Computing Covers under Substring Consistent
Equivalence Relations

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Abstract

Covers are a kind of quasiperiodicity in strings. A string \( C \) is a cover of another string \( T \) if any position of \( T \) is inside some occurrence of \( C \) in \( T \). The literature has proposed linear-time algorithms computing longest and shortest cover arrays taking border arrays as input. An equivalence relation \( \approx \) over strings is called a substring consistent equivalence relation (SCER) iff \( X \approx Y \) implies (1) \( |X| = |Y| \) and (2) \( X[i : j] \approx Y[i : j] \) for all \( 1 \leq i \leq j \leq |X| \). In this paper, we generalize the notion of covers for SCERs and prove that existing algorithms to compute the shortest cover array and the longest cover array of a string \( T \) under the identity relation will work for any SCERs taking the accordingly generalized border arrays.

1 Introduction

Finding regularities in strings is an important task in string processing due to its applications such as pattern matching and string compression. Many variants of regularities in strings have been studied including periods, covers, and seeds. One of the most studied regularities is periods due to their mathematical combinatoric properties and their applications to string processing algorithms \[10\]. Apostolico and Giancarlo \[6\] studied periods on parameterized strings and showed some properties of periodicity of parameterized strings in \[6\].

Covers are another kind of regularities that have extensively been studied. For two strings \( T \) and \( C \), \( C \) is a cover of \( T \) if any position of \( T \) is inside some occurrences of \( C \) in \( T \). For example, \( \text{aba} \) is a cover of \( T = \text{abaababaababaaba} \) because all positions in \( T \) are inside occurrences of \( \text{aba} \). The other covers of \( T \) are \( \text{abaababaababa} \) and \( T \) itself. Apostolico and Ehrenfeucht \[4\] called a string having a cover besides itself quasiperiodic and proposed an algorithm that computes all maximal quasiperiodic substrings of a string. Later, Brodal and Pedersen \[9\] proposed \( O(n \log n) \) time algorithm for this task. Apostolico et al. \[5\] presented an algorithm to test whether a string is quasiperiodic that runs in \( O(n) \) time. Breslauer \[8\] proposed an online linear-time algorithm that computes the shortest covers of all prefixes as the shortest cover array of a string.
Table 1: The time complexity of the proposed algorithm on some SCERs. \( n \) is the length of inputs, \( \Pi \) is a set of parameters in parameterized equivalence, and \( k \) is the number of input strings in permuted equivalence.

| Equivalence relation     | Border     | SCover     | LCover     |
|--------------------------|------------|------------|------------|
| Identity equivalence     | \( O(n) \) | \( O(n) \) | \( O(n) \) |
| Parameterized equivalence| \( O(n \log |\Pi|) \) | \( O(n \log |\Pi|) \) | \( O(n \log |\Pi|) \) |
| Order-isomorphism        | \( O(n \log n) \) | \( O(n \log n) \) | \( O(n \log n) \) |
| Permuted equivalence     | \( O(nk) \) | \( O(nk) \) | \( O(nk) \) |

Moore and Smyth \cite{Moore2000, Smyth2000} proposed a linear-time algorithm to compute all covers of a string. Later, Li and Smyth \cite{Li2004} proposed an online algorithm to compute the longest cover array of a string in linear time. Recently, Matsuoka et al. \cite{Matsuoka2016} introduced the notion of substring consistent equivalence relations (SCERs), which is an equivalence relation \( \approx \) on strings such that \( X \approx Y \) implies (1) \( |X| = |Y| \) and (2) \( X[i : j] \approx Y[i : j] \) for all \( 1 \leq i \leq j \leq |X| \), where \( X[i : j] \) denotes the substring of \( X \) starting at \( i \) and ending at \( j \).

Clearly, the identity relation is an SCER. Moreover, many variants of equivalence relations used in pattern matching are SCERs, such as parameterized pattern matching \cite{Breslauer2000}, order-preserving pattern matching \cite{Matsuoka2015}, and permuted pattern matching \cite{Li2003}. Matsuoka et al. \cite{Matsuoka2016} proposed an algorithm to compute the border array of an input string \( T \) under an SCER, which can be used for pattern matching under SCERs.

In this paper, we generalize the notion of covers, which used to be defined based on the identity relation, to be based on SCERs, and prove that both of the algorithms for the shortest and longest cover arrays by Breslauer \cite{Breslauer2000} and Li and Smyth \cite{Li2004}, respectively, work under SCERs with no changes: just by replacing the input of those algorithms from the border array under the identity relation to the one under a concerned SCER, their algorithms compute the shortest and longest cover arrays under the SCER. Table 1 summarizes implications of our results. The time complexities for computing shortest and longest cover arrays based on various SCERs are the same as those for border arrays. Moreover, if border arrays under an equivalence relation can be computed online, e.g., parameterized equivalence and order-isomorphism, these cover arrays can be computed online by computing border arrays with existing online algorithms at the same time.

## 2 Preliminaries

Let \( \Sigma \) be an alphabet and \( \Sigma^* \) the set of all strings over \( \Sigma \). Let \( \Sigma^k \) denote the set of strings of length \( k \) over \( \Sigma \) and \( \varepsilon \) denote the empty string, the string of length 0. For a string \( T \in \Sigma^* \), \( |T| \) denotes the length of \( T \). Let \( T[i] \) denote the \( i \)-th character of \( T \) and \( T[i : j] \) denote the substring of \( T \) that starts at \( i \) and ends at \( j \), for \( 1 \leq i \leq j \leq |T| \). Let \( T[1 : j] = T[i : j] \) denote the prefix of \( T \) that ends at \( j \) and \( T[i :] = T[i : |T|] \) denote the suffix of \( T \) that starts at \( i \). Throughout the paper, let \( T \) be an input string of length \( n \).

Matsuoka et al. \cite{Matsuoka2016} introduced the notion of substring consistent equivalence relations, generalizing several equivalence relations proposed so far in pattern matching.
Definition 1 (Substring Consistent Equivalence Relation (SCER) ≈). An equivalence relation \( \approx \subseteq \Sigma^* \times \Sigma^* \) is an SCER if for two strings \( X \) and \( Y \), \( X \approx Y \) implies (1) \( |X| = |Y| \) and (2) \( X[i : j] \approx Y[i : j] \) for all \( 1 \leq i \leq j \leq |X| \).

For instance, matching relations in parameterized pattern matching \cite{7}, order-preserving pattern matching \cite{13, 16}, and permuted pattern matching \cite{13} are SCERs, while matching relations in indeterminate string pattern matching \cite{3} and function matching \cite{1} are not.

We show definitions of matching relations in parameterized pattern matching and order-preserving pattern matching.

Definition 2 (Parameterized equivalence \cite{7}). Let \( \Sigma \) and \( \Pi \) be disjoint finite alphabets. Symbols in \( \Sigma \) are called constants, while symbols in \( \Pi \) are called parameters. Two strings \( X \) and \( Y \) on \( (\Sigma \cup \Pi)^* \) are a parameterized match, denoted as \( X \approx Y \), if \( X \) can be transformed into \( Y \) by applying a renaming bijection \( g \) from the symbols of \( X \) to the symbols of \( Y \), such that \( g \) is the identity on the constant symbols.

Definition 3 (Order-isomorphism \cite{14, 16}). For two strings \( X \), \( Y \) of the same length over a linearly ordered alphabet, \( X \) and \( Y \) are order isomorphic, denoted as \( X \approx Y \), if the ranks of the characters of \( X \) and \( Y \) at each position are equal, i.e., \( X[i] \prec X[j] \Leftrightarrow Y[i] \prec Y[j] \) for all \( 1 \leq i, j \leq |X| \), where \( \prec \) denotes the lexicographical order of characters.

Definition 4 (≈-occurrence \cite{18}). For two strings \( T \) and \( P \), a position \( 1 \leq i \leq |T| - |P| + 1 \) is a \( ≈ \)-occurrence of \( P \) in \( T \) if \( P \approx T[i : i + |P| - 1] \). The set of \( ≈ \)-occurrence positions of \( P \) in \( T \) is denoted by \( \text{Occ}_{P,T} \).

Matsuoka et al. \cite{18} defined borders of a string under SCERs and show the following properties.

Definition 5 (≈-border \cite{18}). For a string \( T \) of length \( n \), a string \( B \) of length \( b \) is a \( ≈ \)-border of \( T \) if \( B \approx T[:b] \approx T[n - b + 1:] \). We denote by \( \text{Bord}_{\approx}(T) \) the set of all \( ≈ \)-borders of \( T \). A \( ≈ \)-border \( B \) of \( T \) is called proper if \( |B| < |T| \), and called trivial if \( B = \varepsilon \).

Lemma 1 (\cite{18}). (1) \( B \in \text{Bord}_{\approx}(S) \) and \( B' \in \text{Bord}_{\approx}(B) \) implies \( B' \in \text{Bord}_{\approx}(S) \). (2) \( B, B' \in \text{Bord}_{\approx}(S) \) and \( |B'| \leq |B| \) implies \( B' \in \text{Bord}_{\approx}(B) \).

By using the above properties, Matsuoka et al. \cite{18} proposed an algorithm to compute border arrays under SCERs, which are defined as follows.

Definition 6 (≈-border array). The \( ≈ \)-border array \( \text{Border}_T \) of \( T \) is an array of length \( n \) such that \( \text{Border}_T[i] = \max\{|B| \mid B \text{ is a proper } ≈ \text{-border of } T[:i]\} \) for \( 1 \leq i \leq n \).

Tables 2 and 3 show examples of \( ≈ \)-border arrays. We use the identity relation in Table 2 and the parameterized matching relation in Table 3.

The \( ≈ \)-border arrays have the following property which will be used later in our proof.

Lemma 2. For any \( 1 < i \leq n \), \( \text{Border}_T[i - 1] + 1 \geq \text{Border}_T[i] \).
Table 2: The $-$-border array, the shortest $-$-cover array, and the longest $-$-cover array of $T = \text{abaabaabaabaaba}$. 

| $T$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 |
|-----|---|---|---|---|---|---|---|---|---|----|----|----|----|----|----|----|
| Border$_T$ | a | b | a | b | a | b | a | b | a | a | b | a | b | a | b | a |
| SCover$_T$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 3 | 9 | 5 | 12 | 5 | 3 | 15 | 3 |
| LCover$_T$ | 0 | 0 | 0 | 0 | 0 | 3 | 0 | 3 | 0 | 5 | 6 | 0 | 5 | 6 | 0 | 8 |

Table 3: An example of the $\underline{pr}$ $\approx$-border array, the shortest $\underline{pr}$ $\approx$-cover array, and the longest $\underline{pr}$ $\approx$-cover array of $T = \text{abaabaabaabaaba}$. Notice that $\text{SCover}_T[i] = 1$ for all $i$ because $a \approx b$.

| $T$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 |
|-----|---|---|---|---|---|---|---|---|---|----|----|----|----|----|----|----|
| Border$_T$ | a | b | a | b | a | b | a | b | a | a | b | a | b | a | b | a |
| SCover$_T$ | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| LCover$_T$ | 0 | 1 | 2 | 3 | 3 | 3 | 1 | 5 | 6 | 1 | 5 | 6 | 3 | 8 |

Proof. Let $b = \text{Border}_T[i]$. Since $T[i : b] \approx T[i - b + 1 : i]$ by Definition 3, we have $T[b - 1] \approx T[i + 1 : i - 1]$ by Definition 1. Thus $T[b - 1] \in \text{Border}_T[T[i - 1]]$ by Definition 5. $T[i - 1]$ is a proper $\approx$-border by $b < i$, so we get $\text{Border}_T[i - 1] \geq b - 1$. Therefore $\text{Border}_T[i - 1] + 1 \geq b = \text{Border}_T[i]$. 

3 Covers under SCERs

In this section we define covers under SCERs ($\approx$-covers) and show some properties of $\approx$-covers. Then we prove the algorithms to compute shortest cover arrays and longest cover arrays by Breslauer [8] and Li and Smyth [17] will work under SCERs with no change.

Definition 7 ($\approx$-cover). For a string $T$ of length $n$ and a string $C$ of length $m$, let $\text{Occ}_C(T) = \{x_1, x_2, \ldots, x_k\}$ be the set of all $\approx$-occurrences of $C$ in $T$, where $x_i < x_{i+1}$ for any $i < k$. We say that $C$ is a $\approx$-cover of $T$ if $C \in \text{Border}_T(T)$ i.e. $x_1 = 1$, $x_k = n - m + 1$, and for any $i \neq 1$, $x_{i-1} < x_i \leq x_{i-1} + m$. Moreover, we say that a $\approx$-cover $C$ of $T$ is proper if $m < n$.

We say that a string $S$ is primitive if $S$ does not have any proper $\approx$-cover.

Next, we show the properties between $\approx$-borders and $\approx$-covers.

Lemma 3. For any $C \in \text{Cov}_T(T)$ of length $m$ and $B \in \text{Border}_T(T)$ of length $b$ such that $m \leq b$, $C \in \text{Cov}_T(B)$.

Proof. Since $C \in \text{Cov}_T(T)$, we have $C \in \text{Border}_T(T)$ by Definition 5. Moreover, since $B \in \text{Border}_T(T)$, thus $C \in \text{Border}_T(B)$ by Lemma 1 (2). By Definition 5, we have $1, b - m + 1 \in \text{Occ}_C,B$.

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\footnote{In some references it is called superprimitive, reserving the term "primitive" for strings that cannot be represented as $S^k$ for some string $S$ and integer $k \geq 2$.}
Next, let $y \in \text{Occ}_{C,B} \setminus \{1\}$. Since $C \approx B[y : y + m - 1]$ and $y + m - 1 < b < n$, we have $B[y : y + m - 1] \approx T[y : y + m - 1]$. Thus, $C \approx T[y : y + m - 1]$ and $y \in \text{Occ}_{C,T}$. Moreover, since $C \in \text{Cov}_{\approx}(T)$, there exists $x \in \text{Occ}_{C,T}$ such that $x < y \leq x + m$, i.e. $C \approx T[x : x + m - 1]$. By $x < y$, $C \approx B[x : x + m - 1] \approx T[x : x + m - 1]$ which implies $x \in \text{Occ}_{C,B}$. Therefore, for any $y \in \text{Occ}_{C,B} \setminus \{1\}$ there exists $x \in \text{Occ}_{C,B}$ such that $x < y \leq x + m$.

Next, we show some relation between $\approx$-covers of $T$.

**Lemma 4.** For any $C \in \text{Cov}_{\approx}(T)$, $C' \in \text{Cov}_{\approx}(C)$ implies $C' \in \text{Cov}_{\approx}(T)$.

**Proof.** Let $m$ be the length of $C$ and $m'$ be the length of $C'$. By Definition 7, $C \in \text{Bord}_{\approx}(T)$ and $C' \in \text{Bord}_{\approx}(C)$. Thus, $C' \in \text{Bord}_{\approx}(T)$ by Lemma 1 (2).

From Definition 5 we have $1, n - m' + 1 \in \text{Occ}_{C,T}$. Next, let $z \in \text{Occ}_{C,T} \setminus \{1\}$. There exists $x \in \text{Occ}_{C,T}$ such that (1) $x < z < x + m'$ or (2) $x < z \leq x + m < x + z + m'$. In the case (1), since $C' \in \text{Cov}_{\approx}(C)$, there exists $z' < z \leq z' + m'$. In the case (2), since $C' \in \text{Cov}_{\approx}(C)$, we have $C' \approx C[m - m' + 1]$ and $z' < z \leq z' + m'$. Therefore, we get $C' \in \text{Cov}_{\approx}(T)$.

The following lemma holds by Definition 8 and Lemma 3.

**Lemma 5.** For any two $\approx$-covers $C, C' \in \text{Cov}_{\approx}(T)$ such that $|C'| < |C|$, $C' \in \text{Cov}_{\approx}(C)$.

### 3.1 Shortest $\approx$-cover array

In this section we prove that Algorithm 1 by Breslauer 8 computes the shortest $\approx$-cover array for an input string $T$ based on the $\approx$-border array.

The shortest $\approx$-cover array for a string $T$ is defined as follows.

**Definition 8** (Shortest $\approx$-cover array). The shortest $\approx$-cover array $SCov_{\approx}$ of $T$ is an array of length $n$ such that $SCov_{\approx}[i] = \min\{|C| \mid C \in \text{Cov}_{\approx}(T); i\}$ for $1 \leq i \leq n$.

Algorithm 1 uses an additional array $Reach$ to compute $SCov$. The algorithm updates $Reach$ and $SCov$ incrementally so that $Reach[j]$ shall be the
Lemma 6. For any strings $T$ of length $n$ and $C$, $C$ is a proper $\approx$-cover of $T$ iff $C \in \text{Bord}_\approx(T)$ and $C \in \text{Cov}_\approx(T[:n-i])$ for some $1 \leq i \leq m$, where $m$ is the length of $C$.

Proof. ($\implies$) Since $C$ is a proper $\approx$-cover of $T$, $\text{Occ}_{C,T} \setminus \{1\} \neq \emptyset$ and $C \in \text{Bord}_\approx(T)$ by Definition 7. Let $x_k = \max(\text{Occ}_{C,T}) = n - m + 1$. By Definition 4, there exists $x_{k-1} \in \text{Occ}_{C,T}$ such that $x_{k-1} < x_k \leq x_{k-1} + m$. Clearly $C \in \text{Cov}_\approx(T[:x_{k-1} + m - 1])$ and $n - 1 \geq x_{k-1} + m - 1 \geq n - m$.

($\impliedby$) Assume that $C \in \text{Bord}_\approx(T)$ and $C \in \text{Cov}_\approx(T')$, where $T' = T[:n-i]$ for some $1 \leq i \leq m$. Since $C \in \text{Bord}_\approx(T)$, we have $n - m + 1 \in \text{Occ}_{C,T}$. Moreover, by Definition 1, we have $\text{Occ}_{C,T} \subseteq \text{Occ}_{C,T'}$. Let $x_j = (n-i) - m + 1 \in \text{Occ}_{C,T'}$, we have $x_j < n - m + 1 \leq x_j + m$. Therefore, $C$ is a proper $\approx$-cover of $T$.

In each iteration $i$, the algorithm updates $\text{Reach}$ and $\text{SCover}$ so that they satisfy the following properties at the end of the iteration.

\[\text{A}(i)\text{ At the end of the }i\text{-th iteration of the for loop in Algorithm 1} \text{ Reach}[j] > 0 \text{ iff } T[j] \text{ is primitive for } 1 \leq j \leq i.\]

\[\text{B}(i)\text{ At the end of the }i\text{-th iteration of the for loop in Algorithm 1} \text{ for } 1 \leq j \leq i, \text{ if Reach}[j] \neq 0, \text{ Reach}[j] = \max\{p \mid T[p] \text{ is a } \approx\text{-cover of } T[:p] \text{ and } p \leq i\}.\]

\[\text{C}(i)\text{ At the end of the }i\text{-th iteration of the for loop in Algorithm 1} \text{ SCover}[j] = \min\{|C| \mid C \in \text{Cov}_\approx(T[:j])\} \text{ for } 1 \leq j \leq i.\]

Lemma 7. Suppose $\text{A}(i-1)$, $\text{B}(i-1)$, and $\text{C}(i-1)$ hold. Then $\text{A}(i)$ also holds.

Proof. By the assumption, $\text{Reach}[j] > 0$ iff $T[j]$ is primitive for $1 < j \leq i - 1$. It is enough to show that at the end of the $i$-th iteration, $\text{Reach}[i] > 0$ iff $T[i]$ is primitive. Let $b = \text{Border}[i]$ and $c = \text{SCover}[b]$.

($\implies$) In Algorithm 1, $\text{Reach}[i]$ is updated if either $b = 0$ or $\text{Reach}[c] < i - c$. If $b = 0$, $T[i]$ does not have any proper $\approx$-cover. Thus $T[i]$ is primitive and $\text{A}(i)$ holds. Next, consider the case where $b \neq 0$ and $\text{Reach}[c] < i - c$. Assume $T[i]$ is non-primitive, i.e., there exists a proper $\approx$-cover $T[:c']$ of $T[i]$. From Lemma 3, we have $T[:c'] \in \text{Cov}_\approx(T[:b])$. Since $b \leq i - 1$, $T[:c']$ is the shortest $\approx$-cover of $T[:b]$ by $\text{C}(i-1)$ and $T[:c'] \in \text{Cov}_\approx(T[:c'])$ by Lemma 3. Thus we have $T[:c'] \in \text{Cov}_\approx(T[:i])$ by Lemma 3 and $T[:c]$ is proper, which contradicts the assumption.

($\impliedby$) We prove by contraposition. Assume $\text{Reach}[i] = 0$. In Algorithm 1, $\text{Reach}[i] = 0$ if $b > 0$ and $\text{Reach}[c] \geq i - c$ hold. If $b > 0$, we know that $T[i]$ has a proper $\approx$-border $T[:b]$ and $T[i]$ is the shortest $\approx$-cover of $T[:b]$ by $\text{C}(i-1)$. Thus we have $T[i] \in \text{Bord}_\approx(T[:b])$ by Definition 7 and $T[i]$ is a proper $\approx$-border of $T[:i]$ by Lemma 1. Moreover, from $\text{Reach}[c] \geq i - c$, we have $T[:c] \in \text{Cov}_\approx(T[:i-l])$ for some $1 \leq l \leq c$. Therefore, $T[:c]$ is a proper $\approx$-cover of $T[:i]$ by Lemma 8 which implies $T[i]$ is not primitive.

Lemma 8. Suppose $\text{A}(i)$, $\text{B}(i-1)$, and $\text{C}(i-1)$ hold. Then $\text{B}(i)$ also holds.
Lemma 5. Thus, First, we get the correctness of the algorithm. At the first iteration, since $T[i] = i$ is primitive, i.e., there exists no $j < i$ such that $Reach[j] = i$.

Otherwise, if the condition holds, $Reach[c] = i > 0$ is updated, where $c = SCover[Border[i]]$. Assume there exists $j > c$ such that $Reach[j] = i$ and $T[j]$ is primitive. Since $T[c], T[j] \in Cov_{\approx}(T[i])$ and $T[c] \in Cov_{\approx}(T[j])$ by Lemma 5, thus $T[j]$ is not primitive. Therefore, only $Reach[c]$ needs to be updated.

Lemma 9. Suppose $A(i-1), B(i-1),$ and $C(i-1)$ hold. Then $C(i)$ also holds.

Proof. Let $b = Border[i]$ and $c = SCover[b]$. In the case $b = 0$, $T[i]$ does not have a non-trivial proper border. Thus $T[i]$ does not have a proper $\approx$-cover. Therefore, $SCover[i] = i$ is updated correctly.

In the case $b > 0$, $T[i] \in Border[i]$ is the longest proper border of $T[i]$. If $Reach[c] \geq i - c$, since $T[c] \in Border[T[i]]$ by Lemma 1, we have $T[c] \in Cov_{\approx}(T[i])$ by Lemma 5. Moreover, we have $T[c] \in Cov_{\approx}(T[i])$ by Lemma 5. Since $c < i$ and $Reach[c] \neq 0$, $T[c]$ is primitive by $B(i-1)$. Therefore, $T[c]$ is also the shortest $\approx$-cover of $T[i]$ which implies $SCover[i] = SCover[b]$ is updated correctly.

Otherwise, consider the case $Reach[c] < i - c$. Assume that $T[i]$ has a $\approx$-cover $T[c']$ such that $b \geq c' > c$. By Lemma 3, $T[c']$ is a $\approx$-cover of $T[b]$.

Moreover, by Lemma 5, the shortest $\approx$-cover $T[c']$ of $T[b]$ is a $\approx$-cover of $T[c']$. Therefore, $T[c']$ is a $\approx$-cover of $T[i-j]$ for some $1 \leq j \leq c'$ by Lemma 5. However, it contradicts $B(i-1)$. Therefore, $T[i]$ is primitive which implies $SCover[i] = i$ is updated correctly.

Theorem 1. Given a text $T$ of length $n$ and the $\approx$-border array $Border_T$ of $T$, Algorithm 7 computes the shortest $\approx$-cover array $SCover_T$ of $T$ in $O(n)$ time.

Proof. First, we get the correctness of the algorithm. At the first iteration, since $Border_T[1] = 0$, Algorithm 4 updates $Reach[1] = 1$ which is the maximal. Also, since $T[1]$ does not have a proper $\approx$-cover, $T[1]$ is primitive. Thus $A(1), B(1),$ and $C(1)$ hold. By Lemmas 4 and 5, $SCover$ is updated correctly.

Next, we show the time complexity of the algorithm. The number of iterations of the for loop is $n$ and each operation in the loop can be executed in $O(1)$ time. Therefore, Algorithm 4 runs in $O(n)$ time overall.

Corollary 1. If $Border_T$ can be computed in $\beta(n)$ time, $SCover_T$ can be computed in $O(\beta(n) + n)$ time.

3.2 Longest $\approx$-cover array

In this section, we prove that Algorithm 2 by Li and Smyth [17] returns the longest $\approx$-cover array for an input string $T$ using the $\approx$-border array. The longest $\approx$-cover array for a string $T$ is defined as follows.

Definition 9 (Longest $\approx$-cover array). The longest $\approx$-cover array $LCover_T$ of $T$ is an array of length $n$ such that if $T[i]$ is primitive then $LCover_T[i] = 0$, otherwise $LCover_T[i] = \max{|C| \mid C$ is a proper $\approx$-cover of $T[i]}$ for $1 \leq i \leq n$. 
Algorithm 2: Li and Smyth’s algorithm computing the longest \( \approx \)-cover array

1. let \( T \) be the input text of length \( n \);
2. let \( \text{Border} \) be the border array of \( T \);
3. let \( \text{LCover}[0] = -1 \);
4. \( \text{Dead}[i] \leftarrow \text{False}, \text{LSeedChildren}[i] \leftarrow 0, \text{LongestLSeed}[i] \leftarrow i \) for \( 0 \leq i \leq n; \)
5. for \( 1 \leq i \leq n \) do
   6. if \( \text{Dead}[\text{Border}[i]] = \text{True} \) then
      7. \( \text{LongestLSeed}[\text{Border}[i]] \leftarrow \text{LongestLSeed}[\text{LCover}[\text{Border}[i]]]; \)
      8. \( \text{LCover}[i] \leftarrow \text{LongestLSeed}[\text{Border}[i]]; \)
      9. \( \text{LSeedChildren}[\text{LCover}[i]] \leftarrow \text{LSeedChildren}[\text{LCover}[i]] + 1; \)
   10. if \( i > 1 \) then
        11. \( c_1 \leftarrow i - \text{Border}[i]; \)
        12. \( c_2 \leftarrow (i - 1) - \text{Border}[i - 1]; \)
        13. if \( c_1 \neq c_2 \) then
            14. for \( j \) from \( c_1 - 1 \) downto \( c_2 \) do
                15. SetDead\((j)\);

Algorithm 3: SetDead\((i)\)

1. if \( \text{LSeedChildren}[i] = 0 \) and \( \text{Dead}[i] = \text{False} \) then
2. \( \text{Dead}[i] \leftarrow \text{True}; \)
3. \( \text{LSeedChildren}[\text{LCover}[i]] \leftarrow \text{LSeedChildren}[\text{LCover}[i]] - 1; \)
4. SetDead\((\text{LCover}[i]);\)

Their algorithm involves auxiliary arrays of length \( n \) based on the notion of “live” prefixes. A prefix \( S \) of \( T \) is said to be live if \( T \) can be extended so that \( S \) will be a cover of \( TU \) for some \( U \in \Sigma^* \). To compute the longest \( \approx \)-cover array efficiently, it is useful to maintain such “potential” covers. However, this paper takes an alternative notion on which the auxiliary arrays are based.

Definition 10 (left \( \approx \)-seed). For a string \( T \) of length \( n \), a string \( S \) of length \( m \) and an SCER \( \approx \), \( S \) is said to be a left \( \approx \)-seed of \( T \) if there exist \( k \) and \( l \) such that \( k \leq l \leq m \), \( S \in \text{Cov}_{\approx}(T[n-k:]) \) and \( S[l:k] \approx T[n-l+1:] \). We denote by \( \text{LSeed}_{\approx}(T) \) the set of all left \( \approx \)-seeds of \( T \).

In other words, a left \( \approx \)-seed covers a prefix of \( T \) and the uncovered suffix is included in a bigger suffix of \( T \) which is \( \approx \)-equivalent to a prefix of \( S \). It might first appear that to be live and to be a left \( \approx \)-seed are just equivalent. Indeed it is the case under the identity relation, but the latter property is properly weaker than the former under an SCER. Consider the order-isomorphism relation in Definition 3 on \( \Sigma = \{a, b, c, d\} \) with \( a < b < c < d \). Then \( S = \text{acb} \) is a left \( \approx \)-seed of \( T = \text{abc} \), since \( S \preceq \text{abc} \) and \( S[l:k] \preceq T[n-l+1:] \). However, \( S \) is not live with respect to \( T \). For no character \( u \in \Sigma \), we have \( S \preceq (Tu)[4:6] \), since \( u \) needs to be a character bigger than \( b \) and smaller than \( c \) to make \( S \) live. Although the calculation of the auxiliary arrays in the algorithm need not be
altered due to the change of the mathematical definition of the array, we do need to give a new proof for the correctness of the algorithm under SCERS.

Now we show some properties of \(LCover_T\) and \(LSeed_{\approx}\) that can be used to calculate \(LCover_T\) efficiently. Let \(LCover_T^q[i] = LCover_T[LCover_T^{q-1}[i]]\) for \(q > 1\) and \(LCover_T^1[i] = LCover_T[i]\). The following lemma shows that all covers of \(T\) can be find by accessing \(LCover_T\) recursively.

**Lemma 10.** A prefix \([T; i]\) is a \(\approx\)-cover of \([T; j]\) iff \(i = LCover_T^q[j]\) for some \(q \geq 1\).

**Proof.** \((\Rightarrow)\) If \([T; i]\) is the longest \(\approx\)-cover of \([T; j]\), we have \(i = LCover_T[j] = LCover_T^q[j]\) by the definition. Otherwise, if \([T; i] \in Cov_\approx(T; j')\) for \(i \leq j' < j\), there exists \(q\) for such that \(i = LCover_T^q[j']\). Since \(LCover_T[j] < j\), there exists \(q\) such that \(i = LCover_T^q[LCover_T[j]]\) by the assumption. We have \(i = LCover_T^q[LCover_T[j]] = LCover_T^{q+1}[j]\) by the definition.

\((\Leftarrow)\) If \(q = 1\), then \([T; i] \in Cov_\approx(T; j)\) by the definition. Assume \(i = LCover_T^q[j]\) for some \(q \geq 1\) and \([T; i] \in Cov_\approx(T; j)\). Let \(i' = LCover_T^{q+1}[j] = LCover_T[LCover_T^q[j]] = LCover_T[i]\). By Definition 9 \([T; i'] \in Cov_\approx(T; i)\).

Therefore, \([T; i'] \in Cov_\approx(T; j)\) by Lemma 4.

The following lemma implies that if a left \(\approx\)-seed of \([T; i]\) covers the longest proper \(\approx\)-border of \([T; i]\), then it covers \([T; i]\).

**Lemma 11.** For any \(1 \leq i \leq n\), if \([T; j] \in LSeed_{\approx}(T; i)\), we have \([T; j] \in Cov_\approx(T; i - k)\) for some \(0 \leq k \leq \min(j - 1, \text{Border}_\approx[T; i])\).

**Proof.** Let \(b = \text{Border}_\approx[T; i]\). Assume that \([T; j]\) is a left \(\approx\)-seed of \([T; i]\). By Definition 11, there exists \(l'\) for \(k' \leq l' \leq j\) such that \([T; j] \in Cov_\approx(T; i - k')\) and \([T; l'] \approx [T; i - l' + 1; j]\). Then, \([T; l'] \in \text{Bord}_\approx(T; i)\) by Definition 4.

In the case \(j - 1 < b\), if \(k' \leq j - 1\), we have \([T; j] \in Cov_\approx(T; i - k')\) for some \(0 \leq k' \leq \min(j - 1, b)\). Otherwise, since \(k' > j - 1\) and \(k' \leq l' \leq j\), clearly \(k' = l' = j\). Thus, we have \([T; j] \in Cov_\approx(T; i - j)\) and \([T; j]\) is a proper \(\approx\)-border of \([T; i]\) from \(j \leq b\). Therefore, \([T; j] \in Cov_\approx(T; i)\) by Lemma 6.

Next, we consider the case \(j - 1 \geq b\). If \(l' > b\), we have \(l' = i\) since \([T; l'] \in \text{Bord}_\approx(T; i)\). Moreover, from \(l = j = i\), we have \([T; j] \in Cov_\approx(T; i)\).

Otherwise \(k' \leq l' \leq b\), \([T; j] \in Cov_\approx(T; i - k')\) for some \(0 \leq k' \leq \min(j - 1, b)\).

The following two lemmas are important properties to evaluate whether a prefix is a left \(\approx\)-seeds. The first lemma show the condition of trivial left \(\approx\)-seed prefixes and the other shows the condition for remaining prefixes to be left \(\approx\)-seeds.

**Lemma 12.** For any \(1 \leq i \leq n\), \([T; i - k] \in LSeed_{\approx}(T; i)\) for any \(0 \leq k \leq \text{Border}_\approx[T; i]\).

**Proof.** Let \(b = \text{Border}_\approx[T; i]\). we have clearly \([T; i - k] \in Cov_\approx(T; i - k)\). If \(b \leq i - k\), since \(k \leq b \leq i - k\) and \([T; b] \approx [T; i - b + 1; i]\) by Definition 9, we have \([T; i - k] \in LSeed_{\approx}(T; i)\) by Definition 11.

Otherwise, let \(y_i = j(i - b)\) since \([T; b] \approx [T; i - b + 1; i]\) by Definition 11. \([T; i - k] \approx [T; y_i + 1; y_i + (i - k)]\) by Definition 11 and \([T; i - k] \in \text{Bord}_\approx(T; y_i + (i - k))\) by Definition 5. Thus we have \([T; i - k] \in Cov_\approx(T; y_i + (i - k))\) by \(i - b \leq i - k\) and Lemma 6.
Moreover, \( T[y_1 + 1 : y_1 + i - k] \approx T[y_2 + 1 : y_2 + (i - k)] \) by Definition 11 and \( T[i - k] \in \text{Bord}_t(T[y_2 + (i - k)]) \) by Definition 5. Thus we have \( T[i - k] \in \text{Cov}_t(T[y_2 + (i - k)]) \) by \( i - b \leq i - k \) and Lemma 6. By the processing recursively, let \( m \) such that \( y_m + 1 \leq b \leq y_m + i - k \), we get \( T[i - k] \in \text{Cov}_t(T[y_m + (i - k)]) \) and \( T[y_m + 1 : b] \approx T[b - y_m] \). Therefore \( T[i - k] \in L\text{Seed}_t(T[i]) \) by Definition 10.

**Lemma 13.** For any \( 1 \leq i \leq n \), \( T[j] \) for \( 1 \leq j < i - \text{Border}_t[i] \) is not a left \( \approx \)-seed of \( T[i] \) iff no left \( \approx \)-seed prefix \( T[k] \) of \( T[i] \) that be covered by \( T[j] \) for any \( j < k \leq i \).

**Proof.** \((\Rightarrow)\) We prove the contraposition. Assume there is a left \( \approx \)-seed prefix \( T[k] \) of \( T[i] \) that be covered by \( T[j] \) for some \( j < k \leq i \). By Definition 11 there exist \( k' \) and \( l \) such that \( k' \leq l < k \), \( T[k] \in \text{Cov}_t(T[i-k']) \) and \( T[l] \approx T[i-l+1 : i] \). Thus we have \( T[j] \in \text{Cov}_t(T[i-k']) \) by Lemma 10. Since \( T[l] \approx T[i-l+1 : i] \), \( T[j] \in L\text{Seed}_t(T[l]) \) and \( T[j] \in L\text{Seed}_t(T[i-l+1 : i]) \).

Therefore, we get \( T[l] \in L\text{Seed}_t(T[i]) \) by Definition 10.

\((\Leftarrow)\) We show the contraposition. Assume \( T[l] \approx T[i-l+1 : i] \) for \( 1 \leq j < i - b \) is a left \( \approx \)-seed of \( T[l] \), where \( b = \text{Border}_t[i] \). By Definition 10 there exist \( k' \) and \( l \) such that \( k' \leq l < j \) such that \( T[j] \in \text{Cov}_t(T[i-k']) \) and \( T[l] \approx T[i-l+1 : i] \). Since \( T[i-k'] \in \text{Cov}_t(T[i-k']), T[i-k'] \) is a left \( \approx \)-seed of \( T[i] \).

If \( b < l \leq j, T[l] \in \text{Border}_t(T[i]), \) which implies \( T[l] \) is not the longest proper \( \approx \)-border of \( T[i] \). This makes a contradiction, Thus we have \( l \leq b \). Moreover, from \( k' \leq l \leq b \) and \( j < i - b \leq i - k' \), Therefore, there exists a left \( \approx \)-seed prefix \( T[k' - i] \) of \( T[i] \) that is covered by \( T[j] \) for \( j < i - l \leq l' \).

Next, we introduce a function \( \text{LongestLSeed}_t(j,i) \) that will be used to show the correctness of Algorithm 2 later.

**Definition 11 (LongestLSeed \(_t(j,i)\)).** For a string \( T \), define

\[
\text{LongestLSeed}_t(j,i) = \max\{ \{ l | T[l] \in L\text{Seed}_t(T[i]) \}, T[l] \in \text{Cov}_t(T[j]), 1 \leq l \leq j \} \cup \{0\}.
\]

In other words, \( \text{LongestLSeed}_t(j,i) \) is the longest \( \approx \)-cover \( C \) of \( T[j] \) such that \( C \) is also a left \( \approx \)-seed of \( T[i] \).

The next lemma shows the recursive property of the function \( \text{LongestLSeed}_t \). In Algorithm 2 we use this property to update an array \( \text{LongestLSeed} \) noted below.

**Lemma 14.** For any \( 1 \leq i \leq n \), let \( B = T[i : b] \) be the longest proper \( \approx \)-border of \( T[i] \). If \( B \) is not a left \( \approx \)-seed of \( T[i - 1] \), \( \text{LongestLSeed}_t(b, i) = \text{LongestLSeed}_t(m, i - 1) \), where \( m \) is the length of the longest proper \( \approx \)-cover of \( B \). Otherwise, \( \text{LongestLSeed}_t(b, i) = \text{LongestLSeed}_t(b, i - 1) \).

**Proof.** Consider \( B \) is a left \( \approx \)-seed of \( T[i - 1] \). By Lemma 11 \( B \in \text{Cov}_t(T[i - k - 1]) \) for some \( 0 \leq k \leq b - 1 \). Therefore, by Lemma 6 \( B \in \text{Cov}_t(T[i]) \) which implies \( B \) is a left \( \approx \)-seed of \( T[i] \).

Next, consider \( B \) is not a left \( \approx \)-seed of \( T[i - 1] \). Let \( m' = \text{LongestLSeed}_t(m, i - 1) \) and \( C' = T[i : m'] \). By Lemma 4 and Definition 8 \( C' \in \text{Border}_t(T[i]) \). Since \( C' \) is a left \( \approx \)-seed of \( T[i - 1] \), \( C \in \text{Cov}_t(T[i - k - 1]) \) for some \( 0 \leq k \leq m' - 1 \) by Lemma 11. Therefore, by Lemma 6 \( C' \in \text{Cov}_t(T[i]) \) which implies \( C' \) is a left \( \approx \)-seed of \( T[i] \).
We state the final lemma to compute the longest \( \approx \)-cover, which shows the longest \( \approx \)-cover of \( T[:i] \) is the longest left \( \approx \)-seed of the longest proper \( \approx \)-border of \( T[:i] \).

**Lemma 15.** For any \( 1 \leq i \leq n \), \( \text{LCover}_T[i] = \text{LongestLSeed}_T(b, i) \), where \( b \) is the length of the longest proper \( \approx \)-border of \( T[:i] \).

**Proof.** Let \( l = \text{LongestLSeed}_T(b, i) \). \( T[:l] \in \text{Cov}_\approx(T[:i]) \) from the proof of Lemma 14 so we show \( T[:l] \) is the longest proper \( \approx \)-cover of \( T[:i] \). Assume there exists \( l < m \) such that \( T[:m] \) is a proper \( \approx \)-cover of \( T[:i] \). Clearly \( T[:m] \) is a left \( \approx \)-seed of \( T[:i] \), and \( T[:m] \in \text{Cov}_\approx(T[:b]) \) by Lemma \[13\] thus \( m = \text{LongestLSeed}_T(b, i) \) by Definition \[11\]. This makes a contradiction. Therefore, \( T[:l] \) is the longest proper \( \approx \)-cover of \( T[:i] \). \( \square \)

Algorithm 2 computes the longest \( \approx \)-cover array of \( T \). We use three additional arrays, \text{Dead}, \text{LSeedChildren}, and \text{LongestLSeed}. The algorithm maintains them so that at the end of \( i \)-th iteration they satisfy the following invariants.

1. \( \text{Dead}[0] = 0 \) and for \( 1 \leq j \leq i \), \( \text{Dead}[j] = \text{True} \) iff \( T[:j] \) is not a left \( \approx \)-seed of \( T[:i] \).
2. \( \text{LSeedChildren}[j] = \{ k | T[:k] \text{ is a left } \approx \text{-seed of } T[:i] \text{ such that } j = \text{LCover}_T[k], 1 \leq k \leq i \} \)
3. For \( j \leq \text{Border}_T[i], \text{LongestLSeed}[j] = \text{LongestLSeed}_T(j, i) \).

Li and Smyth describe that \( \text{LongestLSeed}[j] = \text{LongestLSeed}_T(j, i) \) holds for not only \( j \leq \text{Border}_T[i] \), but \( j \leq i \) in [17], however they do not explain its correctness. We give a counterexample to their property. Let \( T = \text{ababab} \) on \( \Sigma = \{a, b, c\} \) under the identity relation. In the end of sixth iteration of their algorithm, \( \text{LongestLSeed} = [0, 0, 3, 4, 5, 0] \) and \( \text{Dead} = [\text{T}, \text{T}, \text{T}, \text{T}, \text{T}, \text{F}] \), where \( T = \text{True} \) and \( F = \text{False} \). It does not hold \( \text{LongestLSeed}[j] = \text{LongestLSeed}_T(j, i) \) for \( j \leq i \), i.e., \( \text{LongestLSeed} = [0, 0, 0, 0, 0, 6] \).

**Theorem 2.** Given a text \( T \) of length \( n \) and the \( \approx \)-border array \( \text{Border}_T \) of \( T \), Algorithm 2 computes the longest \( \approx \)-cover array \( \text{LCover}_T \) of \( T \) in \( O(n) \) time.

**Proof.** First, we prove the correctness of the algorithm by induction. In the end of the first iteration, we know \( \text{Border}[1] = 0 \), \( \text{Dead}[1] = \text{False} \) by the initialization. Since \( T[:1] \) has no proper \( \approx \)-covers, \( \text{LCover}[1] = 0 \) and \( \text{LSeedChildren}[0] = 1 \) are updated clearly by Definition \[9\]. Also, \( \text{LongestLSeed}[0] = 0 \) is updated clearly by Definition \[11\]. Since a left \( \approx \)-seed of \( T[:1] \) is only \( T[:1] \), \( \text{Dead} \) is updated clearly.

Assume that at the end of the \((i-1)\)-th iteration, \( \text{Dead}, \text{LSeedChildren}, \text{and LongestLSeed} \) hold the above properties and \( \text{LCover}[:i-1] \) is updated correctly. Let \( b_i = \text{Border}[i], m = \text{LCover}[:\text{Border}[i]], c_1 = i - b_i \) and \( c_2 = (i-1) - b_{i-1} \). We show \( \text{LongestLSeed} \) and \( \text{LCover} \) is updated correctly in Lines 7-10. In the case \( \text{Dead}[b_i] = \text{True} \), since \( T[:m] \) is a proper \( \approx \)-border of \( T[:i] \) and \( m < b_i \leq b_{i-1} + 1 \) by Lemma \[2\] we have \( \text{LongestLSeed}[m] = \text{LongestLSeed}_T(m, i-1) \). Moreover, we have \( \text{LongestLSeed}[b_i] = \text{LongestLSeed}_T(b_i, i) = \text{LongestLSeed}[m] \) by Lemma \[13\]. Otherwise, if \( \text{Dead}[b_i] = \text{False} \), since \( \text{LongestLSeed}_T(b_i, i-1) = \text{LongestLSeed}[b_i] \), we have \( \text{LongestLSeed}[b_i] = \text{LongestLSeed}_T(b_i, i) \) by...
Next, we show that \( \text{LongestLSeed}[m'] \) for \( m' < b_i \) does not need to be updated. Let \( l = \text{LongestLSeed}[m'] = \text{LongestLSeed}_T(m', i - 1) \neq 0 \). Clearly \( T:[i] \in \text{LSeed}_n(T:[i - 1]) \), thus there exists \( k' \leq l \) such that \( T:[i] \in \text{Cov}_n(T:[i - 1) - k')] \). Since \( b_i \leq i - 1 \), we have \( T:[i] \in \text{LSeed}_n(T:[b_i]) \) and \( T:[i] \in \text{LSeed}_n(T[i - b_i + 1]) \). Therefore, we get \( T:[i] \in \text{LSeed}_n(T[i]) \) by Definition 10.

Next, we show that \( \text{Dead} \) is updated correctly in Lines 11–15. For any prefix \( T[j] \) \( (j \leq i) \), clearly \( T[j] \in \text{LSeed}_n(T[i]) \) implies \( T[j] \in \text{LSeed}_n(T[i - 1]) \). Thus if \( \text{Dead}[j] = \text{True} \) after the \( (i - 1) \)-th iteration, \( \text{Dead}[j] = \text{True} \) is also correct after the \( i \)-th iteration. For \( j, c_1 \leq j \leq i \), clearly \( T:[j] \in \text{LSeed}_n(T:[i]) \) by Lemma 12. For \( j, c_1 \leq j \leq i \) we should check whether \( T[j] \) is a left \( \approx \)-seed of \( T[i] \). We check them directly for \( c_2 \leq j < c_1 \) by Lemma 13 and by recursion for \( j < c_2 \). Therefore, \( \text{Dead} \) is updated correctly by the recursive function \( \text{SetDead} \).

Last, we show \( \text{LSeedChildren} \) is updated correctly in the algorithm. Since \( T[i] \) is clearly a left \( \approx \)-seed of \( T[i] \) and \( T[i] \in \text{LSeed}_n \) is the longest \( \approx \)-cover of \( T[i] \) by the above proof of \( \text{LCover} \), \( \text{LSeedChildren} \) is updated correctly in Line 11 in Algorithm 2. Also, since a prefix which holds the condition of Line 1 in Algorithm 3 will no longer be left \( \approx \)-seed by the above proof of \( \text{LCover} \), \( \text{LSeedChildren} \) is updated correctly in Line 3 in Algorithm 3. Therefore, the algorithm computes the longest \( \approx \)-cover array \( \text{LCover}_T \) of \( T \) correctly.

Next, we show the time complexity of the algorithm. The number of iterations of the outer \textbf{for} loop is \( n \), and each operation to update can be executed in \( O(1) \) time. Moreover the number of iterations of the inner \textbf{for} loop is \( n - \text{Border}[n] \) in total, since \( \text{Border}_T[i - 1] + 1 \geq \text{Border}_T[i] + 1 \) for any \( 1 < j \leq n \) by Lemma 2 and all operations in the loops can be executed in \( O(n) \) time. Therefore, Algorithm 2 runs in \( O(n) \) time overall.

\textbf{Corollary 2.} Suppose \( \text{Border}_T \) can be computed in \( \beta(n) \). \( \text{LCover}_T \) can be computed in \( O(\beta(n) + n) \) time.

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