Absence of renormalization group pathologies in some critical Dyson-Ising ferromagnets

Tom Kennedy
Department of Mathematics
University of Arizona
Tucson, AZ 85721
decimal: tgk@math.arizona.edu
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Abstract

The Dyson-Ising ferromagnet is a one-dimensional Ising model with a power law interaction. When the power is between $-1$ and $-2$, the model has a phase transition. Van Enter and Le Ny proved that at sufficiently low temperatures the decimation renormalization group transformation is not defined in the sense that the renormalized measure is not a Gibbs measure. We consider a modified model in which the nearest neighbor couplings are much larger than the other couplings. For a family of Hamiltonians which includes critical cases, we prove that the first step of the renormalization group transformation can be rigorously defined for majority rule and decimation.

1 Introduction

We consider a one-dimensional ferromagnetic Ising model with a power law interaction. The standard choice for the Hamiltonian is

$$H = - \sum_{i<j} \frac{\sigma_i \sigma_j}{|i-j|^\alpha}$$

(1)

If the power satisfies $1 < \alpha \leq 2$ the model has a phase transition. Van Enter and Le Ny proved that when the inverse temperature is sufficiently large, the decimation renormalization group transformation is not defined. In this paper we will consider a slightly modified model and prove that the first step of the renormalization group transformation is defined in a region of the parameters that includes critical points. Our method applies to both decimation and the majority rule transformations.
For the Hamiltonian (1), the existence of a phase transition was conjectured by Kac and Thompson for $1 < \alpha \leq 2$ [13]. The absence of a phase transition for $\alpha > 2$ was proved by Ruelle [16]. Dyson proved the existence of long-range order at low temperatures for $1 < \alpha < 2$ by comparison with a hierarchical model [7]. Long-range order for the case of $\alpha = 2$ was proved by Fröhlich and Spencer [10]. Further properties for this case were proved in [1,11,12]. Long-range order at low temperatures has also been proved using infrared bounds by Fröhlich, Israel, Lieb and Simon [9] for $1 < \alpha < 2$. Non-rigorous renormalization group treatments of the model can be found in [4,5,8,14]. Bleher carried out a rigorous renormalization group treatment of Dyson’s hierarchical model [3].

Van Enter and Le Ny [18] considered the decimation transformation with a scale factor of 2 for this model. (So every other spin is decimated.) The renormalized measure is always defined; the non-trivial question is whether it is a Gibbs measure. They prove it is not Gibbsian if the temperature is sufficiently low by exhibiting a point of essential discontinuity for the conditional probabilities for the decimated Gibbs measures. We refer to the spins that are fixed in the decimation transformation as the block spins and the spins that are not fixed as the original spins. The “bad” block spin configuration which drives their result is the block spin configuration which alternates between + and −. The interactions between these fixed block spins and a single original spin cancel exactly. What is left is a power law interaction between the original spins. Because the lattice spacing between adjacent original spins is 2, this interaction is weaker than the original interaction by a factor of $2^\alpha$. But if $\beta$ is large enough this conditioned system will still be in the low temperature phase. It is important to note that this argument does not work if the original system is near the critical temperature - the region where one is most interested in applying the renormalization group transformation. The method used in [18] to prove the renormalized measure is not Gibbsian was developed by van Enter, Fernández and Sokal [17].

In this paper we study a slightly modified model. We will take the Gibbs measure to be weighted by $e^{-H}$ rather than $e^{-\beta H}$ and include parameters in $H$ that play the role of inverse temperature. Our Hamiltonian is

$$H = -\gamma \sum_i \sigma_i \sigma_{i+1} - \epsilon \sum_{i<j-1} \frac{\sigma_i \sigma_j}{|i-j|^\alpha}$$

where $\gamma, \epsilon > 0$. The usual model at inverse temperature $\beta$ is obtained by setting $\gamma = \epsilon = \beta$. We focus our attention on the case of small $\epsilon$. Standard methods prove there is no long-range order if both $\gamma$ and $\epsilon$ are sufficiently small. For $1 < \alpha < 2$ existing methods can be used to prove that given an $\epsilon > 0$ there is a $\gamma_0(\epsilon, \alpha)$ such that there is long-range order if $\gamma \geq \gamma_0(\epsilon, \alpha)$. We will sketch a proof of this using infrared bounds in section [3]. Here we show that the original non-rigorous energy-entropy argument for long-range order applies in this case. We impose + boundary conditions on the two ends. Then we consider a segment of − spins of length $L$ which contains the origin. If there are no other − spins in the configuration, then the energy cost compared to the all + configuration is essentially $4\gamma + c\epsilon L^{2-\alpha}$ where $c$ is a constant that...
depends on $\alpha$. The number of such segments is $L$, so summing we have

$$
\exp(-4\gamma) \sum_{L=1}^{\infty} L \exp(-c\epsilon L^{2-\alpha})
$$

(3)

The key point is that this sum is finite for all $\epsilon > 0$ if $\alpha < 2$. So given $\epsilon$ we can take $\gamma$ to be sufficiently large to make the above quantity as small as we like. Of course this argument is a gross over-simplification since it ignores the effect of other islands of $-$ spins which makes estimating the energy cost of a contour quite difficult. In fact, for some configurations the energy cost of the island containing the origin need not behave as $L^{2-\alpha}$. A much more elaborate definition of contours would be needed to turn this into a proof. Contour based Peierls arguments for long-range order which are based on the ideas of [10] can be found in [2, 6, 15].

The main goal of this paper is to prove that there is a small $\epsilon_0$ such that for $\epsilon < \epsilon_0$ and for all $\gamma > 0$ the first step of the RG map is defined. Note that the parameter region in which our result holds includes critical points. Proving that the first step of the RG map is defined is usually easier than proving that subsequent steps are defined since the renormalized Hamiltonian that must be considered in the subsequent steps is considerably more complicated than the original Hamiltonian. That is the case here as well.

Our result applies to both decimation and the majority rule transformations. Standard expansion methods can be used to prove the result for decimation. We provide a sketch of the idea. We denote the original spins by $\sigma_i$ and denote the block spins by $s_i$. We use blocks with two spins. So there is one block spin for every two original spins. The decimation transformation sets $\sigma_{2i} = s_i$ for all $i$. In other words, the even spins in the original system are frozen to the block spin values and the odd spins in the original system are summed out. The renormalized Hamiltonian $H'$ is given by

$$
\exp(-H'(s)) = \sum_{\sigma: odd} \exp(\sum_{i: odd} h_i \sigma_i + \epsilon \sum_{i < j: odd} \frac{\sigma_i \sigma_j}{|i-j|^\alpha})
$$

(4)

where each $h_i$ is a function of the block spins. Note that $h_{2i+1} = \gamma(s_i + s_{i+1}) + O(\epsilon)$ where the $O(\epsilon)$ term is a function of all the block spins. In other words, the original spins which are not fixed to a block spin value have a site-dependent magnetic field which depends on the block spins, and there is a weak pairwise interaction between these original spins. So we can do a high temperature expansion to obtain a convergent expansion for $H'$.

2 Proof of existence of first step of RG

We study the Hamiltonian

$$
H = -\gamma \sum_i \sigma_i \sigma_{i+1} - \epsilon \sum_{i < j} J_{ij} \sigma_i \sigma_j
$$

(5)
where $\epsilon$ is small and $\gamma$ is large. We are primarily interested in $J_{ij} = |i - j|^{-\alpha}$ for $i < j - 1$, but our method works for any translationally invariant $J_{ij}$ such that $\sum_j |J_{0j}| < \infty$. We use blocks with two spins. We denote the original spins by $\sigma_i$ and denote the block spins by $s_i$. Spins $\sigma_{2i}, \sigma_{2i+1}$ are in the same block, and the block spin for that block is $s_i$. We only consider renormalization group kernels of the form

$$K(\sigma, s) = \prod_i k(\sigma_{2i}, \sigma_{2i+1}; s_i)$$

The renormalized Hamiltonian $H'(s)$ is formally defined by

$$\exp(-H'(s)) = \sum_\sigma K(\sigma, s) \exp(-H(\sigma))$$

We require that the kernel satisfy

$$\sum_s K(\sigma, s) = 1, \quad \forall \sigma$$

This condition implies that the partition function is preserved by the renormalization group transformation, i.e.,

$$\sum_s \exp(-H'(s)) = \sum_\sigma \exp(-H(\sigma))$$

For the first part of this section the kernel is general. Later we will consider specific kernels such as majority rule and decimation.

We will use a transfer matrix approach. Suppose we have summed out the spins $\sigma_i$ for $i < 0$. The result depends on the block spins $s_i$ with $i < 0$ and on the original spins $\sigma_i$ with $i \geq 0$. We can write it as

$$\sum_{\sigma_i; i < 0} K(\sigma, s) \exp(-H(\sigma)) = \prod_{i \geq 0} k(\sigma_{2i}, \sigma_{2i+1}; s_i)$$

$$\exp[\sum_{X,Y; X \geq 0, Y < 0} c(X, Y)\sigma(X)s(Y) + \gamma \sum_{i \geq 0} \sigma_i\sigma_{i+1} + \epsilon \sum_{0 \leq i < j} J_{ij}\sigma_i\sigma_j + E]$$

The sum on $X$ is over finite subsets of the sites in the original chain and the sum on $Y$ is over finite subsets of the sites in the renormalized chain. The notation $X \geq 0$ means that $X$ only contains non-negative sites, and the notation $Y < 0$ means that $Y$ only contains negative block sites. The term $E$ does not depend on any $s_i$ or $\sigma_i$. We do not really care about this term since it only contributes a constant to the renormalized Hamiltonian $H'$.

Now consider what happens when we sum over the spins $\sigma_0, \sigma_1$. The resulting function will be the same as the previous one except that all $\sigma_i$ are shifted by two sites and the $s_i$ by one
we have
\[
\sum_{\sigma_0, \sigma_1} \prod_{i \geq 0} k(\sigma_{2i}, \sigma_{2i+1}; s_i) \exp[ \sum_{X, Y : X \geq 0, Y < 0} c(X, Y) \sigma(X) s(Y) + \gamma \sum_{i \geq 0} \sigma_i \sigma_{i+1} + \epsilon \sum_{0 \leq i < j} J_{ij} \sigma_i \sigma_j + E] = \prod_{i \geq 0} k(\sigma_{2i+2}, \sigma_{2i+3}; s_{i+1}) \exp[ \sum_{X, Y : X \geq 0, Y < 0} c(X, Y) \sigma(X + 2) s(Y + 1) + \gamma \sum_{i \geq 0} \sigma_{i+2} \sigma_{i+3}]
\]
\[+ \epsilon \sum_{0 \leq i < j} J_{ij} \sigma_{i+2} \sigma_{j+2} + E']
\]
We use \(X + k\) to denote \(\{i + k : i \in X\}\), i.e., the set of sites in \(X\) with each site shifted by \(k\). After some obvious cancellations and re-indexing of the terms in the right side, this simplifies to
\[
\sum_{\sigma_0, \sigma_1} k(\sigma_0, \sigma_1; s_0) \exp[ \sum_{X, Y : X \geq 0, Y < 0} c(X, Y) \sigma(X) s(Y) + \gamma \sum_{i=0, 1} \sigma_i \sigma_{i+1} + \epsilon \sum_{i=0, 1, j > 0} J_{ij} \sigma_i \sigma_j]
\]
\[= \exp[ \sum_{X, Y : X \geq 2, Y \leq 0} c(X - 2, Y - 1) \sigma(X) s(Y)] \quad (9)
\]
Let \(\mathcal{X}\) be the collection of \(X\) such that \(X \geq 0\) and \(X\) contains at least one of 0, 1. Note that the \(X\) not in \(\mathcal{X}\) are precisely the \(X\) with \(X \geq 2\). Pulling such terms outside the sum on \(\sigma_0, \sigma_1\) we have
\[
\exp[ \sum_{X, Y : X \geq 2, Y < 0} c(X, Y) \sigma(X) s(Y)] \sum_{\sigma_0, \sigma_1} k(\sigma_0, \sigma_1; s_0)
\]
\[
\exp[ \sum_{X, Y : X \in \mathcal{X}, Y < 0} c(X, Y) \sigma(X) s(Y) + \gamma \sum_{i=0, 1} \sigma_i \sigma_{i+1} + \epsilon \sum_{i=0, 1, j > 0} J_{ij} \sigma_i \sigma_j]
\]
\[= \exp[ \sum_{X, Y : X \in \mathcal{X}, Y \leq 0} c(X - 2, Y - 1) \sigma(X) s(Y)] \quad (9)
\]
For a general Hamiltonian of the form
\[
H = \sum_X h(X) \sigma(X) \quad (10)
\]
we define
\[
\hat{H} = \sum_{X \in \mathcal{X}} h(X) \sigma(X) \quad (11)
\]
So for our Hamiltonian
\[
\hat{H} = \gamma \sum_{i=0, 1} \sigma_i \sigma_{i+1} + \epsilon \sum_{i=0, 1, j > 0} J_{ij} \sigma_i \sigma_j \quad (12)
\]
This is the part of the Hamiltonian that appears in (12). We let \( c \) denote the collection \( \{ c(X, Y) : X \in X, Y < 0 \} \). Note that \( \hat{H} \) appears in (9) in exactly the same way that \( c \) does. Now we define \( f(c, X, Y) \) for \( X \geq 2, Y \leq 0 \) by
\[
\sum_{\sigma_0, \sigma_1} k(\sigma_0, \sigma_1; s_0) \exp\left[ \sum_{X, Y : X \in X, Y < 0} c(X, Y) \sigma(X) s(Y) \right] = \exp\left[ \sum_{X, Y : X \geq 2, Y \leq 0} f(c, X, Y) \sigma(X) s(Y) \right]
\]
(13)

Note that the Hamiltonian does not appear in the above. We will include the Hamiltonian by replacing the \( c \) in \( f(c, X, Y) \) by \( c + \hat{H} \). In particular equation (9) becomes
\[
\sum_{X, Y : X \geq 2, Y < 0} c(X, Y) \sigma(X) s(Y) + \sum_{X, Y : X \geq 2, Y \leq 0} f(c + \hat{H}, X, Y) \sigma(X) s(Y) = \sum_{X, Y : X \geq 2, Y \leq 0} c(X - 2, Y - 1) \sigma(X) s(Y)
\]
(14)

Equation (14) implies that
\[
c(X - 2, Y - 1) = c(X, Y) + f(c + \hat{H}, X, Y), \text{ if } X \geq 2, Y < 0
\]
\[
c(X - 2, Y - 1) = f(c + \hat{H}, X, Y), \text{ if } X \geq 2, Y \leq 0, 0 \in Y
\]

This is equivalent to
\[
c(X, Y) = c(X + 2, Y + 1) + f(c + \hat{H}, X + 2, Y + 1), \text{ if } X \geq 0, Y < -1
\]
\[
c(X, Y) = f(c + \hat{H}, X + 2, Y + 1), \text{ if } X \geq 0, Y \leq -1, -1 \in Y
\]

If \( Y \) is non-empty, then by iterating the above we find
\[
c(X, Y) = \sum_{k=1}^{n} f(c + \hat{H}, X + 2k, Y + k), \text{ if } X \geq 0, Y < 0
\]

where \( n \) is the largest (least negative) site in \( Y \). If \( Y = \emptyset \) then we have
\[
c(X, \emptyset) = \sum_{k=1}^{n} f(c + \hat{H}, X + 2k, \emptyset) + c(X + 2n, \emptyset), \text{ if } X \geq 0
\]

Assuming that \( c(X + 2n, \emptyset) \) converges to 0 as \( n \to \infty \), we have
\[
c(X, \emptyset) = \sum_{k=1}^{\infty} f(c + \hat{H}, X + 2k, \emptyset) \text{ if } X \geq 0
\]
If we define \( f(c + \hat{H}, X, Y) \) to be zero when \( Y \) does not satisfy \( Y \leq 0 \), then we have

\[
c(X, Y) = \sum_{k=1}^{\infty} f(c + \hat{H}, X + 2k, Y + k) \quad \text{if} \quad X \geq 0, \ Y < 0
\]  

(15)

for both the case of non-empty \( Y \) and the case of empty \( Y \).

We define for \( X \in \mathcal{X} \) and \( Y < 0 \)

\[
F(c, X, Y) = \sum_{k=1}^{\infty} f(c, X + 2k, Y + k)
\]  

(16)

Then (15) is a fixed point equation for \( c \),

\[
F(c + \hat{H}, X, Y) = c(X, Y), \quad X \in \mathcal{X}, Y < 0
\]  

(17)

The function \( F(c, X, Y) \) is defined for all \( X \) with \( X \geq 0 \). But in the fixed point equation we only use it for \( X \) with \( X \in \mathcal{X} \), i.e., \( X \) contains at least one of 0, 1.

Note that \( f(c + \hat{H}, X, Y) \) is defined when \( X = \emptyset \) and \( Y \leq 0 \), but these functions do not appear in the fixed point equation above. They are terms in the renormalized Hamiltonian \( H' \). The contribution to \( H' \) from summing out \( \sigma_0, \sigma_1 \) is

\[
\sum_{Y:Y\leq0} f(c + \hat{H}, \emptyset, Y) s(Y)
\]

To obtain \( H' \) we must sum this over translations with respect to the block lattice. Recall that we define \( f(c + \hat{H}, X, Y) \) to be zero when \( Y \) does not satisfy \( Y \leq 0 \). So we have

\[
H' = \sum_{k=-\infty}^{\infty} \sum_{Y} f(c + \hat{H}, \emptyset, Y) s(Y + k) = \sum_{Y} h'(c + \hat{H}, Y) s(Y)
\]

where the coefficient of \( s(Y) \) is given by

\[
h'(c + \hat{H}, Y) = \sum_{k=-\infty}^{\infty} f(c + \hat{H}, \emptyset, Y - k) = \sum_{k=-\infty}^{\infty} f(c + \hat{H}, \emptyset, Y + k)
\]  

(18)

At the moment this is a purely formal expression. (We do not know that this infinite series converges.) We will make the definition of the renormalized Hamiltonian rigorous later.

We will prove that Eq. (17) has a fixed point by constructing an approximate fixed point and showing the map is a contraction in a sufficiently large neighborhood of it. Let \( g(X, Y) \) be a function where \( X \) ranges over finite subsets of the original lattice and \( Y \) ranges over finite subsets of the block lattice. We define the norm of \( g \) to be

\[
||g|| = \sum_{X,Y} |g(X, Y)| \exp(\mu|X|)
\]  

(19)
where $|X|$ is the cardinality of $X$, and $\mu \geq 0$. We will only allow functions $g$ with finite $\|g\|$. The functions $c$ that occur in the fixed point equation have an additional property: $c(X, Y) \neq 0$ only for $X \in \mathcal{X}$ and $Y < 0$. The Banach space in which we look for a solution to the fixed-point equation is the set of $c$’s with this property and finite $\|c\|$. We will show that $F(c)$ and its Jacobian $DF(c)$ are defined and continuous on an open subset of this Banach space. We defined the norm $\|\|_{\mathcal{F}}$ for functions on all $X, Y$ for later use.

Given such a function $g(X, Y)$ with $\|g\| < \infty$, we can define a function of $\sigma, s$ by

$$g(\sigma, s) = \sum_{X,Y} g(X, Y)\sigma(X)s(Y)$$

(20)

So we can think of the norm as a norm on functions of $\sigma, s$ of the form $\|g\|$. Given two functions $g_1(X, Y), g_2(Y, Y)$ we can multiply the functions $g_1(\sigma, s)$ and $g_2(\sigma, s)$. Since $\sigma_i^2 = 1$, we have $\sigma(A)\sigma(B) = \sigma(A \Delta B)$ where the symmetric difference is defined by $A \Delta B = A \cup B \setminus (A \cap B)$. Similarly, $\sigma(A)\sigma(B) = \sigma(A \Delta B)$. Thus

$$g_1(\sigma, s) g_2(\sigma, s) = \sum_{X_1,X_2,Y_1,Y_2} g_1(X_1,Y_1)g_2(X_2,Y_2)\sigma(X_1\Delta X_2)s(Y_1\Delta Y_2)$$

$$= \sum_{X,Y} \sigma(X)s(Y) \sum_{X_1,X_2,Y_1,Y_2:X_1\Delta X_2=X,Y_1\Delta Y_2=Y} g_1(X_1,Y_1)g_2(X_2,Y_2)$$

so

$$\|g_1g_2\| \leq \sum_{X_1,X_2,Y_1,Y_2} |g_1(X_1,Y_1)g_2(X_2,Y_2)| \exp(\mu|Y_1\Delta Y_2|)$$

$$\leq \sum_{X_1,X_2,Y_1,Y_2} |g_1(X_1,Y_1)g_2(X_2,Y_2)| \exp(\mu(|Y_1| + |Y_2|)) = \|g_1\|\|g_2\|$$

where we have used $|Y_1\Delta Y_2| \leq |Y_1| + |Y_2|$. We will use this Banach algebra property extensively.

Suppose we have a function $g(\sigma, s)$ which has finite support in the sense that it only depends on finitely many $\sigma_i$ and $s_i$. Then it can be written in the form (20). To compute the coefficients $g(X, Y)$, first consider

$$\sum_{\sigma,s} \sigma(X)s(Y)\sigma(V)s(W) = \sum_{\sigma,s} \sigma(X\Delta V)s(Y\Delta W)$$

where the sum is only over the $\sigma_i$ and $s_i$ in the support of $g$. This sum is 0 unless $X = V$ and $Y = W$, in which case it equals the number of terms in the sum. Letting $N$ denote the number of terms in the sum,

$$g(X, Y) = \frac{1}{N} \sum_{\sigma,s} \sigma(X)s(Y) g(\sigma, s)$$

(21)
If $c$ has finite support, then eqs. (13) and (21) imply

$$f(c, V, W) = \frac{1}{N} \sum_{\sigma, s} \sigma(V)s(W) \ln[\sum_{\sigma_0, \sigma_1} k(\sigma_0, \sigma_1; s_0) \exp(\sum_{X,Y:X,Y<0} c(X,Y)\sigma(X)s(Y))]$$  (22)

If we consider $f(c + \hat{H}, V, W)$ then finite support means that $\hat{H}$ must have finite support as well. We will initially work in the case of finite support and then extend our definitions and bounds to an open subset of $c$ and the full Hamiltonian.

We will extend the definition of $F$ by obtaining bounds on its derivative. Let $DF(c)$ denote the Jacobian of $F$ at $c$. Its matrix elements are $\partial F(c, V, W)/\partial c(X, Y)$. The norm we are using is a weighted $l^1$ norm, so the operator norm of $DF(c)$ is bounded by

$$||DF(c)|| \leq \sup_{X \in X, Y < 0} \sum_{V \in X, W < 0} \left| \frac{\partial F(c, V, W)}{\partial c(X, Y)} \right| \exp(\mu|V| - \mu|X|)$$

As before $X$ denotes the set of finite subsets $X$ with $X \geq 0$ and at least one of 0, 1 is in $X$. From (16) we have

$$\frac{\partial F(c, V, W)}{\partial c(X, Y)} = \sum_{k=1}^{\infty} \frac{\partial f(c, V + 2k, W + k)}{\partial c(X, Y)}$$

Let $g(\sigma, s)$ be a function of the spins $\sigma_i$ with $i \geq 0$ and the block spins $s_i$ with $i < 0$. We define

$$\langle g \rangle_c = \frac{1}{Z} \sum_{\sigma_0, \sigma_1} g(\sigma, s) k(\sigma_0, \sigma_1; s_0) \exp(\sum_{X,Y:X,Y<0} c(X,Y)\sigma(X)s(Y))$$  (23)

where $Z$ is defined by $<1> = 1$. Note that the Hamiltonian does not appear in the definition of $<g>_c$. As before, we include the Hamiltonian by considering $<g>_{c+\hat{H}}$. Note that $<g>_c$ is a function of $\sigma_i$ with $i \geq 2$ and $s_i$ with $i \leq 0$. At this point $<g>_c$ is only defined if $c$ and $g$ have finite support, and so $<g>_{c+\hat{H}}$ is only defined if $c$, $\hat{H}$ and $g$ have finite support.

If $c$, $\hat{H}$ and $g$ have finite support, then eq. (13) implies

$$\frac{\partial f(c, V + 2k, W + k)}{\partial c(X, Y)} = \frac{1}{N} \sum_{\sigma, s} \sigma(V+2k)s(W+k) <\sigma(X)s(Y)>_c$$

Since $X$ contains at least one of 0, 1, the term $\sigma(X)$ contains a factor of either $\sigma_0$, $\sigma_1$ or $\sigma_0\sigma_1$. The rest of $\sigma(X)$ can be factored out of the expectation. So we need to consider $<\sigma_0>_c$, $<\sigma_1>_c$ and $<\sigma_0\sigma_1>_c$. They have expansions of the form

$$<\sigma(A)>_c = \sum_{U,T:U\geq2,T\leq0} d(c, A, U, T)\sigma(U)s(T)$$
where \( A \) is \( \{0\} \), \( \{1\} \) or \( \{0, 1\} \). Note that this sum includes the term with \( U = \emptyset \) and \( T = \emptyset \). The coefficients \( d(c, A, U, T) \) are given by

\[
d(c, A, U, T) = \frac{1}{N} \sum_{\sigma, s} \sigma(U)s(T) < \sigma(A) >_c
\]

For \( c \) with finite support \( d(c, A, U, T) \) is nonzero for only finitely many \( U, T \).

Let \( A = X \cap \{0, 1\} \). So \( \sigma(X) = \sigma(A)\sigma(X \setminus A) \). Using \( \sigma(P)\sigma(Q) = \sigma(P \Delta Q) \) we have

\[
\frac{\partial f(c, V + 2k, W + k)}{\partial c(X, Y)} = \frac{1}{N} \sum_{\sigma, s} \sigma((V + 2k)\Delta(X \setminus A)) s((W + k)\Delta Y) < \sigma(A) >_c
\]

Thus

\[
\frac{\partial F(c, V, W)}{\partial c(X, Y)} = \sum_{k=1}^{\infty} d(c, A, (V + 2k)\Delta(X \setminus A), (W + k)\Delta Y)
\]

So for any \( X, Y \) with \( X \in \mathcal{X}, Y < 0 \),

\[
\sum_{V \in \mathcal{X}, W < 0} \left| \frac{\partial F(c, V, W)}{\partial c(X, Y)} \right| \exp(\mu|V| - \mu|X|) \leq \sum_{V \in \mathcal{X}, W < 0} \sum_{k=1}^{\infty} \sum_{10}^{(25)}
\]

It is trivial to check that for any sets \( P, Q \) we have \(|P| \leq |P \Delta Q| + |Q| \). So we have

\[
|V| = |V + 2k| \leq |(V + 2k)\Delta(X \setminus A)| + |X \setminus A|
\]

We have \(|X \setminus A| = |X| - |A| \). Thus (25) is bounded by

\[
\leq e^{-\mu|A|} \sum_{V \in \mathcal{X}, W < 0} \sum_{k=1}^{\infty} |d(c, A, (V + 2k)\Delta(X \setminus A), (W + k)\Delta Y)| \exp(\mu|(V + 2k)\Delta(X \setminus A)|)
\]

We order this sum as

\[
e^{-\mu|A|} \sum_{V \in \mathcal{X}} \sum_{k=1}^{\infty} \sum_{W < 0} |d(c, A, (V + 2k)\Delta(X \setminus A), (W + k)\Delta Y)| \exp(\mu|(V + 2k)\Delta(X \setminus A)|)
\]

Recall that \( Y \) is fixed here. Now consider a fixed \( k \). The map \( W \to (W + k)\Delta Y \) is one-to-one. As \( W \) ranges over all finite subsets with \( W < 0 \), \( (W + k)\Delta Y \) will include some subsets which
have a site \( \geq 0 \). But the coefficient \( d() \) vanishes for these cases. So we get an upper bound on the above by replacing \( (W + k)\Delta Y \) by just \( W \) and summing over \( W \) subject to \( W \leq 0 \):

\[
\leq e^{-\mu|A|} \sum_{V \in X} \sum_{k=1}^{\infty} \sum_{W \leq 0} |d(c, A, (V + 2k)\Delta(X \setminus A), W)| \exp(\mu|(V + 2k)\Delta(X \setminus A)|)
\]

Now \( V \in X \) implies at least one of 0, 1 is in \( V \). So as we sum over \( V \in X \) and \( k \), \( V + 2k \) will range over all subsets \( \geq 2 \). Since \( X \) is fixed, \( (V + 2k)\Delta(X \setminus A) \) will also range over all subsets \( \geq 2 \). So the above equals

\[
e^{-\mu|A|} \sum_{U \geq 2} \sum_{W \leq 0} |d(c, A, U, W)| \exp(\mu|U|)
\]

Note that this equals \( e^{-\mu|A|}||<\sigma(A)>_c|| \). Define

\[
\mathcal{D}(c) = \max_A e^{-\mu|A|}||<\sigma(A)>_c|| = \max_A e^{-\mu|A|} \sum_{U \geq 2} \sum_{T \leq 0} |d(c, A, U, T)| e^{\mu|U|} \tag{26}
\]

where the max is over non-empty subsets \( A \) of \( \{0, 1\} \). Thus we have proved

**Lemma 1.**

\[
||DF(c)|| \leq \mathcal{D}(c) \tag{27}
\]

We work in the following open subset of the Banach space:

\[
O = \{ c : c = c_0 + \delta, c_0 \ has \ finite \ support, ||\delta|| < \ln 2, \mathcal{D}(c_0) + \rho(||\delta||) < 1 \} \tag{28}
\]

where \( \rho(r) = 2(e^r - 1)/(2 - e^r) \). Note that \( \rho(r) \to 0 \) as \( r \to 0 \).

**Lemma 2.** Let \( g \) be a function of \( \sigma, s \) with finite support. Let \( c \) have finite support with \( \mathcal{D}(c) < 1 \). Define \( <g>_c \) by \( \tag{23} \). Then

\[
||<g>_c|| \leq ||g|| \tag{29}
\]

If \( c_1, c_2 \) have finite support with \( \mathcal{D}(c_i) < 1, i = 1, 2 \) and \( ||c_1 - c_2|| < \ln 2 \), then

\[
||<g>_c - <g>_c|| \leq \rho(||c_1 - c_2||) ||g|| \tag{30}
\]

and

\[
|\mathcal{D}(c_1) - \mathcal{D}(c_2)| \leq \rho(||c_1 - c_2||) \tag{31}
\]

The set of \( c \) with finite support and \( \mathcal{D}(c) < 1 \) is dense in \( O \). Thus there is a unique continuous extension of the definition of \( <g>_c \) to all \( c \in O \) and all \( g \) with \( ||g|| < \infty \). Furthermore \( \tag{29} \) holds for all \( c \in O \) and \( \tag{30}, \tag{31} \) hold for all \( c_1, c_2 \in O \) with \( ||c_1 - c_2|| < \ln 2 \).
Proof: The bound (29) will follow immediately from this bound for the case of \( g(\sigma) = \sigma(V)s(W) \) where \( V, W \) are finite subsets. If \( V \) does not contain either of 0, 1, then \( < \sigma(V)s(W) >_c \) is just \( \sigma(V)s(W) \) and the bound is immediate. Now suppose \( A = V \cap \{0, 1\} \) is non-empty. Then

\[
< \sigma(V)s(W) >_c = \sigma(V \setminus A)s(W) < \sigma(A) >_c \\
= \sigma(V \setminus A)s(W) \sum_{U \geq 2, T \leq 0} d(c, A, U, T) \sigma(U)s(T) \\
= \sum_{U \geq 2, T \leq 0} d(c, A, U, T) \sigma(U \Delta(V \setminus A))s(T \Delta W)
\]

So

\[
|| < \sigma(V)s(W) >_c || \leq \sum_{U \geq 2, T \leq 0} |d(c, A, U, T)| \exp(\mu|U \Delta(V \setminus A)|)
\]

We use

\[
|U \Delta(V \setminus A)| \leq |U| + |V \setminus A| = |U| + |V| - |A|
\]

So the above is

\[
\leq \sum_{U \geq 2, T \leq 0} |d(c, A, U, T)| \exp(\mu(|U| + |V| - |A|)) \\
\leq D(c) \exp(\mu|V|)
\]

Since \( ||\sigma(V)|| = \exp(\mu|V|) \) and \( D(c) < 1 \), the bound follows.

For the bound (30), we denote \( c_1 \) by \( c \) and let \( \delta = c_2 - c_1 \) so \( c_2 = c + \delta \). Then we can express the quantity we need to bound as

\[
< g >_{c+\delta} - < g >_c = \frac{< g \exp(\delta) >_c}{< \exp(\delta) >_c} - < g >_c \\
= \frac{< g \exp(\delta) >_c - < g >_c < \exp(\delta) >_c}{< \exp(\delta) >_c}
\]

Let \( k = \exp(\delta) - 1 \). Since we are in a Banach algebra we have

\[
||k|| = \left| \sum_{n=1}^{\infty} \frac{\delta^n}{n!} \right| \leq \sum_{n=1}^{\infty} \frac{||\delta||^n}{n!} = \exp(||\delta||) - 1
\]

Note that since \( ||\delta|| < \ln 2, ||k|| < 1 \). The quantity we need to bound is

\[
\frac{< g(1+k) >_c - < g >_c < 1+k >_c}{< 1+k >_c} = \frac{< gk >_c - < g >_c < k >_c}{1+ < k >_c}
\]
Now we use (29) and a power series expansion and the fact that we are in a Banach algebra to see
\[ \|(1 + \langle k \rangle)^{-1}\| \leq (1 - \|k\|)^{-1} \]
We use
\[ \| \langle gk \rangle \| \leq \|g\| \|k\| \]
\[ \| \langle g \rangle \| \leq \|g\| \|k\| \]
Thus
\[ \| \langle g \rangle_{c+\delta} - \langle g \rangle_{c} \| \leq 2\|g\| \|k\|(1 - \|k\|)^{-1} \]
\[ \leq 2\|g\| (\exp(\|\delta\|) - 1)(2 - \exp(\|\delta\|))^{-1} = \rho(\|\delta\|) \|g\| \]

The final bound (31) follows from (29) and the definition of \( D(c) \).

**Lemma 3.** Let \( c_1, c_2 \) have finite support with \( \|c_1 - c_2\| < \ln 2 \). Suppose that
\[ D(c_i) + \rho(\|c_1 - c_2\|) \leq 1 \quad (32) \]
for \( i = 1, 2 \). Then
\[ \|DF(c_1) - DF(c_2)\| \leq \rho(\|c_1 - c_2\|) \]
\[ \|F(c_1) - F(c_2)\| \leq \|c_1 - c_2\| \]

**Proof:** The first assertion follows immediately from lemmas [1] and [2]. The first assertion and the hypotheses imply
\[ \|DF(tc_1 + (1 - t)c_2)\| \leq 1 \quad (33) \]
for \( 0 \leq t \leq 1 \). The second assertion then follows from the first by writing \( F(c_1) - F(c_2) \) as the integral of the derivative of \( F(tc_1 + (1 - t)c_2) \) with respect to \( t \).

Recall that we want to solve the fixed point equation \( F(c + \hat{H}) = c \), but the Hamiltonian \( \hat{H} \) does not appear in the lemmas above. Let \( \hat{H}_0 \) be a truncation of the Hamiltonian \( H \) such that \( \hat{H}_0 \) has finite support. Then the above lemmas show \( F(c + \hat{H}_0) \) is defined if \( c + \hat{H}_0 \) is in \( O \), and so \( F(c + \hat{H}) \) is defined if \( \|\hat{H} - \hat{H}_0\| \) is sufficiently small. A sufficient criteria for the existence of a solution \( c \) of \( F(c + \hat{H}) = c \) is given in the next theorem.

**Theorem 1.** Let \( \hat{H}_0 \) be a Hamiltonian such that \( \hat{H}_0 \) has finite support. Let \( h = \|\hat{H} - \hat{H}_0\| \).

Define \( D(c) \) by (26). Suppose there is an approximate fixed point \( c_0 \) such that \( D(c_0 + \hat{H}_0) < 1 \) and
\[ \inf_{r > 0} \frac{\|F(c_0 + \hat{H}_0) - c_0\| + h}{r[1 - D(c_0 + \hat{H}_0) - \rho(r + h)]} < 1 \quad (34) \]
where \( \rho(r) = \frac{2(e^r - 1)}{(2 - e^r)} \). Then there a solution to the fixed point equation \( F(c + \hat{H}) = c \).
Proof: A standard argument shows there is a fixed point if there is an \( r > 0 \) and \( C < 1 \) such that \(|DF(c + \hat{H})|| \leq C\) for \(|c - c_0|| \leq r\) and \(|F(c_0 + \hat{H}) - c_0|| \leq r(1 - C)\). By the hypothesis there is an \( r > 0 \) such that

\[
|F(c_0 + \hat{H}_0) - c_0|| + h < [1 - D(c_0 + \hat{H}_0) - \rho(r + h)]r
\]

(35)

By lemma 3

\[
|F(c_0 + \hat{H}) - c_0|| \leq |F(c_0 + \hat{H}_0) - c_0|| + h
\]

For \( c \) such that \(|c - c_0|| \leq r\), lemma 3 also implies

\[
|DF(c + \hat{H})|| \leq |DF(c_0 + \hat{H}_0)|| + \rho(|c - c_0|| + h) \leq D(c_0 + \hat{H}_0) + \rho(r + h)
\]

(36)

So we can take \( C = D(c_0 + \hat{H}_0) + \rho(r + h)\) and the theorem follows. \( \square \)

The final step is to show that the renormalized Hamiltonian is defined. The renormalized Hamiltonian is formally given by

\[
H' = \sum_W h(c + \hat{H}, W)s(W)
\]

where

\[
h(c + \hat{H}, W) = \sum_{k=-\infty}^{\infty} f(c + \hat{H}, \emptyset, W + k)
\]

(37)

If \( c + \hat{H} \) has finite support, then the series defining \( h(W) \) is finite, so \( h(c + \hat{H}, W) \) is defined. We need to extend it to \( c + \hat{H} \) that do not have finite support.

Theorem 2. Define \( O \) as in (28). Then \( h(c + \hat{H}, W) \) has a unique continuous extension to \( c + \hat{H} \in O \). Moreover, for such \( c + \hat{H} \)

\[
\sum_{W:0 \in W} |h(c + \hat{H}, W)| < \infty
\]

(38)

So the change in the renormalized Hamiltonian from flipping a single spin is finite.

Proof: It suffices to prove the statements in the theorem with \( c + \hat{H} \) replaced by \( c \). We first show that \( h(c, W) \) has a unique continuous extension from the \( c \) with finite support in \( O \) to all of \( O \). For \( c_0 \) with finite support and \( D(c_0) < 1 \), let \( r(c_0) \) be the solution to \( D(c_0) + \rho(r(c_0)) = 1 \). Let \( B_{r(c_0)}(c_0) \) denote the ball centered at \( c_0 \) with radius \( r(c_0) \). Then \( O \) is the union over \( c_0 \) with finite support and \( D(c_0) < 1 \) of \( B_{r(c_0)}(c_0) \). So it suffices to prove there is a unique continuous extension on each of these balls. Suppose \( c_0 + \delta \in B_{r(c_0)}(c_0) \) and \( \delta \) has finite support. Then we can write the difference \( h(c_0 + \delta, W) - h(c_0, W) \) as the integral of the derivative with respect
to $t$ of $h(c_0 + t\delta, W)$. The integrand can be bounded by $||\delta||$ times the following bound on the gradient of $h$:

$$\sup_{0 \leq t \leq 1} \sup_{X \in \mathcal{X}, Y < 0} \left| e^{-\mu|X|} \frac{\partial h(c_0 + t\delta, W)}{\partial c(X, Y)} \right|$$

(39)

Let $A = X \cap \{0, 1\}$. With $d()$ defined as before we have

$$\frac{\partial h(c + t\delta, W)}{\partial c(X, Y)} = \sum_{k=-\infty}^{\infty} d(c_0 + t\delta, A, X \setminus A, (W + k)\Delta Y)$$

(40)

For fixed $W$ and $Y$, the map $k \to (W + k)\Delta Y$ is one to one. So (39) is bounded by

$$\max_{A} e^{-\mu|A|} \sum_{T \leq 0} |d(c_0 + t\delta, A, X \setminus A, T)| \leq D(c_0 + t\delta) \leq 1$$

(41)

were the last inequality follows from (31). Thus $|h(c_0 + \delta, W) - h(c_0, W)| \leq ||\delta||$. Thus $h(c, W)$ has a unique continuous extension to all of $B_{\epsilon(c_0)}(c_0)$.

For the last assertion of the theorem we need to sum over $W$ which contain 0. However, the map from $W, k$ to $W + k$ is not one to one if we allow all $W$ containing 0. (Given $V$, the number of $W, k$ such that $W + k = V$ is $|V|$.) If we restrict the sum to $W$ such that $W \geq 0$ and $0 \in W$ then the map is one to one. So the argument above shows that

$$\sum_{W:0 \in W, W \geq 0} |h(c, W)| < \infty$$

This is weaker than what we want since

$$\sum_{W:0 \in W} |h(c, W)| = \sum_{W:0 \in W, W \geq 0} |h(c, W)||W|$$

We need to modify the norm we use to obtain the stronger result. In place of the norm (19) we define

$$||g|| = \sum_{X,Y} |g(X, Y)| \exp(\mu|X| + \nu|Y|)$$

(42)

where $\mu, \nu \geq 0$. Everything we have done before goes through with this norm. If the hypotheses of theorem (1) hold for some $\mu > 0$ and $\nu = 0$, then they hold for that $\mu$ and sufficiently small $\nu$. This implies

$$\sum_{W:0 \in W, W \geq 0} |h(c, W)|e^{\nu|W|} < \infty$$

for some positive $\nu$. This implies (38).
2.1 Decimation

We now consider specific RG kernels. For decimation we take

\[ k(\sigma_0, \sigma_1; s_0) = \frac{1}{2} + \frac{1}{2}s_0\sigma_1 = \delta_{s_0, \sigma_1} \]  

(43)

This fixes the original spin \( \sigma_1 \) to be equal to the block spin \( s_0 \). We take \( H_0 \) to just be the nearest neighbor part of \( H \):

\[ H_0 = -\gamma \sum_i \sigma_i \sigma_{i+1} \]  

(44)

So \( h = ||\hat{H} - \hat{H}_0|| = c(\alpha)\epsilon \) where

\[ c(\alpha) = 2 \sum_{j=2}^{\infty} j^{-\alpha} \]  

(45)

We take the approximate fixed point \( c_0 \) to just have one term:

\[ c_0 = \gamma s_{-1}\sigma_0 \]  

(46)

Then (13) becomes

\[ \exp\left[ \sum_{X,Y:X\geq2} f(c_0 + \hat{H}_0, X, Y)\sigma(X)s(Y) \right] = \sum_{\sigma_0} \exp[\gamma s_{-1}\sigma_0 + \gamma s_0s_0 + \gamma s_0\sigma_2] \]

\[ = \exp(\gamma s_0\sigma_2) \sum_{\sigma_0} \exp[\gamma s_{-1}\sigma_0 + \gamma s_0s_0] \]

\[ = \exp(\gamma s_0\sigma_2 + c(\gamma)s_{-1}s_0 + E) \]

where \( c(\gamma) = \frac{1}{2}\ln\cosh(2\gamma) \) and \( E \) is a constant we do not care about. The \( \gamma s_0\sigma_2 \) term contributes \( \gamma s_{-1}\sigma_0 \) to \( F(c_0 + \hat{H}_0) \). The \( c(\gamma)s_{-1}s_0 \) term does not contribute to \( F(c_0 + \hat{H}_0) \); it contributes to the renormalized Hamiltonian. So we have \( F(c_0 + \hat{H}_0) = c_0 \). This is reflection of the fact that decimation is trivial for the Hamiltonian with \( \epsilon = 0 \).

With this approximate fixed point

\[ <\sigma_0>_{c_0 + \hat{H}_0} = \frac{d}{2}(s_0 + s_{-1}) \]

\[ <\sigma_1>_{c_0 + \hat{H}_0} = s_0 \]

\[ <\sigma_0\sigma_1>_{c_0 + \hat{H}_0} = \frac{d}{2}(1 + s_0s_{-1}) \]
where \( d = \tanh(2\gamma) \). So

\[
\begin{align*}
|| < \sigma_0 >_{c_0 + \hat{H}_0} || & = d \\
|| < \sigma_1 >_{c_0 + \hat{H}_0} || & = 1 \\
|| < \sigma_0 \sigma_1 >_{c_0 + \hat{H}_0} || & = d
\end{align*}
\]

Noting that \( d \leq 1 \) for all \( \gamma \), this yields

\[ D(c) \leq e^{-\mu} \quad (47) \]

Thus if \( \epsilon \) is sufficiently small we can choose \( \mu > 0 \) so that \( (34) \) is satisfied.

### 2.2 Majority rule

The RG kernel is

\[ k(\sigma_0, \sigma_1; s_0) = \frac{1}{2} + \frac{1}{4} s_0 (\sigma_0 + \sigma_1) \quad (48) \]

Unlike decimation, when \( \epsilon = 0 \) the fixed point equation is not satisfied by a \( c \) with finite support. Nonetheless, the fixed point equation is very well-behaved in this case. For the approximate fixed point it will suffice to only include two terms: \( \sigma_0 s_{-1} \) and \( \sigma_0 s_{-2} \). As we will see, for large \( \gamma \) the coefficient of the first term will be of the form \( \gamma + a \) where \( a \) is essentially constant, and the coefficient of the second term will be essentially constant. So we take the approximate fixed point to be

\[ c_0 = (\gamma + a) \sigma_0 s_{-1} + b \sigma_0 s_{-2} \]

So we consider

\[
\sum_{\sigma_0, \sigma_1} k(\sigma_0, \sigma_1; s_0) \exp[\gamma \sigma_0 \sigma_1 + \gamma \sigma_1 \sigma_2 + (\gamma + a) \sigma_0 s_{-1} + b \sigma_0 s_{-2}]
\]

This is an even function of \( \sigma_2, s_0, s_{-1}, s_{-2} \), so it must be of the form

\[ \exp[c \sigma_2 s_0 + d \sigma_2 s_{-1} + e \sigma_2 s_{-2} + f \sigma_2 s_{-1} s_{-2} + g s_0 s_{-1} + h s_0 s_{-2} + k s_{-1} s_{-2} + E] \quad (49) \]

Note that terms which only contain block spins do not contribute to \( F(c_0 + \hat{H}_0) \). They only contribute to the renormalized Hamiltonian. We have

\[ F(c_0 + \hat{H}_0) = c \sigma_0 s_{-1} + d \sigma_0 s_{-2} + e \sigma_0 s_{-3} + f \sigma_0 s_{-1} s_{-2} s_{-3} \]

So we have

\[ F(c_0 + \hat{H}_0) - c_0 = (c - \gamma - a) \sigma_0 s_{-1} + (d - b) \sigma_0 s_{-2} + e \sigma_0 s_{-3} + f \sigma_0 s_{-1} s_{-2} s_{-3} \quad (50) \]
We find after some algebra that

\[\begin{align*}
c &= \gamma + (-2 \ln 1.5 + x - y + z - w)/8 + O(e^{-\gamma}) \\
d &= (-2 \ln 1.5 - x + y - z + w)/8 + O(e^{-\gamma}) \\
e &= (x - y - z + w)/8 + O(e^{-\gamma}) \\
f &= (-x + y + z - w)/8 + O(e^{-\gamma})
\end{align*}\]

where

\[\begin{align*}
x &= \ln(e^{-a+b} + \frac{1}{2}e^{a-b}) \\
y &= \ln(\frac{3}{2}e^{-a+b} + \frac{1}{2}e^{a-b}) \\
z &= \ln(e^{-a-b} + \frac{1}{2}e^{a+b}) \\
w &= \ln(\frac{3}{2}e^{-a-b} + \frac{1}{2}e^{a+b})
\end{align*}\]

We will choose \(a, b\) to make the coefficients of \(\sigma_0 s_{-1}\) and \(\sigma_0 s_{-2}\) in \((51)\) almost vanish. We let \(a_0, b_0\) be the solution to the equations \(c - \gamma - a = 0\) and \(d - b = 0\) if we ignore the \(O(e^{-\gamma})\) terms. So

\[\begin{align*}
a_0 &= (-2 \ln 1.5 + x - y + z - w)/8 \\
b_0 &= (-2 \ln 1.5 - x + y - z + w)/8
\end{align*}\]

where \(x, y, z, w\) are computed using \(a_0, b_0\) for \(a, b\). Note that these equations are independent of \(\gamma\). With this choice,

\[||F(c_0 + \hat{H}_0) - c_0|| = |e|e^{\mu} + |f|e^{\mu} + O(e^{-\gamma})\]

The solution to \((51)\) is approximately given by

\[\begin{align*}
a_0 &\approx -0.18019161, \quad b_0 \approx -0.02254094
\end{align*}\]

Here and in the following we use \(\approx\) to indicate that we are giving numerical approximations that are accurate to eight decimal places. For a fully rigorous proof we should use interval arithmetic for these computations, but we have not done so. For this choice of \(a, b\) we have

\[\begin{align*}
e &\approx 0.00078810, \quad f \approx -0.00078810
\end{align*}\]

So

\[||F(c_0 + \hat{H}_0) - c_0|| \approx 0.00157619 e^{\mu} + O(e^{-\gamma})\]
The expectations $<\sigma(A)>_{c_0}$ are functions of $\sigma_1, s_0, s_{-1}, s_{-2}$. For large $\gamma$ they are independent of $\gamma$ up to terms of order $e^{-\gamma}$. Their computation is straightforward but tedious. One can give explicit but complicated expressions in terms of $a$ and $b$. We only give the numerical results for $a = a_0$ and $b = b_0$.

$$<\sigma_0>_{c_0} \approx 0.77632018s_0 + 0.22367982s_{-1} - 0.03496197s_2 + 0.03496197s_{-1}s_0$$
$$- 0.00777073s_{-2} + 0.00777073s_{-2}s_{-1}s_0 - 0.00087107s_{-2}s_0s_2 + 0.00087107s_{-2}s_{-1}s_2$$

$$<\sigma_1>_{c_0} \approx 0.69811964s_0 + 0.30188036s_{2} - 0.03145297s_{-1} + 0.03145297s_{-1}s_0$$
$$+ 0.00114994s_{-2} - 0.00114994s_{-2}s_0s_2 - 0.00114994s_{-2}s_{-1}s_0 + 0.00114994s_{-2}s_{-1}s_2$$

$$<\sigma_0\sigma_1>_{c_0} \approx 0.47443982 + 0.26691838s_0s_2 + 0.19222684s_{-1}s_0 + 0.06641495s_{-1}s_2$$
$$- 0.00662078s_{-2}s_0 + 0.00662078s_{-2}s_{-1} - 0.00202101s_{-2}s_{-2}s_0 + 0.00202101s_{-2}s_{-1}s_0$$

Thus by (26)

$$D(c_0) \approx \min\{0.07166608 + 1.01554146e^{-\mu}, 0.33563322 + 0.73187250e^{-\mu}, 0.33737535e^{-\mu} + 0.67990824e^{-2\mu}\}$$

There are many choices of $\mu$ for which the hypothesis of the theorem is satisfied for small $\epsilon$. For example, with $\mu = 1.0$ we have $||F(c_0) - c_0|| \approx 0.00428454$ and $D(c_0) \approx 0.60487407$. The infimum in (31) is approximately 0.25088335 when $\epsilon = 0$ and so it is less than 1 for small $\epsilon$.

3 Proof of LRO by reflection positivity

In this section we sketch a proof that the Hamiltonian [2] has long range order if $\epsilon > 0$ and $\gamma$ is large enough (depending on $\epsilon$).

Proposition 1. Let $1 < \alpha < 2$. Define for positive $\gamma$ and $\epsilon$

$$H = -((\gamma - \epsilon)\sum_i \sigma_i\sigma_{i+1} - \epsilon\sum_{j<k} \frac{\sigma_j\sigma_k}{|j-k|^{\alpha}}$$

For all $\epsilon > 0$ there exists $\gamma_0$ which depends on $\epsilon$ and $\alpha$ such that there is long range order if $\gamma > \gamma_0$.

For consistency with [9] we have included the nearest neighbor terms in the second sum. Note that the Hamiltonian above is the same as [2]. We emphasize that nothing in this section is new. This result follows from the methods in [9]. However, finding exactly the ingredients needed for our case in [9] takes a bit of time, so we highlight these ingredients here. We assume familiarity with the general techniques of [9].

For a positive integer $m$ we let

$$\Lambda = \{1 - m, 2 - m, \cdots - 2, -1, 0, 1, 2, \cdots, m\} \quad (52)$$
We impose periodic boundary conditions so \( m + 1 \) means \( 1 - m \) and \(-m\) means \( m \). Up to a constant, the Hamiltonian can be rewritten as

\[
H = \frac{1}{2} \sum_{j,k \in \Lambda} [(\gamma - \epsilon)N_{j,k} + \epsilon J_{j,k}] (\sigma_j - \sigma_k)^2
\]

where \( J_{j,k} = |j - k|^{-\alpha} \) and \( N_{j,k} \) is 1 if \( j, k \) are nearest neighbors and 0 if they are not. To make this periodic we take \( J_{j,k} \) to be

\[
J_{j,k} = \sum_{n = -\infty}^{\infty} \frac{1}{|j - k + 2mn|^\alpha}
\]

The Hamiltonian depends on \( m \), but we do not make this dependence explicit in the notation.

Define

\[
g_m(p) = \langle \hat{\sigma}_p \hat{\sigma}_{-p} \rangle
\]

where

\[
\hat{\sigma}_p = \frac{1}{\sqrt{2m}} \sum_{j \in \Lambda} e^{ipj} \sigma_j
\]

and \( p \) is an element of the dual lattice \( \Lambda^* = \{0, \frac{2\pi}{m}, \frac{2\pi m}{m}, \cdots, \frac{(2m-1)\pi}{m}\} \). Here \( < \cdots > \) is the Gibbs measure on \( \Lambda \). Let

\[
E(p) = \frac{1}{2} \sum_{j \in \Lambda} (1 - e^{ipj}) [(\gamma - \epsilon)N_{j,0} + \epsilon J_{j,0}] = (\gamma - \epsilon)(1 - \cos p) + \frac{\epsilon}{2} R(p)
\]

where

\[
R(p) = \sum_{n \in \Lambda, n \neq 0} J_{0,n} (1 - \cos (pn))
\]

The infrared bound says that for non-zero \( p \)

\[
g_m(p) \leq \frac{1}{2E(p)}
\]

This bound will follow from the Gaussian domination bound which we now state. For a real valued function \( h_i \) on \( \Lambda \) define

\[
Z(h) = \langle \exp\left(-\frac{1}{2} \sum_{j,k} [(\gamma - \epsilon)N_{j,k} + \epsilon J_{j,k}] |\sigma_j - \sigma_k - h_j + h_k|^2\right) >_0
\]

where \( < \cdots >_0 \) is the sum over the spins \( \sigma_i \) with \( i \in \Lambda \), normalized so that \( < 1 >_0 = 1 \). In the above the sums over \( j, k \) are over \( \Lambda \). Gaussian domination says that for all \( h \)

\[
Z(h) \leq Z(0)
\]
Proposition 2. The Gaussian domination bound \((56)\) implies the infrared bound \((54)\).

Proof: Take \(h_j = \delta e^{ip_j}\) and expand the Gaussian domination bound to second order in \(\delta\). See \[9\] for details.

Since we are in one dimension, the integral of \(1/(1 - \cos p)\) near 0 diverges. The \(R(p)\) term will act as a sort of regularizer.

Proposition 3.

\[
\sup_m \frac{1}{2m} \sum_{p \in \Lambda^*} \frac{1}{R(p)} < \infty \quad (57)
\]

Proof: This can be found in \[9\]. We give a proof since it is central to the proof of LRO. As \(m \to \infty\) the normalized sum on \(p\) converges to a normalized integral over \(p \in [-\pi, \pi]\). Since the integrand is even, we can take the integral just over \([0, \pi]\). So we need to show

\[
\int_0^\pi \frac{dp}{R(p)} < \infty \quad (58)
\]

We get a lower bound on \(R(p)\) by restricting the sum on \(n\) to those with \(\frac{\pi}{2p} \leq n \leq \frac{\pi}{p}\). For such \(n\) we have \(pn \in [\pi/2, \pi]\) so \(1 - \cos(pn) \geq 1\). We have \(J_{0,n} = n^{-\alpha} \geq p^\alpha / \pi^\alpha\). The number of \(n\) in the restricted range goes as \(\pi/(2p)\) as \(p \to 0\). So

\[
R(p) \geq cp^{\alpha - 1}
\]

for some constant \(c\). Since \(1 < \alpha < 2\) the integral of \(1/p^{\alpha - 1}\) is finite.

If we fix \(\epsilon\), then by the dominated convergence theorem,

\[
\lim_{\gamma \to \infty} \int \frac{dp}{E(p)} = \lim_{\gamma \to \infty} \int \frac{dp}{(\gamma - \epsilon)(1 - \cos(p)) + \frac{\pi}{2}R(p)} = 0
\]

So by the usual argument, for large \(\gamma\) there must be an atom at the origin in \(g_\Lambda(p)\) which shows there is long range order.

The key to proving the Gaussian domination bound is the following proposition.

Proposition 4. We write \(Z(h)\) as \(Z(h_{1-m}, h_{2-m}, \cdots, h_{-1}, h_0; h_1, h_2, \cdots, h_m)\) where the semicolon helps us see the reflection point at 1/2. Then

\[
|Z(h_{1-m}, h_{2-m}, \cdots, h_{-1}, h_0; h_1, h_2, \cdots, h_m)|^2 \leq |Z(h_m, h_{m-1}, \cdots, h_2, h_1; h_1, h_2, \cdots, h_m)||Z(h_{1-m}, h_{2-m}, \cdots, h_{-1}, h_0; h_0, h-1, \cdots, h_{1-m})|
\]
One can use this proposition to show that the maximum of $Z(h)$ is attained by an $h$ in which the $h_i$ are all equal. This of course is the same as $Z(0)$, and so proves Gaussian domination. The proof of the above can be found in [9]. The essential ingredient is that $N_{j,k}$ and $J_{j,k}$ are reflection positive. We sketch the proof that $J_{j,k}$ is reflection positive and refer the reader to [9] for the rest of the proof of the proposition.

The reflection is about $1/2$. So terms that cross this reflection point are $J_{1-k}$ where both $j$ and $k$ range from 1 to $m$. Reflection positivity follows from an integral representation for these $J$:

$$J_{j,1-k} = \int_0^1 [\lambda^{j+k-1} (1 - \lambda^{2m})^{-1} + \lambda^{m-j+m-k+1} (1 - \lambda^{2m})^{-1}] \mu(d\lambda)$$

(59)

where $\mu$ is a positive measure on $(0, 1)$. To derive this representation we start with

$$\int_0^\infty e^{-nx} x^{\alpha-1} dx = \frac{\Gamma(\alpha)}{n^\alpha}$$

(60)

Using a change of variables $\lambda = e^{-x}$, this shows there is a positive measure $\mu(d\lambda)$ on $(0, 1]$ such that

$$n^{-\alpha} = \int_0^1 \lambda^{\alpha} \mu(d\lambda)$$

(61)

By taking into account the constraints $1 \leq j, k \leq m$ we can get rid of the absolute values in the definition of $J_{j,1-k}$ in (53) and then sum the two resulting geometric series to obtain (59).

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