Out of Control: Reducing Probabilistic Models by Control-State Elimination

Tobias Winkler, Johannes Lehmann, Joost-Pieter Katoen
Probabilistic Model Checking in Theory

Finite Markov chain or MDP

Property (reachability, expected reward, etc.)

Model Checker

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Probabilistic Model Checking in Practice

High level program description (finite variable domains)

Finite Markov chain or MDP

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Property (reachability, expected reward, etc.)

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High level program description (finite variable domains)

State explosion!

Finite Markov chain or MDP

Model Checker

Property (reachability, expected reward, etc.)
High level program description (finite variable domains)

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MO/TO

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High level program description (finite variable domains)

State explosion!

Finite Markov chain or MDP

Goal: Mitigate state explosion through program transformations

Model Checker

Property (reachability, expected reward, etc.)

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The PRISM Modelling Language and Control-Flow Graphs

```
1 dtmc
2 const int N = 5;
3
4 module coingame
5     x : [0..N+1] init N/2;
6     f : bool init false;
7
8     [] x=0 & x<N & !f -> 0.5 : (x'=x-1) + 0.5 : (f'=true);
9     [] x<N & !f -> 0.5 : (x'=x-1) & (f'=false) + 0.5 : (x'=x+2) & (f'=false);
10     [] x=0 | x=N -> true;
11
12 endmodule
```
The PRISM Modelling Language and Control-Flow Graphs

Resulting Markov chain for N=6:
State Elimination
State Elimination
These Markov chains are equivalent w.r.t. reaching 🟢!
Can we (automatically) achieve such simplifications by manipulating the program?

These Markov chains are equivalent wrt. reaching !
Step 1: Unfold Variable into Control-flow Graph

Unfold $f$
Step 1: Unfold Variable into Control-flow Graph

unfold f
Step 1: Unfold Variable into Control-flow Graph

Variables cannot always be unfolded so easily

- E.g. cannot unfold $f$ if assignment $f = x$ occurs
  - $x$ must be unfolded first
- Most real-world instances have some unfoldable variables
eliminate $f$
se

eliminate $f$

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eliminate $f$
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Elimination Rule

In plain Markov chains:
Elimination Rule

In plain Markov chains:

In the control-flow graph (eliminating a single transition):
Each location (without self-loops) can be eliminated by successively applying the transition-elimination rule to all its incoming transitions.
Location Elimination in general

Each location (without self-loops) can be eliminated by successively applying the transition-elimination rule to all its incoming transitions.

Treat case $l_1 = l_2$ with extra care
Complexity of Location Elimination

Definition (Transition Multiplicity):

\[ \begin{align*}
\phi & \quad \Downarrow \quad m \text{ incoming transitions from same command} \\
\rightarrow & \quad p_1 : u_1 \\
\rightarrow & \quad p_2 : u_2 \\
\rightarrow & \quad \vdots \\
\rightarrow & \quad p_m : u_m \\
\rightarrow & \quad l_1
\end{align*} \]
**Definition (Transition Multiplicity):**

$m$ incoming transitions from same command

\[
\begin{align*}
  p_1 &: u_1 \\
  p_2 &: u_2 \\
  p_m &: u_m \\
\end{align*}
\]
Definition (Transition Multiplicity):

Complexity of Location Elimination

\[ m = 2 \]

Theorem:
Exponentially many (in \( m \)) applications of transition elimination are sufficient and necessary to eliminate location \( l_1 \)
Automization

Heuristics: *Unfold a bit, eliminate reasonably*
Automization

Heuristics: *Unfold a bit, eliminate reasonably*

higher score = generates more self-loop free locations
Automization

Heuristics: Unfold a bit, eliminate reasonably

- Higher score = generates more self-loop free locations
- Don’t blow up the control flow graph
Implementation

- Extension to the probabilistic model checker Storm
- Used as a simplification front end:

Unfolding + Elimination until too many control-flow locations
# Experimental Results

| Name   | Type | Prop. type | Red. type | Params | States | Transitions | Build time | Check time | Total time |
|--------|------|------------|-----------|--------|--------|-------------|------------|------------|------------|
|        |      |            |           |        | orig. | red.        | orig.      | red.       | orig.      | red.       |
| BRP    | dtmc | P          | 134       | 2^{19}/5 | 78.9K | -44%        | 106K       | -33%       | 261 -33%   | 22 -38%    | \(16,418 -46\%\) |
|        |      |            |           |        | 2^{11}/10 | 201K | -45%        | 397K       | -33%       | 1,027 -39% | 101 -46%   |
|        |      |            |           |        | 2^{12}/20 | 1.11M | -46%        | 1.53K       | -33%       | 3,945 -48% | 462 -48%   |
|        |      |            |           |        | 2^{13}/25 | 2.76M | -46%        | 3.8K        | -33%       | 9,413 -47% | 1,187 -47% |
| GOINGAME | dtmc | P          | 35        | \(10^4\) | 20K   | -50%        | 40K        | -50%       | 53 -24%    | 18,500 -79% | \(18,553 -78\%\) |
| DICE5  | mdp  | P          | 671       | n/a    | 371K  | -84%        | 2.01K      | -83%       | 1,709 -82% | 9,538 -99% | \(11,247 -91\%\) |
| BAJS   | mdp  | R          | 223       | \(10^3\) | 194K  | -28%        | 326K       | -1%        | 1,242 -43% | 220 -32%   | \(18,397 -42\%\) |
|        |      |            |           |        | \(10^4\) | 2M    | -28%        | 3.38K       | -1%        | 13,154 -46% | 3,780 -31% |
| GRID   | dtmc | P          | 117       | \(10^4\) | 300K  | -47%        | 410K       | -34%       | 1,082 -57% | 17 -52%    | \(11,716 -52\%\) |
|        |      |            |           |        | \(10^5\) | 3M    | -47%        | 4.1K        | -34%       | 10,430 -53% | 207 -54%   |
| HOSPITAL | mdp  | P          | 37        | n/a    | 160K  | -66%        | 396K       | -27%       | 502 -50%   | 19 -56%    | 521 -39%   |
| NAND   | dtmc | P          | 80        | \(20/4\) | 308K  | -79%        | 476K       | -82%       | 589 -45%   | 108 -75%   | \(84,080 -56\%\) |
|        |      |            |           |        | \(40/4\) | 4M    | -80%        | 6.2M        | -51%       | 8,248 -50% | 1,859 -77% |
|        |      |            |           |        | \(60/2\) | 9.42M | -80%        | 14.9M       | -50%       | 19,701 -49% | 4,685 -76% |
|        |      |            |           |        | \(60/4\) | 18.8M | -80%        | 29.8M       | -50%       | 40,168 -53% | 10,703 -77% |
| ND-NAND | mdp  | P          | 106       | \(20/4\) | 308K  | -79%        | 476K       | -52%       | 618 -36%   | 127 -74%   | \(96,956 -52\%\) |
|        |      |            |           |        | \(40/4\) | 4M    | -80%        | 6.2M        | -51%       | 8,783 -42% | 2,270 -77% |
|        |      |            |           |        | \(60/2\) | 9.42M | -80%        | 14.9M       | -50%       | 21,792 -47% | 5,846 -75% |
|        |      |            |           |        | \(60/4\) | 18.8M | -80%        | 29.8M       | -50%       | 44,409 -48% | 13,312 -76% |
| NEGOTIATION | dtmc | P          | 148       | \(10^3\) | 1.29K | -32%        | 184K       | -26%       | 481 -39%   | 22 -49%    | \(5,631 -39\%\) |
|        |      |            |           |        | \(10^4\) | 1.92M | -32%        | 1.84M       | -26%       | 4,930 -43% | 197 -30%   |
| POLE   | dtmc | R          | 208       | \(10^2\) | 316K  | -46%        | 790K       | -4%        | 1,496 -48% | 26 -42%    | 406 -38%   | \(17,431 -45\%\) |
## Experimental Results

| Name      | Type | Prop. type | Red. time | Params | States  | Transitions | Build | Check | Total time |
|-----------|------|------------|-----------|--------|---------|-------------|--------|-------|------------|
|           |      | orig.      | orig.     |        | orig.   | orig.       | orig.  | orig.  | orig.      |
|           |      | red.       | red.      |        | red.    | red.        | red.   | red.   | red.       |
| BRP       | dtmc | P          | 134       | 2\textsuperscript{10}/5 | 78.9K | 106K         | 261    | 22    | 16,418     |
|           |      | -44\%     | -33\%     |        | -33\%  | -33\%       | -38\% |        | -46\%     |
|           |      | 2\textsuperscript{11}/10 | 201K | 397K | 1,027 | 101 | -39\% | -46\% |
|           |      | -45\%     | -33\%     |        | -33\%  | -39\%       | -46\% |        | -46\%     |
|           |      | 2\textsuperscript{12}/20 | 1.11M | 1.53M | 3,945 | 462 | -48\% | -48\% |
|           |      | -46\%     | -33\%     |        | -33\%  | -48\%       | -48\% |        | -48\%     |
|           |      | 2\textsuperscript{13}/25 | 2.76M | 3.8M | 9,413 | 1,187 | -17\% | -17\% |
|           |      | -46\%     | -33\%     |        | -33\%  | -17\%       | -17\% |        | -17\%     |
| COINGAME  | dtmc | P          | 35        | 10\textsuperscript{4} | 20K | 40K         | 53     | 18,500 | 18,553     |
|           |      | -50\%     | -50\%     |        | -24\%  | -79\%       | -78\% |        | -78\%     |
| DICE5     | mdp  | P          | 671       | n/a    | 371K | 2.01K       | 1,700  | 9,538 | 11,247     |
|           |      | -84\%     | -83\%     |        | -82\%  | -99\%       | -91\% |        | -91\%     |
| BAJS      | mdp  | R          | 223       | 10\textsuperscript{3} | 194K | 326K         | 1,242  | 220   | 18,397     |
|           |      | -28\%     | -1\%      |        | -43\%  | -32\%       | -42\% |        | -42\%     |
|           |      | 10\textsuperscript{4} | 2M | 3.38M | 13,154 | 3,780 | -46\% | -31\% |
| GRID      | dtmc | P          | 117       | 10\textsuperscript{4} | 300K | 410K         | 1,062  | 17    | 11,716     |
|           |      | -47\%     | -34\%     |        | -57\%  | -52\%       | -52\% |        | -52\%     |
|           |      | 10\textsuperscript{5} | 3M | 4.1M | 10,430 | 207    | -54\% | -54\% |
| HOSPITAL  | mdp  | P          | 57        | n/a    | 160K | 396K         | 502    | 19    | 521        |
|           |      | -66\%     | -27\%     |        | -50\%  | -56\%       | -39\% |        | -39\%     |
| NAND      | dtmc | P          | 80        | 20/4   | 308K | 476K         | 589    | 108   | 86,080     |
|           |      | -79\%     | -52\%     |        | -15\%  | -75\%       | -75\% |        | -75\%     |
|           |      | 40/4      | 6.29M | 8,248 | 1,859 | -50\%       | 1,859  | 77\% | -77\%     |
|           |      | 60/2      | 9.62M | 14.9K | 19,701 | 4,685 | -76\% | -76\% |
|           |      | 60/4      | 18.8M | 29.8M | 40,168 | 10,703 | -77\% | -77\% |
| ND-NAND   | mdp  | P          | 106       | 20/4   | 308K | 476K         | 618    | 127   | 96,956     |
|           |      | -79\%     | -52\%     |        | -36\%  | -74\%       | -74\% |        | -74\%     |
|           |      | 40/4      | 6.29M | 8,783 | 2,270 | -42\%       | 2,270  | 77\% | -77\%     |
|           |      | 60/2      | 9.62M | 14.9K | 21,792 | 5,846 | -75\% | -75\% |
|           |      | 60/4      | 18.8M | 29.8M | 44,409 | 13,312 | -76\% | -76\% |
| NEGOTIATION | dtmc | P          | 148       | 10\textsuperscript{4} | 129K | 184K         | 481    | 22    | 5,631      |
|           |      | -32\%     | -26\%     |        | -39\%  | -49\%       | -49\% |        | -49\%     |
|           |      | 10\textsuperscript{5} | 1.29M | 1.84M | 4,930  | 197 | -30\% | -30\% |
| POLE      | dtmc | R          | 208       | 10\textsuperscript{2} | 315K | 790K         | 1,496  | 26    | 17,431     |
|           |      | -46\%     | -4\%      |        | -46\%  | -42\%       | -42\% |        | -42\%     |
|           |      | 10\textsuperscript{3} | 3.16M | 7.9M | 15,508 | 406    | -33\% | -33\% |
## Experimental Results

| Name     | Type | Prop. type | Red. | Param. | States | Transitions | Build time | Check time | Total time |
|----------|------|------------|------|--------|--------|-------------|------------|------------|------------|
|          |      |            |      |        | orig.  | red.        | orig.      | red.       | orig.      | red.       |
| BRP      | dtmc | P          | 134  | 2       | 78.9k  | -44%        | 106k       | -33%       | 261        | -33%       | 22         | -38%       | 16,418     | -46%       |
|          |      | R          | 246  | 2       | 201k  | -45%        | 397k       | -33%       | 1,027      | -39%       | 101        | -46%       | 10,014     | -46%       |
|          |      |            |      |        |        |             | 1,53k      | -33%       | 9,943      | -48%       | 462        | -48%       | 14,985     | -48%       |
|          |      |            |      |        |        |             | 3.8k       | -33%       | 9,413      | -47%       | 1,187      | -47%       | 14,595     | -47%       |
| COINGAME | dtmc | P          | 35   | 10^4    | 20k   | -50%        | 40k        | -50%       | 53         | -24%       | 18,500     | -79%       | 18,553     | -78%       |
| DICE5    | mdp  | P          | 671  | n/a     | 371k  | -84%        | 2.01k      | -83%       | 1,709      | -82%       | 9,538      | -99%       | 11,247     | -91%       |
| BARS     | mdp  | R          | 223  | 10^3    | 194k  | -28%        | 326k       | -1%        | 1,242      | -43%       | 220        | -32%       | 18,397     | -42%       |
|          |      |            |      | 10^4    | 2m   | -28%        | 3.38k      | -1%        | 13,154     | -46%       | 3,780      | -31%       | 11,937     | -31%       |
| GRID     | dtmc | P          | 117  | 10^4    | 300k  | -47%        | 410k       | -34%       | 1,062      | -57%       | 17         | -52%       | 11,716     | -52%       |
|          |      |            |      | 10^8    | 3m   | -47%        | 4.1k       | -34%       | 10,430     | -53%       | 207        | -54%       | 11,716     | -52%       |
| HOSPITAL | mdp  | P          | 57   | n/a     | 160k  | -66%        | 396k       | -27%       | 502        | -50%       | 19         | -56%       | 521        | -39%       |
| NAND     | dtmc | P          | 80   | 20/4    | 308k  | -79%        | 476k       | -82%       | 589        | -45%       | 108        | -75%       | 86,080     | -56%       |
|          |      |            |      | 40/4    | 4m    | -80%        | 6.29k      | -51%       | 8,248      | -50%       | 1,859      | -77%       | 86,080     | -56%       |
|          |      |            |      | 60/2    | 9.62m | -80%        | 14.9k      | -50%       | 19,701     | -49%       | 4,685      | -76%       | 86,080     | -56%       |
|          |      |            |      | 60/4    | 18.8m | -80%        | 29.8k      | -50%       | 40,168     | -53%       | 10,703     | -77%       | 86,080     | -56%       |
| ND-NAND  | mdp  | P          | 106  | 20/4    | 308k  | -79%        | 476k       | -52%       | 618        | -36%       | 127        | -74%       | 96,956     | -52%       |
|          |      |            |      | 40/4    | 4m    | -80%        | 6.29k      | -51%       | 8,783      | -42%       | 2,270      | -77%       | 96,956     | -52%       |
|          |      |            |      | 60/2    | 9.62m | -80%        | 14.9k      | -50%       | 21,792     | -47%       | 5,846      | -75%       | 96,956     | -52%       |
|          |      |            |      | 60/4    | 18.8m | -80%        | 29.8k      | -50%       | 44,409     | -46%       | 13,312     | -76%       | 96,956     | -52%       |
| NEGOTIATION | dtmc | P          | 148  | 10^4    | 129k  | -32%        | 184k       | -26%       | 481        | -39%       | 22         | -49%       | 5,631      | -39%       |
|          |      |            |      | 10^2    | 1.29m | -32%        | 1.84m      | -26%       | 4,930      | -43%       | 197        | -30%       | 5,631      | -39%       |
| POLE     | dtmc | R          | 208  | 10^3    | 315k  | -46%        | 790k       | -4%        | 1,496      | -48%       | 26         | -42%       | 17,431     | -45%       |

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## Control-Flow Reduction and Bisimulation Minimization

| Name | Params. | States Bisim. | States CFR | States both |
|------|---------|---------------|-------------|-------------|
| BRP  | $2^{12}/20$ | 598K | 606K | 344K |
| NAND | 40/4 | 3.21M | 816K | 678K |
| POLE | $10^3$ | 4.06K | 1.72M | 1.2K |
Control-Flow Reduction and Bisimulation Minimization

| Name | Params. | States | Bisim. | CFR. | both |
|------|---------|--------|--------|------|------|
| BRP  | $2^{12}/20$ | 598K   | 606K   | 344K |      |
| NAND | 40/4    | 3.21M  | 816K   | 678K |      |
| POLE | $10^3$  | 4.06K  | 1.72M  | 1.2K |      |

Are orthogonal and can be combined!
Take-Home Messages

In probabilistic model checking …
Take-Home Messages

In probabilistic model checking …

1) **Mechanizable program transformations can reduce the state space**
Take-Home Messages

In probabilistic model checking …

1) Mechanizable program transformations can reduce the state space

2) There are “symmetries” beyond bisimulation
Take-Home Messages

In probabilistic model checking …

1) **Mechanizable program transformations can reduce the state space**

2) **There are “symmetries” beyond bisimulation**

*Thank you!*
Take-Home Messages

In probabilistic model checking …

1) **Mechanizable program transformations can reduce the state space**

2) **There are “symmetries” beyond bisimulation**

*Thank you!*