Distributed Multi-User Wireless Charging
Power Allocation

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Abstract

Wireless power charging enables portable devices to be permanently unplugged. Due to its low transmission power and low transmission efficiency, it requires much longer time slot to charge users compared with that for data transmission in wireless communication networks. Besides, each user’s demand urgency needs to be taken into consideration for power allocation. Therefore, new algorithms are essential for wireless power allocation in multi-user wireless charging networks. In this paper, this problem is formulated as a static noncooperative game. It is shown that there exists a unique Nash equilibrium, which is the static state of the wireless power charging network. A distributed power allocation algorithm is proposed to compute the Nash equilibrium of the game. The main result of the paper consists of rigorous analysis of the distributed algorithm for power allocation. The algorithm is shown to converge to Nash equilibrium of the game with exponentially convergence rate for arbitrary initial value with synchronous scheduling. Moreover, the distributed algorithm is also convergence guaranteed with asynchronous scheduling under communication delay and packet drops. Numerical simulations prove the correctness of the analysis and demonstrate the fast convergence of the algorithm and the robustness to synchronous scheduling.

I. INTRODUCTION

Wireless communication aims at providing communication freedom by bringing network to anywhere at anytime. However, wired charging remains the main way to feed a device battery and the battery endurance shortage has become the major bottleneck that hinders the development of ubiquitous wireless communication systems. The cables supporting power charging has become the only cables that would require removal.

Wireless power charging is being developed to support wireless communication systems. It is expected that wireless power charging will play a role in future wireless networks, including the upcoming Internet of Things systems, wireless sensor networks, RFID networks and small cell networks. It has the potential of powering devices larger than low-power sensors and tags, e.g.,
wearable computing devices and smartphones. Many leading smartphone manufacturers, such as Apple, Samsung and Huawei, have begun to release new-generation devices featured with built-in wireless charging capability. Driven by the practical needs, wireless power charging has gained momentum from theoretics. Existing works on wireless charging communication networks can be categorized in two directions. The first one focuses on exclusive wireless charging, i.e., wireless power transfer and information transmission are separated. The second direction is the research that wireless charging and information transmission are coupled to achieve some tradeoff. In each direction, wireless power charging has been conducted in various contexts, e.g., point-to-point channels [1], [2], relay channels [3], [4], full duplex channels [5], [6].

In general, a wireless power charging system consists of a “transmitter” device connected to a source of power such as mains power lines, which converts the power to a time-varying electromagnetic field, and one or more “receiver” devices which receive the power and convert it back to electric power. From the perspective of transmission range, wireless power charging can be divided into near-field and far-field power charging. The far-field power charging that utilizes diffused RF/microwave as a medium to carry radiant energy has a long history and has recently received attention of researchers from communications society. It is shown that radio frequency signal ambient in the air can be harvested for data transmission, and thus interference signal plays a positive role from the wireless power charging perspective [3].

On the other hand, for the near-field wireless power charging, traditionally, transmission range of centimeters has been realized based on the principle of electromagnetic inductive coupling. For instance, in the Qi standard, the distance between a receiver, e.g., a mobile phone, and its power transmitter should not exceed 4 cm [7]. More recently, to boost the efficiency of power charging, magnetic resonant coupling (MRC) has been extensively studied which has higher power charging efficiency as well as longer operation range up to a couple of meters. MRC effectively avoids power leakage to non-resonant externalities and thus ensures safety to the neighboring environment. In [8], an experiment system with multiple transmitters is successfully delivered which can simultaneously charge a cell phone at distances of about half meter, and work independently how the phone is oriented in user’s pocket and in [9] the authors further prototyped charging, simultaneously, multiple devices including smart phones and tablets.

No matter for which kinds of wireless power charging technology, the number of power transmitter is expected to be small and each transmitter needs to serve multiple users at the same time. Thus how to allocate the charging power to different users needs to be studied
and is still missing. Different from the multi-user communication, multi-user wireless power charging has its unique feature and raise new challenges. First, wireless power charging, due to low transmission power and low transmission efficiency, takes much longer time to full charge users compared with data transmission in wireless communication networks. Besides, the battery capacities for each user may vary and lead to different levels of aspiration for power charging. Therefore, with limited power at the transmitter, multi-user charging power allocation is an important issue to be solved with consideration of different users expected amount and duration.

In this paper, we model the competition of limited charging power among users via game theory. In general, the priority of power allocation should give to a user with low state of charge and tight charging deadline. In order to be able to ensure at which state the multi-user wireless charging networks will effectively operate, we provide rigorous analysis of the Nash equilibrium. The contributions can be summarized as follows

- We propose a game framework for wireless power allocation to multiple users with different charging needs and deadlines. Users compete for the limited power by independently submitting a unit price bid to the power transmitter, and proportional sharing of power follows for the power allocation.
- We analyze this wireless power allocation problem using rigorous game-theoretic analysis. We show the existence and uniqueness of the Nash equilibrium for wireless power allocation. Besides, we show for arbitrary initial values the convergence to the Nash equilibrium with exponential rate.
- In many practical scenarios, the inter-user information exchange is asynchronous since random data packet losses may occur, and different nodes may update at different frequencies. These impacts are considered in computing the Nash equilibrium, which represent a general framework including the synchronous updating as a special case.
- The power allocation algorithm takes the advantage of distributed computation [10]–[12]. The users’ charging capacity and charging deadline is not required to be shared while computing the Nash equilibrium.

The rest of this paper is organized as follows. Section II gives the problem modeling and formulation. Section III rigourously shows the existence and uniqueness of the Nash equilibrium, proposes a distributed algorithm to reach the Nash equilibrium with exponentially convergence rate, which is also robust to communication delay and packet loss. Section IV shows the
simulation results of distributed multi-user wireless charging power allocation. Finally, Section V concludes the paper.

II. SYSTEM MODEL AND PROBLEM FORMULATION

We consider a wireless charging server with total transition power $P$, which serves the charging services of $M$ users in the its effective charging area. $\mathcal{V} = \{1, \ldots, M\}$ denotes the set of users. Similar to the single-input-multiple-output communication systems, the transmitter’s power is wirelessly transferred to multiple users. Let $h_i$ be the wireless power charging efficiency to user $i$, with value varying from 0 to 1, which is defined as the received power at the receiver divided by the input power for charing $i$ at the charging server. Note that with different types of wireless power transmission schemes, $h_i$ has different physical meaning. More specifically, in the MRC based wireless power charging system, $h_i$ depends on the mutual inductance between electromagnetic coils of the transmitter and $i^{th}$ receiver [8]. While in the RF based wireless power transmission schemes, $h_i$ is the product of the transmit efficiency, the free-space propagation efficiency and the receive efficiency [4]. In the state-of-the-art, in order to improve the end-to-end power charging efficiency, researchers efforts were focused on enhancing the wireless power transmission efficiency via designing the non-radiated magnetic field or RF signal array to reduce the side lobes of the beam pattern while keep its main lobe keeps spillover losses to a minimum [9]. In the RF based wireless power transmission schemed, attaining higher receive efficiency was also attempted through the design of high-performance rectifying antennas [13] (i.e. rectennas), which convert the incident RF power back to DC.

In contrast to the high data transmission rate achieved in wireless communication networks, to transmit certain amount of power for wireless charging takes much longer time due to the limited of wireless transmission power and low wireless charging effiency. Therefore, each user $i$ has an expected charging period $[0, D_i]$, after $D_i$ user $i$ has to leave the effective charging area, and $C_i$ is the amount of energy that is needed to fulfill $i$’s power consumption, such as communication, requirement at current time. Each time, user $i$ submits a unit price bid $x_i$ to compete for the amount of power transmitted from the charging server, and proportional sharing mechanism [14] applies, i.e., $P$ is allocated to each user in proportion to their unit price bids. Hence, the received power at user $i$ is

$$y_i = \frac{x_i P}{\sum_{j \in \mathcal{V}} x_j} h_i.$$ (1)
Note that a reasonable unit price bid should be non-negative i.e.,

$$S_i = \{x_i > 0\}. \quad (2)$$

The user’s satisfaction to the wireless charging service relies on whether the charged device can fulfil its requirement. Hence, the satisfaction metric of each user can be defined as

$$s_i = \frac{y_i}{C_i/D_i}. \quad (3)$$

Note that $C_i/D_i$ means the received amount of power that is able to full charge user $i$, which is the expected minimum charging rate for user $i$. Thus, $s_i$ is the ratio of the allocated power charging rate $y_i$ with the minimum expected charging rate. The physical meaning is that if $s_i = 1$, user $i$ is charged with $C_i$ within exactly charging period $D_i$; if $s_i > 1$, it denotes that user $i$ will full fill the charge of $C_i$ power within the charging time $D_i$; and if $s_i < 1$, the charging power rate $x_i$ can not satisfy user $i$’s power charging requirement within $[0, D_i]$ charging period.

In order to obtain energy from the charging server, user $i$ needs to pay for the service. Therefore, the utility function that user $i$ tries to maximize can be formulated as

$$U_i(x_i, x_{V\setminus i}) = s_i - \lambda x_i \frac{P_{x_i}}{\sum_{j \in V} x_j}, \quad i \in V, \quad (4)$$

where $x_i \frac{P_{x_i}}{\sum_{j \in V} x_j}$ denotes the payment required to obtain $\frac{P_{x_i}}{\sum_{j \in V} x_j}$ amount of power, and $\lambda > 0$ implies how to balance between the satisfaction of charging power rate and the payment. The utility function $U_i(x_i, x_{V\setminus i})$ is a measure of user $i$’s preference expressed by the amount of satisfaction he or she receives minus the amount of he/she pays. Note that if the urgency is not take into consideration, i.e., $\lambda = +\infty$, users would bid as low as possible to reduce their payments. Substituting (1) and (3) into (4), the utility function is

$$U_i(x_i, x_{V\setminus i}) = \frac{D_i P h_i x_i}{C_i \sum_{j \in V} x_j} - \lambda \frac{P x_i^2}{\sum_{j \in V} x_j}, \quad i \in V. \quad (5)$$

The multi-user wireless charging problems can then be formalized as a non-cooperative game with the following triplet structure:

$$G = \langle V, \{S_i\}_{i \in V}, \{U_i(x_i, x_{V\setminus i})\}_{i \in V} \rangle, \quad (6)$$

where $V$ denotes the game players; $S_i$ being the range of $x_i$ represents the game strategy, which is a subset of positive real numbers; and $\{U_i(x_i, x_{V\setminus i})\}_{i \in V}$ is the utility function for player $i$. The
concept of a best response is central for problem formulation and analysis with game theory. The formal definition is given as follows.

**Definition 1** The best response function $F_i(x_{V\setminus i})$ of user $i$ is the best strategy for user $i$ given the power allocation of the others, which is then the solution to the following maximization problem:

$$F_i(x_{V\setminus i}) = \arg\max_{x_i} U_i(x_i, x_{V\setminus i}).$$  \hspace{1cm} (7)

Note that for each $i$, the maximum is taken over $x_i$ for a fixed $x_{V\setminus i} \triangleq [x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_M]^T$.

We show in Appendix A that the best response function for user $i$ in analytically form is

$$F_i(x_{V\setminus i}) = \left[\left(\sum_{j \in V\setminus i} x_j\right)^2 + K_i \sum_{j \in V\setminus i} x_j\right]^{1/2} - \sum_{j \in V\setminus i} x_j,$$  \hspace{1cm} (8)

where $K_i = \frac{D_i h_i}{\lambda C_i}$. The best response function (8) reflects the fact that all users take independently their best available actions in order to pursue their own individual objectives as expressed by the particular choice of $x_i$. Since $\sum_{j \in V\setminus i} x_j > 0$, the best response function (8) can be reformulated as

$$F_i(x_{V\setminus i}) = K_i \left[\left(1 + \frac{K_i}{\sum_{j \in V\setminus i} x_j}\right)^{1/2} + 1\right]^{-1},$$  \hspace{1cm} (9)

and it is evident that when some user $j \neq i$ increases, $F_i(x_{V\setminus i})$ will decrease. This observation indicates that what is best for one link depends in general upon actions of other links due to the total energy constraint at the charging server. Thus, an equilibrium for the whole system is reached when every user is unilaterally optimum, i.e., when, given the current strategies of the others, any change in his own strategy would result in a in terms of $F_i(x_{V\setminus i})$. This equilibrium constitutes the celebrated notion of Nash equilibrium. We define the joint best response function as $\mathcal{F} \triangleq \{F_1(x_{V\setminus 1}), \ldots, F_M(x_{V\setminus M})\}$ and $\mathcal{S} = S_1 \times \ldots, \times S_M$, which are a Cartesian product of $F_i(x_{V\setminus i})$ and $S_1, \ldots, S_M$, respectively. Then the corresponding Nash equilibrium for the multi-user wireless charging power allocation game is formally defined as follows.

**Definition 2** A pure strategy profile $x^* = [x^*_1, \ldots, x^*_M] \in \mathcal{S}$ is a Nash equilibrium of the game $\mathcal{G}$ in (6) if $x^* = \mathcal{F}(x^*)$.

In general, the game $\mathcal{G}$ in (6) may admit multiple equilibria. In the next section, we will prove the uniqueness of the Nash equilibrium and we analyze the property of how to converge to such
III. Nash Equilibrium and Distributed Algorithm Analysis

A. Nash Equilibrium Analysis

In this subsection, we will show that there is always a unique Nash equilibrium for the game $G$ in (6) exists. Note that the standard interference functions introduced by Yates [15] have been very influential on the Nash equilibrium analysis and design of distributed power control laws for wireless communication networks. Quite a number of works [16]–[18] has been motivated by [15] which conduct the Nash equilibrium analysis via showing that the corresponding best response function is a standard function. While the standard function framework is powerful, the framework does not shown the existence of fixed-points and there is no guarantees on the rate of convergence of the iterates. Note that, recently, [19]–[21] shows that the distributed belief propagation algorithm equals maximizing local utility function to find a Nash equilibrium and the convergence rate is analyzed. Further, the convergence rate of a variant of standard belief propagation is also analyzed in [22]. However, these conclusions cannot be used in our problem due to the fact that the type of utility functions are different. Motivated by [15] and [19]–[22], we show some unique properties of the best response function in (8) first and then conduct the Nash equilibria and distributed computation algorithm analysis.

Property 1 The updating operator $F(\cdot)$ satisfies the following properties:

P 1.1: For arbitrary $x, y \in S$, $x \leq y$ implies $F(x) \leq F(y)$.

P 1.2 For arbitrary $x \in S$ and $\alpha < 1$, $F(\cdot)$ satisfies $F(\alpha x) > \alpha F(x)$ and $\alpha^{-1} F(x) > F(\alpha^{-1} x)$.

Proof 1 The proof is shown in Appendix B.

Note that Nash equilibrium does not always exist. If the set $S_i$ for all $i \in \mathcal{V}$ was a closed, bounded, and convex subset of the one dimensional Euclidean space, and $U_i(x_{\mathcal{V}\setminus i})$ is a strictly concave function in $S$, then the noncooperative game (6) has a Nash equilibrium [23, p. 173]. This theorem is often used to check the existenceness of Nash equilibrium in literature [24]–[26]. However, $S_i$ is not a compact set as shown in (2), and traditional analysis method fails. Next the existence of Nash equilibria is studied.

Theorem 1 The non-cooperative wireless charging game in (6) admits at least one Nash equilibrium, i.e., $x^* = F(x^*)$. 
Proof 2 Since $\mathcal{F}(\cdot)$ is a continuous function and considering P 1.1, $\mathcal{F}(x^n)$ must be upper bounded by $\mathcal{F}(+\infty 1)$ if $\lim_{x_{\mathcal{V}\setminus i} \to +\infty} F_i(+\infty 1)$ exists. By substituting $x^{(n-1)} = +\infty 1$ into $\mathcal{F}(\cdot)$ in (9), we have

$$u_i = \lim_{x_{\mathcal{V}\setminus i} \to +\infty} = \frac{K_i}{(1 + K_i \sum_{j \in \mathcal{V}\setminus i} x^{(n-1)}_j)^{\frac{1}{2}}} + 1$$

By defining $u = \frac{1}{2} K = \frac{1}{2}[K_1, \ldots, K_i, \ldots, K_M]^T$, we have $\mathcal{F}(x) < u$ for all $x > 0$. Then we choose arbitrary $x^{(0)} > \frac{1}{2} K$, we have $x^{(1)} = \mathcal{F}(x^{(0)}) < u$. By repeatedly applying $\mathcal{F}(\cdot)$ on both sides of the inequality we have $x^{(n)} < x^{(n-1)}$. Therefore $x^{(n)}$ is a monotonically decreasing sequence lower bounded by 0. According to monotonic theorem, it must converge. Thus there exist a fixed point $x^*$ which satisfies $x^* = \mathcal{F}(x^*)$. Thus the non-cooperative power game in (6) admits at least one Nash equilibrium, i.e., $x^*_i = \mathcal{F}_i(x^*_{\mathcal{V}\setminus i})$ for all $i \in \mathcal{V}$.

It is possible that a game has multiple Nash equilibirums. Next, we show the uniqueness of the Nash equilibrium in (6).

Theorem 2 The Nash equilibrium for the non-cooperative wireless charging game in (6) is unique.

Proof 3 The uniqueness property of the converged $x^{(n)}$, denoted as $x^*$ is proved by contradiction. Suppose there are two distinct fixed points $x^*$ and $\tilde{x}^*$. Without loss of generality, there exist $i \in \mathcal{V}$ such that $\tilde{x}^*_i < x^*_i$. It is clear that, there must exist $0 < \alpha_i < 1$ such that $\tilde{x}^*_i = \alpha_i x^*_i$. Let $\alpha$ denote the the smallest $\alpha_i$ for all $i \in \mathcal{V}$, we have

$$\alpha x^* \leq \tilde{x}^*, \quad \alpha < 1. \quad (11)$$

Applying $\mathcal{F}(\cdot)$ on (11), and considering P 1.1, we have

$$\mathcal{F}(\alpha x^*) \leq \mathcal{F}(\tilde{x}^*) = \tilde{x}^*, \quad (12)$$

where the equality follows the definition of fixed point of $x^*$. Besides, as $x^*_j > 0$ and $0 < \alpha < 1$, following P 1.2, we have $\mathcal{F}(\alpha x^*) > \alpha x^*$. According to (12), we obtain

$$\tilde{x}^* > \alpha x^*. \quad (13)$$
Now we can conclude that (11) and (13) is a contradiction, and therefore $x^*$ and $\bar{x}^*$ are the same fixed point. Therefore, $x^{(n)}$ converges to a unique fixed positive value.

B. Distributed Algorithm and Convergence Analysis

In the previous subsection, the existence and uniqueness of the Nash equilibrium have been established by Theorem 1 and Theorem 2 respectively. In order to compute the unique Nash equilibrium, the charging server could gather all the users’ information together to perform the computation. However, the parameter $K_i$, which depends on user $i$’s charging deadline and capacity, is the privacy information of user $i$, and individual user may not be willing to transmit these information to the charging server. In order to reach the Nash equilibrium of the game without sharing individual privacy information, distributed iterative computation procedure is preferred. In this section, we first give the synchronous updating algorithm and then analytically prove that for arbitrary initial value of the distributed computation algorithm, the distance between initial value and the Nash equilibrium decreases exponentially. We will also study how to choose the initial value to make the distributed algorithm converges faster.

We introduce some preliminary definitions to provide a formal description of the distributed computation algorithm. We assume, without any loss of generality, that the set of times at which one or more users update their strategies is the discrete set $T = \{0, 1, 2, \ldots \}$. Let $x_i^{(n)}$ with $n \in T$ denote the unit price bid of user $i$ at the $n^{th}$ iteration. Hence, according to (8), the unit price bid of user $i$ is given by

$$x_i^{(n)} = F_i(x_{\neq i}^{(n-1)}) = \left[ \left( \sum_{j \in \mathcal{V} \setminus i} x_j^{(n-1)} \right)^2 + K_i \sum_{j \in \mathcal{V} \setminus i} x_j^{(n-1)} \right]^{\frac{1}{2}} - \sum_{j \in \mathcal{V} \setminus i} x_j^{(n-1)} \quad (14)$$

It is noteworthy that user $i$ only needs to compute $x_i^{(n)}$ with the updated $x_j^{(n-1)}$ where $j \neq i$, without user $j$ sharing the privacy parameter $K_j$. We summarize the distributed iterative algorithm as follows. The algorithm is started by setting user $i$’s bid as $x_i^{(0)}$. Each node $i$ computes its updated message according to (14) at independent time $n \in T$ with its available $x_{\mathcal{V} \setminus i}^{(n-1)}$. Then the joint distributed computation function is

$$x^{(n)} = F(x^{(n-1)}) \triangleq \{ F_1(x_{\mathcal{V} \setminus 1}^{(n-1)}), \ldots, F_M(x_{\mathcal{V} \setminus M}^{(n-1)}) \}. \quad (15)$$
Mathematically speaking, analyzing the Nash equilibrium of (6) equals to analyze the corresponding fixed point problem of (15) [27]. It is shown in [19], [20] that if an iterative function has some specific properties, the convergence is guaranteed and the convergence rate can be identified. We next show the convergence property of $\mathcal{F}(\cdot)$ follow the proof procedure in [19], [20].

**Theorem 3** $x^{(n)}$ converges to the unique Nash equilibrium $x^*$ of the non-cooperative wireless charging game in (6) for any initial bids $x^{(0)} \in \mathcal{S}$.

**Proof 4** As shown in Theorem 2, the Nash equilibrium is unique. Then the possibility of initial bid $x^{(0)}$ can be categorized into three groups: 1) the bid for each user is larger than its Nash equilibrium, i.e., $x^{*} < x^{(0)}$. 2) the bid for each user is smaller than its Nash equilibrium, i.e., $x^{(0)} < x^{*}$. 3) some of the bid for each user is larger than its Nash equilibrium, and some of the bid for each user is smaller than its Nash equilibrium.

1) $x^{*} < x^{(0)}$: There exist $\alpha \geq 1$ which satisfies $x^{(0)} \leq \alpha x^{*}$. Following the monotonic property, we have $x^{*} = \mathcal{F}(x^{*}) < \mathcal{F}(x^{(0)}) < \mathcal{F}(\alpha x^{*}) \leq \alpha \mathcal{F}(x^{*}) = \alpha x^{*}$. Let sequence $\tilde{x}^{(0)} = \alpha x^{*}$ and $\tilde{x}^{(1)} = \mathcal{F}(\tilde{x}^{(0)})$ then we define a new sequence $\{\tilde{x}^{(t)}\}$. As $\tilde{x}^{(1)} = \mathcal{F}(\tilde{x}^{(0)}) = \mathcal{F}(\alpha x^{*}) \leq \alpha \mathcal{F}(x^{*}) = \tilde{x}^{(0)}$. Then, $\tilde{x}^{(1)} < \tilde{x}^{(0)}$ implies that $\tilde{x}^{(t)}$ is a monotonically decreasing sequence. Thus $\tilde{x}^{(t)}$ converges. Since $x^{*}$ is the unique fixed point, $\tilde{x}^{(t)}$ also converges to $x^{*}$. As $x^{*} < x^{(0)} < \tilde{x}^{(0)}$, $\mathcal{F}(x^{*}) < \mathcal{F}(x^{(0)}) < \mathcal{F}(\tilde{x}^{(0)})$ with $x^{(0)} > x^{*}$, must converges to $x^{*}$.

2) $0 < x^{(0)} < x^{*}$: There exist $1 > \beta > 0$ which satisfies $\tilde{x}^{(0)} = \beta x^{(0)} < x^{*}$, $\tilde{x}^{(1)} = \mathcal{F}(\tilde{x}^{(0)}) = \mathcal{F}(\beta x^{*}) \geq \beta \mathcal{F}(x^{*}) = \beta x^{*}$, $\tilde{x}^{(0)}$ converges to $x^{(e)}$ with $0 < x^{(0)} < x^{*}$.

3) indefinite: There exist $x_{u}$ and $x_{l}$ such that $0 < x_{l} < x^{(0)} < x_{u}$ and $0 < x_{l} < x^{*} < x_{u}$. Following case 1) $\lim_{\ell \to \infty} \mathcal{F}^{(\ell)}(x_{u}) = x^{*}$; and Following case 2) $\lim_{\ell \to \infty} \mathcal{F}^{(\ell)}(x_{u}) = x^{*}$; Thus, $\lim_{\ell \to \infty} \mathcal{F}^{(\ell)}(x_{u}) \leq \lim_{\ell \to \infty} \mathcal{F}^{(\ell)}(x^{(0)}) \leq \lim_{\ell \to \infty} \mathcal{F}^{(\ell)}(x_{u})$.

Another fundamental question is how fast the convergence would be. Define a set

$$C = \{ x^{(n)} | x_{u} \geq x^{(n)} \geq x^{*} + \epsilon \mathbf{1} \}$$

$$\cup \{ x^{(n)} | x^{*} - \epsilon \mathbf{1} \geq x^{(n)} \geq x_{l} \},$$

with $\epsilon > 0$ being an arbitrary small value. Next, we give the definition of the part metric first, and then show that the distance between $\{x^{(n)}\}_{n=1,...}$ and the unique Nash equilibrium $x^{*}$ decreases doubly exponentially fast with respect to part metric in $C$. 

January 23, 2018 DRAFT
For a proof of the fact that \(d(x, y)\) is a metric and for other properties of the part metric we refer to [28]. Then following the proof in [19], we can show the convergence rate is doubly exponential.

**Theorem 4** With the initial bid \(x^{(0)}\) set to be arbitrary value in \(S\), the distance between \(x^{(n)}\) and \(x^*\) decreases doubly exponentially fast with \(n\) increases.

The physical intuition of Theorem 4 is that the sequence \(\{x^{(n)}\}_{l=1}^\infty\) converges at a geometric rate before \(x^{(n)}\) enters \(x^*\’s\) neighborhood, which can be chose arbitrarily small. Next we show how to make sure the monotonic convergence.

**Theorem 5** \(x^{(n)}\) converges monotonically for any initial value \(x^{(0)} > 0\) if the function \(F(\cdot)\) satisfies \(\{x \geq F(x) \cup F(x) \geq x\} \neq \emptyset\).

**Proof 5** On one hand, if \(x \geq F(x)\) holds for some \(x\), we can establish that \(x^{(0)} \geq F(x^{(0)})\) or equivalently, \(x^{(0)} \geq x^{(1)}\). By induction, this relationship can be extended to \(x^{(0)} \geq x^{(1)} \geq \ldots \geq x^{(n)}\). Besides, if \(F(x) \geq x\) holds, it implies \(x^{(0)} \geq x^{(1)} \geq \ldots \geq x^{(n)}\). Therefore, \(\{x \geq F(x) \cup F(x) \geq x\} \neq \emptyset\) implies \(x^{(0)} \ldots x^{(n)}\) must be a monotonic sequence. Since \(x^{(n)} > 0\), for all \(n \geq 0\) and considering P 1.1, we have \(F(x)\) must within the range of \(F(0)\) and \(F(+\infty)\). As shown in [10], we have \(F(+\infty) = u\). On the other hand, substituting \(x^{(n-1)} = 0\) into (14), we obtain \(F(0) = 0\). Hence, the bounded monotonic \(x^{(n)}\) converges.

Note that to guarantee \(\{x \geq F(x) \cup F(x) \geq x\} \neq \emptyset\) we need to choose \(x^{(0)}\) such that \(x^{(0)} \geq x^{(1)}\) or \(x^{(1)} \geq x^{(0)}\). Also, if \(x^{(0)} \geq x^{(1)}\) or \(x^{(1)} \geq x^{(0)}\), we have \(\{x \geq F(x) \cup F(x) \geq x\} \neq \emptyset\). Thereby, Theorem 1 is equivalent to \(x^{(n)}\) converges if \(x^{(0)} \geq x^{(1)}\) or \(x^{(1)} \geq x^{(0)}\) with \(x^{(0)} > 0\). Hence, to guarantee the convergence in a monotonic fashion, we can simply set \(x^{(0)} > [u, x^{(1)}]\) if \(u \geq x^{(1)}\). Next, we will show how to choose the initial value \(x^{(0)}\) to make \(x^{(n)}\) converges faster in a monotonic fashion.

**Theorem 6** \(x^{(n)}\) is convergence guaranteed with arbitrary initial value \(x^{(0)}\) within the set \([u, x^{(1)}]\), and with different \(x^{(0)}\) in this range, \(x^{(n)}\) converges to the same value. With \(x^{(0)} = u\), the iteration (14) converges faster than the same iteration with any other \(x^{(0)} \in (u, +\infty)\).
Proof 6 We denote by $x^{(n)}$ the vector sequence of (14) with initial value $x^{(0)} = u$ and by $y^{(n)}$ the vector sequence with $y^{(0)} \in (u, +\infty)$. We shall prove that

$$x^{(n)} < y^{(n)} \text{ and } ||x^{(n)} - x^*|| < ||y^{(n)} - y^*|| \quad (17)$$

for $n = 0, 1, \ldots$. We prove the above results by induction. At the start, we have $x^{(0)} = u < y^{(0)}$. Assume that $x^{(n)} < y^{(n)}$. According to P 1.1, we obtain $x^* \leq x^{(n+1)} = F(x^{(n)}) < y^{(n+1)} = F(y^{(n+1)})$. Thus, (17) is proved and with $x^{(0)} = u$, then the iteration (14) converges faster than the same iteration with any other $x^{(0)} \in [u, +\infty)$.

We have analyzed the convergence property of synchronous updating $x_i^{(n)}$ for all $i \in \mathcal{V}$. The synchronous updating can be used in centralized computation, where all the information including $K_i$ for all $i \in \mathcal{V}$ are gathered in one computation centre. However, from privacy preserving perspective, users are averse from sharing their charging capacity and charging deadline. The distributed power allocation algorithm takes the advantage of privacy preserving. In distribued algorithm, the updating of $x_i^{(n)}$ at each iteration only occurs after every user receiving updated $x_i^{(n-1)}$ from all its neighbors. However, in practice networks, due to communication packet drops or processing delay the convergence speed of the distributed algorithm would be slow. Next, a totally asynchronous bid updating is analyzed.

C. Asynchronous Power Allocation

With the totally asynchronous updating scheme, all the users compute their individual best response function (8) in a totally asynchronous way. More specifically, some users are allowed to update (8) more frequently than others with outdated information about the updated bid from the others. For example, when node $i$ computes $x_i^{(n)}$, it may only have $x_j^{(n)}$ computed by other node $j$ with $s \leq n - 1$. In order to capture these asynchronous properties of message exchanges, totally asynchronous updating has been adopted in wireless communication networks [27], [29], [30]. Next we introduce the totally asynchronous scheduling definition and formulate the totally asynchronous distributed best response updating.

Definition 3 Totally Asynchronous Scheduling: Let the bid available to node $i$ from user $j$ at time $n$ are $x_j^{\tau_i(n-1)}$, with $\tau_i(n-1)$ satisfying $0 \leq \tau_i(n-1) \leq n$ and $\lim_{n \to \infty} \tau_i(n) = \infty$ for all $i = 1, \ldots, M$. 

January 23, 2018 DRAFT
According to Definition 3, the asynchronous updating version of (14) can be expressed as

\[ x_i^{(n)} = F_i(x_{\mathcal{V} \setminus i}^{\tau(n-1)}) \]

\[ = \left( \sum_{j \in \mathcal{V} \setminus i} x_j^{\tau(n-1)} \right)^2 + K_i \sum_{j \in \mathcal{V} \setminus i} x_j^{\tau(n-1)} \right)^{1/2} - \sum_{j \in \mathcal{V} \setminus i} x_j^{\tau(n-1)}. \]  

(18)

Note that \( x_i^{(n)} \) is the outgoing information from user \( i \), the corresponding available bid used for computation at user \( j \) is denoted by \( x_j^{\tau(n-1)} \), and \( x_{\mathcal{V} \setminus i}^{\tau(n-1)} = [x_1^{\tau(n-1)}, \ldots, x_{i-1}^{\tau(n-1)}, x_{i+1}^{\tau(n-1)}, \ldots, x_M^{\tau(n-1)}]^T \).

The synchronous version of (14) is a special case of asynchronous updating in (14) without considering the possible information loss due to communications and different user may have different updating frequencies. Suppose user \( i \) only update and the time instance \( T_i \), then the outgoing information is

\[ x_i^{(n)} = \begin{cases} 
F_i(x_{\mathcal{V} \setminus i}^{\tau(n-1)}), & \text{if } n \in T_i \\
{x_i^{(n)}}, & \text{otherwise.} 
\end{cases} \]  

(19)

Following the same arguments in the proof of Theorem 1, we have the following convergence properties for the asynchronous updating.

**Corollary 1** With the asynchronous updating as in (19), \( x^{(n)} \) is convergence guaranteed for any initial value \( x^{(0)} \in S \).

**IV. PERFORMANCE EVALUATION**

This section provides simulation results on the tests of the developed synchronous and asynchronous distributed multi-user wireless charging power allocation algorithms proposed in Section III. The related wireless charging parameters are generated according to a wireless charging prototype in [9] which charges users iPhone 4s based on MRC method. More specifically, for different users, the received power \( C_i D_i \) varies within \( \mathcal{R} = [1, 3] \) W The corresponding power transmission efficiency \( h_i \), according to the experiment results is varies within \( \mathcal{H} = [11\%, 19\%] \). The input power \( P \) is set to be 20 W, which is the mean of standardized wireless charging product including Duracell Powermat, Energizer Qi, LUXA2 and WiTricity WiT-500. Note that, with charging methods based on different wireless power charging principle, such as MRC, RF energy transfer and inductive coupling [31], the charging efficiency \( h_i \) is different. Besides, each user may require different amount of energy \( C_i \) to fulfill their devices and have different expected...
charging time $D_i$. Thus there may be different methods to set these simulation parameters. However, the proposed distributed multi-user charging power allocation algorithm and Nash equilibrium analysis is adapt to different charging methods and users. For the asynchronous updating, the probability of user $i$ successfully pass $x_i^{(n)}$ to user $j$ is $p_{i,j}$, with $p_{i,j} \leq 1$. Therefore, $p_{i,j}$ emulate the asynchronous network due to packet loss or communicating delay. The weighting parameter $\lambda$ is set to be 1 without specification.

Fig. 1 shows the convergence behavior of the proposed algorithms for computing the Nash equilibrium $x_i^*$. Two users compete for the wireless power charging is considered. $C_i/D_i$ for $i = 1, 2$ are randomly picked up from $\mathcal{R}$. It is assumed there is no packet loss nor any communication delay, thus $p_{1,2} = p_{2,1} = 1$. We start the distributed synchronous updating algorithm with different values. It shows in Fig. 1 that each user’s $x_i^{(n)}$ converges to the unique Nash equilibrium within 4 iterations. Moreover, Fig. 2 shows the convergence property for asynchronous scheduling with the same parameter set up but a successfully communication probability $p_{1,2} = p_{2,1} = 0.8$. It can be seen that: The distributed approach converges still very rapidly and after convergence the corresponding Nash equilibrium is unique.

Fig. 3 shows the relationship between iteration number upon convergence versus the network size for both synchronous and asynchronous scheduling ($p_{i,j} = 0.8$). It is shown that for both cases, the the iteration number increases slightly when the network size increases. The slow increase of iteration number is due to the exponential convergence rate as shown in Theorem 4. Thus for different network sizes, the iteration number is quite small and almost the same. Fig. 3 also shows with $p_{i,j} = 0.8$, the relationship between iteration number upon convergence versus the network size under asynchronous updating. It can be seen that even with 20% chance of packet loss between each pair of users, the asynchronous algorithm can still reach the Nash equilibrium quickly. Moreover, the computational complexity for the distributed power allocation algorithm is quite low since for each iteration, the computation for the unit price bid $x_i^{(n)}$ in (14) only involves several scalar multiplications.

In Figure 4, we show the Nash equilibrium with respect to the number of users $M$. For each case all the parameters are set to be the same for a fair comparison. For each user, $C_i/D_i$ is set to be 1.2. It is demonstrated that the Nash equilibrium for each user $x_i^*$ monotonically increases as $M$ increases. This is due to the completion among the users gradually become more severe such that the bid offered by each user at the equilibrium also increases.

Next we demonstrate the social welfare behavior of wireless charging user. The social welfare
of the users is defined as

\[ U(x^*) = \sum_{i=1}^{M} U_i(x^*_i, x^*_{\mathcal{V} \setminus i}) = \sum_{i=1}^{M} \frac{D_i P h_i x^*_i}{C_i \sum_{j \in \mathcal{V}} x^*_j} - \lambda \frac{(x^*_i)^2}{\sum_{j \in \mathcal{V}} x^*_j}. \]

For each fixed user number, \( C_i / D_i \) are uniformly generated from \( R \) and \( h_i \) is uniformly generated from \( D \). Each point in the figure is an average of 1000 times simulation. First, Figure 5 shows that the social welfare of the users monotonically decreases for the reason of the increasing competition among the users, which implies the increasing social welfare of the charging server. Second, if as the portion of urgency in the utility function decreases the utility function is increases.
The efficiency of a noncooperative game is defined as the ratio of the highest value of the social welfare to the worse Nash equilibrium of the game. By assuming users are cooperative, the highest value of the social welfare is computed by

$$
\mathbf{x}^P = [x_1^P, \ldots x_M^P] = \arg \min_{\mathbf{x}} \sum_{i=1}^{M} \mathcal{U}_i(x_i, \mathbf{x}_{\setminus i}).
$$

(20)

Besides, since we have shown in Theorem 2 that there is a unique Nash equilibrium in \( \mathcal{G} \), the price of anarchy of the game \( \mathcal{G} \) is

$$
\text{PoA} = \frac{\sum_{i=1}^{M} \mathcal{U}_i(x_i^P, \mathbf{x}_{\setminus i}^P)}{\sum_{i=1}^{M} \mathcal{U}_i(x_i^*, \mathbf{x}_{\setminus i}^*)}.
$$

(21)

Figure 6 shows that the price of anarchy increases slowly with the number of users which implies
Fig. 5. Users’ Social welfare $\mathcal{U}(x^*)$ with respect to users number for different weighting $\lambda$. (ratio = 0.7, synchronous 0.2 success)

Fig. 6. Price of anarchy with respect to the number of users.

the social welfare gap between cooperative optimal and noncooperative optimal increases slowly.

V. CONCLUSION

In this paper, a fully distributed multi-user wireless charging power allocation scheme has been proposed. It has taken into consideration of the impact of charging time on the power allocating problem. The proposed distributed algorithm only involves limited information exchanges among users and avoids sharing privacy information as in the centralized algorithm. We have presented a rigorous analysis for the existence and unique of the Nash equilibrium for both synchronous and asynchronous updating. Moreover, we have proved that the distributed algorithm converges with a exponential rate. Simulations have verified the theatrical analysis and shown the efficiency of the power allocation algorithm.
APPENDIX A

By taking the second derivative of $U_i(x_i, x_{\mathcal{V}\setminus i})$ in (20) with respect to $x_i$, we obtain

$$
\frac{\partial^2 U_i(x_i, x_{\mathcal{V}\setminus i})}{\partial x_i^2} = -2 \frac{D_i P h_i \sum_{j \in \mathcal{V}\setminus i} x_j}{C_i (\sum_{j \in \mathcal{V}} x_j)^3} \\
- 2\lambda P \frac{(\sum_{j \in \mathcal{V}\setminus i} x_j)^2}{C_i (\sum_{j \in \mathcal{V}} x_j)^3}.
$$

(22)

As $x_j > 0$ for all $j \in \mathcal{V}$, it is clear that $\partial^2 U_i(x_i, x_{\mathcal{V}\setminus i})/\partial x_i^2 < 0$. Therefore, the utility function $U_i(x_i, x_{\mathcal{V}\setminus i})$ is strictly concave, and the best response function, according to Definition 1, can be obtained by setting the first derivative of $U_i(x_i, x_{\mathcal{V}\setminus i})$ to be zero, i.e.,

$$
\frac{\partial U_i(x_i, x_{\mathcal{V}\setminus i})}{\partial x_i} = \frac{D_i P h_i \sum_{j \in \mathcal{V}\setminus i} x_j}{C_i (\sum_{j \in \mathcal{V}} x_j)^2} \\
- \lambda P \frac{x_i^2 + 2x_i \sum_{j \in \mathcal{V}\setminus i} x_j}{(\sum_{j \in \mathcal{V}} x_j)^2} = 0.
$$

(23)

After some algebraic manipulations, the best response function for user $i$ can be readily given by

$$
F_i(x_{\mathcal{V}\setminus i}) = \left[ (\sum_{j \in \mathcal{V}\setminus i} x_j)^2 + K_i \sum_{j \in \mathcal{V}\setminus i} x_j \right]^{1/2} - \sum_{j \in \mathcal{V}\setminus i} x_j,
$$

(24)

where $K_i = \frac{D_i h_i}{\lambda C_i}$.

APPENDIX B

By taking the first-order derivative of $F_i(x_{\mathcal{V}\setminus i})$ in (8) with respect to $x_i$, we obtain

$$
\frac{\partial F_i(x_{\mathcal{V}\setminus i})}{\partial x_i} = \frac{\sum_{j \in \mathcal{V}\setminus i} x_j + \frac{1}{2} K_i}{\left[ (\sum_{j \in \mathcal{V}\setminus i} x_j)^2 + K_i \sum_{j \in \mathcal{V}\setminus i} x_j \right]^{1/2}} - 1.
$$

(25)

After performing some algebraic manipulations, the numerator of (25) can be reformulated as

$$
\sum_{j \in \mathcal{V}\setminus i} x_j + \frac{1}{2} K_i = \left[ (\sum_{j \in \mathcal{V}\setminus i} x_j)^2 + K_i \sum_{j \in \mathcal{V}\setminus i} x_j + \left( \frac{1}{2} K_i \right)^2 \right]^{1/2}.
$$

(26)
Therefore,

\[
\sum_{j \in V \setminus i} x_j + \frac{1}{2} K_i > \left[ \left( \sum_{j \in V \setminus i} x_j \right)^2 + \frac{1}{2} K_i \sum_{j \in V \setminus i} x_j \right]^{1/2},
\]

(27)

and hence, it is clear that \( \frac{\partial F_i(x_{V \setminus i})}{\partial x_i} > 0 \). Thus, \( F_i(\cdot) \) or equivalently, \( F(\cdot) \) satisfies that \( x \geq y \) implies \( F(x) \geq F(y) \).

Next, we will show P 1.2. For arbitrary \( x \in \mathcal{S} \) and \( \alpha < 1 \), it can be shown that

\[
\left[ \left( \sum_{j \in V \setminus i} \alpha x_j \right)^2 + K_i \sum_{j \in V \setminus i} \alpha x_j \right]^\frac{1}{2} - \sum_{j \in V \setminus i} \alpha x_j
\]

\[
> \left[ \left( \sum_{j \in V \setminus i} \alpha x_j \right)^2 + K_i \sum_{j \in V \setminus i} \alpha^2 x_j \right]^\frac{1}{2} - \sum_{j \in V \setminus i} \alpha x_j.
\]

(28)

Thus, by the definition of \( F_i(\cdot) \), (28) implies \( F_i(\alpha x_{V \setminus i}) > \alpha F_i(x_{V \setminus i}) = \alpha x_{V \setminus i} \). Thus we have \( F(\alpha x) > \alpha F(x) \). With similar argument, we can also show that \( \alpha^{-1} F(x) > F(\alpha^{-1} x) \).

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