A NEW GAP FOR COMPLETE HYPERSURFACES WITH CONSTANT MEAN CURVATURE IN SPACE FORMS

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Abstract. Let $M$ be an $n$-dimensional closed hypersurface with constant mean curvature and constant scalar curvature in an unit sphere. Denote by $H$ and $S$ the mean curvature and the squared length of the second fundamental form respectively. We prove that if $S > \alpha(n, H)$, where $n \geq 4$ and $H \neq 0$, then $S > \alpha(n, H) + B_n \frac{nH^2}{n-1}$. Here

$$\alpha(n, H) = n + \frac{n^3}{2(n-1)}H^2 - \frac{n(n-2)}{2(n-1)}\sqrt{n^2H^4 + 4(n-1)H^2},$$

$B_n = \frac{1}{5}$ for $4 \leq n \leq 20$, and $B_n = \frac{49}{250}$ for $n > 20$. Moreover, we obtain a gap theorem for complete hypersurfaces with constant mean curvature and constant scalar curvature in space forms.

1. Introduction

In the late 1960’s, Simons [26], Lawson [16] and Chern-do Carmo-Kobayashi [10] proved that if $M$ is a closed minimal hypersurface in $\mathbb{S}^{n+1}$, whose squared length of the second fundamental form satisfies $S \leq n$, then $S \equiv 0$ and $M$ is the great sphere $\mathbb{S}^n$, or $S \equiv n$ and $M$ is the Clifford torus. Moreover, they obtained a rigidity theorem for $n$-dimensional closed minimal submanifolds in $\mathbb{S}^{n+p}$ under the pinching condition $S \leq \frac{n^2}{n+p}$. Afterwards, Li-Li [18] improved Simons’ pinching constant for closed minimal submanifolds to $\max\left\{\frac{n^2}{n+p}, \frac{2n}{3}\right\}$. For minimal submanifolds in spheres, constant scalar curvature implies constant length of second fundamental form. In 1970’s, Chern proposed the following famous conjecture [9, 10], which was listed in the well-known problem section by Yau [40].

Chern Conjecture. Let $M$ be a closed minimal hypersurface with constant scalar curvature in the unit sphere $\mathbb{S}^{n+1}$. Then the set of all possible values of the scalar curvature of $M$ is discrete.

Based on the fact that isoparametric hypersurfaces are the only known examples of closed minimal hypersurfaces with constant scalar curvature in a sphere, and the scalar curvatures of these isoparametric hypersurfaces have only finite number of values, mathematicians conjectured that closed minimal hypersurfaces with constant scalar curvature in spheres must be isoparametric. In [19], Münzner showed that for a closed isoparametric minimal hypersurface in an unit sphere, the number of distinct principal curvatures satisfies $g \in \{1, 2, 3, 4, 6\}$, and $S = (g-1)n$.
During the last three decades, there have been many important progresses on Chern Conjecture (see [4, 14, 15, 25] for more details). In 1983, Peng and Terng [23] made a breakthrough on the Chern conjecture. Peng-Terng proved that if $M^n$ is a closed minimal hypersurface with constant scalar curvature in $S^{n+1}$, and if $S > n$, then $S > n + \frac{1}{12n}$. In particular, for the case $n = 3$, they verified that if $S > 3$, then $S \geq 6$. Afterwards, Yang-Cheng [36, 37, 38] improved the pinching constant $\frac{1}{12n}$ to $\frac{n}{3}$. Suh-Yang [27] improved it to $\frac{3n}{7}$. In 1993, Chang [2] solved Chern conjecture in dimension three. Recently, Deng-Gu-Wei [12] proved that any closed Willmore minimal hypersurface with constant scalar curvature in $S^5$ must be isoparametric. For closed minimal hypersurfaces in spheres, the scalar curvature pinching phenomenon without the assumption of constant scalar curvature was investigated by many authors [6, 13, 17, 24, 29, 35, 41].

Let $M$ be an $n$-dimensional closed submanifold in a sphere. Denote by $H$ and $S$ its mean curvature and the squared norm of its second fundamental form respectively. For hypersurfaces with constant mean curvature, we have the following generalized version of Chern’s conjecture.

**Generalized Chern Conjecture.** Let $M$ be a closed hypersurface with constant mean curvature and constant scalar curvature in the unit sphere $S^{n+1}$. Then for each $n$ and $H$, the set of all possible values of $S$ is discrete.

We set

$$\alpha(n, H) = n + \frac{n^3}{2(n-1)}H^2 - \frac{n(n-2)}{2(n-1)}\sqrt{n^2H^4 + 4(n-1)H^2}. $$

Cheng-Nakagawa [7] and Xu [30] got the following rigidity theorem for closed hypersurfaces with constant mean curvature in a sphere.

**Theorem A.** Let $M$ be an $n$-dimensional closed hypersurface with constant mean curvature in the unit sphere $S^{n+1}$. If the squared length of the second fundamental form satisfies $S \leq \alpha(n, H)$, then $M$ is either a totally umbilical sphere, or a Clifford torus.

More generally, the third author [30, 31] proved the generalized Simons-Lawson-Chern-do Carmo-Kobayashi theorem for compact submanifolds with parallel mean curvature in a sphere.

In 1990, de Almeida and Brito [11] proved that closed 3-dimensional hypersurfaces with constant mean curvature and nonnegative constant scalar curvature in a space form must be isoparametric. In [1], Chang proved that if $M$ is a closed hypersurface with constant mean curvature and constant scalar curvature in the unit sphere $S^4$, then $M$ is isoparametric. Chang [3] also proved that a closed hypersurface with constant mean curvature, constant scalar curvature and three distinct principal curvatures in a sphere must be isoparametric. Later, Cheng and Wan [5] classified complete hypersurfaces with constant mean curvature and constant scalar curvature in $\mathbb{R}^4$. In 2017, Núñez [21] investigated the classification problem for closed hypersurfaces with constant mean curvature and constant scalar curvature in $\mathbb{R}^5$. Recently, Tang-Wei-Yan [28] generalized the theorem of de Almeida and Brito [11] to higher dimensional cases.

Put

$$\beta(n, H) = n + \frac{n^3}{2(n-1)}H^2 + \frac{n(n-2)}{2(n-1)}\sqrt{n^2H^4 + 4(n-1)H^2}. $$
Xu-Tian [32] proved the following second gap theorem for hypersurfaces with constant mean curvature and constant scalar curvature in a sphere, which generalized Suh-Yang’s second gap theorem for minimal hypersurfaces.

**Theorem B.** Let $M$ be a compact hypersurface with constant mean curvature and constant scalar curvature in the unit sphere $S^{n+1}$. There exists a positive constant $\gamma(n)$ depending only on $n$ such that if $|H| < \gamma(n)$, and $\beta(n, H) \leq S \leq \beta(n, H) + \frac{\sqrt{n}}{\sqrt{1 + \mu^2}}$, where $n \geq 4$ and $H \neq 0$, then $S \equiv \beta(n, H)$ and $M = S^1\left(\frac{1}{\sqrt{1 + \mu^2}}\right) \times S^{n-1}\left(\frac{\mu}{\sqrt{1 + \mu^2}}\right)$.

Here $\mu = \frac{n|H| + \sqrt{n^2H^2 + 4(n-1)}}{2(n-k)}$.

For hypersurfaces with small constant mean curvature in a sphere, there have been several scalar curvature pinching theorems [15, 33, 34].

When $H \neq 0$, we set

$$\alpha_k(n, H) = n^3 \frac{2k(n-k)}{2(n-k)} H^2 - n(n-2k) \sqrt{n^2H^2 + 4k(n-k)H^2},$$

for $k = 1, \cdots, n-1$. Then we have

$$\alpha(n, H) = \alpha_1(n, H) < \cdots < \alpha_{n-1}(n, H) = \beta(n, H),$$

and the isoparametric hypersurfaces $S^{n-k}\left(\frac{1}{\sqrt{1 + \lambda_k^2}}\right) \times S^k\left(\frac{\lambda_k}{\sqrt{1 + \lambda_k^2}}\right)$ satisfy

$$S \equiv \alpha_k(n, H),$$

where $\lambda_k = \frac{n|H| + \sqrt{n^2H^2 + 4k(n-k)}}{2(n-k)}$ for $k = 1, 2, \ldots, n-1$.

It is well known that the possible values of the squared length of the second fundamental forms of all closed isoparametric hypersurfaces with constant mean curvature in the unit sphere form a discrete set, which was explicitly given by Muto [20]. In 2014, Xu-Xu [34] showed that there exists a compact isoparametric hypersurface with 3 distinct principal curvatures in the unit sphere $S^{3k+1}$ satisfying $S = 2n + 3nH^2$ for $n = 3k$, where $k = 2, 4, 8$. A direct computation shows that

$$\alpha_{[S]}(n, H) < 2n + 3nH^2.$$  

Motivated by Theorem A and the inequality above, we have the following open problem.

**Open Problem.** Let $M$ be a compact hypersurface with constant mean curvature and constant scalar curvature in the unit sphere $S^{n+1}$. Assume that

$$\alpha(n, H) \leq S \leq \alpha_2(n, H),$$

where $n \geq 4$ and $H \neq 0$. Is it possible to prove that $M$ must be one of the isoparametric hypersurfaces $S^{n-k}\left(\frac{1}{\sqrt{1 + \lambda_k^2}}\right) \times S^k\left(\frac{\lambda_k}{\sqrt{1 + \lambda_k^2}}\right)$ for $k = 1, 2$? Here $\lambda_k = \frac{n|H| + \sqrt{n^2H^2 + 4k(n-k)}}{2(n-k)}$.

In this paper, we first verify the third gap theorem for hypersurfaces in a sphere.

**Theorem 1.** Let $M$ be a closed hypersurface with constant mean curvature and constant scalar curvature in the unit sphere $S^{n+1}$. If $S > \alpha(n, H)$, where $n \geq 4$ and $H \neq 0$, then

$$S > \alpha(n, H) + B_n \frac{nH^2}{n-1},$$
Here

\[ B_n = \begin{cases} \frac{1}{2n}, & \text{for } 4 \leq n \leq 20, \\ \frac{29}{250}, & \text{for } n > 20. \end{cases} \]

Set

\[ \alpha(n, H, c) = nc + \frac{n^3}{2(n-1)} H^2 - \frac{n(n-2)}{2(n-1)} \sqrt{n^2 H^4 + 4(n-1) c H^2}, \]

and \( \hat{\alpha}(n, H, c) = \alpha(n, H, c) - n H^2 \). More generally, we prove the following gap theorem for complete hypersurfaces in space forms.

**Theorem 2.** Let \( M \) be a complete hypersurface with constant mean curvature and constant scalar curvature in the simply connected space form \( \mathbb{F}^{n+1}(c) \) with \( c + H^2 > 0 \). If

\[ 0 \leq S - \alpha(n, H, c) \leq B_n \min \left\{ \frac{nH^2}{n-1}, \hat{\alpha}(n, H, c) \right\}, \]

where \( n \geq 4 \) and \( H \neq 0 \), then \( S = \alpha(n, H, c) \) and \( M \) is the isoparametric hypersurface \( \mathbb{F}^{n-1}(c+\lambda^2) \times \mathbb{F}^1(c+c^2\lambda^{-2}) \). Here

\[ \lambda = \frac{n|H| + \sqrt{n^2 H^2 + 4(n-1)c}}{2(n-1)}, \]

and \( B_n \) is the same as in Theorem 1.

**Remark 1.** Notice that \( \frac{n}{n-1} H^2 < \hat{\alpha}(n, H, c) \) if \( c > 0 \), and \( \frac{n}{n-1} H^2 \geq \hat{\alpha}(n, H, c) \) if \( c \leq 0 \). Thus, Theorem 2 implies Theorem 1.

### 2. Hypersurfaces with Constant Mean Curvature

Let \( \mathbb{F}^{n+1}(c) \) be the \((n+1)\)-dimensional simply connected space form with constant sectional curvature \( c \), and let \( M^n \) be a hypersurface in \( \mathbb{F}^{n+1}(c) \). We denote by \( \nabla \) the Levi-Civita connection of \( \mathbb{F}^{n+1}(c) \), and by \( \nabla \) the connection induced on \( M \). Let \( \nu \) be a unit normal vector field of \( M \). Denote by \( h \) the second fundamental form of \( M \), which is a symmetric bilinear form given by

\[ \nabla_X Y = \nabla_X Y + h(X, Y)\nu. \]

The shape operator \( A : TM \to TM \) is defined by

\[ A(X) = -\nabla_X \nu. \]

The second fundamental form and the shape operator are related by \( h(X, Y) = \langle A(X), Y \rangle \). The eigenvalues of \( A \) are called the principal curvatures of \( M \). If the principal curvatures are all constant, then \( M \) is called an isoparametric hypersurface.

We choose a local orthonormal frame \( \{e_i\} \) for the tangent bundle of \( M \). Set \( h_{ij} = h(e_i, e_j) \). The mean curvature is given by \( H = \frac{1}{n} \sum h_{ii} \). Denote by \( S \) the squared length of the second fundamental form, i.e. \( S = \sum_{i,j} h_{ij}^2 \). We denote by \( h_{ijk} \) the covariant derivative of \( h_{ij} \). Then the Codazzi equation implies that \( h_{ijk} \) is symmetric in \( i, j \) and \( k \).

Suppose that \( M \) has constant mean curvature. Then we have the following Simons’ type formulas.

\[ (\Delta h)_{ij} = -ncH \delta_{ij} + (nc - S) h_{ij} + nH \sum_k h_{ik} h_{jk}, \]
Now we suppose that $\lambda$ is a principal curvature whose multiplicity is 1 at some point $x \in M$. Then $\lambda$ is smooth in a neighborhood of $x$. Let $u$ be the unit eigenvector corresponding to $\lambda$, i.e.

\begin{equation}
\lambda u = A(u).
\end{equation}

Differentiating (3) with respect to a tangent vector $X$, we get

\begin{equation}
X(\lambda)u + \lambda \nabla_X u = \nabla_X A(u) + A(\nabla_X u).
\end{equation}

Let $V$ be the orthogonal complement of $u$ in tangent space $T_xM$. Then $V$ is an $(n-1)$-dimensional $A$-invariant subspace. Since $|u| = 1$, we have $\nabla_X u \in V$. Thus $A(\nabla_X u) \in V$. Let $(\cdot)^V$ denote the projection onto $V$. From (4) we obtain

\begin{equation}
X(\lambda) = \langle \nabla_X A(u), u \rangle = \nabla h(u, u, X)
\end{equation}

and

\begin{equation}
(\lambda \text{id}_V - A)|V| = [\nabla_X A(u)]^V.
\end{equation}

Since $\lambda \text{id}_V - A|V|$ is invertible, we get

\begin{equation}
\nabla_X u = (\lambda \text{id}_V - A|V)^{-1}([\nabla_X A(u)]^V).
\end{equation}

Taking covariant derivative of left and right hand sides of (5) with respect to $Y$, we get

\begin{equation}
\nabla^2 \lambda(X, Y) = \nabla^2 h(u, u, X, Y) + 2 \nabla h(\nabla_Y u, u, X) = \nabla^2 h(u, u, X, Y) + 2 \nabla h((\lambda \text{id}_V - A|V)^{-1}([\nabla_Y A(u)]^V), u, X).
\end{equation}

Taking trace, we obtain

\begin{equation}
\Delta \lambda = \Delta h(u, u) + 2 \sum_i \nabla h((\lambda \text{id}_V - A|V)^{-1}([\nabla_{e_i} A(u)]^V), u, e_i)
\end{equation}

at point $x$. We choose the orthonormal frame $\{e_i\}$ at $x$, such that $h_{ij} = \lambda_i \delta_{ij}$, $\lambda_1 = \lambda$ and $e_1 = u$. Then formula (7) can be written as

\begin{equation}
\Delta \lambda_1 = (\Delta h)_{11} + 2 \sum_i \frac{h_{1ik}^2}{\lambda_1 - \lambda_k}.
\end{equation}

This together with (11) yields

\begin{equation}
\Delta \lambda_1 = -ncH + (nc - S)\lambda_1 + nH\lambda_1^2 + 2 \sum_{k \geq 2} \frac{h_{1ik}^2}{\lambda_1 - \lambda_k}.
\end{equation}

3. SOME ALGEBRAIC INEQUALITIES

Let $\mu_1 \leq \cdots \leq \mu_n$ be $n$ real numbers, which satisfy

\begin{equation}
\sum_i \mu_i = 0 \quad \text{and} \quad \sum_i \mu_i^2 = 1.
\end{equation}
Set
\begin{equation}
\phi = \sum_i \mu_i^3 + \frac{n - 2}{\sqrt{n(n-1)}}, \quad \eta = \sqrt{\frac{n}{n-1}} \mu_1 + 1, \quad \sigma = \left[ \sum_{i \geq 2} \left( \mu_i + \frac{\mu_1}{n-1} \right)^2 \right]^{\frac{1}{2}}.
\end{equation}

Lemma 1. The functions \( \phi, \eta \) and \( \sigma \) satisfy
\begin{enumerate}
\item \( \sqrt{\frac{n(n-1)}{n-2}} \phi \geq \eta \geq \frac{\sigma^2}{2} \),
\item \( \sqrt{n(n-1)} \phi \geq \eta [3n - 3(n+1)\eta - 2\sqrt{n(n-1)}\sigma] \).
\end{enumerate}

Proof. By the definitions, we have
\[
\sigma^2 = \sum_{i \geq 2} \left( \mu_i + \frac{\mu_1}{n-1} \right)^2 = 1 - \frac{n}{n-1} \mu_1^2 = \eta(2 - \eta).
\]
Since \( \mu_1 < 0 \), we get \( \eta < 1 \). Then we have
\[
\eta = 1 - \sqrt{1 - \sigma^2} \geq \frac{\sigma^2}{2}.
\]
By a direct computation, we get
\begin{equation}
\sum_i \left( \mu_i + \frac{\mu_1}{n-1} \right)^2 (\mu_i - \mu_1)
= \sum_i \mu_i^3 - \frac{\mu_1}{n-1} \left( n - 3 + \frac{n}{n-1} \mu_1^2 \right)
= \phi - \frac{n - 2}{\sqrt{n(n-1)}} \eta + \frac{\mu_1}{n-1} \sigma^2.
\end{equation}
Thus we have
\[
\phi \geq \frac{n - 2}{\sqrt{n(n-1)}} \eta.
\]
Hence we prove inequality (i).

From (10), we have
\begin{equation}
\phi \geq \frac{n - 2}{\sqrt{n(n-1)}} \eta - \frac{\mu_1}{n-1} \sigma^2 + \sum_{i \geq 2} \left( \mu_i + \frac{\mu_1}{n-1} \right)^2 (\mu_2 - \mu_1)
= \frac{n - 2}{\sqrt{n(n-1)}} \eta - \frac{\mu_1}{n-1} \sigma^2 + \sigma^2(\mu_2 - \mu_1)
= \frac{n - 2}{\sqrt{n(n-1)}} \eta + \sigma^2 \left( \mu_2 - \frac{n}{n-1} \mu_1 \right).
\end{equation}
Since $\left| \mu_2 + \frac{\mu_1}{n-1} \right| \leq \sigma$, we have

$$\sigma^2 \left( \mu_2 - \frac{n}{n-1} \mu_1 \right) \geq \sigma^2 \left( -\frac{1}{n-1} \mu_1 - \sigma - \frac{n}{n-1} \mu_1 \right) \geq \eta (2 - \eta) \left( \frac{n+1}{\sqrt{n(n-1)}} (1 - \eta) - \sigma \right) \geq \eta \left( \frac{n+1}{\sqrt{n(n-1)}} (2 - 3\eta) - 2\sigma \right) = \frac{1}{\sqrt{n(n-1)}} \eta [(n+1)(2 - 3\eta) - 2\sqrt{n(n-1)}\sigma].$$

(12)

Substituting (12) into (11), we obtain conclusion (ii).

□

Lemma 2. Let $n \geq 4$ and

$$B_n = \begin{cases} \frac{1}{\sqrt{20}}, & \text{for } 4 \leq n \leq 20, \\ \frac{26}{20\eta}, & \text{for } n > 20. \end{cases}$$

If $\phi \leq \frac{B_n}{2} \sqrt{\frac{n}{n-1}}$, then $\eta < 0.0445$, $\sigma < 0.295$ and $\mu_2 - \mu_1 > \frac{2}{3-3\sigma} \sqrt{\frac{n}{n-1}}$.

Proof. Since $\sigma = \sqrt{\eta(2 - \eta)} \leq \sqrt{2\eta}$, we get from Lemma 1(ii) that

$$\sqrt{\frac{n-1}{n}} \phi \geq \eta \left( 3 - \frac{3(n+1)}{n} \eta - 2\sqrt{\frac{2(n-1)\eta}{n}} \right).$$

Let

$$f(\eta) := \eta \left( 3 - \frac{3(n+1)}{n} \eta - 2\sqrt{\frac{2(n-1)\eta}{n}} \right).$$

Then we have $f''(\eta) < 0$.

Case (i). If $4 \leq n \leq 20$, then we have

$$f(\eta) \geq \eta \left( 3 - \frac{15}{4} \eta - \sqrt{\frac{38}{5}} \eta \right).$$

We get from Lemma 1(i) that

$$\eta \leq \sqrt{\frac{n(n-1)}{n-2}} \phi \leq \frac{n}{10(n-2)} \leq \frac{1}{5}.$$

Assume $\eta \geq 0.0445$. From $0.0445 \leq \eta \leq 0.2$, we get

$$f(\eta) \geq \min \{ f(0.0445), f(0.2) \} > \frac{1}{10}.$$
Then we have
\[
\mu_2 - \mu_1 \geq -\frac{\mu_1}{n-1} - \sigma - \mu_1
\]
\[
= \sqrt{\frac{n}{n-1}}(1 - \eta) - \sigma
\]
\[
\geq \sqrt{\frac{n}{n-1}}(1 - \eta - \sqrt{\frac{19}{20}}\sigma)
\]
\[
> 0.667\sqrt{n-1}.
\]

Case (ii). If \( n > 20 \), then we have
\[
f(\eta) \geq \eta \left(3 - \frac{66}{21}\eta - 2\sqrt{2\eta}\right).
\]
We also get from Lemma 1 (i) that
\[
\eta \leq \sqrt{n(n-1)}\phi \leq \frac{49n}{500(n-2)} \leq 0.11.
\]
Assume \( \eta \geq 0.04305 \). From \( 0.04305 \leq \eta \leq 0.11 \), we get
\[
f(\eta) \geq \min \{f(0.04305), f(0.11)\} > 0.098.
\]
This contradicts the condition \( \sqrt{n-1}n \phi \leq \frac{49}{500} \). Therefore, we have \( \eta < 0.04305 \) and \( \sigma = \sqrt{\eta(2 - \eta)} < 0.29026 \).
Then we get
\[
\mu_2 - \mu_1 \geq \sqrt{\frac{n}{n-1}}(1 - \eta - \sigma) > \frac{2}{\sqrt{3}}\sqrt{n-1}.
\]
Combining the two cases, we prove Lemma 2. \( \square \)

For integer \( n > 1 \), and real numbers \( c, H \), we define
\[
\alpha(n, H, c) = nc + \frac{n^3}{2(n-1)}H^2 - \frac{n(n-2)}{2(n-1)}\sqrt{n^2H^4 + 4(n-1)cH^2},
\]
\[
\hat{\alpha}(n, H, c) = \alpha(n, H, c) - nH^2.
\]

**Lemma 3.** If \( H^2 + c > 0 \), then
\[
(n-2)\sqrt{\frac{n}{n-1}}H^2\hat{\alpha}(n, H, c) = n(H^2 + c) - \hat{\alpha}(n, H, c).
\]

**Proof.** Note that
\[
n(H^2 + c) - \hat{\alpha}(n, H, c) = \frac{n(n-2)}{2(n-1)}\left[-(n-2)H^2 + \sqrt{n^2H^4 + 4(n-1)cH^2}\right].
\]
Since
\[
[n^2H^4 + 4(n-1)cH^2] - [(n-2)H^2]^2 = 4(n-1)H^2(H^2 + c) \geq 0,
\]
we get
\[
n(H^2 + c) - \hat{\alpha}(n, H, c) \geq 0.
\]
It follows from (15) that
\[
\left[ n(H^2 + c) - \hat{\alpha}(n, H, c) \right]^2 \\
= \frac{n^2(n - 2)^2}{2(n - 1)} H^2 \left[ 2(n - 1)c + (n^2 - 2n + 2)H^2 - (n - 2)\sqrt{n^2 H^4 + 4(n - 1)cH^2} \right] \\
= \frac{n(n - 2)^2}{n - 1} H^2 \hat{\alpha}(n, H, c).
\]
This proves Lemma 3. □

4. Proof of the main theorem

Suppose \( M \) is an \( n (\geq 4) \)-dimensional hypersurface in \( F^{n+1}(c) \) with constant \( H \) and constant \( S \). Let \( \hat{S} \) denote the squared length of the traceless second fundamental form, i.e. \( \hat{S} = S - nH^2 \). Let \( \hat{\alpha}(n, H, c) \) be the constant given by (14). For ease of notation, we write \( \hat{\alpha} \) instead of \( \hat{\alpha}(n, H, c) \). We choose the orthonormal frame \( \{ e_i \} \), such that \( h_{ij} = \lambda_i \delta_{ij} \), where the principal curvatures \( \lambda_i, 1 \leq i \leq n \) satisfy \( \lambda_1 \leq \cdots \leq \lambda_n \). Assuming \( \hat{S} > 0 \), we put \( \mu_i = (\lambda_i - H)\hat{S}^{-1/2} \). Let \( \phi, \eta \) and \( \sigma \) be functions of \( \mu_i \) defined as (9).

**Proof of Theorem 2.** Without loss of generality, we assume \( H > 0 \). Letting \( \delta = \frac{B_n \min\left\{ \frac{n}{n - 1} H^2, \hat{\alpha} \right\}}{ \hat{S} } \), we have \( \hat{\alpha} \leq \hat{S} \leq \hat{\alpha} + \delta \).

Since \( S \) is constant, the formula (2) implies
\[
0 = |\nabla h|^2 + S(nc - S) - n^2 cH^2 + nH \sum_i \lambda_i^3 \\
= |\nabla h|^2 + \hat{S}^2 + n\hat{S}(H^2 + c) + nH \hat{S}^{3/2} \sum_i \mu_i^3.
\]

This implies
\[
(16) \quad |\nabla h|^2 + nH \hat{S}^{3/2} \phi = \hat{S} \left[ \hat{S} - n(H^2 + c) + (n - 2) \sqrt{\frac{n}{n - 1}} S \right].
\]

By Lemma 3 and the definition of \( \delta \), we get
\[
\hat{S} - n(H^2 + c) + (n - 2) \sqrt{\frac{n}{n - 1}} S \phi \leq \hat{\alpha} + \delta - n(H^2 + c) + (n - 2) \sqrt{\frac{n}{n - 1}} \left( \sqrt{\hat{S}} + \frac{\delta}{2\sqrt{\hat{S}}} \right) H
\]
\[
(17) \quad \delta + (n - 2) \sqrt{\frac{n}{n - 1}} \frac{\delta}{2\sqrt{\hat{S}}} H
\]
\[
\leq B_n \sqrt{\frac{n}{n - 1}} H^2 \hat{\alpha} + B_n (n - 2) \sqrt{\frac{n}{n - 1}} \frac{\sqrt{\hat{S}}}{2} H
\]
\[
\leq B_n \frac{n}{2} \sqrt{\frac{n}{n - 1}} \hat{S} H.
\]

From (16) and (17), we obtain
\[
\phi \leq \frac{B_n}{2} \sqrt{\frac{n}{n - 1}}.
\]
Using Lemma 2, we obtain that the smallest principal curvature $\lambda_1$ has multiplicity one. Thus the equation (8) is valid everywhere on $M$. Inserting $\lambda_i = \mu_i S^2 + H$ and $\mu_1 = \sqrt{\frac{n^{-1}}{n}(\eta - 1)}$ into (8), we have

$$\Delta \lambda_1 = \frac{2}{\sqrt{S}} \sum_{k \geq 2} \frac{h_{1ik}^2}{\mu_1 - \mu_k} - nCH$$

\begin{align*}
&\quad + (nc - S) \left[ \sqrt{\frac{n^{-1}}{n} S} (\eta - 1) + H \right] + nH \left[ \sqrt{\frac{n^{-1}}{n} \hat{S}(\eta - 1) + H} \right]^2 \\
&= \frac{2}{\sqrt{S}} \sum_{k \geq 2} \frac{h_{1ik}^2}{\mu_1 - \mu_k} \\
&\quad + \eta \sqrt{S} \left[ \sqrt{\frac{n^{-1}}{n} (nH^2 + c) - \hat{S}) + (n - 1)(\eta - 2)H \sqrt{\hat{S}} \right] \\
&\quad + \sqrt{\frac{n^{-1}}{n} \hat{S}} \left[ \hat{S} - n(H^2 + c) + (n - 2) \sqrt{\frac{n^{-1}}{n} \hat{S}H} \right].
\end{align*}

This together with (16) yields

$$\Delta \lambda_1 = \frac{2}{\sqrt{S}} \sum_{i} h_{1ik}^2 \frac{1}{\mu_1 - \mu_k} + \sqrt{\frac{n^{-1}}{n} S} |\nabla h|^2 + \sqrt{n(n - 1)} H \sqrt{\hat{S} \phi}$$

\begin{align*}
&\quad + \eta \sqrt{S} \left[ \sqrt{\frac{n^{-1}}{n} (nH^2 + c) - \hat{S}) + (n - 1)(\eta - 2)H \sqrt{\hat{S}} \right]. \\
&\quad \text{(18)}
\end{align*}

By Lemma 2 we have

$$\sum_{k \geq 2} \frac{h_{1ik}^2}{\mu_1 - \mu_k} \geq - \frac{3 - 3^{-9}}{2} \sqrt{\frac{n^{-1}}{n} \sum_{k \geq 2} h_{1ik}^2}$$

\begin{align*}
&\quad \geq - \frac{1 - 3^{-10}}{2} \sqrt{\frac{n^{-1}}{n} \left( 3 \sum_{i} h_{1ik}^2 + \sum_{i,j,k \geq 2} h_{ijk}^2 + h_{111}^2 \right)} \\
&\quad = - \frac{1 - 3^{-10}}{2} \sqrt{\frac{n^{-1}}{n} |\nabla h|^2}.
\end{align*}

(19)
By Lemma 3 and the definition of $\delta$, we have

$$\sqrt{\frac{n-1}{n}}(n(H^2+c) - \hat{S}) + (n-1)(\eta-2)H\sqrt{\hat{S}}$$

$$\geq \sqrt{\frac{n-1}{n}}(n(H^2+c) - \hat{S}) - 2(n-1)H\sqrt{\hat{S}}$$

$$\geq \sqrt{\frac{n-1}{n}}(n(H^2+c) - \hat{\alpha} - \delta) - 2(n-1)H\left(\sqrt{\hat{\alpha}} + \frac{\delta}{2\sqrt{\hat{\alpha}}}\right)$$

(20)

$$\geq -nH\sqrt{\hat{\alpha}} - \sqrt{\frac{n-1}{n}}\delta - (n-1)\frac{H\delta}{\sqrt{\hat{\alpha}}}$$

$$\geq \frac{6}{5}nH\sqrt{\hat{S}}.$$

Combining (18)–(20), we obtain

(21) $$\Delta \lambda_1 \geq 3^{-10} \sqrt{n-1} \frac{\eta}{n\hat{S}} |\nabla h|^2 + \sqrt{n-1} H\hat{S}\phi - \frac{6}{5}n\eta H\hat{S}.$$ 

It follows from Lemma 2 that $3\eta + 2\sigma < \frac{4}{7}$. Using Lemma 1 (ii), we get

(22) $$\sqrt{n(n-1)}\phi \geq \eta[3n - (n+1)(3\eta + 2\sigma)] \geq 2n\eta.$$

By (21) and (22), we obtain

(23) $$\Delta \lambda_1 \geq 3^{-10} \sqrt{n-1} \frac{\eta}{n\hat{S}} |\nabla h|^2 + \frac{2}{5}\sqrt{n(n-1)} H\hat{S}\phi \geq 0.$$

Since $S$ is constant, the principal curvatures and the sectional curvature of $M$ are bounded. Thus we can use the Omori-Yau maximum principle [3, 22, 39]: there exists a sequence of points $\{x_k\} \subset M$, such that $\lim_{k \to \infty} \Delta \lambda_1(x_k) \leq 0$. From (23), we obtain

(24) $$\lim_{k \to \infty} |\nabla h|(x_k) = 0 \quad \text{and} \quad \lim_{k \to \infty} \phi(x_k) = 0.$$

From (16) and (24), we have

$$|\nabla h| \equiv \phi \equiv 0$$

and

$$\hat{S} - n(H^2+c) + (n-2)\sqrt{\frac{n}{n-1}} \hat{S}H = 0.$$ 

By Lemma 3 we get

$$\hat{\alpha} - n(H^2+c) + (n-2)\sqrt{\frac{n}{n-1}} \hat{\alpha}H = 0.$$ 

Comparing the above two equations, we obtain $\hat{S} = \hat{\alpha}$.

Since $\phi \equiv 0$, we get $\eta = \sigma = 0$. Therefore, $M$ is an isoparametric hypersurface with two distinct principal curvatures, and the smallest principal curvature has multiplicity one. $\square$
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