Volley Revolver: A Novel Matrix-Encoding Method for Privacy-Preserving Neural Networks (Inference)

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Abstract

In this work, we present a novel matrix-encoding method that is particularly convenient for neural networks to make predictions in a privacy-preserving manner using homomorphic encryption. Based on this encoding method, we implement a convolutional neural network for handwritten image classification over encryption. For two matrices $A$ and $B$ to perform homomorphic multiplication, the main idea behind it, in a simple version, is to encrypt matrix $A$ and the transpose of matrix $B$ into two ciphertexts respectively. With additional operations, the homomorphic matrix multiplication can be calculated over encrypted matrices efficiently. For the convolution operation, we in advance span each convolution kernel to a matrix space of the same size as the input image so as to generate several ciphertexts, each of which is later used together with the ciphertext encrypting input images for calculating some of the final convolution results. We accumulate all these intermediate results and thus complete the convolution operation.

In a public cloud with 40 vCPUs, our convolutional neural network implementation on the MNIST testing dataset takes $\sim 287$ seconds to compute ten likelihoods of 32 encrypted images of size $28 \times 28$ simultaneously. The data owner only needs to upload one ciphertext ($\sim 19.8$ MB) encrypting these 32 images to the public cloud.

1 Introduction

1.1 Background

Machine learning applied in some specific domains such as health and finance should preserve privacy while processing private or confidential data to make accurate predictions. In this study, we focus on privacy-preserving neural network inference, which aims to outsource a well-trained inference model to a cloud service in order to make predictions on private data. For this purpose, the data should be encrypted first and then sent to the cloud service that should not be capable of having access to the raw data. Compared to other cryptology technologies such as Secure Multi-Party Computation, Homomorphic Encryption (HE) provides the most stringent security for this task.

1.2 Related Work

Combining HE with Convolutional Neural Networks (CNN) inference has been receiving more and more attention in recent years since Gilad-Bachrach et al. [6] proposed a framework called Cryptonets. Cryptonets applies neural networks to make accurate inferences on encrypted data with high throughput. Chanranne et al. [2] extended this work to deeper CNN using a different underlying software library called HElib [7] and leveraged batch normalization and training process to develop better quality polynomial approximations of the ReLU function for stability and accuracy. Chou et al. [4] developed a pruning and quantization approach with other deep-learning optimization
techniques and presented a method for encrypted neural networks inference, Faster CryptoNets. Brutzkus et al. [1] developed new encoding methods other than the one used in CryptoNets for representing data and presented the Low-Latency CryptoNets (LoLa) solution. Jiang et al. [9] proposed an efficient evaluation strategy for secure outsourced matrix multiplication with the help of a novel matrix-encoding method.

1.3 Contributions

Contributions In this study, our contributions are in three main parts:

1. We introduce a novel data-encoding method for matrix multiplications on encrypted matrices, Volley Revolver, which can be used to multiply matrices of arbitrary shape efficiently.
2. We propose a feasible evaluation strategy for convolution operation, by devising an efficient homomorphic algorithm to sum some intermediate results of convolution operations.
3. We develop some simulated operations on the packed ciphertext encrypting an image dataset as if there were multiple virtual ciphertexts inhabiting it, which provides a compelling new perspective of viewing the dataset as a three-dimensional structure.

2 Preliminaries

Let “⊕” and “⊗” denote the component-wise addition and multiplication respectively between ciphertexts encrypting matrices and the ciphertext $ct_P$ the encryption of a matrix $P$. Let $I_{[i][j][m]}$ represent the single pixel of the $j$-th element in the $i$-th row of the $m$-th image from the dataset.

2.1 Fully Homomorphic Encryption

Homomorphic Encryption is one kind of encryption but has its characteristic in that over an HE system operations on encrypted data generate ciphertexts encrypting the right results of corresponding operations on plaintext without decrypting the data nor requiring access to the secret key. Since Gentry [5] presented the first fully homomorphic encryption scheme, tackling the over three decades problem, much progress has been made on an efficient data encoding scheme for the application of machine learning to HE. Cheon et al. [3] constructed an HE scheme (CKKS) that can deal with this technique problem efficiently, coming up with a new procedure called rescaling for approximate arithmetic in order to manage the magnitude of plaintext. Their open-source library, HEAAN, like other HE libraries also supports the Single Instruction Multiple Data (aka SIMD) manner [11] to encrypt multiple values into a single ciphertext.

Given the security parameter, HEAAN outputs a secret key $sk$, a public key $pk$, and other public keys used for operations such as rotation. For simplicity, we will ignore the rescale operation and deem the following operations to deal with the magnitude of plaintext automatically. HEAAN has the following functions to support the HE scheme:

1. $\text{Enc}_{pk}(m)$: For the public key $pk$ and a message vector $m$, HEAAN encrypts the message $m$ into a ciphertext $ct$.
2. $\text{Dec}_{sk}(ct)$: Using the secret key, this algorithm returns the message vector encrypted by the ciphertext $ct$.
3. $\text{Add}(ct_1, ct_2)$: This operation returns a new ciphertext that encrypts the message $\text{Dec}_{sk}(ct_1) \oplus \text{Dec}_{sk}(ct_2)$.
4. $\text{Mul}(ct_1, ct_2)$: This procedure returns a new ciphertext that encrypts the message $\text{Dec}_{sk}(ct_1) \otimes \text{Dec}_{sk}(ct_2)$.
5. $\text{cMul}(C, ct_2)$: This procedure returns a new ciphertext that encrypts the message $\text{Dec}_{sk}(ct_1) \otimes \text{Dec}_{sk}(ct_2)$.
6. $\text{Rot}(ct, l)$: This procedure generates a ciphertext encrypting a new plaintext vector obtained by rotating the original message vector $m$ encrypted by $ct$ to the left by $l$ positions.
We propose a new useful procedure called SumForConv which are all trainable parameters and tuned during the training. Given a greyscale image $I$:

$$Z = \begin{bmatrix}
    z_{11} & z_{12} & \cdots & z_{1f} \\
    z_{21} & z_{22} & \cdots & z_{2f} \\
    \vdots & \vdots & \ddots & \vdots \\
    z_{n1} & z_{n2} & \cdots & z_{nf}
\end{bmatrix}$$

incomplete column shifting

$$\begin{bmatrix}
    z_{12} & z_{13} & \cdots & z_{14} \\
    z_{22} & z_{23} & \cdots & z_{24} \\
    \vdots & \vdots & \ddots & \vdots \\
    z_{n2} & z_{n3} & \cdots & z_{n4}
\end{bmatrix}$$

Han et al. [8] summarize another two procedures, SumRowVec and SumColVec, to compute the summation of each row and column respectively. The results of two procedures on $Z$ are as follows:

$$\text{SumRowVec}(Z) = \begin{bmatrix}
    \sum_{i=1}^{n} z_{i1} & \sum_{i=1}^{n} z_{i2} & \cdots & \sum_{i=1}^{n} z_{if} \\
    \vdots & \vdots & \ddots & \vdots \\
    \sum_{i=1}^{n} z_{ni} & \sum_{i=1}^{n} z_{ni} & \cdots & \sum_{i=1}^{n} z_{ni}
\end{bmatrix}$$

$$\text{SumColVec}(Z) = \begin{bmatrix}
    \sum_{j=1}^{f} z_{1j} & \sum_{j=1}^{f} z_{1j} & \cdots & \sum_{j=1}^{f} z_{1j} \\
    \vdots & \vdots & \ddots & \vdots \\
    \sum_{j=1}^{f} z_{nj} & \sum_{j=1}^{f} z_{nj} & \cdots & \sum_{j=1}^{f} z_{nj}
\end{bmatrix}$$

We propose a new useful procedure called SumForConv to facilitate convolution operation for every image, as shown in Algorithm 1. The computational cost of SumForConv is about $nfn$ additions, constant multiplications, and rotations. Note that rotation operation is comparably expensive than the other two operations due to the need of perform a key-switching operation. Therefore, the complexity can be seen as asymptotically $O(n)$ rotations. Below we illustrate the result of SumForConv on $Z$ taking the example that $n$ and $f$ are both 4 and the kernel size is $3 \times 3$:

$$Z = \begin{bmatrix}
    z_{11} & z_{12} & z_{13} & z_{14} \\
    z_{21} & z_{22} & z_{23} & z_{24} \\
    z_{31} & z_{32} & z_{33} & z_{34} \\
    z_{41} & z_{42} & z_{43} & z_{44}
\end{bmatrix}$$

incomplete convolution shifting

$$\begin{bmatrix}
    s_{11} & s_{12} & 0 & 0 \\
    s_{21} & s_{22} & 0 & 0 \\
    0 & 0 & 0 & 0 \\
    0 & 0 & 0 & 0
\end{bmatrix}$$

where $s_{i,j} = \sum_{p=1}^{i+2} \sum_{q=1}^{j+2} z_{p,q}$ for $1 \leq i, j \leq 2$. In the convolutional layer, SumForConv can help to compute some partial results of convolution operation for an image simultaneously.

### 2.3 Convolutional Neural Networks

Convolutional Neural Networks are neural networks particularly tailored for image recognition, equipped with two distinct kinds of layers: Convolutional layer (CONV) and Pooling layer (POOL) in addition to another two basic kinds of layers: Fully Connected layer (FC) and Activation layer (ACT). A CNN for image classification has a common architecture: $[\text{CONV} \rightarrow \text{ACT}]^p \rightarrow \text{POOL} \rightarrow [\text{CONV} \rightarrow \text{ACT}] \rightarrow [\text{FC} \rightarrow \text{ACT}]^r \rightarrow \text{FC}$, where $p$, $q$, and $r$ are integers usually greater than 1. In our implementation, we use the same CNN architecture as [9]: $[\text{CONV} \rightarrow \text{ACT}] \rightarrow [\text{FC} \rightarrow \text{ACT}] \rightarrow \text{FC}$. Convolutional layer is the fundamental basis of a CNN, which has kernels of size $k \times k$, a stride of $(s, s)$, and a channel (mapcount) of $c$. Each kernel has $k \times k \times c$ parameters besides a kernel bias $k_0$, which are all trainable parameters and tuned during the training. Given a greyscale image $I \in \mathbb{R}^{h \times w}$ and a kernel $K \in \mathbb{R}^{k \times k}$, the result of convolving this input image $I$ with stride of $(1, 1)$ is the output image $I' \in \mathbb{R}^{h' \times w'}$ with $I'_{i', j'} = k_0 + \sum_{i=1}^{k} \sum_{j=1}^{k} K_{i,j} \times I_{i+i', j+j'}$ for $0 < i' \leq h - k + 1$.
Algorithm 1 SumForConv: sum some part results of convolution operation after one element-wise multiplication

**Input**: a ciphertext ct.I encrypting a (convolved) image I of size h \times w, the size k \times k of some kernal K with its bias k_0, and a stride of (1, 1)

**Output**: a ciphertext ct.I_s encrypting a resulting image I_s of the same size as I

1. Set I_s \leftarrow 0 \quad \triangleright I_s \in \mathbb{R}^{(h-k+1) \times (w-k+1)}
2. for i := 1 to (h - k + 1) do
3. \quad for j := 1 to (w - k + 1) do
4. \quad\quad I_s[i][j] \leftarrow k_0
5. end for
6. end for
7. ct.I_s \leftarrow \text{Enc}_{pk}(I_s) \quad \triangleright \text{Accumulate columns (could be computed in parallel)}
8. for pos := 0 to k - 1 do
9. \quad ct.T \leftarrow \text{Rot}(ct.I, \text{pos})
10. \quad ct.I_s \leftarrow \text{Add}(ct.I_s, ct.T)
11. end for
12. \quad \triangleright \text{Accumulate rows (could be computed in parallel)}
13. for pos := 1 to k - 1 do
14. \quad ct.T \leftarrow \text{Rot}(ct.I, \text{pos} \times w)
15. \quad ct.I_s \leftarrow \text{Add}(ct.I_s, ct.T)
16. end for
17. \quad \triangleright \text{Build a new designed matrix to filter out the garbage values}
18. Set M \leftarrow 0 \quad \triangleright M \in \mathbb{R}^{h \times w}
19. for hth := 0 to (h - 1) do
20. \quad for wth := 0 to (w - 1) do
21. \quad\quad \text{if wth mod k = 0 and wth + k \leq \text{width} and hth mod k = 0 and hth + k \leq \text{height} then}
22. \quad\quad\quad M[hth][wth] \leftarrow 1
23. \quad\quad end if
24. \quad end for
25. end for
26. ct.I_s \leftarrow \text{cMul}(M, ct.I_s)
27. return ct.I_s

and 0 < j' \leq w - k + 1. It can be extended to a color image or a convolved image with many channels, refer to [9] for a detail. If there are multiple kernels, the convolutional layer stacks all the convolved results of each kernel and outputs a three-dimensional tensor. For a single input sample, FC layer only accepts a unidimensional vector, which is why the output of previous layers ([CONV \rightarrow ACT]^p \rightarrow POOL) or [CONV \rightarrow ACT]) should be flattened before being fed into FC layer.

### 3 Technical Details

We introduce a novel matrix-encoding method called **Volley Revolver**, which is particularly suitable for secure matrix multiplication. The basic idea is to place each semantically-complete information (such as an example in a dataset) into the corresponding row of a matrix and encrypt this matrix into a single ciphertext. When applying it to private neural networks, **Volley Revolver** puts the whole weights of every neural node into the corresponding row of a matrix, organizes all the nodes from the same layer into this matrix, and encrypts this matrix into a single ciphertext.
3.1 Encoding Method for Matrix Multiplication

Suppose that we are given an $m \times n$ matrix $A$ and a $n \times p$ matrix $B$ and suppose to compute the matrix $C$ of size $m \times p$, which is the matrix product $A \cdot B$ with the element $C[i][j] = \sum_{k=1}^{n} a[i][k] \times b[k][j]$:

$$
A = \begin{bmatrix}
  a_{1}[1] & a_{1}[2] & \cdots & a_{1}[n] \\
  a_{2}[1] & a_{2}[2] & \cdots & a_{2}[n] \\
  \vdots & \vdots & \ddots & \vdots \\
  a_{m}[1] & a_{m}[2] & \cdots & a_{m}[n]
\end{bmatrix},
B = \begin{bmatrix}
  b_{1}[1] & b_{1}[2] & \cdots & b_{1}[p] \\
  b_{2}[1] & b_{2}[2] & \cdots & b_{2}[p] \\
  \vdots & \vdots & \ddots & \vdots \\
  b_{n}[1] & b_{n}[2] & \cdots & b_{n}[p]
\end{bmatrix}.
$$

For simplicity, we assume that each of the three matrices $A$, $B$ and $C$ could be encrypted into a single ciphertext. We also make the assumption that $m$ is greater than $p$, $m > p$. We will not illustrate the other cases where $m \leq p$, which is similar to this one. When it comes to the homomorphic matrix multiplication, Volley Revolver encodes matrix $A$ directly but encodes the padding form of the transpose of matrix $B$, by using two row-ordering encoding maps. For matrix $A$, we adopt the same encoding method that [9] did by the encoding map $\tau_{0} : A \mapsto \bar{A} = (a_{i+1+(k/n)][1+((k/n)%p)])_{0 \leq k < m \times n}$. For matrix $B$, we design a very different encoding method from [9] for Volley Revolver: we transpose the matrix $B$ first and then extend the resulting matrix in the vertical direction to the size $m \times n$. Therefore Volley Revolver adopts the encoding map $\tau_{b} : B \mapsto \bar{B} = (b_{[1+(k/n)][1+((k/n)%p)])_{0 \leq k < m \times n}$, obtaining the matrix from mapping $\tau_{b}$ on $B$:

$$
\begin{bmatrix}
  b_{[1]}[1] & b_{[2]}[2] & \cdots & b_{[1]}[p] \\
  b_{[2]}[1] & b_{[2]}[2] & \cdots & b_{[2]}[p] \\
  \vdots & \vdots & \ddots & \vdots \\
  b_{[n]}[1] & b_{[n]}[2] & \cdots & b_{[n]}[p]
\end{bmatrix} \xrightarrow{\tau_{b}}
\begin{bmatrix}
  b_{[1]}[1] & b_{[2]}[2] & \cdots & b_{[1]}[p] \\
  b_{[1]}[2] & b_{[2]}[2] & \cdots & b_{[2]}[p] \\
  \vdots & \vdots & \ddots & \vdots \\
  b_{[1]}[1] & b_{[2]}[p] & \cdots & b_{[n]}[p] \\
  b_{[1]}[1] & b_{[2]}[1] & \cdots & b_{[n]}[1] \\
  \vdots & \vdots & \ddots & \vdots \\
  b_{[1]}[1+(m-1)%p] & b_{[2]}[1+(m-1)%p] & \cdots & b_{[n]}[1+(m-1)%p]
\end{bmatrix}.
$$

3.2 Homomorphic Matrix Multiplication

We report an efficient evaluation algorithm for homomorphic matrix multiplication. This algorithm uses a ciphertext $ct.R$ encrypting zeros or a given value such as the weight bias of a fully-connected layer as an accumulator and an operation RowShifter to perform a specific kind of row shifting on the encrypted matrix $\bar{B}$. RowShifter pops up the first row of $\bar{B}$ and appends another corresponding already existing row of $\bar{B}$:

$$
\begin{bmatrix}
  b_{[1]}[1] & b_{[2]}[1] & \cdots & b_{[n]}[1] \\
  b_{[1]}[2] & b_{[2]}[2] & \cdots & b_{[n]}[2] \\
  \vdots & \vdots & \ddots & \vdots \\
  b_{[1]}[p] & b_{[2]}[p] & \cdots & b_{[n]}[p] \\
  b_{[1]}[1] & b_{[2]}[1] & \cdots & b_{[n]}[1] \\
  \vdots & \vdots & \ddots & \vdots \\
  b_{[1]}[r] & b_{[2]}[r] & \cdots & b_{[n]}[r]
\end{bmatrix} \xrightarrow{\text{RowShifter}(\bar{B})}
\begin{bmatrix}
  b_{[1]}[2] & b_{[2]}[2] & \cdots & b_{[n]}[2] \\
  \vdots & \vdots & \ddots & \vdots \\
  b_{[1]}[p] & b_{[2]}[p] & \cdots & b_{[n]}[p] \\
  b_{[1]}[1] & b_{[2]}[1] & \cdots & b_{[n]}[1] \\
  \vdots & \vdots & \ddots & \vdots \\
  b_{[1]}[1+(r+1)%p] & b_{[2]}[(r+1)%p] & \cdots & b_{[n]}[(r+1)%p]
\end{bmatrix}.
$$

Algorithm[2] describes how the procedure RowShifter generates a new ciphertext from $ct.\bar{B}$.

For two ciphertexts $ct.A$ and $ct.\bar{B}$, the algorithm for homomorphic matrix multiplication has $p$ iterations. For the $k$-th iteration where $0 \leq k < p$ there are the following four steps:

**Step 1:** This step uses RowShifter on $ct.\bar{B}$ to generate a new ciphertext $ct.\bar{B}_{1}$ and then computes the homomorphic multiplication between ciphertexts $ct.A$ and $ct.\bar{B}_{1}$ to get the resulting product $ct.A\bar{B}_{1}$. When $k = 0$, in this case RowShifter just return a copy of the ciphertext $ct.\bar{B}$. Thus, the computational cost is about $2d$ additions, constant multiplications, and rotations.

**Step 2:** In this step, the public cloud applies SumColVec on $ct.A\bar{B}_{1}$ to collect the summation of the data in each row of $A\bar{B}_{1}$ for some intermediate results, and obtain the ciphertext $ct.D$. The complexity of this procedure is roughly half of the Step 1: $d$ additions, constant multiplications, and rotations.
Algorithm 2 RowShifter: To shift row like a revolver

**Input:** a ciphertext $ct.M$ encrypting a matrix $M$ of size $m \times n$, the number $p$, and the number $idx$ that is determined in the Algorithm 3.

**Output:** a ciphertext $ct.R$ encrypting the resulting matrix $R$ of the same size as $M$.

1: Set $R \leftarrow \mathbf{0}$ \quad $\triangleright R \in \mathbb{R}^{m \times n}$
2: $ct.R \leftarrow \text{Enc}_{pk}(R)$
3: $ct.T \leftarrow \text{Rot}(ct.M, n)$ \quad $\triangleright$ Build a specially designed matrix to filter out the last row
4: Set $F_1 \leftarrow 1$ \quad $\triangleright F_1 \in \mathbb{R}^{m \times n}$
5: for $j := 1$ to $n$ do
6: $F_1[m][j] \leftarrow 0$
7: end for
8: $ct.T_1 \leftarrow \text{cMul}(F_1, ct.T)$
9: $ct.P \leftarrow \text{Rot}(ct.M, n \times ((m\%p + idx + 1)\%p - idx))$ \quad $\triangleright$ Build a specially designed matrix to filter out the last row
10: Set $F_2 \leftarrow 0$ \quad $\triangleright F_2 \in \mathbb{R}^{m \times n}$
11: for $j := 1$ to $n$ do
12: $F_2[m][j] \leftarrow 1$
13: end for
14: $ct.T_2 \leftarrow \text{cMul}(F_2, ct.P)$ \quad $\triangleright$ concatenate
15: $ct.R \leftarrow \text{Add}(ct.T_1, ct.T_2)$
16: return $ct.R$

**Step 3:** This step designs a special matrix $F$ for filtering out the redundancy element in $D$ by one constant multiplication $\text{cMul}(ct.F, ct.D)$, resulting the ciphertext $ct.D_1$. The computational cost of this procedure is about $d$ additions, $2d$ constant multiplications, and $3d$ rotations.

**Step 4:** The ciphertext $ct.R$ is then used to accumulate the intermediate ciphertext $ct.D_1$. The running time of this step is $d$ homomorphic multiplications and additions.

The algorithm will repeat Steps 1 to 4 for $p$ times and finally aggregates all the intermediate ciphertexts, returning the ciphertext $ct.C$. Algorithm 3 shows how to perform our homomorphic matrix multiplication. This implementation of matrix multiplication takes about $5d$ additions, $5d$ constant multiplications, $6d$ rotations, and $d$ multiplications. The complexity of Steps 1 and 2 can be reduced by applying the idea of baby-step/giant-step algorithm. Finally, we perform aggregation and rotation operations to get the final result. This step can be evaluated using a repeated doubling approach, yielding a running time of $\log(d)$ additions and rotations. See Algorithm 3 for an explicit description of homomorphic rectangular matrix multiplication. Table 1 summarizes the complexity and the required depth of each step of Algorithm 3.

![Table 1: Complexity and required depth of Algorithm 3](image)

Figure 1 describes a simple case for Algorithm 3 where $m = 2$, $n = 4$ and $p = 2$.

The calculation process of this method, especially for the simple case where $m = p$, is intuitively similar to a special kind of revolver that can fire multiple bullets at once (The first matrix $A$ is settled still while the second matrix $B$ is revolved). That is why we term our encoding method “Volley Revolver”. In the real-world cases where $m \mod p = 0$, the operation RowShifter can be reduced to only need one rotation $\text{RowShifter} = \text{Rot}(ct, n)$, which is much more efficient and
Algorithm 3 Homomorphic matrix multiplication

Input: ct. $A$ and ct. $B$ for $A \in \mathbb{R}^{m \times n}$, $B \in \mathbb{R}^{n \times p}$ and $B \mapsto \text{Valley Revolver Encoding} \rightarrow \bar{B} \in \mathbb{R}^{m \times n}$

Output: The encrypted resulting matrices ct. $C$ for $C \in \mathbb{R}^{m \times p}$ of the matrix product $A \cdot B$

1: Set $C \leftarrow 0$
2: ct. $C \leftarrow \text{Enc}_{pk}(C)$
3: for $idx := 0$ to $p - 1$ do
4:   ct. $T \leftarrow \text{RowShifter}(ct. \bar{B}, p, idx)$
5:   ct. $T \leftarrow \text{Mul}(ct. A, ct. T)$
6:   ct. $T \leftarrow \text{SumColVec}(ct. T)$
7:   Set $F \leftarrow 0$
8:   for $i := 1$ to $m$ do
9:      $F[i][(i + idx)\%n] \leftarrow 1$
10:  end for
11: ct. $C \leftarrow \text{cMul}(F, ct. T)$
12: end for
13: return ct. $C$

Figure 1: Our matrix multiplication algorithm with $m = 2$, $n = 4$ and $p = 2$

should thus be adopted whenever possible. Corresponding to the neural networks, we can set the number of neural nodes for each fully-connected layer to be a power of two to achieve this goal.

3.3 Homomorphic Convolution Operation

In this subsection, we first introduce a novel but impractical algorithm to calculate the convolution operation for a single grayscale image of size $h \times w$ based on the assumption that this single image can happen to be encrypted into a single ciphertext without vacant slots left, meaning the number $N$ of slots in a packed ciphertext chance to be $N = h \times w$. We then illustrate how to use this method to compute the convolution operation of several images of any size at the same time for a convolutional layer after these images have been encrypted into a ciphertext and been viewed as several virtual ciphertexts inhabiting this real ciphertext. For simplicity, we assume that the image is grayscale and that the image dataset can be encrypted into a single ciphertext.
**An impractical algorithm** Given a grayscale image $I$ of size $h \times w$ and a kernel $K$ of size $k \times k$ with its bias $k_0$ such that $h$ and $w$ are both greater than $k$, based on the assumption that this image can happen to be encrypted into a ciphertext $ct.I$ with no more or less vacant slots, we present an efficient algorithm to compute the convolution operation. We set the stride size to the usual default value $(1,1)$ and adopt no padding technique in this algorithm.

Before the algorithm starts, the kernel $K$ should be called by an operation that we term Kernelspanner to in advance generate $k^2$ ciphertexts for most cases where $h \geq 2 \cdot k - 1$ and $w \geq 2 \cdot k - 1$, each of which encrypts a matrix $P_i$ for $1 \leq i \leq k^2$, using a map to span the $k \times k$ kernel to a $h \times w$ matrix space. For a simple example that $h = 4$, $w = 4$ and $k = 2$, Kernelspanner generates 4 ciphertexts and the kernel bias $k_0$ will be used to generate a ciphertext:

$$\begin{bmatrix} k_1 & k_2 \\ k_3 & k_4 \end{bmatrix}_{\mathbb{R}^{k \times k} \rightarrow \mathbb{R}^{h \times w}} \rightarrow \begin{bmatrix} k_0 & k_0 & k_0 & 0 \\ k_0 & k_0 & k_0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \rightarrow \text{Enc} \rightarrow \begin{bmatrix} k_1 & k_2 & k_1 & k_2 \\ k_3 & k_4 & k_3 & k_4 \\ k_1 & k_2 & k_1 & k_2 \\ k_3 & k_4 & k_3 & k_4 \end{bmatrix} \cdot \begin{bmatrix} 0 & k_1 & k_2 & 0 \\ 0 & k_3 & k_4 & 0 \\ 0 & k_1 & k_2 & 0 \\ 0 & k_3 & k_4 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & k_1 & k_2 & 0 \\ 0 & k_3 & k_4 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Our impractical homomorphic algorithm for convolution operation also needs a ciphertext $ct.R$ to accumulate the intermediate ciphertexts, which should be initially encrypted by the kernel bias $k_0$. This algorithm requires $k \times k$ iterations and the $i$-th iteration consists of the following four steps for $1 \leq i \leq k^2$:

**Step 1:** For ciphertexts $ct.I$ and $ct.P_i$, this step computes their multiplication and returns the ciphertext $ct.I.P_i = \text{Mul}(ct.I, ct.P_i)$.

**Step 2:** To aggregate the values of some blocks of size $k \times k$, this step applies the procedure SumForConv on the ciphertext $ct.I.P_i$, obtaining the ciphertext $ct.D$.

**Step 3:** The public cloud generates a specially-designed matrix in order to filter out the garbage data in $ct.D$ by one constant multiplication, obtaining a ciphertext $ct.D$.

**Step 4:** In this step, the homomorphic convolution-operation algorithm updates the accumulator ciphertext $ct.R$ by homomorphically adding $ct.D$ to it, namely $ct.R = \text{Add}(ct.R, ct.D)$.

Note that Steps 1–3 in this algorithm can be computed in parallel with $k \times k$ threads. We describe how to compute homomorphic convolution operation in Algorithm 4 in detail. In total, Step 1-1 can be homomorphically evaluated with 2d additions, 2d constant multiplications, and 3pd rotations. Step 1-2 can be computed in a similar way using d additions, d constant multiplications, and 2pd rotations. The total computational cost is about 2d rotations and the baby-step/giant-step approach can be used to reduce the complexity; the number of automorphism can be reduced down to 3pd.

Table 2 summarizes the total complexity of Algorithm 4.

| Step | Add | cMult | Rot | Mult | Depth |
|------|-----|-------|-----|------|-------|
| 1    | 0   | 0     | 1   | 1    | 0     |
| 2    | 2k  | 1     | 2k  | 0    | 0     |
| 3    | 0   | 1     | 0   | 0    | 0     |
| 4    | 1   | 0     | 0   | 0    | 0     |
| Total| $O(k^2)$ | $O(k^2)$ | $O(k^2)$ | $O(k^2)$ | 0     |

Figure 2 describes a simple case for the algorithm where $h = 3$, $w = 4$ and $k = 3$.

Next, we will show how to make this impractical homomorphic algorithm work efficiently in real-world cases.
3.4 Encoding Method for Convolution Operation

For simplicity, we assume that the dataset $X \in \mathbb{R}^{m \times f}$ can be encrypted into a single ciphertext $ct.X$. $m$ is a power of two, all the images are grayscale and have the size $h \times w$. Volley Revolver encodes the dataset as a matrix using the database encoding method [10] and deals with any CNN layer with a single formation. In most cases, $h \times w < f$, if this happened, zero columns could be used for padding. Volley Revolver extends this database encoding method [10] with some additional operations to view the dataset matrix $X$ as a three-dimensional structure.

Algorithm [4] is a feasible and efficient way to calculate the secure convolution operation in an HE domain. However, its working-environment assumption that the size of an image is exactly the length of the plaintext, which rarely happens, is too strict to make it a practical algorithm, leaving this algorithm directly useless. In addition, Algorithm [4] can only deal with one image at a time due to the assumption that a single ciphertext only encrypts only one image, which is too inefficient for real-world applications.

To solve these problems, Volley Revolver performs some simulated operations on the ciphertext $ct.X$ to treat the two-dimensional dataset as a three-dimensional structure. These simulated operations together could simulate the first continual space of the same size as an image of each row of the matrix encrypted in a real ciphertext as a virtual ciphertext that can perform all the HE operations. Moreover, the number of plaintext slots is usually set to a large number and hence a single ciphertext could encrypt several images. For example, the ciphertext encrypting the dataset $X \in \mathbb{R}^{m \times f}$ could...
Algorithm 4 Homomorphic convolution operation

**Input:** An encrypted Image \( ct.I \) for \( I \in \mathbb{R}^{h \times w} \) and a kernel \( K \) of size \( k \times k \) with its bias \( k_0 \)

**Output:** The encrypted resulting image \( ct.I_s \) where \( I_s \) has the same size as \( I \)

> The Third Party performs KernelSpanner and prepares the ciphertext encrypting kernel bias

```
1: \( ct.S_i \leftarrow \text{KernelSpanner}(K, h, w) \)  \( \triangleright \) \( 1 \leq i \leq k^2 \)
2: Set \( I_s \leftarrow 0 \)  \( \triangleright \) \( I_s \in \mathbb{R}^{h \times w} \)
3: for \( i := 1 \) to \( h - k + 1 \) do
4: \( k_i[j] \leftarrow k_0 \)
5: for \( j := 1 \) to \( w - k + 1 \) do
6: \( I_s[i][j] \leftarrow k_0 \)
7: end for
8: \( ct.I_s \leftarrow \text{Enc}_{pk}(I_s) \)
   \( \triangleright \) So begins the Cloud its work
9: for \( i := 0 \) to \( k - 1 \) do
10: \( \text{ct.T} \leftarrow \text{Mul}(ct.I, ct.S_i[i \times k^2 + j + 1]) \)
11: \( \text{ct.T} \leftarrow \text{SumForConv}(ct.T) \)
   \( \triangleright \) Design a matrix to filter out the redundant values
12: Set \( F \leftarrow 0 \)  \( \triangleright \) \( F \in \mathbb{R}^{m \times n} \)
13: for \( wth := 0 \) to \( w - 1 \) do
14: if \((wth - i) \mod k = 0 \) and \( wth + k \leq w \) and
15: \((hth - j) \mod k = 0 \) and \( hth + k \leq h \) then
16: \( F[hth][wth] \leftarrow 1 \)
17: end if
18: end for
19: end for
20: \( \text{ct.T} \leftarrow \text{cMul}(F, ct.T) \)
21: end for
22: \( ct.I_s \leftarrow \text{Add}(ct.I_s, ct.T) \)
23: end for
24: end for
25: return \( ct.I_s \)
```

be used to simulate \( m \) virtual ciphertexts \( \text{vct}_i \) for \( 1 \leq i \leq m \), as shown below:

\[
\begin{bmatrix}
  f^{(1)}_{[1][1]} & f^{(1)}_{[1][2]} & \cdots & f^{(1)}_{[1][w]} & 0 & \cdots & 0 \\
  f^{(2)}_{[1][1]} & f^{(2)}_{[1][2]} & \cdots & f^{(2)}_{[1][w]} & 0 & \cdots & 0 \\
  \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
  f^{(m)}_{[1][1]} & f^{(m)}_{[1][2]} & \cdots & f^{(m)}_{[1][w]} & 0 & \cdots & 0
\end{bmatrix}
\]

\[
\rightarrow
\begin{bmatrix}
  \text{vEnc}_{[1][1]} & \text{vEnc}_{[1][2]} & \cdots & \text{vEnc}_{[1][w]} & 0 & \cdots & 0 \\
  \text{vEnc}_{[2][1]} & \text{vEnc}_{[2][2]} & \cdots & \text{vEnc}_{[2][w]} & 0 & \cdots & 0 \\
  \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
  \text{vEnc}_{[m][1]} & \text{vEnc}_{[m][2]} & \cdots & \text{vEnc}_{[m][w]} & 0 & \cdots & 0
\end{bmatrix}
\]

or

\[
\begin{bmatrix}
  f^{(1)}_{[1][1]} & f^{(1)}_{[1][2]} & \cdots & f^{(1)}_{[1][w]} & 0 & \cdots & 0 \\
  f^{(2)}_{[1][1]} & f^{(2)}_{[1][2]} & \cdots & f^{(2)}_{[1][w]} & 0 & \cdots & 0 \\
  \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
  f^{(m)}_{[1][1]} & f^{(m)}_{[1][2]} & \cdots & f^{(m)}_{[1][w]} & 0 & \cdots & 0
\end{bmatrix}
\]

\[
\rightarrow
\begin{bmatrix}
  \text{vEnc}_{[1][1]} & \cdots & \text{vEnc}_{[1][w]} & 0 & \cdots & 0 \\
  \text{vEnc}_{[2][1]} & \cdots & \text{vEnc}_{[2][w]} & 0 & \cdots & 0 \\
  \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
  \text{vEnc}_{[m][1]} & \cdots & \text{vEnc}_{[m][w]} & 0 & \cdots & 0
\end{bmatrix}
\]

Similar to an HE ciphertext, a virtual ciphertext has virtual HE operations: \( \text{vEnc}, \text{vDec}, \text{vAdd}, \text{vMul}, \text{vRescale}, \text{vBootstrapping} \) and \( \text{vRot} \). Except for \( \text{vRot} \), others can be all inherited from the corresponding HE operations. The HE operations, \( \text{Add, Mul, Rescale} \) and \( \text{Bootstrapping} \), result in the same corresponding virtual operations: \( \text{vAdd, vMul, vRescale} \) and \( \text{vBootstrapping} \).
The virtual rotation operation \( \text{vRot} \) is much different from other virtual operations: it needs two rotation operations over the real ciphertext, as described in Algorithm 5. We only need to simulate the rotation operation on these virtual ciphertexts to complete the simulation. The virtual rotation operation \( \text{vRot}(\text{ct}, r) \), to rotate all the virtual ciphertexts dwelling in the real ciphertext \( \text{ct} \) to the left by \( r \) positions, has the following simulation result:

\[
\begin{align*}
\text{Enc} & \left[ \begin{bmatrix} vEnc & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots \\ vEnc & 0 & \cdots & 0 \end{bmatrix} \right] \\
\text{Enc} & \left[ \begin{bmatrix} vEnc & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots \\ vEnc & 0 & \cdots & 0 \end{bmatrix} \right] \\
\downarrow & \text{vRot(ct, r)} \\
\text{Enc} & \left[ \begin{bmatrix} vEnc & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots \\ vEnc & 0 & \cdots & 0 \end{bmatrix} \right]
\end{align*}
\]

Given two sets of virtual ciphertexts \( \text{vct}_{[i]} \) and \( \text{vct}_{[j]} \) that inhabit two ciphertexts \( \text{ct}_{[1]} \) and \( \text{ct}_{[2]} \) respectively, for \( 1 \leq i, j \leq m \), the corresponding virtual HE operation \( \text{vMul(\text{vct}_{[i]}, \text{vct}_{[j]})} \) results:

\[
\begin{align*}
\text{Enc} & \left[ \begin{bmatrix} vEnc & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots \\ vEnc & 0 & \cdots & 0 \end{bmatrix} \right] \\
\text{Enc} & \left[ \begin{bmatrix} vEnc & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots \\ vEnc & 0 & \cdots & 0 \end{bmatrix} \right] \\
\downarrow & \text{vMul(\text{vct}_{[j]}, \text{vct}_{[j]})} \\
\text{Enc} & \left[ \begin{bmatrix} vEnc & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots \\ vEnc & 0 & \cdots & 0 \end{bmatrix} \right]
\end{align*}
\]

The virtual HE operation \( \text{vAdd} \) obtains a similar result:

\[
\begin{align*}
\text{Enc} & \left[ \begin{bmatrix} vEnc & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots \\ vEnc & 0 & \cdots & 0 \end{bmatrix} \right] \\
\text{Enc} & \left[ \begin{bmatrix} vEnc & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots \\ vEnc & 0 & \cdots & 0 \end{bmatrix} \right] \\
\downarrow & \text{vAdd(\text{vct}_{[i]}, \text{vct}_{[j]})} \\
\text{Enc} & \left[ \begin{bmatrix} vEnc & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots \\ vEnc & 0 & \cdots & 0 \end{bmatrix} \right]
\end{align*}
\]

To bring all the pieces together, we can use Algorithm 4 to perform convolution operations for several images in parallel based on the simulation virtual ciphertexts. The most efficient part of
Algorithm 5 vRot: virtual rotation operations on a set of virtual ciphertexts

Input: a ciphertext $ct.X$ encrypting a dataset matrix $X$ of size $m \times n$, each row of which encrypts an image of size $h \times w$ such that $h \times w \leq n$, and the number $r$ of rotations to the left

Output: a ciphertext $ct.R$ encrypting the resulting matrix $R$ of the same size as $X$

1: Set $R \leftarrow 0$ \> $R \in \mathbb{R}^{m \times n}$
2: $ct.R \leftarrow \text{Enc}_{pk}(R)$

[Step 1]: Rotate the real ciphertext to the left by $r$ positions first and then clean up the garbages values
3: $ct.T \leftarrow \text{Rot}(ct.X,r)$
\> Build a specially designed matrix
4: Set $F_1 \leftarrow 0$ \> $F_1 \in \mathbb{R}^{m \times n}$
5: for $i := 1$ to $m$
6: \> $F_1[i][j] \leftarrow 1$
7: end for
8: $ct.T_1 \leftarrow \text{cMul}(F_1, ct.T)$

[Step 2]: Rotate the real ciphertext to the right by $h \times w - r$ positions (the same as to the left by $m \times n - h \times w + r$ positions) first and then clean up the garbage values
9: $ct.P \leftarrow \text{Rot}(ct.X,m \times n - h \times w + r)$
\> Build a special designed matrix
10: Set $F_2 \leftarrow 0$ \> $F_2 \in \mathbb{R}^{m \times n}$
11: for $i := 1$ to $m$
12: \> $F_2[m][j] \leftarrow 1$
13: end for
14: $ct.T_2 \leftarrow \text{cMul}(F_2, ct.P)$
\> concatenate
15: $ct.R \leftarrow \text{Add}(ct.T_1, ct.T_2)$
16: return $ct.R$

These simulated operations is that a sequence of operations on a real ciphertext results in the same corresponding operations on the multiple virtual ciphertexts, which would suffice the real-world applications.

4 Secure Inference

In most real-world applications, the datasets are usually so large that they cannot be encrypted into a single ciphertext. In these cases, we can partition the dataset into several mini-batches, each containing an equal number of data samples. Each mini-batch can then be encrypted into a separate ciphertext. This approach allows us to continue using the algorithms described above, while also achieving better parallelization.

4.1 Parallelized Matrix Multiplication

Algorithm 3 is designed for matrix multiplication based on the assumption that all data can be encrypted into a single ciphertext. This algorithm can be accelerated through improvements in hardware threading capabilities. Suppose we have two matrices $A$ of size $m \times n$ and $B$ of size $n \times p$, where the algorithm requires $m + n$ ciphertext rotations. If we have $m$ hardware threads available, then each thread can work in parallel, requiring only $\log n$ ciphertext rotations per thread.

If the dataset is too large to be encrypted into a single ciphertext, it can be partitioned and encrypted into multiple ciphertexts, represented by two sets, $A_1, \ldots, A_i$ and $B_1, \ldots, B_j$. The proposed algorithm can still be applied under this setup, and the matrix multiplication can be further parallelized at the ciphertext level: each ciphertext in set $A$ needs to be processed with each ciphertext in set $B$ and this can be done in parallel. Mathematically, this approach leverages submatrix computations to ensure theoretical consistency and correctness.
This conveys that even with multiple ciphertexts, the algorithm remains applicable and can achieve parallelism by processing each pair of ciphertexts from the two sets, leveraging the computation principles of submatrices.

4.2 Parallelized Convolution Operation

With the simulation operations on real ciphertexts, we can use Algorithm 4 to perform convolution operations for several images simultaneously on the virtual ciphertexts in parallel.

With the simulation operations on real ciphertexts, we can employ Algorithm 4 to simultaneously perform convolution operations on multiple images using virtual ciphertexts in parallel. Depending on the number of virtual ciphertexts that can be simulated within a single real ciphertext, the algorithm’s time complexity can be further reduced through amortization. For instance, if one real ciphertext can house $m$ virtual ciphertexts, the algorithm can conduct convolution operations on $m$ images at the same time. As a result, the time complexity is reduced to $\frac{1}{m}$ of the normal time complexity for the algorithm.

Recall that the kernel $K$ should be utilized in advance to generate $k^2$ ciphertexts $K_1, \ldots, K_{k^2}$, each of which encrypts an $h \times w$ matrix. Assuming there exists a ciphertext $ct.I$ that encrypts a one-dimensional channel of a gray image of size $h \times w_I$, the partial convolution computations between the ciphertext $ct.I$ and the ciphertexts containing kernel information $K_i$ can be performed in parallel. Moreover, if there are $k^2$ hardware-supported threads available, each thread can perform computations in parallel, thereby accelerating the overall convolution operation.

For color images, unlike those from the MNIST dataset, which have three channels, or for images after convolutional layers that have multiple channels, each channel can be encrypted into a separate single ciphertext, $ct.I_c$, and processed with Algorithm 4. In most cases, a convolutional layer contains multiple kernels, each of which may also have multiple channels. In this case, the ciphertexts encrypting the different channels of the same image perform a ciphertext multiplication with the corresponding ciphertexts encrypting different channels of one kernel. The resulting ciphertexts can then be utilized by Algorithm 4 to compute partial convolution results. A final accumulation of these results via ciphertext addition yields the complete convolution output. The operations for different kernels and the same image can be executed in parallel, enabling the algorithm to achieve high performance through extensive parallelization both within and across ciphertexts.

4.3 Homomorphic CNN Evaluation

An important operation after the convolutional layer is to reform the data structure in the ciphertext, as the convolution computation by SumFor will decrease the image size. Consequently, the resulting...
encrypted image data may be encoded differently from Volley Revolver:

\[
\text{Enc} \left[ \begin{bmatrix}
\text{vEnc} \left[ \tau^{(1)}_{1[1]} \tau^{(1)}_{2[1]} \tau^{(2)}_{1[1]} \tau^{(2)}_{2[1]} \tau^{(1)}_{3[1]} \tau^{(1)}_{3[2]} \tau^{(1)}_{3[3]} \right] 0 \ldots 0
\end{bmatrix}
\right]
\]

A pooling layer would also reduce the size of an image and similarly requires a reformation of the data structure in the ciphertext. Since the baseline didn’t use a pooling layer, we didn’t apply such a layer either.

The Threat Model If the underlying HE scheme can ensure IND-CPA security, meaning that the ciphertexts of any two messages are computationally indistinguishable, then all computations performed on the public cloud will be over encrypted data. This ensures that the cloud server, which operates in a semi-honest model, learns nothing from the encrypted data. Consequently, we can guarantee the confidentiality of the data against such an adversary.

The Usage Model Our approach can be used in several usage scenarios as illustrated in [8]. Whatever the scenario is, the data to be privacy-preserving and the tailored CNN model have to be encrypted before being outsourced to the cloud for its service. In a reasonable usage scenario, there are a few different roles including data owner, the model provider, and the cloud server. In some special scenarios, the first two roles can be the same one who would like to get the service from the cloud server.

Based on this usage model, we can make some plausible assignment preparations for the three roles:

1. Data Owner: Data owner should prepare its dataset, such as cleaning its data and normalizing its data, and even partitioning the dataset into multiple mini-batches of a suitable size if it is too large. Finally, the data owner encrypts its dataset using the database encoding method introduced by [10] and then sends the ciphertext to the cloud.

2. Model Provider: Model provider needs to use Kernelspanner to generate several ciphertexts that encrypt the kernel information, and the Volley Revolver method to encode the weights matrix of fully connected layer and encrypt the resulting matrix later. For the activation layer, the polynomial approximation of ReLU function could be just sent to the cloud as some public parameters without the need for encrypting.

3. Cloud Server: Cloud Server provides the cloud service. To this end, the application (program) to implement the homomorphic CNN algorithm should be deployed on the cloud in advance.
Some public keys that are essential for HE operations should also be sent to the cloud. Such preparations could be done with the coordination and cooperation of the model provider and cloud server.

5 Implementation

We use C++ to implement our homomorphic CNN inference. Our complete source code is publicly available at https://anonymous.4open.science/r/HE-CNNinfer-ECA4/ and https://github.com/petitioner/HE.CNNinfer.

5.1 Neural Networks Architecture

We adopt the same CNN architecture as [9] but with some different hyperparameters. Table 3 gives a description of our neural networks architecture on the MNIST dataset.

| Layer | Description |
|-------|-------------|
| CONV  | 32 input images of size $28 \times 28$, 4 kernels of size $3 \times 3$, stride size of $(1, 1)$ |
| ACT-1 | $x \mapsto -0.00015120704 + 0.4610149 \cdot x + 2.0225089 \cdot x^2 - 1.4511951 \cdot x^3$ |
| FC-1  | Fully connecting with $26 \times 26 \times 4 = 2704$ inputs and 64 outputs |
| ACT-2 | $x \mapsto -1.5650465 - 0.9943767 \cdot x + 1.6794522 \cdot x^2 + 0.5350255 \cdot x^3$ |
| FC-2  | Fully connecting with 64 inputs and 10 outputs |

5.2 Building A Model In The Clear

In order to build a homomorphic model, we follow the normal approach for the machine-learning training in the clear — except that we replace the normal ReLU function with a polynomial approximation: we (1) train our CNN model described in Table 3 with the MNIST training dataset being normalized into domain $[0, 1]$, and then we (2) implement the well-trained resulting CNN model from step (1) using the HE library and HE programming.

For step (1) we adopt the highly customizable library keras with Tensorflow, which provides us with a simple framework for defining our own model layers such as the activation layer to enact the polynomial activation function. After many attempts to obtain a decent CNN model, we finally get a CNN model that could reach a precision of 98.66% on the testing dataset. We store the weights of this model into a CSV file for the future use. In step (2) we use the HE programming to implement the CNN model, accessing its weights from the CSV file generated by step (1). We normalize the MNIST training dataset by dividing each pixel by the floating-point constant 255.

Parameters. We follow the notation of [10] and set the HE scheme parameters for our implementent: $\Delta = 2^{45}$ and $\Delta_c = 2^{20}$; slots = 32768; $\log q = 1200$ and $\log N = 16$ to achieve a security level of 80-bits. (see [8, 9] for more details on these parameters).

5.3 Homomorphic CNN Predictions

At the encryption phase, the data owner uses our database encoding method to batch 32 different images and then encrypt them into a single ciphertext via the SIMD technique. To simplify the HE programming, our CNN inference model takes only one ciphertext each time but could be programmed to deal with multiple ciphertexts at the same time. The data owner sends the ciphertext of the 32 images and the model provider sends the encrypted CNN model parameters, both to the the public cloud.

Encryption of Images. We evaluate our implementation of the homomorphic CNN model on the MNIST dataset to each time calculate ten likelihoods for 32 encrypted images of handwritten digits. The MNIST database includes a training dataset of 60 000 images and a testing dataset of 10 000, each image of which is of size $28 \times 28$. For such an image, each pixel is represented by a 256-level grayscale and each image depicts a digit from zero to nine and is labeled with it.
Encryption of Images the data owner encodes the data with Volley Revolver and encrypts them using the public key of an HE scheme. Suppose that the data owner has 32 grey images of size 28 × 28, that can be encrypted into a single ciphertext \( ct.I \). If the dataset has more grey images, it can be divided into multiple mini-batches of a fixed size and then encrypted into multiple ciphertexts. The resulting ciphertexts \( ct.I \) are sent to the public cloud and then stored in their encrypted form.

**Encryption of Trained Model** Encryption of Trained Model the model provider encrypts the well-trained inference model parameters, including the multiple convolution kernels’ values and the matrix weights of FC layers. The coefficients of polynomials for every activation layer are not so sensitive and can thus be sent to the cloud server in the clear state. The provider begins with the procedure kernelspanner to generate multiple ciphertexts for each of the convolution kernels separately.

**Classifying Encrypted Inputs** We implement our homomorphic CNN inference with the library HEAAN by [3]. Note that before encrypting the testing dataset of images, we also normalize the MNIST testing dataset by dividing each pixel by the floating-point constant 255, just like the normal procedure on the training dataset in the clear.

**Homomorphic convolution layer.** The public cloud takes the ciphertext \( ct.I \) and \( ct.K_{i,j} \) for \( 0 \leq i, j \leq 7 \). We apply the ciphertext multiplication between the encrypted images and the spanned kernel matrices and then the \texttt{SumFor} operation on the resulting ciphertexts. For each \( 0 \leq i, j \leq 7 \), the cloud server performs the following computation:

\[
ct.C_k.
\]

By the definition of the convolution, the resulting ciphertext \( ct_k \) encrypts some flatten convolved result between the image \( I \) and the \( k \)-th kernel \( K(k) \).

**The first activation layer.** The step applies the degree-3 polynomial function to all the encrypted output images of the homomorphic convolution layer in parallel.

Homomorphic Encryption cannot directly compute functions such as the ReLU activation function. We use Octave to generate a degree-three polynomial by the least square method and just initialize all the activation layers with this polynomial, leaving the training process to determine the coefficients of polynomials for every activation layer. Other computation operations, such as matrix multiplication in the fully-connected layer and convolution operation in the convolutional layer, can also be performed by the algorithms we proposed above.

**The FC-1 layer.** This procedure perform a matrix multiplication between a 32 × 2704 input matrix and a 2704 × 64 weight matrix.

**The second activation layer.** This step applies the degree-3 polynomial function to all the output nodes of the first FC layer.

**The FC-2 layer.** This step performs the multiplication algorithm between the output ciphertext \( ct.S \) of the second activation layer and the weight ciphertext \( ct.V \):

Next, the first FC layer is specified by a 2704 × 64 matrix that can be divided into four sub-matrices of size 676 × 64. Each matrix is encrypted into a single ciphertext using our matrix encoding method in Section 5.1.

Finally, the second FC layer can be expressed by a 64 × 10 matrix.

## 5.4 Performance and Comparison

We evaluate the performance of our implementation on the MNIST testing dataset of 10 000 images. Since in this case Volley Revolver encoding method can only deal with 32 MNIST images at one time, we thus partition the 10 000 MNIST testing images into 313 blocks with the last block being padded zeros to make it full. We then test the homomorphic CNN inference on these 313 ciphertexts and finally obtain a classification accuracy of 98.61%. The processing of each ciphertext outputs 32 digits with the highest probability of each image, and it takes \( \sim 287 \) seconds on a cloud server with 40 vCPUs. There is a slight difference in the accuracy between the clear and the encryption, which is due to the fact that the accuracy under the ciphertext is not the same as that under the plaintext. In order to save the modulus, a TensorFlow Lite model could be used to reduce the accuracy in the clear
from float 32 to float 16. The data owner only uploads 1 ciphertext (∼ 19.8 MB) encrypting these 32 images to the public cloud while the model provider has to send 52 ciphertexts (∼ 1 GB) encrypting the weights of the well-trained model to the public cloud.

6 Conclusion

The encoding method we proposed in this work, Volley Revolver, is particularly tailored for privacy-preserving neural networks. There is a good chance that it can be used to assist the private neural networks training, in which case for the backpropagation algorithm of the fully-connected layer the first matrix $A$ is revolved while the second matrix $B$ is settled to be still.

We shifted some work related to the CNN model to the model provider and some data preparation processes to the data owner so as to complete the homomorphic CNN inference. We believe it is all right for privacy-preserving inference due to no sensitive information leaking.

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