A trial symbolic dynamics of the planar three-body problem

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Abstract.
Symbolic dynamics is applied to the planar three-body problem. Symbols are defined on the planar orbit when it experiences a syzygy crossing. If the body \(i\) is in the middle at the syzygy crossing and the vectorial area of the triangle made with three bodies changes sign from \(+\) to \(-\), number \(i\) is given to this event, whereas if the vectorial area changes sign from \(-\) to \(+\), number \(i + 3\) is given. We examine the case of free-fall three-body problem for the first few digits of symbol sequences, and we examine the case with angular momentum only for the first digit of the symbol sequences. This trial experiments show some new aspects of the planar three-body problem.

Key words: planar three-body problem – symbolic dynamics – chaos

1. Introduction
Symbolic dynamics for the planar three-body problem is not yet fully developed. Not many authors are involved in this direction (Chernin et al. 2006; Myllari 2007; Moeckel 2007). We need to find a good procedure to assign symbols comparable to the one-dimensional case (Tanikawa & Mikkola 2000).

In the free-fall three-body problem, we saw that binary collision curves (formed with initial condition points of orbits which experience collision) constitute the boundary of regions in the initial value plane (Tanikawa & Umehara, 1998). We expect this may be the case also for the case with angular momentum. In the present report, we introduce symbols so that the boundaries of different symbol sequences are binary collision curves. After the definition of symbols, we start numerical symbolic dynamics of the planar three-body problem with angular momentum extending the free-fall problem.

2. Introduction of symbols
We introduce symbols in this section. First, we define the signed area of the triangle formed with three bodies. If the three bodies are arranged in the counterclockwise order for body numbers 1, 2, and 3, we consider that the area is positive
(see Fig. 1). If the order is reverse, the area is considered to be negative. The absolute value of the area of the triangle is its usual area.

Using this definition of area, we assign a symbol to a particular event on the orbits. Suppose that the configuration of the three bodies becomes collinear. If the angular momentum of the system is not zero, this configuration cannot be maintained for a finite non-zero time interval except the rectilinear case of the three-body problem. Before and after this configuration, triangles of non-zero area are recovered.

![Figure 1. Vectorial area of triangles](image)

We ask here whether the middle particle may cross or only be tangent to the syzygy line. The latter does not happen since otherwise the trajectory of the middle body is convex to the remaining two bodies when it is on the syzygy. This is impossible because the trajectory of any body should be concave to either or both of the remaining bodies due to the gravitational attraction.

We need to consider the case of binary collision. At binary collision, the configuration becomes collinear. As is well-known, the trajectory of a body relative to the other body is a parabola in the vicinity of binary collision. The third body can be considered to be stand still compared with the high speed of the binary components. This again shows that the collinear configuration cannot be maintained before and after collision.

Finally, we need to make an important remark. At binary collision, an orbit experiences syzygy crossings not only once. The orbit may experience at most three syzygy crossings. In addition, the number of crossings is different depending on whether the collision is considered as a limit of retrograde encounter or prograde encounter. The limit should be taken to keep the continuity to the neighboring initial conditions. The analysis of collision will be given elsewhere. We here make a notice that two or three symbols may be given to the orbit at collision.

Now, suppose that the area changes sign from $+\to -$ at some instant. Then we give symbol '1' if the longest edge is $2\to 3$, that is, the edge connecting body 2 and body 3. Similarly, we give symbol '2' if the longest edge is $3\to 1$, and we give symbol '3' if the longest edge is $1\to 2$. When the area changes sign from $-\to +$, we give symbol '4' if the longest edge is $2\to 3$. Similarly, we give symbol '5' if the
longest edge is 3–1, and we give symbol ‘6’ if the longest edge is 1–2. See Fig. 2.

![Figure 2. Assignment of symbols.](image)

3. Orbits and Symbol sequences

Each time a triple system becomes collinear, a symbol is given according to the rule described in the preceding section except at collision. Therefore, an orbit represented by a continuous curve in the phase space is replaced by a bi-infinite symbol sequence. Here we only consider the future symbol sequence. We denote the present by a point, and symbols by \( s_i \), then a symbol sequence \( s \) can be written as

\[
s = .s_1 s_2 s_2 \ldots
\]  
(1)

3.1. The planar system with angular momentum

Let us briefly explain the initial conditions of numerical integrations. We want to extend the free fall problem with equal masses (Anosova & Orlov 1986; Tanikawa et al. 1995), in which bodies 2 and 3 initially stand still at \((-0.5, 0)\) and \((0.5, 0)\), respectively, whereas body 1 stands still at \((x, y) \in D_{11} = \{x \geq 0, y \geq 0, (x - 0.5)^2 + y^2 \leq 1\} \). \( D_{11} \) is called the initial condition plane (Fig. 3). If the body 1 moves everywhere in \( D_{11} \), all the possible triangles are realized.

Now, we give velocities to the bodies and angular momentum to the system still with equal masses. In the planar three-body problem with angular momentum, there too many degrees of freedom. We need somehow to restrict the initial condition space so as to be able to express the numerical results visibly. Here, we give maximal angular momentum for a given configuration triangle (See Kuwabara & Tanikawa, this issue). In this case, one of the symmetries is lost, so the initial condition space becomes doubled. See the right panel of Fig. 3. This time \( D_{11} \cup D_{12} \) is the initial condition plane (Tanikawa & Kuwabara 2007).
3.2. Boundaries of symbols in the initial condition space

Now, suppose that all the points in the initial condition space have their own symbol sequences (1), that is, the orbits starting at points of the initial condition space are all integrated. If we truncate symbol sequences at the $n$-th digit, there are finite number of possible combinations of symbols in these length-$n$ “words”. The initial condition space is divided by “cylinders” which contain these words. We ask what kind of points, or equivalently, orbits, constitute these boundaries.

For simplicity of discussion, let us consider the case $n = 1$, the first digit. It is apparent from Fig. 4 that initial triangles have positive areas, so only ’1’, ’2’, and ’3’ appear at the first syzygy crossing. These three symbols divide the initial condition space. Suppose the region where the first digit is ’1’. Take orbit $O$ in which body 1 crosses the syzygy between bodies 2 and 3 and at finite non-zero distances both from bodies 2 and 3. Then, all neighboring orbits also experience syzygy crossing of body 1 bewteen bodies 2 and 3. This means that orbit $O$ is inside the region occupied by symbol ’1’ which we call region ’1’. This argument applies as long as body 1 crosses the syzygy at finite distances from both bodies 2 and 3. Then, all neighboring orbits also experience syzygy crossing of body 1 bewteen bodies 2 and 3. This means that orbit $O$ is inside the region occupied by symbol ’1’ which we call region ’1’. This argument applies as long as body 1 crosses the syzygy at finite distances from both bodies 2 and 3. Orbits of the boundaries of region ’1’ should be collision orbits. There can be two boundaries: between regions ’1’ and ’2’ and between regions ’1’ and ’3’. Similarly, there can be two boundaries of region ’2’, and two boundaries of region ’3’. In general, there are six symbols. So, the boundaries of region ’$i$’ ($i = 1, 2, \ldots$) can be more than two.

4. Numerical results

4.1. The free-fall problem

In order to justify the argument of §3.2, we first give results on the free-fall problem (Tanikawa & Umehara 1998). Figure 5 shows how the initial condition plane is divided by collision curves. It is to be noted that the boundaries of the plane, the $x$-axis, $y$-axis, and the circular boundary are all collision curves.
Our numerical results are shown in Fig. 6. In the free-fall case, body 1 passes through between bodies 2 and 3 at the first syzygy crossing. So the whole space is occupied by region '1'. The initial condition space divided by '15' and '16' at the first two digits. The reason is simple. At the right part of the space, bodies 1 and 3 form a binary, so body 3 passes through between bodies 1 and 2, and the vectorial area goes from '-' to '+', giving symbol '6'. At the right part, body 2 passes through the syzygy, giving symbol '5'. The boundary of two regions is the $y$-axis. This is seen in the figure of the left panel of Fig. 6. In the lower-left and lower-right corners of the figure, some different colors are seen. These are considered to be numerical artifacts.

In the middle panel of Fig. 3, the initial plane is divided by regions with the first three digits of symbol sequences. Here we have a new result. The boundary of yellow and orange regions corresponds to the binary collision curve (broken curve) of type 3 (Tanikawa et al. 1995) emanating from point $T_2$ in Fig. 5. In Tanikawa et al. (1995), we could not follow the collision curve down to the
$x$-axis due to numerical difficulty. In the present work, It has been shown that the collision curve actually arrive at the $x$-axis.

![Figure 6. The structure of the initial condition plane. From the left, the first two digits of symbol sequences, the first three digits, and the first four digits.](image)

In the right panel of Fig. 3, the results for the first four digits are shown. New collision curves appear. One is the boundary between light blue and pink regions. This is the collision curve of type 1 emanating from $T_2$. The other two curves between yellow and green regions are a collision curve emanating from $T_{1,1}$ and a curve of type 2 starting at the middle of $T_1$ and $T_{1,1}$ in Fig. 5.

### 4.2. The first digit for systems with angular momentum

In the preceding section, we have numerically shown that the boundaries of regions with different cylinders are formed with binary collision curves. In this section, let us look at the initial condition plane from a different viewpoint. We gradually add angular momentum to triple systems. The parameter is the virial coefficient $k = T/|U|$ where $T$ is the kinetic energy and $U$ is the potential energy of a triple system. As $k$ increases from 0, The contribution of velocities increases relative to a fixed potential energy. At $k = 1$, the energy of the triple system is equal to zero. So we consider the range $0 \leq k \leq 1$.

In this report, we consider only the boundaries of the cylinders of the first one word as a function of $k$. In Fig. 7, results are given for $k = 0, 0.001$ (top), $0.01, 0.1$ (second top), $0.2, 0.3$ (third top), and $0.5, 0.7$ (bottom). Red is for symbol '1', green is for symbol '2', and blue is for symbol '3'.

As mentioned before, the whole plane is occupied by region '1' for $k = 0$. Even for a very small $k > 0$, region '2' appears (top-right figure). This is because near the left boundary of the plane bodies 1 and 2 form a binary, and even for a small angular momentum, this binary revolves each other, hence body 2 instead of body 1 experiences the first syzygy crossing. The boundary of two regions is formed with a binary collision curve of type 3 (collision between bodies 1 and 2). It seems that this curve extends from the center bottom (the Euler point) to the top (the Lagrange point).

As $k$ increases, the boundary collision curve deforms and 'evolves' to the right in the initial condition space (see the second top figures of Fig. 7). Near the top of the plane, region '3' appears. Though this region is visible only for $k \geq 0.1$ in Fig. 7, the region is visible for $k = 0.001$ if we enlarge the neighborhood of
Figure 7. The structure of the initial condition plane for $k = 0, 0.001$ (top), $0.01, 0.1$ (second top), $0.2, 0.3$ (third top), and $0.5, 0.7$ (bottom) for the first digit of symbol sequences. Red is for symbol '1', green is for symbol '2', and blue is for symbol '3'.

the Lagrange point. In addition, the binary collision curve seems to spiral into the Lagrange point as long as the point is unstable (see Fig. 8).

5. Conclusions

We have demonstrated that symbolic dynamics is effective in the planar three-body problem. Main results are

(i) In the free-fall problem, it has been shown that binary collision curves divide the initial condition space. This has been suggested in Tanikawa et al. (1995). In the present work, divisions are more clearly seen.

(ii) It has been shown for the three-body problem with angular momentum that a binary collision curve connects the Euler point and Lagrange point in the initial condition plane, which suggests that the Euler and Lagrange points are connected by an invariant manifold. The collision curve seems to spiral into the Lagrange point if the angular momentum is not zero.
Figure 8. The structure of the initial condition plane around the Lagrange point for the first one digit. \( k = 0.001 \) (left) for \(-0.005 \leq x \leq 0.005, 0.86 \leq y \leq 0.87; \) \( k = 0.01 \) (center) for \(-0.01 \leq x \leq 0.01, 0.85 \leq y \leq 0.87, \) and \( k = 0.1 \) (right) for \(-0.1 \leq x \leq 0.1, 0.76 \leq y \leq 0.96.\)

Acknowledgment

This project was supported by JSPS and RFBR under the Japan–Russia Research Cooperative Program. One of the authors (K.T.) expresses hearty thanks to the Russian members of Japan-Russia Joint Research for the hospitality during his stay in St Petersburg in the end of August, 2007.

References

Agekyan, T.A. & Anosova, J.P. 1968, Sov. Phys.-Astron., 11, 1006 - 1014
Chernin, A., Martynova, A., Mylläri, A., & Orlov, V. 2006, in Few-Body Problem: Theory and Computer Simulations (Ed. C. Flynn), Turku, 4 - 9 July, 2005
Kuwabara, K.H. & Tanikawa, K., 2007, Chaos, 17, 033105 - 033105-13
Moeckel, R. 2007, Regular and Chaotic Dynamics, 12, 449 - 475
Mylläri et al. 2007, in this issue
Kuwabara, K.H. & Tanikawa, K., 2007, in this issue
Tanikawa, K. & Mikkola, S. 2000, Cel.Mech.&Dynam.Astron., 76, 23 - 34
Tanikawa, K., Umehara, H. and Abe, H., 1995, Cel.Mech.&Dynam.Astron., 62, 335 - 362
Tanikawa, K. & Umehara, H. 1998, Cel.Mech.&Dynam.Astron., 70, 167 - 180