Abstract. This paper aims to investigate numerical approximation of a general second order non-autonomous semilinear parabolic stochastic partial differential equation (SPDE) driven by multiplicative noise. Numerical approximations of autonomous SPDEs are thoroughly investigated in the literature, while the non-autonomous case is not yet understood. We discretize the non-autonomous SPDE driven by multiplicative noise by the finite element method in space and the Magnus-type integrator in time. We provide a strong convergence proof of the fully discrete scheme toward the mild solution in the root-mean-square $L^2$ norm. The result reveals how the convergence orders in both space and time depend on the regularity of the noise and the initial data. In particular, for multiplicative trace class noise we achieve convergence order $\mathcal{O}(h^2 (1 + \max(0, \ln (t_m/h^2)) + \Delta t^{1/2})$.

Numerical simulations to illustrate our theoretical finding are provided.

Key words. Magnus-type integrator, Stochastic partial differential equations, Multiplicative noise, Strong convergence, Non-autonomous equations, Finite element method.

1. Introduction. We consider the numerical approximations of the following semilinear parabolic non-autonomous SPDE driven by multiplicative noise

$$\begin{cases} 
    dX = [A(t)X + F(t,X)]dt + B(t,X)dW(t), & \text{in } \Lambda \times (0,T], \\
    X(0) = X_0, & \text{in } \Lambda,
\end{cases} \tag{1.1}$$

in the Hilbert space $L^2(\Lambda)$, where $\Lambda$ is a bounded domain of $\mathbb{R}^d$, $d = 1, 2, 3$ and $T \in (0, \infty)$. The family of unbounded linear operators $A(t)$ are not necessarily self-adjoint. Each $A(t)$ is assumed to generate an analytic semigroup $S_t(s) := e^{A(t)s}$. The nonlinear functions $F$ and $B$ are respectively the drift and the diffusion parts. Precise assumptions on $A(t)$, $F$ and $B$ to ensure the existence of the unique mild solution of (1.1) are given in the next section. The random initial data is denoted by $X_0$. We denote by $(\Omega, \mathcal{F}, \mathbb{P})$ a probability space with a filtration $(\mathcal{F}_t)_{t \in [0,T]} \subset \mathcal{F}$ that fulfills the usual conditions (see [30, Definition 2.1.11]). The noise term $W(t)$ is assumed to be a $Q$-Wiener process defined on a filtered probability space $(\Omega, \mathcal{F}, \mathbb{P}, \{\mathcal{F}_t\}_{t \in [0,T]})$, where the covariance operator $Q : H \to H$ is assumed to be linear, self adjoint and positive definite. It is well known [30] that the noise can be represented as

$$W(t,x) = \sum_{i=0}^{\infty} \sqrt{q_i} e_i(x) \beta_i(t), \tag{1.2}$$

where $(q_i,e_i)_{i \in \mathbb{N}}$ are the eigenvalues and eigenfunctions of the covariance operator $Q$, and $(\beta_i)_{i \in \mathbb{N}}$ are independent and identically distributed standard Brownian motions. The deterministic counterpart of (1.1) finds applications in many fields such as...

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quantum fields theory, electromagnetism, nuclear physics (see e.g. [1] and references therein). It is worth to mention that models based on SPDEs can offer a more realistic representation of the system than models based only on PDEs, due to uncertainty in the input data. In many situations it is very hard to exhibit explicit solutions of SPDEs. For instance the following non-autonomous linear Stratonovich stochastic ordinary differential equation

\[ dy = G_0(t)y \, dt + \sum_{j=1}^{d} G_j(t)y \, dW_j(t), \quad y(0) = y_0 \in \mathbb{R}^m \]  

(1.3)
does not have explicit solution (see e.g. [2, 15]), unless \( G_i \) and \( G_j \) commute for all \( i, j \geq 0 \). Numerical algorithms are therefore excellent tools to provide good approximations. Numerical approximations of (1.1) based on implicit, explicit Euler methods and exponential integrators with \( A(t) = A \), where \( A \) is self-adjoint are thoroughly investigated in the literature, see e.g. [16, 19, 20, 37, 38, 23, 36] and the references therein. If we turn our attention to the case of time independent operator \( A(t) = A \), with \( A \) not necessary self-adjoint, the list of references become remarkably short, see e.g., [22, 20]. To the best of our knowledge numerical approximations of (1.1) with time dependent linear operator \( A(t) \) are not yet investigated in the scientific literature, due to the complexity of the linear operator \( A(t) \) and its semigroup \( S_t(s) := e^{A(t)s} \). Our aim in this paper is to fill that gap and propose an explicit numerical scheme to approximate (1.1). We use the finite element method for spatial discretization and Magnus-type integrator for temporal discretization. Magnus-type integrator is based on a truncation of Magnus expansion, which was first proposed in [25] to represent the solution of non-autonomous homogeneous differential equation in the exponential form. Magnus expansion was further studied in [2, 3, 4]. The first numerical method based on magnus expansion was proposed in [14] for deterministic time-dependent homogeneous Schrödinger equation. The study in [14] was extended in [10] for partial differential equation of the following form

\[ u'(t) = A(t)u(t) + b(t), \quad 0 < t \leq T, \quad u(0) = u_0. \]  

(1.4)

We follow [10] and apply the Magnus-type integrator method to the semi-discrete problem (2.37) and obtain the fully discrete scheme (2.41), called stochastic Magnus-type integrators (SMTI). We investigate the strong convergence of the new fully discrete scheme toward the exact solution. Due to the complexity of the linear operator and the corresponding semi discrete linear operator after space discretisation, novel technical estimates are provided to achieve convergence orders comparable of that of autonomous SPDEs [22, 19, 26]. The result indicates how the convergence orders in both space and time depend on the regularity of the initial data and the noise. In particular for multiplicative trace class noise, we achieve optimal convergence orders of \( O\left(h^3 + \Delta t^{\min(\beta, 1)/2}\right) \), where \( \beta \) is the regularity’s parameter, defined in Assumption 2.1.

The rest of this paper is organised as follows. Section 2 provides the general setting, the fully discrete scheme and the main result. In Section 3 we provide some preparatory results and we present the proof of the main result. Section 4 provides some numerical experiments to confirm our theoretical result.

2. Mathematical setting, numerical scheme and main result.
2.1. Notations and main assumptions. Let \((H, \langle \cdot, \cdot \rangle_H, \|\cdot\|_H)\) be a separable Hilbert space. For a Banach space \(U\), we denote by \(L^2(\Omega, U)\) the Banach space of all equivalence classes of square-integrable \(U\)-valued random variables. Let \(L(U, H)\) be the space of bounded linear mappings from \(U\) to \(H\) endowed with the usual operator norm \(\|\cdot\|_{L(U, H)}\). By \(L_2(U, H) := HS(U, H)\), we denote the space of Hilbert-Schmidt operators from \(U\) to \(H\) equipped with the norm

\[
\|l\|_{L_2(U, H)}^2 := \sum_{i=1}^{\infty} \|l\psi_i\|^2, \quad l \in L_2(U, H),
\]

where \((\psi_i)_{i=1}^{\infty}\) is an orthonormal basis of \(U\). Note that this definition is independent of the orthonormal basis of \(U\). For simplicity, we use the notations \(L(U, U) =: L(U)\) and \(L_2(U, U) =: L_2(U)\). For all \(l \in L(U, H)\) and \(l_1 \in L_2(U, H)\) we have \(ll_1 \in L_2(U, H)\) and

\[
\|ll_1\|_{L_2(U, H)} \leq \|l\|_{L(U, H)} \|l_1\|_{L_2(U)}. \tag{2.2}
\]

The space of Hilbert-Schmidt operators from \(Q^{1/2}(H)\) to \(H\) is denoted by \(L_2^0 := L_2(Q^{1/2}(H), H) = HS(Q^{1/2}(H), H)\). As usual, \(L_2^0\) is equipped with the norm

\[
\|l\|_{L_2^0} := \|lQ^{1/2}\|_{HS} = \left( \sum_{i=1}^{\infty} \|lQ^{1/2}\psi_i\|^2 \right)^{1/2}, \quad l \in L_2^0, \tag{2.3}
\]

where \((\psi_i)_{i=1}^{\infty}\) is an orthonormal basis of \(H\). This definition is independent of the orthonormal basis of \(H\). For an \(L_2^0\)-predictable stochastic process \(\phi : [0, T] \times \Lambda \rightarrow L_2^0\) such that

\[
\int_0^t \mathbb{E}\|\phi Q^{1/2}\|_{HS}^2 ds < \infty, \quad t \in [0, T], \tag{2.4}
\]

the following relation called Itô’s isometry property holds

\[
\mathbb{E} \left\| \int_0^t \phi dW(s) \right\|^2 = \int_0^t \mathbb{E}\|\phi\|_{L_2^0}^2 ds = \int_0^t \mathbb{E}\|\phi Q^{1/2}\|_{HS}^2 ds, \quad t \in [0, T], \tag{2.5}
\]

see e.g. \[29\] Step 2 in Section 2.3.2 or \[30\] Proposition 2.3.5.

In the rest of this paper, we consider \(H = L^2(\Lambda)\). To guarantee the existence of a unique mild solution of (1.1) and for the purpose of the convergence analysis, we make the following assumptions.

**Assumption 2.1.** The initial data \(X_0 : \Omega \rightarrow H\) is assumed to be measurable and satisfies \(X_0 \in L^2(\Omega, D\left((-A(0))^\beta/2\right)), 0 \leq \beta \leq 2\).

**Assumption 2.2.**

(i) As in \[14\] and \[15\], we assume that \(D\left(A(t)\right) = D, 0 \leq t \leq T\) and the family of linear operators \(A(t) : D \subset H \rightarrow H\) to be uniformly sectorial on \(0 \leq t \leq T\), i.e. there exist constants \(c > 0\) and \(\theta \in \left(\frac{\pi}{2}, \pi\right)\) such that

\[
\left\| (\lambda I - A(t))^{-1} \right\|_{L(L^2(\Lambda))} \leq \frac{c}{|\lambda|}, \quad \lambda \in S_\theta, \tag{2.6}
\]

where \(S_\theta := \{ \lambda \in \mathbb{C} : \lambda = \rho e^{i\phi}, \rho > 0, 0 \leq |\phi| \leq \theta\}\). As in \[14\], by a standard scaling argument, we assume \(-A(t)\) to be invertible with bounded inverse.
(ii) Similarly to [17, 18, 19, 20], we require the following Lipschitz conditions: there exists a positive constant $K_1$ such that
\[
\| (A(t) - A(s))(A(0))^{-1} \|_{L(H)} \leq K_1 |t - s|, \quad s, t \in [0, T],
\]
\[
\| (A(0))^{-1} (A(t) - A(s)) \|_{L(D, H)} \leq K_1 |t - s|, \quad s, t \in [0, T].
\]

(iii) Since we are dealing with non smooth data, we follow [32] and assume that
\[
\mathcal{D}((-A(t))^{\alpha}) = \mathcal{D}((-A(0))^{\alpha}), \quad 0 \leq t \leq T, \quad 0 \leq \alpha \leq 1
\]
and there exists a positive constant $K_2$ such that for all $u \in \mathcal{D}((-A(0))^{\alpha})$ the following estimate holds uniformly for $t \in [0, T]$
\[
K_2^{-1} \| (-A(0))^{\alpha} u \| \leq \| (-A(t))^{\alpha} u \| \leq K_2 \| (-A(0))^{\alpha} u \|.
\]

Remark 2.3. As a consequence of Assumption 2.2 (i) and (iii), for all $\alpha \geq 0$ and $\delta \in [0, 1]$, there exists a constant $C_1 > 0$ such that the following estimates hold uniformly for all $t \in [0, T]$
\[
\| (-A(t))^{\alpha} e^{s A(t)} \|_{L(H)} \leq C_1 s^{-\alpha}, \quad s > 0,
\]
\[
\| (-A(t))^{-\delta} \left( I - e^{s A(t)} \right) \|_{L(H)} \leq C_1 s^\delta, \quad s \geq 0,
\]
see e.g. [13, (2.1)].

Proposition 2.4. [28, Theorem 6.1, Chapter 5] Let $\Delta(T) := \{(t, s) : 0 \leq s \leq t \leq T\}$. Under Assumption 2.2 there exists a unique evolution system [28, Definition 5.3, Chapter 5] $U: \Delta(T) \to L(H)$ such that

(i) There exists a positive constant $K_0$ such that
\[
\| U(t, s) \|_{L(H)} \leq K_0, \quad 0 \leq s \leq t \leq T.
\]

(ii) $U(., s) \in C^1([s, T]; L(H))$, $0 \leq s \leq T$,
\[
\frac{\partial U}{\partial t}(t, s) = -A(t)U(t, s), \quad 0 \leq s < t \leq T,
\]
\[
\| A(t)U(t, s) \|_{L(H)} \leq \frac{K_0}{t - s}, \quad 0 \leq s < t \leq T.
\]

(iii) $U(t, .) \in C^1([0, t]; H)$, $0 < t \leq T$, $x \in \mathcal{D}(A(0))$ and
\[
\frac{\partial U}{\partial s}(t, s) = -U(t, s)A(s)x, \quad 0 \leq s \leq t \leq T,
\]
\[
\| A(t)U(t, s)A(s)^{-1} \|_{L(H)} \leq K_0, \quad 0 \leq s \leq t \leq T.
\]

We equip $V_\alpha(t) := \mathcal{D} \left( (-A(t))^{\alpha/2} \right)$, $\alpha \in \mathbb{R}$ with the norm $\| u \|_{\alpha, t} := \| (-A(t))^{\alpha/2} u \|$. Due to (2.9)-(2.10) and for the seek of ease notations, we simply write $V_\alpha$ and $\| . \|_{\alpha}$. We follow [32] and assume the nonlinear operator $F$ to satisfy the following Lipschitz condition.

Assumption 2.5. The nonlinear operator $F: [0, T] \times H \to H$ is assumed to be $\beta/2$-Hölder continuous with respect to the first variable and Lipschitz continuous with respect to the second variable, i.e. there exists a positive constant $K_3$ such that
\[
\| F(s, 0) \| \leq K_3, \quad \| F(t, u) - F(s, v) \| \leq K_3 \left| t - s \right|^{\beta/2} + \| u - v \|,
\]
\[
(2.18)
\]
for all $s, t \in [0, T]$ and $u, v \in H$.

**Assumption 2.6.** We assume the diffusion function $B : [0, T] \times H \to L^2_\Omega$ to be $\beta/2$-Hölder continuous with respect to the first variable and Lipschitz continuous with respect to the second variable, i.e. there exists a positive constant $K_4$ such that

$$
\|B(s, 0)\|_{L^2_\Omega} \leq K_4, \quad \|B(t, u) - B(s, v)\|_{L^2_\Omega} \leq K_4 \left( |t - s|^{\beta/2} + \|u - v\| \right), \quad (2.19)
$$

for all $s, t \in [0, T]$ and $u, v \in H$.

The following theorem ensures the existence of a unique mild solution of (1.1).

**Theorem 2.7.** [32, Theorem 1.3] Let Assumptions 2.1, 2.2 (i)-(ii), 2.5 and 2.6 be fulfilled. Then the non-autonomous SPDE (1.1) has a unique mild solution $X(t) \in L^2(\Omega, D \left( (-A(0))^{\beta/2} \right))$, which takes the following form

$$
X(t) = U(t, 0)X_0 + \int_0^t U(t, s)F(s, X(s))ds + \int_0^t U(t, s)B(s, X(s))dW(s), \quad (2.20)
$$

where $U(t, s)$ is the evolution system of Proposition 2.4. Moreover, there exists a positive constant $K_5$ such that

$$
\sup_{0 \leq t \leq T} \|X(t)\|_{L^2(\Omega, D((-A(0))^{\beta/2}))} \leq K_5 \left( 1 + \|X_0\|_{L^2(\Omega, D((-A(0))^{\beta/2}))} \right). \quad (2.21)
$$

To achieve optimal convergence order in space for multiplicative noise when $\beta \in [1, 2]$, we require the following further assumption, also used in [19, 17, 36, 22, 26].

**Assumption 2.8.** We assume that there exists a positive constant $c_1 > 0$, such that $B \left( s, D((-A(0))^{\beta/2}) \right) \subset HS \left( Q^{1/2}(H), D \left( (-A(0))^{\beta/2} \right) \right)$

$$
\left\| (-A(0))^{\beta/2 - 1} B(s, v) \right\|_{L^2_\Omega} \leq c_1 \left( 1 + \|v\|_{\beta - 1} \right), \quad v \in D \left( (-A(0))^{\beta/2 - 1} \right), \quad s \in [0, T], \quad (2.22)
$$

where $\beta$ comes from Assumption 2.4.

### 2.2. Fully discrete scheme and main result.

For the seek of simplicity, we assume the family of linear operators $A(t)$ to be of second order and has the following form

$$
A(t)u = \sum_{i,j=1}^d \frac{\partial}{\partial x_i} \left( q_{ij}(x, t) \frac{\partial u}{\partial x_j} \right) - \sum_{j=1}^d q_j(x, t) \frac{\partial u}{\partial x_j}, \quad (2.23)
$$

We require the coefficients $q_{i,j}$ and $q_j$ to be smooth functions of the variable $x \in \overline{\mathcal{K}}$ and Hölder-continuous with respect to $t \in [0, T]$. We further assume that there exists a positive constant $c$ such that the following ellipticity condition holds

$$
\sum_{i,j=1}^d q_{ij}(x, t) \xi_i \xi_j \geq c|\xi|^2, \quad (x, t) \in \overline{\mathcal{K}} \times [0, T]. \quad (2.24)
$$

In the abstract form (1.1), the nonlinear functions $F : H \to H$ and $B : H \to HS(Q^{1/2}(H), H)$ are defined by

$$
(F(v))(x) = f(x, v(x)), \quad (B(v)u)(x) = b(x, v(x)).u(x), \quad (2.25)
$$

Indeed the operators $A(t)$ are identified to their $L^2$ realizations given in (2.20) (see [9]).
for all \( x \in \Lambda, v \in H \) and \( u \in Q^{1/2}(H) \), where \( f : \Lambda \times \mathbb{R} \to \mathbb{R} \) and \( b : \Lambda \times \mathbb{R} \to \mathbb{R} \)
are continuously differentiable functions with globally bounded derivatives.
Under the above assumptions on \( q_{ij} \) and \( q_j \), it is well known that the family of linear operators defined by \((2.23)\) fulfills Assumption \((2.22)\) with \( D = H^2(\Lambda) \cap H^1_0(\Lambda) \), see [28, Section 7.6] or [35, Section 5.2]. The above assumptions on \( q_{ij} \) and \( q_j \) also imply that Assumption \((2.22)\) (iii) is fulfilled, see e.g. [32, Example 6.1] or [1, 31].

As in [9, 22], we introduce two spaces \( H \) and \( V \), such that \( H \subset V \), depending on the boundary conditions for the domain of the operator \(-A(t)\) and the corresponding bilinear form. For Dirichlet boundary conditions we take
\[
V = H = H^1_0(\Lambda) = \{ v \in H^1(\Lambda) : v = 0 \text{ on } \partial\Lambda \}. \tag{2.26}
\]
For Robin boundary condition and Neumann boundary condition, which is a special case of Robin boundary condition \((\alpha_0 = 0)\), we take \( V = H^1(\Lambda) \) and
\[
H = \{ v \in H^2(\Lambda) : \partial v / \partial n + \alpha_0 v = 0, \text{ on } \partial\Lambda \}, \quad \alpha_0 \in \mathbb{R}. \tag{2.27}
\]
Using Green’s formula and the boundary conditions, we obtain the corresponding bilinear form associated to \(-A(t)\)
\[
a(t)(u, v) = \int_{\Lambda} \left( \sum_{i,j=1}^d q_{ij}(x, t) \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_j} + \sum_{i=1}^d q_i(x, t) \frac{\partial u}{\partial x_i} v \right) \, dx, \quad u, v \in V,
\]
for Dirichlet boundary conditions and
\[
a(t)(u, v) = \int_{\Lambda} \left( \sum_{i,j=1}^d q_{ij}(x, t) \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_j} + \sum_{i=1}^d q_i(x, t) \frac{\partial u}{\partial x_i} v \right) \, dx + \int_{\partial\Lambda} \alpha_0 uv \, dx.
\]
for Robin and Neumann boundary conditions. Using Gårding’s inequality, it holds that there exist two constants \( \lambda_0 \) and \( c_0 \) such that
\[
a(t)(v, v) \geq \lambda_0 \|v\|^2 - c_0 \|v\|^2, \quad \forall v \in V, \quad t \in [0, T]. \tag{2.28}
\]
By adding and subtracting \( c_0 u \) on the right hand side of \((2.14)\), we obtain a new family of linear operators that we still denote by \( A(t) \). Therefore the new corresponding bilinear form associated to \(-A(t)\) still denoted by \( a(t) \) satisfies the following coercivity property
\[
a(t)(v, v) \geq \lambda_0 \|v\|^2, \quad \forall v \in V, \quad t \in [0, T]. \tag{2.29}
\]
Note that the expression of the nonlinear term \( F \) has changed as we have included the term \(-c_0 u\) in a new nonlinear term that we still denote by \( F \).
The coercivity property \((2.29)\) implies that \( A(t) \) is sectorial on \( L^2(\Lambda) \), see e.g. [21]. Therefore \( A(t) \) generates an analytic semigroup \( S_t(s) = e^{sA(t)} \) on \( L^2(\Lambda) \) such that
\[
S_t(s) = e^{sA(t)} = \frac{1}{2\pi i} \int_C e^{s\lambda}(\lambda I - A(t))^{-1} \, d\lambda, \quad s > 0, \tag{2.30}
\]
where \( C \) denotes a path that surrounds the spectrum of \( A(t) \). The coercivity property \((2.29)\) also implies that \(-A(t)\) is a positive operator and its fractional powers are well
defined and for any \( \alpha > 0 \) we have

\[
\begin{cases}
(-A(t))^{-\alpha} = \frac{1}{\Gamma(\alpha)} \int_0^\infty s^{\alpha-1} e^{sA(t)} ds, \\
(-A(t))^\alpha = \frac{1}{\Gamma(\alpha / 2)} \int_0^\infty s^{\alpha/2} e^{sA(t)} ds,
\end{cases}
\] (2.31)

where \( \Gamma(\alpha) \) is the Gamma function (see [22]). The domain of \((-A(t))^{\alpha/2}\) are characterized in [9, 7, 21] for \( 1 \leq \alpha \leq 2 \) with equivalence of norms as follows.

\[
\mathcal{D}((-A(t))^{\alpha/2}) = H^2_0(\Lambda) \cap H^\alpha(\Lambda) \quad \text{(for Dirichlet boundary condition)}
\]
\[
\mathcal{D}((-A(t))^1) = H^1(\Lambda) \quad \text{(for Robin boundary condition)}
\]
\[
\|v\|_{H^\alpha(\Lambda)} \equiv \|( (-(A(t))^{\alpha/2}v)\|, \quad \forall v \in \mathcal{D}((-A(t))^{\alpha/2}).
\]

The characterization of \( \mathcal{D}((-A(t))^{\alpha/2}) \) for \( 0 \leq \alpha < 1 \) can be found in [27, Theorem 2.1 & Theorem 2.2].

Let us now turn our attention to the space discretization of the problem (1.1). We start by splitting the domain \( \Lambda \) in finite triangles. Let \( T_h \) be the triangulation with maximal length \( h \) satisfying the usual regularity assumptions, and \( V_h \subset V \) be the space of continuous functions that are piecewise linear over the triangulation \( T_h \). We consider the projection \( P_h \) from \( H = L^2(\Lambda) \) to \( V_h \) defined for every \( u \in H \) by

\( \langle P_h u, \chi \rangle_H = \langle u, \chi \rangle_H, \quad \phi, \chi \in V_h. \) (2.32)

For all \( t \in [0, T] \), the discrete operator \( A_h(t) : V_h \rightarrow V_h \) is defined by

\( \langle A_h(t)\phi, \chi \rangle_H = \langle A(t)\phi, \chi \rangle_H - a(t)\phi, \chi \rangle_H, \quad \phi, \chi \in V_h. \) (2.33)

The coercivity property (2.29) implies that there exist constants \( C_2 > 0 \) and \( \theta \in (\frac{1}{2}, \pi) \) such that (see e.g. [21, (2.9)] or [9, 22])

\[ \| (\lambda I - A_h(t))^{-1} \|_{L(H)} \leq \frac{C_2}{|\lambda|}, \quad \lambda \in S_\theta \] (2.34)

holds uniformly for \( h > 0 \) and \( t \in [0, T] \). The coercivity condition (2.29) implies that for any \( t \in [0, T] \), \( A_h(t) \) generates an analytic semigroup \( S^h(s) := e^{sA_h(t)}, s \in [0, T] \). The coercivity property (2.29) also implies that the smooth properties (2.11) and (2.12) hold for \( A_h \) uniformly for \( h > 0 \) and \( t \in [0, T] \), i.e. for all \( \alpha \geq 0 \) and \( \delta \in [0, 1] \), there exists a positive constant \( C_3 \) such that the following estimates hold uniformly for \( h > 0 \) and \( t \in [0, T] \), see e.g. [9, 22]

\[ \left\| (-A_h(t))^{\alpha} e^{sA_h(t)} \right\|_{L(H)} \leq C_3 s^{-\alpha}, \quad s > 0, \] (2.35)
\[ \left\| (-A_h(t))^{-\delta} \left( I - e^{sA_h(t)} \right) \right\|_{L(H)} \leq C_3 s^\delta, \quad s \geq 0. \] (2.36)

The semi-discrete version of (1.1) consists of finding \( X^h(t) \in V_h, t \in [0, T] \) such that \( X^h(0) := P_h X_0 \) and

\[ dX^h(t) = [A_h(t)X^h(t) + P_h F(t, X^h(t))] dt + P_h B(t, X^h(t)) dW(t), \] (2.37)

for \( t \in (0, T] \). Let us consider the following linear system of non-autonomous ordinary differential equations (ODEs)

\[ y'(t) = A(t)y(t), \quad y(0) \quad \text{given.} \] (2.38)
Throughout this paper, we take that the convergence analysis in \[10, 14\] was only done in time for non-autonomous deterministic parabolic partial differential equation (PDE). Note \( \leq (1.1) \), called stochastic Magnus-type integrators (SMTI) simulation is given by

\[
X_{m+1}^h = e^{\Delta t A_{h,m}} X_m^h + \Delta t \varphi_1(\Delta t A_{h,m}) P_h F \left( t_m, X_m^h \right) + e^{\Delta t A_{h,m}} P_h B \left( t_m, X_m^h \right) \Delta W_m, \quad m = 0, \ldots, M, \tag{2.41}
\]

\( X_0^h = P_h X_0 \), where the linear operator \( \varphi_1(\Delta t A_{h,m}) \) is given by

\[
\varphi_1(\Delta t A_{h,m}) := \frac{1}{\Delta t} \int_0^{\Delta t} e^{(\Delta t - s) A_{h,m}} ds, \quad A_{h,m} := A_h \left( t_m \right), \tag{2.42}
\]

and for any \( M \in \mathbb{N} \), \( \Delta t = T/M \), \( t_m = m \Delta t \), \( m = 0, 1, \ldots, M \) and

\[
\Delta W_m := W_{(m+1) \Delta t} - W_{m \Delta t}. \tag{2.43}
\]

Note that the numerical scheme \((2.41)\) can be written in the following integral form, useful for the error analysis

\[
X_{m+1}^h = e^{\Delta t A_{h,m}} X_m^h + \int_{t_m}^{t_{m+1}} e^{(t_{m+1} - s) A_{h,m}} P_h F \left( t_m, X_m^h \right) ds + \int_{t_m}^{t_{m+1}} e^{\Delta t A_{h,m}} P_h B \left( t_m, X_m^h \right) dW(s). \tag{2.44}
\]

We also note that an equivalent formulation of the numerical scheme\((2.41)\), easy for simulation is given by

\[
X_{m+1}^h = X_m^h + P_h B \left( t_m, X_m^h \right) \Delta W_m + \Delta t \varphi_1(\Delta t A_{h,m}) \left[ A_{h,m} \left\{ X_m^h + P_h B \left( t_m, X_m^h \right) \Delta W_m \right\} + P_h F \left( t_m, X_m^h \right) \right]. \tag{2.45}
\]
With the numerical method in hand, we can now state its strong convergence result toward the exact solution, which is in fact our main result. In the rest of this paper $C$ is a generic constant independent of $h$, $m$, $M$ and $\Delta t$ that may change from one place to another.

**Theorem 2.9. (Main result)** Let Assumptions [24, 25, 26] be fulfilled.

(i) If $0 < \beta < 1$, then the following error estimate holds

$$
\left( \mathbb{E} \| X(t_m) - X^h_m \|^2 \right)^{1/2} \leq C \left( h^\beta + \Delta t^{\beta/2} \right).
$$

(ii) If $1 \leq \beta < 2$ and moreover if Assumption [24] is satisfied, then the following error estimate holds

$$
\left( \mathbb{E} \| X(t_m) - X^h_m \|^2 \right)^{1/2} \leq C \left( h^\beta + \Delta t^{1/2} \right).
$$

(iii) If $\beta = 2$ and if Assumption [24] is fulfilled, then the following error estimate holds

$$
\left( \mathbb{E} \| X(t_m) - X^h_m \|^2 \right)^{1/2} \leq C \left[ h^2 \left( 1 + \max(0, \ln(t_m/h^2)) \right) + \Delta t^{1/2} \right].
$$

**3. Proof of the main result.** The proof of the main result needs some preparatory results.

**3.1. Preparatory results.** The following lemma will be useful in our convergence proof.

**Lemma 3.1.** Let Assumption [23] be fulfilled. Then for any $\gamma \in [0, 1]$, the following estimates hold uniformly in $h > 0$ and $t \in [0, T]$

$$
K^{-1} \| (A_h(0))^{-\gamma} v \| \leq \| (A_h(t))^{-\gamma} v \| \leq K \| (A_h(0))^{-\gamma} v \|, \quad v \in V_h, \quad (3.1)
$$

$$
K^{-1} \| (A_h(0))^{-\gamma} v \| \leq \| (A_h(t))^{-\gamma} v \| \leq K \| (A_h(0))^{-\gamma} v \|, \quad v \in V_h, \quad (3.2)
$$

where $K$ is a positive constant independent of $t$ and $h$.

**Lemma 3.2.** Under Assumption [24], the following estimates hold

$$
\| (A_h(t) - A_h(s))^{-1} u^h \| \leq C |t - s| \| u^h \|, \quad r, s, t \in [0, T], \quad u^h \in V_h, \quad (3.3)
$$

$$
\| (A_h(r))^{-1} (A_h(s) - A_h(t)) u^h \| \leq C |s - t| \| u^h \|, \quad r, s, t \in [0, T], \quad u^h \in V_h. \quad (3.4)
$$

**Remark 3.3.** From Lemma [23] and the fact that $\mathcal{D}(A_h(t)) = \mathcal{D}(A_h(0))$, it follows from [28, Theorem 6.1, Chapter 5] that there exists a unique evolution system $U_h : \Delta(T) \rightarrow L(H)$, satisfying [28] (6.3), Page 149]

$$
U_h(t, s) = S^h_s (t - s) + \int_s^t S^h_{s\tau} (t - \tau) R^h (\tau, s) d\tau, \quad (3.5)
$$

where $S^h_s (t) := e^{A_h(s)t}$, $R^h(t, s) := \sum_{m=1}^{\infty} R^h_{m}(t, s)$, with $R^h_{m}(t, s)$ satisfying the following recurrence relation [28] (6.22), Page 153]

$$
R^h_{m+1} = \int_s^t R^h_{m}(t, \tau) R^h_{m}(\tau, s) d\tau, \quad m \geq 1 \quad (3.6)
$$
Lemma 3.4. Under Assumption 2.2, the evolution system

\[ R^h(t, s) = R^h_1(t, s) + \int_s^t R^h_1(t, \tau) R^h(\tau, s) d\tau. \]  

(3.7)

The mild solution of (2.37) is therefore given by

\[ X^h(t) = U_h(t, 0) P_h X_0 + \int_0^t U_h(t, s) P_h F(s, X^h(s)) ds \]

+ \int_0^t U_h(t, s) P_h B(s, X^h(s)) dW(s).  

(3.8)

Lemma 3.5. \([33]\) Let Assumption 2.2 be fulfilled.

(i) The following estimates hold

\[ \|A_h(t) U_h(t, s)\|_{L(H)} \leq \frac{C}{t-s}, \quad 0 \leq s < t \leq T. \]  

(3.9)

\[ \|A_h(t) U_h(t, s) A_h(s)^{-1}\|_{L(H)} \leq C, \quad 0 \leq s < t \leq T. \]  

(3.10)

(ii) For any \(0 \leq \alpha \leq 1, 0 \leq \gamma \leq 1\) and \(0 \leq s \leq t \leq T\), the following estimates hold

\[ \|(-(A_h(r))^{\alpha} U_h(t, s))\|_{L(H)} \leq C(t-s)^{-\alpha}, \quad r \in [0, T], \]  

(3.15)

\[ \|U_h(t, s)(-(A_h(r))^{\alpha})\|_{L(H)} \leq C(t-s)^{-\alpha}, \quad r \in [0, T], \]  

(3.16)

\[ \|(-(A_h(r))^{\alpha} U_h(t, s)(-A_h(s))^{-\gamma})\|_{L(H)} \leq C(t-s)^{-\alpha}, \quad r \in [0, T]. \]  

(3.17)

(iii) For any \(0 \leq s \leq t \leq T\) the following useful estimates hold

\[ \|U_h(t, s) - 1\|_{L(H)} \leq C(t-s)^{\gamma}, \quad 0 \leq \gamma \leq 1, \]  

(3.18)

\[ \|(-(A_h(r))^{-\gamma}(U_h(t, s) - 1))\|_{L(H)} \leq C(t-s)^{\gamma}, \quad 0 \leq \gamma \leq 1. \]  

(3.19)
The following space and time regularity of the semi-discrete problem will be useful in our convergence analysis.

**Lemma 3.6.** Let Assumptions 2.1-2.3 (i)-(ii), 2.6 and 2.10 be fulfilled with the corresponding \( 0 \leq \beta < 1 \). Then for all \( \gamma \in [0, \beta] \) the following estimates hold
\[
\begin{align*}
\|(-A_h(r))^{\gamma/2}X^h(t)\|_{L^2(\Omega, H)} & \leq C, \quad 0 \leq r, t \leq T, \tag{3.20} \\
\|X^h(t_2) - X^h(t_1)\|_{L^2(\Omega, H)} & \leq C(t_2 - t_1)^{\beta/2}, \quad 0 \leq t_1 \leq t_2 \leq T. \tag{3.21}
\end{align*}
\]
Moreover if Assumption 2.8 is fulfilled, then (3.20) and (3.21) hold for \( \beta = 1 \).

**Proof.** We first show that \( \sup_{t \in [0, T]} \|X^h(t)\|_{L^2(\Omega, H)}^2 \leq C \). Taking the norm in both sides of (3.5) and using the inequality \((a + b + c)^2 \leq 3a^2 + 3b^2 + 3c^2\), \( a, b, c \in \mathbb{R}_+ \) yields
\[
\|X^h(t)\|_{L^2(\Omega, H)}^2 \leq 3\|U_h(t, 0)P_hX_0\|_{L^2(\Omega, H)}^2 + 3\|\int_0^t U_h(t, s)P_hF\left(s, X^h(s)\right)ds\|_{L^2(\Omega, H)}^2 \\
+ 3\left\|\int_0^t U_h(t, s)P_hB\left(s, X^h(s)\right)dW(s)\right\|_{L^2(\Omega, H)}^2 := I_0 + I_1 + I_2. \tag{3.22}
\]
Using Lemma 3.5 (i) and the uniformly boundedness of \( P_h \), it holds that
\[
I_0 \leq 3\|X_0\|_{L^2(\Omega, H)}^2 \leq C. \tag{3.23}
\]
Using again Lemma 3.5 (i), Assumption 2.5 and the uniformly boundedness of \( P_h \), it holds that
\[
I_1 \leq 3\left(\int_0^t \|U_h(t, s)P_hF\left(s, X^h(s)\right)\|_{L^2(\Omega, H)}^2\right)^2 \leq C\left(\int_0^t \left(C + \|X^h(s)\|_{L^2(\Omega, H)}\right)ds\right)^2. \tag{3.24}
\]
Using Hölder inequality yields
\[
I_1 \leq C + C\int_0^t \|X^h(s)\|_{L^2(\Omega, H)}^2 ds. \tag{3.25}
\]
Applying the itô-isometry’s property, using Lemma 3.5 (i) and Assumption 2.6 it holds that
\[
I_2 = 3\int_0^t \|U_h(t, s)P_hB\left(s, X^h(s)\right)\|_{L^2(\Omega, H)}^2 ds \leq C + C\int_0^t \|X^h(t)\|_{L^2(\Omega, H)}^2 ds. \tag{3.26}
\]
Substituting (3.20), (3.25) and (3.23) in (3.22) yields
\[
\|X^h(t)\|_{L^2(\Omega, H)}^2 \leq C + C\int_0^t \|X^h(s)\|_{L^2(\Omega, H)}^2 ds. \tag{3.27}
\]
Applying the continuous Gronwall’s lemma to (3.26) yields
\[
\|X^h(t)\|_{L^2(\Omega, H)}^2 \leq C, \quad t \in [0, T]. \tag{3.28}
\]
Let us now prove (3.20). Pre-multiplying (3.8) by \((-A_h(r))^{\gamma/2}\), taking the norm in both sides and using triangle inequality yields
\[
\begin{align*}
\|(-A_h(r))^{\gamma/2}X^h(t)\|_{L^2(\Omega, H)} & \leq \|(-A_h(r))^{\gamma/2}U_h(t, 0)P_hX_0\|_{L^2(\Omega, H)} \\
+ \int_0^t \|(-A_h(r))^{\gamma/2}U_h(t, s)P_hF\left(s, X^h(s)\right)\|_{L^2(\Omega, H)} ds \\
+ \int_0^t \|(-A_h(r))^{\gamma/2}U_h(t, s)P_hB\left(s, X^h(s)\right)dW(s)\|_{L^2(\Omega, H)} ds \\
:= I_0 + I_1 + I_2.
\end{align*}
\]
Inserting \((-A_h(0))^{-\gamma/2}(-A_h(0))^{\gamma/2}\), using Lemma 3.5 (ii) and Lemma 3.1, it holds that

\[
II_0 \leq \|(-A_h(r))^{\gamma/2}U_h(t,0)(-A_h(0))^{-\gamma/2}\|_{L(H)}\|(-A_h(0))^{\gamma/2}X_0\| \leq C. \tag{3.29}
\]

Using Lemmas 3.1, 3.5 (ii), Assumption 2.6 and (3.27) yields

\[
II_1 \leq C \int_0^t \sup_{t \in [0,T]} \|F(s, X^h(s))\| ds \
\leq C \sup_{s \in [0,T]} (1 + \|X^h(s)\|_{L^2(\Omega, H)}) \int_0^t (t-s)^{-\gamma/2} ds \leq C. \tag{3.30}
\]

Applying the Itô-isometry property, using Lemmas 3.1, 3.5 (ii), Assumption 2.5 and (3.27) yields

\[
II_2 = \int_0^t \|(-A_h(0))^{\gamma/2}U_h(t,s)P_h B(s, X^h(s))\|_{L^2_2}^2 ds \
\leq C \sup_{s \in [0,T]} (1 + \|X^h(s)\|_{L^2(\Omega, H)}) \int_0^t (t-s)^{-\gamma} ds \leq C. \tag{3.31}
\]

Substituting (3.31), (3.30) and (3.29) in (3.28) completes the proof of (3.20). The proof of (3.21) follows from (3.3). In fact from (3.8) we have

\[
\|X^h(t_2) - X^h(t_1)\|_{L^2(\Omega, H)} \leq \|(U_h(t_2, 0) - U_h(t_1, 0)) P_h X_0\|_{L^2(\Omega, H)} \
+ \int_0^{t_1} \|U_h(t_2, s) - U_h(t_1, s)\|_{L^2(\Omega, H)} \|P_h F(s, X^h(s))\|_{L^2(\Omega, H)} ds \
+ \int_1^{t_2} \|U_h(t_2, s)P_h F(s, X^h(s))\|_{L^2(\Omega, H)} ds 
+ \int_0^{t_1} U_h(t_2, s) - U_h(t_1, s)\|P_h B(s, X^h(s))\|_{L^2(\Omega, H)} ds \
+ \int_1^{t_2} U_h(t_2, s)P_h B(s, X^h(s))\|_{L^2(\Omega, H)} ds \
:= III_0 + III_1 + III_2 + III_3 + III_4. \tag{3.32}
\]

Inserting an appropriate power of \(-A_h(t_1)\), using Lemmas 3.5 (ii)-(iii) and Lemma 1 yields

\[
III_0 = \|(U_h(t_2, t_1) - I)U_h(t_1, 0)P_h X_0\|_{L^2(\Omega, H)} \
\leq \|(U_h(t_2, t_1) - I)(-A_h(t_1))^{-\beta/2}\|_{L(H)} \
\times \|(A_h(t_1))^{\beta/2}U_h(t_1, 0)(-A_h(t_1))^{-\beta/2}\|_{L(H)} \|(A_h(t_1))^{\beta/2}P_h X_0\|_{L^2(\Omega, H)} \
\leq C(t_2 - t_1)^{\beta/2}. \tag{3.33}
\]
Using Assumption 2.6 (3.20), Lemma 3.5 (ii) and (iii) yields

\[ III_1 \leq \int_0^{t_1} \|(U_h(t_2,t_1) - I)U_h(t_1,s)\|_{L^2(H)} \left\| P_h F \left( s, X^h(s) \right) \right\|_{L^2(O,H)} ds \]

\[ \leq C \int_0^{t_1} \left\| (U_h(t_2,t_1) - I)(-A_h(t_1))^{-\beta/2} \right\|_{L^2(H)} \left\| (-A_h(t_1))^{\beta/2} U_h(t_1,s) \right\|_{L^2(O,H)} ds \]

\[ \leq C \int_0^{t_1} (t_2 - t_1)^{\beta/2} (t_1 - s)^{-\beta/2} ds \]

\[ \leq C(t_2 - t_1)^{\beta/2}. \]  \( (3.34) \)

Using Lemma 3.5 (i) and Assumption 2.5, it holds that

\[ III_2 \leq C \int_{t_1}^{t_2} \sup_{s \in [0,T]} \left\| F \left( s, X^h(s) \right) \right\|_{L^2(O,H)} ds \leq C(t_2 - t_1). \]  \( (3.35) \)

Using the Itô-isometry property, Assumption 2.8 (3.21), Lemma 3.5 (ii)-(iii) and following the same lines as the estimate of \( III_1 \) yields

\[ III_3^2 \leq C(t_2 - t_1)^{\beta}. \]  \( (3.36) \)

Using the Itô-isometry property and following the same lines as that of \( III_2 \) yields

\[ III_4^2 \leq C(t_2 - t_1). \]  \( (3.37) \)

Substituting (3.37), (3.36), (3.35), (3.34) and (3.33) in (3.32) completes the proof of \( (3.21) \).

Let us consider the following deterministic problem: find \( u \in V \) such that

\[ u' = A(t)u, \quad u(\tau) = v, \quad t \in (\tau,T]. \]  \( (3.38) \)

The corresponding semi-discrete problem in space is: find \( u_h \in V_h \) such that

\[ u'_h(t) = A_h(t)u_h, \quad u_h(\tau) = P_h v, \quad t \in (\tau,T], \quad \tau \geq 0. \]  \( (3.39) \)

Let us define the operator

\[ T_h(t,\tau) := U(t,\tau) - U_h(t,\tau)P_h, \]  \( (3.40) \)

so that \( u(t) - u_h(t) = T_h(t,\tau)v \). The following lemma will be useful in our convergence analysis.

**Lemma 3.7.** Let \( r \in [0,2] \) and \( 0 \leq \gamma \leq r \). Let Assumption 2.4 be fulfilled. Then the following error estimate holds for the semi-discrete approximation \( (3.39) \)

\[ \| u(t) - u_h(t) \| = \| T_h(t,\tau)v \| \leq C h^r (t - \tau)^{-(r - \gamma)/2} \| v \|_\gamma, \quad v \in D \left( (-A(0))^{\gamma/2} \right). \]  \( (3.41) \)

**Proposition 3.8.** 

**Space error** Let Assumptions 2.1, 2.2, 2.3 and 2.4 be fulfilled. Let \( X(t) \) and \( X^h(t) \) be the mild solution of \( (3.21) \) and \( (3.39) \) respectively.

(i) If \( 0 < \beta < 1 \), then the following error estimate holds

\[ \| X(t) - X^h(t) \|_{L^2(O,H)} \leq C h^\beta, \quad 0 \leq t \leq T. \]  \( (3.42) \)

(ii) If \( 1 \leq \beta < 2 \) and moreover if Assumption 2.6 is fulfilled, then the following error estimate holds

\[ \| X(t) - X^h(t) \|_{L^2(O,H)} \leq C h^\beta, \quad 0 \leq t \leq T, \]  \( (3.43) \)
(iii) If $\beta = 2$ and moreover if Assumption 2.5 is fulfilled, then the following error estimate holds

$$\|X(t) - X^h(t)\|_{L^2(\Omega, H)} \leq Ch^2 \left(1 + \max \left(0, \ln(t/h^2)\right)\right), \ 0 < t \leq T.$$ (3.44)

**Proof.** Subtracting (3.8) form (2.20), taking the $L^2$ norm and using triangle inequality yields

$$\|X(t) - X^h(t)\|_{L^2(\Omega, H)} \leq \|U(t, 0)X_0 - U_h(t, 0)P_hX_0\|_{L^2(\Omega, H)} + \int_0^t \left\|\left[U(t, s)F(s, X(s)) - U_h(t, s)P_hF\left(s, X^h(s)\right)\right]ds\right\|_{L^2(\Omega, H)}$$

$$+ \int_0^t \left\|\left[U(t, s)B(s, X(s)) - U_h(t, s)P_hB\left(s, X^h(s)\right)\right]dW(s)\right\|_{L^2(\Omega, H)}$$

$$=: IV_0 + IV_1 + IV_2. \quad (3.45)$$

Using Lemma 3.7 with $r = \gamma = \beta$ yields

$$IV_0 \leq Ch^\beta \|X_0\|_{L^2(\Omega, \mathcal{D}(\mathcal{A}(0)^{\beta/2}))} \leq Ch^\beta. \quad (3.46)$$

Using Lemma 3.7 with $r = \beta$, $\gamma = 0$, Assumption 2.5, Lemmas 3.6 and 3.5 yields

$$IV_1 \leq \int_0^t \left\|U(t, s)F(s, X(s)) - U(t, s)F\left(s, X^h(s)\right)\right\|_{L^2(\Omega, H)} ds$$

$$+ \int_0^t \left\|U(t, s)F\left(s, X^h(s)\right) - U_h(t, s)P_hF\left(s, X^h(s)\right)\right\|_{L^2(\Omega, H)} ds$$

$$\leq C \int_0^t \|X(s) - X^h(s)\|_{L^2(\Omega, H)} ds + Ch^\beta \int_0^t (t-s)^{-\beta/2} ds$$

$$\leq Ch^\beta + C \int_0^t \|X(s) - X^h(s)\|_{L^2(\Omega, H)} ds. \quad (3.47)$$

Using the Itô-isometry property, Lemma 3.6, Lemma 3.7 with $r = \beta$ and $\gamma = \frac{\beta-1}{2}$ yields

$$IV_2^2 = \int_0^t \left\|U(t, s)B\left(s, X(s)\right) - U_h(t, s)P_hB\left(s, X^h(s)\right)\right\|_{L^2_2}^2 ds$$

$$\leq \int_0^t \left\|U(t, s)B\left(s, X(s)\right) - U(t, s)B\left(s, X^h(s)\right)\right\|_{L^2_2}^2 ds$$

$$+ \int_0^t \left\|U(t, s)B\left(s, X^h(s)\right) - U_h(t, s)P_hB\left(s, X^h(s)\right)\right\|_{L^2_2}^2 ds$$

$$\leq C \int_0^t \|X(s) - X^h(s)\|_{L^2(\Omega, H)}^2 ds + Ch^{2\beta} \int_0^t (t-s)^{-1+\beta} ds$$

$$\leq Ch^{2\beta} + C \int_0^t \|X(s) - X^h(s)\|_{L^2(\Omega, H)}^2 ds. \quad (3.48)$$

Substituting (3.48), (3.47) and (3.46) in (3.45) yields

$$\|X(t) - X^h(t)\|_{L^2(\Omega, H)}^2 \leq Ch^{2\beta} + C \int_0^t \|X(s) - X^h(s)\|_{L^2(\Omega, H)}^2 ds. \quad (3.49)$$
Applying the continuous Gronwall’s lemma to (3.49) yields

$$\|X(t) - X^h(t)\|_{L^2(\Omega, H)} \leq C h^\beta. \quad (3.50)$$

\[\square\]

For non commutative operators $H_j$ on a Banach space, we introduce the following notation for the composition

$$\prod_{j=1}^k H_j = \begin{cases} H_k H_{k-1} \cdots H_1 & \text{if } k \geq l, \\ H_k & \text{if } k < l. \end{cases} \quad (3.51)$$

The following lemma will be useful in our convergence proof.

**Lemma 3.9.** Let Assumption 2.2 be fulfilled. Then the following estimate holds

$$\left\| \prod_{j=1}^m e^{\Delta t A_{h,j}} (-A_{h,l})^\gamma \right\|_{L(H)} \leq C t_m^{-\gamma}, \quad 0 \leq l < m, \quad 0 \leq \gamma < 1, \quad (3.52)$$

$$\left\| (-A_{h,k})^{\gamma_1} \prod_{j=1}^m e^{\Delta t A_{h,j}} (-A_{h,l})^{-\gamma_2} \right\|_{L(H)} \leq C t_m^{-\gamma_1}, \quad 0 \leq l < m, \quad (3.53)$$

$0 \leq \gamma_1 \leq 1, \quad 0 < \gamma_2 \leq 1$, where $C$ is a positive constant independent of $m, l, h$ and $\Delta t$.

**Lemma 3.10.**

(i) For all $\alpha \geq 0$, the following estimate holds

$$\left\| R^h(t, s)(-A_h(s))^\alpha \right\|_{L(H)} \leq C(t-s)^{-\alpha}, \quad t, s \in [0, T]. \quad (3.54)$$

(ii) For all $\alpha \in [0, 1]$, the following estimate holds

$$\left\| (U_h(t_j, t_{j-1}) - e^{\Delta t A_{h,j-1}} (-A_{h,j-1})^{-\alpha} \right\|_{L(H)} \leq C \Delta t^{1+\alpha}. \quad (3.55)$$

(iii) For all $\alpha \in [0, 1]$, the following estimate holds

$$\left\| (U_h(t_j, t_{j-1}) - e^{\Delta t A_{h,j-1}} (-A_{h,j-1})^\alpha \right\|_{L(H)} \leq C \Delta t^{1-\alpha}. \quad (3.56)$$

(iv) For all $\alpha \in [0, 1]$, the following estimate holds

$$\left\| (-A_{h,j-1})^{-\alpha} (U_h(t_j, t_{j-1}) - e^{\Delta t A_{h,j-1}}) \right\|_{L(H)} \leq C \Delta t^{1+\alpha}. \quad (3.57)$$

**Proof.** From the integral equation (5.7), we have

$$R^h(t, s)(-A_h(s))^\alpha = e^{A_h(s)(t-s)}(-A_h(s))^\alpha + \int_s^t R^h_{1}(t, \tau) R^h(\tau, s)(-A_h(s))^\alpha d\tau. \quad (3.58)$$

Taking the norm in both sides of (3.58), using (2.36) and Lemma 3.5 yields

$$\left\| R^h(t, s)(-A_h(s))^\alpha \right\|_{L(H)} \leq \left\| e^{A_h(s)(t-s)}(-A_h(s))^\alpha \right\|_{L(H)}$$

$$+ \int_s^t \left\| R^h_{1}(t, \tau) \right\|_{L(H)} \left\| R^h(\tau, s)(-A_h(s))^\alpha \right\|_{L(H)} d\tau$$

$$\leq C(t-s)^{-\alpha} + C \int_s^t \left\| R^h(\tau, s)(-A_h(s))^\alpha \right\|_{L(H)} d\tau. \quad (3.59)$$
Applying the continuous Gronwall’s lemma to (3.59) yields
\[ \| R^h(t, s) (-A_h(s))^\alpha \|_{L(H)} \leq C (t - s)^{-\alpha}. \] (3.60)

This completes the proof of (i). From (3.5) and (3.7), we have
\[ A. \text{Tambue and J. D. Mukam} \]

Therefore, from (3.61), for all \( \alpha \in [0, 1] \), using (2.36) and Lemma 3.5 it holds that
\[
\begin{align*}
\| (U_h(t_j, t_{j-1}) - e^{\Delta t A_h(t_{j-1})}) (-A_{h, j-1})^{-\alpha} \|_{L(H)} \\
\leq \int_{t_{j-1}}^{t_j} \| e^{(t_j - \tau) A_h(\tau)} (A_h(\tau) - A_h(t_{j-1})) (-A_{h, j-1})^{-1} \\
\cdot e^{A_{h, j-1}(\tau - t_{j-1})} (-A_{h, j-1})^{-1} \|_{L(H)} d\tau \\
+ \int_{t_{j-1}}^{t_j} \| e^{(t_j - \tau) A_h(\tau)} \|_{L(H)} \left( \int_{t_{j-1}}^{\tau} \| R^h(s, s) R^h(s, t_{j-1}) \|_{L(H)} ds \right) d\tau \\
\leq \int_{t_{j-1}}^{t_j} \| e^{(t_j - \tau) A_h(\tau)} \|_{L(H)} \left( \| (A_h(\tau) - A_h(t_{j-1})) (-A_{h, j-1})^{-1} \|_{L(H)} \\
\times \| e^{A_{h, j-1}(\tau - t_{j-1})} (-A_{h, j-1})^{-1} \|_{L(H)} \right) d\tau \\
+ C \int_{t_{j-1}}^{t_j} ds d\tau \\
\leq C \int_{t_{j-1}}^{t_j} (\tau - t_{j-1})^\alpha d\tau + C \Delta t^2 \leq C \Delta t^{1+\alpha}.
\end{align*}
\] (3.62)

This completes the proof of (ii). The proof of (iii) and (iv) are similar to that of (ii) using (i).

The following lemma can be found in [21]

**Lemma 3.11.** For all \( \alpha_1, \alpha_2 > 0 \) and \( \alpha \in [0, 1) \), there exist two positive constants \( C_{\alpha_1, \alpha_2} \) and \( C_{\alpha, \alpha_2} \) such that
\[
\Delta t \sum_{j=1}^{m} t_{m-j+1}^{-1 + \alpha_1} t_j^{-1 + \alpha_2} \leq C_{\alpha_1, \alpha_2} t_m^{-1 + \alpha_1 + \alpha_2},
\] (3.63)
\[
\Delta t \sum_{j=1}^{m} t_{m-j+1}^{-\alpha} t_j^{-1 + \alpha_2} \leq C_{\alpha, \alpha_2} t_m^{-\alpha + \alpha_2}.
\] (3.64)
Proof. The proof of (3.63) follows from the comparison with the integral
\[ \int_0^t (t-s)^{-1+\alpha_1} s^{-1+\alpha_2} ds. \] (3.65)

The proof of (3.64) is a consequence of (3.63). \( \Box \)

The following lemma is fundamental in our convergence analysis.

**Lemma 3.12.** Let Assumption 3.2 be fulfilled. Then for all \( 1 \leq i \leq m \leq M \).

(i) The following estimate holds
\[ \left\| \left( \prod_{j=i}^m U_h(t_j, t_{j-1}) \right) - \left( \prod_{j=i}^{m-1} e^{\Delta t A_{h,j}} \right) \right\|_{L(H)} \leq C \Delta t^{1-\epsilon}, \] (3.66)

where \( \epsilon > 0 \) is a positive number small enough.

(ii) The following estimate also holds
\[ \left\| \left[ \left( \prod_{j=i}^m U_h(t_j, t_{j-1}) \right) - \left( \prod_{j=i}^{m-1} e^{\Delta t A_{h,j}} \right) \right] (-A_{h,i-1})^{-1} \right\|_{L(H)} \leq C \Delta t. \] (3.67)

**Proof.** First of all note that
\[ \left( \prod_{j=i}^m U_h(t_j, t_{j-1}) \right) - \left( \prod_{j=i}^{m-1} e^{\Delta t A_{h,j}} \right) = \left( \prod_{j=i}^m U_h(t_j, t_{j-1}) \right) - \left( \prod_{j=i}^{m} e^{\Delta t A_{h,j-1}} \right). \] (3.68)

Using the telescopic sum, (3.69) can be rewritten as follows
\[ \left( \prod_{j=i}^m U_h(t_j, t_{j-1}) \right) - \left( \prod_{j=i}^{m} e^{\Delta t A_{h,j-1}} \right) = \sum_{k=1}^{m-i+1} \left( \prod_{j=i+k}^m U_h(t_j, t_{j-1}) \right) \left( e^{\Delta t A_{h,i+k-2}} - e^{\Delta t A_{h,i+k-1}} \right). \] (3.69)

Writing down explicitly the first term of (3.69) gives
\[ \left( \prod_{j=i}^m U_h(t_j, t_{j-1}) \right) - \left( \prod_{j=i}^{m} e^{\Delta t A_{h,j-1}} \right) = \left( \prod_{j=i+1}^m U_h(t_j, t_{j-1}) \right) \left( U_h(t_i, t_{i-1}) - e^{\Delta t A_{h,i-1}} \right) \]
\[ + \sum_{k=2}^{m-i+1} \left( \prod_{j=i+k}^m U_h(t_j, t_{j-1}) \right) \left( e^{\Delta t A_{h,i+k-2}} - e^{\Delta t A_{h,i+k-1}} \right). \] (3.70)
Iterating the mild solution (3.73) yields
\[
\left\| \left( \prod_{j=1}^{m} U_h(t_j, t_{j-1}) \right) - \left( \prod_{j=1}^{m} e^{\Delta t A_h, j-1} \right) \right\|_{L(H)}
\leq \| U_h(t_{m-1}, t_0) \|_{L(H)} \left\| U_h(t_0, t_{i-1}) - e^{\Delta t A_h, i-1} \right\|_{L(H)}
+ \sum_{k=2}^{m-i+1} \| U_h(t_{m-1}, t_{i+k-1}) \|_{L(H)} \left\| U_h(t_{i+k-1}, t_{i+k-2}) - e^{\Delta t A_h, i+k-2} \right\|_{L(H)}
\times \left\| \left( -A_{h,i+k-2} \right)^{1-\epsilon} \left( \prod_{j=1}^{m-k} e^{\Delta t A_h, j-1} \right) \right\|_{L(H)}
\leq C\Delta t + C \sum_{k=2}^{m-i+1} \Delta t^2 \epsilon^{-1+\epsilon}
\leq C\Delta t^2 \epsilon^{-1+\epsilon} \quad (3.71)
\]

This completes the proof of (i). The proof of (ii) is similar to that of (i) using Lemma 3.5 and Lemma 3.11. \( \square \)

With the above preparatory results in hand, we can now prove our main result.

### 3.2. Proof of Theorem [2,9]

Using triangle inequality, we split the fully discrete error in two parts as follows.

\[
\| X(t_m) - X^h_m \|_{L^2(\Omega, H)} \leq \| X(t_m) - X^h(t_m) \|_{L^2(\Omega, H)} + \| X^h(t_m) - X^h_m \|_{L^2(\Omega, H)} =: V + VI.
\]

The space error \( V \) is estimated in Lemma 3.7. It remains to estimate the time error \( VI \). Note that the mild solution of (2.37) can be written as follows.

\[
X^h(t_m) = U_h(t_m, t_{m-1})X^h(t_{m-1}) + \int_{t_{m-1}}^{t_m} U_h(t_m, s)P_h F (s, X^h(s)) \, ds
+ \int_{t_{m-1}}^{t_m} U_h(t_m, s)P_h B (s, X^h(s)) \, dW(s).
\]

Iterating the mild solution (3.73) yields
\[
X^h(t_m)
= \left( \prod_{j=1}^{m} U_h(t_j, t_{j-1}) \right) P_h X_0 + \int_{t_{m-1}}^{t_m} U_h(t_m, s)P_h F (s, X^h(s)) \, ds
+ \int_{t_{m-1}}^{t_m} U_h(t_m, s)P_h B (s, X^h(s)) \, dW(s)
+ \sum_{k=1}^{m-1} \int_{t_{m-k-1}}^{t_m} \left( \prod_{j=m-k+1}^{m} U_h(t_j, t_{j-1}) \right) U_h(t_{m-k}, s)P_h F (s, X^h(s)) \, ds
+ \sum_{k=1}^{m-1} \int_{t_{m-k-1}}^{t_m} \left( \prod_{j=m-k+1}^{m} U_h(t_j, t_{j-1}) \right) U_h(t_{m-k}, s)P_h B (s, X^h(s)) \, dW(s). \quad (3.74)
\]
Iterating the numerical scheme \[2.44\] by substituting \( X^j_h, j = m - 1, \cdots, 1 \) only in the first term of \[2.44\] by their expressions yields

\[
X^h_m = \left( \prod_{j=0}^{m-1} e^{\Delta t A_{h,j}} \right) X^h_0 + \int_{t_{m-1}}^{t_m} e^{(t_m-s)A_{h,m-1}} P_h F \left( t_{m-1}, X^h_{m-1} \right) ds
+ \int_{t_{m-1}}^{t_m} e^{\Delta t A_{h,m-1}} P_h B \left( t_{m-1}, X^h_{m-1} \right) dW(s)
+ \sum_{k=1}^{m-1} \int_{t_{m-k-1}}^{t_{m-k}} \left( \prod_{j=m-k}^{m-1} e^{\Delta t A_{h,j}} \right) e^{(t_{m-k-s)A_{h,m-k-1}}} P_h F \left( t_{m-k-1}, X^h_{m-k-1} \right) ds
+ \sum_{k=1}^{m-1} \int_{t_{m-k-1}}^{t_{m-k}} \left( \prod_{j=m-k}^{m-1} e^{\Delta t A_{h,j}} \right) e^{\Delta t A_{h,m-k-1}} P_h B \left( t_{m-k-1}, X^h_{m-k-1} \right) dW(s)(3.75)
\]

Subtracting \[3.75\] from \[3.74\] yields

\[
X^h(t_m) = X^h_m
= \left( \prod_{j=1}^{m} U_h(t_j, t_{j-1}) \right) P_h X_0 - \left( \prod_{j=0}^{m-1} e^{\Delta t A_{h,j}} \right) P_h X_0
+ \int_{t_{m-1}}^{t_m} U_h(t_m, s) P_h F \left( s, X^h(s) \right) ds - \int_{t_{m-1}}^{t_m} e^{(t_m-s)A_{h,m-1}} P_h F \left( t_{m-1}, X^h_{m-1} \right) ds
+ \int_{t_{m-1}}^{t_m} U_h(t_m, s) P_h B \left( s, X^h(s) \right) dW(s) - \int_{t_{m-1}}^{t_m} e^{\Delta t A_{h,m-1}} P_h B \left( t_{m-1}, X^h_{m-1} \right) dW(s)
+ \sum_{k=1}^{m-1} \int_{t_{m-k-1}}^{t_{m-k}} \left( \prod_{j=m-k}^{m-1} U_h(t_j, t_{j-1}) \right) U_h(t_{m-k}, s) P_h F \left( s, X^h(s) \right) ds
- \sum_{k=1}^{m-1} \int_{t_{m-k-1}}^{t_{m-k}} \left( \prod_{j=m-k}^{m-1} e^{\Delta t A_{h,j}} \right) e^{(t_{m-k-s)A_{h,m-k-1}}} P_h F \left( t_{m-k-1}, X^h_{m-k-1} \right) ds
+ \sum_{k=1}^{m-1} \int_{t_{m-k-1}}^{t_{m-k}} \left( \prod_{j=m-k}^{m-1} U_h(t_j, t_{j-1}) \right) U_h(t_{m-k}, s) P_h B \left( s, X^h(s) \right) dW(s)
- \sum_{k=1}^{m-1} \int_{t_{m-k-1}}^{t_{m-k}} \left( \prod_{j=m-k}^{m-1} e^{\Delta t A_{h,j}} \right) e^{\Delta t A_{h,m-k-1}} P_h B \left( t_{m-k-1}, X^h_{m-k-1} \right) dW(s)
=: VI_1 + VI_2 + VI_3 + VI_4 + VI_5.
\]

Taking the norm in both sides of \[3.70\] yields

\[
\| X^h(t_m) - X^h_m \|_{L^2(\Omega, H)}^2 \leq 25 \sum_{i=1}^{5} \| VI_i \|_{L^2(\Omega, H)}^2.
\]

In what follows, we estimate separately \( \| VI_i \|_{L^2(\Omega, H)}, i = 1, \cdots, 5. \)

### 3.2.1. Estimate of \( VI_1, VI_2 \) and \( VI_3 \)

Using Lemma 3.12, it holds that

\[
\| VI_i \|_{L^2(\Omega, H)} \leq \left\| \left( \prod_{j=1}^{m} U_h(t_j, t_{j-1}) - \prod_{j=0}^{m-1} e^{\Delta t A_{h,j}} \right) \right\|_{L(H)} \| X_0 \|_{L^2(\Omega, H)}
\leq C \Delta t^{1-\epsilon}.
\]
Using triangle inequality, Lemma 3.5, Assumption 2.4 and Theorem 2.4, it holds that

\[
\|V I_2\|_{L^2(\Omega, H)} \leq \int_{t_{m-1}}^{t_m} \left\| U_h(t_m, s) P_h F \left( s, X^h(s) \right) \right\|_{L^2(\Omega, H)} ds + \int_{t_{m-1}}^{t_m} \left\| e^{(t_m-s)A_{h,m-1}} \left[ P_h F \left( t_{m-1}, X^h(t_{m-1}) \right) - P_h F \left( t_{m-1}, X^h(t_{m-1}) \right) \right] \right\|_{L^2(\Omega, H)} ds \]

\[
\leq C \int_{t_{m-1}}^{t_m} ds + C \int_{t_{m-1}}^{t_m} \left\| X^h(t_{m-1}) - X^h(t_{m-1}) \right\|_{L^2(\Omega, H)} ds + C \int_{t_{m-1}}^{t_m} ds.
\]

Applying the Itô-isometry property, using Assumption 2.4, 2.5, Theorem 2.4 and Lemma 3.5 yields

\[
\|V I_3\|_{L^2(\Omega, H)}^2 \leq 9 \int_{t_{m-1}}^{t_m} \left\| U_h(t_m, s) P_h B \left( s, X^h(s) \right) \right\|_{L^2(H)}^2 ds + 9 \int_{t_{m-1}}^{t_m} \left\| e^{\Delta t A_{h,m-1}} \left[ P_h B \left( t_{m-1}, X^h(t_{m-1}) \right) - P_h B \left( t_{m-1}, X^h(t_{m-1}) \right) \right] \right\|_{L^2(H)}^2 ds \]

\[
+ 9 \int_{t_{m-1}}^{t_m} \left\| e^{\Delta t A_{h,m-1}} P_h F \left( t_{m-1}, X^h(t_{m-1}) \right) \right\|_{L^2(H)}^2 ds \]

\[
\leq C \int_{t_{m-1}}^{t_m} ds + C \int_{t_{m-1}}^{t_m} \left\| X^h(t_{m-1}) - X^h(t_{m-1}) \right\|_{L^2(\Omega, H)}^2 ds + C \int_{t_{m-1}}^{t_m} ds.
\]

3.2.2. Estimate of $V I_4$. To estimate $V I_4$, we split it in five terms as follows.

\[
V I_4 = \sum_{k=1}^{m-1} \int_{t_{m-k-1}}^{t_{m-k}} \left( \prod_{j=m-k+1}^m U_h(t_j, t_{j-1}) \right) \left[ P_h F \left( s, X^h(s) \right) - P_h F \left( t_{m-k-1}, X^h(t_{m-k-1}) \right) \right] ds
\]

\[
+ \sum_{k=1}^{m-1} \int_{t_{m-k-1}}^{t_{m-k}} \left( \prod_{j=m-k+1}^m U_h(t_j, t_{j-1}) \right) \left[ U_h(t_m, s) - U_h(t_m, t_{m-k-1}) \right] P_h F \left( t_{m-k-1}, X^h(t_{m-k-1}) \right) ds
\]

\[
+ \sum_{k=1}^{m-1} \int_{t_{m-k-1}}^{t_{m-k}} \left( \prod_{j=m-k}^m U_h(t_j, t_{j-1}) \right) \left[ e^{\Delta t A_{h,j}} - e^{\Delta t A_{h,j}} \right] P_h F \left( t_{m-k-1}, X^h(t_{m-k-1}) \right) ds
\]

\[
+ \sum_{k=1}^{m-1} \int_{t_{m-k-1}}^{t_{m-k}} \left( \prod_{j=m-k}^m e^{\Delta t A_{h,j}} \right) e^{\Delta t A_{h,m-k-1}} P_h F \left( t_{m-k-1}, X^h(t_{m-k-1}) \right) - P_h F \left( t_{m-k-1}, X^h(t_{m-k-1}) \right) ds
\]

\[
= V I_{41} + V I_{42} + V I_{43} + V I_{44} + V I_{45}.
\]

(3.81)
Using Lemma 3.5, Assumption 2.5 and Lemma 3.6 yields

\[ \|VI_1\|_{L^2(\Omega,H)} \]
\[ \leq C \sum_{k=1}^{m-1} \int_{t_{m-k-1}}^{t_{m-k}} \left\| P_h F \left( s, X^h(s) \right) - P_h F \left( t_{m-k-1}, X^h(t_{m-k-1}) \right) \right\|_{L^2(\Omega,H)} ds \]
\[ \leq C \sum_{k=1}^{m-1} \int_{t_{m-k-1}}^{t_{m-k}} (s - t_{m-k-1})^{\beta/2} ds + C \sum_{k=1}^{m-1} \int_{t_{m-k-1}}^{t_{m-k}} \| X^h(s) - X^h(t_{m-k-1}) \|_{L^2(\Omega,H)} ds \]
\[ \leq C \Delta t^{\min(\beta,1)/2}. \tag{3.82} \]

Using Lemma 3.5, Assumption 2.5 and Theorem 2.7 gives

\[ \|VI_2\|_{L^2(\Omega,H)} \]
\[ \leq C \sum_{k=1}^{m-1} \int_{t_{m-k-1}}^{t_{m-k}} \| U_h(t_m, t_{m-k}) U_h(t_{m-k}, s)(I - U_h(s, t_{m-k-1})) \|_{L(H)} ds \]
\[ \times \left\| P_h F \left( t_{m-k-1}, X^h(t_{m-k-1}) \right) \right\|_{L^2(\Omega,H)} \]
\[ \leq C \sum_{k=1}^{m-1} \int_{t_{m-k-1}}^{t_{m-k}} \| U_h(t_m, t_{m-k})(-A_{h,m-k})^{1-\epsilon} \|_{L(H)} \| (-A_{h,m-k})^{-1+\epsilon} U_h(t_{m-k}, s)(-A_{h,m-k})^{1-\epsilon} \|_{L(H)} \]
\[ \times \|(-A_{h,m-k})^{-1+\epsilon}(I - U_h(s, t_{m-k-1}))\|_{L(H)} ds \]
\[ \leq C \sum_{k=1}^{m-1} \int_{t_{m-k-1}}^{t_{m-k}} (t_m - t_{m-k})^{-1+\epsilon}(s - t_{m-k-1})^{1-\epsilon} ds \]
\[ \leq C \Delta t^{1-\epsilon} \sum_{k=1}^{m-1} \int_{t_{m-k-1}}^{t_{m-k}} t_k^{-1+\epsilon} ds \]
\[ \leq C \Delta t^{1-\epsilon} \sum_{k=1}^{m-1} \Delta t_k^{-1+\epsilon} \]
\[ \leq C \Delta t^{1-\epsilon}. \tag{3.83} \]
Using Lemma 3.14, Assumption 2.5, Theorem 2.2, (2.30) and (2.31) yields

\[
\|VI_{43}\|_{L^2(\Omega, H)} \leq \sum_{k=1}^{m-1} \int_{t_{m-k-1}}^{t_{m-k}} \left\| \left( \prod_{j=m-k}^{m-1} e^{\Delta t A_{h,j}} \right) \left( e^{(s-t_{m-k-1})A_{h,m-k-1} - 1} \right) e^{(t_{m-k}-s)A_{h,m-k-1}} \right\|_{L(H)} \\
\times \left\| P_h F(t_{m-k-1}, X^h(t_{m-k-1})) \right\|_{L^2(\Omega, H)} ds \\
\leq C \sum_{k=1}^{m-1} \int_{t_{m-k-1}}^{t_{m-k}} \left\| \left( \prod_{j=m-k}^{m-1} e^{\Delta t A_{h,j}} \right) (-A_{h,m-k-1})^{1-\epsilon} \right\|_{L(H)} \\
\times \left\| (-A_{h,m-k-1})^{-1+\epsilon} \left( e^{(s-t_{m-k-1})A_{h,m-k-1} - 1} \right) \right\|_{L(H)} \left\| e^{(t_{m-k}-s)A_{h,m-k-1}} \right\|_{L(H)} ds \\
\leq C \Delta t^{1-\epsilon} \sum_{k=1}^{m-1} \int_{t_{m-k-1}}^{t_{m-k}} t_k^{-1+\epsilon} (s-t_{m-k-1})^{1-\epsilon} ds \\
\leq C \Delta t^{1-\epsilon}. \quad (3.84)
\]

Using Lemma 3.10, (2.30), (2.36), Assumption 2.5 and Lemma 3.16 yields

\[
\|VI_{44}\|_{L^2(\Omega, H)} \leq \sum_{k=1}^{m-1} \int_{t_{m-k-1}}^{t_{m-k}} \left\| \left( \prod_{j=m-k}^{m-1} e^{\Delta t A_{h,j}} \right) \left( 1 - e^{(s-t_{m-k-1})A_{h,m-k-1}} \right) e^{(t_{m-k}-s)A_{h,m-k-1}} \right\|_{L(H)} \\
\times \left\| P_h F(t_{m-k-1}, X^h(t_{m-k-1})) \right\|_{L^2(\Omega, H)} ds \\
\leq C \sum_{k=1}^{m-1} \int_{t_{m-k-1}}^{t_{m-k}} \left\| \left( \prod_{j=m-k}^{m-1} e^{\Delta t A_{h,j}} \right) (-A_{h,m-k})^{1-\epsilon} \right\|_{L(H)} \\
\times \left\| (-A_{h,m-k})^{-1+\epsilon} \left( 1 - e^{(s-t_{m-k-1})A_{h,m-k-1}} \right) \right\|_{L(H)} \left\| e^{(t_{m-k}-s)A_{h,m-k-1}} \right\|_{L(H)} ds \\
\leq C \sum_{k=1}^{m-1} \int_{t_{m-k-1}}^{t_{m-k}} t_k^{-1+\epsilon} (s-t_{m-k-1})^{1-\epsilon} ds \\
\leq C \Delta t^{1-\epsilon}. \quad (3.85)
\]

Using Lemma 3.19 and Assumption 2.5 yields

\[
\|VI_{45}\|_{L^2(\Omega, H)} \leq C \sum_{k=1}^{m-1} \int_{t_{m-k-1}}^{t_{m-k}} \|X^h(t_{m-k-1}) - X^h_{m-k-1}\|_{L^2(\Omega, H)} \\
\leq C \Delta t \sum_{k=0}^{m-1} \|X^h(t_k) - X^h_k\|_{L^2(\Omega, H)}. \quad (3.86)
\]
Substituting \((3.86), (3.85), (3.84), (3.83)\) and \((3.82)\) in \((3.81)\) yields
\[
\|V I_4\|_{L^2(\Omega, H)} \leq C \Delta t^{\min(\beta, 1)/2} + C \Delta t \sum_{k=0}^{m-1} \|X^h(t_k) - X^h_k\|_{L^2(\Omega, H)}.
\] (3.87)

3.2.3. Estimate of \(VI_5\). To estimate \(VI_5\), we split it in four terms as follows

\[
VI_5 = \sum_{k=1}^{m-1} \int_{t_{m-k-1}}^{t_{m-k}} \left( \prod_{j=m-k+1}^{m} U_h(t_j, t_{j-1}) \right) U_h(t_{m-k}, s) \left[ P_h B \left( s, X^h(s) \right) - P_h B \left( t_{m-k-1}, X^h(t_{m-k-1}) \right) \right] dW(s)
\]

\[
+ \sum_{k=1}^{m-1} \int_{t_{m-k-1}}^{t_{m-k}} \left( \prod_{j=m-k+1}^{m} U_h(t_j, t_{j-1}) \right) \left[ U_h(t_{m-k}, s) - U_h(t_{m-k}, t_{m-k-1}) \right] P_h B \left( t_{m-k-1}, X^h(t_{m-k-1}) \right) dW(s)
\]

\[
+ \sum_{k=1}^{m-1} \int_{t_{m-k-1}}^{t_{m-k}} \left( \prod_{j=m-k}^{m} U_h(t_j, t_{j-1}) \right) \left[ \prod_{j=m-k}^{m} e^{\Delta t A_{h,j}} - \prod_{j=m-k-1}^{m} e^{\Delta t A_{h,j}} \right] P_h B \left( t_{m-k-1}, X^h(t_{m-k-1}) \right) dW(s)
\]

\[
+ \sum_{k=1}^{m-1} \int_{t_{m-k-1}}^{t_{m-k}} \left( \prod_{j=m-k}^{m} e^{\Delta t A_{h,j}} \right) \left[ P_h B \left( t_{m-k-1}, X^h(t_{m-k-1}) \right) - P_h B \left( t_{m-k-1}, X^h(t_{m-k-1}) \right) \right] dW(s)
\]

\[=: VI_{51} + VI_{52} + VI_{53} + VI_{54}.\] (3.88)

Using the Itô-isometry property, Lemma 3.5, Assumption 2.6, and Lemma 3.6 yields

\[
\|VI_{51}\|_{L^2(\Omega, H)}^2 = \sum_{k=1}^{m-1} \int_{t_{m-k-1}}^{t_{m-k}} \mathbb{E} \left| U_h(t_m, s) \left[ P_h B \left( s, X^h(s) \right) - P_h B \left( t_{m-k-1}, X^h(t_{m-k-1}) \right) \right] \right|^2 ds
\]

\[
\leq C \sum_{k=1}^{m-1} \int_{t_{m-k-1}}^{t_{m-k}} (s - t_{m-k-1})^\beta ds + C \sum_{k=1}^{m-1} \int_{t_{m-k-1}}^{t_{m-k}} \|X^h(s) - X^h(t_{m-k})\|_{L^2(\Omega, H)}^2 ds
\]

\[
\leq C \Delta t^\beta + C \sum_{k=1}^{m-1} \int_{t_{m-k-1}}^{t_{m-k}} (s - t_{m-k-1})^{\min(\beta, 1)} ds
\]

\[
\leq C \Delta t^{\min(\beta, 1)}.\] (3.89)
Applying the Itô-isometry property, using Lemma 3.5, Assumption 2.6 and Lemma 3.6 yields

\begin{align*}
\|VI_{52}\|_{L^2(\Omega, H)}^2 &= \sum_{k=1}^{m-1} \int_{tm-k-1}^{tm-k} \mathbb{E}\left[ \left( U_h(t_m, t_{m-k}) U_h(t_{m-k}, s) (I - U_h(s, t_{m-k-1})) P_h B(t_{m-k-1}, X^h(t_{m-k-1})) \right)^2 \right] ds \\
&\leq C \sum_{k=1}^{m-1} \int_{tm-k-1}^{tm-k} \mathbb{E}\left[ \left( U_h(t_m, t_{m-k}) (A_{h, m-k}) \right)^2 \right] \| \left( A_{h, m-k} \right) U_h(t_{m-k}, s) (A_{h, m-k}) \|^2_{L^2(\Omega, H)} ds \\
&\leq C \sum_{k=1}^{m-1} \int_{tm-k-1}^{tm-k} (s - t_{m-k-1}) \Delta t^{-\varepsilon} ds \\
&\leq C \Delta t^{1-\varepsilon}.
\end{align*}

(3.90)

Applying the Itô-isometry property, using Lemma 3.12, Assumption 2.6 and Lemma 3.6 yields

\begin{align*}
\|VI_{53}\|_{L^2(\Omega, H)}^2 &= \sum_{k=1}^{m-1} \int_{tm-k-1}^{tm-k} \mathbb{E}\left[ \left( \prod_{j=m-k}^{m-1} U_h(t_j, t_{j-1}) \right) - \left( \prod_{j=m-k}^{m-1} e^{\Delta t A_{h, j}} \right) \right] ds \\
&\leq C \sum_{k=1}^{m-1} \int_{tm-k-1}^{tm-k} \Delta t^{-\varepsilon} ds \\
&\leq C \Delta t^{1-\varepsilon}.
\end{align*}

(3.91)

Applying the Itô-isometry property, Lemma 3.9 and Assumption 2.6 yields

\begin{align*}
\|VI_{54}\|_{L^2(\Omega, H)}^2 &= \sum_{k=1}^{m-1} \int_{tm-k-1}^{tm-k} \mathbb{E}\left[ \left( \prod_{j=m-k}^{m-1} e^{\Delta t A_{h, j}} \right) \right] ds \\
&\leq C \sum_{k=1}^{m-1} \int_{tm-k-1}^{tm-k} \left[ P_h B(t_{m-k-1}, X^h(t_{m-k-1})) - P_h B(t_{m-k-1}, X^h_{m-k-1}) \right] ds \\
&\leq C \Delta t \sum_{k=0}^{m-1} \left\| X^h(t_k) - X^h_{m-k-1} \right\|_{L^2(\Omega, H)}^2.
\end{align*}

(3.92)

Substituting (3.92), (3.91), (3.90) and (3.89) in (3.88) yields

\begin{align*}
\|VI_5\|_{L^2(\Omega, H)}^2 &\leq C \Delta t^{\min(\beta, 1)} + C \Delta t \sum_{k=0}^{m-1} \left\| X^h(t_k) - X^h_{m-k-1} \right\|_{L^2(\Omega, H)}^2.
\end{align*}

(3.93)
Substituting (3.93), (3.87), (3.80), (3.79) and (3.78) in (3.76) yields
\[
\|X^h(t_m) - X^h_m\|_{L^2(\Omega,H)}^2 \leq C\Delta t \min(\beta,1-t) + C\Delta t \sum_{k=0}^{m-1} \|X^h(t_k) - X^h_k\|_{L^2(\Omega,H)}^2.
\]
Applying the discrete Gronwall’s lemma to (3.94) yields
\[
\|X^h(t_m) - X^h_m\|_{L^2(\Omega,H)} \leq C\Delta t \min(\beta,1-t)/2.
\]
Note that to achieve optimal convergence $1/2 \beta \geq 1$, we only need to re-estimate $\|V_{I52}\|_{L^2(\Omega,H)}$ and $\|V_{I53}\|_{L^2(\Omega,H)}$ by using Assumption 2.8 and Lemma 3.12 (ii). This is straightforward. The proof of Theorem 2.9 is therefore completed.

4. Numerical experiments. We consider the following stochastic reactive dom-
inated advection reaction with constant diagonal diffusion tensor
\[
dX = \left[(1 + e^{-t}) (\Delta X - \nabla \cdot (qX)) - \frac{e^{-t} X}{|X| + 1}\right] dt + X \, dW, \quad X(0) = 0,
\]
with mixed Neumann-Dirichlet boundary conditions on $\Lambda = [0,L_1] \times [0,L_2]$. The Dirichlet boundary condition is $X = 1$ at $\Gamma = \{(x,y) : x = 0\}$ and we use the homogeneous Neumann boundary conditions elsewhere. The eigenfunctions $\{e_{i,j}\} = \{e^{(1)}_i \otimes e^{(2)}_j\}_{i,j \geq 0}$ of the covariance operator $Q$ are the same as for the Laplace operator $-\Delta$ with homogeneous boundary condition, given by
\[
e^{(l)}_0(x) = \sqrt{\frac{1}{L^l}}, \quad e^{(l)}_i(x) = \sqrt{\frac{2}{L^l}} \cos \left(\frac{i \pi L_i}{L_i} x\right), \quad i \in \mathbb{N},
\]
where $l \in \{1,2\}$, $x \in \Lambda$. We assume that the noise can be represented as
\[
W(x,t) = \sum_{(i,j) \in \mathbb{Z}^2} \sqrt{\lambda_{i,j}} e_{i,j}(x) \delta_{i,j}(t),
\]
where $\delta_{i,j}(t)$ are independent and identically distributed standard Brownian motions, $\lambda_{i,j}$, $(i,j) \in \mathbb{Z}^2$ are the eigenvalues of $Q$, with
\[
\lambda_{i,j} = (i^2 + j^2)^{-1/2}, \quad \beta > 0,
\]
in the representation (1.2) for some small $\delta > 0$. To obtain trace class noise, it is enough to have $\beta + \delta > 1$. In our simulations, we take $\beta \in \{1.5,2\}$ and $\delta = 0.001$. In (2.25), we take $b(x,u) = 4u$, $x \in \Lambda$ and $u \in \mathbb{R}$. Therefore, from [17] Section 4 it follows that the operators $B$ defined by (2.25) fulfills Assumption 2.8 and Assumption 2.9.

The function $F$ is given by $F(t,v) = -\frac{e^{-t} v}{1 + \|v\|^2}$, $t \in [0,T]$, $v \in H$ and obviously satisfies Assumption 2.5. The nonlinear operator $A(t)$ is given by
\[
A(t) = (1 + e^{-t}) \left(\Delta v - \nabla \cdot v(\cdot)\right), \quad t \in [0,T],
\]
where $v$ is the Darcy velocity. We obtain the Darcy velocity field $v = (q_i)$ by solving the following system
\[
\nabla \cdot v = 0, \quad v = -k \nabla p,
\]
with Dirichlet boundary conditions on $\Gamma_D = \{0, L_1\} \times [0, L_2]$ and Neumann boundary conditions on $\Gamma_N = (0, L_1) \times \{0, L_2\}$ such that

$$p = \begin{cases} 1 & \text{in } \{0\} \times [0, L_2] \\ 0 & \text{in } \{L_1\} \times [0, L_2] \end{cases}$$

and $-k \nabla p(x,t) \cdot n = 0$ in $\Gamma_N$. Here, we use a constant permeability tensor $k$ and have obtained almost a linear pressure $p$. Clearly $D(A(t)) = D(A(0))$, $t \in [0, T]$ and $D((-A(t))^\alpha) = D((-A(0))^\alpha)$, $t \in [0, T]$, $0 \leq \alpha \leq 1$. The function $q_{ij}(x,t)$ defined in (2.23) is given by $q_{ii}(x,t) = 1 + e^{-t}$, and $q_{ij}(x,t) = 0$, $i \neq j$. Since $q_{ii}(x,t)$ is bounded below by $1 + e^{-T}$, it follows that the ellipticity condition (2.24) holds and therefore as a consequence of Section 2.2 it follows that $A(t)$ is sectorial. Obviously Assumption 2.2 is fulfilled.

![Fig. 4.1](image)

**Fig. 4.1.** Convergence of the implicit scheme for $\beta = 1$, and $\beta = 2$ in (4.3). The order of convergence in time is 0.57 for $\beta = 1$, 0.54 for $\beta = 2$. The total number of samples used is 100.

In Figure 4.1 we can observe the convergence of the stochastic Magnus scheme for two noise’s parameters. Indeed the order of convergence in time is 0.57 for $\beta = 1$ and 0.54 for $\beta = 2$. These orders are close to the theoretical orders 0.5 obtained in Theorem 2.9 for $\beta = 1$ and $\beta = 2$.

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