GAMMA FACTORS OF PAIRS AND A LOCAL CONVERSE THEOREM IN FAMILIES

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In this paper we prove a $GL(n) \times GL(n - 1)$ local converse theorem for $\ell$-adic families of smooth representations of $GL_n(F)$ where $F$ is a finite extension of $\mathbb{Q}_p$ and $\ell \neq p$. Along the way, we extend the theory of Rankin-Selberg integrals, first introduced by Jacquet, Piatetski-Shapiro, and Shalika in [JPSS83], to the setting of families, continuing previous work of the author [Mos].

By a local converse theorem we mean a result along the following lines: given $V_1$ and $V_2$ representations of $GL_n(F)$, if $\gamma(V_1 \times V', X, \psi) = \gamma(V_2 \times V', X, \psi)$ for all representations $V'$ of $GL_{n-1}(F)$, then $V_1$ and $V_2$ are the same. Typically $V_1$, $V_2$, and $V'$ are irreducible admissible generic complex representations, and “the same” means isomorphic. In this setting it is a conjecture of Jacquet that it should suffice to let $V'$ vary over representations of $GL_{\lfloor \frac{n}{2} \rfloor}(F)$, or in other words a $GL(n) \times GL(\lfloor \frac{n}{2} \rfloor)$ converse theorem should hold. In the setting of families, we deal with admissible generic representations whose coefficient rings are more general, and these families are not typically irreducible, so “the same” will mean that $V_1$ and $V_2$ have the same supercuspidal support. Over families, there arises a new dimension to the local converse problem: determining the smallest coefficient ring over which the twisting representations $V'$ can be taken while still having the theorem hold.

Before stating the result, we develop some notation. A family of $GL_n(F)$-representations means an $A[GL_n(F)]$-module $V$ where $A$ is a Noetherian ring in which $p$ is invertible. The development of the theory is facilitated if $A$ is also a $W(k)$-algebra, where $k$ is an algebraically closed field of characteristic $\ell$ and $W$ denotes the Witt vectors (recall that $W(F_\ell) \cong \widehat{\mathbb{Z}}^{nr}_\ell$); this is also the setting of Galois deformations. Given $p$ in $\text{Spec}(A)$ with residue field $\kappa(p) := A_p/pA_p$, the fiber $V \otimes \kappa(p)$ gives a classical representation over $\kappa(p)$.

In this paper we consider admissible generic $A[GL_n(F)]$-modules which are co-Whittaker (Definition 1.5). Each fiber of a co-Whittaker family admits a unique surjection onto an irreducible space of Whittaker functions. Co-Whittaker families are attached to continuous Galois deformations $\text{Gal}(\overline{F}/F) \rightarrow GL_n(A)$ by the local Langlands correspondence in families, conjectured by Emerton and Helm in [EH12]. They conjecture the existence of a map from continuous Galois deformations over $W(k)$-algebras $A$ to co-Whittaker $A[G]$-modules (when $A$ is local Noetherian complete reduced and $\ell$-torsion free). This map is uniquely characterized by requiring that it interpolate (a dualized generic version of) classical local Langlands in characteristic zero ([EH12 Thm 6.2.1]). Their definition is motivated by global constructions: the smooth dual of the $\ell \neq p$ tensor factor of Emerton’s $\ell$-adically completed cohomology is an example of a co-Whittaker module.

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In [Hel12a, Hel12b], Helm has developed the theory of the integral Bernstein center, denoted here by $Z$, which is the endomorphism ring of the identity functor in the category of smooth $W(k)[GL_n(F)]$-modules. By definition, this ring acts on every smooth $W(k)[GL_n(F)]$-module in a compatible way. In [Hel12a, Hel12b], Helm shows that it has a product decomposition which splits the category into full subcategories known as blocks. If $A$ is a $W(k)$-algebra, then smooth $A[GL_n(F)]$-modules can be considered as $W(k)[GL_n(F)]$-modules and analyzed in the context of the integral Bernstein decomposition. Let $V$ be a co-Whittaker $A[GL_n(F)]$-module. $V$ is called primitive if it lives in a single block, or equivalently if it is fixed by a primitive idempotent of the integral Bernstein center.

In the setting of co-Whittaker families, the classical notion of supercuspidal support for representations over a field does not exist. However, the following result of Helm suggests a generalization of the definition of supercuspidal support:

**Theorem 0.1** ([Hel12b], Thm 2.2). Let $\kappa$ be a $W(k)$-algebra that is a field and let $\Pi_1, \Pi_2$ be two absolutely irreducible representations of $G$ over $\kappa$ which live in the same block of the Bernstein center. By Schur’s lemma there are maps $f_1, f_2 : Z \to \kappa$ giving the action of the Bernstein center on $\Pi_1$ and $\Pi_2$. Then $\Pi_1$ and $\Pi_2$ have the same supercuspidal support if and only if $f_1 = f_2$.

Since a co-Whittaker $A[G]$-module satisfies Schur’s lemma ([Hel12b]), there is a map $f_V : Z \to \text{End}_G(V) \xrightarrow{\sim} A$, and we call this map the supercuspidal support of $V$.

The local Rankin-Selberg formal series $\Psi(W, W', X)$ and gamma factors $\gamma(V \times V', X, \psi)$ are established in Sections 2 and 3 for co-Whittaker modules by proving a rationality result and functional equation. Classically the local integrals form elements of $C(q^{-s})$ where $q$ is the order of the residue field of $F$, but since our coefficient rings are not domains, rationality requires more control. As in [Mos], the formal series $\Psi(W, W', X)$ will define an element of the fraction ring $S^{-1}\{A[X, X^{-1}]\}$ where $S$ is the multiplicative subset of $A[X, X^{-1}]$ consisting of polynomials whose first and last coefficients are units. This ring enables us to relate the objects on either side of the functional equation, and also implies that $\Psi(W, W', X)$ will specialize to a rational function at each fiber. The proofs of rationality and the functional equation follow the same overall pattern as the results for the $GL(n) \times GL(1)$ case, which is the subject of [Mos]. In the functional equation for $\Psi(W, W', X)$, there is a term which remains constant as $W$ and $W'$ vary; this defines the gamma factor $\gamma(V \times V', X, \psi).

We are now in a position to state our converse theorem:

**Theorem.** Let $A$ be a finite-type $W(k)$-algebra which is reduced and $\ell$-torsion free, and let $\mathcal{K} = \text{Frac}(W(k))$. Suppose $V_1$ and $V_2$ are two primitive co-Whittaker $A[GL_n(F)]$-modules. There is a finite extension $\mathcal{K}'$ of $\mathcal{K}$ such that, if $\gamma(V_1 \times V', X, \psi) = \gamma(V_2 \times V', X, \psi)$ for all absolutely irreducible generic integral representations $V'$ of $GL_{n-1}(F)$ over $\mathcal{K}'$, then $V_1$ and $V_2$ have the same supercuspidal support.

Thus, in the reduced and $\ell$-torsion free setting, our converse theorem shows it suffices to take the coefficient ring of the twisting representations $V'$ to be no larger than the ring of integers in a finite extension of $\mathcal{K}$. The equality of gamma factors can only occur if $V_1$ and $V_2$ live in the same block of the category $\text{Rep}_{W(k)}(GL_n(F))$, and the finite extension $\mathcal{K}'$ appearing in our converse theorem depends only on this.
block. Finding the smallest possible extension $K'$ for each block will be the subject of future investigation.

If $E$ is a finite extension of $K = \text{Frac}(W(k))$ with ring of integers $O_E$, a representation over $E$ is called integral if it has a $GL_n(F)$-stable $O_E$-lattice $L$. If $V$ is an absolutely irreducible generic integral representation of $GL_n(F)$ over $E$, then in particular its sublattice $L$ is co-Whittaker ([EH12 3.3.2 Prop], [Vig96 1.9.7]), and the supercuspidal support of $L$ determines $V$ up to isomorphism. Thus our converse theorem gives as a special case the following integral converse theorem:

**Corollary 0.2.** Let $V_1, V_2$ be two absolutely irreducible generic integral representations of $GL_n(F)$ over $E$. There is a finite extension $K'/K$ such that if $\gamma(V_1 \times V', X, \psi) = \gamma(V_2 \times V', X, \psi)$ for all absolutely irreducible generic integral representations $V'$ of $GL_{n-1}(F)$ over $K'$, then $V_1 \cong V_2$.

In Section 4 we prove this converse theorem following the method of Henniart in [Hen93] and Jacquet, Piatetski-Shapiro, and Shalika in [JPSS79, Thm 7.5.3]. By employing the functional equation, we establish an equality on the level of Whittaker functions, and this suffices to determine the supercuspidal support for a co-Whittaker family.

There is a key lemma in the setting of complex representations which is more subtle in families. If $N = \left\{ \begin{pmatrix} 1 & \cdots & * \\ \vdots & \ddots & \vdots \\ * & \cdots & 1 \end{pmatrix} \right\}$ and $\psi$ is a nondegenerate character of $N$, this key lemma says that given any smooth compactly supported function $H$ on $GL_n(F)$ with $H(ng) = \psi(n)H(g)$, the vanishing of $H$ is detected by the convolutions of $H$ with the Whittaker functions of a sufficiently large collection of representations. This result was originally proven by Jacquet, Piatetski-Shapiro, and Shalika over $\mathbb{C}$ ([JPSS81 Lemme 3.5]) by using harmonic analysis to decompose a representation as the direct integral of irreducible representations. A purely algebraic analogue of this decomposition was obtained by Bushnell and Henniart in 2003 ([BH03]) by viewing the representation as a sheaf on the spectrum of the Bernstein center. As an application of these algebraic techniques, Bushnell and Henniart give a new proof of this key vanishing lemma ([BH03]). It has been observed by Vigneras in the $\ell$-modular setting [Vig98, Vig04] and more recently by Helm in the integral setting, [Hel12a, Hel12b] that this algebraic approach to Fourier theory and Whittaker models applies to representations over coefficient rings other than $\mathbb{C}$. In Section 5, we apply the theory of the integral Bernstein center, developed by Helm in [Hel12a, Hel12b], to prove the vanishing theorem (and thus the converse theorem) in the case when $A$ is a finite-type $W(k)$-algebra which is reduced and $\ell$-torsion free.

Converse theorems in the complex setting have a long history dating back to Hecke, and for $GL(n)$ in the local setting have been studied over the complex numbers by Chen,Cogdell, Henniart, Jacquet, Langlands, Piatetski-Shapiro, Shalika, among others ([DL70, JPSS79, JPSS83, Hen93, CPS99, Che06]), and in characteristic $\ell$ by Vigneras ([Vig00]).

The Rankin-Selberg convolutions in this paper expand on recent results on Rankin-Selberg convolutions in the $\ell$-modular setting by Kurinczuk and Matringe in [KM13]. In $\ell$-adic families, the analogue of the $L$-factor does not seem to behave well (see [Mos §0]), which is why we focus at present only on the local integral factors $\Psi(W, W', X)$ and the gamma factor.
In the $\ell$-modular setting, where $A = k$, it appears that the approach of Bushnell-Henniart [BH03] would require further knowledge of the relationship between Whittaker models and the Bernstein center modulo $\ell$.

Since gamma factors of pairs determine supercuspidal supports, they determine the action of the Bernstein center on the category. Thus, the methods of this paper may shed light on the ring structure of the integral Bernstein center. Investigations along these lines will be carried out in future research.

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1. Notation and Definitions

We will let $F$ be a finite extension of $\mathbb{Q}_p$, and $k$ an algebraically closed field of characteristic $\ell$, where $\ell \neq p$ is an odd prime. We will denote by $W(k)$ the ring of Witt vectors over $k$. The letter $G$ or $G_n$ will always denote the group $GL_n(F)$.

Throughout the paper $A$ will be a Noetherian $W(k)$-algebra, with additional ring theoretic conditions in various sections of the paper, and for a prime $p$ we denote by $\kappa(p)$ the residue field $A_p/pA_p$. For any locally profinite group $H$, $\text{Rep}_A(H)$ is the category of smooth $A[H]$-modules (smooth means every vector is invariant under a compact open subgroup). An $A[H]$-module is admissible if for every compact open subgroup $K$, the set of $K$-fixed vectors is finitely generated as an $A$-module.

**Definition 1.1.** $V$ in $\text{Rep}_A(H)$ will be called $G$-finite if it is finitely generated as an $A[G]$-module and primitive if there exists a primitive idempotent $e$ in the Bernstein center $Z$ such that $eV = V$.

We denote by $N_n = N$ the subgroup of $G_n$ consisting of all unipotent upper-triangular matrices. Let $\psi : F \to W(k)\times$ be an additive character of $F$ with $\ker \psi = p$. For a $W(k)$-algebra $A$, $\psi_A$ will denote $\psi \otimes_{W(k)} A$. $\psi$ defines a character on any subgroup of $N_n(F)$ by

$$(u)_{i,j} \mapsto \psi(u_{1,2} + \cdots + u_{n-1,n});$$

we will abusively denote this character by $\psi$ as well. $V(N, \psi) \subset V$ is the submodule of $(N, \psi)$-co-invariants.

**Definition 1.2.** For $V$ in $\text{Rep}_A(G_n)$, we say that $V$ is of Whittaker type if $V/V(N, \psi)$ is free of rank one as an $A$-module. If $A$ is a field we say $V$ is generic if $V/V(N, \psi)$ is nonzero.

If $V$ is of Whittaker type, $\text{Hom}_A(V/V(N_n, \psi), A) = \text{Hom}_{N_n}(V, \psi)$ is free of rank one, so we may choose a generator $\lambda$ in $\text{Hom}_{N_n}(V, \psi)$. For any $v$ in $V$, define $W_v \in \text{Ind}_{N_n}^{G_n} \psi$ as $W_v : g \mapsto \lambda(gv)$. This is called a Whittaker function and has the property that $W(nx) = \psi(n)W(x)$ for $n \in N_n$. $v \mapsto W_v$ defines a $G_n$-equivariant homomorphism $V \to \text{Ind}_{N_n}^{G_n} \psi$. The image is an $A[G]$-module independent of the choice of $\lambda$. The map $v \mapsto W_v$ is precisely the generator of $\text{Hom}_{G_n}(V, \text{Ind}_{N_n}^{G_n} \psi)$ corresponding to the generator $\lambda$ of $\text{Hom}_{N_n}(V, \psi)$ under Frobenius reciprocity. The image of the homomorphism $v \mapsto W_v : V \to \text{Ind}_{N_n}^{G_n} \psi$ is called the space of Whittaker functions of $V$ and is denoted $\mathcal{W}(V, \psi)$ or just $\mathcal{W}$.
Note that the map $V \to W(V, \psi)$ is surjective but not necessarily an isomorphism, and different $A[G]$-modules of Whittaker type can have the same space of Whittaker functions. See [Mos, Lemma 1.6] for more details on this point.

For each $m \leq n$, we let $G_m$ denote $GL_m(F)$ and embed it in $G$ via $(\begin{smallmatrix} G_m & 0 \\ 0 & I_{n-m} \end{smallmatrix})$. We let $\{1\} = P_1 \subset \cdots \subset P_n$ denote the mirabolic subgroups of $G_1 \subset \cdots \subset G_n$, which are given $P_m := \{(g_{m-1}^\top \begin{smallmatrix} x \\ 1 \end{smallmatrix}) : g_{m-1} \in G_{m-1}, \ x \in F^{m-1}\}$. We also have the unipotent upper triangular subgroup $U_m := \{(I_{m-1}^\top \begin{smallmatrix} x \\ 1 \end{smallmatrix}) : x \in F^{m-1}\}$ of $P_m$ such that $P_m = U_m G_{m-1}$. In particular, $U_m \sim F^{m-1}$.

We can repeat the Whittaker functions construction for the restriction to $P_n$ of representations $V$ in $\text{Rep}(G_n)$ of Whittaker type. In particular, by restricting the argument of the Whittaker functions $W_v$ to elements of $P_n$, we get a $P_n$-equivariant homomorphism $V \to \text{Ind}_{G_n}^{G} \psi$. The image of the homomorphism $V \to \text{Ind}_{G_n}^{G} \psi : v \mapsto W_v$ is called the Kirillov functions of $V$ and is denoted $\mathcal{K}(V, \psi)$ or just $\mathcal{K}$. It carries a representation of $P_n$ via $pW_n = W_{pn}$. There is a particularly important $P_n$-representation that naturally embeds in the restriction to $P_n$ of any Whittaker type representation $V$ in $\text{Rep}(G_n)$.

In Section 2 we make use of the Bernstein-Zelevinsky derivative functors of [BZ77].

**Definition 1.3.** If $V$ is in $\text{Rep}(P_n)$, the $P_n$ representation $(\Phi^+)^{n-1}V^{(n)}$ is called the Schwarz functions of $V$ and is denoted $\mathcal{S}(V)$.

**Definition 1.4 (Hel12b 3.3).** Let $\kappa$ be a field of characteristic different from $p$. An admissible smooth object $U$ in $\text{Rep}_A(G)$ is said to have essentially $AIG$ dual if it is finite length as a $\kappa[G]$-module, its cosocle $\text{cosoc}(U)$ is absolutely irreducible generic, and $\text{cosoc}(U)^{(n)} = U^{(n)}$ (the cosocle of a module is its largest semisimple quotient).

This condition is equivalent to $U^{(n)}$ being one-dimensional over $\kappa$ and having the property that $W^{(n)} \neq 0$ for any nonzero quotient $\kappa[G]$-module $W$ (see [EH12, Lemma 6.3.5] for details).

**Definition 1.5 (Hel12b 6.1).** An object $V$ in $\text{Rep}_A(G)$ is said to be co-Whittaker if it is admissible, of Whittaker type, and $V \otimes \kappa(p)$ has essentially $AIG$ dual for each $p$.

Co-Whittaker representations satisfy Schur’s lemma ([Hel12b Prop 6.2]) and are generated over $A[G]$ be a single element ([Mos, Lemma 1.17]). And every admissible Whittaker-type $A[G]$-module contains a canonical co-Whittaker submodule ([Mos Prop 1.18]). If $A$ is a Noetherian $W(k)$-algebra and $V$ is an $A[G]$-module, then $V$ is also a $W(k)[G]$-module, and we use the Bernstein decomposition of $\text{Rep}_{W(k)}(G)$ to study $V$.

**Definition 1.6 (Supercuspidal Support).** If $e$ is a primitive idempotent of $Z$, any co-Whittaker $A[G]$ module in the block $e \text{Rep}_{W(k)}(G)$ defines a map $f_V : eZ \to \text{End}_{A[G]}(V) \sim A$, which is called the supercuspidal support of $V$.

Any co-Whittaker $A[G]$-module $V$ is a quotient of the representation $e(\text{c-Ind } \psi) \otimes_{eZ,f_V} A$ and conversely if $A$ is any Noetherian $W(k)$-algebra with an $eZ$-algebra structure, $e(\text{c-Ind } \psi) \otimes_{eZ} A$ is a co-Whittaker $A[G]$-module ([Hel12b Thm 6.3]).
It is possible to define a Haar measure on the space $C_c^\infty(G, A)$ of smooth compactly supported functions $G \to A$ by choosing a filtration $\{H_i\}_{i \geq 1}$ of compact open neighborhoods of 1 in $G$ such that $[H_1 : H_i]$ is invertible in $W(k)$. Then we can set $\mu^*(H_i) = [H_1 : H_i]^{-1}$. This automatically defines integration on the characteristic functions of the $H_i$, and one checks that extending by linearity gives a well-defined Haar integral on $C_c^\infty(G, A)$. Normalizing the Haar measure to be compatible with decompositions of the group requires care when $A$ has characteristic $\ell$; this is dealt with in [KM14, §2.2].

2. Rationality of Rankin-Selberg Formal Series

Let $A$ and $B$ be Noetherian $W(k)$-algebras and let $R = A \otimes_{W(k)} B$. Let $V$ and $V'$ be $A[G_n]$- and $B[G_m]$-modules, respectively, where $m > n$. Suppose both $V$ and $V'$ are of Whittaker type. For $W \in W(V, \psi)$ and $W' \in W(V', \psi)$, we define the formal series with coefficients in $A$:

$$\Psi(W, W', X) := \sum_{r \in \mathbb{Z}} \int_{N_m \{g \in G_m : v(\det g) = r\}} \left(W \begin{bmatrix} g & 0 \\ 0 & I_{n-m} \end{bmatrix} \otimes W'(g)\right) X^r dg$$

and for $0 \leq j \leq n - m - 1$, define

$$\Psi(W, W', X; j) := \sum_{r \in \mathbb{Z}} \int_{M_{j,m}(F) \int_{N_m \{g \in G_m : v(\det g) = r\}}} \left(W \begin{bmatrix} g & I_j \\ I_n & I_{n-m-j} \end{bmatrix} \otimes W'(g)\right) X^r dg dx$$

With $\Psi(W, W', X; 0) = \Psi(W, W', X)$.

**Lemma 2.1.** The formal series $\Psi(W, W', X; j)$ has finitely many nonzero powers of $X^{-1}$, thus forms an element of $R[[X]][X^{-1}]$.

**Proof.** Since [JPSS83, Lemma 6.2] is valid in this context, the proof proceeds exactly as in [Mos, §2.1] after applying the Iwasawa decomposition. The Iwasawa decomposition works in this setting after choosing an appropriate Haar measure, as shown in [KM14, Cor 2.9]. □

**Theorem 2.2.** Suppose $A$ and $B$ are Noetherian $W(k)$-algebras, $V$ is an $A[GL_n(F)]$-module, and $V'$ is a $B[GL_m(F)]$-module, both of Whittaker type and finitely generated over $A[G_n]$ and $B[G_m]$ respectively. Define $S$ to be the multiplicative subset of $R[X, X^{-1}]$ consisting of polynomials whose first and last coefficients are units. For any $W \in W(V, \psi)$, $W' \in W(V', \psi)$, the formal series $\Psi(W, W', X; j)$ lives in $S^{-1}(R[X, X^{-1}])$.

The remainder of this section is devoted to proving Theorem 2.2.

As in [JPSS83] it suffices to consider only the $j = 0$ integral. Using the Iwasawa decomposition as in [JPSS79, JPSS83, KM14], it suffices to prove the theorem when the integration is restricted to the torus $T_m$:

$$\sum_{r \in \mathbb{Z}} \int_{\{a \in T_m : v(\det a) = r\}} \left(W \begin{bmatrix} g & 0 \\ 0 & I_{n-m} \end{bmatrix} \otimes W(a)\right) X^{v(\det a)} da$$

We parametrize the torus $T_m$ by

$$\prod_{i=1}^m F^\times \to T_m : (a_1, \ldots, a_n) \mapsto \begin{pmatrix} a_1 \cdots a_m \\ a_2 \cdots a_m \\ \vdots \\ a_m \end{pmatrix} =: a.$$
In the setting of representations over a field, there is a useful decomposition of any Whittaker function into “finite” functions, which quickly leads to a rationality result (\cite{JPSS79, JPSS83, KM14}). In the setting of rings, such a structure theorem is lacking, but certain elements of its proof can be used to prove rationality even in the setting of rings.

Consider the exterior product representation

\[ \mathcal{W} := \mathcal{W}(V, \psi) \otimes \mathcal{W}(V', \psi) \] in \( \text{Rep}_R(G_n \times G_m) \), and consider the following space of restrictions of products of Whittaker functions to the torus \( T_m \):

\[ \mathcal{V} := \{ W (\begin{smallmatrix} a & 0 \\ 0 & I_{n-m} \end{smallmatrix}) \otimes W'(a) : a \in T_m, W \in \mathcal{W}(V, \psi), W' \in \mathcal{W}(V', \psi) \} \].

There is a natural surjection of \( R \)-modules

\[ \mathcal{W} \twoheadrightarrow \mathcal{V} \]

mapping \( W \otimes W' \) to the restriction \( W (\begin{smallmatrix} a & 0 \\ 0 & I_{n-m} \end{smallmatrix}) \otimes W'(a) \). This map is the restriction of functions on \( G_n \times G_m \) to functions on the subgroup \( T_m \rightarrow T_m \times T_m \rightarrow G_n \times G_m \).

Here, \( T_m \rightarrow G_n \) is the embedding of \( T_m \) within the upper-left block of \( G_n \).

Define

\[ \mathcal{V}_i := \{ \phi \in \mathcal{V} : \phi(a) \rightarrow 0 \text{ uniformly as } v(a_i) \rightarrow \infty \} \].

For \( i \leq m \) let \( N_n(i) \) be the subgroup of \( G_n \) consisting of matrices whose first \( i \) columns are those of the identity matrix and whose last \( n-i \) columns are those of the subgroup \( N_n \). The subgroup \( N_n(i) \subset G_m \) is defined similarly. Define the character \( \psi \) on \( N_n(i) \) (or on \( N_m(i) \)) by the last \( n-i-1 \) superdiagonal entries:

\[ \overline{\psi}(n) = \psi(0 + n_{i+1,i+2} + \cdots + n_{n-1,n}) \];

then we have for any representation \( U \in \text{Rep}(G_n) \) that \( U^{(n-i)} \) is given by the module of coinvariants \( \mathcal{V}_i \).

**Lemma 2.3.** Let \( \theta_i \) denote the composition \( \mathcal{W} \rightarrow \mathcal{V} \rightarrow \mathcal{V}/\mathcal{V}_i \). Then the module of coinvariants \( \mathcal{W}(N_n(i) \times N_m(i), \overline{\psi}) \) is contained in \( \ker(\theta_i) \).

**Proof.** This is a matrix calculation similar to that in the proof of \cite[Lemma 2.2]{Mos}.

Now, since

\[ \frac{\mathcal{W}}{\mathcal{W}(N_n(i) \times N_m(i), \psi)} \cong \mathcal{W}(V, \psi)^{(n-i)} \otimes \mathcal{W}(V', \psi)^{(m-i)} \],

we’ve shown that the map \( \theta_i \) factors through \( \mathcal{W}(V, \psi)^{(n-i)} \otimes \mathcal{W}(V', \psi)^{(m-i)} \).

Let \( \rho_i(\overline{\omega}) \) denote right translation of a function by the diagonal matrix with \( \overline{\omega} \) in the first \( i \) diagonal entries:

\[ \begin{pmatrix} \overline{\omega} \\ \vdots \\ \overline{\omega}_1 \\ \vdots \\ 1 \end{pmatrix} \]

Note that if we’re considering functions on the torus \( T_m \) parametrized as \( \prod_{i=1}^m F^\times \) as above, this translates to

\[ (\rho_i(\overline{\omega}) \phi)(a_1, \ldots, a_m) = \phi(a_1, \ldots, a_{i-1}, a_i \overline{\omega}, a_{i+1}, \ldots, a_m) \].
**Lemma 2.4.** Let $B_i$ be the $R$-subalgebra of $\text{End}_R(\mathcal{V}/\mathcal{V}_i)$ generated by $\rho_i(\overline{\omega})$. Then $B_i$ is finitely generated as a module over $R$.

**Proof.** For any $i$, the operator $\rho_i(\overline{\omega})$ defines a linear endomorphism of the spaces $\mathcal{W}(\mathcal{V}, \psi)^{(n-i)}$ and $\mathcal{W}(\mathcal{V}', \psi)^{(m-i)}$, and so acts diagonally on their tensor product. For each $i$ it preserves the kernel of the surjective map

$$\mathcal{W}(\mathcal{V}, \psi)^{(n-i)} \otimes \mathcal{W}(\mathcal{V}', \psi)^{(m-i)} \to \mathcal{V}/\mathcal{V}_i$$

so in particular the sub-algebra of $\text{End}_R(\mathcal{V}/\mathcal{V}_i)$ generated by $\rho_i(\overline{\omega})$ equals the sub-algebra of $\text{End}_R(\mathcal{W}(\mathcal{V}, \psi)^{(n-i)} \otimes \mathcal{W}(\mathcal{V}', \psi)^{(m-i)})$ generated by $\rho_i(\overline{\omega})$.

But we have an injection

$$\text{End}_{A[G_i]}(\mathcal{W}(\mathcal{V}, \psi)^{(n-i)}) \otimes \text{End}_{B[G_i]}(\mathcal{W}(\mathcal{V}', \psi)^{(m-i)}) \hookrightarrow \text{End}_{R[G_i]}(\mathcal{W}(\mathcal{V}, \psi)^{(n-i)} \otimes \mathcal{W}(\mathcal{V}', \psi)^{(m-i)})$$

as $R$-modules, and the subalgebra $B_i$ we’re considering lands inside the smaller space. By [Mos, Lemma 2.15], it suffices to prove the result in the case where $A$ is a local ring, say with maximal ideal $m$. In this setting, [EHT2, Lemma 2.18] combined with [Mos, Cor 2.13] imply that $\mathcal{W}(\mathcal{V}, \psi)^{(n-i)}$ is admissible and a finite-type $A[G_i]$-module. Hence we can take a finite set $\{w_i\}$ of $A[G_i]$ generators and a sufficiently small compact open subgroup $U$ which fixes them all. Any $A[G_i]$-equivariant endomorphism is uniquely determined by its values on $\{w_i\}$. On the other hand, $G_i$-equivariance means such an endomorphism preserves $U$-invariance, and the $U$-fixed vectors are finitely generated, therefore it is uniquely determined via $A$-linearity from a finite set of values. This shows that the algebra $\text{End}_{A[G_i]}(\mathcal{W}(\mathcal{V}, \psi)^{(n-i)})$ is finitely generated as an $A$-module, hence its sub-algebra defined by $B_i$ is also finitely generated. The same is true for $\mathcal{W}(\mathcal{V}', \psi)^{(m-i)}$, hence their tensor product is finitely generated as a module over $A \otimes B$. \hfill $\Box$

**Lemma 2.5.** There exists a polynomial $f(X_1, \ldots, X_m)$ in $R[X_1, \ldots, X_m]$ such that the operators $\rho_1(\overline{\omega}), \ldots, \rho_m(\overline{\omega})$ in $\text{End}_R(\mathcal{V}/(\cap_i \mathcal{V}_i))$ satisfy $f(\rho_1(\overline{\omega}), \ldots, \rho_m(\overline{\omega})) = 0$.

Moreover, $f(X_1, \ldots, X_m)$ is a product $f_1 \cdots f_m$ of polynomials $f_i$ in $R[X_i]$ such that each $f_i$ is monic in its single variable and has unit constant term.

**Proof.** Proving the lemma means showing that for any $W \in \mathcal{V}$, there exist $N_1, \ldots, N_m$ sufficiently large that

$$(f(\rho_1(\overline{\omega}), \ldots, \rho_m(\overline{\omega}))W)(a_1, \ldots, a_n) = 0$$

whenever any $a_i$ satisfies $v(a_i) > N_i$. The set $\{N_1, \ldots, N_m\}$ must depend only on $W$.

We proceed by induction on $m$. If $m = 1$ then $\cap_i \mathcal{V}_i = \mathcal{V}_1$, so this follows from the $R$-module finiteness of $\langle \rho_1(\overline{\omega}) \rangle \subset \text{End}_R(\mathcal{V}/\mathcal{V}_1)$.

Assume the lemma is true for $m - 1$. Fix $W(a_1, \ldots, a_m)$ an element of $\mathcal{V}$. Since $\rho_m(\overline{\omega})$ is an integral element of the ring $\text{End}(\mathcal{V}/\mathcal{V}_m)$ and $\rho_m(\overline{\omega})$ is invertible, there exists a monic polynomial $f_m(X)$ with unit constant term such that $f_m(\rho_m(\overline{\omega})) = 0$ in $\text{End}(\mathcal{V}/\mathcal{V}_m)$, in other words there exists $N_m$ such that

$$(f_m(\rho_m(\overline{\omega}))W)(a_1, \ldots, a_m) = 0$$

whenever $v(a_m) > N_m$.
Now fix \( b \in F^\times \) and define \( \phi_b : \prod_{i=1}^{m-1} F^\times \to R \) to be the function
\[
(a_1, \ldots, a_{m-1}) \mapsto (f_m(\rho_m(\varpi))W)(a_1, \ldots, a_{m-1}, b).
\]
Note that \( \phi_b \equiv 0 \) when \( v(b) > N_m \).

Let \( \mathcal{V}' \) be the restriction of functions in \( \mathcal{V} \) to the subgroup \( T_{m-1} \cong \prod_{i=1}^{m-1} F^\times \), and let \( \mathcal{V}'_i \) be the subspace defined as before, but for \( \mathcal{V}' \). Previous remarks on \( \mathcal{V} \) hold true for the space of restrictions to \( T_{m-1} \), in particular \( \rho_i(\varpi) \) generates a finite sub-algebra of \( \text{End}_R(\mathcal{V}'/\mathcal{V}'_i) \) for \( i = 1, \ldots, m-1 \). Thus we can apply the induction hypothesis to conclude that there exists a polynomial \( g(X_1, \ldots, X_{m-1}) \in R[X_1, \ldots, X_{m-1}] \) satisfying the required conditions, such that for any \( \phi \in \mathcal{V}' \), there are large integers \( N_1(\phi), \ldots, N_{m-1}(\phi) \), depending on \( \phi \), such that
\[
(g(\rho_1(\varpi), \ldots, \rho_{m-1}(\varpi))\phi)(a_1, \ldots, a_{m-1}) = 0
\]
whenever any one of \( a_1, \ldots, a_{m-1} \) satisfies \( v(a_i) > N_i(b) \).

Since \( \phi_b \) is the restriction of a product of Whittaker functions to \( T_{m-1} \) by construction, we can apply this specifically to \( \phi_b \): there exist large integers \( N_1(b), \ldots, N_{m-1}(b) \), depending on \( b \), such that
\[
(g(\rho_1(\varpi), \ldots, \rho_{m-1}(\varpi))\phi)(a_1, \ldots, a_{m-1}) = 0
\]
whenever any one of \( a_1, \ldots, a_{m-1} \) satisfies \( v(a_i) > N_i(b) \).

We wish to show that we can choose the \( N_i \)'s independently of \( b \). But, since \( \phi_b \equiv 0 \) for \( v(b) > N_m \), and \( \phi_b \) also vanishes when \( v(b) << 0 \) by Lemma 2.1, we have that \( \phi_b \) is only nonzero when \( b \) is confined to a compact subset of \( F^\times \). In particular, since \( f_m(\rho_m(\varpi))W \) is locally constant in each variable, there are only finitely many distinct functions \( \phi_b \) as \( b \) ranges over this compact set. Thus, the sets \( \{ N_i(b) : b \in F^\times \} \) are finite for each \( i \) and we can choose \( N_i \) to be max\( \{ N_i(b) : b \in F^\times \} \).

Now take the polynomial
\[
f(X_1, \ldots, X_m) = g(X_1, \ldots, X_{m-1})f_m(X_m) \in R[X_1, \ldots, X_m],
\]
we have
\[
(f(\rho_1(\varpi), \ldots, \rho_m(\varpi))W)(a_1, \ldots, a_m) = 0
\]
whenever \( v(a_i) > N_i \) for \( i = 1, \ldots, m \).

Since \( f_m(X) \) is a monic polynomial with unit constant term, and \( g(X_1, \ldots, X_{m-1}) \) satisfies the hypotheses of the lemma by induction, the product \( f(X_1, \ldots, X_m) \) will be a product satisfying the required conditions.

We can then deduce rationality of \( \Psi(W, W', X) \) as follows. First we apply \( \Psi(-, -, X) \) to both sides of the following equation:
\[
f(\rho_1(\varpi), \ldots, \rho_m(\varpi))(W \otimes W') = W_0,
\]
for \( W \otimes W' \in \mathcal{V} \) and \( W_0 \in \cap_i \mathcal{V}_i \). In particular, \( \Psi(W_0, X) \in R[X, X^{-1}] \), so we have a polynomial on the right hand side.

Then on the left side we use the transformation property
\[
\Psi(\rho_1(\varpi)^{t_1} \cdots \rho_m(\varpi)^{t_m})(W \otimes W'), X) = X^{t_1 + 2t_2 + \cdots + mt_m} \Psi(W, W', X),
\]
to get that \( \tilde{f}(X) \Psi(W, W', X) \in R[X, X^{-1}] \) where \( \tilde{f} \) is the image of \( f \) in the map
\[
R[X_1, \ldots, X_m] \to R[X]
\]
\[
X_i \mapsto X^i.
\]
Since \(\tilde{f}\) lies in \(S\), this proves the theorem.

**Remark 2.6.** When \(A = B\) we can take the image of the zeta integrals in the map \(S^{-1}_R(R[X,X^{-1}]) \to S^{-1}_A(A[X,X^{-1}])\) induced by the map \(R \to A : a_1 \otimes a_2 \mapsto a_1a_2\) and recover the rationality result that would be desired when both representations live over the same coefficient ring.

### 3. Functional Equation

As in [Moss], we will construct the gamma factor to be what it must in order to satisfy the functional equation for one particular Whittaker function, and then show that the functional equation is satisfied for all Whittaker functions. We will make repeated use of the following Lemma:

**Lemma 3.1.** If \(A\) and \(B\) are reduced \(\ell\)-torsion free \(W(k)\)-algebras, then \(A \otimes_{W(k)} B\) is also a reduced and \(\ell\)-torsion free \(W(k)\)-algebra.

**Proof.** Being \(\ell\)-torsion-free is equivalent to being flat as a module over \(W(k)\). Since the tensor product of two flat modules is again flat, we have that \(A \otimes_{W(k)} B\) is \(\ell\)-torsion free.

To show reducedness first observe that a flat \(W(k)\)-algebra \(C\) is reduced if and only if \(C \otimes_{W(k)} K\) is reduced, where \(K\) denotes Frac\((W(k))\). To see this note that \(R\) embeds in the localization \(S^{-1}R\) where \(S = W(k) \setminus \{0\}\), and thus an element \(\xi\) in the localization is nilpotent if and only if \(r\) is nilpotent.

Applying this to the flat \(W(k)\)-algebra \(S = A \otimes_{W(k)} B\), it suffices to prove that \((A \otimes_{W(k)} B) \otimes_{W(k)} K\) is reduced. But this equals

\[
(A \otimes_{W(k)} K) \otimes_K (B \otimes_{W(k)} K).
\]

We can now apply [Bon07, Ch 5, §15, Thm 3] which says that the tensor product of reduced algebras over a characteristic zero field is again reduced. \(\Box\)

The following fact will be used repeatedly:

**Lemma 3.2.** Let \(K = \prod_p \text{minimal } \kappa(p)\) be the total quotient ring of \(A\). The map \(V \otimes_A K \to \prod_{i=1}^m (V \otimes_A \kappa(p_i))\) is an isomorphism of \(K\)-modules.

**Lemma 3.3.** For \(V\) in Rep\(G_n\) and \(V'\) in Rep\(G_m\) both primitive, there exist sets \(\{W_1, \ldots, W_s\}\) in \(\mathcal{W}(V, \psi)\) and \(\{W'_1, \ldots, W'_s\}\) in \(\mathcal{W}(V', \psi)\) such that \(\sum_i \Psi(W_i, W'_i, X) = 1\).

**Proof.** First suppose that \(A\) and \(B\) are reduced and \(\ell\)-torsion free, so \(\kappa(p)\) has characteristic zero when \(p\) is a minimal prime of \(R\). Hence by [JPSS] (2.7) p.394 there are Whittaker functions \(W_p \in \mathcal{W}(V \otimes \kappa(p), \psi_p)\) and \(W'_p \in \mathcal{W}(V' \otimes \kappa(p), \psi_p)\) such that \(\Psi_p(W_p, W'_p, X) = 1\). Let \(W = W(V, \psi) \otimes W(V', \psi)\). Then if \(T\) is the set of non zero-divisors in \(R\), we have \(T^{-1}W \cong \prod_i W \otimes \kappa(p_i)\) by Lemma 3.2. It follows from the \(R\)-linearity of \(\Psi\) that there is a nonzerodivisor \(r = \sum_i a_i \otimes b_i\) in \(R\) and an element \(W = \sum W_i \otimes W'_i\) of \(W\) such that for all minimal primes \(p\) we have

\[
\sum_{i,j} \Psi(a_j W_i, b_j W'_i, X) \equiv \Psi(W_p, W'_p, X) = 1 \mod p.
\]

Since \(R\) is reduced we have thus found sets \(W_1, \ldots, W_s\) and \(W'_1, \ldots, W'_s\) such that \(\sum_k \Psi(W_k, W'_k, X) = 1\).
When $A$ and $B$ are not reduced and $\ell$ torsion free, we argue as follows. Consider $\mathcal{M} = e \cdot \text{Ind}_{N_m}^G \psi$ and $\mathcal{M}' = e' \cdot \text{Ind}_{N_m}^G \psi'$ where $e$ and $e'$ correspond to the Bernstein blocks containing $V$ and $V'$. Then there exist $\{W_i\}$ and $\{W'_i\}$ satisfying the theorem for $W_i \in \mathcal{W}(\mathcal{M}, \psi)$ and $W'_i \in \mathcal{W}(\mathcal{M}', \psi')$. The surjections $\mathcal{M} \otimes_{\mathbb{Z}_w} A \to V$ and $\mathcal{M}' \otimes_{\mathbb{Z}_w} B \to V'$ induce surjections on the spaces of Whittaker functions, and say $W_i \mapsto w_i$ and $W'_i \mapsto w'_i$. Since the formation of $\Psi$ commutes with change of coefficients, we get that $\sum_i \Psi(w_i, w'_i, X) = (f_V \otimes f_{V'})(\Psi(W_i, W'_i, X)) = 1$ where $f_V : e \mathbb{Z}_n \to A$ and $f_{V'} : e' \mathbb{Z}_m \to B$ are the supercuspidal supports of $V$ and $V'$, respectively.

Since the zeta integrals $\Psi(W, W'; X; j)$ all live in $S^{-1}R[X, X^{-1}]$ we can make sense of both sides of the functional equation. Using Lemma 3.1, we get that the coefficient ring $A \otimes_{W(k)} B$ is reduced and $\ell$-torsion free, hence its minimal primes are precisely its characteristic zero primes and they have trivial intersection. Each characteristic zero point $R \to \kappa$ gives characteristic zero points of $A$ and $B$. For each such point we can take an algebraic closure $\overline{\kappa}$ and choose an isomorphism $\overline{\kappa} \cong \mathbb{C}$, then apply the arguments of [JPSSS] to $V \otimes \overline{\kappa}$ and $V' \otimes \overline{\kappa}$. In this way, the argument of [Moss §3] carries over completely to the setting of gamma factors of pairs $\gamma(V \times V', X, \psi)$. Let $w_{n,m} = \text{diag}(I_{n-m}, w_m)$ where $w_m$ is the antidiagonal $m \times m$ matrix with $1$'s on the diagonal.

**Theorem 3.4.** Suppose $A$ and $B$ are Noetherian $W(k)$-algebras which are reduced and $\ell$-torsion free, and suppose $V$, $V'$ are primary co-Whittaker $A[G_m]$- and $B[G_m]$-modules respectively. Then there exists a unique element $\gamma(V \times V', X, \psi)$ of $S^{-1}(R[X, X^{-1}])$ such that

$$\Psi(W, W'; X; j)\gamma(V \times V', X, \psi)\omega_{V'}(-1)^{n-1} = \Psi(w_{n,m}, W, W', \frac{q^{n-m-1}}{X}; n-m-1-j)$$

for any $W \in \mathcal{W}(V, \psi)$, $W' \in \mathcal{W}(V', \psi)$ and for any $0 \leq j \leq n - m - 1$.

Note that our notation in this theorem is slightly different from [JPSSS], and follows [CPS10] 2.1 Thm].

We now focus on removing the hypothesis that $A$ is reduced and $\ell$-torsion free. To do this we must consider the action of the Bernstein center on $V$ and $V'$ and how they are dominated by the base-change of a universal co-Whittaker module.

Let $\mathcal{Z}$ be the center of $\text{Rep}_{W(k)}(G_n)$ and let $\mathcal{Z}'$ be the center of $\text{Rep}_{W(k)}(G_m)$. It is proved in [Hel12a] that for primitive idempotents $e$ and $e'$ in $\mathcal{Z}$ and $\mathcal{Z}'$ respectively, $e \mathcal{Z}$ and $e' \mathcal{Z}'$ are reduced and $\ell$-torsion free $W(k)$-algebras. Lemma 3.1 implies that $e \mathcal{Z} \otimes_{W(k)} e' \mathcal{Z}'$ is reduced and $\ell$-torsion free, so in particular the hypotheses of the theorem hold for the pair of representations $e \mathcal{M}_n$ and $e' \mathcal{M}_m$. We thus define the universal gamma factor $\Gamma(e \mathcal{M}_n \times e' \mathcal{M}_m, X, \psi) \in S^{-1}(e \mathcal{Z} \otimes e' \mathcal{Z}')[X, X^{-1}]$.

Now, given primitive co-Whittaker modules $V$ in $e \text{Rep}_{W(k)}(G_n)$ and $V'$ in $e' \text{Rep}_{W(k)}(G_m)$ over any coefficient rings $A$ and $B$ which are Noetherian $W(k)$-algebras, we have supercuspidal supports $f_V : e \mathcal{Z} \to A$ and $f_{V'} : e' \mathcal{Z}' \to B$ such that $e \mathcal{M}_n \otimes_{e \mathcal{Z}, f_V} A$ dominates $V$ and $e' \mathcal{M}_m \otimes_{e' \mathcal{Z}', f_{V'}} B$ dominates $V'$.

Because the formation of zeta integrals and gamma factors commute with change of base ring, the image of $\Gamma(e \mathcal{M}_n \times e' \mathcal{M}_m, X, \psi)$ in the map $S^{-1}(e \mathcal{Z} \otimes e' \mathcal{Z}')[X, X^{-1}] \to S^{-1}R[X, X^{-1}]$ induced by $f_V \otimes f_{V'}$ equals $\gamma(V \times V', X, \psi)$. Since $e \mathcal{M}_n \otimes_{e \mathcal{Z}, f_V} A$ dominates $V$, they have the same Whittaker spaces, and thus share all the same zeta integrals, and the same goes for $V'$. Therefore,
\[\gamma(V \times V', X, \psi)\] satisfies the functional equation for all \(W \in \mathcal{W}(V, \psi)\) and all \(W'\) in \(\mathcal{W}(V', \psi)\).

**Corollary 3.5.** For \(V \in \text{Rep}_{A}^{\text{co-Whitt}}(G_{n})\) and \(V' \in \text{Rep}_{B}^{\text{co-Whitt}}(G_{m})\) primitive co-Whittaker modules and \(A, B\) any Noetherian \(W(k)\)-algebras, \(\gamma(V, V', X)\) is a unit in \(S^{-1}(R[X, X^{-1}])\) and

\[\gamma(V \times V', X, \psi)^{-1} = \gamma(V' \times (V')', \frac{q^{n-m-1}}{X}, \psi^{-1}).\]

**Proof.** Let \(\{W_{i}\}\) and \(\{W'_{i}\}\) be the Whittaker functions guaranteed by Lemma 3.3

The original functional equation reads

\[\sum_{i} \Psi(W_{i}, W'_{i}, X) \gamma(V_{i} \times V', X, \psi) \omega_{\tau}(-1)^{t} = \sum_{i} \Psi(W_{i}, W'_{i}, \frac{q^{n-m-1}}{X}).\]

Replacing \(X\) with \(\frac{q^{n-m-1}}{X}\) we have

\[\sum_{i} \Psi(W_{i}, W'_{i}, \frac{q^{n-m-1}}{X}) \gamma(V_{i} \times V', \frac{q^{n-m-1}}{X}, \psi) \omega_{\tau}(-1)^{t} = \sum_{i} \Psi(W_{i}, W'_{i}, X).\]

Now multiplying through by \(\gamma(V'_{i} \times (V')', X, \psi^{-1})\omega_{\tau}(-1)^{t}\) and noticing that \(\omega_{\tau}(-1)^{t} = \omega_{\tau}^{-1}\), we get:

\[\sum_{i} \Psi(W_{i}, W'_{i}, \frac{q^{n-m-1}}{X}) \gamma(V_{i} \times V', \frac{q^{n-m-1}}{X}, \psi) \gamma(V'_{i} \times (V')', X, \psi^{-1}) = \sum_{i} \Psi(W_{i}, W'_{i}, \frac{q^{n-m-1}}{X}),\]

By Lemma 3.3 we have \(\gamma(V_{i} \times V', \frac{q^{n-m-1}}{X}, \psi) \gamma(V'_{i} \times (V')', X, \psi^{-1}) = 1\). \(\square\)

4. **A Converse Theorem for \(GL(n) \times GL(n-1)\)**

Recall that for a co-Whittaker module \(V\), the supercuspidal support of \(V\) is by definition the map \(f_{V} : \mathcal{Z} \to \text{End}_{Q}(V) \cong \tilde{A}\). The main result of this section is that the collection of gamma factors of pairs completely determines the supercuspidal support of a co-Whittaker family.

**Theorem 4.1.** Let \(A\) be a finite-type \(W(k)\)-algebra which is reduced and \(\ell\)-torsion free, and let \(K = \text{Frac}(W(k))\). Suppose \(V_{1}\) and \(V_{2}\) are two primitive co-Whittaker \(A[GL_{n}(F)]\)-modules. There is a finite extension \(K'\) of \(K\) with ring of integers \(O\) such that, if \(\gamma(V_{1} \times V', X, \psi) = \gamma(V_{2} \times V', X, \psi)\) for all co-Whittaker \(O[GL_{n-1}(F)]\)-modules \(V'\), then \(f_{V_{1}} = f_{V_{2}}\).

**Remark 4.2.**

1. Because of the control achieved in Theorem 4.1, it suffices to take in the statement of Theorem 4.1 only those co-Whittaker modules \(V'\) such that \(V' \otimes_{O} K'\) is absolutely irreducible.

2. The equality of gamma factors implies that \(V_{1}\) and \(V_{2}\) must live in the same block of the category \(\text{Rep}_{W(k)}(GL_{n}(F))\). The finite extension \(K'\) depends only on this block.
4.1. Supercuspidal Support and Whittaker Models. In this subsection we investigate the connection between Whittaker spaces and supercuspidal support.

**Lemma 4.3.** Suppose $V_1$ and $V_2$ are co-Whittaker modules. Then $\mathcal{W}(V_1, \psi) = \mathcal{W}(V_2, \psi)$ if and only if $f_{V_1} \equiv f_{V_2}$.

**Proof.** It follows from [4.4 below] that $f_{V_1} = f_{\mathcal{W}(V_1, \psi)} = f_{\mathcal{W}(V_2, \psi)} = f_{V_2}$. □

**Lemma 4.4.** Suppose we have two co-Whittaker modules $V_1$ and $V_2$ such that $V_1$ dominates $V_2$. Then $f_{V_1} = f_{V_2}$.

**Proof.** Suppose $\phi : V_1 \to V_2$ is the dominance map. Choosing a cyclic $A[G]$-generator $v_1 \in V_1$, then $\phi(v_1)$ is an $A[G]$-generator of $V_2$ since its image in $V_2^{(n)}$ is a generator. Denote by $v'_1$ the image of $v_1$ in $V_1 \to V_1^{(n)}$. We have $v'_1$ generates $V_1^{(n)}$ and $\phi^{-1}(v'_1)$ generates $V_2^{(n)}$.

If $z$ is an element of $\mathcal{Z}$, then $z_{V_1} \in \text{End}_G(V_1)$ sends $v_1$ to $f_{V_1}(z)v_1$, where $f_{V_1}(z) \in A$. By definition, the action of the Bernstein center is functorial, hence commutes with the morphism $\phi$, thus

$$z_{V_2}(\phi(v_1)) = \phi(f_{V_1}(z)v_1) = f_{V_1}(z)\phi(v_1).$$

Since $\phi(v_1)$ is an $A[G]$-generator of $V_2$, $z_{V_2}$ is completely determined by where it sends $\phi(v_1)$. This shows that the map

$$f_{V_2} : \mathcal{Z} \to \text{End}_G(V_2) \to A$$

given by $z \mapsto z_{V_2} \mapsto f_{V_2}(z)$ exactly equals the map $f_{V_1}$.

Second proof: use Lemma 4.5 below. □

**Lemma 4.5.** If $V$ is a co-Whittaker module with supercuspidal support $f_V : \mathcal{Z} \to \text{End}_G(V) \to A$, then the map $f_V$ equals its “derivative” $f_V^{(n)} : \mathcal{Z} \to \text{End}_A(V^{(n)}) \to A$ given by $z \mapsto z_V^{(n)} \to A$.

**Proof.** By definition, given $z \in \mathcal{Z}$ the endomorphism $z_V$ is translation by the scalar $f_V(z)$. The derivative morphism $z_V^{(n)} : V^{(n)} \to V^{(n)}$ is translation by that same scalar. The map $\text{End}_A(V^{(n)}) \to A$ is given by choosing a generator (it is free of rank one) and looking at the translation that an endomorphism defines. □

**Remark 4.6.** Any nonzero $G$-equivariant homomorphism between co-Whittaker modules which preserves the top derivative is a surjection.

4.2. Proof of Converse Theorem. For two $W(k)$-algebras $A, B$, $\phi_1 \in \text{c-Ind}_N^G \psi_{A}$ and $\phi_2 \in \text{Ind}_N^G \psi_{B}^{-1}$ we denote by $\langle \phi_1, \phi_2 \rangle$ the element

$$\int_{N \setminus G} \phi_1(x) \otimes \phi_2(x) dx \in A \otimes_{W(k)} B$$

and let $K = \text{Frac} W(k)$. At the heart of the proof of the converse theorem will lie the following result, which is proved in [4.5]

**Theorem 4.7.** Suppose $A$ is a finite-type, reduced, $\ell$-torsion free $W(k)$-algebra. Suppose $H \neq 0$ is an element of $\text{c-Ind} \psi_{A}$. Then there exists a finite extension $K'$ of $K$ with ring of integers $O$ and an absolutely irreducible generic integral $K'$ representation $U'$ with integral structure $U$, such that there is a Whittaker function $W \in \mathcal{W}(U', \psi_{O}^{-1})$ satisfying $\langle H, W \rangle \neq 0$ in $A \otimes_{W(k)} O$. [4.5]
The rest of this section is devoted to proving Theorem \[1]\. Let \(V_1\) and \(V_2\) be co-Whittaker with \(G\)-homomorphisms \(\omega_1 : V_1 \to \text{Ind}_N^G \psi\). Let \(\mathcal{S}(V_i)\) denote the sub-\(A[P_n]\)-module of \(V_i\) consisting of Schwartz functions of \(V_i\).

**Lemma 4.8.** Consider the sub-\(A[P_n]\)-modules \(\omega_1(\mathcal{S}(V_i))\) of \(\text{Ind}_N^G \psi\). If \(r : \text{Ind}_N^G \psi \to \text{Ind}_N^G \psi\) denotes the map given by restriction of functions, then \(r_P(\omega_1(\mathcal{S}(V_i))) = r_P(\omega_2(\mathcal{S}(V_2)))\).

**Proof.** Let \(\omega_{i,p}\) be the maps \(V_i|_P \to \text{Ind}_N^P \psi\) guaranteed by genericity. Then we have \(r_P \circ \omega_i = \omega_{i,p}\) from the definitions.

By [Mos] Prop 1.9 (2) we have \(\omega_1, r_P(\mathcal{S}(V_1)) = \omega_2, r_P(\mathcal{S}(V_2)) = \text{c-Ind}_N^P \psi\) as subsets of \(\text{Ind}_N^P \psi\). This proves the claim. \[\square\]

**Proposition 4.9.** Suppose the gamma factors are equal as in Theorem \[4]\. Take \(W_1 \in \omega_1(\mathcal{S}(V_1))\) and \(W_2 \in \omega_2(\mathcal{S}(V_2))\) such that \(r_P(W_1) = r_P(W_2)\), then \(W_1 = W_2\) as elements of \(\text{Ind}_N^P \psi\).

**Proof.** The proof follows [Hen93]. Let \(\mathcal{S}\) be the subspace of \(\mathcal{W}(V_1, \psi) \times \mathcal{W}(V_2, \psi)\) consisting of pairs \((W_1, W_2)\) such that \(r_{G_m}(W_1) = r_{G_m}(W_2)\), where \(r_{G_m}\) denotes restriction to the subgroup \(G_m\) of \(G_n\) (with \(m = n - 1\)). By the preceding discussion this is nonempty. Let \((W_1, W_2) \in \mathcal{S}\). Then

\[
\Psi(W_1, W', X) = \Psi(W_2, W', X)
\]

for all \(W' \in \mathcal{W}(V', \psi^{-1})\) as \(V'\) varies over all co-Whittaker \(O[G_{r-1}]\)-modules.

By assumption, \(\gamma(V_1 \times V', X, \psi) = \gamma(V_2 \times V', X, \psi)\) for all such \(V'\), whence the equality of the products:

\[
\Psi(W_1, W', X) \gamma(V_1 \times V', X, \psi) = \Psi(W_2, W', X) \gamma(V_2 \times V', X, \psi).
\]

Applying the functional equation with \(j = 0\) and \(m = n - 1\) we thus conclude that

\[
\Psi(\tilde{W}_1, \tilde{W}', \frac{q^{n-m-1}}{X}) = \Psi(\tilde{W}_2, \tilde{W}', \frac{1}{q^{n-m+1}X}),
\]

and furthermore

\[
\Psi(\tilde{W}_1, \tilde{W}', X) = \Psi(\tilde{W}_2, \tilde{W}', X).
\]

For each integer \(m\), denote by \(H_m\) the function on \(G_m\) given by

\[
H_m(g) = 0 \quad \text{if} \quad v_P(\det g) \neq m
\]

\[
H_m(g) = \tilde{W}_1 \left(\begin{smallmatrix} g & 0 \\ 0 & 1 \end{smallmatrix}\right) - \tilde{W}_2 \left(\begin{smallmatrix} g & 0 \\ 0 & 1 \end{smallmatrix}\right) \quad \text{if} \quad v_P(\det g) = m
\]

Then the equality of formal Laurent series \(\Psi(\tilde{W}_1, \tilde{W}', X) = \Psi(\tilde{W}_2, \tilde{W}', X)\) implies that, for each \(m\), we have

\[
\int_{N_m \backslash G_m} H_m(g) \otimes \tilde{W}'(g)dg = 0
\]

for all \(W'\) in the Whittaker spaces \(\mathcal{W}(V', \psi_\mathcal{O})\) of all co-Whittaker \(O[G]\)-modules \(V'\).

Now suppose \(V'\) has the property that \(V' \to V' \otimes \mathcal{K}'\) is an embedding and \(V' \otimes \mathcal{K}'\) is absolutely irreducible. Then \((V')' \otimes \mathcal{K}' \cong (V' \otimes \mathcal{K}')',\) and by [BZ76] Thm 7.3, \((V' \otimes \mathcal{K}')' \cong (V' \otimes \mathcal{K}')',\) where \((-)'\) means pre-composing the \(G\) action with \(g \mapsto g^{-1}\). Thus \(\mathcal{W}((V' \otimes \mathcal{K}')', \psi_{\mathcal{K}}^{-1}) = \mathcal{W}((V' \otimes \mathcal{K}')', \psi_{\mathcal{K}}^{-1})\), so given \(W' \in \mathcal{W}(V' \otimes \mathcal{K}')\),
\[
\mathcal{W}(V', \psi^{-1}) \times \mathcal{W}(V_2', \psi^{-1})
\]
there is an integer \(s\) such that \(\varpi^s W'\) is given by an element \(\tilde{W}\) in \(\mathcal{W}(V', \psi^{-1})\). Therefore
\[
\varpi^s \langle H_m, W' \rangle = \langle H_m, \varpi^s W' \rangle = \langle H_m, \tilde{W} \rangle = 0,
\]
which implies \(\langle H_m, W' \rangle = 0\) since \(A \otimes_{\mathcal{W}(k)} \mathcal{O}\) is flat over \(\mathcal{O}\) (i.e. \(\varpi\)-torsion free).

Therefore we can apply the contrapositive of Theorem 4.10 to conclude that each \(H_m\) is identically zero, for all \(m\). Hence
\[
\tilde{W}_1 \left( \begin{pmatrix} g & 0 \\ 0 & 1 \end{pmatrix} \right) \equiv \tilde{W}_2 \left( \begin{pmatrix} g & 0 \\ 0 & 1 \end{pmatrix} \right).
\]

Let \(\tilde{\mathcal{S}}\) be the subspace of \(\mathcal{W}(V_1', \psi^{-1}) \times \mathcal{W}(V_2', \psi^{-1})\) consisting of pairs \((U_1, U_2)\) whose restrictions to \(G_m \subset G_n\) are equal. Then we have shown that \((\tilde{W}_1, \tilde{W}_2) \in \tilde{\mathcal{S}}\).

**Lemma 4.10.** Let \(W_1, W_2\) be in \(\mathcal{W}(V_1, \psi)\) and \(W_2, W_2\) be in \(\mathcal{W}(V_2, \psi)\). Then \((W_1, W_2)\) is in \(\mathcal{S}\) if and only if \((\tilde{W}_1, \tilde{W}_2)\) is in \(\tilde{\mathcal{S}}\).

**Proof.** We have just proved one direction. By Lemma 3.3 our hypothesis on the equality of gamma factors is equivalent to the equality of the gamma factors
\[
\gamma(V_1' \times (V')', X, \psi^{-1}) = \gamma(V_2' \times (V')', X, \psi^{-1})
\]
for all \((V')'\). Since \((-)^t\) is an exact covariant functor which is additive in direct sums, commutes with base-change, and induces an isomorphism between Whittaker spaces, \(V \mapsto V'\) preserves the property of being co-Whittaker and \(V', (V')'\) are again co-Whittaker. Thus the entire argument works replacing \(V_i\) with \(V'_i\) and \(V'\) with \((V')'\) to get the converse implication.

If we let \(G_n\) act diagonally on \(\mathcal{W}(V_1, \psi) \times \mathcal{W}(V_2, \psi)\) and on \(\mathcal{W}(V_2', \psi^{-1}) \times \mathcal{W}(V_2', \psi^{-1})\), then the subgroup \(P_n\) stabilizes the subspaces \(\mathcal{S}\) and \(\tilde{\mathcal{S}}\). To see this note that for \(g \in G_m\) and \(u \in U_n\) we have \(W_i(gu) = W_i(gug^{-1}g) = \psi^g(u)W_i(g)\), so \(u W_i\)'s restriction to \(G_m\) is completely determined.

A short calculation shows
\[
g \tilde{W}(x) = \tilde{g} W(x).
\]
Combining this with the lemma above, it follows that \(\mathcal{S}\) is stable under \({}^t P\) as well.

Hence \(\mathcal{S}\) is stable under the group generated by \(P\) and \({}^t P\). But this group contains all elementary matrices, hence contains all of \(SL_n(F)\). On the other hand, this group also contains matrices of any determinant. Hence for any \(a \in F^\times\) it contains all matrices in \(GL_n(F)\) with determinant \(a\); in other words this group equals \(G\).

Therefore \(\mathcal{S}\) is stable under the action of all of \(G_n\). Thus given \(W_1\) and \(W_2\) such that \(r_P(W_1) = r_P(W_2)\) we have that \(r_P(gW_1) = r_P(gW_2)\) for any \(g \in G_n\) so we have \(gW_1(1) = gW_2(1)\), i.e. \(W_1(g) = W_2(g)\) for all \(g \in G_n\).

**Corollary 4.11.** If the gamma factors are equal as in Theorem 4.1, \(\omega_1(S(V_1)) = \omega_2(S(V_2))\).

**Proof.** Given \(W_1\) in the left side, there exists \(W_2\) such that \(r_P(W_1) = r_P(W_2)\). The previous lemma then implies \(W_1 = W_2 \in \omega_2(S(V_2))\) which shows one containment. The argument to show the opposite containment is identical.

**Corollary 4.12.** Suppose the gamma factors are equal as in Theorem 4.1 then \(\mathcal{W}(V_1, \psi) = \mathcal{W}(V_2, \psi)\).
Proof. Since $V_i$ is co-Whittaker and surjects onto $\mathcal{W}(V_i, \psi)$, we have that $\mathcal{W}(V_i, \psi)$ is also co-Whittaker. In particular, $\mathcal{W}(V_i, \psi)$ is generated over $A[G]$ by the $A[P]$-submodule consisting of its Schwartz functions, which is the same as $\omega_i(S(V_i))$. But if the gamma factors are equal we have shown that $\omega_1(S(V_1)) = \omega_2(S(V_2))$ and hence this lives inside $\mathcal{W}(V_1, \psi) \cap \mathcal{W}(V_2, \psi)$, the intersection taken within $\text{Ind}_N^G \psi$. Hence $\mathcal{W}(V_1, \psi)$ and $\mathcal{W}(V_2, \psi)$ contain a common $A[G]$-module generating set, hence are equal. □

Combining Corollary [4.12] and Lemma [4.3] concludes the proof of Theorem [4.1]

5. Proof of The Vanishing Theorem

This section is devoted to the proof of Theorem [4.7] Let $\psi : N \to W(k)^\times$ be an additive character, and let $Z$ denote the center of $\text{Rep}_{W(k)}(GL_n(F))$. Denote $\psi_A$ by $\psi \otimes_{W(k)} A$, then $\text{c-Ind}_N^G \psi_A \cong (\text{c-Ind}_N^G \psi) \otimes_{W(k)} A$.

There exists a primitive idempotent $e$ in $Z$ such that $eH \neq 0$. Moreover, there is some compact open subgroup $K$ such that $e_K H = H$, where $e_K$ is the projection $V \to V^K$. Let $e' = e * e_K * e$.

Let $R := eZ \otimes_{W(k)} A$. The $W(k)$-module $e'(\text{c-Ind} \psi_A) \cong e'(\text{c-Ind} \psi) \otimes_{W(k)} A$ carries the structure of an $R$-module, (i.e. we can consider it as an external tensor product). For convenience denote the $R$-module $e'(\text{c-Ind} \psi) \otimes_{W(k)} A$ by $\mathfrak{M}$.

**Lemma 5.1.** $\mathfrak{M}$ is finitely generated and torsion-free as an $R$-module. In particular, $\mathfrak{M}$ embeds in a free $R$-module.

**Proof.** Since $e'(\text{c-Ind} \psi)$ is finitely generated as an $eZ$-module ([Hel12b]), $\mathfrak{M}$ is finitely generated as an $R$-module.

Next, note that $e(\text{c-Ind} \psi)$ is torsion-free as an $eZ$-module. This follows from its torsion-free-ness at characteristic zero primes. Since $A$ and $eZ$ are both reduced and flat over $W(k)$, the ring $R$ is reduced and flat over $W(k)$. Now, a module over a reduced ring is torsion-free if and only if it can be embedded in a free module [Wie92, 1.5, 1.7]. Thus we focus on showing that $\mathfrak{M}$ can be embedded in a free $R$-module.

Since $e(\text{c-Ind} \psi)$ is reduced over $eZ$ there is an embedding of $eZ$-modules $e(\text{c-Ind} \psi) \to (eZ)^r$ for some $r$. Since $W(k) \to A$ is flat, $eZ \to R$ is flat, since flatness is preserved under base-change. Now tensor this embedding with $R$ to get a map of $R$-modules $\mathfrak{M} \cong e(\text{c-Ind} \psi) \otimes_{eZ} R \to (eZ)^r \otimes_{eZ} R \cong R^r$, where the first isomorphism is the canonical one $e(\text{c-Ind} \psi) \otimes_{W(k)} A \cong \left(e(\text{c-Ind} \psi) \otimes_{eZ} eZ\right) \otimes_{W(k)} A \cong e(\text{c-Ind} \psi) \otimes_{eZ} R$.

But since flatness is preserved by base change, $A$ being flat over $W(k)$ implies $R$ flat over $eZ$. Hence, the map $\mathfrak{M} \to R^r$ is an embedding, so $\mathfrak{M}$ is torsion-free. □

**Lemma 5.2.** The set

$$\{ q \in \text{Spec}(R) : eH \in q\mathfrak{M} \}$$

is contained in a closed subset $V$ of $\text{Spec}(R)$ such that $V \neq \text{Spec}(R)$. Moreover, this closed subset does not contain the generic fiber $\{ q \in \text{Spec}(R) : \ell \notin q \}$. 

Proof. From Lemma 5.1, there is an embedding $\mathcal{W} \subset R^r$, so $q\mathcal{W} \subset q^r$. Thus if $eH = (h_1, \ldots, h_n)$ is in $q\mathcal{W}$, each $h_i$ is in $q$. Hence $q$ is in the closed set $V := V(h_1) \cap \cdots \cap V(h_n)$. But $V \neq \text{Spec}(R)$ because some $h_i$ is nonzero (so there is some minimal prime not containing $h_i$, by reducedness).

Thus there is some nonempty open subset $D_{eH} \subset \text{Spec}(R)$ in the generic fiber consisting of points $q$ such that $eH \notin q\mathcal{W}$.

Lemma 5.3. Let $K$ be an infinite field and let $B$ be any infinite subset of $K$. Then the set of points $(X_1 - b_1, \ldots, X_n - b_n)$ such that $b_i \in B$ is dense in $\text{Spec}(K[X_1, \ldots, X_n])$.

Proof. We proceed by induction on $n$. If $n = 1$, we can show that every principal open subset intersects the set of points $\{(X - b)\}$. If $f \in K[X]$ were nonzero, then $f$ could not be divisible by $(X - b)$ for infinitely many $b$, whence there are points $(X - b)$ in $D(f)$.

Suppose the result holds for $n - 1$. We denote by $S$ the subset of points $(X_1 - b_1, \ldots, X_{n-1} - b_{n-1}, X_n)$, and choose an arbitrary $f$ nonzero in $K[X_1, \ldots, X_{n-1}, X_n]$ and consider $V = V(f)$ the set of prime ideals containing $f$. It suffices to show that $S$ cannot be contained in $V$. Consider the map $K[X_1, \ldots, X_n] \to K[X_1, \ldots, X_{n-1} \to X_n \mapsto b]$ for some $b \in B$. This gives the closed immersion $H \to \mathbb{K}$ of the hyperplane $H := \{X_n = b\}$. By the induction hypothesis the subset $T$ of points $(X_1 - b_1, \ldots, X_{n-1} - b_{n-1}, X_n - b)$ is dense in $H$. Suppose $V$ contains $S$, then $V \cap H \supset S \cap H \supset T$, meaning $V \cap H = H$. Since $b$ was arbitrary we’ve shown that $V$ contains every one of the distinct hyperplanes $\{X_n = b\}$ for $b \in B$. In particular this means each $X_n - b$ divides $f$, which is impossible.

Proposition 5.4. Let $\mathcal{K}$ be $\text{Frac}(W(k))$ and $e$ be a primitive idempotent of $\mathbb{Z}$. There exists a finite extension $\mathcal{K}'$ of $\mathcal{K}$, depending on $e$, with ring of integers $\mathcal{O}$ such that the set of points $p$ in $\text{Spec}(e\mathcal{Z})[\frac{1}{p}]$ with an embedding $R/p \hookrightarrow \mathcal{O}$ is dense in $\text{Spec}(e\mathcal{Z})[\frac{1}{p}]$.

Proof. First, note that the proof in Lemma 5.3 carries over for polynomial rings with any number of the variables $X_i$ inverted.

By [Hec12a Prop 11.1], $e\mathcal{Z} \otimes W(k) \cong \prod_{M, \pi} Z_{\mathcal{K}, M, \pi'}$ where $Z_{\mathcal{K}, M, \pi'}$ denotes the center of $\text{Rep}(G)_{M, \pi'}$. From [BDS4] we know $Z_{\mathcal{K}, M, \pi'} \cong (\mathcal{K}[M/M^\varphi]H)^{W(\pi')}$. Thus there exists a complete system primitive orthogonal idempotents $\{f_{M, \pi'}\}$ adding to 1 in $e\mathcal{Z} \otimes W(k) \mathcal{K}$, such that $f_{M, \pi'} e\mathcal{Z} \otimes W(k) \mathcal{K} \cong (\mathcal{K}[M/M^\varphi]H)^{W(\pi')}$. But $f_{M, \pi'}$ lives in $e\mathcal{Z} \otimes \mathcal{K}_i$ for some finite extension $\mathcal{K}_i$ of $\mathcal{K}$, and faithfully flat descent gives that the natural map $f_{M, \pi'} e\mathcal{Z} \otimes W(k) \mathcal{K}_i \to (\mathcal{K}_i[M/M^\varphi]H)^{W(\pi')}$ is an isomorphism (this natural map is described, for example, in [Ber93 p. 74 Rmk]).

Thus there is a finite list $\mathcal{K}_1, \ldots, \mathcal{K}_n$ of finite extensions of $\mathcal{K}$ such that there is a continuous surjection $\bigcup \text{Spec}(\mathcal{K}_i[X_1^{\pm 1}, \ldots, X_n^{\pm 1}]) \to \text{Spec}(e\mathcal{Z}[\frac{1}{p}])$.

Now apply Lemma 5.3 with $\mathcal{K} = \mathcal{K}_i$ and $B = \mathcal{O}_i$ its ring of integers to get a dense subset in the generic fiber of each component of the disjoint union. Since the image of a dense set under a surjective continuous map is dense, the subset of points $p$ in the generic fiber of $\text{Spec}(e\mathcal{Z})$ such that $e\mathcal{Z}/p$ embeds in one of the $O_i$'s is dense (in the generic fiber). Let $\mathcal{K}'$ be a finite extension of $\mathcal{K}$ containing each $\mathcal{K}_i$, and let its ring of integers be $\mathcal{O}$. Then any point $e\mathcal{Z}/p$ which embeds in some $\mathcal{O}_i$ also embeds in $\mathcal{O}$. ∎
In order to have irreducible objects at hand, we will realize irreducibility on an open subset.

**Proposition 5.5.** Suppose \( p \) is a minimal prime of \( e\mathcal{Z} \). Then \( e\mathcal{W} \otimes_{e\mathcal{Z}} \kappa(p') \) is absolutely irreducible for all \( p' \) in an open neighborhood of \( p \).

**Proof.** Let \( \Pi := e(\text{c-Ind} \psi) \). We begin by showing that the locus of points \( p \) such that \( \Pi \otimes \kappa(p) \) is reducible is contained in a closed subset. For a ring \( R \) and \( K \) a compact open subgroup let \( \mathcal{H}(G, K, R) \) be the algebra of smooth compactly supported functions \( G \to R \) which are \( K \)-fixed under right translation. \( \mathcal{H}(G, K, e\mathcal{Z}) \) and \( \Pi \) form sheaves over \( \text{Spec}(e\mathcal{Z}) \), and following [Ber93, IV.1.2], the map \( P_K : \mathcal{H}(G, K, e\mathcal{Z}) \to \text{End}_{e\mathcal{Z}}(\Pi^K) \) which sends \( h \) to \( \Pi(h) \) is a morphism of sheaves. \( \Pi[p] := \Pi \otimes \kappa(p) \) is irreducible if and only if, for any \( K \), \( \Pi[p] \) is either zero or irreducible over \( \mathcal{H}(G, K, \kappa(p)) \). Supposing \( \Pi[p] \) is reducible, there exists a \( K \) such that \( \Pi[p] \) is nonzero and reducible. Since \( \Pi[p] \) is a finite dimensional \( \kappa(p) \)-vector space, a proper \( \mathcal{H}(G, K, \kappa(p)) \)-stable subspace gives a proper submodule of the endomorphism ring containing the image of \( P_K \otimes \kappa(p) \). The set of points \( \mathcal{p} \) where \( (P_K)_{\mathcal{p}} \) fails to be surjective is contained in the support of the finitely generated \( e\mathcal{Z} \)-module \( \frac{\Pi[p]}{\text{im}(P_K)} \), which is closed. For any such point \( \mathcal{p} \), \( \Pi[p] = (\Pi^K)|_{\mathcal{p}} \) must then be reducible by Schur’s lemma. It remains to show that there is at least one point where we have irreducibility.

Suppose \( e = e_{[L, \pi]} \) is the idempotent corresponding to the mod \( \ell \) inertial equivalence class \( [L, \pi] \) in the Bernstein decomposition of \( \text{Rep}_{W(k)}(G) \) (see [Hel12b]). By [Hel12a] Prop 11.1, \( e\mathcal{Z} \otimes_{W(k)} \mathcal{K} \cong \prod_{M, \pi', \sigma} \mathcal{Z}_{M, \pi', \sigma} \) where \( M, \pi' \) runs over inertial equivalence classes of \( \text{Rep}_{\mathcal{K}}(G) \) whose mod \( \ell \) inertial supercuspidal support equals \( (L, \pi) \), and \( \mathcal{Z}_{M, \pi', \sigma} \) denotes the center of \( \text{Rep}_{\mathcal{K}}(G)_{M, \pi', \sigma} \). The ring \( \mathcal{Z}_{M, \pi', \sigma} \) is a Noetherian normal domain. Since \( e\mathcal{Z} \) is reduced and \( \ell \)-torsion free, none of its minimal primes contain \( \ell \). Inverting \( \ell \), this decomposition gives isomorphisms

\[
\prod_{\mathcal{p} \text{ minimal}} [(e\mathcal{Z})/\mathcal{p}] \otimes_{W(k)} \mathcal{K} \cong e\mathcal{Z} \otimes_{W(k)} \mathcal{K} \cong \prod_{M, \pi', \sigma} \mathcal{Z}_{M, \pi', \sigma}.
\]

In particular, for each minimal prime there exists \( M, \pi' \) such that \( [(e\mathcal{Z})/\mathcal{p}] \otimes_{W(k)} \mathcal{K} \cong \mathcal{Z}_{M, \pi', \sigma} \). Hence \( \kappa(p) = \text{Frac}(\mathcal{Z}_{M, \pi', \sigma}) \).

Given such an \( M, \pi' \), we have by [BDN14] that

\[
\mathcal{Z}_{M, \pi', \sigma} := \mathcal{Z}(\text{Rep}_{\mathcal{K}}(G)_{M, \pi'}) \cong (\mathcal{K}[M/M^\circ]^{H})^{W(\pi')},
\]

where \( M^\circ \) is the subgroup generated by all the compact subgroups, which equals the set of \( m \in M \) with \( \det m \in U_F \). Let \( \Psi(M) \) be the linear algebraic group over \( \mathcal{K} \) of unramified characters of \( M \), in other words the ring \( \mathcal{K}[M/M^\circ] \). Then by [Ber93], if \( \pi' \) is our given supercuspidal representation of \( \mathcal{K} \), then \( i_P^G(\pi' \otimes \chi) \) is irreducible for \( \chi \) a generic \( \mathcal{K} \) point of \( \mathcal{K}[M/M^\circ] \). Let \( q \) be a point of \( e\mathcal{Z} \) lying under the point \( \chi \). Since \( i_P^G(\pi' \otimes \chi) \) is cuspidal, \( (i_P^G(\pi' \otimes \chi))^{(n)} \) is one dimensional and therefore we have a map \( e(\text{c-Ind} \psi) \otimes_{\mathcal{Z}_{M, \pi', \sigma}} \kappa(q) \to i_P^G(\pi' \otimes \chi) \) coming from reciprocity. Since \( \pi' \otimes \chi \) is absolutely irreducible this map is surjective. The kernel \( K \) of this map must be zero by the following reasoning. By [EH12 Cor 3.2.14] all the Jordan-Hölder constituents of an essentially AIG representation over \( \mathcal{K} \) have the same supercuspidal support, so the same is true for representations with essentially AIG dual. Therefore, if \( K \) were nonzero it would have all Jordan-Hölder constituents having the same supercuspidal support as \( i_P^G(\pi' \otimes \chi) \), in particular those constituents...
would be irreducible and equivalent to $i_{q}^{G}(\pi' \otimes \chi)$. But then $K^{(n)}$ is nonzero, which contradicts the fact that $e(\text{c-Ind } \psi) \otimes_{Z,M_{\ast}} \kappa(q) \to i_{p}^{G}(\pi' \otimes \chi)$ is a $G$-surjection of generic representations. Hence $e(\text{c-Ind } \psi) \otimes \kappa(q)$ is absolutely irreducible. This proves the claim. \hfill \square

Since the algebra $W(k) \to A$ is flat and finite type, the map $i : eZ \to R$ is flat and finite type. Since these rings are Noetherian, this implies the induced map $\tilde{i} : \text{Spec}(R) \to \text{Spec}(eZ)$ is open. Let $O$ be the finite extension of $W(k)$ given by Prop 5.4, with uniformizer $\pi$. We then have $\tilde{i}(D_{eH})$ forms an open subset of $\text{Spec}(eZ)$, and therefore by Corollary 4.7 and Proposition 5.3 we have an $O$-valued point $f$ satisfying:

(1) $f$ factors as $f : eZ \to eZ/p \to O$ for some $p \in \text{Spec}(eZ)$
(2) The map $\mathcal{M} \to \mathcal{M} \otimes_{eZ,F} O$ does not kill $eH$
(3) The fiber $e(\text{c-Ind } \psi) \otimes_{eZ} \kappa(p)$ is absolutely irreducible.

The first condition can be achieved because $p = i^{-1}(q)$ for a $q \in \text{Spec}(R)$ satisfying $eH \not\in q\mathcal{M}$. Let $f_{A} : R \to O \otimes_{W(k)} A$ be the base change of $f : eZ \to O$ to an $A$-algebra homomorphism. Let $A' = O \otimes_{W(k)} A$. Since $p\mathcal{M} \subset q\mathcal{M}$, $eH$ cannot lie in the submodule $p\mathcal{M}$. We now use this point $p$ to construct a Whittaker function as in Theorem 4.17.

If we define the map $p_{A} : \mathcal{M} \to \mathcal{M}/p\mathcal{M} = \mathcal{M} \otimes_{R,f_{A}} A'$, we have $p_{A}(eH) \neq 0$ by flatness. Also, $p_{A}$ is the base-change $p \otimes 1$ of the map $p$ given by $p : e' \text{c-Ind } \psi \to \frac{e' \text{c-Ind } \psi}{p(e' \text{c-Ind } \psi)} = (e' \text{c-Ind } \psi) \otimes_{eZ,F} O$.

Now denote by $U$ the $O[G]$-module $(e' \text{c-Ind } \psi) \otimes_{eZ,F} O$ and by $U_{A}$ the $A'[G]$-module $\mathcal{M} \otimes_{R,f_{A}} A'$, which is isomorphic to $U \otimes_{W(k)} A$. Let $U_{A}'$ be the smooth $A'$-linear dual of $U_{A}$ and $U^{'\vee}$ be the smooth $O$-linear dual of $U$. If $\pi$ is a uniformizer of $O$ note that condition (3) gives $U[\frac{1}{\pi}]$ is absolutely irreducible, and we have that $U \to U[\frac{1}{\pi}]$ is an embedding because $e(\text{c-Ind } \psi)$ is $\ell$-torsion free. Since $p_{A}(eH) \neq 0$ we can choose $v_{A}' \in U_{A}'$ such that $(v_{A}', p_{A}(eH)) \neq 0$ in $A'$. We will also identify $e' \text{Ind } \psi_{A}^{-1}$ with the $A'$-linear dual $(e' \text{c-Ind } \psi_{A})^{\vee}$ and $e' \text{Ind } \psi_{O}^{-1}$ with the $O$-linear dual of $e' \text{c-Ind } \psi_{W(k)}$. We formulate:

**Lemma 5.6.** The following diagram commutes:

\[ \begin{array}{ccc} \text{Hom}_{Z[G]}(e' \text{c-Ind } \psi, U) \otimes_{W(k)} A & \xrightarrow{\phi} & \text{Hom}_{O[G]}(e' \text{c-Ind } \psi_{A}, U_{A}) \\ \downarrow & & \downarrow \\ \text{Hom}_{O[G]}(U^{'\vee}, e' \text{Ind } \psi_{O}^{-1}) \otimes_{W(k)} A & \xrightarrow{\phi} & \text{Hom}_{A'[G]}((U_{A})^{\vee}, e' \text{Ind } \psi_{A}^{-1}) \end{array} \]

**Proof.** Since $U_{A}' = U^{'} \otimes_{O} A$ and $e' \text{Ind } \psi_{A'} = (e' \text{c-Ind } \psi_{O}) \otimes_{W(k)} A$, the horizontal arrows are maps of $A$-modules given by sending $\phi \otimes 1$ to the map $[h \otimes a \mapsto \phi(h) \otimes a]$. The maps are injective because both $U$ and $e' \text{Ind } \psi_{O}^{-1}$ are finitely generated over $O[G]$. The downward arrows are defined by $\phi \mapsto \phi_{*}$ where $\phi_{*}$ takes a map to its precomposition with $\phi$. 

We now show commutativity:

\[ p \otimes a \mapsto [\phi : h \otimes b \mapsto p(h) \otimes ab] \]
\[ \downarrow \]
\[ p_* \otimes a \mapsto [u^\vee \otimes b \mapsto p_*(u^\vee) \otimes ab] \]

So we must check that \( p_*(u^\vee \otimes b) \) and \( p_*(u^\vee) \otimes ab \) are equal as elements of \((e' c\text{-Ind}_{\psi_A})^\vee \cong (e' c\text{-Ind}_{\psi}) \otimes A\). But given \( h \in e' c\text{-Ind}_{\psi} \) and \( c \in A \) we have \( \phi_*(u^\vee \otimes b)(h \otimes c) = (u^\vee \otimes b)(p(h) \otimes ac) = u^\vee(p(h)) \otimes abc \). On the other hand we have \( (p_*(u^\vee) \otimes ab)(h \otimes c) = u^\vee(p(h)) \otimes abc \), as desired. \( \square \)

The map \( p_A \in \text{Hom}_{R[G]}(\mathfrak{m}, U_A) \) is in the image of the top horizontal map since it is the base change \( p \otimes 1 \). Thus \( (p_A)_* \) equals \( p_\ast \otimes 1 \). Since \( v_A^\vee \) is in \( U^\vee \otimes_A A \) we can expand it as \( v_A^\vee = \sum_i v_i^\vee \otimes a_i \) with \( v_i^\vee \in U^\vee \) and \( a_i \in A \). Then we have

\[
0 \neq \langle v_A^\vee, p_A(eH) \rangle = \langle (p_A)_*(v_A^\vee), eH \rangle \\
= \langle (p_* \otimes 1)(v_A^\vee), eH \rangle \\
= \langle (p_* \otimes 1)(\sum_i v_i^\vee \otimes a_i), eH \rangle \\
= \sum_i p_*(v_i^\vee) \otimes a_i, eH \rangle \\
= \sum_i a_i(p_*(v_i^\vee), eH) 
\]

This implies that not all the terms \( \langle p_*(v_i^\vee), eH \rangle \) are zero. Therefore \( (p_*(v_i^\vee), eH) \neq 0 \) for some \( i \). Since \( p_* : U^\vee \to e' \text{-Ind}_{\psi \otimes Z, J} \mathcal{O} \) is \( \mathcal{O}[G] \)-linear, it is a (the) map to the Whittaker space of \( U^\vee \), so \( p_*(v_i^\vee) \) defines an element of \( \mathcal{W}(U^\vee, \psi^{-1}) \). \( U \) is a co-Whittaker \( \mathcal{O}[G] \)-module, as it equals \( e \text{-Ind}_{\psi \otimes Z, J} \mathcal{O} \). Since the action of the central idempotent \( e \) is self-transpose by Lemma 5.7 below, we have \( (p_*(v_i^\vee), eH) = \langle ep_*(v_i^\vee), H \rangle = \langle p_*(v_i^\vee), H \rangle \) is nonzero. Hence the \( W(k)[G] \)-module \( U \) and the Whittaker function \( p_*(v_i^\vee) \) satisfy the theorem.

**Lemma 5.7.** Let \( \theta \) be an element of \( c\text{-Ind}_N^G \psi \) and let \( \eta \) be an element of \( c\text{-Ind}_N^G \psi \). Let \( z \) be an element of the Bernstein center \( Z(G) \). Then \( \langle z\theta, \eta \rangle = \langle \theta, z\eta \rangle \).

**Proof.** We will use the alternate description of the Bernstein center given in [Ber93 III.4.2], which is as follows. As \( Z(G) \) is the center of the category of nondegenerate \( \mathcal{H}(G) \)-modules, and \( \mathcal{H}(G) \) defines a bi-module over itself, each element \( z \) of the Bernstein center defines an element \( \phi_z \) of \( \text{End}_{G \times G}(\mathcal{H}(G)) \). The map \( Z \to \text{End}_{G \times G}(\mathcal{H}(G)) \) sending \( z \mapsto \phi_z \) is an isomorphism.

Choose a compact open subgroup \( K \) such that \( K = K^t \) and small enough that it fixes both \( \theta \) and \( \eta \). In particular if \( e_K \) is the idempotent in \( \mathcal{H}(G) \) defined by

\[
e_K(x) = \begin{cases} \mu(K)^{-1} & \text{if } x \in K \\ 0 & \text{if } x \notin K \end{cases}
\]
then $e_K \ast \theta = \theta$ and $e_K \ast \eta = \eta$. By the alternate description above ([Ber93 III.4.2]), we have $z \eta = \phi_z(e_K) \ast \eta$. Hence

\[
\langle \theta, z \eta \rangle = \langle \theta, \phi_z(e_K) \ast \eta \rangle = \int_G \phi_z(e_K)(x) \langle \theta, x' \eta \rangle \, dx = \int_G \phi_z(e_K)(x) \langle t x \theta, \eta \rangle \, dx = \langle \phi_z(e_K)^t \ast \theta, \eta \rangle
\]

where $\phi_z(e_K)^t$ denotes the element $g \mapsto \phi_z(e_K)^t(g)$ of $\mathcal{H}(G)$.

But there exists an element $w$ in $G$ such that $t g = w g w^{-1}$. Since $\phi_z$ is a $G \times G$-equivariant endomorphism of $\mathcal{H}(G)$, we have $\phi_z(e_K)^t = \phi_z(e_K^t)$. Since we have chosen $K$ such that $K = K^t$, this is the same as $\phi_z(e_K)$. Therefore,

\[
\langle \phi_z(e_K)^t \ast \theta, \eta \rangle = \langle z \theta, \eta \rangle.
\]

\[\square\]

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