It is well known that a ramified holomorphic covering of a closed unitary disc by another such a disc is given by a finite Blaschke product. The inverse is also true. In this note we give an explicit description of holomorphic ramified coverings of a disc by other bordered Riemann surfaces. The problem of covering a disc by an annulus arises e.g. in multidimensional complex analysis; we show that it may be effectively solved in terms of elliptic theta functions. The covering of a disc by a flat domain is discussed in Chap VI. The machinery used here strongly resembles the description of magnetic configurations in submicron planar magnets.

1 Schottky Double, homologies and real differentials

Preliminary we fix some notations.

Let $X$ be a genus $g$ Riemann surface with $k > 0$ boundary components called (real) ovals. Its Schottky double is a compact borderless surface $X_2$ of genus $g_2 := 2g + k - 1$ obtained from two copies of $X$ identified at the boundaries. The anticonformal involution (reflection) $\tau$ naturally acts on the double interchanging the points on two copies of the bordered surface with the ovals being the set of its fixed points.

1.1 Cycles

The lattice of boundary cycles $H_1(\partial X, \mathbb{Z}) = \mathbb{Z}^k$ is naturally included to the integer homology group $H_1(X, \mathbb{Z})$, however with the loss of the rank: the sum of all boundary ovals with proper orientations is homological to zero. We fix a basis $A_1, \ldots, A_g, B_1, \ldots, B_g, A'_1, \ldots, A'_{k-1}$ in $H_1(X, \mathbb{Z})$: a standard pair of cycles $A_s, B_s$, $s = 1, \ldots, g$, for each handle of the surface and all boundary components $A'_s$, $s = 1, \ldots, k-1$ but one of them $A'_k$. The choice is by no means unique – see Fig. for the possible one.

This basis may be extended to a symplectic basis in the homologies $H_1(X_2, \mathbb{Z})$ of the double

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Figure 1: Fay homology basis on the double $X_2$ of the bordered surface $X$ with $g = 2$ and $k = 3$.

(Cf: [5], Chap. 6):

$$A_s'' = \tau A_s, \quad B_s'' = -\tau B_s, \quad s = 1, \ldots, g; \quad B_j' = B_j^+ - \tau B_j^+, \quad j = 1, \ldots, k - 1,$$

(1)

where $B_j^+$ is the arc in $X$ connecting the oval $A_j'$ to $A_k'$ and disjoint from other arcs of this kind as well as the basic cycles. Thus introduced basis possesses the standard intersection form and the easily checked behaviour under the reflection (Cf: Bobenko, Fay and Vinnikov homology basis):

$$\tau A_s = A_s''; \quad \tau B_s = -B_s'', \quad s = 1, \ldots, g;$$

$$\tau A_j' = A_j'; \quad \tau B_j' = -B_j', \quad j = 1, \ldots, k - 1.$$

(2)

The integer cohomology lattice $H^1(X, \mathbb{Z})$ contains the sublattice $H^1_1(\partial X, \mathbb{Z})$ of functionals vanishing on all boundary cycles of the surface.

1.2 Holomorphic differentials

We consider the space $\Omega^1 \mathbb{R}(X_2) \simeq \mathbb{R}^{2g}$ of real holomorphic differentials on $X$, that is (holomorphic) differentials $d\rho$ on the double surface possessing the reflectional symmetry:

$$\tau d\rho = \overline{d\rho}.$$

(3)

Riemann bilinear relations [4] guarantee that the following pairing is non degenerate:

$$\langle d\rho | C \rangle := Re \int_C d\rho, \quad d\rho \in \Omega^1 \mathbb{R}(X_2), \quad C \in H_1(X, \mathbb{Z}).$$

(4)
Indeed, the symmetry of real differentials (may be meromorphic) implies

\[ \int_C d\rho = \int_C d\rho \]  

(5)

which in turn means that if all the values [4] for the given real differential \( d\rho \) vanish then all periods of it on the double surface are purely imaginary and therefore \( d\rho \) is zero. The last statement means that integer cohomologies \( H^1(X, \mathbb{Z}) \) may be naturally realized as rank \( g_2 \) lattice in the space of real differentials and in particular, there is a unique basis of differentials dual to the above chosen basis in 1-homologies of \( X \):

\[ \text{Re} \int_C d\rho_D = \delta_{CD}, \quad C, D \in \{A_1, \ldots, A_g, B_1, \ldots, B_g, A'_1, \ldots, A'_{k-1}\}. \]  

(6)

Differentials \( d\rho_{A_s}, d\rho_{B_s}, s = 1, \ldots, g \) represent a basis for the mentioned above sublattice \( H^+_1(\partial X, \mathbb{Z}) \) of integer cohomologies.

1.3 Meromorphic differentials

For the construction of meromorphic functions with mirror symmetry we use third kind abelian differentials \( d\eta_{zp} \) with simple poles at points \( z, p \), residues \(+1, 1\) respectively and real normalization: all cyclic periods of \( d\eta_{zp} \) are purely imaginary. For our purpose it is sufficient to take poles on the same real oval only and it is easy to check that abelian differentials \( d\eta_{zp} \) will be real (meromorphic) in this case. This differential may be expressed rather explicitly for a flat domain \( X \) via the solution of the Neumann problem for a harmonic function with two point-like sources on the boundary [3], or via Schottky-Klein prime form [5, 8] in general. The following reciprocity laws are valid for a differential of this kind:

Lemma 1 The Riemann bilinear relations for the \( \text{Re} \)–normalized differentials take the form:

\[ \int_{A_s} d\eta_{zp} = -\pi i \int_p \overline{d\rho_{B_s}}, \]
\[ \int_{B_s} d\eta_{zp} = \pi i \int_p \overline{d\rho_{A_s}}, \quad s = 1, \ldots, g; \]
\[ \int_{A'_j} d\eta_{zp} = 0, \]
\[ \int_{B'_j} d\eta_{zp} = 2\pi i \int_p \overline{d\rho_{A'_j}}, \quad j = 1, \ldots, k - 1. \]  

(7)

On the r.h.s. of the equalities the integrals are taken along the ovals and they are purely real; in the third line the l.h.s integral is taken in the sense of principal value if necessary; in the fourth equality the integration path on the right does not cross the cycle \( B'_j \).

Proof roughly follows the argumentation for the classical reciprocity laws [4, 7, 8] for holomorphic normalization. We give a sketch of it only. W.l.o.g (reshuffle basic cycles otherwise) we
Figure 2: The surface $X_2$ cut along homology basis and an oval arc joining poles $z, p$.

assume that both poles of $d\eta_{zp}$ lie on the oval $A'_k$ of the double. We cut the surface $X_2$ along the chosen above basic cycles as well as the arc of the oval joining the poles $z, p$. In the arising flat surface one can introduce the single valued abelian integral $\eta_{zp}$ – see Fig. 2. Integrating $\eta_{zp} d\rho$ with real differential $d\rho$ from the basis [6] along the boundary of the flat surface we get zero as the integrand has no singularities inside. On the other hand we can assemble the boundary integrals into proper groups and get the above identities.

2 Ramified coverings

The main idea behind the description of ramified coverings $h$ of the closed upper half plane $\hat{H}$ (= disc) by a bordered surface $X$ is to extend the mapping to the double of the surface by reflection. The zeros and poles of $h$ are simple, lie on the real ovals only and alternate on each oval. Moreover, their positions are subjected to a certain lattice relation stemming from Abel’s theorem.

Theorem 1 A. Let $h(u): X \rightarrow \hat{H}$ be a ramified covering and $(h) = (h)^+ - (h)^-$, $(h)^\pm \geq 0$, be the decomposition of its divisor into zeroes and poles, then the covering map admits the representation:

$$h(u) = \exp\left(\int_v d\eta_h\right),$$

with $v \in \partial X$, and $d\eta_h$ being the real-normalized third kind abelian differential with residue divisor $(h)^+ - (h)^-$. For the support of the latter divisor the following restrictions hold:

(i) $|(h)^\pm| \subset \partial X$ and each oval contains at least one zero and one pole

(ii) Zeros and poles are simple and alternate on each oval
(iii) Their positions satisfy the lattice conditions:

\[ \int_{(h)^+} dp \in \left\{ \begin{array}{l l} \mathbb{Z}, & dp \in H^1(X, \mathbb{Z}) \subset \Omega^1 \mathbb{R}(X_2), \\ 2\mathbb{Z}, & dp \in H^1_+(\partial X, \mathbb{Z}) \subset H^1(X, \mathbb{Z}). \end{array} \right. \]  

(B) The converse is also true: once \((h)^+\) and \((h)^-\) are the divisors satisfying the above restrictions (i)-(iii), then formula \((8)\) represents a ramified covering \(X \to \hat{\mathbb{H}}\) provided \(d\eta(v) > 0\) (ovals are oriented as the boundary of \(X\)).

Proof. A. The covering \(h(u)\) may be extended to the double of the surface by reflection: \(h(\tau u) := \overline{h(u)}, \ u \in X_2\). The representation \((8)\) is true for the differential \(d\eta_h := d\log h(u)\) and any \(v \in h^{-1}(1) \subset \partial X\). The differential has periods in \(2\pi i \mathbb{Z}\) and in particular it is real-normalized. Its singularities coincide with the set of zeros and poles of \(h\). The restriction of \(h: \partial X \to \partial \mathbb{H}\) is again a cover map: each oval winds around the real equator of the Riemann sphere, where from statements (i) and (ii) follow. Lattice conditions (iii) follow from the reciprocity laws of Lemma\(1\):

\[ 2\pi i \int_{(h)^-} dp = 2\pi i \sum_{z,p} \int_z^p dp = \int_C d\eta_h = \text{Arg } h(u)|_C \in 2\pi i \mathbb{Z}, \quad dp \in H^1(X, \mathbb{Z}), \]

where \((h) := \sum_{z,p} (z-p)\) and the cycle \(C\) equals to \(2 \sum_{s=1}^g \langle dp|A_s\rangle B_s - \langle dp|B_s\rangle A_s + \sum_{j=1}^{k-1} \langle dp|A_j'\rangle B'_j\).

If \(dp\) annihilates boundary cycles, then \(C \in 2H_1(X, \mathbb{Z})\) and conditions \((9)\) follow.

B. We prove the converse under the assumptions (i)–(iii) about zeros and poles. In that case the following statements are true.

1) Function \(h(u)\) given by formula \((8)\) (with \(Re-\)normalized third kind differential \(d\eta_h\) whose residue divisor is \((h)^+ - (h)^-\)) is single valued on the surface \(X_2\). This is equivalent to

\[ \int_{H_1(X_2, \mathbb{Z})} d\eta_h \subset 2\pi i \mathbb{Z}. \]  

The latter inclusion may be checked on the basic homology cycles. Taking into account the symmetry relation for real differentials \((5)\) and the Fay basis transformation under the reflection \((2)\), it is enough to check the inclusion \((10)\) for the cycles \(A_s, B_s\) and \(B'_j\). For \(C = A_s, B_s\) from the reciprocity laws of Lemma 1 and (iii) it follows:

\[ \int_C d\eta_h = \sum_{z,p} \int_C d\eta_{z,p} = \pm \pi i \sum_{z,p} \int_z^p dp \in 2\pi i \mathbb{Z}, \quad dp \in H^1_+(\partial X, \mathbb{Z}). \]

For \(C = B'_j\) we analogously have

\[ \int_C d\eta_h = \sum_{z,p} \int_C d\eta_{z,p} = 2\pi i \sum_{z,p} \int_z^p dp \in 2\pi i \mathbb{Z}, \quad dp \in H^1(X, \mathbb{Z}). \]

2) Function \(h(u)\) is real since such is the differential \(d\eta_h\) and \(\tau v = v:\)

\[ h(\tau u) := \exp(\int_v^{\tau u} d\eta_h) = \exp(\int_v^u \tau d\eta_h) = \exp(\int_v^u d\eta_h) = \overline{h(u)}. \]
In particular, $h(u)$ maps the boundary of $X$ to the boundary of $\mathbb{H}$.

3) $h^{-1}(\mathbb{R}) = \partial X$. Indeed, the points $h^{-1}\{0, \infty\}$ decompose the ovals into $2\deg h$ closed segments with disjoint interiors. The values of $h(u)$ in the interior of each segment have the same sign, for otherwise additional zero or a pole of $h$ appears. Since $h$ has simple zeros and poles only, the neighbouring segments on ovals are mapped 1-1 to different segments $\pm[0, \infty]$ of the extended real line. Each real value has at least $\deg h$ preimages in the ovals and hence no preimages in the interior of $X$ or its reflection. Since $h$ presents a ramified covering of the Riemann sphere by the double surface $X_2$, its restriction to $X$ will be a cover of $\pm\mathbb{H}$. To change the sign in the target space we just move the initial point $v$ to the neighbouring segment(s) tiling the ovals.

3 Examples

3.1 Disc

For the disc $g = 0$, $k = 1$ and the theorem above gives us a real rational function $R(u)$ with alternating real zeros of numerator and denominator. This function may be transformed to a classical Blaschke product by making a composition $l \circ R \circ l^{-1}$ with linear-fractional $l(u)$ mapping the upper half plane to the unit disc, say $l(u) := (u - i)/(u + i)$. Conjugation by linear fractional map sends $R$ to the same degree rational function with zeros at $l(R^{-1}(i))$ and poles at $l(R^{-1}(-i))$. The latter two sets are mirror symmetric with respect to the unit circle (since $R(u)$ is real) and the first of them lies strictly inside the unit disc (since $\text{Im } R(u)$ is positive in the upper half plane only). Hence we get a finite Blaschke product. Note that the latter may have multiple zeros/poles unlike the representation (8).

3.2 Annulus

The annulus $1 \leq |u| \leq r$ has topological invariants $g = 0$, $k = 2$. We can also represent it as a factor of the vertical strip $0 \leq \text{Re}(x) \leq 1/2$ by a group of translations generated by $iT$, where $T = \pi/\log(r) > 0$. The correspondence of the two models of the ring is given by the explicit formula $u(x) = \exp(2\pi x/T)$.

The generator of $H^1(X, \mathbb{Z})$ is $d\rho := \frac{dx}{iT}$ and $H^1_+(\partial X, \mathbb{Z})$ is empty. Let the sets of zeros/poles be represented by the alternating points $x = z_j$ and $x = p_j$, $j = 1, \ldots, N$ on the sides on the strip. The Abel’s lattice condition [9] takes the form:

$$\frac{1}{iT} \sum_{s=1}^{N} (z_s - p_s) \in \mathbb{Z},$$ (11)
which in terms of the concentric ring model exactly means that the sum of arguments of all poles is equal to the sum of arguments of all zeros modulo $2\pi$. With this choice of representatives, the covering map $X \to \hat{\mathbb{H}}$ is proportional to the following

$$h(x) = \exp(-2\pi imx) \prod_{j=1}^{N} \frac{\theta_1(x - z_j)}{\theta_1(x - p_j)},$$

(12)

here $\theta_1(x) = \exp(-\pi T/4)\sin(\pi x) - \ldots$ is the only odd elliptic theta function of the modulus $iT$ and $m$ is the integer number in the l.h.s. of the inclusion (11).

In the more involved cases formula (8) for the ramified covering map can also be done computationally effective with the help of higher genus Riemann theta functions [8, 5].

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