On system rollback and totalised fields

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April 27, 2021

Abstract

In system operations it is commonly assumed that arbitrary changes to a system can be reversed or ‘rolled back’, when errors of judgement and procedure occur. We point out that this view is flawed and provide an alternative approach to determining the outcome of changes.

Convergent operators are fixed-point generators that stem from the basic properties of multiplication by zero. They are capable of yielding a repeated and predictable outcome even in an incompletely specified or ‘open’ system. We formulate such ‘convergent operators’ for configuration change in the language of groups and rings and show that, in this form, the problem of convergent reversibility becomes equivalent to the ‘division by zero’ problem. Hence, we discuss how recent work by Bergstra and Tucker on zero-totalised fields helps to clear up long-standing confusion about the options for ‘rollback’ in change management.

1 Introduction

The assumption that it is possible to reverse changes, or create generic ‘undo’ buttons in arbitrary software systems, is a persistent myth amongst software developers, system designers, and system operators. The term ‘rollback’ is often used for this, usurped from the original usage in database transaction theory[1] to describe both the undoing of an operation, as well as so-called “time travel” in which one views past snapshots of a database[2]. In current usage, rollback refers to the act of undoing what has been done; it is intimately related to checkpointing[3, 4, 5, 6, 7], version control, and release management.

In single-threaded and parallel software applications, many authors have developed a ‘journaling’ approach to reversibility and rollback (see foregoing references on checkpointing). A stack of state-history can be kept to arbitrary accuracy (and at proportional cost), provided there is sufficient memory to document changes. In more general ‘open’ (or incompletely specified) systems the cost of maintaining history increases without bound as system complexity increases. We shall show that arbitrary choices – which we refer to as policy decisions – are required to choose remedies for incomplete specifications.

A fixed-point model of change was introduced in [8, 9], based on the notion of repairability or ‘maintenance’ of an intended state. This model is realized in the software Cfengine[10], and was further elaborated upon using an alternative formulation in [11]. The crux of this approach is to bring about a certainty of outcome, even in an incompletely specified (or ‘open’) system, and has proved to have several advantages over traditional delta approaches, including that it allows autonomic repair of developing problems. However, this certainty is brought at the expense of a loss of history that would enable the reversal of certain kinds of changes.

In this paper we discuss a formulation of policy-based change management from a different perspective: that of computation with data-types. In particular, we note the relationships to recent work by Bergstra and Tucker on division-safe calculation in algebraic
computation\cite{12, 13}. We show that reversibility in system management and totalization of rational fields are closely related.

The discussion is potentially large, so we set modest goals. We begin by reviewing basic ideas about reversibility, and then recall the notion of 'convergent' or 'desired-state' operations. We explain the relationship of these abstract operators to the zeros of rings and fields and we show how the inverse zero operation $0^{-1}$ can be viewed as an attempt to 'roll back' state from such a convergent operation, in one interpretation of configuration management. This makes a connection between 'calculation' and system configuration, implicit in the encoding of data into types. Finally, noting that zero plays two roles for $+, \cdot$ in ring computation, we compare the remedies for division-safe calculation with options for reversal in change management.

2 Notation

We follow the notation of \cite{8} in writing a generic operators as letters with carets over them, e.g. $\hat{O}_1, \hat{O}_2$, etc, while generic states on which these operators act are written in Dirac notation $|q\rangle$. The resulting state after applying an operator $\hat{O}_1$ to a system in the state $|q\rangle$ is written as $\hat{O}_1|q\rangle$.

The symbol $t$ will represent a time, and $\delta t$ is a time increment. Similarly $\delta X$ will imply a relative change in quantity $X$. $S$ will denote a general set, $R$ a ring and $F$ a field. $G$ is a group with elements $g_1, g_1^{-1}, \ldots, I$, where $I$ is the identity element.

When discussing rings and fields and division-safe calculation, we shall stay close to the notation of Bergstra and Tucker\cite{12, 13}.

3 Modelling configuration parameters

Configuration management is largely viewed as a process of setting and maintaining the values of configuration parameters that control or influence software behavior. A parameter is usually a number or string having a finite (though potentially large) set of useful values. For example, one parameter might be the number of threads to use in a web server, with a typical value of 10. Another might be a ‘yes’ or ‘no’ string determining whether a web server should be started at boot time.

Given that these parameters are viewed as numbers and strings, we propose a field structure for each configuration parameter $X$ by injectively mapping its possible values (as a set $G_X$) to a subset of some field $(F_X, +, \cdot)$, by an injection $\phi : G_X \rightarrow F_X$. There are three possible structures for $G_X$, including sets of rational numbers, finite sets of numbers, and sets of strings. If $G_X = \mathbb{Q}$ is the set of rational numbers, $G_X$ maps to itself. Finite sets of integers $G_X \subset \mathbb{Z}$ containing $n$ possible values can be mapped to the first $n$ integers in $\mathbb{Q}$, starting from 0. String parameter sets containing a finite number of values can be likewise mapped to the first $n$ integers in $\mathbb{Q}$. For example, a string parameter taking the values 'yes' and 'no' might be mapped via:

\begin{align*}
\text{‘yes’} &\rightarrow 1 \\
\text{‘no’} &\rightarrow 0 
\end{align*}

The purpose of $\phi : G_X \rightarrow \mathbb{Q}$ is to impart meaning to the field operations $+$ and $\cdot$ for parameter values in $G_X$. We may extend $G_X$ to a set $G'_X$ by adding potentially meaningless values, and extend $\phi$ to a bijection $\phi' : G'_X \rightarrow F_X$. The exact structure of this mapping does not matter. We may thus define

\begin{align*}
q + r &\equiv \phi'^{-1}(\phi'(q) + \phi'(r)) \\
q \cdot r &\equiv \phi'^{-1}(\phi'(q) \cdot \phi'(r))
\end{align*}
whenever \( \phi'^{-1} \) exists. Since \( F_X \) is a field, + and \( \cdot \) for \( G'_X \) satisfy the usual field axioms:

\[
\begin{align*}
+ & \quad : \quad G'_X \times G'_X \to G'_X, \\
\cdot & \quad : \quad G'_X \times G'_X \to G'_X, \\
\exists 0 & \in G'_X \quad | \quad x + 0 = 0 + x = x, \quad \forall x \in G'_X \\
\exists 1 & \in G'_X \quad | \quad 1 \cdot x = x \cdot 1 = x, \quad \forall x \in G'_X \\
\forall x, \exists -x & \in G'_X \quad | \quad x + (-x) = (-x) + x = 0 \\
\forall x \in G'_X, x \neq 0, \exists x^{-1} & \in G'_X \quad | \quad x \cdot x^{-1} = x^{-1} \cdot x = 0 \\
\forall x, y & \in G'_X \quad | \quad x + y = y + x \\
\forall x, y & \in G'_X \quad | \quad x \cdot y = y \cdot x \\
\forall x, y, z & \in G'_X \quad | \quad (x \cdot y) \cdot z = x \cdot (y \cdot z) \\
\forall x, y, z & \in G'_X \quad | \quad (x + y) + z = x + (y + z) \\
\forall x, y, z & \in G'_X \quad | \quad (x + y) \cdot z = (x \cdot z) + (y \cdot z).
\end{align*}
\]

(3)

The point of this discussion is to make clear that[8]:

**Proposition 1** Without loss of generality, we may consider the potential values \( G_X \) of any configuration parameter \( X \) to have a field structure \( (G'_X, +, \cdot) \) for some \( G'_X \supseteq G_X \), where \( G'_X \) is isomorphic to some field \((F_X, +, \cdot)\).

Usually, the structure of \( \phi' \) is simple. For example, via the mapping in Equation[1]

\[
\begin{align*}
\text{‘yes’} + \text{‘no’} & = \text{‘yes’} \\
\text{‘no’} + \text{‘no’} & = \text{‘no’} \\
\text{‘yes’} \cdot \text{‘yes’} & = \text{‘yes’} \\
\text{‘yes’} \cdot \text{‘no’} & = \text{‘no’}
\end{align*}
\]

(4)

Thus ‘yes’ is the multiplicative unit and ‘no’ is the additive unit of \( G'_X \), respectively.

In the rest of this paper, we will not consider the semantics of \( G'_X \), so there is no need to distinguish between the base field \((F_X, +, \cdot)\) and its image \((G'_X, +, \cdot)\) in parameter space. We will use \((F_X, +, \cdot)\) to refer both to the base field and its isomorphic image in parameter space.

### 4 Modeling parameter changes

Viewing parameter values as a subset of a field (e.g., the rational numbers), with corresponding algebraic structure, allows us to distinguish three approaches to change in the value of a parameter, making precise the notion of change \( q \to q + \delta q \), used in [8]. We call the three approaches relative \((\Delta)\), absolute \((C)\), and multiplicative \((\mu)\) or scale change, and we now wish to separate these, so as to distinguish their properties more clearly.

We partition the field algebra into partial functions using a trick from representation theory to write binary addition in the form of a parameterized unary group multiplication by introducing a tuple form with one extra dimension[13]. We write the parameter \( X \) as a vector \(|X\rangle\):

\[
|X\rangle = \begin{pmatrix} X \\ 1 \end{pmatrix}
\]

(5)

and use standard matrix algebra to express changes, building on + and \( \cdot \) for elements of \( F_X \).

Using this notation, a *multiplicative change* in a parameter \( X \) is the result of a matrix operation of the form:

\[
|X'\rangle = \mu(q) \, |X\rangle = |q \cdot X\rangle
\]

(6)
where \( q \) is an element of the field \((F_X, +, \cdot)\) and \( \mu(q) \) is defined as:

\[
\mu(q) = \begin{pmatrix} q & 0 \\ 0 & 1 \end{pmatrix}
\]

This has the effect of setting an addition operation for \( F_X \) and semantically, is a scaling operation.

An absolute change is the equivalent of setting \( X' = q \cdot X \), and semantically, is a scaling operation.

Any relative change is a linear change. The converse is not true; the linear operators \( \mu(x) \) and \( C(x) \) are not equivalent to relative operators.

Composing combinations of \( \mu(q) \), \( C(q) \), and \( \Delta(q) \) by matrix multiplication always results in a linear operator of the form:

\[
\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}
\]

where \( a \) and \( b \) are elements of \( F_X \).

Note that \( \mu(F_X \setminus \{0\}) = \{\mu(q) \mid q \in F_X \setminus \{0\}\} \) is a (multiplicative) Abelian group, because \( \mu(q) \) has multiplicative inverse \( \mu(q^{-1}) \) for \( q \neq 0 \). Likewise \( \Delta(F_X) = \{\Delta(q) \mid q \in F_X\} \) is a (multiplicative) Abelian group, where \( \Delta(q) \) has multiplicative inverse \( \Delta(-q) \). \( C(q) \), by contrast, is always singular and has no multiplicative inverse, so that \( C(F_X) = \{C(q) \mid q \in F_X\} \) is not a group.

Also note that

\[
\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} \begin{pmatrix} c & d \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} ac & ad + b \\ 0 & 1 \end{pmatrix}
\]

while

\[
\begin{pmatrix} c & d \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} ac & bc + d \\ 0 & 1 \end{pmatrix}
\]

so multiplication of elements in the span of \( \mu(F_X) \cup \Delta(F_X) \cup C(F_X) \) is only commutative if \( ad + b = bc + d \).

Note that the vectors \(|q\rangle\) can still be thought of as a field with operations:

\[
|q\rangle + |r\rangle = \begin{pmatrix} q \\ 1 \end{pmatrix} + \begin{pmatrix} r \\ 1 \end{pmatrix} = \begin{pmatrix} q + r \\ 1 \end{pmatrix} = |q + r\rangle
\]

and

\[
|q\rangle \cdot |r\rangle = \begin{pmatrix} q \\ 1 \end{pmatrix} \cdot \begin{pmatrix} r \\ 1 \end{pmatrix} = \begin{pmatrix} q \cdot r \\ 1 \end{pmatrix} = |q \cdot r\rangle
\]

Thus it is reasonable to write things like \( \delta q = |q_1\rangle - |q_2\rangle \) and \( |q_2\rangle = |q_1\rangle + \delta q \). We will often write the latter as \(|q_1 + \delta q\rangle\) without confusion, and will often switch between additive (\(|q + \delta q\rangle\)) and multiplicative (\(\Delta(\delta q) \mid q\rangle\)) representations of addition.
5 Totalisation and rollback

Obviously, some concept of rollback is possible in the above system only if the effect of each operator can be undone. This is only possible if the operators form a group so that inverses always exist:

Lemma 1 The span of \( \mu(F_X) \cup \Delta(F_X) \cup C(F_X) \) form a multiplicative group iff \( F_X \) is 0-totalized.

Proof 1 An arbitrary operator has the form

\[
\begin{pmatrix}
a & b \\
0 & 1
\end{pmatrix}
\]

(17)

so that it maps \( |X\rangle \) to \( |a \cdot X + b\rangle \). The inverse operation for this mapping (if it exists) maps \( |Y\rangle \) to \( |(Y - b)/a\rangle \). This always exists iff it exists for \( a = 0 \), which in turn holds iff \( F_X \) is 0-totalized.

The operators that are not already invertible in non-totalised fields include \( C(q) \) and \( \mu(0) \); inverting these requires an explicit division by zero.

The work of Bergstra and Tucker views field totalisation as an information problem. Totalising a field is a matter of remembering – for each value 0–h how it came about, so that dividing and multiplying by 0 can be accomplished algebraically. E.g., the fact that \( 0 \cdot X = 0 \) means that for the latter \( 0/0 = 0 \cdot X/0 = X \cdot (0/0) = X \cdot 1 = X \). This can only be done if we somehow remember that the 0 was generated via the operation \( 0 \cdot X \).

Division-safe calculation refers to the situation in which this kind of history is not needed. One of the key contributions of Bergstra and Tucker is to demonstrate just how difficult it is to determine whether information is being lost.

For us, rollback is similarly an information problem. When we employ the operator \( C(q) \), we lose information. Zero-totalizing the base field \( F_X \) is equivalent with saving that (algebraic) information for later use, to compute \( C^{-1}(q) \). In like manner, “division-safe” calculation is analogous to “reversal-safe” rollback, in the sense that the concepts of safety are equivalent. A rollback is reversal-safe exactly when the operations that led to the current state are division-safe.

Our problem is subject, however, to more forms of information loss than zero-totalisation can correct. To model other losses, we must consider issues of time and determinism.

6 Modeling changes over time

So far we have a non-temporal model of change; however, since operators \( C(q) \) do not commute, partial orderings of operations are important and time is the natural expression of sequence at our system level of abstraction. Using our framework we can now say that between any two times \( t \) and \( t + \delta t \), a system state \( |q\rangle \) might change from \( |q\rangle \) to \( |q + \delta q\rangle \).

Starting at a low level, change can be modeled by a finite automaton, where the transitions are operators \( \tilde{O} \) as above. The changes applied after a finite series of steps can be represented as a matrix product of the form \( \tilde{O}_1 \cdots \tilde{O}_n \).

In automaton theory, one makes the distinction between deterministic and non-deterministic automata. A deterministic automaton is a 5-tuple

\[
M_D = (Q, A, |q_i\rangle, Q_f, \Delta_D),
\]

(18)

where \( Q \) is a set of states, \( A \) is an alphabet of input instructions, \( q_i \) is an initial state, \( Q_f \in Q \) a set of possible final states and \( \Delta_D : Q \times A \rightarrow Q \) is a transition function that takes the automaton from a current state to its next state, deterministically in response to a single input symbol from the alphabet. Such a string of operational symbols is the basis for
a ‘journal’ that is intended to track the changes made. In automata which form ‘sufficiently dense’ graphs, the transition function’s effect may also be seen as an evolution operator, driving the system through a path of states

\[ \Delta_D(I) : |q⟩ \rightarrow |q + δtq⟩ \]  

(19)

This mapping might or might not be a bijection; it might or might not possess an inverse. Although automata are considered to be a model for computation or even grammars, they can be used to describe change at any ‘black box’ level of system description, as the model is entirely general.

A non-deterministic automation is almost the same as a deterministic one, except that its transition function \( \Delta_N : Q \times \{ A \cup 0_+ \} \rightarrow Q \), accepts one more pseudo-symbol, \( 0_+ \), which is the empty input string. Thus, a non-deterministic automaton can make transitions spontaneously, unprompted by input. In the language of \([8]\), a deterministic automation is a closed system and a non-deterministic one is an open system.

Non-deterministic changes are common in real systems. Examples of non-deterministic changes include:

- Delete of all files from a computer.
- Remove a firewall, system is infected by virus, replace firewall, system is still infected by virus.
- Checkpointing: erase state and replace with a stored image from time \( t_0 \), all history is lost between \( t_0 \) and now.

Note that there is a one-to-one correspondence between input and output only in the deterministic case. However, there are very few closed deterministic automata in real-world computing environments. Networks of users operating multi-tasked, multi-threaded applications create the high level appearance of many overlapping non-deterministic automata.

**Definition 1** We define a ‘computer system’ to be a non-deterministic automaton, represented as a set of states with types, and data sets that are isomorphic through extension to the rational numbers.

One must assume non-determinism of all actual systems, because in any modern, preemptive operating system high level changes to observable data objects cannot be traced to an alphabet of intentionally applied and documented operations, thus there are apparent transitions that cannot be explained by a journal.

### 7 Journals and histories of change

In any solution generated by a difference equation (or transition function), the conversion of small increments or ‘deltas’ into an absolute state requires the specification of end-points, analogous to the limits of a contour integral in calculus along a well-defined path \( P \):

\[ |q_i⟩ − |q_f⟩ = \int_{P_i}^{P_f} dq. \]  

(20)

This path corresponds to a sequence of input symbols for an automaton, corresponding – in our case – to operators to be applied. The analogue in terms of group transformations is to start from an origin state, or ‘ground state’ \( |0⟩ \) (often called a baseline state in system operations), and to apply relative changes sequentially from this to achieve a final desired outcome.

The choice of the baseline state lies outside of the specification of the change calculus. The origin or baseline state is an ad hoc fixed point of the system, by virtue of an external
specification alone. It is arbitrary, but usually plays a prominent role in system operators’
model of system change. In this work, the choice of a baseline state is part of what we shall
refer to as a calibration of the system, but counter to tradition we shall advocate calibration
of the end state rather than the ad hoc initial state.

To model intended versus actual change, we introduce the notion of a journal, inspired
by the notion of journaling in filesystems. A journal is a documentation of changes applied
to a system intentionally, noting and remembering that – in real systems – this can be
different from what actually takes place.

**Definition 2 (Journal J)** A complete, ordered sequence of all input symbols passed to an
automaton \( \alpha^* \) from an initial time \( t_i \) to a final time \( t_f \) is called the automaton’s journal
\( J = (t_i, t_f) \). Each symbol \( \alpha \) corresponds to a change in system state \( \delta_\alpha \). A journal has
a scope that is known to the user or process that writes the journal. A journal change \( \delta_J \) involves adding or removing symbols in \( \alpha \) to \( J \), and adjusting the times.

Two journals \( J_1 \) and \( J_2 \) may be called congruent if they have the same number of
symbols \( |J_1| = |J_2| \) and every symbol is identically present and in the same order [16].

**Lemma 2** The final state \( |q_f\rangle \) obtained by applying congruent journals of transitions \( J_1, J_2 \)
to identical automata \( M_1, M_2 \) is identical, iff the initial states \( |q_i\rangle \) are identical, and \( M_1 \)
and \( M_2 \) are deterministic.

This follows from the definitions of (non-)deterministic automata which allows spontaneous changes \( \delta_0 \). To record all changes in a non-deterministic system we need to record absolute state even when no input change is made. This brings us to:

**Definition 3 (History)** A complete, ordered stack of all intermediate snapshots of a sys-
tem’s total state \( |q_f\rangle(t) \) output by an automaton at all times \( t \) between an initial time \( t_i \)
to a final time \( t_f \) is called the automaton’s history \( H \). A change \( \delta_H \) involves pushing or
popping the complete current state onto the stack \( H \), and adjusting the times.

The history \( H \) is capable of including states that were not directly affected by the journal transitions \( \delta_\alpha \). We use a stack as a convenient structure to model histories; see for instance [17] and references for a discussion of stacks. The ability to model system configuration by relative changes is affected by the following lemma:

**Lemma 3** For any automaton \( M \), \( |H_M| \geq |J_M| \), and \( |H_M| = |J_M| \) iff \( M \) is a determi-

nistic automaton (closed system).

The proof follows from the form of the transition functions for automata, and the possibility of one or more occurrences of \( 0_\alpha \) in the input of a non-deterministic automaton. In a deterministic system each \( \alpha \) leads to a unique labeled transition \( \delta_\alpha q \), and vice versa. In the non-deterministic case, the history can contain any number of changes \( \delta_0 \) in addition to the \( \alpha \), thus the length of the history is greater than or equal to the length of the journal.

A journal is thus a sequence of intended changes, whereas a history is a sequence of actual changes.

**Definition 4 (Roll-back operation \( J^{-1} \))** The inverse application of a string of inverse jour-
nal operations is called a roll-back operation. The inverse is said to exist iff every operation
symbol in the journal has a unique inverse.

For example, for relative change:

\[
J(q, q') : |q\rangle \mapsto |q + \delta q_1 + \delta q_2 + \delta q_3 \rangle \equiv |q'\rangle \quad (21)
\]

and

\[
J^{-1}(q, q') : |q'\rangle \mapsto |q' - \delta q_3 - \delta q_2 - \delta q_1 \rangle \equiv |q\rangle \quad (22)
\]
Lemma 4 A roll-back journal $J^{-1}(q_i, q_f)$, for automaton $M$, starting from state $|q_f\rangle$ will result in a final state $|q_i\rangle$ iff $M$ is deterministic and $J^{-1}$ exists.

Proof 2 Assume that $M$ is non-deterministic; then the transition to state $q_f$ is only a partial function of the journal $J$, hence $J^{-1}$ has more than one candidate value and thus cannot exist. If $M$ is deterministic then the inverse exists trivially by construction, provided that each operation in the journal exists.

Setting aside technical terminology, the reason for a failure to roll-back is clearly the loss of correspondence between journal and history caused by changes that happen outside the scope of the intended specification. This loss of correspondence can happen in a number of ways, and (crucially) it is likely to happen because today’s computer systems are fundamentally non-deterministic\footnote{In \cite{8}, it was pointed out that this mirrors results in information theory\cite{13} about transmission of data over noisy channels, for which one has the fundamental theorem of channel coding due to Shannon\cite{19} that enables the re-assertion of correspondence between a journal (transmitted data) and actual history (received data) over some time interval. However, we shall not mix metaphors by pursuing this point here.}

Definition 5 (Commit and Restore operations) A commit operation at time $t$ is a system change $\hat{g}$ followed by a push of current history state onto a stack as consecutive operations:

$$\text{commit}(t) : (\hat{g}, \text{push}(|q\rangle(t))) \quad (23)$$

A restore operation is a sequence of one or more operations:

$$\text{restore}(t) : \text{pop}(|q\rangle) \quad (24)$$

These operations are typical of version control schemes, for example. The importance of this construction is that previous states can be recaptured regardless of whether the operation $\hat{g}$ is invertible or not.

Lemma 5 For automaton $M$, $n$ consecutive restore operations starting from $t_f$, are the inverse of $n$ consecutive commit operations ending at $t'_f$, iff the journal of changes between $t_f > t'_f$ and $t'_f$ is empty and $M$ is a deterministic automaton.

The proof, once again, follows from the absence of uncaptured changes. If $t'_f > t_f$ and the journal is empty then the only changes that can have occurred come from symbols $0_+$, but these only occur for non-deterministic $M$. We add the following to this:

Lemma 6 A system journal $J$ cannot be used to restoring system state for arbitrary changes $\hat{g}$.

This result is clear from the independence of the restore operation on $\hat{g}$, and the lack of a stack of actual state in a journal $J$.

What the foregoing discussion tells us is that there is no predictable outcome, either in a forward or a reverse direction, in an open (non-deterministic) system using relative change, and that a journal is quite useless for undoing changes that have no inverse. System configuration is analogous to making calculations in which variables change value spontaneously (as in fact they do without error correction at the hardware level). To make change computation predictable, we need to fix the outcomes rather than the sequences of operations, using ‘singular change operations’ for computing the final state. This was the main observation learned in the development of Cfengine\cite{9,10,20}.
8 Singular transitions and absolute change

In the foregoing cases, the initial choice of state $|q_i\rangle$ was external to the specification of the change, and was the 'origin' of a sequence of changes in a journey from start to finish. This relative ('sequential process') approach to change is deeply ingrained in management and computing culture, but it fails to bring the require predictability due to underlying system indeterminism. The problem is the reliance on the $+$ operation to navigate the state space, so our next step is to suppress it.

Now consider a class of transitions that are not usually considered in classic finite state machines. These are (non-invertible) elements $\hat{p}$ with the property that $\hat{p}|q\rangle = |q_0\rangle$, for any $q$. The final states are 'eigenstates' of these singular group operations: $\hat{p}|q_0\rangle = |q_0\rangle$. These effectively demote the explicit reliance on $+$ and replace it with a linear function.

**Definition 6** A singular transition function $C_{|q_0\rangle}$ is a transition from any state $|q\rangle$ to a unique absorbing state $|q_0\rangle$. It is a many-to-one transition, and is hence non-invertible without a history.

Such transition functions (operators) were introduced in [10] and described in [9], as an alternative to relative change to restore the predictability of outcome. These 'convergent operations' are based on fixed points or eigenstates of a graph. They harness the property of zero elements to ignore the current and historical states and to install a unique state regardless of the history or the determinism of the system. Such parameterized operators form a semi-group $C_{|q_0\rangle}$ with the abstract property:

$$C_{|q_0\rangle}|q\rangle = |q_0\rangle$$

$$C_{|q_0\rangle}|q_0\rangle = |q_0\rangle. \quad (25)$$

For ease of notation in the following, we drop the $|q_0\rangle$ subscript and write $C$ for $C_{|q_0\rangle}$.

The price one pays for this restoration of predictability is an inability to reverse the change. Let us suppose that an object $C^{-1}$ exists such that $C^{-1}C = I$, satisfying the latter equation. Then operating on the left, we may write using (25):

$$C^2|q_0\rangle = C|q_0\rangle$$

$$C^{-1}C^2|q_0\rangle = C^{-1}|q_0\rangle. \quad (26)$$

Thus, at $|q_0\rangle$ we have idempotence and a constraint:

$$C|q_0\rangle = C^{-1}|q_0\rangle \quad (27)$$

$$C|q_0\rangle = C^2|q_0\rangle. \quad (28)$$

The latter result (28) is independent of the existence of an inverse. For a ring, this condition is equivalent to the 'restricted inverse law' used in [12] [13], and it tells us that the inverse would have to be either 0, 1 or $+\infty$.

**Lemma 7** The operators $C_{|q_0\rangle}$ are idempotent and converge on a fixed point final state $|q_0\rangle$.

This follows immediately from eqn (28). The value of these operations is that they can be iterated endlessly, with predictable outcome, in the manner of a highly compressed system error-correction process.

**Example 1:** One can view the state $|q\rangle$ as embodied in the operator $C_{|q\rangle}$ and thus view $C_0$, $|q\rangle$, and $|q_0\rangle$ as elements of the same semigroup. Then we may write:

$$C_0|q\rangle = |q_0\rangle \quad (29)$$

$$C_0|q_0\rangle = |q_0\rangle. \quad (30)$$
Assuming an additive inverse for each element (in the statespace), and subtracting these equations for arbitrary $q$ leads to the conclusion that $C_0 = |q_0⟩ = |0⟩$, thus there is only a single object with this ability to take an arbitrary initial state and render a predictable outcome. Note that, in this representation, $C$ and $|q⟩$ belong to the same semigroup of scalars. So, choosing $|x⟩ = C_0 = |q_0⟩$, we service

$$\quad (x^{-1} \cdot x)x = x$$

using (30). This is the restricted inverse law for fields[12, 13].

The zero plays a fundamental role as an eraser. The uniqueness of zero is not an impediment to using the zero element as a ‘policy operator’ which sets an intended state, as we are free to construct a homomorphism $h(q)$ which calibrates or shifts the absolute location of the solution: e.g.

$$C_0 h(q) = h(q_0)$$
$$C_0 h(q_0) = h(q_0)$$

(32)

to shift the calibration point from $q_0$ to $q_0^*$. Example 2: Consider the tuple form used earlier, and let

$$C_0 \mapsto \begin{pmatrix} 0 & q_0 \\ 0 & 1 \end{pmatrix}$$
$$|q_0⟩ \mapsto \begin{pmatrix} q_0 \\ 1 \end{pmatrix}$$

(33)

Thinking of these as elements of the same semigroup and subtracting these equations leads to a result that is identically true, hence we are free to choose the value of $q_0$ as a matter of policy. However, one observes that $C_0$ does not possess a defined inverse according to the normal rules of fields.

9 Computing and reversing absolute states

Fields have only one element with singular properties: the zero element. It plays two distinct roles: as an identity element for the $+$ operation, and as a fixed point in scaling under $\cdot$. As a fixed point, zero annihilates state, since $0q = 0$ for any $q$. The zero element thus ignores and deletes any history that led us to the state $q$.

It is useful to think of the $C$ operators in the above as a kind of zero-element: they annihilate state in a similar way. The utility of the convergent operations for bringing about absolute change is such that it is useful to embed them in the formalism of a general field structure for computation. One motivation for this is the recent work by Bergstra and Tucker of totalization of fields, and ‘Meadows’, in which they replace the partial function (excluding 0 for division) at the heart of field computation with one that is total, up to constraints. We find their construction intriguing and highly relevant to the matter of reversibility of state. As in Bergstra and Tucker, we reason by first defining the algebraic signatures of structures.

We construct an image of a field $F$, with initial algebra $Alg(\Sigma_F, E_F)$ (as yet a regular field), by introducing a map in three piecewise partial representations $\Phi(F) = \{C(F), \Delta(F), \mu(F)\}$. The signature contains only product explicitly, as addition is concealed as described in section[4]

$$\quad : \Delta \times \Delta \to \Delta$$
$$\quad : \mu \times \mu \to \mu$$
$$\quad : C \times C \to C$$
We define $E_\Phi$ as the image of $E_F$ for the field by,

$$
\Delta(x)\Delta(-x) = I_{\Delta}, \quad \forall x \in F
$$

$$
\mu(1)\mu(x) = I_{\mu}, \quad \forall x \neq 0 \in F
$$

$$
\Delta(x)\Delta(y) = \Delta(y)\Delta(x) \quad \forall x, y \in F
$$

$$
\Delta(x)(\Delta(y)\Delta(z)) = (\Delta(x)\Delta(y))\Delta(z) \quad \forall x, y, z \in F
$$

$$
\mu(x)(\mu(y)\mu(z)) = (\mu(x)\mu(y))\mu(z) \quad \forall x, y, z \neq 0 \in F
$$

$$
C(x)C(y) = C(x) \quad \forall x \in F
$$

$$
C(x)(C(y)C(z)) = (C(x)C(y))C(z) \quad \forall x, y, z \in F
$$

An example in the matrix representation is given by:

$$
C(x) = \begin{pmatrix} 0 & x \\ 0 & 1 \end{pmatrix}
$$

$$
\Delta(x) = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}
$$

$$
\mu(x) = \begin{pmatrix} x & 0 \\ 0 & 1 \end{pmatrix}
$$

The function $C$, in any representation, gives us a way of representing absolute, not relative, changes of state. This is an important ability in maintaining order in a system, and it is the basis on which Cfengine\textsuperscript{10} operates on millions of computers around the world today. Each operation is a function of a field, in which the zero element is mapped to a desired state. The set of all possible parameterized $C(F)$ must therefore span a field, and yet it contains no (multiplicative) inverses at all. This is the interesting paradox which plays into the work of Bergstra and Tucker. The restriction $x \neq 0 \in F$ is prominent.

It is not our intention to reiterate the arguments for totalizing fields, presented by Bergstra and Tucker\textsuperscript{13}. As they point out, there is a number of ways to restore ‘faith’ in the connection between state and history of change after a zero operation, using proof systems, axioms and algebraic properties. Each brings a different kind of merit. As they remark, the issue is not so much about going backwards (reversal) as about going forwards in a way that is unaffected by an ill-defined attempt at reversal.

One remedy relies on proving the outcome of a change was not affected by the result of $0^{-1}$, i.e. the final state is independent of the path taken to evaluate it. Another involves changing the definitions of computation (change) to disallow unsafe operations. Finally one might simply give up on certain requirements so that the outcome satisfies a well defined set of equations (policies) in order to prove that the result is well-defined. Here, we observe by analogy that one may:

- Introduce a stack of history snapshots to some maximum depth\textsuperscript{17}. (This is difficult to do in arithmetic but it is plausible for some system changes.)
- Totalize the data type, using the notion of a totalized field, e.g. set $0^{-1} = 0$, or equivalently, $C^{-1} = C$.
- Perform a naive reversal and then apply some policy equational specification to clean up the result.
- Abandon the attempt to introduce reversals altogether (“rollback does not exist”).
10 Calibration of absolute state

Let us complete the abstract formalization of the operators for absolute change, which builds linear functions on top of the totalized field approach of Bergstra and Tucker, and ends with a vector space. We no longer care about $\Delta$ and $\mu$, but want to embrace the properties of the zero operators to bring predictability in a non-deterministic environment. We start with a signature for the convergent operators based on a different use of commutative rings as a parameterization of the zeroed outcome, and end with non-commutative, non-invertible representations. Let $F$ signify a field, with the usual field axioms.

We use a $\Sigma$-algebra $\Sigma_C = \{\|q\rangle \mid I, 0, \oplus, \circ, C\}$, and this is understood to extend the field algebra $Alg(\sigma_F, E_F)$. Thus, for any index set labels $\alpha, \beta$, labeling the underlying field $Q$, we have signature:

Symbols: $I, 0, |q\rangle, q \in Q$  \hspace{1cm} (52)

Operations:

$I \to F$  \hspace{1cm} (53)
$0 \to F$  \hspace{1cm} (55)

$C : F \to F'$  \hspace{1cm} (56)
$
\oplus : C(F \times F) \to C(F)$  \hspace{1cm} (57)
$
\circ : C(F) \times C(F) \to C(F)$  \hspace{1cm} (58)
$
C(F) \times F' \to F'$  \hspace{1cm} (59)

Equations($E_C$):

$C_\alpha |q\rangle = C(q_\alpha) |q\rangle = |q_\alpha\rangle$  \hspace{1cm} (61)
$C_\alpha \circ C_\beta = C_\alpha$  \hspace{1cm} (62)
$(C_\alpha \circ C_\beta) \circ C_\gamma = C_\alpha \circ (C_\beta \circ C_\gamma)$  \hspace{1cm} (63)
$C_\alpha \oplus C_\beta = C_{\alpha + \beta}$  \hspace{1cm} (64)
$(C_\alpha \oplus C_\beta) \oplus C_\gamma = C_\alpha \oplus (C_\beta \oplus C_\gamma)$  \hspace{1cm} (65)

Naturally, these are true for all $\alpha, \beta, \gamma$, and we are working with $Alg(\Sigma_F \cup \Sigma_C, E_F \cup E_C)$. The opacity of formalism belies a simple structure. Every state $|q\rangle$ is fully specified by a field value $q \in F$. Similarly, every convergent operator $C(q)$ is fully specified by a field value $q \in F$, and results in a new value $q \in F$, which obeys the zero property (61). Thus the $C$ is a transformer which takes any input state and outputs a specific state given by its label (but importantly, only one at a time). This has the ‘zero’ property of ejecting initial state and replacing it wholesale with particular one. Clearly, the operators must be idempotent from (62).

The representation in (33) is useful to see how a tuple-representation quickly captures this algebra. We say that the repeated operation of an operator $C(q_0)$ ‘converges’, as it always returns the system state to its fixed point $|q_0\rangle$. This algebra describes the behaviour of a single ‘convergent operator’ or ‘promise’ in Cfengine[9, 10]. We cannot define $C^{-1}$ because the symbol $0^{-1}$ is not defined in the underlying field $F$, but we may totalize the field[13] with corresponding merits and conditions to assign a meaning to a reversal or ‘roll-back’.

11 Re-calibration - change of policy

There is only a single fixed point for each operator $C(q_0)$. What happens when we want to change the outcome of a ‘promised state’, i.e. change the value of $q_0$? The homomorphism $h$ on states, in eqn. (32) allowed us to calibrate a single singular outcome to any field
value by shifting the zero, but this is less useful than modifying the operators themselves to bring about the desired result. This transformation then has the simple interpretation as an operator the re-calibrates the system baseline.

**Lemma 8** Each operator has only one singularity, i.e. let $F$ be a field, totalized or not, and let $0_1$ and $0_2$ be zero elements for $\cdot$, then $0_1 = 0_2$.

The proof follows by substitution of the field axioms: $0_1 x = 0_1, 0_2 x = 0_2$, setting $x = 0_2$ in the former, implies $0_1 = 0_2$. Hence the zero element is unique in a field.

This means that we cannot have more than one policy fixed point per field. In configuration terms, one cannot have more than one policy for a data item, so any path of changes parameterized by chaining $q_0$ can be uniquely characterised.

This leaves only the possibility of shifting the fixed point by re-calibration, or change of policy. This is no longer a journal of deltas, but a kind of ‘teleportation’ or ‘large transformation’ in the group theoretic sense. Given this, and the utility of formulating policy changes as applied operations, it is useful to reformulate the values in terms of vector spaces. The specification of a vector space is somewhat similar to that of a ring or field except that it is not automorphic.

Let $F$ be a field (totalized or not). A vector space of $F$ is a triple $(S, +, \cdot)$, with the equations:

\[
\begin{align*}
+ & : S \times S \to S, \\
\cdot & : F \times S \to S, \\
0 \in S & | \ x + 0 = 0 + x = x, \ \forall x \in S \\
-x \in S & | \ x + (-x) = (-x) + x = 0, \ \forall x \in S \\
x + y & = y + x, \ \forall x, y \in S \\
(x + y) + z & = x + (y + z), \ \forall x, y, z \in S \\
(\alpha \beta)z & = \alpha(\beta z), \ \forall \alpha, \beta \in F, z \in S \\
1_F & \in F | \ 1x = x1 = x, \ \forall x \in S \\
(\alpha + \beta)x & = \alpha x + \beta x, \ \forall \alpha, \beta \in F, x \in S, \\
\alpha(x + y) & = \alpha x + \alpha y, \ \forall \alpha \in F, x, y \in S,
\end{align*}
\]

(66)

The usefulness of this map is that it involves an external ‘promise’ or ‘policy’ field $F$ from which we may construct the set of $C_\alpha$, not merely an automorphic image of a single set. Thus we can separate policy from changes with convergent, fixed-point zero-operators $0_A \in F_A$, all acting on a single set of states $q \in S$. We thus arrive, by a different route, at the formulation as a vector space in ref. [8].

We note finally that a change of calibration cannot be a commutative ring.

**Lemma 9** Let $C_A$ and $C_B$ be zeros of $F_A$ and $F_B$. Then $C_A$ and $C_B$ cannot commute unless $A = B$.

**Proof 3** The proof is similar to the earlier proof of uniqueness of zero in a ring. We have $C_A q = C_A$, and $C_B q = C_B$ for all $q$. Substituting $q = C_B$ in the former, we have

\[
\begin{align*}
C_A(C_B)q & = C_A C_B = C_A \\
C_B(C_A)q & = C_B C_A = C_B
\end{align*}
\]

(67)

Thus the commutator

\[
[C_A, C_B] = C_A C_B - C_B C_A = C_A - C_B \neq 0 \ (A \neq B)
\]

(68)

This proof does not depend on the representation of $F_A$ and $F_B$, thus it applies equally to higher dimensional tuple formulations also.
12 Predicting outcome with roll-back-safe change

The problems of indeterminism cannot be addressed without absolute change operations, but these do nothing to repair the problem of unsafe reversals. The $C$ operations allow us to basically forget about indeterminism, but not irreversibility. We therefore need to find an approach analogous to that of [13] during non-commutative strings of system recalibrations. We have one advantage here: a lack of commutativity. This is in fact a strength as it makes the need for reversal practically irrelevant. In our view, there is then only one natural choice for $C^{-1}$ or $J^{-1}$ and that is to apply or re-apply the current policy $C(t)$: it is absolute, idempotent and it overrides any previous 'mistakes'.

We have also one disadvantage compared to [13] and that is that time is relevant: we cannot undo the potential consequences of being in a bad state unless we manage to totality of state within the system. For real computers, that might be almost the entire Internet (e.g. during the spread of viruses).

Other weaker arguments can be made for resetting state to a baseline, e.g. (i) Use an arbitrarily chosen baseline state $|q_{\text{initial}}\rangle$ or $|t_0\rangle$ so that an arbitrary journal of convergent changes $\hat{J}$

$$|t_{\text{final}}\rangle = \hat{J}_0|t_{\text{initial}}\rangle = \hat{O}_2\hat{O}_1|t_{\text{initial}}\rangle$$

has an inverse such that

$$J_0^{-1}|t_{\text{final}}\rangle = |t_{\text{initial}}\rangle. \quad (69)$$

Assuming the existence of an operator $\hat{O}_{\text{initial}}$ such that $\hat{O}_{\text{initial}}|q\rangle = |t_{\text{initial}}\rangle$, then clearly

$$J_0^{-1} = \hat{O}_{\text{initial}}. \quad (70)$$

These two choices are both forward-moving absolute changes since they both involve an arbitrary decision and they both move forward in time. However the latter is less natural, since it affects to return to a time in the past which might have nothing directly to do with where one needs to be in the present. Our study was motivated by predictability. The principal advantage of these remedies lies in knowledge of the outcome, in the absence of a complete specification.

13 Multi-dimensional operators

In the discussion above, we have restricted ourselves to the maintenance of a single scalar system-value. The issue of dependencies amongst system changes enters quickly as the complexity of layered models of a system grows. It was shown in [9] that one can develop a spanning set of orthogonal operations that covers the vector space like a coordinate system, simply by embedding in a geometrical tuple-fashion. In the simplest expression, one sees this by extending the matrix representation to higher dimensions.

One way to do this is to consider a system as a controlled by a vector of its individual configuration parameters $X_i$, where each parameter is embedded into the field of rationals and encoded in the obvious way:

$$|X\rangle = \begin{pmatrix} X_1 \\ X_2 \\ \vdots \\ X_n \\ 1 \end{pmatrix} \quad (72)$$
where $|X\rangle$ is ‘Dirac notation’ for state. As optimistic and large as current systems may be, they remain finite and can be modeled by finite vectors. We define relative operators for individual parameters in a state as

$$\Delta_i(q) = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \ddots & \vdots \\ \vdots & \ddots & 1 & q \\ 0 & \cdots & 0 & 1 \end{pmatrix}$$

(73)

(where $q$ appears in the $(n+1)$st column of the $i$th row). Likewise, absolute operators are defined as

$$C_i(q) = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \ddots & \vdots \\ \vdots & \ddots & 0 & q \\ 0 & \cdots & q & 1 \end{pmatrix}$$

(74)

(where $q$ appears again in the $(n+1)$st column of the $i$th row), and multiplicative operators as

$$\mu_i(q) = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \ddots & q \\ \vdots & \ddots & 1 & \vdots \\ 0 & \cdots & 0 & 1 \end{pmatrix}$$

(75)

(Where $q$ appears in the $i$th row and column). Thus we propose to model configuration changes in a system via a set of matrices with rational entries.

This completes the construction of the Cfengine operators. Clearly the zero inverse solution applies independently to each of these diagonal operators in this basis, but becomes rapidly more entangled in other parameterizations, where dependencies occur.

14 Concluding remarks

We have shown that neither the outcome of a journal of changes, nor a reversal (undo operation) is generally meaningful or well-defined in an incompletely specified, or non-deterministic system. A deterministic outcome can only be obtained by grounding a system to a policy defined state, analogous to ‘zero’ in a field.

Neither the ‘restoration of state by roll-back’ nor ‘division safe calculation’, à la Bergstra and Tucker, are about how one goes backwards, but rather about how one recovers meaningfully forwards, given that a poorly defined operation was attempted at some point in the past. A classic answer is ‘well, don’t do that’ – but we know that someone will always attempt to perform ill defined operations and thus our story has a practical meaning, of some importance.

The solutions here mirror way the problem of singularities is handled in other areas of mathematics, e.g. in complex analysis one as analytical continuation\cite{21}, in which a path or history through the states can be defined such that the final result avoids touching
the singular cases. Similarly in algebraic topology, the uniqueness of the result can then depend on the path and cohomology.

The totalization remedies described by Bergstra and Tucker underline an approach to a wider range of problems of incomplete information. The ultimate conclusion of this work is that ‘rollback’ cannot be achieved in any well-defined sense without full system closure. A choice about how to go forward is the only deterministic remedy.

There are plenty of topics we have not touched upon here.

**Problem 1** We have not taken into account fields in which external boundary values are imposed on $S$. Then we would have further fixed points in the total history of a system to contend with:

$$\{\min_S, 0_A, 0_B, \ldots \max_S\}$$

(76)

Each of these might be a reasonable candidate for ‘re-grounding’ the system in an undo. What conditions might be imposed when $0_A$ falls outside the range $[\min_S, \max_S]$.

**Problem 2** We have not taken into account operators that depend on one another in non-orthogonal fashion. Dependencies between operators add potentially severe complications to this account.

There is a deeper issue with roll-back in partial systems. If a system is in contact with another system, e.g. receiving data, or if we have partitioned a system into loosely coupled pieces only one of which is being changed, then the other system becomes a part of the total system and we must write a hypothetical journal for the entire system in order to achieve a consistent rollback.

**Problem 3** The partial restoration can leave a system in an inconsistent state that it has never been in before and is not a state that was ever intended.

The results in this paper are directly applicable to to hands-free automation, or ‘computer immunology’, as demonstrated by Cfengine. Opponents of automation have look for ways of arguing that traditional journaling approaches to system maintenance are necessary, preserving the role of humans in system repair. However, we argue that the role of humans is rather in deciding system policy: it is known that the computational complexity of searching for convergent operations is in PSPACE and NP complete, thus it remains the domain of heuristic methods and system experts to find these convergent in more complex cases.

**Acknowledgment:** This work is dedicated to Jan Bergstra on the occasion of his 60th birthday.

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