Recognition and Teleportation

Karl-Heinz Fichtner
Friedrich-Schiller-Universität Jena,
Institut für Angewandte Mathematik,
07740 Jena, Germany,
E-Mail: fichtner@mathematik.uni-jena.de

Wolfgang Freudenberg
Brandenburgische Technische Universität Cottbus,
Institut für Mathematik, PF 101344, 03013 Cottbus, Germany,
E-Mail: freudenberg@math.tu-cottbus.de

Masanori Ohya
Science University of Tokyo,
Department of Information Science,
Noda City, Chiba 278-8510, Japan,
E-Mail: ohya@is.noda.sut.ac.jp

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1 Introduction

We study a possible function of brain, in particular, we try to describe several aspects of the process of recognition. In order to understand the fundamental parts of the recognition process, the quantum teleportation scheme [3, 2, 6, 7] seems to be useful. We consider a channel expression of the teleportation process that serves for a simplified description of the recognition process in brain. It is the processing speed that we take as a particular character of the brain, so that the high speed of processing in the brain is here supposed to come from the coherent effects of substances in the brain like quantum computer, as was pointed out by Penrose. Having this in our mind, we propose a model of brain describing its function as follows:

The brain system $BS = x$ is supposed to be described by a triple $(B(H), S(H), \Lambda^*(G))$ on a certain Hilbert space $\mathcal{H}$ where $B(H)$ is the set of all bounded operators on $\mathcal{H}$, $S(H)$ is the set of all density operators and $\Lambda^*(G)$ is a channel giving a state change with a group $G$.

Further we assume the following:

(1) $BS$ is described by a quantum state and the brain itself is divided into several parts, each of which corresponds to a Hilbert space so that $\mathcal{H} = \oplus_k \mathcal{H}_k$
and $\varphi = \oplus_k \varphi_k$, $\varphi_k \in \mathcal{S}(\mathcal{H}_k)$. However, in this paper we simply assume that the brain is in one Hilbert space $\mathcal{H}$ because we only consider the basic mechanism of recognition.

(2) The function (action) of the brain is described by a channel $\Lambda^* = \oplus k \Lambda^*_k$. Here as in (1) we take only one channel $\Lambda^*$.

(3) $\mathcal{BS}$ is composed of two parts; information processing part ”$P$” and others ”$O$” (consciousness, memory, recognition) so that $\mathcal{X} = \mathcal{X}_P \otimes \mathcal{X}_O$, $\mathcal{H} = \mathcal{H}_P \otimes \mathcal{H}_O$. Thus in our model the whole brain may be considered as a parallel quantum computer [9], but we here explain the function of the brain as a quantum computer, more precisely, a quantum communication process with entanglements like in a quantum teleportation process. We will explain the mathematical structure of our model.

Let $s = \{s^1, s^2, \ldots, s^n\}$ be a given (input) signal (perception) and $\overline{s} = \{\overline{s}^1, \overline{s}^2, \ldots, \overline{s}^m\}$ the output signal. After the signal $s$ enters the brain, each element $s^j$ of $s$ is coded into a proper quantum state $\rho^j \in \mathcal{S}(\mathcal{H}_P)$, so that the state corresponding to the signal $s$ is $\rho = \otimes_j \rho^j$. This state may be regarded as a state processed by the brain and it is coupled to a state $\rho_O$ stored as a memory (pre-consciousness) in brain. The processing in the brain is expressed by a properly chosen quantum channel $\Lambda^*$ (or $\Lambda^*_P \otimes \Lambda^*_O$). The channel is determined by the form of the network of neurons and some other biochemical actions, and its function is like a (quantum) gate in quantum computer [8, 10]. The outcome state $\overline{s}$ contacts with an operator $F$ describing the work as noema (Husserl’s noema), after the contact a certain reduction of state is occurred, which may correspond to the noesis (Husserl’s) of consciousness. A part of the reduced state is stored in brain as a memory. The scheme of our model is represented in the following figure.
Let us further assume the Hilbert space $H_O$ is composed of two parts, before and after recognition. For notational simplicity, we denote the Hilbert spaces by $H_1, H_2, H_3$ where $H_1$ represents the processing part, $H_2$ the memory before recognition and $H_3 = H_2$ the memory after recognition. Throughout this paper we will have in mind this interpretation of the Hilbert spaces $H_j$ ($j = 1, 2, 3$). However, this is just an illustration of what we are going to do, and the teleportation scheme may be applied to very different situations.

We are mainly interested in the changes of the memory after the process of recognition. For that reason we consider channels from the set of states on $H_1 \otimes H_2$ into $H_3$. Main object to be measured causing the recognition is here assumed to be a self-adjoint operator

$$F = \sum_{k,l=1}^{n} z_{k,l} F_{k,l}$$

on $H_1 \otimes H_2$ where the operators $F_{k,l}$ are orthogonal projections (alternatively, we may take $F_{k,l}$ as an operator valued measure). The channel $\Lambda_{k,l}$ describes the state of the memory after the process of recognition if the outcome of the measurement according to $F$ was $z_{k,l}$ and is given by

$$\Lambda_{k,l}(\rho \otimes \gamma) := \frac{\operatorname{Tr}_{1,2}(F_{k,l} \otimes \mathbb{I})(\rho \otimes J_\gamma J_\gamma^*)(F_{k,l} \otimes \mathbb{I})}{\operatorname{Tr}_{1,2,3}(F_{k,l} \otimes \mathbb{I})(\rho \otimes J_\gamma J_\gamma^*)(F_{k,l} \otimes \mathbb{I})}$$
where $\rho$ and $\gamma$ (denoted $\rho_O$ above) are the state of the processing part and of the memory before recognition and $J$ an isometry extending from $\mathcal{H}_2$ to $\mathcal{H}_2 \otimes \mathcal{H}_3$ and $\mathbb{I}$ denotes the identical operator. The value $\text{Tr}_{1,2,3}(F_{k,l} \otimes \mathbb{I})(\rho \otimes J \gamma)(F_{k,l} \otimes \mathbb{I})$ represents the probability to measure the value $z_{k,l}$. So, obviously, we have to assume that this probability is greater than 0. The state $\Lambda_{k,l}(\rho \otimes \gamma)$ gives the state of the memory after the process of recognition. The elements of a basis $(\langle b_k \rangle)_{k=1}^n$ of $\mathcal{H}_j$ are interpreted as elementary signals.

In this first attempt to our model described above, there appear still a lot of effects being non-realistic for the process of recognition. Some examples (cf. the last section) show that with this model one can describe extreme cases such as storing the full information or total loss of memory, but - as mentioned above - that is still far from being a realistic description.

In this paper we restrict ourselves to finite dimensional Hilbert spaces. Moreover, we assume equal dimension of the Hilbert spaces $\mathcal{H}_j$ ($j = 1, 2, 3$). It seems that infinite dimensional schemes will lead to more realistic models. However, this is just a first attempt to describe the brain function. Moreover, for finite dimensional Hilbert spaces the mathematical model becomes more transparent and one can obtain easily a general idea of the model. To indicate obvious generalizations to more general situations and especially to infinite dimensional Hilbert spaces we sometimes use notions and notations from the general functional analysis. In a forthcoming paper we will discuss a modification of the model using general splitting procedures on a Fock space\cite{4, 5}. We hope to be able to include more realistic effects.

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2 Basic Notions

Let $\mathcal{H}_1, \mathcal{H}_2, \mathcal{H}_3$ be Hilbert spaces with equal finite dimension:

$$\dim \mathcal{H}_j = n, \quad (j \in \{1, 2, 3\}).$$

First we will represent these Hilbert spaces in a way that it seems to be convenient for our considerations. Each of the spaces $\mathcal{H}_1, \mathcal{H}_2, \mathcal{H}_3$ can be identified with the space $\mathbb{C}^n$ of $n$-dimensional complex vectors. The space $\mathbb{C}^n$ again may be identified with the space $\{f : G \to \mathbb{C}\}$ of all complex-valued function on $G := \{1, \ldots, n\}$. The scalar product then is given by

$$\langle f, g \rangle := \sum_{k=1}^n f(k)g(k) = \int \overline{f(k)}g(k)\mu(dk)$$

where $\mu$ is the counting measure on $G$, i.e. $\mu = \sum_{k=1}^n \delta_k$ with $\delta_k$ denoting the Dirac measure in $k$. So, each of the spaces $\mathcal{H}_j$ can be written formally as an $L_2$-space:

$$\mathcal{H}_j = L_2(G, \mu) := L_2(G) \quad (j \in \{1, 2, 3\}).$$
For the tensor product one obtains
\[ f \otimes g(k, l) = f(k)g(l) \quad (f, g \in L^2(G), k, l \in G), \]
and we have
\[ \mathcal{H}_1 \otimes \mathcal{H}_2 = L^2(G \times G, \mu \times \mu) = \mathcal{H}_2 \otimes \mathcal{H}_3. \]
We will abbreviate this tensor product by \( L^2(G^2, \mu^2) \) or just by \( L^2(G^2) \).

By \( B(H) \) we denote the space of all bounded linear operators on a Hilbert space \( H \). In \( B(L^2(G)) \) the operator of multiplication by a function \( g \in L^2(G) \) is given by
\[ (O_g f)(k) = g(k)f(k) \quad (f \in L^2(G), k \in G). \]
Observe that for all \( f, g \in L^2(G) \) one has
\[ O_g f = O_f g, \quad O_f^* = O_{f^*} \]
and for \( f \in L^2(G) \) with \( f(k) \neq 0 \) for all \( k \in G \) it holds \( O_f^{-1} = O_{1/f} \).
The function \( 1, 1(k) = 1 \) for all \( k \in G \), obviously belongs to \( L^2(G) \) and \( \mathbb{I} = O_1 \) is the identity in \( B(L^2(G)) \).
Consequently, an operator of multiplication \( O_f \) is unitary if and only if \( |f(k)| = 1 \) for all \( k \in G \).

Further, we will use the mapping \( J \) from \( L^2(G) \) into \( L^2(G^2) \) given by
\[ (J f)(k, l) = f(k)\delta_{k,l} \quad (f \in L^2(G), k, l \in G) \]
where \( \delta_{k,l} \) denotes the Kronecker symbol. It is immediate to see that \( J \) is an isometry. For the adjoint \( J^* : L^2(G^2) \longrightarrow L^2(G) \) we obtain
\[ (J^* \Phi)(k) = \Phi(k, k) \quad (\Phi \in L^2(G^2), k \in G). \]
Observe that \( G \) equipped with the operation \( \oplus : G \times G \longrightarrow G, k \oplus l := (k+l) \mod n \) is a group. The operation inverse to \( \oplus \) we denote by \( \ominus \). Let us remark that \( k \ominus l = k - l \) in the case \( k > l \) and \( k \ominus l = k - l + n \) if \( k \leq l \). We conclude that for all \( k \in G \) the operator \( U_k \in B(L^2(G)) \) given by
\[ (U_k f)(m) := f(k \oplus m) \quad (f \in L^2(G)) \]
is unitary.

Now, let \( (b_k)_{k=1}^n \) be an orthonormal basis in \( L^2(G) \), and denote by \( (B_k)_{k=1}^n \) the sequence of multiplication operators corresponding to the elements of this basis, i.e. \( B_k := O_{b_k}, k \in G \).

**Lemma 1** For \( k, l \in G \) we put
\[ \xi_{k,l} := (B_k \otimes U_l)J \mathbb{1}. \]

The sequence \( (\xi_{k,l})_{k,l \in G} \) is an orthonormal basis in \( L^2(G^2) \).
Proof: First observe that for all \( k, l \in G \) we have

\[
\xi_{k,l}(m, r) = b_k(m)\delta_{m, r\oplus l} \quad (m, r \in G).
\]  (5)

So one gets

\[
\langle \xi_{i,j}, \xi_{k,l} \rangle = \int \int \xi_{i,j}(m, r) \cdot \xi_{k,l}(m, r)\mu^2(d[m, r])
\]

\[
= \int \int b_i(m)b_k(m)\delta_{m, r\oplus l}\delta_{m, r\oplus l}\mu^2(d[m, r])
\]

Since \( j \neq l \) implies \( \delta_{m, r\oplus j}\delta_{m, r\oplus l} = 0 \), the right side will be equal to 0 in this case. Further, observe that for all \( l, m \in G \) there exists exactly one \( r \in G \) such that \( r \oplus l = m \), namely \( r = m - l \) if \( l < m \) and \( r = m + n - l \) in the case \( l \geq m \). So we may continue the above chain and get for the case \( j = l \)

\[
\langle \xi_{i,l}, \xi_{k,l} \rangle = \int \int b_i(m)b_k(m)\delta_{m, r\oplus l}\mu(dm) = \langle b_i, b_k \rangle
\]

Consequently, \( (\xi_{k,l})_{k,l \in G} \) is an orthonormal system in \( L_2(G^2) \), and since \( \dim L_2(G^2) = n^2 \) it is a basis in \( L_2(G^2) \). □

We denote by \( F_{i,j} \in \mathcal{B}(L_2(G^2)) \) the projection onto \( \xi_{i,j} \), i.e.

\[
F_{i,j} := |\xi_{i,j}\rangle \langle \xi_{i,j}| = (\xi_{i,j}, \cdot)\xi_{i,j}.
\]  (6)

Remark: Sometimes (especially in proofs) the 'scalar product' notation is more convenient, but in some other cases using the 'bra-ket' symbols the statements become more transparent. So we will use both descriptions in the sequel.

Observe that for \( \Phi \in L_2(G^2) \) and \( i, j \in G \) one obtains

\[
F_{i,j} \Phi = \xi_{i,j} \sum_{v=1}^n b_i(v \oplus j)\Phi(v \oplus j, v).
\]  (7)

Indeed, we get from (5) and the definition (6)

\[
F_{i,j} \Phi = \langle \xi_{i,j}, \Phi | \xi_{i,j}
\]

\[
= \xi_{i,j} \int \int b_i(u)\delta_{u, v\oplus j}\Phi(u, v)\mu^2(d[u, v])
\]

\[
= \xi_{i,j} \int \delta_{v \oplus j}\Phi(v \oplus j, v)\mu(dv).
\]

In Section □ we investigate concrete teleportation channels. For this we need explicit expression for the operator \( (F_{i,j} \otimes \mathbb{1})(\mathbb{1} \otimes J) \). Using the definition (6) of the imbedding operator \( J \) and (7) we obtain for all \( k, l, m \in G \)

\[
((F_{i,j} \otimes \mathbb{1})(\mathbb{1} \otimes J)\Phi)(k, l, m) = \xi_{i,j}(k, l)\sum_{v=1}^n b_i(v \oplus j)\Phi(v \oplus j, v)\delta_{v, m}
\]
what leads to
\[ ((F_{ij} \otimes \mathbb{I})(\mathbb{I} \otimes J)\Phi)(k, l, m) = \xi_{i,j}(k, l)b_i(m \oplus j)\Phi(m \oplus j, m) \] (8)
for all \( \Phi \in L_2(G^2) \) and \( i, j, k, l, m \in G \).

Now, we put for \( i, j \in G \)
\[ G_{i,j} := J^*(U_j \otimes \mathbb{I})(B_i \otimes \mathbb{I}) = J^*(U_j B_i^* \otimes \mathbb{I}) \] (9)
where \( B_i^* = \mathcal{O}_{b_i}^* = \mathcal{O}_{b_i}^* \). For \( \Phi \in L_2(G^2) \) and \( m \in G \) we get
\[ (G_{i,j}\Phi)(m) = ((U_j B_i^* \otimes \mathbb{I})\Phi)(m, m) = ((B_i^* \otimes \mathbb{I})\Phi)(m \oplus j, m) = \frac{b_i(m \oplus j)}{b_i} \Phi(m \oplus j, m). \]

The linear operator \( G_{i,j} \) maps from \( L_2(G^2) \) into \( L_2(G) \) (it is not an isometry), and we finally get for \( \Phi \in L_2(G^2) \)
\[ (F_{ij} \otimes \mathbb{I})(\mathbb{I} \otimes J)\Phi = \xi_{i,j} \otimes G_{i,j} \Phi. \] (10)

**Example 2** Consider the orthonormal basis \((b_k)_{k=1}^n = (\Delta_k)_{k=1}^n \) in \( L_2(G) \) given by \( \Delta_k(m) = \delta_{k,m} \). From (8) we get
\[ \xi_{i,j}(u, v) = b_i(u)\delta_{u,v \oplus j} = \Delta_i \otimes \Delta_u(u, v \oplus j) = \Delta_i \otimes \Delta_j(u, v \oplus j) \]
i.e. \( \xi_{i,j} = \Delta_i \otimes \Delta_{i \oplus j} \) and we obtain for \( \Phi \in L_2(G^2) \)
\[ (G_{i,j}\Phi)(m) = b_i(m \oplus j)\Phi(m \oplus j, m) = \Delta_{i \oplus j}(m)\Phi(i, i \oplus j). \]

Summarizing, in this special case we have
\[ (F_{ij} \otimes \mathbb{I})(\mathbb{I} \otimes J)\Phi = \Phi(i, i \oplus j)\Delta_i \otimes \Delta_{i \oplus j} \Delta_{i \oplus j}. \] (11)

### 3 Entangled States

**Definition 3** Let \( \gamma \) be a state on \( H_2 = L_2(G) \) (i.e. \( \gamma \) is a positive trace-class operator with \( \text{Tr}(\gamma) = 1 \)). The state \( e(\gamma) \) on \( L_2(G^2) = H_2 \otimes H_3 \) given by
\[ e(\gamma) = J \gamma J^* \] (12)
where \( J \) is the isometry given by (11) we call the entangled state corresponding to \( \gamma \).

**Example 4** Consider the basis \((b_k)_{k=1}^n = (\Delta_k)_{k=1}^n \) defined in Example 2 and let \( \gamma \) be the pure state \( |1/\sqrt{n}|1 > < (1/\sqrt{n})1 | \) (we recall that \( (1/\sqrt{n})1(k) = 1/\sqrt{n} \) for all \( k \in G \)). For each observable \( A \in \mathcal{B}(L_2(G)) \) one has \( \text{Tr}(\gamma A) = \frac{1}{n} \sum_{k=1}^n A1(k) \). Especially, the quantum expectation of a multiplication operator \( \mathcal{O}_f, f \in L_2(G) \) will be just the arithmetic mean:
\[ \text{Tr}(\gamma \mathcal{O}_f)) = \frac{1}{n} \sum_{k=1}^n f(k). \]
Observe that \((1/\sqrt{n})1 = (1/\sqrt{n})\sum_{m=1}^{n} \Delta_m\) and for all \(\Phi \in L_2(G^2)\) it holds \(\langle \Delta_m, J^*\Phi \rangle = \langle \Delta_m \otimes \Delta_m, \Phi \rangle\). This implies for \(\Phi \in L_2(G^2)\) and \(k, l \in G\)

\[
(J\gamma J^*\Phi)(k, l) = \langle \Delta_m, J^*\Phi \rangle \delta_{k,l}
\]

Consequently, \(e(\gamma)\) is the state on \(L_2(G^2) = \mathcal{H}_2 \otimes \mathcal{H}_3\) given by

\[
|\frac{1}{\sqrt{n}} \sum_{m=1}^{n} \Delta_m \otimes \Delta_m \rangle <|\frac{1}{\sqrt{n}} \sum_{m=1}^{n} \Delta_m \otimes \Delta_m |. 
\]

This state \(e(\gamma)\) is a special representation of the entangled state used for the elementary teleportation model [2].

Now, let \(\rho\) and \(\gamma\) be states on \(\mathcal{H}_1\) resp. \(\mathcal{H}_2\), the state \(e(\gamma)\) (usually denoted by \(\sigma\) in [6]) will be a state on \(\mathcal{H}_2 \otimes \mathcal{H}_3\). Remember that we assumed \(\mathcal{H}_1 = \mathcal{H}_2 = \mathcal{H}_3 = L_2(G)\). The numbering only indicates the meaning of the states (we recall that \(\mathcal{H}_1\) represents the processing part, \(\mathcal{H}_2\) the memory before and \(\mathcal{H}_3\) the memory after the recognition process.) Then \(\rho \otimes e(\gamma)\) is a state on \(\mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \mathcal{H}_3\) and we observe immediately

\[
\rho \otimes e(\gamma) = (\mathbb{I} \otimes J)(\rho \otimes \gamma)(\mathbb{I} \otimes J^*). \tag{13}
\]

In Section 5 we calculate explicitly the trace of

\[
(F_{i,j} \otimes \mathbb{I})(\rho \otimes e(\gamma))(F_{i,j} \otimes \mathbb{I}) = (F_{i,j} \otimes \mathbb{I})(\mathbb{I} \otimes J)(\rho \otimes \gamma)(\mathbb{I} \otimes J^*)(F_{i,j} \otimes \mathbb{I}). \tag{14}
\]

The following proposition will be very useful for this.

**Proposition 5** Let \((g_k)_{k=1}^{n}\) and \((h_k)_{k=1}^{n}\) be orthonormal systems in \(L_2(G)\) and \(\rho\) and \(\gamma\) states on \(L_2(G)\) having the following representations:

\[
\rho = \sum_{k=1}^{n} \alpha_k |g_k \rangle \langle g_k|, \quad \gamma = \sum_{k=1}^{n} \beta_k |h_k \rangle \langle h_k|,
\]

\[
\alpha_k \geq 0, \beta_k \geq 0, \sum_{k=1}^{n} \alpha_k = \sum_{k=1}^{n} \beta_k = 1.
\]

Then for all \(i, j \in G\)

\[
(F_{i,j} \otimes \mathbb{I})(\rho \otimes e(\gamma))(F_{i,j} \otimes \mathbb{I}) = F_{i,j} \otimes \sum_{k,l=1}^{n} \alpha_k \beta_l |G_{i,j} g_k \otimes h_l \rangle \langle G_{i,j} g_k \otimes h_l| \tag{15}
\]

where \(G_{i,j}\) is given by [4].
Proof: Using especially (10) we obtain for \( i, j, k, l \in G \) and \( f_1, f_2, f_3 \in L^2(G) \)

\[
(F_{i,j} \otimes \mathbb{1})(\mathbb{1} \otimes J)(g_k \otimes h_l, \cdot g_k \otimes h_l)(\mathbb{1} \otimes J^*)(F_{i,j} \otimes \mathbb{1})(f_1 \otimes f_2 \otimes f_3)
= \langle (F_{i,j} \otimes \mathbb{1})(\mathbb{1} \otimes J)g_k \otimes h_l, f_1 \otimes f_2 \otimes f_3 \rangle \langle (F_{i,j} \otimes \mathbb{1})(\mathbb{1} \otimes J)g_k \otimes h_l, f_3 \rangle \\
= F_{i,j}(f_1 \otimes f_2)(G_{i,j}g_k \otimes h_l, f_3)G_{i,j}g_k \otimes h_l
\]

Consequently,

\[
(F_{i,j} \otimes \mathbb{1})(\rho \otimes \mathbb{1})(F_{i,j} \otimes \mathbb{1})
= \sum_{k,l=1}^{n} \alpha_k \beta_l \langle (F_{i,j} \otimes \mathbb{1})(\mathbb{1} \otimes J)g_k \otimes h_l, f_1 \otimes f_2 \rangle (\mathbb{1} \otimes J^*)(F_{i,j} \otimes \mathbb{1})
\]

\[
= \sum_{k,l=1}^{n} \alpha_k \beta_l F_{i,j} \otimes (G_{i,j}g_k \otimes h_l, \cdot)G_{i,j}g_k \otimes h_l
\]

\[
= F_{i,j} \otimes \sum_{k,l=1}^{n} \alpha_k \beta_l (G_{i,j}g_k \otimes h_l, \cdot)G_{i,j}g_k \otimes h_l
\]

what ends the proof. \(\Box\)

Example 6 Let us return to Example 2, and suppose \( \rho \) and \( \gamma \) are given as above but with \( g_k = h_k = \Delta_k \). Then

\[
\sum_{k,l=1}^{n} \alpha_k \beta_l G_{i,j} \Delta_k \otimes \Delta_l = \sum_{k,l=1}^{n} \alpha_k \beta_l \Delta_k(i) \Delta_l(i \ominus j) \Delta_{i \ominus j} = \alpha_i \beta_{i \ominus j} \Delta_{i \ominus j}
\]

Consequently,

\[
(F_{i,j} \otimes \mathbb{1})(\rho \otimes \mathbb{1})(F_{i,j} \otimes \mathbb{1}) = \alpha_i \beta_{i \ominus j} |\Delta_i \otimes \Delta_{i \ominus j} \otimes \Delta_{i \ominus j}| \langle \Delta_i \otimes \Delta_{i \ominus j} \otimes \Delta_{i \ominus j} |\Delta_i \otimes \Delta_{i \ominus j} \otimes \Delta_{i \ominus j} \rangle
\]

4 Channels

Denote by \( \mathcal{T} \) the set of all positive trace-class operators on \( L^2(G) \) including the null operator \( \mathbf{0} \),

\[
\mathbf{0}(f) = 0 \quad \text{for} \quad f \in L^2(G).
\]

We fix an operator \( \tau \in \mathcal{T} \) having the representation

\[
\tau = \sum_{k=1}^{n} \gamma_k |h_k > < h_k |
\]
with \((\gamma_k)_{k \in G} \subseteq [0, \infty)\) and \((h_k)_{k \in G}\) being an orthonormal basis in \(L_2(G)\).

The linear mapping \(K_\tau : \mathcal{T} \rightarrow \mathcal{T}\) given by

\[
K_\tau(\rho) := \sum_{k=1}^{n} \gamma_k O_{h_k} \rho O_{h_k}^* \quad (\rho \in \mathcal{T})
\]  

(17)

depends only on the operator \(\tau\) but not on its special representation. Indeed, the following lemma holds

**Lemma 7** Let \(\tau\) have besides (16) a second representation

\[
\tau = \sum_{k=1}^{n} \beta_k |g_k><g_k|
\]

with \((\beta_k)_{k \in G} \subseteq [0, \infty)\) and \((g_k)_{k \in G}\) being an orthonormal basis in \(L_2(G)\). For arbitrary \(\rho \in \mathcal{T}\), it holds

\[
\sum_{k=1}^{n} \gamma_k O_{h_k} \rho O_{h_k}^* = \sum_{k=1}^{n} \beta_k O_{g_k} \rho O_{g_k}^*.
\]  

(18)

**Proof:** It suffices to show (18) for \(\rho \in \mathcal{T}\) of the form \(\rho = |f><f|\) with \(f \in L_2(G)\). Since

\[
\sum_{k=1}^{n} \gamma_k \langle h_k, \cdot \rangle h_k = \sum_{k=1}^{n} \beta_k \langle g_k, \cdot \rangle g_k
\]

one obtains

\[
\sum_{k=1}^{n} \gamma_k O_{h_k} \rho O_{h_k}^* = \sum_{k=1}^{n} \gamma_k \langle h_k f, \cdot \rangle h_k f = \sum_{k=1}^{n} \gamma_k O_f \langle h_k, \cdot \rangle h_k O_f^* \\
= O_f \sum_{k=1}^{n} \gamma_k \langle h_k, \cdot \rangle h_k O_f^* = O_f \tau O_f^* = O_f \sum_{k=1}^{n} \beta_k \langle g_k, \cdot \rangle g_k O_f^* \\
= \sum_{k=1}^{n} \beta_k \langle g_k f, \cdot \rangle g_k f = \sum_{k=1}^{n} \beta_k O_{g_k} \rho O_{g_k}^*.
\]

\[\square\]

**Definition 8** Denote by \(\mathcal{S}\) the set of all states on \(L_2(G)\) and for \(\tau \in \mathcal{T}\) by \(\mathcal{S}_\tau\) the set of all states \(\rho\) from \(\mathcal{S}\) with the property that \(\text{Tr} K_\tau(\rho)\) is positive:

\[
\mathcal{S}_\tau := \{\rho \in \mathcal{S} : \text{Tr} K_\tau(\rho) > 0\}. 
\]  

(19)

For \(\tau \in \mathcal{T}\) the mapping \(\hat{K}_\tau : \mathcal{S}_\tau \rightarrow \mathcal{S}\) given by

\[
\hat{K}_\tau(\rho) := \frac{1}{\text{Tr} K_\tau(\rho)} K_\tau(\rho) \quad (\rho \in \mathcal{S}_\tau)
\]  

(20)

is called the channel corresponding to \(\tau\). The channel corresponding to \(\tau\) is called unitary if there exists an unitary operator \(U\) on \(L_2(G)\) such that \(\hat{K}_\tau(\rho) = U \rho U^*\).
Observe that the channel $\hat{K}_r$ is in general nonlinear. However, in Examples 11 and 12 below the channels are even unitary.

Let us make some remarks on the physical meaning of the channels $K_r$ and $\hat{K}_r$. The channels $K_r$ are mixtures of linear channels of the type

$$K^h(\rho) := O_h \rho O_h^* \quad (\rho \in \mathcal{T})$$

with $h \in L_2(G)$, $||h|| = 1$. Let us consider the more general case

$$||h|| > 0, \quad |h(k)| \leq 1 \quad (k \in G).$$

We define an operator $t_h : L_2(G) \rightarrow L_2(\{1, 2\} \times G)$ by setting for all $f \in L_2(G)$ and $k \in G$

$$(t_h f)(l, k) = \begin{cases} h(k)f(k) & \text{for } l = 1 \\ \sqrt{1 - |h(k)|^2} f(k) & \text{for } l = 2. \end{cases}$$

The operator $t_h$ is an isometry from $L_2(G)$ to $L_2(\{1, 2\} \times G) \cong L_2(\{1, 2\}) \otimes L_2(G)$. Indeed,

$$||t_h f||^2 = \sum_{l=1}^{2} \sum_{k=1}^{n} |(t_h f)(l, k)|^2 = \sum_{k=1}^{n} (|h(k)|^2 + 1 - |h(k)|^2)|f(k)|^2 = ||f||^2.$$ 

Consequently, the mapping $E_h : \mathcal{B}(L_2(\{1, 2\} \times G)) \rightarrow \mathcal{B}(L_2(G))$ given by

$$E_h(B) := t_h^* B t_h$$

is completely positive and identity preserving. The channel $E^*_h(\rho) = t_h \rho t_h^*$ is the corresponding linear channel from the set of states on $L_2(G)$ into the set of states on $L_2(\{1, 2\} \times G)$. The space $L_2(\{1, 2\} \times G)$ has an orthogonal decomposition into $L_2(\{1\} \times G)$ and $L_2(\{2\} \times G)$ both being trivially isomorphic to $L_2(G)$. Performing a measurement according to the projection onto $L_2(\{1\} \times G) \cong L_2(G)$ given the state $E^*_h(\rho)$ one obtains the state $\hat{K}^h(\rho)$. A measurement according to the projection onto $L_2(\{2\} \times G) \cong L_2(G)$ leads to the state $\hat{K}^{\sqrt{1 - |h|^2}}(\rho)$.

Finally, let us mention that from the statistical point of view one could get a deeper insight by considering the second quantization of that procedures. This means especially to replace pure states by the corresponding coherent states and the channel $E^*_h$ by the corresponding beam splitting [5].

**Example 9** Assume $h \in L_2(G)$ fulfills $\inf_{k \in G} |h(k)| \geq c > 0$. Then the function $g \in L_2(G)$ given by

$$g(k) := \frac{c}{h(k)} \quad (k \in G)$$

fulfills the above conditions $||g|| > 0, \quad |g(k)| \leq 1$ for all $k \in G$ and we obtain for all states $\rho$ on $L_2(G)$

$$\hat{K}^g(\hat{K}^h(\rho)) = \rho.$$
Example 10 The identity $\tau = 1$, (i.e. $\tau(g) = g$ for all $g \in L_2(G)$) can be written in the form $\tau = \sum_{k \in G} |g_k><g_k|$ where $(g_k)_{k \in G}$ is an arbitrary orthonormal basis in $L_2(G)$. Now, let $\rho$ be an arbitrary element from $T$, $\rho = \sum_{k \in G} \alpha_k |h_k><h_k|$ where $(h_k)_{k \in G}$ is an orthonormal basis in $L_2(G)$ and $(\alpha_k)_{k \in G} \subseteq [0, \infty)$. Then

$$K_\tau(\rho) = \sum_{k \in G} \mathcal{O}_{g_k} \rho \mathcal{O}_{g_k}^* = \sum_{k, l \in G} \alpha_l \langle g_k h_l, \cdot \rangle g_k h_l$$

$$= \sum_{k, l \in G} \alpha_l \mathcal{O}_{h_l} \langle g_k, \cdot \rangle \mathcal{O}_{h_l}^* = \sum_{l \in G} \alpha_l \mathcal{O}_{h_l} \left( \sum_{k \in G} \langle g_k, \cdot \rangle g_k \right) \mathcal{O}_{h_l}^*$$

$$= \sum_{l \in G} \alpha_l \mathcal{O}_{|h_l|^2}$$

Because of

$$\mathcal{O}_{|h_l|^2} = \sum_{j \in G} |h_l(j)|^2 \langle \Delta_j, \cdot \rangle \Delta_j$$

the chain may be continued and we obtain

$$K_\tau(\rho) = \sum_{j \in G} \gamma_j \langle \Delta_j, \cdot \rangle \Delta_j$$

where

$$\gamma_j = \sum_{l \in G} \alpha_l |h_l(j)|^2.$$

If $\rho$ belongs to $S_\tau$ then $1 = \sum_{j \in G} \alpha_j = \sum_{j \in G} \gamma_j$ and

$$\text{Tr}(K_\tau(\rho)) = \text{Tr}(\rho) = 1.$$

Consequently, $(\alpha_j)_{j \in G}$ is a probability distribution on $G$ and the channel

$$\hat{K}_\tau(\rho) = K_\tau(\rho)$$

transforms each state $\rho \in S_\tau$ into the corresponding "classical" state.

Example 11 If $\tau$ is the pure state corresponding to $(1/\sqrt{n})1$ (cf. Example 4) one gets $K_\tau(\rho) = (1/n) \cdot \rho$ for all $\rho \in T$. For $\rho \in S_\tau$ one obtains $\hat{K}_\tau(\rho) = \rho$.

Example 12 Suppose $\tau = |b><b|$ where $b \in L_2(G)$ satisfies $|b(k)| = 1/\sqrt{n}$ for all $k \in G$ (in Example 17 we assumed $b(k) = 1/\sqrt{n}$). Since $\text{Tr} K_\tau(\rho) = (1/n) \text{Tr}(\rho)$ we obtain for all $\rho \in S_\tau$ $\hat{K}_\tau(\rho) = B\rho B^*$ with $B = \sqrt{n} \mathcal{O}_b$. As we remarked on page 4, $\mathcal{O}_b$ is unitary if and only if $|b(k)| = 1$ for all $k$. Consequently, the channel is unitary.
If \( \tau \) is a mixed state \( \hat{K}_\tau(\rho) \) usually will be mixed even for pure states \( \rho \). Below we give a simple example for this.

**Example 13** Let \( (\Delta_m)_{m \in G} \) denote as in Example 2 the basis in \( L_2(G) \) given by \( \Delta_m(k) = \delta_{m,k} \). Put \( \tau = \frac{1}{\sqrt{2}} (\Delta_1, \cdot) \Delta_1 + \frac{1}{\sqrt{2}} (\Delta_2, \cdot) \Delta_2 \) and let \( \rho \) be the pure state corresponding to \( \frac{1}{\sqrt{2}} (\Delta_1 + \Delta_2) \). Then \( K_\tau(\rho) = \frac{1}{2} \tau \) and \( \hat{K}_\tau(\rho) = \tau \).

**Example 14** Let \( \tau = (f, \cdot) f \) be a pure state, and assume \( f \in L_2(G) \) fulfills \( f(k) \neq 0 \) for all \( k \). Consequently, \( \frac{1}{f} \in L_2(G) \) and for all \( \rho \in S = S_\tau \) \( O_{1/f} \rho O_f^* = \rho \).

This implies \( K_{\tau}^{-1} = K_\tilde{\tau} \) with \( \tilde{\tau} = (\frac{1}{f}, \cdot) \frac{1}{f} \). Normalizing \( \tilde{\tau} \) to a state one could write alternatively \( K_{\tau}^{-1} = \lambda^2 K_\tau \), \( \tilde{\tau} = (g, \cdot) g \) with \( g = \frac{1}{\rho} \cdot \frac{1}{f} \) and \( \lambda = ||\frac{1}{f}|| \).

## 5 The State of the Memory after Recognition

Let us recall that for states \( \rho, \gamma \) on \( L_2(G) \) and \( i, j \in G \)

\[
(F_{i,j} \otimes \mathbb{1})(\rho \otimes \mathbb{1})(F_{i,j} \otimes \mathbb{1})
\]

is a linear operator from \( L_2(G^3) \) into \( L_2(G^2) \), and that (cf. 14) it is equal

\[
(F_{i,j} \otimes \mathbb{1})(\mathbb{1} \otimes J)(\rho \otimes \gamma)(\mathbb{1} \otimes J^*)(F_{i,j} \otimes \mathbb{1}).
\]

In the following we consider the family of channels \( (\Lambda_{i,j})_{i,j} \) from the set of product states \( \rho \otimes \gamma \) on \( H_1 \otimes H_2 \) into the states on \( H_3 \) given by

\[
\Lambda_{i,j}(\rho \otimes \gamma) := \frac{\text{Tr}_{1,2}(F_{i,j} \otimes \mathbb{1})(\rho \otimes \mathbb{1})(F_{i,j} \otimes \mathbb{1})}{\text{Tr}_{1,2,3}(F_{i,j} \otimes \mathbb{1})(\rho \otimes \mathbb{1})(F_{i,j} \otimes \mathbb{1})}
\]

where \( \text{Tr}_{1,2} \) resp. \( \text{Tr}_{1,2,3} \) denotes the partial trace with respect to the first two components resp. the full trace with respect to all three spaces. In the sequel we always will assume that

\[
\text{Tr}_{1,2,3}(F_{i,j} \otimes \mathbb{1})(\rho \otimes \mathbb{1})(F_{i,j} \otimes \mathbb{1}) > 0.
\]

Let \( \rho \) and \( \gamma \) are given as in Proposition 5. Since \( (\xi_{i,j})_{i,j} \) is an orthonormal basis in \( L_2(G^2) \) (Lemma 7) we get from Proposition 5

\[
\text{Tr}_{1,2}(F_{i,j} \otimes \mathbb{1})(\rho \otimes \mathbb{1})(F_{i,j} \otimes \mathbb{1}) = \sum_{k,l=1}^{n} \alpha_k \beta_l (G_{i,j} g_k \otimes h_{l}, \cdot) G_{i,j} g_k \otimes h_{l}
\]

Summarizing, we get the following representation of \( \Lambda_{i,j} \):
**Proposition 15** Let $\rho$ and $\gamma$ be given as in Proposition 5. Further, assume $\Phi \in L_2(G^2)$. Then
\[
\Lambda_{i,j}(\rho \otimes \gamma) = \sum_{k,l=1}^{n} \alpha_k \beta_l \langle G_{i,j}g_k \otimes h_l, \cdot \rangle G_{i,j}g_k \otimes h_l \| G_{i,j}g_k \otimes h_l \|^{-2} \]
where for $\Phi \in L_2(G^2)$
\[
\| G_{i,j} \Phi \|^2 = \sum_{m=1}^{n} |b_i|^2 |(m \otimes j) \Phi(m \otimes j, m)|^2.
\]

**Example 16** Let $\rho$ and $\gamma$ be pure states, $\rho = \langle g, \cdot \rangle g, \gamma = \langle h, \cdot \rangle h$. Then
\[
\text{Tr}_{1,2}(F_{i,j} \otimes \mathbb{1})(\rho \otimes \epsilon(\gamma))(F_{i,j} \otimes \mathbb{1}) = \langle G_{i,j}g \otimes h, \cdot \rangle G_{i,j}g \otimes h = \langle J^*(U_j \mathcal{O}_b^\ast g \otimes h), \cdot \rangle J^*(U_j \mathcal{O}_b^\ast g \otimes h).
\]
Fortunately, we can find expressions for the state $\Lambda_{i,j}(\rho \otimes \gamma)$ of the memory after the recognition process being in many cases simpler. We can express the teleportation channel $\Lambda_{i,j}$ with the help of the channels $K_\tau$ we introduced in the previous section.

**Proposition 17** Let $i, j \in G$ and let $\rho$ be a state from $\mathcal{S}_{|b_i \rangle \langle b_i|}$. Further, let $\gamma$ be a state from $\mathcal{S}$ such that
\[
U_j \mathcal{K}_{|b_i \rangle \langle b_i|} (\rho) U_j^\ast \in \mathcal{S}_\gamma.
\]
Then
\[
\Lambda_{i,j}(\rho \otimes \gamma) = \hat{K}_\gamma \circ K^j \circ \mathcal{K}_{|b_i \rangle \langle b_i|} (\rho)
\]
where $K^j$ denotes the unitary channel given by $K^j(\rho) = U_j \rho U_j^\ast$.

**Proof:** Let $\rho$ and $\gamma$ be given as in Proposition 5. We set $B_i = \mathcal{O}_b$. Thus $B_i^\ast = \mathcal{O}_b^\ast$. From $K_{|b_i \rangle \langle b_i|} (\rho) = B_i^\ast \rho B_i$ we conclude $K^j \circ K_{|b_i \rangle \langle b_i|} (\rho) = U_j B_i^\ast \rho B_i U_j^\ast$ what leads to

\[
K_\gamma \circ K^j \circ K_{|b_i \rangle \langle b_i|} (\rho) = \sum_{k,l=1}^{n} \alpha_k \beta_l \mathcal{O}_h U_j B_i^\ast \rho B_i^\ast U_j^\ast \mathcal{O}_h
\]

\[
= \sum_{k,l=1}^{n} \alpha_k \beta_l \mathcal{O}_h U_j B_i^\ast \langle (g_k, \cdot) g_k \rangle B_i^\ast U_j^\ast \mathcal{O}_h
\]

\[
= \sum_{k,l=1}^{n} \alpha_k \beta_l \langle h_l U_j B_i^\ast g_k, \cdot \rangle h_l U_j B_i^\ast g_k
\]

\[
= \sum_{k,l=1}^{n} \alpha_k \beta_l \langle J^*(U_j B_i^\ast \otimes \mathbb{1}) g_k \otimes h_l, \cdot \rangle J^*(U_j B_i^\ast \otimes \mathbb{1}) g_k \otimes h_l
\]

\[
= \sum_{k,l=1}^{n} \alpha_k \beta_l \langle G_{i,j}g_k \otimes h_l, \cdot \rangle G_{i,j}g_k \otimes h_l
\]
Finally, from
\[
\hat{K}_\gamma \circ K^j \circ K_{|b_i><b_i\rangle}^{B_i} (\rho) = \hat{K}_\gamma \circ K^j \circ \hat{K}_{|b_i><b_i\rangle}^{B_i} (\rho)
\]
we obtain (27). □

In the following let us comment the results and give some examples. Let \( \rho \) be an arbitrary state of the processing part (the brain), and assume the measurement of the incoming signal leads to the value \( z_{i,j} \). Then the input in the memory being in the state \( \gamma \) will be
\[
C \cdot U^*_j B^*_i \rho B_i U^*_j
\]
where \( C \) is the normalizing constant. After the recognition process the brain will be in the state
\[
\tilde{C} \cdot K_\gamma \circ ( U^*_j B^*_i \rho B_i U^*_j )
\]
where \( \tilde{C} \) is again the normalizing constant.

**Example 18** Let us consider the extreme cases that either the processing part or the memory is in the trivial state
\[
\kappa := | \frac{1}{\sqrt{n}} \rangle \langle \frac{1}{\sqrt{n}} |
\]
(cf. Example 4). This state has no experience, no special knowledge, there will be no selection of incoming information. It is easy to check that for all \( \mu \in S \) it holds
\[
\hat{K}_\kappa (\mu) = \mu
\]
(observe that \( K_\kappa (\mu) = (1/n) \cdot \mu \)). On the other hand, for all states \( \mu \) the relation
\[
\hat{K}_\mu (\kappa) = \mu
\]
is true.

Now, we consider the case of the memory being in the state \( \gamma = \kappa \). We obtain from (28) and from (24) for all \( i,j \in G \) and \( \rho \in S_{|b_i><b_i\rangle}^{B_i} \)
\[
\Lambda_{i,j}(\rho \otimes \kappa) = \tilde{C} \cdot U^*_j B^*_i \rho B_i U^*_j.
\]
The memory will store exactly what comes in (the system is able to learn everything - cf. also [2, 7]). Since \( U_j \) and \( \sqrt{n} B_i \) are unitary operators (\( \tilde{C} = n \)) we see that for all \( i,j \) there exists (in the language of teleportation procedures) a unitary key \( V_{i,j} \) to recover \( \rho \), i.e. \( \Lambda_{i,j}(\rho \otimes \kappa) = V_{i,j} \rho V_{i,j}^* \).

Now, let the processing part (the brain) be in the state \( \rho = \kappa \) defined by (29). Then
\[
K^j \circ \hat{K}_{|b_i><b_i\rangle}^{B_i} (\rho) = |U^*_j b_i\rangle < U^*_j b_i|.
\]
For all states of the brain $\gamma$ such that $|U_j b_i > < U_j b_i| \in S_\gamma$ we obtain

$$\Lambda_{i,j}(\rho \otimes \gamma) = \hat{K}_\gamma(|U_j b_i > < U_j b_i|).$$

So (as one could expect) the final state in the memory depends only on the measured value $z_{i,j}$ and the state of the memory (before recognition).

**Example 19** Let $(b_j)_{j \in G}$ be an orthonormal basis fulfilling

$$|b_j(l)|^2 = \frac{1}{n} \quad (j, l, \in G).$$

For the pure state $\rho = \frac{1}{n} I$ we obtain for all $i \in G$

$$K_{|b_i > < b_i|}(\rho) = \frac{1}{n} O_{|b_i|}^2 = \frac{1}{n^2} \sum_{k \in G} |\Delta_k > < \Delta_k| = \frac{1}{n^2} I.$$

Consequently, for $j \in G$

$$K_j \circ K_{|b_i > < b_i|}(\rho) = \frac{1}{n^2} I.$$

For each state $\gamma$ (we use notation as in Proposition 5) and all $i, j \in G$ we finally get

$$\Lambda_{i,j}(\rho \otimes \gamma) = \sum_{k \in G} \frac{\beta_k}{n} O_{|h_k|} = \sum_{l \in G} \gamma_l |\Delta_l > < \Delta_l|$$

with

$$\gamma_l = \sum_{k \in G} \frac{\beta_k |h_k|^2(l)}{n}.$$

So we obtain a classical state with probability distribution $(\gamma_k)_{k \in G}$.

**Example 20** Take $(b_j)_{j \in G}$ as in Example 19 above, but now suppose $\gamma = \frac{1}{n} I$. Let $\rho$ be given as in Proposition 5. For all $i, j \in G$ we get

$$K_j \circ K_{|b_i > < b_i|}(\rho) = U_j B_i^* \rho B_i U_j^* = \sum_{k \in G} \alpha_k |U_j B_i^* g_k \rangle \langle U_j B_i^* g_k|.$$  \(31\)

Observe that for all $i, j$ the sequence $(U_j B_i^* g_k)_{k \in G}$ is an orthogonal system. Indeed,

$$\langle U_j B_i^* g_k, U_j B_i^* g_l \rangle = \frac{1}{n} (U_j g_k, U_j g_l) = \frac{1}{n} < g_k, g_l >.$$

Consequently, as in Example 19 and above we conclude

$$K_\gamma \circ K_j \circ K_{|b_i > < b_i|}(\rho) = \sum_{k \in G} \frac{\alpha_k}{n} O_{|U_j B_i^* g_k|^2}$$
= \sum_{k \in G} \frac{\alpha_k}{n^2} O_{|U_j g_k|^2}
= U_j \sum_{k \in G} \frac{\alpha_k}{n^2} O_{|g_k|^2} U_j^*
= U_j \sum_{k \in G} \gamma_k |\Delta_k \rangle \langle \Delta_k| U_j^*
= \sum_{k \in G} \gamma_k |\Delta_k \Delta_j \rangle \langle \Delta_k \Delta_j|$

with

\[ \gamma_k = \sum_{l \in G} \frac{\alpha_l}{n^2} |g_l|^2(k). \]

Finally we thus get

\[ \Lambda_{i,j}(\rho \otimes \gamma) = \sum_{k \in G} \tilde{\gamma}_k |\Delta_{k \Delta_j} \rangle \langle \Delta_{k \Delta_j}| \]

with

\[ \tilde{\gamma}_l = \sum_{k \in G} \frac{\alpha_l}{n^2} |h_k|^2(l). \]

We see that \( \Lambda_{i,j} \) does not depend on \( i \). Especially, there do not exist unitary keys.

Example 21 We take again \((b_j)_{j \in G}\) as in Examples 19 and 20. Let us further assume that \( \gamma = |h \rangle \langle h| \) is a pure state satisfying \( |h(j)|^2 = \frac{1}{n} \) for all \( j \in G \).

Using (31) we get for arbitrary \( \rho \)

\[ K_\gamma \circ K^j \circ K_{|_\rho \rangle \langle _\rho|}(\rho) = O_h U_j B_i^* \rho B_i U_j^* O_h^* \]

Observe that \( \sqrt{n} O_h \) and \( \sqrt{n} B_i \) are unitaries. Consequently, we get

\[ \Lambda_{i,j}(\rho \otimes \gamma) = V_{i,j} \rho V_{i,j}^* \]

with the unitary key \( V_{i,j} = n O_h U_j B_i^* \).

The choice of the basis \((b_k)_{k \in G}\) is very important in this model. Because of the specially chosen projection operators \( F_{i,j} \) these are the only elementary signals that can be measured. Let us consider the case that the selected basis \((b_k)_{k=1}^n\) is given by \((\Delta_k)_{k=1}^n\). In this case we get an especially simple (but also trivial) output. Let us remark that for all \( r, k, l \in G \) such that \( k \neq l \) it holds \( \Delta_k(r) \Delta_l(r) = 0 \). Thus the elements of the basis fulfill a condition much more stringent than just being orthogonal.

Example 22 In Example 2 we obtain for \( \rho = \sum_{k=1}^n \alpha_k |\Delta_k \rangle \langle \Delta_k|, \gamma = \sum_{k=1}^n \beta_k |\Delta_k \rangle \langle \Delta_k| \)

\[ \text{Tr}_{1,2}(F_{i,j} \otimes \mathbb{I})(\rho \otimes e(\gamma))(F_{i,j} \otimes \mathbb{I}) = \alpha_i \beta_{i \Delta_j} |\Delta_{i \Delta_j} \rangle \langle \Delta_{i \Delta_j}| \]
and if $\alpha_i > 0, \beta_{i\oplus j} > 0$

$$\Lambda_{i,j}(\rho \otimes \gamma) = |\Delta_{i \oplus j} > < \Delta_{i \oplus j}|.$$ 

So if $\alpha_i > 0, \beta_{i\oplus j} > 0$ the state after recognition depends only on the measured value $i \oplus j$. If $\rho$ resp. $\gamma$ cannot occur in the state $\Delta_i$ resp. $\Delta_{i \oplus j}$ no information about the input can be stored in the memory.

**Example 23** Let $\rho = \sum_{k=1}^{n} \alpha_k |g_k > < g_k|$, $\gamma = \sum_{k=1}^{n} \beta_k |h_k > < h_k|$ be arbitrary states. What will be the state of the memory after recognition if only these elementary signals $\Delta_k$ can be measured? Will these elementary signals also damage the states $\rho$ and $\gamma$ in such a way that only the information whether $\alpha_i > 0$ and $\beta_{i \oplus j} > 0$ plays a role? Simple calculations as above lead to the following form (in what follows we omit normalizing constants):

$$\text{Tr}_{1,2}(F_{i,j} \otimes \mathbb{I})(\rho \otimes \mathfrak{e}(\gamma))(F_{i,j} \otimes \mathbb{I}) = \sum_{k,l=1}^{n} \alpha_k \beta_l |A_{k,l} > < A_{k,l}|$$

where

$$A_{k,l}(r) = \Delta_{i \oplus j}(r)g_k (r \oplus j)h_l(r) \quad (r \in G).$$

Now, for each $k \in G$ there exist sequences $(\mu^k_{s=1})^{n}_{s=1}, (\nu^k_{s=1})^{n}_{s=1}$ such that $g_k = \sum_{s=1}^{n} \mu^k_s \Delta_s$ and $h_k = \sum_{s=1}^{n} \nu^k_s \Delta_s$. This implies $A_{k,l} = \mu^k_l \nu^l_{i \oplus j} \Delta_{i \oplus j}$. Consequently,

$$\text{Tr}_{1,2}(F_{i,j} \otimes \mathbb{I})(\rho \otimes \mathfrak{e}(\gamma))(F_{i,j} \otimes \mathbb{I}) = \sum_{k,l=1}^{n} \alpha_k \beta_l |\mu^k_l \nu^l_{i \oplus j}|^2 |\Delta_{i \oplus j} > < \Delta_{i \oplus j}|$$

$$= C \cdot |\Delta_{i \oplus j} > < \Delta_{i \oplus j}|.$$ 

The state $\sigma = \Lambda_{i,j}(\rho \otimes \gamma)$ after recognition will be the same as in the above example. We see that measuring $z_{i,j}$ the state $\sigma$ will be able to store in his memory at most the signal $\Delta_{i \oplus j}$. And this can be done only if there exists at least one pair $(k, l)$ such that $\alpha_k \beta_l |\mu^k_l \nu^l_{i \oplus j}|^2 > 0$.

**Concluding remarks:** The aim of the paper was to touch the problem of finding simplified models for the recognition process. We were interested in how the input signal arriving at the brain is entangled (connected) to the memory already stored and the consciousness that existed in the brain, and how a part of the signal will be finally stored as a memory. Just to achieve simple explicit expressions we illustrated the model on the most simple sequence of signals $(\Delta_k)_{k=1}^{n}$. It is clear that this example is just for illustration and can not serve for describing realistic aspects of recognition. Choosing a more complex basis one obtains expressions depending heavily on the states $\rho$ and $\gamma$. Though the above presented model is only a first attempt it shows that there are possibilities to model the process of recognition. To get closer to realistic models we will try to refine the above models by
- passing over to infinite Hilbert spaces,
- replacing pure states by coherent states on the Fock space,
- considering different Hilbert spaces $\mathcal{H}_1, \mathcal{H}_2$ and $\mathcal{H}_3$,
- making more complex measurements than simple one-dimensional projections $F_{i,j}$,
- replacing the trivial entanglement $J$ by a more complex one based on beam splitting procedures, and finally
- adding an entanglement between the states $\rho$ and $\gamma$ on $\mathcal{H}_1$ and $\mathcal{H}_2$.

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