THE GEOMETRY OF SOME FIBONACCI IDENTITIES IN THE HOSOYA TRIANGLE

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Abstract. The Hosoya triangle is a triangular array where every entry is a product of two Fibonacci numbers. We use the geometry of this triangle to find new identities related to Fibonacci numbers. We give geometric interpretation for some well-known identities of Fibonacci numbers. For instance, the Cassini identity and the Catalan identity. We also extend some identities that hold in the Pascal triangle to the Hosoya triangle. For example, the hockey stick extends from binomials to products of Fibonacci numbers and the rhombus property extends a binomial identity from the Pascal triangle to an identity of products of Fibonacci numbers in the Hosoya triangle.

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1. Introduction

The Hosoya triangle, denoted by $H$, is a triangular array where every entry is a product of two Fibonacci numbers (see [1–5, 7, 8, 10, 12]). Figure 1 Part (a) shows that the fourth entry in row eight is the product $F_5$ and $F_4$.

These types of triangles are ideal to provide geometric interpretation of Fibonacci number identities. For instance, in this paper we give geometric interpretation for the well-known identities: the Cassini identity and the Catalan identity. For example, Figure 1 Part (b) depicts some examples of Cassini identity $F_{n-1}F_{n+1} - F_n^2 = (-1)^n$. From Part (b) we can see that $F_3F_5 - F_4^2 = 10 - 9 = (-1)^4$. Similarly, we can represent geometrically the Catalan identity $F_{n-r}F_{n+r} - F_n^2 = (-1)^{n-r-1}F_r^2$ (see Figure 8).

In this paper we use the geometry of the Hosoya triangle to discover new Fibonacci identities and also we give geometric interpretations of some well-known identities. Since the proofs here are based on the geometry of the triangle, they are simple and easy to understand. For example, in this paper we use the geometric representation of the product of Fibonacci numbers in the triangle to prove these identities.
• If \( a + b = c + d \) and \( a \leq b \), then
\[
\sum_{j=0}^{2k-1} F_{a+j}F_{b+j} = \sum_{j=0}^{2k-1} F_{c+j}F_{d+j} = F_{b+2k}F_{b+2k-1} - F_bF_{a-1},
\]

• if \( m \) is a positive integer, then
\[
\sum_{k=1}^{l} \frac{F_{m-k} + (-1)^kF_{m+k}}{F_m} = \begin{cases} 
-(F_l + F_{l-2} - 1), & \text{if } l \text{ is odd;} \\
5F_{l-1}F_l + 1 + (-1)^{l+1}, & \text{if } l \text{ is even}
\end{cases}
\]

The first of the two identities above is a generalization of this identity (see [12])
\[
\sum_{j=0}^{2k-1} F_{j+1}^2 = F_{2k+1}F_{2k}. 
\]

From [1–4,7,8] we have observed that some properties that hold in the Pascal triangle also extend to other triangles including the Hosoya triangle. In this paper we use the geometric representation, in the Pascal triangle (see [9]), of some binomial identities to discover new identities of Fibonacci numbers. For instance, the hockey stick property is one of the well-known properties that we successfully extend to the Hosoya triangle. The T-stick property in the Pascal triangle gives rise to other triangles including the Hosoya triangle. In this paper we also study some other geometric properties that the Hosoya triangle has. For example we give geometric proofs of the Cassini, Catalan, and Johnson identities.

We have found that if a rectangle is given in a Hosoya triangle, then the differences of two of its corners points is equal to the difference of the remaining corners points. This fundamental property allows us to have geometrical proofs of several identities.

The symmetry present in the Hosoya triangle helps us to explore several patterns, and many identities. The rectangle property gives rise to other geometrical configurations and therefore, more identities associated with those configurations.

2. THE HOSOYA TRIANGLE AND ITS COORDINATE SYSTEM

The construction presented in this section can be found in articles by Flórez et al. [7] and Hosoya [10]. Other similar constructions are also presented in Koshy [12] or in [1–6,8]. The Hosoya sequence \( \{H(r,k)\}_{r,k \geq 0} \) is defined using the double recursion
\[
H(r,k) = H(r-1,k) + H(r-2,k) \quad \text{and} \quad H(r,k) = H(r-1,k-1) + H(r-2,k-2),
\]
with initial conditions \( H(0,0) = 0; \ H(1,0) = 0; \ H(1,1) = 0; \ H(2,1) = 1 \), where \( r > 1 \) and \( 0 \leq k \leq r - 1 \). This sequence gives rise to the Hosoya triangle, where the entry in position \( k \) (taken from left to right) of the \( r \)th row is equa to \( H(r,k) \) (see Tables 1 and 8, and Sloane [13] at A058071). For simplicity in this paper we use \( \mathcal{H} \) to denote the Hosoya triangle.

\[
\begin{array}{ccccccc}
H(0,0) & H(1,0) & H(1,1) & H(2,0) & H(2,1) & H(2,2) \\
H(3,0) & H(3,1) & H(3,2) & H(3,3) & H(4,0) & H(4,1) \\
H(4,2) & H(4,3) & H(4,4) & H(5,0) & H(5,1) \\
H(5,2) & H(5,3) & H(5,4) & H(5,5) & H(6,0) \\
H(6,2) & H(6,3) & H(6,4) & H(6,5) & H(6,6)
\end{array}
\]

Table 1. Hosoya triangle \( \mathcal{H} \).

Proposition 1 ( [10,12]). \( H(r,k) = F_kF_{r-k} \).
Proposition 1 (represented in Figure 1 Part (a)) gives rise to another coordinate system (see also Flórez et al. [7, 8]). If \( P \) is a point in \( \mathcal{H} \), then it is clear that there are two unique positive integers \( r \) and \( k \) such that \( P = H(r, k) \) with \( k \leq r \). From \( H(r, k) = F_k F_{r-k} \) it is easy to see that an \( n \)th diagonal in \( \mathcal{H} \) is the collection of all Fibonacci numbers multiplied by \( F_n(x) \). For example, from Table 1 we can see that the diagonal \( H(4, 1), H(5, 2), H(6, 3), H(7, 4), H(8, 5), H(9, 6), \ldots \) is equal to the diagonal \( 2, 1, 2, 1, 2, 1, \ldots \) in Figure 8 which results from multiplying the Fibonacci sequences by \( F_3 = 2 \).

3. Geometric properties in the Hosoya triangle

A parallel configuration of points in the Hosoya triangle is called a ladder configuration, for simplicity we are going to refer to this as a ladder. A rung is the set of points on a line intersecting both parallel (the right-up) configurations of the ladder. See Figures 2(a), 3, 7, and 13. The length of the rung is the difference of its end points. The absolute length of a rung is the absolute value of its length.

In this section we use the ladder configuration to explore geometric and algebraic properties in the Hosoya triangle. The properties here in this paper can be easily extended to the Hosoya polynomial triangle (see Flórez et al. [6]).

We first prove a lemma that will be helpful in proving several results in this paper. If \( L \) is a horizontal ladder in \( \mathcal{H} \) where its rungs have exactly two points, then a rung sum is a Fibonacci number and it is the same for every rung. The identity (8) in Vajda [14] states that \( G_{n+m} = F_{m-1}G_n + F_mG_{n+1} \), where \( G_n \) is any generalized Fibonacci sequence. This lemma gives a geometric interpretation of the identity in Vajda when \( m = j + 1 \), \( n = k - j \) and \( G_n = F_n \).

**Lemma 2.** If \( k, j \in \mathbb{Z}_{>0} \) and \( j + i + 1 \leq k \), then in \( \mathcal{H} \) this holds

\[
F_j F_{k-j} + F_{j+1} F_{k-j+1} = F_{j+i} F_{k-j-i} + F_{j+i+1} F_{k-j-i+1} = F_{k+1}.
\]

Equivalently, \( H(k, j) + H(k + 2, j + 1) = H(k, j + i) + H(k + 2, j + i + 1) = F_{k+1} \).

**Proof.** First, we take two consecutive rungs of \( L \) forming a square (see Figure 2(a)). Observe that each diagonal (slash and backslash) of this square has three points — two corner points and one inner point. Those two diagonals intersect in the inner point \( p \). From the recursive definition of the entries of \( \mathcal{H} \) and the point \( p \), it is easy to see that the difference of the two corner points of the backslash diagonal of the square is equal to the difference of the corner points of the slash diagonal of the square. This implies that sum of the points in any two consecutive rungs have the same value. Using an inductive argument we can extend the result for any two arbitrary rungs. Since this is true for any rung of \( L \), it is true for the first rung on the left where one of the two points is zero and the other is the Fibonacci number \( F_{k+1} = H(k + 1, 1) \).

An alternate (technical) proof can be found using the recursive definition of the Hosoya triangle and first proving that \( H(r, k) + H(r + 2, k + 1) = H(r, k + 1) + H(r + 2, k + 2) \).

All horizontal rungs in a vertical ladder in \( \mathcal{H} \) have the same length except by the order of their measure (see Figures 2(b) and 3). This result is formally stated in Proposition 3.

**Proposition 3** (Rectangle Property). In \( \mathcal{H} \) it holds that

\[
F_j F_{k-j} - F_{j+i} F_{k-j-i} = (-1)^r (F_{j+r} F_{k+r-j} - F_{j+i+r} F_{k+r-j-i}) = (-1)^{j+1} F_j F_{k-2j-i}.
\]

Equivalently,

\[
H(k, j) - H(k, j + i) = (-1)^r (H(k + 2r, j + r) - H(k + 2r, j + i + r)) = (-1)^{j+1} H(k - 2j, i).
\]

**Proof.** From Figure 3 and Lemma 2 we can observe that

\[
|a_0 - b_0| = |a_1 - b_1| = |a_2 - b_2| = \cdots = |a_i - b_i|.
\]

From Figure 3 we can see that, in particular, if we take \( a_0 = 0 \), then \( b_0 = H(k - 2j, i) \).
The following result provides several identities in the Hosoya triangle. In particular it shows that the alternating sum of the points in a horizontal rung of a vertical ladder and the sum of the points in a vertical rungs of a horizontal ladder is a constant provided that the rung has even number of points in each case. We also see that the absolute length of each rung in a horizontal ladder is the same if there are an odd number of points in the rungs. Finally, if the ladders are oblique (see Figure 7), then the absolute length of each rung is the absolute length of the first rung multiplied by a Fibonacci number and the sum of the points in the oblique rungs equals the sum of the points in the second rung multiplied by a Fibonacci number.

We may also use Proposition 1 to give an algebraic reinterpretation of the results mentioned above in terms of Fibonacci numbers.

**Theorem 4.** In the Hosoya triangle \( \mathcal{H} \) these hold,

1. If \( r, k > 0, j \geq 0 \) and \( 0 < 2n - 1 \leq k \) for some \( n \), then

\[
\sum_{t=0}^{2n-1} (-1)^t H(k, j + t) = \sum_{t=0}^{2n-1} (-1)^t H(k + 2r, j + t + r).
\]

Equivalently,

\[
\sum_{t=0}^{2n-1} (-1)^t F_{j+t} F_{k-j-t} = \sum_{t=0}^{2n-1} (-1)^t F_{j+r+t} F_{k+r-j-t}.
\]

2. If \( m, k > 0, j \geq 0 \) and \( 0 < 2n - 1 \leq k \) for some \( n \), then

\[
\sum_{t=0}^{2n-1} H(k + 2t, j + t) = \sum_{t=0}^{2n-1} H(k + 2t, j + m + t).
\]
Equivalent, \( \sum_{t=0}^{2n-1} F_{j+t}F_{k+t-j} = \sum_{t=0}^{2n-1} F_{j+m+t}F_{k+t-j-m} \).

(3) If \( i \) is a positive even number, then

\[ H(k+2i,j+i) - H(k,j) = H(k+2i,j+n+i) - H(k,j+n) = H(k+2i,i). \]

Equivalently, det \( \begin{bmatrix} F_{j+i} & F_j \\ F_{k-j} & F_{k+i-j} \end{bmatrix} \) = det \( \begin{bmatrix} F_{j+n+i} & F_{j+n} \\ F_{k-j-n} & F_{k+i-j-n} \end{bmatrix} \) = det \( \begin{bmatrix} F_i & 0 \\ 0 & F_{k+i} \end{bmatrix} \).

(4) If \( r, k, \) and \( j \) are positive integers with \( r \geq j \), then

\[ H(r+k,j) - H(r,j) = F_j(H(r+k-j+1,1) - H(r-j+1,1)). \]

Equivalently, det \( \begin{bmatrix} F_j & F_{r-j} \\ F_j & F_{r+k-j} \end{bmatrix} = F_j(F_{r+k-j} - F_{r-j}). \)

(5) If \( i, j, \) and \( k \) are positive integers, then

\[ \sum_{i=0}^{m} H(k+i,j+i) = F_{k-j} \sum_{i=0}^{m} H(j+i+1,1). \]

**Proof.** We prove Part (1) for two consecutive rungs. The general case follows easily using an inductive argument, so we omit it. From Figure 4 and Lemma 2 we have

\[ c_1 - c_0 = d_0 - d_1; \quad c_3 - c_2 = d_2 - d_3; \quad \ldots \quad c_{2i-1} - c_{2i-2} = d_{2i-2} - d_{2i-1}. \]

This implies that \( |\sum_{i=0}^{2n-1} (-1)^i c_i| = |\sum_{i=0}^{2n-1} (-1)^i d_i|. \)

**Figure 4.** The alternating sum of the points in a rung is the same.

Proof of Part (2). From Figure 5 and Lemma 2 we have

\[ c_0 + c_1 = d_0 + d_1; \quad c_2 + c_3 = d_2 + d_3; \quad \ldots \quad c_{2n-2} + c_{2n-1} = d_{2n-2} + d_{2n-1}. \]

This leads to the conclusion.

Proof of Part (3). Let \( a_0 \) and \( a_1 \) be the points of the first rung of the ladder and let \( b_0 \) and \( b_1 \) be the points of the last rung of the ladder, as in Figure 6. We assume that the ladder has an odd number of rungs. We define \( x \) as the sum of all points between \( c_0 \) and \( d_0 \) (left-hand side points), including both of them, and let \( y \) be the sum of all points between \( c_1 \) and \( d_1 \) (right-hand side points), including both of them. From Part (2) \( a_0 + x = a_1 + y \) and \( b_0 + x = b_1 + y. \) From this it is easy to see that \( a_1 - a_0 = x - y = b_1 - b_0. \) If, in particular, we take \( a_0 = H(k,0), \) we have \( b_0 = H(k+2i,1). \)
Proof of parts (4) and (5). Using the coordinate system described in Section 2 and Figure 7 we can see that the points in the first parallel side (the right-up) of the ladder are of the form \( F_k F_i \) where \( F_k \) is fixed and the points in the second parallel side are of the form \( F_k + j F_i \) where \( F_k + j \) is fixed. Therefore, the points in the \( r \)-th rungs are \( F_k F_r \), \( F_k + 1 F_r \), \( F_k + 2 F_r \), \ldots, \( F_k + j F_r \). To prove Part (4) we first note that the length of any rung is given by \( F_k + j F_r - F_k F_r = F_r (F_k + j - F_k) \) where \( F_k + j - F_k \) is the length of the first rung. The proof of Part (5) follows by adding the points of a rung. Thus,

\[
F_k F_r + F_{k+1} F_r + F_{k+2} F_r + \cdots + F_{k+j} F_r = F_r (F_k + F_{k+1} + F_{k+2} + \cdots + F_{k+j}).
\]

Note that \( (F_k + F_{k+1} + F_{k+2} + \cdots + F_{k+j}) \) is the sum of points in the second rung. \( \square \)

In the next part we give a geometric interpretation in the Hosoya triangle of the Cassini, Catalan, and Johnson identities (see [11]).

The length of a rung in a vertical ladder in \( \mathcal{H} \) gives rise to the Cassini identity, if one of the uprights is located in a central vertical line of \( \mathcal{H} \) and the rung has exactly two points (see Figure 8). Thus,

\[
H(2k, k) - H(2k, k - 1) = (-1)^{k-1}.
\]

The same type of ladder as above also gives rise to the Catalan identity. Thus,

\[
H(2k, k) - H(2k, k - j) = (-1)^{k-j} H(2j, j).
\]
The sum of the points of the rungs are proportionally related.

|   |   |   |   |   |   |   |   |   |   |   |
|---|---|---|---|---|---|---|---|---|---|---|
| 1 |   |   |   |   |   |   |   |   |   |   |
| 1 | 1 |   |   |   |   |   |   |   |   |   |
| 2 | 1 | 2 |   |   |   |   |   |   |   |   |
| 3 | 2 | 2 | 3 |   |   |   |   |   |   |   |
| 5 | 3 | 4 | 3 | 5 |   |   |   |   |   |   |
| 8 | 5 | 6 | 6 | 5 | 8 |   |   |   |   |   |
| 13 | 8 | 10 | 9 | 10 | 8 | 13 | 21 |   |   |   |
| 21 | 13 | 16 | 15 | 15 | 16 | 13 | 21 | 34 |   |   |
| 34 | 21 | 26 | 24 | 25 | 24 | 26 | 21 | 34 | 55 |   |
| 55 | 34 | 42 | 39 | 40 | 40 | 39 | 42 | 34 | 55 | 89 |
| 89 | 55 | 68 | 63 | 65 | 64 | 65 | 63 | 68 | 55 | 89 |
| 144 | 89 | 110 | 102 | 105 | 104 | 104 | 105 | 102 | 110 | 89 |
| 233 | 144 | 178 | 165 | 170 | 168 | 169 | 168 | 170 | 165 | 178 |
| 377 | 233 | 288 | 267 | 275 | 272 | 273 | 272 | 275 | 267 | 288 |
| 610 | 377 | 466 | 432 | 445 | 440 | 442 | 441 | 442 | 440 | 445 |
| 987 | 610 | 754 | 699 | 720 | 712 | 714 | 715 | 712 | 720 | 699 |
| 1597 | 987 | 1220 | 1131 | 1165 | 1152 | 1155 | 1155 | 1131 | 1220 | 987 |
| 2584 | 1597 | 1974 | 1830 | 1885 | 1864 | 1870 | 1869 | 1870 | 1864 | 1885 |
| 377 | 233 | 288 | 267 | 275 | 272 | 273 | 272 | 275 | 267 | 288 |
| 610 | 377 | 466 | 432 | 445 | 440 | 442 | 441 | 442 | 440 | 445 |
| 987 | 610 | 754 | 699 | 720 | 712 | 714 | 715 | 712 | 720 | 699 |
| 1597 | 987 | 1220 | 1131 | 1165 | 1152 | 1155 | 1155 | 1131 | 1220 | 987 |

Catalan identity

Cassini identity

Figure 8. Geometry of the Cassini and Catalan identities.

The length of a rung in a vertical ladder in \( H \) gives rise to the d’Ocagne identity if one of them is located on the top of the ladder, and the coordinates of the end points of the second rung are \( H(k+j+1,j) \) and \( H(k+j+1,k) \). Thus,

\[
H(k+j+1,k) - H(k+j+1,j) = (-1)^j H(k+j+1,k-j).
\]

The length of a rung in a vertical ladder in \( H \) gives rise to the Johnson identity. Thus, if \( k+j = r+i \) and \( i < j \), then for every \( l \leq i \) it holds that

\[
H(k+j,j) - H(r+i,i) = (-1)^i (H(k+j-2l,j-l) - H(r+i-2l,i-l))
\]

\[
= (-1)^i H(k+j-2i,j-i).
\]

As a corollary of Theorem 4 Part (3) we have that if the points \( a_i \) and \( b_i \) in the Hosoya triangle are as in Figure 9(e), then \( (a_j + a_{j+1}) - (b_j + b_{j+1}) \) is a constant. This property is analogous to a property in Pascal’s triangle that yields the Catalan numbers (see [15]).

Proposition 5. Let \( a, b, j \) be positive integers with \( j \leq \min\{a, b\} \). If \( A(F_{a-j}, F_{a+j}) \), \( B(F_{b-j}, F_{b+j}) \) and \( C(F_a, F_b) \) are points in the Cartesian plane, then

1. the line passing through \( A \) and \( B \) is parallel to the line passing through \((0,0)\) and \( C \). Thus,

\[
\frac{F_b}{F_a} = \frac{F_{a-j} + (-1)^j F_{a+j}}{F_{b-j} + (-1)^j F_{b+j}}.
\]

2. The triangle with base \( F_{a+j} + (-1)^j F_{a-j} \) and height \( F_b \) has same area as the triangle with base \( F_{b+j} + (-1)^j F_{b-j} \) and height \( F_a \).
The proof of Proposition 5 is easy using Proposition 3, therefore we omit it.

The configuration depicted in Figure 9 Part (a) is called a zigzag. The configuration depicted in Figure 9 Part (b) is called a left zigzag. The configuration depicted in Figure 9 Part (c) is called a right zigzag. The configuration depicted in Figure 9 Part (d) is called a long zigzag. There should be a finite number of points in any zigzag configuration.

The configuration depicted in Figure 12 Part (a) is called a braid (or hourglass). The configuration depicted in Figure 12 Part (b) is called a left braid. The configuration depicted in Figure 12 Part (c) is called a right braid. There should be a finite number of points in any braid configuration.

**Figure 9.** Zigzag configurations.

**Corollary 6.** The sum of alternating points of a long zigzag configuration in $\mathcal{H}$ starting from its second point is equal to the difference of the last point and the first point of the zigzag configuration. Moreover, any column of points forming a rectangle with the column of alternating points has the same sum (see Figure 10(a) and Figure 9 Part (d)). More precisely, if $a, b, c,$ and $d$ are positive integers such that $a + b = c + d$ and $a \leq b$, then for every positive integer $k$ it holds that

$$\sum_{j=0}^{2k-1} F_{a+j} F_{b+j} = \sum_{j=0}^{2k-1} F_{c+j} F_{d+j} = F_{b+2k} F_{b+2k-1} - F_b F_{a-1}.$$  

**Proof.** We prove that the sum of alternating points of a long zigzag configuration in $\mathcal{H}$ starting from its second point is equal to the difference of the last point with the first point of the long zigzag. The last part of this corollary follows from Theorem 4 Part (2).

Suppose $p_1, p_2, p_3, \ldots, p_{n-2}, p_{n-1}, p_n$ are the points of the long zigzag ordered from top to bottom where $p_1$ is the first point on the top and $p_n$ is the last point in the bottom (see Figure 10(a)). We want to show that $p_2 + p_4 + p_6 + \cdots + p_{n-1} = p_n - p_1$. From definition of $\mathcal{H}$ on Page 2 we know that $H(r, k) = H(r-1, k) + H(r-2, k)$ and $H(r, k) = H(r-1, k-1) + H(r-2, k-2)$. This implies
that
\begin{align*}
p_3 &= p_1 + p_2, \quad (1) \\
p_5 &= p_3 + p_4, \quad (2) \\
p_7 &= p_5 + p_6, \quad (3) \\
\vdots & \quad \vdots \\
p_n &= p_{n-2} + p_{n-1}. \quad (4)
\end{align*}

Substituting Equation (1) into \( p_2 + p_4 + p_6 + \cdots + p_{n-1} \) we obtain
\[ p_2 + p_4 + p_6 + \cdots + p_{n-1} = p_1 + p_3 + p_4 + p_6 + \cdots + p_{n-1}. \]
Substituting Equation (2) into the right-hand side of this equality, we obtain
\[ p_2 + p_4 + p_6 + \cdots + p_{n-1} = -p_1 + p_5 + p_6 + \cdots + p_{n-1}. \]
Substituting Equation (3) into the right-hand side of this equality, we obtain
\[ p_2 + p_4 + p_6 + \cdots + p_{n-1} = -p_1 + p_7 + p_8 + \cdots + p_{n-1}. \]
We systematically keep making these substitutions to obtain
\[ p_2 + p_4 + p_6 + \cdots + p_{n-1} = -p_1 + p_n. \]
This completes the proof of the corollary. \( \square \)

The identity in the previous corollary is a generalization of the identities found in Koshy [12] and Vajda [14]. For example, if \( a = b = 1 \), then we obtain
\[ \sum_{j=0}^{2k-1} F_{1+j}^2 = F_{2k}F_{2k+1}. \]
In addition, if \( a = 1, b = 2 \), then
\[ \sum_{j=0}^{2k-1} F_{j+1}F_{j+2} = F_{2k+2}F_{2k+1} \]
and so on.

**Theorem 7** (Zigzag property). If a zigzag configuration with \( 6k + 5 \) points holds in \( \mathcal{H} \), then the sum of all points in its left zigzag is equal to the sum of all points in its right zigzag (see Figure 9 and Figure 10(b)).
Proof. From Figure 10(b) and definition of $\mathcal{H}$ on Page 2 it is easy to see that $p_1+p_2=p_3+p_4=p_5$. Since the zigzag configuration has $6k+5$ points, there remain $6k$ points distributed in three vertical equal sets (see Figure 10(b) for the labeling of points). So, every set has an even number of points. This and Theorem 4 Part (2) imply that $q_1+q_1+\cdots+q_n=t_1+t_1+\cdots+t_n$. This completes the proof. □

As a corollary of Theorem 7 we can see that the Hockey Stick property seen in Figure 11(a) and originally found in the Pascal triangle (see [2, 15]), also holds in $\mathcal{H}$. Thus, the sum of all points in the shaft of a hockey stick is equal to the point on the blade of the hockey stick. The blade of the hockey stick is going to the left or to the right depending on the numbers of points that are on the shaft. If we consider the hockey stick configuration on one side of the Hosoya triangle ($\mathcal{H}$), then (by the symmetry of $\mathcal{H}$) when the same configuration is represented on the other side of $\mathcal{H}$, the blade of the hockey stick changes direction (from left to right or from right to left). We use $s_1, s_2, \cdots, s_t$ to represent the points on the shaft of the hockey stick. We use $b_L$ and $b_R$ to indicate that it is on the left-hand side of the hockey stick and $b_R$ indicates that it is on right side of the hockey stick. (See Figure 11(a).) These give that if $s_1, s_2, \cdots, s_t$ are the points on the shaft of the hockey stick where $s_1$ is a point on one of the edges of $\mathcal{H}$ and $b_t$ is the point on the blade of the hockey stick, with $t \in \{L, R\}$ (see Figure 11(a)), then

1. If the hockey stick is on the left-hand side of $\mathcal{H}$, then $s_1 + s_2 + \cdots + s_{2n} = b_L$, or if the hockey stick is on the right-hand side of $\mathcal{H}$, then $s_1 + s_2 + \cdots + s_{2n} = b_R$. Therefore, we obtain this identity

$$\sum_{i=0}^{k-1} F_{i+1}F_{i+r-1} = F_kF_{k+r-1}.$$ 

2. If the hockey stick is on the left-hand side of $\mathcal{H}$, then $s_1 + s_2 + \cdots + s_{2n+1} = b_R$, or if the hockey stick is on the right-hand side of $\mathcal{H}$, then $s_1 + s_2 + \cdots + s_{2n+1} = b_L$. Therefore, we obtain this identity

$$\sum_{i=0}^{k-1} F_{i+1}F_{i+r-1} = F_{k+1}F_{k+r-2}.$$ 

3. If the hockey stick is in the center of $\mathcal{H}$, then $s_1 + s_2 + \cdots + s_t = b_L = b_R$. Therefore, for every $i$ this holds

$$\sum_{i=0}^{k-1} F_{i+1}^2 = F_kF_{k+1}.$$ 

The first two of the three identities above are generalizations of identities like $\sum_{i=0}^{k-1} F_i = F_k^2$ if we have $r = 1$, $\sum_{i=0}^{k-1} F_{i+1}^2 = F_{k+1}F_k$ if we have $r = 2$) and so on. The third identity (above), is well-known and can be found in [12].

We now formally state the previous results in the following corollary.

Corollary 8 (Hockey Stick property). Let $r, k \in \mathbb{Z}_{>0}$, where $k$ is the number of entries in the shaft of the hockey stick. Then these hold in $\mathcal{H}$.

1. If $k$ is even and the hockey stick is on left-hand side of the median of $\mathcal{H}$, then

$$\sum_{i=0}^{k-1} H(r+2i, i+1) = H(r+2k-1, k).$$
(2) If \( k \) is even and the hockey stick is on right-hand side of the median of \( H \), then
\[
\sum_{i=0}^{k-1} H(r + 2i, r + i) = H(r + 2k - 1, r + k - 1).
\]

(3) If \( k \) is odd and the hockey stick is on left-hand side of the median of \( H \), then
\[
\sum_{i=0}^{k-1} H(r + 2i, i + 1) = H(r + 2k - 1, k + 1).
\]

(4) If \( k \) is odd and the hockey stick is on right-hand side of the median of \( H \), then
\[
\sum_{i=0}^{k-1} H(r + 2i, r + i - 1) = H(r + 2k - 1, r + k - 2).
\]

(5) If the hockey stick is in the median, then for \( k > 0 \) this holds
\[
\sum_{i=0}^{k-1} H(2i + 2, i + 1) = H(2k + 1, k) = H(2k + 1, k + 1).
\]

![Figure 11. Hockey Stick and Braid configurations.](image)

**Proposition 9** (Braid property). If the braid configuration holds in \( H \), then the sum of all points in the left braid is equal to the sum of all points in the right braid (see Figure 12 and Figure 11(b))

**Proof.** In Figure 11(b) we observe that the left braid configuration is formed by all left corner points of the squares of even side length and all corner points in backslash diagonal in the squares of odd length (see for example \( S_2, S_3, S_4, \) and \( S_5 \) in Figure 11(b)). The right braid configuration is formed, similarly, by all right corner points of the squares of even side length and all points in the slash diagonal in the squares of odd side length.

From Proposition 3 it is easy to deduce that in a square configuration in \( H \) with even side length, it holds that the sum of two vertical corner points is equal to the sum of the remaining corner points. If the square configuration has odd side length, then it holds that the sum of two corner points in a diagonal of the square is equal to the sum of the remaining corner points in the other diagonal. Using this property and Figure 11(b) we can see that the sum of left corner points of the innermost square \( S_2 \) is equal to the sum of its right corner points. We now observe that the square \( S_3 \) has odd side length. Therefore, the sum of the corner points in the slash diagonal equals the corner points in the backslash diagonal. The square \( S_4 \) satisfies the property that the sum of the vertical left corner points equals the sum of the right corner points. The square \( S_5 \) satisfies the
property that the sum of the corner points in the slash diagonal equals the sum of the remaining corner points in the backslash diagonal. We can continue this process inductively as long as it is required by the braid configuration embedded in $\mathcal{H}$. From this, Figure 11(b), and the observation given in the first paragraph it is easy to obtain the conclusion of the proposition. □

Figure 12. Braid configurations.

For any square $S$ in Figure 11(b) it holds that the sum of the corner points in one diagonal of $S$ is the additive inverse of the sum of the corner points in the remaining diagonal of $S$. Therefore, using all squares in Figure 11(b) it holds that

$$\sum_{k=0}^{l} H(n-k, m-k) + \sum_{k=1}^{l} (-1)^k H(n+k, m+k) = \sum_{k=0}^{l} H(n-k, m) + \sum_{k=1}^{l} (-1)^k H(n+k, m).$$

If in Figure 11(b) we eliminate the common point —the point that is the intersection of left braid and right braid— we obtain

$$F_{n-m} \sum_{k=1}^{l} (F_{m-k} + (-1)^k F_{m+k}) = F_m \sum_{k=1}^{l} (F_{n-m-k} + (-1)^k F_{n-m+k}).$$

This with $r = n - m$ implies

$$\sum_{k=1}^{l} \frac{F_{m-k} + (-1)^k F_{m+k}}{F_m} = \sum_{k=1}^{l} \frac{F_{r-k} + (-1)^k F_{r+k}}{F_r}.$$

Therefore we have the following corollary.

**Corollary 10.** If $m, k$ are positive integers then,

$$\sum_{k=1}^{l} \frac{F_{m-k} + (-1)^k F_{m+k}}{F_m} = \begin{cases} -(F_l + F_{l-2} - 1), & \text{if } l \text{ is odd;} \\ 5F_{l-1}F_l + 1 + (-1)^{l+1}, & \text{if } l \text{ is even.} \end{cases}$$

This is a previously unknown identity.

4. **Properties of the Pascal triangle that extend to the Hosoya triangle**

In this section we extend a few properties from the Pascal triangle to the Hosoya triangle. These properties of the Pascal triangle may be found in [9].

If we construct an oblique (backslash) ladder with horizontal rungs of length two (see Figure 13), then the ladder gives rise to generalized Fibonacci numbers. Thus, adding the two points of each rung of the ladder gives rise to a sequence of second order which is a generalized Fibonacci sequence. Recall that the generalized Fibonacci sequence is given by $G_n = G_{n-1} + G_{n-2}$ with $G_1 = a$ and $G_2 = b$. Here we note that $a$ and $b$ are the points in the first rung of the oblique ladder, (see Figure 13). In particular, $a$ and $b$ are consecutive Fibonacci numbers with $a > b$. Some of the ladders give rise to certain sequences found in Sloane [13]. In particular, with $a = 1$ and $b = 1$ we...
obtain the Fibonacci number sequence A000045. If \( a = 2 \) and \( b = 1 \), we obtain the Lucas number sequence A000032. If \( a = 3, b = 2 \), we obtain sequence A013655 which is a sequence where each term is obtained by adding a Fibonacci and a Lucas number. The sequence A206610 is obtained with \( a = 13 \) and \( b = 8 \).

![Generalized Fibonacci](image)

**Figure 13.** Generalized Fibonacci.

If we consider two consecutive rungs of an oblique ladder —that has horizontal rungs— we obtain a rhombus property (see Figure 14(b)). This property is an extension of a similar property found in the Pascal triangle (see [9]). Thus, the rhombus property states that the differences of a cross multiplication is always a point of the triangle. That is, the cross multiplication is the determinant of the numbers present in the rungs of the oblique ladder (see Figure 14(b)). Formally we have this proposition.

**Proposition 11.**

1. *Slash ladder.* For a fixed positive integer \( r \) and for any \( n \geq 1 \) it holds that

\[
\begin{vmatrix}
H(n, r) & H(n, r + 1) \\
H(n + 1, r) & H(n + 1, r + 1)
\end{vmatrix} = (-1)^{n-r+1}F_{r}F_{r+1}.
\]

2. *Backslash ladder.* For fixed positive integers \( r \) and \( n \) and for any \( i \geq 1 \) it holds that

\[
\begin{vmatrix}
H(n + i, r + i) & H(n + i, r + i + 1) \\
H(n + i + 1, r + i + 1) & H(n + r + 1, r + i + 2)
\end{vmatrix} = (-1)^{r+i+1}F_{n-r}F_{n-r-1}.
\]

**Proof.** We prove Part (1), the proof of part (2) is similar and we omit it. We observe that the points of two consecutive rungs of a slash ladder are given by \( H(n, r) \), \( H(n, r + 1) \) and \( H(n + 1, r) \), \( H(n + 1, r + 1) \). These points give rise to a rhombus. Subtracting the product of the diagonal points of the rhombus gives \( H(n, r)H(n + 1, r + 1) - H(n, r + 1)H(n + 1, r) \). This, the definition of \( H(n, r) \), and the Cassini identity implies that

\[
H(n, r)H(n + 1, r + 1) - H(n, r + 1)H(n + 1, r) = F_r F_{n-r} F_{r+1} - F_{r+1} F_{n-r-1} F_r F_{n+1-r}
\]

\[
= F_r F_{r+1} (F_{n-r}^2 - F_{n-r-1} F_{n-r+1})
\]

\[
= F_r F_{r+1} (-1)^{n-r+1}.
\]

This completes the proof. \( \square \)

An additional configuration that yields a geometry and an identity that we can explore is a triangle configuration seen in Figure 14(a).

If we take the triangle configuration as in Figure 14 Part (a) then \( a+b-c \) is a Fibonacci number. Note that \( a \) and \( b \) are points constituting the top oblique side of the triangle and the points \( b \) and \( c \) are points along the same vertical line at a distance two from each other. So, if \( a = H(n+1, r-1), \)
Proposition 12. \( a + b − c = (-1)^r H(n−2r + 2, 2) \)

Proof. Since \( a = H(n+1, r−1) \), \( b = H(n, r) \), and \( c = H(n+2, r+1) \), we have
\[
a + b − c = a + b - (H(n+3, r+1) − H(n+1, r+1)) \\
= b + H(n+1, r+1) + a − H(n+3, r+1) \\
= H(n+2, r+2) − H(n+2, r).
\]

This and Proposition 3, imply that \( a + b − c = (-1)^r H(n−2r + 2, 2). \) \( \square \)

Figure 14. Triangle and rhombus properties.

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