Transfer of regularity for Markov semigroups

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Abstract

We study the regularity of a Markov semigroup \((P_t)_{t \geq 0}\), that is, when \(P_t(x, dy) = p_t(x, y)dy\) for a suitable smooth function \(p_t(x, y)\). This is done by transferring the regularity from an approximating Markov semigroup sequence \((P^n_t)_{t \geq 0}, n \in \mathbb{N}\), whose associated densities \(p^n_t(x, y)\) are smooth and can blow up as \(n \to \infty\). We use an interpolation type result and we show that if there exists a good equilibrium between the blow-up and the speed of convergence, then \(P_t(x, dy) = p_t(x, y)dy\) and \(p_t\) has some regularity properties.

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Keywords: Markov semigroups; regularity of probability laws; interpolation techniques.

2010 MSC: 60J25, 46B70.

Acknowledgments. This research is partly funded by the Bézout Labex, funded by ANR, reference ANR-10-LABX-58. L.C. also acknowledges the MIUR Excellence Department Project awarded to the Department of Mathematics, University of Rome Tor Vergata. The Beyond Borders Project “Asymptotic Methods in Probability” is acknowledged by both authors.

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1 Introduction

In this paper we study Markov semigroups, that is, positive semigroups \((P_t)_{t \geq 0}\), such that \(P_1 = 1\). The link with Markov processes is given by a family \(P_t(x, dy)\), \(t \geq 0\), \(x \in \mathbb{R}^d\), of transition probability measures in \(\mathbb{R}^d\) such that

\[
P_t f(x) = \int_{\mathbb{R}^d} f(y) P_t(x, dy), \quad t \geq 0.
\]

We study here the regularity of \(P_t\), which is the property \(P_t(x, dy) = p_t(x, y)dy\), \(t > 0\), for a suitable smooth function \(p_t(x, y)\), by transferring the regularity from an approximating Markov semigroup sequence \((P^n_t)_{t \geq 0}\), \(n \in \mathbb{N}\).

Hereafter we assume that the domain of the Markov semigroup \((P_t)_{t \geq 0}\) contains the Schwartz space \(S(\mathbb{R}^d)\) of the \(C^\infty(\mathbb{R}^d)\) functions all of whose derivatives are rapidly decreasing. We assume that the semigroup is strongly continuous in its domain and we call \(L\) its infinitesimal generator. We suppose also that the domain of \(L\) contains \(S(\mathbb{R}^d)\) and for every \(f \in S(\mathbb{R}^d)\), \(P_tf \in S(\mathbb{R}^d)\), \(t \geq 0\), and \(Lf \in S(\mathbb{R}^d)\).

Let \((P^n_t)_{t \geq 0}, n \in \mathbb{N}\), be a sequence of Markov semigroups:

\[
P^n_t f(x) = \int_{\mathbb{R}^d} f(y) P^n_t(x, dy), \quad t \geq 0, n \in \mathbb{N}.
\]

For every \(n\), we assume that \((P^n_t)_{t \geq 0}\) satisfies the same properties as \((P_t)_{t \geq 0}\): \(S(\mathbb{R}^d)\) is included in the domain of \(P^n_t\) and if \(f \in S(\mathbb{R}^d)\) then \(P_tf \in S(\mathbb{R}^d)\), \(t \geq 0\); \((P^n_t)_{t \geq 0}\) is strongly continuous in its domain; the domain of its infinitesimal operator \(L_n\) contains \(S(\mathbb{R}^d)\) and \(L_n f \in S(\mathbb{R}^d)\) if \(f \in S(\mathbb{R}^d)\).

Classical results (Trotter Kato theorem, see e.g. [14]) assert that, as \(n \to \infty\), if \(L_n \to L\) then \(P^n_t \to P_t\). The problem that we address in this paper is the following. We suppose that \(P^n_t\) has the regularity (density) property \(P^n_t(x, dy) = p^n_t(x, y)dy\) with \(p^n_t \in C^\infty(\mathbb{R}^d \times \mathbb{R}^d)\) and we ask under which hypotheses this property is inherited by the limit semigroup \((P_t)_{t \geq 0}\).

If we know that \(p^n_t\) converges to some \(p_t\) in a sufficiently strong sense, of course we obtain \(P_t(x, dy) = p_t(x, y)dy\). But in our framework \(p^n_t\) does not converge: here, \(p^n_t\) can even “blow up” as \(n \to \infty\). However, if we may find a good equilibrium between the blow-up and the speed of convergence, then we are able to conclude that \(P_t(x, dy) = p_t(x, y)dy\) and \(p_t\) has some regularity properties. This is an interpolation type result.

Roughly speaking our main result is as follows. We assume that the speed of convergence is controlled in the following sense: there exists some \(a \in \mathbb{N}\) such that for every \(q \in \mathbb{N}\)

\[
\|(L - L_n)f\|_{q, \infty} \leq \varepsilon_n \|f\|_{q+a, \infty}.
\]

Here \(\|f\|_{q, \infty}\) is the norm in the standard Sobolev space \(W^{q, \infty}\). In fact we will work with weighted Sobolev spaces, and this is an important point. And also, we will assume a similar hypothesis for the adjoint \((L - L_n)^*\) (see Assumption [2.1] for a precise statement).

Moreover we assume a “propagation of regularity” property: there exist \(b \in \mathbb{N}\) and \(\Lambda_n \geq 1\) such that for every \(q \in \mathbb{N}\)

\[
\|P^n_t f\|_{q, \infty} \leq \Lambda_n \|f\|_{q+b, \infty}.
\]

Here also we will work with weighted Sobolev norms. And a similar hypothesis is supposed to hold for the adjoint \(P^n_*\) (see Assumption [2.2] for a precise statement).
Finally we assume the following regularity property: for every \( t \in (0,1] \), \( P^n_t(x,dy) = p^n_t(x,y)dy \) with \( p^n_t \in C^\infty(\mathbb{R}^d \times \mathbb{R}^d) \) and for every \( \kappa \geq 0, t \in (0,1] \),

\[
\left| \partial_x^\alpha \partial_y^\beta p^n_t(x,y) \right| \leq \frac{C}{(\lambda_n t)^{\theta_0(|\alpha|+|\beta|+\theta_1)}} \times \frac{(1+|x|^2)^{\pi(\kappa)}}{(1+|x-y|^2)^\kappa}.
\] (1.3)

Here, \( \alpha, \beta \) are multi-indexes and \( \partial_x^\alpha, \partial_y^\beta \) are the corresponding differential operators. Moreover, \( \pi(\kappa), \theta_0 \) and \( \theta_1 \) are suitable parameters and \( \lambda_n \to 0 \) as \( n \to \infty \) (we refer to Assumption 2.3). In concrete examples (jump type stochastic differential equations) \( \lambda_n \) is related to the lower eigenvalue of the Malliavin covariance matrix - essentially this is of order \( \lambda_n^{\theta_0} \). And in order to handle the derivatives \( \partial_x^\alpha \) and \( \partial_y^\beta \) we need to make \( |\alpha| \) respectively \( |\beta| \) integrations by parts (which involve \( \lambda_n^{\theta_0} \)). See also Assumption 3.6.

By (1.1)–(1.3), the rate of convergence is controlled by \( \varepsilon_n \to 0 \) and the blow-up of \( p^n_t \) is controlled by \( \lambda_n^{-\theta_0} \to \infty \). So the regularity property may be lost as \( n \to \infty \). However, if there is a good equilibrium between \( \varepsilon_n \to 0 \) and \( \lambda_n^{-\theta_0} \to \infty \) and \( \Lambda_n \to \infty \) then the regularity is saved: we ask that for some \( \delta > 0 \)

\[
\limsup_{n \to \infty} \frac{\varepsilon_n \Lambda_n}{\lambda_n^{\theta_0(a+b+\delta)}} < \infty,
\] (1.4)

the parameters \( a, b \) and \( \theta_0 \) being given in (1.1), (1.2) and (1.3) respectively. Then \( P_t(x,dy) = p_t(x,y)dy \) with \( p_t \in C^\infty(\mathbb{R}^d \times \mathbb{R}^d) \) and the following upper bound holds: for every \( \varepsilon > 0, \kappa \in \mathbb{N} \) and \( R > 0 \), one may find some constant \( C, \pi(\kappa) > 0 \) such that for every \( (x,y) \in \mathbb{R}^d \times \mathbb{R}^d \) with \( |x| < R \) and \( t \in (0,1] \)

\[
\left| \partial_x^\alpha \partial_y^\beta p_t(x,y) \right| \leq \frac{C}{(t^{\theta_0(1+\frac{a+b}{2})}|\alpha|+|\beta|+2d+\varepsilon)} \times \frac{(1+|x|^2)^{\pi(\kappa)}}{(1+|x-y|^2)^\kappa}.
\] (1.5)

This is the “transfer of regularity” that we mention in the title and which is stated in Theorem 2.6. The proof is based on a criterion of regularity for probability measures given in 3, which is close to interpolation spaces techniques.

The regularity criterion presented in this paper is tailored in order to handle the following example (which will be treated in a forthcoming paper). We consider the integro-differential operator

\[
L f(x) = \langle b(x), \nabla f(x) \rangle + \int_E (f(x + c(z,x)) - f(x) - \langle c(z,x), \nabla f(x) \rangle) d\mu(z)
\] (1.6)

where \( \mu \) is an infinite measure on the normed space \( (E, |\cdot|_E) \) such that \( \int_E 1 \wedge |c(z,x)|^2 \, d\mu(z) < \infty \). Moreover, for a sequence \( \delta_n \downarrow 0 \), we denote

\[
A_{n}^{ij}(x) = \int_{\{|z|_E \leq \delta_n\}} c^j(z,x)c^i(z,x) d\mu(z)
\]

and we define

\[
L_n f(x) = \langle b(x), \nabla f(x) \rangle + \int_{\{|z|_E \geq \delta_n\}} (f(x + c(z,x)) - f(x) - \langle c(z,x), \nabla f(x) \rangle) d\mu(z)
\]

\[
+ \frac{1}{2} \text{tr}(A_n(x) \nabla^2 f(x)).
\] (1.7)
By Taylor’s formula,
\[ \|L f - L_n f\|_{3,\infty} \leq \|f\|_{3,\infty} \varepsilon_n \quad \text{with} \quad \varepsilon_n = \sup_x \int_{\{|z|\leq \delta_n\}} |c(z, x)|^3 \, d\mu(z) \]
(recall that \( \| \cdot \|_{3,\infty} \) is the norm in the standard Sobolev space \( W^{3,\infty} \)). Under the uniform ellipticity assumption \( A_n(x) \geq \lambda_n \) for every \( x \in \mathbb{R}^d \), the semigroup \( (P^t_n)_{t \geq 0} \) associated to \( L_n \) has the regularity property \( (1.3) \) with \( \theta_0 \) depending on the measure \( \mu \). The speed of convergence in \( (1.4) \), with \( a = 3 \), is controlled by \( \varepsilon_n \downarrow 0 \). So, if \( (1.4) \) holds, then we obtain the regularity of \( P_t \) and the short time estimates \( (1.5) \).

The semigroup \( (P_t)_{t \geq 0} \) associated to \( L \) corresponds to stochastic equations driven by the Poisson point measure \( N_\mu(dt, dz) \) with intensity measure \( \mu \), so the problem of the regularity of \( P_t \) has been extensively discussed in the probabilistic literature. A first approach initiated by Bismut \[9\], Léandre \[20\] and Bichteler, Gravereaux and Jacod \[8\] (see also the recent monograph of Bouleau and Denis \[10\] and the bibliography therein), is done under the hypothesis that \( E = \mathbb{R}^m \) and \( \mu(dz) = h(z)dz \) with \( h \in C^\infty(\mathbb{R}^m) \). Then one constructs a Malliavin type calculus based on the amplitude of the jumps of the Poisson point measure \( N_\mu \) and employs this calculus in order to study the regularity of \( P_t \). A second approach initiated by Carlen and Pardoux \[12\] (see also Bally and Clément \[6\]) follows the ideas in Malliavin calculus based on the exponential density of the jump times in order to study the same problem. Finally a third approach is due to Picard \[22, 23\], but see also Ishikawa and Kunita \[16\], the contributions of Kunita \[17, 18\] and the recent monograph by Ishikawa \[15\] for many references and developments in this direction. Picard constructs a Malliavin type calculus based on finite differences (instead of standard Malliavin derivatives) and obtains the regularity of \( P_t \) for a general class of intensity measures \( \mu \) including purely atomic measures (in contrast with \( \mu(dz) = h(z)dz \)). We stress that all the above approaches work under different non degeneracy hypotheses, each of them corresponding to the specific noise that is used in the calculus. So in some sense we have not a single problem but three different classes of problems. The common feature is that the strategy in order to solve the problem follows the ideas from Malliavin calculus based on some noise contained in \( N_\mu \). Our approach is completely different because, as described above, we use the regularization effect of \( \text{tr}(A_n(x)\nabla^2) \). This regularization effect may be exploited either by using the standard Malliavin calculus based on the Brownian motion or using some analytical arguments. The approach that we propose in \[5\] is probabilistic, so employs the standard Malliavin calculus. But anyway, as mentioned above, the regularization effect vanishes as \( n \to \infty \) and a supplementary argument based on the equilibrium given in \( (1.4) \) is used. We precise that the non degeneracy condition \( A_n(x) \geq \lambda_n > 0 \) is of the same nature as the one employed by J. Picard so the problem we solve is in the same class.

The idea of replacing “small jumps” (the ones in \( \{|z| \leq \varepsilon_n\} \) here) by a Brownian part (that is \( \text{tr}(A_n(x)\nabla^2) \) in \( L_n \)) is not new – it has been introduced by Asmussen and Rosinski in \[2\] and has been extensively employed in papers concerned with simulation problems: since there is a huge amount of small jumps, they are difficult to simulate and then one approximates them by the Brownian part corresponding to \( \text{tr}(A_n(x)\nabla^2) \). See for example \[11, 7, 13\] and many others. However, at our knowledge, this idea has not been yet used in order to study the regularity of \( P_t \).

The paper is organized as follows. In Section \[2\] we give the notation and the main results mentioned above and in Section \[4\] we give the proof of these results. Section \[3\] is devoted to some preliminary results about regularity. Namely, in Section \[3.1\] we recall and develop some
results concerning regularity of probability measures, based on interpolation type arguments, coming from \[4\]. These are the main instruments used in the paper. In Section 3.2 we prove a regularity result which is a key point in our approach. In fact, it allows to handle the multiple integrals coming from the application of a Lindeberg method for the decomposition of $P_t - P^n_t$.

Finally, in Appendix A and B we prove some technical results used in the paper.

2 Notation and main results

2.1 Notation

For a multi-index $\alpha = (\alpha_1, \ldots, \alpha_m) \in \{1, \ldots, d\}^m$ we denote $|\alpha| = m$ (the length of the multi-index) and $\partial^\alpha$ is the derivative corresponding to $\alpha$, that is $\partial^\alpha = \partial_{x_{\alpha_1}} \cdots \partial_{x_{\alpha_m}}$. For $f \in C^\infty(\mathbb{R}^d \times \mathbb{R}^d)$, $(x,y) \in \mathbb{R}^d \times \mathbb{R}^d$ and two multi-indexes $\alpha$ and $\beta$, we denote by $\partial^\alpha x$ the derivative with respect to $x$ and by $\partial^\beta y$ the derivative with respect to $y$. Moreover, for $f \in C^\infty(\mathbb{R}^d)$ and $q \in \mathbb{N}$ we denote

$$|f|_q(x) = \sum_{0 \leq |\alpha| \leq q} |\partial^\alpha f(x)|.$$ (2.1)

If $f$ is not a scalar function, that is, $f = (f^i)_{i=1,\ldots,d}$ or $f = (f^{i,j})_{i,j=1,\ldots,d}$, we denote $|f|_q = \sum_{i=1}^d |f^i|_q$ respectively $|f|_q = \sum_{i,j=1}^d |f^{i,j}|_q$.

We will work with the weights

$$\psi_\kappa(x) = (1 + |x|^2)^\kappa, \quad \kappa \in \mathbb{Z}. \quad (2.2)$$

The following properties hold:

- for every $\kappa \geq \kappa' \geq 0$,

$$\psi_\kappa(x) \leq \psi_{\kappa'}(x);$$ (2.3)

- for every $\kappa \geq 0$, there exists $C_\kappa > 0$ such that

$$\psi_\kappa(x) \leq C_\kappa \psi_\kappa(y) \psi_\kappa(x - y);$$ (2.4)

- for every $\kappa \geq 0$, there exists $C_\kappa > 0$ such that for every $\phi \in C^\infty_0(\mathbb{R}^d)$,

$$\psi_\kappa(\phi(x)) \leq C_\kappa \psi_\kappa(\phi(0))(1 + \|\nabla \phi\|_\infty^2)^\kappa \psi_\kappa(x);$$ (2.5)

- for every $q \in \mathbb{N}$ there exist $C_q, \underline{C}_q > 0$ such that for every $\kappa \in \mathbb{R}$ and $f \in C^\infty(\mathbb{R}^d)$,

$$C_q \psi_\kappa |f|_q(x) \leq |\psi_\kappa f|_q(x) \leq \underline{C}_q \psi_\kappa |f|_q(x).$$ (2.6)

Note that (2.3)–(2.5) are immediate, whereas (2.6) is proved in Appendix A (see Lemma A.1).

For $q \in \mathbb{N}$, $\kappa \in \mathbb{R}$ and $p \in (1, \infty]$ (we stress that we include the case $p = +\infty$), we set $\| \cdot \|_p$ the usual norm in $L^p(\mathbb{R}^d)$ and

$$\|f\|_{q,\kappa,p} = \|\psi_\kappa f\|_p.$$ (2.7)
We denote $W^{q,\kappa,p}$ to be the closure of $C^\infty(\mathbb{R}^d)$ with respect to the above norm. If $\kappa = 0$ we just denote $\|f\|_{q,p} = \|f\|_{q,0,p}$ and $W^{q,p} = W^{q,0,p}$ (which is the usual Sobolev space). So, we are working with weighted Sobolev spaces. The weighted Sobolev spaces $W^{q,\kappa,p}$ are the natural framework in the paper [4] where the “balance argument” is obtained. There (see Theorem A.2 in [4]) we have used a crucial result of Petrushev and Xu [21] concerning the construction of kernels with polynomial decay at infinity. Then the weights $\psi_n$ appear in a natural way in order to capture the behaviour of the kernel at infinity.

The following properties hold:

- for every $q \in \mathbb{N}$ there exists $\overline{C}_q \geq C_q > 0$ such that for every $\kappa \in \mathbb{R}$, $p > 1$ and $f \in W^{q,\kappa,p}(\mathbb{R}^d)$,
  \[ \overline{C}_q \|\psi_\kappa f\|_q \leq \|f\|_{q,\kappa,p} \leq \overline{C}_q \|\psi_\kappa f\|_p; \]  \hfill (2.8)

- for every $q \in \mathbb{N}$ and $p > 1$ there exists $C_{q,p} > 0$ such that for every $\kappa \in \mathbb{R}$ and $f \in W^{q,\kappa,p}(\mathbb{R}^d)$,
  \[ \|f\|_{q,\kappa,p} \leq C_{q,p} \|f\|_{q,\kappa+d,\infty} \]  \hfill (2.9)
  and if $p > d$,
  \[ \|f\|_{q,\kappa,\infty} \leq C_{q,p} \|f\|_{q+1,\kappa,p}; \]  \hfill (2.10)

- for $\kappa,\kappa' \in \mathbb{R}$, $q, q' \in \mathbb{N}$, $p \in (1,\infty]$ and $U : C^\infty(\mathbb{R}^d) \to C^\infty(\mathbb{R}^d)$, the following two assertions are equivalent: there exists a constant $C_* \geq 1$ such that for every $f$,
  \[ \|Uf\|_{q,\kappa,\infty} \leq C_* \|f\|_{q',\kappa',p} \]  \hfill (2.11)
  and there exists a constant $C_* \geq 1$ such that for every $f$,
  \[ \|\psi_\kappa U\left(\frac{1}{\psi_{\kappa'}}f\right)\|_{q,\infty} \leq C_* \|f\|_{q',p}. \]  \hfill (2.12)

Notice that (2.8) is a consequence of (2.6). The inequality (2.9) is an immediate consequence of (2.6) and of the fact that $\psi_{-d} \in L^p(\mathbb{R}^d)$ for every $p \geq 1$. And the inequality (2.10) is a consequence of Morrey’s inequality (Corollary IX.13 in [11]), whose use gives $\|f\|_{0,\infty} \leq \|f\|_{1,0,p'}$ and of (2.6). In order to prove the equivalence between (2.11) and (2.12), one takes $g = \psi_{\kappa'} f$ (respectively $g = \frac{1}{\psi_{\kappa'}} f$) and uses (2.6) as well.

### 2.2 Main results

We consider a Markov semigroup $(P_t)_{t \geq 0}$ with infinitesimal operator $L$ and a sequence $(P^n_t)_{t \geq 0}$, $n \in \mathbb{N}$, of Markov semigroups with infinitesimal operator $L_n$. We suppose that $\mathcal{S}(\mathbb{R}^d)$ is included in the domain of $(P_t)_{t \geq 0}$, $(P^n_t)_{t \geq 0}$, $L$ and of $L_n$ and we suppose that for $f \in \mathcal{S}(\mathbb{R}^d)$ we have $P_t f, P^n_t f, L f, L_n f \in \mathcal{S}(\mathbb{R}^d)$.

We denote $\Delta_n = L - L_n$. Moreover, we denote by $P^{*,n}_t$ the formal adjoint of $P^n_t$ and by $\Delta^*_n$ the formal adjoint of $\Delta_n$ that is
  \[ \langle P^{*,n}_t f, g \rangle = \langle f, P^n_t g \rangle \quad \text{and} \quad \langle \Delta^*_n f, g \rangle = \langle f, \Delta_n g \rangle, \]  \hfill (2.13)

$(\cdot, \cdot)$ being the scalar product in $L^2(\mathbb{R}^d, dx)$.

We present now our hypotheses. The first one concerns the speed of convergence of $L_n \to L$.
**Assumption 2.1** Let \( a \in \mathbb{N} \), and let \((\varepsilon_n)_{n \in \mathbb{N}}\) be a decreasing sequence such that \( \lim_{n \to \infty} \varepsilon_n = 0 \). We assume that for every \( q \in \mathbb{N}, \kappa \geq 0 \) and \( p > 1 \) there exists \( C > 0 \) such that for every \( n \in \mathbb{N} \) and \( f \in \mathcal{S}(\mathbb{R}^d) \),

\[
\begin{align*}
(A_1) \quad & \| \Delta_n f \|_{q,-\kappa,\infty} \leq C \varepsilon_n \| f \|_{q+a,-\kappa,\infty}, \\
(A_1^*) \quad & \| \Delta_n f \|_{q,\kappa,p} \leq C \varepsilon_n \| f \|_{q+a,\kappa,p}.
\end{align*}
\]  

(2.14) (2.15)

Our second hypothesis concerns the “propagation of regularity” for the semigroups \((P_t^n)_{t \geq 0}\).

**Assumption 2.2** Let \( \Lambda_n \geq 1, n \in \mathbb{N} \), be an increasing sequence such that \( \Lambda_{n+1} \leq \gamma \Lambda_n \) for some \( \gamma \geq 1 \). For every \( q \in \mathbb{N} \) and \( \kappa \geq 0, p > 1 \), there exist \( C > 0 \) and \( b \in \mathbb{N} \), such that for every \( n \in \mathbb{N} \) and \( f \in \mathcal{S}(\mathbb{R}^d) \)

\[
\begin{align*}
(A_2) \quad & \sup_{s \leq t} \| P^n_s f \|_{q,-\kappa,\infty} \leq C \Lambda_n \| f \|_{q+b,-\kappa,\infty}, \\
(A_2^*) \quad & \sup_{s \leq t} \| P^n_s f \|_{q,\kappa,p} \leq C \Lambda_n \| f \|_{q+b,\kappa,p}.
\end{align*}
\]  

(2.16) (2.17)

The hypothesis \((A^*_2)\) is rather difficult to verify so, in Appendix B we give some sufficient conditions in order to check it (see Proposition 13). Our third hypothesis concerns the “regularization effect” of the semi-group \((P_t^n)_{t \geq 0}\).

**Assumption 2.3** We assume that

\[
P^n_t f(x) = \int_{\mathbb{R}^d} p^n_t(x,y)f(y)dy
\]  

(2.18)

with \( p^n_t \in C^\infty(\mathbb{R}^d \times \mathbb{R}^d) \). Moreover, we assume there exist \( \theta_0 > 0 \) and a sequence \( \lambda_n, n \in \mathbb{N} \), with, as \( n \to \infty \),

\[
\lambda_n \downarrow 0, \quad \lambda_n \leq \gamma \lambda_{n+1},
\]  

(2.19)

for some \( \gamma \geq 1 \), such that the following property holds: for every \( \kappa \geq 0, q \in \mathbb{N} \) there exist \( \pi(q, \kappa) \), increasing in \( q \) and in \( \kappa \), a constant \( \theta_1 \geq 0 \), and a constant \( C > 0 \) such that for every \( n \in \mathbb{N}, t \in (0,1] \), for every multi-indexes \( \alpha \) and \( \beta \) with \( |\alpha| + |\beta| \leq q \) and \( (x,y) \in \mathbb{R}^d \times \mathbb{R}^d \)

\[
(A_3) \quad \left| \partial^\alpha_x \partial^\beta_y p^n_t(x,y) \right| \leq C \frac{1}{(\lambda_n t)^{\theta_1(q+\theta_1)}} \times \frac{\psi_{\pi(q, \kappa)}(x)}{\psi_{\kappa}(x-y)}.
\]  

(2.20)

Note that in (2.20) we are quantifying the possible blow-up of \( |\partial^\alpha_x \partial^\beta_y p^n_t(x,y)| \) as \( n \to \infty \). We also assume the following statements hold for the semigroup \((P_t^n)_{t \geq 0}\).

**Assumption 2.4** For every \( \kappa \geq 0, k \in \mathbb{N} \) there exists \( C \geq 1 \) such that

\[
(A_4) \quad \| P_t f \|_{k,-\kappa,\infty} \leq C \| f \|_{k,-\kappa,\infty}.
\]  

(2.21)

**Assumption 2.5** For every \( \kappa \geq 0, k \in \mathbb{N} \) there exists \( C \geq 1, \pi \geq \kappa \) such that

\[
(A_5) \quad P_t \psi_{\kappa}(x) \leq C \psi_{\pi}(x).
\]  

(2.22)

Our main result is the following:
Theorem 2.6 Suppose that Assumption A.1, A.2, A.3, A.4 and A.5 hold. Suppose also that for some \( \delta > 0 \)
\[
limsup_{n \to \infty} \varepsilon_n \Lambda_n < \infty.
\]
Then \( P_t(x, y) = p_t(x, y) \) with \( p_t \in C^\infty(\mathbb{R}^d \times \mathbb{R}^d) \). Moreover, for every \( k \in \mathbb{N}, R \in \mathbb{N}, \varepsilon > 0 \) and every multi-indexes \( \alpha \) and \( \beta \) there exists some constants \( C = C(\delta, R, \kappa, \varepsilon, \alpha, \beta) \) and \( t_0 = t_0(\delta, R, \kappa, \varepsilon, \alpha, \beta) \in (0, 1] \) such that for every \( t \in (0, t_0) \), \( x \in \mathbb{R}^d \) with \( |x| < R \) and \( y \in \mathbb{R}^d \),
\[
\left| \partial_x^\alpha \partial_y^\beta p_t(x, y) \right| \leq C \times t^{-\theta_0(1+\frac{|\alpha|+|\beta|+2d+\varepsilon}}{2d}} \times \frac{1}{\psi_\kappa(x - y)} \tag{2.23}
\]
with \( \theta_0 \) from (2.20).

3 Regularity results

This section is devoted to some preliminary results allowing us to prove the statements resumed in Section 2.2. In Section 3.1 we give an abstract regularity criterion, while in Section 3.2 we prove a regularity result for iterated integrals, that will be useful to handle a Lindeberg type decomposition of \( P_t - P^n_t \).

3.1 A regularity criterion based on interpolation

Let us first recall some results obtained in [4] concerning the regularity of a measure \( \mu \) on \( \mathbb{R}^d \) (with the Borel \( \sigma \)-field). For two signed finite measures \( \mu, \nu \) and for \( k \in \mathbb{N} \) we define the distance
\[
d_k(\mu, \nu) = \sup \left\{ \left| \int f d\mu - \int f d\nu \right| : \|f\|_{k, \infty} \leq 1 \right\}. \tag{3.1}
\]
If \( \mu \) and \( \nu \) are probability measures, \( d_0 \) is the total variation distance and \( d_1 \) is the Fortet-Mourier distance. In this paper we will work with an arbitrary \( k \in \mathbb{N} \). Notice also that \( d_k(\mu, \nu) = \|\mu - \nu\|_{W^{k, \infty}} \) where \( W^{k, \infty}_\nu \) is the dual of \( W^{k, \infty}_\nu \).

We fix now \( k, q, h \in \mathbb{N} \), with \( h \geq 1 \), and \( p > 1 \). Hereafter, we denote by \( p_* = p/(p - 1) \) the conjugate of \( p \). Then, for a signed finite measure \( \mu \) and for a sequence of absolutely continuous signed finite measures \( \mu_n(dx) = f_n(x)dx \) with \( f_n \in C^{2h+q}(\mathbb{R}^d) \), we define
\[
\pi_{k,q,h,p}(\mu, (\mu_n)_n) = \sum_{n=0}^{\infty} 2^{n(k+q+d/p_*)} d_k(\mu, \mu_n) + \sum_{n=0}^{\infty} 2^{nh} \|f_n\|_{2h+q, 2h, p}. \tag{3.2}
\]

The following result is the key point in our approach:

Lemma 3.1 Let \( k, q, h \in \mathbb{N} \) with \( h \geq 1 \), and \( p > 1 \) be given. There exists a constant \( C_* \) (depending on \( k, q, h \) and \( p \) only) such that the following holds. Let \( \mu \) be a finite measure for which one may find a sequence \( \mu_n(dx) = f_n(x)dx \), \( n \in \mathbb{N} \) such that \( \pi_{k,q,h,p}(\mu, (\mu_n)_n) < \infty \). Then \( \mu(dx) = f(x)dx \) with \( f \in W^{q,p} \) and moreover
\[
\|f\|_{q,p} \leq C_* \times \pi_{k,q,h,p}(\mu, (\mu_n)_n). \tag{3.3}
\]

The proof of Lemma 3.1 is given in [4], being a particular case (take \( e = e_p \)) of Proposition A.1 in Appendix A. We give a first simple consequence:
Lemma 3.2 Let \( p_t \in C^\infty(\mathbb{R}^d), t > 0 \), be a family of probability densities such that, for every \( \kappa \in \mathbb{N} \), \( \int \psi(x)p_t(x)dx \leq m_\kappa < \infty \). We assume that for some \( \theta_0 > 0 \) and \( \theta_1 > 0 \) the following holds. For every \( q, \kappa \in \mathbb{N} \) and \( p \geq 1 \) there exists a constant \( C = C(q, \kappa, p, \theta_1) \geq 1 \) such that

\[
\|\psi_n p_t\|_{q,p} \leq C t^{-\theta_0(q+\theta_1)}.
\] (3.4)

Then, for every \( \delta > 0 \) there exists a constant \( C = C(q, \kappa, p, \delta, \theta_1) \) such that

\[
\|\psi_n p_t\|_{q,p} \leq C t^{-\theta_0(q+\frac{d}{p}+\delta)}
\] (3.5)

for every \( t < 2^{-4/\theta_0} \) (so does not matter the value of \( \theta_1 \), one may morally replace it by \( \frac{d}{p} \) in the power of \( t \); however, \( \theta_1 \) appears in the constant \( C \)).

Proof. We take \( n_* \in \mathbb{N} \) and we define \( f_n = 0 \) for \( n \leq n_* \) and \( f_n = \psi_n p_t \) for \( n > n_* \). Notice that \( d_0(\psi_n p_t, 0) = m_\kappa \). Then \( \|\psi_n p_t\|_{q,p} \) with \( k = 0 \) gives

\[
\|\psi_n p_t\|_{q,p} \leq C_\psi \left( m_\kappa \sum_{n=0}^{n_*} 2^{n(q+\frac{d}{p_*})} + \|\psi_n p_t\|_{2h+q,2h,p} \sum_{n=n_*+1}^{\infty} \frac{1}{2^{2n}} \right).
\]

Since \( \|\psi_n p_t\|_{2h+q,2h,p} = \|\psi_{2h+q} p_t\|_{2h+q,p} \), we use (3.4) (with \( q \) replaced by \( 2h+q + \kappa \)) and we obtain

\[
\|\psi_n p_t\|_{q,p} \leq C_\psi \left( m_\kappa 2^{n_* (q+\frac{d}{p_*})} + C_\psi t^{-\theta_0(2h+q+\theta_1)} \frac{1}{2^{2n_*}} \right).
\] (3.6)

We write here \( C_\psi \) to stress the possible dependence on \( h \) of the constant \( C \) in (3.4). We optimize over \( n_* \): we look for \( n_* \) such that

\[
m_\kappa 2^{n_* (q+\frac{d}{p_*})} = C_\psi t^{-\theta_0(2h+q+\theta_1)} \frac{1}{2^{2n_*}}.
\]

Straightforward computations give \( n_* = \lfloor n_* \rfloor \) where

\[
n_* (h) = \frac{\theta_0(2h + q + \theta_1)}{2h + q + d/p_*} \log_2 t^{-1} - \frac{1}{2h + q + d/p_*} \log_2 m_\kappa + \frac{1}{2h + q + d/p_*} \log_2 C_\psi.
\]

In order to successfully insert such \( n_* \) in (3.5), we need that \( n_* \geq 1 \). Notice in fact that if \( n_* = 0 \) then (3.5) would give \( \|\psi_n p_t\|_{q,p} \leq C_\psi (m_\kappa + C_\psi t^{-\theta_0(2h+q+\theta_1)}) \) and this is not the kind of estimate we are looking for.

We take \( h \) sufficiently large in order to have \( 2h \geq \log_2 m_\kappa \) and \( \eta_h \doteq (q + \frac{d}{p_*})/2h \leq 1 \). Then we write

\[
n_* (h) \geq \frac{\theta_0}{1 + \eta_h} \log_2 t^{-1} - \frac{1}{2h} \log_2 m_\kappa \geq \frac{\theta_0}{2} \log_2 t^{-1} - 1.
\]

It follows that the restriction \( n_* (h) \geq 1 \) amounts to \( \theta_0 \log_2 t^{-1} \geq 4 \) which is exactly the restriction we have assumed for \( t \).

We replace \( n_* (h) \) in (3.6) and we obtain

\[
\|\psi_n p_t\|_{q,p} \leq 2C_\psi \times m_\kappa^{\frac{1}{1+\eta_h}} \times C_h^{\frac{\eta_h}{1+\eta_h}} t^{-\theta_0(2h+q+\theta_1)}^{\frac{\eta_h}{1+\eta_h}}.
\]
Then, for every $\delta > 0$ we assume that $C_1$ holds (since $h$ depends on $\delta_1$ so does $C_h$ and consequently the constant in our estimates). \hfill $\Box$

We will also use the following consequence of Lemma 3.1 (the proof is given in [3] and we do not repeat it here):

**Lemma 3.3** Let $k, q, h \in \mathbb{N}$, with $h \geq 1$, and $p > 1$ be given and set

$$\rho_h := \frac{k + q + d/p_*}{2h}. \tag{3.7}$$

We consider an increasing sequence $\theta(n) \geq 1, n \in \mathbb{N}$, such that $\lim_{n \to \infty} \theta(n) = \infty$ and $\theta(n + 1) \leq \Theta \times \theta(n)$ for some constant $\Theta \geq 1$. Suppose that we may find a sequence of functions $f_n \in C^{2h+q}(\mathbb{R}^d), n \in \mathbb{N}$, such that

$$\|f_n\|_{2h+q,2h,p} \leq \theta(n) \tag{3.8}$$

and, with $\mu_n(dx) = f_n(x)dx$,

$$\limsup_{n \to \infty} d_k(\mu, \mu_n) \times \theta^{\rho_h + \varepsilon}(n) < \infty \tag{3.9}$$

for some $\varepsilon > 0$. Then $\mu(dx) = f(x)dx$ with $f \in W^{q,p}$.

Moreover, for $\delta, \varepsilon > 0$ and $n_* \in \mathbb{N}$, let

$$A(\delta) = |\mu| \left(\mathbb{R}^d\right) \times 2^{l(\delta)(1+\delta)(q+k+d/p_*)} \text{ with } l(\delta) = \min\{l : 2^{l \times \frac{\delta}{1+\delta}} \geq 1\}, \tag{3.10}$$

$$B(\varepsilon) = \sum_{l=1}^{\infty} \frac{2^{l(q+k+d/p_*+\varepsilon)}}{2^{2l}}, \tag{3.11}$$

$$C_{h,n_*}(\varepsilon) = \sup_{n \geq n_*} d_k(\mu, \mu_n) \times \theta^{\rho_h + \varepsilon}(n). \tag{3.12}$$

Then, for every $\delta > 0$

$$\|f\|_{q,p} \leq C_*(\Theta + A(\delta)\theta(n_*)^{\rho_h(1+\delta)} + B(\varepsilon)C_{h,n_*}(\varepsilon)), \tag{3.13}$$

$C_*$ being the constant in (3.3) and $\rho_h$ being given in (3.7).

### 3.2 A regularity lemma

We give here a regularization result in the following abstract framework. We consider a sequence of operators $U_j : S(\mathbb{R}^d) \to S(\mathbb{R}^d)$, $j \in \mathbb{N}$, and we denote by $U_j^*$ the formal adjoint defined by $\langle U_j^*f, g \rangle = \langle f, U_jg \rangle$ with the scalar product in $L^2(\mathbb{R}^d)$.

**Assumption 3.4** Let $a \in \mathbb{N}$ be fixed. We assume that for every $q \in \mathbb{N}, \kappa \geq 0$ and $p \in [1, \infty)$ there exist constants $C_{q,\kappa,p}(U)$ and $C_{q,\kappa,\infty}(U)$ such that for every $j$ and $f$,

$$(H_1) \quad \|U_j f\|_{q,\kappa,\infty} \leq C_{q,\kappa,\infty}(U) \|f\|_{q + a,\kappa,\infty}, \tag{3.14}$$

$$(H'_1) \quad \|U_j^* f\|_{q,\kappa,p} \leq C_{q,\kappa,p}(U) \|f\|_{q + a,\kappa,p}. \tag{3.15}$$

We assume that $C_{q,\kappa,p}(U), p \in [1, \infty]$, is non-decreasing with respect to $q$ and $\kappa$. 

---

Since $(2h + q + \theta_1) \frac{\rho_h}{1+\rho_h} \downarrow q + \frac{d}{p_*}$ as $h \to \infty$, $h$ can be chosen large enough in order that (3.5) holds (since $h$ depends on $\theta_1$ so does $C_h$ and consequently the constant in our estimates). \hfill $\Box$
We also consider a semigroup \((S_t)_{t \geq 0}\) of the form
\[
S_t(x, dy) = s_t(x, y)dy \quad \text{with} \quad s_t \in \mathcal{S}(\mathbb{R}^d \times \mathbb{R}^d).
\]

We define the formal adjoint operator
\[
S_t^* f(y) = \int_{\mathbb{R}^d} s_t(x, y) f(x) dx, \quad t > 0.
\]

**Assumption 3.5** If \(f \in \mathcal{S}(\mathbb{R}^d)\) then \(S_t f \in \mathcal{S}(\mathbb{R}^d)\). Moreover, there exists \(b \in \mathbb{N}\) such that for every \(q \in \mathbb{N}, \kappa \geq 0\) and \(p \in [1, \infty)\) there exist constants \(C_{q,\kappa,p}(S)\) such that for every \(t > 0\),
\[
\begin{align*}
(H_2) & \quad \|S_t f\|_{q,-,\kappa,\infty} \leq C_{q,\kappa,\infty}(S) \|f\|_{q+b,-,\kappa,\infty}, & (3.16) \\
(H_2') & \quad \|S_t^* f\|_{q,\kappa,p} \leq C_{q,\kappa,p}(S) \|f\|_{q+b,\kappa,p}. & (3.17)
\end{align*}
\]

We assume that \(C_{q,\kappa,p}(S), \ p \in [1, \infty]\), is non decreasing with respect to \(q\) and \(\kappa\).

We denote
\[
\begin{align*}
C_{q,\kappa,\infty}(U, S) &= C_{q,\kappa,\infty}(U)C_{q,\kappa,\infty}(S), & (3.18) \\
C_{q,\kappa,p}(U, S) &= C_{q,\kappa,p}(U)C_{q,\kappa,p}(S), & (3.19)
\end{align*}
\]

Under Assumptions 3.4 and 3.5 one immediately obtains
\[
\begin{align*}
\| (S_t U_j) f \|_{q,-,\kappa,\infty} & \leq C_{q,\kappa,\infty}(U, S) \| f \|_{q+a+b,-,\kappa,\infty}, & (3.20) \\
\| (S_t^* U_j^*) f \|_{q,\kappa,p} & \leq C_{q,\kappa,p}(U, S) \| f \|_{q+a+b,\kappa,p}. & (3.21)
\end{align*}
\]

In fact these are the inequalities that we will employ in the following. We stress that the above constants \(C_{q,\kappa,\infty}(U, S)\) and \(C_{q,\kappa,p}(U, S)\) may depend on \(a, b\) and are increasing w.r.t. \(q\) and \(\kappa\).

Finally we assume that the (possible) blow-up of \(s_t \to \infty\) as \(t \to 0\) is controlled in the following way.

**Assumption 3.6** Let \(\theta_0, \lambda > 0\) be fixed. We assume that for every \(\kappa \geq 0\) and \(q \in \mathbb{N}\) there exist \(\pi(q, \kappa), \theta_1 \geq 0\) and \(C_{q,\kappa} > 0\) such that for every multi-indexes \(\alpha\) and \(\beta\) with \(|\alpha| + |\beta| \leq q\), \((x, y) \in \mathbb{R}^d \times \mathbb{R}^d\) and \(t \in (0, 1]\) one has
\[
(H_3) \quad \left| \partial_x^\alpha \partial_y^\beta s_t(x, y) \right| \leq \frac{C_{q,\kappa}}{(\lambda t)^{q_0(q+\theta_1)}} \frac{\psi_\pi(q,\kappa)(x)}{\psi_\kappa(x+y)}.
\]

We also assume that \(\pi(q, k)\) and \(C_{q,\kappa}\) are both increasing in \(q\) and \(\kappa\).

This property will be used by means of the following lemma:

**Lemma 3.7** Suppose that Assumption 3.6 holds.

A. For every \(\kappa \geq 0\), \(q \in \mathbb{N}\) and \(p > 1\) there exists \(C > 0\) such that for every \(t \in (0, 1]\) and \(f\) one has
\[
\|S_t^* f\|_{q,\kappa,p} \leq \frac{C}{(\lambda t)^{q_0(q+\theta_1)}} \|f\|_{0,\nu,1},
\]

where \(\nu = \pi(q, \kappa + d) + \kappa + d\).
B. For every $\kappa \geq 0$, $q_1, q_2 \in \mathbb{N}$, there exists $C > 0$ such that for every $t \in (0, 1]$, for every multi-index $\alpha$ with $|\alpha| \leq q_2$ and $f$ one has

$$\left\| \frac{1}{\psi_\eta} S_t(\psi_\kappa \partial^\alpha f) \right\|_{q_1, \infty} \leq \frac{C}{(\lambda t)^{q_0(q_1 + q_2 + \theta_1)}} \|f\|_\infty$$

(3.24)

where $\eta = \pi(q_1 + q_2, \kappa + d + 1) + \kappa$.

**Proof.** In the sequel, $C$ will denote a positive constant which may vary from a line to another and which may depend only on $\kappa$ and $q$ for the proof of A. and only on $\kappa, q_1$ and $q_2$ for the proof of B.

A. Using (3.22) if $|\alpha| \leq q$,

$$|\partial^\alpha S_t^* f(x)| \leq \int |\partial^\alpha s_t(y, x)| \times |f(y)| dy \leq \frac{C}{(\lambda t)^{q_0(q + \theta_1)}} \int \psi_{q(q, \kappa + d)}(y) \psi_{\kappa + d}(x - y) \times |f(y)| dy.$$

By (2.4) $\psi_{\kappa + d}(x)/\psi_{\kappa + d}(x - y) \leq C \psi_{\kappa + d}(y)$ so that

$$\psi_{\kappa + d}(x) |\partial^\alpha S_t^* f(x)| \leq \frac{C}{(\lambda t)^{q_0(q + \theta_1)}} \int \psi_{\kappa + d}(x) \psi_{q(q, \kappa + d)}(y) \psi_{\kappa + d}(x - y) \times |f(y)| dy \leq \frac{C}{(\lambda t)^{q_0(q + \theta_1)}} \int \psi_{q(q, \kappa + d)}(y) \psi_{\kappa + d}(y) \times |f(y)| dy \leq \frac{C}{(\lambda t)^{q_0(q + \theta_1)}} \|f\|_{0, \nu, 1}.$$

We conclude that

$$\|S_t^* f\|_{q_1, \kappa, d, \infty} \leq \frac{C}{(\lambda t)^{q_0(q + \theta_1)}} \|f\|_{0, \nu, 1}.$$

By (2.9) $\|S_t^* f\|_{q_1, \nu, p} \leq C \|S_t^* f\|_{q_1, \kappa, d, \infty}$ so the proof of (3.23) is completed.

B. Let $\gamma$ with $|\gamma| \leq q_1$. Using integration by parts

$$\partial^\gamma S_t(\psi_\kappa \partial^\alpha f)(x) = \int_{\mathbb{R}^d} \partial^\gamma s_t(x, y) \psi_\kappa(y) \partial^\alpha f(y) dy = (-1)^{|\alpha|} \int_{\mathbb{R}^d} \partial_y^\alpha (\partial_x^\gamma s_t(x, y) \psi_\kappa(y)) \times f(y) dy.$$

Using (2.6), (3.22) and (2.4), it follows that

$$|\partial^\gamma S_t(\psi_\kappa \partial^\alpha f)(x)| \leq \int_{\mathbb{R}^d} \left| \partial_y^\alpha (\partial_x^\gamma s_t(x, y) \psi_\kappa(y)) \right| \times |f(y)| dy \leq \int_{\mathbb{R}^d} |s_t(x, y)\psi_\kappa(y)|_{q_1 + q_2} \times |f(y)| dy \leq C \int_{\mathbb{R}^d} |s_t(x, y)|_{q_1 + q_2} \psi_\kappa(y) \times |f(y)| dy \leq \frac{C}{(\lambda t)^{q_0(q_1 + q_2 + \theta_1)}} \|f\|_\infty \int_{\mathbb{R}^d} \psi_{q(q_1 + q_2, \kappa + d + 1)}(x) \psi_{\kappa + d+1}(x - y) \times \psi_\kappa(y) dy \leq \frac{C}{(\lambda t)^{q_0(q_1 + q_2 + \theta_1)}} \|f\|_\infty \int_{\mathbb{R}^d} \psi_{q(q_1 + q_2, \kappa + d + 1) + \kappa}(x) \psi_{\kappa + d+1}(x - y) \times \psi_\kappa(y) dy \leq \frac{C}{(\lambda t)^{q_0(q_1 + q_2 + \theta_1)}} \|f\|_\infty \psi_{q(q_1 + q_2, \kappa + d + 1) + \kappa}(x).$$
This implies \([3.24]\). □

We are now able to give the “regularity lemma”. This is the core of our approach.

**Lemma 3.8** Suppose that Assumption \([3.4, 3.5, 3.6]\) hold. We fix \(t \in (0, 1]\), \(m \geq 1\) and \(\delta_i > 0, i = 1, \ldots, m\) such that \(\sum_{i=1}^{m} \delta_i = t\).

A. There exists a function \(\tilde{\beta}_{\delta_1, \ldots, \delta_m} \in C^\infty(\mathbb{R}^d \times \mathbb{R}^d)\) such that

\[
\prod_{i=1}^{m-1} (S_{\delta_i}U_i) S_{\delta_m} f(x) = \int \tilde{\beta}_{\delta_1, \ldots, \delta_m}(x, y) f(y) dy.
\]  \hspace{1cm} (3.25)

B. We fix \(q_1, q_2 \in \mathbb{N}, \kappa \geq 0, p > 1\) and we denote \(q = q_1 + q_2 + (a + b)(m - 1)\). One may find universal constants \(C, \chi, \tilde{p} \geq 1\) (depending on \(\kappa, p\) and \(q_1 + q_2\)) such that for every multi-index \(\beta\) with \(|\beta| \leq q_2\) and every \(x \in \mathbb{R}^d\)

\[
\|\partial_x^\beta \tilde{\beta}_{\delta_1, \ldots, \delta_m}(x, \cdot)\|_{q_1, \kappa, p} \leq C \left( \frac{2m}{\lambda t} \right)^{\theta_0(q_1 + q_2 + d + 2\theta_1)} \left( C_{q, \chi, \tilde{p}, \infty}(U, S) \left( \frac{2m}{\lambda t} \right)^{\theta_0(a + b)} \right)^{m-1} \psi_{\chi}(x).
\]  \hspace{1cm} (3.26)

**Proof.** A. For \(g = g(x, y)\), we denote \(g^x(y) := g(x, y)\). By the very definition of \(U_i^*\) one has

\[
S_t U_i f(x) = \int_{\mathbb{R}^d} U_i^* s_t^x(y) f(y) dy.
\]

As a consequence, one gets the kernel in \([3.25]\):

\[
\tilde{\beta}_{\delta_1, \ldots, \delta_m}(x, y) = \int_{\mathbb{R}^d \times (m-1)} U_i^* s_t^x(y_1) \left( \prod_{j=2}^{m-1} U_j^* s_t^{y_{j-1} - y_j}(y_j) \right) s_{\delta_m}(y_{m-1}, y) dy_1 \cdots dy_{m-1},
\]

and the regularity immediately follows.

B. We split the proof in several steps.

**Step 1: decomposition.** Since \(\sum_{i=1}^{m} \delta_i = t\) we may find \(j \in \{1, \ldots, m\}\) such that \(\delta_j \geq \frac{t}{m}\). We fix this \(j\) and we write

\[
\prod_{i=1}^{m-1} (S_{\delta_i}U_i) S_{\delta_m} = Q_1 Q_2
\]

with

\[
Q_1 = \prod_{i=1}^{j-1} (S_{\delta_i}U_i) S_{\delta_j} \quad \text{and} \quad Q_2 = S_{\frac{1}{2}\delta_j} U_j \prod_{i=j+1}^{m-1} (S_{\delta_i}U_i) S_{\delta_m} = S_{\frac{1}{2}\delta_j} \prod_{i=j}^{m-1} (U_i S_{\delta_{i+1}}).
\]

Here we use the semi-group property \(S_{\frac{1}{2}\delta_j} S_{\frac{1}{2}\delta_j} = S_{\delta_j}\). We suppose that \(j \leq m - 1\). In the case \(j = m\) the proof is analogous but simpler. We will use Lemma \([3.7]\) in order to estimate the terms corresponding to each of these two operators. As already seen, both \(Q_1\) and \(Q_2\) are given by means of smooth kernels, that we call \(p_1(x, y)\) and \(p_2(x, y)\) respectively.
Step 2. We take $\beta$ with $|\beta| \leq q_2$ and we denote $g^{\beta,x}(y) := \partial_x^\beta g(x,y)$. For $h \in L^1$ we write

$$
\int_{\mathbb{R}^d} h(z) \partial_x^\beta \tilde{\theta}_{\delta_1,\ldots,\delta_m}(x,z) dz = \int_{\mathbb{R}^d} h(z) \int_{\mathbb{R}^d} \partial_x^\beta p_1(x,y)p_2(y,z) dy dz
$$

$$
= \int_{\mathbb{R}^d} \partial_x^\beta p_1(x,y) \int h(z)p_2(y,z) dz dy = \int_{\mathbb{R}^d} \partial_x^\beta p_1(x,y) Q_2 h(y) dy
$$

$$
= \int_{\mathbb{R}^d} Q_2^* \beta(x) h(y) dy.
$$

It follows that

$$
\partial_x^\beta \tilde{\theta}_{\delta_1,\ldots,\delta_m}(x,z) = Q_2^* \beta(x) = \prod_{i=1}^{m-j} (S_{\delta_{m-i+1}}^* U_{m-i}^*) S_{\delta_j}^* p_1^\beta(x)(z).
$$

We will use (3.21) $m - j$ times first and (3.24) then. We denote

$$
q'_1 = q_1 + (m - j)(a + b)
$$

and we write

$$
||\partial_x^\beta \tilde{\theta}_{\delta_1,\ldots,\delta_m}(x,z)||_{q_1,\kappa,p} \leq C_{q'_1,\kappa,p}^{m-j} (U,S) ||S_{\delta_j}^* p_1^\beta||_{q'_1,\kappa,p}
$$

$$
\leq C_{q'_1,\kappa,p}^{m-j} (U,S) C \left( \frac{2m}{\lambda t} \right)^{\theta_0(q'_1 + \theta_1)} ||p_1^\beta||_{0,\nu,1}
$$

(3.27)

with

$$
\nu = \pi(q'_1, \kappa + d) + \kappa + d.
$$

Step 3. We denote $g_z(u) = \prod_{i=1}^d 1_{(0,\infty)}(u_i - \delta_i)$, so that $\delta_0(u - z) = \partial_u^\rho g_z(u)$ with $\rho = (1,2,\ldots,d)$. We take $\mu = \nu + d + 1$ and we formally write

$$
p_1(x,z) = \frac{1}{\psi_{\mu}(z)} Q_1(\psi_{\mu} \partial^\rho g_z)(x).
$$

This formal equality can be rigorously written by using the regularization by convolution of the Dirac function.

We denote

$$
q'_2 = q_2 + (j - 1)(a + b), \quad \eta = \pi(d + q'_2, \mu + d + 1) + \mu
$$

and we write

$$
||p_1^\beta||_{0,\nu,1} \leq C_{\psi}(x) \sup_{z \in \mathbb{R}^d} \left| \frac{1}{\psi_{\eta}} \partial^\beta Q_1(\psi_{\mu} \partial^\rho g_z) \right|_\infty \leq C_{\psi}(x) \sup_{z \in \mathbb{R}^d} \left| \frac{1}{\psi_{\eta}} Q_1(\psi_{\mu} \partial^\rho g_z) \right|_{q_2,\eta,\infty}
$$

$$
\leq C_{\psi}(x) \sup_{z \in \mathbb{R}^d} \left| Q_1(\psi_{\mu} \partial^\rho g_z) \right|_{q_2,\eta,\infty}.
$$

Using (3.20) $j - 1$ times and (3.24) (with $\kappa = \mu$) we get

$$
||Q_1(\psi_{\mu} \partial^\rho g_z) ||_{q_2,\eta,\infty} \leq C_{q_2,\eta,\infty}^{j-1} (U,S) S_{\delta_j}^* (\psi_{\mu} \partial^\rho g_z) ||q_2,\eta,\infty
$$

$$
\leq C_{q_2,\eta,\infty}^{j-1} (U,S) ||g_z||_\infty C \left( \frac{2m}{\lambda t} \right)^{\theta_0(q'_2 + d + \theta_I)}.
$$
Since $\|g_\epsilon\|_\infty = 1$ we obtain
\[
\|P_1^{t,\epsilon}\|_{0,\nu,1} \leq \psi_\eta(x)C^{j-1}\|Q^{2,\eta,\infty}(U, S)\|_\calM \left(\frac{2m}{\lambda t}\right)^{\theta_0(q_2+\theta_1)}.
\]

By inserting in (4.27) we obtain (3.26), so the proof is completed. □

4 Proofs of the main results

In this section we prove Theorem 2.6. But before we give an intermediary result, Theorem 4.1 below, which is more precise concerning constants. Let us introduce some notation. For $\delta > 0$ we denote
\[
\Phi_n(\delta) = \varepsilon_n \Lambda_n \times \lambda_n^{-\theta_0(a+b+\delta)}, \quad (4.1)
\]
We recall that the constants $\varepsilon_n$, $a$, $\Lambda_n$, $b$ and $\lambda_n$ are defined in Assumption 2.1. Assumption 2.2 and Assumption 2.3. Under Assumption 2.3, $\lambda_n \leq \gamma \lambda_{n+1}$ for some $\gamma > 1$, so we have
\[
\Phi_n(\delta) \leq \gamma^{1+\theta_0(a+b+\delta)} \Phi_{n+1}(\delta). \quad (4.2)
\]
For $\kappa \geq 0, \eta \geq 0$ we set
\[
\Psi_{\eta,\kappa}(x, y) := \frac{\psi_{\eta}(y)}{\psi_{\eta}(x)}, \quad (x, y) \in \mathbb{R}^d \times \mathbb{R}^d. \quad (4.3)
\]

Our intermediary result concerning the regularity of the semigroup $(P_t)_{t \geq 0}$ is the following.

**Theorem 4.1** Suppose that Assumption 2.1, 2.2, 2.3, 2.4 hold. Moreover we suppose there exists $\delta > 0$ such that
\[
\limsup_{n \to \infty} \Phi_n(\delta) < \infty, \quad (4.4)
\]
$\Phi_n(\delta)$ being given in (4.4). Then the following statements hold.

**A.** $P_t f(x) = \int_{\mathbb{R}^d} p_t(x, y) dy$ with $p_t \in C^\infty(\mathbb{R}^d \times \mathbb{R}^d)$.

**B.** Let $n_\ast \in \mathbb{N}$ and $\delta_\ast > 0$ be such that
\[
\overline{\Phi}_\ast := \sup_{n \geq n_\ast} \Phi_n(\delta_\ast) < \infty. \quad (4.5)
\]

We fix $q \in \mathbb{N}$, $p > 1$, $\varepsilon_\ast > 0$, $\kappa \geq 0$ and we put $m = 1 + \frac{q+2d/p_\ast}{\delta_\ast}$ with $p_\ast$ the conjugate of $p$. There exist $C \geq 1$ and $\eta_\ast > 1$ (depending on $q, p, \varepsilon_\ast, \delta_\ast, \kappa$ and $\gamma$) such that for every $t \in (0, 1]$
\[
\|\Psi_{\eta,\kappa} p_t\|_{q,p} \leq C \times Q_\ast(q, m) \times t^{-\theta_0((a+b)m+q+2d/p_\ast)(1+\varepsilon_\ast)} \quad (4.6)
\]
\[
Q_\ast(q, m) = \left(\frac{1}{\lambda_n^{-\theta_0(a+b)m+q+2d/p_\ast} + \overline{\Phi}_\ast^m}\right)^{1+\varepsilon_\ast}. \quad (4.7)
\]

**C.** Let $p > 2d$. Set $\bar{m} = 1 + \frac{q+1+2d/p_\ast}{\delta_\ast}$. There exist $C \geq 1$, $\eta_\ast > 1$ (depending on $q, p, \varepsilon_\ast, \delta_\ast, \kappa$) such that for every $t \in (0, 1]$, $x, y \in \mathbb{R}^d$ and for every multi-indexes $\alpha, \beta$ such that $|\alpha| + |\beta| \leq q$,
\[
\left|\partial_x^\alpha \partial_y^\beta p_t(x, y)\right| \leq C \times Q_\ast(q + 1, \bar{m}) \times t^{-\theta_0((a+b)m+q+1+2d/p_\ast)(1+\varepsilon_\ast)} \times \frac{\psi_{\eta,\kappa}(x)}{\psi_{\eta}(x-y)}. \quad (4.8)
\]
Remark 4.2 We stress that in hypothesis (4.9) the order of derivation \( q \) does not appear. However the conclusions (4.6) and (4.8) hold for every \( q \). The motivation of this is given by the following heuristics. The hypothesis (2.20) says that the semi-group \( P^n_t \) has a regularization effect controlled by \( 1/(\lambda_n t)^{\delta_n} \). If we want to decouple this effect \( m_0 \) times we write \( P^n_t = P^n_{t/m_0}...P^n_{t/m_0} \) and then each of the \( m_0 \) operators \( P^n_{t/m_0} \) acts with a regularization effect of order \((\lambda_n \times t/m_0)^{\delta_n^*}\). But this heuristics does not work directly: in order to use it, in the proof we have to develop a Taylor expansion coupled with the interpolation criterion studied in Section 3.

Proof. Step 0: constants and parameters set-up. We first choose some parameters which will be used in the following steps. To begin we stress that we work with measures on \( \mathbb{R}^d \times \mathbb{R}^d \) so the dimension of the space is \( 2d \) (and not \( d \)). We recall that in our statement the quantities \( q, d, p, \delta, \varepsilon, \kappa \) and \( n \) are given and fixed. In the following we will denote by \( C \) a constant depending on all these parameters and which may change from a line to another. We define

\[
m_0 = 1 + \left[ \frac{q + 2d/p_\ast}{\delta_\ast} \right] > 0
\]  

(4.9)

and given \( h \in \mathbb{N} \) we denote

\[
\rho_h = \frac{(a+b)m_0 + q + 2d/p_\ast}{2h}.
\]  

(4.10)

Notice that this is equal to the constant \( \rho_h \) defined in (3.7) corresponding to \( k = (a+b)m_0 \) and \( q \) and to \( 2d \) (instead of \( d \)).

Step 1: a Lindeberg-type method to decompose \( P_t - P^n_t \). We fix (once for all) \( t \in (0, 1] \) and we write

\[
P_t f - P^n_t f = \int_0^t \partial_s (P^n_{t-s} P_s) f ds = \int_0^t P^n_{t-s} (L - L_n) P_s f ds = \int_0^t P^n_{t-s} \Lambda_n P_s f ds.
\]

We iterate this formula \( m_0 \) times (with \( m_0 \) chosen in (4.9)) and we obtain

\[
P_t f(x) - P^n_t f(x) = \sum_{m=1}^{m_0-1} I^n_m f(x) + R^n_{m_0} f(x)
\]  

(4.11)

with (we put \( t_0 = t \))

\[
I^n_m f(x) = \int_0^t dt_1 \int_0^{t_1} dt_2 ... \int_0^{t_{m-1}} dt_m \prod_{i=0}^{m-1} (P^n_{t_i-t_{i+1}} \Lambda_n) P^n_{t_m} f(x), \quad 1 \leq m \leq m_0 - 1,
\]

\[
R^n_{m_0} f(x) = \int_0^t dt_1 \int_0^{t_1} dt_2 ... \int_0^{t_{m_0-1}} dt_{m_0} \prod_{i=0}^{m_0-1} (P^n_{t_i-t_{i+1}} \Lambda_n) P^n_{t_{m_0}} f(x).
\]

In order to analyze \( I^n_m f \) we use Lemma 3.8 for the semigroup \( S_t = P^n_t \) and for the operators \( U_i = \Lambda_n = L - L_n \) (the same for each \( i \)), with \( \delta_i = t_i - t_{i+1}, i = 0, \ldots, m \) (with \( t_{m+1} = 0 \)). So the hypotheses \( (3.11) \) and \( (3.15) \) in Assumption 3.3 coincide with the requests \( (2.14) \) and \( (2.15) \) in Assumption 2.1. And we have \( C_{q,\kappa,\infty}(U) = C_{q,\kappa,p}(U) = C \varepsilon_n \). Moreover the hypotheses \( (3.16) \) and \( (3.17) \) in Assumption 3.3 coincide with the hypotheses \( (2.10) \) and \( (2.17) \) in Assumption 2.2. And we have \( C_{q,\kappa,\infty}(P^n) = C_{q,\kappa,p}(P^n) = \Lambda_n \). Hence,

\[
C_{q,\kappa,\infty, p}(\Lambda_n, P^n) = C \varepsilon_n \times \Lambda_n.
\]  

(4.12)
Finally, the hypothesis \((3.22)\) in Assumption \(3.3\) coincides with \((2.20)\) in Assumption \(2.3\). So, we can apply Lemma \(3.8\) by using \((3.25)\) we obtain

\[
I_n^m f(x) = \int_0^t dt_1 \ldots \int_0^{t_{m-1}} dt_m \int_{\mathbb{R}^d} p_t^{n,m} (x, y) f(y) dy.
\]

We denote

\[
\phi_t^{n,m_0}(x, y) = p_t^n(x, y) + \sum_{m=1}^{m_0-1} \int_0^t dt_1 \ldots \int_0^{t_{m-1}} dt_m p_t^{n,m}(x, y)
\]

so that \((4.11)\) reads

\[
\int f(y) P_t(x, dy) = \int f(y) \phi_t^{n,m_0}(x, y) dy + R_n^{n_0} f(x).
\]

We recall that \(\Psi_{\eta,\kappa}\) is defined in \((4.3)\) and we define the measures on \(\mathbb{R}^d \times \mathbb{R}^d\) defined by

\[
\mu_{\eta,\kappa}(dx, dy) = \Psi_{\eta,\kappa}(x, y) P_t(x, dy) dx \quad \text{and} \quad \mu_n^{n,\kappa,m_0}(dx, dy) = \Psi_{\eta,\kappa}(x, y) \phi_t^{n,m_0}(x, y) dx dy.
\]

So, the proof consists in applying Lemma \((3.3)\) to \(\mu = \mu_{\eta,\kappa}\) and \(\mu_n = \mu_n^{n,\kappa,m_0}\).

**Step 2: analysis of the principal term.** We study here the estimates for \(f_n(x, y) = \Psi_{\eta,\kappa} \phi_t^{n,m_0}(x, y)\) which are required in \((3.8)\).

We first use \((3.26)\) in order to get estimates for \(p_t^{n,m}(x, y)\). We fix \(q_1, q_2 \in \mathbb{N}, \kappa \geq 0, p > 1\) and we recall that in Lemma \((3.8)\) we introduced \(\beta = q_1 + q_2 + (a + b)(m_0 - 1)\). Moreover in Lemma \((3.8)\) one produces \(\chi\) such that \((3.26)\) holds true: for every multi-index \(\beta\) with \(|\beta| \leq q_2\)

\[
\left\| \psi_k \partial_x^\beta p_t^{n,m_0}(x, \cdot) \right\|_{q_1, p} \leq C \left( \frac{1}{\lambda_n t} \right)^{\theta_0(q_1 + q_2 + d + 2\theta_1)} \left( \varepsilon_n A_n \left( \frac{1}{\lambda_n t} \right)^{\theta_0(a + b)} \right)^m \psi_\chi(x).
\]

Denote

\[
\xi_1(q) = q + d + 2\theta_1 + m_0(a + b), \quad \omega_1(q) = q + d + 2\theta_1.
\]

With this notation, if \(|\beta| \leq q_2\) we have

\[
\left\| \psi_k \partial_x^\beta \phi_t^{n,m_0}(x, \cdot) \right\|_{q_1, p} \leq C \left( \frac{1}{\lambda_n t} \right)^{\theta_0(q_1 + q_2 + d + 2\theta_1)} \left( \varepsilon_n A_n \left( \frac{1}{\lambda_n t} \right)^{\theta_0(a + b)} \right)^{m_0} \psi_\chi(x) \quad \text{(4.13)}
\]

\[
= Ct^{-\theta_0 \xi_1(q_1 + q_2)} \left( \frac{1}{\lambda_n t} \right)^{\theta_0 \omega_1(q_1 + q_2)} \Phi_n^{m_0}(0) \psi_\chi(x), \quad \text{(4.14)}
\]

where \(\Phi_n(\delta)\) is the constant defined in \((4.1)\). We take \(l = 2h + q, l' = 2h\) and we take \(q(l) = l + (a + b)m_0\). Moreover we fix \(q_1\) and \(q_2\) (so \(q = q_1 + q_2 \leq l\)) and we take \(\chi\) to be the one in \((4.13)\). Moreover we take \(\eta\) sufficiently large in order to have \(p\eta - 2h - p\chi \geq d + 1\). This guarantees that

\[
\int_{\mathbb{R}^d} dx \psi_{p\eta - p\chi}(x) = C < \infty. \quad \text{(4.15)}
\]
By (2.10) and (4.13) 
\[
\|\Psi_{h,K}^{l,m} \|_{l,p} \leq C \sum_{|\alpha|+|\beta| \leq l} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} |\partial^\alpha_x \partial^\beta_y \phi_t^{l,m}(x,y)|^p \psi_t(x)\psi_t(y) dy dx
\]
\[
= C \sum_{|\alpha|+|\beta| \leq l} \int_{\mathbb{R}^d} \psi_{\eta+\nu}(x,y) \int_{\mathbb{R}^d} |\psi_{\eta+\nu}(y)\partial^\alpha_x \partial^\beta_y \phi_t^{l,m}(x,y)|^p dy dx
\]
\[
\leq C \sum_{|\alpha|+|\beta| \leq l} \int_{\mathbb{R}^d} \psi_{\eta+\nu}(x,y) \psi_{\eta+\nu}(y) \partial^\alpha_x \partial^\beta_y \phi_t^{l,m}(x,y) ||^p_{\beta,p} dx
\]
\[
\leq C(t^{-\theta_0}(l)\lambda_{n-\theta_0\omega_1(l)} \Phi_n(0))^p \int_{\mathbb{R}^d} \frac{dx}{\psi_{\eta+\nu}(x)}.
\]
We conclude that 
\[
\|\Psi_{h,K}^{l,m} \|_{2h,q,2l,p} \leq C t^{-\theta_0}(l+2h) \times \lambda_{n-\theta_0\omega_1(l)} \Phi_n(0) =: \theta(n).
\]
By (4.2) \( \theta(n)^p \uparrow +\infty \) and \( \Theta(n) \geq \theta(n+1) \) with 
\[
\Theta = \gamma^{\theta_0(a+b)l_0+q+2h+2l_1+\omega_1} \geq 1.
\]
In the following we will choose \( h \) sufficiently large, depending on \( \delta, m_0, q, d \) and \( p \). So \( \Theta \) is a constant depending on \( \delta, \), \( m_0, q, d, a, b, \gamma \) and \( p \), as the constants considered in the statement of our theorem.

**Step 3: analysis of the remainder.** We study here \( d_{m_0}(n) := d_{(a+b)m_0}(\mu^{n,\kappa}, \mu^{n,\kappa,m_0}) \) as required in (3.9): we prove that, if \( n \geq k + d + 1 \), then 
\[
d_{m_0}(n) \leq C(\Lambda_n \xi_n)^m \leq \lambda_n^{\theta(a+b+\delta m_0)} \Phi(n) \delta_\kappa.
\]
Using first (A1) and (A2) (see (2.14) and (2.16)) and then (A4) (see (2.11)) we obtain
\[
\left\| \prod_{i=0}^{m_0-1} (P_{i+\tau_i}^n \Delta_n) P_{i,m_0} f \right\|_{0,-\kappa,\infty} \leq C \| f \|_{(a+b)m_0,-\kappa,\infty}(\Lambda_n \xi_n)^{m_0}
\]
which gives
\[
\| R_{m_0} f \|_{0,-\kappa,\infty} \leq C \| f \|_{(a+b)m_0,-\kappa,\infty}(\Lambda_n \xi_n)^{m_0}.
\]
Using now the equivalence between (2.11) and (2.12) we obtain
\[
\left\| \frac{1}{\psi_{\kappa}} R_{m_0} \right\|_{0,-\kappa,\infty} \leq C \| f \|_{(a+b)m_0,\kappa,\infty}(\Lambda_n \xi_n)^{m_0}.
\]
We take now \( g \in C^\infty(\mathbb{R}^d \times \mathbb{R}^d) \), we denote \( g_x(y) = g(x,y) \), and we write
\[
\left\| \int_{\mathbb{R}^d \times \mathbb{R}^d} g(x,y)(\mu^{n,\kappa} - \mu^{n,m_0}) (dx, dy) \right\|
\leq \int_{\mathbb{R}^d} g_x(y) \psi_{\kappa}(y) (P_{i}(x,dy) - \phi_t^{n,m_0}(x,y)) dy
\leq \int_{\mathbb{R}^d} g_x(y) \psi_{\kappa}(y) R_{m_0} \psi_{\kappa}(y) (x) \| \psi_{\kappa}(x) \|_{0,-\kappa,\infty}(\Lambda_n \xi_n)^{m_0}
\leq C \sup_{x \in \mathbb{R}^d} \| g_x \|_{(a+b)m_0,\kappa,\infty}(\Lambda_n \xi_n)^{m_0}.
the last inequality being a consequence of (4.18) and of \( \eta - \kappa \geq d + 1 \). Now (4.17) is proved because \( \sup_{x \in \mathbb{R}^d} \|g_x\|_{(a+b)m_0, \infty} \leq \|g\|_{(a+b)m_0, \infty} \).

**Step 4: use of Lemma 3.3 and proof of A. and B.** We recall that \( \rho_h \) is defined in (4.19) and we estimate

\[
d_{m_0}(n) \times \theta(n) \rho_h \leq C t^{-\theta_0 \xi_2(h)} \theta_0 \omega_2(h) \Phi_n^{m_0(1+\rho_h)}(\delta_*)
\]

with

\[
\xi_2(h) = \rho_h \xi_1(q + 2h) = \rho_h(q + 2h + d + 2\theta_1 + m_0(a + b))
\]

and

\[
\omega_2(h) = (a + b + \delta_*)m_0 - \rho_h(q + 2h + d + 2\theta_1) = \delta_* m_0 - \frac{(a + b)m_0 + q + 2d/p_\ast (q + d + 2\theta_1) - (q + 2d/p_\ast)}{2h}.
\]

By our choice of \( m_0 \) we have

\[
\delta_* m_0 > q + 2d/p_\ast
\]

so, taking \( h \) sufficiently large we get \( \omega_2(h) > 0 \). And we also have \( \xi_2(h) \leq \xi_3 := (a + b)m_0 + q + 2d/p_\ast + \varepsilon_\ast \) and \( \rho_h \leq \varepsilon_\ast \). So we finally get

\[
d_{m_0}(n) \times \theta(n) \rho_h \leq C t^{-\theta_0 \xi_3} \Phi_n^{m_0(1+\varepsilon_\ast)}(\delta_*)
\]

(4.19)

The above inequality guarantees that (3.9) holds so that we may use Lemma 3.3. We take \( \eta > \kappa + d \) and, using (A_4) (see (2.21)) we obtain

\[
|\mu^{n,\kappa}| = \int_{\mathbb{R}^2} \frac{\psi_\eta(x)}{\psi_\kappa(y)} P_t(x, dy) dx \leq C \int_{\mathbb{R}^2} \frac{dx}{\psi_\kappa-\eta(x)} < \infty.
\]

Then, \( A(\delta) < C \) (see (3.10)). One also has \( B(\varepsilon) \leq \infty \) (see (3.11)) and finally (see (3.12))

\[
C_{h, n_\ast}(\varepsilon) \leq C t^{-\theta_0 \xi_3} \Phi_n^{m_0(1+\varepsilon_\ast)}(\delta_*)
\]

We have used here (4.19). For large \( h \) we also have

\[
\theta(n) \rho_h \leq C(\lambda_n t)^{-\theta_0((a+b)m_0 + q + \frac{2d}{p_\ast} + \varepsilon_\ast)} \Phi_n^{\varepsilon_\ast}(0).
\]

Now (4.13) gives (1.6). So A and B are proved.

**Step 5: proof of C.** We apply B. with \( q \) replaced by \( \bar{q} = q + 1 \), so \( \Psi_{\eta,\kappa} p_t \in W^{\bar{q}, p}(\mathbb{R}^d \times \mathbb{R}^d) = W^{\bar{q}, p}(\mathbb{R}^{2d}) \). Since \( \bar{q} > 2d/p \) (here the dimension is 2d), we can use the Morrey’s inequality: for every \( \alpha, \beta \) with \( |\alpha| + |\beta| \leq |\bar{q} - 2d/p| = q \), then \( |\partial_\alpha \partial_\beta \Psi_{\eta,\kappa} p_t(x, y)| \leq C \|\Psi_{\eta,\kappa} p_t\|_{\bar{q}, p} \). By (1.6), one has

\[
|\partial_\alpha \partial_\beta (\Psi_{\eta,\kappa} p_t)(x, y)| \leq C Q_\ast(\bar{q}, \bar{m}) t^{-\theta_0((a+b)m_0 + \bar{q} + 2d/p_\ast)(1+\varepsilon_\ast)}
\]

i.e. (using (2.6)),

\[
|\partial_\alpha \partial_\beta p_t(x, y)| \leq C Q_\ast(\bar{q}, \bar{m}) t^{-\theta_0((a+b)m_0 + \bar{q} + 2d/p_\ast)(1+\varepsilon_\ast)} \times \frac{1}{\Psi_{\eta,\kappa}(x, y)}.
\]
Now, by a standard calculus, $\Psi_{\eta, \kappa}(x, y) \geq C_\kappa \psi_{\eta, \kappa}(x - y)$ (use that $\psi_\kappa(x - y) \leq C_\kappa \psi_\kappa(x) \psi_\kappa(-y) = C_\kappa \psi_\kappa(x) \psi_\kappa(y)$), so (A.8) follows. □

We are finally ready for the

**Proof of Theorem 2.6.** Our assumptions guarantees that $P_t(x, dy) = p_t(x, y)dy$ and $p_t$ satisfies (4.8). We take a cut-off function $F_R \in C^\infty(\mathbb{R}^d)$ such that $1_{B_R(0)} \leq F_R \leq 1_{B_{R+1}(0)}$ ($B_r(0)$ denoting the open ball centered at 0 with radius $r$) and we denote $p_t^R(x, y) = F_R(x) p_t(x, y)$. By (4.8) we know that, for every $\kappa \in \mathbb{N}, \varepsilon > 0$ and every $(x, y) \in \mathbb{R}^d \times \mathbb{R}^d$ one has

$$\left| \partial_x^\alpha \partial_y^\beta p_t^R(x, y) \right| \leq Ct^{-\theta_1(|\alpha| + |\beta| + \theta_1)} \psi_{\eta + \kappa}(x)$$

where $\theta_1 = \theta_0(1 + \frac{a + b}{\delta_0})(1 + \varepsilon)$, $\theta_1$ is computed from (4.8) (the precise value is not important here) and $C$ and $\eta$ both depend on $\kappa, \varepsilon, \delta, |\alpha|$ and $|\beta|$. Since the above left hand side is identically null when $|x| > R + 1$, we can write

$$\left| \partial_x^\alpha \partial_y^\beta p_t^R(x, y) \right| \leq Ct^{-\theta_1(|\alpha| + |\beta| + \theta_1)} \psi_{-\kappa}(x, y)$$

where $C$ is a new constant depending on $R$ as well (we also stress that here $\psi_{-\kappa}(x, y) = (1 + |x|^2 + |y|^2)^{-\kappa}$, so the underlying dimension is $2d$). This allows one to apply Lemma 3.2 for every $p \geq 1, \kappa \in \mathbb{N}$ and $\delta > 0$,

$$\|\psi_{\kappa} p_t^R\|_{q, p} \leq Ct^{-\theta_1(q + 2d + \delta)}, \quad t < 2^{-4/\theta_1}.$$ 

Then by Morrey’s Lemma, for every $p > 2d$

$$\|\psi_{\kappa} p_t^R\|_{q, \infty} \leq \|\psi_{\kappa} p_t^R\|_{q + 1, p} \leq Ct^{-\theta_1(q + 2d + \delta)} \leq C t^{-\theta_1(q + 2d + \varepsilon)}$$

the last inequality being true if we take $p$ close to $2d$ and $\delta < \varepsilon$. And this gives (2.23). □

### A Weights

For $k \in \mathbb{Z}$ and $x \in \mathbb{R}^d$, we denote

$$\psi_k(x) = (1 + |x|^2)^k.$$  

(A.1)

**Lemma A.1** For every multi-index $\alpha$ there exists a constant $C_\alpha$ such that

$$\left| \partial^\alpha \left( \frac{1}{\psi_k} \right) \right| \leq \frac{C_\alpha}{\psi_k}.$$  

(A.2)

Moreover, for every $q$ there is a constant $C_q \geq 1$ such that for every $f \in C^\infty_0(\mathbb{R}^d)$

$$\frac{1}{C_q} \sum_{0 \leq |\alpha| \leq q} \left| \partial^\alpha \left( \frac{f}{\psi_k} \right) \right| \leq \sum_{0 \leq |\alpha| \leq q} \frac{1}{\psi_k} \left| \partial^\alpha f \right| \leq C_q \sum_{0 \leq |\alpha| \leq q} \left| \partial^\alpha \left( \frac{f}{\psi_k} \right) \right|.$$  

(A.3)
Proof. One checks by recurrence that
\[ \partial^\alpha \left( \frac{1}{\psi_k} \right) = \sum_{q=1}^{\lvert \alpha \rvert} P_{\alpha,q} \psi_{k+q} \]
where \( P_{\alpha,q} \) is a polynomial of order \( q \). And since
\[ \frac{(1 + |x|)^q}{(1 + |x|^2)^{q+k}} \leq \frac{C}{(1 + |x|^2)^k} \]
the proof (A.2) is completed. In order to prove (A.3) we write
\[ \partial^\alpha \left( f \psi_k \right) = \frac{1}{\psi_k} \partial^\alpha f + \sum_{(\beta, \gamma) = \alpha \atop |\beta| \geq 1} c(\beta, \gamma) \partial^\beta \left( \frac{1}{\psi_k} \right) \partial^\gamma f. \]
This, together with (A.2) implies
\[ \left| \partial^\alpha \left( \frac{f}{\psi_k} \right) \right| \leq C \sum_{0 \leq |\gamma| \leq |\alpha|} \frac{1}{\psi_k} |\partial^\gamma f| \]
so the first inequality in (A.3) is proved. In order to prove the second inequality we proceed by recurrence on \( q \). The inequality is true for \( q = 0 \). Suppose that it is true for \( q - 1 \). Then we write
\[ \frac{1}{\psi_k} \partial^\alpha f = \partial^\alpha \left( \frac{f}{\psi_k} \right) - \sum_{(\beta, \gamma) = \alpha \atop |\beta| \geq 1} c(\beta, \gamma) \partial^\beta \left( \frac{1}{\psi_k} \right) \partial^\gamma f \]
and we use again (A.2) in order to obtain
\[ \frac{1}{\psi_k} |\partial^\alpha f| \leq \left| \partial^\alpha \left( \frac{f}{\psi_k} \right) \right| + C \sum_{|\gamma| < |\alpha|} \frac{1}{\psi_k} |\partial^\gamma f| \leq C \sum_{0 \leq |\beta| \leq q} \left| \partial^\beta \left( \frac{f}{\psi_k} \right) \right| \]
the second inequality being a consequence of the recurrence hypothesis. □

Remark A.2 The assertion is false if we define \( \psi_k(x) = (1 + |x|)^k \) because \( \partial_i \partial_j |x| = \frac{\delta_{ij}}{|x|} - \frac{x_i x_j}{|x|^2} \) blows up in zero.

We look now to \( \psi_k \) itself.

Lemma A.3 For every multi-index \( \alpha \) there exists a constant \( C_\alpha \) such that
\[ |\partial^\alpha \psi_k| \leq C_\alpha \psi_k. \]
Moreover, for every \( q \) there is a constant \( C_q \geq 1 \) such that for every \( f \in C_c^\infty(\mathbb{R}^d) \)
\[ \frac{1}{C_q} \sum_{0 \leq |\alpha| \leq q} |\partial^\alpha (\psi_k f)| \leq \sum_{0 \leq |\alpha| \leq q} \psi_k |\partial^\alpha f| \leq C_q \sum_{0 \leq |\alpha| \leq q} |\partial^\alpha (\psi_k f)|. \]

Proof. One proves by recurrence that, if \( |\alpha| \geq 1 \) then \( \partial^\alpha \psi_k = \sum_{q=1}^{\lvert \alpha \rvert} \psi_{k-q} P_q \) with \( P_q \) a polynomial of order \( q \). Since \( 1 + |x| \leq 2(1 + |x|^2) \) it follows that \( |P_q| \leq C \psi_q \) and (A.4) follows. Now we write
\[ \psi_k \partial^\alpha f = \partial^\alpha (\psi_k f) - \sum_{(\beta, \gamma) = \alpha \atop |\beta| \geq 1} c(\beta, \gamma) \partial^\beta \psi_k \partial^\gamma f \]
and the same arguments as in the proof of (A.3) give (A.5).
B Semigroup estimates

We consider a semigroup \((P_t)_{t \geq 0}\) on \(C^\infty(\mathbb{R}^d)\) such that \(P_tf(x) = \int f(y)P_t(x,dy)\) where \(P_t(x,dy)\) is a probability transition kernel and we denote by \(P_t^*\) its formal adjoint.

**Assumption B.1** There exists \(Q \geq 1\) such that for every \(t \leq T\) and every \(f \in C^\infty(\mathbb{R}^d)\)

\[
\|P_tf\|_1 \leq Q \|f\|_1. \tag{B.1}
\]

Moreover, for every \(k \in \mathbb{N}\) there exists \(K_k \geq 1\) such that for every \(x \in \mathbb{R}^d\) and \(t \leq T\)

\[
|P_t(\psi_k)(x)| \leq K_k \psi_k(x). \tag{B.2}
\]

**Lemma B.2** Under Assumption [B.1] for every \(t \leq T\) one has

\[
\|\psi_k P_t^*(f/\psi_k)\|_p \leq K_{kp}^{1/p} Q^{1/p} \|f\|_p. \tag{B.3}
\]

**Proof.** Using Hölder’s inequality, the identity \(\psi_k^p = \psi_{kp}\), and (B.2)

\[
|P_t(\psi_k)(x)| \leq |P_t(\psi_k^p)(x)|^{1/p} |P_t(|g|^{p^*})(x)|^{1/p^*} \leq K_{kp}^{1/p} \psi_k(x) |P_t(|g|^{p^*})(x)|^{1/p^*}.
\]

Then, using (B.1)

\[
\left\| \frac{1}{\psi_k} P_t(\psi_k g) \right\|_{p^*} \leq K_{kp}^{1/p} \left\| P_t(|g|^{p^*}) \right\|_{p^*}^{1/p^*} = K_{kp}^{1/p} \left( \||g|^{p^*}\|_1 \right)^{1/p^*}
\]

\[
\leq K_{kp}^{1/p} Q^{1/p^*} \left( \||g|^{p^*}\|_1 \right)^{1/p^*} = K_{kp}^{1/p} Q^{1/p^*} \|g\|_{p^*}.
\]

Using Hölder’s inequality first and the above inequality we obtain

\[
|\langle g, \psi_k P_t^*(f/\psi_k) \rangle| = \left| \left\langle \frac{1}{\psi_k} P_t(g\psi_k), f \right\rangle \right| \leq \|f\|_p \left\| \frac{1}{\psi_k} P_t(g\psi_k) \right\|_{p^*}
\]

\[
\leq K_{kp}^{1/p} Q^{1/p^*} \|g\|_{p^*} \|f\|_p.
\]

\(\square\)

We consider also the following hypothesis.

**Assumption B.3** There exists \(\rho > 1\) such that for every \(q \in \mathbb{N}\) there exists \(D_{(q)}^* (\rho) \geq 1\) such that for every \(x \in \mathbb{R}^d\) and \(t \leq T\)

\[
\sum_{|\alpha| \leq q} |\partial^\alpha P_t^* f(x)| \leq D_{(q)}^* (\rho) \sum_{|\alpha| \leq q} (P_t^* (|\partial^\alpha f|^{p^*})(x))^{1/\rho}. \tag{B.4}
\]

**Proposition B.4** Suppose that Assumption [B.1] and [B.3] hold. Then for every \(k, q \in \mathbb{N}\) and \(p > \rho\) there exists a universal constant \(C\) (depending on \(k\) and \(q\) only) such that for every \(t \leq T\)

\[
\|\psi_k P_t^*(f/\psi_k)\|_{q,p} \leq C K_{kp}^{1/p} Q^{(p-\rho)/\rho p} D_{(q)}^* (\rho) \|f\|_{q,p}. \tag{B.5}
\]
Proof. We will prove (B.5). Let $\alpha$ with $|\alpha| \leq q$. By (B.4)
\[ |\partial^\alpha(\psi_k P_t^*(f/\psi_k)(x))| \leq C \psi_k(x) \sum_{|\gamma| \leq q} |\partial^\gamma (P_t^*(f/\psi_k)(x))| \]
\[ \leq CD^*_q(\rho) \psi_k(x) \sum_{|\beta| \leq q} (\psi_{pk}(x) P_t^*(f/\psi_k)|^p(x))^{1/p} \]
\[ = CD^*_q(\rho) \sum_{|\beta| \leq q} (\psi_{pk}(x) P_t^*(g_{\beta/\psi_{pk}})(x))^{1/p} \]
with
\[ g_{\beta}(x) = \psi_{pk}(x) |\partial^\beta (f/\psi_k)(x)|^p = \psi_k(x) |\partial^\beta (f/\psi_k)(x)|^p. \]
Taking $p > \rho$ and using (B.3)
\[ \left\| (\psi_{pk} P_t^*(g_{\beta/\psi_{pk}}))^{1/p} \right\|_p = \left\| \psi_{pk} P_t^*(g_{\beta/\psi_{pk}}) \right\|^{1/p}_{p/p} \leq K_{kp}^{1/p} Q^{(p-\rho)/p} \left\| g_{\beta} \right\|^{1/p}_{p/p}. \]
And we have
\[ \|g_{\beta}\|^{1/p}_{p/p} = \left( \int \psi_k(x) |\partial^\beta (f/\psi_k)(x)|^p dx \right)^{1/p} \leq C \sum_{|\gamma| \leq q} \left( \int |\partial^\gamma f(x)|^p dx \right)^{1/p} = C \|f\|_{q,p}. \]
We conclude that
\[ \|\psi_k P_t^*(f/\psi_k)\|_{q,p} \leq CK_{kp}^{1/p} Q^{(p-\rho)/p} D^*_q(\rho) \|f\|_{q,p}. \]
\[ \square \]

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