Gamma conjecture II for quadrics

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Abstract
The Gamma conjecture II for the quantum cohomology of a Fano manifold \( F \), proposed by Galkin, Golyshev and Iritani, describes the asymptotic behavior of the flat sections of the Dubrovin connection near the irregular singularities, in terms of a full exceptional collection, if there exists, of \( D^b(F) \) and the \( \hat{\Gamma} \)-integral structure. In this paper, for the smooth quadric hypersurfaces we prove the convergence of the full quantum cohomology and Gamma II. For the proof, we first give a criterion on Gamma II for Fano manifolds with semisimple quantum cohomology, by Dubrovin’s theorem of analytic continuations of semisimple Frobenius manifolds. Then we work out a closed formula of the Chern characters of spinor bundles on quadrics. By the deformation-invariance of Gromov–Witten invariants we show that the full quantum cohomology can be reconstructed by its ambient part, and use this to obtain estimations. Finally we complete the proof of Gamma II for quadrics by explicit asymptotic expansions of flat sections corresponding to Kapranov’s exceptional collections and an application of our criterion.

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1 Introduction

The quantum cohomology, or Gromov–Witten invariants of genus 0, of a smooth projective variety $X$, is a kind of virtual counting of rational curves on $X$. It is then expected that the quantum cohomology of $X$ reflects its geometry. For example, it is known that the semi-simplicity of the quantum cohomology implies the rational connectedness. In his ICM talk [9], Dubrovin proposed a conjecture relating the quantum cohomology of $X$ to the bounded derived category $D^b(X)$ of coherent sheaves on $X$. The qualitative part of the conjecture predicts the equivalence between the semi-simplicity of the quantum cohomology and the existence of full exceptional collections in $D^b(X)$. Dubrovin’s conjecture is refined in various ways (see e.g. [5, 11, 18]). In this paper, we study the Gamma conjecture II (Gamma II for short) proposed by Galkin et al. [14], which can be regarded as a refinement of the quantitative part of Dubrovin’s conjecture. It concerns the asymptotic behavior of flat sections of the Dubrovin connection near its irregular singularities. A fascinating point of Gamma II is the appearance of the Gamma class ([23, 29]; for the definition see (3)), which is related to mirror symmetry (see e.g. [15]).

Let us briefly describe Gamma II. Assuming that the quantum cohomology of $X$ is convergent on an open subset $B$ of $H^{even}(X)$, the associated Dubrovin connection is a PDE system over $B \times \mathbb{P}^1$. We can roughly view it as a family of linear ODE systems on $\mathbb{P}^1$ parameterized by $t \in H^{even}(X)$. Let $z$ be an in-homogeneous coordinate on $\mathbb{P}^1$. For each $t$, the corresponding ODE system has only two singularities: the regular singularity at $z = \infty$ and the irregular singularity at $z = 0$. Let $S_t$ be the space of global (multi-valued) solutions of the ODE system corresponding to $t$. Inspired by mirror symmetry, Iritani used a canonical fundamental solution near $z = \infty$ and the Gamma class to define the $K$-group framing on the space $S_t$, which is a group-homomorphism from $K^{top}(X)$ to $S_t$. The $\hat{\Gamma}$-integral structure is the image of the $K$-group framing, which is a full-rank lattice on $S_t$. Now we assume furthermore the semisimplicity of quantum cohomology. Then the Dubrovin connection has asymptotically exponential fundamental solutions (AEFS for short). Let the parameter $t$ be a semisimple point of the quantum cohomology. For a suitably chosen $\phi \in \mathbb{R}$, an AEFS corresponding to $(t, \phi)$ is a fundamental solution in $S_t$ consisting of flat sections characterized by
their asymptotic behavior near $z = 0$ along the direction $\arg z = \phi$. Loosely speaking, assuming the existence of full exceptional collections in $D^b(X)$, Gamma II states that an AEFS corresponding to $(t, \phi)$ lies in Iritani’s $\hat{\Gamma}$-integral structure of $S_t$, and moreover, it is given by a suitably chosen full exceptional collection of $D^b(X)$. We refer readers to Sect. 2 for a precise statement of the conjecture.

Note that Gamma II in [14] was stated for Fano manifolds, but the statement makes sense without the Fano assumption (see Remark 2.8 for discussions). Galkin et al. [14] also proposed Gamma conjecture I and the underlying Conjecture $\mathcal{O}$ for Fano manifolds, in which we believe the Fano assumption is necessary.

Presently Gamma II is proved in only a few cases: Grassmannians [5, 14], toric Fano manifolds [12], and Hirzebruch surfaces [3], using varied techniques. The main result of this paper is the following.

**Theorem 1.1** (= Theorem 6.1) *The Gamma conjecture II holds for the smooth quadric hypersurfaces in the projective spaces.*

In the rest of this paper, a smooth quadric hypersurface will be called a quadric for short. The strategy of the proof is to build an AEFS from a full exceptional collection via $\hat{\Gamma}$-integral structure. We start with Kapranov’s full exceptional collections for quadrics, and compute their Chern characters by using some knowledge of the spinor bundles. Then, we apply Givental’s mirror theorem to compute the corresponding flat sections in the $\hat{\Gamma}$-integral structure of $S_0$ near $z = \infty$, where $0$ is the origin of the cohomology space. It turns out that these flat sections can be expressed by Meijer’s $G$-functions, and this enables us to find asymptotic expansions of these flat sections near $z = 0$. However, Kapranov’s full exceptional collection does not directly give an AEFS in $S_0$, since these asymptotic expansions are obtained along different directions. Similar difficulty also appears in Galkin, Golyshev and Iritani’s proof of Gamma II for projective spaces, where they solved this problem by showing that suitable mutations of Beilinson’s exceptional collections give AEFS in $S_0$ [14, Section 5.3]. We take another approach, which is inspired by [14, Remark 5.3.3], by establishing a criterion for Gamma II (Theorem 3.8). Then Gamma II for quadrics follows by an application of this criterion.

Probably Theorem 3.8 has independent interests. Roughly speaking, it states the following: assume that for a fundamental solution $\{y_i(t, z)\}$ of the Dubrovin connection (instead of a single ODE system), there exist $t_0 \in H^{\text{even}}(X)$ and real numbers $\phi_i$’s, such that, among other assumptions not spelled out here, for each $i$, $y_i(z) := y_i(t_0, z) \in S_{t_0}$ has a suitable asymptotic behavior near $z = \infty$ along the direction $\arg z = \phi_i$; then there exist $t_1$ and a real number $\phi$, such that $\{\tilde{y}_i(z) := y_i(t_1, z) \in S_{t_1}\}$ is an AEFS corresponding to $(t_1, \phi)$. The point is that the final $\phi$ is common for all $\tilde{y}_i(z)$’s. We give a brief account of the proof of Theorem 3.8, and refer the reader to Sect. 3 for the details. Firstly we can conclude from Lemma 2.7 that each $y_i$ is in an AEFS corresponding to $(t_0, \phi_i)$. Results of [14, Section 2.5] imply that a flat section in an AEFS in $S_t$ is the Laplace transform of a flat section of the dual connection, and vice versa. The Laplace transform [see (10)] is given by a line integral on $\mathbb{C}$ along a half line starting at a singularity of the dual connection, and the singularities are analytic (multivalued) functions of the parameter $t$. This enables us to pictorially view $y_i$ as given by a half line $L_i$ on $\mathbb{C}$ starting at a singularity $u_i$ with phase $\phi_i$; see Fig. 1 for an
The idea is to move the parameter \( t \) starting from \( t_0 \), which corresponds to moving the singularities of the dual connection. At the same time, we need to move the half lines starting at these singularities, and vary their directions continuously if necessary to guarantee that the half lines do not intersect in the moving process, so that at the end we can make these half lines parallel. At first glance, the moving of the parameter \( t \) is only possible in a neighborhood of \( t_0 \) due to the convergence issue.

A result of Dubrovin [10, Section 4] states that a semisimple Frobenius manifold can be meromorphically continued to the universal covering of the configuration space \( C \) of singularities of the dual connection, which is essentially an application of the solution of Riemann–Hilbert problem [32, 34]. So we can move the parameter \( t \) from \( t_0 \in H^{\text{even}}(X) \) to a suitably chosen \( t_1 \in C \) to prove Theorem 3.8.

An important assumption in Theorem 3.8 is the convergence of quantum cohomology. The convergence of the generating functions of Gromov–Witten invariants is a fundamental problem and is unsolved in the general cases. We verify it for the quadrics (Theorem 5.1). The convergence of the ambient part of the quantum cohomology of the quadrics was known [22, 36]. We prove the convergence for invariants with primitive classes. Note that on the cohomology of a smooth quadric there is a \( \mathbb{Z}/2\mathbb{Z} \) monodromy action, which is induced by the whole family of the smooth quadrics. We use the deformation invariance of Gromov–Witten invariants to simplify their WDVV equations, which helps us to reconstruct the full quantum cohomology from its ambient part. This idea was used by the first author to determine the full quantum cohomology of, e.g. the cubic hypersurfaces [20], and also by the second author to verify Conjecture \( O \) for the projective complete intersections [26].

While we were writing this paper, we noticed that Theorem 5.1 can be deduced from a recent result of Cotti [4, Theorem 6.6]. Cotti’s theorem assumes the semisimplicity of the small quantum cohomology, which is the case for quadric hypersurfaces. However, we expect that our approach can be applied to show the convergence in other circumstances without such a semisimplicity assumption. For example, the first named author, using a finer argument than that in this paper, has shown in [21, §5] that the quantum cohomology of a smooth even dimensional complete intersection of two quadrics has a positive convergent radius, while its small quantum cohomology is not semisimple.

The rest of the paper is organized as follows. In Sect. 2, we review some background materials and give a precise statement of Gamma II. In Sect. 3, we prove the abovementioned criterion for Gamma II (Theorem 3.8). In Sect. 4, we compute the Chern characters of the spinor bundles on quadrics, using Riemann–Roch and an inductive feature of the spinor bundles. In Sect. 5, we show that the primary genus zero Gromov–Witten invariants of the smooth quadrics involving primitive classes are determined by the 3-points invariants and the ambient invariants. Then by some inductive estimations we verify a conjecture of Zinger [36, Conjecture 1] in the case of quadrics for the genus zero Gromov–Witten invariants without descendents and thus prove the convergence of the quantum cohomology of them. In Sect. 6, with the help of the result in Sect. 4 we find expressions of the flat sections of the Dubrovin connection, corresponding to Kapranov’s exceptional collections, in terms of Meijer’s \( G \)-functions, then by the asymptotic expansions of the latters and Theorem 3.8 we prove Gamma II for quadrics.
2 Preliminaries

In this section, we collect some known materials and give a precise statement of Gamma II. For simplicity, we only consider quantum cohomology for a Fano manifold $F$ with $H^{\text{odd}}(F) = 0$, and denote $s = \dim H^*(F)$, where $H^*(F) = H^*(F, \mathbb{C})$. In this paper all varieties are defined over $\mathbb{C}$.

2.1 Quantum cohomology and Dubrovin connection

Let $\text{Eff}(F) \subset H_2(F, \mathbb{Z})$ be the set of effective curve classes in $F$. For $d \in \text{Eff}(F)$, let $M_{0,n}(F, d)$ be the moduli space of connected $n$-pointed stable maps from connected nodal curves of arithmetic genus zero to $F$ with degree $d$. Let $e_i$ be the evaluation map at the $i$th marked point. The (genus-zero, descendent) Gromov–Witten invariants of $F$ are defined as

$$
\langle n \prod_{i=1}^{n} \tau_{k_i}(\gamma_i) \rangle_F^d := \int_{[\overline{M}_{0,n}(F,d)]^{\text{vir}}} \prod_{i=1}^{n} \psi_i^{k_i} e_i^* \gamma_i,
$$

where $\gamma_i \in H^*(F)$, $k_i \in \mathbb{Z}_{\geq 0}$, $\psi_i$ is the first Chern class of the cotangent line bundle associated to the $i$th marked point, and $[\overline{M}_{0,n}(F,d)]^{\text{vir}}$ is the virtual fundamental class. Informally speaking, if $\gamma_i$’s are algebraic, then the primary invariant

$$
\langle \gamma_1, \ldots, \gamma_n \rangle_d^F := \left\langle \prod_{i=1}^{n} \tau_0(\gamma_i) \right\rangle^F_d
$$

is the (virtual) number of degree-$d$ rational curves in $F$ intersecting subvarieties Poincaré dual to $\gamma_1, \ldots, \gamma_n$ in general position. We refer readers to [6, Section 7.3, Section 10.1] or [33, Chapter V-VI] for general properties of Gromov–Witten invariants.

Let $T_0 := 1, T_1, \ldots, T_{s-1}$ be a homogeneous basis of $H^*(F)$. For $t = \sum_{i=0}^{s-1} t^i T_i \in H^*(F)$, we formally define the (genus-zero, primary) potential function of $F$ by

$$
\mathcal{F}_0^F(t) := \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{d \in \text{Eff}(F)} \langle \tau_0(t)^n \rangle_d^F.
$$

By the point mapping axiom and the fundamental class axiom, we can formally write

$$
\mathcal{F}_0^F(t) = \sum_{n_0+n_1+\cdots+n_{s-1}=3} \left( \int_F \prod_{i=0}^{s-1} T_i^{n_i} \right)^{s-1} \sum_{i=0}^{s-1} \frac{(i^i)^{n_i}}{n_i!} + \sum_{n_1, \ldots, n_{s-1} \geq 0} \left( \sum_{d \in \text{Eff}(F) \setminus \{0\}} \langle \prod_{i=1}^{s-1} \tau_0(T_i)^{n_i} \rangle_d^F \right) \prod_{i=1}^{s-1} \frac{(i^i)^{n_i}}{n_i!}.
$$

(1)
Since $F$ is Fano, it follows from the degree axiom and [27, Corollary 1.19] that the coefficient of $\prod_{i=1}^{s-1} \frac{(t^i)^{y_i}}{n_i!}$ in (1) is actually a finite sum. So we can see that

$$\mathcal{F}_0^F(t) \in \mathbb{C}[[t^0, t^1, \ldots, t^{s-1}]].$$

For each $t \in H^*(F)$, the quantum product $\bullet_t$ on $H^*(F)$ parameterized by $t$ is formally given by

$$(T_i \bullet_t T_j, T_k) := \partial_i \partial_j \partial_k \mathcal{F}_0^F(t)$$

$$= \int_F T_i \cup T_j \cup T_k + \text{correction terms.}$$

Here $(\cdot, \cdot)^F$ is the Poincaré pairing on $H^*(F)$. Let $g_{pq} := (T_p, T_q)^F$, and let $(g_{pq})^{-1}$ be the inverse matrix of $(g_{pg})$. Then for any $i, j, k, l \in \{0, 1, \ldots, s-1\}$, $\mathcal{F}_0^F(t)$ satisfies the famous WDVV equation:

$$\sum_{p,q} g_{pq} \partial_i \partial_j \partial_q \mathcal{F}_0^F(t) \cdot \partial_p \partial_k \partial_l \mathcal{F}_0^F(t) = \sum_{p,q} g_{pq} \partial_i \partial_j \partial_p \mathcal{F}_0^F(t) \cdot \partial_q \partial_k \partial_l \mathcal{F}_0^F(t).$$

(2)

Assume that the quantum product $\bullet_t$ is analytic for $t$ in a neighborhood $B$ of $0 \in H^*(F)$, i.e. $\mathcal{F}_0^F(t)$ is an absolutely convergent power series of $t^0, t^1, \ldots, t^{s-1}$ for $t \in B$. Then the quantum cohomology $(H^*(F), \bullet_t)$ is a commutative, associative $\mathbb{C}$-algebra with the unit element $1 \in H^*(F)$ for each $t \in B$ (the associativity comes from WDVV).

**Remark 2.1** (i) For a (not necessarily Fano) smooth projective variety $X$, its potential function is not necessarily a formal power series in $t^0, t^1, \ldots, t^{s-1}$, for the coefficient of $\prod_{i=1}^{s-1} \frac{(t^i)^{y_i}}{n_i!}$ in (1) is not necessarily a finite sum. To state the convergence of the potential function one needs to introduce the Novikov variables and consider the large radius limit (see e.g. [6, §6.2.1, §8.5.3] in the case $H^{2,0}(X) = 0$). Moreover, a conjectural estimate of Zinger [36, Conjecture 1] implies the convergence near the large radius limit.

(ii) When $F$ is a Fano manifold of Picard rank 1, the convergence of $\mathcal{F}_0^F(t)$ near $0$ implies the convergence near the large radius limit, for a trivial reason: the degree $d$ of a nonzero invariant $(\prod_{i=1}^{s-1} \tau_0(T_i)^{y_i})^F$ is uniquely determined by its insertions.

The Dubrovin connection [8–10] of $F$ is a meromorphic flat connection $\nabla$ on the trivial $H^*(F)$-bundle over $B \times \mathbb{P}^1$ given as follows: for a point $(t, z) \in B \times \mathbb{P}^1$, we have

$$\nabla_{\partial_i} = \partial_i + \frac{1}{z} (T_i \bullet_t),$$

$$\nabla_{z \partial_z} = z \partial_z - \frac{1}{z} (E \bullet_t) + \mu.$$
Here $E = c_1(F) + \sum_{i=0}^{s-1}(1 - \frac{1}{2} \deg T_i)t^iT_i$ is the Euler vector field, and $\mu \in \text{End}(H^s(F))$ is the Hodge grading operator given by $\mu|_{H^{2p}(F)} := (p - \dim F)\text{id}_{H^{2p}(F)}$. For each $t \in B$, denote by $\nabla_t$ the restriction of $\nabla$ to $(t) \times \mathbb{P}^1$. Then $\nabla_t$ is a meromorphic flat connection on the trivial $H^s(F)$-bundle over $\mathbb{P}^1 = (t) \times \mathbb{P}^1$, given by

$$(\nabla_t)z\partial_z = z\partial_z - \frac{1}{z}(E \bullet t) + \mu.$$ 

Dubrovin proved that $\nabla$ is the isomonodromic deformation of $\nabla_t$ over $B$ [10, Theorem 2.1, 4.4 and 4.6]. The equation $\nabla f = 0$ for a section $f$ is called the quantum differential equation of $F$. The space of global solutions to the quantum differential equation is

$$S_B := \{f \in \Gamma(B \times \widehat{\mathbb{C}}^*, H^s(F)) : \nabla f = 0\},$$

where $\widehat{\mathbb{C}}^*$ is the universal covering of $\mathbb{C}^*$, $H^s(F)$ is the trivial $H^s(F)$-bundle over $B \times \widehat{\mathbb{C}}^*$, and $\nabla$ is the pullback of $\nabla$. The space of global flat sections of $\nabla_t$ is

$$S_t := \{f \in \Gamma((t) \times \widehat{\mathbb{C}}^*, H^s(F)) : \nabla_t f = 0\},$$

where $\overline{H^s(F)}$ is the trivial $H^s(F)$-bundle over $(t) \times \widehat{\mathbb{C}}^*$, and $\nabla_t$ is the pullback of $\nabla_t$.

### 2.2 Flat sections around $z = \infty$

For $t \in H^s(F)$, write $t = (t^{(2)}, t') \in H^2(F) \oplus \bigoplus_{p \neq 1} H^{2p}(F)$, and we define a formal linear endomorphism $L(t, z) \in \text{End}(H^s(F))[[\frac{1}{z}]]$ parameterized by $t$ as follows:

$$\langle L(t, z)T_i, T_j \rangle^F := (e^{-t^{(2)} \over z}T_i, T_j)^F + \sum_{n,m=0}^{\infty} \left(\frac{-1}{z}\right)^{m+1} \frac{1}{n!} \sum_{d \in \text{Eff}(F)} (\tau_m(e^{-t^{(2)} \over z}T_i)\tau_0(t')n\tau_0(T_j))_d e^{f_d t^{(2)}}. $$

Suppose that the quantum product $\bullet_t$ is analytic for $t$ in a neighborhood $B$ of $0 \in H^s(F)$. For any $M \in \text{End}(H^s(F))$, we write $e^{M\log z}$. Let $\rho := (c_1(F) \cup ) \in \text{End}(H^s(F))$. Then we can use $L(t, z)z^{-\mu}z^\rho$ to identify $H^s(F)$ with the space $S_B$. More precisely, the cohomology framing, defined by

$$Z^{coh}_B : H^s(F) \to S_B$$

$$\alpha \mapsto L(t, z)z^{-\mu}z^\rho \alpha,$$

is a linear isomorphism. For $t \in B$, let $Z^{coh}_t : H^s(F) \to S_t$ be the restriction of $Z^{coh}_B$ to $(t) \times \widehat{\mathbb{C}}^*$. Then $Z^{coh}_t$ is also a linear isomorphism. As a consequence, the natural restriction $S_B \to S_t$ is a linear isomorphism of $s$-dimensional vector spaces.
Let $K^\text{top}(F)$ be the Grothendieck group of topological complex vector bundles on $F$. The $K$-group framing is a homomorphism of abelian groups defined by

$$Z^K_B : K^\text{top}(F) \to S_B,$$

$$V \mapsto (2\pi)^{-\frac{\dim C F}{2}} Z^\text{coh}_B \left( \hat{\Gamma}_F \cup \text{Ch}(V) \right).$$

Here $\hat{\Gamma}_F \in H^*(F)$ is the Gamma class of $F$, which is defined by

$$\hat{\Gamma}_F := \prod_{i=1}^{\dim C F} \Gamma(1 + \delta_i),$$

where $\delta_i$’s are Chern roots of the tangent bundle of $F$ and $\Gamma(x)$ is Euler’s Gamma function, and

$$\text{Ch}(V) := \sum_{p=0}^{\infty} (2\pi i)^p \text{ch}_p(V)$$

is the modified Chern character of $V$. For $t \in B$, let $Z^K_B : K^\text{top}(F) \to S_t$ be the restriction of $Z^K_B$ to $(t) \times \mathbb{C}^*$. Iritani’s $\hat{\Gamma}$-integral structure of $S_B$ (resp. $S_t$) is the image of $Z_B$ (resp. $Z_t$), which is a full rank lattice in $S_B$ (resp. $S_t$).

**Remark 2.2** Inspired by the “remarkable identities” found by Hosono et al. [19] from mirror symmetry, Libgober [29] introduced the (inverse) Gamma class. In [23], Iritani introduced the $\hat{\Gamma}$-integral structure, and he showed that for toric Fano manifolds, these structures match the natural integral structures in their Landau–Ginzburg $B$-models. Katzarkov et al. [25] also proposed similar rational structure for quantum cohomology in terms of Gamma classes. We refer the reader to [14, §1.5] for a detailed account of references.

### 2.3 Flat sections around $z = 0$

Assume that the quantum cohomology $(H^*(F), \bullet_t)$ is analytic and semisimple for $t$ in a domain $B$ of $H^*(F)$ ($B$ does not necessarily contain 0). Recall that $(H^*(F), \bullet_t)$ is semisimple if it is isomorphic to a direct sum $\mathbb{C} \oplus \cdots \oplus \mathbb{C}$ as a $\mathbb{C}$-algebra. Let

$$\psi_1(t), \ldots, \psi_s(t)$$

be the idempotent basis of $(H^*(F), \bullet_t)$, i.e. $\psi_i(t) \bullet_t \psi_j(t) = \delta_{ij} \psi_i(t)$. Let

$$\Psi_i(t) := \frac{\psi_i(t)}{\sqrt{\langle \psi_i(t), \psi_i(t) \rangle}}$$

which is called a normalized idempotent (defined up to sign for each $t$). Each $\Psi_i(t)$ is an eigenvector of $(E \bullet_t)$, and we assume that $E \bullet_t \Psi_i(t) = u_i(t) \Psi_i(t)$. We say that a phase $\phi \in \mathbb{R}$ is admissible at $t$ if $e^{i\phi}$ is not parallel to any nonzero difference $u_i(t) - u_j(t)$.

**Definition 2.3** Let $B \subset H^*(F)$ be a domain in which $\bullet_t$ is analytic and semisimple. We say that $B$ is properly-chosen with respect to $\{(u_i, \Psi_i)\}_{1 \leq i \leq s}$, if the maps $u_i : B \to \mathbb{C}$
form analytic coordinates of $B$, and the maps $\Psi_i: B \to H^*(F)$ are single-valued and analytic. In this case, we say that $u_i$’s are canonical coordinates on $B$.

From [10, Theorem 3.1], when the quantum cohomology $(H^*(F), \bullet_t)$ is analytic and semisimple around $t_0 \in H^*(F)$, there exists a properly-chosen neighborhood of $t_0$.

**Proposition 2.4** [14, Proposition 2.5.1] Assume that $\bullet_t$ is analytic and semisimple in a neighborhood $B$ of $t_0 \in H^*(F)$ properly-chosen with respect to $\{(u_i, \Psi_i)\}_{1 \leq i \leq s}$, and $\phi \in \mathbb{R}$ is an admissible phase at $t_0$. Then there exists a unique basis $y_1, \ldots, y_s$ of $S_B$, such that, there exists a neighborhood $B'$ of $t_0$ in $B$ and $\varepsilon > 0$ so that for each $t \in B'$ and $i = 1, \ldots, s$, we have

$$y_i(t, z)e^{u_i(t)z} \to \Psi_i(t), \quad \text{as } z \to 0 \text{ in the sector } |\arg z - \phi| < \frac{\pi}{2} + \varepsilon.$$  

(5)

We call the above basis of $S_B$ the asymptotically exponential fundamental solution (AEFS for short) to the quantum differential equation associated to the phase $\phi$ with respect to $\Psi_1, \ldots, \Psi_s$ over $B'$. We will follow [14, Section 2.5] to sketch the construction of the AEFS in Sect. 3.1.

**Definition 2.5** Let $B \subset H^*(F)$ be properly-chosen with respect to $\{(u_i, \Psi_i)\}_{1 \leq i \leq s}$. For $y \in S_B$, we say that $y$ respects $(u_i, \Psi_i)$ with phase $\phi \in \mathbb{R}$ over $B$, if there exists $\varepsilon > 0$ so that for each $t \in B$, we have

$$y(t, z)e^{u_i(t)z} \to \Psi_i(t),$$

as $z \to 0$ with $|\arg z - \phi| < \frac{\pi}{2} + \varepsilon$.

Recall that $\nabla_t$ is the restriction of $\nabla$ to $\{t\} \times \mathbb{P}^1$.

**Definition 2.6** Let $y \in S_t$ for a fixed $t$. We say that $y$ respects $(u_i(t), \Psi_i(t))$ with phase $\phi \in \mathbb{R}$, if there exists $\varepsilon > 0$ so that

$$y(z)e^{u_i(t)z} \to \Psi_i(t),$$

as $z \to 0$ with $|\arg z - \phi| < \frac{\pi}{2} + \varepsilon$.

For any $y \in S_B$ and $t \in B$, we let $y_t \in S_t$ be the restriction of $y$ to $\{t\} \times \mathbb{C}^*$.

**Lemma 2.7** Under the assumption of Proposition 2.4, let $y \in S_B$.

(i) If $y_{t_0}$ respects $(u_i(t_0), \Psi_i(t_0))$ with phase $\phi$, then $y = y_i$, where $y_i$ is in the resulted AEFS in Proposition 2.4.

(ii) $y_{t_0}$ respects $(u_i(t_0), \Psi_i(t_0))$ with phase $\phi$ if and only if $y$ respects $(u_i, \Psi_i)$ with phase $\phi$ over an open neighborhood of $t_0$ (in the sense of Definition 2.5).
Proof Let $y_1, \ldots, y_s \in \mathcal{S}_B$ be the AEFS in Proposition 2.4. In particular $y_1, \ldots, y_s$ form a basis of $\mathcal{S}_B$. Since the restriction map $\mathcal{S}_B \to \mathcal{S}_{t_0}$ is an isomorphism, $y_1, t_0, \ldots, y_s, t_0$ also form a basis of $\mathcal{S}_{t_0}$. So there exist $k_1, \ldots, k_s \in \mathbb{C}$ such that $y = \sum_{j=1}^s k_j y_j$, and thus $y_{t_0} = \sum_{j=1}^s k_j y_{j, t_0}$. We are going to show

$$k_j = \delta_{ij}, \quad \text{for } 1 \leq j \leq s. \quad (6)$$

If this is done, then $y = y_i$ and the first statement is proved. The second statement is a consequence of the first.

We will prove (6) in the following five steps. To ease the notations, we set $u_{jj} := u_j(t_0) - u_j'(t_0) e^{i\phi}$. Then by (5) we have

$$e^{\frac{u_i(t_0)}{z}} y_{t_0}(z) = \sum_{j=1}^s k_j \exp \left( \frac{u_{ij}}{|z|} e^{i(\phi - \arg z)} \right) \left( \Psi_j(t_0) + g_j(z) \right), \quad (7)$$

where $g_j(z)$ is a holomorphic $H^*(\mathcal{F})$-valued function, and $g_j(z) = o(1)$ as $z \to 0$ in the sector $|\arg z - \phi| < \frac{\pi}{2} + \epsilon$. We will use the following observation:

- Let $V$ be a normed vector space over $\mathbb{C}$, and $v_1, \ldots, v_k \in V$ are linearly independent. Then by the compactness of $(S^1)^k$, for any given $c_1, \ldots, c_k \in \mathbb{C}$ and $\theta_1, \ldots, \theta_k \in \mathbb{R}$, if $c_i$ are not all zero, we have

$$\min_{\theta_j \in \mathbb{R}} \left| \sum_{j=1}^k c_j e^{i\theta_j} v_j \right| > 0. \quad (8)$$

Step 1: Let $M_1 := \max_{j \neq 0} \Im u_{ij}$. We show that if $\Im u_{ij} > 0$, then $k_j = 0$. Suppose this is not true. Then $M_1 > 0$. Let $z \to 0$ with $\arg z = \phi + \frac{\pi}{2}$, then (7) reads

$$|e^{\frac{u_i(t_0)}{z}} y_{t_0}(z)| = \left| \sum_j k_j \exp \left( \frac{\Im u_{ij} - i\Re u_{ij}}{|z|} \right) \left( \Psi_j(t_0) + g_j(z) \right) \right|$$

$$\geq \exp \left( \frac{M_1}{|z|} \right) \left| \sum_{|\Im u_{ij}| = M_1} k_j \exp \left( -i\Re u_{ij} \right) \left( \Psi_j(t_0) + g_j(z) \right) \right|$$

$$- \sum_{|\Im u_{ij}| < M_1} |k_j| \exp \left( \frac{\Im u_{ij} - M_1}{|z|} \right) |\Psi_j(t_0) + g_j(z)|.$$

Note that

$$\sum_{|\Im u_{ij}| < M_1} |k_j| \exp \left( \frac{\Im u_{ij} - M_1}{|z|} \right) |\Psi_j(t_0) + g_j(z)| \to 0,$$
and
\[
\left| \sum_{\{j|\Re u_{ij} = M_1\}} k_j \exp \left(-\frac{\Im u_{ij}}{|z|}\right) (\Psi_j(t_0) + g_j(z)) \right|
= \left| \sum_{\{j|\Re u_{ij} = M_1\}} k_j \exp \left(-\frac{\Im u_{ij}}{|z|}\right) \Psi_j(t_0) \right| + o(1).
\]

Moreover, note that $\Psi_j(t_0)$'s are linearly independent, and by our assumption, there exists $j$ such that $\Re u_{ij} = M_1$ and $k_j \neq 0$. As a consequence, from (8), we have
\[
C_1 := \min \left\{ \left| \sum_{\{j|\Re u_{ij} = M_1\}} k_j \exp(i\theta_j) \Psi_j(t_0) \right| \colon \theta_j \in \mathbb{R} \right\} > 0,
\]
and then
\[
\liminf \left| \sum_{\{j|\Re u_{ij} = M_1\}} k_j \exp \left(-\frac{\Im u_{ij}}{|z|}\right) (\Psi_j(t_0) + g_j(z)) \right| \geq C_1 > 0.
\]

This implies that $|e^{\frac{u_j(t_0)}{z}} y_0(z)| \to \infty$ as $z \to 0$ with $\arg z = \phi + \frac{\pi}{2}$, contradicting the assumption on $y_0(z)$.

**Step 2:** Let $M_2 := \max_{\{j|k_j \neq 0\}}(-\Re u_{ij})$. We show that if $\Re u_{ij} < 0$, then $k_j = 0$.

Suppose this is not true. Then $M_2 > 0$. Now let $z \to 0$ with $\arg z = \phi - \frac{\pi}{2}$, and we have
\[
|e^{\frac{u_j(t_0)}{z}} y_0(z)| = \left| \sum_{j} k_j \exp \left(-\frac{-\Re u_{ij}}{|z|}\right) \left(\Psi_j(t_0) + g_j(z)\right) \right|
\geq \exp \left(-\frac{M_2}{|z|}\right) \left| \sum_{\{j|-\Re u_{ij} = M_2\}} k_j \exp \left(-\frac{\Re u_{ij}}{|z|}\right) \left(\Psi_j(t_0) + g_j(z)\right) \right|
- \sum_{\{j|-\Re u_{ij} < M_2\}} |k_j| \exp \left(-\frac{-\Re u_{ij} - M_2}{|z|}\right) |\Psi_j(t_0) + g_j(z)|.
\]

Note that
\[
\sum_{\{j|-\Re u_{ij} < M_2\}} |k_j| \exp \left(-\frac{-\Re u_{ij} - M_2}{|z|}\right) |\Psi_j(t_0) + g_j(z)| \to 0,
\]
\[
\begin{align*}
\left| \sum_{\{j: -\Im u_{ij} = M_2\}} k_j \exp\left(\frac{i\Re u_{ij}}{|z|}\right) (\Psi_j(t_0) + g_j(z)) \right| \\
= \left| \sum_{\{j: -\Im u_{ij} = M_2\}} k_j \exp\left(\frac{i\Re u_{ij}}{|z|}\right) \Psi_j(t_0) \right| + o(1).
\end{align*}
\]

Moreover, note that \(\Psi_j(t_0)\)'s are linearly independent, and by our assumption, there exists \(j\) such that \(-\Im u_{ij} = M_2\) and \(k_j \neq 0\). As a consequence, from (8), we have

\[
C_2 := \min \left\{ \left| \sum_{\{j: -\Im u_{ij} = M_2\}} k_j \exp\left(\frac{i\Re u_{ij}}{|z|}\right)\Psi_j(t_0) \right| : \theta_j \in \mathbb{R} \right\} > 0,
\]

and then

\[
\liminf \left| \sum_{\{j: -\Im u_{ij} = M_2\}} k_j \exp\left(\frac{i\Re u_{ij}}{|z|}\right) (\Psi_j(t_0) + g_j(z)) \right| \geq C_2 > 0.
\]

This implies that \(|e^{u_{ij}(t_0)/z} y_{t_0}(z)| \to \infty\) as \(z \to 0\) with arg \(z = \phi - \frac{\pi}{2}\), contradicting the assumption on \(y_{t_0}(z)\).

Step 3: Let \(M_3 := \max\{j| k_j \neq 0\} \Re u_{ij}\). From Steps 1 and 2, we know that if \(\Re u_{ij} > 0\), then \(k_j = 0\). Now we show that if \(\Re u_{ij} > 0\), then \(k_j = 0\). Suppose this is not true. Then \(M_3 > 0\). Let \(z \to 0\) with arg \(z = \phi\), and we have

\[
|e^{u_{ij}(t_0)/z} y_{t_0}(z)| = \left| \sum_j k_j \exp\left(\frac{\Re u_{ij}}{|z|}\right) (\Psi_j(t_0) + g_j(z)) \right| \\
\geq \exp\left(\frac{M_3}{|z|}\right) \left\{ \left| \sum_{\{j: \Re u_{ij} = M_3\}} k_j (\Psi_j(t_0) + g_j(z)) \right| \\
- \sum_{\{j: \Re u_{ij} < M_3\}} |k_j| \exp\left(\frac{\Re u_{ij} - M_3}{|z|}\right) |\Psi_j(t_0) + g_j(z)| \right\}.
\]

Note that

\[
\sum_{\{j: \Re u_{ij} < M_3\}} |k_j| \exp\left(\frac{\Re u_{ij} - M_3}{|z|}\right) |\Psi_j(t_0) + g_j(z)| \to 0,
\]

and

\[
\lim \left| \sum_{\{j: \Re u_{ij} = M_3\}} k_j (\Psi_j(t_0) + g_j(z)) \right| = \left| \sum_{\{j: \Re u_{ij} = M_3\}} k_j \Psi_j(t_0) \right|.
\]
Moreover, note that $\Psi_j(t_0)$’s are linearly independent, and by our assumption, there exists $j$ such that $\Re u_{ij} = M_3$ and $k_j \neq 0$. As a consequence, we have

$$\left| \sum_{\{j|\Re u_{ij} = M_3\}} k_j \Psi_j(t_0) \right| > 0.$$  

This implies that $|e^{u_{ij}(t_0)} y_{t_0}(z)| \to \infty$ as $z \to 0$ with $\arg z = \phi$, contradicting the assumption on $y_{t_0}(z)$.

Step 4: Let $M_4 := \max_{\{j|k_j \neq 0\}} (-\Re u_{ij})$. We show that if $\Re u_{ij} < 0$, then $k_j = 0$.

Suppose this is not true. We have $M_4 > 0$. Now fix $\varepsilon' \in (0, \varepsilon) \cap (0, \frac{\pi}{2})$. Then let $z \to 0$ with $\arg z = \phi - (\frac{\pi}{2} + \varepsilon')$, and we have

$$|e^{u_{ij}(t_0)} y_{t_0}(z)| = \left| \sum_j k_j \exp\left(\frac{\Re u_{ij}}{|z|} (-\sin \varepsilon' + i \cos \varepsilon') \right) \left(\Psi_j(t_0) + g_j(z)\right) \right| \geq \exp\left(\frac{M_4}{|z|} \sin \varepsilon'\right) \left| \sum_j k_j \exp\left(i \frac{\Re u_{ij}}{|z|} \cos \varepsilon'\right) \left(\Psi_j(t_0) + g_j(z)\right) \right| - \sum_{\{j|\Re u_{ij} < M_4\}} |k_j| \exp\left(-\frac{\Re u_{ij} - M_4}{|z|} \sin \varepsilon'\right) |\Psi_j(t_0) + g_j(z)|.$$  

Note that

$$\sum_{\{j|\Re u_{ij} < M_4\}} |k_j| \exp\left(-\frac{\Re u_{ij} - M_4}{|z|} \sin \varepsilon'\right) |\Psi_j(t_0) + g_j(z)| \to 0,$$

(where we use that $0 < \varepsilon' < \frac{\pi}{2}$)

and

$$\left| \sum_{\{j|\Re u_{ij} = M_4\}} k_j \exp\left(i \frac{\Re u_{ij}}{|z|} \cos \varepsilon'\right) \left(\Psi_j(t_0) + g_j(z)\right) \right| = \left| \sum_{\{j|\Re u_{ij} = M_4\}} k_j \exp\left(i \frac{\Re u_{ij}}{|z|} \cos \varepsilon'\right) \Psi_j(t_0) \right| + o(1).$$  

Moreover, note that $\Psi_j(t_0)$’s are linearly independent, and by our assumption, there exists $j$ such that $-\Re u_{ij} = M_4$ and $k_j \neq 0$. As a consequence, from (8), we have
\[ C_4 := \min \left\{ \left| \sum_{\{j : -\Re u_{ij} = M \}} k_j \exp(i\theta_j)\Psi_j(t_0) : \theta_j \in \mathbb{R} \right| \right\} > 0 \text{ (we use again that } 0 < \varepsilon' < \frac{\pi}{2}) . \]

and then

\[ \lim \inf \left| \sum_{\{j : -\Re u_{ij} = M \}} k_j \exp\left( i \frac{\Re u_{ij}}{|z|} \cos \varepsilon' \right)(\Psi_j(t_0) + g_j(z)) \right| \geq C_4 > 0. \]

This implies that \( |e^{\frac{\nu_j(t_0)}{z}} y_0(z)| \to \infty \text{ as } z \to 0 \) with \( \arg z = \phi - \left( \frac{\pi}{2} + \varepsilon' \right) \), contradicting the assumption on \( y_0(z) \).

Step 5: We come to the conclusion that

\[ e^{\frac{\nu_j(t_0)}{z}} y_0(z) = \sum_{\{j : |u_{ij}| = 0\}} k_j (\Psi_j(t_0) + g_j(z)). \]

Since

\[ \Psi_i(t_0) = \lim_{z \to 0} e^{\frac{\nu_i(t_0)}{z}} y_0(z) = \sum_{\{j : |u_{ij}| = 0\}} k_j \Psi_j(t_0), \]

it follows from the linear independence of \( \Psi_j(t_0) \)'s that \( k_j = \delta_{ij} \).

\[ \square \]

2.4 Statement of Gamma conjecture II

Roughly speaking, Gamma II expects that an AEFS is in the \( \hat{\Gamma} \)-integral structure. The precise statement [14, Conjecture 4.6.1], [15, Conjecture 4.9] is as follows.

**Gamma conjecture II** Assume that: (i) \( \bullet \) is analytic and semisimple in a neighborhood \( B \) of \( t_0 \in H^*(F) \) properly-chosen with respect to \( \{(u_i, \Psi_i)\}_{1 \leq i \leq s} \); (ii) the bounded derived category of coherent sheaves \( D^b(F) \) admits a full exceptional collection. Let \( \phi \) be an admissible phase at \( t_0 \), and we numbers \( u_i \)'s such that

\[ \Im(e^{-i\phi} u_1(t_0)) \geq \Im(e^{-i\phi} u_2(t_0)) \geq \ldots \geq \Im(e^{-i\phi} u_s(t_0)). \]

Then there exists a full exceptional collection \((E_1, \ldots, E_s)\) of \( D^b(F) \) such that \( Z^K(t_0)(E_i) \) respects \((u_i(t_0), \Psi_i(t_0))\) with phase \( \phi \) for \( 1 \leq i \leq s \).

**Remark 2.8** (i) By the definition of admissible phase, \( \Im(e^{-i\phi} u_i(t_0)) = \Im(e^{-i\phi} u_j(t_0)) \) if and only if \( u_i(t_0) = u_j(t_0) \).

(ii) From Lemma 2.7, the basis \((Z^K_B(E_1), \ldots, Z^K_B(E_s))\) of \( S_B \) is actually the AEFS associated to the phase \( \phi \) with respect to \( \Psi_1, \ldots, \Psi_s \) around \( t_0 \).
(iii) For a (not necessarily Fano) smooth projective variety $X$ one can state Gamma II as long as $X$ satisfies the following conditions: the convergence of quantum cohomology of $X$ near the large radius limit (see Remark 2.1), the semi-simplicity of quantum cohomology of $X$, and the existence of a full exceptional collection in $D^b(X)$; in fact a conjecture of Dubrovin [9] (made more precise in [1, 18]) expects the two latters to be equivalent.

As pointed out in [14, Remark 4.6.3] and [15, Remark 4.13], the validity of Gamma II does not depend on the choice of $(t_0, \phi) \in H^* (F) \times \mathbb{R}$, as long as the quantum cohomology is analytic and semisimple around $t_0$, and $\phi$ is admissible at $t_0$. This is because as $(t_0, \phi)$ varies, the AEFS changes by mutations, and we can consider the corresponding mutations on full exceptional collections. We refer readers to [14, Section 4] for detailed discussions.

3 A sufficient condition for Gamma conjecture II

This section is devoted to proving Theorem 3.8, a criterion for Gamma II, which will be used in Sect. 6. For $u \in \mathbb{C}$ and $\phi \in \mathbb{R}$, define

$$L(u, \phi) := u + \mathbb{R}_{\geq 0} e^{i \phi},$$

i.e. $L(u, \phi)$ is the oriented half line in $\mathbb{C}$ from $u$ to $\infty$ with phase $\phi$. To state Theorem 3.8, we need some preparations.

3.1 AEFS via Laplace transformation

In this subsection, we follow [14, Section 2.5] to sketch the construction of the AEFS in Proposition 2.4. Recall that $B$ is properly-chosen with respect to $\{(u_i, \Psi_i)\}_{1 \leq i \leq s}$.

Let $\hat{\nabla}$ be the Laplace-dual connection of $\nabla$ (see [14, formula (2.5.2)] for the precise definition of $\hat{\nabla}$), which is a meromorphic flat connection on the trivial $H^* (F)$-bundle over $B \times \mathbb{C}_\lambda$, where $\mathbb{C}_\lambda$ is a copy of $\mathbb{C}$ with coordinate $\lambda$.

Let $D_i$ be the smooth divisor in $B \times \mathbb{C}_\lambda$ defined by $\lambda = u_i$, and let $D_\infty := B \times \{ \infty \}$ be the divisor at infinity. Then $\hat{\nabla}$ has only logarithmic singularities at $D_1, \ldots, D_s, D_\infty$.

**Definition 3.1** Let $\hat{y}$ be a $\hat{\nabla}$-flat section near $D_i$. We say that $\hat{y}$ respects $(u_i, \Psi_i)$ over $B$, if we can analytically continue $\hat{y}$ so that $\hat{y}(t, u_i(t)) = \Psi_i(t)$ for all $t \in B$.

Though $\hat{\nabla}$ is singular along the normal crossing divisor $= D_1 \cup \cdots \cup D_s$ in $B \times \mathbb{C}_\lambda$, it was shown in [14, Lemma 2.5.3] that there exists a $\hat{\nabla}$-flat section $\hat{y}_i$ respecting $(u_i, \Psi_i)$ around $t_0$. Now let $\phi$ be an admissible phase at $t_0$, and we define the Laplace transform of $\hat{y}_i$ associated to the phase $\phi$ with respect to $u_i$ by

$$\hat{y}_i(t, z) = \frac{1}{z} \int_{\lambda \in L(u_i(t), \phi)} \hat{y}_i(t, \lambda) e^{-\frac{\lambda}{z}} d\lambda, \quad t \text{ around } t_0, \ |\arg z - \phi| < \frac{\pi}{2}. \ (10)$$
As in [14, Proof of Proposition 2.5.1], by slightly varying the slope of the integration contour in (10), we can analytically continue $\bar{y}_i$ to $|\arg z - \phi| < \frac{\pi}{2} + \epsilon$ for some $\epsilon > 0$. Now one can show that $\bar{y}_1, \ldots, \bar{y}_s$ form the required AEFS.

### 3.2 Analytic continuation of Dubrovin connection

Assume that $B$ is a properly-chosen neighborhood of $t_0 \in H^*(F)$ with respect to $\{(u_i, \Psi_i)\}_{1 \leq i \leq s}$, such that for each $t \in B$, we have $u_i(t) \neq u_j(t)$ for $i \neq j$. Recall that both $t_i$’s and $u_i$’s are coordinates on $B$, and we can view $t_i$’s as functions of the chosen canonical coordinates.

Let $\Delta := \{(u_1, \ldots, u_s) \in \mathbb{C}^s : u_i = u_j \text{ for some } i \neq j\}$, and write $w_0 := (u_1(t_0), \ldots, u_s(t_0)) \in \mathbb{C}^s \setminus \Delta$. Let $C$ be the universal covering of $\mathbb{C}^s \setminus \Delta$ constructed as homotopy classes of paths starting from $w_0$. By abuse of notation we still denote by $w_0$ the point in $C$ corresponding to the trivial loop at $w_0$, and use the chosen canonical coordinates $u_i$’s to identify the open neighborhood $B$ of $t_0$ with a neighborhood of $w_0$ in $C$. Then via the diagram

$$
\begin{array}{ccc}
B & \hookrightarrow & \mathbb{C}^s \setminus \Delta \\
\psi & \downarrow & \psi \\
t_0 & \hookrightarrow & w_0
\end{array}
$$

we regard $u_i$’s as functions of $w \in C$.

From [10, Theorem 4.5, Theorem 4.6], $\nabla_{t_0}$ admits a unique isomonodromic deformation over $C$. More precisely, there is a unique meromorphic flat connection $\nabla$ on the trivial $H^*(F)$-bundle over $C \times \mathbb{P}^1$ of the form

$$
\nabla_{\partial u_i} = \partial u_i + \frac{1}{z} \mathcal{U}_i, \\
\nabla_{\partial z} = z \partial z - \frac{1}{z} \mathcal{U} + \mu,
$$

where $\mathcal{U}_i$’s and $\mathcal{U}$ are $\operatorname{End}(H^*(F))$-valued meromorphic functions on $C$, such that $\nabla$ restricts to the Dubrovin connection on $B \times \mathbb{P}^1$ (via the above mentioned identification $B \hookrightarrow C$). This isomonodromic deformation defines a semisimple Frobenius manifold structure on a dense open subset $\mathcal{B}$ of $C$, such that the complement $C \setminus \mathcal{B}$ is a divisor, and we have $B \hookrightarrow \mathcal{B}$ via the above mentioned identification. In particular, $B$ is connected.

Now we can continue the functions $u_i$’s and $\Phi_i$’s on $B$ to analytic functions on $\mathcal{B}$. Moreover, as functions of chosen canonical coordinates on $B$, from [10, (4.54)], the functions $t_i$’s can be analytically continued to $\mathcal{B}$. So, based on $\mathcal{Z}_{B}^{\text{coh}}$ and $\mathcal{Z}_{B}^{K}$, we can use this analytic continuation to define cohomology framing and $K$-group framing on $\mathcal{B}$. Moreover, the Laplace-dual connection $\hat{\nabla}$ can also be meromorphically continued to $\mathcal{B} \times \mathbb{C}_\lambda$, and we can use Laplace transformation of $\hat{\nabla}$-flat sections to
study asymptotic behavior of $V$-flat sections as in Sect. 3.1, which gives AEFS over an open neighborhood of a point in $\mathcal{B}$.

So we can consider Gamma II for $F$ over $\mathcal{B}$, that is, for $(w, \phi) \in \mathcal{B} \times \mathbb{R}$ with $\phi$ admissible at $w$, matching the corresponding AEFS with flat sections from a full exceptional collection via $\mathcal{Z}_{w}^{K}$. Now it follows from [14, Section 4] that the validity of Gamma II does not depend on the choice of $(w, \phi)$.

3.3 A criterion for Gamma conjecture II

In this subsection, we use notations from Sect. 3.2. The main result of this subsection is Theorem 3.8.

For $w \in \mathcal{B}$, we regard $u_{1}(w), \ldots, u_{s}(w)$ as pairwise distinct points in $\mathbb{C}_{\lambda}$. Let

$$L(u_{i}(w), \phi)^{\circ} := L(u_{i}(w), \phi) \setminus \{u_{i}(w)\}$$

be an open half line in $\mathbb{C}_{\lambda}$, and we define

$$A_{i}(w) := \{\phi \in \mathbb{R} : L(u_{i}(w), \phi)^{\circ} \text{ does not contain any of } u_{1}(w), \ldots, u_{s}(w)\}.$$ 

Then $A_{i}(w)$ is an open subset in $\mathbb{R}$ such that $\mathbb{R} \setminus A_{i}(w)$ is discrete, and the set of admissible phases at $w$ is

$$A_{1}(w) \cap \cdots \cap A_{s}(w).$$

Let $\hat{\nabla}_{w}$ be the restriction of $\hat{\nabla}$ on $\{w\} \times \mathbb{C}_{\lambda}$.

Definition 3.2 Let $\hat{y}$ be a $\hat{\nabla}_{w}$-flat section for a fixed $w$. If $\hat{y}$ is holomorphic near $\lambda = u_{i}(w)$, and $\hat{y}(u_{i}(w)) = \Psi_{i}(w)$, then we say that $\hat{y}$ respects $(u_{i}(w), \Psi_{i}(w))$, and we define the Laplace transform of $\hat{y}$ associated to phase $\phi \in A_{i}(w)$ with respect to $u_{i}(w)$ by

$$y(z) = \frac{1}{z} \int_{\lambda \in L(u_{i}(w), \phi)} \hat{y}(\lambda) e^{-\frac{z}{\lambda}} d\lambda, \quad |\arg z - \phi| < \frac{\pi}{2}.$$ 

One can use arguments in [14, Proof of Proposition 2.5.1] to prove that if a $\hat{\nabla}_{w}$-flat section $\hat{y}$ respects $(u_{i}(w), \Psi_{i}(w))$, then its Laplace transform $y$ is actually $\nabla_{w}$-flat and respects $(u_{i}(w), \Psi_{i}(w))$ with phase $\phi$ (in the sense of Definition 2.6).

The following Lemma 3.3 is well-known to experts, and we state it here for convenience of readers.

Lemma 3.3 Suppose that two phases $\phi$ and $\phi'$ are in the same connected component of $A_{i}(w)$, and $\hat{y}$ is a $\hat{\nabla}_{w}$-flat section respecting $(u_{i}(w), \Psi_{i}(w))$. Let $y^{\phi}, y^{\phi'}$ be Laplace transforms of $y$ with respect to $u_{i}(w)$ associated the phases $\phi, \phi'$ respectively. Then $y^{\phi} = y^{\phi'}$.

Proof Note that $\hat{\nabla}_{w}$ is regular singular at $\lambda = \infty$, which implies that $\hat{y}(\lambda)$ grows at most polynomially as $\lambda \to \infty$, i.e. $|\hat{y}(\lambda)|$ is bounded by a polynomial of $|\lambda|$ as $\lambda \to \infty$. 

$$\square$$ Springer
Lemma 3.4 Let $y$ be the $\nabla_w$-flat section respecting $(u_i(w), \Psi_i(w))$ with phase $\phi \in A_i(w)$. Then there exists a $\nabla_w$-flat section $\hat{y}$ respecting $(u_i(w),\Psi_i(w))$, such that $y$ is the Laplace transform of $\hat{y}$ associated to the phase $\phi$ with respect to $u_i(w)$.

Proof From Definition 2.5, we can choose $\phi' \in A(w)$ near $\phi$, such that the two phases $\phi$ and $\phi'$ are in the same connected component of $A_i(w)$, and that $y$ also respects $(u_i(w), \Psi_i(w))$ with phase $\phi'$. Let $y_1, \ldots, y_s$ be the AEFS associated to the phase $\phi'$ with respect to $\Psi_1, \ldots, \Psi_s$. Then from Lemma 2.7, we have $y = y_i,_{t_0}$. Note that we can use Laplace transforms of $\nabla$-flat sections to construct an AEFS. So from the uniqueness of AEFS, we see that $y$ is the Laplace transform of a $\nabla_w$-flat section $\hat{y}$ associated to the phase $\phi'$ with respect to $u_i(w)$. Now the conclusion follows from Lemma 3.3.

Lemma 3.5 Suppose that two phases $\phi$ and $\phi'$ are in the same connected component of $A_i(w)$, and $y \in S_w$ respects $(u_i(w), \Psi_i(w))$ with phase $\phi$. Then $y$ also respects $(u_i(w), \Psi_i(w))$ with phase $\phi'$.

Proof From Lemma 3.4, $y$ is the Laplace transform of a $\nabla_w$-flat section $\hat{y}$ associated to the phase $\phi$ with respect to $u_i(w)$. Now the required result follows from Lemma 3.3.

Proposition 3.6 Let $y \in S_B$ be such that $y_{w_0}$ respects $(u_i(w_0), \Phi_i(w_0)) = (u_i(t_0), \Phi_i(t_0))$ with phase $\phi_0 \in A_i(w_0)$. Given a path $\{w_t\}_{0 \leq t \leq 1}$ in $B$, assume that there is a continuous map $\phi: [0, 1] \rightarrow \mathbb{R}$ starting from $\phi_0$ such that $\phi(t) \in A_i(w_t)$ for each $t \in [0, 1]$. Then $y_{w_1}$ respects $(u_i(w_1), \Phi_i(w_1))$ with phase $\phi_1$.

Proof From [14, Lemma 2.5.3], for each $t \in [0, 1]$, there exists a $\nabla$-flat section $\hat{y}_t$ respecting $(u_i, \Psi_i)$ around $w_t$, and we let $\tilde{y}_t \in S_B$ be the Laplace transform of $\hat{y}_t$ associated to the phase $\phi(t)$ with respect to $u_i$. Then from Lemma 2.7, for $t' \in [0, 1]$ around $t$, $\tilde{y}_t, w_{t'} \in S_{w_{t'}}$ respects $(u_i(w_{t'}), \Psi_i(w_{t'}))$ with phase $\phi(t)$. Let $(a_t, b_t)$ be the connected component of $A_i(w_t)$ containing $\phi(t)$. Then there exists a connected open neighborhood $N_t$ of $t$ in $[0, 1]$ such that $\phi(t') \in (a_t, b_t)$ for $t' \in N_t$. So from Lemma 3.5, $\tilde{y}_{t, w_{t'}} \in S_{w_{t'}}$ also respects $(u_i(w_{t'}), \Psi_i(w_{t'}))$ with phase $\phi(t')$. From Lemma 2.7, we have $\tilde{y}_t = \tilde{y}_{t'}$ for $t' \in N_t$. Since $[0, 1]$ is compact and connected, it follows that $y = \tilde{y}_t$ for all $t \in [0, 1]$. This finishes the proof of the proposition.

Corollary 3.7 For $1 \leq i \leq s$, let $y_i \in S_B$ be such that $y_{i, w_0} \in S_{w_0}$ respects $(u_i(w_0), \Phi_i(w_0))$ with phase $\phi_{i,0} \in A_i(w_0)$. Given a path $\{w_t\}_{0 \leq t \leq 1}$ in $B$, assume that there are continuous maps $\phi_i: [0, 1] \rightarrow \mathbb{R}$ starting from $\phi_{i,0}$ such that $\phi_i(t) \in A_i(w_t)$ for each $t \in [0, 1]$. Then for each $i$, $y_{i, w_1}$ respects $(u_i(w_1), \Phi_i(w_1))$ with phase $\phi_i(1)$.

Proof This is a consequence of Proposition 3.6.

The following theorem gives a criterion for Gamma conjecture II.
Theorem 3.8  For a Fano manifold $F$, let $E = (E_1, \ldots, E_s)$ be a full exceptional collection of $\mathcal{D}b(F)$, and set $y_i = Z^K_B(E_i) \in S_B$, where $B$ and $Z^K_B(E_i)$ are defined as in Sect. 3.2. Assume that the followings hold:

(i) $y_i$ respects $(u_i, \Psi_i)$ with phase $\phi_i$ in an open neighborhood of $w_0$, with $\phi_1 > \cdots > \phi_s$, $\phi_1 - \phi_s < 2\pi$.

(ii) The half lines $L(u_i(w_0), \phi_i)$ are pairwise disjoint.

Then there exists $w^* \in B$ and $\phi \in \mathbb{R}$ such that:

(1) $\Im(e^{-i\phi}u_1(w^*)) > \Im(e^{-i\phi}u_2(w^*)) > \cdots > \Im(e^{-i\phi}u_s(w^*))$;

(2) $Z^K_{w^*}(E_i)$ respects $(u_i(w^*), \Psi_i(w^*))$ with phase $\phi$ for $1 \leq i \leq s$.

In particular, Gamma conjecture II holds for $F$.

Proof  Without loss of generality, we assume that $2\pi > \phi_1 > \cdots > \phi_s > 0$,

and for any nonzero complex number $\lambda$, we choose its principal argument $\text{Arg}(\lambda) \in [0, 2\pi)$. Here we give a pictorial proof, with the illustrations in Fig. 1. A more formal and more detailed proof will be given after the pictorial proof.

The basic idea is to move all the half lines $L(u_i(w_0), \phi_i)$ to $L(u_i(w^*), \phi_i^*)$ in $\mathbb{C}$, such that $\phi_1^* = \cdots = \phi_s^*$, and $u_i(w^*)$ are located in the fourth quadrant, and such that

$$2\pi > \text{Arg}(u_1(w^*)) > \cdots > \text{Arg}(u_s(w^*)) > \frac{3\pi}{2}, \quad (11a)$$

$$0 < |u_1(w^*)| < \cdots < |u_s(w^*)|, \quad (11b)$$

and thus Gamma II holds. But we require that the movement of the half lines satisfies:

(i) the configuration of the starting points of the half lines lies in $\mathbb{C} \setminus B$;

(ii) the half lines are disjoint in $\mathbb{C}$.

The way to find a movement satisfying both conditions (i) and (ii) are illustrated in the following, where $s$ equals 5 as an example. In the first step we take an open neighborhood $V_i$ of each half line $L_i = L(u_i(w_0), \phi_i)$, such that $V_i$ are pairwise disjoint for $1 \leq i \leq s$, and move the half line $L_i$ towards $\infty$ along the direction $\phi_i$, until its starting point lay on a circle centered at the origin. Recall that the configuration of starting points of these half lines is a point in $\mathbb{C}$. In the process if the configuration meets the divisor $B$, we slightly move these half lines in $V_i$ such that the resulted configuration avoids $B$; this is possible because $B$ has real codimension 2. After this step we obtain $L_i' = L(u_i(w^*), \phi_i')$ for $1 \leq i \leq s$.

In the second step, we take an open neighborhood $U$ of the circle, and rotate successively $L_1', \ldots, L_s'$ counterclockwise, such that all the starting points of $L_i'$ lie in the fourth quadrant part of the thickened circle, and the direction of the half lines...
Fig. 1 Illustration of the proof of Theorem 3.8 in the case $s = 5$
$L'_i$ is the direction of the radius from the center to its starting point. In the process, for the same reason as in the first step, if the configuration of starting points of these half lines meets the divisor $\mathcal{B}$, we slightly move these half lines with their starting points remained in $U$, such that the resulted configuration avoids $\mathcal{B}$. After this step we obtain $L''_i = L(u_i(w''), \phi''_i)$ for $1 \leq i \leq s$.

Write $u_j = x_j + iy_j$ for $1 \leq j \leq s$, where $x_j, y_j \in \mathbb{R}$. In the third step, we move each $L''_i$ along its direction, such that all the starting points of the resulted half lines, for $1 \leq i \leq s$, have the same $x$-coordinates. After this process we also manage to make the $y$-coordinates of the starting points of $L''_i$ satisfy $0 > y_1 > \cdots > y_s$. After this step we obtain $L'''_i = L(u_i(w'''), \phi'''_i)$ for $1 \leq i \leq s$. The resulted positions of $u_i(w''')$, satisfying the above two conditions, enable us to perform the last step.

In the last step we rotate the half lines $L'''_1, \ldots, L'''_s$ counterclockwise successively such that their directions become horizontal. Then finally we obtain the half lines $L(u_i(w^*), \phi^*_i)$ satisfying (11).

More detailed proof of Theorem 3.8 Without loss of generality, we assume

$$2\pi > \phi_1 > \cdots > \phi_s > 0,$$

and for any nonzero complex number $\lambda$, we choose its principal argument $\text{Arg}(\lambda) \in [0, 2\pi)$.

We start by defining certain open subsets and sectors in $\mathbb{C}_\lambda$.

(i) For $i \neq j$, let $d_{ij}$ be the distance between $L(u_i(w_0), \phi_i)$ and $L(u_j(w_0), \phi_j)$, which is nonzero by our assumption. Let

$$d_0 := \frac{1}{3} \min\{d_{ij}: i, j = 1, \ldots, s, i \neq j\}.$$

Then the open subsets $W_i$ in $\mathbb{C}_\lambda$, defined by

$$W_i := \{\lambda_i \in \mathbb{C}_\lambda: d(\lambda_i, L(u_i(w_0), \phi_i)) < d_0\},$$

are pairwise disjoint.

(ii) We find sectors $S_i, S'_i$ and $S''_i$ in $\mathbb{C}_\lambda$ satisfying:

(a) $W_i \setminus S_i$ is a bounded subset of $\mathbb{C}_\lambda$;

(b) $S_i \neq \emptyset$ for $1 \leq i \leq s$, $S_1, \ldots, S_s$ are pairwise disjoint, all lying in the cut plane $\{\text{Arg}(\lambda) \neq 0\}$, and $S_i$ lies in the counter-clockwise side of $S_{i+1}$;

(c) $S''_1 \neq \emptyset, S''_1, \ldots, S''_s$ are pairwise disjoint, all lying in the fourth quadrant $\{\frac{3\pi}{2} < \text{Arg}(\lambda) < 2\pi\}$;

(d) For $1 \leq i \leq s$, $S_i \cap S''_i = \emptyset$, and $S''_i$ lies in the counter-clockwise side of $S_i$;

(e) $S'_i$ is the smallest open sector lying in the cut plane $\{\text{Arg}(\lambda) \neq 0\}$ that contains both $S_i$ and $S'_i$.

For example, putting

$$\varepsilon'_0 := \frac{1}{3} \min ((2\pi - \phi_1) \cup \{\phi_i - \phi_{i+1}: 1 \leq i \leq s - 1\} \cup \{\phi_s\}),$$
and take
\[ \varepsilon_0 < \min \left\{ \varepsilon_0', \frac{2\pi - \phi_1}{3s}, \frac{\pi}{2(3s - 1)} \right\}, \]

one can take the sectors \( S_i, S'_i \) and \( S''_i \) in \( \mathbb{C}_\lambda \) defined by
\[
S_i := \{ \lambda \neq 0; |\text{Arg}(\lambda) - \phi_i| < \varepsilon_0 \}, \tag{12}
\]
\[
S'_i := \{ \lambda \neq 0; \phi_i - \varepsilon_0 < \text{Arg}(\lambda) < 2\pi - (3i - 3)\varepsilon_0 \}, \tag{13}
\]
\[
S''_i := \{ \lambda \neq 0; |\text{Arg}(\lambda) - (2\pi - (3i - 2)\varepsilon_0)| < \varepsilon_0 \}. \tag{14}
\]

By the choice of \( \varepsilon_0 \), these sectors meet all the above requirements (a) to (e). In fact the form of the definition (12) implies (a); \( \varepsilon < \varepsilon_0' \) implies (b); \( \varepsilon_0 < \frac{\pi}{2(3s - 1)} \) implies (c); \( \varepsilon_0 < \frac{2\pi - \phi_1}{3s} \) implies (d).

Now let \( W^{(0)} := W_1 \times \cdots \times W_s \). Then \( W^{(0)} \) is a simply-connected open subset of \( \mathbb{C}^s \setminus \Delta \), and we identify \( W^{(0)} \) with a neighborhood of \( w_0 \) in \( C \). Recall that \( W^{(0)} \setminus \mathcal{B} \) is a divisor. So we can find a path \( \{w_t\}_{0 \leq t \leq 1} \) in \( W^{(0)} \cap \mathcal{B} \) such that \( u_i(w_1) \in W_i \cap S_i \) for each \( i \). For each \( t \in [0, 1] \), since \( u_i(w_t) \in W_i \), it follows that \( \phi_i \in A_i(w_t) \). So from Corollary 3.7, \( y_i \) respects \( (u_i, \Psi_i) \) with phase \( \phi_i \) around \( w_1 \). Since \( u_i(w_1) \in S_i \), it follows that \( \phi_i \) and \( \text{Arg}(u_i(w_1)) \) are in the same connected component of \( A_i(w_1) \). So \( y_i \) also respects \( (u_i, \Psi_i) \) with phase \( \text{Arg}(u_i(w_2)) \) around \( w_2 \).

Similarly, for \( j = 2, 3, \ldots, s - 1 \), assume that we have moved \( w_{j-1} \) to \( w_j \) via a path in \( \mathcal{B} \) such that \( u_i(w_j) \in S''_i \), \( 1 \leq i \leq j \), \( u_i(w_j) \in S_i(j + 1 \leq i \leq s) \), and \( y_i \) respects \( (u_i, \Psi_i) \) with phase \( \text{Arg}(u_i(w_j)) \) around \( w_j \). Let \( W^{(j)} := S'_1 \times \cdots \times S'_{j-1} \times S'_j \times S_{j+1} \times \cdots \times S_s \). Then \( W^{(j)} \) is a simply-connected open subset of \( \mathbb{C}^s \), and we can identify \( W^{(j)} \) with a neighborhood of \( w_j \) in \( C \). Recall that \( W^{(j)} \setminus \mathcal{B} \) is a divisor. So we can find a path \( \{w_t\}_{j \leq t \leq j+1} \) in \( W^{(j)} \cap \mathcal{B} \) such that \( u_j(w_{j+1}) \in S''_j \). Note that for each \( t \in [j, j + 1] \), we have \( \text{Arg}(u_j(w_t)) \in A_i(w_t) \). So from Corollary 3.7, \( y_i \) respects \( (u_i, \Psi_i) \) with phase \( \text{Arg}(u_i(w_{j+1})) \) around \( w_{j+1} \).

So we obtain \( u_{s+1} \in W^{(s+1)} := S''_1 \times \cdots \times S''_s \) such that \( y_i \) respects \( (u_i, \Psi_i) \) with phase \( \text{Arg}(u_i(w_{s+1})) \) around \( w_{s+1} \). Note that \( W^{(j)} \) is a simply-connected open subset of \( \mathbb{C}^s \setminus \Delta \), and we can identify \( W^{(s+1)} \) with a neighborhood of \( w_{s+1} \) in \( C \). Recall that \( W^{(s+1)} \setminus \mathcal{B} \) is a divisor. So we can find a path \( \{w_t\}_{s+1 \leq t \leq s+2} \) in \( W^{(s+1)} \cap \mathcal{B} \) such that for \( i = 1, \ldots, s - 1 \), we have \( |u_{i+1}(w_{s+2})| > |u_{i}(w_{s+2})| \). Note that for each \( t \in [s + 1, s + 2] \), we have \( \text{Arg}(u_i(w_t)) \in A_i(w_t) \). So from Corollary 3.7, \( y_i \) respects \( (u_i, \Psi_i) \) with phase \( \text{Arg}(u_i(w_{s+2})) \) around \( w_{s+2} \).

Now we have
\[ 2\pi > \text{Arg}(u_1(w_{s+2})) > \cdots > \text{Arg}(u_s(w_{s+2})) > \frac{3\pi}{2}, \]
0 < |u_1(w_{s+2})| < \cdots < |u_s(w_{s+2})|.

So we can check that 2\pi and Arg(u_i(w_{s+2})) are in the same connected component of \text{Ai}(w_{s+2}). So from Corollary 3.7, \gamma_i respects (u_i, \Psi_i) with phase 2\pi around w_{s+2}. Then w^* := w_{s+2} satisfies (11). This implies the assertions (1) and (2) in the statement of the theorem. \qed

**Remark 3.9**

(i) Though Theorem 3.8 is stated for Fano manifolds, it also holds for varieties which are not Fano (see Remark 2.8).

(ii) In [14, Section 5.3], Gamma II for projective spaces is proved by finding an AEFS in \mathcal{S}_0 from suitable mutations of flat sections coming from Beilinson’s exceptional collections. One can apply Theorem 3.8 to give another proof without mutations, by moving the parameter from 0 to a suitable point in \mathcal{B}.

### 4 Chern characters of spinor bundles

In this section, we compute the Chern characters of the spinor bundles over quadric hypersurfaces. We denote by \mathcal{Q}^D a smooth quadric hypersurface of dimension \text{dim} \mathcal{Q}^D = D in \mathbb{P}^{D+1}. First we recall some facts on the integral cohomology ring of quadric hypersurfaces [13, p. 315]. Denote by \chi_i be the pullback of the class of a codimension-i linear subspace of the ambient projective space. Then for \text{dim} \mathcal{Q}^D = 2k + 1,

\[ H^*(\mathcal{Q}^{2k+1}, \mathbb{Z}) = \mathbb{Z}1 + \mathbb{Z}h + \cdots + \mathbb{Z}h^k + \mathbb{Z}\frac{1}{2}h^{k+1} + \cdots + \mathbb{Z}\frac{1}{2}h^{2k+1}. \]

For \text{dim} \mathcal{Q}^D = 4k + 2, there are two primitive classes \(e_{2k+1}, e'_{2k+1} \in H^{4k+2}(\mathcal{Q}^{4k+2}, \mathbb{Z})\) such that

\[ H^*(\mathcal{Q}^{4k+2}, \mathbb{Z}) = \mathbb{Z}1 + \mathbb{Z}h + \cdots + \mathbb{Z}h^{2k} + \mathbb{Z}e_{2k+1} + \mathbb{Z}e'_{2k+1} + \mathbb{Z}\frac{1}{2}h^{2k+2} + \cdots + \mathbb{Z}\frac{1}{2}h^{4k+2}, \]

and

\[ e_{2k+1} + e'_{2k+1} = h^{2k+1}, \quad e_{2k+1} \cdot e_{2k+1} = e'_{2k+1} \cdot e'_{2k+1} = 0, \]

\[ e_{2k+1} \cdot e'_{2k+1} = \frac{1}{2}h^{4k+2}, \quad h \cdot e_{2k+1} = h \cdot e'_{2k+1} = \frac{1}{2}h^{2k+2}. \]

For \text{dim} \mathcal{Q}^D = 4k, there are two primitive classes \(e_{2k}, e'_{2k} \in H^{4k}(\mathcal{Q}^{4k}, \mathbb{Z})\) such that

\[ H^*(\mathcal{Q}^{4k}, \mathbb{Z}) = \mathbb{Z}1 + \mathbb{Z}h + \cdots + \mathbb{Z}h^{2k-1} + \mathbb{Z}e_{2k} + \mathbb{Z}e'_{2k} + \mathbb{Z}\frac{1}{2}h^{2k+1} + \cdots + \mathbb{Z}\frac{1}{2}h^{4k}. \]
and

\[ e_{2k} + e'_{2k} = h^{2k}, \quad e_{2k} \cdot e_{2k} = e'_{2k} \cdot e'_{2k} = \frac{1}{2} h^{4k}, \]
\[ e_{2k} \cdot e'_{2k} = 0, \quad h \cdot e_{2k} = h \cdot e'_{2k} = \frac{1}{2} h^{2k+1}. \]  
\hspace{1cm} (16)

Next we collect some facts about the spinor bundles on quadric hypersurfaces from [35, Theorem 1.4, Theorem 2.8, Theorem 2.3]. Denote the spinor bundle on \( Q^{2k+1} \) by \( S_{2k+1} \), and the two spinor bundles on \( Q^k \) by \( S'_{2k} \) and \( S''_{2k} \), \( k \geq 1 \). We have

\[ S_1 = \mathcal{O}_{\mathbb{P}^1}(-1), \quad S'_2 = \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(0, -1), \quad S''_2 = \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(-1, 0), \]
\[ \text{rk}(S_{2k+1}) = 2^k, \quad \text{rk}(S'_{2k}) = \text{rk}(S''_{2k}) = 2^{k-1}, \]
\[ S_{2k+1}|_{Q^{2k}} = S'_2 \oplus S''_2, \quad S'|_{Q^{2k-1}} = S'_2|_{Q^{2k-1}} = S_{2k-1}, \]
\[ S''_{2k+1} \cong S_{2k+1}(1), \]
\[ S_{4k}' \cong S_{4k}'(1), \quad S_{4k}' \cong S_{4k}'(1), \]
\[ S_{4k+2}' \cong S_{4k+2}'(1), \quad S_{4k+2}' \cong S_{4k+2}'(1). \]  
\hspace{1cm} (17a) - (17f)

Here the restriction of a vector bundle on \( Q^D \) to \( Q^{D-1} \) is obtained by imbedding \( Q^{D-1} \) into \( Q^D \) as a smooth hyperplane section. The vanishing (17g) for \( Q^D \) in the range \( 0 \leq i < D \) is [35, theorem 2.3], and for \( i = D \) it follows from Serre duality and (17d) and (17e).

Note that for \( D = 2k \), the presented ring structure (15) or (16) cannot completely determine \( e_k \) and \( e'_k \). Namely, we have a choice to name which one of the two \( e_k \) or \( e'_k \). This freedom will be deprived by the following theorem and eventually by (17a), the restrictions (17c), and a choice of the isomorphism \( Q^2 \cong \mathbb{P}^1 \times \mathbb{P}^1 \).

**Theorem 4.1** (i) For \( k \geq 0 \),

\[ \text{ch}(S_{2k+1}) = \frac{2^{k+1}}{1 + e^h}. \]

(ii) For \( k \geq 1 \), exchanging \( e_{2k} \) and \( e'_{2k} \) if necessary, we have

\[ \text{ch}(S'_{2k}) = \frac{2^k}{1 + e^h} + \frac{e_k - e'_k}{2}, \]
\[ \text{ch}(S''_{2k}) = \frac{2^k}{1 + e^h} - \frac{e_k - e'_k}{2}, \]

and

\[ e_k - e'_k = (-1)^{k-1} \frac{c_k(S'_{2k}) - c_k(S''_{2k})}{(k-1)!}. \]
Proof  By the isomorphisms (17c), there exist rational numbers \(a_i, i = 0, 1, 2, \ldots\), such that
\[
\text{ch}(S'_{2k} \oplus S''_{2k}) = 2^k a_0 + 2^k a_1 h + 2^{k-1} a_2 h^2 + 2^{k-1} a_3 h^3 \\
+ \cdots + 2a_{2k-2} h^{2k-2} + 2a_{2k-1} h^{2k-1} + a_{2k} h^{2k}, 
\]
for any \(k \geq 1\), and
\[
\text{ch}(S_{2k+1}) = 2^k a_0 + 2^k a_1 h + 2^{k-1} a_2 h^2 + 2^{k-1} a_3 h^3 + \cdots + 2a_{2k-2} h^{2k-2} \\
+ 2a_{2k-1} h^{2k-1} + a_{2k} h^{2k} + a_{2k+1} h^{2k+1}, 
\]
for any \(k \geq 0\).

Taking the Chern characters of both sides of (17d) we obtain
\[
2^k a_0 - 2^k a_1 h + 2^{k-1} a_2 h^2 - 2^{k-1} a_3 h^3 \\
+ \cdots + 2a_{2k-2} h^{2k-2} - 2a_{2k-1} h^{2k-1} + a_{2k} h^{2k} - a_{2k+1} h^{2k+1} \\
e^h (2^k a_0 + 2^k a_1 h + 2^{k-1} a_2 h^2 + 2^{k-1} a_3 h^3 \\
+ \cdots + 2a_{2k-2} h^{2k-2} + 2a_{2k-1} h^{2k-1} + a_{2k} h^{2k} + a_{2k+1} h^{2k+1}),
\]
thus
\[
a_{2k+1} = - \frac{2^{k-1} a_0}{(2k + 1)!} - \frac{2^{k-1} a_1}{(2k)!} - \frac{2^{k-2} a_2}{(2k - 1)!} - \frac{2^{k-2} a_3}{(2k - 2)!} - \cdots - a_{2k}.
\]

By (17g) and Riemann–Roch, we have
\[
\int_{\mathcal{Q}^{2k}} \text{Td}(\mathcal{Q}^{2k}) \text{ch}(S'_{2k} \oplus S''_{2k}) = 0, 
\int_{\mathcal{Q}^{2k+1}} \text{Td}(\mathcal{Q}^{2k+1}) \text{ch}(S_{2k+1}) = 0.
\]

Since
\[
\text{Td}(\mathcal{Q}^D) = \frac{h^{D+2}}{(1 - e^{-h})^{D+2}} \cdot \frac{2h}{1 - e^{-2h}} = \frac{1 + e^{-h}}{2} \cdot \left(\frac{h}{1 - e^{-h}}\right)^{D+1},
\]
it follows that, for \(k \geq 1\),
\[
\text{Coeff}_h^{2k} \left( \frac{1 + e^{-h}}{2} \cdot \left(\frac{h}{1 - e^{-h}}\right)^{2k+1} \cdot (2^k a_0 + 2^k a_1 h + 2^{k-1} a_2 h^2 + 2^{k-1} a_3 h^3 \\
+ \cdots + 2a_{2k-2} h^{2k-2} + 2a_{2k-1} h^{2k-1} + a_{2k} h^{2k}) \right) = 0,
\]
(18)
\[
\text{Coeff}_h^{2k+1} \left( \frac{1 + e^{-h}}{2} \cdot \left(\frac{h}{1 - e^{-h}}\right)^{2k+2} \cdot (2^k a_0 + 2^k a_1 h + 2^{k-1} a_2 h^2 + 2^{k-1} a_3 h^3 \\
+ \cdots + 2a_{2k-2} h^{2k-2} + 2a_{2k-1} h^{2k-1} + a_{2k} h^{2k} + a_{2k+1} h^{2k+1}) \right) = 0,
\]
(19)
where \( \text{Coeff}_i g(t) \) denotes the coefficient of \( t^i \) in a formal series \( g(t) \) of \( t \). One directly checks

\[
\text{Coeff}_h \left( \frac{1 + e^{-h}}{2} \cdot \left( \frac{h}{1 - e^{-h}} \right) \cdot a_0 \right) = 1, \tag{20}
\]

and

\[
\text{Coeff}_h \left( \frac{1 + e^{-h}}{2} \cdot \left( \frac{h}{1 - e^{-h}} \right)^2 \cdot (a_0 + a_1 h) \right) = 0. \tag{21}
\]

Let

\[
f(x) = \frac{1 + e^{-x}}{2} \cdot \left( a_0 + a_1 x + \frac{a_2}{2} x^2 + \frac{a_3}{2} x^3 + \cdots \right).
\]

Then (18)–(21) amount to

\[
\text{Res}_{x=0} \frac{f(x)}{(1 - e^{-x})^k} = \begin{cases} 1, & k = 1, \\ 0, & k \geq 2. \end{cases}
\]

Let \( y = 1 - e^{-x} \), we compute by the residue theorem

\[
\text{Res}_{x=0} \frac{f(x)}{(1 - e^{-x})^k} = \frac{1}{2\pi \sqrt{-1}} \oint_{|x| = \epsilon} \frac{f(x)dx}{(1 - e^{-x})^k} = \frac{1}{2\pi \sqrt{-1}} \oint_{|y| = \epsilon} \frac{f(x)e^x dy}{y^k} = \text{Res}_{y=0} \frac{f(x)e^x}{y^k}.
\]

Therefore

\[
f(x)e^x = 1,
\]

i.e.,

\[
a_0 + a_1 x + \frac{a_2}{2} x^2 + \frac{a_3}{2} x^3 + \cdots = \frac{2}{1 + e^x}.
\]

So

\[
\text{ch}(S_{2k+1}) = \frac{2^{k+1}}{1 + e^h},
\]

and

\[
\text{ch}(S_{2k}') = \frac{2^k}{1 + e^h} + \alpha_2', \text{ ch}(S_{2k}'') = \frac{2^k}{1 + e^h} + \alpha_2'',
\]

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where $\alpha'_{2k}, \alpha''_{2k} \in H^{2k}_{\text{prim}}(Q^{2k}, \mathbb{Q})$. Note that

$$c_k(S'_{2k}) = (-1)^{k-1}(k-1)!\alpha'_{2k} + \text{ambient class},$$
$$c_k(S''_{2k}) = (-1)^{k-1}(k-1)!\alpha''_{2k} + \text{ambient class}.$$ 

By Riemann–Roch, we have

$$\sum_{i=0}^{4k} (-1)^i \dim \text{Ext}^i(S'_4, S'_4)$$

$$= \int_{Q^{4k}} \text{Td}(Q^{4k}) \text{ch}(S'_4 \otimes S'_4(1))$$

$$= 2 \text{Coeff}_{h^{4k}} \left( \frac{1}{2} \cdot \left( \frac{h}{1-e^{-h}} \right)^{4k+1} \cdot \frac{2^{4k}}{(1+e^h)^2} \cdot e^h \right)$$

$$+ \int_{Q^{4k}} \alpha'_4 \cup \alpha'_4$$

$$= 2^{4k} \text{Coeff}_{h^{4k}} \left( \left( \frac{h}{1-e^{-h}} \right)^{4k+1} \cdot \frac{1}{1+e^h} \right) + \int_{Q^{4k}} \alpha'_4 \cup \alpha'_4$$

$$= 2^{4k} \text{Res}_{y=0} \frac{1}{(2-y)y^{4k+1}} + \int_{Q^{4k}} \alpha'_4 \cup \alpha'_4$$

$$= \frac{1}{2} + \int_{Q^{4k}} \alpha'_4 \cup \alpha'_4,$$

and

$$\sum_{i=0}^{4k} (-1)^i \dim \text{Ext}^i(S''_4, S''_4) = \sum_{i=0}^{4k} (-1)^i \dim \text{Ext}^i(S''_4, S'_4)$$

$$= \int_{Q^{4k}} \text{Td}(Q^{4k}) \text{ch}(S''_4 \otimes S''_4(1))$$

$$= 2 \text{Coeff}_{h^{4k}} \left( \frac{1}{2} \cdot \left( \frac{h}{1-e^{-h}} \right)^{4k+1} \cdot \frac{2^{4k}}{(1+e^h)^2} \cdot e^h \right)$$

$$+ \int_{Q^{4k}} \alpha''_4 \cup \alpha''_4 = \frac{1}{2} + \int_{Q^{4k}} \alpha'_4 \cup \alpha''_4.$$

But by [24], $S'_4$ and $S''_4$ form an exceptional pair, and thus

$$\sum_{i=0}^{4k} (-1)^i \dim \text{Ext}^i(S'_4, S'_4) = 1, \quad \sum_{i=0}^{4k} (-1)^i \dim \text{Ext}^i(S''_4, S''_4) = 0.$$
Then from the cohomology ring structure (16) of $Q^{4k}$, we obtain

$$
\text{ch}(S'_{4k}) = \frac{2^{2k}}{1 + e^h} \pm \frac{e^{2k} - e'^{2k}}{2}, \quad \text{ch}(S''_{4k}) = \frac{2^{2k}}{1 + e^h} \mp \frac{e^{2k} - e'^{2k}}{2}.
$$

For $Q^{4k+2}$, again by Riemann–Roch, we have

$$
\sum_{i=0}^{4k+2} (-1)^i \dim \text{Ext}^i(S'_{4k+2}, S'_{4k+2}) = \sum_{i=0}^{4k+2} (-1)^i \dim \text{Ext}^i(S''_{4k+2}, S''_{4k+2})
$$

$$
= \int_{Q^{4k}} \text{Td}(Q^{4k+2}) \text{ch}(S''_{4k+2} \otimes S'_{4k+2}(1))
$$

$$
= 2\text{Coeff}_{h^{4k+2}} \left( \frac{1 + e^{-h}}{2} \cdot \left( \frac{h}{1 - e^{-h}} \right)^{4k+3} \cdot \frac{2^{4k+2}}{(1 + e^h)^2} \cdot e^h \right)
$$

$$
+ \int_{Q^{4k+2}} \alpha'_{4k+2} \cup \alpha''_{4k+2}
$$

$$
= \frac{1}{2} + \int_{Q^{4k+2}} \alpha'_{4k+2} \cup \alpha''_{4k+2},
$$

and

$$
\sum_{i=0}^{4k+2} (-1)^i \dim \text{Ext}^i(S''_{4k+2}, S''_{4k+2})
$$

$$
= \int_{Q^{4k}} \text{Td}(Q^{4k+2}) \text{ch}(S''_{4k+2} \otimes S''_{4k+2}(1))
$$

$$
= \frac{1}{2} + \int_{Q^{4k+2}} \alpha''_{4k+2} \cup \alpha''_{4k+2},
$$

and

$$
\sum_{i=0}^{4k+2} (-1)^i \dim \text{Ext}^i(S''_{4k+2}, S'_{4k+2})
$$

$$
= \int_{Q^{4k}} \text{Td}(Q^{4k+2}) \text{ch}(S'_{4k+2} \otimes S'_{4k+2}(1))
$$

$$
= \frac{1}{2} + \int_{Q^{4k+2}} \alpha'_{4k+2} \cup \alpha'_{4k+2}.
$$

So by (15) we get

$$
\text{ch}(S'_{4k+2}) = \frac{2^{2k+1}}{1 + e^h} \pm \frac{e^{2k+1} - e'^{2k+1}}{2}, \quad \text{ch}(S''_{4k+2}) = \frac{2^{2k+1}}{1 + e^h} \mp \frac{e^{2k+1} - e'^{2k+1}}{2}.
$$

$\square$
5 Convergence of quantum cohomology for quadrics

Let $Q^D$ be a $D$-dimensional smooth quadric hypersurface in a projective space. The main result of this section is the following theorem.

**Theorem 5.1** The quantum cohomology of $Q^D$ is analytic in a neighborhood of $0 \in H^*(Q^D)$.

This theorem follows from Propositions 5.2 and 5.10. The growth of invariants with only ambient classes is known from [36, Theorem 1]. So the odd-dimensional case is relatively easy. For the even-dimensional case, we use deformation invariance of GW and WDVV to deduce some estimates on the growth of invariants with primitive classes (Proposition 5.9).

5.1 Odd-dimensional case

**Proposition 5.2** If $D$ is odd, then the quantum cohomology of $Q^D$ is analytic in a neighborhood of $0 \in H^*(Q^D)$.

**Proof** Since $D$ is odd, it follows that $h^0, h^1, \ldots, h^D$ is a homogeneous basis of $H^*(Q^D)$. Recall that $\text{Eff}(Q^D)$ can be naturally identified with $\mathbb{Z}_{\geq 0}$. From [36, Theorem 1], there is a positive number $C$ such that

$$|\left(\prod_{i=1}^{D} \tau_0(h^i)^{n_i}\right)|_{Q^D}^D \leq \left(\sum_{i=1}^{D} n_i\right)! \cdot C \sum_{i=1}^{D} n_i + D, \quad \forall n_1, \ldots, n_D \geq 0.$$

From the degree axiom,

$$\left(\prod_{i=1}^{D} \tau_0(h^i)^{n_i}\right)|_{Q^D}^D \neq 0 \Rightarrow d = \frac{1}{D} \sum_{i=1}^{D} (i - 1) n_i + \frac{3}{D} - 1.$$

Therefore we have

$$\left|\left(\prod_{i=1}^{D} \tau_0(h^i)^{n_i}\right)|_{Q^D}^D \prod_{i=1}^{D} \frac{(t^i)^{n_i}}{n_i!}\right| \leq \left(\sum_{i=1}^{D} \frac{n_1 + \cdots + n_D}{n_1, \ldots, n_D}\right) \cdot \prod_{i=1}^{D} (C^{1 + \frac{i-1}{D}} |t^i|)^{n_i} \cdot C^{\frac{3}{D} - 1}.$$

So for $t = \sum_{i=0}^{D} t^i h^i \in H^*(Q^D)$ satisfying $\sum_{i=1}^{D} C^{1 + \frac{i-1}{D}} |t^i| < 1$, we have

$$\sum_{n_1, \ldots, n_D \geq 0} \left|\sum_{d>0} \left(\prod_{i=1}^{D} \tau_0(h^i)^{n_i}\right)|_{Q^D}^D \prod_{i=1}^{D} \frac{|t^i|^{n_i}}{n_i!}\right| \leq \sum_{n_1, \ldots, n_D \geq 0} \left(\sum_{i=1}^{D} \frac{n_1 + \cdots + n_D}{n_1, \ldots, n_D}\right) \cdot \prod_{i=1}^{D} (C^{1 + \frac{i-1}{D}} |t^i|)^{n_i} \cdot C^{\frac{3}{D} - 1}.$$
\[
\frac{C \frac{3}{x} - 1}{1 - \sum_{i=1}^{D} C^{1+\frac{3}{x}} |t'_i|}.
\]

This implies that \( F_0^{Q^D}(t) \) is an absolutely convergent power series of \( t^0, t^1, \ldots, t^D \) for \( t \) sufficiently close to \( 0 \in H^*(Q^D) \). \( \Box \)

### 5.2 Even-dimensional case

In this subsection, let \( D = 2N \) for a fixed \( N \geq 2 \), and we write \( Q = Q^{2N} \) to ease notations. Then \( \text{Eff}(Q) \) is naturally identified with \( \mathbb{Z}_{\geq 0} \). Let \( e_N, e'_N \in H^{2N}(Q) \) be primitive classes as introduced in Sect. 4, and we set \( \mathcal{Q} := i^N (e_N - e'_N) \). Then \( \mathcal{Q} \cup \mathcal{Q} = h^{2N} \), and \( B := \{ 1, h, h^2, \ldots, h^{2N}, \mathcal{Q} \} \) is a basis of \( H^*(Q) \). For \( \gamma_1, \ldots, \gamma_n \in B \), if the invariant \( \langle \gamma_1, \ldots, \gamma_n \rangle_{Q^D} \) is non-zero, then the degree axiom implies that

\[
\deg_{\mathbb{C}}(\gamma_1) + \cdots + \deg_{\mathbb{C}}(\gamma_n) = 2N - 3 + n + 2Nd.
\] (22)

Here \( \deg_{\mathbb{C}}(\gamma) = p \) for \( \gamma \in H^{2p}(Q) \), and we have identified the effective curve class with a non-negative integer \( d \). For a nonzero invariant \( \langle \gamma_1, \ldots, \gamma_n \rangle_{Q^D} \) with \( \gamma_i \in B \), the class \( d \) is determined by \( \gamma_1, \ldots, \gamma_n \). So we will often use the abbreviated notation

\[
\langle \gamma_1, \ldots, \gamma_n \rangle := \langle \gamma_1, \ldots, \gamma_n \rangle_{d} := \langle \gamma_1, \ldots, \gamma_n \rangle_{Q^D}^{d}, \text{ for } \gamma_i \in B.
\]

The following Proposition 5.3 is a key step to determine the full quantum cohomology from the ambient part.

**Proposition 5.3** An invariant of the form

\[
\langle \mathcal{Q}, \ldots, \mathcal{Q}, h^{k_1}, \ldots, h^{k_n} \rangle^m
\]

is zero unless \( m \) is even.

**Proof** We regard \( Q \) as a special fiber of the family of all smooth quadric hypersurfaces in \( \mathbb{P}^{2N+1} \). The Gauss–Manin connection gives monodromy actions on the cohomology group \( H^*(Q) \). All monodromy actions preserve the ambient classes \( h^k \). By [7, Proposition 5.2 and Remark 5.3], the monodromy actions on the lattice \( H_{\text{prim}}^{2N}(Q) \cap H^{2N}(Q, \mathbb{Z}) \) are generated by the Weyl group of this \( A_1 \)-lattice, i.e. the reflection with respect to the hyperplane in \( H_{\text{prim}}^{2N}(Q) \) perpendicular to the lattice vectors. It follows that there is a monodromy action on \( H^*(Q) \) which transforms \( \mathcal{Q} \) to \( -\mathcal{Q} \). Then the assertion follows from the deformation invariance of Gromov–Witten invariants (see e.g. [28, Theorem 4.2']). \( \Box \)
Lemma 5.4 Suppose $n \geq 4$. Given $\gamma_1, \ldots, \gamma_{n-1} \in B$, there exists at most one class $\gamma_n$ in $B$ such that

$$\langle \gamma_1, \ldots, \gamma_n \rangle \neq 0.$$ 

Proof Suppose $\langle \gamma_1, \ldots, \gamma_n \rangle \neq 0$. By Proposition 5.3, if $\wp$ appears in $\gamma_1, \ldots, \gamma_{n-1}$ odd times, then $\gamma_n$ must be $\wp$. If $\wp$ appears in $\gamma_1, \ldots, \gamma_{n-1}$ even times, then $\gamma_n \in \{1, h, h^2, \ldots, h^{2N}\}$, and there is exactly one such class matching the dimension constraint (22).

Lemma 5.5

$$\langle \wp, \ldots, \wp \rangle_d = 0,$$

$$\langle \wp, \ldots, \wp, h, \ldots, h \rangle_d = 0.$$ 

Proof The first equality holds for even $n$ by the dimension constraint (22), and for odd $n$ by the monodromy reason. If $d > 0$ the second equality follows from the first equality by the divisor axiom. If $d = 0$ the second equality follows from $\wp \cup h = 0$ when $m \geq 1$, and from the dimension reason when $m = 0$.

Now we recall Zinger’s estimate on invariants with only ambient insertions. Define

$$A(k, n) = \frac{k - 2N + 3 - n}{2N}.$$ 

We have

$$A(k_1, n_1) + A(k_2, n_2) = A(k_1 + k_2 - 2N, n_1 + n_2 - 3).$$ 

Proposition 5.6 There exists $C > 0$, such that for all $n \geq 1$ and any $i_1, \ldots, i_n \geq 0$,

$$|\langle h^{i_1}, \ldots, h^{i_n} \rangle| \leq n! \cdot C^{n + A(\sum_{j=1}^n i_j, n)}.$$ 

Proof It suffices to consider non-zero invariants. For a non-zero invariant $\langle h^{i_1}, \ldots, h^{i_n} \rangle_d$, we use the dimension constraint (22) to get $d = A(\sum_{j=1}^n i_j, n)$. Then the inequality follows from [36, Theorem 1].

Lemma 5.7 The non-zero 3-point invariants with insertions in $B$ are

$$\begin{align*}
\langle \wp, \wp, 1 \rangle &= 2, \quad \langle \wp, \wp, h^{2N} \rangle = -4, \\
\langle h^i, h^j, h^k \rangle &= 2, \text{ if } i + j + k = 2N, \\
\langle h^i, h^{2N-i}, h^{2N} \rangle &= 4, \text{ if } 1 \leq i \leq 2N - 1, \\
\langle h^i, h^j, h^k \rangle &= 8, \text{ if } i + j + k = 4N, \text{ and } i, j, k \neq 2N, \\
\langle h^{2N}, h^{2N}, h^{2N} \rangle &= 8.
\end{align*}$$ 

(27)
Proof Let $\circ := \bullet_{t=0}$ be the quantum product on $H^*(Q)$ specialized at $t = 0$. Then 3-point invariants are coefficients of the product $\circ$. From [2, Proposition 1, (1.6), (2.2), (2.3)], the specialized quantum product $\circ$ is given by

\[
\begin{align*}
    h \circ h^i &= \begin{cases} 
        h^{i+1}, & 0 \leq i \leq 2N - 2, \\
        h^{2N} + 2h^0, & i = 2N - 1, \\
        2h, & i = 2N, 
    \end{cases} \\
    h \circ \wp &= 0, \\
    \wp \circ \wp &= h^{2N} - 2h^0.
\end{align*}
\]

We can use these identities to obtain all products. For example, we have

\[
h^{2N} \circ h^{2N} = (h \circ h^{2N-1} - 2h^0) \circ h^{2N} = h^{2N-1} \circ (h \circ h^{2N}) - 2h^{2N} = 2(h^{2N} + 2h^0) - 2h^{2N} = 4h^0.
\]

For one more example, for $1 \leq i \leq 2N - 1$, we have

\[
h^i \circ \wp = h^{i-1} \circ h \circ \wp = 0,
\]

and for $i = 2N$, we have

\[
h^{2N} \circ \wp = (h \circ h^{2N-1} - 2h^0) \circ \wp = h^{2N-1} \circ (h \circ \wp) - 2\wp = -2\wp.
\]

We leave the rest cases to interested readers. □

Denote $\xi_i = h^i$ for $0 \leq i \leq 2N$, and $\xi_{2N+1} = \wp$. Denote the dual coordinates by $t^0, \ldots, t^{2N}, t^{2N+1}$. First let us see how to compute all the genus-zero primary invariants inductively from invariants with only ambient insertions, and the WDVV equation (2)

\[
\sum_{\alpha=0}^{2N+1} \sum_{\beta=0}^{2N+1} \partial_t \partial_{t^\alpha} \partial_{t^\beta} F^Q_0 g^{\alpha\beta} \partial_t \partial_{t^\alpha} \partial_{t^\beta} F^Q_0 = \sum_{\alpha=0}^{2N+1} \sum_{\beta=0}^{2N+1} \partial_t \partial_{t^\alpha} \partial_{t^\beta} F^Q_0 g^{\alpha\beta} \partial_t \partial_{t^\alpha} \partial_{t^\beta} F^Q_0,
\]

where $0 \leq i, j \leq 2N$, and

\[
\begin{align*}
    g^{\alpha\beta} &= \frac{1}{2} \delta_{\alpha+\beta, 2N}, & \text{for } 0 \leq \alpha, \beta \leq 2N, \\
    g^{2N+1, \alpha} &= \frac{1}{2} \delta_{\alpha, 2N+1}, & \text{for } 0 \leq \alpha \leq 2N + 1.
\end{align*}
\]

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By (23) and (24), it suffices to compute invariants of the form
\[
\langle \varphi, \ldots, \varphi, h^{k_1}, \ldots, h^{k_n}, h^l \rangle_d,
\]
which satisfies (22), where \(2m + n + 1 \geq 4\), \(m \geq 1\) and \(l \geq 2\). For a subset \(S \subset \{1, 2, \ldots, 2m - 2 + n\}\), we introduce a convenient notation
\[
\xi_S = \text{the } |S| - \text{tuple } \varphi, \ldots, \varphi, h^{k_1}, \ldots, h^{k_i},
\]
and
\[
\deg_c(\xi_S) = \deg_c(\varphi) \cdot |S \cap \{1, \ldots, 2m - 2\}| + k_1 + \ldots + k_i,
\]
where \(\{i_1, \ldots, i_\sigma\} = S \cap \{2m - 2 + 1, \ldots, 2m - 2 + n\}\).

**Lemma 5.8** Suppose \(2 \leq l \leq 2N\), and take \(1 \leq i, j \leq 2N - 1\) such that \(i + j = l\). Then
\[
\langle \varphi, \ldots, \varphi, h^{k_1}, \ldots, h^{k_n}, h^l \rangle
\]
\[
= - \sum_{S \subset \{1, \ldots, 2m - 2 + n\}} \sum_{\alpha = 0}^{2N + 1} \sum_{\beta = 0}^{2N + 1} \langle \varphi, \varphi, \xi_S, \xi_\alpha \rangle g^{\alpha \beta} \langle \xi_\beta, \xi_{\{1, \ldots, 2m - 2 + n\} - S}, h^i, h^j \rangle
\]
\[+ \sum_{S \subset \{1, \ldots, 2m - 2 + n\}} \sum_{\emptyset \neq S \neq \{1, \ldots, 2m - 2 + n\}} \sum_{\alpha = 0}^{2N + 1} \sum_{\beta = 0}^{2N + 1} \langle \varphi, \xi_S, h^i, \xi_\alpha \rangle g^{\alpha \beta} \langle \xi_\beta, \xi_{\{1, \ldots, 2m - 2 + n\} - S}, \varphi, h^j \rangle.
\]

**Proof** Taking the coefficients of \((t^{2N + 1})^{2m - 2} t^{k_1} \ldots t^{k_n}\) on both sides of (28), we obtain
\[
\sum_{\alpha = 0}^{2N + 1} \sum_{\beta = 0}^{2N + 1} \langle \varphi, \ldots, \varphi, h^{k_1}, \ldots, h^{k_n}, \xi_\alpha \rangle g^{\alpha \beta} \langle \xi_\beta, h^i, h^j \rangle
\]
\[+ \sum_{S \subset \{1, \ldots, 2m - 2 + n\}} \sum_{\alpha = 0}^{2N + 1} \sum_{\beta = 0}^{2N + 1} \langle \varphi, \varphi, \xi_S, \xi_\alpha \rangle g^{\alpha \beta} \langle \xi_\beta, \xi_{\{1, \ldots, 2m - 2 + n\} - S}, h^i, h^j \rangle
\]
\[= \sum_{\alpha = 0}^{2N + 1} \sum_{\beta = 0}^{2N + 1} \langle \varphi, \ldots, \varphi, h^{k_1}, \ldots, h^{k_n}, h^i, h^j \rangle g^{\alpha \beta} \langle \xi_\beta, \varphi, h^j \rangle.
\]
Since $2 \leq l \leq 2N$, we can choose $1 \leq i, j \leq 2N - 1$ such that $i + j = l$. Then by (27),

$$\langle \xi_{\beta}, h^i, h^j \rangle = \begin{cases} 2\delta_{\beta,2N-l}, & \text{if } 2 \leq l \leq 2N-1, \\ 2\delta_{\beta,0} + 4\delta_{\beta,2N}, & \text{if } l = 2N. \end{cases}$$

and

$$\langle \varphi, h^i, \xi_{\alpha} \rangle = 0, \forall \ 0 \leq \alpha \leq 2N + 1.$$

So (32b) and (32c) vanish, and by Lemma 5.4 and (29), the sum (32a) equals

$$\langle \varphi, h^k_1, \ldots, h^k_n, h^l \rangle.$$ 

So we obtain (31).

The invariants appearing on the right handside of (31), either have less $\varphi$, or have the same number of $\varphi$ but have less ambient insertions than that of the left handside. So we can inductively compute all the invariants of the form (30). Now we are ready to make the following estimate.

**Theorem 5.9** There exists $C > 0$ such that for any $n \geq 3$, and for $\gamma_i \in B, 1 \leq i \leq n$,

$$|\langle \gamma_1, \ldots, \gamma_n \rangle| \leq n!C^{n+A(\sum_{i=1}^n \deg_{C}(\gamma_i),n)}.$$ 

**Proof** If $\varphi$ appears exactly odd times in $\gamma_1, \ldots, \gamma_n$, by Proposition 5.3 we have $\langle \gamma_1, \ldots, \gamma_n \rangle = 0$. Suppose there are exactly $2m \varphi$’s in $\gamma_1, \ldots, \gamma_n$, and without loss of generality we assume $n \geq 4$. We prove (33) by induction on $m \geq 0$, and on the length $n$. When $m = 0$, we have Zinger’s estimate (26)

$$|\langle \gamma_1, \ldots, \gamma_n \rangle| \leq n!C^{n+A(\sum_{i=1}^n \deg_{C}(\gamma_i),n)},$$

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for some $C > 0$. Replacing $C$ by $C'$ such that
\[
\left( \frac{C'}{C} \right)^n \geq n(n-1)(n-2), \quad \forall n \geq 3,
\]
we have
\[
|\langle \gamma_1, \ldots, \gamma_n \rangle| \leq (n-3)! C'^n + A \left( \sum_{i=1}^{n} \deg_C(\gamma_i), n \right),
\]
Replacing $C'$ by a larger one again, denoted still by $C$, we have
\[
|\langle \gamma_1, \ldots, \gamma_n \rangle| \leq (n-3)! C^{n-3} + A \left( \sum_{i=1}^{n} \deg_C(\gamma_i), n \right).
\]
Suppose we are given an invariant of the form (30); recall that in (30) we have assumed $l \geq 2$. Now we use (31) to estimate (30), inductively from the invariants with less $\wp$’s or with the same number of $\wp$ but less ambient insertions than that of (30). Therefore
\[
\begin{align*}
|\langle \varphi, \ldots, \varphi, h^{k_1}, \ldots, h^{k_n}, h^l \rangle | & \leq \sum_{S \subseteq \{1, \ldots, 2m-2+n\}} \sum_{\alpha=0}^{2N+1} \sum_{\beta=0}^{2N+1} |\langle \varphi, \varphi, \xi_S, \xi_{\alpha} \rangle g^{\alpha \beta} \langle \xi_{\beta}, \xi_{\{1,\ldots,2m-2+n\}-S}, h^i, h^j \rangle | \\
& + \sum_{\emptyset \neq S \neq \{1, \ldots, 2m-2+n\}} \prod_{\alpha=0}^{2N+1} \prod_{\beta=0}^{2N+1} |\langle \varphi, \xi_S, h^i, \xi_{\alpha} \rangle g^{\alpha \beta} \langle \xi_{\beta}, \xi_{\{1,\ldots,2m-2+n\}-S}, \varphi, h^j \rangle | \\
& = |\langle \xi_{2N}, \xi_{\{1,\ldots,2m-2+n\}}, h^i, h^j \rangle | \\
& + \sum_{\emptyset \neq S \neq \{1, \ldots, 2m-2+n\}} \prod_{\alpha=0}^{2N+1} \prod_{\beta=0}^{2N+1} |\langle \varphi, \xi_S, h^i, \xi_{\alpha} \rangle g^{\alpha \beta} \langle \xi_{\beta}, \xi_{\{1,\ldots,2m-2+n\}-S}, \varphi, h^j \rangle |,
\end{align*}
\]
\[
(34a)
\]
\[
(34b)
\]
\[
(34c)
\]
where we have used (27) to obtain (34a). Moreover, as for the sum (34b) we note that by Lemma 5.4, for every \( S \subseteq \{1, \ldots, 2m - 2 + n\} \), there is at most one pair \((\alpha, \beta) \in \{0, 1, \ldots, 2N + 1\}^2 \) such that

\[
\langle \varphi, \varphi, \xi_S, \xi_\alpha \rangle g^{a_1}g^{b_1} \langle \xi_\beta, \xi_{[1, \ldots, 2m - 2 + n] - S}, h^i, h^j \rangle \neq 0,
\]

and for such a pair \((\alpha, \beta) \), due to (29) we have

\[
(2\deg_C(\varphi) + \deg_C(\xi_S) + \deg_C(\varphi)) + (\deg_C(\beta) + \deg_C(\xi_{[1, \ldots, 2m - 2 + n] - S}) + \deg_C(h^i) + \deg_C(h^j)) = 2mN + k_1 + \ldots + k_n + l + 2N. \tag{35}
\]

The same remark holds for the sum (34c).

**Claim 1**

\[
|\langle \varphi, \varphi, h^{k_1}, \ldots, h^{k_n}, h^l \rangle| \leq \frac{1}{4n - 2} \binom{2n}{n} n!(2C)^n + A(2N + \sum_{a=1}^n k_a + l, n + 3). \tag{36}
\]

Here we set the factors

\[
a_n := \frac{1}{4n - 2} \binom{2n}{n}, \quad n \geq 1,
\]

so that the generating function

\[
f(X) = \sum_{n=1}^{+\infty} a_n X^n = \frac{1 - \sqrt{1 - 4X}}{2}
\]

satisfies

\[
f(X)^2 - f(X) + X = 0,
\]

and thus

\[
a_n = \sum_{i=1}^{n-1} a_i a_{n-i}, \quad n \geq 2. \tag{37}
\]

We are going to prove Claim 1 by induction on \( n \). For \( n = 1 \),

\[
|\langle \varphi, \varphi, h^{k_1}, h^l \rangle| = |\langle h^{2N}, h^{k_1}, h^i, h^j \rangle| \leq C^{1 + A(2N + k_1 + i + j, 4)} = C^{1 + A(2N + k_1 + l, 4)} < (2C)^{1 + A(2N + k_1 + l, 4)}.
\]
Suppose that Claim 1 holds for \( n \leq r - 1 \).

\[
\begin{align*}
|\langle \wp, \wp, h^{k_1}, \ldots, h^{k_r}, h^i \rangle| \\
\leq |\langle h^{2N}, h^{k_1}, \ldots, h^{k_r}, h^i, h^j \rangle| \\
+ \sum_{S \subset \{1, \ldots, r\}} \sum_{\alpha=0}^{2N+1} \sum_{\beta=0}^{2N+1} |\langle \wp, \wp, \xi_S, \xi_{\alpha} \rangle \wp^{\alpha \beta} \langle \xi_{\beta}, \xi_{1, \ldots, r-S} \rangle, h^i, h^j \rangle| \\
+ \sum_{S \subset \{1, \ldots, r\}} \sum_{\alpha=0}^{2N+1} \sum_{\beta=0}^{2N+1} |\langle \wp, \wp, \xi_S, h^i, \xi_{\alpha} \rangle \wp^{\alpha \beta} \langle \xi_{\beta}, \xi_{1, \ldots, r-S} \rangle, h^i, h^j \rangle| \\
+ \sum_{S \subset \{1, \ldots, r\}} \sum_{\alpha=0}^{2N+1} \sum_{\beta=0}^{2N+1} |\langle \wp, \wp, \xi_S, h^i, h^j \rangle \wp^{\alpha \beta} \langle \xi_{\beta}, \xi_{1, \ldots, r-S} \rangle, h^i, h^j \rangle| \\
+ \sum_{S \subset \{1, \ldots, r\}} \sum_{\alpha=0}^{2N+1} \sum_{\beta=0}^{2N+1} |\langle \wp, \wp, \xi_S, h^i, h^j \rangle \wp^{\alpha \beta} \langle \xi_{\beta}, \xi_{1, \ldots, r-S} \rangle, h^i, h^j \rangle| \\
\leq r! C^{r+1} A(2N+\sum_{a=1}^{r} k_a + l, r+3) + r \cdot \frac{1}{2} \cdot (r-1)! C^{r+1} A(2N+\sum_{a=1}^{r} k_a + l, r+3) \\
+ \frac{1}{2} \sum_{k=2}^{r-1} \binom{r}{k} a_k \cdot k! \cdot a_{r-k} \cdot (r-k)! (2C)^{r+1} A(2N+\sum_{a=1}^{r} k_a + l, r+3) \\
+ \frac{1}{2} \sum_{k=1}^{r-1} \binom{r}{k} a_k \cdot k! \cdot a_{r-k} \cdot (r-k)! (2C)^{r+1} A(2N+\sum_{a=1}^{r} k_a + l, r+3) \\
\leq \sum_{k=1}^{r-1} \binom{r}{k} a_k \cdot k! \cdot a_{r-k} \cdot (r-k)! (2C)^{r+1} A(2N+\sum_{a=1}^{r} k_a + l, r+3) \\
= r! (2C)^{r+1} A(2N+\sum_{a=1}^{r} k_a + l, r+3) \cdot \sum_{k=1}^{r-1} a_k a_{r-k}.
\end{align*}
\]

In the above deduction we used (36) for smaller \( n \), with \( l \) possibly being 1. This is legal because by (24) we can take another \( k_i \) as \( l \) if \( l = 1 \). Now combining with (37) we obtain
which is Claim 1. Now we go to the general cases \( m \geq 1 \).

**Claim 2**

\[
\left| \langle \varphi, \varphi, h^{k_1}, \ldots, h^{k_n}, h^l \rangle \right| \\
\leq \frac{1}{4(2m - 2 + n)} \left( \frac{2}{2m - 2 + n} \right) (2m - 2 + n)! . g^{m-1}(2C)^{2m-2+n+A(2mN+\sum_{a=1}^{n} k_a+l,2m+n+1)}.
\]

We are going to prove (38) by induction on \( m \). The case \( m = 1 \) is Claim 1. Suppose that the Claim 2 holds for invariants with less \( \varphi \) or with the same number of \( \varphi \) but smaller \( n \), than that of \( \langle \varphi, \ldots, \varphi, h^{k_1}, \ldots, h^{k_n}, h^l \rangle \). Consider (34b) and (34c).

For \( \emptyset \neq S \neq \{1, \ldots, 2m - 2 + n\} \), when \( |S \cap \{1, \ldots, 2m - 2\}| \) is even, say \( 2c \), the pair \((\alpha, \beta) \in \{0, 1, \ldots, 2N+1\}^2\) such that

\[
\langle \varphi, \varphi, \xi_S, \xi_{\alpha} \rangle g^{a\beta} \langle \xi_{\beta}, \xi_{\{1, \ldots, 2m-2+n\}-S}, h^i, h^j \rangle \neq 0,
\]

lies in \( \{0, 1, \ldots, 2N\}^2 \), so we have, when \( c < m - 1 \),

\[
\sum_{\alpha=0}^{2N+1} \sum_{\beta=0}^{2N+1} |\langle \varphi, \varphi, \xi_S, \xi_{\alpha} \rangle g^{a\beta} \langle \xi_{\beta}, \xi_{\{1, \ldots, 2m-2+n\}-S}, h^i, h^j \rangle| \\
\leq a_{|S|} \cdot |S|! \cdot g^{(c+1)-1} \cdot \frac{1}{2} \cdot a_{2m-2+n-|S|} \cdot (2m - 2 + n - |S|)! \cdot g^{(m-1-c)-1} \\
\cdot (2C)^{2m-2+n+A(2mN+\sum_{a=1}^{n} k_a+l,2m+n+1)},
\]

and when \( c = m - 1 \),

\[
\sum_{\alpha=0}^{2N+1} \sum_{\beta=0}^{2N+1} |\langle \varphi, \varphi, \xi_S, \xi_{\alpha} \rangle g^{a\beta} \langle \xi_{\beta}, \xi_{\{1, \ldots, 2m-2+n\}-S}, h^i, h^j \rangle| \\
\leq a_{|S|} \cdot |S|! \cdot g^{m-1} \cdot \frac{1}{2} \cdot a_{2m-2+n-|S|} \cdot (2m - 2 + n - |S|)! \\
\cdot (2C)^{2m-2+n+A(2mN+\sum_{a=1}^{n} k_a+l,2m+n+1)}.
\]

On the other hand when \( |S \cap \{1, \ldots, 2m - 2\}| \) is odd, say \( 2c + 1 \), the pair \((\alpha, \beta) \in \{0, 1, \ldots, 2N+1\}^2\) such that

\[
\langle \varphi, \varphi, \xi_S, \xi_{\alpha} \rangle g^{a\beta} \langle \xi_{\beta}, \xi_{\{1, \ldots, 2m-2+n\}-S}, h^i, h^j \rangle \neq 0,
\]
is \((2N + 1, 2N + 1)\), so we have

\[
\sum_{\alpha=0}^{2N+1} \sum_{\beta=0}^{2N+1} |\langle \varphi_\alpha, \xi_S, \xi_\alpha \rangle g^{\alpha \beta} \langle \xi_\beta, \xi[1, \ldots, 2m-2+n] - S, h^i, h^j \rangle| \\
\leq a|S| \cdot |S|! \cdot g^{(c+2)-1} \cdot \frac{1}{2} \cdot a_{2m-2+n-|S|} \cdot (2m - 2 + n - |S|)! \cdot g^{(m-1-c)-1} \\
\cdot (2C)^{2m-2+n+A(2mN + \sum_{a=1}^{n} k_a + l, 2m+n+1)}.
\]

In any case, we have

\[
\sum_{\alpha=0}^{2N+1} \sum_{\beta=0}^{2N+1} |\langle \varphi_\alpha, \xi_S, \xi_\alpha \rangle g^{\alpha \beta} \langle \xi_\beta, \xi[1, \ldots, 2m-2+n] - S, h^i, h^j \rangle| \\
\leq \frac{1}{2} a|S| \cdot |S|! \cdot a_{2m-2+n-|S|} \cdot (2m - 2 + n - |S|)! \cdot g^{m-1} \\
\cdot (2C)^{2m-2+n+A(2mN + \sum_{a=1}^{n} k_a + l, 2m+n+1)}.
\]

The same argument shows, for \(\emptyset \neq S \neq \{1, \ldots, 2m - 2 + n\}\),

\[
\sum_{\alpha=0}^{2N+1} \sum_{\beta=0}^{2N+1} |\langle \varphi_\alpha, \xi_S, h^i, \xi_\alpha \rangle g^{\alpha \beta} \langle \xi_\beta, \xi[1, \ldots, 2m-2+n] - S, \varphi, h^j \rangle| \\
\leq \frac{1}{2} a|S| \cdot |S|! \cdot a_{2m-2+n-|S|} \cdot (2m - 2 + n - |S|)! \cdot g^{m-1} \\
\cdot (2C)^{2m-2+n+A(2mN + \sum_{a=1}^{n} k_a + l, 2m+n+1)}.
\]

We need to treat (34b) in the case \(|S| = 1\) individually. In this case, when \(|S \cap \{1, \ldots, 2m - 2\}| = 1\), due to (23) we have

\[
\langle \varphi_\alpha, \varphi, \xi_S, \xi_\alpha \rangle = \langle \varphi_\alpha, \varphi, \varphi, \xi_\alpha \rangle = 0.
\]

So for \(|S| = 1\), only the \(S\) such that \(|S \cap \{1, \ldots, 2m - 2\}| = 0\) contributes to (34b). Thus if \(|S| = 1\),

\[
\sum_{\alpha=0}^{2N+1} \sum_{\beta=0}^{2N+1} |\langle \varphi_\alpha, \xi_S, \xi_\alpha \rangle g^{\alpha \beta} \langle \xi_\beta, \xi[1, \ldots, 2m-2+n] - S, h^i, h^j \rangle| \\
\leq \frac{1}{2} a_a \cdot a_{2m-3+n} \cdot (2m - 3 + n)! \cdot g^{m-2} \\
\cdot (2C)^{2m-2+n+A(2mN + \sum_{a=1}^{n} k_a + l, 2m+n+1)}.
\]
Hence

\[
\begin{align*}
|\{\varphi, \ldots, \varphi, h^{k_1}, \ldots, h^{k_n}, h^{j}\}| \\
\leq |\{\xi_{2N}, \varphi, \ldots, \varphi, h^{k_1}, \ldots, h^{k_n}, h^{i}, h^{j}\}| + \sum_{S \subset \{1, \ldots, 2m - 2 + n\}} \frac{2N + 1}{|S| = 1} \\
\times \sum_{\alpha=0}^{2N+1} \sum_{\beta=0}^{2N+1} |\langle \varphi, \xi \varphi, \xi \alpha \rangle g^{\alpha\beta} \langle \xi \beta, \xi \{1, \ldots, 2m - 2 + n\} - S, h^{i}, h^{j}\rangle| \\
+ \sum_{S \subseteq \{1, \ldots, 2m - 2 + n\}} \sum_{|S| \geq 2} \frac{2N + 1}{2N+1} \\
\times \sum_{\alpha=0}^{2N+1} \sum_{\beta=0}^{2N+1} |\langle \varphi, \xi \varphi, h^{i}, \xi \alpha \rangle g^{\alpha\beta} \langle \xi \beta, \xi \{1, \ldots, 2m - 2 + n\} - S, \varphi, h^{j}\rangle| \\
+ \sum_{\emptyset \neq S \neq \{1, \ldots, 2m - 2 + n\}} \sum_{\alpha=0}^{2N+1} \sum_{\beta=0}^{2N+1} |\langle \varphi, \xi \varphi, h^{i}, \xi \alpha \rangle g^{\alpha\beta} \langle \xi \beta, \xi \{1, \ldots, 2m - 2 + n\} - S, \varphi, h^{j}\rangle| \\
\leq a_{2m-2+n} \cdot (2m - 2 + n)!g^{m-2}(2C)^{2m-2+n} + A(2mN + \sum_{a=1}^{n} k_a + l, 2m+n+1) \\
+ \frac{2m - 2 + n}{2} \cdot a_{2m-3+n} \\
\cdot (2m - 3 + n)!g^{m-2}(2C)^{2m-2+n} + A(2mN + \sum_{a=1}^{n} k_a + l, 2m+n+1) \\
+ \frac{1}{2} \sum_{k=2}^{2m-3+n} \left[ \binom{2m - 2 + n}{k} a_k \cdot k! \cdot a_{2m-2+n-k} \cdot (2m - 2 + n - k)! \cdot g^{m-1} \right. \left. \cdot (2C)^{2m-2+n} + A(2mN + \sum_{a=1}^{n} k_a + l, 2m+n+1) \right] \\
+ \frac{1}{2} \sum_{k=1}^{2m-3+n} \binom{2m - 2 + n}{k} a_k \cdot k! \cdot a_{2m-2+n-k} \cdot (2m - 2 + n - k)! \cdot g^{m-1} \\
\cdot (2C)^{2m-2+n} + A(2mN + \sum_{a=1}^{n} k_a + l, 2m+n+1) \\
\leq \frac{9}{2} \cdot a_{2m-3+n} \cdot (2m - 2 + n)!g^{m-2}(2C)^{2m-2+n} + A(2mN + \sum_{a=1}^{n} k_a + l, 2m+n+1) \\
+ \frac{1}{2} \sum_{k=2}^{2m-3+n} \left[ \binom{2m - 2 + n}{k} a_k \cdot k! \cdot a_{2m-2+n-k} \cdot (2m - 2 + n - k)! \cdot g^{m-1} \right. \left. \cdot (2C)^{2m-2+n} + A(2mN + \sum_{a=1}^{n} k_a + l, 2m+n+1) \right]
\end{align*}
\]
\[ + \frac{1}{2} \sum_{k=1}^{2m-3+n} \binom{2m-2+n}{k} a_k \cdot k! \cdot a_{2m-2+n-k} \cdot (2m-2+n-k)! \cdot 9^{m-1} \]
\[ \cdot (2C)^{2m-2+n+A(2mN+\sum_{a=1}^{n} k_a+t,2m+n+1)} \]
\[ = (2m-2+n)! 9^{m-1} (2C)^{2m-2+n+A(2mN+\sum_{a=1}^{n} k_a+t,2m+n+1)} \sum_{k=1}^{2m-3+n} a_k a_{2m-2+n-k} \]
\[ \leq (2m-2+n)! 9^{m-1} (2C)^{2m-2+n+A(2mN+\sum_{a=1}^{n} k_a+t,2m+n+1)} \cdot a_{2m-2+n}, \]

where in the third inequality we have used
\[ a_{2m-2+n} \leq 4a_{2m-3+n}. \]

Thus we have proved (38). So we have
\[ |\langle \gamma_1, \ldots, \gamma_n \rangle| \leq \frac{1}{4(n-3) - 2} \frac{(2n-6)!}{(n-3)!} (18C)^{n-3+A(\sum_{i=1}^{n} \deg C(\gamma_i),n)} \] (39)
for \( \gamma_i \in B, 1 \leq i \leq n \), if there is at least one \( \wp \) in \( \gamma_1, \ldots, \gamma_n \). Since
\[ \frac{(2n-6)!}{(n-3)!} \leq 4^{n-3} (n-3)!, \]

combining (39) with Zinger’s estimate (26), enlarging \( C \) as the beginning of the proof, we obtain the conclusion. \( \qed \)

**Proposition 5.10** If \( D \) is even, then the quantum cohomology of \( Q^D \) is analytic in a neighborhood of \( 0 \in H^*(Q^D) \).

**Proof** This follows from Theorem 5.9, by a similar computation as in the proof of Proposition 5.2. \( \qed \)

### 6 Proof of Gamma conjecture II for quadrics

The main result of this section is the following theorem.

**Theorem 6.1** For \( D \geq 1 \), a \( D \)-dimensional quadric hypersurface \( Q^D \) satisfies the Gamma conjecture II.

Theorem 6.1 follows from Proposition 6.10 (for \( D \) even) and 6.12 (for \( D \) odd). Note that Gamma II for \( Q^1 \cong \mathbb{P}^1 \) and \( Q^2 \cong \mathbb{P}^1 \times \mathbb{P}^1 \) was already known [5, 12, 14]. So we restrict ourselves to the cases \( D \geq 3 \).

We prove Theorem 6.1 by studying the \( \hat{\Gamma} \)-integral structure of \( S_0 \). Recall that \( S_0 \) is the space of \( (\nabla_0)_{z\partial_z} \)-flat sections, and the restricted Dubrovin connection is given by
\[ (\nabla_0)_{z\partial_z} = z\partial_z - \frac{1}{z}(c_1(Q^D) \circ ) + \mu, \]

where \( \circ := \bullet_{t=0} \) is the quantum product on \( H^*(Q^D) \) specialized at \( t = 0 \).
6.1 Specialized quantum product

We follow the notations in Sect. 4. Recall that $h$ stands for the hyperplane class, and we have $c_1(Q^D) = Dh$. Set

$$\varphi := \begin{cases} \frac{D}{2}(e_D - e'_D), & \text{for } D \text{ even,} \\ 0, & \text{for } D \text{ odd.} \end{cases}$$

Then $\varphi \in H^D(Q^D)$ is a primitive class, and $\varphi \cup \varphi = h^D$ when $D$ is even. A basis of $H^*(Q^D)$ is

$$\begin{cases} h^0, \ldots, h^D, \varphi, & \text{for } D \text{ even,} \\ h^0, \ldots, h^D, & \text{for } D \text{ odd.} \end{cases}$$

The Poincaré pairing on $H^*(Q^D)$ is given by

$$\int_{Q^D} h^i \cup h^j = 2 \delta_{i+j,D}, \quad \int_{Q^D} h^i \cup \varphi = 0, \quad \int_{Q^D} \varphi \cup \varphi = \begin{cases} 2, & \text{for } D \text{ even,} \\ 0, & \text{for } D \text{ odd.} \end{cases}$$

The Hodge grading operator $\mu \in \text{End}(H^*(Q^D))$ is given by

$$\mu(h^i) = \left(i - \frac{D}{2}\right)h^i, \quad \mu(\varphi) = 0.$$ 

Note that $D \geq 3$. So from [2, Proposition 1, (1.6), (2.2), (2.3)], the specialized quantum product $\circ$ is given by

$$h \circ h^i = \begin{cases} h^{i+1}, & 0 \leq i \leq D - 2, \\ h^D + 2h^0, & i = D - 1, \\ 2h, & i = D, \end{cases}$$

$$h \circ \varphi = 0,$$

$$\varphi \circ \varphi = \begin{cases} h^D - 2h^0, & \text{for } D \text{ even,} \\ 0, & \text{for } D \text{ odd.} \end{cases}$$

So the characteristic polynomial of $(c_1(Q^D) \circ) \in \text{End}(H^*(Q^D))$ is

$$\begin{cases} x^{D+2} - 4D^2x^2, & \text{for } D \text{ even,} \\ x^{D+1} - 4D^2x, & \text{for } D \text{ odd.} \end{cases}$$

Let $T := D \cdot \frac{1}{4\pi}$ and $\zeta := \exp\left(\frac{2\pi i}{D}\right)$. Then the spectrum of $(c_1(Q^D) \circ)$ is

$$\begin{cases} 0, 0, T, T\zeta^{-1}, \ldots, T\zeta^{-(D-1)}, & \text{for } D \text{ even,} \\ 0, T, T\zeta^{-1}, \ldots, T\zeta^{-(D-1)}, & \text{for } D \text{ odd.} \end{cases}$$
Set
\[ v_k := \frac{(-1)^k}{\sqrt{2D}} \left( h^0 + 2 \sum_{p=1}^{D} \left( \frac{T \xi^{-k}}{D} \right)^{-p} h^p \right), \quad k \in \mathbb{Z}, \]
and moreover, when \( D \) is even,
\[ v_\pm := i \left( \frac{h^0}{2} - \frac{h^D}{4} \right) \pm \frac{\varphi}{2}, \]
and when \( D \) is odd,
\[ v := \sqrt{2}i \left( \frac{h^0}{2} - \frac{h^D}{4} \right). \]
Then we can verify that
\[
\begin{cases}
  c_1(Q^D) \circ v_\pm = 0, & \text{for } D \text{ even}, \\
  c_1(Q^D) \circ v = 0, & \text{for } D \text{ odd},
\end{cases}
\]
and
\[ c_1(Q^D) \circ v_k = T \xi^{-k} v_k, \quad k \in \mathbb{Z}. \]

**Lemma 6.2** The Frobenius algebra \((H^*(Q^D), \circ)\) is semisimple, and a normalized idempotent basis is
\[
\begin{cases}
  v_+, v_-, v_0, \ldots, v_{D-1}, & \text{for } D \text{ even}, \\
  v, v_0, \ldots, v_{D-1}, & \text{for } D \text{ odd}.
\end{cases}
\]

**Proof** When \( D \) is even, by direct calculation, we can verify that
\[ v_\pm \circ v_\pm = 2i v_\pm, \quad v_\pm \circ v_\mp = 0. \]
So
\[ 0 = (c_1(Q^D) \circ v_\pm) \circ v_k = v_\pm \circ (c_1(Q^D) \circ v_k) = v_\pm \circ T \xi^{-k} v_k \Rightarrow v_\pm \circ v_k = 0, \]
and for \( k \neq k' \),
\[
\begin{align*}
  T \xi^{-k} v_k \circ v_{k'} &= (c_1(Q^D) \circ v_k) \circ v_{k'} = v_k \circ (c_1(Q^D) \circ v_{k'}) \\
  &= T \xi^{-k'} v_k \circ v_{k'} \Rightarrow v_k \circ v_{k'} = 0.
\end{align*}
\]
Note that $v_+, v_-, v_0, \ldots, v_{D-1}$ form a basis of $H^*(Q^D)$, and we can write

$$1 = a_+ v_+ + a_- v_- + a_0 v_0 + \cdots + a_{D-1} v_{D-1}.$$ 

for some constants $a_\pm, a_k (0 \leq k \leq D - 1)$. Then

$$v_k = v_k \circ 1 = v_k \circ a_k v_k \Rightarrow a_k \neq 0 \text{ and } v_k \circ v_k = \frac{1}{a_k} v_k.$$ 

So $v_+, v_-, v_0, \ldots, v_{D-1}$ form an idempotent basis of $(H^*(Q^D), \circ)$. Moreover, we can check that

$$\langle v_\pm, v_\pm \rangle^D_{Q^D} = \langle v_k, v_k \rangle^D_{Q^D} = 1,$$

which implies that $v_+, v_-, v_0, \ldots, v_{D-1}$ form a normalized idempotent basis when $D$ is even.

The proof for $D$ odd is similar. \hfill \Box

### 6.2 $J$-function and $P$-function

Recall that $Z^\text{coh}_0: H^*(F) \to S_0$ is a linear isomorphism. Following [14, Definition 3.6.3], Givental’s $J$-function of $F$ is given by

$$J_F(t) := z^{\dim F} (Z^\text{coh}_0)^{-1}_0 1 = z^{\dim F} z^{-\rho} L(0, z)^{-1} 1, \text{ via } t = z^{-1}.$$ 

More explicitly, we have [14, (3.6.7)]

$$J_F(t) = t^\rho \left[ 1 + \sum_{i=0}^{s-1} T^i \sum_{d \in \text{Eff}(F)} \frac{q^d}{n! h^{m+1}} \langle \tau_m(T_i) \tau_0(t)^n \rangle_{Q^D} \right].$$

where $\{T^i\}$ is the basis in $H^*(F)$ satisfying $\int_F T^i \cup T^j = \delta_{ij}$.

For $F = Q^D$ with $D \geq 3$, it follows from [16, Theorem 9.1] that

$$J_{Q^D}(t) = t^{Dh} \sum_{d=0}^\infty \frac{q^d}{D^{m+1} h^{m+1}} \langle \tau_m(T_i) \tau_0(t)^n \rangle_{Q^D}.$$ 

Here $h \in H^2(Q^D)$ is the hyperplane class.

**Remark 6.3** Recall that the original Givental’s big $J$-function [17, (7)] is a formal function of $t \in H^*(F)$ taking values in $H^*(F)$, given by

$$J_F(t, h; q) = h + t + \sum_{i=0}^{s-1} T^i \sum_{d \in \text{Eff}(F)} \frac{q^d}{n! h^{m+1}} \langle \tau_m(T_i) \tau_0(t)^n \rangle_{Q^D}.$$
It can be reduced to the form

\[ J_F(t, h; q) = 
he^{\frac{1}{2}} \left( \mathbb{1} + \sum_{i=0}^{s-1} T^i \left( \sum_{d \in \text{Eff}(F)} \sum_{m \geq 0} \frac{q^{d - t^{-1}|d|}}{h^{m+2}} \langle \tau_m(T_i) \rangle^F_d \right) \right), \]

for \( t \in H^0(F) \oplus H^2(F) \), after applying the fundamental class axiom and the divisor axiom. Givental’s \( J \)-function in this paper is obtained from the original one by setting

\[ J_F(t) = J_F(t = c_1(F) \log t, h = 1; q = 1) \]

\[ = e^{c_1(X) \log t} \left( \mathbb{1} + \sum_{i=0}^{s-1} T^i \left( \sum_{d \in \text{Eff}(F)} \sum_{m \geq 0} \langle \tau_m(T_i) \rangle^F_d t^{d \cdot c_1(F)} \right) \right). \]

In this note, instead of the well-known \( J_F \), we will need a new function \( P_F \) to understand \( Z_0^{coh} \) (see Lemma 6.5). The new function \( P_F \) is defined as

\[ P_F(t) := z^{\frac{\text{dim} F}{2}} (Z_0^{coh})^* \mathbb{1} = z^{\frac{\text{dim} F}{2}} z^{\rho} z^{\mu} L(0, z)^* \mathbb{1}, \quad \text{via } t = z^{-1}. \]  

(40)

Here for any \( M \in \text{End}(H^*(F)) \), we let \( M^* \in \text{End}(H^*(F)) \) be its dual with respect to the Poincaré pairing, i.e. \( \langle M \alpha, \beta \rangle^F = \langle \alpha, M^* \beta \rangle^F \). For example, we have \( (z^\rho)^* = z^\rho \), and \( (z^\mu)^* = z^{-\mu} \). Since

\[ \langle L(0, z)^* \alpha, \beta \rangle^F = \langle \alpha, L(0, z) \beta \rangle^F = \langle \alpha, \beta \rangle^F + \sum_{d \in \text{Eff}(F)} \sum_{m \geq 0} \left( \frac{1}{z} \right)^{m+1} \langle \tau_m(\beta) \tau_0(\alpha) \rangle^F_d, \]

it follows that

\[ L(0, z)^* \mathbb{1} = \sum_{i=0}^{s-1} \langle L(0, z)^* \mathbb{1}, T_i \rangle^F t^i = \mathbb{1} + \sum_{i=0}^{s-1} T^i \left( \sum_{d \in \text{Eff}(F)} \sum_{m \geq 0} \left( \frac{1}{z} \right)^{m+1} \langle \tau_m(T_i) \tau_0(\mathbb{1}) \rangle^F_d \right). \]

So by the definitions of \( \rho \) and \( \mu \), we have

\[ P_F(t) = z^{\frac{\text{dim} F}{2}} z^{\rho} z^{\mu} L(0, z)^* \mathbb{1} \]

\[ = t^{-\rho} \left[ \mathbb{1} + \sum_{i=0}^{s-1} T^i \left( \sum_{d \in \text{Eff}(F)} \sum_{m \geq 0} \left( -t \right)^{m+1} \langle \tau_m(T_i) \tau_0(\mathbb{1}) \rangle^F_d t^{-\deg T^i} \right) \right] \]

\[ = t^{-\rho} \left[ \mathbb{1} + \sum_{i=0}^{s-1} (-1)^{\deg T^i} T^i \left( \sum_{d \in \text{Eff}(F)} \sum_{m \geq 0} \langle \tau_m(T_i) \tau_0(\mathbb{1}) \rangle^F_d (-t)^d c_1(F) \right) \right]. \]

Here we have used the dimension axiom in the last equality.

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One can observe that \( P_F(t) \) and \( J_F(t) \) are closely related. In fact, we can determine \( P_F(t) \) from \( J_F(t) \). For \( F = Q^D \) with \( D \geq 3 \), we have

\[
P_{Q^D}(t) = t^{-Dh} \sum_{d=0}^{\infty} \frac{\prod_{m=1}^{2d}(-2h + m) + (\prod_{m=1}^{2d}(-h + m))^{D+2}}{\prod_{m=1}^{d}(2h - m)} t^{Dd} = t^{-Dh} \sum_{d=0}^{\infty} \frac{\prod_{m=1}^{2d}(-2h + m) + (\prod_{m=1}^{2d}(-h + m))^{D+2}}{\prod_{m=1}^{d}(2h - m)} t^{Dd}
\]

\[
= \sum_{d=0}^{\infty} \left[ \frac{\Gamma(h - d)}{\Gamma(h)} \right]^{D+2} \frac{\Gamma(2h)}{\Gamma(2h - 2d)} t^{D(d-h)}. \tag{41}
\]

### 6.3 Determination of flat sections \( Z^K_0(\mathcal{O}(k)) \)

The main results of this subsection are Lemmas 6.5 and 6.7. Recall that

\[
(\nabla_0)_{\partial z} = z \partial_z - \frac{1}{z} (c_1(Q^D) \circ) + \mu.
\]

So for a \( H^* (Q^D) \)-valued function \( f(z) = \sum_{i=0}^{D} f_i \zeta^i \zeta^{-D} h^{D-i} + f_\varphi(z) \varphi \), we have

\[
(\nabla_0)_{\partial z} f = z^{-D} \left\{ \sum_{i=0}^{D-2} \left[ z \partial_z f_i - D \cdot f_{i+1} \right] h^{D-i} \right.
\]
\[
\left. + z^{D-1} \left[ z \partial_z f_{D-1} - D \cdot (f_D + 2f_0 \cdot z^{-D}) \right] h^{D-(D-1)} \right.
\]
\[
\left. + z^D \left[ z \partial_z f_D - D \cdot 2f_1 \cdot z^{-D} \right] h^{D-D} \right\} + z(\partial_z f_\varphi) \varphi. \tag{42}
\]

**Lemma 6.4** Let \( f(z) = \sum_{i=0}^{D} f_i \zeta^i \zeta^{-D} h^{D-i} + f_\varphi(z) \varphi \in S_0 \). Then \( (\partial_z f_\varphi) \varphi = 0 \), and \( f(z) - f_\varphi(z) \varphi \) is determined by \( f_0(z) \) via

\[
f_i = D^{-i} (z \partial_z)^i f_0, \quad 0 \leq i \leq D - 1,
\]
\[
f_D = D^{-D} (z \partial_z)^D f_0 - 2f_0 \cdot z^{-D},
\]

and

\[
\left\{ (z \partial_z)^{D+1} - D^D z^{-D} (4z \partial_z - 2D) \right\} f_0 = 0. \tag{43}
\]

**Proof** When \( D \) is even, since \( (\nabla_0)_{\partial z} f = 0 \), it follows from (42) that \( \partial_z f_\varphi = 0 \) and

\[
\begin{aligned}
\frac{1}{\partial z} z \partial_z f_i &= f_{i+1}, & \text{for } 0 \leq i \leq D - 2, \\
\frac{1}{\partial z} z \partial_z f_{D-1} &= f_D + 2f_0 \cdot z^{-D}, \\
\frac{1}{\partial z} z \partial_z f_D &= 2f_1 \cdot z^{-D},
\end{aligned} \tag{44}
\]

which yield the required equations. This finishes the proof for \( D \) even.

When \( D \) is odd, we still get (44) from (42), while \( (\partial_z f_\varphi) \varphi = 0 \) because \( \varphi = 0 \) by definition.
Lemma 6.5 For $\gamma \in H^*(Q^D)$, we have

$$Z_0^{coh}(\gamma) = (h^0 h^1 \ldots h^D \varphi),$$

with

$$f_0(z) = \frac{1}{2} \int_{Q^D} \gamma \cup P_{Q^D}(z^{-1})$$

and $f_{\varphi} = \frac{1}{2} \int_{Q^D} \gamma \cup \varphi$.

(Recall that $P_{Q^D}$ is defined in (40).)

Proof When $D$ is even, from Lemma 6.4, we only need to determine $f_0$ and $f_{\varphi}$. For $f_0$, we have

$$z^{-\frac{D}{2}} f_0(z) = \frac{1}{2} \langle Z_0^{coh}(\gamma), 1 \rangle_{Q^D} = \frac{1}{2} \langle \gamma, (Z_0^{coh})^* 1 \rangle_{Q^D} = \frac{1}{2} \langle \gamma, z^{-\frac{D}{2}} P_{Q^D}(z^{-1}) \rangle_{Q^D},$$

where we have used $\int_{Q^D} h^D = 2$ in the first equality. For $f_{\varphi}$, we have

$$f_{\varphi} = \frac{1}{2} \langle Z_0^{coh}(\gamma), \varphi \rangle_{Q^D}$$

$$= \frac{1}{2} \langle \gamma, z^p z^\mu L(0, z)^* \varphi \rangle_{Q^D}$$

$$= \frac{1}{2} \langle \gamma, z^p z^\mu \left( \varphi + \sum_{d>0} \sum_{m \geq 0} \left( -\frac{1}{z} \right)^{m+1} \langle \tau_m(h^i) \tau_0(\varphi) \rangle_{Q^D} \right) \rangle_{Q^D}$$

$$+ \sum_{d>0} \sum_{m \geq 0} \left( -\frac{1}{z} \right)^{m+1} \langle \tau_m(\varphi) \tau_0(\varphi) \rangle_{Q^D} \varphi \right) \rangle_{Q^D}$$

$$= \frac{1}{2} \langle \gamma, z^p z^\mu \left( \varphi + \sum_{d>0} \sum_{m \geq 0} \left( -\frac{1}{z} \right)^{m+1} \langle \tau_m(\varphi) \tau_0(\varphi) \rangle_{Q^D} \varphi \right) \right) \rangle_{Q^D}$$

$$= \frac{1}{2} \langle \gamma, \varphi + \sum_{d>0} \sum_{m \geq 0} \left( -\frac{1}{z} \right)^{m+1} \langle \tau_m(\varphi) \tau_0(\varphi) \rangle_{Q^D} \varphi \right) \rangle_{Q^D}.$$
where we have used \( \int_{Q^D} \mathcal{P} \cup \mathcal{P} = 2 \) in the first equality, and the fourth equality comes from [31, Lemma 1]. Note that \( f_\mathcal{P} \) is independent of \( z \), which implies that

\[
f_\mathcal{P} = \frac{1}{2} \int_{Q^D} \gamma \cup \mathcal{P}.
\]

This finishes the proof for \( D \) even.

If \( D \) is odd, then (45) still holds, while \( \mathcal{P} = 0 \) by definition. \( \square \)

Now we apply Lemma 6.5 to determine the asymptotic behavior of \( Z_{0}^{K}(\mathcal{O}(k)) \) for \( k \in \mathbb{Z} \). Recall (3) and (4) for the definition of \( \hat{\Gamma} \) and \( \text{Ch} \). Note that

\[
\hat{\Gamma}_{Q^D} \text{Ch}(\mathcal{O}(k)) = \frac{\Gamma(1 + h)^{D+2}}{\Gamma(1 + 2h)} \cdot e^{2k\pi ib},
\]

and we define

\[
g_k(z) := \int_{Q^D} \hat{\Gamma}_{Q^D} \text{Ch}(\mathcal{O}(k)) \cup P_{Q^D}(z^{-1}).
\]

Then from the definition (40) of \( P_{Q^D} \), we have

\[
\langle Z_{0}^{K}(\mathcal{O}(k)) (z), 1 \rangle_{Q^D}^D = \frac{g_k(z)}{(2\pi z)^D}.
\]

**Lemma 6.6** As \( z \to 0 \) in any closed subsector of \( |\arg z + \frac{2k\pi}{D}| < \pi + \frac{\pi}{D} \), we have

\[
e^{\frac{Tz^{-k}}{z}} \frac{g_k(z)}{(2\pi z)^D} \sim (-1)^k \frac{1}{\sqrt{2D}} + \sum_{m=1}^{\infty} b_m^{(k)} z^m, \quad b_m^{(k)} \in \mathbb{C}.
\]

**Proof** From (41), we have

\[
g_k(z) = \sum_{d=0}^{\infty} \int_{Q^D} e^{2k\pi ib} \frac{\Gamma(1 + h)^{D+2}}{\Gamma(1 + 2h)} \left[ \frac{\Gamma(h - d)}{\Gamma(h)} \right]^{D+2} \frac{\Gamma(2h)}{\Gamma(2h - 2d)} z^{D(h - d)}
\]

\[
= \sum_{d=0}^{\infty} \text{Coeff}_{s,D} \left( 2e^{2k\pi ib} \frac{\Gamma(1 + s)^{D+2}}{\Gamma(1 + 2s)} \left[ \frac{\Gamma(s - d)}{\Gamma(s)} \right]^{D+2} \frac{\Gamma(2s)}{\Gamma(2s - 2d)} z^{D(s - d)} \right),
\]

where in the second equality, we have used \( \int_{Q^D} h^D = 2 \). From

\[
\frac{\Gamma(2s)}{\Gamma(1 + 2s)} = \frac{1}{2s}, \quad \frac{\Gamma(1 + s)}{\Gamma(s)} = s,
\]

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and
\[
\frac{1}{\Gamma(2s - 2d)} = \frac{2\sqrt{\pi} \cdot 4^{d-s}}{\Gamma(s - d) \Gamma(s - d + \frac{1}{2})},
\]
we get
\[
g_k(z) = 2\sqrt{\pi} \sum_{d=0}^{\infty} \text{Coeff}_{sD} \left( e^{2k\pi i zD} \frac{\Gamma(s - d)D+1}{\Gamma(\frac{1}{2} + s - d)} \cdot 4^{d-s} \cdot z^{D(s-d)} \right)
\]
\[
= 2\sqrt{\pi} \sum_{d=0}^{\infty} \text{Coeff}_{sD} \left( sD+1 \frac{\Gamma(s - d)D+1}{\Gamma(\frac{1}{2} + s - d)} \left( \frac{e^{2k\pi i}}{4} z^{D-d} \right) \right)
\]
\[
= 2\sqrt{\pi} \sum_{d=0}^{\infty} \text{Res}_{s=d} \Gamma(-s)^{D+1} \frac{\Gamma(\frac{1}{2} - s)}{\Gamma(\frac{1}{2} - s)} \left( \frac{e^{2k\pi i}}{4} z^{D} \right)^{-s}.
\]

Take a path $L$ going from $-i\infty$ to $+i\infty$, such that $d$ lies to the right of $L$ for all nonnegative integers $d$. Then by the residue theorem we get
\[
g_k(z) = 2\sqrt{\pi} \cdot \frac{1}{2\pi i} \int_L \frac{\Gamma(-s)^{D+1}}{\Gamma(\frac{1}{2} - s)} (4e^{-2k\pi i z^{D}})^s ds.
\]

Here the convergence issue is addressed in [30, pp. 144–145]. In fact as a consequence $g_k(z)$ is a multiple of a Meijer’s $G$-function (we refer readers to [30, Section 5.2] for the definition and the notation):
\[
g_k(z) = 2\sqrt{\pi} \cdot G_{1,D+1}^{D+1,0} \left( 4e^{-2k\pi i z^{D}} \right)^s \cdot \left( \frac{1}{\Gamma(\frac{1}{2}, \ldots, 0)} \right).
\]

Now the asymptotic expansion of $g_k(z)$ follows from [30, Section 5.7, Theorem 5].

**Lemma 6.7** As $z \to 0$ in any closed subsector of $|\arg z + \frac{2k\pi}{D}| < \pi + \frac{\pi}{D}$, we have
\[
e^{\frac{\tau_z - k}{\zeta}} Z^K_0 (\mathcal{O}(k))(z) \to v_k.
\]

**Proof** From Lemma 6.5, we have
\[
e^{\frac{\tau_z - k}{\zeta}} Z^K_0 (\mathcal{O}(k))(z) = \left[ -e^{\frac{\tau_z - k}{\zeta}} g_k(z) \frac{h^0}{(2\pi z)^{\frac{D}{2}}} + \frac{1}{2} e^{\frac{\tau_z - k}{\zeta}} g_k(z) \frac{h^1}{(2\pi z)^{\frac{D}{2}}} \right]
\]
\[
+ \sum_{p=1}^{D} \frac{e^{\frac{\tau_z - k}{\zeta}}}{2} \left( \frac{1}{2\pi z} \right)^{\frac{D}{2}} (z^{D})^p g_k(z) h^{D-p}.
\]
Note that

\[
\frac{z^p}{2} e^{\frac{T \zeta - k}{\zeta}} \left( \frac{1}{2\pi \zeta} \right)^{\frac{D}{2}} \left( \frac{z \partial_z}{D} \right)^p g_k(z) \\
\quad = \frac{z^p}{2} e^{\frac{T \zeta - k}{\zeta}} \left( \frac{1}{2\pi \zeta} \right)^{\frac{D}{2}} \left( \frac{z \partial_z}{D} \right)^p \left( e^{\frac{T \zeta - k}{\zeta}} \cdot (2\pi \zeta)^{\frac{D}{2}} \cdot e^{\frac{T \zeta - k}{\zeta}} \frac{g_k(z)}{(2\pi \zeta)^{\frac{D}{2}}} \right) \\
\quad = \sum_{p_1+p_2+p_3=p} \frac{z^p}{2} e^{\frac{T \zeta - k}{\zeta}} \left( \frac{z \partial_z}{D} \right)^{p_1} \left( 2\pi \zeta \right)^{\frac{D}{2}} \cdot \left( \frac{z \partial_z}{D} \right)^{p_2} \left( 2\pi \zeta \right)^{\frac{D}{2}} \cdot \left( \frac{z \partial_z}{D} \right)^{p_3} \left( e^{\frac{T \zeta - k}{\zeta}} \frac{g_k(z)}{(2\pi \zeta)^{\frac{D}{2}}} \right).
\]

We observe that

\[
\frac{z^p}{2} e^{\frac{T \zeta - k}{\zeta}} \left( \frac{z \partial_z}{D} \right)^{p_1} \left( e^{\frac{T \zeta - k}{\zeta}} \right) = \frac{1}{2} \left( \frac{T \zeta - k}{D} \right)^{p_1} z^{p-p_1} + o(z^{p-p_1}),
\]

\[
\left( \frac{1}{2\pi \zeta} \right)^{\frac{D}{2}} \left( \frac{z \partial_z}{D} \right)^{p_2} \left( 2\pi \zeta \right)^{\frac{D}{2}} \left( \frac{z \partial_z}{D} \right)^{p_3} \left( 2\pi \zeta \right)^{\frac{D}{2}} = \frac{1}{2^p},
\]

and from [30, Section 1.3, (3)], we can use Lemma 6.6 to obtain

\[
\left( \frac{z \partial_z}{D} \right)^{p_3} \left( e^{\frac{T \zeta - k}{\zeta}} \frac{g_k(z)}{(2\pi \zeta)^{\frac{D}{2}}} \right) \sim \frac{(-1)^k}{\sqrt{2^D}} \delta_{p_3,0} + \sum_{m=1}^{\infty} \left( \frac{m}{D} \right) p_3 b_m(k) z^m.
\]

So we conclude that

\[
\frac{z^p}{2} e^{\frac{T \zeta - k}{\zeta}} \left( \frac{1}{2\pi \zeta} \right)^{\frac{D}{2}} \left( \frac{z \partial_z}{D} \right)^p g_k(z) = \frac{(-1)^k}{2\sqrt{2^D}} \left( \frac{T \zeta - k}{D} \right)^p + o(1), \quad p \geq 1.
\]

As a consequence, we have

\[
e^{\frac{T \zeta - k}{\zeta}} \mathcal{O}_K^K(O(k))(z) \to u_k.
\]

\[\square\]

6.4 Even-dimensional case

In this subsection, let \( D \geq 4 \) be an even integer. The main result of this subsection is Proposition 6.10.
Let $S_+$ and $S_-$ be the spinor bundles on $Q^D$ with
\[
\text{Ch}(S_\pm) = \frac{2^d}{1 + e^{2\pi i h}} \pm 2\pi \frac{\delta^D}{2} \text{ (see Theorem 4.1)}.
\]

Note that
\[
\hat{\chi}^{Q,D}_\pm \text{Ch}(S_\pm) = \frac{\Gamma(1 + h)^{D+2}}{\Gamma(1 + 2h)} \cdot \frac{2^D}{1 + e^{2\pi i h}} \pm 2\pi \frac{\delta^D}{2},
\]
and we define
\[
g_\pm(z) = \int_{Q^D} \hat{\chi}^{Q,D}_\pm \text{Ch}(S_\pm) \cup P_{Q^D}(z^{-1}).
\]

Then from the definition (40) of $P_{Q^D}$, we have
\[
\langle Z^K_0(S_\pm)(z), 1 \rangle_{Q^D} = \frac{g_\pm(z)}{(2\pi z)^{\frac{D}{2}}},
\]

**Lemma 6.8** As $z \to 0$ in any closed subsector of $|\arg z - \frac{\pi}{D}| < \frac{\pi}{2} + \frac{\pi}{D}$, we have
\[
\frac{g_\pm(z)}{(2\pi z)^{\frac{D}{2}}} \sim \frac{-i}{2} + \sum_{m=1}^{\infty} a_m z^m, \quad a_m \in \mathbb{C}.
\]

**Proof** Note that
\[
\int_{Q^D} \hat{\chi} \cup P_{Q^D}(z^{-1}) = 0.
\]

Then from (41), we have
\[
g_\pm(z) = \sum_{d=0}^{\infty} \int_{Q^D} \frac{\Gamma(1 + h)^{D+2}}{\Gamma(1 + 2h)} \cdot \frac{2^D}{1 + e^{2\pi i h}} \left[ \frac{\Gamma(h - d)}{\Gamma(h)} \right]^{D+2} \frac{\Gamma(2h)}{\Gamma(2h - 2d)} z^{D(h - d)}
\]
\[
= \sum_{d=0}^{\infty} \text{Coeff}_{s^D} \left( 2 \frac{\Gamma(1 + s)^{D+2}}{\Gamma(1 + 2s)} \cdot \frac{2^D}{1 + e^{2\pi i s}} \left[ \frac{\Gamma(s - d)}{\Gamma(s)} \right]^{D+2} \frac{\Gamma(2s)}{\Gamma(2s - 2d)} z^{D(s - d)} \right),
\]
where in the second equality, we have used $\int_{Q^D} h^D = 2$. From (46), we get
\[
g_\pm(z) = 2^D \sum_{d=0}^{\infty} \text{Coeff}_{s^D} \left( \frac{s^{D+1}}{1 + e^{2\pi i s}} \cdot \frac{\Gamma(s - d)^{D+2}}{\Gamma(2s - 2d)} z^{D(s - d)} \right).
\]
Moreover, from the equality $\Gamma(x)\Gamma(1 - x) = \frac{\pi}{\sin \pi x}$, we have

$$\frac{1}{1 + e^{2\pi i s}} = \frac{1}{1 + e^{2\pi i (s - d)}} = e^{-\pi i (s - d)} \frac{e^{-\pi i (s - d)} + e^{\pi i (s - d)}}{e^{-\pi i (s - d)}} \frac{2 \cos \pi(s - d)}{e^{-\pi i (s - d)}} \frac{2 \sin \pi \left(\frac{1}{2} - (s - d)\right)}{2\pi} \Gamma\left(\frac{1}{2} - (s - d)\right) \Gamma\left(\frac{1}{2} + (s - d)\right).$$

and from the equality $\Gamma(x)\Gamma(x + \frac{1}{2}) = \frac{2\sqrt{\pi}}{4^{d-s}} \Gamma(2x)$, we have

$$\frac{1}{\Gamma(2s - 2d)} = \frac{2\sqrt{\pi} \cdot 4^{d-s}}{\Gamma(s - d)\Gamma(s - d + \frac{1}{2})}.$$ 

So we obtain

$$g_{\pm}(z) = \frac{2^D}{\sqrt{\pi}} \sum_{d=0}^{\infty} \text{Coeff}_D \left( s^{D+1} \Gamma\left(\frac{1}{2} - (s - d)\right) \Gamma(s - d)^{D+1} (4e^{\pi i z - D}d^{-s}) \right)$$

$$= \frac{2^D}{\sqrt{\pi}} \sum_{d=0}^{\infty} \text{Res}_{s=d} \Gamma\left(\frac{1}{2} + s\right) \Gamma(-s)^{D+1} (4e^{\pi i z - D})^s.$$

Take a path $L$ going from $-i\infty$ to $+i\infty$, such that $d$ lies to the right of $L$, and $-\frac{1}{2} - d$ lies to the left of $L$, for all nonnegative integers $d$. Then by the residue theorem we get

$$g_{\pm}(z) = \frac{2^D}{\sqrt{\pi}} \cdot \frac{1}{2\pi i} \int_{L} \Gamma(-s)^{D+1} \Gamma\left(\frac{1}{2} + s\right) (4e^{\pi i z - D})^s ds.$$

Here the convergence issue is addressed in [30, p. 144–145]. In fact as a consequence $g_{\pm}(z)$ is a multiple of a Meijer’s $G$-function:

$$g_{\pm}(z) = \frac{2^D}{\sqrt{\pi}} \cdot G_{1, D+1}^{D+1, 1} \left( 4e^{\pi i z - D} \left| \begin{array}{c} \frac{1}{2} \\ 0, \ldots, 0 \end{array} \right| \right).$$

Now the asymptotic expansion of $g_{\pm}(z)$ follows from [30, Section 5.7, Theorem 1]. □
Lemma 6.9 As \( z \to 0 \) in any closed subsector of \( |\arg z - \frac{\pi}{D}| < \frac{\pi}{2} + \frac{\pi}{D} \), we have \( Z^K_0(S_\pm)(z) \to v_\pm \).

**Proof** From Lemma 6.5, we have

\[
Z^K_0(S_\pm)(z) = \left[ -\frac{g_\pm(z)}{(2\pi z)^\frac{D}{2}} + \frac{1}{2} \frac{g_\pm(z)}{(2\pi z)^\frac{D}{2}} h^{D \pm \frac{D}{2}} + \sum_{p=1}^{D} \frac{z^p}{2} \left( \frac{1}{2\pi z} \right)^\frac{D}{2} \left( \frac{z \partial_z}{D} \right)^p g_\pm(z) h^{D-p} \right] =: (I) + (II).
\]

In the limit process in the assertion, the asymptotic expansion in Lemma 6.8 yields \((I) \to v_\pm\). Then it suffices to show that \((II) \to 0\), and to this end, we only need to show that for \( p \geq 1 \),

\[
\frac{1}{(2\pi z)^\frac{D}{2}} \left( \frac{z \partial_z}{D} \right)^p g_\pm(z) = O(1). \tag{47}
\]

By Leibnitz rule one has

\[
\frac{1}{(2\pi z)^\frac{D}{2}} \left( \frac{z \partial_z}{D} \right)^p g_\pm(z) = \frac{1}{(2\pi z)^\frac{D}{2}} \left( \frac{z \partial_z}{D} \right)^p \left( (2\pi z)^\frac{D}{2} \cdot \frac{g_\pm(z)}{(2\pi z)^\frac{D}{2}} \right) = \frac{1}{(2\pi z)^\frac{D}{2}} \sum_{p' = 0}^{p} \binom{p}{p'} \left( \frac{z \partial_z}{D} \right)^{p-p'} \left( (2\pi z)^\frac{D}{2} \right)^{p'} \left( \frac{g_\pm(z)}{(2\pi z)^\frac{D}{2}} \right). \]

Since \( g_\pm(z) \) are analytic, their asymptotic expansions are compatible with taking derivatives [30, Section 1.3, (3)]. Thus by Lemma 6.8 we have, for \( p' \geq 0 \),

\[
\left( \frac{z \partial_z}{D} \right)^{p'} \left( \frac{g_\pm(z)}{(2\pi z)^\frac{D}{2}} \right) \sim -\frac{i}{2} \delta_{p',0} + \sum_{m=1}^{\infty} \left( \frac{m}{D} \right)^{p'} a_m z^m.
\]

This finishes the proof of (47).

**Proposition 6.10** Even-dimensional quadrics satisfy Gamma conjecture II.

**Proof** The case \( Q^2 \cong \mathbb{P}^1 \times \mathbb{P}^1 \) is known from [12]. For \( D \geq 4 \), by Proposition 5.10, the quantum cohomology of \( Q^D \) is analytic and semisimple around \( 0 \in H^* (Q^D) \), and it is known that \( D^b(Q^D) \) admits a full exceptional collection [24]:

\[
(E_1, \ldots, E_{D+2}) = (S_+, S_-, O, \ldots, O(D-1)).
\]
Assume that $B$ is an open neighborhood of 0 properly-chosen with respect to \{$(u_i, \Psi_i)$\}$_{1 \leq i \leq D+2}$, such that
\[
(u_1(0), \Psi_1(0)) = (0, v_+), \quad (u_2(0), \Psi_2(0)) = (0, v_-),
(u_i(0), \Psi_i(0)) = (T \zeta^{-(i-3)}, v_{i-3}), \quad 3 \leq i \leq D+2.
\]
Fix $\phi_\pm \in (0, \frac{\pi}{D})$ such that $\phi_+ > \phi_-$. Let
\[
\phi_1 = \phi_+, \quad \phi_2 = \phi_-, \quad \phi_i = -\frac{2\pi(i-3)}{D}, \quad 3 \leq i \leq D+2.
\]
From Lemmas 6.9 and 6.7, $Z^K_B(E_i)$ respects $(u_i, \Psi_i)$ with phase $\phi_i$ around 0. So we can choose $t_0$ near 0 with $u_i(t_0) \neq u_j(t_0)$ for $i \neq j$, such that $Z^K_B(E_i)$ respects $(u_i, \Psi_i)$ with phase $\phi_i$ around $t_0$. From Theorem 3.8, we see that $Q^D$ satisfies Gamma II.

6.5 Odd-dimensional case

In this subsection, let $D \geq 3$ be an odd integer. Let $S$ be the spinor bundle on $Q^D$. Note that
\[
\hat{\Gamma}_{Q^D} Ch(S) = \frac{\Gamma(1+h)^{D+2}}{\Gamma(1+2h)} \cdot \frac{2^{D+1}}{1 + e^{2\pi i h}},
\]
and we define
\[
g(z) = \int_{Q^D} \hat{\Gamma}_{Q^D} Ch(S) \cup P_{Q^D}(z^{-1}).
\]
Then from the definition (40) of $P_{Q^D}$, we have
\[
\langle Z^K_0(S), 1 \rangle^{Q^D} = \frac{g(z)}{(2\pi z)^{\frac{D}{2}}}.
\]
By similar argument in the proof of Lemma 6.8, we get
\[
g(z) = \frac{2^{D+1}}{\sqrt{\pi}} \cdot G_{1,D+1}^{D+1,1} \left( 4e^{\pi i z^{-D}} \left| \begin{array}{c} \frac{1}{2} \vspace{1mm} \\
\theta_1, \ldots, \theta_{D+1} \end{array} \right| \right),
\]
and therefore, as $z \to 0$ in any closed subsector of $|\arg z - \frac{\pi}{D}| < \frac{\pi}{2} + \frac{\pi}{D}$, we have
\[
\frac{g(z)}{(2\pi z)^{\frac{D}{2}}} \sim -\frac{i}{\sqrt{2}} + \sum_{m=1}^{\infty} a_m z^m, \quad a_m \in \mathbb{C}.
\]
Lemma 6.11 As $z \to 0$ in any closed subsector of $|\arg z - \frac{\pi}{D}| < \frac{\pi}{2} + \frac{\pi}{D}$, we have
\[ Z^K_0(S) \to v. \]

Proof From Lemma 6.5, we have
\[
Z^K_0(S)(z) = \left[ -\frac{g(z)}{(2\pi z)^D} h^0 + \frac{1}{2} \frac{g(z)}{(2\pi z)^D} h^D \right] + \sum_{p=1}^{D} \frac{z^p}{2} \left( \frac{D}{2\pi z} \right)^D (\zeta \frac{\partial}{\partial z})^p g(z) h^{D-p}
\]
\[ =: x(I) + (II). \]

In the limit process in the assertion, the asymptotic expansion (48) yields $I \to v$, and by similar argument in the proof of Lemma 6.9 one can check that $(II) \to 0$. \hfill \qed

Proposition 6.12 Odd-dimensional quadrics satisfy Gamma conjecture II.

Proof The case $Q^1 \cong \mathbb{P}^1$ is known from [5, 14]. For $D \geq 3$, by Proposition 5.2, the quantum cohomology of $Q^D$ is analytic and semisimple around $0 \in H^*(Q^D)$, and it is known that $D^b(Q^D)$ admits a full exceptional collection [24]:
\[(E_1, \ldots, E_{D+1}) = (S, \mathcal{O}, \ldots, \mathcal{O}(D-1)).\]

Assume that $B$ is an open neighborhood of $0$ properly-chosen with respect to $\{(u_i, \Psi_i)\}_{1 \leq i \leq D+1}$, such that
\[(u_1(0), \Psi_1(0)) = (0, v), \quad (u_i(0), \Psi_i(0)) = (T \zeta^{-(i-2)}, v_{i-2}), \quad 2 \leq i \leq D + 1.\]

Fix $\phi_1 \in (0, \frac{\pi}{D})$, and let
\[\phi_i = -\frac{2\pi(i-2)}{D}, \quad 2 \leq i \leq D + 1.\]

From Lemma 6.11 and 6.7, $Z^K_B(E_i)$ respects $(u_i, \Psi_i)$ with phase $\phi_i$ around $0$. From Theorem 3.8, we see that $Q^D$ satisfies Gamma II. \hfill \qed

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