Energy Density and Pressure of Long Wavelength Gravitational Waves

L. R. Abramo∗†

Department of Physics
University of Florida
Gainesville, FL 32611 USA

ABSTRACT

Inflation leads us to expect a spectrum of gravitational waves (tensor perturbations) extending to wavelengths much bigger than the present observable horizon. Although these gravity waves are not directly observable, the energy density that they contribute grows in importance during the radiation- and dust-dominated ages of the universe. We show that the back reaction of tensor perturbations during matter domination is limited from above, since gravitational waves of wavelength $\lambda$ have a share of the total energy density $\Delta \rho(\lambda)/\rho$ during matter domination that is at most equal to the share of the total energy density that they had when the mode $\lambda$ exited the Hubble radius $H^{-1}$ during inflation. This work is to be contrasted to that of Sahni[1], who studied the energy density of gravity waves only insofar as their wavelengths are smaller than $H^{-1}$. Such a cut-off in the spectral energy of gravity waves leads to the breakdown of energy conservation, and we show that this anomaly is eliminated simply by taking into account the energy density and pressure of long wavelength gravitational waves as well as short wavelength ones.

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∗ Address after June 1, 1999: Theoretische Physik, Ludwig Maximilians Universität, Theresienstr. 37, D-80333 MÜNCHEN, GERMANY
† e-mail: abramo@phys.ufl.edu
1 Introduction

Gravitational waves are quantum-mechanically produced in all inflationary scenarios \[2, 3, 4, 5\]. In some models gravitational waves are a crucial component of the cosmic microwave background \[6\], and the prospects that gravitational waves generated during inflation can be directly observed in the future using space-based interferometry are not beyond hope \[7\].

Scales that are now entering our Hubble radius \((H_0^{-1} \approx 10^{28}\text{ cm})\) correspond to physical wavelengths that were equal to the Hubble radius around 65 e-foldings of the scale factor before the end of inflation. Gravity waves that were generated earlier than that time during inflation are today larger than the present Hubble radius and vice-versa. Therefore, unless we fine-tune inflation to happen for no more than about 65 e-foldings, there must be gravitational waves (as well as density perturbations) with physical wavelengths much bigger than \(10^{28}\text{ cm}\).

Obviously, these long wavelength gravitational waves cannot be directly detected, nor have they any impact on observables such as the CMB. Even if the spectrum of gravitational waves on sub-horizon scales is measured, we still could only guess what the spectrum might look like for the super-horizon modes.

Nevertheless, cosmological perturbations of long wavelengths can have an impact on the background in which they propagate, through their self-energy and their gravitational interactions. Tsamis and Woodard \[8\], for example, have investigated the feedback from quantum mechanical pair production in pure gravity with a cosmological constant, and found that the two loop back reaction of the metric fluctuations have the effect of screening the cosmological constant. The one loop back reaction of fluctuations during scalar-field inflation has also been studied by the present author and collaborators, and in this case we found that in some models of inflation the expansion rate of the universe slows down faster due to these feedback effects \[9, 10, 11\].

There is a very simple physical picture for these processes \[8\]: during inflation, virtual pairs are created all the time, and eventually some of them become trapped in the expansion of the universe. As the pair is pulled apart by inflation, the gravitational potential that must exist between the pair fills the intervening space. Even after the pair becomes causally disconnected, the gravitational potential still remains, just as the potential of a particle that falls into a black hole remains after the particle crosses the hole’s horizon.
Since these gravitational interactions are attractive, we expect them to have a tiny impact in slowing the expansion of the universe, as they try to bring the pair back together. The only questions are what is the strength of back reaction, and what is its time dependence (whether back reaction effects grow or decrease in time, and if they can ever become important).

In this paper we examine the energy density and pressure engendered by gravitational waves at lowest order in perturbation theory (one loop). This effective energy-momentum tensor of gravitational waves, when plugged into Einstein’s field equations, is the source for the back reaction of the gravitational waves on the expansion rate of the homogeneous and isotropic universe. We analyze these effective terms during and after inflation, focusing on the contribution from long wavelength modes\textsuperscript{1}. Sahni\textsuperscript{1} considered the energy density due to a spectrum of gravitational waves extending to wavelengths much bigger than the Hubble radius, but in that work a gravitational wave mode is taken into account only if and when the mode becomes smaller than the Hubble radius $H^{-1}(t)$ – an idea first proposed, at least in the context of gravitational waves from inflation, by Allen\textsuperscript{4}. We show that this cut-off leads to the breakdown of energy-momentum conservation, or, equivalently, to a violation of the Bianchi identities. The most natural, and the simplest, way of avoiding this unconventional feature is to include the energy density and pressure of gravitational waves when their wavelengths are bigger than the Hubble radius.

The backgrounds considered for this work are flat Friedmann-Robertson-Walker space-times where the scale factor is a power-law of time, $a(t) \propto t^s$ with $s > 0$. However, we allow the value of $s$ to change during the evolution of the universe: we assume $s > 1$ during inflation ($t < 0$) and $s < 1$ afterwards ($t > 0$).

This paper is organized as follows: in section 2 we determine the perturbative background, then solve (exactly) the equations for the gravitational waves. In section 3 we discuss the nonlinear terms that give rise to back reaction and that were discarded in section 2, and show how gravity waves can impact the background. We also show that the back reaction of long

\textsuperscript{1}Short wavelength gravity waves have been extensively dealt with in the standard textbooks\textsuperscript{3}. The relevant fact in that limit is that the kinetic energy of short wavelength gravitons is much more important than their gravitational interactions, and the gravitons behave essentially like conformally invariant ultra-relativistic particles - i.e., their energy density falls like radiation, $a^{-4}(t)$, where $a(t)$ is the scale factor.
wavelength gravitational waves is not only consistent with, but in fact is demanded by conservation of energy and momentum. We employ the resulting formulas in section 4 for a generic power-law inflationary universe that “reheats” at $t = 0$, and show that the share of the total energy due to long wavelength modes grows during the decelerated expansion phase ($s < 1$), but is limited from above. We conclude in section 5.

## 2 Power-law backgrounds

Consider a flat Friedmann-Robertson-Walker metric,

$$ds^2 = -dt^2 + a^2(t) \delta \cdot d\vec{x},$$

where the scale factor is a power-law of time,

$$a(t) = \left(1 + \frac{H_i t}{s}\right)^s, \quad s > 0.$$  \hspace{1cm} (2)

The expansion rate is given by the Hubble parameter,

$$H(t) = H_i \left(1 + \frac{H_i t}{s}\right)^{-1}$$

where $H_i = H(t = 0)$. From the Einstein equations for the background

$$3H^2 = \frac{\kappa^2}{2} \rho(t),$$

$$-3H^2 - 2\dot{H} = \frac{\kappa^2}{2} p(t),$$

where $\kappa^2 = 16\pi G$ and a dot indicates a time derivative, it follows that

$$\rho(t) \propto \left(1 + \frac{H_i t}{s}\right)^{-2}, \quad \frac{p}{\rho} = -1 + \frac{2}{3s}.$$ \hspace{1cm} (6)

Notice that the de Sitter limit $s \to \infty$ is well defined in this time parameterization.

When we include gravitational waves the metric reads

$$ds^2 = -dt^2 + a^2(t) \left[\delta_{ij} + h_{ij}(\vec{x}, t)\right] dx^i dx^j.$$ \hspace{1cm} (7)
The gravitational waves $h_{ij}(\vec{x}, t)$ (also known as tensor perturbations) are typically expanded in modes as follows:

$$h_{ij}(\vec{x}, t) = \sum_k \left[ \epsilon_{ij}(\vec{k})h_k(t)e^{-i\vec{k} \cdot \vec{x}} + \text{c.c.} \right], \quad (8)$$

where $\epsilon_{ij}$ is the polarization tensor. If one’s interest is in quantizing this system, all that is needed is to promote the fields $h_k(t)$ to quantum operators and to impose the usual canonical commutation relations. For now we are only interested in the time dependence of modes of a given wavenumber. Later we will consider their amplitudes, which arise due to the well-known mechanism of quantum mechanical pair creation in an expanding universe, also known as superadiabatic amplification\cite{2}.

The equation obeyed by the modes $h_k$ is obtained through the linearization of Einstein’s field equations, and is found to be identical to that obeyed by a minimally coupled, massless scalar:

$$\ddot{h}_k + 3H \dot{h}_k + \frac{k^2}{a^2(t)}h_k = 0 . \quad (9)$$

This equation can be exactly solved in power-law backgrounds (see, e.g., \cite{1}). For the sake of clarity we briefly re-derive these solutions in what follows.

It is useful to introduce the variable

$$y(k, t) = k\eta(t) \equiv k \int^t dt' \frac{dt'}{a(t')} = \begin{cases} \frac{k}{H_i} \frac{s}{1-s} \left( 1 + \frac{H_i t}{s} \right)^{1-s} & s \neq 1 , \\ \frac{k}{H_i} \log \left( 1 + \frac{H_i t}{s} \right) & s = 1 , \end{cases} \quad (10)$$

where $\eta$ is the conformal time. Notice also that the far infrared is given by the limit $|y| \to 0$. Indeed, for $s \neq 1$ we have $y(k, t) = \frac{k}{a(t)H(t)} \frac{s}{1-s}$, that is, a mode is infrared if its physical wavelength is bigger than the Hubble radius at time $t$.

If we now write the modes $h_k$ in the form

$$h_k = y^\nu F_\nu(y) \quad \nu(s) = \frac{1}{2} \frac{3s - 1}{s - 1} , \quad (11)$$

then the equation for $F_\nu(y)$ can be reduced to the form

$$y^2 F'' + y F' + (y^2 - \nu^2) F = 0 , \quad (12)$$
which we recognize as Bessel’s equation. Thus, gravitational wave modes in power-law backgrounds are written as

\[ h_k(y) = y^n \left[ A_k H^{(1)}_{|\nu|}(|y|) + B_k H^{(2)}_{|\nu|}(|y|) \right], \quad (13) \]

\[ = y^n \left[ \tilde{A}_k J_{|\nu|}(|y|) + \tilde{B}_k J_{-|\nu|}(|y|) \right], \quad |\nu| \neq \text{integer}, \quad (14) \]

where \( H^{(1,2)}_{|\nu|} \) are Hankel functions of the first and second kind. The positive energy eigenmodes of the gravitational waves are associated with the Hankel functions of the second kind. In going from the first to the second line we used the property that for non-integer \( |\nu| \) the Bessel functions \( J_{|\nu|} \) and \( J_{-|\nu|} \) are linearly independent.

In the ultraviolet limit \( |y| \to \infty \) the dominant behavior of the Bessel functions is an oscillating term \( \exp \left[ -iy(t)/a(t) \right] \). Therefore, in that limit gravitational waves can be regarded as simple plane waves.

The infrared limit is far richer. \( J_{\nu}(|y|) \) has the following asymptotic expansion when \( |y| \to 0 \):

\[ J_{\nu} \approx \left( \frac{|y|}{2} \right)^{\nu} \frac{1}{\Gamma(\nu+1)} \left[ 1 - \frac{1}{4(1+\nu)} |y|^2 + \mathcal{O}(|y|^4) \right], \quad (15) \]

and thus from (11) and (14) we have

\[ h_k(t) \approx C_k |y|^{2\nu} \left[ 1 - \frac{1}{4(1+\nu)} |y|^2 + \mathcal{O}(|y|^4) \right] \quad (16) \]

\[ + D_k |y|^0 \left[ 1 - \frac{1}{4(1-\nu)} |y|^2 + \mathcal{O}(|y|^4) \right]. \]

If \( y < 0 \) (\( s > 1, \nu > 3/2 \)) then \( y \to 0^- \) in time. From Eq. (16) we have that when \( s > 1 \), \( |y| \) becomes smaller with time and therefore the term proportional to \( C_k \) is subdominant.

If, on the other hand, \( y > 0 \) (\( s < 1, \nu < 0.5 \)) then \( y \) grows as a function of time. However, if \( 1/3 < s < 1 \) then \( \nu < 0 \) and the term after \( C_k \) is subdominant again. Lastly, if \( 0 < s < 1/3 \) then \( y \) is growing in time, therefore the first term in (16) looks like it will become dominant with respect to the term after \( D_k \).
At this point we should remind the reader of the cosmological scenario we are considering: initially there is an inflationary phase \( s_1 > 1 \) during which tensor perturbations (gravity waves) are produced and stretched towards the infrared (\(|y|\) decays with time when \( s > 1 \)). Later, when the universe relaxes to a phase \( s_2 < 1 \), gravity waves become less and less infrared (\(|y|\) now grows with time), with some waves eventually re-entering the Hubble radius and becoming effectively ultraviolet.

By the time of the transition from the inflationary phase to the “normal matter” phase (which we can fix at \( t = 0 \)), the only pieces of the gravitational waves that survived from the inflationary phase were the dominant modes \( D_k^{(1)} \). These modes must be glued on the \( t = 0 \) space-like hypersurface to the gravitational wave solutions of the phase \( s_2 < 1 \). By matching the solutions and their first time derivatives on the \( t = 0 \) surface, we get the following expressions for the amplitudes in each mode in the \( s_2 < 1 \) phase:

\[
D_k^{(2)} = D_k^{(1)} \left[ 1 + \mathcal{O} \left( \frac{k}{H_i} \right)^2 \right], \\
C_k^{(2)} = D_k^{(1)} \left( \frac{s_1}{s_1 + 1} - \frac{s_2}{s_2 + 1} \right) \frac{s_2}{3s_2 - 1} \left( 1 - \frac{s_2}{s_1} \right)^{2\nu_2} \left[ 1 + \mathcal{O} \left( \frac{k}{H_i} \right)^2 \right],
\]

where \( \nu_2 \equiv \nu(s_2) \). We see then that for \( 1/3 < s_2 < 1 \) the dominant mode of the inflationary phase \( s_1 > 1 \) is completely transmitted to the dominant mode of the \( s_2 < 1 \) phase. For \( s < 1/3 \) the situation is slightly more complex: the initial amplitude of the growing mode is tiny when compared to the amplitude of the decaying mode. The growing mode only surpasses the decaying mode when \( k/aH \gg 1 \), but by then the wavelength of the gravitational wave is already much smaller than \( H^{-1} \), and the infrared limit is not valid anymore.

We end this section with the general result for the dominant modes in the asymptotic expansion for the gravitational waves in the infrared limit:

\[
h_k(t) \approx D_k \left[ 1 - \frac{1}{2} \frac{s^2}{1 - s^2} \left( \frac{k}{aH} \right)^2 + \ldots \right],
\]
where we have substituted \( y = \frac{s \, k}{\ln a \, a^2} \) into Eq. (19). Although the expression above was derived for non-integer \(|\nu|\), one can show that (19) follows if one takes \(|\nu|\) integer as well.

### 3 The Back Reaction of Gravitational Waves

Gravitational waves have an impact on the background in which they propagate. This nonlinear feedback, called back reaction, has been discussed extensively in the case where the gravitational waves are in the so-called “geometrical optics” limit, that is, when their wavelengths are much smaller than the curvature radius of the background space-time in which they travel [12, 14]. In this limit (the ultraviolet, by our definitions of the last section) the energy density of a gravitational wave mode \( h_{ik}^{UV} = c_k \exp \left[ iy(t)/a(t) \right] \) is purely kinetical, and is given by the pieces of the 0 – 0 component of the Einstein tensor which are quadratic in the amplitude of the metric fluctuations [12]:

\[
\Delta \rho^{UV} \equiv 2 \kappa^2 G^{(2)}_{00} \approx \frac{1}{4 \kappa^2} \left( |\dot{h}_k| + |\vec{\nabla} h_k| \cdot \vec{\nabla} h_k | \right)
\]

\[= \frac{k^2}{2 \kappa^2 a^4(t)} |c_k|^2, \tag{21}\]

that is, the energy density of high frequency gravity waves falls off like the energy density of radiation.

The limit where gravitational waves have long wavelengths is much less discussed in the literature, chiefly, in our opinion, due to the misguided belief that long wavelength fluctuations could not have any impact on the expansion of the universe as measured by a “local” observer.

To gain insight into this question we propose the following thought experiment: consider an inertial observer during inflation that throws his general relativity textbook away, and starts measuring the gravitational potential of the book. Since inflation takes small coordinate distances into enormous physical distances, soon this observer watches as his book falls out of his Hubble radius \( H^{-1} \). The question is: when the book falls out of contact with the observer (that is, when the physical distance between observer and book exceeds \( H^{-1} \)), what happens to the gravitational potential of the book: does it vanish completely, or is there some small potential remaining? We believe
that the only physically acceptable answer, and in particular the only answer consistent with conservation of energy and momentum, is that the observer still measures the gravitational pull from the book.

The obvious analog of this thought experiment is a similar observer that throws his book into a black hole of mass $M$, and measures the gravitational pull of the book with mass $m$. In this case the answer to the question “What happens to the gravitational potential of the book after it falls through the horizon?” is clear: the observer still feels the pull of the book’s gravity, since now the black hole has a mass $\tilde{M} = M + m$. The standard explanation is that the persistent gravitational potential of the book is due to virtual gravitons that were emitted near the horizon just as the book fell through the horizon into the black hole.

Notice that both observers are “local”, that is, they have no knowledge of what goes on beyond either the cosmological or the black hole horizons. Nevertheless, both are able to measure the build-up of the book’s gravitational potentials, even after the book has lost causal contact with the observers. At a much later time neither observer will be able to tell the difference between the pull of the book and that due to the background accelerations. Indeed, all that the observers can measure at that point are accelerations - acceleration towards the black hole in the latter example, cosmic accelerations in the former example. Therefore, neither observer “sees” the book anymore, although they certainly feel the effects of the book’s gravitational pull.

By the same token, long wavelength perturbations can have a gravitational impact on the accelerations (i.e., expansion rate) of the background space-time. The physical picture is as follows: perturbations are generated causally, inside the Hubble radius, and as they are redshifted by inflation their gravitational interactions fill the intervening space.

Rather than “crossing” the Hubble radius, the correct statement is to say that for an inertial observer these fluctuations become exponentially frozen at the Hubble radius as their physical wavelengths become larger than $H^{-1}$, much like the book that falls into the black hole appears to the inertial observer to be frozen near the black hole horizon. The (by now long wavelength) perturbations remain frozen near the Hubble radius for the duration of inflation, and only after reheating they start to defrost (since after inflation $H^{-1}$ grows faster than physical wavelengths). Eventually, as the Hubble radius grows and we have access to larger and larger distances, the perturbations becomes accessible to local observers who can then detect them directly.
At no point in time the inflationary perturbations ever fall completely out of contact with the local observer that witnessed their migration from small-scale fluctuations, to long wavelength perturbations, to cosmological perturbations. This is the crucial distinction between a cosmological scenario where perturbations are generated by a causal process (inflation), and the “old” scenario of the radiation-then dust-dominated ages of the universe where the spectrum of perturbations had to be imposed by fiat: in the “old” scenario there is a true particle horizon in both ages \( R_H = t/3 \) in the radiation phase), whereas in the inflationary scenario the Hubble radius is only an apparent particle horizon – the real particle horizon is the physical size of the quasi-homogeneous region that expanded coherently from the beginning of inflation, and is usually many orders of magnitude larger than the apparent horizon.

It appears therefore that there should be persistent gravitational interactions engendered by inflationary perturbations, regardless of the wavelength of those perturbations. The only questions are what is the magnitude of this effect, and whether it becomes more or less important in time.

The simplest way of estimating the importance of gravitational back reaction is by computing the lowest-order nonlinear (quadratic) corrections to the Einstein field equations, and comparing them to the background energy density and pressure. If we had used quantum mechanics and perturbation theory consistently, and if we also included the fluctuations in the matter fields that drive inflation, this calculation would correspond to computing the one loop effective theory (in that respect see [11] and [15]).

In what follows we calculate the one loop energy-momentum tensor for gravity waves, but do not solve for its back reaction on the metric and the expansion rate of the universe. The distinction should be clear: the former is a source term, while the latter are the actual solutions of the Einstein equations where the quadratic corrections have been taken into account. The reason we avoid writing down and solving these simple equations is purely economical: the effective energy-momentum tensor for gravity waves, at least within the scope of second order perturbation theory, never becomes important when compared to the energy-momentum tensor of the background.

The Einstein field equations to second order give the following expressions for the energy density and pressure of gravitational waves [10]:

\[\]
\[ \frac{\kappa^2}{2} \Delta \rho \equiv G^{(2)}_{00} = \left[ H \dot{h}_{ij} h_{ij} + \frac{1}{8} \left( \dot{h}_{ij}^2 + \frac{h_{ij,k}^2}{a^2} \right) \right], \quad (22) \]

\[ \frac{\kappa^2}{2} \Delta \rho \equiv G^{(2)}_{ii} = \frac{1}{24} \left( -5 \dot{h}_{ij}^2 + 7 \frac{h_{ij,k}^2}{a^2} \right), \quad (23) \]

where the Latin indices are summed with the Euclidean metric. The expression for the pressure was obtained through the assumption (valid at least for inflation-generated tensor fluctuations) that the spectrum of gravity waves does not break the homogeneity and isotropy of the background space-time.

In the ultraviolet limit the first term in the right-hand-side of Eq. (22) can be neglected, and the familiar result (21) follows. It is easy also to calculate the pressure of gravitational waves in this limit and obtain the expected equation of state, \( \Delta p^{UV}/\Delta \rho^{UV} = 1/3 \).

The drag term \( H \dot{h} h \) is an additional interaction of the gravitational waves that is only important for super-horizon waves. Even though \( H \dot{h} h < 0 \), it seems difficult for us to interpret it as a gravitational potential term, since the Newtonian potential does not appear at this order in perturbation theory.

Eqs. (22)-(23) should be consistent with conservation of energy and with the Bianchi identities (which are one and the same thing here). Since we are including quadratic terms in Einstein’s equations, we expect the Bianchi identities to take the usual form with maybe some quadratic corrections. Indeed, if we truncate the perturbative expansion of the Bianchi identities to quadratic order in the gravitational waves, we obtain

\[ 0 = \left[ G^{(2)}_{\nu,\mu} \right] = G^{\mu (2)}_{\nu,\mu} - \Gamma^{(2)}_{\mu \nu} G^{(0)}_{\alpha \alpha} - \Gamma^{(0)}_{\mu \nu} G^{(2)}_{\alpha \alpha} + \Gamma^{(2)}_{\mu \nu} G^{(0)}_{\alpha \alpha} + \Gamma^{(0)}_{\mu \nu} G^{(2)}_{\alpha \alpha}, \quad (24) \]

where we have canceled some terms using that \( G^{(1)}_{\nu \nu}[h] = 0 \) for gravitational waves.

The \( 0 - 0 \) component of this algebraic identity reads

\[ \frac{d}{dt} \Delta \rho + 3H(\Delta \rho + \Delta p) + \frac{1}{2} \dot{h}_{ij} h_{ij}(\rho + p) = 0. \quad (25) \]

It is a short exercise to show that by substituting definitions (22)-(23) into (25) takes us back to the equation of motion for the gravity waves, Eq. (9).
In the effective Einstein field equations we must add the energy density $\Delta \rho$ and pressure $\Delta p$ of gravitational waves to the energy density $\rho$ and pressure $p$ of the background matter. However, since $\Delta \rho$ and $\Delta p$ obey the modified energy conservation law (25), it is useful to define the effective pressure \[ \kappa^2 \Delta p_{\text{eff}} \equiv \frac{\kappa^2}{2} \Delta \rho - \frac{\dot{H}}{3H} h_{ij} h_{ij}, \] (26) where we have used the background relations (4)-(5) to simplify the expression. The last term in (26) is due to the appearance of the additional interaction $H h h$ in (23), and is only important for long wavelength gravitational waves. In terms of this effective pressure, the equation of conservation of energy for the gravity waves takes its usual form, \[ \Delta \dot{\rho} + 3H (\Delta \rho + \Delta p_{\text{eff}}) = 0. \] (27) Eq. (27) is an algebraic constraint on the time dependence of the energy density of gravitational waves. In particular, this constraint is valid even by the time when the physical scales of the gravitational waves are becoming larger than the Hubble radius.

If one believes that the energy density and pressure of gravitational waves effectively disappear after they cross the Hubble radius (i.e., that $\Delta \rho$ and $\Delta p_{\text{eff}}$ are exponentially suppressed, rather than power-law suppressed), then one should at least show that at that point the energy density and effective pressure cancel each other in Eq. (27), $\Delta p_{\text{eff}} \approx -\Delta \rho$. The same is true if one believes that a gravitational wave only acquires energy upon entering the Hubble radius. Otherwise one is forced to the heterodox conclusion that energy is not conserved, $\Delta \dot{\rho} + 3H (\Delta \rho + \Delta p_{\text{eff}}) \neq 0$, and to some modification of Einstein’s equations that includes sources and sinks of particles in order to account for the matter creation entailed by the non-conservation of energy [16].

Furthermore, because the Bianchi identities are integrability conditions on the classical equations of motion in curved space-time, it is difficult for us to understand how one could assign arbitrary energy density and pressure to the super-horizon gravitational waves while still keeping the equations of motion that those waves should obey unchanged.

For example, in Ref. [1] it is easy to see that energy conservation is violated: the integrated energy of short wavelength modes is given by Eq.
(11) of that paper, which is just a sum over momentum modes of the energy per mode given in our expression (21). Since that author chose to neglect modes of physical wavelengths larger than the Hubble radius, he picked an infrared cut-off corresponding to the time-dependent comoving scale $k_0(t) = H(t)$. It is clear that such a time-dependent comoving cut-off injects an extra time dependence into the integrated energy density and violates energy conservation (for ultraviolet modes $\Delta p \approx \Delta p_{\text{eff}}$, so even the naive energy conservation law is violated). What is happening, of course, is that in [1] each mode that comes inside the Hubble radius suddenly starts to contribute to the energy density, that is, for energy accounting purposes that mode is effectively “created” at the time when $\lambda_{\text{phys}} \approx H^{-1}$. By the same token, these modes were effectively “destroyed” when they left the Hubble radius during inflation.

The simplest way to eliminate this puzzling anomaly is to include in the calculations the energy density of gravity waves regardless of their wavelength. If one does that then what we have shown is that energy is conserved as it should, and there is no need to invoke matter creation or any other non-standard physics.

Up to this point the discussion in this section has been generic, and applies to gravitational waves in any background. Now we compute the energy density and pressure contributed by long wavelength gravitational waves in the power-law backgrounds $a = (1 + H_i t/s)^s$. From Eq. (19) we have that

$$\dot{h}_k = -D_k(k) \frac{s}{1 + s} H \left[ \left( \frac{k}{aH} \right)^2 + \mathcal{O} \left( \frac{k}{aH} \right)^4 \right].$$

After substituting Eq. (28) into expressions (22), (23) and (26) and keeping the leading terms in $k/aH$ one obtains for the energy density and pressure of the infrared gravity wave modes:

$$\frac{k^2}{2} \Delta \rho(k) = \frac{|D_k(k)|^2}{8} \frac{7s - 1}{s + 1} \frac{k^2}{a^2} + \ldots,$$

$$\frac{k^2}{2} \Delta p_{\text{eff}}(k) = \frac{|D_k(k)|^2}{24} \frac{7s - 1}{s + 1} \frac{k^2}{a^2} + \ldots,$$

that is, $\Delta p_{\text{eff}} = -\Delta \rho/3$. It can be shown that this result holds for the case $s = 1$ as well.
We remind the reader that $p = -\rho/3$ is the equation of state of curvature, so back reaction of long wavelength gravitational waves can be thought of as a curvature term in addition to the background energy density. We expect then that $\Delta \rho \propto a^{-2}$, as indeed is the case. Notice that $\Delta \rho < 0$, that is, the back reaction of long wavelength gravitational waves tends to slow the expansion rate of the universe (like a positive curvature would slow the expansion). This is so because the (negative) drag term $H\dot{h}h$ in (22) is more important than the (positive) kinetic terms for infrared modes (actually, the spatial gradient and the drag term are of the same order of magnitude, but the drag term has a bigger numerical factor).

Of course, if $s > 1$ the energy density of the background falls like $a^{-2/s}$, thus back reaction never becomes important during inflation. If $s < 1$, on the other hand, $\Delta \rho$ falls less fast than $\rho$, and with time the long wavelength gravity waves increase their share of total energy. Of course, if $s = 1$ [which implies $a(t)H(t) = \text{constant}$] then gravitational waves contribute a constant share of the total energy density, a fact that led to the conjecture[1] that the universe would tend to an equilibrium phase with $s = 1$ if the back reaction of gravitational waves ever became important.

In the next section we show that the growth of the share of energy density contributed by long wavelength gravitational waves during matter domination is elusive, since this share peaks by the time when the gravitational waves come back inside the Hubble radius. We will show that the maximal value of the fraction of the total energy density that is contributed by a gravitational wave in the $s < 1$ phase is the fraction of the total energy density contributed by that wave at the instant when it crossed the Hubble radius during inflation.

4 Back Reaction Before and After Reheating

In this section we study a model in which the universe inflates ($s_1 > 1$) when $t < 0$, then reheats at $t = 0$ and finally expands at a decelerating rate ($s_2 < 1$) for $t > 0$. The scale factor can be conveniently parameterized as

$$a(t) = \begin{cases} \left(1 + \frac{H_i}{s_1}\right)^{s_1} & t \leq 0 \\ \left(1 + \frac{H_i}{s_2}\right)^{s_2} & t \geq 0 , \end{cases}$$

(31)
so that \(a(t = 0) = 1\) and \(H(t = 0) = H_i\).

Consider now the physical wavelength \(\lambda^p_1\) that crosses the Hubble radius at \(t = t_1 < 0\), that is, the scale of the comoving momentum \(k_1 \equiv 2\pi a(t_1)/\lambda^p_1 = H(t_1)a(t_1)\). This scale will cross back into the Hubble radius at some time \(t_2 > 0\) given by the solution of

\[
H(t_2)a(t_2) = H(t_1)a(t_1).
\]

(32)

The energy density of the long wavelength gravitational wave mode \(k_1\) during the \(s_1 > 1\) phase is given by Eq. (29). We write the share of the energy density in the long wavelength gravitational wave as

\[
\delta_1(t) \equiv \frac{\Delta \rho_1(t)}{\rho(t)} = \epsilon(k_1)[a(t)]^{-2+2/s_1} \quad t \leq 0,
\]

(33)

where the constant \(\epsilon(k_1)\) includes the square of the amplitude of mode \(|h(k_1)|^2\) as well as the numerical factors in Eq. (29). The ratio \(\delta_1(t)\) is a measure of the strength of the back reaction of long wavelength gravity waves on the expansion rate. As previously discussed, this ratio decays in time during inflation.

The amplitude of gravitational waves implicit in \(\epsilon(k_1)\) is found by quantizing the metric perturbations, and the well-known result\[2\] is \(h(k_1) \approx \kappa H(t_1)\). This is a very small number as long as inflation happens below the Planck scale, and even for GUT-scale inflation this amplitude is only \(|h| \sim 10^{-6}\). However, the exact value of \(|h(k_1)|\) is irrelevant for our purposes: we just assume that it is some small number.

After the transition to the decelerating phase at \(t = 0\), the share of energy density in super-horizon gravity waves is given by

\[
\delta_2(t) \equiv \frac{\Delta \rho_2(t)}{\rho(t)} = \epsilon(k_1)[a(t)]^{-2+2/s_2} \quad t \geq 0.
\]

(34)

As \(s_2 < 1\), \(\delta_2\) grows with time.

Comparing (33) and (34), the shares of the total energy density contributed by the mode \(k_1\) in each of the two phases are equal when

\[
[a(t \leq 0)]^{-2+2/s_1} = [a(t \geq 0)]^{-2+2/s_2}.
\]

(35)

However, the fraction of the energy density \(\delta_2(t)\) is limited from above, since by the time \(t_2\) the gravitational wave of wavelength \(\lambda^p_1\) crosses back into
the Hubble radius. At that time $t_2$ the energy density in mode $k_1$ starts to
decay as $a^{-4}$ and quickly becomes just another radiation-like component of
the energy density of the universe. From Eq. (32) we see that the time $t_2$ is
defined as

$$[a(t_1)]^{-1+1/s_1} = [a(t_2)]^{-1+1/s_2}.$$  

Therefore, from Eq. (32) the (rather small) fraction of energy density $\delta_1(t_1)$
when the scale $k_1$ crossed the Hubble radius during inflation is equal to the
maximal fraction of energy density due to that gravitational wave during the
decelerating phase, $\delta_2^{\text{max}} = \delta_2(t_2)$.

In summary, we found that when the universe inflates and then reheats,
the energy density in gravitational waves goes through 4 periods: first, when
the gravitational wave is still well inside the Hubble radius during inflation,
its energy density is essentially kinetic and falls like radiation, $\Delta \rho \propto a^{-4}$.
Second, after the wave crosses the Hubble radius, its energy density falls like
$a^{-2}$ and is negative, since the main contribution comes now from the drag
term $H \dot{h} h$. The same behavior $\Delta \rho \propto a^{-2}$ persists in the third phase, when
the universe reheats and starts to expand at a decelerating rate. Therefore,
after inflation the share of total energy density due to long wavelength grav-
itational waves increases. Finally, by the time that the share of the energy
density in gravitational waves approaches the value of the share of the en-
dergy density when the mode first crossed the Hubble radius during inflation
$H^{-1}(t_1)$, the mode crosses back into the Hubble radius $H^{-1}(t_2)$ and starts to
behave like an ordinary radiation component.

5 Conclusions

We have discussed the energy density in gravitational waves, both short and
long wavelength. We found that ignoring the energy density and pressure of
long wavelength gravitational waves is tantamount to violating energy con-
servation. We have also argued that these interactions are not in profanation
of causality or locality: on the contrary, the persistence of gravitational in-
teractions after the wavelength of a perturbation becomes larger than the
Hubble radius is mandated by time-reversal invariance of the classical equa-
tions.
The energy density and pressure of long wavelength gravitational waves tend to slow the expansion rate of the universe, and their share of the total energy density and pressure grows in time during periods of decelerated expansion ($s < 1$). However, this share has an upper limit during the decelerated expansion phase which is the value of that share at the instant when the gravitational wave mode crossed the Hubble radius for the first time during inflation. As soon as this limit is reached, the mode crosses back into the Hubble radius, its energy density begins to fall like $a^{-4}$ and the gravitational wave starts to behave like ordinary radiation.

Our results are in many ways similar to those of Sahni [1], where the energy density of gravitational waves during a decelerated expansion phase is computed but only insofar as the gravitational waves are inside the Hubble radius. We have shown that this implies non-conservation of energy, and we indicated that the most natural way of curing this pathology is simply to include the energy density of gravitational waves irrespective of their wavelength.

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