In this talk we entertain the possibility that the synthesis of general covariance and quantum mechanics requires an extension of the basic kinematical setup of quantum mechanics. According to the holographic principle, regions of space-time bounded by a finite area carry finite entropy. When we in addition assume that the origin of the entropy is a finite dimensional Hilbert space, and apply this to cosmological solutions using a suitable notion of complementarity, we find as a consequence that gravitational effects can lead to dynamical variation in the dimensionality of such Hilbert spaces. This happens generally in cosmological settings like our own universe.

1 Introduction

Quantum mechanics describes the intrinsic fluctuations of systems with a finite number of particles. While the state of a classical system is described as a point in position-momentum phase space, the wavefunction of a quantum mechanical system is a ray in Hilbert space. The synthesis of quantum mechanics with the special theory of relativity forces a generalization – the theory must accommodate transitions between states with different particle number. In particular, Lorentz invariance implies the existence of anti-particles. This leads to field theory, in which states are described as vectors in Fock space, a direct sum of Hilbert spaces describing different numbers of particles.

In this talk we discuss the possibility that the synthesis of quantum mechanics and general covariance requires further innovations in the basic kinematical setup of quantum mechanics. According to the holographic principle, regions of spacetime bounded by a finite area are described by a system with finite entropy. It has been suggested by many authors that this entropy is
associated with an accessible Hilbert space of finite dimensionality. We point out that gravitational effects can lead to dynamical variation in the dimensionality of such Hilbert spaces. This would happen generally in time-dependent cosmological settings including our own universe.

2 Local Gravitational Entropy and Hilbert Space Evolution

It has been known for over 25 years that black holes have an entropy proportional to surface area suggesting a deep connection between the dynamics of gravity and thermodynamics [2]. It has been proposed that the scaling of entropy with surface area rather than volume is a general feature of quantum gravity dubbed the holographic principle [3], wherein the dimension of the Hilbert space required to describe a system enclosed by a closed surface with area $A$ is bounded by $\exp S$ with $S = A/4G_N$. This suggests a breakdown of spacetime locality since the proposed entropy is not extensive in the volume enclosed. A refinement of the proposal says that $\exp S$ is the entropy encoded on converging lightsheets emerging from the closed bounding surface [4].

What does this proposal mean operationally? Perhaps the most remarkable aspect of it is that the entropy is finite, suggesting a finite dimensional Hilbert space of excitations describing the interior of a region bounded by a surface of area $A$. The cosmological implications are particularly intriguing since the finite entropy of, say, the horizon seen by inertial observers in de Sitter space suggests that, operationally, any observer requires a finite dimensional Hilbert space to describe his observable universe [5, 6, 7, 8, 9, 10].

In contrast, in conventional quantum mechanics, a single harmonic oscillator has an infinite dimensional Hilbert space.

1 Note that finite entropy does not always imply a finite dimensional Hilbert space. In quantum mechanics the entropy associated to a density matrix $\rho$ is $S = -\text{tr}(\rho \log \rho)$, and one can have finite $S$ and an infinite dimensional Hilbert space, as is well-known from quantum statistical mechanics. Nevertheless, a finite $S$ typically implies that a finite number of states is operationally accessible. In closed systems that are not coupled to an external heat bath (e.g., the universe as a whole) this implies a finite dimensional Hilbert space. We will assume that finite gravitational entropy generally implies a finite dimensional accessible Hilbert space, but this assumption clearly requires careful scrutiny.

2 This finite dimensional Hilbert space is sufficient to describe the entire universe if there is also a notion of complementarity, which states that the degrees of freedom behind the horizon are the same as the ones seen by the observer [11]. Whether there is indeed such a notion is still a matter of debate, see e.g. [2].
What are the implications of a local notion of holography associated to local observers? Useful insights are gained by invoking the equivalence principle and studying accelerated observers. Indeed, Jacobson has shown that assuming that the thermodynamic laws of horizon mechanics apply to Rindler horizons implies Einstein’s equations [13]. Specifically, consider families of accelerated local observers and assume that their Rindler horizons enjoy a first law of the form

$$\delta E = T \delta S,$$

(1)

where $\delta E = \int T_{\mu\nu} k^\mu k^\nu d\lambda$ is the energy flux of matter through the horizon, $T$ is the local Unruh temperature, $\delta S$ is a change in entropy, and $\lambda$ is the affine parameter of the horizon. Assume a second law relating entropy and area

$$\delta S = \delta A/4G_N.$$

(2)

Here $\delta A = \int \theta d\lambda$ is the change in area of the Rindler horizon, $\theta$ being the expansion of the horizon generators. Combining these equations with the Raychaudhuri equation (here we drop the $\theta^2$ term as well as the term involving the shear and $k^\mu$ is the tangent vector to the horizon generators)

$$\frac{d\theta}{d\lambda} = R_{\mu\nu} k^\mu k^\nu,$$

(3)

yields Einstein’s gravitational field equations, remarkably relating thermodynamic equations of state to dynamical equations of motion [13]. Let us now assume that the entropy is determined by the dimensionality of the accessible Hilbert space $\dim\mathcal{H} = \exp(S)$. Then the Raychaudhuri equation relates spacetime curvature to a change in the dimensionality of the Hilbert space.

Given our assumptions, such changes in the dimensionality of Hilbert spaces are generic consequences of the synthesis of general covariance and the usual interpretation of entropy in quantum mechanics. Consider, for example, any asymptotically AdS$_5$ space $\mathcal{M}$ and the duality between string theory on $\mathcal{M} \times S^5$ and the corresponding deformation of the superconformal $SU(N)$ Yang-Mills theory on $S^3 \times R$ [14]. $\mathcal{M}$, being infinite in size, requires an infinite dimensional Hilbert space to describe it. Suppose we cut off $\mathcal{M}$ at a finite radius – the finite bounding area then implies that a finite dimensional Hilbert space is needed to describe the interior volume. Indeed, the dual description of the interior region is obtained by cutting off the CFT at a certain energy scale, thereby obtaining a finite dimensional Hilbert space [15]. In fact, the renormalization group (RG) equation for the field theory dual to $\mathcal{M} \times S^5$ is identified with the equations of motion governing radial flow of the spacetime
solution with the affine parameter $\lambda$ along the flow identified with the RG scale $\Lambda$ [16]. The Raychaudhuri equation (of which (3) is a special case) applied to the radial flow of surfaces that bound regions of $\mathcal{M}$ translates into an equation for the RG evolution of the trace of the stress tensor of the dual field theory [16]. This quantity can be interpreted as measuring the number of accessible degrees of freedom $c$. Coupled with weak energy condition, the Raychaudhuri equation expresses the monotonicity of the flow of $c$: $c_{UV} > c_{IR}$ [16].

In this example, radial flows in $\mathcal{M}$ are directly related via the renormalization group to changing dimensionality of the accessible Hilbert space.

In Jacobson’s discussion of Rindler horizons the change in the dimensionality of the accessible Hilbert space occurs because matter flows through the horizon. While it is surprising that the change in entropy is reflected in horizon area, it is natural that more states are necessary to describe the space behind the horizon after additional matter falls in. We might say that the dimensionality of the Hilbert space whose entropy is encoded in horizon area changes because there is a flow of degrees of freedom from one side of the horizon to the other. In the AdS example the dimension of the accessible Hilbert space changes with radial flow because we explicitly choose to examine smaller regions of space, requiring fewer degrees of freedom. By contrast, below we will demonstrate that in a cosmological setting the intrinsic dynamics of gravity, rather than a choice of surface, can change the dimension of the Hilbert space accessible to an inertial observer.

3 Holography, Cosmology and Entropy

In the presence of a positive cosmological constant $\Lambda$, the maximally symmetric solution to Einstein’s equations is de Sitter space. Because of the rapid expansion of this universe, inertial observers are surrounded by a cosmological horizon which hides phenomena occurring in the region beyond. This cosmological horizon has a finite and constant area leading to an entropy $S = A/4G \sim \Lambda^{-(n-1)/2}/G_N$ for $(n+1)$-dimensional de Sitter space [5]. Many interpretations of de Sitter entropy have been suggested including: (a) it measures the size of the Hilbert space of the entire universe via $\dim H \sim e^S$, (b) it measures the dimension of the Hilbert space of states that are hidden behind the horizon, (c) it measures the size of the Hilbert space of excitations that the inertial observer can interact with. (See [1] and references in [4, 5, 6, 10].) In
any of these interpretations the relevant Hilbert space is finite dimensional.

An interesting perspective on the number of degrees of freedom required to describe asymptotically de Sitter spaces also arises if there is a holographic duality between such spaces and (deformations of) a Euclidean conformal field theory [7]. If there is such a duality, one would expect a relationship between RG flow of the field theory dual and time evolution in spacetime [8, 9]. As always, in a theory of gravity, the bulk Hamiltonian is zero, and so this is really a map between a holographic RG equation and the Hamiltonian constraint of the bulk gravitation theory, which at the quantum mechanical level becomes the Wheeler-de Witt equation [17].

We can put some flesh on this proposal by following the treatment of holographic RG flows in asymptotically AdS spaces [16]. We fix the gauge so that the bulk metric can be written as

\[ ds^2 = -dt^2 + g_{ij}dx^i dx^j. \]  

(4)

The Hamiltonian constraint reads

\[ \mathcal{H} = 0, \]  

(5)

where in the case of 4d bulk gravity

\[ \mathcal{H} = (\pi^{ij}\pi_{ij} - \frac{1}{2}\pi_{ij}\pi^{ij}) + \frac{1}{2}\pi_I G^{IJ}\pi_J + \mathcal{L}. \]  

(6)

Here \( \pi_{ij} \) and \( \pi_I \) are the canonical momenta conjugate to \( g^{ij} \) and \( \phi^I \) (\( \phi^I \) denotes some background test scalar fields). \( \mathcal{L} \) is a local Lagrangian density and \( G^{IJ} \) denotes the metric on the space of background scalar fields.

As in the context of the AdS/CFT duality [16], the Hamiltonian constraint can be formally rewritten as a renormalization group equation for the dual RG flow

\[ \frac{1}{\sqrt{g}} \left( \frac{1}{2}(g^{ij}\frac{\delta S}{\delta g^{ij}})^2 - \frac{\delta S}{\delta g^{ij}} \frac{\delta S}{\delta g_{ij}} - \frac{1}{2}G^{IJ}\frac{\delta S}{\delta \phi^I} \frac{\delta S}{\delta \phi^J} \right) = \sqrt{g}\mathcal{L}, \]  

(7)

provided the local 4d action \( S \) can be separated into a local and a non-local piece

\[ S(g, \phi) = S_{\text{loc}}(g, \phi) + \Gamma(g, \phi). \]  

(8)

Another interpretation was proposed in the context of a holographic dual description of de Sitter space in terms of an entangled state in a product of two (unconventional) conformal field theories associated with the two temporal infinities of de Sitter space [10]. Here the entropy of the cosmological horizon of one de Sitter observer arose by tracing over the part of the entangled state that is inaccessible to the observer.
Indeed, in that case the Hamiltonian constraint can be formally rewritten as a Callan-Symanzik renormalization group equation

\[ \frac{1}{\sqrt{g}} (g^{ij} \frac{\delta}{\delta g^{ij}} - \beta^I \frac{\delta}{\delta \phi^I}) \Gamma = HO, \tag{9} \]

where \( HO \) denotes higher derivative terms. Here the “beta-function” is defined (in analogy with the AdS situation) to be \( \beta^I = \partial_A \phi^I \), where \( A \) denotes the cut-off of the putative dual Euclidean theory.

Note that the separation of the local and non-local pieces in the 4d action is completely analogous to the way the many-fingered physical time is extracted from the Wheeler-de-Witt equation in the context of quantum cosmology \[17, 18\]. There the non-WKB part of the wave function satisfies the Tomonaga-Schwinger equation, which reduces to the usual time dependent Schrödinger equation for global time variations \[18\].

Some field theories have a ”c-function” which measures the number of accessible degrees of freedom and which decreases during RG flow. Following the AdS literature \[16\], we proposed a holographic c-function for asymptotically de Sitter spaces that is related to the trace of the Brown-York stress tensor \[19\] evaluated on equal time surfaces \[4\]. When the spacetime is four dimensional we would have

\[ c \sim \frac{1}{G \theta^2}, \tag{10} \]

where \( \theta \) is the trace of the extrinsic curvature of equal time surfaces. Note that the trace of the Brown-York stress tensor turns out to be

\[ \langle T^i_i \rangle \sim \theta, \tag{11} \]

up to some terms constructed from local intrinsic curvature invariants of equal time surfaces. Hence the the RG equation of a putative CFT dual to an asymptotically dS space would be given by

\[ \langle T^i_i \rangle \sim \frac{d \Gamma}{d A} = \beta^I \frac{\partial \Gamma}{\partial \phi^I}. \tag{12} \]

The Raychauduri equation then implies the monotonicity of the trace of the Brown-York stress tensor

\[ \frac{d \theta}{dt} \leq 0, \tag{13} \]

An alternative c-function, based on the fact that generic perturbations to de Sitter "lengthen" the Penrose diagram has been given in \[20\].
as long as a form of the weak positive energy condition is satisfied by the background test scalar fields. This in turn guarantees monotonic time evolution of the proposed holographic $c$-function, suggesting that the number of degrees of freedom required to describe an asymptotically de Sitter space can change monotonically with time.

To study this we can also directly examine the area of the cosmological horizon in the generic situation of a scalar field coupled to gravity in $n+1$ dimensions. Below we will show that this system admits solutions which are expanding/contracting universes in which inertial observers see a horizon whose size changes with time. Therefore the associated quantum mechanical Hilbert spaces must be changing in dimension. The equations of motion derived from the action

$$S = \frac{1}{16\pi G_N} \int d^{n+1}x \sqrt{-g} \left[ R - n(n-1)g^{ij}\partial_i\phi\partial_j\phi - n(n-1)V(\phi) \right],$$

with a metric ansatz

$$ds^2 = -dt^2 + a(t)^2 d\Omega^2$$

(where $d\Omega^2$ is the metric on the unit $n$-sphere) imply the Friedmann equations

$$\frac{1 + \dot{a}^2}{a^2} = \ddot{\phi}^2 + V(\phi), \quad \frac{\ddot{a}}{a} = -(n-1)\dot{\phi}^2 + V(\phi).$$

The vacuum energy of the scalar field acts as a cosmological constant $\Lambda$ and we seek a potential and solutions that interpolate between two values of $\Lambda$, implying that $a(t) \to e^{-bt}$ and $a(t) \to e^{ct}$ as $t \to \mp\infty$ respectively. Positivity of $\dot{\phi}^2$ implies

$$1 + \dot{a}^2 - a\ddot{a} \geq 0. \quad (17)$$

It is easy to find a scale factor $a(t)$ and associated potentials that meet these criteria. In order to determine the size of the horizon seen by an inertial observer we introduce conformal time $\eta(t) = \int_{-\infty}^{t} dt/a(t)$ in terms of which

$$ds^2 = a(\eta)^2 \left[-d\eta^2 + d\theta^2 + \sin^2\theta d\Omega_{n-1}^2 \right].$$

The cosmological horizon is obtained by following null geodesics emanating from $\eta = \theta = 0$. This leads to a horizon area and entropy that depends on conformal time as

$$S(\eta) = \frac{[a(\eta)\sin\eta]^{n-1} \text{vol}(\hat{S}_{n-1})}{4G_N},$$

where $\hat{S}_{n-1}$ is the unit $(n-1)$-sphere. Interestingly this entropy evolves monotonically just like a field theory c-function at least up to conformal times of
\[ \eta = \pi/2. \] Indeed (17) implies \[ \eta(t) \geq \pi/2 + \arctan \dot{\alpha}, \]
which in turns implies that \[ \partial S/\partial \eta \geq 0. \] The expression for the entropy (19) may be contrasted with the proposed holographic c-function in de Sitter space (20)

\[ c \sim \left( \frac{a^2}{1 + \dot{\alpha}^2} \right)^{(n-1)/2}, \] in the context of a possible duality between de Sitter space and a Euclidean conformal field theory [4]. The difference between these expressions presumably lies in the fact that while entropy measures the dimension of a Hilbert space, c counts available degrees of freedom.

This example shows that in a cosmological context an inertial observer may face a situation in which the accessible Hilbert space changes in dimension with the passage of time. Since the universe as a whole is an isolated system, this change must be attributed purely to the dynamical expansion of spacetime. Given that our universe probably passed through an era of inflation and appears to be entering a new de Sitter phase [21] these observations may have phenomenological relevance.

It is of course strange to have a Hilbert space whose dimension changes with time. It cannot depend continuously on time, so if one takes this seriously one is almost immediately led to a discretization of space-time as well. A more conservative approach might be to interpret the changing entropy as a change in the entropy of a density matrix. Density matrices can easily depend on continuous parameters, and in this way we don’t have to deal with the ‘quantized’ dimension of a Hilbert space.

4 Conclusion

We have argued that the synthesis of quantum mechanics and general covariance might require innovations in the classic structures of quantum mechanics. According to the holographic principle, regions of space of bounded surface area are described by finite dimensional Hilbert spaces. If so, we have pointed out that inertial observers in a cosmological setting need to describe the world in terms of Hilbert spaces whose dimensions vary in time and that our universe may realize such a situation. The mathematical framework for realizing such structures remains to be uncovered, even though it is tempting to speculate that the relevant mathematics might already have been discussed in the existing literature [22].
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