An interactive version of Lovász local lemma

*Arthur and Merlin implement Moser’s algorithm*

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Abstract

Assume we are given (finitely many) mutually independent variables and (finitely many) “undesirable” events each depending on a subset of the variables of at most \(k\) elements, known as the scope of the event. Assume that the probability of an individual variable belonging to the scope of an occurring event is bounded by \(q\). We prove that if \(ekq \leq 1\) then there exists at least one assignment to the variables for which none of the events occurs. This result is stronger than the classical version of the Lovász local lemma, which is expressed in terms of a bound \(p\) of the probabilities of the individual events, and of \(d\), a bound on the degree of the dependency graph. The proof is through a public coin, interactive implementation of the algorithm by Moser. The original implementation, which yields the classical result, finds efficiently, but probabilistically, an assignment to the events that avoids all undesirable events. Interestingly, the interactive implementation given in this work does not constitute an efficient, even if probabilistic, algorithm to find an assignment as desired under the weaker assumption \(ekq \leq 1\). We can only conclude that under this hypothesis, the interactive protocol will produce an assignment as desired within \(n\) rounds, with probability high with respect to \(n\); however, the provers’ (Merlin’s) choices remain non-deterministic. Plausibly finding such an assignment is inherently hard, as the situation is reminiscent, in a probabilistic framework, of problems complete for syntactic subclasses of TFNP.

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1 Preliminaries

Assume we have mutually independent variables $X_1, \ldots, X_l$, and “undesirable” events $E_1, \ldots, E_m$, each depending on a subset of variables, its scope. The scope of an event $E$ is denoted by $\text{sc}(E)$. Assume all scopes have cardinality at most $k$, with $k$ a positive integer. Assume also that the probability of at least one of the events depending on a variable to occur, is bounded by $q$. Formally, $q$ is such that:

$$\Pr\left[ \bigcup_{E : X \in \text{sc}(E)} E \right] \leq q,$$

for all $X \in \{X_1, \ldots, X_l\}$.

Example 1. As a toy example that we will be used throughout the paper to illustrate various notions, let $X_1, X_2, X_3, X_4$ be random variables, taking values in $\{0, 1\}$ uniformly at random and let $\Omega = \{0, 1\}^4$ be the probability space of all possible assignments of values to the variables. Let also $E_j$, $j = 1, 2, 3, 4, 5$ be the “undesirable” events, where:

$$E_1 = \{X_1 = 1\}, \ E_2 = \{X_2 = 1\}, \ E_3 = \{X_1 = X_2 = 1\},$$

$$E_4 = \{X_3 = 1\} \ \text{and} \ E_5 = \{X_3 = X_4 = 1\}.$$

It is straightforward to check that $\Pr[E_j] = \frac{1}{2}$, for $j \in \{1, 2, 4\}$ and that $\Pr[E_j] = \frac{1}{4}$, for $j = 3, 5$.

Let $E_i$ hold when at least one event depending on $X_i$ occurs, $i = 1, 2, 3, 4$.

Easily:

$$\Pr[E_1] = \Pr[E_2] = \Pr[E_3] = \frac{1}{2}, \ \Pr[E_4] = \frac{1}{4},$$

Thus,

$$q = \frac{1}{2}.$$

Our main result is the following variant of Lovász Local Lemma (LLL):

**Theorem 1.** If $ekq \leq 1$, then there exists an assignment to the variables $X_1, \ldots, X_l$ such that none of the events $E_1, \ldots, E_m$ occurs (we say that such an assignment avoids all undesirable events).

Before delving into the proof let us notice that all classical variants of LLL give sufficient (and in the most general case, the Shearer version [15], necessary) conditions for the existence of an assignment that avoids all undesirable events in terms of the probabilities of the events and the structure of a graph that reflects their mutual dependencies. For example, in one of its simplest forms, LLL states that if $p$ bounds the probabilities of all events and if each event depends on at most $d$ of the others and
if \(e(d + 1)p \leq 1\), then there is an assignment that avoids all undesirable events. Whereas Theorem 1, which is stated in the framework of events dependent on mutually independent variables as described above, gives a sufficient condition in terms of an upper bound for the probability that each individual variable belongs to the scope of at least one occurring event and in terms of a bound on the number of variables in the scope of each event.

To see how Theorem 1 can prove advantageous over the classical result above, observe that if we compute \(kq\) by using union-bound for \(q\), we get a quantity that is close to \((d + 1)p\): indeed by union-bound, \(kq\) is bounded by \(ksp\), where \(s\) is a bound of the number of events that can depend on a particular variable; on the other hand, \(d\) is at most \(k(s - 1)\). Note that in Example 1 above, since \(s = 2\) and \(p = \frac{1}{2}\), by applying union bound we would get the trivial result that \(q = 1\). There are also interesting (non-toy) examples where \(q\) can be bound more cleverly than applying union-bound; it is in such cases where we expect Theorem 1 to prove advantageous. Such cases are, e.g., the problem of estimating the acyclic chromatic number of a graph (least number of colors needed to properly color the vertices of a graph so that no cycle with only two colors exists) as a function of the max degree (see [7] for the best bound to date) or problems related with satisfiability of Boolean formulas. Applying our approach to get improved results for these problems is the subject of ongoing research. We should point out that in [5] a similar method was applied to improve the acyclic chromatic index of a graph (least number of colors needed for a proper coloring of its edges without bichromatic cycles). However, because once its two colors are given, there exists at most one bichromatic cycle starting from an edge, no need for an interactive protocol arose in that case.

For the above classical variant of LLL, much later than the original non-constructive proof and in the independent variable framework described above, Moser [11] provided an efficient probabilistic algorithm that finds an assignment that avoids all undesirable events. Analogous efficient probabilistic algorithms were also given for other similar variants where we assume that the probabilities of events are either given or bounded, see e.g. [4, 10, 12]. Very roughly, such algorithms start from an arbitrary sampling of all variables, and then as long as an event that occurs is still present, the variables in the event’s scope are resampled. This extremely simple process is shown to come to a halt with high probability by first counting the structures that reflect the inter-dependencies of the events chosen for resampling; these structures became known as witness structures. Then, using an entropy argument (see [17]) it is shown that for a large enough \(n\), an assignment for which no undesirable event occurs will be produced within \(n\) steps (for a direct probabilistic argument that avoids the entropic method, see for example Giotis et al. [4]).

This counting (or probability calculation as in Giotis et al. [4]) however strongly depends on knowing at least an upper bound for the probability
of all events. In Theorem 1 above no such bound is assumed to be given. To deal with this situation, we introduce an interactive protocol, with two agents that check the independent variables in a recursive fashion:

- Intuitively, the first agent, Merlin, aims to prove that there is no way to avoid all the undesirable events, i.e. that for each assignment of values to the variables, at least one event occurs. When variable $X$ is examined, he points on an event $E$ depending on $X$ that occurs, as proof that there is still an occurring event.

- The second agent, Arthur, randomly resamples the variables of the event that Merlin pointed to him.

The Interactive Protocol 1 in section 2 describes formally the interaction between Arthur and Merlin. Notice that Merlin makes his choices knowing the current assignment, since in line 7 he chooses an occurring event, so this is a public coin interactive protocol (see [6]). Also we assume that Merlin never cheats, i.e. he faithfully follows the protocol, and that, in particular, if there is an occurring event he can choose, he does so. Finally, let us stress that in contrast to the classical interactive protocols, it is not Merlin's unlimited computing power that is essential here (after all, all we ask from him is to find occurring events under a given assignment to the variables); his essential feature here is his “externality” to the random choices made by Arthur, due to which biases are not introduced to the distribution of variables and so a probabilistic analysis becomes possible (this will become clearer in our second Interactive Protocol 2, since in Interactive Protocol 1, biases do exist, see Remark 1 in Subsection 3.1).

Finally, with respect to how our result compares with other extant stronger variants of LLL (of which there are plenty, for an overview see, e.g. [16]), we do not know if any of them implies or is implied by our result. Plausibly they do not compare, as they are all expressed in terms of the probabilities of individual events, and not in terms of $q$. Also, although there exist efficient algorithmic proofs for the known results, plausibly there is none for ours (see the last paragraphs of Section 2).

2 The Interactive Protocol

Consider the Interactive Protocol 1 below.
Interactive Protocol 1 Arthur & Merlin implement Moser’s Alg.

1: **Arthur:** Sample the variables $X_1, \ldots, X_l$; let $\alpha$ be the assignment thus obtained.
2: **while** Merlin can find a variable with an occurring event dependent on it, let $X_i$ be the least indexed such variable\(^1\) and **do**
3: \hspace{1em} Redo($X_i$)
4: **end while**
5: **halt**
6: **procedure** Redo($X_i$)
7: \hspace{1em} **Merlin:** Choose an occurring event dependent on $X_i$, call it $E$
8: \hspace{1em} **Arthur:** Resample the variables in the scope $\text{sc}(E)$ and change the variables in $\alpha$ accordingly
9: \hspace{1em} **while** Merlin can find a variable in $\text{sc}(E)$ with an occurring event dependent on it, let $X_r$ be the least indexed such variable\(^1\) and **do**
10: \hspace{2em} Redo($X_r$)
11: **end while**
12: **end procedure**

The protocol terminates, if and when it reaches line 5 of its execution. Obviously, if and when that happens, it produces an assignment of values $\alpha$ to the variables, such that no event occurs.

**Example 2.** Consider the setting of Example 1. We will describe an execution of the Protocol, that leads to an assignment of values such that no $E_j$ occurs.

0. Suppose Arthur’s first sampling results in the assignment

$$\alpha_0 = (1,1,1,0).$$

Then, $X_1$ is the variable selected at line 2 of the Protocol.

1. Merlin can choose either $E_1$ or $E_3$. Suppose he chooses $E_3$ and that, after Arthur resamples the variables of its scope, the assignment does not change, i.e.:

$$\alpha_1 = \alpha_0 = (1,1,1,0).$$

$X_1$ is thus selected again.

2. Merlin now chooses $E_1$. Suppose that:

$$\alpha_2 = (0,1,1,0).$$

\(^1\)Instead of choosing the least indexed variable, we could have left the choice to Merlin’s discretion; no essential difference arises.
Now, both events that depend on $X_1$ do not occur. Thus, $X_2$ is now selected at line 9 of the Redo($X_1$) procedure.

3. Merlin chooses $E_2$ and, after Arthur’s resampling of its scope, suppose we get (again):
   \[ \alpha_3 = (0,1,1,0). \]
   $X_2$ is selected.

4. Merlin chooses $E_2$ and, after the resampling, we get:
   \[ \alpha_4 = (0,0,1,0). \]
   The Protocol now exits both the Redo($X_2$) procedure and the Redo($X_1$) procedure. $X_3$ is selected.

5. Merlin can only choose $E_4$ and suppose:
   \[ \alpha_5 = \alpha_4 = (0,0,1,0). \]
   $X_3$ is selected again.

6. Merlin chooses $E_4$ and suppose:
   \[ \alpha_6 = (0,0,0,0). \]
   Under the assignment $\alpha_6$, no undesirable event occurs and the Protocol halts.

**Lemma 1.** Consider an execution of the Interactive Protocol 1 and suppose that at some point, it enters procedure Redo($X_i$), for some $i \in \{1, \ldots, l\}$. Let also $\mathcal{X}$ be the set of variables that have no occurring events depending on them at the beginning of procedure Redo($X_i$). Then, under the assignment produced if and when Redo($X_i$) ends, no event that depends on any variable in $\mathcal{X} \cup \{X_i\}$ occurs.

**Proof.** Let $Y \in \mathcal{X} \cup \{X_i\}$ and suppose that Redo($X_i$) terminates and that $Y$ occurs under the current assignment $\alpha$.

$Y \neq X_i$, else the Redo($X_i$) procedure couldn’t have exited the while-loop at line 9 and thus wouldn’t have terminated.

Thus, there exists some $j \neq i$ such that $Y = X_j$. Since $X_j \in \mathcal{X}$, under the assignment when the Redo($X_i$) procedure was initiated, no event that depended on $X_j$ occurred. So, it must be the case that during this procedure, some resampling of variables caused an event that depends on $X_j$ to occur. Let Redo($X_j$) be the last time this happened and suppose $E$ was the event chosen by Merlin at this procedure. Let also $E'$ be the event depending on $X_j$ that became occurring and thus remained occurring until the end of Redo($X_i$).
By the above, it holds that \( \text{sc}(E) \cap \text{sc}(E') \neq \emptyset \). Let \( Z \in \text{sc}(E) \cap \text{sc}(E') \). Then, \( Z \) is a variable that the protocol checks, during the while-loop of \( \text{REDO}(X_r) \), and has an occurring event depending on it. Thus, \( \text{REDO}(X_r) \) couldn’t have terminated and so neither could have \( \text{REDO}(X_i) \). Contradiction.

Note that Lemma 1 has two implications. First, every \( \text{REDO} \) procedure does indeed make some progress, by adding at least one more variable to the set of those that have no occurring events depending on them. Secondly, this progress is “preserved”, in the sense that once a variable stops having any occurring events that depend on it, no subsequent \( \text{REDO} \) procedure will undo that.

A *round* of the Interactive Protocol 1 is the duration of any \( \text{REDO} \) procedure, initiated either at line 6 or 10 of its execution. Thus, the \( i \)-th round is the duration of the \( i \)-th \( \text{REDO} \) procedure that the Interactive Protocol 1 initiates. Also the execution of line 1 together with the determination of a variable in line 2 constitute round 0. A \( \text{REDO} \) procedure, initiated at line 6 will be called a *root procedure*, while one initiated at line 10 will be called *recursive*.

Complexity considerations below will all be in terms of the number of rounds \( n \).

An immediate corollary of Lemma 1 is the following:

**Corollary 1.** The number of root \( \text{REDO} \) procedures is at most \( l \), the number of variables.

Following a standard practice in interactive protocols, we will sometimes refer to Merlin’s choices as *non-deterministic choices* and to Arthur’s as *random choices* (see e.g. [3]).

Let \( P_n \) be the least real number that bounds the probability that Interactive Protocol 1 lasts for at least \( n \) rounds, uniformly for all choices Merlin makes.

To formally define \( P_n \), consider first sequences \( \mu = (\mu_1, \ldots, \mu_{n-1}) \) of events (intuitively, think of them as Merlin’s successive choices at the first \( n-1 \) rounds of an execution of Interactive Protocol 1). Define \( P_n^\mu \) to be the product of the following probabilities:

- \( p_0 \): the probability that at round 0, Arthur’s random choices are such that there is an occurring event.
- \( p_1 \): the conditional probability that given that Merlin has chosen \( \mu_1 \) at the beginning of round 1, the resampling of the variables of \( \mu_1 \) by Arthur at round 1 is such that there is still an occurring event.
  
  :
\( p_{n-1} \): the conditional probability that given that Merlin has chosen the sequence of events \( \mu_1, \ldots, \mu_{n-1} \) up to the beginning of round \( n - 1 \), the resampling of the variables of \( \mu_{n-1} \) by Arthur at round \( n - 1 \) is such that there is still an occurring event.

Above, if any conditional space turns to be empty, the corresponding probability is understood to be 0.

**Definition 1.** We define \( P_n \) (for \( n \geq 1 \)) to be the maximum of \( P_{\mu_n} \), over all \( \mu = (\mu_1, \ldots, \mu_{n-1}) \).

**Example 3.** We will now explicitly compute \( P_1 \) and \( P_2 \), for the setting described in Example 1.

We will first expand our notation so that \( p_{\mu_i, \mu} = \{\mu_1, \ldots, \mu_{n-1}\} \) will be the probability \( p_i \) as defined above, given that Merlin’s choices up to round \( i - 1 \) were \( \mu_1, \ldots, \mu_{i-1}, i = 1, \ldots, n \) (\( p_0 \) is always the same).

Consider the following subspaces of \( \Omega \):

- \( A_0 = \Omega \) and \( A_1 \) is the subspace of \( A_0 \) such that at least one event occurs.
- \( A_{2i}^\mu, i = 1, \ldots, n, \) is the probability space of round \( i \), before the resampling of the variables of the event Merlin chose in that round and given that, up to round \( i - 1 \), the choices of Merlin were \( \mu_1, \ldots, \mu_{i-1} \).
- \( A_{2i+1}^\mu, i = 1, \ldots, n - 1, \) is the probability space of round \( i \), after the resampling of the variables of \( \mu_i \) and given that, up to round \( i - 1 \), the choices of Merlin were \( \mu_1, \ldots, \mu_{i-1} \).
- \( A_{2i+1}^\mu \) be the subspace of \( A_{2i}^\mu, i = 1, \ldots, n - 1, \) such that at least one event occurs.

**It holds that** \( A_1 = \Omega \setminus \{(0,0,0,0),(0,0,0,1)\} \) and that:

\[
p_0 = \frac{|A_1|}{|A_0|} = \frac{14}{16} = \frac{7}{8}. \tag{1}
\]

Thus, since \( P_1 = p_0 \), it holds that:

\[
P_1 = \frac{7}{8}.
\]

To calculate \( P_2 \), we first need to consider each \( P_{\mu_2}^\mu \), where \( \mu \) is a sequence of length 1. Suppose that \( \mu_j = E_j, j = 1,2,3,4,5 \).

\( P_{\mu_2}^{\mu_1} \):

\[
A_{2}^{\mu_1} = \{(1,0,0,0),(1,0,0,1),(1,0,1,0),(1,1,0,0),
(1,0,1,1),(1,1,0,1),(1,1,1,0),(1,1,1,1)\},
\]
since we want $E_1$ to occur. After the resampling of the variables of $E_1$, we get:

$$A_2^{\mu_1} = \Omega,$$

and thus

$$A_3^{\mu_1} = \Omega \setminus \{(0, 0, 0, 0), (0, 0, 0, 1)\}.$$ 

So, we have:

$$p_{\mu_1}^{\mu_1} = \frac{|A_2^{\mu_1}|}{|A_3^{\mu_1}|} = \frac{7}{8} \quad (2)$$

and, from (1) and (2),

$$P_2^{\mu_1} = p_0 \cdot p_{\mu_1}^{\mu_1} = \frac{7}{8} \cdot \frac{3}{4} = \frac{21}{32} < P_2^{\mu_1}.$$ 

After reampling the variables of $E_2$, we get:

$$A_2^{\mu_2} = \{(0, 1, 0, 0), (0, 1, 1, 0), (0, 1, 1, 0), (0, 1, 1, 1)\}.$$ 

Note that the other four assignments such that $E_2$ occurs, could not have been the result of the first sampling, since then $X_1$ would have been selected by the Protocol, and $E_2$ does not depend on $X_1$.

After reampling the variables of $E_2$, we get:

$$A_2^{\mu_2} = \{(0, 0, 0, 0), (0, 0, 1, 0), (0, 1, 0, 0)$$

$$(0, 0, 1, 1), (0, 1, 0, 1), (0, 1, 1, 0), (0, 1, 1, 1)\}$$

and thus:

$$A_3^{\mu_2} = A_2^{\mu_2} \setminus \{(0, 0, 0, 0), (0, 0, 0, 1)\}.$$ 

So, we have:

$$p_{\mu_2}^{\mu_2} = \frac{|A_2^{\mu_2}|}{|A_3^{\mu_2}|} = \frac{3}{4} \quad (3)$$

and, from (1) and (3),

$$P_2^{\mu_2} = p_0 \cdot p_{\mu_2}^{\mu_2} = \frac{7}{8} \cdot \frac{3}{4} = \frac{21}{32} < P_2^{\mu_1}.$$ 

After reampling the variables of $E_3$, we get:

$$A_2^{\mu_3} = \Omega.$$ 

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and thus

\[ A_3^{\mu_3} = \Omega \setminus \{(0, 0, 0, 0), (0, 0, 0, 1)\}. \]

So, as in (2), we have:

\[ p_{\mu_3}^1 = \frac{7}{8} \tag{4} \]

and, from (1) and (4),

\[ P_{\mu_3}^2 = \frac{7^2}{8^2} = \frac{49}{64} = P_{\mu_1}^2. \]

\[ P_{\mu_4}^2: \]

\[ A_{\mu_4}^2 = \{(0, 0, 1, 0), (0, 0, 1, 1)\}, \]

again since any other assignment such that \(E_3\) occurs, will result in the selection of \(X_1\) or \(X_2\).

After reampling the variables of \(E_4\), we get:

\[ A_{\mu_4}^2 = \{(0, 0, 0, 0), (0, 0, 0, 1), (0, 0, 1, 0), (0, 0, 1, 1)\} \]

and thus:

\[ A_{\mu_3}^2 = \{(0, 0, 1, 0), (0, 0, 1, 1)\}. \]

So, we have:

\[ p_{\mu_4}^1 = \frac{|A_{\mu_4}^2|}{|A_{\mu_4}^2|} = \frac{1}{2} \tag{5} \]

and, from (1) and (5),

\[ P_{\mu_4}^2 = p_0 \cdot p_{\mu_4}^1 = \frac{7}{8} \cdot \frac{1}{2} = \frac{7}{16} < P_{\mu_2}^2. \]

\[ P_{\mu_5}^2: \]

\[ A_{\mu_5}^2 = \{(0, 0, 1, 1)\}. \]

After reampling the variables of \(E_5\), we get:

\[ A_{\mu_5}^2 = \{(0, 0, 0, 0), (0, 0, 0, 1), (0, 0, 1, 0), (0, 0, 1, 1)\} = A_{\mu_4}^2 \]

and thus everything goes through as with \(P_{\mu_4}^2\). So, in the end, we have:

\[ P_{\mu_5}^2 = P_{\mu_2}^2 = \frac{7}{16} < P_{\mu_2}^2. \]
Maximizing over all $P_{2}^{\mu_{j}}$, we get:

$$P_{2} = \max_{j=1,2,3,4,5} \{P_{2}^{\mu_{j}}\} = P_{2}^{\mu_{1}} = \frac{49}{64}.$$ 

Finally, we will find a $P_{n}^{\mu}$ that is 0. This will be the case for $n = 3$ and $\mu = (E_{4}, E_{1})$.

By the analysis above, and the definition of the sets $A_{i}^{\mu}$,

$$A_{3}^{\mu} = \{(0, 0, 1, 0), (0, 0, 1, 1)\}.$$ 

To compute the probability space $A_{4}^{\mu}$, we need to take the subspace of $A_{3}^{\mu}$ such that $E_{1}$ occurs. Since there are no elements in $A_{3}^{\mu}$ such that $E_{1}$ occurs, $A_{4}^{\mu} = \emptyset$ and, by definition, we get $P_{3}^{\mu} = 0$.

The above is not surprising. In fact, it is a direct consequence of Lemma 1, since the first sampling, in order for Merlin to be able to choose $E_{4}$, produced an assignment such that $X_{1}$ had no occurring events depending on it.

We will now prove that if $P_{n} < 1$, then there are choices for the two players that lead the protocol to a halt by round $n$. Formally:

**Lemma 2.** If $P_{n} < 1$ then there exist some $i \in \{1, \ldots, n\}$, some sequence $\mu = (\mu_{1}, \ldots, \mu_{i-1})$, and some random choices for Arthur such that if Merlin chooses $\mu_{1}, \ldots, \mu_{i-1}$, the protocol executes the previous to $i-1$ rounds without halting and at the end of round $i-1$, i.e. after $\mu_{i-1}$ is resampled, there is no occurring event and the protocol halts.

**Proof.** Assume, towards a contradiction, that for all random choices by Arthur at round 0, there is an occurring event (otherwise, take i=1 and $\mu$ be the empty sequence). Therefore, $p_{0} = 1$. Let $\mu_{1}$ be an arbitrary choice for Merlin after some arbitrary execution of round 0 that does not lead to a halt. Assume, again towards a contradiction, that for all executions where Merlin chooses $\mu_{1}$, Arthur’s choices at the resampling of round 1 are such that there is still an occurring event (otherwise take the sequence $\mu = (\mu_{1})$ and $i = 2$). Therefore, the conditional probability $p_{1}$ for $\mu_{1}$ is equal to 1. Since $P_{n} < 1$, this process must stop, i.e. a contradiction should be obtained, lest we get that $p_{0} = \cdots = p_{n-1} = 1$.\[\square\]

The observant reader will notice that from the above proof we also get that independently of Merlin’s choices, Arthur can make random choices that will lead to a halt by round $n$.

We now immediately get:
**Corollary 2.** If for some \( n \), \( P_n < 1 \), there is an assignment that avoids all undesirable events.

In other words, the Interactive Protocol 1 is one-sidedly correct, in the sense that if an assignment as desired does not exist, then for all \( n \), \( P_n = 1 \).

In the following sections we will provide a proof that if \( ekq \leq 1 \), then \( P_n \) is exponentially small (Theorem 2) and therefore, by Corollary 2, we will get Theorem 1.

Unfortunately, proving that \( P_n < 1 \) does not provide an efficient, even if probabilistic, algorithm that finds an assignment to the independent variables so that all undesirable events do not occur. Corollary 2 works much in the way of the classical probabilistic method: if we know that the probability of an assignment that avoids all undesirable events is \( < 1 \), we may conclude that there is such an assignment, even though we do not know how to construct one. Nevertheless, such an assignment is produced by a probabilistic protocol that works interactively with Merlin (who points to occurring events), within \( n \) rounds, with high probability with respect to \( n \). We actually suspect that finding such an assignment is inherently hard, much alike problems complete for syntactic subclasses of TFNP (Total Function Non-deterministic Polynomial), like, e.g., PPAD (Polynomial Parity Argument on Directed Graphs) [13].

Szegedy, in [16], asked the following question:

“Problem: Try to find the best method of selecting a violated constraint in the resample step of RESAMPLE.”

In the terminology above, the violation of a constraint corresponds to the occurring of an undesirable event, whereas “RESAMPLE” corresponds to our “REDO” procedure. We do not answer this question, however we show that Merlin’s help when selecting the next event leads to an improvement. Thus we prove what in [16] was mentioned only as an indication: “Experiments however indicate that the selection process matters.” (we are not familiar where these experiments are presented).

Kolipaka and Szegedy [10] give an example of a graph \( G \) such that for any family of undesirable events in the variable framework that has \( G \) as dependency graph and with a uniform probability bound for all events a number larger than that given by Shearer’s condition [15], an assignment to the variables that avoids all events exists. Recently also Harris [8, 9] has shown that for an application related to the satisfiability problem (a problem that can only be given in the variable framework), a suitable version of LLL yields stronger results than those obtained if Shearer’s version is used. As with these results, our approach can be considered as being in the direction of highlighting the advantages of the variable framework.

Let us also mention at this point that Achlioptas and Iliopoulos [1, 2] have devised a very general abstract setting, with no variables and no probabilities
involved, where they cast Moser’s algorithm for LLL as a process of finding a path in a local search. They give, in this setting, sufficient conditions for the local search to be successful. Their approach however corresponds, if translated to our setting, to working with the probabilities of individual events and not with the probability that a variable is in the scope of an occurring event; expectedly so as their setting does not entail variables.

3 Feasible Forests

In the sequel below, we assume that the scope of each event contains exactly \( k \) variables. If this is not the case, all we have to do is add a number of variables on each scope and assume that the corresponding event “depends” on them as well in an “degenerate” fashion. For notational purposes, we will also assume that, for each event \( E_j, j \in \{1, \ldots, m\}, \) \( \text{sc}(E_j) = \{X_{j_1}, \ldots, X_{j_k}\} \), where \( j_i \in \{1, \ldots, l\} \) and \( j_i < j_{i+1}, i = 1, \ldots, k \). Thus, the \( i \)-th variable that the event \( E_j \) depends on will be \( X_{j_i} \).

**Definition 2.** We call the integer \( i \) the ordinal of the variable \( X_{j_i} \) with respect to \( E_j \)’s scope.

We will depict the succession of variables determined at lines 2 and 9 in an execution of the Interactive Protocol 1 by means of a witness structure whose size is the number of rounds of the execution.

To formally define the witness structure, we first define feasible forests:

**Definition 3.** A feasible forest is a labeled forest with at most \( l \) trees (recall that \( l \) is the number of variables), such that each node of any tree has out-degree at most \( k \) (recall that \( k \) is the number of variables of an event). Nodes are labeled as follows:

- **The roots are labeled with pairwise distinct integers in \( \{1, \ldots, l\} \) (intuitively, these integers correspond to indices of the variables determined at a repetition of the loop at line 2).**

- **The children of a node \( v \) are labeled with pairwise distinct natural numbers in \( \{1, \ldots, k\} \) (intuitively, these integers correspond to the ordinals—with respect to the current event’s scope—of the variables that are determined at the repetitions of the loop at line 9).**

The number of nodes of a feasible forest \( \mathcal{F} \) will be denoted by \( |\mathcal{F}| \). The nodes of \( \mathcal{F} \) are ordered by pre-ordering (depth-first ordering), with the assumption that roots are ordered according to their labels, and siblings that have a common parent are ordered according to their labels again.

**Example 4.** The following is a feasible forest, in the sense of Definition 3, where \( l = 4 \) and \( k = 2 \):
When the nodes of the above forest are ordered, their respective labels are ordered as: 1, 1, 2, 1, 3, 1.

Now, if an execution of the Interactive Protocol 1 lasts for at least \( n \) rounds, we construct a feasible forest \( F \) with \( n \) nodes, called the \( n \)-witness forest. Each node \( v \) of the \( n \)-witness forest is associated with a unique one among the first \( n \) calls of \textsc{Redo} of the execution. Moreover, the structure of the forest reflects the way these calls are nested within each other, meaning that if \textsc{Redo}(\( X_j \)) is called within \textsc{Redo}(\( X_i \)), then the node associated with the former call is a descendant of the node associated with the latter call. Moreover,

- the labels of the roots of the forest are the indices of the variables in the arguments of the root calls of \textsc{Redo}, whereas
- the labels of non-roots of the forest are the ordinals —with respect to the scope of the events chosen by Merlin at the corresponding calls of \textsc{Redo}— of the variables that are the arguments of these calls.

By Lemma 1, it is easy to see that a forest created as above is indeed a feasible one, in the sense of Definition 3. Notice that it is conceivable that two differing sequences of choices by the two players might give rise to the same witness tree.

**Example 5.** Before presenting a witness forest for the setting of Example 1, let us make the necessary changes so that every event depends on exactly \( k \) variables.

We thus introduce three new variables, namely \( X'_1, X'_2, X'_3 \), that take values in \( \{0, 1\} \) uniformly at random. The events \( E_1, E_2, E_4 \) change as follows:

\[
E_1 = \{ X_1 = 1 \, \& \, X'_1 \in \{0, 1\} \}, \\
E_2 = \{ X_2 = 1 \, \& \, X'_2 \in \{0, 1\} \}, \\
E_4 = \{ X_3 = 1 \, \& \, X'_3 \in \{0, 1\} \}.
\]

These new variables will be ignored from now on.
Recall the execution of the Interactive Protocol 1 of Example 2. It is easy to see that the feasible forest of Example 4, is indeed the witness forest of that execution.

We will now describe a different execution of the Protocol, that has the same witness forest. The assignments of values of this execution will be denoted $\alpha'_0, \ldots, \alpha'_6$.

Let $\alpha'_0 = (1, 1, 1, 1)$ and let the protocol function in the exact same way as it did in Example 2, until the beginning of round 4. The produced assignments will thus be:

$$\alpha'_1 = (1, 1, 1, 1), \quad \alpha'_2 = \alpha'_3 = (0, 1, 1, 1) \quad \text{and} \quad \alpha'_4 = (0, 0, 1, 1).$$

This produces the left tree of Example 4.

At this point, $X_3$ will be again selected by the protocol, and now Merlin can choose either $E_4$ or $E_5$. Suppose he chooses the later, and that

$$\alpha'_5 = \alpha'_4 = (0, 0, 1, 1).$$

$X_3$ is chosen again, and the right tree of Example 4 is created.

Merlin now chooses $E_5$ again, and suppose the produced assignment is $\alpha'_6 = (0, 0, 0, 1)$, which is again, an assignment that causes the Protocol to accept.

Note that, although only the last two non-deterministic choices of Merlin differ between these two executions, the random choices of Arthur are different in every round.

Let $P_{\mathcal{F}}$ be the least real number that bounds the probability that Interactive Protocol 1 lasts for at least $|\mathcal{F}|$ rounds and during these rounds, $\mathcal{F}$ is produced as an $|\mathcal{F}|$-witness forest, uniformly for all choices Merlin makes.

To formally define $P_{\mathcal{F}}$, assume $|\mathcal{F}| = n$ and let $(\alpha_1, \ldots, \alpha_n)$ be the sequence of its labels. Let $\mu = (\mu_1, \ldots, \mu_{n-1})$ be a sequence of events (intuitively, think of them as Merlin’s successive choices at the first $n-1$ rounds of an execution of Interactive Protocol 1 that produces $\mathcal{F}$). Define $P_{\mathcal{F}}^\mu$ to be the product $p_0 \cdots p_{n-1}$, where:

- If the $(i+1)$-th node in $\mathcal{F}$ is a root, then $p_i$ is the conditional probability that given that Merlin has chosen the sequence of events $\mu_1, \ldots, \mu_i$ up to the beginning of round $i$, the (re)sampling by Arthur at round $i$ is such that the index of the variable determined at line 2 of round $i$ is $\alpha_{i+1}$ (therefore, in particular, there is an occurring event that contains this variable in its scope), and

- if the $(i + 1)$-th node in $\mathcal{F}$ is not a root, then $p_i$ is the conditional probability that given that Merlin has chosen the sequence of events $\mu_1, \ldots, \mu_i$ up to the beginning of round $i$, the resampling of the variables of $\mu_i$ by Arthur is such that the ordinal of the variable determined
at line 9 of round $i$ within $\mu_i$’s scope is $\alpha_{i+1}$ (therefore, in particular, there is an occurring event that contains this variable in its scope).

Above, if any conditional space turns to be empty, the corresponding probability is understood to be 0. The conditional space at e.g., the second node of $F$ turns out to be empty if, for example, $\mu_1$ does not contain in its scope $X_t$, where $t$ is the index of the first root of the forest (recall what events Merlin chooses, see line 7).

**Definition 4.** We define $P_F$ to be the maximum of $P_{\mu}^F$, over all $\mu = (\mu_1, \ldots, \mu_{n-1})$.

**Lemma 3.** $P_n \leq \sum_{F:|F|=n} P_F$.

*Proof.* Notice first that given Merlin’s choices $\mu_1, \ldots, \mu_{n-1}$, the succession of random choices by Arthur in the first $n$ rounds uniquely determines the $n$-witness forest, so by union-bound on disjoint events:

$$P_{\mu}^F = \sum_{F:|F|=n} P_{\mu}^F. \quad (6)$$

Now

$$P_n = \max_{\mu} P_{\mu}^F = \max_{\mu} \left( \sum_{F:|F|=n} P_{\mu}^F \right) \leq \sum_{F:|F|=n} \max_{\mu} P_{\mu}^F = \sum_{F:|F|=n} P_F,$$

where the first equality is by Definition 1, the second by (6), the third relation (inequality) is obvious, and the fourth is by Definition 4.

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**3.1 The Validation Protocol**

To find an upper bound to $P_n$, we first consider the Interactive Protocol 2, which takes as input a feasible forest $F$ (recall that a feasible forest $F$ was the product of the Interactive Protocol 1; here it is an input).

During the execution of the Interactive Protocol 2, Arthur resamples the variables of the events Merlin chooses. If at any point, Merlin cannot choose an event, the protocol fails. The Validation Protocol 2 succeeds, if it goes through the whole forest given as input without returning failure. In that case, it associates each node $v$ of $F$ with an event $E$, the event Merlin chose during that procedure when the current variable was $X$.

Let $\hat{P}_F$ be the least real number that bounds the probability that Interactive Protocol 2 does not fail on input $F$, uniformly for all choices Merlin makes.

To formally define $\hat{P}_F$, assume $|F| = n$ and let $(\alpha_1, \ldots, \alpha_n)$ be the sequence of its labels. Let $\mu = (\mu_1, \ldots, \mu_{n-1})$ be a sequence of events (intuitively, think of them as Merlin’s successive choices at the first $n-1$ rounds
Interactive Protocol 2 The Validation Protocol

**Input:** Feasible forest $\mathcal{F}$ with $n$ nodes.

1: \( v \leftarrow \emptyset \)
2: **Arthur:** Sample the variables $X_1, \ldots, X_l$
3: for \( i = 1, \ldots, l \) do
4: \[ \text{if there is a root } r \text{ in } \mathcal{F} \text{ labeled by } i \text{ then} \]
5: \( v \leftarrow r \)
6: REDO($X_i$)
7: \[ \text{end if} \]
8: end for
9: return success
10: procedure REDO($X$)
11: \[ \text{if Merlin can find an occurring event depending on } X \text{ then} \]
12: Merlin: Choose an occurring event $E_j$ that depends on $X$.
13: **Arthur:** Resample the variables in $\text{sc}(E_j) = \{X_{j_1}, \ldots, X_{j_k}\}$
14: for \( t = 1, \ldots, k \) do
15: \[ \text{if there is a child } u \text{ of } v \text{ labeled by } j_t \text{ then} \]
16: \( v \leftarrow u \)
17: REDO($X_{j_t}$)
18: \[ \text{end if} \]
19: end for
20: else
21: return failure
22: end if
23: end procedure

of an execution of Interactive Protocol 2 on input $\mathcal{F}$). Define $\hat{P}_\mathcal{F}^\mu$ to be the product $p_0 \cdots p_{n-1}$, where: $p_i$ is the conditional probability that given that Merlin has chosen the sequence of events $\mu_1, \ldots, \mu_i$ with the Interactive Protocol 2 not failing, the Protocol still does not fail after the resampling of the variables of $\mu_i$.

Above, if any conditional space turns to be empty, the corresponding probability is understood to be 0. The conditional space turns out to be empty if, for example, $\mu_1$ does not contain in its scope $X_t$, where $t$ is the index of the first root of the forest.

Observe that the above probabilities are all bounded from above by $q$ (if the corresponding conditional space is not empty). Because all we demand at line 11 of Interactive Protocol 2 in order to avoid failure is that the current variable belongs to at least one occurring event; moreover, the distributions of the variables are not affected by any biases introduced by the fact that certain events already occurred, as the variables of these events
Definition 5. We define $\hat{P}_F$ to be the maximum of $\hat{P}_F^\mu$, over all $\mu = (\mu_1, \ldots, \mu_{n-1})$.

We also define:

$$\hat{P}_n = \sum_{F:|F|=n} \hat{P}_F.$$  \hspace{1cm} (7)

First, we will prove the following Lemma:

Lemma 4. For every feasible forest $F$, $P_F \leq \hat{P}_F$. Thus,

$$P_n \leq \hat{P}_n.$$  

Proof. First notice that $P_F \leq \hat{P}_F$ for any feasible forest $F$: indeed this follows from the fact that, if during the execution of the Validation Protocol 2 on any witness forest $F$, Merlin and Arthur make the exact same choices as in the execution of the Interactive Protocol 1 that produced $F$, then the Validation Protocol 2 will return success.

That $P_n \leq \hat{P}_n$ follows from Lemma 3 and equation (7).

Remark 1. It is conceivable that $P_F$ might not be equal to $\hat{P}_F$. Indeed, for $F$ to be a witness forest of an execution of the Interactive Protocol 1, it must be true that at the leaves of $F$ no occurring event can be chosen, something not necessary in order for the Validation Protocol 2 to return success on input $F$. So, because the “knowledge” obtained at the leaves of $F$ during the execution of the Interactive Protocol 1 is not undone by resampling, biases are introduced into the distribution of the random variables. This is the reason we had to recourse to the Interactive Protocol 2. Another source of biases in protocol 1 is the fact that Merlin always chooses the least indexed variable that belongs to an occurring event. But this could have been avoided, at no essential gain because of the previous source of biases, by letting Merlin make the choice.

Example 6. Suppose that the feasible forest of Example 4, is given as input to the Validation Protocol 2. The following is an execution of the Validation Protocol that returns success:

0. Suppose that the assignment of values that Arthur produces by sampling the variables is:

$$\alpha^n_0 = (1, 1, 1, 1).$$  

The Validation Protocol will interpret the first root of the forest as $X_1$.  

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1. Since there exists an occurring event depending on $X_1$, the Protocol will not fail. Merlin chooses $E_3$ and suppose that, after Arthur resamples its variables, the produced assignment is

$$\alpha''_1 = \alpha''_0.$$ 

Now, the left child of the root will be interpreted as $X_1$ again.

2. The same things happen as in the previous round. Suppose again that:

$$\alpha''_2 = \alpha''_1.$$ 

The right child of the root now will be interpreted as $X_2$.

3. Merlin now chooses $E_2$ and suppose that after the resampling:

$$\alpha''_3 = \alpha''_2.$$ 

The leaf of the left tree will be interpreted as $X_2$.

4. Merlin chooses $E_3$, and the current assignment becomes:

$$\alpha''_4 = (0, 0, 1, 1).$$ 

What remains now is the right tree, whose root is interpreted as $X_3$.

5. Merlin chooses $E_5$, and the suppose new assignment is:

$$\alpha''_5 = (0, 0, 1, 0).$$ 

The leaf of the left tree is interpreted again as $X_3$.

6. Merlin chooses $E_4$, and the assignment that leads the Protocol accepting is:

$$\alpha''_6 = (0, 0, 0, 0).$$ 

We have thus seen a different set of Arthur’s and Merlin’s choices, from both those that lead to the construction of the witness tree of Example 5, that nevertheless resulted in the success of the Validation Protocol.

It is also straightforward to observe that, if these choices are made in an execution of the Interactive Protocol 1, the following is the witness forest of this execution:
By Lemma 4, to finish the proof of Theorem 1, we need to bound $\hat{P}_n$. To do that, first consider the following lemma:

**Lemma 5.** For any feasible forest $\mathcal{F}$ with $n$ nodes, it holds that:

$$\hat{P}_\mathcal{F} \leq q^n.$$

*Proof.* Let $\mu = (\mu_1, \ldots, \mu_{n-1})$ be an arbitrary sequence of events. Since $\hat{P}_\mathcal{F} = p_0 \cdots p_{n-1}$, where $p_i$ are as defined above, and since each $p_i$ is bounded by $q$, we conclude that $\hat{P}_\mathcal{F} \leq q^n$. Since $\mu$ was arbitrary, the required immediately follows. \qed

By equation 6 and Lemma 5, all that remains in order to bind $\hat{P}_n$ is to count the number of feasible forests.

To do that, consider a feasible forest $\mathcal{F}$ with $n$ nodes, comprised of at most $l$ trees and where each node has at most $k$ children. We will add extra leaves to $\mathcal{F}$, creating the feasible forest $\mathcal{F}'$, in the following way:

- Add as many trees to the forest comprised of a single node as needed (each such node will be both the tree’s root and only leaf) and label them so that the set of the roots’ labels becomes $\{1, \ldots, l\}$.

- Add as many leaves hanging from every node of $\mathcal{F}$, internal or not, as needed and label them so that the set of labels of the children of any node becomes $\{1, \ldots, k\}$.

Let $\mathcal{F}'$ be the forest created by the above process. Observe that it has $n$ internal nodes, which are the nodes of $\mathcal{F}$ and it is comprised by $l$ full $k$-ary trees.
Furthermore, observe that the labels of its nodes are uniquely determined from its rooted plane forest structure, when labels are ignored but the ordering imposed by them is retained.

**Example 7.** In the figure below, in the top cell we have the feasible forest $\mathcal{F}$ described in Example 4 and in the bottom cell, the resulting feasible forest $\mathcal{F}'$, after adding leaves (denoted with red circles), as described above, to $\mathcal{F}$.

Thus, to count the feasible forests with $n$ nodes, we can instead count the rooted plane forests with $n$ internal nodes, comprised of $l$ full $k$-ary trees.
Let their number be $f_n$. Easily, it holds that:

$$f_n = \sum_{n_1 + \ldots + n_l = n} t_{n_1} \cdots t_{n_l}, \quad (8)$$

where $t_n$ is the number of full $k$-ary rooted plane trees with $n$ internal nodes.

It is well known that the number $t_n$ of full $k$-ary rooted plane trees with $n$ internal nodes is equal to $\frac{1}{(k-1)n+1} \binom{kn}{n}$, see e.g. [14, Theorem 6.15]. Now by Stirling’s approximation, it easily follows that for some constant $A > 1$, depending only on $k$, we have:

$$t_n < A \left(1 + \frac{1}{k-1}\right)^{k-1} \frac{n}{k}. \quad (9)$$

From (9) and (8) we get:

$$f_n < (An)^l \left(1 + \frac{1}{k-1}\right)^{k-1} \frac{n}{k} < (An)^l (ek)^n. \quad (10)$$

Now, by combining Lemma 4, Lemma 5 and equation (10), we have:

**Theorem 2.** Assuming $q$ and $k$ are constants such that $\left(1 + \frac{1}{k-1}\right)^{k-1} qk < 1$, (and therefore if $ekq \leq 1$), there exists an integer $N$, which depends linearly on $l$, and a constant $c \in (0, 1)$ (depending on $q$ and $k$) such that if $n/\log n \geq N$ then the probability that the Interactive Protocol 1 lasts for at least $n$ rounds is $< c^n$.

By Theorem 2 and the remarks in the last two paragraphs of Section 2, Theorem 1 follows.

**Remark 2.** What prima facie might be considered a little puzzling, is that if we make Merlin part of a non-interactive algorithm, by, e.g., letting him always choose an occurring event according to a prescribed strategy, or even randomly, then the analysis does not go through, as biases are introduced by the revelation of which events occur. We could circumvent the difficulty of biases by considering a non-interactive validation algorithm admitting as input a feasible forest (as is done in [4]), but then we would have to compute the sum of the probabilities of success of the validation algorithm over all feasible forests, and thus loose the advantage we gained by considering $q$ instead of a bound of individual probabilities of the events. It seems we need Merlin!
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