QUASI-MORPHISMS ON THE GROUP OF AREA-PRESERVING DIFFEOMORPHISMS OF THE 2-DISK VIA BRAID GROUPS

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Abstract. Recently Gambaudo and Ghys proved that there exist infinitely many quasi-morphisms on the group $\text{Diff}_\infty^\omega(D^2, \partial D^2)$ of area-preserving diffeomorphisms of the 2-disk $D^2$. For the proof, they constructed a homomorphism from the space of quasi-morphisms on the braid group to the space of quasi-morphisms on $\text{Diff}_\infty^\omega(D^2, \partial D^2)$. In this paper, we study this homomorphism and prove its injectivity.

1. Introduction

For a group $G$, a function $\phi: G \to \mathbb{R}$ is called a quasi-morphism if the real valued function on $G \times G$ defined by

$$(g, h) \mapsto \phi(gh) - \phi(g) - \phi(h)$$

is bounded. The real number

$$D(\phi) = \sup_{g,h \in G} |\phi(gh) - \phi(g) - \phi(h)|$$

is called the defect of $\phi$. We denote the $\mathbb{R}$-vector space of quasi-morphisms on the group $G$ by $\hat{Q}(G)$. By definition, bounded functions on groups are quasi-morphisms. Hence we denote the set of bounded functions on the group $G$ by $C^1_b(G; \mathbb{R})$ and consider the quotient space $Q(G) = \hat{Q}(G)/C^1_b(G; \mathbb{R})$. A quasi-morphism $\phi: G \to \mathbb{R}$ is said to be homogeneous if the equation

$$\phi(g^p) = p \phi(g)$$

holds for any $g \in G$ and $p \in \mathbb{Z}$. For any quasi-morphism $\phi$, a homogeneous quasi-morphism $\tilde{\phi}$ is defined by setting

$$\tilde{\phi}(g) = \lim_{p \to \infty} \frac{1}{p} \phi(g^p).$$

The limit always exists for each element $g$ of $G$. The new function $\tilde{\phi}$ is in fact a quasi-morphism equal to the original quasi-morphism $\phi$ as an element of $Q(G)$. Thus we can identify the vector space of homogeneous quasi-morphisms on the group $G$ with $Q(G)$. Homogeneous quasi-morphisms are invariant under conjugations. Therefore we are interested in $Q(G)$ rather than $\hat{Q}(G)$.

Let $\text{Diff}_\infty^\omega(D^2, \partial D^2)$ be the group of area-preserving $C^\infty$-diffeomorphisms of the 2-disk $D^2$, which are the identity on a neighborhood of the boundary. On the vector space $Q(\text{Diff}_\infty^\omega(D^2, \partial D^2))$, the following theorem is known.
Theorem 1.1 (Entov-Polterovich [2], Gambaudo-Ghys [4]). The vector space \( Q(\text{Diff}^\infty_\Omega(D^2, \partial D^2)) \) is infinite dimensional.

To prove Theorem 1.1, Entov and Polterovich explicitly constructed uncountably many quasi-morphisms on \( \text{Diff}^\infty_\Omega(D^2, \partial D^2) \), which are linearly independent. After that Gambaudo and Ghys constructed countably many quasi-morphisms on \( \text{Diff}^\infty_\Omega(D^2, \partial D^2) \) by a different idea, which is to consider the suspension of area-preserving diffeomorphisms of the disk and average the value of the signature of the braids appearing in the suspension. By generalizing their strategy Brandebursky [1] defined the homomorphism

\[
\Gamma_n : Q(P_n(D^2)) \to Q(\text{Diff}^\infty_\Omega(D^2, \partial D^2)),
\]

which we review in Section 2. Here, \( P_n(D^2) \) denotes the pure braid group on \( n \)-strands.

Let \( B_n(D^2) \) be the braid group on \( n \)-strands. The natural inclusion \( i : P_n(D^2) \to B_n(D^2) \) induces the homomorphism \( Q(i) : Q(B_n(D^2)) \to Q(P_n(D^2)) \). In this paper, we study the homomorphism \( \Gamma_n \) and prove the following theorem.

Theorem 1.2. The composition

\[
\Gamma_n \circ Q(i) : Q(B_n(D^2)) \to Q(\text{Diff}^\infty_\Omega(D^2, \partial D^2))
\]

is injective.

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2. Gambaudo and Ghys’ construction and proof of the main theorem

In this section, we review Gambaudo and Ghys’ construction [4] of quasi-morphisms on the group \( \text{Diff}^\infty_\Omega(D^2, \partial D^2) \) in a generalized form and prove Theorem 1.2.

Let \( X_n(D^2) \) be the configuration space of ordered \( n \)-tuples in the 2-disk \( D^2 \) and \( x^0 = (x^0_1, \ldots, x^0_n) \) its base point. For any \( g \in \text{Diff}^\infty_\Omega(D^2, \partial D^2) \) and for almost alls \( x = (x_1, \ldots, x_n) \in X_n(D^2) \), we define the pure braid \( \gamma(g;x) \) as the following. First we set the loop \( l(g;x) : [0, 1] \to X_n(D^2) \) by

\[
l(g;x)(t) = \begin{cases} 
(1 - 3t)x^0_i + 3tx_i & (0 \leq t \leq \frac{1}{3}) \\
\{g_{3t-1}(x_i)\} & \left(\frac{1}{3} \leq t \leq \frac{2}{3}\right), \\
(3 - 3t)g(x_i) + (3t - 2)x^0_i & \left(\frac{2}{3} \leq t \leq 1\right)
\end{cases}
\]

where \( \{g_t\}_{t \in [0, 1]} \) is a Hamiltonian isotopy such that \( g_0 \) is the identity and \( g_1 = g \). We define the pure braid \( \gamma(g;x) \) to be the braid represented by the loop \( l(g;x) \). For almost every \( x \), the braid \( \gamma(g;x) \) is well-defined. Furthermore, the braid \( \gamma(g;x) \) is independent of the choice of the flow \( \{g_t\} \). This is because of the fact
We may assume that $\beta$ is a morphism $\hat{\Gamma}$ of an area-preserving diffeomorphism $g$. Proof of Theorem 1.2. Let $\hat{\Gamma}_n: \text{Diff}_\Omega^\infty(D^2, \partial D^2) \rightarrow \mathbb{R}$ be the pure braid group $P_n(D^2)$ on $n$-strands, we define the function $\hat{\Gamma}_n(\phi): \text{Diff}_\Omega^\infty(D^2, \partial D^2) \rightarrow \mathbb{R}$ by

$$\hat{\Gamma}_n(\phi)(g) = \int_{x \in X_n(D^2)} \phi(\gamma(g; x)) dx.$$  

For any $\phi \in Q(P_n(D^2))$ and $g \in \text{Diff}_\Omega^\infty(D^2, \partial D^2)$ the function $\phi(\gamma(g; \cdot))$ is integrable and thus the map $\hat{\Gamma}_n: Q(P_n(D^2)) \rightarrow Q(\text{Diff}_\Omega^\infty(D^2, \partial D^2))$ is well-defined [1]. The obtained function $\hat{\Gamma}_n(\phi): \text{Diff}_\Omega^\infty(D^2, \partial D^2) \rightarrow \mathbb{R}$ is also a quasi-morphism and the map $\hat{\Gamma}_n: Q(P_n(D^2)) \rightarrow Q(\text{Diff}_\Omega^\infty(D^2, \partial D^2))$ is clearly $\mathbb{R}$-linear. Moreover, it is easily checked that any bounded function on $P_n(D^2)$ is mapped to a bounded function on $\text{Diff}_\Omega^\infty(D^2, \partial D^2)$ and thus the homomorphism $\hat{\Gamma}_n: Q(P_n(D^2)) \rightarrow Q(\text{Diff}_\Omega^\infty(D^2, \partial D^2))$ induces the homomorphism $\hat{\Gamma}_n: Q(P_n(D^2)) \rightarrow Q(\text{Diff}_\Omega^\infty(D^2, \partial D^2))$.

Remark 2.1. It is easy to see that the homomorphism $\hat{\Gamma}_n: Q(P_n(D^2)) \rightarrow Q(\text{Diff}_\Omega^\infty(D^2, \partial D^2))$ maps the classical linking number homomorphism $\text{lk}_n: B_n(D^2) \rightarrow \mathbb{R}$ on the braid group to a homomorphism on $\text{Diff}_\Omega^\infty(D^2, \partial D^2)$. In fact, the image of $\text{lk}: B_n(D^2) \rightarrow \mathbb{R}$ by the homomorphism $\hat{\Gamma}_n(\text{lk}_n)$ coincides with a constant multiple of the classical Calabi homomorphism on $\text{Diff}_\Omega^\infty(D^2, \partial D^2)$ [3] and in this sense quasi-morphisms obtained in this way can be considered as generalizations of the Calabi homomorphism. By an argument of Brandenbursky, which verify that the homomorphism $\hat{\Gamma}: Q(P_n) \rightarrow Q(\text{Diff}_\Omega^\infty(D^2, \partial D^2))$ is well-defined, it is observed that quasi-morphisms obtained by the homomorphism $\hat{\Gamma}_n: Q(P_n(D^2)) \rightarrow Q(\text{Diff}_\Omega^\infty(D^2, \partial D^2))$ can be defined on the group of area-preserving $C^1$-diffeomorphisms of $D^2$, as well as the Calabi homomorphism.

Now we are ready to prove Theorem 1.2.

Proof of Theorem 1.2. Let us suppose that a homogeneous quasi-morphism $\phi \in Q(B_n(D^2))$ is non-trivial. Then there exists a braid $\beta \in B_n(D^2)$ such that $\phi(\beta) \neq 0$. We may assume that $\beta$ is pure. It is sufficient to prove that the homogeneous quasi-morphism $\hat{\Gamma}_n(\phi) \in Q(\text{Diff}_\Omega^\infty(D^2, \partial D^2))$ is also non-trivial. That is, there exists an area-preserving diffeomorphism $g \in \text{Diff}_\Omega^\infty(D^2, \partial D^2)$ such that

$$\lim_{p \rightarrow \infty} \frac{1}{p} \Gamma_n(\phi)(g^p) \neq 0.$$  

Let $A_{i,j}$ be the pure braid which twists only the $i$-th and the $j$-th strands for $1 \leq i < j \leq n$ (see Figure 1). Since the braid $\beta$ is pure, it can be written as a composition of $A_{i,j}$’s and their inverses. We take $n$ disjoint subsets $U_i$’s of $D^2$. Furthermore, for a pair of $(i, j)$, we take subsets $V_{i,j}$ and $W_{i,j}$ of $D^2$ such that $U_i \cup U_j \subset W_{i,j} \subset V_{i,j}$, $U_i \cap V_{i,j} = \emptyset$ if $k \neq i, j$ and $V_{i,j}$, $W_{i,j}$ are diffeomorphic to $D^2$. Let $\{h_t\}_{t \in [0, 1]}$ be a path in $\text{Diff}_\Omega^\infty(D^2, \partial D^2)$ such that the support of $h_t$ is contained in the interior of $V_{i,j}$ and rotates $W_{i,j}$ once. Taking paths $\{h_t\}$’s constructed above for the all $A_{i,j}$’s which present $\beta$ and composing them, we have a path $\{g_t\}_{t \in [0, 1]}$ in $\text{Diff}_\Omega^\infty(D^2, \partial D^2)$ with $g_0 = id$ which twists $U_i$’s in the form of the pure braid $\beta$. If we set $g = g_1$, then $g$ is the identity on $U_i$’s and $\gamma(g; (x_1, \ldots, x_n)) = \beta$ for $x_i \in U_i$. 


Then by setting $U = U_1 \cup \cdots \cup U_n$, we have
\[
\lim_{p \to \infty} \frac{1}{p} \hat{\Gamma}_n(\phi)(g^p) = \lim_{p \to \infty} \frac{1}{p} \left( \int_{x \in X_n(U)} \phi(\gamma(g^p; x)) \, dx + \int_{x \in X_n(D^2) \setminus X_n(U)} \phi(\gamma(g^p; x)) \, dx \right)
\]
\[
= \int_{x \in X_n(U)} \phi(\gamma(g; x)) \, dx + \lim_{p \to \infty} \frac{1}{p} \int_{x \in X_n(D^2) \setminus X_n(U)} \phi(\gamma(g^p; x)) \, dx.
\]
If we denote the first term of the equation by $Y$ and set $a_i = \text{area}(U_i)$ and $[n] = \{1, \ldots, n\}$, then $Y$ is written as
\[
\int_{x \in X_n(U)} \phi(\gamma(g; x)) \, dx = \sum_{F: [n] \to [n]} \left( \prod_{i=1}^{n} a_{F(i)} \right) x_F,
\]
where $x_F = \phi(\gamma_F)$ and $\gamma_F = \gamma(g; x)$ for $x$ in the case when each $x_i$ is in $U_{F(i)}$. The real numbers $x_F$’s have the following properties.

(i) For two maps $F$ and $G: [n] \to [n]$, if $\#F^{-1}(i) = \#G^{-1}(i)$ for each $1 \leq i \leq n$ then $x_F = x_G$.

(ii) If a map $F: [n] \to [n]$ is bijective, then $x_F$ is non-zero.

The property (i) follows from the invariance of $\phi$ under conjugation and the property (ii) follows because $\phi(\beta)$ is non-zero. Therefore, the coefficient of $a_1 \ldots a_n$ in $Y$ is non-zero. Since the polynomial $Y$ is not identically 0, we can choose $a_i$’s s that $Y$ is non-zero.

Note that if we replace $a_i$’s by bigger ones fixing the ratio of any two of them the term $Y$ stays non-zero. On the other hand, the values $\phi(\gamma(g; x))$ is bounded because of the construction of $g$, and we thus have
\[
\lim_{p \to \infty} \frac{1}{p} \int_{x \notin X_n(U)} \phi(\gamma(g^p; x)) \, dx \to 0 \quad (\text{as } a_1 + \cdots + a_n \to \text{area}(D^2)).
\]
This completes the proof. $\square$

As we noted in Remark 2.1, The homomorphism $\hat{\Gamma}_n$ maps any homomorphism on $P_n(D^2)$ to a homomorphism on $\text{Diff}_\infty^\Omega(D^2, \partial D^2)$. Hence the homomorphism
\[
Q(P_n(D^2))/H^1(P_n(D^2); \mathbb{R}) \to Q(\text{Diff}_\infty^\Omega(D^2, \partial D^2))/H^1(\text{Diff}_\infty^\Omega(D^2, \partial D^2); \mathbb{R})
\]
is also induced. By an argument similar to the proof of Theorem 1.2 the following proposition holds.

**Proposition 2.2.** The map

$$Q(B_n(D^2))/H^1(B_n(D^2); \mathbb{R}) \to Q(\text{Diff}_D^\infty(D^2, \partial D^2))/H^1(\text{Diff}_D^\infty(D^2, \partial D^2); \mathbb{R})$$

induced by the composition $\Gamma_n \circ Q(i): Q(B_n(D^2)) \to Q(\text{Diff}_D^\infty(D^2, \partial D^2))$ is injective.

The homomorphism $\Gamma_n: Q(P_n(D^2)) \to Q(\text{Diff}_D^\infty(D^2, \partial D^2))$ can be defined also for the 2-sphere $S^2$ instead of $D^2$ as Gambaudo and Ghys mentioned in their paper. Let $\text{Diff}_D^\infty(S^2)_0$ be the identity component of the group of area-preserving diffeomorphisms of $S^2$. Then we can choose a pure braid $\gamma(g; x) \in P_n(S^2)$ for any $g \in \text{Diff}_D^\infty(S^2)_0$ and for almost every $x \in X_n(S^2)$ as in the case of the 2-disk. Since the group $\text{Diff}_D^\infty(S^2)_0$ is homotopy equivalent to $SO(3)$ and its fundamental group has order 2, for any element $g$ of $\text{Diff}_D^\infty(S^2)_0$ there exist two homotopy classes of paths connecting the identity and $g$ in $\text{Diff}_D^\infty(S^2)_0$. However, for any homogeneous quasi-morphism $\phi$ on $P_n(S^2)$, the value $\phi(\gamma(g; x))$ is independent of the choice of the path. In fact, from a path which represents the generator of $\pi_1(\text{Diff}_D^\infty(S^2)_0)$ has order 2 and is in the center of $P_n(S^2)$. Hence the homomorphism $\Gamma_n: Q(P_n(S^2)) \to Q(\text{Diff}_D^\infty(S^2)_0)$ is defined. Since the braid group $B_n(S^2)$ of the 2-sphere on $n$-strands can be considered as a quotient group of the braid group $B_n(D^2)$, by an argument similar to the proof of Theorem 1.2 we obtain the following theorem.

**Theorem 2.3.** The composition

$$\Gamma_n \circ Q(i): Q(B_n(S^2)) \to Q(\text{Diff}_D^\infty(S^2)_0)$$

is injective.

The homomorphism $Q(i)$ in the statement of Theorem 2.3 is the one induced from the inclusion $i: P_n(S^2) \to B_n(S^2)$.

3. **Kernel of the homomorphism $\Gamma_n$**

The homomorphism $\Gamma_n: Q(P_n(D^2)) \to Q(\text{Diff}_D^\infty(D^2, \partial D^2))$ itself is not injective although Theorem 1.2 holds. In this section we study the kernel of the homomorphism $\Gamma_n$.

Let $G$ be a group and $H$ its finite index subgroup. We denote by $\overline{\beta}$ the image of an element $\beta \in G$ by the natural projection $G \to G/H$. For each left coset $\sigma \in G/H$ of $G$ modulo $H$, we fix an element $\gamma_\sigma \in G$ such that $\overline{\gamma_\sigma} = \sigma$ and for any $\phi \in Q(H)$ define the function $\hat{T}(\phi): G \to \mathbb{R}$ by

$$\hat{T}(\phi)(\beta) = \frac{1}{[G : H]} \sum_{\sigma \in G/H} \phi(\overline{\gamma_\sigma^{-1}\beta\gamma_\sigma}).$$

Since $\overline{\gamma_\sigma^{-1}\beta\gamma_\sigma}$ is in $H$, the function $\hat{T}(\phi)$ is well-defined on $G$.

**Lemma 3.1.** For any quasi-morphism $\phi$ on $H$, the function $\hat{T}(\phi): G \to \mathbb{R}$ is also a quasi-morphism.
Furthermore, we have the decomposition
\[ G \overset{\gamma_{3,2}^{-1} \beta_1 \beta_2 \gamma_6}{\rightarrow} G \]
holds, we have the inequality
\[ |\hat{T}(\phi)(\beta_1 \beta_2) - \hat{T}(\phi)(\beta_1) - \hat{T}(\phi)(\beta_2)| \]
\[ = \frac{1}{(G : H)} \sum_{\sigma \in G/H} \left\{ \phi((\gamma_{3,2}^{-1} \beta_1 \gamma_{2,2}^{-1} \beta_2 \gamma_6)) - \phi(\gamma_{3,2}^{-1} \beta_1 \gamma_6) - \phi(\gamma_{2,2}^{-1} \beta_2 \gamma_6) \right\} \]
\[ = \frac{1}{(G : H)} \sum_{\sigma \in G/H} \left\{ \phi((\gamma_{3,2}^{-1} \beta_1 \gamma_{2,2}^{-1} \beta_2 \gamma_6)) - \phi(\gamma_{2,2}^{-1} \beta_1 \gamma_{3,2}^{-1} \beta_2) \right\} \]
\[ \leq D(\phi). \]

Hence the function \( \hat{T}(\phi) : G \rightarrow \mathbb{R} \) is also a quasi-morphism.

The map \( \hat{T} : \hat{Q}(H) \rightarrow \hat{Q}(G) \) is clearly \( \mathbb{R} \)-linear and induces a homomorphism \( \mathcal{T} : Q(P_n(D^2)) \rightarrow Q(B_n(D^2)) \). Furthermore, the following proposition holds.

**Proposition 3.2.** The homomorphism \( \mathcal{T} : Q(H) \rightarrow Q(G) \) is independent of the choice of \( \gamma_6 \)'s.

**Proof.** Suppose that \( \phi \) is a homogeneous quasi-morphism on \( H \). If an element \( \beta \) is in \( H \), then \( \gamma_6 \beta = \sigma \) for each \( \sigma \in G/H \). For any \( \beta \in G \) there exists an integer \( k \) such that \( \beta^k \) is in \( H \) and we have
\[ \lim_{p \rightarrow \infty} \frac{1}{p} \hat{T}(\phi)(\beta^p) = \lim_{p' \rightarrow \infty} \frac{1}{kp'} \hat{T}(\phi)(\beta^{kp'}) \]
\[ = \lim_{p' \rightarrow \infty} \frac{1}{(G : H)kp'} \sum_{\sigma \in G/H} \phi(\gamma_6^{-1} \beta^k \gamma_6) \]
\[ = \frac{1}{(G : H)k} \sum_{\sigma \in G/H} \phi(\gamma_6^{-1} \beta^k \gamma_6). \quad (3.1) \]

Since \( \phi \) is invariant under conjugations in \( H \), the value \( \phi(\gamma_6^{-1} \beta^k \gamma_6) \) depends only on \( \sigma \).

Let \( Q(i) : Q(G) \rightarrow Q(H) \) be the homomorphism induced by the inclusion \( i : H \rightarrow G \). As a corollary to Equality (3.1), we have the following.

**Corollary 3.3.** The composition \( \mathcal{T} \circ Q(i) : Q(G) \rightarrow Q(G) \) is the identity on \( Q(G) \). Furthermore, we have the decomposition
\[ Q(H) = \text{Ker}(\mathcal{T}) \oplus \text{Im}(Q(i)) \]
as vector spaces.
Remark 3.4. Of course, the homomorphism $\hat{T}(\phi) : G \to \mathbb{R}$ can be defined using the right coset $H \cdot G$ instead of $G/H$ by

$$\hat{T}(\phi)(\beta) = \frac{1}{(G : H)} \sum_{\sigma \in G/H} \phi(\gamma_{\sigma} \beta \gamma_{\sigma^{-1}}).$$

By an argument similar to the proof of Lemma 3.4 and Proposition 3.2, it is verified that this alternative definition is also well-defined and induces the same homomorphism $T : Q(H) \to Q(G)$.

Remark 3.5. The homomorphism $T : Q(H) \to Q(G)$ is just a straightforward generalization of transfer map, and it is also introduced in [5] and [8].

Since the pure braid groups $P_n(D^2)$ and $P_n(S^2)$ are finite index subgroups of the braid groups $B_n(D^2)$ and $B_n(S^2)$, respectively, the homomorphisms $T : Q(P_n(D^2)) \to Q(B_n(D^2))$ and $T : Q(P_n(S^2)) \to Q(B_n(S^2))$ can be defined and Corollary 3.3 is true for $G = B_n(D^2), H = P_n(D^2)$ and $G = B_n(S^2), H = P_n(S^2)$, respectively.

The following proposition is the main result of this section.

**Proposition 3.6.** The composition

$$\Gamma_n \circ Q(i) \circ T : Q(P_n(D^2)) \to Q(\text{Diff}^+_n(D^2, \partial D^2))$$

coincides with $\Gamma_n$. In particular, Ker($\Gamma_n$) = Ker($T$) and Im($\Gamma_n$) = Im($\Gamma_n \circ Q(i)$).

**Proof.** Let $\mathfrak{S}_n$ be the symmetric group of $n$ symbols. By Equality (3.1), for any homogeneous quasi-morphism $\phi \in Q(P_n(D^2))$ and any area-preserving diffeomorphism $g \in \text{Diff}^+_n(D^2, \partial D^2)$,

$$\lim_{p \to \infty} \frac{1}{p} \Gamma_n \circ Q(i) \circ \hat{T}(\phi)(g^p) = \frac{1}{n!} \sum_{\sigma \in \mathfrak{S}_n} \lim_{p \to \infty} \frac{1}{p} \int_{x \in X_n(D^2)} \phi(\gamma_{\sigma} \gamma(g^p ; x) \gamma_{\sigma}^{-1}) dx.$$

(3.2)

For any $\sigma \in \mathfrak{S}_n$ and almost all $x \in D^2$, we set the path $l : [0, 1] \to X_n(D^2)$ by

$$l(t) = \begin{cases} 
{(1 - 2t)x_0^0 + 2tx_1} & \text{for } 0 \leq t \leq \frac{1}{2}, \\
{(2 - 2t)x_i + (2t - 1)x_0^0} & \text{for } \frac{1}{2} \leq t \leq 1.
\end{cases}$$

Considering the path $l$ as a loop in the quotient space $X_n(D^2)/\mathfrak{S}_n$, we define the braid $\beta(\sigma ; x)$ to be the braid represented by the loop $l$. Then by definition,

$$\beta(\sigma ; x) \gamma(g ; \sigma^{-1}(x)) \beta(\sigma ; g_\ast x)^{-1} = \gamma(g ; x),$$

where the symmetric group $\mathfrak{S}_n$ acts on $X_n(D^2)$ by the permutation

$$\sigma(x_1, \ldots, x_n) = (x_{\sigma(1)}, \ldots, x_{\sigma(n)}).$$

Since the homomorphism $T : Q(P_n(D^2)) \to Q(B_n(D^2))$ is defined independently to the choice of braids $\gamma_{\sigma}$, we may choose $\gamma_{\sigma}$ to be $\beta(\sigma ; x)$. Hence we have

$$\gamma_{\sigma} \gamma(g ; \sigma^{-1}(x)) \gamma_{\sigma}^{-1} = \beta(\sigma ; x) \gamma(g ; \sigma^{-1}(x)) \beta(\sigma ; x)^{-1}
= \gamma(g ; x) \beta(\sigma ; g_\ast(x)) \beta(\sigma ; x)^{-1}.$$
Since the function $\phi(\beta(\sigma; \cdot)) : D^2 \to \mathbb{R}$ is bounded on $D^2$, we have
\[
\lim_{p \to \infty} \frac{1}{p} \int_{x \in X_n(D^2)} \phi(\gamma_\sigma \gamma(g^p; x) \gamma_\sigma^{-1}) dx
= \lim_{p \to \infty} \frac{1}{p} \int_{x \in X_n(D^2)} \phi(\gamma_\sigma \gamma(g^p; \sigma^{-1}(x)) \gamma_\sigma^{-1}) dx
= \lim_{p \to \infty} \frac{1}{p} \int_{x \in X_n(D^2)} \phi(\gamma(g^p; x)) dx.
\]
Therefore, by Equality (3.2),
\[
\lim_{p \to \infty} \frac{1}{p} \tilde{\Gamma}_n \circ Q(i) \circ \hat{T}(\phi)(g^p) = \frac{1}{n!} \sum_{\sigma \in S_n} \lim_{p \to \infty} \frac{1}{p} \int_{x \in X_n(D^2)} \phi(\gamma(g^p; x)) dx
= \lim_{p \to \infty} \frac{1}{p} \tilde{\Gamma}_n(\phi)(g^p)
\]
and thus we have $\Gamma_n \circ Q(i) \circ \mathcal{T} = \Gamma_n$.

Then obviously $\text{Ker}(\mathcal{T}) \subseteq \text{Ker}(\Gamma_n)$ and $\text{Im}(\Gamma_n) = \text{Im}(\Gamma_n \circ Q(i))$ hold. If $\phi \in \text{Ker}(\Gamma_n)$ then
\[
\Gamma_n \circ Q(i) \circ \mathcal{T}(\phi) = \Gamma_n(\phi) = 0
\]
and hence $\mathcal{T}(\phi) = 0$ by Theorem 1.2. Thus we have $\text{Ker}(\Gamma_n) \subseteq \text{Ker}(\mathcal{T})$. □

Remark 3.7. Proposition 3.6 also holds for $P_n(S^2)$ and $\text{Diff}_\infty^\infty(S^2)_0$ instead of $P_n(D^2)$ and $\text{Diff}_\Omega^\infty(D^2, \partial D^2)$, respectively.

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