Study of the Gribov region in Euclidean Yang-Mills theories in the maximal Abelian gauge

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Abstract

The properties of the Gribov region in SU(2) Euclidean Yang-Mills theories in the maximal Abelian gauge are investigated. This region turns out to be bounded in all off-diagonal directions, while it is unbounded along the diagonal one. The soft breaking of the BRST invariance due to the restriction of the domain of integration in the path integral to the Gribov region is scrutinized. Owing to the unboundedness in the diagonal direction, the invariance with respect to Abelian transformations is preserved, a property which is at the origin of the local $U(1)$ Ward identity of the maximal Abelian gauge.
1 Introduction

In recent years, the maximal Abelian gauge has been largely employed in order to investigate nonperturbative aspects of Yang-Mills theories. The dual superconductivity mechanism for color confinement [1, 2, 3], the Abelian dominance hypothesis [4, 5, 6, 7] and the infrared behavior of the two point gluon and ghost correlation functions [8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21] are examples of such nonperturbative aspects.

One important feature of the maximal Abelian gauge is that it possesses a lattice formulation [22, 23], while being a renormalizable gauge in the continuum [24, 25, 26], a property which has provided a useful comparison among results obtained through numerical simulations and theoretical investigations.

As far as the gluon and ghost propagators are concerned, their study in the maximal Abelian gauge has followed a pattern analogous to that employed in the case of the Landau gauge [26, 27, 28]. Due to the existence of the Gribov copies [29], the allowed gauge field configurations are restricted to the Gribov region \( \Omega \), defined as the set of field configurations corresponding to all relative minima of the minimizing functional

\[
F[A] = \int d^4x \ A_a^\mu A_a^\mu, \tag{1}
\]

where the index \( a = 1, 2 \) runs over the off-diagonal components of the gauge field.

In particular, the restriction to the region \( \Omega \) in the Feynman path integral has been achieved by following the framework outlined by Zwanziger in the Landau gauge [27, 28], amounting to add to the Yang-Mills action a nonlocal term, known as the horizon term. Albeit nonlocal, the horizon term can be cast in local form through the introduction of a set of auxiliary fields, leading to a local action which enjoys the property of being renormalizable [19, 20, 21]. This is the starting point for the analytic investigation of the gluon and ghost propagators. We underline that the results obtained so far [21] display a remarkable agreement with the most recent lattice data [10, 11], see Sect.4 for a brief review.

Though, our current knowledge of the properties of the Gribov region in the maximal Abelian gauge has not yet reached the same understanding which has been achieved in the case of the Landau gauge [20, 32, 33]. A better knowledge of this region would be of great help in order to investigate the nonperturbative behavior of nonabelian gauge theories quantized in the maximal Abelian gauge.

This work aims at filling part of this gap. We shall establish a few results on the Gribov region in the maximal Abelian gauge, providing a better understanding of several features displayed by this gauge. This will be the case, for example, of the existence of a local \( U(1) \) Ward identity which has a natural interpretation within the Abelian dominance hypothesis, according to which the relevant degrees of freedom at low energies should correspond to those encoded in the diagonal component of the gauge field. The off-diagonal components are expected to develop a dynamical mass which decouple them in the low momentum region, a feature which has received support from both lattice [8, 9, 10, 11] and analytic investigations [16].

More specifically, we shall see that the Gribov region \( \Omega \) of the maximal Abelian gauge turns out to be bounded in all off-diagonal directions in field space, while it is unbounded in the diagonal one. This feature makes the Gribov region of the maximal Abelian gauge different from that of the Landau gauge, which is known to be bounded in all directions. Moreover, the unboundedness along the diagonal direction turns out to be at the origin of the \( U(1) \) local Ward identity, which holds even in the presence of the horizon function implementing the restriction to the region \( \Omega \). The convexity of the Gribov region \( \Omega \) will be also established. Furthermore, as in the case of the Landau gauge [34], \( BRST \) invariance turns out to be softly broken by the presence of the Gribov horizon in the off-diagonal directions. As we shall see, this breaking originates from the fact that any infinitesimal gauge transformation of the off-diagonal field components gives rise to field configurations lying outside of the Gribov region.

The paper is organized as follows. Sect.2 is devoted to the study of the Gribov region \( \Omega \) in the maximal Abelian gauge. After establishing that the Gribov region is bounded in the off-diagonal directions and unbounded along the diagonal one, we shall face the issue of the convexity of \( \Omega \). Also, Gribov statement’s about infinitesimal copies located near the horizon will be employed to establish that any infinitesimal gauge transformation of the off-diagonal components of a gauge configuration belonging to \( \Omega \) will give rise to a field configuration lying outside of \( \Omega \). In
Sect. 3 we revise the introduction of the horizon function and we discuss the issues of the soft breaking of the BRST invariance and of the $U(1)$ local Ward identity, in the light of the aforementioned properties of the Gribov region. Sect. 4 contains a brief survey of the main results obtained for the gluon and ghost propagators. In Sect. 5 we present our conclusion.

2 Properties of the Gribov region in the maximal Abelian gauge

2.1 Gauge fixing conditions

In this section we discuss the gauge fixing conditions. Let us begin with the standard notation employed in the case of the maximal Abelian gauge. The gauge field $A_\mu$ is decomposed as

$$A_\mu = A_\mu^A T^A = A_\mu^a T^a + A_\mu T^3 ,$$

where $T^3$ stands for the diagonal generator of the $U(1)$ Cartan subgroup of $SU(2)$, while the index $a = 1, 2$ labels the remaining off-diagonal generators $\{ T^a \}$. Similarly to the decomposition of the gauge field $A_\mu$, for the field strength one has

$$F_{\mu \nu} = F_{\mu \nu}^a T^a + F_{\mu \nu} T^3 ,$$

with the off-diagonal and diagonal components given by

$$F_{\mu \nu}^a = D_{\mu \nu}^a - D_{\mu}^a A_{\nu}^b - D_{\nu}^b A_{\mu}^a ,$$

$$F_{\mu \nu}^3 = F_{\mu \nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + g \varepsilon^{ab} A_\mu^a A_{\nu}^b ,$$

$$\varepsilon^{ab} = \varepsilon^3_{ab} ,$$

where we have introduced the covariant derivative $D_{\mu \nu}^{ab}$ with respect to the diagonal components $A_\mu$ of the gauge field, namely

$$D_{\mu \nu}^{ab} = \partial_\mu \partial_\nu - g \varepsilon^{ab} A_{\mu}^a A_{\nu}^b .$$

For the Yang-Mills action in Euclidean space one obtains

$$S_{YM} = \frac{1}{4} \int d^4 x \left( F_{\mu \nu}^a F_{\mu \nu}^a + F_{\mu \nu} F_{\mu \nu} \right) .$$

As it is easily checked, the classical action (6) is left invariant by the gauge transformations

$$\delta A_\mu^a = - D_{\mu \nu}^{ab} \omega^b - g \varepsilon^{ab} A_{\nu}^b ,$$

$$\delta A_\mu = - \partial_\mu \omega - g \varepsilon^{ab} A_\mu^a \omega^b ,$$

where $(\omega^a, \omega)$ stand for the off-diagonal and diagonal infinitesimal gauge parameters, respectively. The maximal Abelian gauge is obtained by demanding that the off-diagonal components $A_\mu^a$ of the gauge field obey the nonlinear condition

$$D_{\mu \nu}^{ab} A_{\mu}^b = 0 ,$$

which follows by requiring that the auxiliary functional

$$\mathcal{F}[A] = \int d^4 x A_\mu^a A_{\mu}^a ,$$

is stationary with respect to the gauge transformations (7). Moreover, as it is apparent from the presence of the covariant derivative $D_{\mu \nu}^{ab}$, equation (8) allows for a residual local $U(1)$ invariance corresponding to the diagonal subgroup of $SU(2)$. This additional invariance has to be fixed by means of a suitable gauge condition on the diagonal component $A_\mu$, which will be chosen to be of the Landau type, also adopted in lattice simulations, namely

$$\partial_\mu A_\mu = 0 .$$
2.2 The Gribov region of the maximal Abelian gauge

In order to introduce the Gribov region $\Omega$ in the maximal Abelian gauge, let us first remind a few properties of the Faddeev-Popov operator, $\mathcal{M}^{ab}$, which is obtained by taking the second variation of the auxiliary functional $\mathcal{F}[A]$:

$$\delta^2\mathcal{F}[A] = 2 \int d^4x \, \omega^a \mathcal{M}^{ab} \omega^b,$$

(11)

where

$$\mathcal{M}^{ab} = -D^{ad}_{\mu} \epsilon^{cb} - g^2 \epsilon^{ac} \epsilon^{bd} A^c_{\mu} A^d_{\mu}.$$  

(12)

The operator $\mathcal{M}^{ab}$ enjoys the property of being Hermitian and, as pointed out in [29], is the difference of two positive semi-definite operators $O^b_1$ and $O^b_2$, namely

$$\mathcal{M}^{ab} = O^b_1 - O^b_2,$$

$$O^b_1 = -D^{ac}_{\mu} \epsilon^{cb},$$

$$O^b_2 = g^2 \epsilon^{ac} \epsilon^{bd} A^c_{\mu} A^d_{\mu},$$

(13)

where we have introduced the notation $\tilde{A}^a_{\mu} = \epsilon^{ac} A^c_{\mu}$. The positivity of both operators $O^b_1$ and $O^b_2$ is easily established. In fact

$$\langle \psi \mid O_1 \mid \psi \rangle = -\int d^4x \, (\psi^a) \dagger D^a_{\mu} \psi^b = \int d^4x \, (D^a_{\mu} \psi^c) \dagger (D^b_{\mu} \psi^b) - \|D^b_{\mu} \psi^b\|^2 \geq 0.$$  

(14)

Analogously

$$\langle \psi \mid O_2 \mid \psi \rangle = \int d^4x \, (\psi^a) \dagger g^2 \tilde{A}^a_{\mu} \tilde{A}^b_{\mu} \psi^b = \|g \tilde{A}^a_{\mu} \psi^a\|^2 \geq 0.$$  

(15)

It is worth noticing that the operator $O_1$ depends only on the diagonal component $A_{\mu}$, $O_1 = O_1(A)$, while $O_2$ contains only the off-diagonal fields $A^a_{\mu}$, $O_2 = O_2(A^a_{\mu})$.

As in the case of the Landau gauge [27] [28], the Gribov region $\Omega$ of the maximal Abelian gauge is defined as the set of all relative minima of the auxiliary functional $\mathcal{F}[A]$, being given by the set of fields fulfilling the gauge conditions [5], [10], and for which the Faddeev-Popov operator $\mathcal{M}^{ab}$ is positive definite, namely

$$\Omega = \{ A_{\mu}, A^a_{\mu}, \partial_{\mu} A_{\mu} = 0, D^{ab}_{\mu} A^b_{\mu} = 0, \mathcal{M}^{ab} = -D^{ac}_{\mu} D^c_{\mu} - g^2 \epsilon^{ac} \epsilon^{bd} A^c_{\mu} A^d_{\mu} > 0 \}.$$  

(16)

In the following, a few properties of the region $\Omega$ will be established.

2.2.1 Properties of the region $\Omega$ along the off-diagonal directions

- **Statement:** The Gribov region $\Omega$ is bounded in all off-diagonal directions

In order to prove this statement we observe that if $(B^a_{\mu}, B^b_{\mu})$ is a field configuration fulfilling the maximal Abelian gauge conditions, $D^{ab}_{\mu} B^b_{\mu} = 0$, $\partial_{\mu} B^a_{\mu} = 0$, then the re-scaled configuration $(\lambda B^a_{\mu}, B^b_{\mu})$, with $\lambda$ a positive constant factor, obeys the same gauge condition. In fact

$$D^{ab}_{\mu} \lambda B^b_{\mu} = \lambda D^{ab}_{\mu} B^b_{\mu} = 0.$$  

(17)

Let now $(A^a_{\mu}, A_{\mu})$ be a field configuration belonging to $\Omega$, i.e.

$$\langle \psi \mid \mathcal{M}(A^a_{\mu}, A_{\mu}) \mid \psi \rangle = \langle \psi \mid O_1(A) \mid \psi \rangle - \langle \psi \mid O_2(A^a_{\mu}) \mid \psi \rangle > 0.$$  

(18)

Let us consider the re-scaled configuration $(\lambda A^a_{\mu}, A_{\mu})$ and let us evaluate $\langle \psi \mid \mathcal{M}(\lambda A^a_{\mu}, A_{\mu}) \mid \psi \rangle$, namely

$$\langle \psi \mid \mathcal{M}(\lambda A^a_{\mu}, A_{\mu}) \mid \psi \rangle = \langle \psi \mid O_1(A) \mid \psi \rangle - \lambda^2 \langle \psi \mid O_2(A^a_{\mu}) \mid \psi \rangle.$$  

(19)

Since both $\langle \psi \mid O_1(A) \mid \psi \rangle$ and $\langle \psi \mid O_2(A^a_{\mu}) \mid \psi \rangle$ are positive definite, it follows that for $\lambda$ large enough the right hand side of eq. (19) will become negative, meaning that one has left the Gribov region $\Omega$. This shows that moving along the off-diagonal directions parametrized by the re-scaled configuration $(\lambda A^a_{\mu}, A_{\mu})$, with $(A^a_{\mu}, A_{\mu})$ belonging to the Gribov region $\Omega$, one always encounters a boundary $\partial \Omega$, i.e. the horizon, where the first vanishing eigenvalue of the Faddeev-Popov operator appears. Beyond $\partial \Omega$, the operator $\mathcal{M}^{ab}$ ceases to be positive definite.
Thus, it turns out that the field configuration can freely move along the diagonal direction in field space. The Faddeev-Popov operator in fact two field configuration property will be established for configurations lying on the same diagonal hyperplane in field space. Let us consider the issue of the convexity of the region \( \Omega \). Due to the nonlinearity of the gauge conditions, this negative, meaning that the region \( \Omega \) is unbounded in the diagonal direction.

\[ \mathcal{M}^{ab}(\bar{0}, A_\mu) = -D^{ac}_{\mu}(A)D^{cb}_{\mu}(A), \] (20)

which is always positive for an arbitrary choice of the transverse diagonal configuration \( A_\mu \). We see thus that one can freely move along the diagonal direction in field space. The Faddeev-Popov operator \( \mathcal{M}^{ab} \) will never become negative, meaning that the region \( \Omega \) is unbounded in the diagonal direction.

### 2.2.3 Convexity of the region \( \Omega \)

Let us face now the issue of the convexity of the region \( \Omega \). Due to the nonlinearity of the gauge conditions, this property will be established for configurations lying on the same diagonal hyperplane in field space. Let us consider in fact two field configuration \((B^a_\mu, A_\mu), (C^a_\mu, A_\mu)\) fulfilling the gauge conditions, i.e.

\[ D^{ab}_{\mu}(A)B^b_\mu = 0, \quad D^{ab}_{\mu}(A)C^b_\mu = 0, \quad \partial_\mu A_\mu = 0, \] (21)

and belonging to the Gribov region \( \Omega \)

\[ \mathcal{M}(B^a_\mu, A_\mu) > 0, \quad \mathcal{M}(C^a_\mu, A_\mu) > 0. \] (22)

Thus, it turns out that the field configuration \((E^a_\mu, A_\mu)\):

\[ E^a_\mu = \alpha B^a_\mu + (1 - \alpha)C^a_\mu, \quad 0 \leq \alpha \leq 1, \] (23)

belongs to \( \Omega \), namely

\[ \mathcal{M}^{ab}(E^c_\mu, A_\mu) > 0. \] (24)

Proof

\[
\begin{align*}
\mathcal{M}^{ab}(E^c_\mu, A_\mu) &= -D^{ac}_{\mu}(A)D^{cb}_{\mu}(A) - g^2 \bar{E}^a_\mu \bar{E}^b_\mu \\
&= -D^{ac}_{\mu}(A)D^{cb}_{\mu}(A) - \alpha^2 g^2 \bar{B}^a_\mu \bar{B}^b_\mu - (1 - \alpha)^2 g^2 \bar{C}^a_\mu \bar{C}^b_\mu - \alpha(1 - \alpha) g^2 \left( \bar{B}^a_\mu \bar{C}^b_\mu + \bar{C}^a_\mu \bar{B}^b_\mu \right) \\
&= \alpha^2 \left( -D^{ac}_{\mu}(A)D^{cb}_{\mu}(A) - g^2 \bar{B}^a_\mu \bar{B}^b_\mu \right) + (1 - \alpha)^2 \left( -D^{ac}_{\mu}(A)D^{cb}_{\mu}(A) - g^2 \bar{C}^a_\mu \bar{C}^b_\mu \right) \\
&\quad + \alpha(1 - \alpha) g^2 \left( \bar{B}^a_\mu \bar{C}^b_\mu + \bar{C}^a_\mu \bar{B}^b_\mu \right). \\
\end{align*}
\] (25)

From

\[ \bar{B}^a_\mu \bar{C}^b_\mu + \bar{C}^a_\mu \bar{B}^b_\mu = \bar{B}^a_\mu \bar{B}^b_\mu + \bar{C}^a_\mu \bar{C}^b_\mu - \left( \bar{C}^a_\mu - \bar{B}^a_\mu \right) \left( \bar{C}^b_\mu - \bar{B}^b_\mu \right), \] (26)

one has

\[
\begin{align*}
\mathcal{M}^{ab}(E^c_\mu, A_\mu) &= \alpha^2 \mathcal{M}^{ab}(B, A) + (1 - \alpha)^2 \mathcal{M}^{ab}(C, A) + \alpha(1 - \alpha) \left( \mathcal{M}^{ab}(B, A) + \mathcal{M}^{ab}(C, A) \right) \\
&\quad + \alpha(1 - \alpha) g^2 \left( \bar{C}^a_\mu - \bar{B}^a_\mu \right) \left( \bar{C}^b_\mu - \bar{B}^b_\mu \right). \\
\end{align*}
\] (27)

Since the operator \( O^a_2(C - B) = g^2 \left( \bar{C}^a_\mu - \bar{B}^a_\mu \right) \left( \bar{C}^b_\mu - \bar{B}^b_\mu \right) \) is positive definite

\[ \langle \psi | O_2(C - B) | \psi \rangle = \left\| g \left( \bar{C}^a_\mu - \bar{B}^a_\mu \right) \psi \right\|^2 \geq 0, \] (28)

it follows that

\[ \mathcal{M}^{ab}(E^c_\mu, A_\mu) > 0, \] (29)

showing that the field configuration \((E^a_\mu, A_\mu)\) belongs to \( \Omega \), thus establishing the convexity of \( \Omega \).
2.2.4 A statement about field configurations belonging to the Gribov region and infinitesimal gauge transformations

In this section we shall discuss how infinitesimal gauge transformations affect the Gribov region. We shall establish that any infinitesimal gauge transformation of a field configuration lying within the region $\Omega$ will give rise to a configuration which is located outside of $\Omega$. This property will be at the origin of the soft breaking of the BRST invariance of the local action implementing the restriction to the region $\Omega$. In order to prove this statement we shall distinguish two cases.

- **First case: the field is not located close to the boundary $\partial \Omega$**

  Let us consider a field configuration $(A_{\mu}^a, A_{\mu})$ belonging to $\Omega$

  \[ \partial_{\mu} A_{\mu} = 0 , \quad D_{\mu}^{\alpha \beta} A_{\mu}^\beta = 0 , \quad M^{\alpha \beta}(A, A^c) > 0 , \tag{30} \]

  and not located close to the boundary $\partial \Omega$. Let us consider an infinitesimal gauge transformation of the configuration $(A_{\mu}^a, A_{\mu})$, namely

  \[
  \tilde{A}_{\mu}^a = A_{\mu}^a - D_{\mu}^{\alpha \beta} A_{\mu}^\beta - g \varepsilon^{\alpha a} A_{\mu}^\alpha \omega , \\
  \tilde{A}_{\mu} = A_{\mu} - \partial_{\mu} \omega - g \varepsilon^{\alpha a} A_{\mu}^\alpha \omega , \tag{31}
  \]

  where $\omega^a$ are the off-diagonal components of the infinitesimal gauge parameter, while $\omega = \omega^3$ is the diagonal component. Suppose now that $(\tilde{A}_{\mu}^a, \tilde{A}_{\mu})$ belongs to the region $\Omega$. Thus, we should have

  \[ \partial_{\mu} \tilde{A}_{\mu} = 0 , \tag{32} \]

  and

  \[ D_{\mu}^{ab} (\tilde{A}) \tilde{A}_{\mu}^b = 0 . \tag{33} \]

  From condition (32) we would get

  \[ \partial^2 \omega = -g \varepsilon^{\alpha a} \partial_{\mu} (A_{\mu}^a \omega^b) , \tag{34} \]

  while from (33) it would follow

  \[
  0 = \partial \tilde{A}_{\mu} - g \varepsilon^{\alpha a} \tilde{A}_{\mu} \tilde{A}_{\mu}^b \\
  = \partial_{\mu} A_{\mu}^a - g \varepsilon^{\alpha a} A_{\mu}^a A_{\mu}^b + g \varepsilon^{\alpha a} A_{\mu}^a D_{\mu}^{bc} \omega^c + g^2 \varepsilon^{\alpha a} A_{\mu}^a \varepsilon^{bc} A_{\mu}^b \\
  + g \varepsilon^{\alpha a} A_{\mu}^a \partial_{\mu} \omega + g^2 \varepsilon^{\alpha a} \varepsilon^{bc} A_{\mu}^a A_{\mu}^b - \partial_{\mu} D_{\mu}^{ab} \omega^b - g \varepsilon^{\alpha a} \partial_{\mu} (A_{\mu}^a \omega) \\
  = -D_{\mu}^{ac} D_{\mu}^{bc} \omega^b - g^2 \varepsilon^{\alpha a} \varepsilon^{bc} A_{\mu}^a A_{\mu}^b \omega^b , \tag{35}
  \]

  where terms of higher orders in the infinitesimal parameters $(\omega^a, \omega)$ have been neglected. Therefore, condition (33) would imply that the Faddeev-Popov operator $M^{ab}$ should possess a zero mode, i.e.

  \[ M^{ab} \omega^b = 0 , \tag{36} \]

  which contradicts the fact that the configuration $(A_{\mu}^a, A_{\mu})$ belongs to the Gribov region $\Omega$. As a consequence, it follows that the gauge transformed configuration $(\tilde{A}_{\mu}^a, \tilde{A}_{\mu})$, eq. (31), is located outside of $\Omega$.

- **Second case: the field is located close to the boundary $\partial \Omega$**

  Let us consider now the case in which the field configuration $(A_{\mu}^a, A_{\mu})$ lies very close to the boundary of the region $\Omega$. Following [17], we can parametrize $(A_{\mu}^a, A_{\mu})$ as

  \[
  A_{\mu}^a = C_{\mu}^a + a_{\mu}^a , \\
  A_{\mu} = C_{\mu} + a_{\mu} , \tag{37}
  \]

  where $(C_{\mu}^a, C_{\mu})$ lies on the boundary $\partial \Omega$, namely

  \[ \partial_{\mu} C_{\mu} = 0 , \quad D_{\mu}^{ab} (C) C_{\mu}^b = 0 , \tag{38} \]
and

\[ \mathcal{M}^{ab}(C_\mu, C_\nu) \varphi^b = 0, \tag{39} \]

where \( \varphi^b \) is the zero mode of the Faddeev-Popov operator \( \mathcal{M}^{ab}(C_\mu, C_\nu) \). The components \( (a_\mu^a, a_\mu) \) in eq. \([37]\) stand for small perturbations. Let us also introduce, for later convenience, the quantity \( \varphi \) defined as

\[ \varphi = -g \varepsilon^{ab} \frac{1}{\partial^2} \partial_\mu \left( C_\mu^a \varphi^b \right), \tag{40} \]

so that

\[ \partial^2 \varphi = -g \varepsilon^{ab} \partial_\mu \left( C_\mu^a \varphi^b \right). \tag{41} \]

From the gauge conditions

\[ \partial_\mu A_\mu = 0, \quad D_\mu^{ab} (A) A^b_\mu = 0, \tag{42} \]

it follows that

\[ \partial_\mu a_\mu = 0, \quad D_\mu^{ab} (C) a^b_\mu - g \varepsilon^{ab} a_\mu c^b_\mu = 0, \tag{43} \]

where we have neglected higher order terms in the small components \( (a_\mu^a, a_\mu) \). Performing now an infinitesimal gauge transformation of the configuration \( (A_\mu^a, A_\mu) \), one gets

\[ \tilde{A}_\mu^a = C_\mu^a + a_\mu^a - D_\mu^{ab} (C) \varphi^b - g \varepsilon^{ab} C_\mu^b \varphi^b, \]
\[ \tilde{A}_\mu = C_\mu + a_\mu - \partial_\mu \varphi - g \varepsilon^{ab} C_\mu^a \varphi^b. \tag{44} \]

We see thus that, unlike the previous case, the new configuration \( \tilde{A}_\mu^a, \tilde{A}_\mu \) can fulfill the gauge conditions, provided one identifies the infinitesimal parameters \( (\omega^a, \varphi) \) with the components of the zero mode, eqs. \([39], [40] \), i.e. \( (\omega^a, \varphi) = (\varphi^a, \varphi) \). Therefore, the configuration

\[ \tilde{A}_\mu^a = C_\mu^a + a_\mu^a - D_\mu^{ab} (C) \varphi^b - g \varepsilon^{ab} C_\mu^b \varphi^b, \]
\[ \tilde{A}_\mu = C_\mu + a_\mu - \partial_\mu \varphi - g \varepsilon^{ab} C_\mu^a \varphi^b, \tag{45} \]

obeys the gauge conditions

\[ \partial_\mu \tilde{A}_\mu = 0, \quad D_\mu^{ab} (\tilde{A}) \tilde{A}^b_\mu = 0. \tag{46} \]

Nevertheless, due to Gribov’s statement\(^1\), the field \( (\tilde{A}_\mu^a, \tilde{A}_\mu) \) lies precisely outside of the region \( \Omega \).

This ends the proof that any infinitesimal gauge transformation of a field configuration belonging to \( \Omega \) gives rise to a configuration which is located outside of \( \Omega \).

3 Soft breaking of the \textit{BRST} invariance due to the restriction to the Gribov region

This section is devoted to discuss the issue of the \textit{BRST} symmetry when implementing the restriction to the Gribov region. We shall see that, in a way completely analogous to the case of the Landau gauge \([34]\), the restriction to the region \( \Omega \) entails a soft breaking of the \textit{BRST} symmetry whose origin can be traced back to the fact that any infinitesimal gauge transformation of a field configuration belonging to \( \Omega \) gives rise to a configuration which lies outside of \( \Omega \).

\(^1\)Let us remind here Gribov’s statement, proven in \([26]\) in the case of the Landau gauge, and extended to the maximal Abelian gauge in \([17]\) (see Appendix A). Statement: for any field configuration \( (A_\mu^a, A_\mu) \) belonging to the Gribov region \( \Omega \) and located close to the boundary \( \partial \Omega \), there exists an equivalent field configuration \( (\tilde{A}_\mu^a, \tilde{A}_\mu) \), given by

\[ \tilde{A}_\mu^a = C_\mu^a + a_\mu^a - D_\mu^{ab} (C) \varphi^b - g \varepsilon^{ab} C_\mu^b \varphi^b, \]
\[ \tilde{A}_\mu = C_\mu + a_\mu - \partial_\mu \varphi - g \varepsilon^{ab} C_\mu^a \varphi^b, \tag{47} \]

which is, however, located on the other side of the boundary, outside of the region \( \Omega \).
3.1 The Faddeev-Popov action and its BRST invariance

Let us start with the Faddeev-Popov action corresponding to the gauge conditions (8), (10), namely

\[ S_{FP} = S_{YM} + S_{MAG} , \]  

(48)

where \( S_{YM} \) is the Yang-Mills action, eq.(6), and \( S_{MAG} \) stands for the gauge fixing term of the maximal Abelian gauge, given by

\[ S_{MAG} = \int d^4 x \left( i b^a D^a_\mu A^b_\mu - \bar{c}^a M^{ab} c^b + g \varepsilon^{ab} c^a (D^b_\mu A^c_\mu) c + i b \partial_\mu A^b_\mu + \bar{c} \partial_\mu (\partial_\mu c + g \varepsilon^{ab} A^a_\mu c^b) \right) , \]  

(49)

where \((b^a, b)\) are the off-diagonal and diagonal Lagrange multipliers enforcing the gauge conditions \( D^a_\mu A^b_\mu = 0, \partial_\mu A^a_\mu = 0 \). The fields \((c^a, \bar{c}^a, c, \bar{c})\) are the off-diagonal and diagonal Faddeev-Popov ghosts, respectively, and \( M^{ab} \) denotes the Faddeev-Popov operator of eq.(12). The action (48) is left invariant by the nilpotent BRST transformation

\[ s A^a_\mu = - (D^a_\mu b^b + g \varepsilon^{ab} A^b_\mu c^b) , \quad s A_\mu = -(\partial_\mu c + g \varepsilon^{ab} A^a_\mu c^b) , \]
\[ sc^a = g \varepsilon^{ab} c^b c^a , \quad sc = \frac{g}{2} \varepsilon^{ab} c^b c^b , \]
\[ s\bar{c}^a = ib^a , \quad s\bar{c} = ib , \]
\[ sb^a = 0 , \quad sb = 0 \]
\[ s^2 = 0 . \]  

(50)

(51)

Notice that the gauge fixing term (49) can be written as an exact BRST variation

\[ S_{MAG} = s \int d^4 x \left( \bar{c}^a D^a_\mu A^b_\mu + \bar{c} \partial_\mu A^b_\mu \right) . \]  

(52)

3.2 Introduction of the horizon function, localization, and soft breaking of the BRST invariance

As already mentioned, the maximal Abelian gauge is affected by the existence of Gribov copies [29], which have to be taken into account in order to properly quantize the theory. To deal with this problem, it is necessary to restrict the domain of integration in the Feynman path integral to the Gribov region \( \Omega \). As in the case of the Landau gauge [27, 28], this restriction is achieved through the introduction of the horizon function \( S_{Hor} \) which, in the case of the maximal Abelian gauge, is given by the following nonlocal expression [19, 21]

\[ S_{Hor} = \gamma g^2 \int d^4 x \varepsilon^{ab} A^c_\mu (M^{-1})^{ac} \varepsilon^{cb} A^c_\mu . \]  

(53)

The parameter \( \gamma \) appearing in the previous expression has the dimension of a mass and is called the Gribov parameter. It is not a free parameter of the theory, being determined in a self-consistent way through the gap equation [17, 19, 21]

\[ \frac{\delta \Gamma}{\delta \gamma^2} = 0 . \]  

(54)

Therefore, for the partition function we write [19, 21]

\[ Z = \int DA DB D\bar{c} DC e^{-(SYM + S_{MAG} + S_{Hor})} . \]  

(55)

The nonlocal term \( S_{Hor} \) can be localized by means of a pair of complex vector bosonic fields, \((\phi^a_\mu, \bar{\phi}^a_\mu)\) according to

\[ e^{-S_{Hor}} = \int D\phi D\bar{\phi} (\det \mathcal{M})^8 \exp \left\{ - \int d^4 x \left[ \phi^a_\mu \mathcal{M}^{ac} \phi^b_\mu + \gamma^2 g \varepsilon^{ab} (\phi^a_\mu - \bar{\phi}^a_\mu) A^c_\mu \right] \right\} , \]  

(56)
where the determinant \((\det \mathcal{M})^8\) takes into account the Jacobian arising from the integration over the fields \((\phi_{\mu}^{ab}, \bar{\phi}^b_{\mu})\). This term can also be localized by means of a pair of complex vector anticommuting fields \((\omega^a_{\mu}, \bar{\omega}^a_{\mu})\), namely

\[
(\det \mathcal{M})^8 = \int \mathcal{D}\bar{\omega} \mathcal{D}\omega \exp \left( \int d^4x \bar{\omega}_{\mu}^a \mathcal{M}^{ac} \omega_{\mu}^c \right) .
\]  

(57)

Moreover, as done in the case of the Landau gauge \([27, 28]\), it will be useful to perform the following shift in the variable \(\omega_{\mu}^{ab}\) \([19, 21]\).

\[
\omega_{\mu}^{ab} \rightarrow \omega_{\mu}^{ab} + (M^{-1})^{ac} (\mathcal{F}^{cd} \phi_{\mu}^{db}) ,
\]

(58)

where the expression \(\mathcal{F}^{ab}\) stands for

\[
\mathcal{F}^{ab} = 2g \varepsilon^{a(c} (\partial_{\mu} c + g \varepsilon^{de} A_{\mu}^d c_e) D_{\mu}^{cb} + \varepsilon^{a} b_{\mu} (\partial_{\mu} c + g \varepsilon^{cd} A_{\mu}^c d) - g^2 (\varepsilon^{ac} \varepsilon_{bd} + \varepsilon^{ad} \varepsilon_{bc}) A_{\mu}^d (D_{\mu}^{ce} + g \varepsilon^{ce} A_{\mu}^e) .
\]

(59)

Therefore, the nonlocal horizon function gives place to a local term \(S_{\text{Local}}\)

\[
e^{-S_{\text{Hor}}} = \int \mathcal{D}\bar{\phi} \mathcal{D}\phi \mathcal{D}\bar{\omega} \mathcal{D}\omega \ e^{-S_{\text{Local}}} ,
\]

\[
S_{\text{Local}} = \int d^4x \left[ \bar{\phi}_{\mu}^{ab} \mathcal{M}^{ac} \phi_{\mu}^{cb} - \bar{\omega}_{\mu}^{ab} \mathcal{M}^{ac} \omega_{\mu}^{cb} + \varepsilon^{ab} M_{\mu}^{ab} \mathcal{F}^{ac} \phi_{\mu}^{cb} + \gamma^2 g \varepsilon^{ab} (\phi_{\mu}^{ab} - \bar{\phi}_{\mu}^{ab}) A_{\mu} \right] ,
\]

(60)

so that we end up with a completely local action \(S\) implementing the restriction to the Gribov region \(\Omega\), namely

\[
S = S_{\text{YM}} + S_{\text{MAG}} + S_{\text{Local}} .
\]

(61)

Let us investigate now if the action \(S\) displays exact \(BRST\) invariance. Following the analysis done in \([34]\), let us first consider the case in which the Gribov parameter \(\gamma\) is set to zero, \(\gamma = 0\). In this case, the action \(S\) reduces to \(S_0\)

\[
S_0 = S_{\text{YM}} + S_{\text{MAG}} + S_{\text{Local}}|_{\gamma = 0} ,
\]

\[
S_{\text{Local}}|_{\gamma = 0} = \int d^4x \left[ \bar{\phi}_{\mu}^{ab} \mathcal{M}^{ac} \phi_{\mu}^{cb} - \bar{\omega}_{\mu}^{ab} \mathcal{M}^{ac} \omega_{\mu}^{cb} + \varepsilon^{ab} M_{\mu}^{ab} \mathcal{F}^{ac} \phi_{\mu}^{cb} \right] ,
\]

(62)

which corresponds to the case in which the restriction to the Gribov region has not been implemented. The physical content of the action \(S_0\) is thus the same as that of the Faddeev-Popov action \(S_{\text{FP}}\), eq.(63). In fact, it is easily established that integration over the auxiliary fields \((\phi_{\mu}^{ab}, \bar{\phi}_{\mu}^{ab}, \omega_{\mu}^{ab}, \bar{\omega}_{\mu}^{ab})\) amounts to introduce a unity factor in the partition function. We expect thus that in this case, the action \(S_0\) displays exact \(BRST\) invariance. In fact, introducing the following nilpotent \(BRST\) transformations of the auxiliary fields \((\phi_{\mu}^{ab}, \bar{\phi}_{\mu}^{ab}, \omega_{\mu}^{ab}, \bar{\omega}_{\mu}^{ab})\)

\[
s \phi_{\mu}^{ab} = \omega_{\mu}^{ab} ,\quad s \omega_{\mu}^{ab} = 0 ,
\]

\[
s \bar{\phi}_{\mu}^{ab} = \bar{\omega}_{\mu}^{ab} ,\quad s \bar{\omega}_{\mu}^{ab} = 0 ,
\]

(63)

it is easily checked that \(S_{\text{Local}}|_{\gamma = 0}\) can be cast in the form of an exact \(BRST\) variations

\[
S_{\text{Local}}|_{\gamma = 0} = s \int d^4x \left( \bar{\omega}_{\mu}^{ab} \mathcal{M}^{ac} \phi_{\mu}^{cb} \right) ,
\]

(64)

so that \(S_0\) displays exact \(BRST\) invariance, i.e.

\[
sS_0 = 0 .
\]

(65)

It is worth remarking here that the auxiliary fields \((\phi_{\mu}^{ab}, \bar{\phi}_{\mu}^{ab}, \omega_{\mu}^{ab}, \bar{\omega}_{\mu}^{ab})\) transform in such a way that the nilpotency of the \(BRST\) operator is preserved. In particular, from eq.(63), it follows that these fields are assembled in \(BRST\) doublets \([35]\). As such, they do not alter the cohomology of the operator \(s\), which is identified by the colorless gauge invariant operators built up with the field strength and its covariant derivatives. In other words, the introduction of the auxiliary fields does not modify the set of observables of the theory.
Let us now consider the case in which $\gamma \neq 0$, corresponding to the implementation of the restriction to the Gribov region $\Omega$. As one can easily checks, the action $S$ of expression (61) is not left invariant by the BRST transformations, eqs. (59), (63). Instead one has the softly broken identity
\[
S = sS_{\text{Local}} = \gamma^2 \Delta_\gamma ,
\]
where $\Delta_\gamma$ is a dimension two soft breaking term, given by
\[
\Delta_\gamma = g \int d^4 x \left( \varepsilon^{ab} \omega^{\mu a}_b A_\mu - \varepsilon^{ab} \left( \phi^{ab}_\mu - \bar{\phi}^{ab}_\mu \right) \left( \partial_\mu c + g \varepsilon^{mn} A_\mu^m c^n \right) \right) .
\]

We see thus that, as in the case of the Landau gauge [34], the restriction to the Gribov region $\Omega$ entails a soft breaking of the BRST invariance. Notice that the right hand side of eq. (66) is proportional to the Gribov parameter $\gamma$. The presence of this breaking is, however, not unexpected. Its origin relies on the properties of the Gribov region, being a consequence of the fact that infinitesimal gauge transformations of field configurations belonging to $\Omega$ give rise to configurations which are located outside of $\Omega$. Therefore, the existence of a soft breaking of the BRST invariance looks rather natural. As already underlined in [34, 36], the presence of this breaking ensures that the Gribov parameter $\gamma$ is a physical relevant parameter of the theory, entering the expression of the gauge invariant correlation functions. This follows by noting that
\[
\frac{\partial S}{\partial \gamma^2} = \Delta_\gamma ,
\]
so that the expression $(\partial S/\partial \gamma^2)$ cannot be written in the form of an exact BRST term, namely
\[
\frac{\partial S}{\partial \gamma^2} \neq s\tilde{\Delta}_\gamma ,
\]
for some local $\tilde{\Delta}_\gamma$. Equation (69) expresses precisely the fact that $\gamma$ is not a gauge parameter of the theory. The BRST soft breaking is necessary in order to ensure that $\gamma$ is a physical parameter of the theory. Suppose in fact that, instead of giving rise to a soft breaking, the term $S_{\text{Local}}$ would be left invariant by the BRST operator, i.e.
\[
sS_{\text{Local}} = 0 .
\]
Therefore, owing to the doublet structure of the auxiliary fields $(\phi^{ab}_\mu, \bar{\phi}^{ab}_\mu, \omega^{ab}_\mu, \bar{\omega}^{ab}_\mu)$, a local functional $\tilde{S}$ should exist such that
\[
S_{\text{Local}} = s\tilde{S} ,
\]
from which it would follow that
\[
\frac{\partial S_{\text{Local}}}{\partial \gamma^2} = s\frac{\partial \tilde{S}}{\partial \gamma^2} ,
\]
which would imply that $\gamma^2$ is an unphysical parameter.

As in the case of the Landau gauge [27, 28, 30, 31, 34], the presence of the soft breaking term, eq. (66), does not spoil the renormalizability of the theory [19, 21]. This remarkable feature relies on the possibility of extending to the maximal Abelian gauge the same procedure outlined by Zwanziger in the case of the Landau gauge [27, 28], amounting to embed $S_{\text{Local}}$ into a generalized action, $S_{\text{inv}}^{\text{Local}}$ which enjoys exact BRST invariance, namely
\[
S_{\text{Local}} \rightarrow S_{\text{inv}}^{\text{Local}} , \quad sS_{\text{inv}}^{\text{Local}} = 0 .
\]
Moreover, the original action action $S_{\text{Local}}$ can be recovered from the generalized action $S_{\text{inv}}^{\text{Local}}$ by demanding that some external sources of $S_{\text{inv}}^{\text{Local}}$ acquire a particular value. Let us elaborate more on this point. Following [19, 21], the generalized BRST invariant action $S_{\text{inv}}^{\text{Local}}$ turns out to be given by the expression
\[
S_{\text{inv}}^{\text{Local}} = s \int d^4 x \left( \tilde{\omega}_{\mu}^{ab} M^{ac} \phi_{\nu}^{cb} - \tilde{N}_{\mu \nu}^{ab} D_{\mu}^{ac} \phi_{\nu}^{cb} + M_{\mu \nu}^{ab} D_{\mu}^{ac} \tilde{\phi}_{\nu}^{cb} \right) = \int d^4 x \left\{ \phi_{\mu}^{ab} M^{ac} \phi_{\nu}^{cb} - \tilde{\mu}_{\mu}^{ab} M^{ac} \omega_{\nu}^{cb} + \tilde{\omega}_{\mu}^{ab} M^{ac} \phi_{\nu}^{cb} + \tilde{\phi}_{\mu}^{ab} M^{ac} \omega_{\nu}^{cb} + M_{\mu \nu}^{ab} D_{\mu}^{ac} \phi_{\nu}^{cb} + N_{\mu \nu}^{ab} D_{\mu}^{ac} \tilde{\phi}_{\nu}^{cb} + g \varepsilon^{ac} (\partial c + g \varepsilon^{de} A_{\mu}^d c^e ) \phi_{\nu}^{cb} + M_{\mu \nu}^{ab} D_{\mu}^{ac} \tilde{\phi}_{\nu}^{cb} + N_{\mu \nu}^{ab} D_{\mu}^{ac} \tilde{\phi}_{\nu}^{cb} \right\} ,
\]
(74)
where \((M_{\mu\nu}^{ab}, \bar{M}_{\mu\nu}^{ab}), (N_{\mu\nu}^{ab}, \bar{N}_{\mu\nu}^{ab})\) are external sources transforming as BRST doublets, i.e.

\[
\begin{align*}
 sM_{\mu\nu}^{ab} &= N_{\mu\nu}^{ab}, & s\bar{M}_{\mu\nu}^{ab} &= 0, \\
 s\bar{N}_{\mu\nu}^{ab} &= -\bar{M}_{\mu\nu}^{ab}, & s\bar{M}_{\mu\nu}^{ab} &= 0.
\end{align*}
\]

(75)

In order to recover \(S_{\text{Local}}\) from the BRST invariant action \(S_{\text{inv}}^{\text{Local}}\) we first take the physical limit of the external sources \((M_{\mu\nu}^{ab}, \bar{M}_{\mu\nu}^{ab}), (N_{\mu\nu}^{ab}, \bar{N}_{\mu\nu}^{ab})\), which is defined by \([19, 21]\)

\[
\begin{align*}
 M_{\mu\nu}^{ab}|_{\text{phys}} &= -\bar{M}_{\mu\nu}^{ab}|_{\text{phys}} = -\delta^{ab}\delta_{\mu\nu}\gamma^2, \\
 N_{\mu\nu}^{ab}|_{\text{phys}} &= -\bar{N}_{\mu\nu}^{ab}|_{\text{phys}} = 0,
\end{align*}
\]

(76)

and then we perform a shift in the variable \(\omega_{\mu}^{ab}\) as \([19, 21]\)

\[
\omega_{\mu}^{ab} \to \omega_{\mu}^{ab} + (\mathcal{M}^{-1})^{ac} \left[ \gamma^2 g\varepsilon^{cb}(\partial_c \Phi + g\varepsilon^{de}A_{\mu}^{de}) \right],
\]

(77)

so that

\[
S_{\text{inv}}^{\text{Local}}|_{\text{phys}} = S_{\text{Local}}.
\]

(78)

Let us conclude by mentioning that the possibility of writing down the generalized action \(S_{\text{inv}}^{\text{Local}}\) enables us to obtain generalized Slavnov-Taylor identities \([19, 21]\) which can be used to establish the renormalizability of the generalized action \(S_{\text{inv}}^{\text{Local}}\) and, in particular, of the action \(S\), eq. (61).

### 3.3 The U(1) local Ward identity

As we have seen in the previous section, the soft breaking of the BRST invariance is deeply related to the properties of the Gribov region and, in particular, to the existence of a boundary \(\partial \Omega\) along the off-diagonal directions. One should also notice that, from eqs. (7), it follows that, when restricted to the diagonal direction, amounting to set to zero the off-diagonal parameters \(\omega^{a}\), the gauge transformations take the form

\[
\begin{align*}
 \delta_{\text{diag}} A_{\mu} &= -\partial_{\mu} \omega , \\
 \delta_{\text{diag}} A_{\mu}^{a} &= -g\varepsilon^{ab} A_{\mu}^{b} \omega ,
\end{align*}
\]

(79)

where \(\omega\) is the diagonal parameter corresponding to the \(U(1)\) Cartan subgroup. From eqs. (79) one sees that the diagonal field \(A_{\mu}\) transforms as an Abelian \(U(1)\) gauge field, while the off-diagonal components \(A_{\mu}^{a}\) play the role of charged matter fields. Moreover, since the Gribov region is unbounded along the diagonal direction, transformations (79) are expected to correspond to an invariance of the action \(S\) of eq. (61). In other words, expression (61) should display a local \(U(1)\) Ward identity, and this in the presence of the horizon term \(S_{\text{Hor}}\), eq. (53). This turns out to be the case. In fact, the action \(S\) enjoys the \(U(1)\) local Ward identity

\[
\partial_{\mu} \frac{\delta S}{\delta A_{\mu}} + g\varepsilon^{ab} \sum_{\phi} \Phi^{a} \frac{\delta S}{\delta \Phi^{b}} = -i\partial^{2} b ,
\]

(80)

where we have set \(\Phi = (A_{\mu}^{a}, b^{a}, \bar{c}^{a}, c^{a}, \bar{\phi}_{\mu}^{ab}, \phi_{\mu}^{ab}, \bar{\omega}_{\mu}^{ab}, \omega_{\mu}^{ab})\) for all off-diagonal fields. The existence of the local \(U(1)\) Ward identity is an important feature of the maximal Abelian gauge, supporting the so called Abelian dominance. It nicely fits with the unboundedness of the Gribov region \(\Omega\) in the diagonal direction.

### 4 An overview on the gluon and ghost propagators

This section provides a short summary of the main results which have been obtained on the gluon and ghost propagators when taking into account the restriction to the Gribov region \([19, 21]\). Let us spend first a few words on dimension two condensates. One should notice that the introduction of the horizon function \(S_{\text{Hor}}\) in its localized form, expression (60), entails the introduction of a dimension two condensate. In fact, the gap equation
Gribov-Zwanziger framework with the most recent lattice data on the gluon and ghost propagators [40, 39]. An analogous operator has been found in the Landau gauge [31, 37, 34], where it allows to reconcile the fields been analysed recently in [38]. Concerning the third operator, eq.(83), we notice that it depends on the auxiliary fields needed for the localization of the horizon function in the Landau gauge and the indices $A, B, C$ belong to the adjoint representation of $SU(N)$, $A, B, C = 1, ..., N^2 - 1$.

Furthermore, in complete analogy with the case of the Landau gauge [31, 37, 34], other dimension two condensates have to be taken into account in the maximal Abelian gauge. More precisely, the following dimension two operators can be introduced in a way which preserves renormalizability of the theory as well as its symmetry content [21]:

$$O_{A^2} = A_{A}^{a} A_{A}^{a},$$

$$O_{\text{ghost}} = g \varepsilon^{ab}c^a c^b,$$

$$O_{ff} = \left(\phi_{\mu}^{ab} \phi_{\mu}^{ab} - \phi_{\mu}^{ab} \phi_{\mu}^{ab} - c^a c^b\right).$$

The operator (81) is related to the dynamical mass generation for off-diagonal gluons, a feature which supports the Abelian dominance hypothesis. Its condensation has been established in [16], where a dynamical off-diagonal gluon mass $m \simeq 2.2 \Lambda_{\text{MS}}$ has been reported. The ghost operator (82) is needed in order to account for the dynamical breaking of the $SL(2, \mathbb{R})$ symmetry present in the ghost sector of the maximal Abelian gauge. Its condensation has been analysed recently in [38].

Concerning the third operator, eq. (83), we notice that it depends on the auxiliary fields $\left(\phi_{\mu}^{ab}, \phi_{\mu}^{ab}, \phi_{\mu}^{ab}, \omega_{\mu}^{ab}\right)$. It is in fact needed to account for the nontrivial dynamics developed by those fields. An analogous operator has been found in the Landau gauge [31, 37, 34], where it allows to reconcile the Gribov-Zwanziger framework with the most recent lattice data on the gluon and ghost propagators [41, 39]. Let us now summarize our results [21] on the tree level gluon and ghost propagators:

- **The off-diagonal gluon propagator:**
  the transverse off-diagonal gluon propagator turns out to be of the Yukawa type

$$\langle A_{\mu}^{a}(-k)A_{\nu}^{b}(k)\rangle = \frac{1}{k^2 + m^2} \left(\delta_{\mu\nu} - \frac{k_\mu k_\nu}{k^2}\right) \delta^{ab},$$

where $m$ is the dynamical mass originating from the condensation of the gluon operator [31].

- **The diagonal gluon propagator:**
  for the diagonal gluon propagator we have obtained an infrared suppressed propagator of the Gribov-Stingl type, namely

$$\langle A_{\mu}(-k)A_{\nu}(k)\rangle = \frac{k^2 + \mu^2}{k^4 + \mu^2 k^2 + 4 \gamma^4 g^2} \left(\delta_{\mu\nu} - \frac{k_\mu k_\nu}{k^2}\right),$$

where $\gamma$ is the Gribov parameter and $\mu$ is a mass parameter related to the condensation of the operator [34]. We observe that expression (85) does not vanish at the origin. It gives rise to a positivity violating propagator in configuration space, a feature usually interpreted as evidence for gluon confinement.

- **The symmetric off-diagonal ghost propagator:**
  for the symmetric off-diagonal ghost propagator we have found

$$\langle \phi_{\mu}^{ab}(-k)\phi_{\mu}^{ab}(k)\rangle_{\text{symm}} = \frac{k^2 + \mu^2}{k^4 + 2 \mu^2 k^2 + (\mu^4 + v^4)} \delta^{ab},$$

where $v$ is a mass parameter related to the condensation of the ghost operator [32]. Notice that expression (86) is suppressed in the infrared and attains a nonvanishing finite value at $k = 0$.

- **The antisymmetric off-diagonal ghost propagator:**
  finally, for the antisymmetric off-diagonal ghost propagator we have

$$\langle \phi_{\mu}^{ab}(-k)\phi_{\mu}^{ab}(k)\rangle_{\text{antisym}} = \frac{v^2}{k^4 + 2 \mu^2 k^2 + (\mu^4 + v^4)} \varepsilon^{ab}. $$

As expected, this behavior is a consequence of the ghost condensate [38], $\langle \varepsilon^{ab}c^a c^b \rangle \sim v^2$. 
It is worth mentioning that the behavior shown above for the gluon and ghost propagators turns out to be in remarkable agreement with the most recent lattice data, as reported in [10, 11].

5 Conclusion

In this work a study of the Gribov region Ω in the maximal Abelian gauge has been performed. Several features of this region have been established. The region Ω has been proven to be bounded in all off-diagonal directions, while it turns out to be unbounded in the diagonal one. The convexity of Ω has also been established. Roughly speaking, the region Ω looks like an infinite cylinder along the diagonal axis in field space.

The Gribov region of the maximal Abelian gauge looks deeply different from the corresponding region of the Landau gauge, which is in fact bounded in all directions in field space.

The results which have been obtained give us a better understanding of several features of the maximal Abelian gauge. This is the case of the soft breaking of the BRST invariance, deeply related to the fact that the region Ω turns out to be bounded in the off-diagonal directions. Moreover, the unboundedness of Ω along the diagonal direction is at the origin of the local $U(1)$ Ward identity (80).

We also point out that our results nicely fit within the hypothesis of the Abelian dominance, which is a key ingredient for the dual superconductivity picture for color confinement in the maximal Abelian gauge. The picture which emerges from our analysis is that the relevant configurations in the low energy nonperturbative region should be those located very close the diagonal axis in field space. This is supported by the following considerations:

- the ghost propagator, eqs. (86), (87), is non-singular and attains a finite value at the origin $k \approx 0$, a result in very good agreement with the lattice data [10] [11]. This suggests that the relevant configurations are not located near the boundary $\partial \Omega$ of the Gribov region, where the ghost propagator becomes divergent.

- all lattice data obtained so far on the gluon propagators [8, 9, 10, 11] give a clear indication of the fact that the off-diagonal gluon propagator is of the Yukawa type, in agreement with expression (84), and turns out to be suppressed in the infrared with respect to the diagonal gluon propagator which, moreover, can be nicely fitted by a Gribov-Stingl propagator [10, 11], in remarkable agreement with expression (86).

- finally, it has to be noted that when an Abelian configuration in the maximal Abelian gauge, i.e. a configuration lying on the diagonal axis in field space, is gauge transformed so as to fulfill the Landau gauge condition, it is mapped into a configuration lying on the horizon $\partial \Omega$ of the Gribov region Ω of the Landau gauge [41, 42]. These configurations are believed to play a relevant role for gluon confinement in the Landau gauge. This observation might have profound consequences in order to consistently relate our current understanding of the confinement mechanism in different gauges.

Much work is still needed in order to unravel the intricacies of the maximal Abelian gauge. Needless to say, a characterization of the properties of the fundamental modular region of the maximal Abelian gauge would be a very relevant achievement, a task which is beyond our present capabilities. Nevertheless, we hope that our present work will stimulate further investigations on the maximal Abelian gauge.

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