Tait coloring and a moduli space

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Abstract

We associate a moduli space $\mathcal{M}(G)$ to a planar trivalent graph $G$. We proved several decomposition properties of $\mathcal{M}(G)$, which implies that the Euler characteristic of $\mathcal{M}(G)$ equals to the number of Tait colorings of $G$ when $G$ is bipartite. Then we interpret $\mathcal{M}(G)$ as a representation space of the fundamental group of $G$ to $SU(3)$.

1 Introduction

Let $G$ be a trivalent graph. A Tait coloring of $G$ is a function from the edges of $G$ to a 3-element set of ”colors” $\{1, 2, 3\}$ such that edges of 3 different colors are incident at each vertex. The four-color theorem is equivalent to the existence of Tait colorings for bridgeless planar trivalent graphs.

There is a similar concept in knot theory. A web is an oriented trivalent planar graph, such that at each vertex the edges are either all oriented in or oriented out. In particular, such graphs are always bipartite. The invariant $P_3(\Gamma)$ ([10][13]) of a web $\Gamma$, which is used to construct the $sl_3$ knot polynomial invariant, is characterized by the properties illustrated in Fig. 1, where $[n] = \frac{q^n - q^{-n}}{q - q^{-1}}$, since a closed bipartite graph contains either a loop, a digon face or a square face.

Let $q = 1$, we see that relations in Fig. 1 ensure that $P_3(\Gamma)(1)$ is the number $|Tait(\Gamma)|$ of Tait colorings of $\Gamma$. This coincidence raised people’s interest on modifying concepts and methods developed for studying $P_n(\Gamma)$ in knot theory to study $|Tait(G)|$ for a general trivalent graph $G$.

In [2], Khovanov defined a link homology theory using webs(oriented closed trivalent graphs) and foams(cobordisms between webs), whose Euler characteristic is the $sl_3$ link polynomial. Kronheimer and Mrowka([7]) defined an instanton homology $J^\#(G)$ for unoriented trivalent spatial graphs $G$, whose dimension is conjectured to be the number of Tait colorings of $G$. The non-vanishing theorem they proved would then, lead to a new proof of the four-color theorem. They also proposed a combinatorial counterpart to $J^\#$ for planar unoriented webs, by imitating the construction of Khovanov’s $sl_3$ homology in [2].

Motivated by the combinatorial construction in [7], Khovanov and Robert developed a homology theory $\langle G \rangle$ for planar unoriented trivalent graphs $G$.
using an unoriented version of the Robert-Wagner foam evaluation, whose rank is conjecturally related to $|Tait(G)|$.

Motivated by the perspective of a potential relationship between the $SU(N)$ knot instanton homology theory [5] [8] and Khovanov-Rozansky’s $sl(N)$ knot homology, the authors of [11] defined a moduli space $\mathcal{M}(\Gamma)$ for any web $\Gamma$, and showed that the Euler characteristic of $\mathcal{M}(\Gamma)$ equals the evaluation of $P_N(\Gamma)$ at 1.

In this paper, we use a modified construction in [11] to associate a moduli space $\mathcal{M}(G)$ to a planar trivalent graph $G$. In Section 2, we defined and proved several decomposition properties of $\mathcal{M}(G)$, which shows that the Euler characteristic $\chi(\mathcal{M}(G))$ behaves the same as the number of Tait colorings $|Tait(G)|$ of $G$ under some decompositions. In particular, this shows that

$$\chi(\mathcal{M}(G)) = |Tait(G)|$$

(1)

for any bipartite graph $G$.

In Section 3, we showed that $\mathcal{M}(G)$ is homeomorphic to the representation space $R_{\Phi}(\pi_1(G); SU(3))$, which is similar to the representation spaces appearing in Kronheimer and Mrowka’s instanton homology [7]. [8].
2 Definitions and Relations

Let $G$ be a planar trivalent graph, $m := |E(G)|$ be the number of edges of $G$.

For each edge $e \in E(G)$, we decorate it by a point $D(e) \in \mathbb{CP}^2$, i.e. a line in $\mathbb{C}^3$ that passes through the origin point. We call this decoration admissible, if at any vertex $v$, the decorations $D(e_i)$ associated to the three edges $e_1, e_2, e_3$ adjacent to $v$ are orthogonal pairwise in $\mathbb{C}^3$.

Definition 1. The set of all admissible decorations of $G$ forms a moduli space which we denote by $\mathcal{M}(G)$. More precisely, a decoration can be viewed as a point in $\bigoplus_{e \in E(G)} \mathbb{P}^2$, and the structure of $\mathcal{M}(G)$ is induced as a subvariety of $\bigoplus_{e \in E(G)} \mathbb{P}^2$.

The main result of this section is:

Theorem 1. For a bipartite trivalent graph $G$ we have

$$\chi(\mathcal{M}(G)) = |\text{Tait}(G)|$$

i.e. the Euler characteristic of $\mathcal{M}(G)$ equals to the number of Tait colorings of $G$.

Question 1. Is Theorem 1 true for any planar trivalent graph?

Motivated by the conjecture in [11] we propose:

Conjecture 1. For any planar trivalent graph $G$ and for $i$ odd, we have $H_i(\mathcal{M}(G)) = 0$.

The conjecture together with Theorem would together imply that $\dim H(\mathcal{M}(G)) = |\text{Tait}(G)|$ for any bipartite $G$.

Lemma 1. 1. Let $U$ be the graph with a single circle. Then $\mathcal{M}(U) \cong \mathbb{P}^2$, therefore $\chi(\mathcal{M}(U)) = |\text{Tait}(U)| = 3$.

2. Let $G_1, G_2$ be two trivalent graph, $G_1 \cup G_2$ their disjoint union, then

$$\mathcal{M}(G_1 \cup G_2) = \mathcal{M}(G_1) \times \mathcal{M}(G_2)$$

In particular,

$$\chi(\mathcal{M}(G_1 \cup G_2)) = \chi(\mathcal{M}(G_1)) \cdot \chi(\mathcal{M}(G_2))$$

Proof. This is immediate from the definition.

Lemma 2. (The bigon relation)

Let $G$ be a trivalent graph containing a bigon, and $G'$ the graph obtained from $G$ by collapsing the bigon (see Figure 2). Then $\mathcal{M}(G)$ is a $\mathbb{CP}^1$-bundle over $\mathcal{M}(G')$. In particular,

$$\chi(\mathcal{M}(G)) = \chi(\mathbb{CP}^1) \cdot \chi(\mathcal{M}(G')) = 3 \chi(\mathcal{M}(G'))$$


Figure 2: The bigon move

Proof. Let \( D \) be a decoration of \( G \) as illustrated in Figure 2. We have \( a = b \) since they are both orthogonal to two different lines \( c, d \). For fixed \( a \), we can choose \( c \) to be any line orthogonal to \( a \), and then \( d \) is the unique line orthogonal to both \( a \) and \( c \). Therefore for each admissible decoration of \( G' \), there corresponds to a \( \mathbb{CP}^1 \)-set of admissible decorations of \( G \).

Lemma 3. (The triangle relation) Let \( G \) be a trivalent graph containing 3 edges which form a triangle. Let \( G' \) be obtained from \( G \) by collapsing the triangle to a single vertex (see Figure 3). Then \( \mathcal{M}(G) \cong \mathcal{M}(G') \).

Proof. Let \( D \) be an admissible decoration of \( G \), as illustrated in Figure 3. Since \( e \neq f \), we have \( a \neq b \), therefore \( a, b, c \) are 3 different edges, and \( d, e, f \) are uniquely determined by \( a, b, c \). It follows that admissible decorarions of \( G \) and \( G' \) are 1-1 identified.

Lemma 4. (The square relation) Let \( G \) be a trivalent graph containing a square: 4 edges connecting 4 vertices in a standard disk. Let \( G' \) and \( G'' \) be obtained from \( G \) as shown in Figure 4. Then

\[
\chi(\mathcal{M}(G)) = \chi(\mathcal{M}(G')) + \chi(\mathcal{M}(G''))
\] (6)
Proof. Let $D$ be an admissible decoration of $G$ as illustrated in the left of Figure 3. If $x \neq z$, then $w = y$, since they are orthogonal to 2 different lines. In this case we have $a = d, b = c$. Similarly if $w \neq y$, then $x = y, a = b, c = d$.

In the case when $a = b \neq c = d$, $x$ and $z$ is uniquely determined, and $w, y$ is determined afterward. So there is a unique choice for the interior decorations. The same result is true when $a = d \neq b = c$. When $a = b = c = d$, we have a $\mathbb{CP}^1$ of choice of, e.g. $w$, and the decorations $x, y, z$ is determined thereafter.

Consider the algebraic maps from $\mathcal{M}(G), \mathcal{M}(G'), \mathcal{M}(G'')$ to $(\mathbb{CP}^2)^4$, defined by evaluating at $a, b, c, d$. Write $V, V_1, V_2$ for the preimages of the subvariety $\Delta$ of $(\mathbb{CP}^2)^4$ given by $a = b = c = d$. Since $\mathcal{M}(G), \mathcal{M}(G'), \mathcal{M}(G'')$ are compact real varieties, we can choose a small enough open set $U \subset (\mathbb{CP}^2)^4$ containing $\Delta$, so that its preimages $\tilde{V}, \tilde{V}_1, \tilde{V}_2$ in $\mathcal{M}(G), \mathcal{M}(G'), \mathcal{M}(G'')$ are respectively homotopy equivalent to $V, V_1, V_2$.

We have shown that $V$ is a $\mathbb{CP}^1$- bundle over $V_1 = V_2$, therefore

$$\chi(\tilde{V}) = \chi(V) = 2\chi(V_1) = \chi(V_1) + \chi(V_2) \quad (7)$$

Since $\mathcal{M}(G) \setminus V, \tilde{V}$ form an open cover of $\mathcal{M}(G)$, using the Mayer-Vietoris sequence we have:

$$\chi(\mathcal{M}(G)) = \chi(\mathcal{M}(G) \setminus V) + \chi(\tilde{V}) - \chi(\tilde{V} \setminus V) \quad (8)$$

Notice that $\mathcal{M}(G) \setminus V$ is the disjoint union of $\mathcal{M}(G_1) \setminus V_1$ and $\mathcal{M}(G_2) \setminus V_2$. 

![Figure 3: The triangle move](image-url)
and $\overline{V}\setminus V$ is the disjoint union of $\overline{V}_1\setminus V_1$ and $\overline{V}_2\setminus V_2$, we have

$$\chi(M(G)) = \chi(M(G_1)\setminus V_1) + \chi(V_1) - \chi(\overline{V}_1\setminus V_1) + \chi(M(G_2)\setminus V_2) + \chi(V_2) - \chi(\overline{V}_2\setminus V_2)$$

$$= \chi(M(G')) + \chi(M(G'')) \tag{9}$$

**Proof of Theorem** We can proceed by induction on the number $n$ of edges of $G$. When $n = 1$, $G$ is a circle, $M(G) = \mathbb{C}P^2$, $\chi(M(G)) = 3 = |\text{Tait}(G)|$. Since a closed bipartite graph contains either a loop, a digon face or a square face, using Lemma 1-4 we can reduce $G$ to graphs with fewer number of edges, for which the inductive hypothesis is true. Notice that properties of $\chi(M(G))$ from Lemma 1-4 are also true for $|\text{Tait}(G)|$, the inductive process works.

### 3 Relation to representation spaces

Let

$$\phi = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \in SU(3) \tag{10}$$
Any matrix $A \in SU(3)$ of order 2 is diagonalizable and is conjugate to $\Phi$. 

Let $G \subset \mathbb{R}^2 \subset \mathbb{R}^3$ be a trivalent planar graph. Choose an orientation on each edge of $G$ so that it induce a meridian around the edge. The meridians generate the fundamental group $\pi_1(G) := \pi_1(\mathbb{R}^3 \setminus G)$ of $G$. 

Define $R_\Phi(\pi_1(G) : SU(3))$ to be the subspace of $Hom(\pi_1(G), SU(3))$ consisting of all homomorphisms 

$$\rho : \pi_1(G) \rightarrow SU(3)$$ 

such that for each meridian $m$, we have that the corresponding $x_m \in \pi_1(G)$ maps to an element $\rho(m)$ that is conjugate to $\Phi$.

**Lemma 5.** Let $S,T \in SU(3)$ be conjugate to $\Phi$, then $ST$ is conjugate to $\Phi$ if and only if the 1-eigenspaces of $S$ and $T$ are orthogonal.

At this time, the 1-eigenspaces of $ST$ is orthogonal to that of $S$ and $T$.

**Proof.** Suppose that $e_1, e_2$ are the 1-eigenvector of $S,T$ respectively, and $e_1 \perp e_2$. Choose $e_3$, so that $e_3 \perp e_1, e_2$. Then $Se_2 = -e_2, Se_3 = -e_3, Te_1 = -e_1, Te_3 = -e_3$. Therefore $STe_1 = -e_1, STe_2 = -e_2, STe_3 = e_3$, which shows that $ST$ is conjugate to $\Phi$.

Conversely, let $ST$ be conjugate to $\Phi$. Let $e_1, e_2, e_3$ be the eigenvectors of $T$, so that under the basis $\{e_1, e_2, e_3\}$, $S,T$ are represented respectively by the matrices 

$$
\begin{pmatrix}
a_{11} & a_{12} & a_{13} \\
a_{21} & a_{22} & a_{23} \\
a_{31} & a_{32} & a_{33}
\end{pmatrix}
= 
\begin{pmatrix}1 & 0 & 0 \\
0 & -1 & 0 \\
0 & 0 & -1
\end{pmatrix}
$$

Then 

$$ST = 
\begin{pmatrix}
a_{11} & -a_{12} & -a_{13} \\
a_{21} & -a_{22} & -a_{23} \\
a_{31} & -a_{32} & -a_{33}
\end{pmatrix}
$$

is conjugate to $\Phi$, $trS = a_{11} + a_{22} + a_{33} = -1$, $tr(ST) = a_{11} - a_{22} - a_{33} = -1$, which implies $a_{11} = -1, a_{22} + a_{33} = 0$. Since $S \in SU(3)$, $|a_{11}|^2 + |a_{12}|^2 + |a_{13}|^2 = 1$, thus $a_{12} = a_{13} = 0$.

Let $xe_1 + ye_2 + ze_3$ be the eigenvector of $S$ with eigenvector 1. Then 

$$
\begin{pmatrix}
1 & 0 & 0 \\
0 & -1 & 0 \\
0 & 0 & -1
\end{pmatrix}
\begin{pmatrix}x \\
y \\
z
\end{pmatrix} = 
\begin{pmatrix}-x \\
\cdots \\
\cdots \\
y \\
z
\end{pmatrix}
$$

therefore $x = 0$, the 1-eigenspace of $S$ is orthogonal to the 1-eigenspace of $T$.

The last argument follows since $T = S^2T = S(ST), S + ST^2 = (ST)T$. \qed

For $\rho \in R_\Phi(\pi_1(G) : SU(3))$, we will define a decoration $D(\rho)$ of $G$ as follows. For each edge $e$ of $G$, there are two meridians along $e$, with reverse orientations. They induce reciprocal elements $x_e, -x_e$ of $\pi_1(G)$, therefore the images $\rho(x_e), \rho(-x_e)$ are inverse to each other. We define the evaluation of $D(\rho)$ at $e$ to be the 1-eigenspace of $\rho(x_e)$, which does not depend on the choice of the orientation of the edge.
Theorem 2. The map described above

\[ R_\Phi(\pi_1(G); SU(3)) \to M(G) \]
\[ \rho \to D(\rho) \] (15)

is a well-defined homeomorphism.

Proof. Let \( e_1, e_2, e_3 \) be adjacent to the same vertex of \( G \). Choose any \( \rho \in R_\Phi(\pi_1(G); SU(3)) \). By the discussion above, we may assume that \( \rho(e_2)\rho(e_1) = \rho(e_3) \). By Lemma 5, \( D(\rho)(e_1), D(\rho)(e_2), D(\rho)(e_3) \) are orthogonal. Therefore \( D(\rho) \) is admissible.

The map is bijective: We can define its inverse

\[ M(G) \to R_\Phi(\pi_1(G); SU(3)) \]
\[ D \to \rho(D) \] (16)

such that \( \rho(D) \) is the unique matrix in \( SU(3) \) that is conjugate to \( \Phi \) and has \( e \) as its 1-eigenvector. Obviously the two maps are continuous and are inverse to each other. \( \square \)

In [7], they defined an invariant of trivalent spatial graphs \( G \) which takes the form of a \( \mathbb{Z}/2 \)-vector space \( J^\#(G) \), using an variant of instanton homology. More precisely, it arises from a Chern-Simons functional whose set of critical points can be identified with the space

\[ \mathcal{R}(K) = \{ \rho : \pi_1(G) \to SO(3) \mid \rho(x_e) \text{ has order } 2 \text{ for each edge } e \} \] (17)

Since a matrix \( A \in SU(3) \) is conjugate to \( \Phi \) if and only if it has order 2, Theorem 2 implies that our moduli space \( M(G) \) is homeomorphic to the representation space

\[ \{ \rho : \pi_1(G) \to SU(3) \mid \rho(x_e) \text{ has order } 2 \text{ for each edge } e \} \] (18)

In the definition of \( M(G) \), if we replace \( \mathbb{C}P^2 \) by \( \mathbb{R}P^2 \), then we get a moduli space \( \tilde{M}(G) \), which is, by iterating the argument of this section, homeomorphic to \( \mathcal{R}(K) \).

Question: What is the Euler characteristic of \( \tilde{M}(G) \)?

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