ON A GENERALIZATION OF BUSEMANN’S INTERSECTION INEQUALITY

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Abstract. Busemann’s intersection inequality gives an upper bound for the volume of the intersection body of a star body in terms of the volume of the body itself. Koldobsky, Paouris, and Zymonopoulou asked if there is a similar result for \( k \)-intersection bodies. We solve this problem for star bodies that are close to the Euclidean ball in the Banach-Mazur distance. We also improve a bound obtained by Koldobsky, Paouris, and Zymonopoulou for general star bodies in the case when \( k \) is proportional to the dimension.

1. Introduction

We say that a set \( K \) in \( \mathbb{R}^n \) is star-shaped if for every \( x \in K \) the closed line segment connecting \( x \) to the origin lies in \( K \). A compact set \( K \) in \( \mathbb{R}^n \) is called a star body if it is star-shaped and its radial function defined by

\[
\rho_K(\xi) = \max\{ a \geq 0 : a\xi \in K \}, \ \xi \in S^{n-1},
\]

is positive and continuous. Geometrically, \( \rho_K(\xi) \) is the distance from the origin to the point on the boundary in the direction of \( \xi \).

Let \( K \) and \( L \) be star bodies in \( \mathbb{R}^n \). Following Lutwak [L], we say that \( L \) is the intersection body of \( K \) if

\[
\rho_L(\xi) = \text{vol}_{n-1}(K \cap \xi^\perp),
\]

for every \( \xi \in S^{n-1} \).

For any star body \( K \) its intersection body always exists and we will denote it by \( I(K) \). The well-known Busemann intersection inequality asserts that

\[
\text{vol}_n(I(K)) \leq \frac{\kappa_n^{n-1}}{\kappa_n^{n-2}} \text{vol}_n(K)^{n-1}
\]

with equality if and only if \( K \) is a centered ellipsoid; see, e.g., [Ga, p. 373]. Here and below,

\[
\kappa_p = \frac{\pi^{\frac{p}{2}}}{\Gamma \left( 1 + \frac{p}{2} \right)},
\]

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which equals the volume of the unit Euclidean ball in $\mathbb{R}^p$ when $p$ is a positive integer.

Koldobsky introduced a generalization of the notion of intersection body; see [K, p. 75]. Let $K$ and $L$ be origin-symmetric star bodies in $\mathbb{R}^n$ and $k$ be an integer, $1 \leq k \leq n - 1$. We say that $L$ is the $k$-intersection body of $K$ if
\[ \text{vol}_k(L \cap H) = \text{vol}_{n-k}(K \cap H^\perp) \]
for every $k$-dimensional subspace $H$ of $\mathbb{R}^n$.

For a given origin-symmetric star body $K$ its $k$-intersection body may not exist, but when it does, we will denote it by $I_k(K)$.

Koldobsky, Paouris, and Zymonopoulou [KPZ] asked whether an analogue of Busemann’s intersection inequality holds for $k$-intersection bodies. Namely, if $K$ is an origin-symmetric star body in $\mathbb{R}^n$ whose $k$-intersection body exists and such that $\text{vol}_n(K) = \text{vol}_n(B_2^n)$, is it true that
\[ \text{vol}_n(I_k(K)) \leq \text{vol}_n(I_k(B_2^n))? \]
(1)

They proved that
\[ \left( \frac{\text{vol}_n(I_k(K))}{\text{vol}_n(I_k(B_2^n))} \right)^{1/n} \leq c \min\{\log n, k \log k\}, \]
(2)
for some absolute constant $c$, under the assumption that $I_k(K)$ is a convex body.

We will slightly modify and extend their conjecture. First of all, let us write (1) somewhat differently. If $L$ is the $k$-intersection body of $K$, and we do not put any restrictions on the volume of $K$, then (1) is equivalent to
\[ (\text{vol}_n(L))^k \leq C_{n,k} (\text{vol}_n(K))^{n-k}, \]
(3)
where $C_{n,k}$ is an appropriate constant so that the latter inequality becomes equality in the case of centered balls.

It follows directly from the definition that if $L$ is the $k$-intersection body of $K$, then $K$ is the $(n-k)$-intersection body of $L$. This means that if inequality (3) is true for $0 < k < n/2$, then the reversed inequality should be true for $n/2 < k < n$.

Further, in terms of the Fourier transform the condition that $L$ is the $k$-intersection body $K$ can be written as follows:
\[ \|\theta\|^k_L = \frac{k}{(2\pi)^{k(n-k)}} \left( \| \cdot \|^n_{-K} \right)^n(\theta), \quad \theta \in S^{n-1}; \]
(4)
see [K, Theorem 4.6]. Here $\| \cdot \|_K$ denotes the Minkowski functional of $K$ and is defined by
\[ \|x\|_K = \min\{a \geq 0 : x \in aK\}, \quad x \in \mathbb{R}^n. \]

Note that relation (4) allows us to write (3) as follows:
\[ \int_{S^{n-1}} \left( \| \cdot \|^n_{-K} \right)^n(\theta) d\theta \leq c_{n,k} (\text{vol}_n(K))^{n-k}, \]
(5)
where \( c_{n,k} \) is a constant that turns \((5)\) into equality for centered balls. The exact value of this constant will be computed later.

As one can see, \((5)\) makes sense even if \(k\) is not an integer. Thus we will write \((5)\) for a larger set of values by using a real number \(p\) instead of \(k\). Let us also note that the assumption that the Fourier transform of \(\| \cdot \|_{K}^{-n+k}\) is a positive continuous function is very restrictive. So this assumption will be dropped.

Summarizing all of the above remarks, let us now state the question in the following form.

**Question 1.** Let \(0 < p < n\) be a real number and \(K\) be an origin-symmetric star body in \(\mathbb{R}^n\). Are the following inequalities true?

\[
\int_{S^{n-1}} \left| (\| \cdot \|_{K}^{-n+p})^\wedge (\theta) \right|^{n/p} d\theta \leq c_{n,p} (\text{vol}_n(K))^{\frac{n}{p} - 1}, \quad \text{if } p < n/2, \tag{6}
\]

and

\[
\int_{S^{n-1}} \left| (\| \cdot \|_{K}^{-n+p})^\wedge (\theta) \right|^{n/p} d\theta \geq c_{n,p} (\text{vol}_n(K))^{\frac{n}{p} - 1}, \quad \text{if } p > n/2, \tag{7}
\]

with equality if and only if \(K\) is a centered ellipsoid.

The case \(p = n/2\) is omitted above since these inequalities become equalities for every origin-symmetric star body \(K\). This is just an application of the spherical version of Parseval’s formula (see [K, p.66]):

\[
\int_{S^{n-1}} \left[ (\| \cdot \|_{K}^{-n/2})^\wedge (\theta) \right]^2 d\theta = (2\pi)^n \int_{S^{n-1}} \| \theta \|_{K}^{-n} d\theta = n(2\pi)^n \text{vol}_n(K).
\]

It is interesting to note that the conjectured inequalities \((6)\) and \((7)\) have connections to other known inequalities. First of all, let us repeat that the case \(p = 1\) corresponds to the Busemann intersection inequality. Thus Question \(\text{[I]}\) has an affirmative answer for \(p = 1\) and \(p = n - 1\). Let us now look at the case when \(p < 1\) and \(p\) is not an even integer. For these values of \(p\) the Fourier transform of \(\| \cdot \|_{K}^{-n+p}\) can be expressed as follows:

\[
(\| \cdot \|_{K}^{-n+p})^\wedge (\theta) = \frac{\pi(n - p)}{2\Gamma(1 - p) \sin(\pi p/2)} \int_K |\langle x, \theta \rangle|^{-p} dx; \tag{8}
\]

see [K, Corollary 3.15].

The reader may recognize such integrals: they appear, for example, in the definition of polar \(q\)-centroid bodies (see [LZ] for \(q \geq 1\) and [YY] for \(-1 < q < 1\)). Let \(K \subset \mathbb{R}^n\) be a star body and \(q > -1, q \neq 0\). The polar \(q\)-centroid body of \(K\) is the star body \(\Gamma^*_q K\) given by

\[
\|x\|_{\Gamma^*_q K} = \left( \frac{1}{\text{vol}_n(K)} \int_K |\langle x, y \rangle|^q dy \right)^{\frac{1}{q}}, \quad x \in \mathbb{R}^n \setminus \{0\}.
\]

Note that in [LZ] the normalization is different.
Lutwak and Zhang [LZ] have shown that if $K \subset \mathbb{R}^n$ is a star body and $q \geq 1$, then
\begin{equation}
\text{vol}_n(K) \text{vol}_n(\Gamma^*_q K) \leq \text{vol}_n(B^*_n) \text{vol}_n(\Gamma^*_q B^*_n),
\end{equation}
with equality if and only if $K$ is an ellipsoid centered at the origin.

By virtue of formula (8) one can check that, in the case when $q$ is not an even integer, the Lutwak-Zhang inequality is equivalent to
\begin{equation}
\int_{S^{n-1}} |\langle \cdot, \theta \rangle|^{-n/q} \, d\theta \leq |c_{n,-q}|(\text{vol}_n(K))^{-\frac{n}{q}-1}.
\end{equation}
Thus, Question 1 can also be viewed as an extension of the Lutwak-Zhang inequality.

In this paper we show that Question 1 has an affirmative answer when the body $K$ is sufficiently close to the Euclidean ball in the Banach-Mazur distance. For general star bodies we will obtain an improvement of inequality (2) when $k > cn/\log^2 n$.

2. Preliminaries

Two of the main tools used in this paper are the Fourier transform of distributions and spherical harmonics. The reader is referred to the books [K] and [Gr] for detailed discussions of such techniques. We will just briefly mention some important facts. Let $f$ be a continuous function on the sphere $S^{n-1}$ and consider its homogeneous extension to $\mathbb{R}^n \setminus \{0\}$ of degree $-n+p$, where $0 < p < n$. We can think of $|x|^n f(x/|x|)$ as a distribution acting on test functions by integration and therefore we can define its Fourier transform in the distributional sense. If $f \in C^\infty(S^{n-1})$, then the Fourier transform of $|x|^n f(x/|x|)$ is equal to a homogeneous of degree $-p$ function, that is infinitely smooth on $\mathbb{R}^n \setminus \{0\}$; see [K, Lemma 3.16]. Thus we can introduce a linear operator $I_p : C^\infty(S^{n-1}) \to C^\infty(S^{n-1})$, that maps a function $f$ to the function $I_p f$ equal to the restriction to the sphere of the Fourier transform of $\frac{\Gamma(\frac{n+2}{2})}{2^n \pi^{n/2} \Gamma(\frac{q}{2})} |x|^{-n+p} f(x/|x|)$. The coefficient in front of the latter function is chosen in such a way that $I_p(1) = 1$, as will be shown later.

When $0 < p < 1$, $I_p$ has the following integral representation, which is just [S] with an appropriate normalization.
\begin{equation}
I_p f(\theta) = \frac{\Gamma(\frac{n+2}{2})}{2^n \pi^{n/2} \Gamma(\frac{q}{2})} \frac{\pi}{2 \Gamma(1-p) \sin(\pi p/2)} \int_{S^{n-1}} |\langle x, \theta \rangle|^{-p} f(x) \, dx
= \frac{\sqrt{\pi} \Gamma(\frac{n+2}{2})}{\Gamma(\frac{q}{2}) \Gamma(\frac{n+2}{2})} \int_{S^{n-1}} |\langle x, \theta \rangle|^{-p} f(x) \, d\sigma(x),
\end{equation}
where $\sigma$ is the rotationally invariant probability measure on the sphere. To compute the coefficient in front of the above integral we used [K, Lemma 2.18].
For a function \( f \) on the sphere, let \( \sum_{m=0}^{\infty} H_m \) denote its spherical harmonic expansion, where each \( H_m \) is a spherical harmonic of degree \( m \) in dimension \( n \). If \( f \) is even, then it has only harmonics of even degrees in its expansion: \( \sum_{m \geq 0, \text{even}} H_m \). Furthermore, \( I_p f \) is also even and its spherical harmonic expansion is given by

\[
\sum_{m \geq 0, \text{even}} \lambda_m(n, p) H_m,
\]

where

\[
\lambda_m(n, p) = \frac{\Gamma \left( \frac{n-p}{2} \right) \Gamma \left( \frac{m+p}{2} \right)}{\Gamma \left( \frac{p}{2} \right) \Gamma \left( \frac{m+n-p}{2} \right)}\;
\]

see, e.g., [GYY] (note that the normalization of \( I_p \) is different there).

Note that \( \lambda_0(n, p) = 1 \) for all \( p \in (0, n) \), and thus \( I_p (1) = 1 \). Another important observation is that \( |\lambda_m(n, p)| > |\lambda_{m+2}(n, p)| \) for all even \( m \geq 0 \) when \( 0 < p < n/2 \) (and the inequality gets reversed when \( n/2 < p < n \)). In particular, for \( 0 < p < n/2 \), we have

\[
\|I_p f\|_2^2 = \sum_{m \geq 0, \text{even}} \lambda_m^2(n, p) \|H_m\|_2 \leq \|f\|_2^2.
\]

Hence, \( I_p \) (when \( 0 < p < n/2 \)) is well defined as a linear operator from \( L^{n/(n-p)}(S^{n-1}) \) to \( L^{n/p}(S^{n-1}) \) with the operator norm equal to \( 1 \).

Here and below we denote by \( \|f\|_q \) the \( L^q(S^{n-1}) \)-norm of \( f \):

\[
\|f\|_q = \left( \int_{S^{n-1}} |f(\theta)|^q d\sigma(\theta) \right)^{1/q}.
\]

Let us finally remark that in terms of the operator \( I_p \) the conjectured inequality (6) can be written as follows.

**Question 1 (Reformulated).** Let \( K \) be an origin-symmetric star body in \( \mathbb{R}^n \) and \( 0 < p < n/2 \). Is it true that

\[
\int_{S^{n-1}} \left| I_p (\| \cdot \|_K^{n-p}(\theta)) \right|^{n/p} d\sigma(\theta) \leq (\kappa_{n})^{1-p} \left( \frac{n}{2p} \right)^{\frac{n}{2p}} (\text{vol}_n(K))^{\frac{n}{2p} - 1},
\]

with equality if and only if \( K \) is a centered ellipsoid?

Below we will not discuss inequality (7), since, at least in the language of \( k \)-intersection bodies, the cases \( 0 < k < n/2 \) and \( n/2 < k < n \) are the same.

3. **Main Results**

Following the discussion in the previous section, we will now show that \( I_p \) can be extended to a bounded linear operator on a larger space of functions.

**Theorem 2.** Let \( 0 < p < n/2 \). \( I_p \) is a bounded linear operator from \( L^{n/(n-p)}_{\text{even}}(S^{n-1}) \) to \( L^{n/p}_{\text{even}}(S^{n-1}) \). Moreover,

\[
\|I_p\|_{L^{n/(n-p)}_{\text{even}}(S^{n-1}) \to L^{n/p}_{\text{even}}(S^{n-1})} \leq \left( \frac{n}{2p} \right)^{\frac{n}{2p}}.
\]
we will distinguish two cases according to the parity of
$\lambda$. Since $\Gamma(z)$ is analytic in the strip $0 \leq \Re(z) \leq 1$, where $z = \frac{n}{2}p$. Thus, the boundary of the strip consists of two lines $z = is$ and $z = 1 + is$, where $s \in \mathbb{R}$. Therefore, when $\Re(z) = 1$, i.e., $p = \frac{n}{2}(1 + is)$, we get

$$\lambda_m\left(n, \frac{n}{2}(1 + is)\right) = \frac{\Gamma\left(\frac{n}{4} - \frac{is}{4}\right) \Gamma\left(\frac{m}{2} + \frac{n}{4} + \frac{is}{4}\right)}{\Gamma\left(\frac{m}{2} + \frac{is}{4}\right) \Gamma\left(\frac{m}{2} + \frac{n}{4} - \frac{is}{4}\right)}.$$

Since $\Gamma(a + ib)$ is the complex conjugate of $\Gamma(a - ib)$, we see that $|\lambda_m(n, \frac{n}{2}(1 + is))| = 1$ for all real $s$, and so

$$\|I_{\frac{n}{2}}(1 + is)f\|_2 = \|f\|_2.$$

Now consider the case $\Re(z) = 0$. Extending $p = \frac{n}{2}is$, $s \in \mathbb{R}$, we get

$$\|I_{\frac{n}{2}}(is)f\|_\infty \leq \frac{\sqrt{\pi}|\Gamma\left(\frac{n-isn/2}{2}\right)|}{\Gamma\left(\frac{\frac{n}{2}}{2}\right) |\Gamma\left(\frac{1-isn/2}{2}\right)|} \int_{S^{n-1}} |f(x)| \, d\sigma(x).$$

Let $A(s)$ be the coefficient in front of the latter integral. To estimate $A(s)$ we will distinguish two cases according to the parity of $n$. If $n$ is odd, then

$$A(s) = \frac{\sqrt{\pi}|\Gamma\left(\frac{n-isn/2}{2}\right)|}{\Gamma\left(\frac{n}{2}\right) |\Gamma\left(\frac{1-isn/2}{2}\right)|}$$

$$= \frac{n-2-isn/2}{n-2} \cdot \frac{n-4-isn/2}{n-4} \cdots \frac{1-isn/2}{1} \cdot |\Gamma\left(\frac{1-isn/2}{2}\right)|$$

$$= \frac{(n-3)/2}{\frac{n-2}{n-2} \cdot \frac{n-4}{n-4} \cdots \frac{1}{1}} \leq \prod_{k=0}^{\infty} \left(1 + \frac{s^2n^2}{4(2k+1)^2}\right)^{1/2} \leq \prod_{k=0}^{\infty} \left(1 + \frac{s^2n^2}{4(2k+1)^2}\right)^{1/2}$$

$$= \left(\cosh \frac{\pi sn}{4}\right)^{1/2} \leq e^{s |\sin|}.$$
Above we used a representation of cosh as an infinite product. See e.g., [C, VII, §5-6] for details on the Weierstrass factorization theorem.

If \( n \) is even, the argument is similar, but we will additionally need the following formulas:

\[
|\Gamma(1 + is)| = \left(\frac{\pi s}{\sinh \pi s}\right)^{1/2}, \quad |\Gamma\left(\frac{1}{2} + is\right)| = \left(\frac{\pi}{\cosh \pi s}\right)^{1/2},
\]

which can be obtained from the Euler reflection formula; see [AAR, p. 9 and p. 22] for some details. The above formulas imply

\[
\frac{|\Gamma\left(\frac{2-isn/2}{2}\right)|}{|\Gamma\left(\frac{1-isn/2}{2}\right)|} = \left(\frac{sn}{4} \coth \frac{\pi sn}{4}\right)^{1/2}.
\]

Then we have, for even \( n \),

\[
A(s) = \sqrt{\pi} \frac{|\Gamma\left(\frac{n-isn/2}{2}\right)|}{\Gamma(n/2)|\Gamma\left(\frac{1-isn/2}{2}\right)|}
= \sqrt{\pi} \frac{|n-2-isn/2| \cdot |n-4-isn/2| \cdots \left|\frac{2-isn/2}{2}\right| \cdot |\Gamma\left(\frac{2-isn/2}{2}\right)|}{\frac{n-2}{2} \cdot \frac{n-4}{2} \cdots \frac{2}{2} \cdot |\Gamma\left(\frac{1-isn/2}{2}\right)|}
= \sqrt{\pi} \frac{|n-2-isn/2| \cdot |n-4-isn/2| \cdots \left|\frac{2-isn/2}{2}\right|}{n-2} \left(\frac{sn}{4} \coth \frac{\pi sn}{4}\right)^{1/2}
\leq \sqrt{\pi} \left(\frac{sn}{4} \coth \frac{\pi sn}{4}\right)^{1/2} \prod_{k=1}^{\infty} \left(1 + \frac{n^2 s^2}{4(2k)^2}\right)^{1/2}
= \sqrt{\pi} \left(\frac{sn}{4} \coth \frac{\pi sn}{4}\right)^{1/2} \prod_{k=1}^{\infty} \left(\frac{4 \sinh \frac{\pi sn}{4}}{\pi sn}\right)^{1/2}
= \left(\cosh \frac{\pi sn}{4}\right)^{1/2} \leq e^{\frac{\pi |sn|}{4}}.
\]

Therefore, regardless of the parity of \( n \) we have \( A(s) \leq e^{\frac{\pi |sn|}{4}} \). Denoting \( A_0(s) = e^{\frac{\pi |sn|}{4}} \) and \( A_1(s) = 1 \), we obtain

\[
\|I_{\nu}f\|_{\infty} \leq A_0(s)\|f\|_1
\]

and

\[
\|I_{\frac{\nu}{2}}(1+is)f\|_2 \leq A_1(s)\|f\|_2
\]

for all \( s \in \mathbb{R} \).

The Stein interpolation theorem now implies that for \( 0 < p < n/2 \) we have

\[
\|I_p f\|_{n/p} \leq C(n,p)\|f\|_{n/(n-p)}, \tag{12}
\]
for some constant $C(n, p)$.

Instead of using Stein’s bound for $C(n, p)$, we will proceed as follows. Consider the function $F(z) = -\frac{n}{4} \Re(z \log z)$, which is clearly harmonic in the strip $0 < \Re(z) < 1$. Now let us compute $F(z)$ on the boundary of the strip. When $z = is$, we get $\log(is) = \log|s| + \frac{i\pi}{2}$ if $s > 0$ and $\log(is) = \log|s| - \frac{i\pi}{2}$ if $s < 0$. Therefore,

$$F(is) = -\frac{n}{4} \Re\left(is \left(\log|s| + \text{sgn}(s)\frac{i\pi}{2}\right)\right) = \frac{\pi n}{8}|s|.$$ 

Observe that $F(is) = \log A_0(s)$.

When $z = 1 + is$, we get

$$F(1 + is) = -\frac{n}{4} \Re\left((1 + is) \left(\log\sqrt{1 + s^2} + i \arctan s\right)\right) = -\frac{n}{8} \log(1 + s^2) + \frac{n}{4} s \arctan s.$$ 

The derivative of the latter function is negative when $s < 0$ and positive when $s > 0$. Therefore the function achieves its minimum at zero, and thus $F(1 + is) \geq 0 = \log A_1(s)$ for all real $s$.

Summarizing, on the boundary of the strip the following holds: $\log A_0(s) = F(is)$ and $\log A_1(s) \leq F(1 + is)$, for all $s \in \mathbb{R}$. We can now conclude that for all $p \in (0, \frac{n}{2})$ we have

$$\log C(n, p) \leq F\left(\frac{2p}{n}\right) = -\frac{p}{2} \log\left(\frac{2p}{n}\right).$$

That is,

$$C(n, p) \leq \left(\frac{n}{2p}\right)^{\frac{p}{2}},$$

which together with (12) yields the result. 

We will now show that (11) holds up to a multiplicative constant (depending on $n$ and $p$) for all origin-symmetric star bodies.

**Theorem 3.** For every $0 < p < n/2$ and every origin-symmetric star body $K$ in $\mathbb{R}^n$ we have

$$\int_{S^{n-1}} \left|I_p(\|\cdot\|_K^{-n+p})(\theta)\right|^{n/p} d\sigma(\theta) \leq \left(\frac{n}{2p}\right)^{\frac{p}{2}} (\kappa_n)^{1 - \frac{n}{p}} (\text{vol}_n(K))^{\frac{n}{p} - 1}.$$
Proof. This is a direct application of Theorem 2 to the function $f = \| \cdot \|_K^{-n+p}$.

\[
\left( \int_{S^{n-1}} I_p(\| \cdot \|_K^{-n+p})(\theta) \right)^{n/p} d\sigma(\theta) \leq \left( \int_{S^{n-1}} (\| \cdot \|_K^{-n+p})^{n/(n-p)} d\sigma(\theta) \right)^{(n-p)/n} = \left( \frac{n}{2p} \right)^{\frac{n}{2}} (\kappa_{-1}^{-1} \text{vol}_n(K))^{(n-p)/n}.
\]

Raising both sides to the power $n/p$, we get the result.

\[ \square \]

**Remark.** Observe that the result above improves inequality (2) when $k > c \log n$. Indeed, let $0 < k < n/2$ and let $K$ be an origin-symmetric star body in $\mathbb{R}^n$ whose $k$-intersection body exists. If we assume that $\text{vol}_n(K) = \text{vol}_n(B^n_2)$, then Theorem 3 yields

\[
\left(\frac{\text{vol}_n(I_k(K))}{\text{vol}_n(I_k(B^n_2))}\right)^{1/n} \leq \sqrt{\frac{n}{2k}}.
\]

Note that we do not require $K$ or $I_k(K)$ to be convex.

Before solving a local version of Question 1 we will prove the following lemma pertaining to the equality case in Question 1.

**Lemma 4.** The conjectured inequality (11) becomes equality if $K$ is a centered ellipsoid.

**Proof.** First, let us show that inequality (11) is invariant under invertible linear transformations. Indeed, let $T \in GL_n(\mathbb{R})$. Applying the transformation $T$ to the right-hand side of inequality (11) yields a factor of $|\det T|^{n/p-1}$. Now let us show that the same happens to the left-hand side of (11).

Consider the origin-symmetric star-shaped set $L$ defined by the formula

\[
\| \theta \|_{L}^{-p} = \left( \| \theta \|_K^{-n+p} \right)^{(n-p)/n}, \quad \theta \in S^{n-1}.
\]

Using the connection between the Fourier transform and linear transformations, we get
\[
\int_{S^{n-1}} \left( \| \cdot \|_{TK}^{n+p} \right)^{\frac{n}{p}} \, d\theta = \int_{S^{n-1}} \left( \| T^{-1}(\cdot) \|_{K}^{n+p} \right)^{\frac{n}{p}} \, d\theta
\]
\[
= | \det T |^{\frac{n}{p}} \int_{S^{n-1}} \left( \| \cdot \|_{K}^{n+p} \right)^{\frac{n}{p}} \, d\theta
\]
\[
= | \det T |^{\frac{n}{p}} \int_{S^{n-1}} \| T^t \|\, d\theta = | \det T |^{\frac{n}{p}} \int_{S^{n-1}} \| \theta \|^n_L \, d\theta
\]
\[
= | \det T |^{\frac{n}{p} - 1} \int_{S^{n-1}} \left( \| \cdot \|_{K}^{n+p} \right)^{\frac{n}{p}} \, d\theta,
\]
where we used \( T^t \) and \( T^{-t} \) to denote the transpose and the inverse of the transpose of \( T \) correspondingly. The calculations above remain valid if the Fourier transform is replaced by the operator \( I_p \) since they are equal up to a constant multiple. Thus, the linear invariance of (11) follows.

It remains to show that (11) turns into equality when \( K \) is a unit ball.

Indeed, we have
\[
\int_{S^{n-1}} I_p(\| \cdot \|_{2}^{n+p}) \, d\sigma(\theta) = 1 = (\kappa_n)^{1 - \frac{2}{p}} (\text{vol}_n(B_2^n))^{\frac{2}{p} - 1}.
\]

We will now prove that a local version of Question 1 has a positive answer.

**Theorem 5.** Let \( 0 < p < n/2 \) and let \( K \) be an origin-symmetric star body in \( \mathbb{R}^n \). If \( K \) is sufficiently close to the Euclidean ball in the Banach-Mazur distance, then
\[
\int_{S^{n-1}} I_p(\| \cdot \|_{K}^{n+p}) \, d\sigma(\theta) \leq (\kappa_n)^{1 - \frac{2}{p}} (\text{vol}_n(K))^{\frac{2}{p} - 1},
\]
with equality if and only if \( K \) is an ellipsoid centered at the origin.

**Proof.** Since \( K \) is close to the Euclidean ball in the Banach-Mazur distance, we can apply an appropriate linear transformation and assume that \( K \) is close to \( B_2^n \) in the Hausdorff distance, that is,
\[
(1 - \varepsilon)B_2^n \subset K \subset (1 + \varepsilon)B_2^n
\]
for some small \( \varepsilon > 0 \).

Applying another linear transformation we can put \( K \) in isotropic position, i.e., a position for which the following holds:
\[
\int_K x_i x_j \, dx = \lambda \delta_{ij},
\]
for some positive constant \( \lambda \) and all \( 1 \leq i, j \leq n \); see [BGVV, Section 2.3.2] for details.

After putting \( K \) in isotropic position, one can check that it is still close to \( B_2^n \) in the Hausdorff distance; see [ANRY, Section 4]. So we can assume that (14) holds (with a different \( \varepsilon \)).
Let us write
\[ \|x\|^{-n+p}_K = H_0(1 + \varphi(x)), \quad x \in S^{n-1}, \tag{15} \]
where \( H_0 \) is a constant (the harmonic of order zero in the spherical harmonic expansion of \( \|x\|^{-n+p}_K \) and \( \int_{S^{n-1}} \varphi(x) d\sigma(x) = 0 \).

Note that (14) implies
\[ (1 + \varepsilon)^{-n+p} \leq \|x\|^{-n+p}_K \leq (1 - \varepsilon)^{-n+p} \]
for all \( x \in S^{n-1} \). Therefore,
\[ (1 + \varepsilon)^{-n+p} \leq H_0 \leq (1 - \varepsilon)^{-n+p}. \]

Since \( H_0 \) is close to one, if we dilate \( K \) by a factor of \( \left(\frac{H_0}{n - p}\right)^{1/(n - p)} \), \( K \) will still be close to \( B^n_2 \) in the Hausdorff metric. So from now on we will assume that
\[ \|x\|^{-n+p}_K = 1 + \varphi(x), \tag{16} \]
where \( \max_{x \in S^{n-1}} |\varphi(x)| < \varepsilon \) and \( \int_{S^{n-1}} \varphi(x) d\sigma(x) = 0 \).

Let
\[ \sum_{m \geq 2, \text{ even}} H_m \]
be the spherical harmonic expansion of \( \varphi \).

Recall that \( K \) is in isotropic position. We will show that this implies that
\[ \|H_2\|_2 \leq C\varepsilon \|\varphi\|_2. \tag{17} \]
Indeed, let \( H(x) = \sum_{i,j=1}^{n} a_{ij} x_i x_j \) be a harmonic quadratic polynomial on \( \mathbb{R}^n \). Observe that we necessarily have \( \sum_{i=1}^{n} a_{ii} = 0 \).

Therefore,
\[ \int_{S^{n-1}} \|x\|^{-n+2}_K H(x) dx = (n + 2) \int_{K} H(x) dx \]
\[ = (n + 2) \sum_{i,j=1}^{n} a_{ij} \int_{K} x_i x_j dx = (n + 2) \sum_{i,j=1}^{n} a_{ij} \delta_{ij} = 0. \]
Thus \( \|x\|^{-n+2}_K \) has no second order harmonic in its spherical harmonic expansion.

Raising (16) to the power \( (n+2)/(n-p) \) and using the Taylor expansion, we get
\[ \|x\|^{-n+2}_K - 1 - \frac{n + 2}{n - p} \varphi(x) \]
\[ \leq C\varepsilon |\varphi(x)|. \]
Taking the \( L^2 \)-norms of both sides and keeping only the second order harmonic in the left-hand side, we obtain (17).

We will now compute the left-hand side of (13). To this end, applying \( I_p \) to both sides of (16) and raising to the power \( n/p \), we get
\[ \left| I_p(\|\cdot\|^{n+p}_K)(\theta) \right|^{n/p} = |1 + I_p \varphi(\theta)|^{n/p}, \tag{18} \]
for all \( \theta \in S^{n-1} \).
To expand the right-hand side of the latter equality, we will need the following observation. For a fixed \( \alpha > 1 \), let \( \zeta(t) = |1 + t|^\alpha - 1 - \alpha t - \frac{\alpha(\alpha - 1)}{2}t^2 \), for \( t \in \mathbb{R} \). We claim that

\[
|\zeta(t)| \leq \begin{cases} 
D|t|^\alpha, & \text{if } 2 < \alpha \leq 3, \\
D(|t|^3 + |t|^\alpha), & \text{if } 3 \leq \alpha,
\end{cases}
\]  

(19)

where \( D \) is a constant (depending on \( \alpha \)). To prove the claim, observe that \( \zeta(t) \) divided by the right-hand side of (19) is a continuous function of \( t \in \mathbb{R} \setminus \{0\} \) with finite limits when \( t \to 0 \) and \( t \to \pm \infty \).

Thus,

\[
\int_{S^{n-1}} |I_p(\| \mathbf{r}^n \|_K + \alpha)^{n/p}| \sigma(	heta)^{n/p} d\sigma(	heta) = \int_{S^{n-1}} |1 + I_p\varphi(\theta)|^{n/p} d\sigma(	heta)
\]

\[
= \int_{S^{n-1}} \left(1 + \frac{n}{p}I_p\varphi(\theta) + \frac{n(n-p)}{2p^2} \varphi^2(\theta) + \zeta(I_p\varphi(\theta))\right) d\sigma(\theta).
\]

Since \( \varphi \) has no spherical harmonic of degree zero, neither does \( I_p\varphi \). That is,

\[
\int_{S^{n-1}} I_p\varphi(\theta) d\sigma(\theta) = 0.
\]

Denoting

\[
R_1 = \int_{S^{n-1}} (I_p\varphi(\theta)) d\sigma(\theta),
\]

we get

\[
\int_{S^{n-1}} |I_p(\| \mathbf{r}^n \|_K + \alpha)^{n/p}| \sigma(\theta)^{n/p} d\sigma(\theta) = 1 + \frac{n(n-p)}{2p^2}\| I_p\varphi \|^2 + R_1.
\]  

(20)

We will now show that \( R_1 = o(\|\varphi\|^2_2) \). By Theorem 2 there is a constant \( C \) (depending on \( n \) and \( p \)) such that

\[
\| I_p\varphi \|_{n/p} \leq C\|\varphi\|_2.
\]

When \( 2 < n/p \leq 3 \), by (19) we have

\[
|R_1| \leq D \int_{S^{n-1}} |I_p\varphi(\theta)|^{n/p} d\sigma(\theta) = D\| I_p\varphi \|_{n/p}^{n/p} \leq C^{n/p}D\|\varphi\|_2^{n/p} = o(\|\varphi\|^2_2).
\]

When \( 3 \leq n/p \), (19) yields

\[
|R_1| \leq D \left( \int_{S^{n-1}} |I_p\varphi(\theta)|^{n/p} d\sigma(\theta) + \int_{S^{n-1}} |I_p\varphi(\theta)|^3 d\sigma(\theta) \right)
\]

\[
= D \left( \| I_p\varphi \|_{n/p}^{n/p} + \| I_p\varphi \|^3_{n/p} \right)
\]

\[
\leq D \left( C^{n/p}\|\varphi\|_2^{n/p} + C^3\|\varphi\|^2_2 \right) = o(\|\varphi\|^2_2).
\]

Thus, in both cases, \( R_1 = o(\|\varphi\|^2_2) \).

We will now compute the right-hand side of (13). Using (16) we get

\[
\| x \|_K^n = (1 + \varphi(x))^{n/(n-p)} = 1 + \frac{n}{n-p}\varphi(x) + \frac{1}{2}\frac{np}{(n-p)^2}\varphi^2(x) + \eta(x),
\]
where
\[ |\eta| \leq c\varepsilon\varphi^2, \]
for some constant \( c \).

Using that the integral of \( \varphi \) over the sphere vanishes, we get
\[
\text{vol}_n(K) = \kappa_n \int_{S^{n-1}} \|x\|^{-n} \, d\sigma(x) = \kappa_n \left( 1 + \frac{np}{2(n-p)^2} \int_{S^{n-1}} \varphi^2(x) \, d\sigma(x) + R_2 \right),
\]
where \( R_2 = o(\|\varphi\|_2^2) \).

Hence,
\[
(n_{\kappa})^{1-\frac{n}{p}} \text{vol}_n(K)^{\frac{n-1}{p}} = 1 + \frac{n}{2(n-p)}\|\varphi\|_2^2 + R_2,
\]
where \( R_2 \) is different from that above, but is still of order \( o(\|\varphi\|_2^2) \).

Let us now compare (20) and (21). Since \( R_1 \) and \( R_2 \) are of order \( o(\|\varphi\|_2^2) \), to finish the proof we need to show that
\[
\frac{n(n-p)}{2p^2} \|I_p\varphi\|_2^2 \leq \frac{n}{2(n-p)}\|\varphi\|_2^2 + o(\|\varphi\|_2^2),
\]
provided \( \|\varphi\|_2 \) is sufficiently small.

Indeed,
\[
\frac{n(n-p)}{2p^2} \|I_p\varphi\|_2^2 = \frac{n(n-p)}{2p^2} \left( \lambda_2^2(n,p)\|H_2\|_2^2 + \sum_{m \geq 4, \text{ m even}} \lambda_m^2(n,p)\|H_m\|_2^2 \right)
\leq \frac{n(n-p)}{2p^2} \left( \lambda_2^2(n,p)\|H_2\|_2^2 + \lambda_4^2(n,p)\sum_{m \geq 4, \text{ m even}} \|H_m\|_2^2 \right)
= \frac{n(n-p)}{2p^2} \left( (\lambda_2^2(n,p) - \lambda_4^2(n,p))\|H_2\|_2^2 + \lambda_4^2(n,p)\sum_{m \geq 4, \text{ m even}} \|H_m\|_2^2 \right)
= o(\|\varphi\|_2^2) + \frac{n(n-p)}{2p^2} \lambda_4^2(n,p)\|\varphi\|_2^2
= o(\|\varphi\|_2^2) + \frac{n}{2(n-p)} \frac{(p+2)^2}{(n-p+2)^2} \|\varphi\|_2^2.
\]

Since \( \frac{(p+2)^2}{(n-p+2)^2} < 1 \), (22) follows.

\( \square \)

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