ONE-FORM ABELIAN GAUGE THEORY AS THE HODGE THEORY

R.P. Malik
Centre of Advanced Studies, Physics Department,
Banaras Hindu University, Varanasi-221 005, India
E-mail address: malik@bhu.ac.in

Abstract: We demonstrate that the two (1 + 1)-dimensional (2D) free 1-form Abelian gauge theory provides an interesting field theoretical model for the Hodge theory. The physical symmetries of the theory correspond to all the basic mathematical ingredients that are required in the definition of the de Rham cohomological operators of differential geometry. The conserved charges, corresponding to the above continuous symmetry transformations, constitute an algebra that is reminiscent of the algebra obeyed by the de Rham cohomological operators. The topological features of the above theory are discussed in terms of the BRST and co-BRST operators. The super de Rham cohomological operators are exploited in the derivation of the nilpotent (anti-)BRST, (anti-)co-BRST symmetry transformations and the equations of motion for all the fields of the theory, within the framework of the superfield formulation. The derivation of the equations of motion, by exploiting the super Laplacian operator, is a completely new result in the framework of the superfield approach to BRST formalism. In an Appendix, the interacting 2D Abelian gauge theory (where there is a coupling between the U(1) gauge field and the Dirac fields) is also shown to provide a tractable field theoretical model for the Hodge theory.

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1 Introduction

There are a host of areas of research in theoretical high energy physics which have provided a meeting-ground for the researchers in the realm of mathematics and investigators in the domain of theoretical high energy physics. Such areas of investigations have enriched the understanding and insights of both the above type of researchers in an illuminating and fruitful fashion. The subject of Becchi-Rouet-Stora-Tyutin (BRST) formalism [1-4] is one such area of research which has found applications in the modern developments in (super)string theories, D-branes, M-theory, etc., that are supposed to be the frontier areas of research in modern-day theoretical high energy physics (see, e.g. [5,6]). In particular, in the context of string field theories, the BRST formalism plays a very decisive role.

One of the most important pillars of strength for the edifice of the ideas behind the BRST formalism is its very fruitful application in the realm of 1-form (non-)Abelian gauge theories in the physical four (3 + 1)-dimensions of spacetime. The latter theories provide the physical basis for the existence of three out of four fundamental interactions of nature. In fact, the above gauge theories are described by the singular Lagrangian densities and they are endowed with the first-class constraints in the language of Dirac’s prescription for the classification scheme of the constraints [7,8]. For such dynamical systems with the first-class constraints, the BRST formalism provides (i) the covariant canonical quantization where the “classical” local gauge symmetry transformations of the original theory are traded with the “quantum” gauge (i.e. BRST) symmetry transformations, (ii) the physical mechanism for the proof of unitarity at any arbitrary given order of perturbative computations for a physical process allowed by the theory [9,10], and (iii) the method to choose the physical states from the total quantum Hilbert space of states which are found to be consistent with the Dirac’s prescription for the quantization scheme of the systems with constraints.

The range and reach of the BRST formalism has been extended to incorporate in its ever widening folds the second-class constraints, too [11]. Its very deep connections with the key notions of the differential geometry and cohomology [12-16], its beautiful application in the context of topological field theories [17-19], its very intimate relations with the key ideas behind the supersymmetry, its cute geometrical origin and interpretation in the framework of the superfield formulation [20-29], its successful applications to the reparametrization invariant theories of free as well as interacting relativistic particle, supergravity, etc., have elevated the key concepts behind the BRST formalism to a very high degree of mathematical sophistication and very useful physical applications.

In our present investigation, we shall concentrate on the application of the BRST formalism to the 2D free 1-form Abelian gauge theory (which is endowed with the first-class constraints). In the BRST approach to any arbitrary $p$-form ($p = 1, 2, 3,...$) gauge theories, the operator form of the first class constraints appear in the physicality condition (i.e. $Q_b |\text{phys} > = 0$) when one demands that the physical states (i.e. $|\text{phys} >$) of the quantum gauge theories are annihilated by the nilpotent ($Q_b^2 = 0$) and conserved ($\dot{Q}_b = 0$) BRST
charge operator $Q_b$. The nilpotency of the BRST charge and the physicality condition are the two key ingredients that allow the BRST formalism to have close connections with the key ideas of the mathematical aspects of differential geometry and cohomology. In fact, two physical (i.e. $Q_b|\text{phys}\rangle\neq |\text{phys}\rangle$) states $|\text{phys}\rangle$ and $|\text{phys}\rangle + Q_b |\xi\rangle$ (for $|\xi\rangle$ being a non-trivial state) are said to belong to the same cohomology class with respect to the BRST charge $Q_b$ if they differ by a BRST exact (i.e. $Q_b|\xi\rangle$) state.

In the language of the differential geometry and differential forms, two closed (i.e. $df_n = 0$, $df_n = 0$) forms $f_n$ and $f_n' = f_n + d g_{n-1}$ of degree $n$ (with $n = 1, 2, 3, ...$) are said to belong to the same cohomology class with respect to the exterior derivative $d = dx^\mu \partial_\mu$ (with $d^2 = 0$) because they differ from each other by an exact form $dg_{n-1}$. Thus, we note that the nilpotent BRST charge $Q_b$, that generates a set of nilpotent BRST transformations for the appropriate and relevant fields of the gauge theories, provides a physical analogue to the mathematically abstract cohomological operator $d = dx^\mu \partial_\mu$. There are two other cohomological operators $\delta = \pm \ast d \ast$ and $\Delta = d\delta + \delta d \equiv \{d, \delta\}$ (where $\ast$ is the Hodge duality operation) that constitute the full set $(d, \delta, \Delta)$ of the de Rham cohomological operators obeying the algebra $d^2 = \delta^2 = 0$, $\Delta = (d + \delta)^2 \equiv \{d, \delta\}$, $[\Delta, d] = [\Delta, \delta] = 0$ (see, e.g., [12-14] for details). The latter two cohomological operators $\delta$ and $\Delta$ are known as the co-exterior derivative and the Laplacian operator, respectively, in the domain of differential geometry.

In terms of the above cohomological operators any $n$-form $f_n$ can be uniquely written as the sum of a harmonic form $h_n$ (with $\Delta h_n = 0$, $dh_n = 0$, $\delta h_n = 0$) an exact form $de_{n-1}$ and a co-exact form $\delta c_{n+1}$ on a compact manifold without a boundary. Mathematically, this statement can be succinctly expressed as (see, e.g. [12-16] for details)

$$f_n = h_n + d e_{n-1} + \delta c_{n+1}. \quad (1.1)$$

The above equation is the statement of the celebrated Hodge decomposition theorem on the compact manifold without a boundary. It has been a long-standing problem, in the framework of the BRST formalism, to obtain the analogue of the cohomological operators $\delta$ and $\Delta$ in the language of the well-defined symmetry properties of the Lagrangian density of any arbitrary $p$-form ($p = 1, 2, ...$) gauge theory in any arbitrary dimension of spacetime. Some attempts [30-34], in this direction, have been made for the physical four $(3 + 1)$-dimensional (4D) (non-)Abelian 1-form gauge theories but the symmetry transformations turn out to be non-local and non-covariant. In the covariant formulation of the above symmetry transformations [35], the nilpotency of the transformations is restored only for a specific value of a parameter (that is introduced by hand in the covariant formulation).

In our earlier set of papers [36-43], we have been able to demonstrate that (i) the 2D free as well as interacting Abelian 1-form gauge theory [36,38-40], (ii) the self-interacting 2D non-Abelian 1-form gauge theory without any interaction with matter fields [37], and (iii) the free 4D Abelian 2-form gauge theory [42-44], provide the tractable field theoretical models for the Hodge theory because all the de Rham cohomological operators $(d, \delta, \Delta)$ are shown to correspond to local, covariant and continuous symmetry transformations. The discrete set of symmetry transformations for the Lagrangian densities of the above theories
are shown to correspond to the Hodge duality $*$ operation of the differential geometry. In fact, the interplay between the continuous and discrete symmetry transformations provides the analogue of the relationship $\delta = \pm \ast d \ast$ that exists in the differential geometry. The point to be emphasized here is the fact that all the above symmetry transformations turn out to be well-defined. In other words, all the symmetry transformations corresponding to the cohomological operators are not found to be non-local or non-covariant for the above theories. The topological features of the above 2D and 4D theories are also discussed, in great detail, by exploiting the nilpotent (anti-)BRST and (anti-)co-BRST symmetry transformations (and their corresponding nilpotent generators) together [41-43].

It is a well-known fact that, for a given single set of local gauge symmetry transformation for a 1-form (non-)Abelian gauge theory, there exist two sets of nilpotent symmetry transformations (and their corresponding nilpotent generators). These nilpotent symmetry transformations are known as the (anti-)BRST symmetry transformations $s_{(a)b}$ which are generated by the conserved and nilpotent (anti-)BRST charges $Q_{(a)b}$. The theoretical reason behind the existence of these couple of nilpotent symmetry transformations, corresponding to a single local gauge symmetry transformation, comes from the super exterior derivative $\tilde{d} = dx^\mu \partial_\mu + d\theta \partial_\theta + d\bar{\theta} \partial_{\bar{\theta}}$ (with $\tilde{d}^2 = 0$) when it is exploited in the celebrated horizontality condition [20-29] on a (D, 2)-dimensional supermanifold on which a D-dimensional 1-form (non-)Abelian gauge theory is considered (see, e.g., Subsec. 4.1 below, for details). This technique, popularly known as the superfield approach to BRST formalism, sheds light on the geometrical origin and interpretation for the (anti-)BRST symmetry transformations ($s_{(a)b}$) and corresponding nilpotent generators ($Q_{(a)b}$). One of the outstanding problems in the superfield approach to BRST formalism has been to tap the potential and power of the super co-exterior derivative $\tilde{\delta} = \pm \ast \tilde{d} \ast$ and the super Laplacian operator $\tilde{\Delta} = \tilde{d} \tilde{\delta} + \tilde{\delta} \tilde{d}$ in the derivation of some physically relevant aspects of a p-form (non-)Abelian gauge theory in D-dimensions of spacetime. In the above, it will be noted that the Hodge duality $*$ operation is defined on the (D, 2)-dimensional supermanifold.

A perfect field theoretical model for the Hodge theory is the one where (i) the analogues of the ordinary de Rham cohomological operators exist in the language of the well-defined symmetry transformations (and their corresponding generators), and (ii) all the super de Rham cohomological operators play very significant and decisive roles in the determination of some of the key features of the field theoretical model. The central purpose of the present investigation is to demonstrate that the 2D free 1-form Abelian gauge theory is a tractable field theoretical model for the Hodge theory because both the above key requirements are fulfilled in a grand and illuminating manner. Moreover, we also mention the physical consequences of our theoretical study in a concise manner. Thus, in our present paper, the mathematical and physical aspects of our 2D 1-form gauge model have been brought together in a cute and complete fashion. First of all, in Sec. 2, we demonstrate that, corresponding to each cohomological operators of the differential geometry, there exists a well-defined symmetry transformation for the Lagrangian density of the 2D free 1-form Abelian gauge
theory. One of the physical consequences of the above symmetry transformations is the fact that, the gauge theory under consideration, is found to be a new type of topological field theory (cf. Sec. 3 below). Parallel to Sec. 2, we demonstrate (in Sec. 4) that the super de Rham cohomological operators \( \tilde{d}, \tilde{\delta} = -\ast \tilde{d} \ast, \tilde{\Delta} = \tilde{d} \tilde{\delta} + \tilde{\delta} \tilde{d} \) play central roles in the appropriate (gauge-invariant) restrictions on the superfield of four (2, 2)-dimensional supermanifold which generate (i) the well-defined nilpotent (anti-)BRST symmetry transformations, (ii) the well-defined nilpotent (anti-)co-BRST symmetry transformations, and (iii) the equations of motion for all the fields of the theory. In the application of the super cohomological operators \( \tilde{\delta} \) and \( \tilde{\Delta} \), we require a proper definition of the Hodge duality \(*\) operation on the four (2, 2)-dimensional supermanifold.

In the language of the symmetry properties of the Lagrangian density (cf. \((2.2)\) below), we have been able to provide the analogue of the Hodge duality \(*\) operation that exists between the exterior derivative \( d \) and the co-exterior derivative \( \delta \) in the well-known relationship \( \delta = -\ast d \ast \) of the differential geometry. In fact, the discrete symmetry transformations (cf. \((2.10)\) and \((2.11)\) below) for the Lagrangian density of the theory plays the role of the Hodge duality \(*\) operation in the relationships (cf. \((2.13)\) and \((2.14)\) below) that exists between the (anti-)co-BRST and (anti-)BRST symmetry transformations. However, we also know that there exists a well-defined meaning of the Hodge duality \(*\) operation on the differential forms (through their proper definition of the inner products) on the 2D spacetime manifold (see, e.g. [12-16]). To obtain the analogy of this Hodge duality \(*\) operation (defined on an ordinary spacetime manifold), the key point is to know the proper definition of the corresponding Hodge duality \(*\) operation on the super differential forms defined on the four (2, 2)-dimensional supermanifold. We have achieved precisely this goal in our earlier work [45]. The materials of our Subsecs. 4.2 and 4.3, where we have exploited the super co-exterior derivative \( \tilde{\delta} = -\ast \tilde{d} \ast \) and super Laplacian operator \( \tilde{\Delta} \) in a specific set of restrictions on the superfields of the above supermanifold, totally depend on the definition of the Hodge duality \(*\) operation on the super forms [45]. The results of the Hodge duality \(*\) operation are found to be correct.

Our present investigation is essential and interesting on the following grounds. First and foremost, our present field theoretical model is one of the simplest examples where the sanctity of our definition of the Hodge duality \(*\) operation on the four (2, 2)-dimensional supermanifold [45] can be tested, particularly, in the application of the super co-exterior derivative \( \tilde{\delta} = -\ast \tilde{d} \ast \) and the super Laplacian operator \( \tilde{\Delta} = \tilde{d} \tilde{\delta} + \tilde{\delta} \tilde{d} \) in some suitable restrictions on the superfields of the above supermanifold. Second, the present model provides the physical meaning of (and theoretical importance to) the ordinary and super de Rham cohomological operators together. The physical implication of the former lies in the proof that the present model is a new kind of TFT. The theoretical importance of the latter cohomological operators is in the derivation of the nilpotent symmetry transformations and equations of motion for the theory. Third, due to the aesthetic reasons, it is nice to note that, for the model under consideration, the continuous and discrete symmetries,
mathematical power of the cohomological operators and their physical consequences, etc., are found to blend together in a beautiful manner. Finally, the present study is a step in the direction to prove that the free 2-form Abelian gauge theory might provide a field theoretical model for the Hodge theory in the physical four dimensions of spacetime where the ordinary as well as the super de Rham cohomological operators would play significant and decisive roles. The physical implication of the former operators has already been shown in the proof that the 4D free 2-form Abelian gauge theory is endowed with some special features and is a model for the quasi-topological field theory [43]. The impact and importance of the latter (i.e. super de Rham cohomological operators) are yet to be seen in the context of theoretical discussions of the above 4D free 2-form Abelian gauge theory.

The contents of our present paper are organized as follows:

In Sec. 2, we discuss the bare essentials of (i) the nilpotent (anti-)BRST symmetry, (ii) the nilpotent (anti-)co-BRST symmetry, and (iii) a non-nilpotent bosonic symmetry transformations for the 2D (anti-)BRST invariant Lagrangian density of a free 1-form Abelian gauge theory. The subtle discrete symmetry transformations for the above Lagrangian density are discussed separately and independently. We pinpoint the deep connections that exist between these (continuous and discrete) symmetry transformations and the cohomological operators of the differential geometry. This exercise provides, at a very elementary level, the proof that the above Abelian gauge theory, described in terms of the BRST invariant Lagrangian density, is a tractable field theoretical model for the Hodge theory.

Section 3 is devoted to demonstrate that the above 1-form gauge theory is a new type of topological field theory which captures a part of the key features associated with the Witten-type topological field theory as well as a part of the salient points connected with the Schwarz-type topological field theory. The existence of the nilpotent (anti-)BRST as well as (anti-)co-BRST symmetry transformations (and their corresponding nilpotent generators) play a pivotal role in this proof. We do not discuss here the topological invariants and their recursion relations which can be found in our earlier works [41,36].

In Sec. 4, we exploit the mathematical power of the super de Rham cohomological operators, in the imposition of some specific (gauge-invariant) restrictions on the superfields of the four (2, 2)-dimensional supermanifold, to derive the nilpotent (anti-)BRST and (anti-)co-BRST symmetry transformations of the theory. We show that super Laplacian operator plays a key role in the derivation of all the equations of motion for all the fields of the theory. This latter result is a completely new result which bolsters up the correctness of our definition of the Hodge duality $\star$ operation on the above supermanifold (see. e.g. [45] for details). The topological features of the above theory are also captured in the language of the superfield approach to BRST formalism. We have focused, for the proof of the topological features, on the form of the Lagrangian density and the symmetric energy-momentum tensor expressed in terms of the superfields defined on the four (2, 2)-dimensional supermanifold.

Finally, we summarize our key results, make some concluding remarks and point out
some promising future directions for further investigations in Sec. 5.

In the Appendix, we show that the 2D interacting 1-form Abelian U(1) gauge theory with Dirac fields is a cute field theoretical model of the celebrated Hodge theory.

2 Cohomological Operators and Symmetries: Lagrangian Formulation

To establish the connection between the key concepts behind the de Rham cohomological operators of the differential geometry and the symmetry properties of the (anti-)BRST invariant Lagrangian density of a given two (1 + 1)-dimensional* (2D) free 1-form Abelian gauge theory, we begin with the following Lagrangian density in the Feynman gauge [46,47]

$$\mathcal{L}_b = -\frac{1}{4} F^{\mu\nu} F_{\mu\nu} - \frac{1}{2} (\partial \cdot A)^2 - i \partial_\mu \bar{C} \partial^\mu C \equiv \frac{1}{2} E^2 - \frac{1}{2} (\partial \cdot A)^2 - i \partial_\mu \bar{C} \partial^\mu C,$$  \hspace{1cm} (2.1)

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is the field strength tensor for the 2D Abelian 1-form $(A^{(1)} = dx^\mu A_\mu)$ gauge field $A_\mu$. It has only one-component $F_{01} = \partial_0 A_1 - \partial_1 A_0 = E$ which turns out to be the electric field $E$ of the theory. There exists no magnetic field in the 2D Abelian gauge theory. The cohomological origin for the existence of the field strength tensor lies in the exterior derivative $d = dx^\mu \partial_\mu$ (with $d^2 = 0$) because the 2-form $F^{(2)} = dA^{(1)} = \frac{1}{2} (dx^\mu \wedge dx^\nu) F_{\mu\nu}$, constructed with the help of $d$ and $A^{(1)}$, defines it. On the other hand, the gauge-fixing term $(\partial \cdot A)$ owes its cohomological origin to the co-exterior derivative $\delta = - * d*$ (with $\delta^2 = 0$) because $\delta A^{(1)} = - * d * A^{(1)} = (\partial \cdot A)$ where $*$ is the Hodge duality operation defined on the 2D Minkowskian spacetime manifold. The (anti-)ghost fields $(\bar{C})C$ are required in the theory for the proof of unitarity for a given physical process at any arbitrary given order of the perturbative computation†. For the 1-form Abelian gauge theory, the “fermionic” (anti-)ghost fields are (i) the spin-zero Lorentz scalar fields, and (ii) they possess anticommuting (i.e. $C^2 = \bar{C}^2 = 0, C \bar{C} + \bar{C} C = 0$) property.

The square terms (i.e. $\frac{1}{2} E^2, -\frac{1}{2} (\partial \cdot A)^2$) corresponding to the kinetic energy term and the gauge-fixing term can be linearized by introducing the auxiliary fields $B$ and $\mathcal{B}$ thereby changing the above Lagrangian density (2.1) in the following equivalent form

$$\mathcal{L}_B = B E - \frac{1}{2} B^2 + B (\partial \cdot A) + \frac{1}{2} \bar{B}^2 - i \partial_\mu \bar{C} \partial^\mu C,$$  \hspace{1cm} (2.2)

where the auxiliary field $B$ is popularly known as the Nakanishi-Lautrup auxiliary field. The above Lagrangian density is endowed with the following off-shell nilpotent ($s_a^2 = 0$)

*We follow here the conventions and notations such that the flat 2D Minkowskian metric $\eta_{\mu\nu} = \text{diag} (+1, -1)$ and the antisymmetric Levi-Civita tensor $\varepsilon_{01} = +1 = -\varepsilon^{01}$ with $\varepsilon^{\mu\nu\mu\nu} = -2!$, $\varepsilon^{\mu\lambda\nu\mu} = -\delta^\lambda_\nu$, etc. Here the Greek indices $\mu, \nu, \lambda, \ldots$ = 0, 1 stand for the time and space directions on the 2D Minkowskian spacetime manifold, respectively. All the local fields of the 2D free 1-form Abelian gauge theory are defined on this spacetime manifold because they are functions of the 2D spacetime variable $x^\mu$.

†The importance of the fermionic (anticommuting) (anti-)ghost fields, in the proof of unitarity for a physical process, comes out in its full blaze of glory in the context of the non-Abelian gauge theory where for each gluon loop diagram (that exists for a given physical process), one requires a loop Feynman diagram constructed with the help of the fermionic (anti-)ghost fields alone (see, e.g. [9,10] for details).
and anticommuting \((s_b s_a + s_a s_b = 0)\) (anti-)BRST symmetry transformations \(s_{(a)b}\) \[^{46,47}\]

\[
\begin{align*}
  s_b A_\mu &= \partial_\mu C, \quad s_b C = 0, \quad s_b \bar{C} = i B, \quad s_b B = 0, \\
  s_b E &= 0, \quad s_b \mathcal{B} = 0, \quad s_b (\partial \cdot A) = \Box C, \quad s_b F_{\mu\nu} = 0, \\
  s_{ab} A_\mu &= \partial_\mu \bar{C}, \quad s_{ab} \bar{C} = 0, \quad s_{ab} C = -i B, \quad s_{ab} B = 0, \\
  s_{ab} E &= 0, \quad s_{ab} \mathcal{B} = 0, \quad s_{ab} (\partial \cdot A) = \Box \bar{C}, \quad s_{ab} F_{\mu\nu} = 0,
\end{align*}
\]

(2.3)

because the above Lagrangian density transforms as: \(s_b \mathcal{L}_B = \partial_\mu [B \partial^\mu C]\) and \(s_{ab} \mathcal{L}_B = \partial_\mu [B \partial^\mu \bar{C}]\), respectively, under the nilpotent (anti-)BRST transformations listed in (2.3). It will be noted that the gauge invariant physical field \(E\) remains invariant under the nilpotent (anti-)BRST transformations in (2.3). We know, however, that the cohomological origin for the above electric field \(E\) is encoded in the exterior derivative \(d\) which generates the 2-form \(F^{(2)} = dA^{(1)}\). The latter, in turn, produces the field strength tensor \(F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu\). Thus, we conclude that the mathematical origin of the nilpotent (anti-)BRST symmetry transformations (e.g. for our present 2D 1-form free Abelian gauge theory) lies in the exterior derivative \(d = dx^\mu \partial_\mu\) of the differential geometry. This observation will be exploited in Subsec. 4.1 where the super exterior derivative, exploited in the so-called horizontality condition \([20-29]\), will generate the nilpotent (anti-)BRST symmetry transformations \(\text{together}\) for all the fields of the above 1-form Abelian gauge theory in the framework of the geometrical superfield approach to BRST formalism.

The Lagrangian density (2.2) is found to be endowed with another off-shell nilpotent (i.e. \(s_{(a)d}^2 = 0\)) symmetry transformations. The latter transformations are christened as the dual(co-) and anti-dual(co-)BRST symmetry transformations \(s_{(a)d}\). In fact, it can be checked that, under the following (anti-co-)BRST symmetry transformations \([36,38,41]\)

\[
\begin{align*}
  s_d A_\mu &= -\varepsilon_{\mu\nu} \partial^\nu \bar{C}, \quad s_d \bar{C} = 0, \quad s_d C = -i \mathcal{B}, \quad s_d \mathcal{B} = 0, \\
  s_d E &= \Box \bar{C}, \quad s_d \mathcal{B} = 0, \quad s_d (\partial \cdot A) = 0, \quad s_d F_{\mu\nu} = [\varepsilon_{\mu\rho} \partial^\rho - \varepsilon_{\nu\rho} \partial_\rho] \partial^\nu \bar{C}, \\
  s_{ad} A_\mu &= -\varepsilon_{\mu\nu} \partial^\nu C, \quad s_{ad} C = 0, \quad s_{ad} \bar{C} = 0, \quad s_{ad} \mathcal{B} = 0, \quad s_{ad} \mathcal{B} = 0, \quad s_{ad} F_{\mu\nu} = [\varepsilon_{\mu\rho} \partial_\rho - \varepsilon_{\nu\rho} \partial_\rho] \partial^\rho C,
\end{align*}
\]

(2.4)

(i) the Lagrangian density (2.2) for the 2D free 1-form Abelian theory changes to a total derivative (i.e. \(s_d \mathcal{L}_B = \partial_\mu \mathcal{B} \partial^\mu C\)), \(s_{ad} \mathcal{L}_B = \partial_\mu \mathcal{B} \partial^\mu C\),

(ii) the anticommuting nature of the nilpotent (anti-co-)BRST symmetry transformations becomes transparent because the operator equation \([s_d s_{ad} + s_{ad} s_d] = 0\) turns out to be true for any arbitrary field \(\Omega\) (i.e. \([s_d s_{ad} + s_{ad} s_d] \Omega = 0\) of the Lagrangian density (2.2),

(iii) the gauge-fixing term (\(\partial \cdot A\)), owing its origin to the nilpotent co-exterior derivative \(\delta = -* d*\), remains invariant under the above (anti-co-)BRST symmetry transformations,

(iv) the gauge-fixing term (\(\partial \cdot A\)) is an on-shell (i.e. \(\Box C = 0, \Box \bar{C} = 0\)) invariant quantity under the nilpotent (anti-)BRST transformations (2.3), and

(v) the cohomological origin for the existence of the (anti-co-)BRST symmetry transformations for the above gauge theory lies in the dual(co) exterior derivative \(\delta\). This observation

\[^{46}\text{We follow here the notations and conventions adopted in [47]. In fact, in its totality, the BRST transformation }\delta_B\text{ is a product }\delta_B = \eta s_b\text{ of an anticommuting (i.e. }\eta C + \bar{C} = 0, \eta \bar{C} + C = 0\text{) spacetime independent parameter }\eta\text{ and the nilpotent }s_b^2 = 0\text{ operator }s_b.\]
will play an important role in the derivation of the (anti-)co-BRST symmetry transformation in the framework of the superfield approach to the BRST formalism (cf. Subsec. 4.2 below).

In fact, we shall see that the cohomological origin of the co-existence of the (anti-)co-BRST symmetry transformations together for the 2D free Abelian gauge theory is encapsulated in the existence of the super co-exterior derivative on the four (2, 2)-dimensional supermanifold which will be exploited in the dual-horizontality condition.

We focus now on the existence of a bosonic (non-nilpotent) symmetry transformation $s_w$ that emerges due to the anticommutation relation between the nilpotent (anti-)BRST and (anti-)co-BRST symmetry transformations (i.e. $s_w = s_b s_d + s_d s_b \equiv s_{ab} s_{ad} + s_{ad} s_{ab}$). Under this symmetry transformation, the fields of the Lagrangian density (2.2) transform as follows (see, e.g. [36,38,41] for details)

$$s_w A_\mu = \partial_\mu B + \epsilon_{\mu\nu} \partial^\nu B, \quad s_w E = -\Box B, \quad s_w (\partial \cdot A) = \Box B,$$

$$s_w C = 0, \quad s_w \bar{C} = 0, \quad s_w B = 0, \quad s_w \bar{B} = 0. \quad (2.5)$$

It can be easily checked that the above transformations entail upon the Lagrangian density (2.2) to change to a total derivative as: $s_w \mathcal{L}_B = \partial_\mu [B \partial^\mu B - B \partial^\rho \bar{B}]$. The algebra followed by the above transformation operators $s_r$ (with $r = b, ab, d, ad, w$) is

$$s_r^2 = 0, \quad s_2 = 0, \quad s_r^2 \neq 0,$$

$$s_r = \{ s_b, s_d \} = \{ s_{ab}, s_{ad} \}, \quad \{ s_b, s_{ab} \} = 0, \quad \{ s_{ad}, s_{ab} \} = 0,$$

$$[s_r, s_{ab}] = [s_r, s_{ad}] = 0. \quad (2.6)$$

This algebra can be compared and contrasted with the algebra obeyed by the de Rham cohomological operators as given below

$$d^2 = 0, \quad \delta^2 = 0, \quad \Delta = \{ d, \delta \} \equiv (d + \delta)^2,$$

$$[\Delta, d] = 0, \quad [\Delta, \delta] = 0, \quad \{ d, \delta \} \neq 0. \quad (2.7)$$

A close look at equations (2.6) and (2.7) establishes a two-to-one correspondence between the symmetry transformation operators $s_r$ (with $r = b, d, ab, ad, w$) and the cohomological operators $(d, \delta, \Delta)$. These are: $(s_b, s_{ab}) \to d, (s_d, s_{ab}) \to \delta, \{ s_b, s_{ab} \} \to [s_d, s_{ab}] \to \Delta$. All the above continuous symmetry transformations are generated by the conserved charges $Q_r (r = b, ab, d, ad, w)$ and their intimate relationship can be succinctly expressed as

$$s_r \Omega = -i \{ \Omega, Q_r \} \pm, \quad \Omega = A_\mu, C, \bar{C}, B, \bar{B}, \quad (2.8)$$

where the $(\pm)$ signs on the square bracket stand for the bracket to be an (anti)commutator for the generic field $\Omega$ of the Lagrangian density (2.2) being (fermionic)bosonic in nature. It should be noted, at this stage, that the conserved charges $Q_r$ (with $r = b, ab, d, ad, w$) obey exactly the same kind of algebra as the one (cf. (2.6)) obeyed by the corresponding symmetry transformation operators $s_r$. Furthermore, the mapping between the conserved charges and the de Rham cohomological operators are found to be exactly the same (i.e. $(Q_b, Q_{ab}) \to d, (Q_d, Q_{ab}) \to \delta$ and $Q_w = \{ Q_b, Q_d \} = \{ Q_{ad}, Q_{ab} \} \to \Delta = \{ d, \delta \}$).
It should be emphasized, as a side remark, that there is no effect of the Laplacian operator $\Delta$ on any of the terms of the Lagrangian density (2.2) because $\Delta A^{(1)} = dx^\mu \Box A_\mu = 0$ leads to the derivation of the equation of motion ($\Box A_\mu = 0$) for the gauge field $A_\mu$, which (i.e. the equation of motion) is not present in the Lagrangian density (2.2) on its own. It emerges, however, from (2.2) due to the application of the Euler-Lagrange equation of motion on it. This observation will be exploited in Subsec. 4.3 where we shall show that the appropriate definition of a super Laplacian operator, in a suitable restriction on the four (2, 2)-dimensional supermanifold, leads to the derivation of the equation of motion for all the fields of the Lagrangian density (2.2) within the framework of the superfield approach to BRST formalism. In fact, the bosonic symmetry (cf. (2.5)) transformations $s_w = \{s_b, s_d\} \equiv \{s_{ad}, s_{ab}\}$ encompass the analogue of the definition of the Laplacian operator in terms of the cohomological operators $d$ and $\delta$ as: $\Delta = \{d, \delta\} \equiv d\delta + \delta d$.

Before we wrap up this Sect., we shall dwell a bit on the existence of the discrete symmetry transformations in the theory. First of all, it can be noted that the (anti-)ghost part of the action (i.e. $S_{F,P} = -i \int d^2x \partial_\mu \bar{C} \partial^\mu C$) remains invariant under the following discrete symmetry transformations (for the 2D 1-form Abelian gauge theory):

$$C \to \pm i \bar{C}, \quad \bar{C} \to \pm i C, \quad \partial_\mu \to \pm i \varepsilon_{\mu \nu} \partial^\nu. \quad (2.9)$$

The existence of the above discrete symmetry transformations is responsible for the derivation of the (anti-)co-BRST symmetry transformations from the (anti-)BRST symmetry transformations [36,38,41]. It is already well known that the ghost action is also invariant under $C \to \pm i \bar{C}, \bar{C} \to \pm i C$ which leads to the derivation of (i) the anti-BRST symmetry transformations from the BRST symmetry transformations, and (ii) the anti-co-BRST symmetry transformations from the co-BRST symmetry transformations. It can be checked explicitly that the Lagrangian density (2.2) remains invariant under the following separate and independent discrete symmetry transformations

$$C \to \pm i \bar{C}, \quad \bar{C} \to \pm i C, \quad \partial_\mu \to \pm i \varepsilon_{\mu \nu} \partial^\nu, \quad A_\mu \to A_\mu, \quad (2.10)$$

$$B \to \mp i B, \quad B \to \mp i B, \quad (\partial \cdot A) \to \pm i E, \quad E \to \pm i (\partial \cdot A), \quad (\partial \cdot A) \to \pm i E, \quad E \to \pm i (\partial \cdot A). \quad (2.11)$$

The above transformations are found to be the analogue of the Hodge duality * operation of the differential geometry for the 1-form free Abelian gauge theory when they are combined with the continuous and nilpotent (anti-)BRST and (anti-)co-BRST symmetry transformations.

To elaborate a bit on the importance of the above discrete symmetry transformations, we, first of all, check the effect of two successive discrete transformations (2.11) on any arbitrary generic field $\Omega$ of the Lagrangian density (2.2). This is required as an essential ingredient for the discussion of any arbitrary duality invariant theory [48]. In fact, this requirement decides the relationship between a physical quantity and its dual. For the
theory under consideration, this requirement is given by the following expression
\[ * (\ast \Omega) = - \Omega, \quad \Omega = C, \bar{C}, A_\mu, B, \mathcal{B}, E, (\partial \cdot A). \] (2.12)
The above (−) sign on the r.h.s. dictates the following interesting relationship
\[ s_{(a)d} (\Omega) = - \ast s_{(a)b} * (\Omega), \] (2.13)
which is a relationship that involves the interplay between the discrete symmetry transformation (2.11) (or (2.10)) and continuous symmetry transformations \( s_{(a)b} \) and \( s_{(a)d} \). It is evident that the above relationship is (i) the analogue of the relationship that exists between the exterior derivative \( d = dx^\mu \partial_\mu \) and the dual(co)-exterior derivative (i.e. \( \delta = - \ast d* \)), and (ii) contains the definition of the Hodge duality \( * \) operation as the discrete symmetry transformations (2.11) (or (2.10)). In more transparent language, the operation of the cohomological differential operators \( \delta \) and \( d \) on a differential form is equivalent to the operation of the nilpotent symmetry transformations \( s_{(a)d} \) and \( s_{(a)b} \) on the generic field \( \Omega \) of the Lagrangian density (2.2) of the 2D 1-form free Abelian gauge theory. This statement has been mathematically expressed in (2.13). It is worthwhile to mention, at this stage, that there is a reciprocal relationship of the above equation, namely;
\[ s_{(a)b} (\Omega) = - \ast s_{(a)d} * (\Omega), \] (2.14)
that also exists on the 2D spacetime manifold. In fact, the duality operation on the 4D and 2D spacetime manifolds are different as can be seen in [48]. For more discussions on the relationships between the Hodge duality \( * \) operation and discrete symmetry transformations of the theory, we refer to the readers our earlier works on this subject [41,38,36].

3. Topological Features of 2D 1-Form Abelian Gauge Theory

In this Sec., first of all, we express the form of the Lagrangian density (2.1) in terms of the conserved and on-shell nilpotent (anti-)BRST and (anti-)co-BRST charges. In Subsec. 3.1, we focus, for our elaborate discussions, only on the Lagrangian density (2.1) because it contains merely the basic dynamical fields of the theory (i.e. there is no presence of any auxiliary fields in it). Our Subsec. 3.2 is devoted to the discussion of the BRST cohomology, physical state condition and topological nature of the theory in terms of the BRST and co-BRST charges and their action on the states of the quantum Hilbert space.

3.1 Topological Aspects: Mathematical Description

In this Subsec., we demonstrate that the form of the Lagrangian density is similar to the Witten-type topological field theory. To corroborate the above assertion, let us concentrate on the Lagrangian density (2.1) and express it as follows
\[ \mathcal{L}_b = \bar{s}_b (iT_1) + \bar{s}_d (iT_2) + \partial_\mu Y^\mu \equiv \bar{s}_{ab} (iP_1) + \bar{s}_{ad} (iP_2) + \partial_\mu Y^\mu. \] (3.1)
The following points, at this stage, are in order now. First, in the above equation, \( \tilde{s}_{(a)b} \) and \( \tilde{s}_{(a)d} \) are the on-shell (\( \Box C = 0, \Box \tilde{C} = 0 \)) nilpotent (\( \tilde{s}_{(a)b}^2 = 0, \tilde{s}_{(a)d}^2 = 0 \)) version of the (anti-)BRST and (anti-)co-BRST symmetry transformations. Explicitly these are

\[
\begin{align*}
\tilde{s}_{(a)b} A_{\mu} &= \partial_{\mu} C, \quad \tilde{s}_{(a)b} C = 0, \quad \tilde{s}_{(a)b} \tilde{C} = -i(\partial \cdot A), \\
\tilde{s}_{(a)b} E &= 0, \quad \tilde{s}_{(a)b} (\partial \cdot A) = \Box C, \quad \tilde{s}_{(a)b} F_{\mu \nu} = 0, \\
\tilde{s}_{(a)d} A_{\mu} &= \partial_{\mu} \tilde{C}, \quad \tilde{s}_{(a)d} \tilde{C} = 0, \quad \tilde{s}_{(a)d} C = +i(\partial \cdot A), \\
\tilde{s}_{(a)d} E &= 0, \quad \tilde{s}_{(a)d} (\partial \cdot A) = \Box \tilde{C}, \quad \tilde{s}_{(a)d} F_{\mu \nu} = 0,
\end{align*}
\tag{3.2}
\]

The above equations (3.2) and (3.3) have been derived from (2.3) and (2.4) by the substitution \( B = -(\partial \cdot A) \) and \( B = E \). Second, the exact expressions for \( T_1, T_2, P_1, P_2 \) and \( Y^\mu \), for the Lagrangian density (2.1), are as follows

\[
\begin{align*}
T_1 &= -\frac{1}{2} (\partial \cdot A) \tilde{C}, \quad T_2 = +\frac{1}{2} E C, \quad P_1 = +\frac{1}{2} (\partial \cdot A) C, \quad P_2 = -\frac{1}{2} E \tilde{C}, \\
Y^\mu &= \frac{i}{2} [ \tilde{C} \partial^\mu C + \partial^\mu \tilde{C} C ] \equiv \frac{i}{2} [ \partial^\mu (\tilde{C} C) ].
\end{align*}
\tag{3.4}
\]

Finally, taking the help of equation (2.8), it can be seen that the Lagrangian density (2.1), modulo some total derivatives, can also be written as the sum of two anticommutators, namely;

\[
\mathcal{L}_b = \{ \tilde{Q}_b, T_1 \} + \{ \tilde{Q}_d, T_2 \} \equiv \{ \tilde{Q}_{ab}, P_1 \} + \{ \tilde{Q}_{ad}, P_2 \},
\tag{3.5}
\]

where \( \tilde{Q}_r \) (with \( r = b, ab, d, ad \)) are the on-shell nilpotent version of the conserved and nilpotent (anti-)BRST and (anti-)co-BRST charges. The exact expressions for these charges are not essential for us at the moment. However, their exact expressions, in terms of the local fields of the theory, can be found in [46,47] and in Subsec. 3.2 (cf. (3.15)).

At this juncture, a few further remarks are in order. First, even though we have two pairs of nilpotent charges (i.e. \( \tilde{Q}_b, \tilde{Q}_d \)) as well as \( \{ \tilde{Q}_{ad}, \tilde{Q}_{ab} \} \) in terms of which the Lagrangian density of the theory can be expressed, still we claim that the Lagrangian density looks like Witten-type topological theory. The key reason behind this assertion is the fact that this field theoretic model is a tractable model for the Hodge theory. As a consequence, we can choose the physical state \( |phys > \) to be the harmonic state in the Hodge decomposition theorem (see, Subsec. 3.2 for details) because this state happens to be the most symmetric state. In fact, the physical state, chosen in such a manner, is (anti-)BRST invariant (i.e. \( \tilde{Q}_{(a)b}|phys > = 0 \)) as well as (anti-)co-BRST invariant (i.e. \( \tilde{Q}_{(a)d}|phys > = 0 \)) \( together \). Precisely speaking, for the Witten type topological field theory, there exist only nilpotent (anti-)BRST charges and the physical states are annihilated by these (i.e. \( \tilde{Q}_{(a)b}|phys > = 0 \) alone. Second, it is evident that the form of the Lagrangian density (3.1) and (3.5) is not like the Schwarz type topological field theory (TFT) because, for such kind of a TFT,
there exists always a piece in the Lagrangian density that can not be expressed as a BRST (anti)commutator. Third, the symmetry transformations for the free 2D 1-form Abelian gauge theory is like the Schwarz type TFT because there exists no local topological shift symmetry for the 1-form Abelian gauge theory (which happens to be the hallmark for a Witten type TFT). Thus, from the symmetry point of view, our prototype field theoretical model of 2D 1-form gauge theory is like the Schwarz type TFT. Finally, we are considering our present 2D 1-form gauge model in the flat Minkowskian spacetime. Thus, there is no non-trivial metric dependence in the theory. As a consequence, the model of our present discussion, once again, is like the Schwarz type TFT. Thus, finally, we conclude that the form of the Lagrangian density of the 2D 1-form gauge theory is like the Witten type TFT. Thus, from the symmetry point of view, our prototype field theoretical symmetry for the 1-form Abelian gauge theory (which happens to be the hallmark for a

One of the decisive features of the TFTs is the absence of any energy excitations in the theory which is mainly governed and dictated by the form of the symmetric energy-momentum tensor \( T_{\mu\nu}^{(s)} = T_{\nu\mu}^{(s)} \) of the theory. This expression for the case of the free 1-form Abelian gauge theory, described by the Lagrangian density (2.1), is

\[
T_{\mu\nu}^{(s)} = \frac{1}{2} \partial_\mu \Omega \frac{\partial \mathcal{L}_b}{\partial \nu \Omega} + \frac{1}{2} \partial_\nu \Omega \frac{\partial \mathcal{L}_b}{\partial \mu \Omega} - \eta_{\mu\nu} \mathcal{L}_b, \tag{3.6}
\]

where the generic field \( \Omega = A_\mu, C, \bar{C} \) for the Lagrangian density (2.1). The exact and explicit expression for the above symmetric tensor is

\[
T_{\mu\nu}^{(s)} = -\frac{1}{2} \left[ \varepsilon_{\mu\rho} E + \eta_{\mu\nu} (\partial \cdot A) \right] (\partial_{\nu} A^\rho) - \frac{1}{2} \left[ \varepsilon_{\nu\rho} E + \eta_{\nu\mu} (\partial \cdot A) \right] (\partial_{\mu} A^\rho)
- i \partial_\mu \bar{C} \partial_{\nu} C - i \partial_\nu \bar{C} \partial_{\mu} C - \eta_{\mu\nu} \mathcal{L}_b. \tag{3.7}
\]

The above expression can be expressed, in terms of the on-shell nilpotent (anti-)BRST charge \( \tilde{Q}_{(a)b} \) and (anti-)co-BRST charge \( \tilde{Q}_{(a)d} \), as

\[
T_{\mu\nu}^{(s)} = \{ \tilde{Q}_{b} V_{\mu\nu}^{(1)} \} + \{ \tilde{Q}_{d} V_{\mu\nu}^{(2)} \} \equiv \{ \tilde{Q}_{ab} \tilde{V}_{\mu\nu}^{(1)} \} + \{ \tilde{Q}_{ad} \tilde{V}_{\mu\nu}^{(2)} \},
\]

\[
V_{\mu\nu}^{(1)} = +\frac{1}{2} \left[ (\partial_{\mu} C) A_{\nu} + (\partial_{\nu} C) A_{\mu} + \eta_{\mu\nu} (\partial \cdot A) \bar{C} \right],
\]

\[
V_{\mu\nu}^{(2)} = +\frac{1}{2} \left[ (\partial_{\mu} C) \varepsilon_{\nu\rho} A^\rho + (\partial_{\nu} C) \varepsilon_{\mu\rho} A^\rho - \eta_{\mu\nu} E C \right],
\]

\[
\tilde{V}_{\mu\nu}^{(1)} = -\frac{1}{2} \left[ (\partial_{\mu} C) A_{\nu} + (\partial_{\nu} C) A_{\mu} + \eta_{\mu\nu} (\partial \cdot A) C \right],
\]

\[
\tilde{V}_{\mu\nu}^{(2)} = -\frac{1}{2} \left[ (\partial_{\mu} C) \varepsilon_{\nu\rho} A^\rho + (\partial_{\nu} C) \varepsilon_{\mu\rho} A^\rho - \eta_{\mu\nu} E C \right]. \tag{3.8}
\]

The form of the symmetric energy-momentum tensor \( (T_{\mu\nu}^{(s)}) \) demonstrates that, when the Hamiltonian density \( T_{00}^{(s)} \) is sandwiched between two physical states, it becomes zero because of the fact that \( \tilde{Q}_{(a)b}|phys > 0 \) and \( \tilde{Q}_{(a)d}|phys > 0 \) when we choose the physical state to be the harmonic state of the Hodge decomposed state in the quantum Hilbert space of states (cf. Subsec. 3.2). In fact, this state is annihilated by all the conserved, hermitian and nilpotent charges of the theory (i.e. \( \tilde{Q}_{(a)b}|phys > 0, \tilde{Q}_{(a)d}|phys > 0 \). To be precise, the conditions \( \tilde{Q}_{b}|phys > 0 \) and \( \tilde{Q}_{d}|phys > 0 \) lead to the restrictions \( (\partial \cdot A)|phys > 0 \) and \( \varepsilon_{\mu\nu} \partial_{\mu} A_{\nu}|phys > 0 \), respectively. This ensures that there are no propagating degrees of freedom in the theory as both the components \( A_0 \) and \( A_1 \) of the 2D
gauge field become conserved quantities (i.e. \( \partial_0 A_0 = \partial_1 A_1, \partial_0 A_1 = \partial_1 A_0 \)). This situation is the most remarkable feature of a TFT. Thus, the topological nature of the 2D free 1-form Abelian gauge theory is confirmed because there are no propagating dynamical degrees of freedom left out with the 2D gauge field \( A_\mu \), after the application of the physicality criteria (i.e. \( \bar{Q}_{(a)b}|\text{phys} >= 0, \bar{Q}_{(a)d}|\text{phys} >= 0 \)). The present theory, however, a new kind of TFT which captures together some of the salient features of both the Witten-type as well as the Schwarz-type TFTs. This key observation is a completely new result [41,36].

3.2 Cohomological Aspects: Topological Features

In this Subsec., we shall discuss the Hodge decomposition theorem and establish the topological nature of the theory due to the consideration of the cohomological aspects of the states in the quantum Hilbert space. To this end in mind, let us begin with the infinitesimal version of the ghost symmetry transformations \( s_g \) for the Lagrangian density (2.1), namely;

\[
s_g A_\mu = 0, \quad s_g C = -\Sigma C, \quad s_g C' = +\Sigma C',
\]

where \( \Sigma \) is a global scale transformation parameter. Under the above symmetry transformation, the Lagrangian density (2.1) remains invariant. The generator (i.e. ghost charge) \( \bar{Q}_g \) of the above transformations is a conserved quantity which obeys the following algebra with the rest of the generators \( \bar{Q}_r \) (with \( r = b, ab, d, ad, w \)) of the theory (see, e.g. [36])

\[
\begin{align*}
\bar{Q}_w, \bar{Q}_g &= 0, \\
i \bar{Q}_g, \bar{Q}_b &= + \bar{Q}_b, \\
i \bar{Q}_g, \bar{Q}_d &= - \bar{Q}_d, \\
i \bar{Q}_g, \bar{Q}_{ab} &= - \bar{Q}_{ab}, \\
i \bar{Q}_g, \bar{Q}_{ad} &= + \bar{Q}_{ad}.
\end{align*}
\]

The above algebra plays a very important role in the discussion of the Hodge decomposition theorem. This is due to the fact that, given any arbitrary state \( |\phi >_n \) with the ghost number \( n \) (i.e. \( i\bar{Q}_g |\phi >_n = n |\phi >_n \)) in the quantum Hilbert space of states, it is clear, from the above algebra, that the following relationships are true, namely;

\[
\begin{align*}
i \bar{Q}_g \bar{Q}_b |\phi >_n &= (n + 1) \bar{Q}_b |\phi >_n, \\
& \quad i \bar{Q}_g \bar{Q}_{ab} |\phi >_n = (n + 1) \bar{Q}_{ab} |\phi >_n, \\
i \bar{Q}_g \bar{Q}_d |\phi >_n &= (n - 1) \bar{Q}_d |\phi >_n, \\
& \quad i \bar{Q}_g \bar{Q}_{ad} |\phi >_n = (n + 1) \bar{Q}_{ad} |\phi >_n, \\
i \bar{Q}_g \bar{Q}_w |\phi >_n &= n \bar{Q}_w |\phi >_n.
\end{align*}
\]

The above equation shows that the ghost number of the states \( \bar{Q}_b |\phi >_n \) and \( \bar{Q}_{ad} |\phi >_n \) is \( (n + 1) \). This situation is like \( df_n \sim \tilde{f}_{n+1} \) in the differential geometry (when the exterior derivative \( d = dx^\mu \partial_\mu \) acts on the n-form \( f_n \)). On the contrary, the states \( \bar{Q}_d |\phi >_n \) and \( \bar{Q}_{ab} |\phi >_n \) carry the ghost number equal to \( (n - 1) \). This feature is like \( \delta f_n \sim \tilde{f}_{n-1} \) due to the operation of the co-exterior derivative \( \delta \) on the n-form \( f_n \). In view of the forms of the algebra illustrated in (2.6) and (2.7) for the symmetry operators and the cohomological operators, it is clear that one can have the analogue of the Hodge decomposition theorem (1.1) in the Hilbert space of states as given below (see, e.g. [41,42] for details)

\[
|\phi >_n \equiv |\omega >_n + \bar{Q}_b |\theta >_{(n-1)} + \bar{Q}_d |\chi >_{(n+1)},
\]

\[
|\phi >_n + \bar{Q}_{ad} |\theta >_{(n-1)} + \bar{Q}_{ab} |\chi >_{(n+1)}.
\]
where \( \hat{Q}_b |\theta >_{(n-1)} \) is the BRST exact state and \( \hat{Q}_d |\chi >_{(n+1)} \) is the BRST co-exact state. There are two ways to write the Hodge decomposition theorem in the quantum Hilbert space of states because there is two-to-one mapping between the conserved charges and the cohomological operators (i.e. \( (\hat{Q}_b, \hat{Q}_{ad}) \rightarrow d, (\hat{Q}_d, \hat{Q}_{ab}) \rightarrow \delta, \{ \hat{Q}_b, \hat{Q}_d \} = \{ \hat{Q}_{ad}, \hat{Q}_{ab} \} \rightarrow \Delta \)). In the above decomposition, the most symmetric state \( |\omega >_{(n)} \) is the harmonic state that is annihilated (i.e. \( \hat{Q}_r |\omega >_{(n)} = 0 \)) by all the charges \( \hat{Q}_r \) (with \( r = b, ab, d, ad, w \)). In other words, the harmonic state is already an (anti-)BRST and (anti-)co-BRST invariant state, to begin with. This is why, it is the most beautiful state of our present 2D free Abelian theory. It will be chosen, therefore, as the physical state on an aesthetic ground.

To observe the key consequences of the above statements, let us begin with our physical state to be the harmonic state of the above decomposition (i.e. \( |\text{phys} >= |\omega > \)). This immediately implies the following

\[
\hat{Q}_b |\text{phys} >= 0, \quad \hat{Q}_d |\text{phys} >= 0, \quad \hat{Q}_w |\text{phys} >= 0. \tag{3.13}
\]

It will be noted that, in the above, we have taken only one set (i.e. \( \hat{Q}_b, \hat{Q}_d, \hat{Q}_w \)) of the conserved charges, as the consequences from this set, would be exactly the same as the ones derived from the other set \( (\hat{Q}_{ad}, \hat{Q}_{ab}, \hat{Q}_w) \). The basic conditions, that emerge from the above restrictions, are as follows

\[
\hat{Q}_b |\text{phys} >= 0 \Rightarrow (\partial \cdot A)|\text{phys} >= 0, \quad \partial_0 (\partial \cdot A)|\text{phys} >= 0, \quad \partial_0 E |\text{phys} >= 0, \quad \partial_0 E |\text{phys} >= 0, \tag{3.14}
\]

where the explicit expressions for the charges \( \hat{Q}_b, \hat{Q}_d, \hat{Q}_w \) are

\[
\begin{align*}
\hat{Q}_b &= \int (dx) \left[ \partial_0 (\partial \cdot A) C - (\partial \cdot A) \dot{C} \right], \\
\hat{Q}_d &= \int (dx) \left[ E \dot{C} - \dot{E} C \right], \\
\hat{Q}_w &= \int (dx) \left[ \partial_0 (\partial \cdot A) E - \dot{E} (\partial \cdot A) \right].
\end{align*}
\tag{3.15}
\]

The conditions generated by \( \hat{Q}_w \) are not new. They are same as the ones emerging due to \( \hat{Q}_b \) and \( \hat{Q}_d \). We shall dwell a bit more on the above conditions (3.14) in the language of the normal mode expansions of the basic fields of the Lagrangian density (2.1). For this purpose, let us have the normal mode expansion (for the equations of motion \( \Box A_\mu = 0, \Box C = 0 \) and \( \Box \dot{C} = 0 \) in the phase space of the theory, as (see, e.g., [47])

\[
\begin{align*}
A_\mu (x, t) &= \int \frac{dk}{(2\pi)^{1/2}(2k_0)^{1/2}} \left[ a_\mu (k)e^{-ik \cdot x} + a_\mu ^\dagger (k)e^{+ik \cdot x} \right], \\
C (x, t) &= \int \frac{dk}{(2\pi)^{1/2}(2k_0)^{1/2}} \left[ c(k)e^{-ik \cdot x} + c^\dagger (k)e^{+ik \cdot x} \right], \\
\dot{C} (x, t) &= \int \frac{dk}{(2\pi)^{1/2}(2k_0)^{1/2}} \left[ b(k)e^{-ik \cdot x} + b^\dagger (k)e^{+ik \cdot x} \right],
\end{align*}
\tag{3.16}
\]

where \( k_\mu = (k_0, k_1 = k) \) is the 2D momentum vector in the phase space and \( a_\mu ^\dagger, c^\dagger \) and \( b^\dagger \) are the creation operators for a photon, a ghost quantum and an anti-ghost quantum, respectively. The corresponding operators, without a dagger, are the annihilation operators.
Taking the help of the nilpotent symmetry transformations of (3.2) and (3.3), we can obtain the following (anti)commutators

\[
\begin{align*}
[\tilde{Q}_b, a_{\mu}^\dagger] &= +k_\mu c^\dagger(k), & [\tilde{Q}_d, a_{\mu}^\dagger] &= -\varepsilon_{\mu\nu} k^\nu b^\dagger(k), \\
[\tilde{Q}_b, a_{\mu}] &= -k_\mu c(k), & [\tilde{Q}_d, a_{\mu}] &= +\varepsilon_{\mu\nu} k^\nu b(k), \\
\{\tilde{Q}_b, c^\dagger(k)\} &= 0, & \{\tilde{Q}_d, c^\dagger(k)\} &= -i\varepsilon_{\mu\nu} k_\mu a_{\nu}^\dagger, \\
\{\tilde{Q}_b, c(k)\} &= 0, & \{\tilde{Q}_d, c(k)\} &= +i\varepsilon_{\mu\nu} k_\mu a_{\nu}, \\
\{\tilde{Q}_b, b^\dagger(k)\} &= +ik_\mu a_{\mu}^\dagger, & \{\tilde{Q}_d, b(k)\} &= 0, \\
\{\tilde{Q}_b, b(k)\} &= -ik_\mu a_{\mu}, & \{\tilde{Q}_d, b^\dagger(k)\} &= 0,
\end{align*}
\]

(3.17)

where (i) we have exploited the expansion given in (3.16), and (ii) we have utilized the formula, given in the infinitesimal nilpotent transformations \(\tilde{s}_r\) (with \(r = b, d\)) and their generators \(\tilde{Q}_r\) (with \(r = b, d\)). It is evident, from the expression for \(\tilde{Q}_w\) in (3.15), that this conserved charge generates the following transformations

\[
\begin{align*}
\tilde{s}_w A_\mu &= \partial_\mu E - \varepsilon_{\mu\nu} \partial^\nu (\partial \cdot A), & \tilde{s}_w E &= +\Box (\partial \cdot A), \\
\tilde{s}_w (\partial \cdot A) &= \Box E, & \tilde{s}_w C &= 0, & \tilde{s}_w \bar{C} &= 0,
\end{align*}
\]

(3.18)

which, ultimately, leads to the following commutation relations

\[
\begin{align*}
[Q_w, a_{\mu}^\dagger(k)] &= +i k^2 \varepsilon_{\mu\nu} (a^\nu(k))^\dagger, & [Q_w, a_{\mu}(k)] &= -i k^2 \varepsilon_{\mu\nu} a^\nu(k), \\
[Q_w, c(k)] &= [Q_w, c^\dagger(k)] = [Q_w, b(k)] = [Q_w, b^\dagger(k)] = 0.
\end{align*}
\]

(3.19)

where, once again, the mode expansion of (3.16) and analogue of the relation (2.8) for \(\tilde{s}_w\) and \(\tilde{Q}_w\) (i.e. \(\tilde{s}_w \Omega = -i[\Omega, \tilde{Q}_w]\)) have been used. For the masslessness condition (i.e. \(k^2 = 0\)), it is evident that the charge \(\tilde{Q}_w\) will become the Casimir operator.

Let us now define the physical vacuum of the theory by the following conditions that are imposed by its \textit{physical} properties and its \textit{harmonic} nature, namely;

\[
\begin{align*}
\tilde{Q}_b |\text{vac} >= 0, & \quad \tilde{Q}_d |\text{vac} >= 0, & \quad \tilde{Q}_w |\text{vac} >= 0, \\
a_{\mu}(k) |\text{vac} >= 0, & \quad c(k) |\text{vac} >= 0, & \quad b(k) |\text{vac} >= 0.
\end{align*}
\]

(3.20)

From the above vacuum state, a single \textit{physical} photon state with the polarization \(e_\mu\) can be created by the application of the creation operator \(a_{\mu}^\dagger\). This can be expressed as:

\(e^\mu a_{\mu}^\dagger |\text{vac} >= |e, \text{vac} >\) (see, e.g. [47]). In an exactly similar fashion, a single photon with momentum \(k^\mu\) can be created from the vacuum state by \(k^\mu a_{\mu}^\dagger |\text{vac} >= |k, \text{vac} >\). The single physical photon state being a harmonic state, the following restrictions emerge due to the application of the conserved charges \(\tilde{Q}_r\) (with \(r = b, d, w\)) on it, namely;

\[
\begin{align*}
\tilde{Q}_b |e, \text{vac} >= [\tilde{Q}_b, e^\mu a_{\mu}^\dagger] |\text{vac} >= 0 & \Rightarrow (k \cdot e) = 0, \\
\tilde{Q}_d |e, \text{vac} >= [\tilde{Q}_d, e^\mu a_{\mu}^\dagger] |\text{vac} >= 0 & \Rightarrow (\varepsilon_{\mu\nu} e^\mu k^\nu) = 0, \\
\tilde{Q}_w |e, \text{vac} >= [\tilde{Q}_w, e^\mu a_{\mu}^\dagger] |\text{vac} >= 0 & \Rightarrow (k^2) = 0.
\end{align*}
\]

(3.21)

The above relations are found to be consistent with one-another. In fact, any two of the above relations imply the third one [36]. In particular, it will be noted that the relations
(i.e. \((k \cdot e) = 0, \varepsilon_{\mu\nu} e^\mu k^\nu = 0\)) emerging from the BRST charge \(\tilde{Q}_b\) as well as co-BRST charge \(\tilde{Q}_d\) are invariant under the gauge transformation \(e_\mu \to e_\mu + \alpha k_\mu\) and the dual gauge transformation \(e_\mu \to e_\mu + \beta \varepsilon_{\mu\nu} k_\nu\) if the masslessness condition \(k^2 = 0\) is taken into account. Here \(\alpha\) and \(\beta\), in the above (dual-)gauge transformations, are spacetime independent constants. The above two symmetry transformations are good enough to gauge away both the degrees of freedom of photon in two \((1+1)\)-dimensions of spacetime. This is the root cause of the 2D photon to be topological in nature because there are no propagating degrees of freedom left out in the free 2D 1-form Abelian gauge theory [36].

We would like to close this subsection with the remark that the existence of the BRST and co-BRST symmetry transformations enables one to decompose both the degrees of freedom of the 2D photon (i.e. \(dx^\mu A_\mu(x) = dx^\mu \partial_\mu \kappa(x) + dx^\mu \varepsilon_{\mu\nu} \partial_\nu \zeta(x)\)) into (i) a component parallel to the momentum vector, and (ii) the other component parallel to the polarization vector. Here the fields \(\kappa(x)\) and \(\zeta(x)\) are the component fields. Thus, a 2D photon has no propagating (dynamical) degrees of freedom. As a consequence, the 2D free 1-form Abelian gauge theory is a (new kind of) topological field theory (see, e.g., [36] for details).

4 Superfield Approach to BRST Formulation of 2D Gauge Theory

In this Sec., we shall tap the power and potential of the super de Rham cohomological operators and demonstrate their usefulness in the derivation of (i) the off-shell nilpotent (anti-)BRST symmetries, (ii) the off-shell nilpotent (anti-)co-BRST symmetries, and (iii) the equations of motion for all the fields of the 2D free 1-form Abelian gauge theory. Furthermore, we shall also capture the topological features of the above theory in the language of the superfield approach to BRST formalism.

4.1 Super Exterior Derivative and (Anti-)BRST Symmetries

As pointed out after equation (2.3), we know that the cohomological origin for the nilpotent (anti-)BRST symmetry transformations lies in the nilpotent exterior derivative. This statement becomes very clear in the framework of the superfield approach to BRST formalism where we exploit the horizontality condition (HC) for the derivation of the (anti-)BRST symmetry transformations for the gauge and (anti-)ghost fields of the 2D Abelian 1-form gauge theory. It turns out that the celebrated HC [20-29], on the four \((2, 2)\)-dimensional supermanifold, owes its origin to the (super) exterior derivatives.

To elaborate a bit on the above assertion, we begin with the super 1-form connection \(\tilde{A}^{(1)} = dZ^M A_M\) where (i) the superspace variable on the four \((2, 2)\)-dimensional supermanifold is \(Z^M = (x^\mu, \theta, \bar{\theta})\) where \(x^\mu\) (with \(\mu = 0, 1\)) are the 2D bosonic spacetime variable and \(\theta, \bar{\theta}\) are the Grassmannian variables (with \(\theta^2 = \bar{\theta}^2 = 0, \theta \bar{\theta} + \bar{\theta} \theta = 0\)), and (ii) the supermultiplet fields \(B_\mu(x, \theta, \bar{\theta}), F(x, \theta, \bar{\theta}), \bar{F}(x, \theta, \bar{\theta})\) on the above supermanifold (as the generalization of the 2D basic fields \(A_\mu(x), C(x), \bar{C}(x)\)) constitute the vector superfield
$A_M(x, \theta, \bar{\theta})$. Similarly, the exterior derivative $d = dx^\mu \partial_\mu$ of the 2D Minkowskian spacetime manifold is generalized to the super exterior derivative $\tilde{d}$ on the four $(2, 2)$-dimensional supermanifold. The above statements can be succinctly expressed, in the mathematical form, as:

\[
\tilde{d} = dZ^M \partial_M = dx^\mu \partial_\mu + d\theta \partial_\theta + d\bar{\theta} \partial_{\bar{\theta}}, \quad \tilde{d}^2 = 0, 
\]

\[
\tilde{A}^{(1)} = dZ^M A_M = dx^\mu B_{\mu}(x, \theta, \bar{\theta}) + d\theta \tilde{F}(x, \theta, \bar{\theta}) + d\bar{\theta} \tilde{F}(x, \theta, \bar{\theta}),
\]

(4.1)

where $\partial_M = (\partial/\partial Z^M) \equiv (\partial_\mu, \partial_\theta, \partial_{\bar{\theta}})$ and $A_M(x, \theta, \bar{\theta}) = (B_{\mu}(x, \theta, \bar{\theta}), \mathcal{F}(x, \theta, \bar{\theta}), \tilde{\mathcal{F}}(x, \theta, \bar{\theta}))$.

The above multiplet superfields can be expanded, along the Grassmannian (i.e. \(\theta, \bar{\theta}\)) directions of the four $(2, 2)$-dimensional supermanifold, as follows

\[
B_\mu(x, \theta, \bar{\theta}) = A_\mu(x) + \theta R_\mu(x) + \bar{\theta} R_\mu(x) + i \theta \bar{\theta} S_\mu(x),
\]

\[
\mathcal{F}(x, \theta, \bar{\theta}) = C(x) + i \theta B_1(x) + i \bar{\theta} B_1(x) + i \theta \bar{\theta} s(x),
\]

\[
\tilde{\mathcal{F}}(x, \theta, \bar{\theta}) = \tilde{C}(x) + i \theta \tilde{B}_2(x) + i \bar{\theta} \tilde{B}_2(x) + i \theta \bar{\theta} \tilde{s}(x).
\]

(4.2)

The points to be noted, at this juncture, are

(i) the above expansion is achieved in terms of the basic fields \((A_\mu, C, \tilde{C})\) and some secondary fields \((R_\mu, \bar{R}_\mu, S_\mu, B_1, B_2, \bar{B}_2, s, \bar{s})\) which are all functions of the 2D spacetime variable \(x^\mu\) alone as they are local fields on the above manifold,

(ii) in the limit \((\theta \to 0, \bar{\theta} \to 0)\), (a) we retrieve the 2D local basic fields \((A_\mu, C, \tilde{C})\) of the Lagrangian density (2.2), and (b) the super exterior derivative \(\tilde{d}\) reduces to the 2D ordinary exterior derivative \(d = dx^\mu \partial_\mu\), and

(iii) the number of the fermionic component fields \((R_\mu, \bar{R}_\mu, C, \tilde{C}, s, \bar{s})\) do match with the number of the bosonic fields \((A_\mu, S_\mu, B_1, B_2, \bar{B}_2)\) in the above expansion.

Now we are all set to exploit the celebrated horizontality condition [20-29] which leads to (i) the exact expression for the secondary fields in terms of the basic fields of the Lagrangian density (2.2), and (ii) the derivation of the off-shell nilpotent \(^\S\) symmetry transformations (2.3). To this end in mind, we first compute the following super 2-form

\[
\tilde{F}^{(2)} = \tilde{d}\tilde{A}^{(1)} = \frac{1}{2!} (dZ^M \wedge dZ^N) F_{MN}.
\]

(4.3)

The explicit form of the above computation is

\[
\tilde{d}\tilde{A}^{(1)} = (dx^\mu \wedge dx^\nu)(\partial_\mu \mathcal{B}_\nu) - (d\theta \wedge d\theta)(\partial_\theta \tilde{\mathcal{F}}) - (d\bar{\theta} \wedge d\bar{\theta})(\partial_{\bar{\theta}} \tilde{\mathcal{F}})
\]

\[
+ (dx^\mu \wedge d\theta)[\partial_\mu \tilde{\mathcal{F}} - \partial_\theta \mathcal{B}_\mu] + (dx^\mu \wedge d\bar{\theta})[\partial_\mu \mathcal{F} - \partial_{\bar{\theta}} \mathcal{B}_\mu] - (d\theta \wedge d\bar{\theta})[\partial_\theta \mathcal{F} + \partial_{\bar{\theta}} \tilde{\mathcal{F}}].
\]

(4.4)

Mathematically, the horizontality restriction on the four $(2, 2)$-dimensional supermanifold requires the equality (i.e. \(\tilde{F}^{(2)} = F^{(2)}\)) of the above super curvature 2-form with the ordinary curvature 2-form \(F^{(2)} = dA^{(1)} = \frac{1}{2!} (dx^\mu \wedge dx^\nu) F_{\mu \nu}\), defined on the ordinary 2D Minkowskian manifold. Physically, this requirement implies that the Abelian U(1) gauge

\(^\S\)The on-shell nilpotent symmetry transformations have been derived in [49-51] where the (anti-)chiral superfields have been taken into account, supplemented with the utility of the equations of motion derived from the Lagrangian density (2.2).
invariant quantity (i.e. electric field $E$ in our case) remains unaffected due to the presence of the Grassmannian variables of the superfield formulation. In other words, one has to set equal to zero all the Grassmannian components of the (anti)symmetric second-rank super tensor $F_{MN}$ of (4.3). This, imposition, leads to the following relationship

$$
R_{\mu} = \partial_{\mu} C, \quad \bar{R}_{\mu} = \partial_{\mu} \bar{C}, \quad S_{\mu} = \partial_{\mu} B, \\
s = \bar{s} = 0, \quad B_1 = \bar{B}_2 = 0, \quad \bar{B}_1 + B_2 = 0,
$$

(4.5)

where we have identified (i.e. $B_2 = B \Rightarrow \bar{B}_1 = -B$) the secondary field $B_2$ with the Nakanishi-Lautrup auxiliary field $B$ of the Lagrangian density (2.2) and we shall remain consistent with this identification in the whole body of our present text.

Taking the help of the nilpotent (anti-)BRST symmetry transformations (2.3), we obtain the expansions for the superfields (4.2) in terms of these (i.e. $s_{(a)b}$) when we insert the specific values (4.5) of the secondary fields. This can be mathematically expressed as

$$
B^{(h)}_{\mu}(x, \theta, \bar{\theta}) = A_{\mu}(x) + \theta (s_{ab} A_{\mu}(x)) + \bar{\theta} (s_{b\bar{a}} A_{\mu}(x)), \\
F^{(h)}(x, \theta, \bar{\theta}) = C(x) + \theta (s_{ab} C(x)) + \bar{\theta} (s_{b\bar{a}} C(x)), \\
\bar{F}^{(h)}(x, \theta, \bar{\theta}) = \bar{C}(x) + \theta (s_{ab} \bar{C}(x)) + \bar{\theta} (s_{b\bar{a}} \bar{C}(x)).
$$

(4.6)

The noteworthy points, at this stage, are

(i) after the application of the horizontality condition, the super 1-form connection

$$
\bar{A}^{(1)}_{(h)} = dZ^M A^{(h)}_M \equiv dx^\mu B^{(h)}_{\mu} + d\theta F^{(h)} + d\bar{\theta} \bar{F}^{(h)}
$$

is such that $dA^{(1)}_{(h)} = dA^{(1)}$,

(ii) the above uniform expressions (4.6) for all the superfields have emerged out because of the fact that we have taken into the considerations $s_b C = 0$ and $s_{ab} \bar{C} = 0$,

(iii) the geometrical interpretations for the nilpotent (anti-)BRST symmetry transformations (and their corresponding generators), as the translational generators along the Grassmannian directions of the supermanifold, are evident from (4.6). Mathematically, they can be expressed as given below (cf. equation (2.8))

$$
s_b \leftrightarrow Q_b \Leftrightarrow \lim_{\theta \to 0} \frac{\partial}{\partial \bar{\theta}}, \quad s_{ab} \leftrightarrow Q_{ab} \Leftrightarrow \lim_{\bar{\theta} \to 0} \frac{\partial}{\partial \theta},
$$

(4.7)

(iv) the nilpotency property of $s_{(a)b}$ (and their corresponding generators $Q_{(a)b}$) is equivalent to the two successive translations of any arbitrary superfields along any particular direction of the supermanifold because $(\partial/\partial \theta)^2 = 0, (\partial/\partial \bar{\theta})^2 = 0$, and

(v) the anticommutativity $s_b s_{ab} + s_{ab} s_b = 0$ of $s_{(a)b}$ (and their corresponding generators, viz. $Q_b Q_{ab} + Q_{ab} Q_b = 0$) is captured by the corresponding relationship between the translational generators (i.e. $(\partial/\partial \theta)(\partial/\partial \bar{\theta}) + (\partial/\partial \bar{\theta})(\partial/\partial \theta) = 0$) on the supermanifold.

### 4.2 Super Co-Exterior Derivative and (Anti-)co-BRST Symmetries

First of all, let us recall our observations (after equation (2.4)) where we have discussed the decisive features of the (anti-)co-BRST symmetry transformations. It is obvious that the
nilpotent (anti-)co-BRST symmetry transformations owe their origin to the nilpotent co-exterior derivative \( \delta = - \star d \star \). This claim becomes very much transparent in the framework of superfield approach to BRST formalism. In fact, we shall exploit here the mathematical power of the super co-exterior derivative \( \tilde{\delta} = - \star \tilde{d} \star \) (with \( \tilde{\delta}^2 = 0 \)) to show the existence of the local, continuous, off-shell nilpotent and anticommuting (anti-)co-BRST symmetry transformations \(^4\) for the free 2D 1-form Abelian gauge theory. Here \( \tilde{d} \) is the super exterior derivative of (4.1) and the \( \star \) corresponds to the Hodge duality operation on the four (2, 2)-dimensional supermanifold (see, e.g. [45] for details).

Let us impose the following dual-horizontality condition || (DHC) on the four (2, 2)-dimensional supermanifold over which our 2D 1-form free Abelian gauge theory is considered:

\[
\delta A^{(1)} = \delta A, \quad \tilde{\delta} = - \star \tilde{d} \star, \quad \delta = - \star d \star, \quad (4.8)
\]

where \( \delta A^{(1)} = (\partial \cdot A) \) on the ordinary 2D Minkowskian spacetime manifold. This condition has a logical backing because the gauge-fixing term \( (\partial \cdot A) \) is an on-shell (i.e. \( \Box C = 0, \Box \tilde{C} = 0 \)) gauge (i.e. (anti-)BRST) invariant quantity on the supermanifold. This is evident from the fact that \( s_b (\partial \cdot A) = \Box C = 0, s_{ab} (\partial \cdot A) = \Box \tilde{C} = 0 \). The l.h.s. of the above equation can be computed, step-by-step, due to the following inputs (see, e.g. [45] for details)

\[
\begin{align*}
\star (dx^\mu) &= \varepsilon^{\mu\nu} (dx_\nu \wedge d\theta \wedge d\bar{\theta}), \\
\star (d\theta) &= \frac{1}{2!} \varepsilon^{\mu\nu} (dx_\mu \wedge dx_\nu \wedge d\bar{\theta}), \\
\star (d\bar{\theta}) &= \frac{1}{2!} \varepsilon^{\mu\nu} (dx_\mu \wedge dx_\nu \wedge d\theta),
\end{align*}
\]

that are required for the computation of \( \star \tilde{A}^{(1)} \) which is present in \( - \star \tilde{d} \star \tilde{A}^{(1)} \). Mathematically, this step, corresponding to the resulting super 3-form, can be expressed as

\[
\star \tilde{A}^{(1)} = \varepsilon^{\mu\nu} (dx_\mu \wedge dx_\nu \wedge d\theta \wedge d\bar{\theta}) B_\mu + \frac{1}{2!} \varepsilon^{\mu\nu} (dx_\mu \wedge dx_\nu \wedge d\bar{\theta}) \bar{F} + \frac{1}{2!} \varepsilon^{\mu\nu} (dx_\mu \wedge dx_\nu \wedge d\theta) F. \quad (4.10)
\]

The application of the super exterior derivative \( \tilde{d} \) on the above super 3-form makes it a super 4-form. The latter is the maximum degree of the super-form that could be supported by the four (2, 2)-dimensional supermanifold. The explicit expression for it is **

\[
\begin{align*}
\tilde{d} \star \tilde{A}^{(1)} &= \varepsilon^{\mu\nu} (dx_\rho \wedge dx_\nu \wedge d\theta \wedge d\bar{\theta}) (\partial^\rho B_\mu) \\
&- \frac{1}{2!} \varepsilon^{\mu\nu} (dx_\mu \wedge dx_\nu \wedge d\theta \wedge d\bar{\theta}) (\partial_\theta \bar{F}) - \frac{1}{2!} \varepsilon^{\mu\nu} (dx_\mu \wedge dx_\nu \wedge d\theta \wedge d\bar{\theta}) (\partial_{\bar{\theta}} \bar{F}) - \frac{1}{2!} \varepsilon^{\mu\nu} (dx_\mu \wedge dx_\nu \wedge d\theta \wedge d\bar{\theta}) (\partial_\theta \bar{F}).
\end{align*}
\]

\(^4\)It will be noted that these symmetry transformations have been shown [30-35] to be non-local and non-covariant for the 4D (non-)Abelian interacting gauge theories with Dirac fields.

\(^\parallel\)We christen this condition as the dual-horizontality condition (DHC) because the (super) dual(co)-exterior derivatives (i.e. \( \delta, \tilde{\delta} \)) are being exploited in the restriction (4.8) on the supermanifold.

**It will be noted that all the super forms having three wedge products of the spacetime differentials (e.g. \( dx_\mu \wedge dx_\nu \wedge dx_\lambda \)) as well as the Grassmannian differentials (e.g. \( d\theta \wedge d\theta \wedge d\bar{\theta} \)) have been dropped in the above computation and this will be followed in the full body of our present text.
The application of a single \((-\star)\) on it would generate a super zero-form on the four \((2, 2)\)-dimensional supermanifold as illustrated below (see, e.g. [45] for details)

\[
\tilde{\delta} \tilde{A}^{(1)} \equiv -\star \tilde{d} \star \tilde{A}^{(1)} = (\partial \cdot B) - \partial \tilde{F} - \partial \bar{\tilde{F}} - s^{\theta \theta}(\partial \tilde{F}) - s^{\bar{\theta} \bar{\theta}}(\partial \bar{\tilde{F}}),
\]

(4.12)

where we have used, besides \(\varepsilon^{\mu \nu} \varepsilon_{\mu \nu} = -2\), \(\varepsilon^{\mu \nu} \varepsilon_{\nu \rho} = -\delta^\mu_\rho\) etc., the following duality relations on the supermanifold (see. e.g. [45] for details)

\[
\begin{align*}
\star (dx_\mu \wedge dx_\nu \wedge d\theta \wedge d\bar{\theta}) &= \varepsilon_{\mu \nu}, \\
\star (dx_\mu \wedge dx_\nu \wedge d\theta \wedge d\bar{\theta}) &= \varepsilon_{\mu \nu} s^{\theta \theta}, \\
\star (dx_\mu \wedge dx_\nu \wedge d\bar{\theta} \wedge d\bar{\theta}) &= \varepsilon_{\mu \nu} s^{\bar{\theta} \bar{\theta}}.
\end{align*}
\]

(4.13)

Here \(s^{\theta \theta}\) and \(s^{\bar{\theta} \bar{\theta}}\) are the symmetric spacetime independent parameters that are required for the derivation of the exact expression for the double Hodge duality operations [45].

All the above computations, combined together, lead to the final equality as given below

\[
(\partial \cdot B) - \partial \tilde{F} - \partial \bar{\tilde{F}} - s^{\theta \theta}(\partial \tilde{F}) - s^{\bar{\theta} \bar{\theta}}(\partial \bar{\tilde{F}}) = (\partial \cdot A).
\]

(4.14)

It is clear that the coefficients of the Grassmannian dependent symmetric parameters \(s^{\theta \theta}\) and \(s^{\bar{\theta} \bar{\theta}}\) would be set equal to zero because there are no such terms on the r.h.s. of (4.14). This entails upon the following conditions on the secondary fields, namely;

\[
\partial \tilde{F} = 0 \Rightarrow B_1 = 0, \quad \bar{s} = 0, \quad \partial \bar{\tilde{F}} = 0 \Rightarrow B_2 = 0, \quad s = 0.
\]

(4.15)

The above substitutions in the expansion for the fermionic superfields \(\mathcal{F}\) and \(\bar{\mathcal{F}}\) lead to the following expressions for the reduced form of the above fields

\[
\begin{align*}
\mathcal{F}(x, \theta, \bar{\theta}) &\rightarrow \mathcal{F}^{(r)}(x, \bar{\theta}) = C(x) + i \bar{\theta} B_1, \\
\bar{\mathcal{F}}(x, \theta, \bar{\theta}) &\rightarrow \bar{\mathcal{F}}^{(r)}(x, \theta) = \bar{C}(x) + i \theta B_2.
\end{align*}
\]

(4.16)

Ultimately, the final equality, that incorporates the above information, is

\[
(\partial \cdot B) - \partial \mathcal{F}^{(r)} - \partial \bar{\mathcal{F}}^{(r)} = (\partial \cdot A).
\]

(4.17)

This equation leads to the following restrictions on the component fields of the expansions in (4.2) when the explicit expressions for (4.2) and (4.16) are substituted in it, namely;

\[
B_1 + \bar{B}_2 = 0, \quad (\partial \cdot R) = 0, \quad (\partial \cdot R) = 0, \quad (\partial \cdot S) = 0.
\]

(4.18)

Making the following judicious choices

\[
R_\mu = -\varepsilon_{\mu \nu} \partial^\nu \bar{C}, \quad \bar{R}_\mu = -\varepsilon_{\mu \nu} \partial^\nu C, \quad S_\mu = +\varepsilon_{\mu \nu} \partial^\nu B, \quad B_1 = -B, \quad \bar{B}_2 = B,
\]

(4.19)

it can be seen that all the superfields of (4.2) can be expressed, in terms of the nilpotent (anti-)co-BRST symmetry transformations (2.4), as

\[
\begin{align*}
\mathcal{B}^{(dh)}_{\mu}(x, \theta, \bar{\theta}) &= A_\mu(x) + \theta (s_{ad} A_\mu(x)) + \bar{\theta} (s_d A_\mu(x)) + \theta \bar{\theta} (s_{ad} s_d A_\mu(x)), \\
\mathcal{F}^{(dh)}(x, \theta, \bar{\theta}) &= C(x) + \theta (s_{ad} C(x)) + \bar{\theta} (s_d C(x)) + \theta \bar{\theta} (s_{ad} s_d C(x)), \\
\bar{\mathcal{F}}^{(dh)}(x, \theta, \bar{\theta}) &= \bar{C}(x) + \theta (s_{ad} \bar{C}(x)) + \bar{\theta} (s_d \bar{C}(x)) + \theta \bar{\theta} (s_{ad} s_d \bar{C}(x)).
\end{align*}
\]

(4.20)
In the above expansion, we have taken into account $s_d \bar{C} = 0$, $s_{ad} C = 0$. Furthermore, a close look at the above expansion provides the geometrical interpretation for the nilpotent symmetry transformations $s(a)d$ (and their corresponding nilpotent generators) as the translational generators along the Grassmannian directions of the four $(2, 2)$-dimensional supermanifold in exactly the same fashion (cf. (4.7)) as that for the nilpotent (anti-)BRST symmetry transformations (and their corresponding nilpotent generators). However, there are clear-cut differences between the HC and DHC which generate them. The key role is played, in this connection, by the nature of the expansions of the fermionic superfields $F$ and $\bar{F}$ (cf. (4.6) and (4.20)). Whereas, after the application of HC, the superfields $F$ and $\bar{F}$ become chiral and anti-chiral, respectively, the same superfields convert themselves to anti-chiral and chiral superfields after the application of the DHC. Furthermore, the results obtained due to the application of HC are mathematically exact but this is not the situation with the DHC. In the case of the latter, one has to make a judicious choice (cf. (4.19)) for the solutions to the restrictions that emerge (cf. (4.18)).

### 4.3 Super Laplacian Operator and Equations of Motion

As remarked earlier in Sec. 2, the consequence of the application of the ordinary Laplacian operator $\Delta$ on the 1-form gauge field (i.e. $\Delta A^{(1)} = dx^\mu \square A_\mu = 0$) is the equation of motion $\square A_\mu = 0$ that emerges from the gauge-fixed Lagrangian density (2.2). One would expect, therefore, that the action of the super Laplacian operator on the super 1-form connection $\bar{A}^{(1)}$ would lead to the derivation of the equations of motion for all the basic fields $A_\mu, C, \bar{C}$ as well as the auxiliary fields $B, \bar{B}$. With the theoretical arsenal of the definition of the Hodge duality $\star$ operation on the four $(2, 2)$-dimensional supermanifold [45], we demonstrate, in this Subsec., that all the other equations of motion $\square C = 0, \square \bar{C} = 0, \square B = 0, \square \bar{B} = 0$ (and their off-shoots $\square E = 0, \square (\partial \cdot A) = 0$) emerge from a single restriction on the gauge superfield of the above supermanifold which owes its origin to the (super) Laplacian operators that are defined on the (super) spacetime manifolds.

To corroborate the above assertion, we begin with the following condition on the four $(2, 2)$-dimensional supermanifold

$$\tilde{\Delta} \tilde{A}^{(1)} = \Delta A^{(1)} = 0, \quad \Delta A^{(1)} = (d\delta + \delta d) A^{(1)} \equiv dx^\mu \square A_\mu = 0,$$

(4.21)

where the super Laplacian operator $\tilde{\Delta} = (\tilde{d}\tilde{\delta} + \tilde{\delta}\tilde{d})$ and super (co-)exterior derivatives $(\tilde{\delta})\tilde{d}$ (with $\tilde{d}^2 = 0, \tilde{\delta}^2 = 0$) are defined earlier. It is clear from the r.h.s. of (4.21) that (i) we obtain the equation of motion for the gauge field (i.e. $\square A_\mu = 0$) due to the ordinary Laplacian operator when we demand that the Laplace equation $\Delta A^{(1)} = 0$ should be satisfied, and (ii) the restriction $\Delta A^{(1)} = 0 \Rightarrow \square A_\mu = 0$ is on-shell $(\square C = 0, \square \bar{C} = 0)$ gauge (i.e. (anti-)BRST) invariant quantity because $s_b(\square A_\mu) = \partial_\mu (\square C) = 0, s_{ab}(\square A_\mu) = \partial_\mu (\square \bar{C}) = 0$. Therefore, its invariance on the supermanifold is physically correct. In other words, we require that the equation of motion $\square A_\mu = 0$ is unaffected due to the presence of the Grassmannian variables in the superfield formulation of the theory.
To compute the exact expression for the l.h.s. (i.e. \( \tilde{d}\tilde{\delta}A^{(1)} + \tilde{\delta}dA^{(1)} \)) of (4.21), let us take the help of our earlier computation in (4.4) (i.e. \( dA^{(1)} \)) and (4.12) (i.e. \( \tilde{\delta}A^{(1)} \)). The simpler computation of \( \tilde{d}\tilde{\delta}A^{(1)} \), in its full blaze of glory, is

\[
\tilde{d}\tilde{\delta}A^{(1)} = (dx^\mu) \left[ \frac{\partial}{\partial \mu} (\partial \cdot B) - (\partial_\mu \partial_\nu \mathcal{F} + \partial_\nu \partial_\mu \mathcal{F}) - s^{\theta \theta} (\partial_\mu \partial_\theta \mathcal{F}) - s^{\theta \theta} (\partial_\mu \partial_\bar{\theta} \mathcal{F}) \right]
+ (d\theta) \left[ \bar{\partial}_\theta (\partial \cdot B) - \partial_\theta \partial_\theta \mathcal{F} - s^{\bar{\theta} \theta} \partial_\theta \mathcal{F} \right]
+ (d\bar{\theta}) \left[ \partial_{\bar{\theta}} (\partial \cdot B) - \bar{\partial}_{\bar{\theta}} \partial_\theta \mathcal{F} - s^{\theta \theta} \partial_{\bar{\theta}} \mathcal{F} \right].
\] (4.22)

In the above computation, the simple definition of \( \tilde{d} \) (cf (4.1)) has been used along with the nilpotency properties of the Grassmannian partial derivatives (viz. \( (\partial_\theta)^2 = 0 \) and \( (\partial_{\bar{\theta}})^2 = 0 \)). Let us now compute, step-by-step, the more complicated term \( \tilde{\delta}dA^{(1)} = -\ast \tilde{d} \ast (\tilde{d}A^{(1)}) \) where we take the help of (4.4) and \( \partial = -\ast \tilde{d} \ast \). In the first step, we compute \( \ast (\tilde{d}A^{(1)}) \) as

\[
\ast (\tilde{d}A^{(1)}) = \varepsilon^{\mu \rho} (d\theta \wedge d\bar{\theta}) (\partial_\mu B_\rho), \quad \ast (dx^\mu \wedge d\theta) (\partial_\mu \mathcal{F} - \partial_\mu B_\rho)
+ \varepsilon^{\mu \rho} (dx_{\mu} \wedge d\theta) (\partial_\mu \mathcal{F} - \partial_\mu B_\rho) - \frac{1}{2!} \varepsilon^{\mu \nu} (dx_{\mu} \wedge dx_{\nu}) (\partial_\nu \mathcal{F} + \partial_\nu \bar{\mathcal{F}})
- \frac{1}{2!} s^{\theta \bar{\theta}} \varepsilon^{\mu \nu} (dx_{\mu} \wedge dx_{\nu}) (\partial_\nu \mathcal{F}).
\] (4.23)

In the derivation of the above expression, the \( \ast \) operations that have been used, are [45]

\[
\ast (dx^\mu \wedge dx^\rho) = \varepsilon^{\mu \rho} (d\theta \wedge d\bar{\theta}), \quad \ast (dx^\mu \wedge d\theta) = \varepsilon^{\mu \rho} (dx_\rho \wedge d\bar{\theta}),
\ast (dx^\mu \wedge d\bar{\theta}) = \varepsilon^{\mu \rho} (dx_\rho \wedge d\theta), \quad \ast (d\theta \wedge d\bar{\theta}) = \frac{1}{2!} \varepsilon^{\mu \nu} (dx_\mu \wedge dx_\nu),
\ast (d\theta \wedge \partial_\rho \partial_\theta) = \frac{1}{2!} s^{\theta \bar{\theta}} \varepsilon^{\mu \nu} (dx_{\mu} \wedge dx_{\nu}).
\] (4.24)

We have taken the above expressions for the \( \ast \) operation on the super 2-forms from [45] just for the sake of this paper to be self-contained.

The application of a \( \tilde{d} = dx^\sigma \partial_\sigma + d\theta \partial_\theta + d\bar{\theta} \partial_{\bar{\theta}} \) on the above expression makes it a super 3-form. In this computation, all the super forms with the wedge products like \( (dx_{\mu} \wedge dx_{\nu} \wedge dx_{\lambda}) \), \( (d\theta \wedge d\theta \wedge d\theta) \), etc., are to be dropped because the four \( (2, 2) \)-dimensional supermanifold cannot support such kind of forms. Physically relevant super 3-form, that emerges from the above operation, is

\[
\tilde{d} \ast \tilde{d} \tilde{A}^{(1)} = L + M + N,
\] (4.25)

where the exact expressions for \( L \), \( M \) and \( N \) are

\[
L = \varepsilon^{\mu \rho} (dx_\sigma \wedge d\theta \wedge d\bar{\theta}) (\partial_\sigma \partial_\mu B_\rho) + \varepsilon^{\mu \rho} (dx_\sigma \wedge dx_{\rho} \wedge d\theta) (\partial_\sigma [ \partial_\mu \mathcal{F} - \partial_\mu B_\rho])
+ \varepsilon^{\mu \rho} (dx_\sigma \wedge dx_{\rho} \wedge d\bar{\theta}) (\partial_\sigma [ \partial_\mu \mathcal{F} - \partial_\mu B_\rho])
\] (4.26)

\[
M = \varepsilon^{\mu \rho} (dx_\mu \wedge d\theta \wedge d\bar{\theta}) (\partial_\theta \partial_\mu B_\rho) + \varepsilon^{\mu \rho} (dx_\mu \wedge d\theta \wedge d\bar{\theta}) (\partial_\theta \partial_\mu \mathcal{F} - \partial_\theta \partial_\mu B_\rho)
- \frac{1}{2!} \varepsilon^{\mu \nu} (dx_\mu \wedge dx_\nu \wedge d\theta) [ \partial_\theta \partial_\nu \mathcal{F} + s^{\theta \bar{\theta}} \partial_\theta \partial_\nu \mathcal{F} ].
\] (4.27)

\[
N = \varepsilon^{\mu \rho} (dx_\mu \wedge d\theta \wedge d\bar{\theta}) (\partial_\theta \partial_\mu \bar{\mathcal{F}}) + \varepsilon^{\mu \rho} (dx_\mu \wedge d\bar{\theta} \wedge d\bar{\theta}) (\partial_\theta \partial_\mu \bar{\mathcal{F}} - \partial_\theta \partial_\mu B_\rho)
- \frac{1}{2!} \varepsilon^{\mu \nu} (dx_\mu \wedge dx_\nu \wedge d\bar{\theta}) [ \partial_\nu \partial_\theta \mathcal{F} + s^{\theta \bar{\theta}} \partial_\nu \partial_\theta \mathcal{F} ].
\] (4.28)
We are now all set to apply a single (−*) operation on the above equations which will yield a super 1-form \( \delta \tilde{d} \bar{A}^{(1)} = - \star \tilde{d} \star (\tilde{d} \bar{A}^{(1)}) \). This final expression is given as follows

\[
- \varepsilon^{\mu \rho} \varepsilon_{\sigma \lambda} \left( \partial^\nu \partial_\mu \bar{B}_\rho \right) - (dx^\mu) \left[ \partial_\rho \partial_\mu \bar{F} + \partial_\rho \partial_\mu \bar{F} + s^\theta \left( \partial_\theta \partial_\mu \bar{F} - \partial_\theta \partial_\mu \bar{B}_\mu \right) + s^\theta \left( \partial_\theta \partial_\mu \bar{F} - \partial_\theta \partial_\mu \bar{B}_\mu \right) \right] + (d\theta) \left[ \Box \bar{F} - \partial_\theta (\partial \cdot \bar{B}) - \partial_\theta \partial_\theta \bar{F} - s^\theta \partial_\theta \partial_\theta \bar{F} \right] + (d\bar{\theta}) \left[ \Box \bar{F} - \partial_{\bar{\theta}} (\partial \cdot \bar{B}) - \partial_{\bar{\theta}} \partial_{\bar{\theta}} \bar{F} - s^\theta \partial_{\bar{\theta}} \partial_{\bar{\theta}} \bar{F} \right].
\]

(4.29)

In the above derivation, the key inputs from [45] have been used for the * operations on the super 3-forms. These relevant expressions are

\[
\star (dx_\sigma \wedge d\theta \wedge d\bar{\theta}) = \varepsilon_{\sigma \lambda} \left( dx^\lambda \right), \quad \star (dx_\sigma \wedge dx_\rho \wedge d\bar{\theta}) = \varepsilon_{\sigma \rho} \left( d\theta \right), \quad \star (dx_\rho \wedge d\theta \wedge d\bar{\theta}) = \varepsilon_{\rho \lambda} \left( dx^\lambda \right) \bar{s}^\theta.
\]

(4.30)

For the sake of the completeness of this paper, we have taken the above expressions from [45]. The exact value of the operation of the super Laplacian operator \( \tilde{\Delta} \) on the super 1-form \( \bar{A}^{(1)} \) is the sum of (4.22) and (4.29).

Let us focus on the terms \( s^\theta (d\theta), s^\theta (d\bar{\theta}), s^\bar{\theta} (d\theta) \) and \( s^\bar{\theta} (d\bar{\theta}) \). These are certainly not present on the r.h.s. of the restriction (4.21). As a consequence, these should be set equal to zero. Mathematically, these statements can be expressed as follows

\[
- s^\theta (d\theta) \left[ \partial_\theta \partial_\theta \bar{F} \right] = 0 \Rightarrow \bar{s} = 0, \quad - s^\theta (d\bar{\theta}) \left[ \partial_{\bar{\theta}} \partial_{\bar{\theta}} \bar{F} \right] = 0 \Rightarrow \bar{s} = 0, \quad - s^\bar{\theta} (d\theta) \left[ \partial_\theta \partial_\theta \bar{F} \right] = 0 \Rightarrow \bar{s} = 0, \quad - s^\bar{\theta} (d\bar{\theta}) \left[ \partial_{\bar{\theta}} \partial_{\bar{\theta}} \bar{F} \right] = 0 \Rightarrow \bar{s} = 0.
\]

(4.31)

The above inputs lead to the reduced form of the fermionic superfields \( \bar{F} \rightarrow \bar{F}^{(r)} \) and \( \bar{F} \rightarrow \bar{F}^{(r)} \) as

\[
\bar{F}^{(r)}(x, \theta, \bar{\theta}) = C(x) + i \theta \tilde{B}_1(x) + i \bar{\theta} \tilde{B}_1(x), \quad \bar{F}^{(r)}(x, \theta, \bar{\theta}) = \tilde{C}(x) + i \theta \tilde{B}_2(x) + i \bar{\theta} \tilde{B}_2(x).
\]

(4.32)

It should be noted that the component fields \( s(x) \) and \( \bar{s}(x) \) are no longer present in the above reduced form of the fermionic superfields. We collect the coefficient of \( d\theta \) and \( d\bar{\theta} \) from (4.22) and (4.29) and set them equal to zero as given below

\[
(d\theta) \left[ \Box \bar{F}^{(r)} - \partial_\theta (\partial \cdot \bar{B}) - \partial_\theta \partial_\theta \bar{F}^{(r)} + \partial_\theta (\partial \cdot \bar{B}) - \partial_\theta \partial_\theta \bar{F}^{(r)} \right] = 0, \quad (d\bar{\theta}) \left[ \Box \bar{F}^{(r)} - \partial_{\bar{\theta}} (\partial \cdot \bar{B}) - \partial_{\bar{\theta}} \partial_{\bar{\theta}} \bar{F}^{(r)} + \partial_{\bar{\theta}} (\partial \cdot \bar{B}) - \partial_{\bar{\theta}} \partial_{\bar{\theta}} \bar{F}^{(r)} \right] = 0.
\]

(4.33)

It is clear that, finally, we obtain the relations \( \Box \bar{F}^{(r)} = 0 \) and \( \Box \bar{F}^{(r)} = 0 \) due to the fact that \( \partial_\theta \partial_\theta + \partial_{\bar{\theta}} \partial_{\bar{\theta}} = 0 \). The above condition entails upon the component secondary fields of (4.32) to obey \( \Box \tilde{B}_1 = 0, \Box \tilde{B}_1 = 0, \Box \tilde{B}_2 = 0, \Box \tilde{B}_2 = 0 \). Our earlier identifications (i.e. \( \tilde{B}_1 = -\tilde{B}_2 \equiv -\bar{B} \) and \( \tilde{B}_1 = -\tilde{B}_2 \equiv -B \)) imply, ultimately, that \( \Box B = 0 \) and \( \Box \bar{B} = 0 \). Furthermore, these equations of motion, in turn, lead to the conclusion that \( \Box (\partial \cdot A) = 0 \) and \( \Box E = 0 \) because of the fact that \( B = - (\partial \cdot A) \) and \( \bar{B} = E \) from the Lagrangian density (2.2). All the above equations of motion are the ones that can be derived from the Lagrangian density (2.2) due to the Euler-Lagrange equations of motion.
We are now well equipped to collect the coefficients of the differentials $dx^\mu, (dx^\mu)s^{\theta\theta}$ and $(dx^\mu)s^{\bar{\theta}\bar{\theta}}$. It is obvious from the equality in (4.21) that the coefficients of $(dx^\mu)s^{\theta\theta}$ and $(dx^\mu)s^{\bar{\theta}\bar{\theta}}$ would be set equal to zero. This statement can be mathematically expressed as

$$-(dx^\mu)s^{\theta\theta}[(\partial_\theta \partial_\theta + \partial_\theta \partial_\mu)\mathcal{F}^{(r)} - \partial_\theta \partial_\mu \mathcal{B}_\mu] = 0, \quad -(dx^\mu)s^{\bar{\theta}\bar{\theta}}[(\partial_\bar{\theta} \partial_\bar{\theta} + \partial_\bar{\theta} \partial_\mu)\bar{\mathcal{F}}^{(r)} - \partial_\bar{\theta} \partial_\mu \bar{\mathcal{B}}_\mu] = 0. \quad (4.34)$$

A close look at the above equations implies that (for $dx^\mu s^{\theta\theta} \neq 0, dx^\mu s^{\bar{\theta}\bar{\theta}} \neq 0$)

$$(\partial_\mu \partial_\theta + \partial_\theta \partial_\mu)\mathcal{F}^{(r)} + (\partial_\mu \partial_\bar{\theta} + \partial_\bar{\theta} \partial_\mu)\bar{\mathcal{F}}^{(r)} = 0. \quad (4.35)$$

We collect finally the coefficient of the spacetime differential $(dx^\mu)$ and equate it with the r.h.s. of (4.21). This is explicitly given as follows

$$(dx^\mu) [\partial_\mu(\partial \cdot \mathcal{B}) - (\partial_\mu \partial_\theta + \partial_\theta \partial_\mu)\mathcal{F}^{(r)} - (\partial_\mu \partial_\bar{\theta} + \partial_\bar{\theta} \partial_\mu)\bar{\mathcal{F}}^{(r)}] - \varepsilon^{\mu\rho}\varepsilon_{\sigma\lambda}(dx^\lambda)(\partial^\sigma \partial_\mu \mathcal{B}_\rho) = dx^\mu \Box A_\mu = 0. \quad (4.36)$$

The expansion of all the terms of the equation (4.36) implies the following

$$(dx^\mu) [\Box \mathcal{B}_\mu - 2i\partial_\mu(B_1 + \bar{B}_2)] = (dx^\mu) \Box A_\mu = 0. \quad (4.37)$$

The consequences, that emerge from the above equation, are

$$\Box A_\mu = 0, \quad \Box R_\mu = 0, \quad \Box \bar{R}_\mu = 0, \quad \Box S_\mu = 0, \quad B_1 + \bar{B}_2 = 0. \quad (4.38)$$

A couple of comments are in order now. First, in the above equation, the last entry is due to the actual restriction $-2i\partial_\mu(B_1 + \bar{B}_2) = 0$. We have chosen, however, the solution to the above restriction as $B_1 + \bar{B}_2 = 0$ because of our earlier experience with the identification $B_1 = -\bar{B}_2 = -\mathcal{B}$. Finally, in a similar fashion, the restriction in (4.35) implies that $\partial_\mu(B_1 + \bar{B}_2) = 0$. Once again, we have chosen the solution to this restriction as $B_1 = -\bar{B}_2 = -\mathcal{B}$ due to our earlier experience with such an identification. With the above inputs, it is clear that the reduced form of the fermionic superfields in (4.32), are

$$\mathcal{F}^{(r)}(x, \theta, \bar{\theta}) = C(x) - i \theta \mathcal{B}(x) + i \bar{\theta} \mathcal{B}(x), \quad \bar{\mathcal{F}}^{(r)}(x, \theta, \bar{\theta}) = \bar{C}(x) - i \theta \bar{\mathcal{B}}(x) + i \bar{\theta} \bar{\mathcal{B}}(x). \quad (4.39)$$

Finally, it is worth pointing out that all the restrictions $\Box R_\mu = 0, \Box \bar{R}_\mu = 0$ and $\Box S_\mu = 0$ are satisfied due to the validity of equations of motions $\Box C = 0, \Box \bar{C} = 0, \Box \mathcal{B} = 0$ and $\Box \bar{\mathcal{B}} = 0$. To make this point clearer, it can be seen that for the (anti-)BRST symmetry transformations, we found that $R_\mu = \partial_\mu C, \bar{R}_\mu = \partial_\mu \bar{C}$ and $S_\mu = \partial_\mu \mathcal{B}$ (cf. (4.5)). All these values satisfy the restrictions listed in (4.38). In an exactly similar manner, all the choices that were made for the derivation of the (anti-)co-BRST symmetry transformations, namely; $R_\mu = -\varepsilon_{\mu\nu}\partial^\nu \bar{C}, \bar{R}_\mu = -\varepsilon_{\mu\nu}\partial^\nu C, S_\mu = +\varepsilon_{\mu\nu}\partial^\nu \mathcal{B}$ (cf. (4.19)) do satisfy the restrictions listed in (4.38). In a more sophisticated mathematical language, the Hodge decomposed version of the 2D vector secondary (fermionic/bosonic) fields, chosen in a specific way, namely;

$$R_\mu = \partial_\mu C - \varepsilon_{\mu\nu}\partial^\nu \bar{C}, \quad \bar{R}_\mu = \partial_\mu \bar{C} - \varepsilon_{\mu\nu}\partial^\nu C, \quad S_\mu = \partial_\mu \mathcal{B} + \varepsilon_{\mu\nu}\partial^\nu \bar{\mathcal{B}}. \quad (4.40)$$
do satisfy the restrictions listed in (4.38) due to the equations of motion \( \square C = \square \bar{C} = 0, \square B = \square \bar{B} = 0 \). In some sense, the above decomposition itself shows that there exist nilpotent (anti-)BRST and (anti-)co-BRST symmetry transformations for the free 1-form Abelian gauge theory. This is due to the fact that, modulo relative signs, the decomposition in (4.40) is a unique decomposition for the vector fields in two \((1 + 1)\)-dimensions.

Before we wrap up this Subsec., we would like to remark, by taking the help of the discussions above (3.8), that the conditions \( (\partial \cdot A) = 0 \) and \( \varepsilon^{\mu\nu} \partial_{\mu} A_\nu = 0 \) provide the solutions to the equations of motion \( \square A_\mu = 0 \) in two \((1 + 1)\)-dimensions of spacetime. These solutions emerge due to the symmetry considerations of the present theory. In an exactly similar fashion, one can provide solutions to the restrictions \( \square R_\mu = 0, \square \bar{R}_\mu = 0 \) and \( \square S_\mu = 0 \). These solutions can be enumerated as (i) \( (\partial \cdot R) = 0, \varepsilon^{\mu\nu} \partial_{\mu} R_\nu = 0 \), (ii) \( (\partial \cdot \bar{R}) = 0, \varepsilon^{\mu\nu} \partial_{\mu} \bar{R}_\nu = 0 \), and (iii) \( (\partial \cdot S) = 0, \varepsilon^{\mu\nu} \partial_{\mu} S_\nu = 0 \), respectively. Furthermore, the above solutions, with the help of the Hodge decompositions (4.40), automatically imply that \( \square C = 0, \square \bar{C} = 0 \), \( \square B = 0 \) and \( \square \bar{B} = 0 \). This is another way to provide the mathematically and physically beautiful solutions to the restrictions (4.38) in the two \((1 + 1)\)-dimensions of spacetime.

### 4.4 Topological Aspects: Superfield Formulation

In this Subsec., we shall express the topological features of the 2D free 1-form Abelian gauge theory in the language of the superfield approach to BRST formalism. To this goal in mind, let us focus on the form of the Lagrangian density expressed as a total derivative of a specific combinations of composite superfields with respect to the Grassmannian variable \( \theta \). Mathematically, this statement can be succinctly written as [51]

\[
\mathcal{L}_b = + \frac{i}{2} \lim_{\theta \to 0} \frac{\partial}{\partial \theta} \left[ \left( \varepsilon^{\mu\nu} \partial_{\mu} \mathcal{B}^{(dh)}_\nu \right) \mathcal{F}^{(dh)} + (\partial \cdot \mathcal{B}^{(h)}) \mathcal{F}^{(h)} \right],
\]

where, for our purposes, the expansions of the superfields, after the application of HC and DHC (cf. (4.6) and (4.20)), can be re-expressed as

\[
\begin{align*}
\mathcal{B}^{(h)}_\mu(x, \theta, \bar{\theta}) &= A_\mu(x) + \theta \left( \partial_{\mu} \bar{C}(x) \right) + \bar{\theta} \left( \partial_{\mu} C(x) \right) + i \theta \bar{\theta} \left( \partial_{\mu} B(x) \right), \\
\mathcal{F}^{(h)}(x, \theta) &= C(x) - i \theta \bar{B}(x), \\
\mathcal{F}^{(h)}(x, \bar{\theta}) &= \bar{C}(x) + i \theta \bar{B}(x), \\
\mathcal{B}^{(dh)}_\mu(x, \theta, \bar{\theta}) &= A_\mu(x) - \theta \varepsilon_{\mu\nu} \partial^\nu C(x) - \bar{\theta} \varepsilon_{\mu\nu} \partial^\nu \bar{C}(x) + i \theta \bar{\theta} \varepsilon_{\mu\nu} \partial^\nu B(x), \\
\mathcal{F}^{(dh)}(x, \theta) &= C(x) - i \theta \bar{B}(x), \\
\mathcal{F}^{(dh)}(x, \bar{\theta}) &= \bar{C}(x) + i \theta \bar{B}(x).
\end{align*}
\]

For the on-shell nilpotent symmetry transformations, given in (3.2) and (3.3), the above expansions can be re-written by inserting the values of the auxiliary fields \( B = -(\partial \cdot A) \) and...
\( \mathcal{B} = E \) that emerge from the Lagrangian density (2.2). Precisely speaking, for the derivation of
the on-shell nilpotent symmetry transformations for the gauge and (anti-)ghost fields, one has to invoke
(anti-)chiral superfields on the three (2, 1)-dimensional sub-manifold(s) of the four (2, 2)-dimensional
supermanifold [49-51]. However, for our discussions, we shall take the appropriate form of the expansions
of the superfields from the above equations. These are re-expressed as follows

\[
\begin{align*}
\mathcal{B}^{(h)}_\mu(x, \theta, \bar{\theta}) &= A_\mu(x) \theta (\partial_\mu \bar{C}(x)) + \bar{\theta} (\partial_\mu C(x)) - i \theta \bar{\theta} (\partial_\mu (\partial \cdot A)(x)), \\
\mathcal{F}^{(h)}(x, \theta) &= C(x) + i \theta (\partial \cdot A)(x), \quad \mathcal{F}^{(h)}(x, \bar{\theta}) = \bar{C}(x) - i \bar{\theta} (\partial \cdot A)(x), \\
\mathcal{B}^{(dh)}_\mu(x, \theta, \bar{\theta}) &= A_\mu(x) - \theta \varepsilon_{\mu\nu} \partial^\nu C(x) - \bar{\theta} \varepsilon_{\mu\nu} \partial^\nu \bar{C}(x) + i \theta \bar{\theta} \varepsilon_{\mu\nu} \partial^\nu E(x), \\
\mathcal{F}^{(dh)}(x, \theta) &= C(x) - i \bar{\theta} E(x), \quad \mathcal{F}^{(dh)}(x, \bar{\theta}) = \bar{C}(x) + i \theta E(x).
\end{align*}
\] (4.44)

The expressions, quoted in (4.44) and (4.45), are to be utilized in the equation (4.41) for
the derivation of the Lagrangian density. It is elementary to check that, modulo a total
derivative, the Lagrangian density (2.1) emerges from the Lagrangian density (4.41). To be
precise, the expression (4.41) is the analogue of the Lagrangian densities (3.1) and (3.5)
which are expressed in terms of (i) the nilpotent symmetry operators \( \tilde{s}_{ab} \) and \( \tilde{s}_{ad} \) (cf. (3.1)),
and (ii) the corresponding nilpotent symmetry generators \( \tilde{Q}_{ab} \) and \( \tilde{Q}_{ad} \) (cf. (3.5)).

The Lagrangian densities (3.1) and (3.5) for the new TFTs, that have been expressed
as the sum of BRST exact and co-exact terms in the language of either the symmetry
transformation operators \( \tilde{s}_b \) and \( \tilde{s}_d \) (or the corresponding nilpotent symmetry generators
\( \tilde{Q}_b \) and \( \tilde{Q}_d \)), can be written as the total derivative with respect to the Grassmannian variable
\( \bar{\theta} \). The precise expression for such a form of the Lagrangian density, in the framework of
the superfield approach to BRST formalism, is

\[
\mathcal{L}_b = -\frac{i}{2} \lim_{\theta \to 0} \frac{\partial}{\partial \bar{\theta}} \left[ (\varepsilon^{\mu\nu} \partial_\mu \mathcal{B}^{(dh)}_\nu) \mathcal{F}^{(dh)} + (\partial \cdot \mathcal{B}^{(h)}) \mathcal{F}^{(h)} \right].
\] (4.46)

It can be easily checked that the substitution of the superfield expansions, given in (4.44)
and (4.45), into the above equation leads to the derivation of the Lagrangian density (2.1),
modulo a total derivative. A close look at the expressions (4.41) and (4.46) provides the
total derivative. A close look at the expressions (4.41) and (4.46) provides the
geometrical interpretation for the Lagrangian density (2.1) of the 2D free 1-form Abelian
gauge theory which happens to be a new TFT. Geometrically, this Lagrangian for the 2D
gauge theory, defined on the ordinary 2D Minkowskian spacetime manifold, corresponds to
the translation of the sum of the composite superfields \( (\partial \cdot \mathcal{B}^{(h)}) \mathcal{F}^{(h)} \) and \( (\varepsilon^{\mu\nu} \partial_\mu \mathcal{B}^{(dh)}_\nu) \mathcal{F}^{(dh)} \)
along the \( \theta \)-direction of the four (2, 2)-dimensional supermanifold over which the above
superfields have been defined. In an exactly analogous manner, the geometrical interpretation
for the Lagrangian density (4.46) can be provided in the language of the translations of
the combination of the superfields \( (\partial \cdot \mathcal{B}^{(h)}) \mathcal{F}^{(h)} \) and \( (\varepsilon^{\mu\nu} \partial_\mu \mathcal{B}^{(dh)}_\nu) \mathcal{F}^{(dh)} \) along the \( \bar{\theta} \)-direction
of the four (2, 2)-dimensional supermanifold. It should be noted here that the combination
of the composite superfields in the Lagrangian densities (4.41) and (4.46) is such that
individually these terms (i.e. the composite superfields) turn out to be the Lorentz scalars.
Besides the superfield formulation of the Lagrangian density (2.1) in terms of translations along \( \theta \)-direction (cf. (4.41)) and \( \bar{\theta} \)-direction (cf. (4.46)) of the four (2, 2)-dimensional supermanifold, there is a third way of expressing the starting Lagrangian density (2.1) in the language of superfield approach to BRST formulation. This new way of expressing the Lagrangian density (2.1), modulo a total spacetime derivative, is

\[
\mathcal{L}_b = \frac{i}{4} \frac{\partial}{\partial \theta} \frac{\partial}{\partial \bar{\theta}} \left[ \mathcal{B}_B^{(h)} \mathcal{B}_A^{(h)} + \mathcal{B}_B^{(dh)} \mathcal{B}_A^{(dh)} \right].
\]

(4.47)

A few comments are in order now. First, it can be noted that the bosonic Lorentz scalar (i.e. \( B^\mu B_\mu \)), on which the Grassmannian derivatives operate, is constructed with the help of a single bosonic superfield \( B_\mu \). Second, the HC and DHC, which are responsible for the derivation of the nilpotent (anti-)BRST and (anti-)co-BRST symmetry transformations in the framework of the superfield formulation, play very important roles in the derivation of the Lagrangian density (4.47). Third, in the language of the geometry on the four (2, 2)-dimensional supermanifold, the above Lagrangian density is nothing but a couple of successive translations for the sum of the composite bosonic Lorentz scalar superfields \( (B_B^{(h)} B^{(h)}) \) and \( (B_B^{(dh)} B^{(dh)}) \) along the Grassmannian \( \theta \)- and \( \bar{\theta} \)-directions of the four (2, 2)-dimensional supermanifold, respectively. Finally, the above Lagrangian density can be expressed in terms of the on-shell nilpotent (anti-)BRST and (anti-)co-BRST symmetry transformations of (3.2) and (3.3) as given below

\[
\mathcal{L}_b = \frac{i}{4} \mathcal{B}_b \mathcal{B}_{ab} \left[ (A_\mu(x) A_\mu(x)) \right] + \frac{i}{4} \mathcal{B}_d \mathcal{B}_{ad} \left[ (A_\mu(x) A_\mu(x)) \right].
\]

(4.48)

We would like to emphasize that the above expression for the starting Lagrangian density is a new one and it has been derived only due to our understanding of the superfield approach to BRST formalism. The crucial role in this derivation has been played by our knowledge of the following beautiful mappings

\[
\begin{align*}
\mathcal{B}_b A_\mu(x) &\Leftrightarrow -i \ [A_\mu(x), \mathcal{Q}_b] \Leftrightarrow \lim_{\theta \to 0} \frac{\partial}{\partial \theta} \mathcal{B}_B^{(h)}, \\
\mathcal{B}_{ab} A_\mu(x) &\Leftrightarrow -i \ [A_\mu(x), \mathcal{Q}_{ab}] \Leftrightarrow \lim_{\theta \to 0} \frac{\partial}{\partial \theta} \mathcal{B}_B^{(h)}, \\
\mathcal{B}_d A_\mu(x) &\Leftrightarrow -i \ [A_\mu(x), \mathcal{Q}_d] \Leftrightarrow \lim_{\theta \to 0} \frac{\partial}{\partial \theta} \mathcal{B}_B^{(dh)}, \\
\mathcal{B}_{ad} A_\mu(x) &\Leftrightarrow -i \ [A_\mu(x), \mathcal{Q}_{ad}] \Leftrightarrow \lim_{\theta \to 0} \frac{\partial}{\partial \theta} \mathcal{B}_B^{(dh)}.
\end{align*}
\]

(4.49)

In terms of the conserved and on-shell nilpotent charges \( \mathcal{Q}_r \) (with \( r = b, ab, d, ad \)), the above Lagrangian density (4.48) can be re-expressed as

\[
\mathcal{L}_b = -\frac{i}{4} \{ \mathcal{Q}_b, [A_\mu(x) A_\mu(x)] \} - \frac{i}{4} \{ \mathcal{Q}_d, [A_\mu(x) A_\mu(x)] \}.
\]

(4.50)

It will be noted that all the expressions for the Lagrangian density, given in (4.47), (4.48) and (4.50), differ from the starting Lagrangian density (2.1) by a total spacetime derivative which is equal to \( \frac{1}{2} \varepsilon^{\mu \nu \alpha} [A_\mu(\partial \cdot A) + \varepsilon_{\mu \nu} A_\nu E] \).
We express the form of the symmetric energy-momentum tensor (3.7) in the language of the superfield approach to BRST formalism. However, before accomplishing this goal, let us express the above symmetric energy momentum tensor in terms of the nilpotent symmetry transformations given in (3.2) and (3.3). The exact expression for (3.7) is

\[
T^{(s)}_{\mu \nu} = \frac{i}{2} \bar{s}_b \, \bar{s}_{ab} \left[ A_\mu A_\nu - \frac{1}{2} \eta_{\mu \nu} A_\rho A^\rho \right] - \frac{i}{2} \bar{s}_d \, \bar{s}_{ad} \left[ \varepsilon_{\mu \rho \varepsilon_{\nu \sigma}} A^\rho A^\sigma + \frac{1}{2} \eta_{\mu \nu} A_\rho A^\rho \right].
\] (4.51)

Exploiting now the mapping given in (4.49), it is easier to express the form of the symmetric energy momentum tensor in terms of the nilpotent superfields along the Grassmannian directions of the four (2, 2)-dimensional supermanifold. This explicit expression, in terms of the superfields, is

\[
T^{(s)}_{\mu \nu} = \frac{i}{2} \frac{\partial}{\partial \theta} \frac{\partial}{\partial \bar{\theta}} \left[ B^{(h)}_\mu B^{(h)}_\nu \right] - \frac{1}{2} \eta_{\mu \nu} B^{(h)}_\rho B^{(h)}_\rho - \varepsilon_{\mu \rho \varepsilon_{\nu \sigma}} B^{(dh)}_\sigma B^{(dh)}_\sigma - \frac{1}{2} \eta_{\mu \nu} B^{(dh)}_\rho B^{(dh)}_\rho \right],
\] (4.52)

where the super expansions of the bosonic superfield \( B_{\mu} \) has to be taken from (4.44) and (4.45) which are obtained after the application of the HC and DHC. Geometrically, the above equation corresponds to a couple of successive translations of the combination of the symmetric composite superfields along the Grassmannian \( \theta \)- and \( \bar{\theta} \)-directions of the four (2, 2)-dimensional supermanifold, respectively. The above Lorentz scalar composite superfields are constructed with the help of the bosonic superfields of the expansions (4.44) and (4.45) alone (that have been obtained after the application of HC and DHC).

The analogue of equations (4.51) and (4.52) can also be written if we take the help of equations (3.7) and (3.8). Taking into account the latter, it can be seen that the symmetric energy momentum tensor can be also expressed as

\[
T^{(s)}_{\mu \nu} = \bar{s}_b (i V^{(1)}_{\mu \nu}) + \bar{s}_d (i V^{(2)}_{\mu \nu}) \equiv \bar{s}_{ab} (i \tilde{V}^{(1)}_{\mu \nu}) + \bar{s}_{ad} (i \tilde{V}^{(2)}_{\mu \nu}),
\] (4.53)

where the explicit expressions for the \( V^{(1,2)}_{\mu \nu} \) and \( \tilde{V}^{(1,2)}_{\mu \nu} \), in terms of the local fields of the Lagrangian density (2.1), are given in (3.8). Exploiting the mappings given in (4.49), it can be seen that the symmetric energy-momentum tensor can also be written in the following total derivative forms

\[
T^{(s)}_{\mu \nu} = \frac{i}{2} \operatorname{Lim}_{\theta \to 0} \frac{\partial}{\partial \theta} \left[ \left\{ \eta_{\mu \nu} (\partial \cdot B^{(h)}) - (\partial_\mu B^{(h)}_\nu + \partial_\nu B^{(h)}_\mu) \right\} \bar{F}^{(h)} \right] + \left\{ \eta_{\mu \nu} \varepsilon^{\rho \sigma} \partial_\rho B^{(dh)}_\sigma - (\varepsilon_{\mu \rho} \partial_\rho B^{(dh)} + \varepsilon_{\nu \rho} \partial_\rho B^{(dh)}) \right\} \bar{F}^{(dh)} \right],
\] (4.54)

\[
T^{(s)}_{\mu \nu} = -\frac{i}{2} \operatorname{Lim}_{\bar{\theta} \to 0} \frac{\partial}{\partial \bar{\theta}} \left[ \left\{ \eta_{\mu \nu} (\partial \cdot B^{(h)}) - (\partial_\mu B^{(h)}_\nu + \partial_\nu B^{(h)}_\mu) \right\} \bar{F}^{(h)} \right] + \left\{ \eta_{\mu \nu} \varepsilon^{\rho \sigma} \partial_\rho B^{(dh)}_\sigma - (\varepsilon_{\mu \rho} \partial_\rho B^{(dh)} + \varepsilon_{\nu \rho} \partial_\rho B^{(dh)}) \right\} \bar{F}^{(dh)} \right],
\] (4.55)

where the expansions for the superfields have to be inserted from (4.44) and (4.45) which have been obtained after the applications of HC and DHC. In the language of the geometry.
on the four (2, 2)-dimensional supermanifold, it can be seen that the symmetrical energy momentum tensor is equivalent to the translations (i) along the Grassmannian $\bar{\theta}$-direction of the supermanifold when a specific combination of the Lorentz scalar composite superfields (cf. (4.54)) is taken into account, and (ii) along the Grassmannian $\theta$-direction of the four (2, 2)-dimensional supermanifold when the same combination of the Lorentz scalar composite superfields (cf. (4.55)) is taken into account, modulo a relative sign factor.

It is very interesting to point out the fact that the local field operators $V_{\mu\nu}^{(1,2)}$ and $\bar{V}_{\mu\nu}^{(1,2)}$ can also be written as the total derivatives of the appropriate composite superfields (taken from the expansions (4.44) and (4.45)) with respect to the Grassmannian variables $\theta$ and $\bar{\theta}$. The explicit expressions for these operators are

$$
V_{\mu\nu}^{(1)} = +\frac{1}{2} \lim_{\theta \to 0} \frac{\partial}{\partial \theta} \left[ \mathcal{B}_{\mu}^{(h)} \mathcal{B}_{\nu}^{(h)} - i \eta_{\mu\nu} \mathcal{F}^{(h)} \bar{\mathcal{F}}^{(h)} \right],
$$

$$
V_{\mu\nu}^{(2)} = -\frac{1}{2} \lim_{\bar{\theta} \to 0} \frac{\partial}{\partial \bar{\theta}} \left[ \mathcal{B}_{\mu}^{(h)} \mathcal{B}_{\nu}^{(h)} - i \eta_{\mu\nu} \mathcal{F}^{(h)} \bar{\mathcal{F}}^{(h)} \right],
$$

$$
\bar{V}_{\mu\nu}^{(1)} = -\frac{1}{2} \lim_{\theta \to 0} \frac{\partial}{\partial \theta} \left[ \mathcal{B}_{\mu}^{(h)} \mathcal{B}_{\nu}^{(h)} - i \eta_{\mu\nu} \mathcal{F}^{(h)} \bar{\mathcal{F}}^{(h)} \right],
$$

$$
\bar{V}_{\mu\nu}^{(2)} = +\frac{1}{2} \lim_{\bar{\theta} \to 0} \frac{\partial}{\partial \bar{\theta}} \left[ \mathcal{B}_{\mu}^{(h)} \mathcal{B}_{\nu}^{(h)} - i \eta_{\mu\nu} \mathcal{F}^{(h)} \bar{\mathcal{F}}^{(h)} \right],
$$

The key consequences of the above explicit expressions are related to yet another way of expressing the symmetric energy-momentum tensor (3.7) in terms of the superfields and Grassmannian derivatives. For this purpose, the geometrical mappings given in (4.49), turn out to be quite handy. Taking the help of (4.56), (4.57) and (4.49), we obtain the following expressions for the symmetric energy-momentum tensor (3.7), namely;

$$
T_{\mu\nu}^{(s)} = +\frac{i}{2} \frac{\partial}{\partial \theta} \left[ \mathcal{B}_{\mu}^{(h)} \mathcal{B}_{\nu}^{(h)} - i \eta_{\mu\nu} \mathcal{F}^{(h)} \bar{\mathcal{F}}^{(h)} \right],
$$

$$
-\varepsilon_{\mu\rho} \varepsilon_{\nu\sigma} \mathcal{B}_{\rho}^{(dh)} \mathcal{B}_{\sigma}^{(dh)} - i \eta_{\mu\nu} \mathcal{F}^{(dh)} \bar{\mathcal{F}}^{(dh)}
$$

(4.58)

$$
T_{\mu\nu}^{(s)} = -\frac{i}{2} \frac{\partial}{\partial \bar{\theta}} \left[ \mathcal{B}_{\mu}^{(h)} \mathcal{B}_{\nu}^{(h)} - i \eta_{\mu\nu} \mathcal{F}^{(h)} \bar{\mathcal{F}}^{(h)} \right],
$$

$$
-\varepsilon_{\mu\rho} \varepsilon_{\nu\sigma} \mathcal{B}_{\rho}^{(dh)} \mathcal{B}_{\sigma}^{(dh)} - i \eta_{\mu\nu} \mathcal{F}^{(dh)} \bar{\mathcal{F}}^{(dh)}
$$

(4.59)

It will be noted that (i) the above expressions are different from their counterpart in (4.52), (ii) the expression in (4.59) is not an independent quantity because it can be derived from (4.58) by exploiting the trivial relationship $\partial_{\theta} \partial_{\bar{\theta}} + \partial_{\bar{\theta}} \partial_{\theta} = 0$, and (iii) the above expressions can be re-written in terms of the on-shell nilpotent (anti-)BRST and (anti-)co-BRST symmetry transformations (3.2) and (3.3) as

$$
T_{\mu\nu}^{(s)} = +\frac{i}{2} \tilde{s}_{\rho} \tilde{s}_{\sigma} \left[ A_{\mu} A_{\nu} - i \eta_{\mu\nu} C \bar{C} \right] - \frac{i}{2} \tilde{s}_{\rho} \tilde{s}_{\sigma} \left[ \varepsilon_{\rho\sigma} A^\rho A^\sigma + i \eta_{\mu\nu} C \bar{C} \right].
$$

(4.60)

It is obvious that the above expression is quite different from its counterpart in (4.51) which has also been expressed in terms of the on-shell nilpotent symmetry transformations.
We close this subsection with the remark that mathematically the Lagrangian density and symmetric energy-momentum tensor of a TFT, in the framework of the superfield approach to BRST formalism, can always be expressed as a total derivative with respect to (i) the Grassmannian variable $\theta$, or (ii) the Grassmannian variable $\bar{\theta}$, and/or (iii) a combination of the Grassmannian variables $\theta$ and $\bar{\theta}$ together. In the language of the geometry on the appropriately chosen supermanifold, the Lagrangian density as well as symmetric energy momentum tensor, of a given TFT, always correspond to the translations(s) of the composite superfields along the Grassmannian directions of the above supermanifold. The above statements are true for the TFTs which have the mathematical structure (i.e the form of the Lagrangian density, energy-momentum tensor etc.) like Witten type TFT.

5. Conclusions

In our present endeavor, we have mainly focused our attention on the continuous as well as discrete symmetry transformations of the Lagrangian density of a given 2D free 1-form Abelian gauge theory in the framework of the BRST formalism. To be specific, we have shown the existence of the continuous nilpotent (anti-)BRST symmetry transformations, nilpotent (anti-)co-BRST symmetry transformations and a non-nilpotent (bosonic) symmetry transformation for the Lagrangian density of the above 2D free 1-form Abelian gauge theory. The above symmetry transformations (and their corresponding generators) have been, in turn, shown to possess a deep connection with the de Rham cohomological operators of the differential geometry. To establish the above connection in its totality, one requires the existence of a specific set of discrete symmetry transformations in the theory. These symmetries have also been shown to exist for the 2D free 1-form Abelian gauge theory. In fact, it is the combination and interplay of the discrete and continuous symmetry transformations that provides the exact analogue of the relationship that exists between the (co-)exterior derivatives $\delta d$ (i.e. $\delta = \pm \ast d \ast$). Precisely speaking, the discrete symmetry transformations of the theory correspond to the Hodge duality $\ast$ operation of the differential geometry in the celebrated relationship $\delta = \pm \ast d \ast$.

We have chosen here the 2D free Abelian 1-form gauge theory as the prototype example for the Hodge theory because, for this tractable field theoretical model, one obtains the local, continuous, covariant and nilpotent (anti-)BRST and (anti-)co-BRST symmetry transformations which is not the case for the 4D 1-form free as well as interacting (non-)Abelian gauge theories. In fact, it turns out that the above (anti-)co-BRST symmetry transformations for the 4D theories are non-local and non-covariant [30-35]. One can obtain the covariant version of the above symmetry transformations by introducing a specific kind of parameter in the theory but the nilpotency of the symmetry transformations disappears. The latter property is restored only for a specific value of the above parameter [35]. As a consequence, the conserved charges (which turn out to be the generators for the above (anti-)co-BRST symmetry transformations) are found to be non-local for the 4D theories.
The above cited problems do not arise for the 2D free as well as interacting 1-form Abelian gauge theory. We have discussed, in our Appendix, these (i.e. nilpotent (anti-)co-BRST) symmetry transformations associated with the interacting 2D 1-form Abelian gauge theory where there is a direct coupling between the U(1) gauge field and the Noether conserved current constructed with the help of the fermionic Dirac fields.

It is precisely due to the existence of the well-defined on-shell nilpotent (anti-)BRST and (anti-)co-BRST symmetry transformations (and their corresponding generators) that the Lagrangian density of the 2D free 1-form Abelian gauge theory has been able to be expressed as the sum of two anticommutators that are constructed with the help of the conserved and on-shell nilpotent (anti-)BRST and (anti-)co-BRST charges. In other words, the Lagrangian density turns our to be the sum of the BRST exact and BRST co-exact terms. This observation, in turn, implies that the Lagrangian density of the above theory has a form which is similar to the Witten type topological field theories. Similarly, the symmetric energy-momentum tensor of the above theory can also be expressed as the sum of the BRST exact and BRST co-exact terms. However, (i) the absence of the local topological shift symmetry transformations, and (ii) the presence of the gauge type (i.e. BRST and co-BRST) symmetry transformations implies that the present 2D 1-form Abelian gauge theory is similar to the Schwarz type of topological field theories. Thus, we conclude that the present 2D free 1-form Abelian gauge theory is a new type of TFT. Its topological nature has been shown through the application of the BRST and co-BRST charges on the harmonic (i.e. physical) state of the Hodge decomposed state in the quantum Hilbert space of states. It turns out that both the dynamical degrees of freedom of the 2D photon \( A_\mu \) field can be gauged away due to the coexistence of the BRST and co-BRST symmetry transformations together. We have established this result for the case of a single physical photon (harmonic) state that is created from the physical vacuum of the theory.

One of the key features of the present field theoretical model is that the model provides a beautiful example for the application of the techniques involved in the geometrical superfield approach to BRST formalism. To be specific and precise, the following points, connected with the superfield formalism, are noteworthy. First, the nilpotent (anti-)BRST symmetry transformations owe their origin to the (super) exterior derivatives \( \tilde{d}d \) which are exploited in the horizontality condition defined on the four (2, 2)-dimensional supermanifold (over which the 2D 1-form Abelian theory is considered). Second, the nilpotent (anti-)co-BRST symmetry transformations are found to be deeply connected with the (super) co-exterior derivatives \( \tilde{\delta}\delta \) which are exploited in the DHC imposed on the above supermanifold. The (super) Laplacian operators \( \tilde{\Delta}\Delta \), however, lead to the derivation of the equations of motion for all the fields of the 2D free Abelian gauge theory from an appropriate restriction (cf. (4.21)) on the above supermanifold. Thus, we note that all the (super) cohomological operators play very significant roles in describing various aspects of the model under consideration. This is why, the 2D free 1-form Abelian gauge theory is a perfect field theoretical model for the Hodge theory. Moreover, the topological features
of the model under consideration have also been shown in terms of the superfields and Grassmannian derivatives.

A very important aspect of our discussion is the significance of the definition of the Hodge duality $\star$ operation on the four (2, 2)-dimensional supermanifold [45]. In fact, our present model provides a very good example where the correctness of the above Hodge duality $\star$ operation [45] has been tested in a meaningful manner. It will be noted that the above definition of the duality plays a key role in the application of the DHC on the supermanifold. Furthermore, the above definition is required in the application of the restriction (4.21) on the four (2, 2)-dimensional supermanifold where the super Laplacian operator $\tilde{\Delta}$ plays a key role. For the present model, we find that the definition of the Hodge duality $\star$ operation, on the appropriately chosen four (2, 2)-dimensional supermanifold, is correct because it leads to the precise and consistent results which have been discussed in Subsec. 4.2 and Subsec. 4.3. It is gratifying to note that, for the model under consideration, the ordinary and super de Rham cohomological operators blend together in a beautiful manner to lead to some very cute theoretical results (thereby rendering the present theory to be a perfect field theoretical model for the Hodge theory).

It would be very interesting endeavor to apply our present ideas to the physical 4D field theories. In this connection, it is worth pointing out that, in our earlier works [42-44], we have been able to show the existence of the local, continuous, covariant and nilpotent (anti-)co-BRST symmetry transformations for the 4D free 2-form Abelian gauge theory. In fact, we have been able to prove that this 4D field theoretical model, in its Lagrangian formulation, is an example of the Hodge theory [44]. However, we have not yet been able to establish the role of the super de Rham cohomological operators in the derivation of the above nilpotent symmetries in the framework of the superfield approach to BRST formalism. One of our key future endeavors is to discuss and derive the above nilpotent (anti-)co-BRST symmetry transformations by imposing the appropriate restrictions on the six (4, 2)-dimensional supermanifold. Furthermore, the application of the key ideas of our present investigation to the 4D gravitational theories is another challenging project that we plan to pursue. In this connection, we would like to point out that the usual superfield formulation has already been applied to the 4D gravitational theories to derive the (anti-)BRST symmetry transformations for the gauge and (anti-)ghost fields [24]. In this attempt, however, the (anti-)BRST symmetry transformations associated with the matter fields of the gravitational theories have not yet been obtained. We can apply the theoretical arsenals of our newly proposed augmented superfield formulation [52, 53, 57-61] to derive the nilpotent symmetry transformations associated with the matter fields. To derive the nilpotent (anti-)co-BRST symmetry transformations for the gravitational theory (in the framework of the superfield approach to BRST formalism) is yet another direction for further investigation. These are some of the promising issues that would be pursued in the future and our results would be reported in our forthcoming publications [62].

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Appended A

We demonstrate, in this Appendix, that the 2D interacting 1-form Abelian U(1) gauge theory, where there is a direct coupling between the U(1) gauge field $A_{\mu}$ and the matter conserved current ($J_{\mu} = -e\bar{\psi}\gamma_{\mu}\psi$) constructed with the help of Dirac fields, is a field theoretical model for the Hodge theory. To this end in mind, let us begin with the following 2D (anti-)BRST invariant Lagrangian density in the Feynman gauge (see, e.g. [39,40])

$$\mathcal{L}^{(m)}_B = B \, E - \frac{1}{2} B^2 + B \, (\partial \cdot A) + \frac{1}{2} B^2 + \bar{\psi} (i\gamma^\mu D_{\mu} - m) \psi - i \partial_{\mu} \bar{\psi} \partial^\mu C, \quad (A.1)$$

where $D_{\mu} \psi = \partial_{\mu} \psi + ieA_{\mu} \psi$ is the covariant derivative on the Dirac field $\psi$ which results in the interaction term ($-e\bar{\psi}\gamma^\mu A_{\mu} \psi$) in the above Lagrangian density. The interaction term is primarily a coupling between the U (1) gauge field $A_{\mu}$ and the conserved matter current $J_{\mu} = -e\bar{\psi}\gamma^\mu \psi$ where $\gamma^\mu$'s are the well known $2 \times 2$ Dirac matrices. The other symbols carry the same meaning as the ones mentioned for the Lagrangian density (2.2). The above Lagrangian density is endowed with the following local, covariant, continuous, nilpotent (i.e. $s_2^{(a)b} = 0$) and anticommuting ($s_bs_{ab} + s_{ab}s_b = 0$) (anti-)BRST symmetry transformations $s_{(a)b}$ (see, e.g. [39,40] for details)

$$s_b A_{\mu} = \partial_{\mu} C, \quad s_b C = 0, \quad s_b \bar{C} = iB, \quad s_b B = 0,$$

$$s_b E = 0, \quad s_b B = 0, \quad s_b (\partial \cdot A) = \Box C, \quad s_b F_{\mu\nu} = 0,$$

$$s_b \psi = -ieC \psi, \quad s_b \bar{\psi} = -ie\bar{\psi} C,$$

$$s_{ab} A_{\mu} = \partial_{\mu} C, \quad s_{ab} \bar{C} = 0, \quad s_{ab} C = -iB, \quad s_{ab} B = 0,$$

$$s_{ab} E = 0, \quad s_{ab} B = 0, \quad s_{ab} (\partial \cdot A) = \Box C, \quad s_{ab} F_{\mu\nu} = 0,$$

$$s_{ab} \psi = -ie\bar{C} \psi, \quad s_{ab} \bar{\psi} = -ie\bar{\psi} C, \quad (A.2)$$

because the above Lagrangian density transforms to a total derivative. In addition to the above nilpotent symmetry transformations, there exists another set of local, covariant, continuous, nilpotent (i.e. $s_2^{(a)d} = 0$) and anticommuting ($s_ds_{ad} + s_{ad}s_d = 0$) (anti-)co-BRST symmetry transformations $s_{(a)d}$ (see, e.g. [39,40] for details)

$$s_d A_{\mu} = -\varepsilon_{\mu\nu} \partial^\nu \bar{C}, \quad s_d \bar{C} = 0, \quad s_d C = -iB, \quad s_d B = 0,$$

$$s_d E = \Box C, \quad s_d B = 0, \quad s_d (\partial \cdot A) = 0, \quad s_d F_{\mu\nu} = [\varepsilon_{\mu\nu} \partial_\nu - \varepsilon_{\nu\rho} \partial_\mu] \partial^\rho \bar{C},$$

$$s_d \psi = -ie\bar{C} \gamma_5 \psi, \quad s_d \bar{\psi} = +ie\bar{\psi} \gamma_5 \bar{C},$$

$$s_{ad} A_{\mu} = -\varepsilon_{\mu\nu} \partial^\nu C, \quad s_{ad} \bar{C} = 0, \quad s_{ad} C = iB, \quad s_{ad} B = 0,$$

$$s_{ad} E = \Box C, \quad s_{ad} B = 0, \quad s_{ad} (\partial \cdot A) = 0, \quad s_{ad} F_{\mu\nu} = [\varepsilon_{\mu\nu} \partial_\nu - \varepsilon_{\nu\rho} \partial_\mu] \partial^\rho C,$$

$$s_{ad} \psi = -ieC \gamma_5 \psi, \quad s_{ad} \bar{\psi} = +ie\bar{\psi} \gamma_5 \bar{C}, \quad (A.3)$$

under which the above Lagrangian density (A.1) remains quasi-invariant because it transforms to a total derivative. In addition, the key requirement for the existence of the above transformations is

\[ \gamma^0 = \sigma_2, \gamma^1 = i\sigma_1 \] so that the $\gamma_5$ matrix is defined as: $\gamma^5 = \gamma^0 \gamma^1 \equiv \gamma_5 = \sigma_3$. It can be checked that the following relationships are satisfied:

\[ \{\gamma^\mu, \gamma^\nu\} = 2\varepsilon^{\mu\nu\rho\sigma} \gamma_{\rho\sigma} = \varepsilon_{\mu\nu} \gamma^5 \] where the 2D Levi-Civita antisymmetric tensor is chosen to satisfy $\varepsilon_{01} = +1 = \varepsilon^{10}$, $\varepsilon_{\mu\nu} \varepsilon_{\mu\lambda} = -\delta^\mu_\lambda$, etc. In the above choices, the $\sigma$’s are the usual Hermitian $2 \times 2$ Pauli matrices.
symmetry transformations is that the mass of the Dirac fields should be zero (i.e. \( m = 0 \)). It should be noted that for the 4D interacting U(1) gauge theory with Dirac fields, the (anti-)co-BRST symmetry transformations are found to be non-local, non-covariant, nilpotent and anticommuting (see. e.g. [30-35] for details). It can also be noted that, for all the fields of the Lagrangian density (A.1), the operator relationships \([sb_{ab} + s_{ab} s_b] \Omega = 0\) and \([s_{ab} s_b + s_b s_{ab}] \Omega = 0\) are valid as can be checked by exploiting the nilpotent (anti-)BRST and (anti-)co-BRST symmetry transformations (A.2) and (A.3). Here \( \Omega \) is the generic field of the Lagrangian density (A.1) of the 2D interacting 1-form gauge theory.

It is interesting to point out that the anticommutation relation between the (anti-)BRST and (anti-)co-BRST symmetry transformations (i.e. \( s_w = s_b s_d + s_d s_b \equiv s_{ab} s_{ad} + s_{ad} s_{ab} \)) defines a bosonic symmetry \( s_w \) (with \( s_w^2 \neq 0 \)). Under this symmetry transformation, the fields of the Lagrangian density (A.1) transform as follows [39,40]

\[
\begin{align*}
  s_w A_\mu &= \partial_\mu B + \varepsilon_{\mu \nu} \partial^\nu B, & s_w E &= -\Box B, & s_w (\partial \cdot A) &= \Box B, \\
  s_w C &= 0, & s_w \bar{C} &= 0, & s_w B &= 0, & s_w \bar{B} &= 0, \\
  s_w \psi &= i e (\gamma_5 B - \bar{B}), & s_w \bar{\psi} &= -i e (\gamma_5 B - \bar{B}).
\end{align*}
\]

(A.4)

The above transformations are also the symmetry transformations for the Lagrangian density (A.1) (only in the case when the mass of the Dirac particle is zero (i.e. \( m = 0 \)). Thus, the operator algebra that is obeyed by all the transformation operators \( s_r \) (with \( r = b, ab, d, ad, w \)) is exactly same as the one given in (2.6). This demonstrates that all the de Rham cohomological operators (cf. equation (2.7)) have found their analogue in terms of the above transformation operators (and their generators) for the 2D interacting 1-form Abelian U(1) gauge theory with Dirac fields.

To establish that the above interacting U(1) gauge theory is a tractable field theoretical model of the Hodge theory, we focus now on the existence of the discrete symmetry transformations of the theory and show their relevance to the Hodge duality * operation of the differential geometry. It can be checked that the following discrete transformations

\[
\begin{align*}
  C &\to \pm i \gamma_5 \bar{C}, & \bar{C} &\to \pm i \gamma_5 C, & B &\to \mp i \gamma_5 \bar{B}, & \psi &\to \psi, \\
  A_0 &\to \pm i \gamma_5 A_1, & A_1 &\to \pm i \gamma_5 A_0, & \bar{\psi} &\to \bar{\psi}, & e &\to \mp i e, \\
  (\partial \cdot A) &\to \pm i \gamma_5 E, & E &\to \pm i \gamma_5 (\partial \cdot A),
\end{align*}
\]

(A.5)

are the symmetry transformations for the Lagrangian density (A.1) because the Lagrangian density remains invariant under it. It should be noted that the above transformations are in the matrix notation. That is to say, the transformations \( C \to \pm i \gamma_5 \bar{C} \), etc, imply that \( C \bar{1} \to \pm i \sigma_3 C \bar{1} \) where \( \bar{1} \) is the 2 \( \times \) 2 unit matrix for the above 2D interacting 1-form U(1) Abelian gauge theory. In other words, \( C \to \pm i \gamma_5 \bar{C} \) implies that the Lagrangian density remains invariant under the transformations: \( C \to \pm i C \) and/or \( C \to \mp i \bar{C} \). In exactly similar manner, rest of the other transformations should be interpreted. In fact, the sign flip in the above transformations is taken care of by the rest of the analogous transformations for the other fields of the theory so that together they become the symmetry transformations of the Lagrangian density (A.1) for the 1-form interacting Abelian gauge theory.
As discussed earlier, the above discrete transformations are found to be the analogue of the Hodge duality $*$ operation of the differential geometry. To see it clearly, it can be seen that any arbitrary generic field $\Omega$ of the Lagrangian density (A.1) has the transformation property $*(\ast \Omega) = \pm \Omega$ under the above two successive operations of the discrete symmetry transformation where the $(\pm)$ sign stands for the generic field $\Omega$ being $\psi$ and $\bar{\psi}$ and $(\mp)$ sign is meant for all the rest of the bosonic fields of the theory. It is gratifying to note that the analogue of the equation (2.13) now becomes

$$s_{(a)d} (\Omega) = \pm \ast s_{(a)b} \ast (\Omega). \quad (A.6)$$

The above equation shows that the interplay of the continuous nilpotent symmetry transformations (cf. (A.2) and (A.3)) and the discrete symmetry transformations (cf. (A.5)) look exactly identical to the relationship $\delta = \pm \ast d\ast$ that exists between the nilpotent co-exterior derivative $\delta(d^2 = 0)$ and the exterior derivative $d(d^2 = 0)$. In (A.6) too, the $(\pm)$ sign stands for $\psi$ as well as $\bar{\psi}$ and the rest of the fields of the theory correspond to $(\mp)$ sign. Moreover, the presence of $\ast$ in the equation (A.6) corresponds to the discrete symmetry transformations quoted in the equation (A.5).

It is worthwhile to point out that the $(\pm)$ signs in the relationship $\delta = \pm \ast d\ast$ of the differential geometry depend on (i) the dimensionality of the manifold on which the Hodge duality $*$ operation is defined (for instance, for the even dimensional manifold, it is always the minus sign that becomes relevant [12,13]), and (ii) the inner product of the degree of the forms that are involved in the definition of the duality. In fact, it is due to the latter condition that, for the odd dimensional manifold, the sign in the relation $\delta = \pm \ast d\ast$ can be positive or negative (see. e.g., [12-16] for details). However, in the relationship (A.6), the positive and negative signs depend on the two successive operations of the discrete transformations (A.5) on a particular field of the Lagrangian density (A.1). In terms of the conserved charges $Q_r$ (with $r = b, ab, d, ad, w$), corresponding to the symmetry transformations $s_r$ (with $r = b, ab, d, ad, w$), the Hodge decomposition theorem can be defined in the quantum Hilbert space of states for the interacting $U(1)$ gauge theory, too. Thus, we conclude that the 2D interacting 1-form Abelian gauge theory is a tractable field theoretical model for the Hodge theory because all the de Rham cohomological operators of the differential geometry are defined in terms of the well-defined symmetry transformations (and their corresponding generators) for the Lagrangian density of the above theory.

In the framework of the augmented superfield formulation, there are ways to derive the nilpotent (anti-)BRST and (anti-)co-BRST symmetry transformations for all the fields of the Lagrangian density (A.1) of the theory [52-61]. The long-standing problem of the derivation of the nilpotent (anti-)BRST and (anti-)co-BRST symmetry transformations for the matter fields $\psi$ and $\bar{\psi}$ of the interacting gauge theory, in the framework of the superfield approach to BRST formalism, has been resolved by taking recourse to (i) the equality of some conserved quantities [52,53,57-61] on the appropriate dimensional supermanifold (e.g. the four (2,2)-dimensional supermanifold for 2D the gauge theory) on top of the horizontality condition, and (ii) a single gauge invariant restriction on the matter superfields.

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of the supermanifold that owes its origin to the (super) covariant derivatives and their connection with the (super) curvature 2-forms [54-56]. However, we shall not elaborate on these issues in our present endeavor because we have already proven that the 2D interacting 1-form Abelian gauge theory is a Hodge theory in the Lagrangian formulation.

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