On the numerical range of generators of symmetric $L_\infty$-contractive semigroups

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Abstract. A result by Liskevich and Perelmuter from 1995 yields the optimal angle of analyticity for symmetric submarkovian semigroups on $L_p$, $1 < p < \infty$. C. Kriegler showed in 2011 that the result remains true without the assumption of positivity of the semigroup. Here we give an elementary proof of Kriegler’s result.

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1. Introduction

Let $(\Omega, \mu)$ be a measure space and let $A$ be a positive self-adjoint operator in $L_2(\Omega, \mu)$. Then $-A$ generates a strongly continuous semigroup $(T(t))_{t \geq 0}$ which extends analytically to a contraction semigroup on the open right half plane. Such a semigroup is called a symmetric $L_\infty$-contractive semigroup\footnote{Such semigroups are called diffusion semigroups in \cite{4} and symmetric contraction semigroups in \cite{2}.} if, in addition, one has

$$
\|T(t)f\|_\infty \leq \|f\|_\infty, \quad \text{for all } f \in L_2(\Omega, \mu) \cap L_\infty(\Omega, \mu).
$$

Then, by symmetry, the semigroup is also $L_1$-contractive, and by interpolation one obtains for each $1 < p < \infty$ a consistent $C_0$-semigroup $(T_p(t))_{t \geq 0}$ of contractions on $L_p(\Omega, \mu)$. Let the generator of this semigroup be denoted by $-A_p$, with domain $\text{dom}(A_p)$.

In order to state the main result we define for $p \in [1, \infty)$ the mapping

$$
F_p : \mathbb{C} \to \mathbb{C}, \quad F_p(z) := \begin{cases} 
|z|^{p-2} & \text{if } z \neq 0, \\
0 & \text{if } z = 0.
\end{cases}
$$
Note that for $f \in L^p(\Omega, \mu)$ with $\|f\|_p = 1$ the function $F_p(f) := F_p \circ f$ has the properties
\[ \|F_p(f)\|_{p'} = 1, \quad \int_\Omega f \cdot \overline{F_p(f)} \, d\mu = 1, \]
where $p'$ is the dual exponent, i.e., $\frac{1}{p} + \frac{1}{p'} = 1$. In case that $p > 1$, $F_p(f)$ is uniquely characterized by these properties, and the numerical range of $A_p$ is the set of numbers
\[ \int_\Omega A_p f \cdot \overline{F_p(f)} \, d\mu, \quad \text{where } f \in \text{dom}(A_p), \|f\|_p = 1. \]

For $0 \leq \varphi \leq \frac{\pi}{2}$ we define the sector
\[ \Sigma(\varphi) := \{ z \in \mathbb{C} \setminus \{0\} : |\arg z| \leq \varphi \} \cup \{0\} \]
and call $\varphi$ its opening angle. For $p \in (1, \infty)$, let $\Sigma_p := \Sigma(\varphi_p)$, where
\[ \varphi_p := \arcsin |1 - \frac{2}{p}|. \]
Note that $\varphi_{p'} = \varphi_p$. Moreover, if $p$ passes from 2 to 1 or to $\infty$, then $|1 - \frac{2}{p}|$ passes from 0 to 1 and the opening angle $\varphi_p = \arcsin |1 - \frac{2}{p}|$ of $\Sigma_p$ passes from 0 to $\frac{\pi}{2}$. We point out that $\varphi_p$ is smaller than the angle $\frac{\pi}{2} |1 - \frac{2}{p}|$ that arises from interpolation between 0 and $\frac{\pi}{2}$. A short computation reveals that $\varphi_p$ has the (often used) alternative representation
\[ \varphi_p = \arcsin |1 - \frac{2}{p}| = \arctan \frac{|p - 2|}{2\sqrt{p - 1}}. \]

Now, here is the main result.

**Theorem 1.** Let $p \in (1, \infty)$ and let $-A_p$ be the generator on $L^p$ of a symmetric $L^\infty$-contractive semigroup on $L^2(\Omega, \mu)$. Then the numerical range of $A_p$ is contained in the sector $\Sigma_p$, and $(T_p(t))_{t \geq 0}$ extends to an analytic contraction semigroup on the sector with opening angle $\arccos |1 - \frac{2}{p}|$.

Under the additional assumption that the semigroup consists of positivity-preserving operators, Theorem 1 is due to Liskevich and Perelmuter [5]. The full result was established by Kriegler in [4] in the framework of noncommutative operator theory. Recently, Theorem 1 has been recovered by Carbonaro and Dragičević in [2] as a corollary of much stronger results. In [3], the first-named author streamlined and extended some of the methods used in [2] and showed that Theorem 1 can be deduced easily without making use of Bellman functions (which feature prominently in Carbonaro and Dragičević’s work).

In the following, we shall present an essentially elementary proof of Theorem 1 extending the arguments from [5]. The relation to the other proofs shall be explained in Section 4 below. We note that the second assertion in Theorem 1 follows from the first by virtue of the Lumer-Phillips theorem and the exponential formula $T_p(t) = \text{s-lim}_{n \to \infty} (I + \frac{t}{n} A_p)^{-n}$, cf. [1, Theorem 3.14]. Hence, it suffices to prove the first assertion.
2. A two-dimensional special case

Consider the special case $\Omega = \{1, 2\}$ with measure $\mu = \delta_1 + \delta_2$ and the matrix

$$A = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}.$$ 

Then $L^p(\Omega, \mu) = \mathbb{C}^2$ with the usual $p$-norm and

$$e^{-tA} = \frac{1}{2} \begin{pmatrix} 1 + e^{-2t} & 1 - e^{-2t} \\ 1 - e^{-2t} & 1 + e^{-2t} \end{pmatrix} \text{ for } t \geq 0.$$ 

Hence, $-A$ generates a (positivity-preserving) symmetric $L_\infty$-contractive semi-group. For this special case, Theorem 1 reduces to the assertion

$$(w - z) \cdot \overline{F_p(w) - F_p(z)} \in \Sigma_p \text{ for all } z, w \in \mathbb{C}, \quad (2.1)$$

which will be established with the next lemma. Moreover, Lemma 2 also shows that the sector $\Sigma_p$ in Theorem 1 is optimal already in this special case.

**Lemma 2.** For all $p \in (1, \infty)$ and $z, w \in \mathbb{C}$ one has

$$\text{cl}\{(w - z) \cdot \overline{F_p(w) - F_p(z)} : z, w \in \mathbb{C}\} = \Sigma_p. \quad (2.2)$$

The inclusion $\subseteq$ in Lemma 2 has been proved originally by Liskevich and Perelmuter [5, Lemma 2.2]. We include a new proof that helps to understand the appearance of the angle $\varphi_p$.

**Proof.** Fix $p \in (1, \infty)$ and write $F = F_p$. To establish (2.1) we can restrict to the case that $0$ is not on the line segment joining $z$ and $w$; otherwise, $(w - z) \cdot \overline{F_p(w) - F_p(z)} \geq 0$. We identify, as usual, $\mathbb{C}$ with $\mathbb{R}^2$, and note that $F$ is continuously $\mathbb{R}$-differentiable on $\mathbb{R}^2 \setminus \{0\}$. Hence, abbreviating $h = w - z$, we obtain

$$F(z + h) - F(z) = \int_0^1 F'(z + th)h \, dt,$$

where $F'$ is the Jacobian matrix of $F$.

Since $\Sigma_p$ is a closed convex cone (note that $\varphi_p \leq \frac{\pi}{2}$), it suffices to prove that

$$h \cdot \overline{F'(y)h} \in \Sigma_p \text{ for all } h \in \mathbb{R}^2, \ y \in \mathbb{R}^2 \setminus \{0\}.$$ 

Now, a short elementary computation yields, for $0 \neq y \in \mathbb{R}^2$,

$$F'(y) = |y|^{p-2} A_y,$$

where $A_y := I + \frac{p-2}{|y|^2} yy^t$. The matrix $A_y$ is symmetric and has eigenvalues $1$ and $p - 1 > 0$. (Indeed, $A_y y = (p - 1)y$ and $A_y z = z$ for all $z \perp y$.) Thus, by Lemma 3 below,

$$h \cdot \overline{F'(y)h} \in \Sigma(\arcsin |y|^{p-2}) = \Sigma_p,$$

for all $h \in \mathbb{R}^2$, and this concludes the proof of (2.1), i.e., the inclusion “$\subseteq$” in (2.2).

For the converse inclusion we denote

$$\Sigma := \text{cl}\{(w - z) \cdot \overline{F(w) - F(z)} : z, w \in \mathbb{C}\}.$$
Since \( F(tz) = t^{p-1}F(z) \) for all \( t > 0 \) and \( z \in \mathbb{C} \) the set \( \Sigma \) is invariant under multiplication with \( t > 0 \), i.e., a cone. Hence, for all \( z, h \in \mathbb{C} \setminus \{0\} \) and \( t > 0 \) we obtain
\[
\frac{1}{t} h \cdot F(z + th) - F(z) = \frac{1}{t^2} (th) \cdot F(z + th) - F(z) \in \Sigma.
\]
Letting \( t \searrow 0 \) we arrive at \( h \cdot F'(z)h \in \Sigma \), and another application of Lemma 3 completes the proof. \( \square \)

**Lemma 3.** Let \( A \in \mathbb{R}^{2 \times 2} \) be a symmetric matrix with eigenvalues \( 1 \) and \( \lambda > 0 \). Then
\[
\{ h \cdot Ah : h \in \mathbb{C} \} = \arcsin \left( \frac{|\lambda-1|}{\lambda+1} \right).
\]
**Proof.** Note that \( \{ h \cdot Ah : h \in \mathbb{C} \} \) is a cone in \( \mathbb{C} \). Thus it suffices to show that
\[
I := \{ \arg(h \cdot Ah) : h \in \mathbb{C} \setminus \{0\} \} = [- \arcsin \left( \frac{|\lambda-1|}{\lambda+1} \right), \arcsin \left( \frac{|\lambda-1|}{\lambda+1} \right]]. \tag{2.3}
\]
Now observe that for \( h \neq 0 \), \( \arg(h \cdot Ah) \) equals \( \angle(Ah, h) \), the signed angle between \( Ah \) and \( h \). As a consequence, one may suppose without loss of generality that \( A = \begin{pmatrix} 1 & 0 \\ 0 & \lambda \end{pmatrix} \). Then \( I = \{ \angle(A, (1, x)) : x \in \mathbb{R} \} \) since \( \angle(A, (0, 1)) = 0 \). Setting \( a := \arctan x \) and \( b := \arctan(\lambda x) \) we obtain
\[
\alpha_x := \angle(A, (1, x)) = \angle((1, x), (1, 1)) = a - b
\]
and, by virtue of the addition formula for the sine,
\[
(1 \pm \lambda)x = \tan a \pm \tan b = \frac{\sin a}{\cos a} \pm \frac{\sin b}{\cos b} = \frac{\sin(a \pm b)}{\cos a \cos b}.
\]
Hence,
\[
\sin \alpha_x = \sin(a - b) = \sin(a + b) \cdot \frac{1 - \lambda}{1 + \lambda}.
\]
Note that the angle \( a + b \) passes from \(-\pi\) to \( \pi \) as \( x \) passes from \(-\infty\) to \( \infty \). Thus we obtain the identity (2.3), and the proof is complete. \( \square \)

**Remark 4.** A more geometric way to prove Lemma 3 consists in applying the law of the sines in the triangles \( \triangle OBC \) and \( \triangle O'B'C \), where the points \( A, B, B', C \) and \( O \) are defined as \( A := (1, 0), B := (1, x), B' := (1, -x), C := (1, \lambda x) \) and \( O := (0, 0) \). (The angle of interest \( \alpha_x \) appears at \( O \) in the triangle \( \triangle BOC \).)

### 3. Proof of Theorem 1

Let us now turn to the proof of Theorem 1

**Proof of Theorem 1** Fix \( p \in (1, \infty) \) and write \( \langle f, g \rangle = \int_\Omega f \overline{g} \, d\mu \) for \( f \in L_p \) and \( g \in L'_p \), \( \frac{1}{p} + \frac{1}{p'} = 1 \). As above, we abbreviate \( F(z) = F_p(z) \).

As noted already, the second assertion of Theorem 1 follows from the first by virtue of the Lumer–Phillips theorem. Hence, we have to show that
\[
\langle A_p f, F(f) \rangle \in \Sigma_p, \quad \text{for all } f \in \text{dom}(A_p).
\]
For this it suffices to show
\[ \langle (I - T_p(t))f, F(f) \rangle \in \Sigma_p, \quad \text{for all } f \in L_p(\Omega, \mu), \ t > 0, \]
since one can divide by \( t \) and let \( t \to 0 \). Moreover, it is sufficient to check this for the dense subset \( D \) of step functions
\[ f = \sum_{j=1}^{n} c_j \mathbb{1}_{B_j} \quad (3.1) \]
where the sets \( B_j \) are pairwise disjoint measurable sets of positive and finite measure and \( c_j \in \mathbb{C} \setminus \{0\} \). (In order to see this, take an arbitrary \( f \in L_p \) and a sequence \( (f_n)_n \) of step functions with \( \|f_n - f\|_p \to 0 \), \( f_n \to f \) almost everywhere and absolutely dominated by some \( 0 \leq g \in L_p \). Then \( F(f_n) \to F(f) \) almost everywhere and absolutely dominated by \( g^{p-1} \), hence in \( L_{p'} \)-norm.\(^2\)

Fix \( t > 0 \) and \( f \) as in (3.1), so that \( F(f) = \sum_k F(c_k) \mathbb{1}_{B_k} \). Define \( d_j := \langle \mathbb{1}_{B_j}, \mathbb{1}_{B_j} \rangle = \mu(B_j) \) and \( a_{kj} = \langle T(t) \mathbb{1}_{B_j}, \mathbb{1}_{B_k} \rangle \) for \( 1 \leq j, k \leq n \). Then
\[
\begin{align*}
\langle (I - T(t))f, F(f) \rangle &= \sum_{jk} c_j \overline{F(c_k)} \langle (I - T(t)) \mathbb{1}_{B_j}, \mathbb{1}_{B_k} \rangle \\
&= \sum_j d_j c_j \overline{F(c_j)} - \sum_{jk} c_j \overline{F(c_k)} a_{kj} \\
&= \sum_j \left( d_j - \sum_k |a_{kj}| \right) c_j \overline{F(c_j)} + \sum_{jk} \left( c_j \overline{F(c_j)} |a_{kj}| - c_j \overline{F(c_k)} a_{kj} \right).
\end{align*}
\]
We claim that the first sum satisfies
\[ \sum_j \left( d_j - \sum_k |a_{kj}| \right) c_j \overline{F(c_j)} > 0. \]
Since \( c_j \overline{F(c_j)} = |c_j|^p \geq 0 \), it suffices to show that \( \sum_k |a_{kj}| \leq d_j \). Choose \( \lambda_{kj} \) such that \( |\lambda_{kj}| = 1 \) and \( a_{kj} = \lambda_{kj} |a_{kj}| \). Then
\[ \sum_k |a_{kj}| = \sum_k |\lambda_{kj} \langle T(t) \mathbb{1}_{B_j}, \mathbb{1}_{B_k} \rangle| = \langle T(t) \mathbb{1}_{B_j}, \sum_k \lambda_{kj} \mathbb{1}_{B_k} \rangle, \]
and hence \( \sum_k |a_{kj}| \leq \|T(t) \mathbb{1}_{B_j}\|_1 \| \sum_k \lambda_{kj} \mathbb{1}_{B_k} \|_\infty \leq \| \mathbb{1}_{B_j} \|_1 = d_j \), since \( T(t) \) is an \( L_1 \)-contraction.

In order to deal with the second sum, we note that, by symmetry,
\[ a_{jk} = \langle T(t) \mathbb{1}_{B_k}, \mathbb{1}_{B_j} \rangle = \langle \mathbb{1}_{B_k}, T(t) \mathbb{1}_{B_j} \rangle = \overline{a_{kj}}. \]

\(^2\)It is also necessary since \(- (I - T(t))\) is again the generator of a symmetric \( L_\infty \)-contractive semigroup on \( L_2(\Omega, \mu) \), see [3] Section 3.1.

\(^3\)Combining this with an argument involving subsequences shows that the mapping \( f \mapsto F(f) \) is continuous from \( L_p \) to \( L_{p'} \).
Therefore and since $\lambda_{kj} F(c_j) = F(\lambda_{kj} c_j)$,
\[
\sum_{j,k} \left( c_j F(c_j | a_{kj}) - c_j F(c_k | a_{kj}) \right)
\]
\[
= \frac{1}{2} \sum_{j,k} \left( c_j F(c_j | a_{kj}) - c_j F(c_k | a_{kj}) + c_k F(c_k | a_{kj}) - c_k F(c_j | a_{kj}) \right)
\]
\[
= \frac{1}{2} \sum_{j,k} \left| a_{kj} \right| (\lambda_{kj} c_j - c_k) (F(\lambda_{kj} c_j) - F(c_k)) \in \Sigma_p
\]
by Lemma 2. This concludes the proof. \qed

4. Relation to the Existing Proofs

Our elementary proof proceeds basically along the same reduction lines as the proof in [3]. In fact, the main ingredient in the proof given above was the fact (established in Lemma 2) that
\[
(\lambda w - z) \cdot \lambda F(w) - F(z) \in \Sigma_p
\]
for all $w, z, \lambda \in \mathbb{C}$ with $|\lambda| = 1$. A short computation reveals that this is actually equivalent to Theorem 1 being valid for the special cases
\[
A = \begin{pmatrix} 1 & -\lambda \\ -\lambda & 1 \end{pmatrix},
\]
where $\lambda \in \mathbb{C}$ with $|\lambda| = 1$.

The main difference to the paper [3] is that here we perform an immediate reduction to a finite atomic measure space similar as in [5], where [3], following [2], takes the detour via a compact model. To make this precise, let us consider as above the function $f$ as in (3.1) and define the atomic measure space $\Omega' := \{1, \ldots, n\}$ with $\mu' = \sum_{j=1}^{n} \mu(B_j) \delta_{\{j\}}$. On $L_2(\Omega', \mu')$ consider the matrix
\[
S = \left( \frac{a_{jk}}{\mu(B_j)} \right)_{j,k}
\]
where $a_{jk} = \langle T(t) \mathbb{1}_{B_k}, \mathbb{1}_{B_j} \rangle$ as above. Let $v := (c_1 \ldots c_n)^t$; then a short computation reveals that
\[
\langle (I - S)v, F(v) \rangle_{L_2(\Omega', \mu')} = \langle (I - T(t))f, F(f) \rangle_{L_2(\Omega, \mu)}.
\] (4.1)
The operator $S$ can be written as $J^* T(t) J$, where
\[
J : L_2(\Omega', \mu') \to L_2(\Omega, \mu), \quad (c_j)_j \mapsto \sum_{j=1}^{n} c_j \mathbb{1}_{B_j}
\]
is the natural isometric lattice embedding and $J$ is its Hilbert space adjoint. (Note that from this observation it is straightforward that $S$ is an $L_1$-contraction, a fact that has been proved in Section 3 by direct computation.)

Identity (4.1) implies that Theorem 1 is true in general if it is true for finite atomic measure spaces. Such spaces are in particular compact, and
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the remaining part of the proof in the previous section is nothing but an adaptation of the proof of [3, Theorem 4.15] to this special situation.

Remark 5. It is straightforward to conjecture that also the general results of [3], Theorem 2.2-2.4, can be proved by a direct reduction to finite atomic measure spaces and avoiding the use of compact models and the sophisticated operator theory presented in Section 4 of [3]. This is indeed true, and will be the topic of a future publication.

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