Stochastic Growth in a Small World

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Summary. We considered the Edwards-Wilkinson model on a small-world network. We studied the finite-size behavior of the surface width by performing exact numerical diagonalization for the underlying coupling matrix. We found that the spectrum exhibits a gap or a pseudo-gap, which is responsible for a finite width in the thermodynamic limit for an arbitrarily weak but nonzero magnitude of the random interactions.

1.1 Introduction

Since the introduction of small-world networks [1,2], a number of well-known models have been studied where the original short-range interaction topology is extended to include a finite number of possibly long-range links per “site” [3]. The common observation is that these systems can undergo a phase transition, even when the random links are added (or rewired) to a one-dimensional original substrate [3–8]. The nature of the transition resembles that of a mean-field one [6–8].

Among the first applications of small-world networks was to study synchronization in dynamical systems [2,9] such as the Kuramoto oscillators [10]. The need for autonomous synchronization for a system with a large number of “agents” (processing elements) also naturally emerges in large-scale parallel discrete-event simulation (PDES) schemes [11] for systems with short-range interactions and asynchronous dynamics [12,13]. Frequent and necessary “local” communications between processing elements (PEs) to ensure the asynchronous causal dynamics of the underlying system will eventually lead to a diverging spread of the progress of the individual PEs [13,14]. This property can seriously hinder efficient data collection for such simulation schemes. An alternative to possibly costly and frequent global synchronizations is to extend the required short-range communication topology to include “weak” random links [14]. Weak in this context refers to the relative timescale of actually using the random connections for synchronization. By directly “simulating the simulations” and using simple coarse-graining arguments, it was demonstrated [13] that the progress of the PEs with only local synchronization exhibits “kinetic roughening” governed by the Kardar-Parisi-Zhang (KPZ) equation. With random links added (a finite number per PE) and invoked at
an arbitrarily small (but non-vanishing) rate, however, the PEs progress in a near-uniform fashion [14].

Here we focus on how critical fluctuations (originally present in the steady state of a one-dimensional system) are suppressed when the interaction topology is extended to include weak interactions facilitated by random links. To this end we study the Edwards-Wilkinson (EW) linear stochastic growth equation on a “substrate” with small-world-like topology. This model is also closely related to phase ordering and synchronization among coupled oscillators in the presence of noise [8] and to the XY-model on a small-world network [6]. We consider the equation

\[ \partial_t h_i = -\left(2h_i - h_{i+1} - h_{i-1} \right) - p \sum_{j=1}^{N} J_{ij} (h_i - h_j) + \eta_i(t) , \quad (1.1) \]

where \( h_i \) is the surface height, \( \eta_i(t) \) is a delta-correlated Gaussian noise with variance 2 (without loss of generality), and we have dropped the \( t \)-dependence from the argument of \( h_i \) for brevity. The matrix \( J_{ij} \) represents the (quenched) random links on top of a one-dimensional lattice of length \( N \) (even for simplicity) with periodic boundary conditions, i.e., \( J_{ij}=1 \) if a random link is present and zero otherwise. The parameter \( p \) is the strength of the interaction through the random links. Our construction of the random links is such that each site has exactly one random link. More specifically, pairs of sites are selected at random and once they are chosen, they cannot be selected again. This somewhat constrained construction of the random network originates from an application to scalable PDES synchronization schemes [14], where fluctuations in the individual connectivity of the PEs are to be avoided.

For a given realization of this small-world network the average surface width characterizing the roughness is defined as

\[ \langle w^2 \rangle_N = \left\langle \frac{1}{N} \sum_{i=1}^{N} (h_i - \bar{h})^2 \right\rangle , \quad (1.2) \]

where \( \bar{h}=(1/N) \sum_{i=1}^{N} h_i \) is the mean height and \( \langle . . \rangle \) denotes an ensemble average over the noise in Eq. (1.1). For \( p=0 \) in Eq. (1.1), we recover the one-dimensional EW model where the steady-state width diverges as \( \langle w^2_N \rangle = N/12 \).

One may wonder how the system would behave if the same total number of links as in the above construction of a small-world network (i.e., \( N/2 \)) were used to connect each site with the one located at the “maximum” possible distance of away from it (\( N/2 \) on a ring with periodic conditions). Elementary calculations show that \( \langle w^2_N \rangle \simeq N/24 \) for large \( N \), i.e., the width would diverge as for a one-dimensional system of size \( N/2 \). Indeed, one can realize that such regularly patterned long-range links make the original system equivalent to a \( 2 \times (N/2) \) system with only nearest-neighbor interactions and shifted periodic boundary conditions. More generally, one can show that, if every
1.2 Roughness and the Density of States

We study the finite-size effects of the width of the surface and also the underlying spectrum (density of states) of the associated random matrix which governs the steady-state height fluctuations. Exploiting that the noise in Eq. (1.2) is Gaussian, the steady-state width for a single realization of the random network can be expressed as

\begin{equation}
\langle w^2 \rangle_N = \frac{1}{N} \sum_{k=1}^{N-1} \frac{1}{\lambda_k}
\end{equation}

where \(\lambda_k\) are the eigenvalues of the real symmetric coupling matrix

\begin{equation}
\Gamma_{ij} = \{(2 + p)\delta_{ij} - \delta_{i+1,j} - \delta_{i-1,j}\} - pJ_{ij}
\end{equation}

as can be read off from Eq. (1.1). Note that we have exploited our specific choice of \(J_{ij}\), resulting in \(\sum_{j=1}^{N} J_{ij} = 1\) for all \(i\). Also, note that since Eq. (1.2) contains the height fluctuations measured from the mean, the eigenvalue \(\lambda_0 = 0\), corresponding to the uniform eigenvector (zero-mode) of \(\Gamma_{ij}\), does not appear in Eq. (1.3). In the limit of \(N \to \infty\) and assuming that the distribution of the eigenvalues of \(\Gamma_{ij}\) becomes self-averaging, the disorder-averaged width can be be written as

\begin{equation}
\left[ \langle w^2 \rangle_N \right] = \left[ \frac{1}{N} \sum_{k=1}^{N-1} \frac{1}{\lambda_k} \right]_{N \to \infty} \int \rho(\lambda) d\lambda \frac{1}{\lambda},
\end{equation}

where \([\ldots]\) stands for averaging over the random-link disorder and \(\rho(\lambda)\) denotes the density of eigenvalues of \(\Gamma_{ij}\). The behavior of \(\rho(\lambda)\) as \(\lambda\) goes to zero determines whether the width remains finite or diverges in the thermodynamic limit. In the pure one-dimensional case, \(\rho(\lambda)\) actually diverges as \(1/(2\pi\sqrt{\lambda})\). If, however, \(\rho(\lambda)\) exhibits a gap or approaches zero fast enough, \(\left[ \langle w^2 \rangle_N \right]\) will be finite. In the context of diffusion on a small-world network, it was found that the density of states exhibits a pseudo-gap (vanishes exponentially fast) [15]. The construction of the small-world graph in Ref. [15] allowed for the existence of arbitrarily long “pure” chain-segments of the network with exponentially small probabilities. These small, but non-vanishing, probabilities were responsible for the pseudo-gap [16]. In our specific construction of the network, where each site has exactly one random link, the above argument does not apply and a true gap may develop. Further, the coupling matrix [Eq. (1.4)] has a realization-independent “mass” term. This
Fig. 1.1. (a) Disorder-averaged surface width \([\langle w_N^2 \rangle]\) as a function of the system size \(N\) for \(p = 1, 10^{-1}, 10^{-2}, 10^{-3}, 10^{-4},\) and \(10^{-5}\) (from bottom to top, respectively, with symbols), obtained by using 1000 independent realizations of the random network (except for \(N=2000\) where only 100 realizations were generated). The solid lines are the analytic forms for the width in the simple “massive” approximation. The straight line corresponds to the “rough” \(p=0\) case. (b) Cumulative (integrated) density of states for \(p = 10^{-1}, 10^{-2}, 10^{-3},\) and \(10^{-4}\) (from right to left, respectively, with bold solid lines) based on 1000 realizations of the random network for \(N=1000\). The thin lines correspond to the analytic form for the infinite system-size “massive” approximation. These curves are not distinguishable on normal scales, except for the \(p=10^{-1}\) case. The inset shows the same on log-log scales to magnify the region near small eigenvalues. We also plot the analytic form for the infinite-system \(p=0\) case (asymptotically a straight line \([\sim (1/\pi)\sqrt{\lambda}]\) for small \(\lambda\).

property would actually allow for a perturbation expansion for small but non-zero values of \(p\) with the term \(-pJ_{ij}\) being the perturbation.

We performed exact numerical diagonalization of the coupling matrix Eq. (1.4) using standard numerical routines [17], and calculated the steady-state width as a function of the system size for various values of \(p\). The results are summarized in Fig. 1(a). We also plotted the analytic form of the width for the simple “massive” coupling matrix, the expression in brackets in Eq. (1.4), as the zeroth-order approximation in a perturbative approach. It appears that for small values of \(p\), the numerically computed (and disorder-averaged) width and this simplest approximation yield the same asymptotic finite-size effects. Fig. 1(b) shows the cumulative eigenvalue distribution \(\int^\lambda \rho(\lambda')d\lambda'\) for \(N=1000\) for various \(p\) values. Whether the spectrum exhibits a true or a pseudo-gap (due to exponentially small likely eigenvalues), cannot be determined by numerics. It is appears, however, that the numerically observed “gap” asymptotically scales linearly with small values of \(p\).
1.3 Conclusions

We carried out exact numerical diagonalization for the coupling matrix representing EW growth on a small-world network. In our construction each site had one random link, i.e., no fluctuations were allowed in the connectivity. We found that the surface width saturates for all nonzero values of the amplitude of the random coupling as a result of the gap or pseudo-gap in the underlying spectrum. We should also note the similarity between the relaxation properties of our model and that of the one-dimensional Ising-like systems with ("annealed") random spin-exchange process [18]. This long-range process creates a mean-field-like environment, in which ordering is possible with other suitable chosen local processes present [18].

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