Minimization to the Zhang’s energy on $BV(\Omega)$ and sharp affine Poincaré-Sobolev inequalities

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Abstract
We prove the existence of minimizers for some constrained variational problems on $BV(\Omega)$, under subcritical and critical restrictions, involving the affine energy introduced by Zhang in [50]. Related functionals have non-coercive geometry and properties like lower semicontinuity and affine compactness are deeper in the weak* topology. As a by-product of our developments, extremal functions are shown to exist for various affine Poincaré-Sobolev type inequalities.

1 An overview and the first statements
Variational problems have been quite studied in the space of functions of bounded variation $BV(\Omega)$, mostly in connection with existence of solutions for equations in the presence of the 1-Laplace operator, as for instance in the famous Cheeger’s problem [10]. Contributions along this line of research are given in the works [7, 8, 14, 15, 28, 30, 31], among others. Part of them focus more specifically on the problem of minimizing the functional

$$\Phi(u) = |Du|(\Omega) + \int_\Omega a|u| \, dx + \int_{\partial \Omega} b|\tilde{u}| \, d\mathcal{H}^{n-1}$$

on the entire $BV(\Omega)$ space or constrained to some subset of it, where $\Omega$ denotes a bounded open in $\mathbb{R}^n$ with Lipschitz boundary, $n \geq 2$, $a \in L^\infty(\Omega)$ and $b \in L^\infty(\partial \Omega)$. Here, $|Du|(\Omega)$, $\tilde{u}$ and $\mathcal{H}^{n-1}$ stand respectively for the total variation measure of $u$ on $\Omega$, the trace of $u$ on $\partial \Omega$ and the $(n-1)$-dimensional Hausdorff measure on $\partial \Omega$.

Two subsets of $BV(\Omega)$ typically considered are:

$$X = \{ u \in BV(\Omega) : \int_\Omega |u|^q \, dx = 1 \} ,$$

$$Y = \{ u \in BV(\Omega) : u \in X, \int_\Omega |u|^{r-1} u \, dx = 0 \}$$

for exponents $1 \leq q, r \leq \frac{n}{n-1}$, and the corresponding minimization problem consists in establishing the existence

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Some subcritical cases have been treated in [28], while a few critical ones have been object of study for example in [7, 8, 11, 12, 15, 31].

The minimization of \( \Phi \) in the sets \( X \) and \( Y \) (with \( r = 1 \)) is also motivated by the existence of extremal functions (i.e. nonzero functions that attain equality) for some classical functional inequalities such as Poincaré, Poincaré-Wirtinger, \( L^q \) Poincaré-Sobolev and \( L^q \) Poincaré-Wirtinger-Sobolev inequalities for \( 1 \leq q \leq \frac{n}{n-1} \). More specifically, their respective sharp versions on \( BV(\Omega) \) state that

- **Poincaré inequality (P):**
  
  There exists an optimal constant \( \lambda_1 > 0 \) such that \( \lambda_1 \|u\|_{L^1(\Omega)} \leq |Du|(|\Omega|) + \|\tilde{u}\|_{L^1(\partial\Omega)} \).

- **Poincaré-Wirtinger inequality (PW):**
  
  There exists an optimal constant \( \mu_1 > 0 \) such that \( \mu_1 \|u - u_{\Omega}\|_{L^1(\Omega)} \leq |Du|(|\Omega|) \).

- **Poincaré-Sobolev inequality (PS):**
  
  There exists an optimal constant \( \lambda_q > 0 \) such that \( \lambda_q \|u\|_{L^q(\Omega)} \leq |Du|(|\Omega|) + \|\tilde{u}\|_{L^1(\partial\Omega)} \).

- **Poincaré-Wirtinger-Sobolev inequality (PWS):**
  
  There exists an optimal constant \( \mu_q > 0 \) such that \( \mu_q \|u - u_{\Omega}\|_{L^q(\Omega)} \leq |Du|(|\Omega|) \),

where \( u_{\Omega} \) denotes the average of \( u \) over \( \Omega \).

Some results on existence of extremal functions are well known. For instance, for (P) we refer to [9, 10], for (PW) to [6, 8], for (PS) to [15] and for (PWS) to [6, 8, 11, 12]. See also [4, 38, 42] for refinements in different directions.

Motivated by the existence problem of extremal functions for sharp affine counterparts, some of which weaker than the above inequalities, we develop a theory of minimization for functionals where the term \( |Du|(|\Omega|) \) gives place to the Zhang’s affine energy.

In the seminal paper [50], Zhang introduced the affine \( L^1 \) energy (or functional) for functions \( u \in W^{1,1}(\mathbb{R}^n) \) given by

\[
\mathcal{E}_{\mathbb{R}^n}(u) = \alpha_n \left( \int_{\mathbb{R}^{n+1}} \left( \int_{\mathbb{R}^n} |\nabla_x u(x)|(dx) \right)^{-n} d\xi \right)^{-\frac{1}{n}},
\]

where \( \alpha_n = (2\omega_{n-1})^{-1}(n\omega_n)^{1+1/n} \). Here, \( \nabla_x u(x) = \nabla u(x) \cdot \xi \) and \( \omega_k \) denotes the volume of the unit ball in \( \mathbb{R}^k \). The “affine” term comes from the invariance property \( \mathcal{E}_{\mathbb{R}^n}(u \circ T) = \mathcal{E}_{\mathbb{R}^n}(u) \) for every \( T \in SL(n) \), where \( SL(n) \) denotes the special linear group of \( n \times n \) matrices with determinant equal to 1.

The main result of [50] ensures that the sharp Sobolev-Zhang inequality

\[
n\omega_n^{1/n} \|u\|_{L^{\frac{n}{n-1}}(\mathbb{R}^n)} \leq \mathcal{E}_{\mathbb{R}^n}(u) \tag{1}
\]

holds for all \( u \in W^{1,1}(\mathbb{R}^n) \), with equality attained at characteristic functions of ellipsoids, that is, images of balls under invertible \( n \times n \) matrices. Actually, characteristic functions are not in \( W^{1,1}(\mathbb{R}^n) \), but rather belong to \( BV(\mathbb{R}^n) \).

The Sobolev-Zhang inequality (1) is weaker than the classical sharp \( L^1 \) Sobolev inequality

\[
n\omega_n^{1/n} \|u\|_{L^{\frac{n}{n-1}}(\mathbb{R}^n)} \leq \|\nabla u\|_{L^1(\mathbb{R}^n)}, \tag{2}
\]
since
\[ E_\mathbb{R}^n (u) \leq \| \nabla u \|_{L^1(\mathbb{R}^n)} \] (3)
(see page 194 of [50]) and also (3) is strict for characteristics of ellipsoids other than balls. Zhang also pointed out that the geometric inequality behind (1) is the Petty projection inequality (e.g. [18, 45, 46]), whereas the geometric inequality behind (2) is the classical isoperimetric inequality.

Later, Wang [49] showed that, like in the Sobolev case (e.g. [17]), the Sobolev-Zhang inequality extends to functions \( u \in BV(\mathbb{R}^n) \), where the affine \( BV \) energy is expressed naturally by
\[
E_{\mathbb{R}^n} (u) = \alpha_n \left( \int_{\mathbb{S}^{n-1}} \left( \int_\Omega |\sigma_u(x) \cdot \xi| \, d(|Du|)(x) \right)^{-n} \, d\xi \right)^{-\frac{1}{n}},
\]
where \( \sigma_u : \Omega \to \mathbb{R}^n \) represents the Radon-Nikodym derivative of \( Du \) with respect to its total variation \( |Du| \) on \( \Omega \), which satisfies \( |\sigma_u| = 1 \) almost everywhere in \( \Omega \) (w.r.t. \( |Du| \)). Moreover, equality in (1) is achieved precisely by multiples of characteristic functions of ellipsoids, and it also remains weaker than the classical prototype once (3) translates to
\[ E_{\mathbb{R}^n} (u) \leq |Du|(\mathbb{R}^n). \] (4)

Since the Zhang’s pioneer work, various improvements and new affine functional inequalities have emerged in a very comprehensive literature. Most of the contributions can be found in the long, but far from complete, list of references [13, 16, 21, 22, 24, 25, 26, 33, 34, 35, 36, 37, 39, 40, 41, 49, 50].

Given a function \( u \in BV(\Omega) \), denote by \( \bar{u} \) its zero extension outside of \( \Omega \). The Lipschitz regularity of \( \partial \Omega \) guarantees that \( \bar{u} \in BV(\mathbb{R}^n) \),
\[ |D\bar{u}|(\mathbb{R}^n) = |Du|(|\Omega| + \|\bar{u}\|_{L^1(\partial\Omega)}) \] (5)
and \( d(D\bar{u}) = \bar{u} \nu \, dH^{n-1} \) \( \mathbb{H}^{n-1} \)-almost everywhere on \( \partial \Omega \), where \( \nu \) denotes the unit outward normal to \( \partial \Omega \) (see e.g. page 38 of [19]). On the other hand, the latter relation implies that
\[
E_{\mathbb{R}^n} (\bar{u}) = \alpha_n \left( \int_{\mathbb{S}^{n-1}} \left( \int_\Omega |\sigma_u(x) \cdot \xi| \, d(|Du|)(x) + \int_{\partial\Omega} |\bar{u}(x)| \, d\nu(x) \cdot dH^{n-1}(x) \right)^{-n} \, d\xi \right)^{-\frac{1}{n}},
\]
This formula along with (4), (5) and a reverse Minkowski inequality yield the following comparisons between the affine \( BV \) energy of zero extended functions and some local terms:

(C1) \( E_{\mathbb{R}^n} (\bar{u}) \leq |Du|(\Omega) + \|\bar{u}\|_{L^1(\partial\Omega)} \) for all \( u \in BV(\Omega) \);

(C2) \( E_{\mathbb{R}^n} (\bar{u}) = \bar{E}(u) \) for all \( u \in BV_0(\Omega) \);

(C3) \( E_{\mathbb{R}^n} (\bar{u}) \geq \bar{E}(u) + E_{\partial\Omega}(\bar{u}) \) for all \( u \in BV(\Omega) \) with \( \bar{u} \neq 0 \) on \( \partial \Omega \) (a.e.) or no restriction provided that \( \partial \Omega \) is non-flat in the sense that \( \nu(x) \cdot \xi \neq 0 \) on \( \partial \Omega \) (a.e.) for every \( \xi \in \mathbb{S}^{n-1} \), where \( BV_0(\Omega) \) denotes the subspace of \( BV(\Omega) \) of functions with zero trace on \( \partial \Omega \),
\[
E_\Omega (u) = \alpha_n \left( \int_{\mathbb{S}^{n-1}} \left( \int_\Omega |\sigma_u(x) \cdot \xi| \, d(|Du|)(x) \right)^{-n} \, d\xi \right)^{-\frac{1}{n}},
\]
and
\[
E_{\partial\Omega} (\bar{u}) = \alpha_n \left( \int_{\mathbb{S}^{n-1}} \left( \int_{\partial\Omega} |\bar{u}(x)| \, d\nu(x) \cdot dH^{n-1}(x) \right)^{-n} \, d\xi \right)^{-\frac{1}{n}}.
\]
Unlike (C1) and (C2), the comparison (C3) is not straightforward (Corollary 3.1). The required geometric condition is clearly satisfied for many domains which include balls. The above expressions are affine invariants in the sense that $\mathcal{E}_\Omega(u \circ T) = \mathcal{E}_{\Omega T}(u)$ and $\mathcal{E}_{\partial \Omega}(u \circ T) = \mathcal{E}_{\partial \Omega T}(u)$ for every $T \in SL(n)$.

It is worth observing from (C1) that the term $\mathcal{E}_{R^n}(\tilde{u})$ weakens the right-hand side of (P) and (PS) and this fact encourages us to investigate the new functional $\Phi_A : BV(\Omega) \to \mathbb{R}$,

$$\Phi_A(u) = \mathcal{E}_{R^n}(\tilde{u}) + \int_\Omega a|u|\,dx + \int_{\partial \Omega} b|\tilde{u}|\,dH^{n-1}.$$  

Evoking the trace embedding and (4), one sees that $\Phi$ is well defined for bounded weights $a$ and $b$.

Consider the least energy levels of $\Phi_A$ on $X$ and $Y$:

$$c_A = \inf_{u \in X} \Phi_A(u) \quad \text{and} \quad d_A = \inf_{u \in Y} \Phi_A(u).$$

Note that, by (3), it may happen that $c_A = -\infty$ or $d_A = -\infty$ depending on the function $b$. However, both levels are finite if one assumes for instance that $b$ is nonnegative.

Minimization problems for $\Phi_A$ are affine variants of those similar for $\Phi$, as can be seen by replacing the term $\mathcal{E}_{R^n}(\tilde{u})$ by $|D\tilde{u}|(\mathbb{R}^n)$ and using (3). On the other hand, inherent to the nature of our variational problems, some intricate points arise in the search for minimizers. We gather the main ones below:

(P1) The geometry of $\Phi_A$ is non-coercive on $BV(\Omega)$;
(P2) The functional $u \mapsto \mathcal{E}_{R^n}(\tilde{u})$ is not convex on $BV(\Omega)$;
(P3) Sequences in $BV(\Omega)$ with bounded affine $BV$ energy can have unbounded total variation;
(P4) The continuous immersion $BV(\Omega) \hookrightarrow L^{\frac{n}{n-1}}(\Omega)$ is no longer compact.

For the points (P1) and (P3) we refer to the example on page 17 of [25]. Already (P2) can indirectly be checked by combining Theorem 3.1 and Proposition 5.1.

The absence of an adequate variational structure leads us to concentrate attention on three central ingredients to be discussed respectively in Sections 3, 4 and 5:

(I1) The functional $u \mapsto \mathcal{E}_{R^n}(\tilde{u})$ is weakly* lower semicontinuous on $BV(\Omega)$;
(I2) The set $B_A(\Omega) = \{u \in BV(\Omega) : \|u\|_{L^q(\Omega)} + \mathcal{E}_{R^n}(\tilde{u}) \leq 1\}$ is compact in $L^q(\Omega)$ for any $1 \leq q < \frac{n}{n-1}$;
(I3) Minimizing sequences to $\Phi_A$ in $X$ are compact in $L^{\frac{n}{n-1}}(\Omega)$, provided that $c_A < n\omega_n^{1/n}$. The same conclusion holds true for $Y$ when one assumes $d_A < n\omega_n^{1/n}$.

Before we go any further, we have a few comments on each above assertion.

Using the solution of the famous $L^1$-Minkowski problem (see e.g. [36]), Ludwig [32] was able to show that $u \mapsto \mathcal{E}_{R^n}(u)$ is strongly continuous on $W^{1,1}(\mathbb{R}^n)$. After regarding the $BV$-Minkowski problem, Wang [49] established the continuity on $BV(\mathbb{R}^n)$ with respect to the strict topology. In Section 3, we prove the strong lower semicontinuity on $L^1_{loc}(\mathbb{R}^n)$ for functions with controlled total variation (Theorem 5.2). The argument bases mainly on a characterization of vanishing of $\mathcal{E}_{R^n}(u)$ (Theorem 3.1) and the theory of relaxation and weak semicontinuity for integral functionals by Goffman and Serrin [20]. As a by-product, $u \mapsto \mathcal{E}_{R^n}(\tilde{u})$ is strictly continuous and weakly* lower semicontinuous on $BV(\Omega)$, which includes (I1) (Corollary 5.2). The affine compactness stated in (I2) (Theorem 4.1), according to (P3), doesn’t follow readily from the Rellich-Kondrachov theorem for $BV$ functions. Its proof makes use of Proposition 4.1 and a result due to Huang and Li [27] which gives a boundedness of the total variation of a function in terms of its affine $BV$ energy unless a suitable affine transformation. Lastly, the assertions (I1) and (I2) along with Proposition 5.1 play an essential role in the study of (I3).

Throughout the work, we assume the general assumptions already mentioned:
(H) \( a \in L^\infty(\Omega), \ b \in L^\infty(\partial\Omega) \) and \( b \geq 0 \) on \( \partial\Omega \).

The first result considers subcritical affine minimization problems.

**Theorem 1.1.** The levels \( c_A \) and \( d_A \) are attained for any \( 1 \leq q, r < \frac{n}{n-1} \).

The next one covers critical cases.

**Theorem 1.2.** The levels \( c_A \) and \( d_A \) are attained for \( q = \frac{n}{n-1} \) and any \( 1 \leq r \leq \frac{n}{n-1} \), provided that \( 0 < c_A < n \omega_n^{1/n} \) and \( 0 < d_A < n \omega_n^{1/n} \), respectively.

The reasoning used in the proof of Theorems 1.1 and 1.2 produces analogous statements on the space \( BV_0(\Omega) \), thanks to its weak* closure in \( BV(\Omega) \) (Proposition 3.2). More precisely, using (C2), the functional \( \Phi_A \), when computed at functions with zero trace in \( BV(\Omega) \), becomes

\[
\Phi_A(u) = \mathcal{E}_\Omega(u) + \int_\Omega a|u| \, dx.
\]

Denote by \( c_{A,0} \) and \( d_{A,0} \) the respective least energy levels of \( \Phi_A \) on the sets \( X_0 = X \cap BV_0(\Omega) \) and \( Y_0 = Y \cap BV_0(\Omega) \).

**Theorem 1.3.** The levels \( c_{A,0} \) and \( d_{A,0} \) are attained for any \( 1 \leq q, r < \frac{n}{n-1} \).

**Theorem 1.4.** The levels \( c_{A,0} \) and \( d_{A,0} \) are attained for \( q = \frac{n}{n-1} \) and any \( 1 \leq r \leq \frac{n}{n-1} \), provided that \( 0 < c_{A,0} < n \omega_n^{1/n} \) and \( 0 < d_{A,0} < n \omega_n^{1/n} \), respectively.

The Sobolev-Zhang inequality on \( BV(\mathbb{R}^n) \) yields the sharp affine variants of (P) and (PW) and also of (PS) and (PWS) for \( 1 \leq q \leq \frac{n}{n-1} \):

- **Affine Poincaré inequality (AP):**
  There exists an optimal constant \( \lambda^A_1 > 0 \) such that \( \lambda^A_1 \|u\|_{L^1(\Omega)} \leq \mathcal{E}_{R^n}(\bar{u}) \);

- **Affine Poincaré-Wirtinger inequality (APW):**
  There exists an optimal constant \( \mu^A_1 > 0 \) such that \( \mu^A_1 \|u - u_{\Omega}\|_{L^1(\Omega)} \leq \mathcal{E}_{R^n}(\bar{u}) \);

- **Affine Poincaré-Sobolev inequality (APS):**
  There exists an optimal constant \( \lambda^A_q > 0 \) such that \( \lambda^A_q \|u\|_{L^q(\Omega)} \leq \mathcal{E}_{R^n}(\bar{u}) \);

- **Affine Poincaré-Wirtinger-Sobolev inequality (APWS):**
  There exists an optimal constant \( \mu^A_q > 0 \) such that \( \mu^A_q \|u - u_{\Omega}\|_{L^q(\Omega)} \leq \mathcal{E}_{R^n}(\bar{u}) \).

As remarked before, (AP) and (APS) are weakened versions of (P) and (PS), respectively. As for (APW) and (APWS), perhaps one would expect that the term \( \mathcal{E}_{R^n}(\bar{u}) \) could be replaced by \( \mathcal{E}_\Omega(u) \) in view of (PW) and (PWS). However, such exchange is not possible, once there are non-constant functions in \( BV(\Omega) \) with zero affine \( BV \) energy on \( \Omega \) (Theorem 2.1). In other words, there exist no positive constant \( A \) so that \( A \|u - u_{\Omega}\|_{L^q(\Omega)} \leq \mathcal{E}_\Omega(u) \) is valid for all \( u \in BV(\Omega) \). This is a sore point of theory that contrasts drastically with the classical case.

It also deserves to be noticed that \( \mathcal{E}_{R^n}(\bar{u}) \) and \(|Du|(\Omega)\) are incomparable via a one-way inequality in \( BV(\Omega) \). In effect, since \( \mathcal{E}_{R^n}(\bar{u}) = \mathcal{E}_\Omega(1) > 0 \) and \(|Dx|\)\(=0\), there is no constant \( C > 0 \) such that \( \mathcal{E}_{R^n}(\bar{u}) \leq C|Du|(\Omega) \) holds for all \( u \in BV(\Omega) \). On the other hand, a reverse inequality also fails in view of the example of [25] in \( BV_0(\Omega) \). Accordingly, (APW) and (APWS) seem to be natural affine counterparts of (PW) and (PWS), respectively.

Nonetheless, the term \( \mathcal{E}_\Omega(u) \) appears on the right-hand side when we restrict ourselves to functions in \( BV_0(\Omega) \). In this space, we denote the respective inequalities by (AP0), (APW0), (APS0) and (APWS0).

A direct application of Theorems 1.1 and 1.3 for \( r = 1 \) is as follows:
Theorem 1.5. The inequalities (AP) and (APW) and also (APS) and (APWS) with \(1 \leq q < \frac{n}{n-r}\) admit extremal functions in \(BV(\Omega)\). The same conclusion holds true in \(BV_0(\Omega)\) for (AP), (APW), (APS) and (APWS).

The study of extremal functions for local affine \(L^p\)-Sobolev type inequalities has been carried out more recently and, as far as we know, it has only been addressed in the papers [14] and [25] for functions with zero trace. More specifically, the first of them furnishes extremals for the affine \(L^2\)-Sobolev inequality on \(W_0^{1,2}(\Omega)\), whereas the second one for the affine \(L^p\)-Poincaré inequality on \(W_0^{1,p}(\Omega)\) for any \(p > 1\) and on \(BV(\Omega)\) for \(p = 1\). In particular, in [25], the authors provide an alternative proof of Theorem 1.5 for (AP) through an elegant approach based on their Lemma 1 and Theorem 9.

In the critical case \(q = \frac{n}{n-r}\), one knows from [14] that characteristic functions of ellipsoids in \(\Omega\) are extremals of (APS), however, exist no extremal for (APWS). The usual argument of nonexistence consists in showing, by means of a standard rescaling, that the optimal constant corresponding to \(BV_0(\Omega)\) is also \(m\omega_n^{1/n}\). The key points are the strict continuity of \(u \mapsto E_{\mathbb{R}^n}(u)\) on \(BV(\mathbb{R}^n)\) (Theorem 4.4 of [15]) and the density of \(BV^\infty(\mathbb{R}^n)\) in \(BV(\mathbb{R}^n)\) (Corollary 3.2 of [15]), where \(BV^\infty(\mathbb{R}^n)\) denotes the space of bounded functions in \(BV(\mathbb{R}^n)\) with compact support.

We close the introduction with an application of Theorems 1.1 and 1.3 for \(L^r(\Omega)\). Of course, \(m_1(u) = u_{\Omega}\) for \(r = 1\). The construction of \(m_r\) is canonical and makes use of basic results as the mean value theorem and dominated and monotone convergence theorems.

The properties satisfied by \(m_r\) together with [14] produce two new affine inequalities for \(1 \leq q, r \leq \frac{n}{n-r}\) that extend (APW), (APWS), (APW) and (APWS).

- Generalized affine Poincaré-Wirtinger-Sobolev inequality (GAPWS) on \(BV(\Omega)\):
  There exists an optimal constant \(\mu_{q,r}^{A} > 0\) such that \(\mu_{q,r}^{A} \|u - m_r(u)\|_{L^r(\Omega)} \leq E_{\mathbb{R}^n}(u)\).

- Generalized affine Poincaré-Wirtinger-Sobolev inequality (GAPWS) on \(BV_0(\Omega)\):
  There exists an optimal constant \(\mu_{q,r}^{A,0} > 0\) such that \(\mu_{q,r}^{A,0} \|u - m_r(u)\|_{L^r(\Omega)} \leq E_{\Omega}(u)\).

Theorem 1.6. The inequalities (GAPWS) and (GAPWS) admit extremal functions for any \(1 \leq q, r < \frac{n}{n-r}\) respectively in \(BV(\Omega)\) and \(BV_0(\Omega)\).

2 Background on the space \(BV(\Omega)\)

This section is devoted to some basic definitions and classical results related to functions of bounded variation. For some complete references on the subject, we refer to the books [3], [17] and [19].

Let \(\Omega\) be a open subset of \(\mathbb{R}^n\) with \(n \geq 2\). A function \(u \in L^1(\Omega)\) is said to be of bounded variation in \(\Omega\), if the distributional derivative of \(u\) is a vector-valued Radon measure \(Du = (D_1u, \ldots, D_nu)\) in \(\Omega\), that is, \(Du\) is a Radon measure satisfying

\[
\int_{\Omega} \varphi D_iu = -\int_{\Omega} \frac{\partial \varphi}{\partial x_i} u \, dx
\]

for every \(u \in C_0^\infty(\Omega)\). The vector space of all functions of bounded variation in \(\Omega\) is denoted by \(BV(\Omega)\).

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The total variation of $u$ is defined by
\[
|Du|(\Omega) = \sup \left\{ \sum_{i=1}^{n} \int_{\Omega} \psi_i D_i u \, dx : \psi = (\psi_1, \ldots, \psi_n) \in C_0^\infty(\Omega, \mathbb{R}^n), |\psi| \leq 1 \right\}
\]
\[
= \sup \left\{ -\int_{\Omega} u \operatorname{div} \psi \, dx : \psi \in C_0^\infty(\Omega, \mathbb{R}^n), |\psi| \leq 1 \right\},
\]
where $|\psi| = (\psi_1^2 + \cdots + \psi_n^2)^{1/2}$. The variation $|Du|$ is a positive Radon measure on $\Omega$. Denote by $\sigma_u$ the Radon-Nikodym derivative of $Du$ with respect to $|Du|$. Then, $\sigma_u : \Omega \to \mathbb{R}^n$ is a measurable field satisfying $|\sigma_u| = 1$ almost everywhere in $\Omega$ (w.r.t. $|Du|$) and $d(Du) = \sigma_u d(|Du|)$.

For $u \in BV(\Omega)$, the Lebesgue-Radon-Nikodym decomposition of the measure $Du$ is given by
\[
Du = \nabla u \mathcal{L} + \sigma_u^* |D^s u|,
\]
where $\nabla u$ and $D^s u$ denote respectively the (density) absolutely continuous part and the singular part of $Du$ with respect to the $n$-dimensional Lebesgue measure $\mathcal{L}$ and $\sigma_u^*$ is the Radon-Nikodym derivative of $D^s u$ with respect to its total variation measure $|D^s u|$. In particular,
\[
|Du| = |\nabla u| \mathcal{L} + |D^s u|.
\]

The space $BV(\Omega)$ is Banach with respect to the norm
\[
\|u\|_{BV(\Omega)} = \|u\|_{L^1(\Omega)} + |Du|(\Omega),
\]
however it is neither separable nor reflexive.

The strict (intermediate) topology is induced by the metric
\[
d(u, v) = | |Du|(\Omega) - |Dv|(\Omega) | + \| u - v \|_{L^1(\Omega)}.
\]

The weak* topology, the weakest of the three ones, is quite appropriate for dealing with minimization problems. A sequence $u_k$ converges weakly* to $u$ in $BV(\Omega)$, if $u_k \to u$ strongly in $L^1(\Omega)$ and $Du_k \rightharpoonup Du$ weakly* in the measure sense, that is,
\[
\int_{\Omega} \varphi Du_k \to \int_{\Omega} \varphi Du
\]
for every $\varphi \in C_0^\infty(\Omega)$.

Assume that $\Omega$ is a bounded open with Lipschitz boundary. We select below some well-known properties that will be used in the next sections:

- Every bounded sequence in $BV(\Omega)$ admits a weakly* convergent subsequence;
- Every weakly* convergent sequence in $BV(\Omega)$ is bounded;
- $BV(\Omega)$ is embedded continuously into $L^q(\Omega)$ for $1 \leq q \leq \frac{n}{n-1}$ and compactly for $1 \leq q < \frac{n}{n-1}$;
- Each function $u \in BV(\Omega)$ admits a boundary trace $\tilde{u}$ in $L^1(\partial\Omega)$ and the trace operator $u \mapsto \tilde{u}$ is continuous on $BV(\Omega)$ with respect to the strict topology;
- For any function $u \in BV(\Omega)$, its zero extension $\tilde{u}$ outside of $\Omega$ belongs to $BV(\mathbb{R}^n)$;
- $\|u\|_{BV(\Omega)} = |Du|(\Omega) + \| \tilde{u} \|_{L^1(\partial\Omega)}$ defines a norm on $BV(\Omega)$ equivalent to the usual norm $\|u\|_{BV(\Omega)}$;
- $W^{1,1}(\mathbb{R}^n)$ is dense in $BV(\mathbb{R}^n)$ with respect to the strict topology.
3 Lower weak* semicontinuity of $\mathcal{E}_{\mathbb{R}^n}$

For an open subset $\Omega \subset \mathbb{R}^n$ and $u \in BV(\Omega)$, consider the affine $BV$ energy

$$
\mathcal{E}_\Omega(u) = \alpha_n \left( \int_{\mathbb{R}^n} \left( \int_{\Omega} |\sigma_u(x) \cdot \xi| \, d|Du|(x) \right)^{-n} \, d\xi \right)^{\frac{1}{n}}.
$$

We start by giving an answer to the question:

When is the affine energy $\mathcal{E}_\Omega(u)$ zero?

For each $\xi \in S^{n-1}$, denote by $\Psi_\xi$ the functional on $BV(\Omega)$,

$$
\Psi_\xi(u) = \int_{\Omega} |\sigma_u(x) \cdot \xi| \, d|Du|(x).
$$

**Theorem 3.1.** Let $u \in BV(\Omega)$. Then, $\mathcal{E}_\Omega(u) = 0$ if, and only if, $\Psi_\xi(u) = 0$ for some $\xi \in S^{n-1}$.

**Proof.** The sufficiency is the easy part. In fact, assume that $\Psi_\xi(u) > 0$ for all $\xi \in S^{n-1}$. Thanks to the continuity of $\xi \in S^{n-1} \mapsto \Psi_\xi(u)$, there exists a constant $c > 0$ so that $\Psi_\xi(u) \geq c$ for all $\xi \in S^{n-1}$. But this lower bound immediately yields $\mathcal{E}_\Omega(u) \geq c \alpha_n(\mathcal{H}_n)^{-1/n} > 0$.

Conversely, we prove that $\mathcal{E}_\Omega(u) = 0$ whenever $\Psi_\xi(u) = 0$ for some $\xi \in S^{n-1}$. Let $m \in \mathbb{N}$ be the maximum number of linearly independent vectors $\xi \in S^{n-1}$ such that $\Psi_\xi(u) = 0$. If $m = n$, then clearly $Du = 0$ in $\Omega$ and thus, by Remark 3.1, we have $\mathcal{E}_\Omega(u) = 0$. Else, choose an orthonormal basis $\{\xi_1, \ldots, \xi_m\}$ of $\mathbb{R}^n$ so that $\Psi_\xi(u) = 0$ for $i = n - m + 1, \ldots, n$, which correspond to the last $m$ vectors of basis with $0 < m < n$.

For $x \in \Omega$ and $\xi \in S^{n-1}$, write

$$
\sigma_u(x) = \sigma_1(x)\xi_1 + \cdots + \sigma_n(x)\xi_n \quad \text{and} \quad \xi = a_1\xi_1 + \cdots + a_n\xi_n.
$$

The condition $\Psi_\xi(u) = 0$ implies that $\sigma_i(x) = 0$ for $i = n - m + 1, \ldots, n$. So, the Cauchy-Schwarz inequality gives

$$
|\sigma_u(x) \cdot \xi| = |\sigma_1(x)a_1 + \cdots + \sigma_{n-m}(x)a_{n-m}| \leq (a_1^2 + \cdots + a_{n-m}^2)^{1/2}.
$$

Set $a(\xi) = (a_1, \ldots, a_{n-m})$ and $a'(\xi) = (a_{n-m+1}, \ldots, a_n)$. Since $0 < m < n$, we get

$$
\int_{S^{n-1}} \left( \int_{\Omega} |\sigma_u(x) \cdot \xi| \, d|Du|(x) \right)^{-n} \, d\xi \geq |Du|(\Omega)^{-n} \int_{S^{n-1}} |a(\xi)|^{-n} \, d\xi
$$

$$
\geq |Du|(\Omega)^{-n} \int_{|a(\xi)| \leq \sqrt{n}/2} |a(\xi)|^{-n} \, d\xi
$$

$$
\geq \frac{m\omega_{m-1}}{2^{m-1}} |Du|(\Omega)^{-n} \int_{|a(\xi)| \leq \sqrt{n}/2} |a(\xi)|^{-n} \, da(\xi)
$$

$$
= (n-m)\omega_{n-m} \frac{m\omega_{m-1}}{2^{m-1}} |Du|(\Omega)^{-n} \int_0^{\sqrt{n}/2} \rho^{-m-1} \, d\rho
$$

$$
= \infty,
$$

and hence $\mathcal{E}_\Omega(u) = 0$.

An interesting application of Theorem 3.1 of independent interest, is

**Corollary 3.1.** Let $\Omega \subset \mathbb{R}^n$ be a bounded open with Lipschitz boundary. Then,

$$
\mathcal{E}_{\partial\Omega}(\tilde{u}) \geq \mathcal{E}_\Omega(u) + \mathcal{E}_{\partial\Omega}(\tilde{u})
$$

for all $u \in BV(\Omega)$ with $\tilde{u} \not= 0$ on $\partial\Omega$ (a.e.) or without any restriction in case $\partial\Omega$ is non-flat, where the definitions of $\mathcal{E}_{\partial\Omega}(\tilde{u})$ and non-flat boundary were given in the comparison (C3) of the introduction.
Proof. Firstly, the identity
\[ E_{\mathbb{R}^n}(\tilde{u}) = \alpha_n \left( \int_{\mathbb{S}^{n-1}} \left( \int_{\Omega} |\sigma_u(x) \cdot \xi| d(|Du|)(x) + \int_{\partial \Omega} |\tilde{u}(x)| |\nu(x) \cdot \xi| d\mathcal{H}^{n-1}(x) \right)^{-\frac{n}{n-1}} d\xi \right)^{-\frac{1}{n-1}} \]
gives \( E_{\mathbb{R}^n}(\tilde{u}) \geq E_\Omega(\tilde{u}) \) and \( E_{\mathbb{R}^n}(\tilde{u}) \geq E_{\Omega_1}(\tilde{u}) \). Therefore, if \( E_\Omega(u) = 0 \) or \( E_{\Omega_1}(\tilde{u}) = 0 \), the conclusion follows.

Assume that \( E_\Omega(u) \) and \( E_{\Omega_1}(\tilde{u}) \) are nonzero. Set \( g(\xi) = \Psi_\xi(u) \) and \( \tilde{g}(\xi) = \tilde{\Psi}_\xi(u) \), where
\[ \Psi_\xi(u) = \int_{\partial \Omega} |\tilde{u}(x)| |\nu(x) \cdot \xi| d\mathcal{H}^{n-1}(x). \]
By Theorem 3.1, we have \( g(\xi) > 0 \) for all \( \xi \in \mathbb{S}^{n-1} \). Moreover, the condition \( E_{\Omega_1}(\tilde{u}) \neq 0 \) and the assumptions assumed in the statement imply that \( \tilde{g}(\xi) > 0 \) for all \( \xi \in \mathbb{S}^{n-1} \). Then, applying the Minkowski inequality for negative parameters, we get
\[ E_{\mathbb{R}^n}(\tilde{u}) = \alpha_n \left( \int_{\mathbb{S}^{n-1}} (g(\xi) + \tilde{g}(\xi))^{-\frac{n}{n-1}} d\xi \right)^{-\frac{1}{n-1}} \]
\[ \geq \alpha_n \left( \int_{\mathbb{S}^{n-1}} g(\xi)^{-\frac{n}{n-1}} d\xi \right)^{-\frac{1}{n-1}} + \alpha_n \left( \int_{\mathbb{S}^{n-1}} \tilde{g}(\xi)^{-\frac{n}{n-1}} d\xi \right)^{-\frac{1}{n-1}} \]
\[ = E_\Omega(u) + E_{\Omega_1}(\tilde{u}). \]

\[ \square \]

The next step is to establish the lower weak* semicontinuity of \( E_{\mathbb{R}^n} \) on \( L^1_{\text{loc}}(\mathbb{R}^n) \) under uniform boundedness of the total variation. The proof makes use, beyond Theorem 3.1, of essential results by Goffman and Serrin (Theorems 2 and 3 of [20]). We also refer to [3] and [5] and references therein for various extensions and improvements of [20].

Let \( f : \mathbb{R}^n \to \mathbb{R} \) be a nonnegative convex function with linear growth, that is, \( f(w) \leq M(|w| + 1) \) for all \( w \in \mathbb{R}^n \), where \( M > 0 \) is a constant. Define the recession function \( f_\infty : \mathbb{R}^n \to \mathbb{R} \) associated to \( f \) by
\[ f_\infty(w) = \limsup_{t \to \infty} \frac{f(tw)}{t}. \]
For \( u \in BV(\mathbb{R}^n) \), write \( Du = \nabla u \mathcal{L} + \sigma_u^*|D^s u| \) and let \( \Psi : BV(\mathbb{R}^n) \to \mathbb{R} \) defined by
\[ \Psi(u) = \int_{\mathbb{R}^n} f(\nabla u(x)) \, dx + \int_{\mathbb{R}^n} f_\infty(\sigma_u^*(x)) \, d(|D^s u|)(x). \]

**Proposition 3.1** (Goffman-Serrin Theorem). The functional \( \Psi \) is strongly lower semicontinuous on \( L^1_{\text{loc}}(\mathbb{R}^n) \).

**Theorem 3.2.** If \( u_k \to u_0 \) strongly in \( L^1_{\text{loc}}(\mathbb{R}^n) \) and \( |Du_k|(\mathbb{R}^n) \) is bounded, then
\[ E_{\mathbb{R}^n}(u_0) \leq \liminf_{k \to \infty} E_{\mathbb{R}^n}(u_k). \]

**Proof.** Let \( u_k \) be a sequence converging strongly to \( u_0 \) in \( L^1_{\text{loc}}(\mathbb{R}^n) \) such that \( |Du_k|(\mathbb{R}^n) \) is bounded. If \( \Psi_\xi(u_0) = 0 \) for some \( \xi \in \mathbb{S}^{n-1} \), by Theorem 3.1 we have \( E_{\mathbb{R}^n}(u_0) = 0 \) and the conclusion follows trivially.

It then suffices to assume that \( \Psi_\xi(u_0) > 0 \) for all \( \xi \in \mathbb{S}^{n-1} \). Set \( f^\xi(w) = |w \cdot \xi| \) for any \( \xi \in \mathbb{S}^{n-1} \). Since \( f^\xi \) is convex, nonnegative, 1-homogeneous and \( f_{\infty}^\xi = f^\xi \), we have
\[ \Psi_\xi(u) = \int_{\mathbb{R}^n} |\sigma_u(x) \cdot \xi| d(|Du|)(x) \]
\[ = \int_{\mathbb{R}^n} f^\xi(\frac{\nabla u(x)}{|\nabla u(x)|}) |\nabla u(x)| \, dx + \int_{\mathbb{R}^n} f^\xi(\sigma_u^*(x)) \, d(|D^s u|)(x) \]
\[ = \int_{\mathbb{R}^n} f^\xi(\nabla u(x)) \, dx + \int_{\mathbb{R}^n} f_{\infty}^\xi(\sigma_u^*(x)) \, d(|D^s u|)(x). \]
Hence, by Proposition 3.1, \( \Psi_\xi \) is strongly lower semicontinuous on \( L_{1,loc}(\mathbb{R}^n) \), and so

\[
\int_{\mathbb{R}^n} |\sigma_{u_0}(x) \cdot \xi| d(|Du_0|(x)) \leq \liminf_{k \to \infty} \int_{\mathbb{R}^n} |\sigma_{u_k}(x) \cdot \xi| d(|Du_k|(x)).
\] (6)

We now ensure the existence of a constant \( c_0 > 0 \) and an integer \( k_0 \in \mathbb{N} \), both independent of \( \xi \in \mathbb{S}^{n-1} \), such that, for any \( k \geq k_0 \),

\[
\int_{\mathbb{R}^n} |\sigma_{u_k}(x) \cdot \xi| d(|Du_k|(x)) \geq c_0.
\] (7)

Otherwise, modulo a renaming of indexes, we get a sequence \( \xi_k \in \mathbb{S}^{n-1} \) such that \( \xi_k \to \tilde{\xi} \) and

\[
\int_{\mathbb{R}^n} |\sigma_{u_k}(x) \cdot \xi_k| d(|Du_k|(x)) \leq \frac{1}{k}.
\]

Using the assumption that \( |Du_k|(\mathbb{R}^n) \) is bounded, we find a constant \( C_1 > 0 \) such that

\[
\int_{\mathbb{R}^n} |\sigma_{u_k}(x) \cdot \tilde{\xi}| d(|Du_k|(x)) \leq C_1\|\xi_k - \tilde{\xi}\| + \frac{1}{k} \to 0.
\]

Therefore, by (6), we obtain the contradiction \( \Psi_\tilde{\xi}(u_0) = 0 \).

Finally, combining (6), (7) and Fatou’s lemma, we derive

\[
\int_{\mathbb{S}^{n-1}} \left( \int_{\mathbb{R}^n} |\sigma_{u_0}(x) \cdot \xi| d(|Du_0|(x)) \right)^{-\frac{1}{n}} d\xi \\
\geq \int_{\mathbb{S}^{n-1}} \limsup_{k \to \infty} \left( \int_{\mathbb{R}^n} |\sigma_{u_k}(x) \cdot \xi| d(|Du_k|(x)) \right)^{-\frac{1}{n}} d\xi \\
\geq \limsup_{k \to \infty} \int_{\mathbb{S}^{n-1}} \left( \int_{\mathbb{R}^n} |\sigma_{u_k}(x) \cdot \xi| d(|Du_k|(x)) \right)^{-\frac{1}{n}} d\xi,
\]

and thus

\[
\mathcal{E}_{\mathbb{R}^n}(u_0) = \left( \int_{\mathbb{S}^{n-1}} \left( \int_{\mathbb{R}^n} |\sigma_{u_0}(x) \cdot \xi| d(|Du_0|(x)) \right)^{-\frac{1}{n}} d\xi \right)^{-\frac{n}{1}} \\
\leq \liminf_{k \to \infty} \left( \int_{\mathbb{S}^{n-1}} \left( \int_{\mathbb{R}^n} |\sigma_{u_k}(x) \cdot \xi| d(|Du_k|(x)) \right)^{-\frac{1}{n}} d\xi \right)^{-\frac{n}{1}} \\
= \liminf_{k \to \infty} \mathcal{E}_{\mathbb{R}^n}(u_k). \]

As an immediate consequence of Theorem 3.2, we have:

**Corollary 3.2.** If \( u_k \to u_0 \) weakly* in \( BV(\Omega) \), then

\[
\mathcal{E}_{\mathbb{R}^n}(\bar{u}_0) \leq \liminf_{k \to \infty} \mathcal{E}_{\mathbb{R}^n}(\bar{u}_k).
\]

This result is the key point towards the lower weak* semicontinuity of the functional \( \Phi_A : BV(\Omega) \to \mathbb{R} \). We recall that

\[
\Phi_A(u) = \mathcal{E}_{\mathbb{R}^n}(\bar{u}) + \int_{\Omega} a|u| \, dx + \int_{\partial\Omega} b|\bar{u}| \, dH^{n-1},
\]

where \( a \in L^\infty(\Omega) \) and \( b \in L^\infty(\partial\Omega) \) is nonnegative. Since the integral functional on \( \Omega \) is clearly weakly* continuous on \( BV(\Omega) \), it only remains to discuss the semicontinuity of the boundary integral term.
Proposition 3.2. If \( u_k \to u_0 \) weakly* in \( BV(\Omega) \), then
\[
\int_{\partial \Omega} b |\tilde{u}_0| \, dH^{n-1} \leq \liminf_{k \to \infty} \int_{\partial \Omega} b |\tilde{u}_k| \, dH^{n-1}.
\]

Proof. Let \( u_k \) be a sequence converging weakly* to \( u_0 \) in \( BV(\Omega) \). For each \( \varepsilon > 0 \), we consider the norm \( \| \cdot \|_\varepsilon \) on \( BV(\Omega) \),
\[
\| u \|_\varepsilon = \varepsilon |Du| (\Omega) + \int_{\partial \Omega} (b + \varepsilon) |\tilde{u}| \, dH^{n-1}.
\]
Since \( b \) is nonnegative, \( \| \cdot \|_\varepsilon \) is equivalent to \( \| \cdot \|_{BV(\Omega)} \) and \( \| \cdot \|_{BV(\Omega)} \), and so
\[
\| u_0 \|_\varepsilon \leq \liminf_{k \to \infty} \| u_k \|_\varepsilon. \tag{8}
\]
Take a constant \( C > 0 \) so that \( \| u_k \|_{BV(\Omega)}' \leq C \) and a subsequence \( u_{k_j} \) such that
\[
\lim_{j \to \infty} \int_{\partial \Omega} b |\tilde{u}_{k_j}| \, dH^{n-1} = \liminf_{k \to \infty} \int_{\partial \Omega} b |\tilde{u}_k| \, dH^{n-1}.
\]
By (8), for \( j \) large, we get
\[
\varepsilon |Du_0| (\Omega) + \int_{\partial \Omega} (b + \varepsilon) |\tilde{u}_0| \, dH^{n-1} - \varepsilon \leq C \varepsilon + \int_{\partial \Omega} b |\tilde{u}_{k_j}| \, dH^{n-1}.
\]
Letting \( j \to \infty \) and after \( \varepsilon \to 0 \), the statement follows as wished. \( \Box \)

Finally, Corollary 3.2 and Proposition 3.2 lead to

Corollary 3.3. The functional \( \Phi_A \) is lower weakly* semicontinuous on \( BV(\Omega) \).

4 Subcritical constrained minimizations on \( BV(\Omega) \)

We present the proof of Theorems 4.1 and 4.3. The main ingredients are Corollary 4.3 and the following Rellich-Kondrachov type compactness theorem:

Theorem 4.1. The affine ball \( B_A(\Omega) \) is compact in \( L^q(\Omega) \) for any \( 1 \leq q < \frac{n}{n-1} \).

Its proof demands in turn two preliminary results. The first of them relates weak* convergence of displacements of bounded sequences in \( BV(\mathbb{R}^n) \) and strong convergence in \( L^q(\mathbb{R}^n) \). Similar results have been established in other spaces, we refer to \( 1 \ 2 \ 43 \ 47 \) where cocompactness of embeddings are studied in depth. We give the proof for the sake of completeness.

Proposition 4.1. Let \( u_k \) be a bounded sequence in \( BV(\mathbb{R}^n) \). Then, \( u_k (\cdot - y_k) \to 0 \) locally weakly* in \( BV(\mathbb{R}^n) \) for any sequence \( y_k \in \mathbb{R}^n \) if, and only if, \( u_k \to 0 \) strongly in \( L^q(\mathbb{R}^n) \) for any \( 1 < q < \frac{n}{n-1} \).

Proof. Assume first that \( u_k \to 0 \) strongly in \( L^q(\mathbb{R}^n) \) for some \( 1 < q < \frac{n}{n-1} \). If \( v_k = u_k (\cdot - y_k) \) doesn’t converge locally weakly* to zero in \( BV(\mathbb{R}^n) \) for some sequence \( y_k \in \mathbb{R}^n \) and \( \varepsilon > 0 \) such that, modulo a subsequence, \( \| v_k \|_{L^q(\Omega)} \geq \varepsilon \) or \( |dv_k (\varphi)| \geq \varepsilon \) for some \( \varphi \in C_0^\infty (\Omega) \), where \( dv (\varphi) = \int_\Omega \varphi \, dv \).

Since \( v_k \) is bounded in \( BV(\mathbb{R}^n) \), one may assume that \( v_k \rightharpoonup v \) weakly* in \( BV(\Omega) \). Thus, letting \( k \to \infty \) in the two cases, one gets \( \| v \|_{L^q(\Omega)} \geq \varepsilon \) or \( |dv (\varphi)| \geq \varepsilon \). On the other hand, one knows that \( v_k \to 0 \) strongly in \( L^q(\mathbb{R}^n) \) and \( v_k \rightharpoonup v \) strongly in \( L^1 (\Omega) \), so \( v = 0 \) in \( \Omega \). But this contradicts the last two inequalities.

Conversely, assume that \( u_k (\cdot - y_k) \to 0 \) locally weakly* in \( BV(\mathbb{R}^n) \) for any sequence \( y_k \in \mathbb{R}^n \). Choose a fixed \( 1 < q < \frac{n}{n-1} \) and consider the n-cube \( Q = (0, 1)^n \).
Using the continuity of the Sobolev immersion $BV(Q) \hookrightarrow L^q(Q)$, we deduce that
\[
\int_{Q+y}|u_k|^q \, dx = \int_Q |u_k(x-y)|^q \, dx
\leq C \|u_k(\cdot - y)\|_{BV(Q)} \left( \int_Q |u_k(x-y)|^q \, dx \right)^{1-\frac{q}{n}}
= C \|u_k\|_{BV(Q+y)} \left( \int_Q |u_k(x-y)|^q \, dx \right)^{1-\frac{q}{n}}
\]
for every $y \in \mathbb{R}^n$, where $C$ is a constant independent of $y$.

By adding the inequality over $y \in \mathbb{Z}^n$, we obtain
\[
\int_{\mathbb{R}^n} |u_k|^q \, dx \leq C \sup_{y \in \mathbb{Z}^n} \left( \int_Q |u_k(x-y)|^q \, dx \right)^{1-\frac{q}{n}}.
\] (9)

We claim that the right-hand side of (9) is finite. Since $u_k$ is bounded in $BV(\mathbb{R}^n)$, it is also bounded in $L^1(\mathbb{R}^n)$ and in $L^{\frac{n}{n-q}}(\mathbb{R}^n)$ by Sobolev inequality. Then, the finiteness follows from the assumption $1 < q < \frac{n}{n-1}$ and a simple interpolation.

Choose $y_k \in \mathbb{Z}^n$ so that
\[
\left( \int_Q |u_k(x-y_k)|^q \, dx \right)^{1-\frac{q}{n}} \geq \frac{1}{2} \sup_{y \in \mathbb{Z}^n} \left( \int_Q |u_k(x-y)|^q \, dx \right)^{1-\frac{q}{n}}.
\]

Hence, (9) gives
\[
\int_{\mathbb{R}^n} |u_k|^q \, dx \leq 2C_1 \left( \int_Q |u_k(x-y_k)|^q \, dx \right)^{1-\frac{q}{n}}
\] (10)
for some constant $C_1$ independent of $k$.

On the other hand, the strict inequality $q < \frac{n}{n-1}$ allows us to apply the Rellich-Kondrachov compactness theorem to the embedding $BV(Q) \hookrightarrow L^q(Q)$ in order to estimate the right-hand side of (10). In fact, module a subsequence, we have $v_k = u_k(\cdot - y_k) \to v$ strongly in $L^1(Q)$. But, by assumption, $v_k \to 0$ locally weakly* in $BV(\mathbb{R}^n)$, and so $v_k \to 0$ strongly in $L^1(Q)$. Therefore, $v = 0$ in $Q$ and, since $q > 1$, we deduce from (10) that $u_k \to 0$ strongly in $L^q(\mathbb{R}^n)$.

As noted in the introduction, exist no upper bound for $|Du|(\mathbb{R}^n)$ in terms of $\mathcal{E}_{\mathbb{R}^n}u$ on $BV(\mathbb{R}^n)$. Nonetheless, Huang and Li (Theorem 1.2 of [27]) proved that such an estimate holds true for functions $u \in W^{1,1}(\mathbb{R}^n)$ unless an adequate affine transformation $T$ depending on $u$. The result is also valid in $BV(\mathbb{R}^n)$, thanks to the necessary tools that were extended by Wang in [49].

**Proposition 4.2** (Huang-Li Theorem). For any $u \in BV(\mathbb{R}^n)$, one has
\[
d_0 \min_{T \in SL(n)} |D(u \circ T)|(\mathbb{R}^n) \leq \mathcal{E}_{\mathbb{R}^n}u,
\]
where $d_0 = 4^{-1} \pi \Gamma(\frac{n+1}{2})\Gamma(n+1)\Gamma(\frac{q}{2} + 1) - \frac{q}{n-1}$.

**Proof of Theorem**. Let $u_k$ be a sequence in $B_{A}(\Omega)$. By Proposition 4.2, there is a matrix $T_k \in SL(n)$ such that $d_0|D(\bar{u}_k \circ T_k)|(\mathbb{R}^n) \leq \mathcal{E}_{\mathbb{R}^n}u_k$. Note also that $\|\bar{u}_k \circ T_k\|_{L^1(\mathbb{R}^n)} = \|u_k\|_{L^1(\Omega)}$, so $v_k = \bar{u}_k \circ T_k$ is bounded in $BV(\mathbb{R}^n)$. We now analyze two possibilities.

Assume first that $|T_k| \to \infty$. Let $y_k$ be an arbitrary sequence in $\mathbb{R}^n$. The boundedness of $v_k(\cdot - y_k)$ in $BV(\mathbb{R}^n)$ implies, module a subsequence, that $v_k(\cdot - y_k) \to \bar{v}$ locally weakly* in $BV(\mathbb{R}^n)$. Since $q < \frac{n}{n-1}$, the Rellich-Kondrachov compactness theorem also gives $v_k(\cdot - y_k) \to \bar{v}$ strongly in $L^q_{loc}(\mathbb{R}^n)$ and $v_k(x-y_k) \to \bar{v}(x)$ almost everywhere in $\mathbb{R}^n$, up to a subsequence.
Consider the set
\[
X = \liminf_{k \to \infty} T_k^{-1}(\Omega + T_k(y_k)) = \bigcup_{m \geq 1} \bigcap_{k \geq m} T_k^{-1}(\Omega + T_k(y_k)).
\]
Since \(|T_k| \to \infty\) and \(\Omega\) is bounded, \(X\) has zero Lebesgue measure (e.g. page 7 of [44]). For \(x \notin X\), we have \(x \notin \bigcap_{k \geq m} T_k^{-1}(\Omega + T_k(y_k))\) for any \(m \geq 1\), which yields \(T_k(x - y_k) \notin \Omega\) for every \(k\), up to a subsequence. Thus,
\[
\bar{v}(x) = \lim_{k \to \infty} v_k(x - y_k) = \lim_{k \to \infty} \bar{u}_k(T_k(x - y_k)) = 0 \quad \text{and hence} \quad v_k(\cdot - y_k) \rightharpoonup 0 \text{ locally weakly}^* \text{ in } BV(\mathbb{R}^n) \text{ for any sequence } y_k \text{ in } \mathbb{R}^n.
\]
By Proposition 3.2, \(\bar{u}_k \rightharpoonup 0\) strongly in \(L^q(\mathbb{R}^n)\) and so \(u_k \rightharpoonup 0\) strongly in \(L^q(\Omega)\).

If \(|T_k| \neq \infty\), then one may assume that \(T_k\) converges to some \(T \in SL(n)\). Choose \(R > 0\) large enough so that \(T^{-1}(\Omega) \subset B_R\) and \(T_k^{-1}(\Omega) \subset B_R\) for every \(k\). Module a subsequence, we know that \(v_k \rightharpoonup v_0\) weakly* in \(BV(B_R)\) and \(u_k \rightharpoonup v_0\) strongly in \(L^q(B_R)\).

Set \(u_0 = v_0 \circ T^{-1}\) in \(\Omega\). Notice that \(u_0 \in BV(\Omega)\) once \(T^{-1}(\Omega) \subset B_R\). Let \(\bar{u}_0 \in BV(\mathbb{R}^n)\) be the extension of \(u_0\) by zero outside of \(\Omega\). Since \(T \circ T_k^{-1}\) converges to the identity \(I\), by the generalized dominated convergence theorem, it follows that \(\|\bar{u}_0 \circ T \circ T_k^{-1} - u_0\|_{L^q(\Omega)} \to 0\). Consequently, since \(T_k^{-1}(\Omega) \subset B_R\), we have
\[
\|u_k - u_0\|_{L^q(\Omega)} \leq \|v_k \circ T_k^{-1} - v_0 \circ T_k^{-1}\|_{L^q(\Omega)} + \|\bar{u}_0 \circ T \circ T_k^{-1} - u_0\|_{L^q(\Omega)} \to 0.
\]

A fact that follows from the proof and deserves to be highlighted is

**Corollary 4.1.** Let \(u_k\) be a sequence in \(B_A(\Omega)\) such that \(u_k \rightharpoonup u_0\) strongly in \(L^q(\Omega)\) for some \(1 \leq q < \frac{n}{n-1}\). If \(u_0 \neq 0\), then \(u_k\) is bounded in \(BV(\Omega)\).

Theorems 1.1 and 1.3 can now be proved by using the previous developments.

*Proof of Theorem 1.1.* Let \(u_k\) be a minimizing sequence of \(\Phi_A\) in \(X\). By Hölder’s inequality, \(u_k\) is bounded in \(L^1(\Omega)\) and, since \(b \geq 0\) on \(\partial \Omega\), the affine energy \(E_{\mathbb{R}^n,u_k}\) is also bounded. Therefore, by Theorem 1.1, there exists \(u_0 \in BV(\Omega)\) such that \(u_k \rightharpoonup u_0\) strongly in \(L^q(\Omega)\). Therefore, \(u_0 \in X\) and, by Corollary 4.1, \(u_0\) is bounded in \(BV(\Omega)\).

Passing to a subsequence, if necessary, one may assume that \(u_k \rightharpoonup u_0\) weakly* in \(BV(\Omega)\). Then, by Corollary 3.3, we derive
\[
\Phi_A(u_0) \leq \liminf_{k \to \infty} \Phi_A(u_k) = c_A,
\]
and thus \(u_0\) minimizes \(\Phi_A\) in \(X\).

The same argument also works for a minimizing sequence \(u_k\) of \(\Phi_A\) in \(Y\). So, \(u_k \rightharpoonup u_0\) weakly* in \(BV(\Omega)\) and \(u_k \rightharpoonup u_0\) strongly in \(L^1(\Omega)\), module a subsequence, and thus \(u_0 \in X\) and
\[
\Phi_A(u_0) \leq \liminf_{k \to \infty} \Phi_A(u_k) = d_A.
\]
It remains to check that \(u_0 \in Y\), which it follows readily from Theorem 1.1 applied to \(L^r(\Omega)\) for \(1 \leq r < \frac{n}{n-1}\).

*Proof of Theorem 1.3.* Applying Proposition 5.2 with \(b = 1\), we conclude that the space \(BV_0(\Omega)\) is weakly* closed in \(BV(\Omega)\). Then, the proof can be performed for the restriction of \(\Phi_A\) to \(BV_0(\Omega)\) exactly as the previous one.
5 Critical constrained minimizations on $BV(\Omega)$

We prove Theorems 1.2 and 1.4 by using Theorem 4.1, Corollary 3.3, Corollary 4.1 and the next result.

Consider the truncation for $h > 0$:

$$T_h(s) = \min(\max(s, -h), h) \quad \text{and} \quad R_h(s) = s - T_h(s).$$

Proposition 2.3 of [7] ensures that $|Du|(\mathbb{R}^n) = |DT_hu|(\mathbb{R}^n) + |DR_hu|(\mathbb{R}^n)$ for every $u \in BV(\mathbb{R}^n)$. Unfortunately, such an identity is not valid within the affine setting, however, using Theorem 3.1 it is still possible to establish an inequality.

**Proposition 5.1.** For any $u \in BV(\mathbb{R}^n)$,

$$\mathcal{E}_{R^n}(u) \geq \mathcal{E}_{R^n}(T_h u) + \mathcal{E}_{R^n}(R_h u).$$

**Proof.** We first prove the inequality for functions $u \in W^{1,1}(\mathbb{R}^n)$. From the definition of $T_h(s)$, we have $T_h u, R_h u \in W^{1,1}(\mathbb{R}^n)$ and $\Psi_\xi(u) = \Psi_\xi(T_h u) + \Psi_\xi(R_h u)$ for all $\xi \in \mathbb{S}^{n-1}$, where

$$\Psi_\xi(u) = \int_{\mathbb{R}^n} |\nabla \xi u(x)| \, dx.$$  

Note that this decomposition implies $\mathcal{E}_{R^n}(u) \geq \mathcal{E}_{R^n}(T_h u)$ and $\mathcal{E}_{R^n}(u) \geq \mathcal{E}_{R^n}(R_h u)$. Thus, the statement follows if $\mathcal{E}_{R^n}(T_h u) = 0$ or $\mathcal{E}_{R^n}(R_h u) = 0$.

Assuming that $\mathcal{E}_{R^n}(T_h u)$ and $\mathcal{E}_{R^n}(R_h u)$ are nonzero, by Theorem 3.1 we have $\Psi_\xi(T_h u), \Psi_\xi(R_h u) > 0$ for all $\xi \in \mathbb{S}^{n-1}$. So, by the Minkowski’s inequality for negative exponents, we get

$$\mathcal{E}_{R^n}(u) = \alpha_n \left( \int_{\mathbb{S}^{n-1}} (\Psi_\xi(T_h u) + \Psi_\xi(R_h u))^{-n} \, d\xi \right)^{-\frac{1}{n}}$$

$$\geq \alpha_n \left( \int_{\mathbb{S}^{n-1}} (\Psi_\xi(T_h u))^{-n} \, d\xi \right)^{-\frac{1}{n}} + \alpha_n \left( \int_{\mathbb{S}^{n-1}} (\Psi_\xi(R_h u))^{-n} \, d\xi \right)^{-\frac{1}{n}}$$

$$= \mathcal{E}_{R^n}(T_h u) + \mathcal{E}_{R^n}(R_h u).$$

Finally, the inequality extends to $BV(\mathbb{R}^n)$ by using both the density of $W^{1,1}(\mathbb{R}^n)$ in $BV(\mathbb{R}^n)$ and the continuity of $u \in BV(\mathbb{R}^n) \mapsto \mathcal{E}_{R^n}(u)$ with respect to the strict topology. 

**Proof of Theorems 1.2 and 1.4.** Thanks to the weak* closure of $BV_0(\Omega)$ in $BV(\Omega)$, it is enough to just prove Theorem 1.2.

Let $u_k$ be a minimizing sequence of $\Phi_A$ in $X$. Proceeding as in the proof of Theorem 1.1 by Theorem 4.1 we have $u_k \rightharpoonup u_0$ strongly in $L^1(\Omega)$, module a subsequence. One may also assume that $u_k \to u_0$ almost everywhere in $\Omega$ and $T_h u_k \to T_h u_0$ weakly in $L^{\infty,n}(\Omega)$.

Using the Sobolev-Zhang inequality on $BV(\mathbb{R}^n)$,

$$n \omega_n^{1/n} \left( \int_{\mathbb{R}^n} |u|^{\frac{n}{n-1}} \, dx \right)^{\frac{n-1}{n}} \leq \mathcal{E}_{R^n}(u),$$

and that $b$ is nonnegative, we derive

$$c_A = \lim_{k \to \infty} \left( \mathcal{E}_{R^n}(\tilde{u}_k) + \int_{\Omega} a|u_k|^dx + \int_{\partial\Omega} b|\tilde{u}_k|d\mathcal{H}^{n-1} \right)$$

$$\geq n \omega_n^{1/n} + \int_{\Omega} a|u_0| \, dx,$$
so the condition $c_A < n\omega_n^{1/n}$ implies that $u_0 \neq 0$. Hence, by Corollaries 3.3 and 4.1 we have $u_k \rightharpoonup u_0$ weakly* in $BV(\Omega)$ and $\Phi_A(u_0) \leq c_A$. It only remains to show that $u_0 \in X$.

By Proposition 5.1 we easily deduce that

\[
c_A = \lim_{k \to \infty} \Phi_A(u_k) \\
\geq \lim_{k \to \infty} (\Phi_A(T_h u_k) + \Phi_A(R_h u_k)) \\
\geq c_A \lim_{k \to \infty} \left( \|T_h u_k\|_{L^{\frac{n}{n-1}}(\Omega)} + \|R_h u_k\|_{L^{\frac{n}{n-1}}(\Omega)} \right).
\]

Applying Lemma 3.1 of [7], we have

\[
c_A \geq c_A \left[ \|T_h u_0\|_{L^{\frac{n}{n-1}}} + \left(1 + \|R_h u_0\|_{L^{\frac{n}{n-1}}} - \|u_0\|_{L^{\frac{n}{n-1}}} \right) \frac{n-1}{n} \right].
\]

Using the condition $c_A > 0$ and letting $h \to \infty$, one obtains

\[
1 \geq \left( \|u_0\|_{L^{\frac{n}{n-1}}} \right) \frac{n-1}{n} + \left(1 - \|u_0\|_{L^{\frac{n}{n-1}}} \right) \frac{n-1}{n},
\]

and thus $u_0 \in X$ because $u_0 \neq 0$.

If the minimizing sequence $u_k$ of $\Phi_A$ is taken in $Y$, the same strategy of proof produces $u_k \to u_0$ almost everywhere in $\Omega$, $u_0 \in X$ and $\Phi_A(u_0) \leq d_A$. On the other hand, the first two properties along with Brezis-Lieb Lemma imply that $u_k \to u_0$ strongly in $L^{\frac{n}{n-1}}(\Omega)$. Finally, since $1 \leq r \leq \frac{n}{n-1}$, it follows that $u_0 \in Y$.

\[\square\]

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