Abstract. Let $M$ be a closed orientable irreducible 3-manifold such that $\pi_1(M)$ is left orderable.

(a) Let $M_0 = M - \text{Int}(B^3)$, where $B^3$ is a compact 3-ball in $M$. We have a process to produce a co-orientable Reebless foliation $\mathcal{F}$ in $M_0$ such that: (1) $\mathcal{F}$ has a transverse $(\pi_1(M), \mathbb{R})$ structure, (2) there exists a simple closed curve in $M$ that is co-orientably transverse to $\mathcal{F}$ and intersects every leaf of $\mathcal{F}$. More specifically, given a pair $(<, \Gamma)$ composed of a left-invariant order “$<$” of $\pi_1(M)$ and a fundamental domain $\Gamma$ of $M$ in its universal cover with certain property (which always exists), we can produce a resulting foliation in $M - \text{Int}(B^3)$ as above, and we can test if it can extend to a taut foliation of $M$.

(b) Suppose further that $M$ is either atoroidal or a rational homology 3-sphere. If $M$ admits an $\mathbb{R}$-covered foliation $\mathcal{F}_0$, then there is a resulting foliation $\mathcal{F}$ of our process in $M - \text{Int}(B^3)$ such that: $\mathcal{F}$ can extend to an $\mathbb{R}$-covered foliation $\mathcal{F}_{\text{extend}}$ of $M$, and $\mathcal{F}_0$ can be recovered from doing a collapsing operation on $\mathcal{F}_{\text{extend}}$. Here, by a collapsing operation on $\mathcal{F}_{\text{extend}}$, we mean the following process: (1) choosing an embedded product space $S \times I$ in $M$ for some (possibly non-compact) surface $S$ such that $S \times \{0\}, S \times \{1\}$ are leaves of $\mathcal{F}_{\text{extend}}$ (notice that $\mathcal{F}_{\text{extend}}|_{S \times I}$ may not be a product bundle), (2) replacing $\mathcal{F}_{\text{extend}}|_{S \times I}$ by a single leaf $S$.

(c) We conjecture that there always exists a resulting foliation of our process in $M - \text{Int}(B^3)$ which can extend to a taut foliation in $M$.

1. Introduction

Throughout this paper, all 3-manifolds are assumed to be orientable, and all foliations and laminations are assumed to be co-orientable.

The L-space conjecture was proposed by Boyer-Gordon-Watson in [BGW] and by Juhász in [J]:

Conjecture 1 (L-space conjecture). Let $M$ be an orientable irreducible rational homology 3-sphere. Then the following statements are equivalent:

1) $M$ is a non-L-space.
2) $\pi_1(M)$ is left orderable.
3) $M$ admits a co-orientable taut foliation.

Conjecture 1 has been proved for graph manifolds in [BC], [Ra], [HRRW]. The implication (3) $\Rightarrow$ (1) is known by [OS], [B], [KR]. In [Ga1], Gabai proved that all closed orientable irreducible 3-manifolds with positive first Betti number admit taut foliations. In [BRW], Boyer, Rolfsen and Wiest proved that for every compact orientable irreducible 3-manifold $M$ with $b_1(M) > 0$, $\pi_1(M)$ is left orderable. Hence (3) and (2) hold for all closed orientable irreducible 3-manifolds with $b_1 > 0$.

Furthermore, connections between foliations and laminations, and orders and group actions (on 1-manifolds) were developed in many works. Through his universal circle construction ([T]), Thurston showed that: for every atoroidal 3-manifold $M$ that admits a taut foliation, there exists an effective action of $\pi_1(M)$ on $S^1$ induced by the taut foliation. In [CD], Calegari and Dunfield showed that tight essential laminations with solid tori guts in every atoroidal 3-manifold $M$ also induce effective actions of $\pi_1(M)$ on $S^1$.

Transverse $(\pi_1, \mathbb{R})$ structure for taut foliations is a way to relate them to left orderability of the fundamental group.
Definition 1.1. Suppose that $M$ is a closed orientable irreducible 3-manifold which admits a taut foliation $\mathcal{F}$. Let $p: \tilde{M} \to M$ be the universal covering of $M$, and let $L$ be the leaf space of the pull-back foliation of $\mathcal{F}$ in $\tilde{M}$. There is a natural action of $\pi_1(M)$ on $L$ induced by the deck transformations of $\tilde{M}$, called the $\pi_1$-action on $L$. A transverse $(\pi_1(M), \mathbb{R})$ structure on $\mathcal{F}$ is an immersion $\tilde{i}_{des}: L \to \mathbb{R}$ that descends the $\pi_1$-action on $L$ to a nontrivial action of $\pi_1(M)$ on $\mathbb{R}$ via homeomorphisms, i.e. $\tilde{i}_{des}$ induces a homomorphism $d: \pi_1(M) \to \text{Homeo}_+(\mathbb{R})$ such that the following diagram commutes (for every $g \in \pi_1(M)$):

$$
\begin{array}{ccc}
L & \xrightarrow{i_{des}} & \mathbb{R} \\
\downarrow{g} & & \downarrow{d(g)} \\
L & \xrightarrow{i_{des}} & \mathbb{R}
\end{array}
$$

Here, by an immersion, we mean a topological immersion (i.e. a local homeomorphism). Furthermore, $\mathcal{F}$ is called $\mathbb{R}$-covered if $L \cong \mathbb{R}$.

Notice that if $\mathcal{F}$ has a transverse $(\pi_1(M), \mathbb{R})$ structure, then $\pi_1(M)$ acts on $\mathbb{R}$ nontrivially, and therefore $\pi_1(M)$ is left orderable (cf. [BRW, Theorem 3.2]). The following question was proposed by Thurston (cf. [Cal3, Question 8.1]):

**Question 2** (Thurston). Let $M$ be a closed orientable irreducible 3-manifold such that $\pi_1(M)$ is left orderable. When does $M$ admit a taut foliation with a transverse $(\pi_1(M), \mathbb{R})$ structure?

See [T] for some motivations for Question 2 and [Cal3, Question 8.1, Remark] for more information on Question 2. It’s clear that every $\mathbb{R}$-covered foliation in a closed 3-manifold has a natural transverse $(\pi_1, \mathbb{R})$ structure. In [Z], Zung showed that a large class of closed 3-manifolds admit (non-$\mathbb{R}$-covered) taut foliations with a similar property.

At first, we consider the related topics for compact 3-manifolds with $S^2$ boundary. Our first result provides foliations in this setting which are analogous to taut foliations in closed 3-manifolds:

**Theorem 1.2.** Let $M$ be a closed orientable irreducible 3-manifold such that $\pi_1(M)$ is left orderable. Let $M_0 = M - \text{Int}(B^3)$, where $B^3$ is a compact 3-ball in $M$. Then there is a Reebless foliation $\mathcal{F}$ in $M_0$, where the leaves of $\mathcal{F}$ may be transverse to $\partial M_0$ or tangent to $\partial M_0$ at their intersections with $\partial M_0$, such that:

1. $\mathcal{F}$ is analogous to “taut” in the following sense:
   - There is a simple closed curve in $M$ that is co-orientably transverse to $\mathcal{F}$ and has nonempty intersection with every leaf of $\mathcal{F}$.

2. $\mathcal{F}$ has a transverse $(\pi_1(M), \mathbb{R})$ structure:
   - Let $L$ denote the leaf space of the pull-back foliation of $\mathcal{F}$ in the universal cover of $M_0$, then there exists an immersion $\tilde{i}_{des}: L \to \mathbb{R}$ descending the $\pi_1$-action on $L$ to an effective action on $\mathbb{R}$.

3. Every transverse intersection component of some leaf of $\mathcal{F}$ and $\partial M_0$ is a circle.

Furthermore, we have a process to produce such foliations. $\mathcal{F}$ is called a resulting foliation of this process.

**Remark 1.3.** (a) There are many equivalent definitions for taut foliations in closed 3-manifolds (cf. [Cal4, Chapter 4], [Cal5]). The property of $\mathcal{F}$ in Theorem 1.2 (1) is only analogous to one of these definitions for taut foliations: there exists a compact 1-manifold (i.e. a finite union of simple closed curves) transverse to the foliation that intersects every leaf.

(b) Theorem 1.2 (2) implies that $\pi_1(M)$ acts on $\mathbb{R}$ effectively. Hence the existence of the foliation $\mathcal{F}$ as given in Theorem 1.2 is a necessary and sufficient condition for $\pi_1(M)$ being left orderable.

(c) $\mathcal{F}$ has the following property: if a simple closed curve in $M$ is co-orientably transverse to $\mathcal{F}$ and has nonempty intersection with some leaves of $\mathcal{F}$, then it is essential in $M$ (cf. Remark 4.19).

It’s clear that Theorem 1.2 is a necessary condition for $M$ to admit a taut foliation with transverse $(\pi_1(M), \mathbb{R})$ structure. We suspect that it is also sufficient. In the following, we show that many resulting foliations of our process can extend to $\mathbb{R}$-covered foliations of $M$. 
1.1. \( \mathbb{R} \)-covered foliations constructed in our process. \( \mathbb{R} \)-covered foliations are studied extensively in atoroidal 3-manifolds in many works, which provide plentiful examples and various geometric and topological properties. See [Fe1], [T], [Cal1], [Cal2], [Fe2] for example. The following question is given in [Cal3, Question 8.3]:

**Question 3.** Given an atoroidal 3-manifold that admits a taut foliation, must it admit \( \mathbb{R} \)-covered foliations?

The following result indicates that: combined with an operation, all \( \mathbb{R} \)-covered foliations in irreducible atoroidal 3-manifolds and rational homology 3-spheres can be constructed by our process:

**Definition 1.4.** Let \( F_0 \) be a co-orientable taut foliation of a closed 3-manifold \( M \). Suppose that there is an embedded product region \( S \times I \) in \( M \) (where \( S \) is a possibly non-compact surface) such that \( S \times \{0\}, S \times \{1\} \) are leaves of \( F_0 \). Notice that \( F_0 \big|_{S \times I} \) may not be a product bundle. We can replace the region \( S \times I \subseteq M \) by a single surface \( S \) and replace \( F_0 \big|_{S \times I} \) by the single leaf \( S \). Then we obtain a foliation \( F'_0 \) of \( M \) which is necessarily co-orientable and taut. We call the process \( F_0 \leadsto F'_0 \) a collapsing operation.

**Theorem 1.5.** Let \( M \) be either an orientable irreducible atoroidal 3-manifold or an orientable rational homology 3-sphere. Suppose that \( M \) admits an \( \mathbb{R} \)-covered foliation \( F_0 \). Then there is a resulting foliation \( F \) of \( M - \text{Int}(B^3) \) obtained from the process in Theorem 1.2, such that:

1. \( F \) can extend to an \( \mathbb{R} \)-covered foliation \( F_{\text{extend}} \) of \( M \).
2. \( F_0 \) can be recovered from doing a collapsing operation on \( F_{\text{extend}}\).

1.2. The test condition for extendability. We provide the input of our process, and we provide the condition to test if a resulting foliation of our process in \( M - \text{Int}(B^3) \) can extend to a taut foliation in \( M \) through the input.

Let \( M \) be a closed orientable irreducible 3-manifold such that \( \pi_1(M) \) is left orderable. Let \( p : \tilde{M} \to M \) be the universal covering of \( M \). We fix a base point \( \tilde{x} \in \tilde{M} \). Let \( x = p(\tilde{x}) \), and let \( G = \pi_1(M,x) \).

**Definition 1.6.** A fundamental domain \( \Gamma \) of \( M \) in \( \tilde{M} \) is called standard if \( p(\partial \Gamma) \) is a standard spine (cf. Definition 2.2). An order-domain pair \( (,\Gamma) \) of \( M \) is a pair of a left invariant order “<” of \( G \) and a standard fundamental domain \( \Gamma \) of \( M \) in \( \tilde{M} \).

In Lemma 3.5, we show that \( M \) always has a standard fundamental domain \( \Gamma \) in \( \tilde{M} \), and \( \Gamma \) can be constructed in finitely many steps. To be convenient, we assume \( \tilde{x} \in \text{Int}(\Gamma) \). For each \( h \in G \), we denote by \( t_h : \tilde{M} \to M \) the deck transformation such that: if \( \eta \) is a path in \( \tilde{M} \) that starts at \( \tilde{x} \) and ends at \( t_h(\tilde{x}) \), then \( p(\eta) \) is a loop in \( M \) with \( [p(\eta)] = h \).

A compact region \( F \subseteq \partial \Gamma \) is called a face of \( \Gamma \) if there is \( h \in G - \{1\} \) such that \( F \) is a component of \( \Gamma \cap t_h(\Gamma) \) and \( F \not\subseteq \Gamma \cap t_g(\Gamma) \) for arbitrary \( g \in G - \{1,h\} \). And we call \( F \) a positive face (resp. negative face) if \( h > 1 \) (resp. \( h < 1 \)).

**Definition 1.7.** An order-domain pair \( (,\Gamma) \) of \( M \) is called a good pair if the union of positive faces of \( \Gamma \) is a single 2-disk. And we call \( (,\Gamma) \) a very good pair if (1) \( (,\Gamma) \) is good, (2) both \( \bigcup_{g \in G, g < 1} t_g(\Gamma) \) and \( \bigcup_{g \in G, g \geq 1} t_g(\Gamma) \) are connected.

The input of our process and the test condition for extendability are as follows:

**Theorem 1.8.** Let \( (,\Gamma) \) be an arbitrary order-domain pair of \( M \).

(a) \( (,\Gamma) \) produces a resulting foliation through the process in Theorem 1.2 (called a resulting foliation of \( (,\Gamma) \)), which is uniquely determined by \( (,\Gamma) \) up to blowing-up/down.

(b) A resulting foliation of \( (,\Gamma) \) can extend to a taut foliation of \( M \) if and only if \( (,\Gamma) \) is a good pair. Moreover, if it can extend to a taut foliation \( F_1 \) of \( M \), then \( F_1 \) has a transverse \( (\pi_1(M),\mathbb{R}) \) structure.
(c) A resulting foliation of $(<, \Gamma)$ can extend to an $\mathbb{R}$-covered foliation of $M$ if and only if $(<, \Gamma)$ is a very good pair.

Combined with Theorem 1.2 and Theorem 1.5, we have

**Corollary 1.9.** Suppose that $M$ is either an orientable irreducible atoroidal 3-manifold or an orientable irreducible rational homology 3-sphere.

(a) $M$ admits an $\mathbb{R}$-covered foliation if and only if there is a very good pair $(<, \Gamma)$ of $M$.

(b) We have a process with input all very good pairs $(<, \Gamma)$ of $M$ and output all $\mathbb{R}$-covered foliations of $M$, up to blowing-up/down, collapsing operation, and its inverse.

Notice that the space of left-invariant orders of $\pi_1(M)$ is either finite or uncountable (cf. [Lin]).

**Corollary 1.10.** Under the assumption of Corollary 1.9, if the space of left-invariant orders of $\pi_1(M)$ is finite, then $M$ admits countably many distinct $\mathbb{R}$-covered foliations, up to blowing-up/down, collapsing operation, and its inverse.

Our result motivates the following conjecture:

**Conjecture 4.** Fix an arbitrary left-invariant order “$<$” of $G$. There is a process to produce a standard fundamental domain $\Gamma$ of $M$ in $\tilde{M}$ such that $(<, \Gamma)$ is a good pair.

1.3. **Organization.** In Section 2, we provide some basic settings of this paper and review some preliminaries on branched surfaces.

For a 3-manifold $M$ as in Theorem 1.2, we construct a branched surface $B$ in $M$ in Section 3. In Subsections 4.1~4.2, we use $B$ to construct the foliation $\mathcal{F}$ as required in Theorem 1.2. In Subsection 4.3, we prove Theorem 1.8. In Subsection 4.4, we discuss the 2-dimensional case. We prove Theorem 1.5 in Section 5.

**Postscript.** After we posted the first version of our paper to arXiv, Baik, Hensel and Wu provided a new method to prove Theorem 1.2 under the assumption that $M$ admits a strongly essential 1-vertex triangulation in [BHW].

1.4. **Acknowledgements.** The author wishes to thank Professor Xingru Zhang for much help and support from him during this work, and for many suggestions and discussions that benefit the author a lot. The author is grateful to Professor Tao Li for answering some questions helpfully.

2. Preliminaries

2.1. **Conventions.** In this paper, all actions on manifolds are assumed to be orientation-preserving. All intervals in $\mathbb{R}$ are assumed to have increasing orientation, and all homeomorphisms between intervals in $\mathbb{R}$ are assumed to be orientation-preserving. By an immersion from a (possibly non-Hausdorff) 1-manifold to $\mathbb{R}$, we will always mean a topological immersion (i.e. a local homeomorphism). For a set $X$, we denote by $|X|$ the cardinality of $X$. For metric spaces $A$ and $B$, we denote by $A \setminus B$ the closure of $A - B$ under the path metric. And we regard two foliations in a 3-manifold as the same one if they are same up to isotopy of the 3-manifold.

For any compact oriented 3-manifold $N$ with nonempty boundary, the positive orientation of $\partial N$ with respect to the orientation of $N$ is assumed to be the orientation on $\partial N$ induced from the orientation of $N$.

A transversal with endpoints of a foliation is a closed interval such that its interior is transverse to the foliation. By a transversal of a foliation, we will always mean a transversal with endpoints. A transversal of a co-oriented foliation is positively oriented (resp. negatively oriented) if its direction is consistent with (resp. opposite to) the co-orientation on the foliation.

Throughout this paper, we will always have the following notations:
Notation 2.1. Let $M$ be a closed, orientable, irreducible 3-manifold with left orderable fundamental group. Let $p: \tilde{M} \to M$ be the universal covering of $M$. We choose an orientation on $M$, which induces an orientation on $\tilde{M}$ such that $p$ is orientation preserving.

(a) We fix a base point $\tilde{x} \in \tilde{M}$, and we denote $p(\tilde{x})$ by $x$. Let $G = \pi_1(M, x)$. For each $h \in G$, we denote by $t_h : \tilde{M} \to \tilde{M}$ the deck transformation of $\tilde{M}$ such that: if $\eta \subseteq \tilde{M}$ is a path that starts at $\tilde{x}$ and ends at $t_h(\tilde{x})$, then $[p(\eta)] = h$.

(b) We fix a left-invariant order “$<$” of $G$.

2.2. Branched surfaces. In this section, we review some background materials on branched surfaces. The branched surface is an important tool to describe taut foliations (cf. [Ga2]) and essential laminations (cf. [GO] and [Li]). Our notations and definitions of branched surfaces follow from [FO], [O], [Ga2], [Li]:

Definition 2.2 (Standard spine). (a) A standard spine ([Cas]) is a 2-complex locally modeled as Figure 1.

(b) Let $S$ be a standard spine. Let $L$ be the set of points in $S$ that have no Euclidean neighborhood in $S$. We call each component of $S \setminus L$ a sector of $S$. And we call each point in $L$ that has no $\mathbb{R}$-neighborhood in $L$ a double point.

Definition 2.3. A branched surface $B$ in $M$ is an embedded standard spine with well-defined cusp directions, which is locally modeled as Figure 2.

Notation 2.4. Let $B$ be a branched surface in $M$.

(a) Let $N(B)$ denote a regular neighborhood of the branched surface $B$ as shown in Figure 3. Then $N(B)$ can be regarded as an interval bundle over $B$, and we call these fibers $I$-fibers (or interval fibers). $N(B)$ is called a fibered neighborhood of $B$.
(b) \( \partial N(B) \) can be decomposed to \( \partial_h N(B) \) (horizontal boundary) and \( \partial_v N(B) \) (vertical boundary) such that: \( \partial_h N(B) \) is transverse to \( I \)-fibers, and \( \partial_v N(B) \) is tangent to \( I \)-fibers.

(c) Let \( \pi : N(B) \to B \) be the projection that takes each \( I \)-fiber to a single point. We call \( \pi \) the collapsing map for \( N(B) \).

(d) A lamination \( \mathcal{L} \) is carried by \( B \) if there exists a fibered neighborhood \( N(B) \) such that \( \mathcal{L} \subseteq N(B) \) and each leaf of \( \mathcal{L} \) is transverse to the \( I \)-fibers of \( N(B) \). \( \mathcal{L} \) is fully carried by \( B \) if it is carried by \( B \) and intersects every \( I \)-fiber of \( N(B) \).

(e) The branch locus of \( B \) is the set of points in \( B \) that have no Euclidean neighborhood in \( B \). We denote it by \( L(B) \). Each component of \( B \setminus L(B) \) is called a branch sector. We call each point in \( L(B) \) that has no \( \mathbb{R} \)-neighborhood in \( L(B) \) a double point of \( L(B) \). Let \( V \) denote the set of double points of \( L(B) \). We call each component of \( L(B) \setminus V \) an edge of \( L(B) \).

**Remark 2.5.** Suppose that \( B \) is a branched surface in \( M \). Let \( \widetilde{B} = p^{-1}(B) \subseteq \widetilde{M} \).

(a) By convention, we also say that \( \widetilde{B} \) is a branched surface in \( \widetilde{M} \).

(b) We’d like to have the fibered neighborhood of \( \widetilde{B} \) be \( \pi_1 \)-equivariant. We call \( \widetilde{N}(B) \) a fibered neighborhood of \( \widetilde{B} \) if there is a fibered neighborhood \( N(B) \) of \( B \) such that \( \widetilde{N}(B) = p^{-1}(N(B)) \), and we will always assume that the \( I \)-fibers of \( N(B) \) are the pull-back of the \( I \)-fibers of \( N(B) \). Let \( \widetilde{\pi} : \widetilde{N}(B) \to \widetilde{B} \) be the map which collapses every \( I \)-fiber of \( \widetilde{N}(B) \) into a point of \( \widetilde{B} \). We also call \( \widetilde{\pi} \) the collapsing map for \( \widetilde{N}(B) \). And we adopt the notations in Notation 2.4 (b), (d), (e) for \( \widetilde{B} \).

2.3. **Blowing-up/down.** Blowing-up/down a leaf of a foliation describes the following operation:

**Definition 2.6 (Blowing-up/down).** Blowing-up a leaf \( \lambda \) of a foliation is to replace it by a product \( \lambda \times I \) foliated with leaves \( \{ \lambda \times \{ t \} \mid t \in I \} \), and its inverse operation is called blowing-down (cf. [Cal4, Example 4.14]). Such a bundle \( \lambda \times I \) is called a pocket of the foliation. The blowing-up operation can be done for a countable collection of disjoint leaves simultaneously, and the blowing-down operation can be done for a countable collection of disjoint pockets simultaneously.

3. **Constructing a branched surface \( B \) of \( M \)**

In this section, we construct a branched surface \( B \) in \( M \) through the following two steps: (1) choosing a fundamental domain \( \Gamma \) of \( M \) in \( \widetilde{M} \) such that \( p(\partial \Gamma) \) is a standard spine (cf. Subsection 3.1), (2) orienting the sectors of \( p(\partial \Gamma) \) through “<” to obtain a branched surface \( B \) in \( M \) (cf. Subsection 3.2).
3.1. **Choosing a fundamental domain \( \Gamma \) of \( M \) in \( \widetilde{M} \).** Our setting for fundamental domains of \( M \) in \( \widetilde{M} \) is as follows:

**Notation 3.1.** (a) We call \( \Gamma \) a *fundamental domain* of \( M \) in \( \widetilde{M} \) if \( \Gamma \subseteq \widetilde{M} \) is a connected, compact, contractible polyhedron with topologically locally flat boundary, \( p |_{\operatorname{Int}(\Gamma)} \) is injective, \( p |_{\Gamma} \) is surjective, and \( \partial \Gamma \) is a finite union of polygons that are identified pairwise by deck transformations.

(b) A compact region \( F \subseteq \partial \Gamma \) is called a *face* of \( \Gamma \) if there is \( h \in G - \{1\} \) such that \( F \) is a component of \( \Gamma \cap t_h(\Gamma) \), and there is no \( g \in G - \{1, h\} \) such that \( F \subseteq \Gamma \cap t_g(\Gamma) \). \( F \) is also called a *common face* of \( \Gamma \) and \( t_h(\Gamma) \).

**Fact 3.2.** \( \widetilde{M} \) is homeomorphic to \( \mathbb{R}^3 \), and every fundamental domain of \( M \) in \( \widetilde{M} \) is homeomorphic to a compact 3-ball.

**Proof.** \( G \) is torsion-free since it is left orderable (cf. [BRW, Proposition 2.1]). The fact follows. \( \square \)

**Fact 3.3.** Let \( \Gamma \) be a fundamental domain of \( M \) in \( \widetilde{M} \). Then every compact set in \( \widetilde{M} \) only meets finitely many members in \( \{t_g(\Gamma) \mid g \in G\} \).

**Proof.** This follows from [Gr, Lemma 1]. \( \square \)

**Notation 3.4.** Let \( \Gamma \) be a fundamental domain of \( M \) in \( \widetilde{M} \). We denote by \( G(\Gamma) \) the set of points in \( p(\partial \Gamma) \) that have no \( \mathbb{R}^2 \)-neighborhood in \( p(\partial \Gamma) \). A point in \( G(\Gamma) \) is called a *vertex* of \( G(\Gamma) \) if it has no \( \mathbb{R}^1 \)-neighborhood in \( G(\Gamma) \). And we call every component of \( G(\Gamma) \setminus \{\text{vertices of } G(\Gamma)\} \) an *edge* of \( G(\Gamma) \). An edge \( e \) of \( G(\Gamma) \) is called *\( k \)-valent* if \( p^{-1}(e) \cap \partial \Gamma \) has \( k \) components. A vertex \( v \) of \( G(\Gamma) \) is called *standard* if \( v \) has a neighborhood in \( p(\partial \Gamma) \) which is homeomorphic to the local model of a double point of a standard spine (cf. the rightmost picture of Figure 1). And we call every component of \( p(\partial \Gamma) \setminus G(\Gamma) \) a *sector* of \( p(\partial \Gamma) \).

**Lemma 3.5.** There exists a fundamental domain \( \Gamma \) of \( M \) in \( \widetilde{M} \) such that \( p(\partial \Gamma) \) is a standard spine.

**Proof.** Let \( \Gamma_0 \) be a fundamental domain of \( M \) in \( \widetilde{M} \). Starting from \( \Gamma_0 \), we repeat the following operation to obtain \( \Gamma_{i+1} \) from \( \Gamma_i \) (\( i \geq 0 \)):

- **(Thicken an edge)** Assume that \( \Gamma_i \) is a fundamental domain of \( M \) in \( \widetilde{M} \), and there is an edge \( e \) of \( G(\Gamma_i) \) which is not 3-valent. Let \( N_e = e \times B^2 \) be a sufficiently small embedded disk bundle over \( e \) in \( M \) such that \( e = N_e \cap G(\Gamma_i) \). Let \( \bar{e} \) be a component of \( p^{-1}(e) \) such that \( \bar{e} \subseteq \partial \Gamma_i \). Let \( \bar{N}_e \) be the component of \( p^{-1}(N_e) \) such that \( \bar{e} \subseteq \bar{N}_e \). Let

\[
\Gamma_{i+1} = \Gamma_i - p^{-1}(\bar{N}_e) \cup \bar{N}_e
\]

(see Figure 4 for the change). Then \( \Gamma_{i+1} \) is still a fundamental domain of \( M \) in \( \widetilde{M} \), and every new edge of \( G(\Gamma_{i+1}) \) produced in this step has valency 3. Thus,

\[
\{|s \text{ is an edge of } G(\Gamma_{i+1}), s \text{ is not 3-valent}| \} < \{|s \text{ is an edge of } G(\Gamma_i), s \text{ is not 3-valent}| \}.
\]

We can repeat the above operation inductively, until we obtain \( \Gamma_k \) (\( k \in \mathbb{N} \)) such that all edges of \( G(\Gamma_k) \) have valency 3. Starting from \( \Gamma_k \), we repeat the following operation to obtain \( \Gamma_{i+1} \) from \( \Gamma_i \) (\( i \geq k \)):

- **(Thicken a vertex)** Assume that \( \Gamma_i \) is a fundamental domain of \( M \) in \( \widetilde{M} \) such that every edge of \( G(\Gamma_i) \) is 3-valent, and there is a vertex \( v \) of \( G(\Gamma_i) \) which is not standard. Let \( N_v \) be a sufficiently small closed regular neighborhood of \( v \) in \( M \) so that: \( N_v \cap p(\partial \Gamma_i) \) is a closed neighborhood of \( v \) in \( p(\partial \Gamma_i) \) that does not contain any other vertices of \( G(\Gamma_i) \), and \( N_v \cap G(\Gamma_i) \) is connected. Let \( \bar{v} \in p^{-1}(v) \) such that \( \bar{v} \in \partial \Gamma_i \). Let \( \bar{N}_v \) be the component of \( p^{-1}(N_v) \) such that \( \bar{v} \in \bar{N}_v \). Let

\[
\Gamma_{i+1} = \Gamma_i - p^{-1}(\bar{N}_v) \cup \bar{N}_v
\]

(see Figure 5 for the change). Then \( \Gamma_{i+1} \) is still a fundamental domain of \( M \) in \( \widetilde{M} \). Each vertex of \( G(\Gamma_{i+1}) \) that lies on \( \partial N_v \) is the intersection of \( \partial N_v \) and an edge of \( G(\Gamma_i) \) (which is necessarily
Figure 4. (a) is the picture near $\tilde{e}$ before the thickening, and (b) is the picture near $\tilde{e}$ after the thickening. Here, the two grey faces in (a) are contained in $\partial \Gamma_i$, and the grey faces in (b) are contained in $\partial \Gamma_{i+1}$.

Figure 5. (a) is the picture near $\tilde{v}$ before the thickening, and (b) is the picture near $\tilde{v}$ after the thickening. Here, the green faces in (a) are contained in $\partial \Gamma_i$, the green faces in (b) are contained in $\partial \Gamma_{i+1}$, and the red curves in (b) are projected to the edges of $G(\Gamma_{i+1})$ which are produced or changed in this step.

3-valent), so it is standard. And each edge of $G(\Gamma_{i+1})$ that lies on $\partial N_v$ is the intersection of $\partial N_v$ and a sector of $p(\partial \Gamma_i)$, so it has valency 3. Hence

$$|\{\text{vertices of } G(\Gamma_{i+1}) \text{ which are not standard}\}| < |\{\text{vertices of } G(\Gamma_i) \text{ which are not standard}\}|,$$

and all new edges produced in this process still have valency 3.

We can repeat this operation inductively. At last, we can obtain a fundamental domain $\Gamma_n$ such that all vertices of $G(\Gamma_n)$ are standard, and all edges of $G(\Gamma_n)$ are still 3-valent. Then $p(\partial \Gamma_n)$ is a standard spine. We can accomplish the proof of Lemma 3.5 by choosing $\Gamma = \Gamma_n$. $\square$
 Guaranteed by Lemma 3.5, we can choose a fundamental domain $\Gamma$ of $M$ in $\widetilde{M}$ such that $p(\partial \Gamma)$ is a standard spine. Without loss of generality, we assume that the base point $\widetilde{x}$ (cf. Notation 2.1 (a)) is contained in $Int(\Gamma)$. And we assume that every member of $\{t_g(\Gamma) \mid g \in G\}$ has an orientation induced from the orientation on $\widetilde{M}$.

3.2. Orienting the sectors of $p(\partial \Gamma)$ to obtain a branched surface of $M$. Recall from Notation 3.1 (b), a common face of $\Gamma$ and $t_h(\Gamma)$ ($h \in G - \{1\}$) is a component of $\Gamma \cap t_h(\Gamma)$ such that there is no $g \in G - \{1, h\}$ with $F \subseteq \Gamma \cap t_g(\Gamma)$.

**Definition 3.6.** Suppose that $F$ is a common face of $\Gamma$ and $t_h(\Gamma)$ for some $h \in G - \{1\}$. We call $F$ a positive face (resp. negative face) of $\Gamma$ if $h > 1$ (resp. $h < 1$). For every $r \in G$ and every face $J$ of the fundamental domain $t_r(\Gamma)$, there is a face $K$ of $\Gamma$ such that $t_r(K) = J$. We call $J$ a positive face (resp. negative face) of $t_r(\Gamma)$ if $K$ is a positive face (resp. negative face) of $\Gamma$.

**Remark 3.7.** The covering map $p : \widetilde{M} \to M$ takes \{faces of $\Gamma$\} to \{sectors of $p(\partial \Gamma)$\}. In particular, \{positive faces of $\Gamma$\} is in one-to-one correspondence with \{sectors of $p(\partial \Gamma)$\}.

For every face in every member of $\{t_g(\Gamma)\}$, “<” determines if it is positive/negative:

**Lemma 3.8.** Suppose that $F$ is a common face of two fundamental domains $t_h(\Gamma)$ and $t_g(\Gamma)$ ($h, g \in G$ and $h \neq g$). Without loss of generality, we assume that $h > g$. Then $F$ is a positive face of $t_g(\Gamma)$ and is a negative face of $t_h(\Gamma)$.

**Proof.** $t_g^{-1}(F)$ is a common face of $\Gamma$ and $t_g^{-1}h(\Gamma)$. We have $g^{-1}h > 1$ since $h > g$. Then $t_g^{-1}(F)$ is a positive face of $\Gamma$, which implies that $F$ is a positive face of $t_g(\Gamma)$.

Similarly, $t_h^{-1}(F)$ is a common face of $\Gamma$ and $t_h^{-1}g(\Gamma)$, and $h^{-1}g < 1$. So $t_h^{-1}(F)$ is a negative face of $\Gamma$, and then $F$ is a negative face of $t_h(\Gamma)$.

Then we orient the faces of $\{t_g(\Gamma)\}$ in a $\pi_1$-equivariant way, which induces a co-orientation on the sectors of $p(\partial \Gamma)$:

**Definition 3.9.** (a) For each $h \in G$ and each face $F$ of $t_h(\Gamma)$, we assign $F$ positive orientation (resp. negative orientation) with respect to the orientation on $t_h(\Gamma)$ if $F$ is a positive face (resp. negative face) of $t_h(\Gamma)$. By Lemma 3.8, every face is a positive face of a member of $\{t_g(\Gamma)\}_{g \in G}$ that contains it, and it is a negative face of the other member of $\{t_g(\Gamma)\}_{g \in G}$ that contains it. Hence the assignment is well-defined.

(b) By Definition 3.6, the assignment in (a) is equivariant under deck transformations. So it induces a co-orientation on the sectors of the standard spine $p(\partial \Gamma)$. We assign $p(\partial \Gamma)$ the cusp directions given by this co-orientation.

At last, we verify that $p(\partial \Gamma)$ is a branched surface.

**Lemma 3.10.** $p(\partial \Gamma)$ is a branched surface.

**Proof.** As shown in Figure 6, the orientations on faces of $\{t_g(\Gamma)\}_{g \in G}$ give well-defined cusp directions to the set of points in $\bigcup_{g \in G} t_g(\partial \Gamma)$ that possess no Euclidean neighborhood. So $\bigcup_{g \in G} t_g(\partial \Gamma)$ is locally modeled as Figure 2. Correspondently, $p(\partial \Gamma)$ is locally modeled as Figure 2, and therefore $p(\partial \Gamma)$ is a branched surface. \qed

We denote this branched surface by $B = p(\partial \Gamma)$. We choose a fibered neighborhood $N(B)$ of $B$. Let $\overline{B} = p^{-1}(B)$. As in Remark 2.5, $\overline{B}$ is a branched surface in $\overline{M}$, and $\overline{B}$ has a fibered neighborhood $\overline{N}(B) = p^{-1}(N(B))$ with $\pi_1$-equivariant interval fibers. We denote by $\overline{\pi} : \overline{N}(B) \to \overline{B}$ the collapsing map for $N(B)$.
Definition 4.1 (Denjoy). Suppose that \( \{r_g : \mathbb{R} \to \mathbb{R} \mid g \in G\} \) is an effective action of \( G \) on \( \mathbb{R} \) such that a point \( v \in \mathbb{R} \) has trivial stabilizer. We blow-up the countable union of points \( \{r_g(v)\}_{g \in G} \) in \( \mathbb{R} \), i.e. replacing every point \( r_g(v) \) \( (g \in G) \) by a blown-up interval \( r_g(v) \times I \). We consider the complement of \( \bigcup_{g \in G}(r_g(v) \times I) \) in \( \mathbb{R} \) to be the same set as \( \mathbb{R} - \bigcup_{g \in G}\{r_g(v)\} \). For each \( h \in G \), let \( s_h : \mathbb{R} \to \mathbb{R} \) be the map such that (1) the restriction of \( s_h \) to \( \mathbb{R} - \bigcup_{g \in G}(r_g(v) \times I) \) is equal to the restriction of \( r_h \) to \( \mathbb{R} - \bigcup_{g \in G}\{r_g(v)\} \), (2) \( s_h \) takes \( r_g(v) \times \{t\} \) to \( r_{h_g}(v) \times \{t\} \) for all \( g, h \in G \), \( t \in I \). Then \( \{s_g : \mathbb{R} \to \mathbb{R} \mid g \in G\} \) is an action of \( G \) on \( \mathbb{R} \). We call it the blowing-up of \( \{r_g : \mathbb{R} \to \mathbb{R} \mid g \in G\} \) at \( v \).

Under the conditions of Definition 4.1, we will always assume \( r_g(v) \times \{0\} < r_g(v) \times \{1\} \) for every \( g \in G \). In the following, we define an action of \( G \) on \( \mathbb{R} \) which is compatible with the left-invariant order “<” on \( G \):

**Proposition 4.2.** There exists an effective action \( \{\rho_g' : \mathbb{R} \to \mathbb{R} \mid g \in G\} \) of \( G \) on \( \mathbb{R} \) and \( N \in \mathbb{R} \) such that: for arbitrary \( g, h \in G \), \( \rho_h'(N) > \rho_g'(N) \) if and only if \( h > g \).

**Proof.** This follows from [Cal4, Lemma 2.43] and [Cal4, Lemma 2.43, Remark]. Precisely, through the process in [Cal4, Lemma 2.43], there is an effective action of \( G \) on \( \mathbb{R} \) with the following additional properties:

- There is an injective map \( e : G \to \mathbb{R} \) such that \( e(g) < e(h) \) for any \( g, h \in G \) with \( g < h \).

- For any \( g, h \in G \), the action \( g : \mathbb{R} \to \mathbb{R} \) transforms \( e(h) \) to \( e(gh) \).

We choose \( \{\rho_g' : \mathbb{R} \to \mathbb{R} \mid g \in G\} \) to be this action and choose \( N = e(1) \). Then the properties of the proposition are satisfied.

**Definition 4.3.** Let \( \{\rho_g : \mathbb{R} \to \mathbb{R} \mid g \in G\} \) be the blowing-up of \( \{\rho'_g : \mathbb{R} \to \mathbb{R} \mid g \in G\} \) at \( N \). We denote by \( N \times I \) the blown-up interval at \( N \). Let \( N_0 = N \times \{0\} \), \( N_1 = N \times \{1\} \).

Note that (1) \( N_1 > N_0 \), (2) \( N_0, N_1 \) have trivial stabilizer under the action \( \{\rho_g : \mathbb{R} \to \mathbb{R} \mid g \in G\} \), (3) for arbitrary \( g, h \in G \) with \( h > g \) and arbitrary \( i, j \in \{0, 1\} \), \( \rho_h(N_i) > \rho_g(N_j) \).
Definition 4.4. (a) Let $\text{Sec}(\tilde{B}) = \{\text{branch sectors of } \tilde{B}\}$.

(b) Let $F \in \text{Sec}(\tilde{B})$. We denote by $s, r \in G$ for which $F$ is a common face of $t_s(\Gamma)$, $t_r(\Gamma)$ and $s < r$. Let $I_F = [\rho_s(N_1), \rho_r(N_0)]$.

Fact 4.5. Let $F \in \text{Sec}(\tilde{B})$, $g \in G$. Then $I_{t_g(F)} = \rho_g(I_F)$.

Proof. We denote by $s, r \in G$ such that $F$ is a common face of $t_s(\Gamma)$, $t_r(\Gamma)$ and $s < r$. Then $t_g(F)$ is a common face of $t_{gs}(\Gamma)$, $t_{gr}(\Gamma)$, and $gr > gs$. By Definition 4.4 (b), we have

$I_{t_g(F)} = [\rho_{gs}(N_1), \rho_{gr}(N_0)] = \rho_g([\rho_s(N_1), \rho_r(N_0)]) = \rho_g(I_F)$.

Recall from Subsection 3.2, a positive face of $\Gamma$ is a common face of $\Gamma$ and $t_h(\Gamma)$ for some $h > 1$.

Definition 4.6. (a) For each positive face $F_0$ of $\Gamma$, we choose an orientation-preserving homeomorphism

$h_{F_0} : I \to I_{F_0}$.

(b) For each $F \in \text{Sec}(\tilde{B})$, there is $r \in G$ and a positive face $F_0$ of $\Gamma$ such that $F = t_r(F_0)$. Let

$h_F = \rho_r \circ h_{F_0} : I \to I_F$.

Fact 4.7. Let $F \in \text{Sec}(\tilde{B})$, $g \in G$. Then $h_{t_g(F)} = \rho_g \circ h_F$.

Proof. There exists $r \in G$ and a positive face $F_0$ of $\Gamma$ such that $F = t_r(F_0)$. Then

$h_{t_g(F)} = h_{t_{gr}(F_0)} = \rho_{gr} \circ h_{F_0} = \rho_g \circ \rho_r \circ h_{F_0} = \rho_g \circ h_F$.

Lemma 4.8. Let $e$ be an edge of the branch locus of $\tilde{B}$. Let $F, X, Y \in \text{Sec}(\tilde{B})$ be the three distinct branch sectors of $\tilde{B}$ such that $e$ is contained in $F, X, Y$ and the cusp direction at $e$ points out of $X, Y$ and points in $F$. We denote by $u, v \in G$ such that (1) $F$ is a common face between $t_u(\Gamma), t_v(\Gamma)$, (2) $Y \subseteq t_u(\Gamma)$, $X \subseteq t_v(\Gamma)$ (cf. Figure 7 (a)). And we assume without loss of generality that $u > v$. Then:

(a) $I_X, I_Y \subseteq I_F$ and $I_X \cap I_Y = \emptyset$.

(b) $\min I_F = \min I_X$, $\max I_F = \max I_Y$.

Proof. There exists exactly one $r \in G$ such that $X, Y \subseteq t_r(\Gamma)$ (then $e \subseteq t_r(\Gamma)$, and the cusp direction at $e$ points out of $t_r(\Gamma)$, see Figure 7 (a)). Then $u > r > v$. By Definition 4.4 (b), we have $I_F = [\rho_v(N_1), \rho_u(N_0)]$, $I_X = [\rho_v(N_1), \rho_r(N_0)]$, $I_Y = [\rho_r(N_1), \rho_u(N_0)]$. Thus

$\min I_F = \min I_X < \min I_Y < \min I_Y = \max I_F$.

So (a), (b) holds.

Definition 4.9. Assume that the conditions of Lemma 4.8 hold. Then $I_X, I_Y \subseteq I_F$. Let

$\delta_{(X,F)} = h_{F}^{-1} \circ h_{X} : I \to I$,

$\delta_{(Y,F)} = h_{F}^{-1} \circ h_{Y} : I \to I$.

Lemma 4.10. Assume that the conditions of Lemma 4.8 hold. Then $\delta_{(X,F)}(I) \cap \delta_{(Y,F)}(I) = \emptyset$, $\delta_{(X,F)}(0) = 0$, $\delta_{(Y,F)}(1) = 1$.

Proof. Follows from Lemma 4.7 directly (cf. Figure 7 (b)).

Next, we show that the maps defined in Definition 4.9 are $\pi_1$-equivariant:

Lemma 4.11. Assume that the conditions of Lemma 4.8 hold. For each $g \in G$, we have $\delta_{(X,F)} = \delta_{(t_g(X),t_g(F))}$ and $\delta_{(Y,F)} = \delta_{(t_g(Y),t_g(F))}$.
If two distinct points in \( F \in \text{Sec}(\widetilde{B}) \) are glued together by Construction 4.12, then their images under \( i_0 \) are equal.
Figure 8. The gluing between $X \times I, Y \times I$ and $F \times I$. We glue $e_X \times I, e_Y \times I$ into $e_F \times I$ and make $\{X \times \{t\}\}_{t \in I}$, $\{Y \times \{t\}\}_{t \in I}$ attach to $\{F \times \{t\}\}_{t \in I}$.

**Proof.** Under the condition of Construction 4.12, we suppose that a segment $e_X \times \{u\} \subseteq X \times I$ ($u \in I$) is glued with a segment $e_F \times \{v\} \subseteq F \times I$ ($v \in I$). Then $v = \delta_{(X,F)}(u)$. By Definition 4.9, we have $h_X(u) = h_F(\delta_{(X,F)}(u)) = h_F(v)$. It follows that $i_0$ takes all points in $e_X \times \{u\}$ and $e_F \times \{v\}$ to the same value in $\mathbb{R}$. Thus, any two points in $\bigsqcup_{F \in Sec(\widetilde{B})}(F \times I)$ are glued together only if they have the same image under $i_0$. 

Let $\mathcal{N}$ denote the resulting space obtained from gluing $\bigsqcup_{F \in Sec(\widetilde{B})}(F \times I)$ as in Construction 4.12. Notice that the collection of horizontal sheets $\{F \times \{t\} \mid F \in Sec(\widetilde{B}), t \in I\}$ are attached edge-by-edge in the gluing. So Construction 4.12 also induces a gluing for $\{F \times \{t\} \mid F \in Sec(\widetilde{B}), t \in I\}$. Let $\mathcal{E}$ be the resulting space obtained from the gluing for $\{F \times \{t\} \mid F \in Sec(\widetilde{B}), t \in I\}$. We have:

**Fact 4.15.** $\mathcal{N}$ is homeomorphic to $\widetilde{N(B)}$, and $\mathcal{E}$ is a foliation of $\mathcal{N}$.

**Proof.** Notice that the gluing $\delta_{(X,F)}: e_X \times I \rightarrow e_F \times I$ as given in Construction 4.12 homeomorphically embeds $e_X \times I$ into $e_F \times I$, and it glues the horizontal sheets edge-by-edge. Next, we focus on the gluings along the interval fibers at the double points of the branch locus of $\widetilde{B}$.

Let $q$ be a double point of the branch locus of $\widetilde{B}$. We illustrate that Construction 4.12 gives compatible gluings between distinct copies of $\{q\} \times I$ in $\bigsqcup_{F \in Sec(\widetilde{B})}(F \times I)$. Let $X, Y \in Sec(\widetilde{B})$ such that $q \in X, Y$. There are two distinct copies of $\{q\} \times I$ in $X \times I, Y \times I$ respectively. We denote them by $\{q_X\} \times I \subseteq X \times I$ and $\{q_Y\} \times I \subseteq Y \times I$. By Fact 4.14, if $a \in \{q_X\} \times I, b \in \{q_Y\} \times I$ are glued together, then $i_0(a)$ must be equal to $i_0(b)$. Notice that $i_0|_{\{q_X\} \times I}, i_0|_{\{q_Y\} \times I}$ are embeddings from $\{q_X\} \times I, \{q_Y\} \times I$ to $\mathbb{R}$. So the gluing between $\{q_X\} \times I$ and $\{q_Y\} \times I$ is a homeomorphism between their subintervals, and this gluing is compatible with other gluings between distinct copies of $\{q\} \times I$ in $\bigsqcup_{F \in Sec(\widetilde{B})}(F \times I)$.

Thus, for every point $t$ in an interval fiber at a double point, the horizontal sheets containing $t$ are glued horizontally at $t$. So $\mathcal{N}$ is homeomorphic to $\widetilde{N(B)}$, and $\mathcal{E}$ is a foliation of $\mathcal{N}$. 

We can regard $\mathcal{N}$ as $\widetilde{N(B)}$. Combining Lemma 4.11 and Construction 4.12, the gluing maps for $\{F \times \{t\} \mid F \in Sec(\widetilde{B}), t \in I\}$ are $\pi_1$-equivariant. So $\mathcal{E}$ is a $\pi_1$-equivariant foliation of $\widetilde{N(B)}$. Let $M_0 = N(B) \cong M - Int(B^3)$. The covering map $p: \tilde{M} \rightarrow M$ descends $\mathcal{E}$ to a foliation of $M_0$. We
Lemma 4.16. (a) $\mathcal{F}$ contains no torus leaf that bounds a solid torus (then $\mathcal{F}$ is Reebless).

(b) There is a simple closed curve in $M$ that is co-orientably transverse to $\mathcal{F}$ and intersects every leaf of $\mathcal{F}$.

The proof of (a). Notice that every leaf of $\mathcal{F}$ is carried by $B$. Since $B$ does not separate $M$, $B$ carries no torus leaf that bounds a solid torus.

The proof of (b). Let $q \in \Gamma - \widetilde{N}(B)$. We can choose a path $\tau$ in $\widetilde{M}$ such that: (1) $\tau$ starts at $q$, (2) for each face $F$ of $\Gamma$, $\tau$ meets at least one member of $\{t_g(F)\}_{g \in G}$, (3) if $\tau$ meets some $F \in \text{Sec}(\widetilde{B})$, then $\tau$ intersects $F$ transversely, and the direction of $\tau$ is consistent with the orientation on $F$ (as a branch sector of $\widetilde{B}$), (4) $\tau$ ends at some point in $\{t_g(q)\}_{g \in G - \{1\}}$. We can isotope $\tau$ (relative to its endpoints) so that $\tau$ intersects $\widetilde{F}$ transversely and $p(\tau)$ is a simple closed curve in $M$. Then $p(\tau)$ is the curve as required in (b).

Next, we show that $\mathcal{F}$ has a transverse $(\pi_1(M_0), \mathbb{R})$ structure. Let $L$ be the leaf space of $\mathcal{F}$. In the following, we will not distinguish any point of $L$ with its corresponding leaf in $\mathcal{F}$.

Recall that we define the map $i_0 : \bigsqcup_{\mathcal{F} \in \text{Sec}(\widetilde{B})} (F \times I) \rightarrow \mathbb{R}$ in Definition 4.13.

Definition 4.17. Let $i_{\text{des}} : L \rightarrow \mathbb{R}$ be the map such that: for each $\bar{\lambda} \in L$, we choose an arbitrary point $v \in \bar{\lambda}$ and define $i_{\text{des}}(\bar{\lambda}) = i_0(v)$.

By Fact 4.14, $i_{\text{des}} : L \rightarrow \mathbb{R}$ is well-defined. And it’s clear that $i_{\text{des}}$ is an immersion.

Next, we show that $i_{\text{des}}$ descends the $\pi_1$-action on $L$ to the action $\{\rho_g : \mathbb{R} \rightarrow \mathbb{R} \mid g \in G\}$ (cf. Definition 4.3). Recall that the $\pi_1$-action on $L$ is induced from the deck transformations on $\widetilde{M}$. For each $g \in G$, we still denote by $t_g : L \rightarrow L$ the transformation on $L$ induced from the deck transformation $t_g : \widetilde{M} \rightarrow \widetilde{M}$, which takes every leaf $\bar{\lambda} \in L$ to $t_g(\bar{\lambda}) \in L$.

Lemma 4.18. $i_{\text{des}}$ descends $\{t_g : L \rightarrow L \mid g \in G\}$ to $\{\rho_g : \mathbb{R} \rightarrow \mathbb{R} \mid g \in G\}$.

Proof. Let $\bar{\lambda} \in L$, $g \in G$. Suppose that $X \times \{s\} (X \in \text{Sec}(\widetilde{B}), s \in I)$ is a horizontal sheet contained in $\bar{\lambda}$. By Definition 4.17 and Definition 4.13, we have $i_{\text{des}}(\bar{\lambda}) = i_0(X \times \{s\}) = h_X(s)$. Combined with Fact 4.7, we have

$$i_{\text{des}}(t_g(\bar{\lambda})) = i_0(t_g(X) \times \{s\}) = h_{t_g(X)}(s) = \rho_g(h_X(s)) = \rho_g(i_{\text{des}}(\bar{\lambda})).$$

It follows that $i_{\text{des}} \circ t_g = \rho_g \circ i_{\text{des}}$, i.e. the following diagram commutes:

$$\begin{array}{ccc}
L & \xrightarrow{i_{\text{des}}} & \mathbb{R} \\
\downarrow t_g & & \downarrow \rho_g \\
L & \xrightarrow{i_{\text{des}}} & \mathbb{R}
\end{array}$$

We have shown that $\mathcal{F}$ satisfies Theorem 1.2 (1), (2). It only remains to show Theorem 1.2 (3).

Let $V$ be a component of the vertical boundary of $\widetilde{N}(B)$ (then $V$ is an annulus). Suppose that $\bar{\lambda}$ is a leaf of $\mathcal{F}$ that meets $V$. Notice that $i_{\text{des}}$ projects every transversal (with endpoints) of $\mathcal{F}$ to a closed interval in $\mathbb{R}$. Thus the two endpoints of any transversal of $\mathcal{F}$ must be contained in distinct leaves of $\mathcal{F}$. If $\bar{\lambda} \cap V$ is not a circle, then there must be a transversal of $\mathcal{F}$ in $V$ such that its two endpoints are both contained in $\bar{\lambda} \cap V$. This is a contradiction. So $\bar{\lambda} \cap V$ is a circle. Hence every transverse intersection component of some leaf of $\mathcal{F}$ and $\partial M_0$ is a circle. So Theorem 1.2 (3) holds.
Remark 4.19. Let \( \tau : I \to M \) be a loop in \( M \) such that: (1) \( \tau \) is positively transverse to \( F \), (2) \( \tau(I) \) has nonempty intersection with some leaves of \( F \). Let \( \bar{\tau} : I \to \bar{M} \) be a lift of \( \tau \) in \( \bar{M} \). Then \( \bar{\tau}(I) \) meets at least two distinct leaves of \( \bar{F} \). We prove \( i_0(\bar{\tau}(1)) > i_0(\bar{\tau}(0)) \) in the following.

It’s clear that \( i_0(\bar{\tau}(t_1)) < i_0(\bar{\tau}(t_2)) \) if \( \bar{\tau}([t_1, t_2]) \subseteq \bar{N}(B) \) (then \( \bar{\tau}([t_1, t_2]) \) is a positively oriented transversal of \( \bar{F} \)). Now assume that \([t_1, t_2]\) is a subinterval of \( I \) with \( \bar{\tau}([t_1, t_2]) \cap \bar{N}(B) = \{\bar{\tau}(t_1), \bar{\tau}(t_2)\} \). We prove \( i_0(\bar{\tau}(t_1)) < i_0(\bar{\tau}(t_2)) \) as follows. Without loss of generality, we can assume \( \bar{\tau}([t_1, t_2]) \subseteq \Gamma \). Then there is a negative face \( E \) of \( \Gamma \) and a positive face \( F \) of \( \Gamma \) such that \( \bar{\tau}(t_1) \in E \times \{1\}, \bar{\tau}(t_2) \in F \times \{0\} \). By Definition 4.4, Definition 4.6 and Definition 4.13, we have \( i_0(E \times \{1\}) = h_E(1) = h_E(0) < i_0(F \times \{0\}) \). It follows that \( i_0(\bar{\tau}(t_1)) < i_0(\bar{\tau}(t_2)) \).

Thus \( i_0(\bar{\tau}(1)) > i_0(\bar{\tau}(0)) \), and therefore \( \bar{\tau}(0), \bar{\tau}(1) \) are distinct, which implies that \( \tau(I) \) is an essential simple closed curve in \( M \).

4.3. Testing if \( F \) can extend to a taut foliation of \( M \). We already have a foliation \( F \) of \( M_0 = \bar{N}(B) \). Now we consider: when can \( F \) extend to a taut foliation of \( M \)?

Proposition 4.20. \( F \) can extend to a taut foliation in \( M \) if and only if either of the following equivalent conditions holds:

1. \( \partial_0 \bar{N}(B) \) is connected (then it is an annulus).
2. The union of positive faces of \( \Gamma \) is a single 2-disk. Then the union of negative faces of \( \Gamma \) is also a single 2-disk (see Figure 9).

Proof. We prove (1) at first. Let \( \mathcal{G} = M - \text{Int}(\bar{N}(B)) \). Then \( \mathcal{G} \) is a compact 3-ball, and each component of \( \partial_0 \bar{N}(B) \) is an annulus that lies on \( \partial \mathcal{G} \). It’s clear that \( F \) can extend to a foliation of \( M \) if and only if \( \partial_0 \bar{N}(B) \) is a connected annulus (then the extending foliation is obtained by filling \( \mathcal{G} \) with horizontal disks). Now suppose that \( F \) can extend to a foliation in \( M \). Similar to Lemma 4.16, we can choose a simple closed curve transverse to the extending foliation that meets every leaf. Hence the extending foliation is taut.

Now we show (1) \( \iff \) (2). Let \( \bar{\mathcal{G}} \) be the component of \( p^{-1}(\mathcal{G}) \) such that \( \bar{\mathcal{G}} \subseteq \Gamma \). Then the map \( \bar{\pi} : \bar{N}(B) \to \bar{B} \) collapses \( \partial_0 \bar{N}(B) \cap \partial \bar{\mathcal{G}} \) into the cusp curves on \( \partial \Gamma \). Thus \( \partial_0 \bar{N}(B) \) is connected if and only if there is exactly one cusp curve on \( \partial \Gamma \). Recall from Definition 3.9, every positive face (resp. negative face) of \( \Gamma \) has positive orientation (resp. negative orientation) with respect to \( \Gamma \), as a branch sector of \( \bar{B} \). Thus

\[
\bigcup_{l \text{ is a cusp curve on } \partial \Gamma} l = \partial \left( \bigcup_{F \text{ is a positive face of } \Gamma} F \right),
\]
i.e. the union of cusp curves on \( \partial \Gamma \) is the boundary of the union of positive faces of \( \Gamma \). It follows that (1) \( \iff \) (2).

**Lemma 4.21.** Suppose that \( \mathcal{F} \) can extend to a taut foliation \( \mathcal{F}_1 \) of \( M \). Then \( \mathcal{F}_1 \) has a transverse \((\pi_1(M), \mathbb{R})\) structure.

**Proof.** Induced from the map \( i_{\text{des}} : L \to \mathbb{R} \) (cf. Lemma 4.18) directly. \( \square \)

As in Definition 1.6, \( (\langle , \Gamma \rangle) \) is an order-domain pair of \( M \). We call \( \mathcal{F} \) a resulting foliation of \( (\langle , \Gamma \rangle) \). We complete the proof of Theorem 1.8 in the remainder of this subsection:

**Theorem 1.8.** (a) \( \mathcal{F} \) is uniquely determined by \( (\langle , \Gamma \rangle) \), up to blowing-up/down.

(b) \( \mathcal{F} \) can extend to a taut foliation of \( M \) if and only if the union of positive faces of \( \Gamma \) is a single 2-disk. Moreover, if \( \mathcal{F} \) can extend to a taut foliation \( \mathcal{F}_1 \) of \( M \), then \( \mathcal{F}_1 \) has a transverse \((\pi_1(M), \mathbb{R})\) structure.

(c) \( \mathcal{F} \) can extend to an \( \mathbb{R} \)-covered foliation of \( M \) if and only if the union of positive faces of \( \Gamma \) is a single 2-disk, and both of \( \bigcup_{g \in G, g < 1} t_g(\Gamma) \), \( \bigcup_{g \in G, g \geq 1} t_g(\Gamma) \) are connected.

To complete Theorem 1.8, it remains to discuss the following: (1) the case where \( \mathcal{F} \) can extend to an \( \mathbb{R} \)-covered foliation of \( M \) (cf. Remark 4.22), (2) \( \mathcal{F} \) is uniquely determined by \( (\langle , \Gamma \rangle) \), up to blowing-up/down (cf. Remark 4.23).

**Remark 4.22.** Suppose that \( \mathcal{F} \) can extend to a taut foliation \( \mathcal{F}_1 \) in \( M \). \( i_{\text{des}} \) takes some leaves of \( \mathcal{F} \) to the point \( N_0 \in \mathbb{R} \) as chosen in Definition 4.3. Notice that the \( \mathcal{F}_1 \) is \( \mathbb{R} \)-covered if and only if \( i_{\text{des}}^{-1}(N_0) \) is a single leaf of \( \mathcal{F} \). Through the proof of Lemma 4.8, we can observe that \( \tilde{\pi} : \mathcal{N}(B) \to \mathcal{B} \) takes \( i_{\text{des}}^{-1}(N_0) \) to \( \partial \bigcup_{g \in G, g < 1} t_g(\Gamma) \). Thus, \( i_{\text{des}}^{-1}(N_0) \) is a single leaf if and only if \( \partial \bigcup_{g \in G, g < 1} t_g(\Gamma) \) is connected, i.e. both of \( \bigcup_{g \in G, g < 1} t_g(\Gamma) \), \( \bigcup_{g \in G, g \geq 1} t_g(\Gamma) \) are connected. Therefore, \( \mathcal{F} \) can extend to an \( \mathbb{R} \)-covered foliation of \( M \) if and only if Proposition 4.20 holds and both of \( \bigcup_{g \in G, g < 1} t_g(\Gamma) \), \( \bigcup_{g \in G, g \geq 1} t_g(\Gamma) \) are connected.

**Remark 4.23.** In our construction, \( \mathcal{F} \) only depends on the following information:

1. The branched surface \( B \) as given in Subsection 3.2.
2. The action \( \{\rho_g : \mathbb{R} \to \mathbb{R} \mid g \in G\} \) and the closed interval \( [N_0, N_1] \subseteq \mathbb{R} \) as given in Definition 4.3, which have following property: for any \( g, h \in G \) with \( g < h \) and \( i, j \in \{0, 1\} \), \( \rho_g(N_i) < \rho_h(N_j) \).

Notice that \( B \) is homeomorphic to \( p(\partial \Gamma) \) and has a co-orientation on its branch sectors which is determined by “\(<\)”. Next, we explain that \( \{\rho_g : \mathbb{R} \to \mathbb{R} \mid g \in G\} \) is determined by “\(<\)”, up to blowing-up/down and conjugation by \( \text{Homeo}_+(\mathbb{R}) \).

Let \( \mathcal{U} \) be the set of components of \( \mathbb{R} - \bigcup_{g \in G} \rho_g([N_0, N_1]) \) of the forms \( [a, b) \), \((a, b) \), \([a, b] \). Notice that all elements of \( \mathcal{U} \) have disjoint closure, and \( \mathcal{U} \) is a countable set (since \( \mathbb{R} \) contains at most countably many disjoint intervals). We blow-down \( \bigcup_{j \in \mathcal{U}} \mathcal{J} \) in \( \mathbb{R} \). Notice that \( \mathcal{U} \) is invariant under the action \( \{\rho_g \mid g \in G\} \), so this blowing-down induces a new action from \( \{\rho_g \mid g \in G\} \). We denote the induced action by \( \{\rho_g^\ast \mid g \in G\} \). We claim that \( \{\rho_g^\ast \mid g \in G\} \) is uniquely determined by “\(<\)”, up to conjugacy in \( \text{Homeo}_+(\mathbb{R}) \). To see this, we describe \( \mathbb{R} - \bigcup_{g \in G} \rho_g^\ast([N_0, N_1]) \) as follows.

Let \( A = \mathbb{R} - \bigcup_{g \in G} \rho_g^\ast([N_0, N_1]) \). Let \( \Omega \) be the set of \( C \subseteq G \) (\( C \neq \emptyset, G \)) such that: if \( h \in C \), then \( g \in C \) for all \( g \in G \) with \( g < h \). Then there is a well-defined map \( f : A \to \Omega \) such that \( f(t) = \{g \in G \mid \rho_g^\ast(N_0) < t\} \) for all \( t \in A \). For arbitrary \( C \subseteq \Omega \), we have the following observation:

1. Assume that both of max \( C \), min \((G - C) \) exist. Then \( f^{-1}(C) \) is an open interval.
2. Assume that exactly one of max \( C \), min \((G - C) \) exists. Then \( f^{-1}(C) \) does not exist (otherwise, \( f^{-1}(C) \) is an interval of the form either \([a, b] \) or \((a, b) \)).
3. Assume that both of max \( C \), min \((G - C) \) do not exist. Then \( f^{-1}(C) \) is a single point (otherwise, \( f^{-1}(C) \) is an interval of the form \([a, b] \)).

By the above discussions, \( A \) is determined by \( \Omega \) (and therefore is determined by “\(<\)”), up to \( \text{Homeo}_+(\mathbb{R}) \). Thus, \( \{\rho_g^\ast \mid g \in G\} \) is uniquely determined by “\(<\)”, up to conjugacy in \( \text{Homeo}_+(\mathbb{R}) \).
Figure 10. (a), (b): Suppose that $S$ is a torus. Then the trivalent graph $p(\partial \Gamma)$ has two 3-valent vertices exactly. We deform $p(\partial \Gamma)$ to the train track $\tau$, then the boundary of $S \setminus \tau$ has two cusps. (c), (d): Suppose that $S$ is a closed orientable surface $S$ of genus 2. Then the trivalent graph $p(\partial \Gamma)$ has six 3-valent vertices. Thus $\tau$ has six cusps, and the boundary of $S \setminus \tau$ has six cusps.

$\{\rho_g^* \mid g \in G\}$ produces a new resulting foliation (denoted $F^*$) of $(<, \Gamma)$ through the process in Subsection 4.2, which is uniquely determined by $(<, \Gamma)$.

At last, we consider the change between $F^*$ and the original resulting foliation $F$. Recall that we blow-down the set $\bigcup_{J \in \mathcal{U}} J$ in $\mathbb{R}$ to make $\{\rho_g \mid g \in G\}$ be $\{\rho_g^* \mid g \in G\}$ and make $F$ be $F^*$. For each $J \in \mathcal{U}$, we have $\rho_g(N_i) \notin J$ for all $g \in G$ and $i \in \{0, 1\}$, and thus $i_{\text{des}}^{-1}(J) \cap \partial \widetilde{N}(B) = \emptyset$. So $i_{\text{des}}^{-1}(J)$ is a union of pockets of leaves in $\tilde{F}$ (which is necessarily a countable union). So the above blowings-down in $\mathbb{R}$ make $\tilde{F}$ be blown-down in $\tilde{M}$ in a $\pi_1$-equivariant way, and therefore make $F$ be blown-down in $M$. As a result, we can blow-down $F$ countably many times to obtain $F^*$.

Thus, all resulting foliations of $(<, \Gamma)$ are same up to blowing-up/down.

Combining Proposition 4.20, Lemma 4.21, Remark 4.22, Remark 4.23, we can complete the proof of Proposition 1.8.

4.4. The 2-dimensional case. Let $S$ be a closed orientable surface of genus $\geq 1$. Then $\pi_1(S)$ is left orderable. Let $p: \tilde{S} \to S$ be the universal cover of $S$. Fix an arbitrary left-invariant order “$<$” of $\pi_1(S)$. We choose a fundamental domain $\Gamma$ of $S$ such that $p(\partial \Gamma)$ is a trivalent graph. Similar to Subsection 3.2, we can orient the trivalent graph $p(\partial \Gamma)$ to a train track $\tau$ through the order “$<$”. Let $D$ be a compact 2-disk on $S$. Through the process in Subsection 4.1~4.2, we can construct an one-dimensional foliation $F$ of $S - \text{Int}(D)$ that has a transverse $(\pi_1(S), \mathbb{R})$ structure.
Next, we consider the following two cases:

(a) Suppose that $S$ is a torus. As shown in Figure 10 (a), $p(\partial \Gamma)$ has two 3-valent vertices. We orient $p(\partial \Gamma)$ to a train track $\tau$, then each 3-valent vertex of $p(\partial \Gamma)$ produces exactly one cusp of $\tau$. So $\tau$ has two cusps, and $S \setminus \tau$ is a disk with two cusps on the boundary (cf. Figure 10 (b)). Similar to Proposition 4.20, we can extend $F$ to a foliation of $S$ by filling $D$ with horizontal segments.

(b) Suppose that $S$ is a closed orientable surface of genus $g \geq 2$. There are $4g - 2$ vertices of $p(\partial \Gamma)$ that have valency 3. Thus $\tau$ has $4g - 2 \geq 6$ cusps, and $S \setminus \tau$ is a disk with at least 6 cusps on the boundary. So $F$ can not extend to a foliation of $S$. On the other hand, we can ensure that $S$ admits no foliation since $\chi(S) \neq 0$. We can contract $D$ to a single point and make $F$ be deformed to a co-orientable singular foliation on $S$ with a single $(4g - 2)$-pronged singular point.

5. $\mathbb{R}$-covered foliations constructed in our process

We prove Theorem 1.5 in this section. To begin with, we review some backgrounds. A taut foliation has hyperbolic leaves if there is a continuously varying leafwise metric on its leaves so that every leaf is locally isometric to $\mathbb{H}^2$ (cf. [Can]). The following theorem is shown by Candel in [Can]:

Theorem 5.1 (Candel). Every taut foliation of an irreducible atoroidal 3-manifold has hyperbolic leaves.

Boyer and Hu prove that taut foliations in rational homology 3-spheres also have hyperbolic leaves (cf. [BH]):

Theorem 5.2 (Boyer-Hu). Every taut foliation of a rational homology 3-sphere has hyperbolic leaves.

Definition 5.3. Let $F$ be an $\mathbb{R}$-covered foliation of a closed 3-manifold $N$. Let $\tilde{F}$ be the pull-back foliation of $F$ in the universal cover of $N$. Let $X$ be a vector field transverse to $F$ and $\tilde{X}$ the pull-back vector field of $X$ in the universal cover of $N$. $X$ is called regulating if every orbit of $\tilde{X}$ intersects every leaf of $\tilde{F}$ exactly once.

In [Cal2], Calegari proves the following theorem:

Theorem 5.4 (Calegari). Every $\mathbb{R}$-covered foliation with hyperbolic leaves admits a regulating transverse vector field.

It follows that

Corollary 5.5. Let $N$ be either an irreducible atoroidal 3-manifold or a rational homology 3-sphere. Then every $\mathbb{R}$-covered foliation of $N$ admits a regulating transverse vector field.

Now we begin to prove Theorem 1.5. Let $M$ be the closed irreducible 3-manifold as given in Notation 2.1. We assume further that $M$ is either an atoroidal 3-manifold or a rational homology 3-sphere, and $M$ admits an $\mathbb{R}$-covered foliation $F_0$. For the reader’s convenience, we restate Theorem 1.5 as follows:

Theorem 1.5. There is a resulting foliation $F$ of $M - \text{Int}(B^3)$ obtained from the process of Theorem 1.2 such that:

1. $F$ can extend to an $\mathbb{R}$-covered foliation $F_{\text{extend}}$ of $M$.
2. $F_0$ can be recovered from doing a collapsing operation on $F_{\text{extend}}$.

This section is organized as follows:

The structure of Section 5. (a) In Subsection 5.1, we show that there exists an $\mathbb{R}$-covered foliation $F_1$ of $M$ such that (1) $F_0$ can be obtained from doing a collapsing operation for $F_1$, (2) $F_1$ has a 2-plane leaf $\lambda$. 

(b) In Subsection 5.2, we blow-up the 2-plane leaf \( \lambda \) of \( F_1 \) to obtain an \( \mathbb{R} \)-covered foliation \( F_2 \). We choose an order-domain pair \( (\prec, \Gamma) \) of \( M \) and show that: there is a resulting foliation \( F \) of \( (\prec, \Gamma) \) in \( M - \text{Int}(B^3) \) that can extend to \( F_2 \). The combination \( F \sim F_2 \sim F_1 \sim F_0 \) is the process as required in Theorem 1.5.

Throughout this section, we adopt the following notations:

**Notation 5.6.** Let \( \tilde{l} \) be a leaf of a \( \pi_1 \)-equivariant foliation of \( \tilde{M} \). We will always denote by \( \text{Stab}_G(\tilde{l}) = \{ g \in G \mid t_g(\tilde{l}) = \tilde{l} \} \) the stabilizer subgroup of \( G \) with respect to \( \tilde{l} \).

**5.1. Perturbing a pocket through an action.** To begin with, we review the existence of fundamental domains for arbitrary surface in its universal cover:

**Lemma 5.7.** Let \( S \) be an orientable surface (\( S \) may be a surface of infinite-type). Let \( \tilde{S} \) be the universal cover of \( S \). There is a (possibly non-compact) fundamental domain \( \Omega \) of \( S \) in \( \tilde{S} \) such that every point in the boundary of \( \Omega \) is contained in finitely many members of \( \{ g(\Omega) \mid g \text{ is a deck transformation of } \tilde{S} \} \).

*Proof.* This follows from [Ri] and [A].

Let \( \tilde{F}_0 \) be the pull-back foliation of \( F_0 \) in \( \tilde{M} \). Let \( l \) be a leaf of \( F_0 \). Let \( \tilde{l} \) be a leaf of \( \tilde{F}_0 \) such that \( p(\tilde{l}) = l \). By Novikov’s Theorem ([N], [Ro]), \( \tilde{l} \) is a universal cover of \( l \) (where the deck transformations on \( \tilde{l} \) are induced from the deck transformations of \( \tilde{M} \) which preserve \( \tilde{l} \)). So there exists a fundamental domain of \( l \) in \( \tilde{l} \) as above.

**Definition 5.8.** Let \( K = \text{Stab}_G(\tilde{l}) \subseteq G \). Then \( K \) acts on \( \tilde{l} \) by the restriction of \( \{ t_g \mid g \in K \} \) to \( \tilde{l} \). We choose a (possibly non-compact) fundamental domain \( \Sigma \) for this action on \( \tilde{l} \) such that every point in the boundary of \( \Sigma \) is contained in finitely many members of \( \{ t_g(\Sigma) \mid g \in K \} \).

**Definition 5.9.** We blow-up the leaf \( l \) of \( F_0 \) to obtain a foliation \( F_0^{\text{blow}} \) of \( M \). Let \( \tilde{F}_0^{\text{blow}} \) be its pull-back foliation in \( \tilde{M} \). Then the leaf \( l \) of \( F_0^{\text{blow}} \) is replaced by a blown-up pocket \( l \times I \) in \( \tilde{F}_0^{\text{blow}} \), and the leaf \( \tilde{l} \) of \( \tilde{F}_0 \) is replaced by a blown-up pocket \( \tilde{l} \times I \) in \( \tilde{F}_0^{\text{blow}} \). We can regard \( \tilde{l} \times I \) as the universal cover of \( l \times I \). As in Definition 5.8, \( \tilde{l} \times I \) can be decomposed to \( \{ t_g(\Sigma) \times I \mid g \in K \} \). Here, \( t_g(\Sigma) \) denotes a region in the fibered surface \( \tilde{l} \times I \) for \( l \times I \).

We fix an effective action \( \{ s_g : I \to I \mid g \in G \} \) of \( G \) on \( I \) such that \( s_g(\frac{1}{2}) \neq \frac{1}{2} \) for every \( g \in G - \{ 1 \} \). Now we glue the copies of \( \{ t_g(\Sigma) \times I \mid g \in K \} \) to make the sheets \( \{ t_g(\Sigma) \times \{ t \} \mid g \in K, t \in I \} \) be glued to a foliation in \( \tilde{l} \times I \):

**Construction 5.10.** We decompose \( \tilde{l} \times I \to \{ t_g(\Sigma) \times I \mid g \in K \} \). Now we consider arbitrary distinct elements \( r, h, j \in K \) with \( t_r(\Sigma) \cap t_h(\Sigma) \neq \emptyset \). Let \( e \) denote \( t_r(\Sigma) \cap t_h(\Sigma) \), and let \( e \times I \) denote \( (t_r(\Sigma) \times I) \cap (t_h(\Sigma) \times I) \). Let \( \delta_{(r,h)} : e \times I \to e \times I \) be the map defined by

\[
\delta_{(r,h)}(u, v) = (u, s_{h^{-1}r}(v)), \forall u \in e, v \in I.
\]

We glue the copy of \( e \times I \) in \( t_r(\Sigma) \times I \) to the copy of \( e \times I \) in \( t_h(\Sigma) \times I \) through the map \( \delta_{(r,h)} \) (cf. Figure 11). Now for each \( t \in I \), the segment \( e \times \{ t \} \subseteq t_r(\Sigma) \times \{ t \} \) is glued with the segment \( e \times \{ s_{h^{-1}r}(t) \} \subseteq t_h(\Sigma) \times \{ s_{h^{-1}r}(t) \} \).

We claim that: for arbitrary three distinct members in \( \{ t_g(\Sigma) \times I \mid g \in K \} \) that have nonempty common-intersection, all the gluing maps we used above for them are compatible in their common-intersection. Now suppose that \( v \) is a point in the fibered surface \( \tilde{l} \times I \) such that \( v \in t_r(\Sigma), t_h(\Sigma), t_j(\Sigma) \) for distinct \( r, h, j \in K \). By Construction 5.10, for each \( t \in I \), the gluing maps between \( t_r(\Sigma) \times I, t_h(\Sigma) \times I, t_j(\Sigma) \times I \) glue the three points \( (v, t) \in t_r(\Sigma) \times I, (v, s_{h^{-1}r}(t)) \in t_r(\Sigma) \times I, (v, s_{j^{-1}r}(t)) \in t_r(\Sigma) \times I \) together (cf. Figure 11). So these gluing maps are compatible in \( v \times I \).
Figure 11. Given \( r, h, j \in K \) such that the three regions \( t_r(\Sigma), t_h(\Sigma), t_j(\Sigma) \) in the fibered surface \( \tilde{l} \) of \( \tilde{l} \times I \) have a common point, then for every \( t \in I \), the three horizontal sheets \( t_r(\Sigma) \times \{t\}, t_h(\Sigma) \times \{s_{h^{-1}r}(t)\}, t_j(\Sigma) \times \{s_{j^{-1}r}(t)\} \) are attached.

We denote by \( t^*_g : \tilde{l} \times I \to \tilde{l} \times I (g \in K) \) the deck transformation of \( \tilde{l} \times I \) which takes \( \Sigma \times I \) to \( t_g(\Sigma) \times I \). And we call \( t_g(\Sigma) \times \{t\} \) a horizontal sheet for all \( g \in K, t \in I \). In the following, we show that all horizontal sheets are glued to a foliation of \( \tilde{l} \times I \) which is equivariant under \( \{t^*_g : \tilde{l} \times I \to \tilde{l} \times I \mid g \in K\} \):

**Lemma 5.11.** \( \{t_g(\Sigma) \times \{t\} \mid g \in K, t \in I\} \) is glued to a foliation (denoted \( \tilde{\mathcal{F}}_{\text{pert}} \)) of \( \tilde{l} \times I \). For every \( t \in I, \bigcup_{g \in K}(t_g(\Sigma) \times \{s_g^{-1}(t)\}) \) is a leaf of \( \tilde{\mathcal{F}}_{\text{pert}} \). And \( t^*_g(\tilde{\mathcal{F}}_{\text{pert}}) = \tilde{\mathcal{F}}_{\text{pert}} \) for every \( g \in K \).

**Proof.** Let \( t \in I \). Let \( h, r \) be distinct elements of \( K \) such that \( t_h(\Sigma) \cap t_r(\Sigma) \neq \emptyset \). Let \( e = t_h(\Sigma) \cap t_r(\Sigma) \). Since \( s_{h^{-1}r}(s_r^{-1}(t)) = s_h^{-1}(t) \), the two horizontal sheets \( t_r(\Sigma) \times \{s_r^{-1}(t)\}, t_h(\Sigma) \times \{s_h^{-1}(t)\} \) are glued along the two segments

\[
e \times \{s_r^{-1}(t)\} \subseteq t_r(\Sigma) \times \{s_r^{-1}(t)\},
\]

\[
e \times \{s_h^{-1}(t)\} \subseteq t_h(\Sigma) \times \{s_h^{-1}(t)\}
\]

as in Construction 5.10. It’s clear that all horizontal sheets in \( \{t_g(\Sigma) \times \{s_g^{-1}(t)\} \mid g \in K\} \) are glued to a leaf (we denote it by \( \tilde{l} \times \{t\} \)). \( \tilde{l} \times \{t\} \mid t \in I \) forms the foliation \( \tilde{\mathcal{F}}_{\text{pert}} \) of \( \tilde{l} \times I \).

For all \( h \in K, t \in I \), we have

\[
t^*_h(\tilde{l} \times \{t\}) = t^*_h(\bigcup_{g \in K}(t_g(\Sigma) \times \{s_g^{-1}(t)\}))
\]

\[
= \bigcup_{g \in K}(t_hg(\Sigma) \times \{s_g^{-1}(t)\})
\]

\[
= \bigcup_{g \in K}(t_g(\Sigma) \times \{s_g^{-1}h(t)\})
\]

\[
= \bigcup_{g \in K}(t_g(\Sigma) \times \{s_g^{-1}(s_h(t))\})
\]

\[
= \tilde{l} \times \{s_h(t)\}. \tag{1}
\]

Thus \( t^*_h(\tilde{\mathcal{F}}_{\text{pert}}) = \tilde{\mathcal{F}}_{\text{pert}} \). □
The $\mathcal{F}_{\text{pert}}$ descends to a foliation $\mathcal{F}_{\text{pert}}$ of $I \times I$. We can replace $\mathcal{F}_{\text{pert}}$ by $\mathcal{F}_{\text{pert}}$ to obtain a foliation of $M$. We denote this foliation by $\mathcal{F}_1$, and we denote its pull-back foliation in $M$ by $\mathcal{F}_1$. For every $g \in G$, the pocket $t_g(\tilde{l} \times I)$ in $\mathcal{F}_{\text{pert}}$ is replaced by $t_g(\mathcal{F}_{\text{pert}})$ in $\mathcal{F}_1$. So $\mathcal{F}_1$ is still an $\mathbb{R}$-covered foliation.

Lemma 5.12. There is a 2-plane leaf of $\mathcal{F}_1$.

Proof. Let $h \in K - \{\tau\}$. By Equation 1 (cf. the proof of Lemma 5.11), we have $t_h^*(l \times \{t\}) = l \times \{s_h(t)\}$ for every $t \in I$. So $t_h^*$ takes the leaf $l \times \{t\}$ to the leaf $l \times \{s_h(t)\}$. Since $s_h(t) \neq t$, we have $t_h^*(l \times \{t\}) \neq l \times \{t\}$. By Novikov’s Theorem ([N], [Ro]), $l \times \{t\}$ descends to a 2-plane leaf of $\mathcal{F}_1$.

Remark 5.13. Recall from Definition 1.4 that a collapsing operation for a taut foliation is to replace a product region $S \times I$ (where $S \times \{0\}, S \times \{1\}$ are leaves) by a single leaf $S$. $\mathcal{F}_0$ can be obtained from doing a collapsing operation for $\mathcal{F}_1$, which replaces $F_1 |_{\tilde{l} \times I}$ by a single leaf $l$.

5.2. The proof of Theorem 1.5. We complete the proof of Theorem 1.5 in this subsection. As shown in Lemma 5.12, there exists a 2-plane leaf of $\mathcal{F}_1$ contained in the region $l \times I$. We denote this 2-plane leaf by $\lambda$.

Fact 5.14. Let $C$ be an embedded compact disk in $\lambda$. Let $\tau$ be a transversal (with endpoints) of $\mathcal{F}_1$. Then $|\tau \cap C|$ is finite.

Proof. Assume that $|\tau \cap C|$ is infinite. Then there is a limit point $q$ of $\tau \cap C$ in $\tau$. Since $\tau, C$ are closed subsets of $M$, $\tau \cap C$ is closed. So $q \in \tau \cap C$. There exists an open neighborhood $U_q$ of $q$ in $C$ and an open neighborhood $V_q$ of $q$ in $\tau$ such that $U_q \cap V_q = \{q\}$. Notice that $C - U_q$ is closed. So there exists an open neighborhood $N_q$ of $q$ in $M$ such that $N_q \cap (C - U_q) = \emptyset$. It’s clear that $q$ is the only intersection point of $C$ and the segment $V_q \cap N_q \subseteq \tau$. So $q$ is an isolated point of $\tau \cap C$ in $\tau$, which is a contradiction. Therefore, $|\tau \cap C|$ is finite.

We blow-up the leaf $\lambda$ of $\mathcal{F}_1$ to obtain a foliation $\mathcal{F}_2$ of $M$. Then $\mathcal{F}_2$ is still an $\mathbb{R}$-covered foliation, and we can still do a collapsing operation on $\mathcal{F}_2$ to obtain $\mathcal{F}_0$. Now the leaf $\lambda$ of $\mathcal{F}_1$ is replaced by a pocket $\lambda \times I$ in $\mathcal{F}_2$. We assume that a transversal of $\mathcal{F}_2$ from $\lambda \times \{0\}$ to $\lambda \times \{1\}$ is positively oriented. Let $\mathcal{L}$ be the essential lamination $\mathcal{F}_2 - \lambda \times (0, 1)$.

Proposition 5.15. There is a branched surface $B$ of $M$ such that (1) $B$ fully carries $\mathcal{L}$, (2) $M \setminus B$ is a compact 3-ball.

Proof. By Corollary 5.5, there is a regulating vector field $X$ in $M$ transverse to $\mathcal{F}_2$. Let $\tilde{X}$ be the pull-back vector field of $X$ in $\tilde{M}$. Let $\mathcal{O} = \{\text{orbits of } X\}$ and $\tilde{\mathcal{O}} = \{\text{orbits of } \tilde{X}\}$. We can assume $\mathcal{O} |_{\tilde{l} \times I} = \{\{q\} \times I \mid q \in \lambda\}$.

Let $\{U_\alpha \mid \alpha \in \Psi\}$ (where $\Psi$ is an index set) be a finite union of product charts of $\mathcal{F}_1$ such that $\bigcup_{\alpha \in \Psi} U_\alpha = \tilde{M}$. For each $\alpha \in \Psi$, we choose a component $U_\alpha$ of $p^{-1}(U_\alpha)$, and we denote $\{\gamma \in \mathcal{O} \mid \gamma \cap U_\alpha \neq \emptyset\}$ by $\tilde{\mathcal{O}}_\alpha$. For any $\alpha \in \Psi$ and $\gamma \in \tilde{\mathcal{O}}_\alpha$, we denote by $\gamma^\perp$ (resp. $\gamma^\parallel$) the component of $\gamma - \tilde{U}_\alpha$ which contains the positive infinity (resp. negative infinity) of $\gamma$. Notice that $X$ is regulating. For each $\alpha \in \Psi$, there is an open connected set $\tilde{X}_\alpha$ in some component of $\bigcup_{\alpha \in \Psi} p^{-1}(\lambda \times \{0\})$ such that: for every $\gamma \in \tilde{\mathcal{O}}_\alpha$, $\gamma^\parallel$ intersects $\tilde{X}_\alpha$. And there is an open connected set $\tilde{Y}_\alpha$ (for each $\alpha \in \Psi$) in some component of $p^{-1}(\lambda \times \{1\})$ such that: for every $\gamma \in \tilde{\mathcal{O}}_\alpha$, $\gamma^\perp$ intersects $\tilde{Y}_\alpha$.

Since $\lambda \times \{0\}, \lambda \times \{1\}$ are 2-plane leaves of $\mathcal{F}_2$, there exists an embedded compact disk $C$ such that: for each $\alpha \in \Psi$, $p(\tilde{X}_\alpha) \subseteq \text{Int}(C) \times \{0\}$, $p(\tilde{Y}_\alpha) \subseteq \text{Int}(C) \times \{1\}$ (where $\text{Int}(C) \times \{0\}, \text{Int}(C) \times \{1\}$ denote regions in $\lambda \times \{0\}, \lambda \times \{1\}$ respectively). Then for arbitrary point $q \in M - \text{Int}(C \times I)$, the ray that starts at $q$ and goes along the element of $\mathcal{O}$ that contains $q$ (through either direction) must meet $\text{Int}(C) \times \{0\} \cup \text{Int}(C) \times \{1\}$ somewhere.
Let $\mathcal{N} = M - \text{Int}(C \times I)$. Let $\mathcal{O}_1$ be the restriction of $\mathcal{O}$ to $\mathcal{N}$. Then every element of $\mathcal{O}_1$ is a closed interval. Let $B = \mathcal{N} / \mathcal{O}_1$, which is a “branched surface” whose branch locus may not only have double-intersection singularities. For simplicity, we still call $B$ a branched surface, and we still adopt Notation 2.4 for $B$. Then $\mathcal{N}$ is a fibered neighborhood of $B$ with $\{\text{interval fibers of } \mathcal{N}\} = \mathcal{O}_1$. In the following, we first explain that every point of $B$ is contained in finitely many branch sectors, and then explain that $B$ has finitely many branch sectors.

Let $N(B) = \mathcal{N}$ with $\{\text{interval fibers of } N(B)\} = \mathcal{O}_1$, and let $\pi : N(B) \to B$ denote the collapsing map for $N(B)$. Then $\partial_h N(B) = (C \times \{0\}) \cup (C \times \{1\})$. Let $v \in B$. Let $J$ be the interval fiber of $N(B)$ such that $v = \pi(J)$. If $v$ is contained in infinitely many branch sectors of $B$, then $J$ must intersect $\partial_h N(B)$ infinitely many times (where the intersections of $\text{Int}(J)$ and $\partial_h N(B)$ are contained in the boundary of $\partial_h N(B)$). By Fact 5.14, $J$ intersects $(C \times \{0\}) \cup (C \times \{1\})$ finitely many times. Thus, $v$ is contained in finitely many branch sectors of $B$. Now assume that $B$ has infinitely many branch sectors. We choose an interior point in each branch sector of $B$, and we denote their union by $P$. Then $\pi^{-1}(P) \cap (C \times \{1\})$ is an infinite set, and so there must be a limit point $q$ of the set $\pi^{-1}(P) \cap (C \times \{1\})$ in $C \times \{1\}$. However, $\pi(q)$ is contained in finitely many branch sectors of $B$. This is a contradiction. Thus $B$ has finitely many branch sectors. At last, we can isotope the branch locus of $B$ so that it only has double-intersection singularities. Then $B$ is a branched surface as in Definition 2.3.

Since $\mathcal{L} \subseteq N(B)$ and $\mathcal{L}$ intersects $\mathcal{O}_1$ transversely, $\mathcal{L}$ is fully carried by $B$. And $M \setminus \partial B \cong M - \text{Int}(N(B)) = C \times I$ is a compact 3-ball. Hence Proposition 5.15 holds.

Henceforth, we will always denote $\mathcal{N}$ by $N(B)$ and assume $\{\text{interval fibers of } N(B)\} = \mathcal{O}_1$. We note that from the proof of Proposition 5.15, $N(B)$ has the following properties: $\partial_h N(B)$ is the union of two disks contained in $\lambda \times \{0\}, \lambda \times \{1\}$ respectively, and $\partial_h N(B)$ is a properly embedded annulus in $\lambda \times I$. Moreover, $\mathcal{F}_2|_{N(B)}$ is a foliation of $N(B)$ such that: the interior of $\partial_h N(B)$ is tangent to $\mathcal{F}_2|_{N(B)}$ and the interior of $\partial_v N(B)$ is transverse to $\mathcal{F}_2|_{N(B)}$.

Let $\tilde{\mathcal{F}}_2$ be the pull-back foliation of $\mathcal{F}_2$ in $\tilde{M}$. Let $\tilde{\lambda} \times I$ be a component of $p^{-1}(\lambda \times I)$, where $p(\tilde{\lambda} \times \{t\}) = \lambda \times \{t\}$ for all $t \in I$. Let $\tilde{\lambda}_0 = \tilde{\lambda} \times \{0\}$ and $\tilde{\lambda}_1 = \tilde{\lambda} \times \{1\}$. Since $\lambda \times \{0\}, \lambda \times \{1\}$ are 2-plane leaves, we have $\text{Stab}_G(\tilde{\lambda}_0) = \text{Stab}_G(\tilde{\lambda}_1) = 1$.

**Definition 5.16.** We denote by $L_2$ the leaf space of $\tilde{\mathcal{F}}_2$. Let $i_{\text{des}} : L_2 \to \mathbb{R}$ be an orientation-preserving homeomorphism. Let $K_0 = i_{\text{des}}(\tilde{\lambda}_0)$, $K_1 = i_{\text{des}}(\tilde{\lambda}_1)$. Let $\{r_g : \mathbb{R} \to \mathbb{R} \mid g \in G\}$ be the action of $G$ on $\mathbb{R}$ defined by $r_g = i_{\text{des}} \circ t_g \circ i_{\text{des}}^{-1}$ for any $g \in G$.

**Remark 5.17.** Since $i_{\text{des}}$ is orientation-preserving, $i_{\text{des}}$ projects positively oriented transversals of $\tilde{\mathcal{F}}_2$ to positively oriented intervals in $\mathbb{R}$. So $K_1 > K_0$. Notice that $K_0, K_1$ have trivial stabilizers under the action $\{t_g : \mathbb{R} \to \mathbb{R} \mid g \in G\}$, and all blown-up regions $\{t_g(\lambda \times I)\}_{g \in G}$ are disjoint in $\tilde{M}$. Hence the images of $[K_0, K_1]$ under the action $\{r_g \mid g \in G\}$ are pairwise disjoint.

**Definition 5.18.** Let “$<$” be the strict linear order of $G$ defined by:

- For any distinct elements $g, h, q \in G$ and $g < h$ if $r_q(K_0) < r_h(K_0)$.

**Fact 5.19.** “$<$” is a left-invariant order of $G$.

**Proof.** Suppose that $g, h, q \in G$ and $g < h$. Then

$$r_{qg}(K_0) = r_q(r_g(K_0)) < r_q(r_h(K_0)) = r_{qh}(K_0).$$

So “$<$” is left-invariant. \qed

Let $\tilde{B} = p^{-1}(B)$, and let $\text{Sec}(\tilde{B}) = \{\text{branch sectors of } \tilde{B}\}$. As in Remark 2.5, we denote by $\mathcal{N}(B) = p^{-1}(N(B))$ the fibered neighborhood of $\tilde{B}$ and denote by $\tilde{\pi} : \mathcal{N}(B) \to \tilde{B}$ the collapsing
Proof. Let the endpoint contained in \( t \in t \) in the base point \( \tilde{x} \in \pi(F) \). Therefore, (a) holds.

(b) Assume that \( I_0 \) has a direction which starts at the endpoint contained in \( t_s(\lambda_1) \) and ends at the endpoint contained in \( t_h(\lambda_0) \). Then \( I_0 \) is a positively oriented transversal of \( \tilde{F}_2 \).

Proof. Let \( \mathcal{G} = M - \text{Int}(N(B)) \) and \( \tilde{\mathcal{G}} = p^{-1}(\mathcal{G}) \cap \Gamma \). Then the two endpoints of \( I_0 \) are contained in \( t_s(\tilde{\lambda}), t_h(\tilde{\lambda}) \) respectively. Thus, one endpoint of \( I_0 \) is contained in \( t_s(\lambda_0) \cup t_s(\lambda_1) \), and the other endpoint of \( I_0 \) is contained in \( t_h(\lambda_0) \cup t_h(\lambda_1) \).

Since distinct elements of \( \{r_g\}_{g \in G} \) take \( [K_0, K_1] \) to disjoint intervals in \( \mathbb{R} \), we have \( r_h(K_1) > r_h(K_0) > r_s(K_1) > r_s(K_0) \), and thus

\[
i_{\text{des}}(t_h(\lambda_1)) > i_{\text{des}}(t_h(\lambda_0)) > i_{\text{des}}(t_s(\lambda_1)) > i_{\text{des}}(t_s(\lambda_0)).\]

It follows that the two endpoints of \( I_0 \) are contained in \( t_h(\lambda_0), t_s(\lambda_1) \) respectively (cf. Figure 12). Therefore, (a) holds.

Since \( i_{\text{des}}(t_h(\lambda_0)) > i_{\text{des}}(t_s(\lambda_1)) \), every transversal of \( \tilde{F}_2 \) from \( t_s(\lambda_1) \) to \( t_h(\lambda_0) \) is positively oriented. So (b) holds.

Recall that Subsection 3.2 provides a process to construct a branched surface given by the order-domain pair \((<, \Gamma)\). We denote this branched surface by \( B^* \). We show \( B^* = B \) as follows.

Proposition 5.21. \( B^* = B \).

Proof. Let \( \tilde{B} = p^{-1}(B^*) \). Since the branch sectors of \( B, B^* \) are same, we have

\[
\{\text{branch sectors of } \tilde{B}^* \} = \text{Sec}(\tilde{B}).
\]

We compare the co-orientations for \( B, B^* \) on their branch sectors as follows.

We assume that every interval fiber of \( N(B) \) has an orientation so that it is a positively oriented transversal of \( \tilde{F}_2 \). \( \tilde{\mathcal{B}} \) has a co-orientation on \( \text{Sec}(\tilde{B}) \) such that every \( F \in \text{Sec}(\tilde{B}) \) has an orientation.

Figure 12. The two endpoints of \( I_0 \) are at \( t_h(\lambda_0), t_s(\lambda_1) \) respectively.
that is consistent with the orientation of the interval fibers of $\tilde{N}(B)$ on $F$, i.e. is consistent with the direction of positively oriented transversals of $\tilde{F}_2$.

Let $F \in \text{Sec}(\tilde{B})$. We denote by $h, s \in G$ for which $F$ is a common face of $t_h(\Gamma)$, $t_s(\Gamma)$ and $h > s$. By Definition 3.9, $F$ has positive orientation with respect to $t_s(\Gamma)$ as a branch sector of $B^*$. By Lemma 5.20 (b), $F$ also has positive orientation with respect to $t_s(\Gamma)$ as a branch sector of $\tilde{B}$. Therefore, $B^* = B$. 

Notice that $\partial_v N(B)$ is connected. Combining Proposition 5.21 and Proposition 4.20 (1), resulting foliations of $(<, \Gamma)$ in $M - \text{Int}(B^3)$ can extend to taut foliations in $M$. Furthermore,

**Proposition 5.22.** There is a resulting foliation $F$ of $(<, \Gamma)$ in $N(B) \cong M - \text{Int}(B^3)$ such that $F = \tilde{F}_2|_{N(B)}$.

**The sketch of the proof of Proposition 5.22.** Recall from Section 4, we first choose an action $\{\rho_g : \mathbb{R} \to \mathbb{R} \mid g \in G\}$ and an interval $[N_0, N_1]$ (cf. Definition 4.3), and then use them to construct a resulting foliation of $(<, \Gamma)$. We set $\{\rho_g : \mathbb{R} \to \mathbb{R} \mid g \in G\}$ to be $\{r_g : \mathbb{R} \to \mathbb{R} \mid g \in G\}$ and set $[N_0, N_1]$ to be $[K_0, K_1]$. Then we have a resulting foliation $F$ of $(<, \Gamma)$ by the process of Section 4. And we can verify $F = \tilde{F}_2|_{N(B)}$. See the following proof for details.

**Proof.** To begin with, we review some ingredients in Section 4:

**Ingredient 1.** In Definition 4.3, we choose an action $\{\rho_g : \mathbb{R} \to \mathbb{R} \mid g \in G\}$ of $G$ on $\mathbb{R}$ and a closed interval $[N_0, N_1] \subseteq \mathbb{R}$ such that (1) $[N_0, N_1]$ has pairwise disjoint images under the action $\{\rho_g\}_{g \in G}$, (2) $\rho_g(N_0) < \rho_h(N_0)$ for any $g, h \in G$ with $g < h$.

**Ingredient 2.** There is a map $i_0 : \bigsqcup_{F \in \text{Sec}(\tilde{B})} (F \times I) \to \mathbb{R}$ (cf. Definition 4.13) with the following properties:

(1) Let $F \in \text{Sec}(\tilde{B})$, $t \in I$. For arbitrary distinct $m, n \in F \times \{t\}$, we have $i_0(m) = i_0(n)$.

(2) For all $F \in \text{Sec}(\tilde{B})$, $t \in I$, we have $\rho_g(i_0(F \times \{t\})) = i_0(t_g(F) \times \{t\})$ (cf. Definition 4.13, Fact 4.7).

(3) Let $F \in \text{Sec}(\tilde{B})$. We denote by $u, v \in G$ for which $F$ is a common face of $t_u(\Gamma), t_v(\Gamma)$ and $v > u$. Then $i_0(F \times \{0\}) = \rho_u(N_1)$, $i_0(F \times \{1\}) = \rho_v(N_0)$ (cf. Definition 4.4, Definition 4.6).

**Ingredient 3.** We decompose $\tilde{N}(B)$ to $\bigsqcup_{F \in \text{Sec}(\tilde{B})} (F \times I)$ and then assign a gluing to it so that the horizontal sheets $\{F \times \{t\} \mid F \in \text{Sec}(\tilde{B}), t \in I\}$ are glued to a $\pi_1$-equivariant foliation in $\tilde{N}(B)$. For arbitrary $a, b \in I$ and distinct $F, X \in \text{Sec}(\tilde{B})$ with $F \cap X \neq \emptyset$, $F \times \{a\}$, $X \times \{b\}$ are attached when $i_0(F \times \{a\}) = i_0(X \times \{b\})$ (cf. Fact 4.14).

We set (1) $\rho_g = r_g$ for every $g \in G$, (2) $N_0 = K_0$, $N_1 = K_1$. Following the process in Section 4, we can construct a $\pi_1$-equivariant foliation $\tilde{F}$ in $\tilde{N}(B)$, which descends to a foliation $F$ of $N(B)$. We verify $F = \tilde{F}_2|_{\tilde{N}(B)}$ in the following.

Similar to Subsection 4.2, we can decompose $\tilde{N}(B)$ to $\bigsqcup_{F \in \text{Sec}(\tilde{B})} (F \times I)$. Then $\tilde{F}_2|_{\tilde{N}(B)}$ is decomposed to a collection of horizontal sheets $\{F \times \{t\} \mid F \in \text{Sec}(\tilde{B}), t \in I\}$. And we assume that: for every $F \in \text{Sec}(\tilde{B})$, a transversal of $\tilde{F}_2$ from $F \times \{0\}$ to $F \times \{1\}$ is a positively oriented. Let $i_{\text{proj}} : \tilde{M} \to \mathbb{R}$ be the combination $\tilde{M} \xrightarrow{\text{proj}} L_2 \xrightarrow{i_{\text{des}}} \mathbb{R}$, where $\tilde{M} \xrightarrow{\text{proj}} L_2$ denotes the projection map from $\tilde{M}$ to the leaf space $L_2$ of $\tilde{F}_2$.

Let $F \in \text{Sec}(\tilde{B})$. We denote by $u, v \in G$ for which $F$ is a common face of $t_u(\Gamma), t_v(\Gamma)$ and $v > u$. Recall from Lemma 5.20,

$$i_{\text{proj}}(F \times \{0\}) = i_{\text{des}}(t_u(\lambda_1)) = r_u(K_1),$$

$$i_{\text{proj}}(F \times \{1\}) = i_{\text{des}}(t_v(\lambda_0)) = r_v(K_0).$$
By Ingredient 2 (c), we have \( i_{proj} (F \times \{0\}) = i_0 (F \times \{0\}) \), \( i_{proj} (F \times \{1\}) = i_0 (F \times \{1\}) \). We can assume that the \( t \)-variable of \( \{ F \times \{t\} \mid t \in I \} \) is parameterized such that 
\[
i_{proj} (F \times \{t\}) = i_0 (F \times \{t\}).
\]

By Ingredient 2 (b), we have \( t_g (F \times \{t\}) = t_g (F \times \{t\}) \) for every \( g \in G \).

At last, we glue \( \bigsqcup_{F \in Sec (\tilde{B})} (F \times I) \) to \( N(B) \) that make \( \{ F \times \{t\} \mid F \in Sec (\tilde{B}), t \in I \} \) be glued to \( \tilde{F}_2 \bigg|_{N(B)} \). To show \( \tilde{F}_2 \bigg|_{N(B)} = \tilde{F} \), we only need to verify that: the gluing relation for this gluing is same as the gluing relation for \( \tilde{F} \) as in Ingredient 3. For any \( F, X \in Sec (\tilde{B}) \) with \( F \cap X \neq \emptyset \) and arbitrary \( a, b \in I \), the two horizontal sheets \( F \times \{a\} , X \times \{b\} \) are contained in the same leaf of \( \tilde{F}_2 \) if and only if \( i_{proj} (F \times \{a\}) = i_{proj} (X \times \{b\}) \), and therefore if and only if \( i_0 (F \times \{a\}) = i_0 (X \times \{b\}) \). This is same as the gluing relation for \( \tilde{F} \) in Ingredient 3. So \( \tilde{F} = \tilde{F}_2 \bigg|_{N(B)} \), and therefore \( F = F_2 \bigg|_{N(B)} \). \( \square \)

Now we already have operations \( F_0 \rightsquigarrow F_1 \rightsquigarrow F_2 \rightsquigarrow F \). Since \( F = F_2 \bigg|_{N(B)} \) is a resulting foliation of \((<, \Gamma)\) and \( F_0 \) can be obtained from doing a collapsing operation for \( F_2 \), \( F \rightsquigarrow F_2 \rightsquigarrow F_0 \) is the process as required in Theorem 1.5.

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