Correlation of rheological parameters with the flow of non-Newtonian fluids

A A Zadorozhnyi¹, ⁵, M P Remarchuk¹, A P Kovrevski², Y V Chovnyuk³ and S A Buhaiievskyi⁴

¹Department of construction, travel and cargo-handling machines, Ukrainian State University of Railway Transport, Feuerbach Square 7, Kharkiv, 61050
²Department of Theoretical Mechanics, Kharkiv National University of Construction and Architecture, Sumska Street 40, Kharkiv, 61002
³Department of Machinery and Equipment Design, National University of Bioresources and Life Sciences of Ukraine, Heroiv Oborony Street15, Kyiv, 03041
⁴Department of Bridges, Structures and Construction Mechanics, Kharkiv National Automobile and Highway University, Petrovskogo Street 25, Kharkiv, 61002
⁵E-mail: zsnj1971@ukr.net

Abstract. When new design and technological solutions are developed for equipment intended for the supply and transportation of non-Newtonian liquids through pipelines, research and design organizations face the problem of accurate understanding and behavior of such media as concrete and mortar mixtures. Currently, dozens of models of various complex fluid systems are known, most of which, however, are only empirical. The reason of this situation is the lack of a sufficiently generalized theory that would allow calculating the characteristics of the mechanical behavior of the flow of various media based on their structure. The specialist literature presents numerous purely empirical and semi-empirical rheological models of limited purpose. In most works, the well-proven and justified rheological Shvedov–Bingham, Bulkley-Herschel, Ostwald-de Waele, Briand, Shulman, and Casson models are not used to a sufficient extent due to the narrow technological specialization of the studied non-Newtonian fluids. Classical rheological models have a strong potential for improvement and rational modification to generalize rheological models of the flow for non-Newtonian fluids.

1. Introduction

Ever increasing global production facilities, such as energy engineering, machine-building, processing of synthetic and natural materials (in particular, the construction industry), oil production and petrochemical industry, pharmacy, food, paper, paint and varnish production, have been stimulating research and engineering developments to study rheological models of complex media, such as concrete and mortar mixtures. The costs associated with transporting non-Newtonian materials in the modern world have reached an immense value. The need for a reasonable choice and optimal estimation for efficient transportation of complex fluids (primarily those with high viscosity) is a relevant problem from an economic point of view. When engineers are designing pipeline transportation equipment, mortar and concrete pipelines, they are faced with the difficult challenge of finding the optimal characteristics (technological diameter of the pipeline, flow velocity of the mixture, etc.) to provide the optimal costs necessary for transporting a non-Newtonian (concrete) mixture to a technologically specified distance. In this case, determining optimal values becomes very
problematic due to the need to correctly account for the non-Newtonian factor, i.e., knowledge of the true rheological “pressure–flow” curve for various fluid flow conditions.

2. Analysis of recent studies and publications
Non-Newtonian fluids, such as concrete and mortar mixtures, are commonly considered as multi-component Shvedov-Bingham media. Such fluids obey the rheological laws of Bingham fluid flow [1, 2, 3, 4, 5 and 6]. Theoretical and experimental studies of the flow processes of Bingham fluids and the construction of mathematical models were carried out by such well-known specialists as: W L Wilkinson, A V Gnoevoy, V Prager, B M Smolsky, Z P Shulman, V V Gorislavets, S S Kutateladze, and many others. The obtained mathematical models are widely used for rigid visco-plastic media, in particular, concrete mixtures. Most of the classical equations of the rheological state of fluids are generalized [7, 8, 9, 10 and 11].

3. Statement of the objective and tasks of the study
The purpose of this work is to conduct a study based on mathematical modeling methods of the known Casson dependence for various rheological power indicators $m$ and $n$, and consequently conduct a comparative analysis; to identify inaccuracies in determining the parameter $\tau_y$. In theoretical plotting of flow curves of Bingham fluids that affect the flow characteristics in concrete pipelines, to approach reasonably estimations in designing technological equipment for transporting mortar and concrete mixtures [12, 13, 14, and 15]. To achieve the purposes of this work, the main task is to determine the productivity $Q$ (relative flow rate) and velocity $U(\rho)$ depending on the parameters of the mixture and dimensionless radius $\rho$.

4. Development of mathematical model
The rheological equation of the Casson-Shulman model has the form

$$\tau^{1/n} = \tau_y^{1/n} + k \left(\frac{\gamma \mu_p}{\tau_y}\right)^{1/m}. \quad (1)$$

Let us consider the influence of inaccuracy in determining the parameter $\tau_y$ on the flow characteristics and flow rate for different $m$ and $n$. Let us introduce the designations for the dimensionless shear stress and the flow velocity $\tilde{\tau}_{02}$ = $\frac{\tau}{\tau_y}$, $\tilde{\tau}_{03}$ = $\frac{\gamma \mu_p}{\tau_y}$.

For $m = n = 2$, equation (1) takes the form

$$\tilde{\tau}_0 = \left(1 + \sqrt{\nu}\right)^3. \quad (2)$$

for $m = n = 3$, equation (1) takes the form

$$\tilde{\tau}_0 = \left(1 + \frac{1}{3\sqrt{\nu}}\right)^3. \quad (3)$$

The graphs (Figure 1 a, b) show the flow curves for large and small values of the dimensionless flow velocity $\nu$ (here $\tilde{\tau}_{02}$ - formula (2), $\tilde{\tau}_{03}$ - formula (3)).

![Figure 1](image)

**Figure 1.** Flow curves based on the Casson model: a), b) – flows at large and small values of dimensionless flow velocity $\nu$, respectively.
To draw up the flow velocity profile, \( u = u(r) \) equation (1) at \( n=m=2 \) is transformed by integration to the form

\[
U(\rho) = \frac{1 - \rho^2}{2} + \tau_0(1 - \rho) - \frac{4}{3} \sqrt{\tau_0} \left(1 - \rho^{3/2}\right),
\]

where \( \rho = \frac{r}{R} \) is the dimensionless radius; \( R \) is the radius of the pipe; \( \tau_0 = \frac{\tau_y}{\tau_w} \) is relative stress; \( \tau_w \) is shear stress on the pipe wall with radius \( R \); \( \tau_y \) is shear stress at the boundary of the shear region.

When the equation is derived, it is assumed that on the walls of the pipe \( \rho = 1, u = 0 \). Graph \( \Delta U(\rho) \) for various \( \tau_0 \) is shown in Figure 2. At \( m=n=3 \), equation 1 is converted to the form

\[
U(\rho) = \frac{1}{2} \frac{\rho^2}{2} - \frac{6}{5} \tau^2 \left(1 - \rho^{3/2}\right) + \frac{9}{4} \sqrt{\tau_0} \left(1 - \rho^{3/2}\right) - \tau_0(1 - \rho).
\]

The corresponding graph of speed profiles \( U(\rho) \) is shown in Figure 3.

In the case of \( m=3 n=2 \), the equation of the Casson model is written as

\[
\tau^{1/2} = \tau^{1/2}_y + k \left( \rho \mu_e \right)^{1/2}.
\]

Here \( k \) is a coefficient that is introduced to "align" the dimensions of the terms.

After equation (6) is written in dimensionless form and integrating by \( \rho \), the expression is obtained for the dimensionless flow velocity \( \tilde{U}(\rho) = u(\rho) \cdot \frac{k \cdot \mu_e}{R \cdot \tau^{1/2}_w} \)

\[
\Delta \tilde{U}(\rho) = 0.4(1 - \rho^{3/2}) + 1.5 \tau_0^{1/2} (\rho^2 - 1) + 2 \tau_0(1 - \rho^{3/2}) + \tau_0^{3/2} (\rho - 1).
\]

Graphs of flow velocity profiles \( U(\rho) \) for different \( \tau_0 \) are shown in Figure 4.

When formulas for velocity profiles are available, it is easy to obtain flow rate formulas for various indicators \( m \) and \( n \). Now we get the formulas for the flow velocity according to the Casson model:
\[ Q = \frac{\pi \cdot R^3 \tau_w}{\mu_p} \left[ \frac{1}{2} + \frac{2}{3} \tau_0 - \frac{4}{7} \tau_0^3 \right] = \frac{\pi \cdot R^3 \tau_w}{\mu_p} \cdot W_1(\tau_0), \tag{8} \]

\[ Q = \frac{\pi \cdot R^3 \tau_w}{\mu_p} \left[ \frac{1}{4} - \frac{6}{11} \tau_0^{1/3} + \frac{9}{10} \tau_0^{2/3} - \frac{1}{3} \tau_0 \right] = \frac{\pi \cdot R^3 \tau_w}{\mu_p} \cdot W_2(\tau_0), \tag{9} \]

\[ Q = B \cdot \left[ \frac{1}{9} - \frac{3}{8} \tau_0^{1/2} + \frac{3}{7} \tau_0 - \frac{1}{6} \tau_0^{3/2} \right] = B \cdot W_3(\tau_0), \tag{10} \]

where \( B = \frac{2\pi \cdot R^3 \tau_w^{3/2}}{k \cdot \mu_p} \).

For the Bingham model of viscous fluid flow, the flow formula has the form [1, 2, 4]

\[ Q = \frac{\pi \cdot R^3 \tau_w}{\mu_p} \left[ \frac{1}{4} - \frac{1}{3} \tau_0 + \frac{\tau_0^4}{12} \right] = \frac{\pi \cdot R^3 \tau_w}{\mu_p} \cdot W_0(\tau_0). \tag{11} \]

Let us consider the effect of a change of \( \tau_0 \) on the maximum flow velocity in the Casson model.

The minimum possible value of \( \tau_y \) is taken as the base value: \( \tau_{ym \min} = \tau_{ym} \). Let the possible change of \( \tau_y \) be \( \Delta \tau_y \), i.e. the current value of the elastic limit will be equal to \( \tau_y = \tau_{ym} + \Delta \tau_y \). The parameter \( \tau_0 = \frac{\tau_y}{\tau_w} \) will be \( \tau_0 = \tau_{0m} + \Delta \tau_0 \). If it is accepted that \( \Delta \tau_y = 0.1 \tau_y \), then \( \tau_0 = 1.1 \tau_{0m} \) where \( \tau_{0m} = \frac{\tau_{ym}}{\tau_w} \). The expression for the maximum velocity (at \( \rho = 0 \)) will have the form:

at \( m=n=2 \) \[ U'(\rho, \tau_0) = 0.5 + \tau_{0m} - 1.333 \sqrt{\tau_{0m}}, \tag{12} \]

at \( m=3, n=2 \) \[ U'(\rho, \tau_0) = 0.4 - 1.5 \tau_{0m}^{1/2} + 2 \tau_{0m} - \frac{\tau_{0m}^{3/2}}{2}. \tag{13} \]

If instead of \( \tau_{0m} \) \( \tau_0 = 1.1 \tau_{0m} \) is used, the expressions for the maximum velocities will take the form:

at \( m=n=2 \) \[ U(\rho, \tau_0) = 0.5 + 1.1 \tau_{0m} - 1.398 \sqrt{\tau_{0m}}, \tag{14} \]
at $m=3$, $n=2$

$$U(\rho, \tau_0) = 0.4 - 1.573\tau_0^{1/2} + 2.2\tau_0 - 1.154\tau_0^{3/2}. \quad (15)$$

The relative value of the maximum velocity is determined by the formula

$$\Delta V = \frac{U(\rho, \tau_0) - U'(\rho, \tau_0)}{U'(\rho, \tau_0)}. \quad (16)$$

Velocity curves are shown in (Figure 5 a, b).

![Velocity curves](image)

**Figure 5.** Maximum relative velocities of profiles according to the Casson model $\Delta V$ at $m=n=2$ and $m=3$, $n=2$: a) – maximum dimensionless speed $\Delta V$ at $m=n=2$; b) – maximum dimensionless speed $\Delta V$ at $m=n=3$.

In the case of $m=3$, $n=2$, the expression for $\Delta V$ at $\tau_0 = 0.45$, $\rho = 0$ and at $\tau_0 = 0.41$, $\rho = 0.1$ the flow curve is broken. Therefore, the flow process can be described using the Casson model at $m=3$, $n=2$ only for small values of $\tau_0$.

Effect of the change of $\tau_0$ on the flow rate. The dependence of the flow rate on the parameter $\tau_0$ is described by the second multipliers in formulas (8 - 11), which are denoted as $\Delta W(\tau_0)$. Let us further consider the corresponding dependencies. The relative change of flow rate is determined by the formula

$$\Delta W = \frac{W(\tau_0) - W'(\tau_0)}{W'(\tau_0)}. \quad (17)$$

here, as in the previous case, $\tau_0 = \tau_{0m} + \Delta \tau_0 = 1.1\tau_{0m}$. The results of estimations are presented in graphical form (Figure 6 a, b).

![Graphical form](image)

**Figure 6.** Relative change in flow rate $\Delta W$ according to the Casson model: a) – dependence of the flow rate $\Delta W$ on the parameter $\tau_0$ at $m=n=2$; b) – dependence of the flow rate $\Delta W$ on the parameter $\tau_0$ at $m=n=3$.

5. **Conclusions**

Velocity profiles do not have any clearly defined horizontal sections at different $\tau_0$, which makes it difficult to determine the value of the parameter $\tau_\gamma$. 
1. The impact of inaccuracy in determining $\tau_y$ on the maximum velocity value is insignificant:
   - for the Bingham model, a 10% increase of $\tau_y$ leads to a 4.9% decrease in maximum velocity;
   - for the Casson model, a 10% increase of $\tau_y$ changes the maximum velocity by no more than 5% ($m=3, n=2$ is a special case).

2. The effect of a change of $\tau_y$ (not more than by 10%) on flow rate is also insignificant – not more than 5%.

References

[1] Wilkinson W L 1964 Nen'yutonovskie zhidkost Non-Newton fluids (Moscow: Mir) p 216
[2] Smolsky B M, Shulman Z P, Gorislavets V M 1970 Rheodynamics and heat transfer of nonlinear viscoplastic materials (Minsk: Science and technology) p 443
[3] Gnoyevoy A V 1969 Theorii techeniy bingamovskikh sred, Basics of theory of flow of Bingham media. (Moscow: FIZMATLIT) p 272
[4] Smolskiy B M, Shulman Z P, Gorislavets V M, 1970 Reodinamika i teploobmen nelineyno -vyazkoplastichnykh materialov Rheodynamics and heat transfer of non-linear viscoplastic materials (Minsk: Nauka i tekhnika) p 240
[5] Prager W 1963 Vvedeniye v mekhaniku splosnykh zhidkosti Introduction into mechanics of continuous media (Moscow: Inostrannaya literatura) p 406
[6] Gorislavets V M, Smolsky V M, Shulman Z P 1968 Convective heat transfer in laminar flow of composite materials in a round tube (Minsk: Science and Technology, in collection of articles "Heat and mass transfer") 1968. Vol 3 pp 182-214
[7] Weipert D 1993 Rheologie der Lebensmittel (Hamburg: Behrs) p 620
[8] Lipscomb G G, Denn M M 1984 Flow of Bingam fluids ill complex geometries (Journal of Non-Newtonian Fluid Mechanics) V 14 p 337-346
[9] Eirich F 1962 Rheology Theory and applications (M Publishing house of Foreign Literature) p 824
[10] Casson N 1959 A Flow Equation for Pigment Oil Suspensions of the In Mill CC Ed Rheology of Disperse Systems (Pergamon Press Oxford) pp 84-102
[11] Krieger I M, Dougherty T J 1959 A Mechanism for non-Newtonian flow in suspensions of rigid spheres (Trans Soc Rheol) Vol 3 pp 137-148
[12] Zadorozhnyi A A, Kovrevski A P 2017 Analysis of the flow of Bingham fluids though circular pipelines (Collection of research papers of UkrDUZT) Issue 168 ISSN(p) 1994-7852 ISSN Online 2413-3795 pp 44-49
[13] Zadorozhnyi A A, Kovrevski A P, Chovnyuk Y V, Remarchuk N P 2018 Features of the flow of liquids of variable viscosity by the pipeline of a various form of tranverse section Collection of scientific works of the international scientific conference (Technology and transport infrastructure Kharkiv UkrDUZT) pp 24-25
[14] Zadorozhnyi A, Kovrevski A, Chovnyuk Y, Remarchuk N 2018 Flow of a Bingham Fluid Through Circular Pipes with Variable Viscosity Coefficient Along the Pipe Length Science Publishing Corporation Publisher of International Academic Journal) Vol 7 No 4.3 Special Issue 3 pp 100-104
[15] Zadorozhnyi A A 2020 Architectural construction using technologies based on knowledge about the flow of Bingham plastic fluids in various, pipelines IOP Conf Series (Materials Science and Engineering) 907 (2020) 012028 IOP Publishing doi:10.1088/1757-899X/907/1/012028 pp 1-6