THE TOPOLOGICAL K-THEORY OF CRYSTALLOGRAPHIC GROUPS WITH HOLONY $\mathbb{Z}/2$

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Abstract. In this note we present a complete computation of the topological K-theory of the reduced C*-algebra of a semidirect product of the form $\Gamma = \mathbb{Z}^n \rtimes_{\rho} \mathbb{Z}/2$ with no further assumptions about of the conjugacy action $\rho$. For this, we use some results for $\mathbb{Z}/2$-equivariant K-theory proved by Rosenberg and previous results of Davis and Luck when the conjugacy action $\rho$ is free outside the origin.

1. Introduction

Let $\Gamma$ be a crystallographic group with holonomy $\mathbb{Z}/2$, it means that $\Gamma$ is defined by an extension

\begin{equation}
0 \to \mathbb{Z}^n \to \Gamma \to \mathbb{Z}/2 \to 0.
\end{equation}

We denote by $C^*_r(\Gamma)$ to the reduced C*-algebra of $\Gamma$. In [DL13] the topological K-theory of $C^*_r(\Gamma)$ is computed, under the assumption of free conjugacy action on $\mathbb{Z}^n - \{0\}$. In this note, we give a complete computation of $K^*(C^*_r(\Gamma))$ avoiding the above assumption. We use the Baum-Connes conjecture for $\Gamma$, reducing the problem, to the computation of the $\Gamma$-equivariant K-homology of the classifying space for proper actions $E\Gamma$. For details on the Baum-Connes conjecture consult [Val02] and for a proof of the Baum-Connes conjecture for $\Gamma$ see [HK01].

The computations are obtained following a very simple idea, firstly, $\Gamma$ can be decomposed as a pullback $\Gamma_1 \times_{\mathbb{Z}/2} \Gamma_2$ over $\mathbb{Z}/2$, where $\Gamma_1$ has trivial conjugacy action and $\Gamma_2$ has free conjugacy action outside the origin. A model for $E\Gamma$ is $\mathbb{R}^n$ with the natural $\Gamma$-action, as $\mathbb{Z}/2$ acts freely we have a natural isomorphism

\[ K^1(E\Gamma) \cong K_{\mathbb{Z}/2}((S^1)^n). \]

The pullback decomposition of $\Gamma$ gives a decomposition of $(S^1)^n$ as a cartesian product $(S^1)^r \times (S^1)^{n-r}$, where $\mathbb{Z}/2$ acts trivially on the first component and non-trivially on the second.

We start computing $\mathbb{Z}/2$-equivariant K-theory of $(S^1)^n$ as is defined in [Seg68], we apply a Kunneth formula for $\mathbb{Z}/2$-equivariant K-theory proved in [Phi87] and some results in [Ros13] to the decomposition above, later we use a Universal Coefficient Theorem for equivariant K-theory proved in [JL13] to obtain the computation of $K^1(E\Gamma)$.

A similar procedure to compute the topological K-theory of crystallographic groups with others holonomy groups is not possible at the moment, because there is no generalizations of results in [Ros13] for finite groups with order greater than 2, mainly because the irreducible real representations of $\mathbb{Z}/n$, ($n > 2$) are actually complex.
2. Kunneth Theorem for $\mathbb{Z}/2$-equivariant K-theory

Throughout this note, K-theory or equivariant K-theory means complex topological K-theory with compact supports for locally compact Hausdorff spaces. Bott periodicity implies that we will regard this theory as being $\mathbb{Z}/2$-graded.

Let $R = R(\mathbb{Z}/2)$ be the representation ring of $\mathbb{Z}/2$, which is isomorphic to $\mathbb{Z}[t]/(t^2 - 1)$, with $t$ representing the 1-dimensional sign representation, this ring is the coefficients for $\mathbb{Z}/2$-equivariant K-theory, let $I = (t - 1)$ be the augmentation ideal and $J = (t + 1)$, each prime ideal $p$ of $R$ contains either $I$ or $J$, and these are the unique minimal prime ideals of $R$. We denote by $R_p$ the localization of $R$ at the prime ideal $p$.

The group $\mathbb{Z}/2$ has exactly two irreducible real representations, the trivial and the sign, denoted by $\mathbb{R}$ and by $\mathbb{R}_-$ respectively. From the sign representation we can define some kind of equivariant twisted K-theory groups

$$K^*_\mathbb{Z}/2, -(X) = K^*_\mathbb{Z}/2(X \times \mathbb{R}_-).$$

The coefficients of this theory are $K^*_\mathbb{Z}/2, -(\bullet) \cong R/J \cong I$ concentrated in even degrees.

Let us consider the inclusion $\{0\} \to \mathbb{R}_-$, it induces for every $\mathbb{Z}/2$-space $X$ a homomorphism of $R$-modules

$$\varphi : K^*_\mathbb{Z}/2, -(X) \to K^*_\mathbb{Z}/2(X),$$

in the other hand, consider the following composition:

$$K^*_\mathbb{Z}/2(X) \xrightarrow{\text{Bott}} K^*_\mathbb{Z}/2(X \times \mathbb{R}_- \times \mathbb{R}_-) \to K^*_\mathbb{Z}/2(X \times \mathbb{R}_- \times \{0\}).$$

It defines a homomorphism of $R$-modules

$$\psi : K^*_\mathbb{Z}/2(X) \to K^*_\mathbb{Z}/2, -(X).$$

**Proposition 2.1.** For any $\mathbb{Z}/2$-space $X$ we have a natural diagram

$$\begin{array}{ccc}
\mathbb{K}^*_\mathbb{Z}/2(X) : K^*_\mathbb{Z}/2(X) & \xrightarrow{\varphi} & K^*_\mathbb{Z}/2, -(X).
\end{array}$$

Where the maps $\varphi$ and $\psi$ preserves the $\mathbb{Z}/2$-grading and the composite in any order is giving by multiplication by $(1 - t)$. Moreover if $p \subseteq R$ is a prime ideal containing $I$, then $\psi$ and $\varphi$ vanish after localizing at $p$.

**Proof.** [Ros13].

We have the following version of the Kunneth formula for $\mathbb{Z}/2$-equivariant K-theory, for a proof see [Phi87].

**Theorem 2.2.** Let $X$ and $Y$ be $\mathbb{Z}/2$-spaces, let $p \subseteq R$ be a prime ideal containing $I$, with $p \neq (J, 2)$ there is short exact sequence of $\mathbb{Z}/2$-graded, $R_p$-modules

$$0 \to K^n_G(X)_p \otimes R_p K^n_G(Y)_p \xrightarrow{\varphi_p} K^{n+m}(X \times Y)_p \to \text{Tor}_1^{R_p}(K^n_G(X)_p, K^{n+1}_G(Y)_p) \to 0.$$

**Remark 2.3.** When $p$ contains $I$ we have a Kunneth formula taking $\mathbb{K}^*_\mathbb{Z}/2, (-)$ instead of $K^*_\mathbb{Z}/2, -(\bullet)$ on Thm. 2.2, this is a result in [Ros13]. On the other hand, as is observed in [Ros13], when $p$ contains $I$, Thm. 2.2 is not true for $K^*_\mathbb{Z}/2, -(\bullet)$ (as can be observed taking $X = Y = \mathbb{Z}/2$ with the transitive action), then it is necessary to consider $\mathbb{K}^*_\mathbb{Z}/2, (-)$. 

3. Crystallographic groups

Let $\Gamma$ be a group defined by the extension\footnote{1} with the conjugation action of $\mathbb{Z}/2$ given by a homomorphism $\rho : \mathbb{Z}/2 \to \text{GL}(n, \mathbb{Z})$. From now on we will suppose that the conjugacy action is not free outside the origin.

Let $H$ be the subgroup of $\mathbb{Z}^n$ where $\mathbb{Z}/2$ acts trivially, then there is $r \geq 1$ and a base $\{v_1, \ldots, v_n\}$ of $\mathbb{Z}^n$ such that $\{v_1, \cdots, v_r\}$ is a base of $H$, in this base, the homomorphism $\rho : \mathbb{Z}/2 \to \text{GL}(n, \mathbb{Z})$ is determined by a matrix (the image of the generator of $\mathbb{Z}/2$) with the following form:

\[
\begin{pmatrix}
I_r & 0 \\
0 & A_{n-r}
\end{pmatrix}.
\]

Where the conjugation action on the last $n - r$ coordinates (denoted by $\rho_{n-r} : \mathbb{Z}/2 \to \text{GL}(n-r, \mathbb{Z})$) is free outside the origin.

Let us consider the canonical action of $\Gamma$ over $\mathbb{R}^n$, as $\mathbb{Z}^n$ acts freely, we have

\[
K^\ast \Gamma(\mathbb{R}^n) \cong K^\ast \mathbb{Z}/2((S^1)^n).
\]

Now we will use Theorem 2.2, considering $X = (S^1)^r$ with the trivial $\mathbb{Z}/2$-action and $Y = (S^1)^{n-r}$ with the $\mathbb{Z}/2$-action determined by $\rho_{n-r}$. Note that $K^\ast \mathbb{Z}/2(X)$ can be computed easily (being $X$ a trivial $\mathbb{Z}/2$-space) and $K^\ast \mathbb{Z}/2(Y)$ was computed in\footnote{1}.

As the $\mathbb{Z}/2$-action on $X$ is trivial, we have an isomorphism of $R$-modules

\[
K^\ast(X) \cong \left\{ \begin{array}{ll}
\mathbb{Z}^{2^r-1} & * = 0 \\
\mathbb{Z}^{2^r-1} & * = 1
\end{array} \right.
\]

Then we have an isomorphism of $R$-modules

\[
K^\ast \mathbb{Z}/2(X) \cong \left\{ \begin{array}{ll}
R^{2^r-1} & * = 0 \\
R^{2^r-1} & * = 1
\end{array} \right.
\]

We need to recall the following result from\footnote{1}.

**Proposition 3.1.** Let $X$ be a locally compact $\mathbb{Z}/2$-space. Then there is a natural 6-term exact sequence

\[
K^1(X) \to K^0_{\mathbb{Z}/2, -}(X) \xrightarrow{\phi} K^0_{\mathbb{Z}/2}(X) \\
\downarrow f \quad \quad \quad \quad \quad \downarrow f \\
K^1_{\mathbb{Z}/2}(X) \xleftarrow{\phi} K^0_{\mathbb{Z}/2, -}(X) \to K^0(X)
\]

where the vertical arrows denoted by $f$ on the left and right are the forgetful maps from equivariant to non-equivariant K-theory.

Now we have to determine the $R$-module structure of $K^\ast \mathbb{Z}/2(Y)$, first note that

\[
K^\ast(Y) \cong \left\{ \begin{array}{ll}
\mathbb{Z}^{2^{n-r}-1} & * = 0 \\
\mathbb{Z}^{2^{n-r}-1} & * = 1
\end{array} \right.
\]

Now we need to recall Thm. 7.1 in\footnote{1}.

**Proposition 3.2.** (i) There is a split short exact sequence of $R$-modules

\[
0 \to K^0(Y/\mathbb{Z}/2) \to K^0_{\mathbb{Z}/2}(Y) \to I^{2^{n-r}} \to 0
\]
(ii) $K^1_{\mathbb{Z}/2}(Y) = 0$.

Where the $R$-module structure in $K^0(Y/(\mathbb{Z}/2))$ is determined by the augmentation map $R \xrightarrow{\varepsilon} \mathbb{Z}$.

**Proof.** It is proved in Thm. 7.1 in [DL13], we only have to remark that the map

$$K^0(Y/(\mathbb{Z}/2)) \rightarrow K^0_{\mathbb{Z}/2}(Y)$$

is the pullback of the quotient $Y \rightarrow Y/(\mathbb{Z}/2)$ and is a homomorphism of $R$-modules, in a similar way the map $K^0_{\mathbb{Z}/2}(Y) \rightarrow I^{2^n-1}$ is induced by the inclusion of representatives of the conjugacy classes of finite non-trivial subgroups of $\Gamma$, then it is a homomorphism of $R$-modules, then we have a short exact sequence of $R$-modules.

A splitting can be defined identifying the $R$-module $K^0(Y/(\mathbb{Z}/2))$ with $K^0(Y)^{\mathbb{Z}/2}$ (it happens because both groups are torsion free as abelian groups), by the forgetful map

$$K^0_{\mathbb{Z}/2}(Y) \rightarrow K^0(Y)^{\mathbb{Z}/2}.$$  

Identifying the $R$-modules $\mathbb{Z}$ with $J$ (it can be done because $t \in R$ acts trivially) and applying the above proposition, we obtain an isomorphism of $R$-modules

$$K^0_{\mathbb{Z}/2}(Y) \cong J^{2^n-1} \oplus I^{2^n-1}.$$  

Now we will use Thm. [DL2] to compute $K^*_p(X \times Y)_p$ for every prime ideal $p \subseteq R$ with $p \supseteq J$ and $p \neq (I, 2)$, in such case we have a short exact sequence

$$0 \rightarrow K^*_p(X \times Y)_p \otimes R_p K^*_p(Y)_p \rightarrow K^*_p(X \times Y)_p \rightarrow \text{Tor}^1_{R_p}(K^*_p(X)_p, K^*_{\mathbb{Z}/2}(Y)_p) \rightarrow 0.$$  

In this specific case as $K^*_p(X)_p$ is a free $R_p$-module, we obtain that

$$\text{Tor}^1_{R_p}(K^*_p(X)_p, K^*_{\mathbb{Z}/2}(Y)_p) = 0,$$

then we have

$$K^*_p(X \times Y)_p \cong K^*_p(X \times Y)_p \otimes R_p K^*_p(Y)_p \cong \begin{cases} (I_p)^{2^n-1} & * = 0 \\ (I_p)^{2^n-1} & * = 1. \end{cases}.$$  

In particular $K^*_p(X \times Y)_p$ is torsion free as abelian group. Now suppose $p \supseteq I$, combining Prop. [DL2] and Prop. [DL1] applied to $X \times Y$ we obtain short exact sequences

$$0 \rightarrow K^1_{\mathbb{Z}/2}(X \times Y)_p \xrightarrow{f} K^1(X \times Y)_p \rightarrow K^0_{\mathbb{Z}/2,-}(X \times Y)_p \rightarrow 0$$

$$0 \rightarrow K^0_{\mathbb{Z}/2}(X \times Y)_p \xrightarrow{f} K^0(X \times Y)_p \rightarrow K^1_{\mathbb{Z}/2,-}(X \times Y)_p \rightarrow 0,$$

where we are considering $K^*(X \times Y)$ as a $R$-module via the augmentation map $\varepsilon : R \rightarrow \mathbb{Z}$, in particular, $K^*_p(X \times Y)_p$ can be considered as a submodule of $K^*(X \times Y)_p$.

On the other hand, as $X \times Y$ is the $n$-torus we have an isomorphism of $R_p$-modules

$$K^*(X \times Y)_p \cong \begin{cases} J^{2^n-1}_p & * = 0 \\ J^{2^n-1}_p & * = 1. \end{cases}.$$  

Then $K^*_{\mathbb{Z}/2}(X \times Y)_p \subseteq K^*(X \times Y)_p$ is torsion free as abelian group.

As our final goal is to obtain the structure as abelian group we only need to prove that the above information implies that $K^*_{\mathbb{Z}/2}(X \times Y)$ is torsion free (as abelian group), it can be done in the following way.

Let $M$ be a $R$-module, consider the $\mathbb{Z}$-torsion module of $M$ defined as

$$\text{ZT}(M) = \{ x \in M \mid \text{there is } n \in \mathbb{Z} - \{0\}, n \cdot x = 0 \}.$$
Note that $\mathbb{Z}T(M)$ is a $R$-submodule of $M$, and moreover $\mathbb{Z}T(M)_p$ can be considered as a submodule of $\mathbb{Z}T(M)$, but the above computations imply that

$$\mathbb{Z}T(K_{2}(X \times Y)) = 0$$

for every prime ideal $p \subseteq R$, then we have

$$\mathbb{Z}T(K_{2}(X \times Y)) = 0,$$

and then $K_{2}(X \times Y)$ is a torsion free $\mathbb{Z}/2$-graded abelian group.

We only need to compute the rank of $K_{2}(X \times Y)$, it can be done by the well known formula proved for example [AS89] or [LO01]

$$\text{rk}(K^*(X \times Y)) = \sum_{g \in \mathbb{Z}/2} \text{rk}(K^*(X^g \times Y^g))$$

for $g \in \mathbb{Z}/2$

$$= \text{rk}(K^*(X)) \text{rk}(K^*(Y))$$

Then we obtain.

**Theorem 3.3.** Let $\Gamma$ a group defined by an extension 1.1, where the action of $\mathbb{Z}/2$ is not free outside the origin, then we have an isomorphism of abelian groups

$$K^*_\Gamma(\mathcal{E} \Gamma) \cong \begin{cases} \mathbb{Z}^{3 \cdot 2^{n-2}} & * = 0 \\ \mathbb{Z}^{3 \cdot 2^{n-2}} & * = 1. \end{cases}$$

4. **Topological K-theory of the reduced group C*-algebra**

Now we can compute $\Gamma$-equivariant K-homology groups of $\mathcal{E} \Gamma$ as is defined for example in [JL13].

As $K^*_\Gamma(\mathcal{E} \Gamma)$ is torsion free, the universal coefficient theorem for equivariant K-theory (Thm. 0.3 in [JL13]) reduces to

$$K^*_\Gamma(\mathcal{E} \Gamma) \cong \text{hom}_\mathbb{Z}(K^*_\Gamma(\mathcal{E} \Gamma), \mathbb{Z}) \cong \begin{cases} \mathbb{Z}^{3 \cdot 2^{n-2}} & * = 0 \\ \mathbb{Z}^{3 \cdot 2^{n-2}} & * = 1. \end{cases}$$

Finally by the Baum-Connes conjecture and previous results in [JL13] we obtain a complete computation of the reduced group C*-algebra of the group $\mathbb{Z}^n \rtimes \mathbb{Z}/2$.

**Theorem 4.1.** Let $\Gamma$ a group defined by an extension 1.1. If the action of $\mathbb{Z}/2$ is not free outside the origin, then we have an isomorphism of abelian groups

$$K_* (C^*_{\text{r}}(\mathbb{Z}^n \rtimes \mathbb{Z}/2)) \cong \begin{cases} \mathbb{Z}^{3 \cdot 2^{n-2}} & * = 0 \\ \mathbb{Z}^{3 \cdot 2^{n-2}} & * = 1, \end{cases}$$

if the action of $\mathbb{Z}/2$ is free outside the origin

$$K_* (C^*_{\text{r}}(\mathbb{Z}^n \rtimes \mathbb{Z}/2)) \cong \begin{cases} \mathbb{Z}^{3 \cdot 2^{n-1}} & * = 0 \\ 0 & * = 1. \end{cases}$$
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