An inverse random source problem in a stochastic fractional diffusion equation

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Abstract
In this work the authors consider an inverse source problem the stochastic fractional diffusion equation. The interested inverse problem is to reconstruct the unknown spatial functions \( f \) and \( g \) (the latter up to the sign) in the source by the statistics of the final time data \( u(x, T) \). Some direct problem results are proved at first, such as the existence, uniqueness, representation and regularity of the solution. Then a reconstruction scheme for \( f \) and \( g \) up to the sign is given. To tackle the ill-posedness, Tikhonov regularization is adopted and some numerical results are displayed.

Keywords: inverse problem, stochastic fractional diffusion equation, random source, Tikhonov regularization, regularity, partial measurements, correlation based imaging

(Some figures may appear in colour only in the online journal)

1. Introduction

In this paper we consider an FDE with a random source term

\[
\begin{align*}
\partial^\alpha_t u(x, t) + Au(x, t) &= F(x, t), \quad (x, t) \in D \times (0, T], \quad \alpha \in (1/2, 1); \\
u(x, t) &= 0, \quad (x, t) \in \partial D \times (0, T]; \\
u(x, 0) &= 0, \quad x \in D,
\end{align*}
\]

(1.1)

where \( \partial^\alpha_t \) is the Djrbashyan–Caputo fractional derivative given by the expression

\[
\partial^\alpha_t u = \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-\tau)^{-\alpha} \frac{\partial u}{\partial t}(x, \tau) \, d\tau \quad \text{for} \quad 0 < \alpha < 1
\]

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and \( \Gamma \) stands for the Gamma function. However, we need a stricter restriction \( \alpha \in (1/2, 1) \) on \( \alpha \) for the regularity estimate, which can be seen in the proof of lemma 3.1. Above, \( D \subset \mathbb{R}^d \) is an open, bounded subdomain with smooth enough boundary, and the operator \( \mathcal{A} \) with the definition

\[
\mathcal{A}u = -\sum_{i,j=1}^{n} (a^{ij}(x) u_{x_{i}x_{j}})_{x_{i}} + c(x)u
\]

with \( a^{ij}, c \in C^{\infty}(\mathbb{R}^d) \) is symmetric, elliptic and positive definite. The random source term has the expression

\[
F(x,t) = f(x)h(t) + g(x)\dot{W}(t),
\]

where the function \( h \in L^{\infty}(\mathbb{R}^d) \) is known and \( \dot{W} \) is the standard Wiener process on a probability space \( (\Omega, \mathcal{F}, P) \). Due to the randomness, we refer to (1.1) as stochastic fractional diffusion equation (SFDE) below. Let us point out that there are alternatives for the definition of the fractional derivative such as the Riemann–Liouville formulation, see [27, chapter 2.1]. However, the Djrbashyan–Caputo derivative is often preferred due to its convenient properties related to initial conditions.

Here, we study the following inverse problem related to correlation based imaging:

\[
given \text{the empirical expectation and correlations of the final time data } u(x, T), \\text{can we recover the unknown functions } f \text{ and } g \text{ up to the sign?}
\]

Notice that the source term \( g\dot{W} \) has an invariant distribution with respect to the sign of \( g \). Therefore, the recovery is considered up to the sign of \( g \). We give a positive answer to this question and demonstrate it by numerical simulations.

### 1.1. Physical background and previous literature

At a microscopic level, the physical phenomenon of diffusion is related to the random motion of individual particles. In one of his celebrated work, Einstein [14] deduced that the density function of particles satisfies the classical diffusion equation under the key assumption that the mean squared displacement over a large number of jumps is proportional to time, i.e. \( (\Delta x)^2 \propto t \). Currently, a large array of physical evidence suggests that there exists also physical diffusion that does not satisfy this assumption [9, 19, 28, 40]. In such anomalous diffusion the rate of mean squared displacement may satisfy \( (\Delta x)^2 \propto t^\alpha \), \( \alpha \neq 1 \). The different rate introduces a modification to the diffusion equation in the form of fractional derivatives and the corresponding equations are often called fractional differential equations (FDEs). The applications of FDEs include, to name a few, the thermal diffusion in media with fractal geometry [43], highly heterogeneous aquifer [1], non-Fickian diffusion in geological formations [7], mathematical finance [6], underground environmental problem [21] and the analysis on viscoelasticity in material science [38, 51, 52].

Fractional differential equations have drawn considerable amount of attention among mathematical community lately. Let us mention the work by Sakamoto and Yamamoto [48] to study the initial and boundary value problems for FDEs and the work by Luchko [36, 37] to establish the maximum principle in FDEs. Moreover, Jin, Lazarov and Zhou [26] gave a numerical scheme to approximate the FDE by the finite element method.

In terms of inverse problems, Cheng et al [12] gave one of the first proofs for a uniqueness theorem in one-dimensional FDE. The article [34] considered an inverse source problem in
an FDE, which was close to this work. The authors in [33, 47] analyzed the distributed differential equations, in which the assumption \((\Delta x)^2 \propto t^\alpha\) was extended to a more general case \((\Delta x)^2 \propto F(t)\), and studied some inverse problems in such equations. For an extensive review of the field we refer to [25] and references therein.

In this work, the source term in (1.2) is separated into a time evolution pattern and a spatial component. Identification of such terms can be valuable in problems related to dispersion of pollution or radiation. Here we assume that the temporal behavior of the source is random although the temporal statistics and drift are known. The spatial components are unknown and, therefore, our objective is the localization of the source.

Time fractional stochastic PDEs have gained attention recently, see e.g. [15, 41, 49, 53] and references therein. Our setup differs slightly from these works: previous studies often assume some spatial randomness of the source, whereas our source term is random only in time. To accommodate the randomness in the spatial variable, one often smooths the source in time. This operation is motivated and well explained in [41]. Let us also mention that one of the first study of inverse source problems for time fractional stochastic PDEs were carried out in [50]. This paper imposes discrete random noise on the measurements, but still uses deterministic source, which is different from our work.

Correlation based imaging has become common in applied inverse problems, where randomness is often an inherent part of the model. If the observational data is extensive but exceptionally corrupted or noisy, it can make more sense to analyze the correlations in the data that connect to the unknown parameters. This paradigm has interesting implications to the inverse problems research, since first, correlation-based imaging can remarkably reduce the ill-posedness of problems where no analytical solution is known (see [10]) and, second, it introduces a new set of analytical problems that need novel mathematical innovations [18, 24]. Such imaging in inverse problems has been considered in applications already for a while and considerable meaningful achievements are created. See e.g. the early work [13] on inverse random source problems. Since then correlation based imaging in random source problems has been considered widely in the framework of different PDE models by Li, Bao and others [2–5, 30–32]. In this regard our paper provides the first study of random source problems in fractional diffusion models. Let us also point out that correlation based imaging has been considered for problems where the randomness is an inherent property of the medium or boundary condition [8, 16–18, 22–24].

1.2. Main results and outline of the paper

Our main contributions in this paper are to build some regularity properties for the model (1.1), and to demonstrate that partial and noisy correlation data under different data acquisition geometries can yield useful information regarding the source terms \(f\) and \(g\). This idea is supported by the following main theorems and numerical evidence.

Before stating the main theorems, we give the definition of the mild solution of equation (1.1).

**Definition 1.1.** A stochastic process \(u : [0, T] \times \Omega \to L^2(D)\) defined by

\[
u(\cdot, t, \omega) = \sum_{n=1}^{\infty} (I_{n,1}(t) + I_{n,2}(t, \omega))\phi_n(\cdot),
\] (1.3)
where
\[ I_{n,1}(t) = f_n \int_0^t (t - \tau)^{\alpha-1} E_{\alpha,\alpha}(-\lambda_n(t - \tau)^\alpha) h(\tau) \, d\tau, \]
\[ I_{n,2}(t, \omega) = g_n \int_0^t (t - \tau)^{\alpha-1} E_{\alpha,\alpha}(-\lambda_n(t - \tau)^\alpha) \, dW(\tau), \]
with
\[ f_n = (f(\cdot), \phi_n(\cdot))_{L^2(D)}, \quad g_n = (g(\cdot), \phi_n(\cdot))_{L^2(D)}, \]
is called a mild solution of equation (1.1).

Moreover, the following assumptions regarding the source term are valid throughout this paper.

**Assumption 1.2.** We assume that \( f, g \in L^2(D) \) with \( \|g\|_{L^2(D)} \neq 0 \), and \( h \in L^\infty(0, T) \) has a positive lower bound, that is, there exists a constant \( C(h) > 0 \) such that \( h \geq C(h) \) a.e. on \( (0, T) \).

Let us next discuss our contribution. First, we consider a priori bound on the final time data. General \( L^2 \)-regularity results are well-known in the literature for noise models arising in the study of thermal memory in materials [11, 35]. In terms of the noise, our straightforward-setup is slightly different to the previous works and the relevant regularity result for this study is given in the following proposition.

**Proposition 1.1.** Provided that \( h \in L^2(0, T) \) and \( f, g \in L^2(D) \), the mild solution \( u(x, t, \omega) \) of equation (1.1), defined in definition 1.1, has the following regularity estimate: there exists \( C > 0 \) such that
\[ E\|u\|^2_{L^2(D \times [0, T])} \leq C\left( \|h\|_{L^2(0, T)}^2 \|f\|^2_{L^2(D)} + T^{2\alpha} \|g\|^2_{L^2(D)} \right). \]

Further, with stronger assumptions \( h \in L^\infty(0, T) \), we can derive the supreme regularities as
\[ \sup_{0 \leq t \leq T} E\|u(\cdot, t)\|^2_{L^2(D)} \leq C\left( \|h\|_{L^\infty(0, T)}^2 \|f\|^2_{L^2(D)} + T^{2\alpha-1} \|g\|^2_{L^2(D)} \right). \]

Similarly, if we have \( g \in H^2(D) \) then
\[ \sup_{0 \leq t \leq T} E\|u(\cdot, t)\|^2_{H^2(D)} \leq C\left( \|h\|_{L^\infty(0, T)}^2 \|f\|^2_{L^2(D)} + T^{2\alpha-1} \|g\|^2_{H^2(D)} \right). \]

Next, for suitably smooth final time data, an \( L^2 \)-bound can be established for the unknowns.

**Theorem 1.2.** Assume that \( E\|u(\cdot, T)\|^2_{H^2(D)} < \infty \), then there exists a constant \( C > 0 \) such that
\[ \|f\|^2_{L^2(D)} + \|g\|^2_{L^2(D)} \leq CE\|u(\cdot, T)\|^2_{H^2(D)} < \infty. \]

For the next theorem, let \( X, Y : \Omega \to \mathbb{R} \) be random variables on some complete probability space and write
\[ \text{Cov}(X, Y) := E(X - E(X))(Y - E(Y)) \]
for the covariance of $X$ and $Y$. In particular, we write $\nabla(X) \equiv \text{Cov}(X, X)$ for variance of $X$. Also, the eigensystem $\{\lambda_n, \phi_n : n \in \mathbb{N}^+\}$ will be introduced in the next section. The unique recovery of the unknowns is characterized by the following theorem.

**Theorem 1.3 (Uniqueness).**  Suppose assumption 1.2 holds, let $N_0$ be an index such that $\langle g, \phi_{N_0} \rangle_{L^2(D)} \neq 0$ and define $u_n(T)$ as

$$u_n(T) := \langle u(\cdot, T), \phi_n(\cdot) \rangle_{L^2(D)}, \quad n \in \mathbb{N}^+. $$

Then the expectation of the final time solution and the correlations at $N_0$, i.e. the quantities

$$\{ \mathbb{E}u_n(T), \text{Cov}(u_{N_0}(T), u_n(T)) : n \in \mathbb{N}^+ \}$$

can determine the source terms $f$ uniquely and $g$ up to the sign.

We come to these conclusions as follows. We first give a construction of the solution to the stochastic direct problem and give suitable regularity estimates given different a priori smoothness of the source terms. Based on these results we show an estimate for recovering $f$ and $g$ (theorem 1.2) and the uniqueness (theorem 1.3) given infinite-precision correlation data of the final time solution $u(x, T)$. Meanwhile, the representation and the properties of Mittag–Leffler function introduced in section 2 yield partial characterization of the ill-posedness of the problem in lemmas 4.1 and 4.2. We demonstrate our results with numerical simulations in section 5. We study different data acquisition geometries to find that satisfying localization of the sources can be achieved even if the observed subdomains are relatively small.

This paper is organized as follows. In section 2 we collect some preliminary material containing the properties of Mittag–Leffler function and the Itô isometry formula, which are crucial in the following proofs. Section 3 includes several results for the forward problem, which support the inverse problem work. We study the inverse problem in section 4, proving theorems 1.2 and 1.3 and an ill-posedness result. Finally, numerical demonstrations are given in section 5.

### 2. Preliminaries

Since $\mathcal{A}$ is a symmetric and elliptic operator with domain $L^2_0(D)$, then its eigensystem $\{ (\lambda_n, \phi_n(x)) : n \in \mathbb{N}^+ \}$ has the following properties: $0 < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n < \cdots$ and $\{ \phi_n : n \in \mathbb{N}^+ \} \subset H^2(D) \cap H^1_0(D)$ constitutes an orthonormal basis of $L^2(D)$. Throughout the paper, we denote the inner product in $L^2(D)$ by $\langle \cdot, \cdot \rangle_{L^2(D)}$. Moreover, we write $\varphi_1 \preceq \varphi_2$ for two functions $\varphi_1, \varphi_2 : X \to \mathbb{R}$ on some domain $X$ if there is a universal constant $C > 0$ such that $\varphi_1(x) \leq C \varphi_2(x)$ for all $x \in X$. Similarly, we write $\varphi_1 \sim \varphi_2$ if both $\varphi_1 \preceq \varphi_2$ and $\varphi_2 \preceq \varphi_1$ hold.

Let us now introduce the Mittag–Leffler function which will play a central role in the following analysis. The Mittag–Leffler function is defined as

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(k\alpha + \beta)}$$

for $z \in \mathbb{C}$. Notice that this expression generalizes the natural exponential function since $E_{1,1}(z) = e^z$. 

Let us next record some well-known properties of the function $E_{\alpha,\beta}$. Below, we study the behaviour of $E_{\alpha,\beta}$ only on the negative real line. However, the statements generalize to the complex plane. For reference, see [27, 45].

**Lemma 2.1 ([45, theorems 1.4 and 1.6]).** Let $0 < \alpha < 2$ and $\beta \in \mathbb{R}$ be arbitrary. Then it holds that

$$|E_{\alpha,\beta}(-t)| \leq \frac{C}{1 + t}$$

for any $t \geq 0$ and for any $p \in \mathbb{N}$ we have the asymptotic formula

$$E_{\alpha,\beta}(-t) = -\sum_{k=1}^{p} \frac{(-t)^{-k}}{\Gamma(\beta - \alpha k)} + \mathcal{O}(t^{-1-p}), \quad t \to \infty.$$

A useful result related to high order differentials of Mittag–Leffler functions is given by Sakamoto and Yamamoto in [48].

**Lemma 2.2 ([48, lemma 3.2]).** For $\lambda > 0$, $\alpha > 0$ and $n \in \mathbb{N}^+$, we have

$$\frac{d^n}{dt^n} E_{\alpha,1}(-\lambda t^\alpha) = -\lambda^{\alpha-n} E_{\alpha,\alpha-n+1}(-\lambda t^\alpha), \quad t > 0.$$

A function $\varphi : (0, \infty) \to \mathbb{R}$ is called completely monotonic if $\varphi \in C^\infty(0, \infty)$ and

$$(-1)^n \varphi^{(n)}(t) \geq 0$$

for all $t \in (0, \infty)$, i.e. the derivatives are alternating in sign. For the proof of the following result, see [46] and [20, lemma 4.25].

**Lemma 2.3.** For $0 < \alpha < 1$, functions $t \mapsto E_{\alpha,1}(-t)$ and $t \mapsto E_{\alpha,\alpha}(-t)$ are completely monotonic.

Lemma 2.3 yields immediately the next corollary.

**Corollary 2.1.** If $0 < \alpha < 1$ and $t > 0$, then $E_{\alpha,\alpha}(-t) > 0$.

Finally, let us recall the well-known Itô isometry formula.

**Lemma 2.4 ([44]).** Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and let $\varphi_j : [0, \infty) \times \Omega \to \mathbb{R}$, $j = 1, 2$ satisfy the following properties

1. $(t, \omega) \to \varphi_j(t, \omega)$ is $\mathcal{B} \times \mathcal{F}$-measurable, where $\mathcal{B}$ denotes the Borel $\sigma$-algebra on $[0, \infty)$;
2. $\varphi_j(t, \omega)$ is $\mathcal{F}_t$-adapted;
3. $\mathbb{E} \int_{S}^{T} \varphi_j^2(t, \omega) dt < \infty$ for some $S, T > 0$.

Then it follows that

$$\mathbb{E} \left[ \left( \int_{S}^{T} \varphi_1(t, \omega) d\mathbb{W}(t) \right) \left( \int_{S}^{T} \varphi_2(t, \omega) d\mathbb{W}(t) \right) \right] = \mathbb{E} \int_{S}^{T} \varphi_1(t, \omega) \varphi_2(t, \omega) dt.$$

(2.1)

Later, we use the identity (2.1) for non-random functions and, consequently, the expectation on the right-hand side becomes trivial.
3. Direct problem

3.1. Regularity estimates

The regularity of (1.3) is proved below in lemma 3.1. Notice also that the term $I_{n,1}(t)$ is fully deterministic and contains only information regarding the deterministic part of the source. Similarly, the term $I_{n,2}$ carries the information related to the stochastic source. In the following, we omit the notation $\omega$ for brevity.

**Remark 3.1.** The mild solutions to more general time-fractional stochastic PDEs have been considered in [49, 53] based on the semigroup approach taken in [15]. Our construction is related but uses the approach introduced by Sakamoto and Yamamoto in [48].

In this part we build the following regularity lemma, which will play a crucial role to solve the inverse problem.

**Lemma 3.1.** The stochastic process $u$ given in (1.3) satisfies

$$
\mathbb{E}\|u\|^2_{L^2(D \times [0,T])} \leq C \left( \|h\|^2_{L^2(0,T)} \|f\|^2_{L^2(D)} + T^{2\alpha} \|g\|^2_{L^2(D)} \right),
$$

$$
\sup_{0 \leq t \leq T} \mathbb{E}\|u(\cdot, t)\|^2_{L^2(D)} \leq C \left( \|h\|^2_{L^2(0,T)} \|f\|^2_{L^2(D)} + T^{2\alpha-1} \|g\|^2_{L^2(D)} \right),
$$

and

$$
\sup_{0 \leq t \leq T} \mathbb{E}\|u(\cdot, t)\|^2_{H^2(D)} \leq C \left( \|h\|^2_{L^2(0,T)} \|f\|^2_{L^2(D)} + T^{2\alpha-1} \|g\|^2_{H^2(D)} \right)
$$

if $g \in H^2(D)$ is provided. Here $C$ is a positive constant.

**Proof.** For the $L^2$ regularity, recall that $\{\phi_n : n \in \mathbb{N}^+\}$ is an orthonormal basis of $L^2(D)$. Now for each $t \in [0, T]$ it holds that

$$
\|u(\cdot, t)\|^2_{L^2(D)} = \left\| \sum_{n=1}^{\infty} (I_{n,1}(t) + I_{n,2}(t)) \phi_n(\cdot) \right\|^2_{L^2(D)} \leq 2 \sum_{n=1}^{\infty} (I_{n,1}(t)^2 + I_{n,2}(t)^2).
$$

Hence we have

$$
\mathbb{E}\|u\|^2_{L^2(D \times [0,T])} = \mathbb{E} \int_0^T \|u(\cdot, t)\|^2_{L^2(D)} \, dt
$$

$$
\leq \mathbb{E} \int_0^T \left( \sum_{n=1}^{\infty} I_{n,1}(t)^2 + \sum_{n=1}^{\infty} I_{n,2}(t)^2 \right) \, dt
$$

$$
= \int_0^T \sum_{n=1}^{\infty} I_{n,1}(t)^2 \, dt + \mathbb{E} \int_0^T \sum_{n=1}^{\infty} I_{n,2}(t)^2 \, dt
$$

$$
= \sum_{n=1}^{\infty} \|I_{n,1}\|^2_{L^2(0,T)} + \int_0^T \sum_{n=1}^{\infty} \mathbb{E}I_{n,2}^2(t) \, dt
$$

$$
:= S_1 + S_2.
$$
First, for the sum $S_1$. We can write the term $I_{n,1}$ as the convolution $I_{n,1}(t) = f_n(G_{\alpha,n} * h)(t)$, where $G_{\alpha,n}(t) = t^{\alpha-1}E_{\alpha,\alpha}(-\lambda_\alpha t^\alpha)$, and therefore, the Young’s convolution inequality yields
$$
\|I_{n,1}\|_{L^2(0,T)} \leq |f_n| \|G_{\alpha,n}\|_{L^1(0,T)} \|h\|_{L^2(0,T)};
$$
while the following result is derived from lemmas 2.2 and 2.3 and corollary 2.1
$$
\|G_{\alpha,n}\|_{L^2(0,T)} = \int_0^T t^{\alpha-1}E_{\alpha,\alpha}(-\lambda_\alpha t^\alpha)dt = \frac{1 - E_{\alpha,1}(-\lambda_\alpha T^\alpha)}{\lambda_\alpha} \leq \frac{1}{\lambda_1}.
$$
In consequence, we can find an upper bound for $S_1$ as follows
$$
S_1 \leq \frac{1}{\lambda_1} \|h\|_{L^2(0,T)}^2 \sum_{n=1}^\infty f_n^2 = C\|h\|_{L^2(0,T)}^2 \|f\|_{L^2(D)}^2.
$$
Second, for $S_2$, fix $t \in [0,T]$ we have
$$
\mathbb{E}f_{n,2}^2(t) = g_n^2 \int_0^t \tau^{2\alpha-2}E_{\alpha,\alpha}(-\lambda_\alpha \tau^\alpha)^2 d\tau \leq g_n^2 \int_0^t \tau^{2\alpha-2}C_2^2 d\tau = C_g n^{2\alpha-1},
$$
where we applied lemmas 2.1 and 2.4 and the restriction $\alpha \in (1/2, 1)$. Thus, the estimate of $S_2$ can be bounded by
$$
S_2 \leq \int_0^T \sum_{n=1}^\infty C_g n^{2\alpha-1} dt = C_T^2 \sum_{n=1}^\infty g_n^2 \leq C_T^{2\alpha} \|g\|_{L^2(D)}^2.
$$
Finally, combining the estimates for $S_1$ and $S_2$ yields the $L^2$ regularity
$$
\mathbb{E}\|u\|_{L^2(D \times [0,T])}^2 \leq C \left( \|h\|_{L^2(0,T)}^2 \|f\|_{L^2(D)}^2 + T^{2\alpha} \|g\|_{L^2(D)}^2 \right).
$$
For the supreme regularity, first we have
$$
\|u(\cdot, t)\|_{L^2(D)}^2 \lesssim \sum_{n=1}^\infty I_{n,1}(t)^2 + \sum_{n=1}^\infty I_{n,2}(t)^2
$$
and
$$
\|u(\cdot, t)\|_{H^1(D)}^2 \simeq \|Au(\cdot, t)\|_{L^2(D)}^2 \lesssim \sum_{n=1}^\infty \lambda_n^2 I_{n,1}(t)^2 + \sum_{n=1}^\infty \lambda_n^2 I_{n,2}(t)^2,
$$
where ‘$\simeq$’ holds since $D$ has smooth boundary. Then it holds that
$$
|I_{n,1}(t)| = |f_n(G_{\alpha,n} * h)(t)| \leq |f_n| \|h\|_{L^\infty(0,T)} \int_0^t G_{\alpha,n}(\tau)d\tau \leq \frac{1}{\lambda_n} |f_n| \|h\|_{L^\infty(0,T)}
$$
and
$$
\mathbb{E}f_{n,2}^2(t) \leq C_g n^{2\alpha-1}.
$$
Hence, we can deduce that
\[
\sup_{0 \leq t \leq T} E\|u(\cdot, t)\|_{L^2(D)}^2 \lesssim \left( \|h\|_{L^\infty(0, T)}^2 \sum_{n=1}^\infty \frac{f_n^2}{\lambda_n} + T^{2\alpha-1} \sum_{n=1}^\infty \frac{\lambda_n^2 \delta_n^2}{\lambda_n^2} \right)^{1/2},
\]
and
\[
\sup_{0 \leq t \leq T} E\|u(\cdot, t)\|_{L^2(D)}^2 \lesssim \left( \|h\|_{L^\infty(0, T)}^2 \sum_{n=1}^\infty \frac{f_n^2}{\lambda_n} + T^{2\alpha-1} \sum_{n=1}^\infty \frac{\lambda_n^2 \delta_n^2}{\lambda_n^2} \right)^{1/2}.
\]

The proof is complete. \qed

The above lemma supports the research on this inverse problem in the sense that it gives the bounds of the moments of the measurements.

**Proof of proposition 1.1.** Proposition 1.1 follows from lemma 3.1 immediately. \qed

4. Reconstruction of \(f\) and \(g\) up to sign from the final time correlations

In this section we consider the inverse problem of reconstructing \(f\) and \(g\). From definition 1.1 and lemma 2.4 it follows that the final time expectation and variance can be formulated as

\[
E u_n(T) = f_n \int_0^T \tau^{\alpha-1} E_{\alpha,\alpha} (-\lambda_n \tau^\alpha) h(T - \tau) \, d\tau, \tag{4.1}
\]

\[
\text{Cov}(u_m(T), u_n(T)) = g_m g_n \int_0^T \tau^{2\alpha-2} E_{\alpha,\alpha} (-\lambda_m \tau^\alpha) E_{\alpha,\alpha} (-\lambda_n \tau^\alpha) \, d\tau \tag{4.2}
\]

for any \(m, n \in \mathbb{N}^+\).

**Lemma 4.1.** For each \(n \in \mathbb{N}^+\), there exists a constant \(C = C(h) > 0\) independent of \(n\) such that

\[
\frac{1}{C} \lambda_n^{-1} \leq \int_0^T \tau^{\alpha-1} E_{\alpha,\alpha} (-\lambda_n \tau^\alpha) h(T - \tau) \, d\tau \leq C \lambda_n^{-1}.
\]

**Proof.** For the first estimate, by lemma 2.2 and assumption 1.2, we obtain

\[
\int_0^T \tau^{\alpha-1} E_{\alpha,\alpha} (-\lambda_n \tau^\alpha) h(T - \tau) \, d\tau \geq C(h) \int_0^T \tau^{\alpha-1} E_{\alpha,\alpha} (-\lambda_n \tau^\alpha) \, d\tau
\]

\[
= C(h) \lambda_n^{-1} (1 - E_{\alpha,1}(-\lambda_n T^\alpha))
\]

\[
\geq C(h) \lambda_n^{-1} (1 - E_{\alpha,1}(-\lambda_1 T^\alpha)).
\]
Similarly, we have that
\[
\int_0^T \tau^{\alpha-1} E_{\alpha,\alpha}(-\lambda_n \tau^\alpha) h(T - \tau) \, d\tau \leq \|h\|_{L^\infty(0,T)} \int_0^T \tau^{\alpha-1} E_{\alpha,\alpha}(-\lambda_n \tau^\alpha) \, d\tau
\]
\[
= \|h\|_{L^\infty(0,T)} \lambda_n^{\alpha-1} (1 - E_{\alpha,1}(-\lambda_n T^\alpha)).
\]
Since the mapping \( t \mapsto E_{\alpha,1}(-t) \) is monotonically decreasing and satisfies \( E_{\alpha,1}(0) = \Gamma(1) = 1 \), we obtain the result.

**Lemma 4.2.** There exists a constant \( C > 0 \) independent of \( m \) and \( n \) such that
\[
\frac{1}{C} (\lambda_m \lambda_n)^{-2} < \int_0^T \tau^{2\alpha-2} E_{\alpha,\alpha}(-\lambda_m \tau^\alpha) E_{\alpha,\alpha}(-\lambda_n \tau^\alpha) \, d\tau \leq C (\lambda_m \lambda_n)^{-1 + \frac{1}{\alpha}}
\]
for all \( m, n \in \mathbb{N}^+ \).

**Proof.** Lemma 2.1 and the fact \( 1/\Gamma(0) = 0 \) give the asymptotic behavior of \( E_{\alpha,\alpha}(-z) \) as \( E_{\alpha,\alpha}(-z) \approx Cz^{-\alpha} + \mathcal{O}(z^{-\alpha-1}) \) as \( z \to \infty \). For the lower bound, lemmas 2.1 and 2.2 yield
\[
\int_0^T \tau^{2\alpha-2} E_{\alpha,\alpha}(-\lambda_m \tau^\alpha) E_{\alpha,\alpha}(-\lambda_n \tau^\alpha) \, d\tau \geq \int_0^T \tau^{2\alpha-2} \, d\tau \geq \int_0^T \lambda_m^{-2T^{2\alpha}} \lambda_n^{-2T^{2\alpha}} \cdot T^{2\alpha-1} 
\]
\[
\geq \lambda_m^{-2} \lambda_n^{-2}. \tag{4.3}
\]
On the other hand, we can let \( \epsilon > 0 \) and use lemma 2.1 to obtain a bound
\[
E_{\alpha,\alpha}(-\lambda_m \epsilon^\alpha) \lesssim \begin{cases} \frac{1}{\lambda_m \epsilon} & \text{for } t < \epsilon \\ \frac{1}{\lambda_m \epsilon^\alpha} & \text{for } t \geq \epsilon, \end{cases}
\]
up to a universal constant. Utilizing this bound, it follows that
\[
\int_0^T \tau^{2\alpha-2} E_{\alpha,\alpha}(-\lambda_m \tau^\alpha) E_{\alpha,\alpha}(-\lambda_n \tau^\alpha) \, d\tau
\]
\[
\lesssim \int_0^\epsilon \tau^{2\alpha-2} \, d\tau + \int_\epsilon^T \tau^{2\alpha-2} \cdot \frac{1}{\lambda_m \epsilon^\alpha} \cdot \frac{1}{\lambda_n \epsilon^\alpha} \, d\tau
\]
\[
\lesssim \epsilon^{2\alpha-1} + \frac{1}{\lambda_m \lambda_n} \left( \frac{1}{\epsilon} - \frac{1}{T} \right). \tag{4.3}
\]
Optimizing \( \epsilon > 0 \) in (4.3) we find that \( \epsilon \approx (\lambda_m \lambda_n)^{-\frac{1}{\alpha}} \), which, consequently, yields the upper bound.

**4.1. Proofs for theorems 1.2 and 1.3**

Theorem 1.2 follows in a straightforward manner.
Proof of theorem 1.2. The lower bounds in lemmas 4.1 and 4.2 yield directly that
\[ f_n^2 + g_n^2 \lesssim \lambda_n^4 \left( (\mathbb{E} u_n(T))^2 + \mathbb{V}(u_n(T)) \right) = \lambda_n^4 \mathbb{E}(u_n(T))^2. \]
Therefore, it follows that
\[ \|f\|^2_{L^2(D)} + \|g\|^2_{L^2(D)} = \sum_{n=1}^{\infty} (f_n^2 + g_n^2) \lesssim \sum_{n=1}^{\infty} \lambda_n^4 \mathbb{E}(u_n(T))^2 \lesssim \mathbb{E}\|u(\cdot,T)\|^2_{H^4(D)} < \infty. \]
The proof is complete.

As discussed above, the stochastic FDE in (1.1) is invariant with respect to the sign of \( g \). Therefore, the observations of the final time do not contain information regarding the sign. However, notice carefully that the observed expectation and variance do not ensure uniqueness for \( |g| \), since each process \( \langle u, \phi_n \rangle_{L^2(D)} \) is invariant to the sign of \( g_n \) independently. As we will see below, the cross-covariance between \( u_n(T) \) and \( u_k(T) \) for \( k \neq n \) adds the crucial information to the system since the random white noise in (1.1) is only time-dependent.

The uniqueness result theorem 1.3 can now be provided as follows.

Proof of theorem 1.3. First, from (4.1), we clearly have
\[ f_n = \frac{\mathbb{E} u_n(T)}{\int_0^T \tau^{\alpha-1} E_{\alpha,\alpha}(-\lambda_n \tau^\alpha) h(T-\tau) \, d\tau}, \]
in which the denominator is nonzero due to assumption 1.2.
Second, recalling (4.2) and the assumption on \( N_0 \), the moment \( \text{Cov}(u_{N_0}(T), u_{N_0}(T)) \) yields the value of \( |g_{N_0}| \). Here note that \( |g_{N_0}| = |\langle g(\cdot), \phi_{N_0}(\cdot) \rangle_{L^2(D)}| \neq \langle |g(\cdot)|, \phi_{N_0}(\cdot) \rangle_{L^2(D)} \). For convenience, we pick the positive solution of \( g_{N_0} \). It follows that
\[ g_n = \frac{\text{Cov}(u_{N_0}(T), u_{N_0}(T))}{g_{N_0} \int_0^T \tau^{2\alpha-2} E_{\alpha,\alpha}(-\lambda_{N_0} \tau^\alpha) E_{\alpha,\alpha}(-\lambda_n \tau^\alpha) \, d\tau}. \]
The integral in the denominator is strictly positive due to \( g_{N_0} > 0 \).

Remark 4.1. Clearly, by identity (4.1) the source function \( f \) can be solved independently of \( g \) only based on the expectation of the data. As a linear problem, the ill-posedness of solving \( f \) is sharply characterized by lemma 4.1. However, the case for solving \( g \) is less obvious. If only partial data is utilized as in the uniqueness proof (theorem 1.3), the ill-posedness is characterized by lemma 4.2 when \( n = N_0 \) is considered fixed.

5. Numerical reconstruction

In this section we illustrate the practical solvability of the inverse problem by numerical demonstrations. We consider to reconstruct \( f \) and \( |g| \) in the finite dimensional space
\[ S_N := \text{Span}\{\phi_n : n = 1, \cdots, N\}, \]
where $\phi_n$ are the eigenfunctions of $A$, and denote the approximations of $f$ and $g$ as

$$f_N(x) = \sum_{n=1}^{N} f_n\phi_n(x), \quad g_N(x) = \sum_{n=1}^{N} g_n\phi_n(x).$$

Also the vector formulations of $f_N$ and $g_N$ can be given as

$$\vec{f}_N = [f_1 \quad f_2 \quad \cdots \quad f_N], \quad \vec{g}_N = [g_1 \quad g_2 \quad \cdots \quad g_N].$$

The domain $D$ is set to be the unit circle in $\mathbb{R}^2$ and we let $A = -\Delta$, then it follows that the eigenfunctions of $A$ are given by

$$\phi_n(r, \theta) = w_n J_m(\sqrt{\lambda_n}r) \cos(m\theta + d_n),$$

where $(r, \theta)$ are the polar coordinates on $D$, the phase $d_n$ is either 0 or $-\pi/2$, $w_n$ is the normalized weight factor and $J_m(z)$ is the first kind Bessel function with degree $m$. The eigenvalues $\{\lambda_n : n \in \mathbb{N}^+\}$ are the squares of the zeros of the class of Bessel functions $\{J_m(z) : m \in \mathbb{N}\}$ and indexed by $n$ with nondecreasing order. Hence, we can see the index $m$ is a function of $n$, i.e. $m = m(n)$. The set $\{\lambda_n : n \in \mathbb{N}^+\}$ can be solved numerically and satisfy $\lambda_j \simeq j$. The data used in all examples below is simulated and the forward solver being used is based on a finite difference scheme. We run the forward solver $10^3$ times for different realizations of the source

Figure 1. Exact solutions of (e1) (top) and (e2) (bottom): $f$ (left), $|g|$ (right).
term and average the final time data \( u(x, T) \) to approximate the exact data \( \hat{E}, \hat{C} \). Here \( \hat{E}, \hat{C} \) stand for the expectation and covariance moments of \( u(x, T) \) discretized in the computational basis, respectively. Their definitions are discussed later. Lastly, we generate the noisy data \( \hat{E}^{\delta}, \hat{C}^{\delta} \) for all examples by adding 1% relative noise.

We consider the two experiments \((e1)\) and \((e2)\), where we use the following source terms:

\[
\begin{align*}
(e1) & : \quad f(r, \theta) &= 10w_1J_{m(1)}(\sqrt{\lambda_1}r) \cos (m(1)\theta) + 5w_2J_{m(2)}(\sqrt{\lambda_2}r) \cos (m(2)\theta) \\
& \quad + 12w_2J_{m(2)}(\sqrt{\lambda_2}r) \sin (m(2)\theta), \\
& \quad g(r, \theta) &= 10w_1J_{m(1)}(\sqrt{\lambda_1}r) \cos (m(1)\theta) + 2w_2J_{m(2)}(\sqrt{\lambda_2}r) \cos (m(2)\theta) \\
& \quad + 13w_2J_{m(2)}(\sqrt{\lambda_2}r) \sin (m(2)\theta); \\
(e2) & : \quad f(x, y) = 6\chi_{[(x-0.1)^2+y-0.5]^2<0.22}^*, \\
& \quad g(x, y) = -3\chi_{[(x+0.3)^2+y-0.5]^2<0.15}^*. 
\end{align*}
\]

The source terms in \((e1)\) and \((e2)\) are represented in figure \(1\).
5.1. Data acquisition and finite-dimensional data

In practise, the data acquisition is unlikely to happen in the basis $\phi_n$ indicated by $A$. For example, the fact that functions $\phi_n$ are not local can be restrictive, if the observations are limited to a strict subset $D_{\text{mea}} \subset D$. To accommodate this thought, suppose our data is given on the basis functions of a finite dimensional subspace $\hat{S}_K \subset L^2(D)$ such that

$$\hat{S}_K = \text{Span}\{\psi_n : n = 1, \ldots, K\},$$

and our data is given by

$$\{\mathbb{E}\hat{u}_n(T), \text{Cov}(\hat{u}_k(T), \hat{u}_\ell(T)) : k \in I, \ell \in J \text{ and } n \in I \cup J\}$$

where $\hat{u}_n(T) = \langle u(T), \psi_n \rangle_{L^2(D)}$ and $I, J \subset \{1, \cdots, K\}$ are some index subsets. For convenience, we assume that $I = J = \{1, \cdots, K\}$ and, therefore, omit denoting the dependence on the index sets. We will require in an example below that $\text{supp}(\psi_n) \subset D_{\text{mea}}$ for all $n = 1, \ldots, K$.

**Source-to-expectation mapping.** Writing $\mathbb{E}\hat{u}_n(T)$ in the $\{\phi_k\}_{k=1}^{\infty}$ basis yields

$$\mathbb{E}\hat{u}_n(T) = \sum_{k=1}^{\infty} \langle \psi_n, \phi_k \rangle \mathbb{E}\langle u(T), \phi_k \rangle = \sum_{k=1}^{\infty} \langle \psi_n, \phi_k \rangle \cdot f_k \int_{0}^{T} \tau^{\alpha-1} E_{\alpha,\alpha}(\lambda_k \tau^\alpha) h(T - \tau) \, d\tau.$$
Therefore, by using notation \( \hat{\mathbf{E}} = (\mathbb{E}\hat{u}_n(T))_{n=1}^K \in \mathbb{R}^K \), we have identity
\[
\hat{\mathbf{E}} = A\mathbf{f},
\]
where the operator \( A : L^2(D) \to \mathbb{R}^K \) is linear and bounded due to lemma 3.1 and satisfies with
\[
(A\mathbf{f})_n = \sum_{k=1}^{\infty} \int_0^T \tau^{\alpha-1} E_{m\alpha} (-\lambda_k \tau^\alpha) h(T - \tau) \mathbf{d}\tau \cdot \langle \psi_m, \phi_k \rangle \phi_k
\]
for \( 1 \leq n \leq K \).

**Source-to-covariance mapping.** We see that we have
\[
\text{Cov}(\hat{u}_m(T), \hat{u}_n(T)) = \psi_m^\top \mathbf{C} \psi_n,
\]
where \( \psi_m = (\langle \psi_m, \phi_k \rangle)_{k=1}^\infty \) and \( \mathbf{C} = (\text{Cov}(u_k, u_{k'}))_{k,k'=1}^\infty \). Therefore, by writing \( \mathbf{R} = (\psi_1, \cdots, \psi_K) \), we have
\[
\hat{\mathbf{C}} = \mathbf{R}^\top \mathbf{C} \mathbf{R}.
\]
Recall now the expression for \( \text{Cov}(u_m(T), u_n(T)) \) in (4.2). On the other hand, we can rewrite (4.2) in the form
\[
\mathbf{C} = \int_0^T \mathbf{g}(\tau) \mathbf{g}(\tau)^\top d\tau,
\]

**Table 1.** Relative \( L^2 \) errors for experiments \((ei), \ i = 1, 2 \) and \((eij), \ i = 1, 2, \ j = a, b, c.\)

| \( ei \)   | \( eij \)   | \( eij \)   |
|------------|-------------|-------------|
| \( e1 \)   | \( 6.06 \times 10^{-2} \) | Not applicable | 2.46 \times 10^{-2} | Not applicable |
| \( e2 \)   | \( 4.88 \times 10^{-1} \) | \( 5.46 \times 10^{-1} \) |
| \( e1a \)  | \( 2.55 \times 10^{-1} \) | \( 1 \times 10^{-10} \) | \( 1.06 \times 10^{-1} \) | \( 1 \times 10^{-12} \) |
| \( e1b \)  | \( 2.77 \times 10^{-1} \) | \( 1 \times 10^{-10} \) | \( 7.54 \times 10^{-2} \) | \( 1 \times 10^{-12} \) |
| \( e1c \)  | \( 3.76 \times 10^{-1} \) | \( 1 \times 10^{-10} \) | \( 1.81 \times 10^{-1} \) | \( 1 \times 10^{-11} \) |
| \( e2a \)  | \( 5.20 \times 10^{-1} \) | \( 1 \times 10^{-10} \) | \( 8.35 \times 10^{-1} \) | \( 1 \times 10^{-16} \) |
| \( e2b \)  | \( 6.16 \times 10^{-1} \) | \( 1 \times 10^{-13} \) | \( 6.54 \times 10^{-1} \) | \( 1 \times 10^{-16} \) |
| \( e2c \)  | \( 6.47 \times 10^{-1} \) | \( 1 \times 10^{-13} \) | \( 1.24 \times 10^{-0} \) | \( 1 \times 10^{-16} \) |

**Figure 4.** Three partial domains \( D_{\text{mea}} \) in section 5.3. In each case the shaded area is observed.
Figure 5. Reconstruction for $f$ with experiment (e1a) (top), (e1b) (middle) and (e1c) (bottom). Numerical approximation (left), difference between exact solution and approximation (right).
Figure 6. Reconstruction for $|g|$ with experiment (e1a) (top), (e1b) (middle) and (e1c) (bottom). Numerical approximation (left), difference between exact solution and approximation (right).
Figure 7. Reconstruction for $f$ with experiment (e2a) (top), (e2b) (middle) and (e2c) (bottom). Numerical approximation (left), difference between exact solution and approximation (right).
Figure 8. Reconstruction for $|g|$ with experiment ($\varepsilon 2a$) (top), ($\varepsilon 2b$) (middle) and ($\varepsilon 2c$) (bottom). Numerical approximation (left), difference between exact solution and approximation (right).
where $g(\tau) = (g_k\tau^{\alpha-1}E_{\alpha,\alpha}(-\lambda_k\tau^{\alpha}))_{k=1}^\infty : [0, T] \to \mathbb{R}^\infty$ and, consequently,

$$
\hat{C} = \int_0^T \mathbf{R}^T g(\tau) g(\tau) \mathbf{R} \, d\tau.
$$

Let us consider now the integrand in (5.1). By simple calculation, we obtain

$$(R^T g(\tau))_m = \left\langle g, \sum_{k=1}^\infty (\psi_m, \phi_k) \tau^{\alpha-1}E_{\alpha,\alpha}(-\lambda_k\tau^{\alpha}) \phi_k \right\rangle.$$

Let us define an operator $B : H^2(D) \to \mathbb{R}^{K \times K}$ as mapping from $g$ to the expression given in equation (5.1). Clearly, due to lemma 3.1 the operator $B$ is bounded. Now we can state the discretized inverse problem

$$Af = \hat{E} \quad \text{and} \quad Bg = \hat{C}$$

for the source terms $f$ and $g$.

5.2. Numerical results with observations on the full domain

Here we investigate the numerical reconstruction with observations on the full domain, i.e. $D_{\text{mea}} = D$, but with correlations based on one fixed point. In other words, we assume that $\{\psi_n\}$ coincide with $\{\phi_n\}$, $J = \{1, \cdots, N\}$ and $I = \{N_0\}$ where $N_0$ is such that $\langle g, \phi_{N_0} \rangle_{L^2(D)} \neq 0$. Moreover, what is interesting, this formulation leads to a linear interpretation of the operator $B$ given above.

The parameters used in these experiments are set as

$$\alpha = 0.8, \quad T = 1, \quad h(t) \equiv 1, \quad N = 36, \quad N_0 = 1.$$  \hspace{1cm} (5.2)

The numerical results are displayed in figures 2 and 3, which show that the method localizes the sources well. The relative $L^2$ errors are collected by table 1. Since the approximation is obtained on the basis functions $\phi_n$, the discontinuities of the true source terms are not exactly recovered. This can be seen from figure 3 and the comparison between the errors of $(e1)$ and $(e2)$.

5.3. Numerical results with observations on partial domain

In this subsection, we consider the numerical reconstruction with partial measurements, i.e. $D_{\text{mea}} \subset D$ and $D_{\text{mea}} \neq D$. Here $\{\psi_n\}$ are set as the characteristic functions on each uniformly partition of $D_{\text{mea}}$ upon the polar coordinates $(r, \theta)$.

Given the noisy data $(\hat{E}_\delta, \hat{C}_\delta)$, for the first equation we set the optimization problem as

$$\arg\min_{\vec{f}_N} \left\{ \|Af_N - \hat{E}_\delta\|_\vec{P}^2 + \gamma_f \|f_N\|_\vec{P}^2 \right\}.$$  \hspace{1cm}

Due to the nonlinearity of the second equation, we choose the Levenberg–Marquardt type iteration [29, 39, 42],

$$\vec{g}_{l+1} = \vec{g}_l + [B'(\vec{g}_l)]^{-1} B'(\vec{g}_l) \hat{C}_\delta - B\vec{g}_l,$$
and the Frechet derivative $B'$ of $B$ is given as

$$B'(g)[h] = \int_0^T R^{\top} \left[ g(\tau)h(\tau) + h(\tau)g(\tau) \right] R \, d\tau.$$  

We try three kinds of subsets of $D$ which are set as the observed area and can be seen in figure 4. (a) is a concentric with radius 1/4, (b) is the annulus between the circles with radius 3/4 and 1, and (c) contains two segments of the annulus in (b) with $\pi/4$ radian span. The exact solutions and the parameter setting (5.2) in experiments $(e1)$ and $(e2)$ are still used but the corresponding notations are changed to $(e1j)$, $(e2j)$, $j = a, b, c$. The results are displayed in figures 5–8, and the relative $L^2$ errors are recorded in table 1. In these experiments, the values of the regularized parameters $\gamma_f, \gamma_g$ are chosen empirically.

Similar to the results of experiments $(e1)$ and $(e2)$, the reconstructions for smooth exact solutions are better than the ones for nonsmooth case. Furthermore, due to the lack of measured data, the performance of experiments $\{ (e1j), (e2j) : j = a, b, c \}$ is worse than $(e1)$ and $(e2)$, and this can be seen in figures 5–8 and table 1. Also, the results for $(e2c)$ show that the observed subdomain (c) in figure 4 for the discontinuous case is close to the limit in terms of noise level and the size of the subdomain of which can ensure a useful localization of the source.

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