Ill-posedness of a double null free evolution scheme for black hole spacetimes

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Abstract

We suggest that “free evolution” integration schemes for the Einstein equations (that do not enforce constraints) may contain exponentially growing modes that render them useless in numerical integrations of black hole spacetimes, independently of how the equations are differenced. As an example we consider the evolution of Schwarzschild and Reissner-Nordström spacetimes in double null coordinates.

I. INTRODUCTION

There are two steps from the Einstein equations to a numerical code for solving them: First one selects the fields of the problem (the metric, its derivatives, connection coefficients etc.) and a subset of the Einstein equations to evolve these from some boundary conditions. This constitutes the “differential problem”. The resulting set of differential equations is
then transformed into a set of difference equations on a numerical grid, constituting the “difference problem”. (Here the term “boundaries” includes spacelike and null as well as timelike boundaries.)

In the first step, one distinguishes between “free” and “constrained” evolution schemes. In a free evolution scheme, the constraints are imposed only at the boundaries, and all fields are propagated by means of the evolution equations. In a constrained scheme, one uses fewer evolution equations, and instead reconstructs some fields at each time step from the constraints. (Here the term “time” refers to the coordinate that labels slices in the numerical evolution, which can be null. Constraints are all equations restricted to a hypersurface.)

In this paper we show that free evolution may already be ill-posed as a differential problem in some cases because it admits exponentially growing perturbations which are solutions of the evolution equations, but which violate the constraints and are therefore unphysical. For analytic purposes this does not matter, as imposing the constraints on the boundaries assures that they are obeyed everywhere, but any finite-differencing scheme introduces small perturbations which do not generically obey the constraints, and serve as initial data for the growing unphysical modes. The amplitude of these modes depends therefore on the choice of difference scheme (and decreases with the grid spacing), but their evolution (for example, the rate of exponential blow-up) is determined by the differential problem alone.

As an example differential problem we consider free evolution in spherical symmetry in double null coordinates. It is sufficiently simple that we can calculate the blowup of the unstable mode analytically, before confirming it in a testbed calculation. This particular instability is a serious problem inside and just outside black holes, harmless far outside black holes, and is not present in perturbations around flat spacetime.

II. DOUBLE NULL COORDINATES IN SPHERICAL SYMMETRY

The metric of any spherically symmetric spacetime can be written as
\[ ds^2 = -4f \, du \, dv + r^2 \, d\Omega^2, \]

where the metric coefficients \( f \) and \( r \) depend on \( u \) and \( v \) only. Our matter will be a minimally coupled massless real scalar field \( \phi \), plus a spherically symmetric Maxwell field. The Maxwell field has no sources, and is therefore constrained to be a pure Coulomb field of constant charge \( q \) anchored at (the singularity) \( r = 0 \).

This model has been used in both analytic \[4\] and numerical \[2,3\] work as a toy model for studying the interior of realistic black holes. Realistic black holes should be spinning, at least slightly, and would therefore be expected to have a Cauchy horizon, like the Kerr solution. On the other hand they would be inevitably perturbed by gravitational wave tails, and these perturbations would blow up on the Cauchy horizon. What really happens has therefore been the subject of prolonged investigation. Replacing the rotation by an electric charge and gravitational waves by the scalar field allows the simplification of spherical symmetry. Although we have chosen a particular matter model here, we shall see that the instability arises in the gravitational part of the equations and is therefore independent of the matter model.

The \( uu \) and \( vv \) components of the Einstein equations are

\[ r_{,uu} - \frac{f_u}{f} r_{,u} + (\phi_{,u})^2 \equiv E_1 = 0, \]

\[ r_{,vv} - \frac{f_v}{f} r_{,v} + (\phi_{,v})^2 \equiv E_2 = 0. \]

The \( \theta\theta \) component is

\[ r_{,uv} + \frac{r_{,u}r_{,v}}{r} - \frac{f}{r} + \frac{q^2 f}{r^3} \equiv H_1 = 0, \]

and the \( uv \) component is

\[ (\ln f)_{,uv} + 2\frac{r_{,uv}}{r} - \frac{2q^2 f}{r^4} + 2\phi_{,u}\phi_{,v} \equiv H_2 = 0. \]

The other components of the Einstein equations are redundant. The massless wave equation for \( \phi \), restricted to spherical symmetry, is
\[
\phi_{uv} + \frac{r_u}{r} \phi_{,u} + \frac{r_v}{r} \phi_{,v} \equiv H_3 = 0.
\] (6)

We then have three hyperbolic equations \(H_1, H_2\) and \(H_3\) and two elliptic equations \(E_1\) and \(E_2\). (The elliptic equations are really ordinary differential equations because of the spherical symmetry.) The elliptic equations are propagated by the hyperbolic equations in the sense of

\[
E_{1,v} = H_{1,u} + 2r \phi_{,u} H_3 - \frac{r_v}{r} E_1 - r_{,u} H_2 + \left( \frac{r_{,u}}{r} - \frac{f_{,u}}{f} \right) H_1 = 0,
\] (7)

while a similar equation holds for \(E_{2,u}\).

III. THE FREE EVOLUTION SCHEME

A possible evolution scheme, perhaps the most natural one, and certainly one easy to implement numerically, is to consider the three hyperbolic equations as evolution equations for \(f, r\) and \(\phi\), and the elliptic equations as constraints which are imposed only on the boundary. The natural initial value problem for these equations, is a double null initial value problem. \(r, f\) and \(\phi\) are given as functions of \(u\) on the null cone \(v = v_0\), subject to the constraint \(E_1 = 0\), and as functions of \(v\) on the intersecting null cone \(u = u_0\), subject to the constraint \(E_2 = 0\). At the intersection point \((u = u_0, v = v_0)\) it is sufficient that the data \(r, f\) and \(\phi\) be continuous. In the free evolution scheme, the constraints are imposed only on the null boundary data, but are not used for the numerical solution inside the numerical domain.

We now examine the stability of this free evolution scheme in the context of black hole physics. As a testbed case we use the Reissner-Nordström solution. It is the unique solution for \(\phi = 0\), and it is known in double null coordinates in (essentially) closed form. In one particular gauge choice (Eddington-Finkelstein coordinates) this solution is

\[
f(u, v) = f(r) = -\frac{1}{4} \left( 1 + \frac{2M}{r} - \frac{q^2}{r^2} \right),
\] (8)

where \(r(u, v)\) is given implicitly by
The implicit equation \( \frac{dr}{dr_*} = f(r) \) can be solved numerically to arbitrary precision so that the solution exists in closed form for all numerical purposes.

In our test case we have set \( \phi = 0 \) because we expect a nonvanishing \( \phi \) to give rise to a physical instability, namely mass inflation, inside the black hole, while we want to check if such instabilities are already contained in the free evolution scheme in a situation when we know that no physical instabilities can be present.

In order to look for instabilities in analytic approximation, it is useful to make a change of variables in order to eliminate first derivatives in the hyperbolic equations. With the new variables

\[
y \equiv \ln f + \ln \rho, \quad x \equiv \rho^2
\]

the evolution equations \( H_1 = 0 \) and \( H_2 = 0 \), now restricted to \( \phi = 0 \), become

\[
x_{,uv} + A(x, y) = 0, \quad y_{,uv} + B(x, y) = 0.
\]

(The evolution equation \( H_3 = 0 \) for \( \phi \) is dropped.) The constraints \( E_1 = 0 \) and \( E_2 = 0 \) become

\[
x_{,uu} - y_{,u} x_{,u} = 0, \quad x_{,vv} - y_{,v} x_{,v} = 0.
\]

Now we linearize the evolution equations, denoting the perturbations of \( x \) and \( y \) by \( \xi \) and \( \eta \):

\[
\xi_{,uv} + A_x \xi + A_y \eta = 0, \quad \eta_{,uv} + B_x \xi + B_y \eta = 0.
\]

The coefficients of these equations on the Reissner-Nordström background are

\[
A = A_y = -2f \left( 1 - \frac{q^2}{r^2} \right), \quad B = B_y = A_x = \frac{f}{r^2} \left( 1 - \frac{3q^2}{r^2} \right), \quad B_x = -\frac{3}{2} \frac{f}{r^4} \left( 1 - \frac{5q^2}{r^2} \right).
\]

To obtain an analytic approximation to the perturbations we now consider \( r \) and \( f \) of the background as slowly varying functions of \( u \) and \( v \). We make a mode ansatz
\[ \xi(u, v) = \xi ke^{i\omega(v+u)+ik(v-u)}, \quad \eta(u, v) = \eta ke^{i\omega(v+u)+ik(v-u)}. \] (15)

The ansatz turns the derivative \( \xi_{uv} \) into the algebraic expression \((k^2 - \omega^2)\xi_k\), and the equations (13) into the local dispersion relation \(\omega^2(k) = k^2 + \lambda\). The \(\lambda\) are the eigenvalues of the two-by-two matrix \([(A_x, A_y), (B_x, B_y)]\). Their value on the Reissner-Nordström background is

\[ \lambda_\pm = \frac{f}{r^2} \left[ (1 - 3\rho) \pm \sqrt{3(1 - \rho)(1 - 5\rho)} \right], \quad \text{where} \quad \rho \equiv q^2/r^2, \] (16)

and the corresponding eigenvectors \((\xi_k, \eta_k)\) are

\[ \eta_{k\pm} = \pm \frac{\sqrt{3}}{2r^2} \sqrt{\frac{1 - 5\rho}{1 - \rho}} \xi_{k\pm}. \] (17)

With \(u\) and \(v\) both increasing to the future. \(u + v\) labels time, and \(u - v\) labels space. \(u + v\) increases away from the null boundary data. (Inside the black hole, evolution towards increasing \(u + v\) is also evolution towards the singularity.) Therefore we have an instability if \(\omega^2(k)\) is negative for any \(k\), that is, for \(\lambda < 0\). We see that for \(q = 0\), \(\lambda_\pm = (1 \pm \sqrt{3})r^{-2}f\). Although \(f\) changes sign at the horizons, one of the \(\lambda\) is always negative for all \(r\). For \(q \neq 0\) the situation is not changed drastically. There is now a region where both \(\lambda\) are positive, namely \(0.74|q| < r < |q|\). But this is only a narrow band inside the black hole, and globally the free evolution scheme is still unstable.

The transformations that leave the form (1) of the metric invariant are \(u \rightarrow U(u)\) and \(v \rightarrow V(v)\), where \(U\) and \(V\) are arbitrary functions. The speed of the exponential blowup itself is gauge-invariant, in the sense that \(\lambda\) transforms correctly under the coordinate transformations \(u \rightarrow U(u)\) and \(v \rightarrow V(v)\). We can see this noting, from (1), that \(f\) transforms in the same way as \(g_{uv}\) for any scalar \(g\), and that \(\lambda\) is the product of \(f\) times a scalar (\(r\) is a scalar), and hence also transforms like \(f\). Our results would therefore also hold in Kruskal coordinates, or any other double null coordinates.

We should stress again that the unstable perturbations we have constructed are not tied to a particular numerical scheme. They are solutions to the hyperbolic part of the Einstein equations. They must arise in any numerical algorithm implementing a free evolution
scheme in null coordinates, simply because they are excited by the discretisation error of any numerical scheme.

Of course these unstable modes are not solutions of the constraints, or elliptic part of the Einstein equations. We can explicitly verify this by constructing the perturbations that do obey the constraints. The linearized constraints are

$$\xi_{,uu} - y_{,u} \xi_{,u} - x_{,u} \eta_{,u} = 0, \quad \xi_{,vv} - y_{,v} \xi_{,v} - x_{,v} \eta_{,v} = 0. \tag{18}$$

Because the Reissner-Nordström solution is unique, any perturbations of (8,9) obeying all the Einstein equations must be infinitesimal coordinate transformations. Perturbatively we write $u \to u + \mu(u)$ and $v \to v + \nu(v)$. The corresponding perturbative changes in the metric variables are

$$\xi = -x_{,u} \mu - x_{,v} \nu, \quad \eta = -y_{,u} \mu - y_{,v} \nu - d\mu/du - d\nu/dv. \tag{19}$$

Clearly these obey the linear constraints and are bounded. The unstable modes we have given do not obey the linear constraints and blow up.

We first encountered this instability in a code modeled on that of reference [2], when we found we were unable to recover the Reissner-Nordström solution in a testbed. Having constructed the unstable modes analytically, we could quantitatively verify their presence in the code. (Once more it should be said that the instability is connected to the free evolution scheme, not to any particular numerical implementation.) As an example we numerically construct a null diamond (that is, a square in $u$ and $v$) of the Schwarzschild solution centered on $r = 3M$. (We have chosen $r \sim 3M$ because there the instability is greatest outside the horizon.) Fig. 1 shows the numerical setup. At $r = 3M$ in the coordinates (8,9), $\lambda_+ \simeq -0.025 M^{-2}$. This corresponds to a blowup of the unstable modes as $\exp(0.32 M^{-1} t)$, where $t = (u + v)/2$ is the usual Schwarzschild time coordinate. Numerically we find that the error grows as $\exp(0.24 M^{-1} t)$ at $r = 3M$. The discrepancy of the exponent may be due to the approximation of constant $r$ and $f$ we have made in the analytic calculation.

There is a second prediction of the analytic model which can be verified at least qualitatively. From the form of the eigenvectors (17), together with the definitions (10) it follows
that the relative error in $f$ is generically of the same order as the relative error in $r$, except for $r \simeq |q|$, where it will much greater (because $1 - \rho \ll 1$). This feature is confirmed by numerical evolutions inside and outside the horizon, for various values of $q$.

IV. CONCLUSIONS

We have given a simple clean example of a numerical instability arising from constraint violation. The amplitude of the unstable mode depends on the discretisation scheme, but its growth rate does not. We have estimated the growth rate analytically, and have confirmed it for a particular discretisation.

The free evolution scheme we have described was used for the numerical study of perturbed Reissner-Nordström black holes in reference [2]. As our investigation shows, this scheme is unable even to evolve the exact Reissner-Nordström solution (setting the scalar field to zero) because of an exponential instability eventually crashing the code. It can therefore not be used for the study of the physical instability triggered by small physical perturbations. (This is the physical instability [1] predicted to lead to mass inflation and the destruction of the Cauchy horizon.) This casts a shadow on the physical results obtained using this scheme.

Other published codes [3–8] using a double null numerical grid use fully constrained evolution. The interior of a charged black hole has been treated with a fully constrained evolution scheme based on coordinates $u$ (retarded time) and $r$ (curvature radius), but evolving them on a double null grid [3]. This kind of algorithm has been pioneered in [4], and has been used extensively since [5–7]. A code actually based on double null coordinates, as well as a null grid, was used in [8], implementing a fully constrained scheme. Here fully constrained means that one uses the maximum number of equations containing only $v$-derivatives.

Another instance of an ill-posedness and instability arising from constraint violation has been found for the nonlinear evolution of weak gravitational waves on a flat background [9].
The growth rate of the instability depended on the discretisation scheme, however, which is puzzling in the light of our results.

Generally our results suggest that free evolution schemes are generally harder (or impossible, as in this case) to make stable than fully constrained schemes. This is not in conflict with the statement [10] that if one if one evolves in free evolution to a certain order in the grid spacing, the constraints are automatically obeyed to that order. The numerical, constraint-violating, error may go down as some power of decreasing step-size, but it can also be growing exponentially with time. If this exponential growth is fast enough, one will, in typical situations, be unable in practice to compensate for the error at late times by using a finer numerical grid.

Enforcing all the constraints at each time step can avoid a blowup only when the solution to be calculated is itself insensitive to small perturbations in the initial data, because then all rapidly growing perturbations must be constraint violations. The solution itself however may be highly sensitive to perturbations in the initial data, and/or may blow up at a spacetime singularity. Examples of such physical instabilities are mass inflation in the perturbed black holes we discussed here, the threshold of black hole formation, or chaos in the mixmaster universe [11,12]. Such examples may not be strictly well-posed but are nevertheless interesting. Then one must enforce the constraints to distinguish physical from unphysical (constraint-violating) perturbations.

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FIG. 1. The numerical grid setup for our example. The partially filled-in numerical grid indicates in what order grid points are completed. The fat lines $u = u_0$ and $v = v_0$ are null boundaries. From a numerical point of view one might call the former the initial time and the latter a boundary, but clearly this is a matter of words. The dotted line marks the boundary of the domain of dependence of the null boundary data. $t$ and $r^*$ are the Schwarzschild time and tortoise radial coordinate.