Topological phases of matter are an exotic phenomenon in modern condensed matter physics, which has attracted much attention due to the unique boundary states and transport properties. Recently, this topological concept in electronic materials has been exploited in many other fields of physics. Motivated by designing and controlling the behavior of electromagnetic waves in optical, microwave, and sound frequencies, topological photonics emerges as a rapid growing research field. Due to the flexibility and diversity of superconducting quantum circuits system, it is a promising platform to realize exotic topological phases of matter and to probe and explore topologically-protected effects in new ways. Here, theoretical and experimental advances of topological photonics on superconducting quantum circuits—via the experimentally demonstrated parametric tunable coupling techniques, including using of the superconducting transmission line resonator, superconducting qubits, and their coupled system—are reviewed. On superconducting circuits, the flexible interactions and intrinsic nonlinearity make topological photonics in this system not only a simple photonic analog of topological effects for novel devices, but also a realm of exotic but less-explored fundamental physics.

1. Introduction

Topology considers the geometric properties of objects that can be preserved under continuous deformations, such as stretching and bending, and quantities remaining invariant under such continuous deformations are denoted as topological invariants.\(^1\)\(^–\)\(^3\) For instance, a 2D closed surface embedded in 3D space has the topological invariant genus which counts the number of the holes within the surface. This topological invariant does not respond to small perturbations as long as the holes are not created by tearing or removed by gluing. Objects which look very different can be equivalent in the sense of topology, and objects with different topologies can be classified into different equivalent classes by their topological invariants. For instance, a sphere and a spoon are equivalent because both of their genus are zero, while a torus and a coffee cup are also topological equivalent as their genus are one.

Historically, the topological concept combined with condensed matter physics through the discovery of the quantum Hall effect (QHE),\(^4\) in which the quantized Hall conductance of 2D electronic materials can be interpreted by the topological Chern number of the filled Landau levels.\(^5\) This exotic phase of matters cannot be explained by the conventional spontaneous symmetry-breaking mechanism. Instead, it should be understood within the framework of the newly-developed topological band theory (TBT).\(^6\)\(^–\)\(^9\) A direct consequence of a nontrivial topological band structure is the emergence of edge state modes (ESMs), that is the bulk-edge correspondence.\(^10\)\(^–\)\(^13\) The existence of the ESMs can be explained by the following simple argument:\(^14\) we put two insulators with different topological band structures together to share a boundary, and away from the boundary the two insulators extend to infinity. Then, these two insulators can not be connected trivially at the boundary region, and thus a topological phase transition has to happen somewhere when the band gaps are closed and reopened, that is there must have ESMs crossing the band gap. The presence of the gapless ESMs can thus be treated as an unambiguous signature of the topological non-trivial band structure of the bulk. The gapless spectra of the ESMs are topologically protected as their existence is guaranteed by the difference of the topologies of the bulk materials on the two sides, and their propagation is robust against perturbations in the materials including disorder and defects.

As TBT mainly considers the band topologies of Hermitian electronic systems in the single-particle picture, the obtained results can be directly applied to interaction-free bosonic systems.\(^15\)\(^–\)\(^17\) However, the practical implementation of the QHE in the condensed matter systems is limited by the presence of impurities or defects that can destroy the topological order. Consequently, the emergence of the ESMs is suppressed, and the gapless spectra of the ESMs is not robust against perturbations. In contrast, the non-Hermitian electronic systems can be used to reproduce the QHE in the condensed matter systems,\(^18\)\(^–\)\(^20\) while the topological order can be experimentally explored through the non-Hermitian systems.

In this article, we review the recent advancements of topological photonics on superconducting quantum circuits, which have been experimentally demonstrated by using the superconducting transmission line resonator, superconducting qubits, and their coupled system. In particular, we focus on the theoretical and experimental advances of topological photonics on superconducting quantum circuits. We discuss the experimental demonstrations of parametric tunable coupling techniques, including using of the superconducting transmission line resonator, superconducting qubits, and their coupled system. We also review the theoretical and experimental advances of topological photonics on superconducting quantum circuits—via the experimentally demonstrated parametric tunable coupling techniques, including using of the superconducting transmission line resonator, superconducting qubits, and their coupled system—are reviewed. On superconducting circuits, the flexible interactions and intrinsic nonlinearity make topological photonics in this system not only a simple photonic analog of topological effects for novel devices, but also a realm of exotic but less-explored fundamental physics.
systems, leading to the emerging research field of topological photonics.[15,16] During the past few years, there are numerous theoretical and experimental explorations of the photonic analogue of the QHE in various artificial photonic metamaterials.[17–20] The motivation lies in both the sides of experimental application and fundamental physics. On integrated optical circuits, the back-reflection induced by disorder and defects is a major hindrance of efficient information transmission. Therefore, it is expected that the unidirectional ESMs, if synthesized, can be used to transmit electromagnetic waves without the back-reflection even in the presence of large disorder. This closely mimics the topologically protected quantized Hall conductance of 2D electronic materials, and such ideal transport may closely mimic the topologically protected quantized Hall conduction of 2D electronic materials, and such ideal transport may offer novel functionalities for photonic systems with topological immunity to fabrication errors or environmental changes. On the other hand, topological photonics also brings new physics of fundamental importance. As the band topology of spinless particles remains trivial as long as the time-reversal symmetry is preserved, effective magnetic fields have to be synthesized for the charge-neutral photons in the metamaterials,[19–21] which is represented by the nontrivial hopping phases on the lattice through Peierls’s substitution. In condensed matter physics, the electrons’ magnetic lengths of the current-achievable strong magnetic field are still far more larger than the lattice spacing.[22,23] Meanwhile, based on the fabrication and manipulation of artificial photonic metamaterials, it is expected to obtain arbitrarily large Aharonov–Bohm phase for hopping photons in a loop containing only few unit-cells, indicating that the synthetic magnetic field for photons can be several orders larger than those for electronic materials. In addition, it should be noticed that the photonic systems is naturally non-equilibrium and bosonic, exhibiting neither chemical potential nor fermionic statistics. This is in stark contrast to the conventional electronic materials. Therefore, from the above two points of view, topological photonics is not only the simple photonic analog of known topological phases of matter, but also a realm of exotic but less-explored fundamental physics.

Here, we focus on reviewing experimental and theoretical advances of topological photonics on superconducting quantum circuits (SQC).[24–29] Including using of the superconducting qubits, superconducting transmission line resonator (TLR), and their coupled system. SQC can be regarded as the realization of atom-photon interaction[30] in on-chip Josephson junction based mesoscopic electronic circuit. It uses the superconducting TLRS[31,32] to replace the standing-wave optical cavities and the superconducting qubits[33,34] to substitute the atoms. Historically, this on-chip superconducting architecture is proposed as a candidate platform for quantum computation. While borrowed a variety of ideas from atomic cavity QED[35] in its early development, SQC has now become an independent research field due to its unprecedented advantages including the strong nonlinearity at the single photon level, the long coherence times, the flexibility in circuit design, the detailed control of atom-photon interaction at the quantum level, and the scalability based on current microelectronic technology.[34–36] Recently, it has been realized that the above mentioned merits are also essential for the simulation of photonic metamaterial in the microwave regime and the quantum simulation of various complicated many-body effects. For the purpose of quantum simulation, individual SQC elements such as superconducting TLRs or qubits play the role of the bosons and/or fermions in lattice models, and the intersite coupling is established through the connection of the SQC elements via various kinds of couplers. From this point of view, a coupler which can support both tunable coupling strength and hopping phase is highly desirable. The former can be exploited to the investigation of the competition between the inter-site hopping and the on-site Hubbard interaction,[37–39] while the latter, being equivalent to the artificial gauge field for the photons on the lattice,[16,40,41] is required to realize non-trivial topological effect, which is our central topic here.

The research of synthesizing gauge fields in artificial models was originated in the ultracold atomic systems,[41–43] while SQC system takes the natural advantages of individual addressing and in situ tunability of circuit parameters.[18,44,35] The other distinct merits of quantum simulation topological phases with SQC are the following aspects. First, the topological state of matters is highly related with the geometry of the lattice. Meanwhile, by taking the advantage of arbitrary wiring, SQC can be used to implement lattice configurations which are difficult for other simulators. Second, the effective strong interaction between photons can be synthesized in a tunable way via many mechanisms, including the electromagnetically induced transparency,[37,44] Jaynes–Cummings–Hubbard nonlinearity,[45] as well as nonlinear Josephson coupling.[46–48] This can be understood by the fact that a superconducting qubit can be treated as a nonlinear resonator in the microwave regime,[49] and thus the topological photonics on SQC can be directly extended to the nonlinear regime, that is, nonlinear topological photonics.[50] In particular, the Hamiltonian of a qubit chain can be described by the Bose–Hubbard model, where the nonlinearities can naturally introduce photonic interactions,[51] and thus interaction induced topological photonics can be possible. As the number of lattice sites grows, the quantum dynamics quickly become intractable even for classical super-computers. Therefore, combining the strong and tunable photonic interaction with the synthetic gauge fields make the SQC system promising in realizing bosonic fractional QHE[52,53] and nontrivial topological ESMs for photons in the microwave regime.[54] Such method can offer facilities in the study of the competition effect between synthetic gauge fields, photon hopping, and Hubbard repulsion. The third point is that, as the control of superconducting devices and their interaction is now very precise and diverse, it is possible to engineer the gain and loss of a system. In this sense, SQC can be exploited to investigate the non-Hermitian induced topological phases which exhibit drastic difference from their Hermitian counterpart.

2. Introduction to TBT

In this section, we briefly review the basic concepts of TBT. Currently there have already been many excellent reviews and tutorials on this subject, see, for example, ref. [8] for the pedagogical tutorial of TBT and ref. [16] for the summary of recent advances in topological photonics.

2.1. The Topological Invariant

Most topological invariants being involved in this review can be understood in the framework of degree of map.[3] For a smooth
map between two closed oriented manifolds with the same dimension, an integer can be defined to characterize the algebraic times an image point is covered. To illustrate this concept, let us take the winding number \( w \) of a 1D closed loop \( M \) parameterized by \( \phi \in [0, 2\pi] \) as an example, see Section 2.3 for the 2D case. As shown in Figure 1, we consider the radical map \( f \) from \( M \) to the 1D unit circle \( S^1 \) parameterized by the azimuthal angle \( \theta \). The rule of \( f \) is that we move each point of \( M \) radically until it strikes \( S^1 \). Since the variation of \( f \) can be equivalently realized by keeping the radical rule but changing the shape of \( M \), the degree of map of \( f \) in this situation is recognized as the winding number of \( w \) of the loop \( M \), which is the Jacobian integration of \( f \) over \( M \),

\[
w = \frac{1}{2\pi} \int_{\phi=0}^{\phi=2\pi} \frac{d\phi}{w} \in \mathbb{Z}
\]  

Here the \( 1/2\pi \) factor indicates the length of the target manifold \( S^1 \). Clearly, \( w \) measure the times that the target manifold \( S^1 \) is covered by the image of \( M \) under \( f \). There is another equivalent discrete form of \( w \). For the loop in Figure 1a, let us choose a point \( q_1 \in S^1 \) and count its inverse images \( p_1, p_2 \), and \( p_3 \). In this counting, every inverse image contributes \( +1 \) or \( -1 \), depending on the direction, that is the sign of the Jacobian of \( f \) at that inverse image. Therefore, \( p_1 \) and \( p_2 \) contribute \( +1 \) each and \( p_3 \) contributes \( -1 \), resulting an overall \( w = 1 \) (see also Figure 1b,c for the \( w = 0 \) and \( w = 2 \) cases). Notice that \( w \) does not depend on the choice of the image point: we can select another image point \( q_2 \), and its single inverse image \( p_4 \) also results in an overall \( +1 \) counting.

The topological nature of \( w \) is manifested by the fact that \( w \) does not alter under the slight deformation of \( M \) as long as the deformation does not hit the origin where the radical rule is ill-defined. We can check the other loops in Figure 1b,c, which have winding numbers 0 and 2, respectively. It is obvious that we can not change their winding numbers until we let some of the point on the loop cross the origin. Such topological robustness is intuitive: as shown in Equation (1), \( w \) is a continuous and integer-valued functional of \( M \). Therefore, under the continuous deformation of \( M \), \( w \) can not change continuously, but has to either remain invariant or change abruptly.

### 2.2. TBT in 1D: SSH Model

A prominent example of closed manifolds in physics is the first Brillouin zone of periodic lattices, and perhaps the most typical and simple model exhibiting nontrivial band topology is the 1D Su–Schrieffer–Heeger (SSH) model describing polyacetylene chain formed by bonded CH groups. As shown in Figure 2, the SSH model describes electrons hopping on a chain with two sublattices and staggered hopping constants. The Hamiltonian takes the form

\[
H_{\text{SSH}} = \sum_n \left[ g_A a^\dagger_{n,A} a_{n,B} + g_B a^\dagger_{n+1,A} a_{n,B} \right] + \text{H.c.}
\]

where \( a_{n,A} \) (\( a^\dagger_{n,A} \)) is the annihilation (creation) operator of the \( n \)th sublattice in the \( n \)th unit-cell, and \( g_A, g_B \) denotes the intra- and inter-unit-cell hopping strength, respectively. In the reciprocal space, \( H_{\text{SSH}} \) has the form

\[
H_{\text{SSH}}(k) = -\mathbf{d}(k) \cdot \sigma, \quad k \in [0, 2\pi]
\]

with \( \mathbf{d}(k) = -(t_1 + t_2 \cos k, t_2 \sin k, 0) \) and \( \sigma = (\sigma^x, \sigma^y, \sigma^z) \) with \( \sigma^{+/-} \) being the Pauli matrices. Thus, \( \mathbf{d}(k) \) defines a closed loop in the \( x - y \) plane, and the Bloch vector \( \mathbf{n}(k) = \mathbf{d}(k)/||\mathbf{d}(k)|| \) of the lower band eigenstate \( |u_-(k)\rangle \) is exactly the radical image of \( \mathbf{d}(k) \).

The SSH model is a one-dimensional system with topological indices and can be classified into two topologically distinct regions: 

- \( |g_A| < |g_B| \): the lower band has a gap and no zero-energy edge modes.
- \( |g_A| > |g_B| \): the lower band has a gap and possesses a pair of zero-energy edge modes.

The existence of the edge states is asymmetric from the dimerized limits: if \( g_A = 0 \) and \( g_B \neq 0 \), both edge states exist, but if \( g_B = 0 \) and \( g_A \neq 0 \), only one edge state exists.

In the two-dimensional case, the SSH chain is a fermionic system that does not have a bulk band gap, as shown in Figure 2, the SSH chain has the form

\[
H_{\text{SSH}}(k) = -\mathbf{d}(k) \cdot \sigma, \quad k \in [0, 2\pi]
\]

with \( \mathbf{d}(k) = -(t_1 + t_2 \cos k, t_2 \sin k, 0) \) and \( \sigma = (\sigma^x, \sigma^y, \sigma^z) \) with \( \sigma^{+/-} \) being the Pauli matrices. Thus, \( \mathbf{d}(k) \) defines a closed loop in the \( x - y \) plane, and the Bloch vector \( \mathbf{n}(k) = \mathbf{d}(k)/||\mathbf{d}(k)|| \) of the lower band eigenstate \( |u_-(k)\rangle \) is exactly the radical image of \( \mathbf{d}(k) \).

The SSH model can be classified into two topologically distinct regions: 

- \( |g_A| < |g_B| \): the lower band has a gap and no zero-energy edge modes.
- \( |g_A| > |g_B| \): the lower band has a gap and possesses a pair of zero-energy edge modes.

The existence of the edge states is asymmetric from the dimerized limits: if \( g_A = 0 \) and \( g_B \neq 0 \), both edge states exist, but if \( g_B = 0 \) and \( g_A \neq 0 \), only one edge state exists.
2.3. TBT in 2D

While the above 1D example is illustrative, historically the combination of the topology concept and the Bloch band theory first brought fruitful results in 2D electronic systems, opening up the rapid growing research field topological insulators.[8] The bulk of topological insulators behave as an insulator while electrons can move along the edge without dissipation or back-scattering even in the presence of defect. The first example of topological insulators is the QHE[4] of 2D electrons in a uniform external magnetic field. It proved out later that the quantized Hall conductance has its root in the nontrivial topology of the bulk band structure.\[57\] A filled energy band below Fermi surface contributes a quantized portion of transverse conductance\[57\]

\[
\sigma_h = \frac{e^2}{2\pi h} C_h
\]

where \( h \) is the band index and \( C_h \) is the corresponding topological Chern number. In particular, \( C_h \) has the form

\[
C_h = \frac{1}{2\pi} \int_{\mathbb{BZ}} \delta^2 k F_{xy}^h(k) \quad (5)
\]

where

\[
F_{xy}^h(k) = i \left[ \left( \frac{\partial u_x(k)}{\partial k_x} \times \frac{\partial u_y(k)}{\partial k_y} \right) - (x \leftrightarrow y) \right] \quad (6)
\]

is the Berry curvature at \( k \) with \( u_i(k) \) being the Bloch function of momentum \( k \) in the \( h \)th band. Such topological invariant is insensitive to the sample sizes, compositions, defects, and local perturbation, and can only be changed through the closing and reopening of the band gaps. Later, it is noted that the emergence of nonzero Chern number is related to the breaking of time-reversal symmetry, and a translational invariant Haldane model with zero net magnetic field is proposed.\[56\]

The 2D Chern number can also be understood in the framework of degree of map. For a two-band system with Hamiltonian \( H_{2D} = -d(k) \cdot \sigma \), the Berry curvature of the lowest band is calculated as\[57\]

\[
F_{xy}(k) = \frac{1}{2|d(k)|} d(k) \cdot \left[ \frac{\partial d(k)}{\partial k_x} \times \frac{\partial d(k)}{\partial k_y} \right]
\]

\[
= \frac{1}{2} n(k) \cdot \left[ \frac{\partial n(k)}{\partial k_x} \times \frac{\partial n(k)}{\partial k_y} \right]
\]

that is, \( F_{xy}(k) \) is the Jacobian of the following map

\[
k = [k_x, k_y] \Rightarrow d(k) = [d_x(k), d_y(k), d_z(k)] \Rightarrow n(k) = \frac{d(k)}{|d(k)|}
\]

from the first Brillouin zone \( T^2 \) to \( S^2 \). In this sense, the integration in Equation (5) counts the times \( T^2 \) wraps around \( S^2 \) under this map. The degeneracy point \( d = 0 \) is equivalent to a fictitious monopole with topological charge 1/2. In this sense, the \( d(k)/|d(k)| \) factor represents the field generated by the monopole, the \( \frac{\partial d(k)}{\partial k_x} \times \frac{\partial d(k)}{\partial k_y} \) factor counts the infinitesimal area on the 2D surface \( d(k) \), and the quantized value of the integration of \( F_{xy}(k) \) coincides exactly with the celebrated Gauss’ law in electromagnetics.

2.4. Topological Photonics

Here, topological photonics means simulating the topological phases on superconducting quantum circuits. However, the transfer from electronics to photonics also imposes several challenges in theory and experiments. An obvious and important issue is how to measure the topological invariants. For electronic systems, the topological invariant of the bands is measured through the quantized Hall conductance in transport experiments. This method is nevertheless not applicable in photonic systems due to the absence of the fermionic statistics. In the optics community, this problem is overcome through the realization of the gedanken adiabatic pumping process proposed by R. B. Laughlin.\[14,58–60\] This method takes the advantage of photonic platforms that the ESMS can be manipulated and visualized in a way much better than in conventional electronic materials. We consider a 2D lattice with a uniform perpendicular magnetic field \( \phi \), as shown in Figure 3, where periodic boundary condition in the y-direction but open edges on the sample in the x-direction are placed. Due to the periodic boundary condition in the y-direction, a thought experiment can be devised in which the...
sample is wrapped in the y-direction into a cylinder, with x being parallel to the axis of the cylinder. Therefore, the uniform magnetic flux \( \phi \) becomes radial to the cylinder and ESMs emerge at the two edges. Through the cylinder and parallel to the x-axis, a magnetic flux \( \alpha \) is inserted. The flux \( \alpha \) can shift the momentum of the ESMs. When one flux quanta are threaded through the Laughlin cylinder exactly, the ESM spectrum returns to its original form with an integer number of ESMs being transferred. This integer is the winding number of the ESMs, which is equivalent to the Chern number of the bulk bands.

3. Basics of Superconducting Quantum Circuits

In this section we briefly introduce the basic concepts of SQC. For the purpose of our review, we put special emphasis on the 2D superconducting TLR and superconducting transmon qubit and the parametric coupling mechanism based on these SQC elements.

3.1. Superconducting TLR

A TLR consists of a coplanar waveguide formed by a center conductor separated on both sides from a ground plane. This planar structure can be regarded as a 2D realization of conventional coaxial cable. The characteristic electromagnetic parameter of the coplanar waveguide are its characteristic impedance \( Z_c = \sqrt{\mu_c/\varepsilon_0} \) and the speed of light in the waveguide \( v_0 = 1/\sqrt{\varepsilon_0\mu_0} \), with \( c_0 \) and \( l_0 \) being the capacitance to ground and the inductance per unit length. Typical values of these parameters are \( Z_c \approx 50 \Omega \) and \( v_0 \approx 1.3 \times 10^{10} \text{ms}^{-1} \). In the continuum limit, the Hamiltonian density in the bulk of the wave guide has the form

\[
H_{TLR} = \frac{1}{2c_0} Q(x,t)^2 + \frac{1}{2l_0} [\frac{\partial}{\partial t} \Phi(x,t)]^2
\]

where \( Q(x,t) \) is the charge density distribution function on the waveguide and the generalized flux \( \Phi(x,t) = \int_{-\infty}^{t} dt' V(x,t') \) is the canonical conjugate of \( Q(x,t) \) with \( V(x,t) \) being the voltage to ground on the center conductor. By using Hamilton’s equations, the wave equation describing the surface plasma propagation along the transmission line,

\[
\frac{v_0}{c_0} \frac{\partial^2 \Phi(x,t)}{\partial x^2} - \frac{\partial^2 \Phi(x,t)}{\partial t^2} = 0
\]

can be derived.

A resonator mode, or a cavity mode, is further established from Equation (9) by imposing boundary conditions at the two end points of the waveguide, which can be achieved by microfabricating a gap in the center conductor. A TLR has a fundamental frequency \( \omega_0 = \frac{\pi v_0}{d} \) and harmonics at \( \omega_n = m \omega_0 \) with \( d \) being the length of the waveguide. Therefore, we get a single-mode \( \lambda/2 \) TLR resonator if we focus only on the lowest mode of the coplanar waveguide. The similar derivation can also be applied to the shorted boundary condition considered in the following section where the two ends are grounded. It can be noticed that such coplanar waveguide geometry is flexible and a large range of
eigenfrequency \( \omega_0 \) can be achieved. Typical SQC experiments use TLR resonators with \( \omega_0 \in [5,15] \text{GHz} \), which is much larger than the thermal temperature in the dilute refrigerator \( \approx 20 \text{mK} \) such that thermal excitation is suppressed, and much smaller than the superconducting energy gap of aluminum or niobium such that the unwanted quasiparticle excitation can be safely neglected. In addition, in experiments it is always advantageous to maximize the internal quality factor of the TLR mode. With current level of technology, the loss of the TLR mode due to its coupling to uncontrollable degrees of freedom in the environment can now be effectively suppressed, and internal \( Q \)-factor in the range \([10^5, 10^6]\) can now be routinely achieved.

3.2. Superconducting Transmon Qubit

The described 2D TLR is a linear circuit. However, nonlinearity is essential to establish two-level qubits in which quantum information is encoded and manipulated. Meanwhile, superconductivity allows the introduction of non-dissipative nonlinearity in mesoscopic electrical circuits as Josephson junction that can be regarded as a nonlinear inductance. For two superconducting metals separated by a thin insulating barrier, a dissipationless supercurrent could flow between them, as described by the relation

\[
I = I_c \cos \varphi
\]

with \( I_c \) being the critical current of the Josephson junction, and \( \varphi \) is the phase difference between the superconducting metals on the two sides of the junction (see Figure 4a). Moreover, the time dependence of \( \varphi \) is related to the voltage across the junction as

\[
\frac{d\varphi}{dt} = \frac{2\pi}{\Phi_0} V
\]

with \( \Phi_0 = \pi \hbar / 2 \) being the flux quantum. Here \( \varphi \) can be associated with the previously introduced generalized flux as

\[
\varphi(t) = 2\pi \Phi(t)/\Phi_0 = 2\pi \int dt' V(t')/\Phi_0.
\]

Therefore, we can regard the Josephson junction as a nonlinear inductance

\[
L_{J}(\Phi) = \frac{\frac{\partial I}{\partial \Phi}}{(2\pi I_c/\Phi_0)^{-1}} = \frac{\Phi_0}{2\pi I_c} \cos \left( \frac{2\pi \Phi/\Phi_0}{(2\pi I_c/\Phi_0)^{-1}} \right)
\]

With the nonlinear Josephson junction inductance, we can build a nonlinear LC resonator by replacing the inductance of a linear LC oscillator with a Josephson junction. In this situation, the energy levels of the nonlinear LC resonator are no longer
equidistant, and we can get an effective two level qubit if we restrict our attention to the lowest two energy levels.

More explicitly, we consider the circuit diagram of a transmon qubit depicted in Figure 4b, which consists of a Josephson junction shunted by a large capacitance $C$. Since the energy stored in the Josephson junction can be written as

$$E = \int IVdt = -E_1 \cos \left( \frac{2\pi}{\Phi_0} \Phi \right)$$  \hspace{1cm} (14)$$

with $E_1 = \Phi_0 I_c/2\pi$ being the Josephson energy of the junction, the quantized Hamiltonian of the capacitively shunted Josephson junction therefore reads

$$H_{\text{Transmon}} = \frac{Q^2}{2C} - E_1 \cos \left( \frac{2\pi}{\Phi_0} \Phi \right) = 4E_C n^2 - E_1 \cos \varphi \hspace{1cm} (15)$$

where $C$ denotes the total capacitance around the transmon qubit, $E_C = e^2/2C$ is the charging energy, and $n = Q/2\pi$ is the canonical momentum of $\varphi$ describing the excess number of Cooper pairs on the metallic island. The spectrum of $H_{\text{Transmon}}$ depends on the ratio $E_1/E_C$. It has been shown that, in the range $E_1/E_C \in [20, 80]$, the energy levels of $H_{\text{Transmon}}$ are insensitive to the perturbation from the low-frequency $1/f$ background charge noise.\(^{[31]}\) In this situation $H_{\text{Transmon}}$ describes a virtual particle with generalized coordinate $\varphi$, generalized momentum $n$, and effective mass $(8E_C)^{-1}$, moving in an anharmonic potential $-E_1 \cos \varphi$. In turn, $H_{\text{Transmon}}$ can have an anharmonic oscillator form

$$H_{\text{Transmon}} \approx \hbar \omega_0 b^\dagger b - \frac{E_C}{2} b^\dagger b b$$  \hspace{1cm} (16)$$

where $b$ ($b^\dagger$) is the annihilation (creation) operator of the anharmonic oscillator, and $\hbar \omega_0 = E_1 - E_0 = \sqrt{8E_C E_f} - E_C$ denotes the energy spacing of the lowest two levels ($E_C$ being the eigen energy of the Fock state $|n\rangle$). The anharmonicity of the resonator is described by the level difference $-E_C = (E_2 - E_1) - (E_1 - E_0)$. Such anharmonicity factor sets the upper speed limit of manipulating the lowest two levels.

In addition, while the critical current $I_c$ depends on the junction size and the material parameters, we can have tunable effective critical current by replacing the single junction by a superconducting quantum interference device (SQUID) loop formed by two identical junctions and penetrated by an external flux $\Phi_{\text{ext}}$, as shown in Figure 4c. In this situation the SQUID can be regarded as a Josephson junction with tunable critical current $I_c(\Phi_{\text{ext}}) = 2I_c \cos(\pi \Phi_{\text{ext}}/\Phi_0)$ with $\Phi_0 = \pi \hbar/2$ being the flux quanta. Such tunability originates from the Aharonov-Bohm interference of the supercurrent flowing across the two junctions. Therefore, replacing the single junction by a SQUID loop yields directly a transmon qubit with flux-tunable eigenfrequency. Moreover, as the flux in the SQUID loop can be tuned with a very high frequency in practice, this additional control knob results in the parametric driving mechanism which we will discuss in the subsequent sections.

### 3.3. Static Couplings

The described TLR resonators and transmon qubits can be connected to form a scaled lattice, and their individual modes can be used to play the role of localized Wannier modes of the lattice. To perform meaningful quantum simulation, one has to induce the coupling between the connected SQC elements. An intuitive idea is to connect them directly by coupling capacitances or inductances. Since the voltages and currents on the SQC elements can be represented by the mode creation and annihilation operators as forms like $V = V_{\text{ZPF}}(a + a^\dagger)$ and $I = I_{\text{ZPF}}(ia - ia^\dagger)$ (ZPF for zero-point fluctuation), this direct coupling mechanism results in exchange type coupling $\approx C_{\text{coupling}} V_1 V_2 = g(a^\dagger b + b^\dagger a)$ between SQC elements. However, it should be noticed that the resulting coupling constant cannot be tuned either in amplitude or in phase. Actually, this kind of coupling results in only static and completely real coupling constants. This is particularly not desirable because complex coupling constants between SQC elements is needed to break the time reversal symmetry and obtain the consequent nontrivial topology. Also, a method exploiting the double electromagnetically induced transparency scheme\(^{[62,63]}\) to realize effective magnetic fields for polaritons in a 2D cavity lattice has been proposed. However, the use of multi-level artificial atoms is still challenging with current experimental technologies.

### 4. Parametric Couplings

As the static coupling is not tunable, we now introduce an alternative method based on the mechanism of parametric frequency conversion (PFC), which can induce coupling between lattice sites with controllable strengths and phases, and discuss a variety of its consequent topological effects. This method is inspired by the laser assisted tunneling technique invented in atomic lattices.\(^{[64]}\) With appropriate modulating frequency imposed on either the couplers between SQC elements or the onsite eigenfrequencies of the SQC elements, effective parametric coupling between SQC elements can be induced, where a hopping phase accompanies during the hopping process, and from the hopping phases, an effective magnetic field on the microwave photons can be induced.

#### 4.1. Parametric Couplings among TLRs

We first consider how to establish the tunable photonic hopping between the lattice sites by using the PFC method.\(^{[65-68]}\) The essential physics can be illustrated intuitively through a toy model of two coupled cavities, assuming $\hbar \equiv 1$ hereafter, with a Hamiltonian of

$$H_{\text{TC}} = H_0 + H_{\text{exc}}^{12}(t)$$  \hspace{1cm} (17)$$

with

$$H_0 = \sum_{i=1}^{2} \omega_i a_i^\dagger a_i$$  \hspace{1cm} (18)$$

$$H_{\text{exc}}^{12}(t) = g_{12}(t) [a_1^\dagger + a_1][a_2^\dagger + a_2]$$  \hspace{1cm} (19)$$
where $a_j/a_j$ and $\omega_j$ are the annihilation/creation operators and the eigenfrequencies of the two cavities, respectively. In addition, the form of $H_{c1}^{12}(t)$ roots from the inductive current-current coupling of two TLRs \(^{68,69}\) and the time-dependent coupling constant $g_{12}(t)$ corresponds to the ac modulation of the coupling SQUID between the TLRs.

Our aim is to implement a controllable inter-TLR photon hopping process. As the two cavities are far off-resonant, the required photon hopping can hardly be achieved if $g_{12}(t)$ is static. Meanwhile, the effective $1 \leftrightarrow 2$ photon hopping can be implemented by modulating $g_{12}(t)$ dynamically as

$$g_{12}(t) = 2J \cos(|\omega_1 - \omega_2|t - \theta_{12})$$

Explicitly, under the assumption of $(\omega_2 + \omega_1) \gg |\omega_2 - \omega_1| \gg J$, the effective Hamiltonian in the interaction picture with respect to $U_0 = \exp[-iH_0 t]$ takes the form

$$H_{\text{eff}}^{12} = U_{0}^{+}H_{\text{ac}}^{12}(t)U_{0}$$

$$= g_{12}(t)[a_1 e^{i\omega_1 t} + a_1^\dagger e^{-i\omega_1 t}] [a_2 e^{i\omega_2 t} + a_2^\dagger e^{-i\omega_2 t}]$$

$$= g_{12}(t)[a_1 a_2^\dagger e^{i(\omega_2 + \omega_1)t} + a_1^\dagger a_2 e^{i(\omega_2 - \omega_1)t} + \text{h.c.}]$$

$$\approx g_{12}(t)[a_1 a_2^\dagger e^{i\omega_2 t} + \text{h.c.}]

= J [a_1^\dagger a_2 e^{i(\omega_2 + \omega_1)t} + \text{h.c.}]$$

$$\approx J a_1^\dagger a_2 e^{i\omega_2 t} + \text{h.c.}$$

with $J$ and $\theta_{12}$ being the effective hopping rate and the hopping phase, respectively. Physically speaking, we can imagine that there is a photon initially placed in the 1st cavity. As $g_{12}(t)$ carries energy quanta filling the gap between the two cavity modes, the photon can absorb the needed energy from the oscillating coupling strength $g_{12}(t)$, convert its frequency to $\omega_2$, and finally jump into the 2nd cavity. During this hopping, the initial phase of $g_{12}(t)$ is adopted by the photon. This hopping process can be further described in a rigorous way by using the rotating wave approximation: in the rotating frame of $H_0$, $H_{c1}^{12}(t)$ reduces to the form in Equation (21), with the fast oscillating $a_1^\dagger a_1 + \text{h.c.}$ term neglected. The essential advantage of the described PFC method is that both the effective hopping strength $J$ and the hopping phase $\theta_{12}$ can be controlled on-demand by the modulating pulse $g_{12}(t)$. In particular, the implementation of the hopping phase $\theta_{12}$ is important as we are using this method to synthesize artificial gauge fields.

From the above derivation, it becomes clear that the importance of realizing this parametric coupling formalism is to realize the tunable coupling constant $g_{12}(t)$. Here we remember that a SQUID can be regarded as a Josephson junction with tunable critical current controlled by the external bias flux. If we can couple the currents from two SQC elements by a SQUID, and if the SQUID is working in the linear region, we can have a tunable coupling between these two SQC elements because in this situation the coupling SQUID can be regarded as a tunable induc-

tance. In the next section we come back to the explicit circuit realization of this idea.

### 4.2. Parametric Tunable Coupling among Qubits

In most quantum information processing tasks, tunable coupling between qubits is of particular importance. However, this tunability is usually achieved at the cost of introducing additional decoherence or circuit complexity.\(^{70-76}\) Alternatively, parametric modulation of qubits’ frequencies can be used to realize this tunability\(^{77-79}\) without coupling devices, and thus simplifies the circuit.

For the parametrically tunable coupling, we here first review the 1D chain case.\(^{78}\) This tunability method can also be directly applicable to the time-dependent tuning\(^{79}\) and 2D\(^{80}\) cases. The system Hamiltonian for $N$ 1D capacitively coupled transmon qubits\(^{33,81}\) is

$$H = \sum_{j=0}^{N-1} \frac{\omega_j}{2} \sigma_j^z + \sum_{j=1}^{N-1} g_j \sigma_j^x \sigma_{j+1}^x$$

where $\sigma_j^z$ are the Pauli operators on the $j$th qubit $Q_j$ with transition frequency $\omega_j$, and $g_j$ is the static coupling strength between qubits $Q_{j-1}$ and $Q_j$, which is fixed. The coupling strength can be fully tuned via modulating the qubits $Q_j$ with $1 \leq j \leq N - 1$ so that the frequencies of the qubits are oscillating as

$$\omega_j = \omega_j + \epsilon_j \sin(\nu_j t + \varphi_j)$$

where $\omega_j$ is the fixed operating qubit-frequency, $\epsilon_j$, $\nu_j$, and $\varphi_j$ are the modulation amplitude, frequency, and phase, respectively. When $\Delta_0 = \omega_{j+1} - \omega_{j-1}$ is $\nu_j$ ($-\nu_j$) for odd (even) $j$, in the interaction picture and ignoring the higher-order oscillating terms, we get a chain of qubits and the interaction among them is the nearest-neighbor resonant XY coupling that is

$$H_j = \sum_{j=1}^{N-1} g'_j \sigma_j^+ \sigma_j^- + \text{H.c.}$$

where $\sigma^\pm = (\sigma^x \pm \sigma^y)/2$ and the effective tunable coupling strengths are

$$g'_j = g_j(\eta_j) \begin{cases} e^{i(\nu_j \epsilon_j / 2)}, & j = 1; \\ J_0(\eta_j) e^{-i(\nu_j \epsilon_j / 2)}, & j \text{ is even}; \\ J_0(\eta_j) e^{i(\nu_j \epsilon_j / 2)}, & j \text{ is odd and } \neq 1, \end{cases}$$

with $\eta_j = \epsilon_j / \nu_j$, $J_m(\cdot)$ being the $m$th Bessel function of the first kind. In this way, the tunability of $g'_j$ is achieved by changing the amplitude $\epsilon_j$ of the modulation to tune $\eta_j$, as experimentally verified in Figure 5 for a two-qubit case.

For tuning the coupling strength via an auxiliary device, the current wisdom is to capacitively couple to two weakly capacitively coupled computational superconducting qubits via a tunable coupler\(^{72}\) which is actually a superconducting qubit, and thus this architecture is easy to scale up. In this scenario, the qubit–qubit coupling will attribute to two paths, one is the original fixed weak capacitively coupling, the other is induced from
the stronger large detuned qubit-coupler interaction, which can be adjustable by tuning the frequency of the coupler. In this way, the total qubit–qubit coupling is still in the form as that of Equation (24), and it can also be fully tuned in a continuous way, even from positive to negative values. Alternatively, the cross-Kerr ZZ-interaction can be obtained for two qubits with large qubit-frequency difference, and, in this case, controlled-phase gates can be implemented. As the total interaction is adjusted from large detuned qubit-coupler interaction, the unwanted qubit interactions can also be completely turned off, and thus high-fidelity quantum operations can be possible.

5. TBT with TLR

In this section, we present the idea of using PFC between TLRs to establish SQC lattice and observe the consequent topological effects.

5.1. TBT in Quasi 1D

We then illustrate explicitly the implementation of the described PFC method in SQC system. In particular, we consider how to synthesize artificial gauge field for the microwave photons with this method, and a variety of the consequent novel topological effects. Our first work in this direction is to investigate the chiral photon flow effect in a TLR necklace, which has been demonstrated experimentally by using three coupler coupled superconducting transmon qubits. As shown in Figure 6a, the proposed circuit consists of three TLRs coupled by three grounding SQUIDs with very large effective Josephson coupling energies and can be described by

\[ H_0 = \sum_{m=1}^{3} \omega_m a_m^\dagger a_m \]  \hspace{1cm} (26)

where \( a_m \) are the annihilation operators of the \( \lambda/2 \) modes of the three TLRs and \( \omega_m \) are the corresponding eigenfrequencies. Here the parameters are specified as \( \{ \omega_1, \omega_2, \omega_3 \} = \{ \omega_0 - \Delta, \omega_0 + \Delta \} \) with \( \omega_0/2\pi \in [10, 15] \text{GHz} \) and \( \Delta/2\pi \in [1, 2] \text{GHz} \). Such configuration is for the following application of the PFC scheme and can be achieved in experiment through the length selection of the TLRs. The structure of the lattice can be explained in an intuitive way: from the point of view of one TLR, its grounding SQUIDs act as shortcuts of its neighbors and itself, as most of the currents from the neighboring TLRs will flow directly to the ground through their common grounding SQUIDs without crossing each other. The separated and localized modes for the TLR lattice can then be approximated by the individual \( \lambda/2 \) modes of the TLRs, as the small inductances of the grounding SQUIDs impose the grounding nodes at the edges of the TLRs.

We turn to the issue of establishing the inter-TLR coupling through the described PFC method. First we take the 1 ↔ 2 coupling as an example. Physically speaking, the common grounding SQUID can be regarded as a tunable mutual inductance between these two TLRs, because currents from these two TLRs flow to the ground through the same grounding SQUID. Therefore, we add an ac flux driving \( \delta \Phi \) to modulate the effective inductance of the 1 ↔ 2 SQUID, such that the parametric coupling Hamiltonian

\[ H_{\text{int},12} = 2 J \cos(\omega_1 - \omega_2)t - \theta_{12}^\dagger a_1 a_2 + \text{h.c.} \]  \hspace{1cm} (27)
can be induced with $J$ being the coupling strength proportional to $\Delta \Phi_{12}$. In the rotating frame of $H_p$ in Equation (26), an effective $1 \leftrightarrow 2$ hopping process described by

$$H_{\text{int},12} = Je^{i\nu_2}a_1^\dagger a_2 + \text{h.c.}$$  \hfill (28)

can then be obtained, which exactly reproduced Equation (21). Moreover, we can add similar pumping pulses on the other two SQUIDs and finally get the effective Hamiltonian

$$H_1 = J_i e^{i\nu_1}a_1^\dagger a_2 + e^{i\nu_3}a_1^\dagger a_3 + e^{i\nu_4}a_i^\dagger a_i + \text{H.c.}$$  \hfill (29)

Here the hopping strength $J$ and the three hopping phases $\nu_1$, $\nu_3$, and $\nu_4$ can be modulated by the amplitudes and the initial phases of the modulating pulses, respectively. In this step, the vector potential $A(x)$ can manifest its presence through the Peierls substitution

$$\theta_\phi = \int A(x) \cdot dx$$  \hfill (30)

and the loop summation of the hopping phases in turn has the physical meaning of the synthetic magnetic flux:

$$\theta_\Sigma = \oint A(x) \cdot dx = \int B(x) \cdot dS$$  \hfill (31)

Figure 7. Chiral photon flow in the TLR necklace. We have considered three situations $\theta_2 = \pi/2$, $\pi$, and $3\pi/2$, and depict the corresponding results in the three panels from top to bottom. In addition, we set the homogeneous coupling strength $g/2\pi = 20$ MHz and decay rate $\kappa/2\pi = 250$ kHz. Reproduced with permission. Copyright 2015, The Authors. Published by Springer Nature Limited.

The synthetic magnetic field leads to the chiral photon flow in this necklace, which is the photonic counterpart of the Lorentz circulation of electrons in an external magnetic field. We assume initially a photon is populated in the 1st mode and numerically simulate its subsequent time evolution by using the master equation approach. Results related with the three situations $\theta_\Sigma = \pi/2$, $\pi$, and $3\pi/2$ are plotted in Figure 7. From the first panel corresponding to $\theta_2 = \pi/2$, we can observe the clear temporal phase delay of the energy population in the three TLRs. Such pattern implies that the photon flow on the lattice is unidirectional, first from TLR 1 to TLR 2 and then from TLR 2 to TLR 3. This chiral character is a consequence of the time-reversal symmetry breaking in this necklace. Here we should emphasize that the chiral character of the photon flow survives although we have chosen the cavity decay rate $\kappa$ much stronger than reported experimental data during this numerical simulation. Similarly, in the third panel which corresponds to the opposite magnetic field $\theta_2 = 3\pi/2$, the chiral photon flow is in the opposite direction. Meanwhile, in the second panel corresponding to the trivial case $\theta_2 = \pi$, the energy transfer exhibit symmetric patterns.

Meanwhile, we can notice that our discussions up to now are mainly restricted to the synthesis and the effect induced by Abelian background gauge field. On the other hand, non-Abelian gauge field, in particular the spin-orbital coupling (SOC) mechanism, has already played important roles in the physics of topological quantum matters. In recent year, synthetic SOC has been experimentally implemented in artificial systems including ultra-cold atoms, exciton-polariton microwaves and coupled pendula chains. Therefore, curious questions arise that whether the proposed PFC method can be extended to synthesize non-Abelian gauge in the SQC architecture, and whether the synthetic non-Abelian gauge field would bring any new physics that can be observed in the near future.

These two issues have been addressed in our recent work in ref. [89] where we derived explicitly that the PFC method can be extended to synthesize U(2) non-Abelian gauge fields in coupled TLR lattices, as shown in Figure 8. The detailed scheme is similar to the described PFC scheme for the Abelian gauge field. For the lattice sites, we merely need to incorporate the $\lambda$ modes of the TLRs into consideration such that the $\lambda/2$ and the $\lambda$ modes of the individual TLRs play the role of the pseudo-spin components. As the lattice sites become multi-component, the linking variables $U_{ij} = \text{exp}(\int_0^L \text{d}x \cdot A(x))$ become matrix-valued as the non-Abelian background potential $A(x)$ is matrix-valued. The establishment of $A(x)$ in turn becomes the establishment of each of the hopping branch defined by the matrix elements of $U_{ij}$. Thus, the generalization of the PFC method to the synthesis of the non-Abelian gauge field is straightforward: for TLRs coupled by common SQUIDs, we only need to guarantee that their eigenfrequencies are far off resonant, see Figure 8d, such that a multi-tone modulation can be applied to manipulate each hopping branch in the linking variables independently.

The physics of Aharonov–Bohm (AB) caging in a periodic 1D rhombic lattice is sketched in Figure 8a with its TLR lattice realization shown in Figure 8b. Here each lattice site consists of $N$ (pseudo)spin modes. When exposed to an U(N) background gauge field $A$, the Hamiltonian of the lattice takes the form

$$H = \sum_{n,m} \kappa_n U_{nm} \beta_m$$  \hfill (32)

where $\kappa_n = [\kappa_{n,1}, \kappa_{n,2}, \ldots, \kappa_{n,N}]$ is the vector of the annihilation operators of the $n$th site in the $n$th unit-cell, and $U_{nm} = \text{exp}(\int_0^L \text{d}x \cdot A(x))$. We then define the non-Abelian AB caging by the nilpotency of interference matrix

$$I = \frac{1}{2}(U_j U_i + U_i U_j)$$  \hfill (33)
with $U_1$, $U_2$, $U_3$, and $U_4$ being the rightward link variables labeled in Figure 8a. For the Abelian situation $N = 1$, AB caging coincides with the $\pi$ magnetic flux penetration in each loop of the lattice. In this situation, a particle initially populated in the site $[n, A]$ cannot move to sites further than $[n \pm 1, A]$ due to the destructive interference of the two up and down paths shown in Figure 8a. For the non-Abelian situation $N > 1$, the matrix feature of $A$ offers much more rich physics: A matrix-formed nonzero interference matrix $I$ can be nilpotent. To illustrate this idea, let us take an $U(2)$ design as the minimal realization of the proposed non-Abelian AB caging. The link variables are set as

$$U_i = U_i = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad U_2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad U_3 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

such that a nilpotent

$$I = \frac{1}{2}(U_2 U_1 + U_4 U_3) = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

can be achieved. That is, an excitation in the $[n, A, 1]$ mode can move only to the $[n + 1, A, 2]$ mode in a rightward way, but then it can move to neither the $[n + 2, A]$ nor the $[n - 1, A]$ sites. Similarly, an excitation in the $[n, A, 2]$ mode can move to the $[n - 1, A, 1]$ mode, but it cannot arrive the $[n - 2, A]$ and $[n + 1, A]$ sites. This is due to the fact that the non-Abelian feature of the synthetic gauge field leads to exotic asymmetric spatial configuration of excitations, which is determined by both the nilpotent power $I$ and the initialization of the photon. These exotic new features stem from the non-Abelian nature of $A$ and have no Abelian analog.

**5.2. TBT in 2D**

With the developed technique of synthesizing gauge fields by the PFC method, we further consider the quantum simulation of integer QHE and the consequent measurement of integer topological quantum number in a 2D photonic lattice. For this aim, we propose a square lattice consisting of TLRs with four different frequencies, placed in a square lattice form and grounded by SQUIDs at their common ends, as shown in Figure 9a.

**Figure 8.** Implementation of a quasi 1D AB caging model. a) Illustration of the model in a periodic rhombic lattice. b) The extension of (a) to a two component non-Abelian case and its implementation with TLR lattice, where the TLRs with three different lengths are connected at the common ends by the grounded coupling SQUIDs. c) Illustration of the effective $[n, \alpha, \sigma_1] \leftrightarrow [m, \beta, \sigma_2]$ coupling induced by the PFC via the ac modulation of the coupling SQUID. d) Spectrum and coupling of two neighboring TLR sites, with $\delta = \omega_{\beta} - \omega_{\alpha}$ being the frequency difference of the neighboring $[n, \alpha]$ and $[m, \beta]$ TLRs. Reproduced with permission. [89] Copyright 2020, American Physical Society.

**Figure 9.** a) Sketch of the TLR square lattice simulator of IQHE. The four TLRs building the lattice are labeled by the four different colors and the grounding SQUIDs are labelled by the black dots. The effective hopping between the TLRs is established through the PFC modulation of the SQUIDs. b) Geometric configuration of the proposed lattice. c) Chiral property of the ESMs located at the inner and outer edges. The lattice sketched in (b) can be regarded as the gluing of a simply-connected plane by its two sides (the upper panel). Such gluing process can intuitively interpret a variety of opposite properties of the inner and the outer ESMs, with the opposite chiral property being the prime example (the arrows in the lower panel). Reproduced under terms of the CC-BY license. [60] Copyright 2016, The Authors, published by Springer Nature Limited.
to ref. [82] the eigenfrequencies of the four kinds of TLRs are specified as \[\omega_0, \omega_0 + \Delta, \omega_0 + 3\Delta, \omega_0 + 4\Delta\], respectively. In each of the grounding SQUIDs, a developed three-tone PFC pulse is applied, such that the hopping strength \(H\) and hopping phase of \(J\) can be independently controlled. Such in situ tunability can in turn lead to the synthesization of the following artificial gauge field for microwave photons on the lattice

\[
A = \left[0, A_y(x), 0\right] \\
B = B e_z = \left[0, 0, \frac{\partial}{\partial x} A_y(x)\right]
\]

For the aim of measuring the integer topological invariants of this system, a nontrivial ring geometry is further endowed to the TLR lattice, that is, we consider an \(N_x \times N_y\) square lattice with an \(n_x \times n_y\) vacancy at its middle, as shown in Figure 9b. An uniform magnetic flux \(\phi\) in each plaquette of the lattice and an extra magnetic flux \(\alpha\) at the central vacancy can be penetrated through the careful setting of the hopping phases of the hopping branches. This configuration closely mimics the previously discussed Laughlin cylinder shown in Figure 3.

Following the bulk-edge correspondence principle, \(^{[14]}\) the ESMs of the lattice provides an efficient handle to probe the topological invariants of the bulk band. Explicitly speaking, the ESMs located in the band gap between the \(h\)th and the \((h + 1)\)th bands can be characterized by the topological winding number \(w_h\), which can be related to the Chern number \(C_h\) of the \(h\)th band as

\[
C_h = w_h - w_{h-1}
\]

Therefore, the measurement of the Chern numbers \(C_h\) of the bulk bands is equivalent to the measurement of the winding numbers \(w_h\) of the ESMs. For the rational magnetic flux situation \(\phi/2\pi = p/q\) with \(p, q\) being co-prime integers, the topological winding numbers \(w_h\) of the ESMs are given by the Diophantine equation

\[
h = s_h q + w_h p, \quad |w_h| \leq q/2
\]

with \(h\) being the gap index and \(w_h, s_h\) being integers.

Since the spatial configuration of the proposed lattice is equivalent to the Laughlin cylinder shown in Figure 3, we can realize the adiabatic Laughlin pumping of the ESMs through controlling the central vacancy flux \(\alpha\). By increasing \(\alpha\) monotonically from 0 to \(2\pi\), we can observe that the spectrum of the lattice varies and returns finally to its original form, with an integer number of ESMs been transferred. This integer is exactly the winding number \(l_h\) of the ESMs. In Figure 10a,d, the ESMs spectrum versus \(\alpha\) is numerically simulated for \(p/q = 1/4\). The guidance of the dashed white lines and the solid yellow arrows clearly indicates the movements of the ESM peaks in the 1st and 3rd gaps, which is in agreement with the calculated topological winding numbers \(w_1 = -w_3 = 1\). A similar situation \(\phi/2\pi = 1/5\) is also calculated and shown in Figure 10b,e, where we observe that the ESMs in the 2nd and 3rd gaps cross two peaks with opposite moving directions during the whole pumping process. This result is in accordance with the calculated \(w_2 = -w_4 = 2\). In addition, Figure 10c,f indicate that the ESMs in the inner and outer edges take the opposite moving directions during the pumping process. In summary, this measurement scheme of the topological invariants can be summarized as follow: during the Laughlin pumping process, we only need to follow the movement of the ESM peaks, as ESMs in the \(h\)th gap will move by \(\left|w_h\right|\) peaks, with the moving direction determined by the sign of \(w_h\). From the obtained \(w_h\), we can

Figure 10. Laughlin pumping in the proposed square lattice. Here the uniform magnetic field is set as \(\phi/2\pi = 1/4\) for (a,c,d,f) and \(\phi/2\pi = 1/5\) for (b,e). In each of the subfigures, the panels correspond to \(\alpha/2\pi = 0, 1/4, 1/2, 3/4, \) and 1 from top to bottom, respectively. For (a,b,d,e), the pumped ESMs are located at the outside edge of the lattice, while for (c) and (f) the pumped ESMs are located at the inner side edge. Reproduced under terms of the CC-BY license. \(^{[60]}\) Copyright 2016, The Authors, published by Springer Nature Limited.
directly calculate the Chern numbers $C_h$ following the relation Equation (37).

The extension of the discussed chiral photon flow on the necklace is the coherent chiral photon flow dynamics of the ESM on the lattice. Such investigation can offer an intuitive insight into the chiral property of the ESMs. We assume that we initialize the lattice in its ground state and then add a driving pulse on an edge site $r_x$ with the pumping frequency $\Omega_{\text{sp}}$ being an ESM eigenfrequency. Then we numerically simulate the time evolution of the lattice and sketch several screenshots of the photon flow dynamics in Figure 11, from which we can observe the unidirectional photon flow around the edge and the direction of the chiral photon flow are determined by both $\Omega_{\text{sp}}$ and $\Omega_{\text{sp}}^*$. If we choose $r_x$ on the outside edge, we can find out that the chiral photon flow is clockwise/counterclockwise if we choose $\Omega_{\text{sp}}$ in the 1st/3rd gap (Figure 11a,c). The different chiralities of the ESMs in different energy gaps can be understood by regarding the chiral photon flow as a macroscopic rotating spin, if we place such a rotating spin in a magnetic field $B$, its energy will split according to its spinning direction. Moreover, from Figure 11a,b (also Figure 11c,d) we can observe that the inner and outer ESMs in the same gap have opposite chiral properties. This can be explained by the spatial topology of the proposed lattice. As shown in the upper panel of Figure 9c, if we tear the lattice apart, we can get a simply-connected plane with only one connected edge. Meanwhile, the inverse of the tearing, that is, the gluing of the two sides marked by dashed lines, cancels the ESM flow (marked by the arrows) on the glued sides, and results in finally two closed circulations located at the inner and outer edges with opposite circulating directions. As each of the hopping branches can be tuned in a site resolved way by the proposed PFC formalism, we can expect the future experimental demonstration of this “tearing-and-gluing” process.

The influence of disorder and defect on the chiral flow is also calculated. In this numerical simulation, we add Gaussian distributed diagonal and off-diagonal disorder with $\sigma(\delta h_0) = \sigma(J_{r'}) = 0.05J$. In addition, we place a $2 \times 2$ hindrance on the upper outer edge with $\delta h_{12;13,21;24}/J = 30$. In this situation, the topological robustness of the ESMs is clearly verified by the survival of the chiral photon flow displayed in Figure 11e.

5.3. Extension: Parametric Amplification

A direct extension of the proposed PFC method is the parametric amplification (PA) process. Compared with the described synthetic gauge field issue, this process is unique for photonic systems. We come back to Equation (20). Now we modulate $g_{12}(t)$ by the summation tone of the two resonators:

$$g_{12}(t) = 2J\cos[(\omega_1 + \omega_2)t]$$

(39)

Then we can get an effective nondegenerate parametric amplification Hamiltonian

$$H_{\text{eff}}^{12} = e^{i\Delta\phi}H_{\text{osc}}^{12}(0)e^{-i\Delta\phi}$$

$$\approx g_{12}^2\Delta_1^2t^2 + h.c.$$  

(40)

which takes the similar form of the fermionic p-wave pairing. From this point of view, the road map of simulating 1D fermionic topological superconductor models is clear: The lattice sites are built by hard-core boson, for example TLRs with strong nonlinearity or superconducting transmon qubits such that the fermionic statistics can be induced by the Jordan–Wigner transformation, while the hopping and pairing terms between the lattice sites are induced by the PFC and parametric amplification modulation of the coupling grounding SQUIDs, respectively. With further modified dispersive amplification modulation method, this architecture can be even exploited to simulate spinful fermionic models. Actually, this idea has already been illustrated in ref. [95] where we proposed a 1D hard-core boson lattice built by superconducting transmon qubits as a faithful simulator of the following fermionic time-reversal invariant DIII model:

$$H_{\text{DIII}} = -\mu(c_{j,\uparrow}^\dagger c_{j,\uparrow} - 1)$$

$$- \sum_{j<}\left(g_{j,\uparrow}^* c_{j+1,\downarrow} + i\Delta c_{j,\downarrow} c_{j,\uparrow} + h.c.\right)$$

(41)

where $c_{j,\sigma}$ is the creation operator of the spin-$\sigma$ fermion on the jth site, $g$, $\mu$, $\Delta$ denote the real-value hopping strength, the chemical potential, and the amplitude of p-wave pairing parameter. The nontrivial coupling between the lattice sites is induced through...
the inductive connection between transmon qubits. In particular, we developed a generalized parametric amplification approach to facilitate the exotic U(1) gauge factor emerged in the Jordan–Wigner bosonization. Furthermore, we proposed that the odd-parity Kramers doublet ground states can be used as the basis of topological qubits, and with these topological bases the universal topological quantum gates can be implemented in principle.

6. TBT with Superconducting Qubits

6.1. TBT in 1D

With the above tunable coupling among qubits, we are readily in the position to simulate topological quantum phases on superconducting quantum circuits. The first example is the 1D SSH model with the Hamiltonian of Equation (2), introduced in Section 2.2 and it can be realized by parametrically tuning the inter-qubit coupling strength in Equation (24) in the required dimerized form. Note that, although the Hamiltonian in Equation (24) is of the bosonic nature, in the single excitation subspace, its topology is equivalent to that of the SSH model, which has two different topological phases with different winding numbers. Specifically, when the qubit couplings are tuned into \( g_A < g_B \) (\( g_A > g_B \)) configuration, with \( g_A \equiv g'_{J+1} \) and \( g_B \equiv g'_{J+2} \), in Equation (24), the winding number will be 0 (1), and then the system is in a topologically trivial (nontrivial) state. Then, by monitoring the quantum dynamics of a single-qubit excitation in the chain, the associated topological quantities can be inferred.

6.2. TBT in 2D

The gauge effect is essential in investigating novel phenomena in modern physics, for example, for exploring exotic quantum many-body physics. In this subsection, we review two schemes to synthesize gauge field by introducing ac modulation on superconducting circuits, and the motivation is to obtain an extremely strong effective magnetic field for bosons, which is hard in conventional solid-state systems. While these pioneering examples are associated with the chiral effect, novel quantum many-body properties induced by the interplay of the synthetic gauge field, bosonic hopping, and interaction of bosons are also deserve further exploration.

We consider a two-leg superconducting circuit consisting of superconducting transmon qubits, as shown in Figure 12a. Extending the parametric coupling in Equation (24) to this two-leg case, the considered model Hamiltonian here is [80]

\[
\hat{H} = \sum_{\nu} \left( t_{\nu} e^{i \varphi_{\nu}} \hat{a}_{\nu} \dagger \hat{a}_{\nu} + \text{H.c.} \right) + \sum_{j} \left( \tilde{t}_{j} e^{i \varphi_{j}} \hat{a}_{j} \dagger \hat{a}_{j} + \text{H.c.} \right)
\]

where \( j \) denotes the number of the rung, \( \nu \in \{ A, B \} \) labels the leg, \( \hat{a}_{\nu} \) (\( \hat{a}_{\nu} \dagger \)) is the creation (annihilation) operator for the \( j \)th site on the \( \nu \)-th leg, \( \varphi_{\nu} = (1)^{j+1} \varphi_{\nu} + \pi / 2 \), \( t_{\nu} = g_{\nu} J_{0}(n_{\nu-1}) J_{1}(n_{\nu}) \), and \( \tilde{t}_{j} = g_{j} J_{0}(n_{\nu}) J_{1}(n_{\nu}) \) with \( g_{\nu} \) is the original static hopping strength between the nearest-neighbor sites along the \( \nu \)-th leg, \( g_{j} \) is the static inter-leg coupling strength at the rung \( j \). In this two-leg lattice model, the non-zero magnetic flux case supports a chiral current follow within the lattice, which is plotted in Figure 12b, c) and Figure 12d, e) single- and two-excitation cases. The solid line in (d) is plotted from the analytical result. Adapted with permission. Copyright 2020, American Physical Society.

Figure 12. The chiral currents for a two-leg quantum circuit. a) Schematic diagram of the two-leg lattice model with an synthetic magnetic flux per plaquette. The blue and red solid circles represent transmon qubits at the A and B legs. The chiral currents between neighboring sites in the single-excitation case for b) \( \varphi = 0.1 \pi \) and c) \( \varphi = 0.9 \pi \) with \( N = 50 \), where the thicknesses of the arrows represent their strengths and the sizes of the circles denote their local densities. The ground-state chiral current as a function of \( \varphi \) for d) single- and e) two-excitation cases. The solid line in (d) is plotted from the analytical result. Adapted with permission. Copyright 2020, American Physical Society.

7. TBT with Dressed State Systems

The above discussed quantum simulation examples on superconducting circuits are limited to the spinless cases, as it is difficult
to engineer the spin degree of freedom in either TLR or qubit systems. To simulate spinful systems, one needs to introduce more degrees of freedom. One of the possible platform is the TLR-qubit coupled system, a typical circuit QED setup, and the dressed-state or polariton of which can mimic the spin degree of freedom. In this platform, the synthetic polaritonic SOC and Zeeman field can be induced simultaneously with fully an in situ tunability and thus it provides a flexible platform to explore topological physics.

### 7.1. The Dressed-State Qubits

We begin with implementing the dressed-state on a conventional circuit QED setup, where a superconducting qubit of the transmon type is capacitively coupled to a 1D TLR. The interaction Hamiltonian is in the Jaynes–Cummings (JC) form of

\[
H_{JC} = \omega_{n}|1\rangle\langle 1| + \omega_{a}a^\dagger a + g(a\sigma^+ + a^\dagger \sigma^-)
\]

(43)

where \(\omega_{n}\) and \(\omega_{a}\) are the frequencies of qubit and cavity, respectively; \(|n\rangle\) labels the Fock state of the TLR, the ground state of which is set to be \(E_g = 0\). As shown in Figure 13a, for \(n \geq 1\), due to the TLR-qubit interaction, the twofold eigenstates for a certain \(n\) are

\[
|\rightarrow, n\rangle = \cos \alpha_n|g, n\rangle - \sin \alpha_n|e, n-1\rangle
\]

(44a)

\[
|\rightarrow, n\rangle = \sin \alpha_n|g, n\rangle + \cos \alpha_n|e, n-1\rangle
\]

(44b)

and corresponding eigenvalues are

\[
E_{\rightarrow \leftarrow} = n\omega_{n} + \frac{1}{2} \left( \delta \pm \sqrt{\delta^2 + 4n\omega_n^2} \right)
\]

(45)

where \(\tan(2\alpha_n) = 2g\sqrt{n}/\delta\) with the detuning \(\delta = \omega_{n} - \omega_{a}\). Note that the transition frequencies among eigenstates are different, and thus selective transition between two target states may be achieved. The three lowest eigenstates are \(|G\rangle, |\rightarrow, 1\rangle\), and \(|\rightarrow, 1\rangle\), which form a V-type artificial atom, as shown in Figure 13a. For the sake of simplicity, we denote the polaritonic states \(|\rightarrow, 1\rangle\) and \(|\rightarrow, 1\rangle\) as \(|\uparrow\rangle\) and \(|\downarrow\rangle\) in the following, mimick a spin 1/2 system, or a spin qubit. Compared with conventional superconducting qubits, this combined definition of qubit has better stability against low-frequency noises.

### 7.2. SOC for Polariton Spin

The described parametric coupling scheme for the microwave TLRs can also be used here with the inclusion of a transmon qubit in each TLR. As the dressed qubits are of the half-TLR plus half-transmon nature, the effective hopping among them can be induced by only coupling the photonic component. Here, we present an example with two dressed qubits case, that is, A- and B-type TLR-qubit coupled system, which are different as they have different eigen-frequencies \((\omega_{a}^g + \omega_{a}^b)\) and coupling strengths \((g_{a}^g + g_{a}^b)\) in the JC Hamiltonian of Equation (43) setting \(\delta = 0\). The energy spectrum and their transitions are shown in Figure 13b when \(\omega_{a}^g - \omega_{a}^b = \Delta > g_{a}^g + g_{a}^b\). The transitions can be induced with the parametric photonic coupling, and the addressing of each transition is obtained when the frequency of the coupling strengths between TLRs, see Equation (20), meet the resonant condition of a target transition. Meanwhile, the energy difference of the four different hoppings are \(2g_{a}^g\) or \(2g_{a}^b\), which can be set to be much larger than the effective hopping strength between the two polariton states, to ensure the selective frequency addressing of each transition. Therefore, the fully tunable spin-preserved hopping and SOC terms can be induced in a same setup, and thus this element can naturally be scaled up into different types of lattices to study virous SOC topological states.

### 7.3. TBT in 1D

We first proceed to present a spin-1/2 chain with adjustable SOC and Zeeman field that can be simulated with the above JC lattice. We set the A- and B-types units to be arranged in an alternate way, as shown in Figure 14a. Then, the energy differences of the four hopping are also set to be much larger than the effective hopping strength to ensure the selective frequency addressing of the target transitions. Moreover, a detuning \(2m\) to the spin-flipped transition tunes, as shown in Figure 14b, can induce a tunable spin splitting \(m\) for each cell, in the rotating frame, as shown in Figure 14c. In this way, neglecting the fast oscillating terms, we obtain

\[
H'_{JC} = \sum_{l=1}^{N} mS_{l}^{z} + \sum_{\ell=1}^{N-1} \sum_{x,\ell} \left( t_{l,\ell,x} e^{i\phi_{l,\ell,x}} \hat{c}_{l,x} \hat{c}_{\ell+1,x} + \text{h.c.} \right)
\]

(46)

where \(S_{l}^{z} = |\uparrow\rangle\langle \uparrow| - |\downarrow\rangle\langle \downarrow|\) is the creation operator for spin-\(\ell\) polariton state at the \(l\)th unit cell, and \(t_{l,\ell,x}\) is the effective hopping strength. Therefore, the Hamiltonian in Equation (46) simulates a 1D tight-binding lattice model consists of spin-1/2 particles, with \(m, t_{l,\ell,x}\) and \(\phi_{l,\ell,x}\) being the corresponding effective Zeeman energy, coupling strength, and phase, which can respectively be tuned via the deliberately chosen of the frequencies, amplitudes, and phases of the external ac driving field. Note that although we used two types of unit cells, in an approximate rotating frame and by adjusting the two types of coupling
The simulation of 1D lattice Hamiltonian in Equation (46) can be extended to a 2D square lattice case, where quantum spin Hall effect can be simulated. Then, we consider a 2D square lattice model with the Hamiltonian of:

$$H = -t_0 \sum_{m,n} \left( c_{m+1,n}^\dagger c_{m,n} + c_{m,n}^\dagger c_{m+1,n} + H.c. \right)$$

where $t_0$ is the neighboring hopping strength; $c_{m,n} = (c_{m,n,1}, c_{m,n,2})^T$ is a 2-component operator for a lattice site $(x = ma, y = nb)$ with $a$ and $b$ being the lattice spacings and $m$ and $n$ being integers; $\hat{\sigma}_x = 2\pi i \alpha \sigma_2$ and $\hat{\sigma}_y = 2\pi k \sigma_1$, with $(j, k)$ being parameters determined by the synthetic magnetic flux and spin mixing; $\chi_{m,n} = (1)^m \chi$ is the on-site potential, which is staggered in $y$-direction. The Hamiltonian in Equation (49) reserves the time reversal symmetry, so it belongs to the $Z_2$ class topological phase, and it can simulate the quantum spin Hall phase, see Figure 17 for numerical results of this quantum simulation, and ref. [106] for details.

8. Experimental Realizations of TBT

The essential advantage of the described parametric coupling formalisms is that they can establish the coupling between SQC elements in a time- and site- resolved way and offer consequently...
unprecedented flexibility to the quantum simulation of various many-body models. In recent years there have been many experimental progresses in this direction, and it is the purpose of this section to briefly introduce them.

For the formalism of coupling superconducting transmission line resonators by grounding SQUIDs, the first experiment in this direction is reported in ref. [76] where three superconducting qubits are connected to form a closed loop geometry. The coupling between the three qubits is established by grounding inductive couplers with tunable inductance, and the synthetic magnetic field penetrated in the loop is obtained from the oscillation phase of the temporal modulation of the three couplers.
which compensate the frequency mismatch between the qubits. From this point of view, this technique falls within the parametric coupling mechanism we have introduced in Section 3.

Due to the long coherence time currently achieved by SQC, experiments in ref. [76] are performed by first preparing the loop in a suitable one- or two-photon Fock state and then following its dynamics without any pump during the evolution. As illustrated in Figure 18, the existence of the synthetic magnetic field is manifested by the directional circulation of photons in the loop with the circulation direction determined simply by the synthetic magnetic field. When the strong nonlinearity of the superconducting qubits is taken into consideration, two interacting photons would circulate in the direction opposite to that of the single photon. This can be explained by the additional $\pi$ phase induced by the Jordan–Wigner transformation. In addition, a few-body interacting ground state can also be prepared by adiabatically tuning up the synthetic magnetic field to the desired value. The same platform has been further exploited to observe the Hofstadter butterfly of one-photon states and the localization effects in the two-photon sector.[51]

On the other hand, with the tunable coupling among qubits in Equation (24) and its demonstration,[78] we are readily in the position to simulate topological quantum phases on superconducting quantum circuits. The first example is the 1D SSH model with the Hamiltonian of Equation (2), introduced in Section 2.2 and experimentally demonstrated in ref. [108] by parametrically tuning the inter-qubit coupling strength in Equation (24) in the required dimerized form. Note that, although the Hamiltonian in Equation (24) is of the bosonic nature, in the single excitation subspace, its topology is equivalent to that of the SSH model, which has two different topological phases with different winding numbers. Specifically, when the qubit couplings are tuned into $g_A < g_B$, the configuration, with $g_A \equiv g^\prime_{2j+1}$ and $g_B \equiv g^\prime_{2j+2}$ in Equation (24), the winding number will be 0 (1), and then the system is in a topologically trivial (nontrivial) state. By monitoring the quantum dynamics of a single-qubit excitation in the chain, the associated topological winding number can be inferred, an experimental verification of which for a four qubits case is presented in Figure 19. The measured value of the topologically trivial configuration is very close to the ideal one while the deviation from the ideal value 1 in the topologically nontrivial case is mainly due to the decoherence effect of the qubits. Similarly, one can also detect the topological magnon ESMs and defect states.[108]

9. Conclusion and Prospects

In summary, in this review we have illustrated the application of the parametric coupling method in investigating topological photonics in SQC system. Such method is a very powerful tool in the sense that it can synthesize artificial gauge field for the microwave photons with fully in situ tunability. Using the adiabatic pumping process, the simulated topologically-protected effects can be probed and further explored in new ways.

Despite the rapid progress in recent years, it is our feeling that we are still in the beginning rather than the end of this research direction. Due to the flexibility of SQC system, we can expect that the PFC architecture allows further incorporations of many other mechanisms, including on-site Hubbard interaction, disorder, and non-Hermicity,[109] and their interplay will definitely bring us into the realm of more fruitful physics. The first of our perspectives is that we should pay extensive attention to the advances in the technical innovation of SQC which may have potential application in quantum simulation. In particular, the reduction of the fabrication error of SQC elements can be exploited.
Figure 19. Measuring the topological winding number in a four qubits chain with a) $g_A = 5$ MHz and $g_B = 1$ MHz and b) $g_A = 1$ MHz and $g_B = 5$ MHz, for the topologically trivial and nontrivial cases, respectively. c,d) The four qubits excitation ($P_{j-1} = |e\rangle_{j-1} \langle e|$) dynamics with an initial state $|\text{egge}\rangle$ for the two prescribed coupling configurations, where dots represent the experimental data while solid lines are plotted by numerically simulations from Hamiltonian in Equation (24). e,f) The time-dependent average of the chiral displacement operator $\bar{P}_j = (P_j^+ - P_j^-) + 2(P_j^+ - P_j^-)$ for the two cases. In the long time limit, the winding number equals twice of the time-averaged $\bar{P}_j(100)$ and thus can be extracted. Dots are experimental record, red dashed lines are from numerical simulations with the black horizontal lines are the corresponding oscillation centers, that is, the time-averaged $\bar{P}_j$. Both experiments agree well with the numerical simulation and theoretical prediction. The measured topological winding numbers are 0.030 and 0.718 for the two cases, respectively. Adapted with permission. Copyright 2019, American Physical Society.

Acknowledgements

The authors thank Dr. J. Liu for proofreading of the manuscript. This work was supported by the National Natural Science Foundation of China (No. 11874156 and No. 11774114), and the Science and Technology Program of Guangzhou (No. 2019050001).

Conflict of Interest

The authors declare no conflict of interest.

Keywords

parametric coupling, superconducting quantum circuits, topological photonics

Received: January 31, 2021
Revised: June 8, 2021
Published online: July 26, 2021

[1] M. A. Armstrong, Basic Topology, 1st ed., Springer, Berlin 1978.
[2] M. Nakahara, Geometry Topology and Physics, 2nd ed., Institute of Physics Publishing, Bristol 2003.
[3] T. Frankel, The Geometry of Physics: An Introduction, 3rd ed., Cambridge University Press, Cambridge 2012.
[4] K. v. Klitzing, G. Dorda, M. Pepper, Phys. Rev. Lett. 1980, 45, 494.
[5] D. J. Thouless, M. Kohmoto, M. P. Nightingale, M. den Nijs, Phys. Rev. Lett. 1982, 49, 405.
[6] M. Z. Hasan, C. L. Kane, Rev. Mod. Phys. 2010, 82, 3045.
[7] X.-L. Qi, S.-C. Zhang, Rev. Mod. Phys. 2011, 83, 1057.
[8] A. B. Bernevig, T. L. Hughes, Topological Insulators and Topological Superconductors, 1st ed., Princeton University Press, Princeton 2013.
[9] A. Bansil, H. Lin, T. Das, Rev. Mod. Phys. 2016, 88, 021004.
[10] Y. Hatsugai, Phys. Rev. Lett. 1993, 71, 3697.
[11] Y. Hatsugai, Phys. Rev. B 1999, 48, 11851.
[12] X.-L. Qi, Y.-S. Wu, S.-C. Zhang, Phys. Rev. B 2006, 74, 045125.
[13] O. Shtanko, L. Levitov, Proc. Natl. Acad. Sci. USA 2018, 115, 5908.
[14] R. B. Laughlin, Phys. Rev. B 1981, 23, 5632.
[15] L. Lu, J. D. Joannopoulos, M. Soljacic, Nat. Photonics 2014, 8, 821.
[16] T. Ozawa, H. M. Price, A. Amo, N. Goldman, M. Hafezi, L. Lu, M. C. Rechtsman, D. Schuster, J. Simon, O. Zilberberg, I. Carusotto, Rev. Mod. Phys. 2019, 91, 015006.
[17] F. D. M. Haldane, S. Raghuram, Phys. Rev. Lett. 2008, 100, 013904.
Zheng-Yuan Xue is a Professor of Physics at South China Normal University. He obtained his Ph.D. in 2009 from the Department of Physics, University of Hong Kong. Then, he joined the current University as a faculty and has been promoted as a Professor in 2015. His current research interests are in quantum computation and quantum simulation on solid-state quantum systems.

Yong Hu is currently a professor at the School of Physics of Huazhong University of Science and Technology. He obtained his Ph.D. in Physics from University of Science and Technology of China in 2007. Currently his research interest is in the field of quantum optics, quantum information, quantum computation based on superconducting quantum circuits.