Employing Non-Markovian effects to improve the performance of a quantum Otto refrigerator

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High-precision control over quantum systems has been providing many proof-of-principle experiments in quantum thermodynamics. The proper extension of thermal machines description to the quantum domain may have important implications to the design of quantum technologies. In general, the stroke-operating Markovian machines consider a complete thermalization of the system in the strokes where heat is exchanged between either the hot or the cold Markovian heat reservoirs. However, current advances in structured thermal reservoir designs have considered the inclusion of memory effects associated to non-Markovian dynamics, which have been receiving special attention in quantum thermodynamics. Here we consider a quantum Otto refrigerator with a particular choice of engineered cold reservoir and discuss its performance providing explicit expressions for the coefficient of performance, injected power, and cooling rate as well as associating these quantities to the total entropy production along the cycle. These relations are general and do not depend on the particular choice of the quantum refrigerant substance. Finally, we also consider a numerical simulation of a spin quantum refrigerator with experimentally feasible parameters to illustrate our results, showing that non-Markovian effects induced by the structured cold reservoir may improve the performance of a quantum Otto refrigerator.

I. INTRODUCTION

The theoretical description of thermal machines was fundamental to the development of classical thermodynamics, providing an operational understanding of the second law as established by William Thomson (Lord Kelvin) [1] and Carnot [2]. Moreover, the thermodynamic characterization of heat engines and refrigerators is essential to engineering since it provides tools for estimating the performance of such machines [3, 4]. In the same perspective, it is expected that the development of quantum thermodynamics will play a similar role in quantum engineering for the development of quantum technologies [5–7].

Quantum thermal machines are excellent platforms to test results from quantum thermodynamics [8–14], transforming thermal energy (heat) into mechanical energy (work). In the quantum heat engine configuration the purpose is to extract an amount of work by absorbing an amount of heat from a hot source. On the other hand, in the quantum refrigerator setting the goal is to absorb heat from a cold source by injecting an amount of work into the system. The performance of the former is characterized by the thermodynamic efficiency and the power output, whereas of the latter by the coefficient of performance, the injected power and the cooling rate. The theoretical underpinnings of the description of quantum thermal machines dates back to the late 1950s with the early works of Scovil and Schulz-Dubois [15, 16]. Since then, several theoretical investigations on quantum heat engines and refrigerators have been carried out [17–38].

From the experimental point of view, two microscopic classical heat engines [39, 40] and a quantum refrigerator [41] and a quantum engine [42] have been recently implemented employing trapped ions. Additionally, a quantum Otto cycle on a nano-beam working medium [43] and a quantum heat engine using an ensemble of nitrogen-vacancy centers in diamonds [44] have been reported. In particular, the experiment reported in Ref. [43] employs a squeezed thermal reservoir and demonstrated that the efficiency of the quantum heat engine (that explores squeezing) may go beyond the standard Carnot’s efficiency, corroborating theoretical expectations [45–56].

Interaction with a squeezed thermal reservoir is not the only generalized process that goes beyond the classical thermodynamic approach. For instance, engineered reservoirs [57] such as including coherence [58], have also been addressed, evidencing that quantum properties can be used to enhance the performance of quantum heat engines and refrigerators when compared with its conventional counterparts.

Reservoir engineering may be useful and important in different physical setups, for instance, for cooling phonons with acoustic reservoir-engineering [59] and in circuit quantum electrodynamics [60]. Quantum fluctuation theorems can be probed by using engineered reservoirs [61]. In the context of quantum thermodynamical processes an approach for reservoir engineering with non-Markovian consequences has been considered in Ref. [62], where the complete reservoir structure is composed of a Markovian part plus a two-level system.

In recent years, advances in quantum technologies and quantum control allow the study of effects beyond the Born-Markov approximation in open systems, enabling tests of memory effects in decoherence dynamics [63]. Although, in general, the non-Markovian aspects of the dynamics are associated with a strong coupling between system and reservoir, a non-Markovian dynamics may be observed in a weak coupling regime, for instance, when the reservoir has a finite-size (structured reservoir) [64]. Recently, non-Markovian aspects in quantum thermodynamics have been studied from different points of view, for instance, in the context of thermodynamic laws and fluctuations theorems [65], in non-equilibrium dynamics [66], and their effects on the entropy production of non-equilibrium protocols [67, 68]. From the perspective of quantum thermal machines, a promising avenue is emerging with recent theoretical results illustrating how to use memory effects to improve the performance on quantum thermodynamic cycles [69–71].

In this work, we are interested in studying and quantifying...
non-Markovian effects (memory effects in the dynamics) resulting from a particular structure of a engineered cold reservoir in quantum refrigerators. With this purpose, we obtain analytical expressions for the coefficient of performance, injected power and cooling rate, characterizing the performance of the refrigerator, in terms of the total entropy production along the thermodynamic cycle, which in turn depends on the non-Markovian dynamics. Employing incomplete thermalization with the engineered cold reservoir (at the finite-time regime), we show that memory effects (non-Markovianity) serve as a resource to increase the performance of the quantum Otto refrigerator, provided we have a sufficiently high control of the parameters involved into the cycle, for instance, the time allocation in each stroke. Then, we consider a numerical simulation of a quantum Otto refrigerator model, in order to illustrate our results.

This work is organized as follows. In section II we discuss the model of engineered cold reservoir employed and the quantum Otto refrigerator. Section III is devoted to present and discuss the performance of the refrigerator, i.e., the coefficient of performance, injected power, and cooling rate in terms of the total entropy production. We also consider a numerical simulation employing experimentally feasible parameters to illustrate that memory effects may improve the performance of a quantum refrigerator. Finally, in the section IV we draw our conclusions and final remarks.

II. QUANTUM OTTO REFRIGERATOR WITH STRUCTURED COLD RESERVOIR

A. Model of structured cold heat reservoir

Before describing a quantum Otto refrigerator, we detail the model of the cold engineered heat reservoir, which induces a non-Markovian dynamics on the refrigerant substance. The cold heat reservoir is composed by two parts, a Markovian heat reservoir and a two-level system (henceforth called auxiliary qubit), as depicted in Fig. 1(a). The system (quantum refrigerator) will be regarded as a qubit, which interacts with the auxiliary qubit, as depicted in Fig. 1(a). The system (quantum refrigerator) will be regarded as a qubit, which interacts with the auxiliary qubit, as depicted in Fig. 1(a). The system (quantum refrigerator) will be regarded as a qubit, which interacts with the auxiliary qubit, as depicted in Fig. 1(a). The system (quantum refrigerator) will be regarded as a qubit, which interacts with the auxiliary qubit, as depicted in Fig. 1(a). The system (quantum refrigerator) will be regarded as a qubit, which interacts with the auxiliary qubit, as depicted in Fig. 1(a). The system (quantum refrigerator) will be regarded as a qubit, which interacts with the auxiliary qubit, as depicted in Fig. 1(a). The system (quantum refrigerator) will be regarded as a qubit, which interacts with the auxiliary qubit, as depicted in Fig. 1(a). The system (quantum refrigerator) will be regarded as a qubit, which interacts with the auxiliary qubit, as depicted in Fig. 1(a). The system (quantum refrigerator) will be regarded as a qubit, which interacts with the auxiliary qubit, as depicted in Fig. 1(a). The system (quantum refrigerator) will be regarded as a qubit, which interacts with the auxiliary qubit, as depicted in Fig. 1(a). The system (quantum refrigerator) will be regarded as a qubit, which interacts with the auxiliary qubit, as depicted in Fig. 1(a). The system (quantum refrigerator) will be regarded as a qubit, which interacts with the auxiliary qubit, as depicted in Fig. 1(a). The system (quantum refrigerator) will be regarded as a qubit, which interacts with the auxiliary qubit, as depicted in Fig. 1(a). The system (quantum refrigerator) will be regarded as a qubit, which interacts with the auxiliary qubit, as depicted in Fig. 1(a). The system (quantum refrigerator) will be regarded as a qubit, which interacts with the auxiliary qubit, as depicted in Fig. 1(a). The system (quantum refrigerator) will be regarded as a qubit, which interacts with the auxiliary qubit, as depicted in Fig. 1(a). The system (quantum refrigerator) will be regarded as a qubit, which interacts with the auxiliary qubit, as depicted in Fig. 1(a). The system (quantum refrigerator) will be regarded as a qubit, which interacts with the auxiliary qubit, as depicted in Fig. 1(a). The system (quantum refrigerator) will be regarded as a qubit, which interacts with the auxiliary qubit, as depicted in Fig. 1(a). The system (quantum refrigerator) will be regarded as a qubit, which interacts with the auxiliary qubit, as depicted in Fig. 1(a). The system (quantum refrigerator) will be regarded as a qubit, which interacts with the auxiliary qubit, as depicted in Fig. 1(a). The system (quantum refrigerator) will be regarded as a qubit, which interacts with the auxiliary qubit, as depicted in Fig. 1(a).

The bosonic modes are in the thermal state it is possible to show that the two-qubit system $SA$ evolves under the Lindblad master equation [62].

$$\frac{d}{dt} \rho_{SA}^{t} = -\frac{i}{\hbar} [H^{SA}, \rho_{SA}^{t}] + \sum_{i=1,2} \gamma^{+} (\epsilon_{i}) \left( L_{i} \rho_{t} L_{i}^{\dagger} - \frac{1}{2} \left\{ L_{i} L_{i}^{\dagger}, \rho_{t} \right\} \right)$$

where $\gamma^{+} (\epsilon_{i}) = \frac{\kappa}{2} \mathcal{J} (\epsilon_{i}) (1 + n_{BE} (\epsilon_{i}))$ and $\gamma^{-} (\epsilon_{i}) = \frac{\kappa}{2} \mathcal{J} (\epsilon_{i}) n_{BE} (\epsilon_{i})$ are the decay rates, with spectral density $\mathcal{J} (\omega) = \kappa / \pi$ and Bose-Einstein distribution $n_{BE} (\omega) = (e^{\beta \hbar \omega} - 1)^{-1}$, and $L_{i}$ are the Lindblad operators (see Appendix A for more details). The quantities $\hbar \beta_{1}$ and $\hbar \beta_{2}$ are the energy gaps between different energy levels of the two-qubit system $SA$ (see Fig. A1).

There are different notions of non-Markovian dynamics for quantum processes [72–74]. Independent of the precise definition employed, non-Markovianity is always related to the concept of memory in the dynamics. Here, we will adopt the following notion: a system undergoes a non-Markovian dynamics if there is a flow of information from the environment to the system. In particular, we consider the trace distance as a measure of distinguishability (information) [73]. For two
arbitrary states $\rho$ and $\sigma$, their trace distance is given by [75]

$$D(\rho, \sigma) = \frac{1}{2} \text{Tr} \left[ \sqrt{(\rho - \sigma)^\dagger (\rho - \sigma)} \right].$$

(4)

We denote by $\lambda_t(\rho(0)) = \rho(t)$ the one-parameter family of dynamical maps. The dynamics is non-Markovian if $dD(\rho_1(t), \rho_2(t))/dt$ becomes positive at any time $t$ and for some pair of initial states $\rho_1(0)$ and $\rho_2(0)$ [73]. Conversely, this means that, in a Markovian dynamics, the distinguishability between any pair of states always decrease monotonically in time. In other words, in a non-Markovian dynamics there always exists some pair of initial states for which the distinguishability (information) increases at a given time. In particular, it is sufficient to assume a pair of initially pure and orthogonal states to witness the non-Markovianity of a qubit system [76]. For that reason, we consider such a pair of states to be the eigenstates of $\sigma_z$, i.e., the states $|+\rangle$ and $|-\rangle$, in order to determine which parameter regimes induce a non-Markovian dynamics into the refrigerant substance [see Fig. 1(b)].

We assume the refrigerant and auxiliary qubits are on resonance, $2\pi \omega^S = 2\pi \omega^A = 2.2$ kHz, the vacuum decay rate $2\pi \kappa = 0.02$ kHz, the temperature $T = 2\hbar \omega^A/k_B \approx 6$ pK for the bosonic mode. These parameters are experimentally achievable in nuclear magnetic resonance setups [79–82]. Using these parameters, in Fig. 1(b), we show for which values of the coupling constant $J$ the Markovian and non-Markovian regimes are achieved.

As explained in the next section, we replace the cold heat reservoir of the Otto cycle by the structured environment described in this section, which in turn will induce a non-Markovian dynamics to the refrigerant. We still refer to this structured environment as a heat reservoir because the asymptotic state (when the interaction time goes to infinity) is the Gibbs state. In other words, this model of non-Markovian environment behaves, in the asymptotic limit, in the exact same way as the typical Markovian heat reservoir.

B. Quantum Otto refrigerator

Let us consider a quantum Otto refrigerator with a single-qubit refrigerant, whose the cycle is comprised by two driven adiabatic and two undriven thermalization strokes. In an Otto refrigerator, work and heat exchange are associated to adiabatic and undriven thermalization strokes, respectively. These thermodynamic quantities can be obtained through the difference between final and initial internal energies for a given stroke, i.e., $\langle U_1 \rangle = \text{Tr}[\rho_1 H(t)]$, where $\rho_1$ is the density operator of the refrigerant (system) and $H(t)$ represents the driven Hamiltonian at a time $t$.

The refrigerant starts the cycle in the hot Gibbs state $\rho_0 = \rho_{eq,h} = e^{-\beta_0 H_0}/Z_0^h$, where $\beta_0 = 1/k_B T_h$ is the hot inverse temperature, $H_0$ is the initial Hamiltonian, and $Z_0^h = \text{Tr}[e^{-\beta_0 H_0}]$ is the associated partition function.

In the first stroke a compression is performed on the refrigerant frequency, with driven Hamiltonian given by

$$H_{\text{com}}(t) = \frac{h\omega(t)}{2}\sigma_z,$$

(5)

where the frequency is changed in a linear ramp as $\omega(t) = \omega_0(1 - t/\tau_1) + \omega_\tau(t/\tau_1)$, $\omega_0$ and $\omega_\tau$ the initial and final frequency of the refrigerant, respectively, with $\omega_0 > \omega_\tau$, and $H_0 = H_{\text{com}}(0) = (h\omega_0/2)\sigma_z$, “com” stands for compression. Although this stroke is performed for a finite time, the reduced density operator of the system will not present coherence into the energy eigenbasis. This is specially due to the structure of the driving (Eq. (5)), which commutes at different times, $[H_{\text{com}}(t), H_{\text{com}}(t')] = 0$. For recent studies of non-commutative driving in quantum heat engines, we refers to [38, 77]. In particular, in Ref. [38], a quantum Otto heat engine that generates coherence in the energy basis due to the non-commutativity of the Hamiltonian was considered. It was shown that the effects of coherence in a finite-time cycle induces fast oscillations in the the figures of merit for the engine performance. In order to focus on the non-Markovian effects due to the engineered cold reservoir, we considered the commutative driving Hamiltonian in (Eq. (5)). The final state is given by $\rho_\tau_1 = \mathcal{U}_{\tau_1,0} \rho_{0_\text{eq},h} \mathcal{U}_{\tau_1,0}^\dagger$, where $\mathcal{U}_{\tau_0} = \Delta > \Delta_0(h\omega_\tau/2)\sigma_z$, such that the resulting work is given by $(W_1) = (\langle \mathcal{U}_{\tau_1} \rangle - \langle \mathcal{U}_{\tau_0} \rangle)$.

In the second stroke the refrigerant system interacts with a cold engineered heat reservoir with an inverse temperature $\beta_0 = 1/k_B T_c$. During this stroke, the refrigerant Hamiltonian is kept fixed at $H(t) = (h\omega_\tau/2)\sigma_z$ along the time $t \in [\tau_1, \tau_2]$. Moreover, some recent developments have pointed out that incomplete (or partial) thermalization can be used to reach a better performance in quantum heat engines [38, 78]. For that reason we consider an incomplete thermalization, denoting by $\rho_{\tau_2}$ the state at the end of this stroke. Furthermore, the structure of the cold reservoir will be set to generate a Markovian or non-Markovian dynamics on the refrigerant substance. This choice is adjusted depending on the ratio of the coupling parameter $J$ between the refrigerant and the structured cold reservoir and the internal coupling $\kappa$ in the cold reservoir [see Fig. 1(b)]. The heat exchanged is given by $(Q_s) = (\langle \mathcal{U}_{\tau_2} \rangle - \langle \mathcal{U}_{\tau_1} \rangle)$.

In the third stroke, the refrigerant is decoupled from the structured cold reservoir and its frequency is increased from $\omega_\tau$ to $\omega_0$ by means of an adiabatic expansion. The responsible Hamiltonian driving this stroke is $H_{\text{exp}}(t) = H_{\text{com}}(\tau_1 + \tau_2 - t)$ for the time interval $t \in [\tau_2, \tau_3]$ and “exp” stands for expansion.

The final state is given by $\rho_{\tau_2} = \mathcal{U}_{\tau_3,\tau_2} \rho_{\tau_2,\tau_3}$, where $\mathcal{U}_{\tau_3,\tau_2} = \mathcal{T}_\tau \exp \left(-i/\hbar \int_{\tau_2}^{\tau_3} H_{\text{exp}}(s) ds\right)$ is the evolution operator associated to the adiabatic expansion. The time duration of the third stroke is assumed to be equal that one for the first stroke. The work in this stroke is $(W_3) = (\langle \mathcal{U}_{\tau_3} \rangle - \langle \mathcal{U}_{\tau_2} \rangle)$.

Finally in the forth stroke, in order to close the cycle, the last stroke comprises a complete thermalization of the refrigerant with the hot reservoir at inverse temperature $\beta_\tau$. During this process the Hamiltonian is kept fixed at $H_{h}(t) = (h\omega_0/2)\sigma_z$ along the time $t \in [\tau_3, \tau_4]$. Once the condition $\tau_4 \gg \tau_{\text{th,rel}} \gg \tau_{\text{th,rel}}$ is fulfilled, the final state is $\rho_{\tau_4} = \rho_{\text{eq},h}$. The heat released to the hot reservoir is given by $(Q_h) = (\langle \mathcal{U}_{\tau_4} \rangle - \langle \mathcal{U}_{\tau_3} \rangle)$.

III. PERFORMANCE OF THE QUANTUM OTTO REFRIGERATOR. NUMERICAL SIMULATION

In this section we discuss the performance of the quantum Otto refrigerator by analyzing the coefficient of performance, injected power into the system and the cooling rate. They are defined by $\epsilon = \langle Q_c \rangle/\langle W_{\text{net}} \rangle = \langle Q_c \rangle/((W_1 + (W_3))$, $\langle P =
\[ (W_{\text{net}})/\tau_{\text{total}}, \text{ and } \langle P \rangle = \langle Q_c \rangle/\tau_{\text{total}}, \text{ respectively, where } \tau_{\text{total}} \text{ stands for the total time of each cycle.} \]

For the finite-time operation of a quantum Otto refrigerator with partial thermalization, the coefficient of performance, the injected power, and the cooling rate can be written as (see Appendix B),

\[
\frac{1}{\epsilon} = \frac{1}{\epsilon_{\text{Carnot}}} + \frac{\langle \Sigma_{\text{total}} \rangle}{\beta_h \langle Q_c \rangle},
\]

(6)

\[
\langle P \rangle = \frac{\langle Q_c \rangle}{\tau_{\text{total}} \epsilon_{\text{Carnot}}} + \frac{\langle \Sigma_{\text{total}} \rangle}{\tau_{\text{total}} \beta_h},
\]

(7)

\[
\langle T \rangle = \frac{\epsilon_{\text{Carnot}} \beta_h \langle Q_c \rangle (W_{\text{net}})}{\tau_{\text{total}} (\beta_h \langle Q_c \rangle + \epsilon_{\text{Carnot}} \langle \Sigma_{\text{total}} \rangle)},
\]

(8)

where \( \epsilon_{\text{Carnot}} = \beta_c / \beta_h - 1 \) is the coefficient of performance for the Carnot refrigerator and \( \langle \Sigma_{\text{total}} \rangle \) is the total entropy production, given by

\[
\langle \Sigma_{\text{total}} \rangle = D (\rho_{\tau_1} || \rho_{\tau_1}^{\text{eq}}) - D (\rho_{\tau_2} || \rho_{\tau_2}^{\text{eq}}) + D (\rho_{\tau_3} || \rho_{\tau_3}^{\text{eq}}). \]

(9)

The expressions in Eqs. (6), (7), (8), and (9) for the coefficient of performance, injected power, cooling rate, and entropy production are general and do not depend on the particular choice for the refrigerant system. In our case, the driving Hamiltonian commutes at different times, \([H_{\text{com}}^m(t), H_{\text{com}}^m(t')] = 0\). This implies an Otto cycle without quantum friction [25, 38, 83–87], and the coefficient of performance of the refrigerator will be the Otto limit irrespective of the choice of time allocation in each stroke, \( \epsilon_{\text{Otto}} = \omega_c / (\omega_h - \omega_c) \). We assume this dynamics for the driving Hamiltonian in order to focus on the non-Markovian aspects in the Otto cycle.

In the following, we performed a numerical simulation of our quantum Otto refrigerator, assuming energy scales compatible with quantum thermodynamics experiments performed in nuclear magnetic resonance setups [42, 79–82]. The initial and final gaps of the compression stroke are chosen as \( \omega_0/2\pi = 3.6 \) kHz and \( \omega_{\tau_1}/2\pi = 2.2 \) kHz, respectively. The cold (\( T_c \)) and hot (\( T_h \)) temperatures are chosen such that \( T_c = 1/(2.5 \omega_{\tau_1}) \) and \( T_h = 1/(2.5 \omega_0) \) with inverse temperatures given by \( \beta_c = 1/T_c \) and \( \beta_h = 1/T_h \). Finally, we assume that the vacuum decay rates of the cold and hot Markovian reservoirs are \( \gamma_c = \gamma_h = 20 \) Hz, where \( \gamma_c = \gamma_h \) from section II.A.

The entropy production \( \langle \Sigma_{\text{total}} \rangle \) covers two different effects of the refrigerator cycle, the finite-time driving in the first and third strokes and the incomplete thermalization with the structured cold reservoir in the second stroke. In Ref. [68] the authors claimed that the entropy production for a non-Markovian dynamics may become negative. For the structured cold reservoir we are employing, the entropy production does not become negative. This can be seen in Fig. 2, which depicts the total entropy production as a function of the cold thermalization time \( \tau_{\text{therm}} \) for three values of the ratio \( J/\kappa \). It is possible to observe two qualitative behaviors: one monotonically increasing and another oscillatory, which are characteristic of a Markovian and non-Markovian dynamics, respectively. Among the non-Markovian regimes, the entropy production rate becomes negative, in agreement with Ref. [68]. The stronger the coupling with the structured cold reservoir the larger the oscillatory behavior. On the other hand, the greatest values of entropy production are close to each other, irrespective of the non-Markovian regimes. In particular, they are near to the asymptotic value of entropy production.

In Fig. 3 we present the cooling rate of the quantum Otto refrigerator model as a function of the cold thermalization time. As expected, the cooling rate \( \langle T \rangle \) decreases for long cold thermalization times, either for the non-Markovian or the Markovian regimes. Similar to the entropy production behavior, the cooling rate also displays an oscillatory profile for the non-Markovian regimes. However, in contrast to the entropy production case, the maximum value of the cooling rate changes with respect to the ratio \( J/\kappa \). In particular, for short cold thermalization time intervals, the stronger the ratio, the larger the cooling rate. Hence, the non-Markovian dynamics induced by the structured cold reservoir may be exploited to enhance the quantum Otto refrigerator performance. The qualitative behavior of the injected power for this numerical example is similar to the cooling rate.
In order to be more detailed in our discussion, we also consider the cooling rate as a function of the coupling constant $J$. In Fig. 4, the cold thermalization times were chosen in order to show that it is possible to obtain a better cooling rate in the non-Markovian case and in short thermalization time.

**IV. CONCLUSIONS**

In this work we investigate a model of a quantum Otto refrigerator where the engineered cold environment is comprised by a Markovian reservoir and an auxiliary two-level system. The coupling between the refrigerant and the structured cold environment determines if the operation regime of the refrigerator is non-Markovian or Markovian.

We present analytical expressions relating the total entropy production along the cycle to the performance of the refrigerator, i.e., the coefficient of performance, the injected power, and the cooling rate. These expressions are valid irrespective of the nature of the quantum refrigerant system. Performing a numerical analysis, we observed an oscillatory behavior for the total entropy production and cooling rate as function of the cold thermalization time in the non-Markovian operation regimes, which is absent in the Markovian regimes. These oscillations arise due to the backflow of information, associated to a non-Markovian dynamics.

The finite-time performance of the present quantum Otto refrigerator may be enhanced by the information backflow provided one has sufficiently control over the time allocated in each stroke. The parameters considered in the numerical simulation can be experimentally realized with current technologies, for instance, in nuclear magnetic resonance setups. Along with other studies addressing non-Markovian effects in quantum thermodynamics, we hope that our analyzes help to unveil the role of memory effects in quantum thermal machines.

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**Appendix A. MASTER EQUATION OF THE NON-MARKOVIAN MODEL**

The master equation of the non-Markovian model employed is given by Eq. (3). Here, we provide some details on the derivation of this master equation. We will follow the approach of Refs. [88, 89]. We begin considering the diagonalization of the two-qubit Hamiltonian [Eq. (2)]

\[
H^{SA} = \sum_{i=S,A} \frac{\hbar \omega_i^2}{2} \sigma_z^i + \hbar J \sigma_z^S \sigma_z^A. \tag{A1}
\]

The eigenvectors and eigenvalues are:

\[
|E_3\rangle = \alpha |00\rangle + \xi |11\rangle \quad \text{and} \quad E_3 = \frac{\hbar}{2} \sqrt{4J^2 + \Omega^2}; \tag{A2}
\]

\[
|E_2\rangle = \eta |01\rangle - \delta |10\rangle \quad \text{and} \quad E_2 = \frac{\hbar}{2} \sqrt{4J^2 + \Delta^2}; \tag{A3}
\]

\[
|E_1\rangle = \delta |01\rangle + \eta |10\rangle \quad \text{and} \quad E_1 = -E_2; \tag{A4}
\]

\[
|E_0\rangle = -\xi |00\rangle + \alpha |11\rangle \quad \text{and} \quad E_0 = -E_3. \tag{A5}
\]

We introduced the parameters $\Delta = \omega^S - \omega^A$, $\Omega = \omega^S + \omega^A$,

\[
\alpha = \frac{\omega^S + \sqrt{4J^2 + \Omega^2}}{\sqrt{4J^2 + (\Omega + \sqrt{4J^2 + \Omega^2})^2}}, \tag{A6}
\]

\[
\xi = \frac{2J}{\sqrt{4J^2 + (\Omega + \sqrt{4J^2 + \Omega^2})^2}}, \tag{A7}
\]

\[
\eta = \frac{\Delta + \sqrt{4J^2 + \Delta^2}}{\sqrt{4J^2 + (\Delta + \sqrt{4J^2 + \Delta^2})^2}}, \tag{A8}
\]

and

\[
\delta = \frac{-2J}{\sqrt{4J^2 + (\Delta + \sqrt{4J^2 + \Delta^2})^2}}. \tag{A9}
\]

The master equation of the two-qubit system is given by [89–91]

\[
\frac{d}{dt} \rho_{ij}^{SA} = -\frac{i}{\hbar} [H^{SA}, \rho_{ij}^{SA}] + D \left( \rho_{ij}^{SA} \right), \tag{A10}
\]
Figure A1. Transition frequencies. Among the six positive frequencies (from a low- to a high-energy level) only four appear on the master equation due to the structure of the interaction Hamiltonian. These are shown in the energy diagram.

where

\[
\mathcal{D}(\rho_t^{SA}) = \sum_{\omega \in \mathcal{F}} \sum_{k,l=1,2} \gamma_{kl} (\omega) \times \left[ A_t (\omega) \rho_t^{SA} A_t^\dagger (\omega) - \frac{1}{2} \left\{ A_t^\dagger (\omega) A_t (\omega), \rho_t^{SA} \right\} \right]
\]  

(A11)

is the dissipator superoperator. We note that we have disregarded the Lamb-shift Hamiltonian since it contributes to an overall energy shift. The set \( \mathcal{F} \) is comprised by all, positive and negative, transition frequencies \( \omega_{nm} = (E_m - E_n) / \hbar \) for \( m, n \in \{0, 1, 2, 3\} \). Since the composite system \( SA \) is a four-level system, there are 12 transition frequencies, 6 positive and 6 negative, \( \omega_{nm} \) and \( \omega_{mn} = -\omega_{nm} \) with \( n < m \), respectively. As we will explain below, among the 6 positive (or negative) transition frequencies only 2 doubly-degenerate ones are relevant to our problem (see Fig. A1). The positive transition frequencies are \( \omega_{01} = \omega_{23} = \epsilon_1 \) and \( \omega_{02} = \omega_{13} = \epsilon_2 \), where

\[
\epsilon_1 = \frac{1}{2} \left[ \sqrt{\Omega^2 + 4J^2} - \sqrt{\Delta^2 + 4J^2} \right]
\]  

(A12)

and

\[
\epsilon_2 = \frac{1}{2} \left[ \sqrt{\Omega^2 + 4J^2} + \sqrt{\Delta^2 + 4J^2} \right].
\]  

(A13)

Before we proceed, we rewrite the interaction Hamiltonian [Eq. (1)] of the two-qubits system and the heat reservoir as

\[
V^{AR} = \int_0^{\omega_{\text{max}}} d\omega \ h (\omega) (\sigma_+ \otimes b_\omega + \sigma_- \otimes b_\omega^\dagger)
\]

\[
= A_1 \otimes B_1 + A_2 \otimes B_2,
\]  

(A14)

where \( \sigma_\pm = (\sigma_x \pm i\sigma_y) / 2 \) and the continuum limit has been taken at the Hamiltonian level [89]. The defined operators are: \( A_1 = \sigma_x^A \), \( A_2 = \sigma_y^A \), \( B_1 = \int_0^{\omega_{\text{max}}} d\omega \ h (\omega) b_\omega + \frac{b_\omega^\dagger}{2} \), and \( B_2 = \int_0^{\omega_{\text{max}}} d\omega \ h (\omega) \frac{b_\omega^\dagger - b_\omega}{2} \). Using the relation \( B_i = \int_0^{\omega_{\text{max}}} d\omega \ h (\omega) B_i (\omega) \), one can obtain the operators \( B_1 (\omega) = \frac{b_\omega h (\omega) b_\omega^\dagger + b_\omega^\dagger h (\omega) b_\omega}{2} \), \( B_1 (-\omega) = \frac{b_\omega h (\omega) b_\omega^\dagger - b_\omega^\dagger h (\omega) b_\omega}{2} \), \( B_2 (\omega) = i \frac{h (\omega) b_\omega^\dagger}{2} \), and \( B_2 (-\omega) = -i \frac{h (\omega) b_\omega^\dagger}{2} \). Since there are two reservoirs operators, \( k, l \in \{1, 2\} \) in Eq. (A11) and, therefore, there are four decay rates for each transition frequency \( \omega \in \mathcal{F} \).

First, we find the decay rates for a given \( \omega \), which are given by the formula [89]

\[
\gamma_{kl} (\omega) = \int_{-\infty}^{+\infty} ds e^{i\omega s} \text{Tr}_R \left[ B_k (s) B_l \rho^{R, eq} \right]
\]

\[
= 2\pi \text{Tr}_R \left[ B_k (\omega) B_l \rho^{R, eq} \right],
\]  

(A15)

where \( \omega \in \mathcal{F} \) and \( \rho^{R, eq} \) is the Gibbs state of the reservoir. Knowing that \( \text{Tr}_R \left[ a_s a_l^\dagger \rho^{R, eq} \right] = [1 + n_{BE} (\omega)] \delta (\omega - \omega') \), \( \text{Tr}_R \left[ a_s a_s^\dagger \rho^{R, eq} \right] = n_{BE} (\omega) \delta (\omega - \omega') \), and \( \text{Tr}_R \left[ a_s a_s^\dagger \rho^{R, eq} \right] = \text{Tr}_R \left[ a_s^\dagger a_s \rho^{R, eq} \right] = 0 \), one can show for the positive transition frequencies \( \omega \in \mathcal{F}_+ = \{ \omega \in \mathcal{F} | \omega \geq 0 \} \) that

\[
\gamma_{11} (\omega) = \frac{\pi}{2} J (\omega) [1 + n_{BE} (\omega)] ,
\]  

(A16)

\[
\gamma_{12} (\omega) = -i \frac{\pi}{2} J (\omega) [1 + n_{BE} (\omega)],
\]  

(A17)

\[
\gamma_{21} (\omega) = i \frac{\pi}{2} J (\omega) [1 + n_{BE} (\omega)],
\]  

(A18)

and

\[
\gamma_{22} (\omega) = \frac{\pi}{2} J (\omega) [1 + n_{BE} (\omega)],
\]  

(A19)

where \( n_{BE} (\omega) = \left( e^{\beta h \omega} - 1 \right)^{-1} \) is the Bose-Einstein distribution and \( J (\omega) = |h (\omega)|^2 \) is the spectral density. On the other hand, for the negative transition frequencies \( \omega \in \mathcal{F}_- = \{ \omega \in \mathcal{F} | \omega < 0 \} \), one finds

\[
\gamma_{11} (-\omega) = \frac{\pi}{2} J (\omega) n_{BE} (\omega),
\]  

(A20)

\[
\gamma_{12} (-\omega) = \frac{\pi}{2} J (\omega) n_{BE} (\omega),
\]  

(A21)

\[
\gamma_{21} (-\omega) = -i \frac{\pi}{2} J (\omega) n_{BE} (\omega),
\]  

(A22)

and

\[
\gamma_{22} (-\omega) = \frac{\pi}{2} J (\omega) n_{BE} (\omega).
\]  

(A23)

Now, we need the operators \( A_k (\omega) \), which come from the system operators \( \sigma_x^A \) and \( \sigma_y^A \). They are given by [89, 90]

\[
A_k (\omega_{nm}) = \sum_{E_m - E_n = \omega_{nm}} \Pi_n A_k \Pi_m ,
\]  

(A24)

where the projection operators are \( \Pi_n = \left| E_n \right> \left< E_n \right| \). Since \( A_1 = 1^S \otimes \sigma_x^A \) and \( A_2 = 1^S \otimes \sigma_y^A \) one can show that

\[
\Pi_n A_1 \Pi_m = \begin{pmatrix}
0 & \alpha \eta - \delta \xi & 0 & \alpha \delta + \eta \xi \\
\alpha \eta - \delta \xi & 0 & 0 & \alpha \delta + \eta \xi \\
0 & 0 & \alpha \delta + \eta \xi & \alpha \eta - \delta \xi \\
0 & \alpha \delta + \eta \xi & \alpha \eta - \delta \xi & 0
\end{pmatrix}
\]  

(A25)
Note how the operators $A_k(\omega)$ associated with the transition frequencies $\omega_{03}$, $\omega_{30}$, $\omega_{12}$, $\omega_{21}$ are identically zero. That is why these transition frequencies are not relevant, i.e., because their associated operators are zero due to the structure of the interaction Hamiltonian. Explicitly, one has

$$A_k(\epsilon_1) = \Pi_0 A_k \Pi_1 + \Pi_2 A_k \Pi_3, \quad (A27)$$

$$A_k(\epsilon_2) = \Pi_0 A_k \Pi_2 + \Pi_1 A_k \Pi_3, \quad (A28)$$

$A_k(-\epsilon_1) = A_k^\dagger(\epsilon_1)$, and $A_k(-\epsilon_2) = A_k^\dagger(\epsilon_2)$, for $k \in \{1, 2\}$. In summary, there are four system operators, two for each nondegenerate transition frequency $\epsilon_1$ and $\epsilon_2$. This means that there will be four dissipative channels in the master equation [see Eq. (3)]. Replacing the decay rates $\gamma_{kl}(\omega)$ and system operators $A_k(\omega)$, one obtains the master equation employed, whose dissipator is given by

$$D \left( \rho_t \right) \dot{=} \gamma_1(\epsilon_1) \left[ L_1(\epsilon_1) \rho_t \frac{S^A}{L_1(\epsilon_1)} - \frac{1}{2} \left\{ L_1(\epsilon_1) \right\} L_1(\epsilon_1), \rho_t \frac{S^A}{L_1(\epsilon_1)} \right] + \gamma_2(\epsilon_2) \left[ L_2(\epsilon_2) \rho_t \frac{S^A}{L_2(\epsilon_2)} - \frac{1}{2} \left\{ L_2(\epsilon_2) \right\} L_2(\epsilon_2), \rho_t \frac{S^A}{L_2(\epsilon_2)} \right]$$

$$+ \gamma_1(\epsilon_1) \left[ L_1(\epsilon_1) \rho_t \frac{S^A}{L_1(\epsilon_1)} - \frac{1}{2} \left\{ L_1(\epsilon_1) \right\} L_1(\epsilon_1), \rho_t \frac{S^A}{L_1(\epsilon_1)} \right] + \gamma_2(\epsilon_2) \left[ L_2(\epsilon_2) \rho_t \frac{S^A}{L_2(\epsilon_2)} - \frac{1}{2} \left\{ L_2(\epsilon_2) \right\} L_2(\epsilon_2), \rho_t \frac{S^A}{L_2(\epsilon_2)} \right], \quad (A29)$$

where $\gamma_1(\epsilon_k) = \frac{\epsilon}{2} J(\epsilon_k) n_{BE}(\epsilon_k)$, $\gamma_2(\epsilon_k) = \frac{\epsilon}{2} J(\epsilon_k) \left[ 1 + n_{BE}(\epsilon_k) \right]$, $L_1(\epsilon_1) = 2\alpha\eta_{\text{E0}}(E_0|E_1) + |E_1\rangle \langle E_1|$, and $L_2(\epsilon_2) = 2\alpha\delta(-|E_0\rangle \langle E_2| + |E_1\rangle \langle E_2|)$. We considered the spectral density to be $J(\omega) = \kappa/\pi$.

### Appendix B. COEFFICIENT OF PERFORMANCE, INJECTED POWER AND COOLING RATE

Here we derive the expressions for the coefficient of performance, injected power, and cooling rate in terms of the total entropy production, eqs.(6), (7) and (8), respectively. Using that the internal energy of the refrigerant in a given moment is $U_t = Tr[\rho_t H_t]$, the coefficient of performance is written as,

$$\epsilon = \frac{(Q_c)}{(W_{\text{net}})} = \frac{U(\rho_{\tau_2}) - U(\rho_{\tau_1})}{U(\rho_{\tau_1}) - U(\rho_{\tau_2}) + U(\rho_{\tau_2}) - U(\rho_{\tau_3})}. \quad (A30)$$

From $D(\rho_t || \rho_t^{\text{eq}}) = \beta[U(\rho_t) - F_{\tau_1}^{\text{eq}}] - S(\rho_t)$, one gets,

$$D(\rho_{\tau_1} || \rho_{\tau_2}^{\text{eq}}) = \beta_c[U(\rho_{\tau_1}) - F_{\tau_1}^{\text{eq}}] - S(\rho_{\tau_1}),$$

$$D(\rho_{\tau_2} || \rho_{\tau_2}^{\text{eq}}) = \beta_c[U(\rho_{\tau_2}) - F_{\tau_2}^{\text{eq}}] - S(\rho_{\tau_2}).$$

Then,

$$[U(\rho_{\tau_2}) - U(\rho_{\tau_1})] = \beta_c^{-1}[D(\rho_{\tau_2} || \rho_{\tau_2}^{\text{eq}}) - D(\rho_{\tau_1} || \rho_{\tau_2}^{\text{eq}})]$$

$$+ \beta_c^{-1}[S(\rho_{\tau_2}) - S(\rho_{\tau_1})]$$

where we assumed that $F_{\tau_2}^{\text{eq}} = F_{\tau_1}^{\text{eq}}$. In a similar way,

$$[U(\rho_{\tau_3}) - U(\rho_{\tau_1})] = \beta_h^{-1}[D(\rho_{\tau_3} || \rho_{\tau_2}^{\text{eq}}) - D(\rho_{\tau_1} || \rho_{\tau_3}^{\text{eq}})]$$

$$+ \beta_h^{-1}[S(\rho_{\tau_3}) - S(\rho_{\tau_1})]$$

where we assumed that $F_{\tau_3}^{\text{eq}} = F_{\tau_1}^{\text{eq}}$. Then,

$$(Q_c) = \beta_c^{-1}[D(\rho_{\tau_2} || \rho_{\tau_2}^{\text{eq}}) - D(\rho_{\tau_1} || \rho_{\tau_2}^{\text{eq}})] + \beta_c^{-1}[S(\rho_{\tau_2}) - S(\rho_{\tau_1})],$$

$$(Q_h) = \beta_h^{-1}[D(\rho_{\tau_3} || \rho_{\tau_2}^{\text{eq}}) - D(\rho_{\tau_1} || \rho_{\tau_3}^{\text{eq}})] + \beta_h^{-1}[S(\rho_{\tau_3}) - S(\rho_{\tau_1})].$$

Note that, from the first law of thermodynamics, $(W_{\tau_2}) + (W_{\tau_3}) = -(Q_h) - (Q_c)$.

We can express the coefficient of performance as,

$$\frac{1}{\epsilon} = -1 - \frac{\beta_c}{\beta_h} \left( \frac{(Q_c)}{(Q_h)} \right) - \frac{(\Sigma_{\text{total}})}{(Q_c)}$$

and using that $S(\rho_{\tau_2}) = S(\rho_{\tau_1})$ and $S(\rho_{\tau_3}) = S(\rho_{\tau_1})$ and $D(\rho_{\tau_3} || \rho_{\tau_2}^{\text{eq}}) = 0$:

$$\frac{1}{\epsilon} = \frac{1}{\epsilon_c} + \frac{(\Sigma_{\text{total}})}{(Q_c)}.$$
[1] W. Thomson, On the Dynamical Theory of Heat, with numerical results deduced from Mr. Joule’s Equivalent of a Thermal Unit, and M. Regnault’s Observation on Steam, Transactions of the Royal Society of Edinburgh. 20, 261 (1851).

[2] S. Carnot, Reflections on the motive power of heat. (J. Wiley & Sons, New York, 1890).

[3] Y. A. Çengel and M. A. Boles, Thermodynamics An Engineering Approach 8th edition (McGraw-Hill Education, New York, 2015).

[4] D. Kondepudi and I. Prigogine, Modern Thermodynamics From Heat Engines to Dissipative Structures 2nd edition (John Wiley & Sons Ltd, Chichester, 2015).

[5] J. P. Dowling, G. J. Milburn, Quantum technology: the second quantum revolution, Philos. Trans. Royal Soc. A 361, 1655 (2003).

[6] I. Georgescu and F. Nori, Quantum technologies: an old new story, Physics World 25, (05) 16 (2012).

[7] J. Jones, The second quantum revolution, Phys. World 26 (08) 40 (2013).

[8] M. Esposito, U. Harbola, and S. Mukamel, Nonequilibrium fluctuations, fluctuation theorems, and counting statistics in quantum systems, Rev. Mod. Phys. 81, 1665 (2009).

[9] M. Campisi, P. Hänggi, and P. Talkner, Colloquium: Quantum fluctuation dynamics and work fluctuations with applications to mag- netic resonance, Am. J. Phys. 84, 948 (2016).

[10] R. Kosloff, Quantum Thermodynamics: A Dynamic Viewpoint, Entropy 15(6), 2100 (2013).

[11] D. Gelbwaser-Klimovsky, W. Niedenzu and G. Kurizki, Chapter Twelve - Thermodynamics of Quantum Systems Under Dy- namical Control, Adv. At. Mol. Opt. Phys. 64, 329 (2015).

[12] J. Millen and A. Xuereb, Perspective on quantum thermody- namics, New J. Phys. 18, 011002 (2016).

[13] S. Vinjanampathy and J. Anders, Quantum Thermodynam- ics, Contemp. Phys. 57, 545 (2016).

[14] W. L. Ribeiro, G. T. Landi and F. L. Semião, Quantum thermodynamics and work fluctuations with applications to mag- netic resonance, Am. J. Phys. 84, 948 (2016).

[15] H. E. D. Scovil and E. O. Schulz-DuBois, Three-Level Masers as Heat Engines, Phys. Rev. Lett. 2, 262 (1959).

[16] J. E. Geusic, E. O. Schulz-DuBios, and H. E. D. Scovil, Quantum Equivalent of the Carnot Cycle, Phys. Rev. 156, 343 (1967).

[17] R. Allicki, The quantum open system as a model of the heat engine, J. Phys. A: Math. Gen. 12, L103 (1979).

[18] R. Kosloff, A quantum mechanical open system as a model of a heat engine, J. Chem. Phys. 80, 1625 (1984).

[19] E. Geva and R. Kosloff, A quantum-mechanical heat engine operating in finite time. A model consisting of spin-1/2 systems as the working fluid, J. Chem. Phys. 96, 3054 (1992).

[20] E. Geva and R. Kosloff, On the classical limit of quantum ther- modynamics in finite time, J. Chem. Phys. 97, 4398 (1992).

[21] E. Geva and R. Kosloff, The quantum heat engine and heat pump: An irreversible thermodynamic analysis of the three- level amplifier, J. Chem. Phys. 104, 7681 (1996).

[22] R. Kosloff, E. Geva, and J. M. Gordon, Quantum refrigerators in quest of the absolute zero, J. Appl. Phys. 87, 8093 (2000).

[23] M. O. Scully, Quantum Afterburner: Improving the Efficiency of an Ideal Heat Engine, Phys. Rev. Lett. 88, 050602 (2002).

[24] N. Sánchez-Salas and A. Calvo Hernández, Harmonic quantum heat devices: Optimum-performance regimes, Phys. Rev. E 70, 046134 (2004).

[25] T. Feldmann and R. Kosloff, Quantum lubrication: Suppres- sion of friction in a first-principles four-stroke heat engine, Phys. Rev. E 73, 025107(R) (2006).

[26] Z. Ting, C. Li-Feng, C. Ping-Xing and L. Cheng-Zu, The Second Law of Thermodynamics in a Quantum Heat Engine Model, Commun. Theor. Phys. 45, 417 (2006).

[27] H. T. Quan, Y.-X. Liu, C. P. Sun, and F. Nori, Quantum thermodynamic cycles and quantum heat engines, Phys. Rev. E 76, 031105 (2007).

[28] A. E. Allahverdyan, R. S. Johal, and G. Mahler, Work extremum principle: Structure and function of quantum heat engines, Phys. Rev. E 77, 041118 (2008).

[29] N. Linden, S. Popescu, and P. Skrzypczyk, How Small Can Thermal Machines Be? The Smallest Possible Refrigerator, Phys. Rev. Lett. 105, 130401 (2010).

[30] N. Brunner, N. Linden, S. Popescu, and P. Skrzypczyk, Virtual qubits, virtual temperatures, and the foundations of thermo- dynamics. Phys. Rev. E 85, 051117 (2012).

[31] J.-Y. Du and F.-L. Zhang, Nonequilibrium quantum absorption refrigerator, New J. Phys. 20, 063005 (2018).

[32] O. Abah and E. Lutz, Optimal performance of a quantum Otto refrigerator, Eur. Phys. Lett. 113, 60002 (2016).

[33] P. A. Erdman, V. Cavina, R. Fazio, F. Taddei, and V. Giovannetti, Maximum Power and Corresponding Efficiency for Two-Level Quantum Heat Engines and Refrigerators, https://arxiv.org/abs/1812.05089 (2018).

[34] J. B. Brask and N. Brunner, Small quantum absorption refrig- erator in the transient regime: Time scales, enhanced cooling, and entanglement, Phys. Rev. E 92, 062101 (2015).

[35] R. Kosloff and A. Levy, Quantum heat engines and refrig- erators: Continuous devices, Annual Rev. Phys.Chem. 65, 1 (2014).

[36] M. T. Mitchison, M. P. Woods, J. Prior and M. Huber, Coherence-assisted single-shot cooling by quantum absorption refrigerators, New J. Phys. 17, 115013 (2015).

[37] A. Levy and R. Kosloff, Quantum Absorption Refrigerator, Phys. Rev. Lett. 108, 070604 (2012).

[38] P. A. Camati, J. F. G. Santos, and R. M. Serra, Coherence effects in the performance of the quantum Otto heat engine, Phys. Rev. A 99, 062103 (2019).

[39] J. Roñkagel, S. T. Dawkins, K. N. Tolazzi, O. Abah, E. Lutz, F. Schmidt-Kaler, K. Singer, A single-atom heat engine, Science 352, 325 (2016).

[40] A. Argun, J. Soni, L. Dabelow, S. Bo, G. Pesce, R. Eichhorn, and G. Volpe, Experimental realization of a minimal microscopic heat engine, Phys. Rev. A 96, 052106 (2017).

[41] G. Maslennikov, S. Ding, R. Hablützel, J. Gan, A. Roulet, S. Nimrichter, J. Dai, V. Scarani, D. Matsukevich, Quantum absorption refrigerator with trapped ions, Nat. Commun. 10, 202 (2019).

[42] J. P. S. Peterson, T. B. Batalhão, M. Herrera, A. M. Souza, R. S. Sathour, I. S. Oliveira, and R. M. Serra, Experimental characterization of a spin quantum heat engine, Phys. Rev. Lett. 123, 240601 (2019).

[43] J. Klaers, S. Faelt, A. Imamoglu, and E. Togan, Squeezed Thermal Reservoirs as a Resource for a Nanomechanical Engine beyond the Carnot Limit, Phys. Rev. X 7, 031044 (2017).

[44] J. Klatzow, J. N. Becker, P. M. Ledingham, C. Weizelt, K. T. Kaczmarek, D. J. Saunders, J. Nunn, I. A. Walmsley, R. Uzdin, Y. Poem, Experimental demonstration of quantum effects in beyond the Carnot Limit, Phys. Rev. X 7, 031044 (2017).

[45] O. Abah and E. Lutz, Efficiency of heat engines coupled to nonequilibrium reservoirs, EPL 106, 20001 (2014).

[46] L. A. Correa, J. P. Palao, D. Alonso and G. Adesso, Quantum- enhanced absorption refrigerators, Sci. Rep. 4, 3949 (2014).

[47] J. Roñkagel, O. Abah, F. Schmidt-Kaler, K. Singer, and E. Lutz, Nanoscale Heat Engine Beyond the Carnot Limit, Phys.
R. Alicki and W. Liu, Performance of quantum Otto refrigerators with squeezing, Phys. Rev. E 91, 062137 (2015).

W. Niedenzu, D. Gelbwaser-Klimovsky, A. G. Kofman, and G. Kurizki, On the operation of machines powered by quantum non-thermal baths, New J. Phys. 18, 083012 (2016).

G. Manzano, F. Galve, R. Zambrini, and J. M. R. Parrondo, Entropy production and thermodynamic power of the squeezed thermal reservoir, Phys. Rev. E 93, 052120 (2016).

R. Kosloff and Y. Rezek, The Quantum Harmonic Otto Cycle, Entropy 19(4), 136 (2017).

C. L. Latune, I. Sinayskiy, F. Petruccione, Extended Carnot bound for autonomous quantum refrigerators powered by non-thermal states, arXiv:1801.10113v1 (2018).

O. Abah and E. Lutz, Efficiency of heat engines coupled to nonequilibrium reservoirs, Eur. Phys. Lett. 116, 20001 (2014).

M. O. Scully, M. S. Zubairy, G. S. Agarwal, H. Walther, Extracting Work from a Single Heat Bath via Vanishing Quantum Coherence, Science 299, 862 (2003).

K. V. Kepesidis, M.-A. Lemonde, A. Norambuena, J. R. Maze, P. Rabl, Cooling phonons with phonons: acoustic reservoir-engineering with silicon-vacancy centers in diamond, Phys. Rev. B 94, 214115 (2016).

M. Haeberlein et al, Spin-boson model with an engineered reservoir in circuit quantum electrodynamics, arXiv:1506.09114 (2015).

C. Elouard, N. K. Bernardes, A. R. R. Carvalho, M. F. Santos Jr., and A. Auffèves, Probing quantumfluctuation theorems in engineered reservoirs, New. J. Phys. 19, 103011 (2017).

S. Hamedani Raja, M. Borrelli, R. Schmidt, J. P.pekola, S. Maniscalco, Thermodynamic fingerprints of non-Markovianity, Phys. Rev. A 97, 032133 (2018).

B.-Heng Liu et al, Efficient suprerdense coding in the presence of non-Markovian noise, Eur. Phys. Lett. 114,10005 (2016).

R. Sampaio, S. Suomela, R. Schmidt, and T. Ala-Nissila, Quantifying non-Markovianity due to driving and a finite-size environment in an open quantum system, Phys. Rev. E 95,022120 (2017).

R. S. Whitney, Non-Markovian quantum thermodynamics: Laws and fluctuation theorems, Phys. Rev. E 98, 085145 (2018).

H.-B. Chen, G.-Y. Chen, Y.-N. Chen, Thermodynamic description of non-Markovian information flux of non-equilibrium open quantum systems, Phys. Rev. A 96,062114 (2017).

S. Bhattacharyya, A. Misra, C. Mukhopadhyay, and A. K. Pati, Exact master equation for a spin interacting with a spin bath: Non-Markovianity and negative entropy production rate, Phys. Rev. A 95,012122 (2017).

S. Marcantoni, S. Alipour, F. Benatti, R. Floreanini and A. T. Rezakhani, Entropy production and non-Markovian dynamical maps, Sci. Rep. 7, 12447 (2017).

G. Thomas, N. Siddharth, S. Banerjee, and S. Ghosh, Thermodynamics of non-Markovian reservoirs and heat engines, Phys. Rev. E 97, 062108 (2018).

P. Abiuso and V. Giovannetti, Non-Markov enhancement of maximum power for quantum thermal machines, Phys. Rev. A 99, 052106 (2019).

O. Abah and M. Paternostro, Implications of non-Markovian dynamics on information-driven engine, https://arxiv.org/abs/1902.06153 (2019).

M. M. Wolf, J. Eisert, T. S. Cubitt, and J. I. Cirac, Assessing Non-Markovian Quantum Dynamics, Phys. Rev. Lett. 101, 150402 (2008).

H.-P. Breuer, E.-M. Laine, and J. Piilo, Measure for the Degree of Non-Markovian Behavior of Quantum Processes in Open Systems, Phys. Rev. Lett. 103, 210401 (2009).

E.-M. Laine, J. Piilo, and H.-P. Breuer, Measure for the non-Markovianity of quantum processes, Phys. Rev. A 86, 062115 (2010).

Nielsen, M. A. & Chuang, I. L. Quantum Computation and Quantum Information (Cambridge University Press, Cambridge, 2010).

S. Wißmann, A. Karlsson, E.-M. Laine, J. Piilo, and H.-P. Breuer, Optimal state pairs for non-Markovian quantum dynamics, Phys. Rev. A 86,062108 (2012).

R. Dann and R. Kosloff, Quantum Signatures in the Quantum Carnot Cycle, New J. Phys. 22, 013055 (2020).

E. Bäumer, M. Perarnau-Llobet, P. Kammerlander, H. Wüling, and R. Renner, Imperfect Thermalizations Allow for Optimal Thermodynamic Processes, Quantum 3, 153 (2019).

T. B. Batalhão, A. M. Souza, L. Mazzola, R. Aurcaise, R. S. Sarthour, I. S. Oliveira, J. Goold, G. De Chiara, M. Paternostro, and R. M. Serra, Experimental Reconstruction of Work Distribution and Study of Fluctuation Relations in a Closed Quantum System, Phys. Rev. Lett. 113, 140601 (2014).

T. B. Batalhão, A. M. Souza, R. S. Sarthour, I. S. Oliveira, M. Paternostro, E. Lutz, and R. M. Serra, Irreversibility and the arrow of time in a quenched quantum system, Phys. Rev. Lett. 115,190601 (2015).

P. A. Camati, J. P. S. Peterson, T. B. Batalhão, K. Micadei, A. M. Souza, R. S. Sarthour, I. S. Oliveira, and R. M. Serra, Experimental Rectification of Entropy Production by Maxwell’s Demon in a Quantum System, Phys. Rev. Lett. 117, 240502 (2016).

K. Micadei, J. P. S. Peterson, A. M. Souza, R. S. Sarthour, I. S. Oliveira, and R. M. Serra, Experimental Reconstruction of Entropy Production by Maxwell’s Demon in a Quantum System, Phys. Rev. Lett. 117, 240502 (2016).

R. Kosloff and T. Feldmann, Discrete four-stroke quantum heat engine exploring the origin of friction, Phys. Rev. E 65, 055102(R) (2002).

T. Feldmann and R. Kosloff, Quantum four-stroke heat engine: Thermodynamic observables in a model with intrinsic friction, Phys. Rev. E 68, 061601 (2003).

A. del Campo, A. Chenu, S. Deng, and H. Wu, Friction-free quantum machines, in Thermodynamics in the Quantum Regime, edited by F. Binder, L. A. Correa, C. Gogolin, J. Anders, and G. Adesso (Springer, New York, 2018).

G. Thomas and R. S. Johal, Friction due to inhomogeneous driving of coupled spins in a quantum heat engine, Eur. Phys. J. B 87, 166 (2014).

F. Plastina, A. Alecce, T. J. G. Apollaro, G. Falcone, G. Francica, F. Galve, N. Lo Gullo, and R. Zambrini, Irreversible Work and Inner Friction in Quantum Thermodynamic Processes, Phys. Rev. Lett. 113, 260601 (2014).

A. Rivas, A. D. K Plato, S. F. Huelga, and M. B. Plenio, Markovian master equations: a critical study, New J. Phys. 12 113032 (2010).

A. Rivas and S. F. Huelga, Open Quantum Systems An Introduction, (Springer, Heidelberg, 2012).

H.-P. Breuer and F. Petruccione, The Theory of Open Quantum Systems, (Oxford University Press, New York, 2002).

R. Alicki and K. Lendi, Quantum Dynamical Semigroups and Applications, (Springer, Heidelberg, 2007).