We give a detailed treatment of the back-reaction effects on the Hawking spectrum in the semiclassical approach to the Hawking radiation. We solve the exact system of non linear equations giving the action of the system, by a rigorously convergent iterative procedure. The first two terms of such an expansion give the \( O(\omega/M) \) correction to the Hawking spectrum.

Keywords: Hawking radiation; semiclassical; late-time.

1. Introduction

The semiclassical treatment of the Hawking radiation was introduced by Kraus and Wilczek\cite{1}. The interest of the approach is to provide a method to compute the back-reaction effect of the radiation on the black hole, an effect which is completely ignored in the external field treatment of the phenomenon. The main idea is to replace the free field modes of the radiation by the semiclassical wave function of a shell of matter or radiation which consistently propagates in the gravitational field generated by the back hole and by the shell itself. The shell dynamics was studied in detail in many papers (see\cite{1–4}). In the original semiclassical treatment\cite{1,5} the spectrum of the Hawking radiation is extracted through the standard Fourier analysis of the regular modes. Later such a treatment was related to the tunneling picture\cite{6} such an approach gave also rise to several proposals and to controversy\cite{7–12}.

We think that the mode analysis is still the clearest and safest way to extract the results in the semiclassical approach.

The present work\cite{2} is devoted to a detailed analysis of the construction of the semiclassical modes and their time Fourier transform. The action related to the modes which are regular on the horizon is defined through mixed boundary conditions, i.e. a condition on the value of the conjugate momentum at \( t = 0 \) and a condition at time \( t \) on the coordinate \( r \). The computation of the action as a function of \( t \) corresponds to the solution of a system of two highly non linear equations where the two unknown are the value \( H \) of the Hamiltonian and the shell position at time \( t = 0, r_0 \) which also depends on the mixed boundary conditions. In\cite{1} a truncated system of equations obtained by keeping only the most singular terms in the exact equations was considered. Through a long chain of approximations the authors reached for the effective temperature, due to the back reaction effects, the value \( 1/(8\pi M(1 - \omega/M)) \). Later Keski-Vakkuri and Kraus\cite{5} using a completely different method obtained for such effective temperature the value \( 1/(8\pi M(1 - \omega/2M)) \).

Here we reconsider the problem along the lines of\cite{1} treating the full exact system.
of equations. We show that such a system of equations is equivalent to another nonlinear equation which can be solved by a convergent iterative procedure. The first two terms of the convergent iterative procedure are sufficient to provide the leading spectrum of the radiation and its back-reaction correction terms of order $\omega/M$ confirming the result of [2].

2. Choice of gauge, the action and the equations of motion

The rotationally invariant Painlevé-Gullstrand metric is given by
\begin{equation}
    ds^2 = -N^2 dt^2 + (dr + N^r dt)^2 + R^2 d\Omega^2
\end{equation}
where all quantities $N, N^r, R$ are functions only of $r$ and $t$. One has still a gauge choice on $R$. In presence of a shell of matter we shall use the “outer gauge” which is defined by $R = r$ for $r \geq \hat{r}$ where $\hat{r}$ denotes the shell position. At $r = \hat{r}$, $R$ is continuous as all the other functions appearing in (1), but its derivative is discontinuous. After solving the constraints the action takes the form in the massless case [1–4]
\begin{equation}
    S = \int_{t_i}^{t_f} \left(p_c \dot{r} - H\right) dt \quad \text{where} \quad p_c = \sqrt{2M\dot{r} - \sqrt{2H\dot{r} - \dot{r}^2 - 2\sqrt{2M}}}.
\end{equation}

At the semiclassical level the modes which are invariant under the Killing vector $\frac{\partial}{\partial t}$ are simply given by
\begin{equation}
    e^{iS/l^2_P} \quad \text{with} \quad S = \int_{t_0}^{t} p_c r \, dt - \frac{H}{2} + \text{const.}
\end{equation}

where $l^2_P = G\hbar$ is the square of the Planck length. As it is well known such modes have the feature of being singular at the horizon. Instead the true vacuum should be described in terms of modes which are regular at the horizon for which an outgoing shell of matter has the following boundary conditions [1, i) at time 0 the conjugate momentum is a given value $k$; ii) at time $t$ the shell position $r$ is a given value $r_1$. The expression for such an action was already given by Kraus and Wilczek in [1]. With the two conditions $p_c(0) = k$ and $r(t) = r_1$ the action is
\begin{equation}
    S(r_1, t, k) = kr_0(r_1, t, k) + \int_{0}^{t} p_c \dot{r} dt - H[r_1, t, k] t.
\end{equation}

$r_0$ denotes the value of $r$ at time 0; also such a quantity depends on the imposed boundary conditions.

In the outer gauge [1,2,4] the equations of motion can be integrated to
\begin{equation}
    t = 4H \log \frac{\sqrt{r_1} - \sqrt{2H}}{\sqrt{r_0} - \sqrt{2H}} + r_1 - r_0 + 2\sqrt{2Hr_1} - 2\sqrt{2Hr_0}.
\end{equation}

with the boundary condition at $t = 0$
\begin{equation}
    0 < k = \sqrt{2Mr_0} - \sqrt{2Hr_0} - r_0 \log \frac{\sqrt{r_0} - \sqrt{2H}}{\sqrt{r_0} - \sqrt{2M}}.
\end{equation}
where \(2M < 2H < r_0 < r_1\), Eq. (6) together with eq. (5) should determine completely \(H\) as a function of \(t\). However with the above mentioned mixed boundary condition in general caustics arise, i.e. in general more that one trajectory in phase space satisfies the mixed boundary conditions. On the other hand we prove\(^2\) that if the end point \(r_1\), is less that a critical value \(r_c\), caustics to not occur.

3. The late-time expansion

The problem now is to solve the system (5,6). After introducing the implicit time \(T = e^{-\frac{t}{4M}}\) and using the notation \(h = \sqrt{2H}, \ m = \sqrt{2M}\) it is possible to rewrite the system of equations (5,6) as

\[
h = m + T f(T) \equiv m + g(T)
\]

with \(g(0) = 0, g'(T) > 0\). It is then possible, starting from \(h_0 = m\) to solve eq. (7) iteratively, where the process converges rigorously at all times.\(^2\) The first two terms give for the semiclassical mode

\[
e^{i\frac{S}{l_P}} = e^{i [q(r_1)-Mt+4M\sqrt{2H}\tau_1 + \tau_2^2]/l_P^2} \text{ with } \tau_1 = c(k,r_1)e^{-t/(4M)}.
\]

Using the modes \(\mathbf{5}\) to compute the Bogoliubov coefficients one obtains\(^2\) for the flux of the Hawking radiation

\[
F(\omega) = \frac{\omega}{2\pi} \frac{1}{e^{\frac{8\pi\omega}{l_P}} (1 - \frac{\omega^2}{M^2}) - 1}.
\]

in agreement with\(^5\)

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