Exclusive Two-Vector-Meson Production from $e^+e^-$ Annihilation

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Abstract

The exclusive production of pairs of vector mesons with $J^{PC} = 1^{--}$ in $e^+e^-$ collisions can proceed through $e^+e^-$ annihilation into two virtual photons. At energies much greater than the meson masses, the cross section is dominated by the independent fragmentation of the virtual photons into the vector mesons. The fragmentation approximation is used to calculate the cross sections and angular distributions for pairs of vector mesons that can be produced at the $B$ factories. The predicted cross sections for $\rho^0 + \rho^0$ and $\rho^0 + \phi$ production agree with recent measurements by the BaBar Collaboration. For the production of two charmonium vector mesons, the nonfragmentation corrections to the cross sections are calculated by using the NRQCD factorization formalism. The predicted cross sections for $J/\psi + J/\psi$ and $J/\psi + \psi(2S)$ production are compatible with upper limits set by the Belle Collaboration.

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I. INTRODUCTION

The production of hadrons in $e^+e^-$ collisions at energies well below the mass of the $Z^0$ proceeds predominantly through the annihilation of $e^+e^-$ into a virtual photon. This process can only produce final states with charge conjugation quantum number $C = -1$. Final states with $C = +1$ can be produced by the annihilation of $e^+e^-$ into two virtual photons, but the cross sections for these processes are suppressed by a factor of $\alpha^2$, where $\alpha \simeq 1/137$ is the QED fine structure constant. Final states with $C = +1$ can also be produced by the annihilation of $e^+e^-$ into a virtual $Z^0$, but the cross sections for these processes are suppressed by a factor of $(s/M_Z^2)^2$ if the $e^+e^-$ center-of-mass energy $\sqrt{s}$ is small compared to $M_Z$. Thus, unless there are enhancement factors to compensate for the factors of $\alpha^2$ or $(s/M_Z^2)^2$, the production cross sections at $B$-factory energies for $C = +1$ states are orders of magnitude smaller than those for $C = -1$ states.

In $e^+e^-$ collisions with large center-of-mass energy $\sqrt{s}$, factors of $\sqrt{s}/m$, where $m$ is a hadronic mass scale, can compensate for the suppression factor of $\alpha^2$. An example of such compensation can be seen in the exclusive double-charmonium production process $e^+e^- \rightarrow J/\psi + J/\psi$. At the beam energy of the $B$ factories, $\sqrt{s}/2 = 5.29$ GeV, the leading-order prediction for the production cross section for the $C = +1$ final state $J/\psi + J/\psi$ has roughly the same order of magnitude as that for the corresponding $C = -1$ final state $J/\psi + \eta_c$. The production cross section for $J/\psi + \eta_c$ scales as $\alpha^2\alpha_s^2m_c^6/s^4$, where $m_c$ is the charm-quark mass. In contrast, the production cross section for $J/\psi + J/\psi$ scales as $\alpha^4/s$ because there is a photon-fragmentation contribution that corresponds to the annihilation process $e^+e^- \rightarrow \gamma^* + \gamma^*$, followed by independent fragmentation processes $\gamma^* \rightarrow J/\psi$. The enhancement factor of $(s/m_c^2)^3$ in the production cross section for $J/\psi + J/\psi$ relative to that for $J/\psi + \eta_c$ partially compensates for the suppression factor of $(\alpha/\alpha_s)^2$.

A similar enhancement mechanism is present in the exclusive production of a pair of light vector mesons $V_1 + V_2$, both of which have quantum numbers $J^{PC} = 1^{--}$. The production process proceeds predominantly through the annihilation process $e^+e^- \rightarrow \gamma^* + \gamma^*$, followed by the independent fragmentation processes $\gamma^* \rightarrow V_i$. We can compare the production cross section for $V_1 + V_2$ to that for $V + P$, where $P$ is a light pseudoscalar meson. In the ratio of the former cross section to the latter, there is an enhancement factor of $(s/\Lambda_{QCD}^2)^3$ that compensates for the suppression factor of $(\alpha/\alpha_s)^2$. In this paper, we calculate the cross
sections and angular distributions for the exclusive production of pairs of vector mesons with \( J^{PC} = 1^{--} \).

The remainder of this paper is organized as follows. In Sec. II we discuss the various contributions to the amplitudes for the production of two vector mesons and estimate their relative sizes. In Sec. III we give expressions for the differential cross sections for the production of two vector mesons in the fragmentation approximation and give numerical results for the cross sections. Sec. IV contains a computation of the nonfragmentation corrections to the cross sections for the production of two charmonium vector mesons. In Sec. V we compare our results with experimental measurements and with previous theoretical calculations. The Appendix contains the expressions for the nonfragmentation corrections to the differential cross sections for two charmonium vector mesons.

II. PRODUCTION AMPLITUDES

The lowest-order QED diagrams that correspond to the creation of the quarks in the two vector mesons \( V_1 \) and \( V_2 \) are shown in Figs. 1 and 2. If the two vector mesons have different quark contents \( q \bar{q} \) and \( q' \bar{q}' \), then only the diagrams in Fig. 1 are present. Once the quarks are created, the formation of the vector mesons involves the exchange of arbitrarily many gluons between the quark lines. Gluons can be exchanged between the quark and antiquark that form a given meson or between quarks or antiquarks that form different mesons. However, factorization theorems and the scaling of various contributions yield a simplification of this picture in the limit \( E_{\text{beam}} \to \infty \). In what follows, we take the beam energy \( E_{\text{beam}} = \sqrt{s}/2 \) to be much greater than the masses \( m_{V_i} \) of the vector mesons, which implies that \( E_{\text{beam}} \) is also much greater than \( \Lambda_{\text{QCD}} \).

Let us first consider the exchange of gluons only between the \( q \) and \( \bar{q} \) within a given meson. Then, the diagrams in Fig. 2 are suppressed in comparison to those in Fig. 1. In the diagrams of Fig. 1 the virtual-photon propagators are \( 1/m_{V_i}^2 \), and the two hadronic electromagnetic currents each give a factor of order \( \Lambda_{\text{QCD}} \) for light mesons and \( m_c \) for charmonium mesons. In the diagrams of Fig. 2 the virtual-photon propagators are of order \( 1/E_{\text{beam}}^2 \), and the two hadronic electromagnetic currents each give factors of order \( E_{\text{beam}} \). This leads to the following suppression factors for the diagrams of Fig. 2 relative to those of Fig. 1: \( \Lambda_{\text{QCD}}^2/E_{\text{beam}}^2 \) for two light mesons and \( m_c^2/E_{\text{beam}}^2 \) for two charmonium mesons. We
have taken the light-meson masses to be of order $\Lambda_{\text{QCD}}$ and the charmonium masses to be of order $m_c$. (The diagrams of Fig. 2 do not contribute in the case of a light meson and a charmonium meson.)

Next, let us consider the exchange of gluons between quarks or antiquarks in different mesons, with some of those gluons soft or, in the case of a light meson, collinear. In the case of two light mesons, one can use standard methods for proving factorization theorems [4] to show that the soft contributions cancel up to corrections of relative order $\Lambda_{\text{QCD}}^4/E_{\text{beam}}^4$ and the collinear contributions cancel up to corrections of relative order $\Lambda_{\text{QCD}}^2/E_{\text{beam}}^2$. Factorization theorems are not well established for the production of heavy-quarkonium mesons. However, on the basis of existing factorization technology, it seems plausible, in the case of one light meson and one charmonium meson, that soft contributions cancel up to terms of relative order $\Lambda_{\text{QCD}}^2(m_c v)^2/E_{\text{beam}}^4$ and that collinear contributions cancel up to terms of relative order $\Lambda_{\text{QCD}}^2/E_{\text{beam}}^2$. Here $v$ is the typical velocity of the charm quark or antiquark in the charmonium rest frame. ($v^2 \approx 0.3$.) In the case of two charmonium mesons, there are no collinear singularities. It is plausible that soft contributions cancel up to terms of relative order $(m_c v)^4/E_{\text{beam}}^4$ in this case.

Finally, let us consider the exchange of hard gluons between quarks or antiquarks in different mesons. (Hard gluons have energies and momenta of order $E_{\text{beam}}$. ) In the case of the diagrams of Fig. 2 exchanges of hard gluons are suppressed as $\alpha_s(E_{\text{beam}})$. In the case of the diagrams of Fig. 1 it is necessary to exchange at least two gluons between the mesons in order to keep the $q\bar{q}$ pair in a color-singlet state. Hence, such exchanges are suppressed as $\alpha_s^2(E_{\text{beam}})$. They are also suppressed by a kinematic factor that arises as follows. In the diagrams of Fig. 1 without hard-gluon exchange, the virtual-photon propagators are $1/m_{\gamma_i}^2$ and the two hadronic electromagnetic currents each give a factor of order $\Lambda_{\text{QCD}}$ for light mesons and $m_c$ for charmonium mesons. In the diagrams of Fig. 1 with hard-gluon exchanges, the virtual-photon propagators are of order $1/E_{\text{beam}}^2$, and the

\[^{1}\] The results that we quote here apply to the case in which one sums over all polarizations of the final-state mesons. If one observes the polarization of one meson, then the soft contributions cancel up to terms of relative order $\Lambda_{\text{QCD}}^3/E_{\text{beam}}^3$, and the collinear contributions cancel up to terms of relative order $\Lambda_{\text{QCD}}/E_{\text{beam}}$. If one observes the polarization of both mesons, then the soft contributions cancel up to corrections of relative order $\Lambda_{\text{QCD}}^2/E_{\text{beam}}^2$ and the collinear contributions cancel up to corrections of relative order $\Lambda_{\text{QCD}}/E_{\text{beam}}$. Similar reductions of the suppression factors for soft and collinear contributions occur in the cases in which one or more charmonium mesons are produced.
two hadronic electromagnetic currents each give factors of order $E_{\text{beam}}$. This leads to the following additional suppression factors for hard-gluon exchanges in the diagrams of Fig. 1: $\Lambda_{\text{QCD}}^2 / E_{\text{beam}}^2$ for two light mesons, $m_c \Lambda_{\text{QCD}} / E_{\text{beam}}^2$ for a light meson and a charmonium meson, and $m_c^2 / E_{\text{beam}}^2$ for two charmonium mesons. Again, we have taken the light-meson masses to be of order $\Lambda_{\text{QCD}}$ and the charmonium masses to be of order $m_c$.

Let us now summarize the results of the preceding analyses. We conclude that we need only consider the fragmentation diagrams in Fig. 1 up to terms of relative order $\Lambda_{\text{QCD}}^2 / E_{\text{beam}}^2$ for two light mesons and $m_c^2 / E_{\text{beam}}^2$ for two charmonium mesons. (The diagrams of Fig. 2 do not contribute in the case of a light meson and a charmonium meson.) We also conclude that we need only consider the QCD corrections to the diagrams of Fig. 1 that correspond to the exchange of gluons between the $q$ and $\bar{q}$ in each meson, up to terms of relative order $\Lambda_{\text{QCD}}^2 / E_{\text{beam}}^2$ for two light mesons, $\Lambda_{\text{QCD}}^2 / E_{\text{beam}}^2$ or $\alpha_s^2(E_{\text{beam}}) (m_c \Lambda_{\text{QCD}} / E_{\text{beam}}^2)$ for a light meson and a charmonium meson, and $(m_c v)^4 / E_{\text{beam}}^4$ or $\alpha_s^4(E_{\text{beam}}) (m_c^2 / E_{\text{beam}}^2)$ for two
TABLE I: Masses, electronic decay widths, and coupling constants $g_{V\gamma}$ for vector mesons $V$. All of the data except for those of $\rho^0$ are taken from Ref. [9]. The data for $\rho^0$ are taken from Ref. [10] in order to maintain consistency between the narrow-width approximation and the nonzero-width parametrization. The uncertainties shown for $g_{V\gamma}$ are those that arise from the uncertainties in the electronic widths of the vector mesons.

| $V$      | Mass (MeV)     | Width (keV)    | $g_{V\gamma}$ (GeV$^2$) |
|----------|----------------|----------------|--------------------------|
| $\rho^0$ | $775.65 \pm 0.64 \pm 0.50$ | $7.06 \pm 0.11 \pm 0.05$ | $0.122 \pm 0.001$       |
| $\omega$ | $782.65 \pm 0.12$       | $0.60 \pm 0.02$       | $0.036 \pm 0.001$       |
| $\phi$   | $1019.46 \pm 0.019$     | $1.27 \pm 0.04$       | $0.078 \pm 0.001$       |
| $J/\psi$ | $3096.916 \pm 0.011$    | $5.55 \pm 0.14$       | $0.860 \pm 0.011$       |
| $\psi(2S)$ | $3686.093 \pm 0.034$   | $2.48 \pm 0.06$       | $0.746 \pm 0.009$       |

The QCD corrections that involve the exchange of gluons between the $q$ and $\bar{q}$ in each meson are precisely those that correspond to the formation of a vector meson $V$ from a $q\bar{q}$ pair that has been created at a point. They can be expressed in terms of a QCD matrix element of the electromagnetic current:

$$J^\mu(x) = e_u \bar{u}(x)\gamma^\mu u(x) + e_d \bar{d}(x)\gamma^\mu d(x) + e_s \bar{s}(x)\gamma^\mu s(x) + e_c \bar{c}(x)\gamma^\mu c(x), \quad (1)$$

where $e_u = e_c = +\frac{2}{3}$ and $e_d = e_s = -\frac{1}{3}$. The matrix element can be used to define a vector-meson-photon coupling constant $g_{V\gamma}$:

$$\langle V(\lambda)|J^\mu(x = 0)|0 \rangle = g_{V\gamma} \epsilon^\mu(\lambda)^*, \quad (2)$$

where $\lambda$ is the helicity of the vector meson. The coupling constant $g_{V\gamma}$ can be determined

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2 There are several recent papers in which the rate for the process $\gamma^*\gamma^* \rightarrow \rho^0\rho^0$ is calculated in the high-energy and diffractive limit $s \gg -t$, where $s$ and $t$ are the Mandelstam variables. In this limit, the dominant contributions to this process involve the exchange of gluons between the quarks and antiquarks that form different mesons. There are analogous contributions to the process $e^+e^- \rightarrow \rho^0\rho^0$, since it also proceeds through $\gamma^*\gamma^* \rightarrow \rho^0\rho^0$. However, as we have discussed above, these contributions are suppressed, and the dominant mechanism in $e^+e^- \rightarrow \rho^0\rho^0$ is independent photon fragmentation. Independent photon fragmentation is not considered in Refs. [5, 6, 7, 8] because it corresponds in the process $\gamma^*\gamma^* \rightarrow \rho^0\rho^0$ to a disconnected diagram.
from the electronic width of the vector meson:

$$\Gamma[V \to e^+e^-] = \frac{4\pi\alpha^2 g_{V\gamma}^2}{3m_V^3}. \quad (3)$$

Using the measured electronic widths that are given in Ref. 9, we obtain the values of $g_{V\gamma}$ that are shown in Table II. The stated uncertainties arise from the electronic widths. The uncertainties in the masses are at most 0.06% and can be neglected.

III. CROSS SECTIONS IN THE FRAGMENTATION APPROXIMATION

We express our results for the cross section for $e^+e^- \to V_1 + V_2$ in terms of dimensionless variables $r_{V_1}$ and $r_{V_2}$, which are defined by

$$r_V = \frac{m_V}{\sqrt{s}}. \quad (4)$$

If we take $\sqrt{s}$ to be 10.58 GeV, the center-of-mass energy of the $B$ factories, then $r_{V_1}$ is small for charmonium ($r_{J/\psi}^2 = 0.086$) and very small for light vector mesons ($r_{\rho}^2 = 0.0054$). The differential cross sections $d\sigma/d\cos\theta$ for each of the helicity states of $V_1$ and $V_2$ are

$$\frac{d\sigma}{dx}[V_1(\lambda_1) + V_2(\lambda_2)] = \frac{16\pi^3\alpha^4 g_{V_1\gamma}^2 g_{V_2\gamma}^2 \lambda^{1/2}(1, r_{V_1}^2, r_{V_2}^2) F_{\lambda_1,\lambda_2}(r_1, r_2, x)}{s^5 r_{V_1}^4 r_{V_2}^4 [(1 - x^2)\lambda(1, r_{V_1}^2, r_{V_2}^2) + 4r_{V_1}^2 r_{V_2}^2]^2}, \quad (5)$$

where $x = \cos\theta$, $r_i = r_{V_i}$, and

$$\lambda(a, b, c) = a^2 + b^2 + c^2 - 2ab - 2bc - 2ca. \quad (6)$$

The functions $F_{\lambda_1,\lambda_2}(r_1, r_2, x)$, which depend on the helicities of the vector mesons, are given by

$$F_{\pm,\mp}(r_1, r_2, x) = f_{+-}(r_1, r_2, x) + f_{-+}(r_1, r_2, -x), \quad (7a)$$
$$F_{\pm,0}(r_1, r_2, x) = f_{+0}(r_1, r_2, x) + f_{0+}(r_1, r_2, -x), \quad (7b)$$
$$F_{0,\pm}(r_1, r_2, x) = f_{+0}(r_2, r_1, x) + f_{0+}(r_2, r_1, -x), \quad (7c)$$
$$F_{\pm,\pm}(r_1, r_2, x) = f_{++}(r_1, r_2, x) + f_{++}(r_1, r_2, -x), \quad (7d)$$
$$F_{0,0}(r_1, r_2, x) = 16r_{V_1}^2 r_{V_2}^2 x^2 (1 - x^2), \quad (7e)$$

where

$$f_{+-}(r_1, r_2, x) = \frac{1}{2}(1 + x)(1 - x)^3 (1 - r_{V_1}^2 - r_{V_2}^2)^2, \quad (8a)$$
$$f_{+0}(r_1, r_2, x) = r_{V_2}^2 (1 - x)^2 [(1 - r_{V_2}^2)(1 + x) - r_{V_1}^2(1 - x)]^2, \quad (8b)$$
$$f_{++}(r_1, r_2, x) = \frac{1}{2}(1 - x^2)[(1 + x)r_{V_2}^2 (1 - r_{V_2}^2) - (1 - x)r_{V_1}^2 (1 - r_{V_1}^2) + 2r_{V_1}^2 r_{V_2}^2 x^2]. \quad (8c)$$
Since \( r_1 \) and \( r_2 \) are small, the cross section is largest if the two vector mesons are transversely polarized with opposite helicities: \( \lambda_1 = -\lambda_2 = \pm 1 \). If the vector meson \( V_1 \) is longitudinally polarized, then the cross section is suppressed as \( r_1^2 \). If the vector mesons have the same helicities, then the cross section is suppressed as four powers of \( r_1 \) or \( r_2 \).

After summing over the helicities \( \lambda_1 \) and \( \lambda_2 \) of the vector mesons, we obtain

\[
\frac{d\sigma}{dx}(m_{V_1}, m_{V_2}) = \frac{32\pi^3 \alpha^4 g_{V_1}^2 g_{V_2}^2 \lambda^{1/2}(1, r_1^2, r_2^2)}{s^5[r_1 r_2 (1 - r_1^2 - r_2^2)]^4[1 - (1 - \Delta)x^2]^2} \left[ 16r_1^2 r_2^2 (r_1^2 + r_2^2) + 2(1 - x^2)(1 + r_1^4 + r_2^4)\lambda(1, r_1^2, r_2^2) - (1 - x^2)^2 \lambda^2(1, r_1^2, r_2^2) \right], \tag{9}
\]

where \( \Delta = 4r_1^2 r_2^2/(1 - r_1^2 - r_2^2)^2 \). The numerical value of \( \Delta \) is very small for both charmonia and light vector mesons. For example, \( \Delta = 0.043 \) for \( J/\psi + J/\psi \) and \( \Delta = 0.00012 \) for \( \rho^0 + \rho^0 \). In the arguments on the left side of Eq. (9), we have indicated the dependences on the masses of the vector mesons, as they will play a rôle in a subsequent discussion of the effect of the width of the \( \rho \) meson. The total cross section \( \sigma \) is obtained by integrating over \( x \) from \(-1\) to \(+1\) if the two vector mesons are distinct. If the two vector mesons are identical, then the integration range of \( x \) is taken to be \( 0 \) to \( 1 \) in order to avoid double counting. The integration formulas that are required to evaluate the definite integrals over \( x \) from \(-a\) to \( a \) analytically are

\[
\int_{-a}^{a} dx \frac{1}{[1 - (1 - \Delta)x^2]^2} = \frac{a}{1 - a^2 + a^2 \Delta} + \frac{1}{1 + a\sqrt{1 - \Delta}} \ln \frac{1 + a\sqrt{1 - \Delta}}{1 - a\sqrt{1 - \Delta}}, \tag{10a}
\]

\[
\int_{-a}^{a} dx \frac{1 - x^2}{[1 - (1 - \Delta)x^2]^2} = -\frac{a \Delta}{(1 - \Delta)(1 - a^2 + a^2 \Delta)} \ln \frac{1 + a\sqrt{1 - \Delta}}{1 - a\sqrt{1 - \Delta}} + \frac{2 - \Delta}{2(1 - \Delta)^{3/2}} \ln \frac{1 + a\sqrt{1 - \Delta}}{1 - a\sqrt{1 - \Delta}}, \tag{10b}
\]

\[
\int_{-a}^{a} dx \frac{(1 - x^2)^2}{[1 - (1 - \Delta)x^2]^2} = \frac{1}{(1 - \Delta)^2} \left( 2a + \frac{a \Delta^2}{1 - a^2 + a^2 \Delta} \right) \ln \frac{1 + a\sqrt{1 - \Delta}}{1 - a\sqrt{1 - \Delta}} + \frac{\Delta(\Delta - 4)}{2(1 - \Delta)^{5/2}} \ln \frac{1 + a\sqrt{1 - \Delta}}{1 - a\sqrt{1 - \Delta}}. \tag{10c}
\]

The cross sections for two vector mesons with \( J^{PC} = 1^{--} \) have also been calculated recently in Ref. [11]. As was pointed out in Ref. [11], at the level of precision of this work, it is necessary to take into account the nonzero width of the \( \rho \) meson. Following Ref. [11], we note that Eq. (9) can be generalized to the case of a continuous mass spectrum:

\[
\left( \frac{d\sigma}{dx} \right)_{cont} = \left( \frac{1}{4\pi^2 \alpha} \right)^2 \int d\tilde{m}_1^2 d\tilde{m}_2^2 \sigma_{e^+ e^- \to H(\tilde{m}_1) \sigma_{e^+ e^- \to H(\tilde{m}_2)}} \frac{d\sigma_{\gamma \gamma^* \gamma^*}}{dx}(\tilde{m}_1, \tilde{m}_2), \tag{11}
\]

8
where $\sigma_{e^+e^-\rightarrow H}(m)$ is the total cross section for the process $e^+e^- \rightarrow$ hadrons at an energy $m$ in the $e^+e^-$ rest frame, and $d\sigma_{\gamma^i\gamma^j}/dx$ is the cross section for $e^+e^-$ annihilation into two massive photons. $d\sigma_{\gamma^i\gamma^j}/dx$ can be related to $d\sigma/dx$ in Eq. (9):

$$
\frac{d\sigma_{\gamma^i\gamma^j}}{dx}(\tilde{m}_1, \tilde{m}_2) = \left(\frac{\tilde{m}_1^2\tilde{m}_2^2}{4\pi\alpha g_{\gamma^i\gamma^j}}\right)^2 \frac{d\sigma}{dx}(\tilde{m}_1, \tilde{m}_2) .
$$

(12)

The ranges of integration of $\tilde{m}_1$ and $\tilde{m}_2$ in Eq. (11) are determined by the physical region for the two-meson final state: $\tilde{m}_1 \leq \sqrt{s} - m_V$, for the case of a $\rho^0$ meson and another meson $V_i$ and $\tilde{m}_1 + \tilde{m}_2 \leq \sqrt{s}$ for the case of two $\rho^0$ mesons. One can recover the narrow-width approximation by writing

$$
\sigma_{\rho^0\rightarrow V_i}(\tilde{m}) = \left(\frac{4\pi\alpha g_{\gamma^i\gamma^j}}{m_V^2}\right)^2 \pi\delta(\tilde{m}^2 - m_V^2).
$$

(13)

We parametrize the contribution of the $\rho^0$ meson to $\sigma_{e^+e^-\rightarrow H}(\tilde{m})$, following Ref. [12], as

$$
\sigma_{e^+e^-\rightarrow \rho}(\tilde{m}) = \frac{8\pi\alpha^2}{3m^3} p_{\pi}(\tilde{m})^2 \theta(\tilde{m} - 2m_{\pi})|F_{\pi}(\tilde{m}^2)|^2,
$$

(14)

where $m_{\pi}$ is the pion mass and $p_{\pi}(s)$ is the pion momentum:

$$
p_{\pi}(s) = \sqrt{s/4 - m_{\pi}^2}.
$$

(15)

The form factor $F_{\pi}(s)$, which depends on the parameters $M_{\rho}$, $\Gamma_{\rho}$, and $\beta$, is

$$
F_{\pi}(s) = \frac{(1 + \beta)^{-1}M_{\rho}^2(1 + dM_{\rho}/M_{\rho})}{M_{\rho}^2 - s + f(s) - iM_{\rho}\Gamma_{\rho}(s)},
$$

(16)

where

$$
d = \frac{3m_{\pi}^2}{\pi p_{\pi}^2(M_{\rho}^2)} \log \frac{M_{\rho} + 2p_{\pi}(M_{\rho}^2)}{2m_{\pi}} + \frac{M_{\rho}}{2\pi p_{\pi}(M_{\rho}^2)} - \frac{m_{\pi}^2M_{\rho}}{\pi p_{\pi}^2(M_{\rho}^2)},
$$

(17)

$\Gamma_{\rho}(s)$ is the energy-dependent width of the $\rho$,

$$
\Gamma_{\rho}(s) = \Gamma_{\rho} \left[\frac{p_{\pi}(s)}{p_{\pi}(M_{\rho}^2)}\right]^3 \left[\frac{M_{\rho}^2}{s}\right]^{1/2},
$$

(18)

and the real part of the correction to the denominator is given by

$$
f(s) = \Gamma_{\rho} \frac{M_{\rho}^2}{p_{\pi}^2(M_{\rho}^2)} \left\{p_{\pi}^2(s)[h(s) - h(M_{\rho}^2)] + (M_{\rho}^2 - s)p_{\pi}^2(M_{\rho}^2)h'(M_{\rho}^2)\right\},
$$

(19a)

$$
h(s) = \frac{2p_{\pi}(s)}{s} \log \frac{\sqrt{s} + 2p_{\pi}(s)}{2m_{\pi}},
$$

(19b)

$$
h'(s) = \frac{dh(s)}{ds}.
$$

(19c)
TABLE II: Cross sections in units of fb for $e^+e^- \rightarrow V_1 + V_2$ at $E_{\text{beam}} = 5.29$ GeV, calculated by using the fragmentation approximation. The uncertainties that are shown are only those that arise from the uncertainties in the electronic widths of the vector mesons. The first five rows are calculated in the narrow-width approximation. The last row is calculated by taking into account the nonzero width of the $\rho$ meson, as is described in the text.

| $V_1 \setminus V_2$ | $\rho^0$          | $\omega$         | $\phi$          | $J/\psi$        | $\psi(2S)$       |
|---------------------|-------------------|------------------|-----------------|-----------------|------------------|
| $\rho^0$            | 139.61 ± 4.82     | 23.47 ± 0.88     | 35.93 ± 1.29    | 41.67 ± 1.27    | 15.60 ± 0.46     |
| $\omega$            | 0.99 ± 0.07       | 3.02 ± 0.14      | 3.50 ± 0.15     | 1.31 ± 0.05     |                  |
| $\phi$              | 2.30 ± 0.15       | 5.20 ± 0.21      | 1.94 ± 0.08     |                 |                  |
| $J/\psi$            | 2.52 ± 0.13       | 1.81 ± 0.06      |                 |                 |                  |
| $\psi(2S)$          |                   | 0.32 ± 0.02      |                 |                 |                  |
| $\rho^0$            | 126.08 ± 4.36     | 22.31 ± 0.84     | 34.18 ± 1.23    | 39.83 ± 1.22    | 14.92 ± 0.44     |

The parameters $M_\rho$ and $\Gamma_\rho$ are obtained from a fit to the data for the $e^+e^- \rightarrow \pi^+\pi^-$ cross section: $M_\rho = 775.65 \pm 0.64 \pm 0.50$ MeV and $\Gamma_\rho = 143.85 \pm 1.33 \pm 0.80$ MeV. (Similar results have been obtained in Ref. 13.) The parameter $\beta$ is not given in Ref. 10. We infer it from the value $\Gamma[\rho \rightarrow e^+e^-] = 7.06 \pm 0.11 \pm 0.05$ MeV that is given in Ref. 10 and the formula for the electronic width that is given in Ref. 12:

$$\Gamma[\rho \rightarrow e^+e^-] = \frac{2\alpha^2 p^3_e (M_\rho^2) (1 + d \Gamma_\rho/M_\rho)^2}{9 M_\rho \Gamma_\rho} \left(1 + \beta^2\right).$$  \hspace{1cm} (20)

Using the values for $M_\rho$ and $\Gamma_\rho$ from Ref. 10, we obtain $\beta = -0.0815234$.

In Table II, we give our results for the integrated cross sections, calculated in the narrow-width approximation, for the production of $V_1 + V_2$ for the vector mesons $V_i = \rho^0$, $\omega$, $\phi$, $J/\psi$, and $\psi(2S)$. We also give integrated cross sections for the production of $\rho^0 + V$ in which the nonzero width of the $\rho$ has been taken into account. The uncertainties that are shown are only those that arise from the uncertainties in the $V - \gamma$ coupling constants that are given in Table I. The effect of the nonzero width of the $\rho$ meson is to decrease the cross sections for $\rho^0 + \rho^0$ by about 10%, for $\rho^0 + \omega$ and $\rho^0 + \phi$ by about 5%, and for $\rho^0 + J/\psi$ and $\rho^0 + \psi(2S)$ by about 4%.

We note that the differences between the zero-width and nonzero-width results arise
FIG. 3: Differential cross sections $d\sigma/d|x|$ for $e^+e^-$ annihilation into $V_1 + V_2$ at $E_{beam} = 5.29$ GeV, where $V_i = \rho^0, \phi,$ or $J/\psi$. The areas under the curves are the total cross sections.

mostly from the fact that the quantity

$$I_\rho = \int_0^\infty d\tilde{m}^2 \sigma_{e^+e^- \rightarrow \rho}(\tilde{m})$$

(21)

differs from the coefficient of the $\delta$ function in the narrow-width approximation of Eq. (13). The quantity $I_\rho$ is about 6% smaller than the coefficient of the $\delta$ function in Eq. (13). (The effect of the nonzero width of the $\rho$ meson itself is to actually increase the cross sections.)

The coefficient of the $\delta$ function in Eq. (13) derives from the electronic width of the $\rho$ meson, which, in turn, is calculated in Ref. [10] by using Eq. (20). Eq. (20) is derived from the vector-meson-dominance model [12]. Hence, the experimental electronic width of the $\rho$ meson that is given in Ref. [10] depends on the assumptions of that model. An alternative definition of the electronic width of the $\rho$ meson can be obtained by equating $I_\rho$ to the coefficient of the $\delta$ function in Eq. (13) and using Eq. (2) to relate $g_{V_i\gamma}^2$ to $\Gamma[\rho \rightarrow e^+e^-]$. This approach leads to the result

$$\Gamma[\rho \rightarrow e^+e^-] = \frac{m_\rho}{12\pi^2} I_\rho.$$  

(22)

It can be shown that the definition in Eq. (22) differs from the one in Eq. (20) by terms of order $\Gamma_\rho^2/m_\rho^2$.

In Fig. 3 we show the differential cross sections $d\sigma/d|x|$ for the production of the identical vector mesons $\rho^0 + \rho^0, \phi + \phi,$ and $J/\psi + J/\psi$ and for the production of the distinct vector
mesons $\rho^0 + \phi$, $\rho^0 + J/\psi$, and $\phi + J/\psi$. The differential cross sections peak sharply near $|x| = 1$ before falling to zero at the endpoint. For example, the maximum values of the differential cross sections are $7.1 \times 10^4$ fb at $x = 0.99994$ for $\rho^0 + \rho^0$ and $13.3$ fb at $x = 0.993$ for $J/\psi + J/\psi$.

We now discuss the theoretical uncertainties in the predictions for the cross sections in Table III. The scaling of the theoretical uncertainties was determined in Section III. In the case of two light vector mesons, the leading theoretical uncertainties arise from the fragmentation approximation and from exchanges of collinear gluons between the mesons. These uncertainties both scale as $(\Lambda_{\text{QCD}}/E_{\text{beam}})^2$. If we take $\Lambda_{\text{QCD}} \approx 0.5$ GeV, then the estimated fractional uncertainty is 0.9%. In the case of a light vector meson and a charmonium meson, there is an additional theoretical uncertainty that arises from the exchange of hard gluons between the mesons that scales as $\alpha_s^2(E_{\text{beam}}) (m_c \Lambda_{\text{QCD}}/E_{\text{beam}}^2)$. If we take $\alpha_s^2(E_{\text{beam}}) = 0.25$ and $m_c = 1.4$ GeV, then the estimated fractional uncertainty is 0.2%. In the case of two charmonium mesons, the leading theoretical uncertainty arises from the fragmentation approximation and scales as $m_c^2/E_{\text{beam}}^2$. If we take $m_c = 1.4$ GeV, then the estimated fractional uncertainty is 7%. This uncertainty can be reduced by calculating the contributions to the cross section from nonfragmentation diagrams. We discuss this calculation in Sec. IV.

IV. NONFRAGMENTATION CORRECTIONS TO THE PRODUCTION OF TWO CHARMONIA

In this section, we calculate corrections to the production cross sections for two charmonium mesons that arise from the nonfragmentation contributions to the amplitudes. The contributions from the nonfragmentation diagrams in Fig. 2 can be calculated using the NRQCD factorization formalism [14]. This formalism allows QCD radiative corrections and relativistic corrections to be taken into account systematically. For example, the expression for the $V - \gamma$ coupling constant defined by Eq. (3), including the leading QCD radiative correction and the leading relativistic correction, is

$$g_{V\gamma} = e_c \sqrt{2m_V} \left(1 - \frac{8\alpha_s}{3\pi} - \frac{1}{6} \langle v^2 \rangle_V \right) \langle V(\lambda)|\psi^\dagger \tilde{\sigma}\chi|0 \rangle \cdot \bar{e}(\lambda). \quad (23)$$

Here $\psi^\dagger$ and $\chi$ are the two-component Pauli operators in the NRQCD formalism that create a heavy quark and antiquark, respectively. $\langle v^2 \rangle_V$ is proportional to a ratio of matrix elements
of NRQCD operators between the state $V$ and the vacuum. The factor $e_c$ comes from the electromagnetic current in Eq. (1). The factor $\sqrt{2m_V}$ takes into account the difference between the standard relativistic and nonrelativistic normalizations of the state $|V(\lambda)\rangle$.

The NRQCD factorization formalism was used in Refs. [1, 2] to calculate the cross sections for double-charmonium production from $e^+e^-$ annihilation into two virtual photons. The calculation of the cross sections and the estimates of theoretical errors were carried out in a way that was as close as possible to a previous calculation of the cross sections for double-charmonium production from $e^+e^-$ annihilation into a single virtual photon [3]. The cross sections were expressed in terms of the coupling constants $\alpha$ and $\alpha_s$, the pole mass $m_c$ of the charm quark, and a factor $\langle O_1 \rangle_V$ that is related to the NRQCD matrix element in Eq. (23):

$$\langle O_1 \rangle_V = \frac{1}{3} \sum \lambda |\langle V(\lambda) | \psi^\dagger \vec{\sigma} \chi |0 \rangle|^2.$$ (24)

The fragmentation terms in the cross sections in Ref. [2] can be recovered by replacing $g^2_{V\gamma}$ in Eq. (5) with $16m_c\langle O_1 \rangle_V/9$ and by replacing $m_V$ with $2m_c$. (In the calculation of Ref. [2], the relative momentum of the $c$ and $\bar{c}$ that form the charmonium and their binding energy were neglected, and so the invariant mass of the $c\bar{c}$ pair was taken to be $2m_c$.) In Table II of Ref. [2], the cross section for $J/\psi + J/\psi$ was given as $6.65 \pm 3.02$ fb, where the uncertainty comes only from varying the charm quark mass over the range $1.2$ GeV $\leq m_c \leq 1.6$ GeV. This result includes both the fragmentation diagrams in Fig. 1 and the nonfragmentation diagrams in Fig. 2. It does not include QCD radiative corrections or relativistic corrections, which are known only for the fragmentation term. As is described in Ref. [2], if the known corrections to the fragmentation term in the cross section are applied to the entire cross section, then the central value for the $J/\psi + J/\psi$ cross section decreases to about 2 fb. This value is reasonably close to the result in Table III but there are large uncertainties that arise from the uncertainty in $m_c$. Furthermore, this procedure suffers from the deficiency that the radiative and relativistic corrections to the fragmentation term in the cross section are not necessarily valid for the entire cross section.

A more correct procedure would be to apply the known corrections to the fragmentation amplitude to that amplitude alone in the calculation of Ref. [2]. However, this approach would still be hampered by a large uncertainty in the fragmentation amplitude that arises from the uncertainty in $m_c$. Furthermore, there are higher-order radiative and relativistic corrections to the fragmentation amplitude that would not be included in this approach.
Both of these drawbacks can be eliminated by writing the fragmentation amplitude in terms of $g_{V\gamma}$ and $m_V$ instead of $\langle O_1 \rangle_V$ and $m_c$. The resulting expression for the fragmentation term in the differential cross section is Eq. (5), which has no explicit dependence on the charm-quark mass. Note that this procedure cannot be used to account correctly for the radiative and relativistic corrections to the nonfragmentation amplitude. These corrections to the nonfragmentation amplitude do not necessarily correspond to those that are contained in $g_{V\gamma}$. Furthermore, it would be inappropriate to set $m_c$ equal to $m_V/2$ in the nonfragmentation amplitude, since the vector-meson mass has little to do with the propagation of a charm quark at short distances of order $1/E_{\text{beam}}$. In the case of production of $J/\psi + \psi(2S)$, no consistent choice would be possible because $m_{J/\psi} \neq m_{\psi(2S)}$.

We now explain precisely how we calculate improved cross sections for two charmonium vector mesons that take into account the nonfragmentation contributions. For each pair of vector meson helicities $\lambda_1$ and $\lambda_2$, the cross section can be expressed as the product of a flux factor $1/(2s)$, the square of a T-matrix element, and a phase-space factor $\lambda^{1/2}(1, r_1^2, r_2^2)/(8\pi)$. We use the physical vector meson masses in the phase-space factor. The T-matrix element is the sum of a fragmentation amplitude and a nonfragmentation amplitude. The cross section is the sum of a fragmentation contribution, a nonfragmentation contribution, and an interference contribution. Our expression for the fragmentation amplitude is proportional to $g_{V_1 \gamma} g_{V_2 \gamma}$, with a coefficient that is a function of the meson masses $m_{V_1}$ and $m_{V_2}$. Hence, the fragmentation contribution to the cross section is given by Eq. (5). Our expression for the nonfragmentation amplitude is proportional to $(4m_{V_1} m_{V_2} \langle O \rangle_{V_1} \langle O \rangle_{V_2})^{1/2}$, with a coefficient that is a function of the quark mass $m_c$. The factors $(2m_{V_i})^{1/2}$ arise from the relativistic normalizations of the meson states. Aside from these normalization factors, the square of the nonfragmentation contribution to the T-matrix is identical to that in Ref. [2]. The interference contribution to the square of the T-matrix element depends on both the meson masses and the quark mass. We calculate the cross sections for each helicity combination and then add them. This approach is convenient for taking into account the difference between the polarization vector for a meson with mass $m_{V_i}$ and the polarization vector for a quark pair with invariant mass $2m_c$. The interference and nonfragmentation contributions to the cross section are given in Eqs. (A5) and (A6) in the Appendix.

Next, let us describe how we estimate the residual theoretical uncertainties. The uncertainties are obtained by adding five theoretical uncertainties in quadrature. The only
significant uncertainties in the fragmentation amplitudes are those that arise from the electronic widths of the vector mesons, which enter through the overall factor of $g_{V1\gamma}g_{V2\gamma}$. These uncertainties affect the fragmentation and interference terms in the cross sections. The most significant uncertainties in the nonfragmentation amplitudes arise from the NRQCD matrix elements, QCD radiative corrections, relativistic corrections, and the charm-quark mass. These uncertainties affect the interference and nonfragmentation contributions to the cross sections. We estimate the error associated with the charm-quark mass by varying $m_c$ in the interference and nonfragmentation terms in the cross section over the range $m_c = 1.4 \pm 0.2$ GeV. Our estimates of the uncertainties in the NRQCD matrix elements $\langle O_1 \rangle_V$ are described below. We assume that the QCD radiative corrections to the nonfragmentation amplitude are of relative size $\alpha_s(2m_c) = 0.25$. We assume that the relativistic corrections are of relative size $(\langle v^2 \rangle_V + \langle v^2 \rangle_{V_i})/2$. Radiative corrections to the fragmentation amplitude that involve the exchange of hard gluons between the two mesons are suppressed as $\alpha_s^2(E_{\text{beam}})(m_c^2/E_{\text{beam}}^2) \approx 0.4\%$ and can be neglected. Corrections that involve the exchange of soft gluons between the two mesons are suppressed as $(m_c v)^4/E_{\text{beam}}^4 \approx 0.04\%$ and can also be neglected.

We compute the NRQCD matrix elements $\langle O_1 \rangle_{J/\psi}$ and $\langle O_1 \rangle_{\psi(2S)}$ from the electronic widths of the charmonia by making use of the formula

$$\Gamma[V \rightarrow e^+e^-] = \frac{8e^2C_F\pi\alpha_s^2}{3} \frac{\langle O_1 \rangle_V}{m_V^2} \left( 1 - \frac{8}{3} \frac{\alpha_s}{\pi} - \frac{1}{6} \langle v^2 \rangle_V \right)^2,$$

which follows from Eq. (23). Note that Eq. (25) differs from the NRQCD factorization formula that is given in Ref. [14]: $m_c$ has been replaced with $m_V/2$ and, consequently, the relativistic correction has changed from $-\frac{2}{3}\langle v^2 \rangle_V$ to $-\frac{1}{6}\langle v^2 \rangle_V$. The formula (25) is less subject to uncertainties in the value of $\langle v^2 \rangle_V$ than the standard NRQCD formula and, through its dependence on $m_V$, resums some corrections of higher order in $v$. In our calculation, we take $\langle v^2 \rangle_{J/\psi} = 0.25 \pm 0.09$ (Ref. [15]), $\langle v^2 \rangle_{\psi(2S)} = 0.45 \pm 0.19$ (Ref. [16]), and $\alpha_s(2m_c) \approx 0.25$. We obtain

$$\langle O_1 \rangle_{J/\psi} = 0.482 \pm 0.049 \text{ GeV}^3,$$

$$\langle O_1 \rangle_{\psi(2S)} = 0.335 \pm 0.080 \text{ GeV}^3.$$  

The error bars are obtained by combining in quadrature the uncertainties from the electronic widths, the uncertainties from the values of $\langle v^2 \rangle_V$, and the estimated errors from
TABLE III: Cross sections in units of fb for $e^+e^− → V_1 + V_2$ at $E_{beam} = 5.29$ GeV for charmonium vector mesons, calculated by including both the fragmentation and nonfragmentation amplitudes, as is described in the text. The four rows give the fragmentation, interference, nonfragmentation, and total contributions to the cross sections. The uncertainties are obtained by combining five uncertainties in quadrature, as is described in the text.

| cross section     | $J/\psi + J/\psi$ | $J/\psi + \psi(2S)$ | $\psi(2S) + \psi(2S)$ |
|-------------------|-------------------|---------------------|------------------------|
| fragmentation     | 2.52 ± 0.13       | 1.81 ± 0.06         | 0.32 ± 0.02            |
| interference      | −0.98 ± 0.48      | −1.09 ± 0.60        | −0.30 ± 0.19           |
| nonfragmentation  | 0.15 ± 0.16       | 0.23 ± 0.29         | 0.09 ± 0.14            |
| total             | 1.69 ± 0.35       | 0.95 ± 0.36         | 0.11 ± 0.09            |

uncalculated radiative and relativistic corrections, which we assume to be of relative size $\alpha_s^2$ and $\langle v^2 \rangle^2$ in the rate, respectively. Note that the values of the NRQCD matrix elements in Eq. (26) are considerably larger than those that were used in Refs. [1, 2].

Our results for the cross sections for the production of $J/\psi + J/\psi$, $J/\psi + \psi(2S)$, and $\psi(2S) + \psi(2S)$ are given in Table III. Note that the error bars for the total cross section are less than the error bars for the fragmentation, interference, and nonfragmentation contributions added in quadrature because the interference uncertainties are 100% anticorrelated with the fragmentation and nonfragmentation uncertainties. The cross section for $J/\psi + J/\psi$ is 1.69 ± 0.35 fb. The central value is 33% smaller than the value in the fragmentation approximation (Table II or the first row of Table III). The 33% correction to the fragmentation approximation is much larger than the estimate $m_c^2/E_{beam}^2 ≈ 7\%$ that was given at the end of Sec. III. The reason for the large relative size of this correction is that radiative and relativistic corrections to the fragmentation contribution to the cross section, which are contained implicitly in the quantities $g_{V_i\gamma}$ and $m_{V_i}$, decrease the fragmentation contribution by about a factor of three from its value without radiative and relativistic corrections. We are able to obtain a reasonably accurate result for the $J/\psi + J/\psi$ total cross section, in spite of the large relative errors in the nonfragmentation and interference terms, because the total cross section is dominated by the fragmentation term.
V. COMPARISON WITH EXPERIMENT AND PREVIOUS THEORETICAL RESULTS

The BaBar Collaboration has recently measured the cross sections for production of $\rho^0 + \rho^0$ and $\rho^0 + \phi$ in $e^+e^-$ collisions at energy $E_{\text{beam}} = 5.29$ GeV [17]:

\begin{align}
\sigma[e^+e^- \rightarrow \rho^0 + \rho^0] &= 20.7 \pm 0.7_{\text{stat}} \pm 2.7_{\text{syst}} \text{ fb,} \\
\sigma[e^+e^- \rightarrow \rho^0 + \phi] &= 5.7 \pm 0.5_{\text{stat}} \pm 0.8_{\text{syst}} \text{ fb.} \tag{27a}
\end{align}

The measured cross sections are subject to the cuts $|\cos \theta| < 0.8, 0.5 \text{ GeV} < m_\rho < 1.1 \text{ GeV,}$ and $1.008 \text{ GeV} < m_\phi < 1.035 \text{ GeV.}$

In Table IV we give our predictions for the cross sections that involve light mesons, integrated over the region $|\cos \theta| < 0.8$. In Table V we give our predictions for the charmonium-charmonium cross sections, integrated over the same angular region. The cross sections in Tables IV and V are much smaller than those in Table II and Table III respectively, because the cut on $\theta$ excludes most of the peak near $|x| = 1$. In the last row of Table IV the $m_\rho$ cut is implemented. We can calculate the effect of the $m_\phi$ cut as follows. The cross section to produce a $\phi$ meson and one of the other vector mesons is a constant, to within 1.5%, over the range $1.008 \text{ GeV} < m_\phi < 1.035 \text{ GeV}$. Hence, we can calculate the fractional correction from the $m_\phi$ cut, with errors of less than 1%, by calculating the integral with respect to $m_\phi^2$ of the $\phi$ line shape over the range $1.008 \text{ GeV} < m_\phi < 1.035 \text{ GeV}$ and comparing it with the integral of the line shape from threshold to infinity. Using a simple Breit-Wigner line shape and taking the values of $m_\phi$ and $\Gamma_\phi$ in Table I we find that the fraction 0.898 of the integral of the line shape is contained in the range $1.008 \text{ GeV} < m_\phi < 1.035 \text{ GeV}$. Therefore, in order to compare the results in Table IV with the BaBar results, one should multiply each entry in Table IV by a factor 0.898 for each $\phi$ meson in the final state. Specifically, we obtain $\sigma[e^+e^- \rightarrow \rho^0 + \phi] = 5.04 \pm 0.18 \text{ fb}$ with the mass cuts on both $m_\rho$ and $m_\phi$. Our predictions for $\rho^0 + \rho^0$ and $\rho^0 + \phi$ in Table IV agree with the BaBar results in Eq. (27) to within the experimental and theoretical uncertainties.

Our results in Tables II and IV differ somewhat from those in Ref. [11]. Some of the differences arise because we are using the 2006 compilation of the Particle Data Group [9], rather than the 2004 compilation [18], for the meson masses and widths. However, once these differences in the input data are taken into account, discrepancies still remain.
TABLE IV: Cross sections in units of fb for $e^+e^- \rightarrow V_1 + V_2$ at $E_{\text{beam}} = 5.29$ GeV, calculated using the fragmentation approximation. The angular cut $|\cos \theta| < 0.8$, which is used in Ref. [17], has been applied. The uncertainties shown are only those that arise from the uncertainties in the electronic widths of the vector mesons. The first three rows are calculated in the narrow-width approximation. The last two rows are calculated by taking into account the nonzero width of the $\rho$ meson, as is described in the text. In the last row, the cut $0.5$ GeV $< m_\rho < 1.1$ GeV, which is used in Ref. [17], has been applied.

| $V_1 \ \backslash \ V_2$ | $\rho^0$ | $\omega$ | $\phi$ | $J/\psi$ | $\psi(2S)$ |
|----------------------|--------|--------|--------|--------|--------|
| $\rho^0$             | 23.30 ± 0.80 | 3.93 ± 0.15 | 6.43 ± 0.23 | 10.92 ± 0.33 | 4.44 ± 0.13 |
| $\omega$             | 0.17 ± 0.01 | 0.54 ± 0.02 | 0.92 ± 0.04 | 0.37 ± 0.02 |
| $\phi$ (no mass cut) | 0.44 ± 0.03 | 1.50 ± 0.06 | 0.61 ± 0.02 |
| $\rho^0$ (mass cut)  | 20.50 ± 0.71 | 3.68 ± 0.14 | 6.03 ± 0.22 | 10.24 ± 0.31 | 4.17 ± 0.12 |
| $\rho^0$ (mass cut)  | 17.71 ± 0.61 | 3.42 ± 0.13 | 5.61 ± 0.20 | 9.52 ± 0.29 | 3.87 ± 0.11 |

TABLE V: Cross sections in units of fb for $e^+e^- \rightarrow V_1 + V_2$ at $E_{\text{beam}} = 5.29$ GeV for charmonium vector mesons, calculated by including both the fragmentation and nonfragmentation amplitudes, as is described in the text. The angular cut $|\cos \theta| < 0.8$ has been applied. The four rows give the fragmentation, interference, nonfragmentation, and total contributions to the cross sections. The uncertainties are obtained by combining five uncertainties in quadrature, as is described in the text.

| cross section | $J/\psi + J/\psi$ | $J/\psi + \psi(2S)$ | $\psi(2S) + \psi(2S)$ |
|---------------|-------------------|-------------------|-------------------|
| fragmentation | 1.20 ± 0.06       | 0.94 ± 0.03       | 0.18 ± 0.01       |
| interference  | −0.72 ± 0.36      | −0.81 ± 0.45      | −0.22 ± 0.15      |
| nonfragmentation | 0.13 ± 0.14     | 0.20 ± 0.25       | 0.08 ± 0.12       |
| total         | 0.60 ± 0.24       | 0.33 ± 0.24       | 0.04 ± 0.06       |

The largest discrepancy is between the results for $\rho^0 + \psi(2S)$ production with nonzero $\rho$-meson width and $|\cos \theta| < 0.8$. This discrepancy is about 27%. Other discrepancies for cross sections computed with the cut $|\cos \theta| < 0.8$ are approximately 4% or less. For cross sections computed with $|\cos \theta| < 1.0$, the discrepancies are as large as 10%.
The production cross sections for two vector-meson charmonium states were also calculated in the fragmentation and narrow-width approximations in Ref. [19]. In this paper, somewhat different values for the electronic widths of the charmonium states were used, and the resulting production cross sections are lower than ours by about 10–30%.

The Belle Collaboration has set an upper limit on the cross section for $J/\psi + J/\psi$ in $e^+e^-$ collisions at energy $E_{\text{beam}} \approx 5.29$ GeV [20]:

$$\sigma[e^+e^- \rightarrow J/\psi + J/\psi] \times B_{>2}[J/\psi] < 9.1 \text{ fb} \quad (90\% \text{ C.L.}), \quad (28a)$$

$$\sigma[e^+e^- \rightarrow J/\psi + \psi(2S)] \times B_{>2}[\psi(2S)] < 5.2 \text{ fb} \quad (90\% \text{ C.L.}), \quad (28b)$$

where $B_{>2}[V]$ is the branching fraction of $V$ into final states with more than two charged tracks. These upper limits are compatible with the predictions in Table III. By adding up exclusive branching fractions for $J/\psi$ decays [9], one can show that the branching fraction in Eq. (28a) satisfies $13\% < B_{>2}[J/\psi] < 80\%$. It should be possible to measure $B_{>2}[J/\psi]$ and $B_{>2}[\psi(2S)]$ at CLEOc or BESIII. The upper limit in Eq. (28a) was obtained with a data sample of 155 fb$^{-1}$ at or near the $\Upsilon(4S)$. The combined data samples of the Belle and BaBar experiments now exceed 1000 fb$^{-1}$. Our prediction for the cross section for $e^+e^- \rightarrow J/\psi + J/\psi$ indicates that there is a possibility that this process can be observed at the $B$ factories.

**APPENDIX A: INTERFERENCE AND NONFRAGMENTATION CONTRIBUTIONS TO THE CROSS SECTIONS**

In this appendix, we give the interference and nonfragmentation contributions to the differential cross section for $e^+e^- \rightarrow V_1(\lambda_1) + V_2(\lambda_2)$, where $V_1$ and $V_2$ are charmonium vector mesons, and $\lambda_1$ and $\lambda_2$ are their helicities. The differential cross section for the fragmentation contribution can be found in Eq. (5). The interference contribution to the differential cross section is

$$\frac{d\sigma^{\text{int}}}{dx}[V_1(\lambda_1) + V_2(\lambda_2)] = -\frac{1024\pi^3\alpha^4 g_{V_1\gamma} g_{V_2\gamma} (4e^4 m_{V_1} m_{V_2} \langle O \rangle_{V_1} \langle O \rangle_{V_2})^{1/2} \lambda_1^{1/2} (1, r_1, r_2)}{3s^5 r_1^2 r_2^2 (1 - r_1^2 - r_2^2)^2 [1 - (1 - \Delta)x^2]} \times F_{\lambda_1,\lambda_2}^{\text{int}}(r_1, r_2, r, x), \quad (A1)$$

where $r = 4m_c/\sqrt{s}$. (Note that $r_1$ and $r_2$ reduce to $r/2$ in the nonrelativistic limit.) In Eq. (A1), the numerator factor in parentheses corresponds to the expression for $g_{V_1\gamma} g_{V_2\gamma}$ at
leading order in $\alpha_s$ and $v$ [Eq. (23)]. Similar factors appear in subsequent equations for cross sections in this Appendix. The functions $F_{\lambda_1,\lambda_2}^{\text{int}}(r_1, r_2, r, x)$ are given by

\[
F_{\pm 1,\pm 1}^{\text{int}}(r_1, r_2, r, x) = r^2 x^2 (1 - x^2)[(1 - r_1^2 - r_2^2) - \lambda(1, r_1^2, r_2^2)], \quad (A2a)
\]
\[
F_{\pm 1,\mp 1}^{\text{int}}(r_1, r_2, r, x) = (1 - r_1^2 - r_2^2)(1 - x^4), \quad (A2b)
\]
\[
F_{\pm 1,0}^{\text{int}}(r_1, r_2, r, x) = rr_2[(1 - r_1^2 - r_2^2)(1 - x^2)(1 - 2x^2) + 4r_1^2 x^4], \quad (A2c)
\]
\[
F_{0,\pm 1}^{\text{int}}(r_1, r_2, r, x) = rr_1[(1 - r_1^2 - r_2^2)(1 - x^2)(1 - 2x^2) + 4r_2^2 x^4], \quad (A2d)
\]
\[
F_{0,0}^{\text{int}}(r_1, r_2, r, x) = 4r_1 r_2(1 + r^2)x^2(1 - x^2). \quad (A2e)
\]

The nonfragmentation contribution to the differential cross section is

\[
\frac{d\sigma^{\text{nf}}}{dx}[V_1(\lambda_1) + V_2(\lambda_2)] = \frac{8192\pi^3 \alpha^4(4e_0^4 m_{V_1} m_{V_2} \langle O \rangle V_1 \langle O \rangle V_2) \lambda^{1/2}(1, r_1^2, r_2^2) F_{\lambda_1,\lambda_2}^{\text{int}}(r, x)}{9s^5 r^4}, \quad (A3)
\]

where

\[
F_{\pm 1,\pm 1}^{\text{nf}}(r, x) = 2r^4 x^2(1 - x^2), \quad (A4a)
\]
\[
F_{\pm 1,\mp 1}^{\text{nf}}(r, x) = 2(1 - x^4), \quad (A4b)
\]
\[
F_{\pm 1,0}^{\text{nf}}(r, x) = F_{0,\pm 1}^{\text{nf}}(r, x) = r^2(1 - 3x^2 + 4x^4), \quad (A4c)
\]
\[
F_{0,0}^{\text{nf}}(r, x) = 2(1 + r^2)^2 x^2(1 - x^2). \quad (A4d)
\]

After summing over helicity states, one finds that the interference and nonfragmentation contributions to the differential cross section become

\[
\frac{d\sigma^{\text{int}}}{dx}(m_{V_1}, m_{V_2}, m_c) = -\frac{2048\pi^3 \alpha^4 g_{V_1} g_{V_2} (4e_0^4 m_{V_1} m_{V_2} \langle O \rangle V_1 \langle O \rangle V_2)^{1/2} \lambda^{1/2}(1, r_1^2, r_2^2)}{3s^5 r^4 r_1^2 r_2^2(1 - r_1^2 - r_2^2)^2[1 - (1 - \Delta)x^2)]
\]
\[
\times \left\{ 4rr_1 r_2(1 + r_2) + (1 - x^2)[r^2(r_1 + r_2)^2(1 - r_1 + r_2)(1 + r_1 - r_2)
- r(r_1 + r_2)(1 - r_1^2 + 8r_1 r_2 - r_2^2) + 2(1 - r_1^2 + r_1 r_2 - r_2^2)]
- (1 - x^2)[1 - r(r_1 + r_2)]^2(1 - r_1 + r_2)(1 + r_1 - r_2) \right\} \quad (A5)
\]

and

\[
\frac{d\sigma^{\text{nf}}}{dx}(m_{V_1}, m_{V_2}, m_c) = \frac{16384\pi^3 \alpha^4 (4e_0^4 m_{V_1} m_{V_2} \langle O \rangle V_1 \langle O \rangle V_2) \lambda^{1/2}(1, r_1^2, r_2^2)}{9s^5 r^4}
\]
\[
\times [4r^2 + (1 - x^2)(3r^2 - 8r^2 + 5) - 3(1 - x^2)^2(1 - r^2)^2]. \quad (A6)
\]
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