ON SELF-SIMILARITY OF WREATH PRODUCTS OF ABELIAN GROUPS

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ABSTRACT. We prove that in a self-similar wreath product of abelian groups $G = B \wr X$, if $X$ is torsion-free then $B$ is torsion of finite exponent. Therefore, in particular, the group $\mathbb{Z} \wr \mathbb{Z}$ cannot be self-similar. Furthermore, we prove that if $L$ is a self-similar abelian group then $L^\omega \wr \mathbb{Z}^2$ is also self-similar.

1. Introduction

A group $G$ is self-similar provided for some finite positive integer $m$, the group has a faithful representation on an infinite regular one-rooted $m$-tree $T_m$ such that the representation is state-closed and is transitive on the tree’s first level. If a group $G$ does not admit such a representation for any $m$ then we say $G$ is not self-similar. In determining that a group is not self-similar we will use the language of virtual endomorphisms of groups. More precisely, a group $G$ is not self-similar if and only if for any subgroup $H$ of $G$ of finite index and any homomorphism $f : H \to G$ there exists a non-trivial subgroup $K$ of $H$ which is normal in $G$ and is $f$-invariant (in the sense $K^f \leq K$).

Which groups admit faithful self-similar representation is an ongoing topic of investigation. The first in depth study of this question was undertaken in [2] and then in book form in [3]. Faithful self-similar representations are known for many individual finitely generated groups ranging from the torsion groups of Grigorchuk and Gupta-Sidki to free groups. Such representations have been also studied for the family of abelian groups [4], of finitely generated nilpotent groups [7], as well as for arithmetic groups [9]. See [8] for further references.

One class which has received attention in recent years is that of wreath products of abelian groups $G = B \wr X$, such as the classical lamplighter group [1] in which $B$ is cyclic of order 2 and $X$ is infinite cyclic. The more general class $G = C_p \wr X$ where $C_p$ is cyclic of prime order $p$ and $X$ is free abelian of rank $d \geq 1$ was the subject of [8] where self-similar groups of this type are constructed for every finite rank $d$.

We show in this paper that the property of $B$ being torsion is necessary to guarantee self-similarity of $G$. More precisely, we prove

Theorem 1. Let $G = B \wr X$ be a self-similar wreath product of abelian groups. If $X$ is torsion free then $B$ is a torsion group of finite exponent. In particular, $\mathbb{Z} \wr \mathbb{Z}$ cannot be self-similar.
Observe that though $G = \mathbb{Z}wr\mathbb{Z}$ is not self-similar, it has a faithful finite-state representation on the binary tree [5].

Next we produce a novel embedding of self-similar abelian groups into self-similar wreath products having higher cardinality.

**Theorem 2.** Let $L$ be a self-similar abelian group and $L^\omega$ an infinite countable direct sum of copies of $L$. Then $L^\omega wr C_2$ is also self-similar.

## 2. Preliminaries

We recall a number of notions of groups acting on trees and of virtual endomorphisms of groups from [4].

### 2.1. State-closed groups.

1. Automorphisms of one-rooted regular trees $T(Y)$ indexed by finite sequences from a finite set $Y$ of size $m \geq 2$, have a natural interpretation as automata on the alphabet $Y$, and with states which are again automorphisms of the tree. A subgroup $G$ of the group of automorphisms $A(Y)$ of the tree is said to have degree $m$. Moreover, $G$ is state-closed of degree $m$ provided the states of its elements are themselves elements of the same group.

2. Given an automorphism group $G$ of the tree, $v$ a vertex of the tree and $l$ a level of the tree, we let $Fix_G(v)$ denote the subgroup of $G$ formed by its elements which fix $v$ and let $Stab_G(l)$ denote the subgroup of $G$ formed by elements which fix all $v$ of level $l$. Also, let $P(G)$ denote the permutation group induced by $G$ on the first level of the tree. We say $G$ is transitive provided $P(G)$ is transitive.

3. A group $G$ is said to be self-similar provided it is a state-closed and transitive subgroup of $A(Y)$ for some finite set $Y$.

### 2.2. Virtual endomorphisms.

1. Let $G$ be a group with a subgroup $H$ of finite index $m$. A homomorphism $f : H \to G$ is called a virtual endomorphism of $G$ and $(G, H, f)$ is called a similarity triple; if $G$ is fixed then $(H, f)$ is called a similarity pair.

2. Let $G$ be a transitive state-closed subgroup of $A(Y)$ where $Y = \{1, 2, \ldots, m\}$. Then the index $[G : Fix_G(1)] = m$ and the projection on the 1st coordinate of $Fix_G(1)$ produces a subgroup of $G$; that is, $\pi_1 : Fix_G(1) \to G$ is a virtual endomorphism of $G$.

3. Let $G$ be a group with a subgroup $H$ of finite index $m$ and a homomorphism $f : H \to G$. If $U \leq H$ and $U^f \leq U$ then $U$ is called $f$-invariant. The largest subgroup $K$ of $H$, which is normal in $G$ and is $f$-invariant is called the $f$-core($H$). If the $f$-core($H$) is trivial then $f$ and the triple $(G, H, f)$ are called simple.

4. Given a triple $(G, H, f)$ and a right transversal $L = \{x_1, x_2, \ldots, x_m\}$ of $H$ in $G$, the permutational representation $\pi : G \to Perm(1, 2, \ldots, m)$ is $g^\pi : i \to j$ which is induced from the right multiplication $Hx_i g = Hx_j$. Generalizing the Kalužnin-Krasner procedure [6], we produce recursively a representation $\varphi : G \to A(Y)$, defined by

\[
g^\varphi = \left( (x_i g \cdot (x(i)g)^{-1})^{f_\varphi} \right)_{1 \leq i \leq m} g^\pi,
\]

seen as an element of an infinitely iterated wreath product of $Perm(1, 2, \ldots, m)$. The kernel of $\varphi$ is precisely the $f$-core($H$) and $G^\varphi$ is state-closed and transitive and $H^\varphi = Fix_G^\varphi(1)$. 

Lemma 1. A group $G$ is self-similar if and only if there exists a simple similarity pair $(H, f)$ for $G$.

3. Proof of Theorem 1

We recall $B$, $X$ are abelian groups, $X$ is a torsion-free group and $G = Bwr X$. Denote the normal closure of $B$ in $G$ by $A = B^G$. Let $(H, f)$ be the similarity pair with respect to which $G$ is self-similar and let $[G : H] = m$. Define

$$A_0 = A \cap H, \quad L = (A_0)^f \cap A, \quad Y = X \cap (AH).$$

Note that if $x \in X$ is nontrivial then the centralizer $C_A(x)$ is trivial. We develop the proof in four lemmas.

Lemma 2. Either $B^m$ is trivial or $(A_0)^f \leq A$. In both cases $A \neq A_0$.

Proof. We have $A^m \leq A_0$ and $X^m \leq H$. As $A$ is normal abelian and $X$ is abelian,

$$[A^m, X^m] \triangleleft G,$$

$$[A^m, X^m] \leq [A_0, X^m] \leq A_0.$$

Also,

$$f : [A^m, X^m] \to [(A^m)^f, (X^m)^f] \leq (A_0)^f \cap G'$$

(1) If $L$ is trivial then $[A^m, X^m] \leq \ker (f)$. Since $f$ is simple, it follows that $\ker (f) = 1$ and $[A^m, X^m] = 1 = [B^m, X^m]$. As $X^m \neq 1$, we conclude $A^m = 1 = B^m$. (2) If $L$ is nontrivial then $L$ is central in $M = A(A_0)^f = A(X \cap M)$ which implies $X \cap M = 1$ and $(A_0)^f \leq A$. (3) If $B$ is a torsion group then $\text{tor}(G) = A$; clearly, $(A_0)^f \leq A$ and $A \neq A_0$. \hfill $\square$

Let $G$ be a counterexample; that is, $B$ has infinite exponent. By the previous lemma $(A_0)^f \leq A$ and so we may use Proposition 1 of \cite{5} to replace the simple similarity pair $(H, f)$ by a simple pair $(\tilde{H}, \tilde{f})$ where $\tilde{H} = A_0 Y$ ($Y \leq X$) and $(Y)^f \leq X$. In other words, we may assume $(Y)^f \leq X$.

Lemma 3. If $z \in X$ is nontrivial and $x_1, \ldots, x_1, z_1, \ldots, z_l \in X$, then there exists an integer $k$ such that

$$z^k \{z_1, \ldots, z_l\} \cap \{x_1, \ldots, x_l\} = \emptyset.$$

Proof. Note that the set $\{k \in \mathbb{Z} | \{z^k z_j\} \cap \{x_1, \ldots, x_l\} \neq \emptyset\}$ is finite, for each $j = 1, \ldots, l$. Indeed, otherwise there exist $k_1 \neq k_2$ such that $z^{k_1 - k_2} = 1$, a contradiction. \hfill $\square$

Lemma 4. If $x \in X$ is nontrivial, then $(x^m)^f$ is nontrivial.

Proof. Suppose that there exists a nontrivial $x \in X$ such that $x^m \in \ker (f)$. Then for each $a \in A$ and each $u \in X$ we have

$$(a^{-mu}a^{mu})^f = (a^{-mu})^f (a^{mu})^f = (a^{-mu})^f ((a^{mu})^f)^f = (a^{-mu})^f (a^{mu})^f = 1.$$

Since $A^m(x^m - 1) \leq \ker (f)$ and is normal in $G$, we have a contradiction. \hfill $\square$
Lemma 5. The subgroup $A^m$ is $f$-invariant.

Proof. Let $a \in A$. Consider $T = \{c_1, \ldots, c_r\}$ a transversal of $A_0$ in $A$, where $r$ is a divisor of $m$. Since $A^m$ is a subgroup of $A_0$ and $A = \oplus_{x \in X} B^x$, there exist $x_1, \ldots, x_t$ such that

$$\langle (c_i^m)^f \mid i = 1, \ldots, r \rangle \leq B^{x_1} \oplus \ldots \oplus B^{x_t}$$

and $z_1, \ldots, z_l \in X$ such that

$$\langle (a^m)^f \rangle \leq B^{z_1} \oplus \ldots \oplus B^{z_l}.$$  

Since $[G : H] = m$, it follows that $X^m \leq Y$. Fix a nontrivial $x \in X$ and let $z = (x^m)^f$.

For each integer $k$, define $i_k \in \{1, \ldots, r\}$ such that

$$a^x c_{i_k}^{-1} \in A_0.$$  

Then

$$\left( (a^{x^m} c_{i_k}^{-1})^m \right)^f = (a^{x^m} c_{i_k}^{-1})^f \in A^m.$$  

But $(a^{x^m} c_{i_k}^{-1})^m = a^{mx^m} c_{i_k}^{-m}$, thus

$$\left( (a^{x^m} c_{i_k}^{-1})^m \right)^f = (a^{mx^m})^f (c_{i_k}^{-m})^f = (a^m)^f z^k c_{i_k}^{-mf}.$$  

By Lemma 4, $z \neq 1$. There exists by Lemma 3 an integer $k'$ such that

$$\{ z^{k'} z_1, \ldots, z^{k'} z_l \} \cap \{ x_1, \ldots, x_t \} = \emptyset,$$

and so,

$$(B^{z^{k'} z_1} \oplus \ldots \oplus B^{z^{k'} z_l}) \cap (B^{x_1} \oplus \ldots \oplus B^{x_t}) = 1.$$  

It follows that

$$(a^m)^f z^{k'} c_{i_k}^{-mf} \in A^m \cap [(B^{z^{k'} z_1} \oplus \ldots \oplus B^{z^{k'} z_l}) \oplus (B^{x_1} \oplus \ldots \oplus B^{x_t})].$$  

But as

$$A^m = \oplus_{x \in X} B^{mx},$$

we conclude, $(a^m)^f z^{k'} \in B^{mx^{k'} z_1} \oplus \ldots \oplus B^{mx^{k'} z_l} \leq A^m$ and $a^m \in A^m$. Hence,

$$(A^m)^f \leq A^m.$$  

With this last lemma, the proof of Theorem 2 is finished.

4. Proof of Theorem 2

Let $L$ be a self-similar abelian group with respect to a simple triple $(L, M, \phi)$; then $\phi$ is a monomorphism. Define $B = \sum_{i \geq 1} L_i$, a direct sum of groups where $L_i = L$ for each $i$. Let $X$ be cyclic group of order 2 and $G = BwrX$, the wreath product of $B$ by $X$. Denote the normal closure of $B$ in $G$ by $A$; then,

$$A = B^X = \left( L_1 \oplus \sum_{i \geq 2} L_i \right) \times B$$

and

$$G = A \cdot X.$$
Define the subgroup of $G$

$$H = \left( M \oplus \sum_{i \geq 2} L_i \right) \times B;$$

an element of $H$ has the form

$$\beta = (\beta_1, \beta_2)$$

where

$$\beta_i = (\beta_{ij})_{j \geq 1}, \beta_{ij} \in L$$

$$\beta_1 = (\beta_{1j})_{j \geq 1}, \beta_{11} \in M.$$  

We note that $[G : H]$ is finite; indeed,

$$[A : H] = [L : M] \text{ and } [G : H] = 2[L : M].$$

Define the maps

$$\phi'_1 : M \oplus \left( \sum_{i \geq 2} L_i \right) \to B, \quad \phi'_2 : B \to B$$

where for $\beta = (\beta_1, \beta_2) = ((\beta_{1j}), (\beta_{2j}))_{j \geq 1}, \beta_{11} \in M$,

$$\phi'_1 : \beta_1 \mapsto (\beta_{1j}^{\phi_1}, (\beta_{2j}^{\phi_1}, \beta_{13}, ..., \beta_{22}, \beta_{21}, \beta_{23}, ...)).$$

Since $L$ is abelian, $\phi'_1$ is a homomorphism and clearly $\phi'_2$ is a homomorphism as well.

Define the homomorphism

$$f : \left( M \oplus \sum_{i \geq 2} L_i \right) \times B \to A$$

by

$$f : (\beta_1, \beta_2) \mapsto \left( (\beta_1)^{\phi'_1}, (\beta_2)^{\phi'_2} \right).$$

Suppose by contradiction that $K$ is a nontrivial subgroup of $H$, normal in $G$ and $f$-invariant and let $\kappa = (\kappa_1, \kappa_2)$ be a nontrivial element of $K$. Since $X$ permutes transitively the indices of $\kappa_i$, we conclude $\kappa_{1i} \in M$ for $i = 1, 2$. Let $s_i$ (call it degree) be the maximum index of the nontrivial entries of $\kappa_i$; if $s_i = 0$ then write $s_i = 0$. Choose $\kappa$ with minimum $s_1 + s_2$; we may assume $s_1$ be minimum among those $s_i \neq 0$. Since

$$\kappa_1 = (\kappa_{1j})_{j \geq 1}, \kappa_{11} \in M,$$

$$\kappa_{1}^{\phi'_1} = (\kappa_{11}^{\phi'_1}, \kappa_{12}, ..., \kappa_{13}, ...),$$

we conclude $(\kappa_{1})^{\phi'_1}$ has smaller degree and therefore

$$\kappa_1 = (\kappa_{11}, e, e, e, ...) \text{ or } \left( (\kappa_{11}, (\kappa_{11}^{\phi'_1}, e, e, ...)) \right).$$
Suppose $\kappa_1 = (\kappa_{11}, e, e, e, \ldots)$. As, $\kappa = (\kappa_1, \kappa_2) \in K$, we have $\kappa_{11} \in M$ and therefore
\[
\kappa^f = \left( (\kappa_{11})^{\phi_1^f}, (\kappa_{2})^{\phi_2^f} \right) \in K,
\]
\[
(\kappa_1)^{\phi'} = \left( (\kappa_{11})^\phi, e, e, e, \ldots \right),
\]
\[
(\kappa_{11})^{\phi} \in M;
\]
\[
\kappa^{f^2} = \left( (\kappa_{11})^{(\phi_1^f)^2}, \kappa_2 \right),
\]
\[
(\kappa_1)^{(\phi_1^f)^2} = \left( (\kappa_{11})^{\phi_2^f}, e, e, e, \ldots \right),
\]
\[
(\kappa_{11})^{\phi_2^f} \in M; \text{ etc.}
\]
By simplicity of $\phi$, this alternative is out. That is,
\[
\kappa_1 = \left( \kappa_{11}, \kappa_{11}^{-\phi}, e, e, \ldots \right), \quad \kappa_{11} \in M.
\]
Therefore
\[
\kappa = (\kappa_1, \kappa_2),
\]
\[
\kappa^x = (\kappa_2, \kappa_1), \quad \kappa^{xf} = \left( (\kappa_2)^{\phi_1^f}, \left( \kappa_{11}^{-\phi}, \kappa_{11}, e, e, \ldots \right) \right),
\]
\[
\kappa^{xfx} = \left( \left( \kappa_{11}^{-\phi}, \kappa_{11}, e, e, \ldots \right), (\kappa_2)^{\phi_1^f} \right)
\]
are elements of $K$ and so, $\kappa_{11}^\phi \in M$. Furthermore,
\[
\kappa^{xfxf} = \left( \left( \kappa_{11}^{-\phi_2}, \kappa_{11}, e, e, \ldots \right), (\kappa_2)^{\phi_1^f \phi_2^f} \right),
\]
\[
\kappa_{11}^{-\phi_2} \kappa_{11} \in M;
\]
successive applications of $f$ to $\kappa^{xfx}$ produces $\kappa_{11}^{\phi_i} \in M$. Therefore, $\left\langle \kappa_{11}^{\phi_i} \mid i \geq 0 \right\rangle$ is a $\phi$-invariant subgroup of $M$; a contradiction is reached.

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