1. Introduction

1.1. Let $X$ be a smooth complex algebraic variety and $f : X \to \mathbb{A}^1_C$ be a non-constant morphism to the affine line. Let $x$ be a singular point of $f^{-1}(0)$, that is, such that $df(x) = 0$.

Fix $0 < \eta \ll \varepsilon \ll 1$. By Milnor’s local fibration Theorem (see [23],[12]) the morphism $f$ restricts to a fibration, called the Milnor fibration,

$$B(x, \varepsilon) \cap f^{-1}(B(0, \eta) \setminus \{0\}) \to B(0, \eta) \setminus \{0\}.$$  

Here $B(a, r)$ denotes the closed ball of center $a$ and radius $r$.

The Milnor fiber at $x$,

$$F_x = f^{-1}(\eta) \cap B(x, \varepsilon),$$

has a diffeomorphism type that does not depend on $\eta$ and $\varepsilon$ and is endowed with an automorphism, defined up to homotopy, the monodromy $M_x$, induced by the characteristic mapping of the fibration. In particular the cohomology groups $H^q(F_x, \mathbb{C})$ are endowed with an automorphism $M_x$, and we can consider the Lefschetz numbers

$$\Lambda(M_x^m) = \text{tr}(M_x^m; H^\bullet(F_x, \mathbb{C})) = \sum_i (-1)^i \text{tr}(M_x^m; H^i(F_x, \mathbb{C})).$$

In [1], N. A’Campo proved that if $x$ is a singular point of $f^{-1}(0)$, then $\Lambda(M_x^1) = 0$ and this was later generalized by Deligne to the statement that $\Lambda(M_x^m) = 0$ for $0 < m < \mu$, with $\mu$ the multiplicity of $f$ at $x$, cf. [2].

In [11], $\Lambda(M_x^m)$ was expressed in terms of arcs in the following way. Set

$$\mathcal{X}_{m,x} = \{ \varphi \in X(\mathbb{C}[t]/t^{m+1}); f(\varphi) = t^m \mod t^{m+1}, \varphi(0) = x \}.$$ 

**Theorem 1.1.1 ([11]).** — For every $m \geq 1,$

$$\chi_c(\mathcal{X}_{m,x}) = \Lambda(M_x^m).$$
Here $\chi_c$ denotes the usual Euler characteristic with compact supports. Note that one recovers Deligne's statement as a corollary since $X_{m,x}$ is empty for $0 < m < \mu$. The original proof in [11] proceeds as follows. One computes explicitly both sides of (1.1.5) on an embedded resolution of $f = 0$ and checks both quantities are equal. The computation of the left hand side relies on the change of variable formula for motivic integration in [8] and the one of the right hand side on A'Campo’s formula in [2]. The problem of finding a geometric proof of Theorem 1.1.1 not using resolution of singularities is raised in [18]. The aim of this paper is to present such a proof.

1.2. Our approach uses étale cohomology of non-archimedean spaces and motivic integration. Nicaise and Sebag introduced in [22] the analytic Milnor fiber $F_x$ of the function $f$ at a point $x$ which is a rigid analytic space over $\mathbb{C}((t))$. Given an embedding of $\mathbb{Q}_\ell$ in $\mathbb{C}$, the étale $\ell$-adic cohomology of the base change of $F_x$ over the algebraic closure of $\mathbb{C}(t)$, once tensored with $\mathbb{C}$, may be identified with the cohomology groups of $F_x$. In this way the right hand side of (1.1.5) may reinterpreted as a trace of the $m$-th iterate of the canonical generator of $\hat{\mu}(\mathbb{C})$ on the étale $\ell$-adic cohomology of a rigid analytic space.

We use motivic integration in the following way. In the paper [15] a canonical morphism

\begin{equation}
\text{EU}_\Gamma : K(VF) \longrightarrow !K(\text{RES})/([A_1] - 1).
\end{equation}

is constructed (see \S 2.5 for a review of the construction). Here $K(VF)$ is the Grothendieck ring of ACVF-definable sets over $\mathbb{C}((t))$ and $!K(\text{RES})$ is a quotient of the Grothendieck ring $K(\text{RES})$ of definable sets over the generalized residue structure $\text{RES}$. In fact, by Proposition 4.3.1, $K(\text{RES})$ may be reinterpreted in terms of the Grothendieck ring $K(\text{Var}_\mathbb{C}; \hat{\mu})$ of $\mathbb{C}$-varieties with $\hat{\mu}$-action. In particular, there is a natural $\hat{\mu}$-action on $!K(\text{RES})/([A_1] - 1)$. Instead of trying to prove directly a Lefschetz fixed points for objects of $VF$, that are infinite dimensional in nature, our strategy is to use the morphism $\text{EU}_\Gamma$ in order to reduce to finite dimensional spaces. To this aim, using étale cohomology of Berkovich spaces, as developed in [3], we construct a natural ring morphism

\begin{equation}
\text{EU}_{\text{ét}} : K(VF) \longrightarrow K(\hat{\mu}-\text{Mod}).
\end{equation}

(where $\hat{\mu}-\text{Mod}$ is the Grothendieck ring of $\hat{\mu}$-modules over $\mathbb{Q}_\ell$, see below.) A fundamental result is that $\text{EU}_{\text{ét}}$ factorizes through $\text{EU}_\Gamma$. Indeed, Proposition 5.4.1 states that the diagram

\begin{equation}
\begin{array}{ccc}
K(VF) & \xrightarrow{\text{EU}_\Gamma} & !K(\text{RES})/([A_1] - 1) \\
\downarrow{\text{EU}_{\text{ét}}} & & \downarrow{\text{Eu}_{\text{ét}}} \\
K(\hat{\mu}-\text{Mod}) & \xleftarrow{\text{Eu}_{\text{ét}}} & K(\hat{\mu}-\text{Mod})
\end{array}
\end{equation}

is commutative, with $\text{Eu}_{\text{ét}}$ corresponding to the cohomology with compact supports on $K(\text{Var}_\mathbb{C}; \hat{\mu})$. In this way, one is able to reduce the proof of Theorem 1.1.1 to
the Lefschetz fixed point theorem for finite order automorphisms acting on complex algebraic varieties.

1.3. In section 7, we explain the connexion between the morphism $EU_\Gamma$ and the motivic Serre invariant of [19]. We show in Proposition 7.2.1 that if $X$ is a smooth proper algebraic variety over $F((t))$ with $F$ a field of characteristic zero, with base change $X(m)$ over $F((t^{1/m}))$, then the motivic Serre invariant $S(X(m))$ can be expressed in terms of the part of $EU_\Gamma(X)$ fixed by the $m$-th power of a topological generator of $\hat{\mu}$. This allows in particular to provide a proof circumventing the use of resolution of singularities of a fixed point theorem originally proved by Nicaise and Sebag in [22]. Section 8 contains additional results of related interest. We show how one can recover the motivic zeta function and the motivic Milnor fiber from a single class in the measured Grothendieck semiring of definable objects over $VF$, namely the class of the set $X_x$ of points $y$ in $X(\mathbb{C}[[t]])$ such that $rvf(y) = rv(t)$ and $y(0) = x$.

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2. Preliminaries on Grothendieck rings of definable sets, after [15]

2.1. We shall consider the theory ACVF(0,0) of algebraically closed valued fields of equal characteristic zero, with two sorts VF and RV. The language on VF is the ring language, and the language on RV consists of abelian group operations · and $(\cdot)^{-1}$, a unary predicate $k^\times$ for a subgroup, an operation $+ : k^2 \to k$, where $k$ is $k^\times$ augmented by a symbol zero, and a function symbol $rv$ for a function $VF^\times \to RV$.

Let $L$ be a valued field, with valuation ring $O_L$ and maximal ideal $M_L$. We set $VF(L) = L$, $RV(L) = L^\times/1 + M_L$, $\Gamma(L) = L^\times/O_L^\times$ and $k(L) = O_L/M_L$. We have an exact sequence

\[(2.1.1) \quad 0 \to k^\times \to RV \to \Gamma \to 0,\]
but be shall view $\Gamma$ as an imaginary sort. We denote by $rv : VF \to RV$, $val : VF \to \Gamma$ and $val_v : RV \to \Gamma$ the natural maps.

2.2. Fix a base structure $L_0$ which is a valued field. For each $\gamma \in \mathbb{Q} \otimes \Gamma(L_0)$, we consider the definable set (a one-dimensional $k$-vector space)

\[(2.2.1) \quad K \mapsto V_{\gamma} = \{0\} \cup \{x \in K; val(x) = \gamma\}/(1 + \mathcal{M}_K)\]
on valued field extensions $K$ of $L_0$. Note that when $\gamma - \gamma' \in \Gamma(L_0)$, $V_{\gamma}$ and $V_{\gamma'}$ are definably isomorphic. For $\gamma = (\gamma_1, \ldots, \gamma_n) \in (\mathbb{Q} \otimes \Gamma(L_0))^n$ we set $V_{\gamma} = \prod_i V_{\gamma_i}$. By a $\gamma$-weighted monomial, we mean an expression $a_{\nu}X^{\nu} = a_{\nu} \prod_i X_i^{\nu_i}$, with $\nu = (\nu_1, \ldots, \nu_n) \in \mathbb{N}^n$ a multi-index, such that $a_{\nu}$ is an $L_0$-definable element of $RV$ with $val_v(a_{\nu}) + \sum_i \nu_i \gamma_i = 0$. A $\gamma$-polynomial is a finite sum of $\gamma$-weighted monomials. Such a $\gamma$-polynomial $H$ gives rise to a function $H : V_{\gamma} \to k$ so we can consider its zero set $Z(H)$. The intersection of finitely such sets is called a generalized algebraic variety over the residue field. The generalized residue structure $RES$ consists of the residue field, together with the collection of one-dimensional vector spaces $V_{\gamma}$, for $\gamma \in \mathbb{Q} \otimes \Gamma(L_0)$ and the functions $H : V_{\gamma} \to k$ associated to each $\gamma$-polynomial.

2.3. We denote by $VF[n]$ the category of definable subsets of $n$-dimensional varieties over $L_0$. An equivalent definition is the following (cf. [15] 3.65): objects of $VF[n]$ are exactly definable subsets $X$ of $VF^* \times RV^*$ such that there exists a definable map $X \to VF^n$ with finite fibers. We use here the notation $S^*$ to mean $S^m$, for some $m$, for a sort $S$.

We denote by $RV[n]$ the category of definable pairs $(X, f)$ with $X \subset RV^*$ and $f : X \to RV^n$ a finite-to-one definable map and by $RES[n]$ the full subcategory consisting of objects with $X$ such that $val_v(X)$ is finite (which is equivalent to the condition $X \subset RES^*$).

We shall denote by $K_+$, resp. $K$, the Grothendieck semi-ring, resp. the Grothendieck ring, of such categories as defined in [15]. The Grothendieck semi-ring $K_+(RV[n])$ is isomorphic to the Grothendieck semi-ring of definable subsets $X$ of $RV^*$ of $RV$-dimension $\leq n$, that is, such that there exists a parametrically definable map with finite fibers to $RV^n$.

We shall set

\[(2.3.1) \quad RV[\leq n] = \oplus_{0 \leq k \leq n} RV[k],\]

\[(2.3.2) \quad RV[*] = \oplus_{0 \leq n} RV[n]\]

and

\[(2.3.3) \quad RES[*] = \oplus_{0 \leq n} RES[n].\]

Also, one defines $\Gamma[n]$ as the category whose objects are definable subsets of $\Gamma^n$, and set $\Gamma[*] = \oplus_{0 \leq n} \Gamma[n]$.

The mapping $X \mapsto val_v^{-1}(X)$ induces a functor $\Gamma[n] \to RV[n]$, hence a morphism $K_+(\Gamma[n]) \to K_+(RV[n])$. We also have a morphism $K_+(RES[n]) \to K_+(RV[n])$. 
induced by the inclusion functor RES[$n$] → RV[$n$]. It is proved in Corollary 10.3 of [15] that the canonical morphism

\[(2.3.4) \quad \Psi : K_+(\text{RES}[\ast]) \otimes_{K_+(\Gamma_{\text{fin}})} K_+(\Gamma[\ast]) \longrightarrow K_+(\text{RV}[\ast]) \]

is an isomorphism, with $K_+(\Gamma_{\text{fin}})$ the Grothendieck semi-ring of finite definable subsets of $\Gamma$.

2.4. One defines

\[(2.4.1) \quad L : \text{ObRV}[n] \longrightarrow \text{ObVF}[n] \]

by sending $(X, f)$ to

\[(2.4.2) \quad L(X, f) = \{(y_1, \ldots, y_n, x) \in (\text{VF}^\times)^n \times X, (\text{rv}(y_i)) = f(x)\}. \]

This mapping induces a morphism of filtered semi-rings

\[(2.4.3) \quad \Lambda : K_+(\text{RV}[\ast]) \longrightarrow K_+(\text{VF}) \]

sending the class of an object $X$ of RV[$n$] to the class of $L(X)$. By Theorem 8.8 of [15], $\Lambda$ is surjective with kernel the semi-ring congruence $I_{sp}$ generated by $[1]_1 \sim [1]_0 + [\text{RV}^{>0}]_1$, with $[1]_1$ and $[1]_0$ the class of the point 1 in $K_+(\text{RV}[1])$ and $K_+(\text{RV}[0])$, respectively, and $[\text{RV}^{>0}]_1$ the class of the set $\{x \in \text{RV}; \text{val}_{\text{rv}}(x) > 0\}$ in $K_+(\text{RV}[1])$.

Thus, by inverting $\Lambda$, one gets a a canonical isomorphism of filtered semi-rings

\[(2.4.4) \quad \oint : K_+(\text{VF}) \longrightarrow K_+(\text{RV}[\ast])/I_{sp}. \]

2.5. Let $I_!$ be the ideal of $K(\text{RES}[\ast])$ generated by the differences $[\text{val}_{\text{rv}}^{-1}(a)] - [\text{val}_{\text{rv}}^{-1}(0)]$ for $a$ running over $\Gamma(L_0) \otimes \mathbb{Q}$. We denote by $!K(\text{RES}[\ast])$ the quotient of $K(\text{RES}[\ast])$ by $I_!$ and by $!K(\text{RES}[n])$ its graded pieces.

We shall now recall the construction of group morphisms

\[(2.5.1) \quad \mathcal{E}_n \text{ and } \mathcal{E}_n' : K(\text{RV}[\leq n])/I_{sp} \longrightarrow !K(\text{RES}[n]) \]

given in Theorem 10.5 of [15].

The morphism $\mathcal{E}_n$ is induced by the group morphism

\[(2.5.2) \quad \gamma : \bigoplus_{m \leq n} K(\text{RV}[m]) \rightarrow !K(\text{RES}[n]) \]

given by

\[(2.5.3) \quad \gamma = \sum_m \beta_m \circ \chi[m], \]

with $\beta_m : !K(\text{RES}[m]) \rightarrow !K(\text{RES}[n])$ given by $[X] \mapsto [X \times \mathbb{A}^{n-m}]$ and $\chi[m] : K(\text{RV}[m]) \rightarrow !K(\text{RES}[m])$ defined as follows. The isomorphism (2.3.4) induces an isomorphism

\[(2.5.4) \quad K(\text{RV}[m]) \simeq \bigoplus_{1 \leq \ell \leq m} K(\text{RES}[m - \ell]) \otimes_{K(\Gamma_{\text{fin}})} K(\Gamma[\ell]), \]

and $\chi[m]$ is defined as $\bigoplus_{1 \leq \ell \leq m} \chi_\ell$ with $\chi_\ell$ sending $a \otimes b$ in $K(\text{RES}[m - \ell]) \otimes_{K(\Gamma_{\text{fin}})} K(\Gamma[\ell])$ to $\chi(b)[G_m]^\ell \cdot a$, where $\chi : K(\Gamma[\ell]) \rightarrow \mathbb{Z}$ is the o-minimal Euler characteristic (cf. Lemma 9.5 of [15]). The definition of $\mathcal{E}_n'$ is similar, replacing $\beta_m$ by the identity
and \( \chi \) by the “bounded” Euler characteristic \( \chi' \), given by \( \chi'(Y) = \lim_{r \to \infty} \chi(Y \cap [-r, r]^n) \) for \( Y \) a definable subset of \( \Gamma^n \).

We will now consider \( K(\text{RES}[n]) \) modulo \([\mathbb{A}^1] - 1\). By the formulas in [15] 10.5 (1) and (3) the morphisms \( \mathcal{E}_n \) and \( \mathcal{E}'_n \) coincide modulo \([\mathbb{A}^1] - 1\), thus they induce the same morphism

\[
E_n : K(\text{RV}[\leq n]) / I_{sp} \longrightarrow !K(\text{RES}[n]) / ([\mathbb{A}^1] - 1).
\]

These morphisms are compatible, thus passing to the limit one gets a morphism

\[
E : K(\text{RV}[\ast]) / I_{sp} \longrightarrow !K(\text{RES}) / ([\mathbb{A}^1] - 1).
\]

In fact, the morphism \( E \) is induced from both the morphisms \( \mathcal{E} \) and \( \mathcal{E}' \) from [15] 10.5 (2) and (4).

The morphism \( E \) maps \( [\text{RV} > 0]_1 \) to 0, and \([X]_k \) to \([X]_k \) for \( X \in \text{RES}[k] \). Composing \( E \) with the morphism \( K(\text{VF}) \to K(\text{RV}[\ast]) / I_{sp} \) obtained by groupification of the morphism \( f \) in (2.4.4) one gets a ring morphism

\[
\text{EU}_\Gamma : K(\text{VF}) \longrightarrow K(\text{RES}) / ([\mathbb{A}^1] - 1).
\]

2.6. The rest of this section is not really needed; it shows however that the introduction of Euler characteristics for \( \Gamma \) can be bypassed in the construction of \( \text{EU}_\Gamma \).

Let \( \text{val} = \text{val}_\nu \) denote the canonical map \( \text{RV} \to \Gamma \). Let \( I'_\Gamma \) be the ideal of \( K(\text{RV}[\ast]) \) generated by all classes \([\text{val}^{-1}(U)]_t\), for any definable \( U \subset \Gamma^m \), \( m \geq 1 \), and let \( I_\ast \) be generated by \( I'_\Gamma \) along with \( I_{sp} \). Since \([\text{RV} > 0] \in I'_\Gamma \), the canonical generator \([\text{RV} > 0]_1 + [1]_0 - [1]_1 \) reduces, modulo \( I'_\Gamma \), to \([1]_0 - [1]_1 \), i.e. the different dimensions are identified. Thus \( K(\text{RV}[\ast]) / I_\ast = K(\text{RV}) / I_{\Gamma} \), where on the right we have the ideal of \( K(\text{RV}) \) generated by all classes \([\text{val}^{-1}(U)] \), for any definable \( U \subset \Gamma^m \), \( m \geq 1 \).

**Lemma 2.6.1.** — The inclusion functor \( \text{RES} \to \text{RV} \) induces an isomorphism \( !K(\text{RES}) / ([\mathbb{A}^1] - 1) \to K(\text{RV}) / I_{\Gamma} \).

**Proof.** — This is already true even at the semiring level, as follows from Proposition 10.2 of [15]. The elements \([\text{val}^{-1}(U)] \) of \( K_+(\text{RV}) \) are those of the form \( 1 \otimes b \) in the tensor product description, with \( b \in K_+(\Gamma[\nu]) \), \( n \geq 1 \). Factoring out the tensor product \( K_+(\text{RES}) \otimes K_+(\Gamma[\ast]) \) by these relations we obtain simply \( K_+(\text{RES}) \otimes K_+(\Gamma[0]) = K_+(\text{RES}) \). Now taking into account the relations of the tensor product amalgamated over \( K_+(\Gamma^{\text{lin}}) \), namely \( 1 \otimes [\gamma]_1 = [\text{rv}^{-1}(\gamma)] \otimes [1]_0 \), as the left hand side vanishes, we obtain the relation \([\text{rv}^{-1}(\gamma)] = 0 \). These are precisely the relations defining \( !K(\text{RES}) \) (namely \([\text{rv}^{-1}(\gamma)] = [\text{rv}^{-1}(\gamma')] \)) along with the relation \( \text{rv}^{-1}(0) = 0 \) (i.e. \([\mathbb{A}^1] - 1 = 0 \)). \( \square \)

**Remark 2.6.2.** — It is also easy to compute that the map

\[
\mathcal{E} : K(\text{RV}[\ast]) / I_{sp} \longrightarrow !K(\text{RES}) / ([\mathbb{A}^1] - 1]
\]

from [15], Theorem 10.5, composed with the natural map \( !K(\text{RES}) / ([\mathbb{A}^1] - 1) \to !K(\text{RES}) / ([\mathbb{A}^1] - 1) \), induces the retraction \( K(\text{RV}) / I_{\Gamma} \to !K(\text{RES}) / ([\mathbb{A}^1] - 1) \) above.
3. Invariant admissible transformations

We continue to work in ACVF(0,0) over a base structure $L_0$ which is a valued field.

3.1. For $\alpha \in \Gamma(L_0)$, one sets $\mathcal{O}_\alpha = \{ x : \text{val}(x) \geq \alpha \}$, and $\mathcal{M}_\alpha = \{ x : \text{val}(x) > \alpha \}$. For $x = (x', x'')$, $y = (y', y'') \in \text{VF}^n \times \text{RV}^m$, write $v(x-y) > \alpha$ if $x'-y' \in (\mathcal{M}_\alpha)^n$. If $f$ is a definable function on a definable subset $X$ of $\text{VF}^n \times \text{RV}^m$, say $f$ is $\alpha$-invariant, resp. $\alpha^+$-invariant, if $f(x+y) = f(x)$ whenever $x, x+y \in X$ and $y \in (\mathcal{M}_\alpha)^n$, resp. $y \in (\mathcal{M}_\alpha)^n$. Say a definable set $Y$ is $\alpha$-invariant, resp. $\alpha^+$-invariant, if the characteristic function $1_Y : \text{VF}^n \times \text{RV}^m \rightarrow \{0, 1\}$ is $\alpha$-invariant, resp. $\alpha^+$-invariant.

Call a definable set of imaginaries non-field if it admits no definable map onto a non empty open disk (over parameters). Any of the geometric sorts other than VF has this property.

In the next Lemma “definable” means $A$-definable, where $A$ is any set of imaginaries.

**Lemma 3.1.1.** — Let $X \subset \text{VF}^n$ be a definable subset bounded and closed in the valuation topology. Let $f : X \rightarrow W$ be definable, where $W$ is a non-field set of imaginaries. Fix $\alpha \in \Gamma(L_0)$. Then there exist $\beta \geq \alpha$, a $\beta^+$-invariant $g : X \rightarrow W$ such that for any $x \in X$, for some $y \in X$, $v(x-y) > \alpha$ and $g(x) = f(y)$.

**Proof.** — We use induction on $\text{dim}(X)$. If $\text{dim}(X) = 0$ we can take $f = g$, and $\beta$ the maximum of $\alpha$ and of the maximal valuative distance between two distinct points of $X$. So assume $\text{dim}(X) > 0$.

Let us start by proving that there exists a definable subset $Y \subset X$, $\text{dim}(Y) < \text{dim}(X)$, $Y$ relatively Zariski closed in $X$, with $f$ locally constant on $X \setminus Y$. Indeed, the locus $Z$ where $f$ is locally constant is definable. If $Z$ does not contain a Zariski dense open set, then its complement contains a non empty open ball $e$. Note that on every non empty open sub-ball of $e$, $f$ is non constant. It follows that the following property holds:

$(\ast)$ the Zariski closure of $e \cap f^{-1}(w)$ is of dimension $< n$ for every $w$ in $W$.

Thus, for any model of ACVF(0,0), there exists $X' \subset \text{VF}^n$ definable bounded and closed in the valuation topology, $f' : X' \rightarrow W'$ definable with $W'$ a non-field set of imaginaries and a non empty open ball $e'$ such that $(\ast)$ holds. It follows that for $p$ large enough there exist such $X'$, $W'$, $f'$ and $e'$ defined over the algebraic closure of $\mathbb{Q}_p$ such that $(\ast)$ holds. Take a finite extension $L$ of $\mathbb{Q}_p$ over which $X'$, $W'$, $f'$ and $e'$ are defined. Since $W'(L)$ is enumerable and, by $(\ast)$, $f^{-1}(w') \cap e'(L)$ is of measure zero, for each $w' \in W'(L)$, it follows that $e'(L)$ is of measure zero, a contradiction.

For each $x$ in $X \setminus Y$, we denote by $\delta(x)$ the valuative radius of the maximal open ball around $x$ contained in $X \setminus Y$ on which $f$ is constant.

By the inductive hypothesis, there exist $\beta' \geq \alpha$, a $\beta'^+$-invariant $g_Y : Y \rightarrow W$ such that for any $y \in X$, for some $z \in Y$, $v(y-z) > \alpha$ and $g_Y(y) = f(z)$.

Let $Y' = \{ x \in X : (\exists y \in Y)(v(x-y) > \beta') \}$. One may extend $g_Y$ to a function $g'$ on $Y'$ by defining $g'(x) = g_Y(y)$ when $v(x-y) > \beta'$. By the $\beta'^+$-invariance of
3.2. Let $\beta = (\beta_1, \ldots, \beta_n) \in \Gamma^n$. Let $\text{VF}^n/\beta\mathcal{O} = \prod_{1 \leq i \leq n} (\text{VF}/\beta_i\mathcal{O})$, and let $\pi = \pi_{\beta} : \text{VF}^n \to \text{VF}^n/\beta\mathcal{O}$ be the natural map. Also write $\pi(x, y) = (\pi(x), y)$ if $x \in \text{VF}^n$ and $y \in \text{RV}^m$. Say $X \subseteq \text{VF}^n \times \text{RV}^m$ is $\beta$-invariant if it is a pullback via $\pi_{\beta}$; and that $f : \text{VF}^n \times \text{RV}^* \to \text{VF}$ is $(\beta, \alpha)$-covariant if it induces a map $\text{VF}^n/\beta\mathcal{O} \times \text{RV}^* \to \text{VF}/\alpha\mathcal{O}$, via $(\pi_{\beta}, \pi_{\alpha})$.

**Definition 3.2.1.** Fix an integer $n$. Let $\mathcal{C}(\beta)$ be the category whose objects are definable subsets $X$ of $\text{VF}^n \times W$ with $W$ a bounded definable set of imaginaries contained in $\text{RV}^m$, and $X_w$ a bounded, $\beta$-invariant subset of $\text{VF}^n$, for $w \in W$. Here $m$ is variable. The morphisms in $\mathcal{C}(\beta)$ are generated by functions of the following type:
(1) Maps

\[(x_1, \ldots, x_n, y_1, \ldots, y_m) \mapsto (x_1, \ldots, x_{i-1}, x_i + a, x_{i+1}, \ldots, x_n, y_1, \ldots, y_m)\]

with \(a = a(x_1, \ldots, x_{i-1}, x_i, y_1, \ldots, y_l)\) a \((\beta, \beta)\)-covariant \(L_0\)-definable function.

(2) Maps of the form \((x_1, \ldots, x_n, y_1, \ldots, y_l) \mapsto (x_1, \ldots, x_n, y_1, \ldots, y_l, h(x_i))\) with \(h\) an \(L_0\)-definable \(\beta\)-invariant function \(VF \to RV\).

Let \(C\) be the union of \(C(\beta)\) over all \(\beta\).

**Proposition 3.2.2.** — Let \(\alpha \in \Gamma^n\) and let \(X\) be an \(\alpha\)-invariant, bounded definable subset of \(VF^n \times RV^b\). Then for some \(\beta \geq \alpha\), \(X\) is the union of finitely many \(\beta\)-invariant definable subsets \(Z\), such that Proposition 4.5 of [15] holds, but with \(C\)-morphisms. More precisely, this means that each \(Z\) is of the following form.

There exist a morphism \(T \in C\), a definable bounded subset \(H\) of \(RV^b\) and a map \(h : \{1, \ldots, n\} \to \{1, \ldots, \ell\}\) such that

\[(3.2.2) \quad T(Z) = \{(a, b); b \in H, rv(a_i) = b_{h(i)}, 1 \leq i \leq n\} .\]

Furthermore, if the projection \(X \to VF^n\) is finite to one, the projection \(H \to RV^n\) given by \(b \mapsto (b_{h(1)}, \ldots, b_{h(n)})\) is finite to one.

**Proof.** — In [15], Lemma 4.2, if \(X\) is \(\alpha^+\)-invariant, the proof gives \(\alpha^+\)-invariant sets \(Z_i\) and transformations \(T_i\). As stated there, the RV sets \(H_i\) are bounded below (since \(X\) is bounded). Since \(X\) is also \(\alpha^+\)-invariant, the sets \(H_i\) (being pullbacks from RV) must be bounded above as well. Next, given a definable map \(\pi : X \to U\), with \(X, U\) bounded and \(X, U, \pi\) all \(\alpha^+\)-invariant, we obtain a partition and transformations of \(X\) over \(U\), such that each fiber becomes an RV-pullback, and each piece of each fiber is \(\alpha^+\)-invariant. Note that the fiber above \(u\) depends only on \(u + (M\alpha)^n\). Note also that \(U\), being \(\alpha^+\)-invariant, is clopen in the valuation topology. Using Lemma 3.1.3, we may modify the partition and the admissible transformations so as to be \(\beta^+\)-invariant, for some \(\beta \geq \alpha\). With this, the inductive proof of [15], Proposition 4.5 goes through to give the invariant result.

\[\square\]

4. Working over \(F((t))\)

4.1. We now work over the base field \(L_0 = F((t))\), with \(F\) a trivially valued algebraically closed field of characteristic zero and \(val(t)\) positive and denoted by 1. Then the sorts of RES are the \(k\)-vector spaces \(V_{k/n} = \{x \in RV : val_v(x) = k/n\} \cup \{0\}\). Since we have a definable bijection \(V_{k/m} \to V_{(k+m)/m}\) given multiplication by \(rv(t)\), it suffices to consider \(V_{k/m}\) with \(0 \leq k < m\).

The group \(\hat{\mu} = \varprojlim \mu_n\) of roots of unity acts on RES by automorphisms. On \(V_{k/n}\), a primitive \(n\)-th root of 1, say \(\zeta\), acts by multiplication by \(\zeta^k\). We have an induced action on \(K(RES)\). The classes \([V_{k/N}]\) are fixed by this action; and so an action is induced on \(K(RES)\).

Given a positive integer \(m\), let \(RES_m\) denote the sorts of RES fixed by \(m\hat{\mu}\), the kernel of \(\hat{\mu} \to \mu_m\), namely, \(V_{k/m}\) for \(k \in \mathbb{Z}\).
Intersection with RES\(m\) provides a canonical morphism
\[
\Delta_m : K_+(\text{RES}) \rightarrow K_+(\text{RES}_m)
\]
inducing
\[
\Delta_m : !K(\text{RES})/([A^1] - 1) \rightarrow !K(\text{RES}_m)/([A^1] - 1),
\]
where \(!K(\text{RES}_m)\) is defined similarly as \(!K(\text{RES})\). One denotes by \(\text{EU}_{\Gamma,m}\) the morphism
\[
\text{EU}_{\Gamma,m} : K(\text{VF}) \rightarrow !K(\text{RES}_m)/([A^1] - 1)
\]
obtained by composing \(\text{EU}_{\Gamma}\) in (2.5.7) and \(\Delta_m\) in (4.1.2).

The following statement is straightforward:

**Lemma 4.1.1.** — Let \(X\) be a definable subset of \(\text{VF}^r\) such that \(\text{EU}_{\Gamma}([X]) = [Y]\), with \(Y\) a definable subset of \(\text{RES}^n\). Then \(\text{EU}_{\Gamma,m}([X])\) is the class of the subset of \(Y\) fixed by \(m\).

4.2. The field \(K_m = F((t^{1/m}))\) does not depend on a particular choice of \(t^{1/m}\), and \(\mu_m\) acts on it. Let \(\beta \in \mathbb{Z}^n \subset \Gamma^n\), and let \(X \subset \text{VF}^n \times \text{RV}^\ell\) be a \(\beta\)-invariant, bounded \(K\)-definable set. Thus, \(X_t\) is \(\beta\)-invariant for each \(t \in \text{RV}^\ell\), the projection of \(X\) to \(\Gamma^\ell\) is contained in a cube \([-a, a]^\ell\), and the projection of \(X\) to \(\text{VF}^n\) is contained in \(c\mathcal{O}^n\) for some \(c\). For notational simplicity, and since this is what we will use, we shall assume \(X \subset \mathcal{O}^n \times \text{RV}^\ell\).

Then the \(K_m\)-points \(X(K_m)\) are the pullback of some subset \(X[m; \beta] \subseteq \prod_{i=1}^n F[t^{1/m}]/t^{\beta_i} \times \text{RV}^\ell\); and the projection \(X[m; \beta] \rightarrow \mathcal{O}^n\) has finite fibers.

We can identify \(F[t^{1/m}]/t^N\) with \(\oplus_{0 \leq k < m} V_{k/m} \cong \oplus_{0 \leq k < m} V_{k/m}^N\). Also, if \(Y\) is definable in \(\text{RV}\) and \(\text{val}_t(Y) \subset [-\alpha, \alpha]\), then
\[
Y(F((t^{1/m}))) \subset \cup \{V_\gamma : \gamma \in (1/m)\mathbb{Z} \cap [-\alpha, \alpha]\}.
\]
Thus \(X[m; \beta]\) can be viewed as a subset of the structure \(\text{RES}_m\) (over \(F\)). Here are two ways to see it is definable. The first one is to say it is definable in \((F((t^{1/m})), t)\); the induced structure on the sorts \(V_{k/m}\) is the same as the structure induced from \(\text{ACVF}\). The second one is to remark that after finitely many invariant canonical transformations, \(X\) becomes a set in standard form, a pullback from \(\Gamma\) or from \(\text{RES}\). These operations induces quantifier free-definable maps on the sets \(X[m; \beta]\); so it suffices to take \(X\) in standard form, and then the statement is clear.

For \(\beta\) and \(\beta'\) in \(\mathbb{N}\), with \(\beta \leq \beta'\), and a \(\beta\)-invariant \(X\), it is clear that
\[
[X[m; \beta']] = [X[m; \beta]] \times [A^{nm(\beta'-\beta)}]
\]
in \(!K(\text{RES})\). Thus \([X[m; \beta]]/[A^{nm\beta}]\) is a well-defined class in the localization \(!K(\text{RES})[[A^1]^{-1}]\), which we denote by \(\tilde{X}[m]\). We will also use the image of this class in \(!K(\text{RES})/([A^1] - 1)\) which we denote by \(X[m]\); both \(\tilde{X}[m]\) and \(X[m]\) depend only on \(X\) and not on any particular choice of \(\beta\).

If \(X \subset \mathcal{O}^n \times \text{RV}^\ell\) is a \(\beta\)-invariant \(K\)-definable set for some \(\beta\), then it is also \(\beta\)-invariant for some \(\beta' \in \mathbb{Z}^n\), so \(\tilde{X}[m]\) and \(X[m]\) are defined.
Let $X \subset \mathcal{O}^n \times RV^\ell$ be a $\beta$-invariant $K$-definable set, and let $f : X \to Y$ be a $\beta$-invariant canonical bijection in $C(\beta)$. Note using Corollary 3.1.2 (2) that $Y$ will remain bounded. Moreover, $f$ induces a bijection $X[m; \beta] \to Y[m; \beta]$. Hence $\bar{X}[m] = \bar{Y}[m]$ and $X[m] = Y[m]$.

**Proposition 4.2.1.** — Let $X$ be a $\beta$-invariant $F((t))$-definable subset of $\mathcal{O}^n \times RV^\ell$, for some $\beta$. Assume the projection $X \to VF^n$ has finite fibers. Then $EU_{\Gamma,m}(X) = X[m]$ as classes in $!K(\text{RES}_m)/([A^1] - 1)$.

**Proof.** — Since both sides are invariant under the transformations of Proposition 3.2.2, we may assume by Proposition 3.2.2 that there exists a definable bounded subset $H$ of $RV^\ell$ and a map $h : \{1, \ldots, n\} \to \{1, \ldots, \ell\}$ such that

$$X = \{(a, b); b \in H, \text{rv}(a_i) = b_h(i), 1 \leq i \leq n\}$$

and the map $r : H \to RV^n$ given by $b \mapsto (b_{h(1)}, \ldots, b_{h(n)})$ is finite to one. According to (2.3.4) we may assume $[H] = \Psi([W] \otimes [\Delta])$ with $W$ in $\text{RES}[\ell]$ and $\Delta$ bounded in $\Gamma[n - \ell]$. By induction on dimension and considering products, it enough to prove the result when $X$ is the lifting of an object of $\Gamma$ or $\text{RES}$. Let us prove that the image of the canonical lift from $\Gamma$ vanishes for both invariants. In the case of $EU_{\Gamma,m}$, the Euler characteristic of any $Z \subset \Gamma^q$ equals the Euler characteristic of some finite $Z_0$; the lift to $K(VF)$ of $Z_0$ vanishes modulo $[A^1] - 1$. In the case of $X[m]$, finitely many points of the value group of $K_m$ in the cube $[0, N]^n$ lie in $Z$; again for each such point, the class of $!K(\text{RES})$ lying above it is divisible by $[A^1] - 1$. On the other hand on $\text{RES}$, both $EU_{\Gamma,m}$ and $X[m]$ correspond to intersection with $\text{RES}_m$. □

**Corollary 4.2.2.** — Let $X$ be a smooth variety over $F$, $f$ a regular function on $X$ and $x$ a closed point of $f^{-1}(0)$. Let

$$X_{t,x} = \{y \in X(\mathcal{O}); f(y) = t \text{ and } \pi(y) = x\}$$

and let

$$X_x = \{y \in X(\mathcal{O}); \text{rv}(f(y)) = \text{rv}(t) \text{ and } \pi(y) = x\}.$$

Then $X_x$ is $\beta$-invariant for $\beta > 0$, and $EU_{\Gamma,m}(X_{t,x}) = X_x[m]$ as classes in $!K(\text{RES})/([A^1] - 1)$.

**Proof.** — The $\beta$-invariance of $X_x$ is clear. Consider the canonical morphism

$$\tilde{\phi} : K_+(VF) \to K_+(RV[\ast])/I_{sp}$$

of (2.4.4). For any $t'$ with $\text{rv}(t') = \text{rv}(t)$, we have

$$\text{tp}(t'/\text{rv}(t), RV) = \text{tp}(t/\text{rv}(t), RV)$$

and

$$\tilde{\phi}[X_{t',x}] = \tilde{\phi}[X_{t,x}].$$

It follows that

$$\tilde{\phi}([X_x]) = \tilde{\phi}[\bigcup X_{t,x}; \text{rv}(t') = \text{rv}(t)] = (\tilde{\phi}[X_{t,x}]) \cdot e,$$

(4.2.4)
where $e$ is the class of an open ball, i.e. $e = [1]$. Applying $EU_{\Gamma}$ we find that
$EU_{\Gamma}(X_e) = EU_{\Gamma}(X_{t,e})$, and the statement follows from Proposition 4.2.1.

4.3. Consider a sequence $t_m \in F((t))^\text{alg}$, $m > 0$, such that $t_1 = t$ and $t_{nm} = t_n$ for every $n$. Set $t_{k/m} = t^m_{k/m}$. So $t_{k/m} = \text{rv}(t_{k/m}) \in V_{k/m}$. Let $X$ be a $F((t))$-definable set over $\text{RES}$. Thus $X \subset \prod_{1 \leq i \leq n} V_{k_i/m}$ for some $m$ and $k_i$'s. The $\bar{\mu}$-action on $X$ factors through a $\mu_m$-action. The image $Y$ of set $X$ by the $F((t^{1/m}))$-definable function $f(x_1, \ldots, x_n) = (x_1/t_{a_1}, \ldots, x_n/t_{a_n})$ is an $F$-definable subset of $k^n$ and is endowed with a $\mu_m$-action coming from the one on $X$. Recall that an $F$-definable subset of $k^n$ is nothing but a constructible subset of $\mathbb{A}_k^n$. Let $K_+(\text{Var}_F; \bar{\mu})$ denote the Grothendieck semi-ring of $F$-varieties with $\bar{\mu}$-action, and $K(\text{Var}_F; \bar{\mu})$ the corresponding ring. Here we impose only the standard cut and paste relations in the definition of $K_+(\text{Var}_F; \bar{\mu})$ and $K(\text{Var}_F; \bar{\mu})$, and not the additional relation considered in [10] and [18]. In this way one gets a morphism of Grothendieck semi-rings

\begin{equation}
\Theta : K_+(\text{RES}) \rightarrow K_+(\text{Var}_F; \bar{\mu}).
\end{equation}

We denote by $!K(\text{Var}_F; \bar{\mu})$ the quotient of $K(\text{Var}_F; \bar{\mu})$ obtained by identifying $[G_m, \rho]$ for the various actions of $\bar{\mu}$ on $G_m$ by multiplication by roots of 1.

**Proposition 4.3.1.** — The morphism $\Theta$ induces an isomorphism

\begin{equation}
\Theta : !K(\text{RES}) \rightarrow !K(\text{Var}_F; \bar{\mu}).
\end{equation}

**Proof.** — Follows directly from the statement and the proof of Lemma 10.7 of [15].

5. Étale Euler characteristics with compact supports

5.1. Étale cohomology with compact supports of semi-algebraic sets. — Let $K$ be a complete non-archimedean normed field. Let $X$ be an affine algebraic variety over $K$ and write $X^{an}$ for its analytification in the sense of Berkovich. A semi-algebraic subset of $X^{an}$, in the sense of [13], is a subset of $X^{an}$ defined by a finite Boolean combination of inequalities $|f| > \lambda |g|$ with $f$ and $g$ regular functions on $X$ and $\lambda \in \mathbb{R}$.

We denote by $\overline{K}$ the completion of a separable closure of $K$ and by $G$ the Galois group $\text{Gal}(\overline{K}/K)$. We set $X^{an} = X^{an} \otimes K$ and for $U$ a semi-algebraic subset of $X$ we denote by $\overline{U}$ the preimage of $U$ in $X^{an}$ under the canonical morphism $X^{an} \rightarrow X^{an}$. Let $\ell$ be a prime number different from the residue characteristic of $K$.

Let $U$ be a locally closed semi-algebraic subset of $X^{an}$. For any finite torsion ring $R$, the theory of germs in [3] provides étale cohomology groups with compact supports $H^i_{\ell}(\overline{U}, R)$ which coincide with the ones defined there when $U$ is an affinoid domain of $X^{an}$. We shall set $H^i_{\ell}(\overline{U}, \mathbb{Q}_\ell) = \mathbb{Q}_\ell \otimes \mathbb{Z}_\ell \lim \ H^i_{\ell}(\overline{U}, \mathbb{Z}/\ell^n)$.

We shall use the following properties of the functor $U \mapsto H^i_{\ell}(\overline{U}, \mathbb{Q}_\ell)$. Detailed proofs of Properties (1) and (2) are provided in [20], while (3) follows directly from the Künneth Formula in [3], Corollary 7.7.3.
Proposition 5.1.1. — Let $U$ be a locally closed semi-algebraic subset of $X^{an}$, with $X$ of dimension $d$.

1. The groups $H^i_c(U, \mathbb{Q}_\ell)$ are finite dimension $\mathbb{Q}_\ell$-vector spaces, endowed with a $G$-action, and $H^i_c(U, \mathbb{Q}_\ell) = 0$ for $i > 2d$.
2. If $V$ is a semi-algebraic subset of $U$ and open in $U$ with complement $F$, there is a functorial long exact sequence

\[ \rightarrow H^{i-1}_c(F, \mathbb{Q}_\ell) \rightarrow H^i_c(U, \mathbb{Q}_\ell) \rightarrow H^i_c(V, \mathbb{Q}_\ell) \rightarrow H^i_c(F, \mathbb{Q}_\ell) \rightarrow \]

3. There are canonical Künneth isomorphisms

\[ \bigoplus_{i+j=n} H^i_c(U, \mathbb{Q}_\ell) \times H^j_c(V, \mathbb{Q}_\ell) \simeq H^n_c(U \times V, \mathbb{Q}_\ell). \]

Remark 5.1.2. — We shall only make use of Proposition 5.1.1 when $X = \mathbb{A}^n$ and $K = k((t))$ with $k$ a field of characteristic zero.

5.2. Definition of $EU_{\text{ét}}$. — We denote by $G^{-}\text{Mod}$ the category of $\mathbb{Q}_\ell[G]$-modules which are finite dimensional as $\mathbb{Q}_\ell$-vector spaces and by $K(G^{-}\text{Mod})$ the corresponding Grothendieck ring. Let $K$ be a valued field endowed with a rank one valuation, that is, with $\Gamma(K) \subset \mathbb{R}$. We can consider the norm $\exp(-\text{val})$ on $K$. Let $U$ be an ACVF$_K$-definable subset of $\text{VF}^n$. By quantifier elimination it is defined by a finite Boolean combination of inequalities $\text{val}(f) > \text{val}(g) + \alpha$ with $f$ and $g$ polynomials and $\alpha$ in $\Gamma(K) \otimes \mathbb{Q}$. Thus, after exponentiating, one can attach canonically to $U$ a semi-algebraic subset $U^{an}$ of $(\mathbb{A}^n_K)^{an}$, for $K$ the completion of $K$, and also a semi-algebraic subset $\overline{U^{an}}$ of $\mathbb{A}^n_K$. When $U^{an}$ is locally closed, we define $EU_{\text{ét}}(U)$ as the class of

\[ \sum_i (-1)^i [H^i_c(\overline{U^{an}}, \mathbb{Q}_\ell)] \]

in $K(G^{-}\text{Mod})$. That this makes sense follows from (1) of Proposition 5.1.1.

Lemma 5.2.1. — Let $U$ be an ACVF$_K$-definable subset of $\text{VF}^n$. Then there exists a finite partition of $U$ into ACVF$_K$-definable subsets $U_i$ such that each $U_i^{an}$ is locally closed.

Proof. — The set $U$ is the union of sets $U_i$ defined by conjunctions $\text{val}(f) < \text{val}(g)$, $f = 0$, or $\text{val}(f) = \text{val}(g)$, with $f$ and $g$ polynomials. Since the intersection of two locally closed sets is locally closed, it suffices to show that each of these basic forms gives a locally closed set. Since $|f|$ is a continuous function for the Berkovich topology with values in $\mathbb{R}_{\geq 0}$, the sets corresponding to $f = 0$ and $\text{val}(f) = \text{val}(g)$ are closed, as well as $\text{val}(f) \leq \text{val}(g)$. The remaining kind of set, $\text{val}(f) < \text{val}(g)$, is the difference between $\text{val}(f) \leq \text{val}(g)$ and $\text{val}(f) = \text{val}(g)$, so is the corresponding set is locally closed.

Proposition 5.2.2. — There exists a unique ring morphism

\[ EU_{\text{ét}} : K(\text{VF}) \rightarrow K(G^{-}\text{Mod}) \]
such that $\text{EU}_\text{ét}([U]) = \text{EU}_\text{ét}(U)$ when $U^{\text{an}}$ is locally closed.

Proof. — Let $U$ be an ACVF $K$-definable subset of $VF^n$. Choose a partition $U_i$ as in Lemma 5.2.1. and set $\text{EU}_\text{ét}(U) = \sum_i \text{EU}_\text{ét}(U_i)$. This is independent of the choice of the partition $U_i$. Indeed, if $U'_j$ is a finer such partition with $(U'_j)^{\text{an}}$ locally closed, $\sum_i \text{EU}_\text{ét}(U_i) = \sum_j \text{EU}_\text{ét}(U'_j)$ by (2) in Proposition 5.1.1, and two such partitions always have a common refinement. Note that $\text{EU}_\text{ét}(U)$ depends only of the isomorphism class of $U$ as a definable set. Indeed, when $f$ is a polynomial isomorphism $f: U \to U''$ (with inverse given by a polynomial function), and $U^{\text{an}}$ and $(U'')^{\text{an}}$ are locally closed, this is clear by functoriality of $H^*_\ell$, and in general one can reduce to this case by taking suitable partitions $U_i$ and $U'_j$ of $U$ and $U'$. The fact that $\text{EU}_\text{ét}$ satisfies the additivity relation and is unique is clear using again (2) in Proposition 5.1.1. For existence, it remains to prove $\text{EU}_\text{ét}$ is multiplicative which follows from (3) in Proposition 5.1.1. Unicity is clear. \hfill \square

5.3. Definition of $\text{Eu}_\text{ét}$. — We now assume again and in §5.4 that $K = F((t))$ with $F$ algebraically closed of characteristic zero. Thus $G$ may be identified with $\hat{\mu}$. Let $X$ be an $F$-variety endowed with a $\hat{\mu}$-action. The $\ell$-adic étale cohomology groups $H^*_\ell(X, \mathbb{Q}_\ell)$ are endowed with a $\hat{\mu}$-action, and we may consider the element

\begin{equation}
\text{Eu}_\text{ét}(X) := \sum_i (-1)^i [H^i_\ell(X, \mathbb{Q}_\ell)]
\end{equation}

in $K(\hat{\mu}-\text{Mod})$. Recall the isomorphism

\begin{equation}
\Theta : !K(\text{RES}) \to !K(\text{Var}_F; \hat{\mu})
\end{equation}

of Proposition 4.3.1. The morphism $\text{Eu}_\text{ét} \circ \Theta$ factors through $!K(\text{RES})/([\mathbb{A}^1] - 1)$ and gives rise to a morphism

\begin{equation}
\text{Eu}_\text{ét} : !K(\text{RES})/([\mathbb{A}^1] - 1) \to K(\hat{\mu}-\text{Mod}).
\end{equation}

5.4. Compatibility. — We have the following basic compatibility property between $\text{EU}_\text{ét}$ and $\text{Eu}_\text{ét}$.

**Proposition 5.4.1.** — The diagram

\begin{equation}
\begin{array}{ccc}
K(VF) & \xrightarrow{\text{EU}_r} & !K(\text{RES})/([\mathbb{A}^1] - 1) \\
\text{EU}_\text{ét} & & \text{Eu}_\text{ét} \\
& K(\hat{\mu}-\text{Mod}) &
\end{array}
\end{equation}

is commutative.

Proof. — It is enough to prove that if $X$ is a definable subset of $VF^n$, then $\text{EU}_\text{ét}(X) = \text{Eu}_\text{ét}(\text{EU}_r([X]))$. Using the notations of (2.4.1) and the isomorphism (2.3.4), we may assume the class of $X$ in $K_+(VF^n)$ is of the form $L(\Psi(a \otimes b))$ with $a$ in $K_+(\text{RES}[m])$ and $b$ in $K_+(\Gamma[r])$. 

If $r \geq 1$, $\text{EU}_\Gamma([X]) = 0$ by construction of $\text{EU}_\Gamma$. Thus, we have to prove that $\text{EU}_\text{ét}(X) = 0$. By multiplicativity of $\text{EU}_\text{ét}$, it is enough to prove that $\text{EU}_\text{ét}(\mathbb{L}(b)) = 0$, which is given by Lemma 5.4.2.

Thus, we may assume $r = 0$ and $[X] = \mathbb{L}(\Psi([Z] \otimes 1))$, with $Z$ a definable subset in $\text{RES}[n]$. Assume first $Z$ is definable over the residue field sort $k$. Thus $Z$ is given with a definable map with finite fibers $f : Z \to \mathbb{A}^n_F$. By additivity, we may assume $Z$ is a smooth variety of dimension $\leq n$ over $F$. In fact, we may even assume $Z$ is smooth of pure dimension $n$. Indeed, if $Z$ is of dimension $\leq n - 1$, since the construction of $\mathbb{L}$ on $\text{RES}[n]$ does not depend of $f$, we may, after maybe replacing $Z$ by a dense Zariski open, replace $f$ by $i \circ g$, with $g$ given by $n - 1$ coordinates such that $g : Z \to \mathbb{A}^n_F$ is finite and $i : \mathbb{A}^{n-1}_F \to \mathbb{A}^n_F$ the inclusion. Thus

\begin{equation}
\text{L}((Z, f)) = \text{L}((Z, i \circ g)) = \text{L}((Z, g)) \times \mathcal{M},
\end{equation}

hence $\text{EU}_\text{ét}([\mathbb{L}((Z, f))]) = \text{EU}_\text{ét}([\mathbb{L}((Z, g))])$ by multiplicativity, since $\text{EU}_\text{ét}([\mathcal{M}]) = 1$. Thus, assume $Z$ is a smooth $F$-variety of pure dimension $n$. Let $\mathcal{Z}$ be the formal completion of $Z \otimes F[[\ell]]$, with generic fiber $\mathcal{Z}_\eta$ and reduction map $\pi : \mathcal{Z}_\eta \to Z$. By Lemma 5.4.3, there is a canonical isomorphism

\begin{equation}
H_c^i(\pi^{-1}(\mathcal{Z}), \mathcal{O}_\mathcal{Z}) \simeq H_c^{2n-i}(Z, \mathcal{O}_\mathcal{Z}(u))^\vee,
\end{equation}

thus $\text{EU}_\text{ét}(\text{EU}_\Gamma([X])) = \text{EU}_\text{ét}([X])$.

Now consider the general case when $Z$ is a definable subset of some $\text{RES}^r$. Recall the notation from §4.3. We write $Z \subset \prod_{1 \leq i \leq n} V_{k_i/m}$ for some $m$ and $k_i$’s and consider the image $Y$ of set $Z$ by the $F((t^{1/m}))$-definable function $f(x_1, \ldots, x_r) = (x_1/t_{a_1}, \ldots, x_r/t_{a_r})$ which is an $F$-definable subset of $\mathbb{A}^r_F$ which is endowed with a $\mu_m$-action coming from the one on $\prod_{1 \leq i \leq r} V_{k_i/m}$ that is, $\zeta \cdot (y_1, \ldots, y_r) = (\zeta^{k_1} y_1, \ldots, \zeta^{k_r} y_r)$, with $(y_1, \ldots, y_r)$ the coordinates on $\mathbb{A}^r_F$. Furthermore $\mathbb{L}(Y)$ is the isomorphic image of $\mathbb{L}(Z)$ under the mapping $(X_1, \ldots, X_r) \mapsto (X_1/t_{a_1}, \ldots, X_r/t_{a_r})$. This induces an isomorphism between the cohomology groups $H^i_c(\mathbb{L}(\mathcal{Z})\text{an}, \mathcal{O}_\mathcal{Z})$ and $H^i_c(\mathbb{L}(\mathcal{Y})\text{an}, \mathcal{O}_\mathcal{Z})$, which becomes $\hat{\mu}$-equivariant since $H^i_c(\mathbb{L}(\mathcal{Y})\text{an}, \mathcal{O}_\mathcal{Z})$ is endowed with the action induced by $\zeta \cdot (x_1, \ldots, x_r) = (\zeta^{k_1} x_1, \ldots, \zeta^{k_r} x_r)$. In this way, the result follows from the former case.

\begin{lemma}
If $Z$ is a definable subset of $\Gamma^r$, with $r \geq 1$, then

\begin{equation}
\text{EU}_\text{ét}(\mathbb{L}(Z)) = 0.
\end{equation}

\end{lemma}

\begin{proof}
By quantifier elimination and cell decomposition in o-minimal structures, cf. [24], we may assume $Z$ is defined by $r$ conditions

\begin{equation}
L_i(x_1, \ldots, x_{i-1}) \square_{i,1} L'_{i}(x_1, \ldots, x_{i-1}),
\end{equation}

with $\square_{i,1}$ and $\square_{i,2}$ either $\leq$, $<$, $=$ or no symbol, $L_i$ and $L'_i$ affine linear forms in variables $x_1, \ldots, x_{i-1}$ with rational coefficients. Thus $\mathbb{L}(Z)$ is the set defined by the $r$ conditions

\begin{equation}
L_i(\text{val}(x_1), \ldots, \text{val}(x_{i-1})) \square_{i,1} \text{val}(x_i) \square_{i,2} L'_{i}(\text{val}(x_1), \ldots, \text{val}(x_{i-1})).
\end{equation}

\end{proof}
Let $Z = (\mathbb{L}(Z))^m$. Let us prove by induction on $r$ that for any finite abelian group $\Lambda$ and any $0 \leq j \leq 2r$, $H^{2r-j}_c(Z, \Lambda)$ is canonically isomorphic to $\Lambda^{(r)}(-r)$. Let $\pi$ be the restriction of the projection onto the first $r-1$ coordinates in $\Gamma^r$ to $Z$. Thus, $Z' = \pi(Z)$ is defined by the first $r-1$ conditions in (5.4.5). Let us still denote by $\pi$ the induced morphism $Z \to Z'$. By the standard computation of the cohomology of annuli, one gets that the sheaves $R^i\pi_!(\Lambda_Z)$ are zero for $i$ different from 1, 2 and are equal to the constant sheaf $\Lambda_Z(-1)$ for $i = 1, 2$. Applying induction to $Z'$ and the Leray spectral sequence 5.2.2 of [3], one gets the claim about $H^{2r-j}_c(Z, \Lambda)$. The statement follows.

**Lemma 5.4.3.** — Let $X$ be a smooth formal scheme over the valuation ring of $K$ with special fiber $X$ of pure dimension $n$ and analytic generic fiber $X_\eta$. Let $\pi : X_\eta \to X$ be the reduction map. Let $S$ be a smooth closed subvariety of $X$. Then there exists canonical isomorphisms

$$H^i_c(\pi^{-1}(S), \mathbb{Q}_\ell) \simeq H^{2n-i}_c(S, \mathbb{Q}_\ell(n))^\vee,$$

with $\vee$ standing for the dual vector space.

**Proof.** — By Corollary 2.5 of [5], for any finite torsion group $\Lambda$, we have a canonical isomorphism

$$R\Gamma_c(\pi^{-1}(S), \Lambda_{X_\eta}) \simeq R\Gamma_S(X, R\psi_\eta(\Lambda_{X_\eta})).$$

By triviality of vanishing cycles for smooth schemes, cf. Corollary 5.4 of [4], $R^q\psi_\eta(\Lambda_{X_\eta}) = 0$ for $q > 0$ and $R^0\psi_\eta(\Lambda_{X_\eta}) = \Lambda_X$, hence it follows that there are canonical isomorphisms

$$H^i_c(\pi^{-1}(S), \Lambda) \simeq H^i_S(X, \Lambda).$$

We may assume $S$ is of pure codimension $r$, hence, by purity, we have canonical isomorphisms $H^i_S(X, \Lambda) \simeq H^{i-2r}_c(S, \Lambda(-r))$, so we get canonical isomorphisms compatible with the $\hat{\mu}$-action. The result follows by Poincaré duality and passing to the limit over torsion coefficients.

**6. Proof of Theorem 1.1.1**

**6.1. Using comparison results.** — Let $X$ be a smooth complex variety and $f$ be a regular function on $X$. Let $x$ be closed point of the fiber $f^{-1}(0)$. We shall use the notation introduced in Corollary 4.2.2. Thus $\pi$ denotes the reduction map $X(\mathcal{O}) \to X(k)$, and we consider the ACVF-definable sets

$$(6.1.1) \quad X_{t,x} = \{ y \in X(\mathcal{O}); f(y) = t \text{ and } \pi(y) = x \}$$

and

$$(6.1.2) \quad X_x = \{ y \in X(\mathcal{O}); rvf(y) = rv(t) \text{ and } \pi(y) = x \}.$$
Note that $X_{t,x}$ is nothing but the underlying set of the analytic Milnor fiber $F_x$ introduced in §9.1 of [22]. Fix an embedding of $\mathbb{Q}_\ell$ in $\mathbb{C}$, and denote by $\varphi$ the canonical topological generator of $\hat{\mu}(\mathbb{C}) = \text{Gal}(\mathbb{C}((t))^a/\mathbb{C}((t)))$ given by the family $(\zeta_n)_{n \geq 1}$ with $\zeta_n = \exp(2i\pi/n)$. By Theorem 9.2 from [22], which is a consequence of the Comparison Theorem 3.5 from [5], there are canonical isomorphisms

\begin{equation}
H^i(F_x, \mathbb{C}) \simeq H^i(F_x \hat{\times} \mathbb{C}((t))^a, \mathbb{Q}_\ell) \otimes_{\mathbb{Q}_\ell} \mathbb{C}
\end{equation}

compatible with the action of $M_x$ and $\varphi$. It follows that

\begin{equation}
\Lambda(M^m_x) = \text{tr}(\varphi^m; H^\bullet(F_x \hat{\times} \mathbb{C}((t))^a, \mathbb{Q}_\ell)).
\end{equation}

By Poincaré Duality as established in §7.3 of [3], there is a perfect duality

$$H^i(F_x \hat{\times} \mathbb{C}((t))^a, \mathbb{Z}/\ell^n\mathbb{Z}) \times H^{2d-i}_c(F_x \hat{\times} \mathbb{C}((t))^a, \mathbb{Z}/\ell^n\mathbb{Z}(d)) \to \mathbb{Z}/\ell^n\mathbb{Z},$$

with $d$ the dimension of $X$, hence after taking the limit over $n$ and tensoring with $\mathbb{Q}_\ell$ one deduces that

\begin{equation}
\Lambda(M^m_x) = \text{tr}(\varphi^m; H^\bullet_c(F_x \hat{\times} \mathbb{C}((t))^a, \mathbb{Q}_\ell)).
\end{equation}

6.2. Lefschetz fixed point for finite order automorphisms. — Let us denote by $\chi_{c,\ell}$ the $\ell$-adic Euler characteristic with compact supports. The following statement is classical and follows in particular from Theorem 3.2 of [6]:

**Proposition 6.2.1.** — Let $X$ a a quasi-projective variety over an algebraically closed field of characteristic zero. Let $T$ be a finite order automorphism of $X$ with fixed point set $X^T$. Then

\begin{equation}
\chi_{c,\ell}(X^T) = \text{tr}(T; H^\bullet_c(X, \mathbb{Q}_\ell)).
\end{equation}

6.3. Proof of Theorem 1.1.1. —

**Step 1.** — By the comparison statement in (6.1.5), we have

\begin{equation}
\Lambda(M^m_x) = \text{tr}(\varphi^m; \text{EU}_\ell(X_{t,x})).
\end{equation}

**Step 2.** — By the compatibility result in (5.4.1), we have

\begin{equation}
\text{tr}(\varphi^m; \text{EU}_\ell(X_{t,x})) = \text{tr}(\varphi^m; \text{EU}_\ell(\text{EU}_\Gamma(X_{t,x}))).
\end{equation}

**Step 3.** — By the Lefschetz fixed point theorem for finite order automorphisms (Proposition 6.2.1) together with Lemma 4.1.1 we have

\begin{equation}
\text{tr}(\varphi^m; \text{EU}_\ell(\text{EU}_\Gamma(X_{t,x}))) = \chi_c(EU_{\Gamma,m}(X_{t,x})).
\end{equation}

**Step 4.** — By Corollary 4.2.2, and with the notation therein,

\begin{equation}
\chi_c(EU_{\Gamma,m}(X_{t,x})) = \chi_c(h(X_x)[m]).
\end{equation}
Step 5. — Finally, one notices that

\[ \chi_c(h(X)[m]) = \chi_c(X_{m,x}). \]

Indeed, for \( m \geq 1 \),

\[ X_{m,x} = \{ \varphi \in X(\mathbb{C}[t]/t^{m+1}); f(\varphi) = t^m \mod t^{m+1}, \varphi(0) = x \} \]

may be rewritten as

\[ \{ \varphi \in X(\mathbb{C}[t^{1/m}]/t^{(m+1)/m}); f(\varphi) = t \mod t^{(m+1)/m}, \varphi(0) = x \} \]

or as

\[ \{ \varphi \in X(\mathbb{C}[t^{1/m}]/t^{(m+1)/m}); \text{rv}(f(\varphi)) = \text{rv}(t), \varphi(0) = x \}. \]

Thus, using the isomorphism \( \Theta \) of Proposition 4.3.1, we have that the class of \( \Theta(h(X)[m]) \) in \( !K(\text{Var}_F; \hat{\mu})/([A^1] - 1) \) is equal to that of \( X_{m,x} \) and the equality of Euler characteristics with compact supports follows.

7. Trace formulas and the motivic Serre invariant

7.1. In this section \( F \) denotes a field of characteristic zero, \( K = F((t)) \), \( K(m) = F((t^{1/m})) \) and \( \bar{K} = \cup_{m \geq 1} K(m) \). When \( F \) is algebraically closed, we identify the Galois group of \( K \) with \( \hat{\mu} \) and fix a topological generator \( \varphi \) of \( \hat{\mu} \).

If \( X \) is an ACVF\( _K \)-definable or an algebraic variety over \( K \), we write \( X(m) \) and \( \bar{X} \) for the objects obtained by extension of scalars to \( K(m) \) and \( \bar{K} \), respectively. We denote by \( \Theta_0 \) the morphism

\[ \Theta_0: !K(\text{RES})/([A^1] - 1) \longrightarrow K(\text{Var}_F)/([A^1] - 1) \]

induced by the morphism \( \Theta \) of Proposition 4.3.1 and by

\[ \chi_c: K(\text{Var}_F)/([A^1] - 1) \longrightarrow \mathbb{Z} \]

the morphism induced by cohomology with compact supports, say \( \ell \)-adic.

Proposition 7.1.1. — Assume \( F \) is algebraically closed. Let \( X \) be an ACVF\(_K\)-definable subset of \( \text{VF}^n \). Then

\[ \text{tr}(\varphi^m; H^\bullet_c(\bar{X}, \mathbb{Q}_\ell)) = \chi_c(\Theta_0 \circ \text{EU}_{\Gamma,m}([X])). \]

Proof. — By (5.4.1),

\[ \text{tr}(\varphi^m; H^\bullet_c(\bar{X}, \mathbb{Q}_\ell)) = \text{tr}(\varphi^{-m}; \text{EU}_{\Gamma}([X])) \]

and by Proposition 6.2.1,

\[ \text{tr}(\varphi^{-m}; \text{EU}_{\Gamma}([X])) = \chi_c(\Theta_0 \circ \text{EU}_{\Gamma,m}([X])). \]

The result follows.
7.2. The motivic Serre invariant. — Let $R$ be a complete discrete valuation ring, with perfect residue field $F$ and field of fractions $K$. Let $X$ be a smooth quasi-compact rigid $K$-variety. In [19], using motivic integration on formal schemes, for any such $X$ a canonical class $S(X) \in K(\text{Var}_F)/([A^1] - 1)$ is constructed, called its motivic Serre invariant. If $X$ is a smooth proper algebraic variety over $K$, one sets $S(X) = S(X^\rig)$, with $X^\rig$ the rigid analytification of $X$.

We have the following comparison between the morphism $\text{EU}_\Gamma$ and the motivic Serre invariant in residue characteristic zero:

**Proposition 7.2.1.** — Let $K = F((t))$ with $F$ a field of characteristic zero. Let $X$ be a smooth proper algebraic variety over $K$. Then, for every $m \geq 1$,

$$\Theta_0(\text{EU}_{\Gamma,m}([X])) = S(X(m)). \quad (7.2.1)$$

**Proof.** — After replacing $F((t))$ by $F((t^{1/m}))$ we may assume $m = 1$. Let $X$ be a weak Néron model of $X$, cf. [19]. Consider the unique definable subset $X_1$ of $X$ such that for any valued field extension $K'$ of $K$, with valuation ring $R'$, $X_1(K')$ is the image of $X(R')$ under the canonical mapping $X(R') \to X(K')$. Let $X_\neq 1$ be the complement of $X_1$ in $X$. By the very construction of $\text{EU}_{\Gamma,1}$, $\Theta_0(\text{EU}_{\Gamma,1}([X_1])) = S(X)$. Thus it is enough to prove that $\text{EU}_{\Gamma,1}([X_\neq 1]) = \emptyset$. Since $X_\neq 1(F'(\{t\})) = \emptyset$ for every field extension $F'$ of $F$ by the Néron property of $X$, this follows from Lemma 7.2.2. \hfill \Box

**Lemma 7.2.2.** — Let $X$ be an $F((t))$-definable subset of $\text{VF}^n$. Assume that $X(F'(\{t\})) = \emptyset$ for every field extension $F'$ of $F$. Then $\text{EU}_{\Gamma,1}([X]) = \emptyset$.

**Proof.** — Using the notations of (2.4.1) and the isomorphism (2.3.4), we may assume $X$ is of the form $[X] = L(\Phi(a \otimes b))$ with $a$ in $K_+(\text{RES}[m])$ and $b$ in $K_+(\Gamma[\ell])$. If $\ell \geq 1$, $\text{EU}_\Gamma([X]) = \emptyset$ by construction of $\text{EU}_\Gamma$. Thus, we may assume $\ell = 0$ and $b = 1$. Write $a = [Z]$ with $Z$ definable in some $\text{RES}^k$. By construction $Z$ and $\text{EU}_\Gamma(X)$ coincide in $K(\text{RES})/([A^1] - 1)$. On the other hand, if $X(F'(\{t\})) = \emptyset$ for every field extension $F'$ of $F$, then $Z \cap \text{RES}^k_1 = \emptyset$. \hfill \Box

In particular, by combining Proposition 7.1.1 and Proposition 7.2.1, we obtain the following:

**Corollary 7.2.3 ([22]).** — Let $K = F((t))$ with $F$ an algebraically closed field of characteristic zero. Let $X$ be a smooth proper algebraic variety over $K$. Then, for every $m \geq 1$,

$$\text{tr}(\varphi^m; H^*_{\text{c}}(\bar{X}, \mathbb{Q}_\ell)) = \chi_c(S(X(m))). \quad (7.2.2)$$

The original proof in Corollary 5.5 [22] of Corollary 7.2.3 uses resolution of singularities, which is not the case of the proof given here.
7.3. Analytic variants. — Assume again $R$ is a complete discrete valuation ring, with perfect residue field $F$ and field of fractions $K$. In [21], the construction of the motivic Serre invariant was extended to the class of generic fibers of generically smooth special formal $R$-schemes. Special formal $R$-schemes are obtained by gluing formal spectra of quotient of $R$-algebras of the form $R\{T_1,\ldots, T_r\}[[S_1,\ldots, S_s]]$, cf. [21]. In particular, if $\mathcal{X}_η$ is such a generic fiber and $K = F((t))$ with $F$ an algebraically closed field of characteristic zero, then it follows from Theorem 6.4 of [21], generalizing Theorem 5.4 of [22], that, with the obvious notations, for every $m \geq 1$,

\begin{equation}
\text{tr}(\varphi^m; H^*_c(\mathcal{X}_η, \mathbb{Q}_l)) = \chi_c(S(\mathcal{X}_η(m))).
\end{equation}

Now, replacing the theory $\text{ACVF}(0, 0)$ by its rigid analytic variant $\text{ACVF}_R(0, 0)$ introduced by Lipshitz in [17], and assuming that the results of §5 extend to $\text{ACVF}_R(0, 0)$-definable sets, which is very likely, all the results in the previous sections would hold for $\text{ACVF}_R(0, 0)$-definable sets. In particular, Proposition 7.2.1 and Corollary 7.2.3 would hold for generic fibers of generically smooth special formal $R$-schemes. This would provide a proof of (7.3.1) which would not use resolution of singularities, unlike the proof in [21].

8. Recovering the motivic zeta function and the motivic Milnor fiber

8.1. Some notations and constructions from [16]. — Let $A$ be an ordered Abelian group and $n$ a non negative integer. We shall consider the categories $\Gamma_A[n]$, $\Gamma^\text{bdd}_A[n]$, $\text{vol} \Gamma_A[n]$ and $\text{vol} \Gamma^\text{bdd}_A[n]$ defined in Definition 2.4 of [16]. Thus, $\Gamma_A[n]$ is the category of $A$-definable subsets of $\Gamma^n$, $\Gamma^\text{bdd}_A[n]$ is the subcategory of bounded subsets, while $\text{vol} \Gamma_A[n]$ has the same objects as $\Gamma_A[n]$ with morphisms $f : X \to Y$, those morphisms in $\Gamma_A[n]$ such that $\sum x_i = \sum y_i$ whenever $(y_1, \ldots, y_n) = f(x_1, \ldots, x_n)$, $\text{vol} \Gamma^\text{bdd}_A[n]$ is the subcategory of $\text{vol} \Gamma_A[n]$ whose objects are bounded below. Finally, we denote by $\text{vol} \Gamma^2\text{bdd}_A[n]$ the subcategory of $\text{vol} \Gamma_A[n]$ whose objects are bounded on both sides.

We shall also consider the corresponding Grothendieck semi-groups $K_+(\Gamma_A[n])$, $K_+(\Gamma^\text{bdd}_A[n])$ and $K_+(\text{vol} \Gamma_A[n])$, $K_+(\text{vol} \Gamma^\text{bdd}_A[n])$. We also set $K_+(\Gamma^\text{bdd}_A[\ast]) = \bigoplus_n K_+(\Gamma^\text{bdd}_A[n])$ with associated ring $K(\Gamma^\text{bdd}_A)$, and similar notation for the measured categories.

Let $[0]_1$ denote the class of 0 in $K_+(\Gamma^\text{bdd}_A[1])$. We set

\begin{equation}
K^\text{df}_+(\Gamma^\text{bdd}_A) = (K_+(\Gamma^\text{bdd}_A[\ast])[0]_{1}^{-1})[0]_{1}^{-1},
\end{equation}

where $(K_+(\Gamma^\text{bdd}_A[\ast])[0]_{1}^{-1})[0]_{1}^{-1}$ is the homogeneous part of the graded semi-ring $K_+(\Gamma^\text{bdd}_A[\ast])[0]_{1}^{-1}$.

In Definition 3.14 of [16], given a base structure $A$, categories $\text{vol} \text{RV}^\text{bdd}_A[n]$ and $\text{vol} \text{RES}_A[n]$ are defined. One defines $\text{vol} \text{RV}^2\text{bdd}_A[n]$ as the subcategory of $\text{vol} \text{RV}^\text{bdd}_A[n]$ whose objects are bounded on both sides. Similar notation as above for the various semi-ring and rings.
We have a map
\[(8.1.2) \quad K_+(\text{volRES}[n]) \to K_+(\text{volRV}^{\text{bdd}}[n])\]
induced by inclusion and a map
\[(8.1.3) \quad K_+(\text{vol}^{\text{bdd}}[n]) \to K_+(\text{volRV}^{\text{bdd}}[n])\]
induced by \(X \mapsto \text{rv}^{-1}(X)\). By §3.4 in [16], taking the tensor product, one gets a canonical morphism
\[(8.1.4) \quad \Psi : K_+(\text{volRES}^*[\bullet]) \otimes K_+(\text{vol}^{\text{bdd}}[\bullet]) \to K_+(\text{volRV}^{\text{bdd}}[\bullet])\]
whose kernel is generated by the elements
\[(8.1.5) \quad [\text{val}^{-1}_{\text{rv}}(\gamma)]_1 \otimes 1 - 1 \otimes [\gamma]_1,\]
with \(\gamma\) in \(\Gamma\) definable. Here the subscript 1 refers to the fact that the classes are considered in degree 1. Note that (8.1.4) restricts to a morphism
\[(8.1.6) \quad \Psi : K_+(\text{volRES}^*[\bullet]) \otimes K_+(\text{vol}^{\text{bdd}}[\bullet]) \to K_+(\text{volRV}^{\text{bdd}}[\bullet]).\]

Similarly, cf. Proposition 10.10 of [15], there is a canonical morphism
\[(8.1.7) \quad \Psi : K_+(\text{volRES}^*[\bullet]) \otimes K_+(\text{vol}\Gamma[\bullet]) \to K_+(\text{volRV}[\bullet])\]
whose kernel is generated by the elements (8.1.5).

Consider the category \(\text{volVF}[n]\) of Definition 3.20 in [16] and its bounded version \(\text{volVF}^{\text{bdd}}[n]\). The mapping \(L\) lifts to a mapping \(L : \text{ObvolRV}[n] \to \text{ObvolVF}[n]\). Let \(I'_{\text{sp}}\) be the congruence generated by \([1]_1 = [\text{RV}^{>0}]_1\) with the constant volume form 0 in \(\Gamma\). By Lemma 3.21 of [16] and Theorems 8.28 and 8.29 of [15], there are canonical isomorphisms
\[(8.1.8) \quad \int : K_+(\text{volVF}[n]) \to K_+(\text{volRV}[n])/I'_{\text{sp}},\]
and
\[(8.1.9) \quad \int : K_+(\text{volVF}^{\text{bdd}}[n]) \to K_+(\text{volRV}^{\text{bdd}}[n])/I'_{\text{sp}}\]
such that \([X] = [L(V)],\) for \(X\) in \(\text{volVF}[n]\) and \(V\) in \(\text{volRV}[n]\) (resp. \(\text{volVF}^{\text{bdd}}[n]\) and \(\text{volRV}^{\text{bdd}}[n]\)) if and only if \(f([X])\) is equal to the class of \([V]\) in \(K_+(\text{volRV}[n])/I'_{\text{sp}}\) (resp. \(K_+(\text{volRV}^{\text{bdd}}[n])/I'_{\text{sp}}\)).

8.2. The morphism \(h_m\). — We go back to the framework of 4.1, thus the base structure the field \(L_0 = F((t))\), with \(F\) a trivially valued algebraically closed field of characteristic zero and \(\text{val}(t)\) positive and denoted by 1.

Let \(\Delta\) be a bounded definable subset of \(\Gamma^n\). For \(\gamma \in \Gamma^n\), let \(w(\gamma) = \sum_{1 \leq i \leq n} \gamma_i\).
For every integer \(m \geq 1\), we set
\[(8.2.1) \quad a_m(\Delta) = \sum_{(\gamma_1, \ldots, \gamma_n) \in \Delta \cap (1/m\mathbb{Z})^n} [A^1]^{-mw(\gamma)}(1 - [A^1]^{-1})^n,\]
in \(!K_+(\text{volRES}[n])([A^1]^{-1}),\) defining a morphism
\[(8.2.2) \quad a_m : K_+(\text{vol}^{\text{bdd}}[\bullet]) \to !K_+(\text{volRES}[\bullet])([A^1]^{-1}).\]
Now consider \((X, f)\) in \(\text{RES}[n]\). Assume \(f(X) \subseteq V_{\gamma_1} \times \cdots \times V_{\gamma_n}\). Set
\[
(8.2.3) \quad b_m(X) = [X][A^1]^{-mw(\gamma)-n}
\]
in \(!K_+(\text{volRES}[n])([A^1]^{-1})\) if \(m(\gamma_1, \ldots, \gamma_n) \in \mathbb{Z}^n\) and \(b_m(X) = 0\) otherwise. Note that \(X = X \cap \text{RES}_m\) in the first case case. This construction extends uniquely to a morphism
\[
(8.2.4) \quad b_m : K_+(\text{volRES}[\star]) \longrightarrow !K_+(\text{volRES}[\star])([A^1]^{-1}).
\]

The morphism
\[
(8.2.5) \quad b_m \otimes a_m : K_+(\text{volRES}[\star]) \otimes K_+(\text{vol}\Gamma^{\text{bdd}}[\star]) \longrightarrow !K_+(\text{volRES}[\star])([A^1]^{-1})
\]
factors through the relations (8.1.5) and gives rise to a morphism
\[
(8.2.6) \quad h_m : K_+(\text{volRV}^{2\text{bdd}}[\star]) \longrightarrow !K_+(\text{volRES}[\star])([A^1]^{-1}).
\]
Indeed, if \(\gamma = i/m\), \(a_m([\gamma]_1) = [A^1]^{-i}(1 - [A^1]^{-1})\) and \([\text{val}_{\nu}^1(\gamma)]_1 = [A^1] - 1\) in \(!K_+(\text{volRES}[1])\), thus \(a_m([\gamma]_1) = b_m([\text{val}_{\nu}^1(\gamma)]_1)\). We still denote by \(h_m\) the morphism induces between the corresponding rings.

Let us now state the analogue of Proposition 4.2.1 in this context.

**Proposition 8.2.1.** — Let \(X\) be a \(\beta\)-invariant \(F((t))\)-definable subset of \(\mathcal{O}^n \times \text{RV}^\ell\), for some \(\beta\). Assume the projection \(X \to \text{VF}^\beta\) has finite fibers. Then \(h_m\) takes the same value \(h_m(f X)\) on all liftings of \(f X\) to \(K_+(\text{volRV}^{2\text{bdd}}[n])\) and
\[
h_m \left( \int X \right) = [\bar{X}[m]]
\]
as classes in \(!K_+(\text{volRES}[n])([A^1]^{-1})\).

**Proof.** — Since both sides are invariant under the transformations of Proposition 3.2.2, we may assume by Proposition 3.2.2 that there exists a definable bounded subset \(H\) of \(\text{RV}^\ell\) and a map \(h : \{1, \ldots, n\} \to \{1, \ldots, \ell\}\) such that
\[
(8.2.7) \quad X = \{(a, b) ; a \in H, \text{rv}(a_i) = b_{h(i)}, 1 \leq i \leq n\}
\]
and the map \(r : H \to \text{RV}^n\) given by \(b \mapsto (b_{h(1)}, \ldots, b_{h(n)})\) is finite to one. According to (8.1.6) we may assume \([H] = \Psi([W] \otimes [\Delta])\) with \(W\) in \(\text{RES}[\ell]\) and \(\Delta\) bounded in \(\Gamma[n - \ell]\). By induction on dimension and considering products, it enough to prove the result when \(X\) is the lifting of \(W\) or \(\Delta\). In both cases, this is clear by construction. \(\square\)

**Remarks 8.2.2.** — 1. Upon adding additive inverses, passing from \(\text{volRES}[n]\) to \(\text{RES}[n]\), and taking the quotient by \([A^1] - 1\), one obtains precisely the equation of Proposition 4.2.1.

2. Let \(V\) be a smooth variety over \(F\), with a volume form \(\omega\) (a nowhere vanishing section of \(\wedge^{\text{top}} \text{TV}\)). Let \(X\) be a bounded, \(\beta\)-invariant \(F((t))\)-definable subset of \(V\). Then \(\int X\) is defined in \(K_+(\text{volRV}^{2\text{bdd}}[n])\), and does not depend on the choice of \(\omega\), as long as \(\omega\) is chosen over \(F\). When \(x\) is a smooth point of \(V\), in the definition of a Milnor fiber fiber one can always pass to a Zariski open
subset of $V$ containing $x$ and carrying a volume form. From now on, we will refer to $f \cdot X$ in this situation, without further mention of the volume form.

3. Let $V$ be a smooth variety over $F$ carrying a volume form. Let $X$ be a bounded, $\beta$-invariant $F((t))$-definable subset of $V$. Then the statement of the Proposition remains valid.

8.3. Expressing the motivic zeta function. — Let $X$ be a smooth connected algebraic variety of dimension $d$ over $F$ and $f$ a non constant regular function $f : X \to \mathbb{A}^1_F$.

Let $\mathcal{X}_{m,x}$ be as in (1.1.4). We consider now $\mathcal{X}_{m,x}$ endowed with its natural $\hat{\mu}$-action that factors through the $\mu_m$-action induced by $t \mapsto \zeta t$.

We still denote by $\Theta$ the isomorphism

$$(8.3.1) \quad \Theta : !K(\text{RES})[[\mathbb{A}^1]]^{-1} \longrightarrow !K(\text{Var}_F; \hat{\mu})[[\mathbb{A}^1]]^{-1}$$

induced by (4.3.2). By composing with the canonical morphism

$$(8.3.2) \quad !K(\text{volRES})[[\mathbb{A}^1]]^{-1} \longrightarrow !K(\text{RES})[[\mathbb{A}^1]]^{-1}$$

one gets a morphism

$$(8.3.3) \quad \tilde{\Theta} : !K(\text{volRES})[[\mathbb{A}^1]]^{-1} \longrightarrow !K(\text{Var}_F; \hat{\mu})[[\mathbb{A}^1]]^{-1}.$$

Recall

$$(8.3.4) \quad \mathcal{X}_x = \left\{ y \in X(\mathcal{O}); \text{rv}(y) = \text{rv}(t) \text{ and } \pi(y) = x \right\}.$$}

We have the following interpretation for the class of $\mathcal{X}_{m,x}$.

**Proposition 8.3.1.** — Let $X$ be a smooth variety over $F$, $f$ a regular function on $X$ and $x$ a closed point of $f^{-1}(0)$. Then, for every integer $m \geq 1$,

$$h_m \left( \int \mathcal{X}_x \right) = [\mathcal{X}_{m,x}] \left[ \mathbb{A}^{md} \right]^{-1}$$

in $!K(\text{Var}_F; \hat{\mu})[[\mathbb{A}^1]]^{-1}$.

**Proof.** — It follows from Proposition 8.2.1, similarly as in Corollary 4.2.2, that

$$(8.3.5) \quad h_m \left( \int \mathcal{X}_x \right) = \bar{X}_x[m].$$

Since, as already observed in Step 5 of 6.3, $\mathcal{X}_{m,x}$ is the image of $\mathcal{X}_x$ modulo $t^{(m+1)/m}$, the result follows.

The motivic zeta function $Z_{f,x}(T)$ attached to $(f, x)$ is the following generating function, cf. [7], [10],

$$(8.3.6) \quad Z_{f,x}(T) = \sum_{m \geq 1} [\mathcal{X}_{m,x}] \left[ \mathbb{A}^{md} \right]^{-1} T^m$$

in $!K(\text{Var}_F; \hat{\mu})[[\mathbb{A}^1]]^{-1}[[T]]$. Thus, by Proposition 8.3.1, $Z_{f,x}(T)$ may be expressed directly in terms of $\mathcal{X}_x$: 

[1] [2] [3] [4] [5] [6] [7] [8] [9] [10]
Corollary 8.3.2. — Let $X$ be a smooth variety over $F$, $f$ a regular function on $X$ and $x$ a closed point of $f^{-1}(0)$. Then,

$$Z_{f,x}(T) = \sum_{m \geq 1} \tilde{\Theta}(h_m\left(\int_{X_x}\right)) T^m.$$ 

Remark 8.3.3. — We could have defined $\tilde{X}[m]$ in §4.2 without going to $!K$ at the cost of inverting the classes of the $V_j$’s. The constructions and statements in this section 8 could also have been performed in $K$ instead of $!K$ after inverting the classes of the $V_j$’s. It is only in 8.4, when we discuss rationality of power series and their consequences that we have to go to $!K$.

8.4. Expressing the motivic Milnor fiber. — It is known that $Z_{f,x}(T)$ is rational and that $\lim_{T \to \infty} Z_{f,x}(T)$ exists, cf. [10], [18]. One sets

(8.4.1) $$S_{f,x} = -\lim_{T \to \infty} Z_{f,x}(T).$$

This is the motivic Milnor fiber consider in [10], [18] (more precisely, it is the image of $S_f$ defined here under the canonical morphism $!K(\Var_F; \hat{\mu})(\mathbb{A}^1 - 1) \to \hat{M}_F^2$ that is considered in [10], [18]). We shall show in Corollary 8.4.2 how one may extract directly $S_{f,x}$ from $\int X_x$.

Consider the morphism

(8.4.2) $$\alpha : K(\vol\Gamma[*]) \to !K(\vol\RES[*])(\mathbb{A}^1 - 1)$$

which send the class of $\Delta$ in $K(\vol\Gamma[*])$ to $\chi(\Delta)(1 - [\mathbb{A}^1]^{-1})^n$, with $\chi$ the o-minimal Euler characteristic and the morphism

(8.4.3) $$\beta : K(\vol\RES[*]) \to !K(\vol\RES[*])(\mathbb{A}^1 - 1)$$

which send the class of $Y$ in $K(\vol\RES[*])$ to $[Y][\mathbb{A}^1]^{-n}$.

Taking the tensor product of $\alpha$ and $\beta$ one gets a morphism

(8.4.4) $$\Upsilon : K(\vol\RV[*]) \to !K(\vol\RES[*])(\mathbb{A}^1 - 1)$$

since the relations (8.1.5) in the kernel of the morphism (8.1.7) are respected.

Proposition 8.4.1. — Let $Y$ be in $K(\vol\RV^{2bdd}[\ast])$. The series

$$Z(Y)(T) = \sum_{m \geq 1} h_m(Y) T^m,$$

in $!K(\vol\RES[*])(\mathbb{A}^1 - 1)([T])$ is rational with denominators products of terms of the form $1 - [\mathbb{A}^1]^a T^b$, $a \in \mathbb{Z}$, $b \geq 1$. Furthermore, $Z(Y)(T)$ has a limit as $T \to \infty$ and

$$\lim_{T \to \infty} Z(Y)(T) = -\Upsilon(Y).$$

Proof. — We may assume $Y$ is of the form $\Psi([W] \otimes [\Delta])$ with $W$ in $\RES[p]$ and $\Delta$ in $\Gamma[q]$. Since $Z(Y)(T)$ is the Hadamard product of $Z(\Psi([W] \otimes 1))(T)$ and $Z(\Psi(1 \otimes \Delta))(T)$, by Propositions 5.1.1 and 5.1.2 of [9], it is enough to prove the statement for $\Psi([W] \otimes 1)$ and $\Psi(1 \otimes \Delta)$. The statement for $\Psi([W] \otimes 1)$ follows from a direct geometric series computation. Now remark that $Z(\Psi(1 \otimes \Delta))(T)$ and
Υ(Ψ(1 ⊗ Δ)) depend only on the image of [Δ]/[0] in K_+^d(Γ^bdd) ⊗ Q. By Proposition 2.20 of [16], K_+^d(Γ^bdd) ⊗ Q is generated as a Q-algebra by the elements [γ]/[0] and [0, γ]/[0], γ in Q, where the index 1 stands for considering the object in Γ[1]. Thus, again by Propositions 5.1.1 and 5.1.2 of [9], it enough to consider the case when q = 1 and Δ is equal to {γ} or to [0, γ). The statement then follows from a direct geometric series computation.

**Corollary 8.4.2.** — Let X be a smooth variety over F, f a regular function on X and x a closed point of f^{-1}(0). Then

$$\tilde{\Theta}\left(\int X_x\right) = S_{f,x}.$$  

**Proof.** — This follows directly from Corollary 8.3.2 and Proposition 8.4.1. □

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