Super Domination in Trees

Wei Zhuang

Received: 7 November 2019 / Revised: 21 July 2021 / Accepted: 9 August 2021 / Published online: 22 December 2021
© The Author(s), under exclusive licence to Springer Japan KK, part of Springer Nature 2021

Abstract
For \( S \subseteq V(G) \), we define \( \overline{S} = V(G) \setminus S \). A set \( S \subseteq V(G) \) is called a super dominating set if for every vertex \( u \in \overline{S} \), there exists \( v \in S \) such that \( N(v) \cap \overline{S} = \{u\} \). The super domination number \( \gamma_{sp}(G) \) of \( G \) is the minimum cardinality among all super dominating sets in \( G \). The super domination subdivision number \( sd_{\gamma_{sp}}(G) \) of a graph \( G \) is the minimum number of edges that must be subdivided in order to increase the super domination number of \( G \). In this paper, we investigate the ratios between super domination and other domination parameters in trees. In addition, we show that for any nontrivial tree \( T \), \( 1 \leq sd_{\gamma_{sp}}(T) \leq 2 \), and give constructive characterizations of trees whose super domination subdivision number are 1 and 2, respectively.

Keywords Super domination number · Super domination subdivision number

1 Introduction
Let \( G = (V, E) \) be a simple graph without isolated vertices, and let \( v \) be a vertex in \( G \). The open neighborhood of \( v \) is \( N(v) = \{u \in V \mid uv \in E\} \) and the closed neighborhood of \( v \) is \( N[v] = N(v) \cup \{v\} \). For \( S \subseteq V(G) \), we define \( \overline{S} = V(G) \setminus S \). The degree of a vertex \( v \) is \( d(v) = |N(v)| \). For two vertices \( u \) and \( v \) in a connected graph \( G \), the distance \( d(u, v) \) between \( u \) and \( v \) is the length of a shortest \( (u, v) \)-path in \( G \). The maximum distance among all pairs of vertices of \( G \) is the diameter of the graph \( G \) which is denoted by \( \text{diam}(G) \). A leaf of \( G \) is a vertex of degree 1, and a support vertex of \( G \) is a vertex adjacent to a leaf. A support vertex that is adjacent to at least two leaves is called a strong support vertex. The corona \( G \circ K_1 \) is the graph obtained from a graph \( G \) by attaching a leaf to each vertex \( v \in V(G) \).
A dominating set (respectively, total dominating set) in a graph $G$ is a set $S$ of vertices of $G$ such that every vertex in $V(G) \setminus S$ (respectively, $V(G)$) is adjacent to at least one vertex in $S$. The domination number (respectively, total domination number) of $G$, denoted by $\gamma(G)$ (respectively, $\gamma_t(G)$), is the minimum cardinality of a dominating set (respectively, total dominating set) of $G$. A dominating set (respectively, total dominating set) of $G$ with cardinality $\gamma(G)$ (respectively, $\gamma_t(G)$) is called a $\gamma(G)$-set (respectively, $\gamma_t(G)$-set). We say a vertex $v$ in $G$ is totally dominated, by a set $D$, if $N(v) \cap D \neq \emptyset$.

The study of super domination in graphs was introduced in [13]. A set $S \subseteq V(G)$ is called a super dominating set if for every vertex $u \in \bar{S}$, there exists $v \in S$ such that $N(v) \cap S = \{u\}$. In particular, we say that $v$ is an external private neighbor of $u$ with respect to $\bar{S}$. For a super dominating set $S$ of $G$, let $P_S(G) = \{v \mid v$ is an external private neighbor of $u$ with respect to $\bar{S}$, for some $u \in \bar{S}\}$, $Q_S(G) = \{v \mid v$ belongs to $\bar{S}$ and $v$ has only one external private neighbor with respect to $\bar{S}\}$ and $U_S(G) = \{v \mid v$ is the unique external private neighbor of $u$ with respect to $\bar{S}$, for some $u \in Q_S(G)\}$. The super domination number $\gamma_{sp}(G)$ of $G$ is the minimum cardinality among all super dominating sets in $G$. A super dominating set of $G$ with cardinality $\gamma_{sp}(G)$ is called a $\gamma_{sp}(G)$-set. More results in this area were investigated in [5, 11, 12] and elsewhere.

The domination subdivision number $sd_c(G)$ of a graph $G$ is the minimum number of edges that must be subdivided (where each edge in $G$ can be subdivided at most once) in order to increase the domination number. The domination subdivision number was first introduced in Velammal’s thesis [16] and since then many results have also been obtained on the parameters $sd_c(G)$, $sd_t(G)$, $sd_t(G)$, $sd_t(g)$, $sd_{cp}(G)$ (see [1–3, 6–10, 15]). One of the purpose of this paper is to initialize the study of the super domination subdivision number. The super domination subdivision number $sd_{sp}(G)$ of a graph $G$ is the minimum number of edges that must be subdivided in order to increase the super domination number of $G$ (each edge in $G$ can be subdivided at most once).

In this paper, we investigate the ratios between super domination and other domination parameters in trees. In addition, we show that for any nontrivial tree $T$, $1 \leq sd_{sp}(T) \leq 2$, and give constructive characterizations of trees whose super domination subdivision number are 1 and 2, respectively.

### 2 On the Ratios Between Super Domination and Other Domination Parameters in Trees

#### 2.1 Preliminary Results

From the definitions of domination number, total domination number and super domination number, we have the following observations.

**Observation 2.1** Let $G$ be a connected graph that is not a star. Then,

1. there is a $\gamma$-set of $G$ that contains no leaf, and
(2) there is a $\gamma_r$-set of $G$ that contains no leaf.

**Observation 2.2** Let $G$ be a connected graph of order at least 2, $v$ be a support vertex of $G$ and $S$ be a $\gamma_{sp}$-set of $G$. Then, at most one of $v$ and its leaf-neighbors belongs to $\overline{S}$.

**Proposition 2.3** Let $G$ be a graph containing the strong support vertices $u_1, u_2, \cdots, u_t$, and each $u_i$ has $c_i$ leaf-neighbors $(1 \leq i \leq t)$. Let $G'$ be a graph obtained from $G$ by deleting $x_i$ $(0 \leq x_i \leq c_i - 1)$ leaf-neighbors of each $u_i$ $(1 \leq i \leq t)$. Then, $\gamma_{sp}(G) = a$ if and only if $\gamma_{sp}(G') = a - \sum_{i=1}^{t} x_i$.

**Proof** Take a strong support vertex of $G$, say $u_1$, and let $u_1'$ be one of its leaf-neighbors. Deleting $u_1'$ from $G$ and denote the resulting graph by $G_1$. Let $S$ be a $\gamma_{sp}$-set of $G$. By Observation 2.2, at most one of $u_1$ and its leaf-neighbors belongs to $\overline{S}$. Set $D = S \setminus \{u_1'\}$ when $u_1' \in S$, and $D = S \setminus \{u_1''\}$ when $u_1'' \in \overline{S}$, where $u_1''$ is a leaf-neighbor of $u_1$ other than $u_1'$. Note that $D$ is a super dominating set of $G_1$. It follows that $\gamma_{sp}(G_1) \leq \gamma_{sp}(G) - 1$.

On the other hand, let $S'$ be a $\gamma_{sp}$-set of $G_1$. It is easy to see that $S' \cup \{u_1'\}$ is a super dominating set of $G$. That is, $\gamma_{sp}(G) \leq \gamma_{sp}(G_1) + 1$. Hence, $\gamma_{sp}(G) = \gamma_{sp}(G_1) + 1$. By a series of similar arguments, we can obtain that $\gamma_{sp}(G) = \gamma_{sp}(G') + \sum_{i=1}^{t} x_i$. $\blacksquare$

**Proposition 2.4** Let $T$ be a tree of order at least 2 and $v$ be a leaf of $T$, there is always a $\gamma_{sp}$-set $S$ of $T$ such that $v \in \overline{S}$.

**Proof** We proceed by induction on the order $n$ of the tree $T$. If $n(T) \leq 4$, the result holds. Assume that the result is true for any tree $T'$ of order $4 \leq n(T') < n(T)$. Let $u$ be a leaf of $T$ at maximum distance from $v$, and $w$ be its support vertex. If $w$ is a strong support vertex, then deleting the leaf $u$ from $T$ and denote the resulting tree by $T'$. By induction, there exists a $\gamma_{sp}$-set $S'$ of $T'$ such that $v \in \overline{S'}$. Let $S = S' \cup \{u\}$. We have that $S$ is a super dominating set of $T$ and $|S| = |S'| + 1$. On the other hand, by Proposition 2.3, $\gamma_{sp}(T) = \gamma_{sp}(T') + 1$. It means that $S$ is a $\gamma_{sp}$-set of $T$, and clearly, $v \in \overline{S}$.

So we assume that $w$ is not a strong support vertex. Combining this with the choice of $u$, we have that $d(w) = 2$. Let $x$ be the neighbor of $w$ other than $u$, and $T'' = T - \{u, w\}$. By induction, there exists a $\gamma_{sp}$-set $S'_1$ of $T''$ such that $v \in \overline{S'_1}$. Set $S_1 = S'_1 \cup \{w\}$ when $x \in S'_1$, and $S_1 = S'_1 \cup \{u\}$ when $x \in \overline{S'_1}$. In either case, $S_1$ is a super dominating set of $T$. This is, $\gamma_{sp}(T) \leq \gamma_{sp}(T'') + 1$.

On the other hand, let $S_2$ be a $\gamma_{sp}$-set of $T$. We have that $|\{u, w, x\} \cap S_2| \leq 2$ (If $\{u, w, x\} \subseteq S_2$, then the set $S_2 \setminus \{u\}$ is also a super dominating set of $T$, contradicting the assumption that $S_2$ is a $\gamma_{sp}$-set of $T$). Next, we consider three cases.

**Case 1.** If $u \in \overline{S_2}$, then it follows from the definition of super domination set that
w, x ∈ S2. And then, S2 \ {w} is a super dominating set of T''.

**Case 2.** If u ∈ S2 and w ∈ S2, then S2 \ {u} is a super dominating set of T''.

**Case 3.** If u, w ∈ S2, combining this with the fact that |{u, w, x} ∩ S2| ≤ 2, we have that x ∈ S2. And moreover, (S2 \ {u, w}) ∪ {x} is a super dominating set of T''.

In either case, we can always obtain a super dominating set of T'' with cardinality γsp(T) − 1. So, γsp(T'') ≤ γsp(T) − 1. And then, γsp(T'') = γsp(T) − 1. Since S1 is a super dominating set of T with cardinality γsp(T') + 1, S1 is a γsp-set of T. And clearly, v ∈ S1.

In the next subsections, we are ready to investigate the ratios between the super domination number and two most important domination parameters (namely the domination number and the total domination number) in trees.

### 2.2 Domination Versus Super Domination in Trees

First, we construct a family \(\mathcal{R}\) which is mentioned in [13].

Given a tree \(T_1 = P_2 = a_1b_1\). Define the tree \(T_i\) (\(i = 2, 3, 4, 5, \ldots\)) such that \(V(T_j) = V(T_{j-1}) \cup \{a_j, b_j\}\) and \(E(T_j) = E(T_{j-1}) \cup \{a_jb_j\} \cup \{e\}\), where \(e \in \{a_ia_j, b_jb_j\}\) for some \(1 ≤ i ≤ j - 1\).

We define the family \(\mathcal{R} = \{T_i \mid i = 1, 2, 3, 4, \ldots\}\).

The following two theorems are useful for our subsequent analysis.

**Theorem 2.5** [14] For a graph \(G\) with even order \(n\) and no isolated vertices, \(\gamma(G) = \frac{n}{2}\) if and only if the components of \(G\) are the cycle \(C_4\) or the corona \(H \circ K_1\) for any connected graph \(H\).

**Theorem 2.6** [13] Let \(T\) be a tree of order at least two. Then, \(\gamma_{sp}(T) = \frac{n}{2}\) if and only if \(T \in \mathcal{R}\).

**Theorem 2.7** For any nontrivial tree \(T\), we have that \(0 < \frac{\gamma(T)}{\gamma_{sp}(T)} ≤ 1\). Further, the nontrivial trees \(T\) satisfying \(\frac{\gamma(T)}{\gamma_{sp}(T)} = 1\) are precisely the corona \(H \circ K_1\) of any tree \(H\).

It is well known that for any graph \(G\) without isolated vertices, we have that \(\gamma(G) ≤ \frac{|G|}{2}\) and \(\gamma_{sp}(G) ≥ \frac{|G|}{2}\). It means that for any tree \(T\), \(\gamma(T) ≤ \gamma_{sp}(T)\). If \(\frac{\gamma(T)}{\gamma_{sp}(T)} = 1\), then \(\gamma(T) = \gamma_{sp}(T) = \frac{n}{2}\). It follows from Theorems 2.5 and 2.6 that \(T\) is the corona \(H \circ K_1\) of any tree \(H\). On the other hand, it is easy to see that for any tree \(H\), both the domination number and the super domination number of the corona \(H \circ K_1\) are \(\frac{n}{2}\).

In addition, given any tree \(T'\), and construct a sequence of trees \(T'_0, T'_1, T'_2, \ldots\), such that \(T'_0 = T'\), the tree \(T'_{i+1}\) is obtained from \(T'_i\) by adding a vertex and joining it to one of the support vertices of \(T'_i\), \(i = 0, 1, 2, \ldots\). By Observation 2.1 and Proposition 2.3, the domination number of the resulting trees have never changed in this process, but the super domination number is constantly increasing. It means that when the number \(n\) is sufficiently large, the ratio \(\frac{\gamma(T'_i)}{\gamma_{sp}(T'_i)}\) is close to 0. It means that the lower bound of Theorem 2.7 is also optimal.
2.3 Super Domination Versus Total Domination in Trees

By a weak partition of a set we mean a partition of the set in which some of the subsets may be empty. We define a labeling of a tree $T$ as a weak partition $S = (S_A, S_B, S_C)$ of $V(T)$ (This idea of labeling the vertices is introduced in [4]). We will refer to the pair $(T, S)$ as a labeled tree. The label or status of a vertex $v$, denoted $\text{sta}(v)$, is the letter $x \in \{A, B, C\}$ such that $v \in S_x$. Next, let $\mathcal{F}$ be the family of labeled trees that: (i) contains $(P_6, S_0)$ where $S_0$ is the labeling that assigns to the two leaves of the path $P_6$ status $C$, to the support vertices status $A$ and to the remaining vertices status $B$ (see Fig. 1a); and (ii) is closed under the operation $\mathcal{O}$, which extend the tree $T'$ to a tree $T$ by attaching a $P_6$ to the vertex $v \in V(T')$.

**Operation $\mathcal{O}$**: Let $v$ be a vertex with $\text{sta}(v) = B$. Add a path $u_1u_2u_3u_4u_5u_6$ and the edge $u_3v$. Let $\text{sta}(u_1) = \text{sta}(u_6) = C$, $\text{sta}(u_2) = \text{sta}(u_5) = A$ and $\text{sta}(u_3) = \text{sta}(u_4) = B$.

The operation $\mathcal{O}$ is illustrated in Fig. 1b.

**Theorem 2.8** Let $T$ be a tree of order at least 3, we have that $0 < \frac{\gamma_{t}(T)}{\gamma_{sp}(T)} \leq \frac{4}{3}$. Further, the trees $T$ of order at least 3 satisfying $\frac{\gamma_{t}(T)}{\gamma_{sp}(T)} = \frac{4}{3}$ are precisely those trees $T$ such that $(T, S) \in \mathcal{F}$ for some labeling $S$.

In Sect. 2.2, we construct a sequence of trees $T_0, T_1, T_2, \ldots$, such that the ratio $\frac{\gamma_{t}(T_n)}{\gamma_{sp}(T_n)}$ is close to 0 when the number $n$ is sufficiently large. By a similar argument, the ratio $\frac{\gamma_{t}(T_n)}{\gamma_{sp}(T_n)}$ is also close to 0 when the number $n$ is sufficiently large. It means that the lower bound of Theorem 2.8 is also optimal.

Let $T$ be a tree containing strong support vertices, and $T'$ be the tree obtained from $T$ by deleting all except one leaf-neighbor of every strong support vertex of $T$.

Fig. 1 The labeled tree $(P_6, S_0)$ and the operation $\sigma$
Then, it follows from Observation 2.1 and Proposition 2.3 that \( \gamma_i(T') = \gamma_i(T) \) and \( \gamma_{sp}(T') \leq \gamma_{sp}(T) \). And then \( \frac{\gamma_i(T)}{\gamma_{sp}(T)} \leq \frac{\gamma_i(T)}{\gamma_{sp}(T)} \). So when we prove Theorem 2.8, we only need to consider the trees containing no strong support vertex.

Before this, we present a preliminary result.

**Observation 2.9** Let \( T \) be a tree and \( S \) be a labeling of \( T \) such that \( (T, S) \in \mathcal{F} \). Then, \( T \) has the following properties:

(a) A vertex is labeled \( C \) if and only if it is a leaf.
(b) A vertex is labeled \( A \) if and only if it is a support vertex.
(c) Every support vertex has degree two, and its two neighbors have status \( C \) and \( B \), respectively.
(d) If a vertex has status \( B \), then all of its neighbors have status \( B \) except one which has status \( A \).
(e) \( |S_A| = |S_B| = |S_C| \).
(f) \( S_A \cup S_B \) is a \( \gamma_t \)-set of \( T \).

**Lemma 2.10** Let \( T \) be a tree and \( S \) be a labeling of \( T \) such that \( (T, S) \in \mathcal{F} \). Then for any leaf \( v \) of \( T \), there exists a set \( X \) of cardinality \( \gamma_i(T) - 1 \) such that \( v \) belongs to \( X \), and each vertex except \( v \) of \( T \) is totally dominated by \( X \).

**Proof** Let \( v \) be a leaf of \( T \), \( u \) be its support vertex and \( w \) be the neighbor of \( u \) other than \( v \). It follows from Observation 2.9(b), (c) and (f) that \( u, w \) have status \( A \) and \( B \), respectively, and \( D = S_A \cup S_B \) is a \( \gamma_t \)-set of \( T \). It is easy to see that \( (D \setminus \{u, w\}) \cup \{v\} \) is the set as we desired.

**Lemma 2.11** Let \( T \) be a tree and \( S \) be a labeling of \( T \) such that \( (T, S) \in \mathcal{F} \). Then, \( \frac{\gamma_i(T)}{\gamma_{sp}(T)} = \frac{4}{3} \).

**Proof** The proof is by induction on the number \( h(T) \) of operations required to construct the tree \( T \). Observe that \( T = P_6 \) when \( h(T) = 0 \), and clearly \( \frac{\gamma_i(T)}{\gamma_{sp}(T)} = \frac{4}{3} \). This establishes the base case. Assume that \( k \geq 1 \) and each labeled tree \( (T', S') \in \mathcal{F} \) with \( h(T') < k \) satisfies the condition that \( \frac{\gamma_i(T')}{\gamma_{sp}(T')} = \frac{4}{3} \). Let \( (T, S) \in \mathcal{F} \) be a labeled tree with \( h(T) = k \). Then \( (T, S) \) can be obtained from a labeled tree \( (T', S') \in \mathcal{F} \) with \( h(T') < k \) by the operation \( \emptyset \). That is, add a path \( u_1u_2u_3u_4u_5u_6 \) and the edge \( u_3v \), where \( v \) is a vertex of \( T' \) which has status \( B \). By induction, \( \frac{\gamma_i(T')}{\gamma_{sp}(T')} = \frac{4}{3} \).

By Observation 2.1(2), we can choose a \( \gamma_t \)-set of \( T \), say \( D \), which contains no leaf. Clearly, \( \{u_2, u_3, u_4, u_5\} \subseteq D \). Since \( v \) has status \( B \), it follows from Observation 2.9(b), (c) and (d) that \( v \) has one neighbor which is a support vertex of degree two, say \( w \). Moreover, \( \{v, w\} \subseteq D \). It means that \( D \setminus \{u_2, u_3, u_4, u_5\} \) is a total dominating set of \( T' \). That is, \( \gamma_i(T') - 4 \geq \gamma_i(T') \). On the other hand, let \( D' \) be a \( \gamma_t \)-set of \( T' \). It is easy to see that \( D' \cup \{u_2, u_3, u_4, u_5\} \) is a total dominating set of \( T' \). That is, \( \gamma_i(T') + 4 \geq \gamma_i(T) \). Therefore, \( \gamma_i(T') + 4 = \gamma_i(T) \).

Let \( R' \) be a \( \gamma_{sp} \)-set of \( T' \). By Proposition 2.4, there exists a \( \gamma_{sp} \)-set \( R \) of \( T \) such that the leaf-neighbor of \( w \) belongs to \( R \). And then \( w \) is its external private neighbor with
respect to $\overline{R}$. Moreover, $v \in R$. It implies that $R \cap V(T')$ is a super dominating set of $T'$. Since $|R \cap \{u_1, u_2, u_3, u_4, u_5\}| \geq 3$, we have that $|R| - 3 \geq |R'|$. On the other hand, let $K = R' \cup \{u_2, u_3, u_6\}$ when $v \in R'$, and $K = R' \cup \{u_1, u_4, u_5\}$ when $v \in \overline{R}$, then $K$ is a super dominating set of $T$. That is, $|R'| + 3 \geq |R|$. Hence, $\gamma_{sp}(T) = \gamma_{sp}(T') + 3$.

In summary, $\frac{\gamma(T)}{\gamma_{sp}(T)} = \frac{\gamma(T) + 4}{\gamma_{sp}(T) + 3} = \frac{4}{3}$.

Below we will prove Theorem 2.8.

**Proof** We proceed by induction on the order $n$ of the tree $T$ (As mentioned above, we only need to consider the case that $T$ has no strong support vertex). The result is immediate for $n \leq 5$. Let $n \geq 6$ and assume that for every tree $T'$ satisfying $3 \leq n(T') < n$, we have $\frac{\gamma(T)}{\gamma_{sp}(T)} \leq \frac{4}{3}$, with equality if and only if $(T', S') \in \mathcal{F}$ for some labeling $S'$.

The result holds when $\text{diam}(T) \leq 5$. Moreover, if $\frac{\gamma(T)}{\gamma_{sp}(T)} = \frac{4}{3}$, then $(T, S) = (P_6, S_0) \in \mathcal{F}$. Hence, we may assume that $\text{diam}(T) \geq 6$. Let $P = v_1v_2 \cdots v_t$ be a longest path in $T$ such that $d(v_3)$ as large as possible. We know that $d(v_2) = 2$. Let $R$ be a $\gamma_{sp}$-set of $T$ such that $v_1 \in \overline{R}$. We now proceed with one claim that we may assume are satisfied by the tree $T$, for otherwise the desired result holds.

**Claim 1.** $d(v_3) = 2$.

If not, assume that $d(v_3) > 2$. Let $T_1 = T - \{v_1, v_2\}$ and $D_1$ be a $\gamma_i$-set of $T_1$ which contains no leaf. Clearly, $v_3 \in D_1$, and $D_1 \cup \{v_2\}$ is a total dominating set of $T$. So, $\gamma_i(T) \leq \gamma_i(T_1) + 1$. On the other hand, let $R_1$ be a $\gamma_{sp}$-set of $T_1$. Note that $v_2$ is the external private neighbor of $v_1$ with respect to $\overline{R}$. And then $R \setminus \{v_2\}$ is a super dominating set of $T_1$. So, $|R| - 1 \geq |R_1|$. By induction, we have that $3\gamma_i(T) \leq 3(\gamma_i(T_1) + 1) = 3\gamma_i(T_1) + 3 \leq 4\gamma_{sp}(T_1) + 3 \leq 4\gamma_{sp}(T) - 1 < 4\gamma_{sp}(T)$. That is, $\frac{\gamma(T)}{\gamma_{sp}(T)} < \frac{4}{3}$.

Next, we consider the vertex $v_4$. We can distinguish two cases as follows.

**Case 1.** $d(v_4) > 2$.

**Subcase 1.1.** $v_4$ is a support vertex.

Let $u$ be the leaf-neighbor of $v_4$, and $R'$ be a $\gamma_{sp}$-set of $T$ such that $u \in \overline{R'}$. Then, $v_4$ is the external private neighbor of $u$ with respect to $\overline{R}$. It implies that $|\{v_1, v_2, v_3\} \cap \overline{R'}| = 1$ and $R' \setminus \{v_1, v_2, v_3\}$ is a super dominating set of $T_1 = T - \{v_1, v_2, v_3\}$. So, $|R'| - 2 \geq |R'|$, where $R'$ is a $\gamma_{sp}$-set of $T_1$. On the other hand, it is easy to see that $\gamma_i(T_1) + 2 \geq \gamma_i(T)$. By induction, we have that $3\gamma_i(T) \leq 3(\gamma_i(T_1) + 2) = 3\gamma_i(T_1) + 6 \leq 4\gamma_{sp}(T_1) + 6 \leq 4\gamma_{sp}(T) - 2 < 4\gamma_{sp}(T)$. That is, $\frac{\gamma(T)}{\gamma_{sp}(T)} < \frac{4}{3}$.

**Subcase 1.2.** $v_4$ has a neighbor outside $P$, say $u_1$, which is adjacent to a support vertex $u_2$. □
From the choice of $P$, $d(u_1) = 2$. Denote the leaf-neighbor of $u_2$ by $u_3$. Note that $\{v_2, v_3\} \cap \overline{R} = \emptyset$. If $v_4 \in \overline{R}$, then $\{u_1, u_2, u_3\} \cap \overline{R} = 1$ and $R \setminus \{u_1, u_2, u_3\}$ is a super dominating set of $T_1 = T - \{u_1, u_2, u_3\}$. If $v_4 \notin \overline{R}$, note that $\{v_1, v_2, v_3\} \cap \overline{R} = 1$ and $R \setminus \{v_1, v_2, v_3\}$ is a super dominating set of $T_1 = T - \{v_1, v_2, v_3\}$. In either case, the proof is similar to that of Subcase 1.1.

**Subcase 1.3.** Every neighbor of $v_4$ outside $P$ is support vertex.

Let $u$ be a neighbor of $v_4$ outside $P$. Suppose that $d(v_4) \geq 4$, and let $T'$ be the component of $T - v_4u$ containing $v_4$. Then, the proof is similar to that of Claim 1.

So we have that $d(v_4) = 3$. Let $T''$ be the component of $T - v_4v_5$ containing $v_5$. It is easy to see that $\gamma_1(T'') + 4 \geq \gamma_1(T)$. On the other hand, let $R_1$ be a $\gamma_{sp}$-set of $T''$. Note that $\{v_2, v_3\} \cap \overline{R} = \emptyset$ and $\{|u, w| \cap \overline{R} \leq 1$, where $w$ is the leaf-neighbor of $u$. If neither $v_4$ nor $v_5$ belongs to $\overline{R}$, then $(R \setminus \{v_2, v_3, w\}) \cup \{v_1, u\}$ is a super dominating set of $T$ whose cardinality is less than $R$, a contradiction. So, $\{v_4, v_5\} \cap \overline{R} \neq \emptyset$.

If $v_4 \in \overline{R}$, then $u \in \overline{R}$, and moreover, $R \setminus \{v_2, v_3, w\}$ is a super dominating set of $T''$. That is, $|R| - 3 \geq |R_1|$. If $v_4 \notin \overline{R}$ and $v_5 \in \overline{R}$, then $\{|v_1, v_5, u, w| \cap \overline{R} = 3$, and $R \setminus \{v_1, v_5, u, w\}$ is the complement of a super dominating set of $T''$. That is, $|R| - 3 \leq |R_1|$. Note that $n(T) = n(T'') + 6$, we have that $|R| - 3 \geq |R_1|$.

In either case, by induction, we have that $3\gamma_1(T) \leq 3(\gamma_1(T'') + 4) = 3\gamma_1(T'') + 12 \leq 4\gamma_{sp}(T'') + 12 \leq 4(\gamma_{sp}(T) - 3) + 12 = 4\gamma_{sp}(T)$. That is, $\frac{\gamma_1(T)}{\gamma_{sp}(T)} \leq \frac{4}{3}$. Suppose next that $3\gamma_1(T) = 4\gamma_{sp}(T)$. Then we have equality throughout the above inequality chain. In particular, $\gamma_1(T) = \gamma_1(T'') + 4$ and $3\gamma_1(T') = 4\gamma_{sp}(T'')$. By induction, $(T'', S') \in \mathcal{F}$ for some labeling $S'$. If $v_5$ is a support vertex, by Lemma 2.10, there exists a set $X$ of cardinality $\gamma_1(T'') - 1$ such that the leaf-neighbor of $v_5$, say $x$, belongs to $X$, and each vertex of $T''$ is totally dominated by $X$ except for $x$. In this case, let $Y = (X \setminus \{x\}) \cup \{v_5, v_2, v_3, u\}$. It is easy to see that $Y$ is a total dominating set of $T$ with cardinality $\gamma_1(T) - 1$, which is impossible. If $v_5$ is a leaf, we can obtain a contradiction through the similar argument. Hence, $v_5$ is neither a leaf nor a support vertex. It follows from Observation 2.9(a) and (b) that $v_5$ has status $B$ in $(T'', S')$. Let $S$ be obtained from the labeling $S'$ by labeling the vertices $v_1$ and $w$ with label $C$, the vertices $v_2$ and $u$ with label $A$, the vertex $v_3$ and $v_4$ with label $B$. Then, $(T, S)$ can be obtained from $(T'', S')$ by operation $\mathcal{O}$. Thus, $(T, S) \in \mathcal{F}$.

**Case 2.** $d(v_4) = 2$.

Let $T'$ be the component of $T - v_4v_5$ containing $v_5$, and $R_1$ be a $\gamma_{sp}$-set of $T'$. It is easy to see that $\gamma_1(T') + 2 \geq \gamma_1(T)$. On the other hand, similar to Subcase 1.3, we have that $|R| - 2 \geq |R_1|$. It follows that $3\gamma_1(T) \leq 3(\gamma_1(T') + 2) = 3\gamma_1(T') + 6 \leq 4\gamma_{sp}(T') + 6 \leq 4\gamma_{sp}(T) - 8 + 6 = 4\gamma_{sp}(T) - 2 < 4\gamma_{sp}(T)$. That is, $\frac{\gamma_1(T)}{\gamma_{sp}(T)} < \frac{4}{3}$.
3 Bound on the Super Domination Subdivision Number of Trees

In this section, we first present an upper bound of \(sd_{sp}(T)\).

**Theorem 3.1** For any tree \(T\) of order at least 2, \(sd_{sp}(T) \leq 2\).

**Proof** It is easy to see that the result holds for a tree of \(\text{diam}(T) \leq 3\), so we assume that \(\text{diam}(T) \geq 4\). Let \(P = u_1u_2u_3 \cdots u_n\) be a longest path of \(T\). Let \(T'\) be obtained from \(T\) by subdividing the edges \(u_2u_3\) and \(u_3u_4\) with vertices \(x\) and \(y\). By Proposition 2.4, there exists a \(\gamma_{sp}\)-set of \(T'\), say \(S'\), such that \(\overline{S'}\) contains the vertex \(u_1\). It follows from the definition of super domination set that \(u_2, x \in S'\). Next, we consider four cases:

**Case 1.** If \(u_3, y \in \overline{S}\), then \(u_4 \in S'\). And the set \(S = (S' \setminus \{u_2, x\}) \cup \{u_1\}\) is a super dominating set of \(T\).

**Case 2.** If \(|\{u_3, y\} \cap S'| = 1\), then let \(S = S' \setminus \{x\}\) when \(u_3 \in S'\), and \(S = (S' \setminus \{x, y\}) \cup \{u_3\}\) when \(y \in S'\). Clearly, \(S\) is a super dominating set of \(T\).

**Case 3.** If \(u_3, y \in S'\) and \(u_4 \in \overline{S}\), then the set \(S = (S' \setminus \{x, y\}) \cup \{u_4\}\) is a super dominating set of \(T\).

**Case 4.** If \(u_3, y, u_4 \in S'\), then the set \(S = S' \setminus \{x, y\}\) is a super dominating set of \(T\).

In either case, we have that \(\gamma_{sp}(T) \leq \gamma_{sp}(T') - 1\). That is, \(sd_{sp}(T) \leq 2\). \(\square\)

Trees are classified as Class 1 or Class 2 depending on whether their super domination subdivision number is 1 or 2, respectively. Next, we are ready to provide the constructive characterizations of trees in Class 1 and Class 2. We introduce the operation as follows.

**Operation \(\mathcal{U}_1\):** Add a star of order at least two and join its central vertex to a vertex \(v\) of \(T'\) when there exists a \(\gamma_{sp}\)-set \(S\) of \(T'\) such that \(N_T[v] \cap \overline{S} = \emptyset\), or \(v \notin \overline{S}\) and \(N_T[v] \cap U_S(T') = \emptyset\).

We define the family \(\mathcal{U}\) as:

\[\mathcal{U} = \{T \mid T\text{ is obtained from a star of order at least three by a finite sequence of operation } \mathcal{U}_1\}\]

We show first that every tree in the family \(\mathcal{U}\) is in Class 2.

**Lemma 3.2** If \(T \in \mathcal{U}\), then \(T\) is in Class 2.

**Proof** The proof is by induction on the number \(h(T)\) of operations required to construct the tree \(T\). Observe that \(T\) is a star of order at least three when \(h(T) = 0\), and the result holds. This establishes the base case. Assume that \(k \geq 1\) and that each tree \(T' \in \mathcal{U}\) with \(h(T') < k\) is in Class 2. Let \(T \in \mathcal{U}\) be a tree with \(h(T) = k\). Then \(T\) can be obtained from a tree \(T' \in \mathcal{U}\) with \(h(T') < k\) by the operation \(\mathcal{U}_1\). In other words, \(T\) is obtained from \(T'\) by adding a star of order at least two and join its central vertex, say \(u\), to a vertex \(v\) of \(T'\), where \(N_{T'}[v] \cap \overline{S} = \emptyset\), or \(v \notin \overline{S}\) and \(N_{T'}[v] \cap U_{S'}(T') = \emptyset\), where \(S'\) is some \(\gamma_{sp}\)-set of \(T'\). By induction, \(T'\) is in Class 2.

Let \(u_1\) be a leaf-neighbor of \(u\) in \(T\), and \(S\) be a \(\gamma_{sp}\)-set of \(T\) such that \(u_1 \in S\). Then, \(u\) is the external private neighbor of \(u_1\) with respect to \(\overline{S}\), and it means that \(\overline{S} \setminus \{u_1\}\) is the complement of a super dominating set of \(T'\), and so \(|\overline{S}| - 1 \leq |\overline{S'}|\). Moreover, note that \(v \notin \overline{S'}\) and \(\overline{S'} \cup \{u_1\}\) is the complement of a super dominating set of \(T\), so
$|S'| + 1 \leq |\bar{S}|$. Hence, $|S'| + 1 = |\bar{S}|$.

Let $e \in E(T)$ and $T^*$ be obtained from $T$ by subdividing the edge $e$ with vertex $x$. Let $S^*$ be a $\gamma_{sp}$-set of $T^*$. Now we can distinguish three cases as follows:

**Case 1.** $e \in E(T^*)$.

Let $T'^*$ be obtained from $T'$ by subdividing the edge $e$ with a vertex, and $S'^*$ be a $\gamma_{sp}$-set of $T'^*$. Similar to the argument as above, we have that $|S'^*| + 1 = |S^*|$. On the other hand, by induction, $\gamma_{sp}(T'^*) = \gamma_{sp}(T')$. And then, $|S'| + 1 = |S'^*|$. It concludes that $|S'| + 2 = |\bar{S}^*|$. $\Box$

**Case 2.** $u$ is a strong support vertex in $T$ and $e = uu_1$.

Suppose that $u_2$ is a leaf-neighbor of $u$ in $T$ other than $u_1$. Note that $v \not\in \bar{S}$, then $\bar{S} \cup \{u_1, u_2\}$ is the complement of a super dominating set of $T^*$. That is, $|\bar{S}| + 2 \leq |\bar{S}|$. On the other hand, by Proposition 2.4, without loss of generality, we may assume that $u_2 \in \bar{S}$. Then, $u$ is the external private neighbor of $u_2$ with respect to $\bar{S}$. Moreover, we have that $u_1 \in \bar{S}$ and $x$ is the external private neighbor of $u_1$ with respect to $\bar{S}$. Therefore, $\bar{S} \setminus \{u_1, u_2\}$ is the complement of a super dominating set of $T^*$. That is, $|\bar{S}| - 2 \leq |\bar{S}|$. So, we have that $|\bar{S}| - 2 = |\bar{S}|$. $\Box$

**Case 3.** $u$ is a strong support vertex in $T$ and $e = uv$, or $u$ is not a strong support vertex in $T$ and $e \in \{uv, uu_1\}$.

We assume that $u$ is not a strong support vertex in $T$ and $e = uu_1$(The other two cases can also be discussed similarly). Let $D = \bar{S} \cup \{u_1, v\}$ when $v \not\in \bar{S}$ and $N_T[v] \cap U_{\bar{S}}(T') = \emptyset$, and $D = \bar{S} \cup \{u, x\}$ when $N_T[v] \cap \bar{S} = \emptyset$. $D$ is the complement of a super dominating set of $T^*$. That is, $|\bar{S}| + 2 \leq |\bar{S}|$. On the other hand, by Proposition 2.4, without loss of generality, we assume that $u_1 \in \bar{S}$. And then $x$ is the external private neighbor of $u_1$ with respect to $\bar{S}$. Among all vertices of $\bar{S} \setminus \{u_1\}$, let $y$ be the vertex at minimum distance from $u$. It is easy to see that $\bar{S} \setminus \{u_1, y\}$ is the complement of a super dominating set of $T'$. That is, $|\bar{S}| - 2 \leq |\bar{S}|$. Hence, $|\bar{S}| - 2 = |\bar{S}|$. $\Box$

In either case, we have that $|\bar{S}| - 2 = |\bar{S}|$. Combining the fact that $|\bar{S}| + 1 = |\bar{S}|$, we have that $|\bar{S}| = |\bar{S}| + 1$. It follows from $n(T^*) = n(T) + 1$ that $\gamma_{sp}(T^*) = \gamma_{sp}(T)$. That is, $T$ is in Class 2. $\Box$

**Lemma 3.3** If a tree $T$ is in Class 2, then $T \in \mathcal{U}$.

**Proof** We know that $T$ is in Class 2, it is a star of order at least three when $\text{diam}(T) \leq 2$, and $T \in \mathcal{U}$. So we consider the case when $\text{diam}(T) \geq 3$. We proceed by induction on the order $n$ of $T$. Assume that the result is true for every tree in Class 2 of order less than $n$. Let $P = v_1v_2 \cdots v_k$ be a longest path in $T$, and $T_1$ be the component of $T - v_2v_3$ containing $v_3$. Let $e \in E(T_1)$ and $T_1^*$ (respectively, $T^*$) be obtained from $T_1$ (respectively, $T$) by subdividing the edge $e$, and $S$ (respectively, $S^*$, $S_1$, $S_1^*$) be a $\gamma_{sp}$-set of $T$ (respectively, $T^*$, $T_1$, $T_1^*$).

Set $D = \bar{S} \cup \{v_1\}$ when $v_3 \not\in \bar{S}$ and $D = \bar{S} \cup \{v_2\}$ when $v_3 \in \bar{S}$. It is easy to see that $D$ is the complement of a super dominating set of $T$. That is, $|\bar{S}| + 1 \leq |\bar{S}|$. On the other hand, without loss of generality, assume that $v_1 \in \bar{S}$, then $\bar{S} \setminus \{v_1\}$ is the
complement of a super dominating set of $T_1$. And so, $|\mathcal{S}| - 1 \leq |\mathcal{S}'_1|$. Hence, $|\mathcal{S}| - 1 = |\mathcal{S}'_1|$. Similarly, we have that $|\mathcal{S}'_1| - 1 = |\mathcal{S}'_1|$. By assumption, we know that $\gamma_{sp}(T) = \gamma_{sp}(T^*)$. That is, $|\mathcal{S}'_1| - 1 = |\mathcal{S}'_1|$. So, $|\mathcal{S}'_1| = |\mathcal{S}| - 1 = |\mathcal{S}'_1| - 2 = |\mathcal{S}'_1| - 1$. It implies that $T_1$ is in Class 2. By induction, $T_1 \in \mathcal{U}$.

Now, let $T'$ be the tree obtained from $T$ by subdividing the edge $v_2v_3$ with vertex $x$, $T''$ be the component of $T' - xv_3$ containing $x$, and let $T'$ be a $\gamma_{sp}(T')$-set. According to the discussion as above, we know that $|\mathcal{S}| = |\mathcal{S}'_1| + 1$ and $|\mathcal{S}'_1| = |\mathcal{S}'_1| - 1$. It concludes that $|\mathcal{S}'_1| = |\mathcal{S}'_1| + 2$. If $|V(T'') \cap \mathcal{S}'| \geq 2$, then we have that $\{x,v_2\} \subset \mathcal{S}'$. It means that $v_3$ is the external private neighbor of $x$ with respect to $\mathcal{S}'$. Moreover, $(N(v_3) \setminus \{x\}) \cap \mathcal{S}' = \emptyset$. Let $D = \mathcal{S}' \setminus \{x,v_2\}$, $V(T') \setminus D$ is a super dominating set of $T_1$. Since $|\mathcal{S}'| = |\mathcal{S}'_1| + 2$, $V(T_1) \setminus D$ is a $\gamma_{sp}(T_1)$-set. And $T$ can be obtained from $T_1$ by operation $\mathcal{U}_1$.

If $|V(T'') \cap \mathcal{S}'| = 1$, without loss of generality, assume that $v_1 \in \mathcal{S}'$. In this case, $v_2, x \notin \mathcal{S}'$, and $(N[v_3] \setminus \{x\}) \cap \mathcal{S}' \neq \emptyset$. Now we can distinguish three cases as follows:

**Case 1.** $v_3 \in \mathcal{S}'$.

Let $D = \mathcal{S}' \setminus \{v_1, v_3\}$. Note that $|\mathcal{S}'| = |\mathcal{S}'_1| + 2$, $H = V(T_1) \setminus D$ is a $\gamma_{sp}(T_1)$-set satisfying $v_3 \notin D$ and $N_{T_1}[v_3] \cup U_H(T_1) = \emptyset$. That is, $T$ can be obtained from $T_1$ by operation $\mathcal{U}_1$.

**Case 2.** $v_3 \notin \mathcal{S}'$ and $v_3 \in P_{\mathcal{S}'}(T')$.

Since $v_3 \in P_{\mathcal{S}'}(T')$, there exists a vertex $y \in (N[v_3] \setminus \{x\}) \cap \mathcal{S}'$ such that $v_3$ is the external private neighbor of $y$ with respect to $\mathcal{S}'$. Let $D = (\mathcal{S}' \setminus \{y, v_1\}) \cup \{x,v_2\}$. Clearly, $V(T') \setminus D$ is also a $\gamma_{sp}(T')$-set and the proof is similar to the case of $|V(T'') \cap \mathcal{S}'| \geq 2$.

**Case 3.** $v_3 \notin \mathcal{S}'$ and $v_3 \notin P_{\mathcal{S}'}(T')$.

In this case, $|(N[v_3] \setminus \{x\}) \cap \mathcal{S}'| \geq 2$. Consider the following subcases.

**Subcase 3.1.** $(N[v_3] \setminus \{x\}) \cap P_{\mathcal{S}'}(T') = \emptyset$.

Take a vertex $z \in (N[v_3] \setminus \{x\}) \cap \mathcal{S}'$. Let $D = (\mathcal{S}' \setminus \{z\}) \cup \{v_3\}$. Note that the proof is similar to Case 1.

**Subcase 3.2.** $(N[v_3] \setminus \{x\}) \cap P_{\mathcal{S}'}(T') \neq \emptyset$.

Assume that $(N[v_3] \setminus \{x\}) \cap P_{\mathcal{S}'}(T') = \{u_1, u_2, \ldots, u_k\}$. For any $i \in \{1, 2, \ldots, k\}$, let $T'_i$ be the component of $T' - v_3u_i$ containing $u_i$. Clearly, $S'_i = S' \cap V(T'_i)$ is a $\gamma_{sp}(T'_i)$-set. By Proposition 2.4, for each $i \in \{1, 2, \ldots, k\}$, there must be a $\gamma_{sp}(T'_i)$-set $S''_i$ such that $u_i \in S''_i$. Then, $(S' \setminus \bigcup_{i=1}^{k} S'_i) \cup \bigcup_{i=1}^{k} S''_i$ is a $\gamma_{sp}(T')$-set and the proof is similar to Subcase 3.1.

As an immediate consequence of Lemmas 3.2 and 3.3 we have the following theorem.

**Theorem 3.4** A tree $T$ is in Class 2 if and only if $T \in \mathcal{U}$.

Let $\mathcal{G} = \{T \mid T$ is a nontrivial tree $\}$, and $\mathcal{P} = \mathcal{G} \setminus \mathcal{U}$. We immediately obtain the following result.

\[ \square \] Springer
Theorem 3.5  A tree $T$ is in Class 1 if and only if $T \in \mathcal{P}$.

Acknowledgements  The research is supported by NSFC (no. 11301440), Natural Science Foundation of Fujian Province (CN) (2015J05017).

References

1. Atapour, M., Khodkar, A., Sheikholeslami, S.M.: Characterization of double domination subdivision number of trees. Discrete Appl. Math. 155, 1700–1707 (2007)
2. Atapour, M., Sheikholeslami, S.M., Hansberg, A., Volkmann, L., Khodkar, A.: 2-domination subdivision number of graphs. AKCE J. Graphs. Combin. 5, 165–173 (2008)
3. Babikir, A., Dettlaff, M., Henning, M.A., Lemańska, M.: Independent domination subdivision in graphs. Graphs Combin. 37, 691–709 (2021)
4. Dorfling, M., Goddard, W., Henning, M.A., Mynhardt, C.M.: Construction of trees and graphs with equal domination parameters. Discrete Math. 306, 2647–2654 (2006)
5. Dettlaff, M., Lemańska, M., Rodríguez-Velázquez, J.A., Zuazua, R.: On the super domination number of lexicographic product graphs. Discrete Appl. Math. 263, 118–129 (2019)
6. Favaron, O., Karami, H., Sheikholeslami, S.M.: Total domination and total domination subdivision numbers. Australas. J. Combin. 38, 229–235 (2007)
7. Favaron, O., Karami, H., Sheikholeslami, S.M.: Paired-domination subdivision numbers of graphs. Graphs Combin. 25, 503–512 (2009)
8. Hao, G., Sheikholeslami, S.M., Chellali, M., Khoelar, R., Karami, H.: On the paired-domination subdivision number of a graph. Mathematics 9, 439 (2021)
9. Haynes, T.W., Henning, M.A., Hopkins, L.S.: Total domination subdivision numbers of trees. Discrete Math. 286, 195–202 (2004)
10. Karami, H., Sheikholeslami, S.M.: Trees whose domination subdivision number is one. Australas. J. Combin. 40, 161–166 (2008)
11. Klein, D.J., Rodríguez-Velázquez, J.A., Yi, E.: On the super domination number of graphs. Commun. Comb. Optim. 5, 83–96 (2020)
12. Krishnakumari, B., Venkatakrishnan, Y.B.: Double domination and super domination in trees. Discrete Math. Algorithms Appl. 8 (2016) UNSP-1650067
13. Lemańska, M., Swaminathan, V., Venkatakrishnan, Y.B., Zuazua, R.: Super dominating sets in graphs. Proc. Natl. Acad. Sci. India Sect. A 85, 353–357 (2015)
14. Payan, C., Xuong, N.H.: Domination balanced graphs. J. Graph Theory 6, 23–32 (1982)
15. Qiang, X., Kosari, S., Shao, Z., Sheikholeslami, S.M., Chellali, M., Karami, H.: A note on the paired-domination subdivision number of trees. Mathematics 9, 181 (2021)
16. Velammal, S.: Studies in Graph Theory: Covering, Independence, Domination and Related Topics, Ph.D. Thesis, Manonmaniam Sundaranar University, Tirunelveli (1997)

Publisher's Note  Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.