Development and verification of the two-layer thick-walled spherical shell’s finite element model under temperature and force exposure

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Abstract. A finite element model of a two-layer thick-walled spherical shell under temperature and force exposure was developed in the ANSYS Mechanical APDL software package. In order to verify the numerical solution of the thermomechanical problem based on the equations of the inhomogeneous bodies’ elasticity theory, an analytical solution of the problem under consideration in spherical coordinates was obtained, the results of which were compared with the numerical solution.

1. Introduction
Thick-walled spherical shells are widely used in industry and nuclear energy. In the present work, we constructed the finite element solution of the idealized problem in the ANSYS Mechanical APDL software package, and then verified it with the solution obtained analytically. In the future, this numerical solution is supposed to be used for more complex problems.

The thick-walled two-layer spherical shell (Figure 1.) is under the action of constant internal and external pressures $p_a$, $p_b$ and stationary temperature field determined by the temperatures of the inner and outer surfaces $T_a$, $T_b$.

![Stresses in a spherical shell and a cross section along the central axis of symmetry with the boundary conditions’ setting.](image)

Figure 1. Stresses in a spherical shell and a cross section along the central axis of symmetry with the boundary conditions’ setting.
According to the section in Figure 1, the inner radius $a = 1.2$ m, bilayer radius $c = 1.5$ m, outer radius $b = 1.8$ m, the internal temperature $T_a = 200 \, ^\circ$C, the external temperature $T_b = 0 \, ^\circ$C, pressure on the inner surface $p_a = 5$ MPa, the external pressure $p_b = 10$ MPa. The physical constants of the considered layers’ materials are shown in Table 1 (materials are arbitrary).

### Table 1. Physical material constants

| Name of the physical parameter                      | 1st layer of material | 2nd layer of material |
|----------------------------------------------------|-----------------------|-----------------------|
| Modulus of elasticity, MPa                         | $E_1 = 2 \cdot 10^4$  | $E_2 = 5 \cdot 10^2$  |
| Poisson’s ratio                                     | $\nu_1 = 0.2$         | $\nu_2 = 0.25$        |
| Coefficient of linear thermal expansion, $1/\degree$C| $\alpha_1 = 2 \cdot 10^{-5}$ | $\alpha_2 = 1 \cdot 10^{-5}$ |
| Thermal Conductivity, W/(m$\cdot$°C)              | $\lambda_1 = 35$      | $\lambda_2 = 17.5$    |

2. **Numerical implementation**

We will build a solution to the thermo-mechanical problem in the ANSYS Mechanical APDL software package in a loosely coupled formulation, sequentially combining several processes based on different physical principles. For each process, its own problem is solved and a different type of finite element is used, and the connection of tasks is ensured through the data exchange (decision fields) [1].

Numerical modeling of the static and temperature state in the ANSYS Mechanical APDL software package is based on the finite element method (FEM) implementation in the matrix form of displacements [2,3]. To solve the basic system of equations, global matrices - are formed: stiffnesses $[K]$, damping $[C]$ and masses $[M]$, as well as the external nodal load vector $\{F\}$.

For the linear static problems, the following system of linear algebraic equations is solved [3]:

$$
[ K^u ] \cdot \{ u \} = \{ F \},
$$

where $\{ u \}$ – is the nodal displacement vector, $[ K^u ]$ – is the global mechanical stiffness matrix defined as:

$$
[ K^u ] = \sum_{i=1}^{N} [ K_{ei}^u ],
$$

where $N$ – defines the number of elements, $[ K_{ei}^u ]$ – shows the mechanical stiffness matrix element.

For the stationary heat conduction problems, the system of linear algebraic equations (1) is solved by substituting a global matrix for $[ K^u ]$ thermal conductivity matrix $[ K^t ]$ instead of the nodal displacement vector $\{ u \}$ – the desired nodal temperature vector $\{ T \}$, instead of the global external nodal load vector $\{ F \}$ – the heat flow vector $\{ Q \}$.

The use of the matrix equations of linear static and thermal conductivity in the framework of one problem will be considered in a loosely coupled formulation. The finite element model of a two-layer thick-walled spherical shell is shown in Figure 2.
The calculation is carried out in a loosely coupled formulation, in which the thermomechanical problem is realized by the Coupled-field analysis [1,3]. The essence of the connection is to expand the matrix equation (1) by adding the dependencies of the mechanical matrix \([K^u]\) and the load vectors \(\{F\}\) from the temperature vector \(\{T\}\), as well as the addition of a thermal conductivity matrix \([K^t]\) and heat flux vectors \(\{Q\}\). Then the matrix equation (1) for the thermomechanical weakly coupled problem in the general form is written as [3]:

\[
\begin{bmatrix}
[K^u(\{u\},\{T\})] & [0] \\
[0] & [K^t(\{u\},\{T\})]
\end{bmatrix}
\begin{bmatrix}
\{u\} \\
\{T\}
\end{bmatrix} = \begin{bmatrix}
\{F(\{u\},\{T\})\} \\
\{Q(\{u\},\{T\})\}
\end{bmatrix},
\]  

(3)

where \([K^u(\{u\},\{T\})]\) – are the displacement-dependent global mechanical stiffness matrix \(u\) and temperature \(T\), \([K^t(\{u\},\{T\})]\) – define the global temperature dependent thermal conductivity matrix \(T\) and displacements \(u\), \(\{F(\{u\},\{T\})\}\) – is the vector of global external load depending on thermal deformation, \(\{Q(\{u\},\{T\})\}\) – is the heat flow vector depending on external heat sources and thermoelastic damping.

Considering that the problem under consideration is solved in a simply connected formulation, the terms containing deformation will be absent in the heat equation. Taking into account the stress-strain state dependence on forced temperature strains and the independence of forced temperature strains on the stress-strain state, the matrix equation (3) can be converted to:

\[
\begin{bmatrix}
[K^u(\{u\},\{T\})] & [0] \\
[0] & [K^t(\{T\})]
\end{bmatrix}
\begin{bmatrix}
\{u\} \\
\{T\}
\end{bmatrix} = \begin{bmatrix}
\{F(\{u\},\{T\})\} \\
\{Q(\{T\})\}
\end{bmatrix},
\]  

(4)

Taking into account that the incoherence of the heat conduction problem with mechanical strains, the consequence of which is the absence in the second row of the matrix equation (4) on the displacement vector \(\{u\}\), this problem can be solved in two ways.

The first method involves adding the heat problem to the mechanics task. In this case, one degree of freedom is added to the mechanical finite element - the temperature \(T\), forming a connected finite element and connecting the global stiffness matrix is built.

In this work, we chose another method - solving the heat conduction problem with further sequential transfer of temperature fields to the strength problem, taking into account the dependence of the mechanical stiffness matrix and the global load vector on temperature in the physical law. A similar method considers the calculations on various grids at both stages of finite element modeling. Such an implementation of the finite element model of a two-layer thick-walled spherical shell under temperature and force exposures can be called a sequential solution in a simply connected formulation.

2.1. Calculation of the temperature field
The first stage of the presented method is characterized by solving the stationary heat conduction problem. The temperature conditions of the inner and outer layers were taken as the boundary conditions ($T_a$ and $T_b$) bilayer spherical shell.

The results of the calculation were the temperature fields’ distribution over the thickness of a two-layer spherical shell, shown in Figure 3.

Figure 3. Temperature distribution in a two-layer spherical shell in degrees °C.

2.2. Stress calculation

Then the strength calculation with boundary conditions follows in the form of symmetry conditions (a quarter of the sphere is considered), force boundary conditions in the form of pressures on the inner and outer layers ($p_a$ and $p_b$) a two-layer spherical shell, as well as a predetermined temperature field obtained from the previous thermal calculation in the form of a nodal temperature distribution in the finite element model.

The results of the strength analysis of a two-layer spherical shell on the temperature and force exposure in a loosely coupled one-sided formulation are presented in Figure 4.

Figure 4. Iso field stresses $\sigma_r$ and $\sigma_\theta$ in a two-layer spherical shell, Pa.

3. Analytical solution

Taking into account that temperature and power effects depend only on a change in radius $r$, we can consider the problem one-dimensional and solve it in spherical coordinates. Considering the symmetry, all the functions (temperature, stress, strain and displacement) will depend on one variable - the radius $r$. 
To verify the finite element model under consideration, we present the analytical solution based on the inhomogeneous bodies’ mechanics theory. We take the temperature distribution $T(r)$, as well as the normal stresses $\sigma_r(r)$ and $\sigma_\theta(r)$ as the verifiable parameters of the thermomechanical problem in a two-layer spherical shell.

3.1. Calculation of the temperature field

For the stationary (not time-varying) temperature field in the absence of heat sources, the Laplace equation [4] in spherical coordinates has the following form:

$$\frac{d^2}{dr^2} T(r) + 2 \frac{d}{dr} T(r) = 0.$$  \hspace{1cm} (5)

The solution to this equation (5), taking into account the piecewise constant inhomogeneity of the spherical shell, is presented as a solution for the inner and outer layers:

$$T_i(r) = A_i + \frac{B_i}{r}, \quad T_o(r) = A_o + \frac{B_o}{r}. \hspace{1cm} (6)$$

The temperature function in the wall of a two-layer spherical shell takes the form of a hyperbolic function, which follows from (6). The four integration constants in (6) are determined from four boundary conditions:

$$r = a, \quad T_i = T_o; \quad r = c, \quad T_i = T_o; \quad r = c, \quad q_i = q_o; \quad r = b, \quad T_o = T_b. \hspace{1cm} (7)$$

Heat flux density $q$ is calculated by the formula:

$$q = -\lambda \frac{dT}{dr},$$  \hspace{1cm} (8)

where $\lambda$ – is the thermal conductivity of the material.

3.2. Stress calculation

Taking the central symmetry of the problem, and also that $\sigma_\theta = \sigma_f$ into consideration, the equilibrium equation in spherical coordinates takes the form [5,6]:

$$\frac{d\sigma_r}{dr} + \frac{1}{r} \sigma_r - \sigma_\theta = 0.$$  \hspace{1cm} (9)

The resolving equation of such a problem is presented in [5-7]:

$$\frac{d^2}{dr^2} \sigma_r(r) + \frac{4}{r} \left( \frac{d}{dr} \varepsilon_r(r) \right) = \frac{2}{r^2} \left( \frac{d}{dr} \varepsilon_r(r) \right) E.$$  \hspace{1cm} (10)

It is worth noting here that the main physical law describing the forced deformations’ effect on the stress-strain state is the Duhamel-Neumann relation [6]:

$$\varepsilon_r = \frac{1}{E} (\sigma_r - 2\nu\sigma_\theta) + \varepsilon_T; \quad \varepsilon_\theta = \frac{1}{E} [(1-\nu)\sigma_\theta - \nu\sigma_r] + \varepsilon_T.$$  \hspace{1cm} (11)

By the forced strains $\varepsilon_T$ in the equations (11) we mean the strains caused by the influence of the temperature field. The main idea of the Duhamel-Neumann relations is the strain tensor $\varepsilon_{ij}$, which components are the sum of thermal deformation $\varepsilon_{ij}^T$ and force elastic deformation $\varepsilon_{ij}^F$ [8]:

$$\varepsilon_{ij} = \epsilon_{ij}^T + \epsilon_{ij}^F.$$  \hspace{1cm} (12)

For example, in simply connected thermomechanical problems, the forced deformations caused by a temperature field are described as unrelated and independent of equations of mechanics:

$$\varepsilon_T = \int_{0}^{T} \alpha_T(T) dT.$$  \hspace{1cm} (13)

Here: $\alpha_T$ – is the coefficient of linear thermal expansion.

When considering stationary or quasi-stationary processes, the temperature distribution $T(r,\theta,\phi)$ can be found from the heat equation, in which there are no terms containing deformation, assuming the
stress-strain state dependence on the forced temperature deformations and the independence of the forced temperature deformations on the stress-strain state (simply connected statement).

Under the conditions of the problem under consideration, the thermal field coupling affecting the stress-strain state of a spherical thick-walled shell will be realized through the forced strain term \( \varepsilon T \) defined by expression (13), which is present in the Duhamel-Neumann physical relations (11), as well as in resolving the equation (10).

The solution of the linear inhomogeneous differential equation (10) for two layers of the spherical shell is sought in the form of the general and particular solutions’ sum:

\[
\sigma_{r1} = \left( \frac{C_1}{r^3} + D_1 \right) + \frac{F_1}{r}; \quad \sigma_{r2} = \left( \frac{C_2}{r^3} + D_2 \right) + \frac{F_2}{r}.
\]  

We have 6 constants: \( C_1, C_2, D_1, D_2, F_1, F_2 \), to be determined. The constants \( F_1 \) and \( F_2 \) are determined by substituting the obtained solutions (14) into the resolving equation (10), the constants \( C_1, C_2, D_1, D_2 \), occurring in (14) are determined from 4 boundary conditions:

\[
r = a, \quad \sigma_{r1} = -p_a; \quad r = c, \quad \sigma_{r1} = \sigma_{r2}; \quad r = c, \quad u_1 = u_2; \quad r = b, \quad \sigma_{r2} = -p_b.
\]  

The functions for stresses \( \sigma_{r1} \) and \( \sigma_{r2} \) in two layers of a spherical shell are determined by substituting (14) into the equilibrium equation (9):

\[
\sigma_{r1} = \frac{2Dr^3 + Fr^2 - C_1}{2r^3}; \quad \sigma_{r2} = \frac{2Dr^3 + Fr^2 - C_2}{2r^3}.
\]  

Thus, the analytical expressions for the temperatures \( T \), the stresses \( \sigma_r \) and \( \sigma_\theta \) in a centrally symmetric thermomechanical problem taking into account piecewise linear inhomogeneity, were obtained.

### 4. Verification

The results of the analytical solution may be interesting from the point of view of the software systems’ verification that implement the numerical methods, in particular, the finite element method. The solution of a simply connected thermomechanical problem verifies both the solution of the static problem taking into account the temperature field and the correctness of the thermal field nodal solution approximation to the finite element grid for static analysis, taking into account the fact that different finite element grids may be required for heat conduction and static problems [9]. The areas of the grids’ thickening may not coincide, since the largest gradients of temperature and force fields are localized in different areas of the computational model and, as a rule, thermal fields have a smaller gradient of change compared to force fields.

For verification, we consider both the qualitative and quantitative characteristics of the functions under consideration. Here are the graphs of function changes \( T(r), \sigma_r \) and \( \sigma_\theta \) by the thickness of the spherical shell, as well as the maximum values of the obtained values. The results of comparing the graphs of functions obtained during finite element implementation in the ANSYS Mechanical APDL software package and are analytically presented in Figures 5-8. A comparison of the quantitative characteristics is presented in Table 2.
**Figure 5.** Temperature distribution in the wall of a two-layer spherical shell (on the left are the results of the analytical solution, on the right are the results of ANSYS).

**Figure 6.** Voltage distribution $\sigma_r$ along the two-layer spherical shell radius (on the left are the results of the analytical solution, on the right are the results of ANSYS).

**Figure 7.** Voltage distribution $\sigma_\theta$ along the two-layer spherical shell radius (on the left are the results of the analytical solution, on the right are the results of ANSYS).
The results of the quantitative characteristics’ comparison are presented in Table 2.

| Verifiable parameter                                    | Analytical solution | ANSYS Mechanical APDL | Relative error δ (%) |
|---------------------------------------------------------|---------------------|-----------------------|----------------------|
| Temperature at the contact of 2 layers (°С)             | 114.3               | 114.3                 | 0                    |
| Maximum voltage σr (MPa)                               | -12.92              | -12.96                | 0.3%                 |
| Maximum voltage σθ (MPa)                               | -53.21              | -53.18                | 0.06%                |

5. Summary
1. The ANSYS Mechanical APDL software package developed a finite element model of a two-layer spherical shell under the action of constant internal and external pressures, as well as a stationary temperature field determined by the internal and external surfaces’ temperatures, implemented by coupled-field analysis in the form of a simply connected thermomechanical problem with the thermal solution thermal fields’ transfer to the strength problem.
2. From the general matrix equation for the loosely coupled problems, a matrix equation in a simply connected statement was obtained by means of which the finite element modeling was performed.
3. The transfer of temperature fields to a static problem makes it possible to carry out the calculations on various grids at both stages of the finite element modeling, which reduces the computation time with a minimum loss of accuracy.
4. A numerical solution was verified with an analytical one based on the mechanics of inhomogeneous bodies in a simply connected formulation for thermomechanical problems.
5. The verification results confirmed the correctness of this finite element model, which allows it to be used later in solving more complex problems, for example, for calculating two-layer thick-walled spherical shells with technological holes.

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