Open String Field Theory on Noncommutative Space

Teruhiko Kawano and Tomohiko Takahashi

Department of Physics, University of Tokyo
Hongo, Tokyo 113-0033, Japan
kawano@hep-th.phys.s.u-tokyo.ac.jp
tomo@hep-th.phys.s.u-tokyo.ac.jp

We study Witten’s open string field theory in the presence of a constant $B$ field. We construct the string field theory in the operator formalism and find that, compared to the ordinary theory with no $B$ field, the vertices in the resulting theory has an additional factor. The factor makes the zero modes of strings noncommutative. This is in agreement with the results in the first-quantized formulation. We also discuss background independence of the purely cubic action derived from the above string field theory and then find a redefinition of string fields to remove the additional factor from the vertex. Furthermore, we briefly discuss the supersymmetric extension of our string field theory.

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1. Introduction

Since the appearance of the seminal paper [1], noncommutative geometry has received much attention in Matrix theory and string theory [2,3]. See [3] for further references. In string theory, we have the familiar antisymmetric tensor field $B_{ij}$ which directly couples to fundamental strings. If we turn on the background $B$ field, spacetime becomes noncommutative on $D$-branes with the nonvanishing $B$ field. $D$-branes can be described by open strings whose ends are on the $D$-branes [4]. By the quantization of the open strings, we have gauge field on the $D$-branes, and the low-energy effective theory of the gauge field are described by the Dirac-Born-Infeld (DBI) action [5]. Therefore, turning on the $B$ field, we can find the DBI action on the noncommutative space.

Recently, Seiberg and Witten have shown that the noncommutative DBI action is equivalent to the ordinary one [3]. To prove the equivalence, they have given a relation between the gauge fields in the noncommutative DBI action and the ordinary one [3]. Some closely related topics have been discussed in [6,7]. However, at present it seems unclear how we can embed the relation into a whole tower of the excitation modes of strings. To uncover such a relation, string field theories seems a natural framework, where we can deal with string fields which include all the excitations as well as the gauge field.

In the paper [8], Witten has constructed, on a commutative flat Minkowski spacetime, a covariant open string field theory based on noncommutative geometry. This noncommutativity comes from the nature of the way that open strings join together to become a new string. Therefore, we may expect that Witten’s string field theory in the background $B$ field has additional noncommutativity. In this paper, we will derive Witten’s open string field theory in the above-mentioned background in the operator formalism [9,10,11]. By solving the overlap conditions, we will show that the string field theory has an additional factor in its vertex. This factor accounts for the noncommutativity of spacetime and is in agreement with the result of [12,3] in the first-quantized formulation. In [13], this factor has also been found in Witten’s open string field theory with a constant background magnetic field $F_{ij}$. By the gauge invariance $B_{ij} \to B_{ij} + \partial_i \Lambda_j - \partial_j \Lambda_i$, $A_i \to A_i + \Lambda_i$, we can see that this background is the same as ours. However, the physical significance of this factor has not been fully realized. Also, open string field theories in general backgrounds have been discussed in terms of the conformal field theory [14]. The string field theory in this paper could be studied in the same way.

Pregeometrical string field theories have been proposed to give a background independent formulation of string theory [15,16]. In particular, the pregeometrical theory given in
[15] is Witten’s open string field theory on a flat Minkowski spacetime without the kinetic term and so it is sometimes referred to as purely cubic action. Therefore, if we drop the kinetic term from our string theory in the background $B$ field, it is tempting to ask whether the resulting theory can be background independent. Since the additional noncommutative factor explicitly depends on the background $B$ field, we may at first think that it cannot be background independent. If we dealt with a particle field theory, this would be true. However, as we will show in this paper, we can remove the noncommutative factor from the three-string vertex by a redefinition of string fields. In addition, we will explicitly demonstrate that the three-string vertex is independent of the background metric which we use to express the vertex in terms of the oscillators of strings. To this end, we will apply the method given in [17] to open string field theory.

This paper is organized as follows: In section 2, we construct Witten’s open string field theory in the background $B$ field by using the operator formalism and solving the overlap conditions for the vertices. In section 3, we explicitly show background independence of our pregeometrical theory in great detail. Section 4 is devoted to discussion. In appendix A, we briefly summarize the operator formalism of the first-quantized string theory [18,19]. In appendix B, we give a derivation of Yoneya’s identities [11,20] of the Neumann coefficients for Witten’s string field theory, which we need in section 3.

When we had almost finished writing this paper, we found a paper [21] given by Sugino, which has considerable overlap with ours. The main difference between that paper and ours is the following two points. First, he argues that the dependence on the $B$ field can be eliminated from our string field theory by a redefinition of string fields. This suggests that we can ‘gauge away’ the background field. We will discuss this point in further detail in section 4. Second, we explicitly show background independence of our pregeometrical theory. In section 4, we will also mention our main results about an open-closed string field theory with the light-cone type interaction [22] in the background we are considering. Furthermore, we will discuss the supersymmetric extension [23] of our string field theory.

2. String Field Theory in Background $B$-Field

We study a bosonic open string field theory proposed by Witten [8] with a constant metric $g_{ij}$ and a constant antisymmetric field $B_{ij}$. The open string field theory in the presence of background fields has been discussed in [14,24]. We show that we can construct the field theory in our background explicitly by using the operator formalism [3,10,11].
this end, it is appropriate to begin with a review of the operator formalism of the first-
qu quantized string theory with the $B$ field \[18,19\]. In appendix A, we give a simple derivation of the result given by \[18,19\] to make this paper self-contained.

In the first-quantized string theory, the worldsheet action is given by

\[
S = -\frac{1}{4\pi\alpha'} \int d\sigma d\tau \left( g_{ij} \eta^{ab} \partial_a X^i \partial_b X^j - 2\pi\alpha' B_{ij} \epsilon^{ab} \partial_a X^i \partial_b X^j \right), \tag{2.1}
\]

From this action, if the Dirichlet boundary condition is not chosen for all the directions of the string coordinates, the boundary condition can be seen to be

\[
g_{ij} X^j + (2\pi\alpha') B_{ij} \dot{X}^j = 0 \quad \text{at } \sigma = 0, \pi,
\]

where we denote the differentiation with respect to $\tau$ and $\sigma$ by the dot $\dot{\cdot}$ and the prime $'\cdot'$, respectively. For simplicity, in this paper, we impose this boundary condition on all the string coordinates $X^i(\tau,\sigma)$. The conjugate momenta of the string coordinates $X^i(\sigma)$ turn out to be

\[
P_i(\sigma) = \frac{1}{2\pi\alpha'} g_{ij} \dot{X}^j(\sigma) + B_{ij} X^j(\sigma).
\]

The authors of \[18,19\] have shown that we can quantize our system by the Dirac quantization procedure if we treat the boundary condition as a constraint. See also appendix A for further details. The resulting commutation relations can be seen to be

\[
[X^i(\sigma), P_j(\sigma')] = i \delta^i_j \delta(\sigma - \sigma'),
\]

\[
[P_i(\sigma), P_j(\sigma')] = \begin{cases} 
  i \theta^{ij}, & (\sigma = \sigma' = 0) \\
  -i \theta^{ij}, & (\sigma = \sigma' = \pi) \\
  0, & (\text{otherwise}),
\end{cases}
\]

where we use the same definitions of the open string metric $G_{ij}$ and the theta parameter $\theta^{ij}$ as those in \[3\]:

\[
G^{ij} = \left( \frac{1}{g + 2\pi\alpha'B} \right)^{ij},
\]

\[
\theta^{ij} = -(2\pi\alpha)^2 \left( \frac{1}{g + 2\pi\alpha'B} \right)^{ij}.
\]

As we can see from appendix A, the mode expansion of the string coordinates $X^i(\sigma)$ turns out to be

\[
X^i(\sigma) = \tilde{X}^i(\sigma) + (\theta G)^i_j Q^j(\sigma), \tag{2.4}
\]

where $\tilde{X}^i(\sigma)$ and $Q^i(\sigma)$ are defined with $l_s = \sqrt{2\alpha'}$ as follows:

\[
\tilde{X}^i(\sigma) = \tilde{x}^i + l_s \sum_{n \neq 0} \frac{i}{n} \alpha^i_n \cos(n\sigma),
\]

\[
Q^i(\sigma) = \frac{1}{\pi} G^{ij} p_j \left( \sigma - \frac{\pi}{2} \right) + \frac{1}{\pi l_s} \sum_{n \neq 0} \frac{1}{n} \alpha^i_n \sin(n\sigma). \tag{2.5}
\]
We can see here that the variables \( \tilde{X}^i(\sigma) \) satisfy the Neumann boundary condition. Similarly, the expansion of the momenta is

\[
P_i(\sigma) = \frac{1}{\pi l_s} \sum_n G_{ij} \alpha_n^j \cos(n\sigma).
\] (2.6)

Note that \( Q^i(\sigma) \) and \( P_i(\sigma) \) have the following relation:

\[
Q^i(\sigma) = \int_{0}^{\frac{\pi}{2}} d\sigma' G^{ij} P_j(\sigma') + \frac{1}{2} G^{ij} (p_{Lj} - p_{Rj}),
\] (2.7)

where we introduce the momentum operators integrated over half a string [25],

\[
p_{Lj} = \int_{0}^{\frac{\pi}{2}} d\sigma P_i(\sigma), \quad p_{Rj} = \int_{\frac{\pi}{2}}^{\pi} d\sigma P_i(\sigma).
\] (2.8)

The commutation relations of these mode variables \( \tilde{x}^i, p_i, \alpha^i \) can be verified [18,19] to be

\[
[\tilde{x}^i, p_j] = i\delta^i_j, \quad [\alpha^i_m, \alpha^j_n] = m\delta_{m+n}G^{ij},
\] (2.9)

and the others vanish, as is seen from appendix A.

The BRS charge in string field theories is necessary to construct their kinetic terms. In order to obtain the BRS charge, we need to know the energy-momentum tensor in the worldsheet theory (2.1). While the contribution from the reparametrization ghosts to the energy-momentum tensor is the same as usual, the energy-momentum tensor from the matter sector, i.e. the string coordinates, can be found to be

\[
T(z) = -\frac{1}{\alpha'} g_{ij} \partial X^i \partial X^j(z) = -\frac{1}{\alpha'} G_{ij} \partial \tilde{X}^i \partial \tilde{X}^j(z),
\]

\[
\tilde{T}(\bar{z}) = -\frac{1}{\alpha'} g_{ij} \partial \bar{X}^i \partial \bar{X}^j(\bar{z}) = -\frac{1}{\alpha'} G_{ij} \partial \tilde{\bar{X}}^i \partial \tilde{\bar{X}}^j(\bar{z}),
\] (2.10)

with \( z = \exp(\tau + i\sigma) \). Note from the boundary condition that \( T(z) = \tilde{T}(\bar{z}) \) for \( z = \bar{z} \). Therefore, we can make use of the doubling technique for open strings to extend the worldsheet of the upper half-plane to a whole complex plane when we define the BRS charge by using the energy momentum tensor in the usual way.

In the rest of this section, we will construct a string field theory with the mid-point interaction in our background. To this end, the reflector and the three-string vertex will be constructed by using the overlap conditions, as usual. From the paper [12,13], we expect that the noncommutativity of spacetime would also appear in our string field theory, in addition
to the usual noncommutativity of an open string field theory. This is actually the case, as we will see below. Although the mode expansion of the string coordinates is different from that with the Neumann boundary condition due to the presence of the background $B$-field, the resulting vertices will be shown to be the same as usual vertices except for one factor. It is this factor that accounts for the noncommutativity of spacetime. By noncommutative spacetime, we mean a spacetime such that, given two arbitrary functions $f(x)$ and $g(x)$ on the space, the product of these functions is given by the Moyal product

$$f \ast g(x) = f(x) \exp \left[ i \frac{\theta^{ij}}{2} \partial_i \partial_j \right] g(x).$$

(2.11)

If we identify the ‘zero mode’ $\tilde{x}$ of string coordinates with coordinates of our spacetime, the above-mentioned factor turns out to be the exponential factor in (2.11), as we will show below.

Besides the BRS charge, in order to obtain a kinetic term in string field theory, we need the reflector $\langle R \rangle$, which is used to give the inner product of string fields. The reflector $\langle R \rangle$ is defined up to an overall normalization by the overlap conditions

$$\langle R \rangle \left( X^{i(1)}(\sigma) - X^{i(2)}(\pi - \sigma) \right) = 0,$n \geq 1
\langle R \rangle \left( P_j^{(1)}(\sigma) + P_j^{(2)}(\pi - \sigma) \right) = 0.$$

(2.12)

Since the ghost part of the reflector remains unchanged even in our case, we will focus on only the matter part of it. This will also be the case later for the three-string vertex. In a case where $\theta = 0$, the matter part of the reflector is thus given by

$$\langle R^x \rangle = (2\pi)^{26} \delta^{26}(p_1 + p_2)_{21} \langle 0 \rangle \exp \left( - \sum_{n \geq 1} \frac{(-)^n}{n} G_{ij} \alpha_n^{(1)} \alpha_n^{(2)} \right),$$

(2.13)

where $21 \langle 0 \rangle$ denotes $2 \langle 0 \rangle_1 \langle 0 \rangle$. We can see that this reflector still satisfies the connection conditions (2.13) even in our case of $\theta \neq 0$. Therefore, using the BRS charge $Q_B$ and the reflector $\langle R \rangle$, we can write the kinetic term of our string field theory. Obviously, this kinetic term does not have any dependence of the theta parameter $\theta^{ij}$.

Now, let us move on to the three-string vertex. The three-string vertex can also be specified up to an overall normalization by the connection equations

$$\langle V_3 \rangle \left( X^{i(r)}(\sigma) - X^{i(r+1)}(\pi - \sigma) \right) = 0, \quad \left( \pi - \frac{r}{2} \right) < \sigma \leq \pi,$n \geq 1
\langle V_3 \rangle \left( P_i^{(r)}(\sigma) + P_i^{(r+1)}(\pi - \sigma) \right) = 0, \quad \left( \pi - \frac{r}{2} \right) < \sigma \leq \pi.$$

(2.14)
where \( r = 1, 2, 3 \) denotes the \( r \)-th string and \( r + 3 \) equals \( r \). Before proceeding to solve these equations, let us consider the three-string vertex with \( \theta = 0 \).

\[
\begin{align*}
\left\langle \tilde{V}_3 \right| (\tilde{X}^i(r)(\sigma) - \tilde{X}^i(r+1)(\pi - \sigma)) &= 0, \quad \left( \frac{\pi}{2} < \sigma \leq \pi \right) \\
\left\langle \tilde{V}_3 \right| (P_i(r)(\sigma) + P_i(r+1)(\pi - \sigma)) &= 0, \quad \left( \frac{\pi}{2} < \sigma \leq \pi \right)
\end{align*}
\] (2.15)

The variables \( \tilde{X}^i(\sigma) \) and \( P_i(\sigma) \) are expressed by the mode expansions of (2.4) and (2.6), and these correspond to strings with the Neumann boundary condition as mentioned above. Therefore, we find that this vertex \( \left\langle \tilde{V}_3 \right| \) agrees with the usual three-string vertex given by

\[
\begin{align*}
\left\langle \tilde{V}_3 \right| &= (2\pi)^{26} \delta^{26} \left( \sum_{r=1}^{3} p^{(r)} \right)_{g21} \langle 0 \left| e_{E123},
E_{123} &= \sum_{m,n \geq 0} \frac{1}{2} N_{mn}^{rs} G_{ij} \alpha_i^{(r)} \alpha_j^{(s)},
\end{align*}
\] (2.16)

where the Neumann coefficients share the same forms as those in the Minkowski spacetime [9,10,11] and \( 321 \langle 0 \left| \) denotes \( 3 \langle 0 \left| 2 \langle 0 \left| 1 \langle 0 \). \)

Using (2.7), (2.4), and (2.15), we can evaluate how the string coordinates \( X^i(r)(\sigma) \) connect with each other on the vertex \( \left\langle \tilde{V}_3 \right| \)

\[
\begin{align*}
\left\langle \tilde{V}_3 \right| (X^i(r)(\sigma) - X^i(r+1)(\pi - \sigma)) &= -\frac{1}{2} \left\langle \tilde{V}_3 \right| \theta^{ij} p_j^{(r+2)}, \quad \left( \frac{\pi}{2} < \sigma \leq \pi \right)
\end{align*}
\] (2.17)

where, due to the momentum conservation on the worldsheet [25],

\[
p_L^{(r)} + p_R^{(r)} = p_i^{(r)}, \quad p_L^{(r+1)} + p_R^{(r)} = 0.
\] (2.18)

are used. From the relation

\[
\left\langle \tilde{V}_3 \right| \left( \sum_{r<s} \theta^{ij} p_i^{(r)} p_j^{(s)} , X^i(t)(\sigma) - X^i(t+1)(\pi - \sigma) \right) = i\theta^{ij} p_j^{(t+2)},
\] (2.19)

the above equation (2.17) leads us to find the three-string vertex with non-zero \( B \)-field \( ^1 \)

\[
\langle V_3 \rangle = \left\langle \tilde{V}_3 \right| \exp \left( -\frac{i}{2} \sum_{r<s} \theta^{ij} p_i^{(r)} p_j^{(s)} \right).
\] (2.20)

\(^1\) A similar expression for the three-string vertex has been discussed in [13].
Thus, the three-string vertex in the background $B$ field can be obtained by multiplying the usual vertex $\langle \tilde{V}_3 \rangle$ by the factor $e^{-\frac{i}{2} \sum \theta^{ij} p^{(r)}_{i} p^{(s)}_{j}}$, which is characteristic of a noncommutative space. Since the BRS charge $Q_B$ can be expressed by the variables $\tilde{X}^i(\sigma)$ and $P_j(\sigma)$ and commute with the zero mode $p_j$ of the momenta, we can see that the three-string vertex satisfies the BRS invariance

$$ (V_3) \sum_{r=1}^{3} Q_B^{(r)} = 0. \quad (2.21) $$

Finally, we find that our string field theory has the following action:

$$ S[\Psi] = \int \left( \frac{1}{2} \Psi \star Q_B \Psi + \frac{1}{3} \Psi \star \Psi \star \Psi \right) $$

$$ = \frac{1}{2} \langle R \mid \Psi \rangle_1 Q_B^{(2)} \mid \Psi \rangle_2 + \frac{1}{3} \langle V_3 \mid \Psi \rangle_1 \mid \Psi \rangle_2 \mid \Psi \rangle_3. \quad (2.22) $$

This $\star$ product is different from the ordinary product by the factor which represents the noncommutativity of space-time, as we have mentioned above. But, except for this factor, the action (2.22) is the same as that of the theory without $B$-field. Note that, in the kinetic term, the ordinary product can be replaced by the $\star$ product due to the momentum conservation. This action can be verified to be invariant under the gauge transformation

$$ \delta \Psi = Q_B \Lambda + \Psi \star \Lambda - \Lambda \star \Psi. \quad (2.23) $$

In the perturbative expansion of this string field theory, if we expand the string field $\Psi$ by its component fields, for example, a tachyon field and a vector field, the product of these component fields in the resulting effective action turns out to be the product of functions on a noncommutative space. Therefore, the low-energy effective theory becomes noncommutative Yang-Mills theory. Also, in this theory, we have the open string metric $G_{ij}$, but not the closed one $g_{ij}$. This is in agreement with the result in [3].

### 3. Background Independence of String Field Theory

As a background independent formulation of string theory, pregeometrical string field theories have been proposed in [13,14], where they dropped the kinetic terms from the actions of the ordinary string field theories on a flat Minkowski space and kept only a cubic term.
If we drop the kinetic term of our string field theory, we may expect the resulting theory to be a pregeometrical theory on the same footing as the theory proposed in [14]. In this section, we will show that this is the case. Although it seems to depend on the theta parameter $\theta_{ij}$, we will find that its background dependence can be absorbed into a redefinition of a string field and that the resulting theory turns out to be the theory in [14]. In addition, we will explicitly show in the oscillator representation that the three-string vertex is also independent of $G_{ij}$. This is an application of the method given by Kugo and Zwiebach [17] to Witten’s open string field theory.

In [17], background independence has been discussed in $\alpha = p^+$ closed HIKKO theory compactified on a torus. Kugo and Zwiebach proposed that $X^i(\sigma)$ and $P_i(\sigma)$ are independent of background fields. We can therefore read the dependence of the oscillators on the background fields, which allows us to explicitly verify in terms of the oscillators that the three-string vertex is background independent.

However, in our open string field theory, the coordinates $X^i(\sigma)$ are no longer universal objects, because the commutation relation of string coordinates itself depends on $\theta$ as in (2.2). What objects should we regard as universal ones? From Eq. (2.4) and (2.7), $X^i(\sigma)$ can be rewritten as

$$X^i(\sigma) = \tilde{X}^i(\sigma) + \int_\frac{\sigma}{2} d\sigma' \theta^{ij} P_j(\sigma') + \frac{1}{2} \theta^{ij} (p_{Lj} - p_{Rj}).$$

Thus, $X^i(\sigma)$ and $P_i(\sigma)$ can be expressed by $\tilde{X}^i(\sigma)$ and $P_i(\sigma)$. Furthermore, their commutation relations

$$\left[\tilde{X}^i(\sigma), P_j(\sigma')\right] = i \delta^i_j \delta(\sigma - \sigma'), \quad [P_i(\sigma), P_j(\sigma')] = 0, \quad \left[\tilde{X}^i(\sigma), \tilde{X}^j(\sigma')\right] = 0. \quad (3.1)$$

have no apparent dependence on background fields. Thus, we propose that $\tilde{X}^i(\sigma)$ and $P_i(\sigma)$ are background independent objects. Namely, under an infinitesimal variation of $G^{ij}$ and $\theta^{ij}$, $\delta \tilde{X}^i(\sigma) = 0$ and $\delta P_i(\sigma) = 0$. Therefore, under the variation, we can obtain the change of the oscillators

$$\delta \alpha^i_n = -\frac{1}{2} G^{ij} \delta G_{jk} \left(\alpha^k_n + \alpha^n_k\right). \quad (3.2)$$

The oscillators $\alpha^i_n$ can be seen to only depend upon the open string metric $G^{ij}$. This means that the theta parameter $\theta^{ij}$ in our theory is only included in the above-mentioned factor of the three-string vertex.
Now, let us consider a purely cubic action with our three-string vertex

\[ S = \int \Psi \star \Psi \star \Psi = 321 \langle V_3 | \Psi \rangle_1 | \Psi \rangle_2 | \Psi \rangle_3. \] (3.3)

If we expand the string field around a classical solution as \( \Psi = Q_L I + \tilde{\Psi} \), we can recover the action Eq. (2.22), as discussed in [15]. Here, \( Q_L \) is the BRS charge density integrated over the left half of a string, and, in terms of the oscillators, \( I \) can be given [10,11] by

\[ |I\rangle = \exp \left( -\sum_{n \geq 1} \frac{(-1)^n}{2n} G_{ij} \alpha^i_n \alpha^j_n \right) |0\rangle (2\pi)^{26} \delta^{26}(p). \] (3.4)

In the following two subsections, we will in turn discuss the dependence of our theory (3.3) on the theta parameter \( \theta^{ij} \) and on the open string metric \( G_{ij} \).

### 3.1. Similarity Transformation of String Fields and the Theta Parameter

The three-string vertex in our theory differs from that in [15] by the noncommutative factor \( \exp[-(i/2) \sum_{r<s} \theta^{ij} p_r^{(r)} p_s^{(s)}] \). Nothing but this factor depends on the theta parameter, as we have seen previously. Therefore, our theory at first seems dependent on the background field \( \theta^{ij} \). If we are dealing with a particle field theory, say \( \phi^3 \) theory, on a noncommutative space, this is true. However, if, in our string theory, we can express the noncommutative factor by a product of operators from each of the three strings, we can eliminate the factor by a redefinition of string fields. Interestingly, this is indeed the case, as we will show below. Therefore, our pregeometrical theory is independent of the theta parameter.

To this end, let us consider the operator \( \sum_{r<s} -(i/2) \theta^{ij} p_r^{(r)} p_s^{(s)} \) on the ordinary three-string vertex \( \langle \tilde{V}_3 \rangle \). This operator can be rewritten as \( \sum_{r=1}^3 (i/2) \theta^{ij} p_{L_i}^{(r)} p_{R_j}^{(r)} \) by using the momentum conservation (2.18) on the vertex \( \langle \tilde{V}_3 \rangle \). Therefore, our three-string vertex \( \langle V_3 \rangle \) can be rewritten as

\[ \langle \tilde{V}_3 \rangle \prod_{r=1}^3 e^{M^{(r)}} = \langle V_3 \rangle, \] (3.5)

where \( M^{(r)} = (i/2) \theta^{ij} p_{L_i}^{(r)} p_{R_j}^{(r)} \).

Since the noncommutative factor can be given by the product of the operators \( e^{M^{(r)}} \) on the three-string vertex, we can eliminate it from the vertex by a redefinition of string fields \( \Psi \rightarrow e^{-M} \Psi \), and we find that our theory turns into the ordinary theory proposed by [15], as we have mentioned before.
Before examining the dependence of our theory on the open string metric, we would like to make some comments about this similarity transformation. When we apply this field redefinition to the theory we have discussed in the last section, we can eliminate the noncommutative factor from the three-string vertex. But this redefinition also affects the kinetic term, and then the BRS charge is transformed into $e^M Q_B e^{-M}$. This transformed BRS charge can be found to have a divergent term in it. Very recently, using an interesting technique, Sugino has argued that the transformed operator indeed remains the original BRS operator $Q_B$ in the kinetic term [21]. This seems to imply that the background $B$ field is physically meaningless. We would like to discuss this puzzle in some detail in section 4.

3.2. Independence of Three-String Vertex from Background Metric

In the string field theory, the reflector and the three-string vertex are defined by the overlap conditions up to an overall normalization. Since the overlap conditions do not include any background fields, we can expect that those vertices are independent of background fields. But we need at least a background metric to concretely construct those vertices in terms of the oscillators. Therefore, it is interesting to examine background independence of the vertices. In this subsection, we will consider the independence of the ordinary three-string vertex $\tilde{V}_3$ from a background metric by using the method given by Kugo and Zwiebach, who applied it to the $\alpha' = p^+$ HIKKO closed string theory. We could also study the background independence by using a general method given by Sen [14].

Before considering the three-string vertex, we will demonstrate the independence of the reflector from the open string metric, as an illustration of the method of [17]. In this subsection, we will focus only on the matter sector.

The Fock vacuum of string fields is defined by $G \langle 0 | \alpha_{-n} = 0$ for $n \geq 1$, where the oscillators $\alpha_n$ depend on the open string metric $G_{ij}$. Thus, the vacuum $G \langle 0 |$ also depends on the metric $G_{ij}$. As we have seen in (3.2), the oscillators change under an infinitesimal variation of $G_{ij}$ by $\delta \alpha_i^n = -\frac{1}{2} G^{ij} \delta G_{jk} \left( \alpha^k_n + \alpha^{-k}_n \right)$. It is useful to introduce an operator

$$B = -\sum_{n \geq 1} \frac{1}{4n} \delta G_{ij} \left( \alpha^i_n \alpha^j_n - \alpha^{-i}_n \alpha^{-j}_n \right),$$

(3.6)

which satisfies $[B, \alpha^i_n] = -\frac{1}{2} G^{ij} \delta G_{jk} \alpha^k_{-n}$. According to the above definition of the Fock vacuum, it is changed under the variation $\delta G_{ij}$ into

$$G + \delta G \langle 0 | = G \langle 0 | - G \langle 0 | B.$$

(3.7)
The part of the reflector relevant to this paper is \( \langle R^x | \sim 21 \langle 0 | e^{E_{12}} \), where \( E_{12} \) is given \[11\] by
\[
E_{12} = -\sum_{n \geq 1} \left( -\frac{n}{n} \right) G_{ij} \alpha_n^i (1) \alpha_n^j (2)
\]
and we will omit the delta function of the zero mode \( p_j \), which, as we have mentioned before, is background independent. By making use of (3.7) and a formula \( \delta (e^{E_{12}}) = [B^{(1)} + B^{(2)}, e^{E_{12}}] \) under the variation, we obtain
\[
\delta \langle R^x | = -\langle R^x | \left( B^{(1)} + B^{(2)} \right).
\] (3.8)

The right-hand side of (3.8) is vanishing, because, on the reflector, the oscillators satisfy
\[
\langle R^x | \left( \alpha_n^i (1) + \left( -\frac{n}{n} \right) \alpha_n^i (2) \right) = 0.
\]

Thus, the reflector is independent of the background \( G_{ij} \).

Similarly, under the variation of the metric, we find the variation of the three-string vertex \( \langle \tilde{V}_3 \rangle \) to be
\[
\delta \langle \tilde{V}_3 \rangle = -321 \langle 0 | e^{E_{123}} \sum_{r=1}^{3} \delta B^{(r)} + 321 \langle 0 | e^{E_{123}} \delta_0 E_{123},
\] (3.9)

where \( \delta_0 E_{123} \) corresponds to the change in the zero-mode parts and is given by
\[
\delta E_{123} = -\frac{1}{2} \sum_{r,s} \bar{N}_{00} \delta G_{ij} \alpha_0^i (r) \alpha_0^j (s) - \frac{1}{2} \sum_{r,s} \sum_{m \geq 1} \bar{N}_{0m} \delta G_{ij} \alpha_0^i (r) \alpha_m^j (s).
\]

The first term of (3.9) can be evaluated to be 321 \( \langle 0 | e^{E_{123}} \) multiplied by
\[
-\frac{1}{4} \sum_{n \geq 1} \sum_{r=1}^{3} \frac{1}{n} \delta G_{ij} \alpha_n^i (r) \alpha_n^j (s) + \frac{1}{4} \sum_{m,l \geq 1} \left( \sum_{n \geq 1} \sum_{t=1}^{3} \bar{N}_{mn} \bar{N}_{nt} \right) \delta G_{ij} \alpha_m^i (r) \alpha_l^j (s)
\]
\[
-\frac{1}{2} \sum_{l \geq 1} \left( \sum_{n \geq 1} \sum_{t=1}^{3} \bar{N}_{0n} \bar{N}_{nt} \right) \delta G_{ij} \alpha_0^i (r) \alpha_l^j (s) - \frac{1}{4} \sum_{r,s} \left( \sum_{n \geq 1} \sum_{t=1}^{3} \bar{N}_{0n} \bar{N}_{nt} \right) \delta G_{ij} \alpha_0^i (r) \alpha_s^j (s)
\]
\[
-\frac{1}{4} \sum_{n \geq 1} \sum_{r=1}^{N} \bar{N}_{nn} \delta G_{ij} G_{ij}.
\] (3.10)

Note that, although our argument is parallel to that in \[17\], the last term of (3.10) is a new term, of which we do not have the counterpart in the closed string case.
To prove the background independence, we can use the identities of the Neumann coefficients \[11,20]\,
\[
\sum_{n \geq 1} \sum_{t=1}^{3} \hat{N}_{mn}^{rt} n \hat{N}_{nl}^{ts} = \frac{1}{m} \delta_{m,l} \delta^{rs}, \quad \sum_{n \geq 1} \sum_{t=1}^{3} \hat{N}_{mn}^{rt} n \hat{N}_{n0}^{ts} = -\hat{N}_{m0}^{rs}, 
\]
\[
\sum_{n \geq 1} \sum_{t=1}^{3} \hat{N}_{0n}^{rt} n \hat{N}_{n0}^{ts} = -2 \hat{N}_{00}^{rs}. 
\]

Note that the second and third equalities need the momentum conservation for the zero-modes, which is guaranteed by the vertex. These identities are proven in appendix B. Therefore, we can see that the second term of (3.9) cancels the first three terms of (3.10). For the last term of (3.10), we need another identity
\[
\sum_{n \geq 1} \sum_{r=1}^{N} n \hat{N}_{mn}^{rr} = 0, 
\]
which is also proved in appendix B. Thus, we can see that the three-string vertex \(\hat{V}_3\) is background independent.

4. Discussion

In this paper, we derived Witten’s open string field theory in a background \(B\) field by using the standard overlap conditions in the operator formalism. The resulting three-string vertex naturally contains an additional factor which gives the Moyal product to the zero modes, compared to the ordinary vertex with no \(B\) field. Thus, the zero modes \(\tilde{x}^i\) can be found to be noncommutative. Besides this noncommutative factor, the three-string vertex can be written by using the open string metric \(G_{ij}\). Therefore, the low-energy effective theory of the gauge field should be described by noncommutative Yang-Mills theory. This result is in agreement with the result in the first-quantization formulation in \[3,12\].

Following the idea of the pregeometrical formulation \[15,16\], we dropped the kinetic term from our string field theory and explicitly demonstrated background independence of the resulting theory by using the method of \[17\].

In order to prove that the three-string vertex is independent of the theta parameter \(\theta^{ij}\), we have shown that the noncommutative factor can be eliminated by field redefinition. If we also apply the redefinition to the theory with the kinetic term, we can easily see that,
besides elimination of the noncommutative factor of the three-string vertex, it also affects the kinetic term and transforms the BRS charge $Q_B$ into $e^M Q_B e^{-M}$. Here the operator $M$ is $(i/2) \theta^{ij} p_L i p_R j$. This transformed BRS charge can be found to have a divergent term in it. This divergent term seems to come from the mid-point $\sigma = \pi/2$ of strings.

Very recently, using an interesting technique, Sugino has argued [21] that the transformed operator indeed remains the original BRS operator $Q_B$ in the kinetic term and that the kinetic term is kept intact under the field redefinition. This seems to imply that the background $B$ field is physically meaningless.

In the paper [3], Seiberg and Witten have shown that the noncommutative Dirac-Born-Infeld (DBI) action is equivalent in the slowly varying field approximation to the commutative DBI action with the background $B$ field. In addition, they emphasized that the $B$ field parallel to $D$-branes cannot be gauged away, due to the gauge invariance $B_{ij} \to B_{ij} + \partial_i \Lambda_j - \partial_j \Lambda_i$, $A_i \to A_i + \Lambda_i$. Here $A_i$ is the gauge field on the D-branes. Therefore, to be consistent with the result in [3], we may expect that, even after the field redefinition, the dependence of the $B$ field remains in the kinetic term of the string field theory and that the $B$ field appears in the low-energy effective action only through the gauge-invariant combination $F_{ij} = B_{ij} + F_{ij}$. However, it apparently seems to conflict with Sugino’s recent result [21]. Thus, we think that we have an interesting puzzle to solve.

To our knowledge, there is no literature which shows that Witten’s open string field theory has the above-mentioned gauge invariance. In addition, it seems difficult to prove it, because the theory has no explicit field from the closed string sector in its action. For this purpose, it may be more suitable to study the gauge invariance and the dependence of $B$ field in an open-closed string field theory with the mid-point interaction in [26].

Apart from the gauge invariance, we would like to discuss the operator $M$ which was used for the field redefinition. If we do not use Sugino’s technique to show that the kinetic term remains intact by the field redefinition, we have to deal with the divergent term in $e^M Q_B e^{-M}$. This divergent term seems to come from the mid-point of strings, as we mentioned above. Since the operator $M$ consists of the half-integrated momenta $p_L, p_R$, we are led to wonder if the singularity may be related to the mid-point interaction. In addition, the operator $M$ seems to be suitable only to the mid-point interaction, because we cannot apply it to the light-cone type interaction. Since the kinetic term of string field theories does not depend on types of interactions of strings, it is desirable to have a
field redefinition which is independent of types of string interactions. Therefore, it may be useful to introduce another candidate

$$\tilde{M} = -\frac{i}{4} \int_0^\pi d\sigma \int_0^\pi d\sigma' \epsilon(\sigma - \sigma')\theta^{ij}P_i(\sigma)P_j(\sigma'),$$  \hspace{1cm} (4.1)$$

where $\epsilon(\sigma)$ is the step function which is 1 for $\sigma > 0$ and $-1$ for $\sigma < 0$. Indeed, rewriting this operator as

$$\frac{i}{2} \theta^{ij} p_{Li} p_{Rj} - \frac{i}{4} \int_\frac{\pi}{2}^\pi d\sigma' \epsilon(\sigma - \sigma')\theta^{ij} \{P_i(\sigma)P_j(\sigma') - P_i(\pi - \sigma)P_j(\pi - \sigma')\}$$  \hspace{1cm} (4.2)$$

and putting the sum $\sum_{r=1}^3 \tilde{M}^{(r)}$ on the usual three-string vertex $\tilde{V}_3$, we can see that the second terms in the operators $\tilde{M}^{(r)}$ cancel each other, due to the overlap condition (2.13). Furthermore, as we will mention just below, this operator $\tilde{M}$ can be used to give the field redefinition to remove the noncommutative factor from the light-cone type interactions.

We have so far been discussing the problem with the mid-point interaction concerning the relation between our string field theory and the ordinary one. However, it is plausible that, if there is such a relation, we can find the same relation in other string field theories. In particular, since we cannot apply the operator $M$ to string field theories with the light-cone type interaction like that given by [22], we can expect that the dependence on the $B$ field cannot completely be removed. Therefore, in order to get some clue to this problem, it may be helpful to study the string field theory with the light-cone type interaction [22] by using the operator $\tilde{M}$. Since this theory explicitly has closed string fields in its Lagrangian as well as open strings, it may also help to find some relation between the condensation of the antisymmetric tensor $B_{ij}$ from the closed string field and the above redefinition of the open string field.

Now, let us just sketch in the main points of our results about the string field theory with the light-cone type interaction [22] in the background $B$ field. These results will be explained in more detail in another paper [27]. As we can verify by the method we have explained in section 2, the light-cone type vertices are also modified to include the noncommutative factor, and there is no other modification due to the background $B$-field. This noncommutative factor can be expressed by a product of the operators $\tilde{M}$ from each string. Since the operator $\tilde{M}$ thus plays an important role, it is useful to make some comments on it.
The commutation relation between the operator $\tilde{M}$ and the string coordinates can be verified to be
\[
[\tilde{M}, X^i(\sigma)] = -\frac{1}{\pi^2 l_s} \sum_{n \neq 0} \frac{1 - (-)^n}{n^2} (\theta G)^i_j \alpha_n^j + \frac{2}{\pi^2} \theta^{ij} p_j \sum_{m \geq 1} \frac{1 - (-)^m}{m^2} \cos(m\sigma)
\]
\[
+ \frac{2}{\pi^2 l_s} \sum_{m \geq 1} \left( \sum_{n \neq \pm m} \frac{1 - (-)^{m+n}}{m^2 - n^2} (\theta G)^i_j \alpha_n^j \right) \cos(m\sigma).
\]

(4.3)

If we naively exchange the order of the summations in the right-hand side of (4.3), we obtain $[\tilde{M}, X^i(\sigma)] = -(\theta G)^i_j Q^j(\sigma)$. Therefore, we would find that $e^\tilde{M} X^i(\sigma) e^{-\tilde{M}} = X^i(\sigma)$. But this consequence must be false because it is inconsistent with the commutation relation $[X^i(\sigma), X^j(\sigma')] \sim \theta^{ij}$, which cannot be changed to be $[\tilde{X}^i(\sigma), \tilde{X}^j(\sigma')] = 0$ by the similarity transformation $e^\tilde{M} X^i(\sigma) e^{-\tilde{M}}$. A closer examination shows that the relation $[\tilde{M}, X^i(\sigma)] = -(\theta G)^i_j Q^j(\sigma)$ holds only for $0 < \sigma < \pi$. Therefore, the transformed BRS operator $e^\tilde{M} Q_B e^{-\tilde{M}}$ naively becomes the ordinary BRS operator with the closed string metric $g_{ij}$, but, due to subtleties from the ends of strings, it has additional terms for which, at least at present, we do not have any interpretations. In spite of the discrepancy between $e^\tilde{M} X^i(\sigma) e^{-\tilde{M}}$ and $\tilde{X}^i(\sigma)$, since the operator $\tilde{M}$ allows us to relate our three-string vertex to the ordinary vertex, regardless of types of string interactions, we are tempted to speculate that the operator $\tilde{M}$ would give us some clue about a relation between the noncommutative string field theory and the ordinary one, like the relation found by Seiberg and Witten [3] between noncommutative DBI theory and commutative one.

Finally, we would like to touch on superstring field theory. We can easily extend our theory to Witten’s superstring field theory [23] by constructing other necessary vertices in a similar way to the bosonic case. In the worldsheet picture of superstring theory, we add to the bosonic sector
\[
S_\psi = -\frac{1}{4\pi\alpha'} \int d^2 z \left( g_{ij} \psi^i \bar{\partial} \psi^j + g_{ij} \bar{\psi}^i \partial \bar{\psi}^j \right),
\]
(4.4)

and the boundary conditions are given by
\[
(g + 2\pi\alpha' B)_{ij} \psi(z) = (g - 2\pi\alpha' B)_{ij} \bar{\psi}(\bar{z}), \quad \text{at } z = \bar{z}.
\]
(4.5)

It is convenient to introduce new fields $\varphi^i(z)$, $\bar{\varphi}^i(z)$ defined by
\[
\varphi^i(z) = G^{ij} (g + 2\pi\alpha' B)_{jk} \psi^k(z)
\]
\[
\bar{\varphi}^i(\bar{z}) = G^{ij} (g - 2\pi\alpha' B)_{jk} \bar{\psi}^k(\bar{z})
\]
(4.6)
These fields play a similar role to the string coordinates $\tilde{X}^i$ in the bosonic case. Since the fields $\varphi^i(z)$ and $\tilde{\varphi}^i(z)$ satisfy the same boundary condition as that with no $B$ field, we have the ordinary mode expansion of these fields $\varphi^i(z)$, $\tilde{\varphi}^i(z)$. Therefore, solving the overlap conditions for these fields, we obtain the ordinary three-string vertex as well as the ordinary reflector. The oscillator expression of the vertices can be found in [28, 11]. From (4.6), we can immediately verify that these vertices also satisfy the overlap conditions for $\psi (z)$, $\tilde{\psi} (z)$. Moreover, the picture changing operators can be expressed by using only $\varphi^i(z)$, $\tilde{\varphi}^i(z)$ and $\tilde{X}^i$, and can be found to have the ordinary expression. Thus, we can extend our string field theory to superstring cases, though we are still faced with the mid-point singularity problem as in [29].

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Appendix A. Operator Formalism of Strings in a Background $B$-Field

We consider the operator formalism of first-quantized string theory in a constant background $B$ field by following the papers [18, 19]. Although Chu and Ho have discussed different methods of quantization in their first and second papers of [18], our strategy is slightly different from both of them; namely, we simplify their methods by combining them.

The worldsheet action is given by

$$S = -\frac{1}{4\pi\alpha'} \int d\sigma \, d\tau \left( g_{ij} \eta^{ab} \partial_a X^i \partial_b X^j - 2\pi\alpha' B_{ij} \epsilon^{ab} \partial_a X^i \partial_b X^j \right), \quad (A.1)$$

Recently, a new approach has been proposed to solve this difficulty [30].
where $g_{ij}$ is a constant background metric and $B_{ij}$ is a constant background antisymmetric tensor. We will also denote $(2\pi\alpha')B_{ij}$ by $b_{ij}$. The signature of the worldsheet metric $\eta^{ab}$ is $(-, +)$ and the invariant antisymmetric tensor $\epsilon^{ab}$ is defined by $\epsilon^{01} = 1$. The equation of motion of the string coordinates is $\partial_a \eta^{ab} \partial_b X^i = 0$. The boundary condition turns out to be
\[ g_{ij}X'^j + 2\pi\alpha' B_{ij}\dot{X}^j = 0 \] (A.2)
at $\sigma = 0, \pi$. The conjugate momenta are given by
\[ P_i = (1/2\pi\alpha')g_{ij}\dot{X}^j + B_{ij}\dot{X}^j, \] where $'$ and $\dot{}$ denote the differentiation with respect to $\tau$ and $\sigma$, respectively. As in the usual way, we can obtain the Hamiltonian density
\[ H = \frac{1}{4\pi\alpha'} \left[ (2\pi\alpha')^2 g^{ij}P_iP_j + (4\pi\alpha')b_{ik}g^{kj}P_j + G_{ij}X'^iX'^j \right]. \] (A.3)
Here, the open string metric $G_{ij}$ is given by
\[ G_{ij} = g_{ij} - (b_g^{-1}b_{ij}). \] The Poisson brackets of the string coordinates and the momenta are all vanishing except $\{X^i(\sigma), P_j(\sigma')\}_P = \delta^i_j\delta(\sigma - \sigma')$.

The idea in the second paper of [18] and in [19] was to deal with the boundary condition (A.2) as a constraint $\phi_i = G_{ij}\dot{X}^j + (2\pi\alpha')b_{ik}g^{kj}\dot{P}_j$ in the operator formalism and to quantize the system by following the Dirac quantization method. By using the consistency condition $\dot{\phi}_i = \{\phi_i, H\}_P \approx 0$ with the Hamiltonian $H = \int d\sigma \mathcal{H}$, all the second class constraints are found [18,19] to be
\[ \frac{\partial^{2n}\phi_i}{\partial\sigma^{2n}}(\sigma) = 0, \quad \frac{\partial^{2n+1}\dot{P}_i}{\partial\sigma^{2n+1}}(\sigma) = 0, \] (A.4)
with $n \geq 0$ at $\sigma = 0, \pi$, and there is no first class constraint. Here, we first solve these constraints (A.4) and find that
\[ \phi_i(\sigma) = -\sum_{n=1}^{\infty} nG_{ij}x^j_n \sin(n\sigma), \quad P_i(\sigma) = \sum_{n=0}^{\infty} p_{ni} \cos(n\sigma). \] (A.5)

Therefore, using $\theta^{ij} = -(2\pi\alpha')(G^{-1}bg^{-1})^{ij}$, we obtain
\[ X^i(\sigma) = \sum_{n=0}^{\infty} x^i_n \cos(n\sigma) + \theta^{ij} \left[ p_{0j}\sigma + \sum_{n=1}^{\infty} \frac{1}{n} p_{nj} \sin(n\sigma) \right]. \] (A.6)

Chu and Ho in their first paper of [18] expressed the string coordinates as a mode expansion in terms of the solutions of the equation of motion and used the invariant symplectic form
\[ \omega = \int d\sigma \left[ -dX^i(\sigma) \wedge dP_i(\sigma) + dP_i(\sigma) \wedge dX^i(\sigma) \right]. \] (A.7)
to find the commutation relations of the modes. Now we apply their idea to our system by using the above mode expansion of string coordinates and the momenta, instead of the mode expansion in terms of the solutions of the equation of motion, to find the commutation relations of \( x_n^i \) and \( p_{nj} \). After this procedure, we will find the equation of motion of these variables \( x_n^i \) and \( p_{nj} \). Substituting the expansions (A.5) and (A.6) into the symplectic form (A.7), we find the Poisson brackets of the modes to be

\[
\{x_0^i, p_{0j}\}_P = \frac{1}{\pi} \delta_j^i, \quad \{x_n^i, p_{mj}\}_P = \frac{2}{\pi} \delta_j^i \delta_{n,m}, \quad \{x_0^i, x_0^j\}_P = \theta_{ij}, \quad (A.8)
\]

and the others are vanishing. Since these variables \( x_n^i \) and \( p_{nj} \) are the solutions of the constraints (A.4), they are all physical variables. Thus, we can obtain the commutation relations of them by the usual prescription \([A, B] = i\{A, B\}_P\) to quantize our system.

Since the time derivative of a physical variable \( O \) can be obtained by \( \dot{O} = \{O, H\}_P \), we can see that \( \dot{\phi}_i(\sigma) = (2\pi \alpha')P_i(\sigma), \quad \dot{p}_i(\sigma) = (1/2\pi \alpha')\phi'_i(\sigma) \). Substituting the mode expansion (A.5) into these equations to get the equations of motion of the variables \( x_n^i \) and \( p_{nj} \) and solving the resulting equations, we find that

\[
x_n^i = i \frac{l_s}{n} (\alpha_n^i e^{-i \tau} - \alpha_{-n}^i e^{i \tau}), \quad p_{nj} = \frac{1}{\pi l_s} G_{jk} (\alpha_k^j e^{-i \tau} + \alpha_{-k}^j e^{i \tau}), \quad (A.9)
\]

with \( l_s = \sqrt{2\alpha'} \), for \( n \neq 0 \). We also have \( x_0^i = x^i + l_s^2 G^{ij} p_j \tau \) and \( p_{0j} = (1/\pi)p_j \). Putting these new mode variables \( \alpha_n^i, x^i, p_i \) into (A.8), we obtain

\[
[x^i, p_j] = i \delta_j^i, \quad [x^i, x^j] = i \theta_{ij}, \quad [\alpha_m^i, \alpha_n^j] = m G^{ij} \delta_{m+n}. \quad (A.10)
\]

These commutation relations are in agreement with the results in [18]. Note here that, if we define new ‘center of mass’ coordinates \( \tilde{x}^i \) by \( \tilde{x}^i = x^i + (1/2)\theta_{ij} p_j \), the coordinates \( \tilde{x}^i \) turn out to be commutative variables; \( [\tilde{x}^i, \tilde{x}^j] = 0 \).

Finally, with the mode variables \( \alpha_n^i, \tilde{x}^i, p_i \), we can express the string coordinates \( X^i(\sigma) \) and the conjugate momenta \( P_i(\sigma) \)

\[
X^i(\sigma) = \tilde{X}^i(\sigma) + \frac{1}{\pi l_s} \theta_{ij} \left[ l_s p_j \left( \sigma - \frac{\pi}{2} \right) + G_{jk} \sum_{n \neq 0} \frac{1}{n} \alpha_n^k e^{-i \tau} \sin(n \sigma) \right],
\]

\[
P_j(\sigma) = \frac{1}{\pi l_s} G_{jk} \sum_{n = -\infty}^{\infty} \alpha_n^k e^{-i \tau} \cos(n \sigma),
\]

where \( \tilde{X}^i(\sigma) = \tilde{x}^i + l_s^2 G^{ij} p_j \tau + l_s \sum_{n \neq 0} (i/n) \alpha_n^i e^{-i \tau} \cos(n \sigma) \) and \( p_j = (1/l_s) G_{jk} \alpha_0^k \).
Appendix B. Identities of the Neumann coefficients

Consider the $N$-strings vertex with a midpoint interaction $\tau_r$. The Neumann function on the strip is given by

$$N(\rho_r, \rho_s') = -\delta_{rs} \left[ 2 \sum_{n \geq 1} \frac{1}{n} e^{-n|\tau_r - \tau_s'|} \cos(n\sigma_r) \cos(n\sigma_s') - 2 \max(\tau_r, \tau_s') \right]$$

$$+ 2 \sum_{m,n \geq 0} \bar{N}_{mn}^{rs} e^{m\tau_r + n\tau_s'} \cos(m\sigma_r) \cos(n\sigma_s').$$

(B.1)

In the case of $\tau_r > \tau_s'$, we find that

$$\frac{\partial}{\partial \tau_r} N(\rho_r, \rho_s') = 2\delta_{rs} \left[ \sum_{n \geq 1} e^{-n|\tau_r - \tau_s'|} \cos(n\sigma_r) \cos(n\sigma_s') + 1 \right]$$

$$+ 2 \sum_{m \geq 1} \sum_{n \geq 0} \bar{N}_{mn}^{rs} e^{m\tau_r + n\tau_s'} \cos(m\sigma_r) \cos(n\sigma_s').$$

The Neumann function and its $\rho$ derivative are continuous at the interaction time $\tau_r = 0$, provided that we use momentum conservation for the zero mode parts. In other words, in order to hold its continuity, it is necessary to multiply the factor $\sum_s p^{(s)}$ to the zero mode terms $2\delta_{rs} \max(\tau_r, \tau_s')$ and $\bar{N}_{n0}^{rs}$.

Using the continuity of the Neumann function, we find the following identity,

$$0 = \sum_{t=1}^{N} \int_0^\pi d\sigma_t'' \sum_{t=1}^{N} \frac{\partial N(i\sigma_t'', \rho_r)}{\partial \tau_t''} N(i\sigma_t'', \rho_s')$$

$$= -2\pi \delta_{rs} \sum_{n \geq 1} \frac{1}{n} e^{n\tau_r + n\tau_s'} \cos(n\sigma_r) \cos(n\sigma_s')$$

$$+ 2\pi \sum_{n \geq 1} \bar{N}_{n0}^{rs} e^{n\tau_r} \cos(n\sigma_r) + 2\pi \sum_{n \geq 1} \bar{N}_{0n}^{rs} e^{n\tau_s'} \cos(n\sigma_s') + 4\pi \bar{N}_{00}^{rs}$$

$$+ 2\pi \sum_{m,l \geq 1} \left( \sum_{t=1}^{N} \bar{N}_{mn}^{rt} \bar{N}_{nl}^{ts} \right) e^{m\tau_r + l\tau_s'} \cos(m\sigma_r) \cos(l\sigma_s')$$

$$+ 2\pi \sum_{m \geq 1} \left( \sum_{t=1}^{N} \bar{N}_{mn}^{rt} \bar{N}_{n0}^{ts} \right) e^{m\tau_r} \cos(m\sigma_r)$$

$$+ 2\pi \sum_{l \geq 1} \left( \sum_{t=1}^{N} \bar{N}_{0n}^{rt} \bar{N}_{nl}^{ts} \right) e^{l\tau_s'} \cos(l\sigma_s').$$
From this identity, we can find (3.11).

To obtain (3.12), we take the limit $\rho_s' \to \rho_r$ in the Neumann function (B.1).

\[
N(\rho_r, \rho_r + \delta) = \ln \delta - \sum_{n \geq 1} \frac{1}{n} \cos(2n\sigma_r) + 2\tau_r \\
+ 2 \sum_{m, n \geq 0} \bar{N}_{nn} e^{(m+n)\tau_r} \cos(m \sigma_r) \cos(n \sigma_r) + O(\delta^2).
\]

From the continuity at the time of interaction, we can obtain (3.12) as follows,

\[
0 = \sum_{r=1}^{N} \int_{0}^{\pi} d\sigma_r \frac{\partial}{\partial \tau_r} N(i\sigma_r, i\sigma_r + \delta) = 2\pi \sum_{n \geq 1} \sum_{r=1}^{N} n\bar{N}_{nn}^{rr},
\]

where we can vanish $\sum_{r=1}^{N} \partial/\partial \tau_r \tau_r$ due to the momentum conservation.

These arguments and identities can also be established for the light-cone type vertices with one interaction time.
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