New approach to $\varepsilon$-entropy and Its comparison with
Kolmogorov’s $\varepsilon$-entropy

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Abstract

Kolmogorov introduced a concept of $\varepsilon$-entropy to analyze information in classical continuous system. The fractal dimension of geometrical sets was introduced by Mandelbrot as a new criterion to analyze the complexity of these sets. The $\varepsilon$-entropy and the fractal dimension of a state in general quantum system were introduced by one of the present authors in order to characterize chaotic properties of general states.

In this paper, we show that $\varepsilon$-entropy of a state includes Kolmogorov $\varepsilon$-entropy, and the fractal dimension of a state describe fractal structure of Gaussian measures.

1 Introduction

The $\varepsilon$-entropy was introduced by Kolmogorov (1963) using the mutual entropy with respect to two random variables $f$ and $g$. The entropy $S(f)$ of a random variable $f$ is usually infinite on a continuous probability space. On the other hand, the $\varepsilon$-entropy $S_{\text{Kolmogorov}}(f;\varepsilon)$ ($S_{\text{K}}(f;\varepsilon)$ for short) can be bounded. Therefore, we can use the $\varepsilon$-entropy to analyze random variables in classical system. This $\varepsilon$-entropy $S_{\text{K}}(f;\varepsilon)$ expresses a degree of information transmission in the $\varepsilon$-neighborhood of a random variable $f$.

By the way, Mandelbrot introduced a new criterion to analyze complexity of geometrical sets, it is so called fractal dimension (Mandelbrot, 1982), which is different from the euclidean dimensions. Usual fractal theory mostly treats only geometrical sets. It is desirable to extend the fractal dimensions in order to characterize some other objects. One of the present authors introduced the notion of $\varepsilon$-entropy $S_{\text{Ohya}}(\mu;\varepsilon)$ ($S_{O}(\mu;\varepsilon)$ for short) for a state in order to formulate the fractal dimension of a state in general quantum system (GQS for short) (Ohya, 1989),(Ohya, 1991),(Ohya and Petz, 1993). Actually, the capacity dimensions, which is one of the fractal dimensions for geometrical sets, was given by the $\varepsilon$-entropy. Namely, it is defined by

$$d_{C}(X) = \lim_{\varepsilon \to 0} \frac{\log N_{X}(\varepsilon)}{\log \frac{1}{\varepsilon}},$$

where $N_{X}(\varepsilon)$ is the minimum numbers of a convex set with diameter $\varepsilon$ covering a set $X$ and $\log N_{X}(\varepsilon)$ is called $\varepsilon$-entropy of a geometrical set $X$ (Kolmogorov and Tihomirov, 1961). The capacity dimension characterize fractal
structure of the geometrical sets as a limiting behavior of \( \varepsilon \)-entropy when \( \varepsilon \) approach to 0.

Our fractal dimension of a state is formulated by extending the concept of the capacity dimension to GQS. That is, the fractal dimension of a state is expressed by the \( \varepsilon \)-entropy of a state instead of the \( \varepsilon \)-entropy of a geometrical set \( X \) and characterize fractal structure of a state.

These \( \varepsilon \)-entropy and fractal dimension provide new criteria describing the complexity of states, so that they can be used to distinguish two states even when they have the same value for the entropy. For instance, we could analyze the complexity for some systems by these criteria (Ohya, 1991),(Matsuoka and Ohya,1995),(Akashi, 1992).

In this paper, we examine the similarity and difference of two \( \varepsilon \)-entropies \( S_K \) and \( S_O \) for Gaussian measures (states) on a Hilbert space. It is shown that our \( \varepsilon \)-entropy and fractal dimension are useful to classify Gaussian measures, in the case of that Kolmogorov’s \( \varepsilon \)-entropy can not be used.

## 2 Kolmogorov’s \( \varepsilon \)-entropy for random variables

In this section, we remind the definition of Kolmogorov’s \( \varepsilon \)-entropy for random variables. Let \((\Omega, \mathcal{F}, \mu)\) be a probability space and \(M(\Omega)\) be the set of all random variables, and \(f, g\) be two random variables on \(\Omega\) with valued on a metric space \((X, d)\). Let \(\mu_f\) be a probability measure associated with a random variable \(f\). Then the mutual entropy \(I(f, g)\) of the random variable \(f\) and \(g\) is defined by (Gelfand and Yaglom, 1959)

\[
I(f, g) = S(\mu_{fg}, \mu_f \otimes \mu_g) = \left\{ \int_X \int_X \frac{d\mu_{fg}}{d\mu_f \otimes \mu_g} \log \frac{d\mu_{fg}}{d\mu_f \otimes \mu_g} d\mu_f \otimes \mu_g \right\},
\]

where \(S(\cdot, \cdot)\) is relative entropy (Kulback-Leibler information), \(\mu_f \otimes \mu_g\) is the direct product probability measure of \(f\) and \(g\), and \(\mu_{fg}\) is the joint probability distribution of \(f\) and \(g\), \(\frac{d\mu_{fg}}{d\mu_f \otimes \mu_g}\) is the Radon-Nikodym derivative of \(\mu_{fg}\) with respect to \(\mu_f \otimes \mu_g\). Moreover the entropy \(S(f)\) of the random variable \(f\) is given by

\[
S(f) = I(f, f). \tag{2}
\]
$S(f)$ is often infinite in continuous case. Kolmogorov introduced the $\varepsilon$-entropy for a random variable $f$ as follows:

$$S_K(f; \varepsilon) = \inf \{I(f, g); g \in M_d(f; \varepsilon)\},$$

(3)

where

$$M_d(f; \varepsilon) = \left\{ g \in M(\Omega) : \sqrt{\int_{X \times X} d(x, y)^2 d\mu_{fg}(x, y)} \leq \varepsilon \right\}.$$  

(4)

### 3 $\varepsilon$-entropy and fractal dimensions of a state

The $\varepsilon$-entropy and the fractal dimension of a state were introduced in Ohya (1991) for GQS. In this section, we review these formulations in the framework of classical measure theory.

In the information theory, an input state is described by a state (probability measure in continuous classical system, density operator in usual quantum system) and it is sent to an output system (receiver) through some channel denoted by $\Lambda^*$. A channel is a transmitter (e.g. optical fiber), mathematically it is a mapping from an input state space to an output state space.

When an input state $\mu$ dynamically changes to an output state $\bar{\mu}(\equiv \Lambda^* \mu)$ under a channel $\Lambda^*$, we ask how much information carried by $\mu$ can be transmitted to the output state through the channel $\Lambda^*$. It is the mutual entropy that represents this amount of information transmitted from $\mu$ to $\bar{\mu}$. Hence, the mutual entropy depends on an input state and a channel. In this scheme, the mutual entropy is formulated as follows:

Let $(\Omega_1, \mathcal{S}_1)$ be an input space, $(\Omega_2, \mathcal{S}_2)$ be an output space and $P(\Omega_k)$ be the set of all probability measures on $(\Omega_k, \mathcal{S}_k)(k = 1, 2)$. We can call the following linear mapping $\Lambda^*$ from $P(\Omega_1)$ to $P(\Omega_2)$ a channel (Markov kernel):

$$\bar{\mu}(Q) = \Lambda^* \mu(Q) = \int_{\Omega_1} \lambda(\omega, Q) \, d\mu(\omega) \quad \mu \in P(\Omega_1),$$

where $\lambda$ is a mapping from $\Omega_1 \times \mathcal{S}_2$ to $[0, 1]$ satisfying the following conditions:

(1) $\lambda(\cdot, Q)$ is a measurable function on $\Omega_1$ for each $Q \in \mathcal{S}_2$. 

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(2) $\lambda(\omega, \ldots) \in P(\Omega_2)$ for each $\omega \in \Omega_1$.

The compound state $\Phi$ of $\mu$ and $\bar{\mu}$ is given by

$$\Phi(Q_1 \times Q_2) = \int_{Q_1} \lambda(\omega, Q_2) \, d\mu(\omega),$$

for any $Q_1 \in \mathcal{F}_1, Q_2 \in \mathcal{F}_2$. The mutual entropy in classical continuous system can be expressed by the relative entropy $S(\cdot, \cdot)$ of the compound state $\Phi$ and the direct product state $\Phi_0 = \mu \otimes \Lambda^* \mu$:

$$I(\mu; \Lambda^*) = S(\Phi, \Phi_0) \equiv I(\mu, \bar{\mu}; \Phi)$$

$$= \left\{ \int_{\Omega_1 \times \Omega_2} \frac{d\Phi}{d\Phi_0} \log \frac{d\Phi}{d\Phi_0} \, d\Phi_0 \right. \begin{cases} \infty & (\Phi \ll \Phi_0) \\ \text{(otherwise)} \end{cases},$$

where $\frac{d\Phi}{d\Phi_0}$ is the Radon-Nikodym derivative of $\Phi$ with respect to $\Phi_0$.

In the following discussion, we consider the case of $(\Omega_1, \mathcal{F}_1) = (\Omega_2, \mathcal{F}_2) \equiv (\Omega, \mathcal{F})$ for simplicity.

We shall give the definition of Kolmogorov’s $\varepsilon$-entropy $S_K(\mu; \varepsilon)$ for a general probability measure $\mu$ on $(\Omega, \mathcal{F})$.

$$S_K(\mu; \varepsilon) \equiv \inf \left\{ I(\mu, \bar{\mu}; \Phi) ; ||\mu - \bar{\mu}|| \leq \varepsilon \right\},$$

where $||\mu||$ is a certain norm of $\mu$.

The $\varepsilon$-entropy of a state $\mu$ is defined as follows (Ohya, 1989).

**Definition 1:** (The $\varepsilon$-entropy of a state $\mu \in P(\Omega)$)

$$S_O(\mu; \varepsilon) \equiv \inf_{\Lambda^*} \left\{ J(\mu; \Lambda^*) ; ||\mu - \Lambda^* \mu|| \leq \varepsilon \right\},$$

where $||\mu||$ is a certain norm of $\mu$ and

$$J(\mu; \Lambda^*) \equiv \sup_{\Gamma^*} \left\{ I(\mu; \Gamma^*) ; \Gamma^* \mu = \Lambda^* \mu \right\},$$

Here $J(\mu; \Lambda^*)$ is called the maximum mutual entropy w.r.t. $\mu$ and $\Lambda^*$.

The $\varepsilon$-entropy $S_O(\mu; \varepsilon)$ is a bit more general than the Kolmogorov $\varepsilon$-entropy $S_K(f; \varepsilon)$ for random variables, more precisely the $\varepsilon$-entropy $S_O(\mu; \varepsilon)$ is different from $S_K(\mu; \varepsilon)$ in the following points.
(1) The definition is based on states (probability measures) not only random variables.

(2) Several possibilities to choose the norm of states.

(3) The concept of the maximum mutual entropy $J(\mu; \Lambda^*)$ is used, which is a very essential as is described in section 6.

This fractal dimension of a state $\mu$ in a classical continuous system is defined by the $\varepsilon$-entropy of states.

Definition 2: (The capacity dimension of a state $\mu \in P(\Omega)$ )

$$d_C^O \equiv \lim_{\varepsilon \to 0} \frac{S_0(\mu; \varepsilon)}{\log \frac{1}{\varepsilon}}$$

(10)

4 Gaussian measure and Gaussian channel on a Hilbert Space

We briefly review the Gaussian communication processes treated by Baker et al (Baker, 1978), (Yanagi, 1988).

Let $\mathcal{B}$ be the Borel $\sigma$-field of a real separable Hilbert space $\mathcal{H}$ and $\mu$ be a Borel probability measure on $\mathcal{B}$ satisfying

$$\int_{\mathcal{H}} \|x\|^2 d\mu(x) < \infty.$$  

(11)

Further, we denote the set of all positive self-adjoint trace class operators on $\mathcal{H}$ by $T(\mathcal{H})_+ \equiv \{R \in \mathcal{B}(\mathcal{H}); R \geq 0, R = R^*, \text{tr} R < \infty\}$ and define the mean vector $m_\mu \in \mathcal{H}$ and the covariance operator $R_\mu \in T(\mathcal{H})_+$ of $\mu$ such as

$$\langle x_1, m_\mu \rangle = \int_{\mathcal{H}} \langle x_1, y \rangle d\mu(y),$$

(12)

$$\langle x_1, R_\mu x_2 \rangle = \int_{\mathcal{H}} \langle x_1, y - m_\mu \rangle \langle y - m_\mu, x_2 \rangle d\mu(y),$$

(13)

for any $x_1, x_2, y \in \mathcal{H}$. A Gaussian measure $\mu$ in $\mathcal{H}$ is a Borel measure such that for each $x \in \mathcal{H}$, there exist real numbers $m_x$ and $\sigma_x(> 0)$ satisfying

$$\mu\{y \in \mathcal{H}; \langle y, x \rangle \leq a\}$$

$$= \int_{-\infty}^{a} \frac{1}{\sqrt{2\pi}\sigma_x} \exp \left\{ -\frac{(t - m_x)^2}{2\sigma_x^2} \right\} dt.$$
The notation $\mu = [m, R]$ means that $\mu$ is a Gaussian measure on $\mathcal{H}$ with a mean vector $m$ and a covariance operator $R$.

Let $(\mathcal{H}_1, \mathcal{B}_1)$ be an input space, $(\mathcal{H}_2, \mathcal{B}_2)$ be an output space and $P_{G}^{(k)}$ be the set of all Gaussian probability measures on $(\mathcal{H}_k, \mathcal{B}_k)(k = 1, 2)$. We consider the case of $(\mathcal{H}_1, \mathcal{B}_1) = (\mathcal{H}_2, \mathcal{B}_2) \equiv (\mathcal{H}, \mathcal{B})$ for simplicity. Moreover, let $\mu \in P(\mathcal{H})$ be a Gaussian measure of the input space and $\mu_0 \in P(\mathcal{H})$ be a Gaussian measure indicating a noise of the channel. Then, a Gaussian channel $\Lambda^*$ from $P(\mathcal{H})$ to $P(\mathcal{H})$ is defined by the following mapping $\lambda : \mathcal{H} \times \mathcal{B} \rightarrow [0, 1]$ such as

$$\overline{\mu}(Q) = \Lambda^* \mu(Q) \equiv \int_{\mathcal{H}} \lambda(x, Q)d\mu(x)$$  \hspace{1cm} (14)

$$\lambda(x, Q) \equiv \mu_0(Q^x),$$  \hspace{1cm} (15)

where $A$ is a linear transformation from $\mathcal{H}$ to $\mathcal{H}$ and $\lambda$ satisfies the following conditions:

1. $\lambda(x, \cdot) \in P(\mathcal{H})$ for each fixed $x \in \mathcal{H}$,
2. $\lambda(\cdot, Q)$ is measurable function on $(\mathcal{H}, \mathcal{B})$ for each fixed $Q \in \mathcal{B}$.

The compound measure $\Phi$ derived form the input measure $\mu$ and the output measure $\overline{\mu}$ is given by

$$\Phi(Q_1 \times Q_2) = \int_{Q_1} \lambda(x, Q_2)d\mu(x)$$  \hspace{1cm} (17)

for any $Q_1, Q_2 \in \mathcal{B}$.

In particular, let $\mu$ be $[0, R] \in P(\mathcal{H})$ and $\mu_0$ be $[0, R_0] \in P(\mathcal{H})$. Then, output measure $\Lambda^* \mu = \overline{\mu}$ can be expressed as

$$\Lambda^* \mu = [0, ARA^* + R_0].$$  \hspace{1cm} (18)

When the dimension of $\mathcal{H}$ is finite, the mutual entropy (information) with respect to $\mu$ and $\Lambda^*$ become

$$I(\mu; \Lambda^*) = \frac{1}{2} \log \frac{|ARA^* + R_0|}{|R_0|},$$  \hspace{1cm} (19)

where $|ARA^* + R_0|$, $|R_0|$ are determinants of $ARA^* + R_0$, $R_0$. 

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5  ε-entropy and fractal dimension of a state for Gaussian measures in the random variable norm

The ε-entropy of states described in section 3 is different from Kolmogorov's definition of the ε-entropy for random variable. In this section, we show that two definitions coincide when \( \mathcal{H} = \mathbb{R}^n \) and the norm of a state \( \mu_f \) is defined by

\[
\|\mu_f\| = \sqrt{\frac{1}{n} \sum_{i=1}^{n} \int_{\Omega} |f_i|^2 d\mu}. \tag{20}
\]

Then the distance between two states \( \mu_f \) and \( \mu_g \) induced by the above norm leads

\[
\|\mu_f - \mu_g\| = \sqrt{\frac{1}{n} \sum_{i=1}^{n} \int_{\Omega} |f_i - g_i|^2 d\mu}. \tag{21}
\]

We call this norm random variable norm (R. V. norm for short) in the sequel. In this paper, we only consider Gaussian measures with the mean 0 and Gaussian channels.

Let an input state \( \mu_f = [0, R] \) be induced from a \( n \)-dimensional random vector \( f = (f_1, \ldots, f_n) \) and its output state \( \Lambda^*\mu_f \) be denoted by \( \mu_g \), where \( g \) is random vector \( g = (g_1, \ldots, g_n) \) induced from \( \Lambda^* \).

Lemma 1: If the distance of two states is given by the above R.V. norm, then

\[
J(\mu_f ; \Lambda^*) = I(\mu_f ; \Lambda^*) \tag{22}
\]

proof: From the assumption, the Gaussian channel \( \Lambda^* \) is represented by a conditional probability density of \( g \) with respect to \( f \) as

\[
p(y|x) = \frac{1}{(2\pi)^{n/2}\sqrt{|R_0|}} \times \exp \left\{ -\frac{1}{2}(y - Ax)R_0^{-1}(y - Ax)^t \right\}
\]

\[
x, y \in \mathbb{R}^n,
\]

where \( R_0 \) is the covariance matrix associated to the channel \( \Lambda^* \). Then the compound state \( \Phi \) of \( \mu_f \) and \( \Lambda^*\mu_f = \mu_g \) is equal to the joint probability measure \( \mu_{fg} = [0, C] \) of \( f \) and \( g \) such as
where $z$ is the $2n$-dimensional random vector $(x, y) = (x_1, \ldots, x_n, y_1, \ldots, y_n)$ and $C$ is the following covariance matrix of $\mu_{fg}$:

$$C = \begin{pmatrix} R & RA^t \\ AR & ARA^t + R_0 \end{pmatrix}.$$  \hfill (23)

where $R$, $RA^t$, $AR$, $ARA^t + R_0$ are $n \times n$ matrix and $(R)_{ij} = E(f_i f_j)$, $(RA^t)_{ij} = E(f_i g_j)$, $(AR)_{ij} = E(g_i f_j)$, $(ARA^t + R_0)_{ij} = E(g_i g_j)$ for each $(i, j)$ $(i, j = 1, \ldots, n)$.

For the channel $\Gamma^*$ satisfying $\Lambda^* \mu_f = \Gamma^* \mu_f$, $\Gamma^* \mu_f$ is a $n$-dimensional Gaussian measure $\mu_h$ induced from a $n$-dimensional random vector $h = (h_1, \ldots, h_n)$, so that we have

$$J(\mu_f; \Lambda^*) = \sup_{\Gamma^*} \{I(\mu_f; \Gamma^*) ; \Lambda^* \mu_f = \Gamma^* \mu_f \} = \sup_{\Gamma^*} \{I(\mu_f; \Gamma^*) ; \|\Lambda^* \mu_f - \Gamma^* \mu_f\| = 0 \} = \sup_{\Gamma^*} \{I(\mu_f; \Gamma^*) ; \|\mu_g - \mu_h\| = 0 \} = \sup_{h} \{I(f, h) ; E[d(g, h)] = 0 \} = \sup_{h} \{I(f, h) ; E[d(g_i, h_i)] = 0 (i = 1, \ldots, n) \} = \sup_{h} \{I(f, h) ; g_i = h_i \text{ a.e.} (i = 1, \ldots, n) \} .$$

From $g_i = h_i \text{ a.e.} (i = 1, \ldots, n)$, we obtain

$$\mu_{fg} = \mu_{fh} . \hfill (24)$$

Therefore,

$$I(\mu_f; \Lambda^*) = I(f, h) = I(\mu_f; \Gamma^*) , \hfill (25)$$
which implies

\[ J(\mu_f; \Lambda^*) = I(\mu_f; \Lambda^*). \]  

Using the above lemma, the following theorem holds.

**Theorem 1:** Under the same assumption as Lemma 1

1. \( S_O(\mu_f; \varepsilon) = S_K(f; \varepsilon) = \frac{1}{2} \sum_{i=1}^{n} \log \left( \frac{\lambda_i}{\theta^2}, 1 \right), \)

where \( \lambda_1, \ldots, \lambda_n \) are the eigenvalues of \( R \) and \( \theta^2 \) is a constant uniquely determined by the equation \( \sum_{i=1}^{n} \min(\lambda_i; \theta^2) = \varepsilon^2. \)

2. \( d_C^0(\mu_f) = n. \)

**proof:** (1) Let \( \hat{C} \) be the set of all Gaussian channels from \( \mathcal{B}(\mathbb{R}^n) \) to \( \mathcal{B}(\mathbb{R}^n) \) and \( \hat{C}(\mu_f; \varepsilon) \) be the set of all Gaussian channels from \( \mathcal{B}(\mathbb{R}^n) \) to \( \mathcal{B}(\mathbb{R}^n) \) satisfying \( \| \mu_f - \Lambda^* \mu_f \| \leq \varepsilon. \) According to Lemma 1, we obtain

\[
S_O(\mu_f; \varepsilon) = \inf \left\{ J(\mu_f; \Lambda^*); \Lambda^* \in \hat{C}(\mu_f; \varepsilon) \right\} = \inf \left\{ I(\mu_f; \Lambda^*); \Lambda^* \in \hat{C}(\mu_f; \varepsilon) \right\},
\]

From (23), we have

\[
S_O(\mu_f; \varepsilon) = \inf \left\{ I(f, g); \mu_{fg} \in \tilde{S}(\mu_f; \varepsilon) \right\} = S_K(f; \varepsilon),
\]

where \( \tilde{S}(\mu_f; \varepsilon) = \left\{ \mu_{fg}; \sqrt{\int_{\mathbb{R}^n \times \mathbb{R}^n} d(x, y)^2 d\mu_{fg}(x, y)} \leq \varepsilon \right\}. \)

The expression of the \( \varepsilon \)-entropy \( \frac{1}{2} \sum_{n=1}^{n} \log \left\{ \max \left( \frac{\lambda_i}{\theta^2}, 1 \right) \right\} \) was obtained by Pinsker (1963).

(2) Since \( S_O(\mu_f; \varepsilon) = \frac{1}{2} \sum_{n=1}^{n} \log \left\{ \max \left( \frac{\lambda_i}{\theta^2}, 1 \right) \right\}, \) we have

\[
d_C^0(\mu_f) = d_C^0(\mu_f) = \lim_{\varepsilon \to 0} \frac{S_O(\mu_f; \varepsilon)}{\log \frac{1}{\varepsilon}} = \lim_{\varepsilon \to 0} \frac{\frac{1}{2} \sum_{i=1}^{n} \log \left\{ \max \left( \frac{\lambda_i}{\theta^2}, 1 \right) \right\}}{\log \frac{1}{\varepsilon}}.
\]
\[ i.e. \sum_{i=1}^{n} \min(\lambda_i, \theta^2) = \varepsilon^2 \]

\[ = \lim_{\varepsilon \to 0} \frac{1}{2} \sum_{i=1}^{n} \log \frac{\lambda_i}{\theta} \left( \sum_{i=1}^{n} \theta^2 = \varepsilon^2 \right) \]

\[ = \lim_{\varepsilon \to 0} \frac{1}{2} \sum_{i=1}^{n} \log \frac{n \lambda_i}{\varepsilon^2} = n. \]

In this case, our \( \varepsilon \)-entropy coincides with Kolmogorov’s \( \varepsilon \)-entropy and the fractal dimension of the state \( \mu_f \) is identical to the dimension of Hilbert space.

6 \( \varepsilon \)-entropy and fractal dimension of a state for Gaussian measures in the total variation norm

In the following discussion, we only consider the case of \( \dim(H) = 1 \), that is, \( H = \mathbb{R} \), the state \( \mu = [0, \sigma^2] \) is a one-dimensional Gaussian measure (distribution) and the distance of two states is given by the total variation norm.

In this section, we show that our \( \varepsilon \)-entropy traces to the fractal property of a Gaussian measure but Kolmogorov’s does not. The difference between \( S_0 \) and \( S_K \) come from the norm of measures taken. We take the norm by the total variation, namely,

\[ \|\mu\| = |\mu|(\mathbb{R}) \quad (27) \]

Let \( H_1 = H_2 = \mathbb{R} \). Then \( A \) becomes a real number \( \beta \) and the noise of a channel is exhibited by one-dimensional Gaussian measure \( \mu_0 = [0, \sigma_0^2] \in P(H) \), so that the output state \( \Lambda^* \mu \) is represented by \( [0, \beta^2 \sigma^2 + \sigma_0^2] \). We calculate the maximum mutual entropy in the following two cases: (1) \( \beta^2 \sigma^2 + \sigma_0^2 \geq \sigma^2 \), (2) \( \beta^2 \sigma^2 + \sigma_0^2 < \sigma^2 \). Since the channel \( \Lambda^* \) depends on \( \beta \) and \( \sigma_0^2 \), we put \( \Lambda^* = \Lambda^*_{(\beta, \sigma_0^2)} \). As the density function for Gaussian measures are error functions, we first give an order estimation for the difference of two Gaussian measures.
Lemma 2: If \( \beta \sigma^2 + \sigma_0^2 \geq \sigma^2 \) and \( \| \mu - \Lambda_{(\beta, \sigma_0^2)}^* \mu \| = \| \mu - \Lambda_{(\beta, \sigma_0^2)}^* \mu \| (\mathbb{R}) = \delta \), then

1. \( \frac{4}{\sqrt{2\pi}} \sqrt{\frac{\beta^2 \sigma^2 + \sigma_0^2 - \sigma}{\sigma}} = \delta + o(\delta) \),

2. \( \{ \Lambda_{(\beta, \sigma_0^2)}^* : \| \mu - \Lambda_{(\beta, \sigma_0^2)}^* \mu \| \leq \varepsilon \} = \{ \Lambda_{(\beta, \sigma_0^2)}^* : \frac{4}{\sqrt{2\pi}} \sqrt{\frac{\beta^2 \sigma^2 + \sigma_0^2 - \sigma}{\sigma}} = \delta + o(\delta), \delta \in M(\varepsilon) \} \),

where \( o(\delta) \) is an order of \( \delta : \lim_{\delta \to 0} o(\delta) = 0 \) and \( M(\varepsilon) = \{ \delta \in \mathbb{R} : 0 \leq \delta \leq \varepsilon \} \).

proof: (1) Let \( p_1, p_2 \) be the density functions of \( \mu, \Lambda^* \mu \), respectively. Then, we have

\[
\| \mu - \Lambda_{(\beta, \sigma_0^2)}^* \mu \| = \int_{\mathbb{R}} |p_1(x) - p_2(x)| dx
\]

Since \( p_i (i = 1, 2) \) are even functions, we obtain

\[
\int_{\mathbb{R}} |p_1(x) - p_2(x)| dx = 2 \int_0^\infty |p_1(x) - p_2(x)| dx
\]

\[
= 2 \left( \int_0^a (p_1(x) - p_2(x)) dx + \int_a^\infty (p_2(x) - p_1(x)) dx \right)
\]

\[
= 4 \int_0^a (p_1(x) - p_2(x)) dx
\]

\[
\left( a = \sqrt{\left( \frac{1}{\sigma^2} - \frac{1}{\beta^2 \sigma^2 + \sigma_0^2} \right)^{-1} \log \frac{\beta^2 \sigma^2 + \sigma_0^2}{\sigma^2}} \right)
\]

\[
\leq \frac{4a}{\sqrt{2\pi}} \left( \frac{1}{\sigma} - \frac{1}{\sqrt{\beta^2 \sigma^2 + \sigma_0^2}} \right)
\]

\[
\leq \frac{4}{\sqrt{2\pi}} \frac{\sqrt{\beta^2 \sigma^2 + \sigma_0^2 - \sigma}}{\sigma}
\]

where \( a \) is a real number satisfying \( p_1(a) = p_2(a) \), and the first inequality is led by a geometrical approximation of the equation (32) and the second inequality is obtained from the inequality \( \log x \leq x - 1 \) for any positive number \( x \).

Since \( \| \mu - \Lambda_{(\beta, \sigma_0^2)}^* \mu \| \) and \( \frac{4}{\sqrt{2\pi}} \sqrt{\frac{\beta^2 \sigma^2 + \sigma_0^2 - \sigma}{\sigma}} \) are monotone decreasing for
\[ \beta^2 \sigma^2 + \sigma_0^2 \rightarrow \sigma^2 \] and

\[
\lim_{\beta^2 \sigma^2 + \sigma_0^2 \rightarrow \sigma^2} \| \mu - \Lambda^*_{(\beta,\sigma_0^2)} \mu \|
= \lim_{\beta^2 \sigma^2 + \sigma_0^2 \rightarrow \sigma^2} \frac{4 \sqrt{\beta^2 \sigma^2 + \sigma_0^2 - \sigma}}{\sqrt{2\pi} \sigma}
= 0,
\]

the above inequality implies

\[
\| \mu - \Lambda^*_{(\beta,\sigma_0^2)} \mu \| = \delta \iff \frac{4 \sqrt{\beta^2 \sigma^2 + \sigma_0^2 - \sigma}}{\sqrt{2\pi} \sigma} = \delta + o(\delta), \quad (35)
\]

where \( o(\delta) \) is an order of \( \delta \).

(2) From Lemma 2(1),

\[
\left\{ \Lambda^*_{(\beta,\sigma_0^2)} ; \| \mu - \Lambda^*_{(\beta,\sigma_0^2)} \mu \| = \delta \right\}
= \left\{ \Lambda^*_{(\beta,\sigma_0^2)} ; \frac{4 \sqrt{\beta^2 \sigma^2 + \sigma_0^2 - \sigma}}{\sqrt{2\pi} \sigma} = \delta + o(\delta) \right\}.
\quad (36)
\]

Let \( M(\epsilon) \) be the set of all \( \delta \in \mathbb{R} \) satisfying \( 0 \leq \delta \leq \epsilon \). From (36), it is clear that

\[
\left\{ \Lambda^*_{(\beta,\sigma_0^2)} ; \| \mu - \Lambda^*_{(\beta,\sigma_0^2)} \mu \| \leq \epsilon \right\} =
\left\{ \Lambda^*_{(\beta,\sigma_0^2)} ; \frac{4 \sqrt{\beta^2 \sigma^2 + \sigma_0^2 - \sigma}}{\sqrt{2\pi} \sigma} = \delta + o(\delta), \quad \delta \in M(\epsilon) \right\}
\quad (37)
\]

Lemma 3: Let \( \Lambda^*_{\delta(\beta,\sigma_0^2)} \) be a channel satisfying \( \beta^2 \sigma^2 + \sigma_0^2 \geq \sigma^2 \) and \( \| \mu - \Lambda^*_{\delta(\beta,\sigma_0^2)} \mu \| = \delta \) for any \( \delta \in M(\epsilon) \). If a Gaussian channel \( \Lambda^*_{\delta(\beta,\sigma_0^2)} \) satisfies the condition \( \beta^2 \leq \frac{C_\delta}{\sigma^2} \), then we have

\[
J(\mu; \Lambda^*_{\delta(\beta,\sigma_0^2)}) = \frac{1}{2} \log \frac{1}{\delta} + \frac{1}{2} \log\sigma^2 \left( 1 + \frac{\sqrt{2\pi}}{4} (\delta + o(\delta)) \right)^2,
\]

where \( C_\delta = \beta^2 \sigma^2 + \sigma_0^2 \) is a constant determined by \( \| \mu - \Lambda^*_{\delta(\beta,\sigma_0^2)} \mu \| = \delta \).
proof: The mutual entropy of $\mu$ with respect to channel $\Lambda^*$ is

$$I(\mu; \Lambda^*) = \frac{1}{2} \log \frac{\beta^2 \sigma^2 + \sigma_0^2}{\sigma_0^2}. \quad (39)$$

Thus, if $\Lambda^*_{\delta(\beta, \sigma_0^2)}$ is any channel satisfying $\|\mu - \Lambda^*_{\delta(\beta, \sigma_0^2)} \mu\| = \delta$, then we have from the above lemma 2

$$I(\mu; \Lambda^*_{\delta(\beta, \sigma_0^2)}) = \frac{1}{2} \log \frac{1}{C_\delta - \beta^2 \sigma^2} + \frac{1}{2} \log \sigma^2 \left( 1 + \frac{\sqrt{2\pi}}{4}(\delta + o(\delta)) \right)^2,$$

The assumption implies,

$$J(\mu; \Lambda^*) = \sup_{\Lambda^*_{\delta(\beta, \sigma_0^2)}} \left\{ I(\mu; \Lambda^*_{\delta(\beta, \sigma_0^2)}); \Lambda^*_{\delta(\beta, \sigma_0^2)} \mu = \Lambda^*_{\delta(\beta, \sigma_0^2)} \mu \right\}$$

$$= \sup_{\Lambda^*_{\delta(\beta, \sigma_0^2)}} \left\{ I(\mu; \Lambda^*_{\delta(\beta, \sigma_0^2)}); \|\Lambda^*_{\delta(\beta, \sigma_0^2)} \mu - \Lambda^*_{\delta(\beta, \sigma_0^2)} \mu\| = 0 \right\}$$

$$= \sup_{\Lambda^*_{\delta(\beta, \sigma_0^2)}} \left\{ I(\mu; \Lambda^*_{\delta(\beta, \sigma_0^2)}); \beta^2 \sigma^2 + \sigma_0^2 = \gamma^2 \sigma^2 + \sigma_0^2 \right\}$$

$$= \sup_{\beta} \left\{ \frac{1}{2} \log \frac{1}{C_\delta - \beta^2 \sigma^2} + \frac{1}{2} \log \sigma^2 \cdot f(\delta); \beta^2 \leq \frac{C_\delta - \delta}{\sigma^2} \right\}$$

$$= \frac{1}{2} \log \frac{1}{\delta} + \frac{1}{2} \log \sigma^2 \cdot f(\delta)$$

where $\gamma^2 = \frac{C_\delta - \delta}{\sigma^2}$ and $f(\delta) = \left( 1 + \frac{\sqrt{2\pi}}{4}(\delta + o(\delta)) \right)^2$.

Lemma 4: Let $\Lambda^*_{\delta(\beta, \sigma_0^2)}$ be a channel satisfying $\beta^2 \sigma^2 + \sigma_0^2 < \sigma^2$ and $\|\mu - \Lambda^*_{\delta(\beta, \sigma_0^2)} \mu\| = \delta$ for any $\delta \in M(\varepsilon)$. If a Gaussian channel $\Lambda^*_{\delta(\beta, \sigma_0^2)}$ satisfies the condition $\beta^2 \leq \frac{C_\delta - \delta}{\sigma^2}$, then we have

$$J(\mu; \Lambda^*_{\delta(\beta, \sigma_0^2)}) = \frac{1}{2} \log \frac{1}{\delta} + \frac{1}{2} \log \frac{\sigma^2}{\left( 1 + \frac{\sqrt{2\pi}}{4}(\delta + o(\delta)) \right)^2}. \quad (40)$$
where \( C_\delta = \beta^2 \sigma^2 + \sigma_0^2 \) is a constant determined by \( \| \mu - \Lambda^*_\delta(\beta, \sigma_0^2) \mu \| = \delta \).

**proof:** Similarly proved as Lemma 2 and Lemma 3.

Using the above those lemmas, we obtain the following theorem.

**Theorem 2:** Under the same conditions of Lemma 3 and 4, we have

1. \( S_\text{O}(\mu; \varepsilon) = \frac{1}{2} \log \frac{1}{\varepsilon} + \frac{1}{2} \log \left( \frac{\sigma^2}{\left(1 + \frac{\sqrt{2\pi}}{4}(\delta + o(\delta))\right)^2} \right) > S_\text{K}(\mu; \varepsilon) = 0 \)

2. \( d_\text{C}^0(\mu) = \frac{1}{2} \)

**proof:** (1) From Lemma 3 and 4, we have

\[
S_\text{O}(\mu; \varepsilon) = \inf_{\Lambda^*} \left\{ J(\mu; \Lambda^*); \| \mu - \Lambda^* \mu \| \leq \varepsilon \right\}
= \inf_{\delta} \left\{ \frac{1}{2} \log \frac{1}{\delta} + \frac{1}{2} \log \left( \frac{\sigma^2}{(1 + \frac{\sqrt{2\pi}}{4}(\delta + o(\delta))} \right)^2 \right\}; \quad \delta \in M(\varepsilon) \}
= \inf_{\delta} \left\{ \frac{1}{2} \log \frac{1}{\varepsilon} + \frac{1}{2} \log \left( \frac{\sigma^2}{(1 + \frac{\sqrt{2\pi}}{4}(\delta + o(\delta))} \right)^2 \right\}; \quad \delta \in M(\varepsilon) \}
= \frac{1}{2} \log \frac{1}{\varepsilon} + \frac{1}{2} \log \left( \frac{\sigma^2}{(1 + \frac{\sqrt{2\pi}}{4}(\delta + o(\delta))} \right)^2 \right\),

because \( \frac{1}{2} \log \frac{1}{\delta} + \frac{1}{2} \log \left( \frac{\sigma^2}{(1 + \frac{\sqrt{2\pi}}{4}(\delta + o(\delta))} \right)^2 \right\) is monotone decreasing with respect to \( \delta \).

(2) From (1), we obtain

\[
d_\text{C}^0(\mu) = \lim_{\varepsilon \to 0} \frac{S_\text{O}(\mu; \varepsilon)}{\log \frac{1}{\varepsilon}}
= \frac{1}{2} \log \frac{1}{\varepsilon} + \frac{1}{2} \log \left( \frac{\sigma^2}{(1 + \frac{\sqrt{2\pi}}{4}(\delta + o(\delta))} \right)^2 \right\)
= \lim_{\varepsilon \to 0} \frac{\log \frac{1}{\varepsilon}}{\log \frac{1}{\varepsilon}}
= \frac{1}{2}.
\]
This result shows that (1) the fractal dimension of a Gaussian measure $\mu$ in the total variation norm describes fractal structure of Gaussian measures, and (2) the fractal dimension of a Gaussian measure $\mu$ is always 0 if we use the Kolmogorov $\varepsilon$-entropy.

We concluded that our fractal dimension of states is a new criterion to study a chaotic aspect of Gaussian measures.

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