Horseshoes and Lyapunov exponents for Banach cocycles over non-uniformly hyperbolic systems

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Abstract. We extend Katok’s result on ‘the approximation of hyperbolic measures by horseshoes’ to Banach cocycles. More precisely, let $f$ be a $C^r (r > 1)$ diffeomorphism of a compact Riemannian manifold $M$, preserving an ergodic hyperbolic measure $\mu$ with positive entropy, and let $A$ be a Hölder continuous cocycle of bounded linear operators acting on a Banach space $X$. We prove that there is a sequence of horseshoes for $f$ and dominated splittings for $A$ on the horseshoes, such that not only the measure theoretic entropy of $f$ but also the Lyapunov exponents of $A$ with respect to $\mu$ can be approximated by the topological entropy of $f$ and the Lyapunov exponents of $A$ on the horseshoes, respectively. As an application, we show the continuity of sub-additive topological pressure for Banach cocycles.

Key words: cocycles, Lyapunov exponents, horseshoes, dominated splitting, entropy
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1. Introduction
Let $f$ be a $C^r (r > 1)$ diffeomorphism of a compact Riemannian manifold $M$, preserving an ergodic hyperbolic measure $\mu$ with positive entropy, and let $A$ be a Hölder continuous cocycle of bounded linear operators acting on a Banach space $X$. If the cocycle satisfies the so-called quasi-compactness condition, then there is a sequence of horseshoes for $f$ and dominated splittings for $A$ on the horseshoes, such that not only the measure theoretic entropy of $f$ but also the Lyapunov exponents of $A$ with respect to $\mu$ can be approximated by the topological entropy of $f$ and the Lyapunov exponents of $A$ on the
horseshoes, respectively. As an application of the main result, we prove the continuity of sub-additive topological pressure of singular value potentials for Banach cocycles. For an explicit statement, see § 2.

This paper is inspired by Katok [16] (or Katok and Hasselblatt [17, Theorem S.5.9]) and Cao, Pesin, and Zhao [7]. The approximation of an ergodic hyperbolic measure by horseshoes was first proved by Katok [16]. In [23], Mendoza showed that for a $C^2$ surface diffeomorphism, an ergodic hyperbolic Sinai–Ruelle–Bowen measure can be approximated by horseshoes with unstable dimensions converging to one. Avila, Crovisier, and Wilkinson [1] observed that the horseshoe constructed by Katok [16] also has a dominated splitting (for $Df$) and the Lyapunov exponents of the hyperbolic measure can be approximated by the exponents on the horseshoe.

In the $C^1$ setting, if a hyperbolic measure has positive entropy and whose support admits a dominated splitting, Gelfert [12] asserted the approximation of ergodic hyperbolic measures by horseshoes. Wang, Zou, and Cao [30] further studied the horseshoe approximation of Lyapunov exponents, which is used to show the arbitrarily large unstable dimension of the horseshoes.

The case of $C^r$ ($r > 1$) maps was considered by Chung [8], Yang [31], and Gelfert [11]. Cao, Pesin, and Zhao [7] constructed repellers such that both the entropy and the Lyapunov exponents can be approximated on the repellers. They also used this result to show the continuity of sub-additive topological pressure for $GL(d, \mathbb{R})$ cocycles, and then gave a lower bound estimate of the Hausdorff dimension of a non-conformal repeller.

For infinite-dimensional dynamical systems, Liu and Young [20, 21] generalized Katok’s results [16] to mappings and semiflows of Hilbert spaces. Recently, Liu and Ma [19] generalized Katok’s result to mappings of Banach spaces. For a Banach cocycle $\mathcal{A}$ over a hyperbolic system $f$, Kalinin and Sadovskaya [14] proved that the upper and lower Lyapunov exponents of $\mathcal{A}$ with respect to an ergodic measure can be approximated in terms of the norms of the values of $\mathcal{A}$ on hyperbolic periodic points of $f$.

The main results of this paper are stated in § 2, together with some notation and preliminaries. In § 3, we prove some preliminary results. The proofs of the main results are given in §§ 4 and 5.

2. Statement of main results

2.1. Cocycles and Lyapunov exponents for cocycles. Let $f$ be a $C^r$ ($r > 1$) diffeomorphism of a compact Riemannian manifold $M$, and $L(\mathcal{X})$ be the space of bounded linear operators on a Banach space $\mathcal{X}$. Assume $A : M \to L(\mathcal{X})$ is a Hölder continuous map. The cocycle over $f$ generated by $A$ is a map $\mathcal{A} : M \times \mathbb{N} \to L(\mathcal{X})$ defined by $\mathcal{A}(x, 0) = 1d$, and $\mathcal{A}(x, n) = A(f^{n-1}x) \ldots A(fx)A(x)$. We also denote $\mathcal{A}(x, -n) := A(f^{-n}x)^{-1} \ldots A(f^{-2}x)^{-1}A(f^{-1}x)^{-1}$ for $n > 0$. Note that $\mathcal{A}(x, -n)$ is not always defined. To be more flexible, we also denote $\mathcal{A}_n := \mathcal{A}(x, n)$.

Let $\mathcal{M}_f(M)$ be the set of $f$-invariant probability measures. Then for a fixed dense subset $\{\varphi_j\}_{j=1}^\infty$ of the unit sphere of $C(M)$, it induces a metric on $\mathcal{M}_f(M)$:

$$D(\mu, \nu) = \sum_{j=1}^\infty \frac{|\int \varphi_j d\mu - \int \varphi_j d\nu|}{2^j}$$

for all $\mu, \nu \in \mathcal{M}_f(M)$. 

$$\sum_{j=1}^\infty \frac{|\int \varphi_j d\mu - \int \varphi_j d\nu|}{2^j}$$
We now study some properties of Lyapunov exponents. The following version of multiplicative ergodic theorem was established by Froyland, LLloyd, and Quas [10], based on the work of [18, 24, 28]. To state the multiplicative ergodic theorem, we introduce some definitions.

Denote by \( B_1 \) the unit ball of \( \mathcal{X} \). Then for any \( T \in L(\mathcal{X}) \), we define the Hausdorff measure of non-compactness of \( T \) by

\[
\| T \|_\kappa := \inf\{ \varepsilon > 0 : T(B_1) \text{ can be covered by a finite number of } \varepsilon\text{-balls} \}.
\]

Then by the definition, we have \( \| T \|_\kappa \leq \| T \|, \) and \( \| \cdot \|_\kappa \) is sub-multiplicative, that is, \( \| T_2 T_1 \|_\kappa \leq \| T_2 \|_\kappa \cdot \| T_1 \|_\kappa \) for any \( T_1, T_2 \in L(\mathcal{X}) \). Let \( \mu \) be an ergodic \( f \)-invariant measure on \( M \). Then by the sub-additive ergodic theorem, the limits

\[
\lambda(A, \mu) := \lim_{n \to +\infty} \frac{1}{n} \int \log \| A^n \| d\mu,
\]

\[
\kappa(A, \mu) := \lim_{n \to +\infty} \frac{1}{n} \int \log \| A^n \|_\kappa d\mu
\]

exist. We say that the cocycle \( A \) is quasi-compact with respect to \( \mu \), if \( \lambda(A, \mu) > \kappa(A, \mu) \).

For a given \( f \)-invariant set \( \Lambda \), a splitting \( \mathcal{X} = E_1(x) \oplus \cdots \oplus E_i(x) \oplus F_i(x) \) on \( \Lambda \) is called an \( A \)-invariant splitting if \( E_j(x) = A^{-1}(E_j(f(x)) \) and \( F_i(x) = A^{-1}F_i(f(x)) \) for every \( x \in \Lambda, j = 1, \ldots, i \).

Remark. In the literature ([9, 18, 28] for instance), the splitting \( \mathcal{X} = E_1(x) \oplus \cdots \oplus E_i(x) \oplus F_i(x) \) is called \( A \)-invariant if \( A(x)E_j(x) = E_j(f(x)) \) and \( A(x)F_i(x) \subset F_i(f(x)) \) for every \( x \in \Lambda, j = 1, \ldots, i \). If \( A(x)|_{E_1(x) \oplus \cdots \oplus E_i(x)} \) is a bijection for every \( x \in \Lambda \), then the two definitions are actually equivalent. Indeed, if \( A(x)E_j(x) = E_j(f(x)) \) and \( A(x)F_i(x) \subset F_i(f(x)) \) for every \( j = 1, \ldots, i \), denote \( H_j(x) = E_j(x) \oplus \cdots \oplus E_i(x) \). Then by Lemma 4.2, \( \mathcal{X} = H_j(x) \oplus A^{-1}F_j(f(x)) \). Hence, \( \text{codim}(A(x)^{-1}F_j(f(x)) = \dim(H_j(x))) = \text{codim}(F_j(x)) \). Then it follows from \( F_i(x) \subset A^{-1}F_i(f(x)) \subset A^{-1}F_i(f(x)) \) that \( F_i(x) = A^{-1}F_i(f(x)) \).

Given two topological spaces \( Y, Z \) and a Borel measure \( \mu \) on \( Y \), a map \( g : Y \to Z \) is called \( \mu \)-continuous if there exists a sequence of pairwise disjoint compact subsets \( Y_n \subset Y \), such that \( \mu(\bigcup_{n \geq 1} Y_n) = 1 \) and \( g\big|_{Y_n} \) is continuous for every \( n \geq 1 \).

We now state a version of multiplicative ergodic theorem used in this paper. To simplify the statement, we write \( \mathbb{N}_k := \{1, 2, \ldots, k\} \) for \( 1 \leq k < +\infty \) and \( \mathbb{N}_{+\infty} := \mathbb{N}_+ \).

**Theorem 2.1.** [10] Let \( f \) be a \( C^r (r > 1) \) diffeomorphism of a compact Riemannian manifold \( M \) preserving an ergodic measure \( \mu \), and let \( A : M \to L(\mathcal{X}) \) be a Hölder continuous map. If \( A \) is quasi-compact with respect to \( \mu \), then there exists an \( f \)-invariant set \( \mathcal{R}^A \subset M \) with \( \mu(\mathcal{R}^A) = 1 \) such that for every \( x \in \mathcal{R}^A \):

(i) there exist \( k_0 \leq +\infty \), finite dimensional subspaces \( E_1(x), E_2(x), \ldots \), and infinite dimensional closed subspaces \( F_1(x), F_2(x), \ldots \) indexed in \( \mathbb{N}_{k_0} \), such that for any \( i \in \mathbb{N}_{k_0} \), there exists an \( A \)-invariant splitting on \( \mathcal{R}^A \)

\[
\mathcal{X} = E_1(x) \oplus \cdots \oplus E_i(x) \oplus F_i(x);
\]
there exist numbers \( \lambda(\mathcal{A}, \mu) = \lambda_1 > \lambda_2 > \cdots > \kappa(\mathcal{A}, \mu) \) indexed in \( \mathbb{N}_{k_0} \) such that for any \( i \in \mathbb{N}_{k_0} \),
\[
\lim_{n \to \pm\infty} \frac{1}{n} \log \| A^n |_{E_i(x)} \| = \lambda_i,
\]
for all \( u \in E_i(x) \setminus \{0\} \),
\[
\lim_{n \to \pm\infty} \frac{1}{n} \log \| A^n |_{F_i(x)} \| = \lambda_{i+1},
\]
where \( \lambda_{k_0 + 1} := \kappa(\mathcal{A}, \mu) \) if \( k_0 < +\infty \);

for any \( i \in \mathbb{N}_{k_0} \), \( E_i(x), F_i(x) \) are Borel measurable \( \mu \)-continuous and the norms of the projection operators \( \pi^\tau f^n |_{\mathcal{X}} \), \( \pi^\tau f^n |_{\mathcal{X}} \) associated with the splitting \( \mathcal{X} = (E_1(x) \oplus \cdots \oplus E_i(x)) \oplus F_i(x) \) are tempered, that is,
\[
\lim_{n \to \pm\infty} \frac{1}{n} \log \| \pi^\tau f^n |_{\mathcal{X}} \| = 0 \quad \text{for all} \quad \tau = E, F. \tag{2.1}
\]

The numbers \( \lambda_1 > \lambda_2 > \cdots \) indexed in \( \mathbb{N}_{k_0} \) are called the Lyapunov exponents of \( \mathcal{A} \) with respect to \( \mu \), and the decomposition \( \mathcal{X} = E_1(x) \oplus \cdots \oplus E_i(x) \oplus F_i(x) \) is called the Oseledets decomposition. Denote \( d_j := \text{dim}(E_j) \).

2.2. Main result. Recall that an ergodic \( f \)-invariant measure is called a hyperbolic measure if it has no zero Lyapunov exponents for \( Df \). We now state the main result of this paper.

**Theorem 2.2.** Let \( f \) be a \( C^r (r > 1) \) diffeomorphism of a compact Riemannian manifold \( M \) preserving an ergodic hyperbolic measure \( \mu \) with \( h_\mu(f) > 0 \), and let \( A : M \to L(\mathcal{X}) \) be a Hölder continuous map such that the generated cocycle \( \mathcal{A} \) is quasi-compact with respect to \( \mu \). Then for any \( i \in \mathbb{N}_{k_0} \), there exists \( \varepsilon_i > 0 \), such that for any \( 0 < \varepsilon < \varepsilon_i \), there exists a hyperbolic horseshoe \( \Lambda \) satisfying the following properties:

(i) \( | h_{\text{top}}(f|_{\Lambda}) - h_\mu(f) | < \varepsilon \);

(ii) \( \Lambda \) is \( \varepsilon \)-close to \( \text{supp}(\mu) \) in the Hausdorff distance;

(iii) for any \( f \)-invariant probability measure \( \nu \) supported in \( \Lambda \), \( D(\mu, \nu) < \varepsilon \);

(iv) there exist \( m \in \mathbb{N} \) and a continuous \( \mathcal{A} \)-invariant splitting of \( \mathcal{X} \) on \( \Lambda \),
\[
\mathcal{X} = E_1(x) \oplus \cdots \oplus E_i(x) \oplus F_i(x),
\]
with \( \dim(E_j(x)) = d_j \) for all \( 1 \leq j \leq i \), such that for any \( x \in \Lambda \), we have
\[
e^{(\lambda_j - \varepsilon)m} \| u \| \leq \| A^m_x |_{E_j(x)} \| \leq e^{(\lambda_j + \varepsilon)m} \| u \| \quad \text{for all} \quad u \in E_j(x), 1 \leq j \leq i,
\]
\[
\| A^m_x |_{F_i(x)} \| \leq e^{(\lambda_{i+1} + \varepsilon)m} \| v \| \quad \text{for all} \quad v \in F_i(x).
\]

The main difficulty in the proof of Theorem 2.2 is to obtain the \( \mathcal{A} \)-invariant splitting in property (iv) of Theorem 2.2. If \( \mathcal{X} \) is finite dimensional, there is a classical argument to obtain the linear subspaces \( E_i(x) \) (see [3, Theorem 6.1.2] for instance). However, this argument relies on the compactness of the unit sphere of \( \mathcal{X} \), which is invalid in the infinite dimensional case. To overcome this difficulty, we obtain the invariant splitting by showing that \( E_i(x) \) is actually a limit of a certain Cauchy sequence in the Grassmannian of closed subspaces of \( \mathcal{X} \).
2.3. Application: continuity of sub-additive topological pressure. For a given continuous map \( f : M \to M \), a sequence of continuous functions \( \Phi = \{\varphi_n\}_{n \geq 1} \) on \( M \) is called sub-additive if
\[
\varphi_{m+n} \leq \varphi_m + \varphi_n \circ f^n \quad \text{for all } m, n \geq 1.
\]
Recall that a subset \( E \subset M \) is called \((n, \epsilon)\)-separated if for any \( x, y \in E \), there exists \( 0 \leq i < n \) such that \( d(f^i x, f^i y) > \epsilon \). Now for a given sub-additive potential \( \Phi = \{\varphi_n\}_{n \geq 1} \), the sub-additive topological pressure of \( \Phi \) is defined by
\[
P(f, \Phi) := \lim_{\epsilon \to 0} \limsup_{n \to \infty} \frac{1}{n} \log P_n(\Phi, \epsilon),
\]
where
\[
P_n(\Phi, \epsilon) = \sup \left\{ \sum_{x \in E} \varphi_n(x) : E \text{ is an } (n, \epsilon) \text{-separated subset of } M \right\}.
\]
The variational principle for sub-additive topological pressure was proved by Cao, Feng, and Huang [5].

**Theorem 2.3.** [5] Let \( f : M \to M \) be a continuous map on a compact metric space \( M \), and \( \Phi = \{\varphi_n\}_{n \geq 1} \) be a sub-additive potential on \( M \). Then,
\[
P(f, \Phi) = \sup \{h_\mu(f) + F_\ast(\Phi, \mu) : \mu \in \mathcal{M}(M, f), F_\ast(\Phi, \mu) \neq -\infty\},
\]
where \( h_\mu(f) \) is the metric entropy of \( f \) with respect to \( \mu \) (for details see [29]) and
\[
F_\ast(\Phi, \mu) = \lim_{n \to \infty} \frac{1}{n} \int \varphi_n \, d\mu = \inf_{n \geq 1} \frac{1}{n} \int \varphi_n \, d\mu.
\]

Let \( A : M \to L(\mathcal{X}) \) be a continuous map. We shall consider the singular value functions of \( A \). The concept of singular values in a general Banach space is not well defined, instead, many different generalizations exist (for details, see [26, §A.3]). We choose the number
\[
\sigma_k(T) = \sup_{\dim(V) = k} \inf_{v \in V \setminus \{0\}} \frac{\|Tv\|}{\|v\|},
\]
to be the singular value of index \( k \) of an operator \( T \). Let
\[
\psi_s(x, n) := \sum_{i=1}^{[s]} \log \sigma_i(A^n_x) + (s - [s]) \log \sigma_{[s]+1}(A^n_x),
\]
and denote \( P(A, s) := P(f, \Psi) \) as the topological pressure of the singular value potential \( \Psi = \{\psi_s(\cdot, n)\}_{n \geq 1} \).

**Remark.** Though \( \Psi \) is not sub-additive, it is equivalent to a sub-additive potential. Indeed, we define a \( k \)-dimensional volume of vectors \((v_1, v_2, \ldots, v_k)\) by
\[
\text{Vol}_k(v_1, v_2, \ldots, v_k) = \left( \prod_{i=1}^{k-1} \text{dist}(v_i, \text{span}(v_{i+1}, \ldots, v_k)) \right) \|v_k\|,
\]

where \( \text{dist}(v_i, \text{span}(v_{i+1}, \ldots, v_k)) = \inf\{\|v - v\| : v \in \text{span}(v_{i+1}, \ldots, v_k)\} \). For \( T \in L(\mathfrak{X}) \), define

\[
V_k(T) = \sup_{\|v_i\|=1} \text{Vol}_k(Tv_1, Tv_2, \ldots, Tv_k).
\]

Then by [18, Lemma 4.7], \( V_k \) is sub-multiplicative. Moreover, by [13], there exists a constant \( C \geq 1 \) which is independent of \( T \in L(\mathfrak{X}) \) such that

\[
C^{-1} \cdot V_k(T) \leq \prod_{i=1}^k \sigma_i(T) \leq C \cdot V_k(T).
\]  

(2.2)

Let

\[
\phi^s(x, n) = (s - \lfloor s \rfloor) \log V_{\lfloor s \rfloor + 1}(A^n_x) + (\lfloor s \rfloor + 1 - s) \log V_{\lfloor s \rfloor}(A^n_x).
\]  

(2.3)

Then it gives that \( \Phi = \{\phi^s(\cdot, n)\}_{n \geq 1} \) is sub-additive and \( |\phi^s(x, n) - \psi^s(x, n)| \leq \log C \), which implies \( P(A, s) = P(f, \Psi) = P(f, \Phi) \).

We shall consider a \( C^r (r > 1) \) diffeomorphism \( f : M \to M \) which satisfies the assumptions:

(A1) the entropy map \( \mu \mapsto h_\mu(f) \) is upper semi-continuous;

(A2) every \( f \)-invariant measure is hyperbolic, that is, the Lyapunov exponents of \( f \) are non-zero for every \( f \)-invariant measure.

Paradigms of diffeomorphisms which satisfy assumptions (A1) and (A2) but not are uniformly hyperbolic are given, for example, by a one-parameter family of \( C^k \) maps constructed by Rios [27], and a family of \( C^2 \) Hénon-like maps constructed by Cao, Luzzatto, and Rios [6].

**Theorem 2.4.** Let \( f : M \to M \) be a \( C^r (r > 1) \) diffeomorphism satisfying the hypothesis of assumptions (A1) and (A2), and let \( A : M \to L(\mathfrak{X}) \) be an \( \alpha \)-Hölder continuous map such that \( A(x) \) is a compact operator for every \( x \in M \). Then for any \( s < k_0, A \mapsto P(A, s) \) is continuous at \( A \), that is, for any \( \epsilon > 0 \), there exist \( \delta > 0 \) such that

\[
|P(B, s) - P(A, s)| < \epsilon
\]

for every \( \alpha \)-Hölder continuous map \( B : M \to L(\mathfrak{X}) \) with \( \|A - B\|_0 < \delta \), where \( k_0 \) is given by Theorem 2.1 and \( \|A - B\|_0 := \max_{x \in M} \|A(x) - B(x)\| \).

If the Banach space \( \mathfrak{X} \) is finite dimensional, then we have the following corollary.

**Corollary 2.5.** Let \( f : M \to M \) be a \( C^r (r > 1) \) diffeomorphism satisfying the hypothesis of assumptions (A1) and (A2), and let \( A : M \to M_d(\mathbb{R}) \) be an \( \alpha \)-Hölder continuous map. Then \( A \mapsto P(A, s) \) is continuous at \( A \), that is, for any \( \epsilon > 0 \), there exist \( \delta > 0 \) such that

\[
|P(B, s) - P(A, s)| < \epsilon
\]

for every \( \alpha \)-Hölder continuous map \( B : M \to M_d(\mathbb{R}) \) with \( \|A - B\|_0 < \delta \), where \( \|A - B\|_0 := \max_{x \in M} \|A(x) - B(x)\| \).
In [7, Theorem 7.1], Cao, Pesin, and Zhao proved a similar result for \( f \) being a sub-shift of finite type. This corollary relaxes the condition of \( f \) to be a totally non-uniformly hyperbolic diffeomorphism, that is, \( f \) satisfies the hypothesis of assumptions (A1) and (A2).

3. Preliminary results

This section is devoted to the preparation of technical tools which shall be used to prove the main theorem. The main result in this section is Lemma 3.5, in which the invariant cones for cocycles are established. This is a necessary step to obtain the dominated splittings in Theorem 2.2. This was also the main step in the previous work [1], as well as [7]. Comparing with the previous result, the main difference in the proof of Lemma 3.5 is to obtain the invariance of the cone \( V_j(x, \theta) \) in item (ii) of Lemma 3.5. This difficulty comes from the fact that \( A^{-n} \) is not a bounded linear operator. We begin by introducing the regular neighborhoods for non-uniformly hyperbolic systems.

3.1. Regular neighborhoods. Let \( f \) be a \( C^r \) \((r > 1)\) diffeomorphism of a compact Riemannian \( d \)-dimensional manifold \( M \), preserving an ergodic hyperbolic measure \( \mu \). Then by Oseledets multiplicative ergodic theorem [24], there exists an \( f \)-invariant set \( R^{Df} \) with \( \mu \)-full measure, a number \( \chi > 0 \), and a \( Df \)-invariant decomposition \( TM = E^u \oplus E^s \) on \( R^{Df} \) such that for any \( x \in R^{Df} \), we have

\[
\lim_{n \to \pm \infty} \frac{1}{n} \log \|D_x f^n(u)\| > \chi \quad \text{for all } u \in E^u(x) \setminus \{0\},
\]

\[
\lim_{n \to \pm \infty} \frac{1}{n} \log \|D_x f^n(v)\| < -\chi \quad \text{for all } v \in E^s(x) \setminus \{0\}.
\]

Denote \( d_u = \dim(E^u), d_s = \dim(E^s) \), and denote by \( B(0, r) \) the standard Euclidean \( r \)-ball in \( \mathbb{R}^d \) centered at 0. We now introduce some properties of regular neighborhoods, see [25] for the proofs.

**Theorem 3.1.** (Pesin) Let \( f \) be a \( C^r \) \((r > 1)\) diffeomorphism of a compact Riemannian manifold \( M \), \( \mu \) be an ergodic hyperbolic measure. Then for any \( \epsilon > 0 \), the following properties hold.

(i) There exists a measurable function \( l' : R^{Df} \to [1, \infty) \) and a collection of embeddings \( \Psi_x : B(0, l'(x)^{-1}) \to M \) for \( x \in R^{Df} \) such that \( \Psi_x(0) = x, l'(x)e^{-\epsilon} \leq l'(fx) \leq l'(x)e^\epsilon \), and the preimages \( \tilde{E}^j(x) = (D_0\Psi_x)^{-1}E^j(x) \) are orthogonal in \( \mathbb{R}^d \), where \( j = u, s \).

(ii) If \( \tilde{f}_x = \Psi_{f^{-1}_x} \circ f \circ \Psi_x : B(0, l'(x)^{-1}) \to \mathbb{R}^d \), then there exist \( A_u \in GL(d_u, \mathbb{R}) \) and \( A_s \in GL(d_s, \mathbb{R}) \) such that \( D_0(\tilde{f}_x) = \text{diag}(A_u, A_s) \) and

\[
\|A_u^{-1}\|^{-1} \geq e^{\chi - \epsilon}, \quad \|A_s\| \leq e^{-\chi + \epsilon}.
\]

(iii) For any \( a, b \in B(0, l'(x)^{-1}) \),

\[
\|D_a(\tilde{f}_x) - D_b(\tilde{f}_x)\|, \quad \|D_a(f_x^{-1}) - D_b(f_x^{-1})\| \leq l'(x)|a - b|^{-1}.
\]

(iv) There exists a constant \( 0 < c_1 < 1 \) such that

\[
\|D(\Psi_x)\| \leq c_1^{-1}, \quad \|D(\Psi_x^{-1})\| \leq l'(x).
\]
So for any \( a, b \in B(0, l'(x)^{-1}) \),
\[
c_1 \cdot d(\Psi_x(a), \Psi_x(b)) \leq |a - b| \leq l'(x) - d(\Psi_x(a), \Psi_x(b)).
\]

The set \( \mathcal{N}(x) := \Psi_x(B(0, l'(x)^{-1})) \) is called a regular neighborhood of \( x \). Let \( r(x) \) be the radius of maximal ball contained in \( \mathcal{N}(x) \). Then Theorem 3.1 implies \( r(x) \geq l'(x)^{-2} \).

By Lusin’s theorem, for any \( \mu(\mathcal{N}(x)) \), there exists a compact subset \( \mathcal{R}^{Df}_\delta \subset \mathcal{R}^{Df} \) with \( \mu(\mathcal{R}^{Df}_\delta) > 1 - \delta \) such that \( x \mapsto \Psi_x, l'(x) \) and the Oseledets splitting \( T_xM = E^u_x \oplus E^s_x \) vary continuously on \( \mathcal{R}^{Df}_\delta \).

### 3.2. \((\rho, \beta, \gamma)\)-rectangles.

Let \( M, f, \mu \) be as above and let \( d_u = \dim(E^u), d_s = \dim(E^s) \). Then \( d_u + d_s = d \). Let \( I = [-1, 1] \), so then we say \( R(x) \subset M \) is a rectangle in \( M \) if there exists a \( C^1 \) embedding \( \Phi_x : I^d \rightarrow M \) such that \( \Phi_x(I^d) = R(x) \) and \( \Phi_x(0) = x \).

A set \( \tilde{H} \) is called an admissible \( u \)-rectangle in \( R(x) \), if there exist \( 0 < \lambda < 1, C^1 \) maps \( \phi_1, \phi_2 : I^{d_u} \rightarrow I^{d_s} \) satisfying \( \|\phi_1(u)\| \geq \|\phi_2(u)\| \) for \( u \in I^{d_u} \) and \( \|D\phi_i\| \leq \lambda \) for \( i = 1, 2 \), such that \( \tilde{H} = \Phi_x(H) \), where
\[
H = \{(u, v) \in I^{d_u} \times I^{d_s} : v = t\phi_1(u) + (1 - t)\phi_2(u), 0 \leq t \leq 1\}.
\]

Similarly, define an admissible \( s \)-rectangle in \( R(x) \).

**Definition 3.2.** Given \( f : M \rightarrow M \) and \( \Lambda \subset M \) compact, we say that \( R(x) \) is a \((\rho, \beta, \gamma)\)-rectangle of \( \Lambda \) for \( \rho > \beta > 0, \gamma > 0 \) if there exists \( \lambda = \lambda(\rho, \beta, \gamma) \) satisfying:

(i) \( x \in \Lambda, B(x, \beta) \subset \text{int } R(x) \) and \( \text{diam}(R(x)) \leq \rho/3 \);

(ii) if \( z, f^mz \in \Lambda \cap B(x, \beta) \) for some \( m > 0 \), then the connected component \( C(z, R(x) \cap f^{-m}R(x)) \) of \( R(x) \cap f^{-m}R(x) \) containing \( z \) is an admissible \( s \)-rectangle in \( R(x) \), and \( f^mC(z, R(x) \cap f^{-m}R(x)) \) is an admissible \( u \)-rectangle in \( R(x) \);

(iii) \( \text{diam } f^kC(z, R(x) \cap f^{-m}R(x)) \leq \rho \cdot e^{-\gamma \min[k, m - k]} \), for \( 0 \leq k \leq m \).

The following lemma is a simplified statement of [17, Theorem S.4.16].

**Lemma 3.3.** Let \( f \) be a \( C^r \) \((r > 1)\) diffeomorphism of a compact Riemannian manifold \( M \) and \( \mu \) be an ergodic hyperbolic measure. Then for any \( \rho > 0, \delta > 0 \), there exists a constant \( \beta = \beta(\rho, \delta) > 0 \) such that for any \( x \in \mathcal{R}^{Df}_\delta \), there exists a \((\rho, \beta, \chi/2)\)-rectangle \( R(x) \).

### 3.3. Lyapunov norm for cocycles.

We now establish some preliminary results for cocycles. Let \( f \) be a \( C^r \) diffeomorphism of a compact Riemannian manifold \( M \) with \( r > 1 \), and \( \mu \) be an ergodic hyperbolic measure for \( f \) with \( h_\mu(f) > 0 \). Let \( A : M \rightarrow L(\mathfrak{X}) \) be an \( \alpha \)-Hölder continuous map such that \( \lambda(A, \mu) > \kappa(A, \mu) \). Then by the multiplicative ergodic theorem stated in §2.1, there are Lyapunov exponents \( \lambda_1 > \lambda_2 > \cdots \) indexed in \( \mathbb{N}_{k_0} \) for some \( k_0 \leq +\infty \).

Fix any \( i \in \mathbb{N}_{k_0}, \epsilon > 0 \), and \( x \in \mathcal{R}^A \). We define the Lyapunov norm \( \| \cdot \|_x = \| \cdot \|_{x,i,\epsilon} \) on \( \mathfrak{X} \) as follows.
For any $u = u_1 + \cdots + u_{i+1} \in \mathcal{X}$, where $u_j \in E_j(x)$ for all $j = 1, \ldots, i$, and $u_{i+1} \in F_i(x)$, we define

$$
\|u\|_x := \sum_{j=1}^{i+1} \|u_j\|_x,
$$

(3.1)

where

$$
\|u_j\|_x = \sum_{n=-\infty}^{+\infty} \|A^n_x(u_j)\| \cdot e^{-\lambda_j n - \varepsilon|n|} \quad \text{for all } j = 1, \ldots, i;
$$

and

$$
\|u_{i+1}\|_x = \sum_{n=0}^{+\infty} \|A^n_x(u_{i+1})\| \cdot e^{-(\lambda_{i+1} + \varepsilon)n}.
$$

(3.2)

Then the following lemma holds.

**Lemma 3.4.** Let $f$, $\mu$, and $A$ be as in Theorem 2.2. Then for any $i \in \mathbb{N}_{k_0}$, $\varepsilon > 0$, the Lyapunov norm $\|\cdot\|_x = \|\cdot\|_{x, i, \varepsilon}$ satisfies the following properties.

(i) For any $x \in \mathcal{R}^A$, we have

$$
e^{\lambda_j - \varepsilon} \|u_j\|_x \leq \|A(x)u_j\|_f \leq e^{\lambda_j + \varepsilon} \|u_j\|_x \quad \text{for all } u_j \in E_j(x), \ j = 1, \ldots, i,
$$

(3.3)

$$
\|A(x)u_{i+1}\|_f \leq e^{\lambda_{i+1} + \varepsilon} \|u_{i+1}\|_x \quad \text{for all } u_{i+1} \in F_i(x).
$$

(3.4)

(ii) There exists an $f$-invariant subset of $\mathcal{R}^A$ with $\mu$-full measure (we may also denote it by $\mathcal{R}^A$), and a measurable function $K(x) = K_{\varepsilon, i}(x)$ defined on $\mathcal{R}^A$ such that for any $x \in \mathcal{R}^A$, $u \in \mathcal{X}$, we have

$$
\|u\| \leq \|u\|_x \leq K(x) \|u\|,
$$

(3.5)

$$
K(x)e^{-\varepsilon} \leq K(fx) \leq K(x)e^{\varepsilon}.
$$

(3.6)

**Proof.** (i) We will prove the inequality

$$
\|A(x)u_j\|_f \leq e^{\lambda_j + \varepsilon} \|u_j\|_x \quad \text{for all } u_j \in E_j(x), \ j = 1, \ldots, i,
$$

the others can be proved analogously. By the definition, we have

$$
\|A(x)u_j\|_f = \sum_{n=-\infty}^{+\infty} \|A^n_{fx}(A^{n-1}_x(u_j))\| \cdot e^{-\lambda_j n - \varepsilon|n|}
$$

$$
= \sum_{n=-\infty}^{+\infty} \|A^{n+1}_x(u_j)\| \cdot e^{-\lambda_j (n+1) - \varepsilon|n+1|} \cdot e^{\lambda_j + \varepsilon(n+1+|n|)}
$$

$$
\leq e^{\lambda_j + \varepsilon} \cdot \|u_j\|_x.
$$

(ii) For any $u = u_1 + \cdots + u_{i+1} \in \mathcal{X}$, by the definition,

$$
\|u\| \leq \|u_1\| + \cdots + \|u_{i+1}\| \leq \|u_1\|_x + \cdots + \|u_{i+1}\|_x = \|u\|_x.
$$
This estimates the lower bound. To estimate the upper bound, we define

\[
M_j(x) := \sup \left\{ \frac{\|A^n_{x}(u_j)\|}{e^{\lambda_j n + \frac{1}{2}\varepsilon |n|} \cdot \|u_j\|} : u_j \in E_j(x), n \in \mathbb{Z} \right\} \quad \text{for all } j = 1, \ldots, i,
\]

\[
M_{i+1}(x) := \sup \left\{ \frac{\|A^n_{x}(u_{i+1})\|}{e^{(\lambda_{i+1} + \frac{1}{2}\varepsilon)n} \cdot \|u_{i+1}\|} : u_{i+1} \in F_{i}(x), n \geq 0 \right\}.
\]

Then,

\[
\|u\| \cdot \|u\|_{x} \leq \sum_{j=1}^{i} \sum_{n=-\infty}^{+\infty} M_j(x) e^{-(1/2)\varepsilon |n|} \cdot \|u_j\| + \sum_{n=0}^{+\infty} M_{i+1}(x) e^{-(1/2)\varepsilon n} \cdot \|u_{i+1}\|
\]

\[
\leq c_0 \cdot \left( \sum_{j=1}^{i} M_j(x) \cdot \|\pi^j_{E}(x) - \pi^{j-1}_{E}(x)\| \cdot \|u\| + M_{i+1}(x) \cdot \|\pi^{j}_{F}(x)\| \cdot \|u\| \right)
\]

\[
=: M(x) \cdot \|u\|,
\]  

(3.7)  

where \(c_0 = \sum_{n=-\infty}^{+\infty} e^{-(1/2)\varepsilon |n|}\) and \(\pi^{0}_{E}(x) = 0\).

**CLAIM.** \(M(x)\) is tempered on an \(f\)-invariant subset of \(\mathcal{R}^{A}\) with \(\mu\)-full measure, that is,

\[
\lim_{n \to \pm \infty} \frac{1}{n} \log M(f^n x) = 0 \quad \text{for } \mu\text{-almost every (a.e.) } x \in \mathcal{R}^{A}.
\]

**Proof of the claim.** Since \(\|\pi^j_{E}(x)\|, \|\pi^j_{F}(x)\|\) are tempered by equation (2.1), it is enough to prove \(M_j(x)\) is tempered for any \(1 \leq j \leq i + 1\). Since

\[
M_{i+1}(x)
\]

\[
= \sup_{u_{i+1} \in F_{i}(x)} \left\{ \frac{\|A^n_{x}(u_{i+1})\|}{e^{\lambda_{i+1} n + (1/2)\varepsilon |n|} \cdot \|A_{x}(u_{i+1})\|} \cdot \frac{\|A_{X}(u_{i+1})\|}{e^{\lambda_{i+1} + \varepsilon / 2 |n| - |n-1|} \cdot \|u_{i+1}\|} ight\}
\]

\[
\leq \max\{c_1 \cdot M_{i+1}(f x), 1\},
\]

where \(c_1 = \max_{x \in M} \|A_{x}\|/e^{\lambda_{i+1} -(1/2)\varepsilon}\), we obtain

\[
(\log M_{i+1}(x) - \log M_{i+1}(f x))^+ \leq \log \max\left\{ c_1, \frac{1}{M_{i+1}(f x)} \right\} \leq \log \max\{c_1, 1\},
\]

and then we conclude \(M_{i+1}(x)\) is tempered on a subset of full measure by [22, Lemma III.8]. The result for \(1 \leq j \leq i\) is obtained similarly.

Let

\[
K(x) := \sum_{n \in \mathbb{Z}} M(f^n x) e^{-\varepsilon |n|} \quad \text{for all } x \in \mathcal{R}^{A},
\]

and then by [3, Lemma 3.5.7], \(K(x)\) satisfies equations (3.5) and (3.6). This completes the proof of the lemma.
Fix any \( i \in \mathbb{N}_{k_0} \). Then by Lusin’s theorem, for any \( \delta > 0 \), there exists a compact subset \( \mathcal{R}^A_\delta \subset \mathcal{R}^A \) such that \( \mu(\mathcal{R}^A_\delta) > 1 - \delta \) and \( K(x) \) is continuous on \( \mathcal{R}^A_\delta \). Denote

\[
l = l_\delta = \sup\{K(x) : x \in \mathcal{R}^A_\delta\}. \tag{3.8}\]

Since the Oseledets decomposition is \( \mu \)-continuous, we may assume the Oseledets decomposition is continuous on \( \mathcal{R}^A_\delta \).

Let \( \mathcal{X} \) be the collection of norms on \( \mathcal{X} \) which are equivalent to \( \| \cdot \| \). Then \( \mathcal{X} \) is a metric space with respect to the metric

\[
\tilde{D}(\varphi, \psi) = \sup_{u \in \mathcal{X}\setminus\{0\}} \frac{|\varphi(u) - \psi(u)|}{\|u\|} \quad \text{for all } \varphi, \psi \in \tilde{X}.
\]

We claim that the function \( x \mapsto \| \cdot \|_x \) is continuous on \( \mathcal{R}^A_\delta \), which will be used in the proof of equations (3.17), (3.18) in Lemma 3.5 and equation (4.9) in Proposition 4.1. Indeed, for any \( x \in \mathcal{R}^A_\delta \), \( u = u_1 + \cdots + u_{i+1} \in \mathcal{X} \), where \( u_j \in E_j(x) \), for all \( j = 1, \ldots, i \), and \( u_{i+1} \in F_i(x) \), we define

\[
\|u\|_{k,x} := \sum_{j=1}^i \sum_{n=-k}^k \|A^n_{ij}(u_j)\| \cdot e^{-\lambda_j n - |n|} + \sum_{n=0}^k \|A^n_{i+1}(u_{i+1})\| \cdot e^{-(\lambda_{i+1} + \varepsilon)n},
\]

and then by equation (3.7),

\[
\|\|u\|_x - \|u\|_{k,x}\| \leq \sum_{|n| > k} e^{-(1/2)|n|} \cdot M(x) \cdot \|u\| \leq \frac{2e^{-(\varepsilon/2)}k}{e^{(\varepsilon/2)} - 1} \cdot K(x) \cdot \|u\| \leq \frac{2l \cdot e^{-(\varepsilon/2)}k}{(e^{(\varepsilon/2)} - 1)} \cdot \|u\|,
\]

that is,

\[
\tilde{D}(\| \cdot \|_x, \| \cdot \|_{k,x}) \leq \frac{2l \cdot e^{-(\varepsilon/2)}k}{(e^{(\varepsilon/2)} - 1)},
\]

which implies \( \| \cdot \|_{k,x} \to \| \cdot \|_x \) uniformly for \( x \in \mathcal{R}^A_\delta \) as \( k \to +\infty \). Since \( x \mapsto \| \cdot \|_{k,x} \) is continuous, we have \( x \mapsto \| \cdot \|_x \) is continuous on \( \mathcal{R}^A_\delta \). This proves the claim.

3.4. Invariant cones for Banach cocycles. We now estimate the growth of vectors in certain invariant cones for cocycles. Let \( f, \mu \), and \( A \) be as in §3.3. Fix any \( i \in \mathbb{N}_{k_0} \), and let \( \varepsilon_i := \min\{\frac{1}{4}\chi \alpha, \chi/4, \chi(r - 1)/2r, (\lambda_1 - \lambda_2)/8, \ldots, (\lambda_i - \lambda_{i+1})/8\} \), where \( \chi \) is given in §3.1. For any \( 0 < \varepsilon < \varepsilon_i, x \in \mathcal{R}^A \), let \( \| \cdot \|_x = \| \cdot \|_{x, i, \varepsilon} \) the Lyapunov norm on \( \mathcal{X} \) defined in equation (3.1). For any \( 1 \leq j \leq i \), we have the Oseledets decomposition \( \mathcal{X} = H_j(x) \oplus F_j(x) \), where \( H_j(x) = E_1(x) \oplus \cdots \oplus E_j(x) \). For any \( \theta > 0, u \in \mathcal{X} \), let \( u = u_H + u_F \), where \( u_H \in H_j(x), u_F \in F_j(x) \). We consider two cones:

\[
U_j(x, \theta) := \{u \in \mathcal{X} : \|u_F\|_x \leq \theta \|u_H\|_x\};
\]

\[
V_j(x, \theta) := \{u \in \mathcal{X} : \|u_H\|_x \leq \theta \|u_F\|_x\}.
\]
For any $\delta > 0$, recall that a sequence $(x_n)_{n \in \mathbb{Z}}$ is a $\rho$-pseudo-orbit of $f^m$ in $\mathcal{R}_\delta^A$ for some $m \in \mathbb{N}$, if $x_n, f^m(x_n) \in \mathcal{R}_\delta^A$ and $d(f^m x_n, x_{n+1}) \leq \rho$ for any $n \in \mathbb{Z}$. Let $l = l_\delta$ be as in equation (3.8). The following lemma is related to a horseshoe $\Lambda^*$ which is given in §4.1. The definition of $\Lambda^*$ using Markov rectangles. In the definition of $\Lambda^*$, the integer $m$ is fixed once for all, and any point $y \in \Lambda^*$ is determined by a sequence $(x_n)_{n \in \mathbb{Z}} \in F_m$, where $F_m \subset \mathcal{R}_\delta^A$ has finite cardinality. More precisely, for any $y \in \Lambda^*$, there exists $\{x_n\}_{n \in \mathbb{Z}} \subset F_m$ such that
\[
d(f^k x_n, f^k (f^{nm} y)) \leq \rho \cdot e^{-(\chi/2) \min[k, m-k]} \quad \text{for all } n \in \mathbb{Z}, k = 0, \ldots, m.
\]
The sequence $(x_n)_{n \in \mathbb{Z}} \subset F_m$ is actually a $\rho$-pseudo-orbit of $f^m$ in $\mathcal{R}_\delta^A$, and the orbit of $y \in \Lambda^*$ shadows $(x_n)_{n \in \mathbb{Z}}$.

**Lemma 3.5.** For any $0 < \varepsilon < \varepsilon_i, \delta > 0$, there exist $\rho_0 > 0$, $\theta_0 > 0$ such that for any $0 < \rho < \rho_0$, $1 \leq j \leq i$, any $\rho$-pseudo-orbit $(x_n)_{n \in \mathbb{Z}}$ of $f^m$ in $\mathcal{R}_\delta^A$ with $m \geq 2 \log l/\varepsilon$, and for any $y \in M$ with $d(f^k x_n, f^k (f^{nm} y)) \leq \rho \cdot e^{-(\chi/2) \min[k, m-k]}$, $k = 0, \ldots, m$, we can find $\eta = \eta(\varepsilon, \delta, m) \in (0, 1)$ such that:

(i) $A^m_{f^{nm+k}y} U_j(f^k x_n, \theta_0) \subset U_j(f^k x_{n+1}, \eta \theta_0)$, and 
\[
e^{(\lambda_j - 4\varepsilon)m} \|u\| \leq \|A^m_{f^{nm+k}y}(u)\| \leq e^{(\lambda_{j+1} + 4\varepsilon)m} \|u\| \quad \text{for all } u \in U_j(f^k x_n, \theta_0);
\]

(ii) $A^m_{f^{nm+k}y} V_j(f^k x_n, \theta_0) \subset V_j(f^k x_{n-1}, \eta \theta_0)$, and 
\[
\|A^m_{f^{(a-1)m+k}y}(v)\| \leq e^{(\lambda_{j+1} + 4\varepsilon)m} \|v\| \quad \text{for all } v \in A^{-m}_{f^{nm+k}y} V_j(f^k x_n, \theta_0).
\]

**Proof.** (i) For the simplicity of notation, we prove $A^m_y U_j(x_0, \theta_0) \subset U_j(x_1, \eta \theta_0)$ and $e^{(\lambda_j - 4\varepsilon)m} \|u\| \leq \|A^m_y(u)\| \leq e^{(\lambda_{j+1} + 4\varepsilon)m} \|u\|$ for all $u \in U_j(x_0, \theta_0)$, $1 \leq j \leq i$.

For any fixed $0 < \varepsilon < \varepsilon_i, \delta > 0$, denote $\theta_0 := e^{\varepsilon_{i+1} - \lambda_1} (e^{\varepsilon_1 - 1} - e^{\varepsilon} - 1)$ and $\eta_0 := \max\{e^{\lambda_{j+1} - \lambda_j + 4\varepsilon} : 1 \leq j \leq i\}$. We have the following claim.

**Claim.** There exist $\tilde{\rho}_0 > 0$, such that for any $x_0, f^m x_0 \in \mathcal{R}_\delta^A$, $y \in M$ with $d(f^k x_0, f^k y) \leq \rho \cdot e^{-(\chi/2) \min[k, m-k]}$ for $k = 0, \ldots, m$ and $0 < \rho < \tilde{\rho}_0$, we have
\[
A(f^k y) U_j(f^k x_0, \theta_0) \subset U_j(f^{k+1} x_0, \eta_0 \theta_0) \quad \text{for all } 0 \leq k \leq m - 1.
\]

**Proof of the claim.** For any $0 \leq k \leq m - 1$, $u = u_H + u_F \in U_j(f^k x_0, \theta_0)$, by equations (3.3) and (3.4), we have
\[
\|A(f^k x_0) u_H\| f^{k+1} x_0 \geq e^{\lambda_j - \varepsilon} \|u_H\| f^k x_0,
\]
\[
\|A(f^k x_0) u_F\| f^{k+1} x_0 \leq e^{\lambda_{j+1} + \varepsilon} \|u_F\| f^k x_0,
\]
Let $w = (A(f^k y) - A(f^k x_0)) u = w_H + w_F$, where $w_H \in H_j(f^{k+1} x_0), w_F \in F_j(f^{k+1} x_0)$. Then
\[
A(f^k y) u = w + A(f^k x_0) u.
\]
Since $A : M \to L(\mathcal{X})$ is $\alpha$-Hölder continuous, there exists a constant $c_0 > 0$ such that
\[
\|A(z_1) - A(z_2)\| \leq c_0 d(z_1, z_2)^\alpha \quad \text{for all } z_1, z_2 \in M.
\]
Since \( x_0, f^m(x_0) \in \mathcal{R}_A^\delta \) and \( K(x) \) is continuous on \( \mathcal{R}_A^\delta \), by equations (3.5) and (3.8),

\[
K(f^{k+1}x_0) \leq \min\{e^{\varepsilon(k+1)}K(x_0), e^{\varepsilon(m-k-1)}K(f^m x_0)\}
\leq le^\varepsilon \min[k+1, m-k-1].
\]

Then by equation (3.6),

\[
\|w_H\|_{f^{k+1}x_0} \leq\|w\|_{f^{k+1}x_0} - \|w_H\|_{f^{k+1}x_0} \leq e^{\lambda j - \varepsilon}\left\|u_H\right\|_{f^{k+1}x_0} - (1 + \theta_0)c_0 e^\varepsilon \rho^\alpha\left\|u_H\right\|_{f^{k+1}x_0},
\]

as \( \varepsilon - (\chi/2)\alpha < 0 \). Similarly,

\[
\|w_F\|_{f^{k+1}x_0} \leq (1 + \theta_0)c_0 e^\varepsilon \rho^\alpha\left\|u_H\right\|_{f^{k+1}x_0}.
\]

Let

\[
A(f^k y)u = (A(f^k y)u)_H + (A(f^k y)u)_F,
\]

where \( (A(f^k y)u)_H \in H_j(f^{k+1}x_0), (A(f^k y)u)_F \in F_j(f^{k+1}x_0) \). Then by equations (3.10), (3.12), and (3.13),

\[
\|(A(f^k y)u)_H\|_{f^{k+1}x_0} \geq e^{\lambda j - \varepsilon}\|u_H\|_{f^{k+1}x_0} - (1 + \theta_0)c_0 e^\varepsilon \rho^\alpha\left\|u_H\right\|_{f^{k+1}x_0},
\]

if \( \rho \) is small enough. Similarly, by equations (3.11), (3.12), and (3.14),

\[
\|(A(f^k y)u)_F\|_{f^{k+1}x_0} \leq e^{\lambda j + \varepsilon}\|u_F\|_{f^{k+1}x_0} - (1 + \theta_0)c_0 e^\varepsilon \rho^\alpha\left\|u_H\right\|_{f^{k+1}x_0},
\]

if \( \rho \) is small enough. Thus,

\[
\|(A(f^k y)u)_F\|_{f^{k+1}x_0} \leq \eta_0 \theta_0\left\|(A(f^k y)u)_H\right\|_{f^{k+1}x_0},
\]

that is, \( A(f^k y)U_j(f^m x_0, \eta_0 \theta_0) \subset U_j(f^{k+1}x_0, \eta_0 \theta_0) \).

The claim implies

\[
A^m_y U_j(x_0, \theta_0) \subset U_j(f^m x_0, \eta_0 \theta_0)
\]

for all \( 1 \leq j \leq i \).

Since, by §3.4, the Lyapunov norm and the Oseledets decomposition are uniformly continuous on the compact set \( \mathcal{R}_A^\delta \), there exists \( \eta_0 < \eta < 1 \) such that

\[
U_j(f^m x_0, \eta_0 \theta_0) \subset U_j(x_1, \eta \theta_0)
\]

for all \( d(f^m x_0, x_1) \leq \rho \),

\[
V_j(x_1, \theta_0) \subset V_j\left(f^m x_0, \frac{\eta}{\eta_0} \theta_0\right)
\]

for all \( d(f^m x_0, x_1) \leq \rho \).
if \( \rho \) is small enough. Hence, for any \( 1 \leq j \leq i \), we have

\[
A^m_y U_j(x_0, \theta_0) \subset U_j(x_1, \eta \theta_0).
\] (3.19)

Moreover, for any \( 0 \leq k \leq m - 1 \) and \( u \in U_j(f^k x_0, \theta_0) \), it follows from equation (3.15) that

\[
\|A(f^k y)u\|_{f^{k+1}x_0} \geq \|(A(f^k y)u) H\|_{f^{k+1}x_0}
\geq e^{\lambda_j - 2\epsilon} \|u_H\|_{f^kx_0}
\geq (1 + \theta_0)^{-1} e^{\lambda_j - 2\epsilon} \|u\|_{f^kx_0}
\geq e^{\lambda_j - 3\epsilon} \|u\|_{f^kx_0}.
\]

Therefore, for any \( 1 \leq j \leq i \) and \( u \in U_j(x_0, \theta_0) \), by equations (3.9) and (3.5), we conclude

\[
\|A^m_y(u)\| \geq \frac{1}{l} \|A^m_y(u)\|_{f^m x_0} \geq \frac{1}{l} e^{(\lambda_j - 3\epsilon)m} \|u\|_{x_0} \geq e^{(\lambda_j - 4\epsilon)m} \|u\|.
\]

This is the first time that the estimate on \( l \) is used. Similar to equation (3.15), we can also get

\[
\|(A(f^k y)u) H\|_{f^{k+1}x_0} \leq \|(A(f^k x_0)u H)\|_{f^{k+1}x_0} + \|w_H\|_{f^{k+1}x_0} \leq e^{\lambda_j + 2\epsilon} \|u_H\|_{f^kx_0}.
\]

Thus, we obtain by using equation (3.16) that

\[
\|A(f^k y)u\|_{f^{k+1}x_0} \leq \|(A(f^k y)u) H\|_{f^{k+1}x_0} + \|(A(f^k y)u) F\|_{f^{k+1}x_0}
\leq e^{\lambda_j + 2\epsilon} \|u_H\|_{f^kx_0} + \theta_0 e^{\lambda_j + 2\epsilon} \|u_H\|_{f^kx_0}
\leq e^{\lambda_j + 3\epsilon} \|u\|_{f^kx_0}.
\]

Hence, for any \( u \in U_j(x_0, \theta_0) \), by equations (3.9) and (3.5), we conclude

\[
\|A^m_y(u)\| \leq \|A^m_y(u)\|_{f^m x_0} \leq e^{(\lambda_j + 3\epsilon)m} \|u\|_{x_0} \leq l e^{(\lambda_j + 3\epsilon)m} \|u\| \leq e^{(\lambda_j + 4\epsilon)m} \|u\|.
\]

This proves the conclusion (i).

(ii) For simplicity, we only prove \( A^{-m}_{f^my} V_j(x_1, \theta_0) \subset V_j(x_0, \eta \theta_0) \) and \( \|A^m_y(v)\| \leq e^{(\lambda_j + 4\epsilon)m} \|v\| \) for all \( v \in A^{-m}_{f^my} V_j(x_1, \theta_0), 1 \leq j \leq i \).

Let \( \theta_1 = (\eta/\eta_0) \theta_0 \), where \( \eta \in (\eta_0, 1) \) is given by equation (3.17). We have the following claim.

CLAIM. There exists \( \tilde{\rho}_0 > 0 \), such that for any \( x_0, f^m x_0 \in \mathcal{R}_{\delta}^A, y \in M \) with \( d(f^k x_0, f^k y) \leq \rho \cdot e^{(\lambda_j/2) \min[k, m-k]} \) for \( k = 0, \ldots, m \) and \( 0 < \rho < \tilde{\rho}_0 \), we have

\[
A(f^{k-1} y)^{-1} V_j(f^k x_0, \theta_1) \subset V_j(f^{k-1} x_0, \eta \theta_1) \quad \text{for all} \quad 1 \leq k \leq m.
\] (3.20)

Proof of the claim. For any \( 1 \leq k \leq m, v \in A(f^{k-1} y)^{-1} V_j(f^k x_0, \theta_1) \), one has \( A(f^{k-1} y)v \in V_j(f^k x_0, \theta_1) \). Let

\[
A(f^{k-1} y)v = (A(f^{k-1} y)v)_H + (A(f^{k-1} y)v)_F \in H_j(f^k x_0) \oplus F_j(f^k x_0),
\]

and

\[
v = v_H + v_F \in H_j(f^{k-1} x_0) \oplus F_j(f^{k-1} x_0).
\]
Denote $w := (A(f^k x) - A(f^k x_0))v$. Since $A(f^k y)v = A(f^k x_0)v + w$, one has
\[
(A(f^k y)v)_H = A(f^k x_0)v_H + w_H, \quad (A(f^k y)v)_F = A(f^k x_0)v_F + w_F.
\]
Similar to the proof of equations (3.10), (3.11), (3.13), and (3.14), we can obtain
\[
\|A(f^k x_0)v_H\|_{f^k x_0} \geq e^{\lambda_j - \varepsilon} \|v_H\|_{f^k x_0},
\]
\[
\|A(f^k x_0)v_F\|_{f^k x_0} \leq e^{\lambda_j + 1 + \varepsilon} \|v_F\|_{f^k x_0},
\]
and
\[
\|w_H\|_{f^k x_0} \leq \|w\|_{f^k x_0} \leq K(f^k x_0) \|A(f^k y) - A(f^k x_0)\| \cdot \|v\|
\leq le^{e \min[k, m-k]} \cdot c_0 e^{(e - (\lambda_j + 1 + \varepsilon)) \|v\|_{f^k x_0}}
\leq c_0 le^{(e - (\lambda_j + 1 + \varepsilon)) \|v\|_{f^k x_0}},
\]
\[
\|w_F\|_{f^k x_0} \leq \|w\|_{f^k x_0} \leq c_0 le^{(e - (\lambda_j + 1 + \varepsilon)) \|v\|_{f^k x_0}}.
\]
Therefore,
\[
\|A(f^k y)v\|_{f^k x_0} \geq \|A(f^k x_0)v_H\|_{f^k x_0} - \|w_H\|_{f^k x_0}
\geq e^{\lambda_j - \varepsilon} \|v_H\|_{f^k x_0} - c_0 le^{\varepsilon} \rho^\alpha \|v\|_{f^k x_0}, \tag*{(3.21)}
\]
and
\[
\|A(f^k y)v\|_{f^k x_0} \leq \|A(f^k x_0)v_F\|_{f^k x_0} + \|w_F\|_{f^k x_0}
\leq e^{\lambda_j + 1 + \varepsilon} \|v_F\|_{f^k x_0} + c_0 le^{\varepsilon} \rho^\alpha \|v\|_{f^k x_0}, \tag*{(3.22)}
\]
Note that $A(f^k y)v \in V_j(f^k x_0, \theta_1)$. It follows that
\[
\|A(f^k y)v\|_{f^k x_0} \leq \theta_1 \|A(f^k y)v\|_{f^k x_0},
\]
Hence, by equations (3.21) and (3.22),
\[
\|v_H\|_{f^k x_0} \leq \theta_1 e^{\lambda_j + 1 - \lambda_j + 2\varepsilon} \|v_F\|_{f^k x_0} + (1 + \theta_1) c_0 le^{\varepsilon} \rho^\alpha \|v\|_{f^k x_0}
\leq \theta_1 e^{\lambda_j + 1 - \lambda_j + 2\varepsilon} \|v_F\|_{f^k x_0} + (1 + \theta_1) c_0 le^{\varepsilon} \rho^\alpha (\|v_H\|_{f^k x_0} + \|v_F\|_{f^k x_0})
\leq \theta_1 e^{\lambda_j + 1 - \lambda_j + 2\varepsilon} \|v_F\|_{f^k x_0} + (1 - e^{-\varepsilon}) \|v_H\|_{f^k x_0},
\]
if $\rho$ is small enough. Thus,
\[
\|v_H\|_{f^k x_0} \leq \theta_1 e^{\lambda_j + 1 - \lambda_j + 2\varepsilon} \|v_F\|_{f^k x_0} = \eta_0 \theta_1 \|v_F\|_{f^k x_0},
\]
that is, $v \in V_j(f^k x_0, \eta_0 \theta_1)$.
\[
\]
This claim implies
\[
A_{f^m}^{-1} V_j(f^m x_0, \theta_1) \subset V_j(x_0, \eta_0 \theta_1) = V_j(x_0, \eta \theta_0) \text{ for all } 1 \leq j \leq i.
\]
By equation (3.18), we obtain
\[ A^{-m}_{m_y} V_j(x_1, \theta_0) \subset V_j(x_0, \eta \theta_0) \quad \text{for all } 1 \leq j \leq i. \]

Now, for any \( v \in A^{-m}_{m_y} V_j(x_1, \theta_0) \), one has
\[ w := A^k_y(v) \in A^{-m+k}_{m_y} V_j(x_1, \theta_0) \subset A^{-m+k}_{m_y} V_j(f^m x_0, \theta_1) \subset V_j(f^k x_0, \theta_0). \]

Let \( w = w_H + w_F \), where \( w_H \in H_j(f^k x_0) \), \( w_F \in F_j(f^k x_0) \). Then,
\[ \| A(f^k x_0) w_H \|_{f^{k+1} x_0} \leq e^{\lambda_1+\varepsilon} \| w_H \|_{f^k x_0}, \quad \| A(f^k x_0) w_F \|_{f^{k+1} x_0} \leq e^{\lambda_j+1+\varepsilon} \| w_F \|_{f^k x_0}. \]

It follows that
\[ \| A(f^k x_0) w \|_{f^{k+1} x_0} \leq \| A(f^k x_0) w_H \|_{f^{k+1} x_0} + \| A(f^k x_0) w_F \|_{f^{k+1} x_0} \leq e^{\lambda_1+\varepsilon} \theta_0 \| w_H \|_{f^k x_0} + e^{\lambda_j+1+\varepsilon} \| w_F \|_{f^k x_0} \leq e^{\lambda_j+1+2\varepsilon} \| w \|_{f^k x_0}. \]

Similar to equation (3.13), we can obtain \( \| A(f^k y) w - A(f^k x_0) w \|_{f^{k+1} x_0} \leq c_0 e^c \rho^a \| w \|_{f^k x_0} \). Therefore,
\[ \| A(f^k y) w \|_{f^{k+1} x_0} \leq \| A(f^k y) w - A(f^k x_0) w \|_{f^{k+1} x_0} + \| A(f^k x_0) w \|_{f^{k+1} x_0} \leq c_0 e^c \rho^a \| w \|_{f^k x_0} + e^{\lambda_j+1+2\varepsilon} \| w \|_{f^k x_0} \leq e^{\lambda_j+1+3\varepsilon} \| w \|_{f^k x_0} \]
if \( \rho \) is small enough. It implies that for any \( v \in A^{-m}_{m_y} V_j(x_1, \theta_0) \),
\[ \| A^m_y(v) \| \leq \| A^m_y(v) \|_{f^m x_0} \leq e^{(\lambda_j+1+3\varepsilon)m} \| v \|_{x_0} \leq e^{(\lambda_j+1+4\varepsilon)m} \| v \|. \]
This completes the proof of the lemma.

4. Proof of Theorem 2.2

4.1. Construction of hyperbolic horseshoes. The aim of this subsection is to construct a hyperbolic horseshoe \( \Lambda \) satisfying the properties listed in Theorem 2.2. This construction resembles Theorem S.5.9 in [17].

We begin with producing a separated set with sufficiently large cardinality. For \( n \geq 1 \), denote by \( d_n(x, y) = \max_{0 \leq k \leq n-1} d(f^k x, f^k y) \) the dynamical distance on \( M \), and denote by \( B_n(x, \rho) = \{ y \in M : d_n(x, y) \leq \rho \} \) the \( d_n \)-balls of radius \( \rho \). Let \( N_\mu(n, \rho, \delta) \) be the minimal numbers of \( d_n \)-balls of radius \( \rho \) whose union has measure at least \( \delta \). Then by [16, Theorem 1.1], for any \( \delta > 0 \),
\[ h_\mu(f) = \lim_{\rho \to 0} \liminf_{n \to +\infty} \frac{1}{n} \log N_\mu(n, \rho, \delta). \]

Given any \( 0 < \delta < 1/2 \), let \( \Lambda_\delta = R^{Df}_{\delta} \cap R^A_{\delta} \cap \text{supp}(\mu) \). Then, \( \mu(\Lambda_\delta) > 1 - 2\delta > 0 \). Take \( \delta = (1/2) \mu(\Lambda_\delta) \), and then for any given \( i \in \mathbb{N}_{k_0}, 0 < \varepsilon < \varepsilon_i \), there exist \( 0 < \rho_1 < \varepsilon/2, N > 1 \), such that for any \( 0 < \rho < \rho_1, n \geq N \), one has
\[ N_\mu(n, \rho, \delta) \geq e^{(h_\mu(f)-\varepsilon)n}. \] (4.1)
Fix a dense subset \( \{ \varphi_j \}_{j=1}^\infty \) of the unit sphere of \( C(M) \). We consider the induced metric on \( M_f(M) \):

\[
D(\mu_1, \mu_2) = \sum_{j=1}^\infty \frac{|\int \varphi_j \, d\mu_1 - \int \varphi_j \, d\mu_2|}{2^j}
\]

for all \( \mu_1, \mu_2 \in M_f(M) \).

Take \( J \) large enough such that \( 1/2^J < \varepsilon/8 \), and take \( \rho < \min\{\rho_0, \rho_1/2\} \) small enough (where \( \rho_0 \) is given by Lemma 3.5) such that

\[
|\varphi_j(x) - \varphi_j(y)| \leq \varepsilon/4 \quad \text{for all } d(x, y) \leq \rho, \ j = 1, \ldots, J.
\] (4.2)

Since \( \Lambda_\delta \subset \mathcal{R}_\delta^{Bf} \), by Lemma 3.3, there exists \( 0 < \beta < \frac{1}{3} \rho \) and finite \((\rho, \beta, \chi/2)\)-rectangles \( R(q_1), \ldots, R(q_t) \), such that \( \bigcup_{j=1}^t B(q_j, \beta) \supset \Lambda_\delta \). We consider a partition \( P = \{P_1, \ldots, P_t\} \) of \( \Lambda_\delta \), where

\[
P_1 = B(q_1, \beta) \cap \Lambda_\delta \quad \text{and} \quad P_k = B(q_k, \beta) \cap \Lambda_\delta \setminus \left( \bigcup_{j=1}^{k-1} P_j \right) \quad \text{for all } 2 \leq k \leq t.
\]

Let

\[
\Lambda_{\delta,n} := \bigg\{ x \in \Lambda_\delta : \text{there exists } k \in [n, (1+\varepsilon)n] \text{ such that } f^k(x) \in P(x),
\]

the collection \( \{f^k(x)\}_{1 \leq k \leq n} \) is \( \varepsilon/2 \)-dense in \( \text{supp}(\mu) \), and

\[
\frac{1}{m} \sum_{p=0}^{m-1} \varphi_j(f^p x) - \int \varphi_j \, d\mu \leq \frac{\varepsilon}{4} \quad \text{for all } m \geq n, 1 \leq j \leq J.
\]

CLAIM. \( \lim_{n \to \infty} \mu(\Lambda_{\delta,n}) = \mu(\Lambda_\delta) \).

Proof of the claim. Let

\[
A_n = \{ x \in \Lambda_\delta : \text{there exists } k \in [n, (1+\varepsilon)n] \text{ such that } f^k(x) \in P(x) \},
\]

and \( A_{n,j} = \{ x \in P_j : \text{there exists } k \in [n, (1+\varepsilon)n] \text{ such that } f^k(x) \in P_j \} \). Then, \( A_n = \bigcup_{j=1}^t A_{n,j} \). Considering \( P_1 \), we may assume \( \mu(P_1) > 0 \). For any \( \tau > 0 \), let

\[
A^\tau_{n,1} = \{ x \in P_1 : \mu(P_1) - \tau \leq \frac{1}{m} \sum_{j=0}^{m-1} \chi_{P_1}(f^j x) \leq \mu(P_1) + \tau \text{ for all } m \geq n \}.
\]

Then the Birkhoff ergodic theorem gives \( \mu(\bigcup_{n \geq 1} A^\tau_{n,1}) = \mu(P_1) \). Since \( A^\tau_{1,1} \subset A^\tau_{2,1} \subset \cdots \), we have

\[
\lim_{n \to \infty} \mu(A^\tau_{n,1}) = \mu(P_1).
\]

Take \( \tau < \varepsilon/(2 + \varepsilon) \mu(P_1) \). Then for any \( x \in A^\tau_{n,1} \), by the definition of \( A^\tau_{n,1} \),

\[
\text{card}\{k \in [n, n + n\varepsilon] : f^k(x) \in P_1\} \geq (\mu(P_1) - \tau)(n + n\varepsilon) - (\mu(P_1) + \tau)n
\]

\[
= n(\mu(P_1)\varepsilon - 2\tau - \tau\varepsilon)
\]

\[
\geq 1,
\]
if \( n \) is taken large enough. Thus, \( x \in A_{n,1} \), that is, \( A^+_n \subseteq A_{n,1} \). Therefore, \( \lim_{n \to \infty} \mu(A_{n,1}) = \mu(P_1) \). Reproduce the proof above for every \( P_j \). Then we conclude

\[
\lim_{n \to \infty} \mu(A_n) = \mu(\Lambda_\delta). \tag{4.3}
\]

Since, by \([29, \text{Theorem 1.7}]\), \( \{x \in M : \overline{O^+(x)} = \text{supp}(\mu)\} \) has \( \mu \)-full measure, letting

\[
B_n = \{x \in \Lambda_\delta : \{f^k(x)\}_{1 \leq k \leq n} \text{ is } \varepsilon / 2 \text{-dense in } \text{supp}(\mu)\},
\]

we have \( \mu(B_n) \to \mu(\Lambda_\delta) \) as \( n \to +\infty \). Let

\[
C_n = \left\{ x \in \Lambda_\delta : \left| \frac{1}{m} \sum_{k=0}^{m-1} \varphi_j(f^k x) - \int \varphi_j \, d\mu \right| \leq \frac{\varepsilon}{4} \text{ for all } m \geq n, 1 \leq j \leq J \right\}.
\]

Then by Birkhoff ergodic theorem, \( \mu(\bigcup_{n \geq 1} C_n) = \mu(\Lambda_\delta) \). Since \( C_1 \subseteq C_2 \subseteq \cdots \), one has

\[
\lim_{n \to \infty} \mu(C_n) = \mu(\Lambda_\delta).
\]

Together with equation (4.3),

\[
\lim_{n \to \infty} \mu(\Lambda_{\delta,n}) = \lim_{n \to \infty} \mu(A_n \cap B_n \cap C_n) = \mu(\Lambda_\delta).
\]

This proves the claim. \( \square \)

Choose \( n > \max\{2 \log l / \varepsilon, N, 1 / \varepsilon \log t, 1 / 4 \varepsilon\} \) large enough such that \( \mu(\Lambda_{\delta,n}) > \frac{1}{2} \mu(\Lambda_\delta) = \bar{\delta} \), and \( n \varepsilon + 1 < e^{\varepsilon n} \), where the constant \((2 \log l / \varepsilon)\) comes from Lemma 3.5. Denote by \( E \) an \((n, 2\rho)\)-separated set of \( \Lambda_{\delta,n} \) of maximum cardinality. Then

\[
\bigcup_{x \in E} B_n(x, 2\rho) \supseteq \Lambda_{\delta,n}.
\]

By equation (4.1),

\[
\text{card}(E) \geq N_\mu(n, 2\rho, \mu(\Lambda_{\delta,n})) \geq N_\mu(n, 2\rho, \bar{\delta}) \geq e^{(h_{\mu}(f) - \varepsilon)n}.
\]

For \( k \in [n, n(1 + \varepsilon)] \), let \( F_k = \{x \in E : f^k(x) \in \mathcal{P}(x)\} \) and take \( m \in [n, n(1 + \varepsilon)] \) satisfying \( \text{card}(F_m) = \max\{\text{card}(F_k) : n \leq k \leq n(1 + \varepsilon)\} \). Then,

\[
\text{card}(F_m) \geq \frac{1}{n \varepsilon + 1} \text{card}(E) \geq \frac{1}{n \varepsilon + 1} e^{(h_{\mu}(f) - \varepsilon)n} \geq e^{(h_{\mu}(f) - 2\varepsilon)n}.
\]

Choose \( P \in \mathcal{P} \) satisfying \( \text{card}(F_m \cap P) = \max\{\text{card}(F_m \cap P_k) : 1 \leq k \leq t\} \). Then,

\[
\text{card}(F_m \cap P) \geq \frac{1}{t} \text{card}(F_m) \geq \frac{1}{t} e^{(h_{\mu}(f) - 2\varepsilon)n}.
\]

Without loss of generality, we may assume

\[
\frac{1}{t} e^{(h_{\mu}(f) - 2\varepsilon)n} \leq \text{card}(F_m \cap P) \leq e^{(h_{\mu}(f) + 2\varepsilon)n}. \tag{4.4}
\]

Otherwise, since \( e^{(h_{\mu}(f) + 2\varepsilon)n} - (1 / t) e^{(h_{\mu}(f) - 2\varepsilon)n} \geq e^{(h_{\mu}(f) - 2\varepsilon)n} (e^{4\varepsilon n} - 1) > 1 \), we can choose a subset of \( F_m \cap P \) such that equation (4.4) holds.

By the definition of the partition \( \mathcal{P} \), there exists \( q \in \{q_1, \ldots, q_t\} \), such that \( P \subseteq B(q, \beta) \cap \Lambda_\delta \). Thus, for any \( x \in F_m \cap P \), since \( x, f^m(x) \in B(q, \beta) \cap \Lambda_\delta \), by Definition 3.2, the connected component \( C(x, R(q) \cap f^{-m} R(q)) \) of \( R(q) \cap f^{-m} R(q) \)
containing \( x \) is an admissible \( s \)-rectangle in \( R(q) \), and \( f^m C(x, R(q) \cap f^{-m} R(q)) \) is an admissible \( u \)-rectangle in \( R(q) \).

We claim that if \( x_1, x_2 \in F_m \cap P \) with \( x_1 \neq x_2 \), then \( C(x_1, R(q) \cap f^{-m} R(q)) \cap C(x_2, R(q) \cap f^{-m} R(q)) = \emptyset \). Indeed, if there is \( y \in C(x_1, R(q) \cap f^{-m} R(q)) \cap C(x_2, R(q) \cap f^{-m} R(q)) \), by Definition 3.2, one sees \( d(f^k x_j, f^k y) \leq \rho \) for any \( 0 \leq k \leq m \) and \( j = 1, 2 \). Thus, \( d_m(x_1, x_2) \leq 2 \rho \). However, since \( F_m \cap P \) is an \((n, 2 \rho)\)-separated set, we obtain \( d_m(x_1, x_2) \geq d_n(x_1, x_2) > 2 \rho \), which is a contradiction. Therefore, there are at least \( \text{card}(F_m \cap P) \) disjoint \( s \)-rectangles in \( R(q) \), mapped by \( f^m \) to \( \text{card}(F_m \cap P) \) disjoint admissible \( u \)-rectangles in \( R(q) \).

Let

\[
\Lambda^* = \bigcap_{n \in \mathbb{Z}} f^{-mn} \left( \bigcup_{x \in F_m \cap P} C(x, R(q) \cap f^{-m} R(q)) \right).
\]

Then \( f^m \vert_{\Lambda^*} \) is conjugate to a full shift in \( \text{card}(F_m \cap P) \)-symbols. Moreover, for any \( y \in \Lambda^* \), any \( n \in \mathbb{Z} \), there exists \( x_n \in F_m \cap P \) such that \( f^{mn}(y) \in C(x_n, R(q) \cap f^{-m} R(q)) \). Note that the orbit of \( y \in \Lambda^* \) remains in the union of \( R(q), \ldots, R(f^m(q)) \). Let

\[
\Lambda = \Lambda^* \cup f(\Lambda^*) \cup \cdots \cup f^{m-1}(\Lambda^*).
\]

Then \( \Lambda \) is a hyperbolic horseshoe.

It remains to show the conclusions (i)–(iv) of Theorem 2.2 hold for this \( \Lambda \).

(i) Since

\[
h_{\text{top}}(f \vert_{\Lambda}) = \frac{1}{m} h_{\text{top}}(f^m \vert_{\Lambda^*}) = \frac{1}{m} \log \text{card}(F_m \cap P),
\]

by equation (4.4), \( h_{\text{top}}(f \vert_{\Lambda}) \geq -1/m \log t + n/m(h_\mu(f) - 2\epsilon) \). Since \((1/\epsilon) \log t < n \leq m \leq n(1 + \epsilon)\),

\[
h_{\text{top}}(f \vert_{\Lambda}) \geq -\epsilon + \frac{1}{1 + \epsilon}(h_\mu(f) - 2\epsilon) \geq h_\mu(f) - (h_\mu(f) + 3\epsilon).
\]

By equation (4.4), we also have

\[
h_{\text{top}}(f \vert_{\Lambda}) \leq \frac{n}{m}(h_\mu(f) + 2\epsilon) \leq h_\mu(f) + 2\epsilon.
\]

(ii) By the construction of \( \Lambda \) and the definition of \( R(q) \), for any \( y \in \Lambda \), there exist \( x \in F_m \cap P \), \( 0 \leq k \leq m - 1 \) such that \( d(y, f^k x) \leq \rho < \epsilon/2 \). Since \( x \in \Lambda_\delta \subset \text{supp}(\mu) \), we conclude \( \Lambda \) is contained in an \( \epsilon/2 \)-neighborhood of \( \text{supp}(\mu) \). However, fix any \( x_0 \in F_m \cap P \subset \Lambda_\delta,m \). By the construction of \( \Lambda_\delta,m \), for any \( z \in \text{supp}(\mu) \), there exists \( 1 \leq k \leq m \) such that \( d(z, f^k x_0) \leq \epsilon/2 \). Take \( y \in \Lambda^* \) such that \( y \in C(x_0, R(q) \cap f^{-m} R(q)) \), so then \( d(f^k x_0, f^k y) \leq \rho < \epsilon/2 \). Therefore, \( \text{supp}(\mu) \) is contained in an \( \epsilon \)-neighborhood of \( \Lambda \). Hence, \( d_H(\Lambda, \text{supp}(\mu)) \leq \epsilon \).

(iii) For any \( f \)-invariant measure \( \nu \) supported on \( \Lambda \), we may assume \( \nu \) is ergodic first. Since
\[
D(\mu, \nu) \leq \sum_{j=1}^{J} \frac{1}{2^j} + \frac{1}{2^{j-1}} \leq \sum_{j=1}^{J} \frac{1}{2^j} + \frac{\varepsilon}{4},
\]

it is enough to show: \(| \int \varphi_j \, d\mu - \int \varphi_j \, d\nu | \leq \frac{3}{4} \varepsilon \) for all \(1 \leq j \leq J\). Take \(y \in \Lambda^* \) and \(s \in \mathbb{N}\) large enough such that

\[
\left| \frac{1}{ms} \sum_{k=0}^{ms-1} \varphi_j(f^k y) - \int \varphi_j \, d\nu \right| \leq \frac{\varepsilon}{4}, \quad 1 \leq j \leq J.
\]

Then there exist \(x_0, x_1, \ldots, x_{s-1} \in F_m \cap P\) such that

\[
d(f^{mk+t} y, f^t x_k) \leq \rho \quad \text{for all} \ 0 \leq k \leq s-1, 0 \leq t \leq m-1.
\]

By equation (4.2) and the construction of \(\Lambda_{\delta, m}\), we obtain

\[
| \int \varphi_j \, d\mu - \int \varphi_j \, d\nu | \leq \frac{3}{4} \varepsilon, \quad 1 \leq j \leq J,
\]

which implies \(D(\mu, \nu) \leq \varepsilon\).

If \(\nu\) is not ergodic, by the ergodic decomposition theorem, \(\nu\)-a.e. ergodic component is supported on \(\Lambda\). Hence,

\[
D(\mu, \nu) = \infty \sum_{j=1}^{J} \frac{1}{2^j} \leq \sum_{j=1}^{J} \frac{1}{2^j} \leq \int D(\mu, \nu_x) \, d\nu(x) \leq \varepsilon.
\]

4.2. 

**Dominated splitting for cocycles.** The conclusion (iv) of Theorem 2.2 is contained in the following proposition.

**Proposition 4.1.** Under the condition of Theorem 2.2, for any \(i \in \mathbb{N} \setminus \{0\}\) and any \(0 < \varepsilon < \varepsilon_i\), let \(\Lambda\) be constructed as above. Then there exists a continuous \(A\)-invariant splitting on \(\Lambda\)

\[
\mathcal{X} = E_1(y) \oplus \cdots \oplus E_i(y) \oplus F_i(y),
\]

with \(\dim(E_j) = d_j \) for \(1 \leq j \leq i\), such that for any \(y \in \Lambda\), one has

\[
e^{(\overline{\lambda}_j - 4\varepsilon)m} \|u\| \leq \|A^m_y (u)\| \leq e^{(\overline{\lambda}_j + 4\varepsilon)m} \|u\| \quad \text{for all} \ u \in E_j(y), 1 \leq j \leq i,
\]

\[
\|A^m_y (v)\| \leq e^{(\overline{\lambda}_{i+1} + 4\varepsilon)m} \|v\| \quad \text{for all} \ v \in F_i(y).
\]

To prove this proposition, we require the following definitions. Let \(\mathcal{G}(\mathcal{X})\) denote the Grassmannian of closed subspaces of \(\mathcal{X}\), endowed with the Hausdorff metric \(d_H\), defined by

\[
d_H(E, F) = \max \left\{ \sup_{u \in S_E} \text{dist}(u, S_F), \sup_{v \in S_F} \text{dist}(v, S_E) \right\}
\]

for all \(E, F \in \mathcal{G}(\mathcal{X})\),

where \(S_F = \{v \in F : \|v\| = 1\}\), \(\text{dist}(u, S_F) = \inf\{\|u - v\| : v \in S_F\}\). Denote by \(\mathcal{G}_j(\mathcal{X}), \mathcal{G}^j(\mathcal{X})\) the Grassmannian of \(j\)-dimensional and \(j\)-codimensional closed subspaces,
respectively. Then by [15, Ch. IV. §2.1], $(\mathcal{G}(X), d_H)$ is a complete metric space, and $\mathcal{G}_j(X), \mathcal{G}^j(X)$ are closed in $\mathcal{G}(X)$. To compute $d_H$ conveniently, we introduce the gap $\hat{\delta}$, defined by

$$\hat{\delta}(E, F) = \max\left\{ \sup_{u \in S_E} \text{dist}(u, F), \sup_{v \in S_F} \text{dist}(v, E) \right\}$$

for all $E, F \in \mathcal{G}(X)$.

Note that the gap is not a metric on $\mathcal{G}(X)$, but

$$\hat{\delta}(E, F) \leq d_H(E, F) \leq 2\hat{\delta}(E, F).$$

See [15, Ch. IV. §2.1] for a proof.

Before proving Proposition 4.1, we give the following lemma first.

**Lemma 4.2.** Let $A \in L(X)$. If there exists a splitting

$$X = A(H) \oplus F$$

and $A|_H$ is a bijection, then

$$X = H \oplus A^{-1}(F).$$

**Proof.** Define $T : X \to X$ by $T = (A|_H)^{-1} \circ \pi_1 \circ A$, where $\pi_1$ is the projection associated with equation (4.5) onto $A(H)$ parallel to $F$. Since $A|_H$ is a bijection, $T$ is well defined. Then it follows from $T$ having image $H$ and kernel $A^{-1}(F)$ that the conclusion holds. □

We now prove Proposition 4.1.

**Proof of Proposition 4.1.** Fix any $i \in \mathbb{N}_{k_0}$, $0 < \varepsilon < \varepsilon_i$, and $1 \leq j \leq i$. We first assume $y \in \Lambda^*$, and then by the definition of $\Lambda^*$, there exist $\{x_n\}_{n \in \mathbb{Z}} \subset F_m$, such that

$$d(f^k x_n, f^k (f^{nm} y)) \leq \rho \cdot e^{-(\chi/2) \min\{k, m-k\}}$$

for all $n \in \mathbb{Z}$, $k = 0, \ldots, m$.

For any $n \in \mathbb{Z}$, let $X = H_j(x_n) \oplus F_j(x_n)$ be the Oseledets decomposition at $x_n$, where $H_j(x_n) = E_1(x_n) \oplus \cdots \oplus E_j(x_n)$. We first prove $\{A_{f^{-m}y}^m \tilde{H}_j(x_{-n})\}_{n \geq 1}$ is a Cauchy sequence in $\mathcal{G}(X)$. Denote $\tilde{H}_j(x_{-n}) = A_{f^{-m}y}^m H_j(x_{-n-1})$. Then,

$$A_{f^{-m}y}^m \tilde{H}_j(x_{-n}) = A_{f^{-m}y}^{m(n+1)} H_j(x_{-n-1}).$$

By Lemma 3.5, $\tilde{H}_j(x_{-n}) \subset U_j(x_{-n}, \theta_0)$, $A_{f^{-m}y}^m \tilde{H}_j(x_{-n}) \subset U_j(x_0, \theta_0)$, and $A_{f^{-m}y}^m$ is injective on $U_j(x_{-n}, \theta_0)$. Thus, we have

$$X = A_{f^{-m}y}^m \tilde{H}_j(x_{-n}) \oplus F_j(x_0).$$

(4.6)

Now, for any $u \in H_j(x_{-n})$. 
By equation (4.6), for $A_{f^{m(n+1)y}}^n(x_n-1) \in \mathcal{X}$, there exists $v_n \in \tilde{H}_j(x_n)$ such that

$$A_{f^{m(n+1)y}}^n \left( \frac{u}{\|u\|} \right) - A_{f^{m(n+1)y}}^n(v_n) \in F_j(x_0) \subset V_j(x_0, \theta_0).$$

Then it follows from equation (4.7) and Lemma 3.5 that

$$\text{dist} \left( \frac{A_{f^{m(n+1)y}}^n(u)}{\|A_{f^{m(n+1)y}}^n(u)\|}, A_{f^{m(n+1)y}}^n \right) \leq e^{(\lambda_j+1-\lambda_j+8\varepsilon)mn} \cdot c.$$  

CLAIM. There exists $c = c(\delta) > 0$, such that $\|u/\|u\| - v_n\| \leq c$.

Proof of the claim. Using equation (4.6) and Lemma 4.2, we obtain

$$\mathcal{X} = \tilde{H}_j(x_n) \oplus A_{y}^{-mn} F_j(x_0).$$  

Let $\pi_2$ be the projection operator associated with equation (4.8) onto $A_{y}^{-mn} F_j(x_0)$ parallel to $\tilde{H}_j(x_n)$. Then,

$$\frac{u}{\|u\|} - v_n = \pi_2 \left( \frac{u}{\|u\|} \right) \in A_{y}^{-mn} F_j(x_0).$$

Since the splitting $\mathcal{X} = H_j(x) \oplus F_j(x)$ is uniformly continuous on $\mathcal{R}_\delta$, there exists $c = c(\delta) > 0$ such that for any $x \in \mathcal{R}_\delta$ and any $H \subset U_j(x, \theta_0), F \subset V_j(x, \theta_0)$ with $\mathcal{X} = H \oplus F$, one has

$$\|\pi^H\| \leq c, \|\pi^F\| \leq c,$$  

where $\pi^H, \pi^F$ are the projections associated with the splitting $\mathcal{X} = H \oplus F$. Therefore,

$$\left\| \frac{u}{\|u\|} - v_n \right\| = \left\| \pi_2 \left( \frac{u}{\|u\|} \right) \right\| \leq c.$$  

This proves the claim. 

The claim gives

$$\text{dist} \left( \frac{A_{f^{m(n+1)y}}^n(u)}{\|A_{f^{m(n+1)y}}^n(u)\|}, A_{f^{m(n+1)y}}^n \right) \leq e^{(\lambda_j+1-\lambda_j+8\varepsilon)mn} \cdot c.$$
Similarly, for any \( v \in H_j(x_{n-1}) \), we have
\[
\text{dist}
\left(\frac{A_{m(n+1)}^{m+1}(x)}{\|A_{m(n+1)}^{m+1}(x)\|}, A_{m(n+1)}^{m+1}H_j(x_n)\right) \leq e^{(\lambda_j+\lambda_j+4\epsilon mn)} \cdot c.
\]
Thus,
\[
\tilde{\delta}(A_{m(n+1)}^{m+1}H_j(x_{n-1}), A_{m(n+1)}^{m+1}H_j(x_{n-1})) \leq e^{(\lambda_j+\lambda_j+4\epsilon mn)} \cdot c,
\]
which implies \( \{A_{m(n+1)}^{m(n+1)}H_j(x_{n-1})\}_{n \geq 1} \) is a Cauchy sequence in \((G(X), d_H)\).

Now we prove \( \{A_{m(n+1)}^{m(n+1)}F_j(x_{n-1})\}_{n \geq 1} \) is a Cauchy sequence in \((G(X), d_H)\). By Lemma 3.5, \( A_{m(n+1)}^{m(n+1)}H_j(x_0) \subset U_j(x_n, \theta_0) \) and \( A_{m(n+1)}^{m(n+1)} \) is injective on \( H_j(x_0) \). Thus, \( X = A_{m(n+1)}^{m(n+1)}H_j(x_0) \oplus F_j(x_n) \). Using Lemma 4.2, we obtain
\[
X = H_j(x_0) \oplus A_{m(n+1)}^{m(n+1)}F_j(x_n) \quad \text{(4.10)}
\]
For any \( u \in A_{m(n+1)}^{m(n+1)}F_j(x_{n+1}) \) with \( \|u\| = 1 \), choose \( v_n \in A_{m(n+1)}^{m(n+1)}F_j(x_n) \) such that \( u - v_n \in H_j(x_0) \).

Then, by Lemma 3.5,
\[
\|A_{m(n+1)}^{m(n+1)}(u - v_n)\| \geq e^{(\lambda_j - 4\epsilon mn)} \|u - v_n\|
\geq e^{(\lambda_j - 4\epsilon mn)} \cdot \inf_{v \in A_{m(n+1)}^{m(n+1)}F_j(x_n)} \|u - v\|
= e^{(\lambda_j - 4\epsilon mn)} \cdot \text{dist}(u, A_{m(n+1)}^{m(n+1)}F_j(x_n)). \tag{4.11}
\]
However, since \( u \in A_{m(n+1)}^{m(n+1)}F_j(x_{n+1}) \subset A_{m(n+1)}^{m(n+1)}V_j(x_n, \theta_0) \) and \( v_n \in A_{m(n+1)}^{m(n+1)}F_j(x_n) \subset A_{m(n+1)}^{m(n+1)}V_j(x_n, \theta_0) \), Lemma 3.5 gives
\[
\|A_{m(n+1)}^{m(n+1)}(u - v_n)\| \leq \|A_{m(n+1)}^{m(n+1)}(u)\| + \|A_{m(n+1)}^{m(n+1)}(v_n)\|
\leq e^{(\lambda_j+1+4\epsilon mn)} + e^{(\lambda_j+1+4\epsilon mn)} \|v_n\|
= e^{(\lambda_j+1+4\epsilon mn)} + e^{(\lambda_j+1+4\epsilon mn)} \|\pi^F(u)\|
\leq e^{(\lambda_j+1+4\epsilon mn)} \cdot (1 + c), \tag{4.12}
\]
where \( \pi^F \) is the projection associated with equation (4.10) onto \( A_{m(n+1)}^{m(n+1)}F_j(x_n) \) parallel to \( H_j(x_0) \) and the constant \( c \) is given by equation (4.9). Therefore, equations (4.11) and (4.12) show that for any \( u \in A_{m(n+1)}^{m(n+1)}F_j(x_{n+1}) \) with \( \|u\| = 1 \),
\[
\text{dist}(u, A_{m(n+1)}^{m(n+1)}F_j(x_n)) \leq e^{(\lambda_j+1+\lambda_j+8\epsilon mn)} \cdot (1 + c).
\]
Similarly, for any \( v \in A_{m(n+1)}^{m(n+1)}F_j(x_n) \) with \( \|v\| = 1 \), we have
\[
\text{dist}(v, A_{m(n+1)}^{m(n+1)}F_j(x_{n+1})) \leq e^{(\lambda_j+1+\lambda_j+8\epsilon mn)} \cdot (1 + c).
\]
Thus,
\[
\hat{\delta}(A_{j}^{-m}F_{j}(x_{n}), A_{f_{m}^{n+1}y}^{-m(n+1)}F_{j}(x_{n+1})) \leq e^{(\lambda_j + 1 - \lambda_j + 8\varepsilon)mn} \cdot (1 + c),
\]
which implies \(\{A_{j}^{-m}F_{j}(x_{n})\}_{n \geq 1}\) is a Cauchy sequence in \((G(\mathcal{X}), d_{H})\).

Let
\[
H_{j}(y) := \lim_{n \to \infty} A_{j}^{-m}H_{j}(x_{-n}), \quad F_{j}(y) := \lim_{n \to \infty} A_{j}^{-m}F_{j}(x_{n}).
\]
Since \(X = A_{y}^{m}H_{j}(x_{0}) \oplus F_{j}(x_{n})\), by Lemma 4.2, \(X = H_{j}(x_{0}) \oplus A_{j}^{-m}F_{j}(x_{n})\). Thus,
\[
\operatorname{codim}(A_{j}^{-m}F_{j}(x_{n})) = D_{j} = \dim(A_{j}^{-m}H_{j}(x_{-n})),
\]
where \(D_{j} = d_{1} + \cdots + d_{j}\). It follows from \(G_{D_{j}}(\mathcal{X})\) and \(G^{D_{j}}(\mathcal{X})\) being closed subsets of \(G(\mathcal{X})\) that \(H_{j}(y) \in G_{D_{j}}(\mathcal{X})\), \(F_{j}(y) \in G^{D_{j}}(\mathcal{X})\). Notice that \(H_{j}(y) \subset U_{j}(x_{0}, \theta_{0})\) and \(F_{j}(y) \subset A_{j}^{-m}V_{j}(x_{1}, \theta_{0}) \subset V_{j}(x_{0}, \theta_{0})\). We conclude
\[
\mathcal{X} = H_{j}(y) \oplus F_{j}(y).
\]
Additionally, by Lemma 3.5, we have
\[
e^{(\lambda_j - 4\varepsilon)m}\|u\| \leq \|A_{y}^{m}(u)\| \leq e^{(\lambda_{1} + 4\varepsilon)m}\|u\| \quad \text{for all } u \in H_{j}(y),
\]
\[
\|A_{y}^{m}(v)\| \leq e^{(\lambda_j + 4\varepsilon)m}\|v\| \quad \text{for all } v \in F_{j}(y).
\]
In general, for any \(z \in \Lambda\), there exists \(0 \leq k \leq m - 1\) and \(y \in \Lambda^{1}\) such that \(z = f^{k}y\). Similar to the proof above, we can also get
\[
H_{j}(z) := \lim_{n \to \infty} A_{j}^{-m}H_{j}(f^{k}x_{n}),
\]
\[
F_{j}(z) := \lim_{n \to \infty} A_{j}^{-m}F_{j}(f^{k}x_{n}),
\]
satisfying \(\mathcal{X} = H_{j}(z) \oplus F_{j}(z), H_{j}(z) \in G_{D_{j}}(\mathcal{X}), F_{j}(z) \in G^{D_{j}}(\mathcal{X})\), and
\[
e^{(\lambda_j - 4\varepsilon)m}\|u\| \leq \|A_{z}^{m}(u)\| \leq e^{(\lambda_{1} + 4\varepsilon)m}\|u\| \quad \text{for all } u \in H_{j}(z), \quad (4.13)
\]
\[
\|A_{z}^{m}(v)\| \leq e^{(\lambda_j + 4\varepsilon)m}\|v\| \quad \text{for all } v \in F_{j}(z). \quad (4.14)
\]
CLAIM. The splitting \(\mathcal{X} = H_{j}(z) \oplus F_{j}(z)\) is \(\mathcal{A}\)-invariant on \(\Lambda\).

Proof of the claim. Before proving the invariance of the splitting, we show that for any \(y \in \Lambda^{1}, 0 \leq k \leq m - 1\) and any \(H_{n,k} \in G_{D_{j}}(\mathcal{X})\) satisfying \(H_{n,k} \subset U_{j}(f^{k}x_{-n}, \theta_{0})\), one has
\[
H_{j}(f^{k}y) = \lim_{n \to \infty} A_{j}^{-m}H_{n,k}.
\]
Indeed, recall that \( X = A_{f_{mn}}(f_{ky}) \mathcal{H}_{m,k} \oplus F_j(\chi_0) \). Thus for any \( u \in H_j(f^k x_{-n}) \), similar to the estimation above, we have

\[
\text{dist}
\left(
\frac{A_{f_{mn}}(f_{ky})(u)}{\|A_{f_{mn}}(f_{ky})(u)\|}, A_{f_{mn}}(f_{ky}) \mathcal{H}_{m,k}
\right)
= \frac{\|u\|}{\|A_{f_{mn}}(f_{ky})(u)\|} \cdot \inf_{v \in \mathcal{H}_{m,k}} \left| A_{f_{mn}}(f_{ky}) \left( \frac{u}{\|u\|} - v \right) \right|
\leq e^{(\lambda_j+1-\lambda_j+8\varepsilon)\lambda_j} c, \]

which implies \( d_H(A_{f_{mn}}(f_{ky}) \mathcal{H}(x_{-n}), A_{f_{mn}}(f_{ky}) \mathcal{H}_{m,k}) \to 0 \). Hence,

\[
H_j(f^k y) = \lim_{n \to \infty} A_{f_{mn}}(f_{ky}) \mathcal{H}_{m,k} \text{ for all } 0 \leq k \leq m - 1. \tag{4.15}
\]

Now for any \( z \in \Lambda \), we may assume \( z = f^k(y) \) for some \( y \in \Lambda^* \) and \( 0 \leq k \leq m - 1 \). If \( 0 \leq k \leq m - 2 \), then by equation (3.9), \( A(f_{-mn}z) \mathcal{H}(f^k x_{-n}) \subset U_j(f^{k+1} x_{-n}, \theta_0) \); if \( k = m - 1 \), then by equation (3.17), \( A(f_{-mn}z) \mathcal{H}(f^{m-1} x_{-n}) \subset U_j(x_{-n+1}, \theta_0) \). Thus, equation (4.15) gives

\[
H_j(f z) = \lim_{n \to \infty} A_{f_{mn}}(f_{ky}) A(f_{-mn}z) \mathcal{H}(f^k x_{-n}).
\]

Therefore, for any \( u \in H_j(z) \),

\[
\text{dist}(A(z)u, A_{f_{mn}}(f_{ky}) A(f_{-mn}z) \mathcal{H}(f^k x_{-n})) \leq \|A(z)\| \cdot \text{dist}(u, A_{f_{mn}}(f_{ky}) \mathcal{H}(f^k x_{-n})) \leq \|A(z)\| \cdot d_H(\mathcal{H}(f z), A_{f_{mn}}(f_{ky}) \mathcal{H}(f^k x_{-n})) \to 0.
\]

which implies \( A(z)u \in H_j(f z) \). Thus, \( A(z) \mathcal{H}(z) \subset H_j(f z) \). It can be proved analogously that \( A(z) F_j(z) \subset F_j(f z) \). Since \( \dim(A(z) \mathcal{H}(z)) = \dim(H_j(f z)) \), we have \( A(z) \mathcal{H}(z) = H_j(f z) \). Then by the remark in §2.1, the splitting \( \mathcal{X} = H_j(z) \oplus F_j(z) \) is \( \mathcal{A} \)-invariant on \( \Lambda \).

**CLAIM.** The splitting \( \mathcal{X} = H_j(z) \oplus F_j(z) \) is continuous on \( \Lambda \).

**Proof of the claim.** Let \( \pi_{H_j}^z, \pi_{F_j}^z \) be the projection operators associated with \( \mathcal{X} = H_j(z) \oplus F_j(z) \). Then for any \( \tilde{z} \in \Lambda, u \in H_j(\tilde{z}) \), let \( w = A_{f_{mn}}^{-1}(u) \in H_j(f^{-mn} \tilde{z}) \), by the invariance of the splitting and equations (4.9), (4.13), and (4.14), we may estimate

\[
\|\pi_{F_j}^z A_{f_{mn}z}^{-1}(w)\| = \|A_{f_{mn}}^{-1} z \pi_{f_{mn}z}^z \| \pi_{F_j}^z \| w \|
\leq e^{(\lambda_j+1+4\varepsilon)\lambda_j} \cdot \| w \|
\leq c \cdot e^{(\lambda_j+1+4\varepsilon)\lambda_j} \| u \|
\leq c \cdot e^{(\lambda_j+1+4\varepsilon)\lambda_j} \| u \|.
\]
Thus,

\[
\|\pi_z^{F_j}(u)\| = \|\pi_z^{F_j} A_{f^{-mn}z}^{mn}(w)\|
\leq \|\pi_z^{F_j} (A_{f^{-mn}z}^{mn} - A_{f^{-mn}z}^{mn})(w)\| + \|\pi_z^{F_j} A_{f^{-mn}z}^{mn}(w)\|
\leq c \cdot (e^{(4\epsilon - \lambda_j)mn} \|A_{f^{-mn}z}^{mn} - A_{f^{-mn}z}^{mn}\| + e^{(\ell_j - 1 - \lambda_j + 8\epsilon)mn}) \|u\|,
\]

which gives

\[
\|\pi_z^{F_j}|_{H_j(z)}\| \leq c \cdot e^{(4\epsilon - \lambda_j)mn} \|A_{f^{-mn}z}^{mn} - A_{f^{-mn}z}^{mn}\| + c \cdot e^{(\ell_j - 1 - \lambda_j + 8\epsilon)mn}.
\]

Similarly, for \( v \in F_j(\tilde{z}) \),

\[
e^{(\lambda_j - 4\epsilon)mn} \|\pi_z^{H_j}(v)\| \leq \|A_{f^{-mn}z}^{mn} \circ \pi_z^{H_j}(v)\|
= \|\pi_z^{H_j} \circ A_{f^{-mn}z}^{mn}(v)\|
\leq \|\pi_z^{H_j} \| \cdot (\|A_{f^{-mn}z}^{mn}(v) - A_{f^{-mn}z}^{mn}(v)\| + \|A_{f^{-mn}z}^{mn}(v)\|)
\leq c \cdot (\|A_{f^{-mn}z}^{mn} - A_{f^{-mn}z}^{mn}\| + e^{(\lambda_j + 4\epsilon)mn}) \|v\|.
\]

Therefore,

\[
\|\pi_z^{H_j}|_{F_j(\tilde{z})}\| \leq c \cdot e^{(4\epsilon - \lambda_j)mn} \|A_{f^{-mn}z}^{mn} - A_{f^{-mn}z}^{mn}\| + c \cdot e^{(\ell_j - 1 - \lambda_j + 8\epsilon)mn}.
\]

Now for any \( \tau > 0 \), take \( n \) large enough such that \( c \cdot e^{(\ell_j - 1 - \lambda_j + 8\epsilon)mn} < \tau \), and then take \( \delta > 0 \) small enough such that for any \( \tilde{z} \in B(z, \delta) \),

\[
c \cdot e^{(4\epsilon - \lambda_j)mn} \cdot \max\{\|A_{f^{-mn}z}^{mn} - A_{f^{-mn}z}^{mn}\|, \|A_{f^{-mn}z}^{mn} - A_{f^{-mn}z}^{mn}\|\} < \tau.
\]

Then we have \( \|\pi_z^{H_j}|_{F_j(\tilde{z})}\| \leq 2\tau, \|\pi_z^{F_j}|_{H_j(\tilde{z})}\| \leq 2\tau \). Thus by [4, Remark 10],

\[
d_H(F_j(z), F_j(\tilde{z})) \leq 4D_j \|\pi_z^{H_j}|_{F_j(\tilde{z})}\| \leq 8D_j \tau,
\]

\[
d_H(H_j(z), H_j(\tilde{z})) \leq 4D_j \|\pi_z^{F_j}|_{H_j(\tilde{z})}\| \leq 8D_2 \tau,
\]

which gives the continuity of \( F_j(z) \) and \( H_j(z) \).

At last, for any \( z \in \Lambda \), \( 1 \leq j \leq i \), let \( E_j(z) := H_j(z) \cap F_{j-1}(z) \), where \( F_0(z) := \mathcal{X} \). Then, \( \dim(E_j) = d_j \). Since

\[
A(z)E_j(z) \subset A(z)H_j(z) \cap A(z)F_{j-1}(z) \subset H_j(fz) \cap F_{j-1}(fz) = E_j(fz)
\]

and \( \dim(A(z)E_j(z)) = d_j = \dim(E_j(fz)) \), we have \( A(z)E_j(z) = E_j(fz) \). Therefore, we obtain a continuous \( A \)-invariant splitting on \( \Lambda \):

\[
\mathcal{X} = E_1(z) \oplus \cdots \oplus E_i(z) \oplus F_i(z),
\]

with \( \dim(E_j) = d_j \) for all \( 1 \leq j \leq i \). Additionally, by equations (4.13) and (4.14), we conclude
\[ e^{(\lambda_j - 4\varepsilon)m} \| u \| \leq \| A^n_m(u) \| \leq e^{(\lambda_j + 4\varepsilon)m} \| u \| \quad \text{for all } u \in E_j(z), \ 1 \leq j \leq i, \]
\[ \| A^n_m(v) \| \leq e^{(\lambda_i+1+4\varepsilon)m} \| v \| \quad \text{for all } v \in F_i(z). \]

This completes the proof. \[ \square \]

5. Proof of Theorem 2.4

We begin by estimating the growth rate of the volume function \( \sigma_k(A^n_m) \), where \( \sigma_k(A^n_m) \) is the singular value of index \( k \) of \( A^n_m \) which is given in §2.3.

**Lemma 5.1.** Under the assumption of Theorem 2.1, for any \( k \geq 1 \) and \( \mu \)-a.e. \( x \in M \), one has
\[ \lim_{n \to \infty} \frac{1}{n} \log \sigma_k(A^n_m) = \chi_k(A, \mu), \]
where \( \chi_k(A, \mu) = \lambda_i \) if \( \dim(E_1 \oplus \cdots \oplus E_{i-1}) < k \leq \dim(E_1 \oplus \cdots \oplus E_i) \), that is, \( \chi_1(A, \mu) \geq \chi_2(A, \mu) \geq \cdots \) are the Lyapunov exponents of \( A \) with respect to \( \mu \) counted with multiplicities.

**Proof.** Suppose that \( \dim(E_1 \oplus \cdots \oplus E_{i-1}) < k \leq \dim(E_1 \oplus \cdots \oplus E_i) \). Denote \( \tilde{E}_i(x) = E_1(x) \oplus \cdots \oplus E_i(x) \) and let \( \mathcal{X} = \tilde{E}_i(x) \oplus F_i(x) \) be the Oseledets splitting given by Theorem 2.1. Let \( \pi_E(x) \) be the projection associated with \( \mathcal{X} = \tilde{E}_i(x) \oplus F_i(x) \) onto \( \tilde{E}_i(x) \) parallel to \( F_i(x) \). To prove \( \lim_{n \to \infty} (1/n) \log \sigma_k(A^n_m) \leq \chi_k(A, \mu) \), for any subspace \( V \subset \mathcal{X} \) with \( \dim(V) = k \), we divide the proof into two cases: \( \dim(\pi_E(x)(V)) = k \) and \( \dim(\pi_E(x)(V)) < k \).

**Case I:** \( \dim(\pi_E(x)(V)) = k \). Since \( k > \dim(E_1 \oplus \cdots \oplus E_{i-1}) \), there exists \( v_0 \in V \) with \( \| v_0 \| = 1 \) such that \( v_E := \pi_E(x)(v_0) \in E_i(x) \). Let \( v_0 = v_E + v_F \). Then
\[
\inf_{v \in V \setminus \{0\}} \frac{\| A^n_m(v) \|}{\| v \|} \leq \| A^n_m(v_0) \| \leq \| A^n_m(v_E) \| + \| A^n_m(v_F) \|
\leq \| A^n_m|_{E_i(x)} \| \cdot \| \pi_E(x)(v_0) \| + \| A^n_m|_{F_i(x)} \| \cdot \| \pi_F(x)(v_0) \|
\leq \| A^n_m|_{E_i(x)} \| \cdot \| \pi_E(x) \| + \| A^n_m|_{F_i(x)} \| \cdot \| \pi_F(x) \|. \]

By Theorem 2.1, for any \( \varepsilon > 0 \), there exist \( N \geq 1 \) such that for any \( n \geq N \),
\[
\| A^n_m|_{E_i(x)} \| \cdot \| \pi_E(x) \| + \| A^n_m|_{F_i(x)} \| \cdot \| \pi_F(x) \|
\leq e^{(\lambda_j+\varepsilon)n} \cdot \| \pi_E(x) \| + e^{(\lambda_i+1+\varepsilon)n} \cdot \| \pi_F(x) \|
= e^{(\lambda_j+\varepsilon)n} (\| \pi_E(x) \| + e^{(\lambda_i+1-\lambda_j)n} \cdot \| \pi_F(x) \|)
\leq e^{(\lambda_i+2\varepsilon)n}.
\]

Hence,
\[
\inf_{v \in V \setminus \{0\}} \frac{\| A^n_m(v) \|}{\| v \|} \leq e^{(\lambda_i+2\varepsilon)n} \quad \text{for all } n \geq N.
\]
Case II: \( \dim(\pi_E(x)(V)) < k \). Then there exists \( 0 \neq v_1 \in F_i(x) \cap V \). Thus,
\[
\inf_{v \in V \setminus \{0\}} \frac{\|A^n_x(v)\|}{\|v\|} \leq \frac{\|A^n_x(v_1)\|}{\|v_1\|} \leq \|A^n_{F_i(x)}\| \leq e^{(\lambda_i+\varepsilon)n} < e^{(\lambda_i+\varepsilon)n} \quad \text{for all } n \geq N.
\]
Therefore, we conclude that
\[
\limsup_{n \to \infty} \frac{1}{n} \log \sigma_k(A^n_x) = \limsup_{n \to \infty} \frac{1}{n} \log \left( \sup_{\dim(V) = k} \inf_{v \in V \setminus \{0\}} \frac{\|A^n_x(v)\|}{\|v\|} \right) \leq \lambda_i + 2\varepsilon.
\]
Since \( \varepsilon > 0 \) is arbitrary,
\[
\limsup_{n \to \infty} \frac{1}{n} \log \sigma_k(A^n_x) \leq \lambda_i = \chi_k(A, \mu).
\]
However, choose \( V_0 \subset \tilde{E}_i(x) \) with \( \dim(V_0) = k \). Then,
\[
\sigma_k(A^n_x) = \sup_{\dim(V) = k} \inf_{v \in V \setminus \{0\}} \frac{\|A^n_x(v)\|}{\|v\|} \geq \inf_{v \in V_0 \setminus \{0\}} \frac{\|A^n_x(v)\|}{\|v\|}
\geq \inf_{v \in \tilde{E}_i(x) \setminus \{0\}} \frac{\|A^n_x(v)\|}{\|v\|}.
\]
Similar to the proof of Thieullen’s paper [28, pp. 68–69], for any \( \varepsilon > 0 \), there exists a measurable function \( C_\varepsilon(x) \) such that
\[
\|A^n_x(v)\| \geq C_\varepsilon(x) \varepsilon \|v\| \quad \text{for all } v \in \tilde{E}_i(x).
\]
It follows that
\[
\liminf_{n \to \infty} \frac{1}{n} \log \inf_{v \in \tilde{E}_i(x) \setminus \{0\}} \frac{\|A^n_x(v)\|}{\|v\|} \geq \lambda_i - \varepsilon.
\]
Since \( \varepsilon > 0 \) is arbitrary, we conclude
\[
\liminf_{n \to \infty} \frac{1}{n} \log \sigma_k(A^n_x) \geq \lambda_i = \chi_k(A, \mu).
\]
This completes the proof. \( \square \)

We now prove Theorem 2.4.

Proof of Theorem 2.4. By [2, Proposition 2.1] and [32, Lemma 2],
\[
P(A, s) = \lim_{n \to \infty} P \left( f, \frac{1}{n} \phi^s(A(\cdot, n)) \right) = \inf_{n \geq 1} P \left( f, \frac{1}{n} \phi^s(A(\cdot, n)) \right),
\]
where \( \phi^s(A(\cdot, n)) = \phi^s(\cdot, n) \) is given by equation (2.3). For every \( n \in \mathbb{N} \), since the map
\[
A \mapsto P \left( f, \frac{1}{n} \phi^s(A(\cdot, n)) \right)
\]
is continuous at \( A \), it gives that \( A \mapsto P(A, s) \) is upper semi-continuous at \( A \).

Therefore, it suffices to prove the lower semi-continuity. Since the entropy map \( \mu \mapsto h_\mu(f) \) is upper semi-continuous, by the variational principle of sub-additive topological pressure (Theorem 2.3), there exists an ergodic measure \( \mu \in \mathcal{E}(M, f) \) such that
\[
P(A, s) = h_\mu(f) + \mathcal{F}_s(\Phi, \mu), \quad (5.1)
\]
where $\mathcal{F}_n(\Phi, \mu) = \lim_{n \to \infty} (1/n) \int \phi^t(A(x, n))d\mu$. By equations (2.2), (2.3), and Lemma 5.1, one has

$$\mathcal{F}_n(\Phi, \mu) = \chi_{1}(A, \mu) + \cdots + \chi_{[s]}(A, \mu) + (s - [s])\chi_{[s]+1}(A, \mu).$$

For any $\varepsilon > 0$ and $i > s$, by Theorem 2.2, one can construct a horseshoe $\Lambda_{\varepsilon}$ such that

1. $h_{\text{top}}(f|_{\Lambda_{\varepsilon}}) > h_{\mu}(f) - \varepsilon$;
2. there is an $\mathcal{A}$-invariant splitting over $\Lambda_{\varepsilon}$,
   $$\mathcal{X} = E_{1}(x) \oplus \cdots \oplus E_{i}(x) \oplus F_{i}(x);$$
3. for any $x \in \Lambda_{\varepsilon}$, we have
   $$\|A^{m}_{x}(u)\| \geq e^{(\lambda_{j} - \varepsilon)m}\|u\| \quad \text{for all } u \in E_{j}(x), 1 \leq j \leq i,$$
where $\lambda_{1} \geq \lambda_{2} \geq \cdots$ are the Lyapunov exponents of $\mathcal{A}$ with respect to $\mu$.

Therefore, if $B : M \to L(\mathcal{X})$ is close to $A$, then:

1. there is an $\mathcal{B}$-invariant splitting over $\Lambda_{\varepsilon}$,
   $$\mathcal{X} = E'_{1}(x) \oplus \cdots \oplus E'_{i}(x) \oplus F'_{i}(x);$$
2. for any $x \in \Lambda_{\varepsilon}$, we have
   $$\|B^{m}_{x}(u)\| \geq e^{(\lambda_{j} - 2\varepsilon)m}\|u\| \quad \text{for all } u \in E'_{j}(x), 1 \leq j \leq i.$$

Take an $f$-invariant ergodic measure $\nu$ on $\Lambda_{\varepsilon}$ such that $h_{\nu}(f) = h_{\text{top}}(f|_{\Lambda_{\varepsilon}})$. Then by the proof of Lemma 5.1, for $\nu$-a.e. $x \in M$,

$$\lim inf_{n \to \infty} \frac{1}{n} \log \sigma_{k}(B^{n}_{x}) \geq \chi_{k}(A, \mu) - 2\varepsilon.$$

Thus, we conclude that

$$P(B, s) \geq P_{\Lambda_{\varepsilon}}(B, s) \geq h_{\nu}(f) + \lim_{n \to \infty} \frac{1}{n} \int \phi^t(B(x, n))d\nu \geq h_{\mu}(f) - \varepsilon + [\chi_{1}(A, \mu) + \cdots + \chi_{[s]}(A, \mu) + (s - [s])\chi_{[s]+1}(A, \mu)] - 2s\varepsilon \geq P(A, s) - (2s + 1)\varepsilon,$$

where $P_{\Lambda_{\varepsilon}}(B, s)$ denotes the topological pressure of $f$ restricted on $\Lambda_{\varepsilon}$. This yields the lower semi-continuity of the topological pressure. \hfill $\square$

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Horseshoes and Lyapunov exponents for Banach cocycles

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