On the Factorization of Nonlinear Recurrences in Modules
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Abstract

For rings $R$ with identity, we define a class of nonlinear higher order recurrences on unitary left $R$-modules that include linear recurrences as special cases. We obtain conditions under which a recurrence of order $k + 1$ in this class is equivalent to a pair, known as a semiconjugate factorization, that consists of a recurrence of order $k$ and a recurrence of order 1. We show that such a factorization is possible whenever $R$ contains certain sequences of units. Further, if the coefficients of the original recurrence in $R$ are independent of the index then we show that the semiconjugate factorization exists if two characteristic polynomials share a common root that is a unit in $R$. We use this fact to show that an overlapping factorization of these polynomials in an integral domain $R$ yields a semiconjugate factorization of the corresponding recurrence in the module. These results are applicable to systems of higher order, nonlinear difference equations in direct products of rings. Such systems may be represented as higher order equations in a module over the ring.

Key Words. nonlinear recurrence, ring, module, polynomial, unit root, semiconjugate factorization

Mathematics Subject Classifications. 12H10, 16D10, 39A10

1 Introduction

Let $R$ be a ring with identity, $M$ a (unitary) left $R$-module and $k$ a non-negative integer. Then for any given sequence of maps $f_n : M^{k+1} \to M$ the difference equation

$$x_{n+1} = f_n(x_n, x_{n-1}, \ldots, x_{n-k})$$

defines a recurrence of order $k + 1$ in $M$; see Section 2 below for a more precise definition. If each $f_n$ is linear then the recurrence is linear, otherwise it is nonlinear.

Methods of linear algebra are fruitfully used in the factorization of linear recurrences. These methods apply generally to linear systems in commutative rings with identity. Such systems include standard unfoldings of higher order linear recurrences in a ring $R$ to first-order recurrences in an $R$-module over the ring. These notions generalize the familiar concept that a difference (or differential) equation of order $k$ can be unfolded to a first-order equation in the (real) vector space of dimension $k$. First-order linear recurrences in modules and algebras with coefficients in rings are studied using

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standard methods; see, e.g., [1], [2], [3], [11], [17], [18], [28], [30]. In particular, the classical operator factorization of difference (and differential) equations follows from these methods.

By contrast, no standard algebraic methods are known for the factorization of nonlinear difference equations, even in concrete cases involving familiar fields such as the real or complex numbers. A method known as semiconjugate factorization (Section 3 below) applies to large classes of nonlinear recurrences as well as to linear ones. For the linear cases, the factorization obtained is the familiar operator factorization from the classical theory of linear difference equations. It coincides with what is obtained from linear algebra as noted above.

More generally, semiconjugate factorization applies when a form symmetry of the given recurrence is identified according to an existing classification scheme that is not limited to linear equations. The form symmetry is used as an order-reducing substitution to break down, or factor, the original equation into two equations of lower orders. This factorization may be repeated as long as form symmetries are identified. For some background on semiconjugate factorization we refer to [25].

In this paper we study a form of semiconjugate factorization that works well for nonlinear recurrences of type

\[ x_{n+1} = \sum_{i=0}^{k} a_{i,n} x_{n-i} + g_n \left( \sum_{i=0}^{k} b_{i,n} x_{n-i} \right) \]

where the coefficients \( a_{i,n}, b_{i,n} \) are in a ring \( R \) with identity. For each \( n \) the map \( g_n : M \rightarrow M \) is defined on a (unitary) left \( R \)-module \( M \) that also contains the variable \( x_n \) (see the next section for more precise definitions). Since each ring \( R \) is a left \( R \)-module over itself, the ideas and results in this paper also apply to recurrences in rings, where \( x_n \in R \) for all \( n \).

When \( M = R \), (2) generalizes linear recurrences since the latter are represented by (2) with \( g_n(u) = 0 \) or \( g_n(u) = c_n u \) for all \( u \in R \) and a given sequence of constants \( c_n \in R \). Non-homogeneous linear recurrences are also special cases of (2) where \( g_n(u) = d_n \) for every \( u \in R \) and a given sequence of constants \( d_n \in R \).

Special cases of Equation (2) on the set of real numbers (\( M = R = \mathbb{R} \)) have often appeared in the applied literature. The classical economic models of the business cycle in mid-twentieth century are among the early occurrences; see, e.g., [12], [21], [22]. Other special cases of (2) occurred later in mathematical studies of biological models ranging from whale populations to neuron activity; see, e.g., [4], [7], [10] and Section 2.5 in [16]. For instance, the global dynamics of the solutions of the following equation are discussed in [10]

\[ x_{n+1} = \alpha x_n + a \tanh \left( x_n - \sum_{i=1}^{k} b_i x_{n-i} \right) \]

with constant real parameters \( 0 \leq \alpha < 1, \ a > 0 \) and \( b_i \geq 0 \). The studies of global dynamics for other special cases of (2) appear in [9] and [14]; also see [16], Section 6.9.
The dynamical properties of the solutions of the second-order case
\[ x_{n+1} = cx_n + g(x_n - x_{n-1}) \] (3)
have been studied in [6], [15], [23]. Further, the bifurcations of solutions of (3), including the Neimark-Sacker type (discrete analog of the Hopf bifurcation) are studied in [19]. A more general form of (3), i.e.
\[ x_{n+1} = ax_n + bx_n + g_n(x_n - cx_{n-1}) \] (4)
is studied in [24]. In particular, [5] presents sufficient conditions on parameters for the occurrence of limit cycles (attracting periodic solutions) and chaos in recurrences of type
\[ x_{n+1} = \frac{ax_n^2 + bx_{n-1}^2 + cx_nx_{n-1} + dx_n + cx_{n-1} + f}{ax_n + \beta x_{n-1} + \gamma} \]
This equation is obtained by using a single rational mapping \( g \) in (4) of type
\[ g(r) = \frac{Ar + B}{Cr + D}. \]
Like linear non-homogeneous equations mentioned above, equations such as (2) are also meaningful in more general algebraic contexts such as rings or modules. Further, in [26] the autonomous version of (2) where the coefficients do not depend on \( n \) and \( g_n = g \) is fixed for all \( n \), is studied for normed algebras over real or complex numbers. Refinements of this study in the case of complex coefficients are discussed in [27]. On the other hand, in [28] we study the linear, non-homogeneous special case of (2) with variable coefficients in rings, along with some examples and applications. Although these studies are largely focused on the dynamics of solutions, the method that is used to reduce (2) to simpler equations is algebraic in nature. We focus on this aspect and discuss the method in the more general setting of left modules on rings with identity.

The layout of this paper is as follows: In Section 2 we discuss some background issues pertaining to higher order nonlinear recurrences in modules and in Section 3 we present the basics of form symmetries and semiconjugate factorization of higher order recurrences. Next, we define the linear form symmetry in Section 4 and obtain the corresponding semiconjugate factorization of (2) over left \( R \)-modules where \( R \) is a ring with identity. In Section 5 we study the case of constant coefficients and establish a connection between semiconjugate factorization of recurrences and polynomial factorization in rings. Extending the above ideas to vector spaces and modules allows the consideration of systems of nonlinear, higher order difference equations in direct products of rings. We clarify this issue in Section 6. In Section 7 we discuss repeated semiconjugate factorizations and conditions for the complete factorization of a higher order recurrence into a system of first-order recurrences. Section 8 starts a discussion of reducibility of (2) in a more general context than polynomial factorization, in particular, the possibility of obtaining a semiconjugate factorization where polynomial factorization is not possible in the underlying ring. This section is open-ended and mainly a starting point for possible future research.
2 Higher order recurrences in modules

In this section we discuss the basics of recurrences in modules. The basics of rings and modules may be found in texts such as [13] and [20]. For a nonempty set $M$ let $S = M^\mathbb{N}$ be the set of all sequences in $M$; here $\mathbb{N}$ is the set of all positive integers. If $R$ is a ring with identity and $M$ is a left (unitary) $R$-module then $S$ is also a left $R$-module under the usual operations of term-wise addition of sequences and multiplying by scalars (elements of $R$).

For all $\{x\} = \{x_1, x_2, \ldots, x_n, \ldots\} \in S$ and each $n \in \mathbb{N}$ we define a projection map $\pi_n : S \to M$ as $\pi_n \{x\} = x_n$ where $x_n$ is the $n$-th term of $\{x\}$. Note that if $\{x\}, \{y\} \in S$ and $\pi_n \{x\} = \pi_n \{y\}$ for all $n \in \mathbb{N}$ then $\{x\} = \{y\}$. Further, if $M$ is a left $R$-module then $\pi_n$ is a left $R$-module epimorphism on $S$ for every $n$.

Next, the (forward) shift map $E : S \to S$ is defined as $\pi_n \circ E = \pi_{n+1}$ for all $n \in \mathbb{N}$; i.e.

$$\pi_n \circ E \{x\} = \pi_{n+1} \{x\} \quad \text{for } \{x\} \in S \text{ and } n \in \mathbb{N}$$

or simply, $E \{x_1, x_2, \ldots\} = \{x_2, x_3, \ldots\}$. $E$ is well-defined in this way since if $\{x\} = E \{x\}$ and also $\{x''\} = E \{x\}$ then $\pi_n \{x\}' = \pi_{n+1} \{x\} = \pi_n \{x''\}$ for all $n \in \mathbb{N}$ so $\{x\}' = \{x''\}$. Further, $E$ is the unique operator on $S$ with this property since if $E' : S \to S$ also satisfies $\pi_n \circ E' = \pi_{n+1}$ for all $n \in \mathbb{N}$ then $E' = E$ because for each $\{x\} \in S$ and all $n$

$$\pi_n \circ E' \{x\} = \pi_{n+1} \{x\} = \pi_n \circ E \{x\}.$$

As $E$ raises each index of a sequence by 1, repeated applications of $E$ define additional shifts via $E^2 = E \circ E$, etc. It is readily verified that $E$ is a left $R$-module epimorphism when $M$ is a module and the kernel of $E$ is the set of all sequences $\{x, 0, 0, \ldots\}$ for $x \in M$. The set of all fixed points of $E$ is the set of all constant sequences $\{x, x, \ldots\}$ in $S$; this set is a left $R$-submodule and a copy of $M$ in $S$.

To define higher order recurrences in left $R$-modules, let $k$ be a non-negative integer and for each $n \in \mathbb{N}$ let $f_n : M^{k+1} \to M$ be a given map. Consider the set $\mathcal{S}$, possibly empty, of all sequences $\{x\} \in S$ that satisfy the equation

$$\pi_n \circ E^{k+1} \{x\} = f_n \left( \pi_n \circ E^k \{x\}, \pi_n \circ E^{k-1} \{x\}, \ldots, \pi_n \circ E \{x\}, \pi_n \{x\} \right)$$

(5)

for every $n \in \mathbb{N}$. If $\mathcal{S}$ is nonempty then we refer to (5) as a recurrence of order $k + 1$ in $M$ and consider each member of $\mathcal{S}$ a solution of (5). In the first-order case where $k = 0$, (5) reduces to the following

$$\pi_n \circ E = f_n \circ \pi_n$$

(6)

A solution of (6) is a sequence in $S$ for which equality in (6) holds for all $n \in \mathbb{N}$.

Note that the two sides of (5) are in $M$ rather than in $S$. To simplify the notation, we write (6) in the abbreviated form

$$x_{n+k+1} = f_n(x_{n+k}, x_{n+k-1}, \ldots, x_n), \quad n \geq 0$$

4
which is equivalent to (1). The same set of initial values $x_0, x_1, \ldots, x_k$ generate identical solutions in both cases, with $n \geq k$ in (1).

In the classical theory a “scalar” recurrence of order $k+1$ in a field such as the real or complex numbers is often unfolded to a first-order recurrence in a vector space of dimension $k+1$ in the following way: functions $F_n : M^{k+1} \rightarrow M^{k+1}$ are defined as

$$F_n(u_0, u_1, \ldots, u_k) = (f_n(u_0, u_1, \ldots, u_k), u_0, \ldots, u_{k-1})$$

and used to recover (1) form

$$(x_{0,n+1}, \ldots, x_{k,n+1}) = F_n(x_{0,n}, \ldots, x_{k,n})
= (f_n(x_{0,n}, \ldots, x_{k,n}), x_{0,n}, \ldots, x_{k-1,n})$$ (7)

where $x_{j,n} = x_{n-j}$ are new variables. In this sense, (1) in the real or complex context is considered a special case of the “vector equation” or system of first-order equations,

$$(x_{0,n+1}, \ldots, x_{k,n+1}) = F_n(x_{0,n}, \ldots, x_{k,n}).$$

However, (5) is actually a higher-order generalization of this first-order system which is the special case $k = 0$ of (5), with $M$ being the vector space of dimension $k+1$. Thus, (1) in the context of modules extends the standard nonlinear theory to systems of higher order equations in direct products of rings; see comments in the Introduction pertaining to the system (45)-(46) and Example 16 below. There is no standard theory comparable to that for first-order systems in the existing literature for analyzing higher order nonlinear systems. The analysis of such a system may be simplified in cases where semiconjugate factorization reduces the order of the system to one.

3 Semiconjugate factorization

In this section we list some general results from [25] that are valid for all recurrences, not only the linear ones. Let $G$ be a nontrivial group and assume in (1) that $f_n : G^{k+1} \rightarrow G$. Let us unfold (1) in the manner described in the preceding section (which still works in this setting) to a first-order recurrence

$$X_{n+1} = \mathcal{F}_n(X_n)$$
on $G^{k+1}$ where $\mathcal{F}_n : G^{k+1} \rightarrow G^{k+1}$. We assume that $k \geq 1$ because we shall define a factorization of a recurrence into lower order recurrences and first-order recurrences are already lowest in order. In analogy with polynomial factorization, we consider first-order recurrences to be trivially irreducible.

Let $1 \leq m \leq k$ and suppose that there is a sequence of maps $\Phi_n : G^{m} \rightarrow G^{m}$ and a sequence of surjective maps $H_n : G^{k+1} \rightarrow G^{m}$ that satisfy the semiconjugate relation

$$H_{n+1} \circ \mathcal{F}_n = \Phi_n \circ H_n$$ (8)
for a given pair of function sequences \( \{ \mathcal{F}_n \} \) and \( \{ \Phi_n \} \). This may be illustrated as follows:

\[
\begin{align*}
G_n(G_{k+1}) & \xrightarrow{F_n} F_n(G_{k+1}) \\
\downarrow H_n & \downarrow H_{n+1} \\
H_n(G_{k+1}) & = \mathcal{G}^m \xrightarrow{\Phi_n} \Phi_n(H_n(G_{k+1})) = H_{n+1}(F_n(G_{k+1}))
\end{align*}
\]

We say that \( \mathcal{F}_n \) is semiconjugate to \( \Phi_n \) for each \( n \) and that the sequence \( \{ H_n \} \) is a form symmetry of (1). Since \( m < k + 1 \), the form symmetry \( \{ H_n \} \) is order-reducing. Note that if \( H_n = H \) for all \( n \) where \( H \) is injective \( (m = k + 1) \) then \( (8) \) is a conjugacy relation between \( \{ \mathcal{F}_n \} \) and \( \{ \Phi_n \} \).

We state the next basic result from [25] as a lemma here without proof.

**Lemma 1** (Semiconjugate factorization) Let \((G, *)\) be a nontrivial group and let \( k \geq 1, 1 \leq m \leq k \) be integers. If \( h_n : G^{k+m-1} \to G \) is a sequence of functions and the functions \( H_n : G^{k+1} \to G^m \) are defined by

\[
H_n(u_0, u_1, \ldots, u_k) = [u_0 * h_n(u_1, \ldots, u_{k-m+1}), \ldots, u_{m-1} * h_n(u_m, \ldots, u_k)]
\]

where \( * \) denotes the group operation in \( G \) then the following statements are true:

(a) The function \( H_n \) is surjective for every \( n \geq 0 \).

(b) If \( \{ H_n \} \) is an order-reducing form symmetry then the difference equation (1) is equivalent to the system of equations

\[
\begin{align*}
t_{n+1} & = \phi_n(t_n, \ldots, t_{n-m+1}), \\
x_{n+1} & = t_{n+1} * h_{n+1}(x_n, \ldots, x_{n-k+m})^{-1}
\end{align*}
\]

whose orders \( m \) and \( k - m + 1 \) respectively, add up to the order of (1).

(c) The map \( \Phi_n : G^m \to G^m \) is the standard unfolding of Eq. (9) for each \( n \geq 0 \).

**Definition 2** The pair of equations (9) and (10) constitute the semiconjugate factorization, or sc-factorization of (1). This pair of equations is a triangular system since (9) is independent of (10). We call (9) the factor equation of (1) and (10) its cofactor equation.

Note that (9) has order \( m \) and (10) has order \( k - m + 1 \). Consider the following special case of \( H_n \) in Lemma 1 with \( m = k \)

\[
H_n(u_0, \ldots, u_k) = [u_0 * h_n(u_1), u_1 * h_n(u_2), \ldots, u_{k-1} * h_n(u_k)]
\]

where \( h_n : G \to G \) is a given sequence of maps. The semiconjugate factorization of (11) in this case is

\[
\begin{align*}
t_{n+1} & = \phi_n(t_n, \ldots, t_{n-k+1}), \\
x_{n+1} & = t_{n+1} * h_{n+1}(x_n)^{-1}
\end{align*}
\]
in which the factor equation has order $k$ and the cofactor equation has order 1.

The next result gives a necessary and sufficient condition for the existence of a form symmetry of type (11); see [25] for the proof.

**Lemma 3** (Invertible-map criterion) Let $(G, *)$ be a nontrivial group and assume that $h_n : G \to G$ is a sequence of bijections. For arbitrary elements $u_0, v_1, \ldots, v_k \in G$ and every $n \geq 0$ define

$$
\zeta_{j,n}(u_0, v_1, \ldots, v_j) = h_{n-j+1}^{-1}(\zeta_{j-1,n}(u_0, v_1, \ldots, v_{j-1})^* v_j).
$$

(14)

Then (1) has the form symmetry (11) if and only if the quantity

$$
f_n(\zeta_{0,n}, \zeta_{1,n}(u_0, v_1), \ldots, \zeta_{k,n}(u_0, v_1, \ldots, v_k)) * h_{n+1}(u_0)
$$

is independent of $u_0$ for every $n \geq 0$. In this case (1) has a semiconjugate factorization into (12) and (13) where the factor functions in (12) are given by

$$
\phi_n(v_1, \ldots, v_k) = f_n(\zeta_{0,n}, \zeta_{1,n}(u_0, v_1), \ldots, \zeta_{k,n}(u_0, v_1, \ldots, v_k)) * h_{n+1}(u_0).
$$

(16)

For a left $R$-module $M$, the group $G$ in the preceding result is the underlying abelian group $(M,+)$ so that $*$ denotes addition $+$ and group inversion is the ordinary negative. Thus (14), (15) and (13) read, respectively, as follows

$$
\zeta_{j,n}(u_0, v_1, \ldots, v_j) = h_{n-j+1}^{-1}(v_j - \zeta_{j-1,n}(u_0, v_1, \ldots, v_{j-1})),
$$

(17)

$$
f_n(\zeta_{0,n}, \zeta_{1,n}(u_0, v_1), \ldots, \zeta_{k,n}(u_0, v_1, \ldots, v_k)) + h_{n+1}(u_0),
$$

(18)

$$
x_{n+1} = t_{n+1} - h_{n+1}(x_n).
$$

(19)

4  sc-factorization relative to the linear form symmetry

To obtain a sc-factorization for the recurrence (2) we identify a suitable form symmetry $H_n$. Unless otherwise noted, we will assume in the rest of the paper that $R$ is a ring with identity and $M$ a left (unitary) $R$-module.

**Definition 4** Let $\{\alpha_n\}$ be a sequence in $R$ such that $\alpha_n \neq 0$ for all $n$. A **linear form symmetry** on $M$ is defined as the special case of (11) with $h_n(u) = -\alpha_{n-1} u$ for all $u \in M$; i.e.,

$$
[u_0 - \alpha_{n-1}u_1, u_1 - \alpha_{n-2}u_2, \ldots, u_{k-1} - \alpha_{n-k}u_k]
$$

(20)

If $\alpha$ is not a zero divisor in $R$ then the mapping $h(u) = -\alpha u$ is clearly injective. In general, $h$ is not surjective even if $R$ contains no zero divisors (e.g. $\alpha \in \mathbb{Z}, \alpha \neq \pm 1$). On the other hand, if each $\alpha$ is a unit in $R$ then each $h$ is a bijection with inverse $h^{-1}(u) = -\alpha^{-1}u$. 7
With the linear form symmetry, Equations (17)-(19) read as follows:

\[
\begin{align*}
\zeta_{j,n}(u_0, v_1, \ldots, v_j) &= -\alpha_{n-j}^{-1}(v_j - \zeta_{j-1,n}(u_0, v_1, \ldots, v_{j-1})), \\
f_n(\zeta_{0,n}, \zeta_{1,n}(u_0, v_1), \ldots, \zeta_{k,n}(u_0, v_1, \ldots, v_k)) &= \alpha_nu_0, \\
x_{n+1} &= t_{n+1} + \alpha_nx_n.
\end{align*}
\]

(21)

(22)

(23)

The following is a straightforward consequence of Lemma 3 and proved by simple induction.

**Lemma 5** Equation (1) has the linear form symmetry (20) on the R-module M if and only if there is a sequence \(\{\alpha_n\}\) of units in R such that the quantity

\[
f_n(u_0, \zeta_{1,n}(u_0, v_1), \ldots, \zeta_{k,n}(u_0, v_1, \ldots, v_k)) - \alpha_nu_0
\]

is independent of \(u_0\) for all \(n\), where for \(j = 1, \ldots, k\),

\[
\zeta_{j,n}(u_0, v_1, \ldots, v_j) = \left(\prod_{i=1}^{j} \alpha_{n-i}\right)^{-1}u_0 - \sum_{i=1}^{j} \left(\prod_{l=i}^{j} \alpha_{n-l}\right)^{-1}v_i.
\]

(24)

(25)

The following is one of the main results of this paper.

**Theorem 6** Equation (2) has the linear form symmetry if and only if there is a sequence of units in R that satisfies both of the following equations:

\[
\begin{align*}
a_{0,n} + a_{1,n}\alpha_{n-1}^{-1} + a_{2,n}(\alpha_{n-1}\alpha_{n-2})^{-1} + \cdots + a_{k,n}(\alpha_{n-1}\alpha_{n-2}\cdots\alpha_{n-k})^{-1} &= \alpha_n \\
b_{0,n} + b_{1,n}\alpha_{n-1}^{-1} + b_{2,n}(\alpha_{n-1}\alpha_{n-2})^{-1} + \cdots + b_{k,n}(\alpha_{n-1}\alpha_{n-2}\cdots\alpha_{n-k})^{-1} &= 0.
\end{align*}
\]

(26)

(27)

If such a unitary sequence say, \(\{\rho_n\}\) exists then the sc-factorization of (2) in M is

\[
\begin{align*}
t_{n+1} &= -\sum_{j=1}^{k} \sum_{i=1}^{j} a_{j,n}(\gamma_{ij})^{-1}t_{n-i+1} + g_n \left(-\sum_{j=1}^{k} \sum_{i=1}^{j} b_{j,n}(\gamma_{ij})^{-1}t_{n-i+1}\right) \\
x_{n+1} &= \rho_nx_n + t_{n+1}
\end{align*}
\]

(28)

(29)

where \(\gamma_{ij} = \prod_{l=i}^{j} \rho_{n-l}\) and its inversion occurs in the group of units of R.

**Proof.** The quantity (24) for (2) is

\[
a_{0,n}u_0 + \sum_{j=1}^{k} a_{j,n}\zeta_{j,n} + g_n \left(b_{0,n}u_0 + \sum_{j=1}^{k} b_{j,n}\zeta_{j,n}\right) - \rho_nu_0
\]

(30)
where \( \zeta_{j,n} \) abbreviates \( \zeta_{j,n}(u_0,v_1,\ldots,v_j) \). Using (25) we obtain

\[
a_{0,n}u_0 + \sum_{j=1}^{k} a_{j,n} \zeta_{j,n} = (a_{0,n} - \rho_n)u_0 + \sum_{j=1}^{k} a_{j,n} \left[ \left( \prod_{i=1}^{j} \rho_{n-i}^{-1} \right) u_0 - \sum_{i=1}^{j} \left( \prod_{l=i}^{j} \rho_{n-l}^{-1} \right) v_i \right]
\]

\[
= \left[ a_{0,n} - \rho_n + \sum_{j=1}^{k} a_{j,n} \left( \prod_{i=1}^{j} \rho_{n-i}^{-1} \right) \right] u_0 - \sum_{j=1}^{k} \sum_{i=1}^{j} a_{j,n} \left( \prod_{l=i}^{j} \rho_{n-l}^{-1} \right) v_i
\]

Now, setting the coefficient of \( u_0 \) equal to zero shows that \( \{\rho_n\} \) satisfies (26). Similarly, for the quantity inside \( g_n \) we obtain

\[
b_{0,n}u_0 + \sum_{j=1}^{k} b_{j,n} \zeta_{j,n} = b_{0,n}u_0 + \sum_{j=1}^{k} b_{j,n} \left[ \left( \prod_{i=1}^{j} \rho_{n-i}^{-1} \right) u_0 - \sum_{i=1}^{j} \left( \prod_{l=i}^{j} \rho_{n-l}^{-1} \right) v_i \right]
\]

\[
= \left[ b_{0,n} + \sum_{j=1}^{k} b_{j,n} \left( \prod_{i=1}^{j} \rho_{n-i}^{-1} \right) \right] u_0 - \sum_{j=1}^{k} \sum_{i=1}^{j} b_{j,n} \left( \prod_{l=i}^{j} \rho_{n-l}^{-1} \right) v_i
\]

Again, setting the coefficient of \( u_0 \) shows that \( \{\rho_n\} \) satisfies (27). Thus, by Lemma 1, (2) has the linear form symmetry if and only if the above \( \{\rho_n\} \) satisfies (26) and (27).

What is left in (30) after all the terms with \( u_0 \) are removed is

\[
\phi_n(v_1,\ldots,v_k) = -\sum_{j=1}^{k} \sum_{i=1}^{j} a_{j,n} \gamma_{ij}^{-1} v_i + g_n \left( -\sum_{j=1}^{k} \sum_{i=1}^{j} b_{j,n} \gamma_{ij}^{-1} v_i \right)
\]

with \( \gamma_{ij} = \prod_{l=i}^{j} \rho_{n-l} \) which yields the factor equation (28). The cofactor equation (29) is clear from Lemma 1. ■

**Example 7** To illustrate the preceding result consider the third-order recurrence

\[
x_{n+1} = a_n x_n + g_n(x_n + x_n - 2)
\]

in a left vector space \( V \) over the (real) quaternions. Here \( a_n \) is a quaternion for each \( n \) and \( g_n : V \to V \) is an arbitrary mapping. To examine possible sc-factorizations of (31) we consider (26) and (27) for this recurrence, where \( a_{0,n} = a_n, b_{0,n} = b_{2,n} = 1 \) and \( a_{1,n} = a_{2,n} = b_{1,n} = 0 \) for every \( n \). We obtain

\[
a_n = \alpha_n, \quad 1 + (\alpha_{n-1} \alpha_{n-2})^{-1} = 0
\]
This implies that the quaternions \( a_n \) must satisfy

\[
a_n a_{n-1} + 1 = 0
\]

for all \( n \geq 2 \) but may otherwise be arbitrary. There are an infinite number of possibilities, including infinitely many constants \( a_n = a \) since the polynomial equation \( a^2 + 1 = 0 \) has infinitely many quaternion solutions. In particular, \( a \) may be any one of the base quaternions \( \pm i, \pm j, \pm k \). There are also infinitely many non-constant solutions of type \( \{a_n\} = \{a, -1/a, a, -1/a, \ldots\} \) for each quaternion \( a \neq 0 \). For all such \( a_n \), Theorem 6 yields the sc-factorization of (31) as

\[
t_{n+1} = g_n(-a_{n-1}t_n - (a_{n-1}a_{n-2})^{-1}t_{n-1}) = g_n(a_n t_n + t_{n-1}),
\]

(33)

\[
x_{n+1} = a_n x_n + t_{n+1}
\]

The following corollary is an immediate consequence of Theorem 6 for vector spaces.

**Corollary 8** Let \( V \) be a vector space over a field \( F \). If \( \{\alpha_n\} \) is a sequence of nonzero elements in \( F \) that satisfies (26) and (27) then (2) has the linear form symmetry and the corresponding sc-factorization in \( V \) into the system (28)-(29). 

**Example 9** The recurrence (31) also has sc-factorizations in complex vector spaces, although the choice of \( a_n \) is more restricted in the complex field \( C \). In particular, rather than an infinite number of constants, the only constant values that work now are \( \pm i \).

Over a real vector space, (31) has no sc-factorization relative to the linear form symmetry with constant \( a_n = a \in \mathbb{R} \). However, (32) has an infinite number of non-constant solutions in \( \mathbb{R} \) of period 2, i.e. \( \{a_n\} = \{a, -1/a, a, -1/a, \ldots\} \) for each real \( a \neq 0 \). Each such solution yields a version of (31) that has a sc-factorization in any vector space over \( \mathbb{R} \).

Each of the equalities (26) and (27) is also a difference equation and \( \{\alpha_n\} \) is their common solution. The existence of this common solution establishes the proper relationship among the coefficients \( a_{i,n} \) and \( b_{i,n} \) for the occurrence of the linear form symmetry, which is independent of the functions \( g_n \). We now identify this relationship explicitly in the second-order case where \( k = 1 \).

**Corollary 10** Consider the following second-order recurrence in \( M \):

\[
x_{n+1} = a_{0,n} x_n + a_{1,n} x_{n-1} + g_n (b_{0,n} x_n + b_{1,n} x_{n-1}).
\]

(34)

If \( b_{0,n}, b_{1,n} \) are units in \( R \) and the following equality holds:

\[
a_{0,n} - a_{1,n} b_{1,n}^{-1} b_{0,n} + b_{0,n+1}^{-1} b_{1,n+1} = 0
\]

(35)

then (34) has a sc-factorization into first-order recurrences in \( M \) as follows:

\[
t_{n+1} = a_{1,n} b_{1,n}^{-1} b_{0,n} t_n + g_n (b_{0,n} t_n)
\]

\[
x_{n+1} = -b_{0,n+1}^{-1} b_{1,n+1} x_n + t_{n+1}.
\]
Proof. A sequence of units \( \{\alpha_n\} \) must satisfy (26) and (27), i.e. both of the following must hold

\[
a_{0,n} + a_{1,n}\alpha_{n-1}^{-1} = \alpha_n, \quad b_{0,n} + b_{1,n}\alpha_{n-1}^{-1} = 0
\]

for all \( n \). If \( b_{0,n}, b_{1,n} \) are units then the second of the above equalities yields \( \alpha_{n-1} = -b_{0,n}^{-1}b_{1,n} \) or equivalently, \( \alpha_n = -b_{0,n+1}^{-1}b_{1,n+1} \). This is also a sequence of units that when substituted in the first equality and terms rearranged, we obtain (35). The sc-factorization now follows readily from Theorem 6.

We note that using (35), an alternative form of the factor equation in Corollary 10 is the following:

\[
t_{n+1} = -(a_{0,n} + b_{0,n+1}^{-1}b_{1,n+1})t_n + g_n(b_{0,n}t_n)
\]

5 Constant coefficients and polynomial roots

If \( a_{i,n}, b_{i,n} \) are independent of \( n \) then (26) and (27) may have a common fixed point (constant solution) in the group of units even if (2) is still explicitly dependent on \( n \) (i.e. is non-autonomous) via \( g_n \). This interesting and important special case is the subject of the following consequence of Theorem 6 that furnishes a sufficient condition for the reducibility of (2) that is generally easier to check than finding a common solution of (26) and (27).

**Theorem 11** Let \( g_n \) be a sequence of functions on the \( R \)-module \( M \) where \( R \) is a commutative ring with identity and let \( a_i, b_i \in R \) for \( i = 0, 1, \ldots k \). If the polynomials

\[
P(\lambda) = \lambda^{k+1} - \sum_{i=0}^{k} a_i\lambda^{k-i}, \quad Q(\lambda) = \sum_{i=0}^{k} b_i\lambda^{k-i}
\]

have a common root \( \rho \) that is a unit in \( R \) then the recurrence

\[
x_{n+1} = \sum_{i=0}^{k} a_i x_{n-i} + g_n \left( \sum_{i=0}^{k} b_i x_{n-i} \right)
\]

(36)

has the sc-factorization

\[
t_{n+1} = -\sum_{i=0}^{k-1} p_i t_{n-i} + g_n \left( \sum_{i=0}^{k-1} q_i t_{n-i} \right)
\]

(37)

\[
x_{n+1} = \rho x_n + t_{n+1}
\]

(38)

where

\[
p_i = \rho^{i+1} - a_0\rho^i - \cdots - a_i \quad \text{and} \quad q_i = b_0\rho^i + b_1\rho^{i-1} + \cdots + b_i
\]
Proof. If with constant coefficients \( a_i \) (26) has a constant solution \( \alpha_n = \alpha \) that is a unit in \( R \) then \( \alpha \) satisfies the equality

\[
a_0 + a_1\alpha^{-1} + a_2(\alpha^2)^{-1} + \cdots + a_k(\alpha^k)^{-1} = \alpha
\]

Multiplication by \( \alpha^k \) shows this equality to be equivalent to

\[
\alpha^{k+1} - a_0\alpha^k - \cdots - a_{k-1}\alpha - a_k = 0
\]

i.e. \( \alpha \) is a root of the polynomial \( P \). Conversely, any unit root of \( P \) is evidently a constant solution of (26). Similarly, \( \alpha \) is a constant solution of (27) with constant coefficients \( b_i \) if and only if it is a root of \( Q \).

Therefore, a common root \( \rho \) of \( P \) and \( Q \) is evidently a common, constant solution of (26) and (27) and if \( \rho \) is a unit in \( R \) then (36) has a sc-factorization by Theorem 6.

Next, to find (37) we start with (28)

\[
t_{n+1} = -k \sum_{j=1}^{k} \sum_{i=1}^{j} a_j (\gamma_{ij})^{-1} t_{n-i+1} + g_n \left( -k \sum_{j=1}^{k} \sum_{i=1}^{j} b_j (\gamma_{ij})^{-1} t_{n-i+1} \right)
\]

where for each \( i = 1, \cdots, k, \)

\[
\sum_{j=1}^{k} \sum_{i=1}^{j} a_j (\gamma_{ij})^{-1} t_{n-i+1} = \sum_{j=1}^{k} \sum_{i=1}^{j} a_j (\rho^{-i+1})^{-1} t_{n-i+1} = \sum_{i=1}^{k} \sum_{j=i}^{k} a_j (\rho^{-1})^{j-i+1} t_{n-i+1} = \sum_{i=1}^{k} (a_i\rho^{k-i+1} + a_{i+1}\rho^{k-i} + \cdots + a_k)(\rho^{-1})^{k-i+1} t_{n-i+1}
\]

Since \( \rho \) is a unit root of the polynomial \( P \) it follows that

\[
\sum_{j=1}^{k} \sum_{i=1}^{j} a_j (\gamma_{ij})^{-1} t_{n-i+1} = \sum_{i=1}^{k} (\rho^{k+1} - a_0\rho^k - \cdots - a_{i-1}\rho^{k-i+1})(\rho^{-1})^{k-i+1} t_{n-i+1} = \sum_{i=1}^{k} p_i t_{n-i+1}
\]

where \( p_i \) is as defined in the statement of the corollary. The last quantity is clearly equivalent to \( \sum_{i=0}^{k-1} p_i t_{n-i} \) as presented in (37). Similarly, since \( \rho \) is also a root of the polynomial \( Q \) it follows that

\[
\sum_{j=1}^{k} \sum_{i=1}^{j} b_j (\gamma_{ij})^{-1} t_{n-i+1} = \sum_{i=0}^{k-1} q_i t_{n-i+1}
\]
where \( q_i \) is as defined in the statement of the corollary. This completes the derivation of (37); finally, (38) is clear from (29). □

**Remark 12**

1. If the coefficients \( a_{j,n} = a_j \) are constants for \( j = 0, 1, \ldots, k \) and all \( n \) then (26) and (27) can be written in the equivalent forms

\[
\alpha_n \alpha_{n-1} \cdots \alpha_{n-k} - a_0 \alpha_{n-1} \alpha_{n-2} \cdots \alpha_{n-k} - a_1 \alpha_{n-2} \cdots \alpha_{n-k} - \cdots - a_k = 0 \quad (39)
\]
\[
b_0 \alpha_{n-1} \alpha_{n-2} \cdots \alpha_{n-k} + b_1 \alpha_{n-2} \cdots \alpha_{n-k} + \cdots + b_k = 0 \quad (40)
\]

These polynomial difference equations resemble the polynomials \( P \) and \( Q \) in the preceding corollary in that the order of each term in them equals the power of \( \lambda \) in the corresponding term of \( P \) or \( Q \).

2. Note also that the factor equation (37) is of the same type as (36) but with order reduced by one. Hence, Theorem 11 can be applied again to reduce the order of (37) further, if the associated polynomials of (37) also share a common unit root. This happens when \( P \) and \( Q \) share more than one unit root; see the next section. The theory for repeated applications of sc-factorization on fields is discussed in [25].

**Example 13** Let \( M \) be an arbitrary module over the finite ring \( \mathbb{Z}_m \) of integers modulo \( m \geq 3 \) and consider the third-order recurrence

\[
x_{n+1} = 2x_n + x_{n-2} + g_n(x_n - x_{n-2})
\]

in \( M \). This recurrence is of type (36) with coefficients \( a_0 = 0, a_1 = 2, a_2 = 1 \) and \( b_0 = 1, b_1 = 0, b_2 = -1 \). These values yield the polynomials

\[
P(\lambda) = \lambda^3 - 2\lambda - 1, \quad Q(\lambda) = \lambda^2 - 1.
\]

These share a unit root \( \rho = -1 \) in \( \mathbb{Z}_m \) so Theorem 11 applies with \( p_0 = \rho = -1, p_1 = \rho^2 - 2 = -1 \) and \( q_0 = 1, q_1 = \rho = -1 \). We obtain the sc-factorization

\[
t_{n+1} = t_n + t_{n-1} + g_n(t_n - t_{n-1}), \quad x_{n+1} = -x_n + t_{n+1}.
\]

The following is the common special case of Corollary 10 and Theorem 11.

**Corollary 14** Let \( g_n \) be a sequence of functions on the \( R \)-module \( M \) where \( R \) is a commutative ring with identity and let \( a_0, a_1, b_0, b_1 \) be fixed elements in the ring \( R \). If \( b_0 \) and \( b_1 \) are units in \( R \) and the following equality holds:

\[
a_0 - a_1 b_1^{-1} b_0 + b_0^{-1} b_1 = 0 \quad (42)
\]

then the recurrence

\[
x_{n+1} = a_0 x_n + a_1 x_{n-1} + g_n(b_0 x_n + b_1 x_{n-1})
\]

(43)
has a sc-factorization in $M$ that is given by

$$t_{n+1} = a_1b_1^{-1}b_0t_n + g_n(b_0t_n)$$

$$x_{n+1} = -b_0^{-1}b_1x_n + t_{n+1}.$$ 

**Proof.** The statements are obviously true by Theorem 11. To see that they are also true by Corollary 10, note that for (43)

$$P(\lambda) = \lambda^2 - a_0\lambda - a_1, \quad Q(\lambda) = b_0\lambda + b_1$$

Now $Q(\lambda) = 0$ if and only if $\lambda = -b_0^{-1}b_1$ which is a unit. Next, $P(-b_0^{-1}b_1) = 0$ if and only if (42) holds and the proof is complete. ■

An application of the above corollary to a system of nonlinear, second-order equations is given in Example 16 below. For linear non-homogeneous equations in rings, Corollary 11 can be applied even to certain types of variable coefficients, as in the next result.

**Corollary 15** Let $a_i, b_i \in R$ for $i = 0, 1, \ldots, k$ and $c_n, d_n \in R$ for $n \geq 1$ where $R$ is a commutative ring with identity. If the polynomials $P$ and $Q$ in Theorem 17 have a common root $\rho$ that is a unit $R$ then the recurrence

$$x_{n+1} = \sum_{i=0}^{k} (a_i + b_ic_n)x_{n-i} + d_n$$

(44)

with variable coefficients has a sc-factorization in the ring $R$ into the following pair of linear non-homogeneous recurrences:

$$t_{n+1} = -\sum_{i=0}^{k-1} (p_i + q_ic_n)t_{n-i} + d_n, \quad x_{n+1} = \rho x_n + t_{n+1}$$

where $p_i, q_i$ are as defined in Theorem 17.

**Proof.** Define $g_n : R \rightarrow R$ as $g_n(r) = c_nr + d_n$. Using these $g_n$ in (43) and re-grouping terms yields (44). Now apply Theorem 11 to conclude the proof. ■

6 sc-factorization of higher order systems

As previously mentioned, a recurrence of order $k$ in a field may be unfolded to a system of first-order recurrences in a $k$-dimensional vector space over the field. Such a representation is not unique but a standard version of it is (17). The reverse process where a first-order system is folded to a higher order recurrence is also possible; see [29] and its bibliography for the method and its background. This folding process is useful when the higher order equation is familiar or more tractable than the system that yields it.
Modules lead to a different kind of folding that we discuss in this section. Certain systems of higher order recurrences in a direct product of a ring with itself may be represented by higher order recurrences in modules. To illustrate this case, consider \((\mathbb{1})\) in a commutative ring \(R\) with identity. Let \(x_n = (x_{1,n}, x_{2,n})\) which is in \(R^2 = R \times R\). We write \(g_n : R^2 \to R^2\) as
\[
g_n(u, v) = (g_n^1(u, v), g_n^2(u, v))
\]
with component functions \(g_n^1, g_n^2 : R^2 \to R\) and obtain the system form of \((\mathbb{1})\) as
\[
\begin{align*}
x_{1,n+1} &= ax_{1,n} + bx_{1,n-1} + g_n^1(x_{1,n} - cx_{1,n-1}, x_{2,n} - cx_{2,n-1}) \\
x_{2,n+1} &= ax_{2,n} + bx_{2,n-1} + g_n^2(x_{1,n} - cx_{1,n-1}, x_{2,n} - cx_{2,n-1})
\end{align*}
\]
(45)
(46)

Note that dependence of \(x_{1,n+1}\) on \(x_{2,n}\) or \(x_{2,n-1}\) (and similarly, of \(x_{2,n+1}\) on \(x_{1,n}\) or \(x_{1,n-1}\)) occurs via the maps \(g_n^i\) and this dependence may occur linearly as well as in a nonlinear way. To highlight this aspect, we extract the possible linear terms from the maps \(g_n^i\) as follows:
\[
g_n^i(u, v) = A_{i,n}u + B_{i,n}v + h_n^i(u, v), \quad i = 1, 2
\]

where \(A_{i,n}, B_{i,n} \in R\) for all \(n\) and \(h_n^i\) do not have linear terms. Now we substitute into (45)-(46) and rearrange terms to obtain the equivalent system
\[
\begin{align*}
x_{1,n+1} &= (a + A_{1,n})x_{1,n} + (b - cA_{1,n})x_{1,n-1} + B_{1,n}x_{2,n} - cB_{1,n}x_{2,n-1} + h_n^1(x_{1,n} - cx_{1,n-1}, x_{2,n} - cx_{2,n-1}) \\
x_{2,n+1} &= A_{2,n}x_{1,n} - cA_{2,n}x_{1,n-1} + (a + B_{2,n})x_{2,n} + (b - cB_{2,n})x_{2,n-1} + h_n^2(x_{1,n} - cx_{1,n-1}, x_{2,n} - cx_{2,n-1})
\end{align*}
\]
(47)
(48)

In the form (47)-(48), we see that the coefficients of the linear terms of the system are less obvious than in the form (45)-(46).

We consider \((\mathbb{1})\) a representation of (47)-(48), or of (45)-(46) in the direct product \(R \times R\) viewed as a left \(R\)-module. The results on semiconjugate factorization discussed above depend on the coefficients \(a, b, c_n, \) etc rather than on the functions \(h_n^i\) or on the modules; therefore, they apply to a broad range of systems, including those in higher dimensions.

Once the equivalent recurrence for a system is found, we obtain the lower order factor and cofactor equations in the module, which we can transform back to systems. This pair of lower order systems is then considered a sc-factorization of the original system. In the next example, we illustrate the application of Corollary \((\mathbb{1})\) to a system of second-order recurrences that can be represented in the form \((\mathbb{1})\).

**Example 16** Let \(c_n, d_n\) be nonzero complex numbers for every \(n\) and consider the following system of higher order difference equations in the vector space \(\mathbb{C}^2\)
\[
\begin{align*}
x_{1,n+1} &= x_{1,n} + \frac{c_n(x_{1,n} - x_{1,n-1})}{x_{2,n} - x_{2,n-1}} \\
x_{2,n+1} &= x_{2,n} + d_n(x_{1,n} - x_{1,n-1})
\end{align*}
\]
(49)
(50)
As this is also a nonlinear system, well-known methods are not available in the literature for analyzing its solutions. In fact, given the possibility that 0 may occur in the denominator of (49) at some iteration n where \(x_{2,n} = x_{2,n-1}\), we need to verify the existence of solutions. The sc-factorization of (49)-(50) not only clarifies the existence of its solutions, but it may also be used to calculate those solutions. In the case of (49)-(50), we define the functions \(g_n: \mathbb{C}^2 \rightarrow \mathbb{C}^2\) as

\[
g_n(u, v) = \left( c_n \frac{u}{v}, d_n u \right)
\]

Then, with \(x_n = (x_{1,n}, x_{2,n})\) the system (49)-(50) can be written as

\[
x_{n+1} = x_n + g_n(x_n - x_{n-1})
\]

which is a recurrence of type (43) with \(a_0 = 1, a_1 = 0, b_0 = 1\) and \(b_1 = -1\). These numbers clearly satisfy (42) so Corollary 14 yields the following sc-factorization for (51):

\[
t_{n+1} = g_n(t_n) \quad (52)
\]

\[
x_{n+1} = x_n + t_{n+1}
\]

Note that each of the factor and cofactor equations is a first-order system; (52) is the system

\[
t_{1,n+1} = c_n \frac{t_{1,n}}{t_{2,n}}, \quad t_{2,n+1} = d_n t_{1,n}
\]

that we call the factor system of (49)-(50) and (53) is

\[
x_{1,n+1} = x_{1,n} + t_{1,n+1}, \quad x_{2,n+1} = x_{2,n} + t_{2,n+1}
\]

i.e. the cofactor system. Each of the above systems is relatively simple to analyze. For instance, if the ratio \(c_n/d_{n-1} = \delta\) is constant then from (52) we obtain

\[
t_{1,n+1} = c_n \frac{t_{1,n}}{t_{2,n}} = c_n \frac{t_{1,n}}{d_{n-1} t_{1,n-1}} = \frac{\delta t_{1,n}}{t_{1,n-1}}
\]

Iterating this recurrence with initial values \(t_{1,1} = x_{1,1} - x_{1,0}, \ t_{1,2} = c_1 t_{1,1}/t_{2,1}\) where \(t_{2,1} = x_{2,1} - x_{2,0}\) and using induction, we obtain the following solution of period 6 for it:

\[
\{t_{1,n}\} = \left\{ t_{1,1}, t_{1,2}, \frac{\delta t_{1,2}}{t_{1,1}}, \frac{\delta^2 t_{1,2}}{t_{1,1}}, \frac{\delta t_{1,1}}{t_{1,1}}, \ldots \right\}
\]

Using this, we then obtain \(t_{2,n} = d_{n-1} t_{1,n-1}\) for \(n \geq 2\), which need not be periodic and further,

\[
x_{1,n} = x_{1,0} + \left[ \frac{n}{6} \right] \left( t_{1,1} + t_{1,2} + \frac{\delta t_{1,2}}{t_{1,1}} + \frac{\delta^2 t_{1,2}}{t_{1,1}} + \frac{\delta t_{1,1}}{t_{1,1}} + \delta t_{1,1} \right) + \sum_{j=1}^{r_n} t_{1,j}
\]
where \([m]\) is the greatest integer less than or equal to \(m\) and \(r_n\) is the remainder of the fraction \(n/6\) (the sum is dropped if \(r_n = 0\)). We calculate \(x_{2,n}\) similarly to obtain the complete solution of the system (49)-(50).

The above calculations show, in particular, that (49)-(50) has solutions as long as \(t_{1,1}, t_{1,2} \neq 0\).

Since
\[
t_{1,2} = c_1 \frac{t_{1,1}}{t_{2,1}} = c_1 \frac{x_{1,1} - x_{1,0}}{x_{2,1} - x_{2,0}}
\]
the existence of solutions is established if the initial values satisfy \(x_{1,1} \neq x_{1,0}\) and \(x_{2,1} \neq x_{2,0}\). A little more calculation shows that this conclusion is valid even for the non-autonomous case where \(c_n/d_n-1\) is not constant and thus \(\{t_{1,n}\}\) may not be periodic.

The next example discusses a variation of the system (49)-(50).

Example 17 Consider the following system in the vector space \(\mathbb{C}^2 = \mathbb{C} \times \mathbb{C}\):

\[
x_{1,n+1} = x_{1,n} + c_n(x_{1,n} - x_{1,n-1})/x_{2,n} - x_{2,n-1} \quad (55)
\]
\[
x_{2,n+1} = x_{2,n-1} + d_n(x_{1,n} - x_{1,n-1}) \quad (56)
\]

For this system we use the form (47)-(48) with \(c = 1\) and parameters defined via the following equations for all \(n\):

\[
a + A_{1,n} = 1, \quad b - A_{1,n} = 0, \quad B_{1,n} = 0
\]
\[
a + B_{2,n} = 0, \quad b - B_{2,n} = 1, \quad A_{2,n} = d_n
\]

From the above equations we obtain

\[
A_{1,n} = b = 1 - a, \quad B_{2,n} = -a, \quad A_{2,n} = d_n, \quad B_{1,n} = 0 \quad (57)
\]

for all \(n\) for arbitrary \(a \in \mathbb{R}\). These values define the functions

\[
g_n(u, v) = \left((1 - a)u + \frac{c_n u}{v}, d_n u - av\right)
\]

and the system (55)-(56) is represented by the recurrence

\[
x_{n+1} = ax_n + (1 - a)x_{n-1} + g_n(x_n - x_{n-1}) \quad (58)
\]

The coefficients in the above equations satisfy (42) with \(a_0 = a\), \(a_1 = 1 - a\), \(b_0 = 1\) and \(b_1 = -1\) so Corollary 14 yields the sc-factorization

\[
t_{n+1} = g_n(t_n) - (1 - a)t_n
\]
\[
x_{n+1} = x_n + t_{n+1}.
\]
The special values \(a = 0, 1\) simplify the various expressions. With \(a = 1\) we obtain

\[
g_n(u, v) = \left( \frac{c_n u}{v}, d_n u - v \right)
\]

and the factor equation in the sc-factorization reduces to \(t_{n+1} = g_n(t_n)\). Note also that in this case, (58) is the same as (57). With \(a = 0\) we have

\[
g_n(u, v) = \left( u + \frac{c_n u}{v}, d_n u \right) = u \left( 1 + \frac{c_n}{v}, d_n \right)
\]

and the factor equation is \(t_{n+1} = g_n(t_n) - t_n\). In this case, (58) is not the same as (57); rather \(x_{n+1} = x_{n-1} + g_n(x_n - x_{n-1})\) is the representing recurrence on \(\mathbb{C}^2\) now.

Note that the preceding example also shows that representations of systems in rings by equations in modules (and thus also the corresponding sc-factorizations) are not unique in general.

The ideas in the examples of this section extend to systems of recurrences of arbitrary order in vector spaces of any dimension (including infinite, as in Example 22 below) as long as it is possible to transform the system to an equation of type (36) in the vector space. We emphasize that this transformation is not possible in general and leave as an open problem a classification of systems that can be represented by recurrences in modules.

7  Polynomial roots and repeated sc-factorization

An important by-product of Theorem 11 is that the factor equation of (36), namely, (37) is of the same type as (36), but has a lower order. This feature, which is also one of the advantages that the linear form symmetry enjoys compared with other form symmetries, suggests that Theorem 11 can be applied repeatedly, each time reducing the order of the original recurrence by one, as long as common, unit polynomial roots exist.

The next result simplifies the task of searching for shared polynomial roots by limiting it to all common roots of the original pair of polynomials \(P\) and \(Q\).

**Lemma 18** Let \(R\) be an integral domain and let \(\rho \in R\) be a common unit root of \(P\) and \(Q\). If \(\rho_1 \in R\) is another shared unit root of \(P\) and \(Q\) in \(R\) then \(\rho_1\) is also a common unit root of both of the following polynomials

\[
P_1(\lambda) = \lambda^k + p_0 \lambda^{k-1} + p_1 \lambda^{k-2} + \cdots + p_{k-1}
\]

\[
Q_1(\lambda) = q_0 \lambda^{k-1} + q_1 \lambda^{k-2} + \cdots + q_{k-1}
\]

where the coefficients \(p_i\) and \(q_i\) are defined in Theorem 11.
Proof. By assumption,
\[ P(\rho_1) = Q(\rho_1) = 0. \] (59)

Now
\[
(\lambda - \rho)P_1(\lambda) = (\lambda - \rho) \left( \lambda^k + \sum_{j=0}^{k-1} p_j \lambda^{k-j-1} \right)
= \lambda^{k+1} + \sum_{j=0}^{k-1} [p_j - \rho p_{j-1}] \lambda^{k-j} - \rho p_{k-1}.
\]
where we define \( p_{-1} = 1 \) to simplify the notation. For each \( j = 0, 1, \ldots, k - 1 \) note that
\[ p_j - \rho p_{j-1} = -a_j \]
and further, since \( P(\rho) = 0, \)
\[ \rho p_{k-1} = \rho \left( \rho^k - a_0 \rho^{k-1} - \cdots - a_{k-1} \right) \]
\[ = P(\rho) + a_k \]
\[ = a_k. \]
Thus
\[ (\lambda - \rho)P_1(\lambda) = P(\lambda) \]
and if \( \rho_1 \neq \rho \) then \( P_1(\rho_1) = 0 \) by (59). If \( \rho_1 = \rho \) then \( \rho \) is a double root of both \( P \) and \( Q \) so that their derivatives are zeros, i.e.,
\[ P'(\rho) = Q'(\rho) = 0. \] (60)

Therefore,
\[ P_1(\rho) = \rho^k + \sum_{j=0}^{k-1} \left( \rho^{j+1} - a_0 \rho^j - \cdots - a_{j-1} \rho - a_j \right) \rho^{k-j-1} \]
\[ = (k+1)\rho^k - \sum_{j=0}^{k-1} (k-j)a_j \rho^{k-j-1} \]
\[ = P'(\rho). \]

In particular, if \( \rho_1 = \rho \) then \( P_1(\rho_1) = 0 \) by (60). Similar calculations show that \( Q_1(\rho_1) = 0, \)
thus completing the proof. \( \blacksquare \)

Theorem 11 and Lemma 18 imply the following result.
Corollary 19  (a) Assume that the polynomials $P$ and $Q$ have two common, unit roots $\rho, \rho_1$ in an integral domain $R$. Then the factor equation (37) of (36) has the linear form symmetry in the $R$-module $M$ with the factor equation

$$r_{n+1} = - \sum_{i=0}^{k-2} p_{1,i} r_{n-i} + g_n \left( \sum_{i=0}^{k-2} q_{1,i} r_{n-i} \right)$$ \hspace{1cm} (61)

where

$$p_{1,i} = \rho_1^{i+1} + p_0 \rho_1^i + \cdots + p_i$$ and $$q_{1,i} = q_0 \rho_1^i + q_1 \rho_1^{i-1} + \cdots + q_i.$$

(b) The recurrence (36) has a secondary or repeated SC factorization that consists of the factor equation (61) and the two cofactor equations

$$t_{n+1} = \rho_1 t_n + r_{n+1}$$

$$x_{n+1} = \rho x_n + t_{n+1}.$$

(c) If $P$ and $Q$ have $m$ common, unit roots in $R$ (counting repeated or multiple roots) where $1 \leq m \leq k$ then (36) can be reduced in order repeatedly $m$ times in $M$.

**Proof.**  (a) The proof applies the same arguments as in the proof of Theorem 11 to the factor equation (37) that was obtained using the shared unit root $\rho$. We simply change $a_i$ to $-p_i$ and $b_i$ to $q_i$ in the proof and make other minor modifications to complete the proof here.

(b) This follows from the general form of cofactors associated with the linear form symmetry.

(c) This follows by induction, using Part (a).  \[\blacksquare\]

**Definition 20**  If $m = k+1$ in Corollary 19(c) then we say that (36) has a complete sc-factorization into a triangular systems of first-order recurrences.

The next example illustrates this concept; see Corollary 23 below for a more general class of recurrences of the same type.

**Example 21**  Let $M$ be a vector space over the finite field $\mathbb{Z}_p$ where $p$ is an odd prime and consider the third-order recurrence

$$x_{n+1} = 2x_{n-1} + x_{n-2} + g_n(x_n - x_{n-1} - x_{n-2})$$ \hspace{1cm} (62)

in $M$. This recurrence is of type (36) with coefficients $a_0 = 0$, $a_1 = 2$, $a_2 = 1$ and $b_0 = 1$, $b_1 = b_2 = -1$. These values yield the polynomials

$$P(\lambda) = \lambda^3 - 2\lambda - 1, \quad Q(\lambda) = \lambda^2 - \lambda - 1$$

Note that $P(\lambda) = (\lambda + 1)(\lambda^2 - \lambda - 1) = (\lambda + 1)Q(\lambda)$. It is known (see, e.g. [3]) that $Q$ has two (nonzero) roots $\rho_1, \rho_2 \in \mathbb{Z}_p$ if and only if $p \equiv 0, 1, 4$ (mod 5). For instance, in $\mathbb{Z}_5$, 3 is a double-root of
Q so \( \rho_1 = \rho_2 = 3 \); in \( Z_{11} \), 4 and 8 are distinct roots. For such primes, Corollary 19 implies that \( (62) \) has at least two sc-factorizations; since \( (62) \) has order 3, it must have a complete sc-factorization.

First, using \( \rho_1 \) we obtain

\[
\begin{align*}
t_{n+1} &= -\rho_1 t_n - (\rho_1 - 1)t_{n-1} + g_n(t_n - (\rho_1 - 1)t_{n-1}) \\
x_{n+1} &= \rho_1 x_n + t_{n+1}
\end{align*}
\]

Next, using \( \rho_2 \) we find the sc-factorization of the factor equation above

\[
\begin{align*}
r_{n+1} &= -r_n + g_n(r_n), \\
t_{n+1} &= \rho_2 t_n + r_{n+1}
\end{align*}
\]

These equations, together with the cofactor equation (63) constitute the complete sc-factorization of \( (62) \) into a triangular system of three first-order recurrences.

In the next example, we find the complete sc-factorization for a third-order recurrence in two different ways.

**Example 22** Consider the following third-order functional recurrence

\[
x_{n+1}(r) = x_{n-1}(r) + \int_0^r \phi_n(\tau, x_n - x_{n-2})(\tau) d\tau, \quad 0 \leq r \leq 1
\]

in the normed vector space \( C[0,1] \) of continuous, complex-valued functions on the interval \( [0,1] \) and \( \phi_n : [0,1] \times C[0,1] \to C[0,1] \) for all \( n \). We may define

\[g_n(x)(r) = \int_0^r \phi_n(\tau, x) d\tau\]

and assume that the initial functions \( x_0(r), x_1(r), x_2(r) \) are given for \( 0 \leq r \leq 1 \). The polynomials corresponding to (64) are

\[P(\lambda) = \lambda^3 - \lambda, \quad Q(\lambda) = \lambda^2 - 1\]

These share two unit roots \( \pm 1 \) in \( \mathbb{C} \) so we may apply Corollary 19. First, with \( \lambda = 1 \) we obtain the sc-factorization

\[
\begin{align*}
t_{n+1}(r) &= -t_n(r) + \int_0^r \phi_n(\tau, t_n + t_{n-1})(\tau) d\tau, \\
x_{n+1}(r) &= x_n(r) + t_{n+1}(r)
\end{align*}
\]

Next, with \( \lambda = -1 \) the factor equation (65) has the sc-factorization

\[
\begin{align*}
s_{n+1}(r) &= \int_0^r \phi_n(\tau, s_n)(\tau) d\tau \\
t_{n+1}(r) &= -t_n(r) + s_{n+1}(r)
\end{align*}
\]

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These equations together with (66) constitute the complete sc-factorization of (64) into first-order recurrences. It is worth mentioning here that a complete sc-factorization for the recurrence (64) may alternatively be obtained as follows. First, write (64) as

\[ x_{n+1}(r) - x_{n-1}(r) = \int_0^r \phi_n(\tau, x_n - x_{n-2})(\tau) d\tau \]

Using this suggestive form, we substitute \( s_n = x_n - x_{n-2} \) to obtain the first-order recurrence

\[ s_{n+1}(r) = \int_0^r \phi_n(\tau, s_n)(\tau) d\tau \]

Note that the cofactor equation, obtained from the above substitution, now has order 2:

\[ x_{n+1}(r) = x_{n-1}(r) + s_{n+1}(r). \tag{67} \]

The last two equations yield an sc-factorization of (64) that is of a different type than what we obtained using Theorem 11; see Chapter 6 in [25]. The cofactor (67) is easily reduced using Theorem 11 (now with \( Q(\lambda) = 0 \)) as

\[ t_{n+1}(r) = -t_n(r) + s_{n+1}(r), \quad x_{n+1}(r) = x_n(r) + t_{n+1}(r) \]

As might be expected, this is the same complete sc-factorization that we obtained via Corollary 19.

Corollary 23 and the above two examples raise a natural question: Are there special cases of (36) that have complete sc-factorizations into a triangular system of \( k + 1 \) first-order recurrences? The following result indicates two such cases; Examples 21 and 22 above illustrate each of these cases.

**Corollary 23** Let \( R \) be an integral domain and \( M \) an \( R \)-module.

(a) Let \( r, b_j \in R, j = 1, \ldots, k, b_k \neq 0 \) and define \( b_0 = 1, b_{k+1} = 0 \). If the polynomial equation

\[ \lambda^k + b_1 \lambda^{k-1} + b_2 \lambda^{k-2} + \cdots + b_k = 0 \tag{68} \]

has \( k \) unit roots in \( R \) then the recurrence

\[ x_{n+1} = \sum_{j=0}^k (rb_j - b_{j+1})x_{n-j} + g_n \left( \sum_{j=0}^k b_j x_{n-j} \right) \tag{69} \]

has a complete sc-factorization into a triangular system of \( k + 1 \) first-order recurrences in \( M \).

(b) Let \( b, a_j \in R, j = 0, \ldots, k - 1 \) and \( a_{k-1} \neq 0 \). If the polynomial equation

\[ \lambda^k - a_0 \lambda^{k-1} - a_1 \lambda^{k-2} - \cdots - a_{k-1} = 0 \tag{70} \]
has k unit roots in R then the recurrence

\[ x_{n+1} = \sum_{j=0}^{k-1} a_j x_{n-j} + g_n \left( b x_n - \sum_{j=1}^{k} a_{j-1} b x_{n-j} \right) \]  

(71)

has a complete sc-factorization into a triangular system of k + 1 first-order recurrences in M.

Proof. (a) We observe that the polynomial in (68) is \( Q(\lambda) \) in this case and from (69), \( P(\lambda) \) is

\[ \lambda^{k+1} - \sum_{j=0}^{k} (rb_j - b_{j+1}) \lambda^{k-j} = (\lambda - r)Q(\lambda) \]

Thus \( P \) and \( Q \) share \( k \) unit roots in \( R \) whenever \( Q \) does and the proof is complete.

(b) Let \( P(\rho) = Q(\rho) = 0 \), i.e.

\[ \rho^{k+1} - a_0 \rho^k - a_1 \rho^{k-1} - \cdots - a_{k-1} \rho = 0 \]  

(72)

\[ b \rho^k - a_0 b \rho^{k-1} - \cdots - a_{k-2} b \rho - a_{k-1} b = 0. \]  

(73)

If \( b \neq 0 \) then since also \( \rho, a_{k-1} \neq 0 \), after cancelling \( \rho \) from (72) and \( b \) from (73), both equations reduce to the polynomial in (70). Thus, every common, unit root of \( P \) and \( Q \) is a zero of the polynomial equation (70) and conversely, every zero of (70) is a root of both \( P \) and \( Q \). Therefore, (71) has \( k \) sc-factorizations by Corollary 19.

If \( b = 0 \) then (73) is true trivially for all \( \rho \in R \). Therefore, again the single equation (70) remains and by Corollary 19 (71) has \( k \) repeated sc-factorizations. Note that in this case, (71) reduces to a linear non-homogeneous equation of order \( k \).

Remark 24 The recurrence (71) has a sc-factorization other than the one in Corollary 23(b). If \( b \neq 0 \) then the substitution

\[ s_{n+1} = x_n - \sum_{j=1}^{k} a_{j-1} x_{n-j} \]

yields the sc-factorization

\[ s_{n+1} = gn(bs_n) \]  

(74)

\[ x_{n+1} = \sum_{j=1}^{k} a_{j-1} x_{n-j+1} + s_{n+1} \]  

(75)

Here the factor equation has order 1 and the cofactor is a linear non-homogeneous equation of order \( k \). For more details on this type of sc-factorization and the corresponding form symmetry we refer to [25] (also see Example 22).
We close this section with the following result on linear recurrences in fields that shows, in particular, if $R$ is an algebraically closed field then (75) has a complete sc-factorization in every vector space over $R$. Recall that if $g_n$ is a sequence of constants then (36) reduces to a non-homogeneous linear equation. In this case, it is no loss of generality to set $b_j = a_j$ for $j = 0, 1, \ldots, k$, so that $Q = P$. Thus the proof of the following is clear.

**Corollary 25** Let $F$ be a nontrivial field, $a_j, c_n \in F$ for $j = 0, 1, \ldots, k$ and all $n \geq k$ and assume that $a_k \neq 0$. The linear non-homogeneous recurrence

$$x_{n+1} = \sum_{i=0}^{k} a_i x_{n-i} + c_n$$

has a complete sc-factorization in $F$ into a triangular system of $k + 1$ first-order, linear non-homogeneous recurrences if the polynomial

$$P(\lambda) = \lambda^{k+1} - \sum_{i=0}^{k} a_i \lambda^{k-i}$$

factors completely in $F$. In particular, every linear non-homogeneous recurrence in an algebraically closed field has a complete sc-factorization.

**8 On the reducibility of recurrences**

The preceding two sections show a close relationship between polynomial factorization and the sc-factorization of (36). Specifically, the existence of common unit roots for the polynomials $P$ and $Q$ in a commutative ring $R$ with identity is sufficient for the sc-factorization of (36) in any $R$-module $M$. In general, we define reducibility for recurrences in modules as follows.

**Definition 26** (Reducibility) The recurrence (2) in a left unitary $R$-module is reducible, relative to the linear form symmetry, if the difference equations (26) and (27) have a common solution in the unit of groups of the ring $R$. If such a common unitary solution does not exist then (2) is irreducible (relative to the linear form symmetry).

Note that the concept of reducibility for recurrences can only be relative to a particular form symmetry because a recurrence may have sc-factorizations relative to some form symmetries but not others; see [25].

While the existence of common roots for $P$ and $Q$ is sufficient for the reducibility of (36) in every $R$-module over a commutative ring $R$ with identity, this condition is not necessary because sc-factorizations may exist even when $P$ or $Q$ have no roots in $R$ at all. We illustrate this possibility with an example.
Example 27 Consider the recurrence
\[ x_{n+1} = -x_{n-1} + g_n(x_n + x_{n+2}) \] (76)
in an arbitrary vector space \( V \) over the field \( \mathbb{R} \) of real numbers with \( g_n : V \to V \) arbitrary functions. In this case, \( P(\lambda) = \lambda^3 + \lambda \) has a single unit root \(-1\) in \( \mathbb{R} \) but \( Q(\lambda) = \lambda^2 + 1 \) has no roots in \( \mathbb{R} \); in particular, \( P \) and \( Q \) share no unit roots in \( \mathbb{R} \). However, (76) does have a complete sc-factorization in vector spaces over \( \mathbb{R} \) which can be determined using Corollary 8. We seek a nonzero sequence \( \alpha_n \) in \( \mathbb{R} \) that satisfies the two equations
\[ \alpha_n = -\alpha_{n-1} = -\frac{1}{\alpha_{n-1}}, \]
\[ 0 = 1 + (\alpha_{n-1}\alpha_{n-2})^{-1} \quad \text{or} \quad \alpha_{n-1} = -\frac{1}{\alpha_{n-2}}. \]

These are identical difference equations, so any nonzero solution, e.g. \( \alpha_n = (-1)^n \) works for \( n \geq 2 \). The resulting sc-factorization is
\[ t_{n+1} = (-1)^n t_n + g_n(t_n + (-1)^{n-1}t_{n-1}) \] (77)
\[ x_{n+1} = (-1)^n x_n + t_{n+1} \] (78)

We may now apply either Corollary 3 or Corollary 10 to (77). Using the latter, it is easy to see that the coefficients of (77) satisfy (35) and we obtain the sc-factorization
\[ s_{n+1} = g_n (s_n) \]
\[ t_{n+1} = (-1)^{n-1} t_n + s_{n+1} \]

This pair of first-order recurrences, together with (78) yield a complete sc-factorization of (76) in any vector space over \( \mathbb{R} \).

We note that (76) is a special case of (71) so it also has the alternative sc-factorization mentioned in Remark 24. This factorization does not use of polynomials, yet it contains only constant coefficients in its factor equation.

The difference equations (39) and (40) reduce to the polynomial equations \( P(\lambda) = 0 \) and \( Q(\lambda) = 0 \) in Theorem 11 when they have a common constant solution, or fixed point, in \( R \). The question as to whether (39) and (40) can have a common solution (a sequence of units) if \( P \) and \( Q \) do not have common unit roots is presently difficult to answer in general.

A deeper understanding of (39) and (40), which are polynomial difference equations, is needed that goes beyond polynomial factorization. A starting point for this more complicated problem is linear non-homogeneous recurrences for which (40) is true trivially so we may focus only on exploring the existence of unitary solutions for (39). A preliminary work in this direction is [28].

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In special cases where common constant solutions of (39) and (40) must exist, the polynomials $P$ and $Q$ play a more decisive role in the study of reducibility of recurrences. We encountered one such case in Corollary 25 for linear recurrences. To discuss a nonlinear case, consider the recurrence

$$x_{n+1} = \sum_{i=0}^{k} a_i x_{n-i} + g_n (x_{n-j} - bx_{n-j-1})$$

(79)

where $0 \leq j \leq k - 1$ is fixed and $a_i$ is in a commutative ring with identity for each $i = 0, 1, \ldots, k$. For this recurrence, (40) reduces to

$$\alpha_{n-j-1} \alpha_{n-j-2} \cdots \alpha_{n-k} - b \alpha_{n-j-2} \cdots \alpha_{n-k} = 0$$

if $j < k - 1$ and to $\alpha_{n-k} - b = 0$ if $j = k - 1$, for all $n \geq k$. Thus, only a constant sequence $\alpha_n = b$ is possible and since $\alpha_n$ is a unit by definition, it follows that $b$ must be a unit. With this choice, (39) reduces to the equation $P(b) = 0$. Therefore, (79) is reducible and we have the following immediate consequence of Theorem 6.

**Corollary 28** Let $R$ be a commutative ring with identity, $a_i, b \in R$ and let $g_n : M \to M$ be a sequence of functions on an $R$-module $M$. The recurrence (79) is reducible in $M$ relative to the linear form symmetry if and only if $b$ is a unit and a root of $P$ in $R$.

From the preceding result, it readily follows that the recurrence

$$x_{n+1} = \sum_{i=0}^{k} a_i x_{n-i} + g_n (x_{n-j} - x_{n-j-1})$$

is reducible relative to the linear form symmetry if and only if

$$\sum_{i=0}^{k} a_i = 1$$

and similarly,

$$x_{n+1} = \sum_{i=0}^{k} a_i x_{n-i} + g_n (x_{n-j} + x_{n-j-1})$$

is reducible relative to the linear form symmetry if and only if

$$\sum_{i=0}^{k} (-1)^{k-i} a_i = (-1)^{k+1}.$$ 

These two statements are also valid for arbitrary $\mathbb{Z}$-modules. The results in this section identify some of the boundary of relevance for the linear form symmetry as far as the sc-factorization of (2) is concerned.
9 Concluding remarks

While the important role that algebra in general, and polynomials in particular play in the factorization of linear recurrences is well-known, the usual methods do not apply to nonlinear recurrences. As the latter are increasingly used in scientific models, it is necessary to develop methods that may be relevant to them. To this end, we presented an algebraic method, namely semiconjugate factorization relative to the linear form symmetry, that can be used to also obtain sc-factorizations of certain nonlinear recurrences in modules. For linear recurrences, this method yields the same results as the standard linear theory.

The preceding study answers a few questions but it leads to many more unanswered questions, some of which are noted in the above discussion. For instance, we noted that the ideas in the examples of Section 6 extend to systems of recurrences of arbitrary order in vector spaces of any dimension including infinite, as in Example 22. This can be achieved as long as it is possible to transform the system to an equation of type (36) in the vector space. However, this transformation is not possible in general a classification of systems that can be represented by recurrences in modules is an open problem.

Finally, a question that goes beyond the boundaries of this paper for possible future study is what types of nonlinear recurrences other than (2) can be fruitfully studied using the linear or other types of form symmetry and the corresponding sc-factorizations. For preliminary ideas that may be useful in such future studies we refer to [25].

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