Symplectic fluctuations for electromagnetic excitations of Hall droplets

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Abstract. We show that the integer quantum Hall effect systems in a plane, sphere or disc can be formulated in terms of an algebraic unified scheme. This can be achieved by making use of a generalized Weyl–Heisenberg algebra and investigating its basic features. We study the electromagnetic excitation and derive the Hamiltonian for droplets of fermions on a two-dimensional Bargmann space (phase space). This excitation is introduced through a deformation (perturbation) of the symplectic structure of the phase space. We show the major role of Moser’s lemma in a dressing procedure, which allows us to eliminate the fluctuations of the symplectic structure. We discuss the emergence of the Seiberg–Witten map and the generation of an Abelian noncommutative gauge field in the theory. As an illustration of our model, we give the action describing the electromagnetic excitation of a quantum Hall droplet in a two-dimensional manifold.
1. Introduction

The notion of deformation, which is very familiar in mathematics and physics, is based on the philosophy of deforming suitable mathematical structures behind the physical theories (e.g., complex, symplectic or algebraic structures). In this connection, the quantum and relativistic mechanics are commonly regarded as the $\hbar$-deformation and $1/c$-deformation, respectively, of classical mechanics. Quantum algebras ($q$-deformation), string theory ($\alpha'$-deformation) and noncommutative field theory ($\theta$-deformation) have been the deformed theories most studied during the last few decades. The deformation often leads to radical changes of the parent theory, inducing new physics, for instance the wave–particle duality in quantum mechanics and $T$-duality in string theory. In this spirit and motivated by string theory arguments [1], the idea of noncommutativity at small length scales [2] has

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Symplectic fluctuations for electromagnetic excitations of Hall droplets drawn much attention in various fields; see for instance [3, 4]. The noncommutative field theory continues to be investigated extensively.

In this paper, we shall first define a generalized Weyl–Heisenberg algebra $W_\kappa$, which can be seen as a deviation from the ordinary boson algebra. This generalization can be understood in the context of the quantum algebras. It can also be related to the concept of deformed boson algebra introduced in the 1970s [5]; see also [6]. The $\kappa$-dependent algebra generalizes the usual boson algebra, which corresponds to the case $\kappa = 0$. This generalized algebra offers an advantage for carrying out a simultaneous study of planar ($\kappa = 0$), spherical ($\kappa = -1$) and hyperbolic ($\kappa = 1$) systems. Using a quantum-classical correspondence, we construct the phase (Bargmann) space equipped with a symplectic structure encoding the dynamics of the system whose relevant symmetry is described by the algebra $W_\kappa$.

The second facet of this work deals with the $\theta$-deformation. This is done by modifying the symplectic structure given by a 2-form $\omega_0$ of the Bargmann space. This provides us with a nice tool for studying the electromagnetic excitations of fermions living in a plane, sphere or disc. This procedure is well known in symplectic geometry and it is deeply related to the noncommutative geometry. It found interesting applications in quantum mechanics and the planar quantum Hall effect; see for instance [7]–[11]. In fact, the interaction of a charged particle can be described in a Hamiltonian formalism without a choice of a potential. The interaction can be introduced through a 2-form $F$ inducing fluctuations of the parent symplectic form.

The modified symplectic 2-form can always be rewritten as $\omega_0$ in a new coordinate system in the phase space. This can be done by using the celebrated Darboux transformation for a flat phase space geometry and for some particular form of the electromagnetic field $F$. We give the explicit form of the dressing transformation, which eliminates the fluctuations of the symplectic form $\omega_0$ and transforms $\omega_0 + F \rightarrow \omega_0$ in curved phase space. This is a refined version of the Darboux transformation recognized in the mathematical literature as the Moser lemma [12]. Interestingly, this transformation turns out to be identical to the Seiberg–Witten map [1]. We show, as a by product, that the dynamics of the system becomes described by a Hamiltonian involving terms encoding the effect of perturbation $F$ and a Chern–Simons-like interaction.

On the other hand, it is well known that the planar fermions in a strong magnetic field are confined in the lowest Landau levels and behave like a rigid droplet of liquid. This is the incompressible quantum fluid picture proposed by Laughlin [13]. This led to new perspectives for using the noncommutative geometry ideas to discuss the quantum Hall phenomenon in the plane [14, 15] and developing later generalizations to other geometries with arbitrary dimensions [16]–[26]. Therefore, we give an illustration of our analysis by studying the effect of the symplectic fluctuations on the quantum Hall droplets and comparing the results obtained with ones recently presented in the literature [27].

The outline of the paper is as follows. In section 2, we introduce a generalized Weyl–Heisenberg algebra. We discuss the corresponding Hilbertian representation and the analytical Bargmann realization. It is remarkable that the realization obtained leads to the Klauder–Perelomov coherent states. Also, it is important to stress that this generalization induces, at a geometrical level, the curvature of the Bargmann space. The flat geometry is recovered in the limit of the standard Weyl–Heisenberg algebra. Using the algebraic structure of the generalized Weyl algebra, we discuss the quantum mechanics
Symplectic fluctuations for electromagnetic excitations of Hall droplets on coset spaces $S_κ^2$ in section 3. This provides us with a unified scheme for quantum mechanically dealing with spherical and hyperbolic systems. Using the standard tools of geometric quantization, we establish one to one correspondence between the lowest energy level (vacuum) wavefunctions and the holomorphic sections (coherent states) defining the Bargmann space. In section 4, we equip the Bargmann space (phase space) with a star product and consequently we define the Moyal brackets. We also give the potential lifting the degeneracies of the vacuum. This is helpful in the semiclassical analysis done in the following sections. In section 5, we discuss the noncommutative dynamics in Bargmann space, which is achieved by modifying the associated symplectic structure. We show that Moser’s lemma offers a nice way to eliminate the fluctuation (modification) of the symplectic 2-form. This can be done by making use of a dressing transformation similar to the Darboux one for flat phase spaces. The effect of the perturbation is then incorporated in the Hamiltonian. We also show that the Moser dressing transformation induces in a non-trivial way a noncommutative gauge field and it coincides with the Seiberg–Witten map. As an illustration, we derive, in section 6, the semiclassical effective action describing the electromagnetic excitation of the two-dimensional quantum Hall droplet. Finally, we conclude with a summary of our results.

2. Generalized Weyl–Heisenberg algebra $W_κ$

The first step in our program is to introduce a generalized Weyl–Heisenberg algebra. We give the corresponding Hilbertian representation and we provide the analytical Bargmann realization.

2.1. Algebraic structures

Let us consider an algebra $W_κ$ characterized by four generators $x_+, x_-, x_0$ and $I$. They satisfy the commutation relations

$$
[x_-, x_+] = I + 2κx_0, \quad [x_0, x_+] = x_+, \quad [x_0, x_-] = -x_- \quad [ , , I] = 0 \quad (1)
$$

where $κ$ is a real parameter. It is clear that the operator $I$ belongs to the center of $W_κ$. When $κ$ is non-null, then it can be rescaled to $+1$ or $-1$; hence we have three representatives values of $κ = -1, 0, +1$. For $κ = 0$, we have the usual harmonic oscillator algebra. The sign of this parameter plays an important role in specifying the representation dimensions associated with $W_κ$ as will be discussed throughout this section. Note that this algebra is relevant in the theory of exactly solvable potentials in one dimension [28] and fractional supersymmetric quantum mechanics [29]. Indeed, in the context of a quantum system evolving in one-dimensional space, $x_+, x_-$ and $x_0$ can physically be interpreted as generalized creation, annihilation and number operators, respectively. It is also important to stress that $W_κ$ can be realized as a class of nonlinear oscillator algebras through the so-called deformed structure function [6]. Accordingly, the generators $x_+, x_-$ and $x_0$ can be realized as

$$
x_+ = a_+ \sqrt{I + κa_+a_-}, \quad x_- = \sqrt{I + κa_+a_-} a_-, \quad x_0 = a_+a_- \quad (2)
$$

in terms of the ordinary creation and annihilation operators of the harmonic oscillator algebra. Finally, we mention that $W_κ$ provides a unified scheme for dealing with the planar, spherical and hyperbolic geometries as will be explained in section 2.2.

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2.2. The Hilbertian representation

A Hilbertian representation corresponding to the algebra $W_\kappa$ can be defined as follows. Let us denote by $\mathcal{F} = \{|s, n\}, n = 0, 1, 2, \ldots, d(\kappa)\}$ the Hilbert–Fock space that the generators $x_+, x_-, x_0$ and $I$ act on. The actions of the elements $I$ and $x_0$ are defined by

$$I|s, n\rangle = 2s|s, n\rangle, \quad x_0|s, n\rangle = n|s, n\rangle$$

and for the remaining generators, we have

$$x_+|s, n\rangle = \sqrt{f_s(n + 1)}|s, n + 1\rangle, \quad x_-|s, n\rangle = \sqrt{f_s(n)}|s, n - 1\rangle$$

where the condition $x^-|0\rangle = 0$ is considered. The parameter $s$ is characterizing the $W_\kappa$ irreducible representations and, for simplicity, we assume $2s \in \mathbb{N}^*$ in the forthcoming analysis. Using the commutation relations (1) and actions (3) and (4), one can check that the functions $f_s(n)$ verify the recurrence relation

$$f_s(n + 1) - f_s(n) = 2s + 2\kappa n$$

implemented through the condition $f_s(0) = 0$. A simple iteration of (5) gives

$$f_s(n) = 2sn + \kappa n(n - 1).$$

This structure function must be positive and therefore leads to the condition $2s + \kappa(n - 1) > 0$ for any quantum number $n > 0$. This determines the dimension $d(\kappa)$ of the irreducible representation space $\mathcal{F}$. Indeed, for $\kappa = +1$, $\mathcal{F}$ is infinite dimensional, i.e. $d(\kappa) = +\infty$, and for $\kappa = -1$, the dimension of $\mathcal{F}$ is finite, i.e. $n = 0, 1, 2, \ldots, 2s$. Recall that, for $\kappa = 0$, we have the usual infinite dimensional bosonic Fock space. Beside the Fock representations of the algebra $W_\kappa$, one can construct an analytical realization of the representation space $\mathcal{F}$.

2.3. The Bargmann realization

The Bargmann realization associated with the algebra $W_\kappa$ uses a suitably defined Hilbert space of entire analytical functions. We represent the Hilbert–Fock states $|s, n\rangle$ as powers of the complex variable $z$, such that

$$|s, n\rangle \longrightarrow C_{s,n} z^n.$$

The generator $x_-$ can be realized as a first-order differential operator with respect to $z$. This is

$$x_- \longrightarrow \frac{\partial}{\partial z}.$$  

Using the action of the annihilation operators on $\mathcal{F}$ and the correspondences (7) and (8), we show that the coefficients $C_{s,n}$ ($n > 0$) take the form

$$C_{s,n} = \frac{1}{\sqrt{n!}} \sqrt{(2s + \kappa(n - 1))(2s + \kappa(n - 2)) \cdots (2s + \kappa)2s} C_{s,0}$$

and for convenience we set $C_{s,0} = 1$. It is easy to see that the differential realization of operator $x_0$ is given by

$$x_0 \longrightarrow z \frac{\partial}{\partial z}.$$  

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To achieve the present realization, we give the differential action of the operator $x_+$. Indeed, by using its actions on the Hilbert–Fock space together with the recursion relation (9), we find
\begin{equation}
    x_+ \rightarrow 2sz + \kappa z^2 \frac{\partial}{\partial z}.
\end{equation}

Clearly, the $W_\kappa$ generators act as first-order linear differential operators.

An arbitrary vector $|\psi\rangle = \sum_n \psi_n |s,n\rangle$ of $\mathcal{F}$ can be mapped as
\begin{equation}
    \psi(z) = \sum_n \psi_n C_{s,n} z^n.
\end{equation}

The inner product of two functions $\psi$ and $\psi'$ is defined by
\begin{equation}
    \langle \psi' | \psi \rangle = \int d^2z \Sigma(s, \bar{z} \cdot z) \psi'^* (z) \psi(z).
\end{equation}

The integration measure $\Sigma$, assumed to be isotropic, can be explicitly determined by choosing two functions $|\psi\rangle = |s,n\rangle$ and $|\psi'\rangle = |s,n'\rangle$. A direct calculation shows that it can be cast, in compact form, as
\begin{equation}
    \Sigma(s, \bar{z} \cdot z) = \frac{2s - \kappa}{\pi} (1 - \kappa \bar{z} \cdot z)^{2\kappa - 2}.
\end{equation}

One can write the function $\psi(z)$ as the product of the state $|\psi\rangle$ with some ket $|\bar{z}\rangle$ labeled by the complex conjugates of the variables $z$. This is
\begin{equation}
    \psi(z) = \mathcal{N} \langle \bar{z} | \psi \rangle
\end{equation}

where $\mathcal{N}$ is a normalization constant to be adjusted later. Taking $|\psi\rangle = |s,n\rangle$, we have
\begin{equation}
    \langle \bar{z} | s,n \rangle = \mathcal{N}^{-1} C_{s,n} z^n.
\end{equation}

This leads to ending up with the states
\begin{equation}
    |z\rangle = \mathcal{N}^{-1} \sum_n C_{s,n} z^n |s,n\rangle
\end{equation}

which converge for $\kappa = +1$ when $\bar{z} \cdot z < 1$. Otherwise, the Bargmann space coincides with the unit disc $\mathcal{D} = \{ z \in \mathbb{C}; \bar{z} \cdot z < 1 \}$. The normalization constant is calculated as
\begin{equation}
    \mathcal{N} = (1 - \kappa \bar{z} \cdot z)^{-\kappa s}
\end{equation}

The states (17) are continuous in the labeling and constitute an overcomplete set, such that
\begin{equation}
    \int d\mu (z, \bar{z}) |z\rangle \langle z| = \sum_n |s,n\rangle \langle s,n|
\end{equation}

with respect to the measure
\begin{equation}
    d\mu (z, \bar{z}) = d^2z \mathcal{N}^2 \Sigma(s, \bar{z} \cdot z) = d^2z \frac{2s - \kappa}{\pi} (1 - \kappa \bar{z} \cdot z)^{-2}.
\end{equation}

Therefore, they define an overcomplete set coherent states. The Bargmann realization derived here turns out to be in one to one correspondence with the lowest landau level wavefunctions for a particle evolving in a two-dimensional manifold in the presence of a high strength magnetic field. This will be clarified in the next section by discussing the quantum mechanics on the manifold that we define by using the algebraic structures of the Weyl–Heisenberg algebra introduced above. In fact, we derive the Abelian gauge field encoded in the geometry of the manifold and determine the wavefunctions in terms of Wigner functions, in particular that corresponding to the lowest energy level.
3. Quantum mechanics on coset space $S^2_\kappa$

Using the generalized Weyl–Heisenberg algebra $W_\kappa$, we define the two-dimensional coset space $S^2_\kappa$. We discuss the quantum mechanics of a particle evolving in this space. In quantizing this manifold, we write down the corresponding wavefunctions and determine the associated spectrum.

3.1. The Abelian connection and Wigner functions

It is well established that for any Lie algebra $\mathcal{G}$, one can construct a geometrical manifold endowed with a symplectic structure. This gives a phase space where the classical trajectories are defined. This manifold is isomorphic to the so-called coset space $G/H$ where $G$ is the covering group of the Lie algebra $\mathcal{G}$ and $H$ the maximal stability subgroup of $G$ with respect to a fixed state in the representation space of the Lie algebra $\mathcal{G}$. More precisely, the manifold is obtained by an exponential mapping. In two-dimensional curved space (with constant curvature), the suitable symmetry groups are $SU(1,1)$ and $SU(2)$.

The algebra $W_\kappa$ provides us with a useful way of dealing with these symmetries in a compact manner by defining a one-parameter family of the Lie group $SU_\kappa(1,1)$ whose Lie algebra is generated by the infinitesimal generators $x_+, x_-$ and $2x_3 = \mathbb{I} + 2\kappa x_0$. Indeed, one can check from the commutation relations and the Hilbertian representations given in the previous section that for $\kappa = +1$, $W_\kappa$ reduces to the $su(1,1)$ algebra. However, in the case where $\kappa = -1$, we get the $su(2)$ Lie algebra. The maximal stability group is $U(1)$ and is generated by the operator $x_3$. We shall focus on the algebra $W_{\kappa \neq 0}$. The limiting case $\kappa = 0$ can be simply recovered by a straightforward contraction procedure.

From the above considerations, one can introduce the coset space $S^2_\kappa = SU_\kappa(1,1)/U(1)$ by carrying out a unitary exponential mapping

$$\eta x_+ - \bar{\eta} x_- \longrightarrow \exp(\eta x_+ - \bar{\eta} x_-)$$

(21)

where $\eta$ is a complex variable. The coset space $SU_\kappa(1,1)/U(1)$ is generated by the elements $g$, which are $2 \times 2$ matrices of the fundamental representation of the group $SU_\kappa(1,1)$. They satisfy the relation

$$\det g = 1, \quad \delta g^\dagger \delta = g^{-1}$$

(22)

with $\delta = \text{diag}(1,-\kappa)$. An adequate parametrization of $g$ can be written as

$$g = \left( \begin{array}{cc} \bar{u}_2 & u_1 \\ \kappa \bar{u}_1 & u_2 \end{array} \right)$$

(23)

where $u_1$ and $u_2$ are the global coordinates of $S^2_\kappa$, which can be mapped in terms of the local coordinates as

$$u_1 = \frac{z}{\sqrt{1 - \kappa \bar{z} \cdot z}}, \quad u_2 = \frac{1}{\sqrt{1 - \kappa \bar{z} \cdot z}}$$

(24)

To write down the symplectic structure associated with the manifold $S^2_\kappa$, we introduce the Maurer–Cartan 1-form $g^{-1} dg$. A straightforward calculation gives

$$g^{-1} dg = -it_+ e_+ dz - it_- e_- d\bar{z} - 2i\theta t_3$$

(25)

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where $t_+, t_-$ and $t_3$ are the $SU_\kappa(1,1)$ generators in the fundamental representation. They can be written in terms of the matrices $(e_{ij})_{kl} = \delta_{ik}\delta_{jl}$ of the algebra $gl(2)$ as

$$t_+ = -e_{12}, \quad t_- = \kappa e_{21}, \quad t_3 = \frac{1}{2}(e_{11} - e_{22}).$$  \tag{26}

The orthonormal 1-forms $e_+$ and $e_-$, in (25), read as

$$e_+ = -\frac{i}{1 - \kappa \bar{z} \cdot z}, \quad e_- = \frac{i}{1 - \kappa \bar{z} \cdot z}.$$  \tag{27}

The $U(1)$ symplectic 1-form, i.e. the $U(1)$ connection, is

$$\theta = i \text{Tr} (t_3 g^{-1} dg) = \frac{i}{2} \frac{\bar{z} \cdot d z - z \cdot d \bar{z}}{1 - \kappa \bar{z} \cdot z}$$  \tag{28}

which is the main quantity for completely specifying the geometry of $SU_\kappa(1,1)/U(1)$ and the corresponding symplectic structure. With the above realization, $S_\kappa^2$ is equipped with the Kahler–Bergman metric

$$d\sigma^2 = \frac{1}{(1 - \kappa \bar{z} \cdot z)^2} dz \ d\bar{z}$$  \tag{29}

as well as a symplectic closed 2-form

$$\omega_0 = d\theta = (\omega_0)_{\bar{z},z} \ dz \wedge d\bar{z} = \frac{i}{(1 - \kappa \bar{z} \cdot z)^2} dz \wedge d\bar{z}.$$  \tag{30}

The manifold $S_\kappa^2$ is symplectic and the associated Poisson bracket is given by

$$\{f_1, f_2\} = i (1 - \kappa \bar{z} \cdot z)^2 \left( \frac{\partial f_1}{\partial z} \frac{\partial f_2}{\partial \bar{z}} - \frac{\partial f_1}{\partial \bar{z}} \frac{\partial f_2}{\partial z} \right)$$  \tag{31}

where $f_1$ and $f_2$ are two functions defined on $SU_\kappa(1,1)$. They can be expanded as

$$f(g) = \sum_{n', n} f_{n', n}^s D_{n', n}^s(g)$$  \tag{32}

and the Wigner $D$-functions $D_{n', n}^s(g)$ on $SU_\kappa(1,1)$ are defined by

$$D_{n', n}^s(g) = \langle s, n'|g|s, n \rangle.$$  \tag{33}

Recall that $s$ labels the discrete irreducible representations derived in section 2. These will allow us to define the wavefunctions of a system living in the coset space.

### 3.2. Geometric quantization and the Kahler vacuum

In the context of geometric quantization, the Kahler vacuum is obtained from the wavefunctions (33) by imposing the so-called polarization condition; see equation (43). This condition gives wavefunctions that are holomorphic (up to a normalization factor) and coincide with the LLL. For more details on the relation between the polarization and the projection on the LLL, we refer the reader to for instance [23,24,26,27].

To show that the polarization condition gives the coherent states derived in section 2.3 and then defines the Kahler vacuum, let us introduce the generators of the right $R_a$ and left $L_a$ translations of $g$, namely

$$R_a g = gx_a, \quad L_a g = x_a g$$  \tag{34}

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where \( a \) runs over \( +, -, 3 \). They act on the Wigner \( D \)-functions as
\[
R_a \mathcal{D}_{n',n}^s(g) = \mathcal{D}_{n',n}^s(g x_a), \quad L_a \mathcal{D}_{n',n}^s(g) = \mathcal{D}_{n',n}^s(x_a g).
\]
(35)

To obtain the Kahler vacuum corresponding to \( S^2_{\kappa} \), we should reduce the degrees of freedom on the manifold \( SU_{\kappa}(1,1) \) to the coset space \( SU_{\kappa}(1,1)/U(1) \). This reduction can be formulated in terms of a suitable constraint on the Wigner \( D \)-functions. Thus, we define the magnetic background by
\[
\omega = dA
\]
where the \( U(1) \) gauge field potential is proportional to the 1-form (28), i.e. \( A = k \theta \) with \( k \) the strength of the magnetic background is a real number. A suitable constraint on the Wigner \( D \)-functions can be established by considering the \( U(1) \) gauge transformation. This is
\[
 g \to gh = g \exp(i x_3 \varphi)
\]
(37)
where \( \varphi \) is the \( U(1) \) parameter (37) that leads to the transformation in the gauge field such that
\[
A \to A + k \, d\varphi.
\]
(38)
It follows that the functions (35) transform as
\[
\mathcal{D}_{n',n}^s(gh) = \exp \left( \int \dot{A} \, dt \right) \mathcal{D}_{n',n}^s(g) = \exp \left( \frac{k}{2} \varphi \right) \mathcal{D}_{n',n}^s(g).
\]
(39)
Therefore, the canonical momentum corresponding to the \( U(1) \) direction is \( k/2 \). Then, the admissible states \( \psi \equiv \mathcal{D}_{n',n}^s(g) \) must satisfy the constraint
\[
R_3 \psi = \frac{k}{2} \psi.
\]
(40)
Thus, the physical states constrained by (40) are the Wigner \( D \)-functions \( \mathcal{D}_{n',n}^s(g) \) where the quantum numbers are connected through the relation \( k/2 = s + \kappa n \). To achieve the derivation of the Kahler vacuum (ground state), we use the polarization condition
\[
R_+ \mathcal{D}_{n',n}^s(g) = 0
\]
(41)
which corresponds to \( n = 0 \), i.e. \( s = k/2 \). This shows that the index \( s \), labeling the \( W_{\kappa} \) irreducible representations, is related to the strength of the magnetic field. It is known in geometric quantization that the constraint (40) and the polarization condition (41) define the Kahler vacuum. This corresponds to the functions \( \mathcal{D}_{n',0}^s(g) \):
\[
\psi_{\text{Kahler}} \equiv \mathcal{D}_{n',0}^s(g) = \langle s, n' | g | s, 0 \rangle
\]
(42)
which are holomorphic in the \( z \)-coordinate. More precisely, in the fundamental representation, we can define \( g \) in terms of the generators \( x_{\pm} \) by
\[
g = \exp(\eta x_+ - x_- \bar{\eta})
\]
(43)
where \( \eta \) is related to the local coordinates via
\[
z = \frac{\eta}{|\eta|} \tan \kappa |\eta| = \frac{\eta e^{\sqrt{\kappa} \eta} - e^{-\sqrt{\kappa} \eta}}{\sqrt{\kappa}|\eta| \left( e^{\sqrt{\kappa} \eta} + e^{-\sqrt{\kappa} \eta} \right)}.
\]
(44)
Using (42), we end up with the required functions
\[ \psi_{\text{Kahler}}(\bar{z}, z) = (1 - \kappa \bar{z} \cdot z)^{\kappa(k/2)} C_{k/2,n'} \bar{z}^{n'} \]  
where \( n' = 0, 1, \ldots, 2s \) for \( \kappa = -1 \) and \( n' \in \mathbb{N} \) for \( \kappa = 1 \). The Kahler vacuum is exactly the Bargmann space constructed in section 2. It is finite (respectively infinite) dimensional for \( \kappa = -1 \) (respectively \( \kappa = 1 \)). Note that the Kahler vacuum coincides with the lowest landau levels of a particle evolving in the manifold \( S^2_\kappa \), which will be clarified in section 3.3.

### 3.3. Energy spectrum solutions

Since the manifold \( S^2_\kappa \) is constructed from the algebra \( W_\kappa \), it is natural to consider the operator
\[ H_\kappa = \frac{1}{2}(R_+ R_- + R_- R_+) \]  
which generalizes the harmonic oscillator Hamiltonian. To establish a relation between the Casimir operator and the Hamiltonian, we may write the eigenvalue equation as
\[ H_\kappa \psi = \frac{1}{2} (x_- x_+ + x_+ x_-) \psi = E \psi. \]

Since the wavefunctions \( \psi \) are the Wigner \( D \)-functions \( D_{n',n}(g) \) satisfying the constraint \( k/2 = s + \kappa n \), the energies are given by
\[ E^n = \frac{k^2}{4} - C_2 \]
where the second-order Casimir operator is
\[ C_2 = x_3^2 - \frac{\kappa}{2} (x_- x_+ + x_+ x_-). \]

It is not difficult to check that the eigenvalues of \( C_2 \) are of the form \( s(s - \kappa) \). Consequently, one can see that those corresponding to (46) are given by
\[ E := E^n = \frac{k}{2} (2n + 1) - \kappa n (n + 1). \]

At this stage, some remarks are in order. Indeed, for \( \kappa = \pm 1 \), we recover the eigenvalues for one particle living in the disc and sphere geometries, respectively. Finally for \( \kappa = 0 \), we have the Landau spectrum on two-dimensional Euclidean space. It is interesting to note also that the Euclidean case can be obtained from (50) for \( k \) large. On the other hand, we notice here that the lowest level energy corresponds to \( n = 0 \). It is \( 2s + 1 \)-fold generated for \( \kappa = -1 \) and infinitely degenerated for \( \kappa = 1 \). This is exactly the Kahler vacuum discussed above.

It is clear from (50) that, for a large \( k \), the spectrum can be rewritten as
\[ E_n^\kappa \sim \frac{k}{2} (2n + 1) \]
which is independent of \( \kappa \), and also the gap between two successive levels, proportional to \( k \), is large. In this situation, the particles are constrained to be accommodated in the lowest energy level. Since it is degenerate, one may fill it with a large number of particles \( N \) such that the density operator is
\[ \rho_0 = \sum_{n=0}^{N} |s, n\rangle \langle s, n|. \]
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Note that the quadratic Hamiltonian \( H_\kappa \) involves only the right translation. An admissible form for \( H_\kappa \) should also be expressed in terms of left translations. Since the right and left generators commute, the operators that lift the degeneracies of the energy levels are functions of the left translations \( L_a \). Since we are interested in studying the excited states, we introduce an excitation potential \( V \) that induces a lifting of the LLL degeneracy [23, 26]. In this respect, it is natural to assume that \( V \) is a function of the left translations \( L_3, L_+ \) and \( L_- \). A simple choice for \( V \), in terms of \( L_3 \), is given by

\[
V = \kappa \left( L_3 - \frac{k}{2} \right).
\]

Then, for \( k \) large, the Hamiltonian governing the dynamics of the system is now

\[
H = E_0^\kappa + V.
\]

The corresponding eigenvalues are

\[
H \psi_{\text{Kahler}} \equiv HD_{n',0}(g) = \left( \frac{k}{2} + n' \right) D_{n',0}(g).
\]

This shows that we have a lifting of the degeneracy.

We close this section by noting that the lowest Landau wavefunctions constitute an overcomplete set. One of the benefits of this property (see section 2) is that it provides us with a simple way to establish a correspondence between operators and classical functions on the phase space of the present systems. In section 4, we develop a strategy similar to that adopted in [23, 24, 26, 27] in order to perform semiclassical calculations, which are valid for high magnetic field.

4. Semiclassical analysis and vacuum excitation

The dynamical description of a large collection of particles confined in the lowest Landau levels can be achieved semiclassically. Some tools are needed in this sense. These concern the star product, the density function and the excitation potential.

4.1. The star product

An important ingredient for performing the semiclassical analysis in the Bargmann space is the star product. In fact, as we will discuss next, for \( s \) large the mean value of the product of two operators leads to the Moyal star product. To show this, with every operator \( O \) acting on the Fock space \( F \) we associate the function

\[
O(\bar{z}, z) = \langle z | O | z \rangle.
\]

An associative star product of two functions \( O_1(\bar{z}, z) \) and \( O_2(\bar{z}, z) \) is defined by

\[
O_1(\bar{z}, z) \star O_2(\bar{z}, z) = \langle z | O_1 O_2 | z \rangle = \int d\mu (\bar{z}', z') \langle z | O_1 | z' \rangle \langle z' | O_2 | z \rangle
\]

where the measure \( d\mu (\bar{z}, z) = d^2z N^2 \Sigma \); see (20). To calculate this, let us exploit the analytical properties of coherent states defined above. Indeed, using (17), one can see that the function defined by

\[
O(\bar{z}', z) = \frac{\langle \bar{z}' | O | z \rangle}{\langle \bar{z}' | z \rangle}
\]

is
satisfies the holomorphic and anti-holomorphic conditions
\[
\frac{\partial}{\partial \bar{z}} \mathcal{O}(\bar{z}', \bar{z}) = 0, \quad \frac{\partial}{\partial z} \mathcal{O}(\bar{z}', \bar{z}) = 0
\]  
when \( z \neq z' \). Consequently, the action of the translation operator on the function \( \mathcal{O}(\bar{z}', \bar{z}) \) gives
\[
\exp \left( z' \frac{\partial}{\partial \bar{z}} \right) \mathcal{O}(\bar{z}', \bar{z}) = \mathcal{O}(\bar{z}', \bar{z} + z').
\]  
This can be used to determine \( \mathcal{O}(\bar{z}, \bar{z}') \) in terms of \( \mathcal{O}(\bar{z}, \bar{z}) \). Indeed, we have
\[
\exp \left( - \bar{z} \frac{\partial}{\partial \bar{z}'} \right) \mathcal{O}(\bar{z}, \bar{z}) = \exp \left( (\bar{z} - z) \frac{\partial}{\partial \bar{z}} \right) \mathcal{O}(\bar{z}, \bar{z}) = \mathcal{O}(\bar{z}, \bar{z}').
\]  
Similarly, one obtains
\[
\exp \left( - \bar{z} \frac{\partial}{\partial \bar{z}'} \right) \mathcal{O}(\bar{z}, \bar{z}) = \mathcal{O}(\bar{z}', \bar{z}).
\]  
Equivalently, (61) and (62) can also be cast in the forms
\[
\exp \left( (\bar{z}' - z) \frac{\partial}{\partial \bar{z}} \right) \mathcal{O}(\bar{z}, \bar{z}) = \mathcal{O}(\bar{z}, \bar{z}')
\]  
as well as
\[
\exp \left( (\bar{z}' - \bar{z}) \frac{\partial}{\partial \bar{z}} \right) \mathcal{O}(\bar{z}, \bar{z}) = \mathcal{O}(\bar{z}', \bar{z}).
\]  
Combining (56)–(58) and (63) and (64), the star product can be rewritten as
\[
\mathcal{O}_1(\bar{z}, \bar{z}) \star \mathcal{O}_2(\bar{z}, \bar{z}) = \int d\mu(\bar{z}', \bar{z}') \exp \left( (\bar{z}' - z) \frac{\partial}{\partial \bar{z}} \right) \mathcal{O}_1(\bar{z}, \bar{z}) |\langle z | z' \rangle|^2 \times \exp \left( (\bar{z}' - \bar{z}) \frac{\partial}{\partial \bar{z}} \right) \mathcal{O}_2(\bar{z}, \bar{z})
\]  
where the overlapping of coherent states is given by
\[
\langle z' | z \rangle = \left[ (1 - \kappa \bar{z}' \cdot \bar{z}')(1 - \kappa \bar{z} \cdot \bar{z}') \right]^{\kappa s}.
\]  
Clearly, the modulus of the kernel (66) possesses the properties \(|\langle z | z' \rangle| = 1\) if and only if \( z = z' \), \(|\langle z | z' \rangle| < 1\) and \(|\langle z | z' \rangle| \to 0\) for \( k \) large (\( k = 2s \)). These are helpful for getting the star product between two functions on the Bargmann space. In this respect, we introduce a function \( \sigma(z', z) \) of the coordinates of two points on the Bargmann space.
\[
\sigma^2(z', z) = - \ln |\langle z | z' \rangle|^2 = 2\kappa s \ln \left( \frac{(1 - \kappa \bar{z}' \cdot \bar{z})(1 - \kappa \bar{z} \cdot \bar{z}')}{(1 - \kappa \bar{z} \cdot \bar{z})(1 - \kappa \bar{z}' \cdot \bar{z}')} \right).
\]  
It verifies the properties \( \sigma(z', z) = \sigma(z, z') \) and \( \sigma(z', z) = 0 \) if and only if \( z' = z \). This function defines the distance between two points \( z \) and \( z' \) on the Bargmann space. Indeed, one can verify that the line element \( d\sigma^2 \), defined as the quadratic part of the decomposition of \( \sigma^2(z, z + dz) \), gives (29) up to a multiplicative factor.
We now come to the evaluation of the star product for $k$ large. Since $\sigma^2(z',z)$ tends to infinity with $k \to \infty$ if $z \neq z'$ and equals zero if $z = z'$, one can conclude that, in that limit, the domain $z \simeq z'$ gives only a contribution to the integral (65). Decomposing the integrand near the point $z \simeq z'$ and going to integration over $\eta = z' - z$, one gets

$$O_1(\bar{z},z) * O_2(\bar{z},z) = \int \frac{d\eta d\bar{\eta}}{\pi} \exp \left( \eta \frac{\partial}{\partial z} \right) O_1(\bar{z},z) \exp \left( -i \omega_\eta \eta \bar{\eta} \right) \exp \left( \bar{\eta} \frac{\partial}{\partial \bar{z}} \right) O_2(\bar{z},z). \tag{68}$$

It follows that the star product between two functions on the Bargmann space can be written as

$$O_1(\bar{z},z) * O_2(\bar{z},z) = O_1(\bar{z},z) O_2(\bar{z},z) - \frac{1}{k} (1 - \kappa \bar{z} \cdot z)^2 \frac{\partial O_1(\bar{z},z)}{\partial \bar{z}} \frac{\partial O_2(\bar{z},z)}{\partial \bar{z}} + O \left( \frac{1}{k^2} \right). \tag{69}$$

Then, the symbol or function associated with the commutator of two operators $O_1$ and $O_2$ is given by

$$\langle z | [O_1, O_2] | z \rangle = \{ O_1(\bar{z},z), O_2(\bar{z},z) \}_* = -\frac{1}{k} (1 - \kappa \bar{z} \cdot z)^2 \left( \frac{\partial O_1(\bar{z},z)}{\partial \bar{z}} \frac{\partial O_2(\bar{z},z)}{\partial \bar{z}} - \frac{\partial O_2(\bar{z},z)}{\partial \bar{z}} \frac{\partial O_1(\bar{z},z)}{\partial \bar{z}} \right) \tag{70}$$

where the quantity

$$\{ O_1(\bar{z},z), O_2(\bar{z},z) \}_* = O_1(\bar{z},z) * O_2(\bar{z},z) - O_2(\bar{z},z) * O_1(\bar{z},z) \tag{71}$$

is the so-called Moyal bracket.

### 4.2. The density matrix and the excitation potential

The symbol function corresponding to the density operator (52) is given by

$$\rho_0(\bar{z},z) = \langle z | \rho_0 | z \rangle = \left( 1 - \kappa \bar{z} \cdot z \right)^N \sum_{n=0}^{N} C_{n,k/2}^2 (\bar{z} \cdot z)^n. \tag{72}$$

It is a simple matter of approximations to see that for $k$ and $N$ large, the classical density (72) behaves like a circular droplet in the Bargmann space. Indeed, one can see that (9), for $k$ large, it reduces to the form

$$C_{n,(k/2)} \sim \frac{k^{n/2}}{\sqrt{n!}}. \tag{73}$$

This leads to the density

$$\rho_0(\bar{z},z) \sim \exp(-k \bar{z} \cdot z) \sum_{n=0}^{N} \frac{(k \bar{z} \cdot z)^n}{n!}. \tag{74}$$

It can be approximated by a step function, such that

$$\rho_0(\bar{z},z) = \Theta(N - k \bar{z} \cdot z), \tag{75}$$

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which corresponds to a circular configuration in the Bargmann space with radius proportional to $\sqrt{N}$. The particles are confined in the interior of the disc $\{z \in \mathbb{C}, k \bar{z} \cdot z \leq N\}$.

The symbol associated with the potential (53) is given by

$$V(\bar{z}, z) = \langle z | V | z \rangle = k \frac{\bar{z} \cdot z}{1 - \kappa \bar{z} \cdot z}$$

which behaves like the harmonic oscillator potential for a strong magnetic field. The star product (69), Moyal bracket (70), density function (75) and excitation potential (76) are the main quantities needed in the derivation of the action describing the electromagnetic excitation of two-dimensional Hall droplets that we derive in section 6.

It is clear that, for $k$ large, a large collection of $N$ particles behaves like Hall droplets of radius $N/k$. The Bargmann space (generated by the LLL wavefunctions) is the phase space of the system. It is equipped with a symplectic 2-form $\omega$ (36). The dynamics of the system is described by the potential $V$ with $\omega$. To describe the electromagnetic excitations of Hall droplets, we now consider the addition of a $U(1)$ magnetic field described by a gauge potential $A$ such that the total potential becomes $a + A$. The new dynamics is then given by the symplectic form $\omega + F$. This induces noncommutative structures in the Bargmann space, which will be discussed in section 5.

5. Noncommutative dynamics in Bargmann space

At this stage, we show how the electromagnetic excitation can be introduced from a purely symplectic scheme and obtain the dynamics of the system by using (75) and (76). Recall that it is well known, in symplectic mechanics, that the coupling of a charged particle with an electromagnetic field can be described in a Hamiltonian formalism without a choice of a gauge potential. This can be achieved through a modification of the symplectic form. With this, the Poisson brackets become deformed leading to noncommutative structure, like for instance the noncommutative positions in the Landau problem. Such an approach dealing with modified structure has been previously considered in connection with the quantum Hall effect [9, 10]. It is also important to stress that due to Moser’s lemma [12], there was developed in [30, 31] (see also [32]) a noncommutative gauge theory in curved spaces which is essentially related to the procedure of symplectic deformations. In this section, we follow arguments similar to those in [30, 31] in order to study the symplectic deformation of Bargmann space.

5.1. Deformed symplectic structure

First, it is more convenient for our purposes to rewrite the symplectic 2-form (36)

$$\omega = \frac{1}{2} \omega_{ij}(\xi) \, d\xi^i \wedge d\xi^j. \quad (77)$$

in terms of the real coordinates $\xi^1$ and $\xi^2$, with $z = (\xi^1 + i\xi^2)/\sqrt{2}$. To introduce the effect of a weak external electromagnetic interaction, let us consider the modified symplectic 2-form. This is

$$\omega + F = \frac{1}{2} \omega_{ij}(\xi) \, d\xi^i \wedge d\xi^j + \frac{1}{2} F_{ij}(\xi) \, d\xi^i \wedge d\xi^j, \quad i, j = 1, 2 \quad (78)$$

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where the closed electromagnetic tensor field is defined by
\[ F = da, \quad a = a_1(\xi^1, \xi^2) d\xi^1 + a_2(\xi^1, \xi^2) d\xi^2. \tag{79} \]
According to this symplectic deformation, the vector fields
\[ Y_{\mathcal{F}} = Y_{\xi^i}^j \frac{\partial}{\partial \xi^i} \tag{80} \]
associated with a given function \( \mathcal{F}(\xi^1, \xi^2) \) are such that the interior contraction of \( \omega + F \) with \( Y_{\mathcal{F}} \) gives
\[ \iota_{Y_{\mathcal{F}}} (\omega + F) = d\mathcal{F}. \tag{81} \]
After straightforward calculation, we obtain
\[ Y_{\xi^i}^j = (\omega + F)^{-1} \partial_{\xi^i} \mathcal{F} \tag{82} \]
where the 2-form \( \omega + F \) is supposed invertible. The Poisson bracket corresponding to this new electromagnetic background is now given by
\[ \{ \mathcal{F}, \mathcal{G} \} = \iota_{Y_{\mathcal{F}}} \iota_{Y_{\mathcal{G}}} (\omega + F) = (\omega + F)^{-1} \partial_{\xi^i} \mathcal{F} \partial_{\xi^j} \mathcal{G} \tag{83} \]
In particular, we obtain
\[ \{ \xi^1, \xi^2 \} = (\omega + F)^{-1} \tag{84} \]
which, for a weak electromagnetic field \( F \), can also be written as
\[ \{ \xi^1, \xi^2 \} = (\omega^{-1})^{12} - (\omega^{-1})^{12} F_{21}(\omega^{-1})^{12} + \frac{1}{2} (\omega^{-1})^{12} F_{21}(\omega^{-1})^{12} F_{21}(\omega^{-1})^{12} + \cdots \tag{85} \]
where \((\omega^{-1})^{ij}\) stands for the matrix elements of the inverse of \( \omega \). The last two terms in (85) encode the effect of the external magnetic excitation and indicate the deformation of the symplectic structure of the Bargmann space. The equations of motion, governing the dynamics of the system, read now
\[ (\omega + F)^{ij} \frac{d\xi^j}{dt} = \frac{\partial \mathcal{H}}{\partial \xi^i}. \tag{86} \]
where \( \mathcal{H} = k/2 + \mathcal{V} \) and \( \mathcal{V} \) is given in (76).

### 5.2. Symplectic dressing and Moser’s lemma

Moser’s lemma [12] (see also [30]) is essentially a refined version of the Darboux theorem according to which there always exists a coordinate transformation for eliminating the electromagnetic force. In fact, it provides a nice way to eliminate the fluctuations induced by the electromagnetic field and is realized by performing the dressing transformation \( \omega + F \rightarrow \omega \). Thus, we have to find a diffeomorphism on the phase space \( f \) relating \( \omega \) and \( \omega + F \) such as
\[ f^*(\omega + F) = \omega \tag{87} \]
where \( f^* \) is the pullback that maps \( \omega + F \) into \( \omega \); more details can be found in [30]. In this respect, Moser’s lemma [12] constitutes the appropriate tool and provides us with an

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elegant way to relate $\omega$ and $\omega + F$. To make this explicit, we start by considering a family of symplectic forms

$$\omega(t) = \omega + tF$$

(88)

interpolating $\omega$ and $\omega + F$ for $t = 0$ and 1, respectively, with $t \in [0, 1]$. We also construct a family of diffeomorphisms

$$f^*(t) \omega(t) = \omega$$

(89)

satisfying $f^*(t = 0) = id$ and the diffeomorphism $f^*(t = 1)$ will then be the required solution of our problem. In order to find $f^*(t)$, we introduce a $t$-dependent vector field $X(t)$ generating $f(t)$ as its flow. Differentiating the last equation, such that $X(t)$ has to satisfy the identity

$$0 = \frac{d}{dt} [f^*(t) \omega(t)] = f^*(t) \left[ L_{X(t)} \omega(t) + \frac{d}{dt} \omega(t) \right]$$

(90)

where $L_{X(t)}$ denotes the Lie derivative of $X(t)$, using the Cartan identity $L_X = \iota_X \circ d + d \circ \iota_X$ and the fact that $d \omega(t) = 0$, we have

$$f^*(t) \left[ \iota_{X(t)} \omega(t) \right] + F = 0.$$  

(91)

Since $F = da$, it follows that $X(t)$ has to satisfy the linear equation

$$\iota_{X(t)} \omega(t) + a = 0$$

(92)

which solves (91). Is is easy to see that the components of $X(t)$ are given by

$$X^i(t) = -a_j (\omega^{-1})^{ji}(t).$$

(93)

As we are dealing with weak electromagnetic perturbation, i.e. $F \ll \omega$, the matrix element in (93) can be obtained from

$$(\omega^{-1})(t) = \omega^{-1} - t(\omega^{-1})F(\omega^{-1}) + t^2(\omega^{-1})F(\omega^{-1})F(\omega^{-1}) + \cdots$$

(94)

The t-evolution of $\omega(t)$ is governed by the first-order differential equation

$$[\partial_t + X(t)] \omega(t) = 0.$$  

(95)

Thus, 2-forms $\omega(t + 1)$ and $\omega(t)$ are related by

$$[\exp(\partial_t + X(t)) \exp(-\partial_t)] \omega(t + 1) = \omega(t).$$  

(96)

Finally, from the last equation, it is easy to see that

$$[\exp(\partial_t + X(t)) \exp(-\partial_t)](t = 0)(\omega + F) = f^*(\omega + F) = \omega$$

(97)

where the diffeomorphism ensuring the dressing transformation is then given by

$$f^* = id + X(0) + \frac{1}{2} (\partial_t X)(0) + \frac{1}{2} X^2(0) + O[(\omega^{-1})^3].$$

(98)

More explicitly, using (93) and (94), one obtains the contribution of the second term in (98) as

$$X(0) = X^i(0) \partial_i = (\omega^{-1})^{ij} a_j \partial_i$$

(99)
which is $\omega^{-1}$-first order. The contribution of the third term in (98) is

$$\frac{1}{2}(\partial_t X)(0) = -\frac{1}{2} [(\omega^{-1}) F(\omega^{-1})]^{ij} a_j \partial_i.$$  \hspace{1cm} (100)

Similarly, the last term in (98) is evaluated to get

$$\frac{1}{2}X^2(0) = \frac{1}{2}[(\omega^{-1})^{ij}a_j \partial_l][(\omega^{-1})^{lj'}a_j \partial_l].$$ \hspace{1cm} (101)

Reporting the contributions (99)--(101) in (98), one can show that the diffeomorphism $f$ transforms the variables $\xi^l$ ($l = 1, 2, \ldots$) as

$$f(\xi^l) = \xi^l + \xi_1^l + \xi_2^l$$ \hspace{1cm} (102)

where $\xi_1^l$ is given by

$$\xi_1^l = (\omega^{-1})^{ij} a_j$$ \hspace{1cm} (103)

and the second one reads as

$$\xi_2^l = -\frac{1}{2}(\omega^{-1})^{lk} F_{kl'}(\omega^{-1})^{lj'} a_j + \frac{1}{2}(\omega^{-1})^{ij} a_j [(\partial_l (\omega^{-1})^{lj}) a_{j'} + \frac{1}{2}(\omega^{-1})^{ij} a_j (\omega^{-1})^{lj'} (\partial_l a_{j'})].$$  \hspace{1cm} (104)

Using the relations

$$\partial_{j'}a_{j'} = (\partial_{j'} \omega_{l'k'}) \xi_{1}^{k'} + \omega_{l'i} (\partial_{j'} \xi_{1}^{i})$$ \hspace{1cm} (105)

$$\partial_{j'}(\omega^{-1})^{lj'} = -(\omega^{-1})^{lj'} (\partial_{l'} \omega_{j'k'}) (\omega^{-1})^{k'j'}$$ \hspace{1cm} (106)

and the antisymmetry property of the symplectic form, one can verify

$$\xi_2^l = -(\omega^{-1})^{lk} F_{kl'}(\omega^{-1})^{mj} a_m + \frac{1}{2}(\omega^{-1})^{lk} (\omega^{-1})^{mj} a_{j'} a_{j'} \partial_m \omega_{nk}$$

$$+ \frac{1}{2}(\omega^{-1})^{lk}(\omega^{-1})^{ms} a_{s} \omega_{mn} \partial_k [(\omega^{-1})^{ns'} a_{s'}].$$ \hspace{1cm} (107)

It is clear that Moser’s lemma is very interesting in the symplectic dressing procedure and provides us with a way to eliminate any fluctuation of the electromagnetic field strength by a simple coordinate redefinition.

### 5.3. The Seiberg–Witten map

It is interesting to note that the dressing transformation based on Moser’s lemma is behind the Seiberg–Witten map. Indeed, one can see from (102) that this transformation can be written as

$$f(\xi^l) = \xi^l + \dot{\alpha}^l$$ \hspace{1cm} (108)

where

$$\dot{\alpha}^l = (\omega^{-1})^{lk} a_k - F_{kl'} (\omega^{-1})^{lj} a_m + \frac{1}{2} (\omega^{-1})^{mj} a_{j'} a_{j'} \partial_m \omega_{nk}$$

$$+ \frac{1}{2} (\omega^{-1})^{ms} a_{s} \omega_{mn} \partial_k [(\omega^{-1})^{ns'} a_{s'}].$$ \hspace{1cm} (109)

The relation (108) is similar to the so-called Susskind map introduced in the context of noncommutative Chern–Simons theory in relation to the fractional quantum Hall effect [14]. It encodes the geometrical fluctuations induced by the external magnetic field $F$. More importantly, (108) coincides with the Seiberg–Witten map in a curved manifold for an Abelian gauge theory [30].

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Under the gauge transformation
\[ a \rightarrow a + d\lambda \]  
the components (109) transform as the noncommutative Abelian gauge field
\[ \hat{a}^l \rightarrow \hat{a}^l + (\omega^{-1})^{lj}\partial_j \hat{\lambda} + \{\hat{a}^l, \hat{\lambda}\} + \cdots \]  
where the noncommutative gauge parameter \( \hat{\lambda} \)
\[ \hat{\lambda} = \lambda + \frac{1}{2}(\omega^{-1})^{ij}a_j \partial_i \lambda + \cdots \]  
is written in terms of the gauge parameter \( \lambda \) and the \( U(1) \) connection \( a \). It is clear that the dressing transformation, using Moser’s lemma, induces a noncommutative gauge field. This establishes a nice correspondence between the symplectic deformations and noncommutative gauge theories.

5.4. The Hamiltonian and the induced Chern–Simons term

As discussed above, we are interested in the droplets (75) on the manifold \( S^2_\kappa \) equipped with the symplectic 2-form \( \omega \). The lowest energy levels are described by the quantization of Bargmann space with this symplectic form. In the situation where the system evolves under the action of an external electromagnetic interaction \( F = da \), the dynamics of the system becomes governed by \( \omega + F \) together with the Hamiltonian \( \mathcal{H} + a_0 \), where \( \mathcal{H} \) is the symbol of \( H \) given by (54) and \( a_0 \) is the time component of the gauge potential \( a \). In this case, a complete description of the dynamics is encoded in the couple \( (\omega + F, \mathcal{H} + a_0) \) and the equations of motion are given by (86). Moser’s lemma provides us with a nice tool for incorporating the external interaction in the Hamiltonian, leaving the symplectic form \( \omega \) unchanged. Indeed, the dressing transformation (102) allows us to describe the dynamics of the system with the couple \( (\omega, \mathcal{H} + a_0) \) where we use the old symplectic form but a new Hamiltonian. This gives
\[ \mathcal{H}_a(\xi^1, \xi^2) = (\mathcal{H} + a_0) \left[ f(\xi^1), f(\xi^2) \right]. \]  
This can be expanded, up to second order, as
\[ \mathcal{H}_a(\xi^1, \xi^2) = \mathcal{H} + a_0 + \xi^i \partial_i (\mathcal{H} + a_0) + \frac{1}{2} \xi^i \xi^j \partial_i \partial_j (\mathcal{H} + a_0) + \xi^i \partial_i (\mathcal{H} + a_0). \]  
The dynamics of the system is now governed by the equations of motion
\[ (\omega^{-1})^{ij} \frac{d\xi^j}{dt} = \frac{\partial \mathcal{H}_a}{\partial \xi^i}. \]  
This shows that the dressing transformation, based on Moser’s lemma, eliminates the fluctuations in the symplectic structure and incorporates the electromagnetic interaction effect in the Hamiltonian. In other words, this means that the dynamics, governed by \( \omega + F \) and \( \mathcal{H} \), can be described by the old (non-deformed) symplectic 2-form \( \omega \) with a new Hamiltonian \( \mathcal{H}_a \) expressed in terms of the electromagnetic field \( F \).

Injecting (103) and (107) into (114) and using the expression of the excitation potential (76), one shows that the new Hamiltonian takes the form
\[ \mathcal{H}_a = \mathcal{H} + a_0 + e^{ij}a_i \xi_j + \frac{1}{2}(\omega^{-1})^{ij} \partial_i \left[ \left( a_0 + e^{kl}a_i \xi_k \right) a_j \right] + \frac{1}{2}(\omega^{-1})^{ij} \left[ a_i \partial_0 a_j - 2a_i \partial_j a_0 - \partial_i (a_0 a_j) \right]. \]
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where we added \( \frac{1}{2}(\omega^{-1})^{ij} a_i \partial_0 a_j \) to ensure the gauge invariance under the change \( a_0 \rightarrow a_0 + \partial_0 \lambda \). Using the expressions of \( \omega \), one can rewrite the expression (116) as

\[
\mathcal{H}_a = \mathcal{H} + a_0 + \epsilon^{ij} a_i \xi_j - \frac{1}{2 \sqrt{\text{det} \omega}} \left\{ \epsilon^{ij} \partial_i \left[ (a_0 + \epsilon^{kl} a_l \xi_k) a_j \right] - \epsilon^{\alpha \beta \gamma} a_\alpha \partial_\beta a_\gamma \right\}
\]

where \( \alpha, \beta, \gamma = 0, 1, 2 \). This form is more suggestive because it involves a Chern–Simons term, i.e. the last contribution in (117).

6. Electromagnetic excitations of a quantum Hall droplet

The analysis developed in the previous section can be applied to derive the electromagnetic interaction of edge excitations of a two-dimensional Hall system.

6.1. The effective WZW action

Once we have determined the spectrum of the lowest Landau levels where the quantum Hall droplet is specified by the density matrix \( \rho_0 \), one may ask about the excited states. The answer can be given by describing the excitations in terms of a unitary time evolution operator \( U \). It contains information concerning the dynamics of the excitations around \( \rho_0 \). Therefore the excited states will be characterized by a density operator given by

\[
\rho = U \rho_0 U^\dagger.
\]

This is basically corresponding to a perturbation of the quantum system.

Thus, the dynamical information, related to the degrees of freedom of the edge states, is contained in the unitary operator \( U \). The action, describing these excitations, in the Hartree–Fock approximation, can be written as [33]

\[
S = \int dt \text{Tr} \left( i \rho_0 U^\dagger \partial_t U - \rho_0 U^\dagger H_a U \right)
\]

where \( H_a \) is the operator associated with the Hamiltonian function given by (117). For a strong magnetic field, i.e. large \( k \), the different quantities occurring in the action can be evaluated as classical functions. To do this, we adopt a method similar to that used in [23]. This is mainly based on the strategy elaborated by Sakita [33] in dealing with a bosonized theory for fermions.

To determine the effective action, we start by calculating the kinetic contribution, i.e. the first term in the rhs of (119). This can be done by setting

\[
U = e^{+i\Phi}, \quad \Phi^\dagger = \Phi.
\]

Using the definition of the star product (57), it is a matter of computation to see that

\[
i \int dt \text{Tr} \left( \rho_0 U^\dagger \partial_t U \right) \simeq \frac{1}{2k} \int d\mu (\bar{z}, z) \{ \phi, \rho_0 \} \partial_t \phi
\]

where we have dropped the terms in \( 1/k^2 \) as well as the total time derivative. In (121), \( \phi \) stands for the classical function associated with the operator \( \Phi \).

The Poisson bracket can be calculated to get

\[
\{ \phi, \rho_0 \} = (\mathcal{L} \phi) \frac{\partial \rho_0}{\partial (\bar{z} \cdot z)}
\]

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and the first-order differential operator $\mathcal{L}$ is given by

$$\mathcal{L} = i (1 - \kappa \bar{z} \cdot z)^2 \left( z \cdot \frac{\partial}{\partial z} - \bar{z} \cdot \frac{\partial}{\partial \bar{z}} \right). \quad (123)$$

This is the angular momentum mapped in terms of the local coordinates of $S^2_\kappa$. Recall that, for large $k$, the density (75) is a step function. Its derivative is a $\delta$-function with a support on the boundary $\partial D = S^1$ of the quantum Hall droplet $D$. By setting $z = re^{i\alpha}$, we show that (123) reduces to $\mathcal{L} = \partial_\alpha$ for large $k$. Therefore, (121) takes the form

$$i \int dt \, \text{Tr} \left( \rho_0 U^\dagger \frac{\partial}{\partial t} U \right) \approx -\frac{1}{2} \int_{S^1 \times \mathbb{R}^+} dt \, (\partial_\alpha \phi) \left( \partial_t \phi \right). \quad (124)$$

Now we come to the semiclassical evaluation of the potential energy term, i.e. the second term in the rhs of (119). By a straightforward calculation, we find

$$\text{Tr} \left( \rho_0 U^\dagger H_a U \right) = \text{Tr} \left( \rho_0 H_a \right) + i \text{Tr} \left( [\rho_0, H_a] \Phi \right) + \frac{1}{2} \text{Tr} \left( [\rho_0, \Phi] [H_a, \Phi] \right). \quad (125)$$

The term in the rhs of (125) is independent of $\Phi$. It is given by

$$\text{Tr} \left( \rho_0 H_a \right) = \int d\mu (\bar{z}, z) \rho_0 \ast H_a. \quad (126)$$

This term gives no contribution when $a = 0$. The second term in (125) can be written in terms of the Moyal bracket as

$$i \text{Tr} \left( [\rho_0, H_a] \Phi \right) \approx -\frac{1}{k} \int d\mu (\bar{z}, z) (\mathcal{L} \phi) \frac{\partial \rho_0}{\partial r} H_a. \quad (127)$$

This term gives no contribution in the absence of electromagnetic interaction. The last term in the rhs of (125) can be evaluated in a similar way to get (127). One obtains

$$\frac{1}{2} \text{Tr} \left( [\rho_0, \Phi] [H_a, \Phi] \right) = -\frac{1}{2k^2} \int d\mu (\bar{z}, z) \left( \mathcal{L} \phi \right) \frac{\partial \rho_0}{\partial r} \left( \mathcal{L} \phi \right) \frac{\partial V}{\partial r}. \quad (128)$$

Note that we have eliminated a term containing the ground state energy $E_0^\kappa$, because it does not contribute to the edge dynamics. Also we ignored in the last equation the contributions coming from the electromagnetic interaction.

The addition of the contributions (121) and (128), which are independent of $a$, gives

$$S_0 = \frac{1}{2k} \int dt \, d\mu (\bar{z}, z) \left( \mathcal{L} \phi \right) \left( \partial_t \phi \right) + \frac{1}{k} \frac{\partial V}{\partial r} \left( \mathcal{L} \phi \right)^2. \quad (129)$$

This is the Wess–Zumino–Witten action describing the edge excitations of quantum Hall droplets in two-dimensional space [33, 34].

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6.2. The total action

The total action is then given by

\[ S = S_0 + S_a \]  \tag{130} \]

where the \( S_a \) part

\[ S_a = - \int dt \, d\mu \left( \bar{z}, z \right) \left[ \rho_0 \star \mathcal{H}_a - \frac{1}{k}(\mathcal{L}_\phi) \frac{\partial \rho_0}{\partial r} \mathcal{H}_a \right] \]  \tag{131} \]

is the sum of (126) and (127) containing the effect of the electromagnetic interaction. More precisely, it is composed of the edge and bulk contributions, such that

\[ S_a = S_a^{\text{bulk}} + S_a^{\text{edge}} \]  \tag{132} \]

where the bulk action

\[ S_a^{\text{bulk}} = - \int dt \, d\mu \left( \bar{z}, z \right) \rho_0 \star \left[ \mathcal{H}_a - \frac{1}{2} (\omega^{-1})^{ij} \partial_i \left( (a_0 + \epsilon^{kl} a_l \xi_k) a_j \right) \right] \]  \tag{133} \]

contains a Chern–Simons action. The edge contribution, arising from the electromagnetic excitations, reads as

\[ S_a^{\text{edge}} = - \int dt \, d\mu \left( \bar{z}, z \right) \frac{1}{2} \rho_0 \star \left( (\omega^{-1})^{ij} \partial_i \left( (a_0 + \epsilon^{kl} a_l \xi_k) a_j \right) \right) - \frac{1}{k}(\mathcal{L}_\phi) \frac{\partial \rho_0}{\partial r} \mathcal{H}_a . \]  \tag{134} \]

It is clear that the second term in the above action is a boundary contribution, i.e. the derivative of the density gives a delta function defined on the edge of the quantum Hall droplet. It is also easy to show that the first term in (134) is an edge contribution. Indeed, using the expression for the measure (20) and the inverse matrix elements of the 2-form \( \omega \), one can verify

\[ \int dt \, d\mu \left( \bar{z}, z \right) \frac{1}{2} \rho_0 \star \left( (\omega^{-1})^{ij} \partial_i \left( (a_0 + \epsilon^{kl} a_l \xi_k) a_j \right) \right) \]

\[ \sim \int dt \, d^2 \xi \, \epsilon^{ij} \xi a_j (a_0 + \epsilon^{kl} a_l \xi_k) a_j \frac{\partial \rho_0}{\partial r} + \cdots . \]  \tag{135} \]

It is remarkable that for all cases \( \kappa = 1 \), \( \kappa = 0 \) and \( \kappa = -1 \), corresponding respectively to hyperbolic, Euclidean and spherical geometry, we obtain the same expression for the WZW action. This is mainly due to the fact that we considered a large magnetic field and so the particles are constrained to be in the lowest landau levels. More importantly, by reporting the Hamiltonian \( \mathcal{H}_a \) given by (117) in the action \( S_a \) (131), it is easily verified that the action \( S \) agrees with the result derived in [27]. This corroborates our claim according to which the electromagnetic coupling of a quantum Hall droplet can be described by a deformation of the symplectic structure of Bargmann phase space associated with the lowest landau levels. In such a description, the key tool is Moser’s lemma which permits us to incorporate the interaction effect in the Hamiltonian. The present application gives a simple way to obtain the effective action describing the electromagnetic interaction of the Kahler vacuum or quantum Hall droplets for other geometries.

Finally, notice that in the absence of electromagnetic field, \( a = 0 \) and for large \( k \), from (130) we recover the effective WZW action describing the edge excitations of a dense collection of fermions in two-dimensional space [33, 34]. Otherwise for \( a = 0 \), the action \( S \) reduces to \( S_0 \).

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7. Concluding remarks

We introduced a generalized Weyl–Heisenberg $W_\kappa$ algebra (in fact a one-parameter family of algebras) that includes the harmonic oscillator, $su(2)$ and $su(1,1)$ algebras. We constructed the corresponding Fock–Bargmann phase space, which provided us with a useful tool for performing semiclassical analysis. We also defined a Lie group $SU_\kappa(1,1)$ involving the aforementioned symmetries and a homogeneous space, which also includes the three two-dimensional surfaces (plane, sphere, disc) in which the quantum system lives. In quantizing the coset space $SU_\kappa(1,1)/U(1)$, we showed that the Landau quantum (or integer quantum Hall) systems in a plane $\kappa = 0$, sphere $\kappa = -1$ and disc $\kappa = 1$ can be described in a unified algebraic scheme using the algebra $W_\kappa$. This unified formulation also provides a nice way to study the electromagnetic excitations of quantum Hall droplets. This is done from a purely symplectic point of view by modifying the symplectic structure of underlying phase spaces.

Subsequently, we showed that through a dressing transformation, based on Moser’s lemma that is a refined version of the celebrated Darboux theorem, one can find a diffeomorphism which eliminates the fluctuation. In this way, the electromagnetic interaction becomes incorporated in the Hamiltonian involving a Chern–Simons-like term. Note also the deep relation between Moser’s lemma and the Seiberg–Witten map. Finally, we gave the effective action governing the electromagnetic excitations of the quantum Hall droplets.

The results of the present work can be extended in many directions. For instance, we may study the higher dimensional phase spaces. Indeed, one can generalize the results obtained so far to a quantum system living in four-dimensional phase space. In this case, on modification of the symplectic 2-form, the position and momentum variables acquire nonvanishing Poisson brackets inducing, upon quantization, noncommutative position and momentum operators. On the other hand, in the four-dimensional case, for some particular forms of the electromagnetic 2-form, it is possible to realize the dressing transformation with the help of the so-called Hilbert–Schmidt orthonormalization procedure. Thus, it will be interesting to compare this method with that based on the Moser lemma.

In another context, in the phase space description of internal degrees of freedom of particles obeying $A_r$ statistics [35,36] they are described by a $2r$-dimensional Bargmann space [37]. The present work gives the main tools for dealing with the electromagnetic interaction of such systems in the semiclassical regime. We hope to report on these issues in a forthcoming work.

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