Characterization of several kinds of quantum analogues of relative entropy

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Received (received date)
Revised (revised date)

Quantum relative entropy $D(\rho\|\sigma) \overset{\text{def}}{=} \text{Tr} \rho (\log \rho - \log \sigma)$ plays an important role in quantum information and related fields. However, there are many quantum analogues of relative entropy. In this paper, we characterize these analogues from information geometrical viewpoint. We also consider the naturalness of quantum relative entropy among these analogues.

Keywords: Autoparallel curve, Divergence, Fisher information, Monotonicity, Additivity

Communicated by: to be filled by the Editorial

1 Introduction

In the quantum information theory, we usually focus on the quantum relative entropy $D(\rho\|\sigma) \overset{\text{def}}{=} \text{Tr} \rho (\log \rho - \log \sigma)$ as a quantum analogue of relative entropy (divergence). However, there are many kinds of quantum analogues of relative entropy. Some of them have been discussed from the viewpoint of operator algebra [5, 4]. In the classical information geometry, the divergence can be defined by using the integral along the autoparallel curve. Since the geometrical approach in classical information systems is very attractive, excellent insights for quantum information system can be expected through the consideration from geometrical viewpoints.

By extending this definition to the quantum system, Nagaoka [2, 10] defined quantum analogues of divergence based on the integral along the parallel translation. $e$-parallel translation and $m$-parallel translation are known as most popular parallel translations in the quantum system as well as in the classical system. These divergences are called $e$-path-divergence and $e$-path-divergence, respectively. In particular, In the classical system, the path-divergences of both translations give the usual relative entropy. On the other hand, Fisher information is unique in the classical system. However, it is not unique in the quantum system. Petz [8] completely characterized its quantum analogues. As famous examples, SLD Fisher information, RLD Fisher information, and Bogoljubov Fisher information are known [7, 8, 6, 1, 9]. Nagaoka showed that the quantum path-divergence concerning $e (m)$-parallel translation coincides with the quantum relative entropy $D(\rho\|\sigma)$ when the quantum Fisher information of interest is Bo-
goljubov Fisher information\[2,10\]. He also calculated the quantum path-divergence with the SLD Fisher information concerning $e$-parallel translation\[2\].

In this paper, we calculate the quantum path-divergence other than the above cases. Then, we succeeded in relating information geometrical path-divergence and an operator-algebraic divergence $D(\rho||\sigma) = \text{Tr} \rho \log(\rho^{\frac{1}{2}}\sigma^{-1}\rho^{\frac{1}{2}})$, was introduced through operator-algebraic context by Belavkin and Staszewski \[11\]. Further, we proved the additivity of quantum path-divergence defined by $e$-parallel translation, and the monotonicity of quantum path-divergence defined by $m$-parallel translation. These two parallel translations are the dual parallel translations of each other. Since these two properties are fundamental, they are expected to be applied in the research field of quantum information.

In the classical system, the divergence also can be defined from a convex function. Hence, divergence is closely related to convex analysis. Amari & Nagaoka \[1\] showed that only Bogoljubov Fisher information has zero-torsion. That is, the geometry of Bogoljubov inner product has the dual flat structure. They also proved the equivalence of the following two conditions. 1) The path-divergences of dual parallel translations can be given from potential function. 2) The dual parallel translation has the dual flat structure. Hence, in the quantum case, we can conclude that only path-divergences of Bogoljubov Fisher information is given by a potential function. This result indicates that the geometry of Bogoljubov Fisher information is closely related to optimization problem in quantum system. In fact, in their proof, the calculations concerning Christoffel symbols were essentially used. However, many quantum information scientists are not familiar to such analysis. In this paper, we give another proof of this argument without any use of Christoffel symbols. This paper can be expected to be a good guidance for quantum information geometry for quantum information scientist.

This paper is organized as follows. In section 2, we review the information geometrical characterization of divergence $D(p||q)$ in the classical system. we also review how the divergence can be defined by the convex function in the classical system. In section 3, we give a review of inner product in quantum systems, which is a fundamental of quantum information geometry. In section 4, two kinds of autoparallel translations and autoparallel curves are reviewed. In section 5, we treat quantum analogues of relative entropy from the operator-algebraic viewpoint. In section 6, we examine quantum path-divergences based on $e$-autoparallel translation, and consider their properties. In section 7, we examine quantum path-divergences based on $m$-autoparallel translation, and consider their properties. In particular, the relation between an operator-algebraic divergence and quantum path-divergences based on $e$ ($m$)-autoparallel translation are derived in section 6 (7).

2 Divergence in Classical Systems

First, we review the information geometrical characterization of divergence $D(p||q)$ in the classical system \[1\]. Let $p(\omega)$ be a probability distribution, and $X(\omega)$ be a random variable. When the family $\{p_\theta|\theta \in \Theta\}$ has the form

$$p_\theta(\omega) = p(\omega)e^{\theta X(\omega) - \mu(\theta)}$$

$$\mu(\theta) \stackrel{\text{def}}{=} \log \sum_\omega p(\omega)e^{\theta X(\omega)},$$

Theorem 2.3. The $\frac{1}{2}$-parallel translation is the dual of the $e$-parallel translation.

Proof. Let $\phi(\omega) = p(\omega)e^{\phi X(\omega)}$. Then, we have

$$\text{Tr} \rho \log(\rho^{\frac{1}{2}}\phi^{-1}\rho^{\frac{1}{2}}) = \text{Tr} \rho \log(\rho^{\frac{1}{2}}\phi^{-1}\rho^{\frac{1}{2}})$$

This completes the proof.

Corollary 2.4. The $\frac{1}{2}$-parallel translation is the dual of the $e$-parallel translation.

Proof. From Theorem 2.3, we have

$$\frac{1}{2} \rho^{\frac{1}{2}}\phi^{-1}\rho^{\frac{1}{2}} = e^{-\phi \mu(\frac{1}{2})}$$

This completes the proof.

Theorem 2.5. The $m$-parallel translation is the dual of the $e$-parallel translation.

Proof. Let $\phi(\omega) = p(\omega)e^{m \theta X(\omega)}$. Then, we have

$$\text{Tr} \rho \log(\rho^{\frac{1}{2}}\phi^{-1}\rho^{\frac{1}{2}}) = \text{Tr} \rho \log(\rho^{\frac{1}{2}}\phi^{-1}\rho^{\frac{1}{2}})$$

This completes the proof.

Corollary 2.6. The $m$-parallel translation is the dual of the $e$-parallel translation.

Proof. From Theorem 2.5, we have

$$m \rho^{\frac{1}{2}}\phi^{-1}\rho^{\frac{1}{2}} = e^{-m \phi \mu(m)}$$

This completes the proof.

Theorem 2.7. The $e$-parallel translation is the dual of the $m$-parallel translation.

Proof. Let $\phi(\omega) = p(\omega)e^{\theta X(\omega)}$. Then, we have

$$\text{Tr} \rho \log(\rho^{\frac{1}{2}}\phi^{-1}\rho^{\frac{1}{2}}) = \text{Tr} \rho \log(\rho^{\frac{1}{2}}\phi^{-1}\rho^{\frac{1}{2}})$$

This completes the proof.

Corollary 2.8. The $e$-parallel translation is the dual of the $m$-parallel translation.

Proof. From Theorem 2.7, we have

$$e \rho^{\frac{1}{2}}\phi^{-1}\rho^{\frac{1}{2}} = e^{-e \phi \mu(e)}$$

This completes the proof.
the logarithmic derivative at respective points equals the logarithmic derivative at a fixed point with the addition of a constant. In this case, the family is called an exponential family, and \( \mu(\theta) \) is called the moment function of \( X \). In particular, since the logarithmic derivative is closely related to exponential families, it is often called the exponential (\( e \)) representation of the derivative. Therefore, we use the superscript (\( e \)) in the inner product \( \langle \cdot, \cdot \rangle_p \). The function \( \mu(\theta) \) is often called a potential function in the context of information geometry. Since the second derivative \( \mu''(\theta) \) is the Fisher information \( J_\theta \geq 0 \), the moment function \( \mu(\theta) \) is a convex function. Therefore, the first derivative \( \mu'(\theta) = \sum_\omega p_\theta(\omega)X(\omega) \) is monotone increasing. That is, we may regard it as another parameter identifying the distribution \( \theta \), and denote it by \( \eta \). The original parameter \( \theta \) is called a natural parameter and the other parameter \( \eta \) is an expectation parameter. For example, in the binomial distribution, the parameterization \( p_\theta(0) = 1/(1 + e^{\theta}) \), \( p_\theta(1) = e^{\theta}/(1 + e^{\theta}) \) is the natural parameter, and the parameterization \( p_\eta(0) = \eta \), \( p_\eta(1) = 1 - \eta \) is the expectation parameter. Hence, the binomial distribution is an exponential family.

Further, let \( X_1(\omega), \ldots, X_k(\omega) \) be \( k \) random variables. We can define a \( k \)-parameter exponential family

\[
p_\theta(\omega) \overset{\text{def}}{=} p(\omega)e^{\sum_i \theta^iX_i(\omega) - \mu(\theta)},
\]

\[
\mu(\theta) \overset{\text{def}}{=} \log \sum_\omega p(\omega)e^{\sum_i \theta^iX_i(\omega)}.
\] (3)

The parameters \( \theta^i \) are natural parameters, and the other parameters \( \eta_i \overset{\text{def}}{=} \frac{\partial \mu}{\partial \theta^i} = \sum_\omega p_\theta(\omega)X_i(\omega) \) are expectation parameters. Since the second derivative \( \frac{\partial^2 \mu(\theta)}{\partial \theta^i \partial \theta^j} \) is equal to the Fisher Information matrix \( J_{\theta i,j} \), the moment function \( \mu(\theta) \) is a convex function.

Let \( \mu(\theta) \) be a twice-differentiable and strictly convex function defined on a subset of the \( d \)-dimensional real vector space \( \mathbb{R}^d \). The divergence concerning the convex function \( \mu \) is defined by

\[
D^\mu(\tilde{\theta}||\theta) \overset{\text{def}}{=} \sum_i \eta_i(\tilde{\theta})(\tilde{\theta}^i - \theta^i) - \mu(\tilde{\theta}) + \mu(\theta),
\]

\[
\eta_i(\theta) \overset{\text{def}}{=} \frac{\partial \mu}{\partial \theta^i}(\theta).
\] (4)

This quantity has the following two characterizations:

\[
D^\mu(\tilde{\theta}||\theta) = \max_{\theta} \frac{\partial \mu}{\partial \theta^i}(\tilde{\theta})(\tilde{\theta}^i - \theta^i) - \mu(\tilde{\theta}) + \mu(\theta)
\]

\[
= \int_0^1 \sum_{i,j} (\tilde{\theta}^i - \theta^i)(\tilde{\theta}^j - \theta^j) \frac{\partial^2 \mu}{\partial \theta^i \partial \theta^j}(\theta + (\tilde{\theta} - \theta)t)tdt.
\] (5)

In the one-parameter case, we obtain

\[
D^\mu(\tilde{\theta}||\theta) = \mu'(\tilde{\theta})(\tilde{\theta} - \theta) - \mu(\tilde{\theta}) + \mu(\theta)
\]

\[
= \max_\theta \mu'(\tilde{\theta})(\tilde{\theta} - \theta) - \mu(\tilde{\theta}) + \mu(\theta) = \int_\theta^\tilde{\theta} \mu''(\tilde{\theta})(\tilde{\theta} - \theta)d\tilde{\theta}.
\] (6)
Since the function \( \mu \) is strictly convex, the correspondence \( \theta^i \leftrightarrow \eta_i = \frac{\partial \mu}{\partial \theta^i} \) is one-to-one. Hence, the divergence \( D^\mu(\tilde{\theta}||\theta) \) can be expressed with the parameter \( \eta \). For this purpose, we define the Legendre transform \( \nu \)

\[
\nu(\eta) \overset{\text{def}}{=} \max_{\tilde{\theta}} \sum_i \eta_i \tilde{\theta}^i - \mu(\tilde{\theta}).
\]

Then, the function \( \nu \) is a convex function, and we can recover the function \( \mu \) and \( \theta \) as

\[
\mu(\theta) = \max_{\eta} \sum_i \theta_i \eta^i - \nu(\eta), \quad \theta^i = \frac{\partial \nu}{\partial \eta^i}.
\]

The second derivative matrix \( \frac{\partial^2 \nu}{\partial \eta \partial \eta} \) of \( \nu \) is equal to the inverse of the matrix \( \frac{\partial^2 \mu}{\partial \theta \partial \theta} \).

In particular, when \( \eta_i = \frac{\partial \mu}{\partial \theta^i}(\theta) \),

\[
\nu(\eta) = \sum_i \eta_i \theta^i - \mu(\theta) = D^\mu(\theta||0) - \mu(0),
\]

\[
\mu(\theta) = \sum_i \theta_i \eta^i - \nu(\eta) = D^\nu(\eta||0) - \nu(0).
\]

Using this relation, we can characterize the divergence concerning the convex function \( \mu \) by the divergence concerning the convex function \( \nu \) as

\[
D^\mu(\tilde{\theta}||\theta) = D^\nu(\eta||\tilde{\eta}) = \sum_i \theta^i(\eta_i - \tilde{\eta}_i) - \nu(\eta) + \nu(\tilde{\eta}).
\]

Now, we apply the discussion about the divergence to a multi-parametric exponential family \( \{p_\theta||\theta \in \mathbb{R}\} \) defined in \([9, 11]\). Then,

\[
D(p_{\tilde{\theta}}||p_\theta) = D^\mu(\tilde{\theta}||\theta) = \sum_i \eta_i(\tilde{\theta}^i - \theta^i) - \mu(\theta) + \mu(\tilde{\theta}).
\]

In particular, applying \([11]\) to a one-parameter exponential family \([11]\), we have

\[
D(p_{\tilde{\theta}}||p_\theta) = D(p_{\eta(\theta)+\epsilon}||p_{\eta(\theta)}) = (\tilde{\theta} - \theta)\eta(\tilde{\theta}) - \mu(\tilde{\theta}) + \mu(\theta)
\]

\[
= \int_{\theta}^{\tilde{\theta}} J_\eta(\tilde{\theta} - \theta) d\theta = \max_{\theta, \tilde{\theta} \geq \theta} (\tilde{\theta} - \theta)(\eta(\tilde{\theta}) + \epsilon) - \mu(\tilde{\theta}) + \mu(\theta).
\]

In the following, we consider the case where \( p \) is the uniform distribution \( p_{\text{mix}} \). Let the random variables \( X_1(\omega), \ldots, X_k(\omega) \) be a CONS of the space of random variables with expectation 0 under the uniform distribution \( p_{\text{mix}} \), and \( Y^1(\omega), \ldots, Y^k(\omega) \) be its dual basis satisfying \( \sum_{i} Y^i(\omega)X_j(\omega) = \delta^i_j \). Then, any distribution can be parameterized by the expectation parameter as

\[
p_{\eta(\theta)}(\omega) = p_{\text{mix}}(\omega) + \sum_i \eta_i(\theta)Y^i(\omega).
\]

From \([10, 9]\),

\[
D(p_{\eta}||p_{\eta}) = D^\nu(\eta||\tilde{\eta}) = \frac{\partial \nu}{\partial \eta^i}(\eta_i - \tilde{\eta}_i) - \nu(\eta) + \nu(\tilde{\eta})
\]

\[
\nu(\eta) = D(p_{\eta}||p_{\text{mix}}) = -H(p_{\eta}) + H(p_{\text{mix}})
\]
because $\mu(0) = 0$. The second derivative matrix of $\nu$ is the inverse of the second derivative matrix of $\mu$, i.e., the Fisher information matrix concerning the natural parameter $\theta$. That is, the second derivative matrix of $\nu$ coincides with the Fisher information matrix concerning the expectation parameter $\eta$. Hence, applying (2) to the subspace $\{(1 - t)p + tq | 0 \leq t \leq 1\}$, we have

$$D(p\|q) = \int_0^1 J_t dt, \quad (12)$$

where $J_t$ is the Fisher information concerning the parameter $t$.

3 Inner Products in Quantum Systems

In this section, in order to define the quantum analogues of divergence, we define as inner products in quantum systems. There are at least three possible ways of defining the product corresponding to $X\rho$:

$$E_{\rho,s}(X) \overset{\text{def}}{=} X \circ \rho \overset{\text{def}}{=} \frac{1}{2} (\rho X + X \rho), \quad (13)$$

$$E_{\rho,b}(X) \overset{\text{def}}{=} \int_0^1 \rho^\lambda X \rho^{1-\lambda} d\lambda,$$

$$E_{\rho,r}(X) \overset{\text{def}}{=} \rho X. \quad (14)$$

Here, $X$ is not necessarily Hermitian. These extensions are unified in the general form

$$E_{\rho,p}(X) \overset{\text{def}}{=} \int_0^1 E_{\rho,\lambda}(X)p(d\lambda), \quad (15)$$

$$E_{\rho,\lambda}(X) \overset{\text{def}}{=} \rho^\lambda X \rho^{1-\lambda}, \quad (16)$$

where $p$ is an arbitrary probability distribution on $[0, 1]$. The case (13) corresponds to the case (15) with $p(1) = p(0) = 1/2$, and the case (14) does to the case (15) with $p(1) = 1$. In particular, the map $E_{\rho,x}$ is symmetric, when $E_{\rho,x}(X)$ is Hermitian if and only if $X$ is Hermitian. Hence, when the distribution $p$ is symmetric, i.e., $p(x) = p(1 - x)$, the map $E_{\rho,p}$ is symmetric. When $\rho > 0$, these maps possess inverses.

Accordingly, we may define these types of inner products

$$\langle Y, X \rangle_{(e)}(\rho, x) \overset{\text{def}}{=} \text{Tr} Y^* E_{\rho,x}(X) \quad x = s, b, r, \lambda, p.$$ 

If $X, Y, \rho$ all commute, these have the same value. These are called the SLD, Bogoljubov, RLD, $\lambda$, and $p$ inner products\cite{6, 3, 1, 9}, respectively (reasons for this will be given in the next section). These inner products are positive semi-definite and Hermitian, i.e.,

$$\left(\|X\|_{(e)}(\rho, x)\right)^2 \overset{\text{def}}{=} \langle X, X \rangle_{(e)}(\rho, x) \geq 0, \quad \langle Y, X \rangle_{(e)}(\rho, x) = \langle X, Y \rangle_{(e)}(\rho, x)^*.$$ 

A dual inner product may be defined $\langle A, B \rangle_{(m)}(\rho, x) \overset{\text{def}}{=} \text{Tr}(E_{\rho,x}^{-1}(A))^* B$ with respect to the correspondence $A = E_{\rho,x}(X)$. Denote the norm of these inner products as $\left(\|A\|_{(m)}(\rho, x)\right)^2 \overset{\text{def}}{=} \langle A, A \rangle_{(m)}(\rho, x)$.

The Bogoljubov inner product is also called the canonical correlation in statistical mechanics. In linear response theory, it is often used to give an approximate correlation between two different physical quantities.
Hence, the inner product $\langle A, B \rangle_{\rho, x}^{(m)}$ is positive semi-definite and Hermitian. Using this inner product, we define quantum analogues of Fisher information as

$$J_{\theta_0, x}^{\text{def}} = \left( \frac{d \rho_{\theta}}{d \theta} (\theta_0) \right)_{(m)}^{(\rho, \theta_0, x)}^2$$

for a one-parameter family $\{\rho_\theta\}$ and $x = s, r, b, \lambda, p$.

4 Autoparallel Curves in Quantum Systems

Next, we define parallel transport and autoparallel curves in quantum systems according to Nagaoka [2] and Amari & Nagaoka [1]. To introduce the concept of a parallel transport, consider an infinitesimal displacement in a one-parameter quantum state family $\{\rho_\theta|\theta \in \mathbb{R}\}$. The difference between $\rho_{\theta + \epsilon}$ and $\rho_\theta$ approximately equals to $\frac{d \rho_{\theta}}{d \theta} (\theta) \epsilon$. Hence, the state $\rho_{\theta + \epsilon}$ can be regarded as the state transported from the state $\rho_\theta$ in the direction $\frac{d \rho_{\theta}}{d \theta} (\theta)$ by an amount $\epsilon$. However, if the state $\rho_{\theta + \epsilon}$ coincides precisely with the state displaced from the state $\rho_\theta$ by $\epsilon$ in the direction of $\frac{d \rho_{\theta}}{d \theta} (\theta)$, the infinitesimal displacement at the intermediate states $\rho_{\theta + \epsilon'}$ ($0 < \epsilon' < \epsilon$) must equal the infinitesimal displacement $\frac{d \rho_{\theta}}{d \theta} (\theta) \Delta$ at $\theta$. Then, the problem is to ascertain which infinitesimal displacement at the point $\theta + \epsilon'$ corresponds to the given infinitesimal displacement $\frac{d \rho_{\theta}}{d \theta} (\theta) \Delta$ at the initial point $\theta$. The rule for matching the infinitesimal displacement at one point to the infinitesimal displacement at another point is called parallel transport. The coefficient $\frac{d \rho_{\theta}}{d \theta} (\theta)$ of the infinitesimal displacement at $\theta$ is called the tangent vector, as it represents the slope of the tangent line of the state family $\{\rho_\theta|\theta \in \mathbb{R}\}$ at $\theta$. Therefore, we can consider the parallel transport of a tangent vector instead of the parallel transport of an infinitesimal displacement.

Commonly used parallel transports can be classified into those based on the $m$ representation (parallel translation) and those based on the $e$ representation (parallel translation). The $m$ parallel translation $\Pi_{\rho_0, \rho_0'}^{(m)}$ moves the tangent vector at one point $\rho_0$ to the tangent vector with the same $m$ representation at another point $\rho_0'$. On the other hand, the $e$ parallel translation $\Pi_{x, \rho_0, \rho_0'}^{(e)}$ moves the tangent vector at one point $\rho_0$ with the $e$ representation $L$ to the tangent vector at another point $\rho_0'$ with the $e$ representation $L - \text{Tr} \rho_0 L$. Of course, this definition requires the coincidence between the set of $e$ representations at the point $\theta$ and that at another point $\theta'$. Hence, this type of $e$ parallel translation is defined only for the symmetric inner product $\langle X, Y \rangle_{\rho_0}^{(e)}$, and its definition depends on the choice of the metric. Indeed, the $e$ parallel translation can be regarded as the dual parallel translation of the $m$ parallel translation concerning the metric $\langle X, Y \rangle_{\rho_0}^{(e)}$ in the following sense:

$$\text{Tr} X^* \Pi_{\rho_0, \rho_0'}^{(m)} (A) = \text{Tr} \Pi_{x, \rho_0', \rho_0}^{(e)} (X)^* A,$$

where $X$ is the $e$ representation of a tangent vector at $\rho_0'$ and $A$ is the $m$ representation of another tangent vector at $\rho_0$.

Further, a one-parameter quantum state family is called a geodesic or an autoparallel curve when the tangent vector (i.e. the derivative) at each point is given as a parallel transport of a tangent vector at a fixed point. Especially, the $e$ geodesic is called a one-parameter exponential family.

For example, in an $e$ geodesic with respect to SLD $\{\rho_\theta|\theta \in \mathbb{R}\}$, any state $\rho_\theta$ coincides with the state transported from the state $\rho_{\theta_0}$ along the autoparallel curve in the direction...
L by an amount θ, where L denotes the SLD e representation of the derivative at ρ₀. We shall henceforth denote the state as Π^θ_{L, s}ρ₀. Similarly, Π^θ_{L, b}ρ₀ denotes the state transported autoparallelly with respect to the Bogoljubov e representation from ρ₀ in the direction L by an amount θ.

When the given metric is not symmetric, the e parallel translation moves the tangent vector at one point θ under the e representation L to the tangent vector at another point θ' with the e representation L' - Tr ρ₀ L' with the condition L + L* = L' + (L')*. That is, we require the same Hermitian part in the e representation. Hence, the e parallel translation Π^{(e)}_{x, ρ₀, ρ₀'} coincides with the e parallel translation Π^{(e)}_{s(x), ρ₀, ρ₀'} with regard to its symmetrized inner product. Therefore, we can define the state transported from the state ρ₀ along the autoparallel curve in the direction with the Hermitian part L by an amount θ with respect to RLD (λ, p), and denote them by Π^θ_{L, r}ρ₀ (Π^θ_{L, λ}ρ₀, Π^θ_{L, p}ρ₀), respectively. However, only the SLD one-parameter exponential family \{Π^θ_{L, s}ρ₀ | s ∈ ℝ\} plays an important role in quantum estimation examined in the next section.

**Lemma 1** Π^θ_{L, s}σ, Π^θ_{L, r}σ, Π^θ_{L, p}σ and Π^θ_{L, x}σ may be written in the following form [3, 2, 1]:

\[
\begin{align*}
Π^θ_{L, s}σ &= e^{-μ_s(θ)} e^{\frac{θ}{2} L} σ e^{\frac{θ}{2} L}, \\
Π^θ_{L, r}σ &= e^{-μ_r(θ)} e^{θ L} \sqrt{σ}, \\
Π^θ_{L, p}σ &= e^{-μ_p(θ)} e^{-θ L} \sqrt{σ}, \\
Π^θ_{L, x}σ &= e^{-μ_{1/2}(θ)} e^{\frac{θ}{2} L} σ e^{\frac{θ}{2} L}.
\end{align*}
\]

where we choose Hermitian matrices L, as L = \(\frac{1}{4}(σ^{-\frac{1}{2}} L r σ^{-\frac{1}{2}} + σ\frac{1}{2} L r σ^{-\frac{1}{2}})\) and L = \(\frac{1}{2}(σ^{-\frac{1}{2}} L σ^{-\frac{1}{2}} + σ\frac{1}{2} L σ^{-\frac{1}{2}})\), respectively, and

\[
\begin{align*}
μ_s(θ) &\equiv \log \text{Tr} e^{\frac{θ}{2} L} σ e^{\frac{θ}{2} L}, \\
μ_r(θ) &\equiv \log \text{Tr} e^{θ L} \sqrt{σ}, \\
μ_p(θ) &\equiv \log \text{Tr} e^{-θ L} \sqrt{σ}, \\
μ_{1/2}(θ) &\equiv \log \text{Tr} e^{\frac{θ}{2} L} σ e^{\frac{θ}{2} L}.
\end{align*}
\]

**Proof.** Taking the derivative of the RHS of [17] and [18], we see that the SLD (or Bogoljubov) e representation of the derivative at each point is equal to the parallel transported e representation of the derivative L at σ. In the RHS of [19], the RLD e representation of the derivative at each point is equal to the parallel transported e representation of the derivative √σ⁻¹ Lr √σ at σ. Further, in the RHS [20], the \(\frac{1}{2}\) e representation of the derivative at each point is equal to the parallel transported e representation of the derivative Lr at σ.

Conversely, from the definition of Π^θ_{L, x}σ, we have

\[
\frac{dΠ^θ_{L, x}σ}{dθ} = E_{ρ₀, x}(L - Tr L ρ₀), \quad x = s, r, \frac{1}{2}.
\]

Since this has only one variable, this is actually an ordinary differential equation. From the uniqueness of the solution of an ordinary differential equation, the only Π^θ_{L, x}σ satisfying
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\[ \Pi_{L,x}^0 \sigma = \sigma \] is the one given above. Since any \( e \) representation \( \sigma \) has the form \( \sqrt{\sigma}^{-1} L \sqrt{\sigma} \) with a Hermitian matrix \( L \), we only discuss \( \rho = \Pi_{L,x}^0 \sqrt{\sigma}^{-1} L \sqrt{\sigma} \). Taking its derivative, we have

\[
\frac{d}{d\theta} \Pi_{L,x}^0 \sqrt{\sigma}^{-1} L \sqrt{\sigma} = \rho_\theta \left( \sqrt{\sigma}^{-1} L \sqrt{\sigma} - \text{Tr} \rho_\theta \sqrt{\sigma}^{-1} L \sqrt{\sigma} \right).
\]

Similarly, from the uniqueness of the solution of an ordinary differential equation, only the state family \( \Pi_{L,x}^0 \sigma \) satisfies this condition.

5 Non-Geometrical Characterization of Divergences in Quantum Systems

First, we briefly characterize quantum analogues of divergence from the non-geometrical viewpoint. A quantity \( \tilde{D}(\rho \parallel \sigma) \) can be regarded as a quantum version of divergence if any commutative states \( \rho \) and \( \sigma \) satisfy

\[
\tilde{D}(\rho \parallel \sigma) = D(p \parallel \tilde{p}),
\]

where \( p \) and \( \tilde{p} \) is the probability distribution consisting of the eigenvalues of \( \rho \) and \( \sigma \). If a relative entropy \( \tilde{D}(\rho \parallel \sigma) \) satisfies the monotonicity for a POVM \( M = \{M_i\} \):

\[
\tilde{D}(\rho \parallel \sigma) \geq D(P_M^\rho \parallel P_M^\sigma), \quad P_M^\rho(i) \overset{\text{def}}{=} \text{Tr} \rho M_i
\]

and the additivity

\[
\tilde{D}(\rho_1 \otimes \rho_2 \parallel \sigma_1 \otimes \sigma_2) = \tilde{D}(\rho_1 \parallel \sigma_1) + \tilde{D}(\rho_2 \parallel \sigma_2),
\]

then Hiai & Petz [5]'s result yields the relation

\[
\tilde{D}(\rho \parallel \sigma) = \lim_{n \to \infty} \frac{D(\rho^\otimes n \parallel \sigma^\otimes n)}{n} \geq \lim_{M} \sup_{n} \frac{D(P_M^\rho^\otimes n \parallel P_M^\sigma^\otimes n)}{n} = D(\rho \parallel \sigma).
\]

That is, the quantum relative entropy \( D(\rho \parallel \sigma) \) is the minimum quantum analogue of relative entropy with the monotonicity for measurement and the additivity.

Further, Hiai & Petz [5] showed the inequality

\[
D(\rho \parallel \sigma) \leq \tilde{D}(\rho \parallel \sigma) \overset{\text{def}}{=} \text{Tr} \rho \log (\rho^{\frac{1}{2}} \sigma^{-1} \rho^{\frac{1}{2}}).
\]

6 Quantum Path-divergences Based on e-Parallel Translation

Now, using the concept of the exponential family, we extend the path-divergence based on the first equation in (14). For any two states \( \rho \) and \( \sigma \), we choose the Hermitian matrix \( L \) such that the exponential family \( \{\Pi_{L,x}^\theta \sigma\}_{\theta \in [0,1]} \) concerning the inner product \( J_{\theta,x} \) satisfies

\[
\Pi_{L,x}^1 \sigma = \rho.
\]

Then, we define the \( x-e \)-divergence as follows:

\[
D_e(x)(\rho \parallel \sigma) = \int_0^1 J_{\theta,x} \theta d\theta,
\]
i.e., the $e$-divergence satisfies the additivity for any inner product.

**Theorem 1** When

$$
L = \begin{cases} 
2 \log \sigma^{-1/2}(\sigma^{1/2}\rho\sigma^{1/2})^{1/2}\sigma^{-1/2} & \text{for } x = s \\
\frac{1}{2} \sigma^{-1/2} \log(\sigma^{-1/2}\rho\sigma^{-1/2})\sigma^{-1/2} & \text{for } x = r \\
\sigma^{-1/2} \log(\sigma^{-1/2}\rho\sigma^{-1/2})\sigma^{-1/2} & \text{for } x = \frac{1}{2} \\
\sigma^{-1/2} \log(\sigma^{-1/2}\rho\sigma^{-1/2})\sigma^{-1/2} & \text{for } x = \frac{1}{2}
\end{cases}
$$

(30)

the condition (27) holds. Hence, we obtain

$$
D_x^{(c)}(\rho||\sigma) = 2 \Tr \rho \log \sigma^{-1/2}(\sigma^{1/2}\rho\sigma^{1/2})^{1/2}\sigma^{-1/2}
$$

(31)

$$
D_b^{(c)}(\rho||\sigma) = \Tr(\rho\log\rho - \log\sigma) = D(\rho||\sigma)
$$

(32)

$$
D_r^{(c)}(\rho||\sigma) = \Tr(\rho\log(\sigma^{-1/2}\rho\sigma^{-1/2})\rho^{-1/2}) = D(\rho||\sigma)
$$

(33)

$$
D_{\frac{1}{2}}^{(c)}(\rho||\sigma) = 2 \Tr(\sigma^{1/2}\rho^{1/2}\sigma^{1/2})(\sigma^{-1/2}\rho^{1/2}\sigma^{-1/2}) \log(\sigma^{-1/2}\rho^{1/2}\sigma^{-1/2}).
$$

(34)

Nagaoka [2] obtained the above results for $x = s, b$.  

**Proof.** When we substitute (30) into $L$, condition (27) can be checked by using Lemma [1]. In this case, $L_r = \log(\sigma^{-1/2}\rho\sigma^{-1/2})$, $L_{\frac{1}{2}} = 2 \log(\sigma^{-1/2}\rho^{1/2}\sigma^{-1/2})$, and we can show that

$$
\frac{d^2 \mu_x(\theta)}{d\theta^2} = J_{\theta, x}.
$$

(35)

Hence, from a discussion similar to (11), we can prove that

$$
D_x^{(c)}(\rho||\sigma) = \frac{d\mu_x(\theta)}{d\theta} \bigg|_{\theta = 1} (1 - \mu_x(1) + \mu_x(0) = \frac{d\mu_x(\theta)}{d\theta} \bigg|_{\theta = 1},
$$

(36)

where $\mu_x(\theta)$ is defined in Theorem 1. Using this relation, we can check (31), (32), and (33). Concerning (33), we obtain

$$
D_r^{(c)}(\rho||\sigma) = \Tr(\rho\sigma^{-1/2}\rho\sigma^{-1/2}) \log(\sigma^{-1/2}\rho\sigma^{-1/2}) = \Tr(\rho\log(\rho\sigma^{-1}\rho^{1/2}),
$$

where the last equation follows from the equation with $AU = \sigma^{-1/2}\rho^{1/2}$ ($A$ is Hermitian and $U$ is unitary):

$$
AUU^*A \log(AUU^*) = AU \log(U^*AAU)U^*A.
$$

□

Now, we compare these quantum analogues of relative entropy given in (31)–(33). As is easily checked, these satisfy the condition (22) for quantum analogues of relative entropy. Let $M$
be a measurement corresponding to the spectral decomposition of $\sigma^{-1/2}(\sigma^{1/2} \rho \sigma^{1/2})^{1/2} \sigma^{-1/2}$. This PVM $M$ satisfies that $D^{(c)}(\rho||\sigma) = D(P^M_\rho || P^M_\sigma)$. Thus, from the monotonicity for measurement concerning the quantum relative entropy $D(\rho||\sigma)$,

$$D(\rho||\sigma) \geq D^{(c)}_s(\rho||\sigma) = 2 \text{Tr} \rho \log \sigma^{-\frac{1}{2}}(\sigma^{\frac{1}{2}} \rho \sigma^{\frac{1}{2}})^{\frac{1}{2}} \sigma^{-\frac{1}{2}}. \quad (37)$$

From (36),

$$D(\rho||\sigma) \leq D^{(c)}_r(\rho||\sigma) = \text{Tr} \rho \log(\rho^{\frac{1}{2}} \sigma^{-1} \rho^{\frac{1}{2}}). \quad (38)$$

Hence, from the inequality (36) and the additivity (29), $D^{(c)}(\rho||\sigma)$ and $D^{(c)}_r(\rho||\sigma)$ do not satisfy the monotonicity even for measurements because the equality in (37) and (38) does not always hold.

7 Quantum Path-divergences Based on $m$-Parallel Translation

Further, we can extend the path-divergence based on the equation (12). For any two states $\rho$ and $\sigma$, the family $\{(1-t)\rho + t\sigma|0 \leq t \leq 1\}$ is the $m$ geodesic joining $\rho$ and $\sigma$. Hence, as an extension of (12), we can define the $x$-$m$ divergence as

$$D^{(m)}_x(\rho||\sigma) \overset{\text{def}}{=} \int_0^1 J_{t,x,t}dt. \quad (39)$$

Since the family $\{(1-t)\kappa(\rho) + t\kappa(\sigma)|0 \leq t \leq 1\}$ is the $m$ geodesic joining $\kappa(\rho)$ and $\kappa(\sigma)$ for any TP-CP map $\kappa$, we have

$$D_x^{(m)}(\rho||\sigma) \geq D^{(m)}_x(\kappa(\rho)||\kappa(\sigma)), \quad (40)$$

i.e., the $m$ divergence satisfies the monotonicity. Since the RLD is the largest inner product,

$$D^{(m)}_r(\rho||\sigma) \geq D^{(m)}_x(\rho||\sigma). \quad (41)$$

We can calculate the $m$ divergence as

$$D^{(m)}_x(\rho||\sigma) = \text{Tr} \rho(\log \rho - \log \sigma) = D(\rho||\sigma) \quad (42)$$

$$D^{(m)}_r(\rho||\sigma) = \text{Tr} \rho \log(\sqrt{\rho} \sigma^{-1} \sqrt{\rho}) = D(\rho||\sigma). \quad (43)$$

In fact, The Bogoljubov case (12) has been obtained by Nagaoka (10), and follows from Theorem 2. Hence, $\text{Tr} \rho \log(\sqrt{\rho} \sigma^{-1} \sqrt{\rho}) = D^{(m)}_r(\rho||\sigma)$ satisfies the monotonicity for TP-CP maps. Also, from (44), we obtain $\text{Tr} \rho \log(\sqrt{\rho} \sigma^{-1} \sqrt{\rho}) \geq D(\rho||\sigma). \quad (44)$

Further, all of $x$-$m$ divergences do not necessarily satisfy the additivity (29). At least, when the inner product $J_{x,\theta}$ is smaller than the Bogoljubov inner product $J_{0,\theta}$, i.e., $J_{0,x} \leq J_{\theta,b}$, we have $D(\rho||\sigma) \geq D^{(m)}_x(\rho||\sigma)$. From (26) and the monotonicity (10), $D^{(m)}_x(\rho||\sigma)$ does not satisfy the additivity (29). For example, SLD $m$ divergence does not satisfy the additivity (29).

We can now verify whether it is possible in two-parameter state families to have states that are $e$ autoparallel transported in the direction of $L_1$ by $\theta^1$, and in the direction $L_2$ by $\theta^2$.

In order to define such a state, we require that the state that is $e$ autoparallel transported first in the $L_1$ direction by $\theta^1$ from $\rho_0$, then further $e$ autoparallel transported in the $L_2$ direction
by \( \theta^2 \) coincides with the state that is \( e \) autoparallel transported in the \( L_2 \) direction by \( \theta^2 \) from \( \rho_0 \), then \( e \) autoparallel transported in the \( L_1 \) direction by \( \theta^1 \). That is, if such a state would be defined, the relation

\[
\Pi_{L_2,x}^{\theta^2} \Pi_{L_1,x}^{\theta^1} \sigma = \Pi_{L_1,x}^{\theta^1} \Pi_{L_2,x}^{\theta^2} \sigma
\]  

(44)

should hold. Concerning this condition, we have the following theorem.

**Theorem 2** The following conditions for the inner product \( J_{\theta,x} \) are equivalent

1. \( J_{\theta,x} \) is the Bogoljubov inner product, i.e., \( x = b \).
2. The condition \( (43) \) holds for any two Hermitian matrices \( L_1 \) and \( L_2 \) and any state \( \rho_0 \).
3. \( D_x^{(r)}(\rho_0||\rho_0) = D^{(r)}(\tilde{\theta}||\theta) \).
4. \( D_x^{(r)}(\rho||\sigma) = D(\rho||\sigma) \).
5. \( D_x^{(m)}(\rho_\eta||\rho_\eta) = D^{(m)}(\eta||\eta) \).
6. \( D_x^{(m)}(\rho||\sigma) = D(\rho||\sigma) \).

Here, the convex functions \( \mu(\theta), \nu(\eta) \) and the states \( \rho_0, \rho_\eta \) are defined by

\[
\rho_\theta \triangleq \exp(\sum_i \theta^i X_i - \mu(\theta)), \\
\mu(\theta) \triangleq \log \text{Tr} e^{\sum_i \theta^i X_i}, \\
\rho_\eta = \rho_{\text{mix}} + \sum_j \eta_j X_j, \\
\nu(\eta) \triangleq D_2^{(m)}(\rho_0||\rho_\eta) = -H(\rho_\eta) + H(\rho_{\text{mix}}),
\]

where \( X_1, \ldots, X_k \) is a basis of the set of traceless Hermitian matrices, and \( Y^1, \ldots, Y^k \) is its dual basis.

This theorem implies that only the quantum path-divergence based on the Bogoljubov Fisher information can be characterized by the convex function among quantum path-divergence based on \( m \)-parallel translation.

**Proof.** First, we prove that \( \circlearrowleft \Rightarrow \circlearrowright \). Theorem 1 guarantees that Bogoljubov \( e \) autoparallel transport satisfies

\[
\Pi_{L_2,b}^{\theta^2} \Pi_{L_1,b}^{\theta^1} \rho = \Pi_{L_1,b}^{\theta^1} \Pi_{L_2,b}^{\theta^2} \rho = e^{-\mu_b(\theta^1, \theta^2)} e^{\log \rho + \theta^1 L_1 + \theta^2 L_2},
\]

where \( \mu_b(\theta) \triangleq \log \text{Tr} e^{\log \rho + \theta^1 L_1 + \theta^2 L_2} \). Hence, we obtain \( \circlearrowright \).

Next, we prove that \( \circlearrowright \Rightarrow \circlearrowleft \). We define \( \tilde{\rho}_\theta \triangleq \Pi_{X_1,x}^{\theta^1} \cdots \Pi_{X_k,x}^{\theta^k} \rho_\eta \) for \( \theta = (\theta^1, \ldots, \theta^k) \). Then, the condition \( \circlearrowright \) guarantees that \( \tilde{\rho}_\theta = \Pi_{L_2,x}^{\theta^2} \Pi_{L_1,x}^{\theta^1} \tilde{\rho}_\theta \). In particular, when \( \theta = 0 \), we obtain \( \tilde{\rho}_0 = \Pi_{L_1,x}^{\theta^1} \tilde{\rho}_0 \). Since \( \sum_i \theta^i X_i \) is commutative with \( \rho_{\text{mix}} \), we can apply the classical observation to this case. Hence, the state \( \tilde{\rho}_0 \) coincides with the state \( \rho_\theta \) defined in (43).
Let $\tilde{X}_{\theta,j}$ be the $x$-$e$ representation of the partial derivative concerning $\theta^j$ at $\rho_\theta$. It can be expressed as

$$\tilde{X}_{\theta,j} = X_j - \text{Tr} \rho_\theta X_j + \bar{X}_{\theta,j},$$

where $\bar{X}_{\theta,j}$ is the skew-Hermitian part. Thus,

$$\frac{\partial \text{Tr} \rho_\theta X_j}{\partial \theta^i} = \text{Tr} \left( \frac{\partial \rho_\theta}{\partial \theta^i} X_j \right) = \text{Tr} \left( \frac{\partial \rho_\theta}{\partial \theta^i} (X_j - \text{Tr} \rho_\theta X_j) \right) = \text{Re} \text{Tr} \left( \frac{\partial \rho_\theta}{\partial \theta^i} (X_j - \text{Tr} \rho_\theta X_j + \bar{X}_{\theta,j}) \right) = \text{Re} J_{\theta,j,i}. $$

Note that the trace of the product of a Hermitian matrix and a skew-Hermitian matrix is an imaginary number. Since $\text{Re} J_{\theta,j,i} = \text{Re} J_{\theta,x;i,j}$, we have $\frac{\partial \text{Tr} \rho_\theta X_j}{\partial \theta^i} = \frac{\partial \text{Tr} \rho_\theta X_i}{\partial \theta^i}$. Thus, there exists a function $\bar{\mu}(\theta)$ such that $\bar{\mu}(0) = \mu(0)$ and

$$\frac{\partial \bar{\mu}(\theta)}{\partial \theta^i} = \text{Tr} \rho_\theta X_i.$$

This function $\bar{\mu}$ satisfies condition $\circled{3}$.

Moreover, since $\text{Tr} \rho_{\text{mix}} X_i = 0$, from the definition $\bar{\mu}$, we have $\bar{\mu}(\theta) - \bar{\mu}(0) = D^\theta(0||\theta)$. Since the state $\rho_{\text{mix}}$ commutes the state $\rho_\theta$, the relation $D(\rho||\rho_\theta) = \mu(\theta) - \mu(0)$ holds. Hence, we obtain $\bar{\mu}(\theta) = \mu(\theta)$.

Further, we have $D(\theta||\theta) = D(\rho||\theta)$. Thus, the equivalence between $\circled{3}$ and $\circled{4}$ is trivial since the limit of $D(\rho||\rho_\theta)$ equals the Bogoljubov inner product $J_{b,\theta}$. Hence, we obtain $\circled{4} \Rightarrow \circled{3}$.

Now, we proceed to the proof of $\circled{1} + \circled{2} \Rightarrow \circled{3} + \circled{4} \Rightarrow \circled{5}$. In this case, the function $\nu(\eta)$ coincides with the Legendre transform of $\mu(\theta)$, and $\eta_i = \frac{\partial \mu}{\partial \theta^i}(\theta)$. Hence, $D(\eta||\eta) = D(\theta||\theta) = D(\rho_{\text{mix}}||\rho_\theta)$. The second derivative matrix $\frac{\partial^2 \nu}{\partial \eta^j \partial \eta^i}$ coincides with the inverse of the second derivative matrix $\frac{\partial^2 \nu}{\partial \theta^j \partial \theta^i}$, which equals the Bogoljubov Fisher information matrix concerning the parameter $\theta$. Since the Bogoljubov Fisher information matrix concerning the parameter $\eta$ equals the inverse of the Bogoljubov Fisher information matrix concerning the parameter $\theta$, the Bogoljubov Fisher information matrix concerning the parameter $\eta$ coincides with the second derivative matrix $\frac{\partial^2 \nu}{\partial \eta^j \partial \eta^i}$. Hence, from $\circled{5}$, we have $D(\eta||\eta) = D(\rho_{\text{mix}}||\rho_\theta)$.

Next, we prove $\circled{5} \Rightarrow \circled{6}$. Since $\rho_{\text{mix}} = \rho_\theta$ commutes with $\rho_\eta$, the m divergence $D(\rho_{\text{mix}}||\rho_\eta)$ coincides with the Bogoljubov $m$ divergence $D(\rho_\theta||\rho_\eta)$, which equals the Legendre transform of $\mu(\theta)$, defined in $\circled{5}$. Thus, $D(\rho_{\text{mix}}||\rho_\eta) = D(\eta||\eta)$ holds. Finally, taking the limit $\eta \rightarrow \eta$, we obtain $J_{\eta,x} = J_{b,\eta}$, i.e., $\circled{6} \Rightarrow \circled{5}$.

8 Concluding Remark

In this paper, we proved the additivity of $e$-divergences and the monotonicity of $m$-divergences. We also found interesting relations between geometrical path-divergences and an operator-algebraic divergence as

$$D^{(e)}(\rho||\sigma) = D^{(m)}(\rho||\sigma) = D(\rho||\sigma).$$

In addition, we obtained the characterization of Bogoljubov inner product as Theorem $\mathbb{E}$ which is a generalization of Amari & Nagaoka $\mathbb{H}$’s characterization. It is expected that these characterizations are applied to quantum information.
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