SUPPLEMENT TO THE ARTICLE “IRREDUCIBLE POLYNOMIALS WITH BOUNDED HEIGHT”

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In this paper we prove that the determinant of a random matrix is unlikely to be a square. Formally,

Theorem. Let $M$ be an $n \times n$ matrix with i.i.d. entries taking the value 0 with probability $\frac{1}{2}$ and the values 1 and $-1$ with probability $\frac{1}{4}$ each. Then

$$\lim_{n \to \infty} \mathbb{P}(\exists k \in \mathbb{Z} \text{ such that } \det M = k^2) = 0.$$ 

We direct the reader to our main paper [1, §4] for motivation for such a result, and in particular why we are interested in squares and not in any other sparse subset of the integers.

Throughout the paper we denote by $\xi_i$ random independent variables taking the value 0 with probability $\frac{1}{2}$ and the values 1 and $-1$ with probability $\frac{1}{4}$ each.

Lemma 1. For any $a_1, \ldots, a_n \in \mathbb{Z}$ and any $x \in \mathbb{Z},$

$$\mathbb{P}\left(\sum_{i=1}^{n} \xi_i a_i = x\right) \leq \mathbb{P}\left(\sum_{i=1}^{n} \xi_i a_i = 0\right).$$

Proof. This follows immediately from the fact that the Fourier transform $\phi$ of the distribution function is positive since then the right-hand side is $\int \phi(t) \, dt$ while the right-hand side is

$$\int \phi(t)e^{-ixt} \, dt \leq \int |\phi(t)| \, dt = \int \phi \, dt.$$ 

To see that $\phi$ is positive, note that it is a product of Fourier transforms of the individual summands, and each one is simply $\frac{1}{2}(1 + \cos(a_i t)).$ \hfill $\square$

Lemma 2. Let $E \subset \{0, \pm 1\}^k$ which is 2-isolated, i.e. for any $v \neq w \in E$ we have that $v$ and $w$ differ in at least two coordinates. Then

$$\mathbb{P}(\xi \in E) \leq \frac{1}{k}$$

where $\xi = (\xi_i)_{i=1}^{k}$ is our usual random vector.
Proof. For every $v \in E$ let $\Omega_v \subset \{0, \pm 1\}^k$ be the set of all $w$ which differ from $v$ by at most one coordinate. Then
\[
P(\xi \in \Omega_v) \geq k P(\xi = v)
\]
and since the $\Omega_v$ for different $v$ are disjoint,
\[
P(\xi \in E) = \sum_{v \in E} P(\xi = v) \leq \frac{1}{k} \sum_{v \in E} P(\xi \in \Omega_v) = \frac{1}{k} P(\xi \in \bigcup \Omega_v) \leq \frac{1}{k}.
\]
\[\square\]

Lemma 3. The sum of $\frac{1}{p}$ over all primes $p$ between 1 and $n$ is less than $C \log \log n$.

Proof. This is a simple corollary from the prime number theorem, which states that there are $(1 + o(1)) n / \log n$ primes up to $n$. Hence for all $k$
\[
\sum_{k} \frac{1}{p} \leq \frac{1}{k} \left( \frac{2k}{\log 2k} - \frac{k}{\log k} + o\left( \frac{k}{\log k} \right) \right) \leq \frac{C}{\log k}.
\]
Summing over $k = 2^l$ for $l$ from 1 to $\log n$ gives the lemma. (in fact, a similar argument shows that this sum is $(1 + o(1)) \log \log n$, but we will have no use for this extra precision). \[\square\]

Proof of the Theorem. The starting point is the result that
\[
P(\det M = 0) \leq 2^{-\delta n}
\]
for some constant $\delta > 0$ (this result is essentially due to of Kahn, Komlós and Szemerédi [3], though formally they only proved the case that the coefficients are $\pm 1$. For a proof for our $\xi_i$ see [2], which bases on [5]. We remark that [2] calculates the correct value of $\delta$, but we have no use for this fact). Expanding the determinant by the first row we get
\[
P\left( \sum_{i=1}^{n} \xi_i d_i = 0 \right) \leq 2^{-\delta n}
\]
where $d_i$ is the determinant of the $(i, 1)^{st}$ minor, i.e. the matrix $M$ with its $i^{th}$ column and first row removed (we suppressed the terms $(-1)^{i+1}$ in the expansion, which we may because the $\xi_i$ are symmetric to taking minus). By lemma 1, for any fixed numbers $d_i$ and for any $x$, $P(\sum \xi_i d_i = x) \leq P(\sum \xi_i d_i = 0)$. Integrating over the $d_i$ gives
\[
P(\det M = x) \leq 2^{-\delta n} \quad \forall x.
\]
(1)
Unfortunately, one cannot simply sum (1) over all squares $x$ in the possible range of values of $\det M$, there are too many of those. So we have to take a more roundabout way.

Let $k$ be some parameter to be fixed later. From (1) we get

$$\mathbb{P}\left(\sum_{i=1}^{n-k} \xi_id_i = 0\right) \leq 2^{k-\delta n}$$

since if the partial sum is 0, then there is a probability of $2^k$ that all remaining $\xi_i$ are zero ($d_i$ are still the $(i,1)$ minors of $M$). Let $\mathcal{A}$ be the event that

$$\mathbb{P}\left(\sum_{i=1}^{n-k} \xi_id_i = 0\right) > 2^{-n\delta/2}$$

where here the $\xi_i$ are independent of $M$, so that $\mathcal{A}$ depends only on rows $2, \ldots, n$ of $M$. By Markov’s inequality, $\mathbb{P}(\mathcal{A}) \leq 2^{k-n\delta/2}$.

Let now $\eta \in \{0, \pm 1\}^n$ and let $\eta'$ be identical to $\eta$ except at one entry, say the $j$th. Assume that

$$\sum_{i=1}^{n} \eta_id_i = A^2 \text{ and } \sum_{i=1}^{n} \eta'_id_i = B^2$$

for integer $A$ and $B$. Then $A^2 - B^2 = (A - B)(A + B)$ must be one of $\{\pm d_j, \pm 2d_j\}$. Therefore every divisor of $2d_j$ corresponds to at most two solutions for $A$ and $B$ (up to the signs of $A$ and $B$). Let therefore $\mathcal{B}$ be the event that for all $j \in \{n - k + 1, \ldots, n\}$ the number of divisors of $2d_j$ is at most $e\sqrt{n}$. By lemma 4 below and Markov’s inequality,

$$\mathbb{P}(\mathcal{B}) \leq Ck^2 (\log n)^2 / \sqrt{n}.$$ 

Like $\mathcal{A}$, $\mathcal{B}$ is an event that depends only on rows $2, \ldots, n$ of $M$. Repeat this argument with $\eta$ and $\eta'$ which defer in two entries, using the “further” clause of lemma 4. Let $\mathcal{C}$ be the corresponding bad event, i.e. the event that for some $j \neq j'$ and some $\tau_1, \tau_2 \in \{\pm 1, \pm 2\}$ we have that $\tau_1d_j + \tau_2d_{j'}$ has more than $e\sqrt{n}$ divisors. Then the conclusion of lemma 4 is that $\mathbb{P}(\mathcal{C}) \leq Ck^2 (\log n)^2 / \sqrt{n}$.

Denote the entries of $M$ by $m_{i,j}$. Let $\mathcal{G}$ be the event that an $\eta$ and an $\eta'$ exist such that

(i) $\eta_i = \eta'_i = m_{i,1}$ for all $1 \leq i \leq n - k$.
(ii) $\eta$ defers from $\eta'$ by either one or two entries
(iii) For some integer $A$ and $B$, (2) holds.
Then $\mathcal{G}$ is an event which depends only on $m_{1,1}, \ldots, m_{1,n-k}$ and $m_{i,j}$ for $i \geq 2$. We now claim that

$$P(\mathcal{G}) \leq 2k^23^k2^{-\delta n/2}e^{\sqrt{n}} + P(\mathcal{A} \cup \mathcal{B} \cup \mathcal{C}). \quad (3)$$

This is just a summary of the previous discussion, but let us do it in details: if $\mathcal{B}$ did not occur then for all $j \in \{n-k+1, \ldots, n\}$, $2d_j$ has no more than $e^{\sqrt{n}}$ divisors. In this case there are only $2e^{\sqrt{n}}$ candidates for $(A, B)$ that satisfy (2) for $\eta$ and $\eta'$ different at any fixed $j$, and summing over the possibilities for $j$ gives a total of $2ke^{\sqrt{n}}$. For each such candidate $(A, B)$

$$P\left(\exists \eta \text{ s.t. } \sum_{i=1}^{n} \eta_i d_i = A^2\right) \leq 3^k \max_{x \in \mathbb{Z}} P\left(\sum_{i=1}^{n-k} m_{i,1} d_i = x\right)$$

where in the left-hand side $\eta$ is assumed to satisfy assumption (i); and where the factor $3^k$ is simply the number of possibilities for $\{m_{i,1}\}_{i=n-k+1}$. Applying Lemma 1 we get that the right-hand side is smaller or equal to $3^kP(\sum_{i=1}^{n-k} m_{i,1} d_i = 0)$, and if $\mathcal{A}$ did not occur then this last probability is smaller than $2^{-\delta n/2}$. We get that for any fixed $A$,

$$P\left(\exists \eta \text{ s.t. } \sum_{i=1}^{n} \eta_i d_i = A^2\right) \leq 3^k2^{-\delta n/2}.$$ 

Summing over the $2ke^{\sqrt{n}}$ candidates finishes the case where $\eta$ differs from $\eta'$ in just one entry. The case of two entries is covered similarly by the event $\mathcal{C}$ (with $k$ replaced by $\binom{k}{2}$ because we sum over two differing coordinates). This shows (3).

The last remaining point is that if $\mathcal{G}$ did not occur, then the probability that $\det M$ is a square is no more than $\frac{1}{k}$, because the set of values of $m_{1,n-k+1}, \ldots, m_{1,n}$ that gives a square is a subset of $\{0, \pm 1\}^k$ which is 2-isolated, and we can apply lemma 2. We conclude:

$$P(\det M \text{ is a square})$$

$$\leq \frac{1}{k} + 2k^23^k2^{-\delta n/2}e^{\sqrt{n}} + 2k^{-\delta n/2} + Ck\frac{(\log n)^2}{\sqrt{n}} + Ck^2\frac{(\log n)^2}{\sqrt{n}}$$

(terms 2-5 being the bound (3) on $P(\mathcal{G})$ with $P(\mathcal{A})$, $P(\mathcal{B})$ and $P(\mathcal{C})$ replaced by their bounds, in this order). Finally we choose $k$, and taking $k = \lfloor n^{1/6} \rfloor$ gives that the right-hand side is $n^{-1/6+o(1)}$, proving the theorem. \qed
Lemma 4. Let $M$ be as in the theorem. Let $X$ be the number of divisors of $\det M$. Then
\[ \mathbb{E} \log X \leq C (\log n)^2. \] (4)
Further, if $d_1$ and $d_2$ are the determinants of two different $n-1 \times n-1$ first row minors of $M$, if $\tau_1$ and $\tau_2$ are in $\{\pm 1, \pm 2\}$ and if $Y$ is the number of divisors of $d_1\tau_1 + d_2\tau_2$, then again we have $\mathbb{E} \log Y \leq C (\log n)^2$.

Proof. The first clause is a simple corollary of Maples [4]. Indeed, theorem 1.1 of [4] gives, for every prime $p$,
\[ \mathbb{P}(p| \det M) = 1 - \prod_{k=1}^{\infty} (1 - p^{-k}) + O(e^{-\epsilon n}) \] (5)
where $\epsilon > 0$ is some absolute constant (the implied constant in $O$ is also absolute, i.e. does not depend on $p$ or $n$). Denote by $k(p)$ the number of times $p$ divides $\det M$ (plus 1) i.e. the $k$ such that $p^{k-1} | \det M$ but $p^k \not| \det M$. Then
\[ \log X = \sum_p \log k(p) \]
Since $| \det M | \leq n!$ we have $k(p) \leq C \log(n!)$, so
\[ \log X \leq C (\log \log n!) |\{p \text{ prime} : p | \det M\}|. \]
For $p < e^{\epsilon n/2}$ we use (5) and get
\[ \mathbb{E}|\{p \text{ prime} : p \leq e^{\epsilon n/2}, p | \det M\}| \]
\[ \leq \sum_{p \leq e^{\epsilon n/2}} \frac{C}{p} + O(e^{-\epsilon n}) \leq C \log \log e^{\epsilon n/2} + O(e^{-\epsilon n/2}) \]
where the last inequality follows from lemma 3. For $p \geq e^{\epsilon n/2}$ we note that the number of such $p$ that divide $\det M$ is no more than
\[ \frac{\log n!}{\log e^{\epsilon n/2}} \leq C \log n. \]
($C$ here depends on $\epsilon$, but $\epsilon$ is an absolute constant anyway). All in all we get
\[ \mathbb{E}|\{p \text{ prime} : p | \det M\}| \leq C \log n \]
which proves (4).

For the second clause (sum of determinants of two minors) we need to examine the proof of [4] a little. Assume for concreteness that $d_i$ is the determinant of the $(i,1)$ minor of $M$ for $i = 1, 2$. Following [4], denote by
$W_k$ the span (over the finite field $\mathbb{F}_p$) of columns $k + 1, \ldots, n$ in the matrix $M$ without its first row (so that $W_k \subset \mathbb{F}_{p^{n-1}}$). By [4, proposition 2.1]

$$\mathbb{P}(\text{codim } W_3 \geq 2) \leq \frac{C}{p^2} + Ce^{-cn}. \quad (6)$$

(in this case, of course, $p$ would divide both $d_1$ and $d_2$ and hence also $d_1 \tau_1 + d_2 \tau_2$).

In the case where $\text{codim } W_3 = 1$, we argue as follows. Let $w_j$ for $j = 2, \ldots, n$ be the determinant of the $n - 2 \times n - 2$ minor of $M$ one gets by removing the first and $j^{\text{th}}$ rows; and the first two columns. Let $\eta > 0$ be some parameter. Let $A$ be the event that

$$\left| \mathbb{P}\left( \sum_{j=2}^{n} (-1)^j \xi_j w_j = x \right) - \frac{1}{p} \right| \leq e^{-\eta m} \quad \forall x \in \mathbb{F}_p. \quad (7)$$

Notice that $A$ depends only on the entries of $M$ different from the first row and the first two columns (the $\xi_j$ in (7) are assumed to be independent of $M$). By [4, §4], for an appropriate choice of $\eta$, independent of $p$ or $n$,

$$\mathbb{P}(A \mid \text{codim } W_3 = 1) > 1 - e^{-cn}. \quad (8)$$

We are now finished, since $d_i = \sum_{j=2}^{n} (-1)^j m_{i,j} w_j$ for both $i = 1, 2$ and the $m_{i,j}$ are independent, so under the event $A$, for any value of $d_1$, the probability that $d_2$ takes the value $-d_1 \tau_1 / \tau_2$ (in $\mathbb{F}_p$, let us assume for a moment $p > 2$) is $\frac{1}{p} + O(e^{-cn})$. With (8) we get

$$\mathbb{P}(p \mid \tau_1 d_1 + \tau_2 d_2 \mid \text{codim } W_3 = 1) = \frac{1}{p} + O(e^{-cn}).$$

Throwing (6) into the mix gives

$$\mathbb{P}(p \mid \tau_1 d_1 + \tau_2 d_2) \leq \frac{C}{p} + Ce^{-cn}.$$ 

This last inequality holds also for $p = 2$, of course, as it become trivial for $C$ sufficiently large. The proof then continues as in the single matrix case. \(\square\)

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