Radiation comb generation with extended Josephson junctions

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We propose the implementation of a Josephson radiation comb generator (JRCG) based on an extended Josephson junction subject to a time dependent magnetic field. The junction critical current shows known diffraction patterns and determines the position of the critical nodes when it vanishes. When the magnetic flux passes through one of such critical nodes, the superconducting phase must undergo a $\pi$-jump to minimize the Josephson energy. Correspondingly a voltage pulse is generated at the extremes of the junction. Under periodic driving this allows us to produce a comb-like voltage pulses sequence. In the frequency domain it is possible to generate up to hundreds of harmonics of the fundamental driving frequency, thus mimicking the frequency comb used in optics and metrology. We discuss several implementations through a rectangular, cylindrical and annular junction geometries.

The field of optical combs has seen a growing interest in recent years. The atomic clocks are extremely stable and have sharp resonances; for these reasons, they are used as time and frequency reference standards. However, their working range is limited to the radio frequency region. This limitation has been overtaken only a decade ago. A combination of technical and conceptual improvements has allowed to extend this accuracy to higher frequency up to the optical region. The key phenomenon is simple: the atoms are manipulated to emit periodic sharp energy pulses which, in the frequency domain, correspond to a comb signal with the harmonics of the fundamental frequency. Since the generated harmonics show very sharp resonances, they can be used as a frequency standard in the optical region. The realization of the optical frequency comb has paved the way to important applications in optical metrology, high precision spectroscopy, and telecommunication technologies.

Recently, it has been shown that a similar frequency comb can be generated with a dc superconducting quantum interference device (SQUID) subject to a time-dependent magnetic field. The driving induces $\pi$-jumps of the superconducting phase which are associated to voltage pulses generated at the extremes of the device. The voltage pulses sequence translates into a radiation comb in frequency domain with up to hundreds of harmonics of the fundamental driving frequency.

Here, we show how similar effect can be obtained in an extended Josephson junction. The underlying physics is similar to that discussed in Ref. 9, but the details of the implementation are different. A setup involving extended junctions opens up the possibility for different geometries and, therefore, for various power spectra of the emitted radiation. In particular, since the generated radiation comb structure depends on the current-magnetic flux relation of the junction, this latter can be properly engineered in order to obtain a desired radiation power spectrum. We discuss the rectangular, cylindrical and annular junction designs with their different strengths and weaknesses as prototypical examples of extended junctions.

Our system is sketched in Fig. 1(a), and consists of an extended Josephson tunnel junction composed of two identical superconducting electrodes. We denote by $\lambda$ and $t$ the London penetration depth and the thickness of the superconductors $S$, respectively, which satisfy the condition $t > \lambda$. Furthermore, $d$ is the insulator thickness, whereas $t_H = 2\lambda + d$ is the magnetic penetration thickness.

For the sake of clarity we focus on a junction in the short limit, i.e., with lateral dimensions much smaller than the Josephson penetration depth. In such a case the self-field generated by the Josephson current in the weak-link can be neglected with respect to the externally applied magnetic field and no traveling solitons are originated. We choose a coordinate system such that the applied magnetic field ($H$), directed along the $x$ direction, is parallel to a symmetry axis of the junction whose electrodes planes lies in the $xy$ plane [see Fig. 1(a)].

For the sake of presentation, we focus on a rectangular junction as in Fig. 1(b), keeping in mind that a similar discussion can be extended to junctions with different geometries [Figs. 1 (c)-(d)]. In the limit of short junctions, the approximate behavior of the local phase is $\phi(y) = \kappa y + \phi$ where $\phi$ is the superconducting phase at the center of the junction, $\kappa = 2\pi\Phi/(\Phi_0 L)$ ($\Phi_0 \simeq 2 \times 10^{-15}$ Wb is the flux quantum), and

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\( L \) is the length of the junction whereas \( \Phi = \mu_0 H_0 L \) is the magnetic flux through the junction and \( \mu_0 \) is the vacuum permeability. By integrating the Josephson current density per unit length \(^{10} I_0(y) \) over the junction length we obtain

\[
I_{j}^{\text{rect}}(\Phi) = \int_{-L/2}^{L/2} dy J_c(y) \sin \phi(y) = I_+ \sin \frac{\pi \Phi/\Phi_0}{\pi \Phi/\Phi_0} \sin \varphi, \tag{1}
\]

where \( I_+ \) is the maximal critical current of the junction. The measurable critical current of the junction as a function of the magnetic flux is \( I_{j}^{\text{rect}}(\Phi) = \max_\varphi I_{j}^{\text{rect}}(\Phi) \). It displays the celebrated Fraunhofer pattern which vanishes at the diffraction nodes appearing at \( \Phi = n\Phi_0 \), where \( n \) is an integer [Fig. 2 (a)].

The phase jump phenomenon we are interested in can be easily described in energetic terms. Assuming that there is no bias current, the energy associated to the Josephson current is

\[
E_J(t) = \int I_J V(t) dt = -E_{J0} f(t) \cos \varphi \quad \text{where} \quad f(t) = \sin(2\pi \Phi/\Phi_0), \quad E_{J0} = \Phi_0 I_+/2\pi. \]

Initially \( f(t = 0) = 1 \) and the minima of the potential energy are found at \( \varphi = 2\pi k \) (with \( k \) integer). When the magnetic flux reaches the diffraction node at \( \Phi = \Phi_0 \), \( E_J \) vanishes, becoming negative for \( \Phi > \Phi_0 \). To remain in a minimum energy state, \( \cos \varphi \) must change sign, which implies that the superconducting phase must undergo a \( \pi \) jump. The original prediction of \( \pi \) jumps \(^7\) has also been recently indirectly confirmed via measurements in heat transport experiments performed in temperature-biased Josephson tunnel junctions \(^{12,13} \). Possible applications of this phenomenon for SQUID devices has been extensively discussed in Ref. \(^6\).

To determine the details of the voltage pulses, such as their shape and amplitude, the above discussed energetic picture is not sufficient. We rely on the so-called resistively and capacitively shunted Josephson junction (RCSJ) model \(^{10,14} \) in which the Josephson junction is modeled as a circuit with a capacitor \( C \), a resistor \( R \), and a non-linear (Josephson) inductance \( L_J \) arranged in a parallel configuration. We consider a sinusoidally-driven magnetic flux with frequency \( \nu \) and amplitude \( \epsilon \), centered in the first node of the interference pattern [Fig. 2 (a)], so that \( \Phi(t) = \Phi_0/2[1 - \epsilon \cos(2\pi \nu t)] \). As a result, the magnetic flux crosses the nodes of the interference pattern at \( t = (2k + 1)/4\nu \), with \( k \) integer. Starting from the RCSJ model we can write an equation of motion for the integrated phase \( \int_{-L/2}^{L/2} dy \varphi(y) \). Because of the symmetry of the problem, this reduces to a RCSJ equation for the phase at the center of the junction \( \varphi \): \(^{11} \)

\[
\frac{\hbar C}{2e} \frac{\partial \Phi}{\partial t} + \frac{\hbar}{2eR} \Phi - I_+ f(t) \sin \varphi = I_B. \tag{2}
\]

We recast the above equation in terms of dimensionless time \( \tau = 2\pi \nu t \) and, using \( \hbar / (2eC) = \Phi_0 / 2\pi \), we obtain \(^{6} \)

\[
c \frac{d^2 \varphi}{d\tau^2} + \frac{d \varphi}{d\tau} - \alpha [f(\tau) \sin \varphi - \delta] = 0, \tag{3}
\]

where \( \delta = I_B / I_+ \), \( c = 2\pi RC \nu \) and \( \alpha = I_R / (\Phi_0 \nu) \). The bias current is supposed to be small (\( \delta \ll 1 \)) and its effect is to impose a preferred direction to the \( \pi \) jumps of the phase. Furthermore, we focus on the limits \( c \ll 1 \) (overdamped regime)

\[
\text{FIG. 2. Diffraction pattern of the critical current for different geometries: (a) rectangular junction, (b) circular junction and (c) annular junction with different numbers } n \text{ of trapped fluxons: } n = 1 \text{ (solid blue line) and } n = 2 \text{ (dashed green line). The red line shows } \Phi(t)/\Phi_0 \text{ oscillating with driving frequency } \nu \text{ and amplitude } \epsilon \text{ around different diffraction nodes.}
\]

\[
\text{and } |I_c/R| \gg 1, \text{ as these two conditions maximize the JRCG performance} \(^6\).
\]

The numerical solution of Eq. (3) is shown in Fig. 3(a) for a rectangular junction made of Nb/AlOx/Nb subject to a 100 MHz driving. As the critical current crosses the diffraction node at \( \Phi = \Phi_0 \), the phase experiences a \( \pi \) jump and a voltage pulse is generated across the junction. The shape of the pulse is determined by the product \( I_c R \): the larger \( I_c R \), the sharper the voltage pulse. Differently to what happens in the SQUID implementation \(^6\), for a rectangular junction the pulse amplitudes are not the same but show an alternating pattern of lower and higher peaks. This is due to the asymmetry in the diffraction pattern of the critical current near the diffraction node [see Fig. 2 (a)].

This real-time voltage comb could be used in several ways. One is as a generator of equally spaced voltage pulses to be used in electronics \(^{14} \). A second one is as a high precision controller. In fact, because of the Josephson relation, the time average of a voltage pulse is actually quantized as a consequence of the \( \pi \) jump of the phase: \( 2e/\hbar \int_{t_1}^{t_2} dt V(t) = \Delta \Phi = \pi \) where \( t_1 \) and \( t_2 \) are the times in which the jump begins and ends, respectively. The phase jumps does not depends on the dynamics or on the speed of the node crossing. The only condition to be satisfied is the crossing of the diffraction node. This makes the pulse generation robust against imperfection in the dynamics of the junction and the driving. A possible application could be the high precise control of a quantum logic gate for superconducting based qubits \(^{15,16} \).

The voltage pulse sequence in Fig. 3 has even more interesting applications as a radiation generator. In fact, in the fre-
quency domain it corresponds to a frequency comb similar to the ones used in optics\(^1\). To test this possible implementation we have calculated the power spectrum \(P\) vs frequency \(\Omega\). We first compute the Fourier transform of the voltage \(V(\Omega) = \int_0^T dt e^{i\Omega t} V(t)\); the power spectral density (PSD) is then PSD(\(\Omega\)) = \(1/T|V(\Omega)|^2\). Finally, the power \(P\) is calculated by integrating the PSD around the resonances \(k\nu\) (where \(\nu\) is the monochromatic driving frequency) and dividing for a standard load resistance of 50 Ohm. This is the power we would measure at a given resonance frequency with a bandwidth exceeding the linewidth of the resonance.

To increase the output power, we have considered an array of identical junctions as done for the metrological standard for voltage based on the Josephson effect\(^6,17–19\). As the voltage scales as the number \(N\) of junctions, the power scales as \(N^2\) allowing us to increase the output power at high frequency.

In our analysis we neglect the coupling between the junctions composing the array. This condition can be realized in practice by a suitable design choice which reduces the cross capacitance and the inductance between neighbor junctions. In this case, the current conservation through any \(i\)-th junction leads immediately to a set of decoupled RCSJ equations of the form (3)\(^6\). Therefore, the dynamics of the junctions is independent and the voltage at the extremes of the array is found by summing up the voltages of the single junctions.

Limitations to this simplified analysis can arise if we must take into account the effects of propagation of the emitted radiation along the chain\(^6\). In fact, in our model we have considered the device as a lumped element and this assumption breaks down as the length of the chain approaches the wavelength of the emitted radiation. As a quantitative estimate, the minimum wavelength \(\lambda_{\text{min}}\) (emitted at 50 GHz) must satisfy the relation \(2\lambda_{\text{min}} \geq L\) where \(L\) is the total length of the device\(^6\). Considering a packing density of the junctions of 5 \(\mu\)m, the above condition is satisfied for \(N = 10^{3}\) junctions. Despite being detrimental for the device performance, the propagation effects can be taken into account and corrected. With a careful design of the device, they could also be exploited to amplify the output power at specific working frequencies.

Figure 4 shows the emitted radiation power spectrum for a chain of \(N = 10^{3}\)\(^{11,20–22}\) Nb/AlOx/Nb rectangular junctions driven by a 100 MHz oscillating magnetic field. As we can see, the device is able to provide a power of about 10 pW at 50 GHz (corresponding to the 500-th harmonic of the driving frequency). Because the voltage pulse is almost rectified, the spectra contain predominantly the even harmonics.

The implementation with extended junctions opens the way also for a geometric optimization. Choosing a different junction geometry affects the critical current of the junction and, therefore, the position and the form of the nodes. For the circular geometry the Josephson current exhibits the known Airy diffraction pattern,

\[
I_{\text{J}}(\Phi) = I_c \frac{J_1(\pi\Phi/\Phi_0)}{(\pi\Phi/2\Phi_0)} \sin \varphi, \tag{4}
\]

where \(J_1(y)\) is the Bessel function of the first kind, \(\Phi = 2\mu_0 H R_{\text{eff}}\) and \(R\) is the junction radius. For the annular junction\(^23,24\), the Josephson current takes the form

\[
I_{\text{J}}^{\text{ann}}(\Phi) = I_c + \frac{2}{1 - \alpha^2} \int_0^{\alpha} dx x J_n(x\pi\Phi/\Phi_0) \sin \varphi, \tag{5}
\]

where \(\Phi = 2\mu_0 H R_{\text{eff}}, \alpha = r/R, J_n(y)\) is the \(n\)-th Bessel function of integer order, \(R\) is the external (internal) radius, and \(n = 0, 1, 2, \ldots\) is the number of trapped fluxons in the junction barrier. The critical currents for these two junction geometries are defined as \(I_{\text{c circ}}(\Phi) = \max \Phi \mu_0 I_{\text{J circ}}(\Phi)\) and \(I_{\text{c ann}}(\Phi) = \max \Phi I_{\text{J ann}}(\Phi)\), respectively, and they are shown in Fig. 2 (b) and (c), respectively. The driving is assumed to have the same periodic behavior, oscillating around the diffraction nodes as shown in Fig. 2.

The power spectrum of the circular junction is similar to the rectangular junction one. It generates smaller output power at high frequency reaching 2 pW at 50 GHz.

Particularly relevant is the annular junction case. Here, we have an additional controllable parameter: the number \(n\) of fluxons trapped in the junction. The most interesting situation is when there is one fluxon trapped in the junction [see Fig. 4 (c)]. In this case, it is possible to modulate the magnetic flux near the vanishing point [see Fig. 2 (c)] making the flux driving easier. In addition, the diffraction pattern is highly symmetric near \(\Phi = 0\). This allows one to generate a very precise voltage pulse patterns that is eventually reflected in a stronger power output at high frequency, as shown in Fig. 4 (c) (0.1nW at 50GHz).

By varying the number of fluxons, the junction diffraction pattern changes [see Fig. 2 (c)]. Correspondingly, the dynamics of the junction is different, generating different emitted radiation power spectra. Figure 4 (d) shows the spectrum generated by an annular junction chain when two fluxons are trapped in each junction. The overall power emitted is smaller and the signal is accessible up to \(\sim 10\)GHz. The spectral features are very different with respect to the single fluxon ones [Fig. 4 (c)]. In particular, the lower harmonics (a few multiple of \(\nu\)) are now suppressed while the output maximum arises around a few GHz.
The main sources of error that can limit the device performance are the imprecisions in the fabrication process. Small differences in the geometry of the junctions, i.e., length $L$ for the rectangular junctions and radii $\mathcal{R}$ and $r$ for the circular and annular junctions, will produce off-sets in the fluxes and delays in the phase jumps. The voltages will still sum up but the total voltage pulse shape will be broadened by these effects. In the frequency domain, this corresponds to an additional cutoff at high frequency. Another potential detrimental factor is the correction to the dynamics due to the intrinsic junction capacitance. However, this effect can be accounted for, minimized or corrected by a proper device design.

The junction array configurations discussed above are suitable for the use as radiation emitters up to 50GHz. The most straightforward way to detect the power in this frequency range is to couple the device to a transmission line and to feed the signal to a commercial spectrum analyzer. To have access to higher frequency we must use different materials (for example, YBCO as discussed in Ref.\textsuperscript{6}) or adopt specific chain design. This change must be accompanied with a new detection schemes, for example, by using antennas coupled to the device electrodes\textsuperscript{8}. Finally, in light of possible implementations, besides the Nb/AlOx/Nb junctions considered in the present work, we signal that other materials could be promising candidates. For instance Nb/HITI/Nb junctions\textsuperscript{25,26}, being SNS-like (superconductor-normal metal-superconductor), would have the advantage of having almost negligible capacitance, despite having a slightly lower $I_c R$ product.

In summary, we have discussed the possibility to realize a Josephson radiation comb generator with \textit{extended} Josephson junctions driven by a time-dependent magnetic field. With an array of $N = 10^3$ Nb/AlOx/Nb junctions and a driving frequency of 100 MHz, we estimate that substantial power [up to $\sim$100 pW] can be generated at 50 GHz (500-th harmonics), opening the way to a number of applications. The device has room for optimization by modeling the geometry of the single junctions, the fabrication materials (see, for example, Ref.\textsuperscript{6}), the driving signal and the array design. The discussed implementation would have the advantage to be built-on-chip and integrated in low-temperature superconducting microwave electronics.

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