Non-analytic dependence of the transition temperature of the homogeneous dilute Bose gas on scattering length

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We show that the shift in the transition temperature of the dilute homogeneous Bose gas is non-analytic in the scattering amplitude, $a$. The first correction beyond the positive linear shift in $a$ is negative and of order $a^2 \ln a$. This non-universal non-analytic structure indicates how the discrepancies between numerical calculations at finite $a$ can be reconciled with calculations of the limit $a \to 0$, since the linearity is apparent only for anomalously small $a$.

Equation [1] is only the beginning of an asymptotic expansion, as one might suspect, since for $a < 0$ the system is unstable. Inclusion of the $a^2 \ln a$ term, when one extrapolates numerical data from finite $a$ values to the limit $a \to 0$, provides a first resolution of the apparent discrepancies between numerical calculations done at finite $a$ [6,10,11] and those valid for $a \to 0$. To obtain a quantitative estimate, we explicitly calculate the logarithmic correction in a model with $N$ internal degrees of freedom, to leading order in $1/N$. The result suggests that the linear increase of $\Delta T_c$ at small but finite $a$ is noticeably suppressed for the physical case of $N = 2$.

We consider a uniform system of bosons of mass $m$ at temperature $T$, and assume that the two-body interaction can be described by the $s$-wave scattering length $a$. Above the critical temperature, the density $n$ is given in terms of a sum over Matsubara frequencies, $z_{\nu} = 2\pi\nu/T$ ($\nu = \pm 1, \pm 2, \ldots$), of the single particle Green’s function $G(k, z)$:

$$n = -T \sum_{\nu} \int \frac{d^3 k}{(2\pi)^3} G(k, z_{\nu}),$$

where ($\hbar = k_B = 1$)

$$G^{-1}(k, z) = z + \mu - \frac{k^2}{2m} - \Sigma(k, z),$$

and $\mu$ is the chemical potential; the condition $\mu = \Sigma(0, 0)$ determines the Bose-Einstein condensation point.

The shift of the critical temperature at fixed density is more conveniently calculated in terms of the shift, $\Delta n_c = n_c(a, T_c) - n_c(0, T_c)$, in the critical density at fixed $T$: the two shifts are related, to the orders of interest (less than $a^2$), by $\Delta T_c/T_c = -(2/3)\Delta n_c/n_c$. As
shown in [7], the leading linear shift $\Delta T^{(1)}_c$ is given solely by the zero Matsubara frequency term:

$$\frac{\Delta T^{(1)}_c}{T^0_c} = \frac{2 T^0_c}{3 n} \int \frac{d^3 k}{(2\pi)^3} \left[ G(k, 0) - G_0(k, 0) \right]$$

$$= \frac{4 A}{3 \pi \zeta(3/2)} \int_0^\infty dk \frac{U(k)}{k^2 + U(k)},$$

(5)

where $A = (2\pi/mT)^{1/2}$ is the thermal wavelength, $\zeta(3/2) = 2.612 \ldots$, $G_0$ is the Green’s function of the ideal Bose gas at $T^0_c$, and

$$U(k) = 2 m(\Sigma(k, 0) - \Sigma(0, 0)).$$

(6)

At the critical temperature, $U(k)$ can be calculated to order $a^2/\lambda^4$ by considering only the $\nu = 0$ sector, which corresponds to a classical field theory. At the transition, $U(k)$ has the scaling structure,

$$U(k) \propto \frac{a^2}{\lambda^4} \sigma(k \lambda^2/a),$$

(7)

from which the linearity of $\Delta T^{(1)}_c$ in $a$ follows [3]. If $U(k)$ is calculated by classical field theory, $\Delta T^{(1)}_c$ is strictly linear in $a$.

The next-to-leading order corrections, $\Delta T^{(2)}_c$, arise in terms with non-zero Matsubara frequencies, both explicitly [Eq. (3)] and in internal loops, in the calculation of $U(k)$. As we show below, the internal loop corrections begin at order $a^2$ and $a^3 \ln a$; the $a^2 \ln a$ terms may be extracted from

$$\frac{\Delta T^{(2)}_c}{T^0_c} = \frac{2 T^0_c}{3 n} \sum_{\nu \neq 0} \int \frac{d^3 k}{(2\pi)^3} \left[ G(k, z_\nu) - G_0(k, z_\nu) \right].$$

(8)

Since the infrared behavior is regular for $\nu \neq 0$, we expand the denominator of $G$ to first order in $\Sigma(k, z_\nu) - \mu$, and write

$$\Delta T^{(2)}_c \approx \frac{2 T^0_c}{3 n} \sum_{\nu \neq 0} \int \frac{d^3 k}{(2\pi)^3} \frac{\Sigma(k, z_\nu) - \Sigma(k, 0)}{z_\nu - k^2/2m}.$$  

(9)

From perturbation theory we know the functional form of $U(k)$ outside the critical region in $k$,

$$U(k) \propto \frac{a^2}{\lambda^4} \ln \frac{k \lambda^2}{a}, \quad k \gg \frac{a}{\lambda^2};$$

(10)

this logarithmic behavior is valid to all orders of perturbation theory, as one can verify by power counting. Since the dominant contribution to the integral in Eq. (9) comes from momenta $k \sim \lambda^{-1}$, the ultraviolet behavior of $U(k)$ generates a logarithmic shift in the critical temperature:

$$\frac{\Delta T^{(2)}_c}{T^0_c} \propto \frac{a^2}{\lambda^4} \ln \frac{a}{\lambda}.$$  

(11)

Since $\Sigma(k, z_\nu) - \Sigma(k, 0)$ tends to zero for large momenta $k$, the contribution of this term in Eq. (8) remains of order $a^2/\lambda^2$. Thus the next-to-leading order to the critical temperature shift is proportional to $a^2 \ln a$ and is always negative for small $a$.

In order to estimate the shift quantitatively, we calculate it in the large $N$ model. The $\nu = 0$ sector, equivalent to a classical $\phi^4$ field theory in three spatial dimensions, is described by the action [13],

$$S\{\phi(r)\} = \int d^3 r \left\{ \frac{1}{2} \sum_{i=1}^N \nabla \phi_i(r)^2 - m u B \left[ \sum_i \phi_i^2(r) \right] \right\},$$

(12)

with $u = 96\pi^2 a/\lambda^2$. The classical field theory suffers from ultraviolet divergences, which can be regularized by introducing a large momentum cutoff $\Lambda$. As shown in Refs. [8], the leading order corrections to the critical density are dominated by long distance properties and $U(k)$ is independent of the cutoff. Therefore one can derive $U(k)$ with a fixed cutoff $\Lambda$ in the action, take the limit $\Lambda \to \infty$, and determine the corrections to the critical density from Eqs. (8) and (9).

Instead of this procedure we will obtain the next-to-leading order corrections in an independent way, which has the advantage of making contact with the numerical $\phi^4$ lattice calculations. Starting from the finite temperature quantum field action, one can derive the effective action of the classical field theory by integrating perturbatively over the non-zero frequency quantum modes, $\nu \neq 0$, which provides a large momentum cutoff $\Lambda \sim \sqrt{mT} \sim 1/\lambda$ and renormalized effective coefficients of the Euclidean action [8]. Following [8], the corrections to the transition temperature are given in this effective field theory by

$$\frac{\Delta T_c}{T_c} = \frac{4 \lambda}{3 \pi \zeta(3/2)} \int_0^\Lambda dk \frac{U_A(k)}{k^2 + U_A(k)},$$

(13)

where the subscript $\Lambda$ indicates the explicit dependence on the ultraviolet cutoff which incorporates the leading effects of non-zero Matsubara frequencies.

In the large $N$ limit $U_A(k)$ is given in terms of the particle-hole bubble, $B(q)$, by

$$U_A(k) = -\frac{N a^2 u}{18} \int_0^\Lambda \frac{d^3 q}{(2\pi)^3} \frac{B(q)}{1 + NuB(q)/6} \times \left[ \frac{1}{(k - q)^2} - \frac{1}{q^2} \right];$$

(14)
to leading order in $1/N$ in three dimensions \[ B(q) = \frac{1}{8q} - \frac{6}{Ng^4} + O(\Lambda^{-2}) \] (15)

where $g^2 = 48\pi^2/N$. Following Ref. \[, we obtain the critical temperature shift

$$\frac{\Delta T_c}{T_c} = \frac{4\lambda}{3\pi(3/2)} \int_0^\Lambda dk \frac{U_\Lambda(k)}{k^2}. \quad (16)$$

As we see from the $\Lambda$-dependence of the bubble, Eq. (15), the $\Lambda$-dependence of $U_\Lambda(k)$ gives rise to higher order corrections in $\Lambda^{-1}$. These terms arise effectively from the internal non-zero Matsubara frequencies, and lead to corrections $\sim \Lambda^{-2} \ln \Lambda$ in $\Delta T_c/T_c$, which we can neglect. Thus

$$\frac{\Delta T_c}{T_c} = -\frac{64\lambda}{3\pi(3/2)} \frac{a}{\Lambda} \int_0^{\Lambda\tau/k} dk \int_0^{\Lambda\tau/k} dx \frac{1}{k(xk + 1)} \times \left[ \frac{x}{2} \log \left| \frac{1 + x}{1 - x} \right| - 1 \right], \quad (17)$$

where $\tau = (Nu/48)^{-1}$. To obtain the leading order corrections we differentiate with respect to $\Lambda \tau$, which allows us to isolate the contributions around $\Lambda \tau \to \infty$. Integrating back we find

$$\frac{\Delta T_c}{T_c} = \frac{8\pi}{3\pi(3/2)} \frac{a}{\lambda^2 \Lambda} \left\{ 1 + 16N \frac{a}{\lambda^2 \Lambda} \ln \frac{Na}{\lambda^2 \Lambda} + O\left( \frac{Na}{\lambda^2 \Lambda} \right) \right\}. \quad (18)$$

The integration constant is given by the large $N$ result for the linear shift, Ref. \[. In contrast to the linear corrections in $a$, the next-order terms depend on the short length scale properties of the system, modelled by the cutoff $\Lambda$, and therefore are not universal.

To estimate the influence of the logarithmic terms we take $N = 2$ and $\Lambda \lambda = (2\pi)^{3/2}$ in Eq. (18). The corrections to the linear behavior of $T_c$ found in this way are precisely those one finds for $N = 2$ by including the particle-hole bubble sum in $U(k)$. The resulting dependence of the transition temperature on $a$ is shown in Fig. 1. For a gas parameter $na^3 \sim 10^{-6}$, corresponding to the experimental region of Bose-Einstein condensation in atomic gases \[ and liquid $^4$He in Vycor \[, and to the lowest density Monte Carlo data of Ref. \[, the nonlinear corrections depress the linear shift by $\sim 50\%$; instead of the $a \to 0$ result $c \simeq 2.33$ in Eq. (1), one obtains a coefficient $c \sim 1.2$ for $an^{1/3} \sim 10^{-2}$. Even if the extrapolation from the large $N$ expansion to $N = 2$ is unjustified, this calculation suggests that the logarithmic terms play an important role for present numerical and experimental parameters. The noticeable depression of $\Delta T_c$ in this parameter regime is also confirmed by self-consistent numerical model calculations \[.

A qualitatively similar strong dependence on $a$ is found in the renormalization group calculations of Ref. \[, which derives an $a \ln a$ correction \[. Such a result would follow from Eq. (1) were $U(k)$ to be linear in $k$ up to an ultraviolet cutoff. However, in the regime $a/\lambda^2 \ll \Lambda$, corresponding to the dilute limit, $a/\Lambda \ll 1$, $U(k)$ is given by the perturbative result \[ for momenta in the region $a/\lambda^2 \ll k \ll \Lambda$ rather than being linear in $k$.

![FIG. 1. Dependence of the transition temperature of a dilute homogeneous Bose gas on scattering length, from Eq. (18) for $N = 2$. The falloff of $\Delta T_c/an^{1/3}$ with increasing $an^{1/3}$ arises from the non-universal next-to-leading order logarithmic corrections. In the region where the curve is represented by a dashed line, higher order corrections begin to become important. The dark circle is the calculation of Ref. \[; the points shown as open circles are the lowest density data from the numerical calculations of Ref. \[. The data point shown by the cross indicates the numerical results of Refs. \[ for a lattice $\phi^4$ theory extrapolated to the continuum.](Image)

The next-to-leading order corrections are important for classical $\phi^4$ calculations as well \[; in the universal region where the influence of the ultraviolet cutoff $\Lambda$ is unimportant ($\Lambda \to \infty$), classical field theory provides perfect scaling, implying a linear shift of $T_c$ for all $a$. Variations in $a$ of $\Delta T_c$ in \[ arise due to non-universal corrections and are sensitive to the details of the scheme used to regularize the classical $\phi^4$ theory \[. Extrapolation to the universal small coupling region, $a/\lambda^2 \Lambda \to 0$, together with finite size scaling to the thermodynamic limit allows one not only to extract the coefficient $c$ of the linear shift in $T_c$, but should also provide the magnitude of the non-universal $a^2 \ln a$ corrections for the physical case $N = 2$.

We dedicate this article to our recently deceased friend, Dominique Vautherin, with whom we had many stimulating discussions on this subject. We are grateful to Henk Stook for stimulating discussions both during his Workshop on Bose-Einstein condensation at the Lorentz Cen-
ter in Leiden, and at the Ecole Normale Supérieure. We also thank Peter Arnold, Nikolay Prokof’ev, and Boris Svistunov for helpful comments on their lattice calculations. Author GB would like to thank the ENS for its hospitality in the course of this work. This research was facilitated by the Cooperative Agreement between the University of Illinois at Urbana-Champaign and the Centre National de la Recherche Scientifique, and supported in part by the NASA Microgravity Research Division, Fundamental Physics Program and by National Science Foundation Grant PHY98-00978. Laboratoire Kastler Brossel de l’Ecole Normale Supérieure is UMR 8552 du CNRS and associé à l’Université Pierre et Marie Curie.

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