FREDHOLM NOTIONS IN SCALE CALCULUS AND HAMILTONIAN FLOER THEORY

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ABSTRACT. We give an equivalent definition of the Fredholm property for linear operators on scale Banach spaces and introduce a (nonlinear) scale Fredholm property with respect to a splitting of the domain. The latter implies the Fredholm property introduced by Hofer-Wysocki-Zehnder in terms of contraction germs, but is easier to check in practice and holds in applications to holomorphic curve moduli spaces. We demonstrate this at the example of trajectory breaking in Hamiltonian Floer theory.

1. INTRODUCTION

Scale calculus has recently been developed by H. Hofer, K. Wysocki and E. Zehnder as part of polyfold theory, which provides an analytic framework for the study of moduli spaces of pseudo-holomorphic curves. Roughly speaking, such moduli spaces are (compactifications of) sets of equivalence classes of smooth maps which satisfy the Cauchy-Riemann equation, where two maps \( u \) and \( v \) are equivalent provided there exists a holomorphic automorphism \( \phi \) of the domain such that \( u = v \circ \phi \). Since these spaces are studied for almost complex structures, there is no readily available algebraic framework, so that they instead are viewed as solution spaces to a nonlinear PDE, modulo a reparametrization action by a usually finite dimensional Lie Group. The fundamental analytic difficulty in this setup is that Fredholm theory for the Cauchy-Riemann operator requires a Banach space completion of the space of smooth maps, while reparametrizations do not act differentiably in any known Banach completion. (In particular, the action of any nondiscrete reparametrization group on an infinite dimensional space of smooth maps is not differentiable in any Hölder or Sobolev norm, see e.g. [McW] for a discussion.)

The novel approach of polyfold theory to this issue is to replace the classical notion of differentiability in Banach spaces by a new notion of scale differentiability on scale Banach spaces, which allows for a natural framework of a scale of Sobolev spaces (a sequence indexed by the differentiability of the maps), in which reparametrizations act scale smoothly. The resulting scale calculus for scale manifolds is rich enough to establish a regularization theorem for suitably defined scale Fredholm sections with compact zero set, which allows to associate cobordism classes of smooth manifolds to e.g. compact moduli spaces of pseudo-holomorphic curves which contain no nodal curves. To deal with the latter, polyfold theory introduces a second fundamentally new concept – generalizing the local models for Banach manifolds to images of scale smooth retractions. More details on polyfold theory can be found in [HWZ1, HWZ2, HWZ3, HWZ4], the surveys [H, FFGW], and its first application to Gromov-Witten theory in [HWZ5].

This note aims to shed some light on the abstract linear and nonlinear Fredholm theory on scale Banach spaces, which obviously are crucial ingredients of the abstract regularization.

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theory provided by polyfolds. In particular, our goal is to explain at examples how the classical Fredholm property of elliptic PDE’s in many cases directly implies a scale Fredholm property – at least in the absence of singularities such as nodes. We give a quick overview of the basic definitions of scale calculus in Section 2 in order to fix notation and make this note (with the exception of the proofs and Section 5) self-contained. In particular, we introduce the notion of a norm scale on an infinite dimensional vector space, which – by completion – is the source of all scale Banach spaces in holomorphic curve applications. Section 3 discusses the notion of a linear Fredholm operator on scale Banach spaces. We introduce a simpler definition, demonstrate why it is naturally satisfied by elliptic operators, and prove that it is equivalent to the definition in [HWZ1]. Section 4 similarly introduces a simpler nonlinear Fredholm property with respect to a splitting, which in practice will be given by splitting off a finite dimensional space of gluing parameters from an infinite dimensional function space. We show that this Fredholm property implies the Fredholm property based on a contraction germ normal form that is introduced in [HWZ1]. We moreover explain the need for a separate nonlinear Fredholm notion in scale calculus, whereas the classical notion of nonlinear Fredholm map is given simply by requiring continuous differentiability and the linear Fredholm property for the linearized maps (at zeros).

Finally, to obtain a Fredholm theory e.g. in the presence of nodes in Gromov-Witten moduli spaces, [HWZ2] introduces a notion of Fredholm map between two images of scale smooth retractions. Here a more fundamental geometric issue is that neither the domain nor the target space has a locally trivial tangent bundle – the fibers can “jump in dimension almost arbitrarily”. Hence the polyfold Fredholm property requires an extension (“filled version”) of the map to a scale smooth Fredholm map between open subsets of the ambient scale Banach spaces, which has the same zero set. If the retractions arise from gluing constructions as the ones used to describe bubbling (nodes) or breaking (building) of pseudoholomorphic curves, then the geometry of the problem suggests a natural filling construction. Section 5 gives an example for the case of Hamiltonian Floer theory.

We demonstrate the strength of our simplified nonlinear Fredholm notion and polyfold theory in general by proving the Fredholm property for the perturbed Cauchy-Riemann operator near a broken Floer trajectory in three short Lemmas 5.3, 5.4, 5.5. Combined with the abstract transversality and implicit function theorem for M-polyfolds in [HWZ2], this Fredholm property replaces the entire transversality and gluing techniques in the classical construction of Hamiltonian Floer theory in the absence of bubbling, as developed by Floer [F]. In fact, it even provides a smooth structure with boundary and corners on perturbations of the higher dimensional compactified moduli spaces of Floer trajectories.

Due to the novelty of scale calculus, we adopt an expository style, in particular repeat all relevant definitions from the work of Hofer-Wysocki-Zehnder (which will be marked by [HWZ1]), provide the basic examples that may also appear elsewhere, and give highly detailed proofs of the functional analytical facts that are not readily available in [HWZ1]. However, in the application to Floer theory, we develop the relevant setting without further explanation of the M-polyfold notions, thus reducing the polyfold Fredholm property to analytic statements, which we then again prove in full detail.

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2. SOME BASIC SCALE CALCULUS

We begin with a new definition, which in practice is applied to spaces of smooth maps to give rise to scaled Banach spaces.

**Definition 2.1.** A norm scale on a vector space \( F \) is a sequence of norms \( (\| \cdot \|_k)_{k \in \mathbb{N}_0} \) on \( F \) such that for each \( k > j \) the identity map

\[
\text{id}_{F,j} : (F, \| \cdot \|_k) \rightarrow (F, \| \cdot \|_j)
\]

is continuous and compact. That is,

- there is a constant \( C_{k,j} \) such that \( \| f \|_j \leq C_{k,j} \| f \|_k \) for all \( f \in F \),
- the \( \| \cdot \|_k \)-ball \( \{ f \in F : \| f \|_k < 1 \} \) has compact closure in \( (F, \| \cdot \|_j) \).

Recall here that precompactness (i.e. compactness after closure) of the \( \| \cdot \|_k \)-unit ball implies precompactness of all other \( \| \cdot \|_k \)-bounded subsets in the \( \| \cdot \|_j \)-topology. Since all norms on a finite dimensional vector space are complete and equivalent, there is no nontrivial example of a norm scale on a finite dimensional vector space. The first two nontrivial examples, with the second one relevant to Gromov-Witten theory, are the following, in which we also list the examples of scale Banach spaces, which are formally introduced in Definition 2.2 below, that result from completion in each scale.

- The \( C^k \)-norms \( (\| \cdot \|_C^k)_{k \in \mathbb{N}_0} \) form a norm scale on \( F = C^\infty(S^1) \). The \( \| \cdot \|_C^k \)-completions of \( C^\infty(S^1) \) then form the scale Banach space

\[
F = \left( F^\| \|_k \right)_{k \in \mathbb{N}_0} = \left( C^k(S^1) \right)_{k \in \mathbb{N}_0}.
\]

- The \( W^{k,p} \) Sobolev-norms \( (\| \cdot \|_{W^{k,p}})_{k \in \mathbb{N}_0} \) form a norm scale on \( F = C^\infty(S^2, \mathbb{C}^n) \) for any \( 1 \leq p \leq \infty, n \geq 1 \). The \( \| \cdot \|_{W^{k,p}} \)-completions of \( C^\infty(S^2, \mathbb{C}^n) \) then form the scale Banach space

\[
F = \left( F^\| \|_k \right)_{k \in \mathbb{N}_0} = \left( W^{k,p}(S^2, \mathbb{C}^n) \right)_{k \in \mathbb{N}_0}.
\]

For the rest of this section we follow [HWZ1] – with some convenient tweaks of notation – in developing the basic language of scale calculus.

**Definition 2.2** ([HWZ1] 2.1). A sc-Banach space \( (E_k)_{k \in \mathbb{N}_0} \) is a Banach space \( (E, \| \cdot \|) \) together with an sc-structure \( (E_k, \| \cdot \|_k)_{k \in \mathbb{N}_0} \), which consists of a sequence of linear subspaces \( E_k \subset E \), each equipped with a Banach norm \( \| \cdot \|_k \), such that the following holds.

(i) We have \( (E, \| \cdot \|) = (E_0, \| \cdot \|_0) \) as Banach space.

(ii) For each \( k > j \) there is an inclusion of subspaces \( E_k \subset E_j \), and the inclusion map \( (E_k, \| \cdot \|_k) \rightarrow (E_j, \| \cdot \|_j) \) is continuous and compact.

(iii) The subspace \( E_\infty := \bigcap_{k \in \mathbb{N}_0} E_k \subset E \) is dense in each \( (E_k, \| \cdot \|_k) \).

**Remark 2.3.** Any finite dimensional Banach space \( E \) carries the trivial sc-structure \( (E_k = E)_{k \in \mathbb{N}_0} \). In particular, for \( n \in \mathbb{N} \) we will denote by \( \mathbb{R}^n \) and \( \mathbb{C}^n \) the real and complex Euclidean spaces with standard norm and trivial sc-structure.

Due to the density requirement (iii), there are no nontrivial sc-structures on finite dimensional spaces; and due to the compactness requirement (ii) there are no trivial sc-structures on infinite dimensional spaces.
At this point, the reader unfamiliar with scale calculus can get familiarized with the concept by checking that the completions with respect to a norm scale always form an sc-Banach space; in particular so do the examples above. Next, the prototypical examples of scale smooth but not classically differentiable maps are the reparametrization actions in the above examples.

- The translation action on functions with domain $S^1 := \mathbb{R}/\mathbb{Z}$,

$$\tau : \mathbb{R} \times C^0(S^1) \rightarrow C^0(S^1), \quad (s, f) \mapsto f(s + \cdot)$$

has directional derivatives at points $(s_0, f_0) \in \mathbb{R} \times C^1(S^1)$, but the derivative $\frac{\partial}{\partial \tau}$ of $f_0 + hF)(s_0 + hS) = S \hat{f}_0(s_0 + \cdot) + F(s_0 + \cdot)$ in direction $(S, F)$ is not even well defined for $f_0 \in C^0(S^1) \setminus C^1(S^1)$. Moreover, $\tau$ is in fact nowhere classically differentiable. However, the restriction of $\tau$ to a map $\mathbb{R} \times C^{k+1}(S^1, \mathbb{R}) \rightarrow C^k(S^1)$ is continuously differentiable for any $k \in \mathbb{N}_0$. In fact, $\tau$ is $sc^\infty$ if we equip $C^0(S^1)$ with the sc-structure $(C^k(S^1))_{k \in \mathbb{N}_0}$. This can be checked by noting that the differential is

$$D_{(s_0, f_0)}\tau(S, F) = S \hat{f}_0(s_0 + \cdot) + F(s_0 + \cdot).$$

(This example secretly uses the product $\mathbb{R} \times E$ of an sc-Banach space with the trivial sc-structure on $\mathbb{R}$, given by the scales $(\mathbb{R} \times E_k)_{k \in \mathbb{N}_0}$; c.f. Remark 3.3 below.)

- The reparametrization action by the group of Möbius transformations $PSL(2, \mathbb{C})$ on $S^2 = \mathbb{C}P^1$,

$$\theta : PSL(2, \mathbb{C}) \times W^{k,p}(S^2, \mathbb{C}^n) \rightarrow W^{k,p}(S^2, \mathbb{C}^n), \quad (\phi, u) \mapsto u \circ \phi$$

has directional derivatives at points $(\phi_0, u_0) \in PSL(2, \mathbb{C}) \times W^{k+1,p}(S^2, \mathbb{C}^n)$, but is differentiable only as map $PSL(2, \mathbb{C}) \times W^{k+1,p}(S^2, \mathbb{C}^n) \rightarrow W^{k,p}(S^2, \mathbb{C}^n)$. However, $\theta$ is $sc^\infty$ on the sc-Banach space $(W^{k,p}(S^2, \mathbb{C}^n))_{k \in \mathbb{N}_0}$.

(This example secretly uses a scale structure on the nonlinear space $PSL(2, \mathbb{C})_k$. Sc-differentiability in this context is defined in local charts of this Lie group in $\mathbb{C}^3$.)

**Definition 2.4 ([HWZ1] 2.3, 2.4, 2.13)** Let $E, F$ be sc-Banach spaces and let $\Phi : U \rightarrow F_0$ be a map defined on an open subset $U \subset E_0$.

(i) $\Phi$ is scale continuous (se$^0$) if $\Phi|_{U \cap E_m} : E_m \rightarrow F_m$ is continuous for all $m \in \mathbb{N}_0$;

(ii) $\Phi$ is scale differentiable if it is se$^0$ and for every $x \in U \cap E_1$ there exists a bounded linear operator $D\Phi(x) : E_0 \rightarrow F_0$ such that

$$\sup_{\|h\|_{E_1} = \epsilon} h^{-1}\|\Phi(x + h) - \Phi(x) - D\Phi(x)h\|_{F_0} \xrightarrow[h \rightarrow 0]{} 0$$

and $D\Phi(x)E_m \subset F_m$ whenever $x \in U \cap E_{m+1}$.

(iii) If $\Phi$ is scale differentiable then its **tangent map** is the map

$$T\Phi : TE|_U \rightarrow TF, \quad (x, h) \mapsto (\Phi(x), D\Phi(x)h),$$

defined on the open subset $TE|_U := (U \cap E_1) \times E_0$ of the sc-Banach space $TE := (E_{m+1} \times E_m)_{m \in \mathbb{N}_0}$, mapping to $TF := (F_{m+1} \times F_m)_{m \in \mathbb{N}_0}$.

1The directional derivative in any fixed direction $(S, F) \in \mathbb{R} \times C^0(S^1)$ exists since uniform continuity of $F$ guarantees $\max_{s \in S^1} |F(s + h) - F(s)| \rightarrow 0$ as $h \rightarrow 0$. However, the unit ball in $C^0(S^1)$ is not equicontinuous, so that differentiability on the normed space, $\sup_{F \in C^0(S^1)} \max_{s \in S^1} |F(s + h) - F(s)| \rightarrow 0$ as $h \rightarrow 0$, fails.

2Note that [HWZ1] does not explicitly define a notion of scale differentiability as in (ii), but rather groups (ii)-(iv) for $k = 1$ into the definition of continuous scale differentiability sc$^1$, which is the relevant notion for most purposes. The purpose of our definition (ii) is to define the tangent map (iii) in maximal generality.
(iv) \( \Phi \) is \( k \)-fold continuously scale differentiable \((sc^k)\) for \( k \geq 1 \) if it is scale differentiable and its tangent map \( T\Phi : TEE|_U \rightarrow TF \) is \( sc^{k-1} \);
(v) \( \Phi \) is scale smooth \((sc^\infty)\) if it is \( sc^k \) for all \( k \in \mathbb{N}_0 \).

Finally, for reference in the nonlinear Fredholm theory, we fix a germ-like notion of scale smoothness at a point.

**Definition 2.5.** Let \( E, F \) be sc-Banach spaces and let \( \Phi : U \rightarrow F_0 \) be a map defined on a neighbourhood \( U \subset E_0 \) of \( e_0 \in E_\infty \). Then we say that \( \Phi \) is **scale smooth at** \( e_0 \) (or **sc-\( \infty \)** at \( e_0 \)) if for every \( k \in \mathbb{N}_0 \) there exists a neighbourhood \( U_k \subset U \) of \( e_0 \) such that \( \Phi|_{U_k} \) is \( sc^k \).

### 3. Fredholm Property for Linear Operators

We begin with a new definition of the Fredholm property for linear maps on scale Banach spaces, which we will then show to be equivalent to the definition of Hofer-Wysocki-Zehnder, which we will denote as “HWZ-Fredholm”.

**Definition 3.1.** Let \( E, F \) be sc-Banach spaces. A **sc-Fredholm operator** \( T : E \rightarrow F \) is a linear map \( T : E_0 \rightarrow F_0 \) that satisfies the following.

(i) \( T \) is \( sc^0 \), that is all restrictions \( T|_{E_m} : E_m \rightarrow F_m \) for \( m \in \mathbb{N}_0 \) are bounded operators.
(ii) \( T \) is **regularizing**, that is \( e \in E_0 \) and \( Te \in F_m \) for some \( m \in \mathbb{N}_0 \) implies \( e \in E_m \).
(iii) \( T : E_0 \rightarrow F_0 \) is a Fredholm operator, that is it has finite dimensional kernel \( kerT \), closed range \( T(E_0) \), and finite dimensional cokernel \( F_0/T(E_0) \).

We will see in Lemma 3.5 below that assumptions (i)-(iii) in this definition in fact also imply that each restriction \( T|_{E_m} : E_m \rightarrow F_m \) is a Fredholm operator. Moreover, the **Fredholm index** of \( T \) is the same on any scale \( m \in \mathbb{N}_0 \),
\[
\text{ind}(T) = \text{ind}(T|_{E_m}) = \dim ker T - \dim(F_m/T(E_m)).
\]
The prototypical examples of sc-Fredholm operators are the following elliptic operators in the examples of Section 2.

- Let \( E := (C^{1+k}(S^1))_{k \in \mathbb{N}_0} \) and \( F := (C^k(S^1))_{k \in \mathbb{N}_0} \), then \( \Phi : C^1(S^1) \rightarrow C^0(S^1) \) is an sc-Fredholm operator \( \Phi : E \rightarrow F \).
- Let \( E := (W^{1+k,p}(S^2, \mathbb{C}^n))_{k \in \mathbb{N}_0} \) for \( 1 < p < \infty \), and denote the \( W^{k,p,*} \)-closure of smooth \((J, j)\)-antilinear \( \mathbb{C}^n \)-valued 1-forms on \( S^2 \) by \( F := (W^{k,p}(S^2, \Lambda^{0,1} \otimes J \mathbb{C}^n))_{k \in \mathbb{N}_0} \).

Then the Cauchy–Riemann operator \( \partial_{j} : W^{1,p}(S^2, \mathbb{C}^n) \rightarrow L^p(S^2, \Lambda^{0,1} \otimes J \mathbb{C}^n) \) with respect to \( J = i \) on \( \mathbb{C}^n \) and \( j = i \) on \( S^2 = \mathbb{C}P^1 \) is given by \( u \mapsto \frac{1}{2}(J \circ du \circ j + du) \). It is an sc-Fredholm operator \( \partial_j : E \rightarrow F \).

The \( sc^0 \)-property of these operators is a formalization of the fact that linear differential operators of degree \( d \) are bounded as operators between appropriate function spaces (e.g. Hölder or Sobolev spaces), with a difference of \( d \) in the differentiability index. The regularizing property, in this context, is simply the statement of elliptic regularity. Finally, the elliptic estimates for an operator and its dual generally hold on all scales similar to the boundedness above, and this implies the Fredholm property on all scales.

We will now compare this scale Fredholm property with the definition of the Fredholm property by Hofer-Wysocki-Zehnder that is based on the notion of direct sums in scale Banach spaces, as follows.

**Definition 3.2 (HWZ) 2.5.** Let \( E \) be an sc-Banach space. Two linear subspaces \( X, Y \subset E_0 \) split \( E \) as a **sc-direct sum** \( E = X \oplus_{sc} Y \) if
Remark 3.3 ([HWZ2] Glossary). Reversing the definition of sc-direct sum, there is a natural product notion $E \times F$ for sc-Banach spaces, such that $E \times F = (E \times \{0\}) \oplus (\{0\} \times F)$. The sc-product $E \times F$ of two sc-Banach spaces $E, F$ is the Cartesian product $E \times F$ with the scale structure $(E \times F)_k := (E_k \times F_k, \| \cdot \|_{E_k} + \| \cdot \|_{F_k})$.

Definition 3.4 ([HWZ1] 2.8.). Let $E, F$ be sc-Banach spaces. A HWZ-Fredholm operator $T : E \to F$ is a linear map $T : E_0 \to F_0$ that satisfies the following.

(i) The kernel $\ker T$ is finite dimensional and has a sc-complement $E = \ker T \oplus_{sc} X$.
(ii) The image $T(E_0)$ has a finite dimensional sc-complement $F = T(E_0) \oplus_{sc} C$.
(iii) The operator restricts to a sc-isomorphism $T|_X : X \to T(E_0)$.

Before we prove the equivalence of the sc-Fredholm and HWZ-Fredholm property, let us show that sc-Fredholm operators are in fact Fredholm on each scale, in particular each image $T(E_m) \subset F_m$ is closed, which is the only strengthening of (iii) that is needed in the proof of equivalence in Lemma 3.6 below.

Lemma 3.5. If $T : E \to F$ is sc-Fredholm, then the restrictions $T|_{E_m} : E_m \to F_m$ are Fredholm for all $m \in \mathbb{N}$ with kernel and cokernel

$$\ker T|_{E_m} = \ker T \subset E_\infty, \quad F_m / T(E_m) \cong F_0 / \lim T,$$

where the latter isomorphism is induced by the inclusion $F_m \subset F_0$. In particular, the Fredholm index of $T|_{E_m}$ is the same on any scale $m \in \mathbb{N}_0$.

Proof. Due to the embedding $E_m \subset E_0$, the kernel of $T|_{E_m}$ is $\ker T \cap E_m$, i.e. finite dimensional since it is a subspace of the kernel $\ker T \subset E_0$ that is finite dimensional by the Fredholm property of $T : E_0 \to F_0$ given by (iii). In fact, the regularization property (ii) for $0 \in F_\infty$ implies $\ker T \subset E_\infty$, so that $\ker T \cap E_m = \ker T$ for all $m \in \mathbb{N}_0$.

Next, we will show that $T(E_m) \subset F_m$ is closed, although this will also follow from finite dimensionality of the cokernel. For that purpose we need to consider any sequence $e_i \in E_m$ which has converging images $T(e_i) \to f_\infty$ in the $F_m$-topology, and show that $f_\infty \in T(E_m)$. Indeed, then closedness of $T(E_0) \subset F_0$ from the Fredholm property (iii) implies $f_\infty = T(e')$ for some $e' \in E_0$, and we have $T(e') = f_\infty \in F_m$ since it is the limit of a sequence in $F_m$, hence the regularization property (ii) implies that $e' \in E_m$ and hence $f_\infty = T(e') \in T(E_m)$ by the boundedness of $T|_{E_m}$ given by the sc$^0$ property (i).

The last part of this argument can be rephrased to say that the regularizing property (ii) together with the boundedness (i) imply $T(E_m) = T(E_0) \cap F_m$. Hence the inclusion $F_m \subset F_0$ induces an injection of cokernels $F_m / T(E_m) \hookrightarrow F_0 / T(E_0)$, of which the latter is finite dimensional by (iii). This proves that $T|_{E_m}$ also has finite dimensional cokernel, and hence is Fredholm as claimed. In fact, since $F_m \subset F_0$ is dense, the image of this injection must also be dense. But in finite dimensions that means equality, as claimed. $\square$

Lemma 3.6. Let $E, F$ be sc-Banach spaces, then a linear map $T : E_0 \to F_0$ is sc-Fredholm iff it is HWZ-Fredholm.

Proof. Given the splittings (iii) for a HWZ-Fredholm operator, the finite dimensional summands are necessarily contained in the “smooth” intersection of all scales, $\ker T \subset E_\infty$ and $C \subset F_\infty$, since otherwise e.g. $C \cap F_\infty$ would be a proper subspace of $C$, which in finite dimensions contradicts the density axiom for sc-Banach spaces. Next, any HWZ-Fredholm
operator $T$ is regularizing by [HWZ1, Proposition 2.9], that is $T(E_0) \cap F_m = T(E_m)$. Together with $\ker T \subset E_\infty$, this indeed implies $T^{-1}(T(E_0) \cap F_m) \subset E_m$. Moreover, each restriction $T|_{E_m}$ can be viewed as operator between the direct sums $T|_{E_m} : \ker T \oplus (X \cap E_m) \to T(E_m) \oplus C$, where by (iii) the further restriction $T|_{X \cap E_m} : X \cap E_m \to T(E_m)$ is an isomorphism. Since $\ker T$ and $C$ are finite dimensional, this implies the classical Fredholm property of $T|_{E_m}$, and hence shows that $T$ is sc-Fredholm.

Conversely, given a sc-Fredholm operator $T$, we have $\ker T \subset E_\infty$ by the regularizing property, and this kernel is finite dimensional by the Fredholm property of $T|_{E_0}$. Then [HWZ1, Proposition 2.7] provides an sc-complement $E = \ker T \oplus_{sc} X$, that is $X \cap E_m$ is a topological complement for $\ker T \subset E_m$ for each $m \in \mathbb{N}_0$. Next, $T(E_0) \subset F_0$ is closed and of finite codimension by the Fredholm property of $T|_{E_0}$, hence has a finite dimensional topological complement in $F_0$. Then [HWZ1, Lemma 2.12] provides a finite dimensional subspace $C \subset F_\infty$ such that $F_0 = T(E_0) \oplus C$. We claim that this in fact induces an sc-direct sum

$$\mathcal{F} = T(E_0) \oplus_{sc} C.$$ 

To check this we first ensure that $T(E_0) \cap F_m$ defines an sc-structure on $T(E_0)$. Indeed, by the regularizing property we have $T(E_0) \cap F_m = T(E_m)$, which is closed by Lemma 3.5, and hence inherits a Banach space structure from $F_m$. Now the embedding $T(E_m) \to T(E_{m+1})$ is compact since it is a restriction of the compact embedding $F_m \subset F_{m+1}$. Moreover, $T(E_0) \cap F_\infty = T(E_\infty)$ is dense in every $T(E_m)$ since $E_\infty \subset E_m$ is dense and $T : E_m \to F_m$ is continuous. Thus we have a scale structure on $T(E_0)$, along with the trivial scale structure $(C_m = C)_{m \in \mathbb{N}_0}$ on $C$. It remains to check that the direct sum isomorphism $\Pi_{T(E_0)} \times \Pi_C : F_0 \to T(E_0) \times C$, given by continuous projection maps, is in fact an sc-isomorphism. On the finite dimensional space $C$ all norms are equivalent, so the continuity of $\Pi_C : F_0 \to C \subset F_0$ and $F_m \subset F_0$ implies continuity of $\Pi_C : F_m \to C \subset F_m$.

$$\|\Pi_C f\|_{F_m} \leq C_m \|\Pi_C f\|_{F_0} \leq C_m C \|f\|_{F_0} \leq C_m C C_{m'} \|f\|_{F_m} \quad \forall f \in F_m.$$ 

Now this implies continuity of $\Pi_{T(E_0)}|_{F_m} = \text{Id}_{F_m} - \Pi_C|_{F_m}$. Hence we have established (1), that is, $C \subset F_m$ is a topological complement of $T(E_m) = T(E_0) \cap F_m$ for each $m \in \mathbb{N}_0$. Finally, the restriction $T|_X : X \to T(E_0)$ is an sc-isomorphism since on every level $T : X \cap E_m \to T(E_m)$ is the restriction of a Fredholm operator to a map between the complement of the kernel and the image. This proves that $T$ is also HWZ-Fredholm. \qed

4. Fredholm property for nonlinear maps

The notion of a nonlinear Fredholm map on scale Banach spaces cannot simply be obtained by adding “sc-” in appropriate places to the classical definition of Fredholm maps, but requires a minor tweaking to obtain an implicit function theorem for scale differentiable maps with surjective linearization. The latter is usually proven by means of a contraction property of the map in a suitable reduction. Since the contraction will be iterated to obtain convergence, it needs to act on a fixed Banach space rather than between different levels of a scale Banach space. However, in classical nonlinear Fredholm theory, this contraction form follows from the continuity of the differential in the operator norm, whereas the differential of a scale smooth map is generally continuous only as operator between different levels. Hofer-Wysocki-Zehnder solve this issue by making the contraction property a part of the definition of Fredholm maps. However, this raises the question of how this property can be proven for a given map. It turns out that in practice, this “contraction germ normal form” is established by proving classical continuous differentiability of the map in all but finitely
many directions and the scale Fredholm property for this partial derivative. We will formalize this approach in an alternative definition of the nonlinear Fredholm property, which is stronger than the following definition from [HWZ1], but is easier to check in practice.

Throughout we restrict our discussion to the Fredholm property at the zero vector $0 \in E$ in a scale Banach space. However, by a simple shift this provides the general Fredholm notion in polyfold theory, namely at any point $e_0 \in E_\infty$ in the dense “smooth” subset.

**Definition 4.1** ([HWZ2] 3.6). Let $\Phi : E \rightarrow F$ be a sc$^\infty$ map between sc-Banach spaces $E$, $F$. Then $\Phi$ is **HWZ Fredholm at 0** if the following holds:

(i) $\Phi$ is regularizing as germ: For every $m \in \mathbb{N}_0$ there exists $\epsilon_m > 0$ such that $\Phi(e) \in F_{m+1}$ and $\|e\|_{E_m} \leq \epsilon_m$ implies $e \in E_{m+1}$.

(ii) $\Phi$ has a contraction germ normal form, that is there exist

- an sc-embedding $h : U \rightarrow \mathbb{R}^k \times \mathbb{W}$ (i.e. an sc$^\infty$ map to an open subset with sc$^\infty$ inverse) for some neighbourhood $U \subset E_0$ of 0, some $k \in \mathbb{N}_0$, and some sc-Banach space $\mathbb{W}$, such that $h(0) = (0, 0)$;

- a germ of strong bundle isomorphism $G = (g_\ell : \mathbb{F} \rightarrow \mathbb{R}^{\ell} \times \mathbb{W})_{\ell \in \mathbb{N}_0}$ for some $\ell \in \mathbb{N}_0$, that is a family of linear bijections $g_\ell : F_0 \rightarrow \mathbb{R}^{\ell} \times W_0$ such that the map

$$G : \left((E_m \cap U) \times F_{m+1}\right)_{m \in \mathbb{N}_0} \rightarrow (\mathbb{R}^\ell \times W_{m+1})_{m \in \mathbb{N}_0}, \quad (e, f) \mapsto g_\ell(f)$$

restricts to sc$^n$ maps for $i = 0, 1$ on neighbourhoods $U_n \subset U$ of 0 for every $n \in \mathbb{N}_0$; such that the transformed map is of the form

$$G \circ (\Phi - \Phi(0)) \circ h^{-1} : (v, w) \mapsto \left(A(v, w), w - B(v, w)\right),$$

where $A : \mathbb{R}^k \times \mathbb{W} \rightarrow \mathbb{R}^\ell$ is any sc$^\infty$ map and $B : \mathbb{R}^k \times \mathbb{W} \rightarrow \mathbb{W}$ is a contraction germ: For every $m \in \mathbb{N}_0$ and $\theta > 0$ there exists $\epsilon_m > 0$ such that for all $v \in \mathbb{R}^k$ and $w_1, w_2 \in \mathbb{W}$ with $|v|_{\mathbb{R}^k}, \|w_1\|_{W_m}, \|w_2\|_{W_m} \leq \epsilon_m$ we have

$$\|B(v, w_1) - B(v, w_2)\|_{W_m} \leq \theta \|w_1 - w_2\|_{W_m}. \tag{2}$$

In classical Fredholm theory, the equivalence to the above contraction germ normal form holds automatically for a continuously differentiable map whose differential at 0 is Fredholm. The following remark explains this in detail and explores the failure of the analogous statement for scale smooth maps with sc-Fredholm differential.

**Remark 4.2.** Suppose that $\Phi : E \rightarrow F$ is a sc$^\infty$ map whose differential $D\Phi(0) : E \rightarrow F$ is Fredholm, and let $\mathbb{W} \subset E$ be a complement of its kernel. Then by Definition 2.4(ii) of the differential we have $\Phi(h) = \Phi(0) + D\Phi(0)h + R(h)$ with $\|R(h)\|_0 \leq \epsilon(||h||_{\mathbb{W}}||h||_1)$, for a function $\epsilon : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ with $\lim_{h \rightarrow 0} \epsilon(h) = 0$. Moreover, let $g = \iota^{-1} \oplus D\Phi(0)^{-1} : F \rightarrow \mathbb{R}^\ell \times \mathbb{W}$ be the isomorphism induced by the direct sum $F = \iota(\mathbb{R}^\ell) \oplus D\Phi(0)(\mathbb{W})$ for a choice of complement $\iota : \mathbb{R}^\ell \hookrightarrow F$ of the image of $D\Phi(0)$. Then, writing $h = v + w \in \ker D\Phi(0) \oplus \mathbb{W}$ we almost obtain the contraction form, namely

$$g(\Phi(v + w)) = g(\Phi(0)) \left(\iota^{-1}(\text{pr}_{\iota(\mathbb{R}^\ell)} R(v + w)), w - B(v, w)\right)$$

where $B(v, w) := -\text{pr}_{D\Phi(0)\mathbb{W}} R(v + w)$ is a contraction with respect to shifted scales from $W_1$ to $W_0$. That is, given $\theta > 0$ we have for $\|v + w_1\|_{E_1}, \|w_2 - w_1\|_{W_1}$ sufficiently small

$$\|B(v, w_1) - B(v, w_2)\|_{W_0} \leq \theta \|w_1 - w_2\|_{W_1}.$$
Indeed, $\Phi : E_1 \to F_0$ is classically continuously differentiable (see e.g. [HWZ4, 2.1]), so that the mean value inequality for $v + w_1, v + w_2 \in E_1$ gives for some $t \in [0,1]$

$$R(v + w_1) - R(v + w_2) = \Phi(v + w_1) - \Phi(v + w_2) + D\Phi(0)(w_2 - w_1)$$

$$= (D\Phi(v + w_1 + t(w_2 - w_1)) - D\Phi(0))(w_1 - w_2).$$

The continuity of the differential $D\Phi : E_1 \to L(E_1, F_0)$ then gives

$$\|D\Phi(v + w_1 + t(w_2 - w_1)) - D\Phi(0)\|_{L(E_1, F_0)} \leq \theta$$

for $\|v + w_1 + t(w_2 - w_1)\|_{E_1} \leq \|v + w_1\|_{E_1} + \|w_2 - w_1\|_{W_1}$ sufficiently small. This proves the above shifted contraction property, and if $\Phi : W_0 \to W_0$ was classically continuously differentiable, then the same estimates would hold with $W_1$ replaced by $W_0$, thus establishing a contraction. In the merely scale differentiable case the analogous estimates can be established on all scales – providing contractions from $W_{m+1}$ to $W_m$ – but this shift prevents us from applying any iteration arguments such as the proof of Banach’s fixed point theorem which would provide an implicit function theorem.

Delving a little deeper into scale differentiability, note (or refer to [HWZ4, 2.1]) that $\Phi$ restricts to classically $C^1$ maps $E_{m+1} \to F_m$ for all $m \in \mathbb{N}_0$, and moreover its differentials extend to bounded linear maps in $L(E_m, F_m)$, which however depend continuously only on $E_{m+1}$ in the weak sense of continuity of the map

$$E_{m+1} \times E_m \to F_m, \quad (x, e) \mapsto d\Phi(x)e.$$

Again considering the mean value inequality for $v, w_1, w_2 \in E_1$,

$$R(v + w_1) - R(v + w_2) = (D\Phi(v + w_1 + t(w_2 - w_1)) - D\Phi(0))(w_1 - w_2),$$

note that if $x \mapsto D\Phi(x)$ is not just continuous in the sense of (3), but in the operator topology as map $E_{m+1} \to L(E_m, F_m)$, which is the case in many applications, then $B$ satisfies the contraction property (2) on the $W_m$-scale for $\|v\|_{W_{m+1}}, \|w_1\|_{W_{m+1}}, \|w_2\|_{W_{m+1}}$ sufficiently small. However, this shift in norms still prevents us from applying Banach’s fixed point theorem to the equation $w = B(v, w)$ since closed $W_{m+1}$-balls are not complete in the $W_m$-norm.

The discussion of this remark shows that an implicit function theorem for maps with surjective differential only follows from standard techniques if $\Phi : E_m \to F_m$ is $C^1$ in the standard sense. However, for Cauchy-Riemann operators, this stronger differentiability will not hold as soon as $E_m$ contains gluing parameters which act on functions by reparametrization. This, however, is usually the only source of non-differentiability, and after splitting off a finite dimensional space of gluing parameters one deals with classical $C^1$-maps on all scale levels. If their differential would depend continuously on the gluing parameters in the operator topology, then the linear transformation of Remark 4.2 would bring $\Phi$ into the contraction germ normal form that is required for $\Phi$ to be HWZ Fredholm. In applications, this is generally not quite the case, but some weaker continuity still holds and suffices to find a nontrivial bundle isomorphism into contraction germ normal. This motivates the following definition, which is just slightly stronger than the definition via contraction germs, but should be more intuitive for Cauchy-Riemann operators in the presence of gluing. In fact, in practice the Fredholm property is implicitly proven via this stronger differentiability, see e.g. [HWZ6, Thm.8.26], [HWZ5, Prop.4.8], and Remark 4.4 below. Here we denote open balls centered at 0 in a level $E_m$ of a scale space by

$$B_r^{E_m} := \{ e \in E_m \mid \| e \|_m < r \} \quad \text{for } r > 0.$$
**Definition 4.3.** Let \( \Phi : E \to F \) be a sc\(^{\infty} \) map between sc-Banach spaces \( E, F \). Then \( \Phi \) is sc-Fredholm at \( 0 \) with respect to the splitting \( E \cong \mathbb{R}^d \times E' \) if the following holds.

(i) \( \Phi \) is regularizing as germ, that is for every \( m \in \mathbb{N}_0 \) there exists \( \epsilon_m > 0 \) such that \( \Phi(e) \in F_{m+1} \) and \( \|e\|_{E_m} \leq \epsilon_m \) implies \( e \in F_{m+1} \).

(ii) \( E \cong \mathbb{R}^d \times E' \) is an sc-isomorphism and for every \( m \in \mathbb{N}_0 \) there exists \( \epsilon_m > 0 \) such that \( \Phi(r, \cdot) : B^E_{\epsilon_m} \to F_m \) is differentiable for all \( |r|_{\mathbb{R}^d} < \epsilon_m \), and its differential \( D_{\mathbb{R}^d} \Phi(r_0, e_0) : E' \to F, e \mapsto \frac{d}{dr} \Phi(r_0, e_0 + te) \) in the direction of \( E' \) has the following continuity properties:

a) For fixed \( m \in \mathbb{N}_0 \) and \( r \in B^{E_m}_{\epsilon_m} \) the differential operator \( B^{E_m}_{\epsilon_m} \to L(E_m', F_m) \), \( e \mapsto D_{\mathbb{R}^d} \Phi(r, e) \) is continuous, and the continuity is uniform in a neighbourhood of \( (r, e) = (0, 0) \). That is, for any \( \delta > 0 \) there exists \( 0 < \epsilon_{m, \delta} \leq \epsilon_m \) such that for all \( (r, e) \in B^{E_m}_{\epsilon_{m, \delta}} \times B^{E_m}_{\epsilon_{m, \delta}} \) we have

\[
\|D_{\mathbb{R}^d} \Phi(r, e)h - D_{\mathbb{R}^d} \Phi(r, e')h\|_{F_m} \leq \delta \|h\|_{E_m'}, \quad \forall \|e' - e\|_{E_m} \leq \epsilon_{m, \delta}, h \in E'_m.
\]

b) For any sequences \( \mathbb{R}^d \ni r^\nu \to 0 \) and \( e^\nu \in B^{E_m}_{\epsilon_m} \) with \( \|D_{\mathbb{R}^d} \Phi(r^\nu, 0)e^\nu\|_{F_m} \to 0 \) there exists a subsequence such that \( \|D_{\mathbb{R}^d} \Phi(0, 0)e^\nu\|_{F_m} \to 0 \).

(iii) The differential \( D_{\mathbb{R}^d} \Phi(0, 0) : E' \to F \) is sc-Fredholm in the sense of Definition 3.1. Moreover, \( D_{\mathbb{R}^d} \Phi(r, 0) : E_0 \to F_0 \) is Fredholm for all \( |r|_{\mathbb{R}^d} < \epsilon_0 \), with Fredholm index equal to that for \( r = 0 \), and weakly regularizing, that is \( \ker D_{\mathbb{R}^d} \Phi(r, 0) \subset E_1 \).

**Remark 4.4.** We can compare the above definition with the analytic properties [HWZ5, Prop.4.23, 4.25] of the Cauchy-Riemann operator in the Gromov-Witten case, from which Hofer-Wysocki-Zehnder deduce its polyfold Fredholm property. They also give an abstract summary in [HWZ5, Prop.4.26], which only came to the authors’ attention after completion of this manuscript.

The differentiability in all but finitely many directions (ii) is the second bullet of [HWZ5, Prop.4.23], but not explicitly assumed in [HWZ5, Prop.4.26]. The continuity in (ii a) corresponds exactly to [HWZ5, Prop.4.25] resp. [HWZ5, Prop.4.26 (3)]. The sc-Fredholm property at \( (0, 0) \) in (iii) is also required by the first bullet of [HWZ5, Prop.4.23] resp. [HWZ5, Prop.4.26 (1)], which moreover require the sc-Fredholm property with the same index for all sufficiently small \( (r, e) \). The latter is stronger than our classical Fredholm, index, and regularization conditions for small \( (r, e) \), but follows from (iii) together with the differentiability (ii) and the techniques of Section 3.

Finally, it remains to compare the continuity in (ii b) with the third bullet of [HWZ5, Prop.4.23] resp. [HWZ5, Prop.4.26 (2)]. The special case of \( K = \{0\} \) and \( z^\nu = 0 \) in the latter is exactly the assertion in our setting that a subsequence of \( e^\nu \) converges in \( E_m \). This then implies (ii b) due to the continuity of \( \mathbb{R}^d \times E_m' \to F_m, (r, e) \mapsto D_{\mathbb{R}^d} \Phi(r, 0)e \). On the other hand, condition (ii b) implies this special case of [HWZ5, Prop.4.23] due to the estimate arising from injectivity of \( D_{\mathbb{R}^d} \Phi(0, 0) \) on a complement of its finite dimensional kernel. The general case of \( D_{\mathbb{R}^d} \Phi(r^\nu, k^\nu) - z^\nu \to 0 \) for nontrivial \( k^\nu \in K \subset \mathbb{E}' \) or \( \|z^\nu\|_{F_{m+1}} \leq 1 \) does not seem to be a direct consequence of our conditions, though it might follow from the contraction germ property.

Using the weak continuity properties of the partial differential in all but finitely many directions, we can extend the techniques of Remark 4.2 to obtain a contraction germ normal form for sc-Fredholm operators, and thus prove that they are essentially HWZ-Fredholm operators. Note here that in applications of the scale Fredholm theory, e.g. the implicit
function theorem of polyfold theory, the contraction germ property is necessary only from some fixed scale onwards, so that Newton iteration can be performed on each sufficiently high scale to find a smooth solution set in the subset of “smooth points” of the polyfold.

**Theorem 4.5.** Let \( \Phi : E \to F \) be a \( sc^\infty \) map. If \( \Phi \) is \( sc \)-Fredholm at 0 with respect to a splitting \( E \cong \mathbb{R}^d \times E' \), then it satisfies all conditions of Definition 4.1 with the possible exception of the contraction (2) for \( m = 0 \). In particular, \( \Phi|_{E_m} : (E_m)_{m \in \mathbb{N}} \to (F_m)_{m \in \mathbb{N}} \) is HWZ-Fredholm at 0 with respect to the induced scale structures on \( E_1 \subset E_0 \) and \( F_1 \subset F_0 \).

**Proof.** Since \( D_{E'} \Phi(0,0) \) is sc-Fredholm, Lemma 3.6 provides sc-direct sums

\[
E' = \ker D_{E'} \Phi(0,0) \oplus_{sc} \mathbb{W}, \quad F = \text{im} D_{E'} \Phi(0,0) \oplus_{sc} C,
\]

such that \( W_m := W_0 \cap E_m \) is an sc-structure on \( W_0 \) and \( C \subset F_\infty \) is finite dimensional. Denote by \( \Pi_C : F \to C \) and \( \Pi_C^\perp := \text{id}_F - \Pi_C : F \to \text{im} \Phi(0,0) \) the \( sc^0 \) projections to the factors, then we claim that for \( \epsilon'^*_m > 0 \) sufficiently small we obtain isomorphisms

\[
\Pi_C^\perp \circ D_{E'} \Phi(r,0)|_{W_0} : W_0 \overset{\approx}{\to} \text{im} D_{E'} \Phi(0,0) \quad \forall |r|_{\mathbb{R}^d} \leq \epsilon'^*_m
\]

satisfying uniform estimates for all \( m \geq 1 \) with some \( \epsilon'_m > 0 \),

\[
\|w\|_{W_m} \leq C_m \|\Pi_C^\perp D_{E'} \Phi(r,0) w\|_{F_m} \quad \forall |r|_{\mathbb{R}^d} \leq \epsilon'_m, w \in W_m.
\]

Both hold by construction for \( r = 0 \), so our claim is that, for a possibly larger constant \( C_m \), they continue to hold for \( |r|_{\mathbb{R}^d} \leq \epsilon'_m \) sufficiently small. Note moreover that due to the finite codimensional restrictions in domain and target and the Fredholm condition (iii) on the map \( \Phi \) – we are dealing with sc-operators of Fredholm index

\[
\text{ind} \Pi_C^\perp D_{E'} \Phi(r,0)|_{W_0} = \text{ind} D_{E'} \Phi(r,0) - \dim \ker D_{E'} \Phi(0,0) + \dim C
\]

\[
= \text{ind} D_{E'} \Phi(0,0) - \dim \ker D_{E'} \Phi(0,0) + \dim \frac{\text{im} F_0}{\text{im} D_{E'} \Phi(0,0)} = 0.
\]

Hence for the isomorphism property (4) is suffice to prove injectivity on \( W_0 \); which follows directly from the estimate (5) for \( m = 1 \) and the weak regularization property in Definition 4.3 (iii). So it remains to prove (5) for \( r \neq 0 \). For that purpose suppose by contradiction that for a fixed \( m \in \mathbb{N} \) there exist sequences \( \mathbb{R}^d \ni r^\prime \to 0 \) and \( \|w^\prime\|_{W_m} = 1 \) such that \( \|\Pi_C^\perp D_{E'} \Phi(r^\prime,0) w^\prime\|_{F_m} \to 0 \). Since \( \Pi_C \) is a bounded map to the finite dimensional subspace \( C \subset F_\infty \), on which all norms are equivalent, we find a subsequence (which we again index by \( \nu \in \mathbb{N} \)) such that \( \Pi_C D_{E'} \Phi(r^\nu,0) w^\nu \to c_\infty \in C \) converges in \( F_m \), and hence \( \|D_{E'} \Phi(r^\nu,0) w^\nu - c_\infty\|_{F_m} \to 0 \). Moreover, since \( W_m \to W_{m-1} \) is compact, we find another subsequence such that \( w^\nu \to w^\infty \) in \( W_{m-1} \) converges in \( W_{m-1} \subset E_{m-1} \).

This means that we have convergence \( (r^\nu,0, w^\nu) \to 0 \) on the \( k = (m - 1) \)-st level of the scale tangent space \( T(\mathbb{R}^d \times E') = (\mathbb{R}^d \times E_{k+1} \times \mathbb{R}^d \times E_k)_{k \in \mathbb{N}_0} \), so that the sc\( d \) regularity of \( \Phi \) implies \( F_{m-1} \)-convergence \( D_{E'} \Phi(r^\nu,0) w^\nu \to D_{E'} \Phi(0,0) w^\infty \). In particular, since \( F_m \subset F_{m-1} \) embeds continuously, we have \( D_{E'} \Phi(0,0) w^\infty = c_\infty \in C \), but since \( w^\nu \in W_{m-1} \) by construction maps to the complement of \( C \), this implies \( c_\infty = 0 \). So we have used the scale smoothness and injectivity of \( \Pi_C^\perp D_{E'} \Phi(0,0) \) to strengthen the assumption to \( \|D_{E'} \Phi(r^\nu,0) w^\nu\|_{F_m} \to 0 \). At this point we can use the continuity property (ii b) to deduce \( \|D_{E'} \Phi(0,0) w^\nu\|_{F_m} \to 0 \) for a subsequence, so that finally (5) for \( r \) \( 0 \) implies \( \|w^\nu\|_{W_m} \to 0 \) in contradiction to the assumption. This proves (5) for \( |r|_{\mathbb{R}^d} \) sufficiently small, in particular, it implies that \( \Pi_C^\perp D_{E'} \Phi(r,0) \) is an injective semi-Fredholm operator, and by the above index calculation, it is in fact an isomorphism which proves (4).

After these preparations, an isomorphism of the base \( h : E \cong \mathbb{R}^d \times E' \to (\mathbb{R}^d \times \ker D_{E'} \Phi(0,0)) \times \mathbb{W} \) is simply given by splitting off the kernel of \( D_{E'} \Phi(0,0) \) from \( E' \).
and adding it to the finite dimensional parameter space. So since the first factor $\mathbb{R}^d \times \ker D_\mathcal{E} \Phi(0, 0)$ is finite dimensional, we can equip it with the $E_0$-norm and find a bounded isomorphism to some $\mathbb{R}^k$. This then also is an sc-isomorphism since all $E_m$-norms restricted to the finite dimensional $\ker D_\mathcal{E} \Phi(0, 0)$ are equivalent. Next, we obtain a bundle isomorphism $G = (g_e)_{e \in \mathcal{U}}$ over $\mathcal{U} := \{ e \in E_0 \mid h(e) \in B^d_{e_0} \times \ker D_\mathcal{E} \Phi(0, 0) \times \mathbb{W} \}$ by

$$G : \mathcal{U} \times \mathbb{F} \to C \times \mathbb{W}, \quad (e, f) \mapsto (\Pi_C f, (\Pi_C^d \circ D_\mathcal{E} \Phi(pr_{\mathbb{R}}(h(e)), 0))^{-1} \Pi_C f).$$

Here again the first factor $C$ is finite dimensional, hence sc-isomorphic to some $\mathbb{R}^d$. Moreover, this map has the required strong sc-infinity regularity as germ near $\{0\} \times \mathbb{F}$ because

$$\left( B^d_{e_0} \times \ker D_\mathcal{E} \Phi(0, 0) \times W_m \times F_{m+i} \right)_{m \in \mathbb{N}_0} \to \left( C \times W_{m+i} \right)_{m \in \mathbb{N}_0},$$

$$\begin{aligned}
(r, k, w, f) &\mapsto (\Pi_C f, (\Pi_C^d \circ D_\mathcal{E} \Phi(0, 0))^{-1} \Pi_C f)
\end{aligned}$$

is independent of $k, w$, so that scale smoothness for any $i \geq -m$ follows by the chain rule from the scale smoothness of $\Pi_C, \Pi_C^d$ (which are constructed as linear sc-operators) and

$$(r, f) \mapsto (\Pi_C^d \circ D_\mathcal{E} \Phi(0, 0))^{-1} f$$

which is a parametrized inverse to an sc-infinity map, so that the usual formula for the derivative of an inverse proves scale smoothness.

These isomorphisms transform the original map $\Phi$ to $G \circ (\Phi - \Phi(0, 0)) \circ h^{-1} : \mathbb{R}^d \times \ker D_\mathcal{E} \Phi(0, 0) \times \mathbb{W} \to C \times \mathbb{W}$ given by

$$(r, k, w) \mapsto g_{h^{-1}(r, k, w)}(\Phi(r, k + w) - \Phi(0, 0)) = (A(r, k, w), w - B(r, k, w)),$$

where

$$A(r, k, w) = \Pi_C(\Phi(r, k + w) - \Phi(0, 0))$$

and

$$w - B(r, k, w) = (\Pi_C^d \circ D_\mathcal{E} \Phi(0, 0))^{-1} \Pi_C(\Phi(r, k + w) - \Phi(0, 0)).$$

By construction, $A : \mathbb{R}^d \times \ker D_\mathcal{E} \Phi(0, 0) \times \mathbb{W} \to C$ is sc-infinity and $B : \mathbb{R}^d \times \ker D_\mathcal{E} \Phi(0, 0) \times \mathbb{W} \to \mathbb{W}$ is sc-infinity at $\{(0, 0)\} \times \mathbb{W}$, so it remains to establish the contraction germ property for $B$ for a fixed $m \in \mathbb{N}$ and $\theta > 0$. Note that we have

$$-\Pi_C^d \circ D_\mathcal{E} \Phi(0, 0) B(r, k, w) = \Pi_C^d (\Phi(r, k + w) - \Phi(0, 0) - D_{\mathcal{E}} \Phi(r, 0) w)$$

and hence we can estimate, using the uniform bound (5),

$$\left\| B(r, k, w_1) - B(r, k, w_2) \right\|_{W_m = W \cap E'_m} \leq C_m \left\| \Pi_C^d (\Phi(r, 0) B(r, k, w_1) - \Phi(r, 0) B(r, k, w_2)) \right\|_{E'_m} \leq C_m \left\| \Pi_C^d (\Phi(r, k + w_1) - \Phi(r, k + w_2) - D_{\mathcal{E}} \Phi(0, 0)(w_1 - w_2)) \right\|_{E'_m} \leq C'_m \left\| \Phi(r, k + w_1) - \Phi(r, k + w_2) - D_{\mathcal{E}} \Phi(0, 0)(w_1 - w_2) \right\|_{E'_m}.$$

To prove the contraction germ property, let $\theta > 0$ be given, then we must bound the last expression by $\theta \left\| w_1 - w_2 \right\|_{W_m}$ for $\left\| (r, k) \right\|_{\mathbb{R}^d \times \ker D_\mathcal{E} \Phi(0, 0)}, \left\| w_1 \right\|_{W_m}$, and $\left\| w_2 \right\|_{W_m}$ sufficiently small. For that purpose we will use the differentiability of $\Phi(r, \cdot) : B^d_{e_0} \to F_m$ for $\left\| r \right\|_{\mathbb{R}^d} < \epsilon_m$. By the triangle inequality, $\left\| k \right\|_{E'_m}, \left\| w_1 \right\|_{E'_m}, \left\| w_2 \right\|_{E'_m} < \frac{\theta}{4} \epsilon_m$ guarantees that $k + \lambda w_1 + (1 - \lambda) w_2 \in B^d_{e_0}$ for all $\lambda \in [0, 1]$. Then $[0, 1] \to F_m, \lambda \mapsto \lambda w_1 + (1 - \lambda) w_2$.
we consider an autonomous Hamiltonian vector field (with the exception of sphere bubbling, that is already treated \[HWZ5\]). For that purpose bation in a simplified geometric setting that nevertheless captures all analytic subtleties setup for Symplectic Field Theory in \[HWZ7\]. The purpose of this section is to demon- 

\[\Phi(r, k + \lambda w_1 + (1 - \lambda) w_2)\] is continuously differentiable, and hence we have

\[
\Phi(r, k + w_1) - \Phi(r, k + w_2) - D_{\mathbb{E}} \Phi(r, 0)(w_1 - w_2) \\
= \int_0^1 \partial_{\lambda} \Phi(r, k + \lambda w_1 + (1 - \lambda) w_2) \, d\lambda - D_{\mathbb{E}} \Phi(r, 0)(w_1 - w_2) \\
= \int_0^1 (D_{\mathbb{E}} \Phi(r, k + \lambda w_1 + (1 - \lambda) w_2) - D_{\mathbb{E}} \Phi(r, 0))(w_1 - w_2) \, d\lambda.
\]

Making use of the continuity properties of the differential in (ii a) we may choose \(0 < \epsilon_m' \leq \frac{1}{2} \epsilon_m\) sufficiently small so that, with the given \(\theta > 0\) and constant \(C_m\) from (5), we have

\[
\|D_{\mathbb{E}} \Phi(r, k + w) - D_{\mathbb{E}} \Phi(r, k)\|_{L(E_{m}, E_{m}')} \leq \theta / C_m' \quad \forall|r|_{\mathbb{R}^d}, \|k\|_{E_m}, \|w\|_{E_{m}'} \leq \epsilon_m'.
\]

If we now have \(\|w_1\|_{E_{m}}, \|w_2\|_{E_{m}'} \leq \epsilon_m\) for all \(\lambda \in [0, 1]\), then we obtain

\[
\|\Phi(r, k + w_1) - \Phi(r, k + w_2) - D_{\mathbb{E}} \Phi(0, 0)(w_1 - w_2)\|_{F_m} \leq \frac{\theta}{C_m} \|w_1 - w_2\|_{E_{m}'}.
\]

Recalling that the \(E_{m}'\) and \(E_{m}\) norms on \(\ker D_{\mathbb{E}} \Phi(0, 0)\) are equivalent, we finally find \(\epsilon_m' \geq \delta_m > 0\) such that \(\|r, k\|_{\mathbb{R}^d \times E_{m}'} < \delta_m\) guarantees \(|r|_{\mathbb{R}^d}, \|k\|_{E_m} \leq \epsilon_m'\). Now the combination of the above estimates proves the contraction property: Given \(m \in \mathbb{N}\) and \(\theta > 0\) we found \(\delta_m > 0\) so that \(\|r, k\|_{\mathbb{R}^d \times E_{m}'}\), \(\|w_1\|_{E_{m}}, \|w_2\|_{E_{m}'} \leq \delta_m\) implies

\[
\|B(r, k, w_1) - B(r, k, w_2)\|_{W_m} \leq C_m' \|\Phi(r, k + w_1) - \Phi(r, k + w_2) - D_{\mathbb{E}} \Phi(0, 0)(w_1 - w_2)\|_{F_m} \\
\leq C_m' \|w_1 - w_2\|_{E_{m}'} = \theta \cdot \|w_1 - w_2\|_{W_m = W \cap E_{m}'}.
\]

This establishes the contraction germ property for \(m \geq 1\) and hence shows that \(\Phi|_{E_1}\) is HWZ-Fredholm at 0.

**Remark 4.6.** Conversely, in order to show that an \(\text{sc}^\infty\) map \(\Phi\) that is HWZ-Fredholm at 0 is also \(\text{sc}\)-Fredholm at 0 with respect to a splitting \(\mathbb{E} \cong \mathbb{R}^k \times \mathbb{W}\), we would need the contraction germ form to hold for this splitting in a stronger sense. Namely, for fixed \(m \in \mathbb{N}_0\) we would need \(\delta_m > 0\) that does not depend on a choice of \(\theta > 0\), such that then for given \(\theta > 0\) there exists \(\epsilon_m > 0\) such that for \(\|v\|_{\mathbb{R}^k}, \|w_1\|_{W_m}, \|w_2\|_{W_m} \leq \delta_m\) we have contraction

\[
\|w_1 - w_2\|_{W_m} \leq \epsilon_m \implies \|B(v, w_1) - B(v, w_2)\|_{W_m} \leq \theta \|w_1 - w_2\|_{W_m}.
\]

This, however, would imply \(\partial_{W_m} B \equiv 0\) in the \(\delta_m\)-neighbourhood. So there seems to be no natural extra condition under which HWZ-Fredholm maps are also \(\text{sc}\)-Fredholm.

### 5. Fredholm property for trajectory breaking in Hamiltonian Floer theory

The full polyfold setup for Hamiltonian Floer theory will be a corollary of the polyfold setup for Symplectic Field Theory in [HWZ7]. The purpose of this section is to demon- 

strate the Fredholm property of the Cauchy-Riemann operator with Hamiltonian perturbation in a simplified geometric setting that nevertheless captures all analytic subtleties (with the exception of sphere bubbling, that is already treated [HWZ5]). For that purpose we consider an autonomous Hamiltonian vector field \(X : \mathbb{C}^n \to \mathbb{C}^n\) with nondegenerate (and hence isolated) critical point \(0 \in \text{Crit}(X)\) and fix an almost complex structure \(J : \mathbb{C}^n \to \text{Hom}(\mathbb{C}^n, \mathbb{C}^n)\), assuming that both have all derivatives uniformly bounded. Then we consider the neighbourhood of a generalized Floer trajectory from \(c_0\) to \(c_0\) that is once
broken at $c_0 : S^1 \rightarrow X, t \mapsto 0$. That is, we consider $\gamma_1, \gamma_2 \in C^\infty(\mathbb{R} \times S^1, \mathbb{C}^n)$ that satisfy the perturbed Cauchy-Riemann equation over $(s, t) \in \mathbb{R} \times S^1$ and have finite energy.

$$\partial_{j, X}\gamma_i = \partial_j \gamma_i + J(\gamma_i)\left(\partial_t \gamma_i - X(\gamma_i)\right) = 0, \quad \int_{\mathbb{R} \times S^1} |\partial_j \gamma_i|^2 + |\partial_t \gamma_i - X(\gamma_i)|^2 < \infty.$$  

Recall from e.g. [S] that finite energy implies exponential decay to Hamiltonian orbits $\gamma_{i, \pm} = \lim_{s \to \pm \infty} \gamma_i(s, \cdot) : S^1 \rightarrow \mathbb{C}^n, \partial_t \gamma_{i, \pm}(s, t) = X(\gamma_{i, \pm})$, and we here simplify to the special case $\gamma_{i, \pm} \equiv 0$. Exponential decay in general means that for some constants $C$ and $\delta > 0$ we have

$$(6) \quad |\partial_s \gamma_i(s, t)| = |\partial_t \gamma_i(s, t) - X(\gamma_i(s, t))| \leq Ce^{-\delta|s|} \quad \forall(s, t) \in \mathbb{R} \times S^1.$$  

To model nearby Floer trajectories, we begin with the pregluing map for some $0 < \delta_0 < \delta$,  

$$(0, \frac{1}{m+42}) \times H^{1, \delta_0}(\mathbb{R} \times S^1, \mathbb{C}^n) \times H^{1, \delta_0}(\mathbb{R} \times S^1, \mathbb{C}^n) \rightarrow H^{1, \delta_0}(\mathbb{R} \times S^1, \mathbb{C}^n) \quad (r, \xi_1, \xi_2) \mapsto \oplus_{\tau_\delta}(\xi_1, \xi_2)$$  

given by a cutoff function $\beta \in C^\infty([0, 1])$ with $\beta([-\infty, -1]) \equiv 0$ and $\beta|_{[1, \infty]} \equiv 1$ as

$$(7) \quad \oplus_{\tau_\delta}(\xi_1, \xi_2) := \beta \cdot \tau_\delta \xi_1 + (1 - \beta) \cdot \tau_{-\delta} \xi_2,$$  

where we denote the shift action by

$$\mathbb{R} \times H^{1, \delta_0}(\mathbb{R} \times S^1, \mathbb{C}^n) \rightarrow H^{1, \delta_0}(\mathbb{R} \times S^1, \mathbb{C}^n) \quad (R, \xi) \mapsto \tau_R \xi : (s, t) \mapsto \xi(R + s, t).$$  

**Remark 5.1.** Since the moduli spaces of Floer trajectories consist of Floer trajectories modulo $\mathbb{R}$-shifts, we also have to fix local slices, e.g. find codimension 1 hyperplanes $\Sigma_1, \Sigma_2 \subset \mathbb{C}^n$ that intersect $\gamma_i$ transversely at $(0, 0) \in \mathbb{R} \times S^1$, then replace $H^{1, \delta_0}$ with $H^{\delta_0, \delta_0}$-neighbourhoods of 0 in the local slice $\{\xi_1 | \xi_1(0, 0) \in T_{\gamma_i}(0, 0) \Sigma_i\}$. Since this does not affect the Fredholm analysis but significantly complicates the notation, we will use the above simplified setup, noting that one obtains a polyfold setup for Hamiltonian Floer theory by making the indicated adjustments in each step.

Throughout, we fix a choice of the cutoff function $\beta$, and will need another similar (not necessarily related) function to define weighted Sobolev spaces that we now organize into scale Banach spaces. (Different choices will induce equivalent norms on the same space.)

**Lemma 5.2.** The weighted Sobolev space with scale structure

$$W^d_{\delta}(\mathbb{R} \times S^1, \mathbb{C}^n) = (W^d_{\delta_m}(\mathbb{R} \times S^1, \mathbb{C}^n))_{m \in \mathbb{N}_0}$$  

defines a sc-Banach space for any $n \in \mathbb{N}, \ell \in \mathbb{N}_0, 1 \leq p < \infty$, and weight sequence $\delta = (\delta_m)_{m \in \mathbb{N}_0}$ with $k > m \Rightarrow \delta_k > \delta_m$ and $\sup_{m \in \mathbb{N}_0} \delta_m < \infty$. It is defined by the weighted Sobolev spaces

$$W^d_{\delta}(\mathbb{R} \times S^1, \mathbb{C}^n) := \{u : \mathbb{R} \times S^1 \rightarrow \mathbb{C}^n \mid (s, t) \mapsto e^{\delta_n(s)}u(s, t) \in W^{k, p}\}$$  

with norm $\|u\|_{W^d_{\delta, p}} := \|e^{\delta_n u}\|_{W^k, p}$ for some choice of smooth function $\eta \in C^\infty(\mathbb{R})$, with $\eta(s) = |s|$ for $|s| \geq 1$ and $0 < \eta(s) < 1$ for $|s| < 1$.

**Proof.** The inclusion $E_k = W^{k, \ell}_{\delta_k}(\mathbb{R} \times S^1, \mathbb{C}^n) \subset W^{m+\ell, p}_{\delta_m}(\mathbb{R} \times S^1, \mathbb{C}^n) = E_m$ for $k > m$ exists since $e^{\delta_k u} \geq e^{\delta_m u}$. It is compact since the restriction $W^{k, \ell}_{\delta_k}(\mathbb{R} \times S^1, \mathbb{C}^n) \rightarrow W^{m+\ell, p}_{\delta_m}([-R, R] \times S^1, \mathbb{C}^n)$ is a compact Sobolev imbedding for any finite $R \geq 1$ (due to the loss of derivatives $k > m$, see [A]) and the restriction $W^{k, \ell}_{\delta_k}(\mathbb{R} \times S^1, \mathbb{C}^n) \rightarrow W^{m+\ell, p}_{\delta_m}([-R, R] \times S^1, \mathbb{C}^n)$.
$W_{\delta_m}^{k+l,p}((\mathbb{R} \setminus [-R, R]) \times S^1, \mathbb{C}^n)$ converges to 0 in the operator norm as $R \to \infty$ (due to the exponential weight $\sup_{|s| \geq R} e^{\delta_m n(s)} e^{-\delta_k n(s)} = e^{-(\delta_k - \delta_m) R}$).

The smooth points $u \in E_{\infty} := \bigcap_{m \in \mathbb{N}} W_{\delta_m}^{k+m,p}(\mathbb{R} \times S^1, \mathbb{C}^n)$ are those smooth maps $u \in C^\infty(\mathbb{R} \times S^1, \mathbb{C}^n)$ whose derivatives decay exponentially, $\sup_{s,t \in \mathbb{R} \times S^1} e^{\delta n(s)}|\partial_{s}^n \partial_{t}^2 u(s, t)| < \infty$ for all $N_1, N_2 \in \mathbb{N}$ and any submaximal weight $\delta < \sup_{m \in \mathbb{N}} \delta_m$. In particular, the compactly supported smooth functions are a subset $C_0^\infty(\mathbb{R} \times S^1, \mathbb{C}^n) \subset E_{\infty}$; and even these are dense in any weighted Sobolev space (for $p < \infty$).

Modulo the adjustments of Remark 5.1, the moduli space of unbroken and broken Floer trajectories near $([\gamma_1], [\gamma_2])$ can now be described as the zero set of a Fredholm section in an $M$-polyfold bundle – more precisely in a local chart, given by a splicing. We begin this description by fixing ambient scale Hilbert spaces from Lemma 5.2 with $p = 2$ and a weight sequence $0 < \delta_0 < \delta_1 < \ldots < \delta_i$.

$$B := \left(B_m := H^{1+m,\delta_m}(\mathbb{R} \times S^1, \mathbb{C}^n) \times H^{1+m,\delta_m}(\mathbb{R} \times S^1, \mathbb{C}^n)\right)_{m \in \mathbb{N}_0},$$

$$E := \left(E_m := H^{0+m,\delta_m}(\mathbb{R} \times S^1, \mathbb{C}^n) \times H^{0+m,\delta_m}(\mathbb{R} \times S^1, \mathbb{C}^n)\right)_{m \in \mathbb{N}_0}.$$  

This will provide the ambient smooth structure of a splicing obtained from the pregluing map (7) together with the complementary anti-pregluing map

$$\left(0, \frac{1}{\ln 42}\right) \times B_0 \longrightarrow H^{1,\delta_0}(\mathbb{R} \times S^1, \mathbb{C}^n)$$

$$(r, \zeta_1, \zeta_2) \mapsto \oplus_{\epsilon^{1/r}} (\zeta_1, \zeta_2) := (1 - \beta) \cdot \tau_{\epsilon^{1/r}} \zeta_1 - \beta \cdot \tau_{-\epsilon^{1/r}} \zeta_2.$$  

To define the splicing, we extend this notation to $r \in \left[0, \frac{1}{\ln 42}\right)$, with $r = 0$ corresponding to gluing parameter $R = \infty$, where we do not glue, i.e. we set $\oplus_{\epsilon^{1/r}} (\zeta_1, \zeta_2) := (\zeta_1, \zeta_2)$ and $\oplus_{\epsilon^{1/r}} (\zeta_1, \zeta_2) := 0$. With that, the ambient splicing core is

$$B := \{ (r, \zeta_1, \zeta_2) \in \left(0, \frac{1}{\ln 42}\right) \times B \mid (\zeta_1, \zeta_2) \in \ker \oplus_{\epsilon^{1/r}} (\zeta_1, \zeta_2) \}.$$  

It arises from the fact that $\oplus_{R} \times \ominus_{R}$ is an isomorphism for each $R \in (42, \infty]$, though only pointwise continuous rather than in the operator norm, so that we obtain a pointwise scale smooth family of projections parametrized via the “gluing profile” $R(r) := e^{1/r}$.

$$\left[0, \frac{1}{\ln 42}\right) \times B \cong E,$$  

$$(r, \zeta_1, \zeta_2) \mapsto \pi_r (\zeta_1, \zeta_2) := (\oplus_{R} \times \ominus_{R})^{-1}(\ominus_{R}(\zeta_1, \zeta_2), 0).$$  

Now one should restrict $B$ to a small neighbourhood of $0 \in \left[0, \frac{1}{\ln 42}\right) \times B$ and think of the neighbourhood of the broken trajectory $(\gamma_1, \gamma_2)$ as given by $\ominus_{R}(\gamma_1 + \xi_1, \gamma_2 + \xi_2)$ for $(r, \zeta_1, \zeta_2) \in B$. Then the moduli space of Floer trajectories near the broken trajectory $([\gamma_1], [\gamma_2])$ is described (up to the adjustments of Remark 5.1) as the zero set of the operator

$$\phi : B \to \mathcal{E},$$  

$$(r, \xi_1, \xi_2) \mapsto (\ominus_{R} \times \ominus_{R})^{-1}\left(\ominus_{R}(\zeta_1 + \xi_1, \zeta_2 + \xi_2), 0\right) \in \mathcal{E}_r.$$  

Here and in the following we will always abbreviate $R = e^{1/r}$. Then the fibers of the bundle $\mathcal{E} = \bigcup_{(r, \xi_1, \xi_2) \in B} \mathcal{E}_r$ are independently of $(\xi_1, \xi_2)$ given by the kernel $\mathcal{E}_r = \ker \ominus_{R} \subset \mathcal{E}$ of the anti-pregluing map $\hat{\ominus}_{R} : \mathcal{E} \to \mathcal{E}$, where the isomorphism $\hat{\ominus}_{R} \times \hat{\ominus}_{R}$ for each $r \in \left[0, \frac{1}{\ln 42}\right)$ is given by the same equations as for the base $\mathcal{B}$ in (7), (8).

Although neither $\mathcal{B}$ nor $\mathcal{E}$ are locally even homeomorphic to Banach spaces, they are given as families of linear subspaces of the scale Hilbert spaces $\mathcal{B}$ and $\mathcal{E}$ parametrized by the gluing parameter $r \in \left[0, \frac{1}{\ln 42}\right)$ such that “base and fiber dimensions jump in the same way” as $r$ converges to 0. This is formalized by the existence of a “filled section”, which restricts to isomorphisms between the complements of the fibers of $\mathcal{B}$ and $\mathcal{E}$, and whose zero set is the same as that of the original section $\phi$. This filling can usually be achieved by acting with a linearized operator on the anti-preglued map. In this case, we use the
linearized operator at the constant Floer trajectory $\tau_0(s, t) = 0$. It gives rise to the operator

$$\Phi : [0, \frac{1}{\ln 42}] \times B \to E$$

given in the cases $r > 0$ and $r = 0$ by

$$\Phi(r, \xi_1, \xi_2) := (\hat{\oplus}_R \times \hat{\oplus}_R)^{-1} \left( \partial_{J,X} \oplus_R (\gamma_1 + \xi_1, \gamma_2 + \xi_2), (D_{\gamma_0} \partial_{J,X} \oplus_R (\xi_1, \xi_2)) \right),$$

$$\Phi(0, \xi_1, \xi_2) := \left( \partial_{J,X}(\gamma_1 + \xi_1), \partial_{J,X}(\gamma_2 + \xi_2) \right).$$

Now $\phi$ is called a **polyfold Fredholm section** of $\mathcal{E} \to B$ if $\Phi$ is a nonlinear Fredholm map in the sense of Definition 4.1. We will prove the latter by applying the criteria of Definition 4.3 and appealing to Theorem 4.5. In fact, the scale smoothness follows from the chain rule and the fact that the shift action is $\mathcal{C}^\infty$, and classical differentiability only fails in the shift action, i.e., in the direction of $[0, \frac{1}{\ln 42}]$. Moreover, the regularization property (i) follows from standard elliptic regularity for the nonlinear Cauchy-Riemann operator to obtain for $r > 0$

$$\gamma_r := \oplus_R(\gamma_1, \gamma_2), \quad \gamma^c_r := \oplus_R(\gamma_1 + e_1, \gamma_2 + e_2), \quad \gamma^{\pm}_r = \gamma_{\pm R} \gamma_r, \quad \gamma^{c\pm}_r = \gamma_{\pm R} \gamma^c_r$$

for $r > 0$ and for $r = 0$ set $\gamma^c_0 := \gamma_0$ which coincides with the pointwise $r \to 0$ resp. $R \to \infty$ limits of $\gamma^{c\pm}_r = \gamma_{\pm R} \gamma^c_r$. With that notation we obtain for $r > 0$

$$D_{E} \Phi(r, \xi)(\xi_1, \xi_2) = \frac{d}{dh}\bigg|_{h=0} (\hat{\oplus}_R \times \hat{\oplus}_R)^{-1} \left( \partial_{J,X} \oplus_R (\gamma_1 + e_1 + h_{\xi_1}, \gamma_2 + e_2 + h_{\xi_2}), (D_{\gamma_0} \partial_{J,X} \oplus_R (\xi_1, \xi_2)) \right)$$

$$= (\hat{\oplus}_R \times \hat{\oplus}_R)^{-1} \left( (\partial_{s} + J(\gamma^c_r) \partial_{t} - D_{\gamma_0}(JX)) \oplus_R (\xi_1, \xi_2) \right)$$

and may then use the formula

$$(\hat{\oplus}_R \times \hat{\oplus}_R)^{-1}(\xi_1, \xi_2) = \begin{pmatrix} \tau_R(\frac{\delta^2}{\delta^2 + (1-\beta)^2} \xi_1 + \frac{1-\beta}{\delta^2 + (1-\beta)^2} \xi_2) \\ \tau_R(\frac{1-\beta}{\delta^2 + (1-\beta)^2} \xi_1 - \frac{\delta^2}{\delta^2 + (1-\beta)^2} \xi_2) \end{pmatrix}$$

to obtain for $r \geq 0$

$$D_{E} \Phi(r, \xi)(\xi_1, \xi_2) = \begin{pmatrix} (D_{\gamma^c_r} \partial_{J,X}) \xi_1 + E_1(r, \xi, \xi_1, \xi_2) \\ (D_{\gamma^c_r} \partial_{J,X}) \xi_2 + E_2(r, \xi, \xi_1, \xi_2) \end{pmatrix}.$$

Here the error terms can be written with the help of further abbreviations

$$B^{(-\infty, R+1]} := \tau_R \left( \frac{\delta^2}{\delta^2 + (1-\beta)^2} \right), \quad B^{(R-1, \infty)} := \tau_R \left( \frac{(1-\beta)^2}{\delta^2 + (1-\beta)^2} \right),$$

$$B^{(\pm R-1, \pm R+1]} := \tau_{\pm R} \left( \frac{(1-\beta)^2}{\delta^2 + (1-\beta)^2} \right), \quad B^{(\pm R-1, \pm R+1]} := \tau_{\pm R} \left( \frac{2(\beta-1)b}{\delta^2 + (1-\beta)^2} \right).$$

\footnote{For the regularization property it is crucial that the weight sequence $\delta$ is chosen between 0 and the exponential decay constant $\delta > 0$. The latter is also closely connected to the asymptotic operators $J(\gamma(\pm \infty, .)) \partial_{t} + D_{\gamma(\pm \infty, .)}(JX)$ in that $\delta$ lies in the spectral gap between 0 and the lowest positive eigenvalue of each of these.}
(where the superscripts indicate the support of functions) for \( r > 0 \) as

\[
E_1(r, \xi_1, \xi_2) = B^{(-\infty, R+1]}(J(\gamma_r^+)) \partial t \xi_1 + B^{[R-1, \infty)}(J(\gamma_r^-)) \partial t \xi_1
\]

\[
- B^{(-\infty, R+1]}(D_{\gamma_r^+}(JX) - D_{\gamma_r^-}(JX)) \xi_1 - B^{[R-1, \infty)}(D_{\gamma_r^+}(JX) - D_{\gamma_r^-}(JX)) \xi_1
\]

\[
+ B^{[R-1, R+1]}(J(\gamma_{r}^-) - J(\gamma_{r}^+)) \tau_{-2R} \partial t \xi_2 - B^{[R-1, R+1]}(D_{\gamma_r^+}(JX) - D_{\gamma_r^-}(JX)) \tau_{-2R} \xi_2
\]

\[
+ B_{h}^{[R-1, R+1]}(J(\gamma_{r}^-)) \tau_{-2R} \xi_2,
\]

and analogously for \( E_2(r, \xi_1, \xi_2) \) with an expression involving the base points \( \gamma_r^+, \gamma_r^-, \xi_0 \) and the vectors \( \xi_2, \tau_{2R} \xi_1 \). For \( r = 0 \) we similarly obtain

\[
E_1(0, \xi_1, \xi_2) = (J(\gamma_1 + \epsilon_1) - J(\gamma_1)) \partial t \xi_1 - (D_{\gamma_1 + \epsilon_1}(JX) - D_{\gamma_1}(JX)) \xi_1,
\]

which coincides with the formulas for \( E_1(r, \xi_1, \xi_2) \) if for \( R = \infty \) we set \( B^{(-\infty, R+1]} \equiv 1, B^{[R-1, \infty)} \equiv 0, B^{[R-1, R+1]} \equiv 0, B_{h}^{[R-1, R+1]} \equiv 0 \). Based on these expressions we will verify the requirements (ii a), (ii b), and (iii) of Definition 4.3 in the following three lemmas. Note here that we need to exclude the case \( m = 0 \) because \( H^1 \) is not closed under multiplication. In the fully adjusted setup according to Remark 5.1, this does not pose a problem since the \( m = 0 \) scale then carries an \( H^3 \)-norm, hence forms a Banach algebra.

**Lemma 5.3.** Property (ii a) of Definition 4.3 holds, that is for any \( m \in \mathbb{N} \) and \( \delta > 0 \) there exists \( c_{m, \delta} > 0 \) such that for all \( r \in (0, c_{m, \delta}), \xi_1, \xi_2 \in \{\epsilon_1, \epsilon_2\} \) \( \forall i : \|e_i\|_{H^{m+1, \delta_m}} < \epsilon_{m, \delta} \) with \( \|e'_i - e_i\|_{H^{m+1, \delta_m}} \leq \epsilon_{m, \delta} \) and any \( \xi_1, \xi_2 \in H^{m+1, \delta_m} \) we have

\[
\|D_{\xi}(\xi_1, \xi_2) - D_{\xi}(\xi'_1, \xi_2)\|_{H^{m, \delta_m}} \leq \delta (\|\xi_1\|_{H^{m+1, \delta_m}} + \|\xi_2\|_{H^{m+1, \delta_m}}).
\]

**Proof.** We begin by noting that the left hand side splits into two similar terms,

\[
\|D_{\xi}(\xi_1, \xi_2) - D_{\xi}(\xi'_1, \xi_2)\|_{H^{m, \delta_m}} = \|E_1(r, \xi_1, \xi_2) - E_1(r, \xi'_1, \xi_2)\|_{H^{m, \delta_m}} + \|E_2(r, \xi_1, \xi_2) - E_2(r, \xi'_1, \xi_2)\|_{H^{m, \delta_m}}.
\]

We establish the estimate representatively for the first factor, where we start off from

\[
\|E_1(r, \xi_1, \xi_2) - E_1(r, \xi'_1, \xi_2)\|_{H^{m, \delta_m}} \leq \|B^{(-\infty, R+1]}\|_{C^m} \left( \|J(\gamma_r^+)' - J(\gamma_r^-)\|_{H^{m, \delta_m}} \right)
\]

\[
+ \left( \|D_{\gamma_r^+}(JX) - D_{\gamma_r^-}(JX)\|_{H^{m, \delta_m}} \right)
\]

\[
+ \left( \|D_{\gamma_r^+}(JX) - D_{\gamma_r^-}(JX)\|_{H^{m, \delta_m}} \right)
\]

\[
+ \left( \|D_{\gamma_r^+}(JX) - D_{\gamma_r^-}(JX)\|_{H^{m, \delta_m}} \right).
\]

Here the \( C^m \)-norms of the modified cutoff functions \( \|B^{(-\infty, R+1]}\|_{C^m} = \|\frac{\beta^2}{\beta^2 + (1-\beta)^2}\|_{C^m} \) and \( \|B^{[R-1, R+1]}\|_{C^m} = \|\frac{\beta^2(1-\beta)}{\beta^2 + (1-\beta)^2}\|_{C^m} \) are bounded independently of \( R \) since, up to a shift, they are given in terms of the fixed function \( \beta \in C^\infty([0, 1]) \). The second of these functions is moreover supported in \([R - 1, R + 1]\) since \( \beta(1 - \beta) \) is supported in \([-1, 1]\). Now we use the Sobolev embeddings \( H^{m+1} \hookrightarrow C^{m-1} \) and \( H^1 \hookrightarrow L^4 \) on 2-dimensional
domains to estimate with some constant $C_m > 0$
\[
\frac{1}{C_m} \| D_{\Omega} \Phi(r, \xi) (\xi_1, \xi_2) - D_{\Omega} \Phi(r, \xi') (\xi_1, \xi_2) \|_{H^{m, \delta_m}}
\leq \| \gamma_r^\ast - \gamma_r^{e^\ast} \|_{C_m^{-1}((-\infty, R+1))} \| \xi_1 \|_{H^{m_{\delta_m}}}
+ \| \nabla^m (\gamma_r^\ast - \gamma_r^{e^\ast}) \|_{L^4((-\infty, R+1))} \| \tau_{-2} R \xi_2 \|_{H^{m_{\delta_m}+1}((R-1, R+1))}
+ \| \tau_{-2} R \xi_2 \|_{H^{m_{\delta_m}+1}((R-1, R+1))}
\]

\[
\leq \| \gamma_r^\ast - \gamma_r^{e^\ast} \|_{H^{m_{\delta_m}+1}((-\infty, 0))} \| \xi_1 \|_{H^{m_{\delta_m}+1, \delta_m}}
+ \| \gamma_r^\ast - \gamma_r^{e^\ast} \|_{H^{m_{\delta_m}+1}((R-1, R+1))} \| \tau_{-2} R \xi_2 \|_{H^{m_{\delta_m}+1, \delta_m}}.
\]

Next, from (7) we read off $\gamma_r^\ast - \gamma_r^{e^\ast} = \tau_{R} \beta \cdot (e_1 - e_1') + (1 - \tau_{R} \beta) \tau_{2R} (e_2 - e_2')$, so that we can make good use of the weighting function $\eta(s) = \| s \|$ for $| s | \geq 1$

\[
\| \gamma_r^\ast - \gamma_r^{e^\ast} \|_{H^{m_{\delta_m}+1}((-\infty, R+1))}
\leq \| e_1 - e_1' \|_{H^{m_{\delta_m}+1}((-\infty, 0))} + \| e_2 - e_2' \|_{H^{m_{\delta_m}+1}((-\infty, 0))}
\leq \| \tau_{-2} R \xi_2 \|_{H^{m_{\delta_m}+1, \delta_m}}
\]

\[
\| \tau_{-2} R \xi_2 \|_{H^{m_{\delta_m}+1, \delta_m}}
= \| \gamma_r^\ast - \gamma_r^{e^\ast} \|_{H^{m_{\delta_m}+1}((-\infty, R+1))}
\leq \| \tau_{-2} R \xi_2 \|_{H^{m_{\delta_m}+1, \delta_m}}
\leq \sup_{s \in (-R+1, -R+1)} \left( e^{\delta_m |s+2R|} - e^{\delta_m |s|} \right) \| \tau_{-2} R \xi_2 \|_{H^{m_{\delta_m}+1}((-\infty, R+1))}
\]

Now we can cancel the $R$-dependent exponential factors in the second term of (10) to obtain
\[
\| D_{\Omega} \Phi(r, \xi) (\xi_1, \xi_2) - D_{\Omega} \Phi(r, \xi') (\xi_1, \xi_2) \|_{H^{m, \delta_m}}
\leq C_m \left( \| e_1 - e_1' \|_{H^{m_{\delta_m}+1, \delta_m}} + \| e_2 - e_2' \|_{H^{m_{\delta_m}+1, \delta_m}} \right) \left( \| \xi_1 \|_{H^{m_{\delta_m}+1, \delta_m}} + e^{2\delta_m \| \xi_2 \|_{H^{m_{\delta_m}+1, \delta_m}}} \right),
\]

which proves the lemma with $\epsilon_{m, \delta} = \delta C_m^{-1} e^{-2\delta_m}$.

For the remaining two parts of the scale Fredholm property, it suffices to consider the partial differential at $\xi = 0$. In that case we decompose the error terms $E_i$ in (9) into first order operators on $\xi_i$ and operators involving a shift on $\xi_j$ for $j \neq 1$,

\[
E_1(r, 0, \xi_1, \xi_2) = Q_1(r) \xi_1 - S_1(r) \xi_2, \quad E_2(r, 0, \xi_1, \xi_2) = Q_2(r) \xi_2 - S_2(r) \xi_1.
\]
For $E_1$, and analogously for $E_2$, and always using the notation $R = e^{1/r}$, these terms are

$$Q_1(r) \xi_1 = B^{[R^{-1}, \infty)} \left( J(\tau_0) - J(\gamma_{-r}^-) \right) \partial_t \xi_1 - (D_{\gamma_{-r}} (JX) - D_{\gamma_{-r}} (JX)) \xi_1 \right) + B_1^{[R^{-1}, \infty)} \xi_1,$$

$$S_1(r) \xi_2 = B_s^{[R^{-1}, \infty)} \tau_{-2R} \xi_2 + B^{[R^{-1}, \infty)} \left( D_{\gamma_{-r}} (JX) - D_{\gamma_{-r}} (JX) \right) \tau_{-2R} \xi_2$$

$$- B^{[R^{-1}, \infty)} \left( J(\gamma_{-r}) - J(\tau_0) \right) \tau_{-2R} \partial_t \xi_2.$$

**Lemma 5.4.** Property (ii b) of Definition 4.3 holds even without allowing a subsequence. That is for any sequences $0 < r^\nu \to 0$ and $\|\xi^\nu_1\|_{H^{m+\delta_m}}$, $\|\xi^\nu_2\|_{H^{m+\delta_m}} < 1$ that satisfy $\|D_{\nu} \Phi(r^\nu, 0)(\xi^\nu_1, \xi^\nu_2)\|_{H^{m,\delta_m}} \to 0$, we also have $\|D_{\nu} \Phi(0, 0)(\xi^\nu_1, \xi^\nu_2)\|_{H^{m,\delta_m}} \to 0$.

**Proof.** Let $r^\nu, \xi^\nu_1, \xi^\nu_2$ be such a sequence, then we have with $R^\nu := e^{1/r^\nu} \to \infty$

$$\|D_{\nu} \Phi(r^\nu, 0)(\xi^\nu_1, \xi^\nu_2) - D_{\nu} \Phi(0, 0)(\xi^\nu_1, \xi^\nu_2)\|_{H^{m,\delta_m}} \leq \|D(\gamma_{-r^\nu} - \partial_{t^\nu} JX - D_{\gamma_{-r^\nu}} JX) \xi^\nu_1\|_{H^{m,\delta_m}} + \|D(\gamma_{-r^\nu} - \partial_{t^\nu} JX - D_{\gamma_{-r^\nu}} JX) \xi^\nu_2\|_{H^{m,\delta_m}}$$

$$+ \|Q_1(r^\nu) \xi^\nu_1\|_{H^{m,\delta_m}} + \|Q_2(r^\nu) \xi^\nu_2\|_{H^{m,\delta_m}} + \|S_1(r^\nu) \xi^\nu_2\|_{H^{m,\delta_m}} + \|S_1(r^\nu) \xi^\nu_2\|_{H^{m,\delta_m}}.$$

The first four terms can be converted to the same sense just by the bounds on $\|\xi^\nu_1\|_{H^{m+\delta_m}}$ and the fact that $\gamma_{1,2} \in H^{m+2,\delta_{m+1}}$. Indeed, for the first term we have $\gamma_{1,2} = (\tau_{-R^\nu \beta} - 1) \gamma_1 + (1 - \tau_{-R^\nu \beta}) \tau_{-2R^\nu} \gamma_2$, and so that we obtain convergence

$$\|\gamma_{1,2} - \tau_{-1,2}\|_{C^m} \leq \|\beta - 1\|_{C^m} \|\gamma_1\|_{C^m} \|\gamma_2\|_{C^m} \to 0$$

since $\sup_{\tau_{R^\nu \beta}} \subset [-R^\nu - 1, -R^\nu + 1]$. Now the estimates of Lemma 5.3 simplify to

$$\|D(\gamma_{-r^\nu} - \partial_{t^\nu} JX - D_{\gamma_{-r^\nu}} JX) \xi^\nu_1\|_{H^{m,\delta_m}} \leq C \|\gamma_{1,2} - \gamma_1\|_{C^m} \|\xi^\nu_1\|_{H^{m+\delta_m}} \to 0,$$

$$\|D(\gamma_{-r^\nu} - \partial_{t^\nu} JX - D_{\gamma_{-r^\nu}} JX) \xi^\nu_2\|_{H^{m,\delta_m}} \leq C \|\gamma_{1,2} - \gamma_2\|_{C^m} \|\xi^\nu_2\|_{H^{m+\delta_m}} \to 0.$$

For the third term (and analogously the fourth) we have

$$\|Q_1(r^\nu) \xi^\nu_1\|_{H^{m,\delta_m}} \leq \|B^{[R^{-1}, \infty)} \left( J(\tau_0) - J(\gamma_{-r}) \right) \partial_t \xi^\nu_1\|_{H^{m,\delta_m}} + \|B^{[R^{-1}, \infty)} \left( D_{\gamma_{-r}} (JX) - D_{\gamma_{-r}} (JX) \right) \xi^\nu_1\|_{H^{m,\delta_m}}$$

$$+ \|B^{[R^{-1}, \infty)} \left( D_{\gamma_{-r}} (JX) - D_{\gamma_{-r}} (JX) \right) \xi^\nu_1\|_{H^{m,\delta_m}} \leq C \|\tau_0 - \gamma_{-1}\|_{C^m} \|\xi^\nu_1\|_{H^{m+\delta_m}} \to 0$$

because

$$\|\tau_0 - \gamma_{-1}\|_{C^m} \|\xi^\nu_1\|_{H^{m+\delta_m}} \to 0.$$
The terms on the right hand side converge to 0 by assumption and by the convergence for the terms proven above, hence the left hand side must also converge to 0 as $\mu > \nu \to \infty$. The analogous argument for the second factor proves the convergence $\|S_2(r^\nu) \xi_1^\nu\|_{H^{m,\delta_m}} \to 0$, so that we have proven $\|D_E \Phi(r^\nu,0)\xi_1^\nu - D_E \Phi(0,0)\xi_1^\nu\|_{H^{m,\delta_m}} \to 0$ as claimed.

\[ \square \]

**Lemma 5.5.** Property (iii) of Definition 4.3 holds, that is the differential $D_E \Phi(0,0) = (\partial_\gamma + J(\gamma) \partial_\tau - D_\gamma(JX))_{i=1,2} : \mathbb{B} \to \mathbb{E}$ is sc-Fredholm in the sense of Definition 3.1. Moreover $D_E \Phi(r,0) : H^{1,\delta_0} \times H^{1,\delta_0} \to H^{0,\delta_0} \times H^{0,\delta_0}$ is a Fredholm operator with $\ker D_E \Phi(r,0) \subset H^{2,\delta_1} \times H^{2,\delta_1}$ for all sufficiently small $0 < r < \epsilon_0$, with the same Fredholm index as $D_E \Phi(0,0)$.

**Proof.** To check that $D_E \Phi(0,0) = (D_\gamma \partial_{JX})_{i=1,2}$ is sc-Fredholm we note that the conditions of Definition 3.1 are fulfilled by the following standard elliptic estimates, regularity, and Fredholm properties for the Cauchy-Riemann operators $D_\gamma \partial_{JX}$ in each factor.

(i) For any $m \in \mathbb{N}_0$ we have bounded restrictions $D_\gamma \partial_{JX} : H^{m+1,\delta_m} \to H^{m,\delta_m}$.

(ii) If $\xi_1 \in H^{1,\delta_1}$ and $D_\gamma \partial_{JX} \xi_2 \in H^{m,\delta_m}$ then $\xi_1 \in H^{m+1,\delta_m}$.

(iii) $D_\gamma \partial_{JX} : H^{1,\delta_0} \to H^{0,\delta_0}$ is a Fredholm operator.\(^5\)

Next, the regularity for the kernel of $\ker D_E \Phi(r,0)$ holds since for $\xi_1, \xi_2 \in H^{1,\delta_0}$ with $D_E \Phi(r,0)\xi_1, \xi_2 = 0$ we have $(D_{\gamma r} \partial_{JX}) \xi_1 = -B_{\tau}^{1-R-1,R+1}(\xi_1 - \tau_2 R \xi_2) \in H^{1,\delta_0}$ for any exponential weight $\delta > 0$, since the right hand side is compactly supported. Now elliptic regularity for the Cauchy-Riemann operator $D_\gamma \partial_{JX}$ as in (ii) implies $\xi_1 \in H^{2,\delta'}$, where in particular we can pick $\delta' = \delta_{m+1} > \delta_m$. Regularity for $\xi_2$ analogously follows from $(D_{\gamma r} \partial_{JX}) \xi_2 = -B_{\tau}^{1-R-1,R+1}(\xi_2 - \tau_2 R \xi_1)$.

Finally, to see that $D_E \Phi(r,0)$ for $r > 0$ sufficiently small is Fredholm with the same index as $D_E \Phi(0,0)$, we note that, since $r > 0$ and hence $R < \infty$ is fixed, we have estimates as in Lemma 5.4,

\[
\left\| \left( (D_{\gamma r} \partial_{JX}, D_{\gamma r} \partial_{JX}) - D_E \Phi(0,0) \right)(\xi_1, \xi_2) \right\|_{H^{0,\delta_0}} \leq \left\| \gamma_r - \gamma_1 \right\|_{C^0} \left\| \xi_1 \right\|_{H^{1,\delta_0}} + \left\| \gamma_r^+ - \gamma_2 \right\|_{C^0} \left\| \xi_2 \right\|_{H^{1,\delta_0}}
\]

\[
\left\| \left( D_E \Phi(0,0) \right)(\xi_1, \xi_2) \right\|_{H^{0,\delta_0}} \leq C_{R,\delta'} \left( \left\| \xi_1 \right\|_{H^{0,\delta'}} + \left\| \xi_2 \right\|_{H^{0,\delta'}} \right)
\]

with the constant $C_{R,\delta'}$ exponentially depending on $R = e^{1/r}$ and the choice of $\delta' > 0$.

The first estimate shows that $(D_{\gamma r} \partial_{JX}, D_{\gamma r} \partial_{JX}) \to D_E \Phi(0,0)$ as $r \to 0$ in the operator topology of $L(H^{1,\delta_0}, H^{0,\delta_0})$. Since the set of Fredholm operators is open and the index is locally constant, this proves that there is $\epsilon_0 > 0$ so that $(D_{\gamma r} \partial_{JX}, D_{\gamma r} \partial_{JX})$ is Fredholm.

\(^5\)Its index is given by the spectral flow of $s \to J(\gamma(s,\cdot))\partial_\tau + D_{\gamma(s,\cdot)}(JX) - \delta_0 \Phi'(s)\Id$, but this is not needed for the proof. In particular, Lemma 3.5 implies that $D_{\gamma r} \partial_{JX} : H^{1+m,\delta_m} \to H^{m,\delta_m}$ has the same Fredholm index. On the other hand, this index also equals to the spectral flow of $s \to J(\gamma(s,\cdot))\partial_\tau + D_{\gamma(s,\cdot)}(JX) - \delta_0 \Phi'(s)\Id$. These index identities are consistent since by choice of the weight sequence, both $\delta_0$ and $\delta_m$ lie in the spectral gap $(0,\delta)$ of the asymptotic operators $J(\gamma(s,\cdot))\partial_\tau + D_{\gamma(s,\cdot)}(JX)$. That choice was necessary to obtain the regularization property for the nonlinear as well as the linear Cauchy-Riemann operator. Indeed, (ii) crucially uses the fact that ker $D_{\gamma r} \partial_{JX} \subset \bigcap_{k \in \mathbb{N}} H^{k,\delta}$ with the exponential decay constant $\delta > 0$ from (6).
with the index of $D_E\Phi(0,0)$ for all $0 < r < \epsilon_0$. Now for any fixed $r < \epsilon_0$, the second estimate with $\delta' > \delta_0$ shows that $D_E\Phi(r,0)$ is a compact perturbation of $(D_{\gamma-r}\partial J,X,D_{\gamma+r}\partial J,X)$, and hence is also Fredholm with the same index, as claimed. □

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