Reduction theory for mapping class groups and applications to moduli spaces

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Abstract

Let \( S = S_{g,p} \) be a compact, orientable surface of genus \( g \) with \( p \) punctures and such that \( d(S) := 3g - 3 + p > 0 \). The mapping class group \( \text{Mod}_S \) acts properly discontinuously on the Teichmüller space \( T(S) \) of marked hyperbolic structures on \( S \). The resulting quotient \( \mathcal{M}(S) \) is the moduli space of isometry classes of hyperbolic surfaces. We provide a version of precise reduction theory for finite index subgroups of \( \text{Mod}_S \), i.e., a description of exact fundamental domains. As an application we show that the asymptotic cone of the moduli space \( \mathcal{M}(S) \) endowed with the Teichmüller metric is bi-Lipschitz equivalent to the Euclidean cone over the finite simplicial (orbi-) complex \( \text{Mod}_S \setminus \mathcal{C}(S) \), where \( \mathcal{C}(S) \) of \( S \) is the complex of curves of \( S \). We also show that if \( d(S) \geq 2 \), then \( \mathcal{M}(S) \) does not admit a finite volume Riemannian metric of (uniformly bounded) positive scalar curvature in the bi-Lipschitz class of the Teichmüller metric. These two applications confirm conjectures of Farb.

Key words: Teichmüller theory, moduli spaces, mapping class groups, reduction theory, asymptotic cones, positive scalar curvature

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1 Introduction and main results

The chief goal of this article is to study the large-scale geometry of moduli spaces of Riemann surfaces endowed with the Teichmüller metric. In this introductory section we recall basic notions and present an outline of our main results.
Let $S = S_{g,p}$ be a compact, orientable surface of genus $g$ with $p$ punctures such that $3g - 3 + p > 0$. This last assumption implies that $S$ carries finite volume Riemannian metrics of constant curvature $-1$ and $p$ cusps. A marked hyperbolic surface is a pair $(X, [f])$ where $X$ is a smooth surface equipped with a complete Riemannian metric of constant curvature $-1$ and where $[f]$ denotes the isotopy class of a diffeomorphism $f : X \to S$ mapping cusps to punctures. Two marked surfaces $(X_1, [f_1])$ and $(X_2, [f_2])$ are equivalent if there is an isometry $h : X_1 \to X_2$ such that $[f_2 \circ h] = [f_1]$. The collection of these equivalence classes is (one possible definition of) the Teichmüller space $\mathcal{T}(S)$ of $S$. The corresponding moduli space $\mathcal{M}(S)$ of isometry classes of hyperbolic surfaces is obtained by forgetting the marking. More precisely, consider the mapping class group $\text{Mod}_S$, i.e., the group of all orientation preserving diffeomorphisms of $S$, which fix the punctures, modulo isotopies which also fix the punctures. Then $\text{Mod}_S$ acts on $\mathcal{T}(S)$ according to the formula $h \cdot x = h \cdot (X, [f]) := (X, [h \circ f])$, for $h \in \text{Mod}_S, x \in \mathcal{T}(S)$, and one has $\mathcal{M}(S) = \Gamma \backslash \mathcal{T}(S)$.

The complex of curves $\mathcal{C}(S)$ of $S$ is an infinite (even locally infinite) simplicial complex of dimension $d(S) - 1$. It was introduced by Harvey in [11] as an analogon in the context of Teichmüller spaces of the (rational) Tits building associated to an arithmetic group. The vertices of $\mathcal{C}(S)$ are the isotopy classes of simple closed curves (called circles) on $S$, which are non-trivial (i.e., not contractible in $S$ to a point or to a component of $\partial S$). We denote the isotopy class of a circle $C$ by $\langle C \rangle$. A set of $k + 1$ vertices $\{\alpha_0, \ldots, \alpha_k\}$ span a $k$-simplex of $\mathcal{C}(S)$ if and only if $\alpha_0 = \langle C_0 \rangle, \ldots, \alpha_k = \langle C_k \rangle$ for some set of pairwise non-intersecting circles $C_0, \ldots, C_k$. For a simplex $\sigma \in \mathcal{C}(S)$ we denote by $|\sigma|$ the number of its vertices. A crucial fact is that $\mathcal{C}(S)$ is a thick chamber complex, i.e., every simplex is the face of a maximal simplex. Moreover the mapping class group $\text{Mod}_S$ acts simplicially on $\mathcal{C}(S)$ and the quotient $\text{Mod}_S \backslash \mathcal{C}(S)$ is a finite (orbi-) complex (see [11], Proposition 1).

In order to simplify exposition and proofs of the present article we work in the framework of manifolds and simplicial complexes rather than orbifolds and orbicomplexes. We thus consider finite index, torsion-free subgroups $\Gamma$ of $\text{Mod}_S$ which in addition consist of pure mapping classes. Recall that a mapping class $h \in \text{Mod}_S$ is called pure if it can be represented by a diffeomorphism $f : S \to S$ fixing (pointwise) some union $\Lambda$ of disjoint and pairwise non-isotopic nontrivial circles on $S$ and such that $f$ does not permute the components of $S \setminus \Lambda$ and induces on each component of the cut surface $S_\Lambda$ a diffeomorphism isotopic to a pseudo-Anosov or to the identity diffeomorphism (see [15], §7.1). It is well-known that such subgroups $\Gamma$ exist. For example one can take $\Gamma = \Gamma_S(m)$, the kernel of the natural homomorphism $\text{Mod}_S \to \text{Aut}(H_1(S, \mathbb{Z}/m\mathbb{Z}))$, $m \geq 3$, defined by the action of diffeomorphisms on
homology (see e.g. [15], § 7.1 or [10], § 1.3).

The first step in our aproach to understand the coarse geometry of moduli space is the construction of a tiling of Teichmüller space $T(S)$, i.e., a $\Gamma$-invariant decomposition into disjoint subsets, which in turn yields a tiling of moduli space $M(S)$. In order to formulate that result, we need length functions. Let $\alpha$ be a vertex of $C(S)$, i.e., $\alpha = (C)$ for a (non-trivial) circle $C$. Since $d(S) > 0$, any point $x \in T(S)$ represents a finite volume Riemann surface of curvature $-1$ with $p$ cusps. On the surface $x$ the isotopy class of $\alpha$ contains a unique geodesic loop; let $l_\alpha(x)$ denote its length. This defines a (smooth) function $l_\alpha : T(S) \to \mathbb{R}_{>0}$ for every vertex $\alpha \in C(S)$.

For $\varepsilon > 0$ we then define the $\varepsilon$-thick part of Teichmüller space

$$\text{Thick}_\varepsilon T(S) := \{ x \in T(S) \mid l_\alpha(x) \geq \varepsilon, \forall \alpha \in C(S) \}.$$ 

This set is $\Gamma$-invariant and its quotient is the $\varepsilon$-thick part $\text{Thick}_\varepsilon M(S)$ of moduli space.

We can now state a concise version of our first main result.

**Theorem A (Tiling of moduli space)** Let $S = S_{g,p}$ be a compact, orientable surface of genus $g$ with $p$ punctures such that $3g - 3 + p > 0$. Let $\Gamma$ be a torsion free, finite index subgroup of the mapping class group of $S$ consisting of pure elements. Further let $M(S) = \Gamma \backslash T(S)$ be the corresponding moduli space of Riemann surfaces and let $E$ be the set of simplices of the finite complex $\Gamma \backslash C(S)$. Then there is $\varepsilon = \varepsilon(S) > 0$ such that the $\varepsilon$-thick part $\text{Thick}_\varepsilon M(S)$ of moduli space is a compact submanifold with corners whose boundary is the union of compact sets $B_\varepsilon(\sigma)$ indexed by $E$ and this structure extends to a thick-thin tiling of the entire moduli space: there is a disjoint union

$$M(S) = \text{Thick}_\varepsilon M(S) \sqcup \bigsqcup_{\sigma \in E} \text{Thin}_\varepsilon (M(S), \sigma)$$

where the $(\varepsilon, \sigma)$-thin part $\text{Thin}_\varepsilon (M(S), \sigma)$ is diffeomorphic to $B_\varepsilon(\sigma) \times \mathbb{R}_{>0}$.

For more details on the structure of the boundary faces $B_\varepsilon(\sigma)$ see Proposition 2.1 in Section 2.1 below. Theorem A (which we prove in Section 2) can be considered as a version of precise reduction theory for mapping class groups. In fact, there are striking parallels to precise reduction theory for arithmetic groups as developed e.g. by Langlands [16], Osborne-Warner [23] and Saper [24] (see also [1]). In Section 2 we further emphasize this analogy using the notion of parabolic subgroups of $\Gamma \subset \text{Mod}_S$ (compare Corollary 2).

In Section 3 we study (coarse) metric properties of the tiling described in Theorem A. We in particular construct a quasi-isometric approximation of moduli space $M(S) = \text{Mod}_S \backslash T(S)$ endowed with the distance function $d_M$ induced by the Teichmüller metric.
This approximation is provided by the *asymptotic cone* (or tangent cone at infinity) defined as the Gromov-Hausdorff limit of rescaled pointed metric spaces:

\[
\text{As-Cone}(\mathcal{M}(S)) := \mathcal{H} - \lim_{n \to \infty} (\mathcal{M}(S), x_0, \frac{1}{n} d_{\mathcal{M}}),
\]

where \(x_0\) is an arbitrarily chosen point of \(\mathcal{M}(S)\). We remark that in contrast to the case considered here, the definition of an asymptotic cone in general involves the use of ultrafilters, and the limit space may depend on the chosen ultrafilter. Various aspects of asymptotic cones of general spaces are discussed in Gromov’s essay [8]. In some cases asymptotic cones are easy to describe. For example, if \(V\) is a finite volume Riemannian manifold of strictly negative sectional curvature and with \(k\) cusps, then As-Cone(\(V\)) is a “cone” over \(k\) points, i.e., \(k\) rays with a common origin. For a Riemannian product \(V = V_1 \times V_2\), where \(V_1, V_2\) are as in the previous example and each has only one cusp, As-Cone(\(V\)) can be identified with the first quadrant in \(\mathbb{R}^2\). Much more intricate are quotients of \(SL(n, \mathbb{R})/SO(n)\) by congruence subgroups of \(SL_n(\mathbb{Z})\) or more general locally symmetric spaces \(V = \Gamma\backslash G/K\) of higher rank. For such \(V\) As-Cone(\(V\)) is *isometric* to the Euclidean cone over the finite simplicial complex given by the quotient of the *rational* Tits building of \(G\) modulo \(\Gamma\) (see [13], [18], [19]).

Theorem A allows us to determine the asymptotic cone of moduli space. Notice the striking similarity of Theorem B below with the result for locally symmetric spaces described above.

**Theorem B (Asymptotic cone of moduli space)** Let \(S\) and \(\Gamma\) be as in Theorem A. Further let the moduli space \(\mathcal{M}(S) = \Gamma\backslash \mathcal{T}(S)\) of Riemann surfaces be endowed with the metric \(d_{\mathcal{M}}\) induced by the Teichmüller metric. Then the asymptotic cone of \((\mathcal{M}(S), d_{\mathcal{M}})\) is bi-Lipschitz equivalent to the Euclidean cone over the finite simplicial complex \(\Gamma\backslash|\mathcal{C}(S)|\), where \(|\mathcal{C}(S)|\) is (a geometric realization of) the complex of curves of \(S\). In particular, \(\dim \text{As-Cone}(\mathcal{M}(S)) = \dim |\mathcal{C}(S)| + 1 = 3g - 3 + p = \frac{1}{2} \dim \mathcal{T}(S)\).

A variant of Theorem B also holds for the orbifold moduli space \(\text{Mod}_S\backslash \mathcal{T}(S)\), which is finitely covered by \(\Gamma\backslash \mathcal{T}(S)\). This confirms (the strong version of) a conjecture of Farb ([7], Conjecture 4.7). The proof of Theorem B, which we give in Section 3, is based on two key ingredients. The first one is McMullen’s modification of the Weil-Petersson metric [21]. In fact, we only need information on the bi-Lipschitz class of the Teichmüller metric and McMullen’s metric belongs to that class. We remark in passing that there are several other complete metrics comparable to the Teichmüller metric (see [20]). The second ingredient of the proof of Theorem B is an expansion of the WP-metric on \((\varepsilon, \sigma)\)-thin parts \(\text{Thin}_\varepsilon(\sigma; \mathcal{M}(S))\) due to Wolpert [26].
Remark. For an arithmetic lattice $\Gamma$ in a semisimple Lie group $G$ with associated locally symmetric space $\Gamma \backslash G/K$ one has $\dim \text{As-Cone}(\Gamma \backslash G/K) = \mathbb{Q}\text{-rank } \Gamma$ (see [17]). On the other hand, by Theorem A, $\dim \text{As-Cone}(\mathcal{M}(S)) = 3g - 3 + p$. The number $d(S) := 3g - 3 + p$, which measures the topological complexity of $S$, might thus be considered as “the” $\mathbb{Q}$-rank of the mapping class group; compare the discussion in [7, § 4.2]. The geometric rank of $\mathcal{T}(S)$ is the maximal dimension of a quasi-isometrically embedded Euclidean space. In [2] it is shown that the Weil-Petersson geometric rank of $\mathcal{T}(S)$ equals $\lfloor\frac{d(S)+1}{2}\rfloor$. Thus WP-rank $\mathcal{T}(S) \leq d(S) = \mathbb{Q}\text{-rank } \text{Mod}_S$. This is in contrast to the case of locally symmetric spaces where $\text{rank } G/K = \mathbb{R}\text{-rank } G \geq \mathbb{Q}\text{-rank } \Gamma$ and rises the question if the Teichmüller geometric rank dominates $\mathbb{Q}\text{-rank } \text{Mod}_S$. Notice that, by Theorem B, there are Euclidean balls of arbitrarily large radius quasi-isometrically embedded in $\mathcal{T}(S)$ and $\mathcal{M}(S)$.

Theorem A together with subresults of its proof also provides an obstruction to a Riemannian metric of positive scalar curvature in the quasi-isometry class of the Teichmüller metric.

**Theorem C (No positive scalar curvature)** Let $S = S_{g,p}$ be a compact, orientable surface with genus $g$ and $p$ punctures such that $d(S) = 3g - 3 + p \geq 2$ (i.e., $\mathbb{Q}\text{-rank } \text{Mod}_S \geq 2$). Further let $\Gamma \subset \text{Mod}_S$ be a torsion free, finite index subgroup. Then the corresponding moduli space $\mathcal{M}(S) = \Gamma \backslash \mathcal{T}(S)$ does not admit a finite volume Riemannian metric of uniformly bounded positive scalar curvature in the quasi-isometry class of the Teichmüller metric.

This result has also been conjectured by Farb ([7, Conjecture 4.6]). For locally symmetric spaces associated to arithmetic groups of $\mathbb{Q}\text{-rank } \geq 2$ the analogon of Theorem C was proven by Chang [9]. We show in Section 4 how the ideas and phenomena underlying his proof can be extended to the present setting of Teichmüller and moduli spaces.

**Notation.** We write $A \asymp B$ (resp. $A \succ B$) if there is a constant $C > 0$ such that $C^{-1}A < B < CA$ (resp. $CA > B$).

## 2 Reduction theory for mapping class groups

The goal of this section is to prove Theorem A of Section 1.
2.1 The complex of curves and submanifolds with corners

The following two lemmata will be used frequently.

Lemma 1 Let $\alpha \in C(S)$ and $x \in T(S)$. Then for any $h \in \text{Mod}_S$ holds

$$l_{\alpha}(h \cdot x) = l_{h^{-1} \cdot \alpha}(x).$$

Proof. If $x = (R, [f])$, with $f : R \longrightarrow S$, then $h \cdot x = (R, [h \circ f])$. By definition, $l_{\alpha}(x)$ is the hyperbolic length of the unique closed geodesic in the isotopy class of $f^{-1}(\alpha)$ in $R$: $l_{\alpha}(x) = L_{hyp}([f^{-1}(\alpha)])$. Hence $l_{\alpha}(h \cdot x) = L_{hyp}([f^{-1} \circ h^{-1}(\alpha)]) = l_{h^{-1} \cdot \alpha}(x)$. \hfill \Box

Lemma 2 There exists a universal constant $c = c(S)$ such that the following holds.

If $\alpha_i$, $0 \leq i \leq k$, are simple geodesic loops on a hyperbolic surface $x \in T(S)$ such that $l_{\alpha_i}(x) < c$ for all $0 \leq i \leq k$, then the loops $\alpha_i$ are disjoint and hence define a $k$-simplex in $C(S)$.

For a proof of Lemma 2 see [1], § 3.3.

Recall that for $\varepsilon > 0$ the $\varepsilon$-thick part of Teichmüller space is the set $\text{Thick}_{\varepsilon} T(S) := \bigcap \{x \in T(S) \mid l_{\alpha}(x) \geq \varepsilon \text{ for all vertices } \alpha \in C(S)\}$. The family of closed sets $\{x \in T(S) \mid l_{\alpha}(x) \leq \varepsilon\}$ turns out to be locally finite if $\varepsilon$ is sufficiently small. For every vertex $\alpha \in C(S)$ we further set $\mathcal{H}_\varepsilon(\alpha) := \{x \in \text{Thick}_\varepsilon T(S) \mid l_{\alpha}(x) = \varepsilon\}$. Then (for fixed $\varepsilon$ sufficiently small) we have $\partial \text{Thick}_\varepsilon T(S) = \bigcup \mathcal{H}_\varepsilon(\alpha).

The following proposition is due to Ivanov (see [14], § 4.6 and also [10], Ch. 3).

Proposition 1 There is $\varepsilon_0 > 0$ depending only on $S$ such that for all $\varepsilon \leq \varepsilon_0$ the following holds: (1) The $\varepsilon$-thick part $\text{Thick}_\varepsilon T(S)$ is a submanifold of $T(S)$ with corners, i.e., locally modelled on a cube $[0,1]^n \subset \mathbb{R}^n$, and invariant under $\Gamma$.

(2) The complex of curves $C(S)$ is the nerve of the covering of $\partial \text{Thick}_\varepsilon T(S)$ by the closed sets $\mathcal{H}_\varepsilon(\alpha)$. In particular there is a one-to-one correspondence between the boundary faces $\bigcap_{\alpha \in \sigma} \mathcal{H}_\varepsilon(\alpha)$ of $\partial \text{Thick}_\varepsilon T(S)$ and the simplices of $C(S)$.

(3) For $\sigma \in C(S)$ the boundary face $\bigcap_{\alpha \in \sigma} \mathcal{H}_\varepsilon(\alpha)$ in $\partial \text{Thick}_\varepsilon T(S)$ is diffeomorphic to $\text{Thick}_\varepsilon T(S_{\sigma}) \times \mathbb{R}^{\lvert \sigma \rvert}$, where $S_{\sigma}$ is the result of cutting $S$ along (non-intersecting) circles from the isotopy classes $\alpha \in \sigma$ and such that the length of each boundary circle is $\varepsilon$.

We emphasize the following simple consequence of Proposition 1 (2).

Lemma 3 To any point $x \in \partial \text{Thick}_\varepsilon T(S)$ there corresponds a unique simplex $\sigma \in C(S)$ such that $x \in \bigcap_{\alpha \in \sigma} \mathcal{H}_\varepsilon(\alpha)$ and $\sigma$ is maximal for this condition.
Proof. In view of Proposition \([1](2)\) it suffices to observe that
\[
\bigcap_{\alpha \in \sigma} \mathcal{H}_\varepsilon(\alpha) \supset \bigcap_{\alpha \in \tau} \mathcal{H}_\varepsilon(\alpha)
\]
if and only if \(\sigma \subset \tau\). \(\square\)

Recall from Section 1 that instead of the mapping class group \(\text{Mod}_S\) itself we consider finite index subgroups \(\Gamma\), which are torsion free and in addition consist of pure mapping classes. This allows us to eliminate all difficulties related to elements of finite order and to possible permutations of components of cut surfaces by reducible diffeomorphisms.

A subgroup \(\Gamma \subset \text{Mod}_S\) with the above properties leaves invariant the \(\varepsilon\)-thick part \(\text{Thick}_\varepsilon T(S)\) of \(\text{Teichm"uller space}\) (by Lemma \([1]\)) and acts freely on it. The corresponding quotient \(\text{Thick}_\varepsilon \mathcal{M}(S) := \Gamma \backslash \text{Thick}_\varepsilon T(S)\), which we call the \(\varepsilon\)-thick part of moduli space, is compact (see \([14]\), § 4). This yields the following corollary. For an analogous result for locally symmetric spaces see \([17]\).

**Corollary 1** There exists \(\varepsilon_0 > 0\) depending only on \(S\), such that there is a \(\Gamma\)-invariant exhaustion of \(\text{Teichm"uller space} T(S) = \bigcup_{\varepsilon \leq \varepsilon_0} \text{Thick}_\varepsilon T(S)\), which induces an exhaustion of moduli space \(\mathcal{M}(S) = \bigcup_{\varepsilon \leq \varepsilon_0} \text{Thick}_\varepsilon \mathcal{M}(S)\).

We choose \(\varepsilon \leq \varepsilon_0\), where \(\varepsilon_0\) is as in Proposition \([1]\). By Proposition \([1](2)\) there is a one-to-one correspondence between the boundary faces of \(\partial \text{Thick}_\varepsilon T(S)\) and the simplices of the complex of curves \(\mathcal{C}(S)\). Clearly, the torsion-free finite index subgroup \(\Gamma\) of the mapping class group acts equivariantly with respect to that correspondence. The quotient \(\Gamma \backslash \mathcal{C}(S)\) is a finite simplicial complex (see \([11]\), Proposition 1). We denote by \(\mathcal{E}\) the set of simplices of \(\Gamma \backslash \mathcal{C}(S)\) or, equivalently, the set of the boundary faces of \(\partial \text{Thick}_\varepsilon \mathcal{M}(S) = \Gamma \backslash \partial \text{Thick}_\varepsilon T(S)\). The next proposition describes the fine structure of the boundary \(\partial \text{Thick}_\varepsilon \mathcal{M}(S)\).

**Proposition 2** Let \(\varepsilon_0 > 0\) be as in Proposition \([1]\). Then, for each \(\varepsilon \leq \varepsilon_0\), the boundary \(\partial \text{Thick}_\varepsilon \mathcal{M}(S)\) consists of a finite number of faces \(\mathcal{B}_\varepsilon(\sigma), \sigma \in \mathcal{E}\). Each \(\mathcal{B}_\varepsilon(\sigma)\) is a (trivial) torus bundle over (a finite covering of) the \(\varepsilon\)-thick part of the moduli space of the cut surface \(S_\sigma\) (i.e., the surface obtained by cutting \(S\) along the circles of \(\sigma\)):

\[
0 \to T|\sigma| \to \mathcal{B}_\varepsilon(\sigma) \to \text{Thick}_\varepsilon \mathcal{M}(S_\sigma) \to 0.
\]

Moreover, the nerve of the covering of \(\partial \text{Thick}_\varepsilon \mathcal{M}(S)\) by these faces is isomorphic to the finite simplicial complex \(\Gamma \backslash \mathcal{C}(S)\). For a simplex \(\tau\) of maximal dimension \(d(S)\), \(\mathcal{M}(S_\tau)\) and hence \(\text{Thick}_\varepsilon \mathcal{M}(S_\tau)\) reduces to a point and \(\mathcal{B}_\varepsilon(\tau)\) is a \(d(S)\)-dimensional torus.
Proof. Given Proposition [1], it remains to determine the stabilizer in \( \Gamma \) of a boundary face \( \bigcap_{\alpha \in \sigma} \mathcal{H}_\varepsilon(\alpha) \) of \( \partial \text{Thick}_\varepsilon \mathcal{T}(S) \). Thus, we assume that for \( x \) in \( \bigcap_{\alpha \in \sigma} \mathcal{H}_\varepsilon(\alpha) \) (with \( \sigma \) maximal as in Lemma [3]) there is \( h \in \Gamma \) such that \( h \cdot x \) also is in \( \bigcap_{\alpha \in \sigma} \mathcal{H}_\varepsilon(\alpha) \). Since \( l_{h^{-1} \cdot \alpha}(x) = l_\alpha(h \cdot x) \) for all \( \alpha \in \sigma \) by Lemma [1], we have \( x \in \bigcap_{\beta \in h^{-1} \cdot \sigma \vee \sigma} \mathcal{H}_\varepsilon(\beta) \) and as \( \sigma \) is maximal for this condition \( h \) must stabilize \( \sigma \): \( h \cdot \sigma = \sigma \).

Now by [12], § 4, in general the stabilizer of \( \sigma \) in \( \text{Mod}_S \) is a group extension of a finite group (a subgroup of the automorphism group, say \( A(\sigma) \), of the decomposition graph associated to the set of circles corresponding to \( \sigma \)) over the normalizer of the twist group \( \text{Tw}(\sigma) \) of \( \sigma \), i.e., the free abelian group generated by the \( |\sigma| \) Dehn twists \( \tau_\alpha \) of circles of \( \sigma \). Since \( \Gamma \) consists of pure mapping classes we have \( \Gamma \cap A(\sigma) = \{ \text{id} \} \). Moreover, we also have \( h^{-1} \cdot \alpha = \alpha \) for all \( \alpha \in \sigma \) and thus \( h^{-1} \tau_\alpha h = \tau_{h^{-1} \cdot \alpha} = \tau_\alpha \) for all \( \alpha \in \sigma \). Hence \( h \in \text{stab}_\Gamma(\sigma) = Z_\Gamma(\text{Tw}(\sigma)) \).

Finally, in order to determine \( Z_\Gamma(\text{Tw}(\sigma)) \), note that a pure mapping class \( h \) which commutes with a Dehn twist “of \( \sigma \)” preserves setwise every circle of \( \sigma \) and every part of \( S \setminus \sigma \) (see [15], 7.5). Thus \( h \) belongs to the direct product \( (\text{Mod}_{S_\sigma} \cap \Gamma) \times \text{Tw}(\sigma) \) and the result follows. \( \square \)

2.2 Tilings of Teichmüller spaces

Recall that the complex of curves \( \mathcal{C}(S) \) is a thick chamber complex. Given a simplex \( \sigma \in \mathcal{C}(S) \) we can thus choose a simplex \( \tau \in \mathcal{C}(S) \) of maximal dimension \( d(S) - 1 = 3g - 4 + p \) containing \( \sigma \). Then there are adapted Fenchel-Nielsen coordinates on Teichmüller space \( \mathcal{T}(S) \) (see e.g., [1]), i.e., a diffeomorphism

\[
\Phi_\tau : \mathcal{T}(S) \longrightarrow \mathcal{T}(S_\sigma) \times \mathbb{R}^{\left|\sigma\right|} \times \mathbb{R}_{>0}^{\left|\sigma\right|} ; \quad p \longmapsto (s(p), \theta(p), l(p)),
\]

where \( s = (\theta_\beta)_{\beta \in \tau \setminus \sigma} \) parametrizes \( \mathcal{T}(S_\sigma) \), \( \theta = (\theta_\alpha)_{\alpha \in \sigma} \) are twist parameters on \( \mathbb{R}^{\left|\sigma\right|} \) and \( l = (l_\alpha)_{\alpha \in \sigma} \) are coordinates on \( \mathbb{R}_{>0}^{\left|\sigma\right|} \) (here and elsewhere \( \left|\sigma\right| \) denotes the number of vertices of \( \sigma \)).

Consider \( x \in \partial \text{Thick}_\varepsilon \mathcal{T}(S) \). By Lemma [3], \( x \) determines a unique (largest) simplex \( \sigma \in \mathcal{C}(S) \). Let \( \tau \in \mathcal{C}(S) \) be a simplex of maximal dimension containing \( \sigma \) and let \( \Phi_\tau \) be adapted FN-coordinates. We define the outer cone \( \text{at} \ x \in \partial \text{Thick}_\varepsilon \mathcal{T}(S) \) as the preimage

\[
CO(x) := \Phi_\tau^{-1}\{ (s(x), \theta(x), (l_\alpha)_{\alpha \in \sigma}) \mid l_\alpha < \varepsilon \text{ for all } \alpha \in \sigma \}.
\]

Note that \( CO(x) \) is diffeomorphic to the open hyperoctant \( \mathbb{R}_{>0}^{\left|\sigma\right|} \).
Lemma 4 This definition of an outer cone $CO(x)$ is independent of the chosen simplex $\tau$ of maximal dimension (resp. its adapted FN-coordinates $\Phi_\tau$). Furthermore, the set of all outer cones of $\{CO(x) \mid x \in \partial\text{Thick}_\varepsilon T(S)\}$ is $\Gamma$-invariant.

Proof. If $\tilde{\tau}$ is another simplex of maximal dimension in $C(S)$ that also contains $\sigma$ we have a diffeomorphism (adapted FN-coordinates)

$$\Phi_\tau : T(S) \rightarrow T(S_\sigma) \times \mathbb{R}^{|\sigma|} \times \mathbb{R}_{>0}^{|\sigma|}; \quad p \mapsto (\tilde{s}(p), \tilde{\theta}(p), (l_\alpha(p))_{\alpha \in \sigma}).$$

Hence

$$\Phi_\tau \circ \Phi^{-1}_{\tilde{\tau}}((\tilde{s}(p), \tilde{\theta}(p), (l_\alpha(p))_{\alpha \in \sigma}) = (s(p), \theta(p), (l_\alpha(p))_{\alpha \in \sigma}), \quad p \in T(S).$$

Since $(l_\alpha)_{\alpha \in \sigma}$ are coordinate functions (for both $\Phi_\tau$ and $\Phi_{\tilde{\tau}}$) the implicit function theorem implies that there is a diffeomorphism

$$\varphi : T(S_\sigma) \times \mathbb{R}^{|\sigma|} \rightarrow T(S_\sigma) \times \mathbb{R}^{|\sigma|}; \quad (\tilde{s}(p), \tilde{\theta}(p)) \mapsto (s(p), \theta(p)).$$

In particular we have $(\tilde{s}, \tilde{\theta}) = \text{const.} \Leftrightarrow (s, \theta) = \varphi((\tilde{s}, \tilde{\theta})) = \text{const.}$ This proves the first claim. The invariance under $\Gamma$ follows from Lemma 11. \hfill \Box

Any of the submanifolds with corners $\text{Thick}_\varepsilon T(S)$ of Teichmüller space as described in Proposition 11 together with its outer cones yields a tiling (or dissection) of $T(S)$ into disjoint subsets.

Proposition 3 Let $\varepsilon_0 > 0$ be as in Proposition 11 and $\varepsilon \leq \varepsilon_0$. Then Teichmüller space can be written as a $\Gamma$-invariant disjoint union of the $\varepsilon$-thick part and all outer cones:

$$T(S) = \text{Thick}_\varepsilon T(S) \sqcup \bigcup_{x \in \partial \text{Thick}_\varepsilon T(S)} CO(x) = \text{Thick}_\varepsilon T(S) \sqcup \bigsqcup_{\sigma \in C(S)} \bigcup_{x \in \bigcap_{\alpha \in \sigma} H_\varepsilon(\alpha)} CO(x).$$

Proof. Pick $y \in T(S) \setminus \text{Thick}_\varepsilon T(S)$. By Lemma 12 there is a maximal simplex $\sigma \in C(S)$ (not necessarily of maximal dimension) such that $l_\alpha(y) < \varepsilon$ for all $\alpha \in \sigma$. Choose a simplex $\tau$ of maximal dimension which contains $\sigma$ as a subsimplex. In adapted FN-coordinates we have $\Phi_\tau(y) = (s(y), \theta(y), (l_\alpha(y))_{\alpha \in \sigma})$. By definition of $\sigma$ we have $l_\beta(y) \geq \varepsilon$ for all $\beta \in C(S) \setminus \sigma$. Thus we find that $x := (s(y), \theta(y), (\varepsilon)_{\alpha \in \sigma}) \in \partial\text{Thick}_\varepsilon T(S)$ and hence that $y \in CO(x)$ by definition of the latter.

We next show that the outer cones are disjoint. Assume that $z \in CO(x) \cap CO(y)$, $x, y \in \partial\text{Thick}_\varepsilon T(S)$. By Lemma 13 there are unique simplices $\sigma$ resp. $\mu$ in $C(S)$ associated to $x$ resp. $y$ and, by definition of outer cones, $l_\alpha(z) < \varepsilon$ for all $\alpha \in \sigma \cup \mu$. By Lemma 12, $\sigma \vee \mu$ is a simplex in $C(S)$ and thus contained in a simplex of
maximal dimension, say $\tau$. We can thus write $CO(x) = \Phi_\tau^{-1}\{(s(x), \theta(x), (l_\alpha)_{\alpha \in \sigma}) \mid l_\alpha < \varepsilon \text{ for all } \alpha \in \sigma\}$. In particular we have $l_\beta(v) \geq \varepsilon$ for all $v \in CO(x)$ and all $\beta \in \tau \setminus \sigma$. Our assumption on $z$ thus implies that $\mu \subseteq \sigma$. Using $CO(y)$ instead of $CO(x)$ we get $\sigma \subseteq \mu$ and hence $\sigma = \mu$. The formulae for outer cones further yield

$$(s(x), \theta(x), (l_\alpha(z))_{\alpha \in \sigma}) = \Phi_\tau(z) = (s(y), \theta(y), (l_\alpha(z))_{\alpha \in \sigma}).$$

This implies that $x = y$ and hence that $CO(x) = CO(y)$ which proves the claim.

In conclusion we have a disjoint union $T(S) = \text{Thick}_\varepsilon T(S) \sqcup \bigsqcup_{x \in \partial T} \text{Thick}_\varepsilon T(S) CO(x)$. The remaining assertion then follows from Proposition 1.

Given $0 < \varepsilon \leq \varepsilon_0$ as in Proposition 3 and $\sigma \in C(S)$ we define the $(\varepsilon, \sigma)$-thin part of Teichmüller space $T(S)$ as a disjoint union of outer cones

$$\text{Thin}_\varepsilon(T(S), \sigma) := \bigsqcup_{x \in \bigcap_{\alpha \in \sigma} H_\varepsilon(\alpha)} CO(x).$$

We can then rewrite the tiling in Proposition 3 as

$$T(S) = \text{Thick}_\varepsilon T(S) \sqcup \bigsqcup_{\sigma \in C(S)} \text{Thin}_\varepsilon(T(S), \sigma).$$

Further note that by Proposition 1 and Proposition 3

$$\text{Thin}_\varepsilon(T(S), \sigma) \cong \text{Thick}_\varepsilon T(S_\sigma) \times \mathbb{R}^{\left|\sigma\right|} \times \mathbb{R}_{\geq 0}^{\left|\sigma\right|}.$$

The $\Gamma$-invariant tiling of Teichmüller space obtained in Proposition 3 induces a corresponding tiling of moduli space as stated in Theorem A in Section 1.

### 2.3 A tiling of moduli space: The proof of Theorem A

We first show that the canonical projection $\pi : T(S) \to M(S)$ restricted to any outer cone is injective. Thus let $CO(x)$ be an outer cone. By its definition it determines a simplex $\sigma \in C(S)$ and we can choose adapted FN-coordinates $\Phi_\tau$ with $\sigma \subseteq \tau$. For two points $p$ and $q$ in $CO(x)$ consider their adapted coordinates, say $\Phi_\tau(p) = (s*, \theta*, (l_\alpha(p))_{\alpha \in \sigma})$ and $\Phi_\tau(q) = (s*, \theta*, (l_\alpha(q))_{\alpha \in \sigma})$. If $\pi(p) = \pi(q)$, i.e., $q = h \cdot p$ for some $h \in \Gamma$, then by Lemma 1

$$l_{h^{-1} \cdot \alpha}(p) = l_\alpha(h \cdot p) = l_\alpha(q) < \varepsilon \quad \text{for all } \alpha \in \sigma.$$

Therefore $l_\beta(p) < \varepsilon$ for all $\beta \in h^{-1} \cdot \sigma \cup \sigma$, so that $h^{-1} \cdot \sigma \cup \sigma$ is a simplex in $C(S)$ (Lemma 2). We now argue as in the proof of Proposition 3 to conclude that $h^{-1} \cdot \sigma = \sigma$. 

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Since $\Gamma$ consists of pure mapping classes we actually have $h^{-1} \cdot \alpha = \alpha$ for all $\alpha \in \sigma$. Hence, by the above equalities, $l_{\alpha}(p) = l_{\alpha}(q)$ for all $\alpha \in \sigma$, which then implies that $\Phi_{\tau}(p) = \Phi_{\tau}(q)$ and thus $p = q$.

Together with the $\Gamma$-invariance of the set of outer cones (see Lemma 4) this allows us to define outer cones in moduli space: for $x \in \text{Thick}_{\varepsilon}M(S)$ we set $CO(x) := \pi(CO(\hat{x}))$ where $\hat{x}$ is any lift of $x$.

From Proposition 2 and Proposition 3 we now get

$M(S) = \text{Thick}_{\varepsilon}M(S) \sqcup \bigcup_{x \in \partial \text{Thick}_{\varepsilon}M(S)} CO(x) = \text{Thick}_{\varepsilon}M(S) \sqcup \bigcup_{\sigma \in \mathcal{E}} \bigcup_{x \in B_{\varepsilon}(\sigma)} CO(x)$. 

Finally, we set $\text{Thin}_{\varepsilon}(M(S), \sigma) := \{CO(x) \mid x \in B_{\varepsilon}(\sigma)\}$. This completes the proof of Theorem A.

We next reformulate Theorem A in a way which emphasizes the remarkable parallels with precise reduction theory for arithmetic groups (see e.g. [23], Theorem 3.4 or [24], Theorem 9.6).

Recall that a rational parabolic subgroup of an algebraic group $G$ defined over $\mathbb{Q}$ fixes pointwise a simplex of the rational Tits building of $G$ (see e.g. [17], Lemma 1.2). Analogously, we call a subgroup $P$ of $\Gamma \subset \text{Mod}_{S}$ parabolic if it fixes pointwise a simplex of the complex of curves $C(S)$. Note that this is in agreement with the characterization of reducible (torsionfree) mapping classes in [12].

**Corollary 2** Let $P_1, \ldots, P_n$ be representatives of conjugacy classes of the parabolic subgroups of $\Gamma$ (corresponding to simplices $\sigma_1, \ldots, \sigma_n \in C(S)$). Then there exist subsets $\omega_i$ and $\Omega_i$, $i = 1, \ldots, n$, of Teichmüller space $T(S)$, where $\omega_i$ is bounded and $\Omega_i$ is diffeomorphic to $\omega_i \times \mathbb{R}_{>0}$, such that

1. The canonical projection $\pi : T(S) \longrightarrow M(S)$ maps each $\Omega_i$ injectively into moduli space $M(S)$.
2. Each image $\pi(\omega_i)$ in $M(S)$ is compact.
3. For $\varepsilon$ sufficiently small, the moduli space can be decomposed into a disjoint union

$$M(S) = \text{Thick}_{\varepsilon}M(S) \sqcup \bigsqcup_{i=1}^{n} \pi(\Omega_i).$$

**Proof.** In view of Theorem A we only have to show the existence of the sets $\omega_i$ and $\Omega_i$. Choose $\varepsilon \leq \varepsilon_0$ as in Proposition 1. By Proposition 2 the boundary face $B_{\varepsilon}(\sigma_i)$ of $\text{Thick}_{\varepsilon}M(S)$ is a torus bundle over $\text{Thick}_{\varepsilon}M(S_{\sigma_i})$ and in particular compact. We can thus choose $\omega_i \subset \bigcap_{\alpha \in \sigma_i} H_{\alpha}$ as a relatively compact fundamental domain for the
action of \((\text{Mod}_S \cap \Gamma) \times \text{Tw}(\sigma)\) (compare the proof of Proposition 2). Finally we set \(\Omega_i := \bigcup_{x \in \omega_i} CO(x)\). The assertions then follow from Theorem A. \(\square\)

Corollary 2 should be compared with the formulation of precise reduction theory for arithmetic groups given in [4], Proposition III.2.21. Notice that in this comparison outer cones are the analoga of Weyl chambers.

### 3 The asymptotic cone of the moduli space

In this section we prove Theorem B of Section 1. In particular, we show that the canonical projection map \(\pi : T(S) \to M(S)\) restricted to certain (coarse) Euclidean cones in \(T(S)\) is bi-Lipschitz.

The asymptotic cone of the moduli space \(M(S)\) (see Section 1) is eventually identified with a Euclidean cone which is constructed as follows: We endow each simplex of the complex of curves \(\mathcal{C}(S)\) with a spherical metric and thus obtain a geometric realization \(|\mathcal{C}(S)|\) of \(\mathcal{C}(S)\) as a spherical complex. This spherical metric induces a distance function \(d_{\mathcal{C}(S)}(x, y)\) on the finite simplical complex \(\Gamma|\mathcal{C}(S)|\). The Euclidean cone \(\text{Cone}(\Gamma|\mathcal{C}(S)|)\) over \(\Gamma|\mathcal{C}(S)|\) is defined as the product \([0, \infty) \times \Gamma|\mathcal{C}(S)|\) with \(\{0\} \times \Gamma|\mathcal{C}(S)|\) collapsed to a point \(O\) and endowed with the cone metric \(d^2_C((a, x), (b, y)) := a^2 + b^2 - 2ab \cos(\min\{\pi, d_{\mathcal{C}(S)}(x, y)\})\) (see [5]).

Alternatively one can construct \(\text{Cone}(\Gamma|\mathcal{C}(S)|)\) as follows. Let \(E\) be the set of simplices of the finite complex \(\Gamma|\mathcal{C}(S)|\). Let \(\tau_i \in E, i = 1, \ldots, m\), be the simplices of maximal dimension \(d(S) - 1\). Let \(B_{\varepsilon_0}(\tau_i)\), \(i = 1, \ldots, m\), be the corresponding minimal boundary faces of \(\partial\text{Thick}_{\varepsilon_0} M(S)\) (which are \(d(S)\)-dimensional tori, see Proposition 2). For each \(i \in \{1, \ldots, m\}\) choose a point \(x_i \in B_{\varepsilon_0}(\tau_i)\) and consider the closure of the (maximal) outer cone \(CO(x_i) \subset M(S)\). Each \(CO(x_i)\) is diffeomorphic to the cone \(\mathbb{R}^{d(S)}_{\geq 0}\) (and actually, by Proposition 4 below, quasi-isometric to this Euclidean hyperoctant with respect to the Teichmüller metric). The faces of the cones \(CO(x_i)\) correspond to subsimplices of \(\tau_i\). Two cones \(CO(x_i)\) and \(CO(x_j)\) are pasted together along faces corresponding to subsimplices \(\sigma_i \subset \tau_i\) and \(\sigma_j \subset \tau_j\) if and only if there is \(h \in \Gamma\) such that \(h \cdot \sigma_i = \sigma_j\) (the action of \(\Gamma \subset \text{Mod}_S\) being that on the complex of curves).

#### 3.1 Approximations of the Teichmüller metric on thin parts

We want to study the coarse geometry of moduli space \(M(S)\) and Teichmüller space \(T(S)\). More precisely, we are interested in properties of the bi-Lipschitz class of the Teichmüller Finsler metric. In what follows we describe explicit representatives of that
bi-Lipschitz class on \((\varepsilon, \sigma)\)-thin parts of \(T(S)\) (and \(M(S)\)). Our approach has two key ingredients. The first is a result of McMullen. He showed that \(M(S)\) is Kähler hyperbolic in the sense of Gromov and thus in particular carries a complete finite volume Riemannian metric of bounded sectional curvature. The McMullen metric is a modification of the (incomplete) Weil-Petersson metric and quasi-isometric to the Teichmüller metric (see [21]). The second key ingredient is an expansion of the Weil-Petersson (WP) metric due to Wolpert (see [25], [26]).

For each length function \(l_\alpha\) we set 
\[
u_\alpha := -\log l_\alpha^{1/2},
\]
Considering this logarithmic root length instead of \(l_\alpha\) itself is suggested by work of Wolpert (see e.g. [25], [26]). Following Wolpert we also set \(\lambda_\alpha := \text{grad} l_\alpha^{1/2}\) (resp. \(\nu_\alpha := -\text{grad} u_\alpha\)) and define the Fenchel-Nielsen-gauge as the differential 1-form 
\[
\rho_\alpha := 2\pi \left( \frac{\langle \lambda_\alpha, \lambda_\alpha \rangle}{\langle J\lambda_\alpha, J\lambda_\alpha \rangle} \right) - 1 \langle J\nu_\alpha, J\nu_\alpha \rangle,
\]
Note that this gauge is normalized such that \(l_\alpha(T_\alpha) = 1\) for the WP-unit infinitesimal FN angle variation. We also set \(\tilde{\rho}_\alpha := l_\alpha^{-1/2} \rho_\alpha\).

**Proposition 4** There is \(\varepsilon_* > 0\) depending only on \(S\), such that for \(\sigma \in C(S), \varepsilon \leq \varepsilon_*\) and the \((\varepsilon, \sigma)\)-thin part 
\[
\text{Thin}_\varepsilon(T(S), \sigma) := \bigcup_{x \in \bigcap_{\alpha \in \sigma} H_\varepsilon(\alpha)} \text{CO}(x) \cong \text{Thick}_\varepsilon T(S_\sigma) \times \mathbb{R}^{1|\sigma} \times \mathbb{R}^{|\sigma|},
\]
the following Finsler metric expansion of the Teichmüller metric with respect to adapted FN-coordinates holds
\[
\|v\|^2_{T(S)} \asymp \|v\|^2_{T(S_\sigma)} + \sum_{\alpha \in \sigma} e^{-6u_\alpha} \tilde{\rho}_\alpha^2 + du_\alpha^2.
\]
The bi-Lipschitz constants involved in this estimate only depend on \(\sigma\) and \(\varepsilon_*\).

On \((\varepsilon, \sigma)\)-thin parts the Teichmüller metric is thus quasi-isometric to a product metric of a lower dimensional Teichmüller space times a product of hyperbolic horoballs. We remark that Minsky also proved a comparison result for such regions but using the sup-metric on the products of horoballs (see [22]).

**Proof.** McMullen’s modification of the WP metric on Teichmüller resp. moduli space is comparable to the Teichmüller metric. More precisely, there is \(\varepsilon_1\) (sufficiently small and depending only on \(S\)) such that for \(\varepsilon \leq \varepsilon_1\) and a tangent vector \(v \in T_x T(S)\) the McMullen metric approximation is given by the formula
\[
\|v\|^2_T \asymp \|v\|^2_{WP} + \sum_{l_\gamma(x) < \varepsilon} |\frac{\partial l_\gamma}{l_\gamma}(v)|^2,
\]
where the sum is taken over all \( \gamma \in \mathcal{C}(S) \) such that \( l_\gamma(x) < \varepsilon \) (see [21], Theorem 1.7). Here the Lipschitz constants only depend on \( \varepsilon_1 \). Note that on the \( \varepsilon \)-thick part there is no modification and in particular \( \|v\|_T \asymp \|v\|_{WP} \) for \( v \in T_\varepsilon \text{Thick}_\varepsilon T(S) \).

In [20] § 1 and § 4, Wolpert provides an expansion of the Weil-Petersson metric on an \((\varepsilon_2, \sigma)\)-thin part \( \text{Thin}_\varepsilon(S, \sigma) \) considered as a neighbourhood of a point of the positive codimension stratum \( S(\sigma) \) of the Weil-Petersson-completion \( \overline{T(S)} \) of the Teichmüller space. This augmented Teichmüller space consists of marked node Riemann surfaces and is a stratified (non locally compact) space (see e.g. [25]). The strata are indexed by the simplices of \( \mathcal{C}(S) \) and are (products of) lower dimensional Teichmüller spaces. A boundary stratum \( S(\sigma) \subset \overline{T(S)} \) consists of those marked Riemann surfaces whose nodes correspond bijectively to the vertices of \( \sigma \) and is isomorphic to the Teichmüller space \( T(S_\sigma) \) of the cut surface \( S_\sigma \) (obtained by cutting \( S \) along the circles of \( \sigma \)). Using the FN-gauges \( \rho_\alpha \) Wolpert’s expansion of the WP-metric can be written as

\[
\langle \cdot, \cdot \rangle_{WP} \asymp \langle \cdot, \cdot \rangle^{T(S_\sigma)} + \sum_{\alpha \in \sigma} |\partial l_\alpha^{1/2}|^2
\]

\[
\asymp \langle \cdot, \cdot \rangle^{T(S_\sigma)} + \sum_{\alpha \in \sigma} (d(l_\alpha^{1/2} \circ J))^2 + (dl_\alpha^{1/2})^2 \asymp \langle \cdot, \cdot \rangle^{T(S_\sigma)} + \sum_{\alpha \in \sigma} l_\alpha^2 \rho_\alpha^2 + (dl_\alpha^{1/2})^2.
\]

The involved Lipschitz constants depend on \( \sigma \) and \( \varepsilon_2 \).

We now apply these estimates to the \((\varepsilon, \sigma)\)-thin part

\[
\text{Thin}_\varepsilon(T(S), \sigma) \cong \text{Thick}_\varepsilon T(S_\sigma) \times \mathbb{R}^{[\sigma]} \times \mathbb{R}_{>0}^{[\sigma]}.
\]

Recall that \( \|\cdot\|_T \asymp \|\cdot\|_{WP} \) on \( \text{Thick}_\varepsilon T(S_\sigma) \). We set \( \varepsilon_* := \min(\varepsilon_1, \varepsilon_2) \) and let \( \varepsilon \leq \varepsilon_* \). Inserting the McMullen formula in the above expansion and using that \( |\partial l_\gamma(v)|^2 = (dl_\gamma(v))^2 + (dl_\gamma(Jv))^2 \) and \( dl_\gamma = 2l_\gamma^{1/2}dl_\gamma^{1/2} \) we obtain

\[
\|\cdot\|_{T(S)}^2 \asymp \|\cdot\|_{T(S_\sigma)}^2 + \sum_{\alpha \in \sigma} (d(l_\alpha^{1/2} \circ J))^2 + \frac{1}{l_\alpha} + (dl_\alpha^{1/2})^2(1 + \frac{1}{l_\alpha})
\]

\[
\asymp \|\cdot\|_{T(S_\sigma)}^2 + \sum_{\alpha \in \sigma} \frac{1}{l_\alpha} (d(l_\alpha^{1/2} \circ J))^2 + \frac{1}{l_\alpha} (dl_\alpha^{1/2})^2 \quad (\text{since } l_\alpha \leq \varepsilon)
\]

\[
\asymp \|\cdot\|_{T(S_\sigma)}^2 + \sum_{\alpha \in \sigma} l_\alpha^2 \rho_\alpha^2 + \frac{1}{l_\alpha} (dl_\alpha^{1/2})^2 \quad (\text{by Wolpert’s expansion above}).
\]

We next set \( \tilde{T}_\alpha := l_\alpha^{1/2}T_\alpha \) for \( T_\alpha \) the WP-unit infinitesimal FN-angle variation. Then \( \|\tilde{T}_\alpha\|_{T(S)} \asymp l_\alpha^{-1/2}\|\tilde{T}_\alpha\|_{WP} = 1 \) and hence \( \rho_\alpha = l_\alpha^{-1/2}\tilde{\rho}_\alpha \) is the renormalized FN gauge: \( \rho_\alpha(T_\alpha) = 1 \). Substituting this together with \( u_\alpha = -\log l_\alpha^{1/2} \) in the previous expression we eventually get the claimed estimate

\[
\|\cdot\|_{T(S)}^2 \asymp \|\cdot\|_{T(S_\sigma)}^2 + \sum_{\alpha \in \sigma} e^{-6u_\alpha} \tilde{\rho}_\alpha^2 + du_\alpha^2.
\]
The Lipschitz constants of this comparison depend only on $\sigma$ and $\varepsilon_*$. \hfill \Box

**Corollary 3** The Finsler metric approximation on the $(\varepsilon, \sigma)$-thin parts $\text{Thin}_\varepsilon(T(S), \sigma)$ of $T(S)$ given in Proposition 4 descends to the $(\varepsilon, \sigma)$-thin parts $\text{Thin}_\varepsilon(M(S), \sigma) \cong B_\varepsilon(\sigma) \times \mathbb{R}_{>0}$ of $M(S)$.

**Proof.** The canonical projection $\pi : T(S) \rightarrow M(S)$ is a Finsler covering for the (induced) Teichmüller Finsler metrics. By Proposition 2 and Theorem A (and the subresults of their proofs) we further have

$$\text{Thin}_\varepsilon(M(S), \sigma) = (\text{Mod}_{S, \sigma} \cap \Gamma) \times \text{Tw}(\sigma) \backslash \text{Thin}_\varepsilon(T(S), \sigma).$$

Finally, note that the group $(\text{Mod}_{S, \sigma} \cap \Gamma) \times \text{Tw}(\sigma)$ leaves the factorization

$$\text{Thin}_\varepsilon(T(S), \sigma) \cong \text{Thick}_\varepsilon T(S_{\sigma}) \times \mathbb{R}^{|\sigma|} \times \mathbb{R}_{>0}$$

used in Proposition 4 invariant. \hfill \Box

**Corollary 4** For adapted FN-coordinates $(s, \theta, u)$ and $\varepsilon > 0$ as in Proposition 4 define the projection map

$$\pi_u : \text{Thin}_\varepsilon(T(S), \sigma) \cong \text{Thick}_\varepsilon T(S_{\sigma}) \times \mathbb{R}^{|\sigma|} \times \mathbb{R}_{>0} \rightarrow \mathbb{R}^{|\sigma|} ; \quad (s, \theta, u) \mapsto u.$$

Then $\pi_u$ is length non-increasing on the the $(\varepsilon, \sigma)$-thin part $\text{Thin}_\varepsilon(S, \sigma)$ with respect to the restrictions of the approximate Teichmüller metric. On $\mathbb{R}_{>0}$ this restriction is comparable with the Euclidean metric.

### 3.2 Bi-Lipschitz images of Euclidean cones and a net in moduli space

Throughout this section we choose a fixed positive real number $\varepsilon_0$ small enough such that both Proposition 1 and Proposition 4 hold. As usual we denote by $\pi : T(S) \rightarrow M(S)$ the canonical projection. The metric on $M(S)$ is given by $d_M(\pi(x), \pi(y)) = \inf_{h \in \Gamma} d_T(x, h \cdot y)$. Recall from Section 2.3 that outer cones in $M(S)$ are projections of outer cones in $T(S)$.

We next point out metric properties of the tiling in Theorem A.

**Proposition 5** Let $\varepsilon \leq \varepsilon_0$ be as in Proposition 4 and let $CO(x)$, $x \in B_\varepsilon(\sigma)$, be an outer cone of $\text{Thick}_\varepsilon M(S)$. Then the restriction of the canonical projection $\pi : T(S) \rightarrow M(S)$ to any lifted outer cone $CO(z) \subset T$, with $\pi(z) = x$, is bi-Lipschitz with respect to the Teichmüller distance functions of $T(S)$ and $M(S)$, respectively.
Proof. In a lift CO(z) of CO(x) to T(S) we pick two arbitrary points v, w. We wish to show that \( d_M(\pi(v), \pi(w)) \succ d_T(v, w) \) (the opposite inequality holds by definition). Let \( c \) be a curve of minimal length in \( M(S) \) between \( \pi(v) \) and \( \pi(w) \), i.e., \( L(c) = d_M(\pi(v), \pi(w)) \). We distinguish two cases: (1) \( c \) does not intersect the compact \( \varepsilon \)-thick part \( \text{Thick}_\varepsilon M(S) \) and (2) \( c \) intersects \( \text{Thick}_\varepsilon M(S) \).

Case (1): According to Theorem A we can decompose the curve
\[
c : [0, d_M(\pi(v), \pi(w))] \longrightarrow M(S) \setminus \text{Thick}_\varepsilon M(S)
\]
into a finite number of segments \( c_i, 1 \leq i \leq l \), say, each of which is contained in a subset of \( M(S) \) diffeomorphic to \( \mathcal{B}_\varepsilon(\sigma) \times \mathbb{R}_{>0}^{[\sigma]} \) where \( \mathcal{B}_\varepsilon(\sigma) \) is a torus bundle over the \( \varepsilon \)-thick part of a moduli space and compact. By its definition each outer cone of \( \mathcal{B}_\varepsilon(\sigma) \) is diffeomorphic to the open hyperoctant \( \mathbb{R}_{>0}^{[\sigma]} \).

By Proposition 3 and Corollary 4 a representative of the bi-Lipschitz class of the Teichmüller metric at points \((b, u) \in \mathcal{B}_\varepsilon(\sigma) \times \mathbb{R}_{>0}^{[\sigma]} \) can be written in the form
\[
ds^2_{(b, u)} \succeq d^2_{(b, u)} + du^2,
\]
where \( du^2 \) is the Euclidean (!) metric on \( \mathbb{R}_{>0}^{[\sigma]} \). For the length of the segment \( c_i(s) = (b_i(s), u_i(s)), s \in [s_i, s_{i+1}] \), we thus have the estimate \( L(c_i) \succeq L(u_i), 1 \leq i \leq l \) (compare also Corollary 4). Hence
\[
d_M(\pi(v), \pi(w)) = L(c) = \sum_{i=1}^{l} L(c_i) \succeq \sum_{i=1}^{l} L(u_i).
\]

In view of the second construction of \( \text{Cone}(\Gamma \setminus |\mathcal{C}(S)|) \) each outer cone of \( \text{Thick}_\varepsilon M(S) \) can be mapped diffeomorphically to a Euclidean cone in \( \text{Cone}(\Gamma \setminus |\mathcal{C}(S)|) \). Denote by \( \overline{v} \) and \( \overline{w} \) the corresponding images of \( v, w \) in \( \text{Cone}(\Gamma \setminus |\mathcal{C}(S)|) \). Then the continuous curve
\[
u := u_1 * u_2 * \cdots * u_l \]can be identified with a curve in \( \text{Cone}(\Gamma \setminus |\mathcal{C}(S)|) \), which connects \( \overline{v} \) to \( \overline{w} \) and thus satisfies
\[
\sum_{i=1}^{l} L(u_i) \geq d_C(\overline{v}, \overline{w}).
\]

By Theorem A we have \( CO(z) \cong CO(x) \) and \( d_C(\overline{v}, \overline{w}) = d_{Euc}(\pi(v), \pi(w)) = d_{Euc}(v, w) \). Finally, by Proposition 4 \( d_{Euc}(v, w) \succeq d_T(v, w) \).

Together with the previous estimate this proves the claim in case (1).

Case (2): We decompose \( c \) into three segments \( c = c_1 * c_2 * c_3 \), such that \( c_1 \) connects \( \pi(v) \) to the first intersection point of \( c \) with \( \text{Thick}_\varepsilon M(S) \) and \( c_3 \) connects the last intersection point of \( c \) with \( \text{Thick}_\varepsilon M(S) \) to \( \pi(w) \). Since \( c \) is minimal we then have
\[
d_M(\pi(v), \pi(w)) = L(c) = L(c_1) + L(c_2) + L(c_3) \geq L(c_1) + L(c_3).
\]
We can now argue as in case (1). We consider the $u$-projections $u_1$ of $c_1$ (resp. $u_3$ of $c_3$) and their images $\overline{u}_1$ (resp. $\overline{u}_3$) in $\text{Cone}(\Gamma \backslash [\mathcal{C}(S)])$. Let also $\overline{v}$ (resp. $\overline{w}$) be the images of $\pi(v)$ (resp. $\pi(w)$) in $\text{Cone}(\Gamma \backslash [\mathcal{C}(S)])$. Then $\overline{u}_1$ (resp. $\overline{u}_3$) joins $\overline{v}$ to the apex $\mathcal{O}$ (resp. $\mathcal{O}$ to $\overline{w}$). Hence as in case (1)

$$L(c_1) + L(c_3) \geq L(u_1) + L(u_3) \geq d_C(\overline{v}, \mathcal{O}) + d_C(\mathcal{O}, \overline{w}) \geq d_C(\overline{v}, \overline{w}) = d_{\text{Euc}}(\pi(v), \pi(w)) = d_{\text{Euc}}(v, w) \geq d_T(v, w).$$

This proves the claim $d_M(\pi(v), \pi(w)) \geq d_T(v, w)$ also in case (2). \hfill \Box

Recall that a subset $\mathcal{N}$ of a metric space $(X, d)$ is called a $(\delta)$-net if there is a positive constant $\delta$ such that $d(p, \mathcal{N}) \leq \delta$ for all $p \in X$; in particular the Hausdorff-distance between $\mathcal{N}$ and $X$ is at most $\delta$.

**Proposition 6** There is a net $\mathcal{N}$ in moduli space $\mathcal{M}(S)$ consisting of finitely many bi-Lipschitz embedded $d(S)$-dimensional Euclidean cones. In particular, if $d(S) \geq 2$, then $\mathcal{M}(S)$ is not Gromov-hyperbolic.

**Proof.** Let $\varepsilon$ be as in Proposition 6. By Theorem A and since Thick$_{\varepsilon}\mathcal{M}(S)$ is compact and thus has finite diameter,

$$\bigcup_{\sigma \in \mathcal{E}} \text{Thin}_{\varepsilon}(\mathcal{M}(S), \sigma) \cong B_{\varepsilon}(\sigma) \times \mathbb{R}^{[\sigma]}_{>0}$$

is a net in $\mathcal{M}(S)$. Note that $B_{\varepsilon}(\sigma)$ is compact and that on $\text{Thin}_{\varepsilon}(\mathcal{M}(S), \sigma)$ one has $u_\alpha \geq -\log \varepsilon^{1/2}$, $\alpha \in \sigma$. Hence the metric expansion in Corollary 3 implies that

$$d_M((b_1, u), (b_2, u)) \propto d_{\mathcal{B}}(b_1, b_2) \leq \text{const}(\varepsilon, \sigma) \quad (b_1, b_2 \in B_{\varepsilon}(\sigma); \; u \in \mathbb{R}^{[\sigma]}_{>0}).$$

Therefore any cone $CO(x_\sigma) \subset \text{Thin}_{\varepsilon}(\mathcal{M}(S), \sigma), x_\sigma \in B_{\varepsilon}(\sigma)$, is a net in Thick$_{\varepsilon}\mathcal{M}(S)$, $\sigma$.

Recall that $\mathcal{E}$ denotes the finite set of simplices in $\Gamma \backslash [\mathcal{C}(S)]$. Given some $\sigma \in \mathcal{E}$ there is a $\tau \in \mathcal{E}$ of maximal dimension $d(S) - 1$ containing $\sigma$ and by Lemma 3, $B_{\varepsilon}(\tau) \subset B_{\varepsilon}(\sigma)$. We can thus choose the above cone $CO(x_\sigma) \cong \mathbb{R}^{[\sigma]}_{>0}$ as a face of a closed cone $\overline{CO(x_\tau)}$ for some $x_\tau \in B_{\varepsilon}(\tau)$. The net $\mathcal{N}$ can now be constructed as follows. Let $\tau_i \in \mathcal{E}, i = 1, \ldots, m, \tau_i$ be the simplices of maximal dimension $d(S) - 1$. Let $B_{\varepsilon}(\tau_i), i = 1, \ldots, m$, be the corresponding minimal boundary faces of $\partial $Thick$_{\varepsilon}\mathcal{M}(S)$ (which are $d(S)$-dimensional tori, see Proposition 2). For each $i \in \{1, \ldots, m\}$ we choose a point $x_i \in B_{\varepsilon}(\tau_i)$ and then we set $\mathcal{N} := \bigcup_{i=1}^{m} \overline{CO(x_i)}$. \hfill \Box
3.3 A quasi-isometric approximation of moduli space

We recall the notion of Gromov-Hausdorff-convergence of (unbounded) pointed metric spaces (see [9], Chapter 3). First, the distortion of a map $f : A \rightarrow B$ of metric spaces $A$ and $B$ is defined as

$$\text{dis}(f) := \sup_{a,b} |d_A(a, b) - d_B(f(a), f(b))|.$$ 

The uniform distance between metric spaces $A$ and $B$ is defined as $\text{u-dist}(A, B) := \inf f \text{dis}(f)$ where the infimum is taken over all bijections $f : A \rightarrow B$. A sequence of metric spaces $(X_n)$ Hausdorff-converges to a metric space $X$ if and only if for every $\delta > 0$ there is a $\delta$–net $X_\delta$ in $X$ which is the uniform limit of $\delta$–nets $(X_n)_\delta$ in $X_n$. We say that a sequence $(X_n, p_n)$ of unbounded, pointed metric spaces Hausdorff-converges to a pointed metric space $(X, p)$ if for every $r > 0$ the balls $B_r(p_n)$ in $X_n$ Hausdorff-converge to the ball $B_r(p)$ in $X$.

Let $x_0$ be an (arbitrary) point of the moduli space $\mathcal{M}(S)$. The asymptotic cone of $\mathcal{M}(S)$ endowed with the Teichmüller distance $d_\mathcal{M}$ is defined as the Gromov-Hausdorff-limit of pointed metric spaces:

$$\text{As-Cone}(\mathcal{M}(S)) := \mathcal{H} - \lim_{n \to \infty} (\mathcal{M}(S), x_0, \frac{1}{n} d_\mathcal{M}).$$

By Proposition 6 there is a net $\mathcal{N}$ in $\mathcal{M}(S)$ consisting of disjoint closed quasi-Euclidean (outer) cones $CO(x_i), i = 1, \ldots, m$. FN-coordinates $(u_\alpha = -\log l_\alpha^{1/2})_{\alpha \in \tau_i}$ give rise to diffeomorphisms $\Phi_i : CO(x_i) \cong \mathbb{R}^{d(S)}_{\geq a} \cong \mathbb{R}^{d(S)}_{\geq 0}, i = 1, \ldots, m$ and $a = -\log \varepsilon^{1/2}$. Using the second construction of $\text{Cone}(\Gamma \setminus |C(S)|)$ we have a map

$$f_1 : \mathcal{N} \rightarrow \text{Cone}(\Gamma \setminus |C(S)|); \quad (u_\alpha)_{\alpha \in \tau_i} \mapsto (\Phi_i(u_\alpha))_{\alpha \in \tau_i}.$$ 

For any $n \in \mathbb{N}$ we then define a map

$$f_n : \mathcal{N} \subset (\mathcal{M}(S), \frac{1}{n} d_\mathcal{M}) \rightarrow \text{Cone}(\Gamma \setminus |C(S)|); \quad f_n(x) := \frac{1}{n} f_1(x).$$

By Proposition 5 (or Theorem A) $f_n$ is a bijection from the interior of $\mathcal{N}$ onto its image in $\text{Cone}(\Gamma \setminus |C(S)|)$. This image is open and dense in $\text{Cone}(\Gamma \setminus |C(S)|)$ for all $n$.

**Proposition 7** There is a constant $D > 0$ such that $\text{dis}(f_n) \leq \frac{1}{n} D$ for all $n \in \mathbb{N}$, i.e., $f_n$ is a quasi-isometry. In particular, $\mathcal{M}(S)$ is quasi-isometric to $\text{Cone}(\Gamma \setminus |C(S)|)$.

**Proof.** We consider a polyhedral exhaustion $\mathcal{M}(S) = \bigcup_{\varepsilon \leq \varepsilon_0} \text{Thick}_\varepsilon \mathcal{M}(S)$ (see Corollary 1). The intersection $\mathcal{N}'$ of the net $\mathcal{N}$ with $\mathcal{M}(S) \setminus \text{Thick}_{\varepsilon_0} \mathcal{M}(S)$ is still a net in
\(M(S)\) since \(\text{Thick}_{\varepsilon_0} M(S)\) is compact. Let \(u, v\) be two points in the interior of \(\mathcal{N}'\). We take a path \(c([0, L])\) in \(\mathcal{M}(S)\) between \(u\) and \(v\) of minimal length and parametrized by arc-length. Since the diameter of \(\text{Thick}_{\varepsilon_0} M(S)\) is finite we can assume that \(c\) lies in \(\mathcal{M}(S) \setminus \text{Thick}_{\varepsilon_0} M(S)\) (taking an additive constant into account). By Theorem A there is a tiling of \(\mathcal{M}(S) \setminus \text{Thick}_{\varepsilon_0} M(S)\) into finitely many \((\varepsilon_0, \sigma)\)-thin parts \(\mathcal{M}_i \cong B_{\varepsilon_0}(\sigma_i) \times \mathbb{R}_{>0}^{k_i}, i = 1, \ldots, q\). We can thus find \(t_i \in [0, L]\), \(0 \leq i \leq n + 1\) such that \(t_0 = 0, t_{n+1} = L\) and \([0, L] = \bigcup_{i=0}^n [t_i, t_{i+1}]\) with \(c([t_i, t_{i+1}]) \subset \mathcal{M}_{j_i}\) for \(j_i \in \{1, \ldots, q\}\). Associated to the path \(c\) we thus get a well-defined string \((j_0, j_1, \ldots, j_n)\). We next replace \(c\) by a path \(\overline{c}\) whose associated string contains an element \(j_k \in \{1, \ldots, q\}\) at most once: Start with \(j_0\) and assume that \(0 \leq l \leq n\) is the greatest index such that \(j_l = j_0\). Geometrically this means that the path \(c\) returns to \(\mathcal{M}_{j_0}\) at \(c(t_l)\). By Proposition 4 and Proposition 5 the shortest curve between two points in \(\mathcal{M} = B_{\varepsilon_0}(\sigma) \times \mathbb{R}_{>0}^{k_i}\) is coarsely equivalent to a straight line in \(\mathbb{R}_{>0}^{k_i}\). We can thus replace \(c([t_0, t_{l+1}])\) by a segment, say \(\overline{c}([s_{j_0}, s_{j_0+1}])\), in \(\mathbb{R}_{>0}^{k_i}\) whose length \(L\) satisfies

\[
L(\overline{c}([s_{j_0}, s_{j_0+1}])) \leq L(c([t_0, t_{l+1}]) + 2D_1,
\]

for \(D_1 := \max_{1 \leq i \leq m} \text{diam}(B_{\varepsilon_0}(\sigma_i))\). Repeating this procedure we eventually get a sequence of at most \(q\) segments \(\overline{c}([s_k, s_{k+1}])\) in \(\mathcal{N}'\) of total length \(\leq L(c) + 2qD_1\). Moreover we can choose \(\overline{c}([s_k, s_{k+1}])\) in such a way that the endpoint \(\overline{c}(s_{k-1}) \in \mathcal{M}_{q_{k-1}}\) and the initial point \(\overline{c}(s_k) \in \mathcal{M}_{q_k}\) are on the same levelset of the exhaustion of \(\mathcal{M}(S)\) we are considering.

We next show that these segments can be modified in such a way that they are mapped by \(f_1\) to a continous path \(\tilde{c}\) in \(\text{Cone}(\Gamma \setminus |\mathcal{C}(S)|)\) from \(f_1(u)\) to \(f_1(v)\). With respect to the representative of the bi-Lipschitz class of the Teichmüller metric described in Proposition 4 each outer cone \(CO(x_j)\) in \(\mathcal{N}\) is isometric to a Euclidean simplicial cone in \(\text{Cone}(\Gamma \setminus |\mathcal{C}(S)|)\). The map \(f_1\) thus yields well defined images of (the interior of) the segments \(\overline{c}([s_k, s_{k+1}]) \in \mathcal{M}_{q_k}\). By construction the endpoint \(\overline{c}(s_{k-1}) \in \mathcal{M}_{q_{k-1}}\) and the initial point \(\overline{c}(s_k) \in \mathcal{M}_{q_k}\) are on the same levelset of the exhaustion and are at most the distance \(2D_1\) apart. By the (second) construction of \(\text{Cone}(\Gamma \setminus |\mathcal{C}(S)|)\) we can join their images by segments in \(\text{Cone}(\Gamma \setminus |\mathcal{C}(S)|)\) of uniformly bounded length \(2D_1\) to obtain the path \(\tilde{c}\). This argument yields the estimate

\[
d_C(f_1(u), f_1(v)) \leq L(\tilde{c}) \leq 4qD_1 + L(c) = 4qD_1 + d_M(u, v).
\]

On the other hand, given a geodesic path \(\tilde{c}\) in \(\text{Cone}(\Gamma \setminus |\mathcal{C}(S)|)\) joining two points \(x\) and \(y\) in the image under \(f_1\) of the interior of \(\mathcal{M}(S) \setminus \text{Thick}_{\varepsilon_0} M(S)\), we can lift the segments contained in the simplicial cones \(C(\tau_j)\), say, in \(\subset \text{Cone}(\Gamma \setminus |\mathcal{C}(S)|)\) to the corresponding outer cones \(CO(x_j)\) via \(f_1^{-1}\). By the same arguments as before the
endpoints of the lifted segments can be joined in $\mathcal{M}(S)$ to form a continuous path $c$ between $f_1^{-1}(x)$ and $f_1^{-1}(y)$ of length

$$d_\mathcal{M}(f_1^{-1}(x), f_1^{-1}(y)) \leq L(c) \leq 2qD_1 + L(\tilde{c}) \leq 2qD_1 + d_C(x, y).$$

Combining (1) and (2) and setting $D := 4qD_1$ we get for all $u, v$ in (the interior of) $\mathcal{M}(S) \setminus \text{Thick}_{\varepsilon_0}\mathcal{M}(S)$ that

$$|d_\mathcal{M}(u, v) - d_C(f_1(u), f_1(v))| \leq D.$$

Finally, since for any $n \in \mathbb{N}$

$$\left| \frac{1}{n}d_\mathcal{M}(u, v) - d_C(f_n(u), f_n(v)) \right| = \left| \frac{1}{n}d_\mathcal{M}(u, v) - \frac{1}{n}d_C(f_1(u), f_1(v)) \right| = \frac{1}{n}|d_\mathcal{M}(u, v) - d_C(f_1(u), f_1(v))| \leq \frac{D}{n},$$

which completes the proof of Proposition 7.

### 3.4 The proof of Theorem B

Given $\delta > 0$ there is $n_\delta \in \mathbb{N}$, such that $\mathcal{N}' := \mathcal{N} \cap (\mathcal{M}(S) \setminus \text{Thick}_{\varepsilon_0}\mathcal{M}(S)) \subset \mathcal{M}(S)$ is an $\delta$–net in $(\mathcal{M}(S), \frac{1}{n}d_\mathcal{M})$ for all $n \geq n_\delta$. Since $f_1(\mathcal{N}')$ is dense in $\text{Cone}(\mathcal{C}(\mathcal{S}))\setminus f_1(\mathcal{N} \cap \text{Thick}_{\varepsilon_0}\mathcal{M}(S))$ it is a $\delta$–net in the latter set for all $\delta > 0$. Then there is $r_0$ such that for $r \geq r_0$ the same assertions are true for the subsets of balls with radius $r$:

$$\mathcal{N}' \cap B_r(v_0) \subset (\mathcal{M}(S), \frac{1}{n}d_\mathcal{M}) \text{ and } f_n(\mathcal{N}') \cap B_r(\mathcal{O}) \subset (\text{Cone}(\mathcal{C}(\mathcal{S})), d_C).$$

From Proposition 7 and its proof we obtain the following uniform estimate:

$$\text{u-dist}(\mathcal{N}' \cap B_r(v_0), f_n(\mathcal{N}') \cap B_r(\mathcal{O})) \leq \frac{D}{n},$$

which completes the proof of Theorem B.

**Corollary 5** The asymptotic cone of moduli space $\mathcal{M}(S)$ endowed with the Teichmüller metric is bi-Lipschitz equivalent to a CAT(0) space and in particular contractible.

**Proof.** By Theorem B, As-Cone$(\mathcal{M}(S))$ is bi-Lipschitz equivalent to the Euclidean cone $\text{Cone}(\mathcal{C}(\mathcal{S})).$ A theorem of Berestovski says that for a geodesic metric space $Y$ the Euclidean cone $C(Y)$ is a CAT(0) space if and only if $Y$ is a CAT(1) space (see [9], Ch. II.4). Since $\mathcal{C}(\mathcal{S})$ is obtained by glueing spherical simplices it is CAT(1) (see [5], Ch II, 4.4). \(\square\)
4 Positive scalar curvature

Block and Weinberger showed that on a locally symmetric space \( \Gamma \backslash G/K \) there exists a complete Riemannian metric of uniformly positive scalar curvature if and only if \( \Gamma \) is an arithmetic group of \( \mathbb{Q} \)-rank \( \geq 3 \) (see [3]). These Riemannian metrics have the quasi-isometry type of rays and one may ask whether there is also a metric that is both uniformly positively curved and quasi-isometric to the symmetric metric. That question was answered negatively for locally symmetric spaces associated to arithmetic groups of \( \mathbb{Q} \)-rank \( \geq 2 \) by Chang (see [6]).

Farb and Weinberger recently announced a result analogous to [3] for \( \mathcal{M}(S) \): A finite cover of the moduli space admits a complete, finite volume Riemannian metric of uniformly bounded positive scalar curvature if and only if \( d(S) \geq 3 \) (see [7], 4.2). Again this metric has the quasi-isometry type of rays and thus, by Theorem B, is not quasi-isometric to the Teichmüller metric. Farb conjectured the analogue of Chang’s result (see [7], Conjecture 4.6). Our Theorem C in Section 1 confirms this conjecture.

4.1 Outline of the proof of Theorem C

Given Theorem A and the subresults of its proof, the proof of Theorem C is along exactly the same lines as that of Theorem 1 in [6]. We therefore only sketch those ingredients which rely on general results and in particular refer to [6] for definitions and more details.

Given a tiling of the moduli space \( \mathcal{M}(S) \) as in Theorem A we consider a (maximal) outer cone \( CO(x) \) of \( \partial \text{Thick}_{\varepsilon_0} \mathcal{M}(S) \) corresponding to a simplex \( \tau \) of maximal dimension in the curve complex \( \mathcal{C}(S) \). Then \( CO(x) \) is contained in a subset of \( \mathcal{M}(S) \) diffeomorphic to \( \mathcal{B}_{\varepsilon_0}(\tau) \times \mathbb{R}^{d(S)}_{\geq 0} \), where (in this case) \( \mathcal{B}_{\varepsilon_0}(\tau) \cong \text{Tw}(\tau) \setminus \mathbb{R}^{d(S)} \cong T^{d(S)} \) is a \( d(S) \)-dimensional torus (see Proposition 2). Using adapted FN-coordinates \((\theta, u)\) on \( \mathcal{B}_{\varepsilon_0}(\tau) \times \mathbb{R}^{d(S)}_{\geq 0} \) we consider the projection

\[
\pi_u : \mathcal{B}_{\varepsilon_0}(\tau) \times \mathbb{R}^{d(S)}_{\geq 0} \longrightarrow \mathbb{R}^{d(S)}_{\geq 0}; \quad (\theta, u) \longmapsto u.
\]

We then select a hypersurface \( Q \) in the hyperoctant \( \mathbb{R}^{d(S)}_{\geq 0} \) sufficiently far from the faces, so that the inverse image under \( \pi_u \) of each point of \( Q \) is a \( d(S) \)-dimensional torus. Then \( V := \pi_u^{-1}(Q) \) is diffeomorphic to \( T^{d(S)} \times \mathbb{R}^{d(S)-1} \). The complement of the hypersurface \( V \) in \( \mathcal{M}(S) \) consists of two components. Let \( A \) be the closure of the component containing the fibre \( \pi_u^{-1}(0) \) and let \( B \) be the closure of \( \mathcal{M}(S) \setminus \text{int} A \). Then the pair \((A, B)\) forms a “coarsely excisive decomposition” of \( \mathcal{M}(S) \) with intersection \( A \cap B = V \) (see [3] for the definition).

One can now proceed exactly as in the proof of Theorem 1 in [6]:
• First note that $\pi_1(V) \cong \text{Tw}(\tau) \cong \mathbb{Z}^{d(S)}$. Hence there is an injection $\pi_1(V) \hookrightarrow \pi_1(\mathcal{M}(S)) = \Gamma$ and a K-theoretic Mayer-Vietoris sequence applies.

• Next let $D$ be the lifted classical Dirac operator on the pullback spinor bundle of $T(S)$. Then one defines a generalized Roe algebra $C^*_\Gamma \mathcal{M}(S)$ and a generalized coarse index $\text{Ind}(D) \in K_*(C^*_\Gamma \mathcal{M}(S))$ as in [6].

• Finally, the fact that the strong Novikov conjecture for nilpotent groups is true, implies that $\text{Ind}(D) \neq 0$. Chang proved in [6] that this is an obstruction to the existence of a uniformly bounded positive scalar curvature metric in the same quasi-isometry class as the given metric (which in the present case is the Teichmüller metric).

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