TWISTED HOCHSCHILD HOMOLOGY AND MACLANE HOMOLOGY

TEIMURAZ PIRASHVILI

Abstract. We prove that $H_i(A, \Phi(A)) = 0$, $i > 0$. Here $A$ is a commutative algebra over the prime field $\mathbb{F}_p$ of characteristic $p > 0$ and $\Phi(A)$ is $A$ considered as a bimodule, where the left multiplication is the usual one, while the right multiplication is given via Frobenius endomorphism and $H_\bullet$ denotes the Hochschild homology over $\mathbb{F}_p$. This result has implications in MacLane homology theory. Among other results, we prove that $\text{HML}_\bullet(A, T) = 0$, provided $A$ is an algebra over a field $K$ of characteristic $p > 0$ and $T$ is a strict homogeneous polynomial functor of degree $d$ with $1 < d < \text{Card}(K)$.

1. Introduction

In this short note we study Hochschild and MacLane homology of commutative algebras over the prime field $\mathbb{F}_p$ of characteristic $p > 0$. Let us recall that MacLane homology is isomorphic to the topological Hochschild homology [13] and to the stable $K$-theory as well [5].

Let $A$ be a commutative algebra over the prime field $\mathbb{F}_p$ of characteristic $p > 0$ and let $\Phi(A)$ be denote $A$-$A$-bimodule, which is $A$ as a left $A$-module, while the right multiplication is given via Frobenius endomorphism. We prove that the Hochschild homology vanishes $H_i(A, \Phi(A)) = 0$, $i > 0$. The proof makes use a simple result on homotopy groups of simplicial rings, which says that if $R_\bullet$ is a simplicial ring such that all rings involved in $R_\bullet$ satisfy $x^m = x$, $m \geq 2$ identity then $\pi_i(R_\bullet) = 0$ for all $i > 0$. These results has implications in MacLane homology theory. We extend the computation of Franjou-Lannes and Schwartz [4] of MacLane (co)homology of finite fields with coefficients in symmetric $S^d$ and divided powers $\Gamma^d$ to arbitrary commutative $\mathbb{F}_p$-algebras, provided that $d > 1$. As a consequence of our computations we show that $\text{HML}_\bullet(A, T) = 0$, provided $T$ is a strict homogeneous polynomial functor of degree $d > 1$ and $A$ is an algebra over a field $K$ of characteristic $p > 0$ with $\text{Card}(K) > d$.

Many thanks to referee for his valuable comments and to Petter Andreas Bergh for his remarks.

2. When it is too easy to compute homotopy groups

It is well-known that the homotopy groups of a simplicial abelian group $(A_\bullet, \partial_\bullet, s_\bullet)$ can be computed as the homology of the normalized chain complex $(N_\bullet(A_\bullet), d)$, where

$$N_n(A_\bullet) = \{x \in A_n | \partial_i(x) = 0, i > 0\}$$

* Partially supported by the grant from MEC, MTM2006-15338-C02-02 (European FEDER support included).
and the boundary map \( N_n(A_\bullet) \rightarrow N_{n-1}(A_\bullet) \) is induced by \( \partial_0 \). Our first result shows that if \( A_\bullet \) has a simplicial ring structure and the rings involved in \( A_\bullet \) satisfy extra conditions then homotopy groups are zero in positive dimensions. This fact is an easy consequence of the following result which is probably well-known

**Lemma 1.** Let \( R_\bullet \) be a simplicial object in the category of not necessarily associative rings and let \( x, y \in N_n(R_\bullet) \) be two elements. Assume \( n > 0 \) and \( x \) is a cycle. Then the cycle \( xy \in N_n(R_\bullet) \) is a coboundary.

**Proof.** Consider the element

\[
z = s_0(xy) - s_1(x)s_0(y).
\]

Then we have

\[
\partial_0(z) = xy - (s_0\partial_0(x))y = xy.
\]

Moreover,

\[
\partial_1(z) = xy - xy = 0,
\]

We also have

\[
\partial_2(z) = (s_0\partial_1(x))(s_0\partial_1(y)) - x(s_0\partial_1(y)) = 0.
\]

Similarly for all \( i > 2 \) we have

\[
\partial_i(z) = (s_0\partial_{i-1}(x))(s_0\partial_{i-1}(y)) - (s_1\partial_{i-1}(x))(s_0\partial_{i-1}(y)) = 0.
\]

Hence \( z \) is an element of \( N_{n+1}(R_\bullet) \) with \( \partial(z) = xy \). \( \square \)

**Corollary 2.** Let \( R_\bullet \) be a simplicial ring. If the rings involved in \( R_\bullet \) satisfy \( x^m = x \) identity for \( m \geq 2 \), then

\[
\pi_n(R_\bullet) = 0, \ n > 0.
\]

**Proof.** Take a cycle \( x \in N_n(R_\bullet), n > 0 \). Then the class of \( x = xx^{m-1} \) in \( \pi_n(R_\bullet) \) is zero. \( \square \)

**Remark.** A more general fact is true. Let \( T \) be a pointed algebraic theory \([15]\) and let \( X_\bullet \) be a simplicial object in the category of \( T \)-models \([15]\). Then \( \pi_i(X_\bullet) \) is a group object in the category of \( T \)-models, while \( \pi_i(X_\bullet) \) are abelian group objects in the category of \( T \)-models for all \( i > 1 \). Thus \( \pi_i(X_\bullet) = 0, i \geq 1 \) provided all group objects are trivial. This is what happens for the category of rings satisfying the identity \( x^m = x, m \geq 2 \). Another interesting case is the category of Heyting algebras \([3]\).

### 3. Hochschild homology with twisted coefficients

In what follows the ground field is the prime field \( \mathbb{F}_p \) of characteristic \( p > 0 \). All algebras are taken over \( \mathbb{F}_p \) and they are assumed to be associative. For an algebra \( R \) and an \( R-R \)-bimodule \( B \) we let \( H_\bullet(R, B) \) and \( H^\bullet(R, B) \) be the Hochschild homology and cohomology of \( R \) with coefficients in \( B \). Let us recall that

\[
H_\bullet(R, B) = \text{Tor}^{R \otimes R^{op}}_\bullet(R, B)
\]

and

\[
H^\bullet(R, B) = \text{Ext}^{\bullet}_{R \otimes R^{op}}(R, B).
\]
Moreover, we let $C_\bullet(R, B)$ be the standard simplicial vector space computing Hochschild homology
\[ \pi_\bullet(C_\bullet(R, B)) \cong H_\bullet(R, B). \]
Recall that $C_n(R, B) = B \otimes R^\otimes n$, while
\[ \partial_0(b, r_1, \cdots, r_n) = (br_1, \cdots, r_n), \]
\[ \partial_i(b, r_1, \cdots, r_n) = (b, r_1, \cdots, r_ir_{i+1}, \cdots, r_n), \quad 0 < i < n \]
and \[ \partial_n(b, r_1, \cdots, r_n) = (r_n b, r_1, \cdots, r_{n-1}). \]
Here $b \in B$ and $r_1, \cdots, r_n \in R$.

Let $n > 1$ be a natural number and let $A$ be a commutative $\mathbb{F}_p$-algebra. The Frobenius homomorphism gives rise to the functors $\Phi^n$ from the category of $A$-modules to the category of $A$-$A$-bimodules, which are defined as follows. For an $A$-module $M$ the bimodule $\Phi^n(M)$ coincides with $M$ as a left $A$-module, while the right $A$-module structure on $\Phi^n(M)$ is given by \[ ma = a^{p^n}m, \quad a \in A, \ m \in M. \]

Having $A$-$A$-bimodule $\Phi^n(M)$ we can consider the Hochschild homology $H_\bullet(A, \Phi^n(M))$. In this section we study these homologies. In order to state our results we need some notation. We let $\psi^n(A)$ be the quotient ring $A/(a - a^{p^n})$, $n \geq 1$ which is considered as an $A$-module via the quotient map $A \twoheadrightarrow \psi^n(A)$. Thus $\psi^n$ is the left adjoint of the inclusion of the category of commutative $\mathbb{F}_p$-algebras with identity $x^m = x, m = p^n$ to the category of all commutative $\mathbb{F}_p$-algebras.

**Example 3.** Let $n > 1$. If $K$ is a finite field with $q = p^d$ element then $\psi^n(K) = K$ if $n = dt, t \in \mathbb{N}$ and $\psi^n(K) = 0$ if $n \neq dt, t \in \mathbb{N}$.

**Lemma 4.** Let $A$ is a commutative algebra over a field $K$ of characteristic $p > 0$ with $\text{Card}(K) > p^n$. Then $\psi^n(A) = 0, n > 1$.

**Proof.** By assumption there exists $k \in K$ such that $k^{p^n} - k$ is an invertible element of $K$. It follows then that the elements of the form $a^{p^n} - a$ generates whole $A$. \(\square\)

**Theorem 5.** Let $A$ be a commutative $\mathbb{F}_p$-algebra and $n > 1$. Then \[ H_i(A, \Phi^n(A)) = 0 \]
for all $i > 0$ and
\[ H_0(A, \Phi^n(A)) \cong \psi^n(A). \]

**Proof.** The proof consists of three steps.

**Step 1.** The theorem holds if $A = \mathbb{F}_p[x]$. In this case we have the following projective resolution of $A$ over $A \otimes A = \mathbb{F}_p[x, y]$:
\[ 0 \to \mathbb{F}_p[x, y] \xrightarrow{\eta} \mathbb{F}_p[x, y] \xrightarrow{\xi} \mathbb{F}_p[x] \to 0. \]
Here $\epsilon(x) = \epsilon(y) = x$ and $\eta$ is induced by multiplication by $(x - y)$. Hence for any $A$-$A$-bimodule $B$, we have $H_i(A, B) = 0$ for $i > 1$ and

$$H_0(A, B) \cong \text{Coker}(u) \quad \text{and} \quad H_1(A, B) \cong \text{Ker}(u),$$

where $u : B \to B$ is given by $u(b) = xb - bx$. If $B = \Phi^n(F_p[x])$, then $u : F_p[x] \to F_p[x]$ is the multiplication by $(x^n - x)$ and we obtain $H_1(A, \Phi^n(A)) = 0$ and $H_0(A, \Phi^n(A)) = \psi^n(A)$.

**Step 2. The theorem holds if $A$ is a polynomial algebra.** Since Hochschild homology commutes with filtered colimits it suffices to consider the case when $A = F_p[x_1, \ldots, x_d]$. By the Künneth theorem for Hochschild homology (see [10], Theorem X.7.4) we have $H_\bullet(A, \Phi^n(A)) = H_\bullet(F[x], \Phi^n(F[x]))^{\otimes d}$ and the result follows.

**Step 3. The theorem holds for arbitrary $A$.** We use the same method as used in the proof of [9], Theorem 3.5.8. First we choose a simplicial commutative algebra $L_\bullet$ such that each $L_n$ is a polynomial algebra, $n \geq 0$ and $\pi_i(L_\bullet) = 0$ for all $i > 0$, $\pi_0(L_\bullet) = A$. Such a resolution exist thanks to [14]. Now consider the bisimplicial vector space $C_\bullet(L_\bullet, \Phi^n(L_\bullet))$. The $s$-th horizontal simplicial vector space is the simplicial vector space $L_\bullet^{\otimes s+1}$. By the Eilenberg-Zilber-Cartier and Künneth theorems it has zero homotopy groups in positive dimensions and $\pi_0(L_\bullet^{\otimes s+1}) = A^{\otimes s+1}$. On the other hand the $t$-th vertical simplicial vector space of $C_\bullet(L_\bullet, \Phi^n(L_\bullet))$ is isomorphic to the Hochschild complex $C_\bullet(L_t, \Phi^n(L_t))$ which has zero homology in positive dimensions by the previous step. Hence both spectral sequences corresponding to the bisimplicial vector space $C_\bullet(L_\bullet, \Phi^n(L_\bullet))$ degenerate and we obtain the isomorphism

$$H_\bullet(A, \Phi^n(A)) \cong \pi_\bullet(\psi^n(L_\bullet)).$$

Now we can use Lemma 2 to finish the proof.

**Corollary 6.** Let $A$ be a commutative $F_p$-algebra, $M$ be an $A$-module and $n > 1$. Then there exist functorial isomorphisms

$$H_\bullet(A, \Phi^n(M)) \cong \text{Tor}_\bullet^A(\psi^n(A), M), \quad n \geq 0$$

and

$$H^\bullet(A, \Phi^n(M)) \cong \text{Ext}_A^\bullet(\psi^n(A), M), \quad n \geq 0.$$

In particular, if $A$ is a commutative algebra over a field $K$ of characteristic $p > 0$ with $\text{Card}(K) > p^n$, then

$$H_\bullet(A, \Phi^n(M)) = 0 = H^\bullet(A, \Phi^n(M)).$$

**Proof.** Observe that $C_\bullet(A, \Phi^n(A))$ is a complex of left $A$-modules. By Theorem 5 it is a free-resolution of $\psi^n(A)$ in the category of $A$-modules. Hence it suffices to note that

$$C_\bullet(A, \Phi^n(M)) \cong M \otimes_A C_\bullet(A, \Phi^n(A)),$$

$$C^\bullet(A, \Phi^n(M)) \cong \text{Hom}_A(C_\bullet(A, \Phi^n(A)), M),$$

where $C^\bullet$ denotes the standard complex for Hochschild cohomology. The last assertion follows from Lemma 4. \qed
Example 7. It follows for instance that $H^i(A, \Phi^n(M)) = 0$, $i > 0$, provided $M$ is an injective $A$-module and $n > 1$. In particular $H^i(A, \Phi^n(A)) = 0$ if $A$ is a self-injective algebra. On the other hand if $A = \mathbb{F}_p[x_1, \cdots, x_d]$ then $H^i(A, \Phi^n(A)) = 0$, $i \neq d$, $n > 1$ and $H^d(A, \Phi^n(A)) = \psi^n(A)$, $n > 1$.

4. Application to MacLane cohomology

We recall the definition of MacLane (co)homology. For an associative ring $R$ we let $\mathcal{F}(R)$ be the category of finitely generated free left $R$-modules. Moreover, we let $\mathcal{F}(R)$ be the category of all covariant functors from the category $\mathcal{F}(R)$ to the category of all $R$-modules. The category $\mathcal{F}(R)$ is an abelian category with enough projective and injective objects. By definition [8] the MacLane cohomology of $R$ with coefficient in a functor $T \in \mathcal{F}(R)$ is given by

$$HML^\bullet(R, T) := \text{Ext}^\bullet_{\mathcal{F}(R)}(I, T),$$

where $I \in \mathcal{F}(R)$ is the inclusion of the category $\mathcal{F}(R)$ into the category of all left $R$-modules. One defines MacLane homology in a dual manner (see [13, Proposition 3.1]). For an $R$-$R$-bimodule $B$, one considers the functor $B \otimes_R (-)$ as an object of the category $\mathcal{F}(R)$. For simplicity we write $HML^\bullet(R, B)$ instead of $HML^\bullet(R, B \otimes_R (-))$. There is a binatural transformation

$$HML^\bullet(R, B) \to H^\bullet(R, B)$$

which is an isomorphism in dimensions 0 and 1. In the rest of this section we consider MacLane (co)homology of commutative $\mathbb{F}_p$-algebras.

Lemma 8. For any commutative $\mathbb{F}_p$-algebra $A$ one has an isomorphism

$$HML_{2i}(A, \Phi^n(A)) = \psi^n(A), \ i \geq 0, n > 1,$$

and

$$HML_{2i+1}(A, \Phi^n(A)) = 0, \ i \geq 0, n > 1.$$

Proof. According to [12, Proposition 4.1] there exists a functorial spectral sequence

$$E^2_{pq} = H_p(A, HML_q(\mathbb{F}_p, B)) \Longrightarrow HML_{p+q}(A, B).$$

Here $B$ is an $A$-$A$-bimodule. By the well-known computation of Breen [2], Bökstedt [1] (see also [4]) we have

$$HML_{2i}(\mathbb{F}_p, B) = B$$

and

$$HML_{2i+1}(\mathbb{F}_p, B) = 0.$$

Now we put $B = \psi^n(A)$ and use Theorem 5 to get $E^2_{pq} = 0$ for all $p > 0$. Hence the spectral sequence degenerates and the result follows. \qed
We now consider MacLane cohomology with coefficients in strict polynomial functors \([6]\). Let us recall that the strict homogeneous polynomial functors of degree \(d\) form an abelian category \(\mathcal{P}_d(A)\) and there exist an exact functor \(i : \mathcal{P}_d(A) \to \mathcal{F}(A)\) \([7]\). For an object \(T \in \mathcal{P}_d(A)\) we write \(\text{HML}_d(A,T)\) instead of \(\text{HML}_d(A,i(T))\). Projective generators of the category \(\mathcal{P}_d\) are tensor products of the divided powers, while the injective cogenerators are symmetric powers. Let us recall that the \(d\)-th divided power functor \(\Gamma^d \in \mathcal{F}(A)\) and \(d\)-th symmetric functors \(S^n\) are defined by

\[
\Gamma^d(M) = (M^\otimes d)^{\Sigma_d}, \quad S^n(M) = (M^\otimes d)^{\Sigma_d}.
\]

Here tensor products are taken over \(A\), \(\Sigma_d\) is the symmetric group on \(d\)-letters, which acts on the \(d\)-th tensor power by permuting of factors, \(M \in \mathcal{F}(A)\) and \(X^G\) (resp. \(X_G\)) denotes the module of invariants (resp. coinvariants) of a \(G\)-module \(X\), where \(G\) is a group.

For a functor \(T \in \mathcal{F}(A)\) we let \(\hat{T} \in \mathcal{F}(\mathbb{F}_p)\) be the functor defined by

\[
\hat{T}(V) = T(V \otimes A).
\]

According to \([13]\) Theorem 4.1] the groups \(\text{HML}_i(\mathbb{F}_p, \hat{T})\) have an \(A\)-\(A\)-bimodule structure. The left action comes from the fact that \(T\) has values in the category of left \(A\)-modules, while the right action comes from the fact that \(T\) is defined on \(\mathcal{F}(A)\). In particular it uses the action of \(T\) on the maps \(l_a : X \to X\), where \(a \in A\), \(X \in \mathcal{F}(A)\) and \(l_a\) is the multiplication on \(a\). Since \(T(l_a) = l_{a^t}\) if \(T\) is a strict homogeneous polynomial functor of degree \(d\) \([6]\), the bimodule \(\text{HML}_i(\mathbb{F}_p, \hat{T})\) is of the form \(\Phi^n(M)\) provided \(d = p^n\).

**Theorem 9.** Let \(d > 1\) be an integer and let \(A\) be a commutative \(\mathbb{F}_p\)-algebra. Then \(\text{HML}_i(A, \Gamma^d) = 0\) if \(d\) is not a power of \(p\). If \(d = p^n\) and \(n > 0\), then

\[
\text{HML}_i(A, \Gamma^d) = 0 \quad \text{if} \quad i \neq 2p^n t, t \geq 0
\]

and

\[
\text{HML}_i(A, \Gamma^d) = \psi^n(A) \quad \text{if} \quad i = 2p^n t, t \geq 0.
\]

In particular \(\text{HML}_i(A, \Gamma^d) = 0\) provided \(A\) is an algebra over a field \(K\) of characteristic \(p > 0\) with \(\text{Card}(K) > d\).

**Proof.** According to \([13]\) Theorem 4.1],\([12]\) there exists a functorial spectral sequence:

\[
E^2_{pq} = H_p(A, \text{HML}_q(\mathbb{F}_p, \hat{T})) \implies \text{HML}_{p+q}(A,T).
\]

For \(T = \Gamma^n_A\) one has \(\hat{T} = \Gamma^n_{\mathbb{F}_p} \otimes A\). Here we used the notation \(\Gamma^n_A\) in order to emphasize the dependence on the ring \(A\). By the result of Franjou, Lannes and Schwartz \([4]\) \(\text{HML}_i(\mathbb{F}_p, \hat{T})\) vanishes unless \(d = p^n\) and \(i = 2p^n t, t \geq 0\). Moreover in these exceptional cases \(\text{HML}_i(\mathbb{F}_p, \hat{T})\) equals to \(\Phi^n(A)\) (as an \(A\)-\(A\)-module). Hence the spectral sequence together with Theorem \([5]\) gives the result.

**Corollary 10.** Let \(A\) be a commutative algebra over a field \(K\) of characteristic \(p > 0\) with \(\text{Card}(K) > d > 1\). If \(T\) is a strong homogeneous polynomial functor of degree \(d\). Then

\[
\text{HML}_\bullet(A, T) = 0 = \text{HML}^\bullet(A, T).
\]
Proof. We already proved that the result is true if $T$ is a divided power. By the well-known vanishing result \[11\] the result is also true if $T = T_1 \otimes T_2$ with $T_1(0) = 0 = T_2(0)$. Since any object of $\mathcal{P}_d$ has a finite resolution which consists with finite direct sums of tensor products of divided powers \[6\] the result follows. □

References

[1] M. Bökstedt. The topological Hochschild homology of $\mathbb{Z}$ and $\mathbb{Z}/p$. Unpublished manuscript.
[2] L. Breen. Extensions du groupe additif. Publl. IHES 48 (1978), 39–125.
[3] L. Esakia. Heyting algebras. I. Duality theory. (Russian) “Metsniereba”, Tbilisi, 1985. 105 pp.
[4] V. Franjou, J. Lannes and L. Schwartz. Autour de la cohomologie de MacLane des corps finis. Invent. Math. 115 (1994), no. 3, 513–538.
[5] V. Franjou, E. Friedlander, T. Pirashvili and L. Schwartz. Rational representations, the Steenrod algebra and functor homology. Panoramas et Synthèses 16. Société Mathématique de France, Paris, 2003. xxi+132 pp.
[6] E. Friedlander and A. Suslin. Cohomology of finite group schemes over a field. Invent. Math. 127 (1997), no. 2, 209–270.
[7] V. Franjou, E. Friedlander, A. Scorichenko and A. Suslin. General linear and functor cohomology over finite fields. Ann. of Math. (2) 150 (1999), no. 2, 663–728.
[8] M. Jibladze and T. Pirashvili. Cohomology of algebraic theories. J. Algebra 137 (1991), no. 2, 253–296.
[9] J.-L. Loday. Cyclic homology. Grundlehren der Mathematischen Wissenschaften, 301. Springer-Verlag, Berlin, 1998.
[10] S. Mac Lane. Homology. Die Grundlehren der mathematischen Wissenschaften, Bd. 114, Springer. Berlin. 1963 x+422 pp.
[11] T. Pirashvili. Higher additivizations. Trudy Tbiliss. Mat. Inst. Razmadze Akad. Nauk Gruzii. SSR 91 (1988), 44–54.
[12] T. Pirashvili. Spectral sequence for MacLane homology. J. of Algebra, 170(1994), 422-427.
[13] T. Pirashvili and F. Waldhausen. MacLane homology and topological Hochschild homology. J. Pure Appl. Algebra 82 (1992), 81–98.
[14] D. Quillen. On the (co-) homology of commutative rings. Proc. Sympos. Pure Math., Vol. XVII, New York, 1968. 65–87.
[15] S. Schwede. Stable homotopy of algebraic theories. Topology 40 (2001), no. 1, 1–41.