Spatial Localization and Relativistic Transformation of Quantum Spins

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The purity of the reduced state for spins which is pure in the rest frame will most likely appear to degrade because spin and momentum become mixed when viewed by a moving observer. We show that such boost-induced decrease in spin purity observed in a moving reference frame is intrinsically related to the spatial localization properties of the wave package observed in the rest frame. Furthermore, we prove that, for any localized pure state with separable spin and momentum in the rest frame, its reduced density matrix for spins inevitably appears to be mixed whenever viewed from a moving reference frame.

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I. INTRODUCTION

One of the most nontrivial and striking observations of relativistic thermodynamics \textsuperscript{1} is that probability distributions can depend on frames. Consequently, entropy and information may change if viewed from different reference frames \textsuperscript{2}. Recently, relativistic quantum information theory has attracted particular interests \textsuperscript{3, 4, 5, 6, 7, 8, 9}. Current investigations show that a single quantum spin is not covariant under Lorentz transformations \textsuperscript{8}, and maximal entanglement between spins in the rest frame will most likely degrade due to mixing with the momentum if viewed from a moving frame \textsuperscript{4, 9}, depending on the initial momentum wave function. For many quantum information protocols \textsuperscript{10}, coherence and entanglement are extremely important and expensive. The observation that coherence and entanglement of the reduced state for spins may degrade when viewed in moving frames implies that there are particular problems for relativistic quantum information processing, particularly for relativistic quantum communication.

It is known that, for a single quantum spin, if and only if we consider the momentum eigenstates (plane waves), can the reduced density matrix for the spin be covariant under Lorentz transformations. But momentum eigenstates are not localized so they may be difficult in feasible applications \textsuperscript{9, 11}. The similar difficulties exist for multipartite states \textsuperscript{12}.

In this paper, we investigate the Lorentz boost-induced decrease in the purity of the reduced density matrix for spins, when a state which has pure reduced state for spins in the rest frame is viewed from a moving reference frame. Taken to the leading order, the decrease in spin purity observed in the moving frame is linear with respect to the momentum mean square deviation observed in the rest frame, which according to the position-momentum uncertainty relationship can be reasonably regarded as a measure of how much the spacial wave package is localized. We also present numerical studies as instance of our general analysis. Furthermore, we prove that, for any localized pure state with separable spin and momentum in the rest frame, its reduced density matrix for spins cannot be covariant under any Lorentz boosts, i.e. it inevitably appears to be mixed when viewed from a moving reference frame. Considering that in practical applications states should be localized, our results may have important consequences for relativistic quantum information processing.

II. GENERAL RELATIONSHIP BETWEEN THE BOOST-INDUCED SPIN DEPURIFICATION AND THE SPATIAL LOCALIZATION

A. A Simple Example: Single Spin-1/2 Massive Particle

We start by briefly recalling Peres et al.’s paper \textsuperscript{8}. Consider a spin half massive particle (of mass \(m\)) that is prepared with spin in the \(z\) direction. The spin state can be represented by the Bloch vector \(\mathbf{n} = (n_x, n_y, n_z)\) with \(n_x = n_y = 0\) and \(n_z = 1\). The momentum wave function is a Gaussian \(g(\mathbf{p}) \propto \exp(-\mathbf{p}^2/2w^2)\). When viewed by an observer moving in the \(x\) direction, the Lorentz-transformed Bloch vector is \(\mathbf{n}' = (n'_x, n'_y, n'_z)\) with \(n'_x = n'_y = 0\) yet \(n'_z < 1\). It is shown in Ref. \textsuperscript{8} that \(1 - n'_z \propto w^2\) to the leading order of \(w/m \ll 1\). By denoting the Lorentz-transformed density matrix for spin as \(\rho'\), the Lorentz boost-induced decrease in its purity is \(1 - \text{tr}(\rho'^2) \propto w^2\). Meanwhile, for this particular case the momentum mean square deviation is \(\langle \Delta p'_\mu^2 \rangle \propto w^2\) (\(\mu = x, y, z\)), hence \(1 - \text{tr}(\rho'^2) \propto \langle \Delta p'_\mu^2 \rangle\). According to the position-momentum uncertainty relationship \(\langle \Delta x'_\mu \rangle \langle \Delta p'_\mu \rangle \geq h^2/4\), the smaller \(\langle \Delta x'_\mu \rangle\) is, the larger \(\langle \Delta p'_\mu \rangle\) is and so is \(1 - \text{tr}(\rho'^2)\). This suggests that the more the wave package is localized in space, the more the boost-induced decrease in spin purity is when viewed in a moving frame.

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B. Generalization to Multiple Massive Particles with Arbitrary Spins

The above observation can indeed generalize to states of multiple massive particles with arbitrary spin quantum number. Consider a pure quantum state with separable spin and momentum in the rest frame. The system consists of \( N \) massive particles, labelled by \( k = 1, \ldots, N \). The spin quantum number of particle \( k \) is \( s_k \), and mass \( m_k > 0 \). The reduced density matrix for spins viewed from the rest frame is denoted by \( \rho \), which is pure with \( \text{tr}(\rho^2) = 1 \). The normalized momentum wave function in the rest frame is denote by \( g(\mathcal{P}) \), where \( \mathcal{P} := (p_{1x}, p_{1y}, p_{1z}, \cdots) \) for compactness of notation. To a moving observer Lorentz transformed by \( \Lambda^{-1} \), the state appears to be transformed by \( \Lambda^{-N} \). Because the Lorentz-transformed state viewed in the moving frame differs from rest-frame one by unitary transformations, the purity will not change provided we do not trace out a part of the state. However, in looking at spins, tracing out over the momentum degrees of freedom is implied. To the Lorentz-transformed observer, the spin and momentum may appear to be entangled, thus the purity of spins may appear to degrade viewed by this observer.

The reduced density matrix for spins viewed by the moving observer is

\[
\rho' = \int |g(\mathcal{P})|^2 U_{\Lambda}(\mathcal{P}) \rho U_{\Lambda}^\dagger(\mathcal{P}) d\mathcal{P},
\]

where, for compactness, we define \( U_{\Lambda}(\mathcal{P}) := U^{s_1}(\Lambda, \mathbf{p}_1) \otimes \cdots \otimes U^{s_N}(\Lambda, \mathbf{p}_N) \) with \( U^{s}(\Lambda, \mathbf{p}) \) being the spin-\( s \) representation of the Wigner rotation \( R(\Lambda, \mathbf{p}) \). \( d\mathcal{P} := dp_1 \cdots dp_N \) is the Lorentz invariant integration measure (defined in Ref. [12]). We represent the Lorentz-transformed spin purity by \( \text{tr}(\rho'^2) \), which can be calculated by

\[
\text{tr}(\rho'^2) = \iint |g(\mathcal{P})|^2 |g(\mathcal{P}')|^2 \Gamma^{\dagger}(\mathcal{P})(\mathcal{P}') U(\mathcal{P})d\mathcal{P}d\mathcal{P}'.
\]

Denoting \( \langle \cdot \rangle \) as the mean value (observed in the rest frame), we define \( \Delta_{\mathcal{P}}(\mathcal{P}') := \langle \mathcal{P}' \rangle - \langle \mathcal{P} \rangle \) (\( \mu = x, y, z \)) and \( \Delta^{\dagger}(\mathcal{P}') = \mathcal{P}(\cdot) - \langle \mathcal{P} \rangle \). Then \( \rho_{\mathcal{P}}(\mathcal{P}, \mathcal{P}') \) can be expanded into power series with respect to \( \Delta^{\dagger} \) and \( \Delta \), noting that \( \Gamma^{\dagger}(\mathcal{P})(\mathcal{P}') = \Gamma^{\dagger}(\mathcal{P}) \mathcal{P}' \):

\[
\Gamma^{\dagger}(\mathcal{P})(\mathcal{P}') = 1 - \frac{1}{2} \left( \Delta(\mathcal{P}), \Delta^{\dagger} \right) \left( \begin{array}{cc} \mathcal{U} & \mathcal{V}^T \\ \mathcal{V} & \mathcal{U} \end{array} \right) \left( \begin{array}{c} \Delta^{\dagger} \\ \Delta \end{array} \right) + \cdots,
\]

where \( 3N \)-dimensional matrices \( \mathcal{U} \) and \( \mathcal{V} \) are real functions of \( \Lambda \) and \( \rho \). When \( \Delta_{\mathcal{P} \mathcal{P}'} = \Delta_{\mathcal{P} \mathcal{P}'} = 0 \), \( \Gamma^{\dagger}(\mathcal{P})(\mathcal{P}') = 1 \). In the r.h.s. of Eq. (4), the zero-order term is 1, the first-order terms vanish, and \( \mathcal{U} \) is positive-semidefinite. Terms of higher than second orders are not explicitly presented here. Using \( \langle \Delta p_{k\mu} \rangle = \langle \Delta p'_{k\mu} \rangle = 0 \) and \( \langle \Delta \mathcal{P} \cdot \mathcal{U} \cdot \Delta \mathcal{P}^T \rangle = \langle \Delta \mathcal{P}' \cdot \mathcal{U} \cdot \Delta \mathcal{P}'^T \rangle \), we see that the boost-induced decrease in spin purity, to the leading order, is

\[
1 - \text{tr}(\rho'^2) \simeq \langle \Delta \mathcal{P} \cdot \mathcal{U} \cdot \Delta \mathcal{P}^T \rangle.
\]

The matrix \( \mathcal{U} \) can be diagonalized as \( \mathcal{U} = \mathcal{M} \cdot \mathcal{D} \cdot \mathcal{M}^T \), where \( \mathcal{M} \) is real and orthogonal, \( \mathcal{D} = \text{diag}(D_1, \cdots, D_{3N}) \) with \( D_\lambda \geq 0 \) (\( \lambda = 1, \cdots, 3N \)) due to that \( \mathcal{U} \) is positive-semidefinite. By denoting \( \mathcal{Q} = \mathcal{P} \cdot \mathcal{M} \) and \( \Delta \mathcal{Q} = \mathcal{Q} - \langle \mathcal{Q} \rangle = \Delta \mathcal{P} \cdot \mathcal{M} \), we have

\[
1 - \text{tr}(\rho'^2) \simeq \sum_{\lambda=1}^{3N} D_\lambda \langle \Delta \mathcal{Q}_\lambda^2 \rangle.
\]

Inspired by \([\bar{x}_{k\mu}, \bar{p}_{\nu}'; \nu'] = i\hbar \delta_{k\mu} \delta_{\nu'\nu} \), we define \( \mathcal{X}_\lambda = (x_{1x}, x_{1y}, x_{1z}, \cdots) \cdot \mathcal{M} \) so that \( \langle \mathcal{X}_\lambda, \bar{\mathcal{Q}}_\lambda \rangle = i\hbar \delta_{\lambda\lambda'} \), leading to the uncertainty relationship \( \langle \Delta \mathcal{X}_\lambda^2 \rangle \langle \Delta \mathcal{Q}_\lambda^2 \rangle \geq \hbar^2/4 \). Hence from Eq. (6) we obtain

\[
\text{tr}(\rho'^2) \leq 1 - \frac{\hbar^2}{4} \sum_{\lambda=1}^{3N} \frac{D_\lambda}{\langle \Delta \mathcal{X}_\lambda \rangle^2}.
\]

Here we shall note that both \( \langle \Delta \mathcal{Q}_\lambda^2 \rangle \) and \( \langle \Delta \mathcal{X}_\lambda^2 \rangle \) are observed in the rest frame, while \( \text{tr}(\rho'^2) \) is the purity observed in the moving frame. In addition, \( \mathcal{D} \) and \( \mathcal{M} \) are functions of \( \Lambda \) and \( \rho \). For any \( \rho \), when \( \Lambda \) degenerate to a pure three-dimensional rotation, we can see in Eq. (6) that \( \Gamma^\Lambda(\mathcal{P}, \mathcal{P}') = 1 \) and then in Eq. (6) that \( D_\lambda = 0 \) for all \( \lambda \), hence obviously \( \text{tr}(\rho'^2) = 1 \). Equation (6) also indicates that the purity of the Lorentz-transformed reduced spin state is (to the leading order) bounded by its spatial localization properties.

Equations (6) and (7) dictate the intrinsic relation between the boost-induced decrease in spin purity (observed in the moving frame) and the spatial localization of the wave package (observed in the rest frame). Indeed, \( \mathcal{X}_\lambda \) is a linear combination of \( (x_{1x}, x_{1y}, x_{1z}, \cdots) \). The more the wave package is localized (the less \( \langle \Delta \mathcal{X}_\lambda^2 \rangle \) is), the more mixed the reduced state for spins becomes when viewed in a moving reference frame. According to the uncertainty relationship \( \langle \Delta \mathcal{X}_\lambda^2 \rangle \langle \Delta \mathcal{Q}_\lambda^2 \rangle \geq \hbar^2/4 \), \( \sum_{\lambda=1}^{3N} D_\lambda \langle \Delta \mathcal{Q}_\lambda^2 \rangle \) can be reasonably regarded as a measure of how much the wave package is localized in space.

We shall note that \( \mathcal{D} \) and \( \mathcal{M} \) do not explicitly depend on the momentum wave function (the implicit dependence is through the mean value \( \langle \mathcal{P} \rangle \) because \( \Delta \mathcal{P} = \mathcal{P} - \langle \mathcal{P} \rangle \)). When the state is not strongly localized, whatever the momentum wave function as long as the localization is the same (\( \mathcal{D} = \mathcal{D}(\mathcal{Q}) \) is the same), the same reduced state for spins suffers the same amount of boost-induced decrease in spin purity when viewed from the moving frame. This provides a general and feasible method to possibly estimate how mixed the reduced spin state would appear by the spatial localization properties, which might be useful in practical relativistic quantum information processing. In addition, for multipartite
states whose spins are not entangled, position (momentum) coordinates of different particles will not be mixed in $\mathcal{X}_\lambda (Q_\lambda)$. The r.h.s. of Eq. (4), as well as that of Eq. (7), turns out to be a sum over each individual particles. While interestingly, if the spins are entangled, position (momentum) coordinates of different particles will in general be mixed in $\mathcal{X}_\lambda (Q_\lambda)$). This implies that, for the present case, in measuring how much the spatial wave package is localized, the correlation (entanglement) between the spins needs also to be taken into account.

C. A Further Illustration: Two Spin-1/2 Massive Particles

As an illustrative example, we numerically study the case of two spin half particles (of mass $m$) with momentum wave function being the “entangled Gaussian” as presented in Ref. [3]:

$$g (p_1, p_2) = \frac{1}{\sqrt{N}} \exp \left[ -\frac{p_1^2 + p_2^2}{4\sigma^2} \right] \exp \left[ -\frac{p_1^2 + p_2^2 - 2x p_1 \cdot p_2}{4\sigma^2 (1-x^2)} \right],$$

(8)

where $N$ is the normalization, $\sigma \geq 0$ is the “width” and $x \in [0,1)$. The Lorentz transformation is chosen to be a pure boost $L(\xi)$ in the $z$ direction, where $\xi$ is the rapidity and denote $\xi = |\xi|$, as defined in Ref. [3]. For this particular momentum wave function, $\langle \Delta p^2_{\mu} \rangle$ is the same for any $k$ and $\mu$. Moreover, when $\langle \Delta p^2_{\mu} \rangle$ and $\langle \Delta p^2_{\nu} \rangle$ are relatively small, $\langle \Delta p_{\mu} \Delta p_{\nu} \rangle$ is (if not zero) proportional to $\langle \Delta p^2_{\mu} \rangle \langle \Delta p^2_{\nu} \rangle$, with the proportion depending upon $x$. Thus for this particular case, Eq. (6) reduces to $1 - \text{tr}(\rho^2) \propto \langle \Delta p^2 \rangle$. Figure 1 depicts the relation between $1 - \text{tr}(\rho^2)$ and $\langle \Delta p^2 \rangle$, for the spin parts being $|\psi^- \rangle$ and $|\psi^+ \rangle$. When the momentum mean square deviation is relatively small, the relation can be well described by linearity, confirming the validity of Eq. (6). As the momentum mean square deviation increases, corresponding to stronger localization, the boost-induced decrease in spin purity increases monotonously. The deviation from linearity is due to terms of higher than second orders, such as $\langle \Delta p^3 \rangle$ and $\langle \Delta p^4 \rangle$ etc. However, such terms can also be regarded in some sense as measures of the spatial localization of the wave package.

D. A Theorem

In the remaining part of this paper, we prove the following theorem.

**Theorem:** When a pure state with separable spin and momentum in the rest frame is viewed from a moving reference frame, its reduced density matrix for spins necessarily appears to be mixed if its spatial wave package is localized.

Here we shall first specify the meaning of being localized. We regard a state localized in the sense that its position wave function could be *completely normalized*, i.e. it is square-integrable. This requirement excludes nonlocalized states, such as momentum eigenstates and the singular ones presented in Ref. [3]. Since momentum wave function is the Fourier transformation of position wave function, being localized means that, in our case, $g(P)$ is square-integrable, i.e. $G(P) := |g(P)|^2 dP/dP \geq 0$ is integrable on $\mathbb{R}^{3N}$ in the sense of Lebesgue integration, and it is normalized as $\int G(P) dP = \int_{\text{supp}(G(P))} G(P) dP = 1$, where $\text{supp}(G(P)) := \{ P \mid G(P) > 0 \}$ is the support of $G(P)$.

We denote $\mathcal{K} = (P, P') \in \mathbb{R}^{6N}$, $T(\mathcal{K}) = \Gamma^A_{\phi}(P, P') \in [0,1]$, and $G(\mathcal{K}) = G(P)G(P') \geq 0$ for compactness. Let $\Omega_\phi = \text{supp}(G(\mathcal{K})) = \{ \mathcal{K} \mid G(\mathcal{K}) > 0 \} = \text{supp}(G(P)) \times \text{supp}(G(P'))$, $\Omega_\phi = \{ \mathcal{K} \mid T(\mathcal{K}) = 1 \}$, and $m(\cdot)$ be the Lebesgue measure in $\mathbb{R}^{6N}$. The integral in Eq. (2) now turns to be the Lebesgue integral over $\Omega_\phi$: $\text{tr}(\rho^2) = \int_{\Omega_\phi} G(\mathcal{K}) T(\mathcal{K}) d\mathcal{K}$.

Since the state is localized, $G(\mathcal{K})$ must be integrable. In addition, $\int_{\Omega_\phi} G(\mathcal{K}) d\mathcal{K} = \left[ \int_{\text{supp}(G(P))} G(P) dP \right]^2 = 1$ implies that $G(\mathcal{K})$ is bounded almost everywhere. Hence one must have $m(\Omega_\phi) > 0$. Otherwise supposing $m(\Omega_\phi) = 0$, one necessarily encounters the contradiction that $\int_{\Omega_\phi} G(\mathcal{K}) d\mathcal{K} = 0$ and $\int_{\text{supp}(G(P))} G(P) dP = 0$.

Denote $x = (p_2, \ldots, p_N, p_1, \ldots, p_N)$ and $\Omega_t(x) = \{ p_1 \mid (p_1, x) \in \Omega_t \}$. It can be easily verified that $\Omega_t(x)$ has Lebesgue measure zero in $\mathbb{R}^3$ for any given $x$. 

![Figure 1: Lorentz boost-induced decrease in spin purity](image-url)
when Λ does not degenerate to a pure three-dimensional rotation. Then, using the Tonelli’s theorem, one obtains
\[ m(\Omega_t) = \int m'(\Omega_t(x))dx = 0 \] (\(m'(\cdot)\) is the Lebesgue measure in \(\mathbb{R}^3\)). Physically, this observation is valid. If there is \(K_0 = (P_0, P_0') \in \Omega_t\), then \(U_0^{1}(P_0)U_{\Lambda}(P_0')\) must have an eigenstate to be exactly \(\rho\) (the rest-frame reduced density matrix for spins). All such \((P_0, P_0')\) occupy only a low dimensional subset in \(\mathbb{R}^{4N}\) for any pure \(\rho\) and non-degenerate \(\Lambda\), thus \(m(\Omega_t) = 0\).

**Proof of the Theorem:** Because \(m(\Omega_g) > 0\) while \(m(\Omega_t) = 0\), we have \(\int_{\Omega_g} \rho(\cdot)d\mathcal{K} = \int_{\Omega_t} \rho(\cdot)d\mathcal{K}\). Therefore
\[
\text{tr}(\rho^2) = \int_{\Omega_g} \mathcal{G}(\mathcal{K})\mathcal{T}(\mathcal{K})d\mathcal{K} = \int_{\Omega_t} \mathcal{G}(\mathcal{K})\mathcal{T}(\mathcal{K})d\mathcal{K} < \int_{\Omega_t} \mathcal{G}(\mathcal{K})d\mathcal{K} = \int_{\Omega_g} \mathcal{G}(\mathcal{K})d\mathcal{K} = 1,
\]
where the inequality is due to that \(\mathcal{T}(\mathcal{K})\) is strictly less than \(1\) on \(\Omega_g/\Omega_t\). Now that \(\text{tr}(\rho^2) < 1\) immediately gives that \(\rho\) is mixed.

The fact that for localized states \(m(\Omega_g) > 0\) is essential. Its meanings can be further clarified by reviewing Eq. \([\text{III}].\) If \(\text{tr}(\rho^2) = 1\), we must have \(D_\lambda(\Delta Q^2_\lambda) = 0\) for all \(\lambda\). Supposing there is \(\lambda_0\) so that \(D_{\lambda_0} \neq 0\), consequently we have \(\langle \Delta Q^2_{\lambda_0} \rangle = 0\). Due to that \(\mathcal{M}\) is orthogonal, there is \(\sigma_0\) so that \(\mathcal{M}_{\sigma_0\lambda_0} \neq 0\). Supposing \(\sigma_0\) corresponds the position coordinate labelled by \(\mu_0\) of particle \(k_0\), we obtain that \(\langle \hat{x}_{k_0\mu_0} \rangle, \langle Q_{\lambda_0} \rangle = i\hbar \mathcal{M}_{\sigma_0\lambda_0}\), which leads to that \(\langle \Delta x^2_{k_0\mu_0} \rangle/\langle \Delta Q^2_{\lambda_0} \rangle \geq \hbar^2 \mathcal{M}_{\sigma_0\lambda_0}^2/4\). However because \(\langle \Delta Q^2_{\lambda_0} \rangle = 0\) here, we see that \(\langle \Delta x^2_{k_0\mu_0} \rangle \rightarrow \infty\).

On the one hand, \(\langle \Delta Q^2_{\lambda_0} \rangle = 0\) implies that \(g(\mathcal{P})\) is an eigenstate of \(\hat{Q}_{\lambda_0}\) and consequently \(m(\Omega_g) = 0\). On the other hand, \(\langle \Delta x^2_{k_0\mu_0} \rangle \rightarrow \infty\) implies that the (reduced) wave packet of particle \(k_0\) is nonlocalized. Inversely, \(m(\Omega_g) > 0\) guarantees that \(g(\mathcal{P})\) is not an eigenstate of any \(\hat{Q}_{\lambda_0}\), so the state can be localized because all \(\langle \Delta x^2_{k\mu} \rangle\) can be finite.

### III. DISCUSSION AND CONCLUSION

We would like to note that the present paper adopts the same notion of spins as in Ref. \([5, 6, 7, 8]\). Beside, there are other possible notions (e.g. see Ref. \([9]\)). We would be interesting to see that when “spin” is defined with respect to projection of Pauli-Lubanski’s vector in a particular null direction of the Lorentz transformation, the reduced density matrix for spins viewed by the moving observer does not depolarize \(\hat{F}\). However, since establishing a perfect shared reference frame requires infinite communication even in non-relativistic situations \([10]\), it might be extremely difficult to acquire precise information about a Lorentz transformation. We argue that in practical applications spin may be defined independently of the particular Lorentz transformation that defines the relative motion between the observers, and the transformation law of such spins would then in general depend upon momentum. Indeed, our results would hold for all such notions of spin, including those adopted in Refs. \([5, 6, 7]\), but Ref. \([8]\).

In conclusion, states one can prepare in real experiments are necessarily localized. Nonlocalized states, e.g. momentum eigenstates and the singular ones presented in Ref. \([5]\), are not practical in reality, although they are useful in theories. We show that, in relativistic applications reduced spin state which is pure in the rest frame unavoidably appears to be mixed whenever viewed from moving reference frames. How much such boost-induced decrease in purity is depends on how much the spatial wave packet is localized. The more the spatial wave packet is localized, the more the purity of the reduced spin state decreases when viewed from moving frames. This observation may be important for relativistic quantum information processing, particularly for relativistic quantum communication.

Although our investigations are based on massive particles, the generalize to massless cases, such as to photons \(\hat{K}\), should be analogous. This may be of interest since most of current experiments in quantum communication are based on photons \(\hat{I}\). Another interesting problem might be to determine how our results generalize to accelerated frames \(\hat{Q}\).

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In Ref. [5], it is shown that a class of states of two spin half particles, other than momentum eigenstates, do have covariant reduced density matrix for spins. However, it can be proved that for such states the reduced state for a single particle is the convex combination of momentum eigenstates. Taking the momentum wave function \( g(p_1, p_2) = |f(p_1)|^2 \delta^3(p_1 - p_2) \) (see Ref. [2]) as an example, and noting that \( g(p_1, p_2) = g(p_2, p_1) \) and \( \delta(x)^{1/2} \equiv \delta(x)/[\delta(0)^{1/2}] \), the reduced state of either particle can be written as a density matrix with elements being

\[
\rho(p, p') = \int g^*(p, q)g(p', q)dq = |f(p)|^2 \delta^3(p - p')/\delta(0)^3 = |f(p)|\delta_{p, p'}.
\]

The similar holds for other singular states presented in Ref. [5].

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