A Note on Existence and Non-existence of Minimal Surfaces in Some Asymptotically Flat 3-manifolds

Pengzi Miao∗†

Abstract

Motivated by problems on apparent horizons in general relativity, we prove the following theorem on minimal surfaces: Let $g$ be a metric on the three-sphere $S^3$ satisfying $\text{Ric}(g) \geq 2g$. If the volume of $(S^3, g)$ is no less than one half of the volume of the standard unit sphere, then there are no closed minimal surfaces in the asymptotically flat manifold $(S^3 \setminus \{P\}, G^4 g)$. Here $G$ is the Green’s function of the conformal Laplacian of $(S^3, g)$ at an arbitrary point $P$. We also give an example of $(S^3, g)$ with $\text{Ric}(g) > 0$ where $(S^3 \setminus \{P\}, G^4 g)$ does have closed minimal surfaces.

1 Introduction

Let $(N^3, g, p)$ be an initial data set satisfying the dominant energy constraint condition in general relativity. It is a fascinating question to ask under what conditions an apparent horizon (of a back hole) exists in $(N^3, g, p)$. Here an apparent horizon is a 2-surface $\Sigma^2 \subset N^3$ satisfying

$$H_\Sigma = \text{Tr}_\Sigma p,$$

where $H_\Sigma$ is the mean curvature of $\Sigma$ in $N$ and $\text{Tr}_\Sigma p$ is the trace of the restriction of $p$ to $\Sigma$.

A fundamental result of Schoen and Yau states that matter condensation causes apparent horizons to be formed [11]. Their result is remarkable not only because it provides a general criteria to the existence question, but also

∗Current address: Department of Mathematics, University of California, Santa Barbara, CA 93106, USA. E-mail: pengzi@math.ucsb.edu

†Address after April 306: School of Mathematical Sciences, Monash University, Victoria 3800, Australia.
because it leads to a refined problem – besides matter fields, what is the pure effect of gravity on the formation of apparent horizons?

To analyze this refined problem, one considers an asymptotically flat initial data set \((N^3, g, p)\) in a vacuum spacetime. As the first step, one assumes \((N^3, g, p)\) is time-symmetric (i.e. \(p \equiv 0\)). In this context, an apparent horizon is simply a minimal surface, and the relevant topological assumption is that \(N^3\) is diffeomorphic to \(\mathbb{R}^3\). (If \(N^3\) has nontrivial topology, a closed minimal surface always exists by [3].)

There is a geometric construction of such an initial data set. Let \([g]\) be a conformal class of metrics on the three-sphere \(S^3\). Recall the Yamabe constant of \((S^3, [g])\) is defined by

\[
Y(S^3, [g]) = \inf_{v \in W^{1,2}(S^3)} \frac{\int_M [8|\nabla v|^2_g + R(g)v^2]dV_g}{\left(\int_M v^6 dV_g\right)^{1/3}},
\]

where \(R(g)\) is the scalar curvature of \(g\). If \(Y(S^3, [g]) > 0\), there exists a positive Green’s function \(G\) of the conformal Laplacian \(\Delta_g - R(g)\) at any fixed point \(P \in S^3\). Consider the new metric \(G^4 g\) on \(S^3 \setminus \{P\}\), it is easily checked that \((S^3 \setminus \{P\}, G^4 g)\) is asymptotically flat with zero scalar curvature. One basic fact about this construction is that the blowing-up manifold \((S^3 \setminus \{P\}, G^4 g)\), up to a constant scaling, depends only on the conformal class \([g]\). Precisely, if one replaces \(g\) by another metric \(\tilde{g} \in [g]\) and let \(\tilde{G}\) be the Green’s function associated to \(\tilde{g}\), then the metric \(\tilde{G}^4 \tilde{g}\) differs from \(G^4 g\) only by a constant multiple. Therefore, it is of interest to seek conditions on \([g]\) that determine whether \((S^3 \setminus \{P\}, G^4 g)\) has a horizon.

So far, no such a conformal invariant condition has been found. However, there are results where conditions in terms of a single metric are given. In [1], Beig and Ó Murchadha studied the behavior of a critical sequence, i.e. a sequence of metrics \(\{g_n\}\) on \(S^3\) converging to a metric \(g_0\) with zero scalar curvature. They showed the blowing-up manifold \((S^3 \setminus \{P\}, G^4 g_n)\) has a horizon for sufficiently large \(n\). Their idea was further explored by Yan [12]. Given a metric \(g\) on \(S^3\), assuming the diameter of \((S^3, g)\) \(\leq D\), the volume of \((S^3, g)\) \(\geq V\) and the Ricci curvature of \(g\) satisfies \(\text{Ric}(g) \geq \mu g\), Yan showed that, for any \(r > \frac{\pi}{2}\), there exists a small positive number \(\delta = \delta(\mu, V, D, r) \leq 1\) such that, if \(R(g) > 0\) and \(||R(g)||_{L^r(S^3, g)} < \delta\), then the blowing-up manifold \((S^3 \setminus \{P\}, G^4 g)\) has a horizon.

One question arising from Yan’s theorem is whether a positive Ricci curvature metric on \(S^3\) can produce a blowing-up manifold with a horizon, as it is unclear whether Yan’s theorem could be applied when \(\mu > 0\). Another motivation to this question is, as a positive Ricci curvature metric can be
deformed to the standard metric on $S^3$ through metrics of positive Ricci curvature, it is of potential interest to study how the horizon disappears in the corresponding deformation of the blowing-up manifold if it exists initially.

In this paper, we focus on conformal classes of metrics with a positive Ricci curvature metric. Our main result is the observation of a volume condition which guarantees non-existence of horizons in the blowing-up manifold. Throughout the paper, $S^3$ denotes $S^3$ with the standard metric of constant curvature $+1$.

**Theorem** Let $[g]$ be a conformal class of metrics on $S^3$ which has a metric of positive Ricci curvature. Consider

$$V_{max}(S^3, [g]) = \sup_{\bar{g} \in [g]} \{Vol(S^3, \bar{g}) \mid \text{Ric}(\bar{g}) \geq 2\bar{g}\},$$

where $Vol(\cdot)$ is the volume functional. If

$$V_{max}(S^3, [g]) \geq \frac{1}{2} Vol(S^3),$$

then the asymptotically flat manifold $(S^3 \setminus \{P\}, G^4g)$ has no horizon.

We also give an example of $(S^3, g)$ with $\text{Ric}(g) > 0$ where $(S^3 \setminus \{P\}, G^4g)$ does have horizons.

### 2 Positive Ricci curvature and maximum volume

We first explain the volume assumption in the Theorem. Let $M^n$ be a smooth, connected, closed manifold of dimension $n \geq 3$. Assume $[g]$ is a conformal class of metrics on $M^n$ which has a metric of positive Ricci curvature. One can define

$$V_{max}(M^n, [g]) = \sup_{\bar{g} \in [g]} \{Vol(M^n, \bar{g}) \mid \text{Ric}(\bar{g}) \geq (n-1)\bar{g}\}. \quad (3)$$

The following result relating $V_{max}(M^n, [g])$ and the Yamabe constant of $(M^n, [g])$ was observed in [5].

**Proposition 1** Let $[g]$ be a conformal class of metrics on $M^n$ which has a metric of positive Ricci curvature. Then the Yamabe constant of $(M^n, [g])$ satisfies

$$Y(M^n, [g]) \geq n(n-1)V_{max}(M^n, [g])^{\frac{2}{n}}. \quad (4)$$
Proof: By definition,

$$Y(M^n, [g]) = \inf_{v \in W^{1,2}(M)} \frac{\int_M [c_n |\nabla v|^2 + R(\bar{g})v^2]dV_{\bar{g}}}{\left(\int_M v^{\frac{2n}{n-2}}dV_{\bar{g}}\right)^{\frac{n-2}{n}}}$$

(5)

for any $\bar{g} \in [g]$, where $c_n = \frac{4(n-1)}{n-2}$.

Now we assume $Ric(\bar{g}) \geq (n-1)\bar{g}$. Then by a result of Ilias [7], which is based on the isoperimetric inequality of Gromov [9], we have

$$\int_M [c_n |\nabla v|^2 + n(n-1)v^2]dV_{\bar{g}} \geq \left(\int_M v^{\frac{2n}{n-2}}dV_{\bar{g}}\right)^{\frac{n-2}{n}} n(n-1)Vol(M^n, \bar{g})^\frac{2}{n}$$

(6)

for any $v \in W^{1,2}(M)$. Note that $R(\bar{g}) \geq n(n-1)$, hence

$$Y(M^n, [g]) \geq \inf_{v \in W^{1,2}(M)} \frac{\int_M [c_n |\nabla v|^2 + n(n-1)v^2]dV_{\bar{g}}}{\left(\int_M v^{\frac{2n}{n-2}}dV_{\bar{g}}\right)^{\frac{n-2}{n}}}$$

$$\geq n(n-1)Vol(M^n, \bar{g})^\frac{2}{n}. \quad (7)$$

Taking the supremum over $\bar{g} \in [g]$ satisfying $Ric(\bar{g}) \geq (n-1)\bar{g}$, we have

$$Y(M^n, [g]) \geq n(n-1)V_{max}(M^n, [g])^\frac{2}{n}. \quad (8)$$

\[\square\]

As an immediate corollary, we see the assumption

$$V_{max}(S^3, [g]) \geq \frac{1}{2} Vol(S^3)$$

in the Theorem implies

$$Y(S^3, [g]) \geq 6 \left(\frac{1}{2}\right)^\frac{2}{3} Vol(S^3)^\frac{2}{3}$$

$$= Y(RP^3, [g_0]), \quad (9)$$

where $RP^3$ is the three dimensional projective space and $g_0$ is the standard metric on $RP^3$ which has constant sectional curvature +1.
3 An upper bound of the Sobolev constant when a horizon is present

One basic fact relating the conformal class \([g]\) on \(S^3\) and the blowing-up metric \(h = G^4 g\) on \(\mathbb{R}^3\) is

\[
Y(S^3, [g]) = 8 S(h),
\]

where \(S(h)\) is the Sobolev constant of the asymptotically flat manifold \((\mathbb{R}^3, h)\) \([3]\). Recall \(S(h)\) is defined by

\[
S(h) = \inf_{u \in W^{1,2}(\mathbb{R}^3, h)} \left\{ \frac{\int_{\mathbb{R}^3} |\nabla u|^2_h \, dV_h}{(\int_{\mathbb{R}^3} u^6 \, dV_h)^{\frac{1}{3}}} \right\}. \tag{11}
\]

The next proposition, which plays a key role in the derivation of the Theorem, was essentially established by Bray and Neves in \([3]\) using the inverse mean curvature flow technique \([6]\). As the statement of Bray and Neves is different from what we need, we include the proof here.

**Proposition 2** Let \(h\) be a complete metric on \(\mathbb{R}^3\) such that \((\mathbb{R}^3, h)\) is asymptotically flat. If \((\mathbb{R}^3, h)\) has nonnegative scalar curvature and has a closed minimal surface, then

\[
S(h) < \frac{1}{8} Y(\mathbb{R}P^3, [g_0]). \tag{12}
\]

**Proof:** Since \((\mathbb{R}^3, h)\) has a closed minimal surface, the outermost minimal surface \(S\) in \((\mathbb{R}^3, h)\), i.e. the closed minimal surface that is not enclosed by any other minimal surface \([2]\), exists and consists of a finite union of disjoint, embedded minimal two-spheres and projective planes. As our background manifold is \(\mathbb{R}^3\), \(S\) must consist of embedded minimal two-spheres alone, furthermore each component of \(S\) necessarily bounds a three-ball.

We fix a component \(\Sigma\) of \(S\) and denote by \(\Omega\) the three-ball that \(\Sigma\) bounds in \(\mathbb{R}^3\). Let \(\phi\) be the weak solution to the inverse mean curvature flow in \((\mathbb{R}^3 \setminus \Omega, h)\) with initial condition \(\Sigma\) \([6]\). \(\phi\) satisfies

\[
\phi \geq 0, \quad \phi|_{\Sigma} = 0, \quad \lim_{x \to \infty} \phi = \infty.
\]

Let \(\Sigma_t\) be the set \(\partial\{u < t\}\) for \(t > 0\) and \(\Sigma_0\) be the starting surface \(\Sigma\), then the family of surfaces \(\{\Sigma_t\}\) satisfies the following properties \([6]\):

1. \(\{\Sigma_t\}\) consists of \(C^{1,\alpha}\) surfaces. For a.e. \(t\), \(\Sigma_t\) has weak mean curvature \(H\) and \(H = |\nabla u|_h\) for a.e. \(x \in \Sigma_t\).
2. $|\Sigma_t| = e^t|\Sigma_0|$, where $|\Sigma_t|$ denotes the area of $\Sigma_t$.

3. Since $(\mathbb{R}^3, h)$ has nonnegative scalar curvature, $\Sigma$ is connected and $\mathbb{R}^3 \setminus \Omega$ is simply connected, the Hawking quasi-local mass of $\Sigma_t$,

$$m_H(\Sigma_t) = \sqrt{\frac{|\Sigma_t|}{16\pi}} \left(1 - \frac{1}{16\pi} \int_{\Sigma_t} H^2 d\mu\right),$$

is monotone increasing. Here $d\mu$ is the induced surface measure.

Now we restrict attention to functions $u \in W^{1,2}(\mathbb{R}^3, h)$ that have the form

$$u(x) = \begin{cases} f(0) & x \in \Omega \\ f(\phi(x)) & x \in \mathbb{R}^3 \setminus \Omega \end{cases} \quad (13)$$

for some $C^1$ functions $f(t)$ defined on $[0, \infty)$. By the coarea formula and Property 1 above, we have

$$\int_{\mathbb{R}^3} |\nabla u|^2_h dV_h = \int_0^\infty f'(t)^2 \left(\int_{\Sigma_t} H d\mu\right) dt \leq \int_0^\infty f'(t)^2 \sqrt{16\pi |\Sigma| \left(e^t - e^t\frac{t}{2}\right)} dt, \quad (14)$$

where the inequality follows from Property 2, 3 and Hölder’s inequality. Similarly, we have

$$\int_{\mathbb{R}^3} u^6 dV_h \geq \int_0^\infty f(t)^6 \left(\int_{\Sigma_t} H^{-1} d\mu\right) dt \geq \int_0^\infty f(t)^6 e^{2t|\Sigma|^2} [16\pi |\Sigma| (e^t - e^t\frac{t}{2})]^{-\frac{1}{2}} dt. \quad (15)$$

Therefore,

$$\frac{\int_{\mathbb{R}^3} |\nabla u|^2_h dV_h}{(\int_{\mathbb{R}^3} u^6 dV_h)^\frac{3}{4}} \leq \left(16\pi\right)^\frac{3}{8} \int_0^\infty f'(t)^2 (e^t - e^t\frac{t}{2})^{\frac{1}{2}} dt \left(\int_0^\infty f(t)^6 e^{2t} (e^t - e^t\frac{t}{2})^{-\frac{1}{2}} dt\right)^{-\frac{3}{2}}. \quad (16)$$

To pick an optimal $f(t)$ that minimizes the right side of (16), we consider the half spatial Schwarzschild manifold

$$(M^3, g_S) = (\mathbb{R}^3 \setminus B_1(0), (1 + \frac{1}{|x|})^4 \delta_{ij})$$
and the quotient manifold \((\tilde{M}^3, \tilde{g}_S)\) obtained from \((M^3, g_S)\) by identifying the antipodal points of \(\{|x| = 1\}\). Up to scaling, \((\tilde{M}^3, \tilde{g}_S)\) is isometric to \((RP^3 \setminus \{Q\}, G^4_0 g_0)\), the blowing-up manifold of \((RP^3, g_0)\) by its Green function at a point \(Q\). Hence, the Sobolev constant \(S(\tilde{g}_S)\) of \((M^3, \tilde{g}_S)\) equals \(\frac{1}{8} Y(RP^3, [g_0])\). On the other hand, \(S(\tilde{g}_S)\) is achieved by a function \(u_0\) that is constant on each coordinate sphere \(\{|x| = t\}\) in \(\tilde{M}\), and the level set of the solution \(\phi_0\) to the inverse mean curvature flow starting at \(\{|x| = 1\}\) in \((M^3, g_S)\) is also given by coordinate spheres. Therefore, lifted as a function on \((\tilde{M}^3, g_S)\), \(u_0\) has the form of

\[
u_0 = f_0 \circ \phi_0
\]

for some explicitly determined function \(f_0(t)\), and

\[
S(\tilde{g}_S) = \frac{\int_{\tilde{M}} \nabla u_0 \tilde{g}_S dV_{g_S}}{\left(\int_{\tilde{M}} u_0^8 dV_{g_S}\right)^{\frac{1}{3}}} = \frac{(16\pi)^{\frac{2}{3}} \int_0^\infty f_0(t)^2(e^t - e^{\frac{t}{2}})\frac{t}{2} dt}{\left(\int_0^\infty f_0(t)^6 e^{2t}(e^t - e^{\frac{t}{2}})^{-\frac{1}{2}} dt\right)^{\frac{3}{2}}},
\]

(17)

where the second equality holds because the Hawking quasi-local mass remains unchanged along the level sets of \(\phi_0\). Now consider \(u = f_0 \circ \phi\) on \(\mathbb{R}^3\). It was verified in [3] that \(u \in W^{1,2}(\mathbb{R}^3, h)\). Therefore, we have

\[
\begin{align*}
S(h) &\leq \frac{\int_{\mathbb{R}^3} |\nabla u_h|^2 dV_h}{\left(\int_{\mathbb{R}^3} u_h^6 dV_h\right)^{\frac{1}{3}}} \\
&\leq \frac{(16\pi)^{\frac{2}{3}} \int_0^\infty f_0(t)^2(e^t - e^{\frac{t}{2}})\frac{t}{2} dt}{\left(\int_0^\infty f_0(t)^6 e^{2t}(e^t - e^{\frac{t}{2}})^{-\frac{1}{2}} dt\right)^{\frac{3}{2}}} \\
&= S(\tilde{g}_S) = \frac{1}{8} Y(RP^3, [g_0]).
\end{align*}
\]

(18)

To show the strict inequality, we assume \(S(h) = \frac{1}{8} Y(RP^3, [g_0])\). Then, \(S(h)\) is achieved by \(u = f_0 \circ \phi\). It follows from the Euler–Lagrange equation of the Sobolev functional (11) that \(u\) satisfies

\[
\triangle_h u + Cu^5 = 0 \quad \text{on} \quad \mathbb{R}^3,
\]

(19)

where \(C = S(h) ||u||_{L^4(\mathbb{R}^3, h)}^4\). However, \(u \equiv f_0(0)\) on \(\Omega\) and \(f_0(0) \neq 0\) (Indeed, up to a constant multiple, \(f_0(t) = (2e^t - e^{\frac{t}{2}})^{-\frac{1}{2}}\)). Hence, \(C = 0\), which contradicts to the fact that \(u\) is not a constant. Therefore, the strict inequality \(S(h) < \frac{1}{8} Y(RP^3, [g_0])\) holds.

\[
\square
\]

Proof of the Theorem: Suppose \((S^3 \setminus \{P\}, G^4g)\) has a horizon, then it follows from [10] and Proposition [2] that

\[
Y(S^3, [g]) < Y(RP^3, [g_0]).
\]

(20)
On the other hand, the assumption $V_{\text{max}}(S^3, [g]) \geq \frac{1}{2} \text{Vol}(S^3)$ implies
\[Y(S^3, [g]) \geq Y(RP^3, [g_0])\] by (9), which is a contradiction. Hence, there are no horizons. 

\[\blacksquare\]

4 An example with horizons

In this section, we provide an example to show that there exist metrics on $S^3$ with positive Ricci curvature such that the blowing-up manifolds do have horizons.

Our example comes from a 1-parameter family of left-invariant metrics $\{g_\epsilon\}$ on $S^3$, commonly known as the Berger metrics. Precisely, we think $S^3$ as the Lie Group $SU(2) = \left\{ \begin{pmatrix} z & -w \\ \bar{w} & \bar{z} \end{pmatrix} : |z|^2 + |w|^2 = 1 \right\}$, where the Lie algebra of $SU(2)$ is spanned by $X_1 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \quad X_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \text{and} \quad X_3 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}.$

Then $\{g_\epsilon\}$ is defined by declaring $X_1, X_2, X_3$ to be orthogonal, $X_1$ to have length $\epsilon$ and $X_2, X_3$ to be unit vectors. Note that scalar multiplication on $S^3 \subset \mathbb{C}^2$ corresponds to multiplication on the left by matrices $\begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix}$ on $SU(2)$, hence $X_1$ is exactly tangent to the circle fiber of the Hopf fibration $\pi : S^3 \rightarrow S^2 = S^3/S^1$

and $g_\epsilon$ shrinks the circle fiber as $\epsilon \rightarrow 0$. One fact of $g_\epsilon$ for small $\epsilon$ is that all sectional curvature of $(S^3, g_\epsilon)$ lies in the interval $[\epsilon^2, 4 - 3\epsilon^2]$ (see [10]), in particular $g_\epsilon$ has positive Ricci curvature.

**Proposition 3** Let $P \in S^3$ be a fixed point and $G_\epsilon$ be the Green’s function of the conformal Laplacian of $g_\epsilon$ at $P$. Then $(S^3 \setminus \{P\}, G_\epsilon^4 g_\epsilon)$ has a horizon for $\epsilon$ sufficiently small.

**Proof.** For each $\epsilon \in (0, 1]$, we consider the rescaled metric $\tilde{g}_\epsilon = \epsilon^{-2} g_\epsilon$ and the Green’s function $\tilde{G}_\epsilon$ associated to $\tilde{g}_\epsilon$ at $P$. Then, with respect to $\tilde{g}_\epsilon$, $X_1$ becomes a unit vector and $X_2, X_3$ have large length $\epsilon^{-1}$ as $\epsilon \rightarrow 0$. Let
$U \subset S^3$ be a fixed neighborhood of $P$ such that $\pi|_{U}$ is a trivial fibration. Let $O$ be a fixed point in the product manifold $S^1 \times \mathbb{R}^2$. By a scaling argument, there exists a family of diffeomorphisms

$$
\Psi_\epsilon : U \rightarrow \Psi_\epsilon(U) \subset S^1 \times \mathbb{R}^2,
$$
such that $\Psi_\epsilon(P) = O \in \Psi_\epsilon(U)$, $\{\Psi_\epsilon(U)\}_{1 \geq \epsilon > 0}$ forms an exhaustion family of $S^1 \times \mathbb{R}^2$ as $\epsilon \rightarrow 0$, and the push forward metrics $\hat{g}_\epsilon = \Psi_\epsilon^{-1*}(\hat{g}|_{U})$ on $\Psi_\epsilon(U)$ converge in $C^2$ norm on compact sets to a flat metric $\hat{g}$ on $S^1 \times \mathbb{R}^2$.

Now fix another point $Q \in \Psi_1(U)$ that is different from $O$ and consider the normalized function

$$
\hat{G}_\epsilon(x) = \frac{\tilde{G}_\epsilon \circ \Psi^{-1}_\epsilon(x)}{\tilde{G}_\epsilon \circ \Psi^{-1}_\epsilon(Q)}
$$

for $x \in \Psi_\epsilon(U) \setminus \{O\}$. Then $\hat{G}_\epsilon$ satisfies

$$
\begin{cases}
8\triangle_{\hat{g}_\epsilon} \hat{G}_\epsilon - R(\hat{g}_\epsilon) \hat{G}_\epsilon = 0 \quad \text{on } \Psi_\epsilon(U) \setminus \{O\}, \\
\hat{G}_\epsilon = 1 \quad \text{at } Q
\end{cases}
$$

(23)

Since $\hat{G}_\epsilon$ is positive and $\hat{g}_\epsilon$ converges to $\hat{g}$ as $\epsilon \rightarrow 0$, it follows from the Harnack inequality that $\hat{G}_\epsilon$ converges to a positive function $\hat{G}$ on $(S^1 \times \mathbb{R}^2) \setminus \{O\}$ in $C^2$ norm on any compact set away from $\{O\}$. Furthermore, $\hat{G}$ satisfies

$$
\begin{cases}
\triangle_{\hat{g}} \hat{G} = 0 \quad \text{on } (S^1 \times \mathbb{R}^2) \setminus \{O\}, \\
\hat{G} = 1 \quad \text{at } Q
\end{cases}
$$

(24)

On the other hand, the fact that the geodesic ball in $(S^1 \times \mathbb{R}^2, \hat{g})$ only has quadratic volume growth implies $(S^1 \times \mathbb{R}^2, \hat{g})$ does not have a positive Green’s function for the usual Laplacian $\triangle_{\hat{g}}$. Therefore, $\hat{G} \equiv 1$ on $(S^1 \times \mathbb{R}^2) \setminus \{O\}$.

Hence, the metrics $\hat{G}_\epsilon^4 \hat{g}_\epsilon$ converge to $\hat{g}$ in $C^2$ norm on any compact set away from $\{O\}$. Now let $V \subset S^1 \times \mathbb{R}^2$ be a small open ball containing $O$ such that $\partial V$ is an embedded two sphere whose mean curvature vector computed with respect to $\hat{g}$ points towards $O$. Then, for sufficiently small $\epsilon$, the mean curvature vector of $\partial V$ computed with respect to $\hat{G}_\epsilon^4 \hat{g}_\epsilon$ still points towards $O$.

As $(\Psi_\epsilon(U), \hat{G}_\epsilon^4 \hat{g}_\epsilon)$ is isometric to $(U, \hat{G}_\epsilon^4 \hat{g}_\epsilon)$, the mean curvature vector of the boundary of $\Psi_\epsilon^{-1}(V)$ in $(S^3 \setminus \{P\}, \hat{G}_\epsilon^4 \hat{g}_\epsilon)$ must point towards the blowing-up point $P$. On the other hand, as $(S^3 \setminus \{P\}, \hat{G}_\epsilon^4 \hat{g}_\epsilon)$ is asymptotically flat, its infinity is foliated by two spheres whose mean curvature vector points away from $P$. Therefore, it follows from standard geometric measure theory that there exists an embedded minimal two sphere in $\Psi_\epsilon(V)$, hence $(S^3 \setminus \{P\}, \hat{G}_\epsilon^4 \hat{g})$ has a horizon. □

Acknowledgment I want to thank Justin Corvino and Rick Schoen for helpful discussions.
References

[1] R. Beig and N. Ó Murchadha. Trapped surfaces due to concentration of gravitational radiation. *Phys. Rev. Lett.*, 66(19):2421–2424, 1991.

[2] Hubert L. Bray. Proof of the Riemannian Penrose inequality using the positive mass theorem. *J. Differential Geom.*, 59(2):177–267, 2001.

[3] Hubert L. Bray and André Neves. Classification of prime 3-manifolds with Yamabe invariant greater than $\mathbb{RP}^3$. *Ann. of Math. (2)*, 159(1):407–424, 2004.

[4] S. Y. Cheng and S. T. Yau. Differential equations on Riemannian manifolds and their geometric applications. *Comm. Pure Appl. Math.*, 28(3):333–354, 1975.

[5] Pengfei Guan and Guofang Wang. Conformal deformations of the smallest eigenvalue of the ricci tensor. *Max Planck Institute Preprint Nr. 43/2005*, 2005.

[6] Gerhard Huisken and Tom Ilmanen. The inverse mean curvature flow and the Riemannian Penrose inequality. *J. Differential Geom.*, 59(3):353–437, 2001.

[7] Saïd Ilias. Constantes explicites pour les inégalités de Sobolev sur les variétés riemanniennes compactes. *Ann. Inst. Fourier (Grenoble)*, 33(2):151–165, 1983.

[8] William Meeks, III, Leon Simon, and Shing Tung Yau. Embedded minimal surfaces, exotic spheres, and manifolds with positive Ricci curvature. *Ann. of Math. (2)*, 116(3):621–659, 1982.

[9] Vitali D. Milman and Gideon Schechtman. *Asymptotic theory of finite-dimensional normed spaces*, volume 1200 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 1986. With an appendix by M. Gromov.

[10] Peter Petersen. *Riemannian geometry*, volume 171 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1998.

[11] Richard Schoen and S. T. Yau. The existence of a black hole due to condensation of matter. *Comm. Math. Phys.*, 90(4):575–579, 1983.

[12] Yu Yan. The existence of horizons in an asymptotically flat 3-manifold. *Math. Res. Lett.*, 12(2-3):219–230, 2005.