Neumann spectral problem in a domain with very corrugated boundary

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Abstract

Let $\Omega$ be a bounded domain in $\mathbb{R}^n$ ($n \geq 2$). We perturb it to a domain $\Omega^\varepsilon$ attaching a family of small domains with so-called "room-and-passage" geometry ($\varepsilon > 0$ is a small parameter), the diameters of the attached domains tend to zero as $\varepsilon \to 0$. Peculiar spectral properties of Neumann problems in such perturbed domains were observed for the first time by R. Courant and D. Gilbert. In the present work we study the case, when the number of attached domains tends to infinity as $\varepsilon \to 0$ and they are $\varepsilon$-periodically distributed along a part of $\partial \Omega$. Our goal is to describe the asymptotic behaviour of the spectrum of the operator $A^\varepsilon = -\left(\rho^\varepsilon\right)^{-1}\Delta_{\Omega^\varepsilon}$, where $\Delta_{\Omega^\varepsilon}$ is the Neumann Laplacian in $\Omega^\varepsilon$, and the positive function $\rho^\varepsilon$ (mass density) is equal to 1 in $\Omega$. We prove that as $\varepsilon \to 0$ the spectrum of $A^\varepsilon$ converges in the Hausdorff sense to the "spectrum" of the problem

$$-\Delta u = \lambda u \text{ in } \Omega, \quad \frac{\partial u}{\partial n} = F(\lambda)u \text{ on } \Gamma, \quad \frac{\partial u}{\partial n} = 0 \text{ on } \partial \Omega \setminus \Gamma,$$

where $\Gamma$ is a perturbed part of $\partial \Omega$, and $F(\lambda)$ is either linear or rational function. In the later case $F(\lambda)$ has exactly one pole, which is a point of accumulation of eigenvalues.

Keywords: perturbed domain, perturbed mass density, Neumann Laplacian, spectrum, Hausdorff convergence, $\lambda$-dependent boundary conditions, boundary homogenization

1. Introduction

Let $\Omega$ be a fixed domain in $\mathbb{R}^n$. We perturb it to a family of domains $\{\Omega^\varepsilon \subset \mathbb{R}^n\}_\varepsilon$, here $\varepsilon > 0$ is a small parameter. It is well known that if the perturbation is smooth (i.e. the boundaries of $\Omega$ and $\Omega^\varepsilon$ are close in $C^1$ sense as $\varepsilon \to 0$, see [14, Chapter VI, § 2.6] for more precise statement), then the $k$-th eigenvalue of the Neumann Laplacian in $\Omega^\varepsilon$ converges to the $k$-th eigenvalue of the Neumann Laplacian in $\Omega$ (the same also true for the Dirichlet or mixed boundary conditions). If the perturbation is only $C^0$ then, in general, this is not true as it is evident from the following example going back to R. Courant and D. Gilbert. Let $\Omega$ be a unit square $\mathbb{R}^2$. We perturb $\Omega$ attaching to it a small domain, which consists of a square $B^\varepsilon$ ("room") with a side length $b^\varepsilon$ and a narrow rectangle $T^\varepsilon$ ("passage") with side lengths $d^\varepsilon$ and $h^\varepsilon$ — see Fig. 1 (left picture). The domain

$$\Omega^\varepsilon = \Omega \cup (B^\varepsilon \cup T^\varepsilon)$$

can be viewed as a $C^0$ perturbation of $\Omega$. We denote by $\Delta_\Omega$ and $\Delta_{\Omega^\varepsilon}$ the Neumann Laplacians in $\Omega$ and $\Omega^\varepsilon$, correspondingly. The first eigenvalues of both $-\Delta_\Omega$ and $-\Delta_{\Omega^\varepsilon}$ are zero. The second eigenvalue of $-\Delta_\Omega$ is strictly positive, while it was shown in [14, Chapter VI, § 2.6] that the second eigenvalue of $-\Delta_{\Omega^\varepsilon}$ tends to zero as $\varepsilon \to 0$ provided $d^\varepsilon = \varepsilon^4$, $b^\varepsilon = h^\varepsilon = \varepsilon$. 

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Later this example was studied in more detail by J.M. Arrieta, J.K. Halle and Q. Han [2] for more general geometry of "rooms" and "passages". Taking almost the same ratios between the "room" diameter, the "passage" cross-section diameter and the "passage" length as those ones in [14] they proved that for $k \geq 2$ the $k$-th eigenvalue $\lambda_k^e$ of $-\Delta_\Omega^e$ converges to the $(k-1)$-th eigenvalue $\lambda_{k-1}^e$ of $-\Delta_\Omega$ as $e \to 0$. Also they generalized this result to the case of finitely many attached "room-and-passage"-like domains proving that for $k \geq 2$ the $k$-th eigenvalue $\lambda_k^e$ converges to the $(k-1)$-th eigenvalue $\lambda_{k-1}^e$ as $e \to 0$. Also they showed that, similarly to the case of one attached "room-and-passage"-like domain, the second eigenvalue of $-\Delta_\Omega^e$ goes to zero.

E. Sanchez-Palencia in his book [32] (see Chapter XII, §4) considered the case, when $\Omega^e$ is obtained by attaching several "room-and-passage"-like domains, whose number goes to infinity as $e \to 0$ (Fig. 1, right picture). He considered the "rooms" and "passages" of the same size as those ones in [14] and proved that for any $\lambda \in \sigma(-\Delta_\Omega)$ there exists $\lambda^e \in \sigma(-\Delta_\Omega^e)$ such that $\lim_{e \to 0} \lambda^e = \lambda$, (1.1)

hereinafter by $\sigma(\cdot)$ we denote the spectrum of an operator. Also he showed that, similarly to the case of one attached "room-and-passage"-like domain, the second eigenvalue of $-\Delta_\Omega^e$ goes to zero.

The goal of the present work is to extend the results obtained in [32] considering various cases of sizes of "rooms" and "passages" and perturbing mass density in the "rooms". Below we sketch the main results of this work.

Let $\Omega$ be a bounded domain in $\mathbb{R}^n$ ($n \geq 2$). It is supposed that the part of $\partial \Omega$ belongs to a $(n-1)$-dimensional hyperplane. We denote this part of $\partial \Omega$ by $\Gamma$. Let $e > 0$ be a small parameter and $b^e, d^e, h^e$ be positive numbers going to zero as $e \to 0$, $d^e \leq b^e \leq e$; also we suppose that $d^e$ tends to zero not too fast, namely

$$d^e \gg \exp(-a/e), \forall a > 0 \text{ (if } n = 2) \quad \text{or} \quad d^e \gg e^{1/n} \text{ (if } n > 2)$$

(1.2)
as $e \to 0$. We attach a family of "room-and-passage"-like domains $e$-periodically along $\Gamma$. Each attached domain consists of two building blocks:

- the "room" $\simeq b^e B$, where $B$ is a fixed domain in $\mathbb{R}^n$,
- the "passage" $\simeq d^e D \times [0, h^e]$, where $D$ is a fixed domain in $\mathbb{R}^{n-1}$.

We denote these "rooms" and "passages" by $B_i^e$ and $T_i^e$, correspondingly (the parameter $i$ counts them). The total number $N(e)$ of "rooms" (or "passages") tends to infinity as $e \to 0$, namely

$$N(e) \sim e^{1-n} |\Gamma|.$$
Hereinafter we use the same notation $|\cdot|$ either for the volume of a domain in $\mathbb{R}^n$ (for example, $|B|$), for the volume of a domain in $\mathbb{R}^{n-1}$ (for example, $|D|$) or for the area of an $(n-1)$-dimensional hypersurface in $\mathbb{R}^n$ (for example, $|\Gamma|$).

We impose also some additional conditions (see (2.1)-(2.4)) guaranteeing that the neighbouring "rooms" are pairwise disjoint, that the $i$-th "room" and the $i$-th "passage" are correctly glued, and that the distance between the neighbouring "passages" is not too small, namely for $i \neq j$ $\text{dist}(T_i, T_j) \geq C\varepsilon$ (here $0 < C < 1$).

![Fig. 2: The domain $\Omega^\varepsilon$](image)

We denote by $\Omega^\varepsilon$ the obtained domain (see Fig. 2),

$$\Omega^\varepsilon = \Omega \cup \left( \bigcup_i (T_i^\varepsilon \cup B_i^\varepsilon) \right)$$

and introduce the operator

$$A^\varepsilon = -\frac{1}{\rho^\varepsilon} \Delta_{\Omega^\varepsilon}.$$ 

Here $\Delta_{\Omega^\varepsilon}$ is the Neumann Laplacian in $\Omega^\varepsilon$, the function $\rho^\varepsilon$ is equal to 1 everywhere except the union of the "rooms", where it is equal to the constant $\varrho^\varepsilon > 0$. The operator $A^\varepsilon$ describes vibrations of the medium occupying $\Omega$ and having the mass density $\rho^\varepsilon$.

Our goal is to study the behaviour of the spectrum $\sigma(A^\varepsilon)$ as $\varepsilon \to 0$ under the assumption that the following limits exist:

$$\lim_{\varepsilon \to 0} \frac{(d^\varepsilon)^{n-1}|D|}{\varepsilon^{n-1}} =: q \in [0, \infty], \quad \lim_{\varepsilon \to 0} \frac{\varrho^\varepsilon (b^\varepsilon)^n |B|}{\varepsilon^{n-1}} =: r \in [0, \infty). \quad (1.4)$$

We note, that the first limit is allowed to be infinite. The finiteness of $r$ implies the uniform (with respect to $\varepsilon$) boundedness of the total mass $m^\varepsilon_B$ of the "rooms", namely, using (1.3), we obtain:

$$m^\varepsilon_B := \int_{\bigcup_i B_i^\varepsilon} \rho^\varepsilon \, dx = \varrho^\varepsilon \sum_i |B_i^\varepsilon| = \varrho^\varepsilon (b^\varepsilon)^n |B| N(\varepsilon) \sim \frac{\varrho^\varepsilon (b^\varepsilon)^n |B| \cdot |\Gamma|}{\varepsilon^{n-1}} \sim r |\Gamma| \text{ as } \varepsilon \to 0.$$ 

Despite the fact that our problem contains many parameters the form of the limit spectral problem depends essentially only on either $q$ is finite or infinite and $r$ is positive or zero.

We present the results in a rather formal way, more precise statements are formulated in the next section using the operator theory language. In what follows the convergence of spectra is understood in the Hausdorff sense, see Definition [2.1]. One has the following four cases:
1) \( q < \infty, \ r > 0 \). In this case \( \sigma(\mathcal{A}^\varepsilon) \) converges as \( \varepsilon \to 0 \) to the union of the point \( q \) and the set of eigenvalues of the spectral problem

\[
\begin{align*}
-\Delta u &= \lambda u \quad \text{in } \Omega, \\
\frac{\partial u}{\partial n} &= \frac{\lambda}{q} u \quad \text{on } \Gamma, \\
\frac{\partial u}{\partial n} &= 0 \quad \text{on } \partial \Omega \setminus \Gamma,
\end{align*}
\]  

(1.5)

where \( n \) is the outward-pointing unit normal to \( \partial \Omega \) (\( \lambda \) is an eigenvalue if (1.5) has a non-trivial solution \( u \)).

The set of eigenvalues of the problem (1.5) consists of two ascending sequences — one of them goes to infinity and the second one goes to \( q \) (in the case \( q = 0 \) the second sequence disappears).

2) \( q < \infty, \ r = 0 \). In this case \( \sigma(\mathcal{A}^\varepsilon) \) converges to the set \( \sigma(-\Delta_\Omega) \cup \{q\} \).

Formally, this result follows from the previous one if we set \( r = 0 \) in (1.5). But, since the spectral properties of (1.5) change drastically under the passage from \( r > 0 \) to \( r = 0 \), we write out these cases separately.

3) \( q = \infty, \ r > 0 \). In this case \( \sigma(\mathcal{A}^\varepsilon) \) converges to the set of eigenvalues of the spectral problem

\[
\begin{align*}
-\Delta u &= \lambda u \quad \text{in } \Omega, \\
\frac{\partial u}{\partial n} &= \lambda u \quad \text{on } \Gamma, \\
\frac{\partial u}{\partial n} &= 0 \quad \text{on } \partial \Omega \setminus \Gamma,
\end{align*}
\]  

(1.6)

4) \( q = \infty, \ r = 0 \). In this case \( \sigma(\mathcal{A}^\varepsilon) \) converges to \( \sigma(-\Delta_\Omega) \).

Obviously, all these cases can be realised. For example, if we take

\[
n = 2, \quad d^\varepsilon = e^{\alpha} (\alpha \geq 1), \quad b^\varepsilon = h^\varepsilon = \varepsilon, \quad q^\varepsilon = \varepsilon^\beta (\beta \geq -1)
\]

then condition (1.2) holds true and the limits (1.4) exist, namely

\[
\begin{align*}
& r > 0, \quad \text{if } \beta = -1, \\
& r = 0, \quad \text{if } \beta > -1,
\end{align*}
\]

and

\[
\begin{align*}
& q > 0, \quad \text{if } \alpha = \beta + 3, \\
& q = 0, \quad \text{if } \alpha > \beta + 3, \\
& q = \infty, \quad \text{if } \alpha < \beta + 3.
\end{align*}
\]

We also prove the following estimate:

\[
\sup_{k \in \mathbb{N}} \left( \lim_{\varepsilon \to 0} \lambda_k^\varepsilon \right) \leq q,
\]  

(1.7)

where \( \lambda_k^\varepsilon \) is the \( k \)-th eigenvalue of \( \mathcal{A}^\varepsilon \). Following [27] we call the quantity staying in the left-hand-side of (1.7) the threshold of low eigenfrequencies. In the case \( q = 0 \) (1.7) implies

\[
\forall k : \lambda_k^\varepsilon \to 0 \text{ as } \varepsilon \to 0.
\]

Note, that the choice of the boundary conditions on unperurbed part of the boundary (i.e. on \( \partial \Omega \setminus \Gamma \)) is inessential – instead of the Neumann boundary conditions we can prescribe, for example, the Dirichlet or mixed ones. These conditions will be inherited by the limit spectral problem.

The paper is organized as follows. In Section 2 we set up the problem and formulate the main results absorbed in Theorem 2.1 also in Section 2 we prove inequality (1.7) (Theorem 2.2).
Theorem 2.1 is proved in the next section: in Subsection 3.1 we establish some auxiliary estimates, Section 3.2 is devoted to the case \( q < \infty \), the case \( q = \infty \) is treated in Subsection 3.3.

In the end of the introduction we would like to make some bibliographical comments.

1. In the "classical" case \( \rho^\varepsilon = 1 \) (i.e. \( \mathcal{A}^\varepsilon = -\Delta_{\Omega^\varepsilon} \)) the property (1.1) follows from the next general result obtained in [24]: let \( \Omega \subset \mathbb{R}^n \) be a fixed domain and \( \{ \Omega^\varepsilon \subset \mathbb{R}^n \}_\varepsilon \) be a family of domains satisfying some mild regularity assumptions and

\[
\Omega \subset \Omega^\varepsilon, \quad \text{meas}(\Omega^\varepsilon \setminus \Omega) \to 0 \quad \text{as} \quad \varepsilon \to 0
\]

(1.8)

(here \( \text{meas}(\cdot) \) stays for the Lebesgue measure in \( \mathbb{R}^n \)), then (1.1) holds true. And indeed, although the number of attached "rooms" and "passages" tends to infinity, their total measure tends to zero.

It may happen, however, that (1.8) holds true, but there exists a sequence \( \lambda^\varepsilon \in \sigma(-\Delta_{\Omega^\varepsilon}) \) converging to a point not belonging to \( \sigma(-\Delta_\Omega) \). As we will see below this can happen for "room-and-passage"-like perturbations. Another important class of such domains are so-called dumbbell-shaped domains. In a simplest case they are defined as follows: let \( \Omega \) be a union of two disjoint domains \( \Omega_j, \ j = 1, 2 \) and \( \Omega^\varepsilon = \Omega \cup T^\varepsilon \), where \( T^\varepsilon \) is a narrow straight channel connecting \( \Omega_1 \) and \( \Omega_2 \) and approaching a 1-dimensional line segment of the length \( L \) as \( \varepsilon \to 0 \). It can be proved that if \( \sigma(-\Delta_{\Omega^\varepsilon}) \ni \lambda^\varepsilon \to \lambda \) as \( \varepsilon \to 0 \) then either \( \lambda \in \sigma(-\Delta_{\Omega^0}) \cup \sigma(-\Delta_{\Omega^\infty}) \) or \( \lambda = \left( \frac{k^2}{L^2} \right) \) for some \( k \in \mathbb{N} \). The spectral properties of boundary value problems posed in dumbbell-shaped domains were studied in a lot of papers – see, e.g., [3] [13]. Some general results allowing to characterize the set of accumulation points, which are not in the spectrum of \( \sigma(-\Delta_\Omega) \), were obtained in [4].

2. One can also study the behaviour of the spectrum of the Dirichlet Laplacian under a perturbation of the boundary of a domain. But in this case the continuity of eigenvalues holds for rather wide set of perturbations. For example (cf. [30]), if \( \Omega \) is a bounded domains and

\[
\text{for every compact set } F \subset \Omega \text{ there exists } \varepsilon_F > 0 \text{ such that } F \subset \Omega^\varepsilon \text{ provided } \varepsilon < \varepsilon_F,
\]

\[
\text{for every open set } O \supset \overline{\Omega} \text{ there exists } \varepsilon_O > 0 \text{ such that } \Omega^\varepsilon \subset O \text{ provided } \varepsilon < \varepsilon_O,
\]

(1.9)

then the \( k \)-th eigenvalue of the Dirichlet Laplacian in \( \Omega^\varepsilon \) converges to the \( k \)-th eigenvalue of the Dirichlet Laplacian in \( \Omega \). It is easy to see that "room-and-passage"-like perturbations described above (with \( \rho^\varepsilon \equiv 1 \)) satisfy conditions (1.9). However, the situation might be more complicated if together with the geometry of a boundary we perturb the mass density near it.

3. Boundary value problems in domains with rapidly oscillating boundary attract a great attention of mathematicians in recent years. Such problems are motivated by various applications in physics and engineering sciences (for example, in scattering of acoustic and electromagnetic waves on small periodic obstacles). We mention here some papers devoted to such problems – [6] [9] [12] [26], more references one can find in [11]. To the best of our knowledge, problems in domains with rapidly oscillating boundary and "room-and-passage"-like geometry of a period cell were considered only in the book [32] mentioned above.

4. The asymptotic behaviour of eigenvibrations of a body with mass density singularly perturbed near its boundary was studied, for example, in [11] [22] [23] [29] and many other papers. The overview of results in this area one can find in the introduction of [13]. Perturbations involving both the perturbation of the boundary and of the mass density were studied in [27] [28], here the domain in \( \mathbb{R}^2 \) is perturbed by attaching a lot of narrow strips of a fixed length (so-called thick junctions), on these strips the mass density is large.

5. As we announced above our limit spectral problem may contain the spectral parameter in boundary conditions. Namely, we get the following boundary conditions:

\[
\frac{\partial u}{\partial n} = F(\lambda) u,
\]

(1.10)
where $\mathcal{F}(\lambda)$ is either linear ($q = \infty$) or rational ($q < \infty$) function. In the later case $\mathcal{F}(\lambda)$ has exactly one pole (the point $q$), which is a point of accumulation of eigenvalues.

The boundary conditions of the form (1.10) appear in some problems with concentrated masses (cf. [22]), also the same affect one can observe in problems involving thick junctions – see, e.g., [26]. In these problems $\mathcal{F}$ is a meromorphic function with a sequence of poles.

Elliptic boundary value problems with boundary conditions containing a spectral parameter were studied in many papers – see, e.g., [5] and references therein.

6. Domains with "room-and-passage"-like geometry are widely used in order to construct examples illustrating various phenomena in Sobolev spaces theory (see, for example, [25]). For instance, it is well-known, that if $\Omega$ is a bounded domain then the imbedding $i_\Omega : H^1(\Omega) \hookrightarrow L_2(\Omega)$ is compact provided $\partial \Omega$ is sufficiently regular. If $\partial \Omega$ is not smooth, then $i_\Omega$ need not be compact. One of the examples demonstrating this can be constructed from a sequence of "rooms", which are joint by a sequence of "passages". The number of the "rooms" and "passages" is infinite, but their diameters decrease in such a way that their union is bounded. It turns out (cf. [16]) that under a special choice of sizes of "rooms" and "passages" the embedding $i_\Omega$ is non-compact.

The non-compactness of $i_\Omega$ leads to occurrence of essential spectrum for $\Delta_\Omega$. In this connection, we mention the following nice result from [17]: for an arbitrary closed set $S \subset [0, \infty)$, $(0) \in S$ and $n \geq 2$ one can construct the domain $\Omega \subset \mathbb{R}^n$ such that the essential spectrum of $-\Delta_\Omega$ is just this set $S$. In their constructions the authors of [17] used "room-and-passage"-like domains.

7. One can consider also the "bulk" analogue of our domain $\Omega^\varepsilon$. Namely, we perturbed the domain $\Omega$ to the family $\{\Omega^\varepsilon\}_{\varepsilon}$, $\Omega^\varepsilon = (\Omega \setminus \cup_i D_i^\varepsilon) \cup (\cup_i B_i^\varepsilon)$ of $n$-dimensional Riemannian manifolds, where $\{D_i^\varepsilon\}_{i}$ is a family of small holes distributed $\varepsilon$-periodically in $\Omega$ (they play the role of "passages"), $\{B_i^\varepsilon\}_{i}$ is a family of spherical surfaces (they play the role of "rooms"), $B_i^\varepsilon$ is glued to the boundary of $D_i^\varepsilon$. When $\varepsilon \to 0$ the number of attached surfaces goes to infinity, while their radii goes to zero. Instead of the usual Laplacian we study the Laplace-Beltrami operator $\widetilde{\Delta}_{\Omega^\varepsilon}$ in $\Omega^\varepsilon$ (the choice of the boundary conditions on $\partial \Omega^\varepsilon = \partial \Omega$ is inessential). Evolution equations involving $\Delta_\Omega^\varepsilon$ were studied in [7][8]. The behaviour of its spectrum as $\varepsilon \to 0$ was studied in [19][21]. In particular, it was shown in [19] that under a suitable choice of sizes of $D_i^\varepsilon$ and $B_i^\varepsilon$ the limit spectral problem in $\Omega$ has the form $-\Delta u = \mathcal{F}(\lambda) u$, where $\mathcal{F}(\lambda)$ is a rational function with one pole; the spectrum of this problem has the same structure as we have in our limit problem, when $q < \infty$, $r > 0$.

2. Setting of the problem and main results

In what follows by $x' = (x_1, \ldots, x_{n-1})$ and $x = (x', x_n)$ we denote the Cartesian coordinates in $\mathbb{R}^{n-1}$ and $\mathbb{R}^n$, correspondingly.

Let $\Omega$ be an open bounded domain in $\mathbb{R}^n$ ($n \geq 2$) with a Lipschitz boundary and satisfying the condition $\Omega \subset \{x \in \mathbb{R}^n : x_n < 0\}$. We denote $\Gamma = \partial \Omega \cap \{x \in \mathbb{R}^n : x_n = 0\}$.

It is supposed that the set $\{x' \in \mathbb{R}^{n-1} : (x', 0) \in \Gamma\} \subset \mathbb{R}^{n-1}$ has non-empty interior.

In the space $L_2(\Omega)$ we introduce the sesquilinear form $\eta$ defined by the formula

$$\eta[u, v] = \int_\Omega \nabla u \cdot \nabla \bar{v} \, dx,$$

with $\text{dom}(\eta) = H^1(\Omega)$. The form $\eta$ is densely defined, closed, positive and symmetric, whence (cf. [31 Theorem VIII.15]) there exists the unique self-adjoint and positive operator $\mathcal{A}$ associated with the form $\eta$, i.e.

$$(\mathcal{A}u, v)_{L_2(\Omega)} = \eta[u, v], \quad \forall u \in \text{dom}(\mathcal{A}), \forall v \in \text{dom}(\eta).$$
The operator $\mathcal{A}$ is the Laplacian in $\Omega$ subject to the Neumann boundary conditions on $\partial\Omega$. Now, we introduce the "rooms" and "passages". Let $\varepsilon > 0$ be a small parameter. We denote by $\{x^{i,\varepsilon}\}_{i \in \mathbb{Z}^{n-1}}$ the family of points $\varepsilon$-periodically distributed on the hyperplane $\{x \in \mathbb{R}^n : x_n = 0\}$:

$$x^{i,\varepsilon} = (\varepsilon i, 0), \ i \in \mathbb{Z}^{n-1}.$$  

For $i \in \mathbb{Z}^{n-1}$ we set:

$$B_i^\varepsilon = \left\{ x \in \mathbb{R}^n : \frac{1}{b^\varepsilon}(x - x^{i,\varepsilon}) \in B, \text{ where } x^{i,\varepsilon} = x_i^{\varepsilon} + (0, h^\varepsilon) \right\} \quad \text{(the } i\text{-th "room"),}$$

$$T_i^\varepsilon = \left\{ x \in \mathbb{R}^n : \frac{1}{d^\varepsilon}(x' - \varepsilon i) \in D, \ 0 \leq x_n \leq h^\varepsilon \right\} \quad \text{(the } i\text{-th "passage"),}$$

where $b^\varepsilon$, $d^\varepsilon$, $h^\varepsilon$ are positive constants, $B$ and $D$ are open bounded domains in $\mathbb{R}^n$ and $\mathbb{R}^{n-1}$, correspondingly, having Lipschitz boundaries and satisfying the conditions

$$B \subset \left\{ x \in \mathbb{R}^n : x' \in \left(-\frac{1}{2}, \frac{1}{2}\right)^{n-1}, \ x_n > 0 \right\},$$

$$\exists R \in \left(0, \frac{1}{2}\right) : \{ x \in \mathbb{R}^n : |x'| < R, \ x_n = 0 \} \subset \partial B,$$

$$\{0\} \in D \subset \{ x' \in \mathbb{R}^{n-1} : |x'| < R \}, \text{ where } R \text{ comes from (2.2)},

$$d^\varepsilon \leq b^\varepsilon \leq \varepsilon,$$

$$h^\varepsilon \to 0 \text{ as } \varepsilon \to 0.$$  

Conditions (2.1)-(2.4) imply that the neighbouring "rooms" are pairwise disjoint and guarantee correct gluing of the $i$-th "room" and the $i$-th "passage" (namely, the upper face of $T_i^\varepsilon$ is contained in $\partial B_i^\varepsilon$). Also, it follows from (2.3)-(2.4) that the distance between the neighbouring "passages" is not too small, namely for $i \neq j$ one has $\text{dist}(T_i, T_j) \geq \varepsilon - 2Rd^\varepsilon \geq 2\varepsilon \left(\frac{1}{2} - R\right)$.

Additionally, we suppose that

$$\varepsilon^{n-1}D^\varepsilon \to 0 \text{ as } \varepsilon \to 0,$$  

where

$$D^\varepsilon = \begin{cases} |\ln d^\varepsilon|, & n = 2, \\ (d^\varepsilon)^{2-n}, & n > 2. \end{cases}$$

(obviously, (2.6) is equivalent to (1.2)).

Attaching the "rooms" and the "passages" to $\Omega$ we obtain the perturbed domain

$$\Omega^\varepsilon = \Omega \cup \left( \bigcup_{i \in I^\varepsilon} (T_i^\varepsilon \cup B_i^\varepsilon) \right),$$

where

$$I^\varepsilon = \left\{ i \in \mathbb{Z}^{n-1} : x_i^{\varepsilon} \in \Gamma \text{ and } \text{dist}(x_i^{\varepsilon}, \partial \Omega \setminus \Gamma) \geq \frac{\sqrt{n}}{2} \right\}.$$  

The domain $\Omega^\varepsilon$ is depicted on Fig. 2.

Now, let us define accurately the operator $\mathcal{A}^\varepsilon$, which will be the main object of our interest. We denote by $\mathcal{H}^\varepsilon$ the Hilbert space of functions from $L^2(\Omega^\varepsilon)$ endowed with a scalar product

$$(u, v)_{\mathcal{H}^\varepsilon} = \int_{\Omega^\varepsilon} u(x)\overline{v(x)}p^\varepsilon(x)dx,$$
where the function \(\rho^\varepsilon(x)\) is defined as follows:

\[
\rho^\varepsilon(x) = \begin{cases} 
1, & x \in \Omega \cup \left( \bigcup_{i \in I^\varepsilon} T^\varepsilon_i \right), \\
\rho^\varepsilon, & x \in \bigcup_{i \in I^\varepsilon} B^\varepsilon_i, 
\end{cases}
\]

\(\rho^\varepsilon > 0\) is a constant.

By \(\eta^\varepsilon\) we denote the sesquilinear form in \(H^\varepsilon\) defined by the formula

\[
\eta^\varepsilon[u, v] = \int_{\Omega^\varepsilon} \nabla u \cdot \nabla \bar{v} \, dx
\]

with dom(\(\eta^\varepsilon\)) = \(H^1(\Omega^\varepsilon)\). The form \(\eta^\varepsilon\) is densely defined, closed, positive and symmetric. We denote by \(\mathcal{A}^\varepsilon\) the operator associated with this form, i.e.

\[
(\mathcal{A}^\varepsilon u, v)_{H^\varepsilon} = \eta^\varepsilon[u, v], \quad \forall u \in \text{dom}(\mathcal{A}^\varepsilon), \; \forall v \in \text{dom}(\eta^\varepsilon).
\]

In other words, the operator \(\mathcal{A}^\varepsilon\) is defined by the operation \(-\frac{1}{\rho^\varepsilon}\Delta u\) in \(\Omega^\varepsilon\) and the Neumann boundary conditions on \(\partial\Omega^\varepsilon\).

The spectrum \(\sigma(\mathcal{A}^\varepsilon)\) of the operator \(\mathcal{A}^\varepsilon\) is purely discrete. The goal of this work is to describe the behaviour of \(\sigma(\mathcal{A}^\varepsilon)\) as \(\varepsilon \to 0\) under the assumption that the following limits exists:

\[q := \lim_{\varepsilon \to 0} q^\varepsilon, \quad r := \lim_{\varepsilon \to 0} r^\varepsilon, \quad q \in [0, \infty], \; r \in [0, \infty),\]

where

\[
q^\varepsilon = \frac{(d^\varepsilon)^{n-1}|D|}{h^\varepsilon \rho^\varepsilon(b^\varepsilon)^n|B|}, \quad r^\varepsilon = \frac{\rho^\varepsilon(b^\varepsilon)^n|B|}{\rho^\varepsilon}.
\]

In order to formulate the main results we introduce additional spaces and operators.

If \(r > 0\) then by \(\mathcal{H}\) we denote the Hilbert space of functions from \(L_2(\Omega) \oplus L_2(\Gamma)\) endowed with the scalar product

\[
(U, V)_\mathcal{H} = \int_{\Omega} u_1(x) \overline{v_1(x)} \, dx + \int_{\Gamma} u_2(x) \overline{v_2(x)} \, dx, \quad U = (u_1, u_2), \; V = (v_1, v_2).
\]

Hereinafter, we use a standard notation \(d\varepsilon\) for the density of the measure generated on \(\Gamma\) (or any other \((n - 1)\)-dimensional hypersurface) by the Euclidean metrics in \(\mathbb{R}^n\).

For \(q < \infty\) we introduce the sesquilinear form \(\eta_{qr}\) in \(\mathcal{H}\) by the formula

\[
\eta_{qr}[U, V] = \int_{\Omega} \nabla u_1 \cdot \nabla \overline{v_1} \, dx + \int_{\Gamma} qr(u_1 - u_2)(\overline{v_1} - \overline{v_2}) \, ds, \quad U = (u_1, u_2), \; V = (v_1, v_2)
\]

with dom(\(\eta_{qr}\)) = \(H^1(\Omega) \oplus L_2(\Gamma)\). Here we use the same notation for the functions \(u_1, v_1\) and their traces on \(\Gamma\). This form is densely defined, closed, positive and symmetric. We denote by \(\mathcal{A}_{qr}\) the self-adjoint operator associated with this form. Formally, the resolvent equation \(\mathcal{A}_{qr} U - \lambda U = F\) (where \(U = (u_1, u_2), \; F = (f_1, f_2)\) ) can be written as follows:

\[
\begin{aligned}
-\Delta u_1 - \lambda u_1 &= f_1 \quad \text{in} \; \Omega, \\
\frac{\partial u_1}{\partial n} + qr(u_1 - u_2) &= 0 \quad \text{on} \; \Gamma, \\
q(u_2 - u_1) - \lambda u_2 &= f_2 \quad \text{on} \; \Gamma, \\
\frac{\partial u_1}{\partial n} &= 0 \quad \text{on} \; \partial\Omega \setminus \Gamma,
\end{aligned}
\]
where \( n \) is the outward-pointing unit normal to \( \Gamma \).

If \( q = 0 \) then \( \mathcal{A}_{qr} \) is a direct sum of the operator \( \mathcal{A} \) and the null operator in \( L_2(\Gamma) \). As a result we have (below by \( \sigma_{\text{ess}} \) and \( \sigma_{\text{disc}} \) we denote the essential and the discrete parts of the spectrum):

\[
\text{if } q = 0 \text{ then } \sigma_{\text{ess}}(\mathcal{A}_{qr}) = \{0\}, \quad \sigma_{\text{disc}}(\mathcal{A}_{qr}) = \sigma(\mathcal{A}) \setminus \{0\}.
\]  

(2.10)

In the case \( q > 0 \) one has the following result.

**Lemma 2.1.** Let \( q > 0 \). Then the spectrum of the operator \( \mathcal{A}_{qr} \) has the form

\[
\sigma(\mathcal{A}_{qr}) = \{q\} \cup \{\lambda_k^+, k = 1, 2, 3\ldots\} \cup \{\lambda_k^-, k = 1, 2, 3\ldots\}.
\]

The points \( \lambda_k^+, k = 1, 2, 3\ldots \) belong to the discrete spectrum, \( q \) is a point of the essential spectrum and they are distributed as follows:

\[
0 = \lambda_1^+ \leq \lambda_2^+ \leq \ldots \leq \lambda_k^+ \leq \ldots \rightarrow q < \lambda_1^- \leq \lambda_2^- \leq \ldots \leq \lambda_k^- \leq \ldots \rightarrow \infty.
\]

(2.11)

We present the proof of this lemma in the end of Subsection 3.2. At this point we only note that if \( \lambda \neq q \) is the eigenvalue of \( \mathcal{A}_{qr} \) and \( U = (u_1, u_2) \) is the corresponding eigenfunction then \( u_1 \neq 0 \) and \( \lambda \) satisfies (1.3) with \( u := u_1 \). Vice versa, if \( \lambda \neq q \) and \( u \neq 0 \) satisfies (1.3) then \( \lambda \) is the eigenvalue of \( \mathcal{A}_{qr} \), \( U = (u, \frac{u}{|\omega|^2}) \) is the corresponding eigenfunction.

Also we introduce in \( \mathcal{H} \) the sesquilinear form

\[
\tilde{\eta}(U, V) = \int_{\Omega} \nabla u_1 \cdot \nabla v_1 \, dx, \quad U = (u_1, u_2), \quad V = (v_1, v_2)
\]

with \( \text{dom}(\tilde{\eta}) = \{ U = (u_1, u_2) \in H^1(\Omega) \oplus L_2(\Gamma) : u_{1|\Gamma} = u_2 \} \), where \( u_{1|\Gamma} \) means the trace of \( u_1 \) on \( \Gamma \). The form \( \tilde{\eta} \) is densely defined, closed, positive and symmetric. We denote by \( \mathcal{A}_t \) the self-adjoint operator associated with this form. The resolvent equation \( \mathcal{A}_t U - \lambda U = F \) (where \( U = (u, u_{1|\Gamma}) \), \( F = (f_1, f_2) \)) can be written as follows:

\[
\begin{align*}
-\Delta u - \lambda u &= f_1 & \text{in } \Omega, \\
\frac{\partial u}{\partial n} - \lambda u &= r f_2 & \text{on } \Gamma, \\
\frac{\partial u}{\partial n} &= 0 & \text{on } \partial\Omega \setminus \Gamma.
\end{align*}
\]

It is clear that \( \mathcal{A}_t \) has compact resolvent in view of the trace theorem and the Rellich embedding theorem. Therefore the spectrum of \( \mathcal{A}_t \) is purely discrete.

Finally, for \( q < \infty \) we denote by \( \mathcal{A}_q \) the operator acting in \( L_2(\Omega) \oplus L_2(\Gamma) \) and defined as follows:

\[
\mathcal{A}_q = \mathcal{A} \oplus qI,
\]

where \( I \) is the identity operator in \( L_2(\Gamma) \). Obviously,

\[
\sigma_{\text{ess}}(\mathcal{A}_q) = \{q\}, \quad \sigma_{\text{disc}}(\mathcal{A}_q) = \sigma(\mathcal{A}) \setminus \{q\}.
\]

(2.12)

In what follows speaking about the convergence of spectra we will use the concept of the Hausdorff convergence.

**Definition 2.1.** The set \( \sigma^e \subset \mathbb{R} \) converges to the set \( \sigma_0 \subset \mathbb{R} \) in the Hausdorff sense if the following conditions hold:

1. if \( \lambda^e \in \sigma^e \) and \( \lim_{e \to 0} \lambda^e = \lambda \) then \( \lambda \in \sigma_0 \),

(A)

2. for any \( \lambda \in \sigma_0 \) there exists \( \lambda^e \in \sigma^e \) such that \( \lim_{e \to 0} \lambda^e = \lambda \).

(B)
Now, we are in position to formulate the main result of this paper.

**Theorem 2.1.** The spectrum of the operator $A_\varepsilon$ converges in the Hausdorff sense as $\varepsilon \to 0$ to the set $\sigma_0$ defined as follows:

$$
\sigma_0 = \begin{cases} 
\sigma(A_q r), & \text{if } q < \infty, \ r > 0, \\
\sigma(A_q), & \text{if } q < \infty, \ r = 0, \\
\sigma(A_r), & \text{if } q = \infty, \ r > 0, \\
\sigma(A), & \text{if } q = \infty, \ r = 0.
\end{cases}
$$

Theorem 2.1 will be proved in the next section. The proof is based on a substitution of suitable test functions into the variational formulation of the spectral problem (see equality (3.21) below) – as in the energy method using for classical homogenization problems (see, e.g., [10, 32]).

Before to proceed to the proof of Theorem 2.1 we obtain an estimate concerning the behaviour of the $k$-th eigenvalue of $A_\varepsilon$.

**Theorem 2.2.** Let $q < \infty$. Then one has

$$
\sup_{k \in \mathbb{N}} \lim_{\varepsilon \to 0} \lambda_k^\varepsilon \le q,
$$

where $\{\lambda_k^\varepsilon\}_{k \in \mathbb{N}}$ is a sequence of eigenvalues of the operator $A_\varepsilon$ written in the ascending order and repeated according to multiplicity.

**Proof.** By the min-max principle (cf. [15, §4.5])

$$
\lambda_k^\varepsilon = \inf_{L \in L_k} \left\{ \sup_{0 \neq v \in L} \frac{\|\nabla v\|^2_{L^2(\Omega^\varepsilon)}}{\|v\|^2_{H_0^1(\Omega^\varepsilon)}} \right\},
$$

where $L_k$ is a set of all $k$-dimensional subspaces in $\text{dom}(\eta^\varepsilon)$.

Let $I_j^\varepsilon \in I^\varepsilon$, $j = 1, \ldots, k$ be arbitrary pairwise non-equivalent indices. We introduce the following functions:

$$
v_j^\varepsilon(x) = \begin{cases} 
1, & x \in B_{I_j^\varepsilon}^\varepsilon, \\
\frac{x_n}{h^\varepsilon}, & x = (x', x_n) \in T_{I_j^\varepsilon}^\varepsilon, \\
0, & \text{otherwise}.
\end{cases}
$$

We denote

$$
L' := \text{span}\{v_j^\varepsilon, \ j = 1, \ldots, k\}.
$$

It is clear that $L' \subset H^1(\Omega)$ and $\dim(L') = k$, hence $L' \in L_k$. Also it is easy to get that

$$
\|\nabla v_j^\varepsilon\|^2_{L^2(\Omega^\varepsilon)} = \frac{(d^\varepsilon)^{n-1}|D|}{h^\varepsilon},
$$

$$
\|v_j^\varepsilon\|^2_{H_0^1} = d^\varepsilon(b^\varepsilon)^n|B| + \frac{1}{3}(d^\varepsilon)^{n-1}|D|h^\varepsilon,
$$

and as a result

$$
\frac{\|\nabla v_j^\varepsilon\|^2_{L^2(\Omega^\varepsilon)}}{\|v_j^\varepsilon\|^2_{H_0^1}} = \frac{(d^\varepsilon)^{n-1}|D|}{h^\varepsilon \left(d^\varepsilon(b^\varepsilon)^n|B| + \frac{1}{3}(d^\varepsilon)^{n-1}|D|h^\varepsilon\right)} = q^\varepsilon \left(1 + \frac{1}{3}q^\varepsilon(h^\varepsilon)^2\right)^{-1}.
$$
Since the supports of \( v^j \) are pairwise disjoint and (2.15)-(2.16) are independent of \( j \), then, obviously, (2.17) is valid for an arbitrary \( v \in L' \) instead of \( v^j \). Using (2.14) and taking into account the finiteness of \( q \) and (2.5), we obtain

\[
\lambda^e_k \leq \frac{\|\nabla v\|_{L^2(\Omega^e)}^2}{\|v\|_{L^2}^2} = q^e \left( 1 + \frac{1}{3} q^e (h^e)^3 \right)^{-1} \sim q \text{ as } \varepsilon \to 0,
\]

which implies the statement of the theorem. \( \square \)

Remark 2.1. In the case \( q > 0, r > 0 \) (2.13) follows easily from Theorem 2.1 and Lemma 2.1.

Corollary 2.1. If \( q = 0 \) then for each \( k \in \mathbb{N} \) \( \lambda^e_k \to 0 \) as \( \varepsilon \to 0 \).

3. Proof of the main results

3.1. Preliminaries

In what follows by \( C, C_1, C_2 \ldots \) we denote generic constants that do not depend on \( \varepsilon \).

If \( G \) is an open domain in \( \mathbb{R}^n \) then by \( \langle u \rangle_G \) we denote the normalized mean value of the function \( u(x) \) in the domain \( G \),

\[
\langle u \rangle_G = \frac{1}{|G|} \int_G u(x)\,dx.
\]

If \( \Sigma \) is an \((n-1)\)-dimensional hypersurface in \( \mathbb{R}^n \) then again by \( \langle u \rangle_\Sigma \) we denote the normalized mean value of the function \( u \) over \( \Sigma \), i.e.

\[
\langle u \rangle_\Sigma = \frac{1}{|\Sigma|} \int_\Sigma u\,ds, \quad |\Sigma| = \int_\Sigma ds.
\]

Next we introduce the following sets for \( i \in I^e \):

- \( \bar{D}^e_i = \{ x \in \partial T^e_i : x_n = 0 \} \),
- \( \tilde{D}^e_i = \{ x \in \partial T^e_i : x_n = h^e \} \),
- \( Y^e_i = \{ x = (x_1, \ldots, x_n) \in \mathbb{R}^n : |x_k - (x^{l, e})_k| < \frac{\varepsilon}{2}, k = 1, \ldots, n-1, -\frac{\varepsilon}{2} < x_n < 0 \} \), where \((x^{l, e})_k\) is the \( k \)-th coordinate of \( x^{l, e} \),
- \( \Gamma^e_i = \{ x = (x_1, \ldots, x_n) \in \mathbb{R}^n : |x_k - (x^{l, e})_k| < \frac{\varepsilon}{2}, k = 1, \ldots, n-1, x_n = 0 \} \).

We have

\[
\bigcup_{i \in I^e} Y^e_i \subset \Omega, \quad (3.1)
\]

\[
\bigcup_{i \in I^e} \Gamma^e_i \subset \Gamma, \quad \lim_{\varepsilon \to 0} \left| \Gamma \setminus \bigcup_{i \in I^e} \Gamma^e_i \right| = 0 \quad (3.2)
\]

(recall that the set \( I^e \) consists of \( i \in \mathbb{Z}^{n-1} \) satisfying \( x^{l, e} \in \Gamma \) and \( \text{dist}(x^{l, e}, \partial \Omega \setminus \Gamma) \geq \varepsilon \frac{\sqrt{n}}{2} \), whence one can easily obtain (3.1)-(3.2)).

Further, we present several estimates which will be widely used in the proof of Theorem 2.1.
Lemma 3.1. One has for \( i \in I^e \):

\[
\forall u \in H^1(\mathcal{Y}^e_\varepsilon) : \quad \left| \langle u \rangle_{\mathcal{D}^e_i} - \langle u \rangle_{\mathcal{Y}^e_\varepsilon} \right|^2 \leq C_1 \mathbf{D}^e \| \nabla u \|^2_{L^2(\mathcal{Y}^e_\varepsilon)}, \tag{3.3}
\]

\[
\forall u \in H^1(B^e_i) : \quad \left| \langle u \rangle_{\mathcal{D}^e_i} - \langle u \rangle_{B^e_i} \right|^2 \leq C_2 \mathbf{D}^e \| \nabla u \|^2_{L^2(B^e_i)}, \tag{3.4}
\]

where \( \mathbf{D}^e \) is defined by formula (2.7).

Proof. We present the proof of (3.3) only. The proof of (3.4) uses the same ideas and needs only some slight modifications.

Using density arguments one concludes that it is enough to prove (3.3) only for smooth functions. Let \( u \in C^1(\mathcal{Y}^e_\varepsilon) \). We denote:

- \( F = \{ x \in \mathbb{R}^n : |x| < \frac{1}{2}, \ x_n < 0 \} \),
- \( D_0 = \{ x \in \mathbb{R}^n : x' \in D, \ x_n = 0 \} \),
- \( F^e_i = \{ x \in \Omega : |x - x^e_i| < \frac{\varepsilon}{2} \} \),
- \( S^e_i = \{ x \in \Omega : |x - x^e_i| = \frac{\varepsilon}{2} \} \),
- \( R^e_i = \{ x \in \Omega : \frac{\varepsilon}{2} < |x - x^e_i| < \frac{3}{4} \} \),
- \( C^e_i = \{ x \in \Omega : |x - x^e_i| = \frac{3}{4} \} \).

In view of (2.3) one has \( D_0 \subset \partial F \). For an arbitrary function \( v \in H^1(F) \) one has the standard trace inequality:

\[
||v||^2_{L^2(D_0)} \leq C ||v||^2_{H^1(F)}. \tag{3.5}
\]

Then via the change of variables \( x \mapsto d^e x + x^e_i \) (mapping \( D_0 \) onto \( \mathcal{D}^e_i \) and \( F \) onto \( F^e_i \)) one can easily obtain from (3.5):

\[
\forall u \in H^1(F^e_i) : ||u||^2_{L^2(\mathcal{D}^e_i)} \leq C \left( (d^e)^{-1} ||u||^2_{L^2(F^e_i)} + d^e ||\nabla u||^2_{L^2(F^e_i)} \right). \tag{3.6}
\]

In a similar way we also get

\[
\forall u \in H^1(F^e_i) : ||u||^2_{L^2(S^e_i)} \leq C \left( (d^e)^{-1} ||u||^2_{L^2(F^e_i)} + d^e ||\nabla u||^2_{L^2(F^e_i)} \right). \tag{3.7}
\]

Then, using the Cauchy inequality, (3.6)-(3.7) and the Poincaré inequality

\[
||u - \langle u \rangle_{F^e_i}||^2_{L^2(F^e_i)} \leq C(d^e)^2 ||\nabla u||^2_{L^2(F^e_i)}
\]

we obtain:

\[
\left| \langle u \rangle_{\mathcal{D}^e_i} - \langle u \rangle_{\mathcal{Y}^e_\varepsilon} \right|^2 \leq 2 \left| \langle u \rangle_{\mathcal{D}^e_i} - \langle u \rangle_{F^e_i} \right|^2 + 2 \left| \langle u \rangle_{S^e_i} - \langle u \rangle_{F^e_i} \right|^2 \leq \frac{2}{|\mathcal{D}^e_i|} ||u - \langle u \rangle_{F^e_i}||^2_{L^2(\mathcal{D}^e_i)} + \frac{2}{|S^e_i|} ||u - \langle u \rangle_{F^e_i}||^2_{L^2(S^e_i)} \leq C(d^e)^{1-n} \left( (d^e)^{-1} ||u - \langle u \rangle_{F^e_i}||^2_{L^2(F^e_i)} + d^e ||\nabla u||^2_{L^2(F^e_i)} \right) \leq C_1(d^e)^{2-n} ||\nabla u||^2_{L^2(F^e_i)}. \tag{3.8}
\]

By \( \Sigma_{n-1} \) we denote \( (n - 1) \)-dimensional unit half-sphere and introduce spherical coordinates \((\varphi, r)\) in \( R^e_i \). Here \( \varphi = (\varphi_1, \ldots, \varphi_{n-1}) \in \Sigma_{n-1} \) are the angular coordinates, \( r \in \left( \frac{\varepsilon}{2}, \frac{3}{2} \right) \) is a distance to \( x^e_i \).
Let \( x = (\varphi, d^e/2) \in S^e \), \( y = (\varphi, \varepsilon/2) \in C^e \). We have
\[
u(y) - u(x) = \int_0^{\frac{\tau}{2} - \frac{d^e}{2}} \frac{\partial u(\xi(\tau))}{\partial \tau} \, d\tau, \text{ where } \xi(\tau) = x + \frac{\tau}{2} \frac{d^e}{2} (x - y).
\]

Then we integrate this equality over \( \Sigma_{n-1} \) with respect to \( \varphi \), divide by \( |\Sigma_{n-1}| \) and square. Using the Cauchy inequality we obtain:
\[
\left| \langle u \rangle_{S^e_i} - \langle u \rangle_{T^e} \right|^2 = \frac{1}{|\Sigma_{n-1}|^2} \int_{\Sigma_{n-1}} \left( \int_0^{\frac{\tau}{2} - \frac{d^e}{2}} \left| \frac{\partial u(\xi(\tau))}{\partial \tau} \right| \, d\tau \right)^2 \, d\varphi \leq C \left\{ \int_0^{\frac{\tau}{2} - \frac{d^e}{2}} \left( \frac{\tau}{2} \frac{d^e}{2} \right)^{1-n} \, d\tau \right\}
\times \left\{ \int_0^{\frac{\tau}{2} - \frac{d^e}{2}} \left| \frac{\partial u(\xi(\tau))}{\partial \tau} \right| \left( \frac{\tau}{2} \frac{d^e}{2} \right)^{n-1} \, d\tau \right\} \leq C_{1} D^e \|\nabla u\|_{L^2(\mathbb{R}^e)}^2.
\] (3.9)

Finally, using the same idea as in the proof of (3.8), we obtain the estimate
\[
\left| \langle u \rangle_{C^e_i} - \langle u \rangle_{T^e} \right|^2 \leq C e^{-2n} \|\nabla u\|_{L^2(\mathbb{R}^e)}^2.
\] (3.10)

Combining (3.8), (3.10) and taking into account that \((d^e)^2 - n + e^{2-n} \leq 2D^e\) we obtain (3.3).

**Lemma 3.2.** One has for \( i \in I^e \):
\[
\forall u \in H^1(T^e) : \left| \langle u \rangle_{D^e_i} - \langle u \rangle_{D^e} \right|^2 \leq \frac{C}{q^e r^e e^{n-1}} \|\nabla u\|_{L^2(\mathbb{R}^e)}^2.
\] (3.11)

where \( q^e \) and \( r^e \) are defined by (2.9).

**Proof.** It is enough to prove (3.11) only for smooth functions. Let \( u \) be an arbitrary function from \( C^1(\overline{T^e_i}) \). Let \( x = (x', 0) \in \overline{D^e_i}, y = (x', h^e) \in \overline{D^e_i} \). One has
\[
u(y) - u(x) = \int_0^{h^e} \frac{\partial u(\xi(\tau))}{\partial \tau} \, d\tau, \text{ where } \xi(\tau) = x + \frac{\tau}{h^e} (x - y).
\]

We integrate this equality over \( D^e_i := d^e D + ie \) with respect to \( x' \), then divide by \( |D^e_i| \) and square. Using Cauchy inequality we obtain:
\[
\left| \langle u \rangle_{\overline{D^e_i}} - \langle u \rangle_{\overline{D^e_i}} \right|^2 = \frac{1}{|D^e_i|^2} \int_{D^e_i} \left( \int_0^{h^e} \left| \frac{\partial u(\xi(\tau))}{\partial \tau} \right| \, d\tau \right)^2 \, dx' \leq C \frac{h^e}{(d^e)^{n-1}} \|\nabla u\|_{L^2(\mathbb{R}^e)}^2 = \frac{C_1}{q^e r^e e^{n-1}} \|\nabla u\|_{L^2(\mathbb{R}^e)}^2
\]
and (3.11) is proved.

**Lemma 3.3.** One has
\[
\forall u \in H^1(\Omega^e) : \sum_{i \in I^e} \|u\|_{L^2(T^e_i)}^2 \leq C_{h^e} \left( \|u\|_{H^1(\Omega)}^2 + \sum_{i \in I^e} \|\nabla u\|_{L^2(T^e_i)}^2 \right).
\] (3.12)
Proof. It is enough to prove the lemma only for smooth functions. Let \( u \) be an arbitrary function from \( C^1(\overline{\Omega}) \). For the sake of simplicity we suppose that there exists \( a > 0 \) such that the set

\[ \Omega_a := \{ x = (x', x_n) \in \mathbb{R}^n : (x', 0) \in \Gamma, \ x_n \in (-a, 0) \} \]

is a subset of \( \Omega \). For the general case the proof need some small modifications.

Let \( x = (x', x_n) = (x', y) \), where \( x' \in D^e_i := a^e D + i\varepsilon, \ x \in (-a, 0), \ y \in (0, h^e) \). One has

\[ u(y) = u(x) + \int_0^{y-x} \frac{\partial u(\xi(\tau))}{\partial \tau} \, d\tau, \]

where \( \xi(\tau) = x + \frac{\tau}{y-x} (y-x) \).

We square this equality, then integrate over \( D^e_i \) with respect to \( x' \), over \((-a, 0)\) with respect to \( x \) and over \((0, h^e)\) with respect to \( y \). We arrive at

\[ a \| u \|^2_{L^2(D^e_i)} = \int_0^{|x' - x|} \int_0^{h^e} \left[ \int_0^{y-x} \frac{\partial u(\xi(\tau))}{\partial \tau} \, d\tau \right]^2 \, dx' \, dy \leq 2h^e \| u \|^2_{L^2(D^e_i)} + 2h^e a(a + h^e) \| \nabla u \|^2_{L^2(D^e_i)} \quad \text{(3.13)} \]

where \( \widehat{D}^e_i = \{ x = (x', x_n) : x' \in D^e_i, \ -a < x_n < 0 \} \). Summing up \( (3.13) \) by \( i \in I^e \) and taking into account, that \( \bigcup_{i \in I^e} \widehat{D}^e_i \subset \Omega_n \subset \Omega \), we obtain the required inequality \( (3.12) \). \( \square \)

3.2. Proof of Theorem 2.1: the case \( q < \infty \)

3.2.1. Proof of the property (A) of Hausdorff convergence

Let \( \lambda^e \in \sigma(\mathcal{A}^e) \) and \( \lambda^e \to \lambda \) as \( \varepsilon \to 0 \). We have to prove that either

\[ \lambda \in \sigma(\mathcal{A}_{q^e}) \text{ if } r > 0 \quad \text{or} \quad \lambda \in \sigma(\mathcal{A}_q) \text{ if } r = 0. \quad (3.14) \]

Recall, that by \( \{ \lambda^e_k \}_{k=1}^{\infty} \) we denote the sequence of eigenvalues of \( \mathcal{A}^e \) written in the ascending order and repeated according to multiplicity. By \( \{ u^e_k \}_{k=1}^{\infty} \) we denote a corresponding sequence of eigenfunctions normalized by the condition \( (u^e_k, u^e_k)_{\mathcal{H}^e} = \delta_{kl} \).

We denote by \( k^e \) the index corresponding to \( \lambda^e \) (i.e. \( \lambda^e = \lambda^e_{k^e} \)). By \( u^e = u^e_{k^e} \in H^1(\Omega) \) we denote the corresponding eigenfunction. One has

\[ \| u^e \|_{\mathcal{H}^e} = 1, \quad \| \nabla u^e \|_{L^2(\Omega)}^2 = \lambda^e. \quad (3.15) \]

In order to describe the behaviour of \( u^e \) on \( \bigcup_{i \in I^e} B^e_i \) as \( \varepsilon \to 0 \) we will use the operator

\[ \Pi^e : L^2 \left( \bigcup_{i \in I^e} B^e_i \right) \to L^2(\Gamma) \]

defined as follows:

\[ \Pi^e u(x) = \begin{cases} (u)_{B^e_i} \sqrt{r^e}, & x \in \Gamma^e_i, \\ 0, & x \in \Gamma \setminus \bigcup_{i \in I^e} \Gamma^e_i. \end{cases} \]

(recall, that \( r^e \) is defined by \( (2.9) \)). Using the Cauchy inequality, \( (3.2) \) and taking into account that

\[ |\Gamma^e_i| = e^{r^e - 1}, \quad |B^e_i| = (h^e)^n |B| \]

for
we obtain
\[ \|\Pi^\varepsilon u\|_{L^2(\Gamma)}^2 \leq r_\varepsilon \sum_{i \in I^\varepsilon} \left[ \frac{\|\nabla^\varepsilon\|_{E_i}}{B_i} \right] \int_{B_i^\varepsilon} |u(x)|^2 \, dx = \sum_{i \in I^\varepsilon} \int_{B_i^\varepsilon} \varphi^\varepsilon |u(x)|^2 \, dx \leq \|u\|_{H^2}^2. \]  
(3.16)

In view of (3.15), (3.16)
\[ \|u^\varepsilon\|_{H^1(\Omega)}^2 + \|\Pi^\varepsilon u\|_{L^2(\Gamma)}^2 \leq C, \]
whence, using the Rellich embedding theorem and the trace theorem, we conclude that there is a subsequence (still denoted by \(\varepsilon\)) and \(u_1 \in H^1(\Omega), u_2 \in L_2(\Gamma)\) such that
\[ u^\varepsilon \to u_1 \text{ in } H^1(\Omega), \]
(3.17)
\[ u^\varepsilon \to u_1 \text{ in } L_2(\Omega), \]
(3.18)
\[ u^\varepsilon \to u_1 \text{ in } L_2(\Gamma), \]
(3.19)
\[ \Pi^\varepsilon u^\varepsilon \to u_2 \text{ in } L_2(\Gamma). \]
(3.20)
as \(\varepsilon \to 0\) (here we use the same notation for the functions \(u^\varepsilon, u_1\) and their traces on \(\Gamma\)).

We start from the case
\[ u_1 \neq 0. \]

We will prove that \(\lambda\) is the eigenvalue of the operator \(\mathcal{A}_q\) if \(r > 0\) (respectively, of the operator \(\mathcal{A}_q\) if \(r = 0\)) and \(U = (u_1, r^{-1/2} u_2)\) (respectively, \(U = (u_1, u_2)\)) is the corresponding eigenfunction.

For an arbitrary \(w \in H^1(\Omega)\) we have
\[ \int_{\Omega^\varepsilon} \nabla u^\varepsilon \cdot \nabla w \, dx = \lambda^\varepsilon \int_{\Omega^\varepsilon} u^\varepsilon w \, dx. \]
(3.21)
The strategy of proof is to plug into (3.21) some specially chosen test-function \(w\) depending on \(\varepsilon\) and then pass to the limit as \(\varepsilon \to 0\) in order to obtain either the equality \(\mathcal{A}_q U = \lambda U \) (\(r > 0\)) or the equality \(\mathcal{A}_q U = \lambda U \) (\(r = 0\)) written in a weak form.

We choose this test-function as follows:
\[ w(x) = w^\varepsilon(x) := \begin{cases} w_1(x) + \sum_{i \in I^\varepsilon} (w_1(x^\varepsilon) - w_1(x)) \varphi^\varepsilon_i(x), & x \in \Omega, \\ \frac{1}{\sqrt{\varepsilon}} w_2(x^\varepsilon) - w_1(x^\varepsilon) & x_n + w_1(x^\varepsilon), & x = (x', x_n) \in T^\varepsilon, \\ \frac{1}{\sqrt{\varepsilon}} w_2(x^\varepsilon), & x \in B^\varepsilon. \end{cases} \]
(3.22)
Here \(w_1 \in C^\infty(\Omega), w_2 \in C^\infty(\Gamma)\) are arbitrary functions, \(\varphi^\varepsilon_i(x) = \varphi\left(\frac{|x - x^\varepsilon_i|}{\varepsilon}\right)\), where \(\varphi : \mathbb{R} \to \mathbb{R}\) is a smooth functions satisfying \(\varphi(t) = 1\) as \(t \leq R\) and \(\varphi(t) = 0\) as \(t \geq \frac{1}{2}\), the constant \(R \in (0, \frac{1}{2})\) comes from (2.2) - (2.3). It is clear that \(w^\varepsilon(x)\) is continuous and piecewise smooth function.

We plug \(w = w^\varepsilon(x)\) into (3.21). Firstly, we study the left-hand-side. Taking into account that \(\text{supp}(\varphi^\varepsilon_i) \subset T^\varepsilon_i\) and \(w^\varepsilon = \text{const} \text{ in } B^\varepsilon_i\) we obtain:
\[ \int_{\Omega^\varepsilon} \nabla u^\varepsilon \cdot \nabla w^\varepsilon \, dx = \int_{\Omega} \nabla u^\varepsilon \cdot \nabla w_1 \, dx + \sum_{i \in I^\varepsilon} \int_{T^\varepsilon_i} \nabla u^\varepsilon \cdot \nabla \left((w_1(x^\varepsilon) - w_1(x)) \varphi^\varepsilon_i(x)\right) \, dx + \sum_{i \in I^\varepsilon} \int_{T^\varepsilon_i} \nabla u^\varepsilon \cdot \nabla w^\varepsilon \, dx. \]
(3.23)
By virtue of (3.17) we get
\[
\int_{\Omega} \nabla u^e \cdot \nabla w_1 \, dx \to \int_{\Omega} \nabla u_1 \cdot \nabla w_1 \, dx \text{ as } \varepsilon \to 0. \quad (3.24)
\]

Using \( \text{supp}(\varphi_i^e) \subset \overline{Y_i} \), one can easily obtain the estimate
\[ \left| \nabla \left( (w_1(x^{1,e}) - w_1) \varphi_i^e \right) \right| \leq C, \]
whence, taking into account that
\[
\sum_{i \in I^e} \varepsilon_i^{n-1} = \sum_{i \in I^e} |\Gamma_i^e| \leq |\Gamma|, \quad (3.25)
\]
we obtain:
\[
\left| \sum_{i \in I^e} \int_{\overline{Y_i}^e} \nabla u^e \cdot \nabla \left( (w_1(x^{1,e}) - w_1) \varphi_i^e \right) \, dx \right|^2 \leq C\| \nabla u^e \|_{L^2(\Omega)}^2 \bigcup_{i \in I^e} Y_i^e \leq C \varepsilon \sum_{i \in I^e} \varepsilon_i^{n-1} \varepsilon \to 0. \quad (3.26)
\]

Now we inspect the third integral in (3.23). Integrating by parts and taking into account that \( \Delta w^e = 0 \) in \( T_i^e \) we get:
\[
\sum_{i \in I^e} \int_{\overline{T_i}^e} \nabla u^e \cdot \nabla w^e \, dx = - \int_{\overline{\hat{D}_i}^e} u^e \frac{\partial w^e}{\partial x_n} \, ds + \int_{\overline{\hat{D}_i}^e} u^e \frac{\partial w^e}{\partial x_n} \, ds
\]
\[
= \sum_{i \in I^e} \frac{(d^e)^{n-1}|D|}{h^e} \left( \langle u^e \rangle_{\hat{D}_i} - \langle u^e \rangle_{\overline{D}_i} \right) \left( w_1(x^{1,e}) - \frac{w_2(x^{1,e})}{\sqrt{r^e}} \right)
\]
\[
= q^e r^e \sum_{i \in I^e} \varepsilon_i^{n-1} \left( \langle u^e \rangle_{\hat{Y}_i} - \langle u^e \rangle_{\overline{B}_i} \right) \left( w_1(x^{1,e}) - \frac{w_2(x^{1,e})}{\sqrt{r^e}} \right) + \delta(e), \quad (3.27)
\]
where \( q^e \) and \( r^e \) are defined by (2.9) and the remainder \( \delta(e) \) vanishes as \( \varepsilon \to 0 \), namely, using the Cauchy inequality, condition (2.6), estimates (3.3), (3.4) and (3.25), we obtain:
\[
|\delta(e)|^2 \leq C (q^e r^e)^{2,2} \sum_{i \in I^e} \varepsilon_i^{n-1} \left( \left| \langle u^e \rangle_{\hat{Y}_i} - \langle u^e \rangle_{\overline{B}_i} \right|^2 + \left| \langle u^e \rangle_{\overline{D}_i} - \langle u^e \rangle_{\overline{B}_i} \right|^2 \right) \cdot \sum_{i \in I^e} \varepsilon_i^{n-1} \frac{1}{r^e}
\]
\[
\leq C_1 (q^e)^2 r^e D^e \varepsilon_i^{n-1} ||\nabla u^e||_{L^2(\bigcup_{i \in I^e} (\overline{\hat{Y}_i} \cup \overline{B}_i))} \leq C_2 D^e \varepsilon_i^{n-1} \to 0 \text{ as } \varepsilon \to 0. \quad (3.28)
\]

We introduce the operator \( Q^e : C^1(\Gamma) \to L^2(\Gamma) \) defined by the formula
\[
Q^e w = \begin{cases} w(x^{1,e}), & x \in \Gamma_i^e, \\ 0, & x \in \Gamma \setminus \bigcup_{i \in I^e} \Gamma_i^e. \end{cases}
\]

It is straightforward to show that
\[
\forall w \in C^1(\Gamma), \ Q^e w \to w \text{ in } L^2(\Gamma). \quad (3.29)
\]
Then, taking into account the definitions of the operators $\Pi^\epsilon$, $Q^\epsilon$ and using (2.8), (3.19), (3.20), (3.28), (3.29), we obtain from (3.27):

\[
\sum_{i\in I} \int_{T_i^\epsilon} \nabla u^\epsilon \cdot \nabla w^\epsilon \, dx = \int_{\Gamma} \left( u^\epsilon - \frac{1}{\sqrt{r^\epsilon}} \Pi^\epsilon u^\epsilon \right) \left( Q^\epsilon w_1 - \frac{1}{\sqrt{r^\epsilon}} Q^\epsilon w_2 \right) \, ds + \delta(\epsilon)
\]

\[
\rightarrow \int_{\Gamma} \left( qru_1w_1 - q \sqrt{r}u_2w_1 - q \sqrt{r}u_1w_2 + quw_2 \right) \, ds \text{ as } \epsilon \to 0. \tag{3.30}
\]

Combining (3.23)-(3.26), (3.30) we arrive at

\[
\int_{\Omega} \nabla u^\epsilon \cdot \nabla w^\epsilon \, dx \to \int_{\Omega} \nabla u_1 \cdot \nabla w_1 \, dx + \int_{\Gamma} \left( qru_1w_1 - q \sqrt{r}u_2w_1 - q \sqrt{r}u_1w_2 + quw_2 \right) \, ds. \tag{3.31}
\]

Now, we study the right-hand-side of (3.21). One has:

\[
\lambda^\epsilon \int_{\Omega^\epsilon} u^\epsilon w^\epsilon \rho^\epsilon \, dx = \lambda^\epsilon \left( \int_{\Omega^\epsilon} u^\epsilon w_1 \, dx + \sum_{i\in I^\epsilon} \int_{T_i^\epsilon} u^\epsilon \left( w_1(x^\epsilon) - w_1 \right) \varphi_i \, dx \right)
\]

\[
+ \sum_{i\in I^\epsilon} \int_{T_i^\epsilon} u^\epsilon w_2 \, dx + \frac{\rho^\epsilon}{\sqrt{r^\epsilon}} \sum_{i\in I^\epsilon} \int_{T_i^\epsilon} u^\epsilon w_2 \, dx \right). \tag{3.32}
\]

It is clear that

\[
\int_{\Omega^\epsilon} u^\epsilon w_1 \, dx \to \int_{\Omega} u_1 w_1 \, dx \text{ as } \epsilon \to 0 \tag{3.33}
\]

and the next two integrals in (3.32) vanishes as $\epsilon \to 0$:

\[
\left| \sum_{i\in I^\epsilon} \int_{T_i^\epsilon} u^\epsilon \left( w_1(x^\epsilon) - w_1 \right) \varphi_i \, dx \right|^2 \leq \sum_{i\in I^\epsilon} \| w_1(x^\epsilon) - w_1 \|_{L^2(T_i^\epsilon)}^2 \leq C \sum_{i\in I^\epsilon} \| u^\epsilon \|_{L^2(T_i^\epsilon)}^2 \to 0, \tag{3.34}
\]

\[
\left| \sum_{i\in I^\epsilon} \int_{T_i^\epsilon} u^\epsilon w_2 \, dx \right|^2 \leq \sum_{i\in I^\epsilon} \| u^\epsilon \|_{L^2(T_i^\epsilon)}^2 \sum_{i\in I^\epsilon} \| w_2 \|_{L^2(T_i^\epsilon)}^2 \leq C \sum_{i\in I^\epsilon} \frac{(\rho^\epsilon)^{p-1}}{r^\epsilon} \leq C_1 \rho^\epsilon \epsilon^{p-1} \to 0. \tag{3.35}
\]

Finally, we inspect the behaviour of the last integral in (3.32). One has:

\[
\sum_{i\in I} \int_{T_i^\epsilon} \frac{\rho^\epsilon}{\sqrt{r^\epsilon}} u^\epsilon w_2 \, dx = \sum_{i\in I^\epsilon} \frac{\rho^\epsilon (h^\epsilon)^{p-1}|B|}{\sqrt{r^\epsilon}} (u^\epsilon)^{p-1} w_2(x^\epsilon) = \int_{\Gamma} \Pi^\epsilon u^\epsilon Q^\epsilon w_2 \, ds \to \int_{\Gamma} u_2 w_2 \, ds. \tag{3.36}
\]

It follows from (3.32)-(3.36) and $\lim_{\epsilon \to 0} \lambda^\epsilon = \lambda$ that

\[
\lim_{\epsilon \to 0} \left( \lambda^\epsilon \int_{\Omega^\epsilon} u^\epsilon w^\epsilon \rho^\epsilon \, dx \right) = \lambda \left( \int_{\Omega} u_1 w_1 \, dx + \int_{\Gamma} u_2 w_2 \, ds \right). \tag{3.37}
\]
Finally, combining (3.21), (3.31) and (3.37), we get

\[ \int_{\Omega} \nabla u_1 \cdot \nabla w_1 \, dx + \int_{\Gamma} (qru_1 w_1 - q \sqrt{ru_2} w_1 - q \sqrt{ru_1} w_2 + qu_2 w_2) \, ds = \lambda \left( \int_{\Omega} u_1 w_1 \, dx + \int_{\Gamma} u_2 w_2 \, ds \right). \]

(3.38)

By the density arguments equality (3.38) is valid for an arbitrary \((w_1, w_2) \in H^1(\Omega) \oplus L_2(\Gamma)\).

If \(r > 0\) then (3.38) is equivalent to equality

\[ \eta_{qr}[U, W] = \lambda(U, W)_{\mathcal{H}}, \quad \text{where} \quad U = (u_1, r^{-1/2} u_2), \quad W = (w_1, r^{-1/2} w_2), \]

whence, evidently,

\[ U \in \text{dom}(\mathcal{A}_{qr}), \quad \mathcal{A}_{qr} U = \lambda U, \]

and therefore, since \(u_1 \neq 0, \lambda\) is the eigenvalue of the operator \(\mathcal{A}_{qr}\). If \(r = 0\) then (3.38) implies

\[ U = (u_1, u_2) \in \text{dom}(\mathcal{A}_q), \quad \mathcal{A}_q U = \lambda U, \]

i.e. \(\lambda\) is the eigenvalue of the operator \(\mathcal{A}_q\).

Now, we inspect the case

\[ u_1 = 0. \]

We will prove that in this instance \(\lambda = q\). Recall (see (2.10), Lemma (2.1) and (2.12)) that the point \(q\) belongs to the essential spectrum of both operators \(\mathcal{A}_{qr}\) and \(\mathcal{A}_q\).

We express the eigenfunction \(u^e\) in the form

\[ u^e = v^e - g^e + w^e, \quad (3.39) \]

where

\[ v^e(x) = \begin{cases} 0, & x \in \Omega, \\ \frac{(u^e)_{x_0}}{\lambda_{x_0}} x_0, & x = (x', x_0) \in T^e, \\ (u^e)_{x_0}, & x \in B^e_i \end{cases} \]

and

\[ g^e = \sum_{i=1}^{k'-1} (v^e, u^e_i)_{H^e_i} u^e_i. \]

(recall, that \(\lambda^e = \lambda_{x_0}^e, u^e = u^e_{x_0}\)). It is clear that \(v^e \in H^1(\Omega^e), g^e \in \text{dom}(\mathcal{A}^e)\) and

\[ v^e - g^e \in \left( \text{span} \{u^e_1, \ldots, u^e_{k'-1} \} \right)^{\perp}. \]

(3.40)

One has the following Poincaré-type inequality:

\[ \sum_{i \in I^e_{x_0}} \int_{B^e_i} |v^e - (u^e)_{B^e_i}|^2 \, dx \leq C g^e(b^e)^2 \sum_{i \in I^e_{x_0}} \|\nabla u^e_i\|_{L_2(B^e_i)}^2 \leq C_1 g^e(b^e)^2. \]

(3.41)

Since \(b^e \geq d^e\) and \(r^e \leq C\) then

\[ \varrho^e(b^e)^2 = C r^e \frac{e^{p-1}}{(d^e)^{n-2}} \leq C r^e \frac{g^e^{p-1}}{(d^e)^{n-2}} = C_1 \left\{ \begin{array}{ll} D^e e^{n-1}, & n \geq 2, \\ e, & n = 2, \end{array} \right. \]

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and therefore in view of (2.6)
\[ g^e(b^e)^2 \to 0 \text{ as } e \to 0. \] (3.42)

It follows from (3.41), (3.42) that
\[ \sum_{i \in I^e} \int_{B_i^e} g^e \left| u^e - \langle u^e \rangle_{B_i^e} \right|^2 \, dx \to 0 \text{ as } e \to 0. \] (3.43)

Using estimate (3.12) we get
\[ \| u^e \|_{L^2(\bigcup_{i \in I^e} T^e_i)} \to 0 \text{ as } e \to 0. \] (3.44)

Taking into account \( \| u^e \|_{H^s} = 1 \) one can easily show that
\[ \sum_{i \in I^e} r^e e^{s-1} \left| \langle u^e \rangle_{B_i^e} \right|^2 = 1 - \| u^e \|_{L^2(\Omega)}^2 - \sum_{i \in I^e} \| u^e \|_{H^s(T^e_i)}^2 - \sum_{i \in I^e} \int_{B_i^e} g^e \left| u^e - \langle u^e \rangle_{B_i^e} \right|^2 \, dx, \]
whence, in view of (3.43), (3.44) and the fact that \( u_1 = 0 \), we obtain
\[ \sum_{i \in I^e} r^e e^{s-1} \left| \langle u^e \rangle_{B_i^e} \right|^2 = 1 + o(1) \text{ as } e \to 0. \] (3.45)

Using (3.45) one has the following asymptotics for the function \( v^e \):
\[ \| \nabla v^e \|_{L^2(\Omega^e)}^2 = q^e \sum_{i \in I^e} r^e e^{s-1} \left| \langle u^e \rangle_{B_i^e} \right|^2 = q + o(1) \text{ as } e \to 0, \] (3.46)
\[ \sum_{i \in I^e} \int_{B_i^e} g^e |v^e|^2 \, dx = \sum_{i \in I^e} r^e e^{s-1} \left| \langle u^e \rangle_{B_i^e} \right|^2 = 1 + o(1) \text{ as } e \to 0, \] (3.47)
\[ \sum_{i \in I^e} \| v^e \|_{L^2(T^e_i)}^2 = \frac{1}{3} q^e (h^e)^2 \sum_{i \in I^e} r^e e^{s-1} \left| \langle u^e \rangle_{B_i^e} \right|^2 = o(1) \text{ as } e \to 0. \] (3.48)

It follows from (3.47), (3.48) and \( v^e = 0 \) in \( \Omega \) that
\[ \| v^e \|_{H^s} = 1 + o(1) \text{ as } e \to 0. \] (3.49)

By virtue of (3.43), (3.44), (3.48) and the fact that \( \| u^e \|_{L^2(\Omega)}^2 \to 0 \) \( \| u_1 \|_{L^2(\Omega)}^2 = 0 \) one gets:
\[ \| u^e - v^e \|_{H^s}^2 \leq \sum_{i \in I^e} \int_{B_i^e} g^e \left| u^e - \langle u^e \rangle_{B_i^e} \right|^2 \, dx + 2 \| u^e \|_{L^2(\bigcup_{i \in I^e} T^e_i)}^2 + 2 \| v^e \|_{L^2(\bigcup_{i \in I^e} T^e_i)}^2 + \| u^e \|_{L^2(\Omega)}^2 \to 0. \] (3.50)

Using the equality \( (u^e, u^e_{k^e})_{H^s} = 0 \) for \( k = 1, \ldots, k^e - 1 \) and the Bessel inequality we obtain:
\[ \| g^e \|_{H^s}^2 = \sum_{k=1}^{k^e-1} \| (v^e, u^e_{k^e})_{H^s} \|^2 \leq \| v^e - u^e \|_{H^s}^2, \]
\[ \| \nabla g^e \|_{L^2(\Omega^e)}^2 = \sum_{k=1}^{k^e-1} \alpha_k^e \| (v^e, u^e_{k^e})_{H^s} \|^2 \leq \sum_{k=1}^{k^e-1} \alpha_k^e \| (v^e - u^e, u^e_{k^e})_{H^s} \|^2 \leq \alpha^e \| v^e - u^e \|_{H^s}^2. \]
and thus in view of (3.50)

$$\|g^e\|^2_{L^2} + \|\nabla g^e\|^2_{L^2} \to 0 \text{ as } \varepsilon \to 0.$$  \hfill (3.51)

Now let us estimate the remainder $w^e$. It is well-known (cf. [31]) that

$$\lambda^e = \inf \left\{ \frac{\|\nabla u\|^2_{L^2(\Omega^e)}}{\|u\|^2_{H^1}} : 0 \neq u \in \left( \text{span}\{u^e, \ldots, u^e_{e-1}\} \right)^\perp \right\}.$$  \hfill (3.52)

We denote $\tilde{v}^e = v^e - g^e$. Taking into account (3.15) and (3.40) we obtain from (3.52):

$$\|\nabla u^e\|^2_{L^2(\Omega^e)} \leq \frac{\|\nabla \tilde{v}^e\|^2_{L^2(\Omega^e)}}{\|\varepsilon\|^2_{H^1}}$$

or, using $u^e = \tilde{v}^e + w^e$,

$$\|\nabla w^e\|^2_{L^2(\Omega^e)} \leq -2(\nabla \tilde{v}^e, \nabla w^e)_{L^2(\Omega^e)} + \|\nabla \tilde{v}^e\|^2_{L^2(\Omega^e)} \left( \|\varepsilon\|^2_{H^1} - 1 \right).$$  \hfill (3.53)

In view of (3.46), (3.49), (3.51)

$$\|\nabla \tilde{v}^e\|^2_{L^2(\Omega^e)} \left( \|\varepsilon\|^2_{H^1} - 1 \right) \to 0 \text{ as } \varepsilon \to 0.$$  \hfill (3.54)

Let us estimate the first term in the right-hand-side of (3.53). One has

$$(\nabla \tilde{v}^e, \nabla w^e)_{L^2(\Omega^e)} = (\nabla \tilde{v}^e, \nabla u^e - \nabla \tilde{v}^e)_{L^2(\Omega^e)} + (\nabla \nabla \tilde{v}^e, \nabla \nabla \tilde{v}^e)_{L^2(\Omega^e)} - (\nabla \tilde{v}^e, \nabla \nabla w^e)_{L^2(\Omega^e)}.$$  \hfill (3.55)

Integrating by parts we obtain:

$$(\nabla \tilde{v}^e, \nabla u^e - \nabla \tilde{v}^e)_{L^2(\Omega^e)} = \sum_{i \in I^e} \int_{Y_i^e} \nabla \tilde{v}^e \cdot \nabla (u^e - \tilde{v}^e) \ dx = \sum_{i \in I^e} \int_{\partial Y_i^e} -\frac{\partial v^e}{\partial n} u^e \ ds + \int_{\partial Y_i^e} \frac{\partial v^e}{\partial n} (u^e - \langle u^e \rangle_{Y_i^e}) \ ds$$

$$= \frac{(q^e)^{n-1}|D|}{h^e} \sum_{i \in I^e} \langle u^e \rangle_{Y_i^e} \left( -\langle u^e \rangle_{Y_i^e} + \langle u^e \rangle_{Y_i^e} \right)$$

$$= q^e \varepsilon \sum_{i \in I^e} \langle u^e \rangle_{Y_i^e} \left( -\langle u^e \rangle_{Y_i^e} + \langle u^e \rangle_{Y_i^e} \right) \varepsilon^{n-1}. \hfill (3.56)$$

Then, using the Cauchy inequality, condition (2.6), estimates (3.3), (3.4), (3.45) and the fact that $u_1 = 0$ (and hence $\|u^e\|_{L^2(\Gamma)} \to 0$ as $\varepsilon \to 0$), we obtain from (3.56):

$$| (\nabla \tilde{v}^e, \nabla u^e - \nabla \tilde{v}^e)_{L^2(\Omega^e)} | \leq q^e \varepsilon \left( \sum_{i \in I^e} \langle u^e \rangle_{Y_i^e}^2 \right)^{\frac{1}{2}} \left( \sum_{i \in I^e} \langle u^e \rangle_{Y_i^e} + \langle u^e \rangle_{Y_i^e} - \langle u^e \rangle_{Y_i^e} \right)^{\frac{1}{2}} \varepsilon^{n-1}$$

$$\leq C_1 \sum_{i \in I^e} \left( \langle u^e \rangle_{Y_i^e}^2 \right)^{\frac{1}{2}} \varepsilon^{n-1} + \left( \langle u^e \rangle_{Y_i^e} + \langle u^e \rangle_{Y_i^e} \right)^{\frac{1}{2}} \varepsilon^{n-1} + \left( \langle u^e \rangle_{Y_i^e} - \langle u^e \rangle_{Y_i^e} \right)^{\frac{1}{2}} \varepsilon^{n-1}$$

$$\leq C_2 \left( \|u^e\|^2_{L^2(\Gamma)} + \varepsilon^{n-1} D^e \|\nabla u^e\|^2_{L^2(\bigcup_{i \in I^e} Y_i^e)} + \varepsilon^{n-1} D^e \|\nabla u^e\|^2_{L^2(\bigcup_{i \in I^e} B_i^e)} \right) \to 0 \text{ as } \varepsilon \to 0.$$  \hfill (3.57)

Further, in view of (3.46), (3.51),

$$\lim_{\varepsilon \to 0} (\nabla \tilde{v}^e, \nabla g^e)_{L^2(\Omega^e)} = 0.$$  \hfill (3.58)
And finally, using (3.15), (3.46) and (3.51), we obtain:

\[ |(\nabla g^\epsilon, \nabla w^\epsilon)|_{L^2(\Omega)} \leq |(\nabla g^\epsilon, \nabla u^\epsilon)|_{L^2(\Omega)} + |(\nabla g^\epsilon, \nabla v^\epsilon)|_{L^2(\Omega)} + \|\nabla g^\epsilon\|^2_{L^2(\Omega^\prime)} \rightarrow 0 \text{ as } \epsilon \rightarrow 0. \quad (3.59) \]

It follows from (3.55), (3.57)-(3.59) that

\[ \lim_{\epsilon \rightarrow 0} |(\nabla v^\epsilon, \nabla w^\epsilon)|_{L^2(\Omega^\prime)} = 0. \quad (3.60) \]

Combining (3.53), (3.54) and (3.60) we conclude that

\[ \lim_{\epsilon \rightarrow 0} \|\nabla w^\epsilon\|^2_{L^2(\Omega^\prime)} = 0 \]

and thus, in view of (3.39), (3.46), (3.51), (3.61), one gets

\[ \lambda^\epsilon = \|\nabla u^\epsilon\|^2_{L^2(\Omega^\prime)} \sim \|\nabla v^\epsilon\|^2_{L^2(\Omega^\prime)} \sim q \text{ as } \epsilon \rightarrow 0. \]

Q.E.D.

3.2.2. Proof of the property (B) of Hausdorff convergence

Let \( \lambda \in \sigma(\mathcal{A}_q) \) if \( r > 0 \) (respectively, \( \lambda \in \sigma(\mathcal{A}_q) \) if \( r = 0 \)). We have to prove that

\[ \exists \lambda^\epsilon \in \sigma(\mathcal{A}^\epsilon) : \lambda^\epsilon \rightarrow \lambda \text{ as } \epsilon \rightarrow 0. \quad (3.62) \]

Proving this indirectly we assume the opposite: a subsequence \( \epsilon_k, \epsilon_k \searrow 0 \) and a positive number \( \delta \) exist such that

\[ (\lambda - \delta, \lambda + \delta) \cap \sigma(\mathcal{A}^\epsilon) = \emptyset \text{ as } \epsilon = \epsilon_k. \quad (3.63) \]

Since \( \lambda \in \sigma(\mathcal{A}_q) \) (respectively, \( \lambda \in \sigma(\mathcal{A}_q) \)) there exists \( F = (f_1, f_2) \in L^2(\Omega) \oplus L^2(\Gamma) \), such that

\[ F \notin \text{im}(\mathcal{A}_q - \lambda I) \text{ (respectively, } F \notin \text{im}(\mathcal{A}_q - \lambda I)). \quad (3.64) \]

We introduce the function \( f^\epsilon \in \mathcal{H}^\epsilon \) by the formula

\[ f^\epsilon(x) = \begin{cases} f_1(x), & x \in \Omega, \\ 0, & x \in \bigcup_{i \in I^\epsilon} T^\epsilon_i, \\ \frac{1}{\sqrt{\epsilon}}(f_2)_i, & x \in B^\epsilon_i, \end{cases} \]

where \( f_2(x) = \sqrt{\epsilon} f_2(x) \) if \( r > 0 \) (respectively, \( f_2(x) = f_2(x) \) if \( r = 0 \)).

One has:

\[ \|f^\epsilon\|^2_{\mathcal{H}^\epsilon} = \|f_1\|^2_{L^2(\Omega)} + \frac{1}{\sqrt{\epsilon}} \sum_{i \in I^\epsilon} \|B^\epsilon_i\|_2 \|f_2\|^2_{L^2(\Gamma)} \leq \|f_1\|^2_{L^2(\Omega)} + \|f_2\|^2_{L^2(\Gamma)} \leq C \|F\|^2_{L^2(\Omega) \oplus L^2(\Gamma)}. \]

In view of (3.63), \( \lambda \) belongs to the resolvent set of \( \mathcal{A}^\epsilon \) as \( \epsilon = \epsilon_k \) and therefore there exists the unique \( u^\epsilon \in \text{dom}(\mathcal{A}^\epsilon) \) such that

\[ \mathcal{A}^\epsilon u^\epsilon - \lambda u^\epsilon = f^\epsilon, \quad \epsilon = \epsilon_k \]

and moreover the following estimates are valid as \( \epsilon = \epsilon_k \):

\[ \|\lambda u^\epsilon\|^2_{\mathcal{H}^\epsilon} \leq \delta^{-1} \|f^\epsilon\|^2_{\mathcal{H}^\epsilon} \leq C_1, \quad (3.65) \]

\[ \|\nabla u^\epsilon\|^2_{L^2(\Omega^\prime)} = \lambda \|u^\epsilon\|^2_{H^\epsilon} + (f^\epsilon, u^\epsilon)_{\mathcal{H}^\epsilon} \leq C_2. \quad (3.66) \]
It follows from (3.65), (3.66) that there exist a subsequence \( \varepsilon_k \subset \varepsilon \) and \( u_1 \in H^1(\Omega) \), \( u_2 \in L_2(\Gamma) \) such that (3.17), (3.20) hold (as \( \varepsilon = \varepsilon_k \to 0 \)).

One has for an arbitrary \( w \in H^1(\Omega^\varepsilon) \):

\[
\int_{\Omega^\varepsilon} \nabla u^\varepsilon \cdot \nabla w \, dx - \lambda \int_{\Omega^\varepsilon} u^\varepsilon w \, dx = \int_{\Omega^\varepsilon} f^\varepsilon w \, dx, \quad \varepsilon = \varepsilon_k. \tag{3.67}
\]

We plug into (3.67) the function \( w = w^\varepsilon(x) \) defined by formula (3.22) and pass to the limit as \( \varepsilon = \varepsilon_k \to 0 \). In the same way as above we obtain that \( (u_1, u_2) \) satisfies the equality

\[
\int_{\Omega} \nabla u_1 \cdot \nabla w_1 \, dx + \int_{\Gamma} (q u_1 w_1 - q \sqrt{r} u_2 w_1 - q \sqrt{r} u_1 w_2 + qu_2 w_2) \, ds - \lambda \left( \int_{\Omega} u_1 w_1 \, dx + \int_{\Gamma} u_2 w_2 \, ds \right) = \int_{\Omega} f_1 w_1 \, dx + \int_{\Gamma} f_2 w_2 \, ds, \tag{3.68}
\]

which holds for an arbitrary \( (w_1, w_2) \in C^\infty(\Omega) \oplus C^\infty(\Gamma) \) (and by the density arguments for an arbitrary \( (w_1, w_2) \in H^1(\Omega) \oplus L_2(\Gamma) \)). It follows easily from (3.68) that

- if \( r > 0 \) then \( U = (u_1, r^{-1/2} u_2) \in \text{dom}(A_{qr}) \) and \( A_{qr} U - \lambda U = F \),
- if \( r = 0 \) then \( U = (u_1, u_2) \in \text{dom}(A_q) \) and \( A_q U - \lambda U = F \).

We obtain a contradiction to (3.64). Thus there is \( \lambda^\varepsilon \in \sigma(\mathcal{A}) \) such that \( \lim_{\varepsilon \to 0} \lambda^\varepsilon = \lambda \) \quad Q.E.D.

3.2.3. Proof of Lemma 2.1

In the proof of Theorem 2.1 we use the fact that

\[
q \in \sigma(\mathcal{A}_{qr}). \tag{3.69}
\]

In this section we prove Lemma 2.1 containing, in particular, the property (3.69).

At first we study the point spectrum of the operator \( \mathcal{A}_{qr} \). Let \( \lambda \neq q \) be an eigenvalue of \( \mathcal{A}_{qr} \) corresponding to the eigenfunction \( U = (u_1, u_2) \neq 0 \). It means that \( \forall V = (v_1, v_2) \in H^1(\Omega) \oplus L_2(\Gamma) \)

\[
\int_{\Omega} \nabla u_1 \cdot \nabla v_1 \, dx + qr \int_{\Gamma} (u_1 - u_2)(v_1 - v_2) \, ds - \lambda \left( \int_{\Omega} u_1 v_1 \, dx + \int_{\Gamma} u_2 v_2 \, ds \right) = 0. \tag{3.70}
\]

One has \( u_1 \neq 0 \) (otherwise, plugging \( u_1 = 0 \) into (3.70) we arrive at \( u_2 = 0 \), that contradicts to \( U \neq 0 \)). Moreover, it is straightforward to show that if \( U = (u_1, u_2) \) satisfies (3.70) then \( u_2 = \frac{qu_1}{q^2 - \lambda} \) and \( u_1 \) satisfies

\[
\int_{\Omega} \nabla u_1 \cdot \nabla \bar{v}_1 \, dx - \frac{qr}{q - \lambda} \int_{\Gamma} u_1 \bar{v}_1 \, ds = \lambda \int_{\Omega} u_1 \bar{v}_1 \, dx, \quad \forall v_1 \in H^1(\Omega). \tag{3.71}
\]

Conversely if \( u_1 \in H^1(\Omega) \) satisfies (3.71) then \( U = (u_1, u_2) \), where \( u_2 = \frac{qu_1}{q^2 - \lambda} \), satisfies (3.70).

Let \( \mu \in \mathbb{R} \). By \( \eta^\mu \) we denote the sesquilinear form in \( L_2(\Omega) \) defined as follows:

\[
\eta^\mu [u, v] = \int_{\Omega} \nabla u \cdot \nabla \bar{v} \, dx - \mu \int_{\Gamma} u \bar{v} \, ds, \quad \text{dom}(\eta^\mu) = H^1(\Omega). \]
We denote by $\mathcal{P}$ the operator generated by this form. Formally the eigenvalue problem $\mathcal{P}u = \lambda u$
can be written as
\[
-\Delta u = \lambda u \text{ in } \Omega, \quad \frac{\partial u}{\partial n} = \mu u \text{ on } \Gamma, \quad \frac{\partial u}{\partial n} = 0 \text{ on } \partial\Omega \setminus \Gamma,
\]
i.e. $\mathcal{P}$ is the Laplacian in $\Omega$ subject to the Robin boundary conditions on $\Gamma$ and the Neumann
tones on $\partial\Omega \setminus \Gamma$.

The spectrum of $\mathcal{P}$ is purely discrete. We denote by $\{\lambda_k(\mu)\}_{k \in \mathbb{N}}$ the sequence of eigenvalues of
$\mathcal{P}$ written in the ascending order and repeated according to their multiplicity.

We denote by $\sigma_p(\mathcal{A})$ the set of eigenvalues of $\mathcal{A}$. It follows from the arguments above that
\[
\sigma_p(\mathcal{A}) \setminus \{q\} = \left\{ \lambda \in \mathbb{R} : \exists k \in \mathbb{N} \text{ such that } \lambda = \lambda_k(\mu), \text{ where } \mu = \frac{\lambda qr}{q - \lambda} \right\}, \tag{3.73}
\]
Using the minimax principle it not hard to prove the following well-known properties of the
eigenvalues of $\mathcal{P}$:

- for each $k \in \mathbb{N}$ the function $\mu \mapsto \lambda_k(\mu)$ is continuous and monotonically decreasing,
- for each $k \in \mathbb{N}$ $\lambda_k(\mu) \to +\infty$ as $\mu \to -\infty$, where $A^D_k$ is the $k$-th eigenvalue of the operator $\mathcal{A}^D$
acting in $L^2(\Omega)$ and generated by the form
\[
\eta^D[u,v] = \int_{\Omega} \nabla u \cdot \nabla v \, dx, \quad \text{dom}(\eta^D) = \{u \in H^1(\Omega) : u = 0 \text{ on } \Gamma\}
\]
(i.e. $\mathcal{A}^D$ is the Laplacian in $\Omega$ subject to the Dirichlet boundary conditions on $\Gamma$ and the
Neumann ones on $\partial\Omega \setminus \Gamma$),
- for each $k \in \mathbb{N}$ $\lambda_k(\mu) \to -\infty$ as $\mu \to +\infty$.

We denote by $\Upsilon$ the curve
\[
\Upsilon = \left\{ (\lambda, \mu) \in \mathbb{R}^2 : \mu = \frac{\lambda qr}{q - \lambda} \right\}.
\]
It consists of two branches $\Upsilon_+ = \{(\lambda, \mu) \in \Upsilon : \pm(q - \lambda) > 0\}$. We also introduce the curves $\Upsilon_k = \{(\lambda, \mu) \in \mathbb{R}^2 : \lambda = \lambda_k(\mu)\}, k \in \mathbb{N}$.

From the properties above we deduce the following:

- For each $k \in \mathbb{N}$ the curve $\Upsilon_k$ intersects the branch $\Upsilon_+$ exactly in one point (we denote the corresponding value of $\lambda$ by $\lambda^+_k$).
- We denote by $k_0$ the smallest integer satisfying $A^D_{k_0} \leq q$ and $A^D_{k_0+1} > q$. Then for each $k \in \mathbb{N}$
the curve $\Upsilon_{k+k_0}$ intersects the branch $\Upsilon_-$ exactly in one point (we denote the corresponding value of $\lambda$ by $\lambda^-_k$). For $k \leq k_0$ the curve $\Upsilon_k$ has no intersections with $\Upsilon_-$.
- [2.11] holds true.

---

\footnote{For the fulfillment of this property it is essential that $n \geq 2$. In the case $n = 1$ this property is violated. Namely, let us consider the problem $-u'' = \lambda u$ on $(0,T), u'(T) = \mu u(T), u'(0) = 0$. Its first eigenvalue $\lambda_1(\mu)$ goes to $-\infty$ as $\mu \to +\infty$, while for $k \geq 2 \lambda_k(\mu)$ goes to the $(k-1)$-th eigenvalue of the problem $-u'' = \lambda u$ on $(0,T), u(T) = 0, u'(0) = 0$.}
Thus, taking into account (3.73), we conclude that
\[
\sigma_p(\mathcal{A}_{qr}) \setminus \{q\} = \{\lambda^+_k, k = 1, 2, 3...\} \cup \{\lambda^-_k, k = 1, 2, 3...\}.
\] (3.74)

Since \(\lambda^+_k \nrightarrow q\) as \(k \to \infty\) then \(q \in \sigma_{\text{ess}}(\mathcal{A}_{qr})\).

It remains to prove that if \(\lambda \notin \mathcal{S} := \left(\bigcup_{k \in \mathbb{N}} \{\lambda^+_k\}\right) \cup \left(\bigcup_{k \in \mathbb{N}} \{\lambda^-_k\}\right)\) and \(\lambda \neq q\) then \(\lambda\) belongs to the resolvent set of \(\mathcal{A}_{qr}\). Namely, we have to show that the problem
\[
\mathcal{A}_{qr}V - \lambda V = F
\] (3.75)
has a solution \(V\) for an arbitrary \(F = (f_1, f_2) \in \mathcal{H}\).

For \(\nu \in \mathbb{R}\) we introduce the operator \(\mathcal{A}^\nu\) \((\nu \in \mathbb{R})\) acting in \(L^2(\Omega) \oplus L^2(\Gamma)\) and generated by the sesquilinear form
\[
\bar{\eta}[U, V] = \int_\Omega \nabla u_1 \cdot \nabla \overline{v_1} \, dx - \nu \int_\Omega u_1 \overline{v_1} \, ds, \quad \text{dom} (\bar{\eta}) = \left\{ U = (u_1, u_2) \in H^1(\Omega) \oplus L^2(\Gamma) : u_1|_\Gamma = u_2 \right\}.
\]
The resolvent equation \(\mathcal{A}^\nu U - \lambda U = G\) (where \(U = (u, u|_\Gamma), G = (g_1, g_2)\)) formally can be written as follows:
\[
\begin{cases}
-\Delta u - \lambda u = g_1 & \text{in } \Omega, \\
\frac{\partial u}{\partial n} - \nu u - \lambda u = g_2 & \text{on } \Gamma, \\
\frac{\partial u}{\partial n} = 0 & \text{on } \partial \Omega \setminus \Gamma.
\end{cases}
\]
The spectrum of \(\mathcal{A}^\nu\) is purely discrete.

Let us consider the set
\[
\mathcal{S} := \left\{ \lambda \in \mathbb{C} : \lambda\ is\ an\ eigenvalue\ of\ \mathcal{A}^\nu,\ where\ \nu = \frac{qr\lambda}{q - \lambda} \right\}.
\]
Evidently, if \(\lambda \notin \mathcal{S} \cup \{q\}\) then for an arbitrary \(G \in L^2(\Omega) \oplus L^2(\Gamma)\) the problem
\[
\mathcal{A}^\nu U - \lambda U = G, \text{ where } \nu = \frac{qr\lambda}{q - \lambda}
\] (3.76)
has a solution.

One can easily see that \(\lambda\) belongs to \(\mathcal{S}\) if and only if \(\lambda\) is an eigenvalue of the operator \(\mathcal{A}^\nu\), where \(\mu = \frac{qr\lambda}{q - \lambda}\). Using this and (3.73)-(3.74) we conclude that \(\mathcal{S} = \mathcal{S}\).

Thus, if \(\lambda \notin \mathcal{S} \cup \{q\}\) then for an arbitrary \(G = (g_1, g_2) \in L^2(\Omega) \oplus L^2(\Gamma)\) the problem (3.76) has a solution \(U = (u_1, u_2)\) (recall, that \(u_1|_\Gamma = u_2\)). Then we take \(g_1 := f_1, g_2 := \frac{qf_2}{q - \lambda}\). It is straightforward to show that \(V := (u_1, \frac{q}{q - \lambda}u_2)\) is a solution of (3.75). Q.E.D.

Obviously, all eigenvalues \(\lambda^+_k\) have finite multiplicity and are isolated points of \(\sigma(\mathcal{A}_{qr})\), whence \(\sigma_p(\mathcal{A}_{qr}) \setminus \{q\} = \sigma_{\text{disc}}(\mathcal{A}_{qr})\).

Lemma 2.1 is proved.

3.3. Proof of Theorem 2.7. the case \(q = \infty\)

We prove the property \(\text{(A)}\) of the Hausdorff convergence. Let \(\lambda^\varepsilon \in \sigma(\mathcal{A}^\varepsilon)\) and \(\lambda^\varepsilon \to \lambda\) as \(\varepsilon \to 0\), we have to show that either
\[
\lambda \in \sigma(\mathcal{A}) \text{ if } r > 0 \quad \text{or} \quad \lambda \in \sigma(\mathcal{A}) \text{ if } r = 0.
\] (3.77)
Again by \( u^\epsilon \) we denote the eigenfunction corresponding to \( \lambda^\epsilon \) and satisfying (3.13). In the same way as in the case \( q < \infty \) we conclude that there is a subsequence (still denoted by \( \epsilon \)) and \( u_1 \in H^1(\Omega) \), \( u_2 \in L_2(\Gamma) \) such that (3.17)-(3.20) hold true.

It is not hard to prove, using the trace inequality and the Poincaré inequality, the following estimate:

\[
\| u - \langle u \rangle_{\Gamma^\epsilon} \|^2_{L_2(\Gamma^\epsilon)} \leq C\epsilon \| \nabla u \|^2_{L_2(\Omega^\epsilon)}, \forall u \in H^1(\Omega^\epsilon).
\] (3.78)

Then, using (3.78) and Lemmata 3.1-3.2, we obtain:

\[
\lim_{\epsilon \to 0} \| \sqrt{r^\epsilon} u^\epsilon - \Pi^\epsilon u^\epsilon \|^2_{L^2(\Gamma)} = \lim_{\epsilon \to 0} \sum_{i \in I^\epsilon} \int_{\Gamma_i^\epsilon} \left| \sqrt{r^\epsilon} u^\epsilon - \sqrt{r^\epsilon} \langle u^\epsilon \rangle_{B_i^\epsilon} \right|^2 \, dx
\]

\[
\leq C \lim_{\epsilon \to 0} \left( r^\epsilon \sum_{i \in I^\epsilon} \| u^\epsilon - \langle u^\epsilon \rangle_{\Gamma_i^\epsilon} \|^2_{L^2(\Gamma_i^\epsilon)} + r^\epsilon \epsilon^{n-1} \sum_{i \in I^\epsilon} \| \langle u^\epsilon \rangle_{\Gamma_i^\epsilon} - \langle u^\epsilon \rangle_{\Gamma_{i-1}^\epsilon} \|^2_{L^2(\Gamma_i^\epsilon)} \right)
\]

\[
+ r^\epsilon \epsilon^{n-1} \sum_{i \in I^\epsilon} \| \langle u^\epsilon \rangle_{\Gamma_i^\epsilon} - \langle u^\epsilon \rangle_{\Gamma_{i-1}^\epsilon} \|^2_{L^2(\Gamma_i^\epsilon)} \right)
\]

\[
\leq C_1 \lim_{\epsilon \to 0} \left( r^\epsilon \sum_{i \in I^\epsilon} \| \nabla u^\epsilon \|^2_{L_2(\Omega^\epsilon)} + r^\epsilon \epsilon^{n-1} \sum_{i \in I^\epsilon} \| \nabla u^\epsilon \|^2_{L_2(\Omega^\epsilon)} \right) = 0,
\]

whence, in view of (3.19)-(3.20),

\[
u_2 = r^{1/2} u_1 \text{ on } \Gamma.
\] (3.79)

Also one has, using the equality \( \| \Pi^\epsilon u^\epsilon \|^2_{L_2(\Omega^\epsilon)} = \sum_{i \in I^\epsilon} g^\epsilon |B_i^\epsilon| \cdot |\langle u^\epsilon \rangle_{B_i^\epsilon}|^2 \),

\[
1 = \| u^\epsilon \|^2_{L_2(\Omega^\epsilon)} + \sum_{i \in I^\epsilon} \| u^\epsilon \|^2_{L_2(\Gamma_i^\epsilon)} + \| \Pi^\epsilon u^\epsilon \|^2_{L_2(\Omega^\epsilon)} + \sum_{i \in I^\epsilon} \int_{B_i^\epsilon} g^\epsilon |u^\epsilon - \langle u^\epsilon \rangle_{B_i^\epsilon}|^2 \, dx.
\] (3.80)

Here the second term tends to zero in view of Lemma 3.3, the last term tends to zero in view of (3.43) (the validity of (3.43) is independent of either \( q \) is finite or infinite). Thus, taking into account (3.79), we obtain from (3.80):

\[
1 = \| u_1 \|^2_{L_2(\Omega)} + r \| u_1 \|^2_{L_2(\Gamma)},
\]

whence,

\[
u_1 \neq 0.
\] (3.81)

For an arbitrary \( w \in H^1(\Omega^\epsilon) \) one has equality (3.21). This time we choose the test-function \( w \) as follows:

\[
w(x) = \tilde{w}^\epsilon(x) := \begin{cases} w(x) + \sum_{i \in I^\epsilon} (w(x^\epsilon_i) - w(x)) \varphi^\epsilon(x), & x \in \Omega, \\
w(x^\epsilon_i), & x \in T_i^\epsilon \cup B_i^\epsilon. \end{cases}
\] (3.82)

Here \( w \in C^\infty(\Omega) \) in an arbitrary function, the cut-off function \( \varphi^\epsilon(x) \) is the same as in (3.22).
We plug \( w = \tilde{w}^\varepsilon(x) \) into (3.21) and pass to the limit as \( \varepsilon \to 0 \). Using (3.17) we obtain
\[
\int_{\Omega^\varepsilon} \nabla u^\varepsilon \cdot \nabla \tilde{w}^\varepsilon \, dx = \int_{\Omega} \nabla u^\varepsilon \cdot \nabla w \, dx + \sum_{i \in I^\varepsilon} \int_{\Gamma_i^\varepsilon} \nabla u^\varepsilon \cdot \nabla \left( \left( w(x^{i,\varepsilon}) - w \right) \varphi_i^\varepsilon \right) \, dx \xrightarrow{\varepsilon \to 0} \int_{\Omega} \nabla u \cdot \nabla w \, dx
\]  
(3.83)

(the second integral vanishes because of the same arguments as those ones in the proof of (3.26)).

Now, let us study the right-hand-side of (3.21). We have:
\[
\lambda^\varepsilon \int_{\Omega^\varepsilon} u^\varepsilon \tilde{w}^\varepsilon \rho^\varepsilon \, dx = \lambda^\varepsilon \left( \int_{\Omega} u^\varepsilon w \, dx + \sum_{i \in I^\varepsilon} \int_{\Gamma_i^\varepsilon} u^\varepsilon \left( w(x^{i,\varepsilon}) - w \right) \varphi_i^\varepsilon \, dx \right.
\]
\[
+ \sum_{i \in I^\varepsilon} \int_{\Gamma_i^\varepsilon} u^\varepsilon w(x^{i,\varepsilon}) \, dx
\]
\[
\left. + \varepsilon \sum_{i \in I^\varepsilon} \int_{\Gamma_i^\varepsilon} u^\varepsilon \varphi_i^\varepsilon \, dx \right). \tag{3.84}
\]

One has, using (3.18),
\[
\int_{\Omega^\varepsilon} u^\varepsilon w \, dx \to \int_{\Omega} u_1 w \, dx \text{ as } \varepsilon \to 0.
\]

The second integral in (3.84) vanishes (here we use the same arguments as in (3.34)), the third integral also tends to zero as \( \varepsilon \to 0 \):
\[
\left| \sum_{i \in I^\varepsilon} \int_{\Gamma_i^\varepsilon} u^\varepsilon w(x^{i,\varepsilon}) \, dx \right|^2 \leq \sum_{i \in I^\varepsilon} \|u^\varepsilon\|^2_{L^2(T_i^\varepsilon)} \sum_{i \in I^\varepsilon} |T_i^\varepsilon| \leq C \varepsilon \|u^\varepsilon\|^2_{H_0^1} \to 0 \text{ as } \varepsilon \to 0.
\]

It remains to study the behaviour of the last integral in (3.84). One has:
\[
\varepsilon \sum_{i \in I^\varepsilon} \int_{\Gamma_i^\varepsilon} u^\varepsilon w(x^{i,\varepsilon}) \, dx = \varepsilon \sum_{i \in I^\varepsilon} \int_{\Gamma_i^\varepsilon} (u^\varepsilon)^\eta_{\varepsilon,i} \tilde{Q} \, dx = \sqrt{\varepsilon} \int_{\Gamma} \tilde{Q}^\varepsilon \tilde{Q}^\varepsilon w \, ds \xrightarrow{\varepsilon \to 0} \sqrt{\varepsilon} \int_{\Gamma} u_2 w \, ds.
\]

Thus, taking into account (3.79), we conclude that
\[
\lim_{\varepsilon \to 0} \lambda^\varepsilon \left( \int_{\Omega^\varepsilon} u^\varepsilon w^\varepsilon \rho^\varepsilon \, dx \right) = \lambda \left( \int_{\Omega} u_1 w \, dx + \int_{\Gamma} u_1 w \, ds \right). \tag{3.85}
\]

Combining (3.21), (3.83) and (3.85) we obtain:
\[
\int_{\Omega} \nabla u_1 \cdot \nabla w \, dx = \lambda \left( \int_{\Omega} u_1 w \, dx + \int_{\Gamma} u_1 w \, ds \right). \tag{3.86}
\]

Since \( u_1 \neq 0 \) then it follows easily from (3.86) that either \( \lambda \) is the eigenvalue of \( A \), if \( r > 0 \) or \( \lambda \) is the eigenvalue of \( A \) if \( r = 0 \). Thus the property (A) of the Hausdorff convergence is proved.

The property (B) of the Hausdorff convergence is proved in the same way as that one for the case \( q < \infty \) (using the test-function \( w = \tilde{w}^\varepsilon(x) \) defined below by (3.82) instead of \( w = w^\varepsilon(x) \) defined by (3.22)).

Theorem 2.1 is proved.
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