Chiral Squaring

S. Nagy

Theoretical Physics, Blackett Laboratory, Imperial College London, London SW7 2AZ, United Kingdom
s.nagy11@imperial.ac.uk

ABSTRACT

We construct the states and symmetries of $\mathcal{N} = 4$ super-Yang-Mills by tensoring two $\mathcal{N} = 1$ chiral multiplets and introducing two extra SUSY generators. This allows us to write the maximal $\mathcal{N} = 8$ supergravity as four copies of the chiral multiplet. We extend this to higher dimensions and discuss applications to scattering amplitudes.
1 Introduction

The idea that supergravity is secretly a double copy of super-Yang-Mills (SYM) theories has its origins in the KLT relations of string theory \[1\], and has found applications in the study of scattering amplitudes \[2\text{–}9\], gravity anomalies from gauge anomalies \[10\] and asymmetric orbifold constructions \[11\]. Recently, this correspondence has been made manifest at a more fundamental level, by giving a dictionary between gravitational and gauge fields, and by constructing the symmetries of the Lagrangian of the former from those of the latter. The global U-duality groups of supergravities built by squaring in all dimensions have been derived from the R-symmetries of the SYM theories \[12\text{–}14\]. At the linearised level, the local symmetries of general covariance, \(p\)-form gauge invariance, local Lorentz invariance and local supersymmetry of the \(\mathcal{N}=1\) gravitational superfield have been obtained by identifying \[15\]

\[
(\mathcal{N}=1)_{SG} = (\mathcal{N}=1)_{SYM} \star (\mathcal{N}=0)_{YM} \star \phi, \tag{1.1}
\]

where \(\star\) denotes a convolution of an \(\mathcal{N}=1\) gauge superfield, a vector field and a scalar living in the biadjoint of the gauge group. This latter term can be identified as the zeroth copy in the BCJ duality \[7\text{–}16\text{–}17\].

Given the simplification in calculations brought about by this decomposition, a natural question is whether we can go further and decompose the gauge superfields in terms of chiral superfields. Though little explored, one can find some hints of the double copy structure of gauge amplitudes in the scattering literature. Of course we can start with the double copy form of gravitational amplitudes and employ supersymmetric Ward identities to find various expressions between gluon and fermion amplitudes; however no systematic understanding of these exists. In 6 dimensions, due to the structure of the little group \((SO(4) = SO(3) \times SO(3))\) and the fact that one can build the states of the \(\mathcal{N} = (1,1)\) SYM multiplet by tensoring together \(\mathcal{N} = (1,0)\) and \(\mathcal{N} = (0,1)\) chiral multiplets, this double copy is revealed when the amplitudes are written in the spinor helicity formalism \[18\text{–}19\]. There is no straightforward analogue in \(D=4\) but a possible hint comes from one-loop calculations in \(\mathcal{N} = 4\) SYM. It is known that the amplitude here is entirely determined by the scalar box functions. The contributions from different particles are related to the scalar contributions via supersymmetric Ward identities. To obtain the contribution from a whole multiplet we sum over all its states and those of \(\mathcal{N} = 4\) SYM and \(\mathcal{N} = 1\) chiral are related via \(\rho^{\mathcal{N}=4} = (\rho^{\mathcal{N}=1})^2\) \[20\].

In this paper we explore the idea of SYM multiplets themselves as a double copy. We proceed as follows. In section 2 we obtain the SUSY transformations of the \(\mathcal{N} = 4\) SYM multiplet from those of the \(\mathcal{N}=1\) chiral multiplet in four dimensions. To achieve this, we introduce extra supersymmetry generators in the chiral multiplet, obtained by a \(U(1)\) rotation of the scalar states in the definition of the original \(Q\)’s. In a sense, this amounts to reversing the process of truncation which breaks an \(\mathcal{N} = 4\) gauge multiplet into an \(\mathcal{N} = 2\) SYM and a \(\mathcal{N} = 2\) hypermultiplet; however, the novelty is that the extra SUSY generators are built entirely from the operators of the \(\mathcal{N} = 1\) theory. This allows us to write the maximal supergravity in four dimensions as four copies of the (enhanced) chiral multiplet. The gauge and R-symmetries are also derived from squaring. We show that the squaring is more straightforward in \(D > 4\), i.e. the chiral multiplets don’t need to be enhanced with extra \(Q\)’s in section 3. We then conclude with possible applications of our dictionary, particularly to scattering amplitudes (where the extra supersymmetry generators will become necessary when using the Ward identities) and more speculatively to off-shell superfield descriptions.

2 \(D=4\) States

The chiral multiplet contains a Weyl spinor and two real (or one complex) scalars, whose on-shell degrees of freedom are multiplied according to the table below:
We begin with the familiar position-space transformations of the \( N = 1 \) chiral multiplet (comprising of a left-handed Weyl fermion and a complex scalar):

\[
\delta \epsilon \phi = e^a \chi_a, \\
\delta \epsilon \chi_a = -i \sigma^a_{\mu \nu} \epsilon^{\nu} \partial_\mu \phi,
\]

and similarly for \( \tilde{\phi} \) and \( \chi^\dagger_A \). It is useful to expand these fields in terms of creation and annihilation operators (since, as we see later, these will be the ones used in the dictionary):

\[
\phi(x) = \int \overline{dp} \left[ \phi_-(p) e^{ipx} + \phi^+_+(p) e^{-ipx} \right],
\]

\[
\chi_a(x) = \sum_{s=\pm} \int \overline{dp} \left[ \chi_s(p) (P_L u_s(p))_a e^{ipx} + \chi^\dagger_s(p) (P_L v_s(p))_a e^{-ipx} \right],
\]

where \( \overline{dp} = \frac{d^3p}{(2\pi)^2 2E_p} \) and \( \phi_\pm, \chi_\pm \) satisfy the usual algebra of bosonic/fermionic creation and annihilation operators. Note that the \( \pm \) labels on the bosonic operators \( \phi \) tell us which of the fermionic operators they are related to via SUSY.

Given that we will be using the raising/lowering operators in our dictionary, we must describe how they transform under supersymmetry. The rules can be read off by combining (2.1) and (2.2) and we get:

\[
\delta \epsilon \chi^+ (p) = [\epsilon p] \phi^+(p) \\
\delta \epsilon \phi^+(p) = \langle \epsilon p \rangle \chi^+ (p) \\
\delta \epsilon \phi^- (p) = [\epsilon p] \chi^- (p) \\
\delta \epsilon \chi^- (p) = \langle \epsilon p \rangle \phi^- (p)
\]

Table 1: \( D = 4, [(N = 1)^L_{\text{chiral}}] \times [(N = 1)^R_{\text{chiral}}] = [(N = 4)_{\text{SYM}}]. \)

The fields in the table can be organized traditionally into an \( N = 2 \) vector \( (A^+, 2\lambda^+, 2\phi, 2\lambda^-, A^-) \) and an \( N = 2 \) hypermultiplet \( (2\lambda^+, 4\phi, 2\lambda^-) \). However, note that we have in fact obtained the field content of \( N = 4 \) SYM.

In the following subsection, we will show how to build the (on-shell) SUSY transformations of the \( N = 4 \) SYM multiplet from those of the \( N = 1 \) chiral multiplet; we will find that we need to introduce two extra generators for SUSY transformations (which are not symmetries) to achieve this\(^1\).

### 2.1 Enhanced chiral multiplet and the extra SUSY generators

We begin with the familiar position-space transformations of the \( N = 1 \) chiral multiplet (comprising of a left-handed Weyl fermion and a complex scalar):

\[
\delta \epsilon \phi = e^a \chi_a, \\
\delta \epsilon \chi_a = -i \sigma^a_{\mu \nu} \epsilon^{\nu} \partial_\mu \phi,
\]

and similarly for \( \tilde{\phi} \) and \( \chi^\dagger_A \). It is useful to expand these fields in terms of creation and annihilation operators (since, as we see later, these will be the ones used in the dictionary):

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\phi(x) = \int \overline{dp} \left[ \phi_-(p) e^{ipx} + \phi^+_+(p) e^{-ipx} \right],
\]

\[
\chi_a(x) = \sum_{s=\pm} \int \overline{dp} \left[ \chi_s(p) (P_L u_s(p))_a e^{ipx} + \chi^\dagger_s(p) (P_L v_s(p))_a e^{-ipx} \right],
\]

where \( \overline{dp} = \frac{d^3p}{(2\pi)^2 2E_p} \) and \( \phi_\pm, \chi_\pm \) satisfy the usual algebra of bosonic/fermionic creation and annihilation operators. Note that the \( \pm \) labels on the bosonic operators \( \phi \) tell us which of the fermionic operators they are related to via SUSY.

Given that we will be using the raising/lowering operators in our dictionary, we must describe how they transform under supersymmetry. The rules can be read off by combining (2.1) and (2.2) and we get:

\[
\delta \epsilon \chi^+ (p) = [\epsilon p] \phi^+(p) \\
\delta \epsilon \phi^+(p) = \langle \epsilon p \rangle \chi^+ (p) \\
\delta \epsilon \phi^- (p) = [\epsilon p] \chi^- (p) \\
\delta \epsilon \chi^- (p) = \langle \epsilon p \rangle \phi^- (p)
\]

\(^1\)We will use a notation similar to [5] - there the authors construct the SUSY transformations of \( N = 8 \) supergravity fields in terms of those of \( N = 4 \) SYM and give a dictionary between the lowering operators of the two theories.

\(^2\)For convenience, we will be using the notation of the spinor-helicity formalism; \( | p \rangle \) and \( \langle p | \) can be thought of as 2-component complex vectors satisfying the massless Weyl equations

\[
p_{ab} | p \rangle^b = 0, \quad p_{ab} = p_{\mu} (\sigma^\mu)_{ab} = \begin{pmatrix} p^{0+} p^3 & p^{1+} p^2 \\ p^{1-} p^2 & -p^{0+} p^3 \end{pmatrix} \\
p^{ab} | p \rangle_b = 0, \quad p^{ab} = p_{\mu} (\tilde{\sigma}^\mu)_{ab} = \begin{pmatrix} p^{0-} p^3 & p^{1+} p^2 \\ -p^{1-} p^2 & p^{0+} p^3 \end{pmatrix}
\]
Figure 1: Susy generators of the chiral multiplet

It will be useful to write down the general form of the SUSY generators $Q_M = (|Q|_M)$ as functions of $\phi$ and $\chi$. One can show that they are

$$|Q|_a = \int \tilde{dp}|p|_a(\phi_{+}(p)\chi_{+}(p) - \chi_{-}(p)\phi_{+}^\dagger(p))$$
$$|Q^\dagger|_{\dot{a}} = \int \tilde{dp}|p|_{\dot{a}}(\phi_{-}(p)\chi_{+}(p) - \chi_{-}(p)\phi_{+}^\dagger(p))$$

(2.5)

Then their action on the lowering operators is

$$[Q, \chi_{+}(p)] = |p|\phi_{+}(p) \quad [Q^\dagger, \chi_{+}(p)] = 0$$
$$[Q, \phi_{+}(p)] = 0 \quad [Q^\dagger, \phi_{+}(p)] = |p|\chi_{+}(p)$$
$$[Q, \phi_{-}(p)] = |p|\chi_{-}(p) \quad [Q^\dagger, \phi_{-}(p)] = 0$$
$$[Q, \chi_{-}(p)] = 0 \quad [Q^\dagger, \chi_{-}(p)] = |p|\phi_{-}(p)$$

(2.6)

In [5], the eight SUSY generators of maximal supergravity came from two copies of the four maximal SYM generators. Although squaring the field content of the $\mathcal{N} = 1$ chiral multiplet does give us the $\mathcal{N} = 4$ SYM multiplet, it seems like we don’t have enough SUSY generators to build the supersymmetry of the gauge multiplet. This can be resolved by noticing that our generator $Q$ only relates ($\phi_{+}$ and $\chi_{+}$) and ($\phi_{-}$ and $\chi_{-}$) separately. One can define a new generator $Q'$ which mixes ($\phi_{+}$ and $\chi_{-}$) and ($\phi_{-}$ and $\chi_{+}$) separately, as shown in Figure 1. Of course, for the purpose of describing the chiral multiplet, this extra generator is redundant, but we will see that it becomes crucial for obtaining the SUSY transformations of $\mathcal{N} = 4$ SYM. One can apply an $SO(2)$ rotation on the bosonic states (keeping the fermionic states unchanged) to obtain a new set of SUSY generators:

$$|Q'|_a = \int \tilde{dp}|p|_a(\phi_{-}(p)\chi_{+}(p) + \chi_{-}(p)\phi_{+}^\dagger(p))$$
$$|Q'|_{\dot{a}} = \int \tilde{dp}|p|_{\dot{a}}(\phi_{+}(p)\chi_{+}(p) + \chi_{-}(p)\phi_{+}^\dagger(p))$$

(2.7)

Note that this $SO(2) = U(1)$ is different from the $U(1)$ R-symmetry group, under which both the bosonic and fermionic states will transform$^3$. This is to be expected, since an R-symmetry rotation cannot give

$^3$This can be seen for example from the interacting Lagrangian

$$\mathcal{L}_I = \frac{1}{2}\phi\dot{\psi} + c.c. - \frac{1}{4}|\phi|^4$$

(2.8)

where it is obvious that $\phi$ must transform with half the $U(1)$ charge of $\phi$. 

4
us new generators. Then the action of the new SUSY generators on the annihilation operators is (see Figure 1):

\[
\begin{align*}
[Q', \chi_+ (p)] &= [p] \phi_-(p) \\
[Q', \phi_+ (p)] &= -[p] \chi_-(p) \\
[Q', \phi_- (p)] &= 0 \\
[Q', \chi_- (p)] &= 0
\end{align*}
\]

\[
[Q''', \chi_+ (p)] = 0 \\
[Q''', \phi_+ (p)] = 0 \\
[Q''', \phi_- (p)] = [p] \chi_+(p) \\
[Q''', \chi_- (p)] = -[p] \phi_+(p)
\]

(2.9)

We can thus rewrite the fields and SUSY generators in our \( \mathcal{N} = 1 \) theory as

\[
\phi^i = (\phi_+, \phi_-) \\
Q^i = (Q, Q')
\]

(2.10)

and it is straightforward to see that the fields will transform under the action of the SUSY generators as

\[
\begin{align*}
[Q^i, \chi_+ (p)] &= [p] \phi^i (p) \\
[Q^i, \chi_-(p)] &= 0 \\
[Q^i, \phi_+ (p)] &= [p] \epsilon^{ij} \chi_-(p) \\
[Q^i, \phi_- (p)] &= [p] \delta^i_j \chi_+(p) \\
[Q^i, \chi_-(p)] &= 0 \\
[Q^i, \chi_+ (p)] &= \epsilon_{ij} \phi^j (p)
\end{align*}
\]

(2.11)

2.2 \( \mathcal{N} = 4 \) SYM

On-shell, the maximal SYM multiplet (with R-symmetry \( SU(4) \)) consists of 1 gluon with helicity \( h = +1 \) (\( B_\pm \)), 4 gluinos with \( h = +\frac{1}{2} \) (\( \lambda^a_\pm \)), 6 scalars with \( h = 0 \) (\( \phi^{ab} \)), 4 gluinos with \( h = -\frac{1}{2} \) (\( \lambda_{abc} \)) and 1 gluon with \( h = -1 \) (\( A_{abcd} \)), with \( a, b, c, d = 1, \ldots, 4 \) being the \( SU(4) \) indices. It is convenient to re-write the states as:

\[
A_+\, , \, \lambda_+^a\, , \, \phi^{ab} = \frac{1}{2} \epsilon^{abcd} \phi_{cd}\, , \, \lambda_a^- = \frac{1}{3!} \epsilon_{abcd} \lambda_{bcd}^-\, , \, A_- = -\frac{1}{4!} \epsilon_{abcd} A_{abcd}
\]

(2.12)

Then, under the action of the SUSY generators, the annihilation operators will transform as

\[
\begin{align*}
[Q^a, A_+ (p)] &= [p] \lambda^a_+ (p) \\
[Q^a, A_- (p)] &= [p] \lambda^-_a (p) \\
[Q^a, \phi^{bc} (p)] &= [p] \epsilon^{abcd} \phi_{cd}^- (p) \\
[Q^a, \lambda_{bc}^- (p)] &= -[p] \delta_a^b \lambda_+^c (p) \\
[Q^a, A_+ (p)] &= [p] \delta_a^b A_- (p) \\
[Q^a, A_- (p)] &= [p] \phi_{ab} (p) \\
[Q^a, \lambda_{bc}^- (p)] &= -[p] \delta_a^b \lambda_+^c (p)
\end{align*}
\]

(2.13)

2.3 Dictionary

We now have all the ingredients necessary to build a dictionary between \( \mathcal{N} = 4 \) SYM and two copies of the \( \mathcal{N} = 1 \) (written as \( \mathcal{N} = 2 \)) chiral multiplet. We will split the \( SU(4) \) indices \( a, b, \ldots = 1, \ldots, 4 \) into left \( SU(2) \) indices \( i, j, \ldots = 1, 2 \) and right \( SU(2) \) indices \( r, s, \ldots = 1, 2 \). The SYM operators can then be built as tensor products of the form \( O_L \otimes \tilde{O}_R \), with \( Q^i \) and \( Q^r \) acting only on the LHS and RHS respectively.
We can then write a dictionary for all the operators described in subsection 2.2:

\[ A_+(p) = 
\begin{cases} 
\chi_+(p) \otimes \tilde{\chi}_+(p) 
\end{cases} 
\]

\[ \lambda_+^a(p) = 
\begin{cases} 
\lambda_+^i(p) = \phi^i(p) \otimes \tilde{\chi}_+(p) 
\end{cases} 
\]

\[ \phi^{ab}(p) = 
\begin{cases} 
\phi^{ij}(p) = \epsilon^{ij} \chi_-(p) \otimes \tilde{\chi}_+(p) 
\end{cases} 
\]

\[ \lambda_-^a(p) = 
\begin{cases} 
\lambda_-(p) = \phi_+(p) \otimes \tilde{\chi}_-(p) 
\end{cases} 
\]

\[ A_-(p) = \chi_-(p) \tilde{\chi}_-(p) \]

It has been checked that for any SYM operator, its SUSY transformation can be obtained from those of the LHS and RHS chiral multiplets.

As an interesting example, let’s look at the action of \( Q_+^i \) on the scalars \( \phi^{ab}(p) \). We know from (2.13) that

\[ [Q_+^i, \phi^{bc}(p)] = |p|(\delta_+^i \lambda_+^b(p) - \delta_+^i \lambda_+^c(p)) \]

We now decompose this in terms of the LHS and RHS indices to get the following set of transformation rules:

- all indices LHS, we expect

\[ [Q_+^i, \phi^{jk}(p)] = |p|(\delta_+^i \lambda_+^k(p) - \delta_+^i \lambda_+^j(p)) \]

To check this, we substitute the dictionary (2.14) and get

\[ [Q_+^i, \epsilon^{jk} \chi_-(p) \otimes \tilde{\chi}_+(p)] = \epsilon^{jk} [Q_+^i, \chi_-(p)] \otimes \tilde{\chi}_+(p) \]

\[ = |p| \epsilon^{jk} \epsilon_{il} \phi^l(p) \otimes \tilde{\chi}_+(p) \]

\[ = |p| (\delta_+^i \lambda_+^k(p) - \delta_+^i \lambda_+^j(p)) \]

as expected.

- two LHS indices, one RHS index, we expect

\[ [Q_+^i, \phi^{jr}] = |p| (\delta_+^i \lambda_+^r(p)) \]

We check this using the dictionary:

\[ [Q_+^i, \phi^j(p) \otimes \tilde{\phi}^r(p)] = [Q_+^i, \phi^j] \otimes \tilde{\phi}^r \]

\[ = |p| \delta_+^i \chi_+(p) \otimes \tilde{\phi}_+^r(p) \]

\[ = |p| \delta_+^i \lambda_+^r(p) \]

as expected. We also have

\[ [Q_+^i, \epsilon^{ij} \chi_-(p) \otimes \tilde{\chi}_+(p)] = 0 \]

and we can again check

\[ [Q_+^i, \epsilon^{ij} \chi_-(p) \otimes \tilde{\chi}_+(p)] = \epsilon^{ij} \chi_-(p) \otimes [Q_+^i, \tilde{\chi}_+] = 0 \]

All the other cases will proceed similarly to the ones above.
2.4 Non-abelian gauge symmetry and R-symmetry

We can allow the states of our $\mathcal{N} = 1$ multiplets to transform in the adjoint of some non-abelian groups $G_{L/R}$:

$$\delta \Phi^\alpha_{L/R} = f^\alpha_{\beta\gamma} \Phi^\beta \theta^\gamma$$  \hspace{1cm} (2.22)

with $\alpha = 1, 2, \ldots, \text{dim}(G_{L/R})$ and $\theta^\alpha$ a global parameter. We will tensor these to obtain linearized SYM, where the abelian local $U(1)$ transformations are decoupled from the non-abelian global ones. Under the latter, the SYM fields will also transform in the adjoint representation and this suggests the natural dictionary:

$$\Phi^\alpha_{SYM} = f^\alpha_{\beta\gamma} \Phi^\beta_L \Phi^\gamma_R$$  \hspace{1cm} (2.23)

which requires $G_L = G_R = G_{SYM}$. Given the absence of local gauge parameters in the chiral multiplet transformations, we see that it will be the field strength, rather than the potential, that is obtained through squaring.

The next question is how the $SU(4)$ R-symmetry is built from transformations of the $\mathcal{N} = 1$ fields. First we notice that after introducing the extra SUSY generators, the automorphism of the SUSY algebra, whose generators act on the $Q$'s via

$$[T_A, Q_a] = (U_A)_a{}^b Q_b, \quad a, b = 1, \ldots, \mathcal{N}$$  \hspace{1cm} (2.24)

is enhanced to $SU(2)$ for our faux $\mathcal{N} = 2$ multiplet. Note that the femionic states will transform trivially under this $SU(2)$ (the $U(1)$ under which they transformed in the traditional chiral multiplet drops out due to CPT self-conjugation) and the bosonic states will transform as

$$[T^c_b, \phi^d] = \delta^c_b \delta^d_s,$$  \hspace{1cm} (2.25)

where $(T^c_b)_d = \delta^c_b \delta^d_s - \frac{1}{2} \delta^c_b \delta^d_s$ and $a, b, c, d = 1, 2^5$. Then $SU(4)$ will be built from $SU(2)_L \times SU(2)_R \times U(1)$: we will have $3_L + 3_R$ $SU(2)$ generators acting separately on the LHS and RHS states as in (2.25), and a $U(1)$ generator $T$ which acts on the supergravity states via:

$$[T, A_+] = 0$$
$$[T, \lambda^i_+] = \lambda^i_+,$$  \hspace{1cm} (2.26)
$$[T, \phi^{ij}] = 2 \phi^{ij}, \quad [T, \phi^r s] = -2 \phi^{rs}, \quad [T, \phi^{ir}] = 0$$

In addition, we have the generators which mix the LHS and RHS states. They are given by $(T^a_r)_b = \delta^a_b \delta^r_s$ and $(T^a_r)_s = \delta^a_b \delta^r_s$ and their action on the supergravity states is

$$[T^r_b, \phi_{is}] = \delta^r_s \phi_{ib}$$  \hspace{1cm} (2.27)

Note that each of the mixed generators can be interpreted as a tensor product of the SUSY operators, where we have formally suppressed the spacetime indices, for example $T^a_s = q^a \otimes q_s$, where $[q_a, \phi_b] = \epsilon_{ab} \lambda_-$.

In conclusion, the action of the R-symmetry generators on the SYM states was built from the action on the states of the chiral multiplet in the same way as the action on supergravity states from the action on SYM states [5].

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4 Note that it is $SU(2)$ rather than $U(2)$ because the introduction of the extra SUSY generator makes the enhanced $\mathcal{N} = 1$ multiplet CPT conjugate. Interestingly, it seems that one can only obtain a CPT self-conjugate multiplet by tensoring two multiplets which are CPT conjugate themselves. This is also observed at the next level of squaring, when we get $\mathcal{N} = 8$ Supergravity from squaring $\mathcal{N} = 4$ SYM.

5 We use the same conventions as in [5].

6 Interestingly, the same interpretation can be given to the terms that mix the LHS and RHS in the global symmetry of supergravities obtained from SYM squaring [14].
Interestingly, the R symmetry of this theory is then built via the same formula that we gave in \[14\] for \( h \), where \( h \) is the maximally compact subalgebra of the U-duality of supergravity theories obtained from squaring\(^7\):

\[
\text{su}(4) = \text{su}(2) \oplus \text{su}(2) \oplus \text{u}(1) + \mathbb{C}[2, 2] = r_L \oplus r_R \oplus \text{u}(1) + q_L \otimes q_R
\]

and \( q_{L/R} \) are obtained from the supercharges as explained above.

### 2.5 Gravity-chiral dictionary

On-shell, the maximal Supergravity multiplet consists of a graviton, 8 gravitini, 28 vectors, 56 fermions and 70 scalars, whose helicity states are represented as:

\[
g_+, \psi^A_+, A^{AB}_+, \chi^{ABC}_+, \phi^{ABCD}, \chi^-_{ABC}, A^-_{AB}, \psi^-_A, g^-
\]

Then the gravity-chiral dictionary is \(^8\) (via \( N = 4 \) SYM):

\[
g = A_+ A'_- = \chi^+ \chi^\prime_-
\]

\[
\psi^A = \begin{cases}
\psi^A_+ = \lambda^A_+ A'_+ = \begin{cases}
\lambda^A_+ = \phi^i \chi^+_i \\
\lambda^\prime_+ = \chi^+_i \phi^i
\end{cases}
\chi^+_i
\end{cases}
\]

\[
\psi^A_+ = A_+ (\lambda')^2_+ = \chi^+_i \chi^+_i
\]

\[
A^{AB}_+ = \begin{cases}
A^{ab}_+ = \phi^{ab} A'_+ = \begin{cases}
\phi^{ij} = \epsilon^{ij} \chi^-_i \\
\phi^{ir} = \phi^i \phi^r \\
\phi^{rs} = \epsilon^{rs} \chi^-_i
\end{cases}
\chi^+_+ \chi^+_i
\end{cases}
\]

\[
\chi_{ABC} = \begin{cases}
\chi_{abc} = \epsilon^{abc} \chi^-_{A'} = \epsilon^{abcd} \begin{cases}
\lambda_+ = \phi_i \chi^-_i \\
\lambda^- = \chi^-_i \phi^i
\end{cases}
\chi^+_+ \chi^+_i
\end{cases}
\]

\[\text{In \[14\], we actually find the general formula for any } 3 \leq D \leq 10:\]

\[
\text{so}(N_L + N_R, D) = [\text{so}(N_L, D) \oplus \text{so}(N_R, D) \oplus \delta_{D,4} \text{u}(1) + \text{D}[N_L, N_R]]
\]

\[\text{where } \text{D} \text{ is the division algebra associated with the spinor representation in dimension } D \text{ and } \text{so}(n, A) \cong \mathfrak{iso}(\mathbb{R}^{n-1})\).

\[\text{Note that the RHS SYM fields are dashed. We use the following convention for the indices:}\]

- supergravity: A, B, C...
- SYM: a,b,... (LHS) and \( \bar{a}, \bar{b},... \) (RHS)
- chiral: \( i(i),j(j) \) (LHS) and \( r(r),s(s) \) (RHS)
The on-shell superfield for the \( N \) where \( A \)

The tensoring table in terms of the little group representations is given by:

Note that we take \( \eta \) and obtain supergravity.

In 6 dimensions, we would like to build the maximal \( N = (2,2) \) supergravity out of 4 copies of the chiral multiplet. The easiest route is to first write the \( N = (2,0) \) tensor multiplet as two copies of the \( N = (1,0) \) multiplet (and similarly for \( N = (0,2) \)). Then it is straightforward to combine the two tensor multiplets and obtain supergravity.

The \( N = (1,0) \) on-shell chiral superfield is

\[
\chi(\eta^{i+}) = \chi^+ + \phi_i \eta^i + \eta^{i\dagger} \eta^{J\dagger} \Omega_{ij} \chi^{-}
\]  

(3.1)

Note that we take \( \eta^{i\dagger} \rightarrow \eta^{i+} \) in order to construct the superspace, breaking the \( SU(2) \) little group symmetry (see [21]). Our states are \( \chi^+ \), \( \phi_i \) and \( \chi^- \), where \( \pm \) are the \( SU(2) \) weights and \( i = 1,2 \) is the \( Sp(1) \) R-symmetry index. Then the action of the SUSY generator on the states is given by

\[
\begin{align*}
\{ \phi_i^A, \chi^+ \} &= \lambda^A_i \\
\{ \phi_i^A, \phi_j \} &= \lambda^A_i \Omega_{ij} \chi^{-} \\
\{ \phi_i^A, \chi^- \} &= 0
\end{align*}
\]  

(3.2)

where \( A = 1, \ldots 4 \) are \( SO(6) = SU(4) \) Lorentz indices in \( D = 6 \) and \( \lambda^A_+ \) is the + component of the \( \lambda^{Aa} \) solution of the Weyl equation in \( D = 6 \) (see [18] for a description of the on-shell spinor-helicity formalism in six dimensions).

Now we want to build the \( N = (2,0) \) tensor multiplet out of two copies of the chiral multiplet above. The tensoring table in terms of the little group representations is given by:

|       | \( \chi \) | \( 2\phi \) |
|-------|--------|--------|
| \( \tilde{\chi} \) | \( (2,1) \) | \( (2,1,1) \) |
| (2,1) | \( B_{\mu\nu} + \phi \) | \( 2\psi \) |
| (2,1) | \( (3,1) + (1,1) \) | \( 2(2,1) \) |
| \( 2\phi \) | \( 2\psi \) | \( 4\phi \) |
| \( 2(1,1) \) | \( 2(2,1) \) | \( 4(1,1) \) |

Table 2: \( D = 6, [(1,0)_{\text{chiral}}^L] \times [(1,0)_{\text{chiral}}^R] = [(2,0)_{\text{Tensor}}] \).

The on-shell superfield for the \( N = (2,0) \) theory is given by [21]

\[
\Phi(\eta^{i+}) = B^+ + \psi^+ \eta^{i+} + \frac{1}{2} \eta^{i+} \eta^{J+}[\phi_{IJ} + \Omega_{IJ} A^0] + \frac{1}{3!} \epsilon_{LJK} \eta^{+I} \eta^{+J} \eta^{+K} \psi^{-L} + (\eta^{+})^4 B^-
\]  

(3.3)
so, in descending order of $SU(2)$ weight, our states are

$$B^+, \quad \psi^+_I, \quad [\phi_{IJ} + \Omega_{IJ} A^0] \equiv A_{IJ}, \quad \psi^{-I}, \quad B^- \quad (3.4)$$

where $I, J = 1, \ldots, 4$ are the $Sp(2)$ R-symmetry indices. The action of the SUSY generators on these states is given by

$$\{q^A_I, B^+\} = \lambda^+_A \psi^+_I \quad (3.5)$$

Then one can write a dictionary for the tensor multiplet as a double copy of the chiral one:

$$B^+ = \chi^+ \otimes \tilde{\chi}^+$$
$$\psi^+_I = \begin{cases} \psi^+_i = \phi_i \otimes \tilde{\chi}^+ \\ \psi^+_r = \chi \otimes \tilde{\phi}_r \end{cases}$$
$$A_{IJ} = \begin{cases} A_{ij} = \Omega_{ij} \chi^- \otimes \tilde{\chi}^+ \\ A_{ir} = \phi_i \otimes \tilde{\phi}_r \\ A_{rs} = \Omega_{rs} \chi^+ \otimes \tilde{\phi}_s \end{cases}$$
$$\psi^{-I} = \begin{cases} \psi^{-i} = \Omega^i \phi_j \otimes \tilde{\chi}^- \\ \psi^{-r} = \Omega^r \chi^- \otimes \tilde{\phi}_s \end{cases}$$
$$B^- = \chi^- \otimes \tilde{\chi}^- \quad (3.6)$$

It has been checked that the SUSY transformations of the tensor multiplet then follow from those of the chiral multiplet, similarly to the situation in $D = 4$. The R-symmetry is again obtained via the formula

$$\text{sp}(2) = \text{sp}(1) \oplus \text{sp}(1) + \Pi[1,1] = r_L \oplus r_R + q_L \otimes q_R \quad (3.7)$$

noting that the representation in $D=6$ is quaternionic. Here $\text{sp}(1) = \text{su}(2)$ is the R-symmetry algebra of the left and right chiral multiplet, and again the total R-symmetry of the resulting multiplet is enhanced via a tensor product of fermionic transformations (which are not symmetries) of the $\mathcal{N} = 1$ states.

We can now perform the more straightforward squaring of the two tensor multiplets of opposite chiralities to get the maximal supergravity, as shown in the table below:
Having constructed the states and symmetries of gauge theories by squaring, one can hope to apply them to simplifying scattering amplitudes. We illustrate the appearance of the double copy structure for 4-point gluon scattering through an example. The $s$-channel 4-point scattering process for two negative-helicity and two positive helicity fermions in Yukawa theory is given by (up to the coupling constant)

$$ A_4(1^-, 2^-, 3^+, 4^+) = A_4^s(f_1 f_2 f_3^+ f_4^+) \propto \frac{\langle 12 \rangle}{\langle 34 \rangle} $$

(4.1)

Since the 3-point fermion scattering process is not allowed, this is the smallest possible amplitude. We note that the scattering amplitude of the four gluons obtained by taking two copies of the fermions above is

$$ A_4(g_1^- g_2^- g_3^+ g_4^+) \propto \frac{\langle 12 \rangle^4}{\langle 12 \rangle \langle 23 \rangle \langle 34 \rangle \langle 41 \rangle} = -\frac{s_{12}}{s_{23}} \left( \frac{\langle 12 \rangle}{\langle 34 \rangle} \right)^2 = -\frac{s_{12}}{s_{23}} [A_4^s(f_1^- f_2^- f_3^+ f_4^+)]^2 $$

(4.2)

after some rearrangements and after imposing momentum conservation on the 4 legs (we use $s_{ij} = (p_i + p_j)^2 = \langle ij \rangle [ij]$). We could potentially find similar relations for a higher number $n$ of external legs, provided $n$ is even, i.e. when we can write the fermion amplitudes.

We can also now write the gravitational 4-point amplitude as four copies of the chiral one via

$$ A_4(h_1^- h_2^- h_3^+ h_4^+) = -s_{12} A_4(g_1^- g_2^- g_3^+ g_4^+) A_4(g_1^- g_2^- g_3^+ g_4^+) \propto -\left( \frac{s_{12}}{s_{23} s_{24}} \right)^3 [A_4^s(f_1^- f_2^- f_3^+ f_4^+)]^4 $$

(4.4)

Again, we expect similar (though more complicated) quadruple copy relations for higher numbers $n = 2k$ of external legs. It would also be interesting to see if there are any connections with (some modified version

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9It is interesting to note that one can also write the scalar-QED amplitude (which is not colour-ordered) as a double copy:

$$ A_4(\phi_1, \phi_2^*, g_3^+, g_4^-) = \frac{\langle 24 \rangle}{\langle 13 \rangle \langle 23 \rangle} = A_4^s(f_1^+ f_2^+ f_3^+ f_4^-) A_4^s(f_1^- f_2^- f_3^- f_4^+) $$

(4.3)

up to coupling constants.
of) the BCJ relations. Additionally, one can make use of the SUSY Ward identities, in conjunction with the gravity-chiral dictionary in (2.31), to find a variety of other quadruple copy relations. Note that to achieve this, we will need to make use of the additional $Q'$ generators introduced in subsection 2.1.

More speculatively, one could investigate whether something can be inferred about the closure of the SUSY algebra of $\mathcal{N} = 4$ SYM from its double copy structure. Note that in [15] we tensored off-shell (super)fields to obtain an off-shell supergravity superfield. It would be interesting to explore whether there are any connections with recent attempts at $\mathcal{N} = 4$ off-shell SUSY via lower-dimensional “holograms” [22].

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