VARIATION OF THE GIESEKER AND UHLENBECK COMPACTIFICATIONS

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Abstract. In this article, we study the variation of the Gieseker and Uhlenbeck compactifications of the moduli spaces of Mumford-Takemoto stable vector bundles of rank 2 by changing polarizations. Some canonical rational morphisms among the Gieseker compactifications are proved to exist and their fibers are studied. As a consequence of studying the morphisms from the Gieseker compactifications to the Uhlebeck compactifications, we show that there is an everywhere-defined canonical algebraic map between two adjacent Uhlenbeck compactifications which restricts to the identity on some Zariski open subset.

1. Introduction

Let $X$ be an algebraic surface with $p_g = 0$ and $H$ an ample divisor over $X$. The moduli space $\mathcal{M}_H^{\mu}$ of the Mumford-Takemoto $H$-stable rank two vector bundles has turned out to be a key ingredient in the Donaldson theory of smooth topology of algebraic surfaces. In fact, Donaldson showed that the moduli space $\mathcal{N}_H$ of $SU(2)$-ASD connections on $X$ with respect to the Hodge metric induced by $H$ is homeomorphic to the moduli space $\mathcal{M}_H^{\mu}$ of Mumford-Takemoto $H$-stable rank two vector bundles. Hence the study of this moduli space is important for the application of the Donaldson theory. It is obvious that the moduli space $\mathcal{M}_H^{\mu} \cong \mathcal{N}_H$ depends on the polarization $H$. The effect on the moduli space of $H$-stable bundles when changing the polarization has been considered before by Donaldson \cite{2}, Friedman-Morgan \cite{12}, Mong \cite{10} and Qin \cite{13}, among others. In particular, Qin \cite{13} gave a very systematic treatment.

However, for many important applications, e.g., computing Donaldson’s polynomials, just considering the open variety $\mathcal{M}_H^{\mu}$ is not sufficient. In fact, Donaldson polynomials are computed on the Uhlenbeck compactification of $\mathcal{N}_H \cong \mathcal{M}_H^{\mu}$. So instead of considering variation of moduli spaces of Mumford-Takemoto $H$-stable rank-two vector bundles $\mathcal{M}_H^{\mu}$ for different $H$, we take the step further to consider the variations of the Gieseker and Uhlenbeck compactifications of the moduli space $\mathcal{M}_H^{\mu}$ of $H$-stable bundles.

Gieseker constructed the moduli space $\mathcal{M}_H$ of $H$-semi-stable torsion-free coherent sheaves and showed that it is a projective scheme. Since $\mathcal{M}_H$ contains the moduli space $\mathcal{M}_H^{\mu}$ as a Zariski open subset, $\mathcal{M}_H$ can be considered as a compactification of $\mathcal{M}_H^{\mu}$. According to the Uhlenbeck weak compactness theorem, $\mathcal{M}_H^{\mu} \cong \mathcal{N}_H$ also admits a Gauge theoretic compactification. This compactification is called the Uhlenbeck compactification, denoted by $\mathcal{N}_H$.

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It appears that the works before only considered variation of $M_{\mu}^H$ and set-theoretic comparison of the moduli spaces $M_{\mu}^H$ by varying $H$ (cf. [13]). However, in this paper, not only we take into account the variation of compactifications of $M_{\mu}^H$ but also our consideration is on the level of morphisms. Namely, we address the existence of morphisms amongst the moduli spaces. In particular, we showed that there are enough canonical algebraic rational maps amongst the Gieseker compactifications and canonical everywhere-defined algebraic maps amongst the Uhlenbeck compactifications. Moreover, we gave some explicit description of these morphisms and maps (see Theorem 5.1 and §7). One of advantages of this consideration is that these maps carry considerable information which may allow one to trace the geometry and topology from one moduli space (and its compactifications) to another.

The main result of this paper may be summarized as follows. Let $C_X$ be the Kähler cone of $X$ which is the closed convex cone in $\text{Num}(X) \otimes \mathbb{R}$ spanned by all ample divisors. There are certain natural wall and chamber structures in $C_X$ such that an ample divisor $H$ lies on a wall if and only if it possesses non-universal strictly MT $H$-semistable bundles. Let $C$ and $C'$ be two adjacent chambers with a common face $F = \overline{C} \cap \overline{C'}$. Pick up divisors $H, H',$ and $H_0$ in $C, C'$ and $F$, respectively. Then there are two canonical rational morphisms $\varphi$ and $\psi$ amongst the Gieseker compactifications which descend to two everywhere-defined algebraic maps $\overline{\varphi}$ and $\overline{\psi}$ amongst the Uhlenbeck compactifications

$$
\begin{array}{ccc}
M_{\mu}(c_2) & \rightarrow & M_{\mu_0}(c_2) \\
\gamma_H | & & | \gamma_H \\
\overline{N}_{\mu}(c_2) & \rightarrow & \overline{N}_{\mu_0}(c_2) \\
\overline{\varphi} & & \overline{\psi} \\
\overline{\gamma}_{H'} | & & | \overline{\gamma}_{H'} \\
\overline{N}_{\mu'}(c_2) & \leftarrow & \overline{N}_{\mu_0'}(c_2)
\end{array}
$$

such that the above diagram commutes (see Theorem 5.1 and Theorem 7.8 for more details). Here the morphisms $\gamma$ are the morphisms from the Gieseker compactifications to their corresponding Uhlenbeck compactifications as constructed by J. Li [8]. Although $\varphi$ and $\varphi$ are just rational maps and hence are not surjective, $\text{Im} \overline{\varphi} \cup \text{Im} \overline{\psi} = M_{\mu_0}(c_2)$.

Another interesting result in this paper is about Uhlenbeck compactifications. Uhlenbeck compactification $\overline{N}_{\mu}(c_2)$ is, in general, a closed subset of $\prod_{j=0}^{c_2} \tilde{N}_{\mu}(j) \times \text{Sym}^{c_2-j}(X)$. It is unknown whether $\overline{N}_{\mu}(c_2) = \prod_{j=0}^{c_2} \tilde{N}_{\mu}(j) \times \text{Sym}^{c_2-j}(X)$. When $p_g(X) = 0$, we are able to give an affirmative answer.

Some of our considerations are inspired by a recent paper of Dolgachev and the first author [1] where they treated the variational problem of geometric invariant theory quotients. However, we would like to point out that the variational problem of the Gieseker compactifications and the Uhlenbeck compactifications is considerably different from that of GIT! Notably, the differences include, amongst other:

(1) in general, there are infinitely many moduli spaces that are distinct to each other in nature, while in the GIT case, the number of naturally distinct quotients is finite;

(2) in general, there only exist rational maps among the Gieseker compactifications, while in the GIT case, morphisms among quotients are always defined everywhere. Quite surprisingly, the maps among the Uhlenbeck compactifications are defined everywhere.
Another inspiration is Jun Li’s paper on the relations between the Uhlenbeck compactification and the Gieseker compactification. Because a morphism from the Gieseker compactification to the Uhlenbeck compactification is constructed, using results of the variation of the Gieseker compactification, we can get some results on the variation of the Uhlenbeck compactification. J. Morgan [11] also studied the map from Gieseker compactification to Uhlenbeck compactification.

We mention that Friedman and Qin [5] obtained stronger relations among the Gieseker compactifications and applied their results to good effect on computing the Donaldson’s invariants. Also, after this work was completed, we learnt the work [9] and received a copy of it. However, neither of [5] and [9] stresses on the Uhlenbeck compactifications.

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2. Background materials

Let $X$ be an algebraic surface.

**Definition 2.1.** Let $V$ be a rank-two torsion-free coherent sheaf over $X$. Let $H$ be an ample divisor on $X$ which will be called a stability polarization (or, a polarization for short). $V$ is said to be Gieseker $H$-stable ($H$-semi-stable) if for any rank one subsheaf $L$ of $V$,

$$
\chi(L(nH)) < (\leq) \frac{1}{2} \chi(V(nH)) \quad \text{for } n \gg 0.
$$

$V$ is strictly Gieseker $H$-semi-stable if in addition there exists $L \subset V$ such that

$$
\chi(L(nH)) = \frac{1}{2} \chi(V(nH)) \quad \text{for } n \gg 0.
$$

There is another notion of stability namely, the Mumford-Takemoto stability.

**Definition 2.2.** $V$ is said to be Mumford-Takemoto $H$-stable ($H$-semi-stable) if for any rank one subsheaf $L$ of $V$,

$$
c_1(L) \cdot H < (\leq) \frac{1}{2} c_1(V) \cdot H.
$$

$V$ is strictly Mumford-Takemoto $H$-semi-stable if in addition there exists rank one subsheaf $L \subset V$ such that

$$
c_1(L) \cdot H = \frac{1}{2} c_1(V) \cdot H.
$$

**Remark 2.3.** In this article, unless otherwise stated, when we say $V$ is $H$-stable ($H$-semi-stable), we shall mean Gieseker $H$-stable ($H$-semi-stable). We abbreviate Mumford-Takemoto $H$-stable ($H$-semi-stable) as M-TH-stable ($H$-semi-stable).

Also, in this paper, the following convention will be adopted. $V$, $V'$, etc. represent rank two torsion free coherent sheaves and $L$, $L'$, $M$, $M'$, etc. represent rank one torsion free coherent sheaves.

Suppose $V$ is strictly $H$-semi-stable. Then following Harder-Narishimhan filtration on semi-stable sheaves, we have that $V$ sits in an exact sequence

$$
0 \to L \to V \to L' \to 0
$$
with
\[ \chi(L(nH)) = \frac{1}{2} \chi(V(nH)). \]
This exact sequence needs not to be unique but \( grV = L \oplus L' \) is uniquely determined by \( V \). We say that two strictly semi-stable bundles \( V \) and \( V' \) are \( s \)-equivalent if \( grV = grV' \) (see \[3\]).

Throughout this paper, we use \( \mathcal{M}_H(c_1, c_2) \), or \( \mathcal{M}_H \) if the Chern classes are obvious from the context, to represent the moduli space of \( H \)-semi-stable sheaves \( V \) over \( X \) with \( c_1(V) = c_1 \) and \( c_2(V) = c_2 \). That is, \( \mathcal{M}_H \) is the set of \( H \)-semi-stable sheaves modulo \( s \)-equivalence. Gieseker \[8\] showed that \( \mathcal{M}_H \) is a projective scheme. We use \( \mathcal{M}_H^{\mathfrak{c}}(c_1, c_2) \) (or \( \mathcal{M}_H^{\mathfrak{c}} \)) to represent \( \text{M-T} \) \( H \)-stable vector bundles \( V \) with \( c_1(V) = c_1 \) and \( c_2(V) = c_2 \).

3. Walls and Chambers

**Definition 3.1.** The Kähler cone \( C_X \) of \( X \) is the closed convex cone in \( \text{Num}(X) \otimes \mathbb{R} \) spanned by ample divisors.

For the purpose of comparing moduli spaces for varying polarizations, we will introduce certain walls in the Kähler cone \( C_X \). These walls arise naturally from semi-stability.

Let \( V \) be a rank 2 torsion-free coherent sheaf and \( L \) be a subsheaf of rank 1. By Riemann-Roch formula, we have
\[
\chi(V(nH)) = \chi(V) + n^2H^2 - nH \cdot K_X + nH \cdot c_1(V),
\]
\[
\chi(L(nH)) = \chi(L) + \frac{n^2}{2} H - \frac{n}{2} H \cdot K_X + nH \cdot c_1(L).
\]

Hence
\[
2\chi(L(nH)) - \chi(V(nH)) = (2\chi(L) - \chi(V)) + n(2c_1(L) - c_1(V)) \cdot H.
\]

Therefore we obtain the following:

(i) \( V \) is \( H \)-stable if and only if for any given subsheaf \( L \) one of the following holds:

(1) \( (2c_1(L) - c_1(V)) \cdot H < 0 \);

(2) \( (2c_1(L) - c_1(V)) \cdot H = 0 \) but \( 2\chi(L) - \chi(V) < 0 \).

(ii) Likewise, \( V \) is strictly \( H \)-semi-stable if and only if for any given subsheaf \( L \) (1) or (2) of the above holds except that for some subsheaves \( L \), we have \( (2c_1(L) - c_1(V)) \cdot H = 0 \) and \( 2\chi(L) - \chi(V) = 0 \).

In fact, in the above, we can always assume that the cokernel \( V/L \) is torsion free. In particular, if \( V \) is strictly \( H \)-semi-stable, \( V \) sits in an exact sequence
\[
0 \rightarrow L \rightarrow V \rightarrow L' \rightarrow 0 \quad (1)
\]
with \( (2c_1(L) - c_1(V)) \cdot H = 0 \) and \( 2\chi(L) = \chi(V) \).

Clearly, that \( V \) is \( \text{M-T} \) \( H \)-stable implies that \( V \) is \( H \)-stable. The converse is not true, however. Notice that \( V \) is strictly \( \text{M-T} \) \( H \)-semi-stable if and only if for some subsheaf \( L \), \( (2c_1(L) - c_1(V)) \cdot H = 0 \), while in the Gieseker case we need to require that \( 2\chi(L) - \chi(V) = 0 \). So the Gieseker stability is finer than \( \text{M-T} \) stability. This is the main feature that distinguishes the variation problem of Gieseker’s stability from that of \( \text{M-T} \) stability.
**Definition 3.2.** Let $\tau \in \text{Num}(X)$ be of the form $2c_1(L) - c_1$ where $L$ is a rank 1 sheaf. Assume further that $-c \leq \tau^2 < 0$ where $c$ is a fixed positive number. We define the hyperplane of type $\tau$ as

$$W^\tau = \{ h \in C_X | \tau \cdot h = 0 \}$$

$W$ is called a $c$-wall (or just a wall).

Let $W$ be the set of $c$-walls in $C_X$. It can be shown that for fixed $c$, the $c$-walls are locally finite. Following [13], we give the following definition.

**Definition 3.3.** A $c$-chamber (or just a chamber) $C$ in $C_X$ is a connected component of the complement, $C_X - \bigcup_{W \in W} W$, of the union of the $c$-walls. A wall $W$ is called a wall of a chamber $C$ if $W \cap C$ contains a non-empty open subset of $W$. In this case, The relative interior of $W \cap C$ is an open face (or just face) of $C$. If $F$ is a face of $C$, then there is unique chamber $C' \neq C$ which also has $F$ as a face. In this case, the chamber $C$ and $C'$ lie on opposite sides of the wall containing the common face $F$.

It is obvious that each chamber (and each of its faces) is a convex cone in $C_X$. In fact, it is a polyhedral cone if its closure is contained entirely (except for the origin) in the interior of $C_X$.

Now we fix $c = 4c_2 - c_2^2$, once and for all.

Suppose $C$ and $C'$ share the same face $F$ lying on a wall $W^\tau$. Then $\tau \cdot H$ is either positive for all $H \in C$ or negative for all $H \in C$. A similar conclusion holds for $H' \in C'$. Thus we may assume that $\tau \cdot C > 0$ and $\tau \cdot C' < 0$. For simplicity, we shall say that $C$ is the upper chamber and $C'$ is the lower chamber. In many places of this paper, we shall use $H \cdot (H')$ to represent an ample divisor in the chamber $C \cdot (C')$ respectively, $H_0$ to represent an ample divisor on the face $F$, and $\tilde{H}$ to represent an arbitrary ample divisor.

The following proposition partially justifies the definition of chambers.

**Proposition 3.4.** Let $C$ be a chamber in the K"{a}hler cone. Then $\mathcal{M}_H = \mathcal{M}_{H_1}$ for any two $H, H_1 \in C$.

**Proof.** We shall prove this proposition by producing contradiction. Without loss of generality, assume that there exists $V \in \mathcal{M}_H \setminus \mathcal{M}_{H_1}$. Then there exists an exact sequence \([1]\) such that either we have $2c_1(L) \cdot H_1 > c_1(V) \cdot H_1$ or we have $2c_1(L) \cdot H_1 = c_1(V) \cdot H_1$ but $2\chi(L) > \chi(V)$. That is, if setting $\tau = 2c_1(L) - c_1(V)$, then $\tau \cdot H_1 \geq 0$ and

$$\text{if } \tau \cdot H_1 = 0, \text{ then } 2\chi(L) > \chi(V). \quad (2)$$

Since $V$ is $H$-semi-stable, we must have $\tau \cdot H \leq 0$ and

$$\text{if } \tau \cdot H = 0, \text{ then } 2\chi(L) \leq \chi(V). \quad (3)$$

Clearly \((2)\) and \((3)\) cannot hold simultaneously. Therefore $\tau$ cannot be numerically trivial.

Now we choose $H_2 = (-\tau \cdot H_1)H_1 + (\tau \cdot H_1)H$. Obviously, $\tau \cdot H_2 = 0$. Because the chamber $C$ is a convex cone, $-\tau \cdot H \geq 0$ and $\tau \cdot H_1 \geq 0$, we obtain that $H_2$ is also an ample line bundle in $C$. Since $\tau$ is not numerically trivial, by Hodge index theorem, $\tau^2 < 0$.
On the other hand, if we calculate the Chern classes from the exact sequence (1), we will get

\[ c_2(V) = c_1(L) \cdot (c_1(V) - c_1(L)) + c_2(L) + c_2(L') \geq c_1(L) \cdot (c_1(V) - c_1(L)). \]

After some simplifications, we get

\[ \tau^2 \geq c_1(V)^2 - 4c_2(V) = -c. \]

So \( \tau \) defines a \( c \)-wall. Hence \( H_2 \) is in the chamber \( C \) as well as on the \( c \)-wall \( W^\tau \), a contradiction.

4. Variation of \( \mathcal{M}_H \) for different polarizations

Let \( C \) and \( C' \) be two chambers with a common face \( F \subset W^\tau \). In this section, we will compare the moduli spaces \( \mathcal{M}_H, \mathcal{M}_{H'} \) and \( \mathcal{M}_{H_0} \) where \( H \in C, H' \in C' \), and \( H_0 \in F \).

**Definition 4.1.** \( V \) is universally stable (semi-stable) if \( V \) is stable (semi-stable) with respect to any polarization.

In this section, we will investigate what kind of \( H \)-stable vector bundles are not \( H_0 \)-semi-stable and so on. We will have a series of propositions of similar nature.

**Proposition 4.2.** Let \( V \) be a \( H \)-semi-stable sheaf of rank 2. Suppose that \( V \) is not universally semi-stable. Then \( V \) must be \( H \)-stable. In another word, every semi-stable sheaves in \( \mathcal{M}_H \) is \( H \)-stable unless it is universally semi-stable.

**Proof.** Suppose \( V \) is \( H \)-semi-stable, only two cases may happen: \( V \) is M-T \( H \)-stable, or \( V \) is strictly M-T \( H \)-semi-stable. If \( V \) is M-T \( H \)-stable, then it must be \( H \)-stable. So assume that \( V \) is strictly M-T \( H \)-semi-stable. Then there exists a subsheaf \( L \subset V \) such that \( V \) sits in the exact sequence (1) with \( 2c_1(L) = c_1(V) \cdot H \). If \( \tau = 2c_1(L) - c_1(V) \) is not numerically trivial, then by Hodge index theorem, \( \tau^2 < 0 \) and \( \tau^2 \geq c_1(V)^2 - 4c_2(V) = -c \). Hence \( \tau \) defines a \( c \)-wall and \( H \) lies on the wall. This contradicts to the assumption that \( H \in C \).

Hence we must have \( 2c_1(L) - c_1(V) \) is numerically trivial. Therefore \( (2c_1(L) - c_1(V)) \cdot H = 0 \) for any ample divisor \( \tilde{H} \), in particular, for the ample divisor \( H \). Since \( V \) is \( H \)-semi-stable, we must have \( 2\chi(L) \leq \chi(V) \). From here, we will show that \( V \) is \( H \)-semi-stable for any \( \tilde{H} \).

In fact, assume that \( M \) is a subsheaf of \( V \). If \( M \) is a subsheaf of \( L \), then \( c_1(M) \cdot \tilde{H} \leq c_1(L) \cdot \tilde{H} \) and \( \chi(M) \leq \frac{1}{2} \chi(V) \). Otherwise, \( M \) admits an injection \( M \lhook\joinrel\rightarrow L' \). Either \( c_1(L') - c_1(M) \) is effective, or \( c_1(L') = c_1(M) \). In the first case, we obtain \( c_1(M) \cdot \tilde{H} < c_1(L) \cdot \tilde{H} = \frac{1}{2} c_1(V) \cdot \tilde{H} \). In the latter case, i.e., \( c_1(L') = c_1(M) \), we take double dual of the exact sequence (1), we get

\[ 0 \rightarrow L^{**} \rightarrow V^{**} \rightarrow L'^{**}I_Z \rightarrow 0, \]

\[ V^{**} \leftrightarrow M^{**} = L'^{**}. \]

Hence \( \ell(Z) = 0 \) and the exact sequence (1) splits, i.e.

\[ V^{**} = L'^{**} \oplus L^{**}. \]

Therefore, the exact sequence (1) splits, i.e. \( V = L \oplus L' \). Since \( 2c_1(L) \cdot H = c_1(V) \cdot H = 2c_1(L') \cdot \tilde{H} \) and \( V \) is \( H \)-semi-stable, we must have \( 2\chi(L) = \chi(V) = 2\chi(L') \).

Hence \( 2c_1(M) \cdot \tilde{H} = c_1(V) \cdot \tilde{H} \) and \( 2\chi(M) \leq 2\chi(L') = \chi(V) \). This implies that \( V \)
is $\tilde{H}$-semi-stable. That is, $V$ is a universally semi-stable sheaf. But this contradicts to the assumption that $V$ is not universally semi-stable. 

**Remark 4.3.** The argument in the proof above to show that the exact sequence (1) splits will be used (or referred) later on.

**Corollary 4.4.** Suppose $2c_1(L) - c_1(V)$ is numerically trivial, then any non-splitting exact sequence (1) gives a universally semi-stable sheaf provided $2\chi(L) \leq \chi(V)$.

**Proposition 4.5.** Let $V$ be strictly $\tilde{H}$-semi-stable sitting in an exact sequence (1) with $2c_1(L) \cdot \tilde{H} = c_1(V) \cdot \tilde{H}$ and $2\chi(L) = \chi(V)$.

(i) If the exact sequence (1) doesn’t split, then the subsheaf $L$ satisfying $2c_1(L) \cdot \tilde{H} = c_1(V) \cdot \tilde{H}$ and $2\chi(L) = \chi(V)$ is unique.

(ii) Any $V$ sitting in (1) satisfying $2c_1(L) \cdot \tilde{H} = c_1(V) \cdot \tilde{H}$ and $2\chi(L) = \chi(V)$ is strictly $\tilde{H}$-semi-stable.

**Proof.** We only need to show the uniqueness of $L$.

Suppose otherwise, we have two exact sequences

$$0 \to L \to V \to L' \to 0,$$

$$0 \to M \to V \to M' \to 0$$

satisfying $2c_1(L) \cdot \tilde{H} = 2c_1(M) \cdot \tilde{H} = c_1(V) \cdot \tilde{H}$, $2\chi(L) = 2\chi(M) = \chi(V)$ and $M \neq L$.

If $M$ is a subsheaf of $L$, since $\chi(L) = \chi(M)$ and $c_1(L) \cdot \tilde{H} = c_1(M) \cdot \tilde{H}$, then $L = M$, a contradiction.

Hence $M$ admits an injection into $L'$. By the similar argument as mentioned in Remark 4.3, the exact sequence splits: $V = L \oplus L'$ and $M = L'$. But we have assume that the sequence (1) does not split. 

From now on, we will be mainly concentrating on non-universally semi-stable sheaves.

**Theorem 4.6.** Let $V$ be a non-universally semi-stable sheaf of rank 2. Assume $V$ is $H_0$-stable. Then one of the following holds.

(i) If $V$ is $M$-$T$ $H_0$-stable, then $V$ is $M$-$T$ $H'$-stable as well as $M$-$T$ $H'$-stable. In particular, $V$ is $H$-stable as well as $H'$-stable.

(ii) If $V$ is not $M$-$T$ $H_0$-stable, then $V$ is either $H$-stable or $H'$-stable, but cannot be both.

**Proof.** Since $V$ is $H_0$-stable, then for subsheaf $L \subset V$ with torsion free cokernel, there exists an exact sequence (1) such that either $2c_1(L) \cdot H_0 < c_1(V) \cdot H_0$ or $2c_1(L) \cdot H_0 = c_1(V) \cdot H_0$ and $2\chi(L) < \chi(V)$.

If $(2c_1(L) - c_1(V)) \cdot H_0 < 0$, then $(2c_1(L) - c_1(V)) \cdot H < 0$ and $(2c_1(L) - c_1(V)) \cdot H' < 0$.

Otherwise there would exist a c-wall separating $H_0$ with $H$ or $H_0$ with $H'$.

If $2c_1(L) \cdot H_0 = c_1(V) \cdot H_0$, then $2\chi(L) < \chi(V)$. Since we assumed that $V$ is not universally stable, hence $2c_1(L) - c_1(V)$ is not numerically trivial. Therefore $2c_1(L) - c_1(V)$ defines the c-wall where $H_0$ lies. Hence

either $(2c_1(L) - c_1(V)) \cdot H > 0$ or $(2c_1(L) - c_1(V)) \cdot H < 0$.

Since $V$ is $H_0$-stable, the exact sequence (1) doesn’t split and subsheaf $L$ satisfying $2c_1(L) \cdot H_0$ is unique.
Assume \((2c_1(L) - c_1(V)) \cdot H < 0\). For any subsheaf \(M\) of \(V\), if \(M\) is a subsheaf of \(L\), we have \(2c_1(M) \cdot H < c_1(V) \cdot H\). Otherwise, \(M\) admits an injection into \(L'\).

If \(c_1(L') - c_1(M)\) is an effective divisor, then \(2c_1(M) \cdot H_0 < 2c_1(L') \cdot H_0 = c_1(V) \cdot H_0\). Hence \((2c_1(M) - c_1(V)) \cdot H_0 < 0\). Therefore \((2c_1(M) - c_1(V)) \cdot H < 0\).

If \(c_1(L') = c_1(M)\), then by the argument mentioned in Remark 4.3, the exact sequence \(\text{(1)}\) splits. Since \(V\) is \(H_0\)-stable, we get a contradiction. Hence \(2c_1(M) \cdot H \cdot < c_1(V) \cdot H\). Therefore \(V\) is \(M\)-\(T\) \(H\)-stable.

Assume \((2c_1(L) - c_1(V)) \cdot H > 0\), then \((2c_1(L) - c_1(V)) \cdot H' < 0\), hence by the similar argument \(V\) is \(M\)-\(T\) \(H'\)-stable.

The proof of (i) and (ii) will follow easily. For example, for (ii), if \(V\) is not \(M\)-\(T\) \(H_0\)-stable, there will exist subsheaf \(L \subset V\) such that \(2c_1(L) \cdot H_0 = c_1(V) \cdot H_0\). Hence if \(2c_1(L) \cdot H < c_1(V) \cdot H\), then \(2c_1(L) \cdot H' > c_1(V) \cdot H'\). In other words, if \(V\) is \(H\)-stable, then \(V\) cannot be \(H'\) stable and vice-versa.

**Theorem 4.7.** Let \(V\) be a sheaf of rank 2 which is not universally semi-stable. Assume that \(V\) is strictly \(H_0\)-semi-stable and sits in the non-splitting exact sequence \(\text{(1)}\). Then \(V\) is either \(H\)-stable or \(H'\)-stable, but can not be both. If the exact sequence \(\text{(1)}\) splits, then \(V\) is neither \(H\)-stable nor \(H'\)-stable.

**Proof.** Assume that \(V\) is not \(H\)-stable nor \(H'\)-stable. Then there exist two exact sequences

\[
0 \rightarrow N \rightarrow V \rightarrow N' \rightarrow 0
\]

\[
0 \rightarrow M \rightarrow V \rightarrow M' \rightarrow 0
\]

such that \(2c_1(N) \cdot H \geq c_1(V) \cdot H\) and \(2c_1(M) \cdot H' \geq c_1(V) \cdot H'\). Hence

\[
2c_1(N) \cdot H_0 \geq c_1(V) \cdot H_0 \quad \text{and} \quad 2c_1(M) \cdot H_0 \geq c_1(V) \cdot H_0
\]

\[
(7)
\]

Since \(V\) is not universally semi-stable, it is easy to show that \(2c_1(N) \cdot H > c_1(V) \cdot H\) and \(2c_1(M) \cdot H' > c_1(V) \cdot H'\). Since \(V\) is strictly \(H_0\)-semi-stable, we must have

\[
2c_1(N) \cdot H_0 \leq c_1(V) \cdot H_0 \quad \text{and} \quad 2c_1(N) \cdot H_0 \leq c_1(V) \cdot H_0
\]

\[
(8)
\]

Combining (7) and (8), we must have

\[
2c_1(N) \cdot H_0 = c_1(V) \cdot H_0 = 2c_1(M) \cdot H_0.
\]

Hence

\[
2c_1(N) \cdot H' < c_1(V) \cdot H' \quad \text{and} \quad 2c_1(M) \cdot H < c_1(V) \cdot H
\]

\[
(9)
\]

Since \(V\) is not universally stable, we have

\[
2c_1(L) \cdot H \neq c_1(V) \cdot H \quad \text{and} \quad 2c_1(L) \cdot H' \neq c_1(V) \cdot H'
\]

\[
(10)
\]

If \(2c_1(L) \cdot H < c_1(V) \cdot H\), then \(N\) cannot be a subsheaf of \(L\), since otherwise, we would have \(2c_1(L) \cdot H > 2c_1(N) \cdot H > c_1(V) \cdot H2c_1(L) \cdot H\), a contradiction. Hence \(N\) admits an injection to \(L'\). Then by the argument mentioned in Remark 4.3, the exact sequence \(\text{(1)}\) splits, a contradiction.

If \(2c_1(L) \cdot H > c_1(V) \cdot H\), then \(L\) cannot be a subsheaf of \(M\), since otherwise, we would have \(c_1(V) \cdot H > 2c_1(M) \cdot H > 2c_1(L) \cdot H > c_1(V) \cdot H\), a contradiction. Hence \(L\) admits an injection to \(M'\). Then by the argument mentioned in Remark 4.3, the exact sequence \(\text{(1)}\) splits, a contradiction.
Hence we proved that $V$ is either $H$-stable or $H'$-stable. It is easy to see that either $2c_1(L) \cdot H > c_1(V) \cdot H$ which implies that $V$ is not $H$-stable, or $2c_1(L) \cdot H < c_1(V) \cdot H$, equivalently, $2c_1(L) \cdot H' > c_1(V) \cdot H'$, which implies that $V$ is not $H'$-stable.

If the exact sequence (1) splits, by the same argument as in the previous paragraph, $V$ is neither $H$-stable nor $H'$-stable.

Next, we give a criterion for strictly $H_0$-semi-stable sheaves.

**Proposition 4.8.** Assume that $V$ is not universally semi-stable. $V$ is strictly $H_0$-semi-stable iff $V$ sits in the exact sequence (1) where $c_1(L) \cdot H_0 = c_1(L') \cdot H_0$ and $\chi(L) = \chi(L')$.

**Proof.** Easy.

**Definition 4.9.** Let $L$, $L'$ be two rank one torsion free coherent sheaves such that $c_1(L) + c_1(L') = c_1$, $c_1(L) \cdot c_1(L') + c_2(L) = c_2$. If $c_1(L) - c_1(L')$ is numerically trivial, then the pair $(L, L')$ is called $U$-pair. Otherwise, it is called NU-pair.

**Proposition 4.10.** Suppose $(L, L')$ is a NU-pair, $c_1(L) \cdot H_0 = c_1(L') \cdot H_0$ and $2c_1(L) \cdot H < c_1 \cdot H$. Then

(i) every non-splitting exact sequence

$$0 \to L \to V \to L' \to 0$$

(11) gives an $H$-stable sheaf $V$.

(ii) $V$ is $H_0$-stable if $\chi(L) < \chi(L')$;

(iii) $V$ is strictly $H_0$-semi-stable if $\chi(L) = \chi(L')$;

(iv) $V$ is $H_0$-unstable if $\chi(L) > \chi(L')$.

**Proof.** Consider a subsheaf $M \subset V$. If $M$ is a subsheaf of $L$, then

$$2c_1(M) \cdot H \leq 2c_1(L) \cdot H < c_1(V) \cdot H.$$

Otherwise, $M$ admits an injection into $L'$. Hence

$$2c_1(M) \cdot H_0 \leq 2c_1(L') \cdot H_0 = c_1(V) \cdot H_0.$$

If $2c_1(M) \cdot H_0 = c_1(V) \cdot H_0 = 2c_1(L') \cdot H_0$, by the argument mentioned in Remark 4.3, the exact sequence (1) splits, a contradiction.

Hence we must have $2c_1(M) \cdot H_0 < c_1(V) \cdot H_0$. Hence $2c_1(M) \cdot H < c_1(V) \cdot H$. The proves of (ii), (iii) and (iv) are quite straight forward.

**Proposition 4.11.** Assume that $V$ is not universally semi-stable.

(i) If $V$ is $H$-stable and strictly $M$-$T$ $H_0$-semi-stable sitting in the exact sequence (1) with $c_1(L) \cdot H_0 = c_1(L') \cdot H_0$, then any sheaf $V'$ sitting in the non-splitting exact sequence

$$0 \to L' \to V' \to L \to 0$$

(12) is $H'$-stable.

(ii) If in addition, $V$ is not $H_0$-semi-stable, then $V'$ is $H_0$-semi-stable.
Proof. Since $V$ is not universally semi-stable, the pair $(L, L')$ is an NU-pair. Because $V$ is $H$-stable and strictly $M$-$T$ $H_0$-semi-stable, we obtain $2c_1(L) \cdot H < c_1 \cdot H$ and $2c_1(L) \cdot H_0 = c_1 \cdot H_0$. Thus we get $2c_1(L) \cdot H' > c_1 \cdot H'$, or equivalently, $2c_1(L') \cdot H' < c_1 \cdot H'$. Apply Proposition 4.10 to the pair $(L', L)$, we get the conclusion. \hfill \Box

**Proposition 4.12.** Fix the first Chern class $c_1$ and a $c$-wall $W^c$ where $c = 4c_2 - c_1^2$. For $c_2' \geq c_2$, $W^c$ is also a $c'$-wall where $c' = 4c_2' - c_1^2$. Suppose that $\tau \cdot H_0 = 0$, $\tau \cdot H < 0$ and $\tau + c_1 = 2c_1(L)$ for some line bundle $L$. Then for $c_2' \gg 0$, there exists a $H$-stable sheaf $V$ with $c_1(V) = c_1$ and $c_2(V) = c_2'$ such that it is not $H_0$-semi-stable.

**Proof.** When $c_2(L') \gg 0$, $Ext^1(L', L) \neq 0$. Hence there exists a non-splitting exact sequence

$$0 \to L \to V \to L' \to 0$$

where $L'$ is a rank one subsheaf such that $c_1(L') = c_1 - c_1(L)$ and $c_2(L') = c_2' - c_2(L) - c_1(L) \cdot c_1(L')$. From Proposition 4.10, $V$ is $H$-stable with $c_2(V) = c_2'$ and $c_1(V) = c_1$.

$$2\chi(L) = c_1^2(L) - c_1(L) \cdot K_X + 2\chi(O_X),$$

$$\chi(V) = \frac{c_1^2 - c_1 \cdot K_X}{2} - c_2' + 2\chi(O_X).$$

If $c_2' \gg 0$, then $2\chi(L) > \chi(V)$. Since $(2c_1(L) - c_1(V)) \cdot H_0 = 0$, hence $V$ is not $H_0$-semi-stable. \hfill \Box

In some propositions above, one may have noticed that we have used the term “non-splitting exact sequence” several times. Suppose we have an exact sequence \([\mathcal{F}]\) with $c_1(L) \cdot H_0 = c_1(L') \cdot H_0$ and $\chi(L) = \chi(L')$, then $V$ is strictly $H_0$-semi-stable and is $s$-equivalent to $gr(V) = L \oplus L'$. However, if $(L, L')$ is a NU-pair and if $V = L \oplus L'$, then $V$ is neither $H$-stable nor $H'$-stable. Therefore if $Ext^1(L', L) = 0$, then there would exist a class in $\mathcal{M}_{H_0}$ represented by $V = L \oplus L'$ such that $V$ is not $s$-equivalent to any $H_0$-semi-stable sheaf which is $H$-semi-stable, nor is it $s$-equivalent to any $H_0$-semi-stable sheaf which is $H'$-semi-stable. In the following, we are going to show that the situation above cannot happen. This fact guarantees (ii) of our main theorem in the next section.

Let $\mathcal{F}$ and $\mathcal{F}'$ be two torsion free coherent sheaves. Define (see [14])

$$\chi(\mathcal{F}', \mathcal{F}) = \sum_{i=0}^{2} \dim Ext^i(\mathcal{F}', \mathcal{F}).$$

**Proposition 4.13.**

$$\chi(\mathcal{F}', \mathcal{F}) = ch(\mathcal{F}')^* \cdot ch(\mathcal{F}) \cdot td(X)_{H^*(X; \mathbb{Z})}$$

where $*$ acts on $H^{2i}(X; \mathbb{Z})$ by $(-1)^i \cdot Id$.

**Corollary 4.14.** Let $L$ and $L'$ be two torsion free rank one sheaves. Let $\tau = c_1(L) - c_1(L')$. Let $V$ sit in the exact sequence \([\mathcal{F}]\). Then

$$\chi(L', L) = \frac{\tau^2}{4} - \frac{K_X \cdot \tau}{2} - c_2(V) + \frac{c_1^2(V)}{4} + \chi(O_X)$$

(13)
Proof. From the exact sequence (1), we get \( \tau = 2c_1(L) - c_1(V) \) and \( c_2(L) + c_2(L') = c_2(V) - c_1(L) \cdot c_1(L') \).

\[
\begin{align*}
\chi(L', L) &= ch(L')^* \cdot ch(L) \cdot td(X) \\
&= (1 - c_1(L') + \frac{c_1(L')^2}{2} - c_2(L')) \cdot (1 + c_1(L) + \frac{c_1(L)^2}{2} - c_2(L)) \\
&= (1 + c_1(L) - c_1(L') \cdot c_1(L') + \frac{c_1(L)^2}{2} + \frac{c_1(L')^2}{2} - c_2(L) - c_2(L')) \\
&= \frac{c_1(L) - c_1(L')}{2} - \frac{K_X \cdot (c_1(L) - c_1(L'))}{2} - c_2(L) - c_2(L') + \chi(\mathcal{O}_X)
\end{align*}
\]

where \( d = 4c_2(V) - c_1(V)^2 - 3\chi(\mathcal{O}_X) \).

\[ \square \]

**Proposition 4.15.** With assumption on \( L \) and \( L' \) as in Proposition 4.14. In addition, assume that \( \tau \cdot H_0 = 0 \) and \( d \), which is the virtual dimension of the moduli space, is non-negative, then

\[-\chi(L, L') - \chi(L', L) > 0.\]

**Proof.** Since \( (L, L') \) is a NU-pair, \( \tau \) is not numerically trivial, hence \( \tau^2 < 0 \). From Corollary 4.14,

\[-\chi(L, L') - \chi(L', L) = \frac{d}{2} + \frac{-\tau^2 - \chi(\mathcal{O}_X)}{2} = \frac{d}{2} + \frac{-\tau^2 - 1 + q}{2} \geq \frac{d}{2} \geq 0.
\]

Suppose \( -\chi(L, L') - \chi(L', L) = 0 \), then \( q = 0, \tau^2 = -1, 4c_2(V) - c_1^2(V) = 3. \)

From the exact sequence (1), we get

\[-\tau^2 + 4c_2(L) + 4c_2(L') = 4c_2(V) - c_1^2(V) = 3.
\]

Hence \( 1 + 4(c_2(L) + c_2(L')) = 3, \) or \( 4(c_2(L) + c_2(L')) = 2, \) impossible. \[ \square \]

**Corollary 4.16.** \( \dim Ext^1(L, L') + \dim Ext^1(L', L) > 0. \) And there either exists a non-splitting exact sequence

\[ 0 \to L \to V \to L' \to 0, \]

or a non-splitting exact sequence

\[ 0 \to L' \to V \to L \to 0. \]

**Proof.** Easy consequence of Proposition 4.17. \[ \square \]
Corollary 4.17. Suppose \((L, L')\) is a NU-pair satisfying \(c_1(L) \cdot H_0 = c_1(L') \cdot H_0\), and \(\chi(L) = \chi(L')\), then there exists a non-splitting exact sequence

\[0 \to L \to V \to L' \to 0\quad \text{or} \quad 0 \to L' \to V \to L \to 0\]

such that \(V\) is strictly \(H_0\)-semi-stable.

Proof. An easy consequence of Corollary 4.16 and Proposition 4.8. \(\Box\)

5. Canonical rational morphisms among the Gieseker compactifications

In this section, we shall draw some conclusions on the variations of Gieseker compactifications following many discussions in the previous section.

Again, we place ourselves in the following situation. Let \(C\) and \(C'\) be two chambers with a common face \(F \subset W_\tau\). We assume that \(C\) is the upper chamber and \(C'\) is the lower chamber with respect to \(\tau\). That is, \(C \cdot \tau > 0\) and \(C' \cdot \tau < 0\).

We will derive some canonical morphisms among the moduli spaces \(\mathcal{M}_H, \mathcal{M}_{H'}\), and \(\mathcal{M}_{H_0}\) where \(H \in C, H' \in C',\) and \(H_0 \in F\).

Theorem 5.1. There are two canonical rational algebraic maps

\[\mathcal{M}_H \xrightarrow{\varphi} \mathcal{M}_{H_0} \xrightarrow{\psi} \mathcal{M}_{H'}\]

with the following properties:

(i) \(\varphi\) and \(\psi\) are isomorphisms over \(\mathcal{M}_{H_0}' = \mathcal{M}_{H_0}\).

(ii) \(\text{Im} \varphi \cup \text{Im} \psi = \mathcal{M}_{H_0}\).

(iii) If \(V \in \mathcal{M}_{H_0}\) is universally semi-stable, then the inverse image \(\varphi^{-1}(\text{gr}\, V)\) or \(\psi^{-1}(V)\) consists of a single point. The same conclusion holds for \(\psi\).

(iv) If \(V \in \mathcal{M}_{H_0}\) is not universally semi-stable and is \(H_0\)-stable, then the inverse image \(\varphi^{-1}(V)\) or \(\psi^{-1}(V)\) consists of a single point or is an empty set; More precisely, if \(V \in \mathcal{M}_{H_0}\) is strictly \(H_0\)-semi-stable, then \(V\) sits in the exact sequence \(\mathcal{E}\);

(a) If \(V\) is \(MT\) \(H_0\)-stable, then \(\varphi^{-1}(V)\) or \(\psi^{-1}(V)\) consists of a single point;

(b) If \(\chi(L) > \chi(L')\), \(\varphi\) is not defined over \(\mathbb{P}(\text{Ext}^1(L, L')) \subset \mathcal{M}_H\), but \(\psi\) sends \(\mathbb{P}(\text{Ext}^1(L, L')) \subset \mathcal{M}_{H_0}\) into \(\mathcal{M}_{H'}\).

(b2) If \(\chi(L) < \chi(L')\), \(\psi\) is not defined over \(\mathbb{P}(\text{Ext}^1(L, L')) \subset \mathcal{M}_{H_0}\), but \(\varphi\) sends \(\mathbb{P}(\text{Ext}^1(L, L')) \subset \mathcal{M}_{H_0}\) into \(\mathcal{M}_H\).

(v) If \(V \in \mathcal{M}_{H_0}\) is not universally semi-stable and is strictly \(H_0\)-semi-stable, then \(V\) sits in the exact sequence \(\mathcal{E}\) with \(\chi(L) = \chi(L')\), the inverse image of \(\text{gr}\, V \in \mathcal{M}_{H_0}\) by \(\varphi\) or \(\psi\) is \(\mathbb{P}(\text{Ext}^1(L, L'))\), and the inverse image of \(\text{gr}\, V \in \mathcal{M}_{H_0}\) by \(\psi\) is \(\mathbb{P}(\text{Ext}^1(L', L'))\).

Proof. First of all, for a given sheaf \(V\) in \(\mathcal{M}_H\), if \(V\) is also a semi-stable sheaf with respect to \(H_0\), then we can define a map which sends \(V \in \mathcal{M}_H\) (or \(\text{gr}\, V\)) to \(V\) (or \(\text{gr}\, V\)) as a point in \(\mathcal{M}_{H_0}\). It is easy to see that this gives rise to a well-defined map \(\varphi\) from a Zariski open subset of \(\mathcal{M}_H\) to \(\mathcal{M}_{H_0}\). Obviously, \(\varphi\) is defined and restricts to the identity over \(\mathcal{M}_{H_0}' = \mathcal{M}_{H_0}\). It remains to show the algebraicity of the map \(\varphi\). The proof is a standard one. So we only brief it. Recall from the construction of the moduli space \(\mathcal{M}_H\) (see \[\text{[3]}\]), \(\mathcal{M}_H\) is the quotient of \(\mathcal{Q}_H^H\) by the group \(\text{PGL}(N)\) (we adopt the notations from \[\text{[3]}\]). By the universality of the quotient scheme, there is a universal quotient sheaf \(\mathcal{F}\) over \(X \times \mathcal{Q}_H^H\) with the usual property. Now by the axiom of the coarse moduli, there is a rational map from \(\mathcal{Q}_H^H\) to \(\mathcal{M}_{H_0}\). Clearly this map respects the group action (send an orbit of \(\text{PGL}(N)\)
to a point by Proposition 4.2), thus by passing to the quotient, we get a rational
map from \( \mathcal{M}_H \) to \( \mathcal{M}_{H_0} \), and this map is by definition the map \( \varphi \). Hence \( \varphi \) is a
morphism.

The other map \( \psi \) can be treated similarly.

Property (i) and (iii) follows immediately from the above explanation.

(iv) and (v) follow as consequences of Proposition 4.10 and Proposition 4.11.

To prove (ii), take a \( H_0 \)-semi-stable sheaf \( V \).

If \( V \) is universally semi-stable, the conclusion follows by definition.

If \( V \) is not universally semi-stable but is \( H_0 \)-stable, then the conclusion follows
from Proposition 4.6.

If \( V \) is not universally semi-stable and is strictly \( H_0 \)-semi-stable, then the con-
clusion follows from Corollary 4.17 and Proposition 4.7.

Proposition 5.2. Let the situation be as in Theorem 5.1. If \( c_2 \gg 0 \), then the
map \( \varphi: \mathcal{M}_H(c_1, c_2) \rightarrow \mathcal{M}_{H_0}(c_1, c_2) \) and \( \psi: \mathcal{M}_H'(c_1, c_2) \rightarrow \mathcal{M}_{H_0}(c_1, c_2) \) are
genuine rational maps (in the sense that they can not be extended to everywhere).

Proof. It follows from Proposition 4.12.

6. From Gieseker’s compactification to Uhlenbeck’s compactification

In this section, we will study the Uhlenbeck compactification of moduli spaces
using its relation with the Gieseker compactification. We will use a technique
established by Jun Li [8] where he compared the Gieseker compactification and the
Uhlenbeck compactification. We assume that \( q = 0 \) and \( c_1 = 0 \) through out this
section. Our analysis relies heavily on the results of Jun Li [8].

Following the notations in [8], let \( H \) be an ample divisor and \( g \) the corresponding
Hodge metric on \( X \). We use \( \mathcal{N}_H(j) \) to represent the moduli space of ASD connections,
with respect to the Riemannian metric \( g \), on an \( SU(2) \) principal bundle \( P \)
over \( X \) with \( c_2(P) = j \), and \( \mathcal{N}_H(j) \) to represent the moduli space of irreducible
ASD connections. \( \mathcal{N}_H(j) \) is known by a Donaldson’s theorem to be homeomorphic
to the moduli space of Mumford-Takemoto \( H \)-stable vector bundles with \( c_1 = 0 \) and
\( c_2 = j \). We adopt the notation \( \overline{\mathcal{N}}_H(c_2) \) to represent the Uhlenbeck compactification.

Uhlenbeck compactification theorem tells us that \( \overline{\mathcal{N}}_H(c_2) \) is a closed subset of
\( \prod_{j=0}^{c_2} \mathcal{N}_H(j) \times Sym^{c_2-j}(X) \). However, we didn’t know whether \( \overline{\mathcal{N}}_H(c_2) \) is the total
space. The main conclusion (Theorem 6.1) of this section will give a complete
answer to this question.

In what follows, we shall quote a useful theorem proved by J. Li (cf. Theorem
0.1, [8]).

Theorem 6.1. [8] There is a complex structure on \( \overline{\mathcal{N}}_H(c_2) \) making it a reduced
projective scheme. Furthermore, if we let \( \mathcal{M}_H^\mu(c_2) \) be the open subset of \( \mathcal{M}_H(c_2) \)
consisting of locally free \( M\bar{T} \) \( H \)-stable sheaves and let \( \overline{\mathcal{M}}_H^\mu(c_2) \) be the closure of
\( \mathcal{M}_H^\mu(c_2) \) in \( \mathcal{M}_H(c_2) \) endowed with reduced scheme structure, then there is a mor-
phism

\[ \gamma: \overline{\mathcal{M}}_H^\mu(c_2) \rightarrow \overline{\mathcal{N}}_H(c_2) \]
extending the homeomorphism between the set of $M$-$T$ $H$-stable rank two vector bundles and the set of gauge equivalent classes of irreducible ASD connections with fixed Chern classes.

It is known that when $c_2$ is large enough, $\mathcal{M}_H(c_2)$ is irreducible ([7]) and thus $\overline{\mathcal{M}}_H(c_2) = \mathcal{M}_H(c_2)$ and $\gamma(\mathcal{M}_H(c_2)) = \gamma(\overline{\mathcal{M}}_H(c_2))$. In §5 of [8], a continuous map

$$\overline{\sigma} : \gamma(\overline{\mathcal{M}}_H(c_2)) \to \prod_{j=0}^{c_2} \mathcal{N}_H(j) \times \text{Sym}^{c_2-j}X$$

is defined and $\overline{\sigma}$ identifies $\gamma(\overline{\mathcal{M}}_H(c_2))$ isomorphically with the Uhlenbeck compactification $\overline{\mathcal{N}}_H(c_2) = \prod_{j=0}^{c_2} \mathcal{N}_H(j) \times \text{Sym}^{c_2-j}X$. We will use $\overline{\sigma}$ to stand for the map from $\gamma(\mathcal{M}_H(c_2))$ to $\prod_{j=0}^{c_2} \mathcal{N}_H(j) \times \text{Sym}^{c_2-j}X$ defined in the proof of Theorem 5 in [8]. If $\mathcal{M}_H(c_2)$ is normal, then this map $\overline{\sigma}$ is simply the map $\overline{\sigma}$. Otherwise, $\overline{\sigma}$ is an extension of $\overline{\sigma}$.

The definition of $\overline{\sigma}$ and $\overline{\sigma}$ are given in J. Li’s paper [8]. It is recommended that the reader consult J. Li’s paper to get familiar with these maps since in this and next section, we make use of these maps a lot.

**Remark 6.2.** In the rest of the paper, rather than directly working on the Uhlenbeck compactification $\overline{\mathcal{N}}_H(c_2)$, we will be working on $\gamma(\mathcal{M}_H(c_2))$ instead. One should keep in mind that $\gamma(\mathcal{M}_H(c_2))$ can be identified via $\overline{\sigma}$ with the Uhlenbeck compactification $\overline{\mathcal{N}}_H(c_2)$ when $\mathcal{M}_H(c_2)$ is irreducible (and this can be ensured by requiring $c_2$ to be large [7]). For small $c_2$, $\overline{\mathcal{N}}_H(c_2)$ is contained in $\gamma(\mathcal{M}_H(c_2))$ via the identification with $\gamma(\overline{\mathcal{M}}_H(c_2))$. In this case, $\gamma(\mathcal{M}_H(c_2))$ is slightly larger than $\mathcal{M}_H(c_2)$.

**Notation 6.3.** Let $Z$ be a zero-cycle. $\text{red}(Z)$ will be the reduced scheme with multiplicity counted at each point.

We are going to prove the following proposition.

**Proposition 6.4.** Assume that $H$ is an ample divisor away from $c$-walls.

(i.) If $c_2 \geq 2$, then $\text{Im} \overline{\sigma} = \prod_{j=0}^{c_2} \mathcal{N}_H(j) \times \text{Sym}^{c_2-j}X$.

(ii.) If $c_2 = 1$, then $\text{Im} \overline{\sigma} = \mathcal{N}_H(1)$. In particular, $\mathcal{N}_H(1)$ is compact.

In order to prove the proposition, we divide into several lemmas.

**Lemma 6.5.** Suppose $V \in \mathcal{M}_H(c_2)$ is strictly $M$-$T$ $H$-semi-stable, then $V$ sits in the exact sequence

$$0 \to I_Z \to V \to I_{Z'} \to 0$$

for some zero-cycles $Z$ and $Z'$ such that

$$\overline{\sigma}(\gamma(V)) = (\mathcal{O}_X \oplus \mathcal{O}_X, \text{red}(Z \cup Z'))$$

**Proof.** Since $V$ is strictly $M$-$T$ $H$-semi-stable, hence there exist torsion free coherent sheaves of rank one $L$ and $L'$ such that $V$ sits in the exact sequence

$$0 \to L \to V \to L' \to 0$$

with $c_1(L) \cdot H = 0$. 
Since $H$ is away from c-walls, $c_1(L)$ is the trivial divisor by Hodge index theorem. Hence we have the exact sequence (14).

By the proof of Lemma 3.3. in [1], we get $I_Z \oplus I_{Z'} \in \Gamma(\gamma(V))$. Therefore

$$\overline{\gamma}(V) = (\mathcal{O}_X \oplus \mathcal{O}_X, \text{red}(Z \cup Z')) \in \mathcal{M}_H(0) \times \text{Sym}^{c_2}(X).$$

Note that $\mathcal{M}_H(0)$ consists of a single point represented by $\mathcal{O}_X \oplus \mathcal{O}_X$.

**Remark 6.6.** Due to the same reason, the universally semi-stable sheaves can only be sheaves sitting in the exact sequence (14).

**Lemma 6.7.** Assume that $c_2 \geq 2$. For any point $(\mathcal{O}_X \oplus \mathcal{O}_X, x)$ in $\mathcal{M}_H(0) \times \text{Sym}^{c_2}(X)$, choose a zero-cycle $Z$ of length $c_2$ such that $\text{red}(Z) = x$. Then there exists a non-splitting exact sequence

$$0 \to I_Z \to V \to \mathcal{O}_X \to 0 \quad (15)$$

such that $V$ is $H$-semi-stable.

**Proof.** Let’s calculate $\dim \text{Ext}^1(\mathcal{O}_X, I_Z) = h^1(I_Z)$.

$$h^1(I_Z) = -\chi(I_Z) + h^0(I_Z) + h^2(I_Z) \geq -\chi(I_Z) = -(c_2 + 1) \geq 1.$$ 

Hence there exists a non-splitting exact sequence (15).

**Claim:** $V$ is $H$-semi-stable.

In fact, let $M$ be a rank one subsheaf of $V$. Notice that

$$2\chi(I_Z) = 2(-\ell(Z) + 1) = -2\ell(Z) + 2 = -2c_2 + 2 < -c_2 + 2 = \chi(V).$$

If $M$ is a subsheaf of $I_Z$, then $c_1(M) \cdot H \leq 0$ and $2\chi(M) \leq 2\chi(I_Z) < \chi(V)$.

Otherwise $M$ is a subsheaf of $\mathcal{O}_X$. Hence $c_1(M) \cdot H \leq 0$. If $c_1(M) \cdot H = 0$, again $c_1(M)$ has to be the trivial divisor. Hence the exact consequence (15) splits, a contradiction. Therefore $c_1(M) \cdot H < 0$.

Thus, we proved the claim.

**Corollary 6.8.** Assume that $H$ is an ample divisor away from c-walls. Then

$$\text{Im}(\overline{\gamma}) \supset \mathcal{M}_H(0) \times \text{Sym}^{c_2}(X).$$

**Lemma 6.9.** Given any element $(A, x) \in \mathcal{N}_H(j) \times \text{Sym}^{c_2-j}X$ where $j > 0$, $A$ is an irreducible ASD corresponding to a $M$-$T$ $H$-stable bundle $V_j$. In particular

$$\mathcal{N}_H(j) = \overline{\mathcal{N}_H(j)}.$$

Choose a zero-cycle $Z$ of length $\ell(Z) = c_2 - j$ such that $\text{red}(Z) = x$. Then the elementary transformation $V$ in

$$0 \to V \to V_j \to \mathcal{O}_Z \to 0$$

is an $M$-$T$ $H$-stable bundle in $\mathcal{M}_H(c_2)$.

**Proof.** Any reducible ASD in $\mathcal{N}_H(j)$ takes form of $L \oplus L^{-1}$ where $L$ is a line bundle with $c_1(L) \cdot H = 0$ and $c_1(L)^2 = -j < 0$. Since $H$ is away from walls, such $L$ doesn’t exist. Hence $A$ corresponds to $M$-$T$ $H$-stable bundle. The second statement is clear.
Corollary 6.10. With assumption and notations as in Lemma 6.9, then \( \tilde{\sigma}(V) = (A, x) \).

Proof. \( V^{**} = V_j \). By definition of \( \tilde{\sigma} \), \( \tilde{\sigma}(V) = (A, x) \).

Combination of Corollary 6.8 and Corollary 6.10 gives a proof of (i) of Proposition 6.4.

Lemma 6.11. Assume \( c_2 = 1 \). Then there is no \( H \)-semi-stable sheaf \( V \) which is strictly \( M \)-\( T \) \( H \)-semi-stable.

Proof. Suppose that \( V \) is strictly \( M \)-\( T \) \( H \)-semi-stable, by Lemma 6.5, \( V \) sits in the exact sequence (14) with \( \ell(Z) + \ell(Z') = c_2 = 1 \). Since \( V \) is also \( H \)-semi-stable, \( 2\chi(I_Z) = -2\ell(Z) + 2 \leq \chi(V) = 1 \). Hence \( \ell(Z) \) has to be one and \( \ell(Z') = 0 \). However,

\[
\dim\text{Ext}^1(O_X, I_Z) = h^1(I_Z) + h^2(I_Z) - \chi(I_Z) = h^2(O_X) - (-\ell(Z) + 1) = p_g = 0.
\]

Hence the exact sequence (1) splits, i.e. \( V = O_X \oplus I_Z \). Since \( 2\chi(O_X) = 2 > \chi(V) = 1 \), \( V \) will not be \( H \)-semi-stable, a contradiction.

This lemma proves (ii) of Proposition 6.4. Hence we finished the proof of Proposition 6.4.

In the following, we consider the case where our polarization is on a face.

Lemma 6.12. Suppose \( H_0 \) is an ample divisor on a face. Suppose \( V \in \mathcal{M}_{H_0}(c_2) \) is strictly \( M \)-\( T \) \( H_0 \)-semi-stable, then either \( V \) sits in (14) and hence satisfies the conclusion of Lemma 6.3, or \( V \) sits in the exact sequence

\[
0 \to L \otimes I_Z \to V \to L^{-1} \otimes I_Z' \to 0
\]

such that \( c_1(L) \cdot H = 0 \) and \( L \) is not the trivial line bundle. Moreover,

\[ \tilde{\sigma}(\gamma(V)) = (L \oplus L^{-1}, \text{red}(Z \cup Z')). \]

Proof. The same as the proof of Lemma 6.3.

For the first case in Lemma 6.12, Lemma 6.8 Lemma 6.7 and Lemma 6.9 still hold with some minor modifications. Let’s prove the following lemma which deals with the latter case in Lemma 6.12.

Lemma 6.13. Without loss of generality, let’s assume \( c_1(L) \cdot K_X \geq 0 \). For any point \((L \oplus L^{-1}, x)\) in \( \tilde{\mathcal{N}}_{H_0} \times \text{Sym}^{c_2-1}(X) \) where \( 0 < j = -c_1(L)^2 \), choose a zero-cycle \( Z \) of length \( c_2 - j \) such that \( \text{red}(Z) = x \). Then there exists an \( H_0 \)-semi-stable sheaf \( V \) in the non-splitting exact sequence

\[
0 \to L \otimes I_Z \to V \to L^{-1} \to 0.
\]

Proof. The proof is similar to that of Lemma 6.7:

\[
\dim\text{Ext}^1(L^{-1}, L \otimes I_Z) = h^1(L^{\otimes 2} \otimes I_Z) \geq -\chi(L^{\otimes 2} \otimes I_Z) = -2c_1(L) \cdot (2c_1(L) - K_X) - \ell(Z) + 1
\]

\[
= -2c_1^2(L) + c_1(L) \cdot K_X + \ell(Z) - 1
\]

\[
= c_2 + c_1(L) \cdot K_X + j - 1 \geq c_2 > 0.
\]

Hence there exists a non-splitting exact sequence (17).
Notice that

\[ 2\chi(L \otimes I_Z) = 2\left(\frac{c_1(L)^2 - c_1(L) \cdot K_X}{2} - \ell(Z) + 1\right) = -j - c_1(L) \cdot K_X - 2\ell(Z) + 2 = -c_2 + 2 - c_1(L) \cdot K_X - \ell(Z) \leq -c_2 + 2 = \chi(V). \]

Let’s prove that \( V \) is \( H_0 \)-semi-stable. Let \( M \) be a rank one subsheaf of \( V \). If \( M \) is a subsheaf of \( L \otimes I_Z \), then

\[ c_1(M) \cdot H_0 \leq c_1(L) \cdot H_0 \quad \text{and} \quad 2\chi(M) \leq 2\chi(L \otimes I_Z) \leq \chi(V). \]

Otherwise, \( M \) is a subsheaf of \( L^{-1} \). Hence \( c_1(L^{-1}) - c_1(M) \) is effective or trivial and \( c_1(M) \cdot H_0 \leq c_1(L^{-1}) \cdot H_0 = 0 \). If \( c_1(M) \cdot H_0 = c_1(L^{-1}) \cdot H_0 \), then \( c_1(L^{-1}) - c_1(M) \) is the trivial divisor, hence the exact sequence splits, a contradiction. Therefore, \( c_1(M) \cdot H_0 < c_1(L^{-1}) \cdot H_0 = 0 \).

Hence \( V \) is \( H_0 \)-semi-stable. \( \square \)

By the similar argument, we get the following proposition

**Proposition 6.14.** Assume that \( H_0 \) is an ample divisor on a face.

(i.) If \( c_2 \geq 2 \), then \( \text{Im} \bar{\sigma} = \prod_{j=0}^{c_2} \overline{\mathcal{N}}_{H_0}(j) \times \text{Sym}^{c_2-j}X \).

(ii.) If \( c_2 = 1 \), then \( \text{Im} \bar{\sigma} = \overline{\mathcal{N}}_{H_0}(1) \). In particular, \( \overline{\mathcal{N}}_{H_0}(1) \) is compact.

**Theorem 6.15.** Let \( \overline{H} \) be an arbitrary polarization. If \( \mathcal{M}_{\overline{H}}(c_2) \) is irreducible, then the Uhlenbeck compactification is the total space, i.e.

\[ \overline{\mathcal{N}}_{\overline{H}}(c_2) = \prod_{j=0}^{c_2} \overline{\mathcal{N}}_{\overline{H}}(j) \times \text{Sym}^{c_2-j}(X) \]

when \( c_2 \geq 2 \); When \( c_2 = 1 \), we have

\[ \overline{\mathcal{N}}_{\overline{H}}(1) = \overline{\mathcal{N}}_{\overline{H}}(1). \]

7. **Canonical regular morphisms among the Uhlenbeck compactifications**

In the following, we are going to study the variation of the Uhlenbeck compactifications.

Let’s recall some notations and results of J. Li [8]. Let \( \mathcal{Q}_H \) be the Grothendieck’s quotient scheme parameterizing all quotient sheaves \( F \) of \( \mathcal{O}_X^N \) with \( \det F = H^\oplus 2n \) and \( c_2(F \otimes H^{-n}) = c_2 \). Let \( \mathcal{Q}_H^\ast \subset \mathcal{Q}_H \) be the open set consisting of all \( M \)-T \( H \)-semi-stable quotient sheaves. Let \( \mathcal{Q}_H^{ss} \subset \mathcal{Q}_H \) be the open set consisting of all \( H \)-semi-stable quotient sheaves.

**Remark 7.1.** J. Li constructed a commutative diagram

\[
\begin{array}{ccc}
\mathcal{Q}_H^{ss} \subset \mathcal{Q}_H^\ast & \xrightarrow{\gamma_{\mathcal{Q}_H}} & \mathbb{P}^k \\
\mathcal{M}_H(c_2) \xrightarrow{\pi} & \equiv & \mathbb{P}^k \\
\end{array}
\]

for some projective space \( \mathbb{P}^k \) such that (Lemma 3.2, [8]) \( \gamma_{\mathcal{Q}_H} (\mathcal{Q}_H^\ast \cap \overline{\mathcal{Q}_H}^{ss}) \) is identical to \( \gamma_H (\mathcal{M}_H(c_2)) \) as sets where \( \overline{\mathcal{Q}_H}^{ss} \) is the closure of \( \mathcal{Q}_H^{ss} \) in \( \mathcal{Q}_H \).
Since we are comparing spaces depending on different stability polarizations, we will use subscripts to distinguish different maps for different spaces, for example, $\gamma_H, \sigma_H, Q_H$, etc.

We also assume throughout this section that the moduli spaces $\mathcal{M}_H(c_2)$ and $\mathcal{M}_{H_0}(c_2)$ are normal. For example, when $c_2$ is sufficiently large, $\mathcal{M}_H(c_2)$ and $\mathcal{M}_{H_0}(c_2)$ are both normal, and generic $H$-stable sheaves are $H_0$-stable, too.

Since we assumed that $\mathcal{M}_H(c_2)$ and $\mathcal{M}_{H_0}(c_2)$ are both normal, the map $\tilde{\sigma}$ defined and discussed in section 7.3 becomes the map $\sigma$ defined by J. Li.

We are going to define a map $\sigma$ from Uhlenbeck compactification of the moduli space $\overline{\mathcal{N}}_H(c_2)$ to $\overline{\mathcal{N}}_{H_0}(c_2)$.

**Notation 7.2.** We use $\ell_x(Q)$ to represent the length of the torsion sheaf $Q$ at the point $x$.

Using Lemma 3.2 and Theorem 4 in [7], we can define a map

$$\sigma: \overline{\mathcal{N}}_H(c_2) \to \overline{\mathcal{N}}_{H_0}(c_2)$$

as follows:

Any element in $\overline{\mathcal{N}}_H(c_2)$ can be represented by $\sigma_H(\gamma_H(V))$ for some $H$-semi-stable sheaf $V \in \mathcal{M}_H(c_2)$. We know that $V$ is either M-T $H_0$-stable or strictly M-T $H_0$-semi-stable. If $V$ is the former, we define

$$\sigma(\sigma_H(\gamma_H(V))) = \sigma_{H_0}(\gamma_{H_0}(V)).$$

If $V$ is the latter, then $V$ sits in an exact sequence

$$0 \to L \otimes I_Z \to V \to L^{-1} \otimes I_{Z^c} \to 0.$$

Then we define

$$\sigma(\sigma_H(\gamma_H(V))) = (L \oplus L^{-1}, \sum(\ell_x(Z)x + \ell_x(Z^c)x)).$$

**Remark 7.3.** This map can be regarded as the induced map from $\sigma$ between Gieseker compactifications.

**Proposition 7.4.** The map $\sigma$ is well-defined.

**Proof.** Suppose an element in $\overline{\mathcal{N}}_H(c_2)$ can be represented by $\sigma_H(\gamma_H(V))$ and $\sigma_H(\gamma_H(V'))$ for some $H$-semi-stable sheaves $V$ and $V'$. Then $V$ and $V'$ sit in the exact sequences

$$0 \to V \to V^{**} \to Q \to 0, \quad 0 \to V \to V'^{*} \to Q' \to 0$$

where $Q$ and $Q'$ are supported at zero-dimensional schemes.

By the definition of $\sigma_H$ and $\gamma_H$ (see [3]), $\sigma_H(\gamma_H(V)) = (V^{**}, \sum \ell_x(Q)x)$ and $\sigma_H(\gamma_H(V')) = (V'^{*}, \sum \ell_x(Q')x)$. Hence $V^{**} = V'^{*}$ and $\sum \ell_x(Q)x = \sum \ell_x(Q')x$.

If $V$ is M-T $H_0$-stable, then $V^{**} = V'^{*}$ is M-T $H_0$-stable. Hence $V'$ is $H_0$-stable. Then

$$\sigma(\sigma_H(\gamma_H(V))) = \sigma_{H_0}(\gamma_{H_0}(V)) = (V^{**}, \sum \ell_x(Q)x) = (V'^{*}, \sum \ell_x(Q')x) = \sigma_{H_0}(\gamma_{H_0}(V)).$$

Otherwise, $V^{**} = V'^{*}$ is strictly M-T $H_0$-semi-stable and $V$ sits in the exact sequences

$$0 \to L \otimes I_{Z_1} \to V \to L^{-1} \otimes I_{Z_2} \to 0 \quad (18)$$
and
\[ 0 \to V \to V^{**} \to Q \to 0. \]
By taking double dual of the exact sequence (18), we get
\[ 0 \to L \to V^{**} = V'^{**} \to L^{-1} \otimes I_Z \to 0. \]
Hence we get
\[ 0 \to L/L \otimes I_Z \to V^{**}/V = Q \to L^{-1} \otimes I_Z/L^{-1} \otimes I_{Z_2} \to 0. \]
Hence
\[ \ell_x(Q) + \ell_x(Z) = \ell_x(Z_1) + \ell_x(Z_2). \]
Therefore
\[ \varphi(\sigma_H(\gamma_H(V))) = (L \oplus L^{-1}, \sum \ell_x(Z_1)x + \sum \ell_x(Z_2)x) \]
and
\[ \varphi(\sigma_H(\gamma_H(V'))) = (L \oplus L^{-1}, \sum \ell_x(Z')x + \sum \ell_x(Z)x). \]

\[ \square \]

Remark 7.5. Since \( \varphi \) is just a rational map, it might be expected that the induced map \( \varphi \) should also be only defined on a Zariski open subset. However, the following two observations may be useful in understanding the differences:

(1) Uhlenbeck compactification losses track of Gieseker strictly semi-stability. It only respects M-T semi-stability (see Lemma 3.3 in [8]).

(2) When we regard the morphism defined by J. Li
\[ \sigma_H \circ \gamma_H: \mathcal{M}_H(c_2) \to \mathcal{N}_H(c_2) \]
as a blowing-down, then although
\[ \mathcal{M}_H(c_2) \xrightarrow{\varphi} \mathcal{M}_{H_0}(c_2) \]
is a rational map, after blowing downs on \( \mathcal{M}_H(c_2) \) and \( \mathcal{M}_{H_0}(c_2) \) respectively, the induced map \( \mathcal{N}_H(c_2) \to \mathcal{N}_{H_0}(c_2) \) becomes a well-defined map. Thus we have the following commutative diagram
\[
\begin{array}{c}
\mathcal{M}_H(c_2) \xrightarrow{\varphi} \mathcal{M}_{H_0}(c_2) \\
\mathcal{N}_H(c_2) \xrightarrow{\varphi} \mathcal{N}_{H_0}(c_2) \\
\end{array}
\]
Next, we are going to show that the map \( \varphi \) is continuous in the classical complex topology.

Theorem 7.6. The map
\[ \varphi: \mathcal{N}_H(c_2) \to \mathcal{N}_{H_0}(c_2) \]
is continuous in analytic topology.
Proof. The argument pretty much follows the argument in the proof of theorem 5 in [8].

Since $\overline{\mathcal{N}}_H$ and $\overline{\mathcal{N}}_{H_0}$ are both compact, it suffices to show that if $\lim s_n = s$ in $\overline{\mathcal{N}}_H$ and $\lim \overline{\varphi}(s_n) = t$ in $\overline{\mathcal{N}}_{H_0}$, then $\overline{\varphi}(s) = t$.

Since $\gamma_H(\overline{\mathcal{N}}_H) = \overline{\mathcal{N}}_H$ is compact, $\mathcal{M}_H^\mu$ is dense in $\overline{\mathcal{M}}_H^\mu$, and generic $H$-stable sheaves are also $H_0$-stable, it suffices to show the following statement: assume that $\{V_i\}$ is a sequence of $H$- and also $H_0$-stable locally free sheaves, $\lim V_i = V$ in $\mathcal{M}_H$, and $\lim \overline{\varphi} \circ \sigma_H \circ \gamma_H(V_i) = t$ in $\overline{\mathcal{N}}_{H_0}$. Then $\overline{\varphi}(V) = t$.

If $V$ is $H$-stable and $H_0$-semi-stable, clearly, the map $\overline{\varphi}$ in the neighborhood of $V$ is induced from $\varphi$. Since $\varphi$ is continuous, $\overline{\varphi}$ is continuous at $V$.

Now suppose $V$ is $H$-stable and not $H_0$ semi-stable. It is clear that $V$ is strictly MT $H_0$-semi-stable. Since continuity is a local problem, we can consider things locally. In classical topology there exists an open subset $U$ of $\mathcal{M}_H$ containing $V$ and a universal sheaf $\mathcal{V}$ over $U \times X$ such that for any $u \in U$, $\mathcal{V}|_u$ represents $u$ in $\mathcal{M}_H$. Since every $H$-stable sheaf is $H_0$-semi-stable, by Cor 1.4 in [8], we know that there exists an integer $N$ such that $h^0(\mathcal{V}|_u(NH_0)) = 0$ for $i \geq 1$ and $H^0(\mathcal{V}|_u(NH_0))$ generates $\mathcal{V}|_u(NH_0)$. By base change theorem, we see that $\mathcal{V}(NH_0)|_{U'}$ is a quotient of $\mathcal{O}_{U', X}$ where $U'$ is an open subset of $U$ containing $V$ and $r = h^0(\mathcal{V}|_u(NH_0))$.

By the universality of the quotient scheme $\mathcal{Q}_{H_0}$, we see that there exists an analytic morphism

$$f : U' \longrightarrow \mathcal{Q}_{H_0},$$

Without loss of generality, we may assume that $V_i \in U'$. Hence

$$\lim_{i \to \infty} f(V_i) = f(V) \quad \text{in } \mathcal{Q}_{H_0}.$$  

Therefore $f(V) \in \mathcal{Q}_{H_0} \cap \overline{\mathcal{Q}}_{H_0}$.

Since $\overline{\varphi} \circ \sigma_H \circ \gamma_H(V_i) = \sigma_{H_0} \circ \gamma_{Q_{H_0}} \circ f(V_i)$, we have

$$\lim_{i \to \infty} \overline{\varphi} \circ \sigma_H \circ \gamma_H(V_i) = \lim_{i \to \infty} \sigma_{H_0} \circ \gamma_{Q_{H_0}} \circ f(V_i) = \overline{\sigma}_{H_0} \circ \gamma_{Q_{H_0}} \circ f(V) = t = \overline{\varphi} \circ \overline{\sigma}_H \circ \gamma_H(V),$$

where the last equality comes from the definition of $\overline{\varphi}$. \hfill \square

Using the following lemma, the above theorem implies immediately the algebraicity of the map $\overline{\varphi} : \overline{\mathcal{N}}_H(c_2) \longrightarrow \overline{\mathcal{N}}_{H_0}(c_2)$.

Lemma 7.7. Let $X$ and $Y$ be two algebraic varieties. Let $U$ be a Zariski dense open subset of $X$ with an algebraic morphism $\varphi_U : U \longrightarrow Y$ which extends to $\varphi : X \longrightarrow Y$ continuously in analytic topology. Then $\varphi$ is an algebraic morphism.

Proof. Consider the graph of the maps $\varphi$ and $\varphi_U$:

$$\text{graph}(\varphi) \subset X \times Y, \quad \text{(19)}$$

$$\text{graph}(\varphi_U) \subset U \times X \subset X \times Y. \quad \text{(20)}$$

Take the closure of $\text{graph}(\varphi_U)$ inside $X \times Y$, call it $\text{graph}(\varphi_U)$. Since $\varphi$ is continuous, it is easy to see that $\text{graph}(\varphi_U) = \text{graph}(\varphi)$.

In fact, for any element $(x, y)$ in $\text{graph}(\varphi_U)$, there exists a sequence $(x_n, y_n) \in \text{graph}(\varphi_U)$ such that $(x_n, y_n) \rightarrow (x, y)$ as $n \rightarrow \infty$. Since $\varphi(x_n) = y_n$,

$$y = \lim_{n \to \infty} y_n = \lim_{n \to \infty} \varphi(x_n) = \varphi(\lim_{n \to \infty} x_n) = \varphi(x). \quad \text{(21)}$$
Hence $\varphi: X \to Y$ can be regarded as a composition of
$$X \xrightarrow{\cong} \text{graph} \varphi = \overline{\text{graph}(\varphi \circ \text{proj})} \to X \times Y \xrightarrow{\text{proj}} Y.$$ 
Since $X$ and $Y$ are all algebraic varieties, $\varphi$ must be an algebraic morphism.

**Corollary 7.8.** The map $\overline{\varphi}: \mathcal{N}_H(c_2) \to \mathcal{N}_{H_0}(c_2)$ is algebraic.

In the rest of the section, we will study the inverse image of the map $\overline{\varphi}$.

**Lemma 7.9.** Let $L$ and $L'$ be line bundles. Suppose $V$ sits in a non-splitting exact sequence
$$0 \to L \otimes I_Z \to V \to L' \otimes I_{Z'} \to 0$$
with $2c_1(L) \cdot H_0 = c_1(V) \cdot H_0$. Then $V$ is strictly $M$-$T$ $H_0$-semi-stable.

If in addition, $2c_1(L) \cdot H < c_1(V) \cdot H$, then $V$ is $M$-$T$ $H$-stable.

**Proof.** That $V$ is strictly $M$-$T$ $H_0$-semi-stable is clear.

Let $M$ be a rank one subsheaf of $V$. If $M$ is a subsheaf of $L \otimes I_Z$, then
$$2c_1(M) \cdot H \leq 2c_1(L) \cdot H \leq c_1(V) \cdot H.$$ 
Otherwise, $M$ is a subsheaf of $L' \otimes I_{Z'}$. Hence
$$2c_1(M) \cdot H_0 \leq 2c_1(L') \cdot H_0 = c_1(V) \cdot H_0.$$ 
If $2c_1(M) \cdot H_0 < 2c_1(L') \cdot H_0 = c_1(V) \cdot H_0$, then $2c_1(M) \cdot H < c_1(V) \cdot H$, since otherwise, $2c_1(M) - c_1(V)$ would define an $c$-wall between $H$ and $H_0$, a contradiction.

If $2c_1(M) \cdot H_0 = 2c_1(L') \cdot H_0$, then $L' = M$. Hence the exact sequence splits, a contradiction.

Now we divide the study of inverse image into several cases.

(i) Suppose $(A, x) = (\mathcal{O}_X \oplus \mathcal{O}_X, x) \in \mathcal{N}_{H_0}(0) \times \text{Sym}^{c_2}(X)$ for $c_2 \geq 2$. Then $\overline{\varphi}^{-1}((A, x)) = (A, x) \in \mathcal{N}_H(0) \times \text{Sym}^{c_2}(X)$. Hence the inverse image of a point in the lowest stratum is just a single point.

(ii) Suppose $(A, x) \in \prod_{j \geq 1} \mathcal{N}_{H_0}(j) \times \text{Sym}^{c_2-j}(X)$, then $A$ corresponds to an $M$-$T$ $H_0$-stable vector bundles $V_j$. $V_j$ is also $M$-$T$ $H$-stable. Then it is easy to see that
$$\overline{\varphi}^{-1}((A, x)) = (A, x) \in \prod_{j \geq 1} \mathcal{N}_H(j) \times \text{Sym}^{c_2-j}(X).$$
Hence the inverse image of $\overline{\varphi}$ of a single point in $\prod_{j \geq 1} \mathcal{N}_H(j) \times \text{Sym}^{c_2-j}(X)$ is also just a single point.

(iii) Suppose $(A, x) = (L \oplus L^{-1}, x)$ with $L \cdot H_0 = 0$ and $c_1(L)^2 = -j$, i.e.
$$A \in \widetilde{\mathcal{N}}_{H_0}(j) - \mathcal{N}_{H_0}(j).$$
Without loss of generality, we assume that $L \cdot H < 0$. Assume that $(A, x) = \overline{\varphi}(\sigma \gamma_H(V))$. By the way how the maps $\overline{\sigma}$ and $\overline{\varphi}$ are defined, it is easy to see that $V$ sits in the non-splitting exact sequence
$$0 \to L \otimes I_Z \to V \to L^{-1} \otimes I_{Z'} \to 0.$$ 
By Lemma 7.9, $V$ is $M$-$T$ $H$-stable.

\[ (23) \]
The inverse image of \((A, x)\) is rather complicated due to the arbitrariness of \(Z\) and \(Z'\) in the exact sequence \((\mathbb{Z})\). We can only give a rough description of the inverse image.

Consider one extreme case where \(V\) sits in a non-splitting exact sequence

\[0 \to L \otimes I_Z \to V \to L^{-1} \to 0.\]

We know that \(V\) is \(H\)-stable. Clearly, \(V'^*\) sits in the exact sequence

\[0 \to L \to V'^* \to L^{-1} \to 0.\]

Hence

\[\overline{\varphi}^{-1}((L \oplus L^{-1}, x)) \supset (\mathbb{P}(H^2(L^{\oplus 2}), x) \subset \mathcal{M}_H(j) \times \text{Sym}^{c_2-j}(X).\]

For other cases, it is easy to see that \(\overline{\varphi}^{-1}((L \oplus L^{-1}, x))\) consists of all

\[(V'^*, x') \in \mathcal{M}_H(j') \times \text{Sym}^{c_2-j'}(X)\]

where \(V\) sits in the exact sequence \((\mathbb{Z})\), \(c_2(V'^*) = j', x' \subset x\), and we have the exact sequence \(0 \to V \to V'^* \to O_{Z''} \to 0\) where \(\text{red}(Z'') = x'\).

In another word, \(\overline{\varphi}^{-1}((L \oplus L^{-1}, x))\) consists of

\[(V'_j, x') \in \mathcal{M}_H(j') \times \text{Sym}^{c_2-j'}(X)\]

where \(c_2(V'_j) = j', x' \subset x\), \(V_j\) is locally free sheaf sitting in the non-splitting exact sequence

\[0 \to L \to V'_j \to L^{-1}I_{Z'} \to 0\]

where \(\text{red}(Z') = x - x'\).

In general, the preimage of the map \(\overline{\varphi}\) may contain points in \(\overline{\mathcal{N}}_H(c_2)\) from every stratum \(\overline{\mathcal{N}}_H(j)\) for \(j' \geq j\). The intersection \(\overline{\varphi}^{-1}((A, x)) \cap \overline{\mathcal{N}}_H(j)\) of the preimage with each stratum may not be closed. But put all these strata together, \(\overline{\varphi}^{-1}((A, x))\) will be closed.

Summarize the above, we have

**Theorem 7.10.** Let the situation be as in Theorem 5.1. Then we have the following commutative diagram

\[
\begin{array}{ccc}
\mathcal{M}_H(c_2) & \xrightarrow{\varphi} & \mathcal{M}_H'_0(c_2) \\
\xrightarrow{\gamma_H} & & \xleftarrow{\gamma_H'} \\
\overline{\mathcal{N}}_H(c_2) & \xrightarrow{\overline{\varphi}} & \overline{\mathcal{N}}_H'_0(c_2)
\end{array}
\]

such that

(i) \(\overline{\varphi}\) and \(\overline{\psi}\) are induced from \(\varphi\) and \(\psi\); both are well-defined everywhere and are algebraic maps;

(ii) \(\overline{\varphi}\) and \(\overline{\psi}\) are homeomorphisms over the Zariski open subset \(\prod_{j=0}^{c_2} \mathcal{N}_H_0(j) \times \text{Sym}^{c_2-j}(X)\);

(iii) Let \((A, x) \in (\overline{\mathcal{N}}_H(j) - \mathcal{N}_H(j)) \times \text{Sym}^{c_2-j}(X)\). The the preimages of \(\overline{\varphi}\) and \(\overline{\psi}\) over \((A, x)\) are contained in \(\prod_{j \geq j} \overline{\mathcal{N}}_H(j') \times \text{Sym}^{c_2-j'}(X)\).
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