Buildings and Classical Groups

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In these notes we describe the classical groups, that is, the linear groups and the orthogonal, symplectic, and unitary groups, acting on finite dimensional vector spaces over skew fields, as well as their pseudo-quadratic generalizations. Each such group corresponds in a natural way to a point-line geometry, and to a spherical building. The geometries in question are projective spaces and polar spaces. We emphasize in particular the rôle played by root elations and the groups generated by these elations. The root elations reflect — via their commutator relations — algebraic properties of the underlying vector space.

We also discuss some related algebraic topics: the classical groups as permutation groups and the associated simple groups. I have included some remarks on K-theory, which might be interesting for applications. The first K-group measures the difference between the classical group and its subgroup generated by the root elations. The second K-group is a kind of fundamental group of the group generated by the root elations and is related to central extensions. I also included some material on Moufang sets, since this is an interesting topic. In this context, the projective line over a skew field is treated in some detail, and possibly with some new results. The theory of unitary groups is developed along the lines of Hahn & O'Meara [15]. Other important sources are the books by Taylor [31] and Tits [32], and the classical books by Artin [2] and Dieudonné [9]. The books by Knus [19] and W. Scharlau [29] should also be mentioned here. Finally, I would like to recommend the surveys by Cohen [7] and R. Scharlau [28].

While most of these matters are well-known to experts, there seems to be no book or survey article which contains these aspects simultaneously. Taylor's book [31] is a nice and readable introduction to classical groups, but it is clear that the author secretly thinks of finite fields — the non-commutative theory is almost non-existent in his book. On the other extreme, the book

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by Hahn & O’Meara [15] contains many deep algebraic facts about classical
groups over skew fields; however, their book contains virtually no geometry
(the key words building or parabolic subgroup are not even in the index!). So,
I hope that this survey — which is based on lectures notes from a course I
gave in 1998 in Würzburg — is a useful compilation of material from different
sources.

There is (at least) one serious omission (more omissions can be found
in the last section): I have included nothing about coordinatization. The
 coordinatization of these geometries is another way to recover some algebraic
structure from geometry. For projective planes and projective spaces, this is
a classical topic, and I just mention the books by Pickert [23] and Hughes
& Piper [17]. For generalized quadrangles (polar spaces of rank 2), Van
Maldeghem’s book [34] gives a comprehensive introduction. Understanding
coordinates in the rank 2 case is in most cases sufficient in order to draw some
algebraic conclusions.

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Preliminaries

We first fix some algebraic terminology. A (left) action of a group $G$ on a set $X$ is a homomorphism $G \rightarrow \text{Sym}(X)$ of $G$ into the permutation group $\text{Sym}(X)$ of $X$; the permutation induced by $g$ is (in most cases) also denoted by $g = [x \rightarrow g(x)]$. A right action is an anti-homomorphism $G \rightarrow \text{Sym}(X)$; for right actions, we use exponential notation, $x \rightarrow x^g$. The set of all $k$-element subsets of a set $X$ is denoted $\binom{X}{k}$.

Given a (not necessarily commutative) ring $R$ (with unit 1), we let $R^\times$ denote the group of multiplicatively invertible elements. If $R^\times = R \setminus \{0\}$, then $R$ is called a skew field or division ring. Occasionally, we use Wedderburn’s Theorem.

**Wedderburn’s Theorem** A finite skew field is commutative and thus isomorphic to some Galois field $\mathbb{F}_q$, where $q$ is a prime power.

For a proof see Artin [2] Ch. I Thm. 1.14 or Grundhöfer [13].

The opposite ring $R^\text{op}$ of a ring $R$ is obtained by defining a new multiplication $a \cdot b = ba$ on $R$. An anti-automorphism $\alpha$ of $R$ is a ring isomorphism $R \rightarrow R^\text{op}$, i.e. $(xy)^\alpha = y^\alpha x^\alpha$. The group consisting of all automorphisms and anti-automorphisms of $D$ is denoted $\text{AAut}(D)$; it has the automorphism group $\text{Aut}(D)$ of $D$ as a normal subgroup (of index 1 or 2). Let $M$ be an abelian group. A right $R$-module structure on $M$ is a ring homomorphism $R^\text{op} \rightarrow \text{End}(M)$; as customary, we write $mrs = \rho(s \cdot r)(m)$ for $r, s \in R$ and $m \in M$ (‘scalars to the right’). The abelian category of all right $R$-modules is defined in the obvious way and denoted $\text{Mod}_R$; the subcategory consisting of all finitely generated right $R$-modules is denoted $\text{Mod}^\text{fin}_R$. If $D$ is a skew field, then $\text{Mod}^\text{fin}_D$ is the category of all finite dimensional vector spaces over $D$. Similarly, we define the category of left $R$-modules $\text{Mod}_R$; given a right $R$-module $M$, we have the dual $M^\vee = \text{Hom}_R(M, R)$ which is in a natural way a left $R$-module.

A right module over a skew field will be called a right vector space or just a vector space. Mostly, we will consider (finite dimensional) right vector spaces (so linear maps act from the left and scalars act from the right), but occasionally we will need both types. Of course, all these distinctions are obsolete over commutative skew fields, but in the non-commutative case one has to be careful.
1 Projective geometry and the general linear group

In this first part we consider the projective geometry over a skew field $D$ and the related groups.

1.1 The general linear group

We introduce the projective geometry $\mathbb{P}(V)$ associated to a finite dimensional vector space $V$ and the general linear group $\text{GL}(V)$, as well as the general semilinear group $\Gamma L(V)$. We describe the relations between these groups and their projective versions $\text{PGL}(V)$ and $\text{P}\Gamma L(V)$. Finally, we recall the 'first' Fundamental Theorem of Projective Geometry.

Let $V$ be a right vector space over a skew field $D$, of (finite) dimension $\dim(V) = n + 1 \geq 2$. The collection of all $k$-dimensional subspaces of $V$ is the Grassmannian

$$\text{Gr}_k(V) = \{ X \subseteq V \mid \dim(X) = k \}.$$  

The elements of $\text{Gr}_1(V)$, $\text{Gr}_2(V)$ and $\text{Gr}_n(V)$ are called points, lines, and hyperplanes, respectively. Two subspaces $X, Y$ are called incident,

$$X \ast Y,$$

if either $X \subseteq Y$ or $Y \subseteq X$. The resulting $n$-sorted structure

$$\mathbb{P}(V) = (\text{Gr}_1(V), \ldots, \text{Gr}_n(V), \ast)$$

is the projective geometry of rank $n$ over $D$.

It is clear that every linear bijection of $V$ induces an automorphism of $\mathbb{P}(V)$. More generally, every semilinear bijection induces an automorphism of $\mathbb{P}(V)$. Recall that a group endomorphism $f$ of $(V, +)$ is called semilinear (relative to an automorphism $\theta$ of $D$) if

$$f(va) = f(v)a^\theta$$

holds for all $a \in D$ and $v \in V$. The group of all semilinear bijections of $V$ is denoted $\Gamma L(V)$; it splits as a semidirect product with the general linear group $\text{GL}(V)$, the groups consisting of all linear bijections, as a normal subgroup,

$$1 \longrightarrow \text{GL}(V) \longrightarrow \Gamma L(V) \longrightarrow \text{Aut}(D) \longrightarrow 1.$$
As usual, we write $GL(D^n) = GL_n(D)$, and similarly for the groups $PGL(V)$ and $PΓL(V)$ induced on $PG(V)$.

The kernel of the action of $GL(V)$ on $PG(V)$ consists of all maps of the form $ρ_c : v \mapsto vc$, where $c \in Cen(D^×)$. Similarly, the kernel of the action of $ΓL(V)$ consists of all maps of the form $ρ_c : v \mapsto vc$, for $c \in D^×$. These groups fit together in a commutative diagram with exact rows and columns

\[
\begin{array}{cccccc}
1 & 1 & 1 & 1 & 1 & 1 \\
1 \rightarrow & \text{Cen}(D^×) \rightarrow & D^× \rightarrow & \text{Int}(D) \rightarrow & 1 \\
1 \rightarrow & \text{GL}(V) \rightarrow & ΓL(V) \rightarrow & \text{Aut}(D) \rightarrow & 1 \\
1 \rightarrow & \text{PGL}(V) \rightarrow & PΓL(V) \rightarrow & \text{Out}(D) \rightarrow & 1 \\
1 \rightarrow & 1 \rightarrow & 1 \rightarrow & 1 \rightarrow & 1
\end{array}
\]

as is easily checked, see Artin [2] II.10, p. 93. Here $\text{Int}(D)$ is the group of inner automorphisms $d \rightarrow da^a = a^{-1}da$ of $D$, where $a \in D^×$, and $\text{Out}(D)$ is the quotient group $\text{Aut}(D)/\text{Int}(D)$. The bottom line shows the groups induced on $PG(V)$, and the top line the kernels of the respective actions. For example, $\text{Out}(\mathbb{H}) = 1$ holds for the quaternion division algebra $\mathbb{H}$ over any real closed field, whence $PGL_{n+1}(\mathbb{H}) \cong PΓL_{n+1}(\mathbb{H})$ for $n \geq 1$. On the other hand, $\text{Int}(D) = 1$ if $D$ is commutative. Recall the Fundamental Theorem of Projective Geometry, which basically says that $\text{Aut}(PG(V)) = PΓL(V)$. The actual statement is in fact somewhat stronger.

### 1.1.1 The Fundamental Theorem of Projective Geometry, I

For $i = 1, 2$, let $V_i$ be vector spaces over $D_i$, of (finite) dimensions $n_i + 1 \geq 3$, and let

$$PG(V_1) \xrightarrow{\phi} PG(V_2)$$

be an isomorphism. Then $n_1 = n_2$, and there exists a skew field isomorphism $θ : D_1 \xrightarrow{\cong} D_2$ and a $θ$-semilinear bijection

$$f : V_1 \rightarrow V_2$$

(i.e. $f(va) = f(v)a^θ$) such that $φ(X) = f(X)$ for all $X \in \text{Gr}_k(V_1)$, for $k = 1, 2, \ldots, n_1$. In particular,

$$\text{Aut}(PG(V)) = PΓL(V).$$
For a proof see Hahn & O’Meara [15] 3.1.C, Artin [2] Ch. II Thm. 2.26, Lüneburg [20] [21], or Faure & Frölicher [10].

The theorem is also valid for infinite dimensional projective spaces; this will be important in the second part, when we consider hermitian forms. Note that for \( n = 1 \), the structure \( \text{PG}(V) = (\text{Gr}_1(V), =) \) is rather trivial. We will come back to a refined version of the projective line in Section 1.8.

### 1.2 Elations, transvections, and the elementary linear group

We introduce elations, transvections, and the elementary linear group \( \text{EL}(V) \) generated by all elations. The projective elementary group is a simple group, except for some low dimensional cases over fields of small cardinality; in the commutative case, the elementary linear group coincides with the special linear group \( \text{SL}(V) \).

As in the previous section, \( V \) is a right vector space over \( D \) of finite dimension \( n + 1 \geq 2 \). Let \( A \in \text{Gr}_n(V) \) be a hyperplane, and let \( z \in \text{Gr}_1(V) \) be a point incident with \( A \). An automorphism \( \tau \) of \( \text{PG}(V) \) which fixes \( A \) pointwise and \( z \) linewise is called an elation or translation, with axis \( A \) and center \( z \). We can choose a base \( b_0, \ldots, b_n \) of \( V \) such that \( A \) is spanned by \( b_0, \ldots, b_{n-1} \) and \( z \) by \( b_0 \), and such that \( \tau \) is represented by a matrix of the form

\[
\begin{pmatrix}
1 & \cdots & a \\
1 & \ddots & \\
& & 1
\end{pmatrix}
\]

for some \( a \in D \). Such an elation is called a \((z, A)\)-elation; the group \( U_{(z, A)} \) consisting of all \((z, A)\)-elations is isomorphic to the additive group \((D, +)\).

There is a coordinate-free way to describe elations. Let

\[ V^\vee = \text{Hom}_D(V, D) \]

denote the dual of \( V \). This is a left vector space over \( D \), so we can form the tensor product \( V \otimes_D V^\vee \); this is an abelian group (even a ring) which is naturally isomorphic to the endomorphism ring of \( V \),

\[ \text{End}_D(V) \cong V \otimes_D V^\vee. \]
We write $x \otimes \phi = x\phi = [v \rightarrow x\phi(v)]$. If the elements of $V$ are represented as column vectors and the elements of $V^\vee$ as row vectors, then $x\phi$ is just the standard matrix product. Now let $(u, \rho) \in V \times V^\vee$ be a pair such that the endomorphism $u\rho \in \text{End}_D(V)$ is nilpotent, i.e. $u\rho u\rho = u\rho(u)\rho = 0$ (which is equivalent to $\rho(u) = 0$). The linear map

$$\tau_{u\rho} = \text{id}_V + u\rho = [v \rightarrow v + u\rho(v)]$$

is called a transvection. Note that

$$\tau_{u\rho} \tau_{u'b\rho} = \tau_{u(a+b)\rho} \quad \text{whence} \quad \tau_{u\rho} \tau_{-u\rho} = \text{id}_V.$$

Such a transvection induces an elation on $\text{PG}(V)$, and conversely, every elation in $\text{PG}(V)$ is induced by a transvection. In fact, suppose that $u\rho \neq 0$. The center of $\tau_{u\rho}$ is $z = uD \in \text{Gr}_1(V)$, and the axis is $A = \ker(\rho)$. The group $(D, +) \cong \{\tau_{u\rho} | a \in D\}$ maps isomorphically onto $U_{z,A}$. Suppose that $\tau_{u\phi} \neq 1 \neq \tau_{v\psi}$ are commuting transvections,

$$\tau_{u\phi} \tau_{v\psi} = \tau_{v\psi} \tau_{u\phi}.$$ 

This implies that $u\phi(v)\psi = v\psi(u)\phi$, and thus $\psi(u) = \phi(v) = 0$.

1.2.1 Lemma Two non-trivial transvections commute if and only if they have either the same axis or the same center. \qed

The group generated by all transvections is the elementary linear group $\text{EL}(V)$. It is a normal subgroup of the group $\text{GL}(V)$ and of $\text{GL}(V)$ (because the conjugate of a transvection is again a transvection). The subgroup in $\text{Aut}(\text{PG}(V))$ generated by all elations is the little projective group of $\text{PG}(V)$; it is normal and an epimorphic image of $\text{EL}(V)$. We denote this group by $\text{PEL}(V)$. We collect a few facts about the group $\text{EL}(V)$.

1.2.2 Lemma The action of $\text{PEL}(V)$ on $\text{Gr}_1(V)$ is 2-transitive and in particular primitive.

Proof. It suffices to show that given a point $p \in \text{Gr}_1(V)$, the stabilizer $\text{PEL}(V)_p$ acts transitively on $\text{Gr}_1(V) \setminus \{p\}$. Also, it is easy to see that given a line $\ell$ passing through $p$, the stabilizer $\text{PEL}(V)_{p,\ell}$ acts transitively on the points lying on $\ell$ and different from $p$ — choose an axis $A$ passing through the center $p$, not containing $\ell$. It is also not difficult to prove that $\text{PEL}(V)_p$ acts transitively on the lines passing through $p$ (here, one chooses an axis containing $p$, but with a different center). The result follows from these observations. \qed

Mutatis mutandis, one proves that $\text{PEL}(V)$ acts 2-transitively on the hyperplanes.
1.2.3 Lemma If $n \geq 2$, then $\text{EL}(V)$ is perfect, i.e. $[\text{EL}(V), \text{EL}(V)] = \text{EL}(V)$. The same is true for $n = 1$, provided that $|D| \geq 4$.

Proof. Assume first that $n \geq 2$. Given $\tau_{u\phi}$, choose $\psi$ linearly independent from $\phi$, such that $\psi(u) = 0$ and $v \in \ker(\phi)$ with $\psi(v) = 1$, then $[\tau_{u\phi}, \tau_{\phi v}] = \tau_{\phi u}$, so every transvection is a commutator of transvections. In the case $n = 1$ one uses some clever matrix identities, and the fact that the equation $x^2 - 1 \neq 0$ has a solution in $D$ if $|D| \geq 4$, see Hahn & O'Meara [15] 2.2.3. □

1.2.4 Proposition If $n \geq 2$ or if $n = 1$ and $|D| \geq 4$, then $\text{PEL}(V)$ is a simple group.

Proof. The proof uses Iwasawa's simplicity criterion, see Hahn & O'Meara [15] 2.2.B. The group $\text{PEL}(V)$ is perfect, and the stabilizer of a point $z$ has an abelian normal subgroup, the group consisting of all elations with center $z$. These are the main ingredients of the proof, see loc.cit. 2.2.13. See also Artin [2] Ch. IV Thm. 4.10. □

We collect some further results about the group $\text{EL}(V)$. Let $H \cong \text{GL}_1(D)$ denote the subgroup of $\text{GL}(V)$ consisting of all matrices of the form

$$
\begin{pmatrix}
1 & & & & \\
& 1 & & & \\
& & & 1 & \\
& & & & \ddots \\
& & & & & 1
\end{pmatrix}
$$

for $a \in D^\times$. One can show that $\text{GL}(V) = H \text{EL}(V)$, see Hahn & O'Meara [15] 1.2.10 (the theorem applies, since a skew field is a euclidean ring in the terminology of loc.cit.). An immediate consequence is the following.

1.2.5 If $D$ is commutative, then $\text{EL}(V) = \text{SL}(V)$.

(Artin [2] denotes the group $\text{EL}(V)$ by $\text{SL}(V)$, even if $D$ is not commutative. Hahn & O'Meara [15] — and other modern books — have a different terminology; in their book, $\text{SL}(V)$ is the kernel of the reduced norm.)

The next result follows from the fact that $\text{EL}(V)$ acts strongly transitively on the building $\Delta(V)$; such an action is always primitive on the vertices of a fixed type, because the maximal parabolics are maximal subgroups. In our case, the vertices of the building $\Delta(V)$ are precisely the subspaces of $V$, see Section 1.5.

1.2.6 The action of $\text{EL}(V)$ on $\text{Gr}_k(V)$ is primitive, for $1 \leq k \leq n$. It is two-transitive if and only if $k = 1, n$. 

8
Finally, we mention some exceptional phenomena.

1.2.7 Suppose that $D \cong \mathbb{F}_q$ is finite, and let $\text{PSL}_m(q) = \text{PEL}_m(\mathbb{F}_q)$. There are the following isomorphisms (and no others, see Hahn & O’Meara [15] p. 81).

$$
\begin{align*}
\text{PSL}_2(2) & \cong \text{Sym}(3) \\
\text{PSL}_2(3) & \cong \text{Alt}(4) \\
\text{PSL}_2(4) & \cong \text{PSL}_2(5) \cong \text{Alt}(5) \\
\text{PSL}_2(7) & \cong \text{PSL}_3(2) \\
\text{PSL}_2(9) & \cong \text{Alt}(6) \\
\text{PSL}_4(2) & \cong \text{Alt}(8)
\end{align*}
$$

In particular, the groups $\text{PSL}_2(q)$ are not perfect for $q = 2, 3$. Note that the groups $\text{PSL}_3(4)$ and $\text{PSL}_4(2)$ have the same order 20160 without being isomorphic.

1.3 $K_1$ and the Dieudonné determinant

We explain the connection between the elementary linear group, the first $K$-group $K_1(D)$ and the Dieudonné determinant.

Recall the category $\mathcal{M}_D^{\text{fin}}$ of all finite dimensional vector spaces over $D$. Let $\mathcal{M} = \{D^n | n \geq 0\}$; these vector spaces together with the linear maps between them form a small and full subcategory of $\mathcal{M}_D^{\text{fin}}$. Every vector space $V$ in $\mathcal{M}_D^{\text{fin}}$ is isomorphic to a unique element $[V] \in \mathcal{M}$. We make $\mathcal{M}$ into an additive semigroup with addition $[V] + [W] = [V \oplus W]$, and neutral element $[0]$. The dimension functor yields an isomorphism

$$(\mathcal{M}, +) \xrightarrow{\text{dim}} (\mathbb{N}, +).$$

Now $K_0(D) = K_0(\mathcal{M}_D^{\text{fin}})$ is defined to be the Grothendieck group generated by the additive semigroup of isomorphism classes of finite dimensional vector spaces over $D$, i.e. $K_0(D) \cong \mathbb{Z}$. (The Grothendieck group of a commutative semigroup $S$ is the universal solution $G(S)$ of the problem

$$
\begin{array}{c}
S \xrightarrow{f} G(S) \\
\downarrow \\
H
\end{array}
$$

where $H$ is any group and $f$ is any semigroup homomorphism.) While all this is rather trivial general nonsense (but only since $D$ is a skew field!), there are higher-rank $K$-groups which bear more information even for skew fields.
For \( n \leq m \), there is a natural inclusion \( \text{GL}_n(D) \rightarrow \text{GL}_m(D) \) (as block matrices in the upper left, with 1s on the diagonal in the lower right). Let \( \text{GL}_{\text{stb}}(D) \) denote the direct limit over these inclusions (this is called the \textit{stable linear group} — not to be confused with stability theory in the model theoretic sense), and let \( \text{EL}_{\text{stb}}(D) \) denote the corresponding direct limit over the groups \( \text{EL}_n(D) \). For \( n \geq 1 \), there are exact sequences

\[
1 \rightarrow \text{EL}_n(D) \rightarrow \text{GL}_n(D) \rightarrow \text{GL}_n(D)/\text{EL}_n(D) \rightarrow 1
\]

\[
1 \rightarrow \text{EL}_{\text{stb}}(D) \rightarrow \text{GL}_{\text{stb}}(D) \rightarrow \text{GL}_{\text{stb}}(D)/\text{EL}_{\text{stb}}(D) \rightarrow 1
\]

We denote these quotients

\[
K_{1,n}(D) = \text{GL}_n(D)/\text{EL}_n(D)
\]

and put \( K_1(D) = \text{GL}_{\text{stb}}(D)/\text{EL}_{\text{stb}}(D) \). The groups \( K_{1,n}(D) \) are \textit{stable}, i.e. independent of \( n \); there are isomorphisms \( K_{1,2}(D) \cong K_{1,3}(D) \cong \cdots \cong K_1(D) \), see Hahn & O’Meara [15] 2.2.4. In this way we have obtained the \textit{first K-group} \( K_1(D) \).

\textbf{1.3.1 Proposition} There is an isomorphism \( K_1(D) \cong D^\times/[D^\times, D^\times] \). The composite

\[
\text{GL}_n(D) \xrightarrow{\det} \text{GL}_{\text{stb}}(D) \xrightarrow{\cong} D^\times/[D^\times, D^\times]
\]

is precisely the \textit{Dieudonné determinant} \( \det \), see Hahn & O’Meara [15] 2.2.2 and Artin [2] Ch. IV Thm. 4.6.

Since the determinant takes values in an abelian group, it is invariant under base change (matrix conjugation); in particular, there is a well-defined (base independent) determinant map

\[
\text{GL}(V) \xrightarrow{\det} D^\times/[D^\times, D^\times].
\]

We end this section with a commutative diagram which compares the linear and the projective actions. Given \( c \in D^\times \), we denote its image in
$D^\times/[D^\times, D^\times]$ by $\bar{c}$. Let $C = \text{Cen}(D^\times)$.

\[
\begin{array}{c}
1 \rightarrow \{ c \in C | \bar{c}^{n+1} = 1 \} \rightarrow C \rightarrow \{ \bar{c}^{n+1} | c \in C \} \rightarrow 1 \\
1 \rightarrow \text{EL}(V) \rightarrow \text{GL}(V) \rightarrow K_1(D) \rightarrow 1 \\
1 \rightarrow \text{PEL}(V) \rightarrow \text{PGL}(V) \rightarrow K_1(D)/\{ \bar{c}^{n+1} | c \in C \} \rightarrow 1
\end{array}
\]

Here are some examples. If $D$ is commutative, then $K_1(D) = D^\times$. Now let $\mathbb{H}$ denote the quaternion division algebra over a real closed field $\mathbb{R}$. The norm $N$ is defined as $N(x_0 + ix_1 + jx_2 + jkx_3) = x_0^2 + x_1^2 + x_2^2 + x_3^2$. Then $\mathbb{H}^\times \xrightarrow{N} \mathbb{R}_{>0}$ has kernel $[\mathbb{H}^\times, \mathbb{H}^\times] = S^3$, whence $K_1(\mathbb{H}) \cong \mathbb{R}_{>0}$.

### 1.4 Steinberg relations and $K_2$

We introduce the Steinberg relations, which are the basic commutator relations for projective spaces, and indicate briefly the connection with higher $K$-theory.

In this section we assume that $\dim(V) = n + 1$ is finite, and that $n \geq 2$. Let $b_0, \ldots, b_n$ be a base for $V$, and let $\beta_0, \ldots, \beta_n \in V^\vee$ be the dual base (i.e. $\beta_i(b_j) = \delta_{ij}$). Let

$$
\tau_{ij}(a) = \tau_{b_ia\beta_j}, \quad \text{for } i \neq j.
$$

Thus $\tau_{ij}(a)$ can be pictured as a matrix with 1s on the diagonal, an entry $a$ at position $(i, j)$ (ith row, jth column), and 0s else. We claim that the elations $\{ \tau_{ij}(a) | a \in D, i \neq j \}$ generate $\text{EL}(V)$. Indeed, it is easy to see that the group generated by these elations acts transitively on incident point-hyperplane pairs; therefore, it contains all elations. The maps $\tau_{ij}(a)$ satisfy the following relations, as is easily checked.

**SR1** $\tau_{ij}(a)\tau_{ij}(b) = \tau_{ij}(a + b)$ for $i \neq j$.

**SR2** $[\tau_{ij}(a), \tau_{kl}(b)] = 1$ for $i \neq k$ and $j \neq l$.

**SR3** $[\tau_{ij}(a), \tau_{jk}(b)] = \tau_{ik}(ab)$ for $i, j, k$ pairwise distinct.
These are the *Steinberg relations*. They show that the algebraic structure of the skew field $D$ is encoded in the little projective group $\text{PEL}_n(D)$.

For each pair $(i,j)$ with $i \neq j$ we fix an isomorphic copy $U_{ij}$ of the additive group $(D, +)$, and an isomorphism $\tau_{ij} : (D, +) \xrightarrow{\cong} U_{ij}$. For $n \geq 2$, we define $\text{St}_{n+1}(D)$ as the free amalgamated product of the $n(n+1)$ groups $U_{ij} = \{ \tau_{ij}(a) | a \in D \}$, factored by the normal subgroup generated by the Steinberg relations $\text{SR2}$, $\text{SR3}$. There is a natural epimorphism

$$\text{St}_{n+1}(D) \twoheadrightarrow \text{EL}_{n+1}(D)$$

whose kernel is denoted $K_{2,n+1}(D)$. Again, there are natural maps

$$K_{2,n+1}(D) \twoheadrightarrow K_{2,m+1}(D)$$

for $m \geq n$, and one can consider the limits, the stable groups $\text{St}_{\text{stab}}(D)$ and $K_2(D)$. Clearly, there are exact sequences

$$1 \twoheadrightarrow K_{2,n+1}(D) \twoheadrightarrow \text{St}_{n+1}(D) \twoheadrightarrow \text{GL}_{n+1}(D) \twoheadrightarrow K_{1,n+1}(D) \twoheadrightarrow 1$$

(and similarly in the limit). The groups $K_2(D)$ bear some information about the skew field $D$. See Milnor [22] for $K_2(D)$ of certain fields $D$; for quaternion algebras, $K_2(D)$ is determined in Alperin-Dennis [11]. One can prove that $K_{2,n+1}(D) \cong K_2(D)$ for $n > 1$, see Hahn & O’Meara [15] 4.2.18 and that $\text{St}_{n+1}(D)$ is a universal central extension of $\text{EL}_{n+1}(D)$, provided that $n \geq 4$, see loc.cit. 4.2.20, or that $n \geq 3$ and that $\text{Cen}(D)$ has at least 5 elements, see Strooker [30] Thm. 1. We just mention the following facts.

(1) If $D$ is finite, then $K_2(D) = 0$, so the groups $\text{SL}_{n+1}(q)$ are centrally closed for $n \geq 4$ (they don’t admit non-trivial central extensions), see Hahn & O’Meara [15] 2.3.10 (in low dimensions over small fields, there are exceptions, see loc.cit.). Also, the Steinberg relations yield a presentation of the groups $\text{SL}_{n+1}(q)$ for $n \geq 4$.

(2) Suppose that $D$ is a field with a primitive $m$th root of unity. Let $\text{Br}(D)$ denote its Brauer group. Then there is an exact sequence of abelian groups

$$1 \longrightarrow [K_2(D)]^m \longrightarrow K_2(D) \longrightarrow \text{Br}(D) \longrightarrow [\text{Br}(D)]^m \longrightarrow 1.$$ 

(where we write $[A]^m = \{ a^m | a \in A \}$ for an abelian group $(A, \cdot)$), see Hahn & O’Meara [15] 2.3.12.

Applications of $K_2$, e.g. in number theory, are mentioned in Milnor [22] and in Rosenberg [27].
1.5 Different notions of projective space, characterizations

We introduce the point-line geometry and the building obtained from a projective geometry and compare the resulting structures. Then we mention the 'second' Fundamental Theorem of Projective Geometry which characterizes projective geometries of rank at least 3. We describe the Tits system (BN-pair) for the projective geometry and the root system, and we explain how the root system reflects properties of commutators of root elations.

A point-line geometry is a structure

\((P, L, *)\),

where \(P\) and \(L\) are non-empty disjoint sets, and \(* \subseteq (P \cup L) \times (P \cup L)\) is a symmetric and reflexive binary relation, such that \(*|_{P \times P} = \text{id}_P\) and \(*|_{L \times L} = \text{id}_L\). Given a projective geometry \(\text{PG}(V) = (\text{Gr}_1(V), \ldots, \text{Gr}_n(V), *)\) of rank \(n \geq 2\), we can consider the point-line geometry

\(\text{PG}(V)_{1,2} = (\text{Gr}_1(V), \text{Gr}_2(V), *)\).

It is easy to recover the whole structure \(\text{PG}(V)\) from this; call a set \(X\) of points a subspace if it has the following property: for every triple of pairwise distinct collinear points \(p, q, r\) (i.e. there exists a line \(\ell \in \text{Gr}_2(V)\) with \(p, q, r \in \ell\)), we have the implication

\((p, q \in X) \implies (r \in X)\).

We define the rank of a subspace inductively as \(\text{rk}(\emptyset) = -1\), and \(\text{rk}(X) \geq k+1\) if \(X\) contains a proper subspace \(Y \subseteq X\) with \(\text{rk}(Y) \geq k\). Clearly, \(\text{Gr}_{k+1}(V)\) can be identified with the set of all subspaces of rank \(k\).

The point-line geometry \((P, L, *) = (\text{Gr}_1(V), \text{Gr}_2(V), *)\) has the following properties.

**PG1** Every line is incident with at least 3 distinct points.

**PG2** Any two distinct points \(p, q\) can be joined by a unique line which we denote by \(p \lor q\).

**PG3** There exist at least 2 distinct lines.
**PG4** If \( p, q, r \) are three distinct points, and if \( \ell \) is a line which meets \( p \lor q \) and \( p \lor r \) in two distinct points, then \( \ell \) meets \( q \lor r \).

Axiom **PG4** is also called *Veblen’s axiom* or the *Veblen-Young property*. A point-line geometry which satisfies these axioms is called a *projective (point-line) geometry*. If there exist two lines which don’t intersect, then it is called a *projective space*, otherwise a *projective plane*.

Veblen’s axiom PG4 is the important ‘geometric’ axiom in this list; the axioms PG1–PG3 exclude only some obvious pathologies. It is one of the marvels of incidence geometry that this simple axiom encodes — by the Fundamental Theorem of Projective Geometry stated below — the whole theory of skew fields, vector spaces, and linear algebra.

### 1.5.1 The Fundamental Theorem of Projective Geometry, II

Let \((\mathcal{P}, \mathcal{L}, \ast)\) be a projective (point-line) space which is not a projective plane. Then there exists a skew field \(D\), unique up to isomorphism, and a right vector space \(V\) over \(D\), unique up to isomorphism, such that

\[
(\mathcal{P}, \mathcal{L}, \ast) = (\text{Gr}_1(V), \text{Gr}_2(V), \ast).
\]

Here, the vector space dimension can be infinite. In fact, the dimension is finite if and only if one the following holds:

1. Every subspace has finite rank.
2. There exists no subspace \(U\) and no automorphism \(\phi\) such that \(\phi(U)\) is a proper subset of \(U\).

The theorem is folklore; we just refer to Lüneburg [20] [21], or to Faure & Fröhlicher [10] for a category-theoretic proof.

Let \(V^{\lor}\) denote the dual of \(V\). This is a right vector space over \(D^{\text{op}}\); it is easy to see that there is an isomorphism

\[
\text{PG}(V^{\lor})_{1,2} \cong (\text{Gr}_n(V), \text{Gr}_{n-1}(V), \ast).
\]
Now consider the following structure (for finite dimension \( \dim(V) = n + 1 \)). Let \( \mathcal{V} = \text{Gr}_1(V) \cup \text{Gr}_2(V) \cup \cdots \cup \text{Gr}_n(V) \) and let \( \Delta \) denote the collection of all subsets of \( \mathcal{V} \) which consist of pairwise incident elements. Such a set is finite and has at most \( n \) elements. Then \( \Delta(V) = (\Delta, \subseteq) \) is a poset, and in fact an abstract simplicial complex of dimension \( n - 1 \). The set \( \mathcal{V} \) can be identified with the minimal elements (the vertices) of \( \Delta \). There is an exact sequence

\[
1 \longrightarrow \text{Aut}(\text{PG}(V)) \longrightarrow \text{Aut}(\Delta(V)) \longrightarrow \text{AAut}(D)/\text{Aut}(D) \longrightarrow 1
\]

The poset \( \Delta(V) \) is the building associated to the projective space \( \text{PG}(V) \). See Brown [4], Garrett [12], Grundhöfer [14] (these proceedings), Ronan [26], Taylor [31], Tits [32]. Here, we view a building as a simplicial complex, without a type function (the type function \text{type} would associate to a vertex \( v \in \text{Gr}_k(V) \) the number \( k \)). Such a type function can always be defined and is unique up to automorphisms; if we consider only type-preserving automorphisms, then we obtain \( \text{Aut}(\text{PG}(V)) \) as the automorphism group. (In Grundhöfer [14], the buildings are always endowed with a type function.) There is no natural way to recover \( \text{PG}(V) \) from \( \Delta(V) \), but we can recover both \( \text{PG}(V) \) and \( \text{PG}(V^\vee) \) simultaneously. The following diagram shows the various ‘expansions’ and ‘reductions’. Only the solid arrows describe natural constructions; the dotted arrows require the choice of a type function, i.e. one has to choose which elements are called points, and which ones are called hyperplanes.

![Diagram](image-url)

This is probably the right place to introduce the Tits system (or \( BN \)-pair) of \( \text{PEL}(V) \). Actually, it is easier if we lift everything into the group \( \text{EL}(V) \) (we could equally well work with the Steinberg group \( \text{St}_n(D) \), or the general linear group \( \text{GL}(V) \)).

1.5.2 The Tits system for \( \text{EL}(V) \) Let \( b_0, \ldots, b_n \) be a base for \( V \), and let \( p_i = b_i D \). Every proper subset of \( \{p_1, \ldots, p_n\} \) spans a subspace; in this way, we obtain a collection

\[
\Sigma^{(0)} = \{V_J = \text{span}\{b_j| j \in J\}| \emptyset \neq J \subseteq \{0, \ldots, n\}\}
\]

of \( 2(2^n - 1) \) subspaces. With the natural inclusion \( \subseteq \), this becomes a poset and an abstract simplicial complex; combinatorially, this is the complex
∂Δ^{n+1} of all proper faces of a \( n+1 \)-simplex; for \( n = 2 \), we have the points and sides of a triangle, and for \( n = 3 \) the points, edges and sides of a tetrahedron.

Now we consider a different simplicial complex, the \textit{apartment} \( \Sigma \). The vertices of \( \Sigma \) are the elements of \( \Sigma^{(0)} \), and the higher rank elements are sets of pairwise incident elements. This complex can be pictured as follows: consider the first barycentric subdivision \( \text{Sd}\Sigma^{(0)} \) of \( \Sigma^{(0)} \); the barycentric subdivision adds a vertex in every face of \( \Sigma^{(0)} \). This flag complex

\[
\Sigma = \text{Sd}\partial\Delta^{n+1}
\]

is the \textit{apartment} spanned by \( p_0, \ldots, p_n \).

Let \( T \subseteq \text{EL}(V) \) denote the pointwise stabilizer of \( p_0, \ldots, p_n \) (equivalently, the elementwise stabilizer of \( \Sigma^{(0)} \) or \( \Sigma \)), and \( N \) the setwise stabilizer of \( \{p_0, \ldots, p_n\} \) (or \( \Sigma^{(0)} \), or \( \Sigma \)). Thus \( N/T \cong \text{Sym}(n+1) \); this quotient is the \textit{Weyl group} for \( \text{PGL}(V) \). Let \( B \subseteq \text{EL}(V) \) denote the stabilizer of the flag

\[
C = (p_0, p_0 \oplus p_2, p_0 \oplus p_1 \oplus p_2, \ldots, p_0 \oplus \cdots \oplus p_{n-1})
\]

With respect to the base \( b_0, \ldots, b_n \), the group \( B \) consists of the upper triangular matrices in \( \text{EL}(V) \), and \( N \) consists of permutation (or monomial) matrices (a permutation matrix has precisely one non-zero entry in every row and column), and \( T = B \cap N \) consists of diagonal matrices with the property that the product of the entries lies in the commutator group \([D^x, D^x]\). For \( 1 \leq i \leq n \), let \( s_i \) be the \( T \)-coset of the matrix

\[
\begin{pmatrix}
1 & & & \\
& \ddots & & \\
& & 1 & \\
0 & 1 & & \\
-1 & 0 & & \\
& & & 1
\end{pmatrix}
\]

which interchanges \( p_i \) and \( p_{i-1} \). Then \( s_i \) is an involution in \( W \), and

\[
(W, \{s_1, \ldots, s_n\})
\]

presents \( W \) as a \textit{Coxeter group}. It is a routine matter to check that these data \((G, B, N, \{s_1, \ldots, s_n\})\) satisfy the axioms of a \textit{Tits system}:

\textbf{TS1} \( B \) and \( N \) generate \( G \).

\textbf{TS2} \( T = B \cap N \) is normalized by \( N \).
The set $S = \{s_1, \ldots, s_n\}$ generates $N/T$ and has the following properties.

$sBs \neq s$ for all $s \in S$.

$BsBwB \subseteq BwB \cup BswB$ for all $s \in S$ and $w \in W = N/T$.

The group $T \subseteq B$ has a normal complement, the group $U$ generated by all elations $\tau_{ij}(a)$, for $i < j$. Thus $B$ is a semidirect product $B = TU$, and $(\text{EL}(V), B, N, S)$ is what is called a (strongly split) Tits system (or BN-pair). (A Tits system is called split if $B$ can be written as a not necessarily semidirect product $B = TU$, such that $U \triangleleft B$ is normal in $B$. Usually, one also requires $U$ to be nilpotent — this is the case in our example. The group $U$ acts transitively on the apartments containing the chamber corresponding to $B$. If $U \cap T = 1$, the Tits system is said to be strongly split.) Note also that $B$ is not solvable if $D$ is not commutative.

Now we describe the root system for $\Delta(V)$.

1.5.3 Thin projective spaces and the root system Let $\mathcal{P} = \{0, \ldots, n\}$ and let $\mathcal{L} = \binom{\mathcal{P}}{2}$ denote the collection of all 2-element subsets of $\mathcal{P}$. The incidence $*$ is the symmetrized inclusion relation. Then $(\mathcal{P}, \mathcal{L}, *)$ is a thin projective geometry of rank $n$, i.e. a projective space where every line is incident with precisely two points. The corresponding thin projective space (the projective geometry over the 'field 0 with one element') is

$$\text{PG}_n(0) = \left( \binom{\mathcal{P}}{1}, \binom{\mathcal{P}}{2}, \ldots, \binom{\mathcal{P}}{n}, \subseteq \right).$$

Now we construct a 'linear model' for this geometry. Consider the real euclidean vector space $\mathbb{R}^{n+1}$ with its standard inner product $\langle -, - \rangle$ and standard base $e_0, e_1, \ldots, e_n$. Let $E$ denote the orthogonal complement of the vector $v = e_0 + \cdots + e_n = (1, 1, \ldots, 1) \in \mathbb{R}^{n+1}$. Let

$$p_i = e_i - v_{-1}^{1/n} \in E,$$

for $i = 0, \ldots, n$.

We identify $p_0, \ldots, p_n$ with the points of $\text{PG}_n(0)$. The subspaces of rank $k$ correspond precisely to (linearly independent) subsets $p_{i_1}, \ldots, p_{i_k}$ of $\mathcal{P}$. We identify such a subspace with the vector $\frac{1}{k}(p_{i_1} + \cdots + p_{i_k})$; this is the barycenter of the convex hull of $\{p_{i_1}, \ldots, p_{i_k}\}$. This is our model of $\text{PG}_n(0)$; the picture
shows the case $n = 2$.

Now we construct the Weyl group. For $i \neq j$ put $\epsilon_{ij} = e_i - e_j$. The reflection $r_{ij}$ at the hyperplane $\epsilon_{ij}^\perp$ in $E$,

$$x \xrightarrow{r_{ij}} x - \langle x, \epsilon_{ij} \rangle \epsilon_{ij},$$

permutes $p_0, \ldots, p_n$; the isometry group $W$ generated by these reflections is the **Weyl group** of type $A_n$ (which is isomorphic to the Coxeter group $N/T$ of the Tits system).

These vectors $\epsilon_{ij}$ form a **root system** of type $A_n$ in $E$, with $\Phi = \{ \epsilon_{ij} \mid i \neq j \}$ as set of roots. We call the set $\Phi^+ = \{ \epsilon_{ij} \mid i < j \}$ the set of **positive roots**; this determines the $n$ **fundamental roots** $\{ \epsilon_{01}, \epsilon_{12}, \ldots, \epsilon_{n-1,n} \}$. The fundamental roots form a base of $E$, and every root is an integral linear combination of fundamental roots, such that either all coefficients are non-negative (this yields the positive roots) or non-positive.

To each root $\epsilon_{ij}$, we attach the group $U_{ij}$ as defined in Section 1.4. From the Steinberg relations, we see the following: if $i < j$ and $k < l$, then $[U_{ij}, U_{kl}] = 0$ if there exists no positive root which is a linear combination $\epsilon_{ij} a + \epsilon_{kl} b$, with $a, b \in \mathbb{Z}_{>0}$. For our root system, the only instance where such a linear combination is a positive root is when $\text{card}\{i, j, k, l\} = 3$, and in this case the Steinberg relations show that the commutator is indeed not trivial. The picture below shows the root system $A_2$ (the case $n = 2$); the fundamental roots are $\epsilon_{01}$ and $\epsilon_{12}$, and the positive roots are $\epsilon_{01}, \epsilon_{12}$, and $\epsilon_{02} = \epsilon_{01} + \epsilon_{12}$. 
The hyperplanes \( \varepsilon_{ij} \subseteq E \) yield a triangulation of the unit sphere \( S^{n-1} \subseteq E \); as a simplicial complex, this is precisely the apartment \( \Sigma \). The half-apartments correspond to the half-spaces \( \{ v \in E \mid \langle v, \varepsilon_{ij} \rangle \geq 0 \} \). From this, it is not hard to see that the groups \( U_{ij} \) are root groups in the building-theoretic sense (as defined in Grundhöfer’s article [14] in these proceedings), see also Ronan [26] Ch. 6.

1.6 The little projective group as a 2-transitive group

We show that the projective space is determined by (and can be recovered from) the action of the elementary linear group on the point set.

Let \( \text{PEL}(V) \subseteq H \subseteq \text{PGL}(V) \) be a subgroup, and assume that \( \dim(V) \geq 3 \). Then \( (H, \text{Gr}_1(V)) \) is a 2-transitive permutation group. Let \( L \subseteq \text{Gr}_1(V) \) be a point row, i.e. the set of all points lying on a line \( \ell \in \text{Gr}_2(V) \). Let \( p, q \in L \) be distinct points. Then \( H_{p,q} \) has precisely four orbits in \( \text{Gr}_1(V) \): the two singletons \( \{p\}, \{q\} \), the set \( L \setminus \{p, q\} \), and \( X = \text{Gr}_1(V) \setminus L \). The set \( X \) has the property that every \( h \) in \( H \) which fixes \( X \) pointwise fixes \( \text{Gr}_1(V) \) pointwise. None of the other three orbits has this property. Thus one can see the line \( L \subset \text{Gr}_1(V) \) from the \( H \)-action, we have a canonical (re)construction

\[
(H, \text{Gr}_1(V)) \rightarrow (H, \text{Gr}_1(V), \text{Gr}_2(V), \ast).
\]

Combining this with the Fundamental Theorem of Projective Geometry [1.1.1], we have the next result.

1.6.1 Proposition For \( i = 1, 2 \), let \( \text{PG}(V_i) \) be projective geometries (of possibly different ranks \( n_i \geq 2 \)) over skew fields \( D_1, D_2 \). Let \( \text{PEL}(V_i) \subseteq H_i \subseteq \text{PGL}(V_i) \) be subgroups. If there exists an isomorphism of permutation groups

\[
(H_1, \text{Gr}_1(V_1)) \xrightarrow{\phi} (H_2, \text{Gr}_1(V_2))
\]

then there exists a semilinear bijection \( F : V_1 \rightarrow V_2 \) which induces \( \phi \) (and \( n_1 = n_2 \)). \( \square \)

The result is also true in dimension 2, but the proof is more complicated, as we will see in Section [1.8]. The problem whether an abstract group isomorphism \( H_1 \xrightarrow{\phi} H_2 \) is always induced by a semilinear map is much more subtle. The result is indeed that such an isomorphism is induced by a linear map, composed with an isomorphism or anti-isomorphism of the skew fields in question, provided that the vector space dimensions are large enough (at least 3), see Hahn & O’Meara [E] 2.2D. The crucial (and difficult) step is to show that \( \phi \) maps transvections to transvections.
1.7 Projective planes

We mention the classification of Moufang planes.

The Fundamental Theorem of Projective Geometry does not apply to projective planes. However, there is the following result. Suppose that \((P, \mathcal{L}, *)\) is a projective plane. Given a flag \(\langle p, \ell \rangle\) (i.e. \(p * \ell\)), the group \(G_{[p,\ell]}\) is defined to be the set of all automorphisms which fix \(p\) linewise and \(\ell\) pointwise (so for \(\text{PG}(D^3)\), we have \(G_{[p,\ell]} = U_{(p,\ell)}\) in our previous notation). Let \(h\) be a line passing through \(p\) and different from \(\ell\). It is easy to see that \(G_{[p,\ell]}\) acts freely on the set \(H' = \{ q \in P \mid q \neq p, q * \ell \}\). If this action is transitive, then \((P, \mathcal{L}, *)\) is called \((p, \ell)\)-homogeneous. The projective plane is called a Moufang plane if it is \((p, \ell)\)-homogeneous for any flag \((p, \ell)\). If \((\mathcal{P}, \mathcal{L}, *) = \text{PGL}(V)\) for some 3-dimensional vector space \(V\), then we have a Moufang plane.

Recall that an alternative field is a (not necessarily associative) algebra with unit, satisfying the following relations.

**AF1** If \(a \neq 0\), then the equations \(ax = b\) and \(ya = b\) have unique solutions \(x, y\).

**AF2** The equalities \(x^2y = x(xy)\) and \(yx^2 = (yx)x\) hold for all \(x, y\).

Clearly, every field or skew field is an alternative field. The structure theorem of alternative fields says that every non-associative alternative field is a central 8-dimensional algebra over a field \(K\), a so-called Cayley division algebra. Not every field \(K\) admits a Cayley division algebra; it is necessary that \(K\) admits an anisotropic quadratic form of dimension 8, so finite fields or algebraically closed fields do not admit Cayley division algebras. Every real closed (or ordered) field admits a Cayley division algebra.

Given an alternative field \(A\), we construct a projective plane \(\text{PG}_2(A)\) as follows. Let \(\infty\) be a symbol which is not an element of \(A\). Let

\[
\mathcal{P} = \{ (\infty) \} \cup \{ (a) \mid a \in D \} \cup \{ (x, y) \mid x, y \in A \}
\]

\[
\mathcal{L} = \{ [\infty] \} \cup \{ [a] \mid a \in D \} \cup \{ [x, y] \mid x, y \in A \}
\]

The incidence \(\ast\) is defined as

\[
(\infty) * [a] * (a, sa + t) * [s, t] * (s) * [\infty] * (\infty)
\]

Note that for a field or skew field \(D\), this is precisely \(\text{PG}(D^3)\).

1.7.1 Theorem (Moufang Planes)

Let \((\mathcal{P}, \mathcal{L}, *)\) be a Moufang plane. The there exists an alternative field \(A\), unique up to isomorphism, such that \((\mathcal{P}, \mathcal{L}, *)\) is isomorphic to \(\text{PG}_2(A)\).
For a proof see Hughes & Piper [17] or Pickert [23]; the structure theorem for non-associative alternative fields is proved in Van Maldeghem [34].

The Moufang planes can also be described by means of Steinberg relations: for each pair \((i, j)\) with \(i \neq j\) and \(i, j \in \{1, 2, 3\}\), fix a copy \(U_{ij} \cong (A, +)\).

Let \(G\) denote the free product of these six groups, factored by the Steinberg relations as given in [14] (note that the Steinberg relations make sense even in the non-associative case). The group \(G\) has a natural Tits system which yields the Cayley plane \(\text{PG}_2(A)\). The group induced by \(G\) on \(\text{PG}_2(A)\) is a \(K\)-form of a simple adjoint algebraic group of type \(E_6\).

1.8 The projective line and Moufang sets

We investigate the action of the linear group as a permutation group on the projective line. This is a special case of a Moufang set.

The projective line \(P = \text{Gr}_1(V)\), for \(V \cong D^2\), is a set without further structure. We add structure by specifying properties of the group \(G = \text{PGL}(V)\) acting on it. This group has two remarkable properties: (1) \(G\) acts 2-transitively on \(P\). (2) The stabilizer \(B = G_p\) of a point \(p = vD \in P\) has a regular normal subgroup \(U_p\), the group induced by maps of the form \(\text{id}_V + v\rho\), where \(\rho\) runs through the collection of all non-zero elements of \(V^\ast\) which annihilate \(v\).

Moufang sets where first defined by Tits in [33]; our definition given below is stated in a slightly different way. We define a Moufang set as a triple \((G, U, X)\), where \(G\) is a group acting on a set \(X\) (with at least 3 elements), and \(U\) is a subgroup of \(G\). We require the following properties.

**MS1** The action of \(G\) on \(X\) is 2-transitive.

**MS2** The group \(U\) fixes a point \(x\) and acts regularly on \(X \setminus \{x\}\).

**MS3** The group \(U\) is normal in the stabilizer \(G_x\).

The properties **MS2** and **MS3** will be summarized in the sequel as \('G_x\ has a regular normal subgroup'\); we will also say that \('U \ makes\ (G, X)\ into\ a\ Moufang\ set'\). Let \(y \in X \setminus \{x\}\), and put \(T = G_{x,y}\). Then clearly \(G_x = TU\) is a semidirect product. (The pair \((G_x, T)\) is a what is called a (strongly) split Tits system (BN-pair) of rank 1 for the group \(G\).)

If \(U = G_x\), then \(T = 1\), so \(G\) is sharply 2-transitive. This case has its own, special flavor. Note that in general, \(U\) is not determined by \(G\) and \(x\). As a counterexample, let \(\mathbb{H}\) denote the quaternion division algebra over a real closed field, let \(X = \mathbb{H}\), and consider the group consisting of maps of the form \([x \mapsto axb + t]\), for \(a, b \in \mathbb{H}^\times\) and \(t \in \mathbb{H}\). The stabilizer of 0 consists of
the maps $[x \mapsto axb]$, and it has two regular normal subgroups isomorphic to $\mathbb{H}^\times$, consisting of the maps $[x \mapsto ax]$ or $[x \mapsto xb]$.

Let $H \subseteq \text{PGL}_2(D)$ be a subgroup containing $\text{PEL}_2(D)$. We identify the projective line $\text{Gr}_1(D^2)$ with $D \cup \{\infty\}$, identifying $(\{1\})D$ with $x$ and $(\{0\})D$ with $\infty$. Let $U_\infty$ denote the group consisting of the maps $[x \mapsto x + t]$, for $t \in D$. So

$$(H, U_\infty, D \cup \{\infty\})$$

is a Moufang set. The stabilizer $T = H_0_\infty$ contains all maps $[x \mapsto axa]$, for $a \in D^\times$. In particular, $T$ is commutative if and only if $D$ is commutative. Now we consider the following problem:

**Is it possible to recover $U_\infty$ from the action of $H$?**

In the commutative case, the answer is easy: $U_\infty$ is the commutator group of $H_\infty$,

$$T \text{ commutative } \implies U_\infty = [H_\infty, H_\infty].$$

Also, an element $1 \neq g \in H_\infty$ is contained in $U_\infty$ if and only if $g$ has no fixed point in $D$. Thus, $U_\infty$ is the only regular normal subgroup of $H_\infty$.

Now suppose that $D$ is not commutative. In this case, we will prove first that the action of $H_\infty$ on $D$ is primitive. We have to show that $T$ is a maximal subgroup of $H_\infty$. Let $g \in H_\infty \setminus T$, and consider the group $K$ generated by $T$ and $g$. Since $H_\infty$ splits as a semidirect product $H_\infty = TU_\infty$, we may assume that $g = [x \mapsto x + t] \in U$, with $t \neq 1$. Since $K$ contains $T$, it contains all maps of the form $[x \mapsto x + taba'^{-1}b^{-1}]$. Using some algebraic identities for multiplicative commutators as in Cohn [8] Sec. 3.9, one shows that the set of all multiplicative commutators generates $D$ additively. Thus $U_\infty \subseteq K$, whence $K = H_\infty$,

$$T \text{ not commutative } \implies H_\infty \text{ primitive}$$

$$\implies U_\infty \text{ unique abelian normal subgroup of } H_\infty$$

(for the last implication see Robinson [25] 7.2.6). As in the commutative case, there is no other way of making the projective line into a Moufang set. Indeed, suppose that $U_\infty \neq U' \leq H_\infty$ is another regular normal subgroup. Then $U_\infty \cap U' \leq H_\infty$ is also normal, so either $U_\infty \cap U' = 1$, or $U' \supseteq U_\infty$. In the latter case, $U_\infty = U'$ since we assumed the action to be regular. So suppose $U_\infty \cap U' = 1$. Then $U'U_\infty$ is a direct product. Define a map $\phi : U_\infty \longrightarrow U'$ by putting $\phi(u) = u'$ if and only if $u(0) = u'(0)$. Then $\phi(u_1u_2)(0) = (u_1u_2)'(0) = u_1u_2(0) = u_2u_1(0) = u_2u_1'(0)$, so $\phi$ is an anti-isomorphism. In particular, $U'$ is abelian, whence $U_\infty = U'$, a contradiction.

**1.8.1 Proposition** Let $V$ be a two-dimensional vector space over a skew field $D$, and assume that $\text{PEL}(V) \subseteq H \subseteq \text{PGL}(V)$. Then $G_x$ contains a
unique regular normal subgroup $U_{\infty}$, i.e. there is a unique way of making $(H, \Gr_1(V))$ into a Moufang set.

We combine this with Hua’s Theorem.

1.8.2 Theorem (Hua) Let

$$(\PEL(V), U_{\infty}, D \cup \{\infty\}) \xrightarrow{\alpha \cong} (\PEL(V'), U'_{\infty}, D' \cup \{\infty'\})$$

be an isomorphism of Moufang sets. Then $\alpha$ is induced by an isomorphism or anti-isomorphism of skew fields.

For a proof see Tits [32] 8.12.3 or Van Maldeghem [33] p. 383–385.

1.8.3 Corollary Let $V, V'$ be 2-dimensional vector spaces over skew fields $D, D'$, let $\PEL(V) \subseteq H \subseteq \PGL(V)$ and $\PEL(V') \subseteq H' \subseteq \PGL(V')$ and assume that

$$(H, \Gr_1(V)) \xrightarrow{\alpha \cong} (H', \Gr_1(V'))$$

is an isomorphism of permutation groups. Then there exists an isomorphism or anti-isomorphism $D \xrightarrow{\theta \cong} D'$ which induces $\alpha$.

Proof. Clearly, $\alpha$ induces an isomorphism of the commutator groups $[H, H] = \PEL(V)$ and $[H', H'] = \PEL(V')$ (we can safely disregard the small fields $\mathbb{F}_2$ and $\mathbb{F}_3$, since here, counting suffices). There is a unique way of making these permutation groups into Moufang sets, and to these Moufang sets, we apply Hua’s Theorem.

The following observation is due to Hendrik Van Maldeghem. Let $V$ be a vector space of dimension at least 3, let $H$ be a group of automorphisms of $\PG(V)$ containing $\PEL(V)$. Then $(H, \Gr_1(V))$ cannot be made into a Moufang set. To see this, let $p \in \Gr_1(V)$ and assume that $U \leq H_p$ is a normal subgroup acting regularly on $\Gr_1(V) \setminus \{p\}$. Let $u \in U$, and let $\tau$ be an elation with center $p$. Then $u\tau u^{-1}$ is also an elation with center $p$, and so is $[u, \tau]$. If we choose $u, \tau$ in such a way that $u$ doesn’t fix the axis of $\tau$ (which is possible, since $\dim(V) \geq 3$), then $[u, \tau] \in U$ is a non-trivial elation with center $p$. Since $U$ is normal, $U$ contains all elations with center $p$. These elations form an abelian normal subgroup of $H_p$ which is, however, not regular on $\Gr_1(V)$.

1.8.4 Lemma Let $\PEL(V) \subseteq H \subseteq \PGL(V)$ and assume that $\dim(V) \geq 3$. Then $H_p$ contains no regular normal subgroup; in particular, $(H, \Gr_1(V))$ cannot be made into a Moufang set.

Combining the results of this section with Proposition 1.6.1, we have the following final result about actions on the point set.
1.8.5 Corollary  For \( i = 1, 2 \), let \( V_i \) be vector spaces over skew fields \( D_i \), of (finite) dimensions \( n_i \geq 2 \). Let \( \text{PEL}(V_i) \subseteq H_i \subseteq \text{PGL}(V_i) \) and assume that

\[
(H_1, \text{Gr}_1(V_1)) \cong (H_2, \text{Gr}_1(V_2))
\]

is an isomorphism of permutation groups. Then \( n_1 = n_2 \), and \( \phi \) is induced by a semilinear isomorphism, except if \( n_1 = 2 \), in which case \( \phi \) may also be induced by an anti-isomorphism of skew fields.

These results are also true for infinite dimensional vector spaces.

2 Polar spaces and quadratic forms

In this second part we consider \((\sigma, \varepsilon)\)-hermitian forms and their generalizations, pseudo-quadratic forms. From now on, we consider also infinite dimensional vector spaces. As a finite dimensional motivation, we start with dualities. Suppose that \( \dim(V) = n + 1 \) is finite, and that there is an isomorphism \( \phi \) between the projective space \( \text{PG}(V) \) and its dual,

\[
\text{PG}(V) \cong \text{PG}(V^\vee).
\]

Now \( V^\vee \) is in a natural way a right vector space over the opposite skew field \( D^{\text{op}} \). By the Fundamental Theorem of Projective Geometry, \( \phi \) is induced by a \( \sigma \)-semilinear bijection \( f : V \cong V^\vee \), relative to an isomorphism \( \sigma : D \cong D^{\text{op}} \), i.e. \( \sigma \) is an anti-automorphism of \( D \). Note also that there is a natural isomorphism

\[
\text{PG}(V^\vee) \cong (\text{Gr}_n(V), \text{Gr}_{n-1}(V), \ldots, \text{Gr}_1(V), *)
\]

Thus, we may view \( \phi \) as an non type-preserving automorphism of the building \( \Delta(V) \). Such an isomorphism is called a duality. If \( \phi^2 = \text{id}_{\Delta(V)} \) (this makes sense in view of the identification above), then \( \phi \) is called a polarity. In this section we study polarities and the related geometries, polar spaces.

2.1 Forms and polarities

We study some basic properties of forms (sesquilinear maps) and their relation with dualities and polarities.

In this section, \( V \) is a (possibly infinite dimensional) right vector space over \( D \). We fix an anti-automorphism \( \sigma \) of \( D \). Using \( \sigma \), we make the dual space \( V^\vee \)
into a right vector space over $D$, denoted $V^\sigma$, by defining $\lambda a = [v \mapsto a^\sigma \lambda(v)]$, for $v \in V$, $\lambda \in V^\vee$ and $a \in D$. We put

$$\text{Form}_\sigma(V) = \text{Hom}_D(V, V^\sigma).$$

Given an element $f \in \text{Form}_\sigma(V)$, we write $f(u, v) = f(u)(v)$; the map $(u, v) \mapsto f(u, v)$ is biadditive (Z-linear in each argument) and $\sigma$-sesquilinear, $f(ua, vb) = a^\sigma f(u, v)b$. Note that $\sigma$ is uniquely determined by the map $f$, provided that $f \neq 0$. Suppose that that

$$F : V \longrightarrow V'$$

is a linear map of vector spaces over $D$. We define

$$\text{Form}_\sigma(V) \xleftarrow{F^*} \text{Form}_\sigma(V')$$

by $F^*(f')(u, v) = f'(F(u), F(v))$. A similar construction works if $V \xrightarrow{F} V'$ is $\theta$-semilinear relative to a skew field isomorphism $D \xrightarrow{\theta} D'$; here, we define $F^*(f')(u, v) = f'(F(u), F(v))^{\theta^{-1}}$ to obtain

$$\text{Form}_\sigma(V) \xleftarrow{F^*} \text{Form}_{\sigma'}(V'),$$

where $\sigma = \theta \sigma' \theta^{-1}$. The group $\text{GL}(V)$ acts thus in a natural way from the right on $\text{Form}_\sigma(V)$, by putting

$$fg = g^s f.$$ 

Forms in the same $\text{GL}(V)$-orbit are called equivalent; the stabilizer of a form $f$ is denoted

$$\text{GL}(V)_f = \{g \in \text{GL}(V) | f(g(u), g(v)) = f(u, v) \text{ for all } u, v \in V\}.$$ 

The assignment $V \xrightarrow{F} V^\sigma$ is a natural cofunctor on $\mathcal{Mod}_D^\text{fin}$ (or $\mathcal{Mod}_D$, if we allow infinite dimensional vector spaces) which we also denote by $\sigma$,

$$V \xrightarrow{F} V' \xrightarrow{\sigma} V'^\sigma.$$ 

Let $V^\vee \xrightarrow{F^\vee} V'^\vee$ be the dual or adjoint of $V \xrightarrow{F} V'$ (i.e. $F^\vee(\lambda) = \lambda F$). Then set-theoretically, $F^\sigma = F^\vee$. There is a canonical linear injection

$$V \xrightarrow{\text{can}} V^{\sigma \sigma}.$$ 

25
which sends $v \in V$ to the map $\text{can}(v) = [\lambda \mapsto \lambda(v)^{\sigma^{-1}}]$. If the dimension of $V$ is finite, then $\text{can}$ is an isomorphism (and the data $\sigma$ and $\text{can}$ make the abelian category $\text{Mod}^{\text{fin}}_D$ into a hermitian category $\text{Herm}^{\text{fin}}_D,\sigma$). Given a form $f$, we have a diagram

$$
\begin{array}{c}
\text{can} \\
\downarrow \\
\text{can} \\
V \xrightarrow{f} V^\sigma \\
\downarrow \\
V^\sigma \xleftarrow{f^\sigma}
\end{array}
$$

Note that $f^\sigma \text{can}(v) = [u \mapsto f(u,v)^{\sigma^{-1}}]$. If both $f$ and $f^\sigma \text{can}$ are injective, then we call the form $f$ non-degenerate (if $\dim(V)$ is finite, then it suffices to require that $f$ is injective). For a subspace $U \subseteq V$, put

$$U^{\perp_f} = \bigcap \{ \ker(f(u)) \mid u \in U \} = \{ v \in V \mid f(u,v) = 0 \text{ for all } u \in U \}$$

$$U^{\perp_f} = \bigcap \{ \ker(f^\sigma \text{can}(u)) \mid u \in U \} = \{ v \in V \mid f(v,u) = 0 \text{ for all } u \in U \}.$$

Thus $f$ is non-degenerate if and only if $V^{\perp_f} = 0 = V^{\perp_f}$.

In the finite dimensional case, a non-degenerate form defines a duality of $\text{PG}(V)$ (by $U \mapsto U^{\perp_f}$, and by the Fundamental Theorem of Projective Geometry 1.1.1 every duality of $\text{PG}(V)$ is obtained in this way). Also, $\perp_f$ determines the anti-automorphism $\sigma$ up to conjugation with elements of $\text{Int}(D)$. The form $f$ itself is, however, not determined by $\perp_f$. Therefore, we introduce another equivalence relation on forms. If $f$ is $\sigma$-sesquilinear, and if $s \in D^\times$, then $sf : (u,v) \mapsto sf(u,v)$ is $\sigma s^{-1}$-sesquilinear,

$$sf(ua,vb) = sa^\sigma f(u,v)b = (sa^\sigma s^{-1})sf(u,v)b = a^\sigma s^{-1}(sf)(u,v)b.$$

The forms $f$ and $sf$ are called proportional, and we say that $sf$ is obtained from $f$ by scaling with $s$; proportional forms induce the same dualities. Scaling with $s$ yields an isomorphism $\text{Form}_\sigma(V) \xrightarrow{\cong} \text{Form}_{\sigma s^{-1}}(V)$.

**2.1.1 Proposition** Suppose that $\dim(V)$ is finite. Let $f \in \text{Form}_\sigma(V)$ and $f' \in \text{Form}_{\sigma'}(V)$ be non-degenerate forms. If $f$ and $f'$ induce the same duality, then $\sigma' s^{-1} \in \text{Int}(D)$, and $f$ and $f'$ are proportional. There is a 1-1 correspondence

$$\begin{array}{c}
\{ \text{Dualities in } \text{PG}(V) \} \\
\xleftarrow{\cong} \xrightarrow{\cong} \\
\{ \text{Proportionality classes of non-degenerate forms} \}
\end{array}$$

\[\square\]
A non-degenerate sesquilinear form which induces a duality has the property that \((U^\perp)^\perp = U\) holds for all subspaces \(U\). This condition makes also sense in the infinite dimensional case if we restrict it to finite dimensional subspaces (although there, no dualities exist), and boils down to \(f\) being reflexive;

\[
f(u, v) = 0 \iff f(v, u) = 0.
\]

In other words, \((U^\perp)^\perp = U^\perp\) holds for all subspaces \(U \subseteq V\). If \(f \neq 0\) is reflexive, then there exists a unique element \(\varepsilon \in D^\times\) such that \(f(u, v) = f(v, u)^\sigma \varepsilon\) for all \(u, v \in V\), see Dieudonné [1] Ch. I §6. Furthermore, this implies that \(\varepsilon^\sigma = \varepsilon^{-1}\) (choose \(u, v\) with \(f(u, v) = 1\), then \(f(v, u) = \varepsilon\)), and \(a^\sigma^2 = a^\varepsilon^{-1}\) for all \(a \in D\) (consider \(f(u, va)\)). A form \(h\) which satisfies the identity

\[
h(u, v) = h(v, u)^\sigma \varepsilon \quad \text{for all } u, v \in V
\]

is called \((\sigma, \varepsilon)\)-hermitian. The collection of all \((\sigma, \varepsilon)\)-hermitian forms is a subgroup of \(\text{Form}_\sigma(V)\) which we denote \(\text{Herm}_{\sigma,\varepsilon}(V)\). A non-degenerate \((\sigma, \varepsilon)\)-hermitian form \(h\) induces thus an involution \(\perp_h\) on the building \(\Delta(V)\), for finite dimensional \(V\). In the finite dimensional setting, we have thus a 1-1 correspondence

\[
\left\{\text{Polarities in } \mathbb{P}G(V)\right\} \leftrightarrow \left\{\text{Proportionality classes of non-degenerate } \left(\sigma, \varepsilon\right)\text{-hermitian forms}\right\}
\]

If a \((\sigma, \varepsilon)\)-hermitian form \(h\) is scaled with \(s \in D^\times\), then the resulting form \(sh\) is \((\sigma s^{-1}, ss^\sigma \varepsilon)\)-hermitian.

### 2.2 Polar spaces

*We introduce polar spaces as certain point-line geometries.*

Assume that \(h\) is non-degenerate \((\sigma, \varepsilon)\)-hermitian. An element \(U \in \text{Gr}_k(V)\) is called *absolute* if it is incident with its image \(U^\perp h\). If \(2k \leq \dim(V)\), then this means that \(U \subseteq U^\perp h\), or, in other words, that \(h|_{U \times U} = 0\). A subspace with this property is called *totally isotropic* (with respect to \(h\)). Let \(\text{Gr}^h_k(V)\) denote the collection of all totally isotropic \(k\)-dimensional subspaces. The maximum number \(k\) for which this set is non-empty is called the *Witt index* \(\text{ind}(h)\) of \(h\) (If the vector space dimension is infinite, then \(\text{ind}(h)\) can of course be infinite). This makes sense also for possibly degenerate \((\sigma, \varepsilon)\)-hermitian forms:

\[
\text{ind}(h) = \max\{\dim(U) \mid U \subseteq V, \; h|_{U \times U} = 0\}
\]
If $h$ is non-degenerate, then $2 \text{ind}(V) \leq \dim(V)$. Suppose that $h$ is non-degenerate and of finite index $m \geq 2$. Let $\text{PG}^h(V)$ denote the structure

$$\text{PG}^h(V) = (\text{Gr}_1^h(V), \ldots, \text{Gr}_m^h(V), \ast),$$

and let $\text{PG}^h(V)_{1,2} = (\text{Gr}_1^h(V), \text{Gr}_2^h(V), \ast)$ denote the corresponding point-line geometry. This is an example of a polar space. This geometry has one crucial property: given a point $p \in \text{Gr}_1^h(V)$ and a line $L \in \text{Gr}_2^h(V)$ which are not incident, $p \not\subseteq L$, there are two possibilities: either there exists an element $H \in \text{Gr}_3^h(V)$ containing both $p$ and $L$, or there exists precisely one point $q$ incident with $L$, such that $p \oplus q \in \text{Gr}_2^h(V)$. Algebraically, this means that either $L \subseteq p^{\perp h}$ or $q = L \cap p^{\perp h}$ (note that $p^{\perp h}$ is a hyperplane, so the intersection has at least dimension 1).

A point-line geometry $(\mathcal{P}, \mathcal{L}, \ast)$ is called a (non-degenerate) polar space if it satisfies the following properties.

**PS1** There exist two distinct lines. Every line is incident with at least 3 points. Two lines which have more than one point in common are equal.

**PS2** Given $p \in \mathcal{P}$ there exists $q \in \mathcal{P}$ such that $p$ and $q$ are not joined by a line.

**PS3** Given a point $p \in \mathcal{P}$ and a line $\ell \in \mathcal{L}$, either $p$ is collinear with every point which is incident with $\ell$, or with precisely one point which is incident with $\ell$.

A subspace of a geometry satisfying PS1–PS3 is a set $X$ of points with the following two properties: given two distinct points $p, q \in X$, there exists a line $\ell$ incident with $p, q$, and if $r$ is also incident with $\ell$, then $r \in X$. It is a (non-trivial) fact that every subspace which contains 3 non-collinear points is a projective space. We define the rank of $(\mathcal{P}, \mathcal{L}, \ast)$ as the maximum of the ranks of the subspaces minus 1.

**PS4** The rank $m$ is finite.
PS5 Every subspace of rank $m - 2$ is contained in at least 3 subspaces of rank $m - 1$.

A structure satisfying the axioms PS1–PS5 is called a thick polar space. It is easy to see that the structure

$$PG^h(V) = (\Gr_1^h(V), \ldots, \Gr_m^h(V), \ast),$$

satisfies PS1–PS4, provided that $h$ is non-degenerate and $\text{ind}(h) = m \geq 2$. Axiom PS5 is more subtle, but is easy to see that every subspace of rank $m - 1$ is contained in at least 2 subspaces of rank $m$. We call such a structure a weak polar space (with thick lines). Our set of axioms is a variation of the Buekenhout-Shult axiomatization of polar spaces given in Buekenhout & Shult [6]. Similarly as Veblen’s axiom in the definition of a projective geometry in Section 1.5, the one axiom which is geometrically important is the Buekenhout-Shult ‘one or all’ axiom PS3. By the Fundamental Theorem of Polar Spaces [2.6.3] this simple axiom encodes the whole body of geometric algebra!

We mention some examples of (weak) polar spaces which do not involve hermitian forms.

2.2.1 Examples

(0) Let $X, Y$ be disjoint sets of cardinality at least 3, put $\mathcal{P} = X \times Y$, and $\mathcal{L} = X \cup Y$. By definition, a point $(x, y)$ is incident with the lines $x$ and $y$. The resulting geometry is a weak polar space of rank 2; every point is incident with precisely 2 lines. This in fact an example of a weak generalized quadrangle, see Van Maldeghem’s article [35] in these proceedings.

(1) Every (thick) generalized quadrangle (see Van Maldeghem’s article [35]) is a thick polar space of rank 2.

(2) Let $V$ be a 4-dimensional vector space over a skew field $D$, put $\mathcal{P} = \Gr_2(V)$ and $\mathcal{L} = \{(p, A) \in \Gr_1(V) \times \Gr_3(V) | p \subseteq A\}$. The incidence is the natural one (inclusion of subspaces). The resulting polar space which we denote by $A_{3,2}(D)$ has rank 3; the planes of the polar space are the points and planes of $\PG(V)$, and every line of this polar space is incident with precisely two planes, so $A_{3,2}(D)$ is a weak polar space.

The next theorem is an important step in the classification of polar spaces of higher rank (the full classification will be stated in Section 2.6).

2.2.2 Theorem (Tits) Let $X$ be a subspace of a polar space $(\mathcal{P}, \mathcal{L}, \ast)$. If $X$ contains 3 non-collinear points, then $X$, together with the set of all lines which intersect $X$ in more than one point, is a self-dual projective space. This projective space is either a Moufang plane over some alternative field $A$, or a desarguesian projective space over some skew field $D$.

For a proof see Tits [32] 7.9, 7.10, and 7.11. □
2.3 Hermitian forms

We continue to study properties of hermitian forms.

We fix the following data: \( D \) is a skew field, \( \sigma \) is an anti-automorphism of \( D \), and \( \varepsilon \in D^\times \) is an element with \( \varepsilon^\sigma \varepsilon = 1 \) and \( x^{\sigma^2} = x^{-\varepsilon} \) as in Section 2.1, and \( \operatorname{Herm}_{\sigma,\varepsilon}(V) \) is the group of all \((\sigma,\varepsilon)\)-hermitian forms on \( V \). We define the set of \((\sigma,\varepsilon)\)-traces as

\[
D_{\sigma,\varepsilon} = \{ c + c^\sigma \varepsilon | c \in D \}
\]

This is an additive subgroup of \((D,+)\), with the property that \( cD_{\sigma,\varepsilon}c \subseteq D_{\sigma,\varepsilon} \), for all \( c \in D \). A form \( h \in \operatorname{Herm}_{\sigma,\varepsilon}(V) \) is called trace valued if \( h(v,v) \in D_{\sigma,\varepsilon} \) holds for all \( v \in V \). We call such a form \( h \) trace \((\sigma,\varepsilon)\)-hermitian. (If \( \text{char}(D) \neq 2 \), then it’s easy to show that every hermitian form is trace hermitian.) A hermitian form \( h \) is trace hermitian if and only if it can be written as

\[
h(u,v) = f(u,v) + f(v,u)^\sigma \varepsilon,
\]

for some \( f \in \operatorname{Form}_\sigma(V) \). We denote the group of all trace hermitian forms by \( \operatorname{TrHerm}_{\sigma,\varepsilon}(V) \).

2.3.1 Lemma Let \( V_0 = \{ v \in V | h(v,v) \in D_{\sigma,\varepsilon} \} \). Then \( V_0 \) is a subspace of \( V \) containing all totally isotropic subspaces of \( V \). \( \square \)

The form induced by \( h \) on \( V_0/(V_0 \cap V_0^\perp_h) \) is thus trace hermitian and non-degenerate. Since we are only interested in the polar space arising from \( h \), we can thus safely assume that \( h \) is trace hermitian.

Suppose now that we scale the form with an element \( s \in D^\times \). Let \( h' = sh \) and put \( \sigma' = \sigma s^{-1} \) and \( \varepsilon' = ss^{-\sigma} \varepsilon \). Then \( D_{\sigma',\varepsilon'} = sD_{\sigma,\varepsilon} \). Thus we can achieve that either \( D_{\sigma,\varepsilon} = 0 \), or that \( 1 \in D_{\sigma,\varepsilon} \). In the first case we have \( \varepsilon' = -1 \) and \( \sigma = \sigma' = \text{id}_D \) (and then \( D \) is commutative), and in the second case \( \varepsilon = 1 \) and \( \sigma'^2 = \text{id}_D \). The study of (non-degenerate) trace hermitian forms is thus — by means of scaling — reduced to the following subcases:

**Trace \( \sigma \)-hermitian forms**

\[
\sigma^2 = \text{id}_D \text{ (here } \sigma = \text{id}_D \text{ is allowed if } D \text{ is commutative) and } h(u,v) = h(v,u)^\sigma \text{ for all } u, v \in V.
\]

If \( \sigma = \text{id}_D \), then we call \( h \) symmetric, and (in the non-degenerate case) \( \operatorname{GL}(V)_h = \operatorname{O}(V,h) \) is the orthogonal group of \( h \); if \( \sigma \neq \text{id}_D \) (and if \( h \) is non-degenerate), then \( \operatorname{U}(V,h) = \operatorname{GL}(V)_h \) is called the unitary group of \( h \).

**Symplectic forms**

\[
\sigma = \text{id}_D \text{ and } h(v,v) = 0 \text{ for all } v \in V \text{ (so } h \text{ is symplectic). The group } \operatorname{GL}(V)_h = \operatorname{Sp}(V,h) \text{ is called the symplectic group} \text{ (in the non-degenerate case).}
\]
Before we consider these groups and forms in more detail, we extend the whole theory to include (pseudo-)quadratic forms.

### 2.4 Pseudo-quadratic forms

We introduce pseudo-quadratic forms, which are certain cosets of sesquilinear forms. In characteristic 2, this generalizes trace hermitian forms.

In the course of the classification of polar spaces, it turns out that in characteristic 2, \((\sigma, \varepsilon)\)-trace hermitian forms are not sufficient; one needs the notion of a pseudo-quadratic form which is due to Tits. We follow the treatment which is now standard and which is based on Bak’s concept [3] of form parameters. This is a modification of Tits’ original approach; however, the characteristic 2 theory of unitary groups over skew fields is entirely due to Tits, a fact which is not always properly reflected in books on classical groups (see e.g. the footnote on p. 190 in Hahn & O’Meara [15]). Let \(\Lambda\) be an additive subgroup of \(D\), with

\[
D_{\sigma, -\varepsilon} = \{ c - c^\sigma \varepsilon \mid c \in D \} \subseteq \Lambda \subseteq D^*_{\sigma, -\varepsilon} = \{ c \in D \mid c^\sigma \varepsilon = -c \},
\]

and with the property that

\[s^\sigma \Lambda s \subseteq \Lambda\]

for all \(s \in D\). Such a subset \(\Lambda\) is called a form parameter. Given an element \(f \in \text{Form}_\sigma(V)\), we define the pseudo-quadratic form \([f] = (q_f, h_f)\) to be the pair of maps

\[
q_f : V \longrightarrow D/\Lambda \quad h_f : V \times V \longrightarrow D
\]

\[v \mapsto f(v, v) + \Lambda \quad (u, v) \mapsto f(u, v) + f(v, u)^\sigma \varepsilon.
\]

It is not difficult to see that the map \(f \mapsto [f]\) is additive; the kernel is

\[
\Lambda-\text{Herm}_{\sigma, -\varepsilon}(V) = \{ f \in \text{TrHerm}_{\sigma, -\varepsilon}(V) \mid f(v, v) \in \Lambda \text{ for all } v \in V \}.
\]

The resulting group of pseudo-quadratic forms is denoted \(\Lambda-\text{Quad}_{\sigma, \varepsilon}(V)\), and we have an exact sequence

\[
0 \longrightarrow \Lambda-\text{Herm}_{\sigma, -\varepsilon}(V) \longrightarrow \text{Form}_\sigma(V) \longrightarrow \Lambda-\text{Quad}_{\sigma, \varepsilon}(V) \longrightarrow 0
\]

which is compatible with the \(\text{GL}(V)\)-action on \(\text{Form}_\sigma(V)\); furthermore, scaling is a well-defined process on pseudo-quadratic forms. Let \(\psi(c) = c + c^\sigma \varepsilon\). Then
\( \ker(\psi) = D^{\sigma,-\varepsilon} \supseteq \Lambda; \) consequently, there is a well-defined map \( D/\Lambda \longrightarrow D \) sending \( c + \Lambda \) to \( \psi(c) \),

\[
\begin{array}{c}
\Lambda \\
\downarrow \\
0 \longrightarrow D^{\sigma,-\varepsilon} \longrightarrow D \psi \longrightarrow D_{\sigma,\varepsilon} \longrightarrow 0.
\end{array}
\]

Thus we have \( \bar{\psi}(q_f(v)) = h(v,v) \); in particular,

\[
q_f(v) = 0 \implies h_f(v,v) = 0.
\]

We call a pseudo-quadratic form non-degenerate if \( h_f \) is non-degenerate (this differs from Tits' notion of non-degeneracy \([32],[5]\); we'll come back to that point later). Similarly as before, a subspace \( U \) is called totally isotropic if \( q_f \) and \( h_f \) vanish on \( U \); the collection of all \( k \)-dimensional totally isotropic subspaces is denoted \( \text{Gr}_k^f(V) \), and the Witt index \( \text{ind}[f] \) is defined in the obvious way. Note that

\[
\text{Gr}_k^f(V) \subseteq \text{Gr}_k^{h_f}(V).
\]

Suppose that \( V, V' \) are vector spaces over \( D \), and that

\[
F : V \longrightarrow V'
\]

is linear. We define a map

\[
\Lambda\text{-Quad}_{\sigma,\varepsilon}(V) \longrightarrow \Lambda\text{-Quad}_{\sigma,\varepsilon}(V'),
\]

by \([f'] \longrightarrow [F^*(f')]\) which we denote also by \( F^* \). A similar construction works if \( V \longrightarrow V' \) is \( \theta \)-semilinear relative to \( D \sim\longrightarrow D' \). The group

\[
U([f]) = \text{GL}(V)_{[f]}
\]

is the group of all isometries of \((V,[f])\). Let \( g \in \Gamma L(V) \). If there exists an element \( s \in D \) such that \( g^*[f] = [sf] \), then \( g \) is called a semi-similitude; if \( g \) is linear, then it is called a similitude. The corresponding groups are denoted

\[
U([f]) \leq \text{GU}([f]) \leq \Gamma U([f]).
\]

Not every automorphism \( \theta \) of \( D \) can appear in \( \Gamma U([f]) \); a necessary and sufficient condition is that

\[
[\theta,\sigma] \in \text{Int}(D);
\]

up to inner automorphisms, \( \theta \) has to centralize \( \sigma \).
2.4.1 The case when \( h_f \) is degenerate

There is one issue which we have to address. In characteristic 2, it is possible that \( V \perp h_f \neq 0 \), while \( q_f^{-1}(0) \cap V \perp h_f = 0 \). Let’s call such a pseudo-quadratic form slightly degenerate. This case can be reduced to the non-degenerate case as follows. Let \( V' = V/V \perp h_f \), and let

\[
\Lambda' = \{ c \in D | c + \Lambda \in q_f(V \perp h_f) \}.
\]

It can be checked that \( \Lambda' \) is a form parameter. Define \( (\tilde{q}, \tilde{h}) \) on \( V' \) by

\[
\tilde{h}(u + V \perp h_f, v + V \perp h_f) = h(u, v) \quad \text{and} \quad \tilde{q}(v + V \perp h_f) = f(v, v) + \Lambda'.
\]

One can check that this pair is a pseudo-quadratic form \( [\tilde{f}] = (\tilde{q}, \tilde{h}) \); there is a canonical bijection

\[
G_{f_k}(V) \to G_{f_k}(V').
\]

Furthermore, there is a corresponding isomorphism \( \text{GL}(V)[f] \cong \text{GL}(V')[\tilde{f}] \).

Here is an example. Let \( D \) be a perfect field of characteristic 2, let \( V = D^5 \), and let \( f \) denote the bilinear form given by the matrix

\[
f \sim \begin{pmatrix}
0 & 1 & & & \\
0 & 0 & 1 & & \\
& & & & 0 \\
& & & & 1
\end{pmatrix}
\]

Thus \( q_f(x) = x_1x_2 + x_3x_4 + x_5^2 \). The associated bilinear form \( h_f \) is symplectic and degenerate; its matrix is

\[
h_f \sim \begin{pmatrix}
0 & 1 & & & \\
1 & 0 & & & \\
& & 0 & 1 & \\
& & 1 & 0 & \\
& & & & 0
\end{pmatrix}
\]

The process above gives us an isomorphism between an orthogonal generalized quadrangle and a symplectic generalized quadrangle, and a group isomorphism

\[
\text{O}(q_f, D) \cong \text{Sp}_4(D) \\
\text{O}(5, 2^k) \cong \text{Sp}(4, 2^k) \text{ for the finite case } D = \mathbb{F}_{2^k}.
\]

In Tits [32] and Bruhat-Tits [33], the chosen form parameter is always the minimal one, \( \Lambda = D_{\sigma, -\varepsilon} \). Therefore, Tits allows his forms to be slightly degenerate.
degenerate (in our terminology). The resulting theory is the same; the choice of a bigger $\Lambda$ makes the vector spaces smaller and avoids degenerate hermitian forms, which is certainly more elegant; the expense is that in this way we don’t really see groups like $O(5, 2^k)$, since they are identified with their isomorphic images belonging to non-degenerate forms, $O(5, 2^k) \cong Sp(4, 2^k)$ — one should keep that in mind.

2.5 Properties of form parameters

We discuss some properties of pseudo-quadratic forms and their form parameters.

In general, we have

$$\Lambda = D^{\sigma,-\varepsilon} \implies \left( q_f(v, v) = 0 \iff h_f(v, v) = 0 \right)$$

so the theory of trace hermitian forms is included in the pseudo-quadratic forms as a subcase. So the question is:

Why pseudo-quadratic forms?

The answer is given by the Fundamental Theorem of Polar Spaces 2.6.3. But first, we mention a few cases where pseudo-quadratic forms are not necessary.

As we mentioned above, this is the case if $\Lambda = D^{\sigma,-\varepsilon}$. Now if $\text{char}(D) \neq 2$, then $D^{\sigma,-\varepsilon} = D^{\sigma,-\varepsilon}$, so in characteristic different from 2, $(\sigma, \varepsilon)$-hermitian forms suffice.

2.5.1 Lemma If $\text{char}(D) \neq 2$, then there is a natural isomorphism

$$\text{TrHerm}_{\sigma, \varepsilon}(V) \cong \Lambda\text{-Quad}_{\sigma, \varepsilon}(V).$$

We consider some more special cases of form parameters. Note that

$$\Lambda = 0 \implies \left( \varepsilon = 1 \text{ and } \sigma = \text{id}_D \text{ and } D \text{ commutative} \right).$$

Suppose now that $D$ is commutative. If $\sigma = \text{id}_D$, then $D_{\text{id}_D,-\varepsilon} = D(1 - \varepsilon)$. So either $\Lambda = D$, or $\varepsilon = 1$. If $D \neq \Lambda \neq 0$, then we have necessarily $\text{char}(D) = 2$, and $\Lambda$ is a $D^2$-submodule of $D$, and $D$ is not perfect.

2.5.2 Lemma Suppose that $D$ is commutative and that $\sigma = \text{id}_D$. If $0 \neq \Lambda \neq D$, then $\text{char}(D) = 2$, the field $D$ is not perfect, and $\Lambda$ is a $D^2$-submodule of $D$. If $\Lambda = D$, then $\varepsilon = -1$. $\square$
Suppose now that $D$ is commutative and that $\sigma \neq \text{id}_D$. Then $\sigma^2 = \text{id}_D$, and $D_{\sigma,-\varepsilon} \neq 0$. Let $K \subseteq D$ denote the fixed field of $\sigma$. After scaling, we may assume that $1 \in D_{\sigma,-\varepsilon}$, which implies that $\varepsilon = -1$. Then $D^{\sigma,1} = K$, and if $\text{char}(D) \neq 2$, then $\Lambda = K$. For $\text{char}(D) = 2$ we put $\psi(c) = c + c^\sigma$; then we have an exact sequence

$$0 \longrightarrow K \longrightarrow D \psi \longrightarrow D_{\sigma,1} \longrightarrow 0$$

of finite dimensional vector spaces over $K$, so $D_{\sigma,1} = K$, regardless of the characteristic.

2.5.3 Lemma Suppose that $D$ is commutative and that $\sigma \neq \text{id}_D$. Then $\sigma^2 = \text{id}_D$, and the form parameters are the left translates $sK$ of the fixed field $K$ of $\sigma$, for $s \in D^\times$; in particular, $\Lambda = D^{\sigma,\varepsilon}$. \hfill $\Box$

2.5.4 Corollary Form parameters and pseudo-quadratic forms over perfect fields (in particular, over finite fields) are not important. \hfill $\Box$

This explains why form parameters and proper pseudo-quadratic forms ($\sigma \neq \text{id}_D$) are not an issue in finite geometry, and why they don’t appear in books on algebraic groups over algebraically closed fields. Note also that if $D$ is algebraically closed and if $\sigma \neq 1$, then the fixed field of $\sigma$ is a real closed field.

The next result is less obvious and was pointed out to me by Richard Weiss.

2.5.5 Proposition (Finite form parameters) Suppose that $\Lambda \neq 0$ is a finite form parameter. Then $D$ is finite.

Proof. If $\sigma = \text{id}_D$, then $D$ is commutative. If $\text{char}(D) = 2$, then $\Lambda \subseteq D$ is a $D^2$-module. If $\text{char}(D) \neq 2$, then $\Lambda = D$; in any case, $\Lambda$ has a subset of the same cardinality as $D$.

If $\sigma \neq \text{id}_D$, then it suffices to consider the minimal case where $\Lambda = D_{\sigma,-\varepsilon}$. We rescale in such a way that $1 \in D_{\sigma,-\varepsilon}$; then $\sigma$ is an involution and $\varepsilon = -1$. Put $\psi(x) = x + x^\sigma$ and let $\lambda = \psi(x) \in \Lambda$. We claim that $\lambda^k \in \Lambda$, for all $k \geq 1$. Indeed, $(x + x^\sigma)^k$ can be written as a sum of $2^k$ monomials of the form

$$x^{(\nu_1)} x^{(\nu_2)} \ldots x^{(\nu_k)},$$

where $\nu_i \in \{0, 1\}$. Now

$$x^{(\nu_1)} x^{(\nu_2)} \ldots x^{(\nu_k)} + x^{(\nu_1 + 1)} x^{(\nu_2 + 1)} \ldots x^{(\nu_k + 1)} = \psi(x^{(\nu_1)} x^{(\nu_2)} \ldots x^{(\nu_k)}).$$

If $\Lambda$ is finite, then all elements $\lambda \in \Lambda \setminus \{0\}$ have thus finite multiplicative order. This implies that $D$ is commutative, see Herstein [16] Cor. 2 p. 116. From Lemma 2.5.3 above, we see that $\text{card}(\Lambda) + \text{card}(\Lambda) = \text{card}(D)$, so $D$ is finite. \hfill $\Box$
(It is a consequence of this proposition that there exist no semi-finite spherical irreducible Moufang buildings: if one panel of such a building is finite, then every panel is finite. This is a problem about Moufang polygons, and, by the classification due to Tits and Weiss (as described by Van Maldeghem in these proceedings [35], the only difficult case is presented by the classical Moufang quadrangles associated to hyperbolic spaces of rank 2; there, the line pencils are parametrized by the set \( \Lambda \)).

This section shows that pseudo-quadratic forms are important only if either \( D \) is a finite field of characteristic 2 and if \( \sigma = \text{id}_D \) (and then we are dealing with quadratic forms), or if \( D \) is a non-commutative skew field of characteristic 2. On the other hand, none of the results about classical groups becomes really simpler if pseudo-quadratic forms are excluded, so we stick with them.

### 2.6 Polar spaces and pseudo-quadratic forms

*We state the classification of polar spaces.*

Suppose that \([f]\) is a non-degenerate pseudo-quadratic form of (finite) index \( \text{ind}[f] = m \geq 2 \). Let \( \text{PG}^{[f]}(V) = (\text{Gr}_1^{[f]}(V), \ldots, \text{Gr}_m^{[f]}(V), *) \) and

\[
\text{PG}^{[f]}(V)_{1,2} = (\text{Gr}_1^{[f]}(V), \text{Gr}_2^{[f]}(V), *).
\]

Then \( \text{PG}^{[f]}(V)_{1,2} \) is a (possibly weak) polar space and a subgeometry of the (possibly weak) polar space \( \text{PG}^{h_{[f]}}(V)_{1,2} \). A polar space isomorphic to such a space is called *embeddable*. Here is the first analogue of the Fundamental Theorem of Projective Geometry.

#### 2.6.1 Theorem (Fundamental Theorem of Polar Spaces, I)

*Let*

\[
\text{PG}^{[f]}(V)_{1,2} \xrightarrow{\Phi} \text{PG}^{[f']}(V')_{1,2}
\]

*be an isomorphism of embeddable (weak) polar spaces of finite ranks \( m, m' \geq 3 \). Then \( m = m' \), and there exists an isomorphism of skew fields \( D \cong D' \) and a \( \theta \)-semilinear isomorphism \( \Phi^*([f']) \) such that the pull-back \( \Phi^*([f']) \) is proportional to \([f]\).*

For a proof see Tits [32] Ch. 8, or Hahn & O’Meara 8.1.5.

This result and the next one are partly due to Veldkamp; the full results were proved by Tits. Cohen [7] and Scharlau [28] are good references for the classification, and for newer results in this area. The situation is more complicated for embeddable polar spaces of rank 2; we refer to Tits [32].
The theorem above deals with embeddable polar spaces. In fact, all polar spaces of higher rank are embeddable.

2.6.2 Theorem (Fundamental Theorem of Polar spaces, II)
Suppose that \((\mathcal{P}, \mathcal{L}, \ast)\) is a (weak) polar space of rank \(m \geq 4\). Then \((\mathcal{P}, \mathcal{L}, \ast)\) is embeddable.

For a proof see Tits [32] 8.21, combined with Thm. 2.2.2. See also Scharlau [28] Sec. 7.

This result is not true for polar spaces of rank 3; there exist polar spaces which have Moufang planes over alternative fields as subspaces, and such a polar space cannot be embeddable. However, this is essentially the only thing which can go wrong.

2.6.3 Theorem (Fundamental Theorem of Polar spaces of rank 3)
Let \((\mathcal{P}, \mathcal{L}, \ast)\) be a polar space of rank 3. If \((\mathcal{P}, \mathcal{L}, \ast)\) is not embeddable, then either there exists a proper alternative field \(A\), and the maximal subspace are projective Moufang planes over \(A\), or \((\mathcal{P}, \mathcal{L}, \ast) \cong A_{3,2}(D)\), for some skew field \(D\).

For a proof see Tits [32] 7.13, p. 176, and 9.1, combined with Thm. 2.2.2. See also Scharlau [28] Sec. 7.

The polar spaces containing proper Moufang planes are related to exceptional algebraic groups of type \(E_7\); these are the only polar spaces of higher rank which do not come from classical groups. If \(D\) is commutative, then \(A_{3,2}(D)\) is related to the Klein correspondence, \(D_3 = A_3\).

Finally, we should mention the following result which is a consequence of Tits’ classification.

2.6.4 Proposition (Tits) Let \((\mathcal{P}, \mathcal{L}, \ast)\) be a weak polar space of rank \(m \geq 3\), such that every subspace of rank \(m - 2\) is incident with precisely two subspaces of rank \(m - 1\). Then either \((\mathcal{P}, \mathcal{L}, \ast) \cong A_{3,2}(D)\), or \((\mathcal{P}, \mathcal{L}, \ast) \cong \text{PG}^h(V)_{1,2}\), where \(V\) is a \(2m\)-dimensional vector space over a field \(D\), and \(h\) is a non-degenerate symmetric bilinear form (i.e. \(\sigma = \text{id}_D\) and \(\varepsilon = 1\)) of index \(m\) (in other words, \(V\) is a hyperbolic module of orthogonal type, see the next section).

Thick polar spaces of rank 2 are the same as generalized quadrangles. Similarly as projective planes, these geometries can be rather ‘wild’ and there is no way to classify them. The classification of the Moufang quadrangles due to Tits and Weiss is a major milestone in incidence geometry. For results about generalized quadrangles we refer to Van Maldeghem’s article [35] in these proceedings. Polar spaces of possibly infinite rank where considered by Johnson [18].
2.7 Polar frames and hyperbolic modules

We show how the classification of pseudo-quadratic forms is reduced to the anisotropic case.

Let $V$ be an $m$-dimensional vector space over $D$, and put

$$H = V \oplus V^\sigma.$$  

We define a form $f \in \text{Form}_\sigma(H)$ by

$$f(((u, \xi), (v, \eta)) = \xi(v).$$

As a matrix, $f$ is represented as

$$f \sim \begin{pmatrix} 0 & 1_m \\ 0 & 0 \end{pmatrix}$$

where $1_m$ denotes the $m \times m$ unit matrix. The space $H$ with the pseudo-quadratic form $[f]$ (relative to a form parameter $(\Lambda, \sigma, \varepsilon)$) is called a hyperbolic module of rank $m$; if $m = 1$ then $H$ is 2-dimensional and we call it a hyperbolic line. (Hyperbolic lines are often called hyperbolic planes; this depends on the viewpoint, linear algebra vs. projective geometry.) The following theorem is crucial.

2.7.1 Theorem Let $[f]$ be a non-degenerate pseudo-quadratic form of finite Witt index $\text{ind}([f]) = m$ in a vector space $V$. Then there exists a hyperbolic module $H$ of rank $m$ in $V$, and $V$ splits as an orthogonal sum

$$V = H \oplus V_0.$$  

where $V_0 = H^{\perp_{[f]}}$. If $H' \subseteq V$ is another hyperbolic module of rank $m$, then there exists an isometry of $V$ which maps $H'$ onto $H$.

This follows from Witt’s Theorem, see Hahn & O’Meara 6.1.12, 6.2.12 and 6.2.13 — the proofs apply despite the fact that Hahn & O’Meara work always with finite dimensional vector spaces. What is needed in their proof is only that $H$ has finite dimension.

Thus, the subspace $V_0$ is unique up to isometry. This subspace (together with the restriction of $[f]$) is called the anisotropic kernel of $[f]$. Since the hyperbolic module has a relatively simple structure, the study of pseudo-quadratic forms is reduced to the anisotropic case; a pseudo-quadratic form is determined its Witt index and its anisotropic kernel (and by $\sigma, \varepsilon, \Lambda$, of course).
The building associated to a polar space \((P, L, \ast)\) is constructed as follows. If \((P, L, \ast)\) is thick, then the vertices are the subspaces of the polar space, and the simplices are sets of pairwise incident vertices. The resulting building has rank \(m\) and type \(C_m\), see Tits [32] Ch. 7. If the polar space is weak, then a new geometry is introduced: the vertices are all subspaces of rank different from \(m - 2\), and two vertices are called incident if one contains the other, or if their intersection has rank \(m - 2\). This is again an \(m\)-sorted structure (there are two classes of subspaces of rank \(m - 1\)), and the resulting simplicial complex is a building of type \(D_m\), see Tits [32] Ch. 6 and 7, 12, 8.10. It is easy (but maybe instructive) to check that this makes the weak polar space \(A_{3,2} (D)\) into the building \(\Delta (D^4)\) obtained from the projective space \(PG(D^4)\). The buildings related to polar spaces are also discussed in Brown [4], Cohen [7], Garrett [12], Ronan [26], Scharlau [28], and Taylor [31].

Finally, we mention the classical groups obtained from non-degenerate pseudo-quadratic forms of index \(m \geq 2\). Scaling the form by a suitable constant \(s \in D^\times\), the following cases appear.

**Symplectic groups**

This is the situation when \((\sigma, \varepsilon, \Lambda) = (\text{id}_D, -1, D)\). Here \(D\) is commutative, \(q_f = 0\) and \(h_f\) is alternating. The dimension of \(V\) is even (and \(V\) is hyperbolic), and \(2 \text{ind}(h_f) = \dim(V)\). The corresponding polar space is thick.

**Orthogonal groups**

This is the situation when \((\sigma, \varepsilon, \Lambda) = (\text{id}_D, 1, 0)\). Here \(D\) is commutative and \(h_f\) is symmetric, and \(2 \text{ind}[f] \leq \dim(V)\). The corresponding polar space is thick if and only if \(2 \text{ind}[f] < \dim(V)\). If \(2 \text{ind}[f] = \dim(V)\), then the corresponding building is the \(D_m\)-building (the oriflamme geometry) described above, and \(V\) is hyperbolic.

**Defective orthogonal groups**

This is the situation when \((\sigma, \varepsilon) = (\text{id}_D, 1)\) and \(0 \neq \Lambda \neq D\). Here \(D\) is commutative and \(h_f\) is symmetric. This occurs only in characteristic 2 over non-perfect fields (in the perfect case, \(\Lambda = D\) and we are in the symplectic case).

**Classical unitary groups**

This is the situation when \(\sigma \neq \text{id}_D = \sigma^2, \varepsilon = 1\) and \(\Lambda = D^{\sigma,-1}\). Here \(D\) need not be commutative and \(h_f\) is \((\sigma, 1)\)-hermitian. Since \(\Lambda\) is maximal, the hermitian form \(h_f\) describes \([f]\) completely and \(q_f\) is not important. The corresponding polar space is thick.
Restricted unitary groups

This is the situation when $\sigma \neq \text{id}_D = \sigma^2$, $\varepsilon = 1$ and $\Lambda < D^{\sigma,-1}$. Here $D$ is of characteristic 2 and not commutative. The corresponding polar space is thick.

2.8 Omissions

By now it should be clear that the theory of pseudo-quadratic forms and the related geometries is rich, interesting, and sometimes difficult. There are many other interesting topics which we just mention without further discussion.

Root elations

Root elations in polar spaces are more complicated than root elations in projective spaces. This is due to the fact that there are two types of half-apartments and, consequently, two types of root groups. One kind is isomorphic to the additive group of $D$, while the other is related to the anisotropic kernel $V_0$ of $V$, and to the form parameter $\Lambda$. These root groups are nilpotent of class 1 or 2. We refer to Van Maldeghem [34] for a detailed description of the root groups. The root elations are Eichler transformations (also called Siegel transformations), which are special products of transvections, and the group generated by these maps is the elementary unitary group $EU([f])$.

K-theory

Starting with the abstract commutator relations for the root groups of a given apartment, one can construct unitary version of the Steinberg groups, and unitary K-groups. Hahn & O’Meara [15] give a comprehensive introduction to the subject (for hyperbolic $V$). There is a natural map $K_0(D) \longrightarrow \text{KU}_0(D)$ whose cokernel is the Witt group of $D$, another important invariant.

Permutation groups

If $\text{ind}[f] \geq 2$, then the action of the unitary group on $\text{Gr}^{[f]}_1(V)$ is not 2-transitive. Instead, one obtains interesting examples of permutation groups of rank 3 (i.e. with 3 orbits in $\text{Gr}^{[f]}_1(V) \times \text{Gr}^{[f]}_1(V)$).

Moufang sets

If $\text{ind}[f] = 1$, then the corresponding unitary group is 2-transitive on $\text{Gr}^{[f]}_1(V)$, and there is a natural Moufang set structure.
Isomorphisms

As in the linear case, one can ask whether two unitary groups can be (abstractly or as permutation groups) isomorphic. Indeed, there are several interesting isomorphisms related to the Klein correspondence and to Cayley algebras. Many results in this direction can be found in Hahn & O’Meara [15].

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