Yang-Lee Zeros of the Ising model on Random Graphs of Non Planar Topology

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Abstract

We obtain in a closed form the 1/N² contribution to the free energy of the two Hermitian N × N random matrix model with non symmetric quartic potential. From this result, we calculate numerically the Yang-Lee zeros of the 2D Ising model on dynamical random graphs with the topology of a torus up to n = 16 vertices. They are found to be located on the unit circle on the complex fugacity plane. In order to include contributions of even higher topologies we calculated analytically the nonperturbative (sum over all genus) partition function of the model \( Z_n = \sum_{h=0}^{\infty} \frac{Z^{(h)}}{\chi_n^{2h}} \) for the special cases of N = 1, 2 and graphs with n ≤ 20 vertices. Once again the Yang-Lee zeros are shown numerically to lie on the unit circle on the complex fugacity plane. Our results thus generalize previous numerical results on random graphs by going beyond the planar approximation and strongly indicate that there might be a generalization of the Lee-Yang circle theorem for dynamical random graphs.

Keywords: Yang-Lee zeros, Lee-Yang theorem, Ising model, random matrix, random surfaces, 2D gravity.

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1 Introduction

Yang and Lee have established long ago [1] that the statistical theory of phase transitions is connected with the distribution of the zeros of the grand partition func-
tion on the complex fugacity plane. They have proved that, in the thermodynamic limit, those zeros circumscribe closed regions on the fugacity plane where physical quantities remain analytic, thus defining different phases of the system. Although complex values for the fugacity are nonphysical, the physical features of the phase transition can be obtained from the study of the distribution of the zeros which increase in number and tend to pinch the positive real axis in the thermodynamic limit.

In \[2\] Lee and Yang have proved that the complex zeros of the partition function of the ferromagnetic Ising model in a complex magnetic field \(H\) lie on the unit circle in the complex \(y\)-plane \((y = e^{-2\beta H})\). This result, known as the circle theorem, makes no assumptions about the details of the lattice like its topology, coordination number, etc. Their starting point is the partition function of the model on a given static lattice which we can assume to be, for instance, a graph \(G_n\) with \(n\) vertices. The corresponding partition function is defined as

\[
Z(G_n) = \sum_{\{\sigma_i\}} e^{\left(\beta \sum_{i,j} G_{ij} \sigma_i \sigma_j + H \sum_{i=1}^n \sigma_i\right)},
\]

(1)

where \(\beta = 1/T\) and \(G_{ij} = 1\) for nearest neighbors and \(G_{ii} = 0\) otherwise. We have absorbed the factor \(\beta\) in the definition of the magnetic field. As usual the configuration of the system is determined by specifying \(\sigma_i = \pm 1\) on each vertex of \(G_n\). Henceforth we assume that \(G_n\) is a closed (no external legs) graph with \(n\) four-legged vertices. It is important to emphasize that \(G_n\) does not need to be regular and one has \(G_{ii} = 1\) for a link that connects a vertex to itself. Defining \(d\) as the number of spins down we can rewrite \(Z(G_n)\) as being proportional to a polynomial in the fugacity \(y\),

\[
Z(G_n) = (cy^{1/2})^{-n} \sum_{d=0}^{n} P_d y^d,
\]

(2)

where \(c = e^{-2\beta}\) and each of the coefficients \(P_d\) corresponds to a partition function of the 2D Ising model on \(G_n\) without magnetic field \((H = 0)\) with \(d\) spins down. Therefore they are all positive real and satisfy \(P_d = P_{n-d}\). Based on this and other properties of \(P_d\) Lee-Yang have proved their theorem \[2\] assuring that all zeros of \(Z(G_n)\) lie on the unit circle: \(|y_k| = 1\), \((k = 1, 2, \ldots, n)\).

Following Kazakov and Boulatov \[3, 4\] we can define the Ising model on dynamical random surfaces (2D Gravity) by treating the lattice \(G_n\) itself as a degree of freedom. Its partition function is obtained from the usual Ising model by summing also over all \(G_n\) with \(n\) vertices:

\[
Z_n = \sum_{\{G_n\}} Z(G_n) = \sum_i Z(G_n^{(i)}).
\]

(3)

Here \(i\) labels the different graphs with \(n\) vertices.

The above sum is not arbitrary. In our case, where we consider four-legged vertices, the weights of the graphs are the corresponding combinatorial factors of a
\( \phi^4 \) field theory. In this case the model of \([3,4]\) is known to have a third order phase transition at the critical temperature \( \beta = \log 2 \) from a disordered (high temperature) to an ordered (low temperature) phase. Clearly \( Z_n \) has the same form of eq.(2) but with new coefficients \( \tilde{P}_d = \sum_i P_d(G_n^{(i)}) \) Although the property \( \tilde{P}_d = \tilde{P}_n - d \) is still satisfied, this is not a sufficient condition for the Lee-Yang circle theorem to hold. In general the zeros of linear combinations of polynomials have a complicated relation to the zeros of the original basic polynomials. However, quite surprisingly it has been observed numerically (see \([3,6]\)) for dynamical planar graphs (spherical topology) with \( 1 \leq n \leq 14 \) vertices that the Yang-Lee zeros are located on the unit circle. It is instructive to illustrate this point for planar graphs with a low number of vertices \( (n = 2, 3, 4) \) where an analytic analysis is simpler. From eq.(1) we easily obtain the partition functions given in Fig. 1. The overall factor \((cy^{1/2})^{-n}\) of a \( G_n \) graph can be canceled by adding a constant to the energy in eq.(1). Since \( 0 \leq c \leq 1 \),

\[
Z(G_n^{(1)}) = c^2 y^{-1} (1 + 2c^2 y + y^2)
\]

\[
Z(G_n^{(2)}) = c^2 y^{-1} (1 + 2c^2 y + y^2)
\]

Fig. 1. Planar graphs and partition functions for \( n = 2 \) vertices.

it is easy to verify that the Lee-Yang theorem holds for \( G_n^{(1)} \) and \( G_n^{(2)} \) separately. This exemplify, in particular, that the theorem does not depend on specific details of the graphs. If we want to treat the graphs as an extra degree of freedom we have to sum over \( G_n^{(1)} \) and \( G_n^{(2)} \) in this case. Taking an arbitrary linear combination \( Z_2(a,b) = aZ(G_2^{(1)}) + bZ(G_2^{(2)}) \) one can easily verify that for \(-2 < b/a < -2(\sqrt{2} - 1)\), the zeros of \( Z_2(a,b) \) will not belong to the unit circle. However, in the Kazakov and Boulatov’s model the combinatorial weights of \( G_2^{(1)} \) and \( G_2^{(2)} \) correspond (see ref. \([7]\) and the planar \( Z_2 \) displayed in \([5]\)) to \((a,b) = (2,16)\), which brings the zeros to the unit circle.

In the next simplest case, \( n = 3 \), we have four planar graphs, as shown in Fig. 2. Each graph gives rise to a polynomial of the form

\[
P_3^{(i)} = 1 + y^3 + 3(y + y^2)a^{(i)}, \quad (4)
\]

where the quantities \( a^{(i)} \) are functions of the temperature which satisfy \( 0 \leq a^{(i)} \leq 1 \), \( (i = 1,2,3,4) \). It is easy to demonstrate that each \( P_3^{(i)} \) has all its roots on the unit circle. Clearly, this is not true for a general linear combination \( \sum_{i=1}^4 k_i P_3^{(i)} \).
Nevertheless, if \( k_i \geq 0 \), the linear combination will have the same basic form of \( P_3^{(i)} \) up to an overall constant, and we will be back to the unit circle. Obviously, the combinatorial factors of the respective graphs (see ref. [7] and the planar \( Z_3 \) displayed in [5]): \( k_i = (32/3, 256/3, 64, 128) \), belong to this subset.

For \( n = 4 \) vertices there are ten planar graphs. In Fig. 3a and 3b we show the diagrams and the corresponding partition functions. Each one corresponds to a polynomial, which can be written as:

\[
P_4^{(i)} = 1 + y^4 + \left[ a_1^{(i)} c^2 + (4 - a_1^{(i)}) c^4 \right] (y + y^3) + 2 \left[ a_2^{(i)} c^2 + a_3^{(i)} c^4 \right] y^2
\]
\[+ 2 \left[ (4 + a_1^{(i)} - 3a_2^{(i)} - 2a_3^{(i)}) c^6 + (2a_2^{(i)} + a_3^{(i)} - 1 - a_1^{(i)}) c^8 \right] y^2 ,
\]

where

\[
0 \leq a_1^{(i)} \leq 4, \quad 0 \leq a_2^{(i)}, a_3^{(i)} \leq 3,
\]
\[4 + a_1^{(i)} - 3a_2^{(i)} - 2a_3^{(i)} \geq 0, \quad 2a_2^{(i)} + a_3^{(i)} - 1 - a_1^{(i)} \geq 0 .
\]

Fig. 2. Planar graphs and partition functions for \( n = 3 \) vertices.

Analogous to the \( n = 3 \) case, each \( P_4^{(i)} \), under conditions (6), possess only unit roots but that is a property that cannot be extended to any linear combination.
\[
\sum_{i=1}^{10} k_i P_4^{(i)}.
\]
However, once again, if \( k_i \geq 0 \) (\( i = 1, \cdots, 10 \)) the conditions (3) and the polynomial form (5) would still hold for the linear combination and we end up with unit roots again. Needless to say the combinatorial factors of the graphs \( G_4^{(i)} \), are all positive and therefore the roots will lie on the unit circle.

It is remarkable that even for larger \( n \) the combinatorial factors of the graphs apparently, as our numerical results indicate, are such that the Yang-Lee zeros still lie on the unit circle.

For an arbitrary number of vertices we were not able to make an analytical analysis of the location of the Yang-Lee zeros for an arbitrary finite temperature but, as one might expect from the results on static lattices, the Yang-Lee zeros of \( Z_n \) should be equally distributed around the unit circle at \( T = 0 \) and coalesce at \( y = -1 \) as \( T \to \infty \). As a self-consistency check we prove that this is indeed the case directly from the two Hermitian matrix model in the next section.

Fig. 3a. First set of diagrams and partition functions for \( n = 4 \) vertices.
key ingredient in those cases is the decoupling of the two Hermitian random matrix model into the one Hermitian random matrix model. In section 3, using orthogonal polynomials, we obtain for arbitrary temperature the $1/N^2$ contribution to the free energy of the two Hermitian matrix model. In section 4 we present the numerical results for the Yang-Lee zeros on graphs of torus topology up to $n = 16$ vertices. In section 5 we calculate, for arbitrary temperature, the non-perturbative partition function of the Ising model on random lattices, including all topologies. For this end, we have to restrict ourselves to the cases of small matrices $N = 1, 2$ and graphs with $n \leq 20$ vertices. Numerical results show that even after summing over all topologies we still have the Yang-Lee zeros on the unit circle.
2 Free Energy at $T = 0$ and $T \to \infty$ for arbitrary topology

By means of two Hermitian matrices $X$ and $Y$ of order $N \times N$, each one associated with a value of $\sigma_i = \mp 1$, one can show that the free energy \[ E(g, c, H) = - \frac{1}{N^2} \log \left( \frac{\int D\mu e^{-\text{Tr}[X^2 + Y^2 - 2cXY + \frac{g}{2}e^H X^4 + e^{-H} Y^4]]}}{\int D\mu e^{-\text{Tr}[X^2 + Y^2 - 2cXY]}} \right) \] (7)
is a generating function for partition functions $Z_n$ of the 2D Ising model in a constant magnetic field $H$ calculated on random graphs $G_n$ of $n$ vertices of 4 links each according to 

\[ E(g, c, H) = - \sum_{n=1}^{\infty} \left[ \frac{-gc}{(1 - c^2)^2} \right]^n Z_n. \] (8)

Notice that $D\mu = DXDY$ is the usual measure for Hermitian matrices. The quadratic terms of the potential in eq.(7) are responsible for the links (propagators) between $<++>$, $<-->$ and $<-+>$ while the quartic terms stand for the two different sites (vertices) $(+)$ and $(-)$ of the lattices (graphs) each one with four links. This corresponds to a $\phi^4$ interaction field theory in which the fields are represented by $N \times N$ matrices.

Taking ratios of the propagators and comparing with the Boltzmann weights of $Z(G_n)$ one identifies the temperature: $c = e^{-2\beta}$. For the specific cases of $c = 0$ ($T = 0$) and $c = 1$ ($T \to \infty$), the matrices in eq.(7) decouple and the free energy $E(g, c, H)$ can be obtained for arbitrary topology in terms of the one Hermitian matrix model free energy ($E_1$) as follows. Taking $c = 0$ ($T = 0$) in eq.(7) we have

\[ E(g, 0, H) = E_1(\frac{g e^H}{4}) + E_1(\frac{g e^{-H}}{4}), \] (9)

where

\[ E_1(g) = - \frac{1}{N^2} \log \left( \frac{\int DX e^{-\text{Tr}(\frac{X^2}{2} + gX^4)}}{\int DX e^{-\text{Tr} \frac{X^2}{2}}} \right) = - \sum_{n=1}^{\infty} (-g)^n a_n. \] (10)

Plugging in eq.(9), we obtain the following expansion in the coupling $g$,

\[ E(g, 0, H) = - \sum_{n=1}^{\infty} a_n(-g)^n(1 + y^n) y^{-n/2}. \] (11)

The coefficients $a_n$ on their turn have a topological expansion

\[ a_n = \sum_{h=0}^{\infty} \frac{q_n^{(h)}}{N^{2h}}, \] (12)
with \( h = 0, 1, \ldots \), corresponding respectively to lattices with spherical topology, torus topology, etc. The coefficients \( a^{(h)}_n \) can be calculated iteratively using orthogonal polynomials for the one matrix model given in (10). From [7, 9] we have, for instance, the sphere and torus contributions:

\[
a^{(0)}_n = \frac{3^n (2n - 1)!}{n! (n + 2)!}, \quad (13)
\]

\[
a^{(1)}_n = \frac{3^n}{24n} \left( 4^n - \frac{(2n)!}{(n!)^2} \right). \quad (14)
\]

In the case \( c \to 1 \ (T \to \infty) \), the decoupling of the two matrices in eq.(7) is less obvious. With the redefinition \( g = \mathcal{F}(1 - c^2)^2 \) we obtain, in the limit \( c \to 1 \), after a change of variables in eq.(7),

\[
E(g, c \to 1, H) = E_1 \left( \frac{\mathcal{F}}{4} y^{1/2} + y^{-1/2} \right) = -\lim_{c \to 1} \sum_{n=1}^\infty a_n \frac{(-g)^n (1 + y)^n y^{-n/2}}{(1 - c^2)^{2n}}. \quad (15)
\]

If we compare eq. (8) with eqs. (11) and (15), we obtain the partition functions

\[
Z_n (T = 0) = a_n \ (1 + y^n) \ y^{-n/2} \quad (16)
\]

and

\[
Z_n (T \to \infty) = a_n \ (1 + y^n) \ y^{-n/2}. \quad (17)
\]

Notice that we have dropped the factor \([c/(1 + c^2)^2]^n\) since it just amounts to a redefinition of the energy by a constant.

We conclude that for arbitrary number of sites the Yang-Lee zeros of the Ising model on dynamical random lattices of arbitrary topology\(^4\) are homogeneously distributed around the unit circle at \( T = 0 \) \((y_k = e^{i \pi (2k - 1)}, \ k = 1, 2, \ldots, n)\) and coalesce at the point \( y_k = -1 \) as \( T \to \infty \). Both cases coincide with the results of the model on a static lattice (see, e.g., [10, 11]).

### 3 Free energy on the Torus at Arbitrary \( T \)

One can reduce the \( 2N^2 \) integrals over \( X_{ij}, Y_{ij} \) to \( 2N \) integrals over their eigenvalues \( x_i, y_i \) obtaining [8, 13] :

\[
E(g, c, H) = -\frac{1}{N^2} \ln \frac{Z(g, c, H)}{Z(0, c, H)}, \quad (18)
\]

\(^4\) Though we have only given \( a^{(0)}_n \) and \( a^{(1)}_n \), the relations (16) and (17) hold for arbitrary topologies.
where
\[ Z(g,c,H) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \prod_{i=1}^{N} dx_i dy_i \omega_i(c,g,H) \prod_{k<j} (x_k - x_j)(y_k - y_j) \]
(19)

with
\[ \omega_i = \exp \left\{ - \left( x_i^2 + y_i^2 - 2cx_iy_i + \frac{ge^H}{N} x_i^4 + \frac{ge^{-H}}{N} y_i^4 \right) \right\}. \]
(20)

We can further decrease the number of degrees of freedom introducing two sets of monic polynomials \[8\],
\[ P_j(x) = x^j + \sum_{k=0}^{j-1} a_k x^k \]
(21)
and
\[ Q_j(y) = y^j + \sum_{k=0}^{j-1} b_k y^k, \]
(22)
which are orthogonal with respect to the weight eq.(20),
\[ \int \int dx \, dy \, \omega(c,g,H) P_i(x)Q_j(y) = \delta_{ij} h_i(c,g,H). \]
(23)

Using the relations
\[ \prod_{k<i} (x_k - x_i)(y_k - y_i) = \det x_i^{j-1} \det y_i^{j-1} = \det P_{j-1}(x_i) \det Q_{j-1}(y_i), \]
(24)
one derives:
\[ E(g,c,H) = -\frac{1}{N^2} \sum_{i=0}^{N-1} \ln \frac{h_i(g,c,H)}{h_i(0,c,H)} \]
\[ = -\frac{1}{N^2} \sum_{i=1}^{N} (N - i) \ln \frac{f_i(g,c,H)}{f_i(0,c,H)} - \frac{1}{N} \ln \frac{h_0(g,c,H)}{h_0(0,c,H)}, \]
(25)
(26)
where \( f_i = h_i/h_{i-1} \). The last term of eq.(26) can be calculated from eq.(23):
\[ h_0(g,c,H) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} dx \, dy \, e^{-\left( x^2 + y^2 - 2cxy + \frac{ge^H}{N} x^4 + \frac{ge^{-H}}{N} y^4 \right)} \]
\[ = h_0(0,c,H) \left[ 1 - \frac{1}{N} \frac{3g \cosh(H)}{2(1 - c^2)^2} + O\left( \frac{1}{N^2} \right) \right]. \]
(27)
(28)

The ratios \( f_i \) and the auxiliary quantities \( r_i, s_i, q_i \) and \( t_i \) defined below,
\[ xP_i(x) = P_{i+1}(x) + r_i P_{i-1}(x) + s_i P_{i-3}(x) + \cdots \]
\[ yQ_i(y) = Q_{i+1}(y) + q_i Q_{i-1}(y) + t_i Q_{i-3}(y) + \cdots \]
(29)
(30)
can be determined altogether by solving a set of coupled equations obtained from integrals of total derivatives involving eq. (23). Since we are going to look at $N \to \infty$ it is more useful to present the continuum version of those equations. Introducing the notation:

\[ x = i/N, \quad \epsilon = 1/N, \quad f_{i+k} = Nf(x + k\epsilon), \]
\[ q_{i+k} = Nq(x + k\epsilon), \quad r_{i+k} = Nr(x + k\epsilon), \]
\[ s_{i+k} = N^2s(x + k\epsilon), \quad t_{i+k} = N^2t(x + k\epsilon), \]

the continuum equations become [4, 8]:

\[ cq'(x) = f(x) \{1 + 2ug [r(x + \epsilon) + r(x) + r(x - \epsilon)]\}, \quad (34) \]
\[ cf(x) + \frac{x}{2} - r(x) = 2ug \{s(x + 2\epsilon) + s(x + \epsilon) + s(x) + r(x) [r(x + \epsilon) + r(x) + r(x - \epsilon)]\}, \quad (35) \]
\[ cs(x) = \frac{2g}{u} f(x)f(x - 2\epsilon)f(x - \epsilon) \]

and three more equations obtained from (34-36) by exchanging $(u, r, s, q, t) \to (1/u, q, t, r, s)$, where $u = e^H$. Using the expansion

\[ f(x) = f_0(x) + \frac{1}{N} f_{1/2}(x) + \frac{1}{N^2} f_1(x) + \cdots \]

and analogous ones for $r(x), q(x), s(x), t(x)$ and their conjugates, we reproduce, collecting the $(1/N)^0$ terms, the planar equation of [4]:

\[ gx = \frac{c^2 z^3}{9} + \frac{2g}{3} \left[ \frac{1}{(1 - z)^2} - c^2 + \frac{Bz}{(1 - z^2)^2} \right] \equiv g(z). \]

Here $B = 2 [\cosh(H) - 1]$ and $z(x) = (6g/c)f_0(x)$. The above equation furnishes $f_0(x)$ parametrically. By further collecting the $(1/N)$ terms we get

\[ f_{1/2} = q_{1/2} = r_{1/2} = 0, \]
\[ f_{1/2} = -\frac{dt_0}{dx}, \quad s_{1/2} = -\frac{ds_0}{dx}. \]

Using such results we derive from the $(1/N)^2$ terms:

\[ f_1(z(x)) = \frac{c g}{2} z (1 - z^2)^4 \left\{ 4(1 + z)^8 + 2B(B + 4) \left[ (1 + z^2)^4 + 32z^4 \right] \right. \]
\[ + 2B^2 + 3c^2 + c^4 + 16Bz + 28c^2z + 14Bc^2z + 75c^2z^2 \]
\[-5 c^4 z^2 + 112 B z^3 + 16 c^2 z^3 + 8 B c^2 z^3 - 233 c^2 z^4
+ 4 c^4 z^4 + 112 B z^5 - 308 c^2 z^5 - 154 B c^2 z^5 + 127 c^2 z^6
+ 28 c^4 z^6 + 16 B z^7 + 512 c^2 z^7 + 256 B c^2 z^7 + 217 c^2 z^8
- 98 c^4 z^8 - 268 c^2 z^9 - 134 B c^2 z^9 - 271 c^2 z^{10} + 154 c^4 z^{10}
- 16 c^2 z^{11} - 8 B c^2 z^{11} + 77 c^2 z^{12} - 140 c^4 z^{12} + 36 c^2 z^{13}
+ 18 B c^2 z^{13} + 5 c^2 z^{14} + 76 c^4 z^{14} - 23 c^4 z^{16} + 3 c^4 z^{18}\]
\times \left\{(1 + z)^4 \left[1 - c^2 (1 - z)^4\right] + 2 B z (1 + z^2)\right\}^{-4}. \tag{40}

Another contribution of order \((1/N)^2\) comes from applying the Euler-Maclaurin summation formula:

\[
\frac{1}{N} \sum_{i=1}^{N} F(i/N) = \int_0^1 dx F(x) + \frac{1}{2N} [F(0) + F(1)] + \frac{1}{12N^2} F'(1)|_0^1 + O(N^{-3}), \tag{41}
\]
on the expression (26) which gives

\[
E(g, c, H) = -\frac{1}{N} \ln h_0(g, c, H) - \int_0^1 dx F(x) dx - \frac{1}{2N} [F(0) + F(1)]
- \frac{1}{12N^2} F'(1)|_0^1 + O(N^{-3}), \tag{42}
\]
where

\[
F(x) = (1 - x) \ln \left[\frac{f(x)}{f(x, g = 0)}\right]. \tag{43}
\]

After using (38), we obtain

\[
g(z(x = 1)) = g \quad \quad g(z(x = 0)) = 0
\]
\[
1 - x = 1 - \frac{g(z)}{g}. \tag{44}
\]

The integral in (12) can be written, after an integration by parts and using (44)

\[
I(g, c, H) \equiv \int_0^1 dx (1 - x) \ln f(x) = I_0(g, c, H) + \frac{1}{N^2} I_1(g, c, H) + O(N^{-4}), \tag{45}
\]
where \(I_0\) is the planar contribution:

\[
I_0(g, c, H) = -\frac{1}{2} \left[(1 - x)^2 \ln f(x)\right]|_0^1 + \frac{1}{2g^2} \int_{z(0)}^{z(1)} \frac{dz'}{z'} [g - g(z')]^2, \tag{46}
\]
while $I_1$ stands for the torus contribution,

$$I_1(g, c, H) = -\frac{1}{2} (1-x)^2 \frac{f_1(x)}{f_0(x)} + \frac{3}{gc} \int_{z(0)}^{z(1)} dz' \left[ g - g(z') \right]^2 \frac{d}{dz'} \left[ \frac{f_1(z')}{z'} \right]. \quad (47)$$

In the above formulas $z(1)$ is the solution of eq.(38) with $x = 1$ that vanishes for $g = 0$ and $z(0) = 0$ since it is the only solution of eq.(38) for $x = 0$ that is real in the whole range of temperatures. After further manipulations we arrive at

$$E(g, c, H) = E^{(0)}(g, c, H) + \frac{1}{N^2} E^{(1)}(g, c, H) + \mathcal{O}(N^{-4}), \quad (48)$$

where

$$E^{(0)}(g, c, B) = -\frac{1}{2} \ln \frac{z}{g} + \frac{1}{g} \int_0^z \frac{dt}{t} g(t) - \frac{1}{2g^2} \int_0^z \frac{dt}{t} g^2(t) - \frac{1}{2} \ln \left( \frac{1-c^2}{3} e^{3/2} \right). \quad (49)$$

Where, following the notation of [4], the integration limit $z \equiv z(x = 1)$ is of course a function of $g, c$ and $B$. This is the known result from [3] for the free energy of the Ising on random planar lattices (sphere). The next term which is not known in the literature is the torus contribution,

$$E^{(1)}(g, c, B) = g \frac{\cosh(H)}{(1-c^2)^2} + \frac{1}{12} \ln \left( \frac{1-c^2}{3} \right) + \frac{1}{12} \ln \frac{z}{g} - 3g R(z) + 6 \int_0^z dt g(t) \frac{d}{dt} \left( \frac{R(t)}{t} \right) - \frac{3}{g} \int_0^z dt g^2(t) \frac{d}{dt} R(t). \quad (50)$$

Here we have define $R(z) \equiv f_1(z)/cz$. Note that the $1/N$ contribution vanishes in (48), which is in agreement with the expected topological expansion [12].

### 4 Yang-Lee Zeros on the Torus

The free energies $E^{(0)}$ and $E^{(1)}$ depend on the coupling $g$ explicitly and implicitly through $z$. Now we can expand $E^{(1)}$ around $g = 0$ using

$$z(x = 1, g) = \sum_{k=1}^{\infty} \frac{1}{k!} h_k g^k, \quad (51)$$

where $h_k = h_k(c, B)$ can be calculated from consecutive derivatives of $g(z) = g$ with respect to $g$ at the point $g = 0$. Moreover, the integral terms in eq.(50) may be combined according to

$$\Lambda(t) \equiv \left[ g^2(t) - 2g g(t) \right] \frac{d}{dt} R(t), \quad (52)$$

and the resulting integral expanded in powers of $z$,
In eq. (53), \( z \) is given by the series in eq. (51), and \( \left[ \partial_t^k \Lambda(t) \right]_{t=0} \) is a function of \( g, c \) and \( B \).

Using eqs. (51) and (53) together with the expansion of \( R(z) \) in powers of \( g \), and comparing with eq. (8), we obtain the partition functions on random toroidal lattices \( Z_n^{(1)} \) up to \( n = 16 \) vertices. The first 8 results are collected in Table 1.

\[
\begin{align*}
Z_1^{(1)} &= \frac{1}{4} c^{-1} y^{-1/2} (1 + y) \\
Z_2^{(1)} &= \frac{5}{8} c^{-2} y^{-1} \left[ 3 + (4c^2 + 2c^4) y + 3y^2 \right] \\
Z_3^{(1)} &= \frac{3}{2} c^{-3} y^{-3/2} (1 + y) \left[ 11 + (-11 + 16c^2 + 17c^4) y + 11y^2 \right] \\
Z_4^{(1)} &= \frac{3}{16} c^{-4} y^{-2} \left[ 837 + (1416c^2 + 1932c^4) (y + y^3) + (936c^2 + 2708c^4 + 1304c^6 + 74c^8) y^2 + 837y^4 \right] \\
Z_5^{(1)} &= \frac{27}{10} c^{-5} y^{-5/2} (1 + y) \left[ 579 + (-579 + 1140c^2 + 1755c^4) (y + y^3) + (579 - 525c^2 + 805c^4 + 2245c^6 + 370c^8) y^2 + 579y^4 \right] \\
Z_6^{(1)} &= \frac{3}{4} c^{-6} y^{-3} \left[ 21411 + (48492c^2 + 79974c^4) (y + y^5) + (23382c^2 + 119160c^4 + 142434c^6 + 36189c^8) (y^2 + y^4) + (18630c^2 + 115452c^4 + 188488c^6 + 92868c^8 + 12522c^{10} + 260c^{12}) y^3 + 21411y^6 \right] \\
Z_7^{(1)} &= \frac{61}{7} c^{-7} y^{-7/2} (1 + y) \left[ 14571 + (-14571 + 37464c^2 + 64533c^4) (y + y^5) + (14571 - 20580c^2 + 34076c^4 + 143444c^6 + 47054c^8) (y^2 + y^4) + (-14571 + 32319c^2 + 56581c^4 + 61278c^6 + 11853c^8 + 41307c^{10} + 2653c^{12}) y^3 + 14571y^6 \right] \\
Z_8^{(1)} &= \frac{61}{32} c^{-8} y^{-4} \left[ 710991 + (2050416c^2 + 3637512c^4) (y + y^7) + (884736c^2 + 5729832c^4 + 9564480c^6 + 3728700c^8) (y^2 + y^6) + (564192c^2 + 5038200c^4 + 14023200c^6 + 14328000c^8 + 5312016c^{10} + 549888c^{12}) (y^3 + y^5) + (487872c^2 + 4742568c^4 + 14910912c^6 + 18385900c^8 + 9308480c^{10} + 1819608c^{12} + 112832c^{14} + 1198c^{16}) y^4 + 710991y^8 \right]
\end{align*}
\]

Table 1: The first eight partition functions on graphs with torus topology.
Fig. 4. Yang-Lee zeros of $Z_{16}^{(1)}$ on torus, for temperatures $c = 0.25(+), 0.45(\diamond)$ and $0.65(\times)$.

The figure 4 shows the flux of zeros on the unit circle, as the temperature $c$ decreases to the critical one, $c = 0.25$. In Fig. 5 we clearly see that the zeros tend to pinch the positive real $y$–axis as the number of sites increases (thermodynamic limit) at the critical temperature $c = 0.25$. It is worth mentioning that the polynomials are different from their planar counterparts, but their zeros still lie on the unit circle. The topology slightly changes the position of the zeros along the unit circle.

5 Non-perturbative Partition Function for Small Matrices

As we have seen in section 3 the exact free energy for spherical and torus contributions were given as integral representations (eqs. (49) and (50)), in the large $N$ limit. These turn out to be the first two contributions of a topological expansion in $N^{-2h}$ \footnote{12}, where $h$ stands for the number of handles of the corresponding surface in the continuum limit. On the other hand, if we consider small values for $N$, it is possible to obtain an explicit closed form for the free energy \footnote{18} including all topologies which contribute to a given number of vertices. Perturbatively in $g$, we have in those cases few gaussian integrals to perform (see \footnote{19}).

It is convenient to write eq. (19) as follows,

$$Z(g, c, H) = Z(0, c, H) + \tilde{Z}(g, c, H).$$

(54)
Fig. 5. Yang-Lee zeros of $Z_n^{(1)}$ at critical temperature $c = 0.25$ for different lattice sizes $n$ on torus: $n = 10(\times), 13(\circ)$ and $16(+)$. 

From eqs. (54) and (18) and, after expanding the logarithm, we obtain the free energy 

$$E(g, c, H) = -\frac{1}{N^2} \sum_{m=1}^{\infty} \frac{(-1)^{m+1}}{m} \left[ \frac{\tilde{Z}(g, c, H)}{Z(0, c, H)} \right]^m. \quad (55)$$

The above expansion can be rewritten in powers of $g$, 

$$\frac{\tilde{Z}(g, c, H)}{Z(0, c, H)} = \sum_{p=1}^{\infty} z_N^{(p)}(c, H) g^p. \quad (56)$$

The coefficients $z_N^{(p)}$ can be easily calculated for small number of eigenvalues. Here we restrict ourselves, for sake of simplicity, to $N = 1$ (no contribution from the vandermonde) and $N = 2$. After rearranging (55) in powers of the coupling $g$ and comparing with the expression (8) we obtain the nonperturbative partition functions $Z_n$ for $N = 1, 2$. We have obtained $Z_n$ explicitly for $n \leq 20$ sites. In tables 2 and 3 we display the nonperturbative partition functions $Z_n$ ($n = 1, 2, ..., 6$) for $N = 1$ and 2, respectively. Finally, in Fig. 6 we show the Yang-Lee zeros for our largest lattice, $n = 20$ sites, for $N = 2$, which are quite close to the ones for $N = 1$. Here, we clearly see the flux of zeros on the unit circle, as the temperature $c$ decreases to the critical one, $c = 0.25$. 

15
\[
Z_1 = \frac{3}{4} c^{-1} y^{1/2} (1 + y)
\]
\[
Z_2 = \frac{3}{2} c^{-2} y^{-1} [2 + (3 c^2 + c^4) y + 2 y^2]
\]
\[
Z_3 = \frac{9}{4} c^{-3} y^{-3/2} (1 + y) [11 + (-11 + 16 c^2 + 17 c^4) y + 11 y^2]
\]
\[
Z_4 = \frac{9}{4} c^{-4} y^{-2} [136 + (198 c^2 + 346 c^4) (y + y^3) + (128 c^2 + 429 c^4 +
242 c^6 + 17 c^8) y^2 + 136 y^4]
\]
\[
Z_5 = \frac{27}{20} c^{-5} y^{-5/2} (1 + y) [3714 + (-3714 + 5440 c^2 + 13130 c^4) (y + y^3) +
(3714 - 2800 c^2 + 1455 c^4 + 16370 c^6 + 3545 c^8) y^2 + 3714 y^4]
\]
\[
Z_6 = \frac{9}{2} c^{-6} y^{-3} [22688 + (33426 c^2 + 102702 c^4) (y + y^5) + (13056 c^2 + 99855 c^4 +
167082 c^6 + 60327 c^8) (y^2 + y^4) + (9801 c^2 + 89211 c^4 + 199254 c^6 +
131184 c^8 + 23601 c^{10} + 709 c^{12}) y^3 + 22688 y^6]
\]

Table 2: The first six all topologies partition functions \(Z_n\) for \(N = 1\).

6 Conclusion

We have obtained a closed and exact expression for the \(1/N^2\) contribution to the free energy of the two Hermitian random matrix model. From this expression one can obtain the partition function \(Z_n^{(1)}\) of the Ising model in the presence of a constant magnetic field \(H\) on random graphs (2D Gravity) of torus topology with \(n\) sites. For arbitrary temperature we have obtained explicitly the partition functions \(Z_n^{(1)}\) with \(n \leq 16\). For the specific temperatures \(c = 0.25, 0.45\) and \(0.65\), we have checked that the zeros of the polynomials \(Z_n^{(1)}\) \((n \leq 16)\) lie on the unit circle \((|y| = 1)\) on the complex fugacity plane.

For the special cases of small matrices \(1 \times 1\) and \(2 \times 2\) we have obtained the non-perturbative partition function \(Z_n\) which includes contributions of all topologies. We have explicit results for \(n \leq 20\) sites and again the Yang-Lee zeros lie on the unit circle. That is quite surprisingly since there is no Lee-Yang circle theorem for dynamical lattices. Taking linear combinations of polynomials in general originates non easily predictable changes in their roots. A similar result has been observed before for the \(Z_n^{(0)}\) \((n \leq 14)\) on dynamical planar graphs. Our calculations on the torus and for higher topologies is thus a strong evidence that the topology of the graph plays no special role what the position of the Yang-Lee zeros is concerned. The same happens on a static lattice since the Lee-Yang theorem (see [2]) is known to be independent on the details of the lattice like its topology, number of nearest neighbors, etc. It should be stressed that our dynamical lattice consists of sums of \(\phi^4\) vacuum to vacuum diagrams where the weight of each diagram corresponds to its combinatorial factor. Nevertheless, those specific weights seem to be immaterial
Table 3: The first six all topologies partition functions for $N = 2$. 

| $Z_1$ | $Z_2$ | $Z_3$ | $Z_4$ | $Z_5$ | $Z_6$ |
|-------|-------|-------|-------|-------|-------|
| $Z_1 = \frac{9}{16} c^{-1} y^{-1/2} (1 + y)$ | $Z_2 = \frac{3}{32} c^{-2} y^{-1} [17 + (28 c^2 + 6 c^4) y + 17 y^2]$ | $Z_3 = \frac{81}{16} c^{-3} y^{-3/2} (1 + y) [7 + (-7 + 12 c^2 + 9 c^4) y + 7 y^2]$ | $Z_4 = \frac{9}{256} c^{-4} y^{-2} [2011 + (3648 c^2 + 4396 c^4) (y + y^3) + (2528 c^2 + 6424 c^4 + 2952 c^6 + 162 c^8) y^2 + 2011 y^4]$ | $Z_5 = \frac{81}{1280} c^{-5} y^{-5/2} (1 + y) [11394 + (-11394 + 21740 c^2 + 35230 c^4) (y + y^3) + (11394 - 9800 c^2 + 14055 c^4 + 44770 c^6 + 11394 c^8) y^2 + 11394 y^4]$ | $Z_6 = \frac{9}{256} c^{-6} y^{-3} [252169 + (501588 c^2 + 1011426 c^4) (y + y^5) + (231528 c^2 + 1299915 c^4 + 1737966 c^6 + 513126 c^8) (y^2 + y^4) + (183438 c^2 + 1229793 c^4 + 2214752 c^6 + 1822392 c^8 + 188238 c^{10} + 4767 c^{12}) y^3 + 252169 y^6]$ |

concerning the unit circle. It is remarkable that replacing those specific weights by arbitrary positive constants, at least for $n \leq 4$ sites, we still have the roots on the unit circle. That happens on the sphere, on the torus and also for linear combinations, with positive constants, of both topologies. In fact this is probably the reason for the location on the unit circle of the zeros of the nonperturbative partition functions that we have obtained for the special cases of $1 \times 1$ and $2 \times 2$ matrices. In those cases we have a linear combination of all topologies with the positive coefficients $N^{-h}$ where $h = 0, 1, \cdots \infty$ labels the different topologies. Such results lead us to conjecture that linear combinations, with positive coefficients, of partition functions of the 2D Ising model calculated on different graphs $G_n$ have all their $n$ Yang-Lee zeros on the unit circle. At this point a word of caution is in order, namely, Lee and Yang have proven their circle theorem for a class of polynomials which includes the partition function of the Ising Model in a constant magnetic field, nevertheless, it is easy to find numerical examples of linear combinations of such polynomials which do not obey the circle theorem even if we take positive coefficients. Therefore it is not obvious how the Lee-Yang theorem can be generalized for sums over the graphs (lattices). It cannot be discarded that the polynomials we have for the dynamical case correspond to a new class of polynomials not encompassed by the Lee-Yang theorem but whose roots still lie on the unit circle. A clear understanding of this issue demands more work which is now in progress. We emphasize that we have only looked at the global location of the Yang-Lee zeros rather than their local distribution close to the positive real axis and its relationship with the topology. This is now under investigation. Finally, it is worth pointing out that we were able
Fig. 6. Yang-Lee zeros of $Z_{20}$ with $N = 2$, for temperatures $c = 0.25(+), 0.45(\circ)$ and $0.65(\times)$.

to sum over all topologies for a given finite number of vertices because the sum is actually finite since not all topologies can contribute to a graph with a finite number of vertices with finite number of links. Thus, we have a natural cut-off.

7 Note Added

After finishing this work we became aware of [14] where for the special case $N = 1$ ("Thin Graphs") it was shown in the thermodynamic limit that the closest zero to the positive real axis is located on the unit circle on the complex fugacity plane. That is clearly in agreement with the expectations from our finite size calculations of all Yang-Lee zeros in the case $N = 1$ (see section 5).

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Figure Captions:
Fig. 1. Diagrams and partition functions for $n = 2$ vertices.
Fig. 2. Diagrams and partition functions for $n = 3$ vertices.
Fig. 3a. First set of diagrams and partition functions for $n = 4$ vertices.
Fig. 3b. Second set of diagrams and partition functions for $n = 4$ vertices.
Fig. 4. Yang-Lee zeros of $Z_{16}^{(1)}$ on torus, for temperatures $c = 0.25(+), 0.45(\diamond)$ and $0.65(\times)$.
Fig. 5. Yang-Lee zeros of $Z_{n}^{(1)}$ at critical temperature $c = 0.25$ for different lattice sizes $n$ on torus: $n = 10(\times), 13(\diamond)$ and $16(+)$.
Fig. 6. Yang-Lee zeros of $Z_{20}$ with $N = 2$, for temperatures $c = 0.25(+), 0.45(\diamond)$ and $0.65(\times)$. 