Complexification of The Vacuum and The Electroweak Gauge Symmetry

Alp Deniz Özer
Ludwig Maximilians University Physics Section, Theresienstr. 37, 80333 Munich Germany

(Dated: March 26, 2022)

PACS numbers: 11.30.Cp, 11.15.Ex, 14.80.Bn

I. INTRODUCTION

The current formulation of the electroweak gauge theory inevitably requires the existence of a scalar field for a few fundamentally important reasons. First of all, the electroweak gauge interactions among elementary particles naturally occur in vacuum, which is understood to be the stable ground state of the scalar field. Historically, to our knowledge, the scalar field has not been incorporated in the electroweak gauge theory to provide an underlying physical continuum for gauge interactions. The main motivation was to utilize the physics in the spontaneous breakdown of the scalar field since it provided masses for the mediators in a gauge invariant and renormalizable way, and also explained the non conservation of isotopic spin.

From the other side, a general fact in nature is that interactions among matter and fields are solely governed by gauge theories, which come along with their respective gauge symmetries. We also demand gauge theories to be locally Lorentz invariant since initial and final particle states are expressed through quantities like energy and momentum which obey the laws of special relativity. We know that Lorentz Invariance is tightly connected with the abstract notion space-time, which makes hardly physical sense unless we consider matter and fields attached to it. The remarkable thing about spacetime, matter and fields is that it constitutes in a completely different fashion the ingredients of special and general relativity. In contrast to the electroweak theory, gravitational interactions or in other words the motion of massive particles governed by the laws of general relativity are accommodated by spacetime.

If it is correct that the Higgs mechanism is the ultimate formalism to create inertial and gravitational masses for elementary particles, we should somehow be able to relate the scalar field, which is responsible for the mass generation mechanism, with spacetime. In this paper we will treat vacuum and spacetime as the two faces of the same medallion and will draw certain consequences out of it, such as the Higgs boson mass and speed of light in the Higgs vacuum. The relation between the two faces will be step wise introduced through out the work, the main steps however are summarized in the conclusion part for clarity. As a starting point it would be most appropriate to consider the global transformation properties of the 4 component scalar field.

Let us start by placing four real valued scalar fields in a column

\[
\phi = \frac{1}{\sqrt{4}} \begin{pmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \\ \phi_0 \end{pmatrix}
\]

(1)

A Lagrangian of the scalar field, in the four vector representation that exhibits a global SO(4) symmetry, is

\[
L_\phi = (\partial_\mu \phi)^T (\partial^\mu \phi) + \mu^2 (\phi^T \phi) - \lambda (\phi^T \phi)^2
\]

(2)
where $\mu^2, \lambda$ are initially chosen to be positive. We characterize the vacuum through these scalar fields. The vacuum state corresponds to the ground state of the scalar field. Prior to any spontaneous symmetry breakdown and phase transition the null vector $\phi \equiv 0$, obviously defines a false vacuum since it pertains to an unstable ground state.

This false vacuum is trivially vacuous since it pertains to an false phase transition the null vector

\[
\phi = \frac{\phi_1 \gamma_1 + \phi_0 \tilde{\gamma}_0}{4} = \frac{1}{\sqrt{8}} \begin{bmatrix}
0 & 0 & \phi_0^0 & \phi_1^+
0 & 0 & -\phi_0^0 - \phi_2^- & 0
-\phi_1^- & 0 & 0 & 0
\end{bmatrix}
\] (10)

Here the upper indices \{+, 0, -\} are showing the charges with respect to the diagonal generator $\Sigma_{12}$ which is obtained from

\[
\Sigma_{ij} = \frac{i}{2} [\gamma_i, \gamma_j] = \epsilon_{ijk} \begin{pmatrix} \sigma_k & 0 \\ 0 & \sigma_k \end{pmatrix}
\] (12)

The lower indices in $\{\phi_0^0, \phi_1^0, \phi_2^0, \phi_3^0\}$ are introduced just to distinguish among the entries. The explicit expressions of these 4 fields in terms of $\{\phi_1, \phi_2, \phi_3, \phi_0\}$ is found to be

\[
\phi_1^0 = (\phi_3 - i \phi_0)/\sqrt{2} \\
\phi_2^0 = (\phi_3 + i \phi_0)/\sqrt{2} \\
\phi_1^+ = (\phi_1 - i \phi_2)/\sqrt{2} \\
\phi_2^- = (\phi_1 + i \phi_2)/\sqrt{2}
\] (13)

From the above expressions it is easy to see that the imaginary sign in front of $\phi_0$ in both $\phi_1^0$ and $\phi_2^0$ stems from $\tilde{\gamma}_0 = -i \gamma_0$. If we had not utilized $\tilde{\gamma}_0$ in the expansion, the fields $\phi_1^0$ and $\phi_2^0$ wouldn’t have had a complex form. An explicit imaginary sign was required to appear in front of $\phi_0$, in this respect. Also note that $Tr[\phi \phi]$ is thereby an $SO(4)$ invariant.

\[
-TR[\phi \phi] = \frac{1}{4} \left( \phi_1^2 + \phi_2^2 + \phi_3^2 + \phi_0^2 \right)
\] (14)

The Lagrangian of the scalar field can equivalently be reexpressed through $\phi$ in eq. (10) as

\[
L_\phi = -Tr \left[ \partial_\mu \phi \partial^\mu \phi \right] - \mu^2 Tr \left[ \phi \phi \right] - \lambda Tr \left[ \phi \phi \right]^2
\] (15)

There is no Hermitian conjugation in the product $\phi \phi$. Note that the $\gamma$ matrices can also be treated algebraically, so that the trace operation can be dropped as well.

In Eq.(13) there are only four independent fields. These fields can be organized into two doublets.

\[
\phi = \begin{pmatrix} \phi_1^+ \\ \phi_0^0 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} \phi_1 - i \phi_2 \\ \phi_3 + i \phi_0 \end{pmatrix}
\] (16)

and

\[
\phi^* = \begin{pmatrix} \phi_2^- \\ \phi_0^0 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} \phi_1 + i \phi_2 \\ \phi_3 - i \phi_0 \end{pmatrix}
\] (17)
The above doublets are complex conjugates of each other, since \( \{ \phi_1, \phi_2, \phi_3, \phi_0 \} \) are real fields. Note that they transform under \( \Sigma_{ij} \), for \( i, j = \{ 1, 2, 3 \} \) independently, so that we can consider them separate \( SU(2) \) doublets. The nice thing about the expansion in Eq. (10) is that it allows each doublet to be grouped so as to include a copy of all four scalar fields. We see that the \( \{ +, 0 \} \) charges in \( \phi \) do not correspond to the \( \sigma_3 \) charges of \( SU(2) \) but are due to \( \Sigma_{12} \). We will come back to this point again.

From the other side the Lagrangian of the scalar field in eq. (14) can sufficiently be reproduced by any of the above doublets, which are indeed equivalent. Since we have put the four fields into a complex representation, the Lagrangian should then be written in a complex form

\[
L_\phi = \frac{1}{2} (\partial_\mu \phi)(\partial^\mu \phi) + \frac{u^2}{2} (\phi^\dagger \phi) - \frac{\lambda}{4} (\phi^\dagger \phi)^2
\]

(18)

Such a restatement imposes an overall \( U(2) \) symmetry on the Lagrangian, which is nothing but an \( SU(2) \times U(1) \) symmetry. The above \( \{ +, 0 \} \) charges in \( \phi \) can be retrieved from this \( SU(2) \times U(1) \) symmetry, such that

\[
\sigma_3 + \frac{Y}{2}
\]

(19)

To obtain the charges \( \{ +, 0 \} \) in the components of \( \phi \), one has to assign +1 to \( Y \), which turns out to be the usual hypercharge of \( \phi \). In this way it becomes easier to understand how \( \phi \) transforms under \( SU(2) \) but carries charges of a larger symmetry. The \( SU(2) \times U(1) \) global symmetry of the scalar field is implied by \( SO(4) \).

After that the scalar field is restated within a complex representation of the \( SU(2) \times U(1) \), we expect that, this symmetry holds also locally, and describes gauge interactions with locally conserved charges. The partial derivatives in eq. (18) should be replaced with the gauge covariant ones to assure local gauge invariance with respect to the local gauge symmetry.

II. COMPLEXIFICATION

Prior to any spontaneous symmetry breakdown and complexification, where the latter will be clearly defined towards the end of this section, we assume that the scalar field \( \phi \) in eq. (14) transforms under \( SO(4) \). Therefore

\[
\phi^T \phi = \frac{1}{4} (\phi_1^2 + \phi_2^2 + \phi_3^2 + \phi_0^2)
\]

(20)

is an \( SO(4) \) invariant of the scalar field. Prior to any spontaneous symmetry breakdown and complexification the null vector which is the unstable ground state, describes a false vacuum and is trivially \( SO(4) \) invariant. We additionally assume that space-time, prior to any spontaneous symmetry breakdown and complexification, transforms under \( SO(4) \) as well.

From the other side we know that, in the spontaneously broken phase of the local \( SU(2) \times U(1) \) electroweak symmetry, space-time transforms under \( SO(3, 1) \). Owing to our assumption that initially both the scalar field and space-time are transforming under \( SO(4) \), we expect analogously, that in the spontaneously broken phase, the scalar field transforms under \( SO(3, 1) \), i.e., it transforms like spacetime. Therefore in the broken phase

\[
\phi^T \phi = \frac{1}{4} (\phi_1^2 + \phi_2^2 + \phi_3^2 + \phi_0^2)
\]

(21)

should be an \( SO(3, 1) \) invariant of the scalar field. The stable ground state of the scalar field corresponds to the true vacuum. Consequently, the true vacuum should also satisfy the above \( SO(3, 1) \) invariant relation.

We postulate that at some stage prior to spontaneous symmetry breakdown, the scalar field must have undergone a phase transition. Note that the transformation

\[
\phi_0 \leftrightarrow i \phi_0
\]

(22)

switches us back and forth between eq. (20) and (21). From the other side, it is formally possible to absorb the phase into the basis, in this way the transformation switches us back and forth between the two sets of gamma matrices given in Eq. (8) and (9). This can be clarified in the following: The complex form of the scalar fields in eq. (16) and (17) are only maintained if the component \( \phi_0 \) has correctly \( i \) as a prefactor. If we let \( \phi_0 \rightarrow i \phi_0 \) then the expansion in eq. (10) should be done over the basis \( \{ \gamma_i, \bar{\gamma}_0 \} \) instead \( \{ \gamma_i, \gamma_0 \} \). This signals us that the initial \( SO(4) \) and final \( SO(3, 1) \) symmetries of the scalar field are related over the phase of \( \phi_0 \). The respective two sets \( \{ \gamma_i, \bar{\gamma}_0 \} \) and \( \{ \gamma_i, \gamma_0 \} \) specify the initial and final metric of spacetime. We will come back to this point later again.

We know that three components of the scalar field are associated with the massless goldstone modes, which can be gauged away (or parameterized) with an \( SU(2) \) transformation. Since \( SU(2) \) is isomorphic to \( SO(3) \), it is appropriate to assign the \( \phi_1, \phi_2, \phi_3 \) fields to \( SU(2) \).

The fourth field \( \phi_0 \) is associated with the Higgs field, we know that the vacuum even after symmetry breakdown preserves the local gauge symmetry in a hidden way, so the \( U(1) \) piece of \( SU(2) \times U(1) \) should be related with the leftover field \( \phi_0 \). Note that, in contrast to the usual assignment in the Higgs mechanism implemented in the electroweak theory, the vacuum expectation value will not be assigned here to \( \phi_3 \) but to \( \phi_0 \), which is a major difference \( 14 \). This choice covers interesting properties as will be later seen. Let us consider the vacuum at a particular minimum resulting from eq. (15) such that

\[
\phi_1 = \phi_2 = \phi_3 = 0
\]

\[
\phi_0 = \frac{\sqrt{2} \mu}{\sqrt{\lambda}} = v
\]

\[
\phi_0 = \left( \begin{array}{c} 0 \\ iv \\ \sqrt{2} \\ i \\ \end{array} \right) = \frac{v}{\sqrt{2}} \left( \begin{array}{c} 0 \\ i \end{array} \right)
\]

(23)
In the rest of this paper, the scalar fields \( \{ \phi_1, \phi_2, \phi_3, \phi_0 \} \) will be parameterized with \( \{ \xi_1, \xi_2, \xi_3, \xi_0 \} \) respectively. All degrees of freedom should be linearly independent and the vacuum should be consistently parameterizable. The above mentioned phase transition described by \( \phi_0 \rightarrow i \phi_0 \) will turn out to be a necessity for a consistent parametrization of the vacuum or more generally of the scalar field. Two cases of parametrization are of interest:

Case I: The scalar field is supposed to transform under the local gauge symmetry, therefore the parametrization of the scalar field around the vacuum state should be done over the generators of the gauge group \( \{ \sigma, Y \} \) together with the expansion parameters of the scalar field \( \{ \xi_1, \xi_2, \xi_3, \xi_0 \} \) respectively. This latter parametrization might suitably called the metric gauge.

Case II: The scalar field is supposed to transform under the SO(4) symmetry, where the \( \gamma \) basis has coordinate dependence. Consistency requires that the scalar field should also be parameterizable around the vacuum state through the basis \( \{ \gamma, \xi_0 \} \) together with the expansion parameters of the scalar field \( \{ \xi_1, \xi_2, \xi_3, \xi_0 \} \) respectively. This condition will turn out to become what we call Complexification, and the local Euclidean symmetry will undergo a change of signature.

Parametrization - Case II: Let us start with the latter parametrization described in case II, by considering the following ground state and the exponentiated transformation:

\[
e^{-i \gamma \xi(x)/v - i \gamma_0 \cdot \xi_0/v} \cdot \left( \begin{array}{cccc}
0 & 1 & 0 & 0 \\
-i \xi_1/v & 0 & -i \xi_2/v & 0 \\
i \xi_1/v & -i \xi_2/v & 0 & 0 \\
i \xi_3/v & -i \xi_0/v & 0 & 0 \\
\end{array} \right)
\]  

(24)

The exponential part if expanded up to first order reads

\[
\left( \begin{array}{cccc}
1 & 0 & -i \xi_1/v - \xi_0/v & -i \xi_1 - i \xi_2 \\
i \xi_1/v + i \xi_2/v & 1 & 0 & 0 \\
i \xi_3/v & -i \xi_0/v & 0 & 0 \\
i \xi_3/v + i \xi_0/v & 0 & 0 & 0 \\
\end{array} \right)
\]

the product of the vacuum state with the above expansion yields

\[
\frac{1}{\sqrt{2}} \left( \begin{array}{cccc}
\xi_1 - i \xi_2 & -\xi_3 - i(\xi_0 + v) \\
\xi_1 + i \xi_2 & -\xi_3 + i(\xi_0 + v) \\
\xi_1 - i \xi_2 & -\xi_3 - i(\xi_0 + v) \\
\xi_1 + i \xi_2 & -\xi_3 + i(\xi_0 + v) \\
\end{array} \right)
\]

It is remarkable to see here that \( \gamma_0 \) behaves formally like the hypercharge \( Y \), and \( \xi_0 \) plays the role of the Higgs field \( H \), because it correctly appears beside \( v \) within the parametrization. Let us consider the last term in the exponential to investigate how the hypercharge acts:

\[
i \gamma_0 \cdot \xi_0 = i (-i \gamma_0) \cdot \xi_0 = \gamma_0 \cdot \xi_0 = -Y \cdot \xi_0 = i Y \cdot (i \xi_0)
\]  

(25)

It is seen in the last line that \( Y \) selects out \( \xi_0 \) as the expansion parameter. Using the same vacuum state, we verify this choice in the forthcoming parametrization mentioned in case I.

Parametrization - Case I: First we demonstrate how the parametrization works, then we identify the \( H \) field. Let us consider the following transformation acting on the minimum

\[
\phi(x) \approx \phi_0(x) = e^{-i \gamma \xi_0(x)/v - i Y \cdot \xi_0(v)/v} \left( \begin{array}{c}
0 \\
1/\sqrt{2} \\
\end{array} \right)
\]

(26)

Expanding the exponential up to first order gives

\[
\left( \begin{array}{cccc}
1 & -i \xi_1/v - i \xi_0/v & -i \xi_1/v - i \xi_2/v \\
-i \xi_1/v + i \xi_2/v & 1 & i \xi_1/v - i \xi_0/v \\
i \xi_3/v - i \xi_0/v & -i \xi_3/v - i \xi_2/v & 0 & 0 \\
i \xi_3/v + i \xi_0/v & 0 & 0 & 0 \\
\end{array} \right) \left( \begin{array}{c}
0 \\
1/\sqrt{2} \\
\end{array} \right)
\]

\[
= \frac{1}{\sqrt{2}} \left( \begin{array}{c}
\xi_1 - i \xi_2 \\
-\xi_3 + i(\xi_0 + v) \\
\end{array} \right)
\]

It is seen that \( \xi_0 \) emerges at the wrong place and does not add up to \( v \), consequently doesn’t operate like the \( H \) field. Since we demand that \( \xi_0 \) should operate like \( H \), we consider the possibility of a phase transition \( \xi_0 \rightarrow i \xi_0 = \tilde{\xi}_0 \) which was previously postulated in eq. (22). Note that this phase transition can be compensated in \( \gamma_0 \cdot \xi_0 \rightarrow -\gamma_0 \cdot \tilde{\xi}_0 \), which preserves invariance, and fulfills the previously stated condition in eq. (23). If we substitute \( \xi_0 \) back in the last line above we obtain the correct form

\[
\phi = \frac{1}{\sqrt{4}} \left( \begin{array}{c}
\phi_1 \\
\phi_2 \\
\phi_3 \\
i \phi_0 \\
\end{array} \right)
\]

(27)

The phase transition is essential and leads the scalar field to undergo a complexification, thereby spacetime changes signature and becomes Minkowski. The four scalar fields should then subsequently be reexpressed as

\[
\phi = \frac{1}{\sqrt{4}} \left( \begin{array}{c}
\phi_1 \\
\phi_2 \\
\phi_3 \\
i \phi_0 \\
\end{array} \right)
\]

Consequently the scalar field satisfies the condition in eq. (24). Physically we will observe the expectation value of \( \phi_0 \) and not its phase, the phase is swallowed by the basis which becomes, \( \{ \gamma_0, \gamma_1, \gamma_2, \gamma_3 \} \) and exhibits a Minkowski signature

\[
2\eta_{\mu \nu} = -\{ \gamma_\mu, \gamma_\nu \} ; \quad (\mu, \nu = 0,1,2,3)
\]
III. LORENTZ INVARIANCE

In the unitary gauge, the photon has no couplings to the Higgs field, nor does it have a mass term, provided over the vacuum expectation value. As a result the photon does encounter no resistance of the Higgs vacuum. A Lorentz invariant is thereby determined as $m^2_H = v^2/2 \approx 123 \text{ GeV}$. Actually the preceding relation should be understood the other way around, conversely $m_H$ and $v$ determine the speed of light in the Higgs vacuum. This follows from a simple consideration that we can normalize the vacuum state $\phi_0$ in eq. (23), into a unit state

$$\begin{pmatrix} 0 \\ i \\ \end{pmatrix}$$

by absorbing the factor $\sqrt{\frac{\gamma_0}{\gamma_0^2}}$ into $v_0$ of the basis $\{\gamma_0, \gamma_2, \gamma_3, \gamma_6\}$, that spans $\{\phi_1, \phi_2, \phi_3, i \phi_0\}$. Furthermore the invariant quantity $Tr[\phi\phi]$ normalizes to $1/4$ (in SI units to $\sqrt{\gamma_0^2}$), if we further absorb $\sqrt{\frac{1}{\gamma_0}}$ into $\gamma_0$. Thereby we redefine $\gamma_0$ for the sake of obtaining a unit vacuum state such that

$$\gamma_0 \to \left(\frac{\gamma}{2m_H}\right)^\frac{1}{2} \cdot \gamma_0$$

The Minkowski signature reveals this non uniformity (in SI units)

$$\eta_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -c^2 \end{pmatrix}$$

where the last entry is departing from 1, and normalizes the fourth spacetime component $x_0 = i c t$. The factor determines the speed of electromagnetic disturbances in the Higgs vacuum:

$$c = \left(\frac{\gamma}{2m_H}\right)^\frac{1}{2} : \text{Speed of Light in Vac.} \quad (33)$$

After the rescaling of $\gamma_0$, which follows from the normalization of the vacuum state, we get

$$-Tr[\phi\phi] = \frac{1}{4}$$

This equality should always hold even when the Higgs field becomes excited and departs from the ground state. In this respect the parametrization allows us to utilize the $\xi_1, \xi_2, \xi_3$ fields, which can be chosen such that the invariant $-\phi^T \phi = 1/4$ is satisfied for any arbitrary value of $\xi_0 \neq 0$, which is by definition the $H$ field. A suitable gauge would be:

$$\xi_0^2 - \sum_{i=1}^{3} \xi_i^2 = 1; \quad \xi_0 = H; \quad \xi_0(0) = 1$$

The excited Goldstone modes $\xi_1, \xi_2, \xi_3$ can be gauged away in the unitary gauge when necessary.
Complexity of the Vacuum: The time variable $t$ is real and thus measurable, but formally can be made to enter the Lorentz transformations with a complex sign, so that the rotation is over a complex angle in a complex plane. A similar situation arose in the vacuum. The ground state in eq. (23) contains a complex sign. However the vacuum expectation value $v$ itself is real and is gained by the component $\phi_0$. The ground state enters the transformation with a complex prefactor just like $t$.

IV. CONCLUSION

We shortly highlight here the underlying steps that lead to the complexification of the vacuum:

(a) The real valued scalar field, can be cast in a complex representation, so that the global $SO(4)$ invariant Lagrangian naturally implies a global $U(2)$ invariant Lagrangian.

(b) The global charges remnant of $SO(4)$ are taken over by the unitary representation $U(1) \times SU(2)$. This symmetry turns out to hold as a local gauge symmetry. The gamma basis should also be spacetime dependent.

(c) A consistent parametrization through the unitary and metric gauges assigns the real fields $\phi_1, \phi_2, \phi_3$ to $SU(2)$ and $\phi_0$ to $U(1)$ and also requires the complexification of the scalar field; $\phi_0 \rightarrow i\phi_0$ which amounts to a change in signature of spacetime; $i \gamma_0 \rightarrow \gamma_0$

(d) The scalar field develops an invariant mass term through the spontaneous breakdown of the unstable ground state to the stable ground state. The spontaneous breakdown is likely to be induced by the complexification; Since the product $-\mu^2(\phi^T \phi)$, contributes for $\phi_0 \rightarrow i\phi_0$ a mass term with the correct sign.

(e) Normalization of the true vacuum state places a constant factor $c = \sqrt{2m_H}$ in front of $\gamma_0$. The rescaled gamma basis inherently defines the speed of light in the Higgs vacuum, over the $SO(3,1)$ invariance of the Maxwell equations.

[1] S. Weinberg, Phys. Rev. Lett. 19, 1264 (1967).
[2] A. Salam, "Originally printed in *Svartholm: Elementary Particle Theory, Proceedings Of The Nobel Symposium Held 1968 At Lerum, Sweden*, Stockholm 1968, 367-377
[3] S. L. Glashow, J. Iliopoulos and L. Maiani, Phys. Rev. D 2, 1285 (1970).
[4] J. Goldstone, Nuovo Cim. 19 (1961) 154.
[5] J. Goldstone, A. Salam and S. Weinberg, Phys. Rev. 127, 965 (1962).
[6] G. 't Hooft and M. J. G. Veltman, Nucl. Phys. B 44, 189 (1972).
[7] The color, Weak and Electromagnetic interactions are correctly described by gauge theories. The former were proposed by H. Fritzsch and M. Gell-Mann in, Proc. of Coral Gables Conf. Vol 2 p.1. Sec(4.2) and the latter two were unified in the electroweak model by several authors in \cite{11}, whose gauge sector is mainly based on pioneering works of H.Weyl (1929) and C.N. Yang and R. Mills (1954).
[8] S. Eidelman et al. [Particle Data Group Collaboration], Phys. Lett. B 592, 1 (2004).
[9] P. W. Higgs, Phys. Lett. 12, 132 (1964).
[10] P. W. Higgs, Phys. Rev. Lett. 13, 508 (1964).
[11] Instead of 4 dimensional spacetime, it would be appropriate to call it 4 dimensional space only, so that time is treated on equal footing with other coordinates.
[12] With the phrase *implied* we mean that the real scalar fields of $SO(4)$ can be cast in a complex field of $U(2)$ . Note that this is not an embedding. We think that in this way can the global charges of $SO(4)$ be taken over by the global $SU(2) \times U(1)$ symmetry. The same applies to the currents as well. Note that non-trivial currents require complex conjugation, and make sense only within unitary representations.
[13] Since the scalar fields are local fields the $\gamma$ basis spanning $\phi$ should be spacetime dependent as well, consequently the $SO(4)$ symmetry should be a local symmetry but not a local gauge symmetry, so that a local metric is defined.
[14] The field $\phi_0$ is by definition initially real, but the $U(1)$ symmetry requires complex representations. This is another hint to see why the scalar field is apt to undergo complexification.
[15] In principle this parametrization is like that in the unitary gauge but differs in the assignment of the expansion parameters as we discussed before. See also eq. \cite{16}
[16] This requirement might be understood in the context of consistent parametrization of the scalar field around the vacuum.