FREE AND PROJECTIVE GENERALIZED MULTINORMED SPACES

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Abstract. The paper investigates free and projective L-spaces, where L is a given normed space. These spaces form a far-reaching generalization of known p-multinormed spaces; in particular, if L = L_p(X), the L-spaces can be considered as p-multinormed spaces, based on arbitrary σ-finite measure spaces X (for “canonical” p-multinormed spaces, X = N with the counting measure). We first describe a “naturally appearing” functor, based on paving L with contractively complemented finite dimensional subspaces. This finite dimensionality is essential; it permits us to describe a free L-space for this functor. As a corollary, we obtain a wide variety of projective L-spaces. For “nice” L (such as the space of simple p-integrable functions on a measure space), we obtain a full description of projective L-spaces.

Introduction

In recent years a new substantial concept appeared in functional analysis. This is the notion of p-multinormed space [2], a natural extension of multinormed and dual multinormed spaces of Dales and Polyakov (where p is 1 or ∞) and, on the other hand, of “operator sequence spaces” of Lambert [11] (with p = 2, and convexity is assumed). These spaces happened to be intimately connected with the theory of Banach lattices, that is with the important part of a wide circle of questions concerning the concept of positivity. More recently, a connection between 2-multinormed spaces and sectorial operators was discovered and investigated by Kalton, Lorist, and Weis in [10].

After the paper [2], the p-multinormed spaces, in their turn, were generalized, in the frame-work of the so-called non-coordinate approach, to what can be considered as “p-multinormed spaces based on arbitrary measures” [6]. Taking L is a normed space (for instance, L_p(X) or L_p^0(X), where X is a σ-finite measure; L_p^0(X) is the span of simple functions in L_p(X)), we define an L-space E as a linear space for which L ⊗ E is equipped with a norm, satisfying certain natural conditions (specified...
in Section 1, where relevant definitions and basic facts are collected). Specializing to \(X := \mathbb{N}\) with the counting measure, we recover \(p\)-multinormed spaces.

The aim of this paper is to describe projective (= “homologically best”) \(L\)-spaces. We consider two versions of this notion. The first one is the metric projectivity which is, roughly speaking, a suitable concretization of the general categorical projectivity. The second one is the extreme projectivity, taking into account specific features of functional-analytic structures, based on the presence of norm. Extremely projective Banach spaces were characterized by Grothendick [5]; much later Blecher [1] described (in different terms) extremely projective operator spaces. Metric projectivity of Banach spaces appeared, under different names, in old papers of Semadeni [17] and Graven [4], and then, for operator spaces, in [7].

Category theory provides us with a valuable tool for studying projective objects: freeness. Our goal is to (i) introduce an appropriate notion of freeness, and (ii) to characterize the free objects. Then projective objects can be described as well, since they are exactly the retracts of free objects. This approach has been applied to quite a few categories in functional analysis, and not only for the metric projectivity, but for the extreme projectivity as well (in the latter case one needs to replace retracts by the so-called near-retracts). For application to operator spaces and modules see [7, 8]; general matricially normed spaces are treated in [9].

In Section 2 of the present paper we introduce a reasonable notion of a free \(L\)-space. Its definition depends on our choice of a family \(L'; \nu \in \Lambda\) (\(\Lambda\) is an index set) of contractively complemented subspaces which “paves” \(L\). We show that free \(L\)-spaces do exist if, and only if, all our subspaces \(L'\) are finite-dimensional (Theorem 2.15). If the latter condition is fulfilled, we obtain a collection of projective \(L\)-spaces. In particular, this is the case when \(L\) is \(L_p(X)\) or \(L^0_p(X)\), and every \(L'\) is a linear span of characteristic functions of several measurable subsets of \(X\).

Addressing freeness, we show that the desirable free \(L\)-spaces are the so-called \(\oplus_1\)-sums of all possible families of some special \(L\)-space, which, in its turn, is the \(\oplus_1\)-sum of all minimal \(L\)-spaces \((L')^*; \nu \in \Lambda\) (Propositions 2.12 and 2.11). The words “minimal \(L\)-space” mean that we consider in \(L \otimes (L')^*\) the injective tensor norm. One of the important steps in the proof consists of establishing a bijection, for a given \(L\)-space \(E\), between the unit ball of \(L \otimes E\) and a specific set of operators from \((L')^*\) into \(E\).

In Section 3 we turn to the study of projectivity. We call an \(L\)-space “well composed” if it is an \(\oplus_1\)-sum of a family of \(L\)-spaces, such that each of them is minimal \(L\)-space \(L^*\) for some finite-dimensional contractively complemented subspace \(L\) of \(L\) (summands may be different). Then every retract, respectively near-retract of a well composed \(L\)-space is metrically, respectively extremely projective. Moreover, if \(L\), like in the case of \(L^0_p(X)\), is the set-theoretical union of its finite-dimensional contractively complemented subspaces, we obtain a full description of relevant \(L\)-spaces: for such an \(L\) a given \(L\)-space is metrically, respectively extremely projective.
if, and only if, it is a retract, respectively near-retract of some well composed \(L\)-space (Theorem 3.7). As a particular case, we obtain (Theorem 3.10) a description of metrically and extremely projective \(p\)-multinormed spaces. The same theorem also provides a “quotient representation” of \(p\)-multinormed spaces. This generalizes and sharpens some of the results, obtained by the second author in [14].

1. \(L\)-spaces and \(L\)-bounded operators

First we fix the notation. If \(G\) and \(E\) are linear spaces, we denote by \(\mathcal{L}(G, E)\) the set of all linear operators from \(G\) to \(E\). If, in addition, \(G\) and \(E\) are normed, denote by \(\mathcal{B}(G, E)\) the space of all bounded operators between them, endowed with the operator norm, and write \(\mathcal{B}(E)\) instead of \(\mathcal{B}(E, E)\). The identity map on a set \(M\) will be denoted by \(1_M\). The symbol \(\otimes\) is used for the algebraic tensor product of linear spaces and linear operators, and also for elementary tensors. The closed and open unit ball in a normed space \(E\) is denoted by \(B(E)\) and \(B^0(E)\), respectively.

Choose and fix (so far arbitrary) normed space \(L\), which we shall call base space.

For an arbitrary base space, \(L \otimes E\) is a left module over the algebra \(\mathcal{B}(L)\) with the outer multiplication “ \(\cdot\) ”, well defined by \(a \cdot (\xi \otimes x) := a\xi \otimes x\).

**Definition 1.1.** Suppose \(E\) is a linear space. A norm on \(L \otimes E\) is called \(L\)-norm on \(E\), if \(L \otimes E\) is contractive as a left \(\mathcal{B}(L)\)-module \(L \otimes E\): the inequality \(|a \cdot u| \leq ||a|| |u|\) holds for any \(a \in \mathcal{B}(L)\) and \(u \in L \otimes E\). This will be referred as contractibility property. The space \(E\), endowed by an \(L\)-norm, is called \(L\)-space.

**Remark 1.2.** Any \(L\)-space \(E\) can be equipped with a norm: for \(e \in E\), define \(|e| := ||\xi \otimes e||\), where \(\xi \in L\) has norm one. Note that, if \(\xi, \eta \in L\) have norm one, then, by Hahn-Banach Theorem, there exists a norm one \(a \in \mathcal{B}(L)\) so that \(a\xi = \eta\); thus, \(|e|\) is well-defined. With this norm on \(E\), the norm on \(L \otimes E\) becomes a cross-norm – that is, \(||\xi \otimes e|| = ||\xi|| |e|\) for any \(\xi \in L\) and \(e \in E\).

Our principal examples of base spaces \(L\) are \(L_p(X),\) where \(X\) is a measure space (always assumed to be \(\sigma\)-finite), \(p \in [1, \infty]\), and its subspace \(L^0_p(X)\), consisting of simple functions. As any bounded operator \(u : L^0_p(X) \rightarrow L^0_p(X)\) extends to a bounded operator \(\tilde{u} : L_p(X) \rightarrow L_p(X)\) (of the same norm), any \(L_p(X)\)-space is also an \(L^0_p(X)\)-space. We claim that the converse is also true. Indeed, suppose \(E\) is an \(L^0_p(X)\)-space; denote the corresponding norm on \(L^0_p(X) \otimes E\) by \(||\cdot||_0\). Let \(\Lambda\) be the net of all finite \(\sigma\)-subalgebras of \(X\). For \(\nu \in \Lambda\), let \(L^\nu \subset L^0_p(X)\) be the set of all \(\nu\)-measurable functions – that is, of functions \(\sum_i \alpha_i x_{A_i}\), with scalars \(\alpha_i\), and \(A_i \in \nu\). Any such \(L^\nu\) is a range of a (contractive) conditional expectation, denoted by \(Q^\nu\). For \(u \in L_p(X) \otimes E\), let \(||u|| := \sup_{\nu \in \Lambda} ||Q^\nu \cdot u||_0\). This quantity is well defined: if \(u = \sum_k \xi_k \otimes x_k\), then \(||u|| \leq \sum_k ||\xi_k|| ||x_k||\). It is easy to see that \(||\cdot||\) is a norm which coincides with \(||\cdot||_0\) on \(L^0_p(X) \otimes E\).
Further, \(\| \cdot \|\) is a \(L_p(X)\)-norm – that is, the inequality \(\| a \cdot u \| \leq \| u \|\) holds for any contraction \(a \in B(L_p(X))\), and any \(u = \sum_{i=k}^n \xi_k \otimes x_k \in L_p(X) \otimes E\). To establish this, fix \(\varepsilon > 0\), and find \(\mu \in \Lambda\) so that \(\sum_{k=1}^n \|\xi_k\||Q^\mu x_k - x_k| < \varepsilon\) (this is possible, due to the density of simple functions in \(L_p(X)\)). For any \(\nu \in \Lambda\),

\[
Q^\nu(a \cdot u) = \sum_{k=1}^n Q^\nu a(Q^\mu)^2 \xi_k \otimes x_k + \sum_{k=1}^n Q^\nu a(\xi_k - Q^\mu \xi_k) \otimes x_k
\]

(recall that \((Q^\mu)^2 = Q^\mu\)). As \(\| \cdot \|_0\) is an \(L_p^0(X)\)-norm, we have

\[
\left\| \sum_{k=1}^n Q^\nu a(Q^\mu)^2 \xi_k \otimes x_k \right\|_0 \leq \left\| Q^\nu aQ^\mu \right\| \sum_{k=1}^n Q^\mu \xi_k \otimes x_k \right\|_0 \leq \| u \|.
\]

From the triangle inequality and Remark 1.2,

\[
\left\| \sum_{k=1}^n Q^\nu a(\xi_k - Q^\mu \xi_k) \otimes x_k \right\|_0 \leq \sum_{k=1}^n \left\| Q^\nu a \right\| \|\xi_k - Q^\mu \xi_k\| \| x_k \| \leq \varepsilon,
\]

hence, by the triangle inequality, \(\| Q^\nu a \cdot u \|_0 \leq \| u \| + \varepsilon\) for any \(\nu \in \Lambda\). Taking the supremum over \(\nu\), and recalling that \(\varepsilon\) can be arbitrarily small, we conclude \(\| a \cdot u \| \leq \| u \|\).

For \(L\) as above, \(L\)-spaces generalize \(p\)-multinormed spaces, investigated in e.g. [2]. Indeed, suppose \(X := \mathbb{N}\) with the counting measure. Then \(L_p^0(X)\) is \(\ell_p^0\) – the space of finite sequences, with the norm inherited from \(\ell_p\). It is easy to see that any \(\ell_p^0\)-norm on \(E\) corresponds to a sequence of cross-norms \(\| \cdot \|_n\) on spaces \(\ell_p^m \otimes E\), with the property that \(\| a \cdot u \|_n \leq \| a \|\| u \|_n\), for any \(u \in \ell_p^m \otimes E\), and \(u : \ell_p^m \rightarrow \ell_p^n\). The preceding paragraph establishes that any \(\ell_p^0\)-space is an \(\ell_p\)-space, and vice versa.

Every linear subspace \(F\) of an \(L\)-space \(E\) will be considered as an \(L\)-space with the \(L\)-norm, induced by the embedding of \(L \otimes F\) into \(L \otimes E\).

If an operator \(\varphi : G \rightarrow E\) between linear spaces is given, we shall use, for the operator \(1_L \otimes \varphi := L \otimes G \rightarrow L \otimes E\), the brief notation \(\varphi_\infty\). Obviously, this is a morphism of left \(B(L)\)-modules.

The following definition is inspired by various “quantizations” (found, for instance, in [3, 2, 7]).

**Definition 1.3.** An operator \(\varphi : G \rightarrow E\) between \(L\)-spaces is called \(L\)-bounded if \(\varphi_\infty\) is bounded. In a similar way we define the notions of a \(L\)-contractive operator, \(L\)-coisometric operator ( = \(L\)-quotient mapping) \(L\)-strictly coisometric operator ( = \(L\)-exact quotient mapping), and so on. In particular, our \(\varphi\) is \(L\)-strictly coisometric, respectively \(L\)-coisometric when \(\varphi_\infty\) maps \(B(L \otimes G)\) onto \(B(L \otimes E)\), respectively \(B^0(L \otimes G)\) onto \(B^0(L \otimes E)\).

The operator norm of \(\varphi_\infty\) is denoted by \(\| \varphi \|_{Lb}\). The set of \(L\)-bounded, respectively \(L\)-contractive operators between \(L\)-spaces \(G\) and \(E\) is denoted by \(CB(G, E)\), respectively \(CC(G, E)\).
Let $P, G, E$ be linear spaces, $\tau : G \to E$, $\varphi : P \to E$ operators. We recall that an operator $\psi : P \to E$ is called a lifting of $\varphi$ across $\tau$, if it makes the diagram
\[
\begin{array}{ccc}
P & \xrightarrow{\varphi} & E \\
\downarrow{\psi} & & \downarrow{\tau} \\
G & & E
\end{array}
\]
commutative.

**Definition 1.4.** An $L$-space $P$ is called *metrically projective*, if for every $L$-spaces $G, E$ every $L$-strictly-coisometric operator $\tau : G \to E$ and every $L$-bounded operator $\varphi : P \to E$ there exists a lifting $\psi : P \to G$ of $\varphi$ across $\tau$, such that $\|\psi\|_{Lb} = \|\varphi\|_{Lb}$.

An $L$-space $P$ is called *extremely projective*, if for $G, E$ and $\varphi$ as before, every $L$-coisometric operator $\tau : G \to E$ and every $\varepsilon > 0$ there exists a lifting $\psi : P \to G$ of $\varphi$ across $\tau$, such that $\|\psi\|_{Lb} < \|\varphi\|_{Lb} + \varepsilon$.

We shall use these definitions in an equivalent form, given by the result below.

**Proposition 1.5.** (i) $P$ is metrically projective if, and only if for every $L$-strictly-coisometric $\tau : G \to E$ and an $L$-contractive $\varphi : P \to E$ there exists an $L$-contractive lifting of $\varphi$ across $\tau$.

(ii) $P$ is extremely projective if, and only if for every $L$-coisometric $\tau : G \to E$ and an $L$-contractive $\varphi : P \to E$ with $\|\varphi\|_{Lb} < 1$ there exists an $L$-contractive lifting of $\varphi$ across $\tau$. $\square$

**Definition 1.6.** An $L$-contractive operator is called *retraction*, if it has a right inverse $L$-contractive operator (which, of course, must be an $L$-isometry). Further, an $L$-contractive operator is a *near-retraction*, if, for every $\varepsilon > 0$, it has a right inverse $L$-bounded operator $\rho$ such that $\|\rho\|_{Lb} < 1 + \varepsilon$. An $L$-space $E$ is called *retract* (near-retract) of an $L$-space $G$ if there is a retraction (respectively, near retraction) from $G$ onto $E$.

In more geometric terms, $E$ is a retract of $G$ if and only if it is a so-called $L$-*direct summand of $G$*, that is $E$ is $L$-isometrically isomorphic to an $L$-subspace $F$ of $G$ such that there is an $L$-contractive projection of $G$ onto $F$. Following [1], we say that $E$ is an *almost $L$-direct summand of $G$* when for every $\varepsilon > 0$ there is a subspace $F$ of $E$ such that there exists a projection $Q$ of $G$ onto $F$ with $\|Q\|_{Lb} < 1 + \varepsilon$ and there exists an $L$-topological isomorphism $I : E \to F$ with $\|I\|_{Lb}, \|I^{-1}\|_{Lb} < 1 + \varepsilon$. Equivalently, for any $\varepsilon > 0$ there exist a contractive operator from $G$ onto $E$, which has a right inverse $\rho$ with $\|\rho\|_{Lb} < 1 + \varepsilon$. Consequently, any near-retract of $G$ must be an almost $L$-direct summand.

**Proposition 1.7.** (i) A retract of a metrically projective $L$-space is itself metrically projective.

(ii) A near-retract of an extremely projective $L$-space is itself extremely projective.

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1Not to be confused with $L$-summands from the theory of $M$-ideals.
Proof. We restrict ourselves to the p. (ii), since p. (i) is simpler. Suppose that 
$P_0$ is extremely projective, $P$ is a near-restrict of $P_0$, while $\tau$, $\varphi$, and $G$ are as in
Proposition 1.5(ii). Fix $\lambda \in (\|\varphi\|_{Lb}, 1)$, and find an $L$-contractive map $\sigma : P_0 \to P$, with a right inverse $\rho$ satisfying $\|\rho\|_{Lb} < 1/\lambda$. As $\|\varphi\|_{Lb} \leq \|\varphi\|_{Lb} < \lambda$, the extreme projectivity of $P_0$ implies the existence of $\psi_0 : P_0 \to G$ with $\tau\psi_0 = \varphi\sigma$, and $\|\psi_0\|_{Lb} < \lambda$. Let $\psi = \psi_0\rho$; then $\tau\psi = \varphi\sigma\rho = \varphi$ (that is, $\psi$ lifts $\varphi$). Further, $\|\psi\|_{Lb} \leq \|\psi_0\|_{Lb}\|\rho\|_{Lb} < 1$. □

Remark 1.8. The proof above shows that any almost $L$-direct summand of an 
 extremely projective $L$-space is extremely projective.

2. The functor $\circ$, and $\circ$-free $L$-spaces

Denote by $\text{LNor}_1$ the category with $L$-spaces as objects and $L$-contractive operators 
as morphisms. As usually, the category of sets and maps is denoted by $\text{Set}$.
Let $\square : \text{LNor}_1 \to \text{Set}$ be (so far) an arbitrary functor.

Definition 2.1. An $L$-contractive operator $\tau$ is called $\square$-admissible, if the map $\square(\tau)$ is surjective (i.e. it is a retraction in $\text{Set}$).

Example 2.2. Consider the functor $\circ : \text{LNor}_1 \to \text{Set}$, taking $E$ to $B(L \otimes E)$, and 
taking the $L$-contractive operator $\varphi : E \to G$ to the restriction of $\varphi$ to $B(L \otimes E)$. 
We see that an $L$-contractive operator $\tau$ is $\circ$-admissible if, and only if it is $L$-strictly 
coisometric.

We shall be mostly concerned with a different functor, better suited for describing 
projective $L$-spaces. In what follows, we suppose that we are given a family $L^\nu; \nu \in \Lambda$, 
where $\Lambda$ is some index set, of contractively complemented subspaces of $L$.

Example 2.3. We are mainly concerned with $L = L_0^0(X)$. Take $\Lambda$ to be the family 
of all finite sets of pairwise disjoint measurable subsets in $X$; let $L^\nu; \nu \in \Lambda$ the 
span of characteristic functions of sets in $\nu$; $Q^\nu$ is the corresponding conditional 
extpectation.

Note that we do not have much latitude in selecting the spaces $L^\nu$. Indeed, any 
contractive projection on $L_0^0(X)$ extends to that on $L_p(X)$ by continuity. Contractive 
projections on $L_p(X)$ are known to be “modified conditional expectations” (see 
e.g. [15, Theorem 4.3]). Further, by [18], contractively complemented subspaces of 
$L_p(X)$ are themselves $L_p$ spaces.

Denote, for brevity, $B_E := B(L \otimes E)$, $B_0 := B^0(L \otimes E), B_E^\nu := B(L^\nu \otimes E), B_0^\nu := B^0(L^\nu \otimes E)$ and consider, for every $L$-space $E$, the set $B_E := X\{B_E^\nu; \nu \in \Lambda\}$, that 
is the Cartesian product of the closed unit balls of the normed spaces $B_E^\nu; \nu \in \Lambda$. 
Thus, its elements can be represented as ‘rows’ $(\ldots, v_\nu, \ldots)_{\nu \in \Lambda}$, with $v_\nu \in B_E^\nu$. Let us 
introduce the functor

$\circ : \text{LNor}_1 \to \text{Set},$
taking $E$ to $B_E$ and an $L$-contractive operator $\phi : G \to E$ to the map $\phi (\varphi) : \circ (G) \to \circ (E), (\ldots , v_\nu , \ldots ) \mapsto (\ldots , \varphi_\infty (v_\nu ) , \ldots )$. (This map is well defined since, for every $v \in B'_G$, we have $\varphi_\infty (v) = \varphi_\infty (Q'' \cdot v) = Q'' \cdot \varphi_\infty (v) \in B''_E$).

Similarly, we define the space $B'_{E} := X\{ B_{E}^{\nu} ; \nu \in \Lambda \}$ and introduce the functor $\circ ^0 : LNor_{1} \to \text{Set}$, taking $E$ to $B'_{E}$ and an $L$-contractive operator $\varphi : G \to E$ to the map $\circ ^0(\varphi) : \circ ^0(G) \to \circ ^0(E)$.

**Proposition 2.4.** Any $L$-strictly coisometric operator is $\circ$-admissible, whereas any $L$-coisometric operator is $\circ ^0$-admissible.

**Proof.** If $\tau : G \to E$ is $L$-strictly coisometric, let us take $u \in B''_E$. Then $u = \tau_\infty (v)$ for some $v \in B_G$. Hence $\tau_\infty (Q'' \cdot v) = Q'' \cdot \tau_\infty v = Q'' \cdot u = u$, $Q'' \cdot v \in L'' \otimes G$ and $\| Q'' \cdot v \| \leq \| Q'' \| \| v \| \leq 1$. The proof in the ‘$L$-coisometric’ case is similar. □

In certain cases, the converse of Proposition 2.4 holds. We say that $L$ is properly presented (relative to the family $L^\nu : \nu \in \Lambda$ of complemented subspaces) if for any $N \in \mathbb{N}$, and $f_1, \ldots , f_N \in L$, there exists $\nu \in \Lambda$ with $f_1, \ldots , f_N \in L^\nu$. Proper presentation occurs, for instance, in the setting of Example 2.3. For $f_1, \ldots , f_N \in L = L^0_p (X)$, let $\nu$ be the (finite) $\sigma$-algebra of subsets of $X$, generated by $f_1, \ldots , f_N$. Clearly $f_1, \ldots , f_N \in L^\nu$.

**Proposition 2.5.** Suppose $L$ is properly presented relative to the family $L^\nu : \nu \in \Lambda$, and let $\circ$ and $\circ ^0$ be the functors arising from this family. Suppose, further, that $G$ and $E$ are $L$-spaces, and $\phi : G \to E$ is a linear operator. Then:

1. $\| \phi \| _{L^0} \leq 1$ if and only if $\circ \phi$ (equivalently, $\circ ^0 \phi$) is a well-defined map on $\text{Set}$.
2. $\phi$ is $\circ$-, respectively $\circ ^0$-admissible if, and only if, it is $L$-strictly coisometric, respectively $L$-coisometric. □

The following definition is a particular case of a well known general-categorical definition (see, e.g., Definition 6 in[9]). In what follows, $\square : LNor_{1} \to \text{Set}$ is an arbitrary functor, $M$ a set.

**Definition 2.6.** An $L$-space $F^\square (M)$ is called a $\square$-free space with the base $M$, if, for every $L$-space $E$ there exists a bijection

$I_E : \text{Set} (M , \square E) \to \text{CC} (F^\square (M) , E)$

between the respective sets of morphisms, and these bijections have the so-called natural property. The latter means that for every $\varphi \in \text{CC}(G,E)$ we have the commutative diagram

\[
\begin{array}{ccc}
\text{Set}(M, \square G) & \xrightarrow{I_G} & \text{CC}(F^\square (M), G) \\
(\square \varphi)^* & & \phi^* \\
\text{Set}(M, \square E) & \xrightarrow{I_E} & \text{CC}(F^\square (M), E).
\end{array}
\]
where $\varphi^*$ acts by composition as $\psi \mapsto \varphi \circ \psi$ and $((\Box \varphi))^* \equiv \rho \mapsto (\Box \varphi) \circ \rho$.

A simple "diagram-chasing" argument shows that the free space of $M$ is unique. Henceforth, we shall write $F(M)$ for $F^2(M)$, if there is no confusion about the functor $\Box$.

The following proposition is well known in the general categorical context of adjoint functors [13].

**Proposition 2.7.** Suppose that for a given $L$-space $E$ there exists a $\Box$-free space $F(\Box E)$ with the base $\Box E$. Then there exists a $\Box$-admissible operator $\pi : F(\Box E) \to E$.

**Proof.** To simplify the notation, denote $F(\Box E)$ by $G$. We show that $\pi := I_E(1_{\Box E})$ is admissible, by establishing that $\Box \pi \circ \psi = 1_{\Box E}$, where $\psi = I_G^{-1}(1_G)$. To this end, observe that, due to the natural property (2.1),

$$I_E(\Box \pi \circ \psi) = \pi \circ I_G(I_G^{-1}(1_G)) = \pi \circ 1_G = \pi,$$

hence

$$\Box \pi \circ \psi = I_E I_E^{-1}(\Box \pi \circ \psi) = I_E^{-1} \pi = 1_{\Box E}.$$

The main result of the rest of this section – Theorem 2.15 – determines the necessary and sufficient condition for the existence of $\circ$-free $L$-spaces.

First we establish that only the case of finite dimensional spaces $L''$ is interesting.

**Proposition 2.8.** Suppose that, for some set $M$, there exists a $\circ$-free $L$-space with the base $M$. Then all subspaces $L''$ are finite-dimensional.

**Proof.** Let $F$ be our $\circ$-free $L$-space. Taking, in the capacity of $E$, the same $F$, we obtain a bijection $I_F : \text{Set}(M, \circ F) \to CC(F, F)$. Let $f = I_F^{-1}(1_F)$. For any $t \in M$, $f(t)$ is a "row," say $(..., u_\nu, ...); \nu \in \Lambda$, where $u_\nu \in B_F$ for all $\nu$.

Fix $\nu \in \Lambda$, and represent $u_\nu \in L'' \otimes F$ as $\sum_{k=1}^n \xi_k \otimes x_k$ for some $\xi_k \in L'', x_k \in F; k = 1, ..., n$. Take an arbitrary non-zero $\eta \in L''$ and $x \in F$ such that $\|\eta \otimes x\| \leq 1$. Consider an arbitrary function $g : M \to B_F$ where $g(t) \in B_F$ is a 'row' with $\eta \otimes x$ on the 'nu-th' place. Then $\varphi := I_F(g)$ is an $L$-contractive operator, acting on $F$. Therefore, by the "natural property", we have

$$I_F(\circ \varphi \circ f) = \varphi I_F(f) = \varphi I_F I_F^{-1}(1_F) = \varphi 1_F = \varphi = I_F(g),$$

hence $\circ \varphi \circ f = g$. This, in particular, means that the 'row' $[\circ \varphi \circ f](t)$ coincides with the 'row' $g(t)$. Looking at the 'nu-th' place in these 'rows', we see that $\varphi_\infty(u_\nu) = \eta \otimes x$, that is $\eta \otimes x = \sum_{k=1}^n \xi_k \otimes \varphi(x_k)$. This implies that $\eta \in \text{span}\{\xi_k : 1 \leq k \leq n\}$. As $\eta \in L''$ is arbitrary, the space $L''$ coincides with the latter span. $\square$

To deal with the functor $\circ$, we shall use a certain family of simpler functors. Fix, for a time, $\nu \in \Lambda$ and consider the functor $\circ_\nu$, taking $E$ to $B_F$ and taking an $L$-contractive operator $\varphi : G \to E$ to the respective restriction $\circ(\varphi) : \circ(G) \to \circ(E)$.
of the operator $\phi_\infty$. Thus, for every $\phi \in \mathcal{C}(E,F)$ and $u = (\ldots, u_\nu, \ldots) \in B_G$, we have
\[
(\circ \phi(u))_\nu = \circ_\nu \phi(u_\nu).
\]

Fix a projection $Q'^* = Q : L \to L'^*$ of norm 1. Equip the dual space $(L'^*)^*$ with the so-called minimal $L$-norm. That is, we supply $L \otimes (L'^*)^*$ with the norm of the injective tensor product of respective normed spaces (see, e.g., [16]). For an $L$-space $E$ we introduce the operator
\[
\Gamma'_E : L \otimes E \to \mathcal{L}((L'^*)^*, E),
\]
assigning to $u \in L \otimes E$ the operator, taking $g \in (L'^*)^*$ to $(g \otimes 1_E)(Q \cdot u)$.

In Propositions 2.9, 2.10, and 2.11 below we suppose that the space $L'^*$ is finite-dimensional. In this case we take an Auerbach basis in $L'$, that is a basis $e_k^* = e_k^*; k = 1, \ldots, n_\nu$ in that space together with a basis $e_k^* = e_k^*$ in $(L'^*)^*$ such that $\phi_k(e_\nu) = \delta_{kl}$, and $\|e_k^*\| = 1 = \|e_k^*\|$ for any $k$ (see, e.g., [12, p. 12]). We fix a distinguished element
\[
w' = w := \sum_k e_k \otimes e_k^* \in L \otimes (L'^*)^*.
\]

As before, $Q'^*$ (or $Q$ for short) is a fixed projection from $L$ onto $L'^*$. 

**Proposition 2.9.** For every $u \in L \otimes E$ the operator $\Gamma'_E(u)$ is $L$-bounded, and we have $\|\Gamma'_E(u)\|_{L^b} = \|Q \cdot u\|$.

**Proof.** Our task is to show that, for any $u \in L \otimes E$, we have $\|\Gamma'(u)\| = \|Q \cdot u\|$. Define the linear map $J : L \otimes (L'^*)^* \to B(L'^*, L)$ by letting $J(\xi \otimes g)(\xi \in L, g \in (L'^*)^*)$ be the operator $\eta \mapsto g(\eta)\xi$ ($\eta \in L'^*$). As $L \otimes (L'^*)^*$ is endowed with the injective tensor product norm, $J$ is an isometry (see, e.g., [16]). Further, $J(w)(e_l) = e_l$ for all $l = 1, \ldots, n$. Therefore $J(w)$ is just the canonical embedding of $L'^*$ into $L$, and hence $\|w\| = 1$.

For $u := \xi \otimes x$ we have $\Gamma'_E(u)(e_k^*) = e_k^*(Q\xi)x$. By linearity, $\Gamma'_E(u)(w) = \sum_k e_k \otimes e_k^*(Q\xi)x$. The equality $\eta = \sum_k e_k^*(\eta)e_k$ is true for any $\eta \in L'^*$, and therefore,
\[
\Gamma'_E(u)(w) = \sum_k e_k \otimes e_k^*(Q\xi)x = Q\xi \otimes x = Q \cdot u
\]
holds for any elementary tensor product $u$. By linearity, $\Gamma'_E(u)(w) = Q \cdot u$ for any $u \in L \otimes E$. To show $\|\Gamma'(u)\| \geq \|Q \cdot u\|$, recall that $\|w\| \leq 1$.

To prove the converse, observe first that $J(v) \cdot (Q \cdot u) = \Gamma'_E(u)(v)$ holds for any $u \in L \otimes E$ and $v \in L \otimes (L'^*)^*$. By linearity, it suffices to consider elementary tensors $u = \xi \otimes x$ and $v = \eta \otimes f$. It is easy to check that then, $J(v) \cdot (Q \cdot u) = f(Q\xi)\eta \otimes x$, whereas $\Gamma'_E(u)(v) = \eta \otimes f(Q\xi)x$. Therefore, taking into account the contractibility property and our choice of norm on $L \otimes (L'^*)^*$, we have
\[
\|\Gamma'_E(u)(v)\| \leq \|J(v) \cdot (Q \cdot u)\| \leq \|J(v)\|\|Q \cdot u\| = \|Q \cdot u\|\|v\|
\]
Taking the supremum over all $v$ of norm not exceeding 1, we conclude that 
\[ \|\mathbf{I}_E^\nu(u)\|_\infty \leq \|Q \cdot u\|. \]

**Proposition 2.10.** The operator $\mathbf{I}_E^\nu$, being restricted to $L^\nu \otimes E$, is an $\mathbf{L}$-isometric isomorphism between the latter space and $(\mathcal{CB}((L^\nu)^*, E))_{\mathbf{L}^b}$.

**Proof.** Since elements of $L^\nu \otimes E$ are exactly those $u \in \mathbf{L} \otimes E$ with $u = Q \cdot u$, Proposition 2.9 implies that our restriction of $\mathbf{I}_E^\nu$ is $\mathbf{L}$-isometric. It is easy to check that the inverse map is given by $\mathcal{CB}((L^\nu)^*, E)) \to L^\nu \otimes E : \varphi \mapsto \varphi_\infty(u)$; this establishes the bijectivity. □

From now on let us agree to identify every map from a singleton to some set with its image in this set.

**Proposition 2.11.** The space $(L^\nu)^*$, supplied with the minimal $\mathbf{L}$-norm, is the $\otimes_\nu$-free $\mathbf{L}$-space whose base is a singleton.

**Proof.** Proposition 2.10 shows that $\mathbf{I}_E^\nu$ implements a bijection between $B^\nu_G$ and the set $\mathcal{CC}((L^\nu)^*, E)$. As to the natural property, it is easy to check that for every $u \in B^\nu_G$ and $\varphi \in \mathcal{CC}(G, E)$ both maps $\varphi^*(\mathbf{I}_E^\nu(u))$ and $\mathbf{I}_E^\nu(\varphi_\nu)(u)$ (cf. diagram (2.1)) take $f \in (L^\nu)^*$ to $(f \otimes \varphi)u$. □

For the rest of this paper, we shall use the following construction. For a family $(E_i)_{i \in \Delta}$ of $\mathbf{L}$-spaces, consider their algebraic sum $\oplus_i E_i$ and make it an $\mathbf{L}$-space in the following way: we identify $\mathbf{L} \otimes (\oplus_i E_i)$ with $\oplus_i (\mathbf{L} \otimes E_i)$ and consider in the latter space the norm of the (non-completed) $l_1$-sum of its direct summands (that is, for an element $u \in \oplus_i (\mathbf{L} \otimes E_i); u = (...) \oplus_i u_i \in \mathbf{L} \otimes E_i$ we set $\|u\|_1 := \sum_i \|u_i\|$). Obviously, we obtain an $\mathbf{L}$-space, called the $\oplus_1$-sum of our spaces and denoted by $(\oplus_i E_i)_1$. Together with the latter, we consider the family of operators $\text{in}_i : E_i \to (\oplus_i E_i)_1 : x \mapsto (...) 0, 0, x_i, 0, 0, ...; x_i = x; i \in \Delta$. These are, of course, $\mathbf{L}$-isometric and hence $\mathbf{L}$-contractive.

Now suppose that for some $\mathbf{L}$-space $E$ we are given a family of $\mathbf{L}$-contractive operators $\varphi_i : E_i \to E; i \in \Delta$. Consider the operator $\oplus_i \varphi_i : (\oplus_i E_i)_1 \to E : x = (...) \mapsto \sum_i \varphi_i(x_i)$, well defined by $(\oplus_i \varphi_i)(\text{in}_i) = \varphi_i$. It is easy to see that $\oplus_i \varphi_i$ is $\mathbf{L}$-contractive, and the map $\{\varphi_i; i \in \Delta\} \mapsto \oplus_i \varphi_i$ is a bijection between the set of all families $\{\varphi_i \in \mathcal{CC}(E_i, E)\}$ and the set $\mathcal{CC}(\oplus_i E_i, E)$.

In the categorical language our construction means that the space $(\oplus_i E_i)_1$, together with the family $\text{in}_i$, is the coproduct of objects $E_i$ in $\mathbf{L} \text{Nor}_1$.

It is easy to see that for given $\varphi_i$ and a $\mathbf{L}$-contractive operator $\psi : E \to F$, where $F$ is another $\mathbf{L}$-space, we have

\[ \psi(\oplus_i \varphi_i) = \oplus_i (\psi \varphi_i) \quad (2.4) \]

Moreover, we obviously have

\[ \|\oplus_i \varphi_i\|_{\mathbf{L}^b} = \sup_i \|\varphi_i\|_{\mathbf{L}^b} \quad (2.5) \]
From now on, we shall denote the $\odot_\nu$-free space of a singleton, constructed in Proposition 2.11, by $F^\nu(*)$.

**Proposition 2.12.** (i) $F(*) := (\oplus_\nu F^\nu(*))_1$ is a $\odot$-free $L$-space with the base $\{\ast\}$.

(ii) For any set $M$, $F(M) := (\oplus_{t \in M} F(t))_1$ is a $\odot$-free $L$-space with $M$ as its base. Here, $F(t)$ is $F(*)$ for $t := \ast$.

In the categorical language p. (ii) means that $F(M)$ is the coproduct of $\text{card}(M)$ copies of $F(*)$.

**Proof.** (i) Identify $\text{Set}(\{\ast\}, B^\nu_E)$ with $B^\nu_E$. Proposition 2.9 shows that, for any $L$-space $E$, the map $I_E^\nu : \text{Set}(\{\ast\}, B^\nu_E) \to \mathcal{C}(F^\nu(*), E)$ is bijective.

For $u = (\ldots, u_\nu, \ldots) \in B_E = \odot(E)$ define $I_E^\nu(u) := \oplus_\nu(I_E^\nu(u_\nu)) \in \mathcal{C}(F(*), E)$. Since $I_E^\nu$ is a bijection for every $\nu \in \Lambda$, the same is obviously true for $I_E^\nu$. To establish the natural property, fix $L$-spaces $E$ and $G$, and $\varphi \in \mathcal{C}(G, E)$. Pick $u = (\ldots, u_\nu, \ldots) \in B_G$. The natural property of bijections $I_E^\nu$, together with (2.2) and (2.4), yields

$$\varphi I_E^\nu(u) = \varphi[\oplus_\nu I_E^\nu(u_\nu)] = \oplus_\nu[\varphi(I_E^\nu(u_\nu))]$$

$$= \oplus_\nu I_E^\nu[\odot\varphi](u_\nu) = \oplus_\nu I_E^\nu((\odot\varphi)(u))_\nu = I_E^\nu(\odot\varphi(u)).$$

(ii) Let $E$ be as before. Suppose a function $f : M \to \odot(E)$, or, what is the same, $f : M \to B_E$, is given. Set $I_E^\nu(f) := \oplus_{t \in M} I_E^\nu(f(t)) \in \mathcal{C}(F(M), E)$, where $I_E^\nu(t \in M)$ are copies of $I_E^\nu$. The map $I_E : \text{Set}(M, \odot(E)) \to \mathcal{C}(F(M), E)$ is bijective, since all “summands” $I_E^\nu$ are. The relevant natural property follows, for $\varphi \in \mathcal{C}(G, E)$ and $f : M \to B_G$, from the equalities

$$\varphi I_E(f) = \varphi[\oplus_\nu I_E(f(t))] = \oplus_\nu[\varphi I_E(f(t))] = \oplus_\nu I_E((\odot\varphi)(f(t)) = I_E(\odot\varphi(f)).$$

**Remark 2.13.** Actually, Proposition 2.12 is but a particular case of a certain general-categorical assertion. Namely, let $\mathcal{K}$ be some category with coproduct. We shall denote the coproduct of a family $Y_i : i \in \Delta$ of objects in $\mathcal{K}$ by $\oplus_i Y_i$. Further, consider a family of functors $\square_\mu : \mathcal{K} \to \text{Set}$, where $\mu$ runs some index set $\Lambda'$. Introduce the “combined” functor $\square : \mathcal{K} \to \text{Set}$, taking $Y$ to the Cartesian product of sets $\square_\mu(Y); \mu \in \Lambda'$. Further, $\square$ takes a morphism $\varphi : Y \to Z$ to the map, transforming a “row” $(\ldots, u_\mu, \ldots); u_\mu \in \square_\mu(Y)$ to $(\ldots, \square_\mu\varphi(u_\mu), \ldots)$. Now suppose that for every $\mu$ there exists a $\square_\mu$-free object in $\mathcal{K}$ with a singleton, say $\{\ast\}$, as its base. Then, if $F^\mu(*)$ is the space constructed above, then:

(i) $F(*) := (\oplus_\mu F^\mu(*))$ is a $\square$-free object in $\mathcal{K}$ with the base $\{\ast\}$.

(ii) For every set $M$ there exists a $\square$-free object in $\mathcal{K}$ with $M$ as its base, and it is $F(M) := (\oplus_{t \in M} F(t))$, where $F(t)$ is $F(*)$ for $t := \ast$.

The proof proceeds as in Proposition 2.12, with obvious modifications.

**Remark 2.14.** Consider the category $L\text{Nor}$ of $L$-spaces and all $L$-bounded operators. Define the functor $\overline{\square} : L\text{Nor} \to \text{Set}$, taking $E$ to $L^\nu \odot E$. Modifying the
proof of Proposition 2.11, one shows that \((L')^*\) is a free \(L\)-space with respect to \(\otimes\). However, this observation does not produce a full analogue of Proposition 2.12(ii): the category \(\text{LNor}\), unlike \(\text{LNor}_1\), has no coproducts of infinite families of \(L\)-spaces.

Combining Propositions 2.11, 2.12 and 2.8, we immediately obtain

**Theorem 2.15.** For every set \(M\) there exists a \(\otimes\)-free \(L\)-space with the base \(M\) if, and only if, all spaces \(L''\) are finite-dimensional.

**Remark 2.16.** In particular, we see that for the functor \(\otimes\) from Example 2.2 \(\otimes\)-free \(L\)-spaces exist only when \(L\) is finite-dimensional. In this setting, our family of subspaces \((L')\) consists of only one space, namely \(L\) itself.

For \(f : M \to B_E : t \mapsto u(t) := (..., u_\nu(t), ...)\) we define \(|||f||| := \sup_{t} \sup_{\nu} ||u_\nu(t)||\) and, for \(\lambda \in [0, 1]\), we set \(\lambda f : M \to B_E : t \mapsto \lambda u(t)\), where \((\lambda u)(t) := (..., \lambda u_\nu(t), ...)\). Of course, for our \(f\) and \(\lambda \in [0, 1]\) we have \(|||\lambda f||| = \lambda |||f|||\).

**Proposition 2.17.** If \(F(M)\) is a \(\otimes\)-free \(L\)-space with the base \(M\), then for every \(L\)-space \(E\), we have \(|||I_E(f)|||_{L_E} = |||f|||\); here, \(I_E : \text{Set}(M, \otimes(E)) \to \text{CC}(F(M), E)\) is the bijection appearing in Definition 2.6.

Thus, our bijections are, in a sense, “isometric” maps, and to prove this we do not have to use a concrete construction of \(F(M)\).

**Proof.** First, for \(\lambda \in [0, 1]\), applying natural property to \(\varphi := \lambda 1_E\), and observing that \(\otimes(\lambda 1_E)(f) = \lambda f\), we have \(I_E(\lambda f) = \lambda I_E(f)\).

Representing every \(f : M \to B_E\) as \(\lambda f_0\) with \(|||f_0||| = 1\), we see that it suffices to show that the desired equality holds provided \(|||f||| = 1\). If it would be not so, then for some \(\lambda \in (0, 1)\), we (still) would have \(\lambda^{-1} I_E(f) \in \text{CC}(F(M), E)\). There exists \(g : M \to B_E\) with \(I_E(g) = \lambda^{-1} I_E(f)\). Then we have \(I_E(\lambda(g)) = I_E(f)\). Hence \(f = \lambda g\) and \(|||f||| = \lambda |||g||| < 1\), a contradiction.

**Remark 2.18.** In Proposition 2.7, we constructed the “canonical” morphism \(\pi : F^\square(\square E) \to E\). For the special case when \(\square\) is our functor \(\otimes\), we can describe \(\pi\) in more detail. By Proposition 2.12, the free \(L\)-space \(F(\otimes E)\) is \(\oplus_{\nu,u}(L'')^*\), where \((\nu, u)\) runs all pairs \((\nu \in \Lambda, u \in \otimes E)\) and \((L'')^*\) is a copy of \((L')^*\) with the minimal \(L\)-norm. In other words, it is the coproduct in \(\text{LNor}_1\) of the family \(\{(L'')^*\}\). Here, \(u\) is the collection of elements \(u_\mu \in B(L'' \otimes E)\), with \(\mu\) running over \(\Lambda\). For such \((\nu, u)\), define \(\varphi_{\nu,u} : (L'')^* \to E\) as \(I_E(u_\nu) : (L')^* \to E\) (recall that \(L'' = L'\)). Following Proposition 2.7, define \(\pi : F(\otimes E) \to E\) to be the coproduct \(\oplus_{\nu,u} \varphi_{\nu,u}\). We show directly that \(\pi\) is \(\otimes\)-admissible, that is, the map \(\otimes \pi : \text{Of}(\otimes E) \to \otimes E\) is surjective.

An elements \(v \in \text{Of}(\otimes E)\) can be represented as “rows” \((..., v_\nu, ...)\) where \(v_\nu \in B(\oplus_{\mu,u} L'' \otimes (L''')^*)\) (with indices \(\mu \in \Lambda, u \in \otimes E)\); \(\oplus\) refers to the \(\ell_1\) sum (coproduct
in Nor₁, as described in the paragraph preceding Proposition 2.12). Then \( \circ \pi(v) = (\ldots, z_\nu, \ldots) \), where
\[
    z_\nu = \sum_{m \in \Lambda, u \in \ominus E} (1_{L^\nu} \otimes \oplus_{\mu,u} \phi_{\mu,u})(v_\nu) \in L^\nu \otimes E.
\]

More specifically, we can write
\[
    v_\nu = (\alpha_{\mu,u}^\nu)_{\mu \in \Lambda, u \in \ominus E},
\]
with
\[
    \alpha_{\mu,u}^\nu \in L^\nu \otimes (L^\mu)^*.
\]

Now suppose we are given
\[
    z = (\ldots, z_\nu, \ldots) \in \ominus E \quad \text{(with } z_\nu \in B(L^\nu \otimes E) \text{ for } \nu \in \Lambda).
\]

Our goal is to find \( v \in \ominus F(\ominus E) \) so that \( \ominus \pi(v) = z \). For \( \nu \in \Lambda \) define by setting
\[
    v_\nu = (\alpha_{\mu,u}^\nu)_{\mu \in \Lambda, u \in \ominus E},
\]
where \( \alpha_{\nu,z_\nu}^\nu = w^\nu \) (\( w^\nu \) was defined in (2.3)), and \( \alpha_{\mu,u}^\nu = 0 \) if \( (\mu, u) \neq (\nu, z_\nu) \). Let \( v = (\ldots, v_\nu, \ldots) \).

By the proof of Proposition 2.9, \( \|w\| = 1 \), hence \( v \in \ominus F(\ominus E) \). It is easy to verify that
\[
    (1_{L^\nu} \otimes \phi_{\nu,u})w^\nu = u \quad \text{for any } u \in L^\nu \otimes (L^\nu)^*,
\]
hence also for \( u = z_n \). Therefore,
\[
    \ominus \pi(v) = z.
\]

### 3. Applications to the projectivity

Suppose, for a moment, that we have an arbitrary base space and an arbitrary functor \( \square : LNor_1 \to \text{Set} \).

**Definition 3.1.** An \( L \)-space \( P \) is called \( \square \)-projective, if for every \( L \)-spaces \( G \) and \( E \), every \( \square \)-admissible operator \( \tau : G \to E \) and every \( L \)-contractive operator \( \varphi : P \to E \) there exists an \( L \)-contractive lifting of \( \varphi \) across \( \tau \). Our \( P \) is called asymptotically \( \square \)-projective, if for every \( G, E \) and \( \tau \) as before, and \( \varphi : P \to E \) with \( \|\varphi\|_{Lb} < 1 \) there exists an \( L \)-contractive lifting \( \psi \) of \( \varphi \) across \( \tau \).

**Proposition 3.2.** Every retract of a \( \square \)-projective \( L \)-space is itself a \( \square \)-projective \( L \)-space. Every near-retract of an asymptotically \( \square \)-projective \( L \)-space is itself an asymptotically \( \square \)-projective \( L \)-space.

**Proof.** Proceed as in Proposition 1.7, with obvious modifications. \( \square \)

**Proposition 3.3.** (i) For every \( M \), the \( \square \)-free \( L \)-space \( F\square(M) \) (if it does exist) is \( \square \)-projective.

(ii) A retract, respectively near-retract of a \( \square \)-free \( L \)-space is \( \square \)-projective, respectively asymptotically \( \square \)-projective.

(iii) If every set is a base of some \( \square \)-free \( L \)-space, then every \( \square \)-projective \( L \)-space is a retract, and every asymptotically \( \square \)-projective \( L \)-space a near-retract of some \( \square \)-free \( L \)-space.
Proof. (i) It is a particular case of a known categorical statement. Suppose $\tau$ is admissible; find a map $\rho : \square E \to \square G$ so that $(\square \tau)\rho = 1_{\square E}$. For $\varphi : F^\square(M) \to E$ let $\psi = I_G(\rho \circ I_E^{-1}(\varphi))$, then $\tau \psi = \varphi$.

(ii) follows from p. (i), Proposition 3.2, and the obvious observation that $\square$–projective $L$–spaces are asymptotically $\square$–projective.

(iii) Let $P$ be a $\square$–projective $L$–space. Proposition 2.7 provides a $\square$–admissible operator $\pi : F(\square P) \to P$. By the projectivity of $P$, $\varphi = 1_P$ has an $L$–contractive lifting across $\pi$, which we denote by $\psi$. In other words, $\psi$ is an $L$–contractive right inverse of $\pi$ – that is, $P$ is a retract of $F(\square P)$.

If $P$ is only asymptotically projective, the preceding proof needs to be modified. For any $\varepsilon \in (0, 1)$, $\varphi := (1 + \varepsilon)^{-1}1_P$ has an $L$–contractive lifting $\psi$ across $\pi$. Then $(1 + \varepsilon)\psi$ is a right inverse of $\pi$, hence $\pi$ is a near retraction. \qed

Return to our special functors $\circ, \circ^0 : L\text{Nor}_1 \to \text{Set}$, corresponding to a (so far arbitrary) family of contractively complemented subspaces $L^\nu$ of $L$. Propositions 2.4 and 2.5 imply:

**Proposition 3.4.** In the above notation, any $\circ$–projective ($\circ^0$–projective) $L$–space is metrically projective (respectively, extremely projective). If $L$ is properly presented relative to the family $L^\nu : \nu \in \Lambda$, then the converse implications also hold.

Next we establish a connection between projectivity and freeness. Part (ii) of Theorem 3.5 below can be deduced from the general theory of asymptotic rigged categories, introduced in [7, 8]. For the sake of clarity, we present a direct approach.

**Theorem 3.5.** Let $M$ be an arbitrary set, and $F(M)$ the $\circ$–free $L$–space with the base $M$ (cf. Theorem 2.15). Then:

(i) $F(M)$ is $\circ$–projective and metrically projective.

(ii) $F(M)$ is asymptotically $\circ^0$–projective and extremely projective.

**Proof.** (i) follows from Propositions 3.3(i) and 3.4 combined.

(ii) Let $G$ and $E$ be $L$–spaces, $\varphi : F(M) \to E$ a completely contractive operator with $\|\varphi\|_{L\text{Nor}} = : \theta < 1$, and $\tau : G \to E$ an $\circ^0$–admissible operator. According to Propositions 1.5 and 3.4, our task is to find an $L$–contractive operator $\psi : F(M) \to G$, making the diagram (1.1) commutative (that is, $\tau \psi = \phi$).

By the condition on $\tau$, for every $\nu \in \Lambda$ the restriction of $\tau_\infty$ to $L^\nu \otimes G$ is a coisometric operator. Define the map $\rho_\nu : B^\nu_E \to B^\nu_G$: if $u \in B^\nu_E$ satisfies $\|u\| \leq \theta$, find $\rho_\nu(u)$ so that $\tau_\infty\rho_\nu(u) = u$. If $\|u\| > \theta$, let $\rho_\nu(u)$ be an arbitrary element of $B^\nu_G$. Now consider the map $\rho : B_E \to B_G$, taking $u = (\ldots, u_\nu, \ldots)$ to $v = (\ldots, \rho_\nu(u_\nu), \ldots)$. We see, in the notation of Proposition 2.17, that if $u : M \to B_E$ satisfies $\|u\| \leq \theta$, then

$$(\circ \tau)\rho(u(t)) = u(t).$$
Now replace $u$ with $I_E^{-1}(\varphi) : M \to B_E$, where $I_E : \text{Set}(M, B_E) \to CC(F(M), E)$ is the bijection from Definition 2.6. Taking into account the aforementioned proposition and (2.5), for every $t \in M$ we obtain
\[(\circ \tau) \rho[I_E^{-1}(\varphi)] = I_E^{-1}(\varphi).\]

Finally, set $\psi := I_G(\rho[I_E^{-1}(\varphi)])$. As our bijections are natural, we conclude that $\tau \psi = I_E[(\circ \tau) \rho[I_E^{-1}(\varphi)]] = \varphi$. \hfill $\square$

Suppose now that $L^\nu; \nu \in \Lambda$ is the family of all contractively complemented finite dimensional subspaces of $L$, and $L$ is properly presented relative to $(L^\nu)$ (this is the case, for instance, when $L = L^0_p(X)$, and the family $(L^\nu)$ is as described in Example 2.3). Let $\circ \cdot$ be the functor corresponding to $L^\nu; \nu \in \Lambda$. Proposition 3.3(i,iii), considered for the case $\square := \circ$, together with Proposition 3.4 gives

**Proposition 3.6.** In the above notation, every metrically projective $L$-space is a retract, and every extremely projective $L$-space a near-retract of some $\circ$-free $L$-space. \hfill $\square$

Now, combining this proposition with the explicit construction of the $\circ$-free $L$-space $F(M)$, provided by Propositions 2.11 and 2.12, we obtain the following description of projective $L$-spaces.

We say that an $L$-space is well composed if it is the $\ell_1$ sum $\oplus_{\mu \in M} \text{MIN}(Z^\mu)$. Here $Z^\mu$ is a finite dimensional contractively complemented subspace of $L$, and $\text{MIN}(Z^\mu)$ refers to the minimal $L$ structure of its dual – that is, we equip $L \otimes Z^\mu_\ast$ with the injective tensor norm. Proposition 2.12 shows that $\circ$-free spaces are well composed; moreover, for every $\mu$ there exists $\nu \in \Lambda$ so that $Z^\mu = L^\nu$, and the cardinality of $\{\mu \in M : Z^\mu = L^\nu\}$ is independent of $\nu$.

**Theorem 3.7.** (i) For every $L$ a retract, respectively near-retract of a well composed $L$-space is metrically, respectively extremely projective.

(ii) If $L$ is properly presented, then every metrically, respectively extremely projective $L$-space is a retract, respectively near-retract of some well composed $L$-space.

(iii) If $L$ is properly presented, then every $L$-space is a strictly coisometric image of a well-composed $L$-space.

**Proof.** (i) We provide a proof for metric projectivity, as extreme projectivity is handled similarly. Suppose $\tau : G \to E$ is a strict $L$-coisometry. It suffices to show that, if $L$ is a finite dimensional subspace of $L$, complemented via a contractive projection $Q$, then any $L$-contractive operator $\varphi : L^* \to E$ admits an $L$-contractive lifting $\psi : L^* \to G$, with $\tau \psi = \varphi$.

For an $L$-space $U$, define the map $I_U : L \otimes U \to L(L^*, U)$ in a manner similar to $I_G$, introduced before Proposition 2.9). For $u \in L \otimes U$, $I_U u$ is the operator $L^* \ni g \mapsto (g \otimes 1_U)(Q \cdot u)$. Proposition 2.9 shows that $\|u\| \geq \|I_U u\|_{Lb}$ fr any...
$u \in L \otimes E$; moreover, $I_U$ implements a bijective isometry from $L \otimes U$ (regarded as a subspace of $L \otimes L$) onto $CB(L^*, U)$. Abusing the notation slightly, we talk about $I_U^{-1} : CB(L^*, U) \to L \otimes U$.

Now fix $\varphi \in CC(L^*, E)$. By the coisometric property of $\tau_\infty$, there exists $u \in B(L \otimes G)$ so that $\tau_\infty u = I_E^{-1} \varphi$. Clearly $Q \cdot u$ has the same properties. Then $\psi := I_G(Q \cdot u)$ is the desired lifting.

(ii) is a direct corollary of Proposition 3.6.

(iii) Suppose $E$ is an $L$-space. Use Theorem 2.15 to find the free object $F(\odot E)$. By Proposition 2.7, there exists an $\odot$-admissible map from $F(\odot E)$ onto $E$. By Propositions 2.5, such a map is $L$-strongly coisometric.

Next we specialize to the case when $L$ is either $L_p(X)$ or $L^0_p(X)$. In the discussion preceding Definition 1.3, we established that the classes of $L_p(X)$-spaces and $L^0_p(X)$-spaces are identical (that is, an $L_p(X)$-space must be an $L^0_p(X)$-space, and vice versa; there is a bijective correspondence between norms as well). We begin our analysis by stating a lemma, which links the notions of $L$-strict coisometricity, and being a (near) retract, for bases $L = L_p(X)$ and $L = L^0_p(X)$.

Lemma 3.8. Suppose $E$ and $G$ are $L^0_p(X)$-spaces, or, equivalently, $L_p(X)$-spaces.

(i) If $\tau : G \to E$ is an $L_p(X)$-coisometric ($L_p(X)$-strictly coisometric) map, then it is also $L^0_p(X)$-coisometric (respectively, $L^0_p(X)$-strictly coisometric).

(ii) If $\tau : G \to E$ is $L^0_p(X)$-coisometric, then it is also $L_p(X)$-coisometric.

(iii) A map $\tau : G \to E$ is a retraction (near-retraction) in the category of $L_p(X)$-spaces if and only if it is a retraction (near-retraction) in the category of $L^0_p(X)$-spaces.

Remark 3.11 below shows that part (ii) of the lemma cannot be improved. Specifically, it provides an example of an $L^0_p(X)$-strictly coisometric map $\tau : G \to E$, which is not $L_p(X)$-strictly coisometric.

Proof. (i) We deal with strict coisometries (the case of coisometries is similar). Fix $u = \sum_{k=1}^N \xi_k \otimes u_k \in L^0_p(X) \otimes E \subset L_p(X) \otimes E$. Due to $\tau$ being $L_p(X)$-strictly coisometric, there exists $v \in L_p(X) \otimes G$ so that $\tau_\infty v = u$, and

$$\|v\|_{L_p(X) \otimes G} = \|u\|_{L_p(X) \otimes E} = \|u\|_{L^0_p(X) \otimes E}.$$ 

Find $\nu \in \Lambda$ so that $\xi_k \in L^\nu$ for all $k$, and let $v' = Q^\nu v$. Clearly $v' \in L^\nu \otimes G \subset L^0_p(X) \otimes G$, $\tau_\infty v' = u$, and

$$\|v'\|_{L^0_p(X) \otimes G} = \|v'\|_{L_p(X) \otimes G} \leq \|v\| = \|u\|,$$

establishing the $L^0_p(X)$-strict coisometricity of $\tau$.

(ii) Consider $u = \sum_{k=1}^N \xi_k \otimes x_k \in L_p(X) \otimes E$. Fix $\varepsilon > 0$; our goal is to find $v \in L_p(X) \otimes G$ so that $\|v\| < \|u\| + \varepsilon$, and $\tau_\infty v = u$. To this end, find $\nu \in \Lambda$
so that \( \sum_k \| \xi_k - Q^\nu \xi_k \| \| x_k \| < \varepsilon/2 \). Further, find \( w \in L^0_p(X) \otimes G \) so that \( \tau_\infty w = \sum_k Q^\nu \xi_k \otimes x_k \), and \( \| w \| < \| u \| + \varepsilon/2 \). For each \( k \) find \( y_k \in G \) so that \( \tau y_k = x_k \), and \( \sum_k \| \xi_k - Q^\nu \xi_k \| \| y_k \| < \varepsilon/2 \). Then \( v = w + \sum_k (\xi_k - Q^\nu \xi_k) \otimes y_k \) is the desired lifting of \( u \).

(iii) This statement follows from the fact that, for any map \( \varphi \) between \( L_p(X) \)-spaces (or, equivalently, \( L^0_p(X) \)-spaces), \( \| \varphi \|_{L_p(X)b} = \| \varphi \|_{L^0_p(X)b} \). \( \square \)

Note that \( L^0_p(X) \) is properly presented, and the spaces \( L^\nu \) involved in this proper presentation are isometric copies of \( \ell^\nu_{p\nu} \). This immediately leads to:

**Lemma 3.9.** An \( L^0_p(X) \)-space is well-composed if and only if it is an \( \ell_1 \) sum \( \oplus_{\mu \in \mathcal{M}} \text{MIN}(\ell^\mu_{p\nu}) \), for some family \( (n_{\mu})_{\mu \in \mathcal{M}} \); here, \( 1/p + 1/q = 1 \). \( \square \)

Having laid the groundwork, we can now characterize projective \( L_p(X) \) and \( L^0_p(X) \) spaces.

**Theorem 3.10.** (i) An \( L^0_p(X) \)-space is metrically (extremely) projective if and only if it is a retract (respectively, near retract) of a well-composed \( L^0_p(X) \)-space.

(ii) An \( L_p(X) \)-space is extremely projective if and only if it is a near retract of a well-composed \( L^0_p(X) \)-space.

(iii) Any \( L_p(X) \)-space which is a retract of a well-composed \( L^0_p(X) \)-space is metrically projective.

(iv) Any \( L^0_p(X) \)-space (\( L_p(X) \)-space) is an image of a well-composed \( L^0_p(X) \)-space under an \( L^0_p(X) \)-strict coisometry (respectively, \( L_p(X) \)-coisometry).

Proposition 3.12 below shows that, in part (iv), an \( L_p(X) \)-space need not be a strictly coisometric image of a well-composed space.

**Proof.** (i) follows from Theorem 3.7(i,ii).

(ii) By Lemma 3.8(i,ii), \( P \) is extremely projective as an \( L_p(X) \)-space if and only if it is extremely projective as an \( L^0_p(X) \)-space. By part (i) of this theorem, the last condition is equivalent to being a near-retract of a well-composed \( L^0_p(X) \)-space. By Lemma 3.8(iii), the notions of being a retract or near-retract in the categories of \( L_p(X) \)-spaces and \( L^0_p(X) \)-spaces coincide.

(iii) Suppose \( E, G, P \) are \( L_p(X) \)-spaces, \( P \) is a retract of a well-composed \( L^0_p(X) \)-space, \( \varphi : P \to E \) is an \( L_p(X) \)-contraction, and \( \tau : G \to E \) is an \( L_p(X) \)-strict coisometry. By Lemma 3.8(i), \( \tau \) is an \( L^0_p(X) \)-strict coisometry as well. Combining part (i) of this theorem with Lemma 3.8(iii), we conclude that \( \varphi \) has an \( L^0_p(X) \)-contractive lifting \( \psi : P \to G \). To complete the proof, note that \( \psi \) must be \( L_p(X) \)-contractive as well.

In (iv), the first statement is a consequence of Theorem 3.7(iii). The second one follows by observing that any \( L^0_p(X) \)-strict coisometry is an \( L_p(X) \)-coisometry. \( \square \)
Remark 3.11. We do not know whether a metrically projective \( L_p(X) \)-space is necessarily metrically projective in the category of \( L^0_p(X) \)-spaces. The following example shows that an \( L^0_p(X) \)-strict coisometry may fail to be an \( L_p(X) \)-strict coisometry.

Let \( I = [0, 2\pi) \). Equip \( G = \ell_1(I) \) and \( E = \ell_2^0 \) with an \( L_1 \)-structure arising from \( L_1(G) \) and \( L_1(E) \) respectively (we use \( L_1 \) and \( L^0_1 \) for \( L_1(0, 1) \) and \( L^0_1(0, 1) \)). These are indeed \( L_1 \)-spaces, by [2, Sections 1.8-9]. Denote the canonical bases of \( G \) and \( E \) by \( g_t \) \((t \in I) \) and \( e_1, e_2 \), respectively.

Define \( \tau : G \to E : g_t \mapsto \cos te_1 + \sin te_2 \), and note it is a strict coisometry on the Banach space level. Indeed, any \( x \in E \) can be written as \( \|x\| (\cos te_1 + \sin te_2) \), with some \( t \in I \). Then \( x = \tau(\|x\|g_t) \).

Next show that \( \tau \) is \( L^0_1 \)-strictly coisometric. Indeed, any element \( u \in L^0_1 \otimes E \) can be written as \( u = \sum_k 1_{A_k} \otimes x_k \), with \( x_k \in E \), and \( A_k \) being disjoint measurable subsets of \((0, 1)\). Then \( \|u\| = \sum_k \lambda(A_k)\|x_k\| \), where \( \lambda \) is the Lebesgue measure. For each \( k \) find \( y_k \in G \) with \( \|x_k\| = \|y_k\|, \tau y_k = x_k \). Let \( v = \sum_k 1_{A_k} \otimes x_k \), and note that \( \tau_\infty v = u \), and \( \|v\| = \|u\| \).

To show that \( \tau \) is not \( L_1 \)-strictly coisometric, note first that, if \( \|y\| = 1 \), and \( \tau y = \cos te_1 + \sin te_2 \) (for some \( t \in I \)), then \( y = g_t \). Indeed, write \( y = \sum_{s \in I} \alpha_s g_s \), where \( \sum_s |\alpha_s| = 1 \) (hence \( \alpha_s = 0 \) for at most countably many values of \( s \)). Then

\[
1 = \langle \cos te_1 + \sin te_2, \tau y \rangle + \sum_s \alpha_s \langle \cos te_1 + \sin te_2, \cos se_1 + \sin se_2 \rangle = \sum_s \alpha_s \cos(t - s).
\]

The only way for this equality to hold is to have \( \alpha_t = 1 \), \( \alpha_s = 0 \) for \( s \neq t \).

Now consider \( u = \xi_1 \otimes e_1 + \xi_2 \otimes e_2 \), where \( \xi_1(s) = \cos 2\pi s \) and \( \xi_2(s) = \sin 2\pi s \) for \( s \in (0, 1) \). Thus, \( u(s) = \cos(2\pi s) e_1 + \sin(2\pi s) e_2 \) for any \( s \). Consequently, \( \|u\|_{L_1(E)} = \int_0^1 \|u(s)\| \ ds = 1 \). We shall show that \( u \) has no norm 1 lifting.

Suppose, for the sake of contradiction, that there exists \( v = \sum_{k=1}^N \xi_k \otimes y_k \in L_1 \otimes G \) so that \( \tau_\infty v = u \), and \( \|v\| = 1 \). In the coordinate form, we must have that \( \|\sum_k \xi_k(s)y_k\| = 1 \), and \( \tau(\sum_k \xi_k(s)y_k) = u(s) = \cos(2\pi s) e_1 + \sin(2\pi s) e_2 \), for any \( s \in S \), where \( S \) has measure 1. Consequently, \( \sum_k \xi_k(s)y_k = g_{2\pi s} \), for any \( s \in S \). This, however, is impossible, as each \( y_k \) has support in \( I \).

We close this paper by an example indicating that the category of \( L_p(X) \)-spaces may not be the “right” one.

Proposition 3.12. There exists an \( L_1(0, 1) \)-space which cannot be represented as an \( L_1(0, 1) \)-strictly coisometric image of a well-composed \( L^0_1(0, 1) \)-space.

Proof. Denote by \( E \) the space \( C(T) \), where \( T \) is the unit circle, equipped with its minimal \( L_1(0, 1) \) structure – that is, the norm on \( L_1(0, 1) \otimes C(T) \simeq C(T, L_1(0, 1)) \) comes from the injective tensor product. Consider normalized independent Gaussian random variables \( g_1, g_2 \in L_1(0, 1) \), and the coordinate functions \( f_1, f_2 \in C(T) \). Let \( u = g_1 \otimes f_1 + g_2 \otimes f_2 \). Then \( \|u\| = 1 \). In fact, \( u \) corresponds to the weak* to norm
As noted above, for every \( Z = \sum_{i} v_{i} \), we construct an operator \( \tilde{u} \) from an \( \tau \)-isometric embedding, and \( q \) is the strictly coisometric adjoint of \( q_{*} : \ell_{2}^{\infty} \to L_{1} : \delta \mapsto g_{i} \). Consequently, \( \tilde{u} \) maps the closed unit ball of \( L_{\infty} \) onto the closed unit ball of \( j(\ell_{2}^{\infty}) \subset L_{1} \).

Suppose, for the sake of contradiction, that there exists an \( L_{1}(0,1) \)-strict coisometry \( \tau \) from an \( \ell_{1} \) sum \( \oplus \text{MIN}(\ell_{2}^{\infty}) \) onto \( C(\mathbb{T}) \). Find \( v \in L_{1}(0,1) \otimes \oplus \text{MIN}(\ell_{2}^{\infty}) \) so that \( \|v\| = 1 \), and \( 1_{L_{1}} \otimes \tau(v) = u \). As \( \oplus \) denotes the algebraic sum, we can write \( v \) as a finite sum \( \oplus_{i \in I} v_{i} \), with \( v_{i} \in L_{1}(0,1) \otimes \ell_{2}^{\infty} \), and \( \|v\| = \sum_{i} \|v_{i}\| \). Based on \( v \), we construct an operator \( \tilde{v} \) from \( L_{\infty}(0,1) \) into the finite dimensional space \( Z = \left( \oplus_{i \in I} \ell_{2}^{\infty} \right)_{1} \). The equality \( u = (1_{L_{1}} \otimes \tau)v \) translates into \( \tilde{u} = \tau|_{Z} \tilde{v} \).

As noted above, for every \( \eta \in \text{ran} \tilde{u} \) we can find \( \xi \in L_{\infty}(0,1) \) so that \( \|\xi\| = \|\eta\| \), and \( \tilde{u}\xi = \eta \). Now let \( Y = \text{ran} \tilde{v} \subset Z \); then, \( \tau(Y) \subset \text{ran} \tilde{u} \), and for every \( \eta \in \text{ran} \tilde{u} \) we can find \( y \in Y \) so that \( \|y\| = \|\eta\| \), and \( \tau y = \eta \). In other words, \( \tau \) acts as a strict coisometry from \( Y \) onto \( \text{ran} \tilde{u} \simeq \ell_{2}^{\infty} \). However, the unit ball of \( Z \) is a polytope, hence the same is true for \( Y \). As a circle is not a cross-section of a polytope, no coisometry from \( Y \) onto a copy of \( \ell_{2}^{\infty} \) exists. \( \square \)

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