Multipliers with inverse square potential and applications I

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Abstract

In this work we give explicit formulas for the Schwartz integral kernels of some multipliers of the Schrödinger operator with inverse square potential on $\mathbb{R}^*_+$. By using the integral transforms connecting these multipliers we obtain old and new formulas involving Bessel and hypergeometric functions.

Key words: Multipliers, Inverse Square Potential, Weighted heat kernel, Weighted resolvent kernels, Bessel functions, Kampe de Feriet generalized hypergeometric functions.

1 Introduction

The inverse square potential is an interesting potential which arises in several contexts, one of them being the Schrödinger equation in non relativistic quantum mechanics [13]. For example the Hamiltonian for a spin zero particle in Coulomb field gives rise to a Schrödinger operator involving the inverse square potential [1]. For the recent papers on the Schrödinger operator with inverse square potential see([1] [8] [9] [15]). The aim of this paper is twofold: first we give explicit formulas for the Schwartz integral kernels of the following multipliers of the Schrödinger operator with inverse square potential: $H^p_{\nu}(t) := e^{tL_{\nu}}(\sqrt{-L_{\nu}})^p$, $R^p_{\nu}(\lambda) := (L_{\nu} + \lambda^2)^{-1}(\sqrt{-L_{\nu}})^p$, $R^p_{\nu,\mu}(\lambda) := (L_{\nu} + \lambda^2)^{-1-\mu}(\sqrt{-L_{\nu}})^p$ and secondly, by using the integral transforms connecting these multipliers we obtain old and new formulas involving Bessel and hypergeometric functions. Explicit formulas for free multipliers on the
Euclidian space $\mathbb{R}^n$ (see \[11\])

First of all we recall the following formulas for the classical heat ([1], p. 68) kernel with inverse square potential

$$H_0^\nu(t, x, x') = \frac{(xx')^{1/2}}{2t}e^{-\frac{(x^2+x'^2)}{4t}} I_\nu(\frac{xx'}{2t})$$ (1.1)

and the classical resolvent kernel \[9\]

$$R_0^\nu(\lambda, x, x') = \frac{i\pi}{2}\sqrt{xx'} \left\{ \begin{array}{ll} J_\nu(\lambda x)H_\nu^{(1)}(\lambda x') & x < x' \\ J_\nu(\lambda x')H_\nu^{(1)}(\lambda x) & x > x' \end{array} \right. \quad (1.2)$$

Using the fact that the resolvent kernel is the Laplace transform of the heat kernel we have for $\Re e \lambda^2 < 0$ and $\nu > -1$

$$\int_0^\infty e^{\lambda^2 t^2}e^{-\frac{(x^2+x'^2)}{4t}} I_\nu(\frac{xx'}{2t})dt = i\pi \left\{ \begin{array}{ll} J_\nu(\lambda x)H_\nu^{(1)}(\lambda x') & x < x' \\ J_\nu(\lambda x')H_\nu^{(1)}(\lambda x) & x > x' \end{array} \right. \quad (1.3)$$

where $I_\nu$ is the first kind modified Bessel function, $J_\nu$ and $H_\nu^{(1)}$ are respectively the first and the third kind Bessel functions (see [5, 10]).

We mention that the absolute convergence of the above integral is assured by the formulas ([10], p. 136)

$$I_\nu(x) \approx \frac{x^\nu}{2^\nu \Gamma(1+\nu)}, x \to 0; I_\nu(x) \approx \frac{e^x}{\sqrt{2\pi x}}, x \to \infty \quad (1.4)$$

The end of this section is devoted to the preliminaries on the Hankel transform on $\mathbb{R}^+$. 

For $\nu > -1$ the Hankel transform of order $\nu$ for a function $f \in C^\infty_0(\mathbb{R}^+)$ is defined by the integral

$$(H_\nu f)(\omega) = \int_0^\infty (x\omega)^{1/2}J_\nu(x\omega)f(x)dx \quad (1.5)$$

where $J_\nu$ is the first order Bessel function of order $\nu$.

**Proposition 1.1.** [12] For $\nu > -1$, we have

i) $H_\nu^2 = 1$

ii) $H_\nu$ is self adjoint

iii) $H_\nu$ is an $L^2$ isometry

iv) $H_\nu L_\nu = -\omega^2 H_\nu$
For more informations on the Hankel transform the reader can consults
the nice book by Davies [2].
Note that we can define $\phi(\sqrt{-L^\nu})$ for $\phi$ a well behaved Borel function by
using the Hankel transform:

**Proposition 1.2.** For $\nu > -1$, the Schwartz kernel of the operator $\phi(\sqrt{-L^\nu})$
is given at last formally by

$$K_\nu(\phi, x, x') = (xx')^{1/2} \int_0^\infty J_\nu(\omega x)J_\nu(\omega x')\phi(\omega)d\omega$$

(1.6)

The proof of this proposition uses essentially the proposition 1. 1 and in
consequence is left to the reader.

Note that using (1.6) with $\phi(x) = \frac{1}{-\omega^2 + \lambda^2}$ we obtain for
$\text{Re}\lambda^2 < 0$ and $\nu > -1$ the following formula

$$\int_0^\infty J_\nu(\omega x)J_\nu(\omega x')\frac{-\omega^2 + \lambda^2}{\omega}d\omega = \frac{i\pi}{2} \left\{ \begin{array}{ll}
J_\nu(\lambda x)H_\nu^{(1)}(\lambda x') & x < x' \\
J_\nu(\lambda x')H_\nu^{(1)}(\lambda x) & x > x'
\end{array} \right.$$  

(1.7)

and the absolute convergence of the above integral is assured by the formulas
(10), p.134)

$$J_\nu(x) \approx \frac{x^\nu}{2^\nu\Gamma(1 + \nu)}, x \to 0; J_\nu(x) \approx \sqrt{\frac{2}{\pi x}}, x \to \infty.$$  

(1.8)

The following lemma give the Laplace transform of the two variables Humbert
confuent hypergeometric function ([5], p.225):

$$\Psi_2(a; c, c', x, y) = \sum_{n,m \geq 0} \frac{(a)n+m}{(c)m(c')n}x^m y^n, |x| < \infty, |y| < \infty$$

(1.9)

in term of the Kampé de Feriét generalized hypergeometric function $F_{A:B}^{C:D}$
given by ([7], p.29). $A + B < C + D + 1$ and $|x| < \infty, |x'| < \infty$:

$$F_{C:D}^{A:B} \left( \begin{array}{c}
a_1, \ldots, a_A; b_1, \ldots, b_B; c_1, \ldots, c_C; d_1, \ldots, d_D; x, x'
\end{array} \right) =$$

$$\sum_{n,m \geq 0} \frac{\prod_{j=1}^A(a_j)m+n}{\prod_{j=1}^C(c_j)m+n} \frac{\prod_{j=1}^B(b_j)_n}{\prod_{j=1}^D(d_j)_n} x^m y^n$$

(1.10)

**Lemma 1.1.** For $\gamma > 0$, $\alpha > 0$, $X, Y \in \mathbb{R}$ we have

$$\int_0^\infty e^{-\gamma t}\nu^{\alpha-1}\Psi_2 \left( a, b_1, b_2, \frac{X}{t}, \frac{Y}{t} \right) dt =$$

$$\frac{\Gamma(\alpha)}{\gamma^\alpha} F_{1:1}^{1:0} \left( \begin{array}{c}
a, -a, b_1, b_2; -\gamma X, -\gamma Y
\end{array} \right)$$

(1.11)
Proof. Replacing the confluent hypergeometric $Ψ_2$ by its series (1.9) in the integral (1.11) and integrating term by term we obtain

$$\int_0^\infty e^{-γt}t^{α-1}Ψ_2\left(a, b_1, b_2, \frac{X}{t}, \frac{Y}{t}\right) dt = γ^{-α} \sum_{n,m≥0} \frac{(a)_{n+m}Γ(α - n - m)}{(b_1)_m(b_2)_n m! n!} (γx)^n (γy)^n$$

using the formula ([14], p. 22) $Γ(α - n)Γ(α) = (-1)^n (1 - α)^n n!$ we can write

$$\int_0^\infty e^{-γt}t^{α-1}Ψ_2\left(a, b_1, b_2, \frac{X}{t}, \frac{Y}{t}\right) dt = \frac{Γ(α)}{γ^α} \sum_{n,m≥0} \frac{(a)_{n+m}}{(1 - α)_{n+m}(b_1)_m(b_2)_n m! n!} (−γx)^n (−γy)^n$$

which gives the result in (1.11) and the proof of the lemma is finished.

The organization of the remaining of the paper is as follows, the Schwartz integral kernel of the weighted heat evolution operator $e^{tL_ν(−L_ν)^p}$ and the weighted Schrödinger evolution operator $e^{itL_ν(−L_ν)^p}$ will be given in section 2. In section 3 we will obtain a closed form of the Schwartz integral kernel of the weighted resolvent operator. The section 4 is devoted to the Schwartz integral kernel of the weighted generalized resolvent operator with inverse square potential on $R^+$.

## 2 Weighted Heat evolution operator with inverse square potential on $R^+$

In this section explicit formulas for the Schwartz integral kernels of the weighted heat and Schrödinger evolution operators $e^{tL_ν(−L_ν)^p}$ and $e^{itL_ν(−L_ν)^p}$ are given in explicit forms.

**Theorem 2.1.** For $Re p > −2(ν + 1)$ and $ν > −1$ The Schwartz integral kernel $H^p_ν(t, x, x')$ of the weighted heat evolution operator with inverse square potential $e^{tL_ν(−L_ν)^p}$ is given as

$$H^p_ν(t, x, x') = \frac{Γ(p/2 + 1 + ν)(x/2)^{ν+1/2}(x'/2)^{ν+1/2}}{[Γ(ν + 1)]^{2p/2+ν+1}} \times \Psi_2\left(p/2 + 1 + ν, ν + 1, ν + 1; x^2/4t, x'^2/4t\right)$$

where $Ψ_2(a, c; c'; x; y)$ denotes the Humbert’s confluent hypergeometric function of two variables given in (1.9).
Proof. Using the formula (1.6) with $\phi(\omega) = e^{-t \omega^2}$ we have

$$H_p^p(t, x, x') = (xx')^{1/2} \int_0^\infty J_\nu(\omega x)J_\nu(\omega x')e^{-t \omega^2} \omega^{p+1} d\omega$$

(2.2)

next we employ the formula ([6] p.187)

$$\int_0^\infty e^{-pt} t^{\nu-1} J_{2\mu_1} (2(a_1 t)^{1/2})J_{2\mu_2} (2(a_2 t)^{1/2}) dt =$$

$$\frac{\Gamma(\nu + M)}{\Gamma(2\mu_1 + 1)\Gamma(2\mu_2 + 1)} p^{-\nu-M} a_1^{\mu_1} a_2^{\mu_2} \Psi_2 (\nu + M, 2\mu_1 + 1, 2\mu_2 + 1, a_1/p, a_2/p)$$

(2.3)

where $M = \mu_1 + \mu_2$, $\Re(\nu + M) > 0$, and $\Psi$ is the two variables confluent hypergeometric function and we arrive at the formula (2.1).

Corollary 2.1. The Schwartz integral kernel $K_p^p(t, x, x')$ of the weighted Schrödinger evolution operator with inverse square potential $e^{itL_\nu(\sqrt{-L_\nu})^p}$ is given in terms of the two variables Humbert’s confluent hypergeometric function for $\Re p > -2(\nu + 1)$ and $\nu > -1$ as

$$K_p^p(t, x, x') = \frac{\Gamma(p/2 + 1 + \nu)(x/2)^{\nu+1/2}(x'/2)^{\nu+1/2}}{\Gamma(\nu + 1)^2 (it)^{p/2 + \nu + 1}} \Psi_2 (p/2 + 1 + \nu, \nu + 1; \nu + 1; x^2/4t, x'^2/4t)$$

(2.4)

By taking $p = 0$ in the theorem 2.1 we have

$$H_0^0(t, x, x') = \frac{(x/2)^{\nu+1/2}(x'/2)^{\nu+1/2}}{\Gamma(\nu + 1)^{\nu+1}} \Psi_2 (\nu + 1, \nu + 1; \nu + 1; x^2/4t, x'^2/4t)$$

(2.5)

and by comparing this with (1.2) we have

$$\Psi_2 (\nu + 1, \nu + 1; \nu + 1; x, y) = \Gamma(\nu + 1)(xy)^{-\nu/2} e^{-x-y} I_{\nu}(2\sqrt{xy})$$

(2.6)

3 Weighted resolvent operator

In this section we give explicit formula for the Schwartz integral kernel of the weighted resolvent operator $R_p^p(\lambda) = (L_\nu + \lambda^2)^{-1} (-L_\nu)^{p/2}$.

We show first that one can write the weighted resolvent in terms of the weighted heat kernel.
Proposition 3.1. Let $R_p^\nu(\lambda, x, x')$ be the Schwartz kernel of the weighted resolvent operator then we have

$$R_p^\nu(\lambda, x, x') = \int_0^\infty e^{\lambda^2 t} H_p^\nu(t, x, x') dt; \quad \Re \lambda^2 < 0. \quad (3.1)$$

where $H_p^\nu(t, x, x')$ is the Schwartz integral kernels of the weighted heat operator.

Proof. The Formula (3.1) is a consequence of the formula $(a^2 + y^2)^{-1} = \int_0^\infty e^{-(a^2+y^2)t} dt$ valid for $\Re a^2 > 0$. \qed

Theorem 3.1. For $\Re \lambda^2 < 0$, $-1 < p/2 + \nu < 0$ and $\nu > -1$ the Schwartz integral kernel for the weighted resolvent operator $(L_\nu + \lambda^2)^{-1} (-L_\nu)^{p/2}$ is given by

$$R_p^\nu(\lambda, x, x') = \frac{\Gamma(p/2+\nu+1)\Gamma(-p/2-\nu)}{\Gamma(\nu+1)^2} (-\lambda^2)^{p/2+\nu} (xx'/4)^{\nu+1/2} F^{0:1}_{0:0} \left( \begin{array}{c} : -\nu+1; \nu+1; \lambda^2 x^2 / 4, \lambda^2 x'^2 / 4 \end{array} \right) \quad (3.2)$$

where $F^{0:1}_{0:0}$ is the Kampe de Feriét hypergeometric function given in (1.10)

The proof of this theorem can be seen as a direct application of the proposition 3.1 theorem 2.1 and of the Lemma 1.1.

Corollary 3.1. For $\Re \lambda^2 < 0$, $-1 < p/2 + \nu < 0$ and $\nu > -1$

$$\int_0^\infty J_\nu(\omega x) J_\nu(\omega x') (-\omega^2 + \lambda^2)^{p+1} d\omega = \frac{\Gamma(p/2+\nu+1)\Gamma(-p/2-\nu)}{2\Gamma(\nu+1)^2} (-\lambda^2)^{p/2+\nu} (xx'/4)^{\nu} F^{0:0}_{0:1} \left( \begin{array}{c} : -\nu+1; \nu+1; \lambda^2 x^2 / 4, \lambda^2 x'^2 / 4 \end{array} \right) \quad (3.3)$$

Proof. Using the proposition 1.2 with $\phi(\omega) = (-\omega^2 + \lambda^2)^{-1} \omega^p$ and the theorem 3.1, where the absolute convergence of the above integral is assured by the formulas (1.8) \qed

By taking $p = 0$ in (3.2) and comparing with (1.2) the following formula is valid for $x < x'$.

$$J_\nu(\lambda x) H^{(1)}_\nu(\lambda x') = \frac{\Gamma(-\nu) (-\lambda^2 xx'/4)^{-\nu}}{i\pi \Gamma(\nu+1)} F^{0:0}_{0:1} \left( \begin{array}{c} : -\nu+1; \nu+1; \lambda^2 x^2 / 4, \lambda^2 x'^2 / 4 \end{array} \right) \quad (3.4)$$

where $F^{A:B}_{C:D}$ is the Kampé de Feriét generalized hypergeometric function given by (1.10)
4 Weighted generalized resolvent operator

In this section we generalize some results of the section 3 by giving an explicit expression of the weighted generalized resolvent kernels $R_{\nu}^{\mu,p}(\lambda) = (L_{\nu} + \lambda^2)^{-1-\mu} (-L_{\nu})^{p/2}$.

Proposition 4.1. We have the following formula connecting the weighted generalized resolvent kernel to the weighted heat kernel

$$R_{\nu}^{\mu,p}(\lambda, x, x') = \frac{1}{\Gamma(\mu + 1)} \int_0^\infty e^{\lambda^2 t^2} H_\nu^p(t, x, x') dt; \quad \Re \lambda^2 < 0.$$  

Proof. We use the formula $(a^2+y^2)^{-1-\mu} = \frac{1}{\Gamma(\mu + 1)} \int_0^\infty e^{-(a^2+y^2)t^2} t^\mu dt$ for $\Re a^2 > 0$.

Theorem 4.1. For $\Re \lambda^2 < 0$, $-1 < p/2 + \nu < \mu$, $\nu > -1$ and $\mu > -1$, the Schwartz integral kernel of the weighted generalized resolvent kernel with inverse square potential is given by

$$R_{\nu}^{\mu,p}(\lambda, x, x') = \frac{\Gamma(\mu - p/2 - \nu)}{\Gamma(\mu + 1)|\Gamma(\nu + 1)|^2}$$

$$(-\lambda^2)^{p/2+\nu-\mu} (xx'/4)^{\nu+1/2} F_{1;1} 1:1 \left( \begin{array}{c} p/2+\nu+1; -1 - \mu/2+\nu+1; \nu+1; x^2; \lambda^2 (xx'/4) \end{array}; -\lambda^2 (xx'/4) \right)$$

where $F_{A:B}^{C:D}$ is the Kampe de Feriét generalized hypergeometric function given by (1.10).

Proof. This theorem is a direct consequence of the proposition 4.1 theorem 2.1 and of the Lemma 1.1.

Corollary 4.1. For $\Re \lambda^2 < 0$, $-1 < p/2 + \nu < \mu$, $\nu > -1$ and $\mu > -1$, we have the following formula

$$\int_0^\infty J_\nu(x\omega) J_\nu(x'\omega) \omega^{p+1} d\omega = \frac{\Gamma(\mu - p/2 - \nu)}{\Gamma(\mu + 1)|\Gamma(\nu + 1)|^2}$$

$$(-\lambda^2)^{p/2+\nu-\mu} (xx'/4)^{\nu+1/2} F_{1;1} 1:1 \left( \begin{array}{c} p/2+\nu+1; -1 - \mu/2+\nu+1; \nu+1; x^2; \lambda^2 (xx'/4) \end{array}; -\lambda^2 (xx'/4) \right)$$

Proof. Using the proposition 1.2 with $\phi(\omega) = (-\omega^2 + \lambda^2)^{-1-\mu} \omega^p$ and the theorem 4.1, and the absolute convergence of the integral is assured by the formulas (1.8)

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By taking \( p = 0 \) in the formula (4.2), we see that the Schwartz integral kernel of generalized resolvent with inverse square potential is given by

\[
R_{\nu}^{\mu,0}(\lambda, x, x') = \frac{\Gamma(\mu - \nu)}{\Gamma(\mu + 1)[\Gamma(\nu + 1)]} (-\lambda^2)^{\nu - \mu} (x x'/4)^{\nu + 1/2} \frac{\chi^2 x^2}{4}, \frac{\chi^2 x'^2}{4}
\]

(4.4)

where \( F_{A:B}^{C:D} \) is the Kampe de Feriét generalized hypergeometric function given by (1.10) and \( \Re \lambda^2 < 0, \nu > -1 \) and \( \mu > -1 \) and \( \mu > \nu \).

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