ARITHMETIC PROPERTIES OF THE SEQUENCE OF DEGREES
OF STERN POLYNOMIALS AND RELATED RESULTS

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Abstract. Let $B_n(t)$ be a $n$-th Stern polynomial and let $e(n) = \deg B_n(t)$ be
its degree. In this note we continue our study started in [10] of the arithmetic
properties of the sequence of Stern polynomials and the sequence $\{e(n)\}_{n=1}^\infty$.
We also study the sequence $d(n) = \text{ord}_2 B_n(t)$. Among other things we
prove that $d(n) = \nu(n)$, where $\nu(n)$ is the maximal power of 2 which divides
the number $n$. We also count the number of the solutions of the equations
$e(m) = i$ and $e(m) - d(m) = i$ in the interval $[1, 2^n]$. We also obtain an
interesting closed expression for a certain sum involving Stern polynomials.

1. Introduction

The Stern sequence (or Stern’s diatomic sequence) $s(n)$ was introduced in [9] and
is defined recursively in the following way

$$s(0) = 0, \quad s(1) = 1, \quad s(n) = \begin{cases} 
    s\left(\frac{n}{2}\right) & \text{if } n \equiv 0 \pmod{2}, \\
    s\left(\frac{n-1}{2}\right) + s\left(\frac{n+1}{2}\right) & \text{if } n \equiv 1 \pmod{2}.
\end{cases}$$

This sequence appears in different mathematical contexts and was an object of
study of many mathematicians like Lehmer [3], Reznick [6] and De Rham [7]. A
comprehensive survey of its properties can be found in [11]. An interesting survey
of known results and applications of the Stern sequence can also be found in [5].

In a recent paper [2] Klavžar, Milutinović and Petr introduced an interesting
polynomial analogue of $s(n)$. More precisely, they define the sequence $\{B_n(t)\}_{n=0}^\infty$
of Stern polynomials as follows: $B_0(t) = 0, B_1(t) = 1$ and for $n \geq 2$ we have

$$B_n(t) = \begin{cases} 
    tB_{\frac{n}{2}}(t) & \text{if } n \equiv 0 \pmod{2}, \\
    B_{\frac{n-1}{2}}(t) + B_{\frac{n+1}{2}}(t) & \text{if } n \equiv 1 \pmod{2}.
\end{cases}$$

The equality $s(n) = B_n(1)$ justifies the name of the sequence $B_n(t)$. In [2] it is shown
that the sequence of Stern polynomials has an interesting connection with
some combinatorial objects. In particular the $i$-th coefficient in $B_n(t)$ counts the
number of hyperbinary representations of $n - 1$ with exactly $i$ occurrences of 1.
Moreover, if $e(n) = \deg B_n(t)$, then the number $e(n)$ is equal to the difference
between the length and the weight of the non-adjacent form of $n$. These two
properties show that the polynomials $B_n(t)$ are an interesting object of study. We
investigated these polynomials and associated sequence $e(n)$ in a recent paper [10].
In this note we continue our study. We give now a short introduction about the
content of the paper.

Key words and phrases. Stern diatomic sequence, Stern polynomials.
In Section 2 we introduce the sequence

\[ d(n) = \text{ord}_{i=0} B_n(t), \]

and study its relations with the sequence \( \{e(n)\}_{n=1}^{\infty} \). We show that \( d(n) \) is just \( \mu(n) \), the maximal power of 2 which divides \( n \). Among other things we also count the number \( e(i, n) \) of solutions of the equation \( e(m) - d(m) = i \) in the interval \([1, 2^n]\).

In Section 3 we study the sum

\[ S_k(n) = \sum_{i: e(i)=n} i^k, \]

where \( k, n \) are given. In particular we give recurrence relations satisfied by the sequence \( G_k(x) = \sum_{n=0}^{\infty} S_k(n)x^n, k = 0, 1, 2, \ldots \).

In Section 4 we give an alternative definition of the polynomial \( B_n(t) \) as a determinant of a certain matrix.

In Section 5 we give a generalization of a certain sum given in Urbiaha’s paper [11]. Finally, in the last section we give some additional results on the sequence \( \{e(n)\}_{n=1}^{\infty} \) and state some open problems and conjectures which appear during our investigations and which we were unable to prove.

2. Relations between \( d(n) \) and \( e(n) \)

We define the following sequences

\[ d(n) = \text{ord}_{i=0} B_n(t), \quad e(n) = \text{deg} B_n(t). \]

The sequence \( e(n), n = 1, 2, \ldots \) was introduced in the paper [2]. We have \( e(1) = 0, e(2) = 1 \) and for \( n \geq 3 \):

\[ e(2n) = e(n) + 1, \quad e(2n + 1) = \max\{e(n), e(n + 1)\} \]

An alternative recurrence relation which is more convenient was obtained in [2] Corollary 13 and has the form

\[ e(2n) = e(n) + 1, \quad e(4n + 1) = e(n) + 1, \quad e(4n + 3) = e(n + 1) + 1. \]

We will use it several times in the sequel. Additional arithmetic properties which will be useful in our investigations were obtained in [10]. Because we will use several times we recall it without proof.

**Theorem 2.1** (Theorem 4.3 in [10]). We have the following equalities:

\[
\begin{align*}
m(n) & = \min\{ e(i) : i \in [2^{n-1}, 2^n] \} = \left\lceil \frac{n}{2} \right\rceil, \quad n \geq 2, \\
M(n) & = \max\{ e(i) : i \in [2^{n-1}, 2^n] \} = n.
\end{align*}
\]

Moreover,

\[
\begin{align*}
m(n) & = \min\{ i : e(i) = n \} = 2^n, \\
M(n) & = \max\{ i : e(i) = n \} = \frac{2^{n+1} - 1}{4}.
\end{align*}
\]

**Theorem 2.2.** We have the equality \( d(n) = \nu(n) \), where \( \nu(n) = \max\{ k : 2^k \text{ divide } n \} \).

**Proof.** First of all let us note that \( d(1) = 0, d(2) = 1, d(3) = 0 \) and \( d(4) = 2 \). We show that the sequence \( d(n) \) satisfies the following relations \( d(2n) = d(n) + 1 \) and \( d(2n + 1) = 0 \). These relations clearly hold for \( n \leq 4 \). From the definition of the sequence \( d(n) \) as \( \text{ord}_{i=0} B_n(t) \) we deduce the following relations

\[ d(2n) = d(n) + 1, \quad d(2n + 1) = \min\{d(n), d(n + 1)\}. \]
Now let us note that
\[ d(4n + 1) = \min\{d(2n), d(2n + 1)\} = \min\{d(n) + 1, \min\{d(n), d(n + 1)\}\} \]
which shows that \(d(4n + 1) = \min\{d(n), d(n + 1)\} = d(2n + 1)\). Similarly, we get that
\[ d(4n + 3) = \min\{d(2n + 1), d(2n + 2)\} = \min\{\min\{d(n), d(n + 1)\}, d(n + 1) + 1\} \]
and thus we get the equality \(d(4n + 3) = d(2n + 1)\). Because \(d(1) = 0\) by induction on \(n\) we get that \(d(2n + 1) = 0\) for all \(n\). This conclusion finishes the proof of our

It is clear that we have an inequality \(d(n) \leq e(n)\) for all \(n\) and thus we can define the

\[ \Phi : \mathbb{N}_+ \ni n \mapsto (d(n), e(n)) \in \{(a, b) \in \mathbb{N} \times \mathbb{N} : a \leq b\}. \]

We have the following.

**Proposition 2.3.** The map \(\Phi\) is onto.

**Proof.** This is very simple. We show that for any pair of nonnegative integers \((p, q)\)
with \(p \leq q\) there exists a natural number \(n\) such that \(\nu(n) = p\) and \(e(n) = q\). If \(p = q\) it is enough to take \(n = 2^p\). We can assume that \(p < q\). In order to prove the demanded property we take \(n = 2^p(2^{q-p+1} + 1)\). We clearly have \(\nu(n) = p\). In

It is an interesting question what can be said about the solutions of the equation
\(e(n) - d(n) = i\), where \(i \in \mathbb{N}\) is given. More precisely we are interested in the

\[ C(i, n) = \{m \in [1, 2^n] : e(m) - d(m) = i\} \]

We define \(c(i, n) = |C(i, n)|\) and note that \(c(i, n)\) exists for \(i \leq n\) which follows from
the properties of the sequence \(e(n)\) presented in Theorem 2.1. Now let us note that
\[ c(i, n + 1) = |\{m \in [1, 2^{n+1}] : e(m) - d(m) = i\}| \]
\[ = |\{m \in [1, 2^n] : e(2m) - d(2m) = i\}| \]
\[ + |\{m \in [0, 2^n - 1] : e(2m + 1) - d(2m + 1) = i\}| \]
\[ = c(i - 1, n) + |\{m \in [0, 2^n - 1] : e(2m + 1) = i\}|, \]
where in the last equality we use the fact that \(d(2m + 1) = \nu(2m + 1) = 0\).
We thus see that in order to compute the \(c(i, n)\) we need to know the value of \(|\{m \in [0, 2^n - 1] : e(2m + 1) = i\}|\). In order to do this we will need some properties of the polynomial

\[
H_n(x) = \sum_{i=1}^{2^n} x^{e(i)} = \sum_{i=0}^{n} e(i, n)x^i,
\]

where the equality \(\deg H_n = n\) follows from the properties of the sequence \(e(n)\), and the number \(e(i, n)\) is the cardinality of the set \(\{m \in [1, 2^n] : e(m) = i\}\).

**Lemma 2.4.** We have \(H_0(x) = 1\), \(H_1(x) = x + 1\) and for \(n \geq 2\) we get that \(H_n(x)\) satisfies the recurrence relation

\[
H_{n+2}(x) = xH_{n+1}(x) + 2xH_n(x) - x^{n+1} + 1.
\]

Moreover, we have the equality \(e(0, n) = 1\) and for \(1 \leq i \leq n\) we have

\[
e(i, n + 2) = e(i - 1, n + 1) + 2e(i - 1, n) - [i = n + 1],
\]

where as usual \([A]\) is equal to 1 if \(A\) is true and 0 otherwise.

**Proof.** We clearly have that \(H_0(x) = 1\) and \(H_1(x) = x + 1\). Let us assume that \(n \geq 2\). Then we have the following chain of equalities

\[
H_{n+2}(x) = \sum_{i=1}^{2^{n+2}} x^{e(i)} = \sum_{i=1}^{2^{n+1}} x^{e(2i)} + 1 + \sum_{i=1}^{2^n} x^{e(4i+1)} - x^{e(2i+2)+1} + \sum_{i=0}^{2^n-1} x^{e(4i+3)}
\]

\[
= xH_{n+1}(x) + 1 + \sum_{i=1}^{2^n} x^{e(i)+1} - x^{n+1} + \sum_{i=0}^{2^n-1} x^{e(i)+1} + 1
\]

\[
= xH_{n+1}(x) + 1 + 2xH_n(x) - x^{n+1}.
\]

This proves the first part of our proposition. Comparing now the coefficients on both sides of the obtained equality we get the second part of the proposition. \(\square\)

As an immediate consequence of the above lemma we get the following.

**Corollary 2.5.** We have \(\sum_{i=0}^{2^n-1} x^{e(2i+1)} = H_{n+1}(x) - xH_n(x)\) and for \(i, n \in \mathbb{N}\) with \(i \leq n\) we get

\[
|\{m \in [0, 2^n - 1] : e(2m + 1) = i\}| = e(i, n + 1) - e(i - 1, n).
\]

From the above corollary we immediately deduce the recurrence relation for \(c(i, n)\). We have \(c(i, n) = c(i - 1, n - 1) + 2c(i, n) - e(i - 1, n - 1)\), which can be rewritten as

\[
c(i, n) - e(i, n) = c(i - 1, n - 1) - e(i - 1, n - 1)
\]

\[
= c(i - 2, n - 2) - e(i - 2, n - 2) - \ldots - c(0, n - i) - e(0, n - i) = 0.
\]

We thus deduce that \(c(i, n) = c(i, n)\) and we left with the problem of computation of the coefficients of the polynomial \(H_n(x)\). In order to do this we start with the following.

**Lemma 2.6.** Let \(n \geq 0\) and consider the polynomial \(H_n(x) = \sum_{i=1}^{2^n} x^{e(i)}\). Then, we have an identity

\[
\mathcal{E}(x, y) = \sum_{n=0}^{\infty} H_n(x)y^n = \frac{1 - xy(1 + y - y^2)}{(1 - y)(1 - xy)(1 - xy - 2xy^2)}.
\]
In particular we have
\[ H_n(x) = \frac{1}{1 - 3x} + \frac{1}{2}x^n + h_n(x), \]
where
\[ h_n(x) = \left(\frac{\sqrt{2x}}{2(1 - 3x)}\right) 8T_n \left( -\frac{\sqrt{2x}}{4} \right) - 3(x + 3)U_n \left( -\frac{\sqrt{2x}}{4} \right) \]
and \( T_n(x) \) (respectively \( U_n(x) \)) is the Chebyshev polynomial of the first kind (respectively of the second kind) and \( i^2 = -1 \).

**Proof.** In order to prove the demanded equality we use the recurrence relation for \( H_n(x) \). We have
\[ E(x, y) = 1 + (1 + x)y + \sum_{n=0}^{\infty} H_n+2(x)y^{n+2} = 1 + (1 + x)y + xy\sum_{n=0}^{\infty} H_{n+1}(x)y^{n+1} \]
\[ + 2xy^2\sum_{n=0}^{\infty} H_n(x)y^n - y\sum_{n=0}^{\infty} x^{n+1}y^{n+1} + \sum_{n=0}^{\infty} y^{n+2} \]
\[ = 1 + (1 + x)y + xy(E(x, y) - 1) + 2xy^2E(x, y) - \frac{xy^2}{1 - xy} + \frac{y^2}{1 - y}. \]
Solving now the obtained (linear) equation with respect to \( E(x, y) \) we get the expression from the statement of the corollary. In order to get the expression for \( H_n(x) \) we use the standard method of decomposition of rational into simple fractions. More precisely we have
\[ E(x, y) = \frac{1}{1 - 3x} \left( \frac{1}{1 - y} + \frac{1}{2(1 - xy)} - \frac{1 + 3x + 4xy}{2(1 - 3x)(1 - xy - 2xy^2)} \right). \]
Let us denote the last term by \( F(x, y) \). We express the \( n \)-th coefficient, say \( h_n(x) \), of the power series expansion of the function \( F(x, y) \) with respect to \( y \) as a certain combination of Chebyshev polynomials of the first and the second kind in the variable \( -\sqrt{2x}/4 \). Let us recall that the \( n \)-th Chebyshev polynomial of the first kind is defined by the relation \( T_n(x) = \cos(n \arccos x) \). The \( n \)-th Chebyshev polynomial of the second kind is defined as \( U_n(x) = \sin((n + 1) \arccos x)/\sin(\arccos x) \). The generating function \( T(x, y) \) (respectively \( U(x, y) \)) for the sequence of Chebyshev polynomials of the first kind (respectively the second kind) is given by
\[ T(x, y) = \frac{1 - xy}{1 - 2xy + y^2}, \quad U(x, y) = \frac{1}{1 - 2xy + y^2}. \]
We express now the function \( F(x, y) \) as a combination of the functions \( T(x, y) \) and \( U(x, y) \). More precisely we have the following equality
\[ F(x, y) = \frac{1}{2(1 - 3x)} \left( 8T \left( -\frac{\sqrt{2x}}{4}, \sqrt{2xy} \right) - 3(x + 3)U \left( -\frac{\sqrt{2x}}{4}, \sqrt{2xy} \right) \right). \]
Comparing now the coefficients on the both sides of the above equality we get the following expression for the rational function \( h_n(x) \):
\[ h_n(x) = \left(\frac{\sqrt{2x}}{2(1 - 3x)}\right) 8T_n \left( -\frac{\sqrt{2x}}{4} \right) - 3(x + 3)U_n \left( -\frac{\sqrt{2x}}{4} \right). \]
This is exactly the expression for the function $h_n(x)$ displayed in the statement of the theorem.

In order to give a closed value of the number $e(i, n)$ we use the expression for the polynomial $H_n(x)$. However, before we do that let us recall how the coefficients of the polynomials $T_n(x)$ and $U_n(x)$ look like:

$$T_n(x) = \frac{n}{2} \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} (-1)^k \frac{(n-k)(2x)^{n-2k}}{n-k},$$

$$U_n(x) = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} (-1)^k C(n-k,k)(2x)^{n-2k},$$

where as usual $C(a, b) = \left(\begin{array}{c}a \\ b\end{array}\right)$ is a binomial coefficient. All properties of the Chebyshev polynomials of both kinds which we have used can be found in [3].

In view of the above identities for $U_n(t)$ and $T_n(t)$ we get the expression for $H_n(x)$ in the following form

$$H_n(x) = \frac{1}{2} x^n + \frac{1}{1 - 3x} \left( 1 + \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \frac{(5k - 3n)2^k}{n-k} C(n-k,k)x^{n-k} \right).$$

Let us put $Z_n(x) = (1 - 3x)(H_n(x) - \frac{1}{2} x^n)$. We thus see that in order to find a formula for $e(i, n)$ it is enough to compute the coefficients of the polynomial $Z_n(x)/(1 - 3x)$. Now, we recall that

$$\sum_{i=0}^{n} a_{i,n} x^{i} = (1 - cx)^{n-1} \sum_{i=0}^{n-1} b_{i,n-1} \iff b_{i,n-1} = c^{i} \sum_{j=0}^{i} a_{j,n} c^{j}.$$

In our situation $c = 3$ and we have the following expression for the coefficients of the polynomial $Z_n(x)$:

$$a_{i,n} = \begin{cases} 1 & \text{for } i = 0, \\ 0 & \text{for } 1 \leq i < n - \lfloor \frac{n}{2} \rfloor, \\ \frac{(2n-5i)2^{n-i}}{i} C(i, n-i) & \text{for } n - \lfloor \frac{n}{2} \rfloor \leq i \leq n. \end{cases}$$

Computing now the expression for $b_{i,n-1}$ using the above equality and performing all necessary simplifications we get the following result.

**Theorem 2.7.** Let $e(i, n) = |\{m \in [1, 2^n] : e(m) = i\}|$. Then we have $H_n(x) = \sum_{i=1}^{n} e(i, n)x^i$, where

$$e(i, n) = \begin{cases} 3^i & \text{for } 0 \leq i \leq n - \lfloor \frac{n}{2} \rfloor - 1, \\ 2^{\lfloor i/2 \rfloor} C(i, n-i) + 3^i \left( 1 + 2^{n} \sum_{j=n-\lfloor \frac{n}{2} \rfloor}^{\lfloor n/2 \rfloor} \frac{2n-5j}{j!} C(j, n-j) \right) & \text{for } n - \lfloor \frac{n}{2} \rfloor \leq i \leq n-1, \\ 1 & \text{for } i = n. \end{cases}$$
We tried to obtain more a convenient expression for the numbers \( e(i, n) \) with \( n - \left\lfloor \frac{n}{2} \right\rfloor \leq i \leq n - 1 \). However we have been unable to do this. This leads to the following

**Problem 2.8.** Find a simpler expression for the number \( e(i, n) \) in case of \( n - \left\lfloor \frac{n}{2} \right\rfloor \leq i \leq n - 1 \).

### 3. Generating function for the sum \( S_k(n) = \sum_{a: e(a) = n} a^k \)

In this section we prove one more interesting result concerning the sequence \( \{e(n)\}_{n=1}^{\infty} \). More precisely, for a fixed integer \( n \) we are interested in the computation of a sum involving all elements of the form \( i^k \), where \( i \) is a solution of the equation \( e(i) = n \) and \( k \) is given. In the recent paper [10] we compute this sum for \( k = 0 \), i.e. we compute the number of solutions of the equation \( e(i) = n \). Let us introduce the set

\[ M(n) = \{ i \in \mathbb{N} : e(i) = n \}. \]

From the cited result we know that the cardinality of the set \( M(n) \) is \( 3^n \). We thus see that the sum we are interested in

\[ S_k(n) = \sum_{a \in M(n)} a^k, \]

is finite for any given \( k \) and \( n \). In order to tackle the problem we introduce the generating function

\[ G_k(x) = \sum_{n=1}^{\infty} n^k x^{e(n)} = \sum_{n=0}^{\infty} S_k(n) x^n. \]

We know that \( G_0(x) = 1/(1 - 3x) \). We show how \( G_k(x) \) can be computed. We prove the following.

**Theorem 3.1.** Let \( k \) be a positive integer and let us consider the function \( G_k(x) \). Then \( G_k(x) \) can be computed as a linear combination of \( G_i(x), (i = 0, 1, \ldots, k - 1) \) with rational functions as coefficients. More precisely we have the following expression:

\[ G_k(x) = \frac{x \sum_{j=0}^{k-1} C(k, j)(4^j + (-1)^k(-4)^j)G_j(x) + 1}{1 - 2^k(2^{k+1} + 1)x}. \]

**Proof.** Using the recurrence relation for the sequence \( e(n) \) and the expression for \( G_0(x) \) we obtain the recurrence relation satisfied by the sequence \( G_k(x) \), \( k = 0, 1, 2, \ldots \). We have the following equality:

\[ G_k(x) = \sum_{i=1}^{\infty} i^k x^{e(i)} = \sum_{i=1}^{\infty} (2i)^k x^{e(2i)} + 1 + \sum_{i=1}^{\infty} (4i + 1)^k x^{e(4i+1)} + \sum_{i=0}^{\infty} (4i + 3)^k x^{e(4i+3)} \]

Now, let us note that \( \sum_{i=1}^{\infty} (2i)^k x^{e(2i)} = 2^k x G_k(x) \). Moreover, we have an equality

\[ \sum_{i=1}^{\infty} (4i + 1)^k x^{e(4i+1)} = x \sum_{i=1}^{\infty} \sum_{j=0}^{k} C(k, j)4^j i^k x^{e(i)} = x \sum_{j=0}^{k} C(k, j)4^j G_j(x), \]
and
\[ \sum_{i=0}^{\infty} (4i + 3)^k x^{(4i+3)} = x \sum_{i=0}^{\infty} \sum_{j=0}^{k} (4(i+1) - 1)^k x^{(i+1)} = x \sum_{i=1}^{\infty} \sum_{j=0}^{k} (4i - 1)^k x^{i(j)} \]
\[ = (-1)^k x \sum_{i=1}^{k} \sum_{j=0}^{k} C(k, j)(-4)^i j! x^{i(j)} = (-1)^k x \sum_{j=0}^{k} C(k, j)(-4)^j G_j(x). \]

Gathering now the obtained expressions together we get
\[ G_k(x) = 2^k xG_k(x) + x \sum_{j=0}^{k} C(k, j)4^j G_j(x) + (-1)^k x \sum_{j=0}^{k} C(k, j)(-4)^j G_j(x) + 1 \]
\[ = 2^k(2^{k+1} + 1)xG_k(x) + x \sum_{j=0}^{k-1} C(k, j)(4^j + (-1)^k(-4)^j)G_j(x) + 1. \]

Solving now the above functional equation with respect to \( G_k(x) \) (this is a linear equation! we get the expression displayed in the statement of the theorem. \( \square \)

From the theorem we have just proved we deduce the following set of corollaries.

**Corollary 3.2.** Let \( k \) be a nonnegative integer and let us consider the generating function \( G_k(x) \).

1. If \( k \) is even then \( G_k(x) \) is a linear combination (with rational functions as coefficients) of the functions \( G_{2i}(x) \) for \( i = 0, 1, \ldots, \frac{k}{2} \).
2. If \( k \) is odd then \( G_k(x) \) is a linear combination (with rational functions as coefficients) of the functions \( G_{2i−1}(x) \) for \( i = 0, 1, \ldots, \frac{k+1}{2} \).

**Proof.** This is an immediate consequence of the expression displayed in the statement of Theorem 3.1. Indeed, if \( k \) is even then the expression for \( G_k(x) \) contains only terms with \( G_{2i} \) for \( i = 0, 1, \ldots, k/2 \). Similar reasoning applies in the case of \( k \) odd. \( \square \)

**Corollary 3.3.** Let \( k, n \in \mathbb{N} \) and let us put \( T_i = 2^i(2^{i+1} + 1) \). Then there exist rational numbers \( \alpha_i \) for \( i = 0, 1, \ldots, k \) such that
\[
S_k(n) = \begin{cases} 
\sum_{i=0}^{\frac{k}{2}} \alpha_{2i}T_{2i}^n & \text{if } k \text{ is even}, \\
\sum_{i=1}^{\frac{k+1}{2}} \alpha_{2i-1}T_{2i-1}^n & \text{if } k \text{ is odd}.
\end{cases}
\]

**Corollary 3.4.** Let \( n \geq 0 \) be given. Then we have the following equalities
\[ S_1(n) = 10^n, \quad S_2(n) = \frac{35 \cdot 36^n - 2 \cdot 3^n}{33}, \quad S_3(n) = \frac{25 \cdot 136^n - 4 \cdot 10^n}{21}. \]

**Proof.** We use the result from the previous theorem. If \( k = 1 \) then we easily get that \( G_1(x) = \frac{1}{1-10x} \). Comparing now the coefficients of the power series expansions of the two functions we get the expression for \( S_1(n) \) displayed in the statement. For \( k = 2 \) we get that
\[ G_2(x) = \frac{1 - x}{(1 - 3x)(1 - 36x)} = \frac{35}{33} \cdot \frac{1}{1 - 36x} + \frac{1}{33} \cdot \frac{1}{1 - 3x}. \]

Using the same method as in the case of \( G_1(x) \) we get the expression for \( S_2(n) \) displayed in the statement of the corollary.
Finally, if \( k = 3 \) then

\[
G_3(x) = \frac{1 + 14x}{(1 - 10x)(1 - 136x)} = \frac{25}{21} \frac{1}{1 - 136x} - \frac{4}{21} \frac{1}{1 - 10x},
\]

and we easily get the expression for \( S_3(n) \).

\[ \square \]

4. A DETERMINANT EXPRESSION FOR THE \( B_n(t) \)

In a recent paper [10, Theorem 2.4] we proved that if \( \mu(n) \) is the highest power of 2 dividing \( n \) then the following identity holds

\[
t^{\mu(n)}(B_{n+1}(t) + B_{n-1}(t)) = (B_{2^{\mu(n)}+1}(t) + B_{2^{\mu(n)}-1}(t))B_n(t).
\]

From the above identity we easily deduce that the sequence \( \{B_n(t)\}_{n=0}^{\infty} \) satisfies a three term recurrence relation with variable coefficients of the form

\[
B_0(t) = 0, \quad B_1(t) = 1, \quad B_{n+1}(t) = A_n(t)B_n(t) - B_{n-1}(t),
\]

where

\[
A_n(t) = t^{-\mu(n)}(B_{2^{\mu(n)}+1}(t) + B_{2^{\mu(n)}-1}(t)).
\]

Because \( B_{2^{\mu(n)}-1}(t) = (t^n - 1)/(t-1) \) and \( B_{2^{\mu(n)}+1}(t) = (t^n - 1)/(t-1)+t \), we get that

\[
A_n(t) = t^{-\mu(n)} \left( 2^{\frac{t^{\mu(n)}-1}{t-1}} + t \right).
\]

We thus see that the function \( A_n(t) \) does not depend on the polynomials \( B_i(t), i = 0, 1, \ldots, n \).

The recurrence relation [11] permits us to give a new definition of the Stern polynomial \( B_n(t) \) as a determinant. Indeed, let us consider [11], with \( n \) replaced by \( n-1 \), as a homogeneous linear equation in three "unknowns" \( B_n, B_{n-1}, B_{n-2} \), namely \( B_n - A_{n-1}B_{n-1} + B_{n-2} = 0 \). Next, we replace \( n \) successively by \( n-1, n-2, \ldots, 3, 2 \). For \( n = 3 \) the equation reads \( B_3 - A_2B_2 = -1 \). For \( n = 2 \) we have \( B_2 - A_1B_1 = 0 \) and finally for \( n = 1 \) we get \( B_1 = 1 \). We solve this system of \( n \) linear equations in \( B_n, B_{n-1}, \ldots, B_1 \) by Cramer’s rule. The determinant of the system is

\[
D = \begin{vmatrix}
1 & -A_{n-1}(t) & 1 & 0 & \ldots & 0 & 0 & 0 \\
0 & 1 & -A_{n-2}(t) & 1 & \ldots & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & 0 & \ldots & 1 & -A_2(t) & 0 \\
0 & 0 & 0 & 0 & \ldots & 0 & 1 & -A_1(t) \\
0 & 0 & 0 & 0 & \ldots & 0 & 0 & 1
\end{vmatrix},
\]

and \( B_n(t)D = D_n \), where \( D_n \) is obtained by replacing the first column of \( D \) by the column vector of the constant terms. The entries of this vector are, as seen, all zeros, except for the last three, which are \(-1\) (from \( B_3 \)), 0 (from \( B_2 \)) and 1 (from \( B_1 \)). Because the matrix associated with \( D \) is triangular we have that \( D = 1 \) and
finally we get the determinant expression for the \( n \)-th Stern polynomial in the form
\[
B_n(t) = \begin{vmatrix}
0 & -A_{n-1}(t) & 1 & 0 & \ldots & 0 & 0 & 0 \\
0 & 1 & -A_{n-2}(t) & 1 & \ldots & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
-1 & 0 & 0 & 0 & \ldots & 1 & -A_2(t) & 0 \\
0 & 0 & 0 & 0 & \ldots & 1 & -A_1(t) & 1 \\
1 & 0 & 0 & 0 & \ldots & 0 & 0 & 1
\end{vmatrix}.
\]

5. An interesting sum involving Stern polynomials

In this section we are interested in the computation of the certain sum involving the reciprocals of products of two consecutive Stern polynomials. This is a variation on a theme of generalization of the (not so well known) identity
\[
\sum_{i=m}^{2m-1} \frac{1}{s(i)s(i+1)} = 1,
\]
where \( s(i) = B_i(1) \) is the \( i \)-th term of the Stern diatomic sequence. The proof of this identity with an interesting discussion can be found in [11].

Before we state the main result in this section we introduce an interesting family of polynomials which will we need in the sequel. Let
\[
S_1(t) = 0, \quad S_2(t) = 0, \quad S_{k+1}(t) = S_k(t) + S_{k+1}(t) + t^\left\lfloor \log_2 k \right\rfloor.
\]
The first few terms of the sequence \( \{S_k(t)\}_{n=0}^\infty \) are contained in the table below

| \( n \) | \( S_n(t) \) | \( n \) | \( S_n(t) \) | \( n \) | \( S_n(t) \) | \( n \) | \( S_n(t) \) |
|---|---|---|---|---|---|---|---|
| 1 | 0 | 2 | 0 | 3 | 1 | 4 | 0 |
| 5 | 1 + t | 6 | \( t \) | 7 | \( 1 + t \) | 8 | \( 0 \) |
| 1 + t + t^2 | \( t(1 + t) \) | \( (1 + t)^2 \) | \( t^2 \) | 13 | \( (1 + t)^2 \) | 14 | \( t(1 + t) \) |
| 15 | \( 1 + t + t^2 \) | 16 | 0 |

In the sequel we will need the following result which connects the sequences \( \{B_n(t)\}_{n=0}^\infty \) and \( \{S_n(t)\}_{n=1}^\infty \).

**Lemma 5.1.** Let \( n \in \mathbb{N}_+ \). Then the following identity holds
\[
(2 - t)(B_n(t)S_{n+1}(t) - S_n(t)B_{n+1}(t)) = t^{\lfloor \log_2 k \rfloor}(B_{n+1}(t) - B_n(t) - t + 1).
\]

**Proof.** In order to prove our theorem we proceed by induction on \( n \). The desired identity is clearly true for \( n = 1 \) (in this case both sides are equal to 0) and for \( n = 2 \) (in this case both sides are equal to \( (2 - t)t \)). Let us suppose that our identity is true for all numbers \( < n \). We prove that it is true for \( n \). We consider two cases: \( n \) even and \( n \) odd.

If \( n \) is even, then \( n = 2m \) for some \( m \in \mathbb{N}_+ \). In order to shorten the notation we put \( B_n := B_n(t) \) and \( S_n := S_n(t) \). Recall now that \( B_{2m} = tB_m \) and \( B_{2m+1} = B_m + B_{m+1} \). Using now the recurrence relation for \( B_n \) and \( S_n \) we have the following
chain of equalities
\[
(2 - t)(B_{2m}S_{2m+1} - S_{2m}B_{2m+1})
= t(2 - t)(B_m(S_m + S_{m+1} + t^{\log_2 m}) - S_m(B_m + B_{m+1}))
= t(2 - t)(B_mS_{m+1} - S_mB_{m+1} + t^{\log_2 m + 1}(2 - t)B_m)
= t^{\log_2 m + 1}(B_{m+1} - B_m - t + 1) + t^{\log_2 m + 1}(2 - t)B_m
= t^{\log_2 (2m)}(B_{m+1} + B_m - tB_m - t + 1)
= t^{\log_2 (2m)}(B_{2m+1} - 2B_{2m} - t + 1).
\]
This finishes the proof of our identity in case of \( n \) even.

If \( n \) is odd, then \( n = 2m + 1 \) for some \( m \in \mathbb{N} \). Using similar reasoning as in the previous case we get the following chain of equalities
\[
(2 - t)(B_{2m+1}S_{2m+2} - S_{2m+1}B_{2m+2})
= t(2 - t)((B_m + B_{m+1})S_{m+1} - (S_m + S_{m+1} + t^{\log_2 m})B_{m+1})
= t(2 - t)(B_mS_{m+1} - S_{m+1}B_m) - (2 - t)t^{\log_2 m + 1}B_{m+1}
= t^{\log_2 m + 1}(B_{m+1} - B_m - t + 1) - (2 - t)t^{\log_2 m + 1}B_{m+1}
= t^{\log_2 m + 1}(B_{m+1} - B_m - t + 1 - 2B_{m+1} + tB_{m+1})
= t^{\log_2 (2m + 1)}(B_{2m+2} - 2B_{2m+1} - t + 1),
\]
where in the last identity we use an obvious fact that \( \lfloor \log_2 m \rfloor + 1 = \lfloor \log_2 (2m + 1) \rfloor \).
This finishes the proof of our identity in case of \( n \) odd. Gathering now what we have we deduce that the identity \( \square \) is true for all \( n \in \mathbb{N} \).

From the above lemma we deduce an interesting corollary which will be important in the proof of the main result of this section.

**Corollary 5.2.** Let \( k \in \mathbb{N}_+ \). Then we have the following identity
\[
(3) \quad \frac{2 - t}{t^{\log_2 k}} S_{2k+1}(t) = \frac{2 - t}{t^{\log_2 k}} B_{2k+1}(t) + \frac{B_{k+1}(t) - (t - 1)(B_k(t) + 1)}{B_k(t)B_{2k+1}(t)}.
\]

**Proof.** This is a simple consequence of the identity from Lemma 5.1. Indeed, we take \( n = 2k \) and note that \( B_{2k}(t) = B_k(t) \) and \( S_{2k}(t) = tS_k(t) \). Moreover, the right side of the identity \( \square \) is clearly equal to \( B_{k+1}(t) - (t - 1)(B_k(t) + 1) \). Dividing now both sides of \( \square \) by \( t^{\log_2 m} B_k(t)B_{2k+1}(t) \) and adding the expression \( \frac{t^{\log_2 k} S_k(t)}{t^{\log_2 k} B_k(t)} \) for both sides we get \( \square \).

We are ready now to prove the following theorem.

**Theorem 5.3.** Let \( k \) be a positive integer. Then the following identity holds:
\[
\sum_{i=2^n}^{k2^n+1} \frac{1}{B_i(t)B_{i+1}(t)} = \frac{2 - t}{t^{n+\log k + 1}} B_k(t) S_k(t) + \frac{1}{t^{n+1} B_k(t)} \left( \frac{1}{B_{k2^n+1}(t)} + 1 \right),
\]
where \( S_1(t) = S_2(t) = 0 \) and for \( k \geq 2 \) we have
\[
S_{2k}(t) = tS_k(t), \quad S_{2k+1}(t) = S_k(t) + S_{k+1}(t) + t^{\log k}.
\]
Proof. Let $P_{k,n}$ (respectively $Q_{k,n}$) denote the left hand side (respectively the right hand side) of the identity which we want to prove. We prove the desired identity in two steps. First we prove that $P_{k,n}$ and $Q_{k,n}$ satisfy the same linear recurrence relation (with respect to $n$). In Step 2 we use similar reasoning in order to prove that $P_{k,0} = Q_{k,0}$ for $k \in \mathbb{N}$. These two facts tied together give the result.

Step 1. We find a recurrence relation satisfied by $P_{k,n}$. In order to shorten the notation we put $B_i := B_i(t)$. We have the following chain of equalities

$$P_{k,n+1} = \sum_{i=k+1}^{k2^n+2} \frac{1}{B_i} - \sum_{i=k^n+1}^{k2^n+1} \frac{1}{B_i B_{i+1}} + \sum_{i=k^n+1}^{k2^n+1} \frac{1}{B_i B_{i+1} B_{i+2}} - \frac{1}{B_{k2^n+2} + 1} \frac{1}{B_{k2^n+2} + 2}$$

We thus find that $P_{k,n+1} = t^{-1} P_{k,n} - 1/t B_{k2^n+1} B_{k2^n+2}$. We show that $Q_{k,n}$ satisfies exactly the same recurrence relation. First of all note that from the definition of $Q_{k,n}$ we have

$$Q_{k,n+1} = \frac{1}{t} Q_{k,n} + \frac{1}{t^{n+2} B_k} \left( \frac{1}{B_{k2^n+2} + 1} - \frac{1}{B_{k2^n+1} + 1} \right).$$

Using the recurrence relation satisfied by $B_n$ we easily get that the expression in the bracket is equal to

$$\frac{B_{k2^n+1} - B_{k2^n+2}}{B_{k2^n+1} B_{k2^n+2}} = \frac{B_{k2^n+1} - (B_{k2^n+1} + B_{k2^n+1})}{B_{k2^n+1} B_{k2^n+2}} = -\frac{t^{n+1} B_k}{B_{k2^n+1} B_{k2^n+2}}.$$

Putting this into the expression for $Q_{k,n+1}$ we easily get that $Q_{k,n}$ satisfies the same recurrence relation as $P_{k,n}$.

Step 2. We prove that $P_{k,0} = Q_{k,0}$. In order to do this we use similar reasoning as in Step 1. We find a recurrence relation for $P_k := P_{k,0}$ and show that $Q_k := Q_{k,0}$ satisfies the same relation. We start with finding a recurrence relation for $P_k$. We note that $P_{2k,n} = P_{k,n+1}$ and thus $P_{2k} = P_{2k,0} = P_{k,1}$ and using the recurrence relation from Step 1 we easily deduce that

$$P_{2k} = \frac{1}{t} P_k - \frac{1}{t B_{2k+1} B_{4k+1}}.$$

Using now the expression for $P_{2k}$ we get

$$P_{2k+1} = P_{2k} + \frac{1}{t B_{2k+1}} \left( \frac{1}{B_{4k+1}} + \frac{1}{B_{4k+3}} - \frac{1}{B_k} \right)$$

$$= \frac{1}{t} P_k - \frac{1}{t B_{2k+1} B_{4k+1}} + \frac{1}{t B_{2k+1}} \left( \frac{1}{B_{4k+1}} + \frac{1}{B_{4k+3}} - \frac{1}{B_k} \right).$$
Using now the recurrence relation for defining the sequence \( \{ B_n \}_{n=0}^{\infty} \) and performing simple but tiresome calculations we arrive at the expression

\[
P_{2k+1} = \frac{1}{t} P_k - \frac{(t+1)B_{k+1}}{tB_kB_{2k+1}B_{4k+3}}.
\]

We consider now the sequence \( Q_k \). Using exactly the same reasoning as in the case of \( P_{2k} \) we deduce that

\[
Q_{2k} = \frac{1}{t} Q_k - \frac{1}{tB_{2k+1}B_{4k+1}}.
\]

We prove that \( Q_{2k+1} \) satisfies the same relation as \( P_{2k+1} \). Using now the result from Corollary 3 and the identity \( \lfloor \log_2(k+1) \rfloor = \lfloor \log_2 k \rfloor + 1 \) we get that

\[
Q_{2k+1} = \frac{2 - t}{t^{\lfloor \log_2 (2k+1) \rfloor + 1}} S_{2k+1} + \frac{1}{tB_{2k+1}} \left( \frac{1}{B_{4k+3}} + 1 \right)
\]

\[
= \frac{1}{t} \left( \frac{2 - t}{t^{\lfloor \log_2 k \rfloor + 1}} S_k + \frac{B_{k+1} - (t - 1)(B_k + 1)}{tB_kB_{2k+1}} \right) + \frac{1}{tB_{2k+1}} \left( \frac{1}{B_{4k+3}} + 1 \right)
\]

We note that

\[
\frac{1}{t} \frac{2 - t}{t^{\lfloor \log_2 k \rfloor + 1}} S_k = \frac{1}{t} Q_k - \frac{1}{t^2 B_k} \left( \frac{1}{B_{2k+1}} + 1 \right).
\]

Using this expression we get that

\[
Q_{2k+1} = \frac{1}{t} Q_k - \frac{1}{t^2 B_k} \left( \frac{1}{B_{2k+1}} + 1 \right) + \frac{B_{k+1} - (t - 1)(B_k + 1)}{tB_kB_{2k+1}} + \frac{1}{tB_{2k+1}} \left( \frac{1}{B_{4k+3}} + 1 \right)
\]

We are thus left with the simplification of the complicated expression in the above identity. One can easily see that the common denominator, say \( D \), of the sum of fractions which arises on the left side in the expression for \( Q_{2k+1} \) is clearly \( D = t^3 B_k B_{2k+1} B_{4k+3} \). The numerator, say \( N \), after simplifications is equal to

\[
N = tB_k - (t - B_k - B_{k+1} + B_{2k+1})B_{4k+3} = t(B_k - B_{4k+3}) = -t(t+1)B_{k+1},
\]

where in the last equality we use the identity \( B_{4k+3} = B_k + (t+1)B_{k+1} \). We thus obtain the relation

\[
Q_{2k+1} = \frac{1}{t} Q_k - \frac{(t+1)B_{k+1}}{tB_k B_{2k+1} B_{4k+3}},
\]

which is exactly the same relation satisfied by \( P_k \). Gathering the information we have obtained we see that \( P_k \) and \( Q_k \) satisfy the same recurrence relation.

In order to finish the proof of our theorem we note that \( P_{1,0} = (t+2)/(t(t+1)) = Q_{1,0} \) and because \( P_k \) and \( Q_k \) satisfy the same recurrence relation with the same initial condition we deduce that \( P_{k,0} = Q_{k,0} = Q_k \) for all \( k \in \mathbb{N}_+ \). Using now the information from Step 1 we get that \( P_{k,n} = Q_{k,n} \) for all \( k \in \mathbb{N}_+ \) and \( n \in \mathbb{N} \). This observation finishes the proof of our theorem.

6. Some additional observations, open questions and conjectures

We start this section with a very simple result on the partial sums of the \( \pm 1 \) sequence \( \{(−1)^{c(i)}\}_{i=1}^{\infty} \).
Theorem 6.1. Let \( e(n) = \deg B_n(t) \) and let us consider the sequence \( S(n) = \sum_{i=1}^{n}(-1)^{e(i)} \). Then we have

\[
\liminf_{n \to +\infty} S(n) = -\infty, \quad \limsup_{n \to +\infty} S(n) = \infty,
\]

Proof. In order to get the demanded equalities we use the value of \( s_n := S(2^n) \) which is just \( H_n(-1) \), where \( H_n \) is the polynomial considered in Section 2. We have \( s_0 = 1, s_1 = 0 \) and for \( n \geq 2 \) we see that the sequence \( \{s_n\}_{n=0}^{\infty} \) satisfies the recurrence relation \( s_{n+2} = -s_{n+1} - 2s_n + 1 + (-1)^n \). Although we have obtained a closed expression for \( H_n(x) \) in Corollary 2.3 it is of little use in our situation. Instead of using it we just solve the recurrence relation for \( s_n \). We have

\[
s_n = \frac{1}{4} + \frac{(-1)^n}{2} + \epsilon_1 p_1^n + \epsilon_2 p_2^n,
\]

where \( p_1 = (1 + \sqrt{-7})/2, p_2 = -(1 + \sqrt{-7})/2 \) and \( \epsilon_1 = (7 - 3\sqrt{-7})/56, \epsilon_2 = (7 + 3\sqrt{-7})/56 \). We introduce a sequence \( \{t_n\}_{n=0}^{\infty} \subset \mathbb{Z} \) defined in the following way

\[
t_n := 4 \left( s_n - \frac{1}{4} - \frac{(-1)^n}{2} \right).
\]

The sequence \( t_n \) starts as follows

\[
1, 1, -3, 1, 5, -7, -3, 17, -11, -23, 45, 1, -91, 89, 93, -271, \ldots.
\]

One can easily check that \( t_{n+2} = -t_{n+1} - 2t_n \). Because \( |p_1| = |p_2| = \sqrt{2} > 1 \), thus from the theory of linear difference equations we know that \( \lim_{n \to +\infty} |t_n| = +\infty \) and clearly the same property holds for the sequence \( s_n \). From the shape of the recurrence relation for \( t_n \) we deduce that \( t_n \) changes sign infinitely often (in fact there is no \( n \in \mathbb{N} \) such that \( t_n, t_{n+1}, t_{n+2} \) are of the same sign). From the equality \( \lim_{n \to +\infty} |t_n| = +\infty \) and the mentioned property of signs we immediately get that

\[
\liminf_{n \to +\infty} t_n = -\infty, \quad \limsup_{n \to +\infty} t_n = \infty,
\]

and clearly the same property holds for \( s_n \). \( \square \)

In the next result we find the functional equation satisfied by the ordinary generating function of the sequence \( \{e(n)\}_{n=1}^{\infty} \).

Theorem 6.2. Let \( \mathcal{E}_1(x) \in \mathbb{Z}[x] \) be an ordinary generating function for the sequence \( \{e(n)\} \), i.e. \( \mathcal{E}_1(x) = \sum_{n=1}^{\infty} e(n)x^n \). Then \( \mathcal{E}_1(x) \) satisfies the following functional equation:

\[
\mathcal{E}_1(x) = \mathcal{E}_1(x^2) + \frac{x^2 + 1}{x} \mathcal{E}_1(x^4) + \frac{x^2}{1 - x}.
\]

Proof. This is an easy consequence of the recurrence relation for the sequence \( e(n) \). Indeed, we have the following chain of equalities

\[
\begin{align*}
\mathcal{E}_1(x) &= \sum_{n=1}^{\infty} e(n)x^n = \sum_{n=1}^{\infty} e(2n)x^{2n} + \sum_{n=0}^{\infty} e(4n + 1)x^{4n+1} + \sum_{n=0}^{\infty} e(4n + 3)x^{4n+3} \\
&= \sum_{n=1}^{\infty} (e(n) + 1)x^{2n} + x \sum_{n=1}^{\infty} (e(n) + 1)x^{4n} + \frac{1}{x} \sum_{n=1}^{\infty} (e(n) + 1)x^{4n} \\
&= \mathcal{E}_1(x^2) + \frac{x^2 + 1}{x} \mathcal{E}_1(x^4) + \frac{x^2}{1 - x}.
\end{align*}
\]
Note that in the second last equality we use the recurrence relation for \( e(n) \). The result follows.

Before we give the next result let us recall two results obtained in [1].

**Proposition 6.3** (Corollary 1 and Lemma 5 from [1]). (1) Let \( \mathbb{F} \) be a field and let \( f(x) \in \mathbb{F}[[x]] \) be a power series satisfying the equation

\[
\sum_{i=0}^{m} a_i(x) f(x^k) = 0,
\]

where \( a_0(x), \ldots, a_m(x) \in \mathbb{F}(x) \) and \( a_0(x) \equiv 1 \). Suppose that there exists a rational function \( r(x) \in \mathbb{F}(x) \setminus \{0\} \) whose poles (in the algebraic closure of \( \mathbb{F} \)) are either zero or roots of unity. If

\[
\frac{r(x^k)}{r(x)} a_i(x) \in \mathbb{F}[x]
\]

for \( i = 1, 2, \ldots, m \), then \( f(x) \) is a \( k \)-regular power series.

(2) If \( f \in \mathbb{Q}[x] \) is a \( k \)-regular power series. Then it is either a rational function or it is transcendental over \( \mathbb{Q}(x) \).

We use the cited result in order to prove the following.

**Corollary 6.4.** The function \( \mathcal{E}_1(x) \) is 2-regular power series and it is transcendental over \( \mathbb{Q}(x) \).

**Proof.** First of all we rewrite the functional equation for \( \mathcal{E}_1(x) \) in the following form

\[
\mathcal{E}_1(x) - \frac{x^2 + x + 1}{x^2} \mathcal{E}_1(x^2) - \frac{x^3 - 1}{x^2} \mathcal{E}_1(x^4) + \frac{(1 + x)(1 + x^4)}{x^4} \mathcal{E}_1(x^8) = 0.
\]

The displayed identity follows from the functional equation for \( \mathcal{E}_1 \) with substitution \( x^2 \) instead of \( x \) and uses the fact that \( \frac{x^4}{1-x} = \frac{x^2}{1+x} \frac{x^2}{1-x} \). From the result of Becker we deduce that \( \mathcal{E}_1 \) is 2-regular by taking \( r(x) = x^4 \).

In order to prove transcendence of \( \mathcal{E}_1(x) \) it is enough to show that \( \mathcal{E}_1(x) \) is not a rational function. So let us suppose that \( \mathcal{E}_1(x) = p(x)/q(x) \) with \( p, q \in \mathbb{Z}[x] \) and \( \gcd(p(x), q(x)) = 1 \). From this we deduce that the power series \( \mathcal{E}(x) := \mathcal{E}_1(x) \pmod{2} \) is rational over \( \mathbb{F}_2 \) (a field with two elements). But then \( \mathcal{E}(x^2) = \mathcal{E}_1(x)^2 \) and thus we get that \( \mathcal{E}(x) \) satisfies an algebraic equation

\[(4) \quad F(x, T) = (1 - x)(1 + x^2)T^4 + x(1-x)T^2 - x(1-x)T + x^2 = 0.\]

Because \( \mathcal{E} \) is rational we know that the equation \( F(x, T) = 0 \) has a rational root. But one can easily check that there is no rational function over \( \mathbb{F}_2 \) for which (4) holds. This contradiction finishes the proof of the transcendence of \( \mathcal{E}_1(x) \).

Using similar reasoning as in the proof of Theorem 6.4 and Proposition 6.2 one can easily deduce the following.

**Theorem 6.5.** Let \( k \) be a nonnegative integer and let us define the function \( \mathcal{E}_k(x) = \sum_{n=1}^{\infty} e(n)x^n \). Then \( \mathcal{E}_0(x) = \frac{1}{1-x} \) and for \( k \geq 1 \) the function \( \mathcal{E}_k(x) \) satisfies the functional equation

\[
\mathcal{E}_k(x) - \mathcal{E}_k(x^2) - \frac{x^2 + 1}{x} \mathcal{E}_k(x^4) = \sum_{j=0}^{k-1} C(k, j) \left( \mathcal{E}_j(x^2) + \frac{x^2 + 1}{x} \mathcal{E}_j(x^4) \right).
\]
We know that \( E_0(x) \) is rational and from Corollary 6.4 we know that the function \( E_1(x) \) is transcendental. It is an interesting question whether the function \( E_k(x) \) for \( k \geq 2 \) is transcendental too. We believe that this is the case and it leads us to the following.

**Conjecture 6.6.** The function \( E_k(x) \) is transcendental for \( k \geq 2 \).

Finally, we state the following conjecture which appeared during our investigations of the sequence of maximal coefficients of the Stern polynomial \( B_n(t) \).

**Conjecture 6.7.** Let \( B_n(t) = \sum_{i=0}^{e(n)} a_{i,c(n)} x^i \) and let us define \( M(n) = \max\{a_{i,c(n)} : i = 1, 2, \ldots, e(n)\} \). Then the following equality holds

\[
\max\{M(m) : m \in [2^{n-1}, 2^n]\} = \max\{C(n, 0), C(n-1, 1), \ldots, C(n-k, k)\},
\]

where \( k \) is equal to \( n/2 \) if \( n \) is even and \( (n-1)/2 \) for \( n \) odd.

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