Solar system tests of brane world models

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Abstract
The classical tests of general relativity (perihelion precession, deflection of light and the radar echo delay) are considered for the Dadhich, Maartens, Papadopoulos and Rezania (DMPR) solution of the spherically symmetric static vacuum field equations in brane world models. For this solution the metric in the vacuum exterior to a brane world star is similar to the Reissner–Nordström form of classical general relativity, with the role of the charge played by the tidal effects arising from projections of the fifth dimension. The existing observational solar system data on the perihelion shift of Mercury, on the light bending around the Sun (obtained using long-baseline radio interferometry), and ranging to Mars using the Viking lander, constrain the numerical values of the bulk tidal parameter and of the brane tension.

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1. Introduction

The idea that our four-dimensional Universe might be a three-brane [1], embedded in a higher dimensional spacetime and inspired by superstring theory, has recently attracted much attention. In this context, the 10-dimensional $E_8 \times E_8$ heterotic string theory, which contains the standard model of elementary particles, could be a promising candidate for the description of the real Universe. This theory is connected with an 11-dimensional theory, the $M$-theory, compactified on the orbifold $R^{10} \times S^1/Z_2$ [2]. According to the brane world scenario, the physical fields in our four-dimensional spacetime, which are assumed to arise as fluctuations of branes in string theories, are confined to the three brane. Only gravity can freely propagate...
in the bulk spacetime, with the gravitational self-couplings not significantly modified. The model originated from the study of a single three-brane embedded in five dimensions, with the 5D metric given by \( ds^2 = e^{-f(y)} \eta_{\mu\nu} \, dx^\mu \, dx^\nu + dy^2 \), which due to the appearance of the warp factor, could produce a large hierarchy between the scale of particle physics and gravity. Even if the fifth dimension is uncompactified, standard 4D gravity is reproduced on the brane. Hence this model allows the presence of large, or even infinite non-compact extra-dimensions. Our brane is identified as a domain wall in a five-dimensional anti-de Sitter spacetime. For a review of dynamics and geometry of brane universes, see [3].

Due to the correction terms coming from the extra-dimensions, significant deviations from the Einstein theory occur in brane world models at very high energies [4, 5]. In particular, gravity is largely modified at the electro-weak scale, 1 TeV. The cosmological and astrophysical implications of the brane world theories have been extensively investigated in the literature [6]. Gravitational collapse can also produce high energies, with the five-dimensional effects playing an important role in the formation of black holes [7].

Note that for standard general relativistic spherical compact objects the exterior spacetime is usually described by the Schwarzschild metric. However, in five-dimensional brane world models, the high-energy corrections to the energy density, together with Weyl stresses from bulk gravitons, imply that on the brane the exterior metric of a static star is no longer the Schwarzschild metric [8]. The presence of the Weyl stresses also means that the matching conditions do not have a unique solution on the brane; knowledge of the five-dimensional Weyl tensor is needed as a minimum condition for uniqueness.

Static and spherically symmetric exterior vacuum solutions of the brane world models were initially proposed by Dadhich et al [8] and Germani and Maartens [9]. The former solutions have the mathematical form of the Reissner–Nordström solution, in which a tidal Weyl parameter plays the role of the electric charge of the general relativistic solution [8]. The solution was obtained by imposing the null energy condition on the three-brane for a bulk having nonzero Weyl curvature, and a specific case was obtained by matching to an interior solution, corresponding to a constant density brane world star. An exact interior uniform-density stellar solution on the brane was also found in [9]. In the latter model, it was found that the general relativistic upper bound for the mass–radius ratio, \( M/R < 4/9 \), was reduced by five-dimensional high-energy effects. It was also found that the existence of brane world neutron stars leads to a constraint on the brane tension, which is stronger than the big-bang nucleosynthesis constraint, but weaker than the Newton-law experimental constraints [9]. We refer the reader to [10, 11] and references therein for further static and spherically symmetric brane world solutions.

There are several possibilities of observationally testing the brane world models at an astrophysical/cosmological scale, such as using the time delay of gamma ray bursts [12] or by using the luminosity distance–redshift relation for supernovae at higher redshifts [13]. The classical tests of general relativity, namely, light deflection, time delay and perihelion shift, have been analyzed, for gravitational theories with large non-compactified extra-dimensions, in the framework of the five-dimensional extension of the Kaluza–Klein theory, using an analog of the four-dimensional Schwarzschild metric in [14]. Solar system data also impose some strong constraints on Kaluza–Klein-type theories. The existence of extra-dimensions and the brane world models can also be tested via the gravitational radiation coming from primordial black holes, with masses of the order of the lunar mass, \( M \sim 10^{-7} M_\odot \), which might have been produced when the temperature of the universe was around 1 TeV. If a significant fraction of the dark halo of our galaxy consists of these lunar mass black holes, a huge number of black hole binaries could exist. The detection of the gravitational waves from these binaries could confirm the existence of extra-dimensions [15].
It is the purpose of the present paper to consider the classical tests (perihelion precession, light bending and radar echo delay) of general relativity for static gravitational fields in the framework of brane world models. To do this we shall adopt for the geometry of the brane outside a compact, stellar-type object, the spherically symmetric, Reissner–Nordström-type, static brane metric obtained by Dadhich, Maartens, Papadopoulos and Rezania (DMPR) [8]. For this metric, we first consider the motion of a particle (planet), and the contributions of the five-dimensional effects to the perihelion precession are calculated.

By considering the motion of a photon in the static brane gravitational field we obtain the corrections, due to the projected bulk Weyl tensor, to the bending of light by massive astrophysical objects and to the radar echo delay, respectively. Existing data on light-bending around the Sun, using long-baseline radio interferometry, ranging to Mars using the Viking lander, and the perihelion precession of Mercury, can all give significant and detectable solar system constraints associated with the extra-dimensional part of the metric. More exactly, the study of the classical general relativistic tests, by taking into account the corrections coming from the extra-dimensions, constrains the tidal bulk parameter and, via the junction conditions, the brane tension. The advance of the perihelion for the (charged) standard general relativistic Reissner–Nordström metric has been considered in [16], where the formula for the shift in the perihelion of a charged particle has been derived. The gravitational lensing by a Reissner–Nordström black hole in the weak field limit has also been analyzed in [17].

The present paper is organized as follows. The static and spherically symmetric vacuum solution on the brane is presented in section 2. In section 3 we consider the classical solar system tests, namely, the perihelion shift, the light deflection and the radar echo delay, for the brane world model stars. We conclude our results in section 4.

2. The DMPR solution of the vacuum field equations on the brane

The five-dimensional Einstein field equation in the bulk is given by $G_{IJ} = k_5^2 T_{IJ}$, where the five-dimensional energy–momentum, $T_{IJ}$, is provided by

$$T_{IJ} = -\Lambda_5 g_{IJ} + \delta(Y) \left( -\lambda g_{IJ} + T_{IJ}^{\text{matter}} \right),$$

and $\Lambda_5$ is the negative vacuum energy in the bulk. In the analysis below, we consider that capital Latin indices run in the range $0, \ldots, 4$, while Greek indices take the values $0, \ldots, 3$.

The effective four-dimensional field equations on the brane (the Gauss equation) take the form [4, 5]

$$G_{\mu\nu} = -\Lambda g_{\mu\nu} + k_4^2 T_{\mu\nu} + k_5^2 S_{\mu\nu} - E_{\mu\nu},$$

where the four-dimensional cosmological constant, $\Lambda$, and the coupling constant, $k_4$, are given by $\Lambda = k_5^2 (\Lambda_5 + k_5^2 \lambda^2 / 6)$ and $k_4^2 = k_5^2 \lambda / 6$. In the limit $\lambda^{-1} \to 0$ we recover standard general relativity, respectively, with $\lambda$ the vacuum energy on the brane.

$S_{\mu\nu}$ is the local quadratic energy–momentum correction, which arises from the extrinsic curvature term in the projected Einstein tensor, and is given by

$$S_{\mu\nu} = \frac{1}{12} TT_{\mu\nu} - \frac{1}{4} T_\mu^a T_{a\nu} + \frac{1}{24} g_{\mu\nu} (3 T^{\alpha\beta} T_{\alpha\beta} - T^2).$$

The term $E_{\mu\nu}$ is the projection of the five-dimensional Weyl tensor $C_{IABJ}$. $E_{\mu\nu} = C_{IABJ} n^I n^J$. The only known property of this nonlocal term is that it is traceless, i.e., $E_\mu^\mu = 0$.

The Einstein equation in the bulk and the Codazzi equation also imply the conservation of the energy–momentum tensor of matter on the brane, $D_\nu T_\mu^\nu = 0$, where $D_\nu$ denotes the brane covariant derivative. Moreover, the contracted Bianchi identities on the brane imply that the projected Weyl tensor should obey the constraint $D_\nu E_\mu^\nu = k_5^2 D_\nu S_\mu^\nu$. 

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Note that the symmetry properties of $E_{\mu\nu}$ imply that in general we can decompose it irreducibly with respect to a chosen 4-velocity field $u^{\mu}$ as
\[
E_{\mu\nu} = -\tilde{k}^4 \left[ U (u_\mu u_\nu + \frac{1}{3} h_{\mu\nu}) + 2Q_\mu u_\nu + P_{\mu\nu} \right],
\]
(4)
where $\tilde{k} = k_5 / k_4$, $h_{\mu\nu} = g_{\mu\nu} + u_\mu u_\nu$ projects orthogonal to $u^{\mu}$, the ‘dark radiation’ term $U = -\tilde{k}^{-4} E_{\mu\nu} u^{\mu} u^{\nu}$ is a scalar, $Q_\mu = \tilde{k}^{-4} h_{\mu\nu} E_{\alpha\beta} u^{\beta}$ is a spatial vector and $P_{\mu\nu} = -\tilde{k}^{-4} \left[ h_\alpha u_\mu u_\nu - \frac{1}{3} h_{\mu\nu} h_\alpha h_\beta \right] E_{\alpha\beta}$ is a spatial, symmetric and trace-free tensor [3].

For the specific case of vacuum, $T_{\mu\nu} = 0$, and consequently $S_{\mu\nu} = 0$, and assuming that $\Lambda = 0$, the field equations describing a static brane take the form
\[
R_{\mu\nu} = -E_{\mu\nu},
\]
(5)
with $R_{\mu\nu} = 0 = E_{\mu}^{\mu}$. For this case $E_{\mu\nu}$ satisfies the constraint $D_{\nu} E_{\mu\nu} = 0$. In a static vacuum $Q_\mu = 0$ and the constraint for $E_{\mu\nu}$ takes the form
\[
\frac{1}{3} D_{\mu} U + \frac{4}{3} U A_{\mu} + D_{\nu} P_{\mu\nu} + A_{\nu} P_{\mu\nu} = 0,
\]
(6)
where $D_{\mu}$ is the projection (orthogonal to $u^{\mu}$) of the covariant derivative and $A_{\mu} = u^\nu D_{\nu} u^{\mu}$ is the 4-acceleration.

In the static spherically symmetric case we may choose $A_{\mu} = A(r) r^{\mu}$ and $P_{\mu\nu} = P(r) \left( r_{\mu} r_{\nu} - \frac{1}{3} h_{\mu\nu} \right)$, where $A(r)$ and $P(r)$ are some scalar functions of the radial distance $r$ and $r_{\mu}$ is a unit radial vector. The choice $U = \tilde{k}^4 Q / r^4 = -P / 2$, where $Q$ is a constant, leads to a Reissner–Nordström-type solution of the static, spherically symmetric field equations on the brane [8]:
\[
ds^2 = -\left( 1 - \frac{2M}{M_p r} + \frac{q}{M_p^2 r^2} \right) dt^2 + \left( 1 - \frac{2M}{M_p r} + \frac{q}{M_p^2 r^2} + r^2 (d\theta^2 + \sin^2 \theta d\phi^2) \right) dr^2 + r^2 (d\theta^2 + \sin^2 \theta d\phi^2),
\]
(7)
where the Planck scales in the brane and in the bulk, $M_p$ and $\tilde{M}_p$, respectively, are related by $M_p = \sqrt{\frac{3}{4\pi}} \tilde{M}_p / \sqrt{\lambda}$, and $q = Q \tilde{M}_p^2$ is the dimensionless tidal parameter.

For this model the expression of the projected Weyl tensor transmitting the tidal charge stresses from the bulk to the brane is [8]
\[
E_{\mu\nu} = -\frac{Q}{r^4} (u_\mu u_\nu - 2r_{\mu} r_{\nu} + h_{\mu\nu}).
\]
(8)

Perturbative studies of the static weak-field regime show that the leading order correction to the Newtonian potential on the brane is given by $\phi = GM (1 + 2l^2 / 3r^2) / r$, where $l$ is the curvature scale of the five-dimensional anti-de Sitter spacetime (AdS$_5$) [1]. However, this result assumes that the bulk perturbations are bounded in conformally Minkowski coordinates and that the bulk is nearly AdS$_5$ [9]. Different bulk geometries could induce different corrections to Newton’s law on the brane. In the following we denote by $r_g = 2M / M^2_p = 2GM / c^2$ the gravitational radius of the brane star.

3. The classical tests of general relativity for the DMPR brane world solution

To determine the trajectory of a massive particle in the metric (7) we use the Hamilton–Jacobi equation,
\[
g^{ik} \frac{\partial S}{\partial x^i} \frac{\partial S}{\partial x^k} - m^2 c^2 = 0,
\]
(9)
where $m \neq 0$ is the mass of the particle [18]. As in every central spherically symmetric field, the motion occurs in a single plane passing through the origin, and without a loss of generality
one may choose the plane with $\theta = \pi/2$. With the use of the metric coefficients from equation (7), we obtain

$$
\left( 1 - \frac{r_e}{r} + \frac{Q}{r^2} \right) \left( \frac{\partial S}{\partial c \partial t} \right)^2 - \left( 1 - \frac{r_e}{r} + \frac{Q}{r^2} \right) \left( \frac{\partial S}{\partial r} \right)^2 - \frac{1}{r^2} \left( \frac{\partial S}{\partial \phi} \right)^2 - m^2 c^2 = 0. \quad (10)
$$

According to the standard procedure for solving the Hamilton–Jacobi equation [19], we chose $S$ in the form

$$
S = -Et + L\phi + S_r(r), \quad (11)
$$

where the energy $E$ and the angular momentum $L$ are constants of motion. Substituting equation (11) into equation (10), we find $S_r(r)$ given by

$$
S_r = \int \sqrt{\frac{E^2}{c^2} \left( 1 - \frac{r_e}{r} + \frac{Q}{r^2} \right)^{-2} - \left( m^2 c^2 + \frac{L^2}{r^2} \right) \left( 1 - \frac{r_e}{r} + \frac{Q}{r^2} \right)^{-1}} \, dr. \quad (12)
$$

Considering a change in the integration variable from $r$ to $r'$ by means of the following transformation,

$$
r (r - r_e) + Q = r'^2, \quad (13)
$$

provides

$$
\frac{r}{r'} \approx 1 + \frac{r_e}{2r'} + \frac{r_e^2}{8r'^2} - \frac{Q}{2r'^2}. \quad (14)
$$

In terms of the new variable, and also by introducing the non-relativistic energy $E_0$ ($E = E_0 + mc^2$), we obtain $S_r$ in the form

$$
S_r = \int \left[ \frac{2E_0m + \frac{E_0^2}{c^2}}{c^2} + \frac{1}{r} (4E_0mr_e + 2Gm^2M) - \frac{L^2}{r^2} \left( 1 - \frac{3m^2 c^2 r_e^2}{2L^2} - \frac{2m^2 c^2 Q}{2L^2} \right) \right]^{1/2} \, dr, \quad (15)
$$

where for brevity the prime on $r'$ has been dropped.

### 3.1. The precession of the perihelion

The trajectory of the particle is defined by the equation $\phi + (\partial S_r/\partial L) = \text{constant} [19]$. Hence, a change of the angle $\phi$ after one revolution of the particle in the orbit is given by $\Delta \phi = - (\partial \Delta S_r/\partial L)$ [18]. Expanding $S_r$ in powers of the small correction to the coefficient of $1/r^2$ we obtain

$$
\Delta S_r = \Delta S_r^{(0)} - \frac{3m^2 c^2 r_e^2 - 2m^2 c^2 Q}{4L} \frac{\partial}{\partial L} \Delta S_r^{(0)}, \quad (16)
$$

where $\Delta S_r^{(0)}$ corresponds to the motion in the closed (Newtonian and unshifted) ellipse. Differentiating this relation with respect to $L$, and taking into account that $\Delta \phi^{(0)} = - (\partial \Delta S_r^{(0)}/\partial L) = 2\pi$, we find

$$
\Delta \phi = 2\pi \left( 1 + \frac{3m^2 c^2 r_e^2 - 2m^2 c^2 Q}{4L^2} \right). \quad (17)
$$

With the use of the Newtonian relation between the angular momentum, the length of the semi-major axis $a$ and the eccentricity $e$ of the ellipse, $L^2/GMm^2 = a(1 - e^2)$ [19], we
obtain the final form of the precession $\delta \varphi = \Delta \varphi - 2\pi$ of the perihelion of a planet moving in the static gravitational field on the DMPR brane:

$$\delta \varphi = -\frac{6\pi GM}{c^2 a(1-e^2)} - \frac{\pi c^2 Q}{GMa(1-e^2)}, \quad \text{(18)}$$

The first term in equation (18) is the well-known general relativistic correction term for the perihelion precession, while the second term gives the correction due to the nonlocal effects arising from the Weyl curvature in the bulk.

The observed value of the perihelion precession of the planet Mercury is $\delta \varphi_{\text{obs}} = 43.11 \pm 0.21 \text{ arcsec per century}$ [20]. The general relativistic formula for the precession, $\delta \varphi_{GR} = 6\pi GM/c^2 a(1-e^2)$, with $M = M_\odot = 1.989 \times 10^{33} \text{ g}$, $c = 2.998 \times 10^{10} \text{ cm s}^{-1}$, $G = 6.67 \times 10^{-8} \text{ cm}^3 \text{ g}^{-1} \text{ s}^{-2}$, $a = 57.91 \times 10^{11} \text{ cm}$ and $e = 0.205615$ [20], gives $\delta \varphi_{GR} = 42.94 \text{ arcsec per century}$. Therefore, the difference $\Delta \varphi = \delta \varphi_{\text{obs}} - \delta \varphi_{GR} = 0.17 \text{ arcsec per century}$ can be attributed to other effects. By assuming that $\Delta \varphi$ is entirely due to the modifications of the general relativistic Schwarzschild geometry as a result of the five-dimensional bulk effects, the observational results impose the following general constraint on the bulk tidal parameter $Q$:

$$|Q| \leq \frac{GM_\odot a(1-e^2)}{\pi c^2} \Delta \varphi. \quad \text{(19)}$$

With the use of the observational data for Mercury, equation (19) gives $|Q| \leq 5.17 \times 10^8 \text{ cm}^3$, or in the natural system of units, with $c = \hbar = G = 1$, $|Q| \leq 1.32 \times 10^{30} \text{ MeV}^{-2}$. On the other hand, for a constant density star, Germani and Maartens [9] have derived the matching conditions at the vacuum boundary of the brane star, implying $Q = -(3GM/c^2)R(\rho/\lambda)$, where $\rho$ is the density of the brane star, $R$ is its radius and $\lambda$ is the brane tension. Therefore, by assuming that the Sun can be described (at least approximatively) as a constant density brane star, we obtain the following solar system observational constraint on the brane tension $\lambda$:

$$\lambda \geq \frac{3\pi R_\odot \rho_\odot}{a(1-e^2)\Delta \varphi}. \quad \text{(20)}$$

The matching conditions for uniform density stars [9] also give

$$\lambda > [GM/c^2/(R - 2GM/c^2)]\rho. \quad \text{(21)}$$

For a typical neutron star with mass $M = 1.4M_\odot$, density $\rho = 2 \times 10^{14} \text{ g cm}^{-3}$ and $R = 10 \text{ km}$, we find $\lambda > 7 \times 10^{13} \text{ g cm}^{-3} (= 2.905 \times 10^{13} \text{ MeV}^4)$. By taking, for the case of the Sun, $R_\odot = 7 \times 10^{10} \text{ cm}$ and for $\rho$ the mean density of the Sun, $\rho = \rho_\odot = 1.41 \text{ g cm}^{-3}$, equation (20) gives, with $\Delta \varphi = 0.17 \text{ arcsec per century}$, $\lambda \geq 8.4 \times 10^7 \text{ g cm}^{-3} (= 3.5 \times 10^2 \text{ MeV}^4)$.

Unfortunately, the observational data on the perihelion precession are strongly affected by the solar oblateness, whose value is poorly known. Solar oblateness introduces a supplementary term of the form $\xi J_2 \delta \varphi_{GR}$ on the right-hand side of equation (18), with $\xi = R_\odot^2 / 2M_\odot a(1-e^2)$ and $J_2$ the solar quadrupole moment [21]. The value of $J_2$ is a subject of debate, but a recent estimate gives $J_2 = (3.64 \pm 2.84) \times 10^{-6}$ [22]. Taking into account the quadrupole correction to $\Delta \varphi$ gives an estimate of the brane tension of the order $\lambda \geq 2 \times 10^{10} \text{ g cm}^{-3} (= 9 \times 10^4 \text{ MeV}^4)$.

### 3.2. Light deflection on the brane

The propagation of light in a centrally symmetric gravitational field is described by the eikonal equation [18]

$$g_{(k)}^{\mu\nu} \frac{\partial \psi}{\partial x^\mu} \frac{\partial \psi}{\partial x^\nu} = 0. \quad \text{(22)}$$
We assume again that the light ray is moving in the plane $\theta = \pi/2$. By representing the eikonal $\psi$ in the form $\psi = -\omega_0 t + L \phi + \psi_r(r)$, where $\omega_0$ is the frequency of the light and $L$ is a constant, it follows that the radial part of the eikonal $\psi_r(r)$ is given by

$$\psi_r(r) = \frac{\omega_0}{c} \int \frac{dr}{\sqrt{r^2 - r_g r + Q^2}} = -\frac{l^2}{r^2 - r_g r + Q}, \quad (23)$$

where we denoted $l = c L / \omega_0$. By means of the transformations given by equations (13) and (14), equation (23) can be written as

$$\psi_r(r) = \frac{\omega_0}{c} \int \frac{1 + \frac{2 r_g}{r} - \frac{l^2 + 2 Q}{r^2}}{\sqrt{1 + 2 Q}} \, dr. \quad (24)$$

Expanding the integrand in powers of $r_g / r$ we obtain

$$\psi_r = \psi_r^{(0)} + \frac{\omega_0 r_g}{c} \int \frac{dr}{\sqrt{r^2 - (l^2 + 2 Q)}} = \psi_r^{(0)} + \frac{\omega_0 r_g}{c} \cosh^{-1} \frac{r}{\sqrt{l^2 + 2 Q}}, \quad (25)$$

where $\psi_r^{(0)}$ corresponds to the classical straight ray, with $r = l / \cos \phi$. The total change in $\psi_r$, during the propagation of the light from a very distant point $R$ to the point $r = l$ nearest to the center and then back to $R$ is $\Delta \psi_r = \Delta \psi_r^{(0)} + (2 \omega_0 r_g / c) \cosh^{-1} (R / \sqrt{l^2 + 2 Q})$.

The change in the polar angle is obtained by differentiating $\Delta \psi_r$ with respect to $L$ [18]:

$$\Delta \phi_{LD} = -\frac{\partial \Delta \psi_r}{\partial L} = -\frac{\partial \Delta \psi_r^{(0)}}{\partial L} + \frac{2 r_g}{l} \left(1 + \frac{2 Q}{l^2}\right)^{-1} \left(1 - \frac{l^2 + 2 Q}{R^2}\right)^{-1/2}. \quad (26)$$

Going to the limit $R \rightarrow \infty$ and taking into account that the straight line corresponds to $\Delta \phi = \pi$, we find that the angle $\delta \phi_{LD} = \Delta \phi_{LD} - \pi$ between the two asymptotes of the light ray differs from $\pi$ by the angle

$$\delta \phi_{LD} = \frac{2 r_g}{l} - \frac{4 Q r_g}{l^3} = \delta \phi^{(GR)}_{LD} \left(1 - \frac{2 Q}{l^2}\right), \quad (27)$$

where $\delta \phi^{(GR)}_{LD} = 4GM/c^2 l$ is the standard general relativistic light deflection term [18].

We consider now the constraints on the brane world models arising from the solar system observations of the light deflections. The best available data come from long baseline radio interferometry [24], which gives $\delta \phi_{LD} = \delta \phi^{(GR)}_{LD} (1 + \Delta_{LD})$, with $\Delta_{LD} \leq 0.0017$. Therefore we have $|Q| \leq \ell^2 \Delta_{LD} / 2$. For light just grazing the Sun’s limb $l = R_{\odot}$, and light deflection in the solar system imposes the restriction $|Q| \leq 4 \times 10^{18} \text{ cm}^2$ on the tidal parameter describing the effect of the five-dimensional bulk on the brane.

Assuming again that the Sun can be modeled as a constant density star, and taking into account the junction condition, we obtain the following limit on the brane tension $\lambda$:

$$\lambda \geq \frac{3 G M}{c^2} \frac{\rho}{R \Delta_{LD}}. \quad (28)$$

As applied to the case of the Sun, equation (28) gives $\lambda \geq 5.2 \times 10^{-3} \text{ g cm}^{-3}$ ($=2.15 \times 10^{-8} \text{ MeV}^4$). However, if we admit that $\Delta_{LD}$ is a universal quantity, giving the absolute deviation from standard general relativity, we can equally apply equation (28) to high density compact objects, such as neutron stars, by using the same $\Delta_{LD}$ as obtained in the case of the solar system. With $M = 1.4 M_{\odot}, \rho = 2 \times 10^{14} \text{ g cm}^{-3}$ and $R = 10^6 \text{ cm}$, equation (28) gives $\lambda \geq 7.3 \times 10^{16} \text{ g cm}^{-3}$ ($=3 \times 10^{11} \text{ MeV}^4$). The light deflection for the DMPR black hole solution has also been analyzed in [23], but using different methods.
3.3. Radar echo delay

A third solar system test of general relativity is the radar echo delay [25]. The idea of this test is to measure the time required for radar signals to travel to an inner planet or satellite in two circumstances: (a) when the signal passes very near the Sun and (b) when the ray does not go near the Sun. The null geodesic equation in the brane world metric (7) for a radar signal traveling in the $\theta = \pi/2$ plane is

$$\left(\frac{dr}{c \, dt}\right)^2 = \left(1 - \frac{r_g}{r} + \frac{Q}{r^2}\right)^2 - r^2 \left(1 - \frac{r_g}{r} + \frac{Q}{r^2}\right) \left(\frac{d\phi}{c \, dt}\right)^2.$$  \hfill (29)

Note that the conservation of the angular momentum and energy implies $r^2 d\phi / d\sigma = \text{constant}$ and $(1 - r_g / r + Q / r^2)(c \, dt / d\sigma) = \text{constant}$, where $\sigma$ is the parameter along the photon path, thus giving $r^2(1 - r_g / r + Q / r^2)^{-1} (d\phi / c \, dt) = b = \text{constant}$. Hence equation (29) becomes

$$\left(\frac{dr}{c \, dt}\right)^2 = \left(1 - \frac{r_g}{r} + \frac{Q}{r^2}\right)^2 - \frac{b^2}{r^2} \left(1 - \frac{r_g}{r} + \frac{Q}{r^2}\right)^3.$$  \hfill (30)

Let PSE be the path of light from the planet, P, to the Earth, E, with S being the point of closest approach to the Sun. At P, S and E the distances are $r = R_P$, $r = R_S$ and $r = R_E$, respectively. The time taken by the light ray to travel from P to E is [26]

$$t = \frac{1}{c} \sum_{i=P,E} \int_{R_i}^{R_E} \frac{dr}{(1 - \frac{r_g}{r} + \frac{Q}{r^2})(\sqrt{1 - \frac{r_g}{r}} + \sqrt{1 - \frac{r_g}{r} + \frac{Q}{r^2}})}.$$  \hfill (31)

Since $dr / dt = 0$ at $r = R_S$ [26], equation (30) fixes the value of the constant $b$ as $b^2 = R_S^2 / (1 - r_g / R_S + Q / R_S^2)$. With this value of $b$ and by using the following approximation

$$1 - b^2(1 - r_g / r + Q / r^2) / r^2 \approx \left[1 - (R_S^2 - Q) / r^2\right][1 - R_S \, r_g / r(r + \sqrt{R_S^2 - Q})],$$

we obtain

$$t = \frac{1}{c} \sum_{i=P,E} \int_{R_i}^{R_E} \frac{1}{\sqrt{1 - \frac{R_i^2 - Q}{r^2}}} \left[1 + \frac{R_S \, r_g}{2r(r + \sqrt{R_S^2 - Q})} + \frac{r_g / r - Q / r^2}{1 - (R_S^2 - Q) / r^2}\right] \, dr.$$  \hfill (33)

In the absence of the standard general relativistic gravitational deflection of light, all terms in $r_g$ vanish and we obtain

$$t_0 = \frac{1}{c} \sum_{i=P,E} \int_{R_i}^{R_E} \frac{1}{\sqrt{1 - \frac{R_i^2 - Q}{r^2}}} \left[1 - \frac{Q}{r^2}\right] \, dr.$$  \hfill (34)

In equation (34) we have assumed that the propagation of light in the vacuum far away from matter sources is influenced by the bulk effects only, characterized by the tidal coefficient $Q$. The standard general relativistic effects are not present in this approximation. Performing the integration gives

$$ct_0 = \sqrt{R_E^2 - R_P^2 + Q} + \sqrt{R_P^2 - R_S^2 + Q} - 2\sqrt{Q}$$

$$+ \frac{Q}{\sqrt{Q - R_E^2}} \ln \frac{R_E^2}{R_E R_P} \left(\sqrt{Q - R_E^2} + \sqrt{Q - R_P^2 + Q}\right) \left(\sqrt{Q - R_S^2} + \sqrt{Q - R_E^2 + Q}\right).$$  \hfill (35)

Equation (35) describes the propagation of a light ray in a brane world vacuum with a Reissner–Nordström-type geometry induced by the five-dimensional bulk effects. In order
that \( c t_0 \) be a real quantity it is necessary that the tidal coefficient \( Q \) be positive, \( Q > 0 \). For \( Q < 0 \), \( t_0 \) is an imaginary quantity. In the limit \( Q \to 0 \) we obtain the classical (Newtonian) result \( c t_0^{(\text{New})} = \sqrt{R_E^2 - R_S^2} + \sqrt{R_P^2 - R_S^2} \).

The time delay \( 2(t - t_0) \) for a round trip of a radar signal traveling to a planet or satellite is

\[
2(t - t_0) = \Delta t_{RD} = 2 \frac{r_S}{c} \ln \frac{(R_E + \sqrt{R_E^2 - R_S^2 + Q})(R_P + \sqrt{R_P^2 - R_S^2 + Q})}{(\sqrt{Q} + R_S)^2} \\
+ \frac{r_S}{c} \frac{R_S}{\sqrt{R_S^2 - Q}} \left( \frac{R_E - \sqrt{R_E^2 - Q}}{R_E + \sqrt{R_E^2 - Q}} + \frac{R_P - \sqrt{R_P^2 - Q}}{R_P + \sqrt{R_P^2 - Q}} - 2 \frac{R_S - \sqrt{R_S^2 - Q}}{R_S + \sqrt{R_S^2 - Q}} \right).
\]

(36)

In the limit \( Q \to 0 \) and by assuming that \( R_S \ll R_E, R_P \) we recover the standard general relativistic expression \( \Delta t_{RD}^{(GR)} \approx 2 r_S \left[ \ln \left( 4 R_E R_P / R_S^2 \right) + 1 \right] \) [25]. The correction term \( \Delta t_{BW} \) to the radar echo delay, due to the effects of the five-dimensional bulk, can be written as

\[
\Delta t_{RD}^{(BW)} \approx 2 \frac{r_S}{c} \left[ \ln \frac{1 + \frac{Q}{4 R_S^2}}{1 + \frac{Q}{4 R_S^2}} + \frac{Q}{2 R_S^2} - \frac{1 - \sqrt{1 - \frac{Q}{4 R_S^2}}}{1 + \sqrt{1 - \frac{Q}{4 R_S^2}}} \right].
\]

(37)

The total delay time for a radar signal traveling in the brane world is \( \Delta t_{RD} = \Delta t_{RD}^{(GR)} + \Delta t_{RD}^{(BW)} \). However, equation (36) for the coordinate time delay is not very useful directly, for it requires knowledge of \( (R_S^2 - R_2)^{1/2} \), etc, to a high degree of accuracy. The differential systems of radial coordinates themselves differ by large amounts. Besides, the electrons in the solar corona affect the time delay by an amount which shows considerable variation with time [26].

The best experimental solar system constraints on time delay so far have come from the Viking lander on Mars [27]. In the Viking mission two transponders landed on Mars and two others continued to orbit round it. The latter two transmitted two distinct bands of frequencies and thus the coronal effect could be corrected for. For the time delay of the signals emitted on the Earth and which graze the Sun one obtains \( \Delta t_{RD} = \Delta t_{RD}^{(GR)} (1 + \Delta_{RD}) \), with \( \Delta_{RD} \leq 0.002 \) [27]. Therefore, the contribution due to the five-dimensional bulk effects to the radar echo delay must satisfy the constraint \( \Delta_{RD}^{(BW)} \leq \Delta_{RD} \Delta_{RD}^{(GR)} \leq 0.002 \Delta_{RD}^{(GR)} \).

For the case of the Earth–Mars–Sun system we have \( R_E = 1.525 \times 10^{13} \text{ cm} \) (the distance Earth–Sun) and \( R_P = 2.491 \times 10^{13} \text{ cm} \) (the distance Mars–Sun). With these values the standard general relativistic radar echo delay has the value \( \Delta t_{RD}^{(GR)} \approx 2.68 \times 10^{-4} \text{ s} \). Hence the numerical value of the bulk tidal parameter \( Q \) can be constrained, by using radar echo delay data, via the relation \( \Delta t_{RD}^{(BW)} \leq 5.36 \times 10^{-7} \), or equivalently,

\[
\ln \left[ \frac{(1 + 1.07 \times 10^{-27} Q)(1 + 4.02 \times 10^{-28} Q)}{(1 + 1.43 \times 10^{-8}\sqrt{Q})^2} \right] + 1.032 \times 10^{-22} Q - 7.18 \times 10^{-12} \sqrt{Q}(1 + 2.58 \times 10^{-23} Q) \leq 0.027.
\]

(38)

By neglecting, in the first approximation, the logarithmic term and taking into account that the term containing \( \sqrt{Q} \) dominates, the radar echo delay results give the following general restriction on the bulk tidal parameter:

\[
|Q| \leq \frac{4 c^2}{r_S^2} (\Delta t_{RD}^{(GR)} \Delta_{RD})^2 R_S^2.
\]

(39)
By using the numerical data we obtain $|Q| \leq 5.78 \times 10^{19}$ cm$^2$, a value which is relatively consistent (taking also into account the approximations we have used) with the similar value obtained from the study of the deflection of light by using long baseline interferometry data.

4. Conclusion

In the present paper, we considered the observational and experimental possibilities for testing at the level of the solar system the DMPR solution of the vacuum field equations in brane world models. The classical tests of general relativity in the solar system give strong constraints on the numerical values of the brane tension and of the bulk tidal parameter, respectively. Perihelion precession, light deflection and radar echo delay all give definite constraints on the numerical value of $Q$. While the two estimates obtained by using electromagnetic waves propagation data (light deflection and radar delay) are relatively consistent with each other, giving $|Q| \leq 10^{18}$–$10^{19}$ cm$^2$, the perihelion precession gives a much stronger constraint $|Q| \leq 6 \times 10^7$–$5 \times 10^8$ cm$^2$. An improvement of one order of magnitude in the observational data on Mercury’s perihelion shift could provide a very precise estimate of the bulk tidal parameter.

Relative to the brane tension, the numerical value of $\lambda$ has been constrained by using big-bang nucleosynthesis data, which gives $\lambda \geq 1$ MeV$^4$ [28]. A much stronger constraint for the brane tension has been obtained by null results of Newton’s law at sub-millimeter scales. Bulk effects lead to the modification of Newton’s law on the brane. The computation of the Newton potential on the brane shows that the correction terms to the Newton potential involve a logarithmic factor [29]. When the distance between two point masses is very small with respect to the AdS radius, the contribution of the Kaluza–Klein spectrum becomes dominant as compared to the usual inverse square law. This type of behavior of the Newtonian gravitational potential may be used to prove the existence of an extra-dimension experimentally [29]. The null results of deviations from Newton’s law give the constraint $\lambda \geq 10^8$ GeV$^4$ [28].

Junction conditions relate the tidal parameter to the brane tension and mass, and radius of a compact astrophysical object. This can be used to calculate the numerical value of the brane tension $\lambda$. However, despite the fact that these relations have been derived for high constant density stars, for which general relativistic effects are extremely strong (which is definitely not the case for the Sun), their application to the case of the solar system can give some approximate estimates of $\lambda$. Stronger estimates, however, are implied from the perihelion precession, giving $\lambda \geq 10^5$ MeV$^4$. All estimates could be very much improved by the use of observational data obtained for high density compact astrophysical objects, like neutron stars.

On the other hand, there are several other vacuum solutions of the spherically symmetric static gravitational field equations on the brane [10]. Indeed, the effects due to the projections of the Weyl tensor specify the deviations of brane world models from general relativity. Since the generic form of the Weyl tensor in the full five-dimensional theory is yet unknown, the effects of known solutions must be studied on a case-by-case basis. While one can in principle constrain the projections, this only yields very mild constraints on the five-dimensional Weyl tensor.

The study of the classical tests of the general relativity could provide a very powerful method for constraining the allowed parameter space of solutions, and to provide a deeper insight into the physical nature and properties of the corresponding spacetime metrics. Therefore, this opens up the possibility of testing brane world models by using astronomical and astrophysical observations at the solar system scale. Of course, this analysis must be extended to the study of all vacuum solutions on the brane, which requires developing general methods for the high precision study of the classical tests in arbitrary spherically symmetric spacetimes.
In the present paper, we have provided some basic theoretical tools necessary for the in-depth comparison of the predictions of the brane world model and of the observational/experimental results.

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