1 Definition and Motivation

In this talk I am going to describe classical solutions which are generating functions for tree form-factors in the corresponding quantum theory. My talk will be based on works done in collaboration with A. Rosly. I should also cite related papers by W. Bardeen and by V. Korepin and T. Oota. These classical solutions have been called perturbiners and have been constructed in various models. I believe that in this audience the most convenient is to start just with a definition.

Definition

Consider a nonlinear field equation:

\[(\partial^2 + m^2)\phi + \lambda \phi^2 + \ldots = 0\]  \hspace{1cm} (1)

(I take here the case of scalar field, generalizations are trivial).

Take a solution of the corresponding free field equation, \((\partial^2 + m^2)\phi = 0\), in the form of linear combination of a set of plane waves,

\[\phi_1(x, {k_j}, {a_j}) = \sum a_j e^{i k_j x} = \sum_j E_j\]

(in the non-scalar case there is a polarization factor etc. in front of the plain waves).

Assume:

a) non-resonantness: \((\sum_j n_j k_j)^2 \neq m^2\), when the sum contains more than one term \((n_j = 0, 1)\);  
b) nilpotency: \(a_j^2 = 0\).

Perturbiner is a complex solution of Eq. (1) of the following type:

\[\phi_{ptb}(x, {k_j}, {a_j}) = \phi^{(1)}(x, {k_j}, {a_j}) + \text{higher order terms in the plane waves} \{E_j\} \text{ entering } \phi^{(1)}\]  \hspace{1cm} (2)

Solution of this type obviously exists and is unique. Due to the nilpotency condition there is a finite number of terms in Eq. (2) and every term is well
defined because the operator \( (\partial^2 + m^2) \) from Eq. (1) is invertible in the space of polynomials in \( \{E_j\}, j = 1, \ldots, N \) in the non-resonantness assumption. In gauge theories the uniqueness takes place after gauge fixing or, equivalently, modulo gauge transformations.

**Motivation**

The perturbiner is a generating function for the so-called form-factors in the tree approximation,

\[
\phi_{ptb}(x, \{k\}, \{a\}) = \sum_{d=1}^{N} \sum_{j_1 \ldots j_d} <k_{j_1}, \ldots, k_{j_d}|\phi(x)|0>_{\text{tree}} a_{j_1} \ldots a_{j_d}.
\]

The nilpotency is equivalent to excluding the form-factors with identical particles (no loss of generality provided the perturbiner is known for any \( N \)).

The non-resonantness warrants that there are no internal lines on-shell.

**Notice:**

Classical text-books on QFT, e.g. books [14], contain chapters about classical solutions generating tree amplitudes but they use different definitions (asymptotic Feynman-type boundary conditions) and do not give any explicit examples.

In the case of our definition the perturbiner is constructed explicitly in all cases when the field equations admit a zero-curvature representation with one-dimensional auxiliary (spectral) space. Actually, in terms of the zero-curvature representation the construction is essentially the same even though the original models look very different (4d Yang-Mills and 2d sin(h)-Gordon).

### 2 Yang-Mills

Yang-Mills equations do not have one-dimensional zero-curvature representation. Therefore we consider not generic Yang-Mills perturbiner, but the one which generates only positive helicity form-factors, \( <^+ k_1, \ldots, ^+ k_N|A_\mu(x)|0> \), that is the solution of Yang-Mills equations of the following type,

\[
A_{\mu}^{ptb}(x, \{k_j\}, \{a_j\}) = \sum_{j=1}^{N} \epsilon^j_\mu t^j a_j e^{ik_j x} + \text{higher order terms in the plane waves } \{E_j = a_j e^{ik_j x}, j = 1, \ldots, N\}, \text{ where } \epsilon^j_\mu \text{ are positive helicity polarizations, } t^j \text{ are color matrixes. Such } A_{\mu}^{ptb} \text{ obeys self-duality equations. Indeed, linearized self-duality equation is obviously equivalent to the positive helicity condition, since both assume that “electric field” is equal to } i \cdot \text{ “magnetic field”. Beyond the linear approximation, solutions of the self-duality equations are solutions of the Yang-Mills equations as well, and for the Yang-Mills equations the solution of the type of perturbiner is unique (modulo gauge transformations, see above). The self-duality equations do have a zero-curvature representation with one-dimensional auxiliary space - the twistor representation.}

Notice: we get integrability not by substituting the theory, we just reduce the set of magnitudes we pretend to compute.
Solving self-duality equations:

It is convenient here to use the spinor notations, so that all vector objects have two spinor indices, e.g. the partial derivative \( \partial_\alpha \dot{\alpha} \), the connection-form \( A_{\alpha \dot{\alpha}} \), and the connection itself, \( \nabla_{\alpha \dot{\alpha}} = \partial_\alpha \dot{\alpha} + A_{\alpha \dot{\alpha}} \), where \( \alpha, \dot{\alpha} = 1, 2 \).

The curvature form, \( F_{\alpha \dot{\alpha} \beta \dot{\beta}} = [\nabla_{\alpha \dot{\alpha}}, \nabla_{\beta \dot{\beta}}] \) (3) has four spinor indices, and being antisymmetric with respect to the transposition of pairs of indices, decomposes as follows:

\[
F_{\alpha \dot{\alpha} \beta \dot{\beta}} = \varepsilon_{\alpha \beta} F_{\dot{\alpha} \dot{\beta}} + \varepsilon_{\dot{\alpha} \dot{\beta}} F_{\alpha \beta}
\] (4)

where \( \varepsilon \)'s are the standard antisymmetric tensors and \( F_{\alpha \beta}, F_{\dot{\alpha} \dot{\beta}} \) are some symmetric tensors. The first term in r.h.s. of Eq. (4) can be identified as self-dual part of the curvature and the second term - as antiself-dual one.

Introduce now a couple of complex numbers, \( \rho^\alpha, \alpha = 1, 2 \), which can be viewed on as homogeneous coordinates on auxiliary \( \mathbb{C}P^1 \) space. Contracting couple of \( \rho \)'s with un-dotted indices in Eq. (4) one sees that the condition \( \rho^\alpha \rho^\beta F_{\alpha \dot{\alpha} \beta \dot{\beta}} = 0 \) identically in \( \rho \), is equivalent to the statement that \( F_{\alpha \dot{\alpha} \beta \dot{\beta}} \) is self-dual. On the other side, contracting couple of \( \rho \)'s with un-dotted indices in Eq. (3), one obtains the zero-curvature representation of the self-duality, \( [\nabla_{\alpha}, \nabla_{\beta}] = 0 \) at any \( \rho^\alpha, \alpha = 1, 2 \) where \( \nabla_\alpha = \rho^\alpha \partial_\alpha \). Thus, if one introduces

\[
A_{\dot{\alpha}} = \rho^\alpha A_{\alpha \dot{\alpha}}, \quad \partial_\dot{\alpha} = \rho^\alpha \partial_{\alpha \dot{\alpha}}
\] (5)

any self-dual connection form can be (locally) represented as

\[
A_{\dot{\alpha}} = g^{-1} \partial_\dot{\alpha} g
\] (6)

where \( g \) is a group valued function of \( \rho \) and \( x \). All the non-triviality of the self-duality equation is now encoded in the condition that \( g \) must depend on \( \rho \) in such a way that \( A_{\dot{\alpha}} \) is a polynomial of degree 1 in \( \rho \), as in Eq. (5). If \( g \) is \( \rho \)-independent, it is a pure gauge transformation, as it is seen from Eq. (6).

The above condition on \( \rho \)-dependence of \( g \) is equivalent to condition that \( g \) is a homogeneous meromorphic function of \( \rho \) of degree 0 such that \( A_{\dot{\alpha}} \) from Eq. (5) is a homogeneous holomorphic function of \( \rho \) of degree 1 (a homogeneous holomorphic function of \( \rho \) of degree 1 is necessary just linear in \( \rho \), as in Eq. (6)). Notice, that nontrivial (not a pure gauge) \( g \) necessary has singularities in \( \rho \), since if it is regular homogeneous of degree 0, then it is just \( \rho \)-independent, that is , a pure gauge.

One-particle solution:

Consider the case when there is only one plane wave, \( A_{\alpha \dot{\alpha}} = \varepsilon_{\alpha \dot{\alpha}} \operatorname{t} a e^{i k_{\beta} x^\beta} \).
The momentum of the particle, \( k_{\beta\dot{\beta}} \), is a light-like four-vector, therefore it decomposes into a product of two spinors:

\[
k_{\alpha\dot{\alpha}} = \alpha_{\dot{\alpha}} \bar{\alpha}_{\alpha}
\]  

(7)

Due to the nilpotency of the parameter \( a \) all equations are automatically linearized in the one-particle case. The linearized self-duality equation,

\[
\varepsilon^{\dot{\alpha}\dot{\beta}} \varepsilon_{(\alpha\dot{\alpha})k_{\beta\dot{\beta}}} = 0,
\]

assumes that the polarization \( \varepsilon_{(\alpha\dot{\alpha})} \) is also light-like four-vector and its decomposition into a product of two spinors contains the same dotted spinor as \( k_{\beta\dot{\beta}} \) in Eq. (7), \( k_{\alpha\dot{\alpha}} = q_{\alpha} \bar{a}_{\dot{\alpha}} \). From the linearized version of Eq. (6), \( A_{\dot{\alpha}} = \partial_{\alpha} g \), one easily finds that

\[
g = 1 + \frac{(\rho q)}{(\rho \bar{a} E)} \hat{E} \]  

(8)

where \( E = a e^{ik_{\beta\dot{\beta}} x_{\beta\dot{\beta}}} \), \( \hat{E} = t E \) and the brackets with two spinors, like \( (\rho \bar{a} E) \), here and below stand for contraction of the spinors with the \( \varepsilon \)-tensor, \( (\rho \bar{a} E) = \varepsilon^{\alpha\dot{\beta}} \rho_{\alpha} \bar{a}_{\dot{\beta}} \) (indices of the spinors are raised and lowered with the \( \varepsilon \)-tensors).

Notice the simple pole of \( g \) in Eq. (8) at \( \rho_{\alpha} = \bar{a}_{\alpha} \) which is absent in \( A_{\dot{\alpha}} \).

**N-particle solution:**

So our problem is to find \( g_{pb}^{ptb}(\rho, \{E_j\}, \{k_j\}) \) - polynomial in \( \{E_j\} \) such that when all but one \( E \) are set to zero it reduces to Eq. (8) and that \( A_{pb}^{ptb} \) defined via \( g_{pb}^{ptb} \) as in Eq. (6) is regular on the auxiliary \( CP^1 \).

Regularity conditions:

One can show that the regularity of \( A_{pb}^{ptb} \) assumes that

a) \( g_{pb}^{ptb} \) has simple poles at \( \rho_{\alpha} = \bar{a}_{\alpha} \), \( j = 1, \ldots , N \) where \( \bar{a}_{\alpha} \) are the spinors which appear in decomposition of momenta of the particles as in Eq.(7);

b) \( g(E_j)^{-1} g_{pb}^{ptb} \) is regular at \( \rho_{\alpha} = \bar{a}_{\alpha} \), where \( g(E_j) \) is the one-particle solution Eq. (8), when only \( j \)-th particle (\( j \)-th plane wave) is present.

These conditions define \( g_{pb}^{ptb} \) up to multiplication by a \( \rho \)-independent matrix on the right, that is up to the gauge freedom. Notice that since \( g(E_j) \) depends on \( \hat{E}_j = t_j E_j \), not on \( t_j \) and \( E_j \) separately, the same will be true about \( g_{pb}^{ptb} \). So \( g_{pb}^{ptb} \) is a polynomial in \( \hat{E}_j, j = 1, \ldots , N \) with the Regularity conditions above. To find it explicitly we use the trick called

**Color ordering:**

Let us assume for a moment that the color matrixes \( t_j, j = 1, \ldots , N \) belong to a free associative algebra (no relation but \( (t_j t_k) t_l = t_j (t_k t_l) \)). Then \( g_{pb}^{ptb} \) is uniquely represented as a sum of ordered monomials in \( \hat{E}_j \)’s:

\[
g_{pb}^{ptb}(\rho, \{E\}) = 1 + \sum_j g_j(\rho) \hat{E}_j + \sum_{j_1, j_2} g_{j_1, j_2}(\rho) \hat{E}_{j_1} \hat{E}_{j_2} + \ldots
\]  

(9)
Then the Regularity conditions become a simple relation on the coefficient functions \( g_{j_1, j_2, \ldots, j_L} \) in Eq. (9) which is easily solved.

The solution:

\[
g_{j_1, j_2, \ldots, j_L}(\rho) = \left( \frac{\rho, q^{j_1}}{\rho, \omega^{j_1}} \right) \left( \frac{\omega^{j_2}, q^{j_1}}{\omega^{j_1}, \omega^{j_2}} \right) \left( \frac{\omega^{j_3}, q^{j_2}}{\omega^{j_2}, \omega^{j_3}} \right) \cdots \left( \frac{\omega^{j_L-1}, q^{j_L}}{\omega^{j_L-1}, \omega^{j_L}} \right)
\]

(10)

Eqs. (9), (10) define the solution of the problem. Of course, it remains to be the solution after specifying the color matrices \( t_j \) to obey some commutation relations.

Since \( g^{ptb} \) is known, one straightforwardly finds \( A_{ptb} \) via relation Eq. (6).

This way one describes all tree form-factors in the self-dual sector of Yang-Mills theory. One can add one antiself-dual plane wave solving linearization of the Yang-Mills equations in the background of \( A_{ptb} \).

3 \( \sin(h) \)-Gordon

Let us now turn to the \( \sin(h) \)-Gordon case (since the perturbiner is anyway complex solution, it does not really matter whether it is \( \sin \) or \( \sin(h) \)):

\[
\bar{\partial} \partial \phi + \frac{m^2}{\beta} \sinh \beta \phi = 0
\]

(11)

where \( \partial = \frac{\partial}{\partial z} \), \( \bar{\partial} = \frac{\partial}{\partial \bar{z}} \), and \( z, \bar{z} \) are two lightcone coordinates. In what follows we put \( m^2 = 1 \) since \( m^2 \)-dependence can easily be restored.

According to the general definition of perturbiner, we are looking for solution of Eq. (11) of the type of Eq. (2), where the plane waves are now

\[ E_j = a_j e^{ik_j z + i\lambda_j \bar{z}}. \]

The key ingredient of the construction of perturbiner in above cases - The zero-curvature representation

- is very well known in the present case, see e.g. the book [4].

\[
A_z = -\frac{\beta}{4} \sigma_1 \partial \phi + \frac{\lambda}{2} \sigma_3 \cosh \frac{\beta \phi}{2} + \frac{\lambda}{2} i \sigma_2 \sinh \frac{\beta \phi}{2}
\]

\[
A_{\bar{z}} = \frac{\beta}{4} \sigma_1 \bar{\partial} \phi - \frac{1}{2\lambda} \sigma_3 \cosh \frac{\beta \phi}{2} + \frac{1}{2\lambda} i \sigma_2 \sinh \frac{\beta \phi}{2}
\]

(12)

where \( \lambda \) is a non-homogeneous coordinate on an auxiliary \( CP^1 \) space, the so-called spectral parameter, and \( \sigma_i \) are Pauli matrixes. The \( \sin(h) \)-Gordon equation (11) is equivalent to \( \partial A_{\bar{z}} - \bar{\partial} A_z + [A_z, A_{\bar{z}}] = 0 \). The connection form Eq. (12) is meromorphic on the auxiliary \( CP^1 \) space with simple poles at \( \lambda = 0 \) and \( \lambda = \infty \). Correspondingly, the zero-curvature condition consists in fact of a
number of equations - at different powers of $\lambda$ - most of which are automatically resolved when the connection form is taken in the form Eq. (12), independently of the field $\phi(z, \bar{z})$. The only nontrivial equation arises at $\lambda^0$ and is equivalent to Eq. (11).

Mikhailov’s reduction:
It would be very inconvenient to look for a flat connection of the particular form Eq. (12). Luckily, due to work we know how those flat connections which produce sin(h)-Gordon are distinguished among all flat connections. Namely, a generic zero-curvature connection with simple poles at $\lambda = 0, \infty$ obeying the “reduction condition” $A(-\lambda) = \sigma_1 A(\lambda) \sigma_1$ is equivalent to the connection Eq. (12) modulo gauge transformations and a choice of coordinates $z, \bar{z}$. The gauge transformations are transformations with $\lambda$-independent $SL(2, \mathbb{C})$ matrix commuting with the reduction condition.

The zero-curvature condition is (locally) solved as $A = g^{-1} dq$ where, $g$ is a nontrivial function of $\lambda$ subject to the condition that the connection form $A(\lambda)$ has simple poles at $\lambda = 0$ and $\lambda = \infty$ and also that $A(\lambda)$ obeys the reduction condition which for $g$ gives $g(-\lambda) = \sigma_1 g(\lambda) \sigma_1$. The gauge transformations act on $g(\lambda)$ as multiplication by a $\lambda$-independent commuting with $\sigma_1$ matrix from the right.

Since $\phi_{ptb}$ is polynomial in the plane waves $\mathcal{E}$, so are $A_{ptb}$ and $g_{ptb}$. A novel thing compared to the Yang-Mills case is that $A_{ptb}$ has a term of zero-th order in $\mathcal{E}$’s which is convenient to split off explicitly:

$$[13]$$

$$A_{ptb}(\lambda, \{E\}) = A^{(0)}(\lambda) + A'(\lambda, \{E\})$$

$$A' = g_{ptb}'^{-1} \nabla^{(0)} g_{ptb}'$$

where the non-derivative term in $\nabla^{(0)}$ acts on $g_{ptb}'$ as commutator.

Further steps of construction are parallel to the Yang-Mills case.

One-particle solution:

$$g'(\lambda, E_j) = 1 + \frac{\beta}{4} E_j \sigma + \frac{\lambda + q_j}{\lambda + ik_j - q_j} \frac{2ik_j}{\lambda - ik_j - q_j} + \frac{\bar{E}_j \sigma}{\lambda - ik_j - q_j} \frac{2ik_j}{\lambda - ik_j - q_j}$$

where $\sigma_\pm = \frac{1}{2}(\sigma_1 \pm i \sigma_2)$. Notice that every particle contributes now two poles ($\lambda = \pm ik_j$) in $g$ which is intimately related with the reduction condition.

To put the problem of constructing the $N$-particle solution:

into a more universal form introduce now some more notations:

$\tilde{j}$ consisting of two indices, $\tilde{j} = (j, s) ; j = 1, \ldots, N; s = \pm$, notations $\tilde{E}_j$, $\tilde{E}_{j, \pm} = \frac{2ik_j}{\lambda - ik_j - q_j} E_j \sigma_\pm$, where $\lambda_j = \lambda_j, \pm = \mp ik_j$, $q_j = q_j, \pm = \mp q_j$ and $g'(\tilde{E}_j) = 1 + \tilde{E}_j \frac{\lambda - q_j}{\lambda - \lambda_j}$. In these notations $g_{ptb}'$ obeys just the same Regularity conditions as $g_{ptb}$ in in Yang-Mills case so
The solution:
for $g_{ptb}$ is given by Eqs. (9), (10):
\[ g'(\lambda) = 1 + \sum_{d=1}^{N} \sum_{j_1,\ldots,j_d} \frac{\lambda - q_{j_1}}{\lambda - \lambda_{j_1}} \frac{\lambda_{j_1} - q_{j_2}}{\lambda_{j_1} - \lambda_{j_2}} \cdots \frac{\lambda_{j_{d-1}} - q_{j_d}}{\lambda_{j_{d-1}} - \lambda_{j_d}} \xi_{j_1} \cdots \xi_{j_d} \]
from which one obtains, finally,
\[ \phi_{ptb} = \sum_{d \text{ odd}} \frac{1}{2} \beta_d^{d-1} \sum_{j_1,\ldots,j_d} \frac{k_{j_1} \cdots k_{j_d}}{(k_{j_1} + k_{j_2}) \cdots (k_{j_d} + k_{j_1})} \xi_{j_1} \cdots \xi_{j_d}. \]  

4 Gravity

Due to lack of space and time I am not able to say anything about gravitational perturbation, I just refer to the original works [4–6].

Acknowledgments

I would like to acknowledge RFBR grant 99-01-10584 and the organizers of the conference for financial support. I would also like to thank the organizers for their kind hospitality and for the beautiful surroundings of the conference.

References

1. K.G.Selivanov, ITEP-21-96, hep-ph/9604206
2. A.A.Rosly and K.G.Selivanov, Phys. Lett. B 399, 135 (1997)
3. K.G.Selivanov, Talk given at International Europhysics Conference on High-Energy Physics (HEP 97), Jerusalem, Israel, 19-26 Aug 1997.
4. A.A.Rosly and K.G.Selivanov, ITEP-TH-56-97, hep-th/9710196
5. K.G.Selivanov, Phys. Lett. B 420, 274 (1998)
6. K.G.Selivanov, Mod. Phys. Lett. A 12 1997 3087
7. A.A.Rosly and K.G.Selivanov, Phys. Lett. B 426, 334 (1998)
8. K.G.Selivanov, ITEP-TH-47-98, hep-th/9809046
9. W.Bardeen, Prog. Theor. Phys. Suppl. 123 (1996) 1
10. V.Korepin, T.Oota, J. Phys. A 29 (1996) 625
11. L.D.Faddeev and A.A.Slavnov, Introduction to the Theory of Quantum Gauge Fields, Nauka, Moscow, 1978
12. C.Itzykson and J.B.Zuber, Quantum Field Theory
New York, USA: Mcgraw-hill (1980)
13. R.S.Ward, Phys. Lett. A 61 (1977) 81
14. L.D.Faddeev, L.A.Takhtajan, Hamiltonian methods in the theory of solitons, Moscow, Nauka, 1986
15. A.Mikhailov, Physica D, 3 (1981) 73