Simplified calculations of plasma oscillations with non-extensive statistics

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Abstract. We use the exponential parametrization of the nonextensive distribution to calculate the dielectric constant in an electron gas obeying the nonextensive statistics. As we show, the exponential parametrization allows us to make such calculations in a straightforward way, bypassing the use of intricate formulas obtained from integral tables and/or numerical methods. For illustrative purposes, we apply first the method to the calculation of the permittivity and the corresponding dispersion relation in the ultrarelativistic limit of the electron gas, and verify that it reproduces in a simple way the results that had been obtained previously by other authors using the standard parametrization of the nonextensive distribution. In the same spirit we revisit the calculation of the same quantities for a non-relativistic gas, in the high frequency limit, which has been previously carried out, first by Lima et al., and subsequently revised by Chen and Li. Our own results agree with those obtained by Chen and Li. For completeness, we also apply the method the low frequency limit in the non-relativistic case, which has been previously considered by Dai et al. in the context of the stream plasma instability. We discuss some features of the results obtained in each case and their interpretation of terms of generalized nonextensive quantities, such as the Debye length $\lambda_D$, the plasma frequency $\omega_p$ and the ultra-relativistic frequency $\Omega_{\text{rel}}$. In the limit $q \to 1$ such quantities reduce to their classical value and the classical result of the dispersion relations are reproduced.

1 Introduction and summary

The formalism of the nonextensive statistics [1–8] have been applied to a variety of physical systems and continues to be an active field of research. This formalism has been studied in physical systems where deviations from local thermodynamic equilibrium occur, for example in the context of heavy ion collisions [9] and high energy hadron collisions [10].

Recently, the formalism has been applied in the context of plasma physics to study the dielectric properties of a collisionless electron gas, assuming that the underlying statistics of the background particles obey the nonextensive, rather than the classical Boltzmann statistics [11–17]. In particular, the nonextensive dielectric permittivity in the high frequency limit, and the corresponding dispersion relation, were calculated in reference [11] for the case of the non-relativistic (NR) electron gas. That calculation has been revised in reference [12], where a different result was obtained for those same quantities. In general, the calculations involving the nonextensive statistics are cumbersome, and usually require the use of intricate formulas from integral tables. This inconvenience becomes more pronounced when conflicting results are obtained, such as those mentioned above.

In the present work we take another look at the calculation of the nonextensive dielectric permittivity and dispersion relation of the collisionless electron gas using a different method. Our approach consists in using an exponential parametrization of the nonextensive momentum distribution function. Specifically, as will be discussed in detail in Section 2, instead of the expression

$$e_q(x) = \frac{1}{\Gamma\left(\frac{1}{1-q}\right)} \int_0^\infty dt \left(\frac{t}{1-t}\right)^{1-1/q} e^{-[1+(1-q)x]t},$$

(1)

that is commonly used [1] for the non-extensive distribution function, written in terms of generic variable $x$, we write $e_q$ in the form

$$e_q(x) = \frac{1}{\Gamma\left(\frac{1}{1-q}\right)} \int_0^\infty dt \left(\frac{t}{1-t}\right)^{1-1/q} e^{-[1+(1-q)x]t}.$$
The parameter \( q \), which characterizes the non-extensive distribution function, must be restricted to be within specific ranges, to be discussed in detail below. As we will see, this allows us to calculate such quantities in a straightforward fashion, in particular avoiding the need of making extensive use of intricate integral table formulas. Our objective is twofold. On one hand, to verify that the previously known results are easily reproduced in this way, and on the other to show how to extend the previous calculations by considering other limiting cases of practical interest.

To this end, we first consider the case of an ultrarelativistic (ER) electron plasma, which has been already studied in references \([14,15]\). Those previous calculations were based on using directly the standard form of the nonextensive distribution (Eq. (1)) provoking either the heavy use of intricate formulas from integral tables or numerical methods. Here we perform the relevant calculations using the exponential parametrization method. As we show, the calculations become straightforward in this manner and illustrate the effectiveness of the method to treat this kind of problem. The results we obtain for the dielectric permittivity and the dispersion relation coincide with those in references \([14,15]\). In the process we identify the generalized relativistic electron plasma frequency \( \Gamma_{e, \text{rel}}^{(q)} \), the temperature \( T_q \), and the dielectric permittivity. All such physical quantities reduce to their classical counterparts in the limit \( q \to 1 \), as expected. This first result illustrates the effectiveness of the method we propose, and provides confidence on its application.

We then consider two other cases, namely the high and the low frequency limit of the non-relativistic (NR) plasma. As we have mentioned, the calculation of the dielectric permittivity in the high frequency limit has already been considered using the standard parametrization of the nonextensive statistics distribution in references \([11,12]\), with conflicting results. Our own result agrees with the result obtained in reference \([12]\) for that same case. For completeness, we also consider the NR case in the low frequency limit, which has been considered before in reference \([16]\) in the context of the stream instability induced by ions and electrons with different drift velocities in a dusty plasma. The dielectric permittivity in that limit can be deduced from the results presented in reference \([16]\) by adapting and restricting the attention to the electron contribution, and our own results agree with it. Since the system considered in reference \([16]\) is a complicated one and the calculation is involved, this case once again illustrates the effectiveness of the exponential parametrization method for this kind of calculation. For example, we show that the next-to-leading order contribution to the dielectric permittivity (in powers of \( \omega/k \)), which is not considered in reference \([16]\), is obtained equally simply. We identify a temperature parameter \( T_q \), in analogy with the high frequency limit case, as well as a parameter \( \lambda_D^q \) that is analogous to the Debye length of a classical plasma, both of which depend on the parameter \( q \) that characterizes the nonextensive distribution. With such identifications, the resulting formula for the dielectric permittivity in the low frequency limit can be expressed in the same form as the classical case, but with the parameters \( T_q \) and \( \lambda_D^q \) replacing the temperature and Debye length, respectively. In the limit \( q \to 1 \), both \( T_q \) and \( \lambda_D^q \) reduce to their classical value, and therefore the well-known classical results for a Maxwellian plasma \([18]\) are recovered, as it should be.

In Section 2 we define the exponential parametrization of the nonextensive distribution. There we also define the analog of the so-called marginal distribution that enters in the calculation of the dielectric permittivity, and we discuss some subtleties that must be kept in mind when using the concept of the marginal distribution in the nonextensive case. In Section 3, we consider the calculation of the (longitudinal) dielectric permittivity in various cases that we have mentioned. We first consider the case of the ultrarelativistic plasma, and then the non-relativistic plasma in both the low and high frequency limits. Finally Section 4 contains our conclusions.

### 2 Exponential parametrization of the nonextensive distribution

Up to a normalization factor \( C_q \), to be determined below, we write the nonextensive momentum distribution function in the form

\[
f_q(p) = C_q \frac{1}{\Gamma(1/q)} \int_0^\infty dt \left(\frac{t}{1-q}\right)^{1/q-1} e^{-[1+(1-q)\beta E_p] t},
\]

where \( E_p \) is the kinetic energy of a particle with momentum \( p \) and \( 1/\beta = k_B T \), with \( T \) being the temperature of the system. In writing equation (3) we have made use of the following identity for the Gamma function,

\[
A^{-\ell} = \frac{1}{\Gamma(\ell)} \int_0^\infty dt \, t^{\ell-1} e^{-At},
\]

which is valid if \( A > 0 \), and for \( A = 1 \) it becomes just the integral formula for the Gamma function. In equation (3) we have used it with the identification \( A = (1+1/q)βE_p \) and \( \ell = 1/(1-q) \). This parametrization is valid in the range \( 0 < q < 1 \). As we show below, the lower limit is further restricted by the requirement that the distribution function can be normalized.

The working assumption involved is that in the calculations of the quantities of interest we can interchange the order of the integration over \( t \) with the integration over the momentum variables of the particle number distributions. It must be kept in mind that, if the limit \( q \to 1 \) is desired, it must be taken after performing the integration over \( t \). The factor \( C_q \) is determined by the normalization condition

\[
n = \int \frac{d^3 p}{(2\pi)^3} f_q(p),
\]

where \( n \) is the number density of the electrons. In the general case, for a given momentum \( p \), the kinetic energy \( E_p \) in equation (3) is given by

\[
E_p = \sqrt{m^2 + p^2} - m,
\]
where \( m \) is the electron mass. Here we consider, separately, either the non-relativistic (NR) or the extremely-relativistic (ER) limit,

\[
E_p = \begin{cases} \frac{p^2}{2m} & \text{(NR)} \\ p^2 & \text{(ER)} \end{cases}
\]

where

\[ p \equiv |p|. \tag{8} \]

### 2.1 NR case

In this case, equation (5) implies that \( C_q \) is given by

\[
\frac{1}{C_q} = \frac{1}{n} \frac{1}{\Gamma(1-q)} \int_0^\infty dt t^{\frac{1}{1-q}-1} \times \int \frac{d^3p}{(2\pi)^3} e^{-[1+(1-q)\beta^2 p^2/t]} t. \tag{9}
\]

Performing the Gaussian integration with respect to each momentum coordinate,

\[
\frac{1}{C_q} = \frac{1}{n} \left( \frac{m}{2\pi\beta} \right)^{3/2} \frac{1}{\Gamma(1-q)} \left( \frac{1-q}{1-q} \right)^{3/2} \times \int_0^\infty dt t^{\left(\frac{1}{1-q} - \frac{3}{2}\right)} e^{-t}, \tag{10}
\]

and finally performing the integration over \( t \) leads to

\[ C_q = n \left( \frac{2\pi\beta}{m} \right)^{3/2} \left( \frac{1}{\Gamma(1-q)} \left( \frac{1-q}{1-q} \right)^{3/2} \right). \tag{11} \]

In summary, in the non-relativistic case, the exponential parametrization of the nonextensive distribution function, properly normalized, is given by

\[
f_q(p) = n \left( \frac{2\pi\beta}{m} \right)^{3/2} \left( \frac{1-q}{\Gamma(1-q)} \left( \frac{1-q}{1-q} \right)^{3/2} \right) \times \int_0^\infty dt t^{\left(\frac{1}{1-q} - 1\right)} e^{-[1+(1-q)\beta^2 p^2/t]} t. \tag{12}
\]

In the context of the calculations of the dielectric permittivity, it is customary to decompose \( \mathbf{p} \) into its components parallel \( (p_\parallel) \) and perpendicular \( (p_\perp) \) to the wave vector \( \mathbf{k} \) of the electromagnetic wave, since the integral formula for the dielectric permittivity depends only on \( p_\parallel \). It is then convenient to introduce the *marginal* distribution (see, e.g., Ref. [12], Eq. (18))

\[
f_q^{\parallel}(p_\parallel) = \int \frac{d^2p_\perp}{(2\pi)^2} f_q(p), \tag{13}
\]

in analogy with the procedure in the standard case (see, e.g., Ref. [18], p. 123).

Using equation (12) and performing the Gaussian integrals over \( p_\perp \), we obtain the following parametrization for the marginal distribution in the nonextensive case

\[
f_q^{\parallel}(p_\parallel) = n \left( \frac{2\pi\beta}{m} \right)^{1/2} \left( \frac{1-q}{\Gamma(1-q/3)} \right) \int_0^\infty dt F_q(p_\parallel, t), \tag{14}
\]

where

\[
F_q(p_\parallel, t) = \left( \frac{1}{1-q} \right) e^{-\left[1+(1-q)\beta p_\parallel^2/(2m)\right]t}. \tag{15}
\]

Equation (14) is the formula that we will use in the calculations. By construction, i.e., equation (13), it satisfies the normalization condition

\[
\int_{-\infty}^\infty \frac{dp_\parallel}{2\pi} f_q^{\parallel}(p_\parallel) = n. \tag{16}
\]

It is appropriate to comment at this point the following. It is not correct to construct \( f_q^{\parallel}(p_\parallel) \) by taking it to be the generalization of the standard marginal distribution, i.e., take \( f_q^{\parallel}(p_\parallel) \) to be

\[
f_q^{\parallel}(p_\parallel) \propto e_q \left( -\beta p_\parallel^2 / 2m \right). \tag{17}
\]

That would give an expression of the form

\[
\left( f_q^{\parallel}(p_\parallel) \right)_{\text{incorrect}} = C_q' \frac{1}{\Gamma(1-q)} \int_0^\infty dt t^{1/2} \times e^{-\left[1+(1-q)\beta p_\parallel^2 / 2m\right]t}, \tag{18}
\]

which is incorrect even if \( C_q' \) is chosen such that it is normalized according to equation (16). This is ultimately a consequence of the fact that the generalized exponential function \( e_q(A) \) does not satisfy the relation

\[ e(A + B) = e(A) e(B), \tag{19} \]

that the exponential function does. Therefore, a calculation in the nonextensive case, based on a marginal distribution assumed to be the generalization of the standard classical marginal distribution leads to incorrect results.

### 2.2 ER case

In this case the normalization condition implies that \( C_q \) is given by

\[
\frac{1}{C_q} = \frac{1}{n} \frac{1}{\Gamma(1-q)} \int_0^\infty t^{\frac{1}{1-q}-1} e^{-t} \int \frac{d^3p}{(2\pi)^3} e^{-\left(1-q)\beta p^2\right} t. \tag{20}
\]

Straightforward evaluation of both integrals then yields

\[
C_q = \frac{n\beta^3}{8\pi} (1-q)^3 \frac{\Gamma(\frac{1}{1-q})}{\Gamma(\frac{1}{1-q} - 3)}, \tag{21}
\]

\(^2\) We use natural units throughout, and therefore we set \( c = 1 \) from now on.
and therefore
\[ f_q(p) = \frac{n\beta^3}{8\pi} \frac{(1 - q)^3}{\Gamma\left(\frac{1}{1 - q} - 3\right)} \int_0^\infty dt t^{\frac{1}{1 - q} - 1} e^{-[1 + (1 - q)\beta p]t}. \] (22)

### 2.3 Valid range of $q$

As already mentioned above, although the exponential parametrization is valid for $q$ in the range $0 < q < 1$, the lower limit is further restricted by the requirement that the distribution function can be normalized. Since, for real $x$, $\Gamma(x)$ is defined provided $x > 0$, looking at equations (12) and (22) we see that $q$ must be restricted such that
\[ \frac{1}{1 - q} - \frac{3}{2} > 0 \] (NR case) (23)
or\[ \frac{1}{1 - q} - 3 > 0 \] (ER case) (24)
otherwise the distribution functions have non-physical value or, stated differently, the distribution function in equation (3) cannot be normalized. Equations (23) and (24) yield the restrictions that we adopt throughout,
\[ q > \frac{1}{3} \] (NR case) (25)
or\[ q > \frac{2}{3} \] (ER case). (26)

### 3 Longitudinal dielectric permittivity

We now consider the application of the exponential parametrization of the nonextensive distribution to the calculation of the longitudinal dielectric permittivity and dispersion relation. The starting point is the standard formula for the longitudinal dielectric permittivity for a collisionless electron plasma for a wave with wave vector $\mathbf{k}$ and frequency $\omega$ [18,19],
\[ \epsilon_\ell = 1 - \frac{4\pi e^2}{k^2} \int \frac{d^3p}{(2\pi)^3} \frac{\mathbf{k} \cdot \nabla_p f_q(p)}{\mathbf{k} \cdot \mathbf{v} - \omega + i\epsilon}, \] (27)
where $\mathbf{v} = \mathbf{p}/m$ is the electron velocity. As usual, the formula includes the factor $i\epsilon$ for defining the function at the pole $\omega = \mathbf{k} \cdot \mathbf{v}$ according to the Landau rule. We consider the evaluation of equation (27), separately depending on whether the gas is non-relativistic or ultrarelativistic.

#### 3.1 Ultra-relativistic (ER) case

In this case we set $v \to 1$ (remembering our convention of using natural units throughout, as we already stated).

Denoting by $\theta$ the angle between $\mathbf{p}$ and $\mathbf{k}$, in this case we then have
\[ \mathbf{k} \cdot \mathbf{v} = k \cos \theta, \] (28)
where
\[ k = |\mathbf{k}|, \] (29)
and
\[ \mathbf{k} \cdot \nabla_p f_q = k \cos \theta \frac{\partial f_q}{\partial p}, \] (30)
where, from equation (22),
\[ \frac{\partial f_q}{\partial p} = -\frac{n\beta^4}{8\pi} \frac{(1 - q)^4}{\Gamma\left(\frac{1}{1 - q} - 3\right)} \int_0^\infty dt t^{\frac{1}{1 - q} - 1} e^{-[1 + (1 - q)\beta p]t}. \] (31)

Therefore, from equation (27)
\[ \epsilon_\ell = 1 - \frac{8\pi^2 e^2}{k^2} \int dp p^2 d(cos \theta) \frac{\partial f_q(p)}{\partial p} \frac{\cos \theta}{k \cos \theta - \omega - i\epsilon} \]
\[ = 1 + \frac{n\pi e^2 \beta^4}{k} \frac{(1 - q)^4}{\Gamma\left(\frac{1}{1 - q} - 3\right)} \left[ \int_{-1}^1 d\cos \theta \frac{\cos \theta}{k \cos \theta - \omega - i\epsilon} \right] \]
\[ \times \int_0^\infty dt t^{\frac{1}{1 - q} - 1} \int_0^\infty dp p^2 e^{-(1 - q)\beta pt}. \] (32)

The integral over $p$ is
\[ \int_0^\infty dp p^2 e^{-(1 - q)\beta pt} = \frac{2}{(1 - q)^3 \beta^3 t^3}, \] (33)
and for the resulting integral over $t$ we use
\[ \int_0^\infty dt t^{\frac{1}{1 - q} - 1} e^{-t} = \Gamma\left(\frac{1}{1 - q} - 2\right). \] (34)

Finally, the integral over the angle $\theta$ is evaluated by rewriting it in the form
\[ \frac{1}{k^2} \int_{-1}^1 dx x - \frac{x}{k} - i\epsilon, \] (35)
and using the identity (see [18])
\[ \frac{1}{(x - a)^2 + i\epsilon} = P \frac{1}{x - a} \pm i\pi \delta(x - a), \] (36)
where $\delta(x - a)$ is the Dirac delta function and $P$ denotes the principal value. Thus we obtain
\[ \epsilon_\ell = 1 + \frac{4\pi e^2 n}{k^2 T_e} \left(3q - 2\right) \left\{ 1 + \frac{\omega}{2k} \log \left|\frac{\omega - k}{\omega + k}\right| + i\frac{\pi \omega}{2k} \right\}, \] (37)
where we have put
\[ T_e \equiv \frac{1}{\beta} = k_B T. \] (38)
As anticipated in Section 1, the result given in equation (37) coincides with the result obtained previously in reference [15] with the commonly-used parametrization of the non-extensive distribution. Equation (37) corresponds to the formula obtained by making the replacement

\[ T \to \frac{T}{3q - 2}, \tag{39} \]

in the well-known result for the classical Maxwell-Boltzman case (see, e.g., Ref. [18], pp. 132). By the same token, the dispersion relations in the present case, defined by the condition \( \epsilon_\ell = 0 \), can be obtained from the corresponding dispersion relations of the classical Maxwell-Boltzman case by making the same replacement given in equation (39). For example, in the limit \( \omega \gg k \),

\[ \omega_k^2 = (\Omega_{\epsilon, \text{rel}}^{(q)})^2 + \frac{3}{5} k^2 \tag{40} \]

where

\[ (\Omega_{\epsilon, \text{rel}}^{(q)})^2 = \frac{4\pi e^2 n}{3T_\epsilon} (3q - 2). \tag{41} \]

Evidently, equation (37) and the dispersion relations reduce to the results for the classical Maxwell-Boltzman case in the limit \( q \to 1 \).

As already mentioned above, the result given in equation (37) coincides with the result obtained previously in reference [15]. The ability to reproduce the result, coupled with the ease with which it has been obtained here, gives us confidence in the capability of the exponential parametrization method for this kind of calculation.

### 3.2 Non-relativistic (NR) case

In this case a useful expression for the dielectric constant in terms of an integral over the marginal distribution can be obtained as follows. Using equation (13), the formula for \( \epsilon_{\ell} \) given in equation (27) becomes

\[ \epsilon_{\ell} = 1 - \frac{4\pi e^2}{k^2} \int_{-\infty}^{\infty} \frac{dp ||}{2\pi} \frac{k \frac{\partial F_q^{(q)}}{\partial p ||}}{kp ||/m - \omega - i\epsilon}, \tag{42} \]

where we have used

\[ k \cdot p = kp ||. \tag{43} \]

Using equations (14) and (15), together with

\[ \frac{\partial F_q}{\partial p ||} = -\frac{(1 - q)\beta p ||}{m} F_q, \tag{44} \]

it follows that

\[ \epsilon_{\ell} = 1 + \frac{4\pi e^2 n\delta}{k^2} \int_0^\infty dt \left( \frac{1}{1 - q} \right) e^{-t} I, \tag{45} \]

where we have defined

\[ I \equiv (2\pi)^{1/2} k \int_{-\infty}^{\infty} \frac{dp ||}{2\pi} \frac{p || e^{-(1-q)\beta \frac{p ||^2}{2m} t}}{kp ||/m - \omega - i\epsilon}, \tag{46} \]

and

\[ \delta = \frac{\left( \frac{\beta}{m} \right)^{3/2} (1 - q)^{3/2}}{\Gamma \left( \frac{1}{1-q} - \frac{3}{2} \right)}. \tag{47} \]

Using the identity equation (36) once again, we obtain

\[ I = I^{(r)} + i \sqrt{\frac{m^2}{2} k} e^{-(1-q)\beta \frac{m^2}{2e^2} t}, \tag{48} \]

with

\[ I^{(r)} = (2\pi)^{1/2} kP \int_{-\infty}^{\infty} \frac{dp ||}{2\pi} \frac{e^{-(1-q)\beta \frac{p ||^2}{2m} t}}{1 - kp ||/m - \omega}. \tag{49} \]

In what follows we focus exclusively on the real part of the dielectric permittivity, which we denote by \( \epsilon^{(r)}_{\ell} \). From equations (45), (48) and (49), it is given by

\[ \epsilon^{(r)}_{\ell} = 1 + \frac{4\pi e^2 n\delta}{k^2} \int_0^\infty dt \left( \frac{1}{1 - q} \right) e^{-t} I^{(r)}, \tag{50} \]

with \( \delta \) given in equation (47).

Equation (50) is our basic integral formula using the exponential parametrization of the nonextensive distribution in the NR case, which can be evaluated explicitly by considering special cases or limits of interest. We consider next the explicit evaluation in the high or low frequency limit.

#### 3.2.1 NR High frequency limit

In this case, \( \omega \gg kp ||/m \). We rewrite equation (49) in the form

\[ I^{(r)} = - (2\pi)^{1/2} \left( \frac{k}{\omega} \right) P \int_{-\infty}^{\infty} \frac{dp ||}{2\pi} \frac{e^{-(1-q)\beta \frac{p ||^2}{2m} t}}{1 - kp ||/m - \omega}. \tag{51} \]

The result obtained in reference [12] for this case corresponds to assume that \( \omega, k \) and the parameters of the distribution function are such that the denominator can be expanded in powers of \( kp ||/m^2 \) and then evaluating the resulting integrals term by term. Thus,

\[ I^{(r)} = - (2\pi)^{1/2} \left( \frac{k}{\omega} \right)^2 \int_{-\infty}^{\infty} \frac{dp ||}{2\pi} \left[ p ||^2 + \left( \frac{k}{m\omega} \right)^2 p || \right], \tag{52} \]
where the odd terms in $p_{\parallel}$ have been dropped since they integrate to zero. Performing the Gaussian integration of the remaining ones then yield

$$I^{(r)} = -\left(\frac{m}{\beta}\right)^{\frac{3}{2}} \frac{k^2}{m\omega^2} \times \left\{ t^{-\frac{5}{2}} \left(1 - q\right)^{\frac{3}{2}} + \frac{3}{2m\omega^2} t^{-\frac{5}{2}} \left(1 - q\right)^{\frac{3}{2}} \right\},$$

(53)

Substituting this result in equation (50) and using the definition of the Gamma function (i.e., Eq. (4) with $A = 1$ and $\ell = \frac{1}{1 - q} - \frac{3}{2}$ or $\frac{1}{1 - q} - \frac{5}{2}$) then yields

$$\epsilon_{\ell}^{(r)} = 1 - \frac{\omega_p^2}{\omega^2} \delta \left(\frac{m}{\beta}\right)^{\frac{3}{2}} \times \left[ \Gamma \left(\frac{1}{1 - q} - \frac{3}{2}\right) + \frac{3}{2m\omega^2} \Gamma \left(\frac{1}{1 - q} - \frac{5}{2}\right) \right],$$

(54)

where $\omega_p$ is the plasma frequency,

$$\omega_p^2 = \frac{4\pi^2n}{m}.$$  

(55)

Using equation (47) and remembering that $\Gamma(z) = \Gamma(z + 1)$ [putting in our case $z = \left(\frac{1}{1 - q} - \frac{5}{2}\right)$] we then obtain,

$$\epsilon_{\ell}^{(r)} = 1 - \frac{\omega_p^2}{\omega^2} \left[ 1 + \frac{3k^2v_T^2}{2m\omega^2} \left(\frac{2}{5q - 3}\right) \right].$$

(56)

with

$$v_T = \sqrt{\frac{1}{m\beta}}.$$  

(57)

The corresponding dispersion relation obtained by the condition $\epsilon_{\ell}^{(r)} = 0$ is then

$$\omega_k^2 = \omega_p^2 + 3k^2v_T^2 \left(\frac{2}{5q - 3}\right).$$

(58)

The results in equations (56) and (58) coincide with those obtained by Chen and Li [12]. In the limit $q = 1$, they reduce to the classical formulas. By making contact with a known result, our purpose here has been to show and convince ourselves once more that the exponential parametrization of the nonextensive distribution provides a succinct method for carrying out this type of calculation. To assess the effectiveness of the exponential parametrization method, the reader is invited to consult the corresponding calculation in reference [12] of the result given in equation (38) of that reference.

### 3.2.2 NR Low frequency limit

We now consider the low frequency limit, that is $\omega \ll \frac{k p_{\parallel}}{m}$. As mentioned in Introduction, the formula for this limit case, can be deduced from the calculations of reference [16], who consider the stream instability induced by ions and electrons with different drift velocities in a dusty plasma, by adapting and restricting the attention to the electron contribution. Our own result confirms the result obtained that way. With the exponential parametrization method the formula for this case also follows straightforwardly, and as a byproduct we show that the next-to-leading order contribution to the dielectric permittivity (in powers of $\omega/k$) is determined equally simply, which is not considered in reference [16].

We rewrite equation (49) in the form

$$I^{(r)} = mP \int_{-\infty}^{\infty} \frac{dp_{\parallel}}{\sqrt{2\pi}} p_{\parallel} e^{-\frac{\left(1 - \frac{m\omega_p^2}{k}\right)^2}{2p_{\parallel}}},$$

(59)

Writing $p_{\parallel} = \left(p_{\parallel} - \frac{m\omega_p^2}{k}\right) + \frac{m\omega_p^2}{k}$ in the numerator,

$$I^{(r)} = m^{3/2} \left(\frac{1}{(1 - q)\beta t}\right)^{1/2} + I_1^{(r)},$$

(60)

where

$$I_1^{(r)} = \frac{m^2 \omega}{k} P \int_{-\infty}^{\infty} \frac{dp_{\parallel}}{\sqrt{2\pi}} e^{-\frac{\left(1 - \frac{m\omega_p^2}{k}\right)^2}{2p_{\parallel}}},$$

(61)

and by the change of variable $p_{\parallel} = \frac{m\omega_p^2}{k} + u$,

$$I_1^{(r)} = \frac{m^2 \omega}{k} e^{-\frac{(1 - q)\frac{m\omega_p^2}{k} u}{2}} \times P \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi} u} e^{-\frac{(1 - \frac{m\omega_p^2}{k}) u^2}{2} - \frac{(1 - q)\frac{m\omega_p^2}{k} u}{2}}.$$  

(62)

Expanding the last exponential factor in equation (62) up to the linear term in $u$,

$$e^{-\frac{(1 - q)\frac{m\omega_p^2}{k} u}{2}} \simeq 1 - \frac{1 - q}{\beta \omega u} \frac{m\omega_p^2}{k},$$

(63)

the $1/u$ term is odd and integrates to zero while the remaining integral is a Gaussian. Thus,

$$I_1^{(r)} = -\left(1 - q\right)^{\frac{3m^2 \omega^2}{k^2}} e^{-\frac{(1 - q)\frac{m\omega_p^2}{k} u}{2}} \sqrt{\frac{1}{m\beta}} \left(\frac{m}{1 - q}\right)^{\frac{3m^2 \omega^2}{k^2}} \left(\frac{m}{1 - q}\right)^{\frac{3m^2 \omega^2}{k^2}} \left(\frac{m}{1 - q}\right)^{\frac{3m^2 \omega^2}{k^2}} = -\frac{m^{5/2} \omega}{k^2} \left[ (1 - \beta t)^{1/2} e^{-\frac{(1 - q)\frac{m\omega_p^2}{k} u}{2}} \right],$$

(64)

and using this result in equation (60), we then have

$$I^{(r)} = m^{3/2} \left(\frac{1}{(1 - q)\beta t}\right)^{1/2} \left[ 1 - \frac{m\omega_p^2}{k^2} (1 - q)\beta t \right],$$

(65)
dropping the terms $O\left(\frac{q^4}{r^4}\right)$. Substituting equation (65) in equation (50),

$$
\epsilon^{(r)}_\ell = 1 + \frac{4\pi e_n^2}{k^2} \frac{\beta(1-q)}{\Gamma\left(1 - \frac{1}{q} \right)} \times \int_0^\infty dt \left(\frac{1}{1-q} - \frac{3}{2}\right) e^{-t} \left[1 - (1-q) \frac{m\omega^2\beta}{k^2} t\right]
$$

$$
= 1 + \frac{4\pi e_n^2}{k^2} \frac{\beta(1-q)}{\Gamma\left(1 - \frac{1}{q} \right)} \times \left[\frac{\Gamma\left(1 - \frac{1}{q} - \frac{1}{2}\right) - (1-q)}{\Gamma\left(1 - \frac{1}{q} + \frac{1}{2}\right)}\right].
$$

(66)

Using once again the property $\Gamma(z+1) = z\Gamma(z)$, after some straightforward algebra this simplifies to

$$
\epsilon^{(r)}_\ell = 1 + \frac{\omega_p^2}{v_T^2 k^2} \left(\frac{3q-1}{2}\right) \left[1 - \frac{\omega^2}{v_T^2 k^2} \left(1 + \frac{q}{2}\right)\right],
$$

(67)

with $\omega_p$ and $v_T$ defined in equations (55) and (57), respectively. The corresponding dispersion relation is then given by

$$
\omega^2(k) = \left(\frac{2}{q+1}\right) k^2 v_T^2 \left[1 + k^2 \lambda_D^2 \left(\frac{2}{3q-1}\right)\right],
$$

(68)

where

$$
\lambda_D \equiv \frac{v_T}{\omega_p},
$$

(69)

is the Debye wavelength. As expected, the results given in equations (67) and (68) reduce to the classical formulas in the limit $q \to 1$ (see e.g., Ref. [18], p. 137).

As already pointed out above, the exponential parametrization method allows us to reproduce the results obtained previously in reference [16], with much less effort, and as further indication of this, we have obtained the next-to-leading term in equation (67).

4 Conclusions and outlook

In this article we have introduced and illustrated the use the exponential parametrization method of the nonextensive distribution in the calculations of the dielectric permittivity and the dispersion relation of a collisionless electron plasma. We have established contact with some of the results that have been obtained previously by other authors by other means in various situations or limiting cases. The results we have obtained illustrate the effectiveness and simplicity of the method, and pave the way for considering the application of the method to other cases of interest, in the context of similar or related systems, such as magnetized and/or anisotropic plasmas.

Author contribution statement

The authors of this paper have contributed equally to all the aspects of this work and its presentation.

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