Phase portraits of a special class of dynamic systems in a Poincare circle

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Abstract. In this paper, authors present results of the original investigation of a special class of dynamic systems with the reciprocal polynomial –cubic and square – right parts on a real plane. The global task was to construct all topologically different phase portraits in a Poincare circle with criteria of them. For such an aim a Poincare method of a central and orthogonal mappings has been used. Eventually above the two hundred of different phase portraits were constructed. Each and every portrait has been described in a table. Each line of a table describes one invariant cell of the phase portrait under consideration, its boundary, a source of its phase flow and a sink of it.

1. Introduction.

The dynamic systems theory is to investigate curves, defined with differential equations. Such a study means to split a phase space into trajectories and investigate a limit behavior of them – to find and classify equilibrium positions, to reveal sinks and sources. Among the mostly important notions here are a notion of the equilibrium states stability, i.e. an ability of a system under small changes of initial data to remain near an equilibrium state, or on a given manifold, for a considerably long interval of time, as well as a notion of a system’s roughness (i.e. saving of properties under the satisfactory tiny changes of a model itself). Rough dynamic systems keep their qualitative character of motion under some arbitrary small changes in parameters.

The present Paper is devoted to the original study, and special research methods, developed in it, being new and effective, are to be used for applied dynamic systems (having polynomial right parts) investigations.

Following to the J.H. Poincare opinion [12 - 14], a second-order normal autonomous differential system on an extended real plane $\mathbb{R}_x^2$, having polynomial right parts, can be fully qualitatively investigated [1]. Further quadratic dynamic systems [2] were studied, as well as systems with nonzero linear terms, systems with nonlinear homogeneous terms of the odd degrees (3, 5, 7) [3], which have a center or a focus in a singular point $O (0, 0)$ [4], and some other particular classes of dynamic systems.

Here a family of systems is studied on a real plane $x, y$

$$\frac{dx}{dt} = X(x, y), \quad \frac{dy}{dt} = Y(x, y), \quad (1)$$

for which $X (x, y), Y (x, y)$ are reciprocal polynomials, $X$ is considered as a cubic, and $Y$ as a square form, such as
\[X(0, 1) > 0, \quad Y(0, 1) > 0.\] (2)

The goal is to construct in a Poincare circle all possible phase portraits for such systems. The Poincare method of consecutive mappings has been applied: a central mapping of the plane \(x, y\) (from the center \((0, 0, 1)\) of a sphere \(\Sigma\)), augmented with a line at infinity (i.e. \(\bar{\mathbb{R}}^2\) plane) onto the sphere \(\Sigma\):
\[x^2 + y^2 + z^2 = 1\] (3)

with diametrically opposite points identified, and after that an orthogonal mapping of the lower enclosed semi sphere of the sphere \(\Sigma\) onto the circle \(\bar{\Omega}\):
\[x^2 + y^2 \leq 1\] (4)

with diametrically opposite points of its boundary \(\Gamma\) also identified.

The circle \(\bar{\Omega}\) will be called the Poincare circle, and the sphere \(\Sigma\) will be called further the Poincare sphere [1].

2. Main notations and definitions

\(\varphi(t, p), \ p = (x, y)\) – a fixed point := a solution (a motion) of an Eq.(1) system with initial data \((0, p)\):

\[L_p: \varphi = \varphi(t, p), \ t \in I_{\max}\] a trajectory of a motion \(\varphi(t, p)\),

\[L_p^{+-} := +(-)\] a semi trajectory of a trajectory \(L_p\).

\(O\)-curve of a system := its semi trajectory \(L'_p(p \neq O, s \in \{+,-\})\), adjoining to a point \(O\) under a condition that \(st \to +\infty\).

\(O^{+-}\) curve of a system := its \(O\)-curve \(L_p^{+-}\).

\(O^{+\pm}\) curve of a system := its \(O\)-curve, adjoining to a point \(O\) from a domain \(x > 0\) (\(x < 0\)).

\(TO\)-curve of a system := its \(O\)-curve, which, being supplemented by a point \(O\), touches some ray in it.

A nodal bundle of \(NO\)-curves of a system := an open continuous family of its \(TO\)-curves \(L'_p\), where \(s \in \{+,-\}\) is a fixed index, \(p \in \Lambda, \Lambda - \) a simple open arc, \(L'_p \cap \Lambda = \{p\}\).

A saddle bundle of \(SO\)-curves of a system, a separatrix of the point \(O\):= a fixed \(TO\) -curve, which is not included into some bundle of \(NO\)-curves of a system.

\(E, H, P\) - \(O\)-sectors of a system: an elliptical sector, a hyperbolic sector, and a parabolic one.

A topological type (T-type) of a singular point \(O\) of a system := a word \(A_O\) consisting of letters \(N, S\) (a word \(B_O\) consisting of letters \(E, H, P\), which describes a circular order of bundles \(N, S\) of its \(O\)-curves (of its \(O\)-sectors \(E, H, P\) ) when traversing the point \(O\) in the «+»-direction, i. e. counter clockwise, starting with some of them.

\[P(u) := X(1, u) \equiv p_0 + p_1u + p_2u^2 + p_3u^3,\] \[Q(u) := Y(1, u) \equiv a + bu + cu^2.\] (5)

Note 1. For every Eq.(1) - system:

1) The topological type of a singular point \(O\) in its forms \(A_O\) and \(B_O\) is easy to transform from one given form into the other.

2) Real roots of the polynomial \(Q(u)\) (the polynomial \(P(u)\)) actually are the angular coefficients of isoclines of a zero (isoclines of the infinity).

3) Writing out the real roots of polynomials \(P(u), Q(u)\), we always number them in the ascending order.

Topological types (T – types) of a singular point \(O\) \((0, 0)\) and indefinitely remote singular points were described in articles [3, 4, 10, 11]. Some applications are given in articles [5 - 9, 15].
3. Systems which right parts contain 3 and 2 multipliers correspondingly
Here we discuss those Eq. (1) - systems, the decompositions of real polynomials \( X(x, y) \), \( Y(x, y) \) into the lower degree forms include 3 and 2 multipliers:

\[
X(x, y) = p_3(y - u_1 x)(y - u_2 x)(y - u_3 x), \quad Y(x, y) = c(y - q_1 x)(y - q_2 x),
\]

where \( p_3 > 0, \ c > 0, \ u_1 < u_2 < u_3, \ q_1 < q_2, \ u_i \neq q_j \) for each \( i \) and \( j \).

The investigation includes the following principle steps \([10, 11]\).

Obviously, 10 different variants of \( RSPQ \) are possible for an Eq. (6) - system, because

\[
C_r^2 = \frac{5!}{2!3!} = 10.
\]

Six of them appear to be independent in pairs in relation to the DC-transformation, and each one of the rest four systems will have among the first six ones the mutually inversed system.

Thus we assign a number \( r \in \{1, ..., 10\} \) to each one of different \( RSPQ \)’s of the Eq. (6) – system so that \( RSPQ \ r = \frac{1}{1,6} \) will be independent in pairs, and \( RSPQ \) with numbers \( r = \frac{7}{10} \) will be mutually inversed to \( RSPQ \) sequences having numbers \( r = \frac{1}{4} \).

An \( r \)-family of Eq. (6) – systems := a totality of systems having the \( RSPQ \) number \( r \).

We study families of Eq. (6)-systems which have numbers \( r = \frac{5}{6} \) following the single plan. For families with numbers \( r = \frac{7}{10} \) we receive information via the DC-transformation of families, \( r = \frac{1}{4} \).

The abovementioned investigation plan for each fixed Eq. (6)– family is the follows.

1) We enlist singular points of systems belong to the selected family in a Poincare circle \( \Omega \). They appear to be a point \( O(0,0) \in \Omega \) and points \( D_i \epsilon (u_i,0) \in \Gamma \). \( i = 0, 3 \). \( u_0 = 0 \). For every point we use notions of a saddle (S) and a node (N) bundles of adjacent to this point semi trajectories, of a separatrix, and of a topo-dynamical type of it.

2) Then we split the considered family into subfamilies numbered with \( s \in \{1, ..., 7\} \). Per each subfamily we find their topo-dynamical types and separatrices.

3) For all singular points of systems belong to the chosen subfamily \( s \) we investigate the behavior of separatrices. The question of uniqueness of continuition of each separatrix from a small singular point’s neighborhood to all its lengths, as well as a question of a mutual arrangement of all separatrices in a Poincare circle \( \Omega \) are important here. We answer these questions for all considered families of dynamic systems.

4) Eventually we construct all phase portraits for a chosen family and describe criteria of every one portrait \([10]\).

4. Conclusion
The family number \( r=1 \) has 25 different types of phase portraits.

For families number 2 and 3 there exist 9 types of phase portraits per each one.

For families with numbers 4 and 5 there were found 7 types of phase portraits for each family.

The systems of the family number \( r=6 \) appear to have 36 different types of phase portraits.

As a result 93 different types for the systems having 3 and 2 different multipliers in the decompositions of their polynomial right parts were revealed. Lots of types for the first view. But it is to keep in mind: every chosen family of such dynamic systems includes an uncountable number of real differential systems \([10, 11]\).

Another types of decompositions of right parts are to be discussed in further articles.

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