Supplementary Materials (Appendices)

Discounting as a double-edged sword: The values of both future goods and present economic growth decrease with the discount rate

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S1 Proof of Theorem 2: Emergence of a Shadow Value due to Endogenous Economic Growth

S1.1 General considerations

In this section, we prove Theorem 2 (main text). Specifically, we show that, under certain assumptions, choosing \( x(t) \) that maximizes social welfare, \( U^T \) (Eq. 1), subject to a given environmental dynamics (Eq. 2), is equivalent to choosing \( x^T \) that maximizes the NPV plus a shadow value that can also be expressed in terms of present values equivalence. We assume, like Ramsey, that \( B(t) \ll c(t) \) for all \( t \). Substituting Eq. 5b into Eq. 2 implies

\[
U^T = \int_0^T u(c) e^{-\rho t} dt = \int_0^T u(c^*) e^{-\rho t} dt + \int_0^T u'(c^*) B(t) e^{-\rho t} dt,
\]

where \( u(c) \) is the instantaneous utility, \( u'(c) \) is its derivative with respect to \( c \), and \( c^* \) is the quasi-equilibrium consumption level given by Eq. 5a with initial conditions \( c^*(0) = c_0 \). It follows that

\[
U^T = U_+^T + U_c^T + U_s^T,
\]

where

\[
U_+^T = \int_0^T u(c) e^{-\rho t} \quad \text{(S3)}
\]

is the base utility that is obtained when \( B(t) = H(t) = 0 \). Namely, \( c_+ \) follows Eq. 5a with \( H \to 0 \):

\[
\frac{1}{c_-(t)} \frac{dc_-(t)}{dt} = g(c_-(t)),
\]

and initial conditions \( c_-(0) = c_0 \).

\[
U_c^T = \int_0^T u'(c^*) B(t) e^{-\rho t} \quad \text{(S5)}
\]

is the contribution to \( U^T \) due to change in consumption due to \( B \), and

\[
U_s^T = \int_0^\infty u(c^*) e^{-\rho t} - U_-^T \quad \text{(S6)}
\]

is the deviation of the total utility from the NPV. Note that \( H \) affects both \( U_s^T \) and \( U_c^T \).

In what follows, we express \( U_c^T \) and \( U_s^T \) in terms of the present values of \( B(t) \) and \( H(t) \), respectively. In other words, we find how much present change in \( B \) (or \( H \)) at time \( t \) would be equivalent
to a ‘present’ change in $B$ (or $H$) at time zero, and thereby calculating the net present value (or shadow value). The net present shadow value plus the net present actual value gives the Generalized Net Present Value (GNPV) that a social planner is aiming at maximizing. In section S1.2, we derive the expression for $U^T_c$ in terms of present values. This is similar to the derivation by (Ramsey 1928, Cass 1965, Koopmans 1965), with the difference that we consider $H$ in addition to the regular growth $g$. Next, in section S1.3, we derive the expression for $U^T_s$ in terms of present values. Finally, in Section 1.4, we integrate the results to complete the derivation of the formula for the GNPV, and we consider the limit $T \to \infty$ and complete the proof of Theorem 2.

**S1.2 Regular discount on short-term changes in consumption: contribution of $B(t)$ to the net present value**

In this subsection, we derive an expression for $U^T_c$ (Eq. (S5)), the value due to direct change in consumption due to $x$ and $P$, $B(x, P)$, in terms of net present equivalent. Namely, we find how much extra consumption does society need at present to achieve an addition of $U^T_c$ to the utility. Note that we can write Eq. (S5) in the form

$$U^T_c = \int_0^T w(t)B(t)e^{-\rho t}dt,$$

where

$$w(t) = u'(c^*(t)).$$

(S8)

It follows that

$$\frac{dw}{dt} = \frac{du'(c^*(t))}{dt} = u''(c^*)\frac{dc^*}{dt} = (g + H)c^*u''(c^*)$$

$$= \frac{c^*u''(c^*(t))}{u'(c^*(t))}(g(c^*(t)) + H(t))w = -\eta(c^*)(g(c^*) + H)w$$

(S9)

where

$$\eta(c) \equiv -\frac{cu''(c)}{u'(c)}$$

(S10)

is elasticity of changes in utility with respect to consumption. Eq. (S9) implies

$$w(t) = w(0)e^{-\int_0^t \eta(c^*(t'))(g(t') + H(t'))dt'},$$

(S11)

where Eq. (S8) implies

$$w(0) = u'(c_0).$$

(S12)
Substituting Eqs. (S11) and (S12) into Eq. (S7) implies

\[ U^T_c = u'(c_0) \int_0^T B(t)e^{-\int_0^t \delta(s)ds} dt, \]  

(S13)

where

\[ \delta(t) = \eta(c^*(t)) [g(c^*(t)) + H(t)] + \rho. \]  

(S14)

This implies that changes in consumption are discounted at a rate \( \delta_c(t) = \eta(c^*) (g(c^*) + H) + \rho \). Finally, note that Eq. (S13) gives the change in utility due to the perturbation \( B(t) \). But since we are interested in the NPV, we need to calculate how much extra consumption at present would result in the same change in utility. Note that a small addition \( \epsilon \) to consumption at time \( t = 0 \) adds \( u(c_0 + \epsilon) - u(c_0) = \epsilon u'(c_0) \) to utility, and therefore,

\[ \text{NPV} = \frac{U^T_c}{u'(c_0)} = \int_0^T B(t)e^{-\int_0^t \delta(s)ds} dt. \]  

(S15)

### S1.3 Contribution of \( H(t) \) to the utility via the shadow value

#### S1.3.1 Constant economic growth \( g \)

**Intuitive considerations**

Considering a constant \( g \) already yields several insights that we examine here, before moving to the more detailed derivation with a variable \( g \). We begin with an informal discussion to highlight the intuition behind the derivation of the expression for \( U^T_s \) in terms of the present equivalent amount of \( H \). Assume that a small amount \( \epsilon \) is subtracted from \( H \) at time \( t_0 \), and the same amount is added to \( H \) at a prior time \( t_0 - \Delta t \) where \( \Delta t \ll 1 \). Then, there is some gain associated with having a greater consumption during the period between \( t_0 - \Delta t \) and \( t_0 \). Specifically, the consumption is greater by \( \epsilon c(t_0) \) during a period of length \( \Delta t \) and the gain to utility is given by

\[ U_{gain} = [u(c(t_0) + \epsilon c(t_0)) - u(c(t_0))]e^{-\rho \Delta t} \Delta t = \epsilon \Delta tc(t_0)u'(c(t_0)) e^{-\rho \Delta t}. \]  

(S16)

To find the \( (t_0 - \Delta t) \)-value of the amount \( \epsilon \) at time \( t_0 \), we need to find by how much we need to discount to compensate for the gain: Instead of adding \( \epsilon \) to \( H \) at time \( t_0 - \Delta t \), we need to add only \( \epsilon \exp(-\delta_s \Delta t) \approx \epsilon (1 - \delta_s \Delta t) \), where the loss due to \( \delta_s \) has to compensate for the gain. A positive discount, \( \delta_s > 0 \), implies that, for any \( t > t_0 \), consumption is reduced by \( \epsilon \delta_s c(t) \) from what it could be without a discount. Therefore,

\[ U_{loss} = \int_{t_0}^T [u(c + \delta_s \epsilon \Delta t c) - u(c)] dt' \approx \delta_g \epsilon \Delta t \int_{t_0}^T u'(c) ce^{-\rho dt}. \]  

(S17)
Alternatively, one may look at the loss as the time it takes until consumption recovers. Since consumption decreases by $\delta_s c$, and since recovery rate is $gc$, the time to recovery is $(\delta_s/g)\Delta t$. $U_{loss}$ comprises three terms: (1) (minus) the value during the time it takes consumption to recover, (ii) the value during the last period before $T$ that is never approached due to the delay, and (iii) if $\rho \neq 0$, the decrease in value after recovery due to the delay:

$$U_{loss} = \frac{1}{g} \left[ u(T)e^{-\rho T} - u(t_0)e^{-\rho t_0} + \rho \int_{t_0}^{T} u(c(t))e^{-\rho t} \right]. \tag{S18}$$

Indeed, for a constant $g$, Eqs. (S17) and (S18) are equivalent as Eq. (S18) can be derived directly from Eq. (S17) via integration by parts ($du/dt = (du/dc) \cdot (dc/dt)$ where $dc/dt = gc$, and thus, $du/dc = (du/dt)/gc$). From setting $U_{gain} = U_{loss}$ using Eqs. (S16) and (S17), it follows that

$$\delta_s^T(t) = \frac{c(t)u'(c(t))}{\int_{t}^{T} u'(c)e^{-\rho(t'-t)}dt'}. \tag{S19}$$

In the following, we derive the expression for $\delta_s^T$ formally and we complete the derivation of the expression for the shadow value. The equivalent expression following from Eq. (S18) is derived formally in Section S3.

**Formal derivation**

To derive the results above and get the expression for $U_{s}^T$, we assume that $\int_{0}^{T} H(t)dt \ll 1$. (this is a special case of the assumption in Theorem 2 where $g$ is constant). It follows that

$$U_{s}^T + U_{-}^T = \int_{0}^{T} u(c)e^{-\rho t}dt = \int_{0}^{T} u\left(1 + \int_{0}^{t} H(t')dt'\right)e^{-\rho t}dt = \int_{0}^{T} u(c)e^{-\rho t} + u'(c)\rho \int_{0}^{t} H(t')dt' dt \tag{S20}$$

It follows that

$$U_{s}^T = \int_{0}^{T} u'(c)\rho \int_{0}^{t} H(t')dt' dt = \int_{0}^{T} H(t)w^T(t)dt, \tag{S21}$$

where

$$w^T(t) = \int_{t}^{T} u'(c)\rho \rho' dt'. \tag{S22}$$

This implies

$$\frac{dw^T}{dt} = u'(c)\rho \rho' = \frac{u'(c)\rho}{v^T(t)}w^T(t), \tag{S23}$$
where
\[ v^T(t) = \int_t^T u'(c_-(t'))c_-(t')e^{-\rho(t'-t)}dt'. \] (S24)

Thus, for all \( T \),
\[ w^T(t) = w^T(0)e^{\int_0^t \delta^T_s(t')dt'} \] (S25)
and \( w^T(0) = v^T(0) \). It follows that
\[ U^T_s = v^T(0) \int_0^\infty H(t)e^{\int_0^t \delta^T_s(t')dt'}dt, \] (S26)

where
\[ \delta^T_s(t) = \frac{U^T_s}{u'(c_0)} = \frac{u'(c_-)c_-}{v^T(t)} \] (S27)

As in section 1.2, we need to divide by \( u'(c_0) \) to obtain the equivalent value in terms of consumption. Therefore,
\[ \text{Shadow Value} = \frac{v^T(0)}{u'(c_0)} \int_0^\infty H(t)e^{\int_0^t \delta^T_s(t')dt'}dt. \] (S28)

**S1.3.2 Stage-dependent economic growth, \( g = g(c^*) \)**

When \( g = g(c^*) \), a change in \( c^* \) at time \( t' \) changes the entire trajectory \( g(c^*(t)) \) for all times \( t > t' \). We would like to find how the function \( H(t) \) affects consumption, \( c^*(t) \), assuming that \( H(t) \) is sufficiently small (and to see what ‘sufficiently small’ means here). We express \( c^* \) as \( c_- \) plus the sum of all contributions by \( H(t') \) for all \( t' < t \). Specifically, we go backward in time from \( t' = t \) to \( t' = 0 \), and, for each \( t' \), we examine how \( g(c^*(t')) \to g(c^*(t')) + H(t') \) affects consumption at time \( t \), given that \( H(t'') = 0 \) for all \( t'' > t' \). In other words, we consider the equality
\[ c^*(t) = c_-(t) + \int_0^t c_-(t')H(t')\frac{d\hat{c}_v(t)}{d\hat{c}_v(t')}dt', \] (S29)

where, for \( t \geq t' \), \( \hat{c}_v(t) \) is given by
\[ \frac{d\hat{c}_v(t)}{dt} = \hat{c}_v(g(\hat{c}_v(t)) + H(t)) \] (S30)
with initial conditions
\[ \hat{c}_v(t') = c_-(t'). \] (S31)

Note that Eq. (S29) is exact and applies to any \( H(t) \) as long as the variational term \( d\hat{c}_v(t)/d\hat{c}_v(t') \) exists for all \( 0 \leq t' \leq t \).
To calculate the variational term, we derive both sides of Eq. (S30). For simplicity, we omit the subscript $t'$ and denote $\dot{c}(t) = \dot{c}_v(t)$ and $g(t) = g(\dot{c}(t))$. For all $t > t'$,

$$
\frac{d}{dt} \left[ \frac{d\dot{c}(t)}{d\dot{c}(t')} \right] = \frac{d\dot{c}(t)}{d\dot{c}(t')} [g(t) + H(t)] + \dot{c}(t) \frac{dg(\dot{c}(t))}{d\dot{c}(t')} = \frac{d\dot{c}(t)}{d\dot{c}(t')} [g(t) + H(t)] + \dot{c}(t) \frac{dg(\dot{c}(t))}{d\dot{c}(t)} \frac{d\dot{c}(t)}{d\dot{c}(t')}.
$$

(S32)

Next, note that the three conditions $g(t) \neq 0$, $|H(t)| \ll |g(t)|$ and $c_0 \neq 0$ together imply that $d\dot{c}(t)/dt \neq 0$. It follows that

$$
\frac{dg(\dot{c}(t))}{d\dot{c}(t)} = \frac{dg(t)/dt}{d\dot{c}(t)} = \frac{dg(t)}{dt} \frac{1}{\dot{c}(t)(g(t) + H(t))},
$$

(S33)

and therefore,

$$
\frac{d}{dt} \left[ \frac{d\dot{c}(t)}{d\dot{c}(t')} \right] = \frac{d\dot{c}(t)}{d\dot{c}(t')} \left[ g(t) + H(t) + \frac{dg(t)}{dt} \frac{1}{g(t) + H(t)} \right].
$$

(S34)

Next, note that $|H(t)| \ll |g(t)|$ implies

$$
\frac{d}{dt} \left[ \frac{d\dot{c}(t)}{d\dot{c}(t')} \right] = \frac{d\dot{c}(t)}{d\dot{c}(t')} \left[ g(t) + H(t) + \frac{dg(t)}{dt} \frac{1}{g(t)} \right],
$$

(S35)

which implies that

$$
\frac{d\dot{c}(t)}{d\dot{c}(t')} = \frac{\dot{c}(t) g(\dot{c}(t))}{\dot{c}(t') g(\dot{c}(t'))}.
$$

(S36)

Substituting Eq. (S36) into Eq. (S29) and noting that $c_v(t') = c_-(t')$ implies

$$
c^*(t) = c_-(t) + \int_0^t c_-(t') H(t') \frac{\dot{c}_v(t') g(t')}{c_v(t') g(t')} dt' = c_-(t) + \int_0^t \dot{c}_v(t') g(\dot{c}_v(t')) \frac{H(t')}{g(c_-(t'))} dt',
$$

(S37)

where $g(c_-(t'))$ appears in the last denominator because $\dot{c}_v(t') = c_-(t')$ (Eq. (S31)). Similarly, for all $t' < t$,

$$
\dot{c}_v(t) = c_-(t) + \int_{t'}^t \dot{c}_v(t') g(\dot{c}_v(t')) \frac{H(t'')}{g(c_-(t''))} dt''.
$$

(S38)

Next, we use the assumption that $H$ is sufficiently small on $[0, T]$. Denote

$$
\epsilon_v(t) = \int_{t'}^t \dot{c}_v(t') g(\dot{c}_v(t')) \frac{H(t'')}{g(c_-(t''))} dt'',
$$

(S39)

namely,

$$
\dot{c}_v(t) = c_-(t) + \epsilon_v(t).
$$

(S40)

Note that, since

$$
g(c_-(t)) \int_0^t \frac{H(t')}{g(c_-(t'))} dt' \ll 1
$$

(S41)
(main text), it follows that

$$\epsilon_v(t) \ll c_-(t)$$  \hspace{1cm} (S42)

for all $0 \leq t' \leq t \leq T$. From the substitution of Eq. (S40) in Eq. (S37), it follows that

$$c^* = c_-(t) + \int_0^t (c_-(t) + \epsilon_v(t)) g(c_-(t) + \epsilon_v(t)) \frac{H(t')}{g(c_-(t'))} dt'$$

$$= c_-(t) + c_-(t)g(c_-(t)) \int_0^t \frac{H(t')}{g(c_-(t'))} dt' + g(c_-(t)) \int_0^t \epsilon_v(t) \frac{H(t')}{g(c_-(t'))} dt' + c_-(t)g'(c_-(t)) \int_0^t \epsilon_v(t) \frac{H(t')}{g(c_-(t'))} dt' + O(\epsilon_v^2(t)) \hspace{1cm} (S43)$$

Now that we calculated $c^*(t)$ in first-order of $\epsilon_0(t)$, we can find the first-order change in welfare. Note that

$$u(c^*) = u(c_-) + u'(c_-)c_-g(c_-) \int_0^t \frac{H(t')}{g(c_-(t'))} dt'$$  \hspace{1cm} (S49)

This implies

$$U_s^T = \int_0^T [u(c^*(t)) - u(c_-(t))] dt = \int_0^T H(t)w^T(t) dt,$$  \hspace{1cm} (S50)

where

$$w^T(t) = \frac{1}{g(c_-(t))} \int_t^T g(c_-(t'))u'(c_-(t'))c_-(t')e^{-\rho t'} dt'.$$  \hspace{1cm} (S51)
This implies
\[ \frac{dw^T(t)}{dt} = \frac{dg(t)}{g(t)} w^T + \frac{u(c_-)c_- e^{-\mu t}}{w^T(t)} w^T = \frac{dg(t)}{g} w^T + \frac{u(c_-)c_-}{v^T(t)} w^T \]  
(S52)

and \( w^T(0) = v^T(0) \), where
\[ v^T(t) = \frac{1}{g(c_-)(t)} \int_0^T g(c_-)u(c_-)e^{-\mu(t'-t)}dt' \]  
(S53)
is the same function given by Eq. 15 (main text). [Note that \( v(t) \) given by Eq. (S24) follows from Eq. (S53) where \( g \) is constant.] Eq. (S52) implies that the shadow value is still given by Eq. (S28) but with
\[ \delta^T_s = \frac{u(c_-)c_-}{v^T(t)} + \frac{dg/dt}{g}. \]  
(S54)
Alternatively, since
\[ e^{-\int_0^t (dg/dt')/g} dt' = g(c_0)/g(t), \]  
(S55)
it follows that
\[ \text{Shadow Value} = \frac{U^T_s}{u'(c_0)} = \frac{v^T(0)g(c_0)}{u'(c_0)} \int_0^\infty \frac{H(t)}{g(c_-(t))} e^{\int_0^T \delta^T_s(v')dv'} dt \]  
(S56)
with
\[ \delta^T_s = \frac{u(c_-)c_-}{v^T(t)}. \]  
(S57)

S1.4 Net Present Effective Value and the Price of \( H \)

Note that \( H \) affects the term \( U^T_c \) in two ways: (1) It alters the discount rate, and (2) it adds another term. Since \( U^T_c \) is independent of \( B \) and \( H \) (and thus independent of the control \( x \)), the objective is to maximize the total change in utility due to the control,
\[ \text{GNPV}(T) = \frac{U^T_c}{u'(c_0)} + \frac{U^T_s}{u'(c_0)}, \]  
(S58)
where \( U^T_c/u'(c_0) \) is the actual change in present value and \( U^T_s/u'(c_0) \) is the shadow value. Substitution of \( U^T_c/u'(c_0) \) (Eq. (S15)) and \( U^T_s/u'(c_0) \) (Eq. (S56)) in Eq. (S58) yields
\[ \text{GNPV}(T) = \int_0^T Be^{-\Delta(t)} dt + \sigma_0 \int_0^T (H/g) e^{\Delta^T(t)} dt, \]  
(S59)
where

\[ \Delta(t) = \int_0^t \delta(t), \quad (S60) \]
\[ \Delta^T_s(t) = \int_0^t \delta^T_s(t), \quad (S61) \]
\[ \sigma_0 = \frac{v^r(0) g(c_0)}{w'(c_0)}, \quad (S62) \]

with \( \delta(t) \) given by Eq. (S14) and \( \delta^T_s(t) \) given by Eq. (S57). Equivalently, we can write

\[ \text{GNPV}(T) = \int_0^T \left( B + \frac{H}{g} \rho(t) \right) e^{-\Delta(t)} dt, \quad (S63) \]

where

\[ \sigma(t) = \sigma_0 e^{\Delta^T_s(t) - \Delta(t)}. \quad (S64) \]

This implies that Eq. 7 holds, where

\[ \Delta_w(t) = \Delta^T_s(t) - \Delta(t) \quad (S65) \]

and

\[ \delta_w = \frac{d\Delta_w}{dt} = \delta^T_s(t) - \delta. \quad (S66) \]

**S1.5 The limit \( T \to \infty \)**

To complete the proof of Theorem 1, consider the limit \( T \to \infty \). Define \( v(t) = \lim_{T \to \infty} v^T(t) \).

If \( v(0) = \infty \), then \( \lim_{T \to \infty} \sigma_0 = \infty \) and \( \lim_{T \to \infty} \delta^T_s = 0 \). This implies that the NPV becomes negligible compared to the shadow value at \( T \to \infty \) and the objective is to maximize

\[ \int_0^T H(x, P, t) g dt \quad (S67) \]

On the other hand, if \( v(0) \) is finite, the objective becomes to maximize the GNPV with some positive \( \delta \) and finite \( \delta_w \) and \( \sigma_0 \). This completes the proof of Theorem 2.
S2 Proof of Theorem 1: Expression for the Shadow Value for General Perturbations, $H(t)$, Constant Elasticity, $\eta$, and Constant Exogenous Growth, $g$

In this section, we prove Theorem 1 (main text). The proof relies on some development from the proof of Theorem 2. Specifically, we do not need to derive again the expression for the NPV (Section 1.2) because that derivation does not rely on our assumption that $H$ is small. We only need to derive an expression for the shadow value in terms of present equivalent changes to economic growth (Eqs. 7-11 in the main text). In Section 2.1, we relax the assumption that $H(t)$ is small, and we derive a general expression for the shadow value allowing any form of $H(t)$. Next, in Section 2.2, we focus on the case where $\eta$ and $g$ are constant and derive a closed-form expression for the discount function, $\delta^T_s(t)$, and the coefficient $\sigma^T$. Finally, in Section 2.3, we consider the limit where $T \to \infty$ and complete the proof.

S2.1 Deriving a general expression for the shadow value for general $H$, $u$ and $g$

First, we express the term

$$U^T_s + U^T = \int_0^T u(c^*(t))e^{-\rho t}dt$$

as a base term without any $H$,

$$U^T = \int_0^T u(c_-(t))e^{-\rho t}dt,$$

plus the sum of all contributions due to $H(t)$ starting at $t = 0$ and going up to $t = T$:

$$\int_0^T u(c^*(t))e^{-\rho t}dt = \int_0^T u(c_-)e^{-\rho t}dt + \int_0^T \frac{d(U_g + U^T)}{dH(t)} \bigg|_{H(t')=0} H(t)dt$$

$$= U^T_s + \int_0^T \frac{d(U_g + U^T)}{dH(t)} \bigg|_{H(t')=0} H(t)dt.$$  \hspace{1cm} (S70)

The meaning of the condition $H(t') = 0$ for any $t' > t$ is that, instead of $c^*$, we consider a consumption function that follows a per-capita growth at a rate $g + H$ if $t' < t$ and $g$ if $t' > t$. Namely, for any $t' > t$,

$$\frac{1}{\tilde{c}_t(t')} \frac{d\tilde{c}_t(t')}{dt} = g(\tilde{c}_t(t')),$$  \hspace{1cm} (S71)
while for any \( t' \leq t \),
\[
\tilde{c}_t(t') = c^*(t').
\] (S72)

[Note the differences between \( \tilde{c}_t(t') \) and \( \hat{c}_t(t') \) considered in Section 1.3.2 (Eqs. (S30) and (S31)). Here we sum the contribution due to \( H(t) \) in ascending order, from \( t = 0 \) to \( t = T \), whereas in Section 1.3.2 we considered the contribution of \( H(t) \) in descending order, from \( t' = t \) to \( t' = 0 \).]

It follows that
\[
U^T_s = \int_0^T H(t) \frac{d}{dH(t)} \left[ \int_0^T u(\tilde{c}_t(t') e^{-\rho t} dt' \right] dt
\]
\[
= \int_0^T H(t) \int_0^T \frac{du(\tilde{c}_t(t'))}{dH(t)} e^{-\rho t} dt' dt,
\] (S73)

where
\[
\frac{du(t')}{dH(t)} = u'(\tilde{c}_t(t')) \frac{d\tilde{c}_t(t')}{dH(t)} = u'(\tilde{c}_t(t')) \frac{\tilde{c}_t(t')}{\tilde{c}_t(t')},
\] (S74)

To further develop the expression for \( U^T_s \), we introduce the assumption that \( g \) and \( \eta \) are constants.

### S2.2 Deriving closed-form expressions for \( \delta_s \) and \( \sigma_0 \) assuming constant \( \eta \) and \( g \)

#### Constant \( g \)

Assuming that \( g \) is a constant (or just assuming that \( g \) depends purely on time \( t \) but not on \( c^* \)) implies that \( c_t(t') \) is given by
\[
c_t(t') = c_-(t') e^{\int_0^t H(u') du'}.
\] (S75)

To calculate the variational term \( d\tilde{c}_t(t')/d\tilde{c}_t(t) \), We derive both sides of Eq. (S71). Assuming that \( g \) is constant, this implies, for all \( t' > t \),
\[
\frac{d}{dt} \left[ \frac{d\tilde{c}_t(t')}{d\tilde{c}_t(t)} \right] = \frac{d\tilde{c}_t(t')}{d\tilde{c}_t(t)} g,
\] (S76)

which implies that
\[
\frac{d\tilde{c}_t(t')}{d\tilde{c}_t(t)} = \frac{\tilde{c}_t(t')}{\tilde{c}_t(t)}.
\] (S77)

[Note that if we do not assume that \( g \) is constant but assume instead that \( g(t) \neq 0 \) for all \( t \), then]
\[
\frac{d\tilde{c}_t(t')}{d\tilde{c}_t(t)} = \frac{\tilde{c}_t(t')}{\tilde{c}_t(t)} \frac{g(\tilde{c}_t(t'))}{g(\tilde{c}_t(t))}
\] (S78)
(see section 1.3.2.) On the other hand, for all \( t' < t \), a change in \( \tilde{c}_i(t) \) has no effect on \( \tilde{c}_i(t') \). It follows that
\[
\frac{du(t')}{dH(t)} = \begin{cases} 
    u'(\tilde{c}_i(t'))\tilde{c}_i(t') & \text{if } t' > t \\
    0 & \text{otherwise}
\end{cases}
\]  
(S79)

Substituting this equation into Eq. (S73) implies
\[
U^T_s = \int_0^T H(t) \int_t^T u'(c_i(t'))c_i(t')e^{-\rho t'} dt' dt.
\]  
(S80)

**Constant \( \eta \)**

To analyze Eq. (S80), we assume that \( \eta \) is constant, namely, \( u'(c) = -\eta cu''(c) \) for all \( c \in (0, \infty) \).

This implies
\[
u'(c) = a_1 c^{-\eta},
\]
(S81)
\[
u'(c)c = a_1 c^{1-\eta},
\]
(S82)

and
\[
u(c) = \begin{cases} 
    a_1 c^{1-\eta} + a_0 & \text{if } \eta \neq 1 \\
    a_1 \ln(c) + a_0 & \text{if } \eta = 1
\end{cases}
\]
(S83)

where \( a_0 \) and \( a_1 \) are constants (note that, since utility is invariant to multiplication and addition of constants, \( a_0 \) and \( a_1 \) have no effect on the results). Substitution of Eqs. (S75) and (S82) into Eq. (S80) implies
\[
U^T_s = \int_0^T H(t) \int_t^T a_1 c_1^{1-\eta} e^{-\int_0^t h(t')dt'} e^{-\rho t'} dt' dt
\]
\[
= \int_0^T H(t) e^{(1-\eta)\int_0^t H(t')dt'} \int_t^T a_1 c_1^{1-\eta} e^{-\rho t'} dt' dt
\]
\[
= \int_0^T H(t) w(t),
\]  
(S84)

where
\[
w^T(t) = e^{(1-\eta)\int_0^t H(t')dt'} \int_t^T a_1 c_1^{1-\eta}(t')e^{-\rho t'} dt'.
\]  
(S85)

It follows that
\[
\frac{dw^T}{dt} = (1 - \eta)H(t)w^T(t) + \frac{u'(c_\infty) c_\infty}{v^T(t)} w^T(t),
\]  
(S86)

and
\[
w^T(0) = v^T(0)
\]  
(S87)
where \( v^T(t) \) is given by Eq. (S53) and by Eq. 15 in the main text. Therefore (see also Section 1.3.2),

\[
\text{shadow value} = \frac{U^T}{u'(c_0)} = \sigma^T \int_0^T H(t)e^{-\int_0^t \delta^T_s(t')dt'} \, dt.
\]  

(S88)

where

\[
\delta^T_s(t) = \frac{u'(c_-)c_-}{v(t)} + (\eta - 1)H(t),
\]  

(S89)

and

\[
\sigma^T = \frac{v^T(0)}{u'(c_0)} = \frac{v^T(0)}{a_1c_0^{1-\eta}}.
\]  

(S90)

Next, to calculate \( v^T(t) \) in the special case where \( g \) and \( \eta \) are constants, note that substitution of Eq. (S81) into Eq. (S22) and assuming constant \( g \) yields

\[
v^T(t) = a_1 \int_t^T c_1^{1-\eta} e^{-\rho(t'-t)} \, dt = a_1 c_0^{1-\eta} e^{\rho t} \int_t^T e^{[(1-\eta)g+\rho]t'} \, dt' = \begin{cases}
\frac{-a_1 c_0^{1-\eta}}{(\eta-1)g+\rho} [1 - e^{[(1-\eta)g+\rho](T-t)}] & \text{if } (1-\eta)g+\rho \neq 0 \\
\frac{a_1 c_0^{1-\eta}}{T-t} & \text{otherwise}.
\end{cases}
\]  

(S91)

This term is becoming simpler as we consider the limit where \( T \to \infty \).

**S2.3 The Limit \( T \to \infty \)**

From Eq. (S91), it follows that

\[
v(t) = \lim_{T \to \infty} v^T(t) = \begin{cases}
\frac{-a_1 c_0^{1-\eta}}{(\eta-1)g+\rho} & \text{if } \eta > 1 - g/\rho \\
\infty & \text{otherwise}.
\end{cases}
\]  

(S92)

Substitution into Eqs. (S89) and (S90) implies

\[
\delta_s(t) = \lim_{T \to \infty} \delta^T_s(t) = \frac{u'(c_-)c_-}{v(t)} + (\eta - 1)H(t) = \begin{cases}
(\eta - 1)(g + H(t)) - \rho & \text{if } \eta > 1 - g/\rho \\
(\eta - 1)H(t) & \text{otherwise}.
\end{cases}
\]  

(S93)

and

\[
\sigma_0 = \lim_{T \to \infty} \sigma^T = \frac{v(0)}{a_1c_0^{1-\eta}} = \begin{cases}
\frac{-c_0}{(\eta - 1)g+\rho} & \text{if } \eta > 1 - \rho/g \\
\infty & \text{otherwise}.
\end{cases}
\]  

(S94)
We therefore distinguish two cases in the limit $T \to \infty$. First, where $\eta > 1 - \rho/g$, $S$ is finite and the objective is to maximize the NPV plus the shadow value where $\delta_s(t) = (\eta - 1)(g + H) + \rho$. Second, where $\eta \leq 1 - \rho/g$, $\sigma^T$ is becoming infinitely large as $T \to \infty$ and the objective becomes to maximize the shadow value, which is proportional to

$$
\int_0^T H(t)e^{(1-\eta)\int_0^t H(t')dt'}dt = \frac{1}{1 - \eta} \int_0^T \frac{d}{dt'} \left[ e^{(1-\eta)\int_0^{t'} H(t'')dt''} \right]_{t'=t} dt = \frac{e^{(1-\eta)\int_0^T H(t')dt}}{1 - \eta}. \quad \text{(S95)}
$$

Therefore, maximizing utility is equivalent to

$$\text{maximize } \int_0^T H(t)dt. \quad \text{(S96)}$$

This completes the proof of Theorem 1. \qed
In this section, we further analyze the function $v(t)$, and, in particular, we analyze the conditions under which it remains finite when $T \to \infty$. We assume that $g(c^*(t))$ is either a positive or a negative differentiable function ($g \neq 0$ for all $t$). We consider the general form

$$v_T(t) = \frac{1}{g(c_-(t))} \int_0^T g(c_-)u(c_-)c_-e^{-\rho(t'-t)}dt'.$$  \hfill (S97)

Note that, since $g(c_-(t)) \neq 0$, it follows that

$$\frac{du(c_-)}{dt} = \frac{du(c_-)}{dc} \frac{dc}{dt} g(c_-).$$  \hfill (S98)

This implies, via integration by parts,

$$v_T(t) = \frac{u(c_-(T))e^{-\rho(T-t)} - u(c_-(t))}{g(t)} + \frac{\rho}{g(t)} U_{-t}^T,$$  \hfill (S99)

where we denote $g(t) = g(c_-(t))$ and

$$U_{-t}^T = \int_t^T u(t')e^{-\rho(t'-t)}dt'$$  \hfill (S100)

is the remaining utility if initial time is set at $t$ (note that $U_T^T = U_{-t}^T$). This gives the formal justification for Eq. (S18) that we derived informally in Section 1.3.1.

In the limit $T \to \infty$, it follows from Eq. (S99) that

$$v(t) = \lim_{T \to \infty} v_T(t) = -\frac{u}{g(t)} + \begin{cases} \frac{u_\infty}{g(t)} & \text{if } \rho = 0 \\ \frac{\rho}{g(t)} U_{-t} + \lim_{T \to \infty} [u(c_-(T))e^{-\rho(T-t)}] & \text{if } \rho > 0 \end{cases}$$  \hfill (S101)

where $u_\infty = \lim_{T \to \infty} u(c(T))$ and $U_{-t} = \lim_{T \to \infty} U_{-t}^T$. Nevertheless, note that

$$\lim_{T \to \infty} \int_t^T u(c_-(t))e^{-\rho t'}dt' < \infty$$  \hfill (S102)

implies that $\lim_{T \to \infty} u(T)e^{-\rho T} = 0$, and therefore,

$$\int_t^\infty u e^{-\rho t'}dt' < \infty$$  \hfill (S103)

is both a sufficient and a necessary condition for $v(t) < \infty$ if $\rho \neq 0$. This implies that $v(t) < \infty$ if and only if

$$\rho = 0 \text{ and } u_\infty < \infty \text{ or } \rho \neq 0 \text{ and } U_- < \infty.$$  \hfill (S104)

If $U_-$ is finite, $\lim_{T \to \infty} [u(c_-(T))e^{-\rho(T-t)}] = 0$, and therefore, in summary,

$$v(t) = -\frac{u}{g(t)} + \begin{cases} \frac{u_\infty}{g(t)} & \text{if } \rho = 0 \\ \frac{\rho}{g(t)} U_{-t} & \text{if } \rho > 0. \end{cases}$$  \hfill (S105)