Torsion Elements in the Mapping Class Group of a Surface

Feng Luo

Abstract Given a finite set of \( r \) points in a closed surface of genus \( g \), we consider the torsion elements in the mapping class group of the surface leaving the finite set invariant. We show that the torsion elements generate the mapping class group if and only if \( (g, r) \neq (2, 5k + 4) \) for some integer \( k \).

§1. Introduction

1.1. The purpose of this paper is to investigate when the mapping class group of a compact surface is generated by torsion elements. Our result gives a complete answer to this question.

Theorem. Suppose \( \Sigma_{g,r} \) is a compact orientable surface of genus \( g \) with \( r \) many boundary components where \( g, r \geq 0 \). Then the mapping class group of the path components of the orientation preserving homeomorphisms of the surface is generated by torsion elements if and only if \( (g, r) \neq (2, 5k + 4) \) for some integer \( k \in \mathbb{Z} \). The torsion elements in the mapping class group of the surface \( \Sigma_{2,5k+4} \) generate an index 5 subgroup.

Furthermore, in the case of \( (g, r) \neq (2, 5k + 4) \), the order \( n \) of the torsion elements generating the group can be chosen as follows: (a) if \( g \geq 3 \), then \( n = 2 \); (b) if \( g = 2 \), then \( n \in \{2, 5\} \); (c) if \( g = 1 \), then \( n \in \{2, 3, 4, \} \); and (d) if \( g = 0 \), then \( n \in \{r - 1, r\} \).

Note that by identifying each boundary component to a point, one sees that the mapping class group in the theorem is the same as the mapping class group of a closed surface leaving a set of \( r \) points invariant.

The existence of the exceptional cases \( (2, 5k + 4) \) is caused by the fact that there is only one non-trivial \( \mathbb{Z}_5 \)-action on the closed surface of genus-2. And this \( \mathbb{Z}_5 \)-action has too few fixed points (only 3 points).

The result is motivated by the example of the torus whose mapping class group is \( SL(2, \mathbb{Z}) = \mathbb{Z}_4 *_{\mathbb{Z}_2} \mathbb{Z}_6 \). In the case of surfaces of genus at least 3, the theorem can be derived easily from the work of Harer [Ha]. A simple derivation of it will be given in §1.3. The main body of the paper is to prove the theorem for surfaces of genus at most 2.

On the related question of Torelli groups, the following can be derived easily from the work of Johnson [Jo] and Powell [Po]. Namely the Torelli group of a closed surface of genus at least 3 is generated by the products of even number of hyperelliptic involutions.

1.2. A basic strategy to prove that a group \( G \) is generated by torsion is to produce a short exact sequence \( 1 \to H \to G \to G/H \to 1 \) so that \( H \) is generated by
torsion elements in $G$ and there are torsion elements in $G$ whose projections in $G/H$ generate the group. In our case, let $\Gamma^*_{g,r}$ be the mapping class group of the surface which is the group of path components of orientation preserving homeomorphisms of the surface. We use the short exact sequence $1 \to [\Gamma^*_{g,r}, \Gamma^*_{g,r}] \to \Gamma^*_{g,r} \to H_1(\Gamma^*_{g,r}) \to 1$. We first show that the mapping class group is generated by Dehn-twists and torsion elements. Then we use it to prove that the commutator subgroup $[\Gamma^*_{g,r}, \Gamma^*_{g,r}]$ is generated by torsion elements. Finally we show that the images of the torsion elements in the first homology group generate the homology.

The most interesting part of the proof is to show that for genus-2 surface $\Sigma_{2,0}$ the projections of a 5-fold symmetry and an involution with two fixed points in the surface generate the first homology group $H_1(\Gamma^*_{2,0}) \simeq \mathbb{Z}_{10}$.

In the process of proof, we show that each compact surface admits a non-trivial $\mathbb{Z}_3$ action. We also calculated the first homology of $\Gamma^*_{g,r}$ using Harer and Powell’s work.

1.3. Using the work of Harer [Ha], we give a simple proof the theorem for surfaces of genus at least 3 in this section. The argument is no longer working for low genus surfaces.

Let $\Gamma_{g,r}$ be the pure mapping class group $\Gamma_{g,r}$ which is the subgroup of $\Gamma^*_{g,r}$ so that it acts trivially on the set of boundary components of the surface. Then there is a short exact sequence $1 \to \Gamma_{g,r} \to \Gamma^*_{g,r} \to S_r \to 1$ where $S_r$ is the permutation group on $r$ boundary components of the surface. It is easy to show that there are torsion elements in the mapping class group $\Gamma^*_{g,r}$ whose projections in $S_r$ generate the permutation group. Thus it remains to show that the pure mapping class group $\Gamma_{g,r}$ is generated by torsion elements in $\Gamma^*_{g,r}$. Note that if $r > 2g + 2$, then the pure mapping class group $\Gamma_{g,r}$ is torsion free.

The fundamental work of Dehn [De] and Lickorish [Li] shows that $\Gamma_{g,r}$ is generated by Dehn-twists. If the genus of the surface is positive, it can be shown that Dehn-twists on non-separating simple loops generate the pure mapping class group. On the other hand, any two Dehn-twists on non-separating simple loops are conjugate. Thus it suffices to show that the Dehn-twist on a non-separating simple loop is a product of torsions in $\Gamma^*_{g,r}$.

Since the genus of the surface is at least three, there is a 4-holed sphere subsurface in $\Sigma_{g,r}$ bounded by four non-separating simple loops $a, b, c, d$ so that the complement of the 4-holed sphere is connected. Take three simple loops $x, y, z$ in the 4-holed sphere forming a lantern position (see figure 4.1(a)). Then the lantern relation gives: $ABCD = XYZ$ where capital letters are the positive Dehn-twist on the small letter simple loops. Thus $A = XB^{-1}YC^{-1}ZD^{-1}$ where each of $(x, b), (y, c)$ and $(z, d)$ is a pair of disjoint non-separating simple loops so that complement of their union in $\Sigma_{g,r}$ is connected. Thus it suffices to show, for instance, that $XB^{-1}$ is a product of involutions. By the choice of $(x, b)$, there is an involution $f$ of the surface sending $x$ to $b$. Thus $B = fXf^{-1}$. This shows that $XB^{-1} = (XfX^{-1})f^{-1}$ which is the product of two involutions $f^{-1}$ and $XfX^{-1}$. In Harer’s proof [Ha], he
used the equation $XB^{-1} = XfX^{-1}f^{-1}$ to show that $A$ is a product of commutators where $f$ is an element in $\Gamma_{g,r}$ and deduced that the first homology of $\Gamma_{g,r}$ is trivial for $g \geq 3$.

The proof for the Torelli group of closed surface of genus at least 3 is the same as above using the work of Johnson [Jo] and Powell [Po]. Indeed, they showed that the Torelli group is generated by a product of Dehn-twists $AB^{-1}$ where the simple loops $a, b$ are disjoint, both non-separating and $a \cup b$ decomposes the surface into two pieces. Thus there is a hyperelliptic involution $f$ sending $a$ to $b$ and $AB^{-1} = (AfA^{-1})f^{-1}$.

1.4. Periodic homeomorphisms and Dehn-twists are the two extremal cases of self homeomorphisms in the sense that the Dehn-twists have the largest support and periodic homeomorphisms have the smallest support. In particular this makes the calculation of a periodic homeomorphism in terms of Dehn-twists quite complicated. In view of the fact that the mapping class group $\Gamma_{1,0}$ of a torus is $SL(2,\mathbb{Z}) = \mathbb{Z}_4 \ast \mathbb{Z}_2 \mathbb{Z}_6$ and also the recent work of Wajnryb [Wa] that the pure mapping class group is generated by two elements, one is attempting to ask what is the minimal number of periodic generators for the mapping class group. Is it possible to generate $\Gamma_{g,r}^*$ by torsion elements where the number of generators is independent of $g, r$?

1.5. The organization of the paper is as follows. In section 2, we recall some well known symmetries of surfaces of low genus and their expressions in Dehn-twists. In section 3 we show that the commutator subgroup of the mapping class group is generated by torsion elements. In section 4, we prove the theorem using the first homology group of the mapping class group.

1.6. Acknowledgment. The work is supported in part by the NSF.

§2. Period Homeomorphisms on Low Genus Surfaces

We introduce notations and terminologies in this section. Also we recall some well known symmetries of surfaces of genus at most 2.

2.1. We start by introducing some notations and conventions. Given a finite set $X$, we use $|X|$ to denote the number of elements in $X$. Surfaces are assumed to be oriented. Subsurfaces have the induced orientation. Simple loops on surfaces will be denoted by $a, b, ..., c$. Positive Dehn-twists on them will be denoted by $D_a, D_b, ..., D_c$ or simply by $A, B, ..., C$ if no confusion will arise. A small regular neighborhood of a 1-dimensional submanifold $s$ will be denoted by $N(s)$. We shall not distinguish homeomorphisms from their isotopy classes. Thus a Dehn-twist sometimes means the isotopy class of a Dehn-twist.

Given an arc $s$ joining two boundary components $\partial_1$ and $\partial_2$ of a surface, the positive half-Dehn-twist (or simply half-twist) along $s$ will be denoted by $D^{1/2}_s$. It is a self-homeomorphism $h$ supported in $N(s \cup \partial_1 \cup \partial_2)$ so that $h$ leaves $s$ invariant and interchanges $\partial_1, \partial_2$ and $h^2$ is the positive Dehn-twist along $\partial N(s \cup \partial_1 \cup \partial_2) - \partial_1 \cup \partial_2$.

Given a compact surface $\Sigma$, let $\Sigma^*$ be the quotient of $\Sigma$ obtained by identifying each boundary component to a point. Clearly each homeomorphism $h$ of $\Sigma$ induces
a canonical homeomorphism \( h^* \) of the closed surface \( \Sigma^* \). An hyperelliptic involution \( h \) on a surface \( \Sigma \) is an involution so that its induced involution \( h^* \) has exactly \( 2g + 2 \) fixed points in \( \Sigma^* \) where \( g \) is the genus. Note that any two hyperelliptic involutions on a closed surface are conjugate. Also the action of a hyperelliptic involution on the first homology of a closed surface is the multiplication by \( -1 \).

2.2. The fundamental work of Dehn [De] and Lickorich [Li] states that the pure mapping class group \( \Gamma_{g,r} \) is finitely generated by Dehn-twists. We shall need a slightly improved version of it. See [Ge] for a proof.

**Proposition.** ([Ge]) For a surface of positive genus, the pure mapping class group is finitely generated by Dehn-twists on non-separating simple loops \( a_1, \ldots, a_n \) so that for each pair of indices \( i, j \), either \( a_i \) is disjoint from \( a_j \) or \( a_i \) intersects \( a_j \) at one point.

For instance the following loops form such a generating set. See figure 2.1.

![Figure 2.1](image)

2.3. The basic symmetries of surfaces of low genus that we shall use are the followings. The \( 2\pi/n \)-rotation \( \tau_n \) of the 2-sphere, the \( \mathbb{Z}_4 \) and \( \mathbb{Z}_6 \) non-free actions on the torus and the \( \mathbb{Z}_5 \) action on the genus-2 surface.

The action of \( \mathbb{Z}_4 \) on the torus is constructed as follows. Consider the torus as the quotient of the square by identifying the opposite sides. Then the generator \( \tau_4 \) of the \( \mathbb{Z}_4 \) action is induced by the \( \pi/2 \)-rotation of the square. It has 2 fixed points corresponding to the center and vertices and has an orbit consisting of two points coming from the mid-points of the four sides. Note that \( \tau_4^2 = \tau_2 \) is a hyperelliptic involution.

The action of \( \mathbb{Z}_6 \) on the torus is constructed as follows. Consider the torus as the quotient of the hexagon by identifying the opposite sides. Then the generator \( \tau_6 \) of the action is induced by the \( 2\pi/6 \)-rotation of the hexagon. It has one fixed
point corresponding to the center of the hexagon. And it has an orbit consisting of three points coming from the mid-points of the six sides and an orbit consisting of two points coming from the six vertices. Note that \( \tau_6^3 = \tau_2 \) is a hyperelliptic involution and that \( \tau_6^2 = \tau_3 \) is a \( \mathbb{Z}_3 \) action having three fixed points.

The action of \( \mathbb{Z}_5 \) on the genus-2 surface is induced by the \( 2\pi/5 \)-rotation of the 10-sided polygon. Here the genus-2 surface is the quotient of the 10-sided polygon by identifying the opposite sides (in fact this shows that \( \mathbb{Z}_{10} \) acts on the surface). The generator \( \tau_5 \) of the action has three fixed points in the genus-2 surface. These fixed points are the quotients of the center and the vertices of the 10-sided polygon.

**Lemma.**  
(a) For \( k = 2, 3, 4, 6 \), each genus-1 surface \( \Sigma_{1,r} \) has an order \( k \) periodic homeomorphism \( f_k \) so that its induced homeomorphism in the torus \( \Sigma_{1,0} = \Sigma_{1,r}^* \) is the standard symmetry \( \tau_k \).

(b) For \( r \neq 5k + 4 \), the surface \( \Sigma_{2,r} \) has an order 5 periodic homeomorphism \( f_5 \) so that its induced homeomorphism on the genus-2 surface is the standard symmetry \( \tau_5 \). There is no \( \mathbb{Z}_5 \) action on the surface \( \Sigma_{2,5k+4} \).

(c) Suppose the genus \( g \) is positive. If \( 0 \leq k \leq 3 \) and \( r - k \) is a non-negative even integer, then there is an involution on the surface \( \Sigma_{g,r} \) so that exactly \( k \) many boundary components of the surface is invariant under the involution.

(d) If \( \tau \) is an involution on a closed surface \( \Sigma_{g,0} \) with at least one fixed point, then there exists an involution in each compact surface \( \Sigma_{g,r} \) which induces \( \tau \) in the closed surface \( \Sigma_{g,0} = \Sigma_{g,r}^* \).

**Proof.** A simple method to construct a period-\( n \) homeomorphism \( f_n \) on the surface \( \Sigma_{g,r} \) inducing a given period-\( n \) homeomorphism \( h_n \) on the closed surface \( \Sigma_{g,0} = \Sigma_{g,r}^* \) is as follows. We take a union of \( h_n \)-orbits consisting of \( r \) points in \( \Sigma_{g,0} \). Let \( \Sigma_{g,r} \) be the surface obtained from \( \Sigma_{g,0} \) by removing an equivariant neighborhood of the union of the orbits and take \( f_n \) to be the restriction. Thus the main issue is a simple arithmetic problem.

To see part (a), since \( \tau_2 \) and \( \tau_3 \) are powers of \( \tau_4 \) and \( \tau_6 \), it suffices to show the result for \( \tau_4 \) and \( \tau_6 \).

For \( \tau_4 \) on the torus, the homeomorphism has two fixed points \( \{a_1, a_2\} \) and an orbit \( \{b_1, b_2\} \) consisting of two points. We choose the union \( X \) of \( \tau_4 \) orbit consisting of \( r \) many points as follows. If \( r \equiv 0 \) mod 4, then \( X \cap \{a_1, a_2, b_1, b_2\} = \emptyset \). If \( r \equiv 1, 2, 3 \) mod 4, then \( X \cap \{a_1, a_2, b_1, b_2\} = \{a_1\} \), \( \{a_1, a_2\} \) and \( \{a_1, b_1, b_2\} \) respectively.

For \( \tau_6 \) on the torus, the homeomorphism has one fixed point \( a \), an orbit \( \{b_1, b_2\} \) consisting of two points, and an orbit \( \{c_1, c_2, c_3\} \) consisting of three points. We choose the union \( X \) of the orbit consisting of \( r \) many points as follows. If \( r \equiv 0 \) mod 6, \( X \cap \{a, b_1, b_2, c_1, c_2, c_3\} \) is empty. If \( r \equiv 1, 2, 3, 4, 5 \) mod 6, we choose \( X \) so that \( X \cap \{a, b_1, b_2, c_1, c_2, c_3\} \) is \( \{a\} \), \( \{b_1, b_2\} \), \( \{c_1, c_2, c_3\} \), \( \{a, c_1, c_2, c_3\} \) and \( \{b_1, b_2, c_1, c_2, c_3\} \) respectively.

To see part (b), we note that \( \tau_5 \) has three fixed points \( \{a_1, a_2, a_3\} \) in \( \Sigma_{2,0} \). Thus, by the same argument as above, we can construct period-5 homeomorphisms.
on \(\Sigma_{2,5k+s}\) when \(0 \leq s \leq 3\). But there are no \(\mathbb{Z}_5\) action on \(\Sigma_{2,5k+4}\). Indeed, if \(f_5\) is such a period-5 homeomorphism, then at least four boundary components of \(\Sigma_{2,5k+4}\) are invariant under \(f_5\). Thus the induced period-5 homeomorphism \(f_5^*\) on the genus-2 surface has \(n \geq 4\) fixed points. By removing an equivariant neighborhood of these fixed points, we obtain a free \(\mathbb{Z}_5\) action on the surface \(\Sigma_{2,n}\) whose quotient is a surface \(\Sigma_{g,n}\) having \(n\) boundary components. But the Euler number multiplies under covering, i.e., \(-2 - n = 5(2 - 2g - n)\) where \(n \geq 4\). A simple inspection shows that this is impossible.

Using the same method, one proves part (d).

To prove part (c), the hyperelliptic involution \(\tau_2\) on \(\Sigma_{g,0}\) has \(2g + 2 \geq 4\) fixed points. Since \(r - k \geq 0\) is even and \(k \leq 3\), we can take a union \(X\) of \(\tau_3\) orbits so that \(X\) contains exactly \(k\) fixed points of \(\tau_2\) and \(|X| = r\). Now take \(\Sigma_{g,r}\) to be \(\Sigma_{g,0}\) with a \(\tau_2\) equivariant neighborhood of \(X\) removed. The restriction of \(\tau\) gives the required involution. \(\square\)

2.4. In this section, we will express the standard symmetries in \(\S 2.3\) in terms of Dehn-twists and half-twists.

**Lemma.** (a) ([De]) Let \(a\) and \(b\) be two simple loops in the torus \(\Sigma_{1,0}\) so that they intersect transversely at one point. Let \(A\) and \(B\) be the positive Dehn-twists on \(a\) and \(b\) respectively. Then the standard symmetries of the torus are the following: the hyperelliptic involution \(\tau_2 = ABABAB\), the 4-fold symmetry \(\tau_4 = ABA\) and the 6-fold symmetry \(\tau_6 = AB\).

(b) (see [Bi]) Let \(a_1, \ldots, a_{r-1}\) be the pairwise disjoint arcs in the planar surface \(\Sigma_{0,r}\) so that \(a_i\) joins the \(i\)-th boundary \(B_i\) to \(B_{i+1}\). Let \(A_i\) be the half-twist about the arc \(a_i\). Then \(\tau_r = A_1 \ldots A_{r-1}\) and \(\tau_{r-1} = A_1 \ldots A_{r-2}\) are \(2\pi/r\) and \(2\pi/(r-1)\)-rotation of the surface sending \(a_i\) to \(a_{i+1}\) for \(1 \leq i \leq r - 3\).

(c) (see [Bi]) Let \(C_1, \ldots, C_5\) be the positive Dehn-twists on the five simple loops \(c_1, \ldots, c_5\) in the genus-2 surface (see figure 2.3). Then the hyperelliptic involution \(\tau_2 = C_1C_2C_3C_4C_5^2C_4C_3C_2C_1\) and the 5-fold symmetry is \(\tau_5 = \tau_2C_1C_2C_3C_4\). There is an involution \(\delta\) with two fixed points on \(\Sigma_{2,0}\) which is a product of 15 or 25 many Dehn-twists on non-separating simple loops.

**Proof.** To see part (a), by the work of Dehn [De], we have \(ABA = BAB\) and also \((ABA)^4 = 1\) and \((AB)^6 = 1\) (this can be verified using the first homology group of the torus). Thus the result follows.

To see part (b), we note that a presentation of the mapping class group \(\Gamma_{0,r}\) was obtained in [Bi] where the generators are \(A_1, \ldots, A_{r-1}\) subject to the relations (1) \(A_iA_j = A_jA_i\) if \(|i - j| \geq 2\); (2) \(A_iA_{i+1}A_i = A_{i+1}A_iA_{i+1}\); (3) \(A_1A_{r-2}A_{r-1}A_{r-2}A_1 = 1\); and (4) \((A_1 \ldots A_{r-1})^n = 1\). The last relation says that \(\tau_r = A_1 \ldots A_{r-1}\) is a \(2\pi/r\)-rotation on the planar surface. One can also see this using braid representation of the mapping class group where the half-twist \(A_i\) corresponds to the standard \(i\)-th switching generator \(\sigma_i\) of the braid group on \(r\)-strings. Now the product \(\sigma_1\sigma_2 \ldots \sigma_{r-1}\) represents a full twist string as shown in figure 2.2.
Thus $\tau_r = A_1 A_2 \ldots A_{r-1}$ corresponds to the $2\pi/r$-rotation. This also shows that the product $\tau_j = A_1 A_2 \ldots A_{j-1}$ leaves the subsurface $N(B_1 \cup \ldots \cup B_j \cup a_1 \cup \ldots \cup a_{j-1})$ invariant and its restriction to the subsurface is a $2\pi/j$-rotation. Now take $j = r - 1$ and observe that the regular neighborhood of $B_1 \cup B_2 \cup \ldots \cup B_{r-1} \cup a_1 \cup \ldots \cup a_{r-2}$ is isotopic to $\Sigma_{0,r}$. We conclude that $\tau_{r-1}$ is a $2\pi/(r - 1)$-rotation on the surface $\Sigma_{0,r}$.

The braid $\sigma_1 \sigma_2 \sigma_3 \ldots \sigma_{r-1}$ representing a $2\pi/r$-rotation

Figure 2.2

To see part (c) we recall that the work of [Li] shows that the mapping class group of the genus-2 closed surface is generated by five Dehn-twists $C_1, \ldots, C_5$ on five simple loops $c_1, \ldots, c_5$ (see figure 2.3). But these five simple loops are invariant under the hyperelliptic involution $\tau_2$ induced by the $\pi$-rotation about the x-axis. Thus the hyperelliptic involution $\tau_2$ is in the center of the mapping class group $\Gamma_{2,0}$ ([Bi], [Vi]). The work of [Bi] and [BH] shows that the quotient of $\Gamma_{2,0}$ by $\tau_2$ is the mapping class group $\Gamma^*_{0,6}$. Thus there is a central extension $1 \to \mathbb{Z}_2 \to \Gamma_{2,0} \to \Gamma^*_{0,6} \to 1$. The quotient homomorphism is constructed as follows. Take an orientation preserving homeomorphism $f$ on $\Sigma_{2,0}$. One may isotope $f$ so that $f(\tau_2(x)) = \tau_2(f(x))$ for all $x$. Thus $f$ induces a homeomorphism $f_*$ on the quotient space $\Sigma_{2,0}/\tau_2$ which is a 2-sphere with six cone points. Thus one may think of $f_*$ as an element in $\Gamma^*_{0,6}$. The work of [Bi] and [BH] shows that the map sending $[f]$ to $[f_*]$ is a well defined epimorphism with kernel generated by the hyperelliptic involution. In particular, we see that the quotient homomorphism sends the Dehn-twist $C_i$ about the simple loop $c_i$ to the half-twist $A_i$ about $a_i$ in $\Sigma_{0,6}$.
The lifts of relation (2) in $\Gamma^*_0$ shows that the hyperelliptic involution $\tau_2 = C_1C_2C_3C_4C_5C_3C_2C_1$. By part (b), the element $A_1A_2A_3A_4$ is a period-4 element in $\Gamma^*_0$. It has two lifts in $\Gamma_{2,0}$ given by $C_1C_2C_3C_4$ and $\tau_2C_1C_2C_3C_4$. The tenth-power of both lifts are the identity. We claim that $\tau_2C_1C_2C_3C_4$ has order 5 and is thus equal to $\tau_5$. To this end, we use the first homology of the surface $\Sigma_{2,0}$. Note that $H_1(\Sigma_{2,0})$ is generated by the homology classes $[c_1],...,[c_4]$ where we choose orientation on $c_i$ so that the algebraic intersection number from $c_i$ to $c_{i+1}$ is one. A simple calculation shows that matrix representative of $C_1C_2C_3C_4$ with respect to the basis $([c_1],...,[c_4])$ is

$$
\begin{pmatrix}
 0 & 1 & 0 & 0 \\
 0 & 0 & 1 & 0 \\
 0 & 0 & 0 & 1 \\
-1 & 1 & -1 & 1
\end{pmatrix}.
$$

The fifth power of the matrix is $-id$. Since the hyperelliptic involution induces the multiplication by $-1$ in homology, thus the fifth power of the periodic element $\tau_2C_1C_2C_3C_4$ is the identity. By Hurwitz theorem that first homology detects periodic homeomorphisms, we see that $\tau_5 = \tau_2C_1C_2C_3C_4$.

To show that there is an involution $\delta$ with two fixed points in $\Sigma_{2,0}$ which is a product of 15 or 25 many Dehn-twists on $c_1,...,c_5$, we draw the surface $\Sigma_{2,0}$ as in figure 2.3 where the hyperelliptic involution $\tau_2$ is induced by the $\pi$-rotation about the $x$-axis and $\delta$ is induced by the $\pi$-rotation about the $z$-axis. We claim that $\delta$ is isotopic to a product of 15 or 25 Dehn-twists on $c_1,...,c_5$. To see this, let us consider the induced homeomorphism $\delta_*$ in the quotient sphere $\Sigma_{2,0}/\tau_2$. The induced involution $\delta_*$ has two fixed points $\{N,S\}$ and leaves the six cone points $\{p_1,...,p_6\}$ invariant. Note that the images of the simple loop $c_i$'s are
the arcs $a_i$ joining $i$-th cone point $p_i$ to $p_{i+1}$. We construct a homeomorphism from the quotient space $\Sigma_{2,0}/\tau_2$ to the Riemann sphere sending $N$ to the infinity and $S$ to the origin, the arc $a_1 \cup \ldots \cup a_5$ into the real axis, and the six points $(p_1, \ldots, p_6)$ to $(-3, -2, -1, 1, 2, 3)$ so that the involution $\delta_*$ becomes $z \rightarrow -z$. Thus a braid representative of $\delta_*$ in the six-string braid group based on $(-3, -2, -1, 1, 2, 3)$ and standard generators $\sigma_1, \ldots, \sigma_5$ is given by $\sigma_5 \sigma_4 \sigma_1 \sigma_3 \sigma_4 \sigma_2 \sigma_5 \sigma_4 \sigma_3 \sigma_1 \sigma_2 \sigma_3 \sigma_1 \sigma_2 \sigma_2$ (see figure 2.4). Thus $\delta_*=A_5 A_4 A_5 A_4 A_5 A_2 A_3 A_4 A_1 A_2 A_3 A_1 A_2 A_1$. Since the lifts of $\delta_*$ are $\delta$ and $\tau_2 \delta$ and $C_i$ is sent to $A_i$, it follows that one of $\delta$ or $\tau_2 \delta$ is $C_5 C_4 C_5 C_3 C_4 C_5 C_2 C_3 C_4 C_1 C_2 C_3 C_1 C_2 C_1$. The result follows. □

The braid is $5453452341233121$ where $i$ stands for the standard $i$-th generator.

Figure 2.4

2.5. Let $S_r$ be the permutation group on $r$ boundary components of the surface. Then there is a short exact sequence $1 \rightarrow \Gamma_{g,r} \rightarrow \Gamma^{*}_{g,r} \rightarrow S_r \rightarrow 1$ obtained by the inclusion of the pure mapping class group into the mapping class group. The lemma below shows that there are torsion elements in $\Gamma^{*}_{g,r}$ whose projections to $S_r$ generate the group.

**Lemma.** (a) If the genus of the surface is positive, then there are involutions in the mapping class group $\Gamma^{*}_{g,r}$ whose projections generate $S_r$.

(b) (see [Bi]) If the genus of the surface is zero, the mapping class group $\Gamma^{*}_{0,r}$ is generated by torsion elements of orders $r-1$ and $r$.

**Proof.** For part (a), note that the permutation group $S_r$ is generated by transpositions $(ij)$. Each transposition $(ij)$ can be expressed as a composition $\alpha \beta$ where $\alpha$ and $\beta$ are involutions in $S_r$ having at most three fixed points. For instance $(12) = [(12)(34)(n-1-n)][(1)(2)(34)(n-1-n)]$ for $n$ even and $(12) = [(12)(34)...(n-1-n-2)(n)][(1)(2)(34)(n)]$ for $n$ odd. Since any two involutions in $S_r$ with the same number of fixed points are conjugate, it suffices to show that an involution in $S_r$ with $k \leq 3$ fixed points can be realized by an involution on the surface. By lemma 2.3(c), the result follows.
Part (b) follows from lemma 2.4(b). Indeed, the mapping class group $\Gamma_{g,r}^{*}$ is generated by $A_1, ..., A_r$ (see [Bi]). Furthermore, any two half-twists are conjugate. Thus it suffices to show a half-twist, say $A_{r-1}$, is a product of periodic homeomorphisms. By lemma 2.4(b), $A_{r-1} = (A_1...A_{r-2})^{-1}(A_1A_2....A_{r-1})$ is a product of $2\pi/(r - 1)$ and $2\pi/r$-rotations. \(\square\)

2.6. **Lemma.** Suppose $a$ and $b$ are two simple loops which intersect transversely at a point in a surface $\Sigma_{g,r}$. Then, if the genus of the surface is one, there is a period-3 homeomorphism sending $a$ to $b$. If the genus of the surface is at least two, there is a product of involutions sending $a$ to $b$.

**Proof.** Note that a regular neighborhood $N$ of $a \cup b$ is a 1-holed torus. If the genus of the surface is at least two, we can find a non-separating simple loop $c$ in the surface $\Sigma_{g,r} - N$. Now $c$ is disjoint from both $a$ and $b$ and the complements of $a \cup c$ and $b \cup c$ in $\Sigma_{g,r}$ are connected. Thus there are involutions in the surface sending $a$ to $c$ and $c$ to $b$ respectively. The result follows.

If the genus of the surface is one, then the completion $\Sigma_{1,r}^{*}$ is the torus. Furthermore, both $a$ and $b$ are still simple loops in the torus intersecting at one point. Let $A$ and $B$ be the Dehn-twists on these two loops. Then the 3-fold symmetry $ABAB$ sends $b$ to $a$ in the torus. By lemma 2.3(a), there is a 3-fold symmetry $f$ in $\Sigma_{1,r}$ inducing the 3-fold symmetry. Thus we have $f(b) = a$. \(\square\)

§3. **The Commutator Subgroup of the Mapping Class Group**

We show in this section that the commutator subgroup $[\Gamma_{g,r}^{*}, \Gamma_{g,r}^{*}]$ of the mapping class group is generated by torsion elements in $\Gamma_{g,r}^{*}$.

By lemma 2.5(b), it suffices to prove the result for positive genus surfaces. By the short exact sequence $1 \to \Gamma_{g,r} \to \Gamma_{g,r}^{*} \to S_r$, proposition 2.2 and lemma 2.5(a), we conclude that the mapping class group is generated by torsion elements and Dehn-twists on non-separating simple loops $a_i$’s so that either $a_i$ is disjoint from $a_j$ or they intersect at one point.

Now if a group $G$ is generated by elements $g_i$’s, then its commutator subgroup $[G, G]$ is normally generated by the commutators $[g_i, g_j]$. On the other hand, if $t$ is a torsion element, then $[s, t] = (stst^{-1})t^{-1}$ is a product of two torsions. Thus it suffices to show that the commutator $[A, B]$ is a product of torsions in $\Gamma_{g,r}^{*}$ when $a$ intersects $b$ in one point and $A = D_a$, $B = D_b$. By Dehn’s relation $ABA = BAB$, we obtain that $[A, B] = ABA^{-1}B^{-1} = B^{-1}A$. Thus it suffices to show that $B^{-1}A$ is a product of torsion elements. But by lemma 2.6, there is a product $f$ of torsion elements sending $a$ to $b$. This implies $B = fA^{-1}f^{-1}$. Thus $[A, B] = f(A^{-1}f^{-1}A)$ is a product of torsion elements.

§4. **The First Homology of the Mapping Class Group**

We finish the proof of main theorem for surfaces of positive genus in this section. By the short exact sequence, $1 \to [\Gamma_{g,r}^{*}, \Gamma_{g,r}^{*}] \to \Gamma_{g,r}^{*} \to H_1(\Gamma_{g,r}^{*}) \to 1$ and the result in §3, it suffices to show that the projections of the torsion elements in the first homology generate the group. To prove this, we shall recall the result on the
first homology of the pure mapping class group. Then we shall calculate the first homology of the mapping class group and identify its generators.

4.1. The first homology of the pure mapping class group $\Gamma_{g,r}$ was shown to be torsion of order dividing 10 ($g > 1$) by Mumford [Mu]. Birman in [Bi1] showed that it is torsion of order dividing 2 for $g > 2$ and Powell [Po] showed the group is vanishing for $g > 2$. The most elegant proof of it is due to Harer in [Ha].

**Theorem.** The first homology group of the mapping class group of positive genus surface is as follows. $H_1(\Sigma_{g,r}) \simeq \mathbb{Z}_{10}$, and $\mathbb{Z}_{12}$ for $g \geq 3$, $g = 2$ and $g = 1$ respectively. Furthermore, a generator of the cyclic group $H_1(\Sigma_{g,r})$ is given by the image of the Dehn-twist on a non-separating simple loop.

A proof of it for low genus surface can be derived from the presentation of the mapping class group obtained in [Lu]. In particular, for positive genus $g$, suppose $\rho$ is the group homomorphism from $\Gamma_{g,r} \to \Gamma_{g,0}$ induced by the inclusion $\Sigma_{g,r} \to \Sigma_{g,0}$. Then the induced homomorphism $\rho_*$ between the first homology groups is an isomorphism.

4.2. In this section, we will use the theorem 4.1 to find the first homology group of the mapping class group $\Gamma^*_{g,r}$.

Let $\phi : \Gamma^*_{g,r} \to \Gamma_{g,0}$ be the group homomorphism induced by the completion construction $\Sigma_{g,0} = \Sigma^*_{g,r}$ and $\psi : \Gamma^*_{g,r} \to S_r$ be the homomorphism induced by the action of the mapping class group on the set of boundary components. Note that $H_1(S_r) = \mathbb{Z}_2$ for $r \geq 2$.

**Proposition.** The group homomorphism $\phi_* \oplus \psi_* : H_1(\Gamma^*_{g,r}) \to H_1(\Gamma_{g,r}) \oplus H_1(S_r)$ is an isomorphism for positive genus surfaces.

We will defer the proof to the last section.

4.3. We now use the proposition 4.2 to prove the remaining part of the main theorem. Namely, for $(g,r) \neq (2,5k + 4)$, the first homology group $H_1(\Gamma^*_{g,r})$ is generated by the images of the torsion elements.

First of all, by lemma 2.5 (a), there are torsion elements whose projections in $H_1(S_r)$ generate the group. Thus it remains to show that images of the torsion elements in $H_1(\Gamma_{g,0})$ generate the group. The result is clear for $g \geq 3$. When $g = 1$, let $A$ and $B$ be the Dehn-twists on two simple loops intersecting at one point. Then group $H_1(\Gamma_{1,0})$ is generated by $[A]$. Now $\tau_3 = ABAB$ and $\tau_4 = ABA$ shows that $A = \tau_3\tau_4^{-1}$. Thus the first homology group is generated by the images of the 3-fold symmetry $\tau_3$ and the 4-fold symmetry $\tau_4$. By lemma 2.3(a), both of these symmetries are induced by some symmetries in $\Sigma_{1,r}$. Thus the result follows for $g = 1$. When the genus is two, the first homology group $H_1(\Gamma_{2,0}) \simeq \mathbb{Z}_{10}$ is generated by the image $[D]$ of a Dehn-twist on a non-separating simple loop. Now by lemma 2.4(c), the 5-fold symmetry $\tau_5$ and the 2-fold symmetry $\delta$ become $4[D]$ and $5[D]$ in $H_1(\Gamma_{2,0})$. Thus $H_1(\Gamma_{2,0})$ is generated by the image of the torsion elements. By lemma 2.3(b) and (d), both of the symmetries $\tau_5$ and $\delta$ are induced by a 5-fold and 2-fold symmetries on $\Sigma_{2,r}$. Thus the result follows for $g = 2$. 

11
Finally, we need to show that the mapping class group $\Gamma^*_{2,5k+4}$ is not generated by torsions. Indeed, if it is generated by torsion elements, then due to $H_1(\Gamma^*_{2,5k+4}) = \mathbb{Z}_5 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2$, there must be torsion elements in $\Gamma^*_{2,5k+4}$ of order 5. This implies that $\mathbb{Z}_5$ acts on $\Sigma_{2,5k+4}$. This contradicts lemma 2.3(b). But our proof above shows that torsion elements generate the kernel of the homomorphism $\theta \phi : \Gamma^*_{2,5k+4} \to \mathbb{Z}_5$ where $\theta : H_1(\Gamma_{2,0}) \to \mathbb{Z}_5$ is the onto homomorphism. This shows that the group generated by torsion elements is a proper subgroup containing an index 5 subgroup. Thus it must be the index 5 subgroup.

4.4. We prove proposition 4.2 in this section.

First of all, we note that the homomorphism $\phi_* \oplus \psi_*$ is an epimorphism. Indeed, clearly both $\phi_*$ and $\psi_*$ are epimorphisms. On the other hand, if $r \geq 2$, a half-twist $D^{1/2}_a$ is in the kernel of $\phi_*$ and is sent to the generator of $H_1(S_r)$ by $\psi_*$. Thus $\phi_* \oplus \psi_*$ is onto.

To show the kernel is trivial, we will prove that the number of elements in $H_1(\Gamma^*_{g,r})$ is at most $|H_1(\Gamma_{g,0})||H_1(S_r)|$. The group $\Gamma^*_{g,r}$ is generated by Dehn-twists on non-separating simple loops and half-twists. Since any two Dehn-twists on non-separating simple loops are conjugate and any two half-twists are conjugate, the first homology group $H_1(\Gamma^*_{g,r})$ is generated by a Dehn-twist $[D_a]$ on a non-separating simple loop and a half-twist $[D^{1/2}]$. By the standard relations on Dehn-twists [De], [Lu], we see that $[D_a]$ has order at most $n$ in $H_1(\Gamma^*_{g,r})$ where $n = |H_1(\Gamma_{g,0})|$. If $r \leq 1$ then we see that $|H_1(\Gamma^*_{g,r})| = |H_1(\Gamma_{g,0})||H_1(S_r)|$. If $r \geq 2$, the composition $D^{1/2}D^{1/2}$ is a Dehn-twist $D_b$ where $b$ is a simple loop bounding a 3-holed sphere with two boundary components $B_1$ and $B_2$ of the surface (see figure 4.1). We claim that the Dehn-twist $D_b$ is a product of commutator in $\Gamma_{g,r}$. Assuming the claim, then $[D^{1/2}]$ has order 2 in the first homology. Thus there is at most $2n$ elements in the homology group $H_1(\Gamma^*_{g,r})$. The result follows.

![Seven curves in lantern position](image)

Figure 4.1

To see that $D_b$ is a product of commutators, we construct a 4-holed sphere bounded by two non-separating simple loops $a', c'$ and $B_1$ and $B_2$. Let $a$ and
c be two simple loops inside the 4-holed sphere so that \( \{a, b, c\} \) forms a lantern pattern. See figure 4.1.(b). Then the lantern relation shows that \( ABC = A'C' \) where capital letters are the Dehn-twists on small letter loops. In particular, \( D_b = B = A'A^{-1}C'C^{-1} \). But each of \( (a, a') \) and \( (c, c') \) is a pair of disjoint non-separating simple loops. Thus there are orientation preserving homeomorphisms \( f \) and \( h \) leaving each boundary component invariant sending \( a' \) to \( a \) and \( c' \) to \( c \) respectively. This shows \( A' = fAf^{-1} \) and \( C' = hCh^{-1} \). Thus \( D_b = [f, A][h, C] \) is a product of commutators in the pure mapping class group. □

4.5. Remarks.

1. It can be shown that a Dehn-twist on a separating simple loop is a product of commutators in the pure mapping class group \( \Gamma_{g,r} \) for positive genus surfaces.

2. Due to homological reason, the period-5 and period-3 elements in the generating set for the mapping class group of surfaces of genus 2 and 1 cannot be dropped.

3. If \( \mathbb{Z}_3 \) acts on a closed genus \( g \) surface, then it has at most \( g + 2 \) fixed points. Indeed, if \( t \) is the number of fixed points, then the quotient space has genus \( g' \) with \( t \) many branched points of order 3. Thus the Euler characteristic calculation shows: \( 2 - 2g - t = 3(2 - 2g' - t) \) or the same \( t = 2 + g - 2g' \). Thus \( t \leq 2 + g \). It is easy to show that there are \( \mathbb{Z}_3 \) actions on the genus \( g \) closed surface with \( g + 2 \) fixed points. Here is one way to see it. Consider the standard \( \mathbb{Z}_3 \) action on the tours with 3 fixed points. Now remove an equivariant neighborhood of two fixed points, we obtain a \( \mathbb{Z}_3 \) action on the 2-holed torus with one fixed point and leaving each boundary component invariant. But the same argument, there is a \( \mathbb{Z}_3 \) action on the 1-holed torus with 2 fixed points. Now each closed surface can be decomposed as a union of two 1-holed tori and \( (g - 2) \) 2-holed tori. See figure 4.2.

![Figure 4.2](image)

We can glue these \( \mathbb{Z}_3 \) actions on the 1-holed tori and 2-holed tori to produce a \( \mathbb{Z}_3 \) action on the closed surface with exactly \( g + 2 \) fixed points. By the work of Nielsen, there is only one \( \mathbb{Z}_3 \) action on the closed genus \( g \) surface with \( g + 2 \) fixed points. These seems to be the analogous action to the hyperelliptic involutions. We do not know the Dehn-twist expression of the generator of the \( \mathbb{Z}_3 \) action. Since
$g + 2 \geq 2$, by the same argument as in lemma 2.3, each compact surface $\Sigma_{g,r}$ admits a $\mathbb{Z}_3$ action with at least $g$ fixed points.

Reference

[Bi] Birman, J.S.: Braids, links, and mapping class groups. Ann. of Math. Stud., 82, Princeton Univ. Press, Princeton, NJ, 1975

[Bi1] Birman, J.S.: Abelian quotients of the mapping class group of a 2-manifold. Bull. Amer. Math. Soc. 76 (1970) 147-150, and Bull. Amer. Math. Soc. 77 (1971) 479.

[BH] Birman, J.S.; Hilden, H.: On isotopies of homeomorphisms of Riemann surfaces. Ann. of Math. (2) 97 (1973), 424-439.

[De] Dehn, M.: Papers on group theory and topology. J. Stillwell (eds.). Springer-Verlag, Berlin-New York, 1987.

[Ge] Gervais, S.: A finite presentation of the mapping class group of an oriented surface, Topology, to appear.

[Har] Harer, J.: The second homology group of the mapping class group of an orientable surface. Invent. Math. 72 (1983), 221-239.

[Jo] Johnson, D.: The structure of the Torelli group. I. A finite set of generators for $T$. Ann. of Math. (2) 118 (1983), no. 3, 42-442.

[Li] Lickorish, R.: A representation of oriented combinatorial 3-manifolds. Ann. Math. 72 (1962), 531-540

[Lu] Luo, F.: A presentation of the mapping class groups. Math. Res. Lett. 4 (1997), no. 5, 735-739.

[Mu] Mumford, D.: Abelian quotients of the Teichmüller modular group. J. Analyse Math. 18 1967 227-244.

[Po] Powell, J.: Two theorems on the mapping class group of a surface. Proc. Amer. Math. Soc. 68 (1978), no. 3, 347-350.

[Vi] Viro, O.: Links, two-fold branched coverings and braids, Soviet Math. Sbornik, 87 no 2 (1972), 216-22.

[Wa] Wajnryb, B.: Mapping class group of a surface is generated by two elements. Topology 35 (1996), no. 2, 377-383.