A NEW WAY TO TABULATE KNOTS

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ABSTRACT. We introduce a new way to tabulate knots by representing knot diagrams using a pair of planar trees. This pair of trees have their edges labeled by integers, they have no valence 2 vertices, and they have the same number of valence 1 vertices. The number of valence 1 vertices of the trees is called the girth of the knot diagram. The classification problem of knots admitting girth 2 and 3 diagrams is studied. The planar tree pair representations of girth ≤ 3 for knot diagrams in Rolfsen’s table are given.

1. Introduction

The first successful tabulation of knots was done in the 1880’s by a Scottish physicist, Peter Guthrie Tait [1]. He was able to list all the alternating knots up to ten crossings. Since then, knots have been primarily tabulated by number of crossings. In 1982, Morwen Thistlethwaite used a computer program to generate a table of all prime knots up to thirteen crossings, up to isomorphism and mirror image. He found that there is only one knot with three crossings, one knot with four crossings, two knots with five crossings, three knots with six crossings, and so on, up to 9988 knots with thirteen crossings. All together he listed 12,965 prime knots. This effort has been continued and the most extensive knot table available currently contains all 1,701,936 primes knots with the minimal number of crossings not exceeding 16 (see [2]). Work is underway on further extension of this table to prime knots with 17 and 18 crossings.

Still, with this method, we can consider only finitely many knots at each level of tabulation. In this paper, we develop a new way to tabulate knots which allows us to study infinitely many knots at each level of tabulation. In this tabulation, a knot is represented by a pair of labeled planar trees. Each edge of this pair of planar trees is labeled by an integer. Furthermore, these two trees have no valence 2 vertices and the same number of valence 1 vertices. Fixing such a pair of planar trees and varying the integer labels, we may get an infinite family of knots and links. The main idea of this paper is that it may be relatively easier to classify knots within such an infinite family. In Section 2, we will describe how to get such a pair of planar trees from a knot diagram. It will be clear that the construction is similar to the construction of a Heegaard splitting of a closed orientable 3-manifold.

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Let us consider a pair of planar trees \((T, T')\) such that \(T\) and \(T'\) have no valence 2 vertices and the same number \(g\) of valence 1 vertices. We will call this number \(g\) the \textit{girth} of \((T, T')\). When \(g = 2\), then the number of edges of \(T\) and \(T'\) are both equal to 1. In this case, the knots associated with such pairs \((T, T')\) are \textit{double twist knots}. We will denote knots in this family by \(K(p, q)\), where \(p, q \in \mathbb{Z}\) are the integer labels of the only edge in \(T\) and the only edge in \(T'\), respectively. We have \(K(p, q) = K(q, p)\) and \(K(\pm 2, q)\) are what we usually call \textit{twist knots}. In Section 3, we will classify double twist knots \(K(p, q)\) with \(p, q\) even using information obtained from the Jones polynomial and the Conway polynomial. Note that these knots \(K(p, q)\) are special 2-bridge knots. So it is known that they can be classified using continuous fractions obtained from \(p, q\). Nevertheless, we present our direct argument here for completeness, and for the reason that the same method will be used in the next case.

The next case is \(g = 3\). Then the number of edges of \(T\) and \(T'\) can only be 3 and the only planar tree of 3 edges is the Y-shaped tree. We will denote the knots associated with a pair of Y-shaped trees by \(K\left(\frac{p}{a}, \frac{q}{b}, \frac{r}{c}\right)\), where \(p, q, r \in \mathbb{Z}\) are the edge labels of one Y-shaped tree and \(a, b, c \in \mathbb{Z}\) are the edge labels of the other Y-shaped tree.

When \(p, q, r, a, b, c\) are all even and positive, the diagram \(K\left(\frac{p}{a}, \frac{q}{b}, \frac{r}{c}\right)\) presents a knot. In this case, the Conway polynomial \(\nabla_{K}\) can be calculated explicitly. Using the explicit form of the Conway polynomial, as well as a certain symmetry in the diagram \(K\left(\frac{p}{a}, \frac{q}{b}, \frac{r}{c}\right)\), we see that for a transposition \(\tau \in S_3\),

\[
\nabla_{K}\left(\frac{p}{a}, \frac{q}{b}, \frac{r}{c}\right) = \nabla_{K}\left(\frac{p}{\tau(a)}, \frac{q}{\tau(b)}, \frac{r}{\tau(c)}\right)
\]

iff

\[
K\left(\frac{p}{a}, \frac{q}{b}, \frac{r}{c}\right) = K\left(\frac{p}{\tau(a)}, \frac{q}{\tau(b)}, \frac{r}{\tau(c)}\right).
\]

In general, the calculation of the Jones polynomial \(J_{K}\) via the Kauffman bracket is more manageable for \(K\left(\frac{p}{a}, \frac{q}{b}, \frac{r}{c}\right)\) than the Conway polynomial. In particular, the difference

\[
J_{K}\left(\frac{p}{a}, \frac{q}{b}, \frac{r}{c}\right) - J_{K}\left(\frac{p}{\tau(a)}, \frac{q}{\tau(b)}, \frac{r}{\tau(c)}\right)
\]

turns out to be quite simple. Thus, we will be able to show that in some other cases, the Jones polynomial can be used to distinguish knots in this family. See Section 4 for these results.

Finally, we give a list of knots in the Rolfsen knot table \([6]\) whose diagram of minimal number of crossings can be put in the form of \(K(p, q)\) or \(K\left(\frac{p}{a}, \frac{q}{b}, \frac{r}{c}\right)\). It turns out that all knots with no more than 7 crossings can be put into these forms, and there is only one knot with 8 crossings, \(8_{18}\), that may not be able to put into this form. In fact, knots and links that admit planar tree pair representations of girth \(\leq 3\) occupy a quite large portion of Rolfsen’s table.
Our method is in the same spirit as Conway’s method of knot tabulation using tangles [2]. Such methods are more structural in their enumeration of knot diagrams. On the other hand, the traditional method of knot tabulation using crossing information, as invented by Dowker [3, 4], has the advantage of lending itself more easily to computer practice. And as we mentioned before, this advantage has been fulfilled with rather speculative success. With such a comparison in mind, it is worthwhile to note that our method also lends itself easily to computer enumeration of knot diagrams.

Special infinite families of knots, like torus knots, alternating knots, and rational knots, etc., are very favorable to knot theorists. Such families of knots all have certain rigidity that some topological quantities are determined by their diagrammatic descriptions. It seems to us that \( K(p \ uparrow q \ uparrow r a \ downarrow b \ downarrow c) \) may give us another such a family of knots. More importantly, the construction of this infinite family of knots \( K(p \ uparrow q \ uparrow r a \ downarrow b \ downarrow c) \) can be generalized to produce systematically other infinite families of knots, and these infinite families of knots can be used to exhaust the entire collection of knot types.

2. Heegaard Decomposition of a Knot

Given a knot diagram \( D_K \) for the knot \( K \), we have the checkerboard coloring of complementary regions of \( D_K \). Recall that \( D_K \) is a generic immersed circle in the plane with the crossing information specified at each double point of the knot diagram. We color the complementary regions of the knot diagram \( D_K \) by black or white such that two regions sharing a common edge will be colored differently. For such a checkerboard coloring, an associated planar graph \( G(D_K) \) can be defined. The set of vertices of \( G(D_K) \) will be the complementary regions of \( D_K \) that are colored black, and there is an edge connecting two black regions if they contain a common double point on the diagram \( D_K \). See Figure 1.

Throughout this paper, we will assume that the knot diagram \( D_K \) is reduced in the sense that \( G(D_K) \) has no vertices of valence 1.
Without loss of generality, we may assume that the knot diagram $D_K$ is contained inside of a small neighborhood of the planar graph $G(D_K)$. Now take a spanning tree $T$ of the graph $D_K$, and let $U$ be a closed neighborhood of $T$ in the plane such that

1. $\partial U$ intersects $D_K$ transversely;
2. crossings of $D_K$ in $U$ are in bijection with edges in $T$; and
3. the number of intersection points of $\partial U$ and $D_K$ is minimal among all choices of $U$ satisfying (1) and (2) above.

Outside of $U$, we have a tree $T'$ dual to $T$, which is a spanning tree of the dual graph $G'(D_K)$ of the planar graph $G(D_K)$. The crossings of $D_K$ not in $U$ are in bijection with edges in $T'$. See Figure 2.

Now we can form a Heegaard decomposition of the knot diagram $D_K$: Redraw $D_K$ in $U$ and outside of $U$ in a unit disk. For the tree $T$, we omit all vertices of valance 2 which are not connected to $\partial U$ by edges of $G(D_K)$ not in $T$. The resulting planar tree is called $\bar{T}$. Similarly, we can get a planar tree $\bar{T}'$ from $T'$. Then redrawings of the diagram $D_K$ in $U$ and outside of $U$ both have a sequence of half twists (crossings) on each edge of $\bar{T}$ and $\bar{T}'$. This integer number of half twists gives a label to that edge of $\bar{T}$ or $\bar{T}'$. See the right hand side of Figure 2, where two unit disks with $\bar{T}$ and $\bar{T}'$, as well as the redrawings of $D_K$ in $U$ and outside of $U$ are shown. The original diagram $D_K$ can be then recovered by gluing these two unit disks together appropriately along the boundary. In the notation specified in Section 4, this knot is denoted by $K(\frac{2}{0}, \frac{2}{-1}, \frac{2}{1})$.

Notice that some vertices of $\bar{T}$ may be connected to $\partial U$ by edges of $G(D_K)$ not in $T$. This is also true for $\bar{T}'$. In the redrawings of $D_K$ in the unit disks, we see parts of these edges as dashed lines. For dashed lines coming out of a vertex of $\bar{T}$ or $\bar{T}'$, the edges of $\bar{T}$ or $\bar{T}'$ coming out of that vertex will divide them into equivalence classes.

It is clear that $\bar{T}$ and $\bar{T}'$ have the same number of equivalence classes of dashed lines.
We call this number the *girth* of this pair of trees \((\bar{T}, \bar{T}')\). The example in Figure 2 has the girth equal to 3. See also Figure 3 (a).

An edge in \(\bar{T}\) or \(\bar{T}'\) is called an *exterior edge* if it has a vertex of valence 1. We can remove or add an exterior edge with label 0. See Figure 3. This is how we should understand the 0’s in the notation like \(K(0 \ 2 \ 2 \ 0\ -1\ -1)\). In that example, the edge number \(e\) of \(\bar{T}\) and \(e'\) of \(\bar{T}'\) are both 2, and the girth is 3. We need to add an exterior edge to both \(\bar{T}\) and \(\bar{T}'\) with label 0 in order to have the notation \(K(0 \ 2 \ 2 \ 0\ -1\ -1)\). Thus, by adding exterior edges with label 0 if necessary, we can make the following assumption:

**Assumption 2.1.** The girth of \((\bar{T}, \bar{T}')\) is equal to the number of valence 1 vertices of \(\bar{T}\) or \(\bar{T}'\), respectively.

Finally, if the girth of the pair \((\bar{T}, \bar{T}')\) is \(g\), the boundary of the unit disk \((\partial U)\) is a union of arcs \(A_1, B_1, \ldots, A_g, B_g\), whose interiors are all disjoint. Furthermore, \(A_1, B_1, \ldots, A_g, B_g\) lie on \(\partial U\) in the given cyclic order, such that each \(A_i\) contains the ends of the dashed lines in a single equivalence class in \(\bar{T}\), and each \(B_i\) contains the ends of the dashed lines in a single equivalence class in \(\bar{T}'\).

This completes our description of the essential features of Heegaard decompositions of a knot diagram. In the next two sections, we are going to study two cases in some detail. In the first case, we have the pair \((\bar{T}, \bar{T}')\) with the girth equal to 2 and the edge number \(e = e' = 1\). In the second case, we have the pair \((\bar{T}, \bar{T}')\) with the girth equal to 3 and the edge number \(e = e' = 3\).

We make the following definition.

**Definition 2.2.** The *girth* of a knot or link \(K\) is the minimal girth of all Heegaard decompositions of knot diagrams of \(K\).
Notice the only knots with girth 2 diagrams are the double twist knots $K(p, q)$ where there are $p$ crossings (half twists) on one tree and $q$ on the other. See Figure 4 for the example of $K(3, -2)$. Figure 4 also fixes our convention for positive and negative crossings with respect to edges of the trees $T$ and $T'$. The sign of crossings is used to determined the integer labels of edges of $T$ and $T'$.

**Lemma 3.1.** We have $K(p, q) = K(q, p)$.

*Proof.* Simply switch the trees $T$ and $T'$, we can see that the knots $K(p, q)$ and $K(q, p)$ are isotopic. \hfill $\square$

There is a special case of a single twist knot with $p$ crossings as shown in Figure 5. We will denote it as $K(p)$.

**Lemma 3.2.** We have $K(p, \pm 1) = K(p \mp 1)$.

*Proof.* Simply look at the diagram of $K(p, \pm 1)$ and we will see that the only crossing on $T'$ can be moved to a crossing on $T$ with the sign changed. \hfill $\square$
We want to calculate the Conway polynomial $\nabla_K$ for $K = K(p)$ and $K = K(p, q)$. Recall the Conway polynomial $\nabla_K$ is an invariant of oriented link $K$. For $K(p)$, we want to illustrate the effect of two different ways to orient its component(s) on its Conway polynomial.

By $K(p)$ with the same direction, we mean to orient $K(p)$ in such a way that both strands of the chain of crossings are pointing in the same direction. By $K(p)$ with opposite directions, we mean to orient $K(p)$ in such a way that the strands of the chain of crossing are pointing in different orientations. The latter case is possible only when $p$ is even.

In the first case of $K(p)$ with the same direction, we may apply the Conway skein relation to $L_+ = K(p)$, $L_- = K(p - 1)$, and $L_0 = K(p - 2)$ (see Figure 5) and get
\[ \nabla_K(p) = z \nabla_K(p-1) + \nabla_K(p-2) \]
for $p \geq 2$.

Now, for simplicity, denote $\nabla_{K(p)}(z)$ as $\nabla_p(z)$.

**Lemma 3.3.** For $p > 0$ and $K(p)$ with the same direction, we have
\[ \nabla_p(z) = i^{p-1} U_{p-1} \left( \frac{-zi}{2} \right), \]
where $U_n$ is the $n$th Chebyshev polynomial of the second kind defined by initial values $U_0(x) = 1$ and $U_1(x) = 2x$, and by the recursion $U_{n+1}(x) = 2x U_n(x) - U_{n-1}(x)$, and $i = \sqrt{-1}$.

If $p$ is odd, then $K(p)$ is a knot. Since the Conway polynomial is independent of orientations for knots, it does not matter whether $K(p)$ has the same or opposite directions. Further, if orientation does not matter, then the sign of $p$ does not matter. So for odd $p$, we have $\nabla_p(z) = \nabla_{-p}(z)$. For $p$ even and $K(p)$ with the same direction, we have $\nabla_p(z) = -\nabla_{-p}(z)$.

**Lemma 3.4.** For $p$ even and $K(p)$ with opposite directions, we have
\[ \nabla_p(z) = \frac{p}{2} z, \quad \text{if } p > 0 \]
and
\[ \nabla_p(z) = \frac{-p}{2} z, \quad \text{if } p < 0. \]

To compare the results of Lemma 3.3 and Lemma 3.4, we quote the following explicit formula for the Chebyshev polynomial of the second kind $U_p$:
\[ U_p(x) = \sum_{m=0}^{|p/2|} \binom{p+1}{2m+1} x^{p-2m} (x^2 - 1)^m. \]

Next, we want to calculate the Conway polynomial for $K(p, q)$. 
Consider all the possibilities for values of $p$ and $q$ with all possible orientations for double twist knots $K(p, q)$, also take into consideration the symmetry $K(p, q) = K(q, p)$, we see that there are the following two cases.

**Case 1:** The $p$-crossings have the same direction and the $q$-crossings have opposite directions. Then $p$ can be either even or odd, and $q$ must be odd.

**Case 2:** Both $p$-crossings and $q$-crossings have opposite directions. Then both $p$ and $q$ must be even. Note in this case, $K(p, q)$ must be a knot.

Since the calculation is straightforward using Conway skein relation, we will only list the results of the calculation in these two cases in the following lemmas.

**Lemma 3.5.** For the double twist knot $K(p, q)$ with $p$ nonzero, $q$ odd, the $p$-crossings having the same direction and the $q$-crossings having opposite directions, the Conway polynomial is given by

$$\nabla_{K(p,q)}(z) = \left(\frac{q - 1}{2}\right) z \nabla_p(z) + \nabla_{p+1}(z)$$

if $q < 0$ and

$$\nabla_{K(p,q)}(z) = \nabla_{p-1}(z) - \left(\frac{q + 1}{2}\right) z \nabla_p(z)$$

if $q > 0$.

Here $\nabla_p(z)$ is given in Lemma 3.3.

**Lemma 3.6.** For the double twist knot $K(p, q)$ with $p, q$ even, the Conway polynomial is given by

$$\nabla_{K(p,q)}(z) = \text{sign}(p) \text{sign}(q) \frac{pq}{4} z^2 + 1.$$
Definition 3.7. For $p > 0$, we define

$$S_p = S_p(A) = \sum_{i=1}^{p} A^{2-i}(-A^3)^{p-i}.$$ 

And we use $S_{-p}(A) = S_p(A^{-1})$ to define $S_p$ for $p < 0$.

Let $\langle K(p,q) \rangle$ be the Kauffman bracket of $K(p,q)$. The following formula is easy to obtain.

Lemma 3.8. We have

$$\langle K(p,q) \rangle = (-A^2 - A^{-2})(S_pA^{-q} + S_qA^{-p}) + S_pS_q + A^{-p-q}.$$ 

Lemma 3.9. If $p,q > 1$, then the lowest degree term of $\langle K(p,q) \rangle$ is $-A^{-p-q}$, and the highest degree term of $\langle K(p,q) \rangle$ is $(-1)^{p+q}A^{3(p+q)-4}$. Thus, the span of the Jones polynomial of $K(p,q)$ is $p+q$.

Proof. Notice that the lowest degree term of $S_pS_q$ is $A^{4-p-q}$, and the lowest degree terms of $S_pA^{-q}$ and $S_qA^{-p}$ are both $A^{2-p-q}$. Thus the lowest degree term of $\langle K(p,q) \rangle$ is $-A^{-p-q}$.

Similarly, notice that the highest degree term of $S_pS_q$ is $(-1)^{p+q}A^{3(p+q)-4}$, and the highest degree terms of $S_pA^{-q}$ and $S_qA^{-p}$ are, respectively, $(-1)^{p-1}A^{3p-2-q}$ and $(-1)^{q-1}A^{3q-2-p}$. We have

$$3(p+q) - 4 - 3p + q = 4q - 4 > 0 \quad \text{and} \quad 3(p+q) - 4 - 3q + p = 4p - 4 > 0.$$ 

Therefore, the highest degree term of $\langle K(p,q) \rangle$ is $(-1)^{p+q}A^{3(p+q)-4}$.

We obtained the Jones polynomial by first orienting $K(p,q)$ and calculating the writhe, then normalizing the Kauffman bracket by a factor $A^w$, where the power $w$ is obtained from the writhe, and finally making the substitution $A^4 = t$. Thus the span of the Jones polynomial of $K(p,q)$ is

$$(3(p+q) - 4 - (-p - q))/4 + 1 = p + q.$$ 

Example 3.10. Consider the knots $K(2,8)$ and $K(4,4)$. They have the same Conway polynomial $1 + 4z^2$. But the spans of the Jones polynomial of these two knots are, respectively, 10 and 8. Therefore $K(2,8) \neq K(4,4)$.

Theorem 3.11. Suppose $p,q,a,b$ are even positive integers. Then $K(p,q) = K(a,b)$ iff $\{p,q\} = \{a,b\}$.

Proof. Since $K(p,q) = K(q,p)$, $\{p,q\} = \{a,b\}$ implies $K(p,q) = K(a,b)$.

Suppose $K(p,q) = K(a,b)$. Then by Lemma 3.9 we have $pq = ab$, and by Lemma 3.9 we have $p + q = a + b$. Thus, $\{p,q\}$ and $\{a,b\}$ are both equal to the pair of
roots of the quadratic equation $(x-p)(x-q) = (x-a)(x-b) = 0$. Therefore $\{p, q\} = \{a, b\}$.

**Remark 3.12.** (1) The mirror image of $K(p, q)$ is $K(-p, -q)$, so if we do not distinguish knots and their mirror images, by Theorem 3.11 knots $K(p, q)$ with $p, q$ even and $pq > 0$ are classified by $\{|p|, |q|\}$.

(2) If $pq < 0$, $K(p, q)$ is an alternating knot. The span of the Jones polynomial in this case is $|p| + |q|$. So if we do not distinguish knots and their mirror images, knots $K(p, q)$ with $p, q$ even and $pq < 0$ are also classified by $\{|p|, |q|\}$.

## 4. Knots with girth 3 diagrams

The only girth 3 tree with no valence 2 vertices is $\overbrace{\quad \quad \quad \quad \quad \quad}^\text{3 vertices}$. So we can denote a knot diagram admitting a Heegaard decomposition of girth 3 as $K(p \, q \, r \atop a \, b \, c)$ where $p, q, r, a, b, c$ correspond to the crossings on edges of $T$ and $T'$ as depicted in Figure 6.

For example, the classical $(p, q, r)$ pretzel knots are of the form $K(p \, q \, r \atop \pm 1 \, \pm 1 \, 0)$ in our representation.

The knot diagram $K(p \, q \, r \atop a \, b \, c)$ has some obvious symmetries. We list these symmetries in the following lemma.

**Lemma 4.1.** (1) $K(p \, q \, r \atop a \, b \, c) = K(a \, b \, c \atop p \, q \, r)$;
(2) $K(p \, q \, r \atop a \, b \, c) = K(q \, r \, p \atop b \, c \, a) = K(r \, p \, q \atop c \, a \, b)$;
(3) $K(p \, q \, r \atop a \, b \, c) = K(p \, r \, q \atop c \, a \, b)$.

**Proof.** The relation (1) corresponds to the isotopy that turns the inner ring $\{a, b, c\}$ of the knot diagram in Figure 6 to the outer ring $\{p, q, r\}$ and vice versa.

For the relations (2) and (3), notice that the symmetric group of the equilateral triangle...
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\[ \begin{array}{c}
\text{Diagram 6}
\end{array} \]

is the dihedral group \( D_3 \). Every \( D_3 \) symmetry can be realized by an isotopy of the knot diagram in Figure 6. Then, the relation (2) corresponds to a rotation of 120°, and the relation (3) corresponds to a reflection. □

We say that \( K(p,q,r,a,b,c) = K(p',q',r',a',b',c') \) by a \( D_3 \) symmetry, if \( K(p,q,r,a,b,c) \) and \( K(p',q',r',a',b',c') \) are interchangeable by the relations (2) and (3) in the lemma above.

Thus, the first problem in the classification of these knots \( K(p,q,r,a,b,c) \) is to decide if

\[ K(p,q,r,a,b,c) = K(p,q,r,\tau(a),\tau(b),\tau(c)) \]

for \( \tau \in S_3 \), the symmetric group of 3 elements.

4.1. The Conway polynomial for \( p,q,r,a,b,c \) even. In the case when \( p,q,r,a,b,c \) are all even, \( K(p,q,r,a,b,c) \) is a knot. When we orient this knot, all the crossings have opposite directions. So, as before, the Conway polynomial has a simpler form in this case.

**Lemma 4.2.** For \( p,q,r,a,b,c \) even and positive, the Conway polynomial of \( K(p,q,r,a,b,c) \) is

\[ \nabla_{K(p,q,r,a,b,c)} = (pq + pr + qr)(ab + ac + bc) \left( \frac{z}{2} \right)^4 + (pa + pc + qa + qb + rb + rc) \left( \frac{z}{2} \right)^2 + 1. \]

Thus, we have

\[ \begin{align*}
(1) \quad \nabla_{K(p,q,r,a,b,c)} - \nabla_{K(p,q,r,b,a,c)} &= (p - r)(a - b) \left( \frac{z}{2} \right)^2 \\
(2) \quad \nabla_{K(p,q,r,a,b,c)} - \nabla_{K(p,q,r,a,c,b)} &= (p - q)(c - b) \left( \frac{z}{2} \right)^2 \\
(3) \quad \nabla_{K(p,q,r,a,b,c)} - \nabla_{K(p,q,r,c,b,a)} &= (q - r)(a - c) \left( \frac{z}{2} \right)^2.
\end{align*} \]

From Equations (1), (2), and (3), we can conclude the following result.

**Lemma 4.3.** Let \( \tau \in S_3 \) be a transposition. For \( p,q,r,a,b,c \) even and positive,

\[ \nabla_{K(p,q,r,a,b,c)} = \nabla_{K(p,q,r,\tau(a),\tau(b),\tau(c))}. \]
iff
\[ K \left( \frac{p}{a} \frac{q}{b} \frac{r}{c} \right) = K \left( \frac{p}{\tau(a)} \frac{q}{\tau(b)} \frac{r}{\tau(c)} \right) \]
by a $D_3$ symmetry.

**Proof.** Suppose
\[ \nabla_{K(p_{a}q_{b}r_{c})} = \nabla_{K(p_{c}q_{b}r_{a})}. \]
Then by Equation (3), we have either $q = r$ or $a = c$. When $q = r$, we use Lemma 4.1 (3) to see that
\[ K \left( \frac{p}{a} \frac{q}{b} \frac{r}{c} \right) = K \left( \frac{p}{c} \frac{q}{b} \frac{r}{a} \right). \]
When $a = c$, we certainly have
\[ K \left( \frac{p}{a} \frac{q}{b} \frac{r}{c} \right) = K \left( \frac{p}{a} \frac{q}{b} \frac{r}{c} \right). \]
Other two cases can be proved similarly. 

When $\tau = (132)$ or $\tau = (123)$, the Conway polynomial sometimes fails to distinguish
\[ K \left( \frac{p}{a} \frac{q}{b} \frac{r}{c} \right) \] and
\[ K \left( \frac{p}{c} \frac{q}{b} \frac{r}{a} \right). \] We have the following lemma.

**Lemma 4.4.** For $p, q, r, a, b, c$ even and positive, we have

\[ \nabla_{K(p_{a}q_{b}r_{c})} - \nabla_{K(p_{c}q_{b}r_{a})} = \left| \begin{array}{ccc} p & q & r \\ c & a & b \\ 1 & 1 & 1 \end{array} \right| \left( \frac{z}{2} \right)^{2} \]

and

\[ \nabla_{K(p_{a}q_{b}r_{c})} - \nabla_{K(p_{b}q_{a}r_{c})} = - \left| \begin{array}{ccc} p & q & r \\ a & b & c \\ 1 & 1 & 1 \end{array} \right| \left( \frac{z}{2} \right)^{2} \]

Therefore, it is possible to have $\nabla_{K(p_{a}q_{b}r_{c})} = \nabla_{K(p_{c}q_{b}r_{a})}$ without having $K(p_{a}q_{b}r_{c}) = K(p_{c}q_{b}r_{a})$, and to have\n\[ \nabla_{K(p_{a}q_{b}r_{c})} = \nabla_{K(p_{b}q_{a}r_{c})} \] without having $K(p_{a}q_{b}r_{c}) = K(p_{b}q_{a}r_{c})$.

**4.2. The Kauffman bracket for general** $p, q, r, a, b, c$. The computation of the Jones polynomial via the Kauffman bracket turns out to be simpler than the Conway polynomial for $K(p_{a}q_{b}r_{c})$.

We define first the following symmetric functions on a triplet $(p, q, r)$ as

\[ S^{1}(p, q, r) = S_{p}A^{-q-r} + S_{q}A^{-p-r} + S_{r}A^{-p-q} \]
\[ S^{2}(p, q, r) = S_{p}S_{q}A^{-r} + S_{p}S_{r}A^{-q} + S_{q}S_{r}A^{-p} \]
\[ S^{3}(p, q, r) = S_{p}S_{q}S_{r} \]
\[ S^{0}(p, q, r) = A^{-p-q-r} \]
Lemma 4.5. The Kauffman bracket for $K \left( \frac{p}{a} \frac{q}{b} \frac{r}{c} \right)$ is
\[
\langle K \left( \frac{p}{a} \frac{q}{b} \frac{r}{c} \right) \rangle = \langle S^0(p, q, r)S^0(a, b, c) + S^2(p, q, r)S^2(a, b, c) \rangle \\
+ S_pS_aA^{-q-r-b-c} + S_pS_cA^{-q-r-a-b} + S_qS_aA^{-p-r-b-c} \\
+ S_qS_bA^{-p-r-a-c} + S_rS_bA^{-p-q-a-c} + S_rS_cA^{-p-q-a-b} \\
+ \langle S^1(p, q, r)S^0(a, b, c) + S^0(p, q, r)S^1(a, b, c) + S^2(p, q, r)S^1(a, b, c) \rangle (A^2 - A^{-2}) \\
+ \langle S^1(p, q, r)S^0(a, b, c) + S^0(p, q, r)S^2(a, b, c) + S^2(p, q, r)S^2(a, b, c) \rangle (A^2 - A^{-2})^2 \\
+ \langle S^1(p, q, r)S^1(a, b, c) + S^3(p, q, r)S^1(a, b, c) \rangle (A^2 - A^{-2})^3
\]

Proof. This is by a straightforward computation (with great patience). \qed

So we see that the Kauffman bracket for $K \left( \frac{p}{a} \frac{q}{b} \frac{r}{c} \right)$ is mostly symmetric with respect to permutations of \{p, q, r\} and \{a, b, c\} except for the
\[
S_pS_aA^{-q-r-b-c} + S_pS_cA^{-q-r-a-b} + S_qS_aA^{-p-r-b-c} \\
+ S_qS_bA^{-p-r-a-c} + S_rS_bA^{-p-q-a-c} + S_rS_cA^{-p-q-a-b}
\]
and
\[
\langle S_pS_bA^{-q-r-a-c} + S_qS_cA^{-p-r-a-b} + S_rS_aA^{-p-q-b-c} \rangle (A^2 - A^{-2})^2
\]
terms.

Lemma 4.6. Let $w = p + q + r + a + b + c$. We have:

6. $\langle K \left( \frac{p}{a} \frac{q}{b} \frac{r}{c} \right) \rangle - \langle K \left( \frac{p}{c} \frac{q}{a} \frac{r}{b} \right) \rangle = A^{-w}(S_pA^p - S_rA^r)(S_aA^a - S_bA^b)(1 - (A^2 - A^{-1})^2)$

7. $\langle K \left( \frac{p}{a} \frac{q}{b} \frac{r}{c} \right) \rangle - \langle K \left( \frac{p}{b} \frac{q}{a} \frac{r}{c} \right) \rangle = A^{-w}(S_pA^p - S_qA^q)(S_cA^c - S_bA^b)(1 - (A^2 - A^{-1})^2)$

8. $\langle K \left( \frac{p}{a} \frac{q}{b} \frac{r}{c} \right) \rangle - \langle K \left( \frac{p}{b} \frac{q}{c} \frac{r}{a} \right) \rangle = A^{-w}(S_qA^q - S_rA^r)(S_aA^a - S_cA^c)(1 - (A^2 - A^{-1})^2)$

9. $\langle K \left( \frac{p}{a} \frac{q}{b} \frac{r}{c} \right) \rangle - \langle K \left( \frac{p}{c} \frac{q}{a} \frac{r}{b} \right) \rangle = A^{-w} \begin{vmatrix} S_pA^p & S_qA^q & S_rA^r \\ S_cA^c & S_aA^a & S_bA^b \\ 1 & 1 & 1 \end{vmatrix} (1 - (A^2 - A^{-1})^2)$

10. $\langle K \left( \frac{p}{a} \frac{q}{b} \frac{r}{c} \right) \rangle - \langle K \left( \frac{p}{b} \frac{q}{c} \frac{r}{a} \right) \rangle = -A^{-w} \begin{vmatrix} S_pA^p & S_qA^q & S_rA^r \\ S_aA^a & S_bA^b & S_cA^c \\ 1 & 1 & 1 \end{vmatrix} (1 - (A^2 - A^{-1})^2)$
Consider now the special case that all \( p, q, r, a, b, c \) are even. In this case, the writhe of the knot \( K(\frac{p}{a} \frac{q}{b} \frac{r}{c}) \) is \( w = p + q + r + a + b + c \). Notice that \( w \) is also the writhe of \( K(\frac{p}{c} \frac{q}{a} \frac{r}{b}) \) as well as the writhe of \( K(\frac{r}{b} \frac{q}{c} \frac{p}{a}) \). So we have the following results.

**Lemma 4.7.** Let \( \tau \in S_3 \) be a transposition. For \( p, q, r, a, b, c \) even,

\[
J_{K(\frac{p}{a} \frac{q}{b} \frac{r}{c})} = J_{K(\frac{p}{\tau(a)} \frac{q}{\tau(b)} \frac{r}{\tau(c)})}
\]

iff

\[
K(\frac{p}{a} \frac{q}{b} \frac{r}{c}) = K(\frac{p}{\tau(a)} \frac{q}{\tau(b)} \frac{r}{\tau(c)})
\]

by a \( D_3 \) symmetry.

**Theorem 4.8.** Suppose \( p, q, r, a, b, c \) are even integers. If \( K(\frac{p}{a} \frac{q}{b} \frac{r}{c}) = K(\frac{p}{b} \frac{q}{c} \frac{r}{a}) \), then

\[
\begin{vmatrix}
S_p A^p & S_q A^q & S_r A^r \\
S_c A^c & S_a A^a & S_b A^b \\
1 & 1 & 1
\end{vmatrix} = 0.
\]

Also, if \( K(\frac{p}{a} \frac{q}{b} \frac{r}{c}) = K(\frac{p}{c} \frac{q}{a} \frac{r}{b}) \), then

\[
\begin{vmatrix}
S_p A^p & S_q A^q & S_r A^r \\
S_a A^a & S_b A^b & S_c A^c \\
1 & 1 & 1
\end{vmatrix} = 0.
\]

**Example 4.9.** Consider \( K(\frac{4}{1} \frac{8}{4} \frac{12}{6}) \) and \( K(\frac{4}{2} \frac{8}{4} \frac{12}{6}) \). Since

\[
\begin{vmatrix}
4 & 8 & 12 \\
2 & 4 & 6 \\
1 & 1 & 1
\end{vmatrix} = 0,
\]

we cannot use the Conway polynomial to distinguish these two knots. But

\[
\begin{vmatrix}
S_4 A^4 & S_8 A^8 & S_{12} A^{12} \\
S_2 A^2 & S_4 A^4 & S_6 A^6 \\
1 & 1 & 1
\end{vmatrix} = \frac{A^{32} - 2A^{40} + 2A^{56} - A^{64}}{(A^2 + A^{-2})^2} \neq 0.
\]

Thus,

\[
K(\frac{4}{1} \frac{8}{4} \frac{12}{6}) \neq K(\frac{4}{2} \frac{8}{4} \frac{12}{6}).
\]

4.3. **Other permutations of** \( p, q, r, a, b, c \). Use the same method, We consider briefly one case of a permutation of \( p, q, r, a, b, c \) and its effect on the knot \( K(\frac{p}{a} \frac{q}{b} \frac{r}{c}) \).

Due to the limitation of our method, we consider only the case that all \( p, q, r, a, b, c \) are even so that the writhe of \( K(\frac{p}{a} \frac{q}{b} \frac{r}{c}) \) is \( w = p + q + r + a + b + c \) and it will not change if we permute \( p, q, r, a, b, c \). The question is, in this case, what happens if
\[ \langle K \left( \frac{p}{a} \frac{q}{b} \frac{r}{c} \right) \rangle = \langle K \left( \frac{a}{p} \frac{q}{b} \frac{r}{c} \right) \rangle \] The difference of the Kauffman bracket is:

\[ \langle K \left( \frac{p}{a} \frac{q}{b} \frac{r}{c} \right) \rangle - \langle K \left( \frac{a}{p} \frac{q}{b} \frac{r}{c} \right) \rangle = (S_p A^{q-r-a-b-c} - S_a A^{-p-q-r-b-c}) \]

\[ \cdot \left( S_b S_c A^{-p-r-a} - S_q A^{-p-a-b-c} \right) \]

\[ \cdot \left( -A^{-2} - A^2 - (-A^{-2} - A^2)^3 \right) \]

\[ + (S_q S_b S_c A^{-p-r-a} + S_r S_b S_c A^{-p-q-a} + S_c A^{-p-q-r-a-b} - S_q S_r S_b A^{-p-a-c} - S_q S_r S_c A^{-p-a-b} - S_q A^{-p-r-a-b-c}) \]

\[ \cdot (1 - (-A^{-2} - A^2)^2). \]

If \( p = a \), then we certainly have \( K \left( \frac{p}{a} \frac{q}{b} \frac{r}{c} \right) = K \left( \frac{a}{p} \frac{q}{b} \frac{r}{c} \right) \). So consider the case when \( p \neq a \). Then, if \( \langle K \left( \frac{p}{a} \frac{q}{b} \frac{r}{c} \right) \rangle = \langle K \left( \frac{a}{p} \frac{q}{b} \frac{r}{c} \right) \rangle \), we have

\[ S_q S_b S_c A^{-r} + S_b S_c A^{-a} + S_c A^{-q-r-b} - S_q S_r S_b A^{-c} - S_q S_r S_c A^{-b} - S_q A^{-r-b-c} \]

\[ = (A^2 + A^{-2})(S_b S_c A^{-q-r} - S_q S_r A^{-b-c}). \] (11)

We have

\[ S_q = A^{2-q} - A^{2-q+4} + A^{2-q+8} - \cdots + (-1)^{q-1} A^{3q-2} \]

\[ = A^{-q}(A^2 - A^6 + A^{10} + \cdots + (-1)^{q-1} A^{2+4(q-1)}). \]

Denote

\[ \hat{S}_q := S_q A^q = A^2 - A^6 + A^{10} + \cdots + (-1)^{q-1} A^{2+4(q-1)} = \frac{1 - A^{4q}}{A^2 + A^{-2}}. \]

We can rewrite Equation (11) as follows:

\[ \hat{S}_q \bar{S}_b \bar{S}_c + \hat{S}_b \hat{S}_c - \hat{S}_q \hat{S}_r \bar{S}_c = \hat{S}_q \hat{S}_r \hat{S}_b - \hat{S}_q \hat{S}_r \hat{S}_c \]

\[ = (A^2 + A^{-2})(\hat{S}_b \hat{S}_c - \hat{S}_q \hat{S}_r) + \hat{S}_q - \hat{S}_c. \] (12)

Put in the fraction form of \( \hat{S}_q \), we get

\[ (1 - A^{4q})(1 - A^{4b})(1 - A^{4c}) + (1 - A^{4r})(1 - A^{4b})(1 - A^{4c}) \]

\[ - (1 - A^{4q})(1 - A^{4r})(1 - A^{4b}) - (1 - A^{4q})(1 - A^{4r})(1 - A^{4c}) \]

\[ = (A^2 + A^{-2})^2 \left[ (1 - A^{4b})(1 - A^{4c}) - (1 - A^{4q})(1 - A^{4r}) + A^{4c} - A^{4q} \right]. \] (13)

Let us take derivative with respect to \( A \) of the both sides of Equation (13) at \( A = 1 \). The left hand side is zero, and the right hand side is zero only when \( q = c \).

So now we assume \( q = c \). Then the equation (3) becomes

\[ (1 - A^{4q})^2(A^{4b} - A^{4r}) = (A^2 + A^{-2})^2(1 - A^{4q})(A^{4b} - A^{4r}). \]

This is true only when \( q = 0 \) or \( b = r \). So we have proved the following theorem.
Theorem 4.10. Suppose that all $p, q, r, a, b, c$ are even integers and $p \neq a$. If

$$K \left( \begin{array}{c} p \\ a \\ b \\ c \end{array} \right) = K \left( \begin{array}{c} q \\ r \\ a \\ b \\ c \end{array} \right),$$

then either (1) $q = c = 0$, or (2) $q = c$ and $b = r$.

We finish this section by proposing two problems.

Problem 4.11. Classify knots $K \left( \begin{array}{c} p \\ q \\ r \\ a \\ b \\ c \end{array} \right)$ for $p, q, r, a, b, c$ all even.

Problem 4.12. Show that the knot $8_{18}$ has girth $g > 3$.

5. Planar Tree Pair Representations of Knots

The following table lists planar tree pair representations of knots with girth $g \leq 3$ from Rolfsen’s knots and links table [6]. The representations are not unique.
| Number | [1]  | [2]  | [3]  | [4]  | [5]  | [6]  |
|--------|-----|-----|-----|-----|-----|-----|
| 1064   | \(0, 3, 2\) & \(2^2\) & (2) & \(8^2\) & \((-1, 0, 2)\) & \(9^2\) |
| 1067   | \((-1, -3, -2)\) & \(2^2\) & (4) & \(9^2\) & \(1, -2, -2\) & \(9^2\) |
| 1082   | \((-2, 0, 1)\) & \(6^2\) & (6) & \(9^2\) & \(1, 1, 5\) & \(9^2\) |
| 1085   | \((-1, -1, -1)\) & \(6^2\) & (1, 1) & \(9^2\) & \(1, 1\) & \(9^2\) |
| 1091   | \((-3, 1, 1)\) & \(6^2\) & (3, -3) & \(9^2\) & \(0, -1, 1\) & \(9^2\) |
| 1094   | \((-1, -3, -2)\) & \(6^2\) & (-1, -2) & \(9^2\) & \(1, -3, 1\) & \(9^2\) |
| 1099   | \((-2, 2, -2)\) & \(7^2\) & (1, 0, -4) & \(9^2\) & \(1, 1, 3\) & \(9^2\) |
| 1102   | \((-3, -1, -1)\) & \(7^2\) & (-1, -3, 0) & \(9^2\) & \(-1, 1, 1\) & \(9^2\) |
| 1108   | \((-2, 3, -2)\) & \(7^2\) & (1, 0, 2) & \(9^2\) & \(-1, 1, 1\) & \(9^2\) |
| 1124   | \((1, -3, -4)\) & \(7^2\) & (-1, -2, -2) & \(9^2\) & \(-1, -3, 1\) & \(9^2\) |
| 1125   | \((-1, -3, -4)\) & \(7^2\) & (0, 2, 1) & \(9^2\) & \(-1, 1, 1\) & \(9^2\) |
| 1126   | \((-2, 3, 1)\) & \(7^2\) & (-1, -3, 1) & \(9^2\) & \(-1, -2, -1\) & \(9^2\) |
| 1128   | \((-3, -1, -1)\) & \(7^2\) & (0, 1, 1) & \(9^2\) & \(-1, 1, 1\) & \(9^2\) |
| 1129   | \((2, -2, -1)\) & \(7^2\) & (0, 2, -1) & \(9^2\) & \(-1, -2, -1\) & \(9^2\) |
| 1140   | \((0, -1, 1)\) & \(8^2\) & (8) & \(9^2\) & \(-1, 1, 1\) & \(9^2\) |
| 1142   | \((3, 3, -2)\) & \(8^2\) & (3, -5) & \(9^2\) & \(-1, 4, 1\) & \(9^2\) |
| 1143   | \((2, -2, 1)\) & \(8^2\) & (0, 2, 3) & \(9^2\) & \(-1, 1, 3\) & \(9^2\) |
| 1157   | \((2, -2, 1)\) & \(8^2\) & (-1, -2, 0) & \(9^2\) & \(-1, -2, 1\) & \(9^2\) |
| 1158   | \((3, -1, -1)\) & \(8^2\) & (1, 1, 0) & \(9^2\) & \(-1, -3, 1\) & \(9^2\) |
| 1163   | \((2, 2, 3)\) & \(8^2\) & (-2, 2, 2) & \(9^2\) & \(-1, 1, 1\) & \(9^2\) |

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