Cost Design in Atomic Routing Games

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Abstract—An atomic routing game is a multiplayer game on a directed graph. Each player in the game chooses a path—a sequence of links that connect its origin node to its destination node—with the lowest cost, where the cost of each link is a function of all players’ choices. We develop a novel numerical method to design the link cost function in atomic routing games such that the players’ choices at the Nash equilibrium minimize a given smooth performance function. This method first approximates the nonsmooth Nash equilibrium conditions with smooth ones, then iteratively improves the link cost function via implicit differentiation. We demonstrate the application of this method to atomic routing games that model noncooperative agents navigating in grid worlds.

I. INTRODUCTION

A routing game is a multiplayer game on a directed graph that contains a collection of nodes and links. Each player in the game chooses a sequence of links, which together form a path, that connect its origin node to its destination node—with the lowest cost, where the cost of each link is a function of all players’ choices of paths. The Nash equilibrium of this game is a collective path choice where no player can obtain a lower cost by unilaterally switching to an alternative path. Routing games are the fundamental mathematical models for predicting the collective behavior of selfish players in communication and transportation networks [1], [2], [3], [4], [5], [6].

Cost design—also known as network design—is the problem of designing the link cost function of a routing game so that the Nash equilibrium satisfies certain desired properties, e.g., matching a desired equilibrium pattern or minimizing the total cost of all players. Cost design is the key to modifying unwanted traffic patterns in congested transportation networks [7], [8], [9], [10], [11], [12].

Existing results on cost design are limited to nonatomic routing games, where the number of players is assumed to be infinite and each player is a negligible fraction of the entire player population [5]. Although reasonable for applications with a large number of players—e.g., predicting the traffic patterns of thousands of vehicles [6]—such an assumption is not valid for applications in multi-agent systems with a medium number of agents, where each player is no longer negligible compared with the entire player population.

We develop a numerical method for cost design in atomic routing games, where the number of players is finite and each player searches for the shortest path in a directed connected graph. We first show that the Nash equilibrium conditions in atomic routing games are equivalent to a set of nonsmooth piecewise linear equations. Next, we develop an approximate projected gradient method to design the link cost function such that the Nash equilibrium of the game minimizes a given smooth performance function. In each iteration, this method first approximates the nonsmooth Nash equilibrium conditions with a set of smooth nonlinear equations, then updates the link cost function via an approximate projected gradient method based on implicit differentiation. We demonstrate the application of this method to atomic routing games that model noncooperative agents navigating in grid worlds.

Our results extend a previous cost design method for matrix games [13] to atomic routing games while remaining scalable. In particular, the results in [13] require enumerating all the options of each player. In an atomic routing game, the number of these options equals the number of all possible paths for all players, which scales exponentially with the number of nodes and links of the underlying graph. In contrast, the proposed method does not require such enumeration, and the number of equations needed scales linearly with the number of nodes and links of the underlying graph.

Notation: We let $\mathbb{R}$, $\mathbb{R}_{\geq 0}$, $\mathbb{R}_{>0}$, and $\mathbb{N}$ denote the set of real, nonnegative real, positive real, and natural numbers, respectively. Given $m$, $n \in \mathbb{N}$, we let $\mathbb{R}^n$ and $\mathbb{R}^{m \times n}$ denote the set of $n$-dimensional real vectors and $m \times n$ real matrices; we let $\mathbf{1}_n$ and $I_n$ denote the $n$-dimensional vector of all 1’s and the $n \times n$ identity matrix, respectively. Given positive integer $n \in \mathbb{N}$, we let $[n] := \{1, 2, \ldots, n\}$ denote the set of positive integers less or equal to $n$. Given $x \in \mathbb{R}^n$ and $k \in [n]$, we let $[x]_k$ denote the $k$-th element of vector $x$, and $\|x\|$ denote the $\ell_2$-norm of $x$. Given a square real matrix $A \in \mathbb{R}^{n \times n}$, we let $A^T$, $A^{-1}$, and $A^{-T}$ denote the transpose, the inverse, and the transpose of the inverse of matrix $A$, respectively; we say $A \succeq 0$ and $A \succ 0$ if $A$ is symmetric positive semidefinite and symmetric positive definite, respectively; we let $\|A\|_F$ denote the Frobenius norm of matrix $A$. Given continuously differentiable functions $f : \mathbb{R}^n \to \mathbb{R}$ and $G : \mathbb{R}^n \to \mathbb{R}^m$, we let $\nabla_x f(x) \in \mathbb{R}^n$ denote the gradient of $f$ evaluated at $x \in \mathbb{R}^n$; the $k$-th element of $\nabla_x f(x)$ is $\frac{\partial f(x)}{\partial x_k}$. Furthermore, we let $\partial_x G(x) \in \mathbb{R}^{m \times n}$ denote the Jacobian of function $G$ evaluated at $x \in \mathbb{R}^n$; the $ij$-th element of matrix $\partial_x G(x)$ is $\frac{\partial G(x)_i}{\partial x_j}$.
II. ATOMIC ROUTING GAMES AND ITS APPROXIMATION VIA ENTROPY REGULARIZATION

We first introduce the mathematical model for atomic routing games on a directed graph, followed by a smooth approximation of the Nash equilibria in atomic routing games. These results provide the foundation of the cost design results in the next section.

A. Directed graphs

A directed graph $\mathcal{G}$ contains a set of nodes $[n]$, a set of directed links $[m]$. Each link is an ordered pair of distinct nodes, where the first and second node are the “tail” and “head” of the link, respectively. We characterize the connection among different nodes in graph $\mathcal{G}$ via the incidence matrix, denoted by $E \in \mathbb{R}^{n \times m}$. The entry $[E]_{ij}$ in matrix $E$ is associated with node $i$ and link $j$ as follows:

$$[E]_{ij} = \begin{cases} 
1, & \text{if node } i \text{ is the tail of link } j, \\
-1, & \text{if node } i \text{ is the head of link } j, \\
0, & \text{otherwise}. 
\end{cases}$$

B. Atomic routing games

Given the incidence matrix $E \in \mathbb{R}^{n \times m}$ of a directed graph $\mathcal{G}$, we consider a game with $p \in \mathbb{N}$ players. Player $i \in [p]$ has an origin node $o_i \in [n]$ and a destination node $d_i \in [n]$ in graph $\mathcal{G}$. We define the key components of this game as follows.

1) Players’ path choices: Each player $i \in [p]$ chooses a path with the lowest cost that connects its origin node $o_i$ and destination node $d_i$. We represent the path chosen by player $i$ via a flow vector $x_i \in \mathbb{R}^m$, where $[x_i]_k \in [0,1]$ denotes the probability for player $i$ to use link $k$ on the path it chooses.

If vector $x_i$ denotes a path that connects node $o_i$ and $d_i$, then it belongs to certain feasible flow set, which we define as follows. Let $r_i \in \mathbb{R}^n$ denote the origin-destination vector of player $i$ such that $[r_i]_j = 1$ if $j = o_i$; $[r_i]_j = -1$ if $j = d_i$; and $[r_i]_j = 0$ if $j \neq o_i$ and $j \neq d_i$. Furthermore, let $s_i \in \mathbb{R}^{n-1}$ denote the reduced origin-destination vector as

$$s_i = \Gamma(r_i, d_i),$$

where $\Gamma(r_i, d_i) \in \mathbb{R}^{n-1}$ is vector obtained from vector $r_i$ by deleting the $d_i$-th element in $r_i$. We also use the notion of reduced incidence matrix for player $i$, defined as follows:

$$E_i := \Gamma(E, d_i),$$

where $E_i \in \mathbb{R}^{(n-1) \times m}$ is the matrix obtained from matrix $E$ by deleting its $d_i$-th row.

Equipped with the above notations, we define the feasible flow set for player $i$, denoted by $\mathbb{P}_i \subset \mathbb{R}^m$, as

$$\mathbb{P}_i = \{ y \in \mathbb{R}^m | E_i y = s_i, y \geq 0 \}$$

for all $i \in [p]$. Notice that if $x_i \in \mathbb{P}_i$ and we let $e_i^\top$ be the $d_i$-th row of matrix $E_i$, then (3) and (1) together imply that

$$e_i^\top x_i = 1_i^\top E x_i - 1_{n-1}^\top E_i x_i = 0 - 1_{n-1}^\top s_i = -1,$$

where the second step is due to the fact that $1_n^\top E = 0_m$. Therefore, $x_i \in \mathbb{P}_i$ is and only if $E x_i = r_i$. In other words, $x_i \in \mathbb{P}_i$ if and only if $x_i$ denotes a unit flow that originates at node $o_i$, disappear at node $d_i$, and is preserved at any other node.

2) The Nash equilibrium conditions: We assume the cost of each link is a quadratic function of all players’ flow vectors, and each player chooses a path that minimizes this quadratic function. In other words, each player $i$ chooses its flow vector $x_i$ as follows:

$$x_i \in \text{argmin}_{y \in \mathbb{P}_i} \left( b_i + \tfrac{1}{2} C_{ii} y + \sum_{j \neq i} C_{ij} x_j \right)^\top y,$$

where vector $b_i \in \mathbb{R}^m$ defines the nominal link cost, which is independent of the players’ strategies; matrices $C_{ij} \in \mathbb{R}^{m \times m}$ for all $i, j \in [p]$ are matrices such that vector $C_{ij} x_j \in \mathbb{R}^m$ denotes the link cost for player $i$ due to its interaction with player $j$. Here we assume that the link cost is a linear function—rather than general polynomials—of the players’ flow vectors to simplify our further computation.

We introduce the notion of Nash equilibrium in an atomic routing game as follows

**Definition 1.** A joint flow $x := [x_1^\top \ldots x_p^\top]^\top$ is a Nash equilibrium if (6) holds for all $i \in [p]$.

C. Computing Nash equilibria via nonlinear programming

We now discuss how to compute the Nash equilibrium in Definition 1. To this end, we denote the joint flow of all players as

$$x := [x_1^\top x_2^\top \ldots x_p^\top]^\top.$$

We also use the following notation:

$$C := \begin{bmatrix} C_{11} & C_{12} & \ldots & C_{1p} \\ C_{21} & C_{22} & \ldots & C_{2p} \\ \vdots & \vdots & \ddots & \vdots \\ C_{p1} & C_{p2} & \ldots & C_{pp} \end{bmatrix}, \quad b := \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_p \end{bmatrix}, \quad s := \begin{bmatrix} s_1 \\ s_2 \\ \vdots \\ s_p \end{bmatrix},$$

$$E_{[1:p]} := \text{blkdiag}(E_1, E_2, \ldots, E_p),$$

where $E_{[1:p]} := \text{blkdiag}(E_1, E_2, \ldots, E_p)$ is the block diagonal matrix obtained by aligning matrices $E_1, E_2, \ldots, E_p$ along the diagonal of matrix $E_{[1:p]}$.

The following lemma shows how to compute the Nash equilibrium in Definition 1 by solving a set of piecewise linear equations.

**Lemma 1.** Suppose that set $\mathbb{P}_i$ is nonempty and $C_{ii} = C_{ii}^\top \succeq 0$ for all $i \in [p]$. Then (6) holds for all $i \in [p]$ if and only if one of the two following set of conditions holds.

1) There exists $v \in \mathbb{R}^{(n-1)}$ and $u \in \mathbb{R}^m$ such that

$$s = E_{[1:p]} x,$$

$$u = b + C x - E_{[1:p]}^\top v, \quad u^\top x = 0, \quad x, u \geq 0_{pm}.$$  

2) There exists $v \in \mathbb{R}^{(n-1)}$ such that

$$s = E_{[1:p]} x,$$

$$0_{pm} = \min \{ x, b + C x - E_{[1:p]}^\top v \}.$$  

**Proof.** We start with the conditions in (9). Since $C_{ii} = C_{ii}^\top \succeq 0$, and set $\mathbb{P}_i$ is nonempty and described by linear
constraints only, (6) holds if and only if its KKT conditions in (9) hold [14, Thm. 27.8].
Next, we prove that the conditions in (9) are equivalent to those in (10). It suffices to show that (9b) holds for some \( u \in \mathbb{R}^{pm} \) if and only if (10b) holds. Notice that (9b) holds for some \( u \in \mathbb{R}^{pm} \) if and only if
\[
[b + Cx - E_{[1,p]}^T v]_k, x_k \geq 0, \quad [b + Cx - E_{[1,p]}^T v]_k[k] = 0
\]
for all \( k \in [pm] \). Since the condition in (11) is equivalent to \( \min \{|x|, [b + Cx - E_{[1,p]}^T v]_k\} = 0 \), we complete our proof.

As a result of Lemma 1, we can compute the Nash equilibrium in Definition 1 by solving either the piecewise linear equations in (10), or the following optimization problem with a bilinear objective function
\[
\begin{aligned}
\text{minimize} & \quad u^T x \\
\text{subject to} & \quad s = E_{[1,p]} x, \\
& \quad u = b + Cx - E_{[1,p]}^T v, \quad x, u \geq 0_{pm}.
\end{aligned}
\]
In particular, one can verify that the conditions in (9) hold for some \( x, u \in \mathbb{R}^{pm} \) and \( v \in \mathbb{R}^{p(n-1)} \)—or equivalently, (10) hold for some \( x \in \mathbb{R}^{pm} \) and \( v \in \mathbb{R}^{p(n-1)} \)—if and only if the optimal value in optimization (12) equals zero.

D. Approximate Nash equilibria via entropy-regularization

We now introduce an approximation of the Nash equilibrium in Definition 1. The idea is to approximate the nonsmooth piecewise linear equations in (10)—which are difficult to solve in general—with smooth nonlinear ones. To this end, we first introduce the following approximation of optimization (6):
\[
x_i \in \arg \min_{y \in P_i} \left( b_i + \frac{1}{2} C_{ii} y + \sum_{j \neq i} C_{ij} x_j \right)^T y + \lambda y^T \ln(y)
\]
where \( \lambda \in \mathbb{R}_{\geq 0} \) is a nonnegative weight, and \( \ln(y) \) denotes the elementwise natural logarithm of \( y \).

The optimization in (13) approximates the one in (6) by adding an entropy regularization term in the objective function. Similar regularization is common in matrix games [13]. The resulting equilibrium is also known as the quantal response equilibrium [15].

The following results give a nonlinear-equations-based characterization of the condition in (13).

Lemma 2. Suppose that set \( \mathbb{P}_i \cap \mathbb{R}_{\geq 0}^{m_i} \) is nonempty, \( C_{ii} = C_{ii}^T \geq 0, \) and \( \lambda > 0 \). Then (13) holds for all \( i \in [p] \) if and only if there exists \( v \in \mathbb{R}^{p(n-1)} \) such that
\[
\begin{aligned}
s &= E_{[1,p]}^T x, \\
x &= \exp\left(\frac{1}{\lambda} \left( E_{[1,p]}^T v - b - Cx \right) - 1_{pm}\right).
\end{aligned}
\]
Proof. Since \( C_{ii} \geq 0 \) for all \( i \in [p] \), the quadratic objective function in (13) is a convex function of \( y \). The KKT conditions of the optimization in (13) are given by
\[
\begin{aligned}
s_i - E_{[1,p]} x_i &= 0_n, \quad x_i \geq 0, \\
b_i + \sum_{j=1}^n C_{ij} x_j + \lambda \ln(x_i) + \lambda 1_m - E_{[1,p]}^T v_i &= 0_m,
\end{aligned}
\]
for all \( i \in [p] \). Notice that the nonnegativity constraints in set \( \mathbb{P}_i \) are redundant since the logarithm function implies that \( x_i \in \mathbb{R}_{\geq 0}^{m_i} \). Since \( \mathbb{P}_i \cap \mathbb{R}_{\geq 0}^{m_i} \) is nonempty, all the linear constraints in optimization (13) can be satisfied, and we know that (13) holds if and only if (15) holds for some \( v_i \in \mathbb{R}^n \) [14, Thm. 27.8]. The rest of the proof is based on the relation between logarithm and exponential function and the assumption that \( \lambda > 0 \).

In the context of the routing game, Lemma 2 shows the effects of an additional tax in each player’s objective function—which corresponds to the entropy term in (13)—on the resulting Nash equilibrium: instead of the nonsmooth equilibrium conditions in (10), we obtain the smooth equilibrium conditions in (14).

E. Computing approximate Nash equilibria via nonlinear least-squares

Lemma 2 shows that solving a smooth approximation of the Nash equilibrium conditions in (14) is equivalent to solving a set of smooth nonlinear equations, or equivalently, the following nonlinear least-squares problem:
\[
\begin{aligned}
\text{minimize}_{x,v} & \quad \left\| x - \exp\left(\frac{1}{\lambda} \left( E_{[1,p]}^T v - b - Cx \right) - 1_{pm}\right) \right\|^2 \\
& \quad + \left\| E_{[1,p]}^T x - s \right\|^2.
\end{aligned}
\]
However, the question remains whether such a solution exists, and if so, whether it is unique or not. To answer these questions, we first make the following assumption (recall the definition of the incidence matrix \( E \) from Section II-A).

Assumption 1. Set \( \mathbb{P}_i \cap \mathbb{R}^m_{\geq 0} \) is nonempty, \( \lambda > 0, C + C^T \geq 0 \), and \( C_{ii} = C_{ii}^T \) for all \( i \in [n] \), and rank \( E = n - 1 \).

Notice that \( C + C^T \geq 0 \) and \( C_{ii} = C_{ii}^T \) together imply that \( C_{ii} \geq 0 \) for all \( i \in [p] \). We note that rank \( E = n - 1 \) if \( E \) corresponds to a directed graph obtained by assigning arbitrary directions to the links of a connected undirected graph [16, Thm. 8.3.11].

The following theorem provides sufficient conditions on matrix \( C \) that ensure that the solution for optimization (16) exists and is unique. The proof is based on the notion of diagonally strictly concavity in games [17].

Theorem 1. Suppose that Assumption 1 holds. There exists unique \( x \in \mathbb{R}^{pm} \) and \( v \in \mathbb{R}^{p(n-1)} \) such that (14) holds.

Proof. We first prove that there exists a unique \( x \) that satisfies (14) for some \( v \in \mathbb{R}^{p(n-1)} \). Since (13) implies that \( x_i \) is elementwise strictly positive (due to the logarithm function), Assumption 1 implies that matrix \( C + C^T + \text{diag}(x)^{-1} \) is positive definite. Hence one can show that any \( x \) that satisfies (13) for all \( i \in [p] \) is the unique Nash equilibrium of a \( n \)-player diagonally strict concave game, whose existence and uniqueness follows from [17, Thm. 1] and [17, Thm. 6], respectively. Finally, under Assumption 1, Lemma 2 states that \( x \) satisfies (13) for all \( i \in [p] \) if and only if there exists \( v \in \mathbb{R}^{p(n-1)} \) such that (14) holds.
Next, we prove the uniqueness of \( v \) by contradiction. Let \( x \) be the unique vector that satisfies (14) for some \( v \) \( \in \mathbb{R}^{p(n-1)} \). Suppose that there exists \( v_a \neq v_b \) such that

\[
x = \exp(\frac{1}{\lambda}(E_{[1,p]}^T v_a - b - Cx) - 1_{pm}) = \exp(\frac{1}{\lambda}(E_{[1,p]}^T v_b - b - Cx) - 1_{pm}),
\]

Then one can verify that

\[
E_{[1,p]}^T (v_a - v_b) = 0 \quad (17)
\]

By using the definition of matrix \( E_{[1,p]} \) in (8), we can show that if (17) holds for some \( v_a \neq v_b \), then there exists \( i \in [n] \) such that \( E_i^T z_i = 0 \) for some \( z_i \in \mathbb{R}^{n-1}, z_i \neq 0_{n-1} \). Hence

\[
[z_i^T \ 0] E_i = 0_{n} = 1_{n} E_i,
\]

where \( e_i \) is the \( d_i \)-th row of matrix \( E \), and the second step is due to the definition of matrix \( E \) in (1). Since \( z_i \neq 0_{n-1} \), vector \( 1_n \) and \( [z_i^T \ 0] \) are linearly independent. Based on the definition of \( E_i \) in (3), we conclude that there exist two linearly independent vectors in the kernel of matrix \( E^T \). Therefore, we must have rank \( E = \text{rank} E^T \leq n-2 \), which contradicts the assumption that rank \( E = n-1 \).

\[ \square \]

### III. Cost Design via Implicit Differentiation

We now introduce the cost design problem in atomic routing games. Our task is to design the value of vector \( b \) and matrix \( C \) such that the Nash equilibrium in Definition 1 optimizes certain performance. In particular, we consider the following optimization problem for cost design:

\[
\begin{align*}
\text{minimize} & \quad \psi(x) \\
\text{subject to} & \quad \text{The conditions in (10) hold, \quad (19)} \\
& \quad b \in \mathbb{B}, \ C \in \mathbb{D},
\end{align*}
\]

where \( \psi : \mathbb{R}^{pm} \to \mathbb{R} \) is a continuously differentiable function that evaluates the performance of the Nash equilibrium \( x \), set \( \mathbb{B} \subseteq \mathbb{R}^{pm} \) and \( \mathbb{D} \subseteq \mathbb{R}^{pm \times pm} \) are closed and convex, representing the candidate values for vector \( b \) and matrix \( C \), respectively.

Due to Lemma 2, one can replace the conditions in (10) with those in (9), and show that optimization (19) is a mathematical program with equilibrium constraints, a non-convex nonsmooth optimization problem notoriously difficult to solve [18]. To overcome this difficulty, we consider the following approximation to (19):

\[
\begin{align*}
\text{minimize} & \quad \psi(x) \\
\text{subject to} & \quad \text{The conditions in (14) hold, \quad (20)} \\
& \quad b \in \mathbb{B}, \ C \in \mathbb{D}.
\end{align*}
\]

The above approximation replaces the nonsmooth conditions in (10) with the smooth ones in (14). As the value of \( \lambda \) decreases, such an approximation becomes more accurate, and a solution for optimization (20) becomes a good approximation of a solution for optimization (19).

Next, we discuss how to solve optimization (20) using an approximate projected gradient method based on implicitly differentiating the conditions in (14).
B. Approximate projected gradient method

We present the approximate projected gradient method for optimization (20) in Algorithm 1, where the projection map \( \Pi_B : \mathbb{R}^m \to \mathbb{R}^m \) and \( \Pi_D : \mathbb{R}^{m \times m} \to \mathbb{R}^{m \times m} \) is given by
\[
\Pi_B(b) = \arg\min_{z \in \mathbb{R}} \| z - b \|, \quad \Pi_D(C) = \arg\min_{X \in D} \| X - C \|_F,
\]
for all \( C \in \mathbb{R}^{m \times m} \). At each iteration, this method first solve the nonlinear least-squares problem in (16)—where \( \lambda \) is a tuning parameter—then update matrix \( C \) using the approximate gradient formulas in (24) and a positive step size \( \alpha \) until convergence.

Algorithm 1 Approximate projected gradient method.

**Input:** \( \psi : \mathbb{R}^{pm} \to \mathbb{R}, \lambda, \alpha, \delta \in \mathbb{R}_{>0}, k_{\text{max}} \in \mathbb{N} \).

1. Initialize \( b = 0_{pm} \) and \( C = 0_{pm \times pm} \).
2. while \( k < k_{\text{max}} \) do
   3. Solve optimization (16) for \( x \).
   4. \( b \leftarrow \Pi_B(b - \alpha \nabla_b \psi(x)) \)
   5. \( C \leftarrow \Pi_D(C - \alpha \nabla_C \psi(x)) \)
   6. \( k \leftarrow k + 1 \)
3. end while

**Output:** Vector \( b \) and matrix \( C \).

IV. NUMERICAL EXAMPLES

Although initially designed for optimization (20), Algorithm 1 also provides good solutions for optimization (19) in practice. We demonstrate this phenomenon via numerical examples on atomic routing games on grid worlds.

A. Atomic routing games setup

We consider atomic routing games in Section II-B defined on 2-dimensional grid worlds. In particular, the directed graph of the game is as follows. Each node in the graph corresponds to a grid. The two nodes are connected by a link if and only if the corresponding two grids are adjacent. In particular, a \( 3 \times 3 \) grid world corresponds to a graph with 9 nodes and 24 links; a \( 5 \times 5 \) grid world corresponds to a graph with 25 nodes and 80 links; see Fig. 1 for an illustration.

Fig. 1: An illustration of the origin nodes of players and the optimal paths of the players at the Nash equilibrium before (solid) and after cost design (dashed).

Next, we focus on two cases of atomic routing games on grid worlds: a two-player atomic routing game in a \( 3 \times 3 \) grid world, and a four-player atomic routing game in a \( 5 \times 5 \) grid world. We illustrate the origin grid of each player in these games along with the paths they choose at the Nash equilibrium in Fig. 1.

B. Cost design setup

We make the following choices of parameters in optimization (19). We let
\[
\begin{align*}
\mathbb{B} & := \{ b \in \mathbb{R}^{pm} | 0_{pm} \leq b \leq \delta 1_{pm} \}, \\
\mathbb{D} & := \{ C \in \mathbb{R}^{pm \times pm} | C + C^T \succeq 0, \| C \|_F \leq \rho \},
\end{align*}
\]
where matrix \( C \) and its \( i \)-th diagonal block \( C_{ii} \) are associated by \( \delta = 0.1 \) and \( \epsilon = 0.01 \) in Algorithm 1; we let \( \pi(b, C) \) denote the solution of optimization (12) computed by IPOPT [21] with default accuracy tolerances. Fig. 2 shows the convergence of \( \pi(b, C) \) to the desired equilibrium \( \hat{x} \), where we fix \( \rho = 0.5 \) in (21) and \( \alpha = 0.005 \) in Algorithm 1. We also illustrate the the cost of the two paths illustrated in Fig. 1 before and after the iterations in Algorithm 1 with \( \lambda = 0.005 \). These results show that Algorithm 1 successfully changes the paths of the players at the Nash equilibrium to the desired ones.

Although our experiments do not explicitly showcase the differences between the solution of optimization (16) and that of optimization (12), they confirmed that optimization (16) is a valid proxy for optimization (12) for the purpose of cost design. In particular, Algorithm 1 designs the value of vector \( b \) and matrix \( C \) by differentiating optimization (16). The resulting vector \( b \) and matrix \( C \), on the other hand, cause the solution of optimization (12)—denoted by \( \pi(b, C) \), which satisfies the exact Nash equilibrium conditions in (9)—to match the desired value \( \hat{x} \), as shown by Fig. 2 and Fig. 4.

Optimization (13) has a numerical limitation. As \( \lambda \) decreases, the numerical values in optimization (13) increases rapidly. As a result, the value of \( \lambda \) is limited to larger than \( 10^{-4} \) in practice to avoid integer overflow. In other words, although optimization (13) provides an arbitrarily accurate approximation for optimization (12) as \( \lambda \) decreases, there is a numerical bottleneck of the quality of this approximation.

We also demonstrate the effects of the parameter \( \rho \) on the Nash equilibrium \( \pi(b, C) \) in Fig. 4, where vector \( b \) and matrix \( C \) are computed by Algorithm 1 with \( \alpha = \lambda = 0.01 \).
These results confirm an intuition: the larger $\rho$ is, the more change in matrix $C$ is allowed in Algorithm 1, the closer to the desired value is the Nash equilibrium.

Fig. 4: The effects of parameter $\rho$ on the Nash equilibrium corresponding to the output of Algorithm 1.

V. CONCLUSION

We developed an approximate projected gradient method to design the link cost function in atomic routing games such that the Nash equilibrium minimizes a given smooth function. However, the current work is limited to linear link cost and routing in deterministic network. For future work, we consider extensions to routing games with polynomial link cost, stochastic dynamic network routing, and stochastic user equilibrium under noisy cost perception.