The collisional resonance function in discrete-resonance quasilinear plasma systems

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A method is developed to analytically determine the resonance broadening function in quasilinear theory, due to either Krook or Fokker-Planck scattering collisions of marginally unstable plasma systems where discrete resonance instabilities are excited without any mode overlap. It is demonstrated that a quasilinear system that employs the calculated broadening functions reported here systematically recovers the nonlinear growth rate and mode saturation levels for near-threshold plasmas previously calculated from kinetic theory. The distribution function is also calculated, which enables precise determination of the characteristic collisional resonance width.

The collisional broadening of resonance lines is a universal phenomenon in physics. For example, in atomic physics, collisions lead to abrupt changes in phase and plane of vibration, thereby destroying phase coherence and leading to uncertainty in the associated photon energy. This leads to broadening of the atoms emission/absorption profile [1, 2]. In plasma physics, decoherence of the orbital motion of resonant particles allows the reduction of reversible equations of motion into a diffusive system of equations that governs the resonant particle dynamics without detailed tracking of the ballistic motion - as is the case in the widely used quasilinear (QL) formulations of [3–5]. In spite of being an essential element of the structure of QL theory, the determination of the appropriate collisional broadening resonance function has not yet been formulated. In this Letter, we show how to calculate the collisional resonance function from first principles and show that its use implies that a QL plasma system automatically replicates the nonlinear growth rate and the wave saturation levels calculated from full kinetic theory near marginality [6, 7].

We shall show how the results of previous works that focused on the dynamics of plasma systems just above the marginal state for instability [6, 7] can be interpreted within the context of QL theory. Ref. [6] developed a method that calculated the transition from the linearly unstable regime to the nonlinear stabilized regime. In their investigation, a cubic nonlinear time delay equation was derived and applied to a wide variety of plasma systems (e.g., the bump-on-tail problem in Q-machine-like devices, alpha particle induced instability that is crucial in burning plasmas [8] and prediction of the emergence of wave frequency chirping in tokamaks [9]). These studies showed that, with stochastic mechanisms present, such as collisions and background turbulence, quasi-steady solutions could be found. Based on these results, a heuristic QL method was developed [10] that replicated the results of these stationary solutions, both near and far from marginal stability. This model was an extension of the collisionless QL theory developed by Kaufman [5]. Berk [10] suggested intuitive rules, relying on an arbitrarily chosen shape, for creating a resonance function (i.e., an envelope function that weights the strength of the resonant interaction) that broadened the singular delta functions that appear in Kaufman’s theory. The aim of the present work is to show that just above the marginal instability state, a systematic QL theory can be developed, where one obtains a resonance function that integrates to unity, as physically expected. Without further assumption, what then emerges is the shape of the resonance function and the mode saturation level, which replicates the results of the original kinetic calculations [6, 7]. The predicted saturation level of the kinetic theory resulted from the derivation of a rather complex time-delayed integro-differential equation, which turned out to be identical to the evolution equation previously derived for a shear flow fluid problem involving Rossby waves [11]. In contrast, the predictions of this new QL theory is derived from a simple set of equations which yields a clear understanding of the physical processes that are taking place. The QL theory that is developed is applicable to complex, multi-dimensional systems. In particular the new theory is being applied to whole-device modeling of multiple Alfvénic instabilities that are driven by energetic beams and fusion products in tokamaks [12].

Resonant particles are described via a distribution function $f(\varphi, \Omega; t)$, where $\varphi$ is a canonical angle, $\Omega$ is a frequency-like variable which is a function of the relevant action $J$ (canonically conjugated to $\varphi$) [13], and $t$ is time. $\Omega = 0$ determines the resonance condition. The kinetic equation for a single resonance is (the generalization of the method for treating multiple non-overlapping resonances is straightforward, and will be presented in a subsequent more expansive publication rather than in this Letter)

$$\frac{\partial f}{\partial t} + \Omega \frac{\partial f}{\partial \varphi} + \Re \left( \omega_b^2 e^{i\varphi} \right) \frac{\partial f}{\partial \Omega} = C[f, F_0],$$

(1)

where the form for the collisional operator $C[f, F_0]$ is taken as either $\nu_K (F_0 - f)$, which are the creation and annihilation terms of the Krook model [14] or $\nu_{\text{scatt}}^2 \partial^2 (f - F_0) / \partial \Omega^2$, which is the diffusive scattering operator [15], and $\nu_K$ and $\nu_{\text{scatt}}$ are the effective collision frequencies. $\omega_b$ is the nonlinear trapping (bounce)
frequency at a given resonance, which is proportional to the square root of the mode amplitude. \( F_0 \) is the distribution function in the absence of wave perturbations. The distribution can be assumed of the form 
\[ f(\varphi, \Omega, t) = F_0(\Omega) + f_0(\Omega, t) + \sum_{n=1}^{\infty} \left( f_n(\Omega, t) e^{in\varphi} + \text{c.c.} \right) \]
with the ordering \(|F_0'| \gg |f_0'(1)| \gg |f_0'(2)|, |f_0'(3)| \) \[16\]. The prime denotes the derivative with respect to \( \Omega \) while the superscript denotes the order in the wave amplitude (equivalently, in orders of \( \omega_0^2 \)). Then, the \( f_0 \) satisfy
\[ \partial f_0/\partial t + i\Omega f_0 + \frac{1}{2} \left( \omega_0^2 |f_0'|^2 + \omega_0^2 |f_0'|^2 \right) f_0 = -\nu_K f_0, \quad \nu_K^{scatt} f_0' = 0. \]
where the brackets on the right hand side denote either Krook or scattering operators. Sufficiently close to the linear instability threshold, with even moderate collisionality, \( \nu_K^{scatt}/(\gamma_{L,0} - \gamma_d) \gg 1 \) is satisfied \( (\gamma_{L,0} \equiv \text{the mode linear growth rate at } t = 0 \text{ and } \gamma_d \equiv \text{the background damping rate}) \). In this case, the detailed time history is not essential for the description of the system’s dynamics \[17\]. Then, to lowest order in \( \omega_0^2/\nu_K^{scatt} \) one can disregard the time derivative in \( \nu_k^{scatt} \). Therefore, the principal time dependency contribution to \( f_0 \) comes from \( \omega_0(t) \) rather than from a delayed time integral over the particle distribution’s time history.

Starting with the Krook case, to first order in \( \omega_0^2/\nu_K^2 \), Eq. \( (2) \) gives
\[ f_1 = \frac{\omega_0^2 F_0}{2(\Omega + \nu_K)} . \]
Noting that the reality constraint implies \( f_{-1} = f_1^{*} \), to second order in \( \omega_0^2/\nu_K^2 \), \( (2) \) gives
\[ \partial f_0 + \frac{1}{2} \left( \omega_0^2 |f_1'|^2 + \omega_0^2 |f_1'|^2 \right) f_0 = -\nu_K f_0. \]
Defining the angle-independent distribution as \( f(\Omega, t) \equiv F_0(\Omega) + f_0(\Omega, t) \) and noting that by construction \( \partial F_0/\partial t = 0 \) and \( |F_0'| \gg |f_0'| \), one then obtains from Eqs. \( (3) \) and \( (4) \) that the relaxation of \( f(\Omega, t) \) is governed by the diffusion equation
\[ \frac{\partial f}{\partial t} - \frac{\pi}{2} \frac{\partial}{\partial \Omega} \left[ |\omega_0^2|^2 \mathcal{R}(\Omega) \frac{\partial f}{\partial \Omega} \right] = C [f, F_0] \]
where, for the Krook case, \( \mathcal{R}(\Omega) \) is
\[ \mathcal{R}_K(\Omega) = \frac{1}{\pi \nu_K (1 + \Omega^2/\nu_K^2)}. \]
A somewhat similar procedure can be employed for the scattering case. To first order in \( \omega_0^2/\nu_{scatt}^2 \), we integrate Eq. \( (2) \) along the characteristics, which gives
\[ f_1 = \frac{i F_0' \omega_0^2 (t)}{2 \nu_{scatt}} \int_0^\infty ds e^{\nu_{scatt}^2 s} e^{s^2/3}. \]
Eq. \( (7) \) is then iterated in \( (2) \) to second order in \( \omega_0^2/\nu_{scatt}^2 \). Again, using that \( \partial F_0/\partial t = 0 \) and \( |F_0'| \gg |f_0'| \), it is readily found that \( f(\Omega, t) \equiv F_0(\Omega) + f_0(\Omega, t) \) for the scattering case also satisfies an equation of the form of Eq. \( (5) \), with
\[ \mathcal{R}_{scatt}(\Omega) = \frac{1}{\pi \nu_{scatt}} \int_0^\infty ds \cos \left( \frac{\Omega s}{\nu_{scatt}} \right) e^{-s^2/3}. \]
The resonance functions \( (6) \) and \( (8) \) are plotted in Fig. 1(a). The property \( \int_{-\infty}^{\infty} F(\Omega) d\Omega = 1 \), expected for functions that replace a delta function, is automatically satisfied by both forms of the resonance function. For a self-consistent description, the QL diffusion Eq. \( (5) \) must be solved simultaneously with the Eq. for amplitude evolution, \( \frac{d |\omega_{0}^2|^2}{dt} = 2(\gamma_L(t) - \gamma_d) |\omega_{0}^2|^2 \), and for the growth rate, \( \gamma_L(t) = \frac{1}{2} \int_{-\infty}^{\infty} ds R(\Omega) \frac{\partial^{2} F_0}{\partial \Omega^{2}} \).
Interestingly, functions similar to \( (6) \) and \( (8) \) appear in the context of broadening of atomic emission lines - their equivalent are Eq. 12 of \[18\] and Eq. 5.68 (with \( p = 1 \) of \[19\], respectively. Eq. \( (8) \) has the same form of the function calculated by Dupree \[20\] in a different context, namely in the study of strong turbulence theory, where a dense spectrum of fluctuations diffuse particles away from their free-streaming trajectories (see Ref. \[21\] for a review covering broadening theories in strong turbulence). In that case, a renormalized average propagator was introduced and the cubic term in the argument of the exponential is proportional to a collisionless diffusion coefficient.

A concern might arise about the physical significance of a resonance function that is negative in a part of its domain, as is shown in Fig. 1(a) for the function \( (8) \). We note that for the problem treated in the present work, the collisional diffusion ensures that the overall diffusion coefficient in Eq. \( (5) \) is always positive. In Dupree’s case, the assumed overlapping turbulent dense spectrum ensures positivity over the entire phase-space domain.

To leading order near marginal instability, there emerges the following higher order steady state distribution functions \( \delta f \equiv f(\Omega, t) - F_0(\Omega) \) from Eq. \( (5) \). For the Krook model, it has the form
\[ \delta f_K = - |\omega_0|^2 \frac{\partial F_0}{\partial \Omega^2} \frac{\Omega}{\nu_K} \frac{1}{(1 + \Omega^2/\nu_K^2)^2} \]
while for the diffusive scattering model,
\[ \delta f_{scatt} = - |\omega_0|^2 \frac{\partial F_0}{\partial \Omega^2} \frac{\Omega}{\nu_{scatt}} \int_0^\infty ds \sin \left( \frac{\Omega s}{\nu_{scatt}} \right) e^{-s^2/3} \]
Fig. 1(b) shows the forms for the marginally unstable \( \delta f \). These forms can be useful for code verification akin to studies reported in Ref. \[22\]. Fig. 1(b) is valid in
the vicinity of the resonance - its behavior far from the resonance would then be determined by the boundary conditions one imposes to Eq. (5).

We now demonstrate that near the instability threshold, the QL theory together with the calculated resonance functions ((6) and (8)) replicates the same saturation levels calculated by nonlinear theory [6, 7]. Let us start with Eq. (5) for the Krook case. To leading order, it can be written as 

\[ -\frac{\pi}{2} \omega_0^2 \int_{-\infty}^{\infty} d\Omega \frac{\partial f}{\partial \Omega} \frac{\partial^2 R}{\partial \Omega^2} d\Omega \]

Differentiating with respect to \( \omega_0 \), then multiplying by \( R \) and integrating over \( \Omega \), we get:

\[ \int_{-\infty}^{\infty} R \frac{\partial^2 R}{\partial \omega_0^2} d\Omega = -\frac{2\nu_K}{\pi} \int_{-\infty}^{\infty} R \left( \frac{\partial F_0}{\partial \Omega} - \frac{\partial f}{\partial \Omega} \right) d\Omega \]

Note that, because \( R \) vanishes at \( \pm \infty \), integration by parts of the left hand side leads to

\[ -\int_{-\infty}^{\infty} \frac{\partial^2 R}{\partial \omega_0^2} d\Omega = -\int_{-\infty}^{\infty} \frac{\partial R}{\partial \Omega} \frac{\partial^2 \Omega}{\partial \Omega^2} d\Omega = -\frac{1}{4\pi v_K} \] (the last equality follows from using the function given in Eq. (6)). Noting that the initial growth rate (at \( t = 0 \)) is defined as \( \gamma_{L,0} = \frac{\pi}{4} \int_{-\infty}^{\infty} d\Omega R \frac{\partial \eta}{\partial \Omega} \) and the dynamical QL growth rate is \( \gamma_L(t) = \frac{\pi}{4} \int_{-\infty}^{\infty} d\Omega R \frac{\partial^2 \Omega(t)}{\partial \Omega^2} \), it follows from Eq. (11) that

\[ \gamma_L(t) = \gamma_{L,0} \left( 1 - \frac{1}{2} \omega_0^2(t)^2 / 8\nu_K^2 \right) \]

At saturation, i.e., when \( \gamma_L = \gamma_d \), then \( |\omega_{b,sat}| = 8^{1/4} (1 - \gamma_d/\gamma_{L,0})^{1/4} \nu_K \), which is the same saturation level as the one predicted by the kinetic time-delayed integral nonlinear equation [6].

A slightly different procedure can be employed for the scattering case, for which the QL diffusion Eq. (5) can be written to leading order as

\[ -\frac{\pi}{2} \omega_0^2 \int_{-\infty}^{\infty} d\Omega \frac{\partial f}{\partial \Omega} \frac{\partial \Omega}{\partial \Omega} = \nu_{scatt}^3 \frac{\partial^2 (f - F_0)}{\partial \Omega^2} \]

Integrating over \( \Omega \), multiplying both sides by \( R \) and integrating over \( \Omega \), one obtains

\[ |\omega_0^2| \int_{-\infty}^{\infty} R^2 d\Omega = \frac{2\nu_{scatt}^3}{\pi F_0} \int_{-\infty}^{\infty} \left( \frac{\partial F_0}{\partial \Omega} - \frac{\partial f}{\partial \Omega} \right) d\Omega \]

The integration on the left hand side can be analytically performed using Eq. (8), which gives

\[ \int_{-\infty}^{\infty} R^2 d\Omega = \frac{2}{\pi \nu_{scatt}} \left( \Gamma \left( \frac{1}{2} \right) \frac{1}{2} \right)^{1/3} \frac{1}{2} \]

Using the definitions \( \gamma_{L,0} = \frac{\pi}{4} \int_{-\infty}^{\infty} d\Omega R \frac{\partial \Omega}{\partial \Omega} \) and \( \gamma_L(t) = \frac{\pi}{4} \int_{-\infty}^{\infty} d\Omega R \frac{\partial^2 \Omega(t)}{\partial \Omega^2} \), one then obtains from Eq. (12) that

\[ \gamma_L(t) = \gamma_{L,0} \left[ 1 - \left( \frac{1}{2} \omega_0^2 \right)^2 \Gamma \left( \frac{1}{2} \right) \left( \frac{1}{2} \right)^{1/3} / (6 \nu_{scatt}^3) \right] \]

At saturation, when \( \gamma_{L,0} = \gamma_d \), then \( |\omega_{b,sat}| \approx 1.18 (1 - \gamma_d/\gamma_{L,0})^{1/4} \nu_{scatt} \), which is the same as what follows from nonlinear kinetic theory [7] QED.

The limit \( \nu_{K,scatt} / (\gamma_{L,0} - \gamma_d) \rightarrow 1 \), when the detailed time history becomes unimportant, allows for the derivation of the analytical expression for the nonlinear growth rate \( \gamma_{NL}(t) = \gamma_{L,0} \left( 1 - \alpha \omega_0^2(t)^2 \right) \) [17], where \( \alpha = (8\nu_K^4)^{-1} \) for the Krook case and \( \alpha = \Gamma \left( \frac{1}{2} \right) \left( \frac{1}{2} \right)^{1/3} / (6 \nu_{scatt}^3) \) for the scattering case. Comparison with the above expressions for the calculated QL growth rates imply that they are equal to the nonlinear growth rate at all times for both collisional cases.

In conclusion, it has been demonstrated that near marginal stability, the systematic QL transport theory we developed replicates the identical growth rates and saturation values as predicted by a significantly more complex nonlinear kinetic theory based on solving a time delayed integro-differential equation. The demonstration did not rely on any assumption for the specific form of the distribution. We note that our demonstration assumed that the overall system is governed by a QL equation that self-consistently embodies collisional effects via a resonance function that was previously determined from first principles ((6) and (8)). However, a QL theory, being a reduced framework, does not contain all the relevant information as to the detailed angle-resolved distribution function. Hence, in work to be shown elsewhere, we have also developed an alternative formal approach, that produces additional structure as part of the perturbed distribution function that is not described by the coarse-grained
QL theory. However, we have shown that this additional structure does not alter the nonlinear corrections to the field amplitude, predicted by the QL theory we report here. A description of the results of this more general approach will be given in a later more detailed paper.

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