WHAT IS AN INTERNAL GROUPOID?

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Abstract. An answer to the question investigated in this paper brings a new characterization of internal groupoids such that: (a) it holds even when finite limits are not assumed to exist; (b) it is a full subcategory of the category of involutive-2-links, that is, a category whose objects are morphisms equipped with a pair of interlinked involutions. This result highlights the fact that even though internal groupoids are internal categories equipped with an involution, they can equivalently be seen as tri-graphs with an involution. Moreover, the structure of a tri-graph with an involution can be further contracted into a simpler structure consisting of one morphism with two interlinked involutions. This approach highly contrasts with the one where groupoids are seen as reflexive graphs on which a multiplicative structure is defined with inverses.

1. Introduction

It is needless to say that internal groupoids are an important class of internal categories which have been thoroughly investigated on a diversity of subtopics in category theory with relevant applications to other areas such as in particular algebra and geometry. A celebrated paper by D. Bourn [5] has certainly contributed to improve and enrich the field on the algebraic side with the subsequent aid of many researchers who became interested in the remarkable properties of the fibration of points and its classifying properties (5 see also [2]). On the side of geometry, the book Topology and Groupoids by R. Brown has been influential in continuing the ideas and work of Grothendieck [12] and Ehresmann [10] on differentiable groupoids and on the fundamental groupoid of a space [8, 9]. Although an internal groupoid is an instance of an internal category, Brandt [7] predates Eilenberg and Mac Lane [11] in delineating an axiomatic portrait of a (connected) groupoid ([21] Remark 19.3.12). Lie groupoids are internal groupoids in the category of smooth manifolds while topological groupoids are internal groupoids in the category of topological spaces. Thus, groupoids are a crossroad between algebra and geometry by virtue of category theory. For a survey see e.g. [13, 22].

It is well known that on some grounds the structure of an internal groupoid simplifies itself due to a coincidence between property and structure. For example, groupoids internal to the category of abelian
groups (or any abelian category) are nothing but group homomorphisms whereas groupoids internal to the category of groups (or any semiabelian category) are the same as internal crossed-modules [15]. One draw-back of working with a category of internal groupoids is the complexity of its plain structure. It involves a reflexive graph, together with a morphism expressing multiplication (or composition) which requires an appropriate pullback as domain (making composable pairs of arrows meaningful), it requires an involution providing inverses, as well as several suitable compatibility conditions, not to mention the iterated pullback necessary to express associativity (Section 2). For that reason various authors have restricted the study of internal groupoids to situations in which the plain structure is significantly reduced due to good properties displayed by the ambient category. That is the case for example of Mal’tsev and weakly Mal’tsev categories in which the category of internal groupoids is a full subcategory of the category of reflexive graphs (see e.g. [19], see also [18], Section 3). Although the algebraically flavoured approach (in which a groupoid is seen as a set equipped with a partial multiplication) is not satisfactory from the point of view of internal structures, it does suggest a simplification on the structure as proposed by Brandt [7].

The purpose of this paper is to show that independently of its ambient, the category of internal groupoids is always a full subcategory of the category whose objects are morphisms equipped with a pair of interlinked involutions. This category, in some respects, is even simpler than the category of reflexive graphs. It will be called the category of involutive-2-links (Section 3).

Throughout this paper we work on an ambient category in which no limits nor colimits are assumed to exist (a good example to bear in mind is the category of smooth manifolds). In particular this means that pullbacks and pushouts are to be understood as properties of commutative squares. When a square is at the same time a pullback and a pushout we call it an exact square (such squares are also called bicartesian squares, Dolittle diagrams or pulation squares [1]). A span which can be completed into an exact square is called an exact span. In the same way, a cospan which can be completed into an exact square is called an exact cospan. A digraph (i.e. a pair of parallel morphisms) which is at the same time an exact span and an exact cospan is called bi-exact. This notion will be needed in the main result (Theorem [1]).

2. The structure of an internal groupoid

An internal groupoid can be presented as a structure of the form

\[
\begin{array}{c}
C_2 \xrightarrow{\pi_2} C_1 \xrightarrow{i} C_0 \\
\pi_1 \end{array}
\]

(1)
such that
\[ \begin{align*}
  de &= 1_{C_1} = ce \\
  dm &= d\pi_2, \quad cm = c\pi_1, \quad d\pi_1 = c\pi_2 \\
  di &= c, \quad ci = d, \quad i^2 = 1_{C_1}, \quad ie = e
\end{align*} \] (2, 3, 4)

and satisfying the following further properties:

1. the commutative square

\[ \begin{array}{ccc}
  C_2 & \xrightarrow{\pi_2} & C_1 \\
  \pi_1 \downarrow & & \downarrow c \\
  C_1 & \xrightarrow{d} & C_0
\end{array} \] (5)

is a pullback square;

2. \( m(1_{C_1}, ed) = 1_{C_1} = m(ec, 1_{C_1}) \);

3. \( m(1_{C_1}, i) = ec, \quad m(i, 1_{C_1}) = ed \);

4. the cospan \( C_2 \xrightarrow{d\pi_2} C_0 \xleftarrow{e} C_1 \) can be completed into a pullback square

\[ \begin{array}{ccc}
  C_3 & \xrightarrow{\pi_2} & C_1 \\
  \pi_1 \downarrow & & \downarrow c \\
  C_2 & \xrightarrow{d\pi_2} & C_0
\end{array} \] (6)

5. \( m(1 \times m) = m(m \times 1), \) where \( (1 \times m), (m \times 1): C_3 \to C_2 \) are morphisms uniquely determined as

\[ \begin{align*}
  \pi_2(m \times 1) &= p_2 \\
  \pi_1(m \times 1) &= m \\
  \pi_2(1 \times m) &= m(\pi_2 p_1, p_2) \\
  \pi_1(1 \times m) &= \pi_2.
\end{align*} \]

There is, clearly, a redundancy in this structure and its properties. For example, as soon as the commutative square \( d\pi_1 = c\pi_2 \), displayed as (5), is required to be a pullback, the morphisms \( \pi_1 \) and \( \pi_2 \) are uniquely determined up to isomorphism. A remarkable observation due to D. Bourn \[ \] shows that the existence of a morphism \( i: C_1 \to C_1 \) such that \( di = c, \ i^2 = 1_{C_1}, \ ie = e \) and \( m(1_{C_1}, i) = ec, \ m(i, 1_{C_1}) = ed \) can be interpreted as a property. Indeed, the existence of the involution morphism \( i \) is equivalent to the property that the commutative square \( dm = d\pi_2 \) is a pullback. This is of course consistent with the general fact that inverses, when they exist, are uniquely determined. However, it raises the question of finding a minimal structure that can carry enough information to encode the notion of internal groupoid.
With a few exceptions (e.g. [6, 14]), most papers so far have consid-
ered the case in which pullbacks are available as a canonical construc-
tion and describe an internal groupoid as a structure of the form
\[ C_2 \xrightarrow{m} C_1 \xrightarrow{d} C_0 \]  
(7)
in which the morphisms \( \pi_1 \) and \( \pi_2 \) are canonically obtained by requiring
the object \( C_2 \) to be the result of taking the pullback of \( c \) along \( d \) as displayed in (5). If the involution morphism \( i \) is explicitly needed then
it is recovered from the pullback square \( dm = d\pi_2 \).

Much work has been done in studying internal groupoids focused on
their underlying reflexive graphs. This is because in some algebraic
contexts such as Mal’tsev or weakly Mal’tsev categories [19] the mor-
phism \( m \) is uniquely determined provided it exists.

The purpose of this paper is to show that a simpler description is
equally possible in general categories if we shift the attention from t he
underlying reflexive graph to the underlying multiplicative structure.

We will show that the relevant information is enclosed in the tri-
graph with involution
\[ C_2 \xrightarrow{\pi_2} \xleftarrow{\pi_1} C_1 \xrightarrow{i} \]  
(8)
from which the square (5) is obtained by taking the pushout of \( \pi_1 \) and
\( \pi_2 \). In fact, we will see that this tri-graph can be further contracte d
into a single morphism \( m: C_2 \to C_1 \) equipped with a pair of involutions
\( \theta, \varphi: C_2 \to C_2 \) satisfying the condition \( \theta \varphi \theta = \varphi \theta \varphi \). Note that the
subgroup generated by \( \theta \) and \( \varphi \) is the Dihedral group of order 6.

The structure of a \textit{link} [20] was introduced to model planar curves
and it consists of a morphism \( f: A \to B \) equipped with an endomor-
phism \( \alpha: A \to A \) with no further conditions (even tough \( \alpha \) is often
required to be an isomorphism). When \( \alpha \) is an involution we refer to it
as an \textit{involutive-link}. For example, the domain morphism \( d: C_1 \to C_0 \)
together with the involution morphism \( i \) is an involutive-link. It is
thus appropriate to speak of an \textit{involutive-2-link} when we have a pair
of involutions \( (\theta, \varphi) \) which are interlinked by the formula \( \theta \varphi \theta = \varphi \theta \varphi \).
The two involutions \( \theta \) and \( \varphi \) that we will consider are determined (in a
unique way) from the structure displayed in (8) by equations (13)–(15)
below. If we write \( m(x, y) \) as \( xy \) and \( i(x) \) as \( x^{-1} \) then \( \theta(x, y) = (x^{-1}, xy) \)
and \( \varphi(x, y) = (xy, y^{-1}) \), assuming of course that \( C_2 \) is the pullback of
\( d \) and \( c \) as illustrated in the structure of an internal groupoid displayed
in (1). A list of examples is provided in Section 4.

The two main observations of this paper are (a) the category of
internal groupoids is a full subcategory of the category of involutive-
2-links and (b) an involutive-2-link is an internal groupoid if and only
if the conditions of Theorem 1 are satisfied. Moreover, straightforward
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generalizations of these results are expected in higher dimensional categorical structures such as Loday’s cat-$n$-groups [16]. Furthermore, applications to the categorical Galois theory of G. Janelidze [14] are expected as well.

3. THE CATEGORY OF INVOLUTIVE-2-LINKS

The category of involutive-2-links is the category whose objects are triples $(\theta, \varphi, m)$ in which $m: A \to B$ is a morphism while $\theta, \varphi: A \to A$ are involutions, i.e., $\theta^2 = \varphi^2 = 1_A$, that satisfy the interlinked condition $\theta\varphi\theta = \varphi\theta\varphi$. Moreover, for convenience, we will restrict our attention to the triples $(\theta, \varphi, m)$ with the property that the three parallel morphisms $m, m\theta$ and $m\varphi$ are jointly monomorphic. This means that any morphism with codomain $A$ is uniquely determined as soon as the result of precomposing it with $m, m\theta$ and $m\varphi$ is known. This is clearly the case if at least one of the three pairs of morphisms $(m, m\theta), (m, m\varphi)$ or $(m\varphi, m\theta)$ are obtained as pullback squares, which will be the case. Indeed, as it will be apparent later on, the triple $(\theta, \varphi, m)$ is used to recover the tri-graph displayed in (8) by setting $\pi_1 = m\varphi$ and $\pi_2 = m\theta$.

A morphism in the category of involutive-2-links, say from a triple $(\theta, \varphi, m: A \to B)$ to a triple $(\theta', \varphi', m': A' \to B')$, consists of a single morphism $f: B \to B'$ with the property that there exists a morphism $\bar{f}: A \to A'$ such that

$$m'\bar{f} = fm$$
$$\theta'\bar{f} = \bar{f}\theta$$
$$\varphi'\bar{f} = \bar{f}\varphi.$$ (11)

It is worth noting that the requirement on the triple $(\theta', \varphi', m')$ asserting that the three parallel morphisms

$$A' \xrightarrow{m'\theta'} m\varphi \xrightarrow{m'} B'$$

are jointly monomorphic has the effect of uniquely determining the morphism $\bar{f}$ as

$$m'\bar{f} = fm$$
$$m'\theta'\bar{f} = fm\theta$$
$$m'\varphi'\bar{f} = fm\varphi.$$ (12)

This restriction on involutive-2-links (of being jointly monomorphic) gives rise to a fully faithful functor that associates a triple $(\theta, \varphi, m)$ to every internal groupoid such as the one displayed in (8), according to
the formulas:

\[ m\varphi = \pi_1, \quad m\theta = \pi_2 \quad (13) \]
\[ \pi_1\varphi = m, \quad \pi_1\theta = i\pi_1 \quad (14) \]
\[ \pi_2\varphi = i\pi_2, \quad \pi_2\theta = m. \quad (15) \]

Let us observe that even when \( i \) is not made explicit, the two involutions \( \theta \) and \( \varphi \) are still uniquely determined because the object \( C_2 \) is not only the pullback of \( d \) and \( c \) but also the kernel pair of \( d \) and the kernel pair of \( c \) and hence both pairs \( (m, \pi_1) \) and \( (m, \pi_2) \) are in particular jointly monomorphic.

**Proposition 1.** There exists a fully faithful functor from the category of internal groupoids to the category of involutive-2-links. The functor takes an internal groupoid such as the one displayed in (1) and converts it into the triple \((\theta, \varphi, m)\) in which \( m: C_2 \to C_1 \) is the morphism carrying the multiplicative structure of the groupoid while the endomorphisms \( \theta, \varphi: C_2 \to C_2 \) are determined as \( \theta = \langle i\pi_1, m \rangle \) and \( \varphi = \langle m, i\pi_2 \rangle \).

Taking into account that \( C_2 \) is the pullback of \( d \) and \( c \).

**Proof.** In some sense the functor keeps the underlying multiplicative structuring morphism \( m: C_2 \to C_1 \) and contracts the relevant information displayed in (8) as two endomorphisms, \( \theta \) and \( \varphi \). It is thus clear that the tri-graph

\[ C_2 \xrightarrow{m \varphi = \pi_1} C_1 \]
\[ C_2 \xrightarrow{m \theta = \pi_2} C_1 \]

is jointly monomorphic. In fact it is much more, as \((\pi_1, \pi_2)\) is obtained by pullback, while \((m, \pi_2)\) and \((\pi_1, m)\) are the kernel pairs of the morphisms \( d \) and \( c \), respectively. It is routine to check that \( \theta \) and \( \varphi \) are involutions and that the interlinked condition \( \theta \varphi \theta = \varphi \theta \varphi \) is satisfied. This also explains why the functor is faithful and well defined on morphisms. To prove that it is full, let us consider two internal groupoids with the relevant tri-graph structure and involution as illustrated in diagram (8), together with a morphism \( f: C_1 \to C_1' \) such that there exists a (unique) \( \tilde{f}: C_2 \to C_2' \) satisfying the three conditions (9)–(11) with \( \theta, \theta', \varphi, \varphi' \) obtained as in (13)–(15), as illustrated.

\[ C_2 \xrightarrow{m \varphi = \pi_1} C_1 \]
\[ \tilde{f} \]
\[ C_2' \xrightarrow{m' \varphi' = \pi'_1} C_1' \]

\[ m\theta = \pi_2 \]
\[ m\theta' = \pi_2' \]

(17)
Under these conditions, it follows that \( f'f = fi \) which is then used in collaboration with the fact that \( C_2 \) and \( C'_2 \) are pullbacks to assert that

\[
\bar{f}(1, ed) = (1, ed)f \\
\bar{f}(ec, 1) = (ec, 1)f
\]

from which we obtain the complete diagram

\[
\begin{array}{ccc}
C_2 & \xrightarrow{i} & C_1 \\
\downarrow{\pi_1} & \downarrow{d} & \downarrow{c} \\
C_0 & \xrightarrow{f} & C_0 \\
\downarrow{\pi_0} & \downarrow{f_0} & \\
C'_2 & \xrightarrow{d'} & C'_1 \\
\downarrow{\pi'_1} & \downarrow{c'} & \\
C'_0 & \xrightarrow{f_0} & C'_0
\end{array}
\]

with \( f_0 \) being uniquely determined by the property that \( f_0d = d'f \) and \( f_0c = c'f \). Moreover, \( fe = e'f_0 \) follows from one of the assertions (18) or (19).

4. Examples

The examples are provided with the notation of sets and maps but are supposed to be defined in any category where the required constructions are possible. In each case we provide a set \( C_1 \), to be interpreted as the set of arrows, a set \( C_2 \), interpreted as the set of composable arrows, one function \( m: C_2 \to C_1 \) and two involutions \( \theta, \varphi: C_2 \to C_2 \) such that \( \theta\varphi \theta = \varphi\theta\varphi \). In addition, the tri-graph

\[
\begin{array}{ccc}
C_2 & \xrightarrow{m\theta} & C_1 \\
\downarrow{m\varphi} & & \\
C_2 & \xrightarrow{m\theta} & C_1
\end{array}
\]

is a ternary relation.

Let \( X \) be a set, \( G = (G, \cdot, 1, (-1)^{-1}) \) a group, \( M = (M, \cdot, 1) \) a monoid, \( S = (S, \cdot, (-)^{-1}) \) and inverse semigroup and \( U = (U_i)_{i \in I} \) an open cover of a smooth manifold:

1. The set \( X \), if considered as a discrete groupoid, becomes an involutive-2-link as:
   - (a) \( C_1 = X, C_2 = X, m(x) = x; \)
   - (b) \( \theta(x) = x, \varphi(x) = x. \)

2. Whereas if the set \( X \) is considered as a co-discrete groupoid, it becomes the involutive-2-link:
   - (a) \( C_1 = X \times X, C_2 = X \times X \times X; \)
   - (b) \( m(x, y, z) = (x, z); \)
   - (c) \( \theta(x, y, z) = (y, x, z); \)
   - (d) \( \varphi(x, y, z) = (x, z, y). \)

3. If an equivalence relation \( R \subseteq X \times X \) is interpreted as a groupoid then it becomes an involutive-2-link as:
(a) $C_1 = R$, $C_2 = \{(x, y, z) \in X \times X \times X \mid xRyRz\};$
(b) $m(x, y, z) = (x, z);$
(c) $\theta(x, y, z) = (y, x, z);$
(d) $\varphi(x, y, z) = (x, z, y).$

(4) The Čech groupoid associated to the open cover $U$ is obtained in the form of an involutive-2-link as:
(a) $C_1 = \bigcup U_{ij},$ $C_2 = \bigcup U_{ijk}$
(b) $m(x_{ijk}) = x_{ik};$
(c) $\theta(x_{ijk}) = x_{jik};$
(d) $\varphi(x_{ijk}) = x_{ikj}.$

(5) The group $G$, considered as a one object groupoid gives rise to an involutive-2-link of the form:
(a) $C_1 = G,$ $C_2 = G \times G;$
(b) $m(a, b, x) = (ab, x);$
(c) $\theta(a, b, x) = (a^{-1}, ab, x);$
(d) $\varphi(a, b, x) = (ab, b^{-1}, \xi(b, x)).$

(6) If $\xi : G \times X \to X$ is a $G$-action on the set $X$, then it can be seen as an involutive-2-link as:
(a) $C_1 = G \times X,$ $C_2 = G \times G \times X;$
(b) $m(a, b, x) = (a^{-1}, ab, x);$
(c) $\theta(a, b, x) = (ab, b^{-1}, \xi(b, x)).$

(7) If $h : G \to M$ is a homomorphism from the group $G$ to the monoid $M$ then it can be seen as an involutive-2-link as:
(a) $C_1 = G \times M,$ $C_2 = G \times G \times M;$
(b) $m(a, b, c) = (ab, c);$
(c) $\theta(a, b, c) = (a^{-1}, ab, c);$
(d) $\varphi(a, b, c) = (ab, b^{-1}, h(b) + c).$

In addition, if $M$ acts on $G$ in the fashion of a crossed-module then the involutive-2-link is an internal structure in the category of monoids. Let us nevertheless observe that although the action $\xi$ on the structure of a crossed-module is necessary to equip the set $C_1 = G \times M$ with the semidirect product $G \rtimes_{\xi} M$, the underlying groupoid is defined as soon as the homomorphism $h : G \to M$ is specified.

(8) The inverse semigroup $S$, if interpreted as a groupoid in the form of an involutive-2-link, becomes:
(a) $C_1 = S,$ $C_2 = \{(x, y) \in S \times S \mid x^{-1}x = yy^{-1}\};$
(b) $m(x, y) = xy;$
(c) $\theta(x, y) = (x^{-1}, xy);$ 
(d) $\varphi(x, y) = (xy, y^{-1}).$

(9) The simplest non-trivial involutive-2-link which is not obtained from a groupoid is:
(a) $C_1 = \{0, 1\},$ $C_2 = \{1, 2, 3\};$
(b) $m : [123] \to [010];$
(c) \( \theta : [123] \mapsto [213] \);
(d) \( \varphi : [123] \mapsto [132] \).

(10) One last construction is obtained by combining several examples above. Let \( B \) be a set and consider two maps \( g : S \to B \) and \( \phi : B \times X \to X \) (with \( S \) and \( X \) as above) such that

\[
\begin{align*}
\phi(g(s^{-1}s), x) &= x = \phi(g(s s^{-1}), x) \\
\phi(g(s's), x) &= \phi(g(s'), \phi(g(s), x)).
\end{align*}
\]

Every subset \( R \subseteq S \times X \) satisfying the following conditions

(i) if \((s, x) \in R\) then \((s^{-1}s, x) \in R\) and \((ss^{-1}, \phi(g(s), x)) \in R\)

(ii) if \((s, x) \in R\) and \((s', \phi(g(s), x)) \in R\) then \((s's, x) \in R\)
gives rise to an involutive-2-link in which:

(a) \( C_1 = R \) while \( C_2 \subseteq S \times S \times X \) consists of those triples \((s', s, x) \in S \times S \times X\) such that \( ss'^{-1}s' = ss^{-1}, (s, x) \in R\) and \((s', \phi(g(s), x)) \in R\)

(b) \( m(s's, s, x) = (s's, x) \)

(c) \( \theta(s', s, x) = (s'^{-1}, ss', x) \)

(d) \( \varphi(s', s, x) = (s's, s^{-1}, \phi(g(s), x)) \).

If \( B \) is a monoid and \( g \) is a homomorphism then we recover example (7) by putting \( X = B \) and \( \phi(b, x) = b + x \). On the other hand, if \( g \) is a bijection then we recover example (6). Examples (5) and (8) are recovered as well.

It is worthwhile noting under which conditions an involutive magma can be seen as an involutive-2-link.

**Proposition 2.** Let \( m : X \times X \to X \) be a magma structure on the set \( X \) together with an involution \( i : X \to X \) such that \( im = m(i \pi_2, i \pi_1) \). The triple \((\theta, \varphi, m)\), with \( \theta = \langle i \pi_1, m \rangle \) and \( \varphi = \langle m, i \pi_2 \rangle \), is an involutive-2-link if and only if:

\[
\begin{align*}
m(i(x), m(x, y)) &= y \\
m(m(x, y), i(y)) &= x
\end{align*}
\]

for all \( x, y \in X \).

**Proof.** The two conditions above are equivalent to the requirement that \( \theta \) and \( \varphi \) are involutions. The interlinked condition \( \theta \varphi \theta = \varphi \theta \varphi \) follows from the hypotheses \( i(m(x, y)) = m(i(y), i(x)) \).

\[\square\]

5. THE MAIN RESULT

In this section we characterize those involutive-2-links which are groupoids, i.e., that are in the image of the functor described in Proposition [1].

Let us first observe some necessary conditions. The notion of contractible pair in the sense of Beck can be found in [17], p. 150.
Proposition 3. If an involutive-2-link \((\theta, \varphi, m)\) is the image of an internal groupoid then the pairs \((m, m\theta)\) and \((m, m\varphi)\) are jointly monomorphic and contractible in the sense of Beck, i.e., there exist morphisms \(e_1, e_2 : C_1 \rightarrow C_2\) such that

\[
me_1 = 1_{C_1}, \quad m\theta e_1 m = m\theta e_1 m\theta \\
me_2 = 1_{C_1}, \quad m\varphi e_2 m = m\varphi e_2 m\varphi.
\]

Proof. It suffices to take \(e_1 = \langle 1, ed \rangle\) and \(e_2 = \langle ec, 1 \rangle\), while observing that \((m, m\theta)\) and \((m, m\varphi)\) are respectively the kernel pairs of \(d\) and \(c\), the domain and codomain morphisms of the groupoid from which the triple \((\theta, \varphi, m)\) is obtained as in Proposition 1.

We say that a digraph (i.e. a pair of parallel morphisms) is bi-exact if when considered as a span it can be completed into a commutative square which is both a pullback and a pushout and moreover, if considered as a cospan, it can be completed into another commutative square which is both a pullback and a pushout. In other words, a digraph such as

\[
\text{Diagram 26}
\]

is bi-exact precisely when the zig-zag

\[
\text{Diagram 27}
\]

can be completed with two commutative squares

\[
\text{Diagram 28}
\]

which are both simultaneously a pullback and pushout.

Proposition 4. If an involutive-2-link \((\theta, \varphi, m)\) is the image of an internal groupoid then the pair \((m\varphi, m\theta)\) is bi-exact.

Proof. If \((\theta, \varphi, m)\) is obtained from an internal groupoid then \(m\varphi = \pi_1\) and \(m\theta = \pi_2\) which can be completed into commutative squares as displayed in (28). The two squares are pullbacks by hypotheses. The
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fact that the two squares are also pushouts is an easy consequence of
the fact that they are split squares. □

The following result presents the desired characterization of those
involutive-2-links which are the image of internal groupoids.

**Theorem 1.** An involutive-2-link \((\theta, \varphi, m: C_2 \to C_1)\) is an inter-

nal groupoid if and only if the pairs \((m, m\theta)\) and \((m, m\varphi)\) are jointly

monomorphic, there exist two morphisms \(e_1, e_2: C_1 \to C_2\) such that

\[
me_1 = 1_{C_1} = me_2 \quad (29)
\]

\[
\theta e_2 = e_2, \quad \varphi e_1 = e_1 \quad (30)
\]

\[
m\theta \varphi e_2 = m\varphi \theta e_1 \quad (31)
\]

\[
m\theta e_1 m\varphi = m\varphi e_2 m\theta \quad (32)
\]

\[
m\theta e_1 m = m\theta e_1 m\theta, \quad m\varphi e_2 m = m\varphi e_2 m\varphi, \quad (33)
\]

\[
the\ \pair\ \(m\varphi, m\theta)\ \is\ bi-exact\ \(\text{as\ illustrated\ in\ diagram\ (28)}\ \with\ m\varphi\ \as\ \pi_1\ \and\ m\theta\ \as\ \pi_2\),\ \and\ the\ \two\ \induced\ \morphisms\ \m_1, m_2: C_3 \to C_2,\ \determined\ by
\]

\[
\pi_1 m_1 = m p_1, \quad \pi_2 m_1 = \pi_2 p_2
\]

\[
\pi_1 m_2 = \pi_1 p_1, \quad \pi_2 m_2 = m p_2
\]

are such that \(mm_1 = mm_2\).

The proof is mainly technical checking and it involves routine cal-
culations, so we omit the details. It is clear that if \((\theta, \varphi, m)\) is obtained
from an internal groupoid then all conditions are satisfied. On the other
hand, the fact that the pairs \((m, m\theta)\) and \((m, m\varphi)\) are jointly

monomorphic uniquely determines the morphisms \(e_1\) and \(e_2\). The in-

volution morphism \(i: C_1 \to C_1\) is obtained by condition (31) either as

\(i = m\theta \varphi e_2\) or as \(i = m\varphi \theta e_1\). The morphism \(e: C_0 \to C_1\) is uniquely
determined by condition (32) as such that \(ed = m\theta e_1\) and \(ec = m\varphi e_2\).

Conditions (33) and (29) assert the contractibility of the pairs \((m, m\theta)\)
and \((m, m\varphi)\) as in Proposition 3. The condition (30) is a central ingre-
dient in the proof and it gives for example \(e_1 e = e_2 e\). The two mor-
p rhisms \(m_1\) and \(m_2\) are well defined because \(dm = d\pi_2\) and \(cm = c\pi_1\)
(with \(m\varphi\) as \(\pi_1\) and \(m\theta\) as \(\pi_2\)) which follows from the conditions in the

Theorem.

6. Conclusion

With the categorical equivalence established by Theorem 1 and Pro-

position 1 between internal groupoids and suitable involutive-2-links,

it is now a straightforward task to extend it to the level of 2-cells. It

is also tempting to explore the possibility of applying these results to

internal categories rather than internal groupoids.

Some advantages of considering an internal groupoid as an involutive-

2-link are: (i) it can be worked out in arbitrary categories, even when
pullbacks are not well-behaved as for smooth manifolds and the study of Lie groupoids; (ii) it suggests a straightforward generalization of n-dimensional groupoid as for Loday’s cat-n-groups; (iii) it allows a systematic study of several properties exhibited by the fibrations and pre-fibrations used in descent theory and categorical Galois theory; (iv) it provides a convenient way of comparing internal groupoids in different ambient categories by investigating whether they are reduced for example to give a crossed module. In addition, several technical results on particular important cases such as pre-ordered groupoids or topological groupoids can be unified using the common language of involutive-2-links. Finally, the category of involutive-2-links certainly deserves to be studied by itself.

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