COURSE 1

2D TRANSONIC HYDRODYNAMICS IN GENERAL RELATIVITY

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Abstract

The goal of my lecture is to present the introduction into the hydrodynamical version of the Grad-Shafranov equation. Although not so well-known as the full MHD one, it allows us to clarify the nontrivial structure of the Grad-Shafranov approach as well as to discuss the simplest version of the 3 + 1-split language – the most convenient one for the description of the ideal flows in the vicinity of a rotating black hole.

1 Introduction

Axisymmetric stationary flows in the vicinity of a central compact body have been studied for a long time in connection with many astrophysical sources. Accretion onto ordinary stars and black holes, axially symmetrical stellar (solar) wind, jets from young stellar objects, outflow from axisymmetric magnetosphere of rotating neutron stars – all of them are flows of the considered type.

Let me stress that the necessity of taking into account the effects of General Relativity is not so obvious for many compact sources. For instance, one can not exclude that the black hole plays only a passive role in the jet formation process, and the effects of General Relativity in this case may be inimportant for flow description in the region of jet formation. At the same time gravitational effects make, apparently, a noticeable contribution to the determination of physical conditions in compact objects. First, this is indicated by the hard spectra and the annihilation line observed in galactic X-ray sources, which are believed to be solar mass black holes. Such characteristics are never observed in the X-ray sources which are firmly established to show accretion not onto a black hole but onto a neutron star.
Another indication comes from superluminal motion in quasars which may be due to the relativistic electron-positron plasma flow ejected along with the weakly relativistic jet [9]. All this testifies in favour of the existence of an additional mechanism for particle creation and acceleration, for which the effects of General Relativity may be of principal importance. So, it is undoubtedly interesting to consider the flow structure in the most general conditions, i.e., in the presence of a rotating black hole.

There are several reasons why I restrict my consideration to the pure hydrodynamic flows. First of all, the hydrodynamical version of the Grad-Shafranov equation is not so popular as the full MHD one. On the other hand, it contains all the features of the full MHD version in the simplest form. In particular, within the hydrodynamic approach one can introduce the $3 + 1$-split language – the most convenient one for the description of the ideal flows in the vicinity of a rotating black hole.

Thus, in this lecture the basic equations describing a steady axisymmetric hydrodynamical flow in the vicinity of Kerr black hole are given. Starting with the well-known set of equations describing the nonrelativistic ideal flow [10], we will go step by step to more complicated cases up to the most general one corresponding to the axisymmetric stationary flow in the Kerr metric. Finally, several examples will be considered which demonstrate how the approach under consideration can be used to obtain the quantitative description of the real transonic flows in the vicinity of rotating black holes.

2 Ideal Hydrodynamics – Some Fundamental Results

2.1 Basic Equations

To start from the very beginning, let us write down the equations of ideal stationary ($\partial/\partial t = 0$) hydrodynamics in a flat space. They are [11]:

- Continuity equation
  \[ \nabla \cdot (n\mathbf{v}) = 0, \]  
  \[ (2.1) \]

- Euler equation
  \[ (\mathbf{v} \nabla)\mathbf{v} = -\frac{\nabla P}{mp_n} - \nabla \phi_g, \]  
  \[ (2.2) \]

- Entropy condition
  \[ \mathbf{v} \cdot \nabla s = 0, \]  
  \[ (2.3) \]

- Equation of state
  \[ P = P(n, s). \]  
  \[ (2.4) \]
The last expression can be rewritten in the form
\[ dP = m_p n dw - nTd s. \] (2.5)

Here \( n \ (\text{cm}^{-3}) \) is the concentration, \( s \) is the entropy per particle (undimensional), \( w \ (\text{cm}^2/\text{s}^3) \) is the specific enthalpy, \( m_p \ (\text{g}) \) is the mass of particle, \( T \ (\text{erg}) \) is the temperature in energetic unit, and \( c_s \ (\text{cm/s}) \) is the velocity of sound. For the polytropic equation of state
\[ P = k(s)n^\Gamma, \] (2.6)
used for simplicity in what follows, one can obtain for \( \Gamma \neq 1 \)
\[ c_s^2 = \frac{1}{m_p} \left( \frac{\partial P}{\partial n} \right)_s = \frac{1}{m_p} \Gamma k(s)n^{\Gamma-1}, \] (2.7)
\[ w = \frac{c_s^2}{\Gamma - 1}, \] (2.8)
\[ T = \frac{m_p}{\Gamma} c_s^2. \] (2.9)

Now one can make several remarks.

- The Euler equation (2.2) together with (2.3) and (2.5) can be rewritten as the energy equation
\[ \nabla \cdot \left[ n v \left( \frac{v^2}{2} + w + \varphi \right) \right] = 0. \] (2.10)

Now using the continuity equation (2.1), one can obtain
\[ v \cdot \nabla E = 0, \] (2.11)
where
\[ E = \frac{v^2}{2} + w + \varphi. \] (2.12)
This is the well-known Bernoulli integral.

- The energy equation (2.10) together with the Euler equation (2.2) can be rewritten as a four-component energy-momentum equation
\[ \nabla_\alpha T^{\alpha \beta} = 0, \] (2.13)
where for \( \varphi = 0 \)
\[ T^{\alpha \beta} = \begin{pmatrix} nm_p v^2/2 + n \varepsilon & nm_p \nu (v^2/2 + w) \\ m_p n v^i & P \delta^{ik} + nm_p v^i v^k \end{pmatrix}. \] (2.14)

Here and below the Greek indices \( \alpha \) and \( \beta \) correspond to four-dimensional values, while the Latin \( i, j, \) and \( k \) are three-dimensional.
As a result, hydrodynamics contains five equations over five unknown values.

**exercise**

1. Prove expressions (2.14)–(2.17).

### 2.2 An Example – Spherically Symmetrical Flow

As a most simple but very important example let us consider the spherically symmetrical flow. Since the basic (ideal) hydrodynamic equations have the form of the conservation laws, one can find

- the continuity equation
  \[ \Phi = 2\pi r^2 n(r)v(r) = \text{const}, \tag{2.15} \]
- the entropy equation
  \[ s = \text{const}, \tag{2.16} \]
- the energy equation
  \[ E = \frac{v^2(r)}{2} + w(r) + \varphi_g(r) = \text{const}. \tag{2.17} \]

As a result, knowing three parameters \( \Phi, s, \) and \( E \), one can determine all the physical characteristics of a flow. Indeed, rewriting the Bernoulli equation (2.17) as

\[ E = \frac{\Phi^2}{8\pi^2 n^2 r^4} + \phi(n, s) + \varphi_g(r), \tag{2.18} \]

we see that this equation contains only one unknown parameter \( n \). Hence, this algebraic equation implicitly determines the concentration \( n \) as a function of the invariants and radius \( r \):

\[ n = n(E, s, \Phi; r). \tag{2.19} \]

Together with the entropy \( s \), it allows us to determine all the other thermodynamic functions and the flow velocity \( v(r) \).

It is necessary to stress that equation (2.18) contains a singularity on the sonic surface. To show this, let us determine the derivative \( \frac{dn}{dr} \):

\[
\frac{dn}{dr} \left[ \left( \frac{\partial w}{\partial n} \right)_s - \frac{\Phi^2}{4\pi^2 n^3 r^4} \right] - \frac{\Phi^2}{2\pi^2 n^2 r^5} + \frac{GM}{r^2} = 0.
\]
As a result, using the thermodynamic relation (2.3), one can obtain for the logarithmic derivative
\[ \eta_1 = \frac{r \frac{dn}{dr}}{n} = \frac{2v^2 - GM/r}{c_s^2 - v^2} = \frac{2 - GM/rv^2}{-1 + c_s^2/v^2} = \frac{N_r}{D}. \] (2.20)

We see that the derivative (2.20) contains the singularity when the velocity is equal to that of sound \( c_s \) \((D = 0)\). It means that to pass through the sonic surface \( r = r_s \) the additional critical condition is to be valid:
\[ N_r(r_s) = 2 - \frac{GM}{r_s c_s^2} = 0. \] (2.21)

In other words, the transonic flows are determined by two invariants only. As shown in Fig. 1, the sonic surface is an X-point on the plane distance – velocity.

**Fig. 1.** Structure of the spherically symmetric accretion for a given \( n_\infty \) and \( c_\infty \), and different stream function \( \Phi \). The transonic flow corresponds to the critical accretion rate \( \Phi = \Phi_{cr} \) (2.34). The curves below the X-point correspond to subsonic accretion with \( \Phi < \Phi_{cr} \).

### exercises

1. **For spherically symmetrical transonic inflow (Bondi accretion)** one can determine the Bernoulli integral \( E \) through the velocity of sound at infinity
\[ E = w_\infty = \frac{c_\infty^2}{\Gamma - 1}. \]

Now using relations (2.13), (2.14) and (2.21), obtain the well-known expressions for the velocity of sound \( c_s \) and the concentration \( n_s \) on
The sonic radius $r_*$:

\begin{align*}
c_*^2 &= \left( \frac{2}{5 - 3\Gamma} \right) c_{\infty}^2, \quad (2.22) \\
n_* &= \left( \frac{2}{5 - 3\Gamma} \right)^{1/(\Gamma - 1)} n_{\infty}, \quad (2.23) \\
r_* &= \left( \frac{5 - 3\Gamma}{4} \right) \frac{GM}{c_{\infty}^2}. \quad (2.24)
\end{align*}

2. Show that

\[ \eta_1(r_*) = -4 \pm \sqrt{10 - 6\Gamma \frac{\Gamma + 1}{\Gamma + 1}}, \quad (2.25) \]

the sign plus corresponding to accretion and minus – to ejection.

3. Find that for the spherically symmetric accretion

- for $r \gg r_*$ (subsonic regime) the flow is approximately incompressible:

\begin{align*}
n(r) &\approx \text{const}, \quad (2.26) \\
v(r) &\propto r^{-2}. \quad (2.27)
\end{align*}

- for $r \ll r_*$ (supersonic regime) the particle motion corresponds to a free fall:

\begin{align*}
n(r) &\propto r^{-3/2}, \quad (2.28) \\
v(r) &\approx (2GM/r)^{1/2}. \quad (2.29)
\end{align*}

4. Find that for the spherically symmetric transonic outflow (Parker ejection \[\square\]):

- Physical parameters on the sound surface $r = r_*$, where again

\[ r_* = \frac{GM}{2c_*^2}, \quad (2.30) \]

have the following relations to the ones on the star surface $r = R$

\begin{align*}
c_*^2 &= \left( \frac{2}{5 - 3\Gamma} \right) c_R^2 + \left( \frac{\Gamma - 1}{5 - 3\Gamma} \right) \left( v_R^2 - \frac{2GM}{R} \right), \quad (2.31) \\
n_* &= n_R \left( \frac{c_*^2}{c_R^2} \right)^{1/(\Gamma - 1)}. \quad (2.32)
\end{align*}
The (radial) velocity on the star surface is to be

\[ v_R = c_\ast \left( \frac{c_\ast^2}{c_R^2} \right)^{1/(\Gamma - 1)} \left( \frac{r_\ast}{R} \right)^2. \]  

(2.33)

Although fairly simple, the radial 1D flow allows us to formulate here several important properties; some of them, as we shall see, retains the common properties of the Grad-Shafranov approach.

- It is possible to pass through the sonic surface in the presence of gravity only. Indeed, the nominator \( N_r \) in (2.20) can vanish for \( D = 0 \) in the presence of the gravity term \( GM/rv^2 \) only.

- Solutions (2.22)–(2.24) and (2.31) have singularity for \( \Gamma = 5/3 \). It means that for \( \Gamma = 5/3 \) the increase/decrease of the velocity of sound as a result of adiabatic heating/cooling equals the change of particle velocity. As a result, in the nonrelativistic case for \( \Gamma \geq 5/3 \) the transonic flow is not realized.

- The transonic problem is two-parametric. It means that to determine the transonic flow it is necessary to specify two boundary conditions, say, the density \( m_p n_\infty \) and the velocity of sound \( c_\infty \) at infinity. In particular, the accretion rate is fixed:

\[ 2\Phi_{cr} = 4\pi r_\ast^2 c_\ast n_\ast = 4\pi \left( \frac{2}{5 - 3\Gamma} \right)^{(5-3\Gamma)/2(\Gamma+1)} \frac{(GM)^2}{c_\infty^3} n_\infty. \]  

(2.34)

On the other hand, for a given \( n_\infty \) and \( c_\infty \) there is an infinite number of subsonic solutions with \( \Phi < \Phi_{cr} \) (see Fig. 1).

- For a given flow structure the number of integrals is enough to determine all the characteristics of a flow from algebraic equations.

The last property is actually the key point of the approach. Indeed, the algebraic relations (2.15)–(2.17), together with the equation of state allow us to determine all the physical parameters of the flow (the flow velocity \( v(r) \), the temperature \( T(r) \), etc.) through the invariants \( E \) and \( s \) and the stream function \( \Phi \). This property remains true for an arbitrary 2D flow structure. But in the general case the flow structure (i.e., the stream function \( \Phi(r, \theta) \)) itself is not known. To determine the stream function, the extra two hydrodynamic equations are to be used.

### 2.3 Flat Potential Flow

To start, let us consider the simplest (and the well-known) case of the flat potential flow without gravity. Then one can introduce potential \( \phi(x, y) \) by
the relation
\[ \mathbf{v} = \nabla \phi(x, y). \] (2.35)

In addition, let us consider the case
\[ E = \text{const}, \quad s = \text{const}. \] (2.36)

Then the continuity equation \( \nabla \cdot (n \mathbf{v}) = 0 \) can be rewritten as
\[ \nabla^2 \phi + \frac{n \cdot \nabla \phi}{n} = 0. \] (2.37)

Finally, using the Euler equation to determine \( \nabla n \cdot \nabla \phi \)
\[ \mathbf{v} \cdot \nabla \left( \frac{v^2}{2} \right) + c_s^2 \frac{n \cdot \nabla \phi}{n} = 0, \]
we obtain
\[ \phi_{xx} + \phi_{yy} + \frac{(\phi_y)^2 \phi_{xx} - 2 \phi_x \phi_y \phi_{xy} + (\phi_x)^2 \phi_{yy}}{(\nabla \phi)^2 D} = 0. \] (2.38)

Here again
\[ D = -1 + \frac{c_s^2}{v^2}. \] (2.39)

This well-known equation can be found in any textbook (e.g., see [11]).

For us it is necessary to stress here the following properties.

- To determine \( c_s^2 \), equation (2.38) is to be supplemented with the Bernoulli equation. For the polytropic equation of state it can be solved for the velocity of sound \( c_s \):
  \[ c_s^2 = (\Gamma - 1)E - \frac{\Gamma - 1}{2} (\nabla \phi)^2. \] (2.40)

- Together with the Bernoulli equation, equation (2.38) contains the potential \( \phi(x, y) \) and the invariant \( E \) only (it does not contain the entropy \( s \) at all, but \( s \) is necessary to determine \( n \)).

- For \( n = \text{const} \) \( (c_s^2 \to \infty) \) the equation becomes linear.

- It is nonlinear in the general case, but linear for second derivatives.

- The equation is elliptical for a subsonic flow \( D > 0 \).

- The equation is hyperbolic for a supersonic flow \( D < 0 \).

- For a given flow structure (for a given \( \phi \), \( E \), and \( s \)) all the physical parameters are determined by algebraic relations.
• Equation (2.38) does not contain coordinates $x$ and $y$.

The latter property was very widely used in the approach of the hodograph transformation, i.e., the transformation from a physical plane $(x, y)$ to a hodograph plane $(v, \theta)$, where $v_x = v \sin \theta$, $v_y = v \cos \theta$. In this case it is possible to introduce another potential $\phi_v(v, \theta)$ so that $\mathbf{r} = \nabla_v \phi_v$. As a result, equation (2.38) can be rewritten as

\[
\frac{\partial^2 \phi_v}{\partial \theta^2} + \frac{v^2}{1-v^2/c_s^2} \frac{\partial^2 \phi_v}{\partial v^2} + v \frac{\partial \phi_v}{\partial v} = 0.
\]

This is the linear Chaplygin equation (1902).

The hodograph transformation approach was the main direction of exploration through the XX-th century [12, 13]. Here I formulate two results obtained in this field, to be used in what follows.

• For the transonic flow it is impossible to solve the direct problem (i.e., to determine the flow structure from the given shape of the boundary, say, knowing the shape of nozzle or wing).

• On the other hand, one can solve the reverse problem. This approach is based on the fundamental theorem: the transonic flow is analytic at the critical point (the point where the sonic surface is orthogonal to the flow line, see Fig. 2) [11].

Let me comment these two statements. The most visible argument clarifying the absence of the regular (not iterative) procedure for the transonic flow is as follows. As is known, the number of boundary conditions $b$ for an arbitrary (not only for a pure hydrodynamical) flow can be determined from the following condition [14, 15]

\[
b = 2 + i - \sigma.
\]

Here $i$ is the number of invariants and $\sigma$ is the number of singular surfaces. In pure hydrodynamics the only singular surface is the sonic one. Hence, for the transonic flow $\sigma = 1$. Furthermore, for planar geometry we have two invariants, $E$ and $s$, so that $i = 2$. Thus, to determine the structure of the transonic flow it is necessary to specify three boundary conditions on a surface. These may be two thermodynamic functions and one component of the velocity. The second (the last in the planar case) component of the velocity is to be determined from the solution. But to solve equation (2.38), it is necessary to know the Bernoulli integral $E = v^2/2 + w$, i.e., both components of the velocity on this surface. Hence, in the general case even the equation describing the flow structure cannot be formulated. For subsonic and supersonic flows $\sigma = 0$ (so that $b = 4$) and this difficulty is absent.
Fig. 2. Structure of the "analytical nozzle" in the vicinity of the critical point $x = y = 0$ - the only point where the sonic surface is orthogonal to the flow line. As the term $xy$ in (2.44) is absent (i.e., $v_y(0, y) = 0$), the plane $x = 0$ corresponds to minimum cross-section of the stream surfaces. The sonic surface $v = c_*$ has the standard parabolic form $x = -\frac{k(\Gamma - 1)}{2c_*}y^2$.

On the other hand, the structure of the transonic flow can be found directly by expansion of the solution in the vicinity of the critical point (where we put $x = y = 0$). Indeed, in addition to the invariants $E$ (which determines the velocity of sound on the sonic surface) and $s$ (which is necessary for the determination of the concentration $n$) one can specify the $x$ component of the velocity $v_x(x, 0)$ along $x$-axis. For our purpose it is enough to know the first two terms in the expansion

$$v_x(x, 0) = c_* + kx + \ldots \quad (2.43)$$

Here $c_*^2 = 2E(\Gamma - 1)/(\Gamma + 1)$. As a result, as one can check directly, the first terms in the expansion of the potential $\phi(x, y)$ look like

$$\phi(x, y) = c_*x + \frac{1}{2}kx^2 + \frac{1}{2c_*}k^2(\Gamma + 1)xy^2 + \frac{1}{24c_*^3}k^3(\Gamma + 1)^2y^4 + \ldots \quad (2.44)$$

Knowing all the coefficients in the expansion (2.43), one can restore the potential $\phi$ with any precision.

Thus, in the general case equation (2.38) cannot be solved directly. Let me stress that this property is common; it takes place for the Grad-Shafranov equation as well. On the other hand, the planar approach has three extra difficulties:

- It is difficult to consider the case $E \neq \text{const}$, $s \neq \text{const}$.
- It is impossible to consider the nonpotential flow with $\nabla \times \mathbf{v} \neq 0$.
- It is impossible to include gravity (which is not planar).
3 Axisymmetric Stationary Flow – Nonrelativistic Case

3.1 Basic Equations

3.1.1 Stream Function

Now let us see how a similar procedure can be realized for the axisymmetric stationary flows. It means that we assume all the values to depend on two variables – \( r \) and \( \theta \). In this sense the flow remains two-dimensional. But now none of the three components of the velocity are equal to zero. Thus, axisymmetric stationary flows are more rich than planar ones.

In the axisymmetric stationary case one can introduce potential \( \Phi(r, \theta) \) which is connected with the poloidal velocity \( v_p \) as

\[
n v_p = \frac{\nabla \Phi \times e_\phi}{2\pi r \sin \theta}.
\]

(3.1)

Such a definition results in the following properties:

- The continuity equation is valid automatically: \( \nabla \cdot (n v) = 0 \).

- On can check that \( d\Phi = n v dS \), where \( S \) is the area. Thus, the potential \( \Phi(r, \theta) \) is the flux through the circuit \( r, \theta, 0 < \varphi < 2\pi \). In particular, the total flux through the sphere of radius \( r \) is \( \Phi_{\text{tot}} = \Phi(r, \pi) \).

- As \( v \cdot \nabla \Phi = 0 \), the relations \( \Phi(r, \theta) = \text{const} \) describe the flow surfaces.

3.1.2 Integrals of motion

As earlier, the components \( \beta = t \) and \( \beta = v_\parallel \) of the energy-momentum conservation law \( \nabla \alpha T^{\alpha\beta} = 0 \) give

\[
E = E(\Phi) = \frac{v^2}{2} + w + \varphi_g,
\]

(3.2)

\[
s = s(\Phi),
\]

(3.3)

But as we see, it is now much easier to describe the case when integrals are different for different flow surfaces.

New information appears from the \( \beta = \varphi \) component of the energy-momentum equation (or, which is the same, from the \( \varphi \) component of the Euler equation)

\[
\nabla_\varphi \left( \frac{v^2}{2} \right) - [v \times (\nabla \times v)]_\varphi + \frac{\nabla \varphi P}{m_v n} + \nabla_\varphi \varphi_g = 0.
\]

(3.4)

As we consider the axisymmetric case, all the gradients vanish. The last term \([v \times (\nabla \times v)]_\varphi \) can be rewritten in the form of the conservation law

\[
v \cdot \nabla (r \sin \theta v_\varphi) = 0.
\]

(3.5)
Hence, in the axisymmetric case the $z$-component of the angular momentum
\[ L(\Phi) = v_\varphi r \sin \theta \]  
(3.6)
is the third integral of motion.

3.1.3 Mathematical Interlude – The Covariant Description

As we are going to generalize our equations to General Relativity, it is convenient to rewrite these relations right now in the covariant form. For this reason, recall that the flat 3D metric $g_{ik}(ds^2 = g_{ik}dx^i dx^k)$ for spherical coordinates $x^1 = r, \ x^2 = \theta, \text{ and } x^3 = \varphi$ has a form
\[ g_{rr} = 1, \quad g_{\theta\theta} = r^2, \quad g_{\varphi\varphi} = r^2 \sin^2 \theta, \]  
(3.7)
all the other components being zero. Using now expression (2.14)
\[ T^k_i = P \delta^k_i + (nm_p) v^k v_i, \]  
(3.8)
one can obtain from the $\varphi$ component of the energy-momentum equation
\[ \nabla_k T^k_\varphi = \nabla_k (\delta^k_\varphi P) + \nabla_k (nm_p v^k v_\varphi) = \frac{1}{r} \frac{\partial P}{\partial \varphi} \]
\[ + \frac{\partial}{\partial x^k} (nm_p v^k v_\varphi) + \Gamma^k_{ik} (nm_p) v^i v_\varphi - \Gamma^k_{i\varphi} (nm_p) v^i v_k = 0. \]  
(3.9)
Here $\Gamma^i_{jk}$ are the Christoffel coefficients.

As one can check, the last term in (3.9) vanishes: $\Gamma^k_{i\varphi} (nm_p) v^i v_k = 0$. The first term vanishes because of axisymmetry of the problem (no $\varphi$ - dependence). Using now the continuity equation
\[ \nabla_k (nm_p v^k) = \frac{1}{\sqrt{g}} \frac{\partial}{\partial x^k} (\sqrt{g} nm_p v^k) = \frac{\partial}{\partial x^k} (nm_p v^k) + \Gamma^k_{ik} nm_p v^i = 0, \]  
(3.10)
where $g = \det g_{ik} = g_{rr} g_{\theta\theta} g_{\varphi\varphi}$, we see that (3.3) can again be written down in the form of the conservation law $\nabla_k T^k_\varphi = (nm_p) v \cdot \nabla v_\varphi$. Hence, the third invariant looks like
\[ L(\Phi) = v_\varphi. \]  
(3.11)

exercises

1. Check Eqns. (3.9)–(3.11).

2. Is there a contradiction between (3.6) and (3.11)?
To understand the difference between (3.6) and (3.11), it is necessary to return to the main definitions of the covariant approach. Up to now we have dealt with the physical components only. Below in the relativistic expressions in Sect. 4 we will mark the physical components by hats, so that \( \hat{v} = v \) is the physical component of the toroidal velocity, and its dimension is \( \text{cm/s} \). But in the covariant expressions (3.9)–(3.11) we encounter other objects – contravariant components \( v^i \) and covariant components \( v_k \). As the definition of the vector length (which is square of the physical component) has the form
\[
(v_{\hat{\varphi}})^2 = g_{\varphi\varphi} (v_{\varphi})^2 = g_{k\varphi} v_k v_{\varphi},
\]
and the same for the other components. Thus,
\[
\begin{align*}
 v_{\varphi} &= \frac{1}{\sqrt{g_{\varphi\varphi}}} v_{\varphi}, \\
 v_{\varphi} &= \sqrt{g_{\varphi\varphi}} v_{\varphi}.
\end{align*}
\]
In particular, it means that the dimension of the covariant and contravariant components may differ from the dimension of the physical one. Comparing now (3.14) with (3.6) and (3.11), we can understand the difference: (3.6) actually involves the physical component of the toroidal velocity while (3.11) includes the covariant one.

### 3.2 The Stream Equation

#### 3.2.1 Grad-Shafranov Equation

To formulate the stream equation (i.e., the equation describing the stream function \( \Phi(r, \theta) \)) it is necessary to return to the poloidal component of the Euler equation. One can check that together with the definitions of the invariants \( E(\Phi) \), \( L(\Phi) \), and \( s(\Phi) \) this vector equation can be written as the product of the scalar equation by the vector \( \nabla \Phi \cdot \) [Euler] \( = [\text{GS}] \cdot \nabla \Phi \).

For this reason, in many MHD papers the stream equation \( [\text{GS}] = 0 \) was obtained as the projection of the poloidal equation onto the gradient \( \nabla \Phi \). The equivalent hydrodynamic expression has the form
\[
\frac{1}{(\nabla \Phi)^2} \nabla \Phi \cdot \left[ (v \nabla)v + \frac{\nabla P}{n m_p} + \nabla \varphi \right] = 0. \tag{3.15}
\]

Using now the definitions (3.3), (3.2), and (3.4), we have
\[
\begin{align*}
- \omega^2 \nabla_k \left( \frac{1}{\omega^2} \nabla^k \Phi \right) + \frac{1}{n} \nabla_k n \cdot \nabla^k \Phi \\
- 4\pi^2 \frac{dL}{d\Phi} + 4\pi^2 \omega^2 n^2 \frac{dE}{d\Phi} - 4\pi^2 \omega^2 n^2 \frac{T}{m_p} \frac{ds}{d\Phi} = 0. \tag{3.16}
\end{align*}
\]
Here and to the very end
\[ \varpi = \sqrt{g_{\varphi \varphi}}, \quad (3.17) \]
so that for the flat metric \( \varpi = r \sin \theta \).

To close the system, i.e., to determine the product \( (\nabla n \cdot \nabla \Phi) \), the stream equation \( (3.16) \) is to be supplemented with the Bernoulli equation \( (3.2) \). It can now be rewritten in the form (cf. \( (2.18) \))
\[ E = \left( \frac{(\nabla \Phi)^2}{8\pi^2 \varpi^2 n^2} + \frac{1}{2} \frac{L^2}{\varpi^2} + w(n, s) + \varphi_\Phi \right). \quad (3.18) \]
As previously, the Bernoulli equation \( (3.18) \) contains, besides \( n \), the invariants \( E, L \), and \( s \), and the stream function \( \Phi \) only. Hence, it again gives the implicit expression for \( n \):
\[ n = n(\nabla \Phi; E, L, s; r, \theta). \quad (3.19) \]
On the other hand, the implicit Bernoulli equation can be presented in the differential form
\[ \nabla_k n = \frac{N_k}{D}, \quad (3.20) \]
where now
\[ D = -1 + \frac{c^2}{u^2}, \quad (3.21) \]
and
\[ N_k = -\frac{\nabla^i \Phi \cdot \nabla_i \nabla_k \Phi}{(\nabla \Phi)^2} + \frac{1}{2} \frac{\nabla_k \varpi^2}{\varpi^2} - 4\pi^2 \varpi^2 n^2 \frac{\nabla_k \varphi_\Phi}{(\nabla \Phi)^2} \]
\[ -4\pi^2 n^2 L \frac{dL}{d\Phi} \frac{\nabla_k \Phi}{(\nabla \Phi)^2} + 2\pi^2 n^2 L^2 \frac{\nabla_k \varpi^2}{\varpi^2(\nabla \Phi)^2} \]
\[ + 4\pi^2 \varpi^2 n^2 \frac{dE}{d\Phi} \frac{\nabla_k \Phi}{(\nabla \Phi)^2} - 4\pi^2 \varpi^2 n^2 \left[ \frac{T}{m_p} + \frac{1}{m_p n} \frac{\partial P}{\partial s} \right] \frac{d}{d\Phi} \frac{\nabla_k \Phi}{(\nabla \Phi)^2}. \quad (3.22) \]
As a result, the stream equation can be written down as
\[ \begin{aligned}
-\varpi^2 \nabla_k \left( \frac{1}{\varpi^2} \nabla^k \Phi \right) &+ \nabla^i \Phi \cdot \nabla_i \nabla_k \Phi \left( \frac{1}{(\nabla \Phi)^2} \right) D + \frac{\nabla \varpi^2 \cdot \nabla \Phi}{2D \varpi^2} \\
-4\pi^2 \varpi^2 n^2 \nabla \varphi_\Phi \cdot \nabla \Phi \left( \frac{1}{D(\nabla \Phi)^2} \right) &- 4\pi^2 n^2 \frac{D + 1}{D} \frac{L}{d\Phi} \frac{dL}{d\Phi} \\
+ 2\pi^2 n^2 \frac{\nabla(\varpi^2)}{D \varpi^2(\nabla \Phi)^2} L^2 &+ 4\pi^2 \varpi^2 n^2 \frac{D + 1}{D} \frac{dE}{d\Phi} \\
-4\pi^2 \varpi^2 n^2 \left[ \frac{D + 1}{D} \frac{T}{m_p} + \frac{1}{D m_p n} \frac{\partial P}{\partial s} \right] \frac{d}{d\Phi} &= 0, \quad (3.23)
\end{aligned} \]
or, in a compact form (cf. [10]), as

\[
-\omega^2 \nabla_k \left( \frac{1}{\omega^2 n} \nabla^k \Phi \right) - 4\pi^2 nL \frac{dL}{d\Phi} + 4\pi^2 \omega^2 n \frac{dE}{d\Phi} - 4\pi^2 \omega^2 n \frac{T}{m_p} \frac{ds}{d\Phi} = 0.
\]

At first glance, the stream equation (3.23) is much more complicated than the planar one (2.38). Nevertheless, one can easily see that these equations have many similarities. As (2.38), the stream equation (3.23) starts with the linear elliptic term and the nonlinear term with an analogous form. The third term does not, of course, exist in (2.38) – it results from non-cartesian geometry. But all the other terms are not complication. They allow us to include into consideration not only gravity, but a much wider class of flows with different invariants onto different flow surfaces.

In other respects the stream equation is quite similar to the planar equation (2.38):

- The stream equation (3.23) is to be supplemented with the Bernoulli equation.
- Together with the Bernoulli equation, equation (3.23) contains the potential \( \Phi(r, \theta) \) and the invariants \( E, L, \) and \( s \) only (i.e., it has the Grad-Shafranov form).
- For \( n = \text{const} \), \( c_s^2 \to \infty \), \( E = \text{const} \), \( s = \text{const} \), and \( L = 0 \) the equation becomes linear.
- It is nonlinear in the general case, but linear for second derivatives.
- The equation is elliptical for a subsonic flow \( D > 0 \).
- The equation is hyperbolic for a supersonic flow \( D < 0 \).
- For a given flow structure (i.e., for a given \( \Phi \)) and for the invariants \( E(\Phi), L(\Phi), \) and \( s(\Phi) \) all the physical parameters are determined by algebraic relations.

The following point should be stressed. The expression for the denominator \( D = -1 + c_s^2/v_p^2 \) (3.21) involves the poloidal rather than the total velocity. It means that the sonic surface exists when the poloidal, not the total velocity becomes equal to that of sound. This fact results from our basic assumption: as we consider axisymmetric flows only, the disturbances (waves) are to have the same symmetry as well. Hence, the disturbances can propagate in the poloidal direction only. For this reason, a singularity appears at the moment when the particle velocity coincides with the velocity of disturbance.
3.2.2 Linear Operator $\varpi^2 \nabla_k (\varpi^{-2} \nabla^k)$

In what follows we shall use the linear operator

$$\hat{\mathcal{L}} = \varpi^2 \nabla_k \left( \frac{1}{\varpi^2} \nabla^k \right) = \frac{\partial^2}{\partial r^2} + \frac{\sin \theta}{r^2} \frac{\partial}{\partial \theta} \left( \frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \right) \quad (3.25)$$

and shall therefore consider it in more detail. First, examine the angular operator

$$\hat{\mathcal{L}}_\theta = \sin \theta \frac{\partial}{\partial \theta} \left( \frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \right). \quad (3.26)$$

It has the following eigenfunctions

$$Q_0 = 1 - \cos \theta, \quad (3.27)$$
$$Q_1 = \sin^2 \theta, \quad (3.28)$$
$$Q_2 = \sin^2 \theta \cos \theta, \quad (3.29)$$
$$\ldots$$
$$Q_m = \frac{2^m m! (m - 1)!}{(2m)!} \sin^2 \theta P'_m (\cos \theta), \quad (3.30)$$

and eigennumber values

$$q_m = -m(m + 1). \quad (3.31)$$

Here $P_m$ are Legendre polynomials and prime means their derivative. As a result, for the full operator $\hat{\mathcal{L}}$ we have the following set of eigenfunctions

1. $m = 1$
   - $\Phi_1^{(1)} = r^2 \sin^2 \theta$ – homogeneous flow,
   - $\Phi_1^{(2)} = \sin^2 \theta/r$ – dipole flow.

2. $m = 2$
   - $\Phi_2^{(1)} = r^3 \sin^2 \theta \cos \theta$ – zero point,
   - $\Phi_2^{(2)} = \sin^2 \theta \cos \theta/r^2$ – quadrupole flow.

3. \ldots

At first glance, this point is absolutely clear so that it is impossible to encounter here any trouble. Nevertheless, it is not so. Indeed, let us consider the eigenfunctions corresponding to $m = 0$. The first one is clear: it is the function

$$\Phi_0^{(1)} = 1 - \cos \theta \quad (3.32)$$

which describes the spherically symmetric accretion/ejection. By the way, this harmonics alone determines the accretion/ejection rate as for all the
other eigenfunctions with \( m > 0 \) we have \( \Phi_m(r, \pi) = 0 \). The uncertainty is due to the second eigenfunction

\[
\Phi^{(2)}_0 = r(1 - \cos \theta)
\]

for which \( \Phi_0(r, \pi) \neq \text{const} \). It means that this harmonics can be realized only if in the volume (not in the origin or at infinity) there are the sources or the sinks of matter. In all other cases the second eigenfunction for \( m = 0 \) is to be dismissed.

### 3.3 Examples

#### 3.3.1 Bondi-Hoyle Accretion

As a first example we consider the accretion onto a moving gravity center (Bondi-Hoyle accretion \([1]\)). This is a classical problem of modern astrophysics. The nature of Active Galactic Nuclei and Quasars, the nature of jets, the activity of some galactic X-ray sources are believed to be associated with the accretion of a gas onto compact objects – neutron stars and black holes \([4, 17, 18, 19]\). Nevertheless, only a few exact solutions describing the accretion flow (even for the simplest adiabatic case) are now known, e.g., the solution for the spherically symmetric flow we have already discussed.

To construct the nonspherical solution, one may assume that a small perturbation of a spherically symmetric flow cannot change strongly the structure of the accretion \([20]\). So, it is possible to seek the solution of the stream equation as a perturbation of the spherically symmetric solution. First of all, let me recall the main results of the qualitative theory. It is more convenient to perform the calculations in the reference frame moving with the gravity center with a velocity \( v_\infty \). Comparing the Bondi accretion rate

\[
2\Phi = 4\pi r^2 n_\infty c_\infty \sim \left(\frac{GM}{r}\right)^2 n_\infty / c_\infty^3
\]

with the flow \( \Phi \sim \pi R_c^2 n_\infty v_\infty \) captured within the capture radius \( R_c \), one can evaluate the \( R_c \) value as

\[
R_c \sim \varepsilon_1^{-1/2} r_*,
\]

where

\[
\varepsilon_1 = \frac{v_\infty}{c_\infty}.
\]

Hence, for \( \varepsilon_1 << 1 \) the capture radius is much larger than the sonic one, and we can assume that for \( r \ll R_c \) the flow structure is similar to the spherically symmetric accretion. Thus, one can seek the solution of the stream equation \((3.23)\) in the form

\[
\Phi(r, \theta) = \Phi_0[1 - \cos \theta + \varepsilon_1 f(r, \theta)].
\]

For a nonmoving gravity center we return to the spherically symmetric flow. As the stream equation \((3.23)\) now contains \( i = 3 \) invariants, so that \( b = 2 + 3 - 1 = 4 \), it is necessary to specify four boundary conditions, say...
1. \( v_\infty = \text{const}, \)
2. \( v_\varphi = 0 \) (and hence \( L = 0 \)),
3. \( s_\infty = \text{const}, \)
4. \( E_\infty = c_r^2/(\Gamma + 1). \)

In the last relation we neglect the terms \( \sim \varepsilon_1^2. \)

As a result, the stream equation can be linearized:

\[
-\varepsilon_1 D \frac{\partial^2 f}{\partial r^2} - \frac{\varepsilon_1}{r^2} (D+1) \sin \theta \frac{\partial}{\partial \theta} \left( \frac{1}{\sin \theta} \frac{\partial f}{\partial \theta} \right) + \varepsilon_1 \left( \frac{2}{r} - \frac{GM}{c_r^2 s_r^2} \right) \frac{\partial f}{\partial r} = 0. \tag{3.37}
\]

This equation has the following properties.

- It is linear.
- The angular operator coincides with \( \hat{L}_\theta \) (3.26).
- As all the terms contain a small \( \varepsilon_1 \) value, the functions \( D, c_s, \text{etc.} \) can be taken from the zero solution.
- As for the spherically symmetric flow the functions \( D, c_s, \text{etc.} \) do not depend on \( \theta \), the solution of equation (3.37) can be expanded in eigenfunctions of the operator \( \hat{L}_\theta \).

Thus, the solution can be presented in the form

\[
f(r, \theta) = \sum_{m=0}^{\infty} g_m(r) Q_m(\theta), \quad \tag{3.38}
\]

the equations for the radial functions \( g_m(r) \) being

\[
r^2 D \frac{d^2 g_m}{dr^2} + \left( \frac{2r - GM}{c_s^2} \right) \frac{dg_m}{dr} + m(m+1)(D+1) g_m = 0. \tag{3.39}
\]

As to the boundary conditions, they can be formulated as follows:

1. No singularity on the sonic surface (where \( r^2 N_r = 2r - GM/c_s^2 = 0, \ D = 0 \))
   \[
g_m(r_*) = 0. \quad \tag{3.40}
\]
2. A homogeneous flow \( \Phi = \pi n_\infty v_\infty r^2 \sin^2 \theta \) at infinity, which gives
   \[
g_1 \rightarrow \frac{1}{2} n_\infty c_s r^2 \frac{r^2}{r_*^2}, \quad g_2, g_3, \ldots = 0. \quad \tag{3.41}
\]
As a result, the complete solution can be presented in the form

$$\Phi(r, \theta) = \Phi_0[1 - \cos \theta + \varepsilon_1 g_1(r) \sin^2 \theta],$$

(3.42)

where the radial function $g_1(r)$ is the solution of the ordinary differential equation (3.39) for $m = 1$ with the boundary conditions (3.40) and (3.41).

At the present level of personal computers it means that we succeed in constructing the analytical solution of the problem in question. It allows us to obtain all the information concerning the flow structure. In particular, the sonic surface now has non-spherical form:

$$r_s(\theta) = r_0 \left( 1 + 2\varepsilon_1 \Gamma + \frac{1}{D} k_2 \cos \theta \right),$$

(3.43)

where the numerical coefficient $k_2 = r_s g'_1(r_0)$ can be obtained from the solution (for more details see [20]). As one can see from Fig. 3, the analytical solution is in full agreement with the numerical calculations [21] although the parameter $\varepsilon_1 = 0.6$ is not too small. Finally, the condition

$$g_0 = 0$$

(3.44)

(to obtain which an additional consideration is necessary) shows that the accretion rate is not changed in this approximation.

Fig. 3. Flow structure and shape of the sonic surface for $\Gamma = 4/3$, $\varepsilon_1 = 0.6$ [20]. Labels on the curves denote the values of $\Phi/\Phi_0$, and dashed curves indicate streamlines and the sonic surface obtained numerically in [21].

In connection with this solution, it is necessary to clarify one point. As one can easily see, outside the capture radius, our main assumption – smallness of the disturbance of the spherically symmetric flow – is not valid.
Nevertheless, the constructed solution is correct. This beautiful property is connected with the already mentioned fact that for an (approximately) constant concentration $n$ the stream equation becomes linear. But as we know from the Bondi accretion (see (2.26)), far from the sonic surface $r \gg r_\ast$ the flow density is actually constant. The same is true for the homogeneous flow. As a result, for $R_c \gg r_\ast$, i.e., for the case $\varepsilon_1 \ll 1$ under consideration, in the vicinity of and outside the capture radius (where the "disturbance" $\sim \varepsilon_1 y_1(r)$ becomes of the same order as the zero approximation $\sim 1$), the stream equation is linear. As a result, the sum of two solutions, homogeneous and spherically symmetric, remains a solution as well.

3.3.2 Ejection from a Slowly Rotating Star

Another interesting nonrelativistic example is the transonic ejection from a slowly rotating star [22,23]. It is necessary to stress from the very beginning that this example is only illustrative because in reality an important role is played by the radiation pressure which cannot be included into consideration within our approach. Nevertheless, the analysis of this problem helped us to clarify some important features of the Grad-Shafranov approach [24].

As a zero approximation we consider the well-known Parker transonic outflow (2.30)–(2.32). It means that we assume all the parameters of the spherically symmetric outflow (the sonic radius $r_\ast$, the velocity of sound on the sonic surface $c_\ast$, the radial velocity $v_R$ on the star surface $r = R$, etc.) to be known. As before, we will seek the solution in the form

$$\Phi(r, \theta) = \Phi_0[1 - \cos \theta + \varepsilon_2^2 f(r, \theta)], \quad (3.45)$$

where the small parameter is now

$$\varepsilon_2^2 = \frac{\Omega^2 R^3}{GM}. \quad (3.46)$$

Here $\Omega$ is the angular velocity of a star.

The problem under consideration needs all the $i = 3$ invariants. Hence, $b = 2 + 3 - 1 = 4$, and it is necessary to specify four boundary conditions on the star surface $r = r_R(\theta)$ which now differs from the sphere

$$r_R(\theta) = R[1 + \varepsilon_2^2 \rho(\theta)]. \quad (3.47)$$

Here we introduce the dimensionless parameter $\rho(\theta) \approx 1$.

Let me stress that at first glance there is a disagreement as we add one degree of freedom (the toroidal rotation with $v_\phi \neq 0$) while it needs two extra functions in comparison with the spherically symmetric outflow. This question will be clarify below.

It is important that for small $\varepsilon_2$ four boundary conditions can be determined through real physical parameters on the star surface, e.g., through
two thermodynamic functions (say, $T$ and $n$) and two components of the velocity (say, $v_r$ and $v_\phi$). As a result, one can express these boundary conditions through four dimensionless functions $\tau(\theta)$, $\eta(\theta)$, $\omega(\theta)$, and $h(\theta)$:

\[
T(r_R, \theta) = T_R[1 + \varepsilon_2^2 \tau(\theta)],
\]

\[
n(r_R, \theta) = n_R[1 + \varepsilon_2^2 \eta(\theta)],
\]

\[
v_\phi(r_R, \theta) = \varepsilon_2 \left(\frac{GM}{R}\right)^{1/2} \omega(\theta) \sin \theta,
\]

\[
v_r(r_R, \theta) = v_R[1 + \varepsilon_2^2 h(\theta)].
\]

Here $\omega(\theta)$, determined as

\[
\Omega(r_R, \theta) = \Omega \omega(\theta),
\]

describes the differential rotation of the star surface.

Using now the thermodynamic relation

\[
ds = \frac{1}{\Gamma - 1} \frac{dT}{T} - \frac{dn}{n},
\]

one can obtain for the first invariant $s(\theta)$

\[
\delta s(\theta) = \varepsilon_2^2 \left[\frac{1}{\Gamma - 1} \tau(\theta) - \eta(\theta)\right].
\]

Accordingly, two other invariants can be determined through the boundary conditions as well

\[
\delta E(\theta) = \varepsilon_2^2 \omega^2 h(\theta) + \varepsilon_2^2 \frac{GM}{2R} \omega^2(\theta) \sin^2 \theta + \frac{\varepsilon_2^2 \Gamma}{\Gamma - 1} \tau(\theta) + \delta \varphi_g(3.55)
\]

\[
L^2(\theta) = \varepsilon_2^2 R^2 \frac{GM}{R} \omega^2(\theta) \sin^2 \theta.
\]

Here we use the following expression for the disturbance of the gravitational potential on the star surface

\[
\delta \varphi_g(r_R, \theta) = \varepsilon_2^2 \frac{GM}{R} \rho(\theta).
\]

It is of great importance that the possibility of taking the next step, i.e., writing down the Grad-Shafranov equation, is connected with the simple geometry of the zero approximation. Since in the zero approximation the stream function is $\Phi = 1 - \cos \theta$, i.e., it is the function of $\theta$ only (and as all the derivatives $dE/d\Phi$, $dL/d\Phi$, and $ds/d\Phi$ as well as $L$ itself vanish for a nonrotating outflow), one can use the relation $d\Phi = \Phi_0 \sin \theta d\theta$. It allows us to determine the derivatives $dE/d\Phi$, $dL/d\Phi$, and $ds/d\Phi$ in the whole space and not only on the star surface.
As a result, the stream equation can be written as

\[-\varepsilon^2 \Phi_0 D \frac{\partial^2 f}{\partial r^2} - \varepsilon^2 \frac{\Phi_0}{r^2} \Phi_0 (D + 1) \sin \theta \frac{\partial}{\partial \theta} \left( \frac{1}{\sin \theta} \frac{\partial f}{\partial \theta} \right) + \varepsilon^2 \Phi_0 N_r \frac{\partial f}{\partial r} =
\]

\[-4\pi^2 n^2 r^2 \Phi_0 \sin^2 \theta (D + 1) \frac{dE}{d\Phi} + 4\pi^2 n^2 (D + 1) L \frac{dL}{d\Phi} \]

\[-4\pi^2 n^2 \cos \theta \frac{\Phi_0}{\sin^2 \theta} L^2 + 4\pi^2 n^2 r^2 \sin^2 \theta \left[ (D + 1) \frac{T}{m_p} + \frac{\Gamma - 1}{\Gamma} c_s^2 \right] ds + \frac{\Gamma}{\Gamma} \nu \frac{\Phi_0}{r^2} c_s^2 \frac{d\nu}{d\Phi}.\]

Here $N_r = 2/r - 4\pi^2 n^2 r^2 GM/\Phi_0^2$.

The properties of this equation are the same as before.

- It is linear.
- The angular operator coincides with $\hat{L}_\theta$ (3.26).
- As all the terms contain a small $\varepsilon^2$ value, the functions $D, c_s, n, etc.$ can be taken from the zero approximation.
- As for the spherically symmetric flow the functions $D, c_s, n, etc.$ do not depend on $\theta$, the solution of equation (3.58) can be expanded in eigenfunctions of the operator $\hat{L}_\theta$.

Hence, we can again seek the solution in the form

\[f(r, \theta) = \sum_{m=0}^{\infty} g_m(r) Q_m(\theta).\] (3.59)

Introducing the dimensionless variables

\[x = \frac{r}{r_*}, \quad u = \frac{n}{n_*}, \quad a = \frac{c_s^2}{c_s},\] (3.60)

one can write down the following ordinary differential equations describing the radial functions $g_m(r)$

\[\left( 1 - x^4 a u^2 \right) \frac{d^2 g_m}{dx^2} + 2 \left( \frac{1}{x} - x^2 u^2 \right) \frac{dg_m}{dx} + m(m + 1)x^2 a u^2 g_m = \]

\[\kappa_m \frac{R^2}{r_*^2} x^4 a u^4 - \lambda_m \frac{R^2}{r_*^2} u^2 - \sigma_m x^6 a u^4 \]

\[+ \frac{1}{\Gamma} \nu_m x^6 a^2 u^4 + \frac{\Gamma - 1}{\Gamma} \nu_m x^2 a u^2,\] (3.61)
where \( \kappa_m, \lambda_m, \sigma_m, \) and \( \nu_m \) values are defined as the expansion coefficients

\[
\sin \theta \frac{dE}{d\theta} = \varepsilon^2 c_s^2 \sum_{m=0}^{\infty} \sigma_m Q_m(\theta), \quad (3.62)
\]

\[
\frac{\cos \theta}{\sin^2 \theta} L^2 = \varepsilon^2 c_s^2 r_s^2 \sum_{m=0}^{\infty} \lambda_m Q_m(\theta), \quad (3.63)
\]

\[
\frac{L}{\sin \theta} \frac{dL}{d\theta} = \varepsilon^2 c_s^2 r_s^2 \sum_{m=0}^{\infty} \kappa_m Q_m(\theta), \quad (3.64)
\]

\[
\sin \theta \frac{ds}{d\theta} = \varepsilon^2 \sum_{m=0}^{\infty} \nu_m Q_m(\theta). \quad (3.65)
\]

Finally, the functions \( a(x) \) and \( u(x) \) corresponding to the spherically symmetric flow for the polytropic equation of state \( (2.6) \) are related as

\[
a = u^{\Gamma - 1} \quad \text{and} \quad u(x) \quad \text{due to} \quad (3.20) - (3.22) \quad \text{can be found from the ordinary differential equation}
\]

\[
\frac{du}{dx} = -\frac{2}{x} \frac{u}{1 - x^3 u^2} \quad (3.66)
\]

with the boundary conditions (cf. \( (2.22) \))

\[
u(1) = 1, \quad \left( \frac{du}{dx} \right)_{x=1} = -\frac{4 + \sqrt{10 - 6\Gamma}}{\Gamma + 1}. \quad (3.67)
\]

As to the boundary condition to the set of equations \( (3.61) \), they are quite similar to the Bondi-Hoyle accretion (for more details see \( [24] \)):

- **Condition on the star surface.** As

\[
\frac{d\Phi}{2 \pi r^2 n_r \sin \theta d\theta} = 2 \pi R^2 n_R v_R [1 + \varepsilon^2 (\eta + h + 2\rho)] \sin \theta d\theta, \quad (3.68)
\]

we have

\[
g_m(R/r_s) = \frac{(2m)!}{2^m(m + 1)!m!} (\eta_m + h_m + 2\rho_m). \quad (3.69)
\]

Here \( \eta_m, h_m, \) and \( \rho_m \) are expansion coefficients in Legendre polynomials, e.g., \( \eta(\theta) = \sum_m \eta_m P_m(\cos \theta) \).

- **The absence of singularity on the sonic surface \( N_\theta = 0 \).** This condition gives

\[
\varepsilon^2 g_m(1) = \frac{(2m)!}{2^m(m + 1)!m!} \left[ \frac{(\delta E)_m}{c_s^2} - \frac{(\delta s)_m}{c_s^2} - \frac{(\delta L/\sin^2 \theta)_m}{2c_s^2 r_s^2} \right], \quad (3.70)
\]

where again \( (\ldots)_m \) means the expansion in Legendre polynomials which can be found from expressions \( (3.54) - (3.56) \).
As a result, equations (3.61) taken together with the boundary conditions (3.69) and (3.70) allow solution of the direct problem, i.e., determination of the flow structure from the physical boundary condition on the star surface.

Here it is necessary to stress two important points:

1. The reason we succeeded in writing the regularity condition $N_{\theta} = 0$ on the sonic surface (and hence to solve the direct problem) is again a very simple geometry of the zero approximation. In particular, in the problem in question the position of the sonic surface can be taken from the spherically symmetric solution. In the general case it is not so and it is impossible to write down the condition $N_{\theta} = 0$ using the known functions $(\delta E)_m$, $(\delta s)_m$, etc. on the star surface. The position of the sonic surface is unknown and is to be found from the solution.

2. We can now understand the nature of the "extra" boundary condition. The point is that for $m = 0$ one can take, as was already stressed, one fundamental solution only, namely, $g_0 = \text{const}$. Another fundamental solution is unphysical. Hence, there is an additional relation

$$g_0(R) = g_0(r_\ast). \quad (3.71)$$

This relation determines $h_0$ which, as we see, is not a free parameter. In other words, we are not fully free in the boundary condition $h(\theta)$: its zero harmonics is to be found from relation (3.71). But as was demonstrated earlier, it is $g_0$ that determines the ejection rate. Hence, the ejection rate is a function of three parameters only, namely, zero harmonics of two thermodynamic functions $\eta_0$ and $\tau_0$, and $v_\phi$. For a spherically symmetric flow $v_\phi = 0$, and we return to two functions which determine the ejection rate. As to higher harmonics with $m > 0$, they are free, and to determine them it is necessary to know four functions on the star surface. Thus, the spherically symmetric case is degenerate and it is necessary to be very careful when extending its properties on the 2D flows.

At the end of this section let me formulate some results which can be obtained under the following simplified assumptions:

- Almost the whole of the star mass is in its center, i.e., $\varphi_\ast = -GM/r$.
- No differential rotation, i.e., $\omega(\theta) = 1$.
- Von Zeipel law: $T(R, \theta) \propto g_{\text{eff}}^{1/4}$, where $\varphi_{\text{eff}} = \varphi_\ast + L^2/r^2$.
- No meridional convection, i.e., $v_\theta(r_R, \theta) = 0$ (it means that we specify here $v_\theta(r_R, \theta)$ instead of $v_r(r_R, \theta)$, i.e., the coefficients $h_0$, $h_1$, etc. are to be found from the solution).
exercises

1. Find that the disturbances of the star radius $\rho(\theta)$ in (3.47) and the temperature $\tau(\theta)$ in (3.48) have the form

$$\rho(\theta) = \frac{1}{2} \sin^2 \theta, \quad \tau(\theta) = -\frac{1}{2} \sin^2 \theta.$$  \hfill (3.72)

2. Show that the only nonzero terms in the expansion (3.59) correspond to $m = 0$ and $m = 2$, the expansion coefficients in (3.62)–(3.65) being

$$\sigma_2 = 2 \frac{r_*}{R} \frac{5 - 3\Gamma}{2(\Gamma - 1)} + \frac{1}{2} \frac{v_R^2}{c_*^2} - 3 \frac{\nu_R^2}{c_*^2} h_2, \quad (3.73)$$

$$\lambda_2 = 2 \frac{R}{r_*}, \quad \kappa_2 = 4 \frac{R}{r_*}, \quad \nu_2 = -\frac{\Gamma}{\Gamma - 1}, \quad (3.74)$$

and $\sigma_0, ..., \nu_0 = 0$. Remember that $h(\theta) = h_0 + h_1 \cos \theta + h_2 P_2(\cos \theta) + ....$

As a result, solving the stream equation (3.61) for $m = 2$, one can find that

1. The ejection rate can be presented as

$$\Phi_{\text{tot}} = 2\Phi_0 \left[ 1 + \frac{\Omega^2 R^3}{GM} (1 + h_0) \right]. \quad (3.75)$$

Here $h_0$ can be obtained from relation (3.73) (see Table 1)

$$h_0 = -\frac{1}{6} + \frac{2}{3} \frac{r_*}{R - R/r_*}.$$  \hfill (3.76)

As we see, the rotation increases the ejection rate.

| Table 1 |
|---|---|---|---|---|---|
| model | $1 + h_0$ | $h_2$ | $q_2$ | $b_0$ | $b_2$ |
| $r_*/R = 1.1, \Gamma = 4/3$ | 2.9 | −0.8 | −0.40 | 2.2 | −0.41 |
| $r_*/R = 2.0, \Gamma = 4/3$ | 3.2 | −3.3 | −0.71 | 2.5 | −0.47 |
| $r_*/R = 10, \Gamma = 4/3$ | 8.0 | −56.0 | −2.18 | 5.7 | −0.92 |
| $r_*/R = 1.1, \Gamma = 1.1$ | 1.7 | −0.8 | −0.15 | 1.8 | −0.37 |
| $r_*/R = 2.0, \Gamma = 1.1$ | 2.1 | −2.3 | −0.18 | 2.2 | −0.40 |
| $r_*/R = 10, \Gamma = 1.1$ | 7.4 | −26.0 | −0.40 | 7.2 | −0.58 |
2. Far from the sonic surface $r \gg r_*$ the stream function has the form

$$
\lim_{r \to \infty} \frac{\Phi(r, \theta)}{\Phi_0} = (1 - \cos \theta) + \frac{\Omega^2 R^3}{GM} (1 + h_0)(1 - \cos \theta)
+ \frac{\Omega^2 R^3}{GM} q_2 \sin^2 \theta \cos \theta,
$$

(3.77)

where the coefficient $q_2$ is tabulated in Table 1 as well. As there is no $r$-dependence, the outflow becomes purely radial at large distances.

3. Accordingly, the asymptotic expression for the concentration $n$ has the form

$$
\lim_{r \to \infty} \frac{n(r, \theta)}{n_*} = \frac{c_*}{v_\infty} \frac{r^2}{r^2_*} \left[ 1 + \frac{\Omega^2 R^3}{GM} b_0 + \frac{1}{2} \frac{\Omega^2 R^3}{GM} b_2 (3 \cos^2 \theta - 1) \right],
$$

(3.78)

where $v_\infty^2 = v_\infty^2 - 2GM/R$. As one can see from Table 1, $b_2 < 0$. It means that the rotation results in the appearance of a dense disk in the equatorial plane; this result is well-known [23], but previously it was obtained by numerical calculations only. Negative $q_2$ values in (3.77) demonstrate that for $\varepsilon_2 \geq 1$ the most mass outflow is concentrated in the vicinity of the equatorial plane as well.

4  
Axisymmetric Stationary Flow – General Relativity

4.1 Basic Equations

We shall show how the Grad-Shafranov approach can be applied to the axisymmetric stationary flows in the vicinity of a rotating black hole. Remember that the main difficulty of the General Relativity is the necessity to work with four-dimensional objects. As a result, we cannot use our three-dimensional intuition in considering the relativistic processes.

But there is a convenient language – 3 + 1-split – which allows work with three-dimensional vectors even in General Relativity [25]. One can find the detailed introduction into this approach in the book "Black Holes. The Membrane Paradigm" by K. Thorne, R. Price, and D. Macdonald [27]. The main idea is that for stationary metric the proper time $\tau$ and "the time at infinity" $t$ are in one-to-one correspondence. It allows the time $t$ to be separated from spatial coordinates $x^i$ ($i = 1, 2, 3$). As a result, all the equations can be written in the simple 3D form, their physical meaning remaining clear. Below I shall give the main relations of this approach (see [24] as well).
4.1.1 Kerr Metric

The Kerr metric is the metric of a rotating black hole. In the Boyer-Lindquist coordinates $t$, $r$, $\theta$, and $\varphi$ it has the form

$$ds^2 = -\alpha^2 dt^2 + g_{ik}(dx^i + \beta^i dt)(dx^k + \beta^k dt),$$

(4.1)

where

$$\alpha = \frac{\rho}{\Sigma} \sqrt{\Delta}$$

(4.2)

($\alpha$ is the lapse function or the red shift), and

$$\beta^r = \beta^\theta = 0, \quad \beta^\varphi = -\omega = -\frac{2aMr}{\Sigma^2}$$

(4.3)

($\omega$ is the Lense–Thirring angular velocity; remember that $\beta^\varphi$ is the contravariant component). Finally, it is important that the 3D metric $g_{ik}$ in (4.1) be diagonal

$$g_{rr} = \frac{\rho^2}{\Delta}, \quad g_{\theta\theta} = \rho^2, \quad g_{\varphi\varphi} = \varpi^2.$$  

(4.4)

Here $M$ and $a$ are respectively the black hole mass and the angular momentum per unit mass ($a = J/M$), and we introduce the standard notation

$$\Delta = r^2 + a^2 - 2Mr, \quad \rho^2 = r^2 + a^2 \cos^2 \theta,$$

$$\Sigma^2 = (r^2 + a^2)^2 - a^2 \Delta \sin^2 \theta, \quad \varpi = \frac{\Sigma}{\rho} \sin \theta.$$  

(4.5)

Units where $c = G = 1$ are used below.

This metric has the following properties.

- The Kerr metric is axisymmetric and stationary. It is precisely what is needed for using the Grad-Shafranov approach.

- The Kerr metric is two-parametric, i.e., it depends on two parameters: the mass $M$ and the rotation parameter $a$.

- It transforms to the Schwarzschild metric for a nonrotating black hole: $g_{rr} = \alpha^{-2}$, $g_{\theta\theta} = r^2$, $g_{\varphi\varphi} = r^2 \sin^2 \theta$. Here $\alpha^2 = 1 - 2M/r$.

- In the limit $r \gg 2M$ Boyer-Lindquist coordinates coincide with the spherical ones: $g_{rr} = 1$, $g_{\theta\theta} = r^2$, $g_{\varphi\varphi} = r^2 \sin^2 \theta$.

- Lapse function $\alpha$
  
  - The lapse function $\alpha$ describes the lapse (the red shift) between the proper time $\tau$ and the time at infinity $t$: $d\tau = \alpha dt$. 

– The condition \( \alpha = 0 \) determines the position of the horizon
\[
 r_g = M + \sqrt{M^2 - a^2}. \tag{4.6}
\]
– Boyer-Lindquist coordinates do not describe the space-time inside the horizon; for \( r = r_g \) the metric has a coordinate singularity.

**Lense-Thirring angular velocity \( \omega \)**

– The Lense-Thirring angular velocity \( \omega \) corresponds to the proper motion of the space around a black hole.
– By definition, \( \Omega_H = \omega(r_g) \) is the black hole angular velocity (does not depend on \( \theta \)).
– \( \omega \propto a, \ \Omega_H r_g = a/(2M) \).

**Convenient reference frame – ZAMO**

– ZAMO (Zero Angular Momentum Observers) [27] are located at a constant radius \( r = \text{const}, \ \theta = \text{const} \), but they rotate with the Lense–Thirring angular velocity \( d\varphi/dt = \omega \).
– For ZAMO the four-dimensional metric \( g_{\alpha\beta} \) is diagonal, the spatial 3D metric \( g_{ik} \) coinciding with (4.4).
– No gyroscope rotation in the local experiment.

To clarify the physical meaning of the values \( \alpha \) and \( \omega \), let us consider the motion of particles in the gravitational field of a rotating black hole. Then, one can rewrite the four-dimensional equation of motion
\[
 \frac{d^2 x^\alpha}{ds^2} + \Gamma^\alpha_{\beta\gamma} \frac{dx^\beta}{ds} \frac{dx^\gamma}{ds} = 0 \tag{4.7}
\]
in the simple 3D form
\[
 \frac{dp_i}{d\tau} = \frac{m_p}{\sqrt{1 - v^2}} g_i + H_{ik} \frac{m_p v^k}{\sqrt{1 - v^2}}, \tag{4.8}
\]
where
\[
 g = -\frac{1}{\alpha} \nabla \alpha, \quad H_{ik} = \frac{1}{\alpha} \nabla_i \beta_k. \tag{4.9, 4.10}
\]
Remember that
• The Greek indices $\alpha$, $\beta$, and $\gamma$ are four-dimensional, while the Latin $i$, $j$, and $k$ are three-dimensional.

• $\tau$ is the proper time, and all three-dimensional vectors are measured by ZAMO.

• $\nabla_i$ means the covariant derivative in the three-dimensional metric (4.4).

In a weak gravitational field, i.e., far from a black hole there is very nice analogy between the gravitational and electrodynamic equations. Indeed, the equation of motion (4.8) can be rewritten as

$$m_p \frac{d^2 \mathbf{r}}{d\tau^2} = m_p \left( \mathbf{g} + \frac{d\mathbf{r}}{d\tau} \times \mathbf{H} \right),$$

where

$$\mathbf{g} = -\nabla \alpha, \quad \mathbf{H} = \nabla \times \beta,$$

(4.12)

$\alpha$ and $\beta$ playing the role of the scalar and vector potentials, respectively. Moreover, the Einstein equations in a weak gravitational field are quite similar to the Maxwell equations

$$\nabla \cdot \mathbf{g} = -4\pi \rho_m,$$

(4.13)

$$\nabla \times \mathbf{g} = 0,$$

(4.14)

$$\nabla \cdot \mathbf{H} = 0,$$

(4.15)

$$\nabla \times \mathbf{H} = -16\pi \rho_m \mathbf{v}.$$  

(4.16)

In other words, the gravitational field $\mathbf{g}$ is analogous to the electric field while the new (so-called gravitomagnetic) field $\mathbf{H}$ – to the magnetic one which is proportional to the angular velocity of a black hole. The sources of the gravitoelectric field $\mathbf{g}$ are masses, and the sources of the gravitomagnetic field $\mathbf{H}$ are mass currents.

For example, for a rotating sphere with mass $M$ and angular momentum $\mathbf{J}$ the fields outside the sphere are

$$\mathbf{g} = -\frac{M}{r^2} \mathbf{e}_r,$$

(4.17)

$$\mathbf{H} = \frac{2\mathbf{J} - 3 \mathbf{e}_z (\mathbf{J} \cdot \mathbf{e}_r)}{r^3},$$

(4.18)

i.e., the rotation induces a dipole gravitomagnetic field around the rotating body. The appearance of an additional gravitomagnetic force is the most important consequence of the black hole rotation.

To summarize, the $3+1$ language allows the description of physical processes in a clear 3D form. If we simultaneously use ZAMO as a reference
frame, no expressions will contain extra terms. In a sense, ZAMO is an inertial frame, at any case in the $\varphi$ direction. As a result, within the $3+1$-split

- All three-dimensional vectors are to be determined from the local experiment by ZAMO.

- All the calculations are to be made in the 3D diagonal metric (4.4), e.g.,

\[
\nabla \cdot A = \frac{1}{\sqrt{g}} \frac{\partial}{\partial x^i} \left( \sqrt{g} A^i \right), \tag{4.19}
\]

\[
\nabla \times A = \frac{1}{\sqrt{g}} \begin{pmatrix}
\sqrt{g_{rr}} e_r & \sqrt{g_{r\theta}} e_{\theta} & \sqrt{g_{r\phi}} e_{\phi} \\
\frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial \phi}
\end{pmatrix} \left( \sqrt{g_{rr}} A_r \sqrt{g_{r\theta}} A_\theta \sqrt{g_{r\phi}} A_\phi \right), \tag{4.20}
\]

- All the vector relations remain the same as in the flat space, e.g., $\nabla \times (\nabla a) = 0$, $\nabla \cdot (\nabla \times A) = 0$.

### 4.1.2 Thermodynamics

On the other hand, all thermodynamic functions within the $3+1$ approach are determined in the comoving reference frame, which is the only invariant one. For this reason, one need no think about the transformation connected with different reference frames. Actually, there is the only complication: in the relativistic case one should work with the relativistic enthalpy $\mu$ including the particle rest mass

\[
\mu = \rho_m + P \approx m_p c^2 + m_p w + \ldots \tag{4.21}
\]

Here $\rho_m$ is the internal energy density. For the polytropic equation of state $P = k(s)n^\Gamma$ (2.4) we have ($c = 1$)

\[
\mu = m_p + \frac{\Gamma}{\Gamma - 1} k(s)n^{\Gamma - 1}, \tag{4.22}
\]

\[
e_s^2 = \frac{1}{\mu} \left( \frac{\partial P}{\partial n} \right)_s = \frac{\Gamma}{\mu} k(s)n^{\Gamma - 1}. \tag{4.23}
\]

Finally, the relativistic energy-momentum tensor has a symmetrical form

\[
T^{\alpha\beta} = \begin{pmatrix}
\varepsilon & S \\
S & T^{ik}
\end{pmatrix} = \begin{pmatrix}
(\rho_m + P) \gamma^2 & (\rho_m + P) \gamma u \\
(\rho_m + P) \gamma u & (\rho_m + P) u^k u^k + Pg^{ik}
\end{pmatrix}. \tag{4.24}
\]

Remember that $\gamma$ is the Lorentz-factor of a flow measured by ZAMO.
Now, using the relativistic version of the energy-momentum conservation law $\nabla_\alpha T^{\alpha \beta} = 0$, one can obtain \[25\]

$$
\begin{align*}
- \frac{1}{\alpha} (\beta \nabla) \varepsilon &= - \frac{1}{\alpha^2} \nabla \cdot (\alpha^2 \mathbf{S}) + H_{ik} T^{ik}, \\
\nabla_k T^k_i + \frac{1}{\alpha} S^\varphi \frac{\partial \omega}{\partial x^i} + \left( \varepsilon b^i_k + T^k_i \right) \frac{1}{\alpha} \frac{\partial \alpha}{\partial x^k} &= 0.
\end{align*}
$$
\tag{4.25}
\tag{4.26}

Here the additional terms in the energy \[4.25\] and momentum \[4.26\] equations are due to the gravitomagnetic forces.

4.1.3 Stream Function, etc.

As in the flat space, one can introduce the stream function $\Phi(r, \theta)$ through the poloidal component of the four-velocity of the flow $u_p$

$$
\alpha n u_p = \frac{\nabla \Phi \times \hat{e}_\varphi}{2\pi \varpi}.
$$
\tag{4.27}

It means the following relations for the physical components

$$
\begin{align*}
\alpha n u_r &= \frac{1}{2\pi \varpi} (\nabla \Phi)_{\theta}, \\
\alpha n u_\theta &= - \frac{1}{2\pi \varpi} (\nabla \Phi)_r.
\end{align*}
$$
\tag{4.28}
\tag{4.29}

The definition \[4.27\] gives the continuity equation in the form

$$
\nabla \cdot (\alpha n \mathbf{u}) = 0.
$$
\tag{4.30}

Let me clarify the extra factor $\alpha$ in \[4.30\]. The point is that the 3D continuity equation \[4.30\] results from the 4D one

$$
\nabla_\beta N^\beta = \frac{1}{\sqrt{g_{tt}}} \frac{\partial}{\partial x^\sigma} (\sqrt{g_{tt}} g^{\beta \sigma} N^\beta),
$$
\tag{4.31}

where $g_{tt}$ is equal to $\alpha^2$.

As a result, using the definitions \[4.24\] and \[4.27\], one can rewrite the energy equation \[4.25\] and the $\varphi$ component of the momentum equation \[4.26\] as

$$
\begin{align*}
\mathbf{u} \cdot \nabla (\alpha \gamma) + \mu \mathbf{u}_\varphi \mathbf{u} \cdot \nabla \omega &= 0, \\
\mathbf{u} \cdot \nabla (\mu \mathbf{u}_\varphi) &= 0.
\end{align*}
$$
\tag{4.32}
\tag{4.33}

Hence, two integrals of motion can now be present as

$$
\begin{align*}
E(\Phi) &= \alpha \gamma \gamma + \mu \omega \varpi \mathbf{u}_\varphi, \\
L(\Phi) &= \mu \varpi \mathbf{u}_\varphi.
\end{align*}
$$
\tag{4.34}
\tag{4.35}
exercise

Obtain expressions (4.32)–(4.34).

Expressions (4.34) and (4.35) are the extension of the nonrelativistic relations (3.2) and (3.6) to the case of a rotating black hole. Indeed, for $\omega = 0$ we have for Bernoulli integral

\[
E = \gamma \mu \alpha \approx \left( 1 + \frac{1}{2} \frac{v^2}{c^2} \right) \left( m_p c^2 + m_p w \right) \left( 1 - \frac{GM}{c^2 r} \right) \\
\approx m_p c^2 + m_p \left( \frac{v^2}{2} + w + \varphi \right) + \ldots
\]  

(4.36)

As to the third invariant, we again have

\[s = s(\Phi)\]  

(4.37)

4.1.4 Grad-Shafranov equation

Using now three invariants $E$, $L$, and $s$, one can write the poloidal component of the relativistic Euler equation [31]

\[
u u_b \nabla_b (\mu u_a) + \nabla_a P - \mu n (u_\varphi)^2 \frac{1}{\omega} \nabla_a \varphi \\
+ \frac{1}{\alpha} \mu \gamma (\omega u_\varphi) \nabla_a \omega + \frac{1}{\alpha} \mu \gamma^2 \nabla_a \alpha = 0
\]  

(4.38)

(here the indices $a$ and $b$ are $r$ and $\theta$ only) as $[\text{Euler}]_p = [\text{GS}] \nabla \Phi$, where the stream equation $[\text{GS}] = 0$ now looks like

\[
-M^2 \left[ \alpha \omega^2 \nabla_k \left( \frac{1}{\alpha \omega^2} \nabla^k \Phi \right) + \frac{\nabla^a \cdot \Phi \nabla^b \cdot \Phi \nabla_a \nabla_b \Phi}{(\nabla \Phi)^2 D} \right] \\
+ \frac{M^2 \nabla_k F \cdot \nabla^k \Phi}{2(\nabla \Phi)^2 D} \\
+ \frac{32\pi^4}{M^2} \frac{\partial}{\partial \Phi} \left[ \omega^2 (E - \omega L)^2 - \alpha^2 L^2 \right] - 16\pi^3 \alpha^2 \omega^2 n T \frac{ds}{d\Phi} = 0.
\]  

(4.39)

Here

\[
F = \frac{64\pi^4}{M^4} \left[ \omega^2 (E - \omega L)^2 - \alpha^2 L^2 - \omega^2 \alpha^2 \mu^2 \right],
\]  

(4.40)

and we introduce the thermodynamic function

\[
M^2 = \frac{4\pi \mu}{n}.
\]  

(4.41)
Next, the operator $\nabla_k^{\prime}$ acts on all the variables except $M^2$, and $\partial/\partial \Phi$ on the invariants $E(\Phi), L(\Phi), \text{and } s(\Phi)$ only. Finally, now the denominator $D$ is

$$D = -1 + \frac{1}{u_p^2} \frac{c_s^2}{1 - c_s^2} \tag{4.42}$$

Here the physical component of the poloidal four-velocity $u_p$ can be found from $[4.27]$. In a compact form the relativistic stream equation looks like

$$-\alpha \pi^2 \nabla_k \left( \frac{M^2}{\alpha \pi^2} \nabla^k \Phi \right) + \frac{32 \pi^4}{M^2} \frac{\partial}{\partial \Phi} \left[ \pi^2 (E - \omega L)^2 - \alpha^2 L^2 \right]$$

$$-16 \pi^3 \alpha^2 \pi^2 n T \frac{ds}{d\Phi} = 0. \tag{4.43}$$

The hydrodynamical version of the Grad-Shafranov equation in the Kerr metric was first formulated in $[28]$ using four-dimensional notation and in $[29]$ within the $3 + 1$-split language.

As before, the Grad-Shafranov equation is to be supplemented with the Bernoulli equation ($\gamma^2 = 1 + u^2_{\phi} + u^2_p$) which can now be written as

$$(E - \omega L)^2 = \alpha^2 \mu^2 + \frac{\alpha^2 L^2}{\pi^2} + \frac{M^4}{64 \pi^4 \omega^2} (\nabla \Phi)^2. \tag{4.44}$$

Remember that in $[4.39]$ and $[4.44]$ the relativistic enthalpy $\mu$ is to be considered as a function of $M^2$ and $s$: $\mu = \mu(M^2, s)$. In the general case the appropriate differential connection is $[29]

$$d\mu = -\frac{c_s^2}{1 - c_s^2} \mu \frac{dM^2}{M^2} + \frac{1}{1 - c_s^2} \left[ \frac{1}{n} \left( \frac{\partial P}{\partial s} \right) + T \right] ds. \tag{4.45}$$

In particular, as in the nonrelativistic case, the Bernoulli equation $[4.44]$ determines $M^2$ in the implicit form through the flux $\Phi$ and three integrals of motion: $M^2 = M^2(\nabla \Phi; E, L, s)$.

Finally, it is convenient to use another form of the poloidal four-velocity $u_p$:

$$u_p^2 = \frac{E^2 - \alpha^2 L^2 / \omega^2 - \alpha^2 \mu^2}{\alpha^2 \mu^2}. \tag{4.46}$$

We see that $u_p \to \infty$ as $\alpha^{-1}$ at the horizon. As was already stressed, it results from our choice of the reference frame which has a coordinate singularity for $r = r_g$. Relation $[4.46]$ suggests a very important conclusion. The flow in a close vicinity of a black hole is to be supersonic ($u_p > c_s$).
4.2 Examples

4.2.1 Exact solutions

1. Spherically symmetric accretion

If the flow velocity at infinity is equal to zero ($\gamma_\infty = 1$) and thermodynamic conditions are homogeneous, then $E = \mu_\infty = \text{const}$ and $s = s_\infty = \text{const}$. So, the two thermodynamic functions at infinity determine two integrals of motion $E$ and $s$. Finally, for a spherically symmetric flow one can put $L = 0$. Under such conditions the stream equation (4.39) has a trivial solution $\Phi = \Phi_0(1 - \cos \theta)$, and the accretion rate $2\Phi_0$ is to be determined from the regularity conditions on the sonic surface $r = r_s$.

As a result, we have the following expression for the sonic radius

$$r_s = \frac{M}{2} \left( \frac{1}{c^2_s} + 3 \right), \quad (4.47)$$

so that for $c^2_s \ll 1$ we return to the nonrelativistic expression (2.30). As to the relation between $c^2_s$ and $c^2_\infty$, it can be found from the condition

$$\frac{1}{[1 - c^2_\infty/(\Gamma - 1)]^2} = \frac{1 - 4c^2_\infty/(1 + 3c^2_\infty)}{[1 - c^2_s/(\Gamma - 1)]^2} \cdot (1 - c^2_s). \quad (4.48)$$

In the limit $c^2_s \ll 1$ we return to the well-known relation (2.22). Next, the values $M^2_s$ and $\mu_s$ on the sonic surface are

$$M^2_s = M^2_\infty \left( \frac{c^2_\infty}{c^2_s} \right)^{(1/(\Gamma - 1)} \left( \frac{\Gamma - 1 - c^2_s}{\Gamma - 1 - c^2_\infty} \right)^{(2-\Gamma)/(\Gamma-1)}, \quad (4.49)$$

$$\mu_s = \frac{\mu_\infty}{\Gamma - 1 - c^2_\infty}. \quad (4.50)$$

Thus, the accretion rate can be written as $2m_p\Phi_0$, where we now have $\Phi_0 = 8\pi^2 r^2 E c_s / M^2_s$.

exercise

Show that in General Relativity a transonic flow takes place even for $\Gamma = 5/3$.

Finally, in the supersonic region $r \ll r_s$ we have

$$\frac{M^2}{M^2_s} \approx 2 \left( \frac{r}{r_s} \right)^{3/2}, \quad (4.51)$$
\[
\frac{c_s^2}{c_*^2} \approx \frac{1}{2^{r-1}} \left( \frac{r}{r_*} \right)^{-3(r-1)/2} .
\] (4.52)

In particular, at the horizon
\[
c_s^2(r_g) = \frac{1}{16^{r-1}} (c_s)^{5-3r} .
\] (4.53)

Hence, for \( c_s^2 \approx c_*^2 \ll 1 \) the velocity of sound remains small (\( c_s \ll 1 \)) up to the horizon.

It is necessary to stress that an accretion onto black holes differs sufficiently from the nonrelativistic case. The point is that all the subsonic trajectories taking place for an accretion onto ordinary stars (see Fig. 1) have unphysical singularity \( v(r \to r_g) = 0, \ n(r \to r_g) = \infty \) on the horizon. In other words, to support the subsonic flow the infinite gravitational force in the vicinity of the horizon is to be balanced by the infinite pressure gradient. Hence, the only physically reasonable regime of the accretion onto a black hole is the transonic one.

2. Accretion of a dust \((P = 0)\)

For a dust, the flow lines must coincide with trajectories of particles freely moving in the gravitational field of a Kerr black hole. For the case of the motion with \( L = 0 \) (\( u_\phi = 0 \)) and zero kinetic energy at infinity \( \gamma_\infty = 1 \), such trajectories are "straight lines" \( \theta = \text{const} \) for an arbitrary rotation parameter \( a \). Moreover, for \( P = 0 \) the density of the flow lines can be arbitrary as well. In other words, the arbitrary function
\[
\Phi = \Phi(\theta) \quad (4.54)
\]

must be a solution to the stream equation. In particular, it means that the accretion rate is free. It is not surprising, for the flow is supersonic in the entire space.

**exercise**

Using the Bernoulli equation (4.44) and the compact form of the relativistic stream equation (4.43), check that for \( E = \mu = \text{const}, \ L = 0, \) and \( s = 0 \) the arbitrary function \( \Phi(\theta) \) is a solution.

3. Accretion of a gas with \( c_s = 1 \) \([30]\)

As one can see from (4.42), for \( c_s = 1 \) we have \( D^{-1} = 0 \). Hence, for \( E = \text{const}, \ L = 0, \) and \( s = \text{const} \) the stream equation becomes linear
\[
\frac{\Delta}{\rho^2} \frac{\partial^2 \Phi}{\partial r^2} + \frac{1}{\rho^2 \sin \theta} \frac{\partial}{\partial \theta} \left( \frac{1}{\sin \theta} \frac{\partial \Phi}{\partial \theta} \right) = 0 .
\] (4.55)
As a result, the solution can again be expanded in eigenfunctions of the operator $\hat{L}_\theta$. For example, for a moving black hole one can obtain
\[
\Phi = \Phi_0 (1 - \cos \theta) + \pi n \infty v_\infty (r^2 - 2r_g r) \sin^2 \theta.
\] (4.56)

Here the accretion rate $2\Phi_0$ is arbitrary for another reason – the flow remains subsonic up to the horizon.

4.2.2 Bondi-Hoyle Accretion – Relativistic Version

First of all, let us consider the relativistic version of the Bondi-Hoyle accretion, i.e., accretion onto a moving nonrotating black hole. The small parameter of the problem is again $\varepsilon_1 = v_\infty / c_\infty$. In the relativistic case the linearized stream equation for the stream function $\Phi = \Phi_0 [1 - \cos \theta + \varepsilon_1 f(r, \theta)]$ can be written as
\[
-\varepsilon_1 \alpha^2 D \frac{\partial^2 f}{\partial r^2} - \frac{\varepsilon_1}{\rho^2} (D + 1) \sin \theta \frac{\partial}{\partial \theta} \left( \frac{1}{\sin \theta} \frac{\partial f}{\partial \theta} \right) + \varepsilon_1 \alpha^2 N_r \frac{\partial f}{\partial r} = 0,
\] (4.57)

where now
\[
N_r = \frac{2}{r} - \frac{\mu^2}{E^2 - \alpha^2 \mu^2} \frac{M}{r^2}.
\] (4.58)

We see that this equation have the same properties as the nonrelativistic equation (3.37), namely
- Equation (4.57) is linear.
- The values of $\mu$, $N_r$, and $D$ should be determined from the unperturbed Schwarzschild metric for a spherically symmetric flow.
- As $\mu = \mu(r)$, $N_r = N_r(r)$, $D = D(r)$, $\alpha^2 = 1 - 2M / r$, and $\rho = r$, one can expand the solution in eigenfunctions of the operator $\hat{L}_\theta$.

But equation (4.57) has another very important property. According to (4.42) and (4.46),
\[
D + 1 = \frac{\alpha^2 \mu^2}{E^2 - \alpha^2 \mu^2} \cdot \frac{c_s^2}{1 - \varepsilon_s^2},
\] (4.59)

so the factor $\alpha^2$ is contained in the all addenda of equation (4.57). Hence, equation (4.57) has no singularity at the horizon. In particular, it means that it is not necessary to specify any boundary conditions for $r = r_g$. It is not surprising because the horizon corresponds to the supersonic region which cannot affect the subsonic flow.

As a result, one can seek the solution of the stream equation in the form (for more details see [20, 14])
\[
\Phi(r, \theta) = \Phi_0 [1 - \cos \theta + \varepsilon_1 g_1(r) \sin^2 \theta],
\] (4.60)
the equation for the radial function \( g_1(r) \) being

\[
-D \frac{d^2 g_1}{dr^2} + N r \frac{dg_1}{dr} + 2 \frac{\mu^2}{E^2 - \alpha^2 \mu^2} \cdot \frac{c_s^2}{1 - c_s^2} \cdot \frac{g_1}{r^2} = 0.
\]

(4.61)

We see that, as in the nonrelativistic case, the accretion rate is not changed in the first order of \( \varepsilon_1 \).

As to the boundary conditions, they are

1. The regularity condition on the sonic surface. It gives \( g_1(r^*) = 0 \).

2. At infinity

\[
g_1(r) \to K(\Gamma) \frac{r^2}{r_*^2},
\]

where (see Table 2)

\[
K(\Gamma) = \frac{1}{2} \frac{M_*^2}{M^2_* c_\infty}.
\]

(4.62)

(4.63)

| \( \Gamma \) | 1.01 | 1.1 | 1.2 | 1.333 | 1.5 | 1.6 |
|-----------|------|-----|-----|-------|-----|-----|
| \( K(\Gamma) \) | 0.49 | 0.09 | 0.07 | 0.044 | 0.016 | 0.003 |
| \( k_1(\Gamma) \) | -3.00 | 0.56 | 0.46 | 0.31 | 0.12 | 0.023 |
| \( K_{in}(\Gamma) \) | -0.74 | -0.09 | -0.03 | 0.025 | 0.0081 | 0.0002 |

Equation (4.61) with boundary conditions 1 and 2 determines the flow structure of the Bondi-Hoyle accretion onto a nonrotating black hole. In particular, as in the nonrelativistic case, the shape of the sonic surface has the form

\[
r_*(\theta) = r_* \left[ 1 + 2\varepsilon_1 \frac{\Gamma + 1}{\mu_1} k_1(\Gamma) \cos \theta \right],
\]

(4.64)

where again \( k_1(\Gamma) = g_1'(r_*) \). The \( k_1(\Gamma) \) values are given in Table 2.

But in general, the flow structure remains the same as for the nonrelativistic accretion (see Fig. 3). Here one can stress only a single interesting feature. For \( r < r_* \) the radial function \( g_1(r) \) has the asymptotics

\[
g_1(r) \approx K_{in}(\Gamma) \left( \frac{r}{r_*} \right)^{-1/2}
\]

(4.65)

\( (K_{in} \) is given in Table 2 as well). Hence, for \( \varepsilon_1 > (M/r_*)^{1/2} \) in the vicinity of the black hole (i.e., for \( r < \varepsilon_1^2 K_{in}^2 r_* \)) the linear approximation is violated, so here one should solve the complete nonlinear equation (4.39). Since the sign of the coefficient \( K_{in}(\Gamma) \) depends on the polytropic index \( \Gamma \), the region of thickening of the flow lines will either appear on the front side for \( \Gamma > 1.27 \) or on the rear side for \( \Gamma < 1.27 \). However, it takes place in the supersonic region which does not affect the subsonic flow for \( r > r_* \).
4.2.3 Accretion onto a Slowly Rotating Black Hole

Let us now consider the accretion of a gas with $L = 0$ (i.e., $\sigma = 2$ and $b = 2 + 2 - 1 = 3$) onto a slowly rotating black hole for which the small parameter is

$$\varepsilon_3 = \frac{a}{M} \ll 1.$$  \hfill (4.66)

In this case the metric $g_{ik}$ \hbox{(4.4)} differs from the Schwarzschild one by the values $\sim \varepsilon_3^2$. One can assume that the thermodynamic functions at infinity, $s_\infty$ and $\mu_\infty$, remain the same as for spherically symmetric accretion. Hence, one can again seek the solution of the stream equation \hbox{(4.39)} in the form

$$\Phi(r, \theta) = \Phi_0[1 - \cos \theta + \varepsilon_3^2 f(r, \theta)],$$  \hfill (4.67)

where $\Phi_0$ is the flux constant corresponding to a spherically symmetrical flow. Inserting this form into the stream equation \hbox{(4.39)}, we have in the first order of $\varepsilon_3^2$

$$-\varepsilon_3^2 \alpha^2 \frac{\partial^2 f}{\partial r^2} - \varepsilon_3^2 \frac{\partial^2 f}{\rho^2} (D + 1) \sin \theta \frac{\partial}{\partial \theta} \left( \frac{1}{\sin \theta} \frac{\partial f}{\partial \theta} \right) + \varepsilon_3^2 \alpha^2 N_r \frac{\partial f}{\partial r} = \frac{a^2}{r^4} \left( 1 - 2 M \right) \left( 1 - 2 \frac{\mu^2}{E^2 - \alpha^2 \mu^2} \frac{M}{r} \right) \sin^2 \theta \cos \theta,$$  \hfill (4.68)

where $N_r$ is given by \hbox{(4.58)}.

The properties of equation \hbox{(4.68)} are quite similar to those of equation \hbox{(4.57)}. In particular, equation \hbox{(4.68)} has no singularity at the horizon. As a result, one can show that the flux $\Phi(r, \theta)$ can be presented as \hbox{(4.69)}

$$\Phi(r, \theta) = \Phi_0[(1 - \cos \theta) + \varepsilon_3^2 \Phi_0(1 - \cos \theta) + \varepsilon_3^2 g_2(r) \sin^2 \theta \cos \theta].$$  \hfill (4.69)

Hence, as for the ejection from a slowly rotating star, the flux $\Phi(r, \theta)$ contains two harmonics, $m = 0$ and $m = 2$, the radial function $g_2(r)$ satisfying equation

$$-D \frac{d^2 g_2}{dr^2} + N_r \frac{dg_2}{dr} + 6 \frac{\mu^2}{E^2 - \alpha^2 \mu^2} \frac{\varepsilon_3^2}{1 - \varepsilon_3^2} \frac{g_2}{r^2} = \frac{M^2}{r^4} \left( 1 - 2 \frac{\mu^2}{E^2 - \alpha^2 \mu^2} \frac{M}{r} \right).$$  \hfill (4.70)

The regularity condition on the sonic surface $N_\theta(r_*) = 0$ gives

$$g_2(r_*) = -\frac{1}{2} \frac{M^2}{r_*^2} \alpha^2(r_*).$$  \hfill (4.71)

On the other hand, the condition at infinity (which is the third boundary condition after $s_\infty$ and $\mu_\infty$) gives $g_2(r \to \infty) = 0$. As a result, the radial function $g_2(r)$ for $r \ll r_*$ can be written in the form

$$g_2(r) = -G(r) \frac{M^2}{r_*^2} \left( \frac{r}{r_*} \right)^{(1 - 3\Gamma)/2},$$  \hfill (4.72)
where $G(r) \sim 1$. Hence, at the horizon $g_2(r_g) \sim (M/r_\ast)^{(5-3\Gamma)/2}$, so that the disturbance is small ($\varepsilon_2^2 g_2(r) \ll 1$) everywhere outside a black hole. On the other hand, as $g_2(r) < 0$, the rotation of a black hole results in the concentration of the flow lines in the equatorial plane. Finally, an additional consideration demonstrates that

$$g_0 = -\frac{2M^3}{r_\ast^3}. \quad (4.73)$$

Thus, the rotation of a black hole diminishes the accretion rate. In reality $c_\infty^2 \ll 1$ and hence $M/r_\ast \ll 1$. As a result, the effects of a black hole rotation in the vicinity of the sonic surface are actually very small.

To summarize the last two sections, one can say that the simple zero approximation, namely, a spherically symmetric transonic accretion, allows us to find analytically 2D structure for very important astrophysical flows. A similar approach was used for the construction of an analytical solution for

- accretion of a gas with small angular momentum $L$ (the small parameter $\varepsilon_4^2 = (L/Er_\ast)^2$) onto a nonrotating black hole [28, 32],
- accretion of a gas with a nonrelativistic temperature ($c_\infty \ll 1$) and without angular momentum ($L = 0$) onto an arbitrary rotating black hole [33].

At present they are the only examples where the analytical solution was found.

### 4.2.4 Thin Transonic Disk

As the last example, let us consider the internal 2D structure of a thin transonic disk. Such a flow can be realized if the intrinsic angular momentum of a gas accreting onto a black hole is large enough ($\varepsilon_4 \gg 1$) [18]. Here for simplicity we consider a nonrotating (Schwarzschild) black hole only (for more details see [34]).

According to the standard disk model [35, 36, 37, 38], for $\varepsilon_4 \gg 1$ the matter forms a thin balanced disk and performs a nearly circular motion with keplerian velocity $v_K(r) \approx (GM/r)^{1/2}$. The disk is thin provided that the accreting gas temperature is sufficiently low and the disk thickness is determined by the pressure gradient

$$H \approx r \frac{c_s}{v_K}. \quad (4.74)$$

Introducing the viscosity parameter $\alpha_{SS} \leq 1$, relating the stress tensor $t_\varphi^r$ and the pressure as $t_\varphi^r = \alpha_{SS} P$, one can obtain

$$\frac{v_r}{v_K} \approx \alpha_{SS} \frac{c_s^2}{v_K^2}. \quad (4.75)$$
Hence, for $c_s \ll v_K$ the radial velocity $v_r$ remains much smaller than both the keplerian velocity $v_K$ and the velocity of sound $c_s$.

The General Relativity effects result in two important properties:

- The absence of stable circular orbits for $r < r_0 = 3r_g$.
- The transonic regime of accretion.

The first point means that the accreting matter passing a marginally stable orbit approaches the black hole horizon sufficiently fast, namely, in the dynamical time $\tau_d \sim [v_r(r_0)/c]^{-1/3}r_g/c$. It is important that such a flow is realized in the absence of viscosity. The second statement results from the fact that according to (4.75) up to the marginally stable orbit the flow is subsonic while at the horizon the flow is to be supersonic.

Up to now in the majority of works the procedure of vertical averaging was used, where the vertical four-velocity $u_\theta$ was assumed to be zero [39]. As a result, the vertical component of the dynamic force $n v^b \nabla_b (\mu u_a)$ in (4.38) was postulated to be inimportant up to horizon. For this reason the disk thickness was determined by the pressure gradient even in the supersonic region near the black hole [40]. Here I am going to demonstrate that the assumption $u_\theta = 0$ is not correct. As in the Bondi accretion, the dynamic force is to be important in the vicinity of the sonic surface.

First of all, let us consider the subsonic region in a close vicinity of the marginally stable orbit $r_0 = 3r_g$ where the poloidal velocity $u_p$ is much smaller than that of sound. Then equation (4.39) can be significantly simplified by neglecting the terms proportional to $D^{-1} \sim u_p^2/c_s^2$. As a result, we have

$$-M^2 \frac{1}{\alpha} \nabla_k \left( \frac{1}{\alpha \omega^2} \nabla^k \Phi \right) + \frac{64 \pi^4}{\alpha^2 \omega^2 M^2} \left( \omega^2 E \frac{dE}{d\Phi} - \alpha^2 L \frac{dL}{d\Phi} \right) - 16 \pi^3 nT \frac{ds}{d\Phi} = 0. \quad (4.76)$$

This equation describing the subsonic flow is elliptical.

To determine the structure of a two-dimensional subsonic flow ($\sigma = 0$, i.e., $b = 2 + 3 - 0 = 5$) one needs to specify five quantities (three velocity components and two thermodynamic functions) on an arbitrary surface $r = r_0(\theta)$. Naturally, we choose it as the surface of the last stable orbit $r_0 = 3r_g$ where $\alpha_0 = \alpha(r_0) = \sqrt{2/3}$, $u_\phi(r_0) = 1/\sqrt{3}$, and $\gamma_0 = \gamma(r_0) = \sqrt{4/3}$ [18]. For the sake of simplicity we consider below the case where the radial velocity is constant on the surface $r = r_0$ and the toroidal velocity is exactly equal to $u_\phi(r_0)$:

$$u_r(r_0, \Theta) = -u_0, \quad (4.77)$$
$$u_\phi(r_0, \Theta) = \Theta u_0, \quad (4.78)$$
$$u_\varphi(r_0, \Theta) = 1/\sqrt{3}. \quad (4.79)$$
Here \( u_\Theta \ll |u_r| \) corresponds to the plane flow at the marginally stable orbit, and we introduced the new angular variable \( \Theta = \pi/2 - \theta \) \((\Theta_{\text{disk}} \sim c_0)\) which is counted off from the equator in the vertical direction. Next, we suppose that the velocity of sound is also a constant on the surface \( r = r_0 \)

\[
c_s(r_0, \Theta) = c_0. \tag{4.80}
\]

For the polytropic equation of state (2.6) it means that both the temperature \( T_0 = T(r_0) \) and the relativistic enthalpy \( \mu_0 = \mu(r_0) \) are also constant on this surface. According to (4.75), one can find that for a nonrelativistic temperature \( c_s \ll 1 \) the small parameter of this problem is

\[
\varepsilon_s = \frac{u_0}{c_0} \sim \alpha_{SS} c_0 \ll 1. \tag{4.81}
\]

Finally, as the last, fifth boundary condition it is convenient to specify the entropy \( s(\Phi) \).

Introducing the values \( E_0 = c_0^2, L_0 = u_\phi(r_0) r_0 = \sqrt{3} g \), one can write the invariants \( E(\Phi) \) and \( L(\Phi) \) in the following form

\[
E(\Phi) = \mu_0 E_0 = \text{const}, \tag{4.82}
\]
\[
L(\Phi) = \mu_0 L_0 \cos \Theta_m. \tag{4.83}
\]

Here \( \Theta_m = \Theta_m(\Phi) \) is the angle for which \( \Phi(r_0, \Theta_m) = \Phi(r, \Theta) \). In other words, the function \( \Theta_m(r, \Theta) \) has the meaning of a theta angle on the last stable orbit. This orbit is connected with a given point \((r, \Theta)\) by a line of flow \( \Phi(r, \Theta) = \text{const} \). In particular, \( \Theta_m(r_0, \Theta) = \Theta \).

First of all, we see that condition \( E = \text{const} \) (4.82) allows us to rewrite equation (4.76) in a simpler form

\[
\frac{\partial^2 \Phi}{\partial r^2} + \frac{\cos \Theta}{\alpha^2 r^2} \frac{\partial}{\partial \Theta} \left( \frac{1}{\cos \Theta} \frac{\partial \Phi}{\partial \Theta} \right) = -4\pi^2 n^2 L \frac{dL}{d\Phi} - 4\pi^2 n^2 r^2 \cos^2 \Theta \frac{T}{\mu} \frac{ds}{d\Phi}. \tag{4.84}
\]

Next, as is shown in Appendix A, for \( r = r_0 \), the r.h.s. of equation (4.84) describes the transverse balance of a pressure gradient and an effective potential, whereas the l.h.s. corresponds to the dynamic term \((v \nabla) v\). At the marginally stable orbit it is of the order of \( u_0^2/c_0^2 \) and may be dropped.

It is therefore natural to choose the entropy \( s(\Phi) \) from the condition of a transverse balance on the surface \( r = r_0 \)

\[
r_0^2 \cos^2 \Theta_m \frac{ds}{d\Theta_m} = \frac{\Gamma}{c_0^2} \frac{L}{\mu^2} \frac{dL}{d\Theta_m}, \tag{4.85}
\]

where \( L(\Theta_m) \) is determined from the boundary condition (4.83). Thus, we have

\[
s(\Theta_m) = s(0) - \frac{\Gamma}{3c_0^2} \ln(\cos \Theta_m). \tag{4.86}
\]
Owing to (4.23), one can show that for \( c_s = \text{const} \) relation (4.86) corresponds to the standard concentration profile

\[
n(r_0, \Theta) \approx n_0 \exp \left( -\frac{\Gamma}{6c_0^2} \Theta^2 \right).
\]

**exercise**

Using relations (2.5)–(2.6), show that the exact expression for \( n(r_0, \Theta) \) is

\[
n(r_0, \Theta) = n_0 (\cos \Theta)^{\Gamma/3} c_0^2.
\]

Finally, the definition (4.27) results in the following connection between functions \( \Phi \) and \( \Theta_m \)

\[
d\Phi = 2\pi\alpha_0 r_0^2 n(r_0, \Theta_m) u_0 \cos \Theta_m d\Theta_m.
\]

Hence, due to (4.83), (4.87), and (4.88), the invariant \( L(\Phi) \) can be directly determined from the boundary conditions as well.

Equation (4.84) together with the boundary conditions (4.83), (4.86), (4.87), and (4.88) and relationship (4.78) specifying the derivative \( \partial \Phi / \partial r \) determines the structure of the inviscid subsonic flow inside the marginally stable orbit. For example, for a nonrelativistic temperature \( c_s \ll 1 \) we obtain

\[
u^2_p = u_0^2 + w^2 + \frac{1}{3} (\Theta_m^2 - \Theta^2) + \frac{2}{\Gamma - 1} (c_0^2 - c_s^2) + \ldots
\]

Here the quantity \( w \), where

\[
w^2(r) = \frac{E_0^2 - \alpha^2 L_0^2/r^2 - \alpha^2}{\alpha^2} \approx \frac{1}{6} \left( \frac{r_0 - r}{r_0} \right)^3,
\]

depending on the radius \( r \) only is a poloidal four-velocity of a free particle having zero poloidal velocity for \( r = r_0 \). As we see, \( w^2 \) increases very slowly when moving away from the last stable orbit. Therefore, the contribution of \( w^2 \) turns out to be negligibly small and the term may be safely dropped in most cases.

An important conclusion can be drown directly from (4.89) in which for the equatorial plane we have \( \Theta_m = \Theta = 0 \). Assuming \( u_p = c_s = c_\ast \) and neglecting \( w^2 \), we find that the velocity of sound \( c_\ast \) on the sonic surface \( r = r_\ast, \Theta = 0 \) is of the same order of magnitude as the velocity of sound on the last stable orbit \( c_0 \)

\[
c_\ast \approx \sqrt{\frac{2}{\Gamma + 1}} c_0.
\]

Since the entropy \( s \) remains constant along the flow lines, the gas concentration remains approximately constant \( (n_\ast \sim n_0) \) as well. In other words,
in agreement with the Bondi accretion, the subsonic flow can be considered as incompressible.

On the other hand, since the density remains almost constant and for $\varepsilon_5 \ll 1$ the radial velocity increases from $u_0$ to $c_* \sim c_0$, i.e., changes over several orders of magnitude, the disk thickness $H$ should change in the same proportion owing to the continuity equation (see Fig. 4)

$$H(r_*) \approx \frac{u_0}{c_0} H(3r_g). \quad (4.92)$$

As a result, a rapid decrease of the disk thickness should be accompanied by the appearance of the vertical component of velocity which also should be taken into account in the Euler equation (4.38).

Indeed, as one can find analyzing the asymptotics of equation (4.84) [34], in the vicinity of the sonic surface located at

$$r_* = r_0 - \Lambda u_0^{2/3} r_0, \quad (4.93)$$

where the logarithmic factor $\Lambda = (3/2)^{2/3} [\ln(c_0/u_0)]^{2/3} \approx 5 - 7$, the components of the velocity and the pressure gradient can be presented as

$$u_\hat{\phi} \to -\frac{c_0}{u_0} \Theta, \quad (4.94)$$
$$u_\hat{r} \to -c_*, \quad (4.95)$$
$$-\frac{\nabla P}{\mu} \to \frac{c_0^2}{u_0^2} \Theta \frac{1}{r}. \quad (4.96)$$

On the other hand, near the sonic surface the radial scale $\delta r$ determining the radial derivatives becomes as small as the transverse dimension of a disk: $\delta r \approx H(r_*) \approx u_0 r_0$, so that

$$\eta_1 = \frac{r}{n} \frac{\partial n}{\partial r} \approx u_0^{-1}. \quad (4.97)$$

As a result, both components of the dynamic force

$$u_\hat{\phi} \frac{\partial u_\hat{\phi}}{\partial \Theta} \to \frac{c_0^2}{u_0^2} \frac{\Theta}{r}, \quad (4.98)$$
$$u_\hat{r} \frac{\partial u_\hat{r}}{\partial r} \to \frac{c_0^2}{u_0^2} \frac{\Theta}{r}. \quad (4.99)$$

become of the order of the pressure gradient (4.96).

To check our conclusions one can consider the flow structure in the vicinity of the sonic surface in more detail. Using again the expansion theorem (the smooth transonic flow is analytical at a singular point), one
Fig. 4. The structure of the thin accretion disk (actual scale) after passing the marginally stable orbit \( r = 3r_g \) obtained by numerical solving equation (4.84) for \( c_0 = 10^{-2}, \ u_0 = 10^{-5} \). The solid lines correspond to the range of parameters \( u_0^2/c_0^2 < 0.2 \), where the solution should not differ greatly from the solution of the complete equation (4.39). The dashed lines indicate an extrapolation of the solution to the sonic-surface region. In the vicinity of the sonic surface the flow has a form of an ordinary nozzle.

\[
\begin{align*}
\Theta/m & = n_\ast \left( 1 + \eta_1 h + \frac{1}{2} \eta_2 \Theta^2 + \ldots \right), \\
\Theta_m & = a_0 \left( \Theta + a_1 h \Theta + \frac{1}{2} \alpha_2 h^2 \Theta + \frac{1}{6} b_0 \Theta^3 + \ldots \right),
\end{align*}
\]

where \( h = (r - r_\ast)/r_\ast \). Here we assume that all three invariants \( E, L, \) and \( s \) are already given, i.e., \( i = 0 \) and \( b = 2 + 0 - 1 = 1 \). Hence, as in the planar case, the problem needs one extra boundary condition. Now comparing the appropriate coefficients in the Bernoulli (4.44) and the full stream equation.
(4.43), one can obtain neglecting terms $\sim u_0^2/c_0^2$

$$a_0 = \left( \frac{2}{\Gamma + 1} \right) ^{(\Gamma + 1)/2(\Gamma - 1)} \frac{c_0}{u_0}, \quad (4.102)$$

$$a_1 = 2 + \frac{1 - \alpha_2^2}{2 \alpha_2^2} \approx 2.25, \quad (4.103)$$

$$a_2 = -(\Gamma + 1) \eta_1^2, \quad (4.104)$$

$$b_0 = \left( \frac{\Gamma + 1}{6} \right) \frac{a_2^2}{c_0^2}, \quad (4.105)$$

$$\eta_3 = -\frac{2}{3}(\Gamma + 1) \eta_1^2 - \left( \frac{\Gamma - 1}{3} \right) \frac{a_2^2}{c_0^2}, \quad (4.106)$$

where $\alpha_2^2 = \alpha^2(r_*) \approx 2/3$. In comparison with the planar case, all the coefficients are expressed here through the radial logarithmic derivative $\eta_1$ (4.97).

Let me stress that it is rather difficult to connect the sonic characteristics $\eta_1 = \eta_1(r_*)$ with physical boundary conditions on the marginally stable orbit $r = r_0$ (for this it is necessary to know all the expansion coefficients in (4.100) and (4.101)). In particular, it is impossible to formulate the restriction on five boundary conditions (4.77)–(4.80) and (4.86) resulting from the critical condition on the sonic surface. Nevertheless, the estimate (4.97) makes us sure that we know the parameter $\eta_1$ to a high enough accuracy. Then, according to (4.102)–(4.106), all the other coefficients can be determined exactly.

The coefficients (4.102)–(4.106) have clear physical meaning. So, $a_0$ gives the compression of flow lines: $a_0 = H(r_0)/H(r_*)$. In agreement with (4.92) we have $a_0 \approx c_0/u_0$. Further, $a_1$ corresponds to the slope of the flow lines with respect to the equatorial plane. As $a_1 > 0$, in a close vicinity of the sonic surface the compression of stream lines finishes, so inside the sonic radius $r < r_*$ the stream lines diverge. On the other hand, as $a_1 \ll u_0^{-1}$, for $r = r_*$ the divergency is still very weak. Hence, in the vicinity of the sonic surface the flow has a form of an ordinary nozzle (see Fig. 2). Finally, as $a_2 \sim \eta_3 \sim b_0 \sim u_0^{-2}$, one can conclude that the transverse scale of the transonic region $H(r_*)$ does the same as the longitudinal one. The latter point suggests a very important consequence that the transonic region is essentially two-dimensional, and so it is impossible to analyze it within the standard one-dimensional approximation.

Using now the expansions (4.100) and (4.101), one can obtain all the other physical parameters of the transonic flow. In particular, we have

$$u_p^2 = c_0^2 \left[ 1 - 2\eta_1 b + \frac{1}{6}(\Gamma - 1) \frac{a_2^2}{c_0^2} \Theta^2 + \frac{2}{3}(\Gamma + 1) \eta_1^2 \Theta^2 \right],$$
The title will be set by the publisher.

\[ c_s^2 = c_s^2 \left[ 1 + (\Gamma - 1) \eta_1 h + \frac{1}{6}(\Gamma - 1) \frac{a_{0}^2}{c_0^2} \Theta^2 - \frac{1}{3}(\Gamma - 1)(\Gamma + 1)\eta_2 \Theta^2 \right]. \]

As a result, the shape of the sonic surface \( u_p = c_s \) has the standard parabolic form

\[ h = \frac{1}{3} \eta_1 \Theta^2. \]  

(4.107)

Thus, the analysis of the hydrodynamic stream equation (4.39) allows us to find a nontrivial structure of the thin transonic disk. As was shown, the diminishing disk thickness inevitably leads to an emergence of the vertical velocity component of the accreting matter. As a result, the dynamic term \((\mathbf{v} \nabla)\mathbf{v}\) in the vertical balance equation cannot be omitted. In this sense the situation is completely analogous to the spherically symmetric Bondi accretion for which the contribution of the dynamic term becomes significant near the sonic surface and dominant for a supersonic flow.

However, there is an important difference. For the Bondi accretion the dynamic term \((\mathbf{v} \nabla)\mathbf{v}\) has only one component \(v_r \partial v_r / \partial r\) which in the vicinity of the sonic surface becomes of the same order of magnitude as both the pressure and the gravity gradients. As to the thin accretion disk, both components of the dynamic term \([ (\mathbf{v} \nabla)\mathbf{v} ]_\theta\), (4.98) and (4.99), become of the same order of magnitude as the pressure gradient, the role of the effective gravity gradient being unimportant: \(\nabla_\theta \varphi_{\text{eff}} \sim \Theta / r\), i.e., it is \(c_{0}^2 / u_{0}^2\) times smaller than the leading terms. As a result, the structure of a thin transonic disk is quite similar to an ordinary planar nozzle shown in Fig. 2. For this reason the critical condition on the sonic surface does not restrict the accretion rate.

5 Conclusion

Thus, in my lecture I have tried to demonstrate the possibilities and difficulties of the Grad-Shafranov approach. As we have seen, in some simple cases it does allow us to construct the analytical solutions. In particular, the Grad-Shafranov approach is very suitable in considering the (analytical) properties of the flow in the vicinity of the sonic surface and in determining the number of boundary conditions.

On the other hand, it has been shown that in the general case the regular procedure does not exist. The point is that the critical conditions are to be specified on singular surfaces which are not known from the very beginning and are themselves to be determined from the solution. Moreover, it is impossible to extend this approach to nonideal, nonstationary, and non axially symmetric flows. For this reason it is not surprising that those interested in astrophysics more than mathematics, passed from the Grad-Shafranov approach to numerical calculations analyzing absolutely another class of equations, namely, the time-dependent ones. The only thing one
can wish is not to forget the basic physical results of the Grad-Shafranov approach which remain true irrespective on the method of calculation.

Appendix

A From Euler to Grad-Shafranov – the Simplest Way

Let me show how the nonrelativistic version of the stream equation (4.84) can be directly derived from the Euler equation. For the sake of simplicity we consider here only a case of nonrelativistic velocities and small angles \( \Theta = \pi/2 - \theta \) in the vicinity of the equatorial plane.

The \( \theta \)-component of the Euler equation is

\[
\frac{v_r \partial v_\theta}{\partial r} + \frac{v_\theta}{r \partial \theta} - \frac{v_r v_\theta}{r^2} - \frac{v_\theta^2}{r} \cot \theta = -\frac{\nabla_\theta P}{m_p n} - \nabla_\theta \varphi_g. \tag{A.1}
\]

Adding \( v_\varphi \partial v_\varphi / r \partial \theta \) to both sides and adding and subtracting \( v_r \partial v_r / r \partial \theta \) in the left-hand side, we get

\[
\frac{v_r \partial v_\theta}{\partial r} - v_r \frac{v_r}{r \partial \theta} + \nabla_\theta \left( \frac{v_\varphi^2}{2} \right) + \frac{v_r v_\varphi}{r} = \frac{v_\varphi^2}{r} \cot \theta + v_\varphi \frac{\partial v_\varphi}{\partial \theta} - \frac{\nabla_\theta P}{m_p n} - \nabla_\theta \varphi_g. \tag{A.2}
\]

Using the definition of the Bernoulli integral \( E = v_\varphi^2 / 2 + w + \varphi_g \) and the thermodynamic relationship \( dP = m_p ndw - nT ds \), we get for \( E = \text{const} \)

\[
\frac{v_r \partial v_\theta}{\partial r} - v_r \frac{v_r}{r \partial \theta} + \frac{v_r v_\varphi}{r} = \frac{rv_\varphi \sin \theta}{m_p} \frac{\partial}{\partial \theta} \left( rv_\varphi \sin \theta \right) + \frac{T}{m_p} \frac{\partial s}{r \partial \theta}. \tag{A.3}
\]

It follows from the definition (3.1) that

\[
v_r = \frac{1}{2\pi nr^2 \sin \theta} \frac{\partial \Phi}{\partial \theta}, \quad v_\theta = -\frac{1}{2\pi nr^2 \sin \theta} \frac{\partial \Phi}{\partial r}. \quad \tag{A.4}
\]

Now, assuming \( n \approx \text{const} \) (this is the case for a subsonic flow), we obtain

\[
-\frac{1}{4\pi^2 n^2} \left( \frac{\partial \Phi}{\partial \theta} \right) \left[ \frac{\partial^2 \Phi}{\partial r^2} + \frac{\sin \theta}{r^2} \frac{\partial}{\partial \theta} \left( \frac{1}{\sin \theta} \frac{\partial \Phi}{\partial \theta} \right) \right] = \frac{rv_\varphi \sin \theta}{m_p} \frac{\partial}{\partial \theta} \left( rv_\varphi \sin \theta \right) + \frac{T}{m_p} \frac{\partial s}{r \partial \theta}. \tag{A.5}
\]

Finally, dividing both sides by \(- (\partial \Phi / \partial \theta)\), we get (4.84).

Hence, whereas the first term in the l.h.s. of (4.84) does correspond to the component \( v_r \partial v_\theta / \partial r \), and the last term in the r.h.s. (for \( c_s = \text{const} \)) corresponds to the pressure gradient, the role of the term \( \propto L \partial L / \partial \Phi \) proves to be less trivial. It contains both the effective potential gradient and, in fact, component \( v_\theta \partial v_\theta / \partial \theta \). The former is the leading one near the marginally stable orbit \( r \approx 3r_g \), whereas the latter becomes important only when approaching the sonic surface.
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