Turbulent cascades for a family of damped Szegö equations

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Abstract
In this paper, we study the transfer of energy from low to high frequencies for a family of damped Szegö equations. The cubic Szegö equation has been introduced as a toy model for a totally non-dispersive degenerate Hamiltonian equation. It is a completely integrable system which develops growth of high Sobolev norms, detecting transfer of energy and hence cascades phenomena. Here, we consider a two-parameter family of variants of the cubic Szegö equation and prove that, adding a damping term unexpectedly promotes the existence of turbulent cascades. Furthermore, we give a panorama of the dynamics for such equations on a six-dimensional submanifold.

Keywords: turbulent cascades, Szegö equation, damped equation

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1. Introduction

An interesting aspect of turbulence is the transfer of energy from long to short-wavelength modes, leading to concentration of energy on small spatial scales. It is usually quantified by growth of Sobolev norms. In this paper, we study turbulent cascades for the family of damped Szegö equations on the one-dimensional torus

\[ i\partial_t u + i\nu(|u|^2 u) = \Pi(|u|^2 u) + \alpha(u|1) - \beta S\Pi(|S^*u|^2 S^*u), \]

(1.1)

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where $\nu > 0$ and $\alpha, \beta \in \mathbb{R}$ are given parameters. The term $\nu$ is the damping term on the smallest Fourier mode $(u|1) = \frac{1}{2\pi} \int_{\mathbb{T}} u(e^{i}) \, dx$. The Szegő projector $\Pi : L^2(\mathbb{T}) \to L^2_+(\mathbb{T})$ is the Fourier multiplier defined by

$$
\Pi(u) = \sum_{k \geq 0} \hat{u}(k) e^{ikx} \in L^2_+(\mathbb{T}).
$$

The shift operator $S : L^2_+(\mathbb{T}) \to L^2_+(\mathbb{T})$ and its adjoint are defined by

$$
Su = e^{ix}u \quad \text{and} \quad S^*u = e^{-ix}(u - (u|1)).
$$

When $\nu = \alpha = \beta = 0$, we recover the usual cubic Szegő equation

$$
i \partial_t u = \Pi(|u|^2u),
$$

which was introduced by the first two authors [GG10] as a toy model of a non-dispersive Hamiltonian system. The $\alpha$-deformation in (1.1) was first introduced by Xu [Xu14] in the $\alpha$-Szegő equation

$$
i \partial_t u = \Pi(|u|^2u) + \alpha(u|1),
$$

and the $\beta$-deformation in (1.1) was introduced by Biasi and Evnin [BE22] in the $\beta$-Szegő equation

$$
i \partial_t u = \Pi(|u|^2u) - \beta S\Pi(|S^*u|^2S^*u).
$$

Observe that, on the Fourier side, this equation corresponds to

$$
id\hat{u}(n)\bigg|_{t} dt = \sum_{n, m, l = 0}^{\infty} C_{nmkl}^{(\beta)} \hat{u}(m)\hat{u}(k)\hat{u}(l), \quad n \in \mathbb{N}
$$

where

$$
C_{nmkl}^{(\beta)} = \begin{cases} 
1 & \text{if } nmkl = 0 \\
1 - \beta & \text{otherwise.}
\end{cases}
$$

When $\beta = 1$, the equation (1.4) is so called the truncated Szegő equation and corresponds to the case where most of the interaction of the Fourier coefficients disappeared. Some other variants of the Szegő equation have also been studied, see e.g. ([Poc11, Thi19]).

### 1.1. Promoted turbulence

The Szegő equation is a completely integrable Hamiltonian system with two Lax pair structures displaying some turbulence cascade phenomenon for a generic set of initial data, in spite of infinitely many conservation laws. Namely, there exists a dense $G_\delta$ subset of initial data in $L^2_+ \cap C^\infty$, such that the solutions of the cubic Szegő equation satisfy, for every $s > \frac{1}{2}$,

$$
\limsup_{t \to +\infty} \|u(t)\|_{H^s} = +\infty, \quad \liminf_{t \to +\infty} \|u(t)\|_{H^s} < +\infty.
$$

Furthermore, this subset has an empty interior, since it does not contain any trigonometric polynomial [GG17].
However, there is no explicit examples of such phenomenon, and even less is known about the existence of solutions with high-Sobolev norms tending to infinity 4.

Later the first two authors [GG20] added a damping term to the cubic Szegő equation

\[ i\partial_t u + iv(u|1) = \Pi(|u|^2 u), \quad \nu > 0. \quad (1.5) \]

Paradoxically, the turbulence phenomenon is promoted by the damping term. Indeed, a nonempty open set of initial data generating trajectories with high-Sobolev norms tending to infinity was observed. In comparison, this is not the case for the damped Benjamin–Ono equation, see [Gas22].

The goal of this paper is to generalise the study of turbulent cascades for the damped Szegő equation to a family of damped Szegő equation (1.1). Biasi and Evnin [BE22] suggested the study of a two-parameter family of equations referred as the \((\alpha, \beta)\)-Szegő equations given by

\[ i\partial_t u = \Pi(|u|^2 u) - \beta S\Pi(|S^* u|^2 S^* u) + \alpha(u|1), \quad \alpha, \beta \in \mathbb{R}. \quad (1.6) \]

Inspired by this, we study the damped \((\alpha, \beta)\)-Szegő equation (1.1). The family is constructed in such a way that part of the Lax-pair structure inherited from the cubic Szegő equation is preserved but the damping term breaks the Hamiltonian structure. In our case, similar to the damped Szegő equation, the damping term promotes the existence of unbounded trajectories.

**Theorem 1.1.** There exists an open subset \(\Omega \subset H^1_{x_t} = H^1_x \cap L^2_x\) independent of \((\alpha, \beta)\) such that, for every \(s > \frac{1}{2}\), the set \(\Omega \cap H^s_x\) is nonempty and, for every \(\beta \neq 1\), every solution \(u\) of (1.1) with \(u(0) \in \Omega \cap H^s_x\) satisfies

\[ \|u(t)\|_{H^s_x} \to +\infty. \]

Furthermore, there exist rational initial data in \(\Omega\) which generate stationary solutions of (1.1) for \(\beta = 1\).

When \(\beta\) is different from 1, the damping term acts on the \((\alpha, \beta)\)-Szegő equations as on the cubic Szegő equation [GG20]. The case \(\beta = 1\) of the damped truncated Szegő equation appears to be more degenerate, and we do not know whether there exists a nonempty open subset of blowing up data. The wave turbulence phenomenon for Hamiltonian systems has been actively studied by mathematicians and physicists in the last decades. Bourgain [Bou00] asked whether there is a solution of the cubic defocusing nonlinear Schrödinger equation on the two-dimensional torus \(\mathbb{T}^2\) with initial data \(u_0 \in H^s(\mathbb{T}^2), s > 1\), such that

\[ \limsup_{t \to +\infty} \|u(t)\|_{H^s_x} = \infty. \]

There is still no complete answer to this question. However, the first mathematical evidence of such behaviour has been exhibited in the seminal work [CKSTT10] in which it is proven that, given any initial data with small Sobolev norm, it is possible to find a sufficiently large time for which the Sobolev norm of the solution is larger than any prescribed constant. This phenomenon also occurs for the half-wave equations on the real line or on the one-dimensional torus, see e.g. [Poc13, GG12]. Based on this, the first author, Lenzmann, Pocovnicu, and

4 In the case of the cubic Szegő equation on the line, a recent result have been obtained in this direction in [GP22] for rational solutions, completing the work in [Poc11].
Raphaël [GLPR18] gave a complete picture for a class of solutions on the real line. Namely, after the transient turbulence, the Sobolev norms of such solutions remains stationary large in time. The turbulence also occurs for two-dimensional incompressible Euler equations, the sharp double exponentially growing vorticity gradient on the disk was constructed by Kiselev and Šverák [KS14] and the existence of exponentially growing vorticity gradient solutions on the torus was shown by Zlatos [Zla15].

1.2. Preliminary observations

For the damped \((\alpha, \beta)\)-Szegő equation (1.1), the momentum

\[
\mathcal{M}(u) = (Du|u) = \sum_{k \neq 0} k|\hat{u}(k)|^2
\]

is preserved by the flow. An easy modification of the arguments in [GG10] shows that (1.1) is globally wellposed on \(H^s\) for every \(s \geq \frac{1}{2}\). Our goal is to study the behaviour of solutions of (1.1) as \(t \to +\infty\), in particular the growth of \(H^s\)-Sobolev norms for \(s > \frac{1}{2}\).

The main property which allows us to do computations, is the existence of a Lax pair. Namely, if \(u\) satisfies (1.1) then

\[
\frac{d}{dt} \tilde{H}_u = [C_u - \beta B_{\tilde{S}^*}u, \tilde{H}_u],
\]

where \(H_u\) is the Hankel operator and \(\tilde{H}_u = \tilde{S}^* H_u\) is the shifted Hankel operator. The operators \(B_u\) and \(C_u\) are the anti-self-adjoint operators appearing in the Lax pairs of the cubic Szegő equation (see section 2.2 for the definitions).

Thanks to this Lax pair, there are invariant manifolds consisting of the functions \(u\) such that \(\text{rank}(\tilde{H}_u) = k, k \geq 0\). From a well known result by Kronecker [Kro81], these manifolds consist of the rational functions

\[
u(x) = \frac{P_1(e^{ix})}{P_2(e^{ix})},
\]

where \(P_1\) and \(P_2\) are polynomials of degrees at most \(k\) with \(\text{deg}(P_1) = k\) or \(\text{deg}(P_2) = k\), no common roots and \(P_2\) has no roots inside the disk \(\{z \in \mathbb{C} | |z| < 1\}\).

1.3. A special case

In this section, we restrict ourselves to the lowest dimensional submanifold where \(\tilde{H}_u\) has rank 1

\[
\mathcal{W} := \left\{ u(x) = b + \frac{c e^{i \alpha}}{1 - p e^{i \alpha}} , \quad b, c, p \in \mathbb{C}, \quad c \neq 0, \quad |p| < 1 \right\}.
\]

We will give a complete picture of (1.1) on \(\mathcal{W}\), which consists of periodic, blow-up and scattering trajectories.

We consider the trajectories with a fixed momentum \(M > 0\), and define

\[
\mathcal{E}_M = \{ u \in \mathcal{W} | \mathcal{M}(u(t)) = M \}, \quad \mathcal{C}_M = \{ u(x) = c e^{i \alpha} | |c|^2 = M \}.
\]

Notice that \(\mathcal{C}_M \subset \mathcal{E}_M\) consists of the periodic trajectories. We write

\[
\mathcal{S}_{\alpha, \beta}(t) u_0 = u(t)
\]
for the solution of the equation (1.1) with initial data \(u_0 \in H^\frac{1}{2}(\mathbb{T})\). Then we have the following theorem.

**Theorem 1.2.** Let \(\nu > 0\) and \(\alpha, \beta \in \mathbb{R}\). There exists a three-dimensional submanifold \(\Sigma_{M,\nu,\alpha,\beta} \subset \mathcal{E}_M\) disjoint from \(\mathcal{C}_M\), invariant under the flow \(S_{\nu,\alpha,\beta}(t)\) and such that \(\Sigma_{M,\nu,\alpha,\beta} \cup \mathcal{C}_M\) is closed and

(a) If \(u_0 \in \mathcal{E}_M \setminus (\Sigma_{M,\nu,\alpha,\beta} \cup \mathcal{C}_M)\), then

\[
\|S_{\nu,\alpha,\beta}(t)u_0\|_{H^{s}} \sim \nu^{s}t^{-\frac{1}{2}}, \quad s > \frac{1}{2},
\]

(b) If \(u_0 \in \Sigma_{M,\nu,\alpha,\beta}\), then \(\text{dist}(S_{\nu,\alpha,\beta}(t)u_0, \mathcal{C}_M) \sim e^{-ct}\), for some \(c > 0\).

Let us emphasize that on this submanifold \(\mathcal{W}\), unlike on higher dimensional submanifolds (see section 3.2), there is no difference between the case \(\beta = 1\) and the other cases.

Compared to theorem 1.1, in theorem 1.2 the open set consisting of the initial data generating blow-up trajectories, is dense in \(\mathcal{W}\). Furthermore, the Sobolev norms of such generating trajectories grow at a uniform polynomial rate \(\sim t^{-\frac{1}{2}}\), independent of \(\nu, \alpha, \beta\). This is consistent with the blow-up rate for the damped Szegö equation (1.5), see [GG20]. In contrast, the initial data in \(\mathcal{W}\) generate only bounded trajectories in the case of the Szegö equation, the \(\alpha\) and \(\beta\)-Szegö equations with negative \(\alpha\) and \(\beta\), see [GG10, Xu14, BE22]. However, if \(\alpha > 0\) and \(\beta > 0\), then even faster blow-up solutions occur for the \(\alpha\) and \(\beta\)-Szegö equations. In this case, there exist trajectories \(u(t)\), whose Sobolev norms grow exponentially in time with

\[
\|u(t)\|_{H^s} \sim e^{ct\left(t^{-\frac{1}{2}}\right)}, \quad s > \frac{1}{2}.
\]

Moreover, if \(\beta > 9\), then there also exists a class of solutions for the \(\beta\)-Szegö equation with the polynomially growing Sobolev norms at the rate (1.7), see [BE22]. In other words, the authors exhibit various strong turbulence phenomena for \(\beta\)-Szegö equations when \(\beta\) is large enough.

One important feature of the damped \((\alpha, \beta)\)-Szegö equations is the existence of the Lyapunov functional

\[
\frac{d}{dt}\|S_{\nu,\alpha,\beta}(t)u_0\|_{L^2}^2 + 2\nu|\{S_{\nu,\alpha,\beta}(t)u_0\}|^2 = 0.
\]

Together with the conserved momentum, one infers the weak limit points of \(u(t)\) as \(t \to +\infty\) in \(H^\frac{1}{2}\). In the second part of theorem 1.2, when \(\beta = 1\), such weak limit points are also strong limit points. Namely, there exists \(u_\infty \in \mathcal{C}_M\) such that

\[
\|S_{\nu,\alpha,1}(t)u_0 - u_\infty\|_{H^s} \to 0, \quad \forall s > \frac{1}{2}.
\]

Let us complete this paragraph by a few more remarks about stationary solutions in the case \(\beta = 1\). On \(\mathcal{W}\), we already observed that these solutions are of the form \(c e^{it}\) with \(c \neq 0\). An elementary calculation shows that such initial data generate periodic solutions in the case \(\beta \neq 1\) and arbitrary \((\alpha, \nu)\). However, as stated in theorem 1.1, one can construct rational stationary solutions in the case \(\beta = 1\) which generate blow up solutions for every \(\beta \neq 1\), \(\nu > 0\) and arbitrary \(\alpha \in \mathbb{R}\).
1.4. Organisation of this paper

Section 2 is devoted to establishing some general properties of the undamped as well as the damped \((\alpha, \beta )\)-Szegö equations. Theorems 1.1 and 1.2 are proved in sections 3 and 4 respectively.

2. Generalities on the damped and undamped \((\alpha, \beta )\)-Szegö equations

In the following, we denote by \(t \mapsto S_{\alpha, \beta} (t) u_0\) (respectively by \(t \mapsto S_{\nu, \alpha, \beta} (t) u_0\)) the solution of the \((\alpha, \beta )\)-Szegö equation (1.6) (respectively of the damped \((\alpha, \beta )\)-Szegö equation (1.1)) with initial datum \(u_0\).

2.1. The Lyapunov functional

As in the case of the damped Szegö equation, an important tool in the study of equation (1.1) is the existence of a Lyapunov functional. Precisely, the following lemma holds.

**Lemma 2.1.** Let \(u_0 \in H_1^{1/2} (\mathbb{T})\). Then, for any \(t \in \mathbb{R}\),

\[
\frac{d}{dt} \| S_{\nu, \alpha, \beta} (t) u_0 \|_{L^2}^2 + 2 \nu \| (S_{\nu, \alpha, \beta} u_0 (t) \|_1 \|^2 = 0. \tag{2.1}
\]

As a consequence, if \(\nu > 0\), \(t \mapsto \| S_{\nu, \alpha, \beta} (t) u_0 \|_{L^2}^2\) is decreasing, and \(|(S_{\nu, \alpha, \beta} (t) u_0 |)\) is square integrable on \([0, +\infty)\), tending to zero as \(t\) goes to \(+\infty\).

**Proof.** Denote by \(u(t) := S_{\nu, \alpha, \beta} (t) u_0\) the solution of (1.1) with \(u(0) = u_0\). Observe first that \(t \mapsto \| u(t) \|_{L^2}^2\) decreases:

\[
\frac{d}{dt} \| u(t) \|_{L^2}^2 = 2 \text{Re}(\partial_t u | u) = 2 \text{Im}(i\partial_t u | u)
\]

\[
= 2 \text{Im}(\Pi(|u|^2 u) | u) - 2 \beta \text{Im}(\Pi(|S^* u|^2 S^* u) | u)
\]

\[
+ 2 \text{Im}((\alpha - i\nu)(|u| |1| |u))
\]

\[
= -2 \nu |(u(t) |1)|^2. 
\]

Hence, \(t \mapsto \| u(t) \|_{L^2}^2\) admits a limit at infinity and since

\[
\| u(t) \|_{L^2}^2 = \| u_0 \|_{L^2}^2 - 2 \nu \int_0^t |u(s) |1|^2 ds
\]

we deduce the finiteness of

\[
\int_0^{\infty} |(u(s) |1)|^2 ds.
\]

On the other hand, we claim that

\[
\frac{d}{dt} |(u(t) |1)|^2
\]
is bounded. Indeed
\[
\frac{d}{dt}|(u(t)|1|^2 = 2 \text{Re}(\partial_t u|1|u))
\]
\[
= 2 \text{Im}((\alpha - i\nu)(u|1|u)) + 2 \text{Im}((|u|^2 u|1|u))
\]
\[
- 2\beta \text{Im}((S|u|^2 S^* u|1|u))
\]
\[
= -2\nu|u|1|^2 + 2 \text{Im}((u^2 u|1|u)
\]
but
\[
|(u(t)|1)| \leq ||u||_{L^2}
\]
and
\[
|(u^2(t)|u(t))| \leq ||u||_{L^2} \times ||u||_{L^2}^2 \leq ||u||_{H^1/2}^2
\]
\[
\leq ||u_0||_{L^2}(\mathcal{M}(u_0) + ||u_0||_{L^2}^2).
\]

From both observations, we conclude that $$|(u(t)|1)|$$ tends to zero as $$t$$ goes to infinity. \(\square\)

From lemma 2.1 and the conservation of the momentum, the $$H^{1/2}$$ norm of $$S_{\nu,\alpha,\beta}(t)u_0$$ remains bounded as $$t \to +\infty$$, hence one can consider limit points $$u_\infty$$ of $$S_{\nu,\alpha,\beta}(t)u_0$$ for the weak topology of $$H^{1/2}$$ as $$t \to +\infty$$. Another general lemma describes more precisely these limit points, according to LaSalle’s invariance principle.

**Proposition 2.2.** Let $$u_0 \in H^{1/2}(\mathbb{T})$$.

Any $$H^{1/2}$$-weak limit point $$u_\infty$$ of $$(S_{\nu,\alpha,\beta}(t)u_0)$$ as $$t \to +\infty$$ satisfies $$(S_{\nu,\alpha,\beta}(t)u_\infty|1) = 0$$ for all $$t$$. In particular, $$S_{\nu,\alpha,\beta}(t)u_\infty$$ solves the $$(\alpha, \beta)$$-Szegö equation—in other words $$S_{\nu,\alpha,\beta}(t)u_\infty = S_{\alpha,\beta}(t)u_\infty$$.

**Proof.** Denote by $$Q$$ the limit of the decreasing non-negative function
\[
t \mapsto ||S_{\nu,\alpha,\beta}(t)u_0||_{L^2}^2.
\]

By the weak continuity of the flow in $$H^{1/2}(\mathbb{T})$$,
\[
u(t + t_n) = S_{\nu,\alpha,\beta}(t)u(t_n) \to S_{\nu,\alpha,\beta}(t)u_\infty
\]
weakly in $$H^{1/2}$$ as $$n \to \infty$$. Hence, thanks to the Rellich theorem,
\[
||u(t + t_n)||_{L^2}^2 \to ||S_{\nu,\alpha,\beta}(t)u_\infty||_{L^2}^2
\]
as $$n$$ tends to infinity. On the other hand, by lemma 2.1,
\[
||u(t + t_n)||_{L^2}^2 \to Q
\]
so eventually, for every $$t \in \mathbb{R},$$
\[
||S_{\nu,\alpha,\beta}(t)u_\infty||_{L^2}^2 = ||u_\infty||_{L^2}^2,
\]
or
\[
\frac{d}{dt}||S_{\nu,\alpha,\beta}(t)u_\infty||_{L^2}^2 = 0.
\]
Recall that, from (2.1),
\[
\frac{d}{dt} \| S_{\nu,\alpha,\beta}(t) u_\infty \|^2_{L^2} = -2\nu |(S_{\nu,\alpha,\beta}(t) u_\infty| 1)^2.
\]

It forces \((S_{\nu,\alpha,\beta}(t) u_\infty| 1) = 0\) for all \(t\). Hence \(S_{\nu,\alpha,\beta}(t) u_\infty = \mathcal{S}_{\alpha,\beta}(t) u_\infty\) is a solution to the \((\alpha, \beta)\)-Szegö equation without damping. \(\Box\)

In order to characterise \(u_\infty\), we need to recall some results about the cubic Szegö equation corresponding to the case \(\alpha = \beta = \nu = 0\) in equation (1.1).

### 2.2. Hankel operators and the Lax pair structure

In this paragraph, we recall some basic facts about Hankel operators and the special structure of the cubic Szegö equation. We keep the notation of [GG17] and we refer to it for details. For \(u \in H^1_{L^2}\), we denote by \(H_u\) the Hankel operator of symbol \(u\) namely
\[
H_u : \left\{ \begin{array}{c}
L^2_+ (\mathbb{T}) \rightarrow L^2_+ (\mathbb{T}) \\
dom f \mapsto \Pi(u f)
\end{array} \right.
\]

It is well known that, for \(u\) in \(H^1_{L^2}\), \(H_u\) is Hilbert–Schmidt with
\[
\text{Tr}(H^2_u) = \sum_{k \geq 0} (k+1)|\hat{u}(k)|^2 = \|u\|^2_{L^2} + \mathcal{M}(u).
\]

One can also consider the shifted Hankel operator \(\tilde{H}_u\) corresponding to \(H_{S^* u}\) where \(S^*\) denotes the adjoint of the shift operator \(S f(x) := e^{ix} f(x)\). This shifted Hankel operator is Hilbert–Schmidt as well, with
\[
\text{Tr}(\tilde{H}^2_u) = \sum_{k \geq 0} k|\hat{u}(k)|^2 = \mathcal{M}(u).
\]

Observe in particular that
\[
\|u\|^2_{L^2} = \text{Tr}(H^2_u) - \text{Tr}(\tilde{H}^2_u). \tag{2.2}
\]

A crucial property of the cubic Szegö equation is its Lax pair structure. Namely, if \(u\) is a smooth enough solution of the cubic Szegö equation, then there exists two anti-selfadjoint operators \(B_u, C_u\) such that
\[
\frac{d}{dt} H_u = H_{-i|u|^2 u} = [B_u, H_u], \quad \frac{d}{dt} \tilde{H}_u = \tilde{H}_{-i|u|^2 u} = [C_u, \tilde{H}_u]. \tag{2.3}
\]

Classically, these equalities imply that \(H_{u(0)}\) and \(\tilde{H}_{u(0)}\) are isometrically equivalent to \(H_{u(0)}\) and \(\tilde{H}_{u(0)}\) (see [GG17] for instance). In particular, both spectra of \(H_u\) and \(\tilde{H}_u\) are preserved by the cubic Szegö flow. It motivated the study of the spectral properties of both Hankel operators that we recall here.

For \(u \in H^1_{L^2}\), let \((s^2)_{j \geq 1}\) be the strictly decreasing sequence of positive eigenvalues of \(H^2_u\) and \(\tilde{H}^2_u\). Following the terminology of [GG17], we say that
• $\sigma^2$ is an $H$-dominant eigenvalue if
\[
\ker(H_u^2 - \sigma^2 I) \neq \emptyset \quad \text{and} \quad u \not\in \ker(H_u^2 - \sigma^2 I).
\]

Respectively,
• $\sigma^2$ is an $\tilde{H}$-dominant eigenvalue if
\[
\ker(\tilde{H}_u^2 - \sigma^2 I) \neq \emptyset \quad \text{and} \quad u \not\in \ker(\tilde{H}_u^2 - \sigma^2 I).
\]

From the fundamental property $\tilde{H}_u^2 = H_u^2 - (|\mu|)u$ and the min–max formula, the first two authors proved that the $s_{j-1}^2$ correspond to $H$-dominant eigenvalues of $H_u^2$ while the $s_j^2$ correspond to $\tilde{H}$-dominant eigenvalues of $\tilde{H}_u^2$. Furthermore, the eigenvalues of $H_u^2$ and of $\tilde{H}_u^2$ interlace and, as a consequence, if $m_j := \dim \ker(H_u^2 - s_j^2 I)$ and $\tilde{m}_j := \dim \ker(\tilde{H}_u^2 - \tilde{s}_j^2 I)$, then
\[
m_{2j-1} = \tilde{m}_{2j-1} + 1 \quad \text{as} \quad \tilde{m}_{2j} = m_{2j} + 1.
\]

To complete the spectral analysis of these Hankel operators, we need to recall the notion of Blaschke product. A function $b$ is a Blaschke product of degree $m$ if
\[
b(x) = e^{i\theta} \prod_{j=1}^{m} \frac{x - p_j}{1 - \overline{p_j}x}
\]
for some $p_j \in \mathbb{C}$ with $|p_j| < 1$, $j = 1$ to $m$. As proved in [GG17]—see also [GP20] for a generalisation to non compact Hankel operators—, for any $H$-dominant eigenvalue $s_{j-1}^2$, there exists a Blaschke product $\Psi_{2j-1}$ of degree $m_{2j-1} - 1$ such that, if $u_0$ denotes the orthogonal projection of $u$ on the eigenspace $\ker(H_u^2 - s_{j-1}^2 I)$, then
\[
\Psi_{2j-1}H_u(u_0) = s_{j-1}u_0.
\]

Analogously, for any $\tilde{H}$-dominant eigenvalue $\tilde{s}_{2j}^2$, there exists a Blaschke product $\Psi_{2k}$ of degree $\tilde{m}_{2j}$ such that, if $u_0$ denotes the orthogonal projection of $u$ on $\ker(\tilde{H}_u^2 - \tilde{s}_j^2 I)$ then
\[
\tilde{H}_u(u_0) = \Psi_{2k}\tilde{s}_{2k}u_0.
\]

The first two authors proved in [GG17] that the sequence $((s_j^2), (\Psi_j))$ characterises $u$, and that it provides a system of action-angle variables for the Hamiltonian evolution. Namely, if $u(0)$ has spectral coordinates $((\xi^2), (\Psi))$ then $u(t)$ has spectral coordinates $((\xi^2), (e^{i\beta t}\xi^2, \Psi))$.

We are now in position to characterise the asymptotics of the damped $(\alpha, \beta)$-Szegö equation. As claimed in the introduction, the equation inherits one Lax pair from the Szegö equation related to the shifted Hankel operator $H_u$. Recall that, from the definition of $\tilde{H}_u = H_{S^\alpha}$, on one hand, the shifted Hankel operator associated to a constant symbol is identically 0 and, on the other hand, $\tilde{H}_{S^\alpha} = H_{S^\alpha} = H_u$. Hence, using (2.3), one obtains for $u(t) := \xi_{t,0,\alpha,\beta}(t)u_0$,
\[
\frac{d}{dt} \tilde{H}_u = \tilde{H}_{S^\alpha} + \beta H_{S^\alpha} + i(\nu - i\alpha)\tilde{H}_{S^\alpha} = [C_u, \tilde{H}_u] + \beta H_{S^\alpha} + i(\nu - i\alpha)\tilde{H}_{S^\alpha} = [C_u, \tilde{H}_u].
\]
Hence,
\[
\frac{d}{dt} \tilde{H}_u = [C_u - \beta B_{S^*} u, \tilde{H}_u]
\]  
(2.4)

where \(B_u\) and \(C_u\) are the anti-selfadjoint operator given by
\[
B_u = -iT_u^2 + \frac{i}{2} \tilde{H}_u^2 \quad \text{and} \quad C_u = -iT_u^2 + \frac{i}{2} \tilde{H}_u^2.
\]  
(2.5)

Here \(T_b\) denotes the Toeplitz operator of symbol \(b\) given on \(L^2_+\) by \(T_b(f) = \Pi(bf)\). As a usual consequence, \(\tilde{H}_{S_{\alpha,\beta}(\alpha_0)}\) is unitarily equivalent to \(\tilde{H}_{\alpha_0}\) and for instance, the class of symbol \(u\) with \(\tilde{H}_u\) of fixed finite rank is preserved by the damped \((\alpha, \beta)\)-Szegö flow. In particular
\[
\forall : \left\{ u(x) = b + \frac{ce^{it}}{1-p e^{-it}}, \ b, c, p \in \mathbb{C}, c \neq 0, \ |p| < 1 \right\}
\]  
(2.6)

is invariant by the flow since it corresponds to the set of symbol whose shifted Hankel operators are of rank 1.

Another consequence is the following result, where the difference between the cases \(\beta \neq 1\) and \(\beta = 1\) clearly appears.

**Theorem 2.3. (i)** Assume \(\beta \neq 1\).

The solutions \(S_{\alpha,\beta}(t) u_0\) of the \((\alpha, \beta)\)-Szegö equation satisfying \((S_{\alpha,\beta}(t) u_0)|1) = 0\) for all \(t\) are characterised by the fact that all the \(H\)-dominant eigenvalues of \(u_0\) are at least of multiplicity 2 and hence, are eigenvalues of \(\tilde{H}_{\alpha_0}\). Furthermore, if \(\{\sigma_k^2\}_k\) denotes the strictly decreasing sequence of the eigenvalues of \(\tilde{H}_{\alpha_0}\), one has
\[
\|S_{\alpha,\beta}(t) u_0\|_{L^2}^2 = \sum_k (-1)^{k-1} \sigma_k^2.
\]

(ii) In the case \(\beta = 1\), the solutions \(S_{\alpha,1}(t) u_0\) of the \((\alpha, 1)\)-Szegö equation satisfying \((S_{\alpha,1}(t) u_0)|1) = 0\) for all \(t\) are characterised by
\[
(u_0)|1) = (H_{\alpha_0}^0 u_0)|1) = 0,
\]

and are stationary solutions.

**Proof.** (i) Write \(u := S_{\alpha,\beta}(\cdot) u_0\). The scheme of the proof is to observe that the closure of the vector space \(< u >_{H_u}\) spanned by \(H_{\alpha_0}^m(u)\), \(m \in \mathbb{N}\) is orthogonal to \(1\). First, as
\[
H_{\alpha_0}^0 u = \tilde{H}_u^2 + (\cdot|u) u,
\]
this space equals the closure of the vector space \(< u >_{\tilde{H}_u^2}\) spanned by \(\tilde{H}_u^m(u)\), \(m \in \mathbb{N}\). We prove by induction on \(m\) that
\[
\forall \ t \in \mathbb{R}, \ (\tilde{H}_u^m(u)|1)(t) = 0.
\]  
(2.7)

For \(m = 0\), it follows from the assumption. For larger \(m\), we observe that \(\tilde{H}_u^2\) is self-adjoint, that \(|S^* u|^2 = |u|^2\) and that \(\tilde{H}_u^2 = H_{\alpha_0}^2\) as \((u)|1) = 0\) by assumption.
For \( m = 1 \), we write
\[
0 = i(\partial_t u|\|) = (\Pi(|u|^2|u|) - (\Pi)(|S^u|u^2|S^u|)|1) = (\Pi(|u|^2|u|) = (u|H^2_0(1)) = (u|H^2_{u}(1)) = (\tilde{H}^2_{u}(u)|1).
\]
Assume now that \((\tilde{H}^2_{u}(u)|1) \equiv 0 \) for any \( j \leq m \) and let us prove
\[(\tilde{H}^{2(m+1)}_{u}(u)|1) \equiv 0.\]

We write
\[
0 = \frac{d}{dt}(\tilde{H}^2_{m+1}(u)|1) = (\tilde{H}^2_{m}(-iT_{|u|^2}u + i\beta ST_{|u|^2}S^u)|1) + (C_u - \beta B S^u, \tilde{H}^2_{m+1}|u|1) \]
\[
= (\tilde{H}^2_{m}(-iT_{|u|^2}u + i\beta ST_{|u|^2}S^u)|1) + (\Pi(|u|^2 + i\beta T_{|u|^2}, \tilde{H}^2_{m+1}|u|1) \]
\[
i(\beta - 1)(T_{|u|^2}(\tilde{H}^2_{m}(u))|1) = i(\beta - 1)(\tilde{H}^2_{m}(u)|u|^2) \]
\[
i(\beta - 1)(\tilde{H}^2_{m}(u)|H^2_0(1)) = i(\beta - 1)(\tilde{H}^2_{m}(u)|H^2_{u}(1)) \]
\[
i(\beta - 1)(\tilde{H}^2_{m+1}(u)|1) \]

Here we used the property that \( \Pi S^u(f) = \Pi(f) - (f)|1 \) so that \( ST_{|u|^2}S^u = T_{|u|^2}u \). Eventually we get (2.7).

Observe that, for any \( H \)-dominant eigenvalue \( \tilde{s}_{j-1}^2 \), the orthogonal projection \( u_j \) of \( u \) onto the eigenspace \( \ker(H^2_0 - \tilde{s}_{j-1}^2 I) \) belongs to the vector space \( < u \geq H^2_0. \) From the preceding result, this space is orthogonal to \( \| \) hence \( u_j \) is orthogonal to \( 1 \). By the spectral analysis of the Hankel operator recalled above, there exists a Blaschke product \( \Psi_{2j-1} \) of degree \( m_{2j-1} - 1 \) with \( s_{2j-1} u_j = \Psi_{2j-1} H^2_{u}(u_j) \). Taking the scalar product with \( \| \) gives
\[
0 = \Psi_{2j-1}(0)(H^2_{u}(u_j)|\|) = \Psi_{2j-1}(0)\|u_j\|_{L^2},
\]
hence \( \Psi_{2j-1}(0) = 0 \) and the degree of \( \Psi_{2j-1} \) is at least \( 1 \). It follows that the multiplicity of \( \tilde{s}_{j-1}^2 \) is at least \( 2 \) (recall that from the definition of \( u_j, \|u_j\|_{L^2} \neq 0 \)). From the interlacement property, this eigenvalue is also an eigenvalue for \( \tilde{H}^2_{u} \). Let \( \{\sigma_k^2\}_k \) denote the strictly decreasing sequence of the eigenvalues of \( \tilde{H}^2_{u} \). We denote by \( m_k \) the multiplicity of \( \sigma_k^2 \) as an eigenvalue of \( \tilde{H}^2_{u} \) and by \( \tilde{m}_k \) its multiplicity as an eigenvalue of \( H^2_{u} \). From the interlacement property, if \( k \) is odd, \( m_k = \tilde{m}_k + 1 \) and if \( k \) is even, \( m_k = \tilde{m}_k - 1 \). We now compute the \( L^2 \) norm of \( S_{\alpha,\beta}(t)u_0 \) using (2.2):
\[
\|S_{\alpha,\beta}(t)u_0\|_{L^2}^2 = \text{Tr} H^2_{u} - \text{Tr} \tilde{H}^2_{u} = \sum m_k \sigma_k^2 - \sum \tilde{m}_k \sigma_k^2 = \sum (-1)^{k-1} \sigma_k^2.
\]

(ii) Next we assume \( \beta = 1 \). Assume \( u_0 \) is such that \((S_{\alpha,1}(t)u_0)|1 = 0 \) for every \( t \). Then obviously \((u_0|1) = 0 \), and the above calculation for \( m = 1 \) implies that \((\tilde{H}^2_{u}(u_0)|1) \equiv 0 \). Conversely, if \((u_0|1) = 0 \) and \((H^2_{u}(u_0)|1) \equiv 0 \), then, as already observed, \(|u_0|^2 = |\tilde{S}^u u_0|^2 \), and \( ST_{|u_0|^2}S^u u_0 = T_{|u_0|^2}u_0 \), hence \( S_{\alpha,1}(t)u_0 = u_0 \). □

Once these basic properties are established, we are in position to prove theorem 1.1.
3. Exploding trajectories in the case $\beta \neq 1$

In this section, we consider trajectories of (1.1) in $H^s$, $s > \frac{1}{2}$, along which the $H^s$ norm of $u(t)$ tends to infinity as $t \to +\infty$. Let us define the functional

$$F(u) = \sum_k (-1)^k \sigma_k^2$$

where $(\sigma_k^2)_k$ is the strictly decreasing sequence of positive eigenvalues of $\hat{H}_u^2$. We prove the following result, which is very similar to the case of the damped Szegö equation [GG20].

**Theorem 3.1.** Assume $\beta \neq 1$. Let $s > \frac{1}{2}$. If $u_0 \in H^s_0$ satisfies

- Either $\|u_0\|_{L^2}^2 < F(u_0)$,
- Or $\|u_0\|_{L^2}^2 = F(u_0)$ and $(u_0|1) \neq 0$,

then the $H^s$-norm of the solution of the damped $(\alpha, \beta)$-Szegö equation

$$\|u(t)\|_{H^s} = \|S_{\alpha,\beta}(t)u_0\|_{H^s}$$

tends to $+\infty$ as $t$ tends to $+\infty$.

**Remark 3.2.** Let us emphasize that the conditions driving to unbounded Sobolev trajectories are independent on $\alpha$, $\beta$.

**Proof.** Let us proceed by contradiction and assume that there exists a sequence $t_n \to +\infty$ such that $u(t_n) := S_{\alpha,\beta}(t_n)u_0$ is bounded in $H^s$. We may assume that $u(t_n)$ is weakly convergent to some $u_\infty$ in $H^s_0$. By the Rellich theorem, the convergence is strong in $H^s_0$, and

$$M(u_\infty) = M(u_0) = \sum_{\sigma^2 \in \Sigma(u_0)} m(\sigma)\sigma^2,$$

where $\Sigma(u_0)$ denotes the set of eigenvalues of $\hat{H}_u^2$, and $m(\sigma)$ the multiplicity of $\sigma^2 \in \Sigma(u_0)$. By the Lax pair structure, the eigenvalues of $\hat{H}_{u_0}^2$ are the same as the eigenvalues of $\hat{H}_{u_\infty}^2$, with the same multiplicities, hence every eigenvalue $\sigma^2$ of $\hat{H}_{u_\infty}^2$ must belong to $\Sigma(u_0)$, with a multiplicity not bigger than $m(\sigma)$. In view of identity (3.1), we infer that

$$\Sigma(u_\infty) = \Sigma(u_0),$$

with the same multiplicities. On the other hand, from proposition 2.2, we know that $u_\infty$ generates a solution of the cubic $(\alpha, \beta)$-Szegö equation which is orthogonal to 1 at every time. Consequently, theorem 2.3 gives

$$\|u_\infty\|_{L^2}^2 = F(u_0).$$

Since the $L^2$-norm of the solution is decreasing by lemma 2.1, $\|u_0\|_{L^2}^2 \geq \|u_\infty\|_{L^2}^2$. Hence, $\|u_0\|_{L^2}^2 \geq F(u_0)$. If $\|u_0\|_{L^2}^2 = F(u_0)$ then $\|S_{\alpha,\beta}(t)u_0\|_{L^2}$ remains constant and necessarily, by the Lyapunov functional identity (2.1), $(S_{\alpha,\beta}(t)u_0|1) = 0$ so that in particular, $(u_0|1) = 0$. Hence, the case $\|u_0\|_{L^2}^2 = F(u_0)$ and $(u_0|1) \neq 0$ drives to an exploding orbit in $H^s$ as well as the case $\|u_0\|_{L^2} < F(u_0)$. It ends the proof of theorem 3.1. □
3.1. An open condition

As a corollary of theorem 3.1, we get the following result, which implies the first part of theorem 1.1.

**Corollary 3.3.** Assume \( \beta \neq 1 \) denote by \( \Omega \) the interior in \( H^1_0 \) of the set of \( u_0 \in H^1_0 \) such that \( \|u_0\|_{L^2} < F(u_0) \). For every \( s > \frac{1}{2} \), \( \Omega \cap H^s_+ \) of \( H^s_+ (\mathbb{T}) \) is not empty, and every solution \( u \) of (1.1) with \( u(0) \in \Omega \cap H^s_+ \) satisfies

\[
\|u(t)\|_{H^s} \to \infty
\]

as \( t \) tends to \( +\infty \).

**Proof.** By elementary perturbation theory, it is easy to prove that function \( F \) is continuous at those \( u \) of \( H^1_0 \) such that \( \tilde{H}^2 u \) has simple non zero spectrum. Furthermore, in the particular case

\[
u(x) = \frac{e^{ix}}{1 - pe^{ix}}, \quad |p| < 1, \quad p \neq 0,
\]

it is easy to check that \( \tilde{H}^2 u \) has rank one with \( \frac{1}{(1 - |p|^2)^2} \) as simple eigenvalue. As \( \|u\|_{L^2}^2 = \frac{1}{1 - |p|^2} \), this function belongs to \( \Omega \), and moreover it belongs to every \( H^s \). In view of theorem 3.1, this completes the proof. \( \square \)

We give a simple class of functions in \( \Omega \).

**Example 1.** The set of functions \( u_0 \) whose nonzero eigenvalues of \( H^2_{u_0} \) and \( \tilde{H}^2_{u_0} \) are all simple, and form the decreasing square summable list

\[
\rho_1 > \sigma_1 > \rho_2 > \sigma_2 > \cdots
\]

with

\[
\sum_j \rho_j^2 < 2 \sum_{k \text{ odd}} \sigma_k^2
\]

is a subset of \( \Omega \).

3.2. Elements of \( \Omega \) generating stationary solutions for \( \beta = 1 \)

Finally, we complete the proof of theorem 1.1 by constructing rational elements in \( \Omega \) which generate stationary solutions of the truncated Szegö equation. This example illustrates the contrast of the case \( \beta = 1 \) with the other cases. More specifically, we give an example of a rational datum \( u \) as in the above example and satisfying

\[
(u|1) = (H^2_n(u)|1) = 0.
\]

From [GG17], there exists a system of coordinates in which any rational function is characterized by a finite sequence \((s_j^2, \Psi_j)\) (see paragraph 2.2) and conversely, to any such finite sequence corresponds a unique rational function. We seek \( u \) such that \( H_n u \) has three simple singular values \( \rho_1, \rho_2, \rho_3 \), and \( \tilde{H}_n u \) has three simple singular values \( \sigma_1, \sigma_2, \sigma_3 \) with

\[
\rho_1 > \sigma_1 > \rho_2 > \sigma_2 > \rho_3 > \sigma_3 = 0.
\]
Denote by $u_j, j = 1, 2, 3$ the projection of $u$ onto $\ker(H_0^2 - \rho_j^2)$. Then we know that there exist $\varphi_j \in T$ such that

$$e^{\imath \varphi_j}H_0(u_j) = \rho_j u_j.$$ Consequently,

$$(u|\|) = \sum_{j=1}^{3} \frac{1}{\rho_j} e^{\imath \varphi_j} \|u_j\|^2_{L^2}, \quad (H_0^3(u)|\|) = \sum_{j=1}^{3} \rho_j e^{\imath \varphi_j} \|u_j\|^2_{L^2}.$$ Choosing $\varphi_1 = \varphi_3 = 0, \varphi_2 = \pi$, the three conditions (3.2) read

$$\frac{1}{\rho_2} \|u_2\|^2_{L^2} = \frac{1}{\rho_1} \|u_1\|^2_{L^2} + \frac{1}{\rho_3} \|u_3\|^2_{L^2},$$

$$\rho_2 \|u_2\|^2_{L^2} = \rho_1 \|u_1\|^2_{L^2} + \rho_3 \|u_3\|^2_{L^2},$$

$$\rho_2^2 + \rho_3^2 < 2\sigma_1^2.$$ Recall that

$$\|u_j\|^2_{L^2} = (\rho_j^2 - \sigma^2_j)(\rho_j^2 - \sigma_2^2) - \rho_j^2 \prod_{1 \leq k \neq j \leq 3} (\rho_j^2 - \rho_k^2).$$

Therefore we can reformulate the problem as finding $\rho_1 > \sigma_1 > \rho_2 > \sigma_2 > \rho_3 > 0$ such that

$$\rho_1(\rho_1^2 - \sigma_1^2)(\rho_1^2 - \sigma_2^2) = -\rho_2(\rho_2^2 - \sigma_1^2)(\rho_2^2 - \sigma_2^2) = \rho_3(\rho_3^2 - \sigma_1^2)(\rho_3^2 - \sigma_2^2),$$

$$\rho_1^2 + \rho_3^2 < 2\sigma_1^2.$$ Let us fix $\sigma_1, \sigma_2$ such that $\sigma_1 > \sigma_2 > 0$, and set

$$P(x) := x(x^2 - \sigma_1^2)(x^2 - \sigma_2^2).$$ Notice that

$$P'(0) = \sigma_1^2 \sigma_2^2 > 0, \quad P'(\sigma_2) = 2\sigma_2^3(\sigma_2^2 - \sigma_1^2) < 0, \quad P'(\sigma_1) = 2\sigma_1^3(\sigma_1^2 - \sigma_2^2) > 0.$$ By the inverse function theorem, for every $\varepsilon > 0$ small enough, there exist $\rho_1(\varepsilon)$ in a $\varepsilon$-neighborhood of $\sigma_1, \rho_2(\varepsilon)$ in a $\varepsilon$-neighborhood of $\sigma_2, \rho_3(\varepsilon)$ in a $\varepsilon$-neighborhood of $0$, such that

$$P(\rho_1(\varepsilon)) = -P(\rho_2(\varepsilon)) = P(\rho_3(\varepsilon)) = \varepsilon.$$ Furthermore,

$$\rho_1(\varepsilon) > \sigma_1 > \rho_2(\varepsilon) > \sigma_2 > \rho_3(\varepsilon) > 0.$$ Consequently,

$$\rho_1(\varepsilon)^2 + \rho_2(\varepsilon)^2 + \rho_3(\varepsilon)^2 = \sigma_1^2 + \sigma_2^2 + o(\varepsilon) < 2\sigma_1^2$$ if $\varepsilon$ is small enough. This completes the proof.
4. A special case of $\mathcal{W}$

In this section, we provide a panorama of the dynamics of the damped $(\alpha, \beta)$-Szegö equations for any fixed $(\alpha, \beta) \in \mathbb{R}^2$ on the six-dimensional submanifold

$$\mathcal{W} := \left\{ u(x) = b + \frac{c e^{ix}}{1 - pe^{ix}}, \ b, c, \ p \in \mathbb{C}, \ c \neq 0, \ |p| < 1 \right\}.$$ 

Recall that $\mathcal{W}$ is preserved by the damped $(\alpha, \beta)$-Szegö flow since it corresponds to the symbol of the shifted Hankel operator of rank 1.

For $u \in \mathcal{W}$, we calculate the mass and the conserved momentum as follows

$$\|u\|^2_{L^2} = |b|^2 + \frac{|c|^2}{1 - |p|^2}, \quad \mathcal{M}(u) = \frac{|c|^2}{(1 - |p|^2)^2}.$$ 

We will repeatedly use the relation between the mass and the momentum

$$\|u\|^2_{L^2} = |b|^2 + \mathcal{M}(u)(1 - |p|^2). \quad (4.1)$$

We will consider the solutions of (1.1) with a fixed momentum $\mathcal{M}(u(t)) = M > 0$. We define two subsets of $\mathcal{W}$

$$\mathcal{E}_M = \{ u \in \mathcal{W} | \mathcal{M}(u) = M \}, \quad \mathcal{C}_M = \{ u(x) = c e^{ix} | |c|^2 = M \}.$$ 

We observe that $\mathcal{C}_M \subset \mathcal{E}_M$ is invariant under the damped $(\alpha, \beta)$-flow (1.1) which consists of the periodic trajectories.

We write the damped $(\alpha, \beta)$-Szegö equations on $\mathcal{E}_M$ in the $(b, c, p)$-coordinate as

$$\begin{cases}
ib' + (\nu - \alpha)b = (|b|^2 + 2M(1 - |p|^2))b + Mc, \\
n\nu c' = 2|b|^2c + 2M(1 - |p|^2)bp + (1 - \beta)Mc, \\
n\nu p' = \sqrt{b} + (1 - \beta)Mp(1 - |p|^2),
\end{cases} \quad (4.2)$$

where $\nu > 0$ is the coefficient of the damping term in (1.1).

We will determine all types of trajectories of the damped $(\alpha, \beta)$-Szegö equations on $\mathcal{E}_M \setminus \mathcal{C}_M \subset \mathcal{W}$, which consists of blow-up and scattering trajectories. By lemma 2.1, the $L^2$ norm of $u(t)$ converges

$$\|u(t)\|^2_{L^2} = |b(t)|^2 + M(1 - |p(t)|^2) \to Q \quad \text{and} \quad |b(t)| = |u(t)| \to 0,$$

as $t \to +\infty$, which implies $M(1 - |p(t)|^2) \to Q$. As a consequence, $|p(t)|$ admits a limit in $[0, 1]$. We claim that this limit can only be 0 or 1, which corresponds to the scattering trajectories or the blow-up trajectories respectively.

We prove that the limit of $|p(t)|$ can only be 0 or 1 by contradiction. Indeed, if $0 < \lim_{t \to +\infty} |p(t)|^2 < 1$, then the trajectory $\{u(t)\}$ is compact in $H^1(\mathbb{T})$. As a consequence, $u(t)$ has a weak limit $u_{\infty} \in \mathcal{W}$. By proposition 2.2,

$$S_{\nu, \alpha, \beta}(t)u_\infty = b_\infty(t) + \frac{c_\infty(t)e^{is}}{1 - p_\infty(t)e^{is}}$$

is a solution of the $(\alpha, \beta)$-Szegö equation (1.6) with $(S_{\nu, \alpha, \beta}(t)u_\infty|1) = b_\infty(t) = 0$. Moreover, the triplet $(0, c_\infty, p_\infty)$ satisfies the ODE system (4.2), which implies

$$\mathcal{M}(c_\infty(t)p_\infty(t)) = 0.$$
On the other hand, the momentum conservation law
\[ \mathcal{M}(S_{\nu,\alpha,\beta}(t)u_0) = \mathcal{M}(u_\infty) = \mathcal{M}(u_0) > 0 \]
ensures that \( c_\infty \) cannot be 0. Therefore, \( p_\infty = 0 \), which contradicts our assumption.

We will show that the case \( |p(t)| \to 0 \) corresponds to trajectories which exponentially converge to \( \mathcal{C}_M \). The conserved momentum \( \mathcal{M}(u) = \frac{\nu^2}{\nu^2 + (1 - \beta)M - \alpha^2} = M \) implies that
\[ |c(t)|^2 \to M. \]

On the other hand, the decay of \( |b(t)| \) (showed in lemma 2.1) implies that
\[ \|u(t)\|_{L^2}^2 = |b(t)|^2 + M(1 - |p(t)|^2) \to M. \]

Since \( \|u(t)\|_{L^2}^2 \) decays monotonically, to study the non-periodic trajectories corresponding to \( |p(t)| \to 0 \), one only needs to study the trajectories \( \{u(t)\} \) disjoint from \( \mathcal{C}_M \) and
\[ \|u(t)\|_{L^2}^2 \geq M, \quad \forall \ t \geq 0. \]

We will show that all the initial values \( u_0 \) with corresponding \( |p(t)| \to 1 \) form a dense open set of \( \mathcal{W} \), on which the growth of the \( H^s \) norm of \( S_{\nu,\alpha,\beta}(t)u_0 \) is of order \( t^{-\frac{1}{2}} \) as \( t \to +\infty \). This case is corresponding to the initial values satisfying
\[ \|S_{\nu,\alpha,\beta}(t)u_0\|_{L^2}^2 < M \tag{4.3} \]
for some \( t \). We remark that (4.3) is equivalent to
\[ \|S_{\nu,\alpha,\beta}(t)u_0\|_{L^2}^2 < F(u_0) \]
for some \( t \), which includes the sufficient conditions in theorem 3.1 for the case \( \beta \neq 1 \). Indeed, \( F(u_0) = \mathcal{M}(u_0) = M \) when \( u_0 \in \mathcal{W} \), and \( \|S_{\nu,\alpha,\beta}(t)u_0\|_{L^2}^2 \) is decreasing in time.

The following theorem of the alternative holds:

**Theorem 4.1.** Let \( \nu > 0 \) and \( \alpha, \beta \in \mathbb{R} \). Then there exists a three dimensional submanifold \( \Sigma_{M,\nu,\alpha,\beta} \subset \mathcal{E}_M \), disjoint from \( \mathcal{C}_M \) and invariant under the flow \( S_{\nu,\alpha,\beta}(t) \), such that \( \Sigma_{M,\nu,\alpha,\beta} \cup \mathcal{C}_M \) is closed and the following holds:

(a) If \( u_0 \in \mathcal{E}_M \setminus (\Sigma_{M,\nu,\alpha,\beta} \cup \mathcal{C}_M) \), then \( \|S_{\nu,\alpha,\beta}(t)u_0\|_{H^s}^2 \) blows up with the rate
\[ \|S_{\nu,\alpha,\beta}(t)u_0\|_{H^s}^2 \sim a^2 t^{2s-1}, \quad s > \frac{1}{2}. \tag{4.4} \]
where
\[ a^2(s, \nu, \alpha, \beta, M) = \Gamma(2s + 1)M^{4s-1} \left( \frac{\nu^2 + (1 - \beta)M - \alpha^2}{2\nu} \right)^{1-2s}. \]

(b) If \( u_0 \in \Sigma_{M,\nu,\alpha,\beta} \), then \( S_{\nu,\alpha,\beta}(t)u_0 \) tends to \( \mathcal{C}_M \) as \( t \to +\infty \), and
\[ \text{dist}(S_{\nu,\alpha,\beta}(t)u_0, \mathcal{C}_M) \sim e^{-\frac{1}{2\nu t^2}}. \]
where \( \sigma = \left( \frac{\nu^2 - \alpha^2 - 4M\alpha}{2} + \sqrt{\frac{\nu^2 - \alpha^2 - 4M\alpha}{2} + 4\gamma^2(2M + \alpha)} \right)^2 \geq \nu. \)

This alternative behavior holds for all \((\nu, \alpha, \beta)\), which is consistent with the dynamics of the damped Szegö equation \((\alpha = \beta = 0)\). Indeed, we will follow a similar argument as for the damped Szegö equation \([GG20]\) to show the above theorem and mainly point out the differences. The first and second parts of the above theorem will be proved in subsections 4.1 and 4.2 respectively.

### 4.1. When \(|p(t)| \to 1\)

We introduce a reduced system with

\[
\eta = |b|^2, \quad \gamma = M(1 - |p|^2), \quad \zeta = Mcbp,
\]

which satisfy the following ODE system

\[
\begin{cases}
\eta' + 2\nu \eta = 2 \text{Im} \zeta, \\
\gamma' = -2 \text{Im} \zeta, \\
\zeta' + (\nu + i(1 - \beta)M - i\alpha)\zeta = i\zeta((3 - \beta)\gamma - \eta) - 2i\eta \gamma M \\
&\quad + i\gamma^2(M - \gamma + 3\eta).
\end{cases}
\]

(4.5)

For \(u \in W\), notice that \(\hat{u}(k) = cp^{k-1}, k \geq 1\), then we have

\[
\|u(t)\|_{H^s}^2 \approx \sum_{k=0}^{\infty} (1 + k^2) |\hat{u}(k)|^2 \\
\approx \frac{\Gamma(2s + 1)M}{(1 - |p(t)|^2)^{2s-1}} = \Gamma(2s + 1)M^{2s} \gamma(t)^{1-2s}.
\]

Hence, we only need to show

\[
\gamma(t) \sim \frac{\kappa}{t}, \quad \kappa = \frac{\nu^2 + ((1 - \beta)M - \alpha)^2}{2\nu M}
\]

(4.6)

to obtain (4.4)

\[
\|u(t)\|_{H^s}^2 \sim a^2 t^{2s-1}
\]

with

\[
a^2(s, \nu, \alpha, \beta, M) = \Gamma(2s + 1)M^{2s} \kappa^{1-2s}.
\]

We observe some facts in this case. The conserved momentum implies that \(1 - |p|^2\) and \(|c|\) decay with the same rate. The integrability and decay of \(|b|\) were given in lemma 2.1. As a consequence, in \((\eta, \gamma, \zeta)\)-coordinate one has

\[
\eta \in L^1(\mathbb{R}_+) \quad \text{and} \quad \zeta = o(\gamma).
\]

(4.7)

To show (4.6), we take the imaginary part of \(\zeta\) equation and use \(\gamma' = -2 \text{Im} \zeta\) to derive
\[
\frac{\text{Im } \zeta'}{\nu + i((1 - \beta)M - \alpha)} - \frac{\gamma'}{2} = \text{Im } f + \text{Im } r, \tag{4.8}
\]

where
\[
f = \frac{i}{\nu + i((1 - \beta)M - \alpha)} \left( M - \gamma + (3 - \beta)\frac{\zeta}{\gamma} \right)
\]
and
\[
r = -\frac{\eta}{\nu + i((1 - \beta)M - \alpha)} (\zeta + 2\gamma M - 3\gamma^2).
\]

The boundedness of \(\eta, \gamma, \zeta\) and the integrability of \(\eta\) ensure that \(r \in L^1(\mathbb{R}_+)\). As a consequence of (4.8), one has \(\text{Im } f \in L^1(\mathbb{R}_+)\). Furthermore, the structure of \(f\) ensures that \(\gamma^2 \in L^1(\mathbb{R}_+)\).

Now, one can integrate the both side of (4.8) to derive
\[
(1 + o(1))\gamma(t) = 2\int_t^\infty \text{Im } f + 2\int_t^\infty \text{Im } r,
\]
where in the left-hand side, we use the fact that \(\zeta = o(\gamma)\). Computing the right-hand side, we have
\[
\int_t^\infty \text{Im } f(s) \, ds = \frac{1}{2\kappa} \left( \int_t^\infty \gamma(s)^2 \, ds \right) (1 + o(1))
\]
and
\[
\int_t^\infty \text{Im } r(s) \, ds = O \left( \int_t^\infty \eta(s) \gamma(s) \, ds \right) = o(\sup_{s \geq t} \gamma(s)),
\]
where the last equality holds due to the integrability of \(\eta\).

Now we arrive at
\[
\gamma(t) = \frac{1}{\kappa} \left( \int_t^\infty \gamma(s)^2 \, ds \right) (1 + o(1)) + o(\sup_{s \geq t} \gamma(s)). \tag{4.9}
\]

We take \(\sup_{s \geq t}\) on the above equality to get
\[
\sup_{s \geq t} \gamma(s) = \frac{1}{\kappa} \left( \int_t^\infty \gamma(s)^2 \, ds \right) (1 + o(1)).
\]
Inserting this equality in the equation (4.9) gives
\[
\gamma(t) = \frac{1}{\kappa} \left( \int_t^\infty \gamma(s)^2 \, ds \right) (1 + o(1)).
\]
Solving this integral equation gives
\[
\gamma(t) = \frac{\kappa}{t} (1 + o(1))
\]
as desired.
4.2. When \(|p(t)| \to 0\)

In this subsection, we investigate the trajectories \(\{u(t)\}\) with momentum \(M\), which do not lie in \(C_M\) but converge to it. As a consequence of lemma 2.1, the \(L^2\) norm of \(u(t)\) decays to the momentum \(M\), namely

\[
\|u(t)\|_{L^2}^2 = |b(t)|^2 + M(1 - |p(t)|^2) \to M,
\]

and \(\|u(t)\|_{L^2}^2 \geq M, \quad \forall \ t \geq 0\).

We first define the set \(\Sigma_{M,p,a,b} \subset E_M\) as following

\[
\Sigma_{M,p,a,b} = \{u_0 \in E_M \setminus C_C | \|S_{p,a,b}(t)u_0\|_{L^2}^2 \geq M, \quad \forall \ t \geq 0\}. \tag{4.10}
\]

At the end of this subsection, we will see that \(\Sigma_{M,p,a,b}\) is a three-dimensional submanifold in \(E_M\). We first observe some facts of the trajectories with \(u_0 \in \Sigma_{M,p,a,b}\):

- Since \(\|u(t)\|_{L^2}^2 = |b|^2 + M(1 - |p|^2) \geq M\), one has
  \[
  |b|^2 \geq M|p|^2. \tag{4.11}
  \]

- One has
  \[
  b(t) \neq 0, \quad \forall \ t \in \mathbb{R}. \tag{4.12}
  \]

Otherwise, \(b\) and \(p\) would cancel at the same time and the trajectory would not be disjoint from \(C_M\).

- By lemma 2.1, the inequality (4.11), and the identity \(|c(t)|^2 = M(1 - |p|^2)^2\), one has
  \[
  |b|^2, \ |p|^2 \in L^1(\mathbb{R}_+) \quad \text{and} \quad |c(t)|^2 \in L^\infty(\mathbb{R}_+). \tag{4.13}
  \]

We are going to show the second part of theorem 4.1 in the following four steps: in step 1, we show that \(u(t)\) converges to \(C_M\) as \(t \to +\infty\); in step 2, we establish a scattering property of a reduced system related to \(b, c, p\); in step 3, the asymptotic behavior of \(u(t)\) can be recovered on the basis of step 2. The geometric structure of \(\Sigma_{M,p,a,b}\) will be discussed in step 4.

**Step 1: convergence of \(c(t)\) and \(\frac{\overline{c(t)} c(t)}{\|c(t)\|^2}\).** We show that there exists \(\theta \in T\) such that

\[
\exp(iM^{1/2}c(t)) \to c_\infty = \sqrt{M} e^{i\theta}
\]

and

\[
\exp(-iM^{1/2}b(t)) \to \left(\frac{\zeta \rho - \alpha + i(\nu - \sigma)}{2M} - 1\right) e^{-i\theta} \tag{4.14}
\]

as \(t \to +\infty\). Here

\[
\zeta = \text{sgn}(\alpha + 2M) \in \{-1, 1\} \tag{4.15}
\]

and \(\sigma, \rho\) are real non negative numbers satisfying

\[
\sigma^2 - \rho^2 = \nu^2 - \alpha^2 - 4\alpha M \quad \text{and} \quad \zeta \rho = \nu(\alpha + 2M). \tag{4.16}
\]
In particular
\[ \sigma = \left( \psi^2 - \alpha^2 - 4M\alpha + \sqrt{(\psi^2 - \alpha^2 - 4M\alpha)^2 + 4\nu^2(\alpha + 2M)^2} \right)^{\frac{1}{2}}. \]

(4.17)

We first derive the convergence of \( c(t) \). The \( c(t) \)-equation in the ODE system (4.2) implies that
\[ i \frac{d}{dt}(e^{iM(1-\beta)}c(t)) = e^{iM(1-\beta)}[2|b(t)|^2 c(t) + 2M(1 - |p(t)|^2)b(t)p(t)]. \]

(4.18)

The fact (4.13) ensures the integrability of the right-hand side of the above ODE, together with the conserved momentum, one obtains that
\[ e^{iM(1-\beta)}c(t) \rightarrow c_{\infty} = \sqrt{M} e^{i\theta}, \quad \theta \in \mathbb{T}. \]

Since \(|b(t)| \rightarrow 0\), we choose \( \varepsilon = \varepsilon \) for some \( T \gg 1 \). As a consequence of (4.11), one has \(|p(T)| \lesssim \sqrt{\varepsilon}\). Furthermore, we claim that
\[ |c(T) - c_{\infty} e^{-iTM(1-\beta)}| = O\left( \int_T^\infty |b(t)|^2 \ dt \right) = O(\varepsilon^2). \]

Indeed, in the above equation, the first equality holds by integration of (4.18). For the second estimate, one integrates the Lyapunov functional (2.1)
\[ \frac{d}{dt}(|b(t)|^2 + M(1 - |p(t)|^2)) = -2\nu|b(t)|^2, \]

from \( T \) to \( \infty \) to get
\[ O(\varepsilon^2) = |b(T)|^2 - M|p(T)|^2 = 2\nu \int_T^\infty |b(t)|^2 \ dt. \]

It proves the second estimate.

We combine these estimates of \( b, c, p \) at time \( T \) and the structure of \( u \) on \( \mathcal{W} \) to obtain the following estimates in any \( H^r(\mathbb{T}) \)
\[ \text{dist}(u(T), c_{\infty} e^{-iTM(1-\beta)} e^{ix}) = O(\varepsilon), \]
\[ \text{dist}(u(T), c_{\infty} e^{-iTM(1-\beta)} e^{ix} + c_{\infty} e^{-iTM(1-\beta)} p(T) e^{ix}) = O(\varepsilon^2). \]

(4.19)

Now we are ready to show the convergence of \( \sqrt{M} \mathcal{L}_2(\mathbb{T}) \) by a linearisation argument. Roughly speaking, we use the convergence (4.19) to linearise the trajectories after time \( T \) and check the \( L^2 \)-norm of the solution. For reader’s convenience, we mention that a baby example of the linearisation procedure around solutions with periodic trajectories for the damped Szegö equation was provided in [GG20].

As a consequence of (4.19), we study the following trajectory
\[ S_{\varepsilon,\psi,\alpha}(t)u(T)(x) = e^{-iTM(1-\beta)}(c_{\infty} e^{-iTM(1-\beta)} e^{ix} + \varepsilon v(t, x) + \varepsilon^2 w(t, x)), \]
where \( w \) is uniformly bounded in the sense
\[ \| w(t) \|_{H^r} \leq C_{\varepsilon, R}, \quad \forall \ t \in [0, R]. \]
To derive the equation $v$ satisfied, we calculate the following quantities

\[ e^{im(1-\beta)}\partial_t u(t) = -iM(1-\beta)(c_{\infty}e^{-it\mathcal{M}(1-\beta)}e^{i\alpha} + \varepsilon v) + \varepsilon \partial_t v + O(\varepsilon^2), \]

\[ e^{im(1-\beta)}(u(t)|t\rangle[1] = \varepsilon(v|1) + O(\varepsilon^2), \]

\[ e^{im(1-\beta)}\Pi(|u(t)|^2u'(t)) = M\alpha e^{-it\mathcal{M}(1-\beta)}e^{i\alpha} + \varepsilon 2Mv \]

\[ + \varepsilon^2 \Pi(e^{-it\mathcal{M}(1-\beta)}e^{i\alpha'}) + O(\varepsilon^2), \]

\[ e^{im(1-\beta)}S\Pi(|S'u'(t)|^2S'u'(t)) = M\alpha e^{-it\mathcal{M}(1-\beta)}e^{i\alpha} + \varepsilon 2M e^{i\alpha'}(e^{-it\mathcal{M}(1-\beta)} - \varepsilon e^{-it\mathcal{M}(1-\beta)}e^{i\alpha'}) + O(\varepsilon^2), \]

Then $v$ satisfies the equation

\[ i\partial_t v + (i\nu - \alpha)(v|1) = \varepsilon^2 e^{-it\mathcal{M}(1-\beta)}\Pi(e^{i\alpha'}) + (\beta + 1)Mv \]

\[ - \beta c_{\infty}^2 e^{-it\mathcal{M}(1-\beta)} e^{i\alpha'}(e^{-it\mathcal{M}(1-\beta)} e^{i\alpha'}) - 2\beta M e^{i\alpha'}(e^{-it\mathcal{M}(1-\beta)} e^{i\alpha'}) + \beta c_{\infty}^2 e^{-it\mathcal{M}(1-\beta)} e^{i\alpha'}(v|1). \]

with the initial value $v(0, x) = \frac{b(T)}{\varepsilon} + c_{\infty} e^{-it\mathcal{M}(1-\beta)} e^{i\alpha'} e^{i\alpha}$. We observe the equation of $v$ and make the ansatz

\[ v(t, x) = q_0(t) + q_1(t)e^{i\alpha} + q_2(t)e^{i\alpha'}, \]

where

\[ iq_0' + (i\nu - \alpha)q_0 = (1 + \beta)Mq_0 + c_{\infty}^2 e^{-it\mathcal{M}(1-\beta)} q_2, \]

\[ iq_1' = (1 - \beta)(c_{\infty}^2 e^{-it\mathcal{M}(1-\beta)} q_1 + Mq_1), \]

\[ iq_2' = (1 - \beta)Mq_2 + c_{\infty}^2 e^{-it\mathcal{M}(1-\beta)} q_0, \]

\[ q_0(0) = \frac{b(T)}{\varepsilon}, \quad q_2(0) = c_{\infty} e^{-it\mathcal{M}(1-\beta)} \frac{b(T)}{\varepsilon}. \]

We take the derivative of the $q_0$-equation and substitute the equation of $q_2$ to derive

\[ q_0'' + (\nu + i(\alpha + 2\beta M))q_0' - ((1 - \beta)(i\nu - \alpha)M + \beta^2 M^2)q_0 = 0, \]

\[ q_0(0) = \frac{b(T)}{\varepsilon}, \]

\[ q_0'(0) = -i(\nu + i(\alpha + 2\beta M)) \frac{b(T)}{\varepsilon} - iM c_{\infty} e^{-it\mathcal{M}(1-\beta)} \frac{b(T)}{\varepsilon}. \]

The characteristic equation of this second-order ODE reads

\[ \lambda^2 + (\nu + i(\alpha + 2\beta M))\lambda - ((1 - \beta)(i\nu - \alpha)M + \beta^2 M^2) = 0. \]
The solutions are given by
\[
\lambda_{\pm} = \frac{-(\nu + i(\alpha + 2\beta M)) \pm (\sigma + i\zeta \rho)}{2},
\]
where \(\zeta, \rho, \sigma\) are defined by equations (4.15) and (4.16).

We will prove in appendix that
\[
\frac{\sigma - \nu}{\sigma + \nu} > 0
\]
so that \(\sigma > \nu\). Hence, the real parts of \(\lambda_{+}\) and \(\lambda_{-}\) admits different signs as
\[
\text{Re}(\lambda_{+}) = \sigma - \nu > 0 \quad \text{and} \quad \text{Re}(\lambda_{-}) = -\sigma - \nu < 0.
\]

And hence, the solution \(q_0\) is given by
\[
q_0(t) = A_+ e^{\lambda_+ t} + A_- e^{\lambda_- t}, \quad (4.20)
\]
where
\[
A_+(T) = \frac{q_0(0) - \lambda_- q_0(0)}{\lambda_+ - \lambda_-}
= \frac{(\nu + i(\alpha + (1 + \beta)M) + \lambda_+) b(T) + iMC_\infty e^{-iTM(1-\beta)p(T)}}{\varepsilon(T)(\sigma + i\zeta \rho)},
\]
\[
A_-(T) = \frac{\lambda_- q_0(0) - q_0(0)}{\lambda_+ - \lambda_-}
= \frac{(\nu + i(\alpha + (1 + \beta)M) + \lambda_-) b(T) + iMC_\infty e^{-iTM(1-\beta)p(T)}}{\varepsilon(T)(\sigma + i\zeta \rho)}.
\]

Now we are ready to check the \(L^2\)-norm of \(S_{\nu,\alpha,\beta}(t)u(T)\), especially the two important features: \(\|S_{\nu,\alpha,\beta}(t)u(T)\|_{L^2}^2 \geq M\) and the Lyapunov functional. We fix \(T\), the following estimate holds for all \(t \in [0, R]\)
\[
0 \leq \|S_{\nu,\alpha,\beta}(t)u(T)\|_{L^2}^2 - M
= \|u(T)\|_{L^2}^2 - M - 2\nu \int_0^t |(S_{\nu,\alpha,\beta}(s)u(T)|1\|)^2 ds
= |b(T)|^2 - M|p(T)|^2 - 2\nu \varepsilon^2 \int_0^t (|v(s)||1\|)^2 + O_\varepsilon \varepsilon) ds.
\]

We divide the above inequality both sides by \(|b(T)|^2\) to derive
\[
1 - \frac{M|p(T)|^2}{|b(T)|^2} - 2\nu \int_0^t (|v(s)||1\|)^2 ds \geq -c_R\varepsilon(T), \quad \forall \ t \in [0, R],
\]
where \(c_R\) is a constant depending only on \(R\). Recall \(q_0(t) = A_+(T)e^{\lambda_+ t} + A_-(T)e^{\lambda_- t}\). By computing the integral of \(q_0\) and using the above estimate, we infer the existence of a constant \(B\) such that
\[
|A_+(T)|^2 e^{(\sigma - \rho)R} \leq c_R\varepsilon(T) + B.
\]
We first take upper limit in $T$ of the above inequality
\[
\limsup_{T \to +\infty} |A(T)|^2 e^{(\sigma - \nu)R} \leq B.
\]
Then we take limit in $R \to \infty$, the above inequality holds only if
\[
\limsup_{T \to +\infty} |A(T)|^2 = 0,
\]
which implies the convergence (4.14)
\[
e^{-iM(1-\beta)\sqrt{\frac{b(T)}{T}} M} \rightarrow \left(\frac{\zeta \rho - \alpha + i(\nu - \sigma)}{2M} - 1\right) e^{-i\theta}
\]
as $T \to +\infty$.

**Step 2: scattering properties for $\eta, \delta, \zeta$.** We write
\[
\eta = |b|^2, \quad \delta = M|p|^2, \quad \zeta = Mc\bar{p}b,
\]
which satisfy the ODE system
\[
\begin{aligned}
\eta' + 2\nu \eta &= 2 \text{Im} \zeta, \\
\delta' &= 2 \text{Im} \zeta, \\
\zeta' + (\nu - i(2M + \alpha))\zeta &= -i((3 - \beta)\delta + \eta)\zeta - 2i\eta\delta(M - \delta) \\
&\quad + i(M - \delta)^2(\delta + \eta).
\end{aligned}
\]  

(4.21)

Due to the decay and boundedness of $(b, c, p)$ in (4.13), we have $(\eta(t), \delta(t), \zeta(t)) \to (0, 0, 0)$ and $\eta \in L^1(\mathbb{R}_+)$. The Lyapunov functional in the $(\eta, \delta, \zeta)$-coordinate can be written as
\[
\eta(t) - \delta(t) = 2\nu \int_t^\infty \eta(s)ds.
\]  

(4.22)

The convergence (4.14) implies that
\[
\frac{\delta(t)}{\eta(t)} \rightarrow \left|\frac{\zeta \rho - \alpha + i(\nu - \sigma)}{2M} - 1\right|^2, \text{ as } t \to +\infty.
\]  

(4.23)

We will prove in **appendix** that
\[
\left|\frac{\zeta \rho - \alpha + i(\nu - \sigma)}{2M} - 1\right|^2 = \frac{\sigma - \nu}{\sigma + \nu}.
\]  

(4.24)

Combining the Lyapunov functional (4.22) and the convergence (4.23), the decay rate of $\eta(t)$ is given by
\[
\frac{\eta(t)}{\int_t^\infty \eta(s)ds} \rightarrow \sigma + \nu, \text{ as } t \to +\infty,
\]  

(4.25)

\[
\log\left(\int_t^\infty \eta(s)ds\right) \rightarrow -(\sigma + \nu)(1 + o(1)), \text{ as } t \to +\infty,
\]  

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and
\[ \eta(t) \leq C \varepsilon e^{-(\sigma + \nu - \varepsilon)t}, \quad t \geq 0, \quad \text{for any } \varepsilon > 0. \]  
\( (4.26) \)

Notice that the estimate (4.26) holds also for \( \delta(t) \) and \( |\zeta(t)| \). We write
\[
X = \begin{pmatrix} \eta(t) \\ \delta(t) \\ \zeta_R(t) = \text{Re} \zeta(t) \\ \zeta_I(t) = \text{Im} \zeta(t) \end{pmatrix} \in \mathbb{R}^4,
\]
then the ODE system of \((\eta, \delta, \zeta)\) can be written as
\[
X' + AX = Q(X), \quad (4.27)
\]
where
\[
A = \begin{pmatrix} 2\nu & 0 & 0 & -2 \\ 0 & 0 & 0 & -2 \\ 0 & 0 & \nu & 2M + \alpha \\ -M^2 & -M^2 & -(2M + \alpha) & \nu \end{pmatrix},
\]
and
\[
Q(X) = \begin{pmatrix} 0 \\ 0 \\ (\eta + (3 - \beta)\delta)\zeta_I \\ -\varepsilon (\eta + (3 - \beta)\delta)\zeta_R - 2M\delta^2 - 4M\eta\delta + \delta^3 + 3\eta\delta^2 \end{pmatrix}.
\]
The matrix \( A \) has the eigenvalues
\[ \nu \pm \sigma, \quad \nu \pm \iota \rho. \]

Now we write the ODE (4.27) as
\[
\frac{d}{dt}(e^{\iota t}X(t)) = e^{\iota t}Q(X(t)),
\]
where the solution \( X(t) \) is given by the Duhamel formula
\[
X(t) = e^{-\iota t}X_\infty - \int_{t}^{\infty} e^{(\iota - \iota t)\iota}Q(X(s)) \, ds. \quad (4.28)
\]
We use estimate (4.26). Since \( \sigma + \nu \) is the largest eigenvalue of \( A \), and \( Q \) is a quadratic-cubic form of \( X \), one has
\[
|X(t)| \lesssim e^{-(\sigma + \nu - \varepsilon)t} M , \\
|e^{\iota t}(Q(X(t)))| \lesssim e^{-(\sigma + \nu - \varepsilon)t} M , \\
|e^{(\iota - \iota t)\iota}Q(X(s))| \lesssim e^{(\sigma + \nu - \varepsilon)(s - t)} M , \quad s, t \geq 0
\]
and
\[
\int_{t}^{\infty} |e^{(\iota - \iota t)\iota}Q(X(s))| \, ds \lesssim e^{2(\sigma + \nu - \varepsilon)t}, \quad t \geq 0.
\]
We substitute the above estimates to (4.28) to obtain
\[ e^{-tA}X_\infty = O(e^{-(\sigma+\nu)t}). \] (4.29)

This shows that \( X_\infty \) is an eigenvector of \( A \) for the eigenvalue \( \sigma + \nu \). Consequently, there exists a constant \( \eta_\infty \in \mathbb{R} \), such that
\[ X_\infty = \eta_\infty \begin{pmatrix} 1 \\ \frac{\sigma - \nu}{\sigma + \nu} \\ \frac{2(M + \alpha)\nu - \sigma}{2\sigma} \\ \frac{\nu - \sigma}{2} \end{pmatrix}. \]

We substitute \( X_\infty \) to (4.28) to conclude that there exists \( \eta_\infty > 0 \) (since \( \eta_\infty(t) > 0 \)) such that
\[ \eta(t) = \eta_\infty e^{-(\sigma+\nu)t}(1 + O(e^{-(\sigma+\nu)t})), \]
\[ \delta(t) = \frac{\sigma - \nu}{\sigma + \nu} \eta_\infty e^{-(\sigma+\nu)t}(1 + O(e^{-(\sigma+\nu)t})), \]
\[ \zeta(t) = \left( \frac{\varsigma \rho - \alpha + i(\nu - \sigma)}{2} - M \right) \eta_\infty e^{-(\sigma+\nu)t}(1 + O(e^{-(\sigma+\nu)t})), \] (4.30)

where the last equality holds since, using (4.16)
\[ (2M + \alpha)(\nu - \sigma) \frac{\nu - \sigma}{2} = \varsigma \rho - \alpha + i(\nu - \sigma) - M. \]

Conversely, we claim that for every \( \eta_\infty > 0 \), there exists a unique triple \((\eta, \delta, \zeta)\) such that
\[ \begin{cases} 
\eta' + 2\nu \eta = 2 \text{Im} \zeta, \\
\delta' = 2 \text{Im} \zeta, \\
\zeta' + (\nu - i(2M + \alpha)) \zeta = -i((3 - \beta) \delta + \eta) \zeta - 2i\nu(\delta - \delta) \\
\text{with } \|X\|_T := \sup_{t \geq T} e^{(\nu+\sigma)t} |X(t)|. 
\end{cases} \] (4.31)

Then the extension to the whole real line is ensured by, say, the identities
\[ |\zeta|^2 = (M - \delta)^2 \eta \delta, \quad (\delta(t) + 2\nu) \int_t^\infty \eta(s) \, ds = \eta(t) \]

which, combined with the first equation, lead to
\[ |\dot{\eta}| = O(\eta). \]

Furthermore, following the same argument as for the damped Szegő equation in [GG20], the lower (upper) bounds of initial value \( \eta(0) \) ensure the lower (upper) bounds for \( \eta_\infty \). Namely, for every \( C > 0 \), there exists \( C' > 0 \) such that
• If \( \eta(0) \geq C^{-1} \), then \( \eta_{\infty} \geq (C')^{-1} \).
• If \( \eta(0) \leq C \), then \( \eta_{\infty} \leq C' \).

**Step 3: asymptotic behavior for \( b, c, p \)**

We first show that there exists a solution \( \eta, \theta, \varphi \in (0, \infty) \times \mathbb{T} \times \mathbb{T} \) such that, as \( t \to +\infty \)

\[
\begin{align*}
b(t) &\sim \sqrt{\eta_{\infty}} e^{-\frac{t}{\eta_{\infty}} - i(2M + \alpha)\frac{\pi}{2} + i\varphi}, \\
c(t) &\sim \sqrt{M} e^{-i[M(1 - \beta) + i\theta]}, \\
p(t) &\sim \sqrt{\frac{\eta_{\infty}}{M}} \left( \frac{\sqrt{\rho - \alpha + i(\nu - \sigma)}}{2M} - 1 \right) e^{-\frac{t}{\eta_{\infty}} + i(2M + \alpha)\frac{\pi}{2} + i(\theta - \varphi)}.
\end{align*}
\]

(4.32)

Notice that, the convergence of \( c(t) \) was shown in (4.2). We combine the equations for \( b \) and \( \eta, p \) and \( \delta \) to derive

\[
\begin{align*}
i \frac{d}{dt} \left( \frac{b}{\sqrt{\eta}} \right) &= \left( \eta - 2\delta + 2M + \alpha + \frac{\text{Re} \, \zeta}{\eta} \right) \frac{b}{\sqrt{\eta}} \\
&= \left( 2M + \alpha \right) \frac{\nu + \sigma}{2\sigma} + O(e^{-(\nu + \sigma)T}) \frac{b}{\sqrt{\eta}} \\
i \frac{d}{dt} \left( \frac{\sqrt{M}p}{\sqrt{\delta}} \right) &= \left( 1 - \beta \right)(M - \delta) + \frac{\text{Re} \, \zeta}{\delta} \frac{\sqrt{M}p}{\sqrt{\delta}} \\
&= -\frac{(\alpha + 2M\beta)\sigma + (2M + \alpha)\nu}{2\sigma} + O(e^{-(\nu + \sigma)T}) \frac{\sqrt{M}p}{\sqrt{\delta}}.
\end{align*}
\]

Then there exist \( \varphi, \psi \) such that

\[
\begin{align*}
b(t) &\sim \sqrt{\eta_{\infty}} e^{-\frac{t}{\eta_{\infty}} - i(2M + \alpha)\frac{\pi}{2} + i\varphi}, \\
p(t) &\sim \sqrt{\frac{\eta_{\infty}}{M}} \left( \frac{\sqrt{\rho - \alpha + i(\nu - \sigma)}}{\sigma + \nu} \right) e^{-\frac{t}{\eta_{\infty}} + i(2M + \alpha)\frac{\pi}{2} + i(\theta - \varphi)}.
\end{align*}
\]

On the other side, we recall the convergence (4.14)

\[
e^{-iM(1 - \beta)} \sqrt{M} \frac{p(t)}{b(t)} \to \left( \frac{\sqrt{\rho - \alpha + i(\nu - \sigma)}}{2M} - 1 \right) e^{-i\theta}
\]

and the equality (4.24)

\[
\left( \frac{\sigma - \nu}{\sigma + \nu} \right)^{\frac{1}{2}} = \left| \frac{\sqrt{\rho - \alpha + i(\nu - \sigma)}}{2M} - 1 \right|.
\]
This implies that
\[ p(t) \sim \frac{\sqrt{\eta_{\infty}}}{M} \left( \frac{\zeta \rho - \alpha + i(\nu - \sigma)}{2M} - 1 \right) e^{-\frac{1}{2M} t + i(\nu + 2M/\sigma + i(\nu - \sigma)) (\theta - \varphi)}. \]

Now we show that the asymptotic behavior (4.32) holds conversely. Namely, for a fixed \((\eta_{\infty}, \theta, \varphi) \in (0, \infty) \times \mathbb{T} \times \mathbb{T}\), there exists a unique trajectory
\[ u(t, \xi) = b(t) + \frac{c(t)e^{it}}{1 - p(t)e^{it}} \]
satisfying the asymptotic behavior (4.32). From step 2, for fixed \((\eta_{\infty}, \theta, \varphi) \in (0, \infty) \times \mathbb{T} \times \mathbb{T}\) there exists a unique triplet \((\eta, \delta, \zeta)\) satisfying (4.21) such that
\[
\begin{align*}
\eta(t) &= \eta_{\infty} e^{-(\sigma + \nu)t}(1 + O(e^{-(\sigma + \nu)t})), \\
\delta(t) &= \frac{\sigma - \nu}{\sigma + \nu} \eta_{\infty} e^{-(\sigma + \nu)t}(1 + O(e^{-(\sigma + \nu)t})), \\
\zeta(t) &= \left(\frac{\zeta \rho - \alpha + i(\nu - \sigma)}{2} - M\right) \eta_{\infty} e^{-(\sigma + \nu)t}(1 + O(e^{-(\sigma + \nu)t})).
\end{align*}
\]

Due to the structure of \(\eta, \delta, \zeta\), there exists a fixed large enough \(T > 0\) such that \(M > \delta(T) > 0\), \(\zeta(T) \neq 0\) and \(\eta(T) > 0\). Then there exists \((b_1, c_1, p_1)\) solving the ODE system (4.2) such that
\[ b_1(T) = \sqrt{\eta(T)}, \quad \sqrt{M} p_1(T) = \sqrt{\delta(T)}, \quad M c_1(T) = \frac{\zeta(T)}{b_1(T) p_1(T)}. \]

Furthermore, due to the uniqueness of the Cauchy problem of the ODE system for \((\eta, \delta, \zeta)\), the above equations hold for all \(t \in \mathbb{R}\). On the other hand, \((b_1, c_1, p_1)\) satisfies the asymptotic behavior (4.32) with a pair \((\theta_1, \varphi_1) \in \mathbb{T} \times \mathbb{T}\), i.e.
\[
\begin{align*}
b_1(t) &\sim \sqrt{\eta_{\infty}} e^{i(\theta_1 - \varphi_1) t + \frac{i(\nu + 2M/\sigma + i(\nu - \sigma))}{2M} (\theta - \varphi)} , \\
c_1(t) &\sim \sqrt{M} e^{iM(1 - \beta) + i\delta_1} , \\
p_1(t) &\sim \frac{\sqrt{\eta_{\infty}}}{M} \left( \frac{\zeta \rho - \alpha + i(\nu - \sigma)}{2M} - 1 \right) e^{i(\nu + 2M/\sigma + i(\nu - \sigma)) (\theta - \varphi) - \frac{1}{2M} t + i(\nu + 2M/\sigma + i(\nu - \sigma)) (\theta - \varphi)}.
\end{align*}
\]

Then the triplet \((b, c, p)\) with
\[
\begin{align*}
b(t) &= e^{i(\nu - \sigma) t} b_1(t) , \\
c(t) &= e^{i(\delta - \varphi_1)} c_1(t) , \\
p(t) &= e^{i(\theta - \delta_1 + \varphi_1) t} p_1(t)
\end{align*}
\]
satisfies the ODE system (4.2) with the desired asymptotic properties.

At the last step, we show the uniqueness of the solution \((b, c, p)\). We assume that \((\tilde{b}, \tilde{c}, \tilde{p})\) is another solution of the ODE system (4.2) with the same asymptotic properties (4.32). Then by the uniqueness of \((\eta, \delta, \zeta)\), for any \(t \in \mathbb{R}\)
\[
|\tilde{b}(t)|^2 = \eta(t), \quad M |\tilde{p}(t)|^2 = \delta(t), \quad M c(t) \bar{b}(t) \bar{p}(t) = \zeta(t)
\]
and hence, \(b - \tilde{b}\) and \(p - \tilde{p}\) satisfy
Since any equation (1.1). This would impose that \( \langle u \rangle = 0 \), hence \( u \in C_M \), which is impossible since \( \Sigma_{M,p,\alpha,\beta} \) is disjoint from \( C_M \).

Step 4: geometric structure of \( \Sigma_{M,p,\alpha,\beta} \subset \mathcal{E}_M \). We define the map
\[
J : (0, \infty) \times \mathbb{T} \times \mathbb{T} \to \mathcal{E}_M, \quad (\eta, \theta, \varphi) \mapsto u(0),
\]
where \( u \) is the unique solution of (1.1) corresponding to the asymptotic behavior (4.32). Step 3 ensures that the range of \( J \) is \( \Sigma_{M,p,\alpha,\beta} \) and its injectivity is given by the conserved momentum. In order to prove that \( \Sigma_{M,p,\alpha,\beta} \) is a three-dimensional submanifold of \( \mathcal{E}_M \), one only need to show that \( J \) is a proper immersion.

Smoothness. The smoothness of \( J \) with respect to \( \eta \) is a consequence of the fixed point argument in step 3. The dependence with respect to \( (\theta, \varphi) \) is much more elementary, since it reflects the gauge and translation invariances, hence it is smooth as well.

Immersion. To prove the immersion property, we just have to check that, for every \( (\eta, \theta, \varphi) \), the three vectors
\[
\partial_{\eta} J(\eta, \theta, \varphi), \partial_{\theta} J(\eta, \theta, \varphi), \partial_{\varphi} J(\eta, \theta, \varphi)
\]
are independent. We claim that the subspace spanned by these three vectors is also spanned by \( \partial_{\eta} u(0), -i \partial_{\theta} u(0), i \partial_{\varphi} u(0) \). Indeed, in view of step 3 and of the invariances of equation (1.1), one easily checks the following identity,
\[
e^{i\sigma \alpha} S_{\alpha, \beta}(t + T) [J(\eta, \theta, \varphi)](x + \theta - \varphi) = S_{\alpha, \beta}(t) [J(\eta, \theta, \varphi)](x),
\]
where
\[
\eta = \eta_0 e^{-(\sigma+\nu)T},
\]
\[
\theta = \theta_0 + \theta - M(1 - \beta)T,
\]
\[
\varphi = \varphi_0 + \varphi - \left( M + \frac{\alpha}{2} \right) \left( 1 + \frac{\nu}{\sigma} \right) T.
\]
If these three vectors were dependent, this would mean that \( u \) is a traveling wave of equation (1.1). This would impose that \( (u|1) \equiv 0 \), hence \( u \in C_M \), which is impossible since \( \Sigma_{M,p,\alpha,\beta} \) is disjoint from \( C_M \).
Closeness. Since $C_M$ is compact, it is enough to prove that the closure of $\Sigma_{M,\nu,\alpha,\beta}$ is contained into $\Sigma_{M,\nu,\alpha,\beta} \cup C_M$. Let $(u_n)$ be a sequence of points of $\Sigma_{M,\nu,\alpha,\beta}$ which tends to $u \in W$. Set $(\eta_n, \theta_n, \varphi_n) := J^{-1}(u_n)$. Since $J$ is a homeomorphism onto its range $\Sigma_{M,\nu,\alpha,\beta}$, the only cases to be studied are $\eta_n \to 0$ and $\eta_n \to +\infty$. We use the last part of step 2 that the lower (upper) bounds of initial value $\eta(0)$ ensure the lower (upper) bounds for $\eta(1)$. As a consequence, in the case $\eta_n \to 0$, we obtain that $(u_n|1) = 0$, and more generally that $(S_{\nu,\alpha,\beta}(t)(u_n|1) = 0$, so that $u \in C_M$.

In the second case, we infer $|(u_n|1)| \to \infty$, which contradicts the fact that $u_n$ is convergent.

The proof of theorem 4.1 is complete.

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Appendix. A calculation

In the following, we give the details of the calculations on

$$Z := \left| \frac{\varsigma \rho - \alpha + i(\nu - \sigma)}{2M} - 1 \right|^2 = \frac{\sigma - \nu}{\sigma + \nu},$$

where $\varsigma, \sigma, \rho$ are given by (4.15) and (4.16). We recall the following equalities.

$$\sigma^2 - \rho^2 = \nu^2 - \alpha^2 - 4\alpha M \quad \text{and} \quad \varsigma \sigma \rho = \nu(\alpha + 2M).$$

Remark first that the case $\sigma = \nu$ is excluded. Indeed, if $\sigma = \nu$ then $\rho^2 = \alpha^2 + 4\alpha M$ which is not compatible with the second condition. We calculate

$$Z = \left| \frac{\varsigma \rho - \alpha + i(\nu - \sigma)}{2M} - 1 \right|^2 = 1 + \frac{1}{4M^2}((\varsigma \rho - \alpha)^2 + (\nu - \sigma)^2 - 4M(\varsigma \rho - \alpha))$$

$$= 1 + \frac{1}{4M^2}(\rho^2 + \alpha^2 + \nu^2 + \sigma^2 - 2\nu \sigma + 4M \alpha - 2\varsigma \rho(\alpha + 2M))$$

$$= 1 + \frac{1}{2M^2}(\rho^2 + \nu^2)\left(1 - \frac{\sigma}{\nu}\right).$$

Then as above

$$(1 - Z)(\nu + \sigma) = \frac{\rho^2 + \nu^2 \sigma - \nu(\sigma + \nu)}{2M^2}$$

$$= \frac{1}{2M^2}(\sigma^2 \rho^2 + \sigma^2 \nu^2 - \nu^2 \rho^2 - \nu^2)$$

$$= \frac{\nu}{2M^2}((\alpha + 2M)^2 + \sigma^2 - \rho^2 - \nu^2)$$

$$= \frac{\nu}{2M^2}((\alpha + 2M)^2 - \alpha^2 - 4M \alpha)$$

$$= 2\nu.$$
According to the above calculation, one has
\[
\frac{2\nu}{1-Z} = \nu + \sigma \quad \text{and} \quad Z = \frac{\sigma - \nu}{\sigma + \nu}
\]

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