Repeated measurements and nuclear spin polarization

Lian-Ao Wu\textsuperscript{1,2}

\textsuperscript{1}Department of Theoretical Physics and History of Science, The Basque Country University (EHU/UPV), PO Box 644, 48080 Bilbao, Spain
\textsuperscript{2}IKERBASQUE, Basque Foundation for Science, 48011 Bilbao, Spain

We study repeated (noncontinuous) measurements on the electron spin in a quantum dot and find that the measurement technique may lead to a different method or mechanism to realize nuclear spin polarization. While it may be used in any case, the method is aimed at the further polarization, providing that nuclear spins have been polarized by the existent electrical or optical methods. The feasibility of the method is analyzed. The existing techniques in electron spin measurements are applicable to this scheme. The repeated measurements deform the structures of the nuclear wave function and can also serve as gates to manipulate nuclear spins.

\textbf{Introduction.}—Quantum dots can host electron spin qubits in a quantum information processor\textsuperscript{[1-4]}, as evidenced by the recent encouraging progress in spin detection, relaxation and coherent manipulation\textsuperscript{[5,7]}. However, random fluctuations in the nuclear spin ensemble of a host quantum dot lead to fast electron spin decoherence\textsuperscript{[8,9]} via the hyperfine coupling. Methods to combat the nuclear spin randomness, nuclear spin polarization, have been proposed (see \textsuperscript{[10,11]} and references therein) and implemented experimentally.

Nuclear spin polarization dates back to 1980s\textsuperscript{[12]}. It has been realized experimentally to some extent. The polarization is achieved either electrically or optically via the hyperfine coupling in particular flip-flop spin exchange\textsuperscript{[10]}. Gammon \textit{et al.} obtain 60\% nuclear spin polarization optically in interface fluctuation GaAs quantum dots\textsuperscript{[13]} (see also\textsuperscript{[14]} and references therein). The recent record is the 80\% polarization in In\textsubscript{0.5}Ga\textsubscript{0.5}As at 5T\textsuperscript{[15]}.

Although significant progress has been made, the full nuclear spin polarization remains far from reach. Nuclear spins distant from the electron spin are subject to rather weak hyperfine coupling\textsuperscript{[14]}. Further electrical and optical polarization becomes more difficult, even impossible. However, proposals to use quantum dots for quantum information processing are based on the fully polarized state\textsuperscript{[11,17]}. Partial polarizations do not yet lead to a significant application in quantum information processing. Therefore, new polarization methods are desired.

Existential polarization methods, both electrical and optical, are dynamic. Here, we explore an alternative possibility of nuclear spin polarization via frequent but noncontinuous measurements. We start with an achieved 80\% polarized equilibrium state. If we measure the electron spin repeatedly but noncontinuously (different from the Zeno effect\textsuperscript{[18]} ) as formulated in\textsuperscript{[19]}, the system may end up with the fully polarized nuclear spin state. While the hyperfine coupling is required in this scheme, the relative stronger nuclear spin correlation plays a positive role. This feature may favor the polarization of those remote nuclear spins subject to weaker hyperfine coupling.

\textbf{Method.}—We consider a single electron confined in a charged quantum dot. The Hamiltonian for the electron spin and its surrounding $K = 10^3 - 10^5$ nuclei spins (spin-$I$) is

$$H = g^* \mu_B B S_z + g_n \mu_n B I_z + H_1 + H_{nuc},$$

(1)

where the first two terms are the Zeeman energies of the electron spin and nuclear spins in a $z$ direction magnetic field $B$. $I_z = \sum_i I^z_i$ is the $z$ component of the total nuclear spin operator. The total angular momentum operator in the $z$ direction, $J_z = S_z + I_z$ with eigenvalue value $J$, is conserved. The hyperfine coupling between nuclear spins and the electron spin is written as

$$H_I = A (A_z S_z + \frac{1}{2} A_\perp S_\perp + \frac{1}{2} A_- S_+),$$

(2)

where $A/\sqrt{2T}$ is an average hyperfine coupling constant. Operators $A_\mu = \sum_i \alpha_i I_\mu^i/\sqrt{2T}$ are expressed by the nuclear spin $I_\mu^i$ ($\mu = z, \perp, -$), where the real numbers $\alpha_i$’s satisfy $\sum_{i=1}^N \alpha_i^2 = 1$\textsuperscript{[11]}. The term $A_z S_z$ provides an effective magnetic field\textsuperscript{[17]}. Overhauser shift for the electron spin, $B_{eff} = B - (g_n \mu_n B + A \sum_i \alpha_i I^z_i)/g^* \mu_B$. The flip-flop spin exchange $\frac{1}{2} A_\perp S_\perp + \frac{1}{2} A_- S_+$ has been used in either electrical or optical nuclear spin polarizations\textsuperscript{[10,14]}. The dominate part of nuclear spin interaction $H_{nuc}$ is the dipole-dipole coupling. However, here we consider the most general nuclear spin interaction $H_{nuc}$\textsuperscript{[14]} preserving the total nuclear spin $I_z$. This general Hamiltonian can be represented by a direct sum of submatrices $H_J$, in the ordered bases \{|$I$\} from $J = -J_m$ to $J_m$

$$H = \bigoplus_{J = -J_m}^{J_m} H_J,$$

where $J_m = KI + 1/2$. Given numbers $J$ and $I$, there are $\Omega(I,N)$ states in the electron and nuclear spin Hilbert space, where $N = (J_m - J)/I$. For instance, $\Omega(1/2, N) = (K + 1)!/((K+1-N)!N!)$ when $I = 1/2$. The representation $H_{nuc}$ is one-dimensional, whose basis $|J_m\rangle = |\Uparrow\rangle |0\rangle$ (|0\rangle = |I, I, ..., I\rangle ) is the fully polarized electron and nuclear spin state. Here $|\Uparrow\rangle$ denotes the electron spin-up state and $|0\rangle = |I, I, ..., I\rangle$ is the fully polarized state of nuclear spins.
Consider the electron spin and its surrounding nuclear spins initially in a separable state \( \rho(0) = |\uparrow\rangle \otimes \rho_n \), where \( \rho_n \) is an initial nuclear spin state. This initial state can be obtained by measuring the electron at \( t = 0 \) with output \( |\uparrow\rangle \) - the spin-up state. The system evolves under the Hamiltonian \((1)\) thereafter. We then measure the electron spin again at time \( t \). If we find again that the electron is in spin-up state, which we call a successful measurement, the nuclear spin state becomes \( \rho_n(t) = V(\tau)\rho_n V^\dagger(\tau)/P_M \), where \( V(\tau) = \langle \uparrow | \exp(-iH\tau) | \uparrow \rangle \). After \( M \) such noncontinuous measurements on electron spin state, given that all outcomes were \( |\uparrow\rangle \), the nuclear spin state \((2)\) becomes

\[
\rho_n(M\tau) = V(\tau)^M\rho_n V^\dagger(\tau)^M/P_M,
\]

where \( P_M = \text{Tr}[V(\tau)^M\rho_n V^\dagger(\tau)^M] \). The matrix representation of the operator \( V(\tau) \) is

\[
V(\tau) = \begin{bmatrix}
e^{-iE_{\text{Jen}}\tau} & 0 \\
0 & V_r(\tau)
\end{bmatrix}
\]

in the ordered basis \( \{|I_z\rangle\} \). \( V(\tau) \) behaves like a non-Unitary gate in the evolution. These measurements preserve the total nuclear spins, \( I_z = J - 1/2 \). We emphasize that measurements are noncontinuous, for the interval is finite and could be long (different from the Zeno effect in quantum dot \((18)\)). The submatrix of \( \tau \) for the fully polarized nuclear state is also one-dimensional, where \( E_{\text{Jen}} \) is the eigenenergy of \( H \). Matrices \( V_{I_z}(\tau)\)'s are in general neither Hermitian nor Unitary. However, one can still find left- and right-eigenvectors with complex eigenvalues, whose modulus can be shown to be bounded between 0 and 1 \((20)\). When \( \tau \) goes to infinite, \( \rho_n(M\tau) \rightarrow |0\rangle \langle 0| \), the nuclear spins are fully polarized, if and only if the moduli of all eigenvalues \( \nu_{I_z}(\tau) \) of \( V_{I_z}(\tau) \) are smaller than one, i.e., \( |\nu_{I_z}(\tau)| < 1 \) for all \( I_z < KI \). However, if there are other eigenstates with moduli equal to one in the submatrix \( V_{I_z}(\tau) \), \( \rho_n(M\tau) \) will end up with a degenerate mixed state of the fully polarized state and these eigenstates. The feasibility of this polarization method relies on whether or not there is degeneracy of \( V(\tau) \) for the general Hamiltonian \((1)\). Since a complete solution for the general Hamiltonian is impossible, the following sections will discuss several limits, numerically and analytically, to verify that there is no priori reason for the general Hamiltonian to have the degeneracy. Before proceeding, we first introduce the initial nuclear spin state \( \rho_n \), whose nature also plays a role in this polarization method.

**Initial state.**—Consider that the nuclear spins have been polarized initially to a polarization rate \( a \) (e.g., equal to 80%), either electrically or optically. An initial state can be a product of individual spin states \( \rho_n = \prod_{i=1}^{K} |I_{z_i}\rangle\langle I_{z_i}| + (1-a)\eta_i \), meaning that each nuclear spin is in its evenly polarized equilibrium state. Here \( \eta_i \) is a normalized state orthogonal to \( |I_{z_i}\rangle \). For simplicity, we set \( K \) even and \( I = 1/2 \) such that \( \eta_i = |-1/2\rangle_i \langle -1/2| \). We expand this initial state as a direct sum,

\[
\rho_n = \sum_{I_z=-KI}^{KI} c(I_z,a) \rho_n^{I_z},
\]

where \( c(I_z,a) = d_{I_z} \delta^{K/2+I_z}(1-a)^{K/2-I_z} \) and \( d_{I_z} = (K/2-I_z)!/(K/2)! \). Here the density matrix \( \rho_n^{I_z} \) is a \( (1/d_{I_z}) \) diagonal and normalized matrix in the basis \( \{|I_z\rangle\} \). In the maximally mixed state \((a = 1/2 \text{ or } 50\%)\), the density matrix \( \rho_n^{I_z} \) with \( I_z = 0 \) is dominant. However, it will be different for a polarized state.

The ratio between coefficients of \( I_z = 0 \) states and fully polarized state \( |0\rangle \langle 0| \) is \( R = (K^{(1-1/a)K/2})/2 \), for instance \( K = 10^5 \), \( R = 2.5 \times 10^{-3} \) when \( a = 80\% \). The fully polarized state \( |0\rangle \langle 0| \) is dominant. The expectation value of \( I_z \) characterizing the polarization degree is

\[
\langle I_z \rangle_M = \sum_{I_z=-KI}^{KI} c(I_z,a) d_{I_z}/d_{KI} \text{Tr}[V_{I_z}(\tau)^M V_{I_z}(\tau)^M],
\]

where the trace runs over all subspaces characterized by \( I_z \). Initially, the expectation value \( \langle I_z \rangle_0 = K(a-1/2) \) and a nuclear spin polarization process means that \( \langle I_z \rangle_M \rightarrow K/2 \) when \( M \) increases.

There is another possible initial state, where the eighty percent of nuclear spins is polarized. The other 20% of spins is distant from the electron and is in thermal equilibrium state \((a = 1/2)\). It is an uneven polarized state. The expectation value of \( I_z \) for this initial state is

\[
\langle I_z \rangle_M = a K I + \langle I_z \rangle_{M}^{'},
\]

where \( \langle I_z \rangle_{M}^{'} \) has the same form as \((5)\) but \( I_z \) in the sum run from \(-(1-a)KI \) to \((1-a)KI \).

**The flip-flop spin exchange.**—Eigenstates \(|m\rangle \) of a Hermitian operator \( h_0 = A_+ A_- \) play crucial roles for the dressed qubit supported by \(|0\rangle_d = |\uparrow\rangle \langle m| \) and \(|1\rangle_d = |\downarrow\rangle \langle \Phi_{m+1}| \). \((11)\). These two states span an invariant space of the flip-flop spin exchange. Note that here we define the state \(|0\rangle = |I, I, \ldots, I\rangle \) as our vacuum state instead of \(|-I, -I, \ldots, -I\rangle \) in ref. \((11)\). Ref. \((11)\) also uses K operators, \( A_k = \sum_{j} \alpha_j^k T_j/\sqrt{27} \), and identifies the collective mode \( k = 0 \) such that \( A_+ = A_{-0} \) and \( \alpha_1 = \alpha_0 \). The set \( \{\alpha_k^k\} \) can be made as a unitary matrix \((11)\) \((22)\) to transform sites \( i \) to modes \( k \). These operators obey the commutation relations

\[
[A_{k+}, A_{k'-}] = \delta_{kk'} - \sum_i \alpha_i^k \alpha_i^{k'} (I - I_z^2)/I.
\]

An eigenstate \(|m\rangle \) of \( h_0 \) can be expressed by a polynomial of product states of the operators \( A_{k'} \)'s,

\[
|m\rangle = \sum_{f} (A_-, A_{1-}, \ldots, A_{K-1-}) |0\rangle.
\]

When \( f \) do not include the collective operator \( A_- \), the states
$|m\rangle = f_{m\sigma}(A_1, \ldots, A_{K-1}) |0\rangle$ are always eigenstates of $h_0$, and $A_\perp |m\rangle = 0$. For instance, in the $I_z = KI - 1$ subspace, the corresponding eigenstates are $|m\rangle = |0\rangle$ and $|m\rangle = |1\rangle = A_{\perp} |0\rangle$.

Now we come to illustrate natures of the operator $V(\tau)$. When $B_{\text{eff}} = 0$ and the nuclear interaction $H_{\text{nuc}}$ is negligible, by calculating $\langle \uparrow | H^n | \uparrow \rangle$ and $\langle \uparrow | e^{-iH\tau} | \uparrow \rangle$ we show that

$$V(\tau) = \cos(\frac{A\tau\sqrt{h}}{4}),$$

where $h = A_\perp A_\perp$. While it is easy to check that $h |0\rangle = 0$, the abovementioned states $|m\rangle$ are also eigenstates with zero eigenvalues, i.e., $h |m\rangle = 0$. In the $I_z = KI - 1$ subspace, $h |1\rangle = 0$. After $M$ measurements with output $\uparrow$, both $|0\rangle$ and the states $|m\rangle$ are project out. This degeneracy makes the full polarization invalid, though the outcoming nuclear state may still be interesting.

If we consider a finite effective magnetic field $B_{\text{eff}}$, a tedious calculation shows that

$$V(\tau) = \cos(\frac{A\tau\sqrt{h}}{4}) e^{-i\frac{\pi T_0^2}{T_{B}^{2}}/4}. \tag{9}$$

The magnetic field does not break the degeneracy either. Therefore, in our method the flip-flop spin exchange cannot fully polarize the nuclear spins, though it is crucial in the electrical and optical polarization methods. The reason is that the flip-flop spin exchange only contains the collective mode $k = 0$.

**General analysis.**—The general Hamiltonian (1) includes all modes $k = 1, \ldots, K - 1$. We can also write it in a way different from eq. (1)

$$H = \frac{A}{2} (A_+ S_- + A_- S_+) + \mathcal{H}, \tag{10}$$

where $\mathcal{H} = (g_{\mu} n_0 B - g_{\mu} \mu_B B) I_z + AA_z (J - I_z) + H_{\text{nuc}}$ acts only on nuclear spins. $J_z$ is conserved and has been replaced by a constant $J$. There does not exist an analytical form of $V(\tau)$ for a general $\mathcal{H}$. However, we may find out its natures in terms of approximation methods, such as perturbation expansions. If the measurement time interval $\tau$ is short, we can expand $\langle \uparrow | e^{-iH\tau} | \uparrow \rangle$ in the power of $\tau$. It is tedious but straightforward to calculate,

$$\langle \uparrow | H^{2m} | \uparrow \rangle = \sum_{k=0}^{m} T(\mathcal{H}^{2k}), \tag{11}$$

where $T(\mathcal{H}^{2k}) = \sum \mathcal{P}(\mathcal{H}^{2k}(A_+ A_\perp))$ and the operation $\mathcal{P}$ is a permutation, for instance, $\mathcal{P}(\mathcal{H}^2 A_+ A_\perp) = A^2 H A_\perp A_\perp$. The sum runs over all possible permutations. Similarly, we also have an expansion for $\langle \uparrow | H^{2m+1} | \uparrow \rangle$, e.g.,

$$\langle \uparrow | H^3 | \uparrow \rangle = \mathcal{H} A^2 A_\perp A_\perp + \mathcal{H} A^2 A_\perp A_\perp + \mathcal{H}^3.$$

The nuclear Hamiltonian $\mathcal{H}$ breaks the degeneracy of $V(\tau)$. For the states $|m\rangle = |1\rangle$, we can see directly that the fourth order term $\mathcal{H} A_\perp A_\perp$ breaks the degeneracy. $\mathcal{H}$ rotates $A_k \rightarrow -A_k$ to a superposition of other operator $A_{k'}$, including the collective operator $A_\perp$ such that $\mathcal{H} A_\perp A_\perp \mathcal{H}$ has an finite expectation value. By using the common eigenstates $|m\rangle$ of $h_0$ and $I_z$, we expect that $\langle m| \mathcal{H} A_\perp A_\perp \mathcal{H} |m\rangle$ is finite for a general $\mathcal{H}$.

**Analysis via Bosonization.**—Consider bosons $B^\dagger_k = \sum_{i} \alpha^+_k \alpha^+_i$, where the set $\{\alpha^+_i\}$ is the same as that for nuclear spins. We denote $B_k = B^0_k$ for the collective bosonic mode. $B_k$ and $B^\dagger_k$ obey the bosonic commutation relations

$$[B_k, B^\dagger_k] = \sum_{i} \alpha^+_i \alpha^+_i \delta_{kk'} \tag{12}.$$
round. The very long coherence time of nuclear spins [17] has implied that nuclear spin correlation may not disturb the process. We can repeat the same steps again and again until the nuclear spins are fully polarized. It is important that each successful step improves the polarization degree of nuclear spins. We can also prepare thousands of copies and repeatedly measure them individually.

Starting from the uneven polarized state, we also calculate \( \langle I_z \rangle_M \) in (6) and find that when \( M = 700 \), it is already fully polarized. The value of \( M \) depends on the assumption of initial states significantly. The discussions for the even polarized initial state are also applicable for the uneven state.

It is not practical to attempt to treat the general Hamiltonian exactly (even numerically) since the configurations for monotonically with \( \tau \). We take parameters \( A \alpha_1 \tau = 8, A \alpha_2 \tau = 4 \) and \( b_{12} \tau = 0.2 \) in the general Hamiltonian, where \( b_{12} \) is the coupling constant of nuclear dipole-dipole interaction. Nuclear spins are fully polarized for both cases after 50 successful measurements.

**Measurements.**—Measurement of single electron spin has been achieved using optical [23] and electrical [5] techniques. The electrical one [5] applies a magnetic field to split the spin-up and spin-down states of the electron via the Zeeman energy. The quantum dot potential is then tuned. If its spin is up, the electron will stay in the dot, otherwise it will leave. The spin state is correlated with the charge state, and the charge on the dot is measured to indicate the original spin. This technique matches the present polarization method. We could exactly follow and repeat the stages of this technique. We first empty the dot and then inject one electron with unknown spin. At time \( t = 0 \), we measure the spin states. If find spin-up, we start processing. After waiting for time \( \tau \), we measure again and if the electron is spin-up, we continue processing. If find spin-down, we inject an electron again and go back to time \( t = 0 \). By repeating this, we will end up with the fully polarized state if we have \( M \) successful measurements. Our analysis has shown that we needs several hundred successful measurements to accomplish the full polarization. However, it is encouraging that each measurement with output \( \uparrow \downarrow \) will increase polarization degrees, as illustrated in Fig. 1 and discussed above. The full polarization may be reached by several uncorrelated measurement procedures in the long coherence period of nuclear spins [17].

**Conclusion.**—Our method is aimed at further polarization, on the basis of the partially polarized states made by the electrical or optical methods. Theoretically, the feasibility of the method relies on eigenvalues of \( V(\tau) \). While the modulus of the eigenvalue of the fully polarized state is always one, we require the moduli of other eigenvalues are smaller than one, i.e., breaking degeneracy of \( V(\tau) \). We analyze the system in several ways. It is interesting to note that while the interaction between nuclei can limit the polarization process in other polarization schemes [24], it plays a positive role on breaking the degeneracy of \( V(\tau) \). Our conclusion is that there is no priori reason that there are other eigenstates with eigenvalues one for the general Hamiltonian. It is noticeable that the existed spin measurement technique fits well with our method. This work also hints that while we explore the feasibility of the polarization method, our study should surely promote the development in applying the noncontinuous measurement scheme to the interesting nuclear spin system, as a new tool.

The author thanks Drs. E. Sherman, X. Hu and W. Yao for helpful discussions. This work was supported by the Ikerbasque Foundation and the Spanish MEC (Project No. FIS2009-12773-C02-02).

---

[1] D. Loss and D. DiVincenzo, Phys. Rev. A 57, 120 (1998).
[2] G. Burkard, H.-A. Engel, and D. Loss, Fortschr. Phys. 48, 965 (2000).
[3] B. Kane, Nature 393, 133 (1998).
[4] L. -A. Wu, D. A. Lidar, and M. Friesen, Phys. Rev. Lett. 93, 030501 (2004).
[5] J. M. Elzerman et al. Nature (Londoan) 430, 431 (2004).
[6] E. Ya. Shermand and D. J. Lockwood, Phys. Rev. B 72, 125340 (2005). D. V. Khomitsky and E. Ya. Sherman, Euro Phys. Lett. 90, 27010 (2010).
[7] J. R. Petta et al., Science 309, 2180 (2005).
[8] A. V. Khaetskii, D. Loss, and L. Glazman, Phys. Rev. Lett. 88, 186802 (2002).
[9] I. A. Merkulov, Al. L. Efros, and M. Rosen, Phys. Rev. B 65, 205309 (2002).
[10] A. Imamoglu et al., Phys. Rev. Lett. 91, 017402 (2003).
[11] L. -A. Wu, Phys. Rev. A 81, 044035 (2010).
[12] Optical Orientation, edited by F. Meier and B. Azkharchenya (North-Holland, Amsterdam, 1984).
[13] D. Gammon, E. S. Snow, B. V. Shanabrook, D. S. Katzer, and D. Park, Phys. Rev. Lett. 76, 3005 (1996).
[14] C. -W. Huang and X. Hu, Phys. Rev. B 81, 205304 (2010).
[15] P. Maletinsky, Ph.D. thesis, ETH, 2008.
[16] X. Xu et al., Nature (London) **459**, 1105 (2009).
[17] J. M. Taylor, C. M. Marcus, and M. D. Lukin, Phys. Rev. Lett. **90**, 206803 (2003).
[18] D. Klauser, W. A. Coish, and Daniel Loss, Phys. Rev. B **78**, 205301 (2008).
[19] H. Nakazato, T. Takazawa, and K. Yuasa, Phys. Rev. Lett. **90**, 060401 (2003).
[20] L. -A. Wu, D. A. Lidar, and S. Schneider, Phys. Rev. A 70, 032322 (2004).
[21] J. M. Taylor, A. Imamoglu, and M. D. Lukin, Phys. Rev. Lett. **91**, 246802 (2003).
[22] Z. Kurucz, et al., Phys. Rev. Lett. **103**, 010502 (2009).
[23] R. Blatt and P. Zoller, Eur. J. Phys. **9**, 250 (1988).
[24] H. Christ, J.I. Cirac, G. Giedke, Solid State Sciences **11**, 965 (2009).