Data Rates for Stabilizing Control under Denial-of-Service Attacks

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Abstract—We study communication-constrained networked control problems for linear time-invariant systems in the presence of Denial-of-Service (DoS) attacks, namely attacks that prevent transmissions over the communication network. Our work aims at exploring the relationship between system resilience and network bandwidth capacity. Given a class of DoS attacks, we first characterize time-invariant bit-rate bounds that are dependent on the unstable eigenvalues of the dynamic matrix of the plant and the parameters of DoS attacks, beyond which exponential stability of the closed-loop system can be guaranteed. Second, we design the time-varying bit-rate protocol and show that it can enable the system to maintain the comparable robustness as the one under the time-invariant bit-rate protocol and meanwhile promote the possibility of transmitting fewer bits especially when the attack levels are low. Our characterization clearly shows the trade-off between the communication bandwidth and resilience against DoS. An example is given to illustrate the proposed solution approach.

I. INTRODUCTION

Cyber-physical systems (CPSs) have attracted much attention due to the advances in automation. Integrating communication and computation technologies, CPSs have a broad spectrum of applications ranging from small local control systems to large-scale systems, some of which are safety-critical. This raises the issue of reliability of CPSs to a considerable important level. Among a variety of aspects in reliability problems, the security of CPSs becomes a challenge from both practical and theoretical points of view. Here the concept of CPSs security mostly concerns the resilience against or protection from malicious attacks, e.g., deceptive attacks and Denial-of-Service (DoS) [1], [2].

This paper deals with resilient control under DoS attacks. We consider a basic problem where the network has limited bandwidth and is subject to DoS attacks, and the intention of the attacker is to induce instability. This implies that the signals transmitted over such a network are subject to both quantization and dropout. It is well known that an insufficient bit rate in the communication channel influences the stability of a networked control system [3], not to mention packet drops [4]. Hence, the topic of networked control under data rate constraints and random packet dropouts has been investigated by many researchers. However, those results may not be applicable in the context of DoS since the communication failures induced by DoS can exhibit a temporal profile quite different from the one induced by genuine packet losses; particularly packet dropouts induced by DoS need not follow a given class of probability distributions [5]. This poses new challenges in theoretical analysis and controller design.

The literature on networked control with bit-rate limitations is large and diverse [6]–[12] and the problem when quantization and genuine packet losses coexist has been well studied, see [13]–[19]. In [8], the authors obtain necessary and sufficient conditions concerning the observability and stabilization for the networked control of a linear time-invariant system under communication constraints. These conditions are independent of information patterns and only rely on the inherent property of the considered plant, i.e., the unstable eigenvalues of the dynamic matrix of the plant. The papers [13], [19] investigate the minimum data rate problem for mean square stability under Markovian packet losses. Necessary and sufficient conditions for stabilization are obtained for both scalar and vector systems.

Recently, systems under DoS attacks have been studied from the control-theoretic viewpoint [20]–[33]. In [20], a framework is introduced where DoS attacks are characterized by frequency and duration. The contribution is an explicit characterization of DoS frequency and duration under which stability can be preserved through state-feedback control. Extensions have been considered dealing with self-triggered networks [23] and nonlinear systems [29]. In [21], the authors generalize this model and consider a scenario where malicious attacks and genuine packet losses coexist, in which the effect of malicious attacks and random packet losses is merged and characterized by an overall packet drop ratio. In [22], the authors investigate launching DoS attacks optimally to a network with genuine packet losses. Specifically, the attacker aims at maximizing the estimation error with constrained energy. In [23], the authors formulate a two-player zero-sum stochastic game framework to consider a remote secure estimation problem, where the signals are transmitted over a multi-channel network under DoS attacks. A game-theory-based model where transmitters and jammers have multiple choices of sending and interfering power is considered in [24]. The recent paper [28] investigates the stabilization problem of a discrete-time output feedback system under quantization and DoS attacks. In the event of the satisfaction of a certain norm condition, a lower bound of quantization level and an upper bound of DoS duration are obtained together guaranteeing stability.

In this paper, we consider the stabilization problem of a linear time-invariant continuous process, possibly open-loop unstable and with complex eigenvalues, where the communication between sensor and controller takes place over a bit-rate limited and unreliable digital channel. Previously,
we have shown that a controller with prediction capability significantly promotes the resilience of a networked control system against DoS in the sense that the missing signals induced by DoS attacks can be reconstructed and then applied for computing the control input [26], [27], [32]. Under proper design, the system can achieve ISS-like robust stability or asymptotic stability in the presence or absence of disturbance and noise, respectively. However, when the network has limited bandwidth, the existing results are not applicable any longer because signal deviation induced by quantization cannot be simply treated as bounded noise, and such signal deviation influences the accuracy of estimation/prediction and hence the resilience of the closed-loop system.

Therefore, there is a trade-off between communication bandwidth and system resilience. An interesting question is to find how large the bit rate must be to ensure the stability of a system under DoS, possibly an open-loop unstable system. We may state this question in another way as how much the limited bit rate degrades the robustness of a networked control system in the context of stabilization. We follow the approach aligned with that for the minimum data rate control problems discussed above. In particular, we recover those results in the case without any DoS. Exploiting the techniques of transformation, we associate the bit rates with the eigenvalues of the dynamic matrix of the process and DoS parameters, and explicitly characterize the relationship between system resilience and bit rates. Specifically, we compute a bit-rate bound element-wise, larger than which the closed-loop system is exponentially stable. This on the other hand reveals the “robustness degradation” induced by quantization. In addition, assuming that the communication protocol is acknowledgment-based, we propose a time-varying bit-rate design where the packet size is time-varying. This enables us to preserve a comparable level of robustness against DoS and meanwhile promotes the possibility of saving transmitted bits especially when the attack levels are low.

This paper is organized as follows. In Section II, we introduce the framework consisting of system transformations, a class of DoS attacks and the contribution of this paper. Section III is the core part of this paper. Considering that the data rate is time-invariant, we accordingly choose the uniform quantizer and design the predictor. The dynamics of quantization range and prediction error are analyzed. Then we conduct the stability analysis. In Section IV, the time-varying bit-rate protocol is introduced, which can stabilize the system and save communication resources. A numerical example is introduced in Section V, and finally Section VI ends the paper with conclusions and possible future research directions.

**Notation.** We denote by $\mathbb{R}$ the set of reals. Given $b \in \mathbb{R}$, $\mathbb{R}_{\geq b}$ and $\mathbb{R}_{> b}$ denote the sets of reals no smaller than $b$ and reals greater than $b$, respectively; $\mathbb{R}_{\leq b}$ and $\mathbb{R}_{< b}$ represent the sets of reals no larger than $b$ and reals smaller than $b$, respectively; $\mathbb{Z}$ denotes the set of integers. For any $c \in \mathbb{Z}$, we denote $\mathbb{Z}_c := \{c, c+1, \cdots\}$. Let $\lfloor x \rfloor$ be the floor function such that $\lfloor x \rfloor = \max\{k \in \mathbb{Z}|k \leq x\}$. Also, let $\lceil x \rceil$ be the ceiling function such that $\lceil x \rceil = \min\{k \in \mathbb{Z}|k \geq x\}$. Given a vector $\beta$, $\|eta\|$ is its Euclidean norm. Given a matrix $\Gamma$, $\|\Gamma\|$ represents its spectral norm and $\Gamma^T$ is its transpose. Given an interval $I$, $|I|$ denotes its length. The Kronecker product is denoted by $\otimes$. Finally, given a signal $F$, $F(t^-)$ denotes the limit from below at $t$.

II. FRAMEWORK

A. System description

Consider the networked control architecture in Figure 1.

The process is a linear time-invariant continuous system given by

$$\dot{x}(t) = Ax(t) + Bu(t)$$

where $t \in \mathbb{R}_{\geq 0}$, $x(t) \in \mathbb{R}^n$ is the state with $x(0)$ arbitrary, $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times u}$, $u(t) \in \mathbb{R}^u$ is the control input and $(A, B)$ is stabilizable. Let $K \in \mathbb{R}^{n_v \times n_u}$ be a matrix such that the real part of each eigenvalue of $A + BK$ is strictly negative. Let $\lambda_r = c_r + d_r i$ be the eigenvalues of $A$ with $c_r, d_r \in \mathbb{R}$, where $c_1, c_2, c_3, \cdots$ are distinct and $i$ represents the imaginary number. If $d_r = 0$ then $\lambda_r$ has only real part and corresponds to a real eigenvalue such that $\lambda_r = c_r$. If $d_r \neq 0$, $\lambda_r$ represents a pair of complex eigenvalues whose real part is $c_r$ and imaginary parts are $d_r i$ and $-d_r i$, respectively. In the following sections, the real part of $\lambda_r$ is denoted by $\alpha_r$, where we do not distinguish if $\alpha_r$ is real or complex. We assume that the state is measurable by sensors.

The measurement channel has limited bandwidth and is moreover subject to DoS attacks. The transmission attempts between the encoder and decoder are carried out periodically with interval $\Delta$, i.e.

$$t_{k+1} - t_k = \Delta$$

where $\{t_k\}_{k \in \mathbb{Z}_0} = \{t_0, t_1, \cdots\}$ denotes the sequence of the instants of transmission attempts. By convention, we let $t_0 = 0$. Moreover, we assume that the network communication protocol is acknowledgment-based (like the TCP protocol) without any delay in terms of both encoded signal and acknowledgment transmissions.

![Fig. 1. Controller and actuator co-location architecture](image-url)

Since we consider a controller-actuator co-location architecture (cf. Figure 1), only the measurement channel is subject to DoS, and the control channel is free from DoS disruptions and always available. Due to DoS attacks, not all the transmission attempts succeed. Hence, we denote by $\{z_m\}_{m \in \mathbb{Z}_0} = \{z_0, z_1, \cdots\} \subseteq \{t_k\}_{k \in \mathbb{Z}_0}$ the sequence of the time instants at which successful transmissions occur.
B. System transformation

In order to facilitate the analysis in Sections III and IV, we carry out two transformations in this subsection.

First, we transform the original process (1) into the real Jordan canonical form. Let $S \in \mathbb{R}^{n_r \times n_r}$ be a transformation matrix such that (1) can be rewritten as

$$\dot{x}(t) = A\dot{x}(t) + Bu(t)$$  \hspace{1cm} (3)

where $\dot{x}(t) = Sx(t)$, $t \in \mathbb{R}_{\geq 0}$ and $\hat{A} \in \mathbb{R}^{n_r \times n_r}$ is the Jordan form of $A$ such that

$$\hat{A} = SAS^{-1} = \text{diag}(A_1, A_2, \cdots, A_p), \quad p \in \mathbb{Z}_1$$  \hspace{1cm} (4)

in which $p$ represents the number of Jordan blocks. Let $r = 1, 2, \cdots, p$. The Jordan block associated with the real eigenvalue $\lambda_r = c_r$ is

$$A_r = \begin{bmatrix} c_r & 1 & \cdots & 0 \\ 1 & c_r & \ddots & \vdots \\ \vdots & \ddots & \ddots & 1 \\ 0 & \cdots & 1 & c_r \end{bmatrix} \in \mathbb{R}^{n_r \times n_r}$$  \hspace{1cm} (5)

where $n_r$ is the order of $A_r$. The Jordan block associated with the complex eigenvalues $\lambda_r = c_r \pm d_r i$ ($d_r \neq 0$) is

$$A_r = \begin{bmatrix} D_r & I \\ I & D_r \\ \vdots & \ddots & \ddots & \ddots \\ I & \cdots & 1 & D_r \end{bmatrix} \in \mathbb{R}^{2n_r \times 2n_r}$$  \hspace{1cm} (6)

with

$$D_r = \begin{bmatrix} c_r & -d_r \\ d_r & c_r \end{bmatrix}, \quad I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$  \hspace{1cm} (7)

where $2n_r$ is the order of $A_r$. Meanwhile we have $\tilde{B} = SB \in \mathbb{R}^{n_r \times n_u}$. If $A$ has only real eigenvalues, the Jordan form of $A$ in (4) with the Jordan blocks in (5) is sufficient for further development. However, in the event of the existence of complex eigenvalues in $A$, we need one more step of transformation, which is carried out by the lemma below.

Lemma 1: Consider the process in (3) where $\hat{A}$ is in the Jordan form as in (4). There exists a transformation $\bar{x}(t) = E(t)\dot{x}(t)$ such that (3) can be transformed into

$$\dot{\bar{x}}(t) = \bar{A}\bar{x}(t) + \bar{B}(t)u(t)$$  \hspace{1cm} (8)

where

$$\bar{A} = E(t)\hat{A}E(t)^{-1} + \dot{E}(t)E(t)^{-1}$$

$$\bar{A} = \begin{bmatrix} \bar{A}_1 \\ \bar{A}_2 \\ \vdots \\ \bar{A}_p \end{bmatrix}, \quad p \in \mathbb{Z}_1$$  \hspace{1cm} (9)

with

$$\bar{A}_r = A_r = \begin{bmatrix} c_r & 1 & \cdots & 0 \\ 1 & c_r & \ddots & \vdots \\ \vdots & \ddots & \ddots & 1 \\ 0 & \cdots & 1 & c_r \end{bmatrix} \in \mathbb{R}^{n_r \times n_r}$$  \hspace{1cm} (10)

corresponding to the real eigenvalue $\lambda_r = c_r$, and

$$\bar{A}_r = A_r = \begin{bmatrix} c_r & 1 & \cdots & 0 \\ 1 & c_r & \ddots & \vdots \\ \vdots & \ddots & \ddots & 1 \\ 0 & \cdots & 1 & c_r \end{bmatrix} \in \mathbb{R}^{2n_r \times 2n_r}$$  \hspace{1cm} (11)

corresponding to the complex eigenvalues $\lambda_r = c_r \pm d_r i$ with $d_r \neq 0$. Besides, $\bar{B}(t) = E(t)\tilde{B}$.

Proof. We refer the readers to the Appendix for the proof including the design of $E(t)$. ■

In [8] and [35], similar techniques of transformation where the transformation matrix is time-varying are used. It is trivial to mention that one can directly transform (1) into [8] by computing $\bar{A} = E(t)\hat{A}E(t)^{-1} + \dot{E}(t)E(t)^{-1}$ and $\bar{B}(t) = E(t)\tilde{B}$. In case of the existence of complex eigenvalues, $E(t)$ is a time-varying matrix. This implies that $\bar{B}(t)$ is time-varying.

C. Time-constrained DoS

We refer to DoS as the phenomenon for which some transmission attempts may fail. We consider a general DoS model that constrains the attacker action in time by only posing limitations on the frequency of DoS attacks and their duration. Let $\{h_n\}_{n \in \mathbb{Z}_0}$ with $h_0 \geq 0$ denote the sequence of DoS off/on transitions, that is, the time instants at which DoS exhibits a transition from zero (transmissions are successful) to one (transmissions are not successful). Hence,

$$H_n := \{h_n\} \cup [h_n, h_n + \tau_n]$$  \hspace{1cm} (12)

represents the $n$-th DoS time-interval, of a length $\tau_n \in \mathbb{R}_{\geq 0}$, over which the network is in DoS status. If $\tau_n = 0$, then $H_n$ takes the form of a single pulse at $h_n$. Given $\tau, t \in \mathbb{R}_{\geq 0}$ with $t \geq \tau$, let $n(\tau, t)$ denote the number of DoS off/on transitions over $[\tau, t]$, and let

$$\Xi(\tau, t) := \bigcup_{n \in \mathbb{Z}_0} H_n \cap [\tau, t]$$  \hspace{1cm} (13)

be the subset of $[\tau, t]$ where the network is in DoS status.

Assumption 1: (DoS frequency). There exist constants $\eta \in \mathbb{R}_{\geq 0}$ and $\tau_D \in \mathbb{R}_{> 0}$ such that

$$n(\tau, t) \leq \eta + \frac{t - \tau}{\tau_D}$$  \hspace{1cm} (14)

for all $\tau, t \in \mathbb{R}_{\geq 0}$ with $t \geq \tau$.

Assumption 2: (DoS duration). There exist constants $\kappa \in \mathbb{R}_{\geq 0}$ and $T \in \mathbb{R}_{> 1}$ such that

$$|\Xi(\tau, t)| \leq \kappa + \frac{t - \tau}{T}$$  \hspace{1cm} (15)

for all $\tau, t \in \mathbb{R}_{\geq 0}$ with $t \geq \tau$.

Remark 1: Assumptions 1 and 2 do only constrain a given DoS signal in terms of its average frequency and duration. Following [36], $\tau_D$ can be defined as the average dwell-time between consecutive DoS off/on transitions, while $\eta$ is the chattering bound. Assumption 2 expresses a similar requirement with respect to the duration of DoS. It expresses the property that, on the average, the total duration over which communication is interrupted does not exceed a certain
fraction of time, as specified by $1/T$. Like $\eta$, the constant $\kappa$ plays the role of a regularization term. It is needed because during a DoS interval, one has $[\xi(h_n, h_n + \tau_n)] = \tau_n > \tau_n/T$. Thus $\kappa$ serves to make (13) consistent. Conditions $\tau_D > 0$ and $T > 1$ imply that DoS cannot occur at an infinitely fast rate or be always active.

The next lemma relates DoS parameters and the time elapsing between successful transmissions.

**Lemma 2:** Consider periodic transmission attempts $t_{k+1} - t_k = \Delta$, along with a DoS attack satisfying Assumptions 1 and 2. If $\frac{1}{T} + \frac{\Delta}{\tau_D} < 1$, then the sequence of successful transmissions satisfies $s_0 \leq Q$ and $s_{m+1} - s_m \leq Q + \Delta$ for all $m \in \mathbb{Z}_0$, where

$$Q := (\kappa + \eta \Delta) \left(1 - \frac{1}{T} - \frac{\Delta}{\tau_D}\right)^{-1}. \quad (16)$$

**Proof.** The proof of Lemma 1 in [22] carries over to this lemma with $s_0$, $s_{r+1}$ and $s_r$ replaced by $z_0$, $z_{m+1}$ and $z_m$, respectively.

The following lemma presents the relationship between DoS parameters, time and the number of successful transmissions therein.

**Lemma 3:** Consider the DoS attacks characterized by Assumptions 1 and 2. The number of successful transmissions within the interval $[z_0, z_m]$, which is denoted by $T_S(z_0, z_m)$, satisfies

$$T_S(z_0, z_m) \geq 1 + \frac{1 - \frac{1}{T} - \frac{\Delta}{\tau_D}}{\Delta} (z_m - z_0) - \kappa \eta \Delta \quad (17)$$

where $z_m \geq z_0$ and $\Delta$ is as in (2).

**Proof.** Consider an interval $[z_0, z_m]$ with $z_m \geq z_0$ and let $H_n$ represent the $n$-th DoS time-interval within $[z_0, z_m]$ here. One can verify that the number of unsuccessful transmissions during $H_n$ is no larger than $\frac{\tau_D}{\tau} + 1$. Hence the number of unsuccessful transmissions during $[z_0, z_m]$ satisfies

$$T_U(z_0, z_m) \leq \sum_{k=0}^{n(z_0, z_m)-1} \left(\frac{T_k}{\Delta} + 1\right) \leq \frac{\Xi(z_0, z_m)}{\Delta} + n(z_0, z_m) \quad (18)$$

Let $T_A(z_0, z_m) = \frac{\tau_D z_m}{\Delta} + 1$ denote the number of total transmission attempts during $[z_0, z_m]$. Note that $T_A(z_0, z_m)$ and $T_S(z_0, z_m)$ are defined corresponding to the intervals $[z_0, z_m]$ and $[z_0, z_m]$, respectively. Therefore it follows from (14), (15), and (18) that $T_S(z_0, z_m)$ satisfies

$$T_S(z_0, z_m) = T_A(z_0, z_m) - T_U(z_0, z_m) - 1 \geq 1 - \frac{1}{T} - \frac{\Delta}{\tau_D} (z_m - z_0) - \kappa \eta \Delta \quad (19)$$

**Remark 2:** In the scenario of a reliable network ($T = \tau_D = \infty$ and $\kappa = \eta = 0$), $Q$ in Lemma 2 becomes zero, and $T_U(z_0, z_m) = 0$ implies $T_S(z_0, z_m) = T_A(z_0, z_m) - 1$. This means that every transmission attempt ends up with a successful transmission. Thus, Lemmas 2 and 3 describe the functioning of a standard periodic transmission policy.

**D. Literature review**

The robustness problem of the structure as in Figure 1 has been investigated in [26] and [27], where we assumed the network has infinite bandwidth and the measurements are not quantized. For the ease of comparison and clarifying the contribution of this paper, we briefly recall the controller and the result in [27]. The control system is given by

$$\begin{cases}
    u(t) = K \xi(t) \\
    \xi(t) = A \xi(t) + Bu(t), \quad \text{if } t \neq z_m \\
    \xi(t) = x(t) + n(t), \quad \text{if } t = z_m
  \end{cases} \quad (20)$$

where $\xi(t)$ is the estimation of $x(t)$ and $n(t)$ represents bounded noises.

**Theorem 1:** [27] Consider the dynamical system as in (1) under a co-located control system as in (20). The closed-loop system is stable for any DoS sequence satisfying Assumptions 1 and 2 with arbitrary $\eta$ and $\kappa$, and with $\tau_D$ and $T$ such that

$$\frac{1}{T} + \frac{\Delta}{\tau_D} < 1 \quad (21)$$

It is trivial that in case $n(t) = 0$, the result above still holds.

**E. Paper contribution**

Exploiting the controller in (20) and the architecture in Figure 1, we first design the encoder and decoder such that they are free of over-flow of quantization range even in the presence of DoS attacks. After fixing the control system’s structure, the number of bits $R_r$ for coding is the only parameter to be taken care of. Given the control framework, the contribution of this paper is mostly in finding the appropriate $R_r$, possibly under the presence of DoS attacks.

The main contribution of this paper is two-fold.

i) The first contribution is to show that the closed-loop system is exponentially stable if the time-invariant bit rate $R_r$ satisfies

$$R_r \begin{cases}
    \geq \frac{1}{1 - \frac{1}{T} - \frac{\Delta}{\tau_D}} c_r \log_2 e, \quad \text{if } c_r \geq 0 \\
    \geq 0, \quad \text{if } c_r < 0
  \end{cases} \quad (22)$$

where $R_r$ represents the number of bits applied to the signals corresponding to the $r$-th block in $\bar{A}$. The condition (22) is general enough in the sense that in the absence of DoS attacks, the result of minimum data rate control is recovered (cf. Remark 4). On the other hand, we characterize the robustness of the system, namely the amount of DoS attacks less than which stability can be still preserved. One can preserve closed-loop stability if the frequency and duration of DoS attacks satisfy

$$\frac{1}{T} + \frac{\Delta}{\tau_D} < 1 - \frac{c_r \log_2 e}{R_r}, \quad \forall c_r \geq 0 \quad (23)$$

where $R_r > 0$. Clearly, the signal inaccuracy due to quantization cannot be simply treated as the one caused by measurement noises in the sense that the noises do not enter the right-hand side of (21), whereas the quantization
degrades the system’s robustness by diminishing the right-hand side of (21) into (23). This implies that some DoS attacks that used to be tolerable would now cause instability.

ii) As a second contribution, we propose the time-varying bit-rate protocol consisting of bit-computing parts and coding parts in both the encoding and decoding systems. The bit-computing parts are able to generate sequences of time instants. By resorting to using acknowledgments, the sequences of time instants can be synchronized in the encoding and decoding systems. Based on the generated time sequences (under the influences of DoS), the number of bits applied for each transmission attempt can be predetermined. If the DoS attack is short, a number of bits no larger than $R_r$ could guarantee the decay of quantization range, and there is no need to apply $R_r$, which leads to the possibility of saving bits. Under suitable choices of parameters, we show that the closed-loop system is stable if the maximum number of bits that the encoding and decoding systems can apply in one transmission attempt satisfies (22).

III. TIME-ININVARIANT BIT RATE

In this section, we introduce the design of the encoding and decoding systems, and the control system, where the number of bits used for coding are time-invariant.

A. Uniform quantizer

The limitation of bandwidth implies that transmitted signals are subject to quantization. Let

$$\chi_l := \frac{e_l}{j_l}$$

be the original $l$-th signal before quantization and $q_{R_l}(\chi_l)$ represents the quantized signal of $\chi_l$ with $R_l$ bits, where $l = 1, 2, 3, \cdots, n_x$. The choices of $R_l$, $e_l \in \mathbb{R}$ and $j_l \in \mathbb{R}_{>0}$ will be specified later. We implement a uniform quantizer such that

$$q_{R_l}(\chi_l) := \begin{cases} \frac{\lfloor 2^{R_l-1} \chi_l \rfloor + 0.5}{2^{R_l-1}}, & \text{if } -1 \leq \chi_l < 1 \\ 0, & \text{if } \chi_l = 1 \end{cases}$$

if $R_l \in \mathbb{Z}_1$ and

$$q_{R_l}(\chi_l) = 0$$

if $R_l = 0$. Note that for any $j_l \in \mathbb{R}_{>0}$ the following property holds:

$$|e_l - j_l q_{R_l}(\frac{e_l}{j_l})| \leq \frac{j_l}{2^{R_l}}, \quad \text{if } \frac{|e_l|}{j_l} \leq 1$$

for both cases, namely $R_l \in \mathbb{Z}_0$ [13], [17]. For the ease of visualizing (25), Figure 2 shows the quantization function with $R_l = 2$. The quantizer applied in the time-varying bit-rate protocol will be presented in Section IV.

![Fig. 2. Example of quantization with $R_l = 2$. For instance, any number falling into $[0, 0.5]$ would be quantized into 0.25.](image)

B. Control architecture

The basic idea of the control system design is that we equip the encoding and decoding systems with prediction capability to properly quantify data and more importantly predict the missing signals that are interrupted by DoS. Specifically, the encoding system outputs quantized signals and transmits them to the decoding system through a DoS-corrupted network. The decoding system attempts to predict future signals based on the received quantized signals. Notice that the following design is based on $\hat{x}(t) = \tilde{A}\hat{x}(t) + \tilde{B}(t)u(t)$.

As shown in Figure 3 on the sensor side the encoding system is embedded with a predictor for predicting $\hat{x}(t)$. Let $\hat{x}(t) = [\hat{x}_1(t) \hat{x}_2(t) \cdots \hat{x}_{n_x}(t)]^T$ denote the prediction of $\bar{x}(t) = [\bar{x}_1(t) \bar{x}_2(t) \cdots \bar{x}_{n_x}(t)]^T$. The error $e_l(t) = [e_{l1}(t) e_{l2}(t) \cdots e_{ln_x}(t)]^T$ describes the discrepancy between $\bar{x}(t)$ and $\hat{x}(t)$, where

$$e_l(t) := \hat{x}_l(t) - \bar{x}_l(t), \quad l = 1, 2, \cdots, n_x.$$ (28)

Furthermore, we will design a dynamic system (see (31)-(32) below), whose state $\tilde{J}(t) = [j_1(t) j_2(t) \cdots j_{n_x}(t)]^T$ is always positive. Namely, it is $j_l(t) > 0$ for $t \in \mathbb{R}_{>0}$, where $j_l(t)$ represents the quantization range that bounds the error, i.e. $j_l(t) \geq |e_{l1}(t)|$ for $t \in \mathbb{R}_{>0}$ as it will be shown in the next subsection. Recalling that $\chi_l(t) := \frac{e_{l1}(t)}{j_l(t)}$, $j_l(t) \geq |e_{l1}(t)|$ for $t \in \mathbb{R}_{>0}$ implies $|\chi_l(t)| = \frac{|e_{l1}(t)|}{j_l(t)} \leq 1$ for $t \in \mathbb{R}_{>0}$, and hence there is no overflow problem and the quantizer (25)-(26) is valid for $t \in \mathbb{R}_{>0}$. More importantly, $j_l(t) \geq |e_{l1}(t)|$ for $t \in \mathbb{R}_{>0}$ would make (27) hold for $t \in \mathbb{R}_{>0}$.

On the actuator side, the decoding system is a copy of the encoding system. Once there is a successful transmission containing the encoded state, it recovers $q_{R_l}(\chi_l(z_m))$ based on the received code and updates the predictor embedded in the decoding system, and sends an acknowledgment back to the encoding system. The acknowledgment would enable the encoding system to know the successful transmission reception. We assume that the encoding and decoding systems have the same initial conditions. Therefore, identical structures and initial conditions, and acknowledgments would guarantee synchronization of all the signals in the encoding and decoding systems.

The predictor in both the encoding and decoding systems predicting $\hat{x}(t)$ is given by

$$\begin{cases} \dot{x}(t) = \tilde{A}\hat{x}(t) + \tilde{B}(t)u(t), \quad t \neq z_m \\ \hat{x}(t) = \hat{x}(t^-) - \Phi(t^-), \quad t = z_m. \end{cases}$$ (29)
As for the input $u(t)$, we have $u(t) = \hat{K}(t)\hat{x}(t)$ where $\hat{K}(t) = K S^{-1} E(t)^{-1} \in \mathbb{R}^{n_x \times n_x}$. In the encoding system, $u(t)$ is applied only to the predictor. In the decoding system, $u(t)$ is applied to both the predictor and actuator (see Figure 3).

The column vector $\Phi(t)$ in \((29)\) is given by

$$\Phi(t) = \begin{bmatrix} j_1(t)q_{R_1}(\chi_1(t)) \\ \vdots \\ j_{n_x}(t)q_{R_{n_x}}(\chi_{n_x}(t)) \end{bmatrix} \quad (30)$$

where $\chi_l(t) = \frac{e_l(t)}{\bar{e}_l(t)}$ and $j_l(t)$ is the $l$-th entry in the column vector $J(t) = [j_1(t) j_2(t) \cdots j_{n_x}(t)]^T$, which is the solution to the impulsive system

$$\begin{cases} \dot{J}(t) = \bar{A} J(t), & t \neq z_m \\ J(t) = H J(t^-), & t = z_m \end{cases} \quad (31)$$

with

$$H = \text{diag}(2^{-R_1} I_1, 2^{-R_2} I_2, \cdots, 2^{-R_p} I_p) \in \mathbb{R}^{n_x \times n_x} \quad (32)$$

where $I_r \in \mathbb{R}^{n_x \times n_x}$ or $I_r \in \mathbb{R}^{2n_x \times 2n_x}$ represents an identity matrix corresponding to $\bar{A}_r$ in (10) or (11), respectively. At the moment of a successful transmission, $J(t)$ in both the encoding and decoding systems is updated according to the second equation in (31). At last, the initial conditions of $\hat{x}$ and $J$ in the encoding and decoding systems are identical and satisfy

$$\begin{cases} \hat{x}_l(0^-) = 0, & l = 1, 2, \cdots, n_x \\ j_l(0^-) > |\hat{x}_l(0^-)|, & l = 1, 2, \cdots, n_x \end{cases} \quad (33)$$

It is worth mentioning that $R_l$ represents the number of bits applied to the $l$-th quantized signal, which is element-wise based. Since the $l$-th quantized signal must be associated with one block $\bar{A}_r$ ($r = 1, 2, \cdots, p$), therefore, in this paper the data rate analysis is based on the index of $\bar{A}_r$, and all the elements corresponding to $\bar{A}_r$ would apply $R_r$ bits. For example, if the $l$-th signal is associated with $\bar{A}_r$, then $R_l = R_r$. In the results of this paper, we will obtain the bounds of $\{R_r\}_{r=1,2,\cdots,p}$, so that $\{R_l\}_{l=1,2,\cdots,n_x}$ can be determined.

### C. Overflow-free quantizer

In this part, our intention is to show that $j_l(t) > |e_l(t)|$ for $t \in \mathbb{R}_{\geq 0}$ with $l = 1, 2, \cdots, n_x$. Exploiting (28)-(30) and the continuity of $\tilde{x}(t)$ such that $\tilde{x}(t) = \tilde{x}(t^-)$, we have

$$\begin{align*} e_l(t) &= \hat{x}_l(t) - \bar{x}_l(t) \\
&= \tilde{x}_l(t^-) - \tilde{x}_l(t^-) - j_l(t^-) q_{R_l} \left( \frac{e_l(t^-)}{j_l(t^-)} \right), \quad t = z_m \\
&= e_l(t^-) - j_l(t^-) q_{R_l} \left( \frac{e_l(t^-)}{j_l(t^-)} \right) \quad (34) \end{align*}$$

where $l = 1, 2, \cdots, n_x$. Hence the dynamics of $e(t)$ obeys

$$\begin{cases} \dot{e}(t) = \bar{A} e(t), & t \neq z_m \\ e(t) = e(t^-) - \Phi(t^-), & t = z_m \end{cases} \quad (35)$$

Moreover, observing $e(t)$ in (35) and $J(t)$ in (31), one has

$$\begin{cases} \dot{e}(t) = \bar{A} e(t), \\ J(t) = \bar{A} J(t) \end{cases} \quad (36)$$

whose solutions are $e(t) = e^{\bar{A}t} e(0)$ and $J(t) = e^{\bar{A}t} J(0)$, respectively, for $0 \leq t < z_0$ (if $z_0 \neq 0$) or $0 \leq t < z_1$ (if $z_0 = 0$), where

$$e^{\bar{A}t} = \text{diag}(U_1(t), U_2(t), \cdots, U_p(t)) \quad (37)$$

with

$$U_r(t) = e^{c_r t} V_r(t) \otimes W, \quad r = 1, 2, \cdots, p \quad (38)$$

where

$$V_r(t) = \begin{bmatrix} 1 & t & \cdots & \frac{n_x-1}{(n_x-1)!} \\ 1 & t & \cdots & \frac{n_x-2}{(n_x-2)!} \\ \vdots & \vdots & \ddots & \vdots \\ \vdots & \vdots & \cdots & 1 \end{bmatrix} \quad (39)$$

and

$$W = \begin{cases} 1, & \text{if } d_r = 0 \\ I, & \text{if } d_r \neq 0 \end{cases} \quad (40)$$

in which $I$ is as in (4).

By $e(t) = e^{\bar{A}t} e(0)$, one can obtain that $|e(t)| \leq e^{\bar{A}|t|} |e(0)|$ holds element-wise, where $| |$ denotes a function that computes the absolute value of each element in a vector, i.e.,

$$|e(t)| = \left[ |e_1(t)| \ e_2(t) \cdots |e_{n_x}(t)| \right]^T.$$  Define the column vector $\varepsilon(t) := J(t) - |e(t)| = [\varepsilon_1(t) \ \varepsilon_2(t) \ \cdots \ \varepsilon_{n_x}(t)]^T$. If $z_0 \neq 0$, one has

$$\varepsilon(t) = J(t) - |e(t)| \geq e^{\bar{A}t} J(0) - e^{\bar{A}|t|} |e(0)|, \quad 0 \leq t < z_0$$

$$= e^{\bar{A}t} \varepsilon(0). \quad (41)$$

By (33), one knows that

$$\varepsilon(0) = J(0) - |e(0)| = J(0) - |\bar{x}(0) - \tilde{x}(0)|$$

$$= J(0^-) - |\bar{x}(0^-) - \tilde{x}(0^-)|$$

$$= J(0^-) - |\bar{x}(0^-)|$$

$$= J(0^-) - |\bar{x}(0^-)| \quad (42)$$
and thus every element in the column vector \( \varepsilon(0) \) is positive, which implies that every element in the column vector \( \varepsilon(t) \) is positive for \( 0 \leq t < z_0 \). Thus, one can infer that \( j_l(z_0^-) = |e_l(z_0^-)| > 0 \), and hence \( j_l(z_0^-) - |e_l(z_0^-)| \geq 0 \). In view of (34), it is clear that

\[
|e_l(z_0^-)| = j_l(z_0^-) - \delta_l(z_0^-) q_{R_l} \leq \frac{j_l(z_0^-)}{2 \delta_l},
\]

where the inequality is implied by (27) and the second equality in (31), from which one obtains that \( |e_l(z_0^-)| \leq j_l(z_0^-) \) and furthermore \( \varepsilon(z_0^-) \geq 0 \). Following the analysis as in (41), one could obtain that \( \varepsilon(t) \geq e^{A(t-z_0^-)\varepsilon(z_0^-)} \) with \( z_0 \leq t < z_1 \). This implies that every element in \( \varepsilon(t) \) is non-negative and \( |e_l(t)| \leq j_l(t) \) for \( z_0 \leq t < z_1 \). By simple induction, we can verify that \( |e_l(t)| \leq j_l(t) \) for \( t \in \mathbb{R}_{\geq 0} \), \( z_0 \neq 0 \).

If \( z_0 = 0 \), we know that \( |e_l(z_0^-)| = |e_l(0^-)| = |e_l(0^-)| < j_l(0^-) = j_l(z_0^-) \), and hence \( j_l(z_0^-) - |e_l(z_0^-)| \geq 0 \). Following (43), one gets \( |e_l(z_0^-)| \leq j_l(z_0^-) \). The remaining part follows the same analysis as in the scenario \( z_0 \neq 0 \) to obtain \( |e_l(t)| \leq j_l(t) \) for \( t \in \mathbb{R}_{\geq 0} \) when \( z_0 = 0 \). Therefore, we conclude that

\[
|e_l(t)| \leq j_l(t), \quad l = 1, 2, \ldots, n_x, \quad t \in \mathbb{R}_{\geq 0}
\]

and thus the quantizer (25) does not undergo any overflow, and (27) always holds. Notice that (44) holds for \( t \in \mathbb{R}_{\geq 0} \), which implies \( |e_l(t)| \) is always bounded by \( j_l(t) \) in the absence or presence of DoS attacks. Without losing generality, we focus the attention from \( z_0 \) onwards.

D. Dynamics of the encoding and decoding systems

Since the evolutions of the signals in the encoding and decoding systems are identical, we would present this part from the view of either the encoding or the decoding system, and omit the other one.

Considering the impulsive system (31)-(32), we obtain that

\[
J(z_m) = H e^{A(z_m^0 - z_m^1)} J(z_{m-1}) = P(z_{m-1}, z_m) J(z_{m-1})
\]

where

\[
P(z_{m-1}, z_m) = H e^{A(z_m^0 - z_m^1)} \]

\[
= \text{diag}(P_1(z_{m-1}, z_m), P_2(z_{m-1}, z_m), \ldots, P_p(z_{m-1}, z_m)).
\]

Note that \( P(z_{m-1}, z_m) \) is a block diagonal matrix in which

\[
P_r(z_{m-1}, z_m) = 2^{-R_r} U_r(\Delta_m)
\]

with \( r = 1, 2, \ldots, p, \Delta_m = z_m - z_{m-1} \) and \( U_r(\Delta_m) \) can be obtained from (38).

Iteratively from (45), we obtain that

\[
J(z_m) = \prod_{k=1}^{m} P(z_{k-1}, z_k) J(z_0) = P(z_0, z_m) J(z_0)
\]

where \( P(z_0, z_m) := \prod_{k=1}^{m} P(z_{k-1}, z_k) \) is a block diagonal matrix given by

\[
P_r(z_0, z_m) = \text{diag}(P_1(z_0, z_m), P_2(z_0, z_m), \ldots, P_p(z_0, z_m))
\]

in which

\[
P_r(z_0, z_m) = \prod_{k=1}^{m} P_r(z_{k-1}, z_k)
\]

with \( r = 1, 2, \ldots, p \).

Recall that \( \{z_m\}_{m \in \mathbb{Z}_0} \) denotes the sequence of time instants of the successful transmissions. Now we introduce a lemma concerning the convergence of \( J(z_m) \).

**Lemma 4:** Consider the dynamics of \( J(t) \) in (31)-(32) and the DoS attacks in Assumptions 1 and 2 satisfying \( \frac{1}{\Delta} + \frac{1}{T} < 1 \) with \( \Delta \) being the sampling interval of the network as in (2).

All the elements in the column vector \( J(z_m) \) converge to zero as \( z_m \to \infty \) if \( R_r \) satisfies

\[
R_r \left\{ \begin{array}{ll}
\frac{1}{1 - \frac{1}{T} \Delta} c_r \Delta \log_2 e, & \text{if } c_r \geq 0 \\
0, & \text{if } c_r < 0 \end{array} \right.
\]

where \( c_r \) is the real part of \( \lambda_r \).

**Proof:** In this proof, we mainly show that \( \|P(z_0, z_m)\| \) converges to zero as \( z_m \to \infty \) if \( \frac{1}{\Delta} + \frac{1}{T} < 1 \) and (50) are satisfied, which implies the convergence of \( J(z_m) \).

According to (47) and (49), we have

\[
P_r(z_0, z_m) = \prod_{k=1}^{m} P_r(z_{k-1}, z_k)
\]

\[
= \prod_{k=1}^{m} (2^{-R_r} U_r(\Delta_k))
\]

\[
= (2^{-R_r}) m U_r(\sum_{k=1}^{m} \Delta_k).
\]

Substituting (38) into (51), we obtain

\[
P_r(z_0, z_m) = e^{r(z_m - z_0)} V_r(z_m - z_0) \otimes W
\]

\[
= e^{r(z_m - z_0)} (z_m - z_0)^{n_r - 1} V_r(z_m - z_0) \otimes W.
\]

It is easy to verify that

\[
V_r(z_m - z_0) \otimes W
\]

\[
\leq \left[ \begin{array}{cccc}
\frac{1}{(z_m - z_0)^{n_r - 1}} & \frac{1}{(z_m - z_0)^{n_r - 2}} & \cdots & \frac{1}{(z_m - z_0)} \\
\frac{1}{(z_m - z_0)^{n_r - 1}} & \frac{1}{(z_m - z_0)^{n_r - 2}} & \cdots & \frac{1}{(z_m - z_0)} \\
\vdots & \vdots & \ddots & \vdots \\
\frac{1}{(z_m - z_0)^{n_r - 1}} & \frac{1}{(z_m - z_0)^{n_r - 2}} & \cdots & \frac{1}{(z_m - z_0)}
\end{array} \right] \otimes W
\]

(53)
which is upper bounded for $z_m - z_0 \geq \Delta$. Meanwhile, exploiting that $m = T_S(z_0, z_m)$ in Lemma 3, we have
\[
e^{\sigma_(z_m - z_0)} \left( \frac{z_m - z_0}{2R_r} \right)^m \leq \theta_r \left( z_m - z_0 \right)^{m - 1} \quad (54)
\]
where $\theta_r := \frac{2^{R_r(\epsilon + \Delta)}}{\gamma}$. If (50) holds and $\frac{1}{\tau} + \frac{\Delta}{\tau \theta_r} < 1$, it is simple to verify that
\[
\alpha_r := \frac{e^{\epsilon \theta_r}}{2R_r \left( 1 + \frac{\Delta}{\tau \theta_r} \right)} < 1. \quad (55)
\]
This implies that there exist a finite number $C_0^f$ and $\mu_r < 0$ such that
\[
e^{\epsilon \theta_r (z_m - z_0)} \left( \frac{z_m - z_0}{2R_r} \right)^m \leq \theta_r \left( \alpha_r \right)^{z_m - z_0} \left( \frac{z_m - z_0}{2R_r} \right)^{m - 1} \leq C_0^f e^{\mu_r (z_m - z_0)}. \quad (56)
\]
In view of (52), (53) and (56), there exists a finite $C_2^f$ such that
\[
\left\| P_r(z_0, z_m) \right\| \leq C_0^f e^{\mu_r (z_m - z_0)} \left\| V_r(z_m - z_0) \right\| \left\| W \right\| \leq C_1^f e^{\mu_r (z_m - z_0)}. \quad (57)
\]
and hence we obtain that there exists finite $C_2$ and $\mu$ such that
\[
\left\| J(z_m) \right\| \leq C_2 e^{\mu (z_m - z_0)} \left\| J(z_0) \right\|. \quad (58)
\]
Finally we obtain the convergence of $J(z_m)$ when $z_m \to \infty$. $\blacksquare$

After proving the convergence of $J(z_m)$, now we introduce another lemma concerning the convergence of $J(t)$ and $e(t)$. 

Lemma 5: Consider $J(t)$ and $e(t)$ whose dynamics are given by (31)-(32) and (35), respectively. Suppose that the DoS attacks in Assumptions 1 and 2 satisfy $\frac{1}{\tau} + \frac{\Delta}{\tau \theta_r} < 1$. If the bit rate $R_r$ satisfies (50), then $J(t)$ and $e(t)$ converge exponentially to the origin.

Proof. According to (31), (58) and Lemma 2 for $z_m \leq t < z_{m+1}$, we have
\[
\left\| J(t) \right\| \leq e^{\bar{\sigma}(t - z_m)} \left\| J(z_m) \right\| \leq e^{\bar{\sigma}(t - z_m)} \left\| J(z_{m+1}) \right\| \leq C e^{\epsilon (t - z_m)} e^{\mu (z_m - z_0)} \left\| J(z_0) \right\| \leq C e^{\epsilon (t - z_m)} e^{\mu (z_m - z_0)} \left\| J(z_0) \right\| \leq \gamma_0 e^{\epsilon (t - z_m)} \left\| J(z_0) \right\| \leq \gamma_1 e^{\epsilon (t - z_m)} \left\| J(z_0) \right\| \quad (59)
\]
where $v = \max\{0, \tilde{v}\}$ with $\tilde{v} = \lambda_{\max} \left( \frac{A + \tilde{A}}{2} \right)$ denoting the logarithmic norm of $\tilde{A}$ and $\gamma_0 := C e^{\epsilon (Q + \Delta)} e^{\mu (Q + \Delta)}$. Since $\gamma_0$ is finite and $\mu < 0$, we conclude that $J(t)$ exponentially converges to the origin when $t \to \infty$. In light of (44), one could also obtain
\[
\left\| e(t) \right\| \leq \left\| J(t) \right\| \leq \gamma_0 e^{\mu (t - z_0)} \left\| J(z_0) \right\| \quad (60)
\]
which implies the convergence of $e(t)$. $\blacksquare$

E. Main result

Now we are ready to present the main result of this paper.

Theorem 2: Consider the linear time-invariant process $\{ t \}$ and its transformed system $\{ z \}$ with control action (29)-(33) under the transmission policy in (2). The transmitted signals are quantized by the uniform quantizer (23)-(26). Suppose that the DoS attacks characterized in Assumptions 1 and 2 satisfy $\frac{1}{\tau} + \frac{\Delta}{\tau \theta_r} < 1$. If the bit rate $R_r$ with $r = 1, 2, \cdots, p$ satisfies (50), then the state of the closed-loop system exponentially converges to the origin.

Proof. Recall the control input $u(t) = \hat{K}(t) \hat{x}(t) = K^{-1}E(t)^{-1} \hat{x}(t) = K\hat{x}_p(t)$, where $\hat{x}_p(t) = S^{-1}E(t)^{-1} \hat{x}(t)$ can be interpreted as the estimation of the original process state $x(t)$ in (1). Then one has the error between the estimation of $x(t)$ (i.e. $\hat{x}_p(t)$) and $x(t)$ such that $e_p(t) := \hat{x}_p(t) - x(t)$. Thus (1) can be rewritten as $\dot{x} = (A + BK)x(t) + BK e_p(t)$, whose solution is
\[
x(t) = e^{(A + BK)(t - z_0)} x(z_0) + \int_{z_0}^{t} e^{(A + BK)(t - \tau)} BK e_p(\tau) d\tau \quad (61)
\]
where $t \in \mathbb{R}_{\geq z_0}$. From the equation above, one sees that the stability of $x(t)$ depends on $e_p(t)$. Thus, we analyze $e_p(t)$ such that
\[
e_p(t) = \hat{x}_p(t) - x(t) = S^{-1}E(t)^{-1} \hat{x}(t) - S^{-1}E(t)^{-1} \hat{x}(t) = S^{-1}E(t)^{-1} \hat{x}(t) - S^{-1}E(t)^{-1} e(t). \quad (62)
\]
If $\frac{1}{\tau} + \frac{\Delta}{\tau \theta_r} < 1$ and $R_r$ satisfies (50), then (60) holds. Then one has
\[
\left\| e_p(t) \right\| \leq \left\| S^{-1}E(t)^{-1} \right\| \left\| e(t) \right\| \leq \left\| S^{-1}E(t)^{-1} \right\| \left\| e_0 e^{\mu (t - z_0)} \right\| \left\| J(z_0) \right\| \leq \gamma_1 e^{\mu (t - z_0)} \left\| J(z_0) \right\|. \quad (63)
\]
Note that such $\gamma_1$ exists and is finite since $\left\| S^{-1}E(t)^{-1} \right\|$ is bounded. Taking the norm of both sides of the solution (61) and applying (63), one has
\[
\left\| x(t) \right\| \leq e^{\sigma (t - z_0)} \left\| x(z_0) \right\| + \int_{z_0}^{t} e^{\sigma (t - \tau)} BK \left\| e_p(\tau) \right\| d\tau \leq e^{\sigma (t - z_0)} \left\| x(z_0) \right\| + \int_{z_0}^{t} e^{\sigma (t - \tau)} BK \left\| e_0 e^{\mu (t - z_0)} \right\| \left\| J(z_0) \right\| d\tau \leq e^{\hat{\sigma} (t - z_0)} \left\| x(z_0) \right\| + \int_{z_0}^{t} e^{\hat{\sigma} (t - \tau)} BK \left\| e_0 e^{\mu (t - z_0)} \right\| \left\| J(z_0) \right\| d\tau \leq e^{\hat{\sigma} (t - z_0)} \left\| x(z_0) \right\| + (t - z_0) e^{\hat{\sigma} (t - z_0)} \left\| BK \right\| \left\| J(z_0) \right\|. \quad (64)
\]
where $\sigma < 0$ is the logarithmic norm of $A + BK$ and $\xi := \max \{\mu, \sigma\} \in \mathbb{R}_{<0}$. Since $\xi < 0$, there exist two finite reals $\delta$ satisfying $\xi < \delta < 0$ and $C_3$ such that $(t - z_0)e^{\xi(t - z_0)} \leq C_3e^{\delta(t - z_0)}$. Then we have

$$
\|x(t)\| \leq e^{\delta(t - z_0)}(\|x(z_0)\| + C_3\gamma_1\|B\||\|J(z_0)\|). \quad (65)
$$

It is immediate to see that $x(t)$ exponentially converges to the origin as $t \to \infty$.

Moreover, in view of (59) and (60), and the fact that $\bar{e}(t) = E(t)Sx(t)$ and $\|\bar{e}(t)\| \leq \|e(t)\| + \|\tilde{e}(t)\|$, we conclude that $J(t)$, $c(t)$, $\bar{e}(t)$, $\tilde{e}(t)$ and $x(t)$ exponentially converge to the origin as $t \to \infty$. This completes the proof.

**Remark 3:** We emphasize that this theorem characterizes how the bit rate affects the system’s resilience. Condition (60) can be rewritten as

$$
\frac{1}{T} + \frac{\Delta}{\tau_D} < 1 - \frac{c_r\Delta \log_2 e}{R_r}, \quad \forall c_r \geq 0 \quad (66)
$$

where $R_r > 0$. The inequality above explicitly quantifies how the data rate affects the robustness, e.g. the larger $R_r$, the smaller $T$ and $\tau_D$ can be, which implies that the system can tolerate more DoS attacks in terms of duration and frequency, and still preserve stability. Figure 4 exemplifies this characterization.

- **Remark 4:** In view of Theorem 2 if the network is reliable ($T = \tau_D = \infty$ and $\kappa = \eta = 0$), one obtains that the closed-loop system is exponentially stable if $R_r$ satisfies

$$
R_r \begin{cases} 
> c_r\Delta \log_2 e, & \text{if } c_r \geq 0, \\
\geq 0, & \text{if } c_r < 0 \end{cases} \quad (r = 1, 2, \cdots, p). \quad (67)
$$

To this end, we almost recover the result (Theorem 6) obtained in [7], where no attacks were considered. By “almost”, we mean that if one omits the disturbance and noise, and considers asymptotic stabilization in [7], then the data rate in (67) and Theorem 6 in [7] are equivalent and minimum, namely they are necessary and sufficient conditions. This is the advantage of the result achieved in this paper in the aspect of recovering the minimum data rate, comparing with the one considering output-feedback scenario in [25].

Under Theorem 2 the average data rate associated with the successfully received packets is

$$
D_d := \lim_{z_m \to \infty} \sum_{t=1}^{z_m} \frac{R_i T_S(z_0, z_m)}{z_m - z_0} > \sum_{k=\{i|c_i \geq 0\}} c_k \log_2 e \quad (68)
$$

which essentially depends on the real parts of the eigenvalues of the dynamic matrix of the process. The average data rate associated with the transmission attempts is

$$
D_e := \lim_{z_m \to \infty} \frac{\sum_{t=1}^{z_m} R_i}{z_m - z_0} = \frac{1}{1 - \frac{\Delta}{\tau_D}} \sum_{k=\{i|c_i \geq 0\}} c_k \log_2 e \quad (69)
$$

which is the corresponding result on packet size under DoS attacks comparing with the achieved result in [7] where genuine packet dropout is considered. Moreover, under a 100% reliable network, one should have $D_e = D_d$, in which case what is sent by the encoder is fully received by the decoder. Due to the presence of DoS attacks, a larger average bit rate associated with transmission attempts is needed, namely $D_e > D_d$, and the lower bound of the average data rate associated with transmission attempts is scaled by $1 - \frac{\Delta}{\tau_D} \in \mathbb{R}_{>1}$ in (69). This reflects the need of redundant communication resources to compensate for the side effect of DoS attacks.

**F: Stability condition over the average data rate**

We have shown that if Theorem 2 holds, then the closed-loop system is stable. The setting there is that the number of bits transmitted at $z_m$ ($m = 0, 1, \cdots$) are identical and equivalent to $R_r$. In this subsection, we loosen the sufficient condition above in the sense that the number of bits transmitted at each successful transmission time ($z_m$) does not have to be identical. Later we will show that if the average value of them is greater than $\frac{1}{1 - \frac{\Delta}{\tau_D}} c_r \Delta \log_2 e$ with $c_r \geq 0$, then the closed-loop system is still stable.

Assume that the number of bits assigned to each transmission attempt is arbitrary, and let $R_r(t_k)$ denote the number of bits applied to each element corresponding to $A_r$ at $t_k$. Notice that $R_r(t_0), R_r(t_1), \cdots$ are not necessarily identical. Due to the physical constraints of communication equipments, it is practical to assume that the maximum number of bits that the network can transmit in one transmission is finite, namely $R_r(t_k) < \infty$ for $k \in \mathbb{Z}_0$. This implies that the average value $R_{r,k} := \frac{r R_r(t_0) + R_r(t_1) + \cdots + R_r(t_k-1)}{r} < \infty$. It is easy to verify that $\{R_r(z_m)\} \subset \{R_{r,k}\}$ and $R_{r,m} := \frac{r R_r(z_0) + R_r(z_1) + \cdots + R_r(z_m-1)}{r} < \infty$ for $m = 1, 2, \cdots, n_m$.

Recall the definition of $\{t_k\}_{k \in \mathbb{Z}_0}$ and $\{z_m\}_{m \in \mathbb{Z}_0}$. The proposition below presents the sufficient condition for stability concerning the average data rate.

**Proposition 1:** Under the transmission policy in (2), consider the process (1) and its transformed system (8) with control action (29)-(33) and the uniform quantizer (25)-(26),
where $R_k = R_r(t_k)$ are arbitrary and finite at each $t_k$. The DoS attacks are characterized as in Assumptions 1 and 2 and satisfy $\frac{1}{T} + \frac{\Delta}{TP} < 1$. If the average value of bits along \(\{z_m\}_{m=1,2,\cdots}\) satisfies
\[
R_{r,m} \left\{ \begin{array}{ll}
g_c r \Delta \log_2 e, & \text{if } c_r \geq 0, \quad r = 1, 2, \cdots, p \\
\geq 0, & \text{if } c_r < 0
\end{array} \right.
\]
then the closed-loop system is stable.

**Proof.** By observing (53), we could obtain that $P_r(z_0, z_m)$ under the average data rate scenario is given by $P_r(z_0, z_m) = U_r(\sum_{k=1}^{m} \Delta_k) \prod_{k=1}^{m} 2^{-R_r(z_{k-1})}$. Then we have
\[
P_r(z_0, z_m) = U_r(\sum_{k=1}^{m} \Delta_k) \prod_{k=1}^{m} 2^{-R_r(z_{k-1})} = \prod_{k=1}^{m} 2^{-R_r(z_{k-1})} = \prod_{k=1}^{m} 2R_r(z_{k-1}) \quad \text{if } \bar{R}_{r,m} = 1
\]
Exploiting that $m = T_S(z_0, z_m)$ in Lemma 5 we have
\[
e^{c_r(z_m-z_0)}(z_m-z_0)^{n_r-1} \leq \theta_{r,m}(\bar{a}_{r,m}, m) \text{ is finite and}
\]
\[
\bar{a}_{r,m} := \frac{c_r}{2R_{r,m} \left(1 + \frac{\Delta}{TP} \right)} < 1
\]
if (70) holds. The rest of the proof can follow the analysis after (55), and we obtain the stability of the closed-loop system.

IV. TIME-VARYING BIT RATE

In this section, we aim at designing a time-varying bit-rate protocol, which preserves a comparable level of resilience against DoS while promoting the possibility of saving bits when the attack levels are low. Recalling $\bar{A}_r$ in Lemma 1, we equip the quantization systems corresponding to $\bar{A}_r (r = 1, 2,\cdots, p)$ with their own “clocks”. Due to the utilization of the acknowledgment-based protocol, the acknowledgments could enable the encoders to update the time sequences soon after the updates of the time sequences in the decoders. This facilitates the quantization systems to use a time-varying bit-rate protocol, in which the number of bits is predetermined before each transmission and depends on the generated time sequences.

We briefly introduce the intuition of the time-varying bit-rate protocol. There are two scenarios.

**Scenario 1:** If the duration of a DoS attack is short, then after the attack, a transmission with fewer bits is enough to guarantee the decay of the quantization range and there is no need to transmit a large number of bits. Actually, it is this mechanism that saves bits compared with the time-invariant bit-rate protocol.

**Scenario 2:** If the duration of a DoS attack is long, then the quantization systems may not be able to obtain the decay of the quantization range with one transmission, even by applying the maximum bit rate. Confronted by this problem, the transmitter attempts to send packets with the maximum number of bits, e.g. $R_r$, for longer time until the quantization range is restored to the level smaller than the one before the “long-time DoS” occurs.

In the time-varying bit-rate protocol, both the encoding and decoding systems consist of two major parts: A) bit-computing parts and B) coding parts. The bit-computing parts pre-determine the number of bits for encoding and decoding (before each transmission attempt instant $t_k$), and the coding parts are responsible for encoding or decoding signals at $t_k$, by applying the pre-determined number of bits.

A. Bit-computing parts

The bit-computing parts mainly generate sequences of time instants and then based on the time sequences, they pre-determine the number of bits for the transmission attempts. Note that both the encoding and decoding systems are equipped with the identical bit-computing parts and coding parts.

We first introduce sequences of time instants generated by the bit-computing parts in the encoding and decoding systems, i.e. $\{s_g^r\} = \{s_0^r, s_1^r, s_2^r, \cdots\} \subseteq \{z_m\}$ with $r = 1, 2,\cdots, p$ and $g \in \mathbb{Z}_0$. In Section IX, we will show that the state corresponding to $\bar{A}_r$ strictly decays along $\{s_g^r\} = \{s_0^r, s_1^r, s_2^r, \cdots\}$ with $r = 1, 2,\cdots, p$. In particular, we have
\[
\left\{ \begin{array}{ll}
s_g = \min\{z_m > s_g^{r-1}, \frac{e^{c_r(z_m-s_g^{r-1})}}{(2R_{r,m} \left(1 + \frac{\Delta}{TP} \right))^{n_r-1}} < 1\} \\
s_0 = 0
\end{array} \right.
\]
where \( R_r \) satisfies Theorem 2. Note that due to the acknowledgments, \( \{s_g^r\} \) can be synchronized in the encoding and decoding systems. Here abusing the notation, \( T_s(s_{g-1}^r, z_m) \) represents the number of successful transmissions during \([s_{g-1}^r, z_m] \). Since the number of successful transmissions during \([s_{g-1}^r, z_m] \) and \([s_{g-1}^r, z_m] \) are the same, then Lemma 3 is still valid when we refer to \([s_{g-1}^r, z_m] \). By applying Lemma 3 we have
\[
e^{c_r(z_m-s_{g-1}^r)} \leq \theta_r(\alpha_r)z_m-s_{g-1}^r. \tag{75}
\]

Note that \( \alpha_r < 1 \) in (55) and \( \theta_r \) is finite if Theorem 2 holds. Then there always exists the smallest and finite \( z_m > s_{g-1}^r \) such that \( e^{c_r(z_m-s_{g-1}^r)} \leq \theta_r(\alpha_r)z_m-s_{g-1}^r < 1 \). Hence according to (74), we have that \( s_{g}^r - s_{g-1}^r \) is finite when Theorem 2 holds.

The pre-determined number of bits in the encoding and decoding systems for the transmission attempts follows
\[
R_t(t_k) = R_r(t_k) = \begin{cases} 
\min\{R_r(t_k), R_r\}, & \text{if } \frac{e^{c_r(t_k-s_{g-1}^r)}}{(2R_rT_s(s_{g-1}^r,t_k)+1)} < 1 \\
R_r, & \text{otherwise}
\end{cases} \tag{76}
\]

where \( R_t(t_k) := [w_r(t_k-s_{g-1}^r) \log_2 e], t_k \in [s_{g-1}^r, s_{g+1}^r], w_r \in \mathbb{R}_{>0}, R_r \) satisfies Theorem 2 and \( r \) is the index of \( A_r \) that the \( t \)-th element corresponds to.

Note that \( R_t(t_k) = R_r(t_k) = R_r(t_k) \) and if \( t_k \) is a successful transmission instant such that \( t_k = z_m \), then \( T_s(s_{g-1}^r, t_k) + 1 = T_s(s_{g-1}^r, z_m) \). By \( T_s(s_{g-1}^r, t_k) + 1 \), it simply means that before the real transmission attempt at \( t_k \), the bit-computing parts first estimate \( e^{c_r(t_k-s_{g-1}^r)} \) by assuming that \( t_k \) would be a successful transmission instant. If it estimates that by using \( R_r \) bits at \( t_k \), the system would have \( \frac{e^{c_r(t_k-s_{g-1}^r)}}{(2R_rT_s(s_{g-1}^r,t_k)+1)} < 1 \), then according to (76) by using \( \min\{R_r(t_k), R_r\} \) bits at \( t_k \), the system would still have
\[
e^{c_r(t_k-s_{g-1}^r)} \leq \frac{R_rT_s(s_{g-1}^r,t_k)+\min\{R_r(t_k), R_r\}}{(2R_rT_s(s_{g-1}^r,t_k)+1)} \tag{77}
\]

which in turn implies the decay of the quantization range (see Section IV. C). Since \( \min\{R_r(t_k), R_r\} \leq R_r \), we achieve the possibility of the reduction of bits. For the ease of visualization, the evolution of \( e^{c_r(t_k-s_{g-1}^r)} \) (where \( c_r > 0 \), \( \{s_g^r\} \) and the applied number of bits are exemplified in Figure 5).

![Fig. 5](image-url)

**Fig. 5.** The evolution of \( e^{c_r(t_k-s_{g-1}^r)} \) and \( \{s_g^r\} \). The instants of successful transmissions \((z_m)\) are indicated by the red lines, at which there are the values of \( e^{c_r(t_k-s_{g-1}^r)} \) being indicated by the blue solid dots and the number of applied bits \((R_r)\) or \((R_r) := \min\{R_r, R_r\}\). Among the instants of successful transmissions, we highlight the sequence \( \{s_g^r\} \), at which the values of \( e^{c_r(t_k-s_{g-1}^r)} \) are smaller than 1. The dashed blue curves represent the evolution due to \( e^{c_r} \) and the solid blue lines represent their drops due to the successful transmissions.

**B. Coding parts**

In the last part, we obtain the pre-determined number of bits applied to each transmission attempt. By applying such a number of bits, the coding parts consisting of the quantizers

\[
e^{c_r(t_k-s_{g-1}^r)} \leq \frac{R_rT_s(s_{g-1}^r,t_k)+\min\{R_r(t_k), R_r\}}{(2R_rT_s(s_{g-1}^r,t_k)+1)} \tag{77}
\]

which in turn implies the decay of the quantization range (see Section IV. C). Since \( \min\{R_r(t_k), R_r\} \leq R_r \), we achieve the possibility of the reduction of bits. For the ease of visualization, the evolution of \( e^{c_r(t_k-s_{g-1}^r)} \) (where \( c_r > 0 \), \( \{s_g^r\} \) and the applied number of bits are exemplified in Figure 5).

\[
\begin{align*}
\Phi(t) &= \begin{bmatrix} \bar{\phi}_1(t) \\ \vdots \\ \bar{\phi}_{n_x}(t) \end{bmatrix} = \begin{bmatrix} \bar{J}_1(t)q_R(t) \chi_1(t) \\ \vdots \\ \bar{J}_{n_x}(t)q_R(t) \chi_{n_x}(t) \end{bmatrix} \\
\end{align*}
\tag{81}
\]

Meanwhile, \( H(t) \) is given by
\[
H(t) = \text{diag}(2^{-R_1(t)}I_1, 2^{-R_2(t)}I_2, \ldots, 2^{-R_p(t)}I_p) \tag{82}
\]

and we let \( H_r(t) := 2^{-R_r(t)}I_r \). Then we have the impulsive system for obtaining the quantization range
\[
\begin{align*}
J(t) &= A_J(t), & \text{if } t \neq z_m \\
J(t) &= H(t)J(t), & \text{if } t = z_m \tag{83}
\end{align*}
\]

At last, the predictor in the time-varying bit-rate design is given by
\[
\begin{align*}
\hat{x}(t) &= \hat{A}\hat{x}(t) + \hat{B}(t)u(t), & t \neq z_m \\
\hat{x}(t) &= \hat{x}(t) - \hat{F}(t), & t = z_m \tag{84}
\end{align*}
\]
where \( u(t) = \bar{K}\hat{x}(t) \) and \( (72) \) holds. By applying a very similar analysis as in Section III, C, one could see that there is no over-flow problem of the quantization systems under the time-varying bit-rate protocol.

In view of \((73)\) and \((76)\) and the coding parts, the mechanism of the time-varying bit-rate protocol can be outlined as

1) Set \( s_0^r = 2\bar{g} \).

2) Let \( t_k \) be the transmission attempt instant after \( s_{r-1}^g \) such that \( s_{r-1}^g < t_k \leq s_g^r \). At \( t_k \), the bit-computing parts in the encoding and decoding systems calculate

\[
e^{c_r(t_k - s_{r-1}^g)}(2\bar{r}_r)^{s_g^r - s_{r-1}^g} > 1.
\]

2.1) If \( e^{c_r(t_k - s_{r-1}^g)}(2\bar{r}_r)^{s_g^r - s_{r-1}^g} < 1 \), then in view of \((76)\), we have \( R_l(t_k^e) = R_r(t_k^e) = \min\{R_l(t_k), R_r\} \), and \( R_r(t_k) = R_r(t_k^e) \) in the bit-computing parts.

2.1.1) If the transmission succeeds (\( t_k = \tilde{t}_k \)), update \( J \) in \((83)\), \( \hat{x} \) in \((84)\) by using the number of bits determined in 2.1, and update \( s_g^r = z_m \) in light of \((73)\) in both the encoding and decoding systems. Then we are at 2) with \( s_{r-1}^g \) becoming \( s_g^r \) and wait for the next \( t_k \) and repeat 2).

2.1.2) If the transmission fails, wait for the next \( t_k \) and repeat 2).

2.2) If \( e^{c_r(t_k - s_{r-1}^g)}(2\bar{r}_r)^{s_g^r - s_{r-1}^g} \geq 1 \), then according to \((76)\), we have \( R_l(t_k^e) = R_r(t_k^e) = \bar{r}_r \), and \( R_r(t_k) = R_r(t_k^e) \).

2.2.1) If the transmission succeeds, update \( J \) in \((83)\), \( \hat{x} \) in \((84)\) by using the number of bits determined in 2.2 in both the encoding and decoding systems. Then wait for the next \( t_k \) and repeat 2).

2.2.2) If the transmission fails, wait for the next \( t_k \) and repeat 2).

C. Stability analysis

For the ease of conveying the ideas, we focus our analysis on the dynamics corresponding to \( A_r \). From \((83)\), it is easy to obtain that

\[
J_r(s_g^r) = \tilde{P}_r(s_{g-1}^r, s_g^r)J_r(s_g^r-1)
\]

\[(85)\]

with \( \tilde{P}_r(s_{g-1}^r, s_g^r) = e^{A_r(s_g^r - s_{g-1}^r)} \prod_{s_{g-1}^r < z_m \leq s_g^r} H_r(z_m) \)

\[(86)\]

where \( H_r(z_m) = 2^{-R_r(z_m)^2}l_r \). \( J_r \) is the subset of \( J \) corresponding to \( A_r \). Since \( e^{A_r(s_g^r - s_{g-1}^r)} = U_r(s_g^r - s_{g-1}^r) \) is upper-triangular whose eigenvalues equal to \( e^{c_r(s_g^r - s_{g-1}^r)} \), and \( \prod_{s_{g-1}^r < z_m \leq s_g^r} H_r(z_m) \) is diagonal, it is easy to obtain the eigenvalues of \( \tilde{P}_r(s_{g-1}^r, s_g^r) \) such that

\[
\lambda_r(\tilde{P}_r(s_{g-1}^r, s_g^r)) = e^{c_r(s_g^r - s_{g-1}^r)} \prod_{s_{g-1}^r < z_m \leq s_g^r} 2^{-R_r(z_m)^2}
\]

\[(87)\]

Iteratively, it is easy to verify that

\[
J_r(s_g^r) = \prod_{k=1}^{g} \tilde{P}_r(s_{k-1}^r, s_k^r)J_r(s_0^r)
\]

\[(88)\]

Recall the definition of \( \{s_g^r\} \) in \((74)\). The next lemma concerns the convergence of \( J_r(s_g^r) \) in \((74)\) and the DoS attacks in Assumptions 1 and 2 satisfying \( 0 < \beta_r < 1 \) such that \( ||J_r(s_g^r)|| \leq C_r^g(\beta_r)^{g-1}||J_r(s_0^r)|| \) in view of \((83)\).

Proof. In the proof, we would like to show that the eigenvalues of \( \tilde{P}_r(s_{g-1}^r, s_g^r) \) satisfy \( \lambda_r(\tilde{P}_r(s_{g-1}^r, s_g^r)) < 1 \) for \( g \in Z_1 \). Let \( z_m^r \) denote the first successful transmission instant after \( s_{g-1}^r \). If \( e^{c_r(s_g^r - s_{g-1}^r)} \geq 1 \), then according to \((76)\), \( \min\{R_l(z_m^r), R_r\} \) would be applied for coding. Hence based on \((87)\), we obtain that

\[
\lambda_r(\tilde{P}_r(s_{g-1}^r, z_m^r)) = e^{c_r(z_m^r - s_{g-1}^r)} \prod_{s_{g-1}^r < z_m^r \leq s_g^r} 2^{-R_r(z_m^r)}
\]

\[
= \frac{e^{c_r(z_m^r - s_{g-1}^r)}}{2^{-R_r(z_m^r)}} \frac{e^{c_r(z_m^r - s_{g-1}^r)}}{2^{-R_r(z_m^r)}} < 1 \tag{89}
\]

where the inequality is implied by the hypothesis. Meanwhile, we see that such \( z_m^r \) qualifies \((74)\) and hence \( s_g^r = z_m^r \). One obtains that \( \lambda_r(\tilde{P}_r(s_{g-1}^r, s_g^r)) < 1 \).

If \( e^{c_r(s_g^r - s_{g-1}^r)} < 1 \), then according to \((76)\), the coding systems would apply \( R_r \) bits at \( z_m^r \) and during \( [z_m^r, z_m^r] \) where \( z_m^r > z_m \). Therefore, according to \((87)\), we obtain that

\[
\lambda_r(\tilde{P}_r(s_{g-1}^r, z_m^r)) = e^{c_r(z_m^r - s_{g-1}^r)} \prod_{s_{g-1}^r < z_m^r \leq z_m^r} 2^{-R_r(z_m^r)}
\]

\[
= \frac{e^{c_r(z_m^r - s_{g-1}^r)}}{2^{-R_r(z_m^r)}} \frac{e^{c_r(z_m^r - s_{g-1}^r)}}{2^{-R_r(z_m^r)}} < 1 \tag{90}
\]

where the rationale of the inequality and the existence of such \( z_m^r \) have been discussed in \((75)\) and the discussion thereafter. Hence such \( z_m^r \) is denoted by \( s_g^r \) and we have \( \lambda_r(\tilde{P}_r(s_{g-1}^r, s_g^r)) < 1 \).

In either case, we have shown that \( \lambda_r(\tilde{P}_r(s_{g-1}^r, s_g^r)) < 1 \), which implies that \( \{\tilde{P}_r(s_{g-1}^r, s_g^r)\}_{g \in Z_1} \) is a sequence of stable matrices and there exist finite \( C_r^g \) and \( 0 < \beta_r < 1 \) such that \( ||J_r(s_g^r)|| \leq C_r^g(\beta_r)^{g-1}||J_r(s_0^r)|| \) in view of \((83)\). Therefore, one can infer that \( J_r(s_g^r) \to 0 \) when \( g \to \infty \) with \( r = 1, 2, \ldots, p \). This completes the proof.

In view of the dynamics of \( J(t) \), we have

\[
||J_r(t)|| \leq C_r e^{c_r(t-s_{g-1}^r)(\beta_r)^{g-1}}||J_r(s_0^r)|| \tag{91}
\]
where \( s_g' \leq t < s_{g-1}' \) and \( v_r = \max\{0, \bar{v}_r\} \) with \( \bar{v}_r = \lambda_{\max}\left(\frac{A - A^T}{2}\right) \). Since \( 0 < \beta_r < 1 \) and \( s_r' - s_{g-1}' \) is finite, one knows that there exist finite \( \gamma_r \) and \( \bar{\mu}_r < 0 \) such that \( \|J_r(t)\| \leq \gamma_r e^{|\bar{\mu}_r| (s_{g-1}' - s_0')} \|J_r(s_0')\| \). This implies that there exists finite \( \tilde{\gamma}_r \) and \( \bar{\mu}_r < 0 \) such that \( \|J(t)\| \leq \gamma r'e^{|\bar{\mu}_r| (s_{g-1}' - s_0')} \|J(s_0')\| \) with \( r = 1, 2, \ldots, p \) by noticing that \( s_0' = s_0 \). Since \( \|e\| \) is upper bounded by \( J \), we obtain that \( \|e(t)\| \leq \|J(t)\| \leq \gamma_0 e^{|\bar{\mu}_r| (s_{g-1}' - s_0')} \|J(s_0')\| \).

**Theorem 3:** Consider the process \( \{1\} \) with control action (31) under the transmission policy in (32). Suppose the DoS attacks characterized as in Assumptions 1 and 2 and satisfy \( \frac{1}{\bar{e}} + \frac{\Delta}{\tau_D} < 1 \). The transmitted signals are quantized by the time-varying-bit quantizer (78)-(79) where \( R_s \) satisfies Theorem 2 and \( w_r > c_r \). Then the closed-loop system is exponentially stable.

**Proof.** Since \( e(t) \) exponentially converges to the origin in view of Lemma 6 and the discussion thereafter, following the very similar calculation as in the proof of Theorem 2 one can obtain the exponential stability of the closed-loop system. ■

**Remark 5:** It is worth mentioning that the reduction of bits is achieved by sacrificing the decay rate of the system, i.e. the system converges in a slower rate compared with the one under the time-invariant bit-rate protocol. This is due to the fact that in the absence of DoS attacks or after short-duration DoS attacks, the time-invariant bit-rate protocol is able to apply \( R_s \) bits, while the time-varying bit-rate protocol can only apply \( \min\{R_s(k), R_r\} \) bits (cf. Figure 5). ■

**Remark 6:** One sees that the system under control is stable if one chooses \( w_r \) and \( R_r \) properly. It is easy to make the design parameter \( w_r > c_r \) and hence we omit the influence of it. Then \( R_r \) is the only parameter affecting the robustness of the system under time-varying bit-rate protocol. In view of Remark 5, we characterize the system’s robustness under time-varying bit-rate protocol such that if (56) holds, then the stability of the closed-loop system can be preserved. ■

V. NUMERICAL EXAMPLE

For simplicity, we consider a process that is in Jordan form and taken from (33). The system to be controlled is open-loop unstable and is characterized by the matrices

\[
A = \bar{A} = \bar{A} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \quad B = \bar{B} = B(t) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}
\]

(92)

The state-feedback matrix is given by

\[
K = \bar{K}(t) = \begin{bmatrix} -2.1961 & -0.7545 \\ -0.7545 & -2.7146 \end{bmatrix}
\]

(93)

The eigenvalues of \( A \) are 1.

The network transmission interval is given by \( \Delta = 0.1s \). We consider a sustained DoS attack with variable period and duty cycle, generated randomly. Over a simulation horizon of 20s, the DoS signal yields \( |\Xi(0, 20)| = 15.52s \) and \( n(0, 20) = 20 \). This corresponds to values (averaged over 20s) of \( \tau_D \approx 0.96 \) and \( T \approx 1.29 \), and \( \approx 80\% \) of transmission failures. It is simple to verify that

\[
\frac{\Delta}{\tau_D} + \frac{1}{T} \approx 0.8793
\]

(94)

According to Theorem 2 we obtain that

\[
R_1 > \frac{c_r \Delta}{1 - \frac{1}{\bar{e}} - \frac{\Delta}{\tau_D}} \log_2 e = 1.1953
\]

(95)

Then we select \( R_1 = 2 \). Meanwhile, considering the choice of \( w_1 \) satisfying \( w_1 > c_1 = 1 \), we let \( w_1 = 2 \) for the time-varying bit-rate protocol. The simulation results of \( x(t) \) are shown in Figure 6. We see that \( x(t) \) converges to the equilibrium in both protocols. In particular, the state in the bottom picture converges with a slightly lower speed. This is due to the fact that in the absence of DoS or after a “short-duration” DoS attack, the network transmits fewer bits (cf. Remark 5). This can be observed from Figure 7. One could see that the convergence of \( J(t) \) under time-varying bit-rate protocol shown in the middle picture of Figure 7 (the numbers of bits applied in the time-varying bit-rate protocol are shown in the bottom picture of Figure 7) is slower than the one under time-invariant bit-rate protocol as shown in the top picture of Figure 7.

In fact, the obtained values of bit rate are conservative in the time-invariant bit-rate protocol. The stability can be still preserved at the lower rate with \( R_1 = 1 \) under the same pattern of DoS attacks. One factor contributing to the conservativeness is that the actual number of successful transmissions is much larger than the theoretical value computed in Lemma 3.

From another viewpoint, if the data rate of the channel is pre-selected as \( R_1 = 2 \), the closed-loop system should be stable under the attacks in this example since the DoS parameters satisfy \( \frac{1}{\bar{e}} - \frac{\Delta}{\tau_D} \approx 0.8793 < 1 - \frac{c_1 \Delta \log_2 e}{R_1} = 0.9279 \).
possibly applied to achieve the corresponding bit-rate bounds under the different packet-drop models considered in [39].

APPENDIX

Proof of Lemma 1. Recall $\hat{A}$ in (4), $A_r$ in (5) and (6) representing the Jordan block associated with real and complex eigenvalues, respectively. Let

$$E(t) = \begin{bmatrix} E_1(t) & E_2(t) & \cdots & E_p(t) \end{bmatrix} \in \mathbb{R}^{n_r \times n_r} \quad (96)$$

with $p \in \mathbb{Z}_+$, where

$$E_r(t) = \begin{bmatrix} 1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 1 \end{bmatrix} \in \mathbb{R}^{n_r \times n_r} \quad (97)$$

corresponds to the real eigenvalue $\lambda_r = c_r$, and

$$E_r(t) = \begin{bmatrix} \varpi_r(t) \\ \varpi_r(t) \\ \vdots \\ \varpi_r(t) \end{bmatrix} \in \mathbb{R}^{2n_r \times 2n_r} \quad (98)$$

with

$$\varpi_r(t) = \begin{bmatrix} \cos(d_r t) & \sin(d_r t) \\ -\sin(d_r t) & \cos(d_r t) \end{bmatrix} \quad (99)$$

corresponds to the complex eigenvalues $\lambda_r = c_r \pm d_r i$ ($d_r \neq 0$).

Since $\tilde{x}(t) = E(t)\tilde{x}(t)$, it is easy to verify that

$$\dot{\tilde{x}}(t) = E(t)\dot{\tilde{x}}(t) + \dot{E}(t)\tilde{x}(t)$$

$$= E(t)(\hat{A}\tilde{x}(t) + \hat{B}u(t)) + \dot{E}(t)\tilde{x}(t)$$

$$= E(t)\hat{A}E(t)^{-1}\tilde{x}(t) + \dot{E}(t)E(t)^{-1}\tilde{x}(t) + E(t)\hat{B}u(t)$$

$$= (E(t)\hat{A}E(t)^{-1} + \dot{E}(t)E(t)^{-1})\tilde{x}(t) + E(t)\hat{B}u(t). \quad (100)$$

Let $\tilde{A} := E(t)\hat{A}E(t)^{-1} + \dot{E}(t)E(t)^{-1} = \text{diag}(\tilde{A}_1, \tilde{A}_2, \ldots, \tilde{A}_p)$ and $\tilde{B}(t) := E(t)\hat{B}$, where

$$\tilde{A}_r := E_r(t)\tilde{A}_rE_r(t)^{-1} + \dot{E}_r(t)E_r(t)^{-1}. \quad (101)$$

If the eigenvalues associated with $A_r$ are real, then $E_r(t)$ is an identity matrix in (101) with order $n_r$ and hence the derivative of $E_r(t)$ is a matrix with only zero entries, which implies that

$$\tilde{A}_r = E_r(t)\tilde{A}_rE_r(t)^{-1} + \dot{E}_r(t)E_r(t)^{-1}$$

$$= \tilde{A}_r = \begin{bmatrix} c_r & 1 \\ 1 & c_r \\ \vdots & \vdots \\ 1 & c_r \end{bmatrix}. \quad (102)$$

If the eigenvalues associated with $A_r$ are complex, i.e. $\lambda_r = c_r \pm d_r i$ with $d_r \neq 0$, then $E_r(t)$ is a time-varying matrix as

VI. CONCLUSIONS

We investigated the data rate problem for stabilizing control of a networked control system under limited bandwidth and Denial-of-Service attacks. It was shown that the sufficient condition of bit rate for stabilization depends on the unstable eigenvalues of the dynamic matrix of the process as well as the DoS parameters. Furthermore, the design of time-varying bit-rate protocol is proven to be effective in saving bits meanwhile maintaining the comparable resilience as the one under time-invariant bit-rate protocol. It is emphasized that the results of the paper clearly indicate the trade-offs between the amount of transmitted data and the robustness against DoS attacks. In particular, the approach is in accordance with the recent studies on the minimum data rate control problems.

In the future, disturbance and noise might be taken into consideration. Moreover, the analysis in this paper can be
in (98), whose derivative is not zero any longer. It is simple to verify that
\[ E_r(t) \dot{A}_r E_r(t)^{-1} = \begin{bmatrix} D_r & I \\ D_r & I \\ & & \ddots & I \\ & & & & I \end{bmatrix} \] (103)
with \( E_r(t) \) being in (98) and recalling that
\[ D_r = \begin{bmatrix} c_r & -d_r \\ d_r & c_r \end{bmatrix}, \quad I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \] (104)
On the other hand, we have
\[ \dot{E}_r(t)/E_r(t)^{-1} = \begin{bmatrix} F_r \\ F_r \\ \ddots \\ F_r \end{bmatrix}, \quad \text{where } F_r = \begin{bmatrix} 0 & d_r \\ -d_r & 0 \end{bmatrix}. \] (105)
Thus,
\[ \dot{A}_r = E_r(t) \dot{A}_r E_r(t)^{-1} + \dot{E}_r(t) E_r(t)^{-1} = \begin{bmatrix} c_r I & I \\ c_r I & I \\ & \ddots & I \\ & & & c_r I \end{bmatrix}. \] (106)
Considering the two scenarios in (102) and (106), we obtain the result as in Lemma [11]. This completes the proof. ■

REFERENCES

[1] P. Cheng, L. Shi, and B. Sinopoli, “Guest editorial special issue on secure control of cyber-physical systems,” IEEE Transactions on Control of Network Systems, vol. 4, no. 1, pp. 1–3, 2017.
[2] A. Teixeira, I. Shames, H. Sandberg, and K. H. Johansson, “A secure control framework for resource-limited adversaries,” Automatica, vol. 51, pp. 135–148, 2015.
[3] P. Antsaklis and J. Baillieul, “Guest editorial special issue on networked control systems,” IEEE Transactions on Automatic Control, vol. 49, no. 9, pp. 1421–1423, 2004.
[4] J. P. Hespanha, P. Naghshtabrizi, and Y. Xu, “A survey of recent results in networked control systems,” Proceedings of the IEEE, vol. 95, no. 1, pp. 138–162, 2007.
[5] S. Amin, A. Cardenas, and S. Sastry, “Safe and secure networked control systems under denial-of-service attacks,” Hybrid systems: Computation and Control, pp. 31–45, 2009.
[6] R. W. Brockett and D. Liberzon, “Quantized feedback stabilization of linear systems,” IEEE Transactions on Automatic Control, vol. 45, no. 7, pp. 1279–1289, 2000.
[7] J. Hespanha, A. Ortega, and L. Vasudevan, “Towards the control of linear systems with minimum bit-rates,” in Proc. of the Int. Symp. on the Mathematical Theory of Networks and Sys., 2002.
[8] S. Tatikonda and S. Mitter, “Control under communication constraints.” IEEE Transactions on Automatic Control, vol. 49, no. 7, pp. 1056–1068, July 2004.
[9] R. W. Brockett and D. Liberzon, “Quantized feedback stabilization of linear systems,” IEEE Transactions on Automatic Control, vol. 45, no. 7, pp. 1279–1289, July 2000.
[10] D. Liberzon, “On stabilization of linear systems with limited information,” IEEE Transactions on Automatic Control, vol. 48, no. 2, pp. 304–307, Feb 2003.
[11] T. Liu and Z. P. Jiang, “Event-triggered control of nonlinear systems with state quantization,” IEEE Transactions on Automatic Control, pp. 1–7, 2018.
[12] P. Talapragada and J. Cortés, “Event-triggered stabilization of linear systems under bounded bit rates,” IEEE Transactions on Automatic Control, vol. 61, no. 6, pp. 1575–1589, 2016.
[13] K. You and L. Xie, “Minimum data rate for mean square stabilizability of linear systems with markovian packet losses,” IEEE Transactions on Automatic Control, vol. 56, no. 4, pp. 772–785, April 2011.
[14] K. Okano and H. Ishii, “Stabilization of uncertain systems with finite data rates and markovian packet losses,” IEEE Transactions on Control of Network Systems, vol. 1, no. 4, pp. 298–307, 2014.
[15] ——, “Stabilization of uncertain systems using quantized and lossy observations and uncertain control inputs,” Automatica, vol. 81, pp. 261–269, 2017.
[16] K. Tsumura, H. Ishii, and H. Hoshina, “Tradeoffs between quantization and packet loss in networked control of linear systems,” Automatica, vol. 45, no. 12, pp. 2963–2970, 2009.
[17] K. You and L. Xie, “Minimum data rate for mean square stabilization of discrete LTI systems over lossy channels,” IEEE Transactions on Automatic Control, vol. 55, no. 10, pp. 2373–2378, 2010.
[18] Q. Ling, “Bit rate conditions to stabilize a continuous-time linear system with feedback drops,” IEEE Transactions on Automatic Control, vol. PP, no. 99, pp. 1–8, 2017.
[19] P. Minero, L. Coviello, and M. Franceschetti, “Stabilization over Markov feedback channels: the general case,” IEEE Transactions on Automatic Control, vol. 58, no. 2, pp. 349–362, 2013.
[20] C. De Persis and P. Tesi, “Input-to-state stabilizing control under denial-of-service,” IEEE Transactions on Automatic Control, vol. 60, no. 11, pp. 2930–2944, 2015.
[21] A. Cetinkaya, H. Ishii, and T. Hayakawa, “Networked control under random and malicious packet losses,” IEEE Transactions on Automatic Control, vol. 62, no. 5, pp. 2434–2449, 2017.
[22] J. Qin, M. Li, L. Shi, and X. Yu, “Optimal denial-of-service attack scheduling with energy constraint over packet-dropping networks,” IEEE Transactions on Automatic Control, vol. PP, no. 99, pp. 1–16, 2017.
[23] K. Ding, Y. Li, D. E. Quevedo, S. Dey, and L. Shi, “A multi-channel transmission schedule for remote state estimation under DoS attacks,” Automatica, vol. 78, pp. 194–201, 2017.
[24] Y. Li, D. E. Quevedo, S. Dey, and L. Shi, “SINR-based DoS attack on remote state estimation: A game-theoretic approach,” IEEE Transactions on Control of Network Systems, vol. 4, no. 3, pp. 632–642, 2017.
[25] M. Wakaiki, A. Cetinkaya, and H. Ishii, “Quantized output feedback stabilization under DoS attacks,” arXiv:1709.08149, 2017.
[26] S. Feng and P. Tesi, “Resilient control under denial-of-service: Robust design,” Automatica, vol. 78, pp. 42–51, 2017.
[27] ——, “Resilient control under denial-of-service: Robust design,” in 2016 American Control Conference, pp. 4737–4742.
[28] D. Senejohnny, P. Tesi, and C. De Persis, “A jamming-resilient algorithm for self-triggered network coordination,” IEEE Transactions on Control of Network Systems, 2017.
[29] C. De Persis and P. Tesi, “Networked control of nonlinear systems under denial-of-service,” Systems & Control Letters, vol. 96, pp. 124–131, 2016.
[30] A. Cetinkaya, H. Ishii, and T. Hayakawa, “Analysis of stochastic switched systems with application to networked control under jamming attacks,” IEEE Transactions on Automatic Control, pp. 1–16, 2018.
[31] A. Y. Lu and G. H. Yang, “Input-to-state stabilizing control for cyber-physical systems with multiple transmission channels under denial-of-service,” IEEE Transactions on Automatic Control, vol. PP, no. 99, pp. 1–8, 2017.
[32] S. Feng and P. Tesi, “Networked control systems under denial-of-service: Co-located vs. remote architectures,” Systems & Control Letters, vol. 108, pp. 40–47, 2017.
[33] Y. Yan, M. Xia, A. Rahnama, and P. Antsaklis, “A passivity-based self-triggered algorithm for control under denial-of-service,” in 2017 IEEE 56th Annual Conference on Decision and Control, Dec 2017, pp. 6082–6087.
[34] L. Perko, Differential Equations and Dynamical Systems. Springer, 2013.
[35] F. Mazenc and O. Bernard, “Interval observers for linear time-invariant systems with disturbances,” Automatica, vol. 47, no. 1, pp. 140–147, 2011.
[36] J. P. Hespanha and A. S. Morse, “Stability of switched systems with average dwell-time,” in Proceedings of the 38th IEEE Conference on Decision and Control, vol. 3, 1999, pp. 2655–2660.
[37] S. Tatikonda and S. Mitter, “Control over noisy channels,” IEEE transactions on Automatic Control, vol. 49, no. 7, pp. 1196–1201, 2004.
[38] F. Forni, S. Galeani, D. Nešić, and L. Zaccarian, “Lazy sensors for the scheduling of measurement samples transmission in linear closed loops over networks,” in IEEE Conference on Decision and Control, Atlanta, USA, 2010.
[39] C. De Persis and P. Tesi, “A comparison among deterministic packet-dropouts models in networked control systems,” *IEEE Control Systems Letters*, vol. 2, no. 1, pp. 109–114, Jan 2018.