Two-mode optical tomograms: a possible experimental check of the Robertson uncertainty relations

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Abstract
An experimental check of the two-mode Robertson uncertainty relations and inequalities for the highest quadrature moments using homodyne photon detection is suggested. The relation between optical tomograms and symplectic tomograms is used to connect the tomographic dispersion matrix and the quadrature components dispersion matrix of the two-mode field states.

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1. Introduction

Recently [¹, ²], the possibility of experimentally checking the Schrödinger–Robertson uncertainty relations [³, ⁴] for conjugate position and momentum by using homodyne photon detection was suggested.

Homodyne photon detection yields the optical tomogram [⁵, ⁶] of the photon quantum state and was implemented in the available experiments [⁷–¹¹] to reconstruct the Wigner function of the state, which is interpreted as measuring the quantum state.

In the probability representation of quantum mechanics [¹²–¹⁶], the state is identified with the tomographic probability which is either an optical [⁵, ⁶] or symplectic tomogram [¹², ¹⁷] or other kinds of tomograms (see the review [¹⁸]).

In view of this, measuring the tomographic probability means measuring the quantum state, since the tomogram contains all the information about the quantum state, including information on the variances and covariances of the conjugate variables (position and momentum or field quadratures).

In this context, reconstructing the Wigner function for calculating other state properties such as the quadrature dispersion matrix is redundant, since all the characteristics can be obtained directly in terms of the measured tomographic probability distributions.

The aim of this work is to suggest possible experiments for obtaining optical tomograms, like for instance in [¹¹] where the states of a two-mode field were studied by measuring the one-mode field states tomogram, and to check uncertainty relations for two-mode light.

The two-mode (multi-mode) uncertainty relations (see [⁴, ¹⁹–²¹]) seem to have never been checked directly in experiments.

Since the uncertainty relations are basic for quantum mechanics it is reasonable to have both experimental confirmation of their validity and knowledge of the experimental accuracy with which the uncertainty relations are checked.

Until now, all the tomographic approaches to measure quantum states have been applied to one-mode light. Only recent experiments [¹¹] were produced by homodyne detecting two-mode light. The experiments can be used to detect the presence or absence of entanglement phenomena in the field under study.

In the two-mode case, the photon states which have been discussed the most have been assumed to be Gaussian. For Gaussian states the photon distribution function was obtained in explicit form [²²] for both the one- and multi-mode cases in terms of Hermite polynomials depending on many variables.

One of our aims is to discuss the photon distribution in a two-mode field by means of the explicit expression given by the multi-variable Hermite polynomials in [²²] for
Gaussian light. This makes possible an additional control of both the Gaussianity property of the photon states and the accuracy of the measurements. We devote particular attention to the uncertainty relations in the tomographic probability representation of Gaussian states.

We also derive the tomographic form of the uncertainty relations for arbitrary quantum states to get formulae where the Robertson uncertainty relations are expressed in terms of two-mode symplectic tomograms which generalize the results of [1] obtained for one-mode states.

The paper is organized as follows. In section 2, we review the Robertson uncertainty relations. In section 3, we briefly review the tomographic probability representation of two-mode quantum states. In section 4, the marginal tomograms of two-mode probability distributions are discussed and the uncertainty relations for photon quadratures are given in the tomographic form appropriate for experimental study. Inequalities for the quadrature highest moments are given in section 5. Then state reconstruction is discussed in section 6. The relations between quadratures and photon statistics are considered in section 7. Conclusions and perspectives are presented in section 8.

2. Two-mode uncertainty relations

The Robertson [23] uncertainty relations for two-mode systems read as a positivity condition for a matrix Σ of the form

\[ Σ = \begin{pmatrix} \sigma_{p_1 p_1} & \sigma_{p_1 Q_1} & \sigma_{p_1 Q_2} & \sigma_{p_2 Q_1} & \sigma_{p_2 Q_2} \\ \sigma_{p_1 p_1} & \sigma_{p_1 Q_1} & \sigma_{p_1 Q_2} & \sigma_{p_2 Q_1} & \sigma_{p_2 Q_2} \\ \sigma_{Q_1 p_1} & \sigma_{Q_1 Q_1} & \sigma_{Q_1 Q_2} & \sigma_{Q_2 Q_1} & \sigma_{Q_2 Q_2} \\ \sigma_{Q_2 p_1} & \sigma_{Q_2 Q_1} & \sigma_{Q_2 Q_2} & \sigma_{Q_2 Q_1} & \sigma_{Q_2 Q_2} \\ \sigma_{Q_1 p_2} & \sigma_{Q_1 Q_1} & \sigma_{Q_1 Q_2} & \sigma_{Q_2 Q_1} & \sigma_{Q_2 Q_2} \end{pmatrix} \]

\[ + \frac{i}{2} \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} . \]

The positivity condition for the above matrix

\[ Σ \geq 0 \]

means that all the principal minors of Σ are non-negative. The dispersion matrix contribution to the matrix Σ is positive (non-negative, in fact) both in the classical and in the quantum domain. The second contribution is due to non-commutativity of the conjugate variables in the quantum domain

\[ Q_j P_k - P_k Q_j = i \delta_{jk} . \]

Thus, one has the obvious inequalities

\[ \sigma_{p_1 p_1} \geq 0, \quad \sigma_{p_1 Q_k} \geq 0, \quad k = 1, 2 \]

accompanied by the Schrödinger–Robertson \([3, 4]\) inequalities for each mode:

\[ \sigma_{p_1 Q_k} \sigma_{Q_k Q_1} - \frac{1}{2} \sigma_{Q_k Q_k} \geq 0, \quad k = 1, 2 . \]

Besides, there are inequalities that are cubic in variances and covariances, as

\[ \det \begin{pmatrix} \sigma_{p_1 p_1} & \sigma_{p_1 p_2} & \sigma_{p_1 Q_1} & \sigma_{p_2 Q_1} & \sigma_{p_2 Q_2} \\ \sigma_{p_1 p_1} & \sigma_{p_1 p_2} & \sigma_{p_2 Q_1} & \sigma_{p_2 Q_1} & \sigma_{p_2 Q_2} \\ \sigma_{Q_1 p_1} & \sigma_{Q_1 p_2} & \sigma_{Q_1 Q_1} & \sigma_{Q_1 Q_2} & \sigma_{Q_2 Q_1} \\ \sigma_{Q_2 p_1} & \sigma_{Q_2 p_2} & \sigma_{Q_2 Q_1} & \sigma_{Q_2 Q_2} & \sigma_{Q_2 Q_1} \\ \sigma_{Q_1 p_2} & \sigma_{Q_1 Q_1} & \sigma_{Q_1 Q_2} & \sigma_{Q_2 Q_1} & \sigma_{Q_2 Q_2} \end{pmatrix} \geq 0 . \]

Also, one has a quartic inequality that is equivalent to the non-negativity of the fourth principal minor of Σ:

\[ \det \begin{pmatrix} \sigma_{p_1 p_1} & \sigma_{p_1 p_2} & \sigma_{p_1 Q_1} & \sigma_{p_2 Q_1} & \sigma_{p_2 Q_2} \\ \sigma_{p_1 p_1} & \sigma_{p_1 p_2} & \sigma_{p_2 Q_1} & \sigma_{p_2 Q_1} & \sigma_{p_2 Q_2} \\ \sigma_{Q_1 p_1} & \sigma_{Q_1 p_2} & \sigma_{Q_1 Q_1} & \sigma_{Q_1 Q_2} & \sigma_{Q_2 Q_1} \\ \sigma_{Q_2 p_1} & \sigma_{Q_2 p_2} & \sigma_{Q_2 Q_1} & \sigma_{Q_2 Q_2} & \sigma_{Q_2 Q_1} \\ \sigma_{Q_1 p_2} & \sigma_{Q_1 Q_1} & \sigma_{Q_1 Q_2} & \sigma_{Q_2 Q_1} & \sigma_{Q_2 Q_2} \end{pmatrix} \geq \frac{1}{16} . \]

These inequalities can be checked by measuring the matrix elements of the dispersion matrix. This can be done in the tomographic approach.

3. Tomograms—symplectic and optical

The two-mode symplectic tomogram was introduced in \([17]\). Let us consider two homodyne quadratures

\[ \hat{X}_1 = \mu_1 Q_1 + \nu_1 P_1, \quad \hat{X}_2 = \mu_2 Q_2 + \nu_2 P_2 , \]

where the Qs and Ps are the usual position and momentum operators. The symplectic tomogram of a two-mode density state \(\hat{\rho}(1, 2)\) depends on six real variables and reads \([18]\)

\[ M (X_1, \mu_1, \nu_1; X_2, \mu_2, \nu_2) = \text{Tr} [\hat{\rho}(1, 2) \delta (X_1 \hat{I} - \hat{X}_1) \delta (X_2 \hat{I} - \hat{X}_2)] . \]

In terms of the Wigner function \(W\) the symplectic tomogram reads

\[ M (X_1, \mu_1, \nu_1; X_2, \mu_2, \nu_2) = \int W (q_1, p_1; q_2, p_2) \delta (X_1 - \mu_1 q_1 - \nu_1 p_1) \]

\[ \times \delta (X_2 - \mu_2 q_2 - \nu_2 p_2) \frac{dq_1 dp_1 dq_2 dp_2}{4\pi^2} \]

and it is non-negative and normalized:

\[ \int M (X_1, \mu_1, \nu_1; X_2, \mu_2, \nu_2) dX_1 dX_2 = 1 . \]

For

\[ \mu_1 = \cos \theta_1, \quad \nu_1 = \sin \theta_1; \quad \mu_2 = \cos \theta_2, \quad \nu_2 = \sin \theta_2, \]

one obtains the optical two-mode tomogram depending on four essential real variables:

\[ W (X_1, \theta_1; X_2, \theta_2) = M (X_1, \mu_1 = \cos \theta_1, \nu_1 = \sin \theta_1; \]

\[ X_2, \mu_2 = \cos \theta_2, \nu_2 = \sin \theta_2) . \]

This tomogram is the joint probability distribution of \(X_1\) and \(X_2\). In view of this, one has

\[ W^{(1)} (X_1, \theta_1) = \int W (X_1, \theta_1; X_2, \theta_2) dX_2 , \]

\[ W^{(2)} (X_2, \theta_2) = \int W (X_1, \theta_1; X_2, \theta_2) dX_1 , \]

where \(W^{(1)} (X_1, \theta_1)\) and \(W^{(2)} (X_2, \theta_2)\) are the optical tomograms of the first and second modes, respectively. In
fact, $W(X_1, \theta_1; X_2, \theta_2)$ is a function of four variables, and integration over one variable would be expected to yield a function of three variables. In contrast, the above formulae show that integration of the two-mode tomogram over a random variable $X_k$ gives a function independent of the associated variable parameter $\theta_k$. However, this property seems obvious in view of the physical meaning of a tomogram as the joint probability density of two random position variables, measured in new reference frames in phase space, rotated by angles $\theta_1$ and $\theta_2$. Since the tomogram of the first mode state is a marginal of the joint probability distribution, the integration over the second mode position washes out any information about the reference frame where the integrated position was measured. Such a property takes place also for other tomographic probability distributions like spin tomograms of multi-qudit states [18].

It is worth noting that for the Wigner function of a one-mode tomogram

$$W^{(1)}(q_1, p_1) = \int W(q_1, p_1; q_2, p_2) \frac{dq_2 dp_2}{2\pi}, \quad \text{(15)}$$

The one-mode tomograms $W^{(k)}$ are related to the corresponding Wigner functions of the states as

$$W^{(k)}(X_k, \theta_k) = \int W^{(k)}(q_k, p_k) \delta(X_k - \cos \theta_k q_k - \sin \theta_k p_k) \frac{dq_k dp_k}{2\pi}$$

with $k = 1, 2$.

From equation (8), one obtains

$$\hat{X}_k^2 = \mu_k^2 Q_k^2 + v_k^2 P_k^2 + 2\mu_k v_k \left( \frac{Q_k P_k + P_k Q_k}{2} \right) \quad (k = 1, 2) \quad \text{(16)}$$

and

$$\hat{X}_k \hat{X}_j = \mu_k \mu_j Q_k Q_j + v_k v_j P_k P_j + \mu_k v_j P_k Q_j + v_k \mu_j P_k Q_j \quad (j \neq k = 1, 2). \quad \text{(17)}$$

Due to the physical meaning of the tomogram as a probability distribution, for homodyne quadratures one has

$$\text{Tr}[\hat{\rho}(1,2)\hat{X}_k^2] = \int X_k^2 W^{(k)}(X_k, \theta_k) dX_k \quad (n = 0, 1, 2, 3, \ldots) \quad \text{(18)}$$

and

$$\text{Tr}[\hat{\rho}(1,2)\hat{X}_k \hat{X}_j] = \int X_k X_j W(X_1, \theta_1; X_2, \theta_2) dX_1 dX_2 \quad (j, k = 1, 2). \quad \text{(19)}$$

Bearing in mind the relations (16) and (17), by means of the optical tomograms $W(X_1, \theta_1; X_2, \theta_2)$, $W^{(1)}(X_1, \theta_1)$, $W^{(2)}(X_2, \theta_2)$, one can express the matrix elements of the dispersion matrix of $\Sigma$ (equation (1)) in terms of integrals on the right-hand side of equations (18) and (19). In fact, one obtains

$$\sigma_{Q_k Q_k} = \int X_k^2 W^{(k)}(X_k, \theta_k = 0) dX_k$$

$$- \left( \int X_k W^{(k)}(X_k, \theta_k = 0) dX_k \right)^2 \quad \text{(20)}$$

and

$$\sigma_{P_k P_k} = \int X_k^2 W^{(k)}(X_k, \theta_k = \pi/2) dX_k$$

$$- \left( \int X_k W^{(k)}(X_k, \theta_k = \pi/2) dX_k \right)^2, \quad \text{(21)}$$

with $k = 1, 2$. Moreover,

$$\sigma_{Q_k Q_k} = \sigma_{X_k X_k} (\pi/2) - \frac{1}{2} \sigma_{X_k X_k}(0) - \frac{1}{2} \sigma_{X_k X_k}(\pi/2). \quad \text{(22)}$$

One can see that it is possible to measure in terms of tomograms all the highest moments of the quadratures

$$(Q_k^m P_k^m) = \text{Tr}[\hat{\rho}(1, 2) Q_k^m P_k^m] \quad (k = 1, 2). \quad \text{(23)}$$

For example, in the case of cubic moments, one has

$$(Q_k^3) = \int X_k^3 W^{(k)}(X_k, \theta_k = 0) dX_k; \quad \text{(24)}$$

$$(P_k^3) = \int X_k^3 W^{(k)}(X_k, \theta_k = \pi/2) dX_k.$$

By using

$$\hat{X}_k^3 = \mu_k^3 Q_k^3 + v_k^3 P_k^3 + 3\mu_k^2 v_k Q_k P_k + 3\mu_k v_k^2 P_k Q_k + 3\mu_k^2 P_k^2 + 3\mu_k v_k^2 Q_k^2 \quad \text{(25)}$$

and the commutation relations of the quadratures, one obtains

$$Q_k P_k Q_k = P_k Q_k^2 + i Q_k, \quad Q_k^2 P_k = P_k Q_k^2 + 2i Q_k. \quad \text{(26)}$$

and similarly,

$$P_k Q_k P_k = P_k Q_k^2 + i P_k, \quad Q_k^2 P_k = P_k Q_k^2 + 2i P_k. \quad \text{(27)}$$

so that

$$\hat{X}_k^3 = \mu_k^3 Q_k^3 + v_k^3 P_k^3 + 3\mu_k^2 v_k Q_k^2 + 3\mu_k v_k^2 P_k^2 + 3\mu_k^2 P_k^2 + 3\mu_k v_k^2 Q_k^2 + 3\mu_k^2 Q_k^2 + 3\mu_k v_k^2 P_k^2.$$ \quad \text{(28)}$

4. Marginals of two-mode tomograms

Let us consider light modes that can be obtained by means of optical devices from two initial ones described by operators $a$ and $b$ satisfying the commutation relations

$$[a, a^\dagger] = [b, b^\dagger] = 1, \quad [a, b] = [a, b^\dagger] = [a^\dagger, b] = 0. \quad \text{(29)}$$

Then one can make symplectic transformations and obtain the modes described by the operators

$$c = \frac{1}{\sqrt{2}} (a + b), \quad d = \frac{1}{\sqrt{2}} (a - b), \quad \text{(30)}$$

$$e = \frac{1}{\sqrt{2}} (a + ib), \quad f = \frac{1}{\sqrt{2}} (a - ib). \quad \text{(31)}$$
One can readily check that
\[ [e, e^\dagger] = [d, d^\dagger] = [f, f^\dagger] = [f^\dagger, f] = 1. \] (31)

The initial modes can be expressed in terms of quadratures:
\[ a = \frac{1}{\sqrt{2}} (Q_1 + iP_1), \quad b = \frac{1}{\sqrt{2}} (Q_2 + iP_2). \] (32)

Besides, one has homodyne quadrature operators for each of these modes
\[ \hat{X}_a (\mu_1, v_1) = (\mu_1 Q_1 + v_1 P_1), \quad \hat{X}_b (\mu_2, v_2) = \mu_2 Q_2 + v_2 P_2, \] (33)

where one can take local oscillator phases to set
\[ \mu_1 = \cos \theta_1, \quad v_1 = \sin \theta_1; \quad \mu_2 = \cos \theta_2, \quad v_2 = \sin \theta_2. \] (34)

Then one can consider the homodyne quadrature operators for all four modes given by (30) in terms of initial quadratures operators
\[
\begin{align*}
\hat{X}_c (\mu_3, v_3) &= \frac{1}{2} \mu_3 (Q_1 + Q_2) + \frac{1}{2} v_3 (P_1 + P_2), \\
\hat{X}_d (\mu_4, v_4) &= \frac{1}{2} \mu_4 (Q_1 - Q_2) + \frac{1}{2} v_4 (P_1 - P_2), \\
\hat{X}_e (\mu_5, v_5) &= \frac{1}{2} \mu_5 (Q_1 - Q_2) + \frac{1}{2} v_5 (Q_2 + P_1), \\
\hat{X}_f (\mu_6, v_6) &= \frac{1}{2} \mu_6 (Q_1 + P_2) + \frac{1}{2} v_6 (P_1 - Q_2).
\end{align*}
\] (35)

We now shift from labels \( a, b, c, d, e, f \) to labels \( 1, 2, 3, 4, 5, 6 \), respectively, so that \( \hat{X}_a (\mu_1, v_1) \rightarrow \hat{X}_1 (\mu_1, v_1), \quad \hat{X}_b (\mu_2, v_2) \rightarrow \hat{X}_2 (\mu_2, v_2), \) and so on. In the above equations the parameters may be chosen as
\[ \mu_k = \cos \theta_k, \quad v_k = \sin \theta_k \quad (k = 3, 4, 5, 6). \]

The above equations (35) can be given a vector form:
\[ \hat{X} = S \hat{Q}, \] (36)

where the vectors \( \hat{X}, \hat{Q} \) have operator components \( \hat{X}_1, \hat{X}_2, \hat{X}_3, \hat{X}_4, \hat{X}_5, \hat{X}_6 \) and \( P_1, P_2, Q_1, Q_2, \) respectively, while the matrix \( S \) reads
\[ S = \frac{1}{2} \begin{pmatrix}
  v_3 & v_3 & \mu_3 & \mu_3 \\
  v_4 & -v_4 & \mu_4 & -\mu_4 \\
 v_5 & -v_5 & \mu_5 & v_5 \\
 v_6 & v_6 & -\mu_6 & \mu_6
\end{pmatrix} \] (37)

and is invertible, as \( \text{det} S \neq 0 \) in the generic case. So one can solve with respect to \( \hat{Q} \) and obtain
\[ \hat{Q} = S^{-1} \hat{X} \] (38)

or, taking mean values,
\[ \langle \hat{Q} \rangle = S^{-1} \langle \hat{X} \rangle, \] (39)

where operators are averaged in the density state \( \rho : \langle \hat{A} \rangle := \text{Tr}(\rho \hat{A}). \)

One-mode homodyne detectors can measure the six optical tomograms
\[ W_k (X_k, \theta_k) = \text{Tr} (\rho \hat{X}_k) \quad (k = 1, 2, \ldots, 6). \] (40)

For each \( k \)-mode one has the Schrödinger–Robertson inequality expressed in terms of measured tomograms (1) associated with the two-mode light state
\[ F (\theta_k) = \left( \int X_k^2 W_k (X_k, \theta_k) dX_k \right) - \left[ \int X_k W_k (X_k, \theta_k) dX_k \right]^2 \] (41)

From the homodyne quadrature operators (33), one obtains
\[ Q_1 = \hat{X}_1 (1, 0), \quad P_1 = \hat{X}_1 (0, 1); \quad Q_2 = \hat{X}_2 (1, 0), \] (42)

\[ P_2 = \hat{X}_2 (0, 1). \]

Moreover,
\[ Q_1^2 = \hat{X}_1^2 (1, 0), \quad P_1^2 = \hat{X}_1^2 (0, 1), \]
\[ \frac{1}{2} \{ Q_1, P_1 \} = \hat{X}_1^2 \left( \frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2} \right) - \frac{1}{2} \{ \hat{X}_1^2 (1, 0) + \hat{X}_1^2 (0, 1) \} \] (43)

and, analogously, turning label 1 into label 2:
\[ Q_2^2 = \hat{X}_2^2 (1, 0), \quad P_2^2 = \hat{X}_2^2 (0, 1), \]
\[ \frac{1}{2} \{ Q_2, P_2 \} = \hat{X}_2^2 \left( \frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2} \right) - \frac{1}{2} \{ \hat{X}_2^2 (1, 0) + \hat{X}_2^2 (0, 1) \}. \] (44)

Also,
\[ Q_1 Q_2 = 2 \hat{X}_3^2 (0, 1) - \frac{1}{2} \{ \hat{X}_1^2 (1, 0) + \hat{X}_2^2 (1, 0) \}, \]
\[ P_1 P_2 = 2 \hat{X}_3^2 (0, 1) - \frac{1}{2} \{ \hat{X}_1^2 (0, 1) + \hat{X}_2^2 (0, 1) \}. \] (45)
\[ Q_1 P_2 = -2 \hat{X}_2^2 (1, 0) + \frac{1}{2} [\hat{X}_1^2 (1, 0) + \hat{X}_2^2 (0, 1)]. \]  
\[ Q_2 P_1 = 2 \hat{X}_2^2 (0, 1) - \frac{1}{2} [\hat{X}_1^2 (0, 1) + \hat{X}_2^2 (1, 0)]. \]  

(46)

The obtained equalities allow us to express the Robertson uncertainty relations in terms of homodyne tomograms which are experimentally measured. There are also relations that are compatible with the properties of the different homodyne quadratures (33) and (35), for example

\[ \hat{X}_3 (1, 0) = \frac{1}{2} \{ \hat{X}_1 (1, 0) + \hat{X}_2 (1, 0) \} \]  

(47)

and many others including quadrature equalities.

The expressions for variances and covariances of the two-mode field quadrature components are, with \( k = 1, 2, \)

\[ \sigma_{Q_k Q_k} = \left( \int X_k^2 \mathcal{W}_k (X_k, \theta_k = 0) dX_k \right) - \left[ \int X_k \mathcal{W}_k (X_k, \theta_k = 0) dX_k \right]^2. \]

\[ \sigma_{P_k P_k} = \left( \int X_k^2 \mathcal{W}_k (X_k, \theta_k = \frac{\pi}{2}) dX_k \right) - \left[ \int X_k \mathcal{W}_k (X_k, \theta_k = \frac{\pi}{2}) dX_k \right]^2. \]

\[ \sigma_{Q_k P_k} = \left( \int X_k^2 \mathcal{W}_k (X_k, \theta_k = \frac{\pi}{4}) dX_k \right) - \left[ \int X_k \mathcal{W}_k (X_k, \theta_k = \frac{\pi}{4}) dX_k \right]^2 \]

\[- \frac{1}{2} (\sigma_{Q_k Q_k} + \sigma_{P_k P_k}). \]  

(48)

The same expressions appear in formula (41) defining \( F (\theta_k) \).

Besides, in view of (45), one has

\[ \sigma_{Q_k Q_k} = 2 \left( \int X_k^2 \mathcal{W}_k (X_k, \theta_k = 0) dX_k \right) - \left[ \int X_k \mathcal{W}_k (X_k, \theta_k = 0) dX_k \right]^2 \]

\[- \frac{1}{2} (\sigma_{Q_k Q_k} + \sigma_{Q_k Q_k}). \]

\[ \sigma_{P_k P_k} = 2 \left( \int X_k^2 \mathcal{W}_k (X_k, \theta_k = \frac{\pi}{2}) dX_k \right) - \left[ \int X_k \mathcal{W}_k (X_k, \theta_k = \frac{\pi}{2}) dX_k \right]^2 \]

\[- \frac{1}{2} (\sigma_{P_k P_k} + \sigma_{P_k P_k}). \]  

(49)

and finally, from (46), one obtains

\[ \sigma_{Q_1 P_1} = -2 \left( \int X_1^2 \mathcal{W}_3 (X_3, \theta_3 = 0) dX_3 \right) - \left[ \int X_3 \mathcal{W}_3 (X_3, \theta_3 = 0) dX_3 \right]^2 \]

\[ - \frac{1}{2} (\sigma_{Q_1 Q_1} + \sigma_{P_1 P_1}). \]

(50)

Inserting the obtained variances and covariances into the matrix \( \Sigma \) defined in equation (1), one can calculate all its principal minors in terms of the measured tomograms \( \mathcal{W}_k (X_k, \theta_k) \) and check by direct experimental data the Robertson uncertainty relations by means of the positivity of such minors. One can tell that the experimental data for all six modes are redundant to obtain the dispersion matrix for the quadratures. One could use other mode combinations. For example, we did not use the mode data associated with the tomogram \( \mathcal{W}_k (X_k, \theta_k) \). Another set of tomograms could be used to obtain the dispersion matrix. This variety is useful in obtaining extra control of the accuracy of the measurements because the dispersion must be the same irrespective of the particular set of tomograms used. Thus one has the dispersion matrix with ‘commutator’ contributions in different permutations of basis, for example

\[ \Sigma = \begin{pmatrix}
\sigma_{P_1 P_1} & \sigma_{P_1 P_1} & \sigma_{P_1 Q_1} - \frac{1}{2} & \sigma_{P_1 Q_2} \\
\sigma_{P_2 P_1} & \sigma_{P_2 P_1} & \sigma_{P_2 Q_1} & \sigma_{P_2 Q_2} - \frac{1}{2} \\
\sigma_{Q_1 P_1} + \frac{1}{2} & \sigma_{Q_1 P_1} & \sigma_{Q_1 Q_1} & \sigma_{Q_1 Q_2} \\
\sigma_{Q_1 P_1} & \sigma_{Q_1 P_1} & \sigma_{Q_1 Q_2} & \sigma_{Q_1 Q_2}
\end{pmatrix} \]

and

\[ \Sigma' = \begin{pmatrix}
\sigma_{Q_1 Q_1} & \sigma_{Q_1 Q_1} & \sigma_{Q_1 P_1} + \frac{1}{2} & \sigma_{Q_1 P_2} \\
\sigma_{P_2 Q_1} & \sigma_{P_2 Q_1} & \sigma_{P_2 P_1} & \sigma_{P_2 P_2} + \frac{1}{2} \\
\sigma_{P_1 P_1} - \frac{1}{2} & \sigma_{P_1 P_1} & \sigma_{P_1 P_1} & \sigma_{P_1 P_2} \\
\sigma_{P_1 P_1} & \sigma_{P_1 P_1} & \sigma_{P_1 P_2} & \sigma_{P_1 P_2}
\end{pmatrix}. \]

All the elements of the matrices are expressed in terms of measured tomograms. One has to check the non-negativity of the principal minors. Although theoretically the non-negativity of the principal minors found in one basis induces their non-negativity in all of the other bases, the checking of the experimental data looks different since the order of the checking depends on the basis. For example, in the case of the matrix \( \Sigma \) the second principal minor provides a check of the Schrödinger–Robertson uncertainty relations, while using \( \Sigma' \) the second leading principal minor

\[ M_2 = \sigma_{Q_1 Q_1} \sigma_{P_1 P_1} - \sigma_{Q_1 Q_2}^2 \geq 0, \]  

(51)

one checks a classical property which does not contain the Planck’s constant influence. Thus, to check the uncertainty
relations and to control the accuracy of the data, it seems reasonable to check the non-negativity of the principal minors in all the bases.

The non-negativity of the third leading principal minor yields the constraints obtained from the matrices $\Sigma$ and $\Sigma'$, respectively:

$$
\det \begin{pmatrix}
\sigma_{p_1, p_1} & \sigma_{p_1, p_2} & \sigma_{p_1, Q_1} - \frac{i}{2} \\
\sigma_{p_1, p_1} & \sigma_{p_1, p_2} & \sigma_{p_1, Q_1} + \frac{i}{2} \\
\sigma_{Q_1, Q_1} & \sigma_{Q_1, p_2} & \sigma_{Q_1, Q_1}
\end{pmatrix} \geq 0
$$

and

$$
\det \begin{pmatrix}
\sigma_{Q_1, Q_1} & \sigma_{Q_1, Q_2} & \sigma_{Q_1, p_1} - \frac{i}{2} \\
\sigma_{Q_1, Q_1} & \sigma_{Q_1, Q_2} & \sigma_{Q_1, p_1} + \frac{i}{2} \\
\sigma_{p_1, p_1} & \sigma_{p_1, p_2} & \sigma_{p_1, p_1}
\end{pmatrix} \geq 0.
$$

The last minor is just the determinant of the matrix $\Sigma$ or $\Sigma'$.

5. Measuring highest moments of quadratures by the homodyne detector

Let us first discuss how to measure highest moments for one-mode light, say the $a$-mode given by equation (33), $\hat{X}_1(\mu_1, \nu_1) = \mu_1 Q_1 + \nu_1 P_1$. Mean values, variances and covariances are given in terms of the tomogram $W(\mu_1, \nu_1)$. Let us construct cubic moments. Then, dropping label 1, we obtain

$$
\langle \hat{X}^3 \rangle(\mu, \nu) = \mu^3 \langle Q^3 \rangle + \nu^3 \langle P^3 \rangle + 3 \mu^2 \nu \langle [PQ^2] + i \langle Q \rangle \rangle + 3 \mu \nu^2 \langle [P^2 Q] + i \langle X \rangle \rangle.
$$

The means of the quadratures read

$$
\langle \hat{X} \rangle(1, 0) = \langle Q \rangle, \quad \langle \hat{X} \rangle(0, 1) = \langle P \rangle.
$$

Besides, one has

$$
\langle \hat{X}^3 \rangle(1, 0) = \langle Q^3 \rangle, \quad \langle \hat{X}^3 \rangle(0, 1) = \langle P^3 \rangle
$$

and

$$
\langle \hat{X}^3 \rangle(\mu, \nu) = \mu^3 \langle Q^3 \rangle(1, 0) + \nu^3 \langle X^3 \rangle(0, 1)
+ 3 \mu^2 \nu \langle [PQ^2] + i \langle X \rangle(1, 0)\rangle + 3 \mu \nu^2 \langle [P^2 Q] + i \langle X \rangle(0, 1)\rangle.
$$

Introducing the function

$$
A(\mu, \nu) := \langle \hat{X}^3 \rangle(\mu, \nu) - \mu^3 \langle \hat{X}^3 \rangle(1, 0) - \nu^3 \langle \hat{X}^3 \rangle(0, 1)
- 3 \mu^2 \nu i \langle X \rangle(1, 0) - 3 \mu \nu^2 i \langle X \rangle(0, 1),
$$

we obtain two linear equations for the remaining two moments:

$$
A(\mu_\alpha, \nu_\alpha) = 3 \mu_\alpha^2 \nu_\alpha \langle P Q^2 \rangle + 3 \mu_\alpha \nu_\alpha^2 \langle P^2 Q \rangle,
$$

$$
A(\mu_\beta, \nu_\beta) = 3 \mu_\beta^2 \nu_\beta \langle P Q^2 \rangle + 3 \mu_\beta \nu_\beta^2 \langle P^2 Q \rangle,
$$

which can be readily solved in terms of the homodyne quadratures given by the tomogram $W(X, \theta)$ only.

The previous construction of the solutions

$$
\langle P Q^2 \rangle = \frac{1}{3} \det \begin{pmatrix}
A(\mu_\alpha, \nu_\alpha) & 3 \mu_\alpha \nu_\alpha^2 \\
A(\mu_\beta, \nu_\beta) & 3 \mu_\beta \nu_\beta^2
\end{pmatrix},
$$

$$
\langle P^2 Q \rangle = \frac{1}{3} \det \begin{pmatrix}
3 \mu_\alpha^2 \nu_\alpha & A(\mu_\alpha, \nu_\alpha) \\
3 \mu_\beta^2 \nu_\beta & A(\mu_\beta, \nu_\beta)
\end{pmatrix},
$$

with

$$
\Delta = \det \begin{pmatrix}
3 \mu_\alpha^2 \nu_\alpha & 3 \mu_\alpha \nu_\alpha^2 \\
3 \mu_\beta^2 \nu_\beta & 3 \mu_\beta \nu_\beta^2
\end{pmatrix},
$$

shows that the same procedure can be applied to obtain all the highest moments $\langle P^n Q^m \rangle$ and $\langle P^m Q^n \rangle$ ($n, m = 0, 1, \ldots$) in terms of the tomogram $W(X, \theta)$ only. It provides the tool to check all the known high moments quantum uncertainty relations [19] in fact both in the one-mode and in the multi-mode case. As an example we derive simple uncertainty relations for cubic moments.

Let us consider the linear forms:

$$
\hat{f} = y_1 Q + y_2 P^2, \quad \hat{f}^* = y_1^* Q + y_2^* P^2.
$$

The obvious inequality for the mean value $\langle \hat{f} \hat{f}^* \rangle \geq 0$ gives a condition of non-negativity for the quadratic form

$$
\langle Q^2 \rangle \langle P^4 \rangle - \langle PQ^3 \rangle \langle PQ \rangle \geq 0.
$$

This inequality can be written in terms of tomograms as

$$
\int X^2 W(X, \theta) \, dX \int X^4 W(X, \theta = \frac{\pi}{2}) \, dX
- \left[ \langle PQ^3 \rangle \langle P^2 Q \rangle \right]_{\theta_0, \omega_0} \geq 0,
$$

where local oscillators, for instance $\theta_0 = \pi/3, \omega_0 = 2\pi/3$, are taken in equation (58), so that the parameters $\langle \mu_\alpha, \nu_\alpha \rangle$ and $\langle \mu_\beta, \nu_\beta \rangle$ are $(\sqrt{3}/2, 1/2)$ and $(1/2, \sqrt{3}/2)$, respectively. Of course, one could use other suitable local oscillator phases, such as $\Delta \neq 0$ in equation (58). The above cubic-in-quadrature uncertainty relation must be satisfied by any of the six modes used in experiments [11].

In view of the generalization proposed in [1] for the Schrödinger–Robertson uncertainty relations, an analogous generalization can be proposed for the above highest-order moments inequality, which can be written in covariant form, i.e. for all the local oscillator phases, as

$$
\int X^2 W(X, \theta) \, dX \int X^4 W(X, \theta = \frac{\pi}{2}) \, dX
- \left[ \langle PQ^3 \rangle \langle P^2 Q \rangle \right]_{\theta_0, \omega_0} \geq 0,
$$

where, as before, equation (58) has to be used with the new values of local oscillator phases, say $\theta + \pi/3, \omega + 2\pi/3$.

6. State reconstruction

The one-mode measurement can be used to obtain complete information about the two-mode state. In fact, the complete information is contained in the symplectic tomogram $M(X_1, \mu_1; X_2, \mu_2, \nu_2)$ or in the optical tomogram...
The knowledge of all moments allows the reconstruction of where the moments explicitly are reads
tomogram or density operator is expressed in terms of all the quadrature moments of two-mode light in terms following one. The marginals give us the possibility to find by measuring several one-mode density operators is the example of a two-mode field.

state tomogram by measuring only a set of suitably chosen that all the highest moments of the multi-mode field can be measured? The answer is positive, in fact. It was shown [24] that a symplectic transformation \( V \) acts on the Wigner function or tomogram of the anti-transformed state \( V^{-1} \). In experiment [11], the one-mode tomograms are measured for such symplectically transformed states. Thus one has the marginal probability distributions depending on a set of parameters sufficient to find out the two-mode state tomogram (the Wigner function or density operator).

This property may be understood in general terms. The quadratures of the multi-mode field close on the Lie algebra of the Weyl–Heisenberg group. All the highest quadrature moments are determined by the elements of the enveloping of this Weyl–Heisenberg algebra. Then, any new basis of the Lie algebra obtained by the initial one by a linear invertible transformation gives rise to the same enveloping algebra. So, having new transformed modes and measuring the corresponding one-mode tomograms makes it possible that all the highest moments of the multi-mode field can be found. This means that one can reconstruct the multi-mode state tomogram by measuring only a set of suitably chosen one-mode tomograms. We now show this procedure through the example of a two-mode field.

The idea of reconstructing the two-mode density operator by measuring several one-mode density operators is the following one. The marginals give us the possibility to find all the quadrature moments of two-mode light in terms of one-mode tomograms only. Then the Wigner function, tomogram or density operator is expressed in terms of moments. For example, the characteristic function of the tomogram, which is the Fourier transform of the tomogram, reads

\[
\hat{W} (K_1, \theta_1; K_2, \theta_2) = \int W (X_1, \theta_1; X_2, \theta_2) \, dX_1 \, dX_2
\]

\[
= \sum_{n, m=0}^{\infty} \frac{(iK_1)^n (iK_2)^m}{n!m!} \langle X^n_1 X^m_2 \rangle (\theta_1, \theta_2),
\]

where the moments explicitly are

\[
\langle X^n_1 X^m_2 \rangle (\theta_1, \theta_2) = \int X^n_1 X^m_2 W (X_1, \theta_1; X_2, \theta_2) \, dX_1 \, dX_2.
\]

The knowledge of all moments allows the reconstruction of the characteristic function. Then the tomogram is given by a Fourier anti-transform:

\[
W (X_1, \theta_1; X_2, \theta_2) = \int \hat{W} (K_1, \theta_1; K_2, \theta_2) \exp [-i (K_1 X_1 + K_2 X_2)] \, dK_1 \, dK_2.
\]

7. Photon statistics

One can formulate the problem of measuring a state in terms of photon statistical properties of the measured two-mode light. For Gaussian states the photon statistics is described by multi-variable Hermite polynomials. For a small number of photons, the expressions can be easily constructed. The photon distributions are determined by the highest moments

\[
\langle \hat{n}_1 \hat{n}_2 \rangle = \text{Tr}(\rho \hat{n}_1 \hat{n}_2).
\]
where  
\[ \hat{n}_1 = \frac{1}{2} \left( Q_1^2 + P_1^2 - 1 \right), \quad \hat{n}_2 = \frac{1}{2} \left( Q_2^2 + P_2^2 - 1 \right). \]  
(73)

Thus, measuring the photon statistics implies measuring the photon quadrature highest moments. Since highest moments satisfy the quantum uncertainty relations, the photon statistics (quantum correlations) demonstrate difference with properties of the classical electromagnetic field. The photon statistics can be studied using measured optical (symplectic) tomograms. An experimental check of the quantum uncertainty relations serves not only to investigate the degree of accuracy with which nowadays the uncertainty relations are known to be fulfilled. Since there is no doubt that the quantum mechanics is a correct theory and the uncertainty relations must be fulfilled, the results of the experiments can serve also to control the correctness of the experimental tools used in homodyne detecting photon states. There exist inequalities in which the highest moments of quadrature components are involved (see, e.g., the review [19]). One can reformulate these highest-order inequalities in terms of tomographic quadrature moments given for example in equation (61) and to obtain extra inequalities expressed in terms of the experimental values of the optical tomogram. Moreover, we suggested the possibility of using the covariant form of equation (61), given by equation (63), which is more suitable for an experimental check. The tomographic probability approach can also be applied for two-mode and multi-mode photon states, especially for Gaussian states for which their properties like photon statistics are sufficiently known.

Thus, the photon distribution function for the two-mode field is explicitly given in [22] in terms of Hermite polynomials of four variables, related to quadrature variances and covariances of the Gaussian field states.

Thus, measuring both photon statistics and optical tomograms provides the possibility of cross checking of the quantum inequalities for the quadrature highest moments.

8. Conclusions

To summarize, we list the main results of this paper. For the two-mode quantum field we express the photon quadrature uncertainty relations, like the Robertson ones, in terms of measurable optical tomograms of a one-mode quantum electromagnetic field. We suggest using the given tomographic expression of Robertson’s inequality to control the accuracy of the homodyne photon state detection. Also, we give examples of inequalities for highest moments of the photon quadratures for a one-mode field. We have expressed all the inequalities in tomographic form, in particular in covariant form; this is suitable for experimental checking. Such checking, to the best of our knowledge, has not yet been done due to the absence of a technique appropriate for the experimental verification of these basic inequalities, which can be violated in the classical domain. We have connected the checking of the photon statistics to the possible suggested experimental checking of the quadrature statistics. The generalization of the tomographic approach to study the Robertson uncertainty relations to the multi-mode field is shown to be straightforward.

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