Complexity Reduction for Systems of Interacting Orientable Agents: Beyond The Kuramoto Model

Sarthak Chandra,1,2 Michelle Girvan,1,3 and Edward Ott1,2,4

1Department of Physics, University of Maryland, College Park, MD, U.S.A.
2Institute for Research in Electronics and Applied Physics, University of Maryland, College Park, MD, U.S.A.
3Institute for Physical Science and Technology, University of Maryland, College Park, MD, U.S.A.
4Department of Electrical and Computer Engineering, University of Maryland, College Park, MD, U.S.A.

Previous results have shown that a large class of complex systems consisting of many interacting heterogeneous phase oscillators exhibit an attracting invariant manifold. This result has enabled reduced analytic system descriptions from which all the long term dynamics of these systems can be calculated. Although very useful, these previous results are limited by the restriction that the individual interacting system components have one-dimensional dynamics, with states described by a single, scalar, angle-like variable (e.g., the Kuramoto model). In this paper we consider a generalization to an appropriate class of coupled agents with higher-dimensional dynamics. For this generalized class of model systems we demonstrate that the dynamics again contain an invariant manifold, hence enabling previously inaccessible analysis and improved numerical study, allowing a similar simplified description of these systems. We also discuss examples illustrating the potential utility of our results for a wide range of interesting situations.

Models of systems of many coupled dynamical agents are useful tools for studying a very wide variety of phenomena1. Examples include flashing fireflies2,3, circadian rhythms of mammals4,5, oscillating neutrinos6, arrays of Josephson junctions7, oscillation of footbridges8, biochemical oscillators9,10, power-grids11,12, collections of neurons13–16, flocking dynamics17–20 and others. In many cases the states of the individual agents can be described by a single angle-like variable, \( \theta \). This class of model systems includes situations for which the dynamical agents are oscillators1, neurons13–16 or robots moving on a two-dimensional plane18, among others. Many such models, such as the Kuramoto model21,22, the Kuramoto–Sakaguchi model23,24, models of theta neurons13–16, among others25, reduce to the form

\[
\dot{\theta}_i = \omega(\eta_i, \{\theta\}, t) + \frac{1}{2\epsilon} [H(\eta_i, \{\theta\}, t)e^{-i\theta_i} - H^*(\eta_i, \{\theta\}, t)e^{i\theta_i}],
\]

where \( \theta_i \) represents the state of the \( i \)-th agent, \( \eta_i \) is a (possibly vector) constant parameter that is associated with the \( i \)-th agent, \( \omega(\eta_i, \{\theta\}, t) \) is its “natural frequency”, \( N \) is the total number of agents, and \( H(\eta_i, \{\theta\}, t) \) is a common field that acts on each agent, dependent on the agent’s parameter \( \eta_i \) and \( \{\theta\} \) indicates a dependence on the set of states \( \{\theta_1, \ldots, \theta_N\} \) in the form of the average over \( i \) of a function of the angle \( \theta_i \).

For example, the well-studied Kuramoto model21,26 can be expressed in the form of Eq. \( \text{(1)} \) by choosing \( H(\eta_i, \{\theta\}, t) = N^{-1} \sum_j \exp(i\theta_j) \), independent of \( \eta_i \), and choosing \( \omega(\eta_i, \{\theta\}, t) \) to independent of \( \{\theta\} \) and \( t \), allowing \( \omega(\eta_i, \{\theta\}, t) \) to be replaced by \( \omega_i \). Reference 27 introduced an ansatz to analytically achieve substantial reductions in the complexity of problems of the type exemplified by Eq. \( \text{(1)} \) in the limit of a large number of agents \( (N \rightarrow \infty) \). Subsequently, this reduction has been applied in studies of a wide variety of systems (e.g., Refs. 28,29).

Several flocking models employ the Kuramoto model (e.g., Refs. 17–19) to describe orientational alignment of the velocities of individuals in a flock. Since the standard Kuramoto model describes the dynamics of scalar angles, these models are restricted to describing flock dynamics in a two-dimensional plane. Other work has shown that the Kuramoto model can be generalized to flocks moving in three and higher-dimensional space30,31. In this case each agent’s state is assumed to be specified by a unit vector \( \mathbf{\sigma}_i(t) \) in the D-dimensional space. Alternately we may think of \( \mathbf{\sigma}_i \) as specifying a point on the unit sphere in D-dimensional space. Reference 30 notes that the vector \( \mathbf{\sigma}_i \) can be thought of as representing the opinion of an individual in a group, or the orientation of the velocity of a member of a flock. (For the case of flocking of birds, fish or flying drones, the generalization to \( D = 3 \) is of most interest.) For \( D = 2 \), the unit vector \( \mathbf{\sigma}_i \) is determined by its scalar orientation angle \( \theta \), specifying a point on the unit circle. References 33,34 have also studied the Kuramoto model and its generalizations to higher dimensions in the contexts of continuous-time consensus protocols, multi-agent rendezvous, distributed control, and coalition formation. In this paper we present a new technique that enables analytic treatment of the dynamics of a large class of systems with higher-dimensional agents, including the aforementioned systems.

In particular, we consider the generalization of Eq. \( \text{(1)} \) to arbitrary dimensions, extend the ansatz of Ref. 27 to analyze the dynamics of this generalized class of systems, and discuss the utility of our extended ansatz.

In a recent paper37, we constructed a generalization of the Kuramoto model to D dimensions. Here we consider an even more general setup, where we consider a
generalization to Eq. (1) to a system in D dimensions,

\[ \sigma_i = [\rho(\eta_i, \{\sigma\}, t) - (\sigma_i, \rho(\eta_i, \{\sigma\}, t)) \sigma_i] + W(\eta_i, \{\sigma\}, t) \sigma_i, \]

where for each i, \( \sigma_i(t) \) is a real D-dimensional unit vector, \( |\sigma_i(0)| = 1 \), \( \rho(\eta_i, \{\sigma\}, t) \) is an arbitrary real D-dimensional vector, which can be thought of as a common field that affects each agent in an \( \eta \) dependent fashion. \( W(\eta_i, \{\sigma\}, t) \) is a real \( D \times D \) antisymmetric matrix, \( \eta \) is a (possibly vector) constant parameter associated with each agent, and, as earlier, \( \{\sigma\} \) indicates a dependence on the set of all states \( \{\sigma_1, \ldots, \sigma_N\} \) in the form of the average over i of a function of the unit vectors \( \sigma_i \), (we further quantify this dependence on \( \{\sigma\} \) later). For example, in the context of flocking agents in D dimensions, \( \sigma_i \) represents the orientation of the \( i \)th agent, \( \rho(\eta_i, \{\sigma\}, t) \) represents a ‘goal’ orientation to which the \( i \)th agent attempts to align itself, and \( W(\eta_i, \{\sigma\}, t) \) represents a fixed bias, or a systematic error to the agent dynamics causing the agent to head in a direction that deviates from the direction of \( \rho(\eta_i, \{\sigma\}, t) \). Note from the form of Eq. (2) that \( \sigma_i \cdot \sigma_i = 0 \), so that \( d|\sigma_i|/dt = 0 \), as required by our identification of \( \sigma \) as a unit vector. Thus the dynamics of each \( \sigma_i \) is restricted to the \( (D - 1) \)-dimensional surface, \( S \), of the unit sphere, \( |\sigma| = 1 \). For \( D = 2 \), choosing \( \sigma_i = (\cos \theta_i, \sin \theta_i)^T \), \( \rho(\eta_i, \{\sigma\}, t) = (\Re[H(\eta_i, \{\theta\}, t)], \Im[H(\eta_i, \{\theta\}, t)])^T \) and

\[ W(\eta_i, \{\sigma\}, t) = \begin{pmatrix} 0 & \omega(\eta_i, \{\theta\}, t) \\ -\omega(\eta_i, \{\theta\}, t) & 0 \end{pmatrix}, \]

reduces Eq. (2) to Eq. (1), thus justifying Eq. (2) as a D-dimensional generalization of Eq. (1).

We now consider the limit of a large number of agents, and denote by \( F(\sigma, \eta, t) \) the distribution of agents on \( S \), such that \( F(\sigma, \eta, t) d^{D-1} \sigma \) is the fraction of agents that lie in the \( (D - 1) \)-dimensional differential element \( d^{D-1} \sigma \) on the surface \( S \) centered at \( \sigma \) at time \( t \), and have an associated parameter \( \eta \) within the differential element \( d\eta \) centered at \( \eta \). Since the associated parameter \( \eta \) for each agent is time independent, we define

\[ g(\eta) = \int_S F(\sigma, \eta, t) d^{D-1} \sigma, \]

and \( f(\sigma, \eta, t) = F(\sigma, \eta, t)/g(\eta) \).

Noting that Eq. (2) specifies the vector field of the flow controlling the dynamics of the distribution \( f \), we write a continuity equation for \( f \),

\[ \partial f(\sigma, \eta, t)/\partial t + \nabla_S \cdot (f(\sigma, \eta, t) v(\sigma, \eta, t)) = 0, \]

where the velocity field \( v(\sigma, \eta, t) \) is given by \( v(\sigma, \eta, t) = (\rho(\eta, t) - (\sigma, \rho(\eta, t)) \sigma) + W(\eta, \sigma) \), and \( \nabla_S \) represents the divergence operator on \( S \). This can be done if the dependence of \( \rho \) and \( W \) on \( \{\sigma\} \) can be specified as a functional of \( F(\sigma, \eta, t) \) that is not explicitly dependent on \( \sigma \). (e.g., the average value of the \( \sigma_i \), which can be written as \( \int f \sigma F(\sigma, \eta, t) d\sigma d\eta \).) Following Appendix B of Ref. [37], Eq. (2) can be rewritten as

\[ \partial f(\sigma, \eta, t)/\partial t + [\nabla_S f(\sigma, \eta, t) - (D - 1)f(\sigma, \eta, t) \sigma] \cdot \rho(\eta, t) + (W(\eta, t) \sigma) \cdot \nabla_S f(\sigma, \eta, t) = 0, \]

where \( \nabla_S \) is the gradient operator on \( S \).

For \( D = 2 \), Refs. [25, 27] demonstrated that the ansatz that \( f(\theta, t) \) is in the form

\[ f(\theta, \eta, t) = \frac{1}{2\pi} \frac{1 - |\alpha(\eta, t)|^2}{|e^{i\theta} - \alpha(\eta, t)|^2}, \]

where \( \alpha(\eta, t) \) is a complex scalar function of \( \eta \) and \( t \), \( |\alpha(\eta, 0)| < 1 \), reduces Eq. (1) to the following \( \theta \)-independent form

\[ \partial \alpha/\partial t + i\alpha + \frac{1}{2} (H^*(\eta, t) \alpha^2(\eta, t) - H(\eta, t)) = 0. \]

The form Eq. (5) represents an invariant manifold in the space of possible distributions \( f \), that satisfy the continuity equation Eq. (3) for \( D = 2 \). Furthermore, previous work [25, 28] has shown that initial conditions for \( f \) are attracted to the invariant manifold Eq. (5) for a large class of possible models of the form Eq. (1). Thus Eq. (5) can be used to greatly simplify the study of the long-term dynamics of these systems.

Here we present an ansatz demonstrating the existence of a similar invariant manifold for Eq. (2) in any dimension \( D \). Noting that \( e^{i\theta} \) can be interpreted as a unit vector in the complex plane and that \( \alpha \) can similarly be interpreted as a two-dimensional vector of its real and imaginary parts, based on Eq. (5) we posit the following guess for the form of \( f(\sigma, \eta, t) \) for arbitrary dimension \( D \),

\[ f(\sigma, \eta, t) = \frac{N_\alpha(\alpha(\eta, t))}{|\sigma - \alpha(\eta, t)|^{2\beta_D}}, \]

where \( \alpha \) is a real \( D \)-dimensional vector such that \( |\alpha(\eta, 0)| < 1 \), \( \beta_D \) is a yet-to-be-determined constant, and \( N_\alpha(\alpha) \) is a scalar normalization chosen to ensure that

\[ \int_S f(\sigma, \eta, t) d^{D-1} \sigma = 1. \]

Inserting Eq. (7) into the continuity equation in Eq. (4), we obtain after some algebra,

\[ (1 + |\alpha|^2 - 2\alpha \cdot \sigma) \partial_t N_\alpha(\alpha) - \beta_D N_\alpha(\alpha) (\alpha \cdot \partial_\alpha - \partial_\sigma) \]

\[ + N_\alpha(\alpha) (\beta_D (\alpha \cdot \rho) + [2(D - 1) - \beta_D] (\alpha \cdot \sigma) (\rho \cdot \sigma) - (D - 1) (\rho \cdot \sigma) (1 + |\alpha|^2) - \beta_D (\alpha \cdot \sigma) (\rho \cdot \sigma), \]

(9)

For our ansatz Eq. (7) to apply, the above equation must hold for all \( \sigma \). Focusing on the term in Eq. (9) that is quadratic in \( \sigma \), i.e., \( N_\alpha(\alpha) [2(D - 1) - \beta_D] (\alpha \cdot \sigma) (\rho \cdot \sigma), \)
since in general \(\alpha\) and \(\rho\) will not be zero for all \(t\), we require that
\[
\beta_D = 2(D - 1).
\] (10)

With \(\beta_D\) in Eq. (7) determined, we use Eq. (8) to calculate the normalization constant \(N_D(\alpha)\), resulting in the form of the ansatz for arbitrary dimensions as
\[
f(\sigma, \eta, t) = K_D \left(1 - |\alpha(\eta, t)|^2\right)^{D-1},
\] (11)
which, for \(D = 2\), agrees with Eq. (9).

To determine whether the ansatz Eq. (11), is consistent with Eq. (8) we insert it into Eq. (9). We find that the ansatz Eq. (9) with \(\beta_D\) given by Eq. (10) indeed is a solution of Eq. (9) and that Eq. (9) reduces to the following equation for \(\alpha\),
\[
\partial_t \alpha = \frac{1}{2} \left(1 + |\alpha|^2\right) \rho - (\rho \cdot \alpha) \alpha + W \alpha.
\] (12)

The key point is that Eq. (12) does not involve \(\sigma\) (and remarkably, also does not involve any dependence on \(D\)). Thus, analogously to Eq. (10), we have a \(\sigma\)-independent description of the dynamics of \(\alpha\). This is our main result.

We now consider a few examples illustrating the utility of the generalized ansatz, Eq. (11), to systems of the form given in Eq. (2). We detail the particular example of the Kuramoto model generalized to \(D\) dimensions \([37]\), as representative of the utility of our main result Eq. (12), and thereafter briefly mention applications of this result to a variety of other systems.

A generalization of the Kuramoto model with homogeneous oscillators to arbitrary dimension was introduced by Olfati-Saber in 2006 \([30]\) in the context of flocking dynamics, consensus protocols, and opinion dynamics. This was later generalized to heterogeneous systems by Chandra et al. \([37]\). For generalization to \(D\) dimensions, a system order parameter, \(z\), can be defined as
\[
z(t) = \frac{1}{N} \sum_i \sigma_i(t).
\] (13)

The magnitude of \(z(t)\) is a measure of the coherence of the set of the agents \(\{\sigma\}\). The common field \(\rho\) is then defined as the \(\eta_i\)-independent function,
\[
\rho(\eta, \{\sigma\}, t) = K z(t) = (K/N) \sum_i \sigma_i(t),
\] (14)
where \(K\) is a coupling constant. By interpreting the vector parameters \(\eta_i\) in \(W(\eta_i, \{\sigma\}, t)\) as the \(D(D - 1)/2\) independent elements of a \(D\)-dimensional antisymmetric matrix \(W_i\), we can replace \(g(\eta)d\eta\) in integrals with \(G(W)dW\) where \(G(W)\) is a distribution of antisymmetric matrices. In cases such as these where \(W(\eta_i, \{\sigma\}, t)\) is independent of \(\{\sigma\}\) and \(t\), we interpret \(W(\eta_i) = W_i\) as the “natural rotation” of \(\sigma_i\).

In the limit \(N \to \infty\), with a distribution of agents given according to Eq. (14),
\[
z(t) = \int S F(\sigma, W, t)\sigma d^{D-1}\sigma dW,
\] (15)
\[
= \int dW G(W)\alpha(W, t)/|\alpha(W, t)|
\times \int_0^\pi K_D \left(1 - |\alpha(W, t)|^2\right)^{D-1} \cos \theta \sin^{D-2} \theta d\theta
\text{ (15)}
\]
For \(D = 2\) (i.e., the original Kuramoto model) Eq. (15) evaluates to give \(\rho(t) = Kz(t) = K\int d\omega \eta(\omega, t)\) or \(F(\eta, \omega, t)\). Equation (12) is then equivalent to Eq. (6) from Ref. [27].

For \(D = 3\), the integral in Eq. (15) gives
\[
\rho = K \int dW G(W)\alpha(W, t)/|\alpha(W, t)|
\times \left[2 |\alpha(1 + |\alpha|^2) + (1 - |\alpha|^2)^2 \log \left(1 + \frac{1}{|\alpha|^2}\right)\right]/4 |\alpha|^2.
\] (16)

This now allows us to use Eq. (12) with some given \(G(W)\) to numerically integrate for the dynamics of \(\alpha\) and \(\rho\).

Using this simplification, we can efficiently simulate the dynamics of the full system of agents governed by Eq. (2). We first focus on the case of homogeneous agents, i.e., identical natural rotations for each agent, \(G(W) = \delta(W - W_0)\), where \(\delta(\cdot)\) is the Dirac-delta function. We can then change to a rotating basis in which the natural rotation term of each agent is zero, \(W_0 \to 0\). This makes the \(W\)-integral in Eq. (15) trivial, allowing a direct representation of \(\rho\) in terms of \(\alpha\). Further, \(\alpha\) is only dependent on time (rather than \(W\) and \(t\)). This represents a very large simplification in the complexity of the dynamics of the system of agents, since Eq. (12) is now a single \(D\)-dimensional ordinary differential equation which represents the collective dynamics of the \(N \to \infty\), \(D\)-dimensional system of coupled differential equations in Eq. (2). The utility of this result is demonstrated for \(D = 3\) in Fig. 1(a), where we show (plotted in black) the time-series for \(|\rho(t)|\) as generated from a system of \(N = 5000\) agents (approximating the \(N \to \infty\) limit), compared with the time-series generated from the theory derived in Eq. (12) (red dashed curve). The initial condition for the full system was chosen such that the agents were uniformly randomly distributed on the sphere. For the theory derived in Eq. (12), i.e., the reduced equations, the initial value of \(\alpha\) was chosen to have magnitude 0.01 in an arbitrary direction. Note the remarkably close agreement between the black and the red dashed curve, demonstrating that the dynamics on the reduced manifold of Eq. (11) indeed gives the large-\(N\) dynamics of the full system of interacting agents.

For the case of heterogeneous agents, \(\alpha\) in Eq. (14) depends on \(W\), and we perform the integral in a Monte-Carlo fashion. We randomly choose \(N_W\) values of \(W\)
from the given distribution \(G(W)\) and simulate the dynamics of the corresponding \(\alpha(W)\)'s. These randomly chosen \(\alpha(W)\)'s are then used as the Monte-Carlo samples to evaluate \(z\) according to Eq. (14), simulating the dynamics of the system in the \(N \rightarrow \infty\) limit by only simulating the dynamics of \(N_W\) variables. Results are shown in Fig. 1(b) for \(D = 3\), where \(N_W = 500\) Monte-Carlo samples were chosen to evaluate the \(|\rho(t)|\) curve via the theory in Eq. (12), and are compared with the curve obtained for simulating the dynamics of the full system of equations in the \(N \rightarrow \infty\) limit, approximated by a simulation of \(N = 5000\) agents. Note how simulating the dynamics of \(N_W \ll N\) Monte-Carlo samples yields a smooth curve approximating the noisy curve generated by simulating the individual dynamics of 5000 agents. Initial conditions for the full system were chosen as a bimodal distribution of \(\sigma_i\)s, independent of the corresponding \(W_i\), with the two peaks being anti-podal to each other, hence representing a distribution explicitly not on the manifold dictated by Eq. (11). The initial condition for the reduced equations, i.e., Eq. (12) were chosen to be uniform on a sphere of radius 0.01, corresponding to an approximately uniform distribution of \(f(\sigma, \eta, t)\) in \(\sigma\). Despite not lying on the invariant manifold described by Eq. (11), we observe that the dynamics of the full system rapidly approach the dynamics as predicted by Eq. (12) for the \(N \rightarrow \infty\) limit for dynamics on the invariant manifold. This indicates that for the case of heterogeneous agents the invariant manifold Eq. (11) is attracting, as has been proven for the case of \(D = 2\). Full system simulations with initial conditions described by a uniform distribution in \(\sigma\) (and hence lying on the invariant manifold Eq. (11) for \(|\alpha| = 0\) yielded a curve that is not discernibly different from the curve presented in Fig. 1(b).

Extensions appropriate to various contexts may be studied using Eq. (11).

For example, each of the agents in the model described above could have a bias towards a particular subspace, such as birds in a flock that prefer to align parallel to the Earth’s surface. In this case, the common field \(\rho\) is defined similar to Eq. (13) as \(\rho = K[(1 - c)z + c\Pi z]\), where \(\Pi\) is the operator that projects onto this preferred subspace, and \(0 \leq c \leq 1\) models the strength of the preference.

Another often-studied extension to the Kuramoto model is the Kuramoto-Sakaguchi model [22], which we can generalize to higher dimensions by defining \(\rho\) as \(\rho = KRz\), where \(R\) is a given fixed rotation matrix.

The ansatz Eq. (11) also facilitates the analysis of interactions with time delay, \(\rho(t) = Kz(t - \tau)\), as studied for \(D = 2\) in Ref. [39].

Also, we note that interactions between multiple communities of Kuramoto-like agents has received attention due to a variety of applications (e.g. Refs. [24, 40, 41]), including the presence of interesting dynamics, such as chimera states [24]. For example, for homogenous natural rotations of \(W_\xi\) within each community \(\xi\),

\[
\frac{\partial}{\partial t} \alpha_\xi = \frac{1}{2}(1 + |\alpha_\xi|^2)\rho_\xi - (\rho_\xi \cdot \alpha_\xi)\alpha_\xi + W_\xi \alpha_\xi, \tag{17}
\]

where the subscript \(\xi\) denotes quantities applying to community \(\xi\). For a case of generalizing the Kuramoto model, we define the order parameter \(z_\xi\) for community \(\xi\) as the average orientation of that community,

\[
\rho_\xi = \sum_{\xi'} K_{\xi,\xi'} z_{\xi'},
\]

with \(K_{\xi,\xi'}\) representing the coupling between community \(\xi\) and \(\xi'\). The order parameters \(z_\xi\) can be written in terms of \(\alpha_\xi\) using Eq. (13) by writing the distribution of rotations for the community \(\xi\) as \(\delta(W - W_\xi)\).

The Kuramoto model with the order parameter defined as Eq. (13) is globally coupled, i.e., each agent is coupled to every other agent. In two dimensions, network-based interaction of agents in Kuramoto-like models have been solved for by application of the ansatz Eq. (5), for a wide range of network topologies, via a mean-field approach [16, 22]. An analogous analysis will apply for our generalised ansatz, Eq. (11), for network-based interactions of \(D\)-dimensional Kuramoto-like units.

FIG. 1: Comparison between the dynamics of the magnitude of the order parameter, \(|z|\), as a function of time via full system modeling of the generalized Kuramoto model with \(D = 3\) (Eqs. 2 for \(\rho\) according to Eq. (11)) using \(N = 5000\) agents shown in black, with the modeling of the reduced differential equation Eq. (12) plotted as the red dashed line. \(K = 2\) for both figures. (a) is the case of homogenous agents, i.e., \(G(W) = \delta(W - W_0)\). (b) is the case of heterogeneous agents, with the distribution \(G(W)\) constructed by choosing each upper triangular element from identical independent normal distributions with zero mean and unit variance, and choosing the remaining elements to make \(W\) antisymmetric. Only \(N_W = 500\) Monte-Carlo samples were required to produce the curve for the reduced system of equations, representing the \(N \rightarrow \infty\) limit of the full system, approximated by the noisy curve generated using \(N = 5000\) agents for the full system. To make the curves for the full system and the reduced equations lie on each other, we shift them in time to align them. See text for details of initial conditions used.
There are some strong differences between the case of $D = 2$ and the case of $D > 2$ that must be considered in general. In the case of $D = 2$, making the additional assumption that $g(\eta)$ is a suitable analytic distribution of the scalar parameter $\eta$ (e.g., a Lorentzian distribution is often employed), allows the integral in Eq. (12) to be performed via a contour integral, and hence requiring the dynamics of $\alpha(\eta)$ according to Eq. (12) to be calculated for only one or a few particular complex values of $\eta$ [27]. In $D = 2$ this implies that many problems of the form Eq. (1) with heterogeneous $\eta$ reduce to a system of a small number of ordinary differential equations in the $N \to \infty$ limit. For our generalization to higher dimension (where $\eta$ is now a vector parameter with at least two components), we are unable to straightforwardly employ contour integration. Thus, while Eq. (11) represents a strong reduction in the dimensionality of the dynamics as compared to the full system in the $N \to \infty$ limit, i.e., Eq. (3), it is still not a ‘low-dimensional system’ in the sense of Ref. [27], since we must still calculate the dynamics of $\alpha(\eta, t)$ as a function of the vector parameter $\eta$ (as opposed to integrating $\eta$ away via $a$, e.g., Lorentzian assumption for $g(\eta)$).

In conclusion, we have developed a technique to tackle the generalization of several Kuramoto-like systems into higher dimensions. However, our analysis has only demonstrated the existence of an invariant manifold to the dynamics of Eq. (3). From numerical experiments we observe for all examined examples of systems given Eq. (2) with a continuous distribution $g(\eta)$ that this manifold is attracting, and initial conditions set up not satisfying Eq. (11) appear to be rapidly attracted towards this invariant manifold. While, in the case of $D = 2$, it has been shown analytically that, for a broad class of models of the form given by Eq. (1), this manifold is a global attractor of the dynamics [22], proof of attraction for $D > 2$ remains an open problem. Given the wide applicability of Eq. (1) and its rich variety of dynamical phenomena, we expect that the generalization to higher dimensions, Eq. (2), may be a useful model system, applicable to diverse situations of interest, while remaining amenable to analysis via the methods developed in this paper.

This work was supported by ONR grant N0001415 and by AFOSR grant FA9550-15-1-0171.

* Electronic address: sarthak@umd.edu

[1] A. Pikovsky, M. Rosenblum, and J. Kurths, Synchronization: a universal concept in nonlinear sciences, vol. 12 (Cambridge university press, 2003).

[2] B. Ermentrout, Journal of Mathematical Biology 29, 571 (1991), ISSN 0303-6812, URL http://link.springer.com/10.1007/BF00164052

[3] J. Buck and E. Buck, Science 159 (1968), URL http://science.sciencemag.org/content/159/3821/1319

[4] T. Antonsen Jr, R. Faghri, M. Girvan, E. Ott, and J. Platig, Chaos: An Interdisciplinary Journal of Nonlinear Science 18, 037112 (2008).

[5] L. M. Childs and S. H. Strogatz, Chaos: An Interdisciplinary Journal of Nonlinear Science 18, 043128 (2008), ISSN 1054-1500, URL http://aip.scitation.org/doi/10.1063/1.3049136

[6] J. Pantaleone, Physical Review D 58, 073002 (1998), ISSN 0556-2821, URL https://link.aps.org/doi/10.1103/PhysRevD.58.073002

[7] S. A. Marvel and S. H. Strogatz, Chaos: An Interdisciplinary Journal of Nonlinear Science 19 (2009), URL http://dx.doi.org/10.1063/1.3087132http://aip.scitation.org/doi/10.1063/1.3087434

[8] M. M. Abidulrehem and E. Ott, Chaos: An Interdisciplinary Journal of Nonlinear Science 19, 013129 (2009), ISSN 1054-1500, URL http://aip.scitation.org/doi/10.1063/1.3087434

[9] S. Yamaguchi, H. Isejima, T. Matsuo, R. Okura, K. Yagita, M. Kobayashi, and H. Okamura, Science 302 (2003), URL http://science.sciencemag.org/content/302/5649/1408.full

[10] I. Z. Kiss, Y. Zhai, and J. L. Hudson, Science 296 (2002), URL http://science.sciencemag.org/content/296/5573/1676

[11] B. A. Carreras, V. E. Lynch, I. Dobson, and D. E. Newman, Chaos: An Interdisciplinary Journal of Nonlinear Science 14, 643 (2004), ISSN 1054-1500, URL http://aip.scitation.org/doi/10.1063/1.1781391

[12] A. E. Motter, S. A. Myers, M. Anghel, and T. Nishikawa, Nature Physics 9, 191 (2013), ISSN 1745-2473, URL http://www.nature.com/doifinder/10.1038/nphys2535

[13] T. B. Luke, E. Barreto, and P. S. Neural Computation 25, 3207 (2013), ISSN 0899-7667, URL http://www.mitpressjournals.org/doi/10.1162/NECO_a_00502

[14] D. Pazó and E. Montbrió, Physical Review X 4, 011009 (2014), URL https://link.aps.org/doi/10.1103/PhysRevX.4.011009

[15] E. Montbrió, D. Pazó, and A. Roxin, Physical Review X 5, 021028 (2015), ISSN 2160-3308, URL https://link.aps.org/doi/10.1103/PhysRevX.5.021028

[16] S. Chandra, D. Hathcock, K. Crain, T. M. Antonsen, M. Girvan, and E. Ott, Chaos: An Interdisciplinary Journal of Nonlinear Science 27, 033102 (2017), ISSN 1054-1500, URL http://aip.scitation.org/doi/10.1063/1.4977514

[17] S.-Y. Ha, E. Jeong, and M.-J. Kang, Nonlinearity 23, 3139 (2010), ISSN 0951-7715, URL http://stacks.iop.org/0951-7715/23/i=12/a=008?key=crossref

[18] N. Moshtagh and A. Jadbabaie, IEEE Transactions on Automatic Control 52, 681 (2007), ISSN 0018-9286, URL http://ieeexplore.ieee.org/document/4154979

[19] J. Zhu, J. Lu, and X. Yu, IEEE Transactions on Circuits and Systems I: Regular Papers 60, 199 (2013), ISSN 1549-8328, URL http://ieeexplore.ieee.org/document/6342895

[20] W. Wang and J.-J. E. Slotine, Biological Cybernetics 92, 38 (2005), URL http://www.math.utah.edu/~palais/pcr/papers/slotine05.pdf

[21] Y. Kuramoto, in International symposium on mathematical problems in theoretical physics (Springer, 1975), pp. 420–422

[22] J. G. Restrepo and E. Ott, EPL (Europhysics Letters) 107, 60006 (2014).

[23] H. Sakaguchi and Y. Kuramoto, Progress of Theoretical
Complexity Reduction for Systems of Interacting Orientable Agents: Beyond The Kuramoto Model

Supplemental Material

DERIVATION OF THE RESULT IN EQ. (11)

To perform the integral in Eq. (8), without loss of generality we take the vector $\mathbf{a}$ to be along the $\hat{z}$ axis. For an arbitrary point $\sigma$ on $S$, we denote the angle between $\sigma$ and $\hat{z}$ by $\theta$. In particular, we note that the distance of the point $\sigma$ from the $\hat{z}$ axis is $\sin \theta$. For a coordinate system on the surface $S$, we use $\theta$ as one of the coordinates, denoting position with respect to $\hat{z}$ on the sphere. From the symmetry of $f$ in Eq. (7) about the direction $\mathbf{a}$, we see that the integrals over these remaining coordinates give the surface area $S_{D-1} \sin^{D-2} \theta$ of the $(D-2)$ dimensional surface of a sphere with radius $\sin \theta$ embedded in $(D-1)$ dimensions, where $S_{D-1} = (2\pi)^{(D-1)/2}/\Gamma((D-1)/2)$ is the area of the sphere of unit radius in $D-1$ dimensional space. Thus Eq. (8) becomes

$$1 = S_{D-1} \int_0^\pi N_D(\mathbf{a}) \sin^{D-2} \theta d\theta.$$  \hspace{1cm} (A1)

This integral can then be evaluated to give

$$1 = K_D^{-1} \frac{N_D(\mathbf{a})}{(1 - |\mathbf{a}|^2)^{D-1}},$$  \hspace{1cm} (A2)

where $K_D$ is a constant dependent only on $D$. The above equation can be rearranged to obtain

$$N_D(\mathbf{a}) = K_D (1 - |\mathbf{a}|^2)^{D-1}.$$  \hspace{1cm} (A3)

Inserting the above form of $N_D(\mathbf{a})$ into Eq. (7), with $\beta_D$ determined in Eq. (10), we obtain the form of our ansatz Eq. (11).
PROOF OF EQ. (12)

Inserting the form of \( f(\sigma, \eta, t) \) from Eq. (11) into Eq. (9) we obtain

\[
[(D - 1)(1 - |\alpha|^2)^{D-2}] \{(1 + |\alpha|^2 - 2\alpha \cdot \sigma)(-2\alpha \cdot \partial_t \alpha) \\
- (1 - |\alpha|^2)(2\alpha \cdot \partial_t \alpha - 2\sigma \cdot \partial_t \sigma) \\
+ (1 - |\alpha|^2) [2(\alpha \cdot \rho) - (\rho \cdot \sigma)(1 + |\alpha|^2) - 2\sigma \cdot W \alpha] \} = 0.
\]  

(A4)

Remarkably, the explicit \( D \) dependence on the differential equation cancels out, and a differential equation involving only terms that are linear and constant in \( \sigma \) remains. For this equation to be identically zero for each direction \( \sigma \), the linear and constant terms must independently be zero. From the constant term we obtain

\[
(1 + |\alpha|^2)(-2\alpha \cdot \partial_t \alpha) - (1 - |\alpha|^2)(2\alpha \cdot \partial_t \alpha) \\
+ (1 - |\alpha|^2)(2(\alpha \cdot \rho)) = 0,
\]

which simplifies to

\[
\alpha \cdot \partial_t \alpha = (1/2)(1 - |\alpha|^2)(\rho \cdot \alpha),
\]

(A5)

or alternately

\[
\partial_t |\alpha| = \left( \frac{1 - |\alpha|^2}{2|\alpha|} \right) (\rho \cdot \alpha).
\]

(A6)

From the \( \sigma \) dependent portion we get

\[
\sigma \cdot [2\alpha(2\alpha \cdot \partial_t \alpha) + (1 - |\alpha|^2)(2\partial_t \alpha) \\
- k(1 - |\alpha|^2)(1 + |\alpha|^2)\rho - 2(1 - |\alpha|^2)W \alpha] = 0.
\]

As discussed earlier, since \( \sigma \) is allowed to be in any direction, we can cancel out the \( \sigma \) and obtain a vector equation that must be satisfied. To further simplify this vector expression, we write \( \partial_t \alpha = \partial_t (|\alpha| \hat{\alpha}) = |\alpha| \partial_t \hat{\alpha} + \hat{\alpha} \partial_t |\alpha| \), where \( \hat{\alpha} \) is a unit vector in the direction of \( \alpha \). We can then use Eqs. (A5) and (A6) to simplify the expression to obtain

\[
\partial_t \hat{\alpha} = \left( \frac{1 + |\alpha|^2}{2|\alpha|} \right) (\rho - (\rho \cdot \hat{\alpha})\hat{\alpha}) + W \hat{\alpha}.
\]

(A7)

Equations (A6) and (A7) can then be combined to obtain Eq. (12).