A NONCOMMUTATIVE MARTINGALE CONVEXITY INEQUALITY

ÉRIC RICARD AND QUANHUA XU

Abstract. Let \( \mathcal{M} \) be a von Neumann algebra equipped with a faithful semifinite normal weight \( \phi \) and \( \mathcal{N} \) be a von Neumann subalgebra of \( \mathcal{M} \) such that the restriction of \( \phi \) to \( \mathcal{N} \) is semifinite and such that \( \mathcal{N} \) is invariant by the modular group of \( \phi \). Let \( \mathcal{E} \) be the weight preserving conditional expectation from \( \mathcal{M} \) onto \( \mathcal{N} \). We prove the following inequality

\[
\|x\|_p^2 \geq \|\mathcal{E}(x)\|_p^2 + (p-1)\|x - \mathcal{E}(x)\|_p^2, \quad x \in L_p(\mathcal{M}), \quad 1 < p \leq 2,
\]

which extends the celebrated Ball-Carlen-Lieb convexity inequality. As an application we show that there exists \( \varepsilon_0 > 0 \) such that for any free group \( F_n \) and any \( q \geq 4 - \varepsilon_0 \)

\[
\|P_t\|_{2 \rightarrow q} \leq 1 \iff t \geq \log \sqrt{q-1},
\]

where \( (P_t) \) is the Poisson semigroup defined by the natural length function of \( F_n \).

1. Introduction

Let \( \mathcal{M} \) be a von Neumann algebra equipped with a faithful semifinite normal weight \( \phi \). The associated noncommutative \( L_p \)-spaces will be simply denoted by \( L_p(\mathcal{M}) \). We refer to [10] for information on noncommutative integration. Recall that if \( \mathcal{N} \) is a von Neumann subalgebra of \( \mathcal{M} \) such that the restriction of \( \phi \) to \( \mathcal{N} \) is semifinite and such that \( \mathcal{N} \) is \( \sigma^\phi \)-invariant (i.e., \( \sigma_n^\phi(\mathcal{N}) = \mathcal{N} \) for all \( t \in \mathbb{R} \)), then there exists a unique \( \phi \)-preserving conditional expectation \( \mathcal{E} \) from \( \mathcal{M} \) onto \( \mathcal{N} \) such that

\[
\mathcal{E}(axb) = a\mathcal{E}(x)b, \quad a, b \in \mathcal{N}, \ x \in \mathcal{M}.
\]

Here \( \sigma^\phi \) denotes the modular group of \( \phi \). Moreover, \( \mathcal{E} \) extends to a contractive projection from \( L_p(\mathcal{M}) \) onto \( L_p(\mathcal{N}) \) for any \( 1 \leq p < \infty \). Below is our main result.

Theorem 1. Let \( \mathcal{M}, \mathcal{N} \) and \( \mathcal{E} \) be as above. If \( 1 < p \leq 2 \), then

\[
(1) \quad \|x\|_p^2 \geq \|\mathcal{E}(x)\|_p^2 + (p-1)\|x - \mathcal{E}(x)\|_p^2, \quad x \in L_p(\mathcal{M}).
\]

If \( 2 < p < \infty \), the inequality is reversed.

Inequality (1) is a martingale convexity inequality. It is closely related to the celebrated convexity inequality of Ball, Carlen and Lieb [2] for the Schatten classes \( S_p \). Namely, for \( 1 < p \leq 2 \) we have

\[
(2) \quad \|x + y\|_p^2 + \|x - y\|_p^2 \geq 2\|x\|_p^2 + 2(p-1)\|y\|_p^2, \quad x, y \in S_p.
\]

In fact, it is easy to see that (2) is a special case of (1) by considering \( \mathcal{M} = B(\ell_2) \oplus B(\ell_2) \). Conversely, the validity of (2) for any noncommutative \( L_p \)-spaces implies (1). Indeed, we will deduce (1) from the following

Theorem 2. Let \( \mathcal{M} \) be any von Neumann algebra. If \( 1 < p \leq 2 \), then

\[
(3) \quad \|x + y\|_p^2 + \|x - y\|_p^2 \geq 2\|x\|_p^2 + 2(p-1)\|y\|_p^2, \quad x, y \in L_p(\mathcal{M}).
\]

If \( 2 < p < \infty \), the inequality is reversed.

What is new and remarkable in (2) or (3) is the fact that \( (p-1) \) is the best constant. In fact, if one allows a constant depending on \( p \) in place of \( (p-1) \), then (3) is equivalent to the known results on the 2-uniform convexity of \( L_p(\mathcal{M}) \). We refer to [2] for more discussion on this point. The optimality of the constant \( (p-1) \) has important applications to hypercontractivity in the noncommutative case. It is the key to the solution of Gross’ longstanding open problem about the optimal hypercontractivity for Fermi fields by Carlen and Lieb [1]. It plays the same rôle in [3].

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and [3]. Note that the optimality of \((p - 1)\) in [3] implies \((p - 1)\) is also the best constant in [1]. It seems that [1] with this best constant is new even in the commutative case.

Clearly, [2] implies [3] for injective \(\mathcal{M}\) (or more generally, QWEP \(\mathcal{M}\)) since then \(L_p(\mathcal{M})\) is finitely representable in \(S_p\). The proof of [2] in [2] goes through a differentiation argument for the function \(t \mapsto \|x + ty\|_p^p\) with selfadjoint \(x\) and \(y\). It seems difficult to directly extend their argument to finite von Neumann algebras. The subtle point is the fact that to be able to differentiate the above function, one needs the invertibility of \(x + ty\) for all \(t \in [0, 1]\) except possibly countably many of them. This invertibility is easily achieved in the matrix algebra case, that is, for \(\mathcal{M} = \mathcal{M}_n\), the algebra of \(n \times n\)-matrices. Instead, we will use a pseudo-differentiation argument which is much less rigid than that of [2]. The main novelty in our argument can be simply explained as follows. We first cut the operator \(x + ty\) by its spectral projections in order to reduce the general case to the invertibility one; to do so we need that \(x + ty\) be of full support for all \(t \in [0, 1]\). We then get this full support property for all \(t\) by adding to \(x + ty\) an independent operator with diffuse spectral measure. Note that by standard perturbation argument, it is easy to insure the full support (or even the invertibility) of \(x + ty\) for one \(t\).

An iteration of Theorem 1 immediately implies the following inequality on noncommutative martingales.

**Corollary 3.** Let \((\mathcal{M}_n)_{n \geq 0}\) be an increasing sequence of von Neumann subalgebras of \(\mathcal{M}\) with \(w^*\)-dense union in \(\mathcal{M}\). Assume that each \(\mathcal{M}_n\) is \(\sigma^w\)-invariant and \(\phi|_{\mathcal{M}_n}\) is semifinite. Let \(\mathcal{E}_n\) be the conditional expectation with respect to \(\mathcal{M}_n\). Then for \(1 < p \leq 2\)

\[
\|x\|_p^p \geq \|\mathcal{E}_0(x)\|_p^p + (p - 1) \sum_{n \geq 1} \|\mathcal{E}_n(x) - \mathcal{E}_{n-1}(x)\|_p^2, \quad x \in L_p(\mathcal{M}).
\]

For \(2 < p < \infty\) the inequality is reversed.

Another possible iteration is the following

**Corollary 4.** Let \((\mathcal{M}_n)_{1 \leq n \leq N}\) be a family of von Neumann subalgebras of \(\mathcal{M}\). Assume that each \(\mathcal{M}_n\) is \(\sigma^w\)-invariant and \(\phi|_{\mathcal{M}_n}\) is semifinite. Let \(\mathcal{E}_n^+\) be the conditional expectation with respect to \(\mathcal{M}_n\) and \(\mathcal{E}_n^- = \text{Id} - \mathcal{E}_n^+\). Then for \(1 < p \leq 2\)

\[
\|x\|_p^p \geq \sum_{(\varepsilon_i) \in \{+1, -1\}^N} (p - 1)^{|(\varepsilon_i) = 0|} \left\| \left( \prod_{i=1}^N \mathcal{E}_{c_i}^\varepsilon_i \right)(x) \right\|_p^2, \quad x \in L_p(\mathcal{M}).
\]

For \(2 < p < \infty\) the inequality is reversed.

Applying it with \(\mathcal{M} = L_\infty(\{\pm 1\}^N)\) and \(\mathcal{M}_n\) the subalgebra of functions independent of the \(n^{th}\) variable gives the classical hypercontractivity for the Walsh system (with operator valued coefficients). Similarly, taking \(\mathcal{M}\) to be the Clifford algebra with \(N\) generators, one obtains the optimal hypercontractivity for Fermi fields as pointed out in [2], [4].

We end the paper by some applications to hypercontractivity for group von Neumann algebras. In particular for the Poisson semigroup of a free group, we obtain the optimal time for the hypercontractivity from \(L_2\) to \(L_q\) for \(q \geq 4\).

2. The proofs

We will prove Theorems 1 and 2. Using the Haagerup reduction theorem as in [4], one can reduce both theorems to the finite case. Thus throughout this section \(\mathcal{M}\) will denote a von Neumann algebra equipped with a faithful tracial normal state \(\tau\). \(L_p(\mathcal{M})\) is then constructed with respect to \(\tau\). We will first prove [3], then deduce [1] from it. \(1 < p < 2\) will be fixed in the sequel.

As explained before, the proof of [3] will be done by a pseudo-differentiation argument. Recall that for a continuous function \(f\) from an interval \(I\) to \(\mathbb{R}\) its pseudo-derivative of second order at \(t\) is

\[
D^2 f(t) = \liminf_{h \to 0^+} \frac{f(t + h) + f(t - h) - 2f(t)}{h^2}.
\]

This pseudo-derivative shares many properties of the second derivative. For instance, if \(D^2 f\) is nonnegative on \(I\), then \(f\) is convex. Indeed, by adding \(\varepsilon t^2\) to \(f\) (with \(\varepsilon > 0\)) we can assume that \(D^2 f(t)\) is positive for all \(t\). If \(f\) was not convex, there would exist \(t_0 < t_1\) in \(I\) such that the
Consequently, D as in [2] by functional calculus, the operator \((a + \tau t b + ba + t^2 b^2)^{-1}\) is twice differentiable in a neighborhood of the origin and hence \((a + \tau t b + ba + t^2 b^2)^{-1} ((ab + ba) + 2tb^2))\) can be adapted to give:

**Lemma 5.** Let \(a, b \in \mathcal{M}\) be selfadjoint elements with a invertible. Then \((D^2_{a,b})\) holds.

**Proof.** As \(a\) is invertible in \(\mathcal{M}\), \(a + \tau t b\) is also invertible for small \(t\). Introduce an auxiliary function \(\psi\) on \(\mathbb{R}\):

\[
\psi(t) = \|a + \tau t b\|_p^2 = \tau \left( (a^2 + t(ab + ba) + t^2 b^2) \right).
\]

\(\psi\) is differentiable in a neighborhood of the origin and

\[
\psi'(t) = \frac{p}{2} \tau \left[ (a^2 + t(ab + ba) + t^2 b^2)^{-1} ((ab + ba) + 2tb^2) \right].
\]

As in [2] by functional calculus, the operator \((a^2 + t(ab + ba) + t^2 b^2)^{-1}\) admits the following integral representation

\[
(a + \tau t b + ba + t^2 b^2)^{-1} = \tau^{-1} \int_0^\infty \left( s + a^2 \right)^{-1} \frac{1}{s + a^2 + t(ab + ba) + t^2 b^2} ds,
\]

where

\[
\tau^{-1} = \int_0^\infty \left( s + a^2 \right)^{-1} \frac{1}{s + 1} ds.
\]

Thus \(\psi\) is twice differentiable at \(t = 0\) and

\[
\psi''(0) = \frac{p}{2} \tau \left[ (a^2 - 2p \psi''(0)) + \frac{2}{p} \|a\|_p^{2-p} \|\psi''(0)\| \right] ds.
\]

It then follows that \(\varphi = \psi \tau\) is also differentiable at \(t = 0\) and

\[
\varphi''(0) = \frac{2}{p} \left( \frac{2}{p} - 1 \right) \|a\|_p^{2-2p} \|\psi''(0)\| \geq \frac{2}{p} \|a\|_p^{2-p} \|\psi''(0)\|.
\]

Hence \((D^2_{a,b})\) will be a consequence of

\[
\frac{1}{p} \|a\|_p^{2-p} \|\psi''(0)\| \geq (p - 1) \|b\|_p^2.
\]

To prove the last inequality we claim that \(\psi''(0)\) increases when \(a\) is replaced by \(|a|\). Indeed, the trace inside the integral in [5] is equal to twice the following sum

\[
\tau \left[ \frac{a}{s + a^2} b \frac{a}{s + a^2} b \right] + \tau \left[ \frac{a^2}{s + a^2} b \frac{1}{s + a^2} b \right].
\]

The second term above depends only on \(|a|\) (recalling that \(a\) is selfadjoint). It remains to show that the first one increases when \(a\) is replaced by \(|a|\). By decomposing \(a\) into its positive and negative parts, we see that the first term is equal to

\[
\tau \left[ \frac{a^+}{s + a^+} b \frac{a^+}{s + a^+} b \right] + \tau \left[ \frac{a^-}{s + a^-} b \frac{a^-}{s + a^-} b \right] - 2\tau \left[ \frac{a^+}{s + a^+} b \frac{a^-}{s + a^-} b \right].
\]

All above traces are nonnegative. Therefore, the above quantity increases when the subtraction is replaced by addition. Then trace back the argument and noting that \(|a| = a^+_+ a^-_\), we get the desired inequality

\[
\tau \left[ \frac{a}{s + a^2} b \frac{a}{s + a^2} b \right] \leq \tau \left[ \frac{|a|}{s + a^2} b \frac{|a|}{s + a^2} b \right].
\]

Returning back to [5], we deduce the claim. Thus in the following we will assume that \(a\) is a positive invertible element of \(\mathcal{M}\).

The positivity of \(a\) will facilitate the calculation of \(\psi''(0)\) as explained in [2]. Since \(a + tb\) is positive for small \(t\), we have

\[
\psi(t) = \tau ((a + tb)^p).
\]
Thus for $t$ close to 0
\[
\psi'(t) = p \tau((a + tb)^{p-1}b).
\]
To calculate the second derivative we use the following integral representation:
\[
(a + tb)^{p-1} = d_p \int_0^\infty s^{p-1} \left[ \frac{1}{s + a + tb} \right] ds.
\]
Consequently,
\[
\psi''(0) = p d_p \int_0^\infty s^{p-1} \tau\left[ \frac{1}{s + a} b \right] ds.
\]
As shown in [2], the function $F : z \mapsto \tau\left[ \frac{1}{s + z} b \right]$ is convex on the positive cone of $\mathcal{M}$.

Let $u$ be the unitary operator in the polar decomposition of $b$ (as $\mathcal{M}$ is finite, the usual partial isometry in this decomposition can be chosen to be a selfadjoint unitary). Then clearly
\[
F(z) = \frac{1}{2} (F(z) + F(uzu)) \geq F\left(\frac{z + uzu}{2}\right).
\]
Now $z' = \frac{z + uzu}{2}$ commutes with $u$, so
\[
F(z) \geq F(z') = \tau\left[ \frac{1}{s + z'} |b| \frac{1}{s + z} |b| \right].
\]
Let $B$ be the abelian von Neumann subalgebra of $\mathcal{M}$ generated by $b$, and let $\mathcal{E}_b$ be the associated trace preserving conditional expectation. Recall that for any $\alpha, \beta \in \mathcal{M}_+$ with $\alpha$ invertible we have
\[
\beta \alpha^{-1} \beta = \inf \left\{ \gamma \in \mathcal{M}_+ : \left[ \begin{array}{c} \alpha \\ \beta \\ \gamma \end{array} \right] \geq 0 \right\}.
\]
It follows that if $\alpha \in B$,
\[
\mathcal{E}_b (\beta \alpha^{-1} \beta) \geq \mathcal{E}_b (\beta) \mathcal{E}_b (\beta) = \mathcal{E}_b (\beta)^2 \alpha^{-1}.
\]
Replacing $\alpha^{-1}$ by $\alpha$, the invertibility assumption on $\alpha$ can be removed in this form. Hence, by the positivity of the trace on products of positive elements,
\[
F(z) \geq \tau\left[ \mathcal{E}_b \left( \frac{1}{s + z'} \right)^2 |b|^2 \right].
\]
Then by the operator convexity of $\frac{1}{\tau}$ we have
\[
\mathcal{E}_b \left( \frac{1}{s + z} \right) \geq \frac{1}{s + \mathcal{E}_b (z')} = \frac{1}{s + \mathcal{E}_b (z)}.
\]
Letting $\tilde{a} = \mathcal{E}_b (a)$, we have just shown
\[
\psi''(0) \geq p d_p \int_0^\infty s^{p-1} \tau\left[ \frac{1}{s + \tilde{a}} b \right] ds = \tilde{\psi}''(0),
\]
where
\[
\tilde{\psi}(t) = \tau((\tilde{a} + tb)^p).
\]
Finally, (7) immediately follows from (7). Indeed, by (7) and the Hölder inequality,
\[
\frac{1}{p} \|a\|_p^{2-p} \psi''(0) \geq \frac{1}{p} \|\tilde{a}\|_p^{2-p} \tilde{\psi}''(0) = (p - 1)\|\tilde{a}\|_p^{2-p} \tau(\tilde{a}^{p-2} b^2) \geq (p - 1)\|b\|_p^2.
\]
This finishes the proof of the lemma.

For $a \in \mathcal{M}$ selfadjoint, we denote by $s(a) = \mathbb{1}_{(0, \infty)}(a)$. $s(a)$ is the support of $a$, i.e., the least projection $e$ of $\mathcal{M}$ such that $ea = a$. We say that $a$ has full support if $s(a) = 1$.

**Lemma 6.** Let $a, b \in \mathcal{M}$ be selfadjoint with $s(a) = 1$. Then $(D_{a,b}^2)$ holds.
Proof. We will reduce this lemma to the previous one by cutting \(a + tb\) with the spectral projections of \(a\). Let \(e\) be a non zero spectral projection of \(a\) and put \(a_e = eae\) and \(b_e = ebe\). Since \(a\) is of full support, \(a_e\) is invertible in the reduced von Neumann algebra \(\mathcal{M}_e = e\mathcal{M}e\). Thus Lemma \(\text{[3]}\) can be applied to the couple \((a_e, b_e)\) in \(\mathcal{M}_e\). Let \(\psi_e(t) = \|a_e + tb_e\|_p^2\) as before. \(\psi_e\) is twice differentiable at \(t = 0\) and \(\text{[3]}\) holds with \(\psi_e\) in place of \(\psi\).

Let \(e^+ = 1 - e\). Then for \(t\) in a neighborhood of the origin, we have (recalling that \(\varphi(t) = \|a + tb\|_p^2\))

\[
\varphi(t) \geq \left(\|e(a + tb)e\|_p^p + \|e^+(a + tb)e^+\|_p^p\right)^\frac{1}{p} \overset{\text{def}}{=} \left(\psi_e(t) + \gamma_e(t)\right)^\frac{1}{p}.
\]

However,

\[
\psi_e(t) = \|a_e\|^p + t\psi_e'(0) + t^2 \psi_e''(0) + o(t^2) \quad \text{as} \quad t \to 0.
\]

Let

\[
\alpha(t) = \|a_e\|^p + \gamma_e(t) = \|a_e + e^+(a + tb)e^+\|_p^p = \|a + te^-1be^-1\|_p^p.
\]

Then

\[
\varphi(t) \geq \left(\alpha(t) + t\psi_e'(0) + \frac{t^2}{2} \psi_e''(0) + o(t^2)\right)^\frac{1}{p} \\
= \alpha(t)^\frac{1}{p} \left(1 + \frac{2t\psi_e'(0)}{p \alpha(t)} + \frac{t^2}{p} \frac{\psi_e''(0)}{\alpha(t)} + \frac{1}{p^2} \frac{t^2 \psi_e''(0)^2}{\alpha(t)^2} + o(t^2)\right) \\
= \alpha(t)^\frac{1}{p} + \frac{2t}{p} \psi_e'(0)\alpha(t)^{-\frac{1}{p}} + \frac{t^2}{p} \psi_e''(0)\alpha(t)^{-\frac{2}{p}} + \frac{1}{p^2} \frac{t^2 \psi_e''(0)^2}{\alpha(t)^{\frac{2}{p}}} + o(t^2) \\
geq \alpha(t)^\frac{1}{p} + \frac{2t}{p} \psi_e'(0)\alpha(t)^{-\frac{1}{p}} + \frac{t^2}{p} \psi_e''(0)\alpha(t)^{-\frac{2}{p}} + o(t^2).
\]

By convexity of norms,

\[
\alpha(t)^\frac{1}{p} + \alpha(-t)^\frac{1}{p} \geq 2\|a\|^\frac{2}{p} = 2\varphi(0).
\]

We then deduce that

\[
\varphi(t) + \varphi(-t) - 2\varphi(0) \geq \frac{2}{p} \psi_e'(0)\alpha(t)^{-\frac{1}{p}} + \frac{2}{p} \psi_e''(0)\alpha(t)^{-\frac{2}{p}} + \frac{1}{p} \psi_e''(0)[\alpha(t)^{-\frac{2}{p}} + \alpha(-t)^{-\frac{2}{p}}] + o(1).
\]

The uniform smoothness of the norm \(\|\cdot\|_p\) implies that the function \(\alpha^{\frac{1}{p}}\) is differentiable at \(t = 0\) and its derivative is equal to

\[
(2 - p)\|a\|^1\tau(\alpha^{\frac{1}{1-p}}) \overset{\text{def}}{=} \delta_e,
\]

where \(\tau\) is the unitary in the polar decomposition of \(a\). It then follows that

\[
D^2\varphi(0) \geq \frac{4}{p} \psi_e'(0)\delta_e + \frac{2}{p} \psi_e''(0)\|a\|^{2-p}.
\]

Hence by \(\text{[3]}\),

\[
D^2\varphi(0) \geq \frac{4}{p} \psi_e'(0)\delta_e + 2(p - 1)\|b\|_p^2.
\]

Thanks to the full support assumption of \(a\), we can let \(e \to 1\) in the above inequality. This limit procedure removes the first extra term, so we finally get

\[
D^2\varphi(0) \geq 2(p - 1)\|b\|_p^2.
\]

Now we are ready to show \(\text{[3]}\).

Proof of Theorem \(\text{[3]}\) First by density, we need only to show \(\text{[3]}\) for \(x, y \in \mathcal{M}\). Then notice that it suffices to do it for selfadjoint elements using a classical \(2 \times 2\)-matrix trick. Indeed, let \(\mathcal{M} = \mathbb{M}_2 \otimes \mathcal{M}\) equipped with the tensor trace. Given \(x, y \in \mathcal{M}\) let

\[
a = \begin{pmatrix} 0 & x \\ x^* & 0 \end{pmatrix} \quad \text{and} \quad b = \begin{pmatrix} 0 & y \\ y^* & 0 \end{pmatrix}.
\]

Then \(a\) and \(b\) are selfadjoint. Moreover, by easy computations, \(\text{[3]}\) for \(x\) and \(y\) is equivalent to the same inequality for \(a\) and \(b\).
To use Lemma 6 we require that $a + tb$ have full support for any $t \in \mathbb{R}$. This is achieved by a tensor product argument. Choose a positive element $c \in L_\infty([0, 1])$ whose spectral measure with respect to Lebesgue measure is diffuse (atomless), say $c(t) = t$ for $t \in [0, 1]$. In other words, considered as a random variable in the probability space $[0, 1]$, the law of $c$ is diffuse. On the other hand, for $t \in \mathbb{R}$, composing the spectral resolution of $a + tb$ with the trace $\tau$, we can view $a + tb$ as a random variable in another probability space $(\Omega, P)$. Now, consider the tensor von Neumann algebra $L_\infty([0, 1]) \otimes M$, it is finite. $M$ and $L_\infty([0, 1])$ are identified as subalgebras of $L_\infty([0, 1]) \otimes M$ in the usual way. Then for any $\varepsilon > 0$, the law of $a + tb + \varepsilon c$ is the convolution of the laws of $a + tb$ and $\varepsilon c$. It is atomless since the law of $c$ is. Consequently, the support of $a + tb + \varepsilon c$ is full in $L_\infty([0, 1]) \otimes M$.

Thus by Lemma 6 applied to the pair $(a + \varepsilon c, b)$, the function $f(t) = \|a + tb + \varepsilon c\|^2 - (p - 1)L_\infty(t)^2\|b\|^2_p$ satisfies $D^2(f)(t) \geq 0$, so is convex. Hence $f(1) + f(-1) \geq 2f(0)$, this is (3) for $a + \varepsilon c$ and $b$. Letting $\varepsilon \to 0$ gives the desired result. 

Finally, we deduce (1) from (3).

Proof of Theorem 2. Given $x \in L_p(M)$ let $a = \mathcal{E}(x)$ and $b = x - \mathcal{E}(x)$. Consider again the function $f$ defined by

$$f(t) = \|a + tb\|^2 - (p - 1)L_\infty(t)^2\|b\|^2_p.$$ 

Then (3) implies $D^2f \geq 0$, so $f$ is convex. On the other hand, The function $g(t) = \|a + tb\|^2$ is also convex and by the contractivity of $\mathcal{E}$ on $L_p(M)$,

$$\|a + tb\|_p \geq \|\mathcal{E}(a + tb)\|_p = \|a\|_p.$$

Hence we conclude that the right derivative $g'(0) \geq 0$, so that $f'(0) \geq 0$ too. Consequently, $f$ is increasing on $\mathbb{R}^+$. In particular, $f(1) \geq f(0)$, which is nothing but (1). 

3. Applications to hypercontractivity

We give in this section some applications to hypercontractivity inequalities on group von Neumann algebras. Let $G$ be a discrete group and $\nu N(G)$ the associated group von Neumann algebra. Recall that $\nu N(G)$ is the von Neumann algebra generated by the left regular representation $\lambda$: $\nu N(G) = \lambda(G)'' \subset B(H_2(G))$. It is equipped with a canonical trace $\tau$, i.e., $\tau(x) = \langle xe, e \rangle$, where $e$ is the identity of $G$. Given a function $\psi: G \to \mathbb{R}_+$ with $\psi(e) = 0$, we consider the associated Fourier-Schur multiplier initially defined on the family $\mathbb{C}[G]$ of polynomials on $G$:

$$P_t: \sum_{g \in G} g(\lambda(g) \mapsto \sum_{g \in G} e^{-t\psi(g) x(g) \lambda(g)}, \ t > 0.$$ 

We will assume that $P_t$ extends to a contraction on $L_p(\nu N(G))$ for every $1 \leq p \leq \infty$. Schoenberg’s classical theorem asserts that if $\psi$ is symmetric and conditionally negative, $P_t$ is a completely positive map on $\nu N(G)$. Since it is trace preserving, $P_t$ defines a contraction on $L_p(\nu N(G)$ for every $1 \leq p \leq \infty$. Thus in this case our assumption is satisfied.

The hypercontractivity problem for the semigroup $(P_t)_{t>0}$ and for $1 < p < q < \infty$ consists in determining the optimal time $t_{p,q} > 0$ such that

$$\|P_t\|_{p \to q} \leq 1, \quad \forall t \geq t_{p,q}.$$ 

We refer to [3, 9] for more information and historical references. It is easy to check that if such a time $t_{p,q}$ exists, then $\psi$ has a spectral gap, namely $\inf_{g \in G \setminus \{e\}} \psi(g) > 0$. After rescaling, we will assume that $\inf_{g \in G \setminus \{e\}} \psi(g) = 1$.

In most known cases the expected optimal time $t_{p,q}$ is attained, namely,

$$t_{p,q} = \log \left( \frac{q - 1}{p - 1} \right).$$ 

It is a particularly interesting problem of determining the optimal time $t_{p,q}$ in the special case where $G = F_n$ is the free group on $n$ generators with $n \in \mathbb{N} \cup \{\infty\}$ and $\psi$ is its natural length function. Some partial results are obtained in [3, 9]. For instance, by embedding $\nu N(F_n)$ into a free product of Clifford algebras, it is proved in [9] that for any $q > 2$

$$t_{2,q} \leq \log \sqrt{q - 1} + \left( \frac{1}{2} - \frac{1}{q} \right) \log \sqrt{2}.$$
On the other hand, [S] shows that for any finite \( n \) there exists \( q(n) \) such that if \( q \geq q(n) \) is an even integer then

\[
t_{2,q} = \log \sqrt{q - 1}.
\]

The proof is combinatoric and based on lengthy calculations.

Here we provide an improvement. We will use Haagerup’s type inequalities [6]. Letting \( S_k \) be the set of words of length \( k \) in \( \mathbb{F}_n \) and for any \( x \in vN(\mathbb{F}_n) \) supported on \( S_k \), the original Haagerup inequality is

\[
\|x\|_\infty \leq (k + 1)\|x\|_2.
\]

For \( q > 2 \) and \( k \in \mathbb{N} \), let \( K_{k,q} \) be the best constant in the following Khintchine inequality for homogeneous polynomials \( x \) of degree \( k \)

\[
\| \sum_{g \in S_k} x(g) \lambda(g) \|_q \leq K_{k,q} \| \sum_{g \in S_k} x(g) \lambda(g) \|_2.
\]

We will need the following:

**Lemma 7.** We have \( K_{k,q} \leq (k + 1)^\frac{4}{q} \).

**Proof.** Denote by \( g_i \) the generators of \( \mathbb{F}_n \) with the convention that \( g_i^{-1} = g_{i+1} \). For a multi-index \( \bar{i} = (i_1, \ldots, i_d) \) with \( i_j + i_{j+1} \neq 0 \), we let \( g_\bar{i} = g_{i_1} \cdots g_{i_d} \) and \( |\bar{i}| = d \). So we may write

\[
x = \sum_{|\bar{i}| = k} \alpha_{\bar{i}} \lambda(g_\bar{i}).
\]

We compute \( x^*x \) according to simplifications that may occur

\[
x^*x = \sum_{0 \leq d \leq k} \sum_{|\bar{i}| = k \atop i_k - \bar{i}_k \neq j_k - d \atop i_k \neq \bar{i}_k} \alpha_{\bar{i},\beta} \lambda(g_\bar{i}^{-1}g_\beta),
\]

where \( \bar{i}, \beta \) denotes the multi-index obtained by superposing the two multi-indices \( \bar{i} \) and \( \beta \). We then deduce that

\[
\|x\|_4^4 = \|x^*x\|_2^2 = \sum_{0 \leq d \leq k} \sum_{|\bar{i}| = k \atop i_k - \bar{i}_k \neq j_k - d \atop i_k \neq \bar{i}_k} \left( \sum_{|\bar{\beta}| = d} \alpha_{\bar{i},\beta} \right)^2 \leq \sum_{0 \leq d \leq k} \sum_{|\bar{i}| = k \atop i_k - \bar{i}_k \neq j_k - d \atop i_k \neq \bar{i}_k} \left( \sum_{|\bar{\beta}| = d} |\alpha_{\bar{i},\beta}|^2 \right) \left( \sum_{|\bar{\beta}| = d} |\alpha_{\bar{i},\beta}|^2 \right) \leq \sum_{0 \leq d \leq k} \left( \sum_{|\bar{\beta}| = d} |\alpha_{\bar{i},\beta}|^2 \right) \left( \sum_{|\bar{\beta}| = d} |\alpha_{\bar{i},\beta}|^2 \right) = (k + 1)\|x\|_2^2.
\]

□

**Remark 8.** Taking \( \alpha_{\bar{i}} = 1 \) and by the free central limit theorem as \( n \to \infty \), one can see that the previous inequality is sharp. Thus \( K_{k,4} = (k + 1)^\frac{2}{4} \). This constant is the \( L_4 \)-norm of the \( k^\text{th} \) Tchebychev polynomial for the semi-circle law.

Using the Hölder inequality we deduce from (S) and the previous lemma that for any \( k \geq 1 \)

\[
K_{k,q} \leq (k + 1)^{1 - \frac{4}{q}}, \quad q \geq 4,
\]

\[
K_{k,q} \leq (k + 1)^{\frac{2}{q} - \frac{1}{4}}, \quad 2 \leq q \leq 4.
\]

We will also use the following elementary folklore:

**Remark 9.** Let \( T : L_p(\mathcal{M}) \to L_q(\mathcal{M}) \) be a bounded linear map. Assume that \( T \) is 2-positive in the sense that \( \text{Id}_{\mathcal{M}_2} \otimes T \) maps the positive cone of \( L_p(\mathcal{M}_2 \otimes \mathcal{M}) \) to that of \( L_q(\mathcal{M}_2 \otimes \mathcal{M}) \). Then

\[
\|T(x)\|_q \leq \|T([x])\|_2 \|T([x^*])\|_q, \quad x \in L_p(\mathcal{M}).
\]
Consequently, 
\[ \|T\| = \sup \{\|T(x)\|_q : x \in L_p(M)^+, \|x\|_p \leq 1\}. \]

Indeed, for any \( x \in L_p(M), \)
\[ \left( \begin{array}{c} |x| \\ x^* \\ x^{*1} \end{array} \right) \geq 0. \]
So the 2-positivity of \( T \) implies
\[ \left( \begin{array}{ccc} T(|x|) & T(x) \\ T(x^*) & T(|x^*|) \end{array} \right) \geq 0. \]
This yields a contraction \( c \in M \) such that \( T(x) = T(|x|)^{1/2} c T(|x^*|)^{1/2} \). Then the Hölder inequality gives the assertion.

**Theorem 10.** There exists \( \varepsilon_0 > 0 \) such that for any free \( \mathbb{F}_n \) and any \( q \geq 4 - \varepsilon_0 \)
\[ \|P_t\|_{2 \rightarrow q} \leq 1 \iff t \geq \log \sqrt{q - 1}. \]

**Proof.** The necessity is clear. The proof of the sufficiency will rely on Remark 3.7 of \[9\]. Let \( \sigma \) be the automorphism of \( vN(\mathbb{F}_n) \) given by \( \sigma(\lambda(y_i)) = \lambda(y_i^{-1}) \). Then \( P_t \) is hypercontractive from \( L_2 \) to \( L_q \) with optimal time on \( vN(\mathbb{F}_n)^\sigma \), the fixed point algebra of \( \sigma \). Let \( \mathcal{E} \) be the conditional expectation on \( vN(\mathbb{F}_n)^\sigma \). Note that \( \mathcal{E} = \frac{Id + \sigma}{2} \) and it commutes with \( P_t \).

Fix \( q > 2 \). To prove \( \|P_t\|_{2 \rightarrow q} \leq 1 \) for \( t \geq \log \sqrt{q - 1} \) it suffices to show \( \|P_t(x)\|_q \leq \|x\|_2 \) for any positive \( x \in C[\mathbb{F}_n] \) by virtue of Remark \[9\] We need one more reduction. Given complex numbers \( \zeta \) of modulus 1, there exists an automorphism \( \pi_\zeta \) of \( vN(\mathbb{F}_n) \) given by \( \pi_\zeta(g) = \zeta g \). It is an isometry on all \( L_p \)'s. Note that \( \pi_\zeta \) and \( P_t \) commute. Thus to prove \( \|P_t(x)\|_q \leq \|x\|_2 \), we may assume that \( x(g_i) \) is real for every generator \( g_i \). We will fix a positive \( x \in C[\mathbb{F}_n] \) with the last property.

Then write \( x = y + z \) where \( y = \mathcal{E}(x). \) Since \( x(g_i) \in \mathbb{R} \), we have that \( z \) does not have constant terms nor of degree 1. By Theorem \[1\] (or Theorem \[3\] and Remark 3.7 of \[9\])
\[ \|P_t(x)\|_q^2 \leq \|P_t(y)\|_q^2 + (q - 1)\|P_t(z)\|_q^2 \leq \|y\|_2^2 + (q - 1)\|P_t(z)\|_q^2. \]
Then for \( t = \log \sqrt{q - 1}, \) decomposing \( z \) according to its homogeneous components \((z_k)\), using the Khintchine inequality and the Cauchy-Schwarz inequality, we get
\[ \|P_t(z)\|_q^2 \leq \left( \sum_{k \geq 2} e^{-tk} \|z_k\|_q^2 \right) \leq \sum_{k \geq 2} K_{k,q}^2 \frac{1}{(q - 1)^k} \|z\|_2^2. \]
We aim to find those \( q > 2 \) for which
\[ R_q = (q - 1) \sum_{k \geq 2} K_{k,q}^2 \frac{1}{(q - 1)^k} \leq 1. \]

For \( q \geq 4 \), by \[1\] we have
\[ R_q \leq \sum_{k \geq 2} (k + 1)^{2(1 - \frac{1}{q})} \frac{1}{(q - 1)^k}. \]
The terms of the sum on the right hand side are decreasing functions of \( q \) if their derivatives are negative, i.e., if
\[ \frac{6(q - 1)}{q^2} \leq \frac{k - 1}{\log(k + 1)}. \]
Noting that the left hand side of the above inequality is decreasing on \( q \), one easily checks that this inequality is true for \( q \geq 4 \) and \( k \geq 3 \). However, it is true for \( k = 2 \) if \( q \geq q_0 \), where \( q_0 = \sqrt{3 \log 3} \left( \sqrt{3 \log 3} + \sqrt{3 \log 3 - 2} \right) \approx 5.36244. \)
We have the following numerical estimates
\[ R_4 \leq 0.92952 \quad \text{and} \quad \frac{3^{2(1 - \frac{1}{q_0})} - 3^{\frac{1}{3}}}{3} \leq 0.02613. \]
Hence if \( q \in [4, q_0] \)
\[ R_q \leq R_4 + \frac{3^{2(1 - \frac{1}{q_0})}}{q_0 - 1} - \frac{3^{\frac{1}{3}}}{3} < 1. \]
We thus conclude that \( R_q < 1 \) for all \( q \geq 4 \).
Since $R_q$ is dominated by a continuous function of $q$, using (10) we get a similar estimate for $q \geq 4 - \varepsilon_0$ for some $\varepsilon_0$. A numerical estimate gives $\varepsilon_0 \approx 0.18$. □

**Remark 11.** Instead of Remark 3.7 of [3], we can equally use Theorem A iii) of [3] in the preceding proof. But the commutation of $P_t$ and the conditional expectation onto the symmetric subalgebra $A_{sym}^+$ is less obvious.

It is likely that $\varepsilon_0 = 2$ but other methods would have to be developed.

Gross’s pioneer work [5] shows that hypercontractivity is equivalent to the validity of log-Sobolev inequalities. In the present situation of free groups, the validity of the hypercontractivity with optimal time in full generality (or equivalently, $\varepsilon_0 = 2$) is equivalent to the following log-Sobolev inequality in $L_q$ for any $q \geq 2$:

$$\text{(SL}_q)\quad \tau(x^q \log x) \leq \frac{q}{2(q-1)} \tau(x^{q-1} L(x)) + \|x\|^q \log \|x\|_q, \quad x \in D^+$$

Here $L$ denotes the negative generator of $(P_t)$ and $D$ is a core for $L$ where the inequality makes sense. It is known that (SL$_2$) implies (SL$_q$) for all $q$ (see [11]). In the same spirit we can show that (SL$_p$) implies (SL$_q$) if $q > p \geq 2$. Let us record this explicitly here since it might be of interest. The semigroup $(P_t)$ can be any completely positive symmetric Markovian semigroup such that $D$ is rich enough.

**Remark 12.** Let $q > p \geq 2$. Then (SL$_p$) implies (SL$_q$).

To check the remark we rewrite (SL$_q$) in a symmetric form with respect to $q$ and its conjugate index $q'$ (provided that $D$ is big enough):

$$\text{(SL}^*_q)\quad \tau(x \log x) \leq \frac{1}{2} q' q \tau(x^{q-1} L(x^{*})) + \tau(x) \log \tau(x), \quad x \in D^+.$$  

Recall that for $y \in \text{Dom}(L)$, $\tau(z L(y)) = \lim_{r \to 0} \frac{1}{r} \tau(z(1 - P_r)(y))$. Let $r > 0$ and $x \in M^+$, we will check that the function $q \mapsto q' q \tau(x^{q-1} (1 - P_r)(x^{*}))$ is increasing for $q \geq 2$; we put $\theta = \frac{1}{2}$. It is known from [3] that there exists a positive symmetric Borel measure $\mu_r$ on $\sigma(x) \times \sigma(x)$ such that

$$\tau(x^{q-1} P_r(x^{*})) = \int_{\sigma(x) \times \sigma(x)} s^{1-\theta} \sigma^\theta d\mu_r(s, t).$$

Hence, by symmetry, it suffices to show that

$$f : \theta \mapsto \frac{1 + u - u^\theta - u^{1-\theta}}{\theta(1 - \theta)}$$

is convex on $[0, 1]$ for $u > 0$ as $f(\theta) = f(1 - \theta)$. One easily checks that

$$f(\theta) = \int_0^1 \log(u) u^{\theta + (1-\theta)(1-t)} - u^{\theta} + u^{1-\theta + \theta t} - u^{(1-\theta)(1-t)} dt,$$

$$f''(\theta) = \int_0^1 \log(u)^2 (u^{\theta + (1-\theta)(1-t)} - u^{\theta}) + (1-t)^2(u^{1-\theta + \theta t} - u^{(1-\theta)(1-t)}) dt \geq 0.$$

Passing to the limit in $r$ gives the result if $D$ is big enough.

We end this section by a more application to general groups $(G, \psi)$. If $\psi$ is symmetric and satisfies the following exponential order growth

$$(11)\quad \{ \{g \in G : \psi(g) \leq R\} \} \leq C \rho^R, \quad \forall R > 0$$

for some $C > 0$ and $\rho > 1$, then one of the main results of [8] shows that for $2 < q < \infty$

$$t_{2,q} \leq \eta \log \sqrt{q-1}$$

for any $\eta > 2$ when $\rho$ is large compared to $C$. Their argument consists in first considering the case $q = 4$ by combinatoric methods and then using Gross’s extrapolation. We will show that the martingale inequality in Theorem [4] easily implies a slight improvement. Note that our estimate on $t_{2,q}$ is as close as to the expected optimal time when $q$ is sufficiently large compared to $\rho$ and $C$.  


Proposition 13. Assume (11) and \(2 < q < \infty\). Then
\[
t_{2,q} \leq \left( \frac{q-2}{q} \log \sqrt{C \rho} + \log \sqrt{q-1} \right) + \log \rho.
\]

Proof. By (11), the range of \(\psi\) is countable. Let \(\psi(G) = \{n_0, n_1, n_2, \ldots\}\) with \(n_0 < n_1 < n_2 < \ldots\). Then \(n_0 = 0\) and \(n_1 = 1\). Let \(x \in vN(G)\) be a polynomial, \(x = \sum x(g)\lambda(g)\), and let \(y = x - x(e)\). By Theorem [11],
\[
\|P_t(x)\|_q^2 \leq \|x(e)^2 + (q-1)\|P_t(y)\|_q^2.
\]

Let \(B_k = \{g \in G : \psi(g) \leq n_k\}\), \(S_k = B_k \setminus B_{k-1}\) and \(y_k = \sum_{g \in S_k} x(g)\lambda(g)\). Then
\[
\|P_t(y)\|_q^2 \leq \left( \sum_{k \geq 1} e^{-tn_k} \|y_k\|_\infty \right)^2 \leq \left( \sum_{k \geq 1} e^{-2tn_k} |S_k| \right) \left( \sum_{k \geq 1} \frac{\|y_k\|_\infty^2}{|S_k|} \right).
\]

Obviously,
\[
\|y_k\|_\infty^2 \leq \left( \sum_{g \in S_k} |x(g)| \right)^2 \leq |S_k| \sum_{g \in S_k} |x(g)|^2.
\]

We get, using the H"older inequality
\[
\|P_t(y)\|_q^2 \leq e^{-\frac{2t}{q}} \left( \sum_{k \geq 1} e^{-2tn_k} |S_k| \right)^{\frac{q-2}{q}} \|y\|_2^q.
\]

Actually exchanging the arguments, one has a slightly better estimate that we won’t use:
\[
\|P_t(y)\|_q^2 \leq \sum_{k \geq 1} e^{-2tn_k} |S_k|^{\frac{2(q-1)}{q}} \|y\|_2^q.
\]

By (11), for \(t > \log \rho\)
\[
\sum_{k \geq 1} e^{-2tn_k} |S_k| \leq \sum_{k \geq 1} (e^{-2tn_k} - e^{-2tn_{k+1}}) |B_k| \leq Ct \int_1^{\infty} e^{-(2t-\log \rho)s} ds
\]
\[
= C \frac{t}{2t-\log \rho} e^{-(2t-\log \rho)} \leq C e^{-2t - \log \rho}.
\]

Hence, if \(2t \geq \frac{2t-\log(C \rho)}{q} \log(q-1)\), we deduce \(\|P_t(x)\|_q \leq \|x\|_2\), whence the assertion. \(\square\)

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References

[1] S. Albeverio and R. Hoegh-Krohn. Dirichlet forms and Markov semigroup on C*-algebras. Commun. Math. Phys. 56 (1977), 173-187.
[2] K. Ball, E.A. Carlen and E.H. Lieb. Sharp uniform convexity and smoothness inequalities for trace norms. Invent. Math. 115 (1994), 463-482.
[3] P. Biane. Free hypercontractivity. Commun. Math. Phys. 184 (1997), 457-474.
[4] E.A. Carlen and E.H. Lieb. Optimal hypercontractivity for Fermi fields and related noncommutative integration inequalities. Comm. Math. Phys. 155 (1993), 27-46.
[5] L. Gross. Logarithmic Sobolev inequalities. Amer. J. Math. 97 (1975), 1061-1083.
[6] U. Haagerup. An example of a nonnuclear C*-algebra, which has the metric approximation property. Invent. Math. 50, 3 (1979/80), 279-293.
[7] U. Haagerup, M. Junge and Q. Xu. A reduction method for noncommutative \(L_p\)-spaces and application Trans. Amer. Math. Soc. 362 (2010), 2125-2165.
[8] M. Junge, C. Palazuelos, J. Parcet and M. Perrin. Hypercontractivity in group von Neumann algebras. Preprint 2013.
[9] M. Junge, C. Palazuelos, J. Parcet, M. Perrin and E. Ricard. Hypercontractivity for free products. Preprint 2012.
[10] G. Pisier and Q. Xu. Noncommutative \(L^p\)-spaces. Handbook of the geometry of Banach spaces Vol. 2, ed. W.B. Johnson and J.Lindenstrauss, 2003, 1459-1517, North-Holland, Amsterdam.
[11] R. Oikawaicz and B. Zegarlinski. Hypercontractivity in noncommutative \(L_p\) spaces. J. Funct. Anal. 161 (1999), 246-285.
Laboratoire de Mathématiques Nicolas Oresme Université de Caen Basse-Normandie 14032 Caen Cedex, France

E-mail address: eric.ricard@unicaen.fr

School of Mathematics and Statistics, Wuhan University, Wuhan 430072, China and Laboratoire de Mathématiques, Université de Franche-Comté, 25030 Besançon Cedex, France

E-mail address: qxu@univ-fcomte.fr