Semi-classical functional calculus on manifolds with ends and weighted $L^p$ estimates

Jean-Marc Bouclet*

Université de Lille 1
Laboratoire Paul Painlevé
UMR CNRS 8524,
59655 Villeneuve d’Ascq

February 2, 2008

Abstract

For a class of non compact Riemannian manifolds with ends $\mathcal{M}, g)$, we give pseudo-differential expansions of bounded functions of the semi-classical Laplacian $h^2\Delta_g$, $h \in (0, 1]$. We then study related $L^p$ boundedness properties and show in particular that, although $\varphi(-h^2\Delta_g)$ is not bounded on $L^p(\mathcal{M}, dg)$ in general, it is always bounded on suitable weighted $L^p$ spaces.

1 Introduction and Results

In this paper we describe semi-classical expansions of functions of the Laplacian on a class of non compact manifolds of bounded geometry. We also derive certain (weighted) $L^p \to L^p$ boundedness properties of such operators. Further applications to Littlewood-Paley decompositions [4] and Strichartz estimates [5] will be published separately. Needless to say, the range of applications of the present functional calculus goes beyond Strichartz estimates; there are many problems which naturally involve spectral cutoffs at high frequencies in linear and non linear PDEs (Littlewood-Paley decompositions, paraproducts) or in spectral theory (trace formulas).

Consider a non compact Riemannian manifold $(\mathcal{M}, g)$ with ends, ie whose model at infinity is a product $(R, +\infty) \times S$ with metric $g = dr^2 + d\theta^2 / w(r)^2$, where $R \gg 1$, $(S, d\theta^2)$ a compact Riemannian manifold and $w(r)$ a bounded positive function. For instance, $w(r) = r^{-1}$ corresponds to conical ends, $w(r) = 1$ to cylindrical ends and $w(r) = e^{-r}$ to hyperbolic ends. We actually consider more general metrics (see Definition 1.1 below for precise statements) but these are the typical examples we have in mind. If $\Delta_g$ denotes the Laplacian on $\mathcal{M}$ and $\varphi$ is a symbol of negative order, we are interested in decompositions of the form

$$\varphi(-h^2\Delta_g) = Q_N(\varphi, h) + h^{N+1} R_N(\varphi, h), \quad h \in (0, 1],$$

(1.1)

*Jean-Marc.Bouclet@math.univ-lille1.fr
where $N \geq 0$ is fixed and arbitrary, $Q_N(\varphi, h)$ has an expansion in powers of $h$ in terms $h$-pseudo-differential operators and $h^{N+1}R_N(\varphi, h)$ is a ‘nice’ remainder. We recall that, for such semi-classical expansions, even the case of $\varphi \in C^\infty_0(\mathbb{R})$ is of interest, by opposition to the classical case ($h = 1$) where $C^\infty_0$ functions of $\Delta_g$ are often treated as negligible operators.

There is a large literature devoted to the pseudo-differential analysis of functions of closed operators on manifolds so we only give references which are either classical or close to our framework. For $h = 1$, the case of compact manifolds (i.e., essentially, the local interior case) was considered by Seeley [19] (see also [20] pp. 917-920). For boundary value problems, we refer to [20] [12] and for non compact or singular manifolds to [13] [11]. We also quote [5] [22] [15] where general manifolds of bounded geometry are studied in connection with the problem of the $L^p \rightarrow L^p$ boundedness of functions the Laplacian (to which we come back below). The semi-classical case is treated for very general operators on $\mathbb{R}^n$ in [14] [17] [11] and in [7] for a compact manifold. Besides, one of our initial motivations is to extend the functional calculus used in [7] to non compact manifolds and thus to provide a convenient tool to prove Strichartz estimates, as for instance in [13] [6].

Although the general picture is quite clear, at least from the $L^2$ point of view, the problem of getting expansions of the form (1.1) requires some care. By opposition to the compact case (or to $\mathbb{R}^n$ for uniformly elliptic operators), one has to take into account certain off diagonal effects possibly leading to the unboundedness of the operators on $L^p(M, dg)$, when $p \neq 2$, if $dg$ denotes the Riemannian measure.

By considering properly supported operators, namely with kernels supported close to the diagonal of $M \times M$, we may insure that the principal part of the expansion $Q_N(\varphi, h)$ is bounded on $L^p(M, dg)$, for all $p \in [1, \infty]$, uniformly with respect to $h$. However, the boundedness of the remainder $R_N(\varphi, h)$ on $L^p(M, dg)$ remains equivalent to the one of the full operator $\varphi(-h^2\Delta_g)$ and it is well known that the latter may fail for non holomorphic $\varphi$, as first noticed by Clerc and Stein [9] for symmetric spaces. The latter question is treated (with $h = 1$) for a large class of manifolds by Taylor in [22] (see also the references therein and the extension [15] to systems of properly supported operators). Taylor proves that, if $A$ denotes the bottom of the spectrum of $-\Delta_g$ and $L = (-\Delta_g - A)^{1/2}$, the boundedness of $\varphi(L)$ on $L^p(M, dg)$ is guaranteed if $\varphi$ is even and holomorphic in a strip of width at least $\kappa[1/p - 1/2]$, with $\kappa$ the exponential rate of the volume growth of balls. This is typically relevant in the hyperbolic case. To illustrate this fact (as well as some of our results), we recall a short proof of the $L^p$-unboundedness of $(z - \Delta_g)^{-1}$ in Appendix A, $\Delta_g$ being the Laplacian on the hyperbolic space.

In summary, our first goal is to provide a fairly explicit and precise description of expansions of the form (1.1). For $h = 1$, this result is essentially contained in [8] [22] but we feel that it is worth giving complete proofs for the semi-classical case too, first because we shall use it extensively in subsequent papers and second because of the subtleties due to $L^p$-unboundedness.

Our second point is to prove weighted $L^p$ estimates on $R_N(\varphi, h)$ or, equivalently, on the resolvent $(z - \Delta_g)^{-1}$. The basic strategy is to use the expansion (1.1) to get $L^p$ estimates on commutators of the resolvent with natural first order differential operators and show that $(z - \Delta_g)^{-1}$ is a pseudo-differential operator, using the Beals criterion. At this stage, the meaning of pseudo-differential operator is rather vague but we emphasize that the point is not (only) to control the singularity of the kernel close to the diagonal but also the decay far from the diagonal. As a consequence of this analysis, we obtain in particular that, although $(z - \Delta_g)^{-1}$ is not necessarily bounded on $L^p(M, dg)$, we always have

$$
\|w(r) \frac{d}{dr} - \frac{d}{dz}(z - \Delta_g)^{-1}w(r) \frac{d}{dz} - \frac{d}{dr} \|_{L^p(\mathcal{M}, dg) \rightarrow L^p(\mathcal{M}, dg)} < \infty,
$$

for all $p \in (1, \infty)$ and $z \notin \text{spec}(\Delta_g)$. More generally, if $W$ is a temperate weight (see Definition 1.6...
below), we have

\[ \|W(r)^{-1} w(r)^{n-1} \left( z - \Delta_g \right)^{-1} w(r)^{n-1} \|_{L^p(M, dg) \to L^p(M, dg)} < \infty. \]

This works in particular for the hyperbolic case where \((z - \Delta_g)^{-1}\) is not bounded on \(L^p(M, dg)\) in general. In the conical case, or more generally if \(w\) itself is a temperate weight, we recover the natural (unweighted) boundedness on \(L^p(M, dg)\) by choosing \(W = w^{n-1} \). The latter boundedness can be seen as a consequence of \([22]\) since, if \(w\) is temperate, the volume growth of balls is polynomial. The above estimates are therefore complementary to the results of \([22]\): if \(z\) is too close to the spectrum of the Laplacian, \((z - \Delta_g)^{-1}\) is maybe not bounded on \(L^p = L^p(M, dg)\) but it is bounded if we accept to replace \(L^p\) by weighted \(L^p\) spaces. Furthermore, these weighted spaces are natural since they contain \(L^p\) itself when \(w\) is temperate (ie essentially if \(w^{-1}\) is of polynomial growth).

Let us now state our results precisely.

In the sequel \(M\) will be a smooth manifold of dimension \(n \geq 2\), without boundary and satisfying the following definition.

**Definition 1.1.** The manifold \((M, g)\) is called almost asymptotic if there exists a compact set \(K \subset M\), a real number \(R\), a compact manifold \(S\), a function \(r \in C^\infty(M, \mathbb{R})\) and a function \(w \in C^\infty((0, +\infty))\) with the following properties:

1. \(r\) is a coordinate near \(M \setminus K\) such that
   \[ r(x) \to +\infty, \quad x \to \infty, \]

2. there is a diffeomorphism
   \[ M \setminus K \to (R, +\infty) \times S, \quad \text{(1.2)} \]
   through which the metric reads, in local coordinates,
   \[ g = G_{\text{unif}}(r, \theta, dr, w(r)^{-1} d\theta) \quad \text{(1.3)} \]
   with
   \[ G_{\text{unif}}(r, \theta, V) := \sum_{1 \leq j, k \leq n} G_{jk}(r, \theta) V_j V_k, \quad V = (V_1, \ldots, V_n) \in \mathbb{R}^n, \]
   if \(\theta = (\theta_1, \ldots, \theta_{n-1})\) are local coordinates on \(S\).

3. The symmetric matrix \((G_{jk}(r, \theta))_{1 \leq j, k \leq n}\) has smooth coefficients such that, locally uniformly with respect to \(\theta\),
   \[ |\partial_\theta^j \partial_\theta^k G_{jk}(r, \theta)| \lesssim 1, \quad r > R, \quad \text{(1.4)} \]
   and is uniformly positive definite in the sense that, locally uniformly in \(\theta\),
   \[ G_{\text{unif}}(r, \theta, V) \approx |V|^2, \quad r > R, \quad V \in \mathbb{R}^n. \quad \text{(1.5)} \]

4. The function \(w\) is smooth and satisfies, for all \(k \in \mathbb{N}\),
   \[ w(r) \lesssim 1, \quad \text{(1.6)} \]
   \[ w(r)/w(r') \approx 1, \quad \text{if } |r - r'| \leq 1 \quad \text{(1.7)} \]
   \[ |d^k w(r)/dr^k| \lesssim w(r), \quad \text{(1.8)} \]
   for \(r, r' \in \mathbb{R}\).
Note that (1.7) is equivalent to the fact that, for some \( C > 0 \),
\[
C^{-1} e^{-C|r-r'|} \leq \frac{w(r)}{w(r')} \leq Ce^{C|r-r'|}.
\]
In particular, this implies that \( w(r) \geq e^{-C|r|} \).

Asymptotically conical manifolds, for which \( g = dr^2 + r^2g_S(r, \theta, d\theta) \) (near infinity), or asymptotically hyperbolic manifolds for which \( g = dr^2 + e^{2r}g_S(r, \theta, d\theta) \), with \( g_S(r, \theta, d\theta) \) a metric on \( S \) depending smoothly on \( r \), satisfy our definition. More precisely, for such asymptotic structures one usually requires that \( g_S(r, \theta, d\theta) \) is a small perturbation of a metric \( g_S^\infty(\theta, d\theta) \) in the sense that \( g_S(r, \theta, d\theta) - g_S^\infty(\theta, d\theta) \to 0 \) as \( r \to \infty \). See for instance [16] for more precise statements. Here we do not require such a condition which is the reason why we use the terminology almost asymptotic.

**Atlas and partition of unit.** We now specify an atlas on \( M \). The diffeomorphism (1.2) is of the form \( \Psi : M \setminus K \to (R, +\infty) \times S \) with
\[
\Psi(\underline{x}) = (r(\underline{x}), \pi_S(\underline{x})), \quad \underline{x} \in M \setminus K,
\]
where \( \pi_S : M \setminus K \to S \) is the ”projection on the manifold at infinity” and \( r \) the ”radial coordinate” used in Definition 1.1. Thus, if we consider a chart on \( S \),
\[
\psi_i : U_i \subset S \to V_i \subset \mathbb{R}^{n-1},
\]
with \( \psi_i(\underline{y}) = (\theta_1(\underline{y}), \ldots, \theta_{n-1}(\underline{y})) \), then the open sets
\[
U_i = \Psi^{-1}((R, +\infty) \times U_i) \subset M, \quad V_i = (R, +\infty) \times V_i \subset \mathbb{R}^n,
\]
and the map
\[
\Psi_i : U_i \to V_i, \quad \text{with} \quad \Psi_i(\underline{x}) = (r(\underline{x}), \psi_i \circ \pi_S(\underline{x})) = (r(\underline{x}), \theta_1(\pi_S(\underline{x})), \ldots, \theta_{n-1}(\pi_S(\underline{x}))),
\]
define a coordinate chart on \( M \setminus K \). With a standard abuse of notation, we will denote for simplicity these coordinates \((r, \theta_1, \ldots, \theta_{n-1})\) or even \((r, \theta)\).

**Definition 1.2.** We call \( U_i \) a coordinate patch at infinity and the triple \((U_i, V_i, \Psi_i)\) a chart at infinity.

Since \( S \) is compact, there is a finite set \( I_{\infty} \) such that the family \((U_i, V_i, \Psi_i)_{i \in I_{\infty}}\) is an atlas on \( M \setminus K \). Choosing another finite collection of coordinate charts for a neighborhood of \( K \), which we denote \( I_{\text{comp}} \) by \((U_i, V_i, \Psi_i)_{i \in I_{\text{comp}}}\) for some finite set \( I_{\text{comp}} \), we get a finite atlas on \( M \) by considering \((U_i, V_i, \Psi_i)_{i \in I} \) with
\[
I = I_{\infty} \cup I_{\text{comp}}.
\]
In particular, we can find a finite partition of unit
\[
\sum_{i \in I} f_i = 1 \quad \text{on} \quad M, \quad \text{(1.10)}
\]
such that, for all \( i \in I \), \( f_i \) is supported in \( U_i \). We also set
\[
\chi_i = f_i \circ \Psi_i^{-1}. \quad \text{(1.11)}
\]
\footnote{We keep the notation \( U_i, V_i, \Psi_i \), but, of course, the corresponding new \( U_i \) and \( V_i \) are not defined by (1.9). In the core of the paper, there should be anyway no confusion for we shall work almost only on \( M \setminus K \).}
If \( \mathcal{U} \) is a patch at infinity, we can assume that \( f_i \) is such that
\[
\chi_i(r, \theta) = \varrho(r) \kappa_i(\theta),
\]
for some smooth functions \( \varrho \) and \( \kappa_i \) such that, for some \( R' > R \),
\[
\varrho(r) = 1 \quad \text{for} \quad r \gg 1, \quad \supp \varrho \subset [R', +\infty), \quad \kappa_i \in C_0^\infty(\mathcal{V}_i).
\]

**Definition 1.3.** For \( m \in \mathbb{N} \), \( \text{Diff}_m^w(\mathcal{M}) \) is the space of differential operators \( P \) of order \( \leq m \), acting on functions on \( \mathcal{M} \), such that, for any chart at infinity \((\mathcal{U}_i, \mathcal{V}_i, \Psi_i)\),
\[
\Psi_i^* P \Psi_i = \sum_{k+|\alpha| \leq m} a_{ka}^k(r, \theta) (w(r)D_\theta)^\alpha D^k_r,
\]
for all \( j, \beta \) and all \( K_\beta \in \mathcal{V}_i \). Here we used the standard notation \( \Psi_i^* u = u \circ \Psi_i \) and \( \Psi_i^* v = v \circ \Psi_i^{-1} \).

By (1.4), (1.8) and (1.14), we see that \( -\Delta_g \in \text{Diff}_2^w(\mathcal{M}) \) and that its principal symbol takes the following form in \( \mathcal{V}_i \), for \( \iota \in I'_\infty \),
\[
p_2^\iota(r, \theta, \rho, w(r)\eta) = G^{11}(r, \theta) \rho^2 + 2G^{1k}(r, \theta) \rho w(r)\eta_k + G^{jk}(r, \theta) w(r)^2 \eta_j \eta_k,
\]
for some smooth \( (g^{jk}(x)) \) such that \( g^{jk}(x) \xi_j \xi_k \gtrsim |\xi|^2 \) for \( \xi \in \mathbb{R}^n \) locally uniformly with respect to \( x \).

**Remark.** Recall that, if \( \iota \in I_\infty \), the principal of \( -\Delta_g \) is given by (1.16) but not by \( p_2^\iota \) itself (see the factor \( w(r) \) in the left hand side of (1.16)). This notation (which is perhaps confusing) will be convenient to state Theorem [1.5].

**Lebesgue spaces.** We now describe volume densities. In coordinates \((r, \theta)\) at infinity, the Riemannian volume density associated to \( g \), denoted by \( dg \), reads
\[
dg = w(r)^{1-n} (\det G_{\text{unit}}(r, \theta))^{1/2} drd\theta,
\]
det \( G_{\text{unit}}(r, \theta) \approx 1 \). (1.19)

Define another density \( \tilde{dg} \) on \( \mathcal{M} \) by

\[
\tilde{dg} = w^{n-1}(r)dg,
\]

we then have

\[
L^p(\mathcal{M}, dg) = w^{\frac{n-1}{p}}(r)L^p(\mathcal{M}, \tilde{dg}), \quad p \in [1, \infty).
\] (1.21)

The map \( u \mapsto u(r)^{(n-1)/2}u \) is unitary from \( L^2(\mathcal{M}, \tilde{dg}) \) on \( L^2(\mathcal{M}, dg) \) and the operator

\[
\tilde{\Delta}_g := w(r)^{\frac{n-1}{2}}\Delta_g w(r)^{\frac{-1}{2}},
\]

is symmetric on \( C^\infty_0(\mathcal{M}) \) with respect to \( \tilde{dg} \). By (1.23), we have

\[
\tilde{\Delta}_g \in \text{Diff}^2_c(\mathcal{M}).
\]

Note that, for all \( \iota \in I_\infty \) and all \( K_i \subset V_i \) (see (1.19)), we have the equivalence of norms

\[
||u||_{L^p(\mathcal{M}, \tilde{dg})} \approx ||u \circ \Psi^{-1}_\iota||_{L^p(\mathbb{R}^n, drd\theta)}, \quad \text{supp}(u \circ \Psi^{-1}_\iota) \subset (R, +\infty) \times K_i,
\]

for \( p \in [1, \infty] \). This is a simple consequence of (1.19). On compact subsets, the same equivalence holds trivially. For the measure \( dg \), we have, if \( \iota \in I_\infty \),

\[
||u||_{L^p(\mathcal{M}, dg)} \approx \left|\left|w^{(1-n)/p}(r)u \circ \Psi^{-1}_\iota\right|\right|_{L^p(\mathbb{R}^n, drd\theta)}, \quad \text{supp}(u \circ \Psi^{-1}_\iota) \subset (R, +\infty) \times K_i. \] (1.24)

\textbf{Pseudo-differential operators.} We now define a class of semi-classical pseudo-differential operators associated to the partition of unit (1.10). We will choose symbols

\[
a^\iota \in S^m_t(V_i \times \mathbb{R}^n),
\]

where \( V_i \subset \mathbb{R}^n \) is defined by (1.9) if \( \iota \in I_\infty \). By definition, this means, if \( \iota \in I_\infty \), that for all \( K_i \subset V_i \),

\[
|\partial^\alpha_x \partial^\beta_\theta \partial^\gamma_\rho \partial^\delta_\eta a^\iota(r, \theta, \rho, \eta)| \leq C(1 + |\rho| + |\eta|)^{m-k-|\beta|}, \quad r > R, \quad \theta \in K_i, \quad \rho \in \mathbb{R}, \quad \eta \in \mathbb{R}^{n-1},
\]

and, if \( \iota \in I_{\text{comp}} \), that for all \( K_i \subset V_i \),

\[
|\partial^\alpha_x \partial^\beta_\xi a^\iota(x, \xi)| \leq C(1 + |\xi|)^{m-|\beta|}, \quad x \in K_i, \quad \xi \in \mathbb{R}^n.
\]

In both cases, the topology of \( S^m_t(V_i \times \mathbb{R}^n) \) is given by the best constants \( C \) which define semi-norms.

We basically would like to use operators of the form

\[
a^\iota(r, \theta, hD_x, hw(r)D_\theta)\chi_\iota, \quad \text{if } \iota \in I_\infty,
\]

\( 2\Delta_g \) and \( \tilde{\Delta}_g \) are respectively essentially self-adjoint on \( L^2(\mathcal{M}, dg) \) and \( L^2(\mathcal{M}, \tilde{dg}) \) from \( C^\infty_0(\mathcal{M}) \) and thus unitarily equivalent.
(see (2.1) below) and
\[ a^i(x, hD_x)\chi_i, \quad \text{if } \iota \in I_{\text{comp}}, \]
where \( \chi_i \) is defined by (1.11) and \( h \in (0, 1] \) is the semi-classical parameter. Actually, we need to consider properly supported operators so we construct first suitable cutoffs near the diagonal. Choose a function \( \zeta \in C_0^\infty(\mathbb{R}^n) \) and \( \varepsilon > 0 \) such that
\[ \zeta(x) = 1 \text{ for } |x| \leq \varepsilon, \quad \zeta(x) = 0 \text{ for } |x| > 2\varepsilon. \] (1.25)
For \( \iota \in I_{\text{comp}} \), the function
\[ \chi^\xi_i(r, \theta, r', \theta') := \chi_i(r', \theta')\zeta((r, \theta) - (r', \theta')) , \] (1.26)
is smooth on \( \mathbb{R}^{2n} \) and, if \( K_i \subset V_i \) is an arbitrarily small neighborhood of \( \text{supp}(\kappa_i) \) (see (1.19)), we may choose \( \varepsilon \) small enough such that
\[ \text{supp}(\chi^\xi_i) \subset \{(R, +\infty) \times K_i\}^2. \] (1.27)
Proceeding similarly for \( \iota \in I_{\text{comp}} \), we obtain a family of functions \( (\chi^\xi_i)_{\iota \in I} \) supported close to the diagonal of \( \mathbb{R}^{2n} \), with also \( \text{supp}(\chi^\xi_i) \subset V_i \times V_i \), and such that
\[ \chi^\xi_{i|\text{diagonal}} = \chi_i. \] (1.28)

**Definition 1.4.** For \( a^i \in S^m_i(V_i \times \mathbb{R}^n) \), the pseudo-differential operator
\[ \text{op}^i_{w,h}(a^i) : C_0^\infty(\mathbb{R}^n) \rightarrow C_0^\infty(V_i) \]
is the operator with kernel
\[ (2\pi)^{-n} \int e^{i(r-r')_\rho+i(\theta-\theta')_\eta} a^i(r, \theta, h\rho, hw(r)\eta)d\rho d\eta \times \chi^\xi_i(r, \theta, r', \theta'), \quad \text{if } \iota \in I_{\text{comp}}, \] (1.29)
\[ (2\pi)^{-n} \int e^{i(x-x')_\xi} a^i(x, h\xi)d\xi \times \chi^\xi_i(x, x'), \quad \text{if } \iota \in I_{\text{comp}}. \] (1.30)

In other words, \( \text{op}^i_{w,h}(a^i) \) is obtained by multiplying the kernel of \( a^i(r, \theta, hD_r, hw(r)D_\theta)\chi_i \) (resp. of \( a^i(x, hD_x)\chi_i \)) by \( \zeta((r, \theta) - (r', \theta')) \) (resp. by \( \zeta(x - x') \)).

If \( m < -n \) the integrals in (1.29) and (1.30) are absolutely convergent, otherwise they must be understood as oscillatory integrals in the usual way. That \( \text{op}^i_{w,h}(a^i) \) maps \( C_0^\infty(\mathbb{R}^n) \) into \( C_0^\infty(V_i) \) follows from the construction of \( \chi^\xi_i \). Note also that
\[ \text{op}^i_{w,h}(1) = \chi_i, \] (1.31)
since the oscillatory integral is the Dirac measure along the diagonal and \( \chi^\xi_i(r, \theta, r', \theta') = \chi_i(r', \theta') \) for \( |r - r'| + |\theta - \theta'| \) small enough.

**Remark.** Note the factor \( w(r) \) in front of \( \eta \) in the amplitude of (1.29). The choice of notation of Definition (1.3) is thus consistent with the expressions of the principal symbol of \( -\Delta_g \) given by (1.10) and (1.17).

We are now ready to state our results. We consider
\[ \varphi \in S^{-\sigma}(\mathbb{R}), \quad \sigma > 0, \]
that is \( |\varphi^{(k)}(\lambda)| \leq C_k(\lambda)^{-\sigma-k} \). The best constants \( C_k \) are semi-norms defining the topology of \( S^{-\sigma}(\mathbb{R}) \).
Theorem 1.5. Let $P$ denote either $-\Delta_g$ or $-\tilde{\Delta}_g$. For all $N \geq 0$, the following holds:

$$
\varphi(h^2P) = \sum_{i \in I} Q^i_N(P, \varphi, h) + h^{N+1} R_N(P, \varphi, h), \quad h \in (0, 1],
$$

where, for all $i \in I$,

$$
\Psi_i Q^i_N(P, \varphi, h) \Psi_i^* = \sum_{j=0}^N h^j \op_{w,h}^i (a_j^i)
$$

with symbols $a_0^i, \ldots, a_N^i$ of the form

$$
a_0^i = \varphi \circ p_2^i, \quad a_j^i = \sum_{k \leq k(j)} d_{jk}^i \varphi^{(k)} \circ p_2^i, \quad j \geq 1,
$$

using the functions $p_2^i$ given by (1.10) for $i \in I_\infty$ and (1.11) for $i \in I_{\text{comp}}$. Here $k(j) < \infty$ and $d_{jk}^i \in S^k_{\text{comp}}(\nu_i \times \mathbb{R}^n)$ is polynomial in the momentum variable ($d_{jk}^i \equiv 0$ if $2k - j < 0$) and independent of $\varphi$.

In addition, for all $m, m' \in \mathbb{N}$, all $A \in \Diff^m_{\text{comp}}(\mathcal{M})$, $B \in \Diff^{m'}(\mathcal{M})$, all $p \in [2, \infty]$ and all $N$ such that $N > n - 2\sigma + m + m'$, we have

$$
\left\| h^m AR_N(-\Delta_g, \varphi, h) h^{m'} B \right\|_{L^2(\mathcal{M}, dg) \to L^p(\mathcal{M}, dg)} \lesssim h^{-n\left(\frac{d-1}{2} - \frac{1}{p}\right)},
$$

and, for $P = -\tilde{\Delta}_g$,

$$
\left\| w(r)^{\frac{n+1}{2} - \frac{n+1}{r}} h^m AR_N(-\tilde{\Delta}_g, \varphi, h) h^{m'} B \right\|_{L^2(\mathcal{M}, dg) \to L^p(\mathcal{M}, dg)} \lesssim h^{-n\left(\frac{d-1}{2} - \frac{1}{p}\right)},
$$

both for $h \in (0, 1]$.

This theorem roughly means that, near infinity, $\varphi(h^2P)$ is well approximated by pseudodifferential operators with symbols of the form $a(r, \theta, \rho, w(r)\eta)$. The principal symbol is for instance

$$
\varphi(p_2^i(r, \theta, \rho, w(r)\eta)).
$$

Note that, when $\varphi \in C^\infty_0(\mathbb{R})$, this symbol is compactly supported with respect to $\rho$ but not uniformly with respect to $\eta$: if $w(r) \to 0$ as $r \to \infty$, $\eta$ is not confined in a fixed compact set, since we only have $|\eta| \lesssim w(r)^{-1}$.

The estimates (1.33) and (1.34) follow from the Sobolev embedding $D((\Delta_g)^k) \subset L^\infty(\mathcal{M})$ for $k > n/4$ (see Proposition 2.11) and, to that extent, Theorem 1.5 is an $L^2$ theorem.

We now consider the $L^p \to L^p$ properties. Recall first a classical definition

Definition 1.6. A function $W : \mathbb{R} \to (0, +\infty)$ is a temperate weight if, for some positive constants $C, M$, 

$$
W(r') \leq CW(r)(1 + |r - r'|)^M, \quad r, r' \in \mathbb{R},
$$

(1.35)
The meaning of this definition is that $W$ can neither grow nor decay too fast. For instance if $d^kw^{-1}/dr^k$ is bounded on $\mathbb{R}$, $w$ is a temperate weight. This is an elementary consequence of Taylor’s formula to order $k$ and of the fact that $|d^kw^{-1}/dr^k| \lesssim w^{-1}$, by (1.8).

The operators $\text{op}_{w, h}(a_j')$ of Theorem 1.5 are bounded on $L^p(M, dg)$, $L^p(M, \tilde{dg})$, or more generally on $L^p(M, W(r)dg)$ and $L^p(M, W(r)d\tilde{g})$ for all temperate weight $W$ and all $p \in [1, \infty]$ (see Proposition 2.3). We therefore focus on the remainder terms $\mathcal{R}_N(P, \varphi, h)$.

**Theorem 1.7.** For all $N \geq 0$, all temperate weight $W$ and all $1 < p < \infty$,

$$\left\| W(r)^{-1}R_N(-\tilde{A}_g, \varphi, h)W(r) \right\|_{L^p(M, \tilde{dg}) \to L^p(M, \tilde{dg})} \leq C_{N, p, \varphi, W}, \quad h \in (0, 1].$$

The constant $C_{N, p, \varphi, W}$ depends (linearly) on a finite number of semi-norms of $\varphi \in S^{-\sigma}(\mathbb{R})$.

**Corollary 1.8.** For all $1 < p < \infty$ and all temperate weight $W$,

$$\left\| W(r)^{-1}\varphi(-h^2\tilde{A}_g)W(r) \right\|_{L^p(M, \tilde{dg}) \to L^p(M, \tilde{dg})} \lesssim 1, \quad h \in (0, 1].$$

Equivalently, we have

$$\left\| W(r)^{-1}w(r)^{\frac{p-1}{p}-\frac{n+1}{2}}\varphi(-h^2\tilde{A}_g)w(r)^{-\frac{n+1}{2}}W(r) \right\|_{L^p(M, dg) \to L^p(M, dg)} \lesssim 1, \quad h \in (0, 1].$$

Observe that Theorem 1.7 and Corollary 1.8 hold in particular if $w(r) = e^{-r}$ in which case $\varphi(-h^2\tilde{A}_g)$ is in general not bounded on $L^p(M, dg)$. Theorem 1.7 is a consequence of a stronger result, namely Proposition 3.8 showing that, in any chart, the resolvent $(z-\tilde{A}_g)^{-1}$ is a pseudodifferential operators whose full symbol belongs to a suitable class. Since this result is of more technical nature, we prefer not to state it in this part.

If the function $w$ itself is a temperate weight, for instance if $w(r) = r^{-1}$ for $r$ large, Theorem 1.7 also implies the following result.

**Corollary 1.9.** If $w$ is a temperate weight, then for all temperate weight $W$, all $N \geq 0$ and all $1 < p < \infty$,

$$\left\| W^{-1}(r)R_N(-\Delta_g, \varphi, h)W(r) \right\|_{L^p(M, dg) \to L^p(M, dg)} \leq C_{N, p, \varphi, W}, \quad h \in (0, 1].$$

The constant $C_{N, p, \varphi, W}$ depends (linearly) on a finite number of semi-norms of $\varphi \in S^{-\sigma}(\mathbb{R})$. In particular,

$$\left\| W(r)^{-1}\varphi(-h^2\Delta_g)W(r) \right\|_{L^p(M, dg) \to L^p(M, dg)} \lesssim 1, \quad h \in (0, 1].$$

Of course, (1.38) holds with $W = 1$. As explained in the introduction, this last result can be considered as essentially well known (see for instance [22] for $h = 1$). We quote it to emphasize the difference with Corollary 1.8 where $w$ is not assumed to be a temperate weight. It follows directly from Theorem 1.7 using (1.21), (1.22) and the fact that products or real powers of temperate weights are temperate weights.

### 2 Parametrix of the resolvent and applications

In the main part of this section, namely until (2.19), we work in coordinate patches $U_\iota$ of the form (1.9) (ie with $\iota \in I_\infty$).
2.1 Elementary pseudo-differential calculus

In this part, we give elementary composition formulas and the related remainder estimates for pseudo-differential operators of the form $\text{op}_w^a(b)$. We will note develop a systematic study of the symbolic calculus but only record the basic results required for the calculation of parametries of $(z - h^2 \Delta_z)^{-1}$ and $(z - h^2 \Delta_z)^{-1}$.

For $\Omega \subset \mathbb{R}^D$, $D \geq 1$, $C^\infty_c(\Omega)$ will denote the space of smooth functions bounded on $\Omega$ as well as their derivatives.

For $b \in \mathcal{S}^m(\mathcal{V}_l \times \mathbb{R}^n)$ and $h \in (0, 1]$, we set

$$[b(r, \theta, hD_r, hw(r)D_\theta)v](r, \theta) = (2\pi)^{-n} \int \int e^{i (r \rho + \theta \eta)} b(r, \theta, h\rho, hw(r)\eta) \hat{v}(\rho, \eta) d\rho d\eta \quad (2.1)$$

with $\hat{v}(\rho, \theta) = \int e^{-ir\rho - i\theta \eta} v(r, \theta) dr d\theta$ the usual Fourier transform. In the special case of a polynomial symbol in $\rho$ and $\eta$, $a(r, y, \rho, \eta) = \sum a_{\alpha}(r, \theta) \rho^\alpha \eta^\beta$, we have

$$a(r, \theta, hD_r, hw(r)D_\theta) = \sum a_{\alpha}(r, \theta)(hw(r)D_\theta)^\alpha (hD_r)^\beta, \quad (2.2)$$

where one must notice that $D_r$ and $w(r)D_\theta$ don’t commute.

We have the following elementary result.

Proposition 2.1. Let $a \in \mathcal{S}^{m_1}(\mathcal{V}_l \times \mathbb{R}^n)$ be polynomial in $(\rho, \eta)$ and let $b \in \mathcal{S}^{m_2}(\mathcal{V}_l \times \mathbb{R}^n)$ with $m_2 \in \mathbb{R}$. We have

$$a(r, \theta, hD_r, hw(r)D_\theta)b(r, \theta, hD_r, hw(r)D_\theta) = \sum_{l=0}^{m_1} h^l(a \# b)_l(r, \theta, hD_r, hw(r)D_\theta) \quad (2.3)$$

where, if we set

$$D_w = D_r + \frac{w'(r)}{w(r)} \eta \cdot D_\eta,$$

the symbol $(a \# b)_k = (a \# b)_k(r, \theta, \rho, \eta) \in \mathcal{S}^{m_1 + m_2 - k}(\mathcal{V}_l \times \mathbb{R}^n)$ is given by

$$(a \# b)_k = \sum_{j+|\beta|=k} \frac{1}{j! |\beta|} w(r)^{|\beta|} (\partial_{\rho}^j \partial_{\eta}^\beta a) \left(D_\rho^j D_\eta^\beta b\right).$$

When $w \equiv 1$, this proposition is of course the usual composition formula for pseudo-differential operators. Note that, since $a$ is polynomial of degree $\leq m_1$, we have $(a \# b)_l \equiv 0$ for $l > m_1$ and the composition formula is exact (there is no remainder term).

Remark. A simple induction shows that the operator $D^l_w$ is a linear combination of

$$\left(\frac{w'(r)}{w(r)}\right)^{(j_1)} \cdots \left(\frac{w'(r)}{w(r)}\right)^{(j_k)} D_r^{j_1} \eta^{\alpha} D_\eta^\beta \quad (2.4)$$

with $j_1 + \cdots + j_k + k + l = j$, $|\alpha| \leq k$ and $k \geq 0$. If $k = 0$ then $(w'/w)^{(j_1)} \cdots (w'/w)^{(j_k)} = 1$. The notation $(w'/w)^{(j_i)}$ stands for the $j_i$-th derivative of $w'/w$ with $j_i \geq 0$.

Proof of Proposition 2.1. Applying the right and side of (2.2) to (2.1), the result follows from the Leibniz rule and the fact that

$$D_r (b(r, \theta, h\rho, hw(r)\eta)) = (D_w b)(r, \theta, h\rho, hw(r)\eta).$$
We omit the standard details of the calculation. That \((a \# b)_k\) belongs to \(S^{m_1 + m_2 - k}(\mathcal{V} \times \mathbb{R}^n)\) follows from (1.8) using (2.4).

We next consider the pseudo-differential quantization \(\text{op}^t_{w,h}(\cdot)\) given by (1.29).

**Proposition 2.2.** Let \(a \in S^m_{c}(\mathcal{V} \times \mathbb{R}^n)\) be polynomial in \((\rho, \eta)\) and let \(b \in S^{m_2}(\mathcal{V} \times \mathbb{R}^n)\) with \(m_2 \in \mathbb{R}\). Let \(W\) be a positive function on \(\mathbb{R}\) such that

\[
W(r) \leq C W(r'), \quad |r - r'| \leq 1.
\]

Then, for all \(N > 0\),

\[
a(r, \theta, hD_r, hw(r)D_\theta)\text{op}^t_{w,h}(b) = \sum_{l=0}^{m_1} h^l \text{op}^t_{w,h}((a \# b)_l) + h^{N+1} R_N(h, a, b),
\]

where, for all \(k_1, k_2 \in \mathbb{N}\), all \(A_1 \in \text{Diff}^{k_1}_w(\mathcal{M})\), \(A_2 \in \text{Diff}^{k_2}_w(\mathcal{M})\) and all \(p \in [1, \infty]\),

\[
\left| \left| W(r)A_1 \Psi^r_N(h, a, b) \Psi_{t\ast} A_2 W(r)^{-1} \right| \right|_{L_p(\mathcal{M}, dg) \rightarrow L_p(\mathcal{M}, dg)} \lesssim 1,
\]

and

\[
\left| \left| w(r) \frac{\partial}{\partial r} W(r)A_1 \Psi^r_N(h, a, b) \Psi_{t\ast} A_2 W(r)^{-1} \right| \right|_{L^2(\mathcal{M}, dg) \rightarrow L^\infty(\mathcal{M})} \lesssim 1,
\]

for \(h \in (0, 1]\). More precisely the norms in (2.6) and (2.7) are controlled by a finite number of semi-norms of \(a\) and \(b\) independent of \(h\).

Note that the condition (2.5) is satisfied if \(W\) is a temperate weight but also by any power of \(w\). In particular, \(W(r) = e^{\gamma r}\) is a possible choice although it is not a temperate weight. In particular, (2.6) and (2.7) are respectively equivalent to

\[
\left| \left| W(r)A_1 \Psi^r_N(h, a, b) \Psi_{t\ast} A_2 W(r)^{-1} \right| \right|_{L_p(\mathcal{M}, dg) \rightarrow L_p(\mathcal{M}, dg)} \lesssim 1,
\]

and

\[
\left| \left| W(r)A_1 \Psi^r_N(h, a, b) \Psi_{t\ast} A_2 W(r)^{-1} \right| \right|_{L^2(\mathcal{M}, dg) \rightarrow L^\infty(\mathcal{M})} \lesssim 1,
\]

They are simply obtained by replacing \(W(r)\) respectively by \(W(r)w(r)^{1-n}\) and \(W(r)w(r)^{\frac{1}{1-n}}\) which both satisfy (2.5).

By opposition to Proposition 2.1 we now have a remainder. It is due to the derivatives of cutoff near the diagonal in the definition of \(\text{op}^t_{w,h}(\cdot)\) but not to the tail of the expansion \(\sum_l h^l (a \# b)_l\) for this sum is finite.

Before proving this proposition, we state two lemmas which will be useful further on and whose proofs are very close to the proofs of the estimates (2.5) and (2.7).

**Lemma 2.3.** Let \(c \in S^m_{c}(\mathcal{V} \times \mathbb{R}^n)\) with \(m < 0\) and let \(W\) be a positive function satisfying (2.5). Then, for all \(p \in [1, \infty]\), we have

\[
\left| \left| W(r)\text{op}^t_{w,h}(c) W(r)^{-1} \right| \right|_{L_p(\mathbb{R}^n) \rightarrow L_p(\mathbb{R}^n)} \lesssim 1, \quad h \in (0, 1].
\]

**Proof.** Consider first the case where \(W \equiv 1\). If \(\hat{c}\) is the Fourier transform of \(c\) with respect to \(\rho, \eta\), the kernel of \(\text{op}^t_{w,h}(c)\) reads

\[
C_t(r, \theta, r', \theta', h) = h^{-n} w(r)^{1-n} \hat{c} \left( r, \theta, \frac{r' - r}{h}, \frac{\theta' - \theta}{hw(r)} \right) W(r) W(r') \lambda_t(r, \theta, r', \theta').
\]
For \((r, \theta) \in V_i\), \(c(r, \theta, \ldots) \in L^{r+n/|m|}(R^n_{p,q})\), with norm uniformly bounded with respect to \((r, \theta)\), thus \(c(r, \theta, \ldots)\) belongs to a bounded subset of \(L^1_{loc}(R^n_{p,q})\) by Young’s theorem. Therefore, for all \(N\) we can write

\[
|\hat{c}(r, \theta, \hat{\rho}, \hat{\eta})| \leq C_N(1 + f_0(r, \theta, \hat{\rho}, \hat{\eta}))(|\hat{\rho}| + |\hat{\eta}| + 1)^{-N}, \quad (r, \theta) \in V_i, \quad \hat{\rho} \in \mathbb{R}, \quad \hat{\eta} \in \mathbb{R}^{n-1}, \quad (2.10)
\]

with \(f_0(r, \theta, \ldots)\) bounded in \(L^{1\,\text{comp}}(R^n_{p,q})\). Thus, the family \(\hat{c}(r, \theta, \ldots)\) is bounded in \(L^1(R^n_{p,q})\). Elementary changes of variables show that

\[
\sup_{(r, \theta) \in \mathbb{R}^n} \int_{\mathbb{R}^{n-1}} |C_i(r, \theta, r', \theta', h)| dr' d\theta' \lesssim 1, \quad \sup_{(r', \theta') \in \mathbb{R}^n} \int_{\mathbb{R}^{n-1}} |C_i(r, \theta, r', \theta', h)| dr d\theta \lesssim 1,
\]

for \(h \in (0, 1]\). Recall that \(C_i\) is globally defined on \(\mathbb{R}^{2n}\) so the above quantities makes sense. The result is then a consequence of the standard Schur lemma. For a general \(W\) the same proof applies since we only have to multiply the kernel \(C_i\) by the bounded function \(W(r)\chi^i_1(r, \theta, r', \theta')W(r')^{-1}\) on the support of which \(r - r'\) is bounded.

\[\square\]

**Lemma 2.4.** Let \(c \in S^{m}_{c}(V_i \times \mathbb{R}^n)\) with \(m < -n/2\) and let \(W\) be a positive function satisfying (2.2). Then

\[
\left\|w(r)^{-\frac{n}{2}} W(r) \text{op}_{w, h}(c) W(r)^{-1}\right\|_{L^2(R^n) \rightarrow L^\infty(R^n)} \lesssim h^{-n/2}, \quad h \in (0, 1].
\]

**Proof.** With the notation of the proof of Lemma 2.3 the result is a direct consequence of the estimate

\[
\sup_{(r, \theta) \in \mathbb{R}^n} \int_{\mathbb{R}^{n-1}} |w(r)^{-\frac{n}{2}} W(r) C_i(r, \theta, r', \theta', h) W(r')^{-1}|^2 dr' d\theta' \lesssim h^{-n}, \quad h \in (0, 1]
\]

which follows again from elementary changes of variables, using that \(\hat{c}(r, \theta, ..)\) belongs to a bounded subset of \(L^2(R^n)\) as \((r, \theta)\) varies and that \(W(r)/W(r')\) is bounded on the support of \(C_i\).

\[\square\]

**Remark.** The proofs of both lemmas still hold if the kernel of \(\text{op}_{w, h}(c)\) is multiplied by a bounded function. We shall use it in the following proof.

**Proof of Proposition 2.2.** We may clearly assume that (2.2) is reduced to one term. Applying this operator to (1.2) (with \(a = b\)) on the \(r, \theta\) variables, we get the kernel of \(\sum_k h^k \text{op}_{w, h}((a\#b)_k)\) (using Proposition 2.1) plus a linear combination of integrals of the form

\[
a_j\alpha(r, \theta) \int e^{i(r-r')^\rho+i(\theta-\theta')\cdot\eta} m(h, \rho, h, w(r)\eta) \frac{d\rho d\eta}{\partial_\theta^\rho \partial_\theta^\eta \chi^i_1(r, \theta, r', \theta')} \]

where \(j_1 + j_2 + j_3 = j, \alpha_1 + \alpha_2 + \alpha_3 = \alpha\) and \(j_3 + |\alpha_3| \geq 1\). The latter implies that \(\partial_\theta^\rho \partial_\theta^\eta \chi^i_1\) is supported in \(|(r, \theta) - (r', \theta')| \geq \varepsilon\) which allows to integrate by parts using \(|(r, \theta) - (r', \theta')|^{-2} \Delta_{\rho, \eta}\). We thus obtain integrals of the form

\[
h^{2N} \int e^{i(r-r')^\rho+i(\theta-\theta')\cdot\eta} c_N(r, \theta, h, w(r)\eta) d\rho d\eta \frac{B_N(r, \theta, r', \theta')}{|(r, \theta) - (r', \theta')|^{2N}}
\]

(2.11) with \(N\) as large as we want, \(c_N \in S^{m+|\alpha|+j-2N}(V_i \times \mathbb{R}^n)\) and \(B_N \in C^\infty(R^{2n})\) with support in \(|\varepsilon \leq |(r, \theta) - (r', \theta')| \leq 2\varepsilon\). With no loss of generality, we may assume that

\[
\Psi_{\varepsilon^*} A_1 \Psi^* = (w(r)D_\theta)^\beta D_r^k, \quad \Psi_{\varepsilon^*} A_2 \Psi^* = (w(r)D_\theta)^{\beta'} D_r^{k'}.
\]

12
Applying \(w(r)D_b \partial^3 D^k\) to (2.11) yields an integral of the same form, using the boundedness of \(w\) and its derivatives. To apply (the transpose of) \((w(r')D_y)\partial^3 D^k\) to the kernel of \(P_N(a, b, h)\), we rewrite this operator as \((w(r')/w(r))||^b|^\partial^3 (w(r)D_y)\partial^3 D^k\). We still obtain integrals of the same form as (2.11) multiplied by derivatives of \((w(r')/w(r))||^b|^\partial^3\). By (1.7), these derivatives are bounded since \(|r - r'| \leq 2\varepsilon\) on the support of \(B_N\). Then (2.6) and (2.7) follow respectively from the proofs of Lemma 2.3 and 2.4.

So far, we have considered composition with differential operators to the left. Since our operators are properly supported, the composition to the right can be also easily considered.

**Proposition 2.5.** Let \(a\) and \(b\) be as in Proposition 2.4, and let \(W\) be a positive function satisfying (2.2). Then, for all \(N > m_1 + m_2 + n\), we have

\[
op_w^\alpha(b)(a(r, \theta, hDr, hw(r)D\psi) = \sum_{i=0}^N b^i \nop_w^\alpha (c_i) + h^{N + 1} P_N^w(h, a, b)
\]

with \(c_i \in S^{m_1, m_2 - l}(\mathcal{V}, \times \mathbb{R}^n)\) depending continuously on \(a\) and \(b\), and \(P_N^w(h, a, b)\) an operator with continuous kernel supported in \(\mathcal{V}, \times \mathcal{V}\). Moreover, for all \(N\), all \(k_1, k_2 \in \mathbb{N}\) such that

\[N > m_1 + m_2 + n + k_1 + k_2\]

all \(A_1 \in \text{Diff}_{w}^{k_1}(\mathcal{M}), A_2 \in \text{Diff}_{w}^{k_2}(\mathcal{M})\) and for all \(p \in [1, \infty]\), we have

\[
\left\| W(r)A_1 \Psi^* P_N^w(h, a, b) \Psi A_2 W(r)^{-1}\right\|_{L^p(\mathcal{M}, \bar{d}g)} \lesssim 1,
\]

\[
\left\| W(r)w(r) \frac{\partial}{\partial r} A_1 \Psi^* P_N^w(h, a, b) \Psi A_2 W(r)\right\|_{L^2(\mathcal{M}, \bar{d}g)} \lesssim 1,
\]

for \(h \in (0, 1]\). More precisely, these norms are controlled by a finite number of semi-norms of \(a\) and \(b\) independent of \(h\).

We will not need the explicit forms of the symbols \(c_i\) since we will only use this proposition for the analysis of some remainder terms.

Note also that the estimates on \(P_N^w(h, a, b)\) have analogues with respect to the measure \(d\gamma\), similar to (2.8) and (2.9).

**Proof.** We have to apply the transpose of \(a(r', \theta', hDr', hw(r')D\psi)\) to the Schwartz kernel of \(\nop_w^\alpha(b)\). For simplicity we assume first that \(a(r', \theta', \rho, \eta) = w(r')\eta_1\). By Taylor’s formula, we have

\[
w(r') = w(r) \left(1 + \sum_{j=1}^N \frac{1}{j!} \frac{w^{(j)}(r)}{w(r)} (r' - r)^j + \frac{(r' - r)^{N + 1}}{N!} \int_0^1 (1 - t)^N \frac{w^{(N + 1)}(r + t(r' - r))}{w(r)} dt\right).
\]

Integrating by parts with respect to \(\rho\) in the kernel of \(\nop_w^\alpha(b)\), the principal part of the Taylor expansion yields the expected expansion with

\[c_1(r, \theta, \rho, \eta) = \frac{1}{j!} D_{\rho}^j b(r, \theta, \rho, \eta) \frac{w^{(j)}(r)}{w(r)} \eta_1.
\]

The remainder is given by two types of terms: first by the derivatives \(D_{\rho}^j\) falling on \(\chi^2(r, \theta, r', \theta')\), which yields kernels of the form (2.11), and second by the remainder in the Taylor formula thanks
to which we can integrate by parts \( N \) times with respect to \( \rho \). In this case, we get a kernel of the form (2.11), with \( N \) instead of \( 2N \) and a symbol \( c_N \in S^{m_1+m_2-N}(V_t \times \mathbb{R}^n) \). Since \( r - r' \) is bounded on the support of \( \chi^\xi \), \( w^{(N)}(r + t(r' - r))/w(r) \) is bounded too, uniformly with respect to \( t \in [0,1] \), and the study of the remainder is similar to the one of Proposition 2.2. By induction, we obtain the result if \( a = (w(r')\eta)^2 \). Derivatives with respect to \( r \) or multiplication operators are more standard and studied similarly.

### 2.2 Parametrix of the resolvent

In this subsection, we construct a parametrix of the semi-classical resolvent of an operator \( P \in \text{Diff}_w^2(\mathcal{M}) \). Recall that this means that \( P \) is a differential operator of order 2 such that, in any chart at infinity,

\[
\Psi_\iota \ast P \Psi_\iota = \sum_{k=0}^{2} p_{2-k}^\iota(r, \theta, D_r, w(r)D_\theta)
\]

with \( p_{2-k}^\iota \in S^{2-k}(V_t \times \mathbb{R}^n) \).

We assume that \( P \) is locally elliptic,

\[
P \text{ is locally elliptic,}
\]

ie, in any chart, its principal symbol \( p_0^\iota(x, \xi) \) satisfies \( |p_0^\iota(x, \xi)| \gtrsim |\xi|^2 \) for \( \xi \in \mathbb{R}^n \), locally uniformly with respect to \( x \). If \( t \in I_\infty \), using the notation (2.12), we furthermore assume that, for all \( K_t \in V_t \) (see (1.9)),

\[
|p_2^\iota(r, \theta, \rho, \eta)| \gtrsim \rho^2 + |\eta|^2, \quad r > R, \quad \theta \in K_t, \quad \rho \in \mathbb{R}, \quad \eta \in \mathbb{R}^{n-1}.
\]

Note that this is not a lower bound for the principal symbol of \( \Psi_\iota \ast P \Psi_\iota^* \), namely \( p_2^\iota(r, \theta, \rho, w(r)\eta) \), whose modulus is only bounded from below by \( \rho^2 + w(r)^2|\eta|^2 \). This is nevertheless the natural (degenerate) global ellipticity condition in this context. We next define \( \mathcal{C} \subset \mathbb{C} \) as

\[
\mathcal{C} = \text{closure of the range of the principal symbol of } P
\]

which is invariantly defined for the principal symbol is a function on \( T^*\mathcal{M} \). We assume that \( \mathcal{C} \neq \mathbb{C} \). In the final applications, with \( P = -\Delta \) or \( -\Delta_g \), we will of course have \( \mathcal{C} = [0, +\infty) \).

We now seek an approximate inverse of \( h^2P - z \), for \( h \in (0,1] \) and \( z \in \mathbb{C} \setminus \mathcal{C} \).

We work first in a patch at infinity. Using the notation of (2.12), we set for simplicity \( p_2 = p_2^\iota - z \), \( p_1 = p_1^\iota \), \( p_0 = p_0^\iota \).

Observe that \( p_0, p_1 \) don't depend on \( z \) but that \( p_2 \) does. We then have

\[
h^2\Psi_\iota \ast P \Psi_\iota^* - z = \sum_{k=0}^{2} h^k p_{2-k}(r, \theta, hD_r, h w(r)D_\theta).
\]

For a given \( N \geq 0 \), we look for symbols \( q_{-2}, q_{-3}, \ldots, q_{-2-N} \) satisfying

\[
\left( \sum_{k=0}^{2} h^k p_{2-k}(r, \theta, hD_r, hw(r)D_\theta) \right) \sum_{j=0}^{N} h^j \text{op}_{w,h}^\iota(q_{-2-j}) = \chi_t + O(h^{N+1}),
\]

(2.16)
where \( \chi \) is defined by (1.12) and where \( O(h^{N+1}) \) will be given a precise meaning below. Of course, we need to find such a family of symbols for each patch, i.e. \( q_{-2-j} \) depends on \( \iota \), but we omit this dependence for notational simplicity. By Proposition 2.6, the left hand side of (2.10) reads

\[
\sum_{k+j+l \leq N} h^{k+j+l} \| \partial_{w,h}^\iota \left( (p_{2-k} \# q_{-2-j})_l \right) + h^{N+1} R^\iota_N (h, z) \]

where

\[
R^\iota_N (h, z) = \sum_{k+j+l \geq N+1} h^{k+j+l-N-1} \| \partial_{w,h}^\iota \left( (p_{2-k} \# q_{-2-j})_l \right) + \sum_{k,j} \overline{R^\iota_N (h, h^k p_{2-k}, h^j q_{-2-j})},
\]

with \( \overline{R^\iota_N} \) defined in Proposition 2.2. In the above sums, we have \( 0 \leq k \leq 2 \), \( 0 \leq j \leq N \) and \( 0 \leq l \leq 2 \). Thus, by (1.12), requiring (2.10) leads to the following equations for \( q_{-2}, \ldots, q_{-2-N} \)

\[
\sum_{k+j+l = \nu} (p_{2-k} \# q_{-2-j})_l = \begin{cases} 1 & \text{if } \nu = 0, \\ 0 & \text{if } \nu \geq 1, \end{cases} \quad 0 \leq \nu \leq N.
\]

This system is triangular and, since \( (a \# b)_0 = ab \), its unique solution is given recursively by

\[
q_{-2} = \frac{1}{p_2}, \quad q_{-2-j} = -\frac{1}{p_2} \sum_{k+j+1 = \nu} (p_{2-k} \# q_{-2-j_i})_l \quad \text{for } j \geq 1.
\]

**Proposition 2.6.** For all \( j \geq 1 \), \( q_{-2-j} \) is a finite sum (with a number of terms \( k(j) \) depending on \( j \) but not on \( z \)) of the form

\[
q_{-2-j} = \sum_{k=1}^{k(j)} d_{jk} \frac{1}{p_2^{1+k}},
\]

where, for each \( k \), \( d_{jk} \in \mathbb{S}_{2k-j}^\iota (\mathbb{V}_1 \times \mathbb{R}^n) \) is a polynomial in \( \rho, \eta \) which is independent of \( z \) (in particular \( d_{jk} \equiv 0 \) when \( 2k-j < 0 \)). More precisely, the coefficients of these polynomials are linear combinations of products of derivatives of \( w, w'/w \) and of the coefficients of \( p_0, p_1 \) and \( \partial^\alpha p_2 \) with \( \alpha \neq 0 \).

**Proof.** This follows from an induction using (2.1) and the fact that, for any multi-index \( \alpha \neq 0 \), \( \partial^\alpha (1/p_2^{1+k}) \) is a linear combination of

\[
\frac{\partial^{\alpha_1} p_2 \cdots \partial^{\alpha_{|\alpha|}} p_2}{p_2^{1+k+k'}},
\]

with \( \alpha_1 + \cdots + \alpha_{k'} = \alpha \), \( 1 \leq k' \leq |\alpha| \) and \( \alpha_i \neq 0 \) for all \( i \in \{1, \ldots, k'\} \).

With the notation (2.17), we set

\[
R^\iota_N (h, z) = \Psi^\ast R^\iota_N (h, z) \Psi_{ss}.
\]

**Lemma 2.7.** Let \( dp \) denote either \( d\rho \) or \( d\eta \). Then, for all positive function \( W \) satisfying (2.5), all \( p \in [1, \infty] \) and all \( N \geq 0 \), there exists \( \nu > 0 \) such that, for all \( A \in \text{Diff}_w^m (\mathcal{M}) \) and \( B \in \text{Diff}_w^m (\mathcal{M}) \) with \( m + m' - N < 0 \), we have

\[
\left\| W (r) h^m A R^\iota_N (h, z) h^{m'} B W (r)^{-1} \right\|_{L^p (\mathcal{M}, dp) \rightarrow L^p (\mathcal{M}, dp)} \lesssim \left( \frac{1 + |z|}{\text{dist}(z, \mathcal{C})} \right)^\nu,
\]

for all \( h \in (0, 1] \) and all \( z \notin \mathcal{C} \).
Proof. We first assume that $A = B = 1$ (and that $m = m' = 0$). By (1.27), the kernel of $R_N^i(h, z)$ is supported in $((R, +\infty) \times K_i)^2$ for some $K_i \in V_i$. Thus, using the equivalence of norms (1.23), the result, with $d \mu = dg$, is a direct consequence of the bound

$$
||W(r)R_N^i(h, z)W(r)^{-1}||_{L^p(\mathbb{R}^n) \rightarrow L^p(\mathbb{R}^n)} \lesssim \left( \frac{1 + |z|}{\text{dist}(z, \mathcal{C})} \right)^\nu, \quad h \in (0, 1), \ z \notin \mathcal{C}, \quad (2.18)
$$

which follows from Proposition 2.2 and Lemma 2.3 once noticed that each semi-norm of $q_{-2-j}$ in $S^{-2-j}_t(V_i \times \mathbb{R}^n)$ is bounded by some power of $(1 + |z|)/\text{dist}(z, \mathcal{C})$. The latter is due to Proposition 2.6 and

$$
\left| 1 + \frac{\rho^2 + \eta^2}{\rho^2 - z} \right| \approx \left| 1 + \frac{p_2^2}{p_2^2 - z} \right| \lesssim \frac{1 + |z|}{\text{dist}(z, \mathcal{C})},
$$

in which we used (2.14). When $d \mu = dg$, we use the equivalence (1.24) so that it is now sufficient to get the bound (2.13) with $R_N^i(h, z)$ replaced by $w(r)^{\frac{j-n}{2n}} R_N^i(h, z) w(r)^{\frac{n-j}{2n}}$. The latter is clear for this amounts to multiply the kernel of $R_N^i(h, z)$ by $(w(r')/w(r))^{(n-1)/p}$ (which is bounded, using the boundedness of $r - r'$ on the support of $\chi_1^i$ and (1.7)) so the (proofs of) Proposition 2.2 and Lemma 2.3 still hold.

For general $A$ and $B$, we use Propositions 2.2 and 2.3 so that we are reduced to the previous case with an operator of the same form as $R_N^i(h, z)$ except that the symbols of the first sum in (2.4.1) now belong to $S^{-N+m+m'}_t(V_i \times \mathbb{R}^n)$. We can apply Lemma 2.3 to this term and the result follows. □

Let us now define

$$
Q^\prime_N(h, z) = \sum_{j=0}^N h^j \sigma_{w, h}(q_{-2-j}), \quad Q_N(h, z) = \Psi_* Q^\prime_N(h, z) \Psi^*.
$$

Then, with $f_i$ given by (1.10), we obtain the relation

$$
(h^2 P - z) Q_N(h, z) = f_i + h^{N+1} R_N^i(h, z).
$$

(2.19)

So far, we have always assumed that $\iota \in I_{\text{comp}}$, ie worked in patches at infinity, but the same analysis still holds for relatively compact patches, ie for $\iota \in I_{\text{comp}}$. We don’t give the details of the construction in the latter case for two reasons: the first is that this is essentially well known for this is like working on a compact manifold and the second is that the proofs are formally the same with the simpler assumptions that $w \equiv 1$ and that $\chi_i$ is compactly supported.

Thus, by setting

$$
Q_N(h, z) = \sum_{\iota \in I} Q^\prime_N(h, z), \quad R_N(h, z) = \sum_{\iota \in I} R_N^i(h, z),
$$

then summing the equalities (2.19) over $I$ and using (1.10), Lemma 2.7 gives the following result where we recall that $\mathcal{C}$ is defined by (2.14).

**Theorem 2.8.** Let $P \in \text{Diff}^{2m}_t(M)$ be a second order differential operator satisfying (2.13) and (2.14). Then, for all $N \geq 0$, we have

$$
(h^2 P - z) Q_N(h, z) = 1 + h^{N+1} R_N(h, z), \quad h \in (0, 1), \ z \notin \mathcal{C}.
$$

(2.20)
If \( d\mu \) denotes either \( dq \) or \( \tilde{d}g \), and \( m,m' \in \mathbb{N} \) satisfy \( m + m' < N \), then for all \( p \in [1,\infty] \) and for all positive function \( W \) satisfying (2.24), there exists \( \nu \geq 0 \) such that, for all \( A \in \text{Diff}_w^m(M) \) and \( B \in \text{Diff}_w^{m'}(M) \), we have

\[
\left\| W(r)h^m A R_N(h,z) h^{m'} B W(r)^{-1}\right\|_{L^p(M,d\mu)\rightarrow L^p(M,d\mu)} \lesssim \left( \frac{1 + |z|}{\text{dist}(z,\mathcal{C})} \right)\nu ,
\]

for all \( h \in (0,1] \) and all \( z \notin \mathcal{C} \).

This theorem gives a parametrix of the resolvent of \( h^2 P \) under the natural ellipticity conditions (2.13) and (2.14) (recall that if \( w \) is not bounded from below, this corresponds to a degenerate ellipticity).

From now on, we assume that

\[ P \text{ is self-adjoint with respect to } d\mu = dq \text{ or } \tilde{d}g. \]

This condition is actually equivalent to the symmetry of \( P \) on \( C_0^\infty(M) \). Indeed, (2.20) and (2.21) implies that \( h^2 P \pm i \) is injective for \( h \) small enough, which shows that \( P \) is essentially self-adjoint.

The resolvent \((h^2 P - z)^{-1}\) is then well defined for all \( z \notin \mathbb{R} \) and

\[
(h^2 P - z)^{-1} = Q_N(z,h) - h^{N+1}(h^2 P - z)^{-1} R_N(z,h), \quad z \notin \mathbb{R}, \quad h \in (0,1].
\]

Theorem 2.8 implies, for \( z = i \), that in the operator norm on \( L^2(M,d\mu) \), we have

\[
(h^2 P - i)^{-1} (1 + O(h^{N+1})) = Q_N(h,i)
\]

and thus, for some \( h_0 > 0 \) small enough and some bounded operator \( B_1 \) on \( L^2(M,d\mu) \), we get

\[
(h_0^2 P - i)^{-1} = Q_N(i,h_0) B_1.
\]

More generally, for \( k \geq 1 \), we can write

\[
(h^2 P - z)^{-k} = \frac{1}{(k-1)!} \partial_z^{k-1} (h^2 P - z)^{-1}
\]

so applying \((k-1)!^{-1}\partial_z^{k-1}\) to (2.22) shows that \((h^2 P - z)^{-k}\) reads

\[
(k-1)!^{-1}\partial_z^{k-1} Q_N(z,h) + h^{N+1}(h^2 P - z)^{-k} \sum_{j=0}^{k-1} \frac{1}{j!} (h^2 P - z)^j \partial_z^j R_N(z,h), \quad (2.23)
\]

using the holomorphy of \( Q_N(z,h) \) and \( R_N(z,h) \) with respect to \( z \in \mathbb{C} \setminus \mathbb{R} \) which standardly follows from Proposition 2.6. Therefore, by choosing \( N \) large enough so that the sum above is bounded on \( L^2 \) (uniformly in \( h \)) and choosing then \( h = h_0 \) small enough, we obtain

\[
(h_0^2 P - i)^{-k} = (k-1)!^{-1}\partial_z^{k-1} Q_N(z,h_0)_{z=i} B_k,
\]

for some operator \( B_k \) bounded on \( L^2(M,d\mu) \).

**Lemma 2.9.** For all \( A \in \text{Diff}_w^{2k}(M) \), \( A\partial_z^{k-1} Q_N(z,h_0) \) is bounded on \( L^2(M,d\mu) \) and \( L^2(M,\tilde{d}g) \).
Proposition 2.10. Let \( \Psi_{\nu} \) be properly supported. Then follows from the same reasoning using since \( AQ_{\nu} \) vailancourt theorem and thus its pullback \( d\mu \). The case of \( \Psi_{\nu} \) be as in Proposition 2.10 and let \( \bar{\Psi}_{\nu} \) is properly supported.

Using (2.24), Lemma 2.9 and the Spectral Theorem, we obtain the following result.

Proposition 2.10. Let \( P \in \text{Diff}_{w}^{\kappa}(\mathcal{M}) \), satisfying (2.13) and (2.14), be self-adjoint with respect to \( d\mu = dg \) or \( \bar{dg} \). Then, for all \( k \geq 1 \) and all \( A \in \text{Diff}_{w}^{\kappa}(\mathcal{M}) \), we have

\[
\|h^{2k}A(h^{2}P - z)^{-k}\|_{L^{2}(\mathcal{M}, d\mu) \rightarrow L^{2}(\mathcal{M}, d\mu)} \lesssim \frac{\langle z \rangle^{k}}{|\text{Im}(z)|^{k}}, \quad z \notin \mathbb{R}, \quad h \in (0, 1].
\]

In the same spirit, we will prove the following Sobolev injections.

Proposition 2.11. Let \( P \) be as in Proposition 2.10 and let \( k > n/4 \) be an integer. Then, if \( P \) is self-adjoint with respect to \( d\mu = dg \), we have

\[
\|w(r)^{\frac{n+1}{2}}(h^{2}P - z)^{-k}\|_{L^{2}(\mathcal{M}, d\mu) \rightarrow L^{\infty}(\mathcal{M})} \lesssim h^{-\frac{n}{4}}\frac{\langle z \rangle^{k}}{|\text{Im}(z)|^{k}}, \quad z \notin \mathbb{R}, \quad h \in (0, 1].
\]

If it is self-adjoint with respect to \( d\mu = \bar{dg} \), we have

\[
\|w(r)^{\frac{n+1}{2}}(h^{2}P - z)^{-k}\|_{L^{2}(\mathcal{M}, \bar{dg}) \rightarrow L^{\infty}(\mathcal{M})} \lesssim h^{-\frac{n}{4}}\frac{\langle z \rangle^{k}}{|\text{Im}(z)|^{k}}, \quad z \notin \mathbb{R}, \quad h \in (0, 1].
\]

Of course, by taking the adjoints, we have the corresponding \( L^{1} \rightarrow L^{2} \) inequalities.

Proof. We assume that \( d\mu = dg \). By (2.24), Lemma 2.4 with \( W(r) = w(r)^{\frac{1-n}{2}} \) and the equivalence of norms (2.24), \( (h^{2}P - i)^{-k} \) is bounded from \( L^{2}(\mathcal{M}, dg) \) to \( L^{\infty}(\mathcal{M}) \). Therefore, using the spectral theorem,

\[
\| (h^{2}P - i)^{-k} \|_{L^{2}(\mathcal{M}, dg) \rightarrow L^{\infty}(\mathcal{M})} \leq C \left\| \frac{(h^{2}P - i)^{k}}{(h^{2}P - i)^{k}} \right\|_{L^{2}(\mathcal{M}, dg) \rightarrow L^{\infty}(\mathcal{M})} \lesssim h^{-k}.
\]

Using (2.23) with \( z = i \) and \( N \) large enough, the estimate above improves to

\[
\| (h^{2}P - i)^{-k} \|_{L^{2}(\mathcal{M}, dg) \rightarrow L^{\infty}(\mathcal{M})} \lesssim h^{-n/2},
\]

using again Lemma 2.4 for the principal part of the expansion. We then obtain the result from the estimate

\[
\left\| \frac{(h^{2}P - i)^{k}}{(h^{2}P - z)^{k}} \right\|_{L^{2}(\mathcal{M}, dg) \rightarrow L^{2}(\mathcal{M}, dg)} \lesssim \frac{\langle z \rangle^{k}}{|\text{Im}(z)|^{k}}.
\]

The case of \( d\mu = \bar{dg} \) is similar.
2.3 Proof of Theorem 1.5

We shall use the classical Helffer-Sjöstrand formula

\[
\varphi(H) = \frac{1}{\pi} \int_{\mathbb{R}^2} \bar{\varphi}(x + iy)(H - x - iy)^{-1}dxdy
\]  \hspace{1cm} (2.25)

with \( \bar{\varphi} = (\partial_x + i\partial_y)/2 \), valid for any self-adjoint operator \( H \). Here \( \bar{\varphi} \in C^\infty(\mathbb{C}) \) is an almost analytic extension of \( \varphi \), ie such that \( \bar{\varphi}|_{\mathbb{R}} = \varphi \) and \( \bar{\varphi}(z) \) vanishes to sufficiently high order on the real axis.

A justification of this formula for \( \varphi \in C^\infty(\mathbb{R}) \) can be found in [11]. It is shown in [11] that, if \( \varphi \in S^{-\sigma}(\mathbb{R}) \) with \( \sigma > 0 \), (2.25) holds with \( \bar{\varphi}_M \) defined by

\[
\bar{\varphi}_M(x + iy) = \chi_0(y/\langle x \rangle) \sum_{k=0}^{M} f^{(k)}(x) \frac{(iy)^k}{k!},
\]  \hspace{1cm} (2.26)

with \( M \geq 1 \) and \( \chi_0 \in C^\infty(\mathbb{R}) \) such that \( \chi_0 \equiv 1 \) near 0. With this choice, one has

\[
|\partial_{\bar{\varphi}_M}(x + iy)| \lesssim |y|^M/|\langle x \rangle|^{\sigma+1+M}, \quad x, y \in \mathbb{R}. \]  \hspace{1cm} (2.27)

This implies in particular that, for all integers \( \nu_1 \geq 1, \nu_2 \geq 0 \) and \( M \geq \nu_1 + \nu_2 \), we have

\[
\int_{\mathbb{R}^2} |\partial_{\bar{\varphi}_M}(x + iy)| \times |y|^{-\nu_1} \left( 1 + \frac{|x|}{|y|} \right)^{\nu_2} dxdy < \infty,
\]  \hspace{1cm} (2.28)

which is easily seen by splitting the integral into two parts, where \(|y| \leq 1 \) or \(|y| > 1 \), using the fact that \(|y/\langle x \rangle| \) is bounded on the support of \( \bar{\varphi}_M \) in the latter case. If \( \sigma > 1 \) and \( M \geq \nu \), we also have

\[
\int_{\mathbb{R}^2} |\partial_{\bar{\varphi}_M}(x + iy)| \left( 1 + \frac{|x|}{|y|} \right)^\nu dxdy < \infty.
\]  \hspace{1cm} (2.29)

Proof of Theorem 1.5 Let \( \nu \in I \). The form of \( \Psi_N(P, \varphi, h)\Psi'_N \), namely (1.32), simply follows by plugging the expansion (2.22) into (2.25) and applying Green’s formula. For the latter we use Proposition 2.10 (recalling that \( p_2 = p_2(z) \)). All the integrals makes sense by (2.28) if we choose \( \bar{\varphi}_M \) with \( M \geq \max_{j \leq N}(k(j) + 1) \).

Let us now prove (1.33) and (1.34). Since the proofs are very similar we only show (1.34) and thus consider \( P = -\Delta_g \). Fix \( N \geq 0 \). For \( N' > N \) and \( M \) large enough, both to be chosen later, we set

\[
R_{N'}(P, \varphi, h) = \frac{1}{\pi} \int_{\mathbb{R}^2} \partial_{x} \bar{\varphi}_M(x + iy)(h^2P - x + iy)^{-1}R_{N'}(x + iy, h)dxdy.
\]  \hspace{1cm} (2.30)

We next fix two integers \( k > n/4, \tilde{m} \geq m/2 \), and rewrite \( h^m A(h^2P - z)^{-1} \) (with \( z = x + iy \)) as

\[
(h^2P - i)^{-k} \left\{ (h^2P - i)^k h^m A(h^2P - i)^{-k + \tilde{m}} \right\} (h^2P - z)^{-1} (h^2P - i)^{k + \tilde{m}}.
\]  \hspace{1cm} (2.31)

Using Proposition 2.10 the term \( \{ \cdots \} \) is bounded on \( L^2(M, d\bar{g}) \) uniformly with respect to \( h \). On the other hand, by Theorem 2.8 there exists \( \nu_2 > 0 \) such that

\[
\left\| (h^2P - i)^{k + \tilde{m}} R_{N'}(z, h) h^m B \right\|_{L^2(M, d\bar{g}) \rightarrow L^2(M, d\bar{g})} \lesssim |z|^{\nu_2}/|\text{Im } z|^{\nu_2},
\]
for \( z \notin \mathbb{R} \) and \( h \in (0, 1] \), provided

\[ N' > m' + 2(k + \hat{m}). \]  

(2.32)

By Propositions 2.10 and 2.11 we therefore get, for \( p \in \{2, \infty\} \),

\[
\left\| w(r)^{\frac{n-1}{2} - \frac{n}{p}} h^m A(h^2 P - z)^{-1} R_N'(z, h) h^{m'} B \right\|_{L^2(\mathcal{M}, \tilde{d}g) \rightarrow L^p(\mathcal{M}, \tilde{d}g)} \lesssim \frac{h^{-n/2}}{|\text{Im} \ z|} \left( \frac{\langle z \rangle}{|\text{Im} \ z|} \right)^{n_2},
\]

where the extra power of \(|\text{Im} \ z|^{-1}\) comes from the term \((h^2 P - z)^{-1}\) in (2.31). Using (2.28), this estimate clearly proves that, for \( p \in \{2, \infty\} \),

\[
\left\| w(r)^{\frac{n-1}{2} - \frac{n}{p}} h^m A\mathcal{R}_N'(P, \varphi, h) h^{m'} B \right\|_{L^2(\mathcal{M}, \tilde{d}g) \rightarrow L^p(\mathcal{M}, \tilde{d}g)} \lesssim h^{-n(1/2 - 1/p)},
\]

if we choose \( M \geq v_2 + 1 \) in (2.30). Then, define \( \mathcal{Q}_N N'(P, \varphi, h) \) by

\[
\sum_{i \in I} \mathcal{Q}_{N'}(P, \varphi, h) = \sum_{i \in I} \mathcal{Q}_N(P, \varphi, h) + h^{N'+1} \mathcal{Q}_N(P, \varphi, h). \tag{2.33}
\]

Using the explicit form of \( \mathcal{Q}_N N'(P, \varphi, h) \), namely the fact that its symbol is a linear combination of terms of the form \( a(r, \theta, \varphi, w(r) \eta) \) with \( a \in S^{-2d-N} \) (this is due to (1.32)), one has

\[
\left\| w(r)^{\frac{n-1}{2} - \frac{n}{p}} h^m A\mathcal{Q}_N N'(\varphi, h) h^{m'} B \right\|_{L^2(\mathcal{M}, \tilde{d}g) \rightarrow L^p(\mathcal{M}, \tilde{d}g)} \lesssim h^{-n(\frac{1}{2} - \frac{1}{p})}, \quad h \in (0, 1],
\]

which is consequence of Propositions 2.2, 2.3 and of Lemmas 2.3 and 2.4. Since

\[
R_N(P, \varphi, h) = h^{N'-N} \mathcal{R}_N'(P, \varphi, h) + \mathcal{Q}_N N'(P, \varphi, h), \tag{2.34}
\]

by choosing \( N' \) such that \( N' - N - 2k \geq -n/2 + n/p \) and (2.32) holds, we get (1.34) for \( p = 2 \) or \( \infty \). The other cases follow by interpolation. \( \square \)

### 3 \( L^p \) bounds for the resolvent

Consider a temperate weight \( W \) in the sense of Definition 1.6. The main purpose of this section is to prove the following theorem.

**Theorem 3.1.** For all \( 1 < p < \infty \), there exists \( \nu_p > 0 \) such that

\[
\| W(r)^{(n-1)}(\frac{1}{r} - \frac{1}{h}) (z - \bar{\Delta}_g)^{-1} W(r)^{(1-n)}(\frac{1}{r} - \frac{1}{h}) W(r)^{-1} \|_{L^p(\mathcal{M}, \tilde{d}g)} \lesssim \left( \frac{\langle z \rangle}{|\text{Im} \ z|} \right)^{\nu_p},
\]

for all \( z \in \mathbb{C} \setminus \mathbb{R} \).

Recall that \( \bar{\Delta}_g \) is defined by (1.22) and is self-adjoint with respect to \( \tilde{d}g \) given by (1.20). Translated in terms of \( \Delta_g \), Theorem 3.1 gives

**Corollary 3.2.** For all \( 1 < p < \infty \), there exists \( \nu_p > 0 \) such that

\[
\| W(r)^{(n-1)}(\frac{1}{r} - \frac{1}{h}) (z - \Delta_g)^{-1} w(r)^{(1-n)}(\frac{1}{r} - \frac{1}{h}) W(r)^{-1} \|_{L^p(\mathcal{M}, dg)} \lesssim \left( \frac{\langle z \rangle}{|\text{Im} \ z|} \right)^{\nu_p},
\]

for all \( z \in \mathbb{C} \setminus \mathbb{R} \).
Theorem 3.1 is a consequence of Proposition 3.8 showing a stronger result, namely that, in local charts, \((z - \Delta_g)^{-1}\) is a pseudo-differential operator with symbol in a class that guarantees the \(L^p\) boundedness on \(L^p(M, \tilde{d}g)\). Using Proposition 3.8 we also obtain the following result.

**Theorem 3.3.** If \(w\) is itself a temperate weight, then for all temperate weight \(W\) and all \(1 < p < \infty\), there exists \(\nu_p > 0\) such that

\[
\|W(r)(z - \Delta_g)^{-1}W(r)^{-1}\|_{L^p(M, \tilde{d}g)} \lesssim \left( \frac{1}{|\text{Im} \ z|} \right)^{\nu_p},
\]

for all \(z \in \mathbb{C} \setminus \mathbb{R}\).

This holds in particular if \(W \equiv 1\).

### 3.1 Reduction

In this subsection, we explain how to reduce Theorem 3.1 to Proposition 3.8 below. This reduction rests on classical results on pseudo-differential operators, namely the Calderón-Zygmund Theorem 3.4 and the Beals Theorem 3.6.

Recall first the definitions of the usual classes of symbols \(S^0\) and \(S^0_0\):

\[
a \in S^0(\mathbb{R}^d \times \mathbb{R}^d) \iff \left|\partial_x^\alpha \partial_\xi^\beta a(x, \xi)\right| \lesssim \langle \xi \rangle^{-|\beta|},
\]

\[
a \in S^0_0(\mathbb{R}^d \times \mathbb{R}^d) \iff \left|\partial_x^\alpha \partial_\xi^\beta a(x, \xi)\right| \lesssim 1.
\]

The following theorem is due to Calderón-Zygmund.

**Theorem 3.4.** Let \(d \geq 1\) and \(a \in S^0(\mathbb{R}^d \times \mathbb{R}^d)\). Then, for all \(1 < p < \infty\),

\[
\|a(x, D)v\|_{L^p(\mathbb{R}^d)} \leq C_p \|v\|_{L^p(\mathbb{R}^d)}, \quad v \in C^\infty_0(\mathbb{R}^d),
\]

where the constant \(C_p\) depends on a finite number of semi-norms of \(a\) in \(S^0\).

For a proof, see for instance [24].

We next introduce the class \(S^0_{0,1} \times \mathbb{R}^n)\) of functions \(b(x_1, x'_1, y, \rho, \eta)\) satisfying

\[
\left|\partial_{x_1}^\alpha \partial_{x'_1}^\gamma \partial_{\rho}^\delta \partial_{\eta}^\beta b(x_1, x'_1, y, \rho, \eta)\right| \leq C_{\alpha \gamma \delta \beta} \langle \rho \rangle^{-2} \langle \eta \rangle^{-|\beta|},
\]

for \(x_1, x'_1 \in \mathbb{R}, y \in \mathbb{R}^{n-1}\), and \((\rho, \eta) \in \mathbb{R} \times \mathbb{R}^{n-1}\). In particular, for fixed \(x_1, x'_1, \rho\), these functions belong to \(S^0(\mathbb{R}^{n-1} \times \mathbb{R}^{n-1})\). Consider the pseudo-differential operator \(B\) defined on \(\mathbb{R}^n\) by the Schwartz kernel

\[
K_B(x_1, y, x'_1, y') = (2\pi)^{-n} \int e^{i(y-y') \cdot \eta} \hat{b}(x_1, x'_1, y, x'_1 - x_1, \eta) d\eta
\]

where \(\hat{b}\) is the Fourier transform of \(b\) with respect to \(\rho\). This kernel is continuous with respect to \(x_1, x'_1\) (with values in \(S'((\mathbb{R}^{n-1} \times \mathbb{R}^{n-1})\)). Integrating by parts with \((x_1 - x'_1)^{-1} \partial_\rho\) in the integral defining \(\hat{b}\), one sees that, for all \(N\) and all \(\alpha, \beta\),

\[
\left|\partial_\rho^\alpha \partial_\eta^\beta \hat{b}(x_1, y, x'_1 - x_1, \eta)\right| \leq C_{N\alpha \beta} \langle x_1 - x'_1 \rangle^{-N} \langle \eta \rangle^{-|\beta|}.
\]

Thus, for all \(1 < p < \infty\) and \(N > 0\), Theorem 3.4 yields the existence of \(C_{NP}\) such that

\[
\|(Bu)(x_1, \cdot)\|_{L^p(\mathbb{R}^{n-1})} \leq C_{NP} \int \langle x_1 - x'_1 \rangle^{-N} \|v(x'_1, \cdot)\|_{L^p(\mathbb{R}^{n-1})} dx'_1,
\]

21
for all \( v \in C_c^\infty(\mathbb{R}^n) \). Denoting by \( p' \) the conjugate exponent to \( p \), Hölder’s inequality yields
\[
\| (Bv) (x,\cdot) \|_{L^p(\mathbb{R}^{n-1})} \leq \left( \int \langle x_1 - x'_1 \rangle^{-N} \, dx'_1 \right)^{\frac{p}{p'}} \left( \int \|v(x'_1,\cdot)\|_{L^p(\mathbb{R}^{n-1})}^p \, dx'_1 \right)^{\frac{1}{p'}}
\]
and thus, if \( N > 1 \), we conclude that
\[
\| Bv \|_{L^p(\mathbb{R}^n)} \leq \left( \int \langle x_1 - x'_1 \rangle^{-N} \|v(x'_1,\cdot)\|_{L^p(\mathbb{R}^{n-1})}^p \, dx'_1 \right)^{\frac{1}{p'}} \|v\|_{L^p(\mathbb{R}^n)}, \quad v \in C_c^\infty(\mathbb{R}^n). \quad (3.7)
\]
More generally, if \( W \) is a temperate weight, estimates of the form (3.5) still hold if we replace \( b(x_1, y, x'_1 - x_1, \eta) \) by \( W(x_1) b(x_1, y, x'_1 - x_1, \eta) W(x'_1)^{-1} \). All this gives the following result.

**Proposition 3.5.** If \( b \in S_{0,1}^{2,0}(\mathbb{R}^{n+1} \times \mathbb{R}^n) \) and \( B \) is defined by the kernel (3.4), then for all temperate weight \( W \), \( W(x_1) B W(x_1)^{-1} \) is bounded on \( L^p(\mathbb{R}^n) \) for all \( 1 < p < \infty \) and its norm depends on a finite number of constants \( C_{j,k,\beta} \) in (3.3).

We shall essentially prove Theorem 3.4 by showing that the pull-backs on \( \mathbb{R}^n \) of \((z - \Delta_y)^{-1}\) by local charts are pseudo-differential operators with symbols in \( S_{0,1}^{-2,0}(\mathbb{R}^{n+1} \times \mathbb{R}^n) \). The main tool to caracterize these pull-backs as pseudo-differential operators on \( \mathbb{R}^n \) is the Beals criterion which we recall in Theorem 3.6 below. Fix first some notation. If \( A \) and \( L \) are operators on suitable spaces, we set
\[
ad_L \cdot A = LA - AL.
\]
In our case, \( L \) will typically belong to
\[
\mathcal{L}_{\mathbb{R}^n} = \{ x_1, \ldots, x_n, \partial_{x_1}, \ldots, \partial_{x_n} \}.
\]

**Theorem 3.6** (Beals). Let \( A : \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}'(\mathbb{R}^n) \) be a continuous linear map. If \( A \) is bounded on \( L^2(\mathbb{R}^n) \) and, more generally for all \( N \) and all \( L_1, \ldots, L_N \in \mathcal{L}_{\mathbb{R}^n} \), if the operator \( ad_{L_1} \cdots ad_{L_N} \cdot A \) is bounded on \( L^2(\mathbb{R}^n) \), then there exists \( a \in S_0^0 \) such that
\[
A = a^W(x, D),
\]
and each semi-norm of \( a \) in \( S_0^0 \) is controlled by a finite number of \( \|ad_{L_1} \cdots ad_{L_N} \cdot A\|_{L^2 \to L^2} \).

Here \( a^W(x, D) \) is the Weyl quantization of \( a \) namely the operator whose kernel is
\[
(2\pi)^{-n} \int e^{ix \cdot x'} e^{i\xi \cdot x}(x + x')/2 \, d\xi.
\]

Theorem 3.6 is for instance proved in [1, 2, 11].

The characterization of operators with symbols in \( S_{0,1}^{-2,0}(\mathbb{R}^{n+1} \times \mathbb{R}^n) \) is easily deduced from this theorem as follows. Recall first the formula
\[
(\partial_x^\beta \partial_\xi^\alpha a)^W(x, D) = i^{-|\beta|} ad_{\partial_x^\beta} a^W \circ a^W(x, D), \quad (3.8)
\]
where \( ad_{\partial_x} = ad_{\partial_x^1} \cdots ad_{\partial_x^n} \) and \( ad_{\partial_\xi} = ad_{\partial_\xi^1} \cdots ad_{\partial_\xi^n} \) (note that \( ad_{L_1} ad_{L_2} = ad_{L_2} ad_{L_1} \) for all \( L_1, L_2 \in \mathcal{L}_{\mathbb{R}^n} \)). On the other hand, we also have
\[
(\xi a)^W(x, D) = D_j a^W(x, D) - \frac{1}{2i}(\partial_{x_j} a)^W(x, D). \quad (3.9)
\]
Proposition 3.7. Let $A : \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}'(\mathbb{R}^n)$ be linear and continuous. Assume that, for all $\alpha, \beta \in \mathbb{N}^n$ and all $\gamma \in \mathbb{N}^n$ such that

$$\gamma_1 \leq 2, \quad \gamma_2 + \cdots + \gamma_n \leq \beta_2 + \cdots + \beta_n,$$

the operator

$$A_{\alpha, \beta}^\gamma := D_x^\gamma (ad^\alpha \cdot \gamma \cdot A)$$

(3.10)
is bounded on $L^2(\mathbb{R}^n)$. Then $A$ is a pseudo-differential operator with symbol $a \in S_{0,1}^{-2,0}(\mathbb{R}^{n+1} \times \mathbb{R}^n)$ (ie has a kernel of the form $\langle \xi, x \rangle^{\gamma}$. Each semi-norm of $a$ in $S_{0,1}^{-2,0}(\mathbb{R}^{n+1} \times \mathbb{R}^n)$ depends on a finite number of operator norms $\|A_{\alpha, \beta}^\gamma\|_{L^2-L^2}$.

Proof. Set $B = (1 + D_x^2)A$. By Theorem 3.6 we can write $B = b^W(x, D)$ for some $b \in S^0_0$. Define then $B_{\alpha, \beta}^\gamma$ similarly to (3.10) with $B$ instead of $A$ and with $\gamma = (0, \gamma_2, \ldots, \gamma_n)$. By (3.8) and (3.9), $B_{\alpha, \beta}^\gamma$ is the sum of

$$i^{-|\beta|} \left( \xi^{\gamma} \partial_x^\alpha \partial_\xi^\beta b \right)^W (x, D),$$

and of a linear combination of operators of the form

$$\left( \xi^{\gamma'} \partial_x^{\alpha'} \partial_\xi^\beta b \right)^W (x, D), \quad \gamma' < \gamma, \quad \alpha' \leq \alpha + \gamma.$$

On the other hand, by Theorem 3.6 again, $B_{\alpha, \beta}^\gamma$ is of the form $(b_{\alpha, \beta}^\gamma)^W(x, D)$ for some $b_{\alpha, \beta}^\gamma \in S^0_0$. Thus

$$b_{\alpha, \beta}^\gamma(x, \xi) = i^{-|\beta|} \xi^{\gamma} \partial_x^\alpha \partial_\xi^\beta b(x, \xi) + \sum_{\alpha' \leq \alpha + \gamma} c_{\gamma', \alpha'} \xi^{\gamma'} \partial_x^{\alpha'} \partial_\xi^\beta b(x, \xi).$$

By induction on $\beta$, we deduce that

$$|\partial_x^\alpha \partial_\xi^\beta b(x, \xi)| \lesssim (1 + |\xi_1| + \cdots + |\xi_n|)^{-\beta_2 - \cdots - \beta_n}. \quad (3.11)$$

Using then the standard fact that any $c^W(y, D_y)$, with $c \in S^0(\mathbb{R}^{n-1} \times \mathbb{R}^{n-1})$, can be written $c_1(y, D_y)$ for some $c_1 \in S^0(\mathbb{R}^{n-1} \times \mathbb{R}^{n-1})$ depending continuously on $c$, we can write $b^W(x, D_x) = b_1(x, D_x)$ for some symbol $b_1(x, D_x)$ satisfying the estimates (3.11) and depending continuously on $b$. Therefore $A = (1 + D_x^2)^{-1} b_1(x, D_x)$ and its symbol $\langle \xi_1 \rangle^{-2} b_1(x, \xi)$ clearly belongs to $S_{0,1}^{-2,0}$. □

Let us now choose, for each $\iota \in I$, three functions $f^{(1)}_\iota, f^{(2)}_\iota, f^{(3)}_\iota \in C^\infty(M)$ such that, if we set also

$$f^{(0)}_\iota = f_\iota,$$

$f_\iota$ being the $\iota$-th element of the partition of unit (1.19), we have

$$f^{(j+1)}_\iota \equiv 1 \thinspace \text{near supp}(f^{(j)}_\iota), \quad j = 0, 1, 2, \quad (3.12)$$

and

$$\text{supp}(f^{(j)}_\iota) \subset U_\iota, \quad j = 1, 2, 3. \quad (3.13)$$

If $\iota \in I_{\text{comp}}$ we may assume that $f^{(j)}_\iota \in C^\infty_0(U_\iota)$ and if $\iota \in I_{\infty}$ we may assume that

$$\Psi_{\iota, j} f^{(j)}_\iota(r, \theta) = g^{(j)}(r)\kappa^{(j)}_\iota(\theta),$$

23
with \( g^{(j)} \) and \( \kappa_{ii}^{(j)} \) supported in small neighborhoods of \( g \) and \( \kappa_i \) respectively (see (1.12)), \( \kappa_{ii}^{(j)} \) being compactly supported and \( g^{(j)}(r) = 1 \) for \( r \) large. Therefore, in all cases,
\[
f_i^{(j)} \in \text{Diff}_w^0(\mathcal{M}).
\]

By (1.10) we can write
\[
(z - \bar{\Delta})^{-1} = \sum_{i \in I} f_i^{(0)}(z - \bar{\Delta})^{-1} f_i^{(2)} + \sum_{i \in I} \sum_{i' \in I} f_i^{(0)}(P - z)^{-1}(1 - f_i^{(2)}) f_i'^{(0)}.
\]

The first sum corresponds to 'diagonal terms' and the second double one to 'off diagonal terms' since \( f_i^{(0)} \) and \( (1 - f_i^{(2)}) f_i'^{(0)} \) have disjoint supports.

By Proposition 3.5, Theorem 3.1 would be a direct consequence of the following proposition.

**Proposition 3.8.** For all \( \iota, \iota' \in I \), the operators
\[
R_{\iota}(z) = \Psi_{\iota,1} f_i^{(0)}(z - \bar{\Delta})^{-1} f_i^{(2)} \Psi_{\iota}^*, \quad z \notin \mathbb{R},
\]
and
\[
R_{\iota,\iota'}(z) = \Psi_{\iota,1} f_i^{(0)}(z - \bar{\Delta})^{-1}(1 - f_i^{(2)}) f_i'^{(0)} \Psi_{\iota'}^*, \quad z \notin \mathbb{R},
\]
have kernels of the form (3.14) with symbols whose semi-norms in \( S_{0,1}^{-2,0}(\mathbb{R}^{n+1} \times \mathbb{R}^n) \) are bounded by \( ((z)/|\text{Im } z|)^\nu \), for some \( \nu \) (depending on the semi-norm).

We shall prove Proposition 3.8 using Proposition 3.7. To compute the commutators with elements of \( \mathcal{L}_{\mathbb{R}^n} \), we start with a few remarks. For \( k = 1, \ldots, n \), we have
\[
x_k \Psi_{\iota} = \Psi_{\iota} x_k', \quad \Psi_{\iota}^* x_k = x_k' \Psi_{\iota}^*,
\]
denoting by \( (x_1', \ldots, x_n') \) the coordinates in the \( \iota \)-th chart. Similarly
\[
\partial_{x_k} \Psi_{\iota,1} = \Psi_{\iota,1} \partial x_k, \quad \Psi_{\iota,1}^* \partial_{x_k} = \partial x_k' \Psi_{\iota,1}^*.
\]

Of course, both (3.14) and (3.15) hold only in coordinate patches. If \( \iota \) and \( \iota' \) belong to \( I_\infty \), (3.14) reads, for \( k = 2, \ldots, n \),
\[
x_k \Psi_{\iota,1} = \Psi_{\iota,1} \theta_{k-1}', \quad \Psi_{\iota,1}^* x_k = \theta_{k-1}' \Psi_{\iota,1}^*,
\]
and for \( k = 1 \),
\[
x_1 \Psi_{\iota,1} = \Psi_{\iota,1} r, \quad \Psi_{\iota,1}^* x_1 = r \Psi_{\iota,1}^*,
\]
where one should note that \( r \) is globally defined on \( \mathcal{M} \). We don’t write the analogous formulas corresponding to (3.15) for \( \iota, \iota' \in I_\infty \) but we recall that \( \partial_r \) is only defined where \( r \) is a coordinate, namely for \( r > R \).

We then note that \( R_{\iota}(z) \) in Proposition 3.8 reads
\[
R_{\iota}(z) = \Psi_{\iota,1} F_i^{(0)} R^M(z) F_i^{(2)} \Psi_{\iota}^*,
\]
with
\[
\text{supp}(F_i^{(0)}) \subset \text{supp}(f_i^{(0)}), \quad \text{supp}(F_i^{(2)}) \subset \text{supp}(f_i^{(2)}),
\]
and
and
\[ R^M(z) = w(r)^{k_1}A_1(z - \Delta_g)^{-1}w(r)^{k_2}A_2(z - \Delta_g)^{-1} \ldots w(r)^{k_N}A_N(z - \Delta_g)^{-1} \]  \hspace{1cm} (3.19)
where \( F^{(0)}_t, F^{(2)}_t \in \text{Diff}^0_w(M), \) \( N \geq 1, \) \( k_1, \ldots, k_N \geq 0 \) and
\[ A_j \in \text{Diff}^{m_j}_w(M), \quad 0 \leq m_j \leq 2, \]
with in particular \( m_1 = 0. \) Similarly,
\[ R_{i_*, t'}(z) = \Psi_{i_*, t'} F^{(0)}_t R^M(z) F^{(2)}_{t'} \Psi^*_{i_*, t'}, \]  \hspace{1cm} (3.20)
with \( F^{(2)}_{t'} \in \text{Diff}^0_w(M) \) such that
\[ \text{supp}(F^{(2)}_{t'}) \subset \text{supp} \left( (1 - f^{(2)}_t) f^{(0)}_t \right). \]  \hspace{1cm} (3.21)
Of course, in (3.17) and (3.20), \( R^M(z) = (z - \Delta_g)^{-1} \) that is (3.19) with \( N = 1, k_1 = 0 \) and \( A_1 = 1. \) By (3.12), (3.15) and (3.18), we have
\[ \partial_k \left( \Psi_{i_*, t'} F^{(0)}_t R^M(z) F^{(2)}_{t'} \Psi^*_{i_*, t'} \right) = \Psi_{i_*, t'} \left[ L_{i, k}, F^{(0)}_t R^M(z) F^{(2)}_{t'} \right] \Psi^*_{i_*, t'} + \left( \Psi_{i_*, t'} F^{(0)}_t R^M(z) F^{(2)}_{t'} \Psi^*_{i_*, t'} \right) \partial_k, \]
with
\[ L_{i, k} = f^{(3)}_t \Psi^*_{i_*, t'} \partial_k \Psi_{i_*, t'}. \]  \hspace{1cm} (3.22)
In particular,
\[ \left[ \partial_k, \Psi_{i_*, t'} F^{(0)}_t R^M(z) F^{(2)}_{t'} \Psi^*_{i_*, t'} \right] = \Psi_{i_*, t'} \left[ L_{i, -i', k}, F^{(0)}_t R^M(z) F^{(2)}_{t'} \right] \Psi^*_{i_*, t'}. \]  \hspace{1cm} (3.23)
For operators like the right hand side of (3.20), we use (3.21), that \( f^{(1)}_t \equiv 1 \) near \( \text{supp}(f^{(0)}_t) \) and that
\[ (1 - f^{(2)}_t) \equiv 0 \quad \text{near} \quad \text{supp}(f^{(1)}_t), \]
\[ (1 - f^{(1)}_t) \equiv \begin{cases} 1 & \text{near} \quad \text{supp}(1 - f^{(2)}_t) \\ 0 & \text{near} \quad \text{supp}(f^{(0)}_t) \end{cases}, \]
which follow from (3.12), to obtain
\[ \partial_k \left( \Psi_{i_*, t'} F^{(0)}_t R^M(z) F^{(2)}_{t'} \Psi^*_{i_*, t'} \right) = \Psi_{i_*, t'} \left[ L_{i, -i', k}, F^{(0)}_t R^M(z) F^{(2)}_{t'} \right] \Psi^*_{i_*, t'}, \]
\[ \left( \Psi_{i_*, t'} F^{(0)}_t R^M(z) F^{(2)}_{t'} \Psi^*_{i_*, t'} \right) \partial_k = \Psi_{i_*, t'} \left[ F^{(0)}_t R^M(z) F^{(2)}_{t'}, L_{i, -i', k} \right] \Psi^*_{i_*, t'}, \]
with
\[ L_{i, -i', k} = f^{(1)}_t \Psi^*_{i_*, t'} \partial_k \Psi_{i_*, t'}, \quad L_{i, -i', k} = (1 - f^{(1)}_t) f^{(1)}_t \Psi^*_{i_*, t'} \partial_k \Psi_{i_*, t'}. \]  \hspace{1cm} (3.24)
The main consequence is that
\[ \left[ \partial_k, \Psi_{i_*, t'} F^{(0)}_t R^M(z) F^{(2)}_{t'} \Psi^*_{i_*, t'} \right] = \Psi_{i_*, t'} \left[ \left[ L_{i, -i', k}, F^{(0)}_t R^M(z) F^{(2)}_{t'} \right] - \left[ F^{(0)}_t R^M(z) F^{(2)}_{t'}, L_{i, -i', k} \right] \right] \Psi^*_{i_*, t'}. \]
With the latter formula, (3.23) and the resolvent identity, namely
\[ \text{ad}_L \cdot (z - \Delta_g)^{-1} = -(z - \Delta_g)^{-1} [L, \Delta_g] (z - \Delta_g)^{-1}, \]  \hspace{1cm} (3.25)
we are equipped to prove the following result.
Proposition 3.9. For all $\alpha \in \mathbb{N}^n$ and all $i \in I$ (resp. all $i, i' \in I$), the operator
\[
ad_{\partial_i}^\alpha R_i(z) \quad \text{(resp. } \ad_{\partial_i}^\alpha R_{i'}(z))\]
is a linear combination (with coefficients independent of $z$) of operators of the form
\[
F_i^{(0)} R^M(z) F_i^{(2)} \quad \text{(resp. } F_{i'}^{(0)} R^M(z) F_{i'}^{(2)})
\]
with $N \leq |\alpha| + 1$ and
\[
A_1 \in \Diff_m^0(M), \quad A_2, \ldots, A_N \in \Diff_m^2(M), \quad k_1 = k_2 = \cdots = k_N = 0.
\]
Proof. It follows from elementary induction once observed that, if $L$ is any of the operators in \eqref{3.22} or \eqref{3.24}, we have
\[
A \in \Diff_m^w(M) \quad \Rightarrow \quad [L, A] \in \Diff_m^w(M).
\]
Indeed, if $L$ is compactly support this is trivial. Otherwise, if it is supported in chart a infinity, this is a consequence of the identities
\[
[\partial_r, \bar{\varphi}(r)\bar{\kappa}(\theta)\partial_{\theta_k}] = \left( \frac{\bar{\varphi}'(r)}{w(r)} \bar{\kappa}(\theta) \right) w(r)\partial_{\theta_k},
\]
\[
[w(r)\partial_{\theta_{k'}}, \bar{\varphi}(r)\bar{\kappa}(\theta)\partial_{\theta_k}] = \left( \bar{\varphi}(r)\partial_{\theta_{k'}} \bar{\kappa}(\theta) \right) w(r)\partial_{\theta_k},
\]
\[
[w(r)\partial_{\theta_k}, \bar{\varphi}(r)\bar{\kappa}(\theta)\partial_r] = \left( w(r) \bar{\varphi}(r) \partial_{\theta_k} \bar{\kappa}(\theta) \right) \partial_r - \left( \frac{w'(r)}{w(r)} \bar{\varphi}(r) \bar{\kappa}(\theta) \right) w(r)\partial_{\theta_{k'}},
\]
where all the brackets in the right hand sides are bounded as well as their derivatives, if $\bar{\varphi}$ and $\bar{\kappa}$ are bounded with compactly supported derivatives, also using \eqref{1.10} and \eqref{1.13}.  \hfill \Box

To compute $\ad_{\partial_i}^\alpha \ad_{\partial_i}^\beta R_i(z)$ and $\ad_{\partial_i}^\alpha \ad_{\partial_i}^\beta R_{i'}(z)$, we need the following lemma.

Lemma 3.10. Let $\bar{\varphi}$ be a smooth function on $\mathbb{R}$ with compactly supported derivative and supported in $r > R$. Let $\bar{\kappa}(\theta)$ be supported in patch of the manifold at infinity. Then, for any $A \in \Diff_m^w(M)$, we have
\[
[A, \bar{\varphi}(r)\bar{\kappa}(\theta)\partial_{\theta_k}] = w(r)A''_k,
\]
for some $A', A'' \in \Diff_m^{m-1}(M)$. Furthermore, for all $F \in C_0^m(M)$ and all $k \in \mathbb{N}$, we can write
\[
[A, F] = w(r)^k A_k,
\]
with $A_k \in \Diff_m^{m-1}(M)$.

Proof. The first two identities follow simply from
\[
[\partial_r, \bar{\varphi}(r)r] = \left( \bar{\varphi}'(r)r + \bar{\varphi}(r) \right),
\]
\[
[\partial_r, \bar{\varphi}(r)\bar{\kappa}(\theta)\partial_{\theta_k}] = w(r) \left( \frac{\bar{\varphi}'(r)}{w(r)} \bar{\kappa}(\theta)\partial_{\theta_k} \right),
\]
\[
[w(r)\partial_{\theta_{k'}}, \bar{\varphi}(r)\bar{\kappa}(\theta)\partial_{\theta_k}] = w(r)\bar{\varphi}(r) \left( \partial_{\theta_{k'}} \bar{\kappa}(\theta) + \delta_{kk'} \bar{\kappa}(\theta) \right),
\]
since all brackets in the right hand sides are smooth and bounded, together with their derivatives. For the third one, we simply observe that $[A, F]$ is a differential operator of order $m - 1$ with compact support and can thus be written $w(r)^k (w(r)^{-k} [A, F])$ since $w$ doesn’t vanish.  \hfill \Box
The main sense of this lemma is that commutators of elements of $\text{Diff}_{\text{c}}^\infty(M)$ with the multiplication operators by coordinates (cut off to be globally defined) are operators in $\text{Diff}_{\text{c}}^{\infty-1}(M)$. More precisely, we get a factor $w(r)$ when commuting with angular coordinates or compactly supported functions. Note also that it is crucial for the first commutator that we commute $A$ with a function of $r$ only. Otherwise, we would have to consider for instance terms like

$$[w(r) \partial_{\theta_r}, \tilde{g}(r) \kappa(\theta)r] = \left( \tilde{g}(r) \partial_{\theta_r} \kappa(\theta) \right) w(r)r,$$

with $w(r)r$ unbounded in general.

To compute the commutators with the multiplication operators $x_1, \ldots, x_n$, we repeat essentially the calculations above Proposition 3.9 with $x_k$ instead of $\partial_k$ except for $x_1$ when we work close to infinity. We proceed as follows. If $\iota \in I_{\text{comp}}$, we define

$$X_{\iota,k} = f^{(3)}_\iota \psi^*_\iota x_k \psi_{\iota,*},$$

$$X_{\iota-\iota',k} = f^{(1)}_\iota \psi^*_\iota x_k \psi_{\iota',*},$$

for $1 \leq k \leq n$. If $\iota' \in I_{\text{comp}}$ and $\iota \in I$, we also set

$$X_{\iota-\iota',k} = (1 - f^{(1)}_\iota) f^{(1)}_{\iota'} \psi^*_\iota x_k \psi_{\iota',*},$$

for $1 \leq k \leq n$. In these cases, $X_{\iota,k}, X_{\iota-\iota',k}$ and $X_{\iota-\iota',k}$ are smooth functions compactly supported in coordinates patches. If $k \geq 2$ and $\iota, \iota' \in I_{\infty}$, we still define $X_{\iota,k}, X_{\iota-\iota',k}$ and $X_{\iota-\iota',k}$ by the right hand sides of (3.26), (3.27) and (3.28). Setting finally

$$X_{\iota,1} = r, \quad \iota \in I_{\infty},$$

$$X_{\iota-\iota',1} = r, \quad \iota, \iota' \in I,$$

$$X_{\iota-\iota',1} = r, \quad \iota \in I, \quad \iota' \in I_{\infty},$$

we have defined $X_{\iota,k}, X_{\iota-\iota',k}$ and $X_{\iota-\iota',k}$ for all $\iota, \iota' \in I$ and all $1 \leq k \leq n$. For operators of the form (3.19) and cutoffs satisfying (3.14) imply that

$$\left[ x_k \left( \psi_{\iota,k} F^{(0)}_\iota R^\mathcal{M}(z) F^{(2)}_\iota \psi^*_\iota \right) \right] = \psi_{\iota,*} F^{(0)}_\iota \left[ X_{\iota,k}, R^\mathcal{M}(z) \right] F^{(2)}_\iota \psi^*_\iota,$$

for all $\iota \in I$ and $1 \leq k \leq n$. For off diagonal terms, namely with right cutoffs satisfying (3.21), we have

$$x_k \left( \psi_{\iota,k} F^{(0)}_\iota R^\mathcal{M}(z) F^{(2)}_{\iota',\iota'} \psi^*_\iota \right) = \psi_{\iota,k} F^{(0)}_\iota X_{\iota-\iota',k} R^\mathcal{M}(z) F^{(2)}_{\iota',\iota'} \psi^*_\iota,$$

$$= \psi_{\iota,k} F^{(0)}_\iota \left[ X_{\iota-\iota',k}, R^\mathcal{M}(z) \right] F^{(2)}_{\iota',\iota'} \psi^*_\iota + \psi_{\iota,k} F^{(0)}_\iota R^\mathcal{M}(z) X_{\iota-\iota',k} F^{(2)}_{\iota',\iota'} \psi^*_\iota,$$

where the last term vanishes if $\iota \in I_{\text{comp}}$ or $k \geq 2$. In the remaining cases, namely $k = 1$ and $\iota \in I_{\infty}$, we have $X_{\iota-\iota',1} = r$ and

$$r F^{(2)}_{\iota',\iota'} \psi^*_\iota \begin{cases} F_{\iota',\iota'} \psi^*_\iota \text{ with } F_{\iota',\iota'} \in C^\infty_0(M) & \text{if } \iota' \in I_{\text{comp}}, \\ F^{(2)}_{\iota',\iota'} \psi^*_\iota x_1 & \text{if } \iota' \in I_{\infty}. \end{cases}$$

Similarly, we have

$$\left( \psi_{\iota,k} F^{(0)}_\iota R^\mathcal{M}(z) F^{(2)}_{\iota',\iota'} \psi^*_\iota \right) x_k = \psi_{\iota,k} F^{(0)}_\iota R^\mathcal{M}(z) X_{\iota-\iota',k} F^{(2)}_{\iota',\iota'} \psi^*_\iota,$$

$$= \psi_{\iota,k} F^{(0)}_\iota \left[ R^\mathcal{M}(z), X_{\iota-\iota',k} \right] F^{(2)}_{\iota',\iota'} \psi^*_\iota + \psi_{\iota,k} F^{(0)}_\iota X_{\iota-\iota',k} R^\mathcal{M}(z) F^{(2)}_{\iota',\iota'} \psi^*_\iota$$

27
where the last term vanishes if $k \geq 2$ or $\iota' \in I_{\text{comp}}$ and

$$
\Psi_\iota X_{\iota' \iota'} = \begin{array}{ll}
F_{\iota' \iota'} \Psi_\iota & \text{with } F_{\iota' \iota'} \in C_0^K(\mathcal{M}) \quad \text{if } \iota \in I_{\text{comp}}, \\
x_1 \Psi_\iota X_{\iota' \iota'} & \text{if } \iota \in I_\infty.
\end{array}
$$

This shows that, unless $\iota, \iota' \in I_\infty$ and $k = 1$, \begin{equation}
\Psi_\iota F_{\iota} \left( \left[ X_{\iota' \iota'}, k, R^M(z) \right] - \left[ R^M(z), X_{\iota' \iota'}, k \right] \right) F_{\iota' \iota'} \Psi_\iota
\end{equation}
and of the same form as $\Psi_\iota F_{\iota} R^M(z) F_{\iota' \iota'} \Psi_\iota$. If $\iota, \iota' \in I_\infty$ and $k = 1$, we simply have

$$
\left[ x_1, \left( \Psi_\iota F_{\iota} R^M(z) F_{\iota' \iota'} \Psi_\iota \right) \right] = \Psi_\iota F_{\iota} \left[ r, R^M(z) \right] F_{\iota' \iota'} \Psi_\iota.
$$

Using lemma 3.10, the resolvent identity (3.25) and a simple induction, we get the following result.

**Proposition 3.11.** For all $\alpha, \beta \in \mathbb{N}^n$ and all $\iota \in I$ (resp. all $\iota, \iota' \in I$), the operator

$$
ad_\alpha^2 \ad_\beta^3 R_\iota(z) \quad (\text{resp. } \ad_\alpha^2 \ad_\beta^3 R_{\iota' \iota'}(z)),
$$
is a linear combination (with coefficients independent of $z$) of operators of the form

$$
F_{\iota} R^M(z) F_{\iota' \iota'},
$$
where $R^M(z)$ is of the form (3.11) with $N \leq |\alpha| + |\beta| + 1$,

$$
A_j \in \text{Diff}^m_{\iota}(\mathcal{M}), \quad 0 \leq m_j \leq 2
$$
and

$$
k_j = 2 - m_j, \quad k_2 + \ldots + k_N = \beta_2 + \ldots + \beta_n.
$$

The next proposition is the final step before being in position to use Proposition 3.7.

**Proposition 3.12.** Fix $\iota \in I$ (resp. $\iota, \iota' \in I$). For all $\alpha, \beta \in \mathbb{N}^n$ and all $\gamma \in \mathbb{N}$ satisfying $\gamma_1 \leq 2$, $\gamma_2 + \ldots + \gamma_n \leq \beta_2 + \ldots + \beta_n$, the operator

$$
D_\alpha^2 \ad_\beta^3 \ad_\gamma^3 R_\iota(z) \quad (\text{resp. } D_\alpha^2 \ad_\beta^3 \ad_\gamma^3 R_{\iota' \iota'}(z)),
$$
is a linear combination (with coefficients independent of $z$) of operators of the form

$$
F_{\iota} R^M(z) F_{\iota' \iota'},
$$
(see (3.11)) with $N \leq |\alpha| + |\beta| + |\gamma| + 1$ and

$$
A_1, \ldots, A_N \in \text{Diff}^m_{\iota}(\mathcal{M}), k_1 = \ldots = k_N = 0.
$$

In particular, they are bounded on $L^2(\mathbb{R}^n)$ with norms controlled by powers of $\langle z \rangle / |\text{Im}(z)|$.  

28
Proof. We treat the case of \( R_\nu(z) \), the one of \( R_{\nu'}(z) \) being completely similar. We start with a simple case. Consider an operator of the form

\[
B(z) := \Psi_{i*} F_{i}^{(0)} (z - \Delta_g)^{-1} w(r) A(z - \Delta_g)^{-1} F_{i}^{(2)} \Psi_{i*}^*,
\]

with \( A \in \text{Diff}^1_w(M) \). Such operators appear in Proposition 3.11 with \( N = 2 \) if \( \beta_2 + \cdots + \beta_n = 1 \) and \( \alpha = 0 \). Compute then \( \partial_B(z) \), with \( k \geq 2 \). We get

\[
\Psi_{i*} \left( \left[ L_{i,k}, F_{i}^{(0)} (z - \Delta_g)^{-1} w(r) \right] A(z - \Delta_g)^{-1} + F_{i}^{(0)} (z - \Delta_g)^{-1} w(r) L_{i,k} A(z - \Delta_g)^{-1} F_{i}^{(2)} \right) \Psi_{i*}^*.
\]

The commutator reads

\[
\left[ L_{i,k}, F_{i}^{(0)} \right] (z - \Delta_g)^{-1} w(r) + F_{i}^{(0)} (z - \Delta_g) \left[ \Delta_g, L_{i,k} \right] (z - \Delta_g)^{-1} w(r) + F_{i}^{(0)} (z - \Delta_g)^{-1} [L_{i,k}, w(r)]
\]

and is bounded on \( L^2(M, \tilde{dg}) \) since \( [L_{i,k}, \Delta_g] \in \text{Diff}^w_w(M) \). The simple and crucial remark is that

\[
w(r) L_{i,k} A \in \text{Diff}^2_w(M),
\]

although \( L_{i,k} A \notin \text{Diff}^2_w(M) \) in general. Therefore \( \partial_B(z) \) is a linear combination of operators of the form (3.10) with \( A_1 \) of order 0. This then implies that \( \partial_B^2 \partial_B(z) \) is also of this form with \( A_1 \) of order 2. Iteration of this argument give the result since Proposition 3.11 show there are at least \( \gamma_2 + \cdots + \gamma_n \) powers \( w(r) \) in the expression of \( a d_{z_x}^2 a d_{\nu_x}^2 R_\nu(z) \) to absorb \( \partial_{x_z}^2 \cdots \partial_{x_{\nu_n}}^2 \). \( \square \)

3.2 Proof of Proposition 3.8

This is a direct consequence of Proposition 3.12 and Proposition 3.7.

3.3 Proof of Theorem 3.1

This is a direct consequence of Proposition 3.5 and Proposition 3.8 using the equivalence of norms (1.23).

3.4 Proof of Theorem 3.3

The boundedness of \( W(r)(z - \Delta_g)^{-1} W(r)^{-1} \) on \( L^p(M, dg) \) is equivalent to the one of

\[
W(r)w(r)^{\frac{n-1}{p}} \Delta_g w(r) \Delta_g^{n-1} w(r)^{-\frac{1}{p}} W(r)^{-1}
\]

on \( L^p(M, \tilde{dg}) \) so the result follows from Proposition 3.5, with the temperate weight \( W \) \( w^{\frac{n-1}{p}} \), and Proposition 3.8. \( \square \)

3.5 Proof of Theorem 1.7

We note first that, by writing \( (z - h^2 \Delta_g)^{-1} = h^{-2} (zh^{-2} - \Delta_g)^{-1} \), Theorem 3.1 implies that

\[
||W(r)(z - h^2 \Delta_g)^{-1} W(r)^{-1}||_{L^p(M, \tilde{dg}) \to L^p(M, \tilde{dg})} \lesssim h^{-2} \frac{(z)^{\nu_p}}{|\text{Im}(z)|^{\nu_p}}, \quad h \in (0, 1], \quad z \in \mathbb{C} \setminus \mathbb{R}. \quad (3.32)
\]

by using the inequality \( |h^{-2}z|/|\text{Im}(h^{-2}z)| \lesssim |z|/|\text{Im}(z)|. \)
Assume next that \( \varphi \in S^{-\sigma}(\mathbb{R}) \) with \( \sigma > 1 \) so that we can use (2.29). By Theorem 2.8 and (3.32), there exists \( \nu_{p, N} \) such that

\[
\left\| W(r)(z - h^2\bar{\Delta}_g)^{-1}R_N(z, h)W(r)^{-1} \right\|_{L^p(M, d\bar{g}) \to L^p(M, d\bar{g})} \lesssim h^{-2} \left( \frac{\langle z \rangle}{|\Im z|} \right)^{\nu_{p, \nu_{N, p}}},
\]

for \( h \in (0, 1] \) and \( z \notin \mathbb{R} \). By choosing \( M \geq \nu = \nu_p + \nu_{N, p} \), the above estimate and (2.29) give the expected estimate up to a factor \( h^{-2} \). The latter is eliminated in the standard way: by pushing the expansion to the order \( h^{N+2} \), we write \( R_N(-\Delta_g, \varphi, h) \) as the sum of properly supported pseudo-differential operators bounded on \( W(r)^{-1}L^p(M, d\bar{g}) \) and of \( h^2R_{N+2}(-\bar{\Delta}_g, \varphi, h) \). This implies (1.36).

If now \( \varphi \in S^{-\sigma}(\mathbb{R}) \) with \( \sigma > 0 \), we cannot use (2.29). We thus write \( \varphi(\lambda) = (\lambda + i)\psi(\lambda) \) with \( \psi \in S^{-\sigma-1}(\mathbb{R}) \) so that

\[
\varphi(-h^2\bar{\Delta}_g) = (i - h^2\bar{\Delta}_g)\psi(-h^2\bar{\Delta}_g).
\]

We then write again \( R_N(-\bar{\Delta}_g, \varphi, h) \) as a finite sum of properly supported pseudo-differential operators bounded on \( W(r)^{-1}L^p(M, d\bar{g}) \) and

\[
h^{N+2} \int \int_{\mathbb{R}^2} \tilde{\Delta}_\psi M(z)(z - h^2\bar{\Delta}_g)^{-1}(i - h^2\bar{\Delta}_g)R_{N+2}(z, h)dx dy
\]

where \( z = x + iy \). By Theorem 2.8 we have

\[
\left\| W(r)(i - h^2\bar{\Delta})R_{N+2}(z, h)W(r)^{-1} \right\|_{L^p(M, d\bar{g}) \to L^p(M, d\bar{g})} \lesssim \left( \frac{\langle z \rangle}{|\Im z|} \right)^{\nu_{N+2, p}},
\]

and we proceed as above. \(\square\)

**A Non \( L^p \to L^p \) boundedness on the hyperbolic space**

Using the hyperboloid model of the hyperbolic space, namely

\[
\mathbb{H}^n = \{ x = (x_0, \ldots, x_n) \in \mathbb{R}^{n+1} \mid x_0^2 - x_1^2 - \cdots - x_n^2 = 1, \ x_0 > 0 \},
\]

we have polar coordinates by considering

\[
x(r, \omega) = (\cosh r, \omega \sinh r), \quad r > 0, \ \omega \in S^{n-1}.
\]

In this parametrization, the distance between \( x = x(r, \omega) \) and \( x' = x(r', \omega') \) reads

\[
d(x, x') = \arccosh (\cosh r \cosh r' - \omega \cdot \omega' \sinh r \sinh r')
\]

\[
= \arccosh \left\{ \left( 1 - \frac{|\omega - \omega'|^2}{4} \right) \cosh(r - r') + \frac{|\omega - \omega'|^2}{4} \cosh(r + r') \right\}
\]

(A.34)

and the volume element is

\[
(\sinh r)^{n-1} dr d\omega,
\]

where \( d\omega \) is the usual Riemannian measure on the sphere. Considering \( n = 3 \) for simplicity, the resolvent

\[
(-\Delta_{\mathbb{H}^3} - 1 + \epsilon^2)^{-1}, \quad \epsilon > 0,
\]

(A.35)
is well defined since, in general, \(-\Delta_{\mathbb{H}^3} \geq (n - 1)^2/4\). Its kernel with respect to the volume element is then given by
\[
\frac{1}{4\pi \sinh d(x, x')} e^{-cd(x, x')}
\]
(see for instance [23, p. 105]).

**Proposition A.1.** Fix \( p \in (1, \infty) \) with \( p \neq 2 \). If \( 0 < \epsilon < \left\| 1 - \frac{2}{p} \right\| \), then \((-\Delta_{\mathbb{H}^3} - 1 + \epsilon^2)^{-1} \) is not bounded on \( L^p(\mathbb{H}^3) \).

We shall proceed by contradiction, using the following simple lemma.

**Lemma A.2.** Let \( K_1, K_2 \) be two locally integrable functions on \((\mathbb{R}_+ \times \mathbb{S}^2)^2\) such that
\[
K_2(r, \omega, r', \omega') \geq |K_1(r, \omega, r', \omega')|.
\] (A.36)
Denote by \( A_j \) be the operator with kernel \( K_j \) with respect to \( drd\omega \) and set \( L^p = L^p(\mathbb{R}_+ \times \mathbb{S}^2, drd\omega) \). Then
\[
||A_1||_{L^p \rightarrow L^p} \leq ||A_2||_{L^p \rightarrow L^p}.
\]
**Proof.** By (A.36), we have, for all \( u \in C_0^\infty(\mathbb{R}_+ \times \mathbb{S}^2) \),
\[
|(A_1 u)(r, \omega)| \leq |(A_2 u)(r, \omega)|
\]
so, taking the \( L^p \) norm, we obtain
\[
||A_1 u||_{L^p} \leq ||A_2 u||_{L^p} \leq ||A_2||_{L^p \rightarrow L^p} ||u||_{L^p} = ||A_2||_{L^p \rightarrow L^p} ||u||_{L^p}
\]
which gives the result. \( \square \)

**Proof of Proposition A.1.** We argue by contradiction and assume that \(( -\Delta_{\mathbb{H}^3} - 1 + \epsilon^2)^{-1} \) is bounded on \( L^p(\mathbb{H}^3) \). This is equivalent to the boundedness on \( L^p(\mathbb{R}_+ \times \mathbb{S}^2, drd\omega) \) of the operator with kernel
\[
K_2(r, \omega, r', \omega') := (\sinh r)^{\frac{2}{p}} \left( \frac{1}{4\pi \sinh d(x, x')} e^{-cd(x, x')} (\sinh r')^{\frac{2}{p}} \right) (\sinh r')^{-\frac{2}{p}}
\]
with respect to \( drd\omega \). Since \( \cosh(r - r') \leq \cosh(r + r') \) for \( r, r' \in \mathbb{R}_+ \), (A.34) gives
\[
d(x, x') \leq r + r'
\]
so, for \( r, r' \geq 1 \), we have
\[
K_2(r, \omega, r', \omega') \geq (e^{r'})^{\frac{2}{p}} \left( \frac{e^{-c(r+r')}}{e^{r+r'}} (e^{r'})^2 (e^{r'})^{-\frac{2}{p}} \right) (e^{r'})^{-\frac{2}{p}} = e^{\left(\frac{2}{p} - 1\right) r} e^{(1 - \frac{2}{p} - \epsilon) r'}. \] (A.37)
Denoting by \( K_1(r, \omega, r', \omega') = K_1(r, r') \) the right hand side of (A.37) multiplied by the characteristic function of \([1, +\infty)^2\), Lemma A.2 implies that the corresponding operator \( A_1 \) is bounded on \( L^p(\mathbb{R}_+ \times \mathbb{S}^2, drd\omega) \). This is clearly not true if \( \frac{2}{p} - 1 > \epsilon \), otherwise \( e^{\left(\frac{2}{p} - 1\right) r} e^{(1 - \frac{2}{p} - \epsilon) r'} \) should belong to \( L^p(\mathbb{R}) \). We also obtain a contradiction if \( 1 - \frac{2}{p} > \epsilon \) by considering the adjoint of \( A_1 \) \( \square \)

Note that the right hand side of (A.37) also reads
\[
e^{\left(\frac{2}{p} - 1\right)(r-r') - \epsilon(r+r')},
\]
31
showing that the above reasoning gives no contradiction for \( p = 2 \) nor by restricting the kernel close to the diagonal.

We also recall that \((n - 1)\left|\frac{1}{p} - \frac{1}{2}\right|\) (ie \(|\frac{2}{p} - 1|\) if \( n = 3 \)) is exactly the width of the strip around the real axis in which \( \varphi \) has to be holomorphic to ensure the boundedness on \( L^p(\mathbb{H}^n) \) of

\[
\varphi \left( (-\Delta_{\mathbb{H}^n} - (n - 1)^2/4)^{1/2} \right),
\]

as proved in [22]. The resolvent (A.35) corresponds to \( \varphi(\lambda) = (\lambda^2 + \epsilon^2)^{-1} \) which is holomorphic for \(|\text{Im}(\lambda)| < \epsilon\).

References

[1] B. Ammann, R. Lauter, V. Nistor, A. Vasy, Complex powers and non compact manifolds, Commun. P.D.E. 29, 671-705 (2004).
[2] R. Beals, Characterization of pseudo-differential operators and applications, Duke Math. J. 44, no. 1, 45-57 (1977) and Correction, Duke Math. J. 46, no. 1, 215, (1979).
[3] J. M. Bony, Caractérisation des opérateurs pseudo-différentiels, Séminaire X-EDP, exp. XXIII (1996-1997).
[4] J. M. Bouclet, Littlewood-Paley decompositions on manifolds with ends, preprint.
[5] ———, Strichartz estimates on asymptotically hyperbolic manifolds, preprint.
[6] J. M. Bouclet, N. Tzvetkov, Strichartz estimates for long range perturbations, Amer. J. Math., to appear.
[7] N. Burq, P. Gérard, N. Tzvetkov, Strichartz inequalities and the non linear Schrödinger equation on compact manifolds, Amer. J. Math. 126, 569-605 (2004).
[8] J. Cheeger, M. Gromov, M. Taylor, Finite propagation speed, kernel estimates for functions of the Laplace operator and the geometry of complete Riemannian manifolds, J. Diff. Geom. 17, 15-53 (1982).
[9] J. L. Clerc, E. M. Stein, \( L^p \)-multipliers for noncompact symmetric spaces, Proc. Nat. Acad. Sci. U.S.A. 71, 3911-3912 (1974).
[10] E. B. Davies, Spectral theory and differential operators, Cambridge University Press (1995).
[11] M. Dimassi, J. Sjöstrand, Spectral asymptotics in the semi-classical limit, London Mathematical Society Lecture Note Series, 268. Cambridge University Press (1999).
[12] G. Grubb, Functionnal calculus of pseudo-differential boundary problems, vol. 65, Birkhäuser, Boston (1986).
[13] A. Hassel, T. Tao, J. Wunsch, A Strichartz inequality for the Schrödinger equation on non-trapping asymptotically conic manifolds, Comm. PDE 30, 157-205 (2004).
[14] B. Helffer, D. Robert, Calcul fonctionnel par la transformation de Mellin et opérateurs admissibles, J. Funct. Analysis 53, 246-268 (1983).
[15] Y. A. Kordyukov, Lp estimates for functions of elliptic operators on manifolds of bounded geometry, Russian J. Math. Phys., Vol. 7, No. 2, 216-229 (2000).

[16] R. B. Melrose, Geometric scattering theory, Stanford lecture, Cambridge Univ. Press (1995).

[17] D. Robert, Autour de l’approximation semi-classique, Progress in mathematics, 68, Birkhäuser (1987).

[18] B. W. Schulze, Pseudo-differential operators on manifolds with singularities, North-Holland, Amsterdam, (1991).

[19] R. T. Seeley, Complex powers of an elliptic operator, Proc. Symp. in Pure Math., vol. 10, 288-307 (1967).

[20] ————, The resolvent of an elliptic boundary problem, Amer. J. Math. 91, 889-920 (1969).

[21] E.M. Stein, Singular integrals and differentiability properties of functions, Princeton Univ. Press (1970).

[22] M. Taylor, Lp estimates on functions of the Laplace operator, Duke Math. J. Vol. 58, No. 3, 773-793 (1989).

[23] ————, Partial Differential Equations II, Linear Equations, Appl. Math. Sci. 116, Springer (1996).

[24] ————, Partial Differential Equations III, Nonlinear Equations, Appl. Math. Sci. 117, Springer (1996).