THE ASYMMETRY OF COMPLETE AND CONSTANT WIDTH BODIES
IN GENERAL NORMED SPACES AND THE JUNG CONSTANT

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ABSTRACT. In this paper we state a one-to-one connection between the Jung constant of a Minkowski space and the maximal Minkowski asymmetry of the complete bodies within that space. This allows to generalize several recent results and to falsify an old assumption of Grünbaum.

1. Introduction

The main result of this paper is a one-to-one relation between the Jung constant of a Minkowski space and the maximal Minkowski-asymmetry of complete bodies within that space.

Geometric inequalities relating different radii of convex bodies, such as Jung’s famous inequality [27] form a central area of research in convex geometry. Starting with [7], in many classical works in convexity, significant parts are devoted to geometric inequalities among the basic radii (see [6, 28, 35, 40]), generalizations (11 10, 15, 25, 33), and between radii and other functionals (3, 8, 26).

There exists a rich variety of asymmetry measures for convex sets (see [20], Sect. 6) for the possibly most comprehensive overview). According to [20], a functional \( \sigma : K^n \to [1, n] \) is an asymmetry measure if \( \sigma \) is affinely invariant and continuous with respect to the Hausdorff distance, s.t. \( \sigma(K) = 1 \) iff \( K = -K \). Frequently it is also required that \( \sigma(K) = n \) iff \( K \) is an \( n \)-dimensional simplex. As it is already claimed in [20], the one that has received by far most interest is the so-called Minkowski asymmetry

\[
s(K) := \inf \{ \rho > 0 : \exists \ c \in \mathbb{R}^n \ \text{s.t.} \ -K \subset c + \rho K \},
\]

where \( K \) may be an arbitrary convex body. It is well known to fulfill all the desired properties above.

For polytopes (given by their vertices or their facets) the Minkowski asymmetry can be computed via Linear Programming (see [2] or [12, Lemma 3.5]). Hence, involving this asymmetry has not only theoretical interest but is useful in computations, too (cf. [11]).

In recent years both, Jung’s inequality as well as the Minkowski asymmetry have drawn renewed attention. Jung’s inequality has been generalized and improved in several ways (see, e.g., [5, 11, 25]) and also the Minkowski asymmetry has been generalized [23]. In [37] the Minkowski asymmetry has been used as a parameter improving geometric inequalities and in [2, 24] especially as a connection between a geometric inequality and its restricted version to symmetric sets. Fundamental results used in the present paper stem from [12]. There, besides others, a sharpened version of Jung’s inequality involving the Minkowski asymmetry has been derived (see Proposition 1.4 below), which is of major interest here, too.

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In [37] Schneider says “as a rule, the first step [...] consists in optimizing a proof of the inequality to make the identification of the equality cases as easy as possible”. We think the following leads in this direction.

Before going into details, some necessary notation has to be stated:

For any $A \subset \mathbb{R}^n$ we write $\text{conv}(A)$ for the convex hull of $A$ and abbreviate by $[x, y] := \text{conv}\{x, y\}$ the line segment with endpoints $x, y \in \mathbb{R}^n$.

For any two sets $A, B \subset \mathbb{R}^n$ and $\rho > 0$, $A + B := \{a + b : a \in A, b \in B\}$ denotes the Minkowski sum of $A$ and $B$ and $\rho A := \{\rho a : a \in A\}$ the $\rho$-dilatation of $A$, abbreviating $-A := (-1)A$.

A Minkowski space $\mathbb{M}^n = (\mathbb{R}^n, \|\cdot\|)$ is an $n$-dimensional real space endowed with a norm $\|\cdot\|$ and we use $\mathbb{B}$ to denote its unit ball. In case the norm is the Euclidean norm, i.e. the Minkowski space is a Euclidean space, we write $\mathbb{E}^n = (\mathbb{R}^n, \|\cdot\|_2)$ and $\mathbb{B}_2$.

Let $\mathcal{K}^n$ be the family of convex bodies $K \subset \mathbb{R}^n$, and $\mathcal{K}_0^n$ its subset formed by centrally symmetric sets, i.e., when $K = -K$. All along the paper we refer to an $n$-dimensional simplex by $S$, and if it is regular (in the Euclidean sense) by $T$.

The support function of a convex body $K \in \mathcal{K}^n$, $h(K, \cdot) : \mathbb{R}^n \to \mathbb{R}$, is defined as $h(K, u) = \max_{x \in K} u^T x$. The outer radius or circumradius $R(K)$ of $K \in \mathcal{K}^n$ is the least dilatation factor $\rho \geq 0$, s.t. a translate of $\rho \mathbb{B}$ contains $K$. In mathematical terms,

$$R(K) := \inf \{\rho > 0 : \exists c \in \mathbb{R}^n \text{ s.t. } K \subset c + \rho \mathbb{B}\} = \inf_{c \in \mathbb{R}^n} \sup_{x \in K} \|x - c\|$$

(see e.g. [17]). Analogously, the inradius $r(K) := \sup \{\rho > 0 : \exists c \in \mathbb{R}^n \text{ s.t. } c + \rho \mathbb{B} \subset K\}$ is the maximal dilatation factor $\rho$, s.t. a translate of $\rho \mathbb{B}$ is contained in $K$. The diameter $D(K)$ of $K$ is defined as $D(K) := 2 \sup_{x, y \in K} R([x, y]) = \sup_{x, y \in K} \|x - y\|$ and the width $w(K)$ of $K$ as $w(K) = \inf_{u \in \mathbb{R}^n} \frac{h(K - K, u)}{h(\mathbb{B}, u)}$.

A set $K \in \mathcal{K}^n$ is said to be of constant width, if $w(K) = \frac{h(K - K, u)}{h(\mathbb{B}, u)}$, independently of the choice of $u$. It is well known that $K$ is of constant width, iff $w(K) = D(K)$, iff $K - K = \mathbb{B}$ [14 (A)]. A set $K$ is called complete if $D(K \cup \{x\}) > D(K)$ for all $x \notin K$ and a set $K^* \supset K$ is a completion of $K$ if $K^*$ is complete and $D(K^*) = D(K)$. We use $\mathcal{K}_{cp}$ and $\mathcal{K}_{cw}$ to denote the subsets of $\mathcal{K}^n$ of all complete bodies and all bodies of constant width, respectively. Again, it is well-known (see e.g. [14]) that $\mathcal{K}_{cw} \subset \mathcal{K}_{cp}$ and $\mathcal{K}_{cw} = \mathcal{K}_{cp}$, in case of $\mathbb{E}^n$. However, these notions are not equivalent (see [32] for many non-coincident properties and several examples) and characterizing Minkowski spaces in which $\mathcal{K}_{cw} = \mathcal{K}_{cp}$, holds is still a major open task in convex geometry (cf. [14, 30]).

For any $K \in \mathcal{K}^n$ we define the Jung ratio $j(K) := R(K)/D(K)$ of $K$ and for any Minkowski space $\mathbb{M}^n$ the Jung constant

$$j(\mathbb{M}^n) := \sup_{K \in \mathcal{K}^n} j(K).$$

It is an easy consequence of the Blaschke Selection Theorem (see e.g. [36]) that this supremum is a maximum. The Jung constant has been widely studied and has been object of many improvements and extensions (e.g. in [5, 6, 11, 12, 25, 27]). The classical inequality of Jung [27] ensures that

$$j(\mathbb{E}^n) = \sqrt{\frac{2}{n(n + 1)}},$$

with equality for $K \in \mathcal{K}^n$, iff $K$ is contained between an $n$-dimensional regular simplex $T$ and its completion $T^*$ [27].

In 1938, Bohnenblust [6] gave an upper bound for the Jung constant for arbitrary Minkowski spaces, which was reproved in [28], where also a full characterization of the equality cases was added.
Proposition 1.1. It holds
\[ j(\mathbb{B}^n) \leq \frac{n}{n + 1}, \]
with equality, iff \( K = S + c' \) and \( \rho(S - S) \subset \mathbb{B} \subset (n + 1)\rho S \) for some \( c' \in \mathbb{R}^n, \rho > 0 \) and some \( n \)-simplex \( S \), which is Minkowski centred (i.e. that (1) is fulfilled with \( c = 0 \)).

Recognize the following: first, \( \mathbb{B} = \rho(S - S) \) means that \( S \) is of constant width in that space, and second that, using the symmetry of \( \mathbb{B} \), we may easily replace the upper bound \( (n + 1)\rho S \) by \( (n + 1)\rho(S \cap (-S)) \).

In [22] the behavior of the Minkowski asymmetry has been studied, when restricted to bodies of constant width in Euclidean spaces. More particularly, the following result has been proved (the characterization of the “only if case” has been left as an open question in [22] and solved separately in [21]).

Proposition 1.2. Let \( K \in \mathcal{K}_{cw} \). Then
\[ s(K) \leq \frac{n + \sqrt{2n(n + 1)}}{n + 2}, \]
with equality if \( K \) is a completion of an \( n \)-dimensional regular simplex.

The following proposition is taken from [12, Corollary 6.3]:

Proposition 1.3. For any Minkowski space and any \( K \in \mathcal{K}^n \) it holds
\[ 2r(K) \leq w(K) \leq (1 + s(K))r(K) \leq r(K) + R(K) \leq \frac{1 + s(K)}{s(K)}R(K) \leq D(K) \leq 2R(K). \]

Proposition 1.3 allows an immediate corollary, summarizing two inequalities we need later. In particular, (3) sharpens Proposition 1.1, but has been restricted to Minkowski space from [12, Theorem 4.1] and (4) generalizes and sharpens an inequality for Euclidean space of Alexander (see [1] and cf. [12, Corollary 6.4]).

Corollary 1.4. For any Minkowski space and any \( K \in \mathcal{K}^n \) it holds
\[ j(K) \leq \frac{s(K)}{s(K) + 1}, \]
and
\[ \frac{r(K)}{D(K)} \leq \frac{1}{s(K) + 1}, \]
and there exist Minkowski spaces and, for which ever value of \( s(K) \) is prescribed, choices of \( K \in \mathcal{K}^n \), s.t. equality holds.

Particularizing in \( \mathbb{E}^n \), (3) also leads to a sharpening of Jung’s inequality [29] (cf. [12, Corollary 5.1]):

Proposition 1.5. In Euclidean spaces \( \mathbb{E}^n \) it holds
\[ j(K) \leq \min \left\{ \sqrt{\frac{n}{2(n + 1)}}, \frac{s(K)}{s(K) + 1} \right\} \]
for any \( K \in \mathcal{K}^n \), with equality for both values inside the minimum at the same time iff \( \text{conv}(T \cup (D(T) - R(T))\mathbb{B}) \subset K \subset T^* \).
The next proposition states several bounds for ratios between different radii of complete sets. The Euclidean result can be found in [14, p. 125], while the result for general Minkowski spaces stems from [30, Theorem 2].

**Proposition 1.6.** Let $M^n$ be a Minkowski space and $K \in \mathcal{K}_{cp}$. Then

$$\frac{r(K)}{D(K)} \geq \frac{1}{n+1}.$$ 

Moreover, if $M^n = E^n$, the bound improves to

$$\frac{r(K)}{D(K)} \geq 1 - \sqrt{\frac{n}{2(n+1)}}.$$ 

Equality holds in the first, if $K$ is an $n$-simplex and $B = K - K$, and in the second case, iff $K$ is a completion of a regular $n$-simplex.

The *Helly dimension* $\text{him}(K)$ of a set $K \in \mathcal{K}^n$ is defined as the smallest positive integer number $k$, s. t. whenever we consider a family of indices $I \neq \emptyset$ with $\cap_{i \in I} (x_i + C) \neq \emptyset$, for any $J \subset I$, $|J| \leq k + 1$, $x_i \in \mathbb{R}^n$ and $i \in I$, it already follows $\cap_{i \in J} (x_i + C) \neq \emptyset$. For a Minkowski space $M^n$, we also use $\text{him}(M^n) := \text{him}(B)$.

In [5, Corollary 1] Boltyanski and Martini improved Bohnenblust inequality by showing the following:

**Proposition 1.7.** For any Minkowski space $M^n$ it holds

$$j(M^n) \leq \frac{\text{him}(M^n)}{\text{him}(M^n) + 1}.$$

We recall that the Banach-Mazur distance between two sets $K, C \in \mathcal{K}^n$ is defined by

$$d_{BM}(K, C) = \min\{\rho > 0 : K \subset A(C) \subset x + \rho K, \text{ with } A \text{ a regular affine map and } x \in \mathbb{R}^n\}.$$ 

Grünbaum in [20] mentioned (but not proved) that $s(K) = \min_{B \in \mathcal{K}^n_0} d_{BM}(K, B)$. For completeness reasons we present a proof of this result and point out that this minimum is attained if $B = K - K$.

**Proposition 1.8.** For any $K \in \mathcal{K}^n$ it holds

$$s(K) = \min_{B \in \mathcal{K}^n_0} d_{BM}(K, B) = d_{BM}(K, K - K).$$

**Proof.** Let $B \in \mathcal{K}^n_0$. If $B \subset K \subset tB$ with $0 < t < s(K)$, then $K \subset tB = -tB \subset -tK$, a contradiction with the definition of $s(K)$. Thus $s(K) \leq \min_{B \in \mathcal{K}^n_0} d_{BM}(K, B)$.

Conversely, since $K \subset -s(K)K$ it follows

$$\frac{1}{1 + s(K)} K - K \subset K \subset \frac{s(K)}{1 + s(K)} K - K$$

and thus $d_{BM}(K, K - K) \leq \frac{s(K)}{1 + s(K)}(1 + s(K)) = s(K)$, which finishes the proof. \qed

The following result is taken from [22, Theorem 2]. Even so it is a direct corollary of Proposition [1.8], it gives a more accurate description of the Banach-Mazur distance when restricted to completions in $E^n$.

**Proposition 1.9.** Let $K \in \mathcal{K}_{cw}$ in the Euclidean space $E^n$. Then $s(K) = d_{BM}(K, B_2)$.

\footnote{Observe that the authors used the equivalent notion of minimal dependance.}
2. Results

Lemma 2.1. Let $\mathbb{M}^n$ be a Minkowski-space. Then

\begin{equation}
    j(\mathbb{M}^n) = \max_{K \in \mathcal{K}_{cp}} j(K)
\end{equation}

and for any $K \in \mathcal{K}^n$ with $j(K) = j(\mathbb{M}^n)$ there exists an $n$-simplex $S \subset K$ and a completion $K^*$ of $K$, s. t. $D(S) = D(K) = D(K^*)$ and $R(S) = R(K) = R(K^*)$.

Proof. In [19] the existence of a completion $K^*$ of any $K \in \mathcal{K}^n$ was shown. Thus choosing $K$, s. t. $j(K) = j(\mathbb{M}^n)$, we obtain $D(K^*) = D(K)$ and $R(K^*) \geq R(K)$, which implies due to the maximality of the Jung constant that $R(K^*) = R(K)$ and $j(K^*) = j(K) = j(\mathbb{M}^n)$, too. Now, due to Helly’s theorem there always exists an $n$-simplex $S \subset K$, s. t. $R(S) = R(K)$ (cf. 2.13) and surely it holds $D(S) \leq D(K)$. As before we obtain from the maximality of the Jung constant that $D(S) = D(K)$ and $j(S) = j(K) = j(\mathbb{M}^n)$. \hfill \Box

Lemma 2.1 implies that there always exists some simplex $S$ with $j(S) = j(\mathbb{M}^n)$. On the other hand $j(K) \geq 1/2$ for all $K$, with equality if $K = -K$. Hence $j$ may be seen as a kind of asymmetry measure, too. An asymmetry measure depending on the Minkowski space it is measured in, differently from the Minkowsky asymmetry, but measuring asymmetry with respect to the way we measure distances seems to make a certain sense.

Example 2.2. An extreme example are those Minkowski spaces $\mathbb{M}^n$ with $\mathbb{B}$ being a parallelotope, since this is equivalent to $\text{him}(\mathbb{B}) = 1$ [39]. It follows from Proposition 1.7 that in that case $j(\mathbb{M}^n) = \frac{1}{2} = j(\mathbb{B})$. In [20] Meissner showed that for any 2-dimensional Minkowski space $\mathbb{M}^2$ it holds $\mathcal{K}_{cp} = \mathcal{K}_{cw}$ and thus we have $j(\mathbb{M}^2) = \max_{K \in \mathcal{K}_{cw}} j(K)$ for all 2-dimensional Minkowski spaces.

Unfortunately, this is not the case when $n \geq 3$. For instance, let $\mathbb{M}^n$ be a Minkowski space with indecomposable unit ball $\mathbb{B}$, i.e. for all $K, L \in \mathcal{K}^n$ with $\mathbb{B} = K + L$, there exist $\lambda, \mu \geq 0$, s. t. $\lambda K = \mu L = \mathbb{B}$ (cf. [36]). This is the case, e.g. if $\mathbb{B}$ is the unit crosspolytope. From the indecomposability it follows on the one side that $\mathcal{K}_{cw} = \{\mathbb{B}\}$ (as $K \in \mathcal{K}_{cw}$ iff $K - K = \gamma \mathbb{B}$ for some $\gamma > 0$) and on the other side that $\text{him}(\mathbb{B}) > 1$ (since otherwise $\mathbb{B}$ is a parallelotope, which is a direct sum of segments). Now we obtain from Proposition 1.7 that $j(\mathbb{M}^n) > 1/2 = j(\mathbb{B}) = \max_{K \in \mathcal{K}_{cw}} j(K)$.

Open Question 2.3. Does there exist a Minkowski space $\mathbb{M}^n$ for which $\mathcal{K}_{cp} \neq \mathcal{K}_{cw}$, but possessing $K \in \mathcal{K}_{cw}$, s. t. $j(\mathbb{M}^n) = j(K)$?

Definition 2.4. Let $\mathbb{M}^n$ be a Minkowski space. Then the asymmetry constant of $\mathbb{M}^n$ is defined as

\[ s(\mathbb{M}^n) := \max_{K \in \mathcal{K}_{cp}} s(K). \]

Theorem 2.5. For any Minkowski space $\mathbb{M}^n$ and any $K \in \mathcal{K}_{cp}$ it holds

\[ j(K) = \frac{s(K)}{s(K) + 1} \]

and thus

\[ j(\mathbb{M}^n) = \frac{s(\mathbb{M}^n)}{s(\mathbb{M}^n) + 1}, \]

with $j(K) = j(\mathbb{M}^n)$ iff $K$ is a completion of an $n$-simplex $S$ with $j(S) = j(\mathbb{M}^n)$. 


Proof. It was shown in [30] (by combining results from [14] and [34]) that $D(K) = r(K) + R(K)$ with concentric in- and circumball for any $K \in K_{\text{cp}}$. Together with [11] Theorem 6.1 this means that $(s(K) + 1)/s(K) = D(K)/R(K)$ or equivalently $j(K) = s(K)/(s(K) + 1)$. Taking maximums on both sides and using (5), we conclude $j(M^n) = s(M^n)/(s(M^n) + 1)$. The characterization then obviously follows from needing $K \in K_{\text{cp}}$ with $j(K) = j(M^n)$ and the existence of such an $n$-simplex $S$ from Lemma 2.1. \hfill \Box

Remark 2.6. An immediate consequence of Theorem 2.5 is that the Jung ratio of a complete set only depends on the Minkowski space in the way that the set has to be complete, but not in its value. This means, if a set $K$ is complete in two different spaces (which happens, e.g. when $K$ is complete but not of constant width, because in this case it is also complete in the Minkowski space with $B = K - K$) the Jung ratio $j(K)$ stays constant. And even more, if $K$ attains the Jung constant in one space then in both spaces.

Example 2.7. Consider again the Minkowski space $M^3$ with $B$ being the unit crosspolytope. Moreover, let $T^3 := \text{conv}\{p^1, p^2, p^3, p^4\}$, with $p^1 = \frac{1}{3}(1,1,1)$, $p^2 = \frac{1}{3}(-1,-1,1)$, $p^3 = \frac{1}{3}(1,-1,-1)$, and $p^4 = \frac{1}{3}(-1,1,-1)$, until $T^3 \subset B$ is a regular 3-simplex with $R(T^3) = R(B) = 1$ and $D(T^3) = \|p^i - p^j\| = \frac{1}{2}$ for all $i, j \in [4], i \neq j$. From Proposition 1.4 it follows $j(M^3) \leq 3/4 = j(T^3)$, thus $T$ attains the Jung constant. Moreover, $T$ fulfills the spherical intersection property, i.e. $T = \cap_{i=1}^4(p^i + D(T)B)$. Thus $T \in K_{\text{cp}}$ (see [14]). On the other hand, in the Minkowski space $M^3$ whose unit ball is $T - T$, we have $T \in K_{\text{cp}}$, and by the same reasons as above $j(M^3) = j(T) = 3/4 = j(M^3)$ holds.

Finally, observe that if $j(M^n) = n/(n+1)$, then $s(M^n) = n$ and thus any simplex $S$ with $j(S) = n/(n+1)$ is complete in $M^n$. What is more, Proposition 1.7 ensures that $S - S \subset B \subset (n+1)S \cap (-S)$, which in dimension 2 implies that $B = S - S$ is the unique unite ball (in fact, $K_{\text{cp}} = K_{\text{cw}}$), but this is not the case if $n \geq 3$ (observe that in the example above $M^3$ and $M^3$ have $4T \cap (-T)$ and $T - T$, respectively, as their unit balls.) However, the cross polytope is a maximal unit ball in the sense of Proposition 1.7, iff $n = 3$.

The inequality shown in the following lemma has been proved for the Euclidean case in [35] for $n = 2$ and in [9] for arbitrary $n$. For general Minkowski spaces it follows from [12] Corollary 6.3, but we would like to add some details about the equality case and present an easy direct proof.

Lemma 2.8. For any Minkowski space $M^n$ and any $K \in K^n$ it holds $R(K) + r(K) \leq D(K)$ and if $R(K) + r(K) = D(K)$ any incenter of $K$ is also a circumcenter and therefore we obtain for the equality case $s(K) = R(K)/r(K)$ or $j(K) = s(K)/(s(K) + 1)$.

Proof. For showing the inequality, assume w.l.o.g. that 0 is an incenter of $K$. Since there exists a point in $K$ which is at least at distance $R(K)$ from the incenter, there must be some $p \in S$ and $\rho \geq R(K)$, s.t. $\rho p \in K$. But, since $r(K)B \subset K$, we also have $r(K)(-p) \in K$ and therefore $D(K) \geq \|\rho p - (-r(K)p)\| \geq R(K) + r(K)$.

The centricity statement then follows directly from the necessity of $\|\rho p - (-r(K,C)p)\| = R(K,C) + r(K,C)$, which means there is no point at a distance bigger than $R(K)$ from any incenter. \hfill \Box

Definition 2.9. For any Minkowski space $M^n$, any $K \in K^n$, and any circumcenter $c$ of $K$, we call $K^+ = \text{conv}(K \cup c + (D(K) - R(K)B))$ a pseudo completion of $K$.

One should remark that for any fixed circumcenter $c$ of $K$ the pseudo completion of $K$ is unique. However, it is quite easy to see that in Minkowski-space with unit balls having segments in their boundary non-unique circumcenters and thus non-unique pseudo completions are possible.
**Lemma 2.10.** For any Minkowski space $\mathbb{M}^n$, any $K \in \mathbb{K}^n$, it holds

a) $D(K^+) = D(K^*) = D(K)$, $R(K^+) = R(K^*) = R(K)$, $r(K^+) = r(K^*)$, and $s(K^+) = s(K^*)$ for all completions $K^*$ of $K$ within any circumball $c + R(K)B$ of $K$.

b) If there exists a simplex $S$ and a completion $S^*$ of $S$ within any circumball $c + R(K)B$ of $K$, s. t. $S \subset K \subset S^*$, $R(K) = R(S)$, and $D(K) = D(S)$, then

$$D(K) = r(K) + R(K) \iff S^+ \subset K.$$

**Proof.** a) First notice that $K \subset K^+ \subset K^* \subset c + R(K)B$, which immediately gives $D(K^*) = D(K^+) = D(K)$ and $R(K^*) = R(K^+) = R(K)$. Moreover, Lemma 2.8 and $c + (D(K) - R(K^*))B \subset K^+$ imply

$$D(K) - R(K) \leq r(K^+) \leq r(K^*) = D(K^*) - R(K^*) = D(K) - R(K)$$

and thus $r(K^+) = r(K^*)$. Hence $D(K^+) = r(K^+) + R(K^*)$ and we obtain from Lemma 2.8 that $s(K^+) = R(K^+)/r(K^+) = R(K^*)/r(K^*) = s(K^*)$.

b) In order to show the “$\Rightarrow$”-direction, since $S \subset K$, it is enough to show that $c + (D(S) - R(S))B \subset K$. However, due to $D(S) - R(S) = D(K) - R(K) = r(K)$, Lemma 2.8 ensures that we can choose a common in- and circumcenter for $K$, and hence, $c + r(K)B \subset K$, as we wanted to show.

For the “$\Leftarrow$-direction”, since $c + (D(K) - R(K))B \subset S^+ \subset K$, we clearly have that

$$D(K) - R(K) \leq r(S^+) \leq r(K) \leq D(K) - R(K),$$

and thus $D(K) = r(K) + R(K)$.

\[\square\]

Applying Theorem 2.5 we obtain some easy but important corollaries, the first below considers the Euclidean case:

**Corollary 2.11.** In Euclidean space it holds

$$s(\mathbb{E}^n) = \frac{j(\mathbb{E}^n)}{1 - j(\mathbb{E}^n)} = \frac{n + \sqrt{2n(n + 1)}}{n + 2},$$

and

$$\forall K \in \mathbb{K}^n : s(K) = s(\mathbb{E}^n) \wedge R(K) + r(K) = D(K) \iff T^+ \subset K \subset T^*,$$

where $T$ may be an arbitrary regular $n$-simplex.

**Proof.** The first statement follows directly from combining Jung’s inequality \[2\] and Theorem 2.5; the second afterwards from combining Proposition 1.5, Theorem 2.5, Lemma 2.1 and Part (b) of Lemma 2.10 \[\square\]

**Remark 2.12.** The latter statement in Corollary 2.11 may be reduced to: for all $K^* \in \mathbb{K}_{cw}^n$ it holds $s(\mathbb{E}^n) = s(K^*)$, iff $K^*$ is a completion of a regular $n$-simplex. Hence Corollary 2.11 does not only imply Proposition 1.2, but also answers the question in \[22\] about a characterization of the “only if case” (and so does Theorem 2.5 even for general Minkowski spaces).

The following lemma generalizes \[11\], Lemma 2.2 (with almost the same proof).

**Lemma 2.13.** Let $\text{him}(\mathbb{M}^n) = k \in [n]$. Then $R(K) = \max \{R(L) : L \subset K, |L| \leq k + 1\}$ for all $K \in \mathbb{K}^n$. Furthermore, if $\dim(K) \geq k$, then there always exists a $k$-simplex $S \subset K$, s. t. $R(S) = R(K)$.

**Proof.** Let $R_k(K) := \max \{R(L) : L \subset K, |L| \leq k + 1\}$ (this is called the $k$-th core radius of $K$ in \[11\]). Surely, for any $L \subset K$ it follows $R(L) \subset R(K)$ and therefore $R_k(K) \leq R(K)$. Now, by definition of $R_k(K)$ any $L \subset K$ with $|L| \leq k + 1$ can be covered by a copy of $R_k(K)B$. Hence for all such $L$ it holds $\bigcap_{x \in L} (x - R_k(K)B) \neq \emptyset$. However, by definition of the Helly
Finally, let \( \mathcal{K} \) and the equality holds iff \( \dim(K) = \dim(M) \). Applying Helly’s theorem within \( \text{aff}(L) \) we may always assume that the set \( L \subset K \) with \( |L| = k + 1 \) and \( R(L) = R(K) \) is affinely independent. Hence, if \( |L| \leq k \leq \dim(K) \), we can complete \( L \) to the vertex set of a \( k \)-simplex.

Applying Proposition \ref{corollary:7} one may also obtain that the asymmetry constant and the Helly dimension of any Minkowski space directly bound each other. Moreover, from applying Lemma \ref{lemma:2.13} also the equality case can be characterized.

**Corollary 2.14.** Let \( M^n \) be a Minkowski space. Then

\[
[s(M^n)] \leq \text{him}(M^n)
\]

and \( s(M^n) = \text{him}(M^n) \) iff there exists a \( \text{him}(M^n) \)-dimensional simplex \( S \), s. t. \( s(S) = s(S^+) = s(S^*) \) for all of its completions \( S^* \). Moreover, in that case it holds \( S = S^+ \cap \text{aff}(S) \).

**Proof.** Since the function \( f(x) = x/(x + 1) \) is increasing whenever \( x > 0 \), the claimed inequality follows directly from combining Proposition \ref{proposition:1.7} with Theorem \ref{thm:2.5}.

Considering the equality case, there exists some \( K \in \mathcal{K}_{cp} \) with \( s(K) = s(M^n) \) and it follows from Lemma \ref{lemma:2.13} that there exists a \( \text{him}(M^n) \)-dimensional simplex \( S \subset K \), s. t. \( R(S) = R(K) \). Now, \( s(K) = s(M^n) \) implies \( j(K) = j(M^n) \). Similar to the proof of Lemma \ref{lemma:2.1} it follows that \( j(S) = j(K) \), \( D(K) = D(S) \), and therefore that \( K \) is a completion of \( S \). Altogether, \( s(S) = s(K) \) is equivalent to \( s(M^n) = \text{him}(M^n) \). Finally, since surely \( S \cup (D(S) - R(S))B \subset -s(S)(S \cup (D(S) - R(S))B \) in general it holds \( s(S^+) \leq s(S) \). From \( s(S^+) = s(S) \) we then obtain that \( s(S^+ \cap \text{aff}(S)) = s(S) = \dim(S) \) and since \( R(S) = R(S^+) \) that \( S^+ \cap \text{aff}(S) = S \). \( \square \)

**Example 2.15.** Considering e. g. \( \mathbb{E}^n \), the inequality in \ref{corollary:2.14} can be strict.

On the other hand, for any given \( k \in [n] \) the inequality is sharp with \( s(M^n) = \text{him}(M^n) = k \), for instance, if we take \( K = S^k \times [0, 1]^{n-k} \), where \( S^k \) denotes a \( k \)-dimensional simplex, and \( \mathbb{B} = K - K \).

Especially, it is known that \( \text{him}(M^n) = 1 \) iff \( \mathbb{B} \) is a parallelotope (see \cite{39}) and this again is equivalent to \( \mathcal{K}_{cp} = \{ \mathbb{B} \} \) (follows from \cite{14} together with \cite{38}). Corollary \ref{corollary:2.14} now says that all this is again equivalent with \( s(M^n) = 1 \) or \( j(M^n) = 1/2 \).

In case of \( s(M^n) = n \), it easy follows that \( s(S^+) = s(S^* \cap \text{aff}(S)) \) iff \( S - S \subset \mathbb{B} \subset (n + 1)S \) (which means the equivalence of the characterizations of the equality cases of Proposition \ref{proposition:1.7} and the “\( \text{him}(M^n) = n \)”-case in Corollary \ref{corollary:2.14} from the fact that both are equivalent to \( S \) being complete (cf. Theorem \ref{thm:2.5}).

Finally, let \( M^3 \) be the space whose unit ball is the hexagonal prism \( \mathbb{B} = (S^2 - S^2) \times [-1, 1] \). Then \( K = S^2 \times [-1/2, 1/2] \in \mathcal{K}_{cv} \) and since \( s(K) = 2 = \text{him}(M^3) \) it follows \( s(M^n) = 2 \). Now, \( S = S^2 \times \{ 0 \} \subset K \), s. t. \( K = S^* \) is the unique completion of \( S \) and \( s(S) = 2 \). However, since \( R(S) = \frac{2}{3} \) and \( D(S) = 1 \) it follows \( S^+ = \text{conv}(S^2 \times [-1/3, 1/3]) \neq S^* \).

The next corollary sharpens and generalizes Proposition \ref{proposition:1.6} (cf. \ref{proposition:1.1}).

**Corollary 2.16.** Let \( M^n \) be a Minkowski space and \( K \in \mathcal{K}_{cp} \). Then

\[
\frac{r(K)}{D(K)} = \frac{1}{s(K) + 1} \geq \frac{1}{s(M^n) + 1}
\]

and equality holds iff \( j(K) = j(M^n) \).
If neither are 

\[ T \]

Remark 2.18. For any 

\[ M^n \]

We observe that property (B) is false for the Minkowski asymmetry:

\[ \text{The inequality follows from the fact that } D(K) = r(K) + R(K) \text{ and } R(K)/r(K) = s(K) \text{ (see Lemma 2.8). Equality holds iff in addition } s(K) = s(M^n), \text{ which by Theorem 2.5 occurs iff } j(K) = j(M^n). \]

Now we state a corollary which generalizes Proposition 1.9 to arbitrary Minkowski space (and sharpens Proposition 1.8).

**Corollary 2.17.** Let \( M^n \) be a Minkowski space and \( K \in \mathcal{K}_{cp} \). Then \( s(K) = d_{BM}(K, B) \).

**Proof.** Since \( K \) is complete, we know from [30] that it has concentric in-and circumball, and it follows \( s(K) = \frac{R(K)}{r(K)} \). Using Proposition 1.8 we obtain

\[
d_{BM}(K, B) \leq \frac{R(K)}{r(K)} = s(K) = \min_{\Xi \in \mathcal{K}_0} d_{BM}(K, \Xi).
\]

Hence \( s(K) = d_{BM}(K, B) \), as required. \[\square\]

Similarly as we have done in Remark 2.6 for Theorem 2.5 we should observe that the above corollary shows that the distance of any complete set and the unit ball of the space does not depend on which unit ball is chosen, as long as the set stays to be complete.

Grünbaum also pointed out that the “supermaximality property” (we changed the original “superminimality” because it matches better with our definition of asymmetry) is a very natural property which should be true for a nice asymmetry measure: (A) an asymmetry \( \pi \) satisfies the supermaximality property if \( \pi(K + L) \leq \max\{\pi(K), \pi(L)\} \), \( K, L \in \mathcal{K}_n \). The asymmetry of Minkowski, e.g., satisfies the maximality property. This property characterizes simplices to be the most asymmetric sets for any asymmetry measure in the plane (cf. [20]), but if this is true in higher dimension is still unknown.

Moreover, since simplices may be (almost) subdimensional and may even converge towards a line segment, it is not incontrovertible if this is a “very natural property”. As already mentioned above one may also interpret \( j \) as an asymmetry measure and argue that the sets with \( j(K) = j(M^n) \) are somehow most asymmetric and the same can be done with the width-inradius ratio or similar coefficients.

As a second property possibly to be fulfilled by a reasonable asymmetry measure Grünbaum suggested the equality case of the supermaximality property: (B) if \( \pi(K + L) = \max\{\pi(K), \pi(L)\} \) then \( K = -K, L = -L \) or \( K \) and \( L \) are similar. However, he did not even clarify if (B) is true for the Minkowski asymmetry, his favourite asymmetry measure. Maybe that was the reason that he also considered the following condition: (B') if \( K \in \mathcal{K}_n \) and \( \pi \) fulfills the supermaximality, then \( \pi(K) = n \) iff \( K \) is an \( n \)-simplex. Surely, this condition is fulfilled by the Minkowsky asymmetry. Moreover, (B') implies (B) when restricted to \( \pi(K + L) = \max\{\pi(K), \pi(L)\} = n \). In fact, assuming w.l.o.g. that \( \pi(K + L) = \pi(K) = n \), it follows from (B') that \( K \) and \( K + L \) are simplices and from the indecomposability of simplices that \( K \) and \( L \) must be even homothetic.

We observe that property (B) is false for the Minkowski asymmetry:

**Remark 2.18.** For any \( n \in \mathbb{N} \) consider the n-dimensional regular simplex \( T \), any Minkowski space \( M^n \) in which \( T^+ \) is not complete (e.g. \( E^n \)), as well as a completion \( T^* \) of \( T \). Since \( s(T^+) = s(T^*) \) and

\[
R(T^+ + T^*) + r(T^+ + T^*) = R(T^+) + r(T^+) + R(T^*) + r(T^*) = D(T^+) + D(T^*) = D(T^+ + T^*),
\]

Lemma 2.8 implies that \( s(T^+ + T^*) = R(T^+ + T^*)/r(T^+ + T^*) = s(T^+) = s(T^*) \), but obviously neither are \( T^+, T^* \) symmetric nor similar, which contradicts property (B).
With the same arguments one can also show that property (B) does not even hold for \( n \geq 3 \) when we restrict to \( K_{cw} \) choosing instead of \( T^+ \) and \( T^* \) two different completions of \( T \) (such as the two 3-dimensional Meissner bodies in \( \mathbb{E}^3 \) – see [7] for a detailed construction and basic properties – or their bodies of evolution in higher dimensions). In 2-space, the Reuleaux triangle is the unique completions of \( T \), but one may easily do the counterproof as above with two less asymmetric bodies of constant width.

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