THE ALGEBRA OF DIFFERENTIAL OPERATORS ON THE CIRCLE AND $W^{(q)}_{\text{KP}}$

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ABSTRACT

Radul has recently introduced a map from the Lie algebra of differential operators on the circle to $W_n$. In this note we extend this map to $W^{(q)}_{\text{KP}}$, a recently introduced one-parameter deformation of $W_{\text{KP}}$—the second hamiltonian structure of the KP hierarchy. We use this to give a short proof that $W_\infty$ is the algebra of additional symmetries of the KP equation.

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§1 Introduction

$W$-algebras (see [1] for a timely review) are one of the most interesting algebraic structures to have appeared in theoretical physics in recent times. Besides the fundamental role they play in physics—both in the theory of integrable systems and in two-dimensional conformal field theory—$W$-algebras are beginning to show up also in mathematics, for example, in Drinfel’d’s approach to the generalized Langlands correspondence [2]. Despite this growing interest on both sides of the fence, a deep understanding of these algebras is to a large extent lacking and fundamental questions concerning their geometrical significance and representation theory remain unanswered (although see [3]). In the absence of a clearer path to follow, one hopes that one can make some progress by relating $W$-algebras to algebraic structures which are better understood. This drive towards determining the right place for $W$-algebras in the algebraic world characterizes much of current research on this topic, and this paper is no exception.

One particularly fruitful method of investigating how a class of algebras fits within other algebraic structures is to try and establish maps (morphisms) between its objects and other well-known objects. In the case of $W$-algebras, examples of such maps are the Miura transformation [4], the (generalized) Drinfel’d–Sokolov reduction [5], and the Radul map [6]. In a few words, the Miura transformation embeds the $W$-algebra in the universal envelope of the algebra of free fields (Heisenberg algebras); whereas the Drinfel’d–Sokolov reduction exhibits the $W$-algebra as a subquotient of the universal enveloping algebra of an affine Lie algebra. On the other hand, the Radul map is a different kind of morphism. $W$-algebras act naturally as infinitesimal canonical transformations on the space of Lax operators and the Radul map is a Lie algebra homomorphism from the differential operators on the circle to the algebra of vector fields on the space of Lax operators. Some of these vector fields generate $W$-transformations.

Brevity demands that we omit a discussion of these maps. There is ample literature already on the first two maps (see, for example, [1] and references therein) and the Radul map is leisurely described in the second reference of [6]. However, the point of this note being to introduce a generalization of the Radul map to the space of generalized pseudodifferential operators, we briefly review this formalism in section 2. Section 3 contains the main result of this paper: the extension of the Radul map to the space of generalized pseudodifferential Lax operators. Section 4 contains an application of this extension to the case of the KP hierarchy. Using the homomorphism property of the generalized Radul map one can make transparent the appearance of $W_\infty$ as the algebra of additional symmetries [7] of the KP hierarchy. Finally, section 5 contains some concluding remarks.
§2 Generalized Pseudodifferential Operators and $W_{\text{KP}}^{(q)}$

In this section we briefly review the extension of the formalism of [8] to encompass generalized pseudodifferential operators. We follow [9] to which the reader is referred for more details. A description of the underlying Lie-Poisson structures can be found in [10].

By a pseudodifferential symbol we mean a formal Laurent series in a parameter $\xi^{-1}$ of the form $P(z, \xi) = \sum_{i=\text{finite}}^{\text{finite}} p_i(z)\xi^i$ whose coefficients we take to be smooth functions on the circle. Symbols have a commutative multiplication given by multiplying the Laurent series; but one can also define a composition law (denoted by $\circ$)

$$P(z, \xi) \circ Q(z, \xi) = \sum_{k \geq 0} \frac{1}{k!} \frac{\partial^k P}{\partial \xi^k} \frac{\partial^k Q}{\partial z^k},$$

making the map

$$P(z, \xi) \mapsto \sum_i p_i(z) \partial^i$$

from pseudodifferential symbols to pseudodifferential operators (ΨDO’s) into an algebra homomorphism. Conversely to every ΨDO we associate its symbol by first writing all $\partial$’s to the right and then substituting $\partial$ with $\xi$.

The advantage of working with symbols is that symbol composition is a well-defined operation on arbitrary smooth functions of $z$ and $\xi$. For example, for $a = a(z)$,

$$\log \xi \circ a = a \log \xi - \sum_{j=1}^{\infty} \frac{(-1)^j}{j} a^{(j)} \xi^{-j},$$

which shows that the commutator (under symbol composition) with $\log \xi$, denoted by $\text{ad log } \xi$, is an outer derivation on the algebra of pseudodifferential symbols. Similarly, if $q$ is any complex number, not necessarily an integer, we find

$$\xi^q \circ a = \sum_{j=0}^{\infty} \left[ \begin{array}{c} q \\ j \end{array} \right] a^{(j)} \xi^{q-j},$$

where we have introduced, for $q$ any complex number, the generalized binomial coefficients $\left[ \begin{array}{c} q \\ j \end{array} \right] \equiv [q]_j/j!$, where $[q]_j \equiv q(q-1)\cdots(q-j+1)$ is the Pochhammer symbol. Conjugation by $\xi^q$ is therefore an outer automorphism of the algebra of pseudodifferential symbols, which is the integrated version of $\text{ad log } \xi$. Hence, via their associated symbols, one can give a well-defined meaning to objects such as the logarithm of the derivative $\log \partial$ and to an arbitrary complex power of the derivative $\partial^q$. From now on, we shall work with these formal objects with the tacit understanding that they are defined via their symbols.

It follows from (2.4) that multiplication (either on the left or on the right) by $\partial^q$ sends ΨDO’s into operators of the form $\sum_{j \leq N} p_j(z)\partial^{q+j}$. Let us denote the set of these operators by $S_q$. It is clear that $S_q$ is a bimodule over the algebra of ΨDO’s, which for $q \in \mathbb{Z}$ coincides
with the algebra itself. In fact, since \( S_q = S_p \) for \( p \equiv q \mod \mathbb{Z} \), we will understand \( S_q \) from now on as implying that \( q \) is reduced modulo the integers. Moreover, symbol composition induces a multiplication \( S_p \times S_q \to S_{p+q} \), where we add modulo the integers. We call their union \( \mathcal{S} = \bigcup_q S_q \), the algebra of generalized pseudodifferential operators (\( q\text{-ΨDO}'s \)).

On \( S_0 \) one can define a trace form as follows. Let us define the residue of a \( q\text{-ΨDO} \ P = \sum_{j \leq N} p_j(z) \partial^j \) by \( \text{res} \ P = P_{-1}(z) \). Then one defines the Adler trace \([11]\) as \( \text{Tr} \ P = \int \text{res} \ P \), where \( \int \) is integration over the circle. As the name suggests, \( \text{Tr} [P, Q] = 0 \), since the residue of a commutator is a total derivative. The Adler trace can be used to define a symmetric bilinear form on \( q\text{-ΨDO}'s \)

\[
\langle A, B \rangle \equiv \text{Tr} \ AB ,
\]

which extends to a symmetric bilinear form on all of \( \mathcal{S} \). Relative to this form the dual space to \( S_q \) is clearly isomorphic to \( S_{-q} \).

The ring \( S_0 \) of \( q\text{-ΨDO}'s \) splits into the direct sum of two subrings \( S_0 = \mathcal{R}_+ \oplus \mathcal{R}_- \), corresponding to the differential and integral operators, respectively. This decomposition is a maximally isotropic split for the bilinear form (2.5), since \( \text{Tr} A_{\pm} B_{\pm} = 0 \), where \( A_{\pm} \) denotes the projection of \( A \) onto \( \mathcal{R}_\pm \) along \( \mathcal{R}_\mp \).

In order to define the generalized Adler map, we need to briefly introduce some formal geometry on the space \( \mathcal{M}_q \) of \( q\text{-ΨDO}'s \) of the form

\[
L = \partial^q + \sum_{j=1}^{\infty} u_j(z) \partial^{q-j} .
\]

The tangent space \( \mathcal{T}_q \) to \( \mathcal{M}_q \) is parametrized by the infinitesimal deformations of \( L \), which are given by \( q\text{-ΨDO}'s \) of the form

\[
A = \sum_{j=1}^{\infty} a_j(z) \partial^{q-j} .
\]

One-forms are parametrized by the dual space \( \mathcal{T}_q^* \) of \( \mathcal{T}_q \) under the bilinear form (2.5). That is, \( \mathcal{T}_q^* \) is made up of \( q\text{-ΨDO}'s \) of the form

\[
X = \sum_{j=1}^{\infty} \partial^{j-q-1} x_j \mod \partial^{-q} \mathcal{R}_- .
\]

In other words, \( \mathcal{T}_q^* \cong \mathcal{S}_{-q}/\partial^{-q} \mathcal{R}_- \). For \( A \) and \( X \) as above,

\[
\text{Tr} AX = \sum_{j=1}^{\infty} \int a_j x_j ,
\]

which is clearly nondegenerate.
The generalized Adler map is defined as follows. If \( X \in S_{-q} \), define

\[
J(X) = (LX)_+ L - L(XL)_+ = L(XL)_- - (LX)_- L .
\]  

(2.10)

Notice that since \( LX \) and \( XL \) belong to \( S_0 \), it makes sense to project onto their integral and/or differential parts. Notice also that the first equation in (2.10) implies that \( \partial^{-q}R_+ \subseteq \ker J \), whereas the second equation tells us that \( \text{Im} J \subseteq T_q \). Furthermore if \( Y \) is another 1-form,

\[
\text{Tr} \ J(X)Y = - \text{Tr} \ XJ(Y) ;
\]

(2.11)

hence \( J \) induces a skewsymmetric linear map \( T^*_q \to T_q \)—also denoted by \( J \). We can therefore use it to define a bracket on the functions on \( M_q \) as follows:

\[
\{ F , G \} = \text{Tr} \ J(dF)dG ,
\]

(2.12)

where for \( F \) a function in \( M_q \), its gradient is defined implicitly by

\[
\text{Tr} \ A dF = \frac{d}{d\epsilon} F[L + \epsilon A] \bigg|_{\epsilon=0} ,
\]

(2.13)

for all tangent vectors \( A \in T_q \).

It is proven in [9] that this is indeed a Poisson bracket, and moreover, that it defines on the coefficients \( \{ u_i \} \) of \( L \) a one-parameter \( W \)-algebra, called \( W_{\text{KP}}^{(q)} \), from which all \( W_n \) can be obtained by reduction. Furthermore, as showed in [12], all hitherto known \( W_\infty \)-type algebras can also be obtained from \( W_{\text{KP}}^{(q)} \) by contractions and/or reductions.

\section{3 The Generalized Radul Map}

Consider the subring \( R_+ \) of differential operators on the circle and give it a Lie algebra structure by the commutator. We call the resulting Lie algebra \( \text{DOP} \). The generalized Radul map is defined as follows. Fix a \( q \)-\( \Psi DO \) \( L \in M_q \) and define \( W : \text{DOP} \to T_q \) by

\[
W(E) = (LEL^{-1})_- L = LE - (LEL^{-1})_+ L .
\]

(3.1)

On \( T_q \) we can define a Lie bracket as follows. Every \( A \in T_q \) of the above form defines a vector field \( \partial_A \) by

\[
\partial_A = \sum_{i=1}^{\infty} \sum_{k=0}^{\infty} a_i^{(k)} \frac{\partial}{\partial u_i^{(k)}} .
\]

(3.2)

In particular, \( \partial_A L = A \). We then define the Lie bracket \( [A, B] \) of two vectors \( A, B \in T_q \) by

\[
\partial_{[A,B]} = [\partial_A, \partial_B] ;
\]

(3.3)

or equivalently

\[
[A, B] = \partial_A B - \partial_B A .
\]

(3.4)

(Notice that this is not the ordinary commutator in \( S_q \), which would send the bracket of \( A, B \) to \( S_{2q} \).)
Theorem 3.5. If $E, F \in DOP$ are independent of $L$ (that is, their coefficients do not depend on the coefficients of $L$) then

$$[W(E), W(F)] = W([E, F]).$$

Proof: Because of (3.3) this is equivalent to

$$[\partial_{W(E)}, \partial_{W(F)}] = \partial_{W([E, F])},$$

which we proceed to prove. Acting on $L$,

$$[\partial_{W(E)}, \partial_{W(F)}]L$$

$$= \partial_{W(E)}W(F) - (E \leftrightarrow F)$$

$$= \partial_{W(E)}(LFL^{-1})_L - (E \leftrightarrow F)$$

$$= (L\partial_{W(E)}F)L + (W(E)LFL^{-1})L$$

$$- (LFL^{-1}W(E)L^{-1})L + (LFL^{-1})W(E) - (E \leftrightarrow F)$$

$$= W(\partial_{W(E)}F) + ((LFL^{-1})_L(LFL^{-1})L$$

$$- (LFL^{-1}(LFL^{-1})_L + (LFL^{-1})(LFL^{-1})_L - (E \leftrightarrow F)$$

$$= W(\partial_{W(E)}F) + ((LFL^{-1})_L(LFL^{-1})_L$$

$$+ (LFL^{-1}(LFL^{-1})_L - (E \leftrightarrow F)$$

$$= W(\partial_{W(E)}F) + (LFL^{-1})_L - (E \leftrightarrow F),$$

whence

$$[\partial_{W(E)}, \partial_{W(F)}]L = W(\partial_{W(E)}F - \partial_{W(F)}E + [E, F]).$$

If $E$ and $F$ are independent of $L$, the variations $\partial_{W(F)}E, \partial_{W(E)}F$ vanish and the theorem follows. \[\square\]

Remark 3.7. It follows from (3.6) that if $E$ and $F$ depend on $L$ then we have to correct the Lie bracket in $DOP$ to preserve the homomorphism. Let us define the improved bracket on $DOP$ by

$$[\_ , \_]_c \equiv \partial_{W(E)}F - \partial_{W(F)}E + [E, F].$$

In [9] it is proven that the image of the generalized Adler map is a subalgebra of $T_q$, and this allows us to pull back the bracket $[,]$ on $T_q$ to a bracket $[,]^*_L$ on $T_q^*$ defined by requiring that the Adler map be a homomorphism. Explicitly, for $X, Y \in T_q^*$,

$$[[J(X), J(Y)] = J([X, Y]_L^*),$$

where

$$[X, Y]_L^* = \partial_{J(X)}Y + X(LY)_L - (XL)_L Y - (X \leftrightarrow Y).$$
Therefore we have two Lie algebra homomorphisms to $T_q$:

$$
\begin{array}{c}
\mathcal{T}_q^* \\
\downarrow J \\
T_q
\end{array} \xrightarrow{W} \begin{array}{c}
\text{DOP} \\
\downarrow J \\
T_q
\end{array}
$$

and it is natural to ask whether the triangle can be completed. It turns out that this is indeed the case.

**Theorem 3.11.** There exists a Lie algebra isomorphism $R : \text{DOP} \rightarrow \mathcal{T}_q^*$ making the following diagram commutative

$$
\begin{array}{c}
\mathcal{T}_q^* \\
\downarrow J \\
\text{DOP} \xrightarrow{W} \mathcal{T}_q
\end{array}
$$

**Proof:** Define $R(E) = -EL^{-1} \mod \partial^{-q}\mathcal{R}_-$. It is clear from (2.10) that $\partial^{-q}\mathcal{R}_- \subset \ker J$; whence

$$J(R(E)) = J(-EL^{-1}) = (LEL^{-1})_L - L(EL^{-1})_L = W(E). \quad (3.12)$$

We now prove that it is bijective. Consider the map $S : S_{-q} \rightarrow \text{DOP}$ given by $S(X) = -(XL)_+$. Its kernel is precisely $\partial^{-q}\mathcal{R}_-$, whence it induces a one-to-one map $R^{-1} : \mathcal{T}_q^* \rightarrow \text{DOP}$, which as the name suggests is inverse to $R$. Indeed,

$$R \circ S(X) = (XL)_+ L^{-1} \mod \partial^{-q}\mathcal{R}_-$$

$$= X - (XL)_- L^{-1} \mod \partial^{-q}\mathcal{R}_-$$

$$= X \mod \partial^{-q}\mathcal{R}_-,$$

whence $R \circ R^{-1} = \text{id}$. Conversely,

$$R^{-1} \circ R(E) = -S(EL^{-1}) = E_+ = E.$$

Finally we prove that $R$ is a Lie algebra homomorphism—whence it will follow that so is $R^{-1}$. Notice that since $J \circ R = W$ and both $J$ and $W$ are homomorphisms, $R$ must be a homomorphism modulo $\ker J$. Nevertheless a short calculation shows that the homomorphism is exact. First notice that $\partial^{-q}\mathcal{R}_-$ is an ideal relative to the extension of $[,]^*_L$ to $S_{-q}$. Therefore,

$$[R(E), R(F)]^*_L = [EL^{-1}, FL^{-1}]^*_L$$

$$= - \partial_{W(E)}(FL^{-1}) + EL^{-1}(LFL^{-1})_- - EFL^{-1} - (E \leftrightarrow F)$$

$$= - \partial_{W(E)}FL^{-1} + FL^{-1}W(E)L^{-1} + EL^{-1}W(F)L^{-1}$$

$$- EFL^{-1} - (E \leftrightarrow F)$$

$$= - (\partial_{W(E)}F - \partial_{W(F)}E + [E, F]) L^{-1}$$

$$= R([E, F]_c),$$

which proves the proposition. $\blacksquare$
Remark 3.13. If we take \( q = N \geq 2 \) and we consider the subspace \( \mathcal{M}_N \subset \mathcal{M} \) consisting only of differential operators, we recover the original Radul map. In fact, let \( \mathcal{T}_N \) denote the tangent space and \( \mathcal{T}_N^* \cong \mathcal{R}_- / \partial^{-N} \mathcal{R}_- \) denote its dual. Then we have Lie algebra homomorphisms

\[
\overline{R} : \text{DOP} \rightarrow \mathcal{T}_N^* \\
E \mapsto -(EL^{-1})_- \mod \partial^{-N} \mathcal{R}_-
\]

and

\[
\overline{W} : \text{DOP} \rightarrow \mathcal{T}_N \\
E \mapsto (LEL^{-1})_- L
\]

making the following diagram commutative

\[
\begin{array}{ccc}
\mathcal{T}_N^* & \longrightarrow & \mathcal{T}_N \\
\overline{R} & \nearrow & \downarrow J \\
\text{DOP} & \overrightarrow{\pi} & \mathcal{T}_N
\end{array}
\]

Unlike in Theorem 3.11, however, \( \overline{R} \) is no longer one-to-one. In fact, for any \( F \in \text{DOP}, \) \( FL \in \ker \overline{R} \).

§4 Dressing Transformations and a Radul-type Map

In this section we discuss an immediate application of the homomorphism property of the Radul map. This concerns the recent identification \([7]\) of \( \mathcal{W}_\infty \) as the algebra of additional symmetries (see, for example, \([8]\) and also \([13]\)) of the KP hierarchy. We start with a brief review of dressing transformations. We follow \([8]\).

The Volterra group \( G = 1 + \mathcal{R}_- \) acts naturally on \( \mathcal{M}_q \) via dressing transformations

\[ L \mapsto \Phi^{-1} L \Phi, \]

where \( G \ni \Phi = 1 + \sum_1^\infty w_i \partial^{-i} \). Since the coefficient \( u_1 \) of \( L \) does not evolve with the KP flows and is invariant under dressing transformations, we will restrict ourselves to those operators with \( u_1 = 0 \). By undressing such an operator \( L \)

\[ L \Phi = \Phi \partial^g \]

we can lift the KP flows to flows on the Volterra group. In the \( q \)-formulation of the KP hierarchy (see \([9]\)) the KP flows are given by

\[ \partial_n L = [L^{n/q}_+, L] = [L, L^{n/q}_-], \]

where \( \partial_n \) stands for \( \frac{\partial}{\partial t_n} \), the derivative along the \( n \)th “time” of the KP hierarchy.

Proposition 4.3. The KP flows (4.2) on \( L \) are induced by the following flows on \( \Phi \):

\[ \partial_n \Phi = -(\Phi \partial^n \Phi^{-1})_- \Phi. \]

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Proof: From (4.1), \( L = \Phi \partial^n \Phi^{-1} \). Hence,
\[
\partial_n L = \partial_n \Phi \partial^n \Phi^{-1} - \Phi \partial^n \Phi^{-1} \partial_n \Phi \Phi^{-1} \\
= - (\Phi \partial^n \Phi^{-1})_L + L(\Phi \partial^n \Phi^{-1})_n \\
= \left[ L, L^\frac{n}{q} \right].
\]

Remark 4.5. In the last line of the proof we have used the fact that if \( A = \Phi B \Phi^{-1} \) then \( A^z = \Phi B^z \Phi^{-1} \) for all \( z \in \mathbb{C} \). This fact follows immediately for \( z \in \mathbb{Q} \). For \( z \in \mathbb{R} \) it follows from continuity of the map \( z \mapsto P^z \) (for all formally elliptic \( \Psi DO \)'s \( P \)) and for \( z \in \mathbb{C} \) it is a consequence of the fact that this map is actually holomorphic (see, for example, [9]).

The expression (4.4) is reminiscent of the Radul map (3.1). In fact, if we define a map \( W' : \text{DOP} \to \text{Lie}(G) \cong \mathcal{R} \) by
\[
W'(E) = (\Phi E \Phi^{-1})_n \Phi, \tag{4.6}
\]
then \( \partial_n \Phi = -W'(\partial^n) \). In general, for \( E \in \text{DOP} \) we define a flow \( \partial_{W'(E)} \) on the Volterra group by \( \partial_{W'(E)} \Phi = W'(E) \). The proof of Theorem 3.5 carries over \textit{mutatis mutandis} to a proof of the analogous result:

Theorem 4.7. For all \( E, F \in \text{DOP} \)
\[
[\partial_{W'(E)}, \partial_{W'(F)}] = \partial_{W'([E, F])_n},
\]
where
\[
[E, F]_c \equiv \partial_{W'(E)} F - \partial_{W'(F)} E + [E, F]. \tag{4.8}
\]

Corollary 4.9. The KP flows (4.2) commute.

Proof: By Proposition 4.3 it is enough to prove that the flows (4.4) commute. But by the theorem, \( [\partial_n, \partial_m] = \partial_{W'([\partial^n, \partial^m])} = 0 \). □

Consider now the formal differential operator
\[
\Gamma = \sum_{j=1}^{\infty} j t_j \partial^{j-1}, \tag{4.10}
\]
where \( t_j \) is the \( j \)-th KP time. It’s clear that \( [\partial_n, \Gamma] = n \partial^{n-1} \), and it follows from (4.2) that \( \partial_1 = \partial \), whence
\[
[\partial_n - \partial^n, \Gamma] = 0. \tag{4.11}
\]
Upon dressing, this relation becomes
\[
[\partial_n - L^{n/q}_+, M] = 0, \tag{4.12}
\]
where \( M \equiv \Phi \Gamma \Phi^{-1} \). Since \( [\partial, \Gamma] = 1 \), the Lie algebra \( \mathcal{A} \) generated by \( \Gamma^k \partial^m \), for \( k \geq 0 \) and \( m \in \mathbb{Z} \) is isomorphic to \( \text{DOP} \) (relative to the commutator). And so is the algebra generated by \( M^k L^{m/q} \), for \( k \geq 0 \) and \( m \in \mathbb{Z} \), since dressing transformations are algebra automorphisms. Explicitly, the isomorphism \( \text{DOP} \to \mathcal{A} \) is given by \( z \mapsto -\partial \) and \( \partial/\partial z \mapsto \Gamma \). It may seem at first more natural to send \( \partial/\partial z \) to \( \partial \) and \( z \) to \( \Gamma \), but in view of the applications we have in mind, we prefer this choice.
Let us define flows on $G$ by

$$\partial_{mk} \Phi = W'(\Gamma^k \partial^m).$$

(Notice that $\partial_{m0} = -\partial_{m}$.) As corollaries of Theorem 4.7 we get two important results concerning these flows.

**Corollary 4.14.** The flows are symmetries of the KP hierarchy; in other words, they commute with the Lax flows (4.2).

**Proof:** By Theorem 4.7,

$$[\partial_{mk}, \partial_n] = \partial_{-W'(\Gamma^k \partial^m) \partial^n};$$

where by (4.8)

$$[\Gamma^k \partial^m, \partial^n]_c = (\partial_n \Gamma^k) \partial^m + \Gamma^k \partial^m \partial^n = \left[\partial_n, \Gamma^k \partial^m\right],$$

which vanishes by (4.11).

**Remark 4.15.** The flows $\partial_{mk}$ for $k \geq 1, m \in \mathbb{Z}$ are the additional symmetries of the KP hierarchy (see, e.g., [8]).

**Corollary 4.16.** (Aoyama–Kodama [7]) The additional symmetries generate a Lie algebra isomorphic to $W_\infty$.

**Proof:** By Theorem 4.7, the algebra of the additional flows is isomorphic to the subalgebra of $\mathcal{A}$ generated by $\Gamma^k \partial^m$, for $k \geq 1$ and $m \in \mathbb{Z}$. But under the isomorphism $\text{DOP} \to \mathcal{A}$ described above, this is isomorphic to the subalgebra of $\text{DOP}$ consisting of differential operators without zeroth order piece—in other words, $W_\infty$.

**Remark 4.17.** Although realized here without it, $W_\infty$ has a natural central extension given, as a subalgebra of $\text{DOP}$, by the Khesin–Kravchenko [14] cocycle. In the KP context, the central extension appears when acting on the $\tau$-functions—equivalently, when we realize $W_\infty$ as free fermion bilinears in a two-dimensional conformal field theory.

§5 A Few Concluding Remarks

In this paper we have exhibited a Lie algebra isomorphism between the algebra $\text{DOP}$ of differential operators on the circle and a subalgebra of the algebra of vector fields on the space of generalized pseudodifferential operators ($q$-$\Psi$DO’s) of the form $\partial^q + \sum_{i=1}^{\infty} u_i(z) \partial^{q-i}$. The subalgebra in question is given by the image of the Adler map and contains $W^{(q)}_{\text{KP}}$ as a subalgebra.

Moreover, a simple and conceptual proof of the fact [7] that $W_\infty$ is the algebra of additional symmetries of the KP hierarchy followed from this result. We did this by using the homomorphism property of a Radul-type map sending $\text{DOP}$ to flows on the Volterra group. This proof extends to the supersymmetric case. For the case of the Manin–Radul SKP hierarchy, the algebra of additional symmetries has already been identified [15]; but from the point of view of dressing transformations, it is the Jacobian SKP hierarchy of Mulase and Rabin which appears more natural. The extension of the results of this paper to the supersymmetric case will appear somewhere else.
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