Combinatorial proof of the transcendence of $L(1, \chi_s)/\Pi$

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Abstract
We give a combinatorial proof of the transcendence of $L(1, \chi_s)/\Pi$, where $L(1, \chi_s)$ (resp. $\Pi$) is the analogue in characteristic $p$ of the function $L$ of Dirichlet (resp. $\pi$). This result has been proven by G. Damamme using the criteria of de Mathan. Our proof is based on the Theorem of Christol and another property of $k$-automatic sequences.

1 Introduction

$$[k] = T^q - T,$$

$$L_k = [k][1], \quad L_0 = 1.$$

$$\Pi = \prod_{j=1}^{\infty} \left( 1 - \frac{[j]}{[j+1]} \right)$$

Theorem 1 (Theorem 2 in [3]). For $s < q$ and $a \in \mathbb{F}_q$,

$$L(1, \chi_s) = \sum_{k=0}^{\infty} (-1)^{k(s-1)} \frac{(T-a)^{q_k-1}}{L_k}. $$

The following Theorem is proved in [3] as a corollary of Theorem 1 using the criteria of De Mathan.

Theorem 2 (Corollary 2 in [3]). For $1 < s < q$, $L(1, \chi_s)/\Pi$ is transcendental over $\mathbb{F}_q(T)$.

Our goal in this article is to give another proof of Theorem 2 starting from the expression of Theorem 1 by means of properties of automatic sequences.

For an integer $k \geq 2$, one of the equivalent definitions of a $k$-automatic sequence is a sequence that can be generated by a $k$-DFAO (deterministic finite automaton with output). We recall here the definition of the latter as we will need it in the proof of Lemma 1:

A $k$-DFAO is a 6-tuple

$$M = (Q, \Sigma_k, \delta, q_0, \Delta, \tau)$$

where $Q$ is a finite set of states, $\Sigma_k$ the input alphabet $\{0, 1, ..., k-1\}$, $\delta : Q \times \Sigma_k \to Q$ the transition function, $q_0 \in Q$ the initial state, $\Delta$ the output alphabet, and $\tau : Q \to \Delta$ the output function.
function. We expand δ to a function from $Q \times \Sigma_k \to Q$ by defining, for a word $w = w_1...w_j$ of length at least 2 in $\Sigma_k^*$, $\delta(q, w) = \delta(\delta(1, w_1), w_1...w_{j-1})$. The sequence $(u(n))_{n \geq 0}$ generated by the automaton $M$ is defined by $u(0) = \tau(q_0)$ and $u(n) = \tau(\delta(q_0, (n)_k))$ for $n > 0$, where $(n)_k$ is the base-$k$ expansion of $n$. In other words, we define $u(n)$ to be the output when we feed the base-$k$ expansion of $n$ to $M$ starting from the least significant digit.

The following theorem reduces the problem of proving the transcendence of a series over $\mathbb{F}_q(T)$ to proving the non-$q$-automaticity of the sequence of its coefficients.

**Theorem 3** (Christol, Kamae, Mendès France, and Rauzy). The formal power series $f(T) = \sum_{n=0}^{\infty} f_n T^{-n} \in \mathbb{F}_q[[\frac{1}{T}]]$ is algebraic over the fraction field $\mathbb{F}_q(T)$ if and only if the sequence $(f_n)_{n \geq 0}$ is $q$-automatic.

The following lemma gives a necessary condition of $k$-automaticity, and therefore a way of proving that a sequence is not $k$-automatic. For a letter $x$ in $\{0, ..., k-1\}$, the notation $x^n$ means the concatenation of $n$ times $x$. For a word $w = w_0...w_n \in \{0, ..., k-1\}^*$, we let $[w]_q$ denote the integer whose base-$q$ expansion is $w$.

**Lemma 1.** Let $(u(n))_{n \geq 0}$ be a $k$-automatic sequence. Then the set of sequences

$$\{ (u([1^n0^j]_k))_{n \geq 1} \mid j \in \mathbb{N} \}$$

is finite.

**Proof.** Let $M = (Q, \Sigma_k, \delta, q_0, \Delta, \tau)$ be a $k$-DFAO that generates $(u(n))_{n \geq 0}$. Then $(u([1^n0^j]_k)) = \tau(\delta(q_0, 0^n)) = \tau(\delta(q_0, 0^j))$. As $\delta(q_0, 0^n) \in Q$ and $Q$ is finite, the set $\{ (u([1^n0^j]_k))_{n \geq 0} \mid j \in \mathbb{N} \}$ is finite. \hfill $\square$

As in [11], we define

$$\alpha = \prod_{j=0}^{\infty} \left( 1 - \frac{Tq^j}{Tq^{j+1}} \right).$$

As $\alpha$ is algebraic over $\mathbb{F}_q(T)$, in order to prove Theorem 2 we only need to prove the transcendence of $\frac{\alpha}{\Pi L(1, \chi_s)}$. From Theorem 1 we deduce the expression that we will use for this article:

$$\alpha \Pi L(1, \chi_s) = \sum_{k=0}^{\infty} (-1)^{k(s-1)} \left( \frac{1}{T} \right)^{(q-s)^k} \left( 1 - \frac{a}{T} \right)^{s^{k-2}} \prod_{j=k+1}^{\infty} \left( 1 - \left( \frac{1}{T} \right)^{q^j-1} \right). \tag{8}$$

In Section 2 we will prove the following proposition:

**Proposition 1.** Let $s$ be an integer such that $1 < s < q$. We denote by $u(n)$ coefficients of $\frac{1}{\Pi L(1, \chi_s)}$. Then for all $j \in \mathbb{N}$, the sequence $(u([1^n0^j]_q))$ is ultimately periodic and the length of the initial non-periodic segment of $(u([1^n0^j]_q))$ is a strictly increasing function with respect to $j$. In particular, the set

$$\{ (u([1^n0^j]_q))_{n \geq 0} \mid j \in \mathbb{N} \}$$

is infinite.

We obtain immediately the following Corollary using Theorem 3 and Lemma 1.

**Corollary 1.** For $1 < s < q$, series $\frac{\alpha}{\Pi L(1, \chi_s)}$ is transcendental over $\mathbb{F}_q(T)$.
2 Proof of Proposition 1

We let $S_k$ denote the $k$-th summand in the expression $(\ast)$. First we observe that for $b \in \mathbb{N}$, the term $T - b$ may appear in $S_k$ for more than one $k$. We want to determine $[T - b]S_k$, the coefficient of $T - b$ in $S_k$. For $1 < s < q$, we denote $q - s$ by $\bar{s}$, then from $(\ast)$ we see that if $[T - b]S_k \neq 0$, then $b$ can be written as

$$b = r_k + \sum_{j=k+1}^{\infty} \varepsilon_j (q^j - 1),$$

(1)

where $r_k \in [[\bar{s}]_q, [1^k0]_q]$, $\varepsilon_j \in \{0, 1\}$ for $j \geq k + 1$ and $\varepsilon_j = 0$ for $j$ big enough. The following Lemma implies that such a decomposition is unique for $b$ and $k$.

**Lemma 2.** i) Let $k$ and $l$ be positive integers such that $l \geq k$, then

$$[1^k0]_q + \sum_{k+1 \leq j \leq l} (q^j - 1) < q^{l+1} - 1.$$

ii) In particular, if $n$ can be written as

$$b = r_k + \sum_{j=k+1}^{\infty} \varepsilon_j (q^j - 1),$$

where $r_k \in [[\bar{s}]_q, [1^k0]_q]$, $\varepsilon_j \in \{0, 1\}$ not all 0 and $\varepsilon_j = 0$ for $j$ big enough, then

$$\max_{j \geq k+1} \{\varepsilon_j = 1\} = \max_{j \in \mathbb{N}} \{q^j - 1 \leq b\}.$$

**Proof.** i)

$$[1^k0]_q + \sum_{k+1 \leq j \leq l} (q^j - 1)$$

\leq [1^k0]_q + \sum_{k+1 \leq j \leq l} q^j

= [1^k0]_q + [1^{l-k}0^{k+1}]q

= [1^l]_q

< [1^{l+1}]_q

\leq q^{l+1} - 1.$$

ii) It is evident that

$$J_1 := \max_{j \geq k+1} \{\varepsilon_j = 1\} \leq \max_{j \in \mathbb{N}} \{q^j - 1 \leq b\} =: J_2.$$

Suppose that the inequality is strict. Then we would have

$$b = r_k + \sum_{j=k+1}^{J_1} \varepsilon_j (q^j - 1)$$

\leq [1^k0] + \sum_{j=k+1}^{J_1} (q^j - 1)

< q^{J_2} - 1

\leq b.$$
For $b \in \mathbb{N}^*$, we can obtain all possible decompositions of $b$ of the form (1) by applying repetitively Lemma 2.

**Input**: positive integer $b$

**Output**: finite sequence $(b)_n$ and a set $I$

\[
i := 1;
\]
\[
I := \emptyset;
\]
\[
b_1 := b;
\]

if $\exists l \in \mathbb{N}$ s.t. $[s^l]_q \leq b_i \leq [1^l]_q$ then

\[
\text{add } i \text{ to } I;
\]
end

while $\exists l \in \mathbb{N}^*$ s.t. $b_i \geq q^l - 1$ do

\[
l_i := \max\{b \geq q^l - 1\};
\]

if $b_i - (q^{l_i} - 1) > [1^{l_i-1}]_q$ then

\[\text{end of procedure;}
\]
else

\[
b_{i+1} := b_i - (q^{l_i} - 1);
\]

\[i + 1;
\]

if $\exists l \in \mathbb{N}$ s.t. $[s^l]_q \leq b_i \leq [1^l]_q$ then

\[
\text{add } i \text{ to } I;
\]
end

end

***Algorithm 1: Decomposition of $b$***

Then all decompositions of $b$ in the form (1) are $b_i + \sum_{k=1}^{i-1} (b_k - b_{k+1})$ for $i \in I$.

As we are interested in the coefficients $u([1^m0^n]_q)$, we define $b_{j,m,1} = [1^m0^j]_q$ for $j, m \in \mathbb{N}^*$. And we define $b_{j,m,n}$ using the procedure above with input $b_{j,m,1}$.

For example, for $j = 2$ and $q = 3$, the base-$q$ expansion of $b_{j,m,n}$ is as follows, the symbol $*$ means that $b_{j,m,n}$ is not defined:

|   | 1   | 2   | 3   | 4   | 5   |
|---|-----|-----|-----|-----|-----|
| 1 | 100 | 1   | *   | *   | *   |
| 2 | 1100| 101 | 2   | 0   | *   |
| 3 | 11100|1101|102  |10  | *   |
| 4 | 111100|1101|1102 |110 | *   |
|   |     |     |     |     |     |

Table 1: $b_{2,m,n}$ for $q = 3$

We can observe some patterns from the table above, which we summarize in the following Lemma:

**Lemma 3.** For $j \geq 2$, the statement $P(n)$ is true for $1 \leq n \leq q^{j-1} + 1$ and the statements $Q(n)$ and $R(n)$ are true for $1 \leq n \leq q^{j-1}$:

- **$P(n)$:** For all $m \in \mathbb{N}^*$ and $m \geq n - 1$, $b_{j,m,n}$ is defined and $b_{j,m+1,n} = b_{j,m,n} + q^{j+m+1-n}$.
- **$Q(n)$:** For all $m \geq n$, $l_{j,m,n} := \max\{b_{j,m,n} \geq q^l - 1\} = j + m - n$, and $b_{j,m,n} - (q^{j,m,n} - 1) \leq q^{j,m,n}$ for all $m \geq n$. The proof of these statements is omitted for brevity.
For Corollary 2.

Proof. We prove by induction on $n$.

For $n = 1$, $P(1)$ is true by definition of $b_{j,m,1}$.

To prove $Q(1)$ we use induction on $m$. First,

$$l_{j,1,1} = \max_{l \in \mathbb{N}^*} \{ b_{j,1,1} \geq q^l - 1 \} = \max_{l \in \mathbb{N}^*} \{ [10^l]_1 \geq q^l - 1 \} = j + 1 - 1.$$  

And

$$b_{j,1,1} = (q^{j+1,1} - 1) = 1 \leq [1^{j+1,1-1}0]_q.$$ 

Suppose that the statements are true for $m$, using $P(1)$ we have

$$l_{j,m+1,1} = \max_{l \in \mathbb{N}^*} \{ b_{j,m+1,1} \geq q^l - 1 \} = \max_{l \in \mathbb{N}^*} \{ b_{j,m,1} + q^{j+m+1,1-1} \geq q^l - 1 \} = l_{j,m,1} + 1 = j + m + 1 - 1,$$

and

$$b_{j,m+1,1} = (q^{j+m+1,1-1} - 1) = (b_{j,m,1} + q^{j+m+1,1-1}) - q^{j+m+1} + 1 = b_{j,m,1} + 1 = b_{j,m,1} - (q^{j+m-1} - 1) + q^{j+m-1} \leq [1^{j+m-1-1}0]_q + q^{j+m-1} = [1^{j+m-1}0]_q,$$

which proves $Q(1)$.

From $P(1)$ and $Q(1)$ follows $R(1)$.

Suppose that for $n < q^{l-1}$, we have proven $P(n')$, $Q(n')$ and $R(n')$ for all $n' \in \{1, \ldots, n\}$. Let us prove $P(n + 1)$, $Q(n + 1)$ and $R(n + 1)$.

First, $P(n + 1)$ can be deduced immediately from $P(n)$ and $R(n)$.

For $Q(n + 1)$, we prove by induction on $m \geq n + 1$. By $R(1), \ldots, R(n)$ we have $b_{j,n+1,n+1} = b_{j,1,1} + n \leq q^l + q^{l-1} - 1$. Therefore

$$l_{j,n+1,n+1} = \max_{l \in \mathbb{N}^*} \{ b_{j,n+1,n+1} \geq q^l - 1 \} \leq \max_{l \in \mathbb{N}^*} \{ q^l + q^{l-1} - 1 \geq q^l - 1 \} = j,$$

on the other hand,

$$l_{j,n+1,n+1} = \max_{l \in \mathbb{N}^*} \{ b_{j,n+1,n+1} \geq q^l - 1 \} \geq \max_{l \in \mathbb{N}^*} \{ q^l \geq q^l - 1 \} = j.$$

Therefore $l_{j,n+1,n+1} = j = j + (n + 1) - (n + 1)$. Besides,

$$b_{j,n+1,n+1} - (q^{j,n+1,1-1} - 1) \leq q^j + q^{j-1} - 1 - (q^j - 1) = q^{j-1} \leq [1^{j-1}0]_q,$$

which proves $Q(n + 1)$. From $P(n + 1)$ and $Q(n + 1)$ follows $R(n + 1)$.

Finally, $P(q^{l-1} + 1)$ can be deduced from $Q(q^{l-1})$ and $R(q^{l-1})$.

Corollary 2. For $j \geq 2$, $1 \leq n \leq q^{j-1} + 1$ and $m \geq n$, $b_{j,m,n} \in \{ [q^{j+m-n}]_q, \ldots, [1^{j+m-n}0]_q \}$.  

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Now we look at the table of $b_{j,m,n}$ for $j = 4$ and $q = 3$.

Table 2: $b_{3,m,n}$ for $q = 3$

We notice that starting from $m = 9$ and $n = 10$, the subtable is the same as that of $j = 2$ and $q = 3$. It is the case in general that the table of $b_{j,m,n}$ occurs at the end of the table of $b_{j+1,m,n}$.

**Lemma 4.** For $j, m, n \in \mathbb{N}^*$, $n \geq q^i + 1$ and $m \geq q^j$, $b_{j+1,m,n}$ is defined if and only if $b_{j,m-q^i+1,n-q^i}$ is defined. When they are defined they have the same value.

**Proof.** By the definition of $b_{j,m,n}$, the first two columns determine the rest of the table. Therefore we only need to prove that for all $m \geq q^j$,

\[ b_{j+1,m,q^i+1} = b_{j,m-q^i+1,1} \quad (2) \]

and

\[ b_{j+1,m,q^i+2} = b_{j,m-q^i+1,2} \quad (3) \]

By applying Lemma 3 we have

\[ b_{j+1,q^i,q^i+1} = b_{j+1,1,1,2} + q^j - 1 = q^j = b_{j,1,1}, \]

and for $k \in \mathbb{N}$,

\[ b_{j+1,q^i+k+1,q^i+1} = b_{j+1,q^i+k,q^i+1} = q^j+k+2. \]

Thus for $m \geq q^j$

\[ b_{j+1,m,q^i+1} = [1^{m+1-q^i}0^1]^q_j = b_{j,m+1-q^i,1} \]

which proves (2).

For $m \geq q^j$, according to the second point Lemma 3

\[ b_{j+1,m,q^i+1} = b_{j+1,m,q^i+2} = b_{j+1,m,q^i+1} - b_{j+1,m,q^i+2} = q^j+1+m-(q^i+1) - 1 = q^j+(m-q^i+1)-1 = b_{j,m-q^i+1,1} - b_{j,m-q^i+1,2}. \]

which proves (3).

To calculate the coefficient of $T^{-n}$ in $S_k$, we define $k_{j,m,n}$ and $c_{j,m,n}$ as follows: When $b_{j,m,n}$ is defined and there exists $k \in \mathbb{N}$ such that $b_{j,m,n} \in \{[s^k]^q_j, \ldots, [1^k0]^q_j\}$, $k_{j,m,n}$ is defined to be $k$. Otherwise $k_{j,m,n}$ is not defined. When $k_{j,m,n}$ is defined, $c_{j,m,n}$ is defined to be $b_{j,m,n} - [s^k_{j,m,n}]_q$. Finally we define $N_{j,m}$ to be $\{n \in \mathbb{N}^* \text{ such that } c_{j,m,n} \text{ is defined}\}$. From the expression (*) we see that:

**Lemma 5.** For $j, m \in \mathbb{N}^*$,

\[ \left(T^{[1^{n-q^i}]_j}\right) \alpha \prod L(1, \chi_s) = \sum_{n \in N_{j,m}} (-1)^{k_{j,m,n}(s-1)} \left(\frac{s^{k_{j,m,n}}}{c_{j,m,n}}\right)_q (-a)^{c_{j,m,n}} \cdot (-1)^{n-1}. \]
For $n \in N_{j,m}$, we denote by $d_{j,m,n}$ the quantity $(-1)^{k_{j,m,n}(s-1)} \begin{bmatrix} k_{j,m,n} \\ c_{j,m,n} \end{bmatrix} (-a)^{c_{j,m,n}} (-1)^{n-1}$.

For other $n \in N^*$, we define $d_{j,m,n}$ to be 0 for convenience.

In order to calculate the coefficients we need the following Theorem:

**Theorem 4** (Lucas). Let $p$ be a prime number and $q = p^k$ for $k \in N^*$. Let $m = \sum_i m_i q^i$, $n = \sum_j n_j q^j$ be two integers, where $m_i, n_j \in \{0, 1, ..., q-1\}$. Then

$$\binom{m}{n} = \prod_i \binom{m_i}{n_i} \mod p.$$  

Let us look at an example of $c_{j,m,n}$ and $d_{j,m,n}$ with $j = 2$, $q = 3$ and $s = 2$:

| $m$ | 1 | 2 | 3 | 4 | 5 |
|-----|---|---|---|---|---|
| 1   | 12| 0 | * | * | * |
| 2   | 212| 20| 1 | 0 | * |
| 3   | 2212| 220| 21| 2 | * |
| 4   | 22212| 2220| 221| 22| * |

Table 3: $c_{2,m,n}$ for $q = 3$ and $s = 2$

| $m$ | 1 | 2 | 3 | 4 | 5 |
|-----|---|---|---|---|---|
| 1   | $2(-a)^4$| 1 | 0 | 0 | 0 |
| 2   | $-2(-a)^5$| $(-a)^2$| $-2(-a)$| -1| 0 |
| 3   | $2(-a)^7$| $(-a)^4$| $2(-a)^3$| $(a)^2$| 0 |
| 4   | $-2(-a)^9$| $(-a)^6$| $-2(-a)^5$| $(-a)^4$| 0 |
| 5   | $2(-a)^{11}$| $(-a)^8$| $2(-a)^7$| $(a)^6$| 0 |

Table 4: $d_{2,m,n}$ for $q = 3$ and $s = 2$

From the table we observe that $(d_{j,m,n})_{m \geq n}$ seems to be periodic. Indeed, we have:

**Lemma 6.** For $j \in N^*$ and $1 \leq n \leq q^{j-1} + 1$, the sequence $(d_{j,m,n})_{m \geq n}$ is periodic.

**Proof.** Throughout this proof we suppose that $1 \leq n \leq q^{j-1} + 1$ and $m \geq n$.

First, we know from Corollary 2 that $k_{j,m,n}$ and thus $d_{j,m,n}$, are defined and $k_{j,m,n} = j+m-n$.

From Lemma 3 we see that

$$0 < c_{j,m,n} \leq q^{j+m-n} - 1$$

and

$$c_{j,m+1,n} = c_{j,m,n} + s \cdot q^{j+m-n}.$$  

Therefore

$$d_{j,m+1,n} = (-1)^{k_{j,m+1,n}(s-1)} \begin{bmatrix} k_{j,m+1,n} \\ c_{j,m+1,n} \end{bmatrix} (-a)^{c_{j,m+1,n}} (-1)^{n-1}$$

$$= (-1)^{(j+m+1-n)(s-1)} \begin{bmatrix} k_{j,m+1,n} + c_{j,m,n} \\ s \cdot q^{j+m-n} \end{bmatrix} (-a)^{s \cdot q^{j+m-n} + c_{j,m,n}} (-1)^{n-1}$$

$$= (-1)^{s-1}(-1)^{k_{j,m,n}(s-1)} \begin{bmatrix} s \cdot q^{j+m-n} \\ c_{j,m,n} \end{bmatrix} (-a)^{s \cdot q^{j+m-n} - a} (-1)^{n-1}$$

$$= (-1)^{s-1}(-a)^s \cdot d_{j,m,n}.$$
As \((-1)^{s-1}(-a)^s\) is an element in a finite field, if \(a \neq 0\), the sequence \((d_{j,m,n})_{m \geq n}\) is periodic. If \(a = 0\), as \(c_{j,n,n} \neq 0\), the sequence \((d_{j,m,n})_{m \geq n}\) is always 0, therefore also periodic. \(\square\)

For an ultimately periodic sequence \((a_n)_n\) we define \(IN((a_n)_n)\) to be the index of the earliest term from which the sequence is periodic. That is,

\[
IN((a_n)_n) = \min_i\{(a_n)_{n \geq i} \text{ is periodic}\}.
\]

The idea of the proof of Proposition \[\] is that the sequences \((d_{j,m,n})_{m \geq 1}\) are ultimately periodic and the \(IN((d_{j,m,n})_{m \geq 1})\) increases with \(n\) for \(n \leq n_0 := \sum_{i=0}^{j-1} q^i\). For \(n > n_0\), the sequence \((d_{j,m,n})_{m \geq 1}\) is zero. We have \(u \big([1^m0^3]_q\big) = \sum_{n=1}^{n_0} d_{j,m,n}\) and \(IN\left((u \big([1^m0^3]_q\big))_{m \geq 1}\right)\) is not far from the \(IN((d_{j,m,n_0})_{m \geq 0})\). In order to justify the last point, we need to take a closer look at the table of \(d_{2,m,n}\), which according to Lemma \[\] occurs at the end of the table of \(d_{j,m,n}\) for \(j \geq 3\).

From the proof of Lemma \[\] and the definition of \(b_{j,m,n}\) and \(d_{j,m,n}\) it is easy to give an explicit expression of \(d_{2,m,n}\):

**Lemma 7.** For \(n = 1,\)

\[
d_{2,m,n} = \left(\frac{s}{s-1}\right)(-a)^{s+1+(m+1-n)s}(1)^{(s-1)(m-n)}(-1)^{n-1}.
\]

For \(2 \leq n \leq s,\) \(d_{2,m,n} = 0\) for all \(m \in \mathbb{N}^*\).

For \(s+1 \leq n \leq q,\)

\[
d_{2,m,n} = \begin{cases} 0 & \text{if } m < n-1 \\ \left(\frac{s}{s-1}\right)(-a)^{(n-1-s)+(m+1-n)s}(1)^{(s-1)(m-n)}(-1)^{n-1} & \text{if } m \geq n-1 \\
\end{cases}
\]

For \(n = q+1,\)

\[
d_{2,m,n} = \begin{cases} 0 & \text{if } m < n-2 \\ (-a)^{(m+2-n)s}(1)^{(s-1)(m-n)}(-1)^{n-1} & \text{if } m \geq n-2 \\
\end{cases}
\]

For \(n > q + 1,\) \(d_{2,m,n} = 0\) for all \(m \in \mathbb{N}^*\)

From the table of \(d_{2,m,n}\) and \(d_{4,m,n}\) we can see easily that the following Lemma is true. We provide nonetheless a proof for the sake of completeness.

**Lemma 8.** \(IN\left((u \big([1^m0^3]_q\big))_m\right) \geq q^2.\)

**Proof.** We divide the argument into two cases.

Case 1: \(1 - s \cdot q^{s-1} \neq 0\) or \(q > 3.\)

In this case we prove that \(IN\left((u \big([1^m0^2]_q\big))_m\right) \geq 2.\) Thus by Lemma \[\] and Lemma \[\] we have \(IN\left((u \big([1^m0^3]_q\big))_m\right) \geq q^2.\)

Case 1.1: If \(a = 0,\) then from Lemma \[\] we know that

\[
\max_m\{\exists n \in \mathbb{N}^* \text{ s. t. } d_{2,m,n} \neq 0\} = q - 1.
\]

Therefore \(IN\left((u \big([1^m0^2]_q\big))_m\right) = q - 1.\)

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Case 1.2 If \( a \neq 0 \) and \( 1 - s \cdot a^{s-1} \neq 0 \), we rewrite the expressions in Lemma 7 for \( n = q + 1 \) as

\[
d_{2,m,q} = \begin{cases} 
0 & \text{if } m < q - 1 \\
 s \cdot (-a)^{(s-1)+(m+1-q)} \cdot (-1)^{(s-1)(m-q)} (1)^{q-1} & \text{if } m \geq q - 1.
\end{cases}
\]

\[
d_{2,m,q+1} = \begin{cases} 
0 & \text{if } m < q - 1 \\
 (-a)^{(m+1-q)} \cdot (-1)^{(s-1)(m-q-1)} (1)^{q} & \text{if } m \geq q - 1.
\end{cases}
\]

Therefore

\[
d_{2,m,q} + d_{2,m,q+1} = \begin{cases} 
0 & \text{if } m < q - 1 \\
 (1 - s \cdot a^{s-1}) (-a)^{(m+1-q)} \cdot (-1)^{(s-1)(m-q-1)} (1)^{q} & \text{if } m \geq q - 1.
\end{cases}
\]

Since \( 1 - s \cdot a^{s-1} \neq 0 \), \( IN((d_{2,m,q} + d_{2,m,q+1})_m) = q - 1 \). By Lemma 8, \( IN((d_{2,m,n})_m) \leq q - 2 \) for all \( 1 \leq n \leq q - 1 \). Therefore \( IN((u([1^{m0^{q}]}_q])_m) = q - 1 \geq 2 \).

Case 1.3 If \( 1 - s \cdot a^{s-1} = 0 \) and \( q \geq 4 \), \( IN((d_{2,m,q} + d_{2,m,q+1})_m) = 1 \). Since \( s > 1 \), \( s + 1 \leq q - 1 \). Therefore by Lemma 6 we have \( IN(d_{2,m,q-1}) = q - 2 > 1 \) since \( q \geq 4 \). By Lemma 4 for \( 1 \leq n < q - 1 \), \( IN(d_{2,m,n}) \geq \max(1, \ldots, q - 3) \). Therefore \( IN((u([1^{m0^{q}]}_q])_m) = q - 2 \geq 2 \).

Case 2: If \( 1 - s \cdot a^{s-1} = 0 \) and \( q = 3 \), using the formula in Lemma 7 we find that \( d_{2,m,n} = 0 \) for \( m, n \in \mathbb{N}^+ \). So we look at the table of \( d_{3,m,n} \). With similar calculation we find for \( n \leq q^2 \), the sequences \( (d_{3,m,n})_m \) are periodic from \( m = q^2 - 1 \). But the sum of the last four columns \( n = q^2 + 1, \ldots, q^2 + 4 \) is only periodic from \( m = q^2 \). Therefore \( IN((u([1^{m0^{q}]}_q])_m) = q^2 \).

\[ \square \]

**Proof of Proposition 1** We prove by induction on \( j \) that \( IN((u([1^{m0^{q}]}_q])_m) \geq q^{j-1} \) for \( j \geq 3 \).

By Lemma 9 we know that \( IN((u([1^{m0^{q}]}_q])_m) \geq q^2 \).

Suppose that for \( j \) we have proven \( IN((u([1^{m0^{q}]}_q])_m) \geq q^{j-1} \). We define \( n_0 := \sum_{i=0}^{j} q^i \) and \( n_1 = \sum_{i=0}^{j-1} q^i \). Then

\[
u([1^{m0^{q+1}}_q]) = \sum_{n=1}^{n_0} d_{j+1,m,n} + \sum_{n=1}^{q^j} d_{j+1,m,n} + \sum_{n=1}^{n_1} d_{j+1,m,n+q^j}.
\]

By Lemma 8 we know that \( IN(\sum_{n=1}^{q^j} d_{j+1,m,n}) \leq q^j \). By Lemma 9 we know that

\[
\sum_{n=1}^{n_1} d_{j+1,m,n+q^j-1,n+q^j} = \sum_{n=1}^{n_1} d_{j,m,n} \cdot (-1)^{q} = \nu([1^{m0^{q}}_q]) \cdot (-1)^{q}.
\]

By the hypothesis of induction we have

\[
IN((\sum_{n=1}^{n_1} d_{j+1,m,n+q^j})_m) = q^j - 1 + IN((\nu([1^{m0^{q}}_q])_m) \geq q^j + 2.
\]

When we have two ultimately periodic sequences \( u \) and \( v \) such that \( IN(u) > IN(v) \), we have \( IN(u+v) = IN(u) \). Therefore

\[
IN((\nu([1^{m0^{q+1}}_q])) = IN(\sum_{n=1}^{n_1} d_{j+1,m,n+q^j}) \geq q^j + 2.
\]

This completes the proof. \[ \square \]
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