Sublinear-Time Algorithms for Computing & Embedding Gap Edit Distance

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Abstract

In this paper, we design new sublinear-time algorithms for solving the gap edit distance problem and for embedding edit distance to Hamming distance. For the gap edit distance problem, we give a greedy algorithm that distinguishes in time $\tilde{O}(\frac{n}{k} + k^2)$ between length-$n$ input strings with edit distance at most $k$ and those with edit distance more than $4k^2$. This is an improvement and a simplification upon the main result of [Goldenberg, Krauthgamer, Saha, FOCS 2019], where the $k$ vs $\Theta(k^2)$ gap edit distance problem is solved in $\tilde{O}(\frac{n}{k} + k^3)$ time. We further generalize our result to solve the $k$ vs $\alpha k$ gap edit distance problem in time $\tilde{O}(\frac{n}{\alpha} + k^2 + \frac{k}{\alpha} \sqrt{nk})$, strictly improving upon the previously known bound $\tilde{O}(\frac{n}{\alpha} + k^3)$. Finally, we show that if the input strings do not have long highly periodic substrings, then the gap edit distance problem can be solved in sublinear time within any factor $\alpha > 1$. Specifically, if the strings contain no substring of length $\ell$ with the shortest period of length at most $2k$, then the $k$ vs $(1 + \epsilon)k$ gap edit distance problem can be solved in time $\tilde{O}(\frac{n}{\epsilon^2} + k^2 \ell)$.

We further give the first sublinear-time algorithm for the probabilistic embedding of edit distance to Hamming distance. Our $\tilde{O}(\frac{n}{k})$-time procedure yields an embedding with distortion $k^2 p$, where $k$ is the edit distance of the original strings. Specifically, the Hamming distance of the resultant strings is between $\frac{k}{p}$ and $k^2$ with good probability. This generalizes the linear-time embedding of [Chakraborty, Goldenberg, Koucký, STOC 2016], where the resultant Hamming distance is between $k$ and $k^2$. Our algorithm is based on a random walk over samples, which we believe will find other applications in sublinear-time algorithms.

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1 Introduction

The edit distance, also known as the Levenshttein distance [Lev66], is a basic measure of sequence similarity. For two strings $X$ and $Y$ over an alphabet $\Sigma$, the edit distance $ED(X,Y)$ is defined as the minimum number of character insertions, deletions and substitutions required for converting $X$ into $Y$. A natural dynamic programming computes edit distance of two strings of total length $n$ in $O(n^2)$ time. While for many applications running a quadratic-time algorithm is prohibitive, the Strong Exponential Time Hypothesis (SETH) [IP01] implies that there is no truly subquadratic-time algorithm that computes edit distance exactly [BI18].

The last two decades have seen a surge of interest in designing fast approximation algorithms for edit distance computation [GKS19, BR20, KS20, CDG18, BEG+18, AO12, AKO10, BES06, BJKK04, BEK+03]. A breakthrough result of Chakraborty, Das, Goldenberg, Koucký, and Saks provided the first constant-factor approximation of edit distance in subquadratic time [CDG+18]. Nearly a decade earlier, Andoni, Krauthgamer, and Onak showed a polylogarithmic-factor approximation for edit distance in near-linear time [AKO10]. Recently, the result of [CDG+18] was improved to provide a constant-factor approximation in near-linear time, both by Brakensiek and Rubinstein [BR20], as well as by Koucký and Saks [KS20] for the regime of near-linear edit distance. Designing fast algorithms for edit distance has also been considered in other models, such as in the quantum and massively parallel framework [BEG+18], and when independent preprocessing of each string is allowed [GRS20].

In this paper, we focus on sublinear-time algorithms for edit distance, the study of which was initiated by Batu, Ergür, Kilian, Magen, Raskhodnikova, Rubinfeld, and Sami [BEK+03], and then continued in [AN10, AO12, NSS17, GKS19]. Here, the goal is to distinguish, in time sublinear in $n$, whether the edit distance is below $k$ or above $k'$ for some $k' > k$. This is known as the gap edit distance problem. In computational biology, before an in-depth comparison of new sequences is performed, a quick check to eliminate sequences that are not highly similar can save a significant amount of resources [DKF99]. In text corpora, a super-fast detection of plagiarism upon arrival of a new document can help to save both time and space. In these applications, we have relatively small $k$, the sequences are highly repetitive, and a sublinear-time algorithm with $k'$ relatively close to $k$ could be very useful.

Results on Gap Edit Distance. The algorithm of Batu et al. [BEK+03] distinguishes between $k = n^\eta$ and $k' = \Omega(n)$ in $O(n^{\max\{2\eta-1,\eta/2\}})$ time. However, their algorithm crucially depends on $k'$ being $\Omega(n)$, and cannot distinguish between, say, $n^{3/4}$ and $n^{0.99}$. A more recent algorithm by Andoni and Onak [AO12] resolves this issue and can distinguish between $k = n^\eta$ and $k' = n^{\beta}$, with $\beta > \eta$, in $O(n^{2+\eta-2\beta+o(1)})$ time. However, if we want to distinguish between $k$ vs $k^2$, then the algorithm of [AO12] achieves sublinear time only when $k = \omega(n^{1/3})$. (Setting $k' = k^2$ yields a natural test case for the gap edit distance problem since the best that one can currently distinguish in linear time is $k$ vs $k^2$.) In a recent work, Goldenberg et al. [GKS19] gave an algorithm solving the $k$ vs $k^2$ gap edit distance in $\tilde{O}(\frac{n}{k^3})$ time\footnote{The $\tilde{O}$ notation hides factors polylogarithmic in the input size and, in case of Monte Carlo randomized algorithms, in the inverse error probability.}, thereby providing a sublinear-time algorithm for the quadratic gap edit distance as long as $k = o(n^{1/3})$ and $k = \omega(1)$. Bar-Yossef, Jayram, Krauthgamer, and Kumar [BJKK04] introduced the term gap edit distance and solved the quadratic gap problem for non-repetitive strings. Their algorithm computes a constant-size sketch but still requires a linear-time pass over the data. This result was later generalized to arbitrary sequences [CGK16] via embedding edit distance into Hamming distance, but again in linear time. Nevertheless, already the algorithm of Landau and Vishkin [LV88] solves the edit distance problem exactly in $O(n + k^2)$ time, and thus also solves the quadratic gap edit distance problem in linear time. Given the prior works, Goldenberg et al. [GKS19] raised a question whether it is possible to solve the quadratic gap edit problem in sublinear time...
for all $k = \tilde{\omega}(1)$. In particular, the running times of the algorithms of Goldenberg et al. [GKS19] and Andoni and Onak [AO12] algorithms meet at $k = n^{1/3}$, when they become nearly-linear. In light of the $O(n + k^2)$-time exact algorithm [LV88], the presence of a $k^3$ term in the time complexity of [GKS19] is undesirable, and it is natural to ask if the dependency can be reduced. In particular, for $k = O(n^{1/3})$ if the polynomial dependency on $k$ can be reduced to $k^2$, then the contribution of that term can be neglected compared to $\frac{n}{k}$.

**Quadratic Gap Edit Distance.** We give a simple greedy algorithm solving the quadratic gap edit distance problem in $\tilde{O}(\frac{n^2}{k} + k^2)$ time. This resolves an open question posed in [GKS19] as to whether a sublinear-time algorithm for the quadratic gap edit distance exists for $k = n^{1/3}$. Our algorithm improves upon the main result of [GKS19] and simultaneously provides a conceptual simplification.

**k vs $\alpha k$ Gap Edit Distance.** Combining the greedy approach with the structure of computations in [LV88], we can solve the $k$ vs $\alpha k$ gap edit distance problem in $\tilde{O}(\frac{n^2}{\alpha k} + k^2 + \frac{k}{\alpha} \sqrt{kn})$ time. This improves upon the $\tilde{O}(\frac{n^2}{\alpha} + k^3)$ time bound of [GKS19] for all values of $\alpha$ and $k$.

**$k$ vs $(1 + \varepsilon)k$ Gap Edit Distance.** We can distinguish edit distance at most $k$ and at least $(1 + \varepsilon)k$ in $\tilde{O}(\frac{n^2}{(1 + \varepsilon)k} + \ell k^2)$ time as long as there is no length-$\ell$ substring with the shortest period of length at most $2k$. Previously, sublinear-time algorithms for distinguishing $k$ vs $(1 + \varepsilon)k$ were only known for the very special case of Ulam distance, where each character appears at most once in each string [AN10, NSS17]. Note that not only we can allow character repetition, we get an $(1+\varepsilon)$-approximation as long as the same periodic structure does not continue for more than $\ell$ consecutive positions, or the period length is large. This is the case with most text corpora, and for biological sequences with interspersed repeats.

**Embedding to Hamming Distance.** Along with designing fast approximation algorithms for edit distance, a parallel line of works have investigated how edit distance can be embedded into other metric spaces, especially to the Hamming space [ADG+03, BES06, OR07, CGK16, CGKK18]. Indeed such embedding results have led to new approximation algorithms for edit distance (e.g., the embedding of [OR07, BES06] applied in [AO12, BES06]), new streaming algorithms and document exchange protocols (e.g., the embedding of [CGK16] applied in [CGK16, BZ16]). In particular, Chakraborty, Goldenberg, and Koucký [CGK16] provided a probabilistic embedding of edit distance to Hamming distance with quadratic distortion. Their algorithm runs in linear time, and if the original edit distance between two given sequences were $k$, then the Hamming distance between the resultant sequences is bounded between $k$ and $k^2$.

The embedding is based on performing an interesting one-dimensional random walk which had also been used previously to design fast approximation algorithms for a more general language edit distance problem [Sah14]. So far, we are not aware of any sublinear-time metric embedding algorithm from edit distance to Hamming distance. In this paper, we design one such algorithm.

**Random Walk over Samples** We show that it is possible to perform a random walk similar to [Sah14, CGK16] over a suitably crafted sequence of samples. This leads to the first sublinear-time algorithm for embedding edit distance to Hamming distance: Given any parameter $p \geq 1$, our embedding algorithm processes any length-$n$ string in $\tilde{O}(\frac{n}{p})$ time, and guarantees that (with good probability) the Hamming distance of the resultant strings is between $\frac{k}{p}$ and $k^2$, where $k$ is the edit distance of the input strings. That is, we maintain the same expansion rate as [CGK16] and allow additional contraction by a factor at most $p$. Just like the algorithm of [CGK16] has been very influential (see its applications in [BZ16, BGZ17, ZZ17, Hae19]), we believe the technique of random walk over samples will also find other usages in designing sublinear-time and streaming algorithms.
The classic exact algorithm by Landau and Vishkin [LV88] for testing if $\text{ED}(X, Y) \leq k$ runs in $O(n + k^2)$ time, where $n = |X| + |Y|$. Since any character $X[x]$ can only be matched to $Y[y]$ with $y \in [x - k \ldots x + k]$, it is sufficient to identify for each diagonal $j \in [-k \ldots k]$ and each distance $i \in [0 \ldots k]$, a value $d_{i,j}$ defined as the maximum index $x$ such that $\text{ED}(X[0 \ldots x], Y[0 \ldots x+j]) \leq i$. After a linear-time preprocessing, these values can be computed in $O(1)$ time each using a dynamic-programming approach. This results in $O(n + k^2)$ running time.

Our algorithm for the quadratic gap edit distance problem follows the basic framework of [LV88]. However, instead of computing each value $d_{i,j}$, it only computes values $d_i$ for $i = [0 \ldots k]$, interpreted as relaxed version of $\max_{j=-k}^k d_{i,j}$ allowing for an $O(k)$-factor underestimation of the number of edits $i$. Testing whether the value $d_k$ satisfies $d_k < |X|$ is sufficient to distinguish $k$ vs $k^2$. In addition to uniform sampling at a rate of $O(\frac{1}{k+1})$, identifying each of these $O(k)$ values $d_i$ requires reading $O(k)$ additional characters. This yields $\hat{O}(\frac{n}{k} + k^2)$ total running time. Our algorithm not only improves upon the main result of Goldenberg et al. [GKS19], but also provides a conceptual simplification. Indeed, the algorithm of [GKS19] also follows the basic ideas of [LV88], but it identifies each value $d_{i,j}$ (relaxed in a similar way), paying an extra $\Theta(k)$ time per $\Theta(k^2)$ values. In order to do so, the algorithm follows a more complex row-by-row approach of an online version of the Landau–Vishkin algorithm [LMS98].

To improve the gap to $k$ vs $\alpha k$ for $\alpha < k$, greedily computing the values $d_i$ for $i \in [0 \ldots k]$ is not sufficient. (It is enough, though, for $\alpha \geq k$, where the time complexity becomes $\hat{O}(\frac{n}{k} + k^2)$ if we simply decrease the sampling rate to $\hat{O}(\frac{1}{\alpha+1})$.) Each shift between diagonals $j \in [-k \ldots k]$, which happens for each $i \in [1 \ldots k]$ as we determine $d_i$ based on $d_{i-1}$, involves up to $2k$ insertions or deletions, whereas to distinguish $k$ vs $\alpha k$ for $\alpha < k$, we would like to approximate the number of edits within a factor $\alpha$. In order to do so, we decompose the entire set of $2k+1$ diagonals into $O(\frac{n}{k})$ groups each consisting of consecutive $\alpha$ diagonals. Within each of these wide diagonals, we compute the (relaxed) maxima of $d_{i,j}$ following our greedy algorithm. This approximates the true maxima with $\alpha$ approximation ratio on the number of edits. Computing each of the $O(k)$ values for each wide diagonal requires reading $O(\alpha)$ extra characters. Therefore, for each wide diagonal, the running time is $\hat{O}(\frac{n}{\alpha} + \alpha k)$, and across the $O(\frac{n}{k})$ wide diagonals, the total time complexity becomes $\hat{O}(\frac{n}{\alpha} + k^2)$. This bound is incomparable to $\hat{O}(\frac{n}{\alpha} + k^2)$ of [GKS19]: While we pay less on the second term, the uniform sampling rate increases. In order to decrease the first term, instead of computing over each wide diagonal independently, we provide a synchronization mechanism so that overall, the global uniform sampling rate remains $\hat{O}(\frac{1}{\alpha})$. This leads to an implementation with running time $\hat{O}(\frac{n}{\alpha} + \frac{\sqrt{kn}}{\alpha})$, which already improves upon [GKS19]. However, synchronizing only over smaller groups of wide diagonals, we can further improve the running time, achieving $\hat{O}(\frac{n}{\alpha} + k^2 + \frac{\sqrt{kn}}{\alpha}, \sqrt{kn})$, which subsumes both $\hat{O}(\frac{n}{\alpha} + k^2)$ and $\hat{O}(\frac{n}{\alpha} + \frac{\sqrt{kn}}{\alpha})$.

Our algorithm distinguishing $k$ vs $(1+\varepsilon)k$ for strings without length-$\ell$ substrings with the shortest period of length at most $2k$ follows a very different approach, inspired by the existing solutions for estimating the Ulam distance [AN10, NSS17]. This method consists of three ingredients. First, we decompose $X = X_0 \ldots X_m$ and $Y = Y_0 \ldots Y_m$ into phrases of length $O(\ell k)$ such that if $\text{ED}(X, Y) \leq k$, then $\sum_{i=0}^m \text{ED}(X_i, Y_i) \leq k$ holds with good probability (note that $\text{ED}(X, Y) \geq \text{ED}(X_i, Y_i)$ is always true). The second ingredient estimates $\text{ED}(X_i, Y_i)$ for any given $i$. This subroutine is then applied for a random sample of indices $i$ by the third ingredient, which distinguishes between $\sum_{i=0}^m \text{ED}(X_i, Y_i) \leq k$ and $\sum_{i=0}^m \text{ED}(X_i, Y_i) > (1+\varepsilon)k$ relying on the Chernoff bound. The assumption that $X$ does not contain long periodic substrings is needed only in the first step: it lets us uniquely determine the beginning of the phrase $Y_i$, assuming that the initial $\ell$ positions of the phrase $X_i$ are aligned without mismatches in the optimal edit distance alignment (which is true with good probability for a random decomposition of $X$). We did not optimize the $\hat{O}(\ell k^2)$ term in our running time $\hat{O}(\frac{n}{\alpha} + \ell k^2)$. This helps to

\footnote{For $\ell, r \in \mathbb{Z}$, we denote $[\ell \ldots r] = \{j \in \mathbb{Z} : \ell \leq j < r\}$ and $[\ell \ldots r] = \{j \in \mathbb{Z} : \ell \leq j \leq r\}$.
keep our implementation of the other two ingredients much simpler than their counterparts in [AN10, NSS17].

A simple random deletion process, introduced in [Sah14], can solve the $k$ vs $k^2$ gap edit distance problem in linear time. The algorithm starts with $X[0]$ and $Y[0]$. If the two symbols match, they are aligned, and the algorithm proceeds to $X[1]$ and $Y[1]$. Otherwise, one of the symbols is deleted uniformly at random (the algorithm proceeds to $X[0]$ and $Y[1]$ or to $X[1]$ and $Y[0]$). This process, continued over all positions, can be interpreted as a one-dimensional random walk, and the hitting time of the random walk provides the necessary upper bound on the edit distance. In order to conduct a similar process in sublinear time, we sample positions in $s_0 < s_1 < \cdots$ in $[0 .. n]$ uniformly with probability $\tilde{O}(\frac{1}{\sqrt{n}})$. We now start with $X[s_0 + \delta_x]$ and $Y[s_0 + \delta_y]$, where $\delta_x$ and $\delta_y$ are both initialized to 0. If $X[s_i + \delta_x]$ matches with $Y[s_i + \delta_y]$, then we proceed to $X[s_{i+1} + \delta_x]$ and $Y[s_{i+1} + \delta_y]$. Otherwise, we delete one of $X[s_i]$ and $Y[s_i]$, which is interpreted as incrementing $\delta_x$ or $\delta_y$, and we still proceed to $X[s_{i+1} + \delta_x]$. The process is continued over all the sampled points. We show that performing this random walk over samples is sufficient to distinguish $k$ vs $k^2p$.

In the above, the positions of $X$ are sampled non-adaptively, while the positions of $Y$ are sampled adaptively based on $X$. In order to compute an embedding, we sample each $X$ and $Y$ adaptively but separately via shared randomness. Each sequence has information about the $k$ that one can solve the $k$ vs $k^2$ gap edit distance problem by running the standard Landau–Vishkin algorithm [LV88] suitably over the sampled sequences, $S$.

Organization. After introducing the main notations in Section 2, we provide the algorithm and analysis of the quadratic gap edit distance problem in Section 3. In Section 4, we elaborate on our results on $k$ vs $ak$ gap edit distance problem for $\alpha < k$. The $(1 + \varepsilon)$ approximation to gap edit distance problem without the presence of long periodic strings is given in Section 5. The procedure of random walk over samples is described in Section 6. Finally, the embedding result is provided in Section 7.

Further Remarks. We have recently been aware of an independent work that uses a greedy algorithm similar to ours for the sublinear gap edit distance problem and achieves a running time of $\tilde{O}(\frac{n}{\sqrt{k}})$ to distinguish $k$ vs $\Theta(k^2)$ [BCR20]. This is in contrast to our bound of $\tilde{O}(\frac{n}{k} + k^2)$, which is superior for $k \leq n^{2.5}$. At the same time, for $k \geq n^{2+\alpha(1)}$, the algorithm of Andoni and Onak [AO12] has a better running time. We also remark here that there exists an even simpler algorithm (by now folklore) that has query complexity $\tilde{O}(\frac{n}{\sqrt{k}})$. The algorithm samples both sequences $X$ and $Y$ independently with probability $\tilde{O}(\frac{1}{\sqrt{k}})$ so that $\text{Prob}(X[x] \text{ and } Y[x+d] \text{ are sampled }) = \tilde{O}(\frac{1}{k})$ for all $x \in [1 .. n]$ and $d \in [-k .. k]$. Then by running the standard Landau–Vishkin algorithm [LV88] suitably over the sampled sequences, one can solve the $k$ vs $k^2$ gap edit distance problem. Nevertheless, this is significantly worse to the bounds achieved here. It remains open to characterize a tight lower bound for the quadratic gap edit distance problem.

2 Preliminaries

A string $X$ is a finite sequence of characters from an alphabet $\Sigma$. The length of $X$ is denoted by $|X|$ and, for $i \in [0 .. |X|]$, the $i$th character of $X$ is denoted by $X[i]$. A string $Y$ is a substring
of a string $X$ if $Y = X[\ell]X[\ell + 1] \cdots X[r - 1]$ for some $0 \leq \ell \leq r \leq |X|$. We then say that $Y$ occurs in $X$ at position $\ell$. The set of positions where $Y$ occurs in $X$ is denoted $\text{Occ}(Y, X)$. The occurrence of $Y$ at position $\ell$ in $X$ is denoted by $X[\ell \ldots r]$ or $X[\ell \ldots r - 1]$. Such an occurrence is a fragment of $X$, and it can be represented by (a pointer to) $X$ and a pair of indices $\ell \leq r$. Two fragments (perhaps of different strings) match if they are occurrences of the same substring. A fragment $X[\ell \ldots r)$ is a prefix of $X$ if $\ell = 0$ and a suffix of $X$ if $r = |X|$.

A positive integer $p$ is a period of a string $X$ if $X[i] = X[i + p]$ holds for each $i \in [0 \ldots |X| - p]$. We define $\text{per}(X)$ to be the smallest period of $X$. The following result relates periods to occurrences:

**Fact 2.1** (Breslauer and Galil [BG95, Lemma 3.2]). If strings $P, T$ satisfy $|T| \leq \frac{3}{2}|P|$, then $\text{Occ}(P, T)$ forms an arithmetic progression with difference per($P$).

**Hamming distance and edit distance.** The Hamming distance between two strings $X, Y$ of the same length is defined as the number of mismatches. Formally, $\text{HD}(X, Y) = |\{i \in [0 \ldots |X|] : X[i] \neq Y[i]\}|$. The edit distance between two strings $X$ and $Y$ is denoted $\text{ED}(X, Y)$.

**LCE queries.** Let $X, Y$ be strings and let $k$ be a non-negative integer. For $x \in [0 \ldots |X|]$ and $y \in [0 \ldots |Y|]$, we define $\text{LCE}^{X,Y}_k(x, y)$ as the largest integer $\ell$ such that $\text{HD}(X[x \ldots x + \ell], Y[y \ldots y + \ell]) \leq k$ (in particular, $\ell \leq \min(|X| - x, |Y| - y)$ so that $X[x \ldots x + \ell]$ and $Y[y \ldots y + \ell]$ are well-defined). We also set $\text{LCE}^{X,Y}_k(x, y) = 0$ if $x \notin [0 \ldots |X|]$ or $y \notin [0 \ldots |Y|]$.

Our algorithms rely on two notions of approximate LCE queries. The first variant is sufficient for distinguishing between $\text{ED}(X, Y) \leq k$ and $\text{ED}(X, Y) > k(3k + 5)$ in $\tilde{O}(\frac{n}{k^2} + k^2)$ time, while a more general algorithm distinguishing between $\text{ED}(X, Y) \leq k$ and $\text{ED}(X, Y) > ak$ is based on the more subtle second variant.

**Definition 2.2.** Let $X, Y$ be strings and let $k \geq 0$ be an integer parameter. For integers $x, y$, we define $\text{LCE}^{X,Y}_{\leq k}(x, y)$ as any value satisfying $\text{LCE}^{X,Y}_k(x, y) \leq \text{LCE}^{X,Y}_{\leq k}(x, y) \leq \text{LCE}^{X,Y}_k(x, y)$.

**Definition 2.3.** Let $X, Y$ be strings and let $r > 0$ be a real parameter. For integers $x, y$, we define $\text{LCE}^{X,Y}_r(x, y)$ as any random variable satisfying the following conditions:

- $\text{LCE}^{X,Y}_r(x, y) \geq \text{LCE}^{X,Y}_0(x, y)$,
- $\mathbb{P}[\text{LCE}^{X,Y}_r(x, y) > \text{LCE}^{X,Y}_k(x, y)] \leq \exp(-\frac{k^2}{2r^2})$ for each integer $k \geq 0$.

Note that $\text{LCE}^{X,Y}_r(x, y)$ for $r = \frac{k + 1}{k^2}$ satisfies the conditions on $\text{LCE}^{X,Y}_{\leq k}(x, y)$ with probability $1 - \frac{1}{k}$. Thus, $\text{LCE}^{X,Y}_r$ queries with sufficiently small $r = \tilde{O}(k + 1)$ yield $\text{LCE}^{X,Y}_{\leq k}$ queries with high probability.

## 3 Improved Gap Edit Distance

The classic exact algorithm by Landau and Vishkin [LV88] for testing if $\text{ED}(X, Y) \leq k$ is given below as Algorithm 1. The key property of this algorithm is that $d^e_{i,j} = \max\{x : \text{ED}(X[0 \ldots x], Y[0 \ldots x + j]) \leq i\}$ holds for each $i \in [0..k]$ and $j \in [-k..k]$. Since $\text{LCE}^{X,Y}_0$ queries can be answered in $O(1)$ time after linear-time preprocessing, the running time is $O(|X| + k^2)$.

The main idea behind the algorithm of Goldenberg et al. [GKS19] is that if $\text{LCE}_0$ queries are replaced with $\text{LCE}_{\leq k}$ queries, then the algorithm is still guaranteed to return YES if $\text{ED}(X, Y) \leq k$ and NO if $\text{ED}(X, Y) > k(k + 2)$. The cost of their algorithm is $O\left(\frac{1}{k^2}|X|\right) + \tilde{O}(k)$ per $\text{LCE}_{\leq k}$ query, which yields $\tilde{O}\left(\frac{1}{k^2} |X| + k^3\right)$ in total. Nevertheless, their implementation is tailored to the specific structure of LCE queries in Algorithm 1, and it requires these queries to be asked and answered in a certain order, which makes them use an online variant of the Landau–Vishkin algorithm [LMS98].
By definition of LCE, every

\[ d \]

Since

\[ (3 \]

We proceed by induction on

\[ f \]

further edits (insertions or deletions) to change the

\[ X \]

We characterize the values

\[ d \]

Algorithm 1: The algorithm of Landau and Vishkin [LV88]

1. foreach \( i \in [0 \ldots k] \) and \( j \in [-k-1 \ldots k+1] \) do \( d_{i,j} := d'_{i,j} := -\infty \);
2. \( d_{0,0} := 0; \)
3. for \( i := 0 \) to \( k \) do
   4. for \( j := -k \) to \( k \) do
      5. if \( d_{i,j} \neq -\infty \) then
         6. \( d'_{i,j} := d_{i,j} + LCE_{0}^{X,Y}(d_{i,j}, d_{i,j} + j); \)
   7. for \( j := -k \) to \( k \) do
      8. \( d_{i+1,j} := \min(|X|, \max(d'_{i,j-1}, d'_{i,j} + 1, d'_{i,j+1} + 1)); \)
5. if \( ||X| - |Y|| \leq k \) and \( d'_{k,|Y| - |X|} = |X| \) then return YES;
6. else return NO;

One of the auxiliary results of this paper is that \( LCE_{\leq k}^{X,Y}(x, y) \) queries with \( |x - y| \leq k \) can be answered in \( \tilde{O}(k) \) time after \( \tilde{O}(\frac{k+1}{k+1}|X|) \) preprocessing which immediately yields a cleaner implementation of the algorithm of [GKS19]. In fact, we show that \( \tilde{O}(k) \) time is sufficient to answer all queries \( LCE_{\leq k}^{X,Y}(x, y) \) with a given index \( x \) and arbitrary index \( y \in [x-k \ldots x+k] \).

Unfortunately, this does not give a direct speed-up, because the values \( d_{i,j} \) in Algorithm 1 might be different. However, given that relaxing \( LCE_{0} \) queries to \( LCE_{\leq k} \) queries yields a cost of up to \( k \) edits (mismatches) for every \( LCE_{\leq k}^{X,Y}(x, y) \) query, the algorithm may as well pay \( O(k) \) further edits (insertions or deletions) to change the shift \( j = y - x \) arbitrarily. As a result, we do not need to consider each shift \( j \) separately. This results in a much simpler Algorithm 2.

Algorithm 2: Simple algorithm

1. \( d_{0} := 0; \)
2. for \( i := 0 \) to \( k \) do
   3. \( d'_{i} := d_{i} + \max_{k-i}^{k} LCE_{\leq k}^{X,Y}(d_{i}, d_{i} + \delta); \)
   4. \( d_{i+1} := \min(|X|, d'_{i} + 1); \)
5. if \( ||X| - |Y|| \leq k \) and \( d'_{k} = |X| \) then return YES;
6. else return NO;

Lemma 3.1. Algorithm 2 returns YES if \( ED(X, Y) \leq k \) and NO if \( ED(X, Y) > (3k + 5)k \).

Proof. We characterize the values \( d_{i} \) and \( d'_{i} \) using two claims with inductive proofs.

Claim 3.2. Each \( i \in [0 \ldots k] \) satisfies the following two properties:

(a) \( ED(X[0 \ldots d'_{i}], Y[0 \ldots y]) \leq (3k + 1)i + k \) for every \( y \in [d_{i} - k \ldots d_{i} + k] \cap [0 \ldots |Y|]; \)
(b) \( ED(X[0 \ldots d'_{i}], Y[0 \ldots y]) \leq (3k + 1)i + 4k \) for every \( y \in [d'_{i} - k \ldots d'_{i} + k] \cap [0 \ldots |Y|]. \)

Proof. We proceed by induction on \( i \). Our base case is Property (a) for \( i = 0 \). Since \( d_{0} = 0 \), for every \( y \in [d_{0} - k \ldots d_{0} + k] \cap [0 \ldots |Y|], \) we have \( ED(X[0 \ldots d_{0}], Y[0 \ldots y]) = y \leq k. \)

Next, we shall prove Property (b) for \( i \geq 0 \) assuming that Property (a) is true for \( i \).

By definition of \( LCE_{\leq k} \) queries, we have \( d'_{i} \leq d_{i} + LCE_{\leq k}^{X,Y}(d_{i}, y') \) for some position \( y' \in [d_{i} - k \ldots d_{i} + k] \cap [0 \ldots |Y|], \) and thus \( HD(X[d_{i} \ldots d'_{i}], Y[y' \ldots y' + d'_{i} - d_{i}]) \leq k. \) The assumption yields \( ED(X[0 \ldots d_{i}], Y[0 \ldots y']) \leq (3k + 1)i + k, \) so we have \( ED(X[0 \ldots d'_{i}], Y[0 \ldots y' + d'_{i} - d_{i}]) \leq (3k + 1)i + 2k. \) Due to \( |y' + d'_{i} - d_{i} - y| \leq 2k, \) we conclude that \( ED(X[0 \ldots d'_{i}], Y[0 \ldots y]) \leq (3k + 1)i + 4k. \)

Finally, we shall prove Property (a) for \( i > 0 \) assuming that Property (b) is true for \( i - 1. \) Since \( d_{i} \leq d'_{i-1} + 1, \) the assumption yields \( ED(X[0 \ldots d_{i} - 1], Y[0 \ldots y - 1]) \leq (3k + 1)(i - 1) + 4k, \) and therefore \( ED(X[0 \ldots d_{i}], Y[0 \ldots y]) \leq 1 + (3k + 1)(i - 1) + 4k = (3k + 1)i + k. \)
In particular, if the algorithm returns YES, then $ED(X,Y) \leq (3k + 5)k$.

**Claim 3.3.** If $ED(X|0\ldots x), Y|0\ldots y) = i$ for $x \in [0\ldots |X|], y \in [0\ldots |Y|]$, and $i \in [0\ldots k]$, then $x \leq d'_i$.

**Proof.** We proceed by induction on $i$. Both in the base case of $i = 0$ and the inductive step of $i > 0$, we shall prove that $x \leq d_i + \max_{k=0}^{k-1} LCE_{0}^{X,Y}(d_i, d_i + \delta)$. Since $d'_i \geq d_i + \max_{k=0}^{k-1} LCE_{0}^{X,Y}(d_i, d_i + j)$ holds by definition of $LCE_{\leq k}$ queries, this implies the claim.

In the base case of $i = 0$, we have $X|0\ldots x) = Y|0\ldots y$ and $d_0 = 0$. Consequently, $x \leq LCE_{0}^{X,Y}(0,0) \leq d_0 + \max_{k=0}^{k-1} LCE_{0}^{X,Y}(d_0, d_0 + \delta)$.

For $i > 0$, we consider an optimal alignment between $X|0\ldots x)$ and $Y|0\ldots y)$, and we distinguish its maximum prefix with $i - 1$ errors. This yields positions $x', x'' \in [0\ldots x]$ and $y', y'' \in [0\ldots y]$ with $x'' - x' \in \{0,1\}$ and $y'' - y' \in \{0,1\}$ such that $ED(X|0\ldots x'), Y|0\ldots y')) = i - 1$ and $X|x''\ldots x) = \widetilde{Y}|y''\ldots y)$. The inductive assumption yields $x' \leq d'_{i-1}$, which implies $x'' \leq \min(x, d'_{i-1} + 1) \leq d_i$. Due to $X|x''\ldots x) = \widetilde{Y}|y''\ldots y)$, we have $LCE_{0}^{X,Y}(x'', y'') \geq x - x''$.

By $x'' \leq d_i$, this implies $LCE_{0}^{X,Y}(d_i, d_i + y - x) \geq x - d_i$. Since $|y - x| \leq k$, we conclude $x = d_i + (x - d_i) \leq d_i + LCE_{0}^{X,Y}(d_i, d_i + y - x) \leq d_i + \max_{k=0}^{k-1} LCE_{0}^{X,Y}(d_i, d_i + \delta)$.

In particular, if $ED(X,Y) \leq k$, then the algorithm returns YES.

A data structure computing $LCE_{\leq k}^{X,Y}(x,y)$ for a given $x$ and all $y \in [x - k\ldots x + k]$ is complicated, but Algorithm 2 only requires a simpler stated below and proved in Section 3.1.

**Proposition 3.4.** Given strings $X$ and $Y$, an integer $k \geq 0$, an index $i$, and a range of indices $J$, a value $\ell := \max_{j \in J} LCE_{\leq k}^{X,Y}(i, j)$ can be computed with high probability in $\widetilde{O}(\frac{|X|}{k+1} + |J|)$ time.

**Theorem 3.5.** There is an algorithm that, given strings $X$ and $Y$ and an integer $k \geq 0$, returns:

- YES if $ED(X,Y) \leq k$,
- NO if $ED(X,Y) > (3k + 5)k$.

With high probability, the algorithm is correct and its running time is $\widetilde{O}(\frac{1}{k+1}|X| + k^2)$.

**Proof.** The algorithm is given in Algorithm 2. Queries $LCE_{\leq k}$ are implemented using Proposition 3.4. With high probability, all the queries are answered correctly. Conditioned on this assumption, Lemma 3.1 yields that Algorithm 2 is correct with high probability. It remains to analyze the running time. The cost of instructions other than $LCE_{\leq k}$ queries is $O(k)$. By Proposition 3.4, the cost of computing $d'_i$ is $O(\frac{1}{k+1}(d'_i - d_i) + k)$. Since $d'_i \leq d_{i+1}$ and $d'_i \leq |X|$, this sums up to $\widetilde{O}(\frac{1}{k+1}|X| + k^2)$ across all queries.

**3.1 Proof of Proposition 3.4**

Our implementation of $LCE_{\leq k}$ queries in Proposition 3.4 relies on two auxiliary subroutines. The first one either certifies that a given string is approximately periodic or finds a witness fragment proving that the string is not exactly periodic.

**Lemma 3.6.** There is an algorithm that, given a string $X$ and integers $q, k \geq 0$ with $|X| \geq 2q$, returns either:

- a length-$2q$ fragment $Y$ of $X$ such that $per(Y) > q$, or
- $\perp$, certifying that $p := per(X|0\ldots 2q)) \leq q$ and $|\{i \in [0\ldots |X|) : X[i] \neq X[i \mod p]\}| \leq k$.

With high probability, the algorithm is correct and its running time is $\widetilde{O}(\frac{1}{k+1}|X| + q)$.

**Proof.** A procedure $\text{FindBreak}(X, q, k)$ implementing Lemma 3.6 is given as Algorithm 3.

First, the algorithm computes the shortest period $p = per(X|0\ldots 2q))$. If $p > q$, then the algorithm returns $Y := X|0\ldots 2q)$, which is correct due to $per(Y) = p$. 

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Otherwise, the algorithm tries to check if $\perp$ can be returned. If we say that a position $i \in [0..|X|]$ is compatible when $X[i] = X[i \mod p]$, then $\perp$ can be returned provided that there are at most $k$ incompatible positions. The algorithm samples a subset $S \subseteq [0..|X|]$ with a sufficiently large rate $\tilde{O}\left(\frac{1}{k+1}\right)$. Such sampling rate guarantees that if there are at least $k + 1$ incompatible positions, then with high probability at least one of them belongs to $S$. Consequently, the algorithm checks whether all positions $i \in S$ are compatible (Line 5), and, if so, returns $\perp$ (Line 13); this answer is correct with high probability.

In the remaining case, the algorithm constructs a fragment $Y$ based on an incompatible position $i \in S$ (Lines 6–12). The algorithm performs a binary search maintaining positions $b, e$ with $2q \leq b \leq e < |X|$ such that $e$ is incompatible and positions in $[b-2q..b]$ are all compatible. The initial choice of $b := 2q$ and $e := i$ satisfies the invariant because positions in $[0..2q]$ are all compatible due to $p = \per(X[0..2q])$. While $b < e$, the algorithm chooses $m := \lceil \frac{b+e}{2} \rceil$. If $[m-2q..m]$ contains an incompatible position $j$, then $j \geq b$ (because $j \geq m-2q \geq b-2q$ and positions in $[b-2q..b]$ are all compatible), so the algorithm maintains the invariant setting $e := j$ for such a position $j$ (Line 10). Otherwise, all the positions in $[m-2q..m]$ are compatible. Due to $m \leq e$, this means that the algorithm maintains the invariant setting $b := m$ (Line 11). Since $e-b$ decreases at least twice in each iteration, after $\tilde{O}(\log |X|)$ iterations, the algorithm obtains $b = e$. Then, the algorithm returns $Y := X[b-2q+1..b]$.

We shall prove that this result is correct. For a proof by contradiction, suppose that $p' := \per(Y) \leq q$. Then, $p'$ is also period of $X[b-2q+1..b]$. Moreover, the invariant guarantees that positions in $[b-2q+1..b]$ are all compatible, so also $p$ is a period of $X[b-2q+1..b]$. Since $p+p'-1 \leq 2q-1$, the periodicity lemma [FW65] implies that also gcd($p, p'$) is a period of $X[b-2q+1..b]$. Consequently, $X[b] = X[b-p'] = X[b-p] = X[(b-p) \mod p] = X[b \mod p]$, i.e., $b$ is compatible. However, the invariant assures that $b$ is incompatible. This contradiction proves that $\per(Y) > q$. Since $|Y| = 2q$, the fragment $Y$ forms a correct result.

It remains to analyze the running time. Determining $\per(X[0..2q])$ in Line 1 costs $O(q)$ time using a classic algorithm [MP70]. The number of sampled positions is $|S| = \tilde{O}\left(\frac{1}{k+1}|X|\right)$ with high probability, so the test in Line 5 costs $\tilde{O}\left(\frac{1}{k+1}|X|\right)$ time in total. Binary search (the loop in Line 7) has $O(\log |X|) = \tilde{O}(1)$ iterations, each implemented in $O(q)$ time. Consequently, the total running time is $\tilde{O}\left(\frac{1}{k+1}|X| + q\right)$.

The second auxiliary procedure can be seen as a restricted decision version of Proposition 3.4. Namely, it allows deciding whether $\max_{j \in J} \text{LCP}_{\leq k}^{p,T}(0,j) = |P|$. The strings are dubbed $P$ and $T$ because this task is intimately related to finding certain exact or approximate occurrences of $P$ in $T$.

\textbf{Algorithm 3: FindBreak($X, q, k$)}

\begin{verbatim}
1 p := \per(X[0..2q]);
2 if p > q then return X[0..2q];
3 Let S \subseteq [0..|X|) with each element sampled independently with probability \tilde{O}\left(\frac{1}{k+1}\right);
4 foreach i \in S do
5   if X[i] \neq X[i \mod p] then
6     b := 2q; e := i;
7     while b < e do
8       m := \lceil \frac{b+e}{2} \rceil;
9       for j := m-2q to m-1 do
10       if X[j] \neq X[j \mod p] then e := j;
11       if e \geq m then b := m;
12     return X[b-2q+1..b]
13 return \perp
\end{verbatim}

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Lemma 3.7. There is an algorithm that, given non-empty strings \( P \) and \( T \), an integer range \( J \), and an integer \( k \), returns YES if \( \max_{j \in J} \text{LCE}^{P,T}_0(0,j) = |P| \), and NO if \( \max_{j \in J} \text{LCE}^{P,T}_k(0,j) < |P| \). With high probability, the algorithm is correct and its running time is \( \tilde{O}(\frac{1}{\epsilon} |P| + |J|) \).

Proof. A procedure \( \text{PM}(P, T, J, k) \) implementing Lemma 3.7 is given as Algorithm 4.

The algorithm starts with a few simple reductions in Lines 1–4. First, the algorithm restricts \( J \) to be a subset of \([0\ldots|T| - |P|]\). This is correct, because \( \text{LCE}^{P,T}_0(0,j) < |P| \) for \( j \notin [0\ldots|T| - |P|] \). Next, the algorithm trims the prefix \( T[0\ldots\min J] \) and the suffix \( T[\max J + |P|\ldots|T|] \), which cannot influence \( \text{LCE}^{P,T}_0(0,j) \) and \( \text{LCE}^{P,T}_k(0,j) \) for any \( j \in J \). Technically, the algorithm sets \( T \leftarrow T[\min J\ldots\max J + |P|] \), and (due to the shift of indices) the interval \( J \) implicitly becomes \([0\ldots\Delta] \), where \( \Delta \) is set to \(|T| - |P| \) in Line 4.

Now, the task of the algorithm can be expressed as a pattern matching problem: the algorithm must return YES if \( P \) has an exact occurrence in \( T \), and NO if \( P \) has no \( k \)-mismatch occurrence in \( T \), i.e., if \( T \) contains a fragment at Hamming distance at most \( k \) from \( P \).

If \( |P| < 3\Delta \), then a classic pattern matching algorithm is used to check if \( P \) has an exact occurrence in \( T \) (Line 6), and the answer is given based on that. If \( |P| \geq 3\Delta \), then a classic pattern matching algorithm is used to identify exact occurrences of the prefix \( P[0\ldots3\Delta] \) of \( P \) within a prefix \( T[0\ldots4\Delta] \) of \( T \). The algorithm returns NO if there are no such occurrences (Line 8), which is correct due to \( \text{Occ}(P,T) \subseteq \text{Occ}(P[0\ldots3\Delta],T[0\ldots4\Delta]) \). Otherwise, the algorithm retrieves the leftmost occurrence of \( P[0\ldots3\Delta] \) in \( T[0\ldots4\Delta] \) and trims the prefix of \( T \) preceding this occurrence (Line 9). Due to \( \text{Occ}(P,T) \subseteq \text{Occ}(P[0\ldots3\Delta],T[0\ldots4\Delta]) \), we do not lose any exact occurrence of \( P \) in \( T \). Since we do not gain any \( k \)-mismatch occurrence either,

Algorithm 4: \( \text{PM}(P, T, J, k) \)

1. \( J := J \cap [0\ldots|T| - |P|] \);
2. if \( J = \emptyset \) then return NO;
3. \( T := T[\min J\ldots\max J + |P|] \);
4. \( \Delta := |T| - |P| \);
5. if \( |P| < 3\Delta \) then
6. if \( \text{Occ}(P,T) \neq \emptyset \) then return YES;
7. else return NO;
8. if \( \text{Occ}(P[0\ldots3\Delta],T[0\ldots4\Delta]) = \emptyset \) then return NO;
9. \( T := T[\min \text{Occ}(P[0\ldots3\Delta],T[0\ldots4\Delta])\ldots|T|] \);
10. \( \Delta := |T| - |P| \);
11. \( Y_p := \text{FindBreak}(P, \Delta, \lfloor \frac{1}{2}\Delta \rfloor) \);
12. \( Y_T := \text{FindBreak}(T[0\ldots|P|], \Delta, \lfloor \frac{1}{2}\Delta \rfloor) \);
13. if \( Y_T = T[\ell\ldots r] \) with \( r \leq 3\Delta \) then \( Y_p := P[\ell\ldots r] \);
14. else \( Y_T \neq \bot \) then
15. \( P[\ell\ldots r] := Y_p \);
16. if \( \text{Occ}(Y_p, T[\ell\ldots r + \Delta]) = \emptyset \) then return NO;
17. \( \delta := \min \text{Occ}(Y_p, T[\ell\ldots r + \Delta]) \);
18. else \( Y_T \neq \bot \) then
19. \( T[\ell\ldots r] := Y_T \);
20. if \( \text{Occ}(Y_T, P[\ell - \Delta\ldots]) = \emptyset \) then return NO;
21. \( \delta := \Delta - \min \text{Occ}(Y_T, P[\ell - \Delta\ldots]) \);
22. else return YES;
23. for each \( i \in \mathcal{S} \) do
24. if \( P[i] \neq T[i + \delta] \) then return NO;
25. return YES;
this reduction is correct. The algorithm also updates \( \Delta = |T| - |P| \) in Line 10. Since this may only decrease \( \Delta \), at this point we are guaranteed that \( T[0..3\Delta] = P[0..3\Delta] \).

The algorithm applies the procedure \textbf{FindBreak} of Lemma 3.6 to \( P \) and \( T[0..|P|] \), both with \( q = \Delta \) and a decreased threshold \( \lfloor \frac{1}{2}k \rfloor \) instead of \( k \), yielding \( Y_P \) and \( Y_T \), respectively. These calls succeed with high probability, and the analysis below is conditioned on that event.

If \( Y_T = T[\ell..r] \) with \( r \leq 3\Delta \), then (for technical reasons) the algorithm replaces \( Y_P \) with \( P[\ell..r] \). This is a valid transformation because \( P[\ell..r] = T[\ell..r] \) is guaranteed to be a fragment of length \( 2\Delta \) with \( \text{per}(P[\ell..r]) > \Delta \). Then, the algorithm considers three cases depending on whether \( Y_P = \perp \) and \( Y_T = \perp \).

If \( Y_P = \perp \) and \( Y_T = \perp \), then the algorithm returns YES in Line 22. We shall prove that this is correct. Since \( P[0..2\Delta] = T[0..2\Delta] \), these two prefixes have a common shortest period \( p \leq \Delta \). Lemma 3.6 guarantees that \( P[i] \neq P[i \mod p] \) holds for at most \( \lfloor \frac{1}{2}k \rfloor \) positions \( i \in [0..|P|] \) and that \( T[i] \neq T[i \mod p] \) holds for at most \( \lfloor \frac{1}{2}k \rfloor \) positions \( i \in [0..|P|] \). Thus, \( P[i] = P[i \mod p] = T[i \mod p] = T[i] \) holds for the remaining at least \( |P| - 2\lfloor \frac{1}{2}k \rfloor \) positions \( i \in [0..|P|] \). In other words, \( \text{HD}(P,T[0..|P|]) \leq 2\lfloor \frac{1}{2}k \rfloor \leq k \), that is, \( P \) has a \( k \)-mismatch occurrence as a prefix of \( P \). Hence, the algorithm may indeed return YES.

In the remaining two cases, the algorithm uses \( Y_P \neq \perp \) or \( Y_T \neq \perp \) to derive \( \text{Occ}(P,T) = \emptyset \) or to find a single position \( \delta \) such that \( \text{Occ}(P,T) \subseteq \{ \delta \} \). First, suppose that \( Y_P = P[\ell..r] \neq \perp \).

Observe that if \( j \in \text{Occ}(P,T) \), then \( j + \ell \in \text{Occ}(Y_P,T) \) or, equivalently, \( j \in \text{Occ}(Y_T,T[\ell..r+\Delta]) \) (because \( \text{Occ}(P,T) \subseteq [0..\Delta] \)). Consequently, a classic pattern matching algorithm is used to identify exact occurrences of \( Y_T \) within \( T[\ell..r+\Delta] \). If there are no such occurrences, then \( \text{Occ}(P,T) = \emptyset \), and the algorithm correctly returns NO in Line 16. Otherwise, due to \( |Y_P| = 2\Delta \) and \( |T[\ell..r+\Delta]| = 3\Delta \), by Fact 2.1, the starting positions form an arithmetic progression with difference \( \text{per}(Y_P) > \Delta \). Since these starting positions belong to \([0..\Delta] \), this means that there is only one such position, and the algorithm retrieves it as \( \delta \) in Line 17. The discussion above shows that \( \text{Occ}(P,T) \subseteq \{ \delta \} \) indeed holds in this case.

Next, suppose that \( Y_P = \perp \) and \( Y_T = T[\ell..r] \neq \perp \). Due to the technical transformation in Line 13, we have \( \ell = r - 2\Delta > \Delta \). Thus, if \( j \in \text{Occ}(P,T) \), then \( \ell - j \in \text{Occ}(Y_T,T[\ell..r]) \) or, equivalently, \( \Delta - j \in \text{Occ}(Y_T,P[\ell - \Delta..r]) \) (because \( \text{Occ}(P,T) \subseteq [0..\Delta] \)). Consequently, a classic pattern matching algorithm is used to identify exact occurrences of \( Y_T \) within \( P[\ell - \Delta..r] \). If there are no such occurrences, then \( \text{Occ}(P,T) = \emptyset \), and the algorithm correctly returns NO in Line 20. Otherwise, as in the previous case, there is exactly one occurrence, and the algorithm retrieves its starting position \( \Delta - \delta \) in Line 21. The discussion above shows that \( \text{Occ}(P,T) \subseteq \{ \delta \} \) indeed holds in this case.

In the final step, the algorithm tries to distinguish between \( \text{HD}(P,T[\delta..\delta + |P|]) \) and \( \text{HD}(P,T[\delta..\delta + |P|]) > k \). For this, the algorithm samples a subset \( S \subseteq [0..|P|] \) with a sufficiently large rate \( \tilde{O}(\frac{1}{k+1}) \). Such sampling rate guarantees that if there are at least \( k + 1 \) mismatches between \( P \) and \( T[\delta..\delta + |P|] \), then with high probability the position of at least one of them belongs to \( S \). Consequently, the algorithm checks whether \( P[i] = T[i+\delta] \) holds for all positions \( i \in S \) (Line 25), and, if so, returns YES (Line 26); this answer is correct with high probability due to a \( k \)-mismatch occurrence of \( P \) at position \( \delta \) in \( T \).

On the other hand, if the algorithm finds a mismatch \( P[i] \neq T[i+\delta] \), then \( \delta \notin \text{Occ}(P,T) \). Due to \( \text{Occ}(P,T) \subseteq \{ \delta \} \), this means that \( P \) has no exact occurrences in \( T \). Consequently, the algorithm correctly returns NO (Line 25).

It remains to analyze the running time. Exact pattern matching in Lines 6, 8, 16 and 20 can be implemented in \( \mathcal{O}(\Delta) \) time using a classic algorithm [MP70]. The calls to \textbf{FindBreak} from Lemma 3.6 cost \( \tilde{O}(\frac{1}{k+2} |P| + \Delta) = \tilde{O}(\frac{1}{k+1} |P| + \Delta) \) time with high probability. Finally, the number of sampled positions is \( |S| = \tilde{O}(\frac{1}{k+1} |X|) \) with high probability, so the test in Line 25 costs \( \tilde{O}(\frac{1}{k+1} |X|) \) time in total. Hence, the total running time is \( \tilde{O}(\frac{1}{k+1} |P| + \Delta) \) with high probability. This running time can be expressed as \( \tilde{O}(\frac{1}{k+1} |P| + |J|) \) because the reductions in Lines 1 and 9 may only decrease \( |J| \) and \( \Delta \). \(\square\)
Lemma 4.1. For any integers $k \geq 0$ and $\alpha \geq 1$, Algorithm 5 returns YES if $\text{ED}(X,Y) \leq k$ and NO if $\text{ED}(X,Y) > k + 3(k+1)(\alpha - 1)$.

Proof. As in the proof of Lemma 3.1, we characterize the values $d_{i,j}$ and $d'_{i,j}$ using two claims.

Claim 4.2. Each $i \in [0..k]$ and $j \in [\lfloor \frac{i}{\alpha} \rfloor \ldots \lfloor \frac{i}{\alpha} \rfloor]$ satisfies the following two properties:

Algorithm 5: Improved algorithm

```
1 foreach i ∈ [0..k] and j ∈ [\lfloor \frac{i}{\alpha} \rfloor − 1..\lfloor \frac{i}{\alpha} \rfloor + 1] do d_{i,j} := d'_{i,j} := −∞;
2 d_{0,0} := 0;
3 for i := 0 to k do
4 for j := [\lfloor \frac{i}{\alpha} \rfloor] to [\lfloor \frac{i}{\alpha} \rfloor] do
5 if $d_{i,j} \neq −∞$ then
6 \quad $d'_{i,j} := d_{i,j} + \max_{\delta=1}^{\alpha-1} \text{LCE}_{\alpha-1}^{X,Y}(j+1,\delta)$;
7 for j := [\lfloor \frac{i}{\alpha} \rfloor] to [\lfloor \frac{i}{\alpha} \rfloor] do
8 \quad $d_{i+1,j} := \min(||X||,\max(d'_{i,j}+1,1))$;
9 j := \lfloor\frac{1}{\alpha}||Y||−||X||\rfloor;
10 if $||X||−||Y|| ≤ k$ and $d'_{k,j} = ||X||$ then return YES;
11 else return NO;
```

Finally, we derive Proposition 3.4 as a pretty simple reduction to Lemma 3.7.

Proposition 3.4. Given strings $X$ and $Y$, an integer $k \geq 0$, an index $i$, and a range of indices $J$, a value $\ell := \max_{j \in J} \text{LCE}_{\leq k}^{X,Y}(i,j)$ can be computed with high probability in $O(\frac{n}{k+1} + |J|)$ time.

Proof. It suffices to give an oracle returning YES if $\max_{j \in J} \text{LCE}_{0}^{X,Y}(i,j) ≥ \ell$ and NO if $\max_{j \in J} \text{LCE}_{k}^{X,Y}(i,j) < \ell$, to be used in an exponential search for the resulting value.

If $||X|| + ||X|| < \ell$, then the answer is clearly NO. Otherwise, the oracle forwards the answer of the procedure $\text{PM}$ of Lemma 3.7 with $P := X[i..i+\ell]$, $T := Y$, and the original values of $J$ and $k$. If $\text{LCE}_{0}^{X,Y}(i,j) ≥ \ell$, then $\text{LCE}_{0}^{P,T}(0,j) = |P|$, so $\text{PM}$ returns YES. Conversely, if $\text{PM}$ returns YES, then $\text{LCE}_{k}^{P,T}(0,j) = |P|$ for some $j \in J$, and consequently, $\text{LCE}_{k}^{X,Y}(i,j) ≥ \ell$. Thus, conditioned on the success of $\text{PM}$ (which happens with high probability), the reduction is correct. Since $|P| = \ell$, the running time is $O(|J| + \frac{n}{k+1} + \ell)$ with high probability. 

4 Improved Approximation Ratio

Goldenberg et al. [GKS19] generalize their algorithm in order to solve the $k$ vs $\alpha k$ gap edit distance problem in $O(\frac{n}{\alpha} + k^3)$ time for any $\alpha \geq 1$. This transformation is quite simple, because Algorithm 1 (the Landau–Vishkin algorithm) with LCE$_0$ queries replaced by LCE$_{\leq \alpha-1}$ queries returns YES if ED$(X,Y) \leq k$ and NO if ED$(X,Y) > k + (\alpha-1)(k+1)$.

However, if we replace LCE$_k$ queries with LCE$_{\leq \alpha-1}$ queries in Algorithm 2, then we are guaranteed to get a NO answer only if ED$(X,Y) > 2k(k+2) + (\alpha-1)(k+1)$. As a result, with an appropriate adaptation of Proposition 3.4, Algorithm 2 yields an $O(\frac{n}{\alpha} + k^3)$-time solution to the $k$ vs $\alpha k$ gap edit distance problem only for $\alpha = \Omega(k)$. The issue is that Algorithm 2 yields a cost of up to $\Theta(k)$ edits for up to $\Theta(k)$ arbitrary changes of the shift $y-x$ within queries LCE$_{\leq \alpha-1}^{X,Y}(x,y)$. On the other hand, in Algorithm 1 no such changes are performed, but this results in LCE$_{\leq \alpha-1}^{X,Y}(x,y)$ queries asked for up to $\Theta(k^2)$ distinct positions $x$, which is the reason behind the $O(k^3)$ term in the running time $O(\frac{n}{\alpha} + k^3)$ of [GKS19].

Nevertheless, since each LCE$_{\leq \alpha-1}^{X,Y}(x,y)$ query yields a cost of up to $\alpha-1$ edits (mismatches) it is still fine to pay $O(\alpha-1)$ further edits (insertions or deletions) to change the shift $y-x$ by up to $\alpha-1$. Hence, we design Algorithm 5 as a hybrid of Algorithms 1 and 2.

Lemma 4.1. For any integers $k \geq 0$ and $\alpha \geq 1$, Algorithm 5 returns YES if $\text{ED}(X,Y) \leq k$ and NO if $\text{ED}(X,Y) > k + 3(k+1)(\alpha - 1)$.

Proof. As in the proof of Lemma 3.1, we characterize the values $d_{i,j}$ and $d'_{i,j}$ using two claims.
(a) \( \text{ED}(X[0 \ldots d_{i,j}], Y[0 \ldots y]) \leq i + (3i + 1)(\alpha - 1) \) for every \( y \in [d_{i,j} + j\alpha \ldots d_{i,j} + (j + 1)\alpha) \cap [0 \ldots |Y|] \); 
(b) \( \text{ED}(X[0 \ldots d'_{i,j}], Y[0 \ldots y]) \leq i + 3(i + 1)(\alpha - 1) \) for every \( y \in [d'_{i,j} + j\alpha \ldots d'_{i,j} + (j + 1)\alpha) \cap [0 \ldots |Y|] \).

**Proof.** We proceed by induction on \( i \). Our base case is Property (a) for \( i = 0 \). Due to \( d_{i,j} = -\infty \) for \( j \neq 0 \), the only non-trivial subcase is \( j = 0 \). Since \( d_{0,0} = 0 \), for every \( y \in [d_{0,0} \ldots d_{0,0} + \alpha) \cap [0 \ldots |Y|] \), we have \( \text{ED}(X[0 \ldots d_0], Y[0 \ldots y]) = y \leq \alpha - 1 \).

Next, we shall prove Property (b) for \( i \geq 0 \) assuming that Property (a) is true for \( i \). By definition of \( \text{LCE}_{\leq \alpha - 1} \) queries, we have \( d'_{i,j} \leq d_{i,j} + \text{LCE}_{\alpha - 1}(d_{i,j}, y') \) for some position \( y' \in [d_{i,j} + j\alpha \ldots d_{i,j} + (j + 1)\alpha) \cap [0 \ldots |Y|] \), and thus \( \text{HD}(X[d_{i,j} \ldots d'_{i,j}], Y[y' \ldots y' + d'_{i,j} - d_{i,j}]) \leq \alpha - 1 \). The assumption yields \( \text{ED}(X[0 \ldots d_{i,j}], Y[0 \ldots y']) \leq i + (3i + 1)(\alpha - 1) \), so we have \( \text{ED}(X[0 \ldots d'_{i,j}], Y[0 \ldots y' + d'_{i,j} - d_{i,j}]) \leq i + 3i(\alpha - 1) \). Due to \( |y' + d'_{i,j} - d_{i,j} - y| \leq \alpha - 1 \), we conclude that \( \text{ED}(X[0 \ldots d'_{i,j}], Y[0 \ldots y]) \leq i + 3(i + 1)(\alpha - 1) \).

Finally, we shall prove Property (a) for \( i > 0 \) assuming that Property (b) is true for \( i - 1 \). We consider three subcases: If \( d_{i,j} \leq d'_{i-1,j-1} \), then the assumption yields \( \text{ED}(X[0 \ldots d_{i,j}], Y[0 \ldots y - \alpha]) \leq (i - 1) + 3i(\alpha - 1) \), and therefore \( \text{ED}(X[0 \ldots d_{i,j}], Y[0 \ldots y]) \leq i + (i - 1) + 3i(\alpha - 1) = i + (3i + 1)(\alpha - 1) \). If \( d_{i,j} \leq d'_{i-1,j} + 1 \), then the assumption yields \( \text{ED}(X[0 \ldots d_{i,j}], Y[0 \ldots y - \alpha]) \leq (i - 1) + 3i(\alpha - 1) \), and therefore \( \text{ED}(X[0 \ldots d_{i,j}], Y[0 \ldots y]) \leq i + (i - 1) + 3i(\alpha - 1) = i + 3i(\alpha - 1) \). If \( d_{i,j} \leq d'_{i-1,j} + 1 \), then the assumption yields \( \text{ED}(X[0 \ldots d_{i,j}], Y[0 \ldots y]) \leq i + (i - 1) + 3i(\alpha - 1) = i + 3i(\alpha - 1) \).

In particular, if the algorithm returns YES, then \( \text{ED}(X, Y) \leq k + 3(k + 1)(\alpha - 1) \).

**Claim 4.3.** If \( \text{ED}(X[0 \ldots x], Y[0 \ldots y]) = i \) for \( x \in [0 \ldots |X|] \), \( y \in [0 \ldots |Y|] \), and \( i \in [0 \ldots k] \), then \( x \leq d'_{i,j} \) holds for \( j = \lfloor \frac{i}{\alpha} \rfloor \).

**Proof.** We proceed by induction on \( i \). Both in the base case of \( i = 0 \) and in the inductive step of \( i > 0 \), we prove that \( x \leq d_{i,j} + \max_{\delta = j\alpha}(d_{i,j} + \delta) \cdot \text{LCE}_{\alpha - 1} \cdot \text{LCE}_{\alpha - 1} \). This implies the claim since \( d'_{i,j} \geq d_{i,j} + \max_{\delta = j\alpha}(d_{i,j} + \delta) \), holds by definition of \( \text{LCE}_{\leq \alpha - 1} \) queries.

For \( i > 0 \), we consider an optimal alignment between \( X[0 \ldots x] \) and \( Y[0 \ldots y] \), and we distinguish its maximum prefix with \( i - 1 \) errors. This yields positions \( x', x'' \in [0 \ldots x] \) and \( y', y'' \in [0 \ldots y] \) with \( x'' - x' \in \{0, 1\} \) and \( y'' - y' \in \{0, 1\} \) such that \( \text{ED}(X[0 \ldots x'], Y[0 \ldots y']) = i - 1 \) and \( X[x'' \ldots x] = Y[y'' \ldots y] \). The inductive assumption yields \( x'' \leq d''_{i-1,j'} = \lfloor \frac{1}{\alpha}(y'' - x') \rfloor \) satisfies \( j - j' \leq 1 \). We shall prove that \( x'' \leq d_{i,j} \) by considering two possibilities. If \( j' \geq j \), then \( x'' \leq \min(x', x'' + 1) \leq \min(x', d'_{i-1,j} + 1) \leq d_{i,j} \). If \( j' < j \), on the other hand, then \( y' - x' < y'' - x'' \) implies \( x'' = x' \leq d''_{i-1,j} = d'_{i-1,j} - 1 \leq d_{i,j} \). Due to \( X[x'' \ldots x] = Y[y'' \ldots y] \), we have \( \text{LCE}_{\alpha - 1} \geq x'' - x'' \). By \( x'' \leq d_{i,j} \), this implies \( \text{LCE}_{\alpha - 1} \geq x'' - d_{i,j} \). By definition of \( j' \), we conclude \( x = d_{i,j} + (x - d_{i,j}) \leq d_{i,j} + \text{LCE}_{\alpha - 1} \).

In particular, if \( \text{ED}(X, Y) \leq k \), then the algorithm returns YES.

If we use Proposition 3.4 to implement \( \text{LCE}_{\leq \alpha - 1} \) queries in Algorithm 5, then the cost of computing \( d'_{i,j} \) is \( O(\alpha + \frac{1}{\alpha}(d_{i,j} - d_{i,j})) \), with high probability. This query is performed only \( d_{i,j} \geq 0 \), and it results in \( d'_{i,j} \leq |X| \). Since \( d'_{i,j} \leq d_{i+1,j} \), the total query time for fixed \( j \) sums up to \( \tilde{O}(\frac{1}{\alpha}|X| + k\alpha) \) across all queries. Across all \( O(\frac{1}{\alpha}|X| + k\alpha) \) steps, this gives \( \tilde{O}(\frac{1}{\alpha}|X| + k^2) \) time with high probability, which is not comparable to \( \tilde{O}(\frac{1}{\alpha}|X| + k^3) \) of [GKS19], unfortunately.
However, we can obtain a faster algorithm using the following data structure described in Section 4.1. In particular, if we set $\Delta = [-k .. k]$, then $\text{LCE}_{x_k}^{X,Y}(x, y)$ queries with $|x - y| \leq k$ can be answered in $\tilde{O}(k^4)$ time after $\tilde{O}(\frac{k}{k+1} |X|)$ preprocessing, as promised in Section 3.

**Proposition 4.4.** There is a data structure that, initialized with strings $X$ and $Y$, an integer $k \geq 0$, and an integer range $\Delta$, answers the following queries: given an integer $x$, return $\text{LCE}_{x_k}^{X,Y}(x, x + \delta)$ for all $\delta \in \Delta$. The initialization costs $\tilde{O}(\frac{1}{k+1} |X|)$ time with high probability, and the queries cost $\tilde{O}(|\Delta|)$ time with high probability.

Since the $\text{LCE}_{x_k}^{X,Y}(x, y)$ queries in Algorithm 5 are asked for $O(\frac{k^2}{\alpha})$ positions $x$ and for positions $y$ satisfying $|y - x| = O(k)$, a straightforward application of Proposition 4.4 yields an $O(\frac{|X|}{\alpha} + \frac{k^2}{\alpha^2})$-time implementation of Algorithm 5, which is already better the running time of [GKS19]. However, the running time of a more subtle solution described below subsumes both $O(\frac{k}{\alpha} |X| + \frac{k^2}{\alpha^2})$ and $\tilde{O}(\frac{k}{\alpha^2} |X| + k^2)$ (obtained using Proposition 3.4).

**Theorem 4.5.** There is an algorithm that, given strings $X$ and $Y$ and an integer $k \geq 0$, and a positive integer $\alpha = O(k)$, returns:

- **YES** if $\text{ED}(X, Y) \leq k$,
- **NO** if $\text{ED}(X, Y) > k + 3(k + 1)(\alpha - 1)$.

With high probability, the algorithm is correct and its running time is $\tilde{O}(\frac{1}{\alpha} |X| + k^2 + \frac{k^2}{\alpha} \sqrt{|X| k})$.

**Proof.** We define an integer parameter $b$, $1 \leq b \leq \lceil \frac{k}{\alpha} \rceil$ (to be fixed later) and initialize $O(\frac{k}{\alpha})$ instances of the data structure of Proposition 4.4 for answering $\text{LCE}_{x_k}^{X,Y}$ queries. The instances are indexed with $j' \in [\lceil \frac{k}{\alpha} \rceil, \lfloor \frac{k}{\alpha} \rfloor]$, and the $j'$th instance has interval $\Delta_{j'} = [j' \alpha b .. (j' + 1)\alpha b)$. This way, the value $d'_{i,j}$ can be retrieved from the values $\text{LCE}_{\lfloor \frac{k}{\alpha} \rfloor}^{X,Y}(d_{i,j}, d_{i,j} + \delta)$ for $\delta \in \Delta_{\lfloor \frac{k}{\alpha} \rfloor}$, that is, from a single query to an appropriate instance of the data structure of Proposition 4.4.

Correctness follows from Lemma 4.1 since with high probability all $\text{LCE}_{x_k}^{X,Y}$ queries are answered correctly. The total preprocessing cost is $\tilde{O}(\frac{k}{\alpha} \cdot \frac{1}{\alpha} |X|) = \tilde{O}(\frac{k}{\alpha^2} |X|)$ with high probability, and each value $d'_{i,j}$ is computed in $\tilde{O}(\alpha b)$ time with high probability. The number of such queries is $O(\frac{k^2}{\alpha^2})$, so the total running time is $\tilde{O}(\frac{k}{\alpha^2} |X| + k^2 b)$ with high probability. Optimizing for $b$ yields $\tilde{O}(\frac{k}{\alpha^2} \sqrt{|X| k})$. Due to $1 \leq b \leq \lceil \frac{k}{\alpha} \rceil$, we get additional terms $\tilde{O}(k^2)$ and $\tilde{O}(\frac{1}{\alpha} |X|)$. □

### 4.1 Proof of Proposition 4.4

While there are many similarities between the proofs of Propositions 3.4 and 4.4, the main difference is that in the proof of Proposition 4.4, we heavily rely on $\text{LCE}_r$ queries. The following fact illustrates their main advantage compared to $\text{LCE}_{x_k}$ queries: *composability.*

**Fact 4.6.** Let $P, P', T$ be strings, let $r > 0$ be real parameter, and let $j \in [0 .. |T| - |P|]$. Suppose that $\overline{\text{LCE}}^{P,T}_r(0, j)$ and $\text{LCE}^{P,T}_r(0, j + |P|)$ are independent random variables, and define

$$
\ell := \begin{cases} 
\text{LCE}^{P,T}_r(0, j) & \text{if } \text{LCE}^{P,T}_r(0, j) < |P|, \\
|P| + \text{LCE}^{P,T}_r(0, j + |P|) & \text{otherwise}.
\end{cases}
$$

Then $\ell$ satisfies the conditions for $\overline{\text{LCE}}^{P',T}_r(0, j)$.

**Proof.** Define $d = \text{HD}(P, T[j .. j + |P|])$ and note that the following equality holds for each $k \geq 0$:

$$
\text{LCE}_{k}^{P',T}(0, j) = \begin{cases} 
\text{LCE}_{k}^{P,T}(0, j) & \text{for } k < d, \\
|P| + \text{LCE}_{k-d}^{P,T}(0, j + |P|) & \text{for } k \geq d.
\end{cases}
$$

Let us first prove that $\ell \geq \text{LCE}_{d}^{P',T}(0, j)$. If $\text{LCE}^{P,T}_r(0, j) < |P|$, then $\overline{\text{LCE}}^{P,T}_r(0, j) \geq \text{LCE}^{P,T}_r(0, j)$ implies $d > 0$, and therefore $\ell = \text{LCE}^{P,T}_r(0, j) \geq \text{LCE}^{P,T}_r(0, j) = \text{LCE}^{P',T}_r(0, j)$. 

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Otherwise, \( \ell = |P| + \text{LCE}^{P',T}_r(0, j + |P|) \geq |P| + \text{LCE}^{P',T}_r(0, j + |P|) \geq \text{LCE}^{P',T}_0(0, j) \). Hence, the claim holds in either case.

Next, let us bound the probability \( \mathbb{P}[\ell > \text{LCE}^{P',T}_k(0, j)] \) for some \( k \geq 0 \). We consider two cases. If \( k < d \), then \( \text{LCE}^{P',T}_k(0, j) < |P| \) and

\[
\mathbb{P}[\ell > \text{LCE}^{P',T}_k(0, j)] 
\leq \mathbb{P}[\text{LCE}^{P',T}_r(0, j) > \text{LCE}^{P',T}_k(0, j)] 
= \mathbb{P}[\text{LCE}^{P',T}_r(0, j) > \text{LCE}^{P',T}_k(0, j)] 
\leq \exp(-\frac{k+1}{r}).
\]

On the other hand, if \( k \geq d \), then \( \text{LCE}^{P',T}_k(0, j) \geq |P| > \text{LCE}^{P,T}_{d-1}(0, j) \). Hence, the independence of \( \text{LCE}^{P',T}_r(0, j) \) and \( \text{LCE}^{P,T}_{d-1}(0, j + |P|) \) yields

\[
\mathbb{P}[\ell > \text{LCE}^{P',T}_k(0, j)] 
\leq \mathbb{P}[\text{LCE}^{P',T}_r(0, j) > |P| \text{ and } \text{LCE}^{P',T}_r(0, j + |P|) > \text{LCE}^{P',T}_{d-1}(0, j + |P|)]
\leq \mathbb{P}[\text{LCE}^{P',T}_r(0, j) > |P|] \cdot \mathbb{P}[\text{LCE}^{P',T}_r(0, j + |P|) > \text{LCE}^{P',T}_{d-1}(0, j + |P|)]
\leq \exp(-\frac{d}{r}) \cdot \exp(-\frac{k-d+1}{r})
= \exp(-\frac{k+1}{r}).
\]

This completes the proof.

Next, we show that a single value \( \text{LCE}^{P,T}_r(0, j) \) can be computed efficiently. We also require that the resulting value \( \ell \) satisfies the required conditions.

**Fact 4.7.** There is an algorithm that given strings \( P \) and \( T \), a real parameter \( r > 0 \), and an integer \( j \), returns a value \( \ell = \text{LCE}^{P,T}_r(0, j) \) such that \( P[\ell] \neq T[j + \ell] \) or \( \ell = \min(|P|, |T| - j) \). The algorithm takes \( \tilde{O}(\frac{1}{r}|P|) \) time with high probability.

**Proof.** If \( r \leq 1 \), then the algorithm returns \( \ell = \text{LCE}^{P,T}_0(0, j) \) computed naively in \( O(|P|) \) time.

It is easy to see that this value satisfies the required conditions.

If \( r > 1 \), then the algorithm samples a subset \( S \subseteq [0 \ldots \min(|P|, |T| - j)] \) so that the events \( i \in S \) are independent with \( \mathbb{P}[i \in S] = \frac{1}{r} \). If \( P[i] = T[i + j] \) for each \( i \in S \), then the algorithm returns \( \ell = \min(|P|, |T| - j) \). Otherwise, the algorithm returns \( \ell = \min\{i \in S : P[i] \neq T[i + j]\} \). This way, \( \ell \geq \text{LCE}^{P,T}_r(0, j) \), and \( P[\ell] \neq T[j + \ell] \) or \( \ell = \min(|P|, |T| - j) \) with high probability, the total running time is \( \tilde{O}(\frac{1}{r}|P|) \) with high probability.

We are now ready to describe a counterpart of Lemma 3.6.

**Lemma 4.8.** There is an algorithm that, given a string \( X \), a real parameter \( r > 0 \), and a positive integer \( q \leq \frac{1}{r}|X| \) such that \( p := \text{per}(X[0\ldots, 2q]) \leq q \), returns \( \ell \in [2q \ldots |X|] \) such that

\begin{itemize}
  \item \( \ell = \text{LCE}^{X',X'}_r(0, 0) \), where \( X' \) is an infinite string with \( X'[i] = X[i \mod p] \) for \( i \geq 0 \), and
  \item \( \ell = |X| \) or \( \text{per}(X[\ell - 2q \ldots \ell]) > q \).
\end{itemize}

The algorithm takes \( \tilde{O}(\frac{1}{r}|X| + q) \) time with high probability.

**Proof.** A procedure \( \text{FindBreak2}(X, r, q) \) implementing Lemma 4.8 is given as Algorithm 6.

First, the algorithm computes the shortest period \( p := \text{per}(X[0\ldots, 2q]) \) (guaranteed to be at most \( q \) by the assumption) and constructs an infinite string \( X' \) with \( X'[i] = X[i \mod p] \) for each \( i \geq 0 \); note that random access to \( X' \) can be easily implemented on top of random access to \( X \).
Next, the algorithm computes $\ell' := \text{LCE}_{r, X'}^T(0, 0)$ using Fact 4.7. If $\ell' = |X|$, then $|X|$ satisfies both requirements for the resulting value $\ell$, so the algorithm returns $\ell := |X|$ (Line 1).

Otherwise, the algorithm tries to find a position $\ell \leq \ell'$ such that $\text{per}(X(\ell - 2q \ldots \ell)) > q$ (Lines 5–11). This step is implemented as in the proof of Lemma 3.6. We call a position $i \in [0 \ldots |X|)$ compatible if $X[i] = X'[i]$. The algorithm performs a binary search maintaining positions $b, e$ with $2q \leq b \leq e < |X|$ such that $e$ is incompatible and positions in $[b - 2q \ldots b]$ are all compatible. The initial choice of $b := 2q$ and $e := \ell$ satisfies the invariant because positions in $[b \ldots 2q]$ are all compatible due to $p = \text{per}(X[0 \ldots 2q])$. While $b < e$, the algorithm chooses $m := \lceil \frac{b + e}{2} \rceil$. If $[m - 2q \ldots m)$ contains an incompatible position $j$, then $j \geq b$ (because $j \geq m - 2q \geq b - 2q$ and positions in $[b - 2q \ldots b]$ are all compatible), so the algorithm maintains the invariant setting $e := j$ for such a position $j$ (Line 9). Otherwise, all the positions in $[m - 2q \ldots m)$ are compatible. Due to $m \leq e$, this means that the algorithm maintains the invariant setting $b := m$ (Line 10). Since $e - b$ decreases at least twice in each iteration, after $O(\log |X|)$ iterations, the algorithm obtains $b = e$. Then, the algorithm returns $\ell := b$.

We shall prove that this result is correct. For a proof by contradiction, suppose that $p' := \text{per}(X(b - 2q \ldots b)) \leq q$. Then, $p'$ is also period of $X(b - 2q \ldots b)$. Moreover, the invariant guarantees that positions in $[b - 2q \ldots b]$ are all compatible, so also $p$ is a period of $X(b - 2q \ldots b)$. Since $p + p' - 1 \leq 2q - 1$, the periodicity lemma [FW65] implies that also gcd($p, p'$) is a period of $X[b - 2q + 1 \ldots b]$. Consequently, $X[b] = X[b - p'] = X[b - p] = X'[b - p] = X'[b]$, i.e., $b$ is compatible. However, the invariant assures that $b$ is incompatible. This contradiction proves that $\text{per}(X(b - 2q \ldots b)) > q$. The incompatibility of $b$ guarantees that $\text{LCE}_{0, X'}^T(0, 0) \leq b$. Moreover, since $b \leq \ell'$, we have $P[b > \text{LCE}_{0, X'}^T(0, 0)] \leq P[\ell' > \text{LCE}_{0, X'}^T(0, 0)] \leq \exp(-\frac{b - e}{k})$ for each $k \geq 0$. Thus, $b$ satisfies the requirements for $\text{LCE}_{0, X'}^T(0, 0)$.

It remains to analyze the running time. Determining $\text{per}(X[0 \ldots 2q])$ in Line 1 costs $O(q)$ time using a classic algorithm [MP70]. The application of Fact 4.7 costs $O(\lceil |X| \rceil)$ time with high probability. Binary search (the loop in Line 6) has $O(\log |X|) = O(1)$ iterations, each implemented in $O(q)$ time. Consequently, the total running time is $O(\frac{1}{r}|X| + q)$ with high probability.

Next, we develop a counterpart of Lemma 3.7.

**Lemma 4.9.** There is an algorithm that given strings $P$ and $T$, a real parameter $r > 0$, and a range of positions $J$, returns $\text{LCE}_{P, T}^r(0, j)$ for each $j \in J$. The algorithm costs $\tilde{O}(\frac{1}{r}|P| + |J|)$ time with high probability.

**Proof.** A procedure $\text{PM}(P, T, r, J)$ implementing Lemma 4.9 is given as Algorithm 7.

First, the algorithm sets $\Delta := \max J - \min J$ and computes $\min(\text{LCE}_{0, T}^r(0, j), 2\Delta)$ for each $j \in J$. Implementation details of this step are discussed later on. Then, the algorithm sets

---

**Algorithm 6: FindBreak2($X, r, q$)**

1. $p := \text{per}(X[0 \ldots 2q]);$
2. Define $X'[0 \ldots \infty)$ with $X'[i] = X[i \mod p];$
3. $\ell' := \text{LCE}_{X, X'}^T(0, 0);$ \hspace{1cm} \triangleright computed using Fact 4.7
4. if $\ell' = |X|$ then return $|X|;$
5. $b := 2q; e := \ell';$
6. while $b < e$ do
7. \hspace{1cm} $m := \lceil \frac{b + e}{2} \rceil;$
8. \hspace{1cm} for $j := m - 2q$ to $m - 1$ do
9. \hspace{2cm} if $X[j] \neq X'[j]$ then $e := j;$
10. \hspace{1cm} if $e \geq m$ then $b := m;$
11. return $b$
Claim 4.10. For each \( j \in J' \), the value \( \ell_j = \min(\ell^P, \ell^T - j + \min J') \) set in Line 11 satisfies
\[
P[\ell_j > \text{LCE}_k^{P,T}(0,j)] \leq \exp(-\frac{2\Delta}{k})\] for every integer \( k \geq 0 \).

Proof. Note that \( \ell_j \leq \min(|P|, |T| - j) \) due to \( \ell^P \leq |P| \) and \( \ell^T \leq |T| \). Consequently, if \( \text{LCE}_k^{P,T}(0,j) = \min(|P|, |T| - j) \), then the claim holds trivially. In the following, we assume that \( d := \text{LCE}_k^{P,T}(0,j) < \min(|P|, |T| - j) \) so that \( P[0..d] \) and \( T[j..j+d] \) are well-defined fragments with \( \text{HD}(P[0..d], T[j..j+d]) = k+1 \).

Consider the infinite string \( P' \) with \( P'[i] = P[i \mod p] \) for each \( i \geq 0 \), and define \( k_1 = \text{HD}(P[0..d], P'[0..d]) \) as well as \( k_2 = \text{HD}(T[j..j+d], T'[0..d]) \), observing that the triangle inequality yields \( k_1 + k_2 \geq k + 1 \). Due to \( \ell^P = \text{LCE}_k^{P,T}(0,0) \) (by Lemma 4.8), we have
\[
P[\ell^P > d] \leq \exp(-\frac{2\Delta}{k_1}),\] because \( d \geq \text{LCE}_k^{P,T}(0,0) \).

Next, consider the infinite string \( T' \) with \( T'[i] = T[i \mod p] \) for \( i \geq 0 \) and observe that \( T' = P' \) due to \( T[0..2\Delta] = P[0..2\Delta] \). Consequently, \( k_2 = \text{HD}(T[j - \min J'..j - \min J' + d], T'[0..d]) \). Since \( J' \) forms an arithmetic progression with difference \( p \) and \( j \in J' \), we further have \( k_2 = \text{HD}(T[j - \min J'..j - \min J' + d], T'[j - \min J'..j - \min J' + d]) \).

Algorithm 7: \( \text{PM2}(P, T, r, J) \)

1. \( \Delta := \max J - \min J; \)
2. foreach \( j \in J \) do \( \ell_j := \min(\text{LCE}_0^{P,T}(0,j), 2\Delta); \)
3. \( J' := \{ j \in J : \ell_j = 2\Delta \}; \)
4. if \( |J'| \leq 1 \) then
5.   foreach \( j \in J' \) do \( \ell_j := \text{LCE}_0^{P,T}(0,j); \) \( \triangleright \text{computed using Fact 4.7} \)
6. else
7.   \( T := T[\min J'..\min(\max J' + |P|, |T|)]; \)
8.   \( \ell^P := \text{FindBreak2}(P, r, 2\Delta); \)
9.   \( \ell^T := \text{FindBreak2}(T, r, 2\Delta); \)
10. foreach \( j \in J' \) do
11.    \( \ell_j := \min(\ell^P, \ell^T - j + \min J'); \)
12.    if \( \ell_j < \min(|P|, |T| - j) \) and \( P[\ell_j - 2\Delta..\ell_j] = T[\ell_j - 2\Delta..\ell_j] \)
13.    then \( \ell_j := \text{LCE}_0^{P,T}(0,j); \) \( \triangleright \text{computed using Fact 4.7} \)
14. return \( (\ell_j)_{j \in J} \).
per($\mathcal{T}[0..2\Delta]$) yields $\mathcal{T}[0..j - \min J'] = \mathcal{T}'[0..j - \min J']$, and therefore $k_0 = \text{HD} (\mathcal{T}[0..j - \min J' + d], \mathcal{T}'[0..j - \min J' + d])$. We conclude that $j - \min J + d \geq \LCE_{k_2-1} (0,0)$. Due to $\ell^T = \LCE_{k_2} (0,0)$ (by Lemma 4.8), we thus have $\mathbb{P}[\ell^T > j - \min J' + d] = \mathbb{P}[\ell^T > j - \min J' + d] \leq \exp(-\frac{2\Delta}{k_2})$.

Finally, since the calls to Lemma 4.8 use independent randomness (and thus $\ell^P$ and $\ell^T$ are independent random variables), we conclude that

$$\mathbb{P}[\ell_j > \LCE_{k_2} (0,j)] = \mathbb{P}[\ell_j > d] = \mathbb{P}[\ell^P > d]$$

$$= \mathbb{P}[\ell^P > d] \cdot \mathbb{P}[\ell^T - j + \min J' > d] \leq \exp(-\frac{k_2}{d}) \cdot \exp(-\frac{2\Delta}{k_2}) = \exp(-\frac{k_2+2\Delta}{d}) \leq \exp(-\frac{k_2+2\Delta}{\ell^P}) \leq \exp(-\frac{k+1}{\ell^T}),$$

which completes the proof.  \[\square\]

For each $j \in J'$, after setting $\ell_j$ in Line 11, the algorithm performs an additional check in Line 12; its implementation is discussed later on. If the check succeeds, then the algorithm falls back to computing $\ell_j = \LCE_{k_2} (0,j)$, using Fact 4.7, which by definition results in a correct value. Otherwise, $\ell_j \geq |P|$, $\ell_j \geq |T| - j$, or $P(\ell_j - 2\Delta \ldots \ell_j) \neq T(j + \ell_j - 2\Delta \ldots j + \ell_j)$. Each of these condition yields $\LCE_{k_2} (0,j) \leq \ell_j$. Hence, due to Claim 4.10, returning $\ell_j$ as $\LCE_{k_2} (0,j)$ is then correct. This completes the proof that the values returned by Algorithm 7 are correct.

However, we still need to describe the implementation of Line 2 and Line 12. The values $\min (\LCE_{k_2} (0,j), 2\Delta)$ needed in Line 2 are determined using an auxiliary string $X = P[0..2\Delta] \$ T[\min J .. \max J + 2\Delta]$, where $\$ and each out-of-bounds character does not match any other character. The PREF table of $X$, with $\text{PREF}_X [x] = \text{LCE}_X \left(X, x \right)$ for $x \in [0 \ldots |X|]$, can be constructed in $\mathcal{O}(|X|) = \mathcal{O}(|J|)$ time using a textbook algorithm [CR03] and satisfies $\min (\LCE_{k_2} (0,j), 2\Delta) = \text{PREF}_X [2\Delta + 1 + j - \min J]$ holds for each $j \in J$.

Our approach to testing $P(\ell_j - 2\Delta \ldots \ell_j) = T(j + \ell_j - 2\Delta \ldots j + \ell_j)$ in Line 12 depends on whether $\ell_j = \ell^P$ or not. Positions $j \in J'$ with $\ell_j = \ell^T$ need to be handled only if $\ell^P < |P|$. In this case, $P(\ell_j - 2\Delta \ldots \ell_j) = T(j + \ell_j - 2\Delta \ldots j + \ell_j)$ holds if and only if $P(\ell^P - 2\Delta \ldots \ell^P)$ has an occurrence in $T$ at position $j + \ell^P - 2\Delta + 1$. Hence, a linear-time pattern matching algorithm is used to find the occurrences of $P(\ell^P - 2\Delta \ldots \ell^P)$ starting in $T$ between positions $\min J' + \ell^P - 2\Delta + 1$ and $\max J' + \ell^P - 2\Delta + 1$, inclusive. Due to $\max J' - \min J' \leq \Delta$, this takes $\mathcal{O}(\Delta)$ time. Moreover, since $\text{per}(P(\ell^P - 2\Delta \ldots \ell^P)) > \Delta$ holds by Lemma 4.8, Fact 2.1 implies that there is at most one such occurrence, i.e., at most one position $j$ with $\ell_j = \ell^P$ passes the test in Line 12.

Next, consider positions $j \in J'$ with $\ell_j \neq \ell^P$. Since these positions satisfy $\ell_j = \ell^T - j + \min J'$, the condition $\ell_j < |T| - j$ implies $\ell^T - j + \min J' < |T|$. Moreover, the condition $\ell_j < |P|$ implies $\ell^T < |P| + j - \min J' \leq |P| + \max J' - \min J'$. Consequently, such $j \in J'$ need to be handled only if $\ell^T < \min(|T| - \min J', |P| + \max J' - \min J') = |T|$. Our observation is that $P(\ell_j - 2\Delta \ldots \ell_j) = T(j + \ell_j - 2\Delta \ldots j + \ell_j)$ if and only if $T(\ell^T - 2\Delta \ldots \ell^T)$ has an occurrence in $P$ at position $\ell^T - j + \min J' - 2\Delta + 1$. Hence, a linear-time pattern matching algorithm is used to find the occurrences of $T(\ell^T - 2\Delta \ldots \ell^T)$ starting in $T$ between positions $\ell^T - \max J' + \min J' - 2\Delta + 1$ and $\ell^T - 2\Delta + 1$, inclusive. Due to $\max J' - \min J' \leq \Delta$, this takes $\mathcal{O}(\Delta)$ time. Moreover, since $\text{per}(T(\ell^T - 2\Delta \ldots \ell^T)) > \Delta$ holds by Lemma 4.8, Fact 2.1 implies that there is at most one such occurrence, i.e., at most one position $j$ with $\ell_j \neq \ell^T$ passes the test in Line 12.

We conclude that Line 12 (across all $j \in J'$) can be implemented in $\mathcal{O}(\Delta)$ time and that Line 13 needs to be executed for at most two indices $j \in J'$. Consequently, the overall cost of executing Lines 11–13 is $\tilde{\mathcal{O}}(\frac{1}{\ell^T} |P| + \Delta)$ with high probability. Due to $|T| \leq |P| + \Delta$, the cost of calls to FindBreak2 of Lemma 4.8 is also $\tilde{\mathcal{O}}(\frac{1}{\ell^T} |P| + \Delta)$ with high probability. As explained above, executing Line 2 costs $\mathcal{O}(|J|)$ time. The cost of Line 5 is $\tilde{\mathcal{O}}(\frac{1}{\ell^P} |P|)$ with high probability. Due to $\Delta = |J| - 1$, the total running time is therefore $\tilde{\mathcal{O}}(\frac{1}{\ell^T} |P| + |J|)$ with high probability.  \[\square\]

We are now ready to prove a counterpart of Proposition 4.4 for $\LCE_r$ queries.
Lemma 4.11. There is a data structure that, initialized with strings $X$ and $Y$, a real parameter $r > 0$, and an integer range $\Delta$, answers the following queries: given an integer $x$, return $\text{LCE}_{r}^{X,Y}(x, x + \delta)$ for each $\delta \in \Delta$. The initialization costs $\tilde{O}(\frac{1}{r} |X|)$ time with high probability, and the queries cost $\tilde{O}(|\Delta|)$ time with high probability.

Algorithm 8: Implementation of the data structure of Lemma 4.11

```plaintext
Construction(X, Y, r, \Delta) begin

1. $q := \lceil r |J| \rceil$;
2. $x := |X|$;
3. foreach $\delta \in \Delta$ do $\ell_{x,\delta} := 0$;
4. while $x \geq q$ do
5. \hspace{1em} $x := x - q$;
6. \hspace{2em} $(v_{\delta})_{\delta \in \Delta} := \text{PM2}(X[.\ldots x + q], Y, r, \{x + \delta : \delta \in \Delta\})$;
7. \hspace{1em} foreach $\delta \in \Delta$ do
8. \hspace{2em} if $v_{\delta} < q$ then $\ell_{x,\delta} := v_{\delta}$;
9. \hspace{2em} else $\ell_{x,\delta} := q + \ell_{x+q,\delta}$;

Query(x) begin
10. if $x \not\in [0 \ldots |X|]$ then return $(0)_{\delta \in \Delta}$;
11. $x' := x + (|X| - x \mod q)$;
12. if $x' \not= x$ then
13. \hspace{1em} $(v_{\delta})_{\delta \in \Delta} := \text{PM2}(X[.\ldots x'], Y, r, \{x + \delta : \delta \in \Delta\})$;
14. \hspace{1em} foreach $\delta \in \Delta$ do
15. \hspace{2em} if $v_{\delta} < x' - x$ then $\ell_{x,\delta} := v_{\delta}$;
16. \hspace{2em} else $\ell_{x,\delta} := x' - x + \ell_{x',\delta}$;
17. return $(\ell_{x,\delta})_{\delta \in \Delta}$;
```

Proof. Procedures Construction(X, Y, r, \Delta) and Query(x) implementing Lemma 4.11 are given as Algorithm 8.

The construction algorithm precomputes the answers for all $x \in [0 \ldots |X|]$ satisfying $x \equiv |X|$ (mod $q$), where $q = \lceil r |\Delta| \rceil$. More formally, for each such $x$, the data structure stores $\ell_{x,\delta} = \text{LCE}_{r}^{X,Y}(x, x + \delta)$ for all $\delta \in \Delta$. First, the values $\ell_{x,\delta}$ are set to 0. In subsequent iterations, the algorithm computes $\ell_{x,\delta}$ based on $\ell_{x+q,\delta}$. For this, the procedure $\text{PM2}(X[.\ldots x + q], Y, r, \{x + \delta : \delta \in \Delta\})$ of Lemma 4.9 is called. The resulting values $\text{LCE}_{r}^{X,Y}(0, x + \delta)$ are then combined with $\ell_{x+q,\delta} = \text{LCE}_{r}^{X,Y}(0, x + q + \delta)$ based on Fact 4.6, which yields $\ell_{x,\delta}$.

The cost of a single iteration is $\tilde{O}(\frac{2}{r} + |\Delta|) = \tilde{O}(\frac{2}{r})$ with high probability, and the number of iterations is $\tilde{O}(\frac{1}{r} |X|)$, so the total preprocessing time is $\tilde{O}(\frac{1}{r} |X|)$ with high probability.

To answer a query for a given integer $x$, the algorithm needs to compute $\text{LCE}_{r}^{X,Y}(x, x + \delta)$ for all $\delta \in \Delta$. If $x \not\in [0 \ldots |X|]$, then these values are equal to 0 by definition. Otherwise, the algorithm computes the nearest integer $x' \geq x$ with $x' \equiv |X|$ (mod $q$). If $x' = x$, then the sought values have already been precomputed. Otherwise, the algorithm proceeds based the values $\ell_{x',\delta}$. For this, the procedure $\text{PM2}(X[.\ldots x'], Y, r, \{x + \delta : \delta \in \Delta\})$ of Lemma 4.9 is called. The resulting values $\text{LCE}_{r}^{X,Y}(0, x + \delta)$ are then combined with $\ell_{x',\delta} = \text{LCE}_{r}^{X,Y}(0, x' + \delta)$ based on Fact 4.6, which yields the sought values $\text{LCE}_{r}^{X,Y}(x, x + \delta)$.

The cost of a query is $\tilde{O}(\frac{2}{r} + |\Delta|) = \tilde{O}(\Delta)$ with high probability due to $x' \leq x + q$. \hfill $\square$

Finally, we recall that $\text{LCE}_{r}^{X,Y}(x, y)$ satisfies the requirements for $\text{LCE}_{\leq k}^{X,Y}(x, y)$ with probability at least $1 - \exp(-\frac{k+1}{r})$. Consequently, taking sufficiently small $r = \Theta(k + 1)$ guarantees success with high probability. Thus, Lemma 4.11 yields Proposition 4.4, which we restate below.
Proposition 4.4. There is a data structure that, initialized with strings $X$ and $Y$, an integer $k \geq 0$, and an integer range $\Delta$, answers the following queries: given an integer $x$, return $\text{LC}=X,Y(x+x+\delta)$ for all $\delta \in \Delta$. The initialization costs $O(\frac{1}{k+1}|X|)$ time with high probability, and the queries cost $O(|\Delta|)$ time with high probability.

5 PTAS for Strings without Long Highly Periodic Substrings

In this section, we design an algorithm distinguishing between $\text{ED}(X,Y) \leq k$ and $\text{ED}(X,Y) > (1+\varepsilon)k$, assuming that $X$ does not have a length-$\ell$ substring with period at most $2k$. The high-level approach of our solution is based on the existing algorithms for Ulam distance [AN10, NSS17]. The key tool in these algorithms is a method for decomposing $X = X_0 \cdots X_m$ and $Y = Y_0 \cdots Y_m$ into short phrases such that $\text{ED}(X,Y) = \sum_{i=0}^{m} \text{ED}(X_i,Y_i)$ if $\text{ED}(X,Y) \leq k$. While designing such a decomposition in sublinear time for general strings $X$ and $Y$ remains a challenging open problem, the lack of long highly periodic substrings makes this task feasible.

Lemma 5.1. There is an algorithm that, given strings $X$ and $Y$, integers $k$ and $\ell$ such that $\text{per}(X[i..i+\ell]) > 2k$ for each $i \in [0..|X|-\ell]$, and a real parameter $0 < \delta < 1$, returns factorizations $X = X_0 \cdots X_m$ and $Y = Y_0 \cdots Y_m$ with $m = O(\frac{\delta}{\ell(k+1)}|X|)$ such that $|X_i| \leq \lceil \delta^{-1}(k+1)\ell \rceil$ for each $i \in [0..m]$ and, if $\text{ED}(X,Y) \leq k$, then $\mathbb{P}[\text{ED}(X,Y) = \sum_{i=0}^{m} \text{ED}(X_i,Y_i)] \geq 1 - \delta$. The running time of the algorithm is $O(\frac{\delta}{k+1}|X|)$.

Proof. Let $q = \lfloor \delta^{-1}(k+1)\ell \rfloor$. If $|X| \leq q$, then the algorithm returns trivial decompositions of $X$ and $Y$ with $m = 0$. In the following, we assume that $q < |X|$. The algorithm chooses $r \in [0..q]$ uniformly at random and creates a partition $X = X_0 \cdots X_m$ so that $|X_0| = r$, $|X_i| = q$ for $i \in [1..m]$, and $|X_m| \leq q$. This partition clearly satisfies $m = O(\frac{\delta}{q}|X|) = O(\frac{\delta}{\ell(k+1)}|X|)$ and $|X_i| \leq q = \lfloor \delta^{-1}(k+1)\ell \rfloor$ for each $i \in [0..m].$

Let us define $x_i$ for $i \in [0..m+1]$ so that $X_i = X[x_i..x_{i+1}]$ for $i \in [0..m]$. For each $i \in [1..m]$, the algorithm finds the occurrences of $X[x_i..x_{i+1}]$ in $Y$ with starting positions between $x_i - k$ and $x_i + k$. If there is no such occurrence (perhaps due to $|X_i| < \ell$ for $i = m$), then the algorithm declares a failure and returns a partition $Y = Y_0 \cdots Y_m$ with $Y_0 = Y$ and $Y_i = \varepsilon$ for $i \geq 1$. Otherwise, due to the assumption that $\text{per}(X[x_i..x_{i+1}]) > 2k$, there is exactly one occurrence, say, at position $y_i$. (Recall that the distance between two positions in $\text{Occ}(P,T)$ is always a period of $P$.) The algorithm defines $Y_i = Y[y_i..y_{i+1}]$ for $i \in [0..m]$, where $y_0 = 0$ and $y_{m+1} = |Y|$. This approach can be implemented in $O(m\ell) = O(\frac{\delta}{k+1}|X|)$ time using a classic linear-time pattern matching algorithm [MP70].

We shall prove that the resulting partition $Y = Y_0 \cdots Y_m$ satisfies the requirements. Assuming that $\text{ED}(X,Y) \leq k$, let us fix an optimal alignment between $X$ and $Y$. We need to prove that, with probability at least $1 - \delta$, the fragments $X[x_i..x_{i+1}]$ are all matched against $Y[y_i..y_{i+1}]$. By optimality of the alignment, this will imply $\mathbb{P}[\text{ED}(X,Y) = \sum_{i=0}^{m} \text{ED}(X_i,Y_i)] \geq 1 - \delta.$

We say that a position $x \in [0..|X|]$ is an error if $x = |X|$ or (in the alignment considered) the position $X[x]$ is deleted or matched against a position $Y[y]$ for $y \in [0..|Y|]$ such that $Y[y] \neq X[x]$ or $Y[y+1]$ is inserted. Each edit operation yields at most one error, so the total number of errors is at most $k+1$. Moreover, if $x_i..x_{i+1}$ does not contain any error, then $X[x_i..x_{i+1}]$ is matched exactly against a fragment of $Y$, and that fragment must be $Y[y_i..y_{i+1}]$ (since we considered all starting positions in $x_i - k \cdots x_{i+k}$). Hence, if the algorithm fails, then there is an error $x \in [x_i..x_{i+1}]$ for some $i \in [1..m]$. By definition of the decomposition $X = X_0 \cdots X_m$, this implies $x \bmod q \in [r..r+\ell] \bmod q$, or, equivalently, $r \in (x-\ell..x) \bmod q$. The probability of this event is $\frac{\ell}{q} \leq \frac{\delta}{k+1}$. The union bound across all errors yields an upper bound of $\delta$ on the failure probability. \qed

Next, we describe a subroutine that will be applied to individual phrases of the decompositions obtained in Lemma 5.1. Given that the phrases are short, we can afford using the classic
Landau–Vishkin algorithm [LV88] whenever we find out that the corresponding phrases do not match exactly.

**Lemma 5.2.** There is an algorithm that, given strings $X$ and $Y$, and a non-negative integer $k$, computes $\mathbb{E}[X] > 0$ exactly, taking $\tilde{O}(|X| + \mathbb{E}[X, Y]^2)$ time, or certifies that $\mathbb{E}[X, Y] \leq k$ with high probability, taking $\tilde{O}(1 + \frac{1}{k+1}|X|)$ time with high probability.

**Proof.** The algorithm first checks if $|X| = |Y|$, and then it samples $X$ with sufficiently large rate $\tilde{O}(\frac{1}{k+1})$ checking whether $X[i] = Y[i]$ for each sampled position $i$. If the checks succeed, then the algorithm certifies that $\mathbb{E}[X, Y] \leq k$. This branch takes $\tilde{O}(1 + \frac{1}{k+1}|X|)$ time with high probability. Otherwise, $\mathbb{E}[X, Y] > 0$, and the algorithm falls back to a procedure of Landau and Vishkin [LV88], whose running time is $\tilde{O}(|X| + |Y| + \mathbb{E}[X, Y]^2) = \tilde{O}(|X| + \mathbb{E}[X, Y]^2)$.

For the correctness, it suffices to show that if the checks succeeded, then $\mathbb{E}[X, Y] \leq k$ with high probability. We shall prove a stronger claim that $\text{HD}(X, Y) \leq k$. For a proof by contradiction, suppose that $\text{HD}(X, Y) \geq k + 1$ and consider some $k + 1$ mismatches. Notice that the sampling rate is sufficiently large that at least one of these mismatches is sampled with high probability. This completes the proof. \hfill $\square$

The next step is to design a procedure which distinguishes between $\sum_{i=0}^{m} \mathbb{E}[X_i, Y_i] \leq k$ and $\sum_{i=0}^{m} \mathbb{E}[X_i, Y_i] \geq (1 + \varepsilon)k$. Our approach relies on the Chernoff bound: we apply Lemma 5.2 to determine $\mathbb{E}[X_i, Y_i]$ for a small sample of indices $i$, and then we use these values to estimate the sum $\sum_{i=0}^{m} \mathbb{E}[X_i, Y_i]$.

**Lemma 5.3.** There is an algorithm that, given strings $X_0, \ldots, X_m, Y_0, \ldots, Y_m$, and a real parameter $0 < \varepsilon < 1$, returns:

- **YES** if $\sum_{i=0}^{m} \mathbb{E}[X_i, Y_i] \leq k$,
- **NO** if $\sum_{i=0}^{m} \mathbb{E}[X_i, Y_i] \geq (1 + \varepsilon)k$.

The algorithm succeeds with high probability, and its running time is $\tilde{O}(qk + k^2 + \frac{n}{\varepsilon^2(k+1)})$, where $q = \max_{i=0}^{m} |X_i|$ and $n = \sum_{i=0}^{m} (|X_i| + |Y_i|)$.

**Proof.** If $k = 0$, then the algorithm naively checks if $X_i = Y_i$ for each $i$, which costs $\mathcal{O}(n)$ time. In the following, we assume that $k > 0$.

For $i \in [0 \ldots m]$ and $j \in [0 \ldots |X_i| + |Y_i|]$, let us define an indicator $r_{i,j} = [\mathbb{E}[X_i, Y_i] > j]$. Observe that $\sum_{i=0}^{m} \mathbb{E}[X, Y] = \sum_{i=0}^{m} \sum_{j=0}^{(|X_i| + |Y_i|) - 1} r_{i,j}$. The algorithm samples independent random variables $R_1, \ldots, R_r$ distributed as a uniformly random among the $n$ terms $r_{i,j}$, where $r = \tilde{O}(\varepsilon^{-2} \frac{k}{n})$ is sufficiently large, and returns YES if and only if $\sum_{i=1}^{m} R_i \leq (1 + \frac{\varepsilon}{2}) \frac{k}{n}$.

Before we provide implementation details, let us prove the correctness of this approach. If $\sum_{i=0}^{m} \mathbb{E}[X, Y] \leq k$, then $\mathbb{E}[R_i] \leq \frac{k}{n}$. Consequently, the multiplicative Chernoff bound implies $\mathbb{P}[\sum_{i=1}^{m} R_i \geq (1 + \varepsilon) \frac{k}{n}] \leq \exp(-\frac{\varepsilon^2 k}{12n})$. Since $r = \tilde{O}(\varepsilon^{-2} \frac{k}{n})$ is sufficiently large, the complementary event holds with high probability. Similarly, if $\sum_{i=0}^{m} \mathbb{E}[X, Y] \geq (1 + \varepsilon)k$, then $\mathbb{E}[R_i] \geq \frac{(1 + \varepsilon)k}{n}$. Consequently, the multiplicative Chernoff bound implies $\mathbb{P}[\sum_{i=1}^{m} R_i \leq (1 + \frac{\varepsilon}{2}) \frac{k}{n}] \leq \mathbb{P}[\sum_{i=1}^{m} R_i \leq (1 - \varepsilon) \frac{(1 + \varepsilon)k}{n}] \leq \exp(-\frac{\varepsilon^2 r(1 + \varepsilon)k}{48n})$. Since $r = \tilde{O}(\varepsilon^{-2} \frac{k}{n})$ is sufficiently large, the complementary event holds with high probability. This finishes the correctness proof.

Evaluating each $R_i$ consists in drawing a term $r_{i,j}$ uniformly at random and testing if $r_{i,j} = 1$, that is, whether $\mathbb{E}[X_i, Y_i] > j$. For this, the algorithm makes a call to Lemma 5.2. If this call certifies that $\mathbb{E}[X_i, Y_i] \leq j$, then the algorithm sets $R_i = 0$. Otherwise, the call returns the exact value $\mathbb{E}[X_i, Y_i] > 0$, and the algorithm sets $R_i = 1$ if and only if $\mathbb{E}[X_i, Y_i] > j$. Moreover, the algorithm stores the value $\mathbb{E}[X_i, Y_i]$ so that if $i' = i$ for some $t' > t$, then the algorithm uses the stored value to evaluate $r_{i',j}$, instead of calling Lemma 5.2 again. Whenever the sum of the stored values exceeds $k$, the algorithm terminates and returns NO. Similarly, a call to Lemma 5.2 is terminated preemptively (and NO is returned) if the call takes too much time, indicating that $\mathbb{E}[X_i, Y_i] > k$. Consequently, since the function $x \mapsto x^2$ is convex, the
total running time of the calls to Lemma 5.2 that compute ED(X_{i}, Y_{i}) is \( \tilde{O}(gk + k^2) \). The total cost of the remaining calls can be bounded by \( \tilde{O}(\sum_{i=1}^{r} |X_{i}|) \). Since \( E[\frac{1}{r+1} |X_{i}| | i = i] \leq \ln(|X| + |Y|) + 1 \leq \ln n + 1 \), the total expected running time of these calls is \( \tilde{O}(r) = \tilde{O}(\frac{k}{\epsilon}) \). When \( \sum_{i=1}^{r} \frac{1}{j+1} |X_{i}| \) exceeds twice the expectation, the whole algorithm is restarted; with high probability, the number of restarts is \( \tilde{O}(1) \).

Finally, we obtain the main result of this section by combining Lemma 5.1 with Lemma 5.3.

**Theorem 5.4.** There is an algorithm that, given strings \( X \) and \( Y \), integers \( k \) and \( \ell \) such that \( \text{per}(X[i \ldots i + \ell]) > 2k \) for each \( i \in [0 \ldots |X| - \ell] \), and a real parameter \( 0 < \epsilon < 1 \), returns:

- **YES** if \( \text{ED}(X, Y) \leq k \),
- **NO** if \( \text{ED}(X, Y) \geq (1 + \epsilon)k \).

With high probability, the algorithm is correct and its running time is \( \tilde{O}(\frac{1}{\epsilon^2(k+1)} |X| + k^2 \ell) \).

**Proof.** The algorithm performs logarithmically many iterations. In each iteration, the algorithm calls Lemma 5.1 (with \( \delta = \frac{1}{k} \)) to obtain decompositions \( X = X_0 \cdots X_m \) and \( Y = Y_0 \cdots Y_m \). Then, the phrases are processed using Lemma 5.3. If this subroutine returns YES, then the algorithm also returns YES, because \( \text{ED}(X, Y) \leq \sum_{i=0}^{m} \text{ED}(X_i, Y_i) < (1 + \epsilon)k \) with high probability. On the other hand, if each call to Lemma 5.3 returns NO, then the algorithm returns NO. If \( \text{ED}(X, Y) \leq k \), then with high probability \( \text{ED}(X, Y) = \sum_{i=0}^{m} \text{ED}(X_i, Y_i) \) holds in at least one iteration; thus, \( \text{ED}(X, Y) > k \) holds with high probability if the algorithm returns NO.

As for the running time, the calls to Lemma 5.1 cost \( \tilde{O}(\frac{1}{\epsilon^2} |X|) \) time, and the calls to Lemma 5.3 cost \( \tilde{O}\left((k + 1)^2 \ell + \frac{1}{\epsilon^2(k+1)}(|X| + |Y|)\right) \) time because \( \max |X_i| = O((k + 1)\ell) \).

### 6 Random Walk over Samples

In this section, we describe the process of random walks over samples. This will be used in Section 7 to obtain an embedding of Edit Distance to Hamming distance in sublinear time.

Given strings \( X, Y \in \Sigma^n \), and parameters \( k \geq 0 \) and \( p > 0 \), we perform a random walk as follows. First, we sample every index in \( X \) with probability \( \frac{1}{p} \), and let \( S \) be the set of sampled indices. We can assume (without loss of generality) that \( X \) and \( Y \) ends with a special symbol $\$, and both 0 and \( |X| = n \) are in \( S \).

The algorithm pseudocode is given in Algorithm 9. The algorithm initializes \( X_{\text{Start}} \) and \( Y_{\text{Start}} \) with \( S[0] \), and sets \( \text{shift} = 0 \). If \( X[X_{\text{Start}}] \) matches with \( Y[Y_{\text{Start}}] \) then it proceeds to \( S[1] \) without changing the shift, and updates \( X_{\text{Start}} = S[1], Y_{\text{Start}} = S[1] + \text{shift} \). Otherwise, an unbiased coin is tossed (Line 4). If the coin returns head (indicating \( X[X_{\text{start}}] \) must be deleted), this is reflected by setting \( \text{shift} = \text{shift} - 1 \) (Line 6: \( r = +1 \) if the coin returns head). On the other hand, if the coin returns tail (indicating \( Y[Y_{\text{Start}}] \) must be deleted), this is reflected by setting \( \text{shift} = \text{shift} + 1 \) (Line 6: \( r = -1 \) if the coin returns tail). Finally, the algorithm updates \( X_{\text{Start}} = S[1], Y_{\text{Start}} = S[1] + \text{shift} \). This process continues until all the sampled indices of \( X \) is explored or at least \( k^2 \) coin tosses are made. In the first case, the algorithm returns YES. In the later case, the algorithm returns NO.

We now prove the following theorem.

**Theorem 6.1.** Given input strings \( X, Y \in \Sigma^n \) and parameter \( k \geq 0 \), \( p = \Omega(\log n) \), the algorithm Sampled Random Walk requires \( O(\frac{1}{p} + k^2) \) queries and time complexity, and satisfies the following:

- If \( \Delta_e(X, Y) \leq 2k \), it outputs YES with probability \( \frac{1}{p} \).
- If \( \Delta_e(X, Y) > k^2p \), it outputs NO with probability \( 1 - \frac{1}{n} \).

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YES Case. First, we prove the YES case. In order to do so, we only consider insertions and deletions as viable edit operations. Note that this may increase the total edit cost by a factor of two compared to when insertions, deletions and substitutions are allowed. Moreover, allowing insertions and deletions is equivalent to only allowing deletions on both $X$ and $Y$. Henceforth, we only consider deletions. We append a unique symbol $\$ at the end of both $X$ and $Y$ so that in any optimal alignment, they will be matched.

We fix an optimal alignment $A(X, Y)$, and the remaining analysis of this section is based on this fixed alignment. A symbol in $X$ (similarly in $Y$) is either deleted or is matched with some symbol in $Y$. In the later case, we say the two symbols have been aligned.

Optimal Comparison Process Let us consider the following process that generates the alignment $A(X, Y)$. It starts by comparing $X[0]$ and $Y[0]$, if they are aligned in $A(X, Y)$, then it moves to compare $X[1]$ and $Y[1]$. If $X[0]$ is deleted in $A(X, Y)$ but not $Y[0]$, then it moves to compare $X[1]$ and $Y[0]$. Similarly, If $Y[0]$ is deleted in $A(X, Y)$ but not $X[0]$, then it moves to compare $X[0]$ and $Y[1]$. Otherwise, if both are deleted, then it moves to compare $X[1]$ and $Y[0]$ (that is we consider deletion on $X$ happens first). We continue this process until we reach the $\$ symbol in both $X$ and $Y$. In this process each element of $X$ is compared to one or more elements of $Y$ and vice-versa.

We can view this process as follows. Consider a semi-circular dial with equally-spaced markings $-n, -n + 1, \ldots, 0, 1, \ldots, n - 1, n$, and a moving hand $h_O$ that starts at 0. If $X[0]$ and $Y[0]$ are matched, then the hand does not move. Otherwise, if the next comparison is $X[1]$ with $Y[0]$ then the hand moves one step to the left, that is to $-1$ since it started at 0. Else, if the next comparison is $X[0]$ with $Y[1]$ then the hand moves one step to the right, that is to $+1$. Hence at any point in the process if $X[i]$ is being compared with $Y[j]$, the hand points to $(j - i)$. Let $h_O[x]$ denote the position of the hand right when the optimal alignment moves to $X[x + 1]$.

Comparisons in Sampled Random Walk. While comparing $X[XStart]$ with $Y[YStart]$, if they do not match, the algorithm tosses a coin and set $r = +1$ and $-1$ with equal probability. Based on the choice of $r$, we can view the algorithm as incrementing (equivalent to deletion) either $XStart$, or $YStart$, and then moving to the next sampled position in $X$.

We can similarly define a moving hand $h_S$ for our algorithm. The hand starts at position 0. If $X[XStart]$ is deleted ($r = +1$), it moves one step left, and if $Y[YStart]$ is deleted ($r = -1$), it moves one step towards right. Otherwise, it does not move.

Let $S = \{i_1, i_2, \ldots, i_{|S|}\}$. $h_S[i_j]$ denote the position of the hand $h_S$ right after processing $i_j$.

Correct Choice of $r$ for Sampled Random Walk Sampled Random Walk selects $r \in \{+1, -1\}$ randomly when comparing $X[XStart]$ with $Y[YStart]$ if they do not match, and

| Algorithm 9: Sampled Random Walk($X, Y, S, k$) |
|---|
| 1 $XStart := 0, YStart := 0, index := 1, shift := 0, Cost := 0;$ |
| 2 while $index \leq |S|$ and $Cost \leq k^2$ do |
| 3 if $X[XStart] \neq Y[YStart]$ then |
| 4 $r := random(+1, -1)$; |
| 5 Cost := Cost + 1; |
| 6 shift := shift $- r$; |
| 7 $XStart := S[index]$; |
| 8 $YStart := S[index] + shift$; |
| 9 index := index + 1; |
| 10 if $Cost \leq k^2$ then return YES; |
| 11 else return NO; |

---
update \( h_S \) accordingly. Let us initiate \( h_O[i-1] = h_S[i-1] = 0 \). Consider \( i_j \). Suppose \( X[i_j] \) is compared with \( Y[i_i, ..., i_i + a] \) \((a > 0) \) is possible when \( Y[i_i, ..., i_i + a - 1] \) are all deleted during the comparison with \( X[i_j]\). Define \( h_O^*[i_j - 1] = (i_i + a) - i_j \), and initialize \( h_O^*[1] = 0 \).

We say the choice of \( r \) is correct at \( i_j \) if

\[
|h_O[i_j] - h_S[i_j]| \leq |h_O^*[i_j - 1] - h_S[i_j-1]|
\]

Note that if \( a = 0 \) (that is when \( X[i_j] \) is deleted, or aligned with \( Y_i \)) then \( h_O^*[i_j - 1] = h_O[i_j - 1] \).

**Lemma 6.2.** If \( \Delta_e(X, Y) \leq k \), and Sampled Random Walk always makes correct choices for \( r \), then the cost incurred by the algorithm is at most \( 2k \).

**Proof.** Let \( i_1, i_2, ..., i_s \) denote the sampled positions. We can assume \( i_1 = 0 \) and \( i_{s} = n \). Sampled Random Walk starts with \( X_{\text{Start}} = i_1 \) on \( X \), and then sequentially moves to each sampled positions in \( S \). That is for \( i_j \in S, j = 1, 2, ..., |S| \), it compares \( X[i_j] \) with the appropriate entry in \( Y \) as dictated by the random choices.

Let \( \text{Sample}[i_j] \) denote the total cost incurred by Sampled Random Walk right after processing \( i_j \). Similarly, let \( \text{OPT}[i_j] \) denote the total number of deletions performed by \( A(X, Y) \) up to the point when the optimal comparison process moves to \( X[i_j + 1] \).

We next show by induction on \( j \) if the choice of \( r \) is always correct, then \( \text{Sample}[i_j] \) is at most \( \text{OPT}[i_j] - |h_O[i_j] - h_S[i_j]| \).

**Claim 6.3.** For all \( j = 1, 2, ..., |S| \), \( \text{Sample}[i_j] \leq \text{OPT}[i_j] - |h_O[i_j] - h_S[i_j]| \)

Clearly Lemma 6.2 follows from the above claim, since the claim establishes the total cost incurred by Sampled Random Walk is at most the optimum cost, whereas the optimum cost is at most \( 2k \).

We now prove the claim.

**Base Case.** Sampled Random Walk starts at \( X_{\text{Start}} = 0 \) and \( Y_{\text{Start}} = 0 \), and so does the optimal comparison process.

- If in \( A(X, Y) \), \( X[0] \) matches \( Y[0] \) then so do they in Sampled Random Walk. In that case, both \( h_O(0) \) and \( h_S(0) \) stays at 0, and none incurs any cost.

- Now suppose the optimal comparison process matches \( X[0] \) with \( Y[j], j \geq 1 \). Then \( h_O^*[1] = j \) and \( \text{OPT}[i_1] = j \). The correct choice for Sampled Random Walk in this case is to select \( r = -1 \). Thus \( |h_O[0] - h_S[0]| = j - 1 \) and \( \text{Sample}[i_1] = 1 \). Thus, we have \( \text{Sample}[i_1] = \text{OPT}[i_1] - |h_O[i_1] - h_S[i_1]| \).

- Finally, assume \( X[0] \) is deleted while comparing with \( Y[0] \). Then the correct choice for Sampled Random Walk is to select \( r = +1 \) since \( h_O[0] = -1 \). In that case, \( |h_O[i_1] - h_S(i_1)| = 0 \) and they both incur a cost of 1.

Hence the base case holds.

**Induction.** Suppose by the induction hypothesis, the claim is true up to \( i_{j-1} \), and we now consider \( i_j \). Therefore, we have

\[
\text{Sample}[i_{j-1}] \leq \text{OPT}[i_{j-1}] - |h_O[i_{j-1}] - h_S[i_{j-1}]| \]

Let between \( i_{j-1} + 1 \) to \( i_j \), \( A(X, Y) \) makes \( d_j \) deletions.

- If \( d_j = 0 \) and \( |h_O[i_{j-1}] - h_S[i_{j-1}]| = 0 \), then note that both Sample Random Walk and the optimum comparison process compares the same \( X[i_j] \) and \( Y[i_j + h_S[i_{j-1}]] \), and they both must be matched since \( d_j = 0 \). Thus, \( \text{Sample}[i_j] \) remains at most the cost of \( \text{OPT}[i_j] \).

Moreover, \( |h_O[i_j] - h_S[i_j]| \) remains 0. Thus the induction holds.

Now assume \( d_j = 0 \), but \( |h_O[i_{j-1}] - h_S[i_{j-1}]| > 0 \). Note that since \( d_j = 0 \), all the symbols \( X[i_{j-1} + 1, ..., i_j] \) must be aligned. Thus, \( h_O^*[i_j - 1] = h_O[i_j - 1] = h_O[i_j] = h_O[i_{j-1}] \). Since
Sample Random Walk takes a correct move, $|h_O[i_j] - h_S[i_j]| = |h_O[i_{j-1}] - h_S[i_{j-1}]| - 1$ if Sample Random Walk incurs a cost. Otherwise, if Sample Random Walk does not incur a cost then $|h_O[i_j] - h_S[i_j]| = |h_O[i_{j-1}] - h_S[i_{j-1}]|$. In both cases, the induction holds.

- If $d_j > 0$ then $OPT[i_j] = OPT[i_{j-1}] + d_j$. We consider two cases based on whether Sampled Random Walk incurs a cost or not.
  - If Sampled Random Walk does not incur any cost then $h_S[i_j] = h_S[i_{j-1}]$, and the following holds $|h_O[i_j] - h_S[i_j]| = |h_O[i_{j-1}] - h_S[i_{j-1}]|$. Thus, if Sampled Random Walk does not incur any cost at $i_j$, then the induction holds.
  - Therefore, assume Sampled Random Walk incurs a cost, and since it takes a correct move, $|h_O[i_j] - h_S[i_j]| = |h_O[i_{j-1}] - h_S[i_{j-1}]|$. Let us consider the positions of $h_O[i_{j-1}]$ and $h_O[i_j]$.
    (i) If $h_O[i_{j-1}] = h_O[i_j]$, then $X[i_j]$ must be matched with $Y[i_j + h_O(i_j)]$. In that case, if $h_S[i_{j-1}] = h_O[i_{j-1}]$, then Sampled Random Walk will not incur any cost. Else if $h_S[i_{j-1}] = h_O[i_{j-1}]$ and Sampled Random Walk incurs a cost, then since the move is correct $|h_O[i_j] - h_S[i_j]| = |h_O[i_j] - h_S[i_{j-1}]| - 1$ as $h_O[i_{j-1}] = h_O[i_j]$. Hence, $|h_O[i_j] - h_S[i_j]| - |h_O[i_{j-1}] - h_S[i_{j-1}]| \leq d_j - 1$. Thus

$$Sample[i_j] = Sample[i_{j-1}] + 1 \leq OPT[i_{j-1}] - |h_O[i_{j-1}] - h_S[i_{j-1}]| + 1 \leq OPT[i_{j-1}] - |h_O[i_{j-1}] - h_S[i_{j-1}]|.$$ 

(ii) Now let us consider $h_O[i_{j-1}] \neq h_O[i_j]$. Then it must be the case that $X[i_j]$ is deleted. If $X[i_j]$ is deleted, then $h_O[i_{j-1}] = h_O[i_j - 1]$. We have $h_O[i_j] = h_O[i_{j-1}] - 1$, and $OPT[i_{j-1}] = OPT[i_{j-1}] + d_j - 1$. Then $|h_O[i_{j-1}] - h_S[i_{j-1}]| - |h_O[i_{j-1}] - h_S[i_{j-1}]| \leq d_j - 1$. Since the move is correct, if $h_O[i_{j-1}] = h_S[i_{j-1}]$, then we must have $|h_O[i_{j-1}] - h_S[i_{j-1}]| = 0$. Thus, the induction holds since $d_j \geq 1$.

Otherwise, $h_O[i_{j-1}] \neq h_S[i_{j-1}]$. If $h_O[i_j] = h_S[i_{j-1}]$, then $|h_O[i_{j-1}] - h_S[i_{j-1}]| = 1$. Since, we take a correct move, we maintain $|h_O[i_{j-1}] - h_S[i_{j-1}]| = 1$. If $d_j \geq 2$, then again the induction holds. Else $d_j = 1$, that is the only deletion that happened is $X[i_j]$. In that case $h_O[i_{j-1}] = h_O[i_{j-1}]$.

We have

$$Sample(i_{j}) = Sample(i_{j-1}) + 1 \leq OPT(i_{j-1}) - |h_O[i_{j-1}] - h_S[i_{j-1}]| + 1 \leq OPT(i_{j-1}) - |h_O[i_{j-1}] - h_S[i_{j-1}]|.$$ 

For all other cases, $h_O[i_j] \neq h_S[i_{j-1}]$ and $h_O[i_{j-1}] \neq h_S[i_{j-1}]$. We have $|h_O[i_j] - h_S[i_j]| = |h_O[i_{j-1}] - h_S[i_{j-1}]| - 1$. Since $|h_O[i_{j-1}] - h_S[i_{j-1}]| - |h_O[i_{j-1}] - h_S[i_{j-1}]| \leq d_j$, we have $|h_O[i_{j-1}] - h_S[i_{j-1}]| - |h_O[i_{j-1}] - h_S[i_{j-1}]| \leq d_j - 1$. Thus, the induction holds.

This completes the prove of the claim, and the lemma.

**Lemma 6.4.** If $\Delta_s(X, Y) \leq k$, and Sampled Random Walk always makes $w$ wrong choices, then the cost incurred by the algorithm is at most $2w + 2k$.

**Proof.** Here the proof follows by observing that if Sampled Random Walk takes a wrong move at $i_j$, then $|h_O[i_j] - h_S[i_j]| = |h_O[i_j] - h_S[i_{j-1}]| + 1$. The effect of this can be nullified by taking at most one correct move. Therefore, the total cost incurred is at most $2w$ to nullify the effect of the wrong choices and at most $2k$ from Lemma 6.2.

**Mapping to a Random Walk** In order to bound the number of wrong moves $w$, we observe that at every step of Sampled Random Walk, a correct choice is made with probability at least $1/2$. Note that $|h_O[i_{j-1}] - h_O[i_j]| \leq 1$. If $h_O[i_{j-1}] = h_O[i_j] = h_S[i_{j-1}]$ then Sampled Random Walk does not incur any cost, and no random choice is made. In all other cases, with probability at least $1/2$, a correct choice is made.

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Therefore, $w$ can be bounded by the hitting time of a one-dimensional random walk starting at $2k$ that with equal probability moves one step towards left or right at every step. The following lemma is quoted from [Sah14].

**Lemma (Lemma 13 [Sah14]).** $\mathbb{P}(w \geq k^2) \leq 1/6$

This completes the proof of the YES case. \hfill $\Box$

**NO Case.** First, we prove a simple lemma. When comparing a symbol $X[i]$ with $Y[j]$, if they do not match, we call it a *mismatch*. The following lemma asserts we cannot miss too many mismatches due to sampling.

**Lemma 6.5.** Suppose the sampling probability is $\Theta(\frac{\log n}{p})$. Given $i \in [1, n]$ and $d \in [-k, k]$, let $i' \geq i$ be the smallest index such that $X[i]X[i+1] \cdots X[i']$ and $Y[i+d]Y[i+1+d] \cdots Y[i'+d]$ have at least $p$ mismatches. Let $i \leq j_1, j_2, \ldots, j_p = i'$ be the indices such that $X[j_i] \neq Y[j_i+d]$. Define a bad event $B(i, d)$ to be the event that none of these $p$ mismatch indices are sampled. Then $\mathbb{P}(\text{Bad}(i, d)) \leq 1 - \frac{1}{n}$. Moreover, all bad events are avoided with probability at least $1 - \frac{1}{n}$.

**Proof.** Since the sampling probability is $\Theta(\frac{\log n}{p})$, the expected number of points sampled from $j_1, j_2, \ldots, j_p$ is $\Theta(\log n)$. Now, by the Chernoff bound, the probability that none of them are sampled can be made to be $1 - \frac{1}{n}$ (by choosing the constants in the sampling probability appropriately). Then by a union bound over all $i \in [1, n]$ and $d \in [-k, k]$, with probability $\geq 1 - \frac{1}{n}$ none of the bad events $\text{Bad}(i, d)$ happens. \hfill $\Box$

We write a corollary of Lemma 6.5 with identical proof to be used later.

**Corollary 6.6.** Suppose the sampling probability is $\Theta(\frac{\log n}{p})$. Given $i \in [1, n]$ and $d \in [-k, k]$, let $i' \geq i$ be the smallest index such that $X[i]X[i+1] \cdots X[i']$ and $Y[i+d]Y[i+1+d] \cdots Y[i'+d]$ have at least $p$ mismatches. Let $i \leq j_1, j_2, \ldots, j_p = i'$ be the indices such that $X[j_i] \neq Y[j_i+d]$. Given a fixed integer vector $a = \{a_1, a_2, \ldots, a_p\}$, define a bad event $B^a(i, d)$ to be the event that none of $j_h - a_h$, $h = 1, 2, \ldots, p$ indices are sampled. Then $\mathbb{P}(\text{Bad}^a(i, d)) \leq 1 - \frac{1}{n}$. Moreover, all bad events are avoided with probability at least $1 - \frac{1}{n}$.

Henceforth, we can assume all bad events $B(i, d)$ are avoided as given in Lemma 6.5.

Now suppose $\Delta_s(X, Y) > k^2p$, however Sampled Random Walk incurs at most $k^2$ cost. Then there are at most $k^2$ indices in $S$, where the cost is incurred. Suppose they are $i_{t_1} < i_{t_2} < \cdots < i_{s \leq s \leq k^2}$. Note that in between $X[i_{t_1} + 1]$ to $X[i_{t_2} - 1]$, we have $h_S(i_{t_1}) = h_S(i_{t_2}) = \cdots = h_S(i_{t_2} - 1)$. Since $\text{Bad}([i_{t_1} + 1, h_s(i_{t_1})])$, then the Hamming distance between $X[i_{t_1} + 1] \cdots X[i_{t_2} - 1] = h_s(i_{t_1})]$ to $X[i_{t_2} - 1 + h_s(i_{t_1})]$ is at most $p$. Thus, we obtain an alignment of $X$ and $Y$ with a total cost of at most $k^2p$ with probability at least $1 - \frac{1}{n}$. This contradicts our assumption of the NO case.

This completes the proof of Theorem 6.1. \hfill $\Box$

### 7 Sublinear Time Embedding of Edit Distance to Hamming Distance

Given any two strings $X$ and $Y$, a randomized embedding of edit distance to Hamming distance is given by a function $f$ so that with high probability over the randomness $R$, $\text{HD}(f(X, R), f(Y, R)) \approx \text{ED}(X, Y)$. The distortion of an embedding is defined as the product of expansion and contraction, where expansion is the maximum possible ratio of $\text{HD}(f(X, R), f(Y, R))$, and contraction is the maximum possible ratio of $\frac{\text{ED}(X, Y)}{\text{HD}(f(X, R), f(Y, R))}$ over all $X$ and $Y$.

Chakraborty, Goldenberg and Koucký gave one such randomized embedding with quadratic distortion [CGK16]. Specifically, they proved the following theorem.
Theorem (1.1. [CGK16]). For any integer \( n > 0 \), there is \( \ell = O(\log n) \) and a function \( f : \{0, 1\}^n \times \{0, 1\}^\ell \to \{0, 1\}^{3n} \) such that for any pair of strings \( X, Y \in \{0, 1\}^n \)

\[
\frac{1}{2} \text{ED}(X, Y) \leq \text{HD}(f(X, R), f(Y, R)) \leq O(\text{ED}^2(X, Y))
\]

with probability at least \( \frac{2}{3} \) over the choice of random string \( R \in \{0, 1\}^\ell \). Moreover, the function \( f \) can be computed in linear time.

The algorithm works as follows. The algorithm has access to \( 3n \) hash functions \( h_1, h_2, \ldots, h_{3n} \) mapping \( \Sigma \) to \( \{0, 1\} \). It reads each element of \( X \) sequentially, and at time \( i \) if it is at \( X[i] \), it appends the embedding with \( X[i] \) and increment \( i \) by \( h_i(X[i]) \). This can be viewed as tossing an unbiased coin, and depending on its outcome either insert the same symbol \( X[i] \), or move to the next symbol \( X[i+1] \). That is, in contrast to the random deletion method performed in the previous section, the algorithm here performs random insertion. The mapping of \( Y \) is obtained in a similar manner. The above algorithm requires linear number of random bits which can be reduced to \( O(\log n) \) by using Nisan’s pseudorandom number generator.

In this section by utilizing random walk over samples, we provide the first sublinear time algorithm for obtaining a randomized embedding of edit to Hamming distance. Given a sampling parameter \( p \), we sample each index between \( [1, n] \) independently with probability \( \frac{1}{p} \). Let the sampled indices be referred to as \( S \). Consider \( S = S \cup \{0\} \), and select \( s = |S| \) hash functions \( h_1, h_2, \ldots, h_s \) as follows. Consider two functions \( g_1 : 0 \to 0, 1 \to 1 \), and \( g_2 : 0 \to 1, 1 \to 0 \). Each \( h_i \) is set to either \( g_1 \) or \( g_2 \) with equal probability. The shared randomness \( R \) consists of \( S \) and the hash functions \( h_1, h_2, \ldots, h_s \). Also assume both \( X \) and \( Y \) are appended at the end with a block of \( $$$ \) of length \( n \). We prove the following theorem.

**Theorem 7.1.** For any integer \( n > 0 \) and a parameter \( p = \Omega(\log n) \), there is \( \ell = O(\log \frac{n}{p}) \) and a function \( f : \{0, 1\}^n \times \{0, 1\}^\ell \to \{0, 1\}^{\frac{3n}{p}} \) such that for any pair of strings \( X, Y \in \{0, 1\}^n \)

\[
\frac{\text{ED}(X, Y) - 1}{p + 1} \leq \text{HD}(f(X, R), f(Y, R)) \leq O(\text{ED}^2(X, Y))
\]

with probability at least \( \frac{2}{3} \) over the choice of random string \( R \in \{0, 1\}^\ell \). Moreover, the function \( f \) can be computed in \( O(\frac{n}{p}) \) time.

The pseudocode of the embedding is given below.

**Algorithm 10: Sublinear Embedding**

```plaintext
Output := 0; gap := 0;
Xstart := S[1] + gap;
for i = 1 to s do
    Output := Output · X[Xstart];
    if h_i(X[Xstart]) = 1 then gap := gap + 1;
    Xstart := S[i + 1] + gap;
return Output
```

**Interpretation through random walk & Correctness.** The algorithm can be interpreted as follows. Let us compare this algorithm to the one described in the last section. Once the analogy is clear, the correctness proof of this algorithm follows immediately from the last section. In particular, the bound on expansion follows from the YES case analysis of the previous section, whereas the bound on contraction follows from the NO case analysis of the previous section.
YES Case. Suppose the embedding of $X$ is given by $X_{i_1}X_{i_2}...X_{i_{|S|}}$ where we can assume $i_1 = 0$. Construct $T = \{i_1, i_2, ..., i_{|S|}\}$. We will use $T$ as the sampled indices for Sampled Random Walk in the last section.

While constructing the embedding of $X$ and $Y$, the algorithm initially starts with both $X_{\text{Start}} = 0$ and $Y_{\text{Start}} = 0$. If $X[X_{\text{Start}}] = Y[Y_{\text{Start}}]$, then $h_1$ evaluates to the same value. Hence, the embedding either increments both $X_{\text{Start}}$ and $Y_{\text{Start}}$, or none of them are incremented. Finally, they move to the next exact same sampled location without changing the shift. This is analogous to Sampled Random Walk: after processing $X[i_1]$ and $Y[i_1]$, it moves to $X[i_2]$ and $Y[i_2]$. If $X[X_{\text{Start}}]$ and $Y[Y_{\text{Start}}]$ do not match, then their hashed value under $h_1$ must be different. Based on whether $h_1$ equals $g_1$ or $g_2$, we can view this as incrementing either $X_{\text{Start}}$, or $Y_{\text{Start}}$, and then moving to the next sampled position in $T$. If $X_{\text{Start}}$ is incremented (same for the other case), then this is analogous to have $r = +1$ in Sampled Random Walk. In that case, Sampled Random Walk will next compare $X[i_2]$ and $Y[i_2 - 1]$, and these are exactly the second bit of $f(X, R)$ and $f(Y, R)$ in the embedding. Assuming, the analogy holds till the $(j - 1)$th bit, then by the exact same analysis, if Sampled Random Walk compares $X[i_j]$ with $Y[i_j + s]$, then the $j$th bit of $f(X, R)$ and $f(Y, R)$ are respectively $X[i_j]$ and $Y[i_j + s]$.

Hence by the YES case analysis of Sampled Random Walk, $\text{HD}(f(X, R), f(Y, R)) \leq k^2$.

NO Case. Suppose if possible $\text{HD}(f(X, R), f(Y, R)) < \frac{k - 1}{p + 1}$. Then there are at most $k' - 1 = \frac{k - 1}{p + 1} - 1$ locations in the embedding that $f(X, R)$ and $f(Y, R)$ differ. Let us use $X'$ and $Y'$ as shorthand of $f(X, R)$ and $f(Y, R)$. Let these locations in $X'$ be $X'[h_1], X'[h_2], ..., X'[h_{k'}]$, and in $Y'$ be $X'[g_1], X'[g_2], ..., X'[g_{k'}]$. Then between $h_1 + 1$ to $h_2 - 1$, the evaluations of the hash functions matched both in $X$ and $Y$, thus producing the same values of gap (which dictates the actual sampled bit based on $S$). Then by Corollary 6.6, $\text{HD}(X[h_i + 1, h_{i+1} - 1], Y[g_i + 1, g_{i+1} - 1]) \leq p$ for $i = 1, 2, ..., k' - 1$. Finally, again by Corollary 6.6, $\text{HD}(X[0, h_1 - 1], Y[0, g_1 - 1]) \leq p$ and $\text{HD}(X[h_{k'} + 1, n], Y[g_{k'} + 1, g_{k'} + n - X[h_{k'}]]) \leq p$. Finally, an extra $k'$ deletions may be necessary on either $X$ or $Y$ since the shift at $X'[i']$ could be at most $k'$. Hence, there exists an alignment of $X$ and $Y$ with total edit cost at most $k'p + k' = k - 1$, a contradiction.

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