Asymptotic behavior of solutions toward the strong contact discontinuity for compressible Navier-Stokes equations with Cauchy problem

Tingting Zheng ∗ †

Computer and Message Science College, Fujian Agriculture and Forest University,
Fuzhou 350001, P. R. China

Abstract. In this paper, we consider the nonisentropic ideal polytropic Navier-Stokes equations to the Cauchy problem. The asymptotic stability of contact discontinuity is established under the condition that the initial perturbations are partly small but the strength of contact discontinuity can be suitably large. With this conditions, the bounds of density and temperature can be obtained from the complicated structure of Navier-Stokes equations. The proofs are given by the elementary energy method.

AMS Subject Classifications (2000). 35B40, 35B45, 76N10, 76N17

Keywords: Cauchy problem, Compressible Navier-Stokes equations, Strong contact discontinuity, Asymptotic stability.

1 Introduction

This paper is concerned with the one-dimensional compressible viscous heat-conducting flows in the whole space $\mathbb{R} = (-\infty, +\infty)$, which is governed by the following initial value problem in Eulerian coordinate $(\tilde{x}, t)$:

\[
\begin{aligned}
\rho_t + (\rho \rho u)_x &= 0, \quad (\tilde{x}, t) \in \mathbb{R} \times \mathbb{R}_+, \\
(\rho u)_t + (\rho u^2 + p)_x &= \mu \rho u_{xx}, \\
\left( \rho \left( \tilde{e} + \frac{\tilde{u}^2}{2} \right) \right)_t + \left( \rho \rho u \left( \tilde{e} + \frac{\tilde{u}^2}{2} \right) + \tilde{p} \tilde{u} \right)_x &= \kappa \tilde{u} \tilde{u}_{xx} + (\rho \rho u_{xx})_x, \\
(\rho, \tilde{u}, \tilde{\theta})|_{t=0} &= (\rho_0, \tilde{u}_0, \tilde{\theta}_0)(\tilde{x}) \to (\rho_\pm, 0, \theta_\pm) \quad \text{as} \quad \tilde{x} \to \pm \infty,
\end{aligned}
\]

where $\rho$, $\tilde{u}$ and $\tilde{\theta}$ are the density, the velocity and the absolute temperature, respectively, while $\mu > 0$ is the viscosity coefficient and $\kappa > 0$ is the heat-conductivity coefficients, respectively. It is assumed throughout the paper that $\rho_\pm$ and $\theta_\pm$ are prescribed positive constants and $\mathbb{R}_+ = (0, +\infty)$, $\mathbb{R}_- = (-\infty, 0)$ with $\| (\rho_0 - \rho_\pm, \tilde{u}_0, \tilde{\theta}_0 - \theta_\pm) \|_{L^2(\mathbb{R}_\pm)}$ suitably small but permitting $|\rho_+ - \rho_-|$, $|\theta_+ - \theta_-|$ and $\| (\rho_{0\Omega}, \tilde{u}_{0\Omega}, \tilde{\theta}_{0\Omega}) \|_{L^2(\mathbb{R})}$ not small. We shall focus our interests on the case of

∗Corresponding author email: asting16@sohu.com(T.Zheng).
†This work was partially supported by the Youth Natural Science Foundation of Fujian Province, China (Grant No. 2017J05001).
viscous polytropic ideal gases, so that, the pressure \( \bar{p} = \bar{p}(\bar{\rho}, \bar{\theta}) \) and the internal energy \( \bar{e} = \bar{e}(\bar{\rho}, \bar{\theta}) \) are related by the second law of thermodynamics:

\[
\bar{p} = R \bar{\rho} \bar{\theta}, \quad \bar{e} = \frac{R}{\gamma - 1} \bar{\theta} + \text{const.,}
\]

where \( \gamma > 1 \) is the adiabatic exponent and \( R > 0 \) is the gas constant.

The purpose is to prove the solvability and stability of the problem (1.1) at \( t \in [0, +\infty) \).

As there is a local theorem of existence [1] and its references, the main difficulty in studying the problem in the whole is related to obtaining the a priori estimate, the constants in which depend only on the coefficients and the initial data. In this case the local solution can be extended onto the whole of the length \( [0, +\infty) \). When deducing global estimates, it is convenient to transform (1.1) to the problem in the Lagrangian coordinate and then make use of a coordinate transformation to reduce the initial value problem (1.1) into the following form:

\[
\begin{cases}
  v_t - u_x = 0, & (x, t) \in \mathbb{R} \times \mathbb{R}_+,
  \\
  u_t + \left( \frac{R \theta}{v} \right)_x = \mu \left( \frac{u_x}{v} \right),
  \\
  \frac{R}{\gamma - 1} \theta_t + R \frac{\theta}{v} u_x = \kappa \left( \frac{\theta_x}{v} \right) + \mu \frac{u^2}{v},
  \\
  (v, u, \theta)|_{t=0} = (v_0, u_0, \theta_0) \to (v_{\pm}, 0, \theta_{\pm}) \quad \text{as} \quad x \to \infty,
\end{cases}
\]

where \( v_{\pm} = \rho_{\pm}^{-1} \) and \( \theta_{\pm} \) are given positive constants, \( \|(v_{0x}, u_{0x}, \theta_{0x})\|_{L^2(\mathbb{R})} \) not small and \( v_0, \theta_0 > 0 \). Without loss of generality, we set \( 1 = \theta_+ > \theta_- > 0 \). Here \( v = v(x, t) \), \( u = u(x, t) \), \( \theta = \theta(x, t) \) are the specific volume, velocity and temperature as in (1.1).

Up to the present time, some deep results have been obtained on the asymptotic stability toward nonlinear waves, viscous shock profiles and viscous rarefaction, for quite general perturbation of the Navier-Stokes system (1.3) and general systems of viscous strictly hyperbolic conservation laws (see [2–12, 14, 16–19]). It was observed in [13, 15], where the metastability of contact waves was studied for viscous conservation laws with artificial viscosity dominates the large-time behavior of solutions. The nonlinear stability of contact discontinuity for the (full) compressible Navier-Stokes equations was then investigated in [21, 20] for the free boundary value problem and [23, 22] for the Cauchy problem. Recently, some problems are call stability of strong viscous waves (see [18, 27, 30]). These stability results are shown with some special conditions. Especially, zero dissipation result is shown in [26] and \( \gamma \to 1 \) in [18] or [25]. Base on small oscillation, initial smallness perturbation or zero dissipation (and so on), Navier-Stokes equations stability results can be obtained with special help. However to our best knowledge, there is no any mathematical literature known for the large-time behaviors of solutions to the general Cauchy problem (1.3) due to various difficulties. To conquer these difficulties, we find that the important crucial step is to improve the time estimates of \( (V_x, U_x, \Theta_x) \) in [29]. This step make the inequality of Lemma 1.1 be simpler than [24, 29], then when we use similar skills as [30], we can obtain our uniform estimates.

The main purpose of this paper is to justify this unknown problem, i.e., we will show that for a general initial value of the Cauchy problem (1.3), it is possible to be resolved and stable. Furthermore, the solution approximate the contact discontinuity \( (v_{\pm}, 0, \theta_{\pm}) \). To deduce the desired stability result by the elementary energy method, as describe in [1, 24] and [29], it is sufficient to deduce certain uniform (with respect to the time variable \( t \)) energy type
estimates on the solution \((v, u, \theta)\) and to establish the upper boundaries of \((v, v^{-1}, \theta, \theta^{-1})\), also the Poincaré type inequality in Lemma 4.1 without the smallness of \(|\theta_+ - \theta_-|\) is important, where the arguments employed in \([19, 21, 22, 23, 24, 25, 29]\) use both smallness \(|\theta_+ - \theta_-|\) and \(N(t) = \sup_{0 \leq r \leq t} \|(\varphi, \psi, \zeta)\|_{H^1}\) to overcome these difficulties.

Based on the analysis above, the remainder of this paper is organized as follows. In section 2, we construct a pair of viscous functions \((V, U, \Theta)(x, t)\) and check that it is nearly close to \((v_\pm, 0, \theta_\pm)\). In section 3, we reformulate the problem and give the precise statement of our main theorem. Finally, we complete the proof of the main results by the global a priori estimates established in Section 4.

Throughout this paper, we shall denote \(H^l(\omega)\) the usual \(l\)-th order Sobolev space with the norm \(\|f\|_{L^1(\omega)} = (\sum_{j=0}^l \|\partial^j_x f\|^2)^{1/2}, \| \cdot \| := \| \cdot \|_{L^2(\omega)}\).

For simplicity, we also use \(C \text{ or } C_i (i = 1, 2, 3, \ldots)\) to denote the various positive generic constants; \(C(\delta_0)\) or \(C_i(\delta_0) (i = 1, 2, 3, \ldots)\) to denote one small constant about \(\delta_0^q (q > 0)\). And \(\partial_x^i = \frac{\partial^i}{\partial x^i}, C_v = \frac{R}{\gamma - 1}\).

### 2 Preliminaries

In this section, to study the asymptotic behavior of the solution to the Cauchy problem (1.3), we provide some preliminary lemmas that are important for the proof of Theorem 3.1.

First of all, let \(\frac{v_+}{\theta_+} = v_\pm = \frac{v_+}{\theta_+}\) and

\[
P(V, \Theta) = R \frac{\Theta}{V} = p_+, \quad U(x, t) = \frac{\kappa(\gamma - 1)\Theta_x}{\gamma R \Theta}.
\]

Here \(\Theta(x, t) \ ((x, t) \in \mathbb{R} \times \mathbb{R}_+)\) is the solution of the following problem

\[
\begin{aligned}
\Theta_t &= a(\ln \Theta)_{xx}, \quad a = \frac{\kappa p_+(\gamma - 1)}{\gamma R^2} > 0, \\
\Theta(x, 0) &= \Theta_0(x) \to \theta_\pm, \\
\Theta_0(x) &= \left(\frac{1}{\sqrt{\pi}}(\theta_+^{1/\delta_0} - \theta_-^{1/\delta_0})\int_0^{\ln(x + \sqrt{1 + x^2})} \exp\{-y^2\}dy + \frac{\theta_-^{1/\delta_0} + \theta_+^{1/\delta_0}}{2}\right)^{\delta_0}.
\end{aligned}
\]

In this paper, we ask \(\delta_0\) is a suitably small positive constant and \(1/\delta_0\) is an integer.

According to the smallness \(\delta_0\), we can find that the properties of \(\Theta_0(x)\) can be listed as follows.

**Lemma 2.1**

\[
\begin{aligned}
\|\Theta_0x\|_{L^1(\mathbb{R})} &\leq C, \quad |\Theta_0x| \leq C\delta_0, \quad \|\Theta_0x\|^2 \leq C\delta_0^2, \quad \|\Theta_0 - \theta_\pm\|_{L^1(\mathbb{R}_+)} \leq C; \\
\|\ln(\Theta_0)_{xx}\|^2 &\leq C\delta_0^2, \quad \|\ln(\Theta_0)_{xxx}\|^2 \leq C.
\end{aligned}
\]
Proof. In fact, if $K(x) = \ln(x + \sqrt{1 + x^2})$, we can get
\[
\int_{\mathbb{R}_+} \exp\{-K^2(x)\} dx = \int_{\mathbb{R}_+} \exp\{-K^2(x)\} \frac{1 + x^2}{\sqrt{1 + x^2}} dx
\]
\[
= \int_{\mathbb{R}_+} \exp\{-K^2(x)\} \sqrt{1 + x^2} dK(x)
\]
\[
\leq \int_{\mathbb{R}_+} \exp\{-K^2(x)\} \exp\{|K(x)|\} dK(x)
\]
\[
\leq \int_{\mathbb{R}_+} \exp\{-K^2(x) + |K(x)|\} dK(x) \leq C.
\] (2.4)

Set
\[
H(x) = \Theta_0^{1/\delta_0} = \frac{\theta_+^{1/\delta_0} + \theta_-^{1/\delta_0}}{2} + \frac{\theta_+^{1/\delta_0} - \theta_-^{1/\delta_0}}{\sqrt{\pi}} \int_0^{K(x)} \exp\{-x^2\} dx,
\] (2.5)

from $K(x) = (1 + x^2)^{-1/2}$ and
\[
\Theta_{0x} = \delta_0 H^{\delta_0-1}(x) H_x(x) = \delta_0 H(x)^{\delta_0-1} \frac{\theta_+^{1/\delta_0} - \theta_-^{1/\delta_0}}{\sqrt{\pi}} K(x) \exp\{-K^2(x)\} > 0, \quad x \in \mathbb{R},
\] (2.6)

we can get \(\|V_{0x}\|_{L^1(\mathbb{R})} = \int_{\mathbb{R}} \delta_0 H^{\delta_0-1}(x) H_x(x) dx < C\).

When $x > 0$, $K(x) > 0$, we can know
\[
\frac{\theta_+^{1/\delta_0} - \theta_-^{1/\delta_0}}{H(x)} \leq C \frac{\theta_+^{1/\delta_0} - \theta_-^{1/\delta_0}}{\theta_+^{1/\delta_0} + \theta_-^{1/\delta_0}} \leq C.
\] (2.7)

From (2.5),
\[
\theta_-^{1/\delta_0} \leq H(x) \leq \theta_+^{1/\delta_0},
\] (2.8)

from (2.6) and $K(x) = \frac{1}{\sqrt{1 + x^2}}$, we can get $|\Theta_{0x}| \leq C \delta_0$. Also from (2.4), (2.5), (2.6) and (2.7) we can get $\|\Theta_{0x}\|_{L^2(\mathbb{R}_+)} \leq C \delta_0$. Similar as above estimates, it is easy to check $\|\Theta_{0xx}\|_{L^2(\mathbb{R}_+)} \leq C \delta_0^2$ and $\|\Theta_{0xxx}\|_{L^2(\mathbb{R}_+)} \leq C$.

When $\delta_0 = \frac{1}{2k + 1}$, $k \in \mathbb{N}$ is a suitably large constant, from the equality $a^n - b^n = (a - b) \sum_{i=0}^{n-1} a^{n-i} b^i$, $\forall a > 0$, $b > 0$, $n \in \mathbb{N}_+$ and (2.4), (2.6), (2.7), we can get that
\[
\int_{\mathbb{R}_+} |\Theta_0 - \theta_+| dx = \int_{\mathbb{R}_+} |H^{\delta_0} - \theta_+| dx
\]
\[
= \int_{\mathbb{R}_+} \frac{|H - \theta_+^{1/\delta_0}|}{\sum_{i=0}^{2k} H^{(2k-i)/(2k+1)} \theta_+^{i/(2k+1)}} dx
\]
\[
\leq C \int_{\mathbb{R}_+} \frac{(\theta_+^{1/\delta_0} - \theta_-^{1/\delta_0}) \exp\{-CK^2(x)\}}{\sum_{i=0}^{2k} H^{(2k-i)/(2k+1)} \theta_+^{i/(2k+1)}} dx
\]
\begin{equation}
\leq C\left(\theta_+^{1/\delta_0} - \theta_-^{1/\delta_0}\right) \sup_{x \in \mathbb{R}_+} H^{1/(2k+1)-1} \leq C(\theta_+ + \theta_-). \tag{2.9}
\end{equation}

When $x < 0$, $K(-x) = -K(x) > 0$. We can obtain from (2.5) that

\begin{equation}
H(x) = \Theta_0^{1/\delta_0} = \frac{\theta_+^{1/\delta_0} - \theta_-^{1/\delta_0}}{\sqrt{\pi}} \int_{-\infty}^{K(x)} \exp\{-y^2\} dy + \theta_-^{1/\delta_0}. \tag{2.10}
\end{equation}

From

\begin{equation}
\int_{-\infty}^{K(x)} \exp\{-x^2\} dx = \left(\int_{-\infty}^{\sqrt{2}\delta_0} \int_{-\infty}^{\sqrt{2}\delta_0} \exp\{-x^2 - y^2\} dx dy\right)^{1/2} \geq \sqrt{\pi}/2 \exp\{-r^2/2\} \geq \sqrt{\pi}/2 \exp\{-K^2(x)\}, \tag{2.11}
\end{equation}

we can obtain

\begin{equation}
H(x)^{-1}(\theta_+^{1/\delta_0} - \theta_-^{1/\delta_0}) \exp\{-K^2(x)\} \leq C. \tag{2.12}
\end{equation}

Combine with (2.4), (2.6), (2.8) and (2.12), we can get

\begin{equation}
0 < \Theta_0 x \leq C\delta_0 \pi^{-1/2} K_x(x) H^{\delta_0}, \tag{2.13}
\end{equation}

then

\begin{equation}
|\Theta_0 x| \leq C\delta_0, \quad \int_{\mathbb{R}_-} |\Theta_0 x|^2 dx \leq C\delta_0^2.
\end{equation}

Also

\begin{equation}
\|\Theta_0 xx\|_{L^2(\mathbb{R}_-)}^2 \leq C\delta_0^2, \quad \|\Theta_0 xx\|_{L^2(\mathbb{R}_-)}^2 \leq C.
\end{equation}

Because $1 = \theta_+ > \theta_- > 0$, similar as (2.9) we can get

\begin{equation}
\int_{\mathbb{R}_-} |\Theta_0 - \theta_-| dx = \int_{\mathbb{R}_+} |H^{\delta_0} - \theta_-| dx
\end{equation}

\begin{equation}
= \int_{\mathbb{R}_-} \frac{|H - \theta_{+}^{1/\delta_0}|}{\sum_{i=0}^{2k} H^{(2k-i)/(2k+1)} \theta_{i}^{(2k+1)}} dx
\end{equation}

\begin{equation}
\leq C \int_{\mathbb{R}_-} \left(\theta_+^{1/\delta_0} - \theta_-^{1/\delta_0}\right) \exp\{-3/4K^2(x)\} dx
\end{equation}

\begin{equation}
\leq \sum_{i=0}^{2k} H^{(2k-i)/(2k+1)} \theta_{i}^{(2k+1)} \leq C(\theta_+^{1/\delta_0} - \theta_-^{1/\delta_0}) \exp\{-1/2K^2(x)\} \sup_{x \in \mathbb{R}_-} (H\theta_-)^{1/2\delta_0 - 1/2}.
\end{equation}

Combine with (2.12) we can get

\begin{equation}
\int_{\mathbb{R}_-} |\Theta_0 - \theta_-| dx \leq C\sqrt{\theta_+^{1/\delta_0} - \theta_-^{1/\delta_0}} \leq C.
\end{equation}

So we can get

\begin{equation}
\int_{\mathbb{R}_-} |\Theta_0 - \theta_-| dx \leq C.
\end{equation}

We finish this lemma. □
In summary, from (2.1) and (2.2) we have constructed a pair of functions \((V, U, \Theta)\) satisfies
\[
\begin{align*}
R \frac{\Theta}{V} &= p_+, \\
V_t &= U_x, \\
U_t + (R\Theta/V)_x &= \mu \left( \frac{U_x}{V} \right) + F, \\
R \frac{\Theta_t + R \Theta}{V} U_x &= \kappa \left( \frac{\Theta_x}{V} \right) + \mu \frac{U_x^2}{V} + G, \\
(V, U, \Theta)(x, 0) &= (V_0, U_0, \Theta_0) = \left( \frac{R}{p_+} \Theta_0, \frac{k(\gamma - 1) \Theta_0x}{\gamma R} \Theta_0, \Theta_0 \right) \to (v_\pm, 0, \theta_\pm), \text{ as } x \to \infty.
\end{align*}
\] (2.14)

where
\[
\begin{align*}
G(x,t) &= -\mu \frac{U_x^2}{V} = O((\ln \Theta)_{xx}^2), \\
F(x,t) &= \kappa \frac{(\gamma - 1)}{\gamma R} \left\{ (\ln \Theta)_{xt} - \mu \left( \frac{\ln \Theta}_{xx} \right)_x \right\} \\
&= \frac{ka(\gamma - 1) - \mu p_+ \gamma}{R\gamma} \left( \frac{\ln \Theta}_{xx} \right)_x.
\end{align*}
\] (2.15)

Furthermore, from (2.1) and (2.14), we obtain
\[
||V_x + U|| \leq C||\Theta_x||, \quad ||\Theta_x||^2 \leq C ||(\ln \Theta)_x|| ||(\ln \Theta)_{xx}||, \quad ||U_x||^2 \leq C ||(\ln \Theta)_{xx}|| ||(\ln \Theta)_{xxx}||. \quad (2.16)
\]

When the time \(t \to \infty\), it is easily check that \((V, U, \Theta)\) is nearly close to \((v_\pm, 0, \theta_\pm)\). We can proof this result by the following lemma.

**Lemma 2.2**

\[
\begin{align*}
||\ln \Theta||^2 + \int_0^t ||(\ln \Theta)_{xx}||^2 \, dt &\leq C\delta_0^2. \quad (2.17) \\
\int_0^t ||(\ln \Theta)_x||^2 \, dt &\leq C(1 + t)^{1/3}. \quad (2.18) \\
||\ln \Theta||^2 &\leq C(1 + t)^{-2/3}. \quad (2.19) \\
||\ln \Theta_{xx}||^2 &\leq C(1 + t)^{-5/3}. \quad (2.20) \\
||\ln \Theta_{xxx}||^2(1 + t) + \int_0^t ||\ln \Theta||^2(1 + t) \, dt &\leq C\delta_0^2. \quad (2.21) \\
||\ln \Theta||^2 &\leq C(1 + t)^{-8/3}. \quad (2.22) \\
||\theta_x||^2_{L^\infty(R_\pm)} &\leq C\delta_0^{1/4}(1 + t)^{-1/24}. \quad (2.23)
\end{align*}
\]
Proof. Set
\[
\theta_2(x,t) = \int_{-\infty}^{+\infty} (4\pi at)^{-1/2} \Theta_0(h) \exp\left\{-\frac{(h-x)^2}{4at}\right\} \, dh,
\]
\[
\theta_{2t} = a\theta_{2x},
\]
\[
\theta_2(x,0) = \Theta_0(x) \rightarrow \theta_{\pm}, \tag{2.24}
\]
and
\[
\theta_{2x} = \int_{-\infty}^{+\infty} (4\pi at)^{-1/2} \Theta_0(z) \exp\left\{-\frac{(z-x)^2}{4at}\right\} \frac{z-x}{2at} \, dz
\]
\[
= \int_{-\infty}^{+\infty} (4\pi at)^{-1/2} \Theta_0(z) \exp\left\{-\frac{(z-x)^2}{4at}\right\} \, dz
\]
\[
= \int_{-\infty}^{+\infty} (4\pi at)^{-1/2} (\Theta_0(z) - \Theta_0(x)) \exp\left\{-\frac{(z-x)^2}{4at}\right\} \frac{z-x}{2at} \, dz. \tag{2.25}
\]
By using Hölder inequality, Fubini Theorem, (2.25) and \(|\Theta_0 - \theta_{\pm}|_{L^1(\mathbb{R})} < C, \|\theta_{2x}\| \leq C(\delta_0)\), we can get that
\[
\int_0^t \int_{-\infty}^{+\infty} \theta_{2x}^2 \, dx \, dt
\]
\[
\leq C \int_1^t \int_{-\infty}^{+\infty} (4\pi at)^{-1} \int_{\mathbb{R}^\pm} |\Theta_0(z) - \theta_{\pm}| \exp\left\{-\frac{(z-x)^2}{4at}\right\} \frac{|z-x|}{2a\sqrt{t}} \, t^{-1/2} \, dz \, dx \, dt
\]
\[
+ C \int_1^t \int_{-\infty}^{+\infty} (4\pi at)^{-1} \int_{\mathbb{R}^\pm} |\theta_{\pm} - \Theta_0(x)| \exp\left\{-\frac{(z-x)^2}{4at}\right\} \frac{|z-x|}{2a\sqrt{t}} \, t^{-1/2} \, dz \, dx \, dt
\]
\[
+ \int_0^1 \|\theta_{2x}\|^2 \, d\tau
\]
\[
\leq C \ln(1+t) + C \leq C(1+t)^{1/3} + C \leq C(1+t)^{1/3}. \tag{2.26}
\]

Now, let’s consider the estimates about \(\partial_x^i \Theta\) \((i = 1, 2, 3)\) of (2.2). In fact from (2.2) we can get
\[
(\ln \Theta)_t = a \frac{(\ln \Theta)_{xx}}{\Theta}.
\]
Both side of it multiply by \((\ln \Theta)_{xx}\) and integrate it with respect to \(\mathbb{R} \times (0, t)\) we can get
\[
\|(\ln \Theta)_x\|^2 + \int_0^t \|(\ln \Theta)_{xx}\|^2 \, dt \leq C\|(\ln \Theta_0)_{x}\|^2. \tag{2.27}
\]
When combine with (2.3), we can get
\[
\|(\ln \Theta)_x\|^2 + \int_0^t \|(\ln \Theta)_{xx}\|^2 \, dt \leq C\delta_0^2. \tag{2.28}
\]
On the other hand, integrate \((2.2) - (2.24)) \times (\Theta - \theta_2)\) in \(\mathbb{R} \times (0, t)\) and combine with Cauchy-Schwarz inequality, we can get
\[
\|\Theta - \theta_2\|^2 + \int_0^t \|(\ln \Theta)_x\|^2 \, dt \leq C \int_0^t \|\theta_{2x}\|^2 \, dt. \tag{2.29}
\]
Insert (2.26) and (2.27) to (2.29) we can get
\[
\int_0^t \|(\ln \Theta)_x\|^2 dt \leq C(1 + t)^{1/3}.
\] (2.30)
That is (2.18).

Next, integrate (2.21) \times \Theta^{-1}(\ln \Theta)_{xx}(1 + t) in \mathbb{R} \times (0, t), we can get
\[
0 = a \int_0^t \int_{\mathbb{R}} \frac{(\ln \Theta)^2_{xx}}{\Theta} (1 + t) \, dxdt + \int_0^t \int_{\mathbb{R}} ((\ln \Theta)^2_x)_t (1 + t) \, dxdt.
\] (2.31)
So
\[
(1 + t)\|(\ln \Theta)_x\|^2 + \int_0^t \int_{\mathbb{R}} (1 + t)(\ln \Theta)^2_{xx} \, dx \, dt\, dt
\leq C\|\Theta_{0x}\|^2 + \int_0^t \int_{\mathbb{R}} (\ln \Theta)^2_x \, dx \, dt.
\] (2.32)
Combine with (2.30) we can get
\[
(1 + t)\|(\ln \Theta)_x\|^2 + \int_0^t \int_{\mathbb{R}} (1 + t)(\ln \Theta)^2_{xx} \, dx \, dt \leq C(1 + t)^{1/3}.
\] (2.33)
That means \|(\ln \Theta)_x\|^2 \leq C(1 + t)^{-2/3}, which is (2.19).

Again from (2.21) we can get
\[
(\ln \Theta)_xt = a \left( \frac{(\ln \Theta)_{xx}}{\Theta} \right)_x.
\] (2.34)
Both side of (2.34) multiply by \(\partial^2_x \ln \Theta\) and get
\[
((\ln \Theta)_xt\partial^2_x (\ln \Theta))_x - 1/2(\partial^2_x \ln \Theta)_t = a \left( \frac{(\ln \Theta)_{xx}}{\Theta} \right)_x \partial^2_x (\ln \Theta).
\] (2.35)
Both side of (2.35) multiply by \((1 + t)^2\), integrate it with respect to \(\mathbb{R} \times (0, t)\) and combine with Cauchy-Schwarz inequality and (2.33) we have
\[
\|(\ln \Theta)_{xx}\|^2(1 + t)^2 + \int_0^t \int_{\mathbb{R}} (1 + t)^2(\ln \Theta)^2_{xxx} \, dx \, dt \leq C(1 + t)^{1/3},
\] (2.36)
which means
\[
\|(\ln \Theta)_{xx}\|^2 \leq C(1 + t)^{-5/3}.
\] (2.37)
So we finish (2.20).

If both side of (2.35) multiply by \((1 + t)\), similar as the proof of (2.36), when combine with (2.37) we can get
\[
\|(\ln \Theta)_{xx}\|^2(1 + t) + \int_0^t \int_{\mathbb{R}} (1 + t)(\partial^2_x \ln \Theta)^2 \, dx \, dt \leq C\delta^2_0,
\] (2.38)
which means (2.21).
From Lemma 2.1, (2.43), (2.17) and (2.19) we can get
\[ \partial_t (\ln \Theta)_{xx} = a \partial_x^2 \left( \frac{(\ln \Theta)_{xx}}{\Theta} \right). \]  
(2.39)

When both side of (2.39) multiply by \((\partial_x^4 \ln \Theta)(1+t)^3\), then integrate it with respect to \(\mathbb{R} \times (0,t)\), we can get that there exists a constant \(\epsilon > 0\) such that (2.39) can be change to
\[
\| \partial_x^2 \ln \Theta \|^2 (1+t)^3 + C \int_0^t (1+t)^3 \| \partial_x^4 \ln \Theta \|^2 \, dt \\
\leq C + C \int_0^t \int_{\mathbb{R}} \partial_x^3 (\partial_x^4 \ln \Theta) (1+t)^3 \, dx \, dt + C \int_0^t \int_{\mathbb{R}} \partial_x \ln \Theta (1+t)^3 \, dx \, dt \\
+ C \int_0^t \int_{\mathbb{R}} (\partial_x^3 \ln \Theta)^2 (1+t)^2 \, dx \, dt + C \int_0^t \int_{\mathbb{R}} (\partial_x^3 \ln \Theta)^2 (1+t)^2 \, dx \, dt \\
\leq C \int_0^t \| \ln \Theta \|_x \| \partial_x^3 \ln \Theta \| \| \partial_x^1 \ln \Theta \| (1+t)^3 \, dt + C \int_0^t \| \ln \Theta \|_x \| \partial_x^2 \ln \Theta \| (1+t)^3 \, dt \\
+ C \int_0^t \| \ln \Theta \|_x \| \partial_x^2 \ln \Theta \| (1+t)^3 \, dt + C(1+t)^{1/3} + C \\
\leq \epsilon \int_0^t \| \partial_x^4 \ln \Theta \|^2 (1+t)^3 \, dt + C/\epsilon \int_0^t \| \partial_x^2 \ln \Theta \|^2 (1+t)^2 \, dt \\
+ C/\epsilon \int_0^t \| \partial_x^3 \ln \Theta \|^2 (1+t) \, dt + C(1+t)^{1/3}.
\]

By using (2.33) and (2.36) we can get
\[
\| \partial_x^3 \ln \Theta \|^2 (1+t)^3 + \int_0^t (1+t)^3 \| \partial_x^4 \ln \Theta \|^2 \, dt \leq C(1+t)^{1/3}.
\]  
(2.40)

This means (2.22) finished.

When we change \((1+t)^3\) to \((1+t)^2\) and combine with (2.21), we can get
\[
\| \partial_x^3 \ln \Theta \|^2 (1+t)^2 + \int_0^t (1+t)^2 \| \partial_x^4 \ln \Theta \|^2 \, dt \leq C.
\]  
(2.41)

From (2.2) we can get
\[
(\Theta - \Theta_0)_L(\Theta - \Theta_0) = a ((\ln \Theta)_x(\Theta - \Theta_0))_x - a(\ln \Theta)_x(\Theta - \Theta_0)_x.
\]
When integrate both sides of it in \(\mathbb{R} \times [0,t]\), we can get
\[
\| \Theta - \Theta_0 \|^2 \leq C \| \Theta_0 x \|_{L^1} \int_0^t \| \Theta_0 x \|_{L^\infty} \, d\tau \leq C \| \Theta_0 x \|_{L^1} \int_0^t \| \Theta_0 x \|^{1/2} \| \Theta_{xx} \|^{1/2} \, d\tau.
\]  
(2.42)

From (2.42), Lemma 2.1 (2.19) and (2.20) we can obtain
\[
\| \Theta - \Theta_0 \|^2 \leq C(1+t)^{5/12}.
\]  
(2.43)

From Lemma 2.1 (2.43), (2.17) and (2.19) we can get
\[
\| \Theta - \theta_\pm \|^2_{L^\infty(\mathbb{R}^\pm)} \leq C \| \Theta - \theta_\pm \|_{L^2(\mathbb{R}^\pm)} \| \Theta_x \|^{3/4} \| \Theta_x \|^{1/4}
\]
we can get that the asymptotic stability results to \((v,u,\theta)\) Theorem 3.1. Our main results of this paper now reads as follows.

I

\(2.1, 2.16\) and Lemma 2.2 that for \(x \in \mathbb{R}_\pm\),

\[
|v(x,t) - V(x,t)| \leq C \|(V - v_\pm, U, \Theta - \theta_\pm)\|_{L^2(\mathbb{R}_\pm)} \|(v_\pm, U, \Theta_x)\|, \tag{2.14}
\]

Combining (2.14) and (1.3), the original problem can be reformulated as

\[
\begin{aligned}
    \varphi(x,t) &= v(x,t) - V(x,t), \\
    \psi(x,t) &= u(x,t) - U(x,t), \\
    \zeta(x,t) &= \theta(x,t) - \Theta(x,t).
\end{aligned}
\tag{3.1}
\]

Combining (2.14) and (1.3), the original problem can be reformulated as

\[
\begin{aligned}
    \varphi_t &= \psi_x, \\
    \psi_t - \left(\frac{R\Theta}{vV} \varphi\right)_x + \left(\frac{R\zeta}{v}\right)_x &= -\mu(U_x + \varphi)_x + \mu(\psi_x)_x - F, \\
    \frac{R}{\gamma - 1} \zeta_t + \frac{R\Theta}{v}(\psi_x + U_x) - \frac{R\Theta}{V}U_x &= k\left(\frac{\zeta_x}{v}\right)_x - k\left(\frac{\Theta_x \varphi}{vV}\right)_x + \mu\left(\frac{u^2}{v} - \frac{U}{V}\right) - G,
\end{aligned}
\tag{3.2}
\]

and

\[
\begin{aligned}
    (\varphi_0, \psi_0, \zeta_0) &= (v(x,0) - V(x,0), u(x,0) - U(x,0), \theta(x,0) - \Theta(x,0)) \tag{3.3}
\end{aligned}
\]

From (2.14), it is easy to check that the initial values in (3.3) satisfy

\[
(\varphi, \psi, \zeta)(x,0) \rightarrow (0, 0, 0), \quad \text{as} \quad x \rightarrow \pm \infty.
\]

Moreover, for an interval \(I \in (0, \infty)\), we define the function space

\[
X(I) = \{(\varphi, \psi, \zeta) \in C(I, H^1) | \varphi_x \in L^2(I; L^2), (\psi_x, \zeta_x) \in L^2(I; H^1)\}.
\]

Our main results of this paper now reads as follows.

\textbf{Theorem 3.1} If \((v_0 - v_\pm, u_0, \theta_0 - \theta_\pm) \in H^2(\mathbb{R}_\pm) \cap L^1(\mathbb{R}_\pm)\), \((\varphi_0, \psi_0, \zeta_0)\) is suitably small, \(\frac{v_-}{\theta_-} = \frac{v_+}{\theta_+}\) and \(|\theta_+ - \theta_-|\) not small, (3.2) has a global solution \((\varphi, \psi, \zeta)\) satisfying \((\varphi, \psi, \zeta) \in X([0, \infty)), \) and when \(t \rightarrow \infty, \)

\[
\|(\varphi, \psi, \zeta)\|_{L^\infty(\mathbb{R}_\pm)} \rightarrow 0.
\]
We prove Theorem 3.1 by combining the local existence and the global-in-time a priori estimates. The local existence of the solution is well known (e.g., see [1, 21]), so we omit it here for brevity. To prove the global existence part of Theorem 3.1, the same as the asymptotic stability result, we need to establish the following a priori estimate.

**Proposition 3.1** (A priori Estimate) Let \((\varphi, \psi, \zeta)(x, t) \in X([0, T])\) be a solution of problem (3.2) for a constant \(T > 0\). Set \(C\) is a positive constant only depending on \(C_v, R, \mu, \theta_{\pm}, v_{\pm}\) and \(||(\varphi_0, \psi_0, \zeta_0)||_1\). If \(||(\varphi_0, \psi_0, \zeta_0)||_{L^2(\mathbb{R})}\) is suitably small and

\[
1 < N_1(T) = \sup_{t \in [0, T]} \{m_v^{-1}, M_v, m_\theta^{-1}, M_\theta, ||(\varphi, \psi, \zeta)||_1\} \leq 2 \left(C|||\varphi_0, \psi_0, \zeta_0|||_1^2 + C + 1\right)^{1/2}
\]

with \(0 < m_v = v^{-1}(x, t) \leq v(x, t) \leq M_v, 0 < m_\theta \leq \theta(x, t) \leq M_\theta\), then when \(t \in [0, +\infty)\), \((\varphi, \psi, \zeta)(t)\) satisfies

\[
\sup_{t \in [0, +\infty)} ||(\varphi, \psi, \zeta)||_1^2 + \int_0^t \left\{||\varphi_x||^2 + ||\psi_x, \zeta_x||^2\right\} d\tau \leq C\left(|||\varphi_0, \psi_0, \zeta_0|||_1^2 + C\right).
\]

This proposition means that for any \(T > 0\), all the properties of \(X([0, T])\) have uniform bounds, so the solution’s time interval \([0, T]\) can be extend onto \([0, +\infty)\).

4 Proof of Theorem 3.1

Before establishing (3.4), we must obtain the upper and lower boundaries of \(v(x, t)\) and \(\theta(x, t)\). Here, we set the initial data of (1.3) are sufficiently smooth functions and set

\[
\Phi(z) = z - \ln z - 1, \\
\Psi(z) = z^{-1} + \ln z - 1,
\]

where \(\Phi'(1) = \Psi(1) = 0\) is a strictly convex function around \(z = 1\). Similar to the proof in [1] or [21], we can get

**Lemma 4.1** If \(C\) is a positive constant independent of \(x\) and \(t\), when \(\delta_0\) is a small constant, we can get

\[
\int_0^t \int_{\mathbb{R}} \Theta^2_x(\varphi^2 + \zeta^2) dx d\tau \leq C(\delta_0) \int_0^t ||(\varphi_x, \zeta_x)||^2 d\tau + C(\delta_0).
\]

**Proof.**

\[
\int_0^t \int_{\mathbb{R}} \Theta^2_x(\zeta^2 + \varphi^2) dx d\tau \\
\leq \int_0^t \int_{\mathbb{R}} \Theta^2_x(||\zeta|| ||\zeta_x|| + ||\varphi|| ||\varphi_x||) dx d\tau \\
\leq C \int_0^t (||\zeta|| + ||\varphi||)^2 ||\Theta_x||^{1/4} d\tau + C \int_0^t ||\Theta_x||^{15/4} d\tau.
\]

From (2.17) and (2.19) we can get

\[
\int_0^t \int_{\mathbb{R}} \Theta^2_x(\zeta^2 + \varphi^2) dx d\tau \leq C(\delta_0) \int_0^t (||\zeta|| + ||\varphi||)^2 d\tau + C(\delta_0).
\]

That we finish this lemma. $$\square$$
Lemma 4.2 If $C$ is a positive constant independenting of $x$ and $t$, when $\delta_0$ is a small constant, we can get

$$
\int_{\mathbb{R}} \left( R\Theta\Phi \left( \frac{v}{V} \right) + \frac{1}{2} v^2 + C_v \Theta\Phi \left( \frac{\theta}{\Theta} \right) \right) dx + \int_0^t \| \psi_x / \sqrt{v\theta}, \zeta_x / (\theta \sqrt{v}) \|^2 \, d\tau \\
\leq C(\delta_0) + C(\delta_0) \int_0^t \| \varphi_x \|^2 \, d\tau + C(\varphi_0, \psi_0, \zeta_0) ^2.
$$

Proof. Similar to the proof in [1, 21] and use the definition of (4.1), we deduce from (3.2) that

$$
\left( \frac{\psi^2}{2} + R\Theta\Phi \left( \frac{v}{V} \right) + C_v \Theta\Phi \left( \frac{\theta}{\Theta} \right) \right)_t \\
+ \mu \theta \psi^2_v + \kappa \Theta \zeta^2_v + H_x + Q = \mu \left( \frac{\psi_x}{v} \right)_x - F \psi - \zeta G,
$$

where

$$
H = R \frac{\zeta \psi}{v} - R \frac{\Theta \varphi \psi}{v V} + \mu \frac{U_x \varphi \psi}{v V} - \kappa \zeta \zeta_x + \kappa \Theta \varphi \zeta_x,
$$

and

$$
Q = p_+ \Phi \left( \frac{V}{v} \right) U_x + \frac{p_+}{\gamma - 1} \Phi \left( \frac{\varphi}{\theta} \right)_x - \frac{\zeta}{\Theta} (p_+ - p) U_x - \mu \frac{U_x \varphi \psi_x}{v V} \\
- \frac{\Theta \zeta \zeta_x}{v \theta^2 V} \varphi_x - \frac{\Theta \zeta \zeta_x}{v \theta^2 V} \varphi_x - 2 \mu \frac{U_x \zeta \psi_x}{v \theta^2 V} \varphi \zeta + \frac{U_x \zeta \varphi \zeta_x}{v \theta^2 V}.
$$

Note that $p = R\theta / v$, $p_+ = R\Theta / V$, combine with the definition of $U, \tilde{N}_1$ with [21], use integration by parts and Cauchy-Schwarz inequality, we can get

$$
Q_1 + Q_2 = Ra \left( \Phi \left( \frac{V}{v} \right) (\ln \Theta)_x \right) + \frac{Ra}{\gamma - 1} \left( \Phi \left( \frac{\varphi}{\theta} \right) (\ln \Theta)_x \right)_x \\
- aR (\ln \Theta)_x \left( \frac{V \varphi_x - V_x \varphi^2}{v \theta^2} \right) \\
- a \frac{p_+}{\gamma - 1} (\ln \Theta)_x \left( \frac{\Theta \zeta \zeta_x - \Theta \zeta_x^2}{\Theta \theta^2} \right) \\
\geq \left( p_+ \Phi \left( \frac{V}{v} \right) U + \frac{p_+}{\gamma - 1} \Phi \left( \frac{\varphi}{\theta} \right) U \right)_x \\
- C^{1/2}(\delta_0) \left( \frac{\zeta_x^2}{v \theta^2} + \varphi_x^2 \right) - C^{-1/2}(\delta_0) \tilde{N}_1^2 \Theta_x^2 \left( \zeta^2 + \varphi^2 \right).
$$

Using $p - p_+ = \frac{R \zeta - p_+ \varphi}{v}$ and the definition of $\tilde{N}_1, U$ and $V$, we can get

$$
Q_3 \geq \frac{R \zeta - p_+ \varphi}{v} \left( \frac{\zeta_x}{v \theta} U_x \right) \geq \left( \frac{R \zeta_x^2 U}{v \theta} - \frac{p_+ \zeta \varphi U}{v \theta} \right)_x - C^{1/2}(\delta_0) \left( \frac{\zeta_x^2}{v \theta^2} + \varphi_x^2 \right) - C^{-1/2}(\delta_0) \tilde{N}_1^2 \Theta_x^2 \left( \zeta^2 + \varphi^2 \right).
$$
And again from the definition of $\bar{N}_1$, Cauchy-Schwarz inequality we know

\[
(Q_4 + Q_7) + (Q_5 + Q_6 + Q_8) + Q_9 \geq -C^{-1/2}(\delta_0)\bar{N}_1^4|\theta_0 \frac{1}{\sqrt{\theta}}| \geq 0
\]

At the end we use the definition of $F$ and $G$ in [2.15] then combine the definition of $\bar{N}_1$ with the general inequality skills as above to get

\[
-F\psi - G\frac{\zeta}{\theta} = -\frac{\kappa a(\gamma - 1) - \mu p + \gamma}{R\gamma} \left( \frac{\theta_0}{\Theta} \right)^2 \psi
\]

\[
+ \frac{\mu_\psi}{R\Theta} \left( \frac{\kappa a(\gamma - 1)}{R\gamma} \left( \frac{\theta_0}{\Theta} \right)^2 \right) \frac{\zeta}{\theta}
\]

\[
\leq -\frac{\kappa a(\gamma - 1) - \mu p + \gamma}{R\gamma} \left( \frac{\theta_0}{\Theta} \right)^2 \psi + \frac{\kappa a(\gamma - 1) - \mu p + \gamma}{R\gamma} \frac{\theta_0}{\Theta} \psi
\]

\[
+ \frac{\mu_\psi}{R\Theta} \left( \frac{\kappa a(\gamma - 1)}{R\gamma} \left( \frac{\theta_0}{\Theta} \right)^2 \right) \frac{\zeta}{\theta}
\]

\[
\leq -\frac{\kappa a(\gamma - 1) - \mu p + \gamma}{R\gamma} \left( \frac{\theta_0}{\Theta} \right)^2 \psi + C^{1/2}(\delta_0) \frac{\psi^2}{\theta}
\]

\[
+ C^{-1/2}(\delta_0)\bar{N}_1^2(\ln \Theta)^2_2.
\]

Integrating each term from (4.2) to (4.3) in $\mathbb{R} \times (0, t)$, using Lemma 4.1 and Lemma 2.2 at the end, we get that for a small $C(\delta_0) > 0$,

\[
\int\left( R\Theta \frac{\psi}{\sqrt{v}} + \frac{1}{2} \psi^2 + C_0 \Theta \frac{\theta}{\Theta} \right) dx + \int_0^t \|\psi_x / \sqrt{v\theta}, \zeta_x / \sqrt{v\theta^2}\|^2 d\tau
\]

\[
\leq C(\delta_0)(1 + \bar{N}_1^8) + C(\delta_0) \int_0^t \|\varphi_x\|^2 d\tau + C\|\varphi_0, \psi_0, \zeta_0\|^2.
\]

Then we finish this lemma. $\square$

**Lemma 4.3** If $\|\varphi_0, \psi_0, \zeta_0\|$ and $C(\delta_0) > 0$ are suitably small and $\|\varphi_0x\|$ not small, we can get

\[
\|\varphi_x / v\|^2 + \int_0^t \|\varphi_x / v \sqrt{R\theta} / v\|^2 d\tau \leq C\|\varphi_0, \varphi_0x, \psi_0, \zeta_0\|^2 + C(\delta_0),
\]

with $C$ independent of $x$ and $t$.

**Proof.** Set $\bar{v} = \frac{v}{\sqrt{v}}$, take it into (3.2)_1, (3.2)_2 (p = R\theta / v) to get

\[
\psi_t + p_x = \frac{\mu}{v} \left( \frac{\bar{v}_x}{\bar{v}} \right)_t - F,
\]

Both sides of last equation multiply by $\bar{v}_x / \bar{v}$ to get

\[
\left( \frac{\mu}{2} \left( \frac{\bar{v}_x}{\bar{v}} \right)^2 - \psi \frac{\bar{v}_x}{\bar{v}} \right)_t + \frac{R\theta}{v} \left( \frac{\bar{v}_x}{\bar{v}} \right)_t + \left( \psi \frac{\bar{v}_t}{\bar{v}} \right)_x
\]

13
\[
= \frac{\psi_x^2}{v} + U_x \left( \frac{1}{v} - \frac{1}{V} \right) \psi_x + \frac{R\zeta_x}{v} \frac{\psi_x}{v} - \frac{R\theta}{v} \left( \frac{1}{\Theta} - \frac{1}{\bar{\Theta}} \right) \Theta_x \frac{\psi_x}{v} + F \frac{\bar{\psi}_x}{v}.
\] \tag{4.4}

On the other hand if we integrate (4.4) in \( R \times (0, t) \), the right side of it is less than

\[
C \left( \int_0^t \| (\zeta_x / \sqrt{v\theta}, \psi_x / \sqrt{v}) \|^2 \, d\tau + \bar{N}_1^2 \int_0^t \int_{\mathbb{R}} \Theta_x \zeta^2 \, dx \, d\tau \right)
+ C\bar{N}_1 \int_0^t \int_{\mathbb{R}} U_x \varphi^2 \, dx \, d\tau + C\bar{N}_1^2 \int_0^t \int_{\mathbb{R}} |F|^2 \, dx \, d\tau + 1/4 \int_0^t \int_{\mathbb{R}} \frac{R\theta}{v} \left( \frac{\bar{\psi}_x}{v} \right)^2 \, dx \, d\tau.
\]

Use Lemma 4.1 to the term \( \int_0^t \int_{\mathbb{R}} \Theta_x \zeta^2 \, dx \, d\tau \) and combine with the definition of \((V, U, \Theta)\), \( \int_0^t \int_{\mathbb{R}} (1+\bar{\psi}_x/v)^2 \, dx \, d\tau \) can be change to

\[
\int_0^t \left\| \sqrt{R\theta} \frac{\psi_x}{v} \right\|^2 \, d\tau + \left\| \frac{\bar{\psi}_x}{v} \right\|^2 - C\| \psi \|^2 - C\| \psi_0 \|^2 - C \int_{\mathbb{R}} \frac{\bar{\psi}_x}{v}(x, 0)^2 \, dx
\]
\[
\leq C \left( \int_0^t \left\| (\zeta_x / \sqrt{v\theta}, \psi_x / \sqrt{v}) \right\|^2 \, d\tau + C(\delta_0)\bar{N}_1^2 \int_0^t \int_{\mathbb{R}} \zeta_x^2 \, dx \, d\tau + C(\delta_0)\bar{N}_1^2 \right) + C\| \varphi_0 \|^2.
\]

Insert \( C_1 (\varphi_x / v)^2 - C\bar{N}_1^2 v_x^2 \leq (\frac{\bar{\psi}_x}{v})^2 \leq C_3 (\varphi_x / v)^2 + C\bar{N}_1^2 v_x^2 \) to last inequality and combine with Lemma 4.2, Lemma 2.2, and the definition of \( \bar{N}_1 \) we can obtain that

\[
\int_0^t \left\| \varphi_x / v \right\|^2 \, d\tau + \left\| \varphi_x / v \right\|^2
\]
\[
\leq C \left( \left\| (\varphi_0, \varphi_0, \psi_0, \zeta_0) \right\|^2 + C(\delta_0)(1 + \bar{N}_1^8) + C \int_0^t \left\| (\zeta_x / \sqrt{v\theta}, \psi_x / \sqrt{v}) \right\|^2 \, d\tau.\right.
\]

So we finish this lemma. \( \square \)

When \( C(\delta_0) \) and \( \left\| (\varphi_0, \psi_0, \zeta_0) \right\| \) are suitably small and use it to control the terms about \( \bar{N}_1 \), insert Lemma 4.3 to Lemma 4.2 we obtain the first a priori energy estimate

\[
\sup_{t \in [0, +\infty)} \int_{\mathbb{R}} (\varphi^2 + \psi^2 + \zeta^2) \, dx + \int_0^t \left\| (\psi_x, \zeta_x) \right\|^2 \, d\tau \leq C \left( \left\| (\varphi_0, \psi_0, \zeta_0) \right\|^2 + C(\delta_0) \right). \tag{4.5}
\]

**Lemma 4.4** For a suitably small constant \( C(\delta_0) > 0 \) and \( \left\| (\psi_0, \zeta_0) \right\| \) not small, the solution of (3.2) satisfies

\[
\left\| (\psi_x, \zeta_x) \right\|^2 + \int_0^t \left\| (\psi_{xx}, \zeta_{xx})(\tau) \right\|^2 \, d\tau \leq C \left( \left\| (\varphi_0, \psi_0, \zeta_0) \right\|^2 + C(\delta_0) \right),
\]

with \( C \) independent of \( x \) and \( t \).

**Proof.** Multiplying (3.2) and (3.2) by \( \psi_{xx} \) and \( \zeta_{xx} \), respectively, and summing up the resulting equations, we find

\[
\left( \frac{1}{2} \psi_x^2 + \frac{R}{2(\gamma - 1)} \zeta_x^2 \right)_t + \mu \frac{\psi_x^2}{v} + \kappa \frac{\zeta_x^2}{v}.
\]
We now estimate using (4.5), Lemma 4.1, Cauchy-Schwarz inequalities and Sobolev inequalities, we infer that
\[ \frac{1}{2} \left( \int_{t}^{\infty} \int_{\mathbb{R}} \frac{U_{x} \phi_{x}}{v} \right) \] which, combined with gives
\[ \int_{0}^{t} \int_{\mathbb{R}} \frac{U_{x} \phi_{x}}{v} \] which, integrated over \( \mathbb{R} \times (0, t) \), gives
\[ \| (\psi_{x}, \zeta_{x})(t) \|^2 + \int_{0}^{t} \| (\psi_{x}, \zeta_{x})(\tau) \|^2 d\tau \]
\[ \leq C \| (\psi_{0x}, \zeta_{0x}) \|^2 + C \frac{1}{\epsilon} \int_{0}^{t} \int_{-\infty}^{\infty} I_{1} dx d\tau. \quad (4.6) \]

We now estimate \( \int I_{1} dx d\tau \) (i = 1, \ldots, 5). First, using Cauchy-Schwarz inequality, we have
\[ \int_{0}^{t} \int_{-\infty}^{\infty} |I_{1}| dx d\tau \leq \epsilon \int_{0}^{t} \| (\psi_{x}, \zeta_{x}) \|^2 d\tau + \frac{C}{\epsilon} \int_{0}^{t} \int_{-\infty}^{\infty} (\psi_{x}^{2} + \phi_{x}^{2})(\psi_{x}^{2} + \zeta_{x}^{2}) dx d\tau \]
\[ \leq \epsilon \int_{0}^{t} \| (\psi_{x}, \zeta_{x}) \|^2 d\tau + \frac{C}{\epsilon} \int_{0}^{t} \left( \| (\psi_{x}, \zeta_{x}) \|^2 + \| (\psi_{x}, \zeta_{x}) \|^2_{L^{\infty}} \| \phi_{x} \|^2 \right) d\tau, \]
which, combined with
\[ \| (\psi_{x}, \zeta_{x}) \|^2_{L^{\infty}} \leq C \| (\psi_{x}, \zeta_{x}) \| \| (\psi_{x}, \zeta_{x}) \|, \]
yields
\[ \int_{0}^{t} \int_{-\infty}^{\infty} |I_{1}| dx d\tau \leq 2 \epsilon \int_{0}^{t} \| (\psi_{x}, \zeta_{x})(\tau) \|^2 d\tau + \frac{C}{\epsilon} \left( C(\delta_{0}) + N^4(T) \right) \int_{0}^{t} \| (\psi_{x}, \zeta_{x})(\tau) \|^2 d\tau. \quad (4.7) \]

Noting that
\[ |I_{2}| \leq C \left( |U_{xx}| |\varphi| + |U_{x}| |\varphi_{x}| + |U_{x}| |\varphi| |\varphi_{x}| + |U_{x}| |V_{x}| |\varphi| + |\Theta_{x}| |\varphi| + |\varphi_{x}| + |\varphi| + |\varphi_{x}| + |V_{x}| |\varphi| + |\zeta| |\varphi_{x}| + |V_{x}| |\zeta| |\varphi_{x}| \right) |\psi_{xx}| \]
\[ \leq C \left( |\varphi_{x}| + |\zeta_{x}| + |U_{x}| |\varphi_{x}| + |U_{x}| |\varphi| |\varphi_{x}| + |\varphi| |\varphi_{x}| + |\zeta| |\varphi_{x}| \right) |\psi_{xx}| \]
\[ + C \left( |U_{xx}| + |U_{x}| |V_{x}| + |\Theta_{x}| + |V_{x}| \right) (|\varphi| + |\zeta|) |\psi_{xx}|, \]
using (4.5), Lemma 4.1 Cauchy-Schwarz inequalities and Sobolev inequalities, we infer that
\[ \int_{0}^{t} \int_{-\infty}^{\infty} |I_{2}| dx d\tau \leq \epsilon \int_{0}^{t} \| (\varphi_{x}, \zeta_{x})(\tau) \|^2 d\tau + \frac{C}{\epsilon} \int_{0}^{t} \| (\varphi_{x}, \zeta_{x})(\tau) \|^2 d\tau \]
\[ + \frac{C}{\epsilon} \int_{0}^{t} \int_{-\infty}^{\infty} \Theta_{x}^{2} (\varphi^{2} + \zeta^{2}) dx d\tau \]
\[ + \frac{C}{\epsilon} \int_{0}^{t} \| (\varphi, \zeta) \| \left( \| \partial_{x}^{2} (\ln \Theta) \|^2 + \| \varphi_{x} \|^2 \right) d\tau \]
\[ \leq \epsilon \int_{0}^{t} \| \psi_{xx} \|^2 d\tau + \frac{C}{\epsilon} \left( \| (\varphi_{0}, \psi_{0}, \zeta_{0}) \|^2 + \| \varphi_{0x} \|^2 + C(\delta_{0}) \right). \quad (4.8) \]
In a similar manner, we also have

\[
\int_0^t \int_{-\infty}^{\infty} |I_5| \, dx \, d\tau \leq \frac{C}{\varepsilon} \int_0^t \left( \|F\|^2 + \|G\|^2 \right) \, d\tau + \varepsilon \int_0^t \left( \|\psi_{xx}\|^2 + \|\zeta_{xx}\|^2 \right) \, d\tau.
\]

(4.10)

Noting that

\[
\int_0^t \int_{-\infty}^{\infty} I_5 \, dx \, d\tau = 0,
\]

(4.11)

inserting the inequalities from (4.7) to (4.11) into (4.6) and choosing \( \varepsilon > 0 \) suitably small, recalling the definition of \( G \), \( F \), combining Lemma 4.3, 4.5 and Lemma 2.2, we conclude that for a \( M_v(t) = \|v(x, t)\|_{L^\infty(\mathbb{R})} \)

\[
\|\psi_{xx}, \zeta_{xx}\|^2 + \int_0^t \|\psi_{xx}, \zeta_{xx}\|^2 \, d\tau \leq C M_v(t) \left( \|(\varphi_0, \psi_0, \zeta_0)\|^2 + 1 \right).
\]

(4.12)

Let (4.12) combine with (4.5) and Lemma 4.3 we can obtain that for small \( \|(\varphi_0, \psi_0, \zeta_0)\| \) and \( C(\delta_0) \) we can get

\[
|v - V| \leq C \|\varphi\| \|\varphi_x\| < C N^3 \left( \|(\varphi_0, \psi_0, \zeta_0)\| + C(\delta_0) \right)
\]

which means there exist positive constant \( C_5 \) and \( C_6 \) independent of \( x \) and \( t \) such that \( C_5 < v < C_6 \). Now we inserting this bounds of \( v \) into (4.12) we can get for a positive constant \( C \) independent of \( x \) and \( t \) such that

\[
\|\psi_x, \zeta_x\|^2 + \int_0^t \|\psi_{xx}, \zeta_{xx}\|^2 \, d\tau \leq C \left( \|(\varphi_0, \psi_0, \zeta_0)\|^2 + 1 \right).
\]

We finish this lemma. □

In all, from (4.5), Lemma 4.3 and Lemma 4.4 we finish Proposition 3.1.

To finish Theorem 3.1 now let’s consider the stability result.

In fact from Proposition 3.1 we can get

\[
\int_0^\infty \left( \frac{d}{dt} \|\psi_x(t)\|^2 \right) \, d\tau \leq \frac{C}{\varepsilon} \int_0^t \left( \|\psi_{xx}(t)\|^2 \right) \, d\tau + \left( \|\varphi_x(t)\|^2 \right) \, d\tau \leq C \left( \|(\varphi_0, \psi_0, \zeta_0)\|^2 + 1 \right).
\]

(4.13)

It means

\[
\|(\varphi, \psi, \zeta)(t)\|_{L^\infty}^2 \leq 2 \|(\varphi, \psi, \zeta)(t)\| \|(\varphi_x, \psi_x, \zeta_x)(t)\| \to 0 \quad \text{when} \quad t \to \infty.
\]

(4.14)

Combine with (2.45) we finish the stability result of the theorem.
References

[1] S.N.Antonsev, A.V.Kashikhov and V.N.Monakhov, Boundary value problems in mechanics of nonhomogeneous fluids, North-Holland Amsterdam.New York.Oxford.Tokyo,1990.

[2] J.Smoller:Shock waves and reaction-diffusion equations, Berlin, Heidelberg, New York, Springer,1982.

[3] A. Matsumura, K. Nishihara, Asymptotics toward the rarefaction waves of the solutions of a one-dimensional model system for compressible viscous gas. Japan J. Appl. Math., 3 (1985) 1-13.

[4] A. Matsumura, K. Nishihara, On the stability of travelling wave solutions of a one-dimensional model system for compressible viscous gas. Japan J. Appl. Math., 2 (1985) 17-25.

[5] A. Matsumura, S. Kawashima, Asymptotic stability of traveling wave solutions of systems for one-dimensional gas motion. Commun. Math. Phys., 101 (1985) 97-127.

[6] A. Matsumura, K. Nishihara, S. Kawashima, Asymptotic behavior of solutions for the equations of a viscous heat-conductive gas. Proc. Japan Acad. Ser. A, 62 (1986) 249-252.

[7] A. Matsumura, K. Nishihara, Global stability of the rarefaction wave of a one-dimensional model system for compressible viscous gas. Commun. Math. Phys., 165 (1992) 325-335.

[8] A. Matsumura, K. Nishihara, Large-time behaviors of solutions to an inflow problem in the half space for a one-dimensional system of compressible viscous gas, Comm.Math.Phys.222(2001)449-474.

[9] A. Matsumura, Inflow and outflow problems in the half space for a one-dimensional isentropic model system of compressible viscous gas. Meth. Appl. Anal., 8(4) (2001) 645-666.

[10] A.Szepessy, Z.P.Xin, Nonlinear stability of viscous shock waves, Arch.Ration.Mech.Anal.,122(1993)53-103.

[11] A.Szepessy, K.Zumbrun, Stability of rarefaction waves in viscous media. Arch.Ration.Mech.Anal., 133(1996)249-298.

[12] T.P. Liu, Nonlinear stability of shock waves for viscous conservation laws, Mem. Amer.Math.Soc.,56(328),(1985).

[13] T.P. Liu, Z.P. Xin, Pointwise decay to contact discontinuities for systems of viscous conservation laws. Asian J. Math., 1 (1997) 34-84.

[14] Goodman J. Nonlinear asymptotic stability of viscous shock profiles for conservation laws. Arch Ration Mech Anal, (95)325-344, 1986.

[15] Z.P. Xin, On nonlinear stability of contact discontinuities. Proceeding of 5th International Conferences on Hyperbolic Problems: Theory, Numerics and Applications. Ed. Glimm, etc., World Sci. Publishing, River Edge, NJ, 1996.
[16] S. Kawashima, S. Nishibata, P.C. Zhu, Asymptotic stability of the stationary solution to the compressible Navier-Stokes equations in the half space. Comm. Math. Phys., 240 (2003) 483-500.

[17] S. Kawashima, P.C. Zhu, Asymptotic stability of nonlinear wave for the compressible Navier-Stokes equations in the half space. J. Diff. Eqn., 244 (2008), 3151–3179.

[18] K. Nishihara, T. Yang, H.J. Zhao, Nonlinear stability of strong rarefaction waves for compressible Navier-Stokes equations. SIAM J. Math. Anal., 35 (2004) 1561-1597.

[19] F.M. Huang, A. Matsumura, X. Shi, Viscous shock wave and boundary-layer solution to an inflow problem for compressible viscous gas, Comm.Math.Phys. 239(2003) 261-285.

[20] F.M. Huang, H.J. Zhao, On the global stability of contact discontinuity for compressible Navier-Stokes equations. Rend. Sem. Mat. Univ. Padova, 109 (2003) 283-305.

[21] F.M. Huang, A. Matsumura, X. Shi, On the stability of contact discontinuity for compressible Navier-Stokes equations with free boundary. Osaka J. Math., 41 (1) (2004) 193-210.

[22] F.M. Huang, A. Matsumura, Z.P.Xin, Stability of contact discontinuity for the 1-D compressible Navier-Stokes Equations. Arch. Ration Mech. Anal., 179 (2005) 55-77.

[23] F.M. Huang, Z.P. Xin, T.Yang, Contact discontinuity with general perturbations for gas motions. Adv. Math., 219 (2008) 1246-1297.

[24] F.M. Huang, J. Li, A. Matsumura, Asymptotic stability of combination of viscous contact wave with rarefaction waves for one-dimensional compressible Navier-Stokes system. Arch.Ration.Mech.Anal., 197(1)(2010) 89-116.

[25] Hakho Hong, Global stability of viscous contact wave for 1−D compressible Navier-Stokes equations, JDE., 252(2012)3482–3505.

[26] Ma S. X. , Zero dissipation limit to strong contact discontinuity for the 1−D compressible Navier-Stokes equations, JDE., 248(2010) 95–110.

[27] Tao Wang; Huijiang Zhao; Qingyang Zou, One-dimensional compressible Navier-Stokes equations with large density oscillation, Kinetic and Related Models, 6(3)(2013)649-670.

[28] T. Zheng, JW. Zhang, JN. Zhao, Asymptotic stability of viscous contact discontinuity to an inflow problem for compressible Navier-Stokes equations, Nonlinear Analysis: Nonlinear Anal. Theor., 74(17)(2011) 6617-6639.

[29] T. Zheng and J. Zhao, On the stability of contact discontinuity for Cauchy problem of compress Navier-Stokes equations with general initial data. Sci. China. Math., 55(10)(2012) 2005-2026.

[30] T. Zheng, Stability of strong viscous contact discontinuity in a free boundary problem for compressible Navier-Stokes equations. Nonlinear Anal. Real World App., 25(2015) 238C275.