Marginal densities of the “true” self-repelling motion

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Abstract

Let $X(t)$ be the true self-repelling motion (TSRM) constructed in [20], $L(t, x)$ its occupation time density (local time) and $H(t) := L(t, X(t))$ the height of the local time profile at the actual position of the motion. The joint distribution of $(X(t), H(t))$ was identified in [18] in somewhat implicit terms. Now we give explicit formulas for the densities of the marginal distributions of $X(t)$ and $H(t)$. The distribution of $X(t)$ has a particularly surprising shape (see Picture 1 below): It has a sharp local minimum with discontinuous derivative at $0$. As a consequence we also obtain a precise version of the large deviation estimate of [4].

1 Introduction and main results

1.1 Introduction

In the present paper, we study some marginal densities of a self-interacting one dimensional process called the true self-repelling motion (TSRM), defined in [20]. The true self-repelling motion is a continuous real-valued process $(X(t), t \geq 0)$ that is locally self-interacting with its past occupation-time: It is pushed away from the places it has visited the most in its past. This process has a completely singular behavior compared
to diffusions: For example, it has a finite variation of order $3/2$, which contrasts with the finite quadratic variation of semi-martingales.

A crucial property of the TSRM is the existence of a continuous density for its occupation-time measure, denoted $L(t, x)$ and named again “local time” by analogy with semi-martingales (see Theorem 4.2 of [20]). This enables to prove that the TSRM is driven by the negative gradient of its own occupation time density $L(t, x)$. Written formally:

$$\partial_t X(t) = -\partial_x L(t, X(t)), \quad \partial_t L(t, x) = \delta(x - X(t))$$

$$X(0) = 0, \quad L(0, x) = 0. \quad (1)$$

The driving mechanism (1) is to be considered in a properly regularized form. Making proper mathematical sense of this mechanism is the main content of [20].

The local time at the current position of the TSRM is denoted by $H(t) := L(t, X(t))$ and is called the height process. The processes $X(t)$, the local time $L(t, x)$ and the height $H(t)$ scale as follows:

$$X(at) \sim a^{2/3} X(t), \quad L(at, a^{2/3} x) \sim a^{1/3} L(t, x), \quad H(at) \sim a^{1/3} H(t).$$

We will examine in this paper the distributions of the marginals $X(t)$ and $H(t)$ at the fixed time $t$. It turns out that it is more convenient to work at first with an exponential random time of parameter $s$ independent of the process instead of with a fixed time. With Feynman-Kac formulas applied for the Brownian motion, it is possible to deduce explicit expressions for its marginals (in terms of Airy function, confluent hypergeometric functions and Mittag-Leffler density). It leads to the Pictures 1 and 2 for $t = 1$. We found the shape of the of the density of the position $X(t)$ rather unusual and surprising. Indeed, it has a sharp wedge-like local minimum at $x = 0$ with discontinuous derivative, and global maxima away from zero. It means that at any positive time, the process still remembers its starting point and it is strongly pushed away from it. These results are also confirmed by numerical simulations (see Figures 3. and 4.).

1.2 Review of background and notations

The true self-avoiding walk (TSAW) is a nearest neighbor random walk with long memory on $\mathbb{Z}^d$ pushed by the negative gradient of its own occupation time measure (local time). For historical background see [2], [13], [14] or the introductions to [18] or [20]. In the present paper we are interested only in the one-dimensional case.

A discrete time version considered and analyzed in [18] is the following. Let $n \mapsto S(n) \in \mathbb{Z}$ be a nearest neighbor walk on $\mathbb{Z}$. The occupation time on lattice bonds, lattice
sites, and the negative (discrete) gradient of bond occupation time are in turn:

$$\ell(n,j+1/2) := \#\{0 \leq m < n : (S(m) + S(m+1))/2 = j + 1/2\},$$
$$\ell(n,j) := (\ell(n,j-1/2) + \ell(n,j+1/2))/2,$$
$$\delta(n,j) := \ell(n,j-1/2) - \ell(n,j+1/2),$$

where $j \in \mathbb{Z}$ label lattice sites, $j \pm \frac{1}{2} \in \mathbb{Z} + \frac{1}{2}$ label lattice bonds and $n \in \mathbb{Z}_+$ is the (discrete) time of the process.

The TSAW (with bond repulsion) is defined by the law:

$$P(S(n+1) = j \pm 1 \mid \mathcal{F}_n, S(n) = j) = \frac{w(\pm \delta(n,j))}{w(\delta(n,j)) + w(-\delta(n,j))},$$

where $w : \mathbb{R} \to (0, \infty)$ is a fixed rate function which is assumed nondecreasing and non-constant. In [18] a limit theorem was proved for the distribution of the displacement of the TSAW, with time-to-the-two-thirds scaling. Later, the convergence was established for the whole process [20] [12]. In order to expose this and also prepare our present work we need some notation and preliminary concepts.

Let $\mathbb{R} \ni y \mapsto B(y) \in \mathbb{R}_+$ be a two sided Brownian motion, starting from level $h \geq 0$, and let $\omega'$ be the first hitting time of zero by the backward Brownian motion, $\omega_x$ the first one after time $x \geq 0$ of the forward Brownian motion:

$$\omega' := \sup\{y \leq 0 : B(y) = 0\}, \quad \omega_x := \inf\{y \geq x : B(y) = 0\}, \quad x \geq 0,$$

and denote:

$$T_x = \int_{\omega'}^{\omega_x} |B(t)|dt.$$

The random variable $T_x$ is a.s. positive and finite and has an absolutely continuous distribution with density

$$\varrho(t;x,h) := \partial_t P(T_x < t \mid B(0) = h), \quad t \in \mathbb{R}_+, \ x \in \mathbb{R}_+, \ h \in \mathbb{R}_+.$$

The Laplace transform of this density with respect to the variable $t$ is

$$\hat{\varrho}(s;x,h) := s \int_0^\infty e^{-st} \varrho(t;x,h)dt = sE(e^{-sT_x} \mid B(0) = h).$$

We extend $\varrho$ and $\hat{\varrho}$ to $x \in \mathbb{R}$ as even functions: $\varrho(t;x,h) = \varrho(t;|x|,h)$ and $\hat{\varrho}(s;x,h) = \hat{\varrho}(s;|x|,h)$ for all $x \in \mathbb{R}$, $h \geq 0$, $s \geq 0$ and $t \geq 0$. Let us introduce the following marginals:

$$\varrho_1(t;x) := \int_0^\infty \varrho(t;x,h)dh, \quad \varrho_1(t;x) := \int_{-\infty}^\infty \varrho(t;x,h)dx,$$
$$\hat{\varrho}_1(s;x) := \int_0^\infty \hat{\varrho}(s;x,h)dh, \quad \hat{\varrho}_1(s;x) := \int_{-\infty}^\infty \hat{\varrho}(s;x,h)dx.$$
Brownian scaling \( (B(a \cdot) \mid B(0) = \sqrt{ah}) \sim (\sqrt{a}B(\cdot) \mid B(0) = h) \) implies the following scaling relations:

\[
\begin{align*}
\varrho(t; x, h) &= t^{-1} \nu(t^{-2/3}x, t^{-1/3}h), & \varrho_1(t; x) &= t^{-2/3} \nu_1(t^{-2/3}x), & \varrho_2(t; h) &= t^{-1/3} \nu_2(t^{-1/3}h), \\
\hat{\varrho}(s; x, h) &= s \nu(s^{2/3}x, s^{1/3}h), & \hat{\varrho}_1(s; x) &= s^{2/3} \nu_1(s^{2/3}x), & \hat{\varrho}_2(s; h) &= s^{1/3} \nu_2(s^{1/3}h),
\end{align*}
\]

where

\[
\begin{align*}
\nu(x, h) &:= \varrho(1; x, h), & \nu_1(x) &:= \varrho_1(1; x), & \nu_2(h) &:= \varrho_2(1; h), \\
\hat{\nu}(x, h) &:= \hat{\varrho}(1; x, h), & \hat{\nu}_1(x) &:= \hat{\varrho}_1(1; x), & \hat{\nu}_2(h) &:= \hat{\varrho}_2(1; h).
\end{align*}
\]

We recall the main results of [18] and [20] and more recent large deviation estimates from [11] which are necessary background material for the present paper:

- **Theorem 2 of [18]:** \((x, h) \mapsto \nu(x, h)\) and \((x, h) \mapsto \hat{\nu}(x, h)\) are probability densities on \(\mathbb{R} \times \mathbb{R}_+\):

\[
\int_{-\infty}^{\infty} \int_{0}^{\infty} \nu(x, h)dhdx = 1 = \int_{-\infty}^{\infty} \int_{0}^{\infty} \hat{\nu}(x, h)dhdx. \tag{8}
\]

- **Theorem 3 of [18]:** Let \(s > 0\) be fixed and \(\theta_n\) a sequence of geometrically distributed stopping times with \(E(\theta_n) = n/s\), independent of the TSAW \(S(\cdot)\). The following limit theorem holds:

\[
((\alpha n)^{-2/3}S(\theta_n), (\alpha n)^{-1/3} \ell(\theta_n, S(\theta_n))) \Rightarrow (\hat{X}_s, \hat{H}_s),
\]

where the density of the joint distribution of \((\hat{X}_s, \hat{H}_s)\) is \(\varrho(s; x, h)\). The constant \(\alpha\) in the norming factors on the left hand side depend only on the rate function \(w(\cdot)\) and is given by formula (1.23) in [18].

Actually, in [18] only the case \(w(z) = e^{-\beta z}\) was considered and only the limit theorem for the displacement was stated explicitly. However, the proofs implicitly contain these extensions.

- **Remark after Theorem 3 of [18]:** Assume that the sequence of random variables \((n^{-2/3}S(n), n^{-1/3} \ell(n, S(n))) \in \mathbb{R} \times \mathbb{R}_+\) is tight. Let \(t > 0\) be fixed. Then

\[
((\alpha n)^{-2/3}S([nt]), (\alpha n)^{-1/3} \ell([nt], S([nt]))) \Rightarrow (X(t), H(t)),
\]

where the density of the joint distribution of \((X(t), H(t))\) is \(\varrho(t; x, h)\).

See also [19] for similar results for a continuous time version with on-site-repulsion.
Finally, in [20] the true self-repelling motion (TSRM) introduced in the first paragraph of this paper,
\[ t \mapsto X(t) \in \mathbb{R} \]
was constructed. This is the scaling limit of the properly rescaled TSAW, \( t \mapsto (\alpha n)^{-2/3} S([nt]) \), as \( n \to \infty \). It implies that the density of the joint distribution of the position and height processes \( (X(t), H(t)) \) is exactly
\[
\partial^2_{xh} \mathbb{P} (X(t) < x, H(t) < h) = g(t; x, h).
\]
Nevertheless, it is interesting to note that it can be proved directly in the continuous setting thanks to the local time properties established in [20]. Indeed, TSRM’s construction immediately gives a strong “Ray-Knight” Theorem concerning its local times at certain random times (see Theorem 4.3 in [20]). Here is a weaker version:

For every \((x, h) \in \mathbb{R} \times \mathbb{R}^+\), let us denote by \( \tau_{x,h} \) the first hitting time of \((x, h)\) by the process \((X(t), H(t))\). Then the local time at time \( \tau_{x,h} \) has the distribution of a two sided Brownian motion starting at time \( x \) at the level \( h \) reflected above the \( x \)-line between 0 and \( x \) and absorbed by the \( x \)-line outside this interval. In particular, it implies that \( \tau_{x,h} \) has the distribution of \( T_x \) conditionally on \( B(0) = h \). Therefore, with the measure preserving change of variable \( t = \tau_{x,h} \), we can write for every bounded Borel function \( f \) on \( \mathbb{R} \times \mathbb{R}^+ \),
\[
\mathbb{E} (f(X(\theta_s), H(\theta_s))) = s \int_{-\infty}^{+\infty} \int_{0}^{+\infty} f(x, h) \mathbb{E} (e^{-sT_x} \mid B(0) = h) \, dh \, dx.
\]
The scaling property readily gives that the density of \( (X(t), H(t)) \) at \((x, h) \in \mathbb{R} \times \mathbb{R}^+\) is equal to the density of \( T_{x,h} \) at time \( t \) i.e. \( g(t; x, h) \).

By choosing the initial conditions
\[
X(0) = 0, \quad L(0, x) = B(x),
\]
where \( x \mapsto B(x) \) is a two sided 1d Brownian motion, rather than [2], we obtain a version of the TSRM with stationary increments. Denote this process \( X_{st}(t) \), its local time \( L_{st}(t, x) \), and the height process \( H_{st}(t) := L_{st}(t, X_{st}(t)) \). The scaling of the stationary version is similar to that of the TSRM starting from initial conditions [2]:
\[
X_{st}(at) \sim a^{2/3} X_{st}(t), \quad L_{st}(at, a^{2/3}x) \sim a^{1/3} L_{st}(t, x), \quad H_{st}(at) \sim a^{1/3} H_{st}(t),
\]
From the construction of TSRM it follows directly that
\[
X_{st}(1) \sim 2^{-1/3} X(1).
\]
But no distributional similarities of this sort hold for the 1d marginal \( H_{st}(1) \).
In [4], using TSRM’s construction, asymptotics for the tails of \(X(1)\) and \(H(1)\) are found. It is shown that:

\[
\begin{align*}
\mathbb{P}(X(1) > x) &= \exp\left(-\frac{4\delta_1^3 x^3}{27} + O(\ln(x))\right), \\
\mathbb{P}(H(1) > h) &= \exp\left(-\frac{8h^3}{9} + O(\ln(h))\right),
\end{align*}
\]

where \(\delta_1^3\) is the absolute value of the first negative zero of the derivative of a renormalized Airy function. It is precisely defined in paragraph 3.2.

In the stationary case, the tails of the distributions are also computed. In this case, due to (9), the constant in front of \(x^3\) is multiplied by 2 for \(X_{st}(1)\). The height \(H_{st}(1)\) behaves differently as the initial local time helps to have large values of \(|H_{st}(1)|\). It is shown that there exists \(\eta > 0\) such that for every large \(h\):

\[
\exp\left(-\frac{1}{\eta h^{3/2}}\right) \leq \mathbb{P}(\pm H_{st}(1) > h) \leq \exp\left(-\eta h^{3/2}\right).
\]

### 1.3 Main results

In the present paper, our main goal is to identify the marginal distributions, \(\nu_1(x)\) and \(\nu_2(h)\) more explicitly than it was done in [18]. Our main results are collected in the following theorem:

**Theorem 1.** (i) The second marginal densities \(\hat{\nu}_2(h)\) and \(\nu_2(h)\) are

\[
\hat{\nu}_2(h) = -2u(h)u'(h) = -\partial_h \left(u^2\right)(h),
\]

\[
\nu_2(h) = \frac{2 \cdot 6^{1/3} \sqrt{\pi}}{\Gamma(1/3)^2} e^{-\left(8h^3\right)/9} U\left(1/6, 2/3; (8h^3)/9\right),
\]

where \(u : \mathbb{R}_+ \to \mathbb{R}\) is the (rescaled and normed) Airy function, as defined in subsection 3.2, (42), (43), and \(U(1/2, 4/3; \cdot)\) is the confluent hypergeometric function of the second kind (or Tricomi function), given by the integral representation (52) in subsection 3.4.

(ii) The first marginal densities \(\hat{\nu}_1(x)\) and \(\nu_1(x)\) are

\[
\hat{\nu}_1(x) = \sum_{k=1}^{\infty} p_k \frac{\delta_k^3}{2} \exp(-\delta_k^3 |x|),
\]

\[
\nu_1(x) = \sum_{k=1}^{\infty} p_k \frac{\delta_k^3}{2} f_{2/3}(\delta_k^3 |x|),
\]

where \(f_{2/3}\) is the Mittag-Leffler density function as defined in subsection 3.3, the scaling factors \(\delta_k^3\) are the zeros of the derivative of the (rescaled and normed) Airy function \(u(\cdot)\), and the coefficients \(p_k\) of the convex combinations are those given in subsection 3.2, (48).
It is worth to compare the computed graphs of these density functions with histograms of the empirical distributions on $X(1)$ and $H(1)$ (see Figures 3 and 4).

Figure 3: The histogram represents a numerical simulation of the TSRM at time one $X(1)$ with a sample of $10^5$ independent realizations. Each realization is computed using the discrete “toy-model” introduced in [20] with a step of $2 \cdot 10^{-6}$. The plain blue line is the graph of the density of $X(1)$.

Figure 4: The histogram represents a numerical simulation of the height process at time one $H(1)$ (sample of $10^5$ independent realizations using discrete “toy-model” with a step of $2 \cdot 10^{-6}$). The plain blue line is the graph of the density of $H(1)$.

The moments of the marginal distributions collected in Theorem 1 are as follows

$$
E(H(1)^n) = \int_0^{\infty} h^n \nu_2(h) dh = \Gamma(5/6) \frac{(2 \cdot 3^{1/3})^{-n} n!}{\Gamma(n/3 + 1) \Gamma(n/3 + 5/6)}
$$

(16)

$$
E(|X(1)|^n) = 2 \int_0^{\infty} x^n \hat{\nu}_1(x) dx = \sum_{k=1}^{\infty} p_k \frac{n!}{(\delta_k)^n \Gamma(2n/3 + 1)}
$$

(17)
As a direct consequence of (13) and (15) we also obtain the precise tail asymptotics of $\nu_2(h)$ and $\nu_1(x)$ as $h \gg 1$, respectively, as $|x| \gg 1$:

**Corollary 1.**

$$\lim_{h \to \infty} h^{-3} \log P(H(1) > h) = -\frac{8}{9},$$

$$\lim_{x \to \infty} x^{-3} \log P(X(1) > x) = -\frac{4}{27} \delta_1'.$$

(18)

In view of (9) the tail asymptotics (18) also implies

$$\lim_{x \to \infty} x^{-3} \log P(X_{st}(1) > x) = -\frac{8}{27} \delta_1'.$$

Note that this is matching with the large deviation results of (4). More precision in the asymptotics can be derived from the expression of the densities.

The structure of further parts of this note is as follows: In section 2 we present the proofs of the main statements. Section 3 is an Appendix which contains classical ingredients: the Feynman-Kac formulas used, and collections of facts about Airy function, Mittag-Leffler distributions and confluent hypergeometric functions. We do not prove the statements collected in the Appendix but give precise reference for all.

## 2 Proofs

### 2.1 Preliminaries

Recall the definitions of $T_x$, $\omega_x$ and $\omega'(x \geq 0)$ in (3), (5), (4), and (3). We write

$$T_x = S + T^1_x + T^2_x,$$

where

$$S := \int_0^0 |B(y)| \, dy, \quad T^1_x := \int_0^x |B(y)| \, dy, \quad T^2_x := \int_x^{\omega_x} |B(y)| \, dy,$$

(see Picture 5).

The main ingredients of our investigations are the following three functions:

$$u(h) := E(e^{-S} \mid B(0) = h), \quad h \geq 0,$$

$$\varphi(x, h) := E(e^{-T^1_x - T^2_x} \mid B(0) = h), \quad x \geq 0, \ h \geq 0,$$

$$w(x) := \varphi(x, 0) = E(e^{-T^2_x} \mid B(0) = 0) \quad x \geq 0.$$
Figure 5: Representation of $T_x = S + T_x^1 + T_x^2$

We also define the Laplace transforms (in the variable $x \geq 0$).

\[
\tilde{\varphi}(\lambda, h) := \int_0^\infty e^{-\lambda x} \varphi(x, h) dx, \quad (21)
\]
\[
\tilde{w}(\lambda) := \int_0^\infty e^{-\lambda x} w(x) dx.
\]

**Remark.** Note that this is a different type of Laplace transform than the one we used for the time variable. That is why we use different notations (hat for the time transform, tilde for this last one).

In the following Proposition we collect the basic starting identities. All these follow from straightforward applications of infinitesimal conditioning and/or Feynman-Kac formula.

**Proposition 1.** (i) The function $u : \mathbb{R}_+ \rightarrow [0, 1]$ defined in (19) is the unique bounded solution of the boundary value problem

\[
\partial_x^2 u(h) = 2hu(h), \quad u(0) = 1. \quad (22)
\]

That is: It is exactly the normalized Airy function defined and treated in subsection 3.2, restricted to $h \in [0, \infty)$.

(ii) The function $\varphi : \mathbb{R}_+ \times \mathbb{R}_+ \rightarrow [0, 1]$ defined in (20) is the unique bounded solution of the parabolic Cauchy problem

\[
\partial_x \varphi(x, h) = \frac{1}{2} \partial_h^2 \varphi(x, h) - h\varphi(x, h), \quad (23)
\]

with initial and boundary conditions

\[
IC : \quad \varphi(0, h) = u(h), \quad BC : \quad \partial_h \varphi(x, 0) = 0, \quad \text{for } x > 0. \quad (24)
\]
(iii) The function \( \hat{\nu} : \mathbb{R}_+ \times \mathbb{R}_+ \to [0, 1] \) is the unique bounded solution of the parabolic Cauchy problem
\[
\partial_x \hat{\nu}(x, h) = \frac{1}{2} \partial_h \left( u(h)^2 \partial_h \left( u(h)^{-2} \hat{\nu}(x, h) \right) \right),
\]
with initial and boundary conditions
\[
\text{IC : } \hat{\nu}(0, h) = u(h)^2, \quad \text{BC : } \hat{\nu}(x, 0) = w(x) \quad \text{for } x > 0.
\]

Proof of Proposition 1
(i) (22) follows from usual “infinitesimal conditioning”.
(ii) By strong Markov property of the Brownian motion
\[
\varphi(x, h) = \mathbb{E}\left( e^{-\int_0^x |B(y)|dy u(|B(x)|)} \big| B(0) = h \right),
\]
and thus, we apply the Feynman-Kac formula from subsection 3.1, with \( V(h) = |h| \) and \( f(h) = u(|h|) \). Since for all \( x > 0 \), \( h \mapsto \varphi(x, h) \) is even and smooth, the boundary condition in (24) follows.
(iii) Note that
\[
\hat{\nu}(x, h) = u(h) \varphi(x, h),
\]
and use (22), (23) and (24).

2.2 Computation of \( \hat{\nu}_2 \)

Since
\[
\hat{\nu}_2(h) = 2 \int_0^\infty \hat{\nu}(x, h)dx,
\]
from (25) and (26) we readily get
\[
(u(h)^2 (u(h)^{-2} \hat{\nu}_2(h)))' = -4u(h)^2.
\]
Using the following identities (that directly come from integration by parts and the differential equation of \( u \)):
\[
\int_0^h u(\chi)^2 d\chi = hu(h)^2 - u'(h)^2/2 + u'(0)^2/2,
\]
\[
\int_0^h u(\chi)^{-2} u'(\chi)^2 d\chi = -u'(h)/u(h) + u'(0) + h^2,
\]
we deduce that the general solution of (28) is
\[
-2u(h)u'(h) + C_1 u(h)^2 + C_2 u(h)^2 \int_0^h u(\chi)^{-2} d\chi.
\]
From the asymptotics (45) it follows that
\[ u(h)^2 \int_0^h u(\chi)^{-2} d\chi \approx h^{-1/2}, \quad \text{as } h \to \infty. \]

Since, by (38),
\[ \int_0^\infty \hat{\nu}_2(h) dh = 1, \quad (30) \]
we must choose \( C_2 = 0 \). (Otherwise \( \nu_2 \) wouldn’t be integrable.) Finally, due to (30) again, \( C_1 = 0 \), too. The proof of (12) is completed.

**Remark.** The interchange between integration and differentiation used to obtain (28) is standard. One can justify it properly by taking the integral over a compact interval of the type \([0, x]\) first and deduce an expression for this function of a similar kind of (29). Taking the limit \( x \to \infty \) gives the result.

### 2.3 Computation of \( \nu_2 \)

Due to (6) and (7) \( \nu_2(h) \) and \( \hat{\nu}_2(h) \) are related by
\[ \hat{\nu}_2(h) = \int_0^\infty e^{-t^{1/3}} \nu_2(t^{-1/3} h) dt. \quad (31) \]

The goal of the present subsection is inverting this integral transform.

By straightforward computations – performing a change of variables and an integration by parts – form (12) and (31) we obtain
\[ u^2(z^{1/3}) = \frac{1}{3} \int_0^\infty e^{-s} \nu_2(s^{-1/3}) s^{-4/3} ds, \]
and hence
\[ \nu_2(h) = 3h^{-4} (f * f)(h^{-3}), \quad (32) \]
where \( f \) is the inverse Laplace transform of the function \( z \mapsto u(z^{1/3}) \):
\[ u(z^{1/3}) = \int_0^\infty e^{-hs} f(s) ds, \]
and \( * \) stands for convolution on \([0, \infty)\).

In the following computations we denote generically by \( C \) a multiplicative constant. The value of this may change from line to line. These constants could be written explicitly in each step, but this wouldn’t be much illuminating. The value of the norming constant in (13) will be identified at the very end of the computations.
The function \( f \) is explicitly known, see [11]:

\[
f(s) = Cs^{-4/3}e^{-2/(9s)}.
\]

Thus,

\[
f \ast f(s) = C \int_0^s (t(s - t))^{-4/3} e^{-2/(9st(1-t))} dt
\]

\[
= Cs^{-5/3} \int_0^1 (t(1 - t))^{-4/3} e^{-2/(9st(1-t))} dt
\]

\[
= Cs^{-5/3}e^{-8/(9s)} \int_0^\infty (1 + z)^{-1/6}z^{-1/2}e^{-8z/(9s)} dz
\]

\[
= Cs^{-5/3}e^{-8/(9s)}U(1/2, 4/3; 8/(9s)).
\]

\[
= Cs^{-4/3}e^{-8/(9s)}U(1/6, 2/3; 8/(9s)). \tag{33}
\]

In the third line we perform the change of integration variable \((t(1-t))^{-1} =: 4(1+z)\). Finally, in the last step we apply Kummer’s identity (53).

From (32) and (33) we readily get

\[
\nu_2(h) = Ce^{-(8h^3)/9}U(1/6, 2/3; (8h^3)/9).
\]

Identification of the norming constant \( C \) in the last expression follows from (12), (44) and (31):

\[
2^{4/3}3^{1/3}\Gamma(2/3)\Gamma(1/3)^{-1} = \hat{\nu}_2(0) = \Gamma(2/3)\nu_2(0)
\]

\[
= C\Gamma(2/3)U(1/6, 2/3; 0) = C\Gamma(2/3)\Gamma(1/3)/\Gamma(1/2).
\]

Hence (13).

### 2.4 Computation of \( \hat{\nu}_1 \)

Recall the definition of \( \hat{\nu}_1 \):

\[
\hat{\nu}_1(x) = \int_0^\infty \hat{\nu}(x, h)dh.
\]

Let us examine the case \( x > 0 \) (the function \( \hat{\nu}_1 \) is even). Using (27) we readily obtain

\[
\partial_x \hat{\nu}_1(x) = \frac{1}{2} \int_0^\infty \partial_h (u(\chi)\partial_h \varphi(x, \chi) - u'(\chi)\varphi(x, \chi)) d\chi
\]

\[
= \frac{1}{2} (u'(0)w(x) - u(0)\partial_h \varphi(x, 0)). \tag{34}
\]
Remark. Again, the interchange between integration and differentiation is classical. Note first that for every $x \geq 0$, $\partial_h \varphi(x, h) \to 0$ when $h \to \infty$: It admits a finite or infinite limit (integrating (23) with respect to $h$, one can see that the terms involved admits finite or infinite limits – recall $\partial_x \varphi(x, h)$ is negative) and its integral is finite. One can then integrate $\partial_h \tilde{v}(x, h)$ first on $[0, h]$ and obtain an expression for $\int_0^h u(x) \varphi(x, \chi) d\chi$. Taking $h \to \infty$ and using $\lim_{h \to \infty} \partial_h \varphi(x, h) = 0$ rigorously prove (34).

Notice that we have $\partial_h \varphi(x, 0) = 0$ for every $x$ (see (24)). The following lemma will serve to compute $w(x)$ on the right hand side of (34).

Lemma 1. The function $\tilde{\varphi} : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}$ defined in (21) is expressed as

$$
\tilde{\varphi}(\lambda, h) = \frac{-1}{u(\lambda)u'(\lambda)} \left\{ u(\lambda + h) \int_0^\infty u(\lambda + \chi) u(\chi) d\chi 
+ u(\lambda + h) \int_0^h v_\lambda(\lambda + \chi) u(\chi) d\chi 
+ v_\lambda(\lambda + h) \int_h^\infty u(\lambda + \chi) u(\chi) d\chi \right\},
$$

where $v_\lambda : \mathbb{R} \to \mathbb{R}$ is the solution of the Airy equation (42) with initial conditions

$$v_\lambda(\lambda) = u(\lambda), \quad v_\lambda'(\lambda) = -u'(\lambda).$$

Proof of Lemma 1. Let $v_\lambda : \mathbb{R} \to \mathbb{R}$ be the solution of (42) with boundary conditions (36) and

$$
\phi_+(\lambda, h) := u(\lambda + h) \mathbb{I}\{h \geq 0\} + v_\lambda(\lambda - h) \mathbb{I}\{h < 0\},
$$

$$
\phi_- (\lambda, h) := \phi_+(\lambda, -h).
$$

Then $\phi_{\pm}(\lambda, \cdot)$ are exactly the unique solutions of (40), with $V(h) = |h|$, such that $\lim_{h \to \pm\infty} \phi_{\pm}(\lambda, h) = 0$. Applying (41) we readily obtain (35). □

Using now (35), we obtain

$$
\tilde{w}(\lambda) = \tilde{\varphi}(\lambda, 0) = \frac{-2}{u'(\lambda)} \int_0^\infty u(\lambda + \chi) u(\chi) d\chi
= \frac{-1}{\lambda u'(\lambda)} \int_0^\infty (u''(\lambda + \chi) u(\chi) - u(\lambda + \chi) u''(\chi)) d\chi
= \frac{-1}{\lambda u'(\lambda)} \int_0^\infty (u'(\lambda + \chi) u(\chi) - u(\lambda + \chi) u'(\chi))' d\chi
= \frac{u'(\lambda) - u'(0) u(\lambda)}{\lambda u'(\lambda)}.
$$

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In the first step we have used (35), in the second step the Airy equation (42).

Now, using the key identity (46) and the trace formula (47), from (37) we derive

\[ \tilde{\omega}(\lambda) = \frac{|u'(0)|}{2} \sum_{k=1}^{\infty} (\delta_k')^{-2} \frac{1}{\delta_k' + \lambda}, \]

and hence

\[ \omega(x) = \frac{|u'(0)|}{2} \sum_{k=1}^{\infty} (\delta_k')^{-2} e^{-\delta_k' x}. \quad (38) \]

Finally, (34), (26) and (38) yield (14).

### 2.5 Computation of \( \nu_1 \)

Due to (6) and (7) \( \nu_1(x) \) and \( \hat{\nu}_1(x) \) are related by

\[ \hat{\nu}_1(x) = \int_0^\infty e^{-t} t^{-2/3} \nu_1(t^{-2/3} x) dt. \]

The goal of the present subsection is inverting this integral transform. Since \( \hat{\nu}_1(x) \) is convex combination of scaled exponentials, we only have to compute the inverse transform of the exponential density. Assume

\[ e^{-x} = \int_0^\infty e^{-t} t^{-2/3} \phi(t^{-2/3} x) dt, \quad x \geq 0. \]

Writing the integer moments of both sides, by elementary computations we obtain

\[ \int_0^\infty \phi(x) x^n dx = \frac{n!}{\Gamma(2n/3 + 1)}. \]

This identifies \( \phi \) as the Mittag-Leffler density \( f_{2/3} \) from (51) and (54) thanks to (50). Hence (15).

### 3 Appendix:

#### 3.1 Feynman-Kac formulas

According to the present context we formulate the Feynman-Kac formulas only for the one-dimensional Brownian motion. For the general theory of Feynman-Kac formulas see e.g. [9], [10], or [17].

Let \( y \mapsto B(y) \) be a standard 1-dimensional Brownian motion which starts from level \( B(0) = h \in \mathbb{R} \), and \( V : \mathbb{R} \to \mathbb{R}_+ \) and \( f : \mathbb{R} \to \mathbb{R} \) be continuous functions. Assume that \( f \) is also bounded. Define

\[ g(x, h) := \mathbb{E}(e^{-\int_0^x V(B(y)) dy} f(B(x)) \mid B(0) = h), \quad x \geq 0, \]

\[ \tilde{g}(\lambda, h) := \int_0^\infty e^{-\lambda x} g(x, h) dx, \quad \lambda > 0. \]

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Theorem. [The Feynman-Kac formula for 1-d Brownian motion]

(i) \((x, h) \mapsto g(t, x)\) is the unique bounded solution of the parabolic Cauchy problem

\[ \partial_x g(x, h) = \frac{1}{2} \partial_h^2 g(x, h) - V(h)g(x, h), \quad g(0, h) = f(h). \]

(ii) For \(\lambda > 0\) fixed, \(h \mapsto \tilde{g}(\lambda, h)\) is the unique bounded solution of the elliptic PDE

\[ \frac{1}{2} \partial_h^2 \tilde{g}(\lambda, h) = (V(h) + \lambda)\tilde{g}(\lambda, h) - f(h). \tag{39} \]

(iii) Fix \(\lambda > 0\) and let \(\phi_{\pm}(\lambda, \cdot) : \mathbb{R} \to \mathbb{R}\) be the unique solutions of the ODE

\[ \frac{1}{2} \partial_h^2 \phi(h) = (V(h) + \lambda)\phi(h), \tag{40} \]

for which \(\lim_{h \to \pm \infty} \phi_{\pm}(\lambda, h) = 0.\) (The convergence is actually at least exponentially fast.) Then the unique bounded solution of (39) is expressed as:

\[ \tilde{g}(\lambda, h) = 2 \left( \phi_+(\lambda, 0)\phi'_-(\lambda, 0) - \phi_-(\lambda, 0)\phi'_+(\lambda, 0) \right)^{-1} \]

\[ \times \left\{ \phi_+(\lambda, h) \int_{-\infty}^{h} \phi_-(\lambda, \chi)f(\chi)d\chi - \phi_-(\lambda, \chi) \int_{h}^{\infty} \phi_+(\lambda, \chi)f(\chi)d\chi \right\}. \tag{41} \]

For a full proof of these statements see e.g. [8].

3.2 Airy functions

In the present Appendix we collect the necessary minimum information about Airy functions needed and used in this paper. For an exhaustive treatment of Airy functions see [1], [6], [11], [16].

Consider the second order linear ODE for \(f : \mathbb{R} \to \mathbb{R}:\)

\[ f''(h) = 2hf(h), \tag{42} \]

commonly called Airy’s equation. We denote by \(u : \mathbb{R} \to \mathbb{R}\) the unique solution of (42) with boundary conditions

\[ u(0) = 1, \quad \lim_{h \to \infty} u(h) = 0. \tag{43} \]

In terms of the conventionally defined Airy function of the first kind \(\text{Ai}(\cdot)\) we have

\[ u(h) = 3^{2/3} \Gamma(2/3) \text{Ai}(2^{1/3} h), \]
and we also have
\[ u'(0) = -\frac{6^{1/3}\Gamma(2/3)}{\Gamma(1/3)}, \quad (44) \]
\[ u(h) \sim \frac{3^{2/3}\Gamma(2/3)}{2^{13/12}\sqrt{\pi}} h^{-1/4} \exp \left( -\left(\frac{2\sqrt{3}}{3}\right) h^{3/2} \right), \quad \text{as } h \to \infty \quad (45) \]
\[ u(h) \sim \frac{3^{2/3}\Gamma(2/3)}{2^{13/12}\sqrt{\pi}} |h|^{-1/4} \sin \left( \left(\frac{2\sqrt{3}}{3}\right) |h|^{3/2} + \frac{\pi}{4} \right), \quad \text{as } h \to -\infty. \]

The function \( u : \mathbb{R} \to \mathbb{R} \) is convex and decreasing on \([0, \infty)\) and oscillates indefinitely on \((-\infty, 0]\). The function \( u \) extends as an entire function to \( \mathbb{C} \ni z \mapsto u(z) \in \mathbb{C} \).

We denote by \( \delta'_k, k = 1, 2, \ldots \) the consecutive zeros of \( h \mapsto u'(-h) \). It is known that
\[ 0 < \delta'_1 < \delta'_2 < \cdots < \delta'_k < \cdots, \]
and their asymptotics, as \( k \to \infty \), are
\[ \delta'_k \sim \frac{1}{2}(3\pi k)^{2/3}. \]

(Finer asymptotics are also available but not needed for our purposes in this paper.)

The following key identity holds
\[ \frac{u''(z)}{u'(z)} = -\sum_{k=1}^{\infty} \frac{1}{\delta'_k} \cdot \frac{z}{\delta'_k + z}, \quad z \notin \{-\delta'_k : k = 1, 2, \ldots \}. \quad (46) \]

It is easily seen that the sum on the right hand side is absolutely convergent and the two sides have simple poles at \(-\delta'_k, k = 1, 2, \ldots \), with the same residues. Thus, the two sides can differ in an entire function only. For a full proof of this formula see [6].

From (46) an infinite family of trace formulas for \( \sum_{k=1}^{\infty} (\delta'_k)^{-n} \) \( (n \geq 2) \) are expressed in terms of \( u'(0) \). We will use only the first three of these:
\[ \sum_{k=1}^{\infty} (\delta'_k)^{-2} = \frac{-2}{u'(0)}, \quad \sum_{k=1}^{\infty} (\delta'_k)^{-3} = 2, \quad \sum_{k=1}^{\infty} (\delta'_k)^{-4} = \frac{2}{u'(0)^2}. \quad (47) \]
These are easily checked by differentiating both sides of (47) at $z = 0$ and using (42). We will denote
\[ p_k := \frac{u'(0)^2}{2} (\delta'_k)^{-4}. \] (48)

Note that $\sum_{k=1}^{\infty} p_k = 1$.

### 3.3 Mittag-Leffler distributions

Following the terminology of [5] and [3], we will call Mittag-Leffler distribution of index $\alpha \in [0, 1]$ the probability distribution $F_\alpha : \mathbb{R}_+ \to [0, 1]$ whose moment generating function (Laplace transform) is
\[ \int_0^\infty e^{-yx} dF_\alpha(x) = \sum_{k=0}^{\infty} \frac{(-y)^k}{\Gamma(\alpha k + 1)} =: E_\alpha(-y). \] (49)

The function $y \mapsto E_\alpha(y)$ defined by the power series in (49) is the Mittag-Leffler function of index $\alpha \in [0, 1]$. It is proved in [15] – where reference is also made to unpublished alternative proof of W. Feller – that, for $\alpha \in [0, 1]$, $[0, \infty) \ni y \mapsto E_\alpha(-y)$ is indeed completely monotone, and thus the Laplace transform of a probability density function on $\mathbb{R}_+$.

The moments of the Mittag-Leffler distributions are
\[ \int_0^\infty x^m dF_\alpha(x) = \frac{m!}{\Gamma(\alpha m + 1)}. \] (50)

$\alpha = 0, \frac{1}{2}, 1$ are special cases:

\[ F_0(x) = 1 - e^{-x}, \quad F_{1/2}(x) = \sqrt{\frac{2}{\pi}} \int_0^x e^{-y^2/2} dy, \quad F_1(x) = \mathbb{1}\{x > 1\}. \]

For $\alpha \in [0, 1)$ the distribution function $F_\alpha$ is smooth and the following power series expansion holds for the density function $f_\alpha(x) := F'_\alpha(x)$, see [15]:
\[ f_\alpha(x) = \frac{1}{\pi} \sum_{k=0}^{\infty} \frac{\sin(k+1)\alpha\pi)\Gamma((k+1)\alpha)}{k!} (-x)^k \] (51)

This power series defines actually an entire function on $\mathbb{C}$. See the picture below for a representation of those functions with $\alpha = 0, 1/2$ and $2/3$.

We are interested in the case $\alpha = 2/3$. This particular case is also expressed in terms of a confluent hypergeometric function, see (54) below.

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Figure 7: Mittag-Leffler density functions for \( \alpha = 0 \) (dotted), \( \alpha = 1/2 \) (plain) and \( \alpha = 2/3 \) (thick)

### 3.4 Confluent hypergeometric functions

See [7], [1] for details on confluent hypergeometric functions. We collect here only a few facts needed for our purposes.

Let \( a > 0 \) and \( b \in \mathbb{R} \) be fixed parameters. We define the confluent hypergeometric functions of the second kind, or Tricomi functions

\[
U(a, b; \cdot) : \mathbb{R}_+ \mapsto \mathbb{R}
\]

by the integrals

\[
U(a, b; z) = \frac{1}{\Gamma(a)} \int_0^\infty e^{-zs} s^{a-1} (1 + s)^{b-a-1} ds.
\]  \hspace{1cm} (52)

(For more general definitions see [7], [1].) In our formulas the particular cases \( U(1/6, 2/3; \cdot), U(1/2, 4/3; \cdot), \) and \( U(1/6, 4/3; \cdot) \) will occur.

**Kummer’s identity**

\[
U(a, b; z) = z^{1-b} U(1 + a - b, 2 - b, z)
\]  \hspace{1cm} (53)

holds if \( b > a + 1 \).

The Mittag-Leffler density function \( f_{2/3} \), which appears in [15], is also expressed in terms of \( U(1/6, 4/3; \cdot) \) as follows:

\[
f_{2/3}(z) = \frac{2^{1/3}}{\sqrt{3\pi}} z e^{-(4z^3)/27} U(1/6, 4/3; (4z^3)/27).
\]  \hspace{1cm} (54)

Computing the moments of the right hand side of (54) we obtain the expressions (50).

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References

[1] M. Abramowitz, I.A. Stegun: *Handbook of Mathematical Functions*. Dover, New York, 1964

[2] D. Amit, G. Parisi, L. Peliti: Asymptotic behavior of the ‘true’ self-avoiding walk. *Phys. Rev. B* 27: 1635–1645 (1983)

[3] D. A. Darling, M. Kac: On occupation times for Markoff processes. *Trans. Amer. Math. Soc.* 84: 444-458 (1957)

[4] L. Dumaz: Large deviations and path properties of the true self-repelling motion. submitted (2011), [http://arxiv.org/abs/1105.2948](http://arxiv.org/abs/1105.2948)

[5] W. Feller: Fluctuation theory of recurrent events. *Trans. Amer. Math. Soc.* 67: 98-119 (1949)

[6] P. Flajolet, G. Louchard: Analytic variations on the Airy distribution. *Algorithmica*, 31: 361–377 (2001)

[7] I.S. Gradshteyn, I.M. Ryzhik: *Table of Integrals, Series, and Products*. Seventh edition. Academic Press, 2007. (Original Russian edition: Moscow, 1946)

[8] S. Janson: Brownian excursion area, Wright’s constants in graph enumeration, and other Brownian areas. *Probab. Surveys*, 4: 80-145 (2007)

[9] M. Kac: On distributions of certain Wiener functionals. *Trans. Amer. Math. Soc.* 65: 1–13 (1949)

[10] I. Karatzas, S. E. Shreve: *Brownian motion and stochastic calculus*, second edition, *Graduate Texts in Mathematics*, bf vol. 113, Springer-Verlag, New York, 1991.

[11] M. J. Kearney, S. N. Majumdar: On the area under a continuous time Brownian motion till its first-passage time. *J. Phys. A*, 38: 4097–4104 (2005)

[12] C. M. Newman, K. Ravishankar: Convergence of the Tóth lattice filling curve to the Tóth-Werner plane filling curve. *ALEA. Lat. Am. J. Probab. Math. Stat.* 1: 333-346 (2006)

[13] S.P. Obukhov, L. Peliti: Renormalisation of the “true” self-avoiding walk. *J. Phys. A*, 16: L147–L151 (1983)

[14] L. Peliti, L. Pietronero: Random walks with memory. *Riv. Nuovo Cimento*, 10: 1–33 (1987)

[15] H. Pollard: The completely monotonic character of the Mittag-Leffler function $E_{\alpha}(-x)$. *Bull. Amer. Math. Soc.* 54: 1115-1116 (1948)
[16] W. H. Reid: Integral representations for products of Airy functions. Z. Angew. Math. Phys. 46: 159–170 (1995)

[17] B. Simon: Functional integration and quantum physics. Second edition. American Mathematical Society, 2005

[18] B. Tóth: The 'true' self-avoiding walk with bond repulsion on Z: limit theorems. Ann. Probab., 23: 1523-1556 (1995)

[19] B. Tóth, B. Vető: Continuous time 'true' self-avoiding random walk on Z. ALEA. Lat. Am. J. Probab. Math. Stat. 8: 59-75 (2011)

[20] B. Tóth, W. Werner: The true self-repelling motion. Probab. Theory Related Fields 111: 375-452 (1998)

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