The Pfaffian state in an electron gas with small Landau level gaps

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Landau level mixing plays an important role in the even denominator fractional quantum Hall states. In ZnO the Landau level gap is essentially an order of magnitude smaller than that in a GaAs quantum well. We introduce the screened Coulomb interaction in a single Landau level to deal with that situation. Here we study the incompressibility and the overlap of the ground state and the Pfaffian (or anti-Pfaffian) state at filling factors $\nu = \frac{2}{5}$ and $\frac{2}{3}$ with a general screened Coulomb interaction. For small Landau level gaps, the overlap is strongly system size-dependent and the screening can stabilize the incompressibility of the ground state with particle-hole symmetry which suggests a newly proposed particle-hole symmetry Pfaffian ground state. When the ratio of Coulomb interaction to the Landau level gap $\kappa$ varies, we find a possible topological phase transition in the range $2 < \kappa < 3$, which was actually observed in an experiment. We then study how the width of quantum well combined with screening influences the system.

The even-denominator fractional quantum Hall effect (FQHE) was observed [1, 2] and studied in great detail in GaAs. It is believed that the concept of pairing of electrons is behind this unique quantum Hall state [3–7], though the nature of this state is still unclear. A strong candidate for the ground state is the Moore-Read Pfaffian state which contains non-abelian excitations and chiral edge modes [3, 4, 7]. However this topological state has not been observed directly, perhaps because the mobility of GaAs is still not high enough for this state to be detected. In other systems, such as cold atoms, the circuit and cavity QED systems [8–11], in theory it is possible to emulate this unique topological ground state by tuning the Hamiltonian to approximate the parent Hamiltonian of the Pfaffian state. The even-denominator FQHE, has been studied and observed in graphene systems [12–21]. Recently, FQHE was observed again in the ZnO/MgZnO heterointerface [22–24]. The Pfaffian states and its topological properties can be potentially observed in this new system, albeit its low mobility. Surprisingly, in ZnO the well-known $\nu = \frac{2}{5}$ FQHE was found to go missing while its spinful electron-hole conjugate $\nu = \frac{2}{3}$ FQHE survived [24, 25]. Tilted-field studies [26] also unveiled some interesting results in this system [24, 27].

The ZnO system is distinctly different from the GaAs and graphene systems [16–21] since the effective mass in ZnO is very large and the Landau level (LL) gap is very small. The ratio of Coulomb interaction to the LL gap is $\kappa = 25.1/\sqrt{B}$ for the magnetic field $B$, which is one order of magnitude higher than that in GaAs or in graphene systems. As a result, the electron-electron interaction would definitely drive the transport of the electrons uniquely in many aspects [28]. Since the LL gaps are very small in ZnO, the Landau level mixing (LLM) is too strong to be negligible. However, it would be a major computational challenge to include even only a few LLs. In graphene or other small $\kappa$ systems, perturbative theories which involve renormalized two-body and three-body interactions, have been developed in more accurate calculations [29]. However, the limitation of that approach is that $\kappa$ cannot be too large. In experiments, $\kappa$ is 0.5 ~ 0.8 (it depends on the dielectric constant of the substrate) for graphene and could be smaller than unity (in a high magnetic field) for GaAs. But $\kappa$ cannot be too small than unity in ZnO unless the magnetic field reaches 630T. Consequently, we have proposed that the screened Coulomb interaction in which all the other LLs are integrated out in the random phase approximation (RPA) replaces the bare one, so that the correlations of the electrons become very different at different filling factors. This seems to explain the extraordinary phenomenon observed in the experiment [25, 27].

Trial wavefunctions have been proposed to describe this special even-denominator FQHE state [3, 31–32]. It appears that the spin polarized Pfaffian state [33, 34] is the most probable candidate. Its particle-hole (PH) conjugate, the anti-Pfaffian state [35] is also likely since the two-body Coulomb interaction cannot break the PH symmetry in a half-filled Landau level. Recently, a new Pfaffian-like state with PH symmetry [36] was proposed and may be valid for strong LLM or disorder. It is convenient to employ the Haldane pseudopotential to study the overlap between the Pfaffian state and the ground state of an even-electron system in the spherical geometry [37, 38].

The rotational symmetry is preserved so that the ground state of the incompressible state is uniquely located at the total angular momentum $L = 0$. In contrast, the ground state is degenerate in the toroidal geometry due to the translational symmetry. In particular, the ground state is quasi-triply degenerate for the even-electron system at $\nu = \frac{2}{3}$ FQHE in toroidal geometry, due to the topological property of the Pfaffian state.

When the ratio $\kappa$ is very large the Coulomb interaction may not be renormalized by the perturbation theory. Further, the PH symmetry may be stabilized by the LLM [36]. It means that the two-body screened interaction in the RPA, which preserves the PH symmetry of the Hamiltonian, should have essentially included some of the most important information of the LLM especially in strong LLM, although it does not include all the correlations. We have previously demonstrated in the torus geometry that the collective modes are not stable...
for the $\nu = \frac{5}{2}$ FQHE in ZnO in an odd-electron system. However, in our previous works the nature of the ground states and the relation between the screening and incompressibility still remained to be understood. In this work the ground states are studied at different flux in a spherical geometry to indicate how the ground states evolve with different screening.

The Pfaffian state is the zero-energy ground state of its parent Hamiltonian with a three-body interaction \cite{4}. The pseudopotentials could be approximated instead for the planar case

$$V_m = \int \frac{dq}{2\pi \ell^2} \frac{V(q)}{\epsilon_s(q)} \left[ L_n \left( \frac{q^2}{2} \right) \right]^2 L_m \left( \frac{q^2}{2} \right) e^{-q^2},$$

where $n$ is the LL index, $m$ is the momentum index, $\ell = \sqrt{\hbar/eB}$ is the magnetic length, and we add the screening effect into the system. The screened pseudopotential was also used (although a simplified version of it) in \cite{14}. As in the torus geometry, the two-dimensional Coulomb potential $V(q) = \frac{2\pi\kappa}{\nu^2}$ must be screened by all the other LLs with the dielectric constant $\epsilon$. The static dielectric function is $\epsilon_s(q) = 1 - V(q)\lambda^0_{\nu,n}(q)$. The pseudopotentials for different screening are shown in Figs. 1(b) to (d). For $\nu = \frac{7}{2}$, the overlap is decreased by half when $N_e$ increases to 12. In contrast, for $\nu = \frac{3}{2}$ the Pfaffian overlap dramatically increases at $N_e = 12$. But when the system size increases for up to 16 electrons the overlap of $\frac{7}{2}$ decreases. From our numerical works the overlap appears to be very size-dependent (much more than for the unscreened case). On the other hand, the $(V_1/V_5) - (V_3/V_5)$ curve for $\nu = \frac{5}{2}$ is also very different from $\nu = \frac{3}{2}$. In the strong LLM region the ground state for $\nu = \frac{3}{2}$ even enters into the compressible area extrapolated from Ref. \cite{14}, although we do not find any compressibility for up to 16 electrons. However, the ground state should be compressible in the thermodynamic limit since the energy gap in this case is very size dependent and oscillates very much in Fig. 2(a), which agrees with our previous works in the less size-dependent torus geometry \cite{23}.

More importantly, in all cases the overlaps of the Pfaffian trial wave function are not very high, but at most 0.65. This could be due to our choice of planar screened dielectric function. It is also possible that the ground state itself is another Pfaffian-like wave function with PH symmetry \cite{6,30}, especially for large $\kappa$, the strong LLM region. In the PH symmetric case where the shift is $S = -1$ and the flux is $N_{PF} = 2N_e - 1$ on a sphere, the ground state is compressible and located at $L = 2$ for $N_e = 10$ without screening and disorder. So this PH symmetric flux was not considered in decades.

To confirm the possibility of the PH symmetric ground state in LLM, we consider the PH symmetric flux. We calculate the energy spectrum for $N_e = 10$ at this PH

![FIG. 1: (Color online) (a) The Pfaffian overlap at $\nu = 5/2$ and 7/2 with screening in ZnO, comparing with the unscreened case $\kappa = 0$. (b) The trajectories of different screened pseudopotentials at $\nu = 7/2$ and 7/5 fillings in the pseudopotential $(V_1/V_5) - (V_3/V_5)$ plane with $\kappa \in [1.08, 1.13]$. The compressible area is an extrapolation of the Ref. \cite{14}. (c) and (d) are the pseudopotentials for different screening.](image-url)
symmetric flux. In contrast, the ground states are surprisingly incompressible for $\kappa > 0.6$ at $\nu = \frac{5}{2}$ and for $\kappa > 0.93$ at $\nu = \frac{7}{2}$ when the screening is included in Fig. 2(b). More generally, for $N_e = 4 \sim 14$, all the ground states are stable and incompressible with screening. Comparing with [44], all the pseudopotentials $V_m$ here are modified by screening. $V_m$ with higher angular momentum also plays very important roles, since both the overlap and the excitation gap are completely changed when $V_m > 5 = 0$. Moreover, if we only tune $V_1$ slightly and leave others unchanged, the ground state still favors the Pfaffian shift $S = -3$. Only when all the pseudopotentials are screened, then the particle-hole symmetric ground state can be stabilized and incompressible. Hence, we assert here that screening helps to stabilize the incompressibility of the PH symmetric states.

When the screening is weak (small $\kappa$), the excitation gaps at the Pfaffian flux $N_{PF}$ are a few times larger than those at PH flux $N_{PH}$ in Fig. 2(b). So the Pfaffian state should be more stable. In the ZnO system ($\kappa \approx 13 \sim 15$) for both flux $N_{PF}$ and $N_{PH}$, the lowest energy gaps of $\frac{7}{2}$ are strongly size-dependent, while those gaps for $\frac{5}{2}$ vary very smoothly as shown in Fig. 2(a). That is a very strong indication that the $\frac{7}{2}$ state is compressible in the thermodynamic limit. It seems that the $\frac{7}{2}$ FQHE state would be closer to this newly proposed Pfaffian-like state since the screening (the strength of LLM) of $\nu = \frac{7}{2}$ is always stronger than that of $\nu = \frac{5}{2}$. Hence, the overlap of the PH symmetry breaking Pfaffian or anti-Pfaffian state is not high. On the other hand, our screened Coulomb interaction does not break the PH symmetry in a single LL.

We artificially tune the pseudopotential $V_1$ to study how the pseudopotential influences the overlap of the topological state in Fig. 3. For $N_e = 10$, when $V_1$ is varied the $\frac{5}{2}$ state would have much higher overlap than that of $\frac{7}{2}$, and the high-overlap window is much wider than that of $\frac{5}{2}$. For $N_e = 12$, the overlap of $\frac{7}{2}$ state is generally higher than that of the $\frac{5}{2}$ state. For $N_e = 14$, the overlap of $\frac{5}{2}$ gets steep rise when $V_1/V_1^{Coulomb} \approx 1.01$. It is interesting that the pseudopotential studies also provide indications that the two states would be distinguished at different filling factors.

We also calculate the Pfaffian overlaps with different screening since screening is closely related to the LLM. Here we suppose the two filling factors are all at $B = 3.75T$ and $\kappa = 13$. However, the results are also size-dependent, as shown in Fig. 4. For $N_e = 12$, in the ZnO region the overlap of $\nu = 7/2$ is higher than that of $\nu = 5/2$. The overlap of $\nu = 7/2$ decreases in general, while at $\nu = 5/2$ for small $\kappa$ the overlap increases a little with screening and falls rapidly in the region of $\kappa \in [2, 3]$. It is possible that there is a topological phase transition between $\kappa = 2$ and $3$. It seems that the numerical results

FIG. 2: (Color online) (a) The lowest energy gaps of the collective modes for flux $N_{PF}$ and $N_{PH}$ at fillings $\frac{5}{2}$ and $\frac{7}{2}$ in ZnO. (b) Energy gaps vs. $\kappa$ for different flux with 10 electrons.

FIG. 3: (Color online) The overlap changes with the change of pseudopotential $V_1$ at (a) 10, (b) 12 and (c) 14 electrons.

FIG. 4: (Color online) (a) The overlaps for $N_e = 12$ at different filling factors are different. It suggests a phase transition between $\kappa = 2$ and 3 for $\nu = 5/2$. (b) For $N_e = 16$, the overlap of $\nu = 5/2$ increase up to 0.94 when the screening is weak. The rapid drop occurs at $\nu = 7/2$, also in the region $2 < \kappa < 3$. 
are compatible with the experiment in a doped GaAs system [40], where the energy gap decreases dramatically at $\kappa \approx 2.6$, and decreases close to zero at $\kappa \approx 2.9$. The decreased energy gap is the signal of instability of the system. Results in Fig. 2(a) indicate that in the same region $\kappa \in [2, 3]$, the relation between the ground state and the topological incompressible Pfaffian state becomes gradually weaker. In Fig. 2(b), for $N_z = 16$ the overlaps increase significantly up to 0.94, when the screening is not strong ($\kappa = 1.08$). It decreases when the screening is strong. It is interesting that this time the $\frac{1}{2}$ overlap dramatically drops in the same region $\kappa \in [2, 3]$.

To confirm the stability of the ground states with different screening strength ($\kappa$), we explore the scenario in torus geometry [25, 47, 48]. It is interesting that when $\kappa > 2.6$, the ground state for $\nu = \frac{1}{2}$ becomes unstable due to the softening of the collective modes. This shows the geometry independence of the results. However, we do not find any instability at $\nu = \frac{1}{2}$, though the gaps of the collective modes are small. It is also possible that the $\frac{1}{2}$ state turns into a PH Pfaffian-like state when the screening is increased, so that the ground state is still incompressible.

All of these studies suggest that the ground state should be very close to the Pfaffian state with large LL gaps. However, when the LLM becomes stronger, i.e. $\kappa$ increases, the ground state could evolve into another state (perhaps the PH symmetric Pfaffian state) and a topological phase transition may occur. ZnO would be an ideal platform to experimentally study the PH symmetric Pfaffian state at $\nu = \frac{1}{2}$ due to the presence of strong LLM [21]. However, more experiments, such as the thermal Hall conductance related to the topological order of the bulk state [35, 36], are necessary to identify this property.

We also consider finite well thickness effect and consequently suggest that the electron gas is trapped in an infinite square well with width $L_z$. The $z$-component wave function is $\psi(z) = \sqrt{2/L_z} \sin(n\pi z/L_z)$. We suppose that the well is not very wide so that only the lowest subband dominates the system since the LL gap is very small in ZnO. The Coulomb interaction is modified by multiplying a thickness factor $V_z(q)$ [39]. The screened dielectric function which still preserves the rotational symmetry approximately becomes

$$\epsilon'_s(q) = 1 - V_z(q) V(q) \chi^0_{\nu n}(q).$$

Thickness alone can not transform the Pfaffian ground state to the PH symmetric one (when $L_z > 5.8\ell$ it is possible, but for a wide well the case should be different), yet with screening the situation would change significantly. For the Pfaffian shift $S = -3$, the $\frac{1}{2}$ overlap monotonically increases with the increase of the width. But the overlap of the $\frac{1}{2}$ state decreases when the width begins to increase. Moreover, the $\frac{1}{2}$ overlap is also very size dependent, in contrast to the monotonic increase in the absence of LLM [33]. For the PH symmetric shift $S = -1$, the excitation gaps of $\frac{1}{2}$ are not sensitive to the width. The gaps of $\frac{1}{2}$ become more smooth and less size dependent when the width increases, which agrees with the results in the torus geometry that the $\frac{1}{2}$-FQHE is more stable in a wider ZnO quantum well [27]. Hence this agreement also supports the idea that the ground state is more likely at the $-1$ shift for strong LLM.

To summarize, we have studied different topological states on a sphere in a large region of $\kappa$. The screened Coulomb potential obtained by the polarizability of all other LLs in the RPA indeed offers important information about the LLM. It gives us a clue to the nature of the even-denominator FQHE with small LL gaps. Since there is no direct indication [49] of a phase transition between the Pfaffian and anti-Pfaffian states in experiments, and further, the screening and the thickness is able to stabilize the incompressibility at the PH symmetric flux, it reveals that a newly proposed PH symmetric Pfaffian state [8, 30] may dominate the system in strong LLM. An obvious drop of the Pfaffian overlap in the region of $\kappa \in [2, 3]$ strongly suggests a topological phase transition from a Pfaffian state to another. If the strong LLM could be mapped onto the $s$-wave scattering tunable [50] or fast rotating cold atom systems (similar to the LLM effect) [51], then the PH symmetric Pfaffian state may be studied in those systems, which would open up different venues for exploring new topological states of matter.

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