ON THE ENTANGLED ERGODIC THEOREM

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Abstract. Let \( U \) be a unitary operator acting on the Hilbert space \( \mathcal{H} \), and \( \alpha : \{1, \ldots, m\} \mapsto \{1, \ldots, k\} \) a partition of the set \( \{1, \ldots, m\} \). We show that the ergodic average

\[
\frac{1}{N^k} \sum_{n_1, \ldots, n_k=0}^{N-1} U^{n_\alpha(1)} A_1 U^{n_\alpha(2)} \cdots U^{n_\alpha(m-1)} A_{m-1} U^{n_\alpha(m)}
\]

converges in the weak operator topology if the \( A_j \) belong to the algebra of all the compact operators on \( \mathcal{H} \). We write explicitly the formula for these ergodic averages in the case of pair–partitions. Some results without any restriction on the operators \( A_j \) are also presented in the almost periodic case.

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1. Introduction

An entangled ergodic theorem was formulated in [1] in connection with the quantum central limit theorem. Namely, let \( U \) be a unitary operator on the Hilbert space \( \mathcal{H} \), and for \( m \geq k \), \( \alpha : \{1, \ldots, m\} \mapsto \{1, \ldots, k\} \) a partition of the set \( \{1, \ldots, m\} \). The entangled ergodic theorem concerns the convergence in the strong, or merely weak (s–limit, or w–limit for short) operator topology, of the multiple Cesaro mean

\[
\frac{1}{N^k} \sum_{n_1, \ldots, n_k=0}^{N-1} U^{n_\alpha(1)} A_1 U^{n_\alpha(2)} \cdots U^{n_\alpha(2k-1)} A_{m-1} U^{n_\alpha(m)}
\]

\( A_1, \ldots, A_{m-1} \) being bounded operators acting on \( \mathcal{H} \).

Expressions like (1.1) naturally appear also in [3], in the study of the multiple mixing. The entangled ergodic theorem is a generalization of

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1A partition \( \alpha : \{1, \ldots, m\} \mapsto \{1, \ldots, k\} \) of the set made of \( m \) elements in \( k \) parts is nothing but a surjective map, the parts of \( \{1, \ldots, m\} \) being the preimages \( \{\alpha^{-1}(\{j\})\}_{j=1}^k \).
the well–known mean ergodic theorem due to von Neumann (cf. [4])

\[
\lim_{N} \frac{1}{N} \sum_{n=0}^{N-1} U^n = E_1,
\]

\(E_1\) being the selfadjoint projection onto the eigenspace of the invariant vectors for \(U\).

Therefore, it is natural to address the systematic investigation of the conditions under which the entangled ergodic theorem holds true. A first attempt was done in [2], where some facts concerning the structure of the above ergodic average were pointed out. Unfortunately, there is yet no general result on the entangled ergodic theorem.

In the present note, we show that the ergodic average (1.1) converges in the weak operator topology if the \(A_k\) belong to \(K(\mathcal{H})\), the algebra of all the compact operators acting on \(\mathcal{H}\). We write down the formula for those ergodic averages in the case of pair–partitions as

\[
\lim_{N} \left\{ \frac{1}{N^k} \sum_{n_1, \ldots, n_k=0}^{N-1} U^{n_{\alpha(1)}} A_1 U^{n_{\alpha(2)}} \cdots U^{n_{\alpha(2k-1)}} A_{2k-1} U^{n_{\alpha(2k)}} \right\}
= \sum_{z_1, \ldots, z_k \in \sigma_{pp}(U)} E_{z_\alpha(1)}^\# A_1 E_{z_\alpha(2)}^\# \cdots E_{z_\alpha(2k-1)}^\# A_{2k-1} E_{z_\alpha(2k)}^\#
\]

(see below for the precise definition of the r.h.s. of this formula), and conjecture that it holds true for each set \(A_1, \ldots, A_{2k-1}\) of bounded operators.

Finally, we present some results on the entangled ergodic theorem relative to the case when the dynamics induced by the unitary \(U\) on \(\mathcal{H}\) is almost periodic, that is when \(\mathcal{H}\) is generated by the eigenvectors of \(U\) (cf. [3]) without any restriction on the operators \(A_i\).

2. THE ENTANGLED ERGODIC THEOREM FOR COMPACT OPERATORS

Let \(U \in \mathcal{B}(\mathcal{H})\) be a unitary operator, and \(\alpha : \{1, \ldots, 2k\} \mapsto \{1, \ldots, k\}\) a pair–partition of the set \(\{1, \ldots, 2k\}\). Define

\[
\sigma_{pp}^a(U) := \{z \in \sigma_{pp}(U) \mid zw = 1 \text{ for some } w \in \sigma_{pp}(U)\}
\]

where \(\sigma_{pp}(U) = \{z \in \mathbb{T} \mid z \text{ is an eigenvalue of } U\}\), see e.g. [4].

Consider, for each finite subset \(F \in \sigma_{pp}^a(U)\) and \(\{A_1, \ldots, A_{2k-1}\} \subset \mathcal{B}(\mathcal{H})\), the following operator

\[
S_{\alpha; A_1, \ldots, A_{2k-1}}^F := \sum_{z_1, \ldots, z_k \in F} E_{z_\alpha(1)}^\# A_1 E_{z_\alpha(2)}^\# \cdots E_{z_\alpha(2k-1)}^\# A_{2k-1} E_{z_\alpha(2k)}^\#
\]
together with the sesquilinear form

$$s^F_{\alpha;A_1,\ldots,A_{2k-1}}(x,y) := \langle S^F_{\alpha;A_1,\ldots,A_{2k-1}} x, y \rangle,$$

where the pairs $z_{\alpha(i)}^\#$ are alternatively $z_j$ and $\bar{z}_j$ whenever $\alpha(i) = j$, and $E_z$ is the selfadjoint projection on the eigenspace corresponding to the eigenvalue $z \in \sigma_{pp}(U)$.

**Lemma 2.1.** We have for the above sesquilinear form,

$$|s^F_{\alpha;A_1,\ldots,A_{2k-1}}(x,y)| \leq \|x\|\|y\| \prod_{j=1}^{2k-1} \|A_j\|,$$

uniformly for $F$ finite subsets of $\sigma_{pp}(U)$.

**Proof.** To simplify matter, we deal with a particular case. The computation can be easily generalized to all the situations. Consider for example the pair–partition $\alpha$ of six elements given by $\{1,2,1,3,2,3\}$.

\[\text{If for example, } \alpha \text{ is the pair–partition } \{1,2,1,2\} \text{ of four elements, we write}\]

$$S^F_{\alpha;A,B,C} = \sum_{z,w \in F} E_z A E_w B E_z C E_{\bar{w}}.$$  

If we have the trivial pair–partition of two elements, we write $S^F_A = \sum_{z \in F} E_z A E_{\bar{z}}$, and $S_A = \sum_{z \in \sigma_{pp}^+(U)} E_z A E_{\bar{z}}$ for its limit in the strong operator topology (cf. Lemma 2.2), omitting the subscript $\alpha$.\[\text{2}\]
We get
\[
|s^F_{\alpha; A_1, \ldots, A_5}(x, y)|^2 \leq \| y \|^2 \left( \sum_{z \in F} E_z A_1 \sum_{w \in F} E_w A_2 E_z A_3 \sum_{\zeta \in F} E_\zeta A_4 E_\zeta A_5 E_\zeta x \right)^2
\]
\[
= \| y \|^2 \sum_{z \in F} \| E_z A_1 \sum_{w \in F} E_w A_2 E_z A_3 \sum_{\zeta \in F} E_\zeta A_4 E_\zeta A_5 E_\zeta x \|^2
\]
\[
\leq (\| y \| \| A_1 \|)^2 \sum_{z \in F} \left( \sum_{w \in F} E_w A_2 E_z A_3 \sum_{\zeta \in F} E_\zeta A_4 E_\zeta A_5 E_\zeta x \right)^2
\]
\[
= \| y \| \| A_1 \| \| A_2 \|^2 \sum_{w, \zeta \in F} \| E_\zeta A_4 E_\zeta A_5 E_\zeta x \|^2
\]
\[
\leq \| y \| \prod_{j=1}^3 \| A_j \|^2 \sum_{w, \zeta \in F} \| E_\zeta A_4 E_\zeta A_5 E_\zeta x \|^2
\]
\[
\leq \| y \| \prod_{j=1}^4 \| A_j \|^2 \sum_{\zeta \in F} \sum_{w \in F} \| E_w A_5 E_\zeta x \|^2
\]
\[
\leq \| y \| \prod_{j=1}^5 \| A_j \|^2 \| E_\zeta x \|^2 \leq (\| y \| \| A_1 \| \prod_{j=1}^4 \| A_j \|)^2,
\]
where the previous computation follows from the Schwarz and Bessel inequalities, and Pythagoras theorem. \(\square\)

**Lemma 2.2.** The net \( \{ \sum_{z \in F} E_z A E_z \mid F \text{ finite subset of } \sigma_{pp}^* (U) \} \) converges in the strong operator topology.

**Proof.**
\[
\left\| \sum_{z \in F} E_z A E_z x - \sum_{z \in G} E_z A E_z x \right\| \leq \left\| \sum_{z \in F \setminus G} E_z A E_z x \right\| + \left\| \sum_{z \in G \setminus F} E_z A E_z x \right\|.
\]

Thus, as the strong operator topology is complete, it is enough to prove that for \( \varepsilon > 0 \), there exists a finite set \( G_\varepsilon \), such that \( \left\| \sum_{z \in H} E_z A E_z x \right\| < \frac{\varepsilon}{3} \).
whenever \( H \subset G_x \). But,
\[
\| \sum_{z \in H} E_z A E_z x \|^2 = \sum_{z \in H} \| E_z A E_z x \|^2 \leq \| A \|^2 \sum_{z \in H} \| E_z x \|^2 .
\]
The proof follows as the last sum is convergent. \( \Box \)

**Proposition 2.3.** The net \( \{ S_{\alpha; A_1, \ldots, A_{2k-1}}^F \mid F \text{ finite subset of } \sigma_{pp}(U) \} \) converges in the weak operator topology for each finite set \( \{ A_1, \ldots, A_{2k-1} \} \subset \mathcal{B}(\mathcal{H}) \).

**Proof.** By Lemma 2.1 and Theorem II.1.3 of [5], it is enough to show that the \( \{ s_{A_1, \ldots, A_{2k-1}}^F(x, y) \} \) converges for each \( x, y \in \mathcal{H} \). We can also suppose without loss of generality that \( x \in \mathcal{H} \) is an eigenvector of \( U \) with eigenvalue \( z_0 \). The proof is by induction on \( k \). By Lemma 2.2 it is enough to show that the assertion holds for the pair–partition \( \beta : \{ 1, \ldots, 2k+2 \} \rightarrow \{ 1, \ldots, k+1 \} \), whenever it is true for any pair–partition \( \alpha : \{ 1, \ldots, 2k \} \rightarrow \{ 1, \ldots, k \} \). Let \( k_\beta \in \{ 1, \ldots, 2k+2 \} \) be the first element of the pair \( \beta^{-1}(\{ k+1 \}) \), and \( \alpha_\beta \) the pair–partition of \( \{ 1, \ldots, 2k \} \) obtained by deleting \( \beta^{-1}(\{ k+1 \}) \) from \( \{ 1, \ldots, 2k+2 \} \), and \( k+1 \) from \( \{ 1, \ldots, k \} \). We obtain
\[
s_{\beta; A_1, \ldots, A_{2k+1}}^F(x, y) = s_{\alpha_\beta; A_1, \ldots, A_{2k-1} E_{k_\beta} A_{k_\beta+1} \ldots, A_{2k}}^F(A_{2k+1} x, y)
\]
whenever \( F \) is big enough, such that \( z_0 \in F \). Thus in our situation,
\[
\lim_{F \uparrow \sigma_{pp}(U)} s_{\beta; A_1, \ldots, A_{2k+1}}^F(x, y) = \langle S_{\alpha_\beta; A_1, \ldots, A_{2k-1} E_{k_\beta} A_{k_\beta+1} \ldots, A_{2k}}^F A_{2k+1} x, y \rangle ,
\]
\( S_{\alpha; A_1, \ldots, A_{2k-1}} \) being the limit in the weak operator topology of \( S_{\alpha; A_1, \ldots, A_{2k-1}} \) which exists by hypothesis. \( \Box \)

**Proposition 2.3** together with (2.1) allow us to define, and write symbolically for each finite subset \( \{ A_1, \ldots, A_{2k-1} \} \subset \mathcal{K}(\mathcal{H}) \),
\[
(2.2) \quad S_{\alpha; A_1, \ldots, A_{2k-1}} := \text{w} - \lim_{F \uparrow \sigma_{pp}(U)} S_{\alpha; A_1, \ldots, A_{2k-1}}^F
\]
\[
= \sum_{z_{\alpha(1)} \ldots z_{\alpha(2k)} \in \sigma_{pp}(U)} E_{z_{\alpha(1)}} A_{z_{\alpha(2)}} \ldots E_{z_{\alpha(2k-1)}} A_{2k-1} E_{z_{\alpha(2k)}} ,
\]
where in (2.2) the pairs \( z_{\alpha(i)} \) are alternatively \( z_j \) and \( \bar{z}_j \) whenever \( \alpha(i) = j \) as in (2.1).

**Proposition 2.4.** If \( A \in \mathcal{K}(\mathcal{H}) \), then
\[
\text{w} - \lim_{N} \frac{1}{N} \sum_{n=0}^{N-1} U^n A U^n = S_A .
\]
Proof. Consider \( \left\langle \frac{1}{N} \sum_{n=0}^{N-1} U^n A U^n x, y \right\rangle \). By Lemma 2.1, we can suppose without loss of generality that \( A = \langle \cdot, \eta \rangle \xi \). We can also suppose that some of the vectors \( x, y, \xi, \eta \) are eigenvectors of \( U \) if needed (see below).

We have

\[
\left\langle \frac{1}{N} \sum_{n=0}^{N-1} U^n U^n x, y \right\rangle = \frac{1}{N} \sum_{n=0}^{N-1} \langle U^n x, \eta \rangle \langle U^n \xi, y \rangle.
\]

We first suppose that \( x, y, \xi, \eta \in H_{\text{cont}} \), the last being the subspace of \( H \) made of all vectors with continuous spectral measure on the unit circle (cf. [4], Section VII.2). In this situation, we compute

\[
\frac{1}{N} \sum_{n=0}^{N-1} \langle U^n x, \eta \rangle \langle U^n \xi, y \rangle = \int \int_{T^2} \left( \frac{1}{N} \sum_{n=0}^{N-1} (zw)^n \right) f_{x,\eta}(z) f_{\xi,y}(w) \, d|\mu_{x,\eta}|(z) \, d|\mu_{\xi,y}|(w),
\]

where \( |\mu_{x,\eta}|, |\mu_{\xi,y}| \) are atomless positive bounded measures.\(^3\) As it was shown in [2], \( \frac{1}{N} \sum_{n=0}^{N-1} (zw)^n \) converges pointwise to indicator of the antidiagonal of the two–dimensional torus \( T^2 \), the last being negligible w.r.t. the product measure \( |\mu_{x,\eta}| \times |\mu_{\xi,y}| \). This means that if \( x, y, \xi, \eta \in H_{\text{cont}} \), the ergodic mean under consideration is zero. The same happens if only one of the pairs \( x, \eta \) or \( \xi, y \) belongs to \( H_{\text{cont}} \). Namely, we suppose without loss of generality that \( x, \eta \in H_{\text{cont}} \) and, say \( y \) is a eigenvector of \( U \) with eigenvalue \( w_0 \). We have

\[
\frac{1}{N} \sum_{n=0}^{N-1} \langle U^n x, \eta \rangle \langle U^n \xi, y \rangle = \langle \xi, y \rangle \int \left( \frac{1}{N} \sum_{n=0}^{N-1} (zw_0)^n \right) f_{x,\eta}(z) \, d|\mu_{x,\eta}|(z).
\]

In this situation, \( \frac{1}{N} \sum_{n=0}^{N-1} (zw_0)^n \) converges pointwise to the indicator of \( w_0 \) which is negligible w.r.t. the measure \( |\mu_{x,\eta}| \). Thus, we first

\(^3\) The measures \( |\mu_{x,y}| \) and the measurable functions \( f_{x,y} \), \( x, y \in H \) are the total variation measures of \( d\mu_{x,y}(z) := \langle E(dz) x, y \rangle \) and the corresponding densities, \( \{ E(z) \mid z \in T \} \) being the resolution of the identity of the unitary \( U \).
conclude that
\[
\frac{1}{N} \sum_{n=0}^{N-1} \langle U^n x, \eta \rangle \langle U^n \xi, y \rangle = 0 = \sum_{z \in \sigma_{pp}(U)} \langle E_z x, \eta \rangle \langle E_\bar{z} \xi, y \rangle = \langle S_A x, y \rangle
\]
if at least one of the pairs \(x, \eta\) or \(\xi, y\) belongs to \(\mathcal{H}_{\text{cont}}\). Second, \(\frac{1}{N} \sum_{n=0}^{N-1} \langle U^n x, \eta \rangle \langle U^n \xi, y \rangle \) can be nonnull only if at least one of the elements of the pairs \(x, \eta\) or \(\xi, y\), say \(\xi, \eta\), belong to \(\mathcal{H}_{pp}\). As previously explained, we can suppose that \(\xi, \eta\) are eigenvectors of \(U\) with eigenvalues \(z_0, w_0\). In this situation,
\[
\frac{1}{N} \sum_{n=0}^{N-1} \langle U^n x, \eta \rangle \langle U^n \xi, y \rangle = \langle x, \eta \rangle \langle \xi, y \rangle \frac{1}{N} \sum_{n=0}^{N-1} (z_0 w_0)^n
\]
\[
\rightarrow \langle x, \eta \rangle \langle \xi, y \rangle \delta_{z_0, w_0} = \sum_{z \in \sigma_{pp}(U)} \langle E_z x, \eta \rangle \langle E_\bar{z} \xi, y \rangle = \langle S_A x, y \rangle.
\]

The proof follows by orthogonality, as the cases considered above exhaust all the possibilities. \(\square\)

Here, there is the announced entangled ergodic theorem for pair–partitions and compact operators.

**Theorem 2.5.** Let \(\{A_1, \ldots, A_{2k-1}\} \subset \mathcal{K}(\mathcal{H})\), and \(\alpha : \{1, \ldots, 2k\} \mapsto \{1, \ldots, k\}\) a pair–partition of the set \(\{1, \ldots, 2k\}\). Then
\[
w^{-}\lim_N \left\{ \frac{1}{N^k} \sum_{n_1, \ldots, n_k = 0}^{N-1} U^{n_\alpha(1)} A_1 U^{n_\alpha(2)} \ldots U^{n_\alpha(2k-1)} A_{2k-1} U^{n_\alpha(2k)} \right\} = S_{\alpha; A_1, \ldots, A_{2k-1}},
\]
where \(S_{\alpha; A_1, \ldots, A_{2k-1}}\) is given in \((2.2)\).

**Proof.** We start by noticing that
\[
\left\| \frac{1}{N^k} \sum_{n_1, \ldots, n_k = 0}^{N-1} U^{n_\alpha(1)} A_1 U^{n_\alpha(2)} \ldots U^{n_\alpha(2k-1)} A_{2k-1} U^{n_\alpha(2k)} \right\| \leq \prod_{j=1}^{2k-1} \|A_j\|.
\]

Thanks to this and Lemma 2.1 as \(\{A_1, \ldots, A_{2k-1}\} \subset \mathcal{K}(\mathcal{H})\), we can suppose that the \(A_j\) are rank one. Indeed, put \(K : = \max_{1 \leq j \leq 2k-1} \|A_j\|\). Choose finite rank operators \(A_j^\epsilon\), such that \(\|A_j^\epsilon\| \leq K\) and \(\|A_j - A_j^\epsilon\| <\)
Theorem 2.6. For $m \geq k$, let $\alpha : \{1, \ldots, m\} \mapsto \{1, \ldots, k\}$ be a partition of the set $\{1, \ldots, m\}$. If $\{A_1, \ldots, A_{m-1}\} \subset K(\mathcal{H})$, then the ergodic average

\begin{equation}
\frac{1}{Nk} \sum_{n_1, \ldots, n_k=0}^{N-1} U^{n_\alpha(1)} A_1 U^{n_\alpha(2)} \ldots U^{n_\alpha(m-1)} A_{m-1} U^{n_\alpha(m)}
\end{equation}

for general partitions of any finite set $\{1, \ldots, 2k-1\}$. Then

\[
\left| \left\langle \frac{1}{N^k} \sum_{n_1, \ldots, n_k=0}^{N-1} U^{n_\alpha(1)} A_1 U^{n_\alpha(2)} \ldots U^{n_\alpha(2k-1)} A_{2k-1} U^{n_\alpha(2k)} x, y \right\rangle \right|
\]

\[
= \left| \left\langle \frac{1}{N^k} \sum_{n_1, \ldots, n_k=0}^{N-1} U^{n_\alpha(1)} A_1 U^{n_\alpha(2)} \ldots U^{n_\alpha(2k-1)} A_{2k-1} U^{n_\alpha(2k)} x, y \right\rangle \right|
\]

Thus, for rank one operators $A_j$, we obtain

\[
\left\langle \frac{1}{N^k} \sum_{n_1, \ldots, n_k=0}^{N-1} U^{n_\alpha(1)} A_1 U^{n_\alpha(2)} \ldots U^{n_\alpha(2k-1)} A_{2k-1} U^{n_\alpha(2k)} x, y \right\rangle
\]

\[
= \prod_{j=1}^{k} \left\langle \frac{1}{N} \sum_{n_j=0}^{N-1} U^{n_j} x_j, y_j \right\rangle \left\langle U^{n_j} \xi_j, \eta_j \right\rangle \equiv \prod_{j=1}^{k} \frac{1}{N} \sum_{n_j=0}^{N-1} \left\langle U^{n_j} B_j U^{n_j} \xi_j, \eta_j \right\rangle
\]

with $B_j = \langle \cdot, \eta_j \rangle x_j$. Here, $\{x_j, y_j, \xi_j, \eta_j\}_{j=1}^{k}$ are suitable vectors depending on the $A_i$ and $x, y$. By Proposition 2.4, we obtain

\[
\left\langle \frac{1}{N^k} \sum_{n_1, \ldots, n_k=0}^{N-1} U^{n_\alpha(1)} A_1 U^{n_\alpha(2)} \ldots U^{n_\alpha(2k-1)} A_{2k-1} U^{n_\alpha(2k)} x, y \right\rangle
\]

\[
= \prod_{j=1}^{k} \frac{1}{N} \sum_{n_j=0}^{N-1} \left\langle U^{n_j} B_j U^{n_j} \xi_j, y_j \right\rangle \longrightarrow \prod_{j=1}^{k} \left\langle S_B \xi_j, y_j \right\rangle
\]

We end the present section by proving the entangled ergodic theorem for general partitions of any finite set $\{1, \ldots, m\}$, and for compact operators $\{A_1, \ldots, A_{m-1}\}$.
converges in the weak operator topology.

Proof. As before, it is enough to show that

\[
\left\langle \frac{1}{N^k} \sum_{n_1,\ldots,n_k=0}^{N-1} U_{n_1}(1) A_1 U_{n_2}(2) \cdots U_{n_{m-1}}(m-1) A_{m-1} U_{n_m}(m) x, y \right\rangle
\]

converges for every \( x, y \in \mathcal{H} \), whenever the \( A_j \) are rank one operators. But, in this situation,

\[
\left\langle \frac{1}{N^k} \sum_{n_1,\ldots,n_k=0}^{N-1} U_{n_1}(1) A_1 U_{n_2}(2) \cdots U_{n_{m-1}}(m-1) A_{m-1} U_{n_m}(m) x, y \right\rangle
\]

\[
= \prod_{j=1}^{k} \left( \int \cdots \int_{\Gamma} \left( \frac{1}{N} \sum_{n_j=0}^{N-1} \prod_{p \mid \alpha(p)=j} z_p \right)^n j \right) \prod_{p \mid \alpha(p)=j} \langle E(dz_p) x_{p,j}, y_{p,j} \rangle
\]

where we have used the Lebesgue dominated convergence theorem. Here, the \( x_{p,j}, y_{p,j} \) are vectors uniquely determined by the rank one operators \( A_1, \ldots, A_{m-1} \) and vectors \( x, y \), and \( \chi_{\Gamma} \) denotes the indicator of the set \( \Gamma \).

We notice that it seems difficult to provide an explicit formula for the weak limit of (2.3) similar to that in Theorem 2.5 relative to the case of pair-partitions.

3. The Almost Periodic Case

The present section deals with some cases relative to the almost periodic situation, without any restriction relative to the operators appearing in (1.1). We then suppose that \( \mathcal{H} \) is generated by the eigenvectors of \( U \).

Proposition 3.1. In the almost periodic case,

\[
s-\lim_N \frac{1}{N} \sum_{n=0}^{N-1} U^n A U^n = S_A
\]

for each \( A \in \mathcal{B}(\mathcal{H}) \).
Proof. By our assumptions, we can suppose that $x$ is an eigenvector of $U$ with eigenvalue $z_0$. We have

$$
\frac{1}{N} \sum_{n=0}^{N-1} U^n U x = \frac{1}{N} \sum_{n=0}^{N-1} (z_0 U)^n A x
$$

$$
= \left( \frac{1}{N} \sum_{n=0}^{N-1} (z_0 U)^n \right) A x \rightarrow_N E_{z_0} A x \equiv S_A x.
$$

Here, we have used the mean ergodic theorem (cf. [4]), and as usual, $E_z \equiv E(\{ z \})$ is the selfadjoint projection onto the eigenspace corresponding to the eigenvalue $z \in \mathbb{T}$.

Now we treat the cases relative to all the pair partitions of four elements.

**Theorem 3.2.** Suppose that the dynamics induced by the unitary $U$ on $\mathcal{H}$ is almost periodic. Then

$$
\lim_{N \rightarrow \infty} \left\{ \frac{1}{N^2} \sum_{n_1,n_2=0}^{N-1} U^{n_{\alpha(1)}} A U^{n_{\alpha(2)}} B U^{n_{\alpha(3)}} C U^{n_{\alpha(4)}} x, y \right\}
$$

$$
=S_{\alpha;A,B,C}
$$

for each pair–partition $\alpha$ of four elements, and every $\{A, B, C\} \subset \mathcal{B}(\mathcal{H})$.

Proof. As previously explained, we can suppose that $x, y \in \mathcal{H}$ are eigenvectors of $U$ with eigenvalues $z_0, w_0$, respectively.

Suppose that $\alpha$ is the partition $\{1, 1, 2, 2\}$. Then, by Proposition 3.1

$$
\left\langle \frac{1}{N^2} \sum_{k,n=0}^{N-1} U^k A U^n B U^n C U^n x, y \right\rangle
$$

$$
= \left\langle \left( \frac{1}{N} \sum_{k=0}^{N-1} U^k A U^k \right) B \left( \frac{1}{N} \sum_{n=0}^{N-1} U^n C U^n \right) x, y \right\rangle
$$

$$
\rightarrow_N \left\langle S_A B S_C x, y \right\rangle \equiv \left\langle S_{\alpha;A,B,C} x, y \right\rangle.
$$
Let $\alpha$ be the partition $\{1, 2, 2, 1\}$. Then, again by Proposition 3.1,

$$\left\langle \frac{1}{N^2} \sum_{k,n=0}^{N-1} U^k A U^n B U^n C U^k x, y \right\rangle$$

$$= \left( \frac{1}{N} \sum_{k=0}^{N-1} (z_0 w_0)^k \right) \left\langle \frac{1}{N} \sum_{n=0}^{N-1} A U^n B U^n C x, y \right\rangle$$

$$\rightarrow_N \delta_{z_0,w_0} \left\langle A S_B C x, y \right\rangle \equiv \left\langle S_{\alpha;A,B,C} x, y \right\rangle.$$

Finally, if $\alpha$ is the partition $\{1, 2, 1, 2\}$, then by the mean ergodic theorem,

$$\left\langle \frac{1}{N^2} \sum_{k,n=0}^{N-1} U^k A U^n B U^n C U^k x, y \right\rangle$$

$$= \left\langle A \left( \frac{1}{N} \sum_{k=0}^{N-1} (z_0 U)^k \right) B \left( \frac{1}{N} \sum_{n=0}^{N-1} (w_0 U)^n \right) C x, y \right\rangle$$

$$\rightarrow_N \left\langle A E_{z_0} B E_{w_0} C x, y \right\rangle \equiv \left\langle S_{\alpha;A,B,C} x, y \right\rangle.$$

□

To end the present section by noticing that a general entangled ergodic theorem is not yet available even in the almost periodic case. However, Proposition 3.1 and Theorem 3.2 allow us to treat, always in the almost periodic case, other situations relative to pair–partitions of sets with more than four elements.

4. OUTLOOK

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