Renormalisation and the density of prime pairs

G.H. Gadiyar and R. Padma

(a) Ramanujan Institute for Advanced Study in Mathematics,
University of Madras, Chennai 600 005 INDIA

Abstract

Ideas from physics are used to show that the prime pairs have the density conjectured by Hardy and Littlewood. The proof involves dealing with infinities like in quantum field theory.

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1E-mail: padma@imsc.ernet.in
2E-mail: padma@imsc.ernet.in
This article may be considered as an invitation to theoretical physicists to enter the field of additive number theory. For sometime now there have been serious attempts to cross fertilise the disciplines of physics and number theory. It seems strange that on the one hand the most practical of disciplines, namely, physics has connections with the most aracane of disciplines, namely, number theory. However, surprising connections have appeared between number theory and physics as can be seen in [1], [3] and [9]. The work of Ramanujan in particular has had surprising connections with string theory, conformal field theory and statistical physics.

For sometime now the authors one of whom is a theoretical physicist and the other a number theorist have been trying to understand problems in additive number theory using ideas from both fields. One such problem is the distribution of prime pairs. Prime pairs are numbers which are primes differing by some even integer. For example, 3, 5; 5, 7; 11, 13; 17, 19 and so on are all prime pairs with common difference 2. The question is whether such prime pairs are infinite, if so, what is the density?

We will now summarise the standard method used to attack the problem which is the circle method. For technical reasons, the von Mangoldt function $\Lambda(n)$ (which is defined to be $\log p$ if $n = p^m$ where $p$ is a prime and 0 otherwise) is used instead of the characteristic function on the primes. Hence in the circle method

\[
\int_0^1 \left| \sum_{n=1}^{\infty} \Lambda(n) e^{2\pi i n \theta} \right|^2 e^{-2\pi i h z} dz
\]
gives the number of prime pairs with common difference $h$, an even integer. The interval of integration $[0, 1)$ in the above expression is partitioned by certain rational numbers called Farey fractions. The circle method consists in proving that the sum of the integrals over the intervals that correspond to the fractions with small denominators (called major arc) gives the conjectural main term and the contribution from the remaining part of the interval $[0, 1)$ (called minor arc) is small.

The authors noted in an earlier preprint [4] that the entire method could be replaced by proving a Wiener–Khintchine formula for arithmetical functions having Ramanujan–Fourier expansion. Ramanujan [7] using extremely simple arguments showed that the commonly known arithmetical functions all have Ramanujan–Fourier expansions. These are typically expansions of the form

$$ a(n) = \sum_{q=1}^{\infty} a_q c_q(n) , \quad (1) $$

where

$$ c_q(n) = \sum_{(k,q)=1}^{q} e^{2\pi i k n / q} , $$

and is known as the Ramanujan sum. For example he showed that

$$ d(n) = -\sum_{q=1}^{\infty} \log \left( \frac{q}{q} \right) c_q(n) , $$

$$ \sigma(n) = \frac{\pi^2}{6} n \sum_{q=1}^{\infty} \frac{c_q(n)}{q^2} , $$

$$ r(n) = \pi \sum_{q=1}^{\infty} \frac{(-1)^{q-1}}{2q-1} c_{2q-1}(n) $$
where \( d(n) \) is the number of divisors of \( n \), \( \sigma(n) \) their sum and \( r(n) \) the number of ways \( n \) can be expressed as the sum of two squares. Hardy in an extremely elegant paper \[5\] elucidated the arithmetical properties of \( c_q(n) \). He proved that \( c_q(n) \) is multiplicative. That is,

\[
c_{qq'}(n) = c_q(n)c_{q'}(n), \text{ if } (q, q') = 1
\]

and using this property he proved that

\[
\frac{\phi(n)}{n} \Lambda(n) = \sum_{q=1}^{\infty} \frac{\mu(q)}{\phi(q)} c_q(n)
\]

where \( \mu(q) \) is the Möbius function defined as follows: \( \mu(1) = 1, \mu(n) = 0 \), if \( n \) has a squared factor and \( \mu(p_1p_2...p_l) = (-1)^l \) if \( p_1, p_2, ..., p_l \) are different primes and \( \phi(q) \) is the Euler - totient function defined as the number of positive integers less than and prime to \( n \). The next step was taken by Carmichael \[2\] who essentially showed that \( e^{2\pi i \frac{k}{q} n} \) are almost periodic functions defined on the integers. This leads to orthogonality relations and a method for evaluating the Ramanujan - Fourier coefficient. Denote by \( M(g) \) the mean value of \( g \), that is,

\[
M(g) = \lim_{N \to \infty} \frac{1}{N} \sum_{n \leq N} g(n).
\]

For \( 1 \leq k \leq q, (k, q) = 1 \), let \( e_k(n) = e^{2\pi i \frac{k}{q} n}, (n \in \mathbb{N}) \). If \( a(n) \) is an arithmetical function with the Ramanujan - Fourier expansion (1) then

\[
a_q = \frac{1}{\phi(q)} M(ac_q) = \frac{1}{\phi(q)} \lim_{N \to \infty} \frac{1}{N} \sum_{n \leq N} a(n)c_q(n).
\]
Also
\[ M(e \frac{k}{q}, e \frac{k'}{q'}) = \begin{cases} 1, & \text{if } \frac{k}{q} = \frac{k'}{q'}, \\ 0, & \text{if } \frac{k}{q} \neq \frac{k'}{q'}. \end{cases} \] (3)

All this is well known in the field of almost periodic functions.

Hardy and Littlewood in [6] conjectured that there are infinitely many prime pairs \(p, p+h\) for every even integer \(h\) and if \(P_h(N)\) denotes the number of pairs less than \(N\), then
\[
P_h(N) \sim 2 \prod_{p>2} \left(1 - \frac{1}{(p-1)^2}\right) \frac{N}{\log^2 N} \prod_{p|h, p>2} \left(\frac{p-1}{p-2}\right).
\]

This is equivalent to
\[
\sum_{n \leq N} \Lambda(n) \Lambda(n+h) \sim 2N \prod_{p>2} \left(1 - \frac{1}{(p-1)^2}\right) \prod_{p|h, p>2} \left(\frac{p-1}{p-2}\right).
\]

Numerical calculations done by Mrs. Streatfeild [6] strengthen the validity of this conjecture.

Using (2) and (3) one can give the heuristic argument

\[
\lim_{N \to \infty} \frac{1}{N} \sum_{n \leq N} \frac{\phi(n)}{n} \Lambda(n) \frac{\phi(n+h)}{n+h} \Lambda(n+h) = \prod_{p>2} \left(1 - \frac{1}{(p-1)^2}\right) \prod_{p|h, p>2} \left(\frac{p-1}{p-2}\right)
\]

\[
= \sum_{q=1}^\infty \sum_{k=1}^{\infty} \frac{\mu(q)}{\phi(q)} e^{2\pi i \frac{k}{q}} M(e, e \frac{k'}{q'}) e^{-2\pi i \frac{k'}{q'}(n+h)}
\]
\[
\sum_{q=1}^{\infty} \frac{\mu^2(q)}{\phi^2(q)} c_q(h) ,
\]
\[
= 2 \prod_{p>2} \left( 1 - \frac{1}{(p-1)^2} \prod_{p|h} \frac{(p-1)}{p-2} \right) .
\]

However the proof of the Wiener - Khintchine formula is not possible for the von Mangoldt function due to the poor estimates available for

\[
\frac{1}{N} \sum_{n \leq N} e^{2\pi i \left( \frac{k}{q} - \frac{k'}{q'} \right) n} \text{ where } 1 \leq k \leq q \text{ and } (k, q) = 1 \text{ and } 1 \leq k' \leq q' \text{ and } (k', q') = 1 \text{ and } \frac{k}{q} \neq \frac{k'}{q'}.
\]

In this paper we show that replacing \(\sum_{n=0}^{\infty} a_n \sigma \) by \(\sum_{n=-\infty}^{\infty} a_n \sigma \) (note that the limits of summations are different) makes it unnecessary to have to undertake error term analysis.

Consider

\[
\sum_{N=1}^{\infty} \sum_{n \leq N} \frac{\phi(n)}{n} \Lambda(n) \frac{\phi(n+h)}{n+h} \Lambda(n+h) x^N
\]
\[
= \sum_{N=1}^{\infty} \sum_{n \leq N} \sum_{q,q'=1}^{\infty} \frac{\mu(q) \mu(q')}{\phi(q) \phi(q')} c_q(n) c_{q'}(n+h) x^N
\]
\[
= \sum_{q, q'=1}^{\infty} \sum_{k=1}^{q} \sum_{k'=1}^{q'} \sum_{n=1}^{\infty} \frac{\mu(q) \mu(q')}{\phi(q) \phi(q')} \frac{e^{2\pi i \left( \frac{k}{q} - \frac{k'}{q'} \right) n - \frac{k}{q} h}}{\phi(q) \phi(q')} x^N
\]
\[
= \frac{1}{1-x} \sum_{q, q'=1}^{\infty} \sum_{k=1}^{q} \sum_{k'=1}^{q'} \sum_{n=1}^{\infty} \frac{\mu(q) \mu(q')}{\phi(q) \phi(q')} e^{2\pi i \left( \frac{k}{q} - \frac{k'}{q'} \right) n - \frac{k}{q} h} x^n
\]
\[
= \frac{x}{(1-x)^2} \sum_{q=1}^{\infty} \frac{\mu^2(q)}{\phi^2(q)} c_q(h) \text{ + cross terms} .
\]
Thus we have,
\[
\sum_{n\leq N} \frac{\phi(n)}{n} \Lambda(n) \frac{\phi(n+h)}{n+h} \Lambda(n+h) = N \sum_{q=1}^{\infty} \frac{\mu^2(q)}{\phi^2(q)} c_q(h) + \text{cross terms}.
\]

Note that the first term is the main term as conjectured by Hardy and Littlewood. However, the toughest part is to show that the main term which are few in number is greater than the error terms which are greater in number. For this even assuming hypotheses like those of Riemann or Montgomery and Vaughan do not seem to suffice. *Analytical number theorists spend most of their time and effort in estimates of these error terms.*

In this paper we just make a small change in the ideas of Ramanujan, Hardy and Littlewood. We note first the Poisson summation formula in the distributional form [8].

\[
\sum_{n=-\infty}^{\infty} e^{in\theta} = \sum_{k=-\infty}^{\infty} \delta(\theta - 2\pi k)
\]

It is immediately obvious that in (4), summation from 1 to \(\infty\) gives the cross terms which contribute to error term. *But in this case because one has to integrate product of \(\delta\) functions, there are no cross terms. However this leads to a new difficulty which seems tractable using ideas from physics like renormalisation.*

Let \(a(n)\) be an arithmetical function having the Ramanujan - Fourier expansion (1). Since \(c_q(n) = c_q(-n)\), we take \(a(-n) = a(n)\). Then
\[ \sum_{n=-\infty}^{\infty} a_n a_{n+h} \]

\[ = \frac{1}{2\pi} \int_0^{2\pi} \left( \sum_{n=-\infty}^{\infty} a_n e^{i\theta n} \right) \left( \sum_{m=-\infty}^{\infty} a_m e^{-i\theta m} \right) e^{-ih\theta} d\theta \]

\[ = \frac{1}{2\pi} \int_0^{2\pi} \sum_{q,q'=1}^{\infty} \sum_{k=1}^{q} \sum_{k'=1}^{q'} a_q a_{q'} \delta(\theta - 2\pi \left( \frac{k}{q} - m \right)) \delta(-\theta - 2\pi \left( \frac{k'}{q'} - n \right)) e^{-ih\theta} d\theta \]

\[ = \sum_{q=1}^{\infty} a_q^2 c_q(h) \sum_{m=-\infty}^{\infty} 1 \]

We write this as

\[ \sum_{n=-\infty}^{\infty} a_n a_{n+h} \sum_{m=-\infty}^{\infty} 1 = \sum_{q=1}^{\infty} a_q^2 c_q(h). \]

This formula has ratio of two infinite quantities which we interpret as follows. The denominator is the number of integers and the numerator is a suitable function on the integers. Hence the ratio is in some sense a density. Hence this quantity is the density of twin primes if we take \(a(n) = \Lambda(n) \phi(n) / n\).

We note that though we are dealing with the ratio of two infinite quantities which is exactly like renormalisation in quantum field theory the argument can be made rigorous by a limiting process which should be akin to regularisation in physics. The authors have not been able to find a neat and clever regularisation which would then make the argument both simple and precise.
However, it should be emphasized that absolutely no hard analysis of error terms is necessary in this case. It would be important to get simple regularisation schemes as large number of problems in additive number theory fall in this category. As this method makes error term analysis unnecessary, sharpening this argument would mean real progress in this field.

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