Prediction of multi-parameters in the inverse heat conduction problems

Talaat Abdelhamid\textsuperscript{1,2,3}, Rongliang Chen\textsuperscript{1,4}, and Md. Mahbub Alam\textsuperscript{2}

\textsuperscript{1}Shenzhen Institutes of Advanced Technology, Chinese Academy of Science, Shenzhen, China.
\textsuperscript{2}Institute for Turbulence-Noise-Vibration Interaction and Control, Shenzhen Graduate School, Harbin Institute of Technology, Shenzhen, China.
\textsuperscript{3}Physics and Mathematical Engineering Department, Faculty of Electronic Engineering, Menoufiya University, Menouf, Egypt.
\textsuperscript{4}Shenzhen Key Laboratory for Exascale Engineering and Scientific Computing, Shenzhen, China.

E-mail: Talaat.2008@yahoo.com, alam@hit.edu.cn

Abstract. This paper studies the prediction of the spatial-dependent heat transfer coefficient $\gamma(x)$ and heat flux $q(x)$, using the modified conjugate gradient method (MCGM). The mathematical formulation of the problem well defined and the existence of the minimizer is investigated. We establish the sensitivity and adjoint equations for computing the gradient with respect to $\gamma(x)$ and $q(x)$. The proposed algorithm is derived for reconstructing $\gamma(x)$ and $q(x)$ using the MCGM. The numerical experiments are examined to show the efficiency and accuracy of the proposed method. Finally, some conclusions and remarks are given.

1. Introduction

Heat transfer coefficient and heat flux are of significant practical interest in thermal and heat conduction problems such as the design of gas turbine blades and nuclear reactors [1]. Robin boundary condition describes the physical phenomenon that the heat flux exchange on the boundary also depends on the boundary temperature. This problem arises in several areas in engineering [2]. Jiang and Abdelhamid [3] introduced a numerical study for reconstructing the Robin coefficient and heat flux in the elliptic system using the Levenberg-Marquardt method to change the non-convex minimization into convex. Yun-Jie, M. [4] proposed an iterative method for solving a nonlinear inverse problem of identifying the time-dependent Robin coefficient from boundary temperature measurement. He investigated the convergence of the method with respect to the amount of noise in the data. Abdelhamid [5] introduced the numerical identification of the Spatio-temporal heat transfer coefficient $\gamma(x,t)$ and spatially dependent heat flux $q(x)$, respectively using the MCGM in a parabolic system.

Jin and Zou [6] have investigated a variational approach to the nonlinear stochastic inverse problem of probabilistically calibrating the Robin coefficient from boundary measurements for the steady-state heat conduction problem. They have used the nonlinear conjugate gradient method for the optimization system. Furthermore, Abdelhamid et al. [7] studied the numerical identification of the time-dependent $\gamma(t)$ and $q(t)$, simultaneously. The space-time dependent of identifying multi-unknown parameters has been considered by Abdelhamid et al. [8]. To the
best of our knowledge, a little work in the literature exists on the mathematical and numerical justification for reconstructing multi-parameters, which is the concern of this paper. The reconstruction of the spatial dependence of the heat transfer coefficient $\gamma(x)$ and heat flux $q(x)$, is the focus of this study. Herein, we aim to solve the problems under-investigation (estimating the unknown heat fluxes and heat transfer coefficients using the measured temperature data on the boundary) via applying the Levenberg-Marquardt method (L-M) and the modified conjugate gradient method (MCGM).

This paper is organized as follows: Section 2 describes the variational formulation and existence of minimizer. Section 3, establishes the sensitivity and adjoint equations for computing the gradient with respect to $\gamma(x)$ and $q(x)$. Section 4, introduces algorithm 4.1 for reconstructing $\gamma(x)$ and $q(x)$ using the modified conjugate gradient method. In section 5, the numerical experiments are discussed and examined to show the efficiency and accuracy of the proposed method. Finally, we give some conclusions and remarks in section 6.

2. Mathematical formulation

Let $\Omega \subset \mathbb{R}^d$ be a bounded and connected polyhedral domain. Consider the following parabolic system

$$
\begin{align*}
\frac{\partial u}{\partial t} - \Delta u &= f(x, t) \quad \text{in} \quad \Omega \times (0, T), \\
\frac{\partial u}{\partial n} + \gamma(x)u(x, t) &= g(x, t) \quad \text{on} \quad \Gamma_1 \times (0, T), \\
\frac{\partial q}{\partial n} &= q(x)\gamma(x, t) \quad \text{on} \quad \Gamma_2 \times (0, T), \\
\frac{\partial q}{\partial n} &= 0 \quad \text{on} \quad \Gamma_3 \times (0, T), \\
u(x, 0) &= 0 \quad \text{in} \quad \Omega,
\end{align*}
$$

(2.1)

we assume that the whole boundary $\partial \Omega$ consists of three parts i.e. $\partial \Omega = \Gamma_1 \cup \Gamma_2 \cup \Gamma_3$ which is a finite collection of disjoint and smooth $(d-1)$-dimensional polyhedral domain such that $\partial \Omega$ refers to the interface. Here, $\gamma(x)$ and $q(x)$ are the spatial-dependent heat transfer coefficient and heat flux, respectively, contained in the following constrained sets

$$
K_1 = \{ \gamma(x) \in L^2(\Gamma_1) : 0 < \gamma_1 \leq \gamma(x) \leq \gamma_2 \text{ a.e. on } \Gamma_1 \},
$$

$$
K_2 = \{ q(x) \in L^2(\Gamma_2) : 0 < q_1 \leq q(x) \leq q_2 \text{ a.e. on } \Gamma_2 \}.
$$

Let $u(x, t)$ is the solution of the system equation (2.1), and $z^\delta$ be the exact solution on a single part of the boundary $\Gamma_3 \times (0, T)$ which is used as measured data. Let $\Omega$ be an open bounded and connected with the boundary $\partial \Omega$, the given source strength $f(x, t) \in L^2(0, T; L^2(\Omega))$, $g(x, t) \in L^2(0, T; L^2(\Gamma_1))$ and $q(x)\gamma(x, t) \in L^2(0, T; L^2(\Gamma_2))$. The bounded constraints $\gamma_1, \gamma_2$ and $q_1, q_2$ on $\gamma(x)$ and $q(x)$, respectively, are assumed to be given and have positive values.

By minimizing the nonlinear Tikhonov regularization [9] which is adapted to deal with the numerical instability of the inverse problem as follows:

$$
J(\gamma(x), q(x)) = \int_0^T \| u(\gamma, q) - z^\delta \|^2_{L^2(\Gamma_1)} dt + \beta \| \gamma(x) \|^2_{L^2(\Gamma_1)} + \eta \| q(x) \|^2_{L^2(\Gamma_2)},
$$

(2.2)

where $u \equiv u(\gamma, q)(x, t) \in L^2(0, T; H^1(\Omega))$ satisfies $u(x, 0) = 0$ in $\Omega$, and

$$
\int_0^T \int_\Omega \partial_t u v dx dt + \int_0^T \int_\Omega \nabla u \cdot \nabla v dx dt + \int_0^T \int_{\Gamma_1} \gamma \nu v ds dt = \int_0^T \int_\Omega f v dx dt
$$

$$
+ \int_0^T \int_{\Gamma_1} g v ds dt + \int_0^T \int_{\Gamma_2} q v ds dt \quad \forall \ v \in L^2(0, T; H^1(\Omega)).
$$

(2.3)
The following theorem investigates the existence of the minimizer to the inverse problem equations (2.2)-(2.3), we have previously proved in (Theorem 2.1, [3]).

**Theorem 2.1.** There exists at least a minimizer to the optimization problem equations (2.2)-(2.3).

3. The optimization framework

In this section, the partial Fréchet derivatives of the forward solution $u(\gamma, q)$ with respect to $\gamma$ and $q$ and their adjoint equations are established. We denote the sensitivity equations as $u_\gamma^1 = u_\gamma^1(\gamma, q)d$ and $u_q^1 = u_q^1(\gamma, q)p$, the partial Fréchet derivatives of the forward solution with respect to the heat transfer coefficient $\gamma(x)$ and heat flux $q(x)$ in any directions $d \in L^2(\Gamma_1)$ and $p \in L^2(\Gamma_2)$, respectively. Hence, $u_\gamma^1$ and $u_q^1$ satisfy the following systems

\[
\begin{aligned}
\frac{\partial u_\gamma^1}{\partial t} - \Delta u_\gamma^1 &= 0 & \text{in } \Omega \times (0, T), \\
\frac{\partial u_\gamma^1}{\partial n} + \gamma u_\gamma^1 &= -du(\gamma, q) & \text{on } \Gamma_1 \times (0, T), \\
\frac{\partial u_\gamma^1}{\partial n} &= 0 & \text{on } \Gamma_2 \times (0, T), \\
\frac{\partial u_q^1}{\partial n} &= 0 & \text{on } \Gamma_3 \times (0, T), \\
u_\gamma^1(x, 0) &= 0 & \text{in } \Omega,
\end{aligned}
\]

and

\[
\begin{aligned}
\frac{\partial u_q^1}{\partial t} - \Delta u_q^1 &= 0 & \text{in } \Omega \times (0, T), \\
\frac{\partial u_q^1}{\partial n} + \gamma u_q^1 &= 0 & \text{on } \Gamma_1 \times (0, T), \\
\frac{\partial u_q^1}{\partial n} &= p\gamma & \text{on } \Gamma_2 \times (0, T), \\
\frac{\partial u_q^1}{\partial n} &= 0 & \text{on } \Gamma_3 \times (0, T), \\
u_q^1(x, 0) &= 0 & \text{in } \Omega,
\end{aligned}
\]

which are linear with respect to $d$ and $p$.

**Lemma 3.1.** (Lions and Magenes [10]) For any $(\gamma, q) \in K_1 \times K_2$, there exists a unique solution $u(\gamma, q)(x, t) \equiv u(\gamma, q) \in L^2(0, T; H^1(\Omega))$ to equation (2.1). Then, it satisfies the following priori estimate

\[
\|u(\gamma, q)\|_{L^2(0, T; H^1(\Omega))} \leq C (\|f\|_{L^2(0, T; L^2(\Omega))} + \|g\|_{L^2(0, T; L^2(\Gamma_1))} + \|\gamma\|_{L^2(0, T; L^2(\Gamma_2))}).
\]

where the constant $C = C(\gamma_1, \gamma_2, q_1, q_2)$.

**Theorem 3.1.** (Abdelhamid et al. [7]) For any $\gamma, \gamma + d \in K_1$ and $q, q + p \in K_2$. Then, the mapping $(\gamma, q) \rightarrow u(\gamma, q)$ from $K_1 \times K_2 \rightarrow L^2(0, T; H^1(\Omega))$ holds Lipschitz continuous and Fréchet differentiability such that

\[
\lim_{\|d\|_{L^\infty(\Gamma_1)} \rightarrow 0} \frac{\|u(\gamma + d, q) - u(\gamma, q) - u_\gamma^1\|_{L^2(0, T; H^1(\Omega))}}{\|d\|_{L^\infty(\Gamma_1)}} = 0,
\]
and
\[
\lim_{\|p\|_{L^\infty(\Gamma_2)} \to 0} \frac{\|u(\gamma, q+p) - u(\gamma, q) - u_1^p\|_{L^2(0,T;H^1(\Omega))}}{\|p\|_{L^\infty(\Gamma_2)}} = 0. \tag{3.4}
\]

For a given \((\gamma, q) \in K_1 \times K_2\), we can apply the linearization:
\[
\begin{align*}
  u(\gamma + d, q) &= u(\gamma, q) + u'_\gamma(\gamma, q)d + O(\|d\|_{L^\infty(\Gamma_1)}), \\
  u(\gamma, q + p) &= u(\gamma, q) + u'_q(\gamma, q)p + O(\|p\|_{L^\infty(\Gamma_2)}).
\end{align*}
\]

The partial Gâteaux derivatives \(\omega^*_\gamma \equiv u'_\gamma(\gamma, q)^*\nu\) and \(\omega^*_q \equiv u'_q(\gamma, q)^*\zeta\) for any directions \(\nu\) and \(\zeta\), respectively, define the adjoint equations which satisfy the following system
\[
\begin{aligned}
  &\frac{\partial \omega^*_\gamma}{\partial t} + \Delta \omega^*_\gamma = 0 \quad \text{in } \Omega \times (0, T), \\
  &\frac{\partial \omega^*_\gamma}{\partial n} + \gamma \omega^*_\gamma = 0 \quad \text{on } \Gamma_1 \times (0, T), \\
  &\frac{\partial \omega^*_\gamma}{\partial n} = 0 \quad \text{on } \Gamma_2 \times (0, T), \\
  &\frac{\partial \omega^*_\gamma}{\partial n} = -\nu \quad \text{on } \Gamma_3 \times (0, T), \\
  &\omega^*_\gamma(x, T) = 0 \quad \text{in } \Omega,
\end{aligned}
\tag{3.5}
\]
and
\[
\begin{aligned}
  &\frac{\partial \omega^*_q}{\partial t} + \Delta \omega^*_q = 0 \quad \text{in } \Omega \times (0, T), \\
  &\frac{\partial \omega^*_q}{\partial n} + \gamma \omega^*_q = 0 \quad \text{on } \Gamma_1 \times (0, T), \\
  &\frac{\partial \omega^*_q}{\partial n} = 0 \quad \text{on } \Gamma_2 \times (0, T), \\
  &\frac{\partial \omega^*_q}{\partial n} = \zeta \quad \text{on } \Gamma_3 \times (0, T), \\
  &\omega^*_q(x, T) = 0 \quad \text{in } \Omega.
\end{aligned}
\tag{3.6}
\]

Now, we can show the differentiability of the objective functional \(J(\gamma, q)\) with respect to \(\gamma(x)\) and \(q(x)\).

**Theorem 3.2.** The objective functional \(J(\gamma, q)\) is Fréchet differentiable, and its Fréchet derivative \(J'_\gamma(\gamma, q)\) and \(J'_q(\gamma, q)\) are given as follows
\[
\begin{aligned}
  J'_\gamma[d] &= -2 \int_0^T \int_{\Gamma_1} u(\gamma, q) \omega^*_\gamma d\gamma ds dt + 2\beta \int_{\Gamma_1} \gamma ds, \\
  J'_q[p] &= 2 \int_0^T \int_{\Gamma_2} h \omega^*_q pd\gamma ds dt + 2\eta \int_{\Gamma_2} q pd\gamma ds.
\end{aligned}
\tag{3.7}
\]
and
\[
\begin{aligned}
  J'_\gamma[d] &= -2 \int_0^T \int_{\Gamma_1} u(\gamma, q) \omega^*_\gamma d\gamma ds dt + 2\beta \int_{\Gamma_1} \gamma ds, \\
  J'_q[p] &= 2 \int_0^T \int_{\Gamma_2} h \omega^*_q pd\gamma ds dt + 2\eta \int_{\Gamma_2} q pd\gamma ds.
\end{aligned}
\tag{3.8}
\]
where, \((\gamma(x), q(x)) \in K_1 \times K_2\) in the directions \((d, p)\), respectively.
Proof. By using Theorem 3.1, we find that:

\[ ||u_q^1||_{L^2(0,T;L^2(G_1))} \leq C ||d||_{L^\infty(G_1)} \quad \text{and} \quad ||u_q^1||_{L^2(0,T;L^2(G_2))} \leq C ||p||_{L^\infty(G_2)}, \]

we have:

\[ J_0(\gamma, q) = \int_0^T \int_{\Gamma_3} (u(\gamma, q) - \delta)^2 dsdt. \quad (3.9) \]

Then, we derive the residual of \( \gamma(x) \) with respect to \( J(\gamma, q) \)

\[ R_\gamma = J_0(\gamma + d, q) - J_0(\gamma, q) \]

\[ = \int_0^T \int_{\Gamma_3} (u(\gamma + d, q) - \delta)^2 dsdt - \int_0^T \int_{\Gamma_3} (u(\gamma, q) - \delta)^2 dsdt \]

\[ = \int_0^T \int_{\Gamma_3} \left\{ (u(\gamma, q) + u_q^1 + O(||d||_{L^\infty(G_1)}) - \delta)^2 - (u(\gamma, q) - \delta)^2 \right\} dsdt \]

\[ = 2 \int_0^T \int_{\Gamma_3} \left\{ (u(\gamma, q) - \delta)^u_q^1 + O(||d||_{L^\infty(G_1)}) + (u_q^1 + O(||d||_{L^\infty(G_1)}) \right\} dsdt \]

\[ = 2 \int_0^T \int_{\Gamma_3} (u(\gamma, q) - \delta)^u_q^1 dsdt + O(||d||_{L^\infty(G_1)}). \]

Neglecting the last term, we obtain

\[ J_{q,0}'[d] = 2 \int_0^T \int_{\Gamma_3} (u(\gamma, q) - \delta)^u_q^1 dsdt. \quad (3.10) \]

Similarly, the Fréchet derivative \( J_{q,0}' \) in the direction \( p \) can be given as:

\[ J_{q,0}'[p] = 2 \int_0^T \int_{\Gamma_3} (u(\gamma, q) - \delta)^u_q^1 dsdt. \quad (3.11) \]

By taking \( \varphi = u_q^1, \psi = \omega^*_q \) and multiplying equation (3.1) by \( \psi \), equation (3.5) by \( \varphi \), and applying Green’s second identity by subtracting the two equations, we obtain:

\[ \int_0^T \int_\Omega (\psi \nabla \cdot \nabla \varphi - \varphi \nabla \cdot \nabla \psi) dx dt = \int_0^T \int_{\partial \Omega} \left( \frac{\partial \varphi}{\partial n} \psi - \frac{\partial \psi}{\partial n} \varphi \right) ds dt = 0. \quad (3.12) \]

By substituting the boundary conditions for \( \varphi \) and \( \psi \), we also obtain:

\[ - \int_0^T \int_{\Gamma_3} d\psi ds dt = \int_0^T \int_{\Gamma_3} \nu \varphi ds dt. \quad (3.13) \]

Furthermore, taking \( \tilde{\varphi} = u_q^1, \tilde{\psi} = \omega^*_q \) and multiplying equation (3.2) by \( \tilde{\psi} \) and equation (3.6) by \( \tilde{\varphi} \), and similarly applying the Green’s second identity, we get:

\[ \int_0^T \int_\Omega (\tilde{\psi} \nabla \cdot \nabla \tilde{\varphi} - \tilde{\varphi} \nabla \cdot \nabla \tilde{\psi}) dx dt = \int_0^T \int_{\partial \Omega} \left( \frac{\partial \tilde{\varphi}}{\partial n} \tilde{\psi} - \frac{\partial \tilde{\psi}}{\partial n} \tilde{\varphi} \right) ds dt = 0. \quad (3.14) \]

By substituting the boundary conditions for \( \tilde{\varphi} \) and \( \tilde{\psi} \), we obtain

\[ \int_0^T \int_{\Gamma_2} p \tilde{\psi} ds dt = \int_0^T \int_{\Gamma_3} \zeta \tilde{\varphi} ds dt. \quad (3.15) \]
Therefore, from substituting equation (3.13) into equation (3.10), we deduce
\[ J'_\gamma[d] = -2 \int_0^T \int_{\Gamma_1} u(\gamma, q)(\omega^*_d) ds dt + 2\beta \int_{\Gamma_1} (\gamma d) ds, \] (3.16)
such that
\[ J'_{\gamma,0}[d] = 2 \int_0^T \int_{\Gamma_1} du(\gamma, q)\omega^*_d ds dt. \]
Similarly, the Fréchet derivative of \( J_0 \) with respect to \( q \) in the direction \( p \)
\[ J'_q[p] = 2 \int_0^T \int_{\Gamma_2} p\omega^*_q ds dt + 2\eta \int_{\Gamma_2} qp ds, \] (3.17)
and
\[ J'_{q,0}[p] = 2 \int_0^T \int_{\Gamma_2} p\omega^*_q ds dt. \]
This completes the proof of Theorem 3.2.

4. Numerical algorithm

First, we present the MCGM throughout the following steps for solving the inverse problem
(2.2)-(2.3). Such that, each iteration requires solving the sensitivity and adjoint equations
to compute the gradient formulas and step lengths, for reconstructing \( \gamma(x) \) and \( q(x) \). The
reconstructed values of \( \gamma(x) \) depends continuously on the computed \( q(x) \).

**Algorithm 4.1.**
- Choose an initial guess \((\gamma^0, q^0), (d^0_\gamma, d^0_q), \varepsilon_1, \varepsilon_2 \geq 0\), and set \( k := 0 \).
- Solve the direct problem equation (2.1) \( u(\gamma, q)(x, y; t) = u(\gamma^k, q^k)(x, y; t) \), and compute the residual error at \( k \)th step
  \[ r_q = u(\gamma^k, q^k) - z^\delta \quad \text{on} \quad \Gamma_3 \times (0, T). \]
- Solve the adjoint problem equation (3.6) for \( \omega^*_q(\gamma^k, q^k) \) with the boundary conditions
- Compute the gradient in equation (3.8)
  \[ J'_q(\gamma^k, q^k) = 2 \int_0^T b^k \omega^*_q(\gamma^k, q^k) dt + 2\eta q^k. \]
- The conjugate coefficient \( \beta^k_q \) can be computed by:
  \[ \beta^k_q = \frac{\|J'_q(\gamma^k, q^k)\|_{L^2(\Gamma_2)}}{\|J'_q(\gamma^{k-1}, q^{k-1})\|_{L^2(\Gamma_2)}}. \]
- Compute the descent direction for \( q(x) \)
  \[ d^{k+1}_q = -J'_q(\gamma^k, q^k) + \beta^k_q d^k_q. \]
- Solve the sensitivity problem equation (3.2) for \( u^1_q(\gamma^k, q^k) \).
The initial guess values should behold the following conditions with respect to the identification of previous works with ill-posed inverse problems guarantee its convergence (cf. [3, 11, 16]). We have reduced the constrained optimization problem to a sequence of unconstrained optimization problem by adding the Tikhonov regularization to Newton’s method to find a point where the gradient of the objective function vanishes (see [14, 15]). The step lengths parameters of 
\[ \beta_q \]
and 
\[ \gamma \]
were examined by trial and error method, \( \alpha_k \) and \( \alpha^k \) may not guarantee its convergence. However, the numerical experiments in this work and the previous works with ill-posed inverse problems guarantee its convergence (cf. [3, 11, 16]). The initial guess values should behold the following conditions with respect to the identification parameters \( \gamma \) and \( q \), respectively; \( \gamma_1 \leq \gamma(x) \leq \gamma_2 \) on \( \Gamma_1 \) and \( q_1 \leq q(x) \leq q_2 \) on \( \Gamma_2 \). The values of \( \beta \) and \( \eta \) were examined by trial and error method, \( \beta = \eta = 10^{-3} \) were thus chosen.

• Compute the step length

\[ \alpha_q^k = \frac{\langle r_k, u_q^k(\gamma^k, q_k, d_q^{k+1}) \rangle}{\| u_q^k(\gamma^k, q_k, d_q^{k+1}) \|^2_{L^2(\Gamma_1)}} + \eta \| d_q^{k+1} \|^2_{L^2(\Gamma_2)}. \]

• Update the heat flux \( q(x) \) by

\[ q^{k+1} = q^k - \alpha_k d_q^{k+1}. \]

• Solve the forward equation (2.1) \( u(\gamma, q) = u(\gamma^k, q^{k+1}) \), and compute the residual for \( \gamma \)

\[ r_{\gamma}^k = u(\gamma^k, q^{k+1}) - \varepsilon \delta \text{ on } \Gamma_3 \times (0, T). \]

• Solve the adjoint problem equation (3.5) for \( \omega_{\gamma}^* (\gamma^k, q^{k+1}) \) with the boundary conditions

• Compute the gradient in equation (3.7)

\[ J_{\gamma}^* (\gamma^k, q^{k+1}) = -2 \int_0^T u(\gamma^k, q^{k+1}) \omega_{\gamma}^* (\gamma^k, q^{k+1}) dt + 2 \beta \gamma^k. \]

• The conjugate coefficient

\[ \beta_{\gamma}^k = \frac{\| J_{\gamma}^* (\gamma^k, q^{k+1}) \|^2_{L^2(\Gamma_1)}}{\| J_{\gamma}^* (\gamma^{k-1}, q^{k+1}) \|^2_{L^2(\Gamma_1)}}. \]

• Compute the descent direction for \( \gamma(x) \)

\[ d_{\gamma}^{k+1} = -J_{\gamma}^* (\gamma^k, q^{k+1}) + \beta_{\gamma}^k d_{\gamma}^k. \]

• Solve the sensitivity problem equation (3.1) for \( u_{\gamma}^k (\gamma^k, q^{k+1}) \).

• Compute the step length

\[ \alpha_{\gamma}^k = \frac{\langle r_k, u_{\gamma}^k(\gamma^k, q_k, d_{\gamma}^{k+1}) \rangle}{\| u_{\gamma}^k(\gamma^k, q_k, d_{\gamma}^{k+1}) \|^2_{L^2(\Gamma_1)}} + \beta \| d_{\gamma}^{k+1} \|^2_{L^2(\Gamma_1)}. \]

• Update the heat transfer coefficient \( \gamma(x) \) by

\[ \gamma^{k+1} = \gamma^k - \alpha_{\gamma}^k d_{\gamma}^{k+1}. \]

• If \( \frac{\| q^{k+1} - q^k \|_{L^2(\Gamma_2)}}{\| q^k \|_{L^2(\Gamma_2)}} \leq \varepsilon_1 \), and \( \frac{\| \gamma^{k+1} - \gamma^k \|_{L^2(\Gamma_1)}}{\| \gamma^k \|_{L^2(\Gamma_1)}} \leq \varepsilon_2 \) stop; otherwise \( k := k + 1 \), and go to Step 2.

Herein, \( x \) refers to \( (x, y) \). Conjugate gradient method has been implemented for various ill-posed inverse problems (see [11, 12, 13]). We have reduced the constrained optimization problem to a sequence of unconstrained optimization problem by adding the Tikhonov regularization terms to Newton’s method to find a point where the gradient of the objective function vanishes (see [14, 15]). The step lengths parameters \( \alpha_q^k \) and \( \alpha^k \) are determined by a quadratic approximation of the objective function. Theoretically, the computation of the step sizes \( \alpha_q^k \) and \( \alpha^k \) may not guarantee its convergence. However, the numerical experiments in this work and the previous works with ill-posed inverse problems guarantee its convergence (cf. [3, 11, 16]).
5. Numerical experiments and discussions

The considered solution domain Ω is rectangular as Ω = (0, 1) × (0, 2) which is discretized using triangular mesh generated by dividing each element of the regular rectangular mesh into two triangles. The domain boundaries consist of three parts, Γ₁ = {(x, y) : x = 0, 0 ≤ y ≤ 1}, Γ₂ = {(x, y) : y = 0, 0 ≤ x ≤ 1}, and Γ₃ = ∂Ω \ (Γ₁ ∪ Γ₂). The number of triangular finite elements can be calculated from N × M, and t ∈ (0, 3), the time step size Δt = 0.5.

The noisy data zₜ are generated by adding uniformly distributed random variable R that varies in [−1, 1] and R realised by using the MATLAB function rand(·). We apply the formula zₜ = u + δRu on Γ₃ × (0, T) where δ refers to the level of noise. We set the initial guesses of the directions (dₒ, dᵢ) to be zeros vectors and tolerance parameters ε₁ = ε₂ = 2 × 10⁻³.

Now, we introduce numerical examples for reconstructing the unknown parameters γ(x) and q(x), and present the performance of the proposed methods with respect to the exact data from equation (2.1). We assume that, the given functions are defined as follows:

\[ f(x, y; t) = (xy + 1)t \quad \text{in} \quad Ω × (0, T), \]

\[ g(x, y; t) = 1 - y - 3t \quad \text{on} \quad Γ₁ × (0, T), \]

\[ h(x, y; t) = x² + t \quad \text{on} \quad Γ₂ × (0, T). \]

**Remark 5.1.** The relative error of heat transfer coefficient is given by \( RE_γ = \frac{\|γ^k - γ\|_{L^2(Γ₁)}}{\|γ\|_{L^2(Γ₁)}} \) and relative error of the heat flux is given by \( RE_q = \frac{\|q^k - q\|_{L^2(Γ₂)}}{\|q\|_{L^2(Γ₂)}} \).

**Example 5.1.** Consider the exact heat flux is given by \( q(x) = 1 + (x - 1)² \) on Γ₂ and heat transfer coefficient is given by \( γ(x) = \frac{1}{4}(4 - y²) \) on Γ₁.

**Example 5.2.** Consider the exact heat flux is given by \( q(x) = -0.2x + 1 \) on Γ₂ and heat transfer coefficient is given by \( γ(x) = \sin(\frac{π}{2}y - \frac{π}{2}) + 2 \) on Γ₁.

Figure 1 shows the obtained reconstruction using the MCGM according to the exact solutions for \( γ(x) \) and \( q(x) \). Table 1 presents the numerical reconstruction of \( γ(x) \) and \( q(x) \) at different levels of noise in the measured data using the MCGM. Moreover, the elapsed time for each example was calculated to show the efficiency of the proposed methods as well as the relative errors for the reconstructed parameters \( γ(x) \) and \( q(x) \), \( RE_γ \) and \( RE_q \). Increasing the noise level investigates the robustness of the proposed methods. The \( RE_γ \) and \( RE_q \) increase gradually with increasing noise in the measured data. Figure 2 shows the numerical convergence of the method by drawing the relationship between the residual and relative errors with the number of iterations \( k \) using the MCGM. The numerical results show that the reconstruction is quite satisfactory, in spite of the highly ill-posedness and nonlinearity of the optimization problem equation (2.2).

**Table 1.** Reconstructing the \( γ(x) \) and \( q(x) \) by MCGM at different levels of noise in the measured data

| Example | δ(%) | k | \( RE_γ \) | \( RE_q \) | δ(%) | k | \( RE_γ \) | \( RE_q \) | Elapsed time (sec) |
|---------|------|---|-----------|-----------|------|---|-----------|-----------|-----------------|
| 5.1     | 1    | 5 | 0.047     | 0.0096    | 3    | 7 | 0.0456    | 0.03      | 9.0             |
| 5.2     | 1    | 7 | 0.037     | 0.012     | 3    | 7 | 0.0417    | 0.009     | 8.04            |
6. Conclusions and remarks

In this paper, the proposed inverse optimization problem for reconstructing the unknown parameters spatial-dependent- heat transfer coefficient $\gamma(x)$ and- heat flux $q(x)$ is investigated. The variational formulation, existence of the minimizer, and a nonlinear Tikhonov regularization approach are investigated. We derived the objective functional that is differentiable, using the adjoint approach to obtain the gradient formulas for multi-unknown parameters. The modified conjugate gradient method MCGM was adapted to solve the inverse problem. The efficiency and accuracy of the proposed method are investigated with introducing numerical experiments. Implementation of the numerical study shows that the proposed methods display convergent and stable results for the heat transfer coefficient and heat flux with relatively low error.
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