Matching three-point functions
of BMN operators at weak and strong coupling

Gianluca Grignani, A. V. Zayakin

1 Dipartimento di Fisica, Università di Perugia,
I.N.F.N. Sezione di Perugia,
Via Pascoli, I-06123 Perugia, Italy

2 Institute of Theoretical and Experimental Physics,
B. Cheremushkinskaya ul. 25, 117259 Moscow, Russia

grignani@pg.infn.it, a.zayakin@gmail.com

Abstract

The agreement between string theory and field theory is demonstrated in the leading order by providing the first calculation of the correlator of three two-impurity BMN states with all non-zero momenta. The calculation is performed in two completely independent ways: in field theory by using the large-$N$ perturbative expansion, up to the terms subleading in finite-size, and in string theory by using the Dobashi-Yoneya 3-string vertex in the leading order of the Penrose expansion. The two results come out to be completely identical.
1 Introduction and motivations

A very fascinating progress in computing three-point functions for $\mathcal{N} = 4$ Super Yang-Mills (SYM) operators has taken place during the last two years [1–31]. The global aim of these efforts would be to provide the full set of three-point correlators for arbitrary number of constituent fields, number of colours $N$ and t’Hooft coupling $\lambda$. Given the conformal invariance of $\mathcal{N} = 4$ SYM, this would be equivalent to a full solution of the theory. In fact, two point correlation functions of local operators can be determined by the anomalous dimensions of the gauge theory operators and all the higher point correlation functions can be reconstructed using OPE’s with the three point function structure constants. This ambitious project is far from having been completed. While we have a complete understanding of the spectrum of anomalous dimensions of SYM operators, which can be extracted from the Thermodynamic Bethe Ansatz of Refs. [32–34], there is still a lot of work to do to get an acceptable comprehension of the three point functions. Yet correlators in some specific sectors of the theory, i.e. in several well-defined limits, have been obtained in the form of expansions in $\frac{1}{\lambda}$ ($J$ being one of the $R$-charges), $\frac{1}{N}$ and $\lambda' = \frac{\lambda}{J^2}$, i.e. as finite-size, non-planarity and loop expansions correspondingly. For some “protected” cases the results hold actually as exact ones. The well-studied sectors of the theory are an important playground for getting a more complete holographic understanding of the three-point functions. On the field-theory side the computation of the three-point functions is available for small coupling and follows the planar perturbation theory. On the string theory side two different approaches are feasible. One is valid mostly for strong coupling and is based upon the semiclassical methods, which amounts to a calculation of a world-sheet with three singularities that is equivalent to a three-point correlator in the field theory. The other string-theoretical method is the string field vertex construction, which on $AdS_5 \times S^5$ is only known in the pp-wave limit. An important feature of all the three calculations is that there exists a limit in which all of them may be valid, namely, in the so called Frolov-Tseytlin limit of small $\lambda' = \frac{\lambda}{J^2}$ [35, 36]. Expansion in $\lambda'$ resembles a weak coupling expansion in $\lambda$, yet is not identical to it, since while $\lambda'$ is taken to be small, $\lambda$ can be either small or large. The Frolov-Tseytlin limit was originally conceived as a bridge between weak and strong coupling for the non-protected operators in the spectral problem. For the string states with angular momentum $J$ on $S^5$, the energies can be expanded in a limit of large $J$ around a BPS solution with $\lambda' = \frac{\lambda}{J^2}$ fixed. This expansion can then be compared in the Frolov-Tseytlin limit to the loop expansion on the gauge theory side. The energies match the anomalous dimensions of the corresponding operators on the gauge theory side up to and including the second order in the expansion parameter, i.e. two-loops on the gauge theory side, but the matching breaks down at three-loops [37]. In [38] it is shown that the one-loop match is a consequence of the suppression of quantum corrections to the string near the BPS point, allowing a regime where the classical action of the string is large even if a weak coupling expansion in $\lambda$ is considered.

The first results for the three-point functions have shown that the leading order calculations in string theory and field theory do coincide, and this is already a non-trivial state-
However there are some cases where the gauge theory and string theory results have structure similarities but do not match perfectly even at the leading order [13]. This can be presumably interpreted as the inability of the basis chosen to describe gauge theory operators to interpolate between weak and strong coupling.

Thus the aim of this work is to provide extra evidence for the gauge/string theory comparison in three point functions, using operators for which the gauge and string identifications is very well established [39]. We perform in fact the analysis of the BMN [39] correlators with all three momenta non-zero (the so-called fully dynamical correlators) in the Frolov-Tseytlin limit. To the best of our knowledge, our work is the first where this analysis is carried out for the operators with all three momenta being non-zero.

To which of the classes – heavy, light or intermediate – do our operators belong to? Since \( \Delta - J \sim \sqrt{1 + \lambda n^2} \), at a fixed \( \lambda' \) these operators represent an interesting example of operators already heavy but still without an adequate semiclassical description: taking \( \lambda \) large, the anomalous dimension \( \Delta \sim J \equiv \sqrt{\lambda/\lambda'} \) can be made scale as \( \sqrt{\lambda} \). Thus we claim that at a fixed \( \lambda' \) our BMN operators are rather large. This will eventually, as we hope, provide a solid ground to compare the correlator of the (field-theory/pp-wave string-theory) BMN operators/string states with a semiclassical correlator of giant magnons, the latter being the “heaviest” objects available in all possible senses of the definition.

It has recently been observed in [27] that there is a discrepancy in the next-order \( \lambda' \) expansion for three-point correlators. The reason for this discrepancy is not yet known, it might be due to the subtleties in the computation on the gauge theory side.

For example, an apparent mismatch observed in an early stage of three-point correlator studies [40] was successfully resolved by finding a next-order in \( 1/N \) correction to the operator-state identification rule – a mixing of single-trace with double trace operators was detected, since the single-trace operators happened not to be the exact matches for the string states.

In any case, understanding the cause for this mismatch is of direct interest now. In doing so, extra evidence from other states and sectors of the theory is of primary importance, since it can possibly help us to distinguish between different causes: state mixing, wrongly interpreted limits or, much less likely, some fundamental problems with the duality conjecture.

The work is structured as follows: in Section 2.1 we perform a field-theoretical calculation of the correlator, and in Section 2.2 we compute the same correlator from string theory via the Dobashi-Yoneya 3-string vertex using the asymptotic Neumann matrices in the pp-wave limit and make sure the two results do agree. In the final Section 3 we comment on the agreement between the two calculations and suggest possible future directions of research in (dis)establishing the equivalence.

2 Correlators of BMN operators

Three-point correlators can be classified by the weights of the operators involved, these can be light (L), intermediate and heavy (H). By definition, heavy state anomalous dimensions
scale as $\sqrt{\lambda}$

$$\Delta_{\text{heavy}} \sim \sqrt{\lambda}, \quad (2.1)$$

“intermediate” states scale as

$$\Delta_{\text{intermediate}} \sim \lambda^{\frac{1}{4}}, \quad (2.2)$$

and the light states have

$$\Delta_{\text{light}} \sim 1. \quad (2.3)$$

The three-point correlators are then classified in the simplest approximation into LLL, LLH, LHH and HHH combinations. We can say we know almost everything about the LLL; correlators from quite some time [40–50] the HLL correlators are a bit exotic, they are mostly known from the gauge theory side [23]; the HHL starting from [3, 5] have recently been and continue to be an object of thorough research on both integrability/field theory and string theory sides; there have been some very promising attempts to construct also HHH correlators both from string [21, 25, 51, 52] and gauge theory sides [9, 26, 29–31].

The object of our novel analysis are the fully dynamical correlators of three BMN operators. They take an intermediate position between the heavy and the light operators, since on the one hand they do not possess a proper semiclassical description, on the other hand at constant $\lambda'$ they scale as heavy operators. Thus, for large and small $\lambda$ they make a perfect bridge towards the yet undisclosed domain of the HHH correlators made of three giant magnons. For some reasons there is a gap in the literature for BMN state correlators. Namely, the results for the correlators of two BMN with one BPS are abundant, whereas three BMN with three non-zero momenta have not been calculated either on the gauge theory side (from the $1/N$ expansion of Feynman diagrams [40]), or on string theory side using Neumann matrices provided by [53]. These are the calculations done in the sections 2.1 and 2.2 respectively.

There are however already some very interesting results on BMN correlators. The topic was started from the string-theoretic point of view in [51, 53–62] and from field theory in [40, 63, 64]. The three-point functions for two dynamical BMN and one static (zero-momentum) operators on the field theory side up to first order in $\lambda'$ were calculated in [65]. Full agreement with string theory has been found. An “intermediate-intermediate-intermediate” correlator of BMN vacuum, BMN fermion-and-scalar excitation, BMN fermion-and-scalar-and-an-R-charge excitation was calculated by Dobashi in [66] who pointed out the equality between string and gauge theory results.

## 2.1 BMN correlators from field theory

Here we consider the computation of the three-point correlation function of BMN operators with non-zero momentum in the weak coupling leading order in $1/N$, the leading and the next-to-leading order in the $1/J$ expansion. The operators we are interested in are single trace scalar operators defined as

$$O_{ij,n}^J = \frac{1}{\sqrt{JN^{J+2}}} \sum_{l=0}^{J} \text{Tr} \left( \phi_i Z^l \phi_j Z^{J-l} \right) \psi_{n,l}, \quad (2.4)$$
which belong to the three irreducible representations of $SO(4)$

$$4 \otimes 4 = 1 + 6 + 9,$$

(2.5)

where $1$ is the trace (T), $6$ is the antisymmetric (A), $9$ is the symmetric traceless representation (S). The orthonormal basis therefore is

$$A_{ij} = \frac{1}{\sqrt{2}} (O_{ij} - O_{ji}),$$

$$S_{ii} = \frac{2}{\sqrt{3}} (O_{ii} - \frac{1}{4} \sum_{i'} O_{i'i'}),$$

$$\tilde{S}_{ij} = \frac{1}{\sqrt{2}} (O_{ij} + O_{ji}),$$

$$T = \frac{1}{2} \sum_{i'} O_{i'i'}.$$  

(2.6)

To simplify the notation we omit the momentum indices $n_i$. We shall be interested in the leading-order $1/N$ behavior solely, therefore, we do not take into account the mixing of single-trace with double trace operators that takes place at the next-order. The wave-functions for different representations are

$$\psi^S_{n,l} = \cos \left( \frac{(2l+1)\pi n}{J+1} \right),$$

$$\psi^A_{n,l} = \sin \left( \frac{2(l+1)\pi n}{J+2} \right),$$

$$\psi^T_{n,l} = \cos \left( \frac{2(l+3)\pi n}{J+3} \right).$$

(2.7)

We consider the correlation function of three BMN fully dynamical operators which is given by

$$C_{i_1j_1;i_2j_2;i_3j_3,n_1;n_2;n_3}^{J_1J_2J} = \langle O_{i_1j_1,n_1}^J O_{i_2j_2,n_2}^{J_2} \bar{O}_{i_3j_3,n_3}^{J_3} \rangle,$$

(2.8)

where no extra overall normalization has been introduced since the operators $O_{ij,n}^J$ are already unity-normalized. We denote the correlator of three BMN operators taken with non-zero momentum as “fully dynamical”, unlike e.g. the three-point correlator mentioned in [40], which, having one vanishing momentum, can be denoted as “partially dynamical”, being a correlator of two BMN and one chiral primary. As already mentioned, the obvious generalization to a fully dynamical correlator has not yet been considered in the literature. The R-charges $J$ have to be conserved, therefore $J = J_1 + J_2$. For convenience below we shall use the notation

$$J_1 = rJ, \quad J_2 = (1-r)J,$$

(2.9)

where the parameter $r$ is understood as a finite fixed quantity, $0 \leq r \leq 1$, and we consider the large $J$ limit. We are interested only in the contribution to the 3-point correlation function coming from the connected diagrams. An example of such a diagram is given in Fig. (1). This diagram is evaluated as prescribed in [40]. One first contracts the impurity operators and this leads to two decoupled single-trace vacuum diagrams, as that shown in Fig. (2),
Figure 1. Connected diagrams contributing to the three-point function.

and its counterparts with respect to the transformation $l \rightarrow J - l$. The diagram Fig. (2) corresponds to the quantity

$$
\left\langle \text{Tr} \left( Z^{l_1} Z^{l_2} \bar{Z}^{l_3} \right) \text{Tr} \left( Z^{J_1 - l_1} Z^{J_2 - l_2} \bar{Z}^{J_3 - l_3} \right) \right\rangle.
$$

(2.10)

The $l_1(J_1 - l_1)$ and $l_2(J_2 - l_2)$ $Z$-operators are separated to recall from which operators they originally came from. Since we work in the leading-order approximation in $1/N$, the diagram is evaluated as disconnected and simply equals to $N^{J+2}$. Disconnectedness of this diagram imposes the condition $l_3 = l_1 + l_2$. There are 4 diagrams in total like those in Fig. (2). Let the diagram in Fig. (2) be equal to $f(l_1, l_2, l_3)$. The full contribution to the correlator is then

$$
C_{IJK} \sim f(l_1, l_2, l_1 + l_2) + f(l_1, l_2, J_1 - l_1 + l_2) + f(l_1, l_2, J_2 + l_1 - l_2) + f(l_1, l_2, J - l_1 - l_2).
$$

(2.11)

The answer for the correlator is given by a convolution of the three corresponding wavefunctions with this expression.

Due to $SO(4)$ charge conservation there are two possible types of the $(i_1 j_1)(i_2 j_2)(i_3 j_3)$ indices that can contribute, as shown in Fig. (3): (iii), (ii), (ii) (Fig. (a)) and (ii)(ij)(ij) (Fig. (b)).

There are four $SO(4)$ irrep structures corresponding to Fig. (3) (a): $SSS, SST, STT, TTT$ and six $SO(4)$ irrep structures corresponding to Fig. (3) (b): $S\bar{S}\bar{S}, S\bar{S}A, SAA, T\bar{S}\bar{S}, T\bar{S}A, TAA$. The irrep combinatorics is supplemented by permutations of $n_1, n_2, n_3$. Since $n_1 \leftrightarrow n_2, J_1 \leftrightarrow J_2$ is a trivial symmetry of the three-point function under consideration, the total amount of combinations can be handled, and we do not show the correlators that differ only by a permutation of the two first operators. In Table 1 we list all of the remaining structures. The first letter refers to the wavefunction with momentum quantum number $n_1$, the second with $n_2$, the third with $n_3$. One immediately sees that all correlators where the third operator is an antisymmetric one vanish due to the property of the antisymmetric wave functions. Also
Figure 2. Two decoupled single-trace vacuum diagrams, \( l_3 = l_1 + l_2 \). The \( l_1 (J_1 - l_1) \) and \( l_2 (J_2 - l_2) \) \( Z \)-operators are divided to recall from which operators they originally came from.

Figure 3. Two possible types of the \((i_1j_1)(i_2j_2)(i_3j_3)\) indices that can contribute.

the \( \tilde{S} \) and \( S \) states after the internal \( ij \) lines have been contracted differ only by a constant multiplicative factor. We can summarize the table (1) in terms of few simpler objects using the orthonormal basis defined above:

\[
SSS = \frac{3}{8}n_s^3c_{SSS},
\]

\[
SST = \frac{3}{4}n_s^2n_tC_{SST}, \quad TSS = \frac{3}{4}n_s^2n_tC_{TSS},
\]

\[
STT = 0, \quad TTS = 0,
\]

\[
TTT = 4n_t^3c_{TTT},
\]

(2.13)
Table 1. Possible configuration of the three-point functions.

\[\begin{align*}
SSS & \\
SST, TSS & \\
STT, TTS & \\
TTT & \\
S\tilde{S}\tilde{S} & \\
S\tilde{S}A, S\tilde{A}, \tilde{S}AS & \\
SAA, AAS & \\
T\tilde{S}\tilde{S} & \\
T\tilde{S}A, TA\tilde{S}, \tilde{S}AT & \\
TAA, AAT & \\
\end{align*}\]

\[(2.12)\]

\[S\tilde{S}\tilde{S} = \frac{1}{2}n_a n_s^2 c_{SSS},\]

\[A\tilde{S}\tilde{S} = \frac{1}{2}n_a n_s n_{\tilde{s}} c_{ASS},\]

\[A\tilde{S}S = \frac{1}{2}n_a n_s n_{\tilde{s}} c_{ASS},\]

\[AAS = \frac{1}{2}n_s^2 n_{\tilde{s}} c_{AAS},\]

\[T\tilde{S}\tilde{S} = 4n_t n_s^2 c_{TSS},\]

\[T\tilde{S}A = 4n_t n_s n_{\tilde{s}} c_{TAS},\]

\[\tilde{S}AT = 4n_t n_s n_{\tilde{s}} c_{SAT},\]

\[AAT = 4n_s^2 n_t c_{AAT}.\]

Here the norms are \(n_a = n_{\tilde{s}} = \frac{1}{\sqrt{2}}\), \(n_s = \frac{2}{\sqrt{3}}\), \(n_t = 2\). The coefficients \(c_{IJK}\) are defined as the correlators of the operators: \(O_{I1}\) (where \(I = S, T\)) and \(O_{I2}\) (where \(I = \tilde{S}, A\)). The order of letters reflects the cardinal numbers of the momenta \(n_1, n_2, n_3\). Calculating the \(c_{IJK}\) directly we find that there are only four non-zero contributions: \(SSS, SST, TTS\) and \(TTT\). The correlators are known to us in the leading \(1/N\) order and up to the subleading \(1/J\) order. Defining

\[c_{IJK} = \frac{J^{1/2}}{N} \left( c_{IJK}^0 + \frac{1}{J} c_{IJK}^1 \right),\]

after the calculation we see that

\[c_{SSS}^0 = c_{SST}^0 = c_{STT}^0 = c_{TTT}^0 = \frac{-4n_2^3 r^{3/2}(1-r)^{3/2} \sin(\pi n_3 r)^2}{\pi^2 (n_2^2 - (1-r)^2 n_3^2)(n_1^2 - r^2 n_3^2)},\]

and for the subleading part one gets the structures

\[c_{IJK}^1 = \frac{1}{\pi^2 (n_2^2 - (1-r)^2 n_3^2)(n_1^2 - r^2 n_3^2)^2} c_{IJK}^1,\]
which for $|n1| = r|n3|$ or $|n2| = (1 - r)|n3|$ is singular since it is multiplied by the following regular numerators

$$c_{SSS}^1 = 4n_3^2(r - 1)r \sin(\pi n_3 r) (\pi n_3 r (2r^2 - 3r + 1) (n_3^2(r - 1)^2 - n_2^2) (n_3^2r^2 - n_1^2) \cos(\pi n_3 r) +$$

$$+ 2(n_1^2(n_2^2(r - 1)^2 + n_3^2(r - 1)^2) - n_3^2 r^3 (n_3^2(r - 1)^3 + n_2^2)) \sin(\pi n_3 r)),\quad (2.17)$$

$$c_{SST}^1 = 4n_3^2(r - 1)r \sin(\pi n_3 r) (3\pi n_3 r (2r^2 - 3r + 1) (n_3^2(r - 1)^2 - n_2^2) (n_3^2r^2 - n_1^2) \cos(\pi n_3 r) +$$

$$+ 2(n_1^2(n_2^2(3r^2 - 3r + 1) + n_3^2(r - 1)^3) - n_3^2 r^3 (3n_3^2(r - 1)^3 + n_2^2)) \sin(\pi n_3 r)),\quad (2.18)$$

$$c_{STT}^1 = -4n_3^2(r - 1) \sin(\pi n_3 r) (\pi (r - 1) (n_2^2 - n_3^2(r - 1)^2) (n_3^2r^4(2r - 1) -$$

$$- n_3^2 n_3^2 r^2(2r + 1) + n_1^4) \cos(\pi n_3 r) +$$

$$+ 2n_3 r (n_3^2 r^3 (n_3^2(r - 1)^3 + n_2^2) - n_1^2(n_2^2(r^2 - 3r + 3) + 3n_3^2(r - 1)^3)) \sin(\pi n_3 r)),\quad (2.19)$$

$$c_{TTT}^1 = 4n_3 \sin(\pi n_3 r) (6n_3(r - 1)r (n_1^2(n_2^2(r - 1)^2 + n_3^2(r - 1)^3) -$$

$$- n_3^2 r^3 (n_3^2(r - 1)^3 + n_2^2)) \sin(\pi n_3 r) -$$

$$- \pi (n_2^2 - n_3^2(r - 1)^2) (n_3^2r^2 - n_1^2) (r^2 (3n_3^2(r - 1)^2(2r - 1) + 2n_3^2) - 2n_1^2(r - 1)^2) \cos(\pi n_3 r)) .\quad (2.20)$$

Note that all four structures are different from each other in the subleading order. To make some sense from these illegible expressions let us expand for small momenta, $n_i \to 0$. This will correspond to the near-BMN limit. We get then

$$c_{SSS}^1 = 4\pi^2 n_1^2 n_2^2 n_3^4 r^3 (4r^3 - 9r^2 + 8r - 3),\quad (2.21)$$

$$c_{SST}^1 = 4\pi^2 n_1^2 n_2^2 n_3^4 r^3 (12r^3 - 27r^2 + 20r - 5),$$

$$c_{STT}^1 = 4\pi^2 n_1^2 n_2^2 n_3^4 r^3 (4r^3 - 11r^2 + 12r - 5),$$

$$c_{TTT}^1 = 12\pi^2 n_1^2 n_2^2 n_3^4 r^3 (4r^3 - 9r^2 + 8r - 3).$$

The leading order part of our results resembles (perhaps not surprisingly) the expressions obtained in [18]. Now one could consider comparing these expressions to semiclassical calculations. They must not necessarily coincide, since the above calculation has been performed at weak coupling. Therefore such a comparison will be highly non-trivial. The closest objects on the strong coupling side to our BMN operators are the giant magnons. They require a full two-dimensional analysis of the worldsheet configurations, unlike the long BPS operators considered by [18, 25] that effectively reduced the classical worldsheet to a combination of
geodesics. We postpone this truly semiclassical analysis to a successive work, and now proceed in Section 2.2 to a doable yet nontrivial comparison with the matrix elements of the string interaction Hamiltonian 3-vertex in the pp-wave limit.

2.2 The three-point BMN correlator from string theory

About a decade ago a very advanced technique was developed for calculating the light-cone string-theory three-point matrix elements of the interaction Hamiltonian. The general idea of the calculation is that a matrix element

$$H_{IJK} = \langle IJK | H | 0 \rangle$$

is obtained from the construction

$$H_{IJK} = \langle IJK | P | V \rangle,$$  \hspace{1cm} (2.22)

where the exponential factor is

$$|V\rangle = e^{\frac{i}{2} \sum_{a,b,i,j} N_{a,b,i,j} a_i^\dagger a_j^\dagger |0\rangle},$$  \hspace{1cm} (2.23)

the matrices $N_{a,b}$ are the Neumann matrices, the indices $a, b$ running through 1 to 3 and corresponding to the states $IJK$, the indices $i, j$ corresponding to the oscillator modes. The most advanced three string vertex in the pp-wave limit \cite{61, 62} was found by Dobashi and Yoneya \cite{59} as a linear combinations with equal weight of the vertices proposed in \cite{54, 58}. The prefactor $P$ is organized as

$$P = \frac{\omega_{1,n_1}}{\mu r} \left( 2 + \alpha_{1,n_1}^\dagger \alpha_{1,-n_1} + \alpha_{1,-n_1}^\dagger \alpha_{1,n_1} \right) + \frac{\omega_{2,n_2}}{\mu (1 - r)} \left( 2 + \alpha_{2,n_2}^\dagger \alpha_{2,-n_2} + \alpha_{2,-n_2}^\dagger \alpha_{2,n_2} \right) - \frac{\omega_{3,n_3}}{\mu} \left( 2 + \alpha_{3,n_3}^\dagger \alpha_{3,-n_3} + \alpha_{3,-n_3}^\dagger \alpha_{3,n_3} \right).$$  \hspace{1cm} (2.24)

Here $\mu$ is the expansion parameter of the Penrose limit, $\mu \sim \frac{1}{\sqrt{\lambda}}$. The frequencies $\omega$ are

$$\omega_{1,n} = \sqrt{n^2 + \mu^2 r^2},$$  

$$\omega_{2,n} = \sqrt{n^2 + \mu^2 (1 - r)^2},$$  \hspace{1cm} (2.25)

$$\omega_{3,n} = \sqrt{n^2 + \mu^2}.$$  

We do not discuss here the fermionic contribution to the prefactor, which caused a lot of dispute in the literature, where at least three different types of vertices have been compared \cite{62}. This discussion is so far irrelevant to us since all our states are bosonic. The matrices $N_{a,b}$ are taken by us from \cite{53, 61, 62}. Their behaviour for the positive and the negative values of the mode numbers is essentially different. For the positive modes $m, n$ the leading-order in $\mu$
(up to $O(\mu^1)$) is

$$N_{m,n} = \begin{pmatrix}
\frac{(-1)^{m+n}}{2\mu \pi r} & -\frac{(-1)^m}{2\mu \pi \sqrt{(1-r)r}} & -\frac{2(-1)^{m+n}m r^{3/2} \sin(n\pi r)}{\pi(n^2 r^2 - m^2)} \\
-\frac{(-1)^n}{2\mu \pi \sqrt{(1-r)r}} & -\frac{1}{2\mu \pi (r-1)} & -\frac{2(-1)^m n (1-r)^{3/2} \sin(n\pi r)}{\pi(m^2 - n^2 (1-r)^2)} \\
-\frac{2(-1)^{m+n}m r^{3/2} \sin(m\pi r)}{\pi(m^2 - n^2)} & -\frac{2(-1)^m n (1-r)^{3/2} \sin(m\pi r)}{\pi(n^2 - m^2 (1-r)^2)} & 0
\end{pmatrix}.$$  

(2.26)

For negative modes $-m, -n$ the Neumann matrix becomes

$$N_{-m,-n} = \begin{pmatrix}
0 & 0 & -\frac{2(-1)^{m+n}m \sqrt{r} \sin(n\pi r)}{\pi(m^2 - n^2 r^2)} \\
0 & 0 & -\frac{2(-1)^m n (r-1)^{1/2} \sin(n\pi r)}{\pi(n^2 (r-1)^2 - m^2)} \\
-\frac{2(-1)^{m+n}m \sqrt{r} \sin(m\pi r)}{\pi(m^2 - n^2)} & -\frac{2(-1)^m n (r-1)^{1/2} \sin(m\pi r)}{\pi(n^2 (r-1)^2 - m^2)} & \frac{2(-1)^{m+n} \sin(m\pi r) \sin(n\pi r)}{\mu \pi}
\end{pmatrix}.$$  

(2.27)

The idea behind the comparison between the correlation function and Hamiltonian matrix element is the conjecture

$$\langle iH_{jk} \rangle \sim \mu (\Delta_i - \Delta_j - \Delta_k) C_{ijk},$$  

(2.28)

where the correlator $C_{ijk}$ is exactly what we have just calculated in the previous section

$$C_{ijk} = \langle \mathcal{O}_i \mathcal{O}_j \mathcal{O}_k \rangle.$$  

(2.29)

It is supposed that the string states are identified in some well-defined way with the single-trace gauge theory operators. This is not really true [64], due to mixing with double-trace operators $^1$, but we omit here this discussion, since in the leading $\mu$ and $\lambda$ order it is irrelevant.

In the next-leading order in $\mu$ the operator redefinition will have to be taken into account. When identifying gauge theory operators with the string theory states we should also note the different oscillator bases used. Namely, the natural spin chain/gauge theory creation operator is given by $a^\dagger_n$, whereas the natural string theory operators are denoted by $a^\dagger$. The relation between them is

$$\alpha_n = \frac{a_{[n]} - i \text{sign}(n) a_{-[n]}}{\sqrt{2}}.$$  

(2.30)

String theory states in the matrix element $H_{123}$ are defined as $a^\dagger |0\rangle$. Field theory oscillators in $C_{123}$ are defined as $\alpha^\dagger |0\rangle$. With all normalizations taken into account, the conjecture (2.28) will boil down to the following working formula that is given by Dobashi and Yoneya [59] and rewritten in our notations at the leading-order in the large $\mu$ limit as (see their eq. (2.2), (3.9))

$$C_{123} = \frac{1}{2\mu} \frac{\sqrt{J_1 J_2 J}}{N} \left( \frac{J}{4\pi \mu} \right)^{-1} H_{123}.$$  

(2.31)

$^1$We specially thank Gordon Semenoff for a discussion of this point.
We have already taken into account here that \( \Delta_3 - \Delta_2 - \Delta_1 = 2 \). The matrix element \( H_{123} \) is organized as

\[
H_{123} = \frac{1}{8} PN^3,
\]

(2.32)

where we symbolically denote by \( P \) the prefactor contribution, by \( N^3 \) the exponential contribution; the \( 1/8 \) factor comes from the operator redefinition (2.30). For simplicity we take the case of three similar excitations, like the SSS case in the previous section. Doing the elementary algebra we get

\[
P = 4,
\]

(2.33)

and

\[
N^3 = -N^{12}_{n_1,n_2} N^{23}_{n_2,n_3} N^{31}_{n_3,n_1} - N^{11}_{n_1,n_1} N^{23}_{n_2,n_3} N^{23}_{n_2,n_3} -
\]

\[
+ N^{12}_{n_1,-n_2} N^{23}_{-n_2,-n_3} N^{31}_{-n_3,-n_1} + N^{11}_{-n_1,-n_1} N^{23}_{-n_2,-n_3} N^{23}_{-n_2,-n_3},
\]

(2.34)

where we have taken into account the combinatorial factor 48 (8 possible choices of pairings \( \times 6 \) permutations), canceled with the factor \( 1/48 \) coming from the exponent. Noticing that the piece with \( N^{11} \) exactly corresponds to a disconnected diagram, the connected sector, equivalent to the diagram in Fig. (3) is simply given by

\[
N^3_{\text{connected}} = -N^{12}_{n_1,n_2} N^{23}_{n_2,n_3} N^{31}_{n_3,n_1} + N^{12}_{-n_1,-n_2} N^{23}_{-n_2,-n_3} N^{31}_{-n_3,-n_1},
\]

(2.35)

Gathering all the coefficients and expanding the product of Neumann matrices, we get

\[
C_{123} = \frac{-4r^{3/2}(1-r)^{3/2}\sin^2 \pi ny}{\pi^2 (n_2^2 - (1-r)^2 n_3^2) (n_1^2 - r^2 n_3^2)},
\]

(2.36)

fully agreeing to the leading-order of our correlator in the previous section (2.15). This agreement is the main result of our work.

### 3 Conclusions and outlook

In this work we have demonstrated that the fully dynamical correlator of three BMN states, each with two impurities and with a non-zero momentum, as calculated field-theoretically with the procedure of [64] completely agrees with the string-theoretical calculation of the 3-string vertex matrix elements, as proposed in [53, 56–59].

In [27] a remarkable discrepancy at the next order in \( \lambda' \) was found for three-point correlators from strings and from field theory. Our leading-order result establishes a firm ground for the next order, a comparison which will be the next logical step to be done. Another extension that naturally follows from our work would be to look at the finite-size corrections, as done for the three-point correlators in [19, 67]. Finite-size corrections often do provide non-trivial tests of the AdS/CFT correspondence, e.g. as for a different sector of observables was done in our work [68].

While the two suggested further research directions – doing the next-order in \( \lambda' \) and \( \frac{1}{J} \) are in principle feasible, going along the path suggested in [5, 18, 25] and performing a true
strongly-coupled semiclassical calculation in order to move to the giant magnon end of the asymptotic space, would be a true challenge and a step into the terra incognita for our class of states. One would certainly be interested in proceeding to the correlator of giant-magnon-like heavy states with all momenta non-zero from our “heavy” (at fixed $\lambda'$) BMN ones. To achieve that goal several conceptual problems have to be solved, the most important of them is how to match the three giant magnons world-sheets\(^2\).

Such heavy-heavy-heavy correlators are certainly the most mysterious and the least known beasts in the three-point bestiary. Yet taking the existence of a smooth transition from giant magnons to simply heavy magnons, and the miracles observed for the correspondence between small $\lambda'$ and large $\lambda'$ expansions, our calculations would provide at least a starting point for comparison of correlators of three giant magnons. In view of the alleged universality of $\lambda'$ asymptotics for both large and small $\lambda$, the asymptotic way to the semiclassic regime seems to be feasible.

**Acknowledgments**

We thank Agnese Bissi, Troels Harmark, Tristan McLoughlin, Marta Orselli, Gordon Semenoff and Arkady Tseytlin for interesting and stimulating discussions. This work was supported in part by the MIUR-PRIN contract 2009-KHZKRX. The work of A.Z. is supported in part by the Ministry of Education and Science of the Russian Federation under contract 14.740.11.0081, NSh 3349.2012.2, the RFBR grants 10-01-00836 and 10-02-01483.

**References**

[1] R. A. Janik, P. Surowka, and A. Wereszczynski, “On correlation functions of operators dual to classical spinning string states,” *JHEP* 1005 (2010) 030, arXiv:1002.4613 [hep-th].

[2] E. Buchbinder and A. Tseytlin, “On semiclassical approximation for correlators of closed string vertex operators in AdS/CFT,” *JHEP* 1008 (2010) 057, arXiv:1005.4516 [hep-th].

[3] M. S. Costa, R. Monteiro, J. E. Santos, and D. Zoakos, “On three-point correlation functions in the gauge/gravity duality,” *JHEP* 1011 (2010) 141, arXiv:1008.1070 [hep-th].

[4] R. Roiban and A. Tseytlin, “On semiclassical computation of 3-point functions of closed string vertex operators in $AdS_5 \times S^5$,” *Phys.Rev.* D82 (2010) 106011, arXiv:1008.4921 [hep-th].

[5] K. Zarembo, “Holographic three-point functions of semiclassical states,” *JHEP* 1009 (2010) 030, arXiv:1008.1059 [hep-th].

[6] R. Hernandez, “Three-point correlation functions from semiclassical circular strings,” *J.Phys.A* A44 (2011) 085403, arXiv:1011.0408 [hep-th].

[7] D. Arnaudov and R. Rashkov, “On semiclassical calculation of three-point functions in $AdS_4 \times CP^3$,” *Phys.Rev.* D83 (2011) 066011, arXiv:1011.4669 [hep-th].

[8] G. Georgiou, “Two and three-point correlators of operators dual to folded string solutions at strong coupling,” *JHEP* 1102 (2011) 046, arXiv:1011.5181 [hep-th].

\(^2\)We specially thank Tristan McLoughlin for an interesting correspondence on that point.
[9] J. Escobedo, N. Gromov, A. Sever, and P. Vieira, “Tailoring Three-Point Functions and Integrability,” *JHEP* 1109 (2011) 028, arXiv:1012.2475 [hep-th].

[10] C. Park and B.-H. Lee, “Correlation functions of magnon and spike,” *Phys.Rev.* D83 (2011) 126004, arXiv:1012.3293 [hep-th].

[11] J. Russo and A. Tseytlin, “Large spin expansion of semiclassical 3-point correlators in $AdS_5 \times S^5$, “ *JHEP* 1102 (2011) 029, arXiv:1012.2760 [hep-th].

[12] D. Bak, B. Chen, and J.-B. Wu, “Holographic Correlation Functions for Open Strings and Branes,” *JHEP* 1106 (2011) 014, arXiv:1103.2024 [hep-th].

[13] A. Bissi, C. Kristjansen, D. Young, and K. Zoubos, “Holographic three-point functions of giant gravitons,” *JHEP* 1106 (2011) 085, arXiv:1103.4079 [hep-th].

[14] D. Arnaudov, R. Rashkov, and T. Vetsov, “Three and four-point correlators of operators dual to folded string solutions in $AdS_5 \times S^5$, “ *Int.J.Mod.Phys.* A26 (2011) 3403–3420, arXiv:1103.6145 [hep-th].

[15] R. Hernandez, “Three-point correlators for giant magnons,” *JHEP* 1105 (2011) 123, arXiv:1104.1160 [hep-th].

[16] X. Bai, B.-H. Lee, and C. Park, “Correlation function of dyonic strings,” *Phys.Rev.* D84 (2011) 026009, arXiv:1104.1896 [hep-th].

[17] C. Ahn and P. Bozhilov, “Three-point Correlation functions of Giant magnons with finite size,” *Phys.Lett.* B702 (2011) 286–290, arXiv:1105.3084 [hep-th].

[18] T. Klose and T. McLoughlin, “A light-cone approach to three-point functions in $AdS_5 \times S^5$, “ arXiv:1106.0495 [hep-th].

[19] D. Arnaudov and R. Rashkov, “Quadratic corrections to three-point functions,” arXiv:1106.0859 [hep-th].

[20] M. Michalcik, R. C. Rashkov, and M. Schimpf, “On semiclassical calculation of three-point functions in $AdS_5 \times T(1,1)$,” arXiv:1107.5795 [hep-th].

[21] R. A. Janik and A. Wereszczynski, “Correlation functions of three heavy operators: The AdS contribution,” arXiv:1109.6262 [hep-th].

[22] J. Escobedo, N. Gromov, A. Sever, and P. Vieira, “Tailoring Three-Point Functions and Integrability II. Weak/strong coupling match,” *JHEP* 1109 (2011) 029, arXiv:1104.5501 [hep-th].

[23] N. Gromov, A. Sever, and P. Vieira, “Tailoring Three-Point Functions and Integrability III. Classical Tunneling,” arXiv:1111.2349 [hep-th].

[24] G. Georgiou, “SL(2) sector: weak/strong coupling agreement of three-point correlators,” *JHEP* 1109 (2011) 132, arXiv:1107.1850 [hep-th].

[25] E. Buchbinder and A. Tseytlin, “Semiclassical correlators of three states with large $S^5$ charges in string theory in $AdS_5 \times S^5$, “ arXiv:1110.5621 [hep-th].

[26] O. Foda, “N=4 SYM structure constants as determinants,” arXiv:1111.4663 [math-ph]. 26 pages. Added more introductory material and more references, improved the presentation and corrected (a large number of) typos, particularly the references to the figures. Result is unchanged.
[27] A. Bissi, T. Harmark, and M. Orselli, “Holographic 3-point function at one loop,” arXiv:1112.5075 [hep-th].

[28] G. Georgiou, V. Gili, A. Grossardt, and J. Plefka, “Three-point functions in planar N=4 super Yang-Mills Theory for scalar operators up to length five at the one-loop order,” arXiv:1201.0992 [hep-th]. 45, 5 tables, many figures. arXiv admin note: substantial text overlap with arXiv:1007.2356.

[29] N. Gromov and P. Vieira, “Quantum Integrability for Three-Point Functions,” arXiv:1202.4103 [hep-th]. 4 pages, 3 figures.

[30] D. Serban, “A note on the eigenvectors of long-range spin chains and their scalar products,” arXiv:1203.5842 [hep-th].

[31] I. Kostov, “Classical Limit of the Three-Point Function from Integrability,” arXiv:1203.6180 [hep-th].

[32] N. Gromov, V. Kazakov, and P. Vieira, “Exact Spectrum of Anomalous Dimensions of Planar N=4 Supersymmetric Yang-Mills Theory,” Phys.Rev.Lett. 103 (2009) 131601, arXiv:0901.3753 [hep-th].

[33] N. Gromov, V. Kazakov, A. Kozak, and P. Vieira, “Exact Spectrum of Anomalous Dimensions of Planar N = 4 Supersymmetric Yang-Mills Theory: TBA and excited states,” Lett. Math. Phys. 91 (2010) 265–287, arXiv:0902.4458 [hep-th].

[34] N. Gromov, V. Kazakov, S. Leurent, and D. Volin, “Solving the AdS/CFT Y-system,” arXiv:1110.0562 [hep-th]. * Temporary entry *.

[35] S. Frolov and A. A. Tseytlin, “Rotating string solutions: AdS / CFT duality in nonsupersymmetric sectors,” Phys.Lett. B570 (2003) 96–104, arXiv:hep-th/0306143 [hep-th].

[36] M. Kruczenski, “Spin chains and string theory,” Phys.Rev.Lett. 93 (2004) 161602, arXiv:hep-th/0311203 [hep-th].

[37] J. Callan, Curtis G., H. K. Lee, T. McLoughlin, J. H. Schwarz, I. Swanson, et al., “Quantizing string theory in AdS(5) x S**5: Beyond the pp wave,” Nucl.Phys. B673 (2003) 3–40, arXiv:hep-th/0307032 [hep-th].

[38] T. Harmark, K. R. Kristjansson, and M. Orselli, “Matching gauge theory and string theory in a decoupling limit of AdS/CFT,” JHEP 0902 (2009) 027, arXiv:0806.3370 [hep-th].

[39] D. E. Berenstein, J. M. Maldacena, and H. S. Nastase, “Strings in flat space and pp waves from N=4 superYang-Mills,” JHEP 0204 (2002) 013, arXiv:hep-th/0202021 [hep-th].

[40] C. Kristjansen, J. Plefka, G. Semenoff, and M. Staudacher, “A New double scaling limit of N=4 superYang-Mills theory and PP wave strings,” Nucl.Phys. B643 (2002) 3–30, arXiv:hep-th/0205033 [hep-th].

[41] S. Lee, S. Minwalla, M. Rangamani, and N. Seiberg, “Three point functions of chiral operators in D = 4, N=4 SYM at large N,” Adv.Theor.Math.Phys. 2 (1998) 697–718, arXiv:hep-th/9806074 [hep-th].

[42] D. Z. Freedman, S. D. Mathur, A. Matusis, and L. Rastelli, “Correlation functions in the
CFT(d) / AdS(d+1) correspondence,” *Nucl.Phys.* **B546** (1999) 96–118, arXiv:hep-th/9804058 [hep-th].

[43] G. Arutyunov and S. Frolov, “Some cubic couplings in type IIB supergravity on AdS(5) x S**5 and three point functions in SYM(4) at large N,” *Phys.Rev.* **D61** (2000) 064009, arXiv:hep-th/9907085 [hep-th].

[44] G. Arutyunov, S. Frolov, and A. C. Petkou, “Operator product expansion of the lowest weight CPOs in N=4 SYM(4) at strong coupling,” *Nucl.Phys.* **B586** (2000) 547–588, arXiv:hep-th/0005182 [hep-th].

[45] F. Bastianelli and R. Zucchini, “Three point functions of universal scalars in maximal SCFTs at large N,” *JHEP* **0005** (2000) 047, arXiv:hep-th/0003230 [hep-th].

[46] P. Heslop and P. S. Howe, “OPEs and three-point correlators of protected operators in N=4 SYM,” *Nucl.Phys.* **B626** (2002) 265–286, arXiv:hep-th/0107212 [hep-th].

[47] E. D’Hoker and A. V. Ryzhov, “Three point functions of quarter BPS operators in N=4 SYM,” *JHEP* **0202** (2002) 047, arXiv:hep-th/0109065 [hep-th].

[48] M. Bianchi, M. Prisco, and W. Mueck, “New results on holographic three point functions,” *JHEP* **0311** (2003) 052, arXiv:hep-th/0310129 [hep-th].

[49] K. Okuyama and L.-S. Tseng, “Three-point functions in N = 4 SYM theory at one-loop,” *JHEP* **0408** (2004) 055, arXiv:hep-th/0404190 [hep-th].

[50] G. Georgiou, V. L. Gili, and R. Russo, “Operator mixing and three-point functions in N=4 SYM,” *JHEP* **0910** (2009) 009, arXiv:0907.1567 [hep-th].

[51] S. Dobashi, H. Shimada, and T. Yoneya, “Holographic reformulation of string theory on AdS(5) x S**5 background in the PP wave limit,” *Nucl.Phys.* **B665** (2003) 94–128, arXiv:hep-th/0209251 [hep-th]. To the memory of Prof. Bunji Sakita.

[52] Y. Kazama and S. Komatsu, “On holographic three point functions for GKP strings from integrability,” arXiv:1110.3949 [hep-th].

[53] Y.-H. He, J. H. Schwarz, M. Spradlin, and A. Volovich, “Explicit formulas for Neumann coefficients in the plane wave geometry,” *Phys.Rev.* **D67** (2003) 086005, arXiv:hep-th/0211198 [hep-th].

[54] M. Spradlin and A. Volovich, “Superstring interactions in a pp wave background. 2.,” *JHEP* **0301** (2003) 036, arXiv:hep-th/0206073 [hep-th].

[55] J. Pearson, M. Spradlin, D. Vaman, H. L. Verlinde, and A. Volovich, “Tracing the string: BMN correspondence at finite J**2/N,” *JHEP* **0305** (2003) 022, arXiv:hep-th/0210102 [hep-th].

[56] A. Pankiewicz and J. Stefanski, B., “PP wave light cone superstring field theory,” *Nucl.Phys.* **B657** (2003) 79–106, arXiv:hep-th/0210246 [hep-th].

[57] A. Pankiewicz and J. Stefanski, B., “On the uniqueness of plane wave string field theory,” arXiv:hep-th/0308062 [hep-th].

[58] P. Di Vecchia, J. L. Petersen, M. Petrini, M. Russo, and A. Tanzini, “The Three string vertex and the AdS / CFT duality in the PP wave limit,” *Class.Quant.Grav.* **21** (2004) 2221–2240, arXiv:hep-th/0304025 [hep-th].
[59] S. Dobashi and T. Yoneya, “Impurity non-preserving 3-point correlators of BMN operators from PP-wave holography. I. Bosonic excitations,” Nucl.Phys. B711 (2005) 54–82, arXiv:hep-th/0409058 [hep-th].

[60] H. Shimada, “Holography at string field theory level: Conformal three point functions of BMN operators,” Phys.Lett. B647 (2007) 211–218, arXiv:hep-th/0410049 [hep-th].

[61] G. Grignani, M. Orselli, B. Ramadanovic, G. W. Semenoff, and D. Young, “Divergence cancellation and loop corrections in string field theory on a plane wave background,” JHEP 0512 (2005) 017, arXiv:hep-th/0508126 [hep-th].

[62] G. Grignani, M. Orselli, B. Ramadanovic, G. W. Semenoff, and D. Young, “AdS/CFT versus string loops,” JHEP 0606 (2006) 040, arXiv:hep-th/0605080 [hep-th].

[63] N. R. Constable, D. Z. Freedman, M. Headrick, S. Minwalla, L. Motl, et al., “PP wave string interactions from perturbative Yang-Mills theory,” JHEP 0207 (2002) 017, arXiv:hep-th/0205089 [hep-th].

[64] N. Beisert, C. Kristjansen, J. Plefka, G. Semenoff, and M. Staudacher, “BMN correlators and operator mixing in N=4 superYang-Mills theory,” Nucl.Phys. B650 (2003) 125–161, arXiv:hep-th/0208178 [hep-th].

[65] C.-S. Chu, V. V. Khoze, and G. Travaglini, “Three point functions in N=4 Yang-Mills theory and pp waves,” JHEP 0206 (2002) 011, arXiv:hep-th/0206005 [hep-th].

[66] S. Dobashi, “Impurity Non-Preserving 3-Point Correlators of BMN Operators from PP-Wave Holography. II. Fermionic Excitations,” Nucl.Phys. B756 (2006) 171–206, arXiv:hep-th/0604082 [hep-th].

[67] P. Bozhilov, “More three-point correlators of giant magnons with finite size,” JHEP 1108 (2011) 121, arXiv:1107.2645 [hep-th].

[68] D. Astolfi, G. Grignani, E. Ser-Giacomi, and A. Zayakin, “Strings in AdS4 × CP3: finite size spectrum vs. Bethe Ansatz,” arXiv:1111.6628 [hep-th].