A LOWER BOUND ON GOWERS’ FIN$_k$ THEOREM

ALEXANDER P. KREUZER

Gowers’ FIN$_k$ theorem, also called Gowers’ pigeonhole principle or Gowers’ theorem, is a Ramsey-type theorem. It first occurred in the study of Banach space theory [3], and is a natural generalization of Hindman’s theorem. In this short note, we will show that Gowers’ FIN$_k$ theorem does not follow from ACA$_0$.

1. HINDMAN’S THEOREM

Hindman’s theorem is the following statement. As the name suggest it was established by Neil Hindman, see [4].

**Theorem 1** (Hindman’s theorem, HT). If the natural numbers are colored with finitely many colors then there is an infinite set $S \subseteq \mathbb{N}$ such that the non-repeating, finite sums of $S$

$$FS(S) := \left\{ \sum_{i \in I} s_i \mid I \in \mathcal{P}_{\text{fin}}(\mathbb{N}) \setminus \{\emptyset\} \right\}$$

where $(s_i)$ is the enumeration of $S$ are colored with only one color.

In the context of reverse mathematics HT was first investigated by Blass, Hirst, Simpsons in [2]. There it was shown that it follows from ACA$_0^+$, that is ACA$_0$ plus the statement that for all $X$ the $\omega$-jump $X^{(\omega)}$ exists. The best known lower bound is ACA$_0$, see also [2]. It is one of the big open questions of reverse mathematics what is the exact strength of HT and whether it is equivalent to ACA$_0$. There has been some partial process on this question. However no definite answer could be given. See [5, 11, 8] and [6, Section 2.3].

As already said Gower’s pigeonhole principle is a generalization of HT. Below we will see that it is not provable in ACA$_0$.

To understand the statement of Gower’s pigeonhole principle we will first look at the following finite unions variant of Hindman’s theorem. It is not to difficult to see that it is equivalent (relative to RCA$_0$) to HT, see [2].

**Theorem 2** (Hindman’s theorem, finite unions variant). If the finite subsets of the natural numbers $\mathcal{P}_{\text{fin}}(\mathbb{N})$ are colored with finitely many colors there exists a infinite set $S \subseteq \mathcal{P}_{\text{fin}}(\mathbb{N})$ consisting of pairwise disjoint sets and such that the non-empty finite unions of $S$

$$\text{NU}(S) := \{ s_1 \cup \cdots \cup s_n \mid n \in \mathbb{N} \setminus \{0\}, s_i \in S, \max s_i < \min s_{i+1} \}$$

are colored with only one color.

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2. Gowers’ FIN\(_k\) theorem

Before we can formulate Gowers’ FIN\(_k\) theorem we have to introduce some notations. The following definitions will be made in RCA\(_0\).

Let \(k \in \mathbb{N}\) and let \(p: \mathbb{N} \rightarrow [0, k]\). We call the set the support of \(p\). The space FIN\(_k\) we be the following.

\[
\text{FIN}_k := \left\{ p \in [0, k]^{\mathbb{N}} \mid |\text{supp}(p)| < \omega, \exists n p(n) = k \right\} = \left\{ p \in [0, k]^{\mathbb{N}} \mid \exists n < \text{th}(p) (p(n) = k) \right\}
\]

This space will play the role of \(P_{\text{fin}}(\mathbb{N})\) in Hindman’s theorem. For \(k = 2\) it is actually isomorphic to \(P_{\text{fin}}(\mathbb{N})\).

On FIN\(_k\) we define the following order

\[
p < q \quad \text{iff} \quad \max \text{supp}(p) < \min \text{supp}(q)
\]

and the following partial addition for comparable elements

\[
p + q \quad \text{for} \quad p < q
\]

which will be equal to the usual pointwise addition (if \(p < q\)).

On FIN\(_k\) we will make use of the following, so called “tetris” operation \(T\)

\[
T: \text{FIN}_k \rightarrow \text{FIN}_{k-1} \quad T(p)(n) = p(n) - 1.
\]

A block sequence \(B\) is an infinite increasing sequence \(B = (b_n)_{n \in \mathbb{N}}\) in FIN\(_k\). The combinatorial space \(\langle B \rangle\) generated by \(B\) is the smallest subsets of FIN\(_k\) containing \(B\) and closed under addition and tetris, i.e.,

\[
\langle B \rangle := \left\{ \sum_{n \in \mathbb{N}} T^{k-f(n)}(b_n) \mid f \in \text{FIN}_k \right\}.
\]

(Note that the above sum is finite since the support of \(f\) is finite.)

We can now formulate Gowers’ FIN\(_k\) theorem.

**Theorem 3** (Gowers’ FIN\(_k\) theorem, FIN\(_{<\infty}\)). For any \(k \in \mathbb{N}\) and any finite coloring of FIN\(_k\) there exists a combinatorial subspace colored by only one color.

We will denote full version of of Gowers’ FIN\(_k\) theorem by FIN\(_{<\infty}\) and the restriction to a particular \(k\) by FIN\(_k\). It is clear that for \(k = 2\) this theorem is the same as Hindman’s theorem since a combinatorial subspace in FIN\(_2\) is just as the set given in (1). So in other words

\[
\text{RCA}_0 \vdash \text{HT} \leftrightarrow \text{FIN}_2.
\]

Moreover, it is clear that FIN\(_k\) \(\rightarrow\) FIN\(_l\) if \(k > l\) since FIN\(_l\) can be embedded into FIN\(_k\) via the following mapping \(i(p)(n) = p(n) + k - l\).

**Remark 4** (RCA\(_0\)). A combinatorial subspace \(\langle B \rangle \subseteq \text{FIN}_k\) is isomorphic to FIN\(_k\) via the isomorphism \(\Theta_B(f) := \sum_{n \in \mathbb{N}} T^{k-f(n)}(b_n)\).

Thus, FIN\(_k\) remains true if one colors only a combinatorial subspace instead of FIN\(_k\). This variant is equivalent to FIN\(_k\).
3. A LOWER BOUND ON $\text{FIN}_k$

**Theorem 5.** There exists a recursive coloring of $\text{FIN}_{k+1}$ with $2^k$ many colors, such that each monochromatic combinatorial subspace computes $\emptyset^{(k)}$.

Before we will come to the proof we will fix some notation and state a proposition. We will need computable approximations of the $n$-fold Turing jump. For this we shall write

$$\emptyset^{(n)}_{s_n, s_{n-1}, \ldots, s_1}$$

for

$$\cdots (\emptyset'_{s_1})'_{s_2} \cdots s_n$$

(In other words the $s_n$-step approximation of the Turing jump of the $s_{n-1}$-step approximation of the Turing jump ... of the $s_1$-step approximation of the relativized Turing jump.) If the tuple $(s_n, s_{n-1}, \ldots, s_m)$ is shorter than $n$, we shall take the true Turing jump for the missing indexes.

**Proposition 6.** For each $n, m$ there exists a finite sequence $(m_1, \ldots, m_n)$ such that

$$\emptyset^{(n)} \cap [0; m] = \emptyset^{(n)}_{m_n, \ldots, m_1} \cap [0; m].$$

Moreover, we may assume that $(m_1, \ldots, m_n)$ is such that, taking $m_{n+1} := m$,

$$(2) \quad \emptyset^{(i)} \cap [0; m_{i+1}] = (\emptyset^{(i-1)})'_s \cap [0; m_{i+1}] \quad \text{for} \ s > m_i$$

where $i \in [1; n]$.

**Proof.** Apply the limit lemma—relative to $\emptyset^{(n-1)}$—we obtain an $m_n$ such that

$$\emptyset^{(n)} \cap [0; m_{n+1}] = (\emptyset^{(n-1)})'_s \quad \text{for all} \ s > m_n.$$  

Iterating this process we obtain $(m_1, \ldots, m_n)$ satisfying (2). This sequence automatically satisfies the other statement of the proposition. $\square$

For an $f \in \text{FIN}_k$ with $k \geq 2$ we shall write $\mu_i(f) := \max \{n \mid f(n) = i\}$ and $\lambda_i(f) := \min \{n \mid f(n) = i\}$ and $\mu(f) := \max \{n \mid f(n) \neq 0\}$ and $\lambda(f) := \min \{n \mid f(n) \neq 0\}$. Note that $\mu_i, \lambda_i$ are undefined if $i$ is not in the image of $f$. However $\mu, \lambda$ is by definition of $\text{FIN}_k$ always defined.

**Proof of Theorem 5.** This proof is inspired by Theorem 2.2 of [2].

Let $f \in \bigcup_{k \leq k} \text{FIN}_{k+1}$. Fix an $i \geq 1$ and let $(n_0, \ldots, n_i)$ be the indexes (in ascending order) where $f(n_j) = i$. We call $(n_j, n_{j+1})$ a short gap if

$$\emptyset^{(i)} \cap [0; n_j] \neq (\emptyset^{(i-1)})'_{n_{j+1}} \cap [0; n_j].$$

We will write $SG_i(f)$ for the number of short gaps in $f$. Note that in general $SG_i(f)$ is not computable.

We will now construct computable approximation of short gaps. Let $i$, $(n_0, \ldots, n_i)$ as above. We call $(n_j, n_{j+1})$ a very short gap if

$$(3) \quad \emptyset_{\mu_i(f), \mu_{i-1}(f), \ldots, \mu_1(f)}^{(i)} \cap [0; n_j] \neq \emptyset_{\mu_i(f), \mu_{i-1}(f), \ldots, \mu_2(f), n_{j+1}}^{(i)} \cap [0; n_j].$$

(We treat $\mu_i(f)$ as if it were 0 if it is undefined.) E.g. for $i = 1$ we call $(n_j, n_{j+1})$ a very short gap if

$$\emptyset'_{\mu_i(f)} \cap [0; n_j] \neq \emptyset'_{n_{j+1}} \cap [0; n_j].$$

Note that $VSG_i(f)$ is computable.
We color $\FIN_{k+1}$ with the following coloring.

$$c(f) := \sum_{i=1}^{k} 2^{i-1} \cdot (\VSG_i(f) \mod 2).$$

By $\FIN_{k+1}$ there is a homogeneous combinatorial subspace $B$.

We will write $\langle B \rangle$ for $\bigcup_{i=0}^{k} T^i(B)$.

**Claim 1:** For each $f \in \langle B \rangle$ there exists $g \in \langle B \rangle$ with $f < g$ such that every short gap in $f$ is a very short gap in $f + g$ and such that between $f$ and $g$ no gap is (very) short gap and not very short.

**Proof of Claim 1:**

Recursively build a sequence $(g_i)_{i \leq k} \subseteq \langle B \rangle$ with

1. $g_0 > f$ and $g_{i+1} > g_i$,
2. $\emptyset^{(k-i')} \cap [0, \mu(g_i)] = (\emptyset^{(k-i'-1)})'_s \cap [0, \mu(g_i)]$
   for $i' \leq i < k$ and $s \geq \lambda(g_{i+1})$.

The proof proceeds in the same way as the proof of Proposition 6. Suppose we have chosen $g_0, \ldots, g_{i-1}$, then by the limit lemma there exists an $m$ such that the equation in (ii) is true for all $s \geq m$. Choose $g_i$ such that $g_i > g_{i-1}$ and such that $\lambda(g_i) > s'$. Setting

$$m_{k-i} := \mu_{k+1}(g_i) \quad \text{for } i < k$$

we recover a sequence $(m_1, \ldots, m_k)$ as in Proposition 6 From this we get in the same way that

$$\emptyset^{(i)}_{m_1, \ldots, m_{i+1}} \cap [0, m_{i+1}] = \emptyset^{(i)} \cap [0, m_{i+1}].$$

Now consider

$$g := \sum_{i=0}^{k} T^i(g_i).$$

By definition we have that $\mu_{k-i}(g) = \mu_{k-i}(f + g) = m_{k-i}$. Therefore, the right hand side of (ii) for $f + g$ is equal $\emptyset^{(i)}$ for $[0; n_1] \subseteq [0; \mu(f)]$. With this the claim is satisfied.

**Claim 2:** $\SG_i(f)$ is even for each $f \in \langle B \rangle$.

**Proof of Claim 2:** Assume that $f \in \langle B \rangle$ and take again $g$ as in Claim 1. We get

$$\VSG_i(f + g) = \SG_i(f) + \VSG_i(g).$$

Since $f + g, g \in \langle B \rangle$, the parity of $\VSG_i(f + g), \VSG_i(g)$ is the same by assumption to $B$. Therefore, $\SG_i(f)$ must be even.

We now show by induction that one can compute $\emptyset^{(i)}$ for $i \leq k$ from $B$. Assume that we already have an algorithm which computes $\emptyset^{(i-1)}$. To compute whether $x$ is contained in $\emptyset^{(i)}$ or not search for an $f, g \in \langle B \rangle$ with

1. $f < g$,
2. $x < \lambda(f)$, and
3. the image of $f, g$ contains $i$ (this can always be achieved by searching for $f_1 < f_2 \in \langle B \rangle$ and taking $f := f_1 + T^{k+1-i}(f_2)$).

By Claim 2, we know that $\SG_i(f), \SG_i(g), \SG_i(f + g)$ is even. Thus, $(\mu_i(f), \lambda_i(g))$ is not a short gap. (For this argument we use (ii) and the consequence that $\mu_i(f), \lambda_i(g)$ are defined.) Therefore,

$$x \in \emptyset^{(i)} \iff x \in (\emptyset^{(i-1)})'_{\lambda_i(g)}$$
By induction hypothesis $\emptyset^{(i-1)}$ is computable relative to $B$. Therefore $x \in \emptyset^{(i)}$ is computable in $B$, too. □

**Remark 7.** The above proof formalizes in $I\Sigma_{k+2}^0$. The critical steps where induction is used are Claim 1, and the verification of the algorithm. In Claim 1, $BS^0_2$ relative to $\emptyset^{(i)}$ for $i < k$ is used to find the $m$. The analysis of this the same as in the limit lemma and it is equivalent to $B\Sigma_2$. In the verification of the algorithm we have to show that, calling the algorithm for the $i$-Turing jump $e_i$, that

$$\forall x \left( \Phi^B_{e_i}(x) = 1 \leftrightarrow x \in \emptyset^{(i)} \right)$$

for all $i \leq k$. Since this statement is $\Pi^0_{k+2}$, $I\Sigma_{k+2}^0$ is sufficient.

**Corollary 8.**

(i) For all $k$ we have that $\text{RCA}_0 + \text{FIN}_k + \Sigma_{k+2}^0$-IND proves that $\emptyset^{(k)}$ exists.

(ii) $\text{ACA}_0 + \Delta^1_1$-IND + $\text{FIN}_{<\infty} \vdash \forall X \forall k X^{(k)}$ exists. In other words, the above theory proves $\text{ACA}'_0$.

(iii) $\text{ACA}_0 \not\vdash \text{FIN}_{<\infty}$.

**Proof.** (i) is just a reformulation of Remark 7. (ii) follows from (i) by noting that $\Delta^1_1$-IND implies $\Sigma^0_n$-IND uniformly for each $n$ using Skolemization. (iii) follows from the following. First, $\forall X, k X^{(k)}$ exists can be written as a $\Pi^1_2$-statement, i.e.,

$$\forall X, k \exists Y (Y_0 = X \land \forall i < k Y_{i+1} = \text{TJ}(Y_i))$$

Now if $\text{FIN}_{<\infty}$ would be provable in $\text{ACA}_0$ then the above statement would be provable already in $\text{ACA}_0 + \Delta^1_1$-IND and a fortiori in $\text{ACA}_0 + \Sigma^0_\infty$-CA. However, this theory is $\Pi^1_2$-conservative over $\text{ACA}_0$, see [7, IX.4.4], which leads to the contradiction $\text{ACA}_0 \not\vdash \text{ACA}'_0$. □

4. **Conclusion**

We could show that the generalization of Hindman’s theorem (HT), Gowers’ FIN$_k$ theorem (FIN$_{<\infty}$) is stronger than the best known lower bound for HT. It remains open to find a matching upper bound for FIN$_{<\infty}$. It seems to be in general very difficult since to the knowledge of the author any known proofs of FIN$_{<\infty}$ makes use of special ultrafilters (or similar objects). Of course by Shoenfield absoluteness FIN$_{<\infty}$ must be provable without the axiom of choice.

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Department of Mathematics, Faculty of Science, National University of Singapore, Block S17, 10 Lower Kent Ridge Road, Singapore 119076

E-mail address: matkaps@nus.edu.sg

URL: [http://aleph.one/matkaps/](http://aleph.one/matkaps/)