Planewave Response of a Simple Lorentz-Nonreciprocal Medium with Magnetoelastic Gyrotropy

Akhlesh Lakhtakia

The Pennsylvania State University, Department of Engineering Science and Mechanics, University Park, PA 16802, USA
akhlesh@psu.edu

Abstract

The simple Lorentz-nonreciprocal medium described by the constitutive relations $\mathbf{D} = \varepsilon_0 \varepsilon_r \mathbf{E} - \mathbf{\Gamma} \times \mathbf{H}$ and $\mathbf{B} = \mu_0 \mu_r \mathbf{H} - \mathbf{\Gamma} \times \mathbf{E}$ is inspired by a specific spacetime metric, $\mathbf{\Gamma}$ being the magnetoelastic-gyrotropy vector. Field representations in this medium can be obtained from those for the isotropic dielectric-magnetic medium. When a plane wave is incident on a half space occupied by the Lorentz-nonreciprocal medium with magnetoelastic gyrotropy, theory shows that the transverse component of the magnetoelastic-gyrotropy vector is responsible for a rotation about the normal axis; furthermore, left/right reflection asymmetry is exhibited. Additionally, left/right transmission asymmetry is exhibited by a planar slab composed of the Lorentz-nonreciprocal medium with magnetoelastic gyrotropy. The left/right asymmetries are of interest for one-way devices.

**Keywords:** left/right asymmetry, Lorentz nonreciprocity, magnetoelastic gyrotropy, Plebanski theorem

1 Introduction

According to the Plébanski theorem [1,2], relativistic spacetime can be replaced by a bianisotropic continuum with constitutive relations

\[
\begin{align*}
\mathbf{D} &= \varepsilon_0 \gamma \cdot \mathbf{E} - (\mathbf{\Gamma} \times \mathbf{I}) \cdot \mathbf{H} \\
\mathbf{B} &= \mu_0 \gamma \cdot \mathbf{H} + (\mathbf{\Gamma} \times \mathbf{I}) \cdot \mathbf{E}
\end{align*}
\]

where $\mathbf{I}$ is the identity dyadic [3], the dyadic $\gamma$ and the vector $\mathbf{\Gamma}$ emerge from the gravitational metric, $\mu_0$ is the permeability of gravitationally unaffected free space, and $\varepsilon_0$ is the permittivity of gravitationally unaffected free space. Several composite materials have been theoretically formulated [4–10] to simulate certain characteristics of relativistic spacetime.

Whereas the dyadic $\gamma$ is real symmetric, the non-zero magnitude $\mathbf{\Gamma} = |\mathbf{\Gamma}|$ of the magnetoelastic-gyrotropy vector $\mathbf{\Gamma}$ indicates that bianisotropic continuum is nonreciprocal in the Lorentz sense [11,12] because the dyadic $\mathbf{\Gamma} \times \mathbf{I}$ is antisymmetric. For certain spacetime metrics, $\gamma$ is a diagonal dyadic, which inspires a medium with constitutive relations [13]

\[
\begin{align*}
\mathbf{D} &= \varepsilon_0 \varepsilon_r \mathbf{E} - \mathbf{\Gamma} \times \mathbf{H} \\
\mathbf{B} &= \mu_0 \mu_r \mathbf{H} + \mathbf{\Gamma} \times \mathbf{E}
\end{align*}
\]

1
The electromagnetic response characteristics of this medium, with isotropic relative permittivity \(\varepsilon_r\) and relative permeability \(\mu_r\) but with anisotropic magnetoelectric properties, are the subject of this theoretical paper.

Section 2 provides solutions for rectilinear propagation in the Lorentz-nonreciprocal medium with magnetoelectric gyrotropy. Section 3 is devoted to reflection of a plane wave by a half space occupied by this medium, and Sec. 4 to reflection and transmission of a plane wave by a planar slab composed of the same medium. The speed of light in free space, the free-space wavenumber, the free-space wavelength, and the intrinsic impedance of free space are denoted by \(c_0 = 1/\sqrt{\varepsilon_0\mu_0}\), \(k_0 = \omega/c_0\), \(\lambda_0 = 2\pi/k_0\), and \(\eta_0 = \sqrt{\mu_0/\varepsilon_0}\), respectively, with \(\omega\) as the angular frequency. Vectors are in boldface, with Cartesian unit vectors identified as \(\hat{x}, \hat{y}\), and \(\hat{z}\). Dyadics are double underlined. Matrixes are double underlined and enclosed in square brackets. Complex quantities are exemplified by \(\zeta = \zeta' + i\zeta''\), where \(\zeta' = \text{Re}(\zeta)\) and \(\zeta'' = \text{Im}(\zeta)\).

## 2 Rectilinear propagation

Electromagnetic field phasors in a medium described by Eqs. (2) can be obtained after defining the auxiliary field phasors [14]

\[
\begin{align*}
\bar{E}(r) &= E(r) \exp(-i\omega\Gamma \cdot r) \\
\bar{H}(r) &= H(r) \exp(-i\omega\Gamma \cdot r)
\end{align*}
\]  

(3)

These auxiliary field phasors are governed by the differential equations

\[
\begin{align*}
\nabla \cdot \bar{E}(r) &= 0 \\
\nabla \cdot \bar{H}(r) &= 0 \\
\nabla \times \bar{E}(r) &= i\omega\mu_0\mu_r\bar{H}(r) \\
\nabla \times \bar{H}(r) &= -i\omega\varepsilon_0\varepsilon_r\bar{E}(r)
\end{align*}
\]  

(4)

in source-free regions.

Plane-wave solutions of Eqs. (4) are straightforward to find [3,15]. If the direction of propagation is aligned with the unit vector \(\hat{a}\), then

\[
\begin{align*}
\bar{E}(r) &= \left(A_b \hat{b} + A_c \hat{c}\right) \exp(ik\hat{a} \cdot r) \\
\bar{H}(r) &= \left(\eta_0\eta_r\right)^{-1} \left(A_b \hat{c} - A_c \hat{b}\right) \exp(ik\hat{a} \cdot r)
\end{align*}
\]  

(5)

where \(\eta_r = \sqrt{\mu_r/\varepsilon_r}\) is the relative impedance, \(k = k_0\sqrt{\mu_r\varepsilon_r}\) is the wavenumber, \(A_b\) and \(A_c\) are amplitudes, the unit vector \(\hat{b} \perp \hat{a}\), and the unit vector \(\hat{c} = \hat{a} \times \hat{b}\). Equations (3) then
yield

\[ E(r) = \left( A_b \hat{b} + A_c \hat{c} \right) \exp \left[ i \left( k\hat{a} + \omega \Gamma \right) \cdot r \right] \]
\[ H(r) = \left( \eta_0 \eta_r \right)^{-1} \left( A_b \hat{c} - A_c \hat{b} \right) \exp \left[ i \left( k\hat{a} + \omega \Gamma \right) \cdot r \right] \] \tag{6}

for rectilinear propagation.

Whereas Eqs. (5) describe transverse-electromagnetic plane waves, Eqs. (6) generally do not. This can be seen by decomposing \( \Gamma \) into components parallel and perpendicular to \( \hat{a} \) as follows:

\[ \Gamma = (\Gamma \cdot \hat{a}) \hat{a} + \Gamma \cdot (\hat{I} - \hat{a} \hat{a}) \] \tag{7}

Then,

\[ \exp \left[ i \left( k\hat{a} + \omega \Gamma \right) \cdot r \right] = \exp \left[ i \left( k + \omega \Gamma \cdot \hat{a} \right) \hat{a} \cdot r \right] \exp \left[ i\omega \Gamma \cdot (\hat{I} - \hat{a} \hat{a}) \cdot r \right] , \] \tag{8}

thereby indicating spatial variation in directions perpendicular to \( \hat{a} \) in addition to spatial variation in the direction parallel to \( \hat{a} \).

The medium is lossless if (i) \( \varepsilon'' = 0 \), (ii) \( \mu'' = 0 \), and (iii) \( \Gamma'' = 0 \). The medium is dissipative if (i) \( \varepsilon'' > 0 \), (ii) \( \mu'' > 0 \), and (iii) \( 0 < c_0 |\Gamma''| < \sqrt{\varepsilon'' \mu''} \) \cite{16}. Thus, while \( \Gamma = |\Gamma| \neq 0 \) is responsible for Lorentz nonreciprocity, \( \Gamma' \) adds phase and \( \Gamma'' \) adds attenuation.

### 3 Reflection and refraction by a half space

#### 3.1 Theory

Suppose that the half space \( z < 0 \) is vacuous but the half space \( z > 0 \) is occupied by a homogeneous medium with the constitutive relations (2).

A plane wave, propagating in the half space \( z < 0 \) at an angle \( \theta_0 \in [0, \pi/2) \) with respect to the \( z \) axis and an angle \( \psi_0 \in [0, 2\pi) \) with respect to the \( x \) axis in the \( xy \) plane, is incident on the plane \( z = 0 \). Thus, the wave vector of the incident plane wave can be stated as

\[ \mathbf{k}_{\text{inc}} = k_0 [\hat{x} \cos \psi_0 + \hat{y} \sin \psi_0 \sin \theta_0 + \hat{z} \cos \theta_0] . \] \tag{9}

The electromagnetic field phasors associated with the incident plane wave are represented as

\[ E_{\text{inc}}(r) = (a_s \mathbf{s} + a_p \mathbf{p}_+) \exp (i\mathbf{k}_{\text{inc}} \cdot \mathbf{r}) \]
\[ H_{\text{inc}}(r) = \eta_0^{-1} (a_s \mathbf{p}_+ - a_p \mathbf{s}) \exp (i\mathbf{k}_{\text{inc}} \cdot \mathbf{r}) \] \tag{10}

where the unit vectors

\[ \mathbf{s} = -\hat{x} \sin \psi_0 + \hat{y} \cos \psi_0 \]
\[ \mathbf{p}_\pm = \mp (\hat{x} \cos \psi_0 + \hat{y} \sin \psi_0 \cos \theta_0 + \hat{z} \sin \theta_0) \] \tag{11}
identify linear-polarization states. The amplitudes of the s- and the p-polarized components of the incident plane wave, denoted by \(a_s\) and \(a_p\), respectively, are assumed given.

The reflected electromagnetic field phasors are expressed as

\[
\begin{align*}
E_{\text{refl}}(r) &= (r_s s + r_p p) \exp(i k_{\text{refl}} \cdot r), \quad z < 0, \\
H_{\text{refl}}(r) &= \eta_0^{-1}(r_s p - r_p s) \exp(i k_{\text{refl}} \cdot r) 
\end{align*}
\]

where the wave vector of the reflected plane wave is

\[
k_{\text{refl}} = k_0 \left[ (\hat{x} \cos \psi_0 + \hat{y} \sin \psi_0) \sin \theta_0 - \hat{z} \cos \theta_0 \right].
\]

The reflection amplitudes \(r_s\) and \(r_p\) have to be determined in terms of the incidence amplitudes \(a_s\) and \(a_p\) by the solution of a boundary-value problem.

In order to represent the refracted field phasors, we resort to Eqs. (4) to obtain

\[
\begin{align*}
\tilde{E}_{\text{refr}}(r) &= (\rho_s \sigma + \rho_p \tau_+) \exp[i (q_s x + q_y y + \alpha z)] \\
\tilde{H}_{\text{refr}}(r) &= (\eta_0 \eta_r)^{-1} (\rho_s \tau_+ - \rho_p \sigma) \exp[i (q_s x + q_y y + \alpha z)] 
\end{align*}
\]

wherein the refraction amplitudes \(\rho_s\) and \(\rho_p\) are unknown, the polarization-state vectors

\[
\sigma = \begin{pmatrix} \hat{x} q_y + \hat{y} q_x \\ q \end{pmatrix}, \\
\tau_\pm = \mp \left( \hat{x} q_x + \hat{y} q_y \right) \frac{\alpha + z q}{k} \frac{1}{k},
\]

and the wavenumbers \(q = +\sqrt{q_x^2 + q_y^2}\) and \(\alpha = +\sqrt{k^2 - q^2}\) with \(\text{Im}(\alpha) \geq 0\). By virtue of Eqs. (3), we get

\[
\begin{align*}
E_{\text{refr}}(r) &= (\rho_s \sigma + \rho_p \tau_+) \exp[i (q_s x + q_y y + \alpha z + \omega \Gamma \cdot r)] \\
H_{\text{refr}}(r) &= (\eta_0 \eta_r)^{-1} (\rho_s \tau_+ - \rho_p \sigma) \exp[i (q_s x + q_y y + \alpha z + \omega \Gamma \cdot r)] 
\end{align*}
\]

The standard boundary conditions across the interface \(z = 0\) can be satisfied only if the tangential components of the electric and magnetic field phasors are matched in phase across the plane \(z = 0\); accordingly,

\[
\begin{align*}
q_x &= k_0 (\sin \theta_0 \cos \psi_0 - c_0 \Gamma_x) \\
q_y &= k_0 (\sin \theta_0 \sin \psi_0 - c_0 \Gamma_y)
\end{align*}
\]

Thus, the wave vector of the refracted plane wave is given by

\[
k_{\text{refr}} = k_{\text{inc}} + (\alpha + \omega \Gamma z - k_0 \cos \theta_0) \hat{z}.
\]
This equation indicates that the magnetoelectric-gyrotopic vector $\Gamma$ affects the bending of light upon refraction in two ways. The first is directly through the presence of $\Gamma_x$ in the difference $\text{Re}(k_{\text{refr}}) - k_{\text{inc}}$. The second is indirectly through the presence of $\Gamma_x$ and $\Gamma_y$ in

$$
\alpha^2 = k^2 - k_0^2 \left[ (\sin \theta_0 \cos \psi_0 - c_0 \Gamma_x)^2 + (\sin \theta_0 \sin \psi_0 - c_0 \Gamma_y)^2 \right].
$$

In order to appreciate the two ways, suppose first that $\varepsilon_r = \mu_r = 1$ and $\Gamma = 0$, i.e., the refracting medium is also free space. Reflection is then impossible and the refracted plane wave is the same as the incident plane wave. Next, suppose that $\varepsilon_r = \mu_r = 1$ and $\Gamma^\theta = 0$ but $\Gamma \neq 0$, so that the refracting medium is the simplest Lorentz-nonreciprocal medium [18]. Then the time-averaged Poynting vector and the phase velocity of the refracted plane wave are not parallel to each other; in other words, the ray vector

$$
u_{\text{refr}} = k_0^{-1}k_{\text{inc}}$$

of the refracted wave is not collinear with $k_{\text{refr}}$. Additionally, the z-directed components of $k_{\text{refr}}$ and $k_{\text{inc}}$ differ.

Enforcement of the continuities of the tangential components of the electric and magnetic field phasors across the plane $z = 0$ yield the solutions

$$
\left\{
\begin{array}{l}
r_s = r_{ss} a_s + r_{sp} a_p , \\
r_p = r_{ps} a_s + r_{pp} a_p ,
\end{array}
\right.
\quad \rho_s = \rho_{ss} a_s + \rho_{sp} a_p , \quad \rho_p = \rho_{ps} a_s + \rho_{pp} a_p
$$

where the products

$$
\Delta \cdot r_{ss} &=& -(1 - \eta_r^2)k q^2 \alpha \cos \theta_0 - \eta_r \left[ (k^2 - \alpha^2 \cos^2 \theta_0)(q_x \sin \psi_0 - q_y \cos \psi_0)^2 \\
&+& (\alpha^2 - k^2 \cos^2 \theta_0)(q_x \cos \psi_0 + q_y \sin \psi_0)^2 \right],
$$

$$
\Delta \cdot r_{ps} &=& -\Delta \cdot r_{sp} \\
&=& 2\eta_r q^2(q_x \sin \psi_0 - q_y \cos \psi_0)(q_x \cos \psi_0 + q_y \sin \psi_0) \cos \theta_0 ,
$$

$$
\Delta \cdot r_{pp} &=& (1 - \eta_r^2)k q^2 \alpha \cos \theta_0 - \eta_r \left[ (k^2 - \alpha^2 \cos^2 \theta_0)(q_x \sin \psi_0 - q_y \cos \psi_0)^2 \\
&+& (\alpha^2 - k^2 \cos^2 \theta_0)(q_x \cos \psi_0 + q_y \sin \psi_0)^2 \right],
$$

$$
\Delta \cdot \rho_{ss} &=& 2\eta_r k q(\alpha \eta_r + k \cos \theta_0)(q_x \cos \psi_0 + q_y \sin \psi_0) \cos \theta_0 ,
$$

$$
\Delta \cdot \rho_{ps} &=& 2\eta_r k q(k \eta_r + k \cos \theta_0)(q_x \sin \psi_0 - q_y \cos \psi_0) \cos \theta_0 ,
$$

$$
\Delta \cdot \rho_{sp} &=& -2\eta_r k q(k + \alpha \eta_r \cos \theta_0)(q_x \sin \psi_0 - q_y \cos \psi_0) \cos \theta_0 ,
$$

$$
\Delta \cdot \rho_{pp} &=& 2\eta_r k q(\alpha + k \eta_r \cos \theta_0)(q_x \cos \psi_0 + q_y \sin \psi_0) \cos \theta_0 ,
$$

involve

$$
\Delta &=& (1 + \eta_r^2)k q^2 \alpha \cos \theta_0 + \eta_r \left[ (k^2 + \alpha^2 \cos^2 \theta_0)(q_x \sin \psi_0 - q_y \cos \psi_0)^2 \\
&+& (\alpha^2 + k^2 \cos^2 \theta_0)(q_x \cos \psi_0 + q_y \sin \psi_0)^2 \right].
$$
The foregoing expressions reduce to standard results [17] when \( \Gamma_x = \Gamma_y = 0 \). If the refracting medium is matched in impedance to free space, i.e., \( \eta_r = 1 \), Eqs. (22)–(29) yield: \( r_{ss} = r_{pp} \), \( \rho_{ss} = \rho_{pp} \), and \( \rho_{ps} = -\rho_{sp} \).

The four reflection coefficients in Eqs. (21) are denoted by \( r_{ab} \) and the four refraction coefficients by \( \rho_{ab} \), \( a \in \{p, s\} \) and \( b \in \{p, s\} \). Co-polarized coefficients have both subscripts identical, but cross-polarized coefficients do not. Four reflectances are defined as \( R_{ab} = |r_{ab}|^2 \). The principle of conservation of energy requires that

\[
\begin{align*}
0 & \leq R_{ss} + R_{ps} \leq 1 \\
0 & \leq R_{pp} + R_{sp} \leq 1
\end{align*}
\]  

The difference \( 1 - (R_{ss} + R_{ps}) \) is the fraction of the incident energy that is refracted into the half space \( z > 0 \) if the incident plane wave is \( s \) polarized, and \( 1 - (R_{pp} + R_{sp}) \) is the analogous quantity if the incident plane wave is \( p \) polarized.

### 3.2 Numerical results and discussion

Whereas \( \Gamma_z \) does not appear in Eqs. (22)–(29) because the bi-medium interface is the plane \( z = 0 \), both \( \Gamma_x \) and \( \Gamma_y \) do. One way to quantify their effect is via

\[ \psi_q = \tan^{-1}(q_y / q_x), \]

which is a real angle if \( \Gamma''_x = \Gamma''_y = 0 \). In that case, \( \psi_q \neq \psi \) unless \( \Gamma'_x = \Gamma'_y = 0 \) also. Thus, the magnetoelectric-gyrotropy vector can rotate the refracted light about the \( z \) axis even if the refracting medium is the simplest Lorentz-nonreciprocal medium [18].

Figure 1 shows the variations of the four reflectances \( R_{ab} \) with \( \theta_0 \) and \( \psi_0 \), when \( \varepsilon_r = 5 \), \( \mu_r = 1.1 \), and \( c_0 \Gamma = 2(\hat{x}\cos \varphi + \hat{y}\sin \varphi) \). Data are presented for \( \varphi \in \{15^\circ, 75^\circ\} \). For any fixed \( \theta_0 \), the reflectances calculated for \( \varphi = 75^\circ \) are shifted on the \( \psi_0 \) axis by \( 60^\circ \) in relation to the reflectances for \( \varphi = 15^\circ \). Equations (22)–(29) predict this shift because all reflection and refraction coefficients are functions of \( \psi_0 - \varphi \), but not of \( \psi_0 \) and \( \varphi \) separately, when \( \Gamma \) can be expressed in the form \( \Gamma_{xy}(\hat{x}\cos \varphi + \hat{y}\sin \varphi) + \Gamma_z\hat{z} \), even if \( \Gamma_{xy} \) is complex.

The prominent bulbous features in Fig. 1 arise because \( \alpha \) changes from purely real to purely imaginary and then to purely real again, when \( \theta_0 \) is kept fixed but \( \psi_0 \) is varied continuously. These features do not exist if \( \Gamma_{xy} \) is reduced to a sufficiently small and real value, as is exemplified by the reflectance plots in Fig. 2 for \( c_0 \Gamma_{xy} = 0.9 \). The bulbous features also vanish when the refracting medium is dissipative.

Both \( \Gamma_x \) and \( \Gamma_y \), singly and jointly, lead to depolarization upon reflection and refraction. Expressions for \( r_{ps} = -r_{sp} \), \( \rho_{ps} \), and \( \rho_{sp} \) are directly proportional to the term \( (q_x \sin \psi_0 - q_y \cos \psi_0) \), as is clear from Eqs. (23), (26), and (27). But \( q_x \sin \psi_0 \rightarrow q_y \cos \psi_0 \) as \( \Gamma_x \rightarrow 0 \) and \( \Gamma_y \rightarrow 0 \), according to Eqs. (17).
Figure 1: Reflectances as functions of the incidence angles $\theta_0$ and $\psi_0$, when a plane wave propagating in free space is reflected and refracted by a half space occupied by a Lorentz-nonreciprocal medium described by Eqs. (2) with $\varepsilon_r = 5$, $\mu_r = 1.1$, and $c_0\Gamma = 2(\hat{x}\cos\varphi + \hat{y}\sin\varphi)$. The reflectances are unaffected by $\Gamma_z$. Top row: $\varphi = 15^\circ$. Bottom row: $\varphi = 75^\circ$. The white line around the bubble in each plot is a numerical artifact arising from the discrete values of $\theta_0$ and $\psi_0$ used to generate the plot.

Also, both $\Gamma_x$ and $\Gamma_y$, singly and jointly, cause asymmetry in reflection with respect to the reversal of projection of the propagation direction of the incident plane wave on the illuminated plane $z = 0$. The phenomenon of left/right reflection asymmetry [19] can be quantitated through

$$\Delta R_{ab}(\theta_0, \psi_0) = R_{ab}(\theta_0, \psi_0) - R_{ab}(\theta_0, \psi_0 + \pi), \quad a \in \{p, s\}, \quad b \in \{p, s\}. \quad (32)$$

Left/right reflection asymmetry exists if $\Delta R_{ab}(\theta_0, \psi_0) \neq 0$ for at least one of the four combinations of the subscripts $a$ and $b$.

Although left/right reflection asymmetry can be visually inferred from Figs. 1 and 2, it can be grasped more quickly from the plots of $\Delta R_{ab}(\pi/3, \psi_0)$ versus $\psi_0 \in [0, \pi]$ in Fig. 3 when the refracting medium is taken to be dissipative. Left/right asymmetry in cross-polarized reflectances is considerably weaker than in co-polarized reflectances. Furthermore, the degrees of left/right asymmetry of $R_{ss}$ and $R_{pp}$ can be quite different. Thus, the simple Lorentz-nonreciprocal medium with magnetoelectric gyrotropy is promising for one-way optical devices that could be used to reduce backscattering noise as well as instabilities in communication networks and various imaging devices for microscopy and tomography [20, 21].

7
4 Reflection by and transmission through a slab

4.1 Theory

Suppose that the half spaces $z < 0$ and $z > L$ are vacuous but the slab $0 < z < L$ is composed of a homogeneous medium with the constitutive relations (2).

As in Sec. 3, a plane wave, propagating in the half space $z < 0$ at an angle $\theta_0 \in [0, \pi/2)$ with respect to the $z$ axis and at an angle $\psi_0 \in [0, 2\pi)$ with respect to the $x$ axis in the $xy$ plane, is incident on the plane $z = 0$. Equations (10) and (12) still apply in the half space $z < 0$ with unknown reflection amplitudes $r_s$ and $r_p$, but the incidence amplitudes $a_s$ and $a_p$ are presumed known. The wave vector of the plane wave transmitted into the half space
$z > L$ is $k_{inc}$, whereas the associated electromagnetic field phasors can be stated as

$$
\begin{align*}
\mathbf{E}_{inc}(r) &= (t_s \mathbf{s} + t_p \mathbf{p}_+) \exp [i k_{inc} \cdot (r - L \hat{z})], \\
\mathbf{H}_{inc}(r) &= \eta_0^{-1} (t_s \mathbf{p}_+ - t_p \mathbf{s}) \exp [i k_{inc} \cdot (r - L \hat{z})],
\end{align*}
$$

where the transmission amplitudes $t_s$ and $t_p$ are unknown.

A standard $4 \times 4$-matrix procedure to solve the boundary-value problem yields [22]

$$
\begin{bmatrix}
t_s \\
t_p \\
0 \\
0
\end{bmatrix} = \left[ K \right]^{-1} \cdot \exp \left\{ i [P] L \right\} \cdot \left[ K \right] \cdot \begin{bmatrix}
a_s \\
a_p \\
r_s \\
r_p
\end{bmatrix},
$$

where the $4 \times 4$ matrixes

$$
\left[ K \right] =
\begin{bmatrix}
-\sin \psi_0 & -\cos \psi_0 \cos \theta_0 & -\sin \psi_0 & \cos \psi_0 \cos \theta_0 \\
\cos \psi_0 & -\sin \psi_0 \cos \theta_0 & \cos \psi_0 & \sin \psi_0 \cos \theta_0 \\
-\eta_0^{-1} \sin \psi_0 \cos \theta_0 & \eta_0^{-1} \sin \psi_0 & \eta_0^{-1} \cos \psi_0 \cos \theta_0 & \eta_0^{-1} \sin \psi_0 \\
-\eta_0^{-1} \sin \psi_0 \cos \theta_0 & -\eta_0^{-1} \cos \psi_0 \cos \theta_0 & \eta_0^{-1} \sin \psi_0 \cos \theta_0 & -\eta_0^{-1} \cos \psi_0
\end{bmatrix}
$$

and

$$
\left[ P \right] =
\begin{bmatrix}
\omega \Gamma_z & 0 & \frac{q_s q_y}{\omega \varepsilon_0 \varepsilon_r} & \omega \mu_0 \mu_r - \frac{q_s^2}{\omega \varepsilon_0 \varepsilon_r} \\
0 & \omega \Gamma_z & -\omega \mu_0 \mu_r + \frac{q_s^2}{\omega \varepsilon_0 \varepsilon_r} & -\frac{q_s q_y}{\omega \varepsilon_0 \varepsilon_r} \\
-\frac{q_s q_y}{\omega \mu_0 \mu_r} & -\omega \varepsilon_0 \varepsilon_r + \frac{q_s^2}{\omega \mu_0 \mu_r} & \omega \Gamma_z & 0 \\
\omega \varepsilon_0 \varepsilon_r - \frac{q_s^2}{\omega \mu_0 \mu_r} & \frac{q_s q_y}{\omega \mu_0 \mu_r} & 0 & \omega \Gamma_z
\end{bmatrix}.
$$

Equation (34) can be solved by numerical means [23, 24] to determine the four reflection coefficients $r_{ab}$ and the four transmission coefficients $t_{ab}$, $a \in \{p, s\}$ and $b \in \{p, s\}$, that appear in the following relations:

$$
\begin{align*}
\begin{cases}
    r_s &= r_{ss} a_s + r_{sp} a_p, \\
    t_s &= t_{ss} a_s + t_{sp} a_p, \\
    r_p &= r_{ps} a_s + r_{pp} a_p, \\
    t_p &= t_{ps} a_s + t_{pp} a_p,
\end{cases}
\end{align*}
$$

Four reflectances are defined as $R_{sp} = \left| r_{sp} \right|^2$, etc., and four transmittances as $T_{sp} = \left| t_{sp} \right|^2$, etc. The principle of conservation of energy requires that

$$
\begin{align*}
0 &\leq R_{ss} + R_{ps} + T_{ss} + T_{ps} \leq 1 \\
0 &\leq R_{pp} + R_{sp} + T_{pp} + T_{sp} \leq 1
\end{align*}
$$

The differences $1 - (R_{ss} + R_{ps} + T_{ss} + T_{ps})$ and $1 - (R_{pp} + R_{sp} + T_{pp} + T_{sp})$ indicate the fraction of the incident energy that is absorbed in the slab $0 < z < L$. 

A word of caution: The reflection coefficients appearing in Eqs. (37) are not the same as the reflection coefficients appearing in Eqs. (21). The obvious difference between the two sets of reflection coefficients is that the ones in Eqs. (37) depend on $L$ and $\Gamma_z$ but the ones in Eqs. (21) do not.

### 4.2 Numerical results and discussion

Figure 4 shows the co-polarized reflectances and transmittances calculated as functions of the incidence angles $\theta_0$ and $\psi_0$, when $L = \lambda_0$, $\varepsilon_r = 5 + i0.04$, $\mu_r = 1.1 + i0.01$, and $c_0\Gamma = (0.3 + i0.01)(\hat{x}\cos\varphi + \hat{y}\sin\varphi) + (0.5 + i0.01)\hat{z}$ with $\varphi = 15^\circ$. The cross-polarized reflectances and transmittances are $< 10^{-3}$ in magnitude and therefore are not shown. Calculations for Fig. 5 were made with the same parameters as for Fig. 4, except that $\varphi$ was increased to $75^\circ$. A comparison of the two figures quickly reveals that, for any fixed $\theta_0$, all remittances calculated for $\varphi = 75^\circ$ are shifted on the $\psi_0$ axis by almost $60^\circ$ in relation to their counterparts calculated for $\varphi = 15^\circ$. Thus, the ability of the transverse component $(\Gamma - \Gamma_z\hat{z})$ of the magnetoelastic-gyrotropy vector $\Gamma$ to cause rotation about the $z$ axis is evinced, just as in Sec. 3, for reflection and transmission by a slab.

![Figure 4](image_url)

**Figure 4:** Co-polarized reflectances and transmittances as functions of the incidence angles $\theta_0$ and $\psi_0$, when a plane wave propagating in free space is incident on a 1-wavelength-thick slab (i.e., $L = \lambda_0$) composed of a Lorentz-nonreciprocal medium described by Eqs. (2) with $\varepsilon_r = 5 + i0.04$, $\mu_r = 1.1 + i0.01$, and $c_0\Gamma = (0.3 + i0.01)(\hat{x}\cos\varphi + \hat{y}\sin\varphi) + (0.5 + i0.01)\hat{z}$ with $\varphi = 15^\circ$. The cross-polarized reflectances and transmittances are $< 10^{-3}$ in magnitude and therefore are not presented.

Left/right asymmetry is evident for the reflectances in Figs. 4 and 5, just as in Sec. 3. Left/right asymmetry is evident also for the transmittances in both figures. The degree of
left/right transmission asymmetry, as quantified by

\[ \Delta T_{ab}(\theta_0, \psi_0) = T_{ab}(\theta_0, \psi_0) - T_{ab}(\theta_0, \psi_0 + \pi), \quad a \in \{p, s\}, \quad b \in \{p, s\}, \]  

(39)
is comparable to the degree of left/right reflection asymmetry. This becomes clear from the plots of \( \Delta R_{ab}(\pi/3, \psi_0) \) and \( \Delta T_{ab}(\pi/3, \psi_0) \) presented in Fig. 6.

The effect of \( \Gamma_z \) can be appreciated when the Lorentz-nonreciprocal medium with magnetolectric gyrotropy occupies a finite extent along the \( z \) axis, as in this section. The plots in Fig. 7 were obtained with the same parameters as those in Fig. 4, except that \( L \) was increased from \( \lambda_0 \) to \( 2\lambda_0 \), so that the slab for the former figure is twice as thick as for the latter figure. The transmittances are smaller for the higher value of \( L/\lambda_0 \), and so are the reflectances in general. Let us recall, however, that all reflectances, transmittances, and absorptances have an upper limit of unity.

5 Concluding remarks

The Plébanski theorem of general relativity inspires a bianisotropic medium with: (i) isotropic permittivity, (ii) isotropic permeability, and (iii) magnetoelectric gyrotropy. This is a simple Lorentz-nonreciprocal medium that can be transformed into an isotropic dielectric-magnetic medium. Accordingly, field representations in the Lorentz-nonreciprocal medium can be obtained from those for the isotropic dielectric-magnetic medium.

When a plane wave is incident on a half space occupied by the Lorentz-nonreciprocal medium with magnetoelectric gyrotropy, theory shows that the transverse component of the magnetolectric-gyrotropy vector is responsible for a rotation about the normal axis.
Figure 6: Left/right reflection asymmetry and transmission asymmetry in relation to $\psi_0 \in [0, \pi]$ when a plane wave propagating in free space is incident on a 1-wavelength-thick slab (i.e., $L = \lambda_0$) composed of a Lorentz-nonreciprocal medium described by Eqs. (2) with $\varepsilon_r = 5 + i0.04$, $\mu_r = 1.1 + i0.01$, and $c_0 \Gamma = (0.3 + i0.01)\hat{x}\cos\varphi + \hat{y}\sin\varphi + (0.5 + i0.01)\hat{z}$ with $\varphi = 15^\circ$; $\theta_0 = \pi/3$.

Furthermore, left/right reflection asymmetry is exhibited. A planar slab of the Lorentz-nonreciprocal medium with magnetoelectric gyrotropy exhibits left/right transmission asymmetry in addition to left/right reflection asymmetry. Thus, magnetoelectric gyrotropy is a constitutive mechanism that delivers left/right asymmetry in both reflection and transmission, a phenomenon of major interest to realize one-way devices in optics.

**Acknowledgment.** The author is grateful to the Charles Godfrey Binder Endowment at Penn State for ongoing support of his research.
Figure 7: Same as Fig. 4, except that $L = 2\lambda_0$.

References

[1] J. Plébanski, Electromagnetic waves in gravitational fields, Phys. Rev. 118 (1960) 1396–1408.

[2] F. de Felice, On the gravitational field acting as an optical medium, Gen. Relativity Grav. 2 (1971) 347–357.

[3] H. C. Chen, Theory of Electromagnetic Waves: A Coordinate-free Approach, McGraw–Hill, New York, NY, USA,1985.

[4] M. Li, R.-X. Miao, Y. Pang, Casimir energy, holographic dark energy and electromagnetic metamaterial mimicking de Sitter, Phys. Lett. B 689 (2010) 55–59.

[5] T. G. Mackay, A. Lakhtakia, Towards a realization of Schwarzschild-(anti-)de Sitter spacetime as a particulate metamaterial, Phys. Rev. B 83 (2011) 195424.

[6] R.-X. Miao, R. Zheng, M. Li, Metamaterials mimicking dynamic spacetime, D-brane and noncommutativity in string theory, Phys. Lett. B 696 (2011) 550–555.

[7] T. G. Mackay, A. Lakhtakia, Towards a piecewise-homogeneous metamaterial model of the collision of two linearly polarized gravitational plane waves, IEEE Trans. Antennas Propagat. 62 (2014) 6149–6154.

[8] D. V. Khveshchenko, Analogue holographic correspondence in optical metamaterials, Europhys. Lett. 109 (2015) 61001.

[9] I. Fernández-Núñez, O. Bulashenko, Anisotropic metamaterial as an analogue of a black hole, Phys. Lett. A 380 (2016) 1–8.
[10] D. G. Pires, J. C. A. Rocha, P. A. Brandão, Ergoregion in metamaterials mimicking a Kerr spacetime, J. Opt. (Bristol) 20 (2018) 025101.

[11] H. A. Lorentz, Het theorema van Poynting over de energie in het electromagnetisch veld en een paar algemeene stellingen over de voortplanting van liet licht, Versl. K. Akad. W. Amsterdam 4 (1896) 176–187.

[12] C. M. Krowne, Electromagnetic theorems for complex anisotropic media, IEEE Trans. Antennas Propagat. 32 (1984) 1224–1230.

[13] A. D. U. Jafri, A. Lakhtakia, Light scattering by magneto-electrically gyro-tropic sphere with unit relative permittivity and relative permeability, J. Opt. Soc. Am. A 31 (2014) 2489–2494.

[14] A. Lakhtakia, W. S. Weigloher, On electromagnetic fields in a linear medium with gyro-tropic-like magneto-electric properties, Microw. Opt. Technol. Lett. 15 (1997) 168–170.

[15] M. Born, E. Wolf, Principles of Optics, 6th ed., Cambridge University Press, Cambridge, United Kingdom, 1980.

[16] T. G. Mackay, A. Lakhtakia, Electromagnetic Anisotropy and Bia-isotropy, World Scientific, Singapore, 2010, Sec. 1.7.2.2.

[17] M. F. Iskander, Electromagnetic Fields and Waves, 2nd ed., Waveland Press, Long Grove, IL, USA, 2013, Sec. 6.3.

[18] H. M. Alkhoori, A. Lakhtakia, J. K. Breakall, C. F. Bohren, Scattering by a three-dimensional object composed of the simplest Lorentz-nonreciprocal medium,” J. Opt. Soc. Am. A 35 (2018) at press.

[19] A. Lakhtakia, T. G. Mackay, Left/right asymmetry in reflection and transmission by a planar anisotropic dielectric slab with topologically insulating surface states, J. Nanophotonics 10 (2016) 020501.

[20] D. Jalas, A. Petrov, M. Eich, W. Freude, S. H. Fan, Z. F. Yu, R. Baets, M. Popovic, A. Melloni, J. D. Joannopoulos, M. Vanwolleghem, C. R. Doerr, H. Renner, What is—and what is not—an optical isolator, Nat. Photonics 7 (2013) 579–582.

[21] C. Sayrin, C. Junge, R. Mitsch, B. Albrecht, D. O'Shea, P. Schneeweiss, J. Volz, A. Rauschenbeutel, Nanophotonic optical isolator controlled by the internal state of cold atoms, Phys. Rev. X 5 (2015) 041036.

[22] A. Lakhtakia, W. S. Weigloher, Further results on light propagation in helicoidal bia-isotropic mediums: Oblique propagation, Proc. R. Soc. Lond. A 453 (1997) 93–105.

[23] Y. Jaluria, Computer Methods for Engineering, Taylor & Francis, Washington, DC, USA, 1996.
[24] S. C. Chapra, R. P. Canale, *Numerical Methods for Engineers*, 4th ed., McGraw–Hill, New York, NY, USA, 2002.

**Declarations of interest:** none