DISCRETE PERIOD MATRICES AND RELATED TOPICS

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ABSTRACT. We continue our investigation of Discrete Riemann Surfaces with the discussion of the discrete analogs of period matrices, Riemann’s bilinear relations, exponential of constant argument, series and electrical moves. We show that given a refining sequence of critical maps, the discrete period matrix converges to the continuous one.

1. Introduction

The notion of discrete Riemann surfaces was defined in [1, 2]. The interesting paper [3] initiated a renewed interest in the subject. In their paper, R. Costa-Santos and B. McCoy observed numerically that certain pfaffians intervening in a dimer or critical Ising model converge at the thermodynamic limit to a certain (power of a) theta function at the origin. They computed the period matrix needed to define the theta function using discrete holomorphy method. The present paper aims at putting their work in a more general theoretical framework. Most of the results in this paper are a straightforward application of the continuous theory [4, 5], together with the results in [1, 2, 6], to which we refer for details. We define the discrete period matrix, which is twice as large as in the continuous case: the periods of a holomorphic form on the graph and on its dual are in general different, but the continuous limit theorem, given a refining sequence of critical maps, ensures that they converge to the same value. The main tool is the same as in the continuous case, the Riemann bilinear relations. We define the discrete exponential of a constant argument on a critical map and explore its properties as well as its link with series. The tools for that purpose, which needs more investigations in its own right, are the electrical moves.

2. Discrete Riemann Surfaces

We recall in this section basic definitions and results from [2] where the notion of discrete Riemann surfaces was defined. We are interested in discrete surfaces given by a cellular decomposition of dimension 1.
two, where all faces are quadrilaterals (a quad-graph [7]). It defines, up to homotopy and away from the boundary, two dual cellular decompositions $\Gamma$ and $\Gamma^*$. Edges $\Gamma^*_1$ are dual to edges $\Gamma_1$, faces $\Gamma^*_2$ are dual to vertices $\Gamma_0$ and vice-versa. Their union is denoted the double $\Lambda = \Gamma \sqcup \Gamma^*$. A discrete conformal structure on $\Lambda$ is a real positive function $\rho$ on the unoriented edges satisfying $\rho(e^*) = 1/\rho(e)$. It defines a genuine Riemann surface structure on the discrete surface: Choose a length $\delta$ and realize each quadrilateral by a lozenge whose diagonals have a length ratio given by $\rho$. Gluing them together provides a flat riemannian metric with conic singularities at the vertices, hence a conformal structure [8]. It leads to a straightforward discrete version of the Cauchy-Riemann equation. A function on the vertices is discrete holomorphic iff for every quadrilateral $(x, y, x', y') \in \Diamond_2$,

$$f(y') - f(y) = i \rho(x, x') (f(x') - f(x)).$$

We recall elements of de-Rham cohomology, doubled in our context: The complex of chains $C(\Lambda) = C_0(\Lambda) \oplus C_1(\Lambda) \oplus C_2(\Lambda)$ is the vector space span by vertices, edges and faces. It is equipped with a boundary operator $\partial : C_k(\Lambda) \to C_{k-1}(\Lambda)$, null on vertices and fulfilling $\partial^2 = 0$. The kernel $\ker \partial =: Z_k(\Lambda)$ of the boundary operator are the closed chains or cycles. Its image are the exact chains. It provides the dual spaces of forms, called cochains, $C^k(\Lambda) := \text{Hom}(C_k(\Lambda), \mathbb{C})$ with a coboundary $d : C^k(\Lambda) \to C^{k+1}(\Lambda)$ defined by Stokes formula:

$$\int_{(x, x')} df := f(\partial(x, x')) = f(x') - f(x), \quad \int\int_F d\alpha := \oint_{\partial F} \alpha.$$

A cocycle is a closed cochain and we note $\alpha \in Z^k(\Lambda)$. 

0. The vertex dual to a face. 1. Dual edges. 2. The face dual to a vertex.

**Figure 1.** Duality.
Figure 2. The discrete Cauchy-Riemann equation.

Duality of complexes allows us to define a Hodge operator $\ast$ on forms by

\[
\begin{align*}
\ast : C^k(\Lambda) & \rightarrow C^{2-k}(\Lambda) \\
C^0(\Lambda) \ni f & \mapsto \int \int_F \ast f := f(F^*) \\
C^1(\Lambda) \ni \alpha & \mapsto \ast \alpha := -\rho(e^*) \int e^* \alpha, \\
C^2(\Lambda) \ni \omega & \mapsto \ast \omega := (\ast \omega)(x) := \int \int x^* \omega.
\end{align*}
\]

It fulfills $\ast^2 = (-\text{Id})^k$ and defines $\Delta := -d \ast d = -\ast d \ast d$, the usual discrete Laplacian. Its kernel are the harmonic forms. The discrete holomorphic forms are special harmonic forms: a 1-form

\[
\alpha \in C^1(\Lambda) \text{ is holomorphic iff } d\alpha = 0 \text{ and } \ast \alpha = -i\alpha,
\]

that is to say if it is closed and of type $(1,0)$. We will note $\alpha \in \Omega^1(\Lambda)$. A function $f : \Lambda_0 \rightarrow \mathbb{C}$ is holomorphic iff $df$ is holomorphic and we note $f \in \Omega^0(\Lambda)$.

In the compact case, the Hodge theorem orthogonally decomposes forms into exact, coexact and harmonic; harmonic forms are the closed and co-closed ones; and harmonic 1-form are the orthogonal sum of holomorphic and anti-holomorphic ones. We studied discrete Hodge theory for higher dimensional complexes in [1].

We construct a wedge product on $\diamondsuit$ such that $d_{\diamondsuit}$ is a derivation for this product $\wedge : C^k(\diamondsuit) \times C^l(\diamondsuit) \rightarrow C^{k+l}(\diamondsuit)$. It is defined by the
following formulae, for \( f, g \in C^0(\Diamond) \), \( \alpha, \beta \in C^1(\Diamond) \) and \( \omega \in C^2(\Diamond) \):

\[
(f \cdot g)(x) := f(x) \cdot g(x) \quad \text{for } x \in \Diamond_0,
\]

\[
\int_{(x,y)} f \cdot \alpha := \frac{f(x) + f(y)}{2} \int_{(x,y)} (x,y)\alpha \quad \text{for } (x,y) \in \Diamond_1,
\]

\[
\int\int_{(x_1,x_2,x_3,x_4)} \alpha \wedge \beta := \frac{1}{4} \sum_{k=1}^{4} \int_{(x_{k-1},x_k)} (x_k,x_{k+1}) \int_{(x_k,x_{k+1})} \alpha \int_{(x_{k+1},x_k)} \beta - \int_{(x_{k-1},x_k)} \alpha \int_{(x_k,x_{k+1})} \beta
\]

\[
\int\int_{(x_1,x_2,x_3,x_4)} f \cdot \omega := \frac{f(x_1)+f(x_2)+f(x_3)+f(x_4)}{4} \int\int_{(x_1,x_2,x_3,x_4)} \omega
\]

for \((x_1, x_2, x_3, x_4) \in \Diamond_2\).

A form on \( \Diamond \) can be averaged into a form on \( \Lambda \): This map \( A \) from \( C^\bullet(\Diamond) \) to \( C^\bullet(\Lambda) \) is the identity for functions and defined by the following formulae for 1 and 2-forms:

\[
\int_{(x,x')} A(\alpha_\Diamond) := \frac{1}{2} \left( \int_{(x,y)} + \int_{(y,x')} + \int_{(x,y')} + \int_{(y',x')} \right) \alpha_\Diamond,
\]

\[
\int\int_{x^*} A(\omega_\Diamond) := \frac{1}{2} \sum_{k=1}^{d} \int_{(x_k,x_{k+1})} \omega_\Diamond
\]

where notations are made clear in Fig. 3. The map \( A \) is neither injective nor surjective in the non simply-connected case, so we can neither define a Hodge star on \( \Diamond \) nor a wedge product on \( \Lambda \). Its kernel is \( \text{Ker}(A) = \text{Vect}(d_\Diamond \varepsilon) \), where \( \varepsilon \) is the biconstant, \( +1 \) on \( \Gamma \) and \( -1 \) on \( \Gamma^* \). But \( d_\Lambda A = Ad_\Diamond \) so it carries cocycles on \( \Diamond \) to cocycles on \( \Lambda \) and its image are these cocycles of \( \Lambda \) verifying that their holonomies along cycles of \( \Lambda \) only depend on their homology on the combinatorial surface:

Given a 1-cocycle \( \mu \in Z^1(\Lambda) \) with such a property, a corresponding 1-cocycle \( \nu \in Z^1(\Diamond) \) is built in the following way, choose an edge \((x_0, y_0) \in \Diamond_1\); for an edge \((x, y) \in \Diamond_1\) with \( x \) and \( x_0 \) on the same leaf of \( \Lambda \), choose two paths \( \lambda_{x,x_0} \) and \( \lambda_{y_0,y} \) on the double graph \( \Lambda \), from \( x \) to \( x_0 \) and \( y_0 \) to \( y \) respectively, and define

\[
\int_{(x,y)} \nu := \int_{\lambda_{x,x_0}} \mu + \int_{\lambda_{y_0,y}} \mu - \oint_{[\gamma]} \mu
\]

where \( [\gamma] = [\lambda_{x,x_0} + (x_0, y_0) + \lambda_{y_0,y} + (y, x)] \) is the class of the full cycle in the homology of the surface. Changing the base points change \( \mu \) by a multiple of \( d_\Diamond \varepsilon \).
It follows in the compact case that the dimensions of the harmonic forms on ♦ (the kernel of ∆A) modulo dε, as well as the harmonic forms on Λ with same holonomies on the graph and on its dual, are twice the genus of the surface, as expected. Unfortunately, the space \( \text{Im } A = \mathcal{H}^⊥ \oplus \text{Im } d \) is not stable by the Hodge star \(*\). We can nevertheless define holomorphic 1-forms on ♦ but their dimension can be much smaller than in the continuous, namely the genus of the surface. Criticality provides conditions which ensure that the space \(*\text{Im } A\) is “close” to \(\text{Im } A\).

\[\text{(2.4)}\]

\[\text{(2.5)}\]

\[\text{Figure 3. Notations.}\]

We construct an heterogeneous wedge product for 1-forms: with \(\alpha, \beta \in C^1(Λ)\), define \(\alpha \wedge \beta \in C^1(♦)\) by

\[\int\int (x,y,x',y') \alpha \wedge \beta := \int (x,x') \alpha \int (y,y') \beta + \int (y,y') \alpha \int (x',x) \beta.\]  

\[\text{(2.7)}\]

It verifies \(A(\alpha ♦) \wedge A(\beta ♦) = \alpha ♦ \wedge \beta ♦\), the first wedge product being between 1-forms on Λ and the second between forms on ♦. The usual scalar product on compactly supported forms on Λ reads as expected:

\[\langle \alpha, \beta \rangle := \sum_{e \in Λ_1} \rho(e) \left( \int_e \alpha \right) \left( \int_e \beta \right) = \int\int ♦_2 \alpha \wedge *\beta\]

\[\text{(2.8)}\]

3. Period matrix

We use the convention of Farkas and Kra [4], chapter III, to which we refer for details. Consider \((♦, \rho)\) a discrete compact Riemann surface.
3.1. **Intersection number, on \( \Lambda \) and on \( \Diamond \).** In [1], for a given simple (real) cycle \( C \in Z_1(\Lambda) \), we constructed a harmonic 1-form \( \eta_C \) such that \( \oint_A \eta_C \) counts the algebraic number of times \( A \) contains an edge dual to an edge of \( C \). It is the solution of a Neumann problem on the surface cut open along \( C \). It follows from standard homology technique that \( \eta_C \) depends only on the homology class of \( C \) (all the cycles which differ from \( C \) by an exact cycle \( \partial A \)) and can be extended linearly to all cycles; it fulfills, for a closed form \( \theta \),

\[
\oint_C \theta = \int_\Diamond \eta_C \wedge \theta,
\]

and a basis of the homology provides a dual basis of harmonic forms on \( \Lambda \). Beware that if the cycle \( C \in Z_1(\Gamma) \) is purely on \( \Gamma \), then this form \( \eta_C|_{\Gamma} = 0 \) is null on \( \Gamma \).

The **intersection number** between two cycles \( A, B \in Z_1(\Lambda) \) is defined as

\[
A \cdot B := \int_\Diamond \eta_A \wedge \eta_B.
\]

It is obviously linear and antisymmetric, it is an integer number for integer cycles. Let’s stress again that the intersection of a cycle on \( \Gamma \) with another cycle on \( \Gamma \) is always null. A cycle \( C \in Z_1(\Diamond) \) defines a pair of cycles on each graph \( C_\Gamma \in Z_1(\Gamma) \), \( C_{\Gamma^*} \in Z_1(\Gamma^*) \) which are homologous to \( C \) on the surface, composed of portions of the boundary of the faces on \( \Lambda \) dual to the vertices of \( C \). They are uniquely defined if we require that they lie “to the left” of \( C \) as shown in Fig.4. By the procedure (2.6) applied to \( \eta_C, \eta_C_{\Gamma^*} \), we construct a 1-cocycle \( \eta_C \in Z^1(\Diamond) \) unique up to \( d\varepsilon \), and since \( \forall \theta, d\varepsilon \wedge \theta = 0 \), Eq. (3.2) defines an intersection number on \( Z_1(\Diamond) \). Unlike the intersection number on \( \Lambda \), this one has all the usual expected properties. In particular Eq. (3.2) holds for \( A, B \in Z_1(\Diamond) \).

3.2. **Canonical dissection, fundamental polygon.** The complex \( \Diamond \) being connected, consider a maximal tree \( T \subset \Diamond_1 \), that is to say \( T \) is a \( \mathbb{Z}_2 \)-homologically trivial chain and every edge added to \( T \) forms a cycle. A **canonical dissection** or cut-system \( \mathcal{N} \) of the genus \( g \) discrete Riemann surface \( \Diamond \) is given by a set of oriented edges \( (\varepsilon_k)_{1 \leq k \leq 2g} \) such that the cycles \( \mathcal{N} \subset (T \cup \varepsilon_k) \) form a basis of the homology group \( H_1(\Diamond) \) verifying, for \( 1 \leq k, \ell \leq g \)

\[
\mathcal{N}_k \cdot \mathcal{N}_\ell = 0, \quad \mathcal{N}_{k+g} \cdot \mathcal{N}_{\ell+g} = 0, \quad \mathcal{N}_k \cdot \mathcal{N}_{\ell+g} = \delta_{k,\ell}.
\]

\[
(3.3)
\]
They actually form a basis of the fundamental group $\pi_1(\diamondsuit)$ and the defining relation among them is (noted multiplicatively)

$$
\prod_{k=1}^{g} N_k N_{k+g} N_k^{-1} N_{k+g}^{-1} = 1.
$$

The construction of such a basis is standard and we won’t repeat the procedure. What is less standard is the interpretation of Eq. (3.4) in terms of the boundary of a fundamental domain, discretization introduces some subtleties.

Considering $T \cup e_k$ as a rooted graph, we can prune it of all its pending branches, leaving a simple closed loop $\mathcal{N}_k$, attached to the origin $O$ by a simple path $\lambda_k$ (see Fig. 5), yielding the cycle $\mathcal{N}_k$. These three cycles are deformation retract of one another, $\mathcal{N}_k^{-1} \subset \mathcal{N}_k \subset T \cup e_k$ hence are equal in homology.

In the continuous case, a basis of the homology can be realized by $2g$ simple arcs, transverse to one another and meeting only at the base point. It defines an isometric model of the surface as a fundamental domain homeomorphic to a disc and bordered by $4g$ arcs to identify pairwise. In the discrete case, by definition, the set $\diamondsuit \setminus \mathcal{N}$ of the cellular
complex minus the edges taking part into the cycles basis is homeomorphic to a disc hence the surface is realized as a polygonal fundamental domain $\mathcal{M}$ whose boundary edges are identified pairwise.

But it is sometimes impossible to choose a basis of the homology verifying (3.3) by simple discrete cycles which are transverse to one another. If the path $\lambda_k$ is not empty, the cycle $\mathcal{N}_k$ is not even simple. Moreover, some edges may belong to several cycles. In this case, the edges on the boundary of this fundamental polygon can not be assigned a unique element of the basis or its inverse, and therefore can not be grouped into only $4g$ continuous paths to identify pairwise but more than $4g$.

In fact, the information contained into the basis $\mathcal{N}$ is more than simply this polygon, the set of edges composing the concatenated cycle (3.5)  
$$(\mathcal{N}_1, \mathcal{N}_{g+1}, \mathcal{N}_1^{-1}, \mathcal{N}_{g+1}^{-1}, \mathcal{N}_2, \ldots, \mathcal{N}_g^{-1}, \mathcal{N}_{2g}^{-1})$$
encodes a cellular complex $\mathcal{M}_+$ which is not a combinatorial surface and consists of the fundamental polygon $\mathcal{M}$ plus some dangling trees, corresponding to the edges which belong to more than one cycle or participate more than once in a cycle (the paths $\lambda_k$), as exemplified in Fig. 4. By construction, the edge $e_k$ belongs to the cycle $\mathcal{N}_k$ only, hence these trees are in fact without branches, simple paths whose only leaf is the base point $O$. To retrieve the surface, the edges of this structure $\mathcal{M}_+$ are identified group-wise, an edge participating $k$ times in cycles will have $[k/2] + 2$ representatives to identify together, two on the fundamental polygon and the rest as edges of dangling trees.

Eliminating repetition, that is to say looking at (3.5) not as a sequence of edges but as a simplified cycle (or a simplified word in edges), thins $\mathcal{M}_+$ into $\mathcal{M}$, pruning away the dangling paths. The fundamental polygon boundary loses its structure as $4g$ arcs to be identified pairwise, in general a basis cycle will be disconnected around the fundamental
Figure 6. Three different fundamental polygons of a decomposition of the torus ($g = 1$) by three quadrilaterals: 1. The standard fundamental domain where the $4g$ paths are not adapted to $\diamondsuit$. 2. $M_+$ is composed of edges of $\diamondsuit$ composing $4g$ arcs (which may have portions in common) to identify pairwise, each edge corresponds to an element of the basis $\mathscr{N}$ or its inverse, except for edges of “dangling trees” which are associated with two such elements. 3. $M$ is composed of edges of $\diamondsuit$ composing more than $4g$ arcs to identify pairwise, there is no correspondence with a basis of cycles.

This peculiarity gives a more complex yet well defined meaning to the contour integral formula for a 1-form $\theta$ defined on the boundary edges of $M_+$,

$$\oint_{\partial M} \theta = \sum_{k=1}^{2g} \oint_{\mathscr{N}_k} \theta + \oint_{\mathscr{N}_k^{-1}} \theta.$$  \hspace{1cm} (3.6)

This basis gives rise to cycles $\mathscr{N}^\Gamma$ and $\mathscr{N}^{\Gamma^*}$ whose homology classes form a basis of the group for each respective graph, that we compose into $\mathbf{N}^\Lambda$ defined by

$$\begin{align*}
\mathbf{N}_k^\Lambda &= \mathbf{N}_k^\Gamma, \\
\mathbf{N}_{k+2g}^\Lambda &= \mathbf{N}_{k+2g}^{\Gamma^*}, \\
\mathbf{N}_{k+g}^\Lambda &= \mathbf{N}_{k+g}^{\Gamma^*}, \\
\mathbf{N}_{k+3g}^\Lambda &= \mathbf{N}_{k+3g}^{\Gamma}.
\end{align*}$$  \hspace{1cm} (3.7)

for $1 \leq k \leq g$ so that while the intersection numbers matrix on $\diamondsuit$ is given by the $2g \times 2g$ matrix

$$\begin{align*}
(\mathbf{N}_k \cdot \mathbf{N}_\ell)_{k,\ell} &= \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}.
\end{align*}$$  \hspace{1cm} (3.8)
the intersection numbers matrix on $\Lambda$ is the $4g \times 4g$ matrix with the same structure

\[ (\mathcal{N}^\Lambda_k \cdot \mathcal{N}^\Lambda_{\ell})_{k,\ell} = \begin{pmatrix} \Gamma & \Gamma^* & \Gamma & \Gamma \\ 0 & 0 & I & 0 \\ 0 & 0 & 0 & I \\ -I & 0 & 0 & \Gamma^* \\ 0 & -I & 0 & 0 \end{pmatrix} \]

\[(3.9)\]

3.3. Basis of harmonic forms, basis of holomorphic forms. We define $\alpha^\Lambda$, the basis of real harmonic 1-forms, dual to the homology basis $\mathcal{N}^\Lambda$,

\[ \alpha^\Lambda_k := \eta_{\mathcal{N}^\Lambda_{k+2g}} \quad \text{and} \quad \alpha^\Lambda_{k+2g} := -\eta_{\mathcal{N}^\Lambda_k} \quad \text{for } 1 \leq k \leq 2g \]

which verify

\[ \oint_{\mathcal{N}^\Lambda_k} \alpha_{\ell} = \delta_{k,\ell}, \]

\[ \oint_{\mathcal{N}^\Lambda_{k+2g}} \alpha_{\ell+2g} = \delta_{k,\ell}, \]

\[(3.11)\]

and dually, the intersection matrix elements are given by

\[ \mathcal{N}^\Lambda_k \cdot \mathcal{N}^\Lambda_{\ell} = \int \int_\Diamond \alpha^\Lambda_k \wedge \alpha^\Lambda_{\ell} = (\alpha^\Lambda_k, - \ast \alpha^\Lambda_{\ell}). \]

\[(3.12)\]

On $\Diamond$, the elements $\alpha^\Diamond_k := \eta_{\mathcal{N}^\Diamond_{k+g}}$ and $\alpha^\Diamond_{k+g} := -\eta_{\mathcal{N}^\Diamond_k}$ for $1 \leq k \leq g$, defined up to $d\varepsilon$, verify $A(\alpha^\Diamond_k) = \alpha^\Lambda_k + \alpha^\Lambda_{k+g}$, $A(\alpha^\Diamond_{k+g}) = \alpha^\Lambda_{k+2g} + \alpha^\Lambda_{k+3g}$ and form a basis of the cohomology on $\Diamond$ dual to $\mathcal{N}$ as well,

\[ \alpha^\Diamond_k := \eta_{\mathcal{N}^\Diamond_{k+g}} \quad \text{and} \quad \alpha^\Diamond_{k+g} := -\eta_{\mathcal{N}^\Diamond_k} \quad \text{for } 1 \leq k \leq g, \]

\[(3.13)\]

they fulfill the first identity in Eq.(3.12) but the second is meaningless in general since $\ast$ can not be defined on $\Diamond$. We will drop the mention $\Lambda$ when no confusion is possible.

**Proposition 3.1.** The matrix of inner products on $\Lambda$,

\[ (\alpha_k, \alpha_{\ell})_{k,\ell} = \int \int_\Diamond \alpha_k \wedge \ast \alpha_{\ell} = \begin{cases} + \int_{\mathcal{N}^\Lambda_{k+2g}} \ast \alpha_{\ell}, & 1 \leq k \leq 2g, \\ - \int_{\mathcal{N}^\Lambda_{k-2g}} \ast \alpha_{\ell}, & 2g < k \leq 4g. \end{cases} =: \begin{pmatrix} A & D \\ B & C \end{pmatrix} \]

is a real symmetric positive definite matrix.
Proof 3.1. It is real because the forms are real, and symmetric because the scalar product (2.8) is skew symmetric. Definition Eq. (3.10) and Eq. (3.1) lead to the integral formulae. Positivity follows from the bilinear relation Eq. (4.2): for $\theta = \sum_{k=1}^{4g} \xi_k \alpha_k$, with $\xi_k \in \mathbb{C}$, $\sum_{k=1}^{4g} \xi_k^2 > 0$, 

$$\| \theta \|^2 = \sum_{j=1}^{2g} \left[ \int_{\mathbb{R}_j} \theta \int_{\mathbb{R}_{2g+j}} \ast \theta - \int_{\mathbb{R}_{2g+j}} \theta \int_{\mathbb{R}_j} \ast \theta \right]$$

$$= \sum_{k,\ell=1}^{4g} \xi_k \xi_\ell \sum_{j=1}^{2g} \left[ \int_{\mathbb{R}_j} \alpha_k \int_{\mathbb{R}_{2g+j}} \ast \alpha_\ell - \int_{\mathbb{R}_{2g+j}} \alpha_k \int_{\mathbb{R}_j} \ast \alpha_\ell \right]$$

$$= \sum_{k,\ell=1}^{4g} \xi_k \xi_\ell (\alpha_k, \alpha_\ell) > 0.$$ (3.15)

The form $\alpha_k$ is supported by only one of the two graphs $\Gamma$ or $\Gamma^*$, the form $\ast \alpha_k$ is supported by the other one, and the wedge product $\theta_\Gamma \wedge \theta_{\Gamma^*} = 0$ is null for two 1-forms supported by the same graph. Therefore the matrices $A$ and $C$ are $g \times g$-block diagonal and $B$ is anti-diagonal.

(3.16)

$$A = \begin{pmatrix} A_\Gamma & 0 \\ 0 & A_{\Gamma^*} \end{pmatrix}, \quad B = \begin{pmatrix} 0 & B_{\Gamma, \Gamma^*} \\ B_{\Gamma^*, \Gamma} & 0 \end{pmatrix}, \quad C = \begin{pmatrix} C_{\Gamma^*} & 0 \\ 0 & C_{\Gamma} \end{pmatrix}.$$  

The matrices of intersection numbers (3.9) and of inner products differ only by the Hodge star $\ast$. Because $\ast$ preserves harmonic forms and the inner product, we get its matrix representation in the basis $\alpha$,

(3.17)

$$\ast = \begin{pmatrix} -D & A \\ -C & B \end{pmatrix}$$

and because $\ast^2 = -1$,

(3.18) $B^2 - C \cdot A + I = 0$

(3.19) $A \cdot B = {\ast} B \cdot A$

(3.20) $C \cdot {\ast} B = B \cdot C.$
forms modulo \(d\varepsilon\), to which the set \((\alpha_k^\diamondsuit)\) belong:

\[
(\alpha^\diamondsuit, \beta^\diamondsuit) := (A(\alpha^\diamondsuit), A(\beta^\diamondsuit))
\]

\[
= \sum_{(x,y,x',y') \in \mathcal{O}_2, \rho=\rho(x,x'), \rho'=\rho(y,y')} t \begin{pmatrix}
\int_{(x,y)} \alpha & +\rho+\rho^* & -\rho-\rho^* & -\rho+\rho^* \\
\int_{(y,x')} \alpha & +\rho-\rho^* & -\rho+\rho^* & -\rho+\rho^* \\
\int_{(x',y')} \alpha & -\rho-\rho^* & +\rho+\rho^* & +\rho+\rho^* \\
\int_{(y',x')} \alpha & -\rho+\rho^* & -\rho-\rho^* & +\rho+\rho^*
\end{pmatrix} \begin{pmatrix}
\int_{(x,y)} \beta \\
\int_{(y,x')} \beta \\
\int_{(x',y')} \beta \\
\int_{(y',x')} \beta
\end{pmatrix}.
\]

and it yields

\[
(\alpha_k^\diamondsuit, \alpha_{\ell}^\diamondsuit)_{k,\ell} = \begin{pmatrix}
A_{1\Gamma} + A_{1\Gamma^*} & tB_{1\Gamma^*} + tB_{1\Gamma} \\
B_{1\Gamma^*} + B_{1\Gamma} & C_\Gamma + C_{\Gamma^*}
\end{pmatrix},
\]

which, in general, can not be understood as the periods of a set of forms on \(\diamondsuit\) along the basis \(\mathcal{K}\).

Let’s decompose the space of harmonic forms into two orthogonal supplements,

\[
\mathcal{H}^1(\Lambda) = \mathcal{H}^1_\parallel \oplus \mathcal{H}^1_\perp
\]

where the first vector space are the harmonic forms whose holonomies on one graph are equal to their holonomies on the dual, that is to say

\[
\mathcal{H}^1_\parallel := \text{Vect} (\alpha_k + \alpha_{k+g}, 1 \leq k \leq g \text{ or } 2g < k \leq 3g).
\]

Definition (3.10) and Eq. (3.1) imply that

\[
\mathcal{H}^1_\perp = \text{Vect} (*\alpha_k - *\alpha_{k+g}, 1 \leq k \leq g \text{ or } 2g < k \leq 3g).
\]

These elements in the basis \((\alpha_k + \alpha_{k+g}, \alpha_k - \alpha_{k+g})\) for \(1 \leq k \leq g\) and \(2g < k \leq 3g\), are represented by the following invertible matrix:

\[
\begin{pmatrix}
I & 0 & tB_{1\Gamma^*} - tB_{1\Gamma} & A_{1\Gamma} - A_{1\Gamma^*} \\
0 & I & C_{1\Gamma} - C_{1\Gamma^*} & B_{1\Gamma^*} - B_{1\Gamma} \\
0 & 0 & tB_{1\Gamma^*} + tB_{1\Gamma} & A_{1\Gamma} + A_{1\Gamma^*} \\
0 & 0 & C_{1\Gamma} + C_{1\Gamma^*} & B_{1\Gamma^*} + B_{1\Gamma}
\end{pmatrix}.
\]

It implies in particular that the lower right \(g \times g\) block is invertible, therefore so is Eq. (3.21).

3.4. Period matrix.

**Proposition 3.2.** The matrix \(\Pi = C^{-1} \cdot (i - B)\) is the period matrix of the basis of holomorphic forms

\[
\zeta_k := (i - *) \sum_{\ell=1}^{2g} C_{k,\ell}^{-1} \alpha_{\ell+2g}
\]
in the canonical dissection $\mathfrak{N}$, that is to say

$$
\oint_{\mathfrak{N}_k} \zeta_\ell = \begin{cases}
\delta_{k,\ell} & \text{for } 1 \leq k \leq 2g, \\
\Pi_{k-2g,\ell} & \text{for } 2g < k \leq 4g,
\end{cases}
$$

and $\Pi$ is symmetric, with a positive definite imaginary part.

The proof is essentially the same as in the continuous case [4] and we include it for completeness.

**Proof 3.2.** Let $\omega_j := \alpha_j + i \alpha_j$ for $1 \leq j \leq 4g$. These holomorphic forms fulfill

$$
P_{k,j} := \frac{1}{2}(\omega_k, \omega_j) = (\alpha_k, \alpha_j) + i (\alpha_k, - \alpha_j)
$$

$$
= \begin{cases}
-i \int_{\mathfrak{N}_{j+2g}} \omega_k, & 1 \leq j \leq 2g, \\
i \int_{\mathfrak{N}_{j-2g}} \omega_k, & 2g < j \leq 4g.
\end{cases}
$$

$P$ is the period matrix of the forms $(\omega)$ in the homology basis $\mathfrak{N}$. The first $2g$ forms $(\omega_j)_{1 \leq j \leq 2g}$ are a basis of holomorphic forms. It has the right dimension and they are linearly independent:

$$
\sum_{j=1}^{2g} (\lambda_j + i \mu_j)(\alpha_j + i \alpha_j) = \sum_{j=1}^{2g} \left( (\lambda_j + \sum_{k=1}^{2g} \mu_k B_{j,k}) \alpha_j + \sum_{k=1}^{2g} \mu_k C_{j,k} \alpha_{2g+j} \right)
$$

$$
+ i \sum_{j=1}^{2g} \left( (\mu_j + \sum_{k=1}^{2g} \lambda_k B_{j,k}) \alpha_j + \sum_{k=1}^{2g} \lambda_k C_{j,k} \alpha_{2g+j} \right)
$$

is null, for $\lambda, \mu \in \mathbb{R}$ only when $\lambda = \mu = 0$ because $C$ is positive definite. Similarly for the last $2g$ forms. The change of basis $i C^{-1}$ on them provides the basis of holomorphic forms $(\zeta)$. The last $2g$ rows of $P$ is the $2g \times 4g$ matrix $(B - i I, C)$ hence the periods of $(\zeta)$ in $\mathfrak{N}$ are given by $(I, \Pi)$. \hfill \Box

The first identity in Eq.(3.27) uniquely defines the basis $\zeta$ and a holomorphic 1-form is completely determined by whether its periods on the first $2g$ cycles of $\mathfrak{N}$, or their real parts on the whole set.

Notice that because $C$ is $g \times g$ block diagonal and $B$ is anti-diagonal, $\Pi$ is decomposed into four $g \times g$ blocks, the two diagonal matrices form $i C^{-1}$ and are pure imaginary, the other two form $-C^{-1} \cdot B$ and are real.

$$
\Pi = \begin{pmatrix}
\Pi_{\ast \ast} & \Pi_{\ast r} \\
\Pi_{r \ast} & \Pi_{r r}
\end{pmatrix} = \begin{pmatrix}
i C_{\ast \ast}^{-1} & -C_{\ast r}^{-1} \cdot B_{r \ast} \\
-C_{r \ast}^{-1} \cdot B_{\ast r} & i C_{r r}^{-1}
\end{pmatrix}.
$$
Therefore the holomorphic forms $\zeta_k$ are real on one graph and pure imaginary on its dual,

\begin{align}
1 \leq k \leq g & \Rightarrow \zeta_k \in C^1_R(\Gamma) \oplus iC^1_R(\Gamma^*) \\
g < k \leq 2g & \Rightarrow \zeta_k \in C^1_R(\Gamma^*) \oplus iC^1_R(\Gamma).
\end{align}

We will call

\begin{equation}
\Pi_{\Gamma} = \Pi_r + \Pi_i
\end{equation}

the period matrix on the graph $\Gamma$ the sum of the real periods of $\zeta_k$, $1 \leq k \leq g$, on $\Gamma$, with the associated pure imaginary periods on the dual $\Gamma^*$, and similarly for $\zeta_k$, $g < k \leq 2g$, the period matrix on $\Gamma^*$.

It is natural to ask how close $\Pi_{\Gamma}$ and $\Pi_{\Gamma^*}$ are from one another, and whether their mean can be given an interpretation. Criticality [1, 2] redefined in Sec. 5, answers partially the issue:

**Theorem 3.1.** In the genus one critical case, the period matrices $\Pi_{\Gamma}$ and $\Pi_{\Gamma^*}$ are equal to the period matrix $\Pi_{\Sigma}$ of the underlying surface $\Sigma$. For higher genus, given a refining sequence $(\kappa_k, \rho_k)$ of critical maps of $\Sigma$, the discrete period matrices $\Pi_{\Gamma_k}$ and $\Pi_{\Gamma^*_k}$ converge to the period matrix $\Pi_{\Sigma}$.

**Proof**. The genus one case is postponed to Sec. 3.3. The continuous limit comes from techniques in [1, 2] to be developed in [3] which prove that, given a refining sequence of critical maps, any holomorphic function can be approximated by a sequence of discrete holomorphic functions. Taking the real parts, this implies as well that any harmonic function can be approximated by discrete harmonic functions. In particular, the discrete solutions $f_k$ to a Dirichlet or Neumann problem on a simply connected set converge to the continuous solution $f$ because the latter can be approximated by discrete harmonic functions $g_k$ and the difference $f_k - g_k$ being harmonic and small on the boundary, converge to zero. In particular, each form in the basis $(\alpha^\kappa)$, provides a solution to the Neumann problem Eq. (3.13) and a similar procedure, detailed afterwards, define a converging sequence of forms $\zeta_\ell^\kappa$, yielding the result.

We can try to replicate the work done on $\Lambda$ on the graph $\downarrow$. A problem is that $A_{\Gamma} + A_{\Gamma}$ and $C_{\Gamma} + C_{\Gamma}$ need not be positive definite. Moreover, the Hodge star $*$ doesn’t preserve the space $(A(\alpha^\kappa))$ of harmonic forms with equal holonomies on the graph and on its dual, so we can not define the analogue of $\alpha + i \ast \alpha$ on $\downarrow$. We first investigate what happens when we can partially define these analogues:

Assume that for $2g < k \leq 3g$, the holonomies of $\ast \alpha_k$ on $\Gamma$ are equal to the holonomies of $\ast \alpha_{k+g}$ on $\Gamma^*$, that is to say $C_{\Gamma} = C_{\Gamma^*} =: \frac{1}{2} C_\diamond$ and
\textbf{3.34} \quad \zeta_k^\Diamond = \sum_{\ell=1}^{g} C_{k,\ell}^{-1} \left( \alpha_{k+g} - i \beta_{k+g}^\Diamond \right), \quad 1 \leq k \leq g

verify \( A(\zeta_k^\Diamond) = \frac{n_k + n_{k+g}}{2} \) and have periods on \( \mathbb{H}^\Diamond \) given by the identity for the first \( g \) cycles and the following \( g \times g \) matrix, mean of the period matrices on the graph and on its dual:

\textbf{(3.35)} \quad \Pi^\Diamond = C_{C,\Diamond}^{-1}(i - B_{\Diamond}) = \frac{\Pi_{\Gamma} + \Pi_{\Gamma^*}}{2}.

The same reasoning applies when the periods of the forms \(*\alpha_k\) on the graph and on its dual are not equal but close to one another. In the context of refining sequences, we said that the basis \((\alpha_{\ell}^\Diamond)\), converges to the continuous basis of harmonic forms defined by the same Neumann problem Eq. (3.13). Therefore

\textbf{(3.36)} \quad C_{\Gamma} - C_{\Gamma^*} = o(1), \quad B_{\Gamma^*} - B_{\Gamma^*} = o(1).

A harmonic form \( \nu_{k+g} = o(1) \) on \( \Gamma^* \) can be added to \(*\alpha_{k+g}\) such that there exists \( \beta_{k-g}^\Diamond \in Z^1(\Diamond) \) with \( A(\beta_{k-g}^\Diamond) = *\alpha_{k+g} + \nu_{k+g} \), yielding forms \( \zeta_k^\Diamond \), verifying \( A(\zeta_k^\Diamond) = \frac{1}{2}(\zeta_k + \zeta_{k+g}) + o(1) \) and whose period matrix is \( \Pi^\Diamond + o(1) \). Since the periods of \( \alpha_k \) converge to the same periods as its continuous limit, this period matrix converges to the period matrix \( \Pi_{\Sigma} \) of the surface. Which is the claim of Th. 3.1.

In the paper [3], R. Costa-Santos and B. McCoy define a period matrix on a special cellular decomposition \( \Gamma \) of a surface by squares. They don’t consider the dual graph \( \Gamma^* \). Their period matrix is equal to one of the two diagonal blocks of the double period matrix we construct in this case. They don’t have to consider the off-diagonal blocks because the problem is so symmetric that their period matrix is pure imaginary.

\textbf{3.5. Genus one.} Criticality solves partially the problem of having two different \( g \times g \) period matrices instead of one since they converge to one another in a refining sequence. However, on a genus one critical torus, the situation is simpler: The overall curvature is null and a critical map is everywhere flat. Therefore the cellular decomposition is the quotient of a periodic cellular decomposition of the plane by two independant periods. They can be normalized to \((1, \tau)\). The continuous
period matrix is the $1 \times 1$-matrix $\tau$. A basis of the two dimensional holomorphic 1-forms is given by the real and imaginary parts of $dZ$ on $\Gamma$ and $\Gamma^*$ respectively, and the reverse. The discrete period matrix is the $2 \times 2$ matrix $\begin{pmatrix} \text{Im} \ \tau & \text{Re} \ \tau \\ \text{Re} \ \tau & \text{Im} \ \tau \end{pmatrix}$ and the period matrices on the graph and on its dual are both equal to the continuous one.

For illustration purposes, the whole construction, of a basis of harmonic forms, then projected onto a basis of holomorphic forms, yielding the period matrix, can be checked explicitly on the critical maps of the genus 1 torus decomposed by square or triangular/hexagonal lattices:

Consider the critical square (rectangular) lattice decomposition of a torus $\hat{\diamondsuit} = (\mathbb{Z}e^{i\theta} + \mathbb{Z}e^{-i\theta})/(2pe^{i\theta} + 2qe^{-i\theta})$, with horizontal parameter $\rho = \tan \theta$ and vertical parameter its inverse. Its modulus is $\tau = \frac{2}{p}e^{2i\theta}$. The two dual graphs $\Gamma$ and $\Gamma^*$ are isomorphic. An explicit harmonic form $\alpha_1^\Gamma$ is given by the constant $\frac{1}{2p}$ on horizontal and downwards edges of the graph $\Gamma$ and 0 on all the other edges. Its holonomies are 1 and 0 on the $p$, resp. $q$ cycles. Considering $\frac{1}{2}q$ and the dual graph, we construct in the same fashion $\alpha_2^\Gamma, \alpha_1^{\Gamma^*}, \alpha_2^{\Gamma^*}$. The matrix of inner products is

$$\begin{pmatrix} \alpha_k, \alpha_\ell \end{pmatrix}_{k,\ell} = \frac{1}{\sin 2\theta} \begin{pmatrix} \frac{q}{p} & \frac{q}{p} \\ \cos 2\theta & \cos 2\theta \end{pmatrix} \begin{pmatrix} \frac{\cos 2\theta}{\cos 2\theta} & \frac{\cos 2\theta}{\cos 2\theta} \\ \frac{\cos 2\theta}{\cos 2\theta} & \frac{\cos 2\theta}{\cos 2\theta} \end{pmatrix}$$

using $\frac{\rho + 1/\rho}{2} = 1/\sin 2\theta$ and $\frac{\rho - 1/\rho}{2} = -1/\tan 2\theta$ so that the period matrix is

$$\Pi = \frac{q}{p} \begin{pmatrix} i \sin 2\theta & \cos 2\theta \\ \cos 2\theta & i \sin 2\theta \end{pmatrix}.$$

Therefore there exists a holomorphic form which has the same periods on the graph and on its dual, it is the average of the two half forms of Eq. (3.27) and its periods are $(1, \frac{2}{p}e^{2i\theta})$ along the $p$, resp. $q$ cycles, yielding the continuous modulus. This holomorphic form is simply the normalized fundamental form $\frac{dz}{pe^{-i\theta}}$.

In the critical triangular/hexagonal lattice, we just point out to the necessary check by concentrating on a tile of the torus, composed of two triangles, pointing up and down respectively. We show that there exists an explicit holomorphic form which has the same shift on the graph and on its dual, along this tile. Let $\rho_-, \rho_\backslash$, and $\rho_\backslash$ the three parameters around a given triangle. Criticality occurs when $\rho_- \rho_\backslash + \rho_\backslash \rho_\backslash + \rho_\backslash \rho_- = 1$. The form which is 1 on the rightwards and South-West edges and 0 elsewhere is harmonic on the triangular lattice. Its pure imaginary
companion on the dual hexagonal lattice exhibits a shift by $i \rho$ in the horizontal direction and $i (\rho + \rho_-)$ in the North-East direction along the tile. Dually, on the hexagonal lattice, the form which is $\rho \rho_-$ along the North-East and downwards edges and $1 - \rho \rho_-$ along the South-East edges, is a harmonic form. Its shift in the horizontal direction is 1, in the North-East direction 0, and its pure imaginary companion on the triangular lattice exhibits a shift by $i \rho$ in the horizontal direction and $i (\rho + \rho_-)$ in the North-East direction along the tile as before. Hence their sum is a holomorphic form with equal holonomies on the triangular and hexagonal graphs and the period matrix it computes is the same as the continuous one. This simply amounts to pointing out that the fundamental form $dz$ can be explicitly expressed in terms of the discrete conformal data.

4. Bilinear relations

**Proposition 4.1.** Given a canonical dissection $\mathbb{R}$, for two closed forms $\theta, \theta' \in Z^1(\diamond)$,

\[
\int \int_\diamond \theta \wedge \theta' = \sum_{j=1}^{g} \left( \oint_{R_j} \theta \oint_{R_{j+g}} \theta' - \oint_{R_{j+g}} \theta \oint_{R_j} \theta' \right);
\]

for two closed forms $\theta, \theta' \in Z^1(\Lambda)$,

\[
\int \int_\diamond \theta \wedge \theta' = \sum_{j=1}^{2g} \left( \oint_{R_j} \theta \oint_{R_{j+2g}} \theta' - \oint_{R_{j+2g}} \theta \oint_{R_j} \theta' \right).
\]

**Proof [4.1].** Each side is bilinear and depends only on the cohomology classes of the forms. Decompose the forms onto the cohomology basis $(\alpha_k)$. On $\Lambda$, use Eq (3.12) for the LHS and the duality property Eq. (3.11) for the RHS. On $\diamond$, use their counterparts. Notice that for a harmonic form $\theta \in H^1(\Lambda)$, the form $*\theta$ is closed as well, therefore its norm is given by

\[
\theta \in H^1(\Lambda) \implies \|\theta\|^2 = \sum_{j=1}^{2g} \left( \oint_{R_j} \theta \oint_{R_{j+2g}} *\theta - \oint_{R_{j+2g}} \theta \oint_{R_j} *\theta \right).
\]

5. Criticality and exponentials

5.1. **Integration at criticality.** We call a local map $Z : \diamond \supset U \to \mathbb{C}$ from a connected, simply connected subcomplex to the euclidean plane, **critical** iff it is locally injective, orientation preserving and the faces of $\diamond$ are mapped to rhombi such that the ratio of the diagonal euclidean lengths is given by $\rho$. The common length $\delta$ of these rhombi
is a characteristic of the map. We showed in [1, 2] that a converging sequence of discrete holomorphic forms, on a refining (the lengths $\delta$ go to zero and faces don’t collapse) sequence of conformal maps of the same Riemann surface, converge to a genuine holomorphic form. The converse is also true, every holomorphic form on a Riemann surface can be approximated by a converging sequence of discrete holomorphic forms given a refining sequence of critical maps. And such a refining sequence exists for every Riemann surface.

Of course, a very natural atlas of maps of a discrete Riemann surface is given by the applications from every face, considered as a subcomplex, to a well shaped rhombus in $\mathbb{C}$. Such an application is only well defined if two edges of the same face are not identified. They then form an atlas of the combinatorial surface and provide it with a genuine structure of Riemann surface, namely a riemannian flat metric with conic singularities [8]. A discrete Riemann surface is called critical or critically flat if its conic singularities are all multiple of $2\pi$.

We’ve seen that given a function $f$ and a 1-form $dZ$, the residue theorem implies that if they are both holomorphic then $fdZ$ is a closed 1-form. The key point about criticality is that, given a critical map $Z$ and a discrete holomorphic function $f$, one can construct the holomorphic 1-form $f dZ$

$$\int_{(x,y)} f dZ = \frac{f(x) + f(y)}{2} (Z(y) - Z(x)). \quad (5.1)$$

Reciprocally, every holomorphic 1-form is locally of such a form, unique up to $\varepsilon$. The change of map acts as expected (see Prop. (7.3)) so for a critical discrete Riemann surface, where the conic singularity angles are multiple of $2\pi$, $f dZ$ can be continued as a well defined 1-form on the universal covering of $\Diamond$ wherever $f$ is defined. Be careful that for another holomorphic form $\theta \in \Omega(\Diamond)$, the 1-form $f \theta \in Z^1(\Diamond)$ is only closed but not holomorphic in general.

5.2. **Exponential.** As an interesting example of discrete holomorphic function we present here the discrete exponential of a constant argument. It is the first step in trying to define the exponential map needed to provide Abel’s correspondence between divisors and meromorphic functions.

Different attempts have been made to define a discrete exponential, see [9, 10] and references therein.

**Definition 5.1.** Let $Z$ a critical map and $O \in \Lambda_0$ such that $Z(O) = 0$, the origin. For $\lambda \in \mathbb{C}$, define the holomorphic function exponential
Exp(λ) ∈ Ω^0(Λ), denoted \( z \mapsto \text{Exp}(\lambda; z) \), by

\[
\begin{align*}
\text{Exp}(\lambda; O) &= 1 \\
\text{dExp}(\lambda; z) &= \lambda \text{Exp}(\lambda; z) \, dz.
\end{align*}
\]

(5.2)

**Proposition 5.2.** If \(|\lambda| \neq 2/\delta\), \(\text{Exp}(\lambda;)\) is a well defined holomorphic function.

**Proof 5.2.** If \((x, y) \in \Diamond_1\) is an edge, Eq.(5.2) reads

\[
\text{Exp}(\lambda; y) - \text{Exp}(\lambda; x) = \lambda \frac{\text{Exp}(\lambda; y) + \text{Exp}(\lambda; x)}{2} \left( Z(y) - Z(x) \right)
\]

so that

\[
\text{Exp}(\lambda; y) = \frac{2 + \lambda (Z(y) - Z(x))}{2 - \lambda (Z(y) - Z(x))} \text{Exp}(\lambda; x).
\]

(5.3)

As the map \(Z\) is critical, on the face \((x, y, x', y') \in \Diamond_2\), \(Z(y') - Z(x) = Z(x') - Z(y)\) as well as \(Z(y') - Z(x) = Z(x') - Z(y)\) so that the product of the four terms along the face

\[
\frac{2 + \lambda(y - x)}{2 - \lambda(y - x)} \frac{2 + \lambda(x' - y)}{2 - \lambda(x' - y)} \frac{2 + \lambda(y' - x')}{2 - \lambda(y' - x')} = 1
\]

(5.4)

where we wrote \(Z(z)\) as \(z\) for readability. So \(\text{Exp}(\lambda;)\) is a well defined function on the simply connected sub-complex \(U \subset \Diamond\), its value at a point is the product of the contributions of each edge along a given path connecting it to the origin, and the result doesn’t depend on the path. It is also holomorphic as

\[
\frac{\text{Exp}(\lambda; y') - \text{Exp}(\lambda; y)}{\text{Exp}(\lambda; x') - \text{Exp}(\lambda; x)} = \frac{2 + \lambda(y' - x)}{2 - \lambda(y' - x)} \frac{2 + \lambda(y - x)}{2 - \lambda(y - x)} = \frac{2 + \lambda(x' - y)}{2 - \lambda(x' - y)} \frac{2 + \lambda(y' - x)}{2 - \lambda(y' - x)} - 1
\]

(5.5)

\[
= \frac{y' - y}{x' - x} = i\rho(x, x').
\]

(5.6)

We note that it is logical to state \(\varepsilon = \text{Exp}(\infty;) = \pm 1\) on \(\Gamma\) and \(\Gamma^*\) respectively.

On the rectangular lattice \(\Diamond = \delta(e^{-i\theta}Z + e^{i\theta}Z)\), the exponential is given explicitly, for \(z = \delta(ne^{-i\theta} + me^{i\theta})\) by

\[
\text{Exp}(\lambda; z) = \left( \frac{1 + \frac{\lambda}{2} e^{i\theta}}{1 - \frac{\lambda}{2} e^{i\theta}} \right)^n \left( \frac{1 + \frac{\lambda}{2} e^{-i\theta}}{1 - \frac{\lambda}{2} e^{-i\theta}} \right)^m
\]

(5.7)

\[= \exp(\lambda z) + O(\delta^2)\]
as $\delta$ goes to zero keeping $z \in \mathbb{C}$ fixed, because $(1+x)^n = \exp(n \log(1+x)) = \exp(nx) + O((nx)^2)$. Similar results are obtained for other lattices, see Fig. 4 for the comparison of the discrete and continuous exponentials on the triangular/hexagonal double. The discrepancy between the two increases with the modulus of the parameter and decreases as the mesh is refined. More generally, we will prove in [6] the same behavior for any refining sequence of critical maps of a critical discrete Riemann surface.

5.3. Change of base point. Given another critical map $\zeta : V \to \mathbb{C}$ with an origin $b \in \Lambda_0$, there exists a complex number $a$ such that $\zeta = a(Z - b)$ on a connected component of the intersection $U \cap V$. If the discrete Riemann surface is itself critical and if $U \cup V$ is contained in a ball then this number $a$ is the same for all connected components. As $d\zeta = a dZ$, it follows that

\begin{equation}
\text{Exp}_\zeta(:\lambda : z) = \frac{\text{Exp}_Z(a \lambda : z)}{\text{Exp}_Z(a \lambda : b)}
\end{equation}

as expected. Hence, on a critical discrete Riemann surface, the exponential can be continued as a well defined holomorphic function on the universal covering of the discrete surface.

5.4. Train-tracks. For $\lambda$ with a norm equal to $2/\delta$ in a given critical map, the associated exponential may not be defined straightforwardly. We prove that in the compact case, in general it can only be defined as zero:

We name the equivalence class of an (oriented) edge, generated by the equivalence relation relating opposite sides of the same rhombi, a train-track, medial cycle or thread as it can be seen graphically as a train-track succession of parallel edges of rhombi, graph theoretically as a cycle in the medial graph [1, 11] or knot theoretically as a thread in the projection of a link.

Given the train-track of an edge $(x, y) \in \Diamond$ in a critical map $Z$, it defines a complex number $\lambda = 2/(Z(y) - Z(x))$, of norm $2/\delta$, which verifies that $\text{Exp}(:\lambda :)$ is null on one side of the train-track, $\text{Exp}(:-\lambda :)$ on the other. If the sum of train-tracks parallel to this one is non null in homology on the universal covering, then $\text{Exp}(:\pm \lambda :) \equiv 0$ everywhere. If it were null however, $\text{Exp}(:\lambda :)$ would be non trivial and defined independently on each connected component.

**Proposition 5.3.** On a critical discrete Riemann surface, a train-track is never null in homology.
Figure 7. The image of the first sextant of the triangular/hexagonal lattice under the exponential, both discrete and continuous, for growing parameters vertically and finer meshing horizontally.
Proof 5.3. Refining by splitting each quadrilateral in four doesn’t change neither criticality nor the homology class of a thread, so we can suppose that if a thread is homotopic to zero then we have a finite domain which completely contains it. Because all the conic angles are multiple of $2\pi$, this domain is a branched covering of a region of the euclidean complex plane. The train-track is characterized by the conserved angle of each parallel side of its rhombi. As all the angles of the rhombi are positive, the train-track follows a directed walk and can not backtrack to close. Hence no train-track is null in homology.

In the compact case, the fundamental domain is glued back into a closed flat surface with conic singularities multiple of $2\pi$ hence the fundamental group acts by a discrete subgroup $G$ of the Galilean group of translations and rotations. A corollary of the proposition is that for all the angles appearing in this fundamental domain and their image by $G$ (a discrete subset of the circle), the associated $\text{Exp}(\lambda \cdot)$ with $\lambda$ of modulus $2/\delta$ with these given arguments must be defined as zero (or more precisely infinity).

We define a region $R \subset \Diamond_2$ of a discrete Riemann surface as convex iff it is connected and for every pair of faces in $R$ along the same thread, if the thread is open then every face in between is also in $R$ and if it is closed, the same property holds for at least one of the two halves of the thread.

Given a non convex set $D \subset \Diamond_2$ contained in a disc, there is a well defined procedure to find its minimal convex hull by adding all the faces that are missing. This ultimately reduces the number of edges on the boundary as it doesn’t introduce new threads by the Jordan theorem so the procedure reduces the dimension of the space of holomorphic functions, resp. 1-forms on it which are explicitly $\partial D_0/2 + 1$, resp. $\partial D_0/2 - 1$. Notice that $\partial D_0/2$ is the number of open threads segments contained in $D$. In the tensed case (see below), a basis of holomorphic functions is provided by the set of functions which are non null on one side of a given thread segment $[12]$ and $\varepsilon$.

**Conjecture 5.4.** On a simply connected convex, the exponentials generate the space of holomorphic functions.

What is clear is that on a non convex, they don’t because in general, not all the holomorphic functions defined on a non convex can be continued into a holomorphic convex on the convex hull; however,

**Proposition 5.5.** In the simply connected case, every holomorphic function on a connected non convex which can be continued into a holomorphic function on the convex hull has a unique continuation; and for
every convex, there exists a maximal convex containing it where every holomorphic function on the smaller convex can be uniquely analytically continued.

To prove this we need the notion of electrical moves. We could also prove Prop. 5.3 by implementing homotopy through Reidemeister-like moves with spectral parameters. They are called inversion relation and star-triangle relation (linked to the Yang-Baxter equation) in the context of integrable statistical models [13] or electrical moves in the context of electrical networks [12].

6. Electrical moves

Figure 8. The electrical moves.

6.1. Definition. The three types of electrical moves are presented in Figure 8. They can be understood as moving, splitting and merging the conic singularities around, keeping constant the overall curvature, sum on the vertices of $2\pi$ minus the conic angle.

The first move involves the elimination of a quadrilateral with any discrete conformal parameter $\rho$ such that two of its adjacent edges are identified. Let’s say $\rho$ labels the edge closed in a loop, pictured as a dashed arc. Before the move there are three vertices, of angle $\theta$ at the black vertex, $\theta'$ at the white vertex and $2 \arctan 1/\rho$ at the cone summit. After the move, where the cone disappears, the angles are $\theta - 2 \arctan 1/\rho$ and $\theta' - 4 \arctan 1/\rho$ respectively. As $4 \arctan \rho + 4 \arctan 1/\rho = 2\pi$, the total curvature has not changed. In statistical mechanics, this move amounts to normalizing the energy by discarding autonomous self interaction.

The second move replaces two quadrilaterals, with conformal parameters $\rho_1$ and $\rho_2$ along their respective parallel dashed diagonals, identified along two adjacent edges, by a single quadrilateral of conformal parameter the sum $\rho_1 + \rho_2$. Dually, one replaces the series $\rho_1^*$ and $\rho_2^*$ by $\frac{\rho_1^* \rho_2^*}{\rho_1^* + \rho_2^*}$. These formulae ring a bell to anyone versed in electrical networks as the parallel and series replacement of two conductances by a single one. The conic angles change by $2 \arctan (\rho_1 + \rho_2) - 2 \arctan \rho_1 - 2 \arctan \rho_2$ for the white vertices and $2 \arctan (1/\rho_i) - 2 \arctan \frac{1}{\rho_1 + \rho_2}$ for
each black vertex. It is called *inversion relation* in statistical mechanics.

The third move is the star-triangle transformation. To three quadrilaterals arranged in a hexagon, whose diagonals form a triangle of conformal parameters $\rho_1, \rho_2$ and $\rho_3$, one associates a configuration of three other quadrilaterals whose diagonals form a three branched star with conformal parameters $\rho'_i$ verifying

\[
\rho_i \rho'_i = \rho_1 \rho_2 + \rho_2 \rho_3 + \rho_3 \rho_1 = \frac{\rho'_1 \rho'_2 \rho'_3}{\rho'_1 + \rho'_2 + \rho'_3}.
\]

It is the *star-triangle relation*, leading to the Yang-Baxter equation in statistical mechanics. Notice that if the central vertex is flat, then the equation Eq. (6.1) is equal to 1 and the conic angles at either of the six external vertices don’t change, hence flatness and criticality are preserved by this move. The relations for non flat angles are more complicated and left to the reader.

Given two discrete Riemann surfaces related by an electrical move, a holomorphic function or form defined on all the shared simplices uniquely defines it on the simplices which are different:

The value of a holomorphic function

- at the conic singularity appearing in the type I move has to be equal to the value at the opposite vertex in the quadrilateral;
- at the middle point in the type II move is the \( \left\{ \frac{\rho_2}{\rho_1 + \rho_2}, \frac{\rho_1}{\rho_1 + \rho_2} \right\} \) weighted average of the left and right values;
- at the central point in the three pointed star configuration is the \( \left\{ \frac{\rho_i}{\rho_1 + \rho_2 + \rho_3} \right\} \) weighted average of the three boundary values.

Moreover, the norm of the two related 1-forms (or the coboundary of the related functions) are equal. Hence the spaces of holomorphic forms for two electrically equivalent discrete Riemann surfaces are isomorphic, and isometric on 1-forms.

The notion of convexity is stable by electrical move as well.

A discrete Riemann surface where no moves of type I or II are possible is called *tensed* [12]. On a disc, electrically equivalent tensed discrete Riemann surfaces can be obtained from one another through a series of type III moves only but it is no longer the case for non trivial topology. Using the Dirichlet theorem and the maximum principle, one shows [12] that tensed discrete Riemann surfaces have the property that for every edge of the double graph $\Lambda$, there exists a holomorphic form which is not zero on that edge. It also provides with a basis of holomorphic functions in the case of a topological disc, together with $\varepsilon$, for each open thread, there is a holomorphic function which is null on
one side and nowhere null on the other, unique up to a multiplicative constant \[12\].

6.2. Demonstration of Prop. 5.5. Proof 5.5. As electrical moves respect convexity and yield isomorphic holomorphic spaces, we can assume that the disc is tensed. If there were several analytic continuations then it would contradict the fact that a basis of holomorphic functions is given by the set of functions which are non null on one side of a given thread segment. The information contained in the value of a holomorphic function on a given convex allows to compute the value of the function on a greater convex. Wherever the value of the function is known at three vertices of a rhombi, it can be determined on the fourth, recursively closing the inwards corners. The result is a convex set where each vertex on the boundary is adjacent to at least two faces which are not in the set. On the square lattice \[\diamondsuit = \mathbb{Z}^2\] for example, knowing the value of a holomorphic function on the horizontal and vertical axis form the origin up to \((x,0)\) and \((0,y)\) allows to compute its values on the whole rectangle \(((0,0),(x,0),(x,y),(0,y))\).

The procedure of recursively closing the corners (and more elaborate conditions as well, implying more than three points) allows to try and extend a given holomorphic function defined on a non convex set but for every thread which is locally convexified, a condition for the values of the function has to be satisfied, if it is not satisfied then the function can be analytically continued into a meromorphic functions in an essentially non unique way.

6.3. Continuous moves. These moves, while presented as discrete, are in fact continuous. A quadrilateral with a small or dually large conformal parameter \(\rho\) is associated with a thin rhombus where two vertices are close to one another hence the limit \(\rho = 0\) has to be understood as an absent quadrilateral, where the edges on both sides are identified by pairs and dually, an infinite conformal parameter along an edge means that its two ends are identified into a single vertex.

Allowing for zero and infinite conformal parameters not only enables to understand electrical moves as continuous, it also enables to recover the first and second moves as special cases of the third. Notice that these moves do not preserve in general the complex structure of the surface coded by the flat Riemannian metric associated with the conic singularities. However we will see that criticality ensures that it is the case, the Riemann structure is unchanged by critical electrical moves.
\[
\frac{\rho'_D}{\rho_U} = \frac{\rho_D - \rho'_L}{\rho'_R - \rho_R} = \frac{1 + \rho_U \rho_D}{\rho_U} + \frac{\rho_D}{\rho_U}
\]

(6.2) \[= \frac{1}{(\rho'_R - \rho_R) \frac{1 + \rho'_U \rho'_D}{\rho'_D} + \frac{\rho'_U}{\rho'_D}}.\]

**Figure 9.** The continuous electrical move.

### 6.4. Reidemeister moves.

To recover isotopy and the usual Reidemeister moves, one has to allow yet another type of discrete conformal structure, namely negative parameters. Geometrically speaking, a negative parameter corresponds to a negative angle, or a change in the orientation, so the associated quadrilateral takes part in a fold, an overhanging wrinkle in the fabric of the surface. Notice that the change of all the signs corresponds to the complex conjugation at the level of functions or forms: a holomorphic function on a graph with negative parameters is an anti-holomorphic function on the same graph with positive parameters. We will call a graph with parameters of either signs a virtual discrete Riemann surface. A holomorphic function on it can then be understood as locally holomorphic, locally anti-holomorphic on a non virtual discrete Riemann surface but it is not harmonic for this positive discrete conformal structure so we won’t make use of this remark. Notice that the total curvature, sum on all the vertices of \(2\pi\) minus the conic angle, is still unchanged by virtual electrical moves.

Let’s restrict ourselves to the case when all parameters on the graph \(\Gamma\) are equal or opposite, \(\pm \rho\). We associate to the planar graph with signed edges \((\Gamma, \rho)\), a link in the tubular neighborhood of the surface [14]: The sign of each edge allows the resolutions of the crossings occurring in the thread associated to a given train-track into positive or negative regular projection crossings. There is of course an overall ambiguity of sign. The Reidemeister moves for this link are then recovered when we restrict the second and third moves in a certain way:

We allow the second move only as the disappearance and appearance of a pair of quadrilaterals of opposed conformal parameters \(\rho, -\rho\), leaving a collapsed quadrilateral of parameter zero.

In terms of catastrophe theory, the whole quadrilateral with negative parameter is the unstable sheet of a cusp and its edges are the bifurcation lines. In this picture, the orientation of the surface is only locally
defined and changes across bifurcation lines where it is not defined at all.

We restrict the third move to configurations where two parameters are equal and opposite to the third.

\[ I \rightarrow II \rightarrow III \]

Figure 10. The Reidemeister moves.

Notice that the usual non virtual discrete Riemann surfaces correspond to alternating links. Alternating links are special links so not all the virtual discrete Riemann surfaces are Reidemeister equivalent to a non virtual discrete Riemann surface.

Isotopy on the other hand is more free than the Reidemeister moves, the signs and spectral parameters should be irrelevant. But in fact, on a critical map, we can keep track of them in a consistent way. We no longer restrict ourselves to maps where the parameters are all equal or opposite and we free the restriction on the third move, but we keep the restriction on the second move and forbid the first move. We call this set of allowed moves and the associated equivalence relation critical isotopy. With these moves, isotopy can be performed keeping criticality along the way. These moves also have the property to be stable by refining, two critical virtual discrete conformal maps related by critical isotopy remain related if all their quadrilaterals are cut into four (or more) smaller ones. Hence, refining to the continuous limit, the conformal structure of a class of virtual discrete conformal maps (containing at least one genuine discrete conformal map) is well defined. Moreover, a given thread on a map yields two homotopic threads on the refined map obtained by splitting a quadrilateral in four.

Notice that the first move can not occur on a critical map as it involves a conic singularity of a given conic angle between $-\pi$ and $\pi$. Notice as well that the sum of angles at a degree three vertex, like the one appearing in a move of type $III$, can only be 0 or $\pm 2\pi$ on a virtual critical surface because it is the sum of three angles between $-\pi$ and $\pi$ and has to be null mod $2\pi$ by criticality.

7. Series

As a first attempt to define divisors, and relate them to the degrees of zeros and poles, we discuss here, after [5], discrete polynomials.
7.1. Polynomials. In a critical map $Z$, with $Z(O) = 0$, we define recursively the powers

\[ Z^k := \int_O^Z k Z^{k-1} dZ \]

with $Z^0 = 1$ and $Z^1 = Z$.

Consider for example $\Diamond$ containing the chain \{0, $\frac{1}{n}$, $\frac{2}{n}$, \ldots, 1\}, the first $Z^k(x)$ with $x = \frac{i}{n}$ are listed in Table 1 and you show in general, for a critical map of characteristic length $\delta$,

\[ |Z^k(x) - x^k| \leq \lambda_k |x|^{k-2} \delta^2 \]

with $\lambda_k$ independent of $x$ and $\delta$. But this constant is growing very rapidly with $k$, namely $\lambda_k = \frac{k!}{2} \left( \frac{1}{\sin \eta} \right)^{k-2}$ where $\eta$ is the smallest rhombus angle in the discrete Riemann surface. It is for example not true that for a point close enough to the origin $Z^k$ will tend to zero with growing $k$, on the contrary, if $x$ is a neighbor of the origin with $(O, x) \in \Diamond_1$ and $k \geq 1$, then $Z^k(x) = \frac{k!}{2k^2} x^k$ in fact diverges with $k$. If $y$ is a next neighbor of the origin, with the rhombi $(O, x, y, x') \in \Diamond_2$ having a half angle $\theta$ at the origin, $Z^k(y) = \frac{k!}{2k^2} \frac{\sin k \theta}{\sin \theta} y^k$ has the same behavior and so has every point at a finite distance of the origin. It’s only in the scaling limit with the proper balance given by criticality that one recovers the usual behavior $|x| < 1 \implies |x^k| \xrightarrow{k \to \infty} 0$.

See Fig. 11 for the comparison of the discrete and continuous powers on the triangular/hexagonal double. The discrepancy between the two increases with the degree (is null for degree up to two) and decreases as the mesh is refined.

7.2. Discrete Exponential as a series. The discrete exponential $\Exp(\lambda \cdot)$ is equal to the series $\sum_{k=0}^{\infty} \frac{\lambda^k Z^k}{k!}$ whenever the latter is defined as its derivatives fulfills the right equation and its value at the origin is 1. The great difference with the continuous case is that the series is absolutely convergent only for bounded parameters, $|\lambda| < \frac{2}{7}$. This

| $k$ | 3 | 4 | 5 | 6 | 7 |
|-----|---|---|---|---|---|
| $Z^k(x)$ | $x^3 + \frac{x}{2n^2}$ | $x^4 + \frac{2x^2}{n^2}$ | $x^5 + \frac{5x^4}{n^2} + \frac{3x}{2n^2}$ | $x^6 + \frac{10x^5}{n^2} + \frac{23x^3}{n^4} + \frac{3x}{2n^2}$ | $x^7 + \frac{35x^6}{n^4} + \frac{49x^4}{4n^6}$ |

Table 1. The first powers on the interval $[0, 1]/n$ for $x = i/n$. 

28 CHRISTIAN MERCAT
Figure 11. The image of the first sextant of the triangular/hexagonal lattice under the map $z \mapsto z^k$ both discrete and continuous, for degree $1 \leq k \leq 8$ vertically and finer meshing horizontally.
suggests that asking for a product such that \( \exp(\lambda) \cdot \exp(\mu) = \exp(\lambda + \mu) \) may not be the right choice.

7.3. **Ramification Number.** Each rhombus \( F = (x, y, x', y') \in \mathcal{D} \) is mapped by a holomorphic function \( f \in \Omega^0(\Lambda) \) to a quadrilateral \( (f(x), f(y), f(x'), f(y')) \) in the euclidean complex plane and generally a cycle to a polygon. If this polygon doesn’t contain the origin \( 0 \in \mathbb{C} \), we define the *ramification number* of \( f \) at a cycle \( \gamma \) to be its winding number around \( 0 \)

\[
(7.3) \quad b_f(\gamma) := \oint_{f(\gamma)} \frac{dz}{z}.
\]

The ramification number of a face can only be \( \{-1, 0, +1\} \) because it is only four sided. If the ramification numbers at the faces inside an exact cycle \( \gamma = \partial B \) are all defined, then the ramification of \( f \) at \( \gamma \) is the sum of these numbers. The ramification number of \( Z \) at any loop is its winding number around \( O \), twice for \( Z^2 \). Unfortunately, things get more complicated for higher degrees.

**Conjecture 7.1.** *The ramification number of a polynomial of degree \( k \) is at most \( k \), for all polynomial of degree \( k \), there exists a cycle around which it has ramification number \( k \).*

It is true at the continuous limit, it can only change by a unit, numerical evidence suggests that \( Z^k \) has only non negative ramification numbers but I don’t know how to prove it in general.

7.4. **Holomorphic maps.** A rhombus is mapped by a holomorphic function to a quadrilateral of the plane which may or may not be immersed. It is immersed whenever the images of the diagonals whether cross or the line going through one of them crosses the other (a kite), it is not immersed when the line going through them cross at a point outside both of the segments. Because its two diagonals are mapped to vectors forming an orthogonal direct basis, the only possibility for the ramification number to be \( -1 \) is if the quadrilateral is not immersed and the origin is inside the triangle winding the other way round (see Fig.12). This case appears in the discrete analog of \( 1/z \), in the derivative \( \varepsilon' \), or in polynomials of degree 2 or more.

If we don’t allow this situation to occur, we can define the notion of holomorphic map between discrete Riemann surfaces. We need a more relaxed definition than criticality, namely semi-criticality \[ where rhombi are replaced by (not necessarily convex) quadrilaterals. Note that this condition is sufficient to ensure the continuous limit theorem.
Let $(\Lambda, \rho)$ a discrete Riemann surface and $\Sigma$ the associated topological surface. It is *semi-critical* for a flat riemannian metric on $\Sigma$ iff the conic singularities are among the vertices of $\Lambda$ and every face of $\triangledown$ is realized by a linear quadrilateral flattened so that its diagonals are orthogonal and have a ratio of lengths governed by $\rho$.

If the quadrilateral is convex then its diagonals are well defined as there exists a geodesic on the surface going from one of its four vertices to the opposite one, staying inside the quadrilateral, a usual linear segment. When the quadrilateral is in the “kite” conformation however, one of the diagonals would be outside the flat quadrilateral interior and its geometry may be disturbed by conic singularities, it may not be a single segment anymore, it may even be a single point, in particular the notion of orthogonality may not be well defined: To compute an angle between two segments of geodesics, they first have to be parallel transported so that they cross, turning around a singularity alters this angle. It is why the lengths and orthogonality we are talking about are not those of geodesics on the surface but those computed in the quadrilateral isometrically flattened in the euclidean plane.

A discrete *holomorphic map* $f$ between two semi-critical discrete Riemann surfaces $(\Lambda, \rho), (\Lambda', \rho')$ is a cellular complex map, orientation preserving, locally injective outside the vertices (where it can have a branching point) preserving the conformal parameters. It induces a genuine holomorphic map between the underlying Riemann surfaces. Reciprocally, a holomorphic map to a Riemann surface associated with a semi-critical double $(\Lambda, \rho)$ from another Riemann surface such that the branching points are among the vertices of $\Lambda$ induces a semi-critical double on the pre-image, hence a discrete holomorphic map.

In that case, the ramification number is well defined, so that with $B := \sum_{x \in \Lambda_0} b_f(x)$ the total ramification number, $n = |f^{-1}(y)|$ for any $y$ in the image (counting multiplicities) the degree of the map, $g$ and $g'$ the genuses of the two Riemann surfaces, the Riemann-Hurwitz relation
applies and
\[ g = n(g' - 1) + 1 + B/2. \] (7.4)

This notion allows for the composition of discrete holomorphic maps.

**Proposition 7.2.** The exponential \( \text{Exp}(\lambda) \), for \( |\lambda| < 2/\delta \) in a critical map \( Z : U \to \mathbb{C} \), is a holomorphic map from the discrete Riemann surface \( U \cap \triangle_2 \) to the semi-critical discrete Riemann surface \( Z(U) \).

It suffices to check that every rhombus is mapped to a quadrilateral, whether convex or of a kite form, it will imply local injectivity as well. One can assume that a given rhombus under consideration contains the origin. By inspecting the map
\[ C \to C, \quad z \mapsto \frac{1 + z}{1 - z}, \] (7.5)
one sees that the quadrilateral \((1, \frac{1 + z'}{1 - z}, \frac{1 + z'}{1 - z}, \frac{1 + z'}{1 - z})\) with \( |z| = |z'| < 1 \) is always of that type, hence the result.

### 7.5. Change of coordinate.

We are now going to consider the change of coordinate for a critical map \( Z \). If \( \zeta \) is another critical map with \( \zeta = a(Z - b) \) on their common definition set, the change of map for \( Z^k \) is not as simple as the Leibnitz rule \((a(z - b))^k = a^k \sum_{j=0}^{k} \binom{k}{j} z^{k-j}(-b)^j\) but is a deformation of it. The problem is that pointwise product is not respected, \( Z^{k+l}(z) \neq Z^k(z) \times Z^l(z) \), in particular the first is holomorphic and not the second, for each partition of \( k \) into a sum of integers, there is a corresponding monomial of degree \( k \). An easy inference shows that the result is still a polynomial in \( Z \):

**Proposition 7.3.** The powers of the translated critical map \( \zeta = a(Z - b) \) are given by
\[ \zeta^k = a^k \sum_{j=0}^{k} \binom{k}{j} (-1)^j Z^{k-j} B^j(b) \] (7.6)

where \( B^j(b) \) corresponding to \( b^j \) is a sum over all the degree \( j \) monomials in \( b \), defined recursively by \( B^0 = 1 \) and
\[ B^k(b) := \sum_{j=0}^{k-1} \binom{k}{j} (-1)^{k+j+1} Z^{k-j}(b) B^j(b) \] (7.7)

The calculation is easier to read using Young diagrams, note the product of pointwise multiplication monomial
\[ (Z^{k_1}(z))^{\ell_1} (Z^{k_2}(z))^{\ell_2} \ldots (Z^{k_n}(z))^{\ell_n} \] (7.8)
with \( k_1 > k_2 > \ldots > k_n \) as a Young diagram \( Y \), coding columnwise the
partition of the integer given by the total degree \( k = \sum_{j=1}^{n} k_j \times \ell_j \) into
the sum of \( \ell = \sum_{j=1}^{n} \ell_j \) integers. For example the following monomial
of degree 15 = 3 \times 2 + 2 \times 4 + 1 is noted
\[(7.9) \quad (Z^3(z))^2 (Z^2(z))^4 Z^1(z) =: .\]
Then, \( B_j(b) = \sum_Y c(Y)Y(b) \) where the sum is over all Young diagrams
of total degree \( j \), \( c(Y) \) is an integer coefficient that we are going to
define and \( Y(b) \) is the pointwise product of the monomials coded by
\( Y \) at \( b \). The coefficient of the Young diagram \( Y \) above is given by the
multinomials
\[(7.10) \quad c(Y) = (-1)^{k+\ell} \frac{k!}{(k_1)! \ell_1! (k_2)! \ell_2! \ldots (k_n)! \ell_n!} \frac{\ell!}{\ell_1! \ell_2! \ldots \ell_n!}.\]
For example, the first Young diagrams have the coefficients \( c(\ ) = \frac{n!}{1! \ldots 1!} = n!, \ c(\ ) = (-1)^{n+1}, \ c(\ ) = (-1)^{n+1} \frac{(n+1)!}{n!} \frac{2!}{1!} = (-1)^{n+1} 2(n + 1)\)
and \( c(\ ) = -\frac{(n+1)!}{2!} \frac{n!}{1! (n-1)!} = -\frac{(n+1)! n}{2}. \) The first few terms are
listed explicitly in Table 2.

It is to be noted that the formula doesn’t involve the shape of the
graph, the integer coefficients for each partition are universal constants
and add up to 1 in each degree. As a consequence, since \( Z^k(z)Z^\ell(z) = Z^{k+\ell}(z) + O(1/\delta^2), \) \( k, \ell, z \) fixed, the usual Leibnitz rule is recovered
in \( O(1/\delta^2) \). Let’s stress again that these functions \( B^k \) are discrete
functions on the graph \( \Lambda \) which are not holomorphic.
The general change of basis of a given series however possible in theory is nevertheless complicated and the information on the convergence of the new series is difficult to obtain, even though there are some exceptions like the exponential, if $\zeta = a(Z - b)$, (see Eq.(5.8)):

$$
\sum_{k=0}^{\infty} \frac{\lambda^k}{k!} \zeta^k \propto \sum_{k=0}^{\infty} \frac{(a \lambda)^k}{k!} Z^k.
$$

(7.11)

8. Conclusion

The theory of discrete Riemann surfaces is shown to share a lot of theorems and properties with the continuous theory, extending the previous results to the matters related to the period matrix and holonomies of holomorphic forms. It begs for a good definition of the order of a pole or a zero which would allow to prove the analog of several crucial theorems of the continuous theory, namely Riemann-Roch theorem, Abel’s theorem and the Jacobi Inversion problem. The main challenge is to define a good discrete analog of the exponential of a discrete holomorphic function so that the bilinear relations would provide Abel’s correspondences between divisors and meromorphic functions. The study of the discrete exponential $\text{Exp}(\lambda; Z)$ for a critical map $Z$ and a constant $\lambda$ is a first step in that direction. Electrical moves are a crucial and powerful tool to investigate the matter. These combinatorial moves are clearly what is needed to make the connection with Kasteleyn theory of pfaffians, on the way to define discrete theta functions.

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