FORMALITY OF POSITIVE QUATERNION KÄHLER MANIFOLDS

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ABSTRACT. Positive Quaternion Kähler Manifolds are Riemannian manifolds with holonomy contained in $\text{Sp}(n) \text{Sp}(1)$ and with positive scalar curvature. Conjecturally, they are symmetric spaces. We offer a new approach to this field of study via Rational Homotopy Theory, thereby proving the formality of Positive Quaternion Kähler Manifolds. This result is established by means of an in-depth investigation on how formality behaves under spherical fibrations.

INTRODUCTION

Quaternion Kähler Manifolds settle in the highly remarkable class of special geometries. Hereby one refers to Riemannian manifolds with special holonomy among which Kähler manifolds, Calabi–Yau manifolds or Joyce manifolds are to be mentioned as the most prominent examples. Quaternion Kähler Manifolds have holonomy contained in $\text{Sp}(n) \text{Sp}(1)$; they are called \textit{positive}, if their scalar curvature is positive.

Positive Quaternion Kähler Geometry lies in the intersection of very classical yet rather different fields in mathematics. Despite its geometrical setting which involves fundamental definitions from Riemannian geometry, it was soon discovered to be accessible by methods from (differential) topology, symplectic geometry and complex algebraic geometry even.

In this article we shall present a different approach using Rational Homotopy Theory. To the knowledge of the author, this is the first one of its kind.

Rational Homotopy Theory is a very elegant and easily-computable version of homotopy theory at the expense of losing information on torsion. It provides a transition from topology to algebra by encoding the rational homotopy type of a space in a commutative differential graded algebra. In particular, rational homotopy groups as well as Massey products can be derived from the algebra structure. Likewise, the rational cohomology algebra of the space is the homology algebra of the corresponding commutative differential graded algebra.

The concept of formality features prominently among the properties of topological spaces as this property reduces the study of the rational homotopy type entirely to the problem of merely understanding the rational cohomology algebra. Or in other terms: We may derive the rational cohomology from the rational homotopy type, however, is the information contained in the rational cohomology already sufficient to reconstruct the

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rational homotopy type? If the answer is in the positive, the space is called formal.

On the one hand, Positive Quaternion Kähler Manifolds are conjectured to be symmetric spaces. On the other hand, the name itself presumes a certain proximity to the field of Kähler Geometry. Indeed, (Positive) Quaternion Kähler Manifolds can be considered a quaternionic analogue of (the complex) Kähler Manifolds. Both symmetric spaces as well as Kähler manifolds are formal spaces, i.e. their rational homotopy type can be “formally” derived from the rational cohomology ring. As our main result we shall prove the same for Positive Quaternion Kähler Manifolds:

**Theorem A.** A Positive Quaternion Kähler Manifold is a formal space.

Note that the concept of formality seems to be a recurring theme within the field of special geometries.

The proof will make use of the so-called *twistor fibration* \( S^2 \hookrightarrow Z \rightarrow M \) which relates the properties of the Positive Quaternion Kähler Manifold \( M \) to the ones of its *twistor space* \( Z \)—a Fano contact Kähler Einstein manifold.

Kähler manifolds have been found to be formal spaces by joint work of Deligne, Griffiths, Morgan and Sullivan, i.e. the total space of this *spherical fibration* is formal. The concept of formality can be extended to maps and we shall prove

**Theorem B.** The twistor fibration (for Positive Quaternion Kähler Manifolds) is a formal map.

In order to establish these “geometric” results—theorems A,B—we shall investigate to which extent the formality of the total space in such a spherical fibration suffices to derive the formality of the base space. Thus the theorems will be a consequence of the following algebraically topological results:

**Theorem C.** Let

\[
F \hookrightarrow E \xrightarrow{p} B
\]

be a spherical fibration of simply-connected (and path-connected) topological spaces with rational homology of finite type. Let the fibre satisfy \( F \cong_{\mathbb{Q}} S^n \) for an even \( n \geq 2 \). Suppose further that the rationalised Hurewicz homomorphism \( \pi_*(B) \otimes \mathbb{Q} \to H_*(B) \) is injective in degrees \( n \) and \( 2n \).

Then we obtain: If the space \( E \) is formal, so is \( B \).

**Theorem D.** Let

\[
F \hookrightarrow E \xrightarrow{p} B
\]

be a spherical fibration of simply-connected (and path-connected) topological spaces. Let the fibre satisfy \( F \cong_{\mathbb{Q}} S^n \) for an even \( n \geq 2 \).

If \( B \) and consequently (cf. lemma 4.1) also \( E \) is a formal space, then the fibration \( p \) is a formal map.

Moreover, we shall see that the discussion for the case of even-dimensional fibres is completely distinct from the one with odd-dimensional fibres, where hardly no relations between the formality of the base space and the one of the total space can be found.
Let us daringly suggest and sketch two interpretations that might serve as a motivation for proving formality. On the one hand formality is an obstruction to geometric formality (cf. [10]) which consists in the property that the product of harmonic forms is harmonic again. Geometric formality enforces strong restrictions on the topological structure of the underlying manifold. For example, the Betti numbers of the manifold $M^n$ are restricted from above by the Betti numbers of the $n$-dimensional torus—cf. [10], theorem 6. (This result was even improved on Kähler manifolds by Nagy—cf. [15], corollary 4.1.) Symmetric spaces are geometrically formal (cf. [11], [18]). Thus it is tempting to conjecture the same for Positive Quaternion Kähler Manifolds.

On the other hand the Bott conjecture speculates that simply-connected compact Riemannian manifolds with non-negative sectional curvature are rationally elliptic. In the quaternionic setting there are theorems (cf. theorem A on [6], p. 150, formula [3].14.42b, p. 406) that suggest that positive scalar curvature might be regarded as a substitute for positive sectional curvature to a certain extent.

In the case of Positive Quaternion Kähler Manifolds—mainly because rational cohomology is concentrated in even degrees only—we suggest to see formality as a very weak substitute for ellipticity. Indeed, if Positive Quaternion Kähler Manifolds were elliptic spaces—e.g. like simply-connected homogeneous spaces—then they would be $F_0$-spaces, which are formal. If one is willing to engage with this point of view, the formality of Positive Quaternion Kähler Manifolds may be seen as heading towards a quaternionic Bott conjecture.

**Structure of the article.** In section 1 we shall give a very brief introduction to Positive Quaternion Kähler Geometry whilst we do similar for Rational Homotopy Theory in section 2. Sections 3 and 4 feature the proofs of theorems C and D respectively. In section 5 the main results, theorems A and B will be derived. Finally, in section 6, we conclude with a depiction of several counter-examples—for possible statements similar to theorem C—in the case of odd-dimensional fibre spheres.

We remark that more elaborate introductions to Positive Quaternion Kähler Manifolds and Rational Homotopy Theory as well as detailed proofs can be found in [2].

**Note that throughout this article cohomology will be taken with rational coefficients whenever coefficients are suppressed; all commutative differential graded algebras are algebras over the rationals unless stated differently. All spaces and commutative differential graded algebras are supposed to be simply-connected (and path-connected) unless indicated differently.**

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1. Positive Quaternion Kähler Manifolds

Due to Berger’s celebrated theorem the holonomy group Hol\((M, g)\) of a simply-connected, irreducible and non-symmetric Riemannian manifold \((M, g)\) is one of \(\text{SO}(n), \text{U}(n), \text{SU}(n), \text{Sp}(n), \text{Sp}(n)\text{Sp}(1), \text{G}_2\) and \(\text{Spin}(7)\).

A connected oriented Riemannian manifold \((M^n, g)\) is called a Quaternion Kähler Manifold if

\[
\text{Hol}(M, g) \subseteq \text{Sp}(n)\text{Sp}(1) = \text{Sp}(n) \times \text{Sp}(1)/\langle -\text{id}, -1 \rangle
\]

(In the case \(n = 1\) one additionally requires \(M\) to be Einstein and self-dual.) Quaternion Kähler Manifolds are Einstein (cf. \([3]\).14.39, p. 403). In particular, their scalar curvature is constant.

**Definition 1.1.** A Positive Quaternion Kähler Manifold is a Quaternion Kähler Manifold with complete metric and with positive scalar curvature.

For an elaborate depiction of the subject we recommend the survey articles \([16]\) and \([17]\). We shall content ourselves with mentioning a few properties that will be of importance throughout the article:

Foremost, we note that Positive Quaternion Kähler Manifolds \(M\) clearly are not necessarily Kählerian, as the name might suggest. Moreover, the manifold \(M\) is compact and simply-connected (cf. \([16]\), p. 158 and \([16]\).6.6, p. 163).

The only known examples of Positive Quaternion Kähler Manifolds are given by the so-called Wolf-spaces, which are all symmetric and the only homogeneous examples due to Alekseevski. Indeed, they are given by the infinite series \(\mathbb{H}\text{P}^n\), \(\text{Gr}_2(\mathbb{C}^{n+2})\) and \(\text{Gr}_4(\mathbb{R}^{n+4})\) (the Grassmanian of oriented real 4-planes) and the exceptional spaces \(\text{G}_2/\text{SO}(4), \text{F}_4/\text{Sp}(3)\text{Sp}(1), \text{E}_6/\text{SU}(6)\text{Sp}(1), \text{E}_7/\text{Spin}(12)\text{Sp}(1), \text{E}_8/\text{E}_7\text{Sp}(1)\). Besides, it is known that in each dimension there are only finitely many Positive Quaternion Kähler Manifolds (cf. \([12]\).0.1, p. 110). This endorses the fundamental conjecture

**Conjecture 1.2** (LeBrun, Salamon). Every Positive Quaternion Kähler Manifold is a Wolf space.

A confirmation of the conjecture has been achieved in dimensions four (Hitchin) and eight (Poon–Salamon, LeBrun–Salamon). Thus the motivating question throughout this article will be: How close are Positive Quaternion Kähler Manifolds to symmetric spaces? We shall give a partial answer from the viewpoint of Rational Homotopy Theory.

It is an astounding fact that the theory of Positive Quaternion Kähler Manifolds may completely be transcribed to an equivalent theory in complex geometry. This is done via the twistor space \(Z\) of the Positive Quaternion Kähler Manifold \(M\). This Fano contact Einstein manifold may be constructed as follows:

Locally the structure bundle with fibre \(\text{Sp}(n)\text{Sp}(1)\) may be lifted to its double covering with fibre \(\text{Sp}(n) \times \text{Sp}(1)\). So locally one may use the standard representation of \(\text{Sp}(1)\) on \(\mathbb{C}^2\) to associate a vector bundle \(H\). In general, \(H\) does not exist globally but its complex projectivisation \(Z = \mathbb{P}_\mathbb{C}(H)\)
does. In particular, we obtain the twistor fibration
\[ \mathbb{C}P^1 \hookrightarrow \mathbb{P}_{\mathbb{C}}(H) \to M \]

Alternatively, the manifold \( Z \) may be considered as the unit sphere bundle \( S(E') \) associated to the 3-dimensional subbundle \( E' \) of the vector bundle \( \text{End}(TM) \) generated locally by the almost complex structures \( I, J, K \) which behave like the corresponding unit quaternions \( i, j \) and \( k \). That is, the twistor fibration is just
\[ S^2 \hookrightarrow S(E') \to M \]
(Comparing this bundle to its version above we need to remark that clearly \( \mathbb{C}P^1 \cong S^2 \).)

As an example one may observe that on \( \mathbb{H}P^n \) we have a global lift of \( \text{Sp}(n)\text{Sp}(1) \) and that the vector bundle associated to the standard representation of \( \text{Sp}(1) \) on \( \mathbb{C}^2 \) is just the tautological bundle. Now complex projectivisation of this bundle yields the complex projective space \( \mathbb{C}P^{2n+1} \) and the twistor fibration is just the canonical projection.

More generally, on Wolf spaces one obtains the following: The Wolf space may be written as \( G/K_{\text{Sp}(1)} \) (cf. the table on [3], p. 409) and its corresponding twistor space is given as \( G/K_{\text{U}(1)} \) with the twistor fibration being the canonical projection.

Using twistor theory a variety of remarkable results have been obtained. Let us mention the ones we shall make use of in the following:

**Theorem 1.3 (Strong rigidity).** Let \((M, g)\) be a Positive Quaternion Kähler Manifold. Then we have
\[
\pi_2(M) = \begin{cases} 
0 & \text{iff } M \cong \mathbb{H}P^n \\
\mathbb{Z} & \text{iff } M \cong \text{Gr}_2(\mathbb{C}^{n+2}) \\
\text{finite with } \mathbb{Z}_2\text{-torsion contained in } \pi_2(M) & \text{otherwise}
\end{cases}
\]

**Proof.** See theorems [12].0.2, p. 110, and [17].5.5, p. 103. \( \square \)

**Theorem 1.4.** Odd-degree Betti numbers vanish, i.e. \( b_{2i+1} = 0 \) for \( i \geq 0 \).

**Proof.** See theorem [16].6.6, p. 163, where it is shown that the Hodge decomposition of the twistor space is concentrated in terms \( H^{p,p}(Z, \mathbb{R}) \). \( \square \)

2. **RATIONAL HOMOTOPY THEORY**

As a main reference we want to recommend the book [7]. Throughout this article we shall follow its conceptual approach and its notation unless indicated differently. We shall not attempt to provide an introduction to the theory and it is recommended to the interested reader to familiarise herself/himself with the notions of commutative differential graded algebras, polynomial differential forms \( A_{\text{PL}} \), Sullivan (minimal) models and further essential concepts.

Let us now collect some concepts that are of essential importance for us and/or not covered by [7].
Definition 2.1. A fibration
\[ F \hookrightarrow E \xrightarrow{p} B \]
is called a spherical fibration if the fibre \( F \) is path-connected and has the rational homotopy type of a connected sphere, i.e. \( F \cong \mathbb{Q} \mathbb{S}^n \) for \( n \geq 1 \).

Remark 2.2. Let \((\bigwedge V, d)\) be a minimal model for \( B \). Following example [7],15.4, p. 202 one sees that for a spherical fibration with \( n \) even one obtains the following quasi-isomorphism of cochain algebras:
\[ (\bigwedge V \otimes \bigwedge \langle \tilde{e}, e' \rangle, d) \xrightarrow{\cong} (A_{PL}(E), d) \]
with \( d\tilde{e} = 0, \, de' = \tilde{e}^2 - u, \, u \in \bigwedge V, \, du = 0 \).

If \( n \) is odd there is the quasi-isomorphism
\[ (\bigwedge V \otimes \bigwedge \langle e \rangle, d) \xrightarrow{\cong} (A_{PL}(E), d) \]
with \( de =: u \in \bigwedge V \).

In each case we shall refer to this morphism as the model of the fibration.

Definition 2.3. The spherical fibration \( p \) is called primitive, if \( u \) is not decomposable, i.e. \( u \notin \bigwedge >0 V \cdot \bigwedge >0 V \). Otherwise, the fibration \( p \) will be called non-primitive.

The next proposition gives a way to identify Sullivan algebras.

Proposition 2.4. Let a cochain algebra \((\bigwedge V, d)\) satisfy the properties
\[ V = V^{\geq 2} \quad \text{and} \quad \text{im} d \subseteq \bigwedge >0 V \cdot \bigwedge >0 V \]
Then this algebra is necessarily a minimal Sullivan algebra.

Proof. See [7], example 12.5, p. 144.

A topological space \( X \) is called \( n \)-connected if for \( 0 \leq i \leq n \) it holds \( \pi_i(X) = 0 \). In this vein we shall call \( X \) rationally \( n \)-connected if for \( 0 \leq i \leq n \) it holds \( \pi_i(X) \otimes \mathbb{Q} = 0 \). Thus for the minimal model of a simply-connected space \( X \) with homology of finite type this is equivalent to \( V^i = 0 \) for \( 0 \leq i \leq n \).

Remark 2.5. Recall the Hurewicz homomorphism \( \text{hur} : \pi_*(X) \to H_*(X, \mathbb{Z}) \) (cf. [7], p. 58). We have \( H^{>0}(\bigwedge V, d) = \ker d/\text{im} d \), where \( \text{im} d \subseteq \bigwedge ^{\geq 2} V \) by minimality. Let \( \zeta : H^{>0}(\bigwedge V, d) \to V \) be the linear map defined by forming the quotient with \( \bigwedge ^{\geq 2} V \), i.e. the natural projection
\[ \zeta : \ker d/\text{im} d \to \ker d \big/ \left( \left( \bigwedge ^{\geq 2} V \right) \cap \ker d \right) \subseteq V \]

There is a commutative square (cf. [7], p. 173 and the corollary on [7], p. 210)
\[
\begin{array}{ccc}
H^{>0}(\bigwedge V, d) & \xrightarrow{\cong} & H^{>0}(X) \\
\zeta \downarrow & & \downarrow \text{hur}^* \\
V & \xrightarrow{\cong} & \text{Hom}(\pi^{>0}(X), \mathbb{Q})
\end{array}
\]
where $\text{hur}^*$ is the dual of the rationalised Hurewicz homomorphism. Thus by this square $\zeta$ and $\text{hur}^*$ are identified.

In particular, we see that if $\text{hur} \otimes \mathbb{Q}$ is injective, then $\text{hur}^*$ is surjective and so is $\zeta$. In such a case every element $x \in V$ is closed and defines a homology class.

Let us now describe the concept of formality.

**Definition 2.6.** The commutative differential graded algebra $(A, d)$ (over a field $\mathbb{K} \supseteq \mathbb{Q}$) is called *formal* if it is weakly equivalent to the cohomology algebra $(H(A, \mathbb{K}), 0)$ (with trivial differential).

We call a path-connected topological space *formal* if its rational homotopy type is a formal consequence of its rational cohomology algebra, i.e. if $(A_{\text{PL}}(X), d)$ is formal. In detail, the space $X$ is formal if there is a weak equivalence $(A_{\text{PL}}(X), d) \simeq (H^*(X), 0)$, i.e. a chain of quasi-isomorphisms

$$(A_{\text{PL}}(X), d) \simeq \ldots \simeq \ldots \simeq \ldots \simeq (H^*(X), 0)$$

**Theorem 2.7.** Let $X$ have rational homology of finite type. The algebra $(A_{\text{PL}}(X), \mathbb{K})$ is formal for any field extension $\mathbb{K} \supseteq \mathbb{Q}$ if and only if $X$ is a formal space.

**Proof.** See [7], p. 156 and theorem 12.1, p. 316. \hfill $\square$

Thus we need not worry about field extension and it suffices to consider rational coefficients only.

**Example 2.8.**
- $H$-spaces are formal (cf. [7], example 12.3, p. 143).
- Symmetric spaces of compact type are formal (cf. [7],.12.3, p. 162).
- $N$-symmetric spaces are formal (cf. [18], Main Theorem, p. 40, for the precise statement, [11]).
- Compact Kähler manifolds are formal (cf. [5], Main Theorem, p. 270). \hfill $\square$

The following theorem gives the characterisation of formality we shall use as a main tool.

**Theorem 2.9.** A minimal model $(\bigwedge V, d)$ is formal if and only if there is in each $V^i$ a complement $N^i$ to the subspace of $d$-closed elements $C^i$ with $V^i = C^i \oplus N^i$ and such that any closed form in the ideal $I(\bigoplus N^i)$ generated by $\bigoplus N^i$ in $\bigwedge V$ is exact.

**Proof.** See [5], theorem 4.1, p. 261. \hfill $\square$

We shall leave the proof of the next two standard characterisations to the reader.

**Proposition 2.10.**
- A minimal model $(\bigwedge V, d)$ of a commutative differential graded algebra $(A, d)$ is formal if and only if $(A, d)$ is formal.
- A minimal Sullivan algebra $(\bigwedge V, d)$ is formal if and only if there is a quasi-isomorphism

$$\mu : (\bigwedge V, d) \simeq (H(\bigwedge V, d), 0)$$

- The quasi-isomorphism $\mu$ from the last point may be taken to be the identity in cohomology.
This quasi-isomorphism then has the property that
\[ \mu : x \mapsto [x] \]
for \( d \)-closed elements \( x \).

The next lemma is similar to theorem [5].4.1, p. 261. It can be considered as a refinement of proposition 2.10.

**Lemma 2.11.** Suppose \((\wedge V, d) \xrightarrow{\sim} (A, d)\) is a minimal model. Let \((A, d)\) be a formal algebra. Let \( V = C \oplus N \) be the decomposition from theorem 2.9 with \( C = \ker d|_V \). Then there is a quasi-isomorphism
\[ \mu_A : (\wedge V, d) \xrightarrow{\sim} (H(\wedge V, d), 0) \]
with the property that \( \mu_A(x) = [x] \) if \([x]\) is closed and with \( \mu(x) = 0 \) for \( x \in N \).

Let us eventually define the notion of a formal map. We shall present the version of formality of a map which was proposed and used in [1]—cf. definition [1].2.7.20, p. 123—as it turns out to be most suitable for our purposes. (Contrast this version with the definition in [5].4, p. 260.)

Let \((A, d), (B, d)\) be commutative differential graded algebras with minimal models
\[ m_A : (\wedge V, d) \xrightarrow{\sim} (A, d) \]
\[ m_B : (\wedge W, d) \xrightarrow{\sim} (B, d) \]
A morphism \( f : (A, d) \to (B, d) \) induces the map \( f_* : H(A) \to H(B) \) and the Sullivan representative \( \hat{f} : (\wedge V, d) \to (\wedge W, d) \) uniquely defined up to homotopy (cf. [7].12, p. 154). Indeed, the morphism \( \hat{f} \) is defined by lifting the diagram
\[
\begin{array}{ccc}
(\wedge V, d) & \xrightarrow{f} & (\wedge W, d) \\
\cong & & \cong \\
(A, d) & \xrightarrow{f} & (B, d)
\end{array}
\]
Completing the square by \( \hat{f} \) makes it a diagram which is commutative up to homotopy (cf. [7], p. 149).

Suppose now that \( A \) and \( B \) are formal. By theorem 2.10 there are quasi-isomorphisms
\[ \phi_A : (\wedge V, d) \xrightarrow{\sim} (H^*(A), 0) \]
\[ \phi_B : (\wedge W, d) \xrightarrow{\sim} (H^*(B), 0) \]
By composing such a model with a suitable automorphism of the cohomology algebra we may suppose that \((m_A)_* = (\mu_A)_*\) and \((m_B)_* = (\mu_B)_*\).
Then the morphism $f$ is said to be formal, if there is a choice of such quasi-isomorphisms $\mu_A$ and $\mu_B$ (with $(m_A)_* = (\mu_A)_*$ and $(m_B)_* = (\mu_B)_*$) which makes the diagram

\[
\begin{array}{ccc}
(\bigwedge V, d) & \xrightarrow{f} & (\bigwedge W, d) \\
\mu_A & & \mu_B \\
(H(A), 0) & \xrightarrow{f^*} & (H(B), 0)
\end{array}
\]

commute up to homotopy.

Call a continuous map $f : M \to N$ between two formal topological spaces $M$, $N$ formal if the map $A_{PL}(f)$ induced on polynomial differential forms is formal.

The next property of formal spaces can be found in the literature with varying formulation and more or less detailed proofs (cf. proposition [4].5, p. 335 or lemma [8].2.11, p. 7 for example). We leave it to the reader to provide the necessary arguments.

**Theorem 2.12.** Let $(A, d)$ and $(B, d)$ be differential graded algebras. The product algebra $(A \otimes B, d)$ is formal if and only if so are both $A$ and $B$. In this case the canonical inclusion $(A, d) \hookrightarrow (A \otimes B, d)$ given by $a \mapsto (a, 1)$ is a formal map.

\[\square\]

**3. Proof of Theorem C**

In this section we shall see that under slight technical prerequisites it is possible to relate the formality of the base space to the formality of the total space in a fibration when the (rational) fibre sphere is (homologically) even-dimensional. So let us prove the main topological theorem.

**Proof of Theorem C.** Choose a minimal Sullivan model $(\bigwedge V_B, d_B) \cong A_{PL}(B)$. Recall (cf. 2.2) the following quasi-isomorphism of cochain algebras:

\[
(\bigwedge V_B \otimes \bigwedge \langle z, z' \rangle, d) \cong (A_{PL}(E), d)
\]

with $dz = 0$, $dz' = z^2 - u$, $u \in \bigwedge V_B$, $du = 0$ and $d|_{\bigwedge V_B} = d_B$. Recall from definition 2.3 that we call the fibration $p : E \to B$ primitive if $u$ is not decomposable.

We shall prove the theorem in the following manner:

**Case 1:** The fibration is primitive.

**Step 1:** We construct a minimal Sullivan algebra from (1).

**Step 2:** We show that it extends to a minimal Sullivan model of $E$ by proving the

1.: **Surjectivity** and

2.: **Injectivity** of the morphism induced in cohomology.

**Step 3:** We show the formality of the model for $B$ by means of the one for $E$.

**Case 2:** The fibration is non-primitive.

**Step 1:** We prove that (1) is a minimal Sullivan model of $E$. 
Step 2.: We show the formality of the model.

Let us commence the proof:

Case 1. We suppose the fibration to be primitive.

Step 1. If the fibration satisfies \( dz' = z^2 \), i.e. \( u = 0 \), then remark 2.2 yields that the fibration is rationally trivial. This just means that \( E \cong Q \) \( B \times F \) or, equivalently, that there is a weak equivalence \( A_{PL}(E) \cong A_{PL}(B) \otimes A_{PL}(F) \). Theorem 2.12 then applies and yields that the formality of \( E \) is equivalent to the formality of \( B \), since spheres are formal spaces.

Therefore it suffices to assume \( dz' \neq z^2 \), i.e. \( u \neq 0 \). Since the fibration is primitive, we may write \( u = u' + v \) with \( u' \neq 0 \) \( \in V_B \) and \( v \in \bigwedge^2 V_B \) both homogeneous with respect to degree.

The condition of primitivity contradicts the property of minimality for Sullivan algebras. Thus we shall construct a minimal Sullivan algebra \( \bigwedge V_E, d_E \) out of \( \bigwedge V_B \otimes \langle z, z' \rangle, d \).

This will be done as follows: Let \( u', \{ b_i \}_{i \in J} \) denote a (possibly infinite yet countable) basis of \( V_B \) homogeneous with respect to degree. Set \( V_E := \langle z, \{ b_i \}_{i \in J} \rangle \) with the previous grading. Let

\[
\phi : \bigwedge V_B \otimes \langle z, z' \rangle \to \bigwedge V_E
\]

be the (surjective) morphism of commutative graded algebras defined by \( \phi(b_i) := b_i \) for \( i \in J \), \( \phi(z) := z \), \( \phi(z') := 0 \), \( \phi(u') := z^2 - \phi(v) \) and extended linearly and multiplicatively. (This is well-defined, since there is a subset \( J' \subseteq J \) such that \( v \in \bigwedge \langle \{ b_i \}_{i \in J'} \rangle \) by degree. Thus when we define \( \phi(u') \), the element \( \phi(v) \) is already defined.)

We now define a differential \( d_E \) on \( \bigwedge V_E \) by \( d_E z := 0 \), by \( d_E b_i := \phi(d(b_i)) \) for \( i \in J \) and by making it a derivation. We shall see that this really defines a differential, i.e. that \( d_E^2 = 0 \): This will be a direct consequence of the property

\[
d_E \circ \phi = \phi \circ d
\]

which is immediate from the definition. Now for \( x \in \bigwedge V_E \) there is \( y \in V_B \otimes \langle z, z' \rangle \) with \( \phi(y) = x \) by the surjectivity of \( \phi \). Applying (3) twice yields

\[
d_E^2(x) = d_E(\phi(d(y))) = \phi(d^2(y)) = 0
\]

which proves that \( \bigwedge V_E, d_E \) is a commutative differential graded algebra, even a cochain algebra.

One easily sees that it is even a minimal Sullivan algebra. The minimality follows from the minimality of \( \bigwedge V_B, d_B \) (which is a differential subalgebra of \( \bigwedge V_B \otimes \langle z, z' \rangle, d \)), property (3), the multiplicativity of \( \phi \) and the fact that \( d_E(z) = 0 \).

As \( B \) is simply-connected, it holds that \( V_B^1 = 0 \). Thus proposition 2.4 implies that \( \bigwedge V_E, d_E \) is a minimal Sullivan algebra.
Step 2. We shall now exhibit $\phi$ as a quasi-isomorphism. This will make $(\bigwedge V_E, d_E)$ a minimal Sullivan model of $A_{PL}(E)$ due to step 1, the quasi-isomorphism (1) and the weak equivalence

$$(\bigwedge V_E, d_E) \xrightarrow{\phi} (\bigwedge V_B \otimes \bigwedge (z, z'), d) \xrightarrow{\sim} (A_{PL}(E), d)$$

In fact, basically using the lifting lemma, i.e. proposition [7].12.9, p. 153, we then obtain a quasi-isomorphism

$$(\bigwedge V_E, d_E) \xrightarrow{\phi} (A_{PL}(E), d)$$

We shall prove that $\phi$ is a quasi-isomorphism by separately proving the surjectivity and the injectivity of the induced map $\phi_*$ in homology.

**Surjectivity.** Let $x \in (\bigwedge V_E, d_E)$ be an arbitrary closed element. Thus it defines a homology class. We shall construct a $d$-closed preimage of $x$; this will prove surjectivity.

Define a map of commutative graded algebras

$$\psi : \bigwedge V_E \rightarrow \bigwedge V_B \otimes \bigwedge (z, z')$$

by $b_i \mapsto b_i$ for $i \in J$, by $z \mapsto z$ and by extending it linearly and multiplicatively. Then $\psi$ is injective, since

(4) \quad $\phi \circ \psi = \text{id}$

The main idea to prove surjectivity will be the following: We have $\phi(\psi(x)) = x$. Now vary $\psi(x)$ additively by a form $\tilde{x} \in \ker \phi$ such that $d(\psi(x) + \tilde{x}) = 0$.

Let $I \in (\mathbb{Q} \times \mathbb{N}^N)^N$ be a finite tuple $I = (a_1, \ldots, a_s)$ of finite tuples $a_i = (k_i, a_{i,1}, \ldots, a_{i,t_i})$. We shall use the multi-multiindex notation

$$b_I := \sum_{i=1}^s k_i \prod_{j=1}^{t_i} b_{a_{i,j}}$$

in order to denote arbitrary linear combinations of products in the $b_i$. We construct a set $\tilde{I}$ from $I$ in such a way that the summands of $b_I$ change sign according to their degree: Let $\delta(i) := \deg \prod_{j=1}^{t_i} b_{a_{i,j}}$. With the coefficients of $b_I$ we obtain

$$b_I = \sum_{i=1}^s (-1)^{\delta(i)} k_i \prod_{j=1}^{t_i} b_{a_{i,j}}$$

With this terminology we come back to our problem. There is a finite family $(I_{(i,j)})_{i,j}$ of such tuples with the property that

$$d(\psi(x)) = (b_{I_{(0,0)}}) + b_{I_{(1,0)}} u + b_{I_{(2,0)}} u^2 + \ldots$$

$$+ (b_{I_{(0,1)}}) + b_{I_{(1,1)}} u + b_{I_{(2,1)}} u^2 + \ldots)z$$

$$+ (b_{I_{(0,2)}}) + b_{I_{(1,2)}} u + b_{I_{(2,2)}} u^2 + \ldots)z^2$$

$$+ \ldots$$
Note that every coefficient $b_{I(i,j)}$ is uniquely determined. Set

$$\tilde{x}_{i,j} := b_{I(i,j)} \cdot z' \cdot z \cdot \sum_{l+2m=2i-2} z'^{l} u^{m}$$

and

$$\tilde{x} := \sum_{i>0,j} \tilde{x}_{i,j} \in (\bigwedge V_{B} \otimes \bigwedge \langle z, z' \rangle, d) \quad (5)$$

We shall prove in the following that $\tilde{x}$ is indeed the element we are looking for, i.e. $\psi(x) + \tilde{x}$ is $d$-closed and maps to $x$ under $\phi$. This last property actually is self-evident, since

$$\tilde{x} \in z' \bigwedge \langle \{b_{i}\}_{i \in J}, z, u' \rangle$$

and since $\phi(z') = 0$ by definition. Thus $\phi(\psi(x) + \tilde{x}) = \phi(\psi(x)) = x$.

Let us now see that $\psi(x) + \tilde{x}$ is $d$-closed: For this we shall prove first that the term $d(\psi(x) + \tilde{x})$ does not possess any summand that has $u'z^j$ as a factor for any $i, j \in \mathbb{N} \times \mathbb{N}_{0}$. (Actually, such a summand is replaced by one in $z^{2i+j}$ and by one more in $z'$.) This is due to the computation

$$u^{i}z^{j} + (z^{2} - u)z^{j} \sum_{l+2m=2i-2} z'^{l} u^{m}$$

$$= u^{i}z^{j} + z^{j}(z^{2} - u)(u^{i-1} + z^{2}u^{i-2} + z^{4}u^{i-3} + \ldots + z^{2(i-1)})$$

$$= z^{2i+j} \quad (6)$$
for \(i > 0\) and finally—the sums run over \((i, j) \in \mathbb{N}_0 \times \mathbb{N}_0\)—by

\[
\begin{align*}
d(\psi(x) + \tilde{x}) &= \sum_{i,j} b_{I(i,j)} u^i z^j + \\
&= \sum_{i,j} d(b_{I(i,j)} z' z^j \sum_{l+2m=2i-2} z'^l u^m) \\
&= \sum_{i,j} b_{I(i,j)} u^i z^j + \\
&= \sum_{i>0,j} b_{I(i,j)} u^i z^j + \\
&= \sum_{i>0,j} b_{I(i,j)} (z^2 - u) z^j \sum_{l+2m=2i-2} z'^l u^m + \\
&= \sum_{i,j} b_{I(i,j)} z' z^j \sum_{l+2m=2i-2} z'^l u^m \\
&= \sum_{i,j} b_{I(i,j)} z' z^j + \\
&= \sum_{i>0,j} (b_{I(i,j)} z^{2i+j} + d(b_{I(i,j)}) z' z^j \sum_{l+2m=2i-2} z'^l u^m) + \\
&= \sum_{i,j} b_{I(i,j)} z^{2i+j} + \\
&= \sum_{i>0,j} (d(b_{I(i,j)}) z' z^j \sum_{l+2m=2i-2} z'^l u^m)
\end{align*}
\]

Consequently, the term \(d(\psi(x) + \tilde{x})\) is in the “affine subalgebra”

\[
\bigwedge(\{b_i\}_{i \in J}, z) \oplus z' \bigwedge(\{b_i\}_{i \in J}, z, u')
\]

More precisely, we see that

\[
\begin{align*}
(d(\psi(x) + \tilde{x}))|_{\bigwedge(\{b_i\}_{i \in J}, z)} &= \sum_{i,j} b_{I(i,j)} z^{2i+j} \\
&= \phi(d(\psi(x))) \\
&= d_E(\phi(\psi(x))) \\
&= d_E(x) \\
&= 0
\end{align*}
\]
The last equation holds by assumption, the second one by the definition of \(d(\psi(x))\). Thus it suffices to prove that also the second direct summand

\[
(d(\psi(x) + \tilde{x}))|_{x' \wedge \langle \{b_i\}_{i \in J}, z, u' \rangle}
\]
equals zero. Using the vanishing of the first one we compute

\[
0 = d^2(\psi(x) + \tilde{x}) = d\left(d(\psi(x) + \tilde{x})|_{x' \wedge \langle \{b_i\}_{i \in J}, z, u' \rangle}\right)
\]
equals

\[
(z^2 - u) \bigwedge \langle \{b_i\}_{i \in J}, z, u' \rangle \oplus z' \bigwedge \langle \{b_i\}_{i \in J}, z, u' \rangle
\]

So let \(z'y := (d(\psi(x) + \tilde{x}))|_{x' \wedge \langle \{b_i\}_{i \in J}, z, u' \rangle}\). We have

\[
0 = d(z'y) = (z^2 - u)y - z'dy
\]

Since \(y, dy \in \bigwedge \langle \{b_i\}_{i \in J}, z, u' \rangle\), we have \((z^2 - u)y \neq z'dy\) unless both sides vanish. The term

\[
z^2 - u = z^2 - u' - v \in \bigwedge V_B \otimes \bigwedge \langle z, z' \rangle
\]
is not a zero-divisor, since \(u' \in V_B^{\text{even}}\) is not. This necessarily implies \(y = 0\). So we obtain

\[
d(\psi(x) + \tilde{x})|_{x' \wedge \langle \{b_i\}_{i \in J}, z, u' \rangle} = 0
\]

Combined with (8) this yields

\[
d(\psi(x) + \tilde{x}) = 0
\]

This finishes the proof of the surjectivity of \(\phi_x\).

**Injectivity.** We shall now prove that \(\phi_x\) is also injective, i.e. an isomorphism in total. For this we prove that its kernel is trivial. So let \(y \in \left(\bigwedge V_B \otimes \bigwedge \langle z, z' \rangle, d\right)\) be a closed element with \(\phi_x([y]) = 0\). Thus there is an element \(x \in \left(\bigwedge V_E, d_E\right)\) with \(d_E(x) = \phi(y)\). We shall construct an element \(\tilde{x} \in \left(\bigwedge V_B \otimes \bigwedge \langle z, z' \rangle, d\right)\) with the property that \(d(\psi(x) + \tilde{x}) = \tilde{y}\). Hereby \(\tilde{y}\) will be a newly constructed closed element with the property that \([y] = [\tilde{y}]\). This will show that \([\tilde{y}] = 0\) and will prove injectivity.

There are families \((I_{(i,j)})_{i,j}\) and \((I'_{(i,j)})_{i,j}\) as above—cf. the surjectivity part—such that the element \(y\) may be written in the form

\[
y = \sum_{i,j} b_{I_{(i,j)}} u^i z^j + z' \sum_{i,j} b_{I'_{(i,j)}} u^i z^j
\]

Again coefficients are uniquely determined. Once more we form elements

\[
\tilde{y}_{i,j} := b_{I_{(i,j)}} \cdot z' \cdot z^j \sum_{l+2m=2i-2} z^l u^m
\]

(for \(i, j \geq 0\)) and

\[
\tilde{y} := \sum_{i>0,j} d\tilde{y}_{i,j}
\]

A computation analogous to (7) yields

\[
y + \tilde{y} = \sum_{i,j} b_{I_{(i,j)}} z^{2i+j} + z' \left(\sum_{i,j} b_{I'_{(i,j)}} u^i z^j - d(b_{I_{(i,j)}}) z^j \sum_{l+2m=2i-2} z^l u^m\right)
\]
Since \( \tilde{y} \) is an exact form by construction, we see that this additive variation has no effect on the homology class of \( y \), i.e. \( y + \tilde{y} \) is d-closed with \( [y] = [y + \tilde{y}] \).

Consequently, by this reasoning we have found a representative \( y + \tilde{y} \) of the homology class of \( y \) with

\[
\tilde{y} := y + \tilde{y} \in \bigwedge (\{b_i\}_{i \in J}, z) \oplus z' \bigwedge (\{b_i\}_{i \in J}, z, u')
\]

Since \( \phi \) commutes with differentials (cf. (3)), we have

\[
d_E \left( x + \phi \left( \sum_{i > 0, j} \tilde{y}_{i,j} \right) \right) = d_E(x) + \phi(\tilde{y}) = \phi(\bar{y})
\]

Moreover, since for every \( i, j \in \mathbb{N}_0 \) the term \( \tilde{y}_{i,j} \) has \( z' \) as a factor (with \( \phi(z') = 0 \)), we see that

\[
\phi(\sum_{i > 0, j} \tilde{y}_{i,j}) = 0.
\]

So we obtain

\[
\phi(y) = d_E(x) = \phi(\bar{y})
\]

It is easy to see that a d-closed element

\[
a \in \bigwedge (\{b_i\}_{i \in J}, z) \oplus z' \bigwedge (\{b_i\}_{i \in J}, z, u')
\]

is completely determined by its first direct summand. Clearly, this is equivalent to the assertion that there is no non-trivial closed element of this kind with vanishing first summand. So we suppose additionally that \( a|_{\bigwedge (\{b_i\}_{i \in J}, z)} = 0 \). Comparing coefficients in \( d(a) = 0 \) yields the result.

So let us eventually prove that the cohomology class \( [y] = [y + \tilde{y}] \) equals zero by constructing an element that maps to \( y + \tilde{y} \) under \( d \). As in the surjectivity part we may deform \( \psi(x) \) additively by the special element

\[
\bar{x} \in z' \bigwedge (\{b_i\}_{i \in J}, z, u') \subseteq \ker \phi
\]

from (5). Since \( \phi \circ \psi = \text{id} \) by (4), once more we obtain

\[
\phi(\psi(x) + \bar{x}) = x
\]

and computation (7) shows that

\[
d(\psi(x) + \bar{x}) \in \bigwedge (\{b_i\}_{i \in J}, z) \oplus z' \bigwedge (\{b_i\}_{i \in J}, z, u')
\]

So we obtain a commutative diagram

\[
\begin{array}{ccc}
\bigwedge (\{b_i\}_{i \in J}, z, z', u')_{\psi(x) + \bar{x}} & \phi & \bigwedge (\{b_i\}_{i \in J}, z) \\
\downarrow \psi(x) + \bar{x} & \phi(\bar{y}) & \downarrow d_E \\
\bigwedge (\{b_i\}_{i \in J}, z) & \phi(\bar{y}) & \bigwedge (\{b_i\}_{i \in J}, z)
\end{array}
\]
where the commutativity in the upper left corner of the outer square has to be interpreted as
\[
\phi(\psi(x) + \tilde{x}) \in \bigwedge \langle \{b_i\}_{i \in J}, z \rangle
\]
and
\[
d(\psi(x) + \tilde{x}) \in \bigwedge \langle \{b_i\}_{i \in J}, z \rangle \oplus \langle z', \{b_i\}_{i \in J}, z, u' \rangle
\]
It remains to prove the correctness of the arrow tagged by question marks: We know that \( \bar{y} \mid_{\bigwedge \langle \{b_i\}_{i \in J}, z \rangle} = \psi(\phi(\bar{y})) \), since \( \phi \mid_{\bigwedge \langle \{b_i\}_{i \in J}, z \rangle} \) is the “identity” by definition. A standard computation thus shows
\[
d(\psi(x) + \tilde{x}) \mid_{\bigwedge \langle \{b_i\}_{i \in J}, z \rangle} = \bar{y} \mid_{\bigwedge \langle \{b_i\}_{i \in J}, z \rangle}
\]
The class \( \bar{y} \) is closed by construction, the class \( d(\psi(x) + \tilde{x}) \) is closed as it is exact. Since both classes coincide on the first direct summand, the uniqueness property we proved previously yields
\[
\bar{y} = d(\psi(x) + \tilde{x})
\]
So \( \bar{y} \) is exact. Hence we have that \( \bar{y} = [\bar{y}] = 0 \) and that \( \phi_* \) is injective.

**Step 3.** Let us now work with the minimal model \((\bigwedge V_B, d)\) in order to prove formality for \( B \). We shall make essential use of the characterisation given in theorem 2.9. So we shall split \( V_B \) as in the theorem, which will yield a similar decomposition for \( V_B \). This will enable us to derive the formality of the minimal model of \( B \). By theorem 2.10 this is equivalent to the formality of \( B \) itself.

Since \( E \) is formal, we may decompose \( V_B = C_E \oplus N_E \) with \( C_E = \ker d_E \mid_{V_B} \) and \( N_E \) being a complement of \( C_E \) in \( V_B \) as in theorem 2.9.

Without restriction we may assume that there is a subset \( J' \subseteq J \) such that
\[
C_E = \langle z, \{b_i\}_{i \in J'} \rangle
\]
where \( \{b_i\}_{i \in J}, u' \) is the homogeneous basis we chose for \( V_B \).

It is straightforward to prove that the morphism
\[
\phi : \left( \bigwedge V_B \otimes \bigwedge \langle z, z' \rangle, d \right) \to \left( \bigwedge V_E, d_E \right)
\]
restricts to an isomorphism of commutative differential graded algebras
\[
\phi \mid_{\bigwedge V_B} : \left( \bigwedge \langle \{b_i\}_{i \in J}, u' \rangle, d_B \right) \to \left( \bigwedge \langle z^2, \{b_i\}_{i \in J} \rangle, d_E \right)
\]
i.e. to see that it is well-defined, bijective and commuting with differentials. (For the last property it will suffice to show that \( \langle z^2, \{b_i\}_{i \in J} \rangle \) is respected by \( d_E \).)

We remark that clearly \( d \mid_{\bigwedge V_B} = d_B \) which makes the left hand side a well-defined differential subalgebra of the domain of \( \phi \). By abuse of notation—the element \( z^2 \) has word-length 2—we write \( \bigwedge \langle z^2, \{b_i\}_{i \in J} \rangle \) in order to denote the subalgebra generated by \( z^2, \{b_i\}_{i \in J} \) in \( \bigwedge \langle z, \{b_i\}_{i \in J} \rangle \).

We shall make use of this isomorphism by constructing a decomposition for \( V_B \). We start by setting
\[
C_B := \ker d_B \mid_{V_B}
\]
In the assertion we assume that the rationalised Hurewicz homomorphism 
\( \pi_*(B) \otimes \mathbb{Q} \to H_*(B) \) is injective in degree \( n \). This guarantees (cf. remark 2.5) that there are no relations in this degree, i.e. \( V_B^n = C_B^n \), which itself implies that \( N_E^n = 0 \) by the construction of \( (\bigwedge V_E, d_E) \). So every complement of \( C_E \) in \( V_E \) necessarily lies in \( \langle \{b_i\}_{i \in J} \rangle \). Hence we may assume without restriction that

\[
N_E = \langle \{b_i\}_{i \in J \setminus J'} \rangle
\]

(For this it might be necessary to change the basis \( \langle \{b_i\}_{i \in J} \rangle \).) Set

\[
N_B := \psi|_{\bigwedge (\langle \{b_i\}_{i \in J} \rangle)} N_E = \langle \{b_i\}_{i \in J \setminus J'} \rangle \subseteq V_B
\]

Let us now prove that \( V_B = C_B \oplus N_B \) and that every closed element in 
\( I(N_B) \subseteq \bigwedge V_B \) is exact: Due to (9), since \( \phi|_{\bigwedge V_B} \) commutes with differentials (cf. (3)) and by our results on \( \phi|_{\bigwedge V_B} \), we know that \( \langle \{b_i\}_{i \in J'} \rangle \subseteq \ker d_B \).

Moreover, the second (non-technical) assumption asserts that the rationalised Hurewicz homomorphism \( \pi_*(B) \otimes \mathbb{Q} \to H_*(B) \) is injective in degree \( 2n \). This again guarantees—cf. 2.5—that \( V_B^{2n} = C_B^{2n} \), which itself implies that \( u' \in C_B^{2n} \), as \( u' \in V_B^{2n} \).

Thus we obtain that \( \langle \{b_i\}_{i \in J'}, u' \rangle \subseteq C_B \), which leads to a direct sum decomposition

\[
V_B = C_B \oplus N_B
\]

with

\[
C_B = \langle \{b_i\}_{i \in J'}, u' \rangle \quad \text{and} \quad N_B = \langle \{b_i\}_{i \in J \setminus J'} \rangle
\]

Now assume there is a closed element \( y \in I(N_B) \). We shall construct an element \( \bar{x} \in (\bigwedge V_B, d_B) \) with \( d_B \bar{x} = y \). This will finally prove formality in case 1.

The element \( \phi|_{\bigwedge V_B} (y) \in (\bigwedge \langle z^2, \{b_i\}_{i \in J} \rangle, d_E) \) is also closed. Moreover, it is easily seen to lie in \( I(N_E) \).

Thus by the formality of \( E \) there is an element \( x \in (\bigwedge V_E, d_E) \) with \( d_E x = \phi|_{\bigwedge V_B} (y) \). As \( x \in (\bigwedge V_E, d_E) \), we may write

\[
x = x_0 + x_1 z + x_2 z^2 + x_3 z^3 + \ldots
\]
as a finite sum with \( x_i \in \bigwedge (\{b_i\}_{i \in J}) \). Since \( z \) is \( d_E \)-closed, we compute

\[
\langle z^2, \{b_i\}_{i \in J} \rangle \supseteq \phi|_{\bigwedge V_B} (y) = d_E(x) = d_E(x_0) + d_E(x_1) z + d_E(x_2) z^2 + \ldots
\]

As \( (\bigwedge \langle z^2, \{b_i\}_{i \in J} \rangle, d_E) \) is a differential subalgebra of \( (\bigwedge V_E, d_E) \), we see that a term \( d_E(x_i) z^i \) lies in \( z^i \cdot \bigwedge \langle z^2, \{b_i\}_{i \in J} \rangle \) again. In particular, the parity of the power of \( z \) is preserved by differentiation. Thus—by comparison of coefficients in \( z \)—we obtain

\[
\sum_{i \geq 0} d_E(x_{2i+1}) = 0
\]

In particular, we see that already

\[
d_E(x_0 + x_2 z^2 + x_4 z^4 + \ldots) = \phi|_{\bigwedge V_B} (y)
\]
Hence, without restriction, we may assume \( x \) to consist of monomials with even powers of \( z \) only, i.e. \( x_{2i+1} = 0 \) for \( i \geq 0 \). Thus it holds that \( x \in \langle z^2, \{b_i\}_{i \in J} \rangle \). As \( \phi|_{\wedge V_B} \) is bijective, there is the well-defined element \( \bar{x} = (\phi|_{\wedge V_B})^{-1}(x) \in \wedge V_B \) and we eventually obtain
\[
\text{d}_B(\bar{x}) = (\phi|_{\wedge V_B})^{-1}(\text{d}_E(x)) = (\phi|_{\wedge V_B})^{-1}(\phi|_{\wedge V_B}(y)) = y
\]

**Case 2.** We now shall prove the assertion under the assumption that the fibration is non-primitive. Again we shall rely on the quasi-isomorphism
\[
(\wedge V_B \otimes \wedge \langle z, z' \rangle, \text{d}) \xrightarrow{\sim} (\text{AP}_l(E), \text{d})
\]
where the left hand side contains the minimal Sullivan algebra \((\wedge V_B, \text{d}_B)\) as a differential subalgebra. Again this last algebra is supposed to be a minimal model for the space \( B \).

**Step 1.** We show that the algebra generated by \( V_E := V_B \oplus \langle z, z' \rangle \) is a minimal Sullivan algebra: We have
\[
\wedge V_B \otimes \wedge \langle z, z' \rangle = \wedge V_E
\]
Set \( \text{d}_E := \text{d} \). So \((\wedge V_E, \text{d}_E)\) is a commutative differential graded algebra.
Moreover, it holds that \( V_E^1 = 0 \), since \( B \) is simply-connected by assumption.

By the definition of (non-)primitivity \( u \in \wedge V_B \) is decomposable, i.e. \( u \in \wedge^{>0} V_B \cdot \wedge^{>0} V_B \). Thus we have that \( \text{d}_E z' = z^2 - u \in \wedge^{>0} V_E \cdot \wedge^{>0} V_E \).
Thus proposition 2.4 yields that \((\wedge V_E, \text{d}_E)\) is a minimal Sullivan model of \( E \).

**Step 2.** By assumption \( E \) is formal. Theorem 2.9 then yields the existence of a decomposition \( V_E = C_E \oplus N_E \) with \( C_E = \ker \text{d}_E|_{V_E} \). Clearly, we have \( z \in C_E \) and without restriction we may assume that
\[
C_E = \langle z, \{b_i\}_{i \in J'} \rangle
\]
for a subset \( J' \subseteq J \), where \( \{b_i\}_{i \in J} \) now is a homogeneous basis of \( V_B \). (Recall that non-primitivity implies that the element \( u \) lies in \( \wedge^{>2}(\{b_i\}_{i \in J}) \).

Since \( \text{d}_E z' \neq 0 \) and since \( z' \in V_E \), there is an element \( c \in C \) with the property that \( z' + c \in N_E \). Set \( z'' := z' + c \). Thus we have that \( z'' \in N_E \).
The change of basis caused by \( z' \mapsto z'' \) induces a linear automorphism of the vector space \( V_E \) which is the identity on \( V_B \oplus \langle z \rangle \). This automorphism itself induces an automorphism \( \sigma \) of the commutative differential graded algebra \( \wedge V_E \). By construction this automorphism has the property that \( \sigma|_{\wedge(V_B \oplus \langle z \rangle)} = \text{id} \). Moreover, we know that \( \text{d}_E(\wedge(V_B \oplus \langle z \rangle)) \subseteq \wedge(V_B \oplus \langle z \rangle) \).
Hence one easily checks that \( \sigma \) is also compatible with the differential. Thus up to this isomorphism of differential graded algebras we may assume that there is a decomposition of \( V_E = C_E \oplus N_E \) with \( C_E = \ker \text{d}_E|_{V_E} \) and with \( z' \in N_E \). Moreover, up to a change of basis we hence may assume that
\[
N_E = \langle \{b_i\}_{i \in J \setminus J'}, z' \rangle
\]

We use this to split \( V_B \) in the following way: Set \( C_B := \ker \text{d}_B|_{V_B} \). We recall that \((\wedge V_B, \text{d}_B)\) is a differential subalgebra of \((\wedge V_E, \text{d}_E)\). Thus we
realise that

\[ C_B = \langle \{b_i\}_{i \in J} \rangle \]

and we set

\[ N_B : = \langle \{b_i\}_{i \in J \setminus J'} \rangle \]

We directly obtain \( V_B = C_B \oplus N_B \).

As for the formality of \( B \), by theorem 2.9 it remains to prove that every closed element in the ideal \( I(N_B) \) generated by \( N_B \) in \( \bigwedge V_B, d_B \) is exact in \( \bigwedge V_B, d_B \). So let \( y \in I(N_B) \) with \( d_B y = 0 \). Since \( N_B \subseteq N_E \) we obtain that \( I(N_B) \subseteq I(N_E) \). Thus by the formality of \( E \) we find an element \( x \in \bigwedge V_E \) with \( d_E x = y \). Recall the notation we introduced in the surjectivity part of case 1, step 2 and write \( x \) as a finite sum

\[ x = b_{I_0} + b_{I_2}^2 z + b_{I_2} z^2 + \cdots + b_{I_0}^{i_0} z^{i_0} + b_{I_2}^{i_2} z^{i_2} + b_{I_2} z^{i_2} + \cdots \]

with \( b_{I_i}, b_{I_i}' \in \bigwedge V_B \). We compute

\[
y = d_E x = d_E b_{I_0} + (d_E b_{I_1}) z + (d_E b_{I_2}) z^2 + \cdots + ((d_E b_{I_1}') z' + b_{I_0}' (z^2 - u)) + ((d_E b_{I_1}')(z^2 - u)) + \cdots + \sum_{i \geq 0} ((d_E b_{I_i}) - b_{I_i} u + b_{I_i-2} u) z^i + \sum_{i \geq 0} (d_E b_{I_i}) z^i \]

(10)

Since \( d_E(b_i) \in \bigwedge V_B \) for \( 1 \leq i \leq m \) and since \( u \in \bigwedge V_B \), we shall proceed with our reasoning by comparing coefficients in \( z^i \) respectively in \( z^i z' \). That is, the element \( y \) lies in \( \bigwedge V_B \). Thus it cannot contain a summand with \( z^i \) (for \( i > 0 \)) or with \( z^i z' \) (for \( i \geq 0 \)) as a factor. So we see that

\[ y = d_E x = d_E b_{I_0} - b_{I_0} u \]

In particular, \( (d_E b_{I_2}) - b_{I_2} u + b_{I_0} u = 0 \). Thus the element \( b_{I_0} + b_{I_2} u \in \bigwedge V_B \) satisfies

\[ d_E(b_{I_0} + b_{I_2} u) = d_E b_{I_0} + (d_E b_{I_2}) u = d_E b_{I_0} + b_{I_2} u^2 - b_{I_0} u = y + b_{I_2} u^2 \]

Below we prove that \( b_{I_2} \) is exact, i.e. there is some \( a \in \bigwedge V_B \) with \( d_E a = b_{I_2} \). This yields the exactness of \( b_{I_2} u^2 \) by \( d_E(a u^2) = b_{I_2} u^2 \). So we shall finally obtain

\[ d_E(b_{I_0} + b_{I_2} u - a u^2) = (y + b_{I_2} u^2) - b_{I_2} u^2 = y \]

Since this is a computation in \( \bigwedge V_B \) and since \( d_E \) restricts to \( d_B \) on \( \bigwedge V_B \) we obtain that \( y \) is exact in \( \bigwedge V_B \). This will provide formality.

So let us eventually prove that \( b_{I_2} \) is exact. Since we are dealing with finite sums only, there is an even \( i_0 \geq 2 \) with \( b_{I_0} = 0 \). From (10) we again see that

\[ 0 = (d_E b_{I_0}) - b_{I_0} u + b_{I_0-2} = (d_E b_{I_0}) + b_{I_0-2} \]
This implies $d_B(-b_{i_0}) = b_{j'}$ and $b_{j'}$ is exact in $(\wedge V_B, d_B)$. We shall continue this iteratively. That is, as a next step we have

$$0 = (d_Eb_{i_0-2}) - b_{j'} - (d_Eb_{i_0-2}) - d_B(-b_{i_0}) + b_{j'_{i_0-4}}$$

and $b_{j'_{i_0-4}}$ is $d_B$-exact. As $i_0$ is even, this iterative procedure will finally end up with proving that $b_{j'}$ is exact.

This finishes the proof of the formality of $B$. \(\square\)

We already did so in the proof and we shall continue to do so: We refer to the two conditions in the assertion of the theorem that involve the Hurewicz homomorphism as “non-technical” conditions. (The other prerequisites are mainly due to the general setting of the theory.)

### 4. Proof of Theorem D

It is of importance to remark that $S^{2n} (n \geq 1)$ is an $F_0$-space. Thus the question whether the formality of the base space implies formality for the total space is related to the Halperin conjecture, which has been established on even-dimensional spheres.

More precisely, a consequence of theorem [14].3.4 is that the formality of the base space of a spherical fibration with (rationally) even-dimensional fibre implies the formality of the total space. A proof for is a direct consequence of the next lemma, which we shall actually need for the proof of theorem D. In order to formulate it we need the following considerations.

We use the notation and conventions as above and again we require $E$ and $B$ to be simply-connected. Choose a minimal model $m_B : (\wedge V_B, d_B) \xrightarrow{\sim} (\Apl(B), d)$.

As $B$ is formal, we have the quasi-isomorphism

$$\mu_B : (\wedge V_B, d_B) \xrightarrow{\sim} (H^*(B), 0)$$

from lemma 2.11. For this we fix a complement $N$ of $\ker d_B|V_B$ in $V_B$. We obtain that $\mu_B(N)$=0. Form the model of the fibration $p$

$$\tilde{m}_E : (\wedge V_B \otimes \wedge \langle z, z' \rangle, d) \xrightarrow{\sim} (\Apl(E), d)$$

as in remark 2.2. We use the terminology of the proof of theorem C for the model of the fibration, i.e. $d(z') = u \in \wedge V_B$ and $\{b_i\}_{i \in J}$ denotes a homogeneous basis of $V_B$. (Here we do not make use of a distinguished basis element $u'$.) We now define a morphism of commutative graded algebras

$$\tilde{\mu}_E : (\wedge V_B \otimes \wedge \langle z, z' \rangle, d) \to (H^*(E), 0)$$

$$b_i \mapsto \mu_B(b_i)$$

$$z \mapsto [z]$$

$$z' \mapsto 0$$

by extending this assignment linearly and multiplicatively. Hence it is a morphism of commutative graded algebras.

**Lemma 4.1.** The morphism $\tilde{\mu}_E$ is a quasi-isomorphism.
Proof. Our main tool for proving this will be that every closed element of $(\bigwedge V_B \otimes \bigwedge \langle z, z' \rangle, d)$ maps to its homology class under $\tilde{\mu}_E$. Let us establish this result first. So suppose $y \in (\bigwedge V_B \otimes \bigwedge \langle z, z' \rangle, d)$ is such a $d$-closed element. It then may be written as

$$y = y_0 + z'y_1$$

with $y_0, y_1 \in \bigwedge (V_B \oplus \langle z \rangle)$.

Step 1. First we shall prove that $y_1$ is exact. We have

$$0 = dy = dy_0 + (z^2 - u)y_1 - z'dy_1$$

By construction of $(\bigwedge V_B \otimes \bigwedge \langle z, z' \rangle, d)$ we know that $d(\bigwedge V_B \oplus \langle z \rangle) \subseteq (\bigwedge V_B \oplus \langle z \rangle)$. This implies that $dy_1 = 0$. Thus we have

$$dy_0 = -(z^2 - u)y_1 \quad (13)$$

Hence we write

$$y_0 = y_{0,0} + y_{0,1}z + y_{0,2}z^2 + y_{0,3}z^3 + \cdots + y_{0,k}z^k$$

and

$$y_1 = y_{1,0} + y_{1,1}z + y_{1,2}z^2 + y_{1,3}z^3 + \cdots + y_{1,k-2}z^{k-2}$$

as $z$-graded elements with $y_{0,i}, y_{1,i} \in \bigwedge V_B$. As $z$ is $d$-closed, the grading is preserved under differentiation. Equation (13) implies in this graded setting that

$$d_B y_{0,i} = uy_{1,i} - y_{1,i-2}$$

Starting in top degree $k$ we see that $d_B y_{0,k} = -y_{1,k-2}$. Set $x_{k-2} := -y_{0,k}$. This directly leads to

$$d_B y_{0,k-2} = uy_{1,k-2} - y_{1,k-4}$$

where $uy_{1,k-2}$ is $d_B$-exact by $d_B(-uy_{0,k}) = uy_{1,k-2}$. Thus $y_{1,k-4}$ is exact. Set

$$x_{k-4} := -y_{0,k-2} - uy_{0,k}$$

Hence, inductively continuing in this fashion, we obtain that each $y_{1,i}$ with $i \equiv k \mod 2$ is exact. The same line of argument applied from degree $k - 1$ downwards yields finally that also all the $y_{1,i}$ with $i \equiv k - 1 \mod 2$ are $d_B$-exact. In total, for each $0 \leq i \leq k - 2$ there exists an $x_i \in \bigwedge V_B$ with $dx_i = y_{1,i}$. So the element $y_1$ itself is exact: Set

$$x := x_0 + x_1 z + x_2 z^2 + \cdots + x_{k-2} z^{k-2}$$
and obtain
\[ d(x) = d(x_0 + x_1 z + x_2 z^2 + \cdots + x_{k-2} z^{k-2}) \]
\[ = d_B x_0 + d_B(x_1) z + d_B(x_2) z^2 + \cdots + d_B(x_{k-2}) z^{k-2} \]
\[ = y_{1,0} + y_{1,1} z + y_{1,2} z^2 + y_{1,3} z^3 + \cdots + y_{1,k-2} z^{k-2} \]
\[ = y_1 \]

Moreover, the \( x_i \) are constructed iteratively within the algebra \( \bigwedge V_B \). So we obtain \( x_i \in \bigwedge V_B \) for \( 0 \leq i \leq k - 2 \) and
\[ x \in \bigwedge (V_B \oplus \langle z \rangle) \quad (15) \]

**Step 2.** This will enable us to see that \( y \) is mapped to its homology class. Due to the formality of \( B \) we decomposed \( V_B = C \oplus N \) at the beginning of the section. This decomposition is homogeneous with the property that \( \ker d_B|_V \) and such that every closed element in \( I(N) \), the ideal generated by \( N \) in \( \bigwedge V_B \), is exact—cf. theorem 2.10. We clearly obtain
\[ \bigwedge V_B = \bigwedge (C \oplus I(N)) \]

For each \( 1 \leq i \leq k \) we split \( y_{0,i} =: y_{0,i}^0 + y_{0,i}^n \) with \( y_{0,i}^0 \in \bigwedge C \) (being d-closed) and \( y_{0,i}^n \in I(N) \). The morphism \( \mu_B \) was chosen to map a closed element to its homology class and to vanish on \( I(N) \). Thus we see that \( \tilde{\mu}_E(y_{0,i}^0) = [y_{0,i}^0] \) and that \( \tilde{\mu}_E(y_{0,i}^n) = 0 \). We then need to show that
\[ \tilde{\mu}_E(y) = \sum_{i \geq 0} [y_{i,i}^0] [z]^{1 \cdot \frac{1}{i}} = [y] \]

This will be a consequence of the following reasoning: We suppose \( y_{0,i} = y_{0,i}^n \) for all \( 0 \leq i \leq k \) and show that
\[ \tilde{\mu}_E(y) = [0] \equiv [y] \]

By construction we have
\[ y_0 \in N \cdot \bigwedge (V \oplus \langle z \rangle) \quad (16) \]

We shall now show that the element
\[ y + d(z' x) = y_0 + z' y_1 + (z^2 - u) x - z' y_1 = y_0 + (z^2 - u) x \]
is exact, which will imply the exactness of \( y \) itself.

The whole commutative algebra splits as a direct sum
\[ \bigwedge V_B \otimes \bigwedge (z, z') = \left( \bigwedge (C \oplus \langle z \rangle) \right) \oplus \left( (N \oplus \langle z' \rangle) \cdot \left( \bigwedge V_B \otimes \bigwedge (z, z') \right) \right) \]

Thus the element \( x \) may be written as \( x = c + n \) with \( c \in \bigwedge (C \oplus \langle z \rangle) \) and \( n \in (N \oplus \langle z' \rangle) \cdot (\bigwedge V_B \otimes \bigwedge (z, z')) \). We conclude that \( d(c) = 0 \) and that \( d(x - c) = dn = dx \). Hence, without restriction, we may assume \( x = n \). Combining this with (15) thus yields
\[ x \in N \cdot \bigwedge (V_B \oplus \langle z \rangle) \]
Thus, writing \( x \) in its \( z \)-graded form, we see that each coefficient \( x_i \) (for \( 0 \leq i \leq k - 2 \)) lies in \( I(N) = N \cdot \wedge V_B \). By (16) every coefficient \( y_{0,i} \) (for \( 0 \leq i \leq k \)) in the \( z \)-grading of \( y_0 \) equally lies in \( I(N) \). Hence so does every \( z \)-coefficient

\[
y_{0,i} - ux_i + x_{i-2} \in I(N)
\]

of

\[
y_0 + (z^2 - u)x = \sum_{i \geq 0} (y_{0,i} - ux_i + x_{i-2})z^i
\]

Since \( y \) was assumed to be \( d \)-closed and since \( y_0 + (z^2 - u)x = y + d(z'x) \), the element \( y_0 + (z^2 - i)x \) also is \( d \)-closed. In particular, every \( z \)-coefficient is a \( d_B \)-closed element in \( I(B) \). By the formality of \( B \) and by theorem 2.9 every such coefficient is \( d_B \)-exact then. An argument analogous to (14) then shows that \( y \) is \( d \)-exact. In total, this proves that \( \tilde{\mu}_E \) maps a closed element to its cohomology class.

As for the properties of \( \tilde{\mu}_E \), which now follow easily from this feature, we observe: The morphism is compatible with differentials as \( \tilde{\mu}_E (d(x)) = 0 \). (The closed and exact class \( d(x) \) maps to its cohomology class.) Eventually, it is a quasi-isomorphism as the morphism induced in homology is just the identity. \( \square \)

As we remarked, the formality of \( B \) implies the one of \( E \) and we can give the

**Proof of theorem **D**. We shall use the terminology and the results from theorem C. **By**

\[
m_B : (\bigwedge V_B, d_B) \xrightarrow{\sim} A_{PL}(B, d)
\]

we denote the minimal model. We let \( \{b_i\}_{i \in J} \) denote a basis of \( V_B \) (and do not make use of a distinguished element \( u' \)).

**Case 1.** Suppose the fibration \( p \) to be primitive. In the notation of the proof of theorem C we then have \( dz' = z^2 - u \) with primitive \( u \) in the model of the fibration. In the case \( u = 0 \) we obtained the weak equivalence

\[
E \simeq_{Q} B \times F
\]

Theorem 2.12 applies and makes it possible to derive the formality of \( p \) in this case.

So we may assume \( u \neq 0 \). We then have constructed the minimal model

\[
m_E : (\bigwedge V_E, d_E) \xrightarrow{\sim} A_{PL}(E, d)
\]

in case 1, step 2 of the proof of theorem C. By formality we have the quasi-isomorphism

\[
\mu_B : (\bigwedge V_B, d_B) \xrightarrow{\sim} (H^*(B), 0)
\]
which we now may assume to satisfy \((m_B)_* = (\mu_B)_*\) (by composing it with an automorphism of \(H^*(B)\)). Once more, for a spherical fibration we form the model of the fibration

\[
\tilde{m}_E : (\bigwedge V_B \otimes \bigwedge \langle z, z' \rangle, d) \xrightarrow{\cong} (A_{PL}(E), d)
\]

as in remark 2.2. Consider the following diagram

\[
\begin{array}{ccc}
\bigwedge V_B, d_B & \xrightarrow{m_B} & (A_{PL}(B), d) \\
\bigwedge V_B \otimes \bigwedge \langle z, z' \rangle, d & \xrightarrow{\phi} & (A_{PL}(E), d)
\end{array}
\]

where \(\phi\) is the quasi-isomorphism from (2), which we construct in the proof of theorem C, case 1, step 2. (Note that until case 1, step 3 of the theorem we did not need any of the “non-technical” assumptions which we did not require for this theorem.)

We shall now prove that diagram (17) commutes up to homotopy. Hereby the minimal model \(m_E\) was constructed by means of a diagram of liftings and isomorphisms extending

\[
\begin{array}{ccc}
(A_{PL}(E), d) & \xrightarrow{\tilde{m}_E} & \bigwedge V_B \otimes \bigwedge \langle z, z' \rangle, d \\
(A_{PL}(B), d) & \xrightarrow{\hat{p}} & (A_{PL}(E), d)
\end{array}
\]

Thus this diagram is commutative up to homotopy (cf. [7], 12.9, p. 153) and so is the right hand triangle of (17).

Since the model of the fibration comes out of a relative Sullivan algebra (cf. [7], 14 and [7], 15, p. 196) for \(A_{PL}(p)\)—composed with the minimal model \((\bigwedge V_B, d_B) \xrightarrow{m_B} A_{PL}(B, d)\)—the algebra \((\bigwedge V_B, d_B)\) includes into \((\bigwedge V_B \otimes \bigwedge \langle z, z' \rangle, d)\) by \(x \mapsto (x, 1)\) as a differential subalgebra (cf. definition [7], p. 181) and the square in the diagram commutes. (This holds although we use a slightly altered model as established in remark 2.2.)

As a consequence, both the following diagrams commute up to homotopy—the second one as the Sullivan representative \(\hat{p}\) is constructed as a lifting.

\[
\begin{array}{ccc}
\bigwedge V_B, d_B & \xrightarrow{m_B} & (A_{PL}(B), d) \\
\bigwedge V_E, d_E & \xrightarrow{m_E} & (A_{PL}(E), d)
\end{array}
\]

Since the Sullivan representative \(\hat{p}\) is uniquely defined up to homotopy, we know that \(\hat{p}\) and \(\phi|_{\bigwedge V_B}\) are homotopic morphisms (cf. [7], p. 149). Thus—in order to prove the assertion in this case—it is sufficient to find a minimal
model $\mu_E$ with $(\mu_E)_* = (m_E)_*$ for which the diagram

$$
\begin{array}{ccc}
(\bigwedge V_B, d_B) & \xrightarrow{\phi|_{\bigwedge V_B}} & (\bigwedge V_E, d_E) \\
\mu_B & & \mu_E \\
(H^*(B), 0) & \xrightarrow{\mu_B^*} & (H^*(E), 0)
\end{array}
$$

commutes up to homotopy.

We shall start to do so by foremost recalling the quasi-isomorphism

$$
\tilde{\mu}_E : (\bigwedge V_B \otimes \bigwedge \langle z, z' \rangle, d) \xrightarrow{\approx} (H^*(E), 0)
$$

from (12) and lemma 4.1. In the following we shall identify $H^*(E)$ with $p^* H^*(B) \oplus p^* H^*(B)[z]$—which clearly is not intended to denote the splitting as an algebra. So we obtain $(\tilde{m}_E)_* = (p^* \circ (m_B)_*, p^* \circ (m_B)_*)$ and $(m_B)_* = (\mu_B)_*$ implies that $(\tilde{m}_E)_* = p^* \circ (\tilde{\mu}_E)_*$.

Now form the diagram

$$
\begin{array}{ccc}
(\bigwedge V_B, d_B) & \xrightarrow{\phi|_{\bigwedge V_B}} & (\bigwedge V_E, d_E) \\
\mu_B & & \mu_E \\
(H^*(B), 0) & \xrightarrow{\mu_B^*} & (H^*(E), 0)
\end{array}
$$

where the map $\mu_E$ is defined by means of liftings and isomorphisms that make the right hand triangle in (18) homotopy commute. As $\tilde{\mu}_E$ and $\phi$ are quasi-isomorphisms, so becomes $\mu_E$. Again the square in the diagram commutes. So the whole diagram commutes up to homotopy. Since $p^* \circ (\tilde{\mu}_E)_* = (\tilde{m}_E)_*$, we derive that $(\mu_E)_* = (m_E)_*$ by commutativity. This combines with the homotopy commutativity of diagram (18) and yields that $p$ is a formal map.

**Case 2.** We treat the case when $p$ is not primitive. In case 2, step 1 of the proof of theorem C we have seen that the model of the fibration

$$
m_E : (\bigwedge V_E, d_E) \xrightarrow{\approx} (A_{PL}(E), d)
$$

where $V_E = V_B \oplus \langle z, z' \rangle$ and $d_E|_{V_B} = d_B$, $d_E z = 0$, $d_E z' = z^2 - u$ is a minimal model of $E$ already. As this model comes out of a relative Sullivan algebra over $(\bigwedge V_B, d_B)$, the induced map $\hat{p}$ is just the canonical inclusion as a differential graded subalgebra. As in case 1 we use the quasi-isomorphism $\tilde{\mu}_E$ from (12) with the property that $p^* \circ (\tilde{\mu}_E)_* = (m_E)_*$ as seen in case 1.
Thus it is obvious that the diagram

\[
\begin{array}{ccc}
\bigwedge V_B, d_B & \xrightarrow{\hat{\mu}} & \bigwedge V_E, d_E \\
\mu_B & \downarrow & p^* \circ \hat{\mu}_E \\
(H^*(B), 0) & \xrightarrow{p^*} & (H^*(E), 0)
\end{array}
\]

commutes. So \( p \) is formal in this case, too. \( \square \)

5. Consequences and the proofs of Theorems A and B

We shall now state some results related to theorems C and D. This will lead us to the proof of theorems A and B. We begin with the following apparent consequence:

**Corollary 5.1.** Let

\[ F \hookrightarrow E \xrightarrow{p} B \]

be a spherical fibration of simply-connected (and path-connected) topological spaces with rational homology of finite type. Let the fibre satisfy \( F \cong \mathbb{Q} S^n \) for an even \( n \geq 2 \). Suppose further that the space \( B \) is rationally \( (2n - 1) \)-connected.

Then we derive:

- The space \( E \) is formal if and only if so is \( B \).
- If this is the case, the map \( p \) is formal, too.

**Proof.** Since \( B \) is rationally \( (2n - 1) \)-connected, we have

\[ \pi_i(B) \otimes \mathbb{Q} = 0 = H^i(B) \quad \text{for} \quad i \leq 2n - 1 \]

So we obtain in particular that the rationalised Hurewicz homomorphism \( \pi_*(B) \otimes \mathbb{Q} \to H_*(B) \) is injective in degree \( n \). By the Hurewicz theorem we also obtain that the rationalised Hurewicz homomorphism \( \pi_*(B) \otimes \mathbb{Q} \to H_*(B) \) is an isomorphism in degree \( 2n \) —and injective in this degree, in particular. Thus theorems C and D yield the result. (Recall the remark concerning the fact that the formality of \( B \) implies the one of \( E \).) \( \square \)

We remark that this corollary only needs case 1 of the proof of theorem C, since by the connectivity-assumption the fibration necessarily is primitive.

Let us formulate another observation.

**Theorem 5.2.** Let

\[ F \hookrightarrow E \xrightarrow{p} B \]

be a fibration of simply-connected (and path-connected) topological spaces with rational homology of finite type. Suppose the rational cohomology algebra \( H^*(F) \) is generated (as an algebra) by exactly one (non-trivial) element of even degree \( n \). Suppose further that the space \( B \) is rationally \( n' \)-connected, where

\[ n' := n \cdot (\dim \mathbb{Q} H^*(F)) - 1 \]

Then we obtain: The space \( E \) is formal if and only if so is \( B \). In this case the map \( p \) is formal, too.
Proof. Set \( d := \dim Q H^*(F) \). The cohomology algebra of the fibre \( F \) is given by the polynomial algebra \( H^*(F) = \mathbb{Q}[z]/z^d \). The minimal model \( (\bigwedge V_F, d_F) \) of \( F \) is given by \( V_F = V^n \oplus V^{n-d-1} \) with \( V^n = \langle z \rangle \) and \( V^{n-d-1} = \langle z' \rangle \) and by the differentials \( d_F z = 0 \) and \( d_F z' = z^d \).

Let \( (\bigwedge V_B, d_B) \) be a minimal model of the base space \( B \). We form the model of the fibration:

\[
(\bigwedge V_B \otimes \bigwedge V_F, d) \cong (\mathbb{A} \text{PL}(E), d)
\]

(cf. the corollary on [7], p. 199). The differential \( d \) of the model equals \( d_B \) on the subalgebra \( \bigwedge V_B \). It is now easy to see that \( dz = 0 \) and that \( dz' = z^d - u \) for some closed \( u \in V_B^{d,n} \).

The result now follows by imitating the proofs of theorems C and D in a completely analogous fashion. (The fibre can easily be seen to satisfy the Halperin conjecture, so the reverse implication to C holds again.) Just replace \( z^2 \) by \( z^d \) and adapt the calculations. Some iterative constructions will need to be done not only on even and odd indices separately, but for each class modulo \( d \). In theorem C clearly only case 1 has to be considered. In an analogous fashion to the proof of corollary 5.1 the connectedness assumption can be seen to suffice to derive all the “non-technical” conditions in theorem C.

Clearly, this theorem directly applies to fibrations with fibre \( F \) having the rational homotopy type of either \( \mathbb{CP}^n \) or \( \mathbb{HP}^n \)—always supposed that all the other assumptions (including the connectivity-assumption) are fulfilled.

Proposition 5.3. Let

\[
F \hookrightarrow E \xrightarrow{p} B
\]

be a spherical fibration of simply-connected (and path-connected) topological spaces with rational homotopy of finite type. Let the fibre satisfy \( F \cong \mathbb{Q} S^n \) for an even \( n \geq 2 \). Suppose further that the rationalised Hurewicz homomorphism \( \pi_*(B) \otimes \mathbb{Q} \to H_*(B) \) is injective in degree \( n \). Additionally assume the space \( B \) to be rationally \( k \)-connected with \( 3k + 1 \geq 2n \).

Then we obtain: If the space \( E \) is formal, so is \( B \).

Proof. In order to be able to apply theorem C we need to modify its proof slightly: In case 1, step 3 we used that the rationalised Hurewicz homomorphism was supposed to be injective in degree 2. The purpose of this assumption was to show that \( V^{2n} = C_B^{2n} \) or actually merely that \( u' \in C_B^{2n} \).

We know that \( dz' = z^2 - u \) with \( u-u' \in \bigwedge^{\geq 2} \langle \{b_i\}_{i \in J} \rangle \) in the model of the fibration and with \( u \) being \( d_B \)-closed (cf. 2.11). We have to show that our new condition is apt to replace the former one, i.e. to equally yield \( u' \in C_B^{2n} \).

Since \( B \) is rationally \( k \)-connected, it holds that \( V^{2k} = 0 \). This directly leads to \( d_B V^{\leq 2k} = 0 \) by the minimality of \( (\bigwedge V_B, d_B) \). As \( (2k+1)+(k+1) = 3k+2 \), this itself implies that

\[
d_B \left( \left( \bigwedge^{\geq 2} V_B \right)^{\leq 3k+1} \right) = 0
\]
Thus by the assumption $3k + 1 \geq 2n$ we derive that the element $u - u' \in (\bigwedge V^2)^{2n}$ is closed, i.e. $d_B(u - u') = 0$. However, we know that $d_B u = 0$, whence $d_B u' = 0$ and $u' \in C_B^{2n}$. From here the proof of the theorem can be continued without any change. \hfill $\square$

**Corollary 5.4.** Let
\[ S^2 \hookrightarrow Z \overset{p}{\rightarrow} M \]
be a fibre bundle of simply-connected smooth closed manifolds. If $Z$ is formal, so is $M$. \hfill $\square$

Let us now prove theorems A and B. As for the latter we see that it is a direct consequence of theorem D. We shall give two slightly different proofs of theorem A. The first one basically relies the strong rigidity (cf. 1.3); the second one will need clearly less structure theory of Positive Quaternion Kähler Geometry. Both proofs make essential use of the twistor fibration.

**FIRST PROOF OF THEOREM A.** For every Positive Quaternion Kähler Manifold $M$ there is the twistor fibration
\[ S^2 \hookrightarrow Z \rightarrow M \]
with $Z$ a compact Kähler manifold. Positive Quaternion Kähler Manifolds are simply-connected. Thus so is $Z$. (As compact manifolds both $M$ and $Z$ have finite-dimensional (rational) homology.) The twistor space $Z$ is a compact Kähler manifold. Thus it is a formal space (cf. example 2.8).

Positive Quaternion Kähler Manifolds have vanishing odd-degree Betti numbers—cf. theorem 1.4. In particular, $b_3(M) = 0$. A Positive Quaternion Kähler Manifold $M^n$ satisfying $b_2(M) > 0$ is necessarily isometric to the complex Grassmannian $\mathbf{Gr}_2(\mathbb{C}^{n+2})$—cf. theorem 1.3. The latter is a symmetric space. Consequently, it is formal (cf. example 2.8). Thus, without restriction, we may focus on the case when $b_2(M) = 0$.

In total, we may assume $M$ to be rationally 3-connected. Now corollary 5.1 applies and yields the formality of $M$. \hfill $\square$

**SECOND PROOF OF THEOREM A.** As in the first proof we use the twistor fibration (19) and apply corollary 5.4 to it. Since $Z$ is a formal space, the corollary directly yields the result. \hfill $\square$

Note that we do not use $b_2(M) = b_3(M) = 0$ in the second proof. Needless to mention that formality may be interpreted as some sort of rigidity result: Whenever $M$ has the rational cohomology type of a Wolf space, it also has the rational homotopy type of the latter.

For the definition of the following numerical invariants see the definition on [7].28, p. 370, which itself relies on various definitions on the pages 351, 360 and 366 in [7].27.

**Corollary 5.5.** A rationally 3-connected Positive Quaternion Kähler Manifold $M$ satisfies
\[ \frac{\dim M}{4} = c_0(M) = c_0(M) = \text{cat}_0(M) = \text{cl}_0(M) \]
Proof. On a Positive Quaternion Kähler Manifold $M^n$ there is a class $u \in H^4(M)$ with respect to which $M$ satisfies the analogue of the Hard-Lefschetz theorem. In particular, there is the volume form $u^n \neq 0$. Thus the rational cup-length of a rationally 3-connected Positive Quaternion Kähler Manifold $M$ is $c_0(M) = \frac{\dim M}{4}$. The rest of the equalities is due to formality (cf. [7], example 29.4, p. 388). □

Corollary 5.6. A compact simply-connected Non-Negative Quaternion Kähler Manifold is formal. The twistor fibration is formal.

Proof. A simply-connected Quaternion Kähler Manifold with vanishing scalar curvature is hyperKählerian and Kählerian in particular. So it is a formal space. The twistor fibration in this case is the canonical projection $M \times S^2 \to M$. □

Remark 5.7. • In general, Positive Quaternion Kähler Manifolds are not coformal, i.e. their rational homotopy type is not necessarily determined by their rational homotopy Lie algebra or, equivalently, their minimal Sullivan models do not necessarily have a strictly quadratic differential. A counterexample is clearly $\mathbb{HP}^n$ for $n \geq 2$.

• In low dimensions, i.e. in dimensions 12 to 20, relatively simple proofs for formality can be given using the concept of s-formality developed in [8]. Alternatively, one may use the existence of isometric $S^1$-actions on 12-dimensional and 16-dimensional Positive Quaternion Kähler Manifolds and further structure theory to apply corollary [13].5.9, p. 2785, which yields formality.

6. Odd-dimensional fibre spheres

In the following we shall deal with the case of a spherical fibration

$$F \hookrightarrow E \to B$$

(20)

where the fibre has odd (homological) dimension. We shall see that a priori no general statement can be made on how the formality of the base space and the total space are related. That is, we shall see that the formality of $E$ does not necessarily imply the one of $B$, neither does the formality of $B$ necessarily imply formality for $E$. The examples arise as geometric realisations of algebraic constructions. We remark that it is possible to use non-formal homogeneous spaces (as found e.g. in [9], [19]) to produce $S^1$-fibrations with non-formal total space and formal base space.

Before doing so we remark the following facts that can easily be proved using the usual methods: If the fibration (20) is totally non-cohomologous to zero, i.e. if the map induced in cohomology of the fibre-inclusion is surjective, then the space $E$ is formal if and only if $B$ is formal—always supposed that the spaces are simply-connected and with homology of finite type. Equally, the fibration is formal then. Indeed, one even obtains $E \simeq_\mathbb{Q} B \times F$.

Let us now illustrate the appropriate negative examples. First we shall deal with the case of a formal base $B$ and a non-formal total space $E$. On
[5], p. 261, the 3-dimensional compact Heisenberg manifold is given as the simplest non-formal compact manifold. It comes along with a fibre bundle

$$S^1 \to M^3 \to T^2$$

with $T^2 = S^1 \times S^1$. However, $M$ is proved to be non-formal in [5], whereas $B := T^2$ is formal.

A higher dimensional and simply-connected analogue of this example may be constructed algebraically by the minimal Sullivan algebra $(\bigwedge V, d)$ defined as follows: Let $V$ be generated by elements $x, y, z$ with $\deg x = \deg y = n$, $\deg z = 2n - 1$, $n$ odd, $dx = dy = 0$ and $dz = xy$—cf. [7], example 1 on page 157, for a similar example. By [5], lemma 3.2, p. 258, we may give a geometric realisation of this example as a fibration: We may use spatial realisation (cf. [7],17) to construct a CW-complex for the algebra $(\bigwedge \langle x, y \rangle, d)$. The algebra $\bigwedge \langle x, y \rangle \otimes \bigwedge \langle z \rangle$ then forms an elementary extension (cf. [5], p. 249), since $dz \in \bigwedge \langle x, y \rangle$. So the lemma realises this example with total space having the rational homotopy type of $(\bigwedge V, d)$, base space a realisation of $(\bigwedge \langle x, y \rangle, d)$ and fibre a realisation of $(\bigwedge \langle z \rangle, 0)$.

In all these examples the fibration is non-primitive as $dz = xy$ is decomposable. (The algebra $(\bigwedge \langle x, y \rangle, d)$ is a minimal Sullivan algebra already.)

So we have seen:

**Remark 6.1.** Assume the fibration (20) to be non-primitive: The formality of $B$ does not necessarily imply the formality of $E$. ❭

Next we shall give a similar negative example under the assumption that (20) is primitive. Consider the commutative differential graded algebra $(\bigwedge V, d)$ over the graded vector space $V$ which we define as follows. We shall indicate generators with degree; moreover, we indicate the differential.

\[
\begin{align*}
2 : & \quad y \mapsto 0 \\
3 : & \quad b \mapsto 0 \quad c \mapsto 0 \\
4 : & \quad u \mapsto 0 \\
5 : & \quad n \mapsto bc + uy
\end{align*}
\]

Extend the differential to $\bigwedge V$ as a derivation. By definition the algebra is minimal and hence a minimal Sullivan algebra, as it is simply-connected (cf. 2.4). Form the algebra $(\bigwedge V \otimes \bigwedge \langle z \rangle, d)$ with $dz = u$, $\deg z = 3$. Since this algebra is an elementary extension of $(\bigwedge V, d)$, by lemma [5],3.2 we may realise it as the total space of a fibration with fibre rationally a sphere $S^3$. Since $(\bigwedge V, d)$ is minimal and since $z$ is mapped to $u$ under $d$, the fibration is primitive. It is a straightforward computation to show that $(\bigwedge V, d)$ is formal, whereas $(\bigwedge V \otimes \bigwedge \langle z \rangle, d)$ is not.

**Remark 6.2.** Assume the fibration (20) to be primitive: The formality of $B$ does not necessarily imply the formality of $E$. ❭

We shall conclude with a simple example of a fibration with formal total space and non-formal base space. We construct an algebra the realisation of which will be the base space $B$. The algebra is $(\bigwedge V, d)$, where $V$ will be
given by

\[
\begin{align*}
3 : & \quad b \mapsto 0 \\
4 : & \quad c \mapsto 0 \\
6 : & \quad n \mapsto bc
\end{align*}
\]

The algebra is minimal by construction. So it is a minimal Sullivan algebra as it is simply-connected (cf. 2.4). It is rather obvious that the algebra cannot be formal: According to theorem 2.9 there is a unique decomposition \( C = \langle b, c \rangle, N = \langle n \rangle \) by degree. We compute \( d(nb) = bc = 0 \) and \( nb \in I(N) \) is a closed but obviously non-exact element.

Form the algebra \( (\bigwedge V \otimes \bigwedge \langle z \rangle, d) \) with \( dz = c, \deg z = 3 \). Since this algebra is an elementary extension of \((\bigwedge V, d)\), by lemma [5]3.2 we may realise it as the total space of a fibration with fibre rationally an \( S^3 \). A minimal model for \( (\bigwedge V \otimes \bigwedge \langle z \rangle, d) \) is given by \( (\bigwedge \langle b, n \rangle, d) \) with \( \deg b = 3, \deg n = 6, db = dn = 0 \) (and where the \( n \) here corresponds to \( n + zb \) in the model of the fibration). Obviously, this model is formal, as it is free. This proves

**Remark 6.3.** The formality of \( E \) does not necessarily imply the formality of \( B \).

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