On normalized differentials on hyperelliptic
curves of infinite genus

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Abstract

We develop a new approach for constructing normalized differentials on hyperelliptic curves of infinite genus and obtain uniform asymptotic estimates for the distribution of their zeros.

1 Introduction

Much of the analysis of closed Riemann surfaces is based on the Riemann bilinear relation. Given a canonical basis $A_1, B_1, \ldots, A_g, B_g$, of $H_1(\Sigma, \mathbb{Z})$ of a closed Riemann surface $\Sigma$ of genus $g$, it reads

$$\int_{\Sigma} \omega \wedge \eta = \sum_{m=1}^{g} \left( \int_{A_m} \omega \int_{B_m} \eta - \int_{A_m} \eta \int_{B_m} \omega \right)$$

where $\omega, \eta$ are arbitrary smooth closed 1-forms on $\Sigma$. As a consequence one obtains the following vanishing theorem: for any holomorphic 1-form $\omega$ on $\Sigma$ with vanishing $A$-periods, $\int_{A_m} \omega = 0 \forall 1 \leq m \leq g$, one has $\int_{\Sigma} \omega \wedge \overline{\omega} = 0$, hence $\omega = 0$. Furthermore it follows from Hodge theory that the space of holomorphic differentials on $\Sigma$ is a complex vector space of dimension $g$ admitting a basis $\omega_1, \ldots, \omega_g$ such that $\int_{A_m} \omega_n = \delta_{mn}$ for any $1 \leq m, n \leq g$. By the Riemann bilinear relation, this basis is unique. The period matrix $R_\Sigma = (\int_{B_j} \omega_i)_{1 \leq i,j \leq g}$, which enters the definition of the theta function associated to $\Sigma$, is known to be symmetric and has the property that $\text{Im} R_\Sigma$ is definite.

However, for many applications to integrable PDEs one needs to consider open Riemann surfaces of infinite genus, a subject pioneered by Ahlfors and Nevanlinna – see the monographs [2] and [4] as
well as references therein. Unfortunately, it is not sufficiently developed for our purposes. In particular for application to the nonlinear Schrödinger (NLS) equation we need to establish a vanishing theorem for holomorphic 1-forms on two sheeted open Riemann surfaces of infinite genus which are not necessarily $L^2$-integrable. With an eye towards such applications to the focusing nonlinear Schrödinger equation we formulate our results for the specific Riemann surfaces involved. Our method is quite general and can be directly applied for studying Riemann surfaces related to other non-linear equations.

Consider the NLS system of equations in one space dimension with periodic boundary conditions,

\[
\begin{align*}
    i\partial_t \varphi_1 &= -\partial_x^2 \varphi_1 + 2\varphi_1^2 \varphi_2, \\
    -i\partial_t \varphi_2 &= -\partial_x^2 \varphi_2 + 2\varphi_2^2 \varphi_1,
\end{align*}
\]

where $\varphi = (\varphi_1, \varphi_2)$ is in $L^2_c := L^2(\mathbb{T}, \mathbb{C}) \times L^2(\mathbb{T}, \mathbb{C})$ and $\mathbb{T} = \mathbb{R}/\mathbb{Z}$. If $\varphi_2 = \varphi_1$ [$\varphi_2 = -\varphi_1$], the system is referred to as defocusing [focusing] NLS equation. By Zakharov and Shabat [21] the NLS system admits a Lax pair formulation, $\partial_t L = [B, L]$ where $L(\varphi)$ denotes the Zakharov-Shabat (ZS) operator

\[
L(\varphi) := i \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \partial_x + \begin{pmatrix} 0 & \varphi_1 \\ \varphi_2 & 0 \end{pmatrix}.
\]

Associated to this operator is the following curve,

\[
C_\varphi := \{ (\lambda, w) \in \mathbb{C}^2 \mid w^2 = \Delta^2(\lambda, \varphi) - 4 \}
\]

where $\Delta(\lambda, \varphi)$ is the discriminant of $L(\varphi)$ (cf. Section 2). It is known (see e.g. [5]) that for any given $\varphi \in L^2_c$, the entire function $\Delta^2(\lambda, \varphi) - 4$ vanishes at $\lambda \in \mathbb{C}$ iff $\lambda$ is a periodic eigenvalue of $L(\varphi)$, i.e., an eigenvalue of $L(\varphi)$, considered on the interval $[0, 2]$ with periodic boundary conditions. In addition, the algebraic multiplicity of $\lambda$ as a root of $\Delta(\lambda, \varphi)^2 - 4$ coincides with the algebraic multiplicity of it as a periodic eigenvalue of $L(\varphi)$ (see [10]). Note that $L(\varphi)$ has a compact resolvent and hence its spectrum is discrete. In this paper we consider potentials $\varphi$ in $L^2_c$ so that each periodic eigenvalue $\lambda$ of $L(\varphi)$ has algebraic multiplicity $m(\lambda)$ at most two. Denote the subset of such elements by $L^2_\bullet$. Then $L^2_\bullet$ is open, dense, and contains the zero potential ([10]). In the case of the defocusing NLS equation, $L(\varphi)$ is self-adjoint. Then the algebraic multiplicity of an eigenvalue coincides with the geometric one which is at most two, and hence $\varphi \in L^2_\bullet$ in
the defocusing case. For \( \varphi \in L^2 \), the periodic eigenvalues of \( L(\varphi) \) can be listed as a sequence of distinct pairs, \( \lambda_k^-(\varphi), \lambda_k^+(\varphi), k \in \mathbb{Z} \), so that 
\[
\lambda_k^\pm(\varphi) = k\pi + \ell^2(k), \text{ i.e., } \sum_{k \in \mathbb{Z}} |\lambda_k^\pm(\varphi) - k\pi|^2 < \infty, \quad \text{and } \lambda_k^- = \lambda_k^+ \iff \lambda_k^- \text{ has algebraic multiplicity two, } m(\lambda) = 2 \text{ (Section 2)}.
\]

Let \( Z_\varphi \) denote the subset of periodic double eigenvalues of \( L(\varphi) \), 
\[
Z_\varphi := \{ \lambda \in \text{spec } L(\varphi) \mid m(\lambda) = 2 \}
\]

Then
\[
C^*_\varphi := C_\varphi \setminus \{ (\lambda, 0) \mid \lambda \in Z_\varphi \}
\]
is a two sheeted open Riemann surface. Generically it is a surface of infinite genus \((10)\). Our aim is to prove a vanishing theorem for holomorphic differentials on \( C^*_\varphi \) which are not necessarily \( L^2 \)-integrable and to construct a family of normalized holomorphic differentials \( \omega_n, n \in \mathbb{Z} \), on \( C^*_\varphi \) with respect to an appropriately chosen infinite set of cycles on \( C^*_\varphi \). In addition we want to get asymptotic estimates of the zeroes of these differentials. The cycles are defined as follows: for any given potential \( \psi \in L^2 \) list its periodic eigenvalues in pairs, \( \lambda_k^-, \lambda_k^+, k \in \mathbb{Z} \), as discussed above. It is shown in Section 2 that there exist an open neighborhood \( W \) of \( \psi \) in \( L^2 \) and a family of simple, closed, smooth, counterclockwise oriented curves \( \Gamma_m, m \in \mathbb{Z} \), so that the closures of the domains in \( C \), bounded by the \( \Gamma_m \) are pairwise disjoint and for any \( \varphi \in W \) and \( m \in \mathbb{Z} \) the domain bounded by \( \Gamma_m \) contains the pair \( \lambda_m^-(\varphi), \lambda_m^+(\varphi) \) but no other periodic eigenvalues of \( L(\varphi) \). Denote by \( A_m \) the cycle on the canonical sheet \( C^\epsilon \) of \( C_\varphi \) (cf. Section 2) so that \( \pi(A_m) = \Gamma_m \) where \( \pi : C_\varphi \to C, (\lambda, w) \mapsto \lambda \).

We are then looking for holomorphic differentials \( \omega_n \) on \( C^*_\varphi \) so that 
\[
\int_{A_m} \omega_n = 2\pi \delta_{mn} \quad \text{for any } m, n \in \mathbb{Z}.
\]
In addition we want to prove a vanishing theorem for holomorphic differentials on \( C^*_\varphi \) with vanishing \( A \)-periods which are not necessarily \( L^2 \)-integrable. For an arbitrary entire function \( \varsigma : C \to C \) with \( \varsigma \mid_{Z_\varphi} = 0 \), let
\[
\omega_\varsigma := \frac{\varsigma(\lambda)}{\sqrt{\Delta^2(\lambda, \varphi) - 4}} d\lambda.
\]

Then \( \omega_\varsigma \) is a holomorphic 1-form on \( C^*_\varphi \) which is locally square integrable. More precisely, for any \( r > 0 \),
\[
V(r) := -\frac{1}{2i} \int_{X_r} \omega_\varsigma \wedge \overline{\omega_\varsigma} < \infty
\]
where
\[
X_r := \pi^{-1}\left(\{\lambda \in \mathbb{C} \mid |\lambda| \leq r\}\right) \cap \mathcal{C}_\varphi^*
\]
and where the orientation is induced by the complex structure on \(\mathbb{C}\).

Using polar coordinates \((\rho, \theta)\) on \(\mathbb{C}\) and taking into account that \(\mathcal{C}_\varphi\) is a two sheeted curve one has
\[
V(r) = 2 \int_0^r \int_0^{2\pi} \frac{|\zeta(\rho e^{i\theta})|}{\sqrt{\Delta^2(\rho e^{i\theta}) - 4}} \rho \, d\theta \, d\rho.
\]
In particular, if \(\zeta \not\equiv 0\) then \(V(r) > 0\) for any \(r > 0\) and \(V(r)\) is strictly increasing. Note that \(\omega_\zeta\) might not be \(L^2\)-integrable as it could happen that \(\lim_{r \to \infty} V(r) = \infty\).

**Theorem 1.1** Let \(\varphi \in W\) and \((A_m)_{m \in \mathbb{Z}}\) be as above and assume that \(\zeta : \mathbb{C} \to \mathbb{C}\) is an entire function with \(\zeta \not\equiv 0\) and \(\zeta \mid_{\mathcal{C}_\varphi} = 0\). If \(\int_{A_m} \omega_\zeta = 0\) for any \(m \in \mathbb{Z}\), then there exists \(C > 0\) so that
\[
V(r) \geq Cr^{2/\pi}
\]
for any \(r \geq 1\).

Theorem 1.1 leads to the following vanishing theorem.

**Theorem 1.2** Let \(\varphi \in W\) and \((A_m)_{m \in \mathbb{Z}}\) be as above and let \(\zeta : \mathbb{C} \to \mathbb{C}\) be entire with \(\zeta \mid_{\mathcal{C}_\varphi} = 0\). If \(\int_{A_m} \omega_\zeta = 0\) for any \(m \in \mathbb{Z}\) and
\[
V(m\pi) = o\left(m^{2/\pi}\right)
\]
as \(m \to \infty\), then \(\zeta \equiv 0\) and hence \(\omega_\zeta \equiv 0\).

**Remark 1.1** The conclusion of Theorem 1.2 no longer holds when the assumption \(V(m\pi) = o\left(m^{2/\pi}\right)\) is dropped – see the form \(\hat{\omega}\) discussed below.

**Remark 1.2** As \(V(r)\) is increasing, the conditions \(V(m\pi) = o\left(m^{2/\pi}\right)\) as \(m \to \infty\) is equivalent to \(V(r) = o\left(r^{2/\pi}\right)\) as \(r \to \infty\).

Next we state our result on normalized differentials on \(\mathcal{C}_\varphi^*\) and describe features of them needed in our studies of the focusing NLS equation.
Theorem 1.3 There exist an open neighborhood $\mathcal{W} \subseteq L^2_*$, cycles $\Gamma_m, A_m, m \in \mathbb{Z}$, as above and analytic functions $\zeta_n : \mathbb{C} \times \mathcal{W} \to \mathbb{C}, n \in \mathbb{Z}$, so that for any $\varphi \in \mathcal{W}$ and $n \in \mathbb{Z}$, the holomorphic differential $\omega_n = \zeta_n(\cdot, \varphi) \frac{\Delta^2(\lambda, \varphi) - 4}{\Delta^2(\lambda, \varphi)} d\lambda$ on $C_\varphi^*$ satisfies

$$\frac{1}{2\pi} \int_{A_m} \omega_n = \delta_{nm} \quad \forall m \in \mathbb{Z}.$$ 

In addition, there exists $N \geq 1$ such that for any $n \in \mathbb{Z}$ and for any $\varphi \in \mathcal{W}$ the entire function $\zeta_n(\cdot, \varphi)$ admits infinitely many zeroes, $\sigma_k^n$, $k \in \mathbb{Z}\setminus\{n\}$, so that:

(i) for any $|k| > N, k \neq n$, $\sigma_k^n$ is the only zero of $\zeta_n(\cdot, \varphi)$ in the disk $D_k(\pi/4)$ and the map $\sigma_k^n : \mathcal{W} \to D_k(\pi/4)$ is analytic;

(ii) for any $|k| \leq N, k \neq n$, $\sigma_k^n \in D_0(N\pi + \pi/4)$. There are no other zeroes of $\zeta_n(\cdot, \varphi)$ in $\mathbb{C}$;

(iii) for any $|k| > N, k \neq n$,

$$\sigma_k^n = \frac{\lambda_k^+ + \lambda_k^-}{2} + (\lambda_k^+ - \lambda_k^-)^2 \ell^2(k)$$

uniformly in $n \in \mathbb{Z}$ and locally uniformly in $\mathcal{W}$.

Moreover, $\zeta_n(\cdot, \varphi)$ admits the product representation

$$\zeta_n(\lambda, \varphi) = \frac{-2}{\pi_n} \prod_{k \neq n} \frac{\sigma_k^n - \lambda}{\pi_k}$$

where

$$\prod_{k \neq n} \frac{\sigma_k^n - \lambda}{\pi_k} := \lim_{K \to \infty} \prod_{|k| \leq K, k \neq n} \frac{\sigma_k^n - \lambda}{\pi_k}$$

and

$$\pi_k := \begin{cases} k\pi, & k \neq 0 \\ 1, & k = 0 \end{cases}.$$ 

For any $\lambda_k^+ \in \mathbb{Z}_\varphi$ with $k \neq n$, $\zeta_n(\lambda_k^+, \varphi) = 0$, hence $\omega_n$ is $L^2$-integrable in sufficiently small punctured neighborhoods of the point $(\lambda_k^+, 0)$ in $C_\varphi^*$. If, however $\lambda_k^+ \in \mathbb{Z}_\varphi$, then $\omega_n$ is not $L^2$-integrable in any punctured neighborhood of the point $(\lambda_k^+, 0)$ in $C_\varphi^*$.

Remark 1.3 Using the product representation for $\zeta_n(\lambda)$, the asymptotic estimates for the $\sigma_k^n$’s, and estimates on infinite products in [3, Lemma C.5] one can show that for any $n \in \mathbb{Z}$ and $\varphi \in \mathcal{W}$ there exists $C > 0$ so that $-\frac{1}{2\pi} \int_{X_r \setminus X_{|n|\pi + \pi/4}} \omega_n \wedge \overline{\omega_n} \geq C \log r$ for any $r \geq |n|\pi + \pi/4$. Hence $\omega_n$ is never $L^2$-integrable.
Remark 1.4 The uniformity statement in the asymptotic formula (2) means that for any $\varphi \in \mathcal{W}$ there is a neighborhood $\mathcal{V}$ of $\varphi$ in $\mathcal{W}$ and a constant $C > 0$ so that for any $n \in \mathbb{Z}$ and for any $\varphi \in \mathcal{V}$ there are constants $(c_k^n)_{|k| > N, k \neq n}$, $c_k^n \geq 0$, so that for any $|k| > N$, $k \neq n$,

$$\left| \sigma_k^n - \frac{\lambda_k^+ + \lambda_k^-}{2} \right| \leq |\lambda_k^- - \lambda_k^+|^2 c_k^n$$

and $\sum_{k \neq n} |c_k^n|^2 \leq C$.

Consider the holomorphic form on $\mathcal{C}_\varphi$

$$\hat{\omega} := \frac{\hat{\Delta}(\lambda)}{\sqrt{\Delta^2(\lambda) - 4}} \, d\lambda,$$

where $\hat{\Delta}(\lambda) := \partial_\lambda \Delta(\lambda, \varphi)$. One easily verifies that for any $m \in \mathbb{Z}$,

$$\int_{A_m} \hat{\omega} = 0. \quad (3)$$

This follows as $\partial_\lambda \left[ \log \left( \Delta(\lambda) - \sqrt{\Delta^2(\lambda) - 4} \right) \right] = -\frac{\hat{\Delta}(\lambda)}{\sqrt{\Delta^2(\lambda) - 4}}$ where the expression $\sqrt{\Delta^2(\lambda) - 4}$ is the canonical root $\Omega$ defined in Section 2. Note that $\Delta(\lambda) - \sqrt{\Delta^2(\lambda) - 4}$ is a holomorphic function on $\mathbb{C} \setminus (\cup_{k \in \mathbb{Z}} G_k)$ where $G_k$, $k \in \mathbb{Z}$, are the cut-off curves defined in Section 2. The function $\Delta(\lambda) - \sqrt{\Delta^2(\lambda) - 4}$ never vanishes as it is proportional to one of the Floquet multipliers of the ZS operator $L(\varphi)$. In particular, we see that $\log \left( \Delta(\lambda) - \sqrt{\Delta^2(\lambda) - 4} \right)$ is a holomorphic function on $\mathbb{C} \setminus (\cup_{k \in \mathbb{Z}} G_k)$. As the cycles $A_m$, $m \in \mathbb{Z}$, lie in $\mathbb{C} \setminus (\cup_{k \in \mathbb{Z}} G_k)$ we get that (3) holds

Clearly, the differentials $\omega_n$ of Theorem 1.3 are unique within the class of holomorphic differentials obtained by perturbations of the type defined by Theorem 1.2. However, without any conditions on the behaviour of the differentials near infinity, one cannot expect uniqueness. Indeed, any sequence of holomorphic differentials of the form $\tilde{\omega}_n := \omega_n + c_n \hat{\omega}$, $c_n \in \mathbb{C}$, satisfies $\frac{1}{2\pi} \int_{A_m} \tilde{\omega}_n = \delta_{mn}$ for any $m, n \in \mathbb{Z}$.

Besides Theorem 1.2 a key ingredient into the proof of Theorem 1.3 is a novel ansatz for the entire function $\zeta_n$ leading to a linear system of equations. A detailed outline of the proof of Theorem 1.3 is given in Section 4.
Applications: In [8], Theorem 1.2 is used to construct action-angle variables for the focusing NLS equation, significantly extending previous results in this direction obtained in [7] near the zero potential. See [1] for related results for 1-gap and 2-gap potentials and [3], [6], [18], [20] for finite gap potentials. Such coordinates allow to obtain various results concerning well-posedness for these equations and study their (Hamiltonian) perturbations – see e.g. [11], [13] – where results in this direction have been obtained for the KdV equation.

Related results: (1) In [5] (cf. also [15]) the case of the defocusing NLS equation is treated, i.e., where \( \varphi = (\varphi_1, \varphi_2) \) satisfies \( \varphi_2 = \overline{\varphi_1} \). In this case \( \varphi \in L^2_\ast \) and \( L(\varphi) \) is self-adjoint, hence its periodic spectrum real. More precisely the eigenvalues can be listed in such a way that

\[
\cdots < \lambda_k^- \leq \lambda_k^+ < \lambda_{k+1}^+ < \cdots \quad \text{and} \quad \lambda_k^\pm = k\pi + \ell^2(k).
\]

It then follows that zeroes of \( \zeta_n \) are confined to the closed gaps \([\lambda_k^-, \lambda_k^+]\) with \( k \neq n \). More precisely as \( \Delta(\lambda) \) is real valued and \( |\Delta(\lambda)| > 2 \) on the open gaps \((\lambda_k^-, \lambda_k^+)\), \( k \in \mathbb{Z} \), one easily deduces that for any \( n \in \mathbb{Z} \), \( \zeta_n(\lambda) \) must have a zero \( \sigma_k^\pm \) in any closed gap \([\lambda_k^-, \lambda_k^+]\) with \( k \neq n \). Using the implicit function theorem it is shown in [5] that differentials can be constructed in a sufficiently small neighborhood \( W \subseteq L^2_\ast \) of such a \( \varphi \). In [9] we consider finite gap potentials \( \varphi \in L^2_\ast \). Listing the periodic eigenvalues of \( L(\varphi) \) in pairs \( \lambda_k^\pm(\varphi), \ k \in \mathbb{Z} \), as above it means that the subset \( J(\varphi) \) consisting of all \( k \in \mathbb{Z} \) with \( \lambda_k^+(\varphi) \neq \lambda_k^-(\varphi) \) is finite. Using again the implicit function theorem it is shown in [9] that differentials can be constructed in a sufficiently small neighborhood of such a finite gap potential.

Note that if \( \varphi \in L^2_\ast \) is arbitrary we have no a priori knowledge on the zeroes of \( \zeta_k \) and hence we cannot apply the implicit function theorem in the way as above.

(2) In the case of the KdV equation on the circle, the relevant operator of its Lax pair is the Hill operator. For potentials with sufficiently small imaginary part, normalized differentials have been constructed in [11], using the implicit function theorem. In the case where the potential is real valued and the corresponding Hill operator has simple periodic spectrum, the corresponding spectral curve is a Riemann surface of infinite genus and the existence of normalized holomorphic differentials can be proved by Hodge theory using the fact that in this case these differentials are \( L^2 \)-integrable – see [4], [15]. Note however that these arguments do not provide the asymptotic estimates of the
zeroes nor the analytic dependence on the potential. But even for the existence part this approach would not work for the ZS operator as the differentials of Theorem 1.3 are never $L^2$-integrable on $C^\phi$.

**Organization:** In the preliminary Section 2 we introduce additional notation and review the spectral properties of ZS operators needed throughout the paper. In Section 3 we prove Theorem 1.1 and Theorem 1.2 whereas in Section 4 we give an outline of the proof of Theorem 1.3. Its details are then presented in the remaining sections.

## 2 Preliminaries

In this section we introduce some more notation and review properties of the Zakharov-Shabat operator $L(\varphi)$, introduced in Section 1. For $\varphi \in L^2_c$ and $\lambda \in \mathbb{C}$, let $M(x) \equiv M(x, \lambda, \varphi)$ denote the fundamental $2 \times 2$ matrix of $L(\varphi)$, $L(\varphi)M(x) = \lambda M(x)$, satisfying the initial condition $M(0, \lambda, \varphi) = 1d_{2 \times 2}$. Let $\mathbb{R}_0^+ := \{x \in \mathbb{R} \mid x \geq 0\}$. It is well known that $M : \mathbb{R}_0^+ \times \mathbb{C} \times L^2_c \to \mathbb{C}^{2 \times 2}$ is continuous and for any $x$ fixed, $M(x, \cdot, \cdot) : \mathbb{C} \times L^2_c \to \mathbb{C}^{2 \times 2}$ is analytic – see e.g. [5], Chapter I.

**Periodic spectrum:** Denote by $\text{spec} L(\varphi)$ the spectrum of $L(\varphi)$ with domain

$$\text{dom}_{\text{per}} L(\varphi) := \{F \in H^1_{loc} \times H^1_{loc} : F(2) = F(0)\}$$

where $H^1_{loc} = H^1_{loc}(\mathbb{R}, \mathbb{C})$. As $L(\varphi)$ has a compact resolvent, the periodic spectrum of $L(\varphi)$ is discrete. It has been analyzed in great details.

The discriminant $\Delta(\lambda) \equiv \Delta(\lambda, \varphi)$ of $L(\varphi)$ is defined to be the trace of $M(1, \lambda, \varphi)$, $\Delta(\lambda) = trM(1, \lambda)$. It is straightforward to see that $\lambda$ is a periodic eigenvalue of $L(\varphi)$ iff $\Delta^2(\lambda, \varphi) - 4 = 0$. Clearly, $\Delta : \mathbb{C} \times L^2_c \to \mathbb{C}$ is analytic.

We say that $a, b \in \mathbb{C}$ are lexicographically ordered, $a \preceq b$, iff $\text{Re}(a) < \text{Re}(b)$ or $\text{Re}(a) = \text{Re}(b)$ and $\text{Im}(a) \leq \text{Im}(b)$. Similarly, $a \prec b$ iff $a \preceq b$ and $a \neq b$. The following two propositions are well known – see e.g. [5, Section 3] or references therein. For any $k \in \mathbb{Z}$ and $r > 0$ denote by $D_k(r)$ the disk

$$D_k(r) := \{\lambda \in \mathbb{C} \mid |\lambda - k\pi| < r\}.$$
Proposition 2.1  For any $\psi \in L^2$ there exist an open neighborhood $W$ of $\psi$ in $L^2$ and an integer $N_0 \geq 1$ such that for any $\varphi \in W$ the following holds:

(i) For any $k \in \mathbb{Z}$ with $|k| \geq N_0$, the disk $D_k(\pi/6)$ contains precisely two periodic eigenvalues $\lambda_k^+(\varphi) \not\ll \lambda_k^-(\varphi)$ of $L(\varphi)$ and one zero $\dot{\lambda}_k(\varphi)$ of $\dot{\Delta}(\lambda, \varphi) = \partial_\lambda \Delta(\lambda, \varphi)$ (all counted with their algebraic multiplicities).

(ii) The disk $D_0((N_0 - 3/4)\pi)$ contains precisely $4N_0 - 2$ periodic eigenvalues of $L(\varphi)$ and $2N_0 - 1$ zeroes of $\dot{\Delta}(\lambda, \varphi)$ (all counted with their algebraic multiplicities).

(iii) There are no other periodic eigenvalues of $L(\varphi)$ and no other zeroes of $\dot{\Delta}(\lambda, \varphi)$ than the ones listed in (i) and (ii).

Proposition 2.2  Let $W \subseteq L^2$ be given by Proposition 2.1. For any $\varphi \in W$, the periodic eigenvalues $(\lambda_k^\pm)_{|k| \geq N_0}$ and the zeroes $(\dot{\lambda}_k)_{|k| \geq N_0}$ satisfy the asymptotic estimates

$$\lambda_k^\pm = k\pi + \ell^2(k) \text{ and } \dot{\lambda}_k = k\pi + \ell^2(k)$$

locally uniformly in $W$, e.g., $\sum_{|k| \geq N_0} |\dot{\lambda}_k - k\pi|^2$ is locally bounded in $W$.

Take $\psi \in L^2$ and construct a neighborhood $W \subseteq L^2$ of $\psi$ in $L^2$ so that the statements of Proposition 2.1 and Proposition 2.2 hold$^1$. For any $\varphi \in W$, in addition to the periodic eigenvalues $(\lambda_k^\pm)_{|k| \geq N_0}$ of the ZS operator $L(\varphi)$ there are $4N_0 - 2$ periodic eigenvalues in the disk $D_0((N_0 - 3/4)\pi)$. We list these eigenvalues in pairs $\lambda_k^-, \lambda_k^+$, $|k| < N_0$, in an arbitrary way except that any double eigenvalue is listed as a pair and for any $|k| < N_0$ the eigenvalues $\lambda_k^-$ and $\lambda_k^+$ are lexicographically ordered $\lambda_k^- \ll \lambda_k^+$. For all integer $|k| < N_0$, choose simple, closed smooth, counterclockwise oriented curves $\Gamma_k$ contained in the disk $D_0((N_0 - 3/4)\pi)$ so that the closures of the (open) domains in $\mathbb{C}$, bounded by the $\Gamma_k$ are pairwise disjoint and for any $|k| < N_0$, the domain bounded by $\Gamma_k$ contains the pair $\lambda_k^\pm$, but no other periodic eigenvalue of $L(\varphi)$. For each $|k| < N_0$ choose a closed smooth curve $\Gamma'_k$ in the interior of $\Gamma_k$ so that dist$(\Gamma_k, \Gamma'_k) > 0$ and $\lambda_k^\pm$ are inside $\Gamma'_k$. The choice of $\Gamma'_k$, $|k| < N_0$, can be done uniformly in $W$, i.e., for any $\varphi \in W$ and for any $|k| < N_0$ the domain bounded by $\Gamma'_k$ contains precisely

$^1$Recall that $L^2_\bullet$ is open and dense in $L^2_\circ$.  

9
two periodic eigenvalues of $L(\varphi)$, $\lambda_k^- (\varphi) \preceq \lambda_k^+ (\varphi)$. Furthermore for any $\varphi \in W$ and $|k| < N_0$ chose a continuously differentiable simple curve $G_k \equiv G_k(\varphi)$ inside $\Gamma'_k$ connecting $\lambda_k^- (\varphi)$ with $\lambda_k^+ (\varphi)$. In the case when $\lambda_k^- (\varphi) = \lambda_k^+ (\varphi)$, $G_k(\varphi)$ is chosen to be the constant curve $\lambda_k^- (\varphi)$. For $|k| \geq N_0$, we choose $\Gamma_k$ to be the counterclockwise oriented boundary of the disk $D_k(\pi/4)$ and $G_k$ to be the straight line,

$$G_k : [0, 1] \to D_k(\pi/4), \ t \mapsto (1 - t)\lambda_k^-(\varphi) + t\lambda_k^+(\varphi).$$

Furthermore define for $k \in \mathbb{Z}$ and $\varphi \in W$ denote

$$\tau_k(\varphi) = (\lambda_k^- (\varphi) + \lambda_k^+ (\varphi))/2$$

and

$$\gamma_k(\varphi) := \lambda_k^+ (\varphi) - \lambda_k^- (\varphi).$$

**Infinite products:** We say that an infinite product $\prod_{k \in \mathbb{Z}} (1 + a_k)$ with $a_k \in \mathbb{C}$, $k \in \mathbb{Z}$, converges if $\lim_{K \to \infty} \prod_{|k| \leq K} (1 + a_k)$ exists. The limit is then also denoted by $\prod_{k \in \mathbb{Z}} (1 + a_k)$. The infinite product $\prod_{k \in \mathbb{Z}} (1 + a_k)$ converges absolutely if $\prod_{k \in \mathbb{Z}} (1 + |a_k|)$ converges.

**Product representations:** For any $\varphi \in L^2_c$, $\Delta^2 (\lambda, \varphi) - 4$ and $\hat{\Delta}(\lambda, \varphi)$ admit product representations. For any given element in $L^2_c$, choose $N_0$ and $W$ as in Proposition 2.1 According to Proposition 2.1 for any $\varphi \in W$, $\Delta(\lambda, \varphi)$ admits $2N_0 - 1$ zeroes in the disk $D_0((N - \frac{3}{4})\pi)$. For convenience list them in lexicographic order, $\hat{\lambda}_{-N_0+1}(\varphi) \preceq \hat{\lambda}_{-N_0+2}(\varphi) \preceq \ldots \preceq \hat{\lambda}_{-N_0-1}(\varphi)$. The remaining zeroes are listed as in Proposition 2.1. The proof of the following statement can be found in [5, Lemma 6.5, Lemma 6.8].

**Proposition 2.3** For any $\varphi \in W$ and $\lambda \in \mathbb{C}$

$$\Delta^2 (\lambda, \varphi) - 4 = -4 \prod_{k \in \mathbb{Z}} \frac{(\lambda_k^+ (\varphi) - \lambda)(\lambda_k^- (\varphi) - \lambda)}{\pi_k^2}$$

and

$$\hat{\Delta}(\lambda, \varphi) = 2 \prod_{k \in \mathbb{Z}} \frac{\dot{\lambda}_k(\varphi) - \lambda}{\pi_k}.$$

**Standard and canonical roots:** Denote by $\sqrt[\pm 1]{z}$ the branch of the square root defined on $\mathbb{C} \setminus \{z \in \mathbb{R} : z \leq 0\}$ by $\sqrt[\pm 1]{1} = 1$. For any $a, b \in \mathbb{C}$, we
define the standard root of \((a - \lambda)(b - \lambda)\) by the following relation

\[
\sqrt{(a - \lambda)(b - \lambda)} = -\lambda \sqrt{\left(1 - \frac{a}{\lambda}\right)\left(1 - \frac{b}{\lambda}\right)}
\]  

(4)

for all \(\lambda \in \mathbb{C}\setminus\{0\}\) such that \(\left|\frac{a}{\lambda}\right|, \left|\frac{b}{\lambda}\right| \leq 1/2\). Let \(G_{[a,b]}\) be an arbitrary continuous simple curve connecting \(a\) and \(b\). By analytic extension, (4) uniquely defines a holomorphic function on \(\mathbb{C}\setminus G_{[a,b]}\) that we call the standard root of \((a - \lambda)(b - \lambda)\) on \(\mathbb{C}\setminus G_{[a,b]}\). One has the asymptotic formula

\[
\sqrt{(a - \lambda)(b - \lambda)} \sim -\lambda \text{ as } |\lambda| \to \infty.
\]

For any \(\varphi \in \mathcal{W}\) and \(\lambda \in \mathbb{C}\setminus \bigcup_{k \in \mathbb{Z}} G_k\) with \(\mathcal{W}\) and \(G_k, k \in \mathbb{Z}\), given as above, we define the canonical root of \(\Delta^2(\lambda, \varphi) - 4\) as

\[
\sqrt{\Delta^2(\lambda, \varphi) - 4} := 2i \prod_{k \in \mathbb{Z}} \frac{\sqrt{\lambda_+^k(\varphi) - \lambda(\lambda_-^k(\varphi) - \lambda)}}{\pi_k}. 
\]  

(5)

To simplify notation, we occasionally will write \(\mathcal{R}(\lambda, \varphi)\) for \(\Delta^2(\lambda, \varphi) - 4\),

\[
\mathcal{R}(\lambda, \varphi) := \Delta^2(\lambda, \varphi) - 4.
\]

The proof of the following lemma is straightforward and hence omitted.

**Lemma 2.1** Let \(\mathcal{W}\) be given as above. For any \(\varphi \in \mathcal{W}\), the canonical root (5) defines a holomorphic function on \(\mathbb{C}\setminus \left(\cup_{k \in \mathbb{Z}} G_k\right)\).

For any \(\varphi \in \mathcal{W}\), define the canonical sheet (or canonical branch) of the open Riemann surface \(C^\bullet_{\varphi}\),

\[
C^\varphi_{\varphi} := \{(\lambda, w) \in \mathbb{C}^2 \mid \lambda \in \mathbb{C}\setminus \left(\cup_{k \in \mathbb{Z}} G_k\right), \ w = \sqrt{\Delta^2(\lambda, \varphi) - 4}\}.
\]  

(6)

As in the Introduction, denote by \(A_k, k \in \mathbb{Z}\), the cycles on the canonical sheet \(C^\varphi_{\varphi}\) s.t. for any \(k \in \mathbb{Z}\),

\[
\pi(A_k) = \Gamma_k,
\]

where \(\pi : C_{\varphi} \to \mathbb{C}, (\lambda, w) \mapsto \lambda\).
3 Proof of Theorems 1.1 and 1.2

The aim of this section is to prove Theorem 1.1 and Theorem 1.2. Let $W \subseteq L^2$ be the neighborhood constructed in Section 2. Throughout this section we fix $\varphi \in W$ and define the cycles $(A_m)_{m \in \mathbb{Z}}$ as in Section 2. Without further reference we will use the terminology introduced in Section 1 and Section 2.

Let $\zeta : \mathbb{C} \to \mathbb{C}$ be entire so that $\zeta$ vanishes on the set $Z_{\varphi}$ of double eigenvalues of $L(\varphi)$. It then follows that the differential $\omega_{\zeta} = \frac{\zeta(\lambda)}{\sqrt{\Delta^2(\lambda) - 4}} d\lambda$ is locally $L^2$-integrable. Using Stokes’ theorem we will estimate $V(r) := \frac{i}{2} \int_{X_r} \omega_{\zeta} \wedge \overline{\omega_{\zeta}}$ where $X_r = \pi^{-1} \left( \{ \lambda \in \mathbb{C} \mid |\lambda| < r \} \right) \cap C^\circ$. In view of Proposition 2.1 and Proposition 2.2 one can choose $0 < \varepsilon_m < \pi/4$, $m > N_0$, with $\sum_{m > N_0} \varepsilon_m^2 < \infty$ so that $\lambda_m^\pm \in D_m(\varepsilon_m)$ and $\lambda_m^\pm \in D_{-m}(\varepsilon_m)$ for any $m > N_0$. Choose $r > 0$ so that for some $m > N_0$

$$\pi m + \varepsilon_m \leq r \leq (m + 1)\pi - \varepsilon_{m+1}. \quad (7)$$

On the Riemann surface $X_r$ consider the points $p^+ := \pi^{-1}(r) \cap C^c_{\varphi}$ and $p^- := \pi^{-1}(r) \setminus \{ p^+ \}$ and choose a simple $C^1$-smooth curve $B_*$ on $X_r$ (i.e., a $C^1$-smooth map $[0, 1] \to X_r$ without self intersections) that connects $p^+$ with $p^-$ and changes sheets once its projection $\pi(B_*)$ passes through the 0-th curve $G_0$. If $\lambda_0^+ = \lambda_0^- = \tau_0$ and hence $G_0$ is the constant curve $\tau_0$ we allow the curve $B_*$ to pass through the point $(\tau_0, 0) \in C_{\varphi}$ that is excluded from $X_r$. The inverse image $\pi^{-1}(\partial D_0(r))$ consists of two simple closed curves with images

$$C_r^+ := \pi^{-1} \left( \{ \lambda \in \mathbb{C} \mid |\lambda| = r \} \right) \cap C^c_{\varphi}, \quad (8)$$

and

$$C_r^- := \pi^{-1} \left( \{ \lambda \in \mathbb{C} \mid |\lambda| = r \} \right) \setminus C_r^+$$

that we orient so that their projections $\pi(C_r^\pm) \subseteq \mathbb{C}$ has counterclockwise orientation. Furthermore, for any $1 \leq |k| \leq m$, denote by $B_k'$ a simple $C^1$-smooth curve in $X_r$ that starts and ends at $p^+$ and its projection $\pi(B_k')$ changes sheets twice – first passing through $G_k$ and then through $G_0$. If the image of $G_k$ is a point we proceed as above and allow the curve $B_k'$ to pass through the point $(\tau_k, 0) \in C_{\varphi}$. Similarly, for any $1 \leq |k| \leq m$, denote by $A_k'$ a simple $C^1$-smooth curve in $X_r$ that starts and ends at $p^+$ and that is homologous to $A_k$. The curves $B_*, C_r^+, A_k'$, and $B_k', 1 \leq |k| \leq m$, considered above are chosen so that they intersect each other only at $p^+$. Denote by
Then $\tilde{X}_r$ is a disk and its boundary $\partial \tilde{X}_r$ can be represented as a composition of the following curves (composed in the order of their appearance): $C_+^r$, $B^*$, $C_-^r$, $(B^*)^{-1}$, $B_1^r$, $A_1^r$, $(B_1^*)^{-1}$, $(A_1^*)^{-1}$, ..., $B_m^r$, $(B_m^*)^{-1}$, $(A_m^*)^{-1}$.

Consider the function $F : \tilde{X}_r \to \mathbb{C}$ given by

$$F(p) := \int_{p^+}^p \omega_\zeta$$

for $p \in \tilde{X}_r$. Note that the integral is independent of the choice of the path and hence $F$ is well defined on $\tilde{X}_r$. Furthermore introduce $a_k = \int_{A_k^r} \omega_\zeta (0 \leq |k| \leq m)$, $b_k := \int_{B_k^r} \omega_\zeta$, $(1 \leq |k| \leq m)$, $b_s = \int_{B_s} \omega_\zeta$ and $c_{r}^\pm = \int_{C_{r}^\pm} \omega_\zeta$. By Stokes’ theorem

$$-2iV(r) = \int_{\tilde{X}_r} d(F \omega_\zeta) = \int_{\partial \tilde{X}_r} F \overline{\omega_\zeta}$$

$$= \int_{C_r^+} F \overline{\omega_\zeta} + \int_{C_r^-} F \overline{\omega_\zeta} - c_r^- b_s - \sum_{1 \leq |k| \leq m} (a_k b_k - \overline{a_k} b_k)$$

where we used that $\int_{A_k^r} \omega_\zeta = a_k$ for any $1 \leq |k| \leq m$ as $A_k^r$ and $A_k$ are homologous. Note that for $z^- \in C_r^-$

$$F(z^-) = \int_{C_r^+} \omega_\zeta + \int_{B_s} \omega_\zeta + \int_{p^-} \omega_\zeta = c_r^+ + b_s + \int_{p^+} -\omega_\zeta$$

where $z^+ \in C_r^+$ is determined by $\pi(z^+) = \pi(z^-)$ and the minus sign stems from passing to the canonical sheet. Hence

$$\int_{C_r^-} F(z^-) \overline{\omega_\zeta} = -\int_{C_r^+} (c_r^+ + b_s - F(z^+)) \overline{\omega_\zeta}$$

$$= -|c_r^+|^2 - b_s c_r^+ + \int_{C_r^+} F \overline{\omega_\zeta}$$

yielding

$$-2iV(r) = 2 \int_{C_r^+} F \overline{\omega_\zeta} - |c_r^+|^2 - c_r^- b_s - b_s c_r^+ - \sum_{1 \leq |k| \leq m} (a_k b_k - \overline{a_k} b_k).$$

Now we use the assumption that $a_k = 0$ for $0 \leq |k| \leq m$. As $\sum_{0 \leq |k| \leq m} A_k$ is homologous to $C_r^+$ it then also follows that $c_r^+ = 0$. 

13
and as $c_r^{-} = -c_r^{+}$ one also has $c_r^{-} = 0$. We thus have proved that for any $m\pi + \varepsilon_m \leq r \leq (m+1)\pi - \varepsilon_{m+1}$ with $m > N_0$

$$V(r) = i \int_{C_r^+} F \overline{\zeta}.$$  \hfill (10)

We use this identity to prove

**Lemma 3.1** Let $\zeta : \mathbb{C} \to \mathbb{C}$ be entire with $\zeta |_{Z_{\phi}} = 0$ and $\int \omega \zeta = 0 \forall j \in \mathbb{Z}$. Then for any $m\pi + \varepsilon_m \leq r \leq (m+1)\pi - \varepsilon_{m+1}$ with $m > N_0$

$$V(r) \leq \frac{1}{2} \left( r \int_0^{2\pi} \left| \frac{\zeta(re^{i\theta})}{\sqrt{R(re^{i\theta})}} \right| d\theta \right)^2$$  \hfill (11)

and

$$V'(r) \geq \frac{2}{r\pi} V(r).$$  \hfill (12)

**Proof of Lemma 3.1.** Using polar coordinates one has, with $R(\lambda) = \Delta^2(\lambda) - 4$ we get from (11),

$$\int_{C_r^+} F \overline{\zeta} = \int_0^{2\pi} F(re^{i\theta}) \left( \frac{\zeta(re^{i\theta})}{\sqrt{R(re^{i\theta})}} \right) d(re^{i\theta})$$

$$= r^2 \int_0^{2\pi} \left( \int_0^\theta \frac{\zeta(re^{i\theta_1})}{\sqrt{R(re^{i\theta_1})}} e^{i\theta_1} d\theta_1 \right) \left( \frac{\zeta(re^{i\theta})}{\sqrt{R(re^{i\theta})}} \right) (-i)e^{-i\theta} d\theta,$$

where $\sqrt{R(\lambda)}$ denotes the canonical root \footnote{See}. Hence

$$V(r) = \left| \int_{C_r^+} F(p) \overline{\zeta} \right|$$

$$\leq r^2 \int_0^{2\pi} \left( \int_0^\theta \left| \frac{\zeta(re^{i\theta_1})}{\sqrt{R(re^{i\theta_1})}} \right| d\theta_1 \right) d\left( \int_0^\theta \left| \frac{\zeta(re^{i\theta_1})}{\sqrt{R(re^{i\theta_1})}} \right| d\theta_1 \right)$$

$$= \frac{r^2}{2} \left( \int_0^{2\pi} \left| \frac{\zeta(re^{i\theta})}{\sqrt{R(re^{i\theta})}} \right| d\theta \right)^2.$$  \hfill (11)

This proves the estimate \footnote{See}. To get \footnote{See} note that

$$\frac{r^2}{2} \left( \int_0^{2\pi} \left| \frac{\zeta(re^{i\theta})}{\sqrt{R(re^{i\theta})}} \right| d\theta \right)^2 \leq \pi r^2 \int_0^{2\pi} \left| \frac{\zeta(re^{i\theta})}{\sqrt{R(re^{i\theta})}} \right|^2 d\theta$$

$$= \pi r \frac{d}{dr} \left( \int_0^r \int_0^{2\pi} \left| \frac{\zeta(re^{i\theta})}{\sqrt{R(re^{i\theta})}} \right|^2 \rho \, d\theta d\rho \right) = \pi r \frac{V'(r)}{2}. $$
Hence \( V(r) \leq \frac{\pi r^2}{2} V'(r) \) as claimed. \( \square \)

Estimate (12) is now used to prove Theorem 1.1.

Proof of Theorem 1.1. Assume that \( \zeta \not\equiv 0 \) is an entire function satisfying \( \int_{A_m} \omega_\zeta = 0 \) for any \( m \in \mathbb{Z} \). Then \( V(r) \neq 0 \forall r > 0 \) and in view of (12), for any \( m > N_0 \) and for any \( m\pi + \epsilon_m \leq r \leq (m+1)\pi - \epsilon_{m+1} \),
\[
V(r) \neq 0 \quad \forall \ r > 0 \quad \text{and in view of (12), for any } m > N_0 \text{ and for any } m\pi + \epsilon_m \leq r \leq (m+1)\pi - \epsilon_{m+1},
\]
\[
(V'(r))^2 \geq 2 \pi r.
\]

Integrating this inequality over the interval \([m\pi + \epsilon_m, (m+1)\pi - \epsilon_{m+1}]\), we obtain that
\[
\frac{V((m+1)\pi - \epsilon_{m+1})}{V(m\pi + \epsilon_m)} \geq e^{2\pi \left( \log((m+1)\pi - \epsilon_{m+1}) - \log(m\pi + \epsilon_m) \right)}.
\]

This implies that for any \( m \geq m_0 > N_0 \),
\[
V((m+1)\pi) \geq V(m_0\pi) e^{2\pi S(m,m_0)},
\]
where
\[
S(m,m_0) := \sum_{j=m_0}^m \log \left( \frac{(j+1)\pi - \epsilon_{j+1}}{(j+1)\pi} \right) - \log \left( \frac{(j)\pi + \epsilon_j}{j\pi} \right)
\]
\[
= \log \frac{m+1}{m_0} + \sum_{j=m_0}^m \log \left( 1 - \frac{\epsilon_{j+1}}{(j+1)\pi} \right) - \sum_{j=m_0}^m \log \left( 1 + \frac{\epsilon_j}{j\pi} \right)
\]
\[
\geq \log \frac{m+1}{m_0} - O\left( \sum_{j=m_0}^{m+1} \frac{\epsilon_j}{j\pi} \right).
\]

As \( \sum_{j=m_0}^m \frac{\epsilon_j}{j\pi} = O\left( \sum_{j=m_0}^m \frac{2^j}{j^{1/2}} \right) \) one then concludes that
\[
V((m+1)\pi) \geq C(m+1)^{2/\pi}
\]
where \( C > 0 \) depends on \( m_0 > N_0 \) but not on \( m \geq m_0 \). \( \square \)

Proof of Theorem 1.2. Theorem 1.2 is an immediate consequence of Theorem 1.1. \( \square \)
4 Outline of proof of Theorem 1.3

In this section we describe the main steps in the proof of Theorem 1.3. We consider a neighborhood \( \mathcal{W} \subseteq L^2_{\bullet} \) as in Section 2 with \( N_0 \geq 1 \) so that Proposition 2.1 and Proposition 2.2 hold. Let \( (A_m)_{m \in \mathbb{Z}} \) be the cycles on the canonical branch of \( C_{\bullet} \) introduced in Section 2. Without further explanations, for any given \( \varphi \in \mathcal{W} \) and \( n \in \mathbb{Z} \) consider the following ansatz for the holomorphic differentials of Theorem 1.3

\[
\Omega^n := \Omega - \omega^n. \tag{13}
\]

To make sure that \( \Omega^n \) is holomorphic we distinguish the cases \( |n| > N_0 \) and \( |n| \leq N_0 \) in the definition of \( \Omega^n \) and \( \omega^n \):

\[
\Omega^n := \begin{cases} 
\frac{\Delta(\lambda)}{\lambda - \lambda_n} \frac{d\lambda}{\sqrt{R(\lambda)}}, & |n| > N_0 \\
\frac{\Delta(\lambda)}{\lambda - \lambda_{N_0}} \frac{d\lambda}{\sqrt{R(\lambda)}}, & |n| \leq N_0
\end{cases} \tag{14}
\]

and for any given \( \beta = (\beta_k)_{k \neq n} \in \ell^1_\mathbb{C} \equiv \ell^1(\mathbb{Z}\backslash\{n\}, \mathbb{C}) \)

\[
\omega^n := \frac{\xi^n_\beta(\lambda)}{R(\lambda)} d\lambda \tag{15}
\]

\[
\xi^n_\beta(\lambda) := \begin{cases} 
\sum_{|j| > N_0, j \neq n} \frac{\beta_j}{\lambda - \lambda_j} + \frac{p^n_\beta(\lambda)}{\prod_{k \leq N_0}(\lambda - \lambda_k)} \frac{\Delta(\lambda)}{\lambda - \lambda_n}, & |n| > N_0 \\
\sum_{|j| > N_0} \frac{\beta_j}{\lambda - \lambda_j} + \frac{p^n_\beta(\lambda)}{\prod_{k \neq N_0}(\lambda - \lambda_k)} \frac{\Delta(\lambda)}{\lambda - \lambda_{N_0}}, & |n| \leq N_0
\end{cases} \tag{16}
\]

and

\[
p^n_\beta(\lambda) = \begin{cases} 
2N_0 \sum_{j=0}^{n+N_0-1} \beta_{j-N_0}\lambda^j, & |n| > N_0 \\
\sum_{j=0}^{n+N_0-1} \beta_{j-N_0}\lambda^j + \sum_{j=n+N_0}^{2N_0-1} \beta_{j-N_0+1}\lambda^j, & |n| \leq N_0.
\end{cases}
\]

Note that \( \xi^n_\beta(\lambda) \) is entire. In the case \( |n| \leq N_0 \), it is convenient to write the polynomial \( p^n_\beta(\lambda) \) in the following alternative way

\[
p^n_\beta(\lambda) = \left( \sum_{|j| \leq N_0, j \neq n} \beta_j \lambda^{j - \varepsilon^n_j} \right) \lambda^{N_0} \]

where

\[
\varepsilon^n_j := \begin{cases} 
1, & j \geq n \\
0, & j < n.
\end{cases} \tag{17}
\]
We want to find $\beta = (\beta_k)_{k \neq n} \in \ell^1_n$ so that
\[
\frac{1}{2\pi} \int_{A_m} \Omega^n_\beta = \delta_{nm} \quad \forall m \in \mathbb{Z}. \tag{18}
\]
The following proposition is proved in Section 6.

**Proposition 4.1** For any $n \in \mathbb{Z}$, $\varphi \in \mathcal{W}$, and $\beta \in \ell^1_n$
\[
\sum_{m \in \mathbb{Z}} \frac{1}{2\pi} \int_{A_m} \Omega^n_\beta = \lim_{K \to \infty} \sum_{|m| \leq K} \frac{1}{2\pi} \int_{A_m} \Omega^n_\beta = 1. \tag{19}
\]
In particular, for $\beta = 0$, $\Omega^n_\beta \equiv \Omega^n$ satisfies
\[
\sum_{m \in \mathbb{Z}} \frac{1}{2\pi} \int_{A_m} \Omega^n = 1.
\]
In view of Proposition 4.1, the system of equations (18) is equivalent to
\[
\int_{A_m} \Omega^n_\beta = \int_{A_m} \Omega^n \quad \forall m \neq n. \tag{20}
\]
By multiplying the right and left hand side of the above equation by $m \pi - n \pi$ (if $|n| > N_0$) or $m \pi - N_0 \pi$ (if $|n| \leq N_0$), we arrive to the following linear system for $\beta$,
\[
T^n_\beta = b^n, \quad T^n = (T^n_{mj})_{m,j \neq n}, \quad b^n = (b^n_m)_{m \neq n} \tag{21}
\]
where for any $m \neq n$,
\[
b^n_m := \begin{cases} \frac{1}{2\pi} \int_{\Gamma_m} \frac{m \pi - n \pi}{\lambda - \lambda_n} \frac{\Delta(\lambda)}{\sqrt{R(\lambda)}} \, d\lambda, & |n| > N_0 \\ \frac{1}{2\pi} \int_{\Gamma_m} \frac{m \pi - N_0 \pi}{\lambda - \lambda_{N_0}} \frac{\Delta(\lambda)}{\sqrt{R(\lambda)}} \, d\lambda, & |n| \leq N_0 \end{cases}
\]
for $|n| > N_0$,
\[
T^n_{mj} := \begin{cases} \frac{1}{2\pi} \int_{\Gamma_m} \frac{m \pi - n \pi}{\lambda - \lambda_n} \frac{1}{\sqrt{R(\lambda)}} \, d\lambda, & |j| > N_0, j \neq n \\ \frac{1}{2\pi} \int_{\Gamma_m} \frac{m \pi - n \pi}{\lambda - \lambda_n} \frac{\Delta(\lambda)}{\sqrt{R(\lambda)}} \, d\lambda, & |j| \leq N_0 \end{cases}
\]
and, for $|n| \leq N_0$,
\[
T^n_{mj} := \begin{cases} \frac{1}{2\pi} \int_{\Gamma_m} \frac{m \pi - n \pi}{\lambda - \lambda_n} \frac{1}{\sqrt{R(\lambda)}} \, d\lambda, & |j| > N_0 \\ \frac{1}{2\pi} \int_{\Gamma_m} \frac{(m \pi - N_0 \pi)}{\lambda - \lambda_{N_0}} \frac{\Delta(\lambda)}{\sqrt{R(\lambda)}} \, d\lambda, & |j| \leq N_0 \end{cases}
\]
Using Proposition 6.1 – an application of Theorem 1.2 – we prove in Section 7 the following

17
Proposition 4.2 For any \( n \in \mathbb{Z} \) and \( \varphi \in \mathcal{W} \) we have:

(i) \( b^n \in \ell^1_h \);

(ii) \( T^n : \ell^1_h \rightarrow \ell^1_h \) is a linear isomorphism.

Denote by \( \beta^n \equiv \beta^n(\varphi) \in \ell^1_h \) the unique solution of (21), guaranteed by Proposition 4.2 and define,

\[
\zeta_n(\lambda, \varphi) := \begin{cases} 
\frac{\dot{\Delta}(\lambda)}{\lambda - \lambda_n} - \xi^n_{\beta_n}(\lambda), & |n| > N_0 \\
\frac{\dot{\Delta}(\lambda)}{\lambda - \lambda_{N_0}} - \xi^n_{\beta_n}(\lambda), & |n| \leq N_0,
\end{cases}
\]

where \( \xi^n_{\beta_n} \) is given by (16) with \( \beta^n \) substituted for \( \beta \). The following proposition is proved in Section 7.

Proposition 4.3 For any \( n \in \mathbb{Z} \), \( \beta^n : \mathcal{W} \rightarrow \ell^1_h \) and \( \zeta_n : \mathbb{C} \times \mathcal{W} \rightarrow \mathbb{C} \) are analytic maps. Furthermore, for any \( \varphi \in \mathcal{W} \) and \( n \in \mathbb{Z} \),

\[
\frac{1}{2\pi} \int_{\Gamma_m} \frac{\zeta_n(\lambda, \varphi)}{\sqrt{R(\lambda, \varphi)}} d\lambda = \delta_{nm} \quad \forall m \in \mathbb{Z}.
\]

To obtain uniform in \( n \in \mathbb{Z} \) and locally uniformly in \( \mathcal{W} \) estimates of zeroes of \( \zeta_n \) we consider the following "limiting" linear system for \( \beta = (\beta_k)_{k \in \mathbb{Z}} \in \ell^1 \equiv \ell^1_\mathbb{C} \),

\[
T^* \beta = 0
\]

(22)

where \( T^* = (T^*_{mj})_{m, j \in \mathbb{Z}} \) is given by

\[
T^*_{mj} = \begin{cases} 
\frac{1}{2\pi} \int_{\Gamma_m} \frac{\dot{\Delta}(\lambda)}{\lambda - \lambda_j} \frac{1}{\sqrt{R(\lambda)}} d\lambda, & |j| > N_0 \\
\frac{1}{2\pi} \int_{\Gamma_m} \frac{\dot{\Delta}(\lambda)}{\lambda - \lambda_{N_0+1}} \frac{1}{\sqrt{R(\lambda)}} d\lambda, & |j| \leq N_0.
\end{cases}
\]

This linear system is equivalent to the condition

\[
\int_{\Gamma_m} \omega^*_\beta = 0 \quad \forall m \in \mathbb{Z}
\]

where \( \beta \in \ell^1 \) and the holomorphic 1-form \( \omega^*_\beta \) on \( C^*_\varphi \) is given by

\[
\omega^*_\beta := \frac{\xi^*_\beta(\lambda)}{\sqrt{R(\lambda)}} d\lambda,
\]

with

\[
\xi^*_\beta(\lambda) := \left( \sum_{|j| > N_0} \frac{\beta_j}{\lambda - \lambda_j} + \prod_{|j| \leq N_0} (\lambda - \lambda_j) \right) \dot{\Delta}(\lambda),
\]

(23)
and
\[ p_\beta^*(\lambda) := \sum_{j=0}^{2N_0} \beta_{-N_0+j} \lambda^j. \]

Note that \( \xi_\beta^* \) is an entire function of \( \lambda \) and \( p_\beta^*(\lambda) \) is a polynomial of degree at most \( 2N_0 \). Using Proposition 5.1 – another application of Theorem 1.2 – we prove in Section 7 the following

**Proposition 4.4** For any \( \varphi \in \mathcal{W} \), \( T^* : \ell^1 \to \ell^1 \) is a linear isomorphism.

Proposition 4.4 is used in Section 8 to prove the uniform estimates of the zeroes of \( \zeta_n \), stated in Theorem 1.3 – see Proposition 8.1 and Lemma 8.3 – and the product representation of \( \zeta_n \) – see Corollary 8.1.

Finally, in Lemma 8.4, it is proved that \( \zeta_n(\cdot, \varphi) \) vanishes on the set \( \mathbb{Z}_\varphi \setminus \{\lambda_n^\pm(\varphi)\} \). Combining the results described above, the proof of Theorem 1.3 is complete.

## 5 Vanishing Lemma

Let us fix \( \varphi \in \mathcal{W} \subseteq L^2_\phi \) where \( \mathcal{W} \) is the neighborhood constructed in Section 2. In this section we prove the following

**Proposition 5.1** Let \( \beta \in \ell^1 \) be arbitrary. If \( \int_{A_m} \omega_\beta^* = 0 \) for any \( m \in \mathbb{Z} \), then \( \beta = 0 \).

We prove Proposition 5.1 with the help of Theorem 1.2. To this end we prove the following lemmas.

**Lemma 5.1** If \( \int_{A_m} \omega_\beta^* = 0 \) for any \( m \in \mathbb{Z} \) then \( \xi_\beta^* \mid_{\mathbb{Z}_\varphi} = 0 \).

**Proof of Lemma 5.1.** Assume that for some \( k \in \mathbb{Z} \), \( \lambda_k^- = \lambda_k^+ = \tau_k \). Then in view of (5) for \( (\lambda, w) \in C^c \varphi \) near \( (\tau_k, 0) \in C^c \varphi \),
\[
\omega_\beta^* = \xi_\beta^*(\lambda) \frac{h(\lambda)}{\lambda - \tau_k} d\lambda
\]  
where \( h(\lambda) \) is a holomorphic function that is defined in an open neighborhood of \( \tau_k \) and satisfies \( h(\tau_k) \neq 0 \). As by assumption \( \int_{A_k} \omega_\beta^* = 0 \) we conclude from (24) that \( \xi_\beta^*(\tau_k) = 0 \).

Lemma 5.1 implies that for any \( \beta \in \ell^1 \) as in Proposition 5.1 and for any \( r > 0 \)
\[
V_\beta(r) := \frac{i}{2} \int_{X_r} \omega_\beta^* \wedge \overline{\omega_\beta^*} < \infty.
\]
Lemma 5.2 If \( \int_{A_j} \omega^*_\beta = 0 \) for any \( j \in \mathbb{Z} \), then for any \( \delta > 0 \),

\[
V_\beta(m\pi) = O(m^\delta) \quad \text{as } m \to \infty.
\] (25)

Proof of Lemma 5.2 Assume that \( \int_{A_j} \omega^*_\beta = 0 \) for any \( j \in \mathbb{Z} \). Then by (11) in Lemma 3.1 for \( r_m = (m + \frac{1}{2})\pi \) with \( m > N_0 \),

\[
V_\beta(m\pi) \leq \frac{r_m^2}{2} \left( \int_0^{2\pi} \frac{|\xi^*_\beta(r_m e^{i\theta})|}{\sqrt{R(r_m e^{i\theta})}} d\theta \right)^2.
\]

By [5, Lemma C.5] and Proposition 2.2, \( \frac{\Delta(r_m e^{i\theta})}{\sqrt{R(r_m e^{i\theta})}} = O(1) \) as \( m \to \infty \) uniformly in \( 0 \leq \theta < 2\pi \). This together with the definition of \( \xi^*_\beta \) implies

\[
\left| \frac{\xi^*_\beta(r_m e^{i\theta})}{\sqrt{R(r_m e^{i\theta})}} \right| \leq C \left( \sum_{|j| > N_0} \frac{|\beta_j|}{|r_m e^{i\theta} - \lambda_j|} + \frac{|p^*_\beta(r_m e^{i\theta})|}{\prod_{|j| \leq N_0} |r_m e^{i\theta} - \lambda_j|} \right)
\]

with a constant \( C > 0 \) independent of \( m > N_0 \) and \( 0 \leq \theta < 2\pi \). As \( p^*_\beta(\lambda) \) is a polynomial in \( \lambda \) of degree at most \( 2N_0 \) it follows that

\[
\int_0^{2\pi} \frac{|p^*_\beta(r_m e^{i\theta})|}{\prod_{|j| \leq N_0} |r_m e^{i\theta} - \lambda_j|} d\theta = O\left( \frac{1}{m} \right) \quad \text{as } m \to \infty.
\] (27)

For any \( m > 2N_0 \), we split the sum \( \sum_{|j| > N_0} = \sum_{j \in J_1(m)} + \sum_{j \in J_2(m)} \) where \( J_1(m) \) is the set

\[
\left\{ j > N_0 \left| |j - (m + \frac{1}{2})| \leq \frac{m}{2} \right\} \cup \left\{ j < -N_0 \left| |j + (m + \frac{1}{2})| \leq \frac{m}{2} \right\}
\]

and \( J_2(m) := \mathbb{Z} \setminus ([N_0, N_0] \cup J_1(m)) \). Then for any \( j \in J_2(m) \) with \( m > 2N_0 \) and any \( 0 \leq \theta < 2\pi \)

\[
|r_m e^{i\theta} - \lambda_j| \geq \left| (m + \frac{1}{2})\pi - j\pi \right| - \frac{\pi}{4} \geq \frac{(m + 1)\pi}{2} - \frac{\pi}{4} \geq \frac{m\pi}{2}
\]

implying that

\[
\int_0^{2\pi} \sum_{j \in J_2(m)} \frac{|\beta_j|}{|r_m e^{i\theta} - \lambda_j|} d\theta = O\left( \frac{1}{m} \right).
\] (28)
To estimate \( \int_0^{2\pi} \sum_{j \in J_1(m)} \frac{|\beta_j|}{|r_m e^{i\theta} - \lambda_j|} \, d\theta \) the integral \( \int_0^{2\pi} \) is split up as follows: For \( 0 < \alpha < 1 \), one has uniformly in \( j \in J_1(m) \),

\[
\int_{\pi - \frac{1}{m^\alpha}}^{\pi} \frac{d\theta}{|r_m e^{i\theta} - \lambda_j|} + \int_{\pi + \frac{1}{m^\alpha}}^{\pi + \frac{1}{m^\alpha}} \frac{d\theta}{|r_m e^{i\theta} - \lambda_j|} = O\left( \frac{1}{m^\alpha} \right)
\]

as \( |r_m e^{i\theta} - \lambda_j| \geq \pi/4 \). By choosing \( N_1 \geq 2N_0 \) sufficiently large we can ensure that for any \( m \geq N_1, \theta \in \left[ \frac{1}{m^\alpha}, \pi - \frac{1}{m^\alpha} \right] \) and \( j \in J_1(m) \)

\[
|r_m e^{i\theta} - \lambda_j| \geq |r_m \sin \theta - \text{Im} \dot{\lambda}_j| \\
\geq r_m \sin \theta - \frac{\pi}{4} \geq \left( m + \frac{1}{2} \right) \sin \left( \frac{1}{m^\alpha} \right) - \frac{1}{4} \geq 1.
\]

Note that \( r_m \sin \theta - \frac{\pi}{4} \) is the distance from \( r_m e^{i\theta} \) to the horizontal line \( \text{Im} z = \frac{\pi}{4} \). Using (29), \( \sin \theta \geq \frac{2}{\pi} \theta \) for \( 0 \leq \theta \leq \pi/2 \) and taking \( N_1 \geq 2N_0 \) larger if necessary we get

\[
\int_{\pi - \frac{1}{m^\alpha}}^{\pi} \frac{d\theta}{r_m e^{i\theta} - \lambda_j} \leq 2 \int_{\frac{1}{m^\alpha}}^{\pi} \frac{d\theta}{r_m \sin \theta - \frac{\pi}{4}} \\
= \frac{\pi}{r_m} \int_{\frac{1}{m^\alpha}}^{\pi/2} \frac{d\theta}{\theta - \frac{\pi^2}{8r_m}} = O\left( \frac{\log m}{m} \right)
\]

and similarly

\[
\int_{\pi + \frac{1}{m^\alpha}}^{2\pi - \frac{1}{m^\alpha}} \frac{d\theta}{r_m e^{i\theta} - \lambda_j} = O\left( \frac{\log m}{m} \right)
\]

uniformly in \( j \in J_1(m) \). Hence

\[
\int_0^{2\pi} \sum_{j \in J_1(m)} \frac{|\beta_j|}{|r_m e^{i\theta} - \lambda_j|} \, d\theta = O\left( \frac{1}{m^\alpha} \right) \text{ as } m \to \infty. \quad (30)
\]

Combining (26), (27), (28) and (30) yields

\[
V_{\beta}(m\pi) = O\left( m^{2-2\alpha} \right)
\]

for any \( 0 < \alpha < 1 \). This completes the proof of the lemma.

\( \square \)

**Proof of Proposition 5.1.** By assumption \( \omega^*_\beta \) with \( \beta \in \ell^1_1 \) satisfies \( \int_{A_m} \omega^*_\beta = 0 \) for and \( m \in \mathbb{Z} \). Then Lemma 5.1 implies that \( \xi^*_\beta \big|_{Z^\alpha} = 0 \). Hence we can apply Theorem 1.2 and Lemma 5.2 to conclude that
\( \xi_0^\ast \equiv 0 \). Evaluating \( \xi_0^\ast \) at \( \lambda = \dot{\lambda}_k \) with \( |k| > N_0 \) we get from (23) and Proposition 2.3 that

\[
0 = \xi_0^\ast(\dot{\lambda}_k) = 2\beta_k \prod_{m \neq k} \frac{\dot{\lambda}_k - \dot{\lambda}_m}{\pi_m}.
\]

As \( |k| > N_0 \), \( \dot{\lambda}_k \) is a simple zero of \( \dot{\Delta}(\lambda) \) (cf. Proposition 2.1) and hence \( \prod_{m \neq k} \frac{\dot{\lambda}_k - \dot{\lambda}_m}{\pi_m} \neq 0 \). We therefore conclude that \( \beta_k = 0 \) for any \( |k| > N_0 \) and thus in view of (23),

\[
\xi_0^\ast(\lambda) = p_0^\ast(\lambda) \prod_{|m| > N_0} \frac{\lambda - \dot{\lambda}_m}{\pi_m} \prod_{|m| \leq N_0} \frac{\pi_m}{\pi_m}.
\]

As \( \xi_0^\ast \equiv 0 \) it follows that \( p_0^\ast \equiv 0 \) implying that \( \beta_k = 0 \) for \( |k| \leq N_0 \). We thus have proved that \( \beta = 0 \) as claimed. \( \square \)

For any given \( n \in \mathbb{Z} \) and \( \beta = (\beta_k)_{k \neq n} \in \ell_1^n \) consider the holomorphic 1-form \( \omega_\beta^n = \frac{\xi_0^\ast(\dot{\lambda})}{\sqrt{R(\lambda)}} d\lambda \) defined in (15). Arguing in the same way as in the proof of Proposition 5.1 one obtains

**Proposition 5.2** Let \( n \in \mathbb{Z} \) and \( \beta = (\beta_k)_{k \neq n} \in \ell_1^n \) be arbitrary. If \( \int_{A_m} \omega_\beta^n = 0 \) for any \( m \in \mathbb{Z} \), then \( \beta = 0 \).

### 6 Proof of Proposition 4.1

The aim of this section is to prove Proposition 4.1 concerning the identity of the sum of all \( A \)-periods of a holomorphic differential of the form \( \Omega_\beta^n \).

**Proof of Proposition 4.1.** As the proof in the two cases \( |n| \leq N_0 \) and \( |n| > N_0 \) are similar, we consider the case \( |n| > N_0 \) only. Recall that for any \( |n| > N_0 \) and \( \beta \in \ell_1^n \)

\[
\Omega_\beta^n = \frac{1 - \eta_\beta^n(\lambda)}{\lambda - \dot{\lambda}_n} \frac{\Delta(\lambda)}{\sqrt{R(\lambda)}} d\lambda
\]

where

\[
\eta_\beta^n(\lambda) = \sum_{|j| > N_0, j \neq n} \frac{\beta_j}{\lambda - \dot{\lambda}_j} + \frac{p_\beta^n(\lambda)}{\prod_{|k| \leq N_0} (\lambda - \dot{\lambda}_k)}
\]

(31)
is a meromorphic function that can have poles only at the points \( \lambda_j, j \in \mathbb{Z} \). Let \( r_m := m\pi + \pi/2 \) for \( m > N_0 \) with \( N_0 \) as in Proposition 2.1. As \( \sum_{|k| \leq m} A_k \) is homologous to \( C_r^+ \) (cf. (3)) one has for any \( m > N_0 \)

\[
\sum_{|k| \leq m} \frac{1}{2\pi} \int_{A_k} \frac{1}{2\pi} \int_{C_r^+} \frac{1}{\lambda^n - \lambda^m} \frac{1}{|\lambda^n|^{\beta}} d\lambda d\lambda = C + r_m \Omega^n_{\beta}.
\]

(32)

In view of [5, Lemma C.5], Proposition 2.1, Proposition 2.2 and the definition of the canonical root (5)

\[
\frac{\Delta(r_m e^{i\theta})}{\sqrt{R(r_m e^{i\theta})}} = \frac{1}{i(1 + o(1))} \quad \text{as } m \to \infty
\]

uniformly for \( 0 \leq \theta < 2\pi \). This together with (31) implies that uniformly for \( \lambda \in C_r^+ \)

\[
\Omega^n_{\beta} = \frac{1}{i} \left( \frac{1}{\lambda - \lambda^n} + O\left( \frac{\eta^n_\beta(\lambda)}{m} \right) + o\left( \frac{1}{m} \right) \right) d\lambda \quad \text{as } m \to \infty
\]

(33)

with constants independent of \( \lambda \in C_r^+ \) and \( m > N_0 \). Combining (32) with (33) one gets for \( m > \max\{n, N_0\} \)

\[
\sum_{|k| \leq m} \frac{1}{2\pi} \int_{A_k} \Omega^n_{\beta} = 1 + O\left( \max_{C_r^+} |\eta^n_\beta(\lambda)| \right) + o(1)
\]

(34)

with constants uniform in \( m > \max\{n, N_0\} \). Arguing in a similar way as in the proof of Lemma 5.2 one sees that

\[
\max_{\lambda \in C_r^+} |\eta^n_\beta(\lambda)| \to 0 \quad \text{as } m \to \infty.
\]

This combined with (34) yields Proposition 4.1

□

As an immediate Corollary of Proposition 4.1 we get the following result for \( \omega^n_\beta = \Omega^n - \Omega^n_\gamma = \Omega^n_\gamma |_{\gamma=0} - \Omega^n_\beta \).

**Corollary 6.1** For any \( n \in \mathbb{Z} \) and any \( \beta \in \ell_1^\mathbb{N} \),

\[
\sum_{m \in \mathbb{Z}} \frac{1}{2\pi} \int_{A_m} \omega^n_\beta = \lim_{K \to \infty} \sum_{|m| \leq K} \frac{1}{2\pi} \int_{A_m} \omega^n_\beta = 0.
\]

Corollary 6.1 can be combined with Proposition 5.2 yielding the following

**Proposition 6.1** Let \( n \in \mathbb{Z} \) and \( \beta = (\beta_k)_{k \neq n} \in \ell_1^\mathbb{N} \) be arbitrary. If \( \int_{A_m} \omega^n_\beta = 0 \) for any \( m \in \mathbb{Z} \setminus \{n\} \), then \( \beta = 0 \).
7 Existence of normalized differentials

The aim of this section is to study the operators $T^n$ and $T^*$, introduced in Section 4, and to prove Proposition 4.2, Proposition 4.3, and Proposition 4.4. We begin with the study of $T^*$.

Lemma 7.1. Locally uniformly on $\mathcal{W}$, the coefficients $T^*_m j$ of $T^*$ satisfy the following estimates

$$T^*_m j = \begin{cases} O\left(\frac{1}{|m|} \right), & j \in \mathbb{Z}, |m| > N_0, |m| \neq j \\ 1 + \ell^2(m), & |j| > N_0, m = j \\ O\left(\frac{1}{1 + |j|} \right), & j \in \mathbb{Z}, |m| \leq N_0 \end{cases}$$

Proof. Case $|j| > N_0$ and $\lambda^+_m = \lambda^-_m$. Recall that by Proposition 2.2 and the definition of the canonical root

$$\frac{\Delta(\lambda)}{\sqrt[\uparrow]{R(\lambda)}} = \frac{1}{i} \prod_{k \in \mathbb{Z}} \frac{\dot{\lambda}_k - \lambda}{\sqrt[\downarrow]{(\lambda^+_k - \lambda)(\lambda^-_k - \lambda)}}$$

and that for $|j| > N_0$, $T^*_m j$ is given by

$$T^*_m j = \frac{1}{2\pi} \int_{\Gamma_m} \frac{1}{\lambda - \dot{\lambda}_m} \frac{\Delta(\lambda)}{\sqrt[\uparrow]{R(\lambda)}} d\lambda.$$ 

As $\lambda^+_m = \lambda^-_m = \tau_m = \dot{\lambda}_m$ one has by the definition of the standard root, $\sqrt[\uparrow]{(\lambda^+_m - \lambda)(\lambda^-_m - \lambda)} = \tau_m - \lambda$, and hence, if in addition $m \neq j$, $\frac{1}{\lambda - \dot{\lambda}_m} \frac{\Delta(\lambda)}{\sqrt[\uparrow]{R(\lambda)}}$ is holomorphic near $\tau_m$ and thus $T^*_m j = 0$. If $m = j$ (and hence $|m| > N_0$) one gets

$$T^*_m j = \frac{1}{2\pi} \int_{\Gamma_m} \frac{1}{\lambda - \dot{\lambda}_m} \frac{\Delta(\lambda)}{\sqrt[\uparrow]{R(\lambda)}} d\lambda = \frac{1}{2\pi i} \int_{\Gamma_m} \frac{1}{\lambda - \tau_m} \prod_{k \neq m} \frac{\dot{\lambda}_k - \lambda}{\sqrt[\downarrow]{(\lambda^+_k - \lambda)(\lambda^-_k - \lambda)}} d\lambda.$$ 

Therefore, by the residue theorem and the product estimate in [5, Lemma C.3]

$$T^*_m m = \prod_{k \neq m} \frac{\dot{\lambda}_k - \tau_m}{\sqrt[\downarrow]{(\lambda^+_k - \tau_m)(\lambda^-_k - \tau_m)}} = 1 + \ell^2(m)$$

24
locally uniformly in $W$.

*Case $|j| > N_0$ and $\lambda_j^+ \neq \lambda_j^-$:* If $m = j$, one uses again [5, Lemma C.3] to see that
\[
T_{mm}^* = \frac{1}{2\pi i} \int_{\Gamma_m} \frac{-1}{\sqrt{(\lambda_m^+ - \lambda)(\lambda_m^- - \lambda)}} \prod_{k \neq m} \frac{\hat{\lambda}_k - \lambda}{\sqrt{(\lambda_k^+ - \lambda)(\lambda_k^- - \lambda)}} d\lambda
\]
\[
= -\frac{1}{2\pi i} \int_{\Gamma_m} \frac{d\lambda}{\sqrt{(\lambda_m^+ - \lambda)(\lambda_m^- - \lambda)}} + \ell^2(m).
\]

A direct calculation shows that $\frac{1}{2\pi i} \int_{\Gamma_m} \frac{d\lambda}{\sqrt{(\lambda_m^+ - \lambda)(\lambda_m^- - \lambda)}} = -1$ yielding the claimed estimate $T_{mm}^* = 1 + \ell^2(m)$ in this case. If $|m| > N_0$ but $m \neq j$, then
\[
T_{mj}^* = \frac{1}{2\pi i} \int_{\Gamma_m} \frac{B_{mj}(\lambda)}{\sqrt{(\lambda_m^+ - \lambda)(\lambda_m^- - \lambda)}} d\lambda
\]
where
\[
B_{mj} := -\frac{\hat{\lambda}_m - \lambda}{\sqrt{(\lambda_j^+ - \lambda)(\lambda_j^- - \lambda)}} \prod_{k \neq m,j} \frac{\hat{\lambda}_k - \lambda}{\sqrt{(\lambda_k^+ - \lambda)(\lambda_k^- - \lambda)}}.
\]

By deforming the contour $\Gamma_m$ to the straight interval $G_m$ (taken twice) one then sees (cf. [5, Lemma 14.3]) that
\[
|T_{mj}^*| \leq \max_{\lambda \in G_m} |B_{mj}(\lambda)|.
\]

For $\lambda \in G_m$, $|\hat{\lambda}_m - \lambda| \leq |\hat{\lambda}_m - \tau_m| + |\gamma_m|$ and
\[
\left(\sqrt{(\lambda_j^+ - \lambda)(\lambda_j^- - \lambda)}\right)^{-1} = O\left(\frac{1}{j - m}\right)
\]
whereas again with [5, Lemma C.3], $\prod_{k \neq m,j} \frac{\hat{\lambda}_k - \lambda}{\sqrt{(\lambda_k^+ - \lambda)(\lambda_k^- - \lambda)}} = O(1)$ uniformly on $W$. Altogether we thus conclude that in the case considered $T_{mj}^* = O\left(\frac{1}{j - m} + |\gamma_m|\right)$. Finally if $|m| \leq N_0$ and hence $m \neq j$ (as we assume $|j| > N_0$), one has that (37) and (38) hold. Hence,
\[
|T_{mj}^*| \leq \max_{\lambda \in \Gamma_m} \left| \frac{B_{mj}(\lambda)}{\sqrt{(\lambda_m^+ - \lambda)(\lambda_m^- - \lambda)}} \right| \text{length}(\Gamma_m)/2\pi
\]
where length($\Gamma_m$) is the Euclidean length of $\Gamma_m$. Using that $\lambda_m^\pm$ are inside $\Gamma'_m$ and hence uniformly in $W$ separated from $\Gamma_m$ and as by definition, different contours are apart by a uniform constant one concludes from the estimate $(\sqrt{(\lambda_j^+ - \lambda)(\lambda_j^- - \lambda})^{-1} = O\left(\frac{1}{j} \right)$ and [11, Lemma C.3] that $T_{mj}^* = O\left(\frac{1}{j} \right)$ uniformly on $W$.

**Case $|j| \leq N_0$:** In this case the coefficient $T_{mj}^*$ is given by the formula

$$T_{mj}^* = \frac{1}{2\pi} \int_{\Gamma_m} \lambda^{N_0+j} \Delta(\lambda) \prod_{|k| \leq N_0} \frac{\hat{\lambda}_k - \lambda}{\sqrt{\mathcal{R}(\lambda)}} \frac{d\lambda}{\sqrt{\mathcal{R}(\lambda)}}$$

$$= \frac{1}{2\pi i} \int_{\Gamma_m} \lambda^{N_0+j} \prod_{|k| \leq N_0} \sqrt{(\lambda_k^+ - \lambda)(\lambda_k^- - \lambda)} \prod_{|k| > N_0, k \neq m} \sqrt{(\lambda_k^+ - \lambda)(\lambda_k^- - \lambda)} \frac{d\lambda}{\sqrt{\mathcal{R}(\lambda)}}$$

If $|m| > N_0$ we apply [11, Lemma 14.3] to conclude that,

$$|T_{mj}^*| \leq \max_{\lambda \in G_m} \left| \prod_{|k| \leq N_0} \sqrt{(\lambda_k^+ - \lambda)(\lambda_k^- - \lambda)} \prod_{|k| > N_0, k \neq m} \sqrt{(\lambda_k^+ - \lambda)(\lambda_k^- - \lambda)} \right|$$

If $|m| > N_0$,

$$\max_{\lambda \in G_m} \left| \prod_{|k| \leq N_0} \sqrt{(\lambda_k^+ - \lambda)(\lambda_k^- - \lambda)} \right| = O\left(\frac{1}{m} \right)$$

and, again by the product estimate [5, Lemma C.3], we get that $T_{mj}^* = O\left(\frac{1}{m} \right)$ uniformly on $W$. If $|m| \leq N_0$ we use again that $\lambda_m^\pm$ are inside $\Gamma'_m$, hence uniformly in $W$ separated form $\Gamma_m$, and that different contours are apart by a uniform constant, to see that $T_{mj}^* = O(1)$. The claimed estimates for $T_{mj}^*$ are proved. □

From Lemma 7.1 it immediately follows that $T^*$ defines a bounded linear operator, $T^* : \ell^1 \to \ell^1$.

**Proof of Proposition 4.4.** Take $\varphi \in W$. By Proposition 5.1 $T^*$ is injective. We claim that $T^* - Id$ is a compact operator on $\ell^1$. Therefore, $T^*$ is Fredholm and thus $T^*$ is a linear isomorphism. To see that $T^* - Id$ is compact introduce for any $N > N_0$ the operator $K_N : \ell^1 \to \ell^1$,

$$K_N := \Pi_N \circ (T^* - Id)$$
where \( \Pi_N : \ell^1 \to \ell^1 \) is the projection, 
\[
(\beta_k)_{k \in \mathbb{Z}} \mapsto (\ldots, 0, \beta_{-N}, \ldots, \beta_N, 0, \ldots).
\]
Note that \( K_N \) is an operator of finite rank and therefore compact. By Lemma 7.1 we have for any \( N \geq N_0 \),
\[
(T^* - \text{Id} - K_N)_{m,j} = \begin{cases} 
O(\frac{|\lambda_m - \tau_m| + |\gamma_m|}{|j - m|}), & j \in \mathbb{Z}\backslash\{m\}, \ |m| > N \\
\ell^2(m), & |m| > N, \ j = m \\
0, & j \in \mathbb{Z}, \ |m| \leq N.
\end{cases}
\]
Hence there exists a constant \( C > 0 \) independent of \( N \geq N_0 \) so that for any \( N \geq N_0 \) and any \( \beta \in \ell^1 \),
\[
\|(T^* - \text{Id} - K_N)\beta\|_{\ell^1} \leq C \sum_{|m| > N} \sum_{j \neq m} |\lambda_m - \tau_m| + |\gamma_m| \ |\beta_j| 
+ \sum_{|m| > N} \ell^2(m) |\beta_m|.
\]
Clearly,
\[
\sum_{|m| > N} \ell^2(m) |\beta_m| = \sup_{|m| > N} |\ell^2(m)||\beta||_{\ell^1} \to 0
\]
as \( N \to \infty \). By changing the order of summation we get from Cauchy-Schwartz inequality
\[
\sum_{|m| > N} \sum_{j \neq m} |\lambda_m - \tau_m| + |\gamma_m| \ |\beta_j| \leq C \|\beta\|_{\ell^1} (\sum_{|m| > N} |\lambda_m - \tau_m|^2 + |\gamma_m|^2)^{1/2}.
\]
Altogether it then follows that the operator norm,
\[
\|T^* - \text{Id} - K_N\|_{\mathcal{L}(\ell^1)} \to 0 \quad \text{as} \quad N \to \infty,
\]
showing that \( T^* - \text{Id} \) is compact. \( \square \)

Next we want to prove Proposition 4.2. First we establish the following two lemmas

**Lemma 7.2** For any \( n \in \mathbb{Z} \) and for any \( \varphi \in \mathcal{W} \) the coefficients \( b^n_m \), \( m \neq n \), of \( b^n \) satisfy
\[
b^n_m = \begin{cases} 
O(\frac{|\lambda_m - \tau_m| + |\gamma_m|}{|m - n|}), & |m| > N_0, \ m \neq n \\
O(\frac{1}{n}), & |m| \leq N_0, \ m \neq n
\end{cases}
\]
uniformly in \( n \in \mathbb{Z} \) and locally uniformly in \( \mathcal{W} \). In particular, it follows that \( b^n \) is in \( \ell^1_\varphi \).
Proof. As the cases \(|n| > N_0\) and \(|n| \leq N_0\) can be treated in the same way, we only consider the case \(|n| > N_0\).

Case \(|m| > N_0\): Taking into account that \(\int_{\Gamma_m} \frac{\Delta(\lambda)}{\sqrt{R(\lambda)}} d\lambda = 0\) for any \(m \in \mathbb{Z}\), the coefficient \(b^n_m = \frac{1}{2\pi i} \int_{\Gamma_m} \frac{(m\pi - n\pi) - 1}{\lambda - \lambda_n} \frac{\lambda_m - \lambda}{(\lambda_m - \lambda)(\lambda_m - \lambda)} \Pi_m(\lambda) d\lambda\)
can be written as

\[
b^n_m = \frac{1}{2\pi i} \int_{\Gamma_m} \frac{(m\pi - n\pi) - 1}{\lambda - \lambda_n} \frac{\dot{\lambda}_m - \lambda}{(\lambda_m - \lambda)(\lambda_m - \lambda)} \Pi_m(\lambda) d\lambda,
\]

where \(\Pi_m := \prod_{k \neq m} \frac{\lambda_k - \lambda}{\sqrt{(\lambda_k^+ - \lambda)(\lambda_k^- - \lambda)}}\). If \(\lambda_m^+ = \lambda_m^-\), one has \(\dot{\lambda}_m = \tau_m\) and \(\sqrt{(\lambda_m^+ - \lambda)(\lambda_m^- - \lambda)} = \tau_m - \lambda\). As \(m \neq n\) one then concludes from the analyticity of the integrand in \(D_m(\pi/4)\) that \(b^n_m = 0\). If \(\lambda_m^+ \neq \lambda_m^-\) we first note that

\[
\frac{m\pi - n\pi}{\lambda - \lambda_n} - 1 = \frac{(m\pi - \lambda) + (\dot{\lambda}_n - n\pi)}{\lambda - \lambda_n} = O\left(\frac{1}{m - n}\right)
\]

where we used that \(\dot{\lambda}_n - n\pi = O(1)\) by Proposition 2.1. Furthermore, by [5, Lemma C.3], \(\Pi_m = O(1)\). By deforming the contour \(\Gamma_m\) to the straight interval \(G_m\) (taken twice) and by using that for \(\lambda \in G_m\), \(|\dot{\lambda}_m - \lambda| \leq |\hat{\lambda}_m - \tau_m| + |\gamma_m|\), one sees from [11, Lemma 14.3] that \(b^n_m = O\left(\frac{1}{m - n}\right)\).

Case \(|m| \leq N_0\): We can argue similarly as above to conclude that \(b^n_m = O\left(\frac{1}{m}\right)\) (cf. Lemma 7.1). Going through the arguments of the proof one verifies that the estimates obtained are uniform in \(n \in \mathbb{Z}\) and locally uniform on \(W\).

The coefficients \(T^n_{mj}\) can be estimated using Lemma 7.1 by writing

\[T^n_{mj} = T^n_{mj} + R^n_{mj} (m, j \neq n)\] where \(R^n_{mj}\) is defined for \(|n| > N_0\) as follows

\[
R^n_{mj} := \begin{cases} 
\frac{1}{2\pi} \int_{\Gamma_m} \left(\frac{m\pi - n\pi}{\lambda - \lambda_n} - 1\right) \frac{\lambda_{n+j} - \lambda}{\lambda - \lambda_n} \frac{\Delta(\lambda)}{\sqrt{R(\lambda)}} d\lambda, & |j| \leq N_0 \\
\frac{1}{2\pi} \int_{\Gamma_m} \left(\frac{m\pi - n\pi}{\lambda - \lambda_n} - 1\right) \frac{\Delta(\lambda)}{\lambda - \lambda_j} \frac{d\lambda}{\sqrt{R(\lambda)}}, & |j| > N_0
\end{cases}
\]
whereas in the case $|n| \leq N_0$\footnote{See [17] for the definition of $\varepsilon_j^n$.}

$$R_{mj}^n := \begin{cases} 
\frac{1}{2\pi} \int_{\Gamma_m} \frac{(m\pi - N_0 \pi - 1)}{\lambda - \lambda_{n-j}^m} d\lambda, & |j| \leq N_0 \\
\frac{1}{2\pi} \int_{\Gamma_m} \frac{(m\pi - N_0 \pi - 1)}{\lambda - \lambda_{n-j}^m} \frac{\lambda^{N_0+j+\varepsilon_j^n}}{\sqrt{R(\lambda)}} d\lambda, & |j| > N_0.
\end{cases}$$

Note that by Proposition 2.2 for $\lambda \in \Gamma_m$ and $|n| > N_0$,

$$\frac{m\pi - n\pi}{\lambda - \lambda_n} - 1 = \frac{(m\pi - \lambda) + (\lambda_n - n\pi)}{\lambda - \lambda_n} = O\left(\frac{1}{m - n}\right).$$

It is convenient to rewrite $R_{mj}^n$ in the case $|n| > N_0$ as follows

$$\begin{cases} 
\frac{1}{2\pi} \int_{\Gamma_m} \frac{(m\pi - \lambda) + (\lambda_n - n\pi)}{\sqrt{(\lambda_n - \lambda)(\lambda_n - \lambda)}} \prod_{|k| \leq N_0} \sqrt{\lambda_k - \lambda} \Pi_n(\lambda) d\lambda, & |j| \leq N_0 \\
\frac{1}{2\pi} \int_{\Gamma_m} \frac{(m\pi - \lambda) + (\lambda_n - n\pi)}{\sqrt{(\lambda_n - \lambda)(\lambda_n - \lambda)}} \prod_{|k| > N_0} \sqrt{\lambda_k - \lambda} \Pi_{nj}^n(\lambda) d\lambda, & |j| > N_0
\end{cases}$$

where

$$\Pi_n := \prod_{|k| \geq N_0, k \neq n} \frac{\lambda_k - \lambda}{\sqrt{(\lambda_k^+ - \lambda)(\lambda_k^- - \lambda)}}$$

and

$$\Pi_{nj} := \prod_{k \neq n,j} \frac{\lambda_k - \lambda}{\sqrt{(\lambda_k^+ - \lambda)(\lambda_k^- - \lambda)}}.$$  \hfill (39)

In the case $|n| \leq N_0$, similar identities hold.

**Lemma 7.3.** For any $n \in \mathbb{Z}$ and for any $\varphi \in W$, the coefficients $R_{mj}^n$ satisfy the following estimates

$$R_{mj}^n = \begin{cases} 
O\left(\frac{\lambda_m \tau_m + |\gamma_m|}{(m-n)(j-m)}\right), & j \in \mathbb{Z} \setminus \{n\}, |m| > N_0, m \neq j, n \\
O\left(\frac{\tau^2(m) + \gamma^2(n)}{m-n}\right), & |j| > N_0, m = j, m \neq n \\
O\left(\frac{1}{(m-n)(j-m)}\right), & j \in \mathbb{Z} \setminus \{n\}, |m| \leq N_0, m \neq n.
\end{cases}$$

**Proof.** As cases $|n| > N_0$ and $|n| \leq N_0$ are proved in the same way we will only consider the case $|n| > N_0$.

Throughout the proof we assume that $m \neq n$ and $j \neq n$. The proof is very similar to the one of Lemma 7.1.
Lemma 7.1 to conclude that hence $R_{mn} = 0$. If $m = j$ (and hence $|m| > N_0$) one gets from the residue theorem and [3] Lemma C.3 that

$$R_{mn} = \frac{(m\pi - \tau_m) + (\hat{\lambda}_n - n\pi)}{\sqrt{\lambda_n^+ - \tau_m}(\lambda_n^- - \tau_m)} \prod_{k \neq m,n} \frac{\hat{\lambda}_k - \tau_m}{\sqrt{\lambda_k^+ - \tau_m}(\lambda_k^- - \tau_m)}$$

$$= O\left(\frac{(m\pi - \tau_m) + (\hat{\lambda}_n - n\pi)}{m - n}\right).$$

By Proposition 2.2, $m\pi - \tau_m = \ell^2(m)$ and $\hat{\lambda}_n - n\pi = \ell^2(n)$, hence $R_{mn} = \frac{\ell^2(m) + \ell^2(n)}{m - n}$.

Case $|j| > N_0$ and $\lambda_m^+ \neq \lambda_m^-$: If $j = m$ (and hence $|m| > N_0$) one deforms the contour $\Gamma_m$ to the straight interval $G_m$ (taken twice) and obtains from [3] Lemma 14.3 and [3] Lemma 14.3 that

$$R_{mn} = \frac{1}{2\pi i} \int_{\Gamma_m} \frac{(m\pi - \lambda) + (\hat{\lambda}_n - n\pi)}{\sqrt{\lambda_n^+ - \lambda}(\lambda_n^- - \lambda)} \frac{1}{\sqrt{\lambda_m^+ - \lambda}(\lambda_m^- - \lambda)} \Pi_{mn}(\lambda) d\lambda$$

$$= \frac{\ell^2(m) + \ell^2(n)}{m - n}$$

where $\Pi_{mn}$ is defined in (10). If $|m| > N_0$, but $m \neq j$, one argues similarly: Deforming the contour $\Gamma_m$ to $G_m$ (taken twice) one sees that

$$|R_{mn}| \leq \max_{\lambda \in G_m} \left|\frac{(m\pi - \lambda) + (\hat{\lambda}_n - n\pi)}{\sqrt{\lambda_n^+ - \lambda}(\lambda_n^- - \lambda)} \frac{(\hat{\lambda}_m - \lambda)}{\sqrt{\lambda_m^+ - \lambda}(\lambda_m^- - \lambda)} \Pi_{mjn}(\lambda)\right|$$

where $\Pi_{mjn} := \prod_{k \neq m,j,n} \frac{\hat{\lambda}_k - \lambda}{\sqrt{\lambda_k^+ - \lambda}(\lambda_k^- - \lambda)}$. As for $|m| > N_0$ and $\lambda \in G_m$,

$$|\hat{\lambda}_m - \lambda| \leq |\hat{\lambda}_m - \tau_m| + |\gamma_m|$$

and

$$\left(\sqrt{\lambda_j^+ - \lambda}(\lambda_j^- - \lambda)^{-1}\right)^{-1} = O\left(\frac{1}{m-n}\right),$$

one concludes that $R_{mj} = O\left(\frac{|\hat{\lambda}_m - \tau_m| + |\gamma_m|}{(m-n)(1+|j|)}\right)$. Finally, if $|m| \leq N_0$ and hence $m \neq j$ (as we assume $|j| > N_0$), one argues as in the proof of Lemma 7.1 to conclude that

$$R_{mj} = O\left(\frac{1}{(m-n)(1+|j|)}\right).$$
Case $|j| \leq N_0$: In this case $R_{mj}^n$ is given by the second equation in (39). If $|m| > N_0$ note that

$$\max_{\lambda \in G_m} \left| \frac{\lambda^{N_0+j}}{\prod_{|k| \leq N_0} \sqrt[|k|-\lambda+\lambda_k^-}} \right| = O\left( \frac{1}{m} \right),$$

and $|\dot{\lambda}_m - \lambda| \leq |\dot{\lambda}_m - \tau_m| + |\gamma_m|$ for any $\lambda \in G_m$. This together with [5, Lemma 14.3] implies that,

$$R_{mj}^n = O\left( \frac{|\dot{\lambda}_m - \tau_m| + |\gamma_m|}{(m-n)(j-m)} \right).$$

In the remaining case $|m| \leq N_0$ we argue again as in the proof of Lemma 7.1 to see that $R_{mj}^n = O(1)$. The claimed estimates for $R_{mj}^n$ are thus proved. Going through the arguments of the proofs one sees that the derived estimates hold locally uniformly on $W$. \qed

From Lemma 7.1 and Lemma 7.3 it immediately follows that for each $n \in \mathbb{Z}$, $T^n$ defines a bounded linear operator, $T^n : \ell_1^n \to \ell_1^n$.

Proof of Proposition 4.2. By Proposition 6.1, $T^n$ is injective. Arguing as in the proof of Proposition 4.4 one sees that $T^n - Id$ is a compact operator on $\ell_1^n$. Therefore $T^n$ is Fredholm and thus $T^n$ a linear isomorphism. Finally, by Lemma 7.2, $b^n \in \ell_1^n$. \qed

Now let us turn to the proof of Proposition 4.3. Recall that for any $n \in \mathbb{Z}$ and for any given $\varphi$ in $W$ we denote by $\beta^n$ the unique solution of $T^n \beta = b^n$. In this way we obtain maps

$$\beta^n : W \to \ell_1^n$$

and

$$\zeta_n : \mathbb{C} \times W \to \mathbb{C}, \ (\lambda, \varphi) \mapsto \zeta^n_{\varphi}(\lambda).$$

We need to show that these maps are analytic.

Lemma 7.4. For any $n \in \mathbb{Z}$, (i) $b^n : W \to \ell_1^n$ and (ii) $T^n : W \to L(\ell_1^n)$ are analytic maps.
Remark 7.1 Arguing as in the proof of Lemma 7.4 one can show that $T^* : \mathcal{W} \to \mathcal{L}(\ell^1)$ is analytic.

Proof. (i) Let us first consider the case $|n| > N_0$. According to the definition of $b^n = (b^n_m)_{m \neq n}$ in Section 4,

$$b^n_m = \frac{1}{2\pi} \int_{\Gamma_m} \frac{m\pi - n\pi}{\lambda - \lambda_n} \frac{\dot{\Delta}(\lambda)}{\sqrt{\Delta^2(\lambda) - 4}} \, d\lambda.$$ 

By [5, Theorem A.3] it suffices to show that $b^n$ is locally bounded and weakly analytic. By Lemma 7.2 and Cauchy-Schwartz estimate, $b^n$ is locally bounded. As the dual of $\ell^1_{\hat{n}}$ is $\ell^\infty_{\hat{n}} = \ell^\infty(\mathbb{Z}\setminus\{n\}, \mathbb{C})$, in view of Montel’s theorem, the weak analyticity of $b^n$ then follows once we prove that each component $b^n_m$, $m \neq n$, of $b^n$ is analytic on $\mathcal{W}$. To this end let us analyze the integrand in the definition of $b^n$. Recall that $\Delta, \dot{\Delta} : \mathbb{C} \times L^2_\mathbf{c} \to \mathbb{C}$ are analytic maps. As $|n| > N_0$ by assumption, $\lambda_n$ is a simple zero of $\dot{\Delta}(\lambda)$ (cf. Proposition 2.1) and hence we obtain from the implicit function theorem that $\lambda_n : \mathcal{W} \to \mathbb{C}$ is analytic. By construction, $\sqrt{\Delta^2(\lambda) - 4}$ is analytic on $G \setminus \bigcup_{k \in \mathbb{Z}} G_k$. In view of the definition of $\Gamma_m$ and $G_m$, $m \in \mathbb{Z}$ (Section 2) there exists $\varepsilon > 0$, independent of $m$ so that $\sqrt{\Delta^2 - 4}$ is analytic on $U_\varepsilon(\Gamma_m) \times \mathcal{W}$, where $U_\varepsilon(\Gamma_m)$ is the $\varepsilon$-tubular neighborhood of $\Gamma_m$, $U_\varepsilon(\Gamma_m) := \{ \lambda \in \mathbb{C} \mid \text{dist}(\lambda, \Gamma_m) < \varepsilon \}$. It then follows that for any $m \neq n$, $b^n_m : \mathcal{W} \to \mathbb{C}$ is analytic.

In the case $|n| \leq N_0$, $b^n_m$ is defined for any $m \neq n$ by

$$b^n_m = \frac{1}{2\pi} \int_{\Gamma_m} \frac{m\pi - N_0\pi}{\lambda - \lambda_n} \frac{\dot{\Delta}(\lambda)}{\sqrt{\Delta^2(\lambda) - 4}} \, d\lambda.$$ 

By the choice of $N_0$ in Proposition 2.1, for any $\varphi \in \mathcal{W}$, $\lambda_{N_0}$ is the only root of $\Delta(\cdot, \varphi)$ in $D_{N_0}(\pi/4)$. Hence arguing as above, $\lambda_{N_0} : \mathcal{W} \to \mathbb{C}$ is analytic. Using the same arguments as in the case $|n| > N_0$ one sees that $b^n : \mathcal{W} \to \mathbb{C}$ is analytic also in this case.

(ii) As the proofs for $|n| > N_0$ and $|n| \leq N_0$ are similar, we consider the case $|n| > N_0$ only. In this case, the coefficients $T^n_{mj}(m, j \in \mathbb{Z}\setminus\{n\})$ of $T^n$ are given by

$$T^n_{mj} = \begin{cases} \frac{1}{2\pi} \int_{\Gamma_m} \frac{m\pi - n\pi}{\lambda - \lambda_n} \frac{1}{\lambda - \lambda_j} \frac{\dot{\Delta}(\lambda)}{\sqrt{\Delta^2(\lambda) - 4}} \, d\lambda, & |j| > N_0 \\ \frac{1}{2\pi} \int_{\Gamma_m} \frac{m\pi - n\pi}{\lambda - \lambda_n} \frac{\dot{\Delta}(\lambda)}{\prod_{k \leq N_0} (\lambda - \lambda_k)} \frac{\lambda^{N_0 + j}}{\sqrt{\Delta^2(\lambda) - 4}} \, d\lambda, & |j| \leq N_0 \end{cases}.$$
Again by [5 Theorem A.3] it suffices to show that $T^n$ is locally bounded and weakly analytic. By Lemma 7.3 and Lemma 7.3, $T^n$ is locally bounded (see the arguments in the proof of Proposition 4.4).

It remains to show that $T^n$ is weakly analytic. First note that by arguing as in (i) one sees that for any $m, j \in \mathbb{Z}\setminus\{n\}, T^n_{mj} : W \to \mathbb{C}$ is analytic. Since $T^n : W \to \mathcal{L}(\ell^1_h)$ is locally bounded so is for any $m \neq n$ the map

$$T^n_m : \ell^\infty_n, \varphi \mapsto (T^n_{mj}(\varphi))_{j \neq m}.$$ 

Using that the components $T^n_{mj}$ of $T^n_m$ are analytic it then follows from [11, Theorem A.3] that $T^n_m : \ell^1_n \to \ell^1_n, \varphi \mapsto (T^n(\varphi))_{j \neq m}$ is analytic. As a consequence, for each $N > |n|$, the map $\Pi_N \circ T^n : W \to \mathcal{L}(\ell^1_n)$ is analytic, where $\Pi_N : \ell^1_n \to \ell^1_n$ denotes the projection,

$$(\beta_j)_{j \neq n} \mapsto (\cdots, 0, \beta_{-N}, \cdots, \beta_N, 0, \cdots).$$

To show that $T^n : W \to \mathcal{L}(\ell^1_n)$ is weakly analytic it suffices to show that on any disk $D(\varphi, h) = \{ \varphi + zh \mid z \in \mathbb{C}, |z| < 1 \}$ with closure $\overline{D(\varphi, h)} \subseteq W$, $\Pi_N \circ T^n$ converges in $\mathcal{L}(\ell^1_n)$ to $T^n$ locally uniformly in $D(\varphi, h)$ as $N \to \infty$. Indeed, if this is the case it follows from [11 Theorem A.3] (Weierstrass theorem) that $T^n|_{\overline{D(\varphi, h)}} : D(\varphi, h) \to \mathcal{L}(\ell^1_n)$ is analytic, establishing in this way that $T^n$ is weakly analytic. To see that $\Pi_N \circ T^n$ converges locally uniformly in $D(\varphi, h)$ to $T^n$ as $N \to \infty$, observe that $\overline{D(\varphi, h)}$ is a compact subset of $W$. The claimed convergence thus follows from the estimates Lemma 7.3 and Lemma 7.3 (cf. the proof of Proposition 4.4).

Proof of Proposition 4.3. Take $n \in \mathbb{Z}$. Let us begin by showing that $\beta^n = (T^n)^{-1}b^n, \beta^n : W \to \ell^1_n,$ is analytic. As by Lemma 7.3 (ii) $T^n : W \to \mathcal{L}(\ell^1_n)$ is analytic and by Proposition 4.2 (ii), $\forall \varphi \in W, T^n(\varphi) \in \mathcal{L}(\ell^1_n)$ is a linear isomorphism, it follows that $(T^n)^{-1} : W \to \mathcal{L}(\ell^1_n), \varphi \mapsto (T^n(\varphi))^{-1}$ is analytic as well. This combined with the analyticity of $b^n$ (Lemma 7.3 (i)) implies that $\beta^n$ is analytic.

Let us now turn towards $\zeta_n$. As the cases $|n| > N_0$ and $|n| \leq N_0$ are proved in the same way we consider only $|n| > N_0$. Then $\zeta_n$ is given by

$$\zeta_n : \mathbb{C} \times W \to \mathbb{C}, (\lambda, \varphi) \mapsto \frac{\Delta(\lambda)}{\lambda - \lambda_n} - \xi^n_{\beta^n}(\lambda).$$
As $|n| > N_0$, $\lambda_n : \mathcal{W} \to \mathbb{C}$ is analytic and so is $\mathbb{C} \times \mathcal{W} \to \mathbb{C}, (\lambda, \varphi) \mapsto \frac{\hat{\Delta}(\lambda)}{\lambda - \lambda_n}$, as $\lambda - \hat{\lambda}_n$ is a factor in the product representation for $\hat{\Delta}(\lambda)$ (Proposition 2.3). Recall that for any $\beta \in \ell^1_\mathbb{R}$ and $|n| > N_0$,

$$\xi^n_\beta(\lambda) = \sum_{|j| > N_0} \beta_j \frac{\hat{\Delta}(\lambda)}{(\lambda - \hat{\lambda}_j)(\lambda - \lambda_n)} + \left( \sum_{j=0}^{2N_0} \beta_{j-N_0} \lambda^j \right) \frac{\hat{\Delta}(\lambda)}{(\lambda - \hat{\lambda}_n)(\lambda - \hat{\lambda}_j)} \prod_{|k| \leq N_0} (\lambda - \hat{\lambda}_k).$$

As above one argues that $\frac{\hat{\Delta}(\lambda)}{(\lambda - \hat{\lambda}_j)(\lambda - \lambda_n)}$ and $\frac{\hat{\Delta}(\lambda)}{(\lambda - \hat{\lambda}_n)(\lambda - \hat{\lambda}_j)} \prod_{|k| \leq N_0} (\lambda - \hat{\lambda}_k)$ are analytic on $\mathbb{C} \times \mathcal{W}$ and uniformly in $j$ locally bounded. By [11, Theorem A.3] it then follows that the mapping $\mathbb{C} \times \mathcal{W} \to \ell^\infty_\mathbb{R}$, that assigns to any $(\lambda, w) \in \mathbb{C} \times \mathcal{W}$ the sequence,

$$\left( \left( \frac{\lambda^{N_0+j}}{\prod_{|k| \leq N_0} (\lambda - \hat{\lambda}_j) \lambda - \lambda_n) \right) \right)_{|j| \leq N_0} \left( \left( \frac{\hat{\Delta}(\lambda)}{(\lambda - \hat{\lambda}_n)(\lambda - \hat{\lambda}_j)} \right) |_{j > N_0, j \neq n} \right) \right) \in \ell^\infty_\mathbb{R},$$

is analytic. This combined with the result above, saying that $\beta^n : \mathbb{C} \times \mathcal{W} \to \ell^1_\mathbb{R}$ is analytic it follows that

$$\xi^n_{\beta^n} : \mathbb{C} \times \mathcal{W} \to \mathbb{C}, \ (\lambda, \varphi) \mapsto \xi^n_{\beta^n}(\lambda, \varphi)$$

is analytic. Finally, by construction, the identity $\frac{1}{2\pi} \int_{\mathcal{A}_m} \frac{\xi_n(\lambda)}{\sqrt{R(\lambda)}} d\lambda = \delta_{nm}$ holds for any $n, m \in \mathbb{Z}$. □

To finish this section we prove results for the limiting behavior of $b^n, T^n,$ and $\beta^n$ as $|n| \to \infty$. To this end introduce the maps $\hat{b}^n : \mathcal{W} \to \ell^1, \hat{T}^n : \mathcal{W} \to \mathcal{L}(\ell^1), \hat{\beta}^n : \mathcal{W} \to \ell^1$,

$$\hat{b}^n_m := \begin{cases} b^n_m, & m \neq n \\ 0, & m = n \end{cases}, \quad \hat{\beta}^n_j := \begin{cases} \beta^n_j, & j \neq n \\ 0, & j = n \end{cases},$$

$$\hat{T}^n_{mj} := \begin{cases} T^n_{mj}, & j, m \in \mathbb{Z}\{n}\} \\ 0, & (j, m) \in (\mathbb{Z}\{n}\} \times \{n\} \cup \{n\} \times (\mathbb{Z}\{n}\} \\ 1, & (j, m) = (n, n). \end{cases}$$

**Lemma 7.5** One has locally uniformly in $\mathcal{W}$,
\( \lim_{|n| \to \infty} \| \hat{b}^n \|_{\ell^1} = 0; \)

\( \lim_{|n| \to \infty} \| \hat{T}^n - T^* \|_{\mathcal{L}(\ell^1)} = 0; \)

\( \lim_{|n| \to \infty} \| \hat{\beta}^n \|_{\ell^1} = 0. \)

**Remark 7.2** The statement on local uniformity means, e.g. in the case of \( \beta^n \), the following: For any \( \varphi_0 \in W \) there exists a neighborhood \( V \) of \( \varphi_0 \) in \( W \) such that for any \( \varepsilon > 0 \) there exist \( n_0 > N_0 \) so that \( \| \beta^n(\varphi) \|_{\ell^1} < \varepsilon \) for any \( n \geq n_0 \) and \( \varphi \in V \).

**Proof.** (i) By Lemma 7.2

\[
|\hat{b}^n_m| \leq \begin{cases} 
C \frac{\lambda_m - \tau_m + |\gamma_m|}{|m-n|}, & |m| > N_0, \ m \neq n \\
C \frac{1}{|m-n|}, & |m| \leq N_0, \ m \neq n.
\end{cases}
\]

By Proposition 2.2 \( |\dot{\lambda}_m - \tau_m| - |\gamma_m| = \ell^2(m) \) locally uniformly in \( W \). Therefore, \( |\hat{b}^n_m| \leq C \frac{a_m}{|m-n|} \) where \( (a_m)_{m \in \mathbb{Z}} \in \ell^2 \) locally uniformly in \( W \). Thus,

\[
\sum_{m \in \mathbb{Z}} |\hat{b}^n_m| \leq C \sum_{|m-n| \leq |\frac{1}{2}|} \frac{a_m}{|m-n|} + C \sum_{|m-n| > |\frac{1}{2}|} \frac{a_m}{|m-n|}
\]

\[
\leq C \left( \sum_{|m-n| \leq |\frac{1}{2}|} a_m \right)^{1/2} \left( \sum_{k \neq 0} \frac{1}{k^2} \right)^{1/2} + C \left( \sum_{m \in \mathbb{Z}} a_m^2 \right)^{1/2} \left( \sum_{k \neq 0} \frac{1}{k^{4/3}} \right)^{1/2} \left( \frac{2}{|n|} \right)^{1/3}
\]

\[
(41)
\]

Using [5, Lemma A.3] and Montel’s theorem one sees as in the proof of Lemma 7.2 (i) that the maps \( W \to \ell^1, \varphi \mapsto (\gamma_m^2)_{|m| \geq N_0} \), and \( W \to \ell^1, \varphi \mapsto ((\dot{\lambda}_m - \tau_m)^2) \) are analytic. This together with (41) implies that

\( \lim_{|n| \to \infty} \| \hat{b}^n \|_{\ell^1} = 0 \) locally uniformly on \( W \).

(ii) Using the same arguments as in the prove of (i) one concludes from Lemma 7.3 that the claimed convergence holds.

(iii) It follows from Proposition 4.2 and Proposition 4.3 that \( \hat{T}^n \) and \( T^* \) are linear isomorphisms in \( \ell^1 \). Hence (ii) implies that \( (\hat{T}^n)^{-1} \to (T^*)^{-1} \) in \( \mathcal{L}(\ell^1) \) as \( |n| \to \infty \). Furthermore note that \( \hat{\beta}^n = (\hat{T}^n)^{-1} \hat{b}^n \)

\[
\| \hat{\beta}^n \|_{\ell^1} \leq \| (\hat{T}^n)^{-1} - (T^*)^{-1} \|_{\mathcal{L}(\ell^1)} \| \hat{b}^n \|_{\ell^1} + \| (T^*)^{-1} \|_{\mathcal{L}(\ell^1)} \| \hat{b}^n \|_{\ell^1},
\]

35
yielding that \( \lim_{|n| \to \infty} \| \hat{\beta}^n \|_{\ell^1} = 0 \). Going through the arguments of the proof one verifies that the convergence holds locally uniformly in \( \mathcal{W} \).
\[ \square \]

8 Estimates of the zeroes

In this section we prove that the zeroes of the analytic function \( \zeta_n : \mathbb{C} \times \mathcal{W} \to \mathbb{C} \), introduced in Section 4, satisfy the properties stated in Theorem 1.3. The ansatz we have chosen for the \( \zeta_n \) is well suited to obtain these claimed estimates. Recall that for any \( |n| > N_0 \)

\[ \zeta_n(\lambda) \equiv \zeta_n(\lambda, \varphi) = (1 - \eta_n(\lambda)) \frac{\Delta(\lambda)}{\lambda - \lambda_n} \]

where

\[ \eta_n(\lambda) := \sum_{|j| > N_0, j \neq n} \frac{\beta^n_j}{\lambda - \lambda_j} + \frac{p_n(\lambda)}{\prod_{|j| \leq N_0} (\lambda - \lambda_j)} \]

and \( p_n(\lambda) := p^n_{\beta^n}(\lambda, \varphi) \) is the polynomial introduced in Section 4 with \( \beta \) given by \( \beta^n \equiv \beta^n(\varphi) \) of Proposition 4.3. First note that for any \( n \in \mathbb{Z} \)

\[ \frac{1}{2 \pi i} \int_{\Gamma_m} \partial_\lambda \log \left( \frac{\Delta(\lambda)}{\lambda - \lambda_n} \right) d\lambda = 1 - \delta_{nm} \quad \forall |m| > N_0 \quad (42) \]

and

\[ \dot{\lambda}_m = \frac{1}{2 \pi i} \int_{\Gamma_m} \lambda \partial_\lambda \log \left( \frac{\Delta(\lambda)}{\lambda - \lambda_n} \right) d\lambda \quad \forall |m| > N_0 \quad (43) \]

whereas for any \( N \geq N_0 \)

\[ \frac{1}{2 \pi i} \int_{\partial D_0(N\pi + \frac{\pi}{4})} \partial_\lambda \log \left( \frac{\Delta(\lambda)}{\lambda - \lambda_n} \right) d\lambda = \begin{cases} 2N, & |n| \leq N \\ 2N + 1, & |n| > N. \end{cases} \quad (44) \]

Viewing \( \zeta_n(\lambda) \) as a perturbation of \( \frac{\Delta(\lambda)}{\lambda - \lambda_n} \) we want to argue in a similar fashion for \( \zeta_n(\lambda) \). First we need to establish some auxiliary estimates.

**Lemma 8.1** For any \( \varphi \in \mathcal{W} \), \( \beta \in \ell^1 \), \( N > N_0 \geq 1 \), \( n \in \mathbb{Z} \), and \( |m| > 2N \), one has

36
Lemma 8.2

For any $r$ uniformly in $|\lambda - \lambda_j|$.

Proof. For any $|m| > 2N$ and $\lambda \in \Gamma_m$ one has by Proposition 2.1, $1/|\lambda - \lambda_m| \leq \pi/12$ as dist($\lambda_m, \Gamma_m$) $\geq \pi/12$. In view of Proposition 2.1 it then follows that for $|m| > 2N$ and $\lambda \in \Gamma_m$

$$\sum_{|j| > N_0} \frac{|\beta_j|}{|\lambda - \lambda_j|} \leq C \left( |\beta_m| + \sum_{|j| > N_0,j \neq m} \frac{|\beta_j|}{|j - m|} \right) + \sum_{N_0 < |j| \leq N} \frac{|\beta_j|}{|\lambda - \lambda_j|}$$

$$\leq C \left( |\beta_m| + \sum_{|j| > N_0,j \neq m} \frac{|\beta_j|}{|j - m|} \right) + C \|\beta\|_{\ell^1} \frac{1}{m}$$

where $C > 0$ can be chosen uniformly in $|m| > 2N$ and $\varphi \in \mathcal{W}$. Towards (ii) note that as $|\lambda| \geq 1$ one has,

$$\frac{|p_2^2(\lambda)|}{\prod_{|j| \leq N_0} |\lambda - \lambda_j|} \leq \frac{1}{|\lambda|} \frac{\sum_{|j| \leq N_0} |\beta_j|}{\prod_{|j| \leq N_0} \left|1 - \lambda_j^*\right|},$$

uniformly in $|n| > N_0$ and $\varphi \in \mathcal{W}$. In addition $|\lambda_j^*| < (N_0 + 1)\pi$ for any $|j| \leq N_0$ and $|\lambda| \geq m\pi - \pi/4$. Hence, $\prod_{|j| \leq N_0} \left|1 - \lambda_j^*\right| \geq C$ where the constant $C > 0$ can be chosen uniformly in $\varphi \in \mathcal{W}$. This implies that,

$$\sup_{\lambda \in \Gamma_m} \left( \frac{|p_2^2(\lambda)|}{\prod_{|j| \leq N_0} |\lambda - \lambda_j|} \right) \leq C \|\beta\|_{\ell^1} \frac{1}{m},$$

uniformly in $|n| > N_0$, $|m| > 2N$, and $\varphi \in \mathcal{W}$. Item (iii) is proved in a similar fashion.

Next we want to estimate $\eta_n(\lambda)$ on $\partial D_0(r_{2N})$ where for any $m \in \mathbb{Z}$, $r_m := m\pi + \pi/2$.

Lemma 8.2 For any $\varphi \in \mathcal{W}$, $\beta \in \ell^1 \cap \mathcal{W}$, $N > N_0$, and $n \in \mathbb{Z}$, one has

(i) $\sup_{\lambda \in \partial D_0(r_{2N})} \left( \sum_{|j| > N_0} \frac{|\beta_j|}{|\lambda - \lambda_j|} \right) \leq C \frac{\|\beta\|_{\ell^1}}{N} + C \sum_{|j| > N} |\beta_j|$. 

37
(ii) \[ \sup_{\lambda \in \partial D_0(r_{2N})} \left( \frac{|p_0^j(\lambda)|}{\prod_{|j| \leq N_0} |\lambda - \lambda_j|} \right) \leq C \|\beta\|_{\ell^1} \frac{1}{N}, \quad \text{if } |n| > N_0; \]

(iii) \[ \sup_{\lambda \in \partial D_0(r_{2N})} \left( \frac{|p_0^j(\lambda)|}{\prod_{|j| \leq N_0, j \neq n} |\lambda - \lambda_j|} \right) \leq C \|\beta\|_{\ell^1} \frac{1}{N}, \quad \text{if } |n| \leq N_0, \]

where \( C > 0 \) can be chosen uniformly for \( n \in \mathbb{Z}, N > N_0, \) and \( \varphi \in \mathcal{W}. \)

**Proof.** To prove item (i) we split the sum \( \sum_{|j| > N_0} \) into two parts: \( \sum_{N_0 < |j| \leq N} \) and \( \sum_{|j| > N}. \) Clearly, for any \( \lambda \in \partial D_0(r_{2N}), \)

\[ \sum_{N_0 < |j| \leq N} \frac{|\beta_j|}{|\lambda - \lambda_j|} \leq C \|\beta\|_{\ell^1}/N \]

and

\[ \sum_{|j| > N} \frac{|\beta_j|}{|\lambda - \lambda_j|} \leq \left( \sum_{|j| > N} |\beta_j|^2 \right)^{1/2} \left( \sum_{|j| > N} \frac{1}{|\lambda - \lambda_j|^2} \right)^{1/2} \leq C \sum_{|j| > N} |\beta_j| \]

where \( C > 0 \) can again be chosen independently in \( n \in \mathbb{Z}, N > N_0, \)

and \( \varphi \in \mathcal{W}. \) The estimates (ii) and (iii) are proved in the same way as items (ii) respectively (iii) of Lemma 8.1. \( \square \)

Lemma 8.1 and Lemma 8.2 can be used to localize the zeroes of \( \zeta_n(\cdot, \varphi) \) for any \( n \in \mathbb{Z} \) and \( \varphi \in \mathcal{W}. \) Indeed, according to these lemmas and as \( \lim_{|n| \to \infty} \|\hat{\beta}^n(\varphi)\|_{\ell^1} = 0 \) locally uniformly in \( \varphi \in \mathcal{W} \) by Lemma 7.5 and as \( \hat{\beta}^n : \mathcal{W} \to \ell^1 \) is analytic by Proposition 4.3 and the definition of \( \hat{\beta}^n, \) there exists \( N_1 > N_0 \) so that for any \( N \geq N_1, \)

\[ \sup_{\lambda \in \Gamma_m} |\eta_n(\lambda)| \leq 1/2 \quad (45) \]

and

\[ \sup_{\lambda \in \partial D_0(r_{2N})} |\eta_n(\lambda)| \leq 1/2 \quad (46) \]

locally uniformly for \( \varphi \in \mathcal{W}. \) It then follows that for any \( \lambda \in \Gamma_m, \)

\[ |m| > 2N, \]

and for any \( \lambda \in \partial D_0(r_{2N}), \)

\[ |n| > N_0 \]

\[ \frac{\hat{\Delta}(\lambda)}{\lambda - \lambda_n} - \zeta_n(\lambda) \leq |\eta_n(\lambda)| \frac{\hat{\Delta}(\lambda)}{\lambda - \lambda_n} \leq \frac{1}{2} \left| \frac{\hat{\Delta}(\lambda)}{\lambda - \lambda_n} \right|, \quad |n| > N_0 \]

\[ \frac{\hat{\Delta}(\lambda)}{\lambda - \lambda_{N_0}} - \zeta_{N_0}(\lambda) \leq |\eta_n(\lambda)| \frac{\hat{\Delta}(\lambda)}{\lambda - \lambda_{N_0}} \leq \frac{1}{2} \left| \frac{\hat{\Delta}(\lambda)}{\lambda - \lambda_{N_0}} \right|, \quad |n| \leq N_0. \]
Hence by Rouché’s theorem and formulas (42) and (44) one has for any \( |m| > 2N \)
\[
\frac{1}{2\pi i} \int_{\Gamma_m} \partial\lambda (\log \zeta_n(\lambda)) \, d\lambda = 1 - \delta_{nm}
\]
and
\[
\frac{1}{2\pi i} \int_{\partial D_0(r_{2N})} \partial\lambda (\log \zeta_n(\lambda)) \, d\lambda = \begin{cases} 4N, & |n| \leq 2N \\ 4N + 1, & |n| > 2N. \end{cases}
\]
For any \( |m| > 2N \) we denote the zero of \( \zeta_n(\lambda) \) inside \( \Gamma_m \) by \( \sigma^n_m \). By the argument principle one has (cf. (43))
\[
\sigma^n_m = \frac{1}{2\pi i} \int_{\Gamma_m} \lambda \partial\lambda (\log \zeta_n(\lambda)) \, d\lambda
\]
\[
= \dot{\lambda}_m + \frac{1}{2\pi i} \int_{\Gamma_m} \lambda \partial\lambda (\log(1 - \eta_n(\lambda))) \, d\lambda. \tag{47}
\]
Integration by parts leads to
\[
\sigma^n_m = \dot{\lambda}_m - \frac{1}{2\pi i} \int_{\Gamma_m} \log(1 - \eta_n(\lambda)) \, d\lambda.
\]
Using (45) and Lemma 8.1 one sees that for \( |m| > 2N \),
\[
\sup_{\lambda \in \Gamma_m} \left| \log(1 - \eta_n(\lambda)) \right| \leq 2 \sup_{\lambda \in \Gamma_m} \left| \eta_n(\lambda) \right| \leq C_1 \left( |\beta^n_m| + \sum_{|j| > N_1, j \neq m,n} \frac{|\beta^n_j|}{|j - m|} + \|\beta^n\|_{\ell^1_n} / m \right)
\]
where \( C_1 > 0 \) is independent of \( n \in \mathbb{Z}, \varphi \in \mathcal{W} \), and \( |m| > 2N \). By using the Cauchy-Schwartz inequality and then changing the order of summation in the double sum be get
\[
\sum_{|m| > 2N} \left( \sum_{j \neq m,n} \frac{|\beta^n_j|}{|j - m|} \right)^2 \leq \sum_{|m| > 2N} \|\beta^n\|_{\ell^1_n} \sum_{j \neq m,n} \frac{|\beta^n_j|}{|j - m|^2} =
\]
\[
= \|\beta^n\|_{\ell^1_n} \sum_{|m| > 2N} \sum_{m \neq j, j \neq n} \frac{1}{|j - m|^2} \leq C_2 \|\beta^n\|^2_{\ell^1_n}
\]
where \( C_2 = 2 \sum_{k \geq 1} \frac{1}{k^2} \). Hence, as \( \beta^n : \mathcal{W} \to \ell^1_n \) is analytic (Proposition 4.3) and as \( \lim_{|n| \to \infty} \|\beta^n\|_{\ell^1_n} = 0 \) locally uniformly in \( \mathcal{W} \) (Lemma 7.5) we get that
\[
\sum_{|m| > 2N} |\sigma^n_m - \dot{\lambda}_m|^2 \leq C
\]
where $C > 0$ can be chosen uniformly in $n \in \mathbb{Z}$ and locally uniformly in $\varphi \in W$. Note that as $\zeta_n : \mathbb{C} \times W \to \mathbb{C}$ is analytic by Proposition 4.3, the identity (47) also shows that $\sigma_n^m : W \to \mathbb{C}$ is analytic for any $|m| > 2N, m \neq n$. By denoting $2N$ again by $N$, we get

Proposition 8.1 There exists $N \geq N_0$ so that for any $n \in \mathbb{Z}$ and for any $\varphi \in W$, the entire function $\zeta_n(\lambda)$ has precisely $2N + 1$ (or $2N$) zeroes inside $D_0(r_N)$ if $|n| > N$ ($|n| \leq N$). For any $|m| > N, m \neq n$, $\zeta_n(\lambda)$ has precisely one zero, denoted by $\sigma_n^m(\varphi)$, in $D_m(\pi/4)$. There are no other zeroes of $\zeta_n(\lambda)$ in $\mathbb{C}$. Moreover, $\sigma_n^m = \ell_m + \ell^2(m)$, $|m| > N$, uniformly in $n \in \mathbb{Z}$ and locally uniformly in $W$, and for any $|m| > N$, $\sigma_n^m : W \to \mathbb{C}$ is analytic.

Proposition 8.1 implies that $\zeta_n(\lambda) \equiv \zeta_n(\lambda, \varphi)$ has in fact a product representation. List the roots of $\zeta_n(\lambda, \varphi)$ inside $D_0(r_N)$ in lexicographic order and with their multiplicities, $\sigma_n^j, |j| \leq N, m \neq n$.

Corollary 8.1 For any $n \in \mathbb{Z}$ and $\varphi \in W$,

$$
\zeta_n(\lambda) = -\frac{2}{\pi n} \prod_{j \neq n} \frac{\sigma_j^n - \lambda}{\pi_j}.
$$

Proof. Take $\varphi \in W$. As the cases $|n| > N_0$ and $|n| \leq N_0$ are proved in the same way let us consider the case $|n| > N_0$. By [5, Theorem 2.2], $\hat{\Delta}(\lambda)$ is an entire function of order 1 and so is $\frac{\hat{\Delta}(\lambda)}{\lambda - \lambda_n}$. By Proposition 4.3, $\zeta_n(\lambda)$ is an entire function and by Lemma 8.2

$$
\sup_{\lambda \in \partial D_0(r_{2N})} |1 - \eta_n(\lambda)| = 1 + o(1) \text{ as } N \to \infty.
$$

It then follows that $\zeta_n(\lambda) = (1 - \eta_n(\lambda)) \frac{\hat{\Delta}(\lambda)}{\lambda - \lambda_n}$ is an entire function of order 1. Moreover, by Proposition 2.2 and Proposition 8.1 the exponent of convergence of the zeroes of $\zeta_n(\lambda)$ is equal to 1 and the series $\sum_{|j| > N_0} \frac{1}{|\sigma_j|}$ diverges. This implies that the genus of $\zeta_n(\lambda)$ is equal to 1. By Hadamard’s factorizations theorem

$$
\zeta_n(\lambda) = \lambda^{\nu_n} e^{\tilde{a}_n \lambda + \tilde{b}_n} \prod_{\sigma_k^m \neq 0} E\left(\frac{\lambda}{\sigma_k^m}, 1\right)
$$

where $\nu_n$ is the order of vanishing of $\zeta_n(\lambda)$ at $\lambda = 0$, $\tilde{a}_n, \tilde{b}_n \in \mathbb{C}$ are constants independent of $\lambda \in \mathbb{C}$, and $E(z, 1)$ is the canonical factor.
\( E(\lambda, 1) := (1 - \lambda)e^{\lambda} \). For \(|m| > n\) we pair the factors \( E\left(\frac{\lambda}{\sigma_m}, 1\right) \cdot E\left(\frac{1}{\sigma_m}, 1\right) \) and conclude from Proposition 2.2 and Proposition 8.1 that \( \zeta_n(\lambda) \) has a product representation of the form

\[
e^{a_n \lambda + b_n} \prod_{k \neq n} \frac{\sigma_k^n - \lambda}{\pi_k} = e^{a_n \lambda + b_n} \lim_{K \to \infty} \prod_{|k| \leq K, k \neq n} \frac{\sigma_k^n - \lambda}{\pi_k}.
\]

On the other hand, by Proposition 2.3, \( \frac{\Delta(\lambda)}{\lambda - \lambda_n} = -\frac{2}{\pi_n} \prod_{k \neq n} \frac{\lambda_k - \lambda}{\pi_k} \) and by [5, Lemma C.5], on the circles \(|\lambda| = r_{2N}^n\),

\[
\prod_{k \neq n} \frac{\sigma_k^n - \lambda}{\pi_k} = 1 + o(1) \quad \text{as} \quad N \to \infty.
\]

By Lemma 8.2 and Proposition 8.1 on the circle \(|\lambda| = r_{2N}^n\),

\[
e^{a_n \lambda + b_n} \prod_{k \neq n} \frac{\sigma_k^n - \lambda}{\pi_k} - \frac{2}{\pi_n} \prod_{k \neq n} \frac{\lambda_k - \lambda}{\pi_k} = 1 + o(1) \quad \text{as} \quad N \to \infty.
\]

It then follows that \( a_n = 0 \) and \( e^{b_n} = -\frac{2}{\pi_n} \), yielding the claimed formula \( \zeta_n(\lambda) = -\frac{2}{\pi_n} \prod_{k \neq n} \frac{\sigma_k^n - \lambda}{\pi_k} \). \( \square \)

The refined asymptotics of the zeroes \((\sigma_m^n)_{m \neq n}\) of \( \zeta_n \) sated below are proved in the same way as in [5, Lemma 14.12] and hence we omit its proof.

**Lemma 8.3** There exist \( N \geq N_0 \) so that

\[
\sigma_m^n = \tau_m + \gamma_m^2 \ell_m^2 \quad \forall |m| > N
\]

uniformly in \( n \in \mathbb{Z} \) and locally uniformly in \( \mathcal{W} \).

Finally we prove the following

**Lemma 8.4** For any \( n \in \mathbb{Z} \) and \( \varphi \in \mathcal{W} \subseteq L^2_\varphi \), the entire function \( \zeta_n(\lambda) \) vanishes at \( \lambda \in Z_\varphi \setminus \{\lambda_n^{\pm}(\varphi)\} \). If \( \lambda_n^{\pm} \in Z_\varphi \) then \( \zeta_n(\lambda) \) does not vanish at \( \lambda_n^{\pm} \).

**Proof.** Take \( \varphi \in \mathcal{W} \). To see that \( \zeta_n(\lambda, \varphi) \) vanishes on \( Z_\varphi \setminus \{\lambda_n^{\pm}(\varphi)\} \) one argues as in the proof of Lemma 5.1. If \( \lambda_n^{\pm} \in Z_\varphi \), then \( \lambda_n^{\pm} \) is a zero of order two of \( \mathcal{R}(\lambda) \) and hence \( \frac{\zeta_n(\lambda)}{\sqrt[4]{\mathcal{R}(\lambda)}} \) has a pole of order \( \leq 1 \) at \( \lambda_n^{\pm} \).

As \( \int_{\Gamma_n} \frac{\zeta_n(\lambda)}{\sqrt[4]{\mathcal{R}(\lambda)}} d\lambda = 2\pi \) we conclude that \( \zeta(\lambda_n^{\pm}) \neq 0 \). \( \square \)
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