On some Hölder type trace inequalities for operator weighted geometric mean

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Abstract. We obtain some Hölder type trace inequalities for operator weighted geometric mean. Some vector inequalities are also given.

1. Introduction

If \( \{ e_i \}_{i \in I} \) is an orthonormal basis of a Hilbert space \( H \), then we say that \( A \in \mathcal{B}(H) \) is a trace class provided

\[
\| A \|_1 := \sum_{i \in I} \langle |A| e_i, e_i \rangle < \infty.
\]

The definition of \( \| A \|_1 \) does not depend on the choice of the orthonormal basis \( \{ e_i \}_{i \in I} \). We denote by \( \mathcal{B}_1(H) \) the set of trace class operators in \( \mathcal{B}(H) \).

The following properties are also well known:

(i) for any \( A \in \mathcal{B}_1(H) \) we have

\[
\| A \|_1 = \| A^* \|_1 ;
\]

(ii) \( \mathcal{B}_1(H) \) is an operator ideal in \( \mathcal{B}(H) \), i.e.,

\[
\mathcal{B}(H) \mathcal{B}_1(H) \mathcal{B}(H) \subseteq \mathcal{B}_1(H) ;
\]

(iii) \( (\mathcal{B}_1(H), \| \cdot \|_1) \) is a Banach space.

We define the trace of a trace class operator \( A \in \mathcal{B}_1(H) \) to be

\[
\text{tr} (A) := \sum_{i \in I} \langle Ae_i, e_i \rangle .
\] (1.1)

Note that this coincides with the usual definition of the trace if \( H \) is finite-dimensional. We observe that the series (1.1) converges absolutely and it is independent from the choice of basis.

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We collect some properties of the trace:

(i) if \( A \in \mathcal{B}_1(H) \), then \( A^* \in \mathcal{B}_1(H) \) and
\[
\text{tr} (A^*) = \text{tr} (A);
\]
(ii) if \( A \in \mathcal{B}_1(H) \) and \( T \in \mathcal{B}(H) \), then \( AT, TA \in \mathcal{B}_1(H) \),
\[
\text{tr} (AT) = \text{tr} (TA), \quad \text{and} \quad |\text{tr} (AT)| \leq \|A\|_1 \|T\|;
\]
(iii) \( \text{tr} (\cdot) \) is a bounded linear functional on \( \mathcal{B}_1(H) \) with \( \|\text{tr}\| = 1 \);
(iv) \( \mathcal{B}_{fin}(H) \), the space of operators of finite rank, is a dense subspace of \( \mathcal{B}_1(H) \).

Now, for the finite dimensional case, it is well known that the trace functional is submultiplicative, that is, for positive semidefinite matrices \( A \) and \( B \) in \( M_n(\mathbb{C}) \),
\[
0 \leq \text{tr}(AB) \leq \text{tr}(A) \text{tr}(B).
\]
Therefore,
\[
0 \leq \text{tr}(A^k) \leq [\text{tr}(A)]^k,
\]
where \( k \) is any positive integer.

In 2000, Yang [22] proved a matrix trace inequality
\[
\text{tr} [(AB)^k] \leq (\text{tr} A)^k (\text{tr} B)^k,
\] (1.2)
where \( A \) and \( B \) are positive semidefinite matrices over \( \mathbb{C} \) of the same order \( n \), and \( k \) is any positive integer.

If \( (H, \langle \cdot, \cdot \rangle) \) is a separable infinite-dimensional Hilbert space, then the inequality (1.2) is also valid for any positive operators \( A, B \in \mathcal{B}_1(H) \). This result was obtained by L. Liu in 2007, see [12].

In 2001, Yang et al. [23] improved (1.2) as follows:
\[
\text{tr} [(AB)^m] \leq [\text{tr} (A^{2m}) \text{tr} (B^{2m})]^{1/2},
\]
where \( A \) and \( B \) are positive semidefinite matrices over \( \mathbb{C} \) of the same order and \( m \) is any positive integer.

In [18] the authors have proved many trace inequalities for sums and products of matrices. For instance, if \( A \) and \( B \) are positive semidefinite matrices in \( M_n(\mathbb{C}) \), then
\[
\text{tr} [(AB)^k] \leq \min \left\{ \|A\|^k \text{tr} (B^k), \|B\|^k \text{tr} (A^k) \right\}
\]
for any positive integer \( k \). Also, if \( A, B \in M_n(\mathbb{C}) \), then for \( r \geq 1 \) and \( p, q > 1 \) with \( 1/p + 1/q = 1 \) we have the following Young type inequality:
\[
\text{tr} (|AB^*|^r) \leq \text{tr} \left[ \left( \frac{|A|^p}{p} + \frac{|B|^q}{q} \right)^r \right].
\] (1.3)
Ando [1] proved a strong form of Young’s inequality. It was shown that if $A$ and $B$ are in $M_n(\mathbb{C})$, then there is a unitary matrix $U$ such that

$$|AB^*| \leq U \left( \frac{1}{p} |A|^p + \frac{1}{q} |B|^q \right)^{1/p} \left( \frac{1}{q} |A|^q + \frac{1}{p} |B|^p \right)^{1/q} U^*,$$

where $p, q > 1$ with $1/p + 1/q = 1$. This gives immediately the trace inequality

$$\text{tr} (|AB^*|) \leq \frac{1}{p} \text{tr} (|A|^p) + \frac{1}{q} \text{tr} (|B|^q).$$

This inequality can also be obtained from (1.3) by taking $r = 1$.

The following Hölder’s type inequality has been proved by Ruskai [16]:

$$|\text{tr} (AB)| \leq \text{tr} (|AB|) \leq \left[ \text{tr} (|A|^p) \right]^{1/p} \left[ \text{tr} (|B|^q) \right]^{1/q},$$

where $p, q > 1$ with $1/p + 1/q = 1$, and $A, B \in B(H)$ with $|A|^p, |B|^q \in B_1(H)$.

In particular, for $p = 2$ we get the Schwarz inequality

$$|\text{tr} (AB)| \leq \text{tr} (|AB|) \leq \left[ \text{tr} \left( |A|^2 \right) \right]^{1/2} \left[ \text{tr} \left( |B|^2 \right) \right]^{1/2}$$

with $|A|^2, |B|^2 \in B_1(H)$.

For the theory of trace functionals and their applications the reader is referred to [20].

For some classical trace inequalities see [4], [6], [14] and [24], which are continuations of the work of Bellman [2]. For related works the reader can refer to [1], [3], [4], [9], [11], [12], [13], [17] and [21].

2. Some Hölder type trace inequalities

Assume that $A, B$ are positive invertible operators on a complex Hilbert space $(H, \langle \cdot, \cdot \rangle)$. We use the notation

$$A_{\nu}^B := A^{1/2} \left( A^{-1/2} B A^{-1/2} \right)^\nu A^{1/2}$$

for the weighted geometric mean. When $\nu = 1/2$, we write $A_{\frac{1}{2}}^B$ for brevity.

We have the following Hölder type trace inequality.

**Theorem 1.** If $A, B$ are positive invertible operators, $p, q > 1$ with $1/p + 1/q = 1$, and $A^p, B^q \in B_1(H)$, then $B^{q_{\frac{1}{p}}}A^p \in B_1(H)$ and

$$\text{tr} (B^{q_{\frac{1}{p}}}A^p) \leq \left[ \text{tr} (A^p) \right]^{1/p} \left[ \text{tr} (B^q) \right]^{1/q}.$$

In particular, if $A^2, B^2 \in B_1(H)$, then $B^{2\frac{1}{p}}A^2 \in B_1(H)$ and

$$\left[ \text{tr} (B^{2\frac{1}{p}}A^2) \right]^2 \leq \text{tr} (A^2) \text{tr} (B^2).$$
Proof. In [8], the authors obtained the following Hölder’s type inequality for the weighted geometric mean:

$$\langle B^q_{1/p} A^p x, x \rangle \leq \langle A^p x, x \rangle^{1/p} \langle B^q x, x \rangle^{1/q} \tag{2.1}$$

for any \(x \in H\).

Let \(\{e_i\}_{i \in I}\) be an orthonormal basis of \(H\). Then by (2.2) and Hölder’s inequality we have

\[
\text{tr} \left( B^q_{1/p} A^p \right) = \sum_{i \in I} \langle B^q_{1/p} A^p e_i, e_i \rangle \\
\leq \sum_{i \in I} \langle A^p e_i, e_i \rangle^{1/p} \langle B^q e_i, e_i \rangle^{1/q} \\
\leq \left( \sum_{i \in I} \left[ \langle A^p e_i, e_i \rangle^{1/p} \right]^p \right)^{1/p} \left( \sum_{i \in I} \left[ \langle B^q e_i, e_i \rangle^{1/q} \right]^q \right)^{1/q} \\
= \left( \sum_{i \in I} \langle A^p e_i, e_i \rangle \right)^{1/p} \left( \sum_{i \in I} \langle B^q e_i, e_i \rangle \right)^{1/q} \\
= [\text{tr} (A^p)]^{1/p} [\text{tr}(B^q)]^{1/q},
\]

which proves the desired inequality (2.1). \(\square\)

Corollary 1. If \(A_k, B_k\) are positive invertible operators, \(p, q > 1\) with \(1/p + 1/q = 1\), and \(A_k^p, B_k^q \in B_1(H)\) for \(k \in \{1, \ldots, n\}\), then \(B^q_{1/p} A^p_k \in B_1(H)\) for \(k \in \{1, \ldots, n\}\), and for any \(p_k \geq 0, k \in \{1, \ldots, n\}\), we have

\[
\text{tr} \left( \sum_{k=1}^n p_k B^q_{1/p} A^p_k \right) \leq \left( \text{tr} \left( \sum_{k=1}^n p_k A^p_k \right) \right)^{1/p} \left( \text{tr} \left( \sum_{k=1}^n p_k B^q_k \right) \right)^{1/q}. \tag{2.3}
\]

In particular, if \(A_k^2, B_k^2 \in B_1(H)\) for \(k \in \{1, \ldots, n\}\), then \(B^q_{1/p} A^p_k \in B_1(H)\) for \(k \in \{1, \ldots, n\}\), and for any \(p_k \geq 0, k \in \{1, \ldots, n\}\), we have

\[
\left[ \text{tr} \left( \sum_{k=1}^n p_k B^q_{1/p} A^p_k \right) \right]^2 \leq \text{tr} \left( \sum_{k=1}^n p_k A^2_k \right) \text{tr} \left( \sum_{k=1}^n p_k B^2_k \right).
\]

Proof. Using Hölder’s weighted discrete inequality, we have

\[
\text{tr} \left( \sum_{k=1}^n p_k B^q_{1/p} A^p_k \right) = \sum_{k=1}^n p_k \text{tr} \left( B^q_{1/p} A^p_k \right) \leq \sum_{k=1}^n p_k \left[ \text{tr} (A^p_k) \right]^{1/p} \left[ \text{tr}(B^q_k) \right]^{1/q} \\
\leq \left( \sum_{k=1}^n p_k \left[ \text{tr} (A^p_k) \right]^{1/p} \right)^{1/p} \left( \sum_{k=1}^n p_k \left[ \text{tr}(B^q_k) \right]^{1/q} \right)^{1/q}
\]
and the inequality (2.3) is proved. □

**Theorem 2.** If $A, B$ are positive invertible operators, $p, q > 1$ with $1/p + 1/q = 1$, and $C \in B_1(H)$, $C \geq 0$, then $CA^p, CB^q, C(B^{q_1/p_1}A^p) \in B_1(H)$ and

$$\text{tr} \left(C \left(B^{q_1/p_1}A^p\right)\right) \leq \left[\text{tr} \left( CA^p \right) \right]^{1/p} \left[\text{tr} \left( CB^q \right) \right]^{1/q}. \quad (2.4)$$

In particular, if $C \in B_1(H)$, then $CA^2, CB^2, C(B^{q_1}A^2) \in B_1(H)$ and

$$[\text{tr} \left(C \left(B^{q_1}A^2\right)\right)]^2 \leq \text{tr} \left( CA^2 \right) \text{tr} \left( CB^2 \right).$$

**Proof.** From the inequality (2.2) we have

$$\left\langle B^{q_{11}/p_{11}}A^pC^{1/2}x, C^{1/2}x \right\rangle \leq \left\langle A^pC^{1/2}x, C^{1/2}x \right\rangle^{1/p} \left\langle B^qC^{1/2}x, C^{1/2}x \right\rangle^{1/q}$$

for any $x \in H$, which is equivalent to

$$\left\langle C^{1/2}B^{q_{11}/p_{11}}A^pC^{1/2}x, x \right\rangle \leq \left\langle C^{1/2}A^pC^{1/2}x, x \right\rangle^{1/p} \left\langle C^{1/2}B^qC^{1/2}x, x \right\rangle^{1/q} \quad (2.5)$$

for any $x \in H$.

Let $\{e_i\}_{i \in I}$ be an orthonormal basis of $H$. Then by (2.5) and Hölder’s inequality we have

$$\text{tr} \left(C \left(B^{q_1}A^p\right)\right)
= \text{tr} \left(C^{1/2} \left(B^{q_{11}/p_{11}}A^p\right) C^{1/2}\right) = \sum_{i \in I} \left\langle C^{1/2} \left(B^{q_{11}/p_{11}}A^p\right) C^{1/2}e_i, e_i \right\rangle
\leq \sum_{i \in I} \left\langle C^{1/2}A^pC^{1/2}e_i, e_i \right\rangle^{1/p} \left\langle C^{1/2}B^qC^{1/2}e_i, e_i \right\rangle^{1/q}
\leq \left( \sum_{i \in I} \left[ \left\langle C^{1/2}A^pC^{1/2}e_i, e_i \right\rangle^{1/p} \right] \right)^{1/p} \left( \sum_{i \in I} \left[ \left\langle C^{1/2}B^qC^{1/2}e_i, e_i \right\rangle^{1/q} \right] \right)^{1/q}
= \left( \sum_{i \in I} \left\langle C^{1/2}A^pC^{1/2}e_i, e_i \right\rangle \right)^{1/p} \left( \sum_{i \in I} \left\langle C^{1/2}B^qC^{1/2}e_i, e_i \right\rangle \right)^{1/q}
= \left[ \text{tr} \left(C^{1/2}A^pC^{1/2}\right) \right]^{1/p} \left[ \text{tr} \left(C^{1/2}B^qC^{1/2}\right) \right]^{1/q} \equiv \left[\text{tr} \left( CA^p \right) \right]^{1/p} \left[\text{tr} \left( CB^q \right) \right]^{1/q},$$

which proves the desired result (2.4). □
Corollary 2. If $A_k, B_k$ are positive invertible operators, $p, q > 1$ with $1/p + 1/q = 1$, and $C_k \in B_1(H)$, $C_k \geq 0$ for $k \in \{1, \ldots, n\}$, then $C_k A_k^p, C_k B_k^q, C_k (B_k^{*^2}_{1/p} A_k^p) \in B_1(H)$ for $k \in \{1, \ldots, n\}$ and we have

$$\text{tr} \left( \sum_{k=1}^{n} C_k (B_k^{*^2}_{1/p} A_k^p) \right) \leq \left( \text{tr} \left( \sum_{k=1}^{n} C_k A_k^p \right) \right)^{1/p} \left( \text{tr} \left( \sum_{k=1}^{n} C_k B_k^q \right) \right)^{1/q}.$$  

In particular, $C_k A_k^2, C_k B_k^2, C_k (B_k^{*^2}_2 A_k^2) \in B_1(H)$ for $k \in \{1, \ldots, n\}$ and

$$\left[ \text{tr} \left( \sum_{k=1}^{n} C_k (B_k^{*^2}_2 A_k^2) \right) \right]^2 \leq \text{tr} \left( \sum_{k=1}^{n} C_k A_k^2 \right) \text{tr} \left( \sum_{k=1}^{n} C_k B_k^2 \right).$$

The proof follows by (2.4) making use of a similar argument to the one in the proof of Corollary 1.

3. Some reverse vector inequalities

We have the following reverse of Hölder’s vector inequality for operators.

Theorem 3. Let $A$ and $B$ be two positive invertible operators, $p, q > 1$ with $1/p + 1/q = 1$ and let $m, M > 0$ be such that

$$m^p B^q \leq A^p \leq M^p B^q.$$

Then

$$(A^p x, x)^{1/p} (B^q x, x)^{1/q} \leq \exp \left[ \frac{1}{2pq} \left( \left( \frac{M}{m} \right)^p - 1 \right)^2 \right] \langle B^q x, A^p x \rangle$$

for any $x \in H$.

Proof. In [7] we proved the following double inequality that provides a refinement and a reverse of the arithmetic mean - geometric mean inequality:

$$\exp \left[ \frac{1}{2} \nu (1 - \nu) \left( 1 - \frac{\min \{a, b\}}{\max \{a, b\}} \right)^2 \right] \leq \frac{(1 - \nu) a + \nu b}{a^{1 - \nu} b^{\nu}} \leq \exp \left[ \frac{1}{2} \nu (1 - \nu) \left( \frac{\max \{a, b\}}{\min \{a, b\}} - 1 \right)^2 \right]$$

for any $a, b > 0$ and $\nu \in [0, 1]$.

If $a, b \in [t, T] \subset (0, \infty)$ and since

$$0 < \frac{\max \{a, b\}}{\min \{a, b\}} - 1 \leq \frac{T}{t} - 1,$$

we have

$$\left( \frac{\max \{a, b\}}{\min \{a, b\}} - 1 \right)^2 \leq \left( \frac{T}{t} - 1 \right)^2.$$
Therefore, by (3.3) we get
\[
(1 - \nu) a + \nu b \leq a^{1-\nu} b^\nu \exp \left[ \frac{1}{2} \nu (1 - \nu) \left( \frac{T}{t} - 1 \right)^2 \right], \tag{3.4}
\]
for any \( a, b \in [t, T] \) and \( \nu \in (0, 1) \).

Now, if \( C \) is an operator with \( tI \leq C \leq TI \), then for \( p > 1 \) we have \( t^p I \leq C^p \leq T^p I \). Using the functional calculus, we get from (3.4) for \( \nu = \frac{1}{p} \) that
\[
\left( 1 - \frac{1}{p} \right) d + \frac{1}{p} C^p \leq \exp \left[ \frac{1}{2pq} \left( \left( \frac{T}{t} \right)^p - 1 \right)^2 \right] d^{1-\frac{1}{p}} C,
\]
namely, the vector inequality
\[
\left( 1 - \frac{1}{p} \right) d + \frac{1}{p} \langle C^p y, y \rangle \leq \exp \left[ \frac{1}{2pq} \left( \left( \frac{T}{t} \right)^p - 1 \right)^2 \right] \langle C^p y, y \rangle ^{1-\frac{1}{p}} \langle Cy, y \rangle, \tag{3.5}
\]
for any \( y \in H, \|y\| = 1 \) and \( d \in [t^p, T^p] \).

Since \( d = \langle C^p y, y \rangle \in [t^p, T^p] \) for any \( y \in H, \|y\| = 1 \), and hence by (3.5) we have
\[
\left( 1 - \frac{1}{p} \right) \langle C^p y, y \rangle + \frac{1}{p} \langle C^p y, y \rangle \leq \exp \left[ \frac{1}{2pq} \left( \left( \frac{T}{t} \right)^p - 1 \right)^2 \right] \langle C^p y, y \rangle ^{1-\frac{1}{p}} \langle Cy, y \rangle,
\]
which is equivalent to
\[
\langle C^p y, y \rangle \leq \exp \left[ \frac{1}{2pq} \left( \left( \frac{T}{t} \right)^p - 1 \right)^2 \right] \langle C^p y, y \rangle ^{1-\frac{1}{p}} \langle Cy, y \rangle,
\]
and by division with \( \langle C^p y, y \rangle ^{1-\frac{1}{p}} > 0, y \in H, \|y\| = 1 \), to
\[
\langle C^p y, y \rangle ^{1/p} \leq \exp \left[ \frac{1}{2pq} \left( \left( \frac{T}{t} \right)^p - 1 \right)^2 \right] \langle Cy, y \rangle. \tag{3.6}
\]

If \( z \in H \) with \( z \neq 0 \), then by taking \( y = \frac{z}{\|z\|} \) in (3.6) we get
\[
\langle C^p z, z \rangle ^{1/p} \leq \exp \left[ \frac{1}{2pq} \left( \left( \frac{T}{t} \right)^p - 1 \right)^2 \right] \langle Cz, z \rangle, \tag{3.7}
\]
for any \( z \in H \).

Now, from (3.1) by multiplying both sides with \( B^{-\frac{2}{p}} \), we have \( m^p I \leq B^{-\frac{2}{p}} A^p B^{-\frac{2}{p}} \leq M^p I \), and by taking the power \( 1/p \) we get \( mI \leq \left( B^{-\frac{2}{p}} A^p B^{-\frac{2}{p}} \right)^{\frac{1}{p}} \leq MI \).
Writing the inequality (3.7) for $C = \left( B^{-\frac{q}{2}} A^{p} B^{-\frac{q}{2}} \right)^{\frac{1}{p}}$, $t = m$, $T = M$ and $z = B^{\frac{q}{2}} x$, with $x \in H$, we have

$$\left\langle B^{-\frac{q}{2}} A^{p} B^{-\frac{q}{2}} B^{\frac{q}{2}} x, B^{\frac{q}{2}} x \right\rangle^{1/p} \left\langle B^{\frac{q}{2}} x, B^{\frac{q}{2}} x \right\rangle^{1/q} \leq \exp \left[ \frac{1}{2pq} \left( \left( \frac{M}{m} \right)^{p} - 1 \right) \right] \left\langle B^{-\frac{q}{2}} A^{p} B^{-\frac{q}{2}} \right\rangle^{\frac{1}{p}} \left\langle B^{\frac{q}{2}} x, B^{\frac{q}{2}} x \right\rangle,$$

namely

$$\langle A^{p} x, x \rangle^{1/p} \langle B^{q} x, x \rangle^{1/q} \leq \exp \left[ \frac{1}{2pq} \left( \left( \frac{M}{m} \right)^{p} - 1 \right) \right] \left\langle B^{\frac{q}{2}} \left( B^{-\frac{q}{2}} A^{p} B^{-\frac{q}{2}} \right)^{\frac{1}{p}} B^{\frac{q}{2}} x, x \right\rangle,$$

for any $x \in H$. The inequality (3.2) is proved. □

**Remark 1.** We observe, for two positive invertible operators $A$ and $B$, that the condition (3.1) is equivalent to condition

$$mI \leq \left( B^{-\frac{q}{2}} A^{p} B^{-\frac{q}{2}} \right)^{\frac{1}{p}} \leq MI.$$

If we assume that $r B^{q} \leq A^{p} \leq RB^{q}$, then by (3.2) we have the inequality

$$\langle A^{p} x, x \rangle^{1/p} \langle B^{q} x, x \rangle^{1/q} \leq \exp \left[ \frac{1}{2pq} \left( \frac{M}{m} \right)^{p} - 1 \right] \left\langle B^{\frac{q}{2}} \left( B^{-\frac{q}{2}} A^{p} B^{-\frac{q}{2}} \right)^{\frac{1}{p}} B^{\frac{q}{2}} x, x \right\rangle,$$

for any $x \in H$.

The following particular case is related to Schwarz’s trace inequality.

**Corollary 3.** Let $A$ and $B$ be two positive invertible operators and let $m$, $M > 0$ be such that

$$mI \leq \left( B^{-\frac{q}{2}} A^{p} B^{-\frac{q}{2}} \right)^{\frac{1}{p}} \leq MI.$$

Then we have

$$\langle A^{2} x, x \rangle^{1/2} \langle B^{2} x, x \rangle^{1/2} \leq \exp \left[ \frac{1}{8} \left( \left( \frac{M}{m} \right)^{2} - 1 \right) \right] \left\langle A^{2} x B^{2} x, x \right\rangle,$$

for any $x \in H$.

Under more suitable conditions for the operators involved, we have the following result.

**Corollary 4.** Assume that $A$ and $B$ satisfy the conditions

$$m_{1} I \leq A \leq M_{1} I, \ m_{2} I \leq B \leq M_{2} I$$
for some \(0 < m_1 < M_1\) and \(0 < m_2 < M_2\). Then we have

\[
\langle A^p x, x \rangle^{1/p} \langle B^q x, x \rangle^{1/q} \leq \exp \left[ \frac{1}{2pq} \left( \left( \frac{M_1}{m_1} \right)^p - 1 \right) \left( \frac{M_2}{m_2} \right)^q - 1 \right] \langle B^q \sharp_1 A^p x, x \rangle,
\]

for any \(x \in H\).

In particular, we have

\[
\langle A^2 x, x \rangle^{1/2} \langle B^2 x, x \rangle^{1/2} \leq \exp \left[ \frac{1}{8} \left( \left( \frac{M_1 M_2}{m_1 m_2} \right)^2 - 1 \right) \right] \langle A^2 \sharp B^2 x, x \rangle,
\]

for any \(x \in H\).

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