Extractable work, the role of correlations, and asymptotic freedom in quantum batteries

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We investigate a quantum battery made of \( N \) two-level systems, which is charged by an optical mode via an energy-conserving interaction. We quantify the fraction \( \varepsilon^{(N)}_B \) of energy stored in the battery that can be extracted in order to perform thermodynamic work. We first demonstrate that \( \varepsilon^{(N)}_B \) is highly reduced by the presence of correlations between the charger and the battery or between the two-level systems composing the battery. We then show that the correlation-induced suppression of extractable energy, however, can be mitigated by preparing the charger in a coherent optical state. We conclude by proving that the charger-battery system is asymptotically free of such locking correlations in the \( N \to \infty \) limit.

Introduction.—The possibility of using quantum phenomena for technological purposes is currently a very active research field. In this context, an interesting research topic is that of “quantum batteries” (QBs) [1–9], i.e. quantum mechanical systems which behave as efficient energy storage devices. This is motivated by the fact that genuine quantum effects, such as entanglement or squeezing, can typically boost the performances of classical protocols, e.g. by speeding up the underlying dynamics [10, 11]. The advantage provided by quantum correlations in the charging (or discharging) process of a QB has been discussed in a fully abstract fashion [1–4] and, more recently, for concrete models that could be implemented in the laboratory [5–7]. Up to now, research efforts have been mostly focused on maximizing the stored energy, minimizing the charging time or maximizing the average charging power [3–7]. A “good” QB, however, should not only store a relevant amount of energy, but also have the capability to fully deliver such energy in a useful way which, said in thermodynamic terms, is the capability of performing work. This observation is not a negligible subtlety, since in quantum information theory it is well known that correlations and entanglement may induce limitations on the task of energy extraction [1, 12–16]. We are therefore naturally led to face a somewhat frustrating situation in which quantum correlations have simultaneously both a positive and a negative effect in the process of energy storage. On the one hand, they can speed up the charging time of QBs, while, on the other hand, they can pose a severe limit on the work that can be actually extracted from it.

By analyzing a prototypical model made of \( N \) two-level systems (qubits) charged by a single optical mode, in this work we shed some light on the competition between the aforementioned positive and negative aspects of quantum correlations. Our findings show that in the case of QBs involving a small number of qubits the energy locked by correlations can be large and must be taken into account for a rigorous and fair analysis of the performance of the QB itself. Luckily, however, this negative effect can be strongly reduced by an optimization over the initial state of the charging system, i.e. by properly preparing the initial state of the charger. Moreover, in the thermodynamic \( N \to \infty \) limit of many qubits, the fraction
of locked energy becomes negligible, independent of the initial state of the charger. We argue that this is a general property of quantum charging processes of closed Hamiltonian systems, which is ultimately linked to the integrability of the dynamics and does not depend on the details of the underlying microscopic model.

Mean Energy versus Extractable work.— We start by defining a general model for the charging process of a QB, schematically represented in Fig. 1(a). Here a first quantum system A acts as the energy “charger” for a second quantum system B that instead acts as the battery of the model. They are characterized by local Hamiltonians $\mathcal{H}_A$ and $\mathcal{H}_B$ respectively, that for the sake of convenience are selected to have both zero ground-state energy. Later on we shall also assume B to be composed of non-mutually interacting elements: for the model as a pure factorized state $|\psi\rangle_A \otimes |0\rangle_B$, with $|0\rangle_B$ being the ground state of $\mathcal{H}_B$, and $|\psi\rangle_A$ having mean local energy $E_A(0) := \lambda \langle \psi | \mathcal{H}_A | \psi \rangle_A > 0$. By switching on a coupling Hamiltonian $\mathcal{H}_1$ between the two systems, our aim is to transfer as much energy as possible from A to B, in some finite amount of time $\tau$, the charging time of the protocol. For this purpose we write the global Hamiltonian of the model as

$$\mathcal{H}(t) := \mathcal{H}_A + \mathcal{H}_B + \lambda(t)\mathcal{H}_1,$$  \hspace{1cm} (1)

where $\lambda(t)$ is a classical parameter that represents the external control we exert on the system, and which we assume to be given by a step function equal to 1 for $t \in [0, \tau]$ and zero elsewhere. Accordingly, indicating with $|\psi(t)\rangle_{AB}$ the evolved state of the system at time $t$, its total energy $E(t) := \text{tr}[\mathcal{H}\rho(t)|\psi(t)\rangle_{AB}\langle\psi(t)|\rho(t)]_{AB}$ is constant at all times with the exception of the switching points, $t = 0$ and $t = \tau$, where some non-zero energy can be passed on AB by the external control (see Ref. [7] for a detailed analysis on the energy cost of modulating the interaction). For the sake of simplicity we set these contributions equal to zero by assuming $\mathcal{H}_1$ to commute with the local terms $\mathcal{H}_A + \mathcal{H}_B$ [17]. Under this condition, the energy that moves from A to B can be expressed in terms of the mean local energy of the battery at the end of the protocol, i.e. the quantity

$$E_B(\tau) := \text{tr}[\mathcal{H}_B\rho_B(\tau)],$$  \hspace{1cm} (2)

$\rho_B(\tau)$ being the reduced density matrix of the battery at time $\tau$. The next question to ask is which part of $E_B(\tau)$ can be extracted from B without having access to the charger (a reasonable scenario in any relevant practical applications where the charger A is not available to the end user), and what is instead locked by the correlations AB have established during the charging process. A proper measure for this quantity is provided by the entropic [18] of the state $\rho_B(\tau)$. We remind that given a quantum system X characterized by a local Hamiltonian $\mathcal{H}$, the ergotropy $\mathcal{E}(\rho, \mathcal{H})$ is a functional which measures the maximum amount of energy that can be extracted from a density matrix $\rho$ of X without wasting into heat. A closed expression for this quantity can be obtained in terms of the difference

$$\mathcal{E}(\rho, \mathcal{H}) = E(\rho) - E(\hat{\rho}),$$  \hspace{1cm} (3)

between the mean energy $E(\rho) = \text{tr}[\mathcal{H}\rho]$ of the state $\rho$ and of the mean energy $E(\hat{\rho}) = \text{tr}[\mathcal{H}\hat{\rho}]$ of the passive counterpart $\hat{\rho}$ of $\rho$ [18][24]. The latter is defined as the density matrix of X which is diagonal on the eigenbasis of $\mathcal{H}$ and whose eigenvalues correspond to a proper reordering of those of $\rho$, i.e. $\hat{\rho} = \sum_n r_n |\epsilon_n\rangle\langle\epsilon_n|$ with $\rho = \sum_n r_n |\epsilon_n\rangle\langle\epsilon_n|$, $\mathcal{H} = \sum_n \epsilon_n |\epsilon_n\rangle\langle\epsilon_n|$, with $r_0 \geq r_1 \geq \cdots$ and $\epsilon_0 \leq \epsilon_1 \leq \cdots$, yielding

$$E(\hat{\rho}) = \sum_n r_n \epsilon_n.$$  \hspace{1cm} (4)

Notice that, if we set the ground-state energy to zero ($\epsilon_0 = 0$) and if the state is pure then $E(\hat{\rho}) = 0$ and the ergotropy coincides with the mean energy of $\rho$, i.e. $\mathcal{E}(\rho, \mathcal{H}) = E(\rho)$. On the contrary, if the state is mixed, the extractable work is in general smaller than the mean energy, i.e. $\mathcal{E}(\rho, \mathcal{H}) < E(\rho)$. For the problem we are considering since the global system dynamics of AB is unitary and the initial state of the charger-battery system is pure, it remains pure at all times. However, the local state of the battery $\rho_B(\tau)$ will be in general mixed because of its entanglement with the charger introducing a non trivial gap between its ergotropy

$$\mathcal{E}_B(\tau) := \mathcal{E}(\rho_B(\tau), \mathcal{H}_B),$$  \hspace{1cm} (5)

and the energy $E_B(\tau)$ of Eq. [2] it stores at the end of the charging process. As we will show below, the former can be much smaller than the latter for the experimentally relevant case of a system composed by a small number of battery elements [25][26].

Results.—For the sake of concreteness and the feasibility of its experimental realization, in the remaining of the paper we focus on a definite model in which the charger A is a photonic cavity coupled to a array of $N$ non-mutually interacting qubits that acts as the battery B [6]. The microscopic Hamiltonian is therefore the one of the Tavis-Cummings model [27][28]. $\mathcal{H}_A = \omega_0 a^\dagger a$, $\mathcal{H}_B = \omega_0 \sum_{i=1}^{N} \sigma_i^+ \sigma_i^-$, $\mathcal{H}_1 = g \sum_{i=1}^{N} (a \sigma_i^+ + a^\dagger \sigma_i^-)$, where $a (a^\dagger)$ is a bosonic annihilation (creation) operator, $\sigma_i^\pm$ are raising/lowering spin operators for the $i$-th qubit, $\omega_0$ is the characteristic frequency of both subsystems, and $g$ the coupling strength ($\hbar = 1$ throughout this work). In this setting we compare the final maximum extractable work measured by the ergotropy $\mathcal{E}_B^{(N)}(\tau)$ and the mean energy $E_B^{(N)}(\tau)$ of the battery with respect to different initial states $|\psi\rangle_A$ of the charger (the label N being added...
FIG. 2. (Color online) The energy $E_B^{(N)}(\tau)$ (solid black line), the ergotropy $\mathcal{E}_B^{(N)}(\tau)$ (dashed red line), and the energy $E_A^{(N)}(\tau)$ (dotted blue line) are shown as functions of $\sqrt{N}g\tau$. All quantities are measured in units of $N\omega_0$. Numerical results in this figure have been obtained by choosing a coherent state for $N = 8$.

We restrict ourselves to three typical quantum optical states [29]: a Fock state, a coherent state, and a squeezed vacuum state, all having the same input energy $E_A^{(N)}(0)$ that we set equal to $N\omega_0$ in order to ensure that it matches the full energy capacity of the battery. In Fig. 2 we show the stored energy $E_B^{(N)}(\tau)$, the energy of the charger $E_A^{(N)}(\tau) := \text{tr}[H_A\rho_A(\tau)]$ and ergotropy $\mathcal{E}_B^{(N)}(\tau)$ as functions of the duration of the charging protocol $\tau$ for the case of the input coherent state. We clearly see that for $\sqrt{N}g\tau \lesssim \pi/4$ the difference between ergotropy and energy is relatively small. Conversely, at large times, correlations between A and B are developed and energy and ergotropy are significantly different.

We now focus on the main point of this work, i.e. a comparison between the fraction of extractable work with respect to the total mean energy of the battery. Consistently with previous approaches already used in the literature [3, 6], we fix the duration of the protocol to the value $\tau = \bar{\tau}$ which ensures the maximum value for the average charging power $P_B^{(N)}(\tau) := E_B^{(N)}(\tau)/\tau$, i.e. $P_B^{(N)}(\bar{\tau}) \leq P_B^{(N)}(\tau)$.

We start by observing that, as explicitly discussed in the Supplemental Material [30] all initial states exhibit the same $P_B^{(N)}(\bar{\tau}) \propto N^{3/2}$ scaling reported in Ref. [6] where only Fock states were considered. This corresponds to a $\bar{\tau} \propto 1/\sqrt{N}$ collective speed-up of the charging time which is independent of the initial state of A, and valid in particular for a semi-classical coherent state.

highly non-classical initial states are therefore not necessary for optimizing the charging part of the protocol.

Next, in Fig. 3(a) we illustrate the dependence of the ratio $\mathcal{E}_B^{(N)}(\bar{\tau})/E_B^{(N)}(\bar{\tau})$ between the extractable work and the mean energy of the battery on the number $N$ of qubits and for the three selected initial states. We clearly see two important facts: (i) for small $N$, the extractable work can be much smaller than the mean energy of the battery and coherent input states appear to be optimal; (ii) for large values of $N$, almost all the mean energy of the battery becomes extractable. The latter result justifies a posteriori previous asymptotic approaches [3, 6] to QBs...
in which only the mean energy was considered as a figure of merit.

Fig. 3(b) shows the amount of energy that can be extracted from a fraction of \( M \leq N \) qubits (and normalized by \( M \)) divided by the same quantity evaluated for all \( N \) qubits (and normalized by \( N \)), i.e., \( [\mathcal{E}_B^{(M)}(\tau)]/M \) divided by \( [\mathcal{E}_B^{(N)}(\tau)]/N \). This ratio describes the fraction of energy that can be extracted when only operations on a subset of \( M \) qubits are allowed. This is of interest because performing operations on all qubits may be experimentally challenging. Our results show, however, that this is in general not necessary. Indeed, our illustrative results for \( N = 8 \) demonstrate that operating on a subset of just \( M = 4 \) is already sufficient to extract \( \geq 3/4 \) of all the available work. We further note that also in this case the coherent states are optimal. The fraction of extractable work from these initial states is weakly affected by the limitation to local operations on \( M \leq N \) qubits, and is practically constant and \( \approx 1 \). These makes coherent states ideal initial charging states for QBs. In the Supplemental Material \[31\] we further elaborate on the fact that coherent states are the optimal choice for work extraction.

Discussion and summary.—We now discuss on more qualitative grounds the physics behind the two main results emerging from our numerical analysis, i.e. the optimality of coherent states for small \( N \) and the asymptotic freedom of the charger-battery system from locking correlations in the \( N \gg 1 \) limit.

From the definition of ergotropy in Eq. \( 3 \), it is clear that the more mixed a state is, the more difficult it is to extract its energy, a fact which is analogous to the difficulty of extracting work from a classical thermodynamic system with large entropy. Since the state of the charger-battery is pure, in our quantum model the entropy of the reduced state of the battery is a consequence of its entanglement with the charger. We can therefore say that, for what concerns the capability of work extraction, it is convenient to produce as little entanglement as possible between the charger and the battery. From this argument, we naturally conclude that highly non-classical initial states of the charger (such as Fock or squeezed states), which induce a complex and entangling dynamics, are not optimal for work extraction. On the contrary, we expect semi-classical states like coherent states, which are well known in quantum optics for producing small entanglement under energy-conserving interactions, to be optimal for maximizing the final ergotropy of the battery (while maintaining the collective speed-up of the charging time). This argument provides a simple yet natural qualitative explanation of our numerical results.

Asymptotic freedom from locking correlations in the \( N \rightarrow \infty \) limit is not a peculiar feature of our model but rather a much more universal fact stemming from the integrability of the dynamics. In order to understand this point we start again from our previous observation that the charger-battery entanglement is the main limiting factor for the task of work extraction. It is well known that the entanglement of the subsystems of an integrable system usually fails to scale with their size. This phenomenon is also known under the name of area law \[31,33\]. On the contrary, the energy is an extensive quantity, which grows linearly with the size of our battery. For this reason, we expect that the relative ratio between the locked and the extractable energy is negligible in the \( N \rightarrow \infty \) limit.

In the specific case of our integrable \[34\] Tavis-Cummings QB, we have an optical mode with mean energy \( N \omega_0 \) which is used to charge \( N \) qubits. Independent of the initial state of the charger, we can well approximate its infinite Hilbert space by truncating it up to the first \( d \) levels, where \( d \) scales at most linearly with \( N \) (see e.g. Refs. \[6\] and \[35\]), i.e.

\[
\frac{\text{# of relevant eigenstates}}{d} \leq \alpha N , \tag{6}
\]

where \( \alpha \) is a positive number. Now, the interaction Hamiltonian \( \mathcal{H}_1 \) considered in the charging protocol is energy-conserving and so the number of relevant eigenvalues of the final state of the charger is also upper bounded by \( d \). This is a simple consequence of the fact that the energy of each Fock state of the optical mode can only be reduced by the interaction with the battery, initially in its ground state. Since the global state of the complete system is pure, the spectrum of \( \rho_A(\tau) \) is equal to the spectrum of \( \rho_B(\tau) \) \[30\] and therefore the number of non-negligible eigenvalues of \( \rho_B(\tau) \) is also at most \( d \leq \alpha N \). As we now show, this implies that the corresponding passive state has non-extensive energy. Indeed, for a battery comprising \( N \) qubits, the degeneracy of the first and second excited states scales like \( N \) and \( N^2 \), respectively. The corresponding subspaces spanned by these states are large enough to contain the relevant spectral components of \( \rho_B(\tau) \). Therefore, there exists a unitary operation mapping the final state of the battery having extensive mean energy \( E(\rho_B(\tau)) \propto N \) into a passive state living approximately within the subspace spanned by the first and second excited energy levels, yielding \( E(\rho_B(\tau)) \leq 2\omega_0 \). Thus, from the definition of ergotropy and in particular from Eq. \( 3 \), we get the following asymptotic limit,

\[
\lim_{N \rightarrow \infty} \frac{\mathcal{E}_B^{(N)}}{\mathcal{E}_B^{(N)}} = 1 . \tag{7}
\]

The physical meaning of Eq. \( 7 \) is that, for “large” batteries charged according to the general protocol considered in this work, almost all the mean energy can be extracted by unitary operations. On the other hand, as we have seen from the previous numerical analysis, this is not the case for small-sized batteries, where there can be a relevant gap between mean energy and extractable work.
In summary, in this work we studied the performance of quantum batteries in terms of their extractable work and we quantitatively compared this figure of merit to the mean energy of the battery. We tackled this problem by addressing a specific but instructive model in which a charging system consisting of a quantized optical field is coupled to a battery made of $N$ qubits, for a finite amount of time $\tau$. The interaction process has two consequences. On the one end, the energy of the charger flows into the battery, as desired. On the other end, however, some entanglement between the charger and the battery is generated. The latter phenomenon implies that the reduced state of the battery can be so mixed that a significant fraction of its energy is actually locked, in the sense that it is not possible to use it to perform work.

We quantified the extractable energy in terms of the ergotropy and we studied its behavior for different initial states and for different values of $N$. For small values of $N$, which is, as discussed in Refs. 6 and 17, a regime that can be reached with current state-of-the-art solid-state technology, the extractable energy can be significantly smaller than the mean energy. This negative effect strongly depends on the choice of the initial state and we found that coherent states are optimal for mitigating this phenomenon. For large values of $N$, instead, we found that ergotropy and mean energy coincide. We also argued that this a rather universal phenomenon characterizing all charger-battery systems in which the amount of entanglement is not extensive with respect to the size $N$ of the battery.

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Supplemental Material for “Extractable work, the role of correlations, and asymptotic freedom in quantum batteries”

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In this Supplemental Material we provide additional information on the explicit form of the initial states mentioned in the main text and the scaling of the average charging power with \( N \) for these states. Finally, we also elaborate on why the coherent state is optimal for the ergotropy.

Explicit form of the three initial states of the charger

We here provide the explicit form of the three initial states studied in our work, i.e. a Fock state, a coherent state, and a squeezed state:

\[
|n\rangle_A = \frac{(a^\dagger)^n}{\sqrt{n!}} |0\rangle, \\
|\alpha\rangle_A = \exp \left( \alpha a^\dagger - \alpha^* a \right) |0\rangle, \\
|z\rangle_A = \exp \left( za^2 - z^*(a^\dagger)^2 \right) |0\rangle,
\]

where \(|0\rangle\) is the vacuum of the cavity. The three parameters \( n, \alpha, \) and \( z \), are fixed by the requirement to have input energy equal to \( E_A^{(N)}(0) = N\omega_0 \). We therefore have \( n = N, \alpha = \sqrt{N}, \) and \( z = \text{arcsinh}(\sqrt{N}) \).

Scaling of the maximum average charging power

Here, we study the maximum average charging power \( P_B^{(N)}(\bar{\tau}) \) as a function of \( N \), for the three initial states introduced in the main text and in the previous section of this file. In Fig. [S1] we report a log-log scale plot of \( P_B^{(N)}(\bar{\tau})/N \) as a function of \( N \). A simple inspection of this plot shows that \( P_B^{(N)}(\bar{\tau})/N \propto \sqrt{N} \), independently of the initial state. Now, by definition, \( P_B^{(N)}(\bar{\tau}) = E_B^{(N)}(\bar{\tau})/\bar{\tau} \). Since \( E_B^{(N)}(\bar{\tau}) \) is an extensive quantity, the collective advantage \( P_B^{(N)}(\bar{\tau})/N \propto \sqrt{N} \) stems from the scaling \( \bar{\tau} \propto 1/\sqrt{N} \) of the optimal time.

Optimality of coherent states for work extraction

In this Section we offer an argument that explains why coherent states with an high number \( N \) of average excitations create weak correlations between \( A \) and \( B \) for relatively small times \( \sqrt{Ng\bar{\tau}} \lesssim \pi/3 \), \( B \) being initially in the ground state. We start by rewriting the Tavis-Cummings

\[
H_A = \omega_0 a^\dagger a, \\
H_B = \omega_0 \left[ J_z^{(N)} + \frac{N}{2} \right], \\
H_1 = g \left[ a J_+^{(N)} + a^\dagger J_-^{(N)} \right].
\]
We now use the Holdstein-Primakoff transformation [S1] to express the collective spin operators in terms of auxiliary harmonic oscillator operators $b$ and $b^{\dagger}$: $J_{z}^{(N)} = (b^{\dagger}b - N/2)$ and $J_{+}^{(N)} = b^{\dagger}\sqrt{N}\sqrt{1 - b^{\dagger}b/N}$. If we are interested only in the first few excitations of the spectrum we can neglect terms like $b^{\dagger}b/N$ (since $N \gg 1$). In this case, we have $J_{+}^{(N)} \approx b^{\dagger}\sqrt{N}$, obtaining

$$
\begin{align*}
\mathcal{H}_{A} &= \omega_{0}a^{\dagger}a , \\
\mathcal{H}_{B} &= \omega_{0}b^{\dagger}b , \\
\mathcal{H}_{1} &\approx g\sqrt{N}(ab^{\dagger} + a^{\dagger}b) . 
\end{align*}
$$

(S5)

The total Hamiltonian is now approximately the one of two harmonic oscillators coupled via a quadratic term. When this approximation holds, an initial coherent state remains a coherent state under time evolution, i.e.

$$
|\Psi(t)\rangle = \exp(-i\mathcal{H}_t) |\sqrt{N}\rangle_A \otimes |0\rangle_B \\
= |\sqrt{N}\cos(g_N t)\rangle_A \otimes |i\sqrt{N}\sin(g_N t)\rangle_B ,
$$

(S6)

where $g_N = \sqrt{Ng}$ and $|\alpha(t)\rangle$ is the coherent state defined by the displacement parameter $\alpha(t)$. The energy stored in $B$, as calculated from Eq. (S5), is $E_B(\tau) \approx N\omega_0 \sin^2(g_N t)$ and is independent of the initial state.

The large-$N$ bosonic approximation is good only for small times, i.e. for $\sqrt{Ng}\tau \ll 1$, when the battery is poorly charged and highly excited states are empty. Furthermore, we can verify a posteriori the condition $b^{\dagger}b/N \ll 1$ by calculating the occupation number in $B$ within the approximation. This yields $\langle b^{\dagger}b \rangle /N = \sin^2(g_N t) \ll 1$, which works for $\sqrt{Ng}\tau \ll 1$.

In Fig. (S2) we compare the energy $E_B^{(N)}$ calculated within the large-$N$ bosonic approximation (black solid line) with that calculated from the exact dynamics. In addition, we indicate by filled symbols the value $E_B^{(N)}(\bar{\tau})$ evaluated at the optimal time $\bar{\tau}$. We clearly see that at the optimal time $\bar{\tau}$ the large-$N$ bosonic approximation is qualitatively correct.

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