\} \\
\leq C \frac{\sigma_2 \sqrt{s(\theta)}}{1 + \sqrt{1 - \sigma_2^2/s(\theta)}} \\
\leq C|\sigma_2|.
\]

Next, the second term can be estimated as follows
\[
|\sigma_1 \partial_{\sigma_1} \tilde{P}(t\sigma_1, \sigma_2, x, z')|^2 \\
= \frac{\sigma_1^2}{4|\tilde{P}(t\sigma_1, \sigma_2, x, z')|^2} \left\{ 4(t\sigma_1)^2 \cos^2(2\theta) + \sin^2(2\theta) \left( \frac{(t\sigma_1)^2}{\sqrt{1 + (t\sigma_1)^2}} + \sqrt{1 + (t\sigma_1)^2} \right)^2 \right\} \\
\leq C \frac{\sigma_1^2}{|t|} \\
\leq C \frac{|\sigma_1|}{|t|},
\]

by using (2.33). Then,
\[
\left| \sigma_1 \int_0^1 \partial_{\sigma_1} \tilde{P}(t\sigma_1, \sigma_2, x, z') \ dt \right| \leq \tilde{C}|\sigma_1|^{1/2} \int_0^1 \frac{1}{\sqrt{t}} \ dt = 2\tilde{C}|\sigma_1|^{1/2}.
\]

Thus we have (2.18) for \(|\sigma| \leq 2\) and \(z' \in \mathbb{R}^2 \setminus U_1(x)\). Using this, we can follow the argument getting the estimates for (2.14) and (2.15). Thus, we obtain
\[
|E_{t,0}^{s,\gamma,-}(x, z')| \leq Ce^{-\gamma |x-z'|/\sqrt{\tau}} \quad (x \in \overline{D}, \ z' \in \mathbb{R}^2 \setminus U_1(x)), \quad (2.34)
\]
\[
|E_{t,k}^{s,\gamma,-}(x, z')| \leq Ct_{\gamma}e^{-\gamma |x-z'|/\sqrt{\tau}} \quad (x \in \overline{D}, \ z' \in \mathbb{R}^2 \setminus U_1(x), \ k = 1, 2, 3). \quad (2.35)
\]
From now, we shall estimate $E_{\tau,k}^{m,\gamma}(x,z')$. From (2.30), the integral $I_{\tau,k}^m(x-z',\zeta_2)$ ($k = 0, \ldots, 3$) can be expressed by

$$I_{\tau,0}^m(x-z',\zeta_2) = -8\sqrt{\gamma_1} \sqrt{1 + \zeta_2^2} \int_{b_0(\zeta_2)}^{\sin \theta} G(w,\zeta_2) e^{-\tau r_0(w) \theta} \, dw,$$

$$I_{\tau,k}^m(x-z',\zeta_2) = 8\sqrt{\gamma_1} (1 + \zeta_2^2) \int_{b_0(\zeta_2)}^{\sin \theta} w G(w,\zeta_2) e^{-\tau r_0(w) \theta} \, dw, \quad (k = 1, 2),$$

$$I_{\tau,3}^m(x-z',\zeta_2) = 8\sqrt{\gamma_1} (1 + \zeta_2^2) \int_{b_0(\zeta_2)}^{\sin \theta} \sqrt{1 - w^2} G(w,\zeta_2) e^{-\tau r_0(w) \theta} \, dw,$$

where

$$G(w,\zeta_2) = \frac{a_0^2 \sqrt{1 - w^2} \sqrt{b_0(\zeta_2)^2}}{a_0^4 (1 - w^2) + |w^2 - b_0(\zeta_2)^2|}.$$  

Since $0 \leq G(w,\zeta_2) \leq 1/2$, we have

$$|I_{\tau,0}^m(x-z',\zeta_2)| \leq 4\sqrt{\gamma_1} \sqrt{1 + \zeta_2^2} \int_{\sin \theta_0}^{\sin \theta} e^{-\tau r(w \sin \theta + \sqrt{1-w^2} \cos \theta)} \, dw,$$

where $\sin \theta_0 = a_0 < b_0(\zeta_2)$ is used. Moreover, from the change of variable $w = \sin \alpha$ and the relation $w \sin \theta + \sqrt{1-w^2} \cos \theta = \sin \alpha \sin \theta + \cos \alpha \cos \theta = \cos(\theta - \alpha)$ it follows that

$$|I_{\tau,0}^m(x-z',\zeta_2)| \leq 4\sqrt{\gamma_1} \sqrt{1 + \zeta_2^2} \int_{\theta_0}^{\theta} \cos \alpha e^{-\tau T(\alpha) \sqrt{1+\zeta_2^2}} \, d\alpha,$$

where we used expression (2.12) of $T(\alpha) = T_{x,z}(\alpha)$ defined by (2.11). In the same way as above, we have

$$|I_{\tau,k}^m(x-z',\zeta_2)| \leq 4\sqrt{\gamma_1} (1 + \zeta_2^2) \int_{\theta_0}^{\theta} \sin \alpha \cos \alpha e^{-\tau T(\alpha) \sqrt{1+\zeta_2^2}} \, d\alpha \quad (k = 1, 2),$$

$$|I_{\tau,3}^m(x-z',\zeta_2)| \leq 4\sqrt{\gamma_1} (1 + \zeta_2^2) \int_{\theta_0}^{\theta} \cos^2 \alpha e^{-\tau T(\alpha) \sqrt{1+\zeta_2^2}} \, d\alpha.$$

Since $2\theta - \theta_0 - \alpha \geq \alpha - \theta_0 > 0$ for $\theta_0 \leq \alpha \leq \theta$, noting $\sin t \geq 2t/\pi$ for $0 \leq t \leq \pi/2$, we obtain

$$T(\alpha) - T(\theta_0) = \frac{2|z' - x|}{\sqrt{\gamma_1}} (\cos(\theta - \alpha) - \cos(\theta - \theta_0))$$

$$= \frac{2|z' - x|}{\sqrt{\gamma_1}} \sin \left(\frac{2\theta - \theta_0 - \alpha}{2}\right) \sin \left(\frac{\alpha - \theta_0}{2}\right)$$

$$\geq \frac{2|z' - x|}{\pi^2 \sqrt{\gamma_1}} (2\theta - \theta_0 - \alpha)(\alpha - \theta_0)$$

$$\geq \frac{2|z' - x|}{\pi^2 \sqrt{\gamma_1}} (\alpha - \theta_0)^2.$$

Hence, we have

$$|I_{\tau,k}^m(x-z',\zeta_2)| \leq C (1 + \zeta_2^2)^{k_0+1} e^{-\tau T(\theta_0) \sqrt{1+\zeta_2^2}} \int_{\theta_0}^{\theta} e^{-\frac{2|z' - x|}{\pi^2 \sqrt{\gamma_1}} (\alpha - \theta_0)^2} \, d\alpha.$$
\[ C_\tau^{-1/2}(1 + \zeta_2^2)^{2k_0+1} e^{-\tau T(\theta_0)\sqrt{1+\zeta_2^2}}, \]

where \( k_0 = 0 \) for \( k = 0 \) and \( k_0 = 1 \) for \( k = 1, 2, 3 \). Thus we have

\[ \left| \int_\mathbb{R} I_{r,k}^m(x - z', \zeta_2) \, d\zeta_2 \right| \leq \tilde{C}_\tau^{-1/2} \int_0^\infty e^{-\tau T(\theta_0)\sqrt{1+\zeta_2^2}}(1 + s^2)^{(2k_0+1)/4} \, ds. \]

Since \( \sqrt{1 + s^2} \geq 1 + s^2/3 \) for \( 0 \leq s \leq 1 \), it follows that

\[ \int_0^1 e^{-\tau T(\theta_0)\sqrt{1+\zeta_2^2}}(1 + s^2)^{(2k_0+1)/4} \, ds \\
\leq \int_0^1 2 e^{-\tau T(\theta_0)}(1 + s^2)^{2s^2/3} \, ds + \int_1^\infty (1 + s^2)^{(2k_0+1)/4} \sqrt{2s} e^{-\tau T(\theta_0)\sqrt{1+\zeta_2^2}} \, ds \\
\leq \frac{2(2k_0-3)/4}{\sqrt{\tau T(\theta_0)}} + \sqrt{2} \int_1^\infty e^{-\tau T(\theta_0)\tilde{s}^{k_0+1/2}} \, d\tilde{s} (\tilde{s} = \sqrt{1 + s^2}) \\
\leq C_\tau^{-1} e^{-\tau T(\theta_0)} + \sqrt{2} \int_1^\infty e^{-\tau T(\theta_0)(\sqrt{2} + \tilde{s})^{k_0+1/2}} \, d\tilde{s} \\
\leq \tilde{C}_\tau^{-1/2} e^{-\tau T(\theta_0)}. \]

Thus we have

\[ \left| \int_\mathbb{R} I_{r,k}^m(x - z', \zeta_2) \, d\zeta_2 \right| \leq C_\tau^{-1} e^{-\tau T(\theta_0)}, \]

which means that

\[ |E_{\tau,k}^m(x, z')| \leq C_\tau^{-k_0} e^{-\tau T(\theta_0)} \quad (k = 0, 1, 2, 3) \] (2.36)

from (2.31) and (2.32). Since \( \frac{|x - z'|}{\sqrt{\gamma_-}} \geq \frac{|x - z'|}{\sqrt{\gamma_-}} \cos(\theta - \theta_0) = T(\theta_0) \), Proposition 2.3 is proved by (2.34) - (2.36).

\[ \square \]

## 3 The optical distance and asymptotics of \( \Phi_\tau(x, y) \)

For \((x, y) \in \mathbb{R}_-^3 \times \mathbb{R}_+^3\), we define \( \tilde{l}_{x,y}(z') \) by

\[ \tilde{l}_{x,y}(z') = \begin{cases} 
    l_{x,y}(z') & (z' \in U_1(x)), \\
    |x_3| \cos \theta_0 \frac{|x' - z'| + |z' - y|}{\sqrt{\gamma_-}} & (z' \in \mathbb{R}^3 \setminus U_1(x)),
\end{cases} \]

(3.1)

where \( 0 < \theta_0 < \pi/2 \) is given by \( \sin \theta_0 = a_0 < 1 \). Note that

\[ T(\theta_0) + \frac{|z' - y|}{\sqrt{\gamma_+}} = \frac{1}{\sqrt{\gamma_-}} \left( |x_3| \cos \theta_0 + |z' - x'| \sin \theta_0 \right) + \frac{|z' - y|}{\sqrt{\gamma_+}} = \tilde{l}_{x,y}(z') \]

(3.2)

for \( z' \in \mathbb{R}^3 \setminus U_1(x) \).
Proposition 2.3 shows that $\tilde{l}_{x,y}(z')$ gives the time in which the waves travel from $x$ to $y$ via $\tilde{z} \in \partial \mathbb{R}^3_+$ if $z' \in \mathbb{R}^2 \setminus \mathcal{U}_1(x)$. This arrival time $\tilde{l}_{x,y}(z')$ is different from $l_{x,y}(z')$, which is caused by the total reflection phenomena.

Let us explain the meaning of $\tilde{l}_{x,y}(z')$ for $z' \in \mathbb{R}^2 \setminus \mathcal{U}_1(x)$. Since $|x' - z'|/|x - z'| > \sin \theta_0$, there exists a point $z'_0 = z'_0(x, z') \in \mathbb{R}^2$ on the line segment $x'z'$ such that $|x' - z'_0|/|x - z'_0| = \sin \theta_0$ and $|x' - z'| = |x' - z'_0| + |z'_0 - z'|$. Note that $\tilde{l}_{x,y}(z')$ is written by

$$\tilde{l}_{x,y}(z') = \frac{\cos \theta_0}{\sqrt{\gamma_-}} |x - z'_0| + \frac{|x' - z'_0| + |z'_0 - z'| + |\tilde{z}' - y|}{\sqrt{\gamma_+}}$$

$$= \frac{\cos^2 \theta_0}{\sqrt{\gamma_-}} |x - z'_0| + \frac{\sqrt{\gamma_-} |x - z'_0|}{\sqrt{\gamma_+} + \sqrt{\gamma_-}} + \frac{|z'_0 - z'|}{\sqrt{\gamma_+}} + \frac{|\tilde{z}' - y|}{\sqrt{\gamma_+}}$$

$$= \frac{|x - z'_0|}{\sqrt{\gamma_-}} + \frac{|z'_0 - z'| + |\tilde{z}' - y|}{\sqrt{\gamma_+}}. \quad (3.3)$$

This means that if the total reflection is caused, i.e. $z' \in \mathbb{R}^2 \setminus \mathcal{U}_1(x)$, the waves emanating from $x$ and arriving at $y$ via $\tilde{z}'$ go to $z'_0 \in \partial \mathbb{R}^3_+$ first, move to $\tilde{z}'$ along the transmission boundary $\partial \mathbb{R}^3_+$, and travel to $y$ in $\mathbb{R}^3_+$.

To obtain Proposition 1.4, we need to find $\inf_{z' \in \mathbb{R}^2} \tilde{l}_{x,y}(z')$. From Lemma 4.1 in [10], $l(x, y)$ in (1.2) is attained by only one point $z'(x, y)$ which is $C^\infty$ for $(x, y) \in \mathbb{R}_-^3 \times \mathbb{R}_+^3$. Note that this point $z'(x, y)$ is determined by Snell’s law

$$\frac{\sin \theta_-}{\sqrt{\gamma_-}} = \frac{\sin \theta_+}{\sqrt{\gamma_+}}, \quad (3.4)$$

where $0 \leq \theta_+ < \pi/2$ is taken by

$$\sin \theta_- = \frac{|z'(x, y) - x'|}{|z'(x, y) - x|}, \quad \sin \theta_+ = \frac{|z'(x, y) - y'|}{|z'(x, y) - y|}. \quad (3.5)$$

As in the proof of Lemma 4.1 in [10],

$$|z(x, y) - x'| \leq |x' - y'|, \quad |z(x, y) - y'| \leq |x' - y'|, \quad (3.6)$$

since $l(x, y) = \inf \{l_{x,y}(z') \mid z' \in \mathbb{R}^2, |z' - x'| \leq |x' - y'|, |z' - y'| \leq |x' - y'|\}$.

Here, we show the following properties of $z'(x, y)$ and the function $\tilde{l}_{x,y}(z')$.

**Lemma 3.1** (1) For any $x, y \in \mathbb{R}^3$ with $x_3 < 0$ and $y_3 > 0$, $\inf_{z' \in \mathbb{R}^2} \tilde{l}_{x,y}(z') = l(x, y)$, and this infimum is attained at only $z' = z'(x, y)$.

(2) There exists a constant $0 < \delta_0 < 1$ such that for any $(x, y) \in \overline{D \times B}$, $z'(x, y) \in \overline{U_{\delta_0}}(x)$, that is

$$|x' - z'(x, y)| \leq a_0 \delta_0 |x - z'(x, y)| \quad ((x, y) \in \overline{D \times B}), \quad (3.7)$$

and for any $\delta_1 > 0$ with $\delta_0 < \delta_1$, there exists a constant $c_0 > 0$ such that

$$\tilde{l}_{x,y}(z') \geq l(x, y) + c_0 |z' - z'(x, y)| \quad ((x, y) \in \overline{D \times B}, z' \in \mathbb{R}^2 \setminus \mathcal{U}_{\delta_1}(x)). \quad (3.8)$$
Proof. From Lemma 4.1 in [10], (1) of Lemma 3.1 is obvious if (2) of Lemma 3.1 is obtained. To show (2) of Lemma 3.1, we take constants $A > 0$ and $L > 0$ satisfying
\[ |x' - y'| \leq L, A \leq |x_3| \leq A^{-1}, A \leq |y_3| \leq A^{-1} \quad (x \in \overline{D}, y \in \overline{B}). \tag{3.9} \]

From Snell’s law (3.4), for any $x \in \overline{D}$ and $y \in \overline{B}$, $\theta_{\pm}$ defined by (3.5) satisfies $\sin \theta_- = \sqrt{\frac{L}{\gamma_+}} \sin \theta_+ = \sin \theta_0 \sin \theta_+ = a_0 \sin \theta_+$. From (3.5), (3.6) and (3.9), it follows that
\[ 0 \leq \sin \theta_+ = \frac{|y' - z'|}{\sqrt{|y' - z'|^2 + y_3^2}} \leq \frac{|y - z'|}{\sqrt{|y - z'|^2 + A^2}} \leq \frac{L}{\sqrt{L^2 + A^2}} < 1 \]

since $t \mapsto \frac{t}{\sqrt{t^2 + A^2}}$ is monotone increasing for $t \geq 0$. Choose $0 < \theta_{\text{max}} < \pi/2$ satisfying $\sin \theta_{\text{max}} = \frac{L}{\sqrt{L^2 + A^2}}$. Then for any $x \in \overline{D}$ and $y \in \overline{B}$, we have $\sin \theta_- = a_0 \sin \theta_+ \leq a_0 \sin \theta_{\text{max}} < a_0$. Hence, putting $\delta_0 = \sin \theta_{\text{max}}$, we obtain
\[ z'(x, y) \in \overline{U}_{\sin \theta_{\text{max}}}(x) = \overline{U}_{\delta_0}(x) \quad (x \in \overline{D}, y \in \overline{B}), \tag{3.10} \]

which gives (3.7).

It suffices to show (3.8) for $\delta_1$ with $0 < \delta_0 < \delta_1 < 1$, since $\overline{U}_{\delta_1}(x) \subset \overline{U}_{\delta_0}(x)$ for $\delta_1 < \delta_2$. Take any $\delta_1$ with $1 > \delta_1 > \delta_0$ and put $\mathcal{K} = \{(x, y, z') \in \overline{D} \times \overline{B} \times \mathbb{R}^2 \mid z' \in \overline{U}_{1}(x) \setminus \overline{U}_{\delta_1}(x)\}$. Noting (2.10), we obtain
\[ \frac{a_0 \delta_1}{\sqrt{1 - a_0^2 \delta_1^2}} |x_3| \leq |x' - z'| \leq \frac{a_0}{\sqrt{1 - a_0^2}} |x_3| \quad (z' \in \overline{U}_{1}(x) \setminus \overline{U}_{\delta_1}(x)). \tag{3.11} \]

Compactness of $\overline{D} \times \overline{B}$, (3.9) and (3.11) imply that $\mathcal{K}$ is compact. From (3.7), (3.9), (3.10) and (3.11), it follows that there exists a constant $c_1 > 0$ such that
\[ \frac{l_{x, y}(z') - l(x, y)}{|z' - z'(x, y)|} \geq c_1 \quad ((x, y, z') \in \mathcal{K}), \tag{3.12} \]

since the function in the above is positive and continuous on $\mathcal{K}$.

Next, take an arbitrary $(x, y, z') \in \overline{D} \times \overline{B} \times \mathbb{R}^2$ with $z' \in \mathbb{R}^2 \setminus \overline{U}_{1}(x)$. We take $z'_0 = z_0'(x, z')$ as in (3.3). Since $y \neq z'$ and $z' \neq z'_0$, from
\[ |y - z'|^2 = |y - z'_0| - (z' - z'_0) \cdot \frac{y - z'_0}{|y - z'_0|} |y - z'_0| + |z' - z'_0|^2 \left(1 - \left(\frac{z' - z'_0}{|z' - z'_0|} \cdot \frac{y - z'_0}{|y - z'_0|}\right)^2\right), \]
we have
\[ |y - z'| + |z' - z'_0| \geq |y - z'_0| + |z' - z'_0| \left(1 - \left(\frac{z' - z'_0}{|z' - z'_0|} \cdot \frac{y - z'_0}{|y - z'_0|}\right)^2\right). \tag{3.13} \]

Since $z'_0 \in \overline{U}_{1}(x)$, from (3.9) and (2.10), it follows that
\[ |z'_0 - y'| \leq |x' - z'_0| + |x' - y'| \leq L + \frac{a_0}{\sqrt{1 - a_0^2}} |x_3| \leq R, \tag{3.14} \]
where \( R = L + \frac{a_0}{A\sqrt{1-a_0^2}} > 0 \). From (3.14) and (3.9), it follows that
\[
\frac{|\tilde{z}' - \tilde{z}'_0| - |\tilde{z}' - \tilde{z}'_0|}{|\tilde{z}' - \tilde{z}'_0|} = \frac{|(z' - z'_0) \cdot (z'_0 - y')|}{|z' - z'_0||\tilde{z}'_0 - y|} \leq \frac{|z'_0 - y'|}{\sqrt{|\tilde{z}'_0 - y|^2 + y_3^2}} \leq \frac{R}{\sqrt{R^2 + A^2}}.
\]
since \( t \mapsto t/\sqrt{t^2 + A^2} \) is monotone increasing. Combining this with (3.13), we obtain
\[
|y - \tilde{z}'| + |z' - \tilde{z}'_0| \geq |y - \tilde{z}'_0| + c_2|z' - \tilde{z}'_0|
\]
where \( c_2 = \frac{A^2}{\sqrt{R^2 + A^2(\sqrt{R^2 + A^2})}} > 0 \). From (3.3), it follows that
\[
\bar{l}_{x,y}(z') = \frac{|x - \tilde{z}'_0|}{\sqrt{\gamma_-}} + \frac{|z'_0 - z'| + |\tilde{z}' - y|}{\sqrt{\gamma_+}} \geq \frac{|x - \tilde{z}'_0|}{\sqrt{\gamma_-}} + \frac{|y - \tilde{z}'_0| + c_2|z' - \tilde{z}'_0|}{\sqrt{\gamma_+}} = l_{x,y}(z'_0) + \frac{c_2}{\sqrt{\gamma_+}}|z' - \tilde{z}'_0|.
\]
Since \( x \in \mathbb{R}^3 \) and \( \tilde{z}'_0 \in \overline{\mathcal{U}_1(x) \setminus \mathcal{U}_{\delta_0}(x)} \), (3.12) implies
\[
l_{x,y}(z'_0) \geq l(x,y) + c_1|z'_0 - \tilde{z}'(x,y)|.
\]
Combining these estimates and taking \( c_0 = \min\{c_1, \frac{c_2}{\sqrt{\gamma_+}}\} \), we obtain
\[
\bar{l}_{x,y}(z') \geq l(x,y) + c_0(|z'_0 - \tilde{z}'(x,y)| + |z'_0 - \tilde{z}'|) \geq l(x,y) + c_0|z' - \tilde{z}'(x,y)| \quad (x \in \overline{D}, y \in \overline{B}, z' \in \mathbb{R}^2 \setminus \mathcal{U}_1(x)).
\]
This estimate and (3.12) imply (3.8), which completes the proof of Lemma 3.1. □

![Figure 3: Propagation from the lower half-space](image)

**Figure 3: Propagation from the lower half-space**

**Proof of Proposition 1.4.** For \( 0 < \delta_0 < 1 \) given in Lemma 3.1, take \( \delta_1 \) with \( \delta_0 < \delta_1 < 1 \) and put \( \varepsilon_0 = \frac{4}{6}(a_0\delta_1/\sqrt{1-a_0^2\delta_1^2} - a_0\delta_0/\sqrt{1-a_0^2\delta_0^2}) > 0 \). First, we show
\[
\{z' \in \mathbb{R}^2 ||z' - \tilde{z}'(x,y)|| \leq 3\varepsilon_0 \} \subset \mathcal{U}_{\delta_1}(x) \quad (x \in \overline{D}, y \in \overline{B}). \tag{3.15}
\]
Choose any \( x \in \overline{D} \) and \( y \in \overline{B} \). Since (3.7) is equivalent to \( \tilde{z}'(x,y) \in \overline{\mathcal{U}_{\delta_0}(x)} \), noting (2.10) we have
\[
|x' - \tilde{z}'(x,y)| \leq \frac{a_0\delta_0}{\sqrt{1-a_0^2\delta_0^2}}|x_3|, \tag{3.16}
\]
If $z'$ satisfies $|z' - z'(x, y)| \leq 3\varepsilon_0$, from (3.9) it follows that

$$|z' - z'(x, y)| \leq 3\varepsilon_0 A^{-1}|x_3| \leq \frac{|x_3|}{2} \left( \frac{a_0\delta_1}{\sqrt{1 - a_0^2\delta_1^2}} - \frac{a_0\delta_0}{\sqrt{1 - a_0^2\delta_0^2}} \right).$$

This and (3.16) imply

$$|x' - z'| \leq |z' - z'(x, y)| + |x' - z'(x, y)|$$

$$\leq \frac{1}{2} \left( \frac{a_0\delta_0}{\sqrt{1 - a_0^2\delta_0^2}} + \frac{a_0\delta_1}{\sqrt{1 - a_0^2\delta_1^2}} \right) |x_3|$$

$$< \frac{a_0\delta_1}{\sqrt{1 - a_0^2\delta_1^2}} |x_3|,$$

which yields (3.15) by noting (2.10).

For $\delta_1$ given in (3.15), we divide the integral in (2.1) into three parts:

$$\nabla^k \Phi_\tau(x, y) = \frac{\tau}{4\pi\gamma_+} (I_{k,\tau}(x, y) + J^{(1)}_{k,\tau}(x, y) + J^{(2)}_{k,\tau}(x, y)) \quad (k = 0, 1),$$

(3.17)

where

$$I_{k,\tau}(x, y) = \int_{U_{\delta_1}(x)} \nabla^k E^{\gamma_-}_\tau(x, z') e^{-\tau|z' - y|/\sqrt{\gamma_+}} |z' - y| dz',$$

and $J^{(1)}_{k,\tau}(x, y)$ and $J^{(2)}_{k,\tau}(x, y)$ are the integrals defined by replacing the integrated region $U_{\delta_1}(x)$ in $I_{k,\tau}(x, y)$ with $\overline{U_1(x)} \setminus U_{\delta_1}(x)$ and $\mathbb{R}^2 \setminus U_1(x)$ respectively. Taking a cutoff function $\phi \in C_0^\infty(\mathbb{R}^2)$ with $0 \leq \phi \leq 1$, $\phi(z') = 1$ for $|z'| \leq \varepsilon_0$ and $\phi(z') = 0$ for $|z'| \geq 2\varepsilon_0$, we define

$$I^{(0)}_{k,\tau}(x, y) = \int_{U_{\delta_1}(x)} \phi(z' - z'(x, y))) \nabla^k E^{\gamma_-}_\tau(x, z') e^{-\tau|z' - y|/\sqrt{\gamma_+}} |z' - y| dz',$$

$$I^{(-\infty)}_{k,\tau}(x, y) = \int_{U_{\delta_1}(x)} (1 - \phi(z' - z'(x, y))) \nabla^k E^{\gamma_-}_\tau(x, z') e^{-\tau|z' - y|/\sqrt{\gamma_+}} |z' - y| dz'.$$

Note that Lemma 2.1 and Proposition 2.2 imply that there exists a constant $C > 0$ such that

$$|\nabla^k E^{\gamma_-}_\tau(x, z')| \leq C \tau^k e^{-\tau|x - z'|/\sqrt{\gamma_+}} \quad (x \in \overline{D}, z' \in \overline{U_1(x)}, k = 0, 1).$$

(3.18)

Hence, for $y \in \overline{D}$ and $k = 0, 1$, from (3.9) and (3.8), it follows that

$$|J^{(1)}_{k,\tau}(x, y)| \leq C \tau^k \int_{U_1(x) \setminus U_{\delta_1}(x)} |\nabla^k E^{\gamma_-}_\tau(x, z')| e^{-\tau|z' - y|/\sqrt{\gamma_+}} |z' - y| dz'$$

$$\leq C \tau^k \int_{U_1(x) \setminus U_{\delta_1}(x)} e^{-\tau|x - z'|/\sqrt{\gamma_+}} e^{-\tau|z' - y|/\sqrt{\gamma_+}} |z' - y| dz'$$

$$= C \tau^k \int_{U_1(x) \setminus U_{\delta_1}(x)} e^{-\tau|I_{x,y}(z')|} dz'.$$
follows that there exists a constant
This and (3.18) imply
Hence, similarly to getting (3.20), we obtain
Since (3.15) yields
\[ |J_{k,\tau}^{(2)}(x, y)| \leq \int_{\mathbb{R}^2 \setminus \mathcal{U}_{\delta_1}(x)} |\nabla_x E_\tau^{\gamma}(x, z')| e^{-\tau|z' - y|/\sqrt{\tau}} dz' \]
\[ \leq C \tau^k \int_{\mathbb{R}^2 \setminus \mathcal{U}_{\delta_1}(x)} e^{-\tau T(\xi_0)} e^{-\tau|z' - y|/\sqrt{\tau}} dz' \]
\[ = C \tau^k \int_{\mathbb{R}^2 \setminus \mathcal{U}_{\delta_1}(x)} e^{-\tau t_{x,y}(z')} dz' \]
\[ \leq C \tau^k e^{-\tau l(x,y)} \int_{\mathbb{R}^2 \setminus \mathcal{U}_{\delta_1}(x)} e^{-c_0 \tau|z' - z(x,y)|} dz'. \]
Thus, we obtain
\[ |J_{k,\tau}^{(1)}(x, y)| + |J_{k,\tau}^{(2)}(x, y)| \leq C \tau^k e^{-\tau l(x,y)} \int_{\mathbb{R}^2 \setminus \mathcal{U}_{\delta_1}(x)} e^{-c_0 \tau|z' - z(x,y)|} dz'. \tag{3.19} \]
Since (3.15) yields \(|z' - z(x,y)| \geq 3\varepsilon_0\) for any \(x \in \overline{D}, y \in \overline{B}\) and \(z' \in \mathbb{R}^2 \setminus \mathcal{U}_{\delta_1}(x)\), it follows that
\[ \int_{\mathbb{R}^2 \setminus \mathcal{U}_{\delta_1}(x)} e^{-c_0 \tau|z' - z(x,y)|} dz' \leq e^{-c_0 \varepsilon_0 \tau} \int_{\mathbb{R}^2} e^{-c_0 \tau|z' - z(x,y)|/2} dz' \]
\[ \leq \frac{8\pi}{c_0^2 \tau^2} e^{-c_0 \varepsilon_0 \tau} \quad (x \in \overline{D}, y \in \overline{B}). \]
This and (3.19) imply that there exists a constant \(C_2 > 0\) such that
\[ |J_{k,\tau}^{(1)}(x, y)| + |J_{k,\tau}^{(2)}(x, y)| \leq C_2 \tau^{-k-2} e^{-c_0 \varepsilon_0 \tau} e^{-\tau l(x,y)} \quad (x \in \overline{D}, y \in \overline{B}, k = 0, 1). \tag{3.20} \]
Since the set \(\{(x, y, z') \in \overline{D} \times \overline{B} \times \mathbb{R}^2 \mid z' \in \overline{\mathcal{U}_{\delta_1}(x)}, |z' - z'(x, y)| \geq \varepsilon_0\}\) is compact, it follows that there exists a constant \(c_1 > 0\) such that
\[ \frac{l_{x,y}(z') - l(x,y)}{|z' - z'(x,y)|} \geq 2c_1 \quad (x \in \overline{D}, y \in \overline{B}, z' \in \overline{\mathcal{U}_{\delta_1}(x)}, \phi(z' - z'(x, y)) \neq 1). \]
This and (3.18) imply
\[ |I_{k,\tau}^{(-\infty)}(x, y)| \leq C \tau^k e^{-\tau l(x,y)} \int_{\{z' \in \mathcal{U}_{\delta_1}(x) \mid |z' - z'(x,y)| \geq \varepsilon_0\}} e^{-2c_1 \tau|z' - z'(x,y)|} dz'. \]
Hence, similarly to getting (3.20), we obtain
\[ |I_{k,\tau}^{(-\infty)}(x, y)| \leq C \tau^{-k-2} e^{-c_1 \varepsilon_0 \tau} e^{-\tau l(x,y)} \quad (x \in \overline{D}, y \in \overline{B}, k = 0, 1). \tag{3.21} \]
For $I_{0,\tau}^{(0)}(x, y)$, from Lemma 2.1, it follows that

$$I_{0,\tau}^{(0)}(x, y) = \frac{1}{4\pi \gamma_0^2} \sum_{j=0}^{N-1} \tau^{-j} \int_{U_{0j}(x)} e^{-\tau l_{x,y}(z')} f_j(z'; x, y) \phi(z' - z(x, y)) dz'$$

$$+ \frac{1}{4\pi \gamma_0^2} \int_{U_{0j}(x)} \frac{e^{-\tau l_{x,y}(z')}}{|x - \tilde{z}'|} \tilde{E}_N(x, z'; \tau) \phi(z' - z'(x, y)) dz',$$

where

$$f_j(z'; x, y) = \frac{\gamma_j^2}{|x - \tilde{z}'|^j |z' - y|} E_j(x - \tilde{z}') \quad (j = 0, 1, \ldots).$$

From Lemmas 2.1 and 3.1, the integral $I_N$ containing the remainder term $\tilde{E}_N(x, z'; \tau)$ in (3.22) is estimated by

$$|I_N| \leq C_{N,\delta_1} \frac{e^{-\tau l(x,y)}}{\tau^N} \int_{R^2} \frac{dz'}{|x - \tilde{z}'|^N |z' - y|} \leq C_{N,\delta_1} \frac{e^{-\tau l(x,y)}}{\tau^N} \quad (x \in D, y \in \overline{D}, \tau \geq 1).$$

From (3.15) and Lemma 3.1, we can handle the integrals containing $f_j$ in (3.22) as in the proof of Proposition 1 in [10]. Hence, $I_{0,\tau}^{(0)}(x, y)$ has the same asymptotic expansion as given in Proposition 1.4. Similarly, we can treat $I_{1,\tau}^{(0)}(x, y)$. Combining these facts with (3.17), (3.20) and (3.21), we obtain Proposition 1.4.

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