Description of Incomplete Financial Markets for the Discrete Time Evolution of Risk Assets.

N.S. Gonchar
Bogolyubov Institute for Theoretical Physics of NAS of Ukraine.

Abstract
In the paper, the martingales and super-martingales relative to a regular set of measures are systematically studied. The notion of local regular super-martingale relative to a set of equivalent measures is introduced and the necessary and sufficient conditions of the local regularity of it in the discrete case are founded. The regular set of measures play fundamental role for the description of incomplete markets. In the partial case, the description of the regular set of measures is presented. The notion of completeness of the regular set of measures have the important significance for the simplification of the proof of the optional decomposition for super-martingales. Using this notion, the important inequalities for some random values are obtained. These inequalities give the simple proof of the optional decomposition of the majorized super-martingales. The description of all local regular super-martingales relative to the regular set of measures is presented. It is proved that every majorized super-martingale relative to the complete set of measures is a local regular one. In the case, as evolution of a risk asset is given by the discrete geometric Brownian motion, the financial market is incomplete and a new formula for the fair price of super-hedge is founded.

Keywords: Random process; Regular set of measures; Optional Doob decomposition; Local regular super-martingale; martingale; Discrete geometric Brownian motion.

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1 Introduction
This paper is a continuation of the paper [1]. In it, a new method of investigation of martingales and super-martingales relative to the regular set of measures is developed. A notion of the local regular super-martingale relative to the regular set of measures is introduced and the necessary and sufficient conditions are found under that the above defined super-martingale is a local regular one. The last fact allowed us to describe the local regular super-martingales. On a measurable space, a notion of the set of equivalent measures consistent with the filtration is introduced. Such a set of measures guarantee the existence of the sufficient set of nonnegative super-martingales. The next important fact is the existence of a martingale on such a measurable space. Further, we introduce the important notion of the regular set of measures. In partial cases, we describe completely the set of regular measures. An important notion of the completeness of the regular set of measures is introduced. To prove that the regular set of measures for the local regular martingale is a complete one we describe the set of equivalent measures to a given measure, which satisfy the condition: expectation of a given random value relative to every measure from this set of measures equals zero. The representation for every measure of this set of measures and a notion of the exhaustive decomposition for the $\sigma$-algebra gives us the possibility to prove the statement that the set of equivalent martingale measures for the

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regular martingale is a complete one. This notion is very important, since it permits us to find some important inequalities for a certain class of random variables. These inequalities simplify the proof of the optional decomposition for the class of majorized super-martingales.

The notion of the completeness of the regular set of measures permits us to give a new proof of the optional decomposition for a nonnegative super-martingale. This proof does not use the no-arbitrage arguments and the measurable choice \[2\], \[3\], \[4\], \[5\].

First, the optional decomposition for diffusion processes super-martingale was opened by El Karoui N. and Quenez M. C. \[6\]. After that, Kramkov D. O. and Follmer H. \[2\], \[3\] proved the optional decomposition for the nonnegative bounded super-martingales. Follmer H. and Kabanov Yu. M. \[4\], \[5\] proved analogous result for an arbitrary super-martingale. Recently, Bouchard B. and Nutz M. \[7\] considered a class of discrete models and proved the necessary and sufficient conditions for the validity of the optional decomposition.

The optional decomposition for super-martingales plays the fundamental role for the risk assessment in incomplete markets \[2\], \[3\], \[6\], \[8\], \[9\], \[10\], \[11\]. Considered in the paper problem is a generalization of the corresponding one that appeared in mathematical finance about the optional decomposition for a super-martingale and which is related with the construction of the super-hedge strategy in incomplete financial markets.

At last, we consider an application of the results obtained to find the new formula for the fair price of super-hedge in the case, as the risk asset evolves by the discrete geometric Brownian motion.

\section{Local regular super-martingales relative to a set of equivalent measures.}

We assume that on a measurable space \(\{\Omega, \mathcal{F}\}\) a filtration \(\mathcal{F}_m \subset \mathcal{F}_{m+1} \subset \mathcal{F}, \ m = 0, \infty\), and a set of equivalent measures \(M\) on \(\mathcal{F}\) are given. Further, we assume that \(\mathcal{F}_0 = \{\emptyset, \Omega\}\) and the \(\sigma\)-algebra \(\mathcal{F} = \sigma(\bigvee_{n=1}^{\infty} \mathcal{F}_n)\) is a minimal \(\sigma\)-algebra generated by the algebra \(\bigvee_{n=1}^{\infty} \mathcal{F}_n\).

A random process \(\psi = \{\psi_m\}_{m=0}^{\infty}\) is said to be adapted one relative to the filtration \(\{\mathcal{F}_m\}_{m=0}^{\infty}\), if \(\psi_m\) is a \(\mathcal{F}_m\) measurable random value, \(m = 0, \infty\).

**Definition 1.** An adapted random process \(f = \{f_m\}_{m=0}^{\infty}\) is said to be a super-martingale relative to the filtration \(\mathcal{F}_m, \ m = 0, \infty\), and the family of equivalent measures \(M\), if \(E^P|f_m| < \infty, \ m = 1, \infty, P \in M, and the inequalities\)

\[E^P\{f_m|F_k\} \leq f_k, \ 0 \leq k \leq m, \ m = 1, \infty, \ P \in M, \quad (1)\]

are valid.

Further, for an adapted process \(f\) we use both the denotation \(\{f_m, \mathcal{F}_m\}_{m=0}^{\infty}\) and the denotation \(\{f_m\}_{m=0}^{\infty}\).

**Definition 2.** A super-martingale \(\{f_m, \mathcal{F}_m\}_{m=0}^{\infty}\) relative to a set of equivalent measures \(M\) is a local regular one, if \(\sup_{P \in M} E^P|f_m| < \infty, \ m = 1, \infty, \) and there exists an adapted nonnegative increasing random process \(\{g_m, \mathcal{F}_m\}_{m=0}^{\infty}\), \(g_0 = 0, \sup_{P \in M} E^P|g_m| < \infty, \ m = 1, \infty,\) such that \(\{f_m + g_m, \mathcal{F}_m\}_{m=0}^{\infty}\) is a martingale relative to every measure from \(M\).

The next elementary Theorem \[1\] will be very useful later.
Theorem 1. Let a super-martingale \( \{f_m, F_m\}_{m=0}^{\infty} \), relative to a set of equivalent measures \( M \) be such that \( \sup_{P \in M} E^P |f_m| < \infty, \ m = 1, \infty \). The necessary and sufficient condition for it to be a local regular one is the existence of an adapted nonnegative random process \( \{g^0_m, F_m\}_{m=0}^{\infty} \), \( \sup_{P \in M} E^P |g^0_m| < \infty, \ m = 1, \infty \), such that

\[
f_{m-1} - E^P \{f_m | F_{m-1}\} = E^P \{g^0_m | F_{m-1}\}, \quad m = 1, \infty, \ P \in M. \tag{2}\]

Proof. The necessity. If \( \{f_m, F_m\}_{m=0}^{\infty} \) is a local regular super-martingale, then there exist a martingale \( \bar{M}_m, F_m \) and let \( \xi \in \Omega \), a set of measures and let \( \bar{M}_m, F_m \) be a set of equivalent measures and let \( \bar{M}_m, F_m \) be a set of equivalent measures, then there exist a martingale \( \bar{M}_m, F_m \) and a non-decreasing nonnegative random process \( \bar{M}_m, F_m \) such that

\[
f_m = \bar{M}_m - g^0_m, \quad m = 1, \infty. \tag{3}\]

From here, we obtain the equalities

\[
E^P \{f_{m-1} - f_m | F_{m-1}\} = E^P \{g^0_{m-1} | F_{m-1}\} = E^P \{g^0_m | F_{m-1}\}, \quad m = 1, \infty, \ P \in M, \tag{4}\]

where we introduced the denotation \( g^0_m = g_m - g_{m-1} \geq 0 \). It is evident that \( E^P g^0_m \leq \sup_{P \in M} E^P g_m + \sup_{P \in M} E^P g_{m-1} < \infty \).

The sufficiency. Suppose that there exists an adapted nonnegative random process \( \bar{g}^0 = \{g^0_m\}_{m=0}^{\infty} \), \( \bar{g}_0^0 = 0 \), \( E^P \bar{g}^0_m < \infty, \ m = 1, \infty \), such that the equalities (2) hold. Let us consider the random process \( \{\bar{M}_m, F_m\}_{m=0}^{\infty} \), where

\[
\bar{M}_0 = f_0, \quad \bar{M}_m = f_m + \sum_{i=1}^{m} \bar{g}^0_i, \quad m = 1, \infty. \tag{5}\]

It is evident that \( E^P |\bar{M}_m| < \infty \) and

\[
E^P \{\bar{M}_{m-1} - \bar{M}_m | F_{m-1}\} = E^P \{f_{m-1} - f_m - \bar{g}^0_m | F_{m-1}\} = 0. \tag{6}\]

Theorem 1 is proved. \( \square \)

Lemma 1. Any super-martingale \( \{f_m, F_m\}_{m=0}^{\infty} \) relative to a family of measures \( M \) for which there hold equalities \( E^P f_m = f_0, \ m = 1, \infty, \ P \in M, \) is a martingale with respect to this family of measures and the filtration \( F_m, m = 1, \infty \).

Proof. The proof of Lemma 1 see [12]. \( \square \)

In the next Lemma, we present the formula for calculation of the conditional expectation relative to another measure from \( M \).

Lemma 2. On the measurable space \( \{\Omega, \mathcal{F}\} \) with the filtration \( \mathcal{F}_n \) on it, let \( M \) be a set of equivalent measures and let \( \xi \) be an integrable random value. Then, the following formulas

\[
E^{P_1} \{\xi | \mathcal{F}_n\} = E^{P_2} \{\xi \varphi_n^{P_1} | \mathcal{F}_n\}, \quad n = 1, \infty, \tag{7}\]

are valid, where

\[
\varphi_n^{P_1} = \frac{dP_1}{dP_2} \left[ E^{P_2} \left\{ \frac{dP_1}{dP_2} | \mathcal{F}_n \right\} \right]^{-1}, \quad P_1, \ P_2 \in M. \tag{8}\]

Proof. The proof of Lemma 2 is evident. \( \square \)
3 Local regular super-martingales relative to a set of equivalent measures consistent with the filtration.

**Definition 3.** On a measurable space \( \{ \Omega, \mathcal{F} \} \) with a filtration \( \mathcal{F}_n \) on it, a set of equivalent measures \( M \) we call consistent with the filtration \( \mathcal{F}_n \), if for every pair of measures \((Q_1, Q_2) \in M^2\) the set of measures

\[
R^k_s(A) = \int_A \frac{E^{Q_1}\left\{\frac{dQ_2}{dQ_1} | \mathcal{F}_k\right\}}{E^{Q_1}\left\{\frac{dQ_2}{dQ_1} | \mathcal{F}_s\right\}} dQ_1, \quad A \in \mathcal{F}, \quad k \geq s \geq n, \quad n = 0, \infty,
\]  

(9)

belongs to the set \( M \), where \( M^2 \) is a direct product of the set \( M \) by itself.

**Lemma 3.** On the measurable space \( \{ \Omega, \mathcal{F} \} \) with the filtration \( \mathcal{F}_n \) on it, the set of measures \( M = \{ Q, Q(A) = \int_A \alpha(\omega) dP, \ A \in \mathcal{F}, \ Q(\Omega) = 1 \} \)

(10)
is a consistent one with the filtration \( \mathcal{F}_n \), if \( P \) is a measure on \( \{ \Omega, \mathcal{F} \} \) and a random value \( \alpha(\omega) \) runs over all nonnegative random values, satisfying the condition \( P(\{ \omega, \alpha(\omega) > 0 \}) = 1 \).

**Proof.** Suppose that \((Q_1, Q_2) \) belongs to \( M^2 \). Then, \( \frac{dQ_2}{dQ_1} = \frac{\alpha_2(\omega)}{\alpha_1(\omega)} \) and \( P(\{ \omega, \frac{dQ_2}{dQ_1} > 0 \}) = 1 \), since the equalities \( P(\{ \omega, 0 < \alpha_1(\omega) < \infty \}) = 1 \), \( P(\{ \omega, 0 < \alpha_2(\omega) < \infty \}) = 1 \) are true. It is evident that

\[
R^k_s(A) = \int_A \frac{E^{Q_1}\left\{\frac{dQ_2}{dQ_1} | \mathcal{F}_k\right\}}{E^{Q_1}\left\{\frac{dQ_2}{dQ_1} | \mathcal{F}_s\right\}} dQ_1 = \int_A \frac{E^{Q_1}\left\{\frac{dQ_2}{dQ_1} | \mathcal{F}_k\right\}}{E^{Q_1}\left\{\frac{dQ_2}{dQ_1} | \mathcal{F}_s\right\}} \alpha_1(\omega) dP, \quad A \in \mathcal{F}, \quad k \geq s \geq n, \quad n = 0, \infty.
\]

(11)

It is easy to see that

\[
P(\{ \omega, E^{Q_1}\left\{\frac{dQ_2}{dQ_1} | \mathcal{F}_k\right\} \alpha_1(\omega) > 0 \}) = 1, \quad k \geq s,
\]

(12)
since

\[
P(\{ \omega, E^{Q_1}\left\{\frac{dQ_2}{dQ_1} | \mathcal{F}_k\right\} > 0 \}) = 1, \quad k \geq s,
\]

(13)

\[
P(\{ \omega, 0 < E^{Q_1}\left\{\frac{dQ_2}{dQ_1} | \mathcal{F}_s\right\} < \infty \}) = 1, \quad s \geq n, \quad n = 0, \infty.
\]

(14)

The last equality follows from the equivalence of the measures \( Q_1, Q_2 \) and \( P \). Altogether, it means that the set of measures \( R^k_s, \ k \geq s \geq n, \ n = 0, \infty \), belongs to the set \( M \). The same is true for the pair \((Q_2, Q_1) \in M^2\). Lemma 3 is proved. \( \square \)
Theorem 2. On the measurable space \( \{\Omega, \mathcal{F} \} \) with the filtration \( \mathcal{F}_n \) on it, let the set of equivalent measures \( M \) be consistent with the filtration \( \mathcal{F}_n \). Then, for every nonnegative random value \( \xi \) such that \( \sup_{P \in M} E^P \xi < \infty \), the random process \( \{f_n, \mathcal{F}_n\}_{n=0}^\infty \) is a supermartingale relative to the set of measures \( M \), where \( f_n = \text{ess sup}_{P \in M} E^P \{\xi|\mathcal{F}_n\} \), \( n = 0, \infty \).

Proof. Let \( Q \in M \), then, due to Lemma 2, for every \( P \in M \)

\[
E^P \{\xi|\mathcal{F}_n\} = E^Q \left\{ \xi \left| \frac{dP}{dQ} \right. |\mathcal{F}_n \right\} .
\]

(15)

If to put instead of the measure \( P \) the measure \( R^k_s \), \( k \geq s \geq n \), for the pair of measures \( (Q, P) \) we obtain

\[
E^{R^k_s} \{\xi|\mathcal{F}_n\} = E^Q \left\{ \xi \left| \frac{dR^k_s}{dQ} \right. |\mathcal{F}_n \right\} = E^Q \left\{ \xi \left| \frac{E^Q \{dP/dQ|\mathcal{F}_k\}}{E^Q \{dP/dQ|\mathcal{F}_s\}} \right. |\mathcal{F}_n \right\} ,
\]

(16)

where we took into account the equality

\[
E^Q \left\{ \frac{dR^k_s}{dQ} \right. |\mathcal{F}_n \right\} = E^Q \left\{ \frac{E^Q \{dP/dQ|\mathcal{F}_k\}}{E^Q \{dP/dQ|\mathcal{F}_s\}} \right. |\mathcal{F}_n \right\} = 1, \quad k \geq s \geq n.
\]

(17)

From the formula (16), it follows the equality

\[
\text{ess sup}_{P \in M} E^P \{\xi|\mathcal{F}_n\} = \text{ess sup}_{T \in R_n} E^P \{\xi|\mathcal{F}_n\},
\]

(18)

where \( R_n \) is a set of martingales \( T = \{T_m\}_{m=0}^\infty \) relative to the measure \( Q \) such that \( T_m = 1, m \leq n, T_m = \frac{E^Q \{dP/dQ|\mathcal{F}_m\}}{E^Q \{dP/dQ|\mathcal{F}_1\}} \), \( m \geq s \geq n \), \( P \in M \). The definition of ess sup for the uncountable set of random values see [14]. It is evident that \( T_n \subseteq T_{n-1} \). Let us consider

\[
E^Q \{\text{ess sup}_{P \in M} E^P \{\xi|\mathcal{F}_n\}|\mathcal{F}_{n-1}\} = E^Q \{\text{ess sup}_{T \in R_n} E^P \{\xi|\mathcal{F}_n\}|\mathcal{F}_{n-1}\} =
\]

\[
E^Q \{\text{sup}_{i \geq 1} E^P \{\xi T_i|\mathcal{F}_n\}|\mathcal{F}_{n-1}\} = E^Q \{\lim_{k \to \infty} \max_{1 \leq i \leq k} E^P \{\xi T_i|\mathcal{F}_n\}|\mathcal{F}_{n-1}\} =
\]

\[
\lim_{k \to \infty} E^Q \{\max_{1 \leq i \leq k} E^P \{\xi T_i|\mathcal{F}_n\}|\mathcal{F}_{n-1}\} \leq E^P \{\xi T_{n-1}|\mathcal{F}_{n-1}\} \leq \text{ess sup}_{T \in R_n} E^Q \{\xi T|\mathcal{F}_{n-1}\} = E^Q \{\xi|\mathcal{F}_{n-1}\},
\]

(19)

where

\[
\tau_1 = 1,
\]

(20)

\[
\tau_i = \begin{cases} \tau_{i-1}, & E^P \{\xi T_{n-1}|\mathcal{F}_n\} > E^P \{\xi T_i|\mathcal{F}_n\}, \\ i, & E^P \{\xi T_{n-1}|\mathcal{F}_n\} \leq E^P \{\xi T_i|\mathcal{F}_n\}, \end{cases} \quad i = 2, k.
\]

(21)

Lemma 2 is proved. \( \Box \)
Theorem 3. On the measurable space \( \{\Omega, \mathcal{F}\} \), \( \mathcal{F} = \sigma(\bigvee_{i=1}^{\infty} \mathcal{F}_i) \), let \( M \) be a set of equivalent measures being consistent with the filtration \( \mathcal{F}_n \). If there exists a nonnegative random value \( \xi \neq 1 \) such that \( E^P\xi = 1 \), \( P \in M \), then \( E^P\{\xi|\mathcal{F}_n\}, P \in M \), is a local regular martingale.

Proof. Due to Lemma 2, the random process \( \{f_n, \mathcal{F}_n\}_{n=0}^{\infty} \), where \( f_n = \text{ess sup}_{P \in M} E^P\{\xi|\mathcal{F}_n\} \), \( n = 0, \infty \), is a super-martingale relative to the set of measures \( M \), that is,

\[
E^Q\{\text{ess sup}_{P \in M} E^P\{\xi|\mathcal{F}_n\}|\mathcal{F}_{n-1}\} \leq \text{ess sup}_{P \in M} E^P\{\xi|\mathcal{F}_{n-1}\}, \quad Q \in M, \quad n = 0, \infty. \tag{22}
\]

From the inequality (22), it follows the inequality

\[
E^Q\text{ess sup}_{P \in M} E^P\{\xi|\mathcal{F}_n\} \leq 1, \quad n = 0, \infty. \tag{23}
\]

Since \( E^Q\text{ess sup}_{P \in M} E^P\{\xi|\mathcal{F}_n\} \geq E^Q E^Q\{\xi|\mathcal{F}_n\} = 1 \), we have

\[
E^Q\text{ess sup}_{P \in M} E^P\{\xi|\mathcal{F}_n\} = 1, \quad Q \in M, \quad n = 0, \infty. \tag{24}
\]

The inequalities (22) and the equalities (24) give the equalities

\[
E^Q\{\text{ess sup}_{P \in M} E^P\{\xi|\mathcal{F}_n\}|\mathcal{F}_{n-1}\} = \text{ess sup}_{P \in M} E^P\{\xi|\mathcal{F}_{n-1}\}, \quad Q \in M, \quad n = 1, \infty, \tag{25}
\]

which are true with the probability 1. The last means that \( \{f_n, \mathcal{F}_n\}_{n=0}^{\infty} \) is a martingale relative to the set of measures \( M \), where \( f_n = \text{ess sup}_{P \in M} E^P\{\xi|\mathcal{F}_n\}, \quad n = 0, \infty \). With the probability 1, \( \lim_{n \to \infty} \text{ess sup}_{P \in M} E^P\{\xi|\mathcal{F}_n\} = f_\infty \), where the random value \( f_\infty \) is \( \mathcal{F} \) measurable one. From the inequality (23) and Fatou Lemma [13], [14], we obtain

\[
E^P f_\infty \leq 1, \quad P \in M. \tag{26}
\]

Prove that \( f_\infty = \xi \). Going to the limit in the inequality

\[
\text{ess sup}_{P \in M} E^P\{\xi|\mathcal{F}_n\} \geq E^{P_1}\{\xi|\mathcal{F}_n\}, \tag{27}
\]

as \( n \to \infty \), we obtain the inequality

\[
f_\infty \geq \xi. \tag{28}
\]

From the inequality (26) and the inequality (28), we obtain the inequalities \( 1 \geq E^P f_\infty \geq E^P\xi = 1 \). Or, \( E^P f_\infty = 1 \). The equalities \( E^P f_\infty = 1 \), \( E^P\xi = 1 \) and the inequality (28) give the equality \( f_\infty = \xi \) with the probability 1. Lemma 3 is proved. \[\square\]

Lemma 4. On the measurable space \( \{\Omega, \mathcal{F}\} \) with the filtration \( \mathcal{F}_n \) on it, let there exist \( k \) equivalent measures \( P_1, \ldots, P_k, k > 1 \), and a nonnegative random value \( \xi_0 \neq 1 \) be such that

\[
E^{P_i}\{\xi_0|\mathcal{F}_n\} = E^{P_1}\{\xi_0|\mathcal{F}_n\}, \quad E^{P_i}\xi_0 = 1, \quad i = 2, \ldots, k, \quad n = 0, \infty. \tag{29}
\]

Then, there exists the set of equivalent measures \( M \) consistent with the filtration \( \mathcal{F}_n \), satisfying the condition \( E^P\xi_0 = 1, P \in M \).
Proof. Let us consider the set of equivalent measures $M$, satisfying the condition
\[ E^P \{ \xi_0 | \mathcal{F}_n \} = E^{P_1} \{ \xi_0 | \mathcal{F}_n \}, \quad n = 0, \infty, \quad P \in M. \] (30)
Such a set of measures is a nonempty one. Suppose that $Q_1, Q_2 \in M$, then
\[ E^{Q_1} \{ \xi_0 | \mathcal{F}_n \} = E^{Q_2} \{ \xi_0 | \mathcal{F}_n \}, \quad n = 0, \infty. \] (31)
Let us prove that the formula
\[ E^{Q_1} \{ E^{Q_2} \{ \xi_0 | \mathcal{F}_s \} | \mathcal{F}_n \} | \mathcal{F}_n \} = E^{Q_1} \{ \xi_0 | \mathcal{F}_n \}, \quad n \leq s \leq k, \quad n = 0, \infty, \] (32)
is valid. Let $s \geq n$. Then, from the equalities (31), we have
\[ E^{Q_1} \{ E^{Q_2} \{ \xi_0 | \mathcal{F}_s \} | \mathcal{F}_n \} = E^{Q_1} \{ \xi_0 | \mathcal{F}_n \}. \]
Let $k \geq s$. Then,
\[ E^{Q_1} \{ E^{Q_2} \{ \xi_0 | \mathcal{F}_s \} | \mathcal{F}_n \} = E^{Q_1} \{ E^{Q_2} \{ \xi_0 | \mathcal{F}_k \} | \mathcal{F}_s \} | \mathcal{F}_n \} =
\[ E^{Q_1} \{ E^{Q_2} \{ \xi_0 | \mathcal{F}_k \} \frac{dQ_2}{dQ_1} | \mathcal{F}_s \} | \mathcal{F}_n \} =
\[ E^{Q_1} \{ E^{Q_2} \{ \xi_0 | \mathcal{F}_k \} \frac{dQ_2}{dQ_1} | \mathcal{F}_s \} | \mathcal{F}_n \} =
\[ E^{Q_1} \{ E^{Q_1} \{ \xi_0 | \mathcal{F}_k \} \frac{dQ_2}{dQ_1} | \mathcal{F}_s \} | \mathcal{F}_n \} =
\[ E^{Q_1} \{ E^{Q_1} \{ \xi_0 | \mathcal{F}_k \} \frac{dQ_2}{dQ_1} | \mathcal{F}_s \} | \mathcal{F}_n \} =
\[ E^{Q_1} \{ E^{Q_1} \{ \xi_0 | \mathcal{F}_k \} \frac{dQ_2}{dQ_1} | \mathcal{F}_s \} | \mathcal{F}_n \}.
\]
This proves the formula (32). To finish the proof of Lemma 4 it needs to prove that the set of measures
\[ R^k_s(A) = \int_A E^{Q_1} \{ \frac{dQ_2}{dQ_1} | \mathcal{F}_k \} E^{Q_1} \{ \frac{dQ_2}{dQ_1} | \mathcal{F}_s \} dQ_1, \quad A \in \mathcal{F}, \quad k \geq s \geq n, \quad n = 0, \infty, \] (33)
belongs to the set $M$. Really,
\[ E^{R^k_s} \{ \xi_0 | \mathcal{F}_n \} = E^{Q_1} \left\{ \xi_0 \frac{dR^k_s}{dQ_1} \frac{dQ_2}{dQ_1} | \mathcal{F}_n \} \right\} =
\]
\[
E_{Q_1} \left\{ \frac{dQ_k}{dQ_1} \bigg| F_n \right\} = E_{Q_1} \left\{ \xi_0 \bigg| F_n \right\},
\]
where we took into account the equality
\[
E_{Q_1} \left\{ \frac{dR_k}{dQ_1} \bigg| F_n \right\} = E_{Q_1} \left\{ \frac{dQ_k}{dQ_1} \bigg| F_n \right\} = 1, \quad k \geq s \geq n.
\]

From this, it follows that the set of measures \( R_k \in M \). This proves the consistence with the filtration of the set of measures \( M \). \( \Box \)

On a probability space \( \{ \Omega, \mathcal{F}, P \} \), let \( \xi \) be a random value, satisfying the conditions
\[
0 < P(\{ \omega, \xi > 0 \}) < 1, \quad 0 < P(\{ \omega, \xi < 0 \}).
\]

Denote \( \Omega^+ = \{ \omega, \xi(\omega) > 0 \} \), \( \Omega^- = \{ \omega, \xi(\omega) \leq 0 \} \) and let \( \mathcal{F}^- \), \( \mathcal{F}^+ \) be the restrictions of the \( \sigma \)-algebra \( \mathcal{F} \) on the sets \( \Omega^- \) and \( \Omega^+ \), correspondingly. Suppose that \( P^- \) and \( P^+ \) are the contractions of the measure \( P \) on the \( \sigma \)-algebras \( \mathcal{F}^- \), \( \mathcal{F}^+ \), correspondingly. Consider the measurable space with measure \( \{ \Omega^- \times \Omega^+, \mathcal{F}^- \times \mathcal{F}^+, \mu \} \), which is a direct product of the measurable spaces with measures \( \{ \Omega^-, \mathcal{F}^-, P^- \} \) and \( \{ \Omega^+, \mathcal{F}^+, P^+ \} \), where \( \mu = P^- \times P^+ \). Introduce the denotations
\[
\xi^+(\omega) = \left\{ \begin{array}{ll}
\xi(\omega), & \omega \in \{ \xi(\omega) > 0 \}, \\
0, & \omega \in \{ \xi(\omega) \leq 0 \},
\end{array} \right.
\]
\[
\xi^-(\omega) = \left\{ \begin{array}{ll}
-\xi(\omega), & \omega \in \{ \xi(\omega) \leq 0 \}, \\
0, & \omega \in \{ \xi(\omega) > 0 \}.
\end{array} \right.
\]

Then, \( \xi(\omega) = \xi^+(\omega) - \xi^-(\omega) \).

On the measurable space \( \{ \Omega^- \times \Omega^+, \mathcal{F}^- \times \mathcal{F}^+, P^- \times P^+ \} \), we assume that the set of nonnegative measurable functions \( \alpha(\omega_1, \omega_2) \), satisfying the conditions
\[
\mu(\{(\omega_1, \omega_2) \in \Omega^- \times \Omega^+, \alpha(\omega_1, \omega_2) > 0 \}) = P(\Omega^+)P(\Omega^-),
\]
\[
\int_{\Omega^-} \int_{\Omega^+} \alpha(\omega_1, \omega_2) \frac{\xi^-(\omega_1)\xi^+(\omega_2)}{\xi^-(\omega_1) + \xi^+(\omega_2)} d\mu(\omega_1, \omega_2) < \infty,
\]
\[
\int_{\Omega^-} \int_{\Omega^+} \alpha(\omega_1, \omega_2) d\mu(\omega_1, \omega_2) = 1,
\]
is a nonempty set. Such assumptions are true for the nonempty set of bounded random values \( \alpha(\omega_1, \omega_2) \), for example, if the random value \( \xi \) is an integrable one relative to the measure \( P \).
Lemma 5. On the probability space \( \{\Omega, \mathcal{F}, P\} \), let a random value \( \xi \) satisfy the conditions (36) and let a measure \( Q \) be equivalent to the measure \( P \) and such that \( E^Q \xi = 0 \). Then, for the measure \( Q \) the following representation

\[
Q(A) = \int_{\Omega^-} \int_{\Omega^+} \chi_A(\omega_1, \omega_2) \frac{\xi^+ (\omega_2)}{\xi^- (\omega_1) + \xi^+ (\omega_2)} d\mu(\omega_1, \omega_2) + \int_{\Omega^-} \int_{\Omega^+} \chi_A(\omega_2, \omega_1) \frac{\xi^- (\omega_1)}{\xi^- (\omega_1) + \xi^+ (\omega_2)} d\mu(\omega_1, \omega_2), \quad A \in \mathcal{F},
\]

is valid for those random value \( \alpha(\omega_1, \omega_2) \) that satisfy the conditions (39) - (41).

Every measure \( Q \), given by the formula (42), with the random value \( \alpha(\omega_1, \omega_2) \), satisfying the conditions (39) - (41) is equivalent to the measure \( P \) and is such that \( E^Q \xi = 0 \).

For the measure \( Q \), the canonical representation

\[
Q(A) = \int_{\Omega^-} \int_{\Omega^+} \chi_A(\omega_1, \omega_2) \frac{\xi^+ (\omega_2)}{\xi^- (\omega_1) + \xi^+ (\omega_2)} d\mu(\omega_1, \omega_2) + \int_{\Omega^-} \int_{\Omega^+} \chi_A(\omega_2, \omega_1) \frac{\xi^- (\omega_1)}{\xi^- (\omega_1) + \xi^+ (\omega_2)} d\mu(\omega_1, \omega_2), \quad A \in \mathcal{F},
\]

is valid, where

\[
\alpha(\omega_1, \omega_2) = \frac{\psi_1(\omega_1) \psi_2(\omega_2) [\xi^- (\omega_1) + \xi^+ (\omega_2)]}{d}, \quad (\omega_1, \omega_2) \in \Omega^- \times \Omega^+,
\]

\[
\psi_1(\omega_1) = \int_{\Omega^+} \alpha(\omega_1, \omega_2) \frac{\xi^+ (\omega_2)}{\xi^- (\omega_1) + \xi^+ (\omega_2)} dP(\omega_2), \quad \omega_1 \in \Omega^-.
\]

\[
\psi_2(\omega_2) = \int_{\Omega^-} \alpha(\omega_1, \omega_2) \frac{\xi^- (\omega_1)}{\xi^- (\omega_1) + \xi^+ (\omega_2)} dP(\omega_1), \quad \omega_2 \in \Omega^+.
\]

\[
d = \int_{\Omega^-} \xi^- (\omega_1) \psi_1(\omega_1) dP(\omega_1) = \int_{\Omega^+} \xi^+ (\omega_2) \psi_2(\omega_2) dP(\omega_2).
\]

Proof. From the Lemma 5 conditions,

\[
Q(A) = \int_A \psi(\omega) dP, \quad P(\{\omega, \psi(\omega) > 0\}) = 1, \quad \int_{\Omega} \psi(\omega) \xi(\omega) dP(\omega) = 0.
\]

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The condition (49) means
\[ \int_{\Omega^+} \psi_2(\omega_2) \xi^+(\omega_2) dP(\omega_2) = \int_{\Omega^-} \psi_1(\omega_1) \xi^- (\omega_1) dP(\omega_1) = d > 0, \] (50)
where
\[ \psi_1(\omega) = \begin{cases} \psi(\omega), & \omega \in \Omega^-; \\ 0, & \omega \in \Omega^+, \end{cases} \] (51)
\[ \psi_2(\omega) = \begin{cases} \psi(\omega), & \omega \in \Omega^+; \\ 0, & \omega \in \Omega^- \end{cases}. \] (52)
Let us put
\[ \alpha(\omega_1, \omega_2) = \frac{\psi_1(\omega_1) \psi_2(\omega_2) [\xi^- (\omega_1 + \xi^+(\omega_2))]}{d}, \quad (\omega_1, \omega_2) \in \Omega^- \times \Omega^+. \] (53)
Then, for such \( \alpha(\omega_1, \omega_2) \) the equality (39) is true. Moreover,
\[ \int_{\Omega^-} \int_{\Omega^+} \alpha(\omega_1, \omega_2) \frac{\xi^- (\omega_1) \xi^+(\omega_2)}{\xi^- (\omega_1) + \xi^+(\omega_2)} d\mu(\omega_1, \omega_2) = d^2 < \infty, \] (54)
\[ \int_{\Omega^-} \int_{\Omega^+} \alpha(\omega_1, \omega_2) d\mu(\omega_1, \omega_2) = \] \[ \int_{\Omega^-} \psi_1(\omega_1) dP(\omega_1) + \int_{\Omega^+} \psi_2(\omega_2) dP(\omega_2) = 1, \] (55)
\[ E_Q \xi = \int_{\Omega^-} \int_{\Omega^+} \alpha(\omega_1, \omega_2) \xi(\omega) \frac{\xi^+(\omega_2)}{\xi^- (\omega_1) + \xi^+(\omega_2)} d\mu(\omega_1, \omega_2) + \] \[ \int_{\Omega^-} \int_{\Omega^+} \alpha(\omega_1, \omega_2) \xi(\omega_2) \frac{\xi^- (\omega_1)}{\xi^- (\omega_1) + \xi^+(\omega_2)} d\mu(\omega_1, \omega_2) = 0, \] (56)
since \( \xi(\omega_1) = -\xi^- (\omega_1), \ \omega_1 \in \Omega^-, \ \xi(\omega_2) = \xi^+(\omega_2), \ \omega_2 \in \Omega^+. \)

Let us prove the last statement of Lemma 5. Suppose that the representation (42) for the measure \( Q \), satisfying the conditions (39) - (41), is valid. Taking into account the denotations (45) - (47), we obtain
\[ Q(A) = \int_{\Omega^-} \chi_A(\omega_1) \psi_1(\omega_1) dP(\omega_1) + \int_{\Omega^+} \chi_A(\omega_2) \psi_2(\omega_2) dP(\omega_2), \] (57)
\[ 0 = E_Q \xi = \int_{\Omega^-} \xi(\omega_1) \psi_1(\omega_1) dP(\omega_1) + \int_{\Omega^+} \xi(\omega_2) \psi_2(\omega_2) dP(\omega_2) = \]
\[- \int_{\Omega^-} \xi^- (\omega_1) \psi_1 (\omega_1) dP (\omega_1) + \int_{\Omega^+} \xi^+ (\omega_2) \psi_2 (\omega_2) dP (\omega_2). \tag{58}\]

If to introduce the denotation

\[\psi (\omega) = \begin{cases} \psi_1 (\omega), & \omega \in \Omega^-; \\ \psi_2 (\omega), & \omega \in \Omega^+; \end{cases} \tag{59}\]

then we obtain the representation

\[Q (A) = \int_A \psi (\omega) dP (\omega), \tag{60}\]

where \(P (\psi_1 (\omega) > 0) = P (\Omega^-), \ P (\psi_2 (\omega) > 0) = P (\Omega^+)\).

The last formula proves the equivalence of the measures \(Q\) and \(P\). At last, to prove the canonical representation \((43)\) it is sufficient to substitute the expression \((44)\) for \(Q\) into the expression \((43)\) for \(Q (A)\). We obtain the expression \((57)\) for \(Q (A)\). Then, if to substitute the expressions \((45), (46)\) for \(\psi_1 (\omega_1), \psi_2 (\omega_2)\) into the expression \((57)\) for \(Q (A)\), we obtain that the canonical representation for \(Q (A)\) is true. This proves Lemma 5.

Let \(\Omega, F, P\) be a probability space and let \(G\) be a sub-\(\sigma\)-algebra of the \(\sigma\)-algebra \(F\).

**Lemma 6.** On the probability space \(\Omega, F, P\), let a random value \(\xi\) satisfy the conditions \((56)\) and let it be an integrable one relative to the measure \(P\). A measure \(Q\), being equivalent to the measure \(P\), satisfies the condition

\[E^Q \{\xi | G\} = 0 \tag{61}\]

if and only if for every \(B \in G\) such that \(P (B) > 0\) for the measure \(Q\) the representation

\[Q (A) = \int_{\Omega^B, - } \int_{\Omega^B, + } \chi_A (\omega_1) \alpha_1 (\omega_1, \omega_2) \frac{\zeta^{B, +} (\omega_2)}{\zeta^{B, -} (\omega_1) + \zeta^{B, +} (\omega_2)} d\mu (\omega_1, \omega_2) + \int_{\Omega^B, - } \int_{\Omega^B, + } \chi_A (\omega_2) \alpha_1 (\omega_1, \omega_2) \frac{\zeta^{B, -} (\omega_1)}{\zeta^{B, -} (\omega_1) + \zeta^{B, +} (\omega_2)} d\mu (\omega_1, \omega_2), \ A \in F, \tag{62}\]

is true and the equalities

\[\alpha_1 (\omega_1, \omega_2) = \frac{\psi_1 (\omega_1) \psi_2 (\omega_2) [\zeta^{B, -} (\omega_1 + \zeta^{B, +} (\omega_2))]}{dB}, \tag{63}\]

\[(\omega_1, \omega_2) \in \Omega^{B, -} \times \Omega^{B, +}, \]

\[dB = \int_{\Omega^{B, -}} \zeta^{B, -} (\omega_1) \psi_1 (\omega_1) dP (\omega_1) = \int_{\Omega^{B, +}} \zeta^{B, +} (\omega_2) \psi_2 (\omega_2) dP (\omega_2), \tag{64}\]

are valid, where

\[\zeta^{B, +} (\omega) = \begin{cases} \xi (\omega), & \omega \in B \cap \{\xi (\omega) > 0\} \\ 0, & \omega \in (\Omega \setminus B) \cup \{\xi (\omega) \leq 0\} \end{cases}, \tag{65}\]
\[
\zeta^{B,-}(\omega) = \begin{cases} 
-\xi(\omega), & \omega \in B \cap \{\xi(\omega) \leq 0\}, \\
0, & \omega \in (\Omega \setminus B) \cup \{\xi(\omega) > 0\}, 
\end{cases}
\] (66)

\[
\psi_1(\omega) = \begin{cases} 
\psi(\omega), & \omega \in \Omega^{B,-}, \\
0, & \omega \in \Omega^{B,+}, 
\end{cases}
\] (67)

\[
\psi_2(\omega) = \begin{cases} 
\psi(\omega), & \omega \in \Omega^{B,+}, \\
0, & \omega \in \Omega^{B,-}, 
\end{cases}
\] (68)

\[
\Omega^{B,+} = B \cap \{\xi(\omega) > 0\}, \quad \Omega^{B,-} = (\Omega \setminus B) \cup \{\xi(\omega) \leq 0\},
\] (69)

\[
Q(A) = \int_A \psi(\omega) dP(\omega), \quad A \in \mathcal{F}, \quad P(\{\omega, \psi(\omega) > 0\}) = 1.
\] (70)

**Proof.** The necessity. Suppose that the condition (61) is true. Then, for every \(B \in G\), \(P(B) > 0\), we have

\[
\int_B \xi(\omega) \psi(\omega) dP(\omega) = 0,
\] (71)

or,

\[
\int_{B \cap \{\xi(\omega) > 0\}} \xi(\omega) \psi(\omega) dP(\omega) = -\int_{B \cap \{\xi(\omega) \leq 0\}} \xi(\omega) \psi(\omega) dP(\omega).
\] (72)

From the equality \(P(B) = P(B \cap \{\xi(\omega) > 0\}) + P(B \cap \{\xi(\omega) \leq 0\})\) and the equalities (70), (72), it follows that \(P(B \cap \{\xi(\omega) > 0\}) > 0\) and \(P(B \cap \{\xi(\omega) \leq 0\}) > 0\). Therefore, the equality (72) can be written in the form

\[
0 < d^B = \int_{\Omega^{B,+}} \zeta^{B,+}(\omega_2) \psi_2(\omega_2) dP(\omega_2) = \int_{\Omega^{B,-}} \zeta^{B,-}(\omega_1) \psi(\omega_1) dP(\omega_1).
\] (73)

Define \(\alpha_1(\omega_1, \omega_2)\) by the formula (63) and prove that the formula (62) coincide with the formula (70) for all \(A \in \mathcal{F}\). But, if to substitute the expression for \(\alpha_1(\omega_1, \omega_2)\) defined by the formula (63) into the formula (62) and to take into account the expression for \(d^B\), we obtain

\[
Q(A) = \int_{\Omega^{B,-}} \chi_A(\omega_1) \psi_1(\omega_1) dP(\omega_1) + \int_{\Omega^{B,+}} \chi_A(\omega_2) \psi_2(\omega_2) dP(\omega_2) = \\
\int_{A \cap \Omega^{B,-}} \psi(\omega) dP(\omega) + \int_{A \cap \Omega^{B,+}} \psi(\omega) dP(\omega) = \int_A \psi(\omega) dP(\omega).
\] (74)

The last proves the necessity.
The sufficiency. From the equality
\[ \chi_B \xi(\omega) = \zeta^{B,+}(\omega) - \zeta^{B,-}(\omega) \] (75)
for the measure \( Q \), given by the formula (62), it follows the equality
\[ E^Q \chi_B \xi(\omega) = 0, \quad B \in G. \] (76)
The last means that \( E^Q \{ \xi(\omega)|G \} = 0 \). Lemma 3 is proved. \( \square \)

For further investigations, the next Theorem 4 is very important.

**Theorem 4.** The necessary and sufficient conditions of the local regularity of the non-negative super-martingale \( \{ f_m, \mathcal{F}_m \}_{m=0}^{\infty} \) relative to a set of equivalent measures \( M \) are the existence of \( \mathcal{F}_m \)-measurable random values \( \xi_m^0 \in A_0, \ m = \overline{1, \infty} \), such that
\[ \frac{f_m}{f_{m-1}} \leq \xi_m^0, \quad E^P \{ \xi_m^0|\mathcal{F}_{m-1} \} = 1, \quad P \in M, \quad m = \overline{1, \infty}. \] (77)

**Proof.** The necessity. Without loss of generality, we assume that \( f_m \geq a \) for a certain real number \( a > 0 \). Really, if it is not so, then we can come to the consideration of the super-martingale \( \{ f_m + a, \mathcal{F}_m \}_{m=0}^{\infty} \). Thus, let \( \{ f_m, \mathcal{F}_m \}_{m=0}^{\infty} \) be a nonnegative local regular super-martingale. Then, there exists a nonnegative adapted random process \( \{ g_m \}_{m=0}^{\infty}, \ g_0 = 0, \) such that \( \sup_{P \in M} E^P g_m < \infty \),
\[ f_{m-1} - E^P \{ f_m|\mathcal{F}_{m-1} \} = E^P \{ g_m|\mathcal{F}_{m-1} \}, \quad P \in M, \quad m = \overline{1, \infty}. \] (78)
Let us put \( \xi_m^0 = \frac{f_m + g_m}{f_{m-1}}, \ m = \overline{1, \infty} \). Then, \( \xi_m^0 \in A_0 \) and from the equalities (78) we obtain
\[ E^P \{ \xi_m^0|\mathcal{F}_{m-1} \} = 1, \quad P \in M, \quad m = \overline{1, \infty} \). It is evident that the inequalities (77) are valid.

The sufficiency. Suppose that the conditions of Theorem 4 are valid. Then, \( f_m \leq f_{m-1} + \xi_m^0 - 1 \). Introduce the denotation \( g_m = -f_m + f_{m-1} \xi_m^0 \). Then, \( g_m \geq 0 \),
\[ \sup_{P \in M} E^P g_m \leq \sup_{P \in M} E^P f_m + \sup_{P \in M} E^P f_{m-1} < \infty, \quad m = \overline{1, \infty} \). The last equality and the inequalities give
\[ f_m = f_0 + \sum_{i=1}^{m} f_{i-1}(\xi_i^0 - 1) - \sum_{i=1}^{m} g_i, \quad m = \overline{1, \infty}. \] (79)
Let us consider the random process \( \{ M_m, \mathcal{F}_m \}_{m=0}^{\infty} \), where \( M_m = f_0 + \sum_{i=1}^{m} f_{i-1}(\xi_i^0 - 1) \).
Then, \( E^P \{ M_m|\mathcal{F}_{m-1} \} = M_{m-1}, \quad P \in M, \quad m = \overline{1, \infty} \). Theorem 4 is proved. \( \square \)

### 4 Completeness of the regular set of measures.

In the next two Lemma, we investigate the closure of a convex set of equivalent measures presented in Lemma 5 by the formula (42) that play the fundamental role in the definition of the completeness of the regular set of measures. First, we consider the countable case.

Suppose that \( \Omega_1 \) contains the countable set of elementary events and let \( \mathcal{F}_1 \) be a \( \sigma \)-algebra of all subsets of the set \( \Omega_1 \). Let \( P_1 \) be a measure on the \( \sigma \)-algebra \( \mathcal{F}_1 \). We assume
that $P_1(\omega_i) = p_i > 0, \ i = 1, \infty$. On the probability space $\{\Omega_1, \mathcal{F}_1, P_1\}$, let us consider a nonnegative random value $\xi_1$, satisfying the conditions

$$0 < P_1(\{\omega \in \Omega_1, \eta_1(\omega) < 0\}) < 1, \ 0 < P_1(\{\omega \in \Omega_1, \eta_1(\omega) > 0\}),$$

$$E^{P_1}|\eta_1(\omega)| < \infty, \quad (80)$$

where we introduced the denotation $\eta_1(\omega) = \xi_1(\omega) - 1$. On the measurable space $\{\Omega_1, \mathcal{F}_1\}$, let us consider the set of measures $M_1$, which are equivalent to the measure $P_1$ and are given by the formula

$$Q(A) = \sum_{\omega_1 \in \Omega_1^-} \sum_{\omega_2 \in \Omega_1^+} \chi_A(\omega_1)\alpha(\omega_1, \omega_2) \frac{\eta_1^+(\omega_2)}{\eta_1^- (\omega_1) + \eta_1^+ (\omega_2)} P_1(\omega_1) P_1(\omega_2) +$$

$$\sum_{\omega_1 \in \Omega_1^-} \sum_{\omega_2 \in \Omega_1^+} \chi_A(\omega_2)\alpha(\omega_1, \omega_2) \frac{\eta_1^- (\omega_1)}{\eta_1^- (\omega_1) + \eta_1^+ (\omega_2)} P_1(\omega_1) P_1(\omega_2), \quad A \in \mathcal{F}_1, \quad (81)$$

where $\eta(\omega) = \eta_1^+(\omega) - \eta_1^-(\omega), \ \Omega_1^+ = \{\omega, \eta_1(\omega) > 0\}, \ \Omega_1^- = \{\omega, \eta_1(\omega) \leq 0\}$. Introduce the denotations $\mathcal{F}_1^+ = \Omega_1^+ \cap \mathcal{F}_1, \ \mathcal{F}_1^- = \Omega_1^- \cap \mathcal{F}_1$. Let $P_1^+$ be a contraction of the measure $P_1$ on the $\sigma$-algebra $\mathcal{F}_1^+$ and let $P_1^+$ be a contraction of the measure $P_1$ on the $\sigma$-algebra $\mathcal{F}_1^-$. On the probability space $\{\Omega_1^- \times \Omega_1^+, \mathcal{F}_1^- \times \mathcal{F}_1^+, P_1^- \times P_1^+\}$, the set of random value $\alpha(\omega_1, \omega_2)$ satisfy the conditions

$$P_1 \times P_1(\{(\omega_1, \omega_2) \in \Omega_1^- \times \Omega_1^+, \alpha(\omega_1, \omega_2) > 0\}) = P_1(\Omega_1^+) P_1(\Omega_1^-), \quad (82)$$

$$\sum_{\omega_1 \in \Omega_1^-} \sum_{\omega_2 \in \Omega_1^+} \alpha(\omega_1, \omega_2) \frac{\eta_1^- (\omega_1)\eta_1^+(\omega_2)}{\eta_1^- (\omega_1) + \eta_1^+ (\omega_2)} P_1(\omega_1) P_1(\omega_2) < \infty, \quad (83)$$

$$\sum_{\omega_1 \in \Omega_1^-} \sum_{\omega_2 \in \Omega_1^+} \alpha(\omega_1, \omega_2) P_1(\omega_1) P_1(\omega_2) = 1. \quad (84)$$

On the probability space $\{\Omega_1^- \times \Omega_1^+, \mathcal{F}_1^- \times \mathcal{F}_1^+, P_1^- \times P_1^+\}$, all the bounded random values $\alpha(\omega_1, \omega_2)$ the above conditions satisfy. Introduce into the set of all measures on $\{\Omega_1, \mathcal{F}_1\}$ the metrics

$$\rho(Q_1, Q_2) = \sum_{i=1}^{\infty} |Q_1(\omega_i) - Q_2(\omega_i)|. \quad (85)$$

**Lemma 7.** The closure of the set of measures $M_1$ in metrics (83) contains the set of measures

$$\mu_{\omega_1, \omega_2}(A) = \chi_A(\omega_1) \frac{\eta_1^+(\omega_2)}{\eta_1^- (\omega_1) + \eta_1^+ (\omega_2)} + \chi_A(\omega_2) \frac{\eta_1^- (\omega_1)}{\eta_1^- (\omega_1) + \eta_1^+ (\omega_2)} \quad (86)$$

for $\omega_1 \in \Omega_1^-, \omega_2 \in \Omega_1^+, A \in \mathcal{F}_1$. For every bounded random value $f(\omega)$, the closure of the set of points $E^Q f, \ Q \in M_1$, in metrics $\rho(x, y) = |x - y|, \ x, y \in R^1$, contains the points $E^{\mu_{\omega_1, \omega_2}} f, \ (\omega_1, \omega_2) \in \Omega_1^- \times \Omega_1^+$.
Proof. Let us choose the set of equivalent measures $Q^{\varepsilon}$ defined by $\alpha^{\varepsilon}(\omega_1, \omega_2), 0 < \varepsilon < 1$, and given by the law:

$$\alpha^{\varepsilon}(\omega_1, \omega_2) = \frac{1 - \varepsilon}{P(\omega_1^0)P(\omega_2^0)}, \quad \omega_1^0 \in \Omega^{-}, \quad \omega_2^0 \in \Omega^{+},$$

$$\alpha^{\varepsilon}(\omega_1, \omega_2) = \varepsilon \alpha^{0}(\omega_1, \omega_2), \quad \alpha^{0}(\omega_1, \omega_2) = \frac{1}{\sum_{\omega_1 \neq \omega_1^0} \sum_{\omega_2 \neq \omega_2^0} P(\omega_1)P(\omega_2)^0}, \quad (\omega_1, \omega_2) \neq (\omega_1^0, \omega_2^0),$$

$$\omega_1 \in \Omega^{-}, \quad \omega_2 \in \Omega^{+}.$$  

It is evident that $\alpha^{\varepsilon}(\omega_1, \omega_2) > 0, (\omega_1, \omega_2) \in \Omega^{-} \times \Omega^{+},$ for every $1 > \varepsilon > 0$, and satisfy the equality

$$\sum_{(\omega_1, \omega_2) \in \Omega^{-} \times \Omega^{+}} \alpha^{\varepsilon}(\omega_1, \omega_2)P_1(\omega_1)P_2(\omega_2) = 1. \quad (87)$$

Then,

$$Q^{\varepsilon}(\omega_1^0) = \sum_{\omega_2 \in \Omega^{+}} \alpha^{\varepsilon}(\omega_1^0, \omega_2) \frac{\eta^{+}(\omega_2)}{\eta^{-}(\omega_1^0) + \eta^{+}(\omega_2)} P_1(\omega_1^0)P_2(\omega_2), \quad (88)$$

$$Q^{\varepsilon}(\omega_2^0) = \sum_{\omega_1 \in \Omega^{-}} \alpha^{\varepsilon}(\omega_1, \omega_2^0) \frac{\eta^{-}(\omega_1)}{\eta^{-}(\omega_1^0) + \eta^{+}(\omega_2^0)} P_1(\omega_1^0)P_2(\omega_2^0). \quad (89)$$

$$Q^{\varepsilon}(\omega_1^0) = (1 - \varepsilon) \frac{\eta^{+}(\omega_2^0)}{\eta^{-}(\omega_1^0) + \eta^{+}(\omega_2^0)} +$$

$$\varepsilon \sum_{\omega_2 \in \Omega^{+} \setminus \omega_2^0} \alpha^{\varepsilon}(\omega_1^0, \omega_2) \frac{\eta^{+}(\omega_2)}{\eta^{-}(\omega_1^0) + \eta^{+}(\omega_2)} P_1(\omega_1^0)P_2(\omega_2), \quad (90)$$

$$Q^{\varepsilon}(\omega_2^0) = (1 - \varepsilon) \frac{\eta^{+}(\omega_2^0)}{\eta^{-}(\omega_1^0) + \eta^{+}(\omega_2^0)} +$$

$$\varepsilon \sum_{\omega_1 \in \Omega^{-} \setminus \omega_1^0} \alpha^{\varepsilon}(\omega_1, \omega_2^0) \frac{\eta^{-}(\omega_1)}{\eta^{-}(\omega_1^0) + \eta^{+}(\omega_2^0)} P_1(\omega_1^0)P_2(\omega_2^0). \quad (91)$$

If $\omega_1 \neq \omega_1^0, \omega_2 \neq \omega_2^0$, then

$$Q^{\varepsilon}(\omega_1) = \varepsilon \sum_{\omega_2 \in \Omega^{+}} \alpha^{\varepsilon}(\omega_1, \omega_2) \frac{\eta^{+}(\omega_2)}{\eta^{-}(\omega_1^0) + \eta^{+}(\omega_2)} P_1(\omega_1)P_2(\omega_2), \quad (92)$$

$$Q^{\varepsilon}(\omega_2) = \varepsilon \sum_{\omega_1 \in \Omega^{-}} \alpha^{\varepsilon}(\omega_1, \omega_2) \frac{\eta^{-}(\omega_1)}{\eta^{-}(\omega_1^0) + \eta^{+}(\omega_2^0)} P_1(\omega_1)P_2(\omega_2). \quad (93)$$
The distance between the measures \( Q^\varepsilon \) and \( \mu_{\omega_1^0,\omega_2^0} \) is given by the formula

\[
\rho(Q^\varepsilon, \mu_{\omega_1^0,\omega_2^0}) = \varepsilon + \sum_{\omega_2 \in \Omega_+^1, \omega_2 \neq \omega_2^0} \alpha_0^\varepsilon(\omega_1,\omega_2) \frac{\eta_1^+ (\omega_2)}{\eta_1^- (\omega_1) + \eta_1^+ (\omega_2)} P_1(\omega_1) P(\omega_2) + \\
\varepsilon \sum_{\omega_1 \in \Omega_+^1, \omega_1 \neq \omega_1^0} \alpha_0^\varepsilon(\omega_1,\omega_2) \frac{\eta_1^- (\omega_1)}{\eta_1^- (\omega_1) + \eta_1^+ (\omega_2)} P_1(\omega_1) P(\omega_2) + \\
\varepsilon \sum_{\omega_1 \in \Omega_+^1, \omega_1 \neq \omega_1^0} \sum_{\omega_2 \in \Omega_+^1} \alpha_0^\varepsilon(\omega_1,\omega_2) \frac{\eta_1^+ (\omega_2)}{\eta_1^- (\omega_1) + \eta_1^+ (\omega_2)} P_1(\omega_1) P(\omega_2) + \\
\varepsilon \sum_{\omega_2 \in \Omega_+^1, \omega_2 \neq \omega_2^0} \sum_{\omega_1 \in \Omega_+^1} \alpha_0^\varepsilon(\omega_1,\omega_2) \frac{\eta_1^- (\omega_1)}{\eta_1^- (\omega_1) + \eta_1^+ (\omega_2)} P_1(\omega_1) P(\omega_2).
\] 

(94)

Since

\[
\sum_{\omega_2 \in \Omega_+^1, \omega_2 \neq \omega_2^0} \alpha_0^\varepsilon(\omega_1,\omega_2) \frac{\eta_1^+ (\omega_2)}{\eta_1^- (\omega_1) + \eta_1^+ (\omega_2)} P_1(\omega_1) P(\omega_2) + \\
\sum_{\omega_1 \in \Omega_+^1, \omega_1 \neq \omega_1^0} \alpha_0^\varepsilon(\omega_1,\omega_2) \frac{\eta_1^- (\omega_1)}{\eta_1^- (\omega_1) + \eta_1^+ (\omega_2)} P_1(\omega_1) P(\omega_2) \leq 1,
\]

\[
\sum_{\omega_1 \in \Omega_+^1, \omega_1 \neq \omega_1^0} \sum_{\omega_2 \in \Omega_+^1} \alpha_0^\varepsilon(\omega_1,\omega_2) \frac{\eta_1^+ (\omega_2)}{\eta_1^- (\omega_1) + \eta_1^+ (\omega_2)} P_1(\omega_1) P(\omega_2) \leq 1,
\]

\[
\sum_{\omega_2 \in \Omega_+^1, \omega_2 \neq \omega_2^0} \sum_{\omega_1 \in \Omega_+^1} \alpha_0^\varepsilon(\omega_1,\omega_2) \frac{\eta_1^- (\omega_1)}{\eta_1^- (\omega_1) + \eta_1^+ (\omega_2)} P_1(\omega_1) P(\omega_2) \leq 1,
\]

we obtain

\[
\rho(Q^\varepsilon, \mu_{\omega_1^0,\omega_2^0}) \leq 4\varepsilon.
\]

Let us prove the second part of Lemma 7. It is evident that the inequality

\[
|E^{Q^\varepsilon} f - E^{\mu_{\omega_1^0,\omega_2^0}} f| \leq 4\varepsilon \sup_{\omega \in \Omega_1} |f(\omega)|
\]

(95)

is true. Due to arbitrariness of the small \( \varepsilon \), Lemma 7 is proved. \( \square \)

**Definition 4.** Let \( \{\Omega_1, F_1\} \) be a measurable space. The decomposition \( A_{n,k} \), \( n, k = 1, \infty \), of the space \( \Omega_1 \) we call exhaustive one if the following conditions are valid:

1) \( A_{n,k} \in F_1 \), \( A_{n,k} \cap A_{n,s} = \emptyset \), \( k \neq s \), \( \bigcup_{k=1}^\infty A_{n,k} = \Omega_1 \), \( n = 1, \infty \);

2) the \( (n + 1) \)-th decomposition is a sub-decomposition of the \( n \)-th one, that is, for every \( j, A_{n+1,j} \subseteq A_{n,k} \) for a certain \( k = k(j) \);

3) the minimal \( \sigma \)-algebra containing all \( A_{n,k} \), \( n, k = 1, \infty \), coincides with \( F_1 \).

The next Remark is important for the construction of the filtration having the exhaustive decomposition.
Remark 1. Suppose that the measurable spaces \( \{ \Omega_1, \mathcal{F}_1 \} \) and \( \{ \Omega_2, \mathcal{F}_2 \} \) have the exhaustive decompositions \( A_{n,k}^{1}, n, k = 1, \infty, \) and \( A_{m,s}^{2}, m, s = 1, \infty, \) then the measurable space \( \{ \Omega_1 \times \Omega_2, \mathcal{F}_1 \times \mathcal{F}_2 \} \) also have the exhaustive decomposition \( B_{n,ks}^{1}, n = 1, \infty; k, s = 1, \infty, \) \( B_{n,ks} = A_{n,k}^{1} \times A_{n,s}^{2}, k, s = 1, \infty, n = 1, \infty. \) Really,

1) \( A_{n,k}^{1} \times A_{n,s}^{2} \in \mathcal{F}_1 \times \mathcal{F}_2, A_{n,k}^{1} \times A_{n,s}^{2} \cap A_{n,k}^{1} \times A_{n,s}^{2} = \emptyset, (k, s) \neq (t, r), \)

\[ \bigcup_{k,s=1}^{\infty} B_{n,ks} = \Omega_1 \times \Omega_2, n = 1, \infty; \]

2) the \((n+1)\)-th decomposition is a sub-decomposition of the \(n\)-th one, that is, for every \(k, s\) \( B_{n+1,ks} \subseteq B_{n,ij}\) for a certain \(i = i(k), j = j(s);\)

3) the minimal \(\sigma\)-algebra containing all \(B_{n,ks}, n, k, s = 1, \infty, \) coincides with \(F_1 \times F_2.\)

Lemma 8. Let a measurable space \(\{ \Omega, \mathcal{F} \}\) have an exhaustive decomposition and let \(\xi\) be an integrable random value relative to the measure \(P,\) satisfying the conditions \(36\). Then, the closure of the set of measure \(Q,\) given by the formula \(42,\) relative to the pointwise convergence of measures contains the set of measures

\[ \nu_{(\omega_1, \omega_2)}(A) = \chi_A(\omega_1) \frac{\xi^+(\omega_2)}{\xi^-(\omega_1) + \xi^+(\omega_2)} + \]

\[ \chi_A(\omega_2) \frac{\xi^-(\omega_1)}{\xi^-(\omega_1) + \xi^+(\omega_2)}, \quad A \in \mathcal{F}, \quad (\omega_1, \omega_2) \in \Omega^- \times \Omega^+, \quad (96) \]

for those \((\omega_1, \omega_2) \in \Omega^- \times \Omega^+\) which have the full measure \(\mu = P^- \times P^+.\) For every integrable finite valued random value \(f(\omega)\) relative to all measures \(Q,\) the closure in metrics \(\rho(x_1, x_2) = |x_1 - x_2|, x_1, x_2 \in \mathbb{R}^1,\) of the set of real numbers

\[ \int \int_{\Omega^- \times \Omega^+} f(\omega_1) \alpha(\omega_1, \omega_2) \frac{\xi^+(\omega_2)}{\xi^-(\omega_1) + \xi^+(\omega_2)} d\mu(\omega_1, \omega_2) + \]

\[ \int \int_{\Omega^- \times \Omega^+} f(\omega_2) \alpha(\omega_1, \omega_2) \frac{\xi^-(\omega_1)}{\xi^-(\omega_1) + \xi^+(\omega_2)} d\mu(\omega_1, \omega_2), \quad (97) \]

when \(\alpha(\omega_1, \omega_2)\) runs over all random values satisfying the conditions \(39, 41,\) contains the set of numbers

\[ f(\omega_1) \frac{\xi^+(\omega_2)}{\xi^-(\omega_1) + \xi^+(\omega_2)} + \]

\[ f(\omega_2) \frac{\xi^-(\omega_1)}{\xi^-(\omega_1) + \xi^+(\omega_2)}, \quad (\omega_1, \omega_2) \in \Omega^- \times \Omega^+. \quad (98) \]

Proof. On a probability space \(\{ \Omega, \mathcal{F}, P \},\) let \(\xi\) be an integrable random value, satisfying the conditions \(36.\) As before, let \(\Omega^+ = \{ \omega, \xi(\omega) > 0 \}, \Omega^- = \{ \omega, \xi(\omega) \leq 0 \}\) and let \(\mathcal{F}^-, \mathcal{F}^+\) be the restrictions of the \(\sigma\)-algebra \(\mathcal{F}\) on the sets \(\Omega^-\) and \(\Omega^+,\) correspondingly. Suppose that \(P^-\) and \(P^+\) are the contractions of the measure \(P\) on the \(\sigma\)-algebras \(\mathcal{F}^-\) and \(\mathcal{F}^+,\) correspondingly. Consider the probability space \(\{ \Omega^- \times \Omega^+, \mathcal{F}^- \times \mathcal{F}^+, P^- \times P^+ \}\) which is a direct product of the probability spaces \(\{ \Omega^-, \mathcal{F}^-, P^- \}\) and \(\{ \Omega^+, \mathcal{F}^+, P^+ \}\). Due to Lemma 8 and Remark 1, the measurable space \(\{ \Omega^- \times \Omega^+, \mathcal{F}^- \times \mathcal{F}^+ \}\) has the
exhaustive decomposition $B_{n,ks}$, $k, s = 1, \infty$, $n = 1, \infty$. Denote $\mathcal{F}_n$ the minimal $\sigma$-algebra generated by decomposition $B_{n,ks}$, $k, s = 1, \infty$. It is evident that $\mathcal{F}_n \subset \mathcal{F}_{n+1}$. Moreover, $
abla \{ \mathcal{F}_n \} = \mathcal{F}^- \times \mathcal{F}^+$. On the probability space $\{ \Omega^- \times \Omega^+, \mathcal{F}^- \times \mathcal{F}^+, P^{-} \times P^+ \}$, for every integrable finite valued random value $f(\omega_1, \omega_2)$ the sequence $E[ f(\omega_1, \omega_2) | \mathcal{F}_n]$ converges to $f(\omega_1, \omega_2)$ with probability one, as $n \to \infty$, since it is a regular martingale. It is evident that for those $B_{n,ks}$ for which $\mu(B_{n,ks}) \neq 0$

$$E[ f(\omega_1, \omega_2) | \mathcal{F}_n] = \frac{\int f(\omega_1, \omega_2) d\mu}{\mu(B_{n,ks})}, \quad (\omega_1, \omega_2) \in B_{n,ks}. \quad (99)$$

Denote $D_0 = \bigcup_{n, k, s, \mu(B_{n,ks}) = 0} B_{n,ks}$. It is evident that $\mu(D_0) = 0$. For every $(\omega_1, \omega_2) \in \Omega^- \times \Omega^+ \setminus D_0$, the formula (99) is well defined and is finite. Let $D_1$ be the subset of the set $\Omega^- \times \Omega^+ \setminus D_0$, where the limit of the left hand side of the formula (99) does not exists. Then, $\mu(D_1) = 0$. For every $(\omega_1, \omega_2) \in \Omega^- \times \Omega^+ \setminus (D_0 \cup D_1)$, the right hand side of the formula (99) converges to $f(\omega_1, \omega_2)$. For $(\omega_1, \omega_2) \in \Omega^- \times \Omega^+ \setminus (D_0 \cup D_1)$, denote $A_n = A_n(\omega_1, \omega_2)$ those set $B_{n,ks}$ for which $(\omega_1, \omega_2) \in B_{n,ks}$ for a certain $k, s$. Then, for every integrable finite valued $f(\omega_1, \omega_2)$

$$\lim_{n \to \infty} \frac{\int f(\omega_1, \omega_2) d\mu}{\mu(A_n)} = f(\omega_1, \omega_2). \quad (100)$$

Let us consider the sequence

$$\alpha_n^{\varepsilon n}(\omega_1, \omega_2) = (1 - \varepsilon_n) \chi_{A_n}(\omega_1, \omega_2) + \varepsilon_n \frac{\chi_{\Omega^- \times \Omega^+ \setminus A_n}(\omega_1, \omega_2)}{\mu(\Omega^- \times \Omega^+ \setminus A_n)} \quad (101)$$

where $0 < \varepsilon_n < 1$, $\lim_{n \to \infty} \varepsilon_n = 0$. Such a sequence $\alpha_n^{\varepsilon n}(\omega_1, \omega_2)$ satisfy the conditions (39) - (41) and

$$Q_n^{\varepsilon n}(A) = \int_{\Omega^- \times \Omega^+} \chi_{A}(\omega_1) \alpha_n^{\varepsilon n}(\omega_1, \omega_2) \frac{\xi^+(\omega_2)}{\xi^-(\omega_1) + \xi^+(\omega_2)} d\mu(\omega_1, \omega_2) + \int_{\Omega^- \times \Omega^+} \chi_{A}(\omega_2) \alpha_n^{\varepsilon n}(\omega_1, \omega_2) \frac{\xi^-(\omega_1)}{\xi^-(\omega_1) + \xi^+(\omega_2)} d\mu(\omega_1, \omega_2) = \int \chi_{A}(\omega_1) \frac{\xi^+(\omega_2)}{\xi^-(\omega_1) + \xi^+(\omega_2)} d\mu(\omega_1, \omega_2) \frac{1 - \varepsilon_n}{\mu(A_n)} + \int \chi_{A}(\omega_2) \frac{\xi^-(\omega_1)}{\xi^-(\omega_1) + \xi^+(\omega_2)} d\mu(\omega_1, \omega_2) \frac{(1 - \varepsilon_n) A_n}{\mu(A_n)} + \int \chi_{A}(\omega_1) \frac{\xi^+(\omega_2)}{\xi^-(\omega_1) + \xi^+(\omega_2)} d\mu(\omega_1, \omega_2) \frac{(1 - \varepsilon_n) A_n}{\mu(A_n)} + \int \chi_{A}(\omega_2) \frac{\xi^-(\omega_1)}{\xi^-(\omega_1) + \xi^+(\omega_2)} d\mu(\omega_1, \omega_2) \frac{\varepsilon_n}{\mu(\Omega^- \times \Omega^+ \setminus A_n)}.$$
\[
\int_{\Omega^- \times \Omega^+ \setminus A_n} \chi_A(\omega_2) \frac{\xi^-(\omega_1)}{\xi^-(\omega_1) + \xi^+(\omega_2)} d\mu(\omega_1, \omega_2) \frac{\xi^+ (\omega_2)}{\mu(\Omega^- \times \Omega^+ \setminus A_n)}.
\] (102)

From the formula (102), we obtain
\[
\lim_{n \to \infty} Q^n_\varepsilon(A) = \chi_A(\omega_1) \frac{\xi^+ (\omega_2)}{\xi^-(\omega_1) + \xi^+(\omega_2)} + \chi_A(\omega_2) \frac{\xi^-(\omega_1)}{\xi^-(\omega_1) + \xi^+(\omega_2)}.
\]
\(
A \in \mathcal{F}, (\omega_1, \omega_2) \in \Omega^- \times \Omega^+ \setminus (D_0 \cup D_1).
\) (103)

Further,
\[
E^{Q^n_\varepsilon} f(\omega) = \int \int_{\Omega^- \times \Omega^+} f(\omega_1) \alpha_n(\omega_1, \omega_2) \frac{\xi^+(\omega_2)}{\xi^-(\omega_1) + \xi^+(\omega_2)} d\mu(\omega_1, \omega_2) + \int \int_{\Omega^- \times \Omega^+} f(\omega_2) \alpha_n(\omega_1, \omega_2) \frac{\xi^-(\omega_1)}{\xi^-(\omega_1) + \xi^+(\omega_2)} d\mu(\omega_1, \omega_2) \]
\[
\int f(\omega_1) \frac{\xi^+(\omega_2)}{\xi^-(\omega_1) + \xi^+(\omega_2)} d\mu(\omega_1, \omega_2)
\]
\[
(1 - \varepsilon_n)^A_n \mu(A_n)
\]
\[
\int f(\omega_2) \frac{\xi^-(\omega_1)}{\xi^-(\omega_1) + \xi^+(\omega_2)} d\mu(\omega_1, \omega_2)
\]
\[
(1 - \varepsilon_n)^A_n \mu(A_n)
\]
\[
\int \int_{\Omega^- \times \Omega^+ \setminus A_n} f(\omega_1) \xi^+(\omega_2) d\mu(\omega_1, \omega_2)
\]
\[
\varepsilon_n \frac{\xi^-(\omega_1)}{\xi^-(\omega_1) + \xi^+(\omega_2)} d\mu(\omega_1, \omega_2)
\]
\[
\int \int_{\Omega^- \times \Omega^+ \setminus A_n} f(\omega_2) \xi^-(\omega_1) d\mu(\omega_1, \omega_2)
\]
\[
\varepsilon_n \frac{\xi^+(\omega_2)}{\mu(\Omega^- \times \Omega^+ \setminus A_n)}
\] (104)

From the formula (104), we obtain
\[
\lim_{n \to \infty} E^{Q^n_\varepsilon} f(\omega) = f(\omega_1) \frac{\xi^+(\omega_2)}{\xi^-(\omega_1) + \xi^+(\omega_2)} + f(\omega_2) \frac{\xi^-(\omega_1)}{\xi^-(\omega_1) + \xi^+(\omega_2)}.
\]
\[
A \in \mathcal{F}, (\omega_1, \omega_2) \in \Omega^- \times \Omega^+ \setminus (D_0 \cup D_1).
\) (105)

Lemma 8 is proved.

The next Theorem 5 is a consequence of Lemma 5.
Theorem 5. On the probability space \( \{\Omega, \mathcal{F}, P\} \), for the nonnegative random value \( \xi \neq 1 \) the set of measures \( M_0 \) on the measurable space \( \{\Omega, \mathcal{F}\} \), being equivalent to the measure \( P \), satisfies the condition

\[
E^Q \xi = 1, \quad Q \in M_0, \tag{106}
\]

if and only if as for \( Q \in M_0 \) the representation

\[
Q(A) = \int_{\Omega^-} \int_{\Omega^+} \chi_A(\omega_1) \alpha(\omega_1, \omega_2) \frac{(\xi - 1)^+(\omega_2)}{(\xi - 1)^-(\omega_1) + (\xi - 1)^+(\omega_2)} d\mu(\omega_1, \omega_2) + \\
\int_{\Omega^-} \int_{\Omega^+} \chi_A(\omega_2) \alpha(\omega_1, \omega_2) \frac{(\xi - 1)^-(\omega_1)}{(\xi - 1)^-(\omega_1) + (\xi - 1)^+(\omega_2)} d\mu(\omega_1, \omega_2), \quad A \in \mathcal{F}, \tag{107}
\]

is true, where on the measurable space \( \{\Omega^- \times \Omega^+, \mathcal{F}^- \times \mathcal{F}^+, P^- \times P^+\} \), the random value \( \alpha(\omega_1, \omega_2) \) satisfies the conditions

\[
\mu(\{(\omega_1, \omega_2) \in \Omega^- \times \Omega^+, \alpha(\omega_1, \omega_2) > 0\}) = P(\Omega^+) P(\Omega^-), \tag{108}
\]

\[
\int_{\Omega^-} \int_{\Omega^+} \alpha(\omega_1, \omega_2) \frac{(\xi - 1)^-(\omega_1)(\xi - 1)^+(\omega_2)}{(\xi - 1)^-(\omega_1) + (\xi - 1)^+(\omega_2)} d\mu(\omega_1, \omega_2) < \infty, \tag{109}
\]

\[
\int_{\Omega^-} \int_{\Omega^+} \alpha(\omega_1, \omega_2) d\mu(\omega_1, \omega_2) = 1. \tag{110}
\]

We introduced above the following denotations: \( \mu = P^- \times P^+ \), \( P^- \) is a contraction of the measure \( P \) on the set \( \Omega^- = \{\omega \in \Omega, \xi - 1 \leq 0\} \), \( P^+ \) is a contraction of the measure \( P \) on the set \( \Omega^+ = \{\omega \in \Omega, \xi - 1 > 0\} \), \( \mathcal{F}^- = \Omega^- \cap \mathcal{F}, \mathcal{F}^+ = \Omega^+ \cap \mathcal{F} \).

It is evident that the set of measure \( M_0 \) is a nonempty one, since it contains those measures \( Q \), for which the random value \( \alpha(\omega_1, \omega_2) \) is bounded, since \( E^Q |\xi - 1| < \infty \).

Theorem 6. On the probability space \( \{\Omega, \mathcal{F}, P\} \) with the filtration \( \mathcal{F}_n \) on it, the set of measures \( M_0 \), given by the formula (107), is consistent with the filtration \( \mathcal{F}_n \), if and only if, as \( E^Q \{\xi |\mathcal{F}_n\} \), \( Q \in M_0 \), is a local regular martingale.

Proof. The necessity. Let the set of measures \( M_0 \) be consistent with the filtration. Then, due to Theorem 3 \( E^Q \{\xi |\mathcal{F}_n\}, Q \in M_0 \), is a local regular martingale.

The sufficiency. Suppose that \( E^Q \{\xi |\mathcal{F}_n\}, Q \in M_0 \), is a local regular martingale. Let us prove that, if \( Q_1, Q_2 \in M_0 \), then the set of measures

\[
R^k_s(A) = \int_A E^{Q_2} \{\frac{dQ_1}{dQ_2} |\mathcal{F}_k\} dQ_2, \quad A \in \mathcal{F}, \quad k \geq s \geq n, \quad n = 0, \infty, \tag{111}
\]

belongs to the set \( M_0 \). For this, it is to prove that \( E^{R^k_s}(\xi - 1) = 0 \), or \( E^{R^k_s} \xi = 1 \). Really, if \( E^{Q_1} \xi = 1 \), \( E^{Q_2} \xi = 1 \), then

\[
E^{R^k_s} \xi = E^{Q_2} \xi \frac{E^{Q_2} \{\frac{dQ_1}{dQ_2} |\mathcal{F}_k\}}{E^{Q_2} \{\frac{dQ_1}{dQ_2} |\mathcal{F}_s\}} = E^{Q_2} \xi E^{Q_2} \{\xi |\mathcal{F}_k\} \frac{dQ_1}{dQ_2} |\mathcal{F}_k\} =
\]

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for those \((\Omega, \mathcal{F}, P)\) with the filtration \(\mathcal{F}_n\) on it, the set of measures \(M_0\), given by the formula \((107)\), is consistent with the filtration \(\mathcal{F}_n\), if and only if there exists not depending on \((\omega_1, \omega_2) \in \Omega^- \times \Omega^+\) the random process \(\{m_n, \mathcal{F}_n\}_{n=0}^\infty\) such that

\[
E^{Q_2} E^{Q_2} \{E^{Q_2} \{\xi|\mathcal{F}_n\} | \mathcal{F}_s\} \frac{dQ_1}{dQ_2} = E^{Q_2} E^{Q_1} \{E^{Q_2} \{\xi|\mathcal{F}_k\} | \mathcal{F}_s\} = E^{Q_2} E^{Q_1} \{\xi|\mathcal{F}_s\} = E^{Q_2} \xi = 1.
\]  

(112)

Theorem \(6\) is proved.

**Theorem 7.** On the probability space \(\{\Omega, \mathcal{F}, P\}\) with the filtration \(\mathcal{F}_n\) on it, the set of measures \(M_0\), given by the formula \((107)\), is consistent with the filtration \(\mathcal{F}_n\), if and only if there exists not depending on \((\omega_1, \omega_2) \in \Omega^- \times \Omega^+\) the random process \(\{m_n, \mathcal{F}_n\}_{n=0}^\infty\) such that

\[
E^{\nu_{\omega_1, \omega_2}} \{\xi|\mathcal{F}_n\} = m_n, \quad n = 1, \infty,
\]  

(113)

for those \((\omega_1, \omega_2) \in \Omega^- \times \Omega^+\) that have the full measure \(\mu = P^- \times P^+\), where

\[
\nu_{\omega_1, \omega_2} (A) = \chi_A(\omega_1) \frac{(\xi-1)^+(\omega_2)}{(\xi-1)^-(\omega_1) + (\xi-1)^+(\omega_2)}, \quad A \in \mathcal{F}, \quad (\omega_1, \omega_2) \in \Omega^- \times \Omega^+.
\]  

(114)

**Proof.** The necessity. Suppose that the set of measures \(M_0\), given by the formula \((107)\), is consistent with the filtration \(\mathcal{F}_n\). Due to Theorem \(6\) \(E^Q \{\xi|\mathcal{F}_n\}, Q \in M_0\), is a local regular martingale. Then, \(E^Q \{\xi|\mathcal{F}_n\} = m_n\). Using Lemma \(8\) we obtain \(E^{\nu_{\omega_1, \omega_2}} \{\xi|\mathcal{F}_n\} = m_n\) for those \((\omega_1, \omega_2) \in \Omega^- \times \Omega^+\) that have the full measure \(\mu\).

The sufficiency. If the formula \((113)\) is true, then \(E^Q \{\xi|\mathcal{F}_n\} = m_n, \quad Q \in M_0\). From this, it follows that \(E^Q \{\xi|\mathcal{F}_n\}, \quad Q \in M_0\), is a local regular martingale. Theorem \(7\) is proved.

**Definition 5.** On the probability space \(\{\Omega, \mathcal{F}, P\}\) with the filtration \(\mathcal{F}_n\) on it, the consistent with the filtration \(\mathcal{F}_n\) subset of the measures \(M\) of the set of the measures \(M_0\) that is generated by the nonnegative random value \(\xi \neq 1\), \(E^Q \xi = 1, \quad Q \in M_0\), we call the regular set of measures.

Let \(\{\Omega, \mathcal{F}, P\}\) be a probability space. On the measurable space \(\{\Omega, \mathcal{F}\}\) with the filtration \(\mathcal{F}_n\) on it, let \(M \subseteq M_0\) be a set of regular measures, where the set \(M_0\) is generated by the nonnegative random value \(\xi \neq 1\). Denote by \(\{m_n, \mathcal{F}_n\}_{n=0}^\infty\) the regular martingale, where \(m_n = E^Q \{\xi|\mathcal{F}_n\}, Q \in M, \quad n = 1, \infty\). Assume that \(M_0\) is a contraction of the set of regular measures \(M\) onto the \(\sigma\)-algebra \(\mathcal{F}_n\). Every \(Q^n \in M_0\) is equivalent to \(P^n\), where \(P^n\) is a contraction of the measure \(P\) on the \(\sigma\)-algebra \(\mathcal{F}_n\). For every \(Q^n \in M_0\), we have \(E^{Q^n} [m_n - m_{n-1}] = 0\). Therefore, for the measure \(Q_n \in M_n\) the representation

\[
Q_n (A) = \int_{\Omega_+^n \times \Omega_+^n} \chi_A(\omega_1) \frac{\alpha_n(\omega_1, \omega_2) [m_n - m_{n-1}]^+(\omega_2)}{[m_n - m_{n-1}]^-(\omega_1) + [m_n - m_{n-1}]^+(\omega_2)} d\mu_n(\omega_1, \omega_2) + \int_{\Omega_+^n \times \Omega_+^n} \chi_A(\omega_2) \frac{\alpha_n(\omega_1, \omega_2) [m_n - m_{n-1}]^-(\omega_1)}{[m_n - m_{n-1}]^-(\omega_1) + [m_n - m_{n-1}]^+(\omega_2)} d\mu_n(\omega_1, \omega_2), \quad A \in \mathcal{F}_n,
\]  

(115)

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where we introduced the denotation 
\( \eta \), 
\( \xi \), the filtration. On the probability space 
\( \{ \Omega_n^-, \Omega_n^+, \mathcal{F}_n^- \times \mathcal{F}_n^+ \} \), the random value \( \alpha_n(\omega_1, \omega_2) \) satisfies the conditions

\[
\mu_n(\{(\omega_1, \omega_2) \in \Omega_n^- \times \Omega_n^+, \alpha_n(\omega_1, \omega_2) > 0\}) = P_n(\Omega^+)P_n(\Omega^-),
\]

(116)

\[
\int \int_{\Omega_n^- \times \Omega_n^+} \alpha_n(\omega_1, \omega_2) \frac{[m_n - m_{n-1}]^-(\omega_1)[m_n - m_{n-1}]^+(\omega_2)}{[m_n - m_{n-1}]^-(\omega_1) + [m_n - m_{n-1}]^+(\omega_2)} \, d\mu_n(\omega_1, \omega_2) < \infty,
\]

(117)

\[
\int \int_{\Omega_n^- \times \Omega_n^+} \alpha_n(\omega_1, \omega_2) \, d\mu_n(\omega_1, \omega_2) = 1.
\]

(118)

Here, the measure \( \mu_n = P_n^+ \times P_n^- \) is given on the measurable space \( \{ \Omega_n^- \times \Omega_n^+, \mathcal{F}_n^- \times \mathcal{F}_n^+ \} \) and it is a direct product of the measures \( P_n^- \) and \( P_n^+ \), where the measure \( P_n^+ \) is a contraction of the measure \( P_n \) on the \( \sigma \)-algebra \( \mathcal{F}_n^+ = \Omega_n^+ \cap \mathcal{F}_n \) and \( P_n^- \) is a contraction of the measure \( P_n \) on the \( \sigma \)-algebra \( \mathcal{F}_n^- = \Omega_n^- \cap \mathcal{F}_n \).

**Definition 6.** We say that the regular set of measures \( M \) is complete one, if for every \( n = 1, \infty \) the set of measures \( Q_n \) contains the measures of the kind (115) for the random values \( \alpha_n(\omega_1, \omega_2) \) of the kind \( C_1^n \chi_A(\omega_1, \omega_2) + C_2^n \chi_{\Omega_n^- \times \Omega_n^+ \setminus A}(\omega_1, \omega_2) \), as \( A \) runs all sets from the \( \sigma \)-algebra \( \mathcal{F}_n^- \times \mathcal{F}_n^+ \), where \( C_1^n \mu_n(A) + C_2^n \mu_n(\Omega_n^- \times \Omega_n^+ \setminus A) = 1 \), \( C_1^n \geq 0 \), \( C_2^n \geq 0 \).

It is evident that the regular set of measures \( M \) is a convex set of measure. On the probability space \( \{ \Omega, \mathcal{F}, P \} \) with the filtration \( \mathcal{F}_n \) on it, let us introduce into consideration the set \( A_0 \) of all integrable nonnegative random values \( \zeta \) relative to the set of regular measures \( M \), satisfying the conditions

\[
E^P \zeta = 1, \quad P \in M.
\]

(119)

It is evident that the set \( A_0 \) is a nonempty one, since it contains the random value \( \zeta = 1 \). The more interesting case is as \( A_0 \) contains more then one element. So, further we consider the regular set of measure \( M \) with the set \( A_0 \), containing more then one element.

The set \( A_0 \) can contain more then two elements. Then, for every element \( \eta \in A_0 \) \( E^Q \{ \eta | \mathcal{F}_n \} \), \( Q \in M \), forms the local regular martingale.

In the next Lemma 8 using Lemma 5, we construct a set of measures consistent with the filtration. On the probability space \( \{ \Omega_1^0, \mathcal{F}_1^0, P_1 \} \), let us consider a nonnegative random value \( \xi_1 \), satisfying the conditions

\[
0 < P_1(\{ \omega \in \Omega_1^0, \eta_1(\omega) < 0 \}) < 1,
\]

\[
0 < P_1(\{ \omega \in \Omega_1^0, \eta_1(\omega) > 0 \}),
\]

(120)

where we introduced the denotation \( \eta_1(\omega) = \xi_1(\omega) - 1 \). Described in Lemma 5 the set of equivalent measures to the measure \( P_1 \) and such that \( E^Q \eta_1(\omega) = 0 \), we denote by \( M_1 \). Let us construct the infinite direct product of the measurable spaces \( \{ \Omega_i^0, \mathcal{F}_i^0 \}, i = 1, \infty \), where \( \Omega_i^0 = \Omega_1^0, \mathcal{F}_i^0 = \mathcal{F}_1^0 \). Denote \( \Omega = \prod_{i=1}^{\infty} \Omega_i^0 \). On the space \( \Omega \), under the \( \sigma \)-algebra...
where the random values \( a \) we understand the minimal \( \sigma \)-algebra, generated by the sets \( \prod_{i=1}^{\infty} G_i, \, G_i \in \mathcal{F}_i^0 \), where in the last product only the finite set of \( G_i \) do not equal \( \Omega_i^0 \). On the measurable space \( \{\Omega, \mathcal{F}\} \), under the filtration \( \mathcal{F}_n \) we understand the minimal \( \sigma \)-algebra generated by the sets \( \prod_{i=1}^{\infty} G_i, \, G_i \in \mathcal{F}_i^0 \), where \( G_i = \Omega_i^0 \) for \( i > n \). We consider the probability space \( \{\Omega, \mathcal{F}, P\} \), where \( P = \prod_{i=1}^{\infty} P_i, \, P_i = P_1, \, i = 1, \infty \).

On the measurable space \( \{\Omega, \mathcal{F}\} \), we introduce into consideration the set of measures \( M \), where \( Q \) belongs to \( M \), if \( Q = \prod_{i=1}^{\infty} Q_i, \, Q_i \in M_1 \). We denote by \( M^{Q_0} \) a subset of the set \( M \) of those measures \( Q = \prod_{i=1}^{\infty} Q_i, \, Q_i \in M_1 \), for which only the finite set of \( Q_i \) does not coincide with the measure \( Q_0 \in M_1 \).

**Lemma 9.** On the measurable space \( \{\Omega, \mathcal{F}\} \) with the filtration \( \mathcal{F}_n \) on it, there exists consistent with the filtration \( \mathcal{F}_n \) the set of measures \( M_0 \) and the nonnegative random variable \( \xi_0 \) such that \( E^Q \xi_0 = 1, \, Q \in M_0 \), if the random value \( \xi_1 \), satisfying the conditions \( (120) \), is bounded.

**Proof.** To prove Lemma 9 we need to construct a nonnegative bounded random value \( \xi_0 \) on the measurable space \( \{\Omega, \mathcal{F}\} \) and a set of equivalent measures \( M_0 \) on it, such that \( E^Q \xi_0 = 1, \, Q \in M_0 \), and to prove that the set of measures \( M_0 \) is consistent with the filtration \( \mathcal{F}_n \). From the Lemma 9 conditions, the random value \( \eta_i(\omega_i) = \xi_i(\omega_i) - 1 \) is also bounded. Let us put

\[
\xi_0 = \prod_{i=1}^{\infty} [1 + a_i(\omega_1, \ldots, \omega_{i-1}) \eta_i(\omega_i)], \quad (121)
\]

where the random values \( a_i(\omega_1, \ldots, \omega_{i-1}) \) are \( \mathcal{F}_{i-1} \)-measurable, \( i = 1, \infty \), they satisfy the conditions \( 0 < a_i(\omega_1, \ldots, \omega_{i-1}) \leq b_i < 1 \). The constants \( b_i \) are such that \( \sum_{i=1}^{\infty} b_i < \infty \), the random value \( \eta_i(\omega_i) \) is given on \( \{\Omega_i^0, \mathcal{F}_i^0, P_i\} \) and is distributed as \( \eta_i(\omega_i) \) on \( \{\Omega_i^0, \mathcal{F}_i^0, P_i\} \).

From this, it follows that the random value \( \xi_0 \) is bounded by the constant \( \prod_{i=1}^{\infty} [1 + Cb_i] \), where \( C > 0 \) and it is such that \( \vert \eta_i(\omega_i) \vert < C, \, i = 1, \infty \). It is evident that \( E^Q \xi_0 = 1, \, Q \in M^{Q_0} \). Really,

\[
E^Q \prod_{i=1}^{n} [1 + a_i(\omega_1, \ldots, \omega_{i-1}) \eta_i(\omega_i)] = \prod_{i=1}^{n-1} [1 + a_i(\omega_1, \ldots, \omega_{i-1}) \eta_i(\omega_i)] \times E^Q_{n-1} [1 + a_{n-1}(\omega_1, \ldots, \omega_{n-1}) \eta_{n}(\omega_n)], \quad (122)
\]

where \( Q = \prod_{i=1}^{\infty} Q_i, \, Q^{n-1}_n = \prod_{i=1}^{n-1} Q_i, \)

\[
E^Q_n [1 + a_{n-1}(\omega_1, \ldots, \omega_{n-1}) \eta_{n}(\omega_n)] =
\]
\[ [1 + a_{n-1}(\omega_1, \ldots, \omega_{n-1})E^{Q^n}\eta_n(\omega_n)] = 1. \] (123)

From the last equality, we have

\[ E^Q \prod_{i=1}^{n} [1 + a_i(\omega_1, \ldots, \omega_{i-1})\eta_i(\omega_i)] = 1. \] (124)

Since \( \xi_0 = \lim_{n \to \infty} \prod_{i=1}^{n} [1 + a_i(\omega_1, \ldots, \omega_{i-1})\eta_i(\omega_i)] \), from the equality (124) and the possibility to go to the limit under the mathematical expectation, we prove the needed statement. Let us prove the existence of the set of measures \( M_0 \) consistent with the filtration \( F_n \). If \( Q \in M_0^Q_0 \), then

\[ E^Q \{ \xi_0 | F_n \} = \prod_{i=1}^{n} [1 + a_i(\omega_1, \ldots, \omega_{i-1})\eta_i(\omega_i)] \quad Q \in M_0^Q_0. \] (125)

Due to Lemma 4, there exists a set of measures \( M_0 \) such that it is consistent with the filtration and \( M_0 \supseteq M_0^Q_0 \), \( E^Q \xi_0 = 1 \), \( Q \in M_0 \). The set \( M_0 \) is a linear convex span of the set \( M_0^Q_0 \). It means that the set of measures \( M_0 \) is consistent with the filtration. Lemma 4 is proved. \( \square \)

**Remark 2.** The boundedness of the random value \( \xi_1 \) is not essential. For applications, the case, as \( a_i(\omega_1, \ldots, \omega_{i-1}) = 0 \), \( i \geq n + 1 \), is very important (see Section 8). In this case, Lemma 4 is true as the random value \( \eta_1 \) is an integrable one. The random value \( \xi_0 \) is also integrable one relative to every measures from the set \( M_0 \) and it is \( F_n \)-measurable one.

Below, we describe completely the regular set of measures in the case as \( \xi_0 = \prod_{i=1}^{N} [1 + a_i(\omega_1, \ldots, \omega_{i-1})\eta_i(\omega_i)] \), \( N < \infty \), \( 0 < a_i(\omega_1, \ldots, \omega_{i-1}) \leq 1 \), \( i = 1,N \), and the random value \( \xi_1 \) is an integrable one relative to the measure \( P_1 \). For this purpose, we introduce the denotations: \( \Omega_i^- = \{ \omega_i \in \Omega_i^0, \eta_i(\omega_i) \leq 0 \} \), \( \Omega_i^+ = \{ \omega_i \in \Omega_i^0, \eta_i(\omega_i) > 0 \} \), \( P_i^- \) is a contraction of the measure \( P_1 \) on the \( \sigma \)-algebra \( F_i^- \), \( P_i^+ \) is a contraction of the measure \( P_1 \) on the \( \sigma \)-algebra \( F_i^+ \).

Denote \( U_1 = \Omega_1^- \times \Omega_1^+ \) and introduce the measure \( \mu_1 = P_1^- \times P_1^+ \) on the \( \sigma \)-algebra \( G_1 = F_1^- \times F_1^+ \). Let us introduce the measurable space \( \{ \mathcal{V}, \mathcal{L}, \mu \} \), where \( \mathcal{V} = \prod_{i=1}^{N} U_i \), \( U_i = U_1 \), \( i = 1,N \), is a direct product of the spaces \( U_i = \Omega_i^- \times \Omega_i^+ \), \( \Omega_i^- = \Omega_1^- \), \( \Omega_i^+ = \Omega_1^+ \), \( \mathcal{L} = \prod_{i=1}^{N} G_i \) is a direct product of the \( \sigma \)-algebras \( G_i = G_1 \), \( i = 1,N \). At last, let \( \mu = \prod_{i=1}^{N} \mu_i \) be a direct product of the measures \( \mu_i = \mu_1 \), \( i = 1,N \), and let \( \nu_v = \prod_{i=1}^{N} \nu_{v_i} \), \( v = \{(\omega_1^1,\omega_1^2), \ldots, (\omega_N^1,\omega_N^2)\} \), be a direct product of the measures \( \nu_{\omega_i^1,\omega_i^2} \), \( i = 1,N \), which is a countable additive function on the \( \sigma \)-algebra \( \mathcal{F}_N \) for every \( v \in \mathcal{L} \), where

\[ \nu_{\omega_i^1,\omega_i^2}(A_i) = \chi_{A_i}(\omega_1^1) \frac{\eta_i^+(\omega_i^2)}{\eta_i^-(\omega_i^2)} + \chi_{A_i}(\omega_1^2) \frac{\eta_i^-(\omega_i^1)}{\eta_i^+(\omega_i^1)} + \chi_{A_i}(\omega_1^1) \frac{\eta_i^-(\omega_i^1)}{\eta_i^+(\omega_i^1)} + \chi_{A_i}(\omega_1^2) \frac{\eta_i^+(\omega_i^2)}{\eta_i^-(\omega_i^2)} \] (126)

for \( \omega_i^1 \in \Omega_i^- \), \( \omega_i^2 \in \Omega_i^+ \), \( A_i \in \mathcal{F}_i^0 \).

In the next Theorem 8 we assume that the random value \( \eta_1(\omega_1) \) is an integrable one.
Theorem 8. On the measurable space \( \{\Omega, F\} \) with the filtration \( F_n \) on it, every measure \( Q \) of the regular set of measures \( M \) for the random value \( \xi_0 = \prod_{i=1}^{N}[1 + a_i(\omega_1, \ldots, \omega_{i-1})\eta_i(\omega_i)] \), \( N < \infty, 0 < a_i(\omega_1, \ldots, \omega_{i-1}) \leq 1, i = 1, N, \) has the representation

\[
Q(A) = \int_{\mathcal{V}} \alpha(v)\nu_v(A)d\mu(v),
\]

where the random value \( \alpha(v) \) satisfies the conditions

\[
\mu(\{v \in \mathcal{V}, \alpha(v) > 0\}) = [P_1(\Omega_1^-)P_1(\Omega_1^+)]^N, 
\]

\[
\int_{\mathcal{V}} \alpha(v)\prod_{i=1}^{N} \frac{\eta_i^- (\omega_i^1)\eta_i^+ (\omega_i^2)}{\eta_i^- (\omega_i^1) + \eta_i^+ (\omega_i^2)}d\mu(v) < \infty,
\]

\[
\int_{\mathcal{V}} \alpha(v)d\mu(v) = 1.
\]

Proof. To prove Theorem, it needs to prove that the countable additive measure \( \nu_v(A) \) at every fixed \( v \in \mathcal{V} \) is a measurable map from the measurable space \( \{\mathcal{V}, \mathcal{L}\} \) into the measurable space \( \{[0, 1], B([0, 1])\} \) for every fixed \( A \in F_N \). For \( A = \prod_{i=1}^{N} A_i, A_i \in F_i^0, \) \( \nu_v(A) \) is a measurable map from the measurable space \( \{\mathcal{V}, \mathcal{L}\} \) into the measurable space \( \{[0, 1], B([0, 1])\} \). The family of sets of the kind \( \bigcup_{i \in I} E_i, E_i = \prod_{s=1}^{N} A_i^s, A_i^s \in F_s^0, \) where \( E_i \cap E_j = \emptyset, \) the set \( I \) is an arbitrary finite set, forms the algebra of the sets that we denote by \( U_0 \). From the countable additivity of \( \nu_v(A), \nu_v(\bigcup E_i) = \sum_{i \in I} \nu_v(E_i) \) is a measurable map from the measurable space \( \{\mathcal{V}, \mathcal{L}\} \) into the measurable space \( \{[0, 1], B([0, 1])\} \). Let \( T \) be a class of the sets from the minimal \( \sigma \)-algebra \( \Sigma \) generated by \( U_0 \) for every subset \( E \) of that \( \nu_v(E) \) is a measurable map from the measurable space \( \{\mathcal{V}, \mathcal{L}\} \) into the measurable space \( \{[0, 1], B([0, 1])\} \). Let us prove that \( T \) is a monotonic class. Suppose that \( E_i \subset E_{i+1}, i = 1, \infty, E_i \in T \). Then, \( \nu_v(E_i) \leq \nu_v(E_{i+1}) \). From this, it follows that \( \lim_{i \to \infty} \nu_v(E_i) \) is a measurable map from the measurable space \( \{\mathcal{V}, \mathcal{L}\} \) into the measurable space \( \{[0, 1], B([0, 1])\} \). But, \( \nu_v(E_{i+1} \setminus E_i) = \nu_v(E_{i+1}) - \nu_v(E_i) \) is a measurable map from \( \{\mathcal{V}, \mathcal{L}\} \) into \( \{[0, 1], B([0, 1])\} \). From this equality, it follows that the set \( E_{i+1} \setminus E_i \) belongs to the class \( T \). Since \( \bigcup_{i=1}^{\infty} E_i = E_1 \cup \bigcup_{i=1}^{\infty} E_{i+1} \setminus E_i \), we have

\[
\lim_{n \to \infty} \nu_v(E_n) = \nu_v(E_1) + \lim_{n \to \infty} \sum_{i=1}^{n} \nu_v(E_{i+1} \setminus E_i) =
\]

\[
\nu_v(E_1) + \sum_{i=1}^{\infty} \nu_v(E_{i+1} \setminus E_i) = \nu_v(E_1 \cup \bigcup_{i=1}^{\infty} (E_{i+1} \setminus E_i)) = \nu_v(\bigcup_{i=1}^{\infty} E_i).
\]
The equalities (131) mean that \( \bigcup_{i=1}^{\infty} E_i \) belongs to \( T \), since \( \nu_v(\bigcup_{i=1}^{\infty} E_i) \) is a measurable map of \( \{ \mathcal{V}, \mathcal{L} \} \) into \( \{ [0,1], \mathcal{B}([0,1]) \} \). Suppose that \( E_i \supset E_{i+1}, \ E_i \in T, \ i = 1, \infty \). Then, this case is reduced to the previous one by the note that the sequence \( E_i = \bigcap_{i=1}^{N} \Omega_i^{0} \setminus E_i, \ i = 1, \infty \) is monotonically increasing. From this, it follows that \( E = \bigcup_{i=1}^{\infty} E_i \in T \). Therefore, \( \bigcap_{i=1}^{\infty} E_i = \bigcap_{i=1}^{N} \Omega_i^{0} \setminus \bigcup_{i=1}^{\infty} E_i \in T \). Thus, \( T \) is a monotone class. But, \( U_0 \subset T \). Hence, \( T \) contains the minimal monotone class generated by the algebra \( U_0 \), that is, \( m(U_0) = \Sigma \), therefore, \( \Sigma \subset T \). Thus, \( \nu_v(E) \) is a measurable map of \( \{ \mathcal{V}, \mathcal{L} \} \) into \( \{ [0,1], \mathcal{B}([0,1]) \} \) for \( A \in \Sigma \). The fact that the random value \( \alpha(v) \) satisfies the conditions (128 - 130) means that \( Q \), given by the formula (127), is a countable additive function of sets and \( E^Q \xi_0 < \infty \). Moreover, \( E^Q \xi_0 = 1 \). It is evident that \( E^Q \xi_0 | \mathcal{F}_n = \prod_{i=1}^{n} [1 + a_i(\omega_1, \ldots, \omega_{i-1})\eta_i(\omega_i)], \ Q \in M \). Due to Lemma 4, this proves that the set \( M \) is a regular set of measure. Theorem 8 is proved.

Remark 3. The representation (127) for the regular set of measures \( M \) means that \( M \) is a convex set of equivalent measures. Since the random value \( \alpha(v) \) runs all bounded random value, satisfying the conditions (128 - 130), it is easy to show that the set of measures \( \nu_v(A), \ v \in \mathcal{V}, \ A \in \mathcal{F}_n \), is the set of extreme points for the set \( M \). Moreover, since in the representation (127) for the regular set of measures \( M \) \( \alpha(v) \) runs all bounded random values, satisfying the conditions (128 - 130), then \( M \) is a complete set of measures.

Theorem 9. On the probability space \( \{ \Omega, \mathcal{F}, P \} \) with the filtration \( \mathcal{F}_n \) on it, let \( M \) be a complete set of measures. If every \( \sigma \)-algebra \( \mathcal{F}_n, \ n = 1, \infty \), has an exhaustive decomposition, then the closure of the set of points \( E^Q f_n(\omega), \ Q \in M_n \), in metrics \( \rho(x, y) = |x - y|, \ x, y \in R^1 \), contains the set of points

\[
\begin{align*}
    f_n(\omega_1) \left[ \frac{[m_n - m_{n-1}]^+(\omega_2)}{[m_n - m_{n-1}]^-(\omega_1) + [m_n - m_{n-1}]^+(\omega_2)} + \frac{[m_n - m_{n-1}]^-(\omega_1)}{[m_n - m_{n-1}]^-(\omega_1) + [m_n - m_{n-1}]^+(\omega_2)} \right] \\
    f_n(\omega_2) \left[ \frac{[m_n - m_{n-1}]^-(\omega_1)}{[m_n - m_{n-1}]^-(\omega_1) + [m_n - m_{n-1}]^+(\omega_2)} + \frac{[m_n - m_{n-1}]^+(\omega_2)}{[m_n - m_{n-1}]^-(\omega_1) + [m_n - m_{n-1}]^+(\omega_2)} \right]
\end{align*}
\]

(132)

for every integrable relative to every measure \( Q \in M_n \) the finite valued \( \mathcal{F}_n \)-measurable random value \( f_n(\omega) \), where \( \Omega^-_n = \{ \omega_1 \in \Omega, [m_n - m_{n-1}]^-(\omega_1) \leq 0 \}, \ \Omega^+_n = \{ \omega_2 \in \Omega, [m_n - m_{n-1}]^+(\omega_2) > 0 \} \).

Proof. Since

\[
E^Q f_n(\omega) = \int_{\Omega^-_n \times \Omega^+_n} f_n(\omega_1) \frac{\alpha_n(\omega_1, \omega_2)[m_n - m_{n-1}]^+(\omega_2)}{[m_n - m_{n-1}]^-(\omega_1) + [m_n - m_{n-1}]^+(\omega_2)} d\mu_n(\omega_1, \omega_2) + \\
\int_{\Omega^-_n \times \Omega^+_n} f_n(\omega_2) \frac{\alpha_n(\omega_1, \omega_2)[m_n - m_{n-1}]^-(\omega_1)}{[m_n - m_{n-1}]^-(\omega_1) + [m_n - m_{n-1}]^+(\omega_2)} d\mu_n(\omega_1, \omega_2).
\]

(133)

Then, all arguments, used in the proof of Lemma 8, can be applied for the proof of Theorem 9, since \( E^{P_n}[m_n - m_{n-1}] < \infty \). Theorem 9 is proved.
Theorem 10. On the probability space \( \{\Omega, \mathcal{F}, P\} \) with the filtration \( \mathcal{F}_n \) on it, let \( M \) be a complete set of measures and let every \( \sigma \)-algebra \( \mathcal{F}_n, n = 1, \infty \), have an exhaustive decomposition. Suppose that \( f_n(\omega) \) is a nonnegative integrable \( \mathcal{F}_n \)-measurable random value, satisfying the condition \( E^{Q_n} f_n(\omega) \leq 1, Q_n \in M_n \). Then, there exists a constant \( \alpha_n \), depending on \( f_n(\omega) \), such that

\[
 f_n(\omega) \leq 1 + \alpha_n [m_n - m_{n-1}](\omega), \quad \omega \in \Omega. \tag{134}
\]

Proof. Due to the completeness of the set of measures \( M \), let us denote a local regular martingale by \( \{m_n, \mathcal{F}_n\}_{n=0}^{\infty}, m_n = E^{Q_n} \{\xi_0|\mathcal{F}_n\}, Q_n \in M, \xi_0 \in A_0, \xi_0 \neq 1 \). From the completeness of the set of measures \( M \), we obtain the inequality

\[
 f_n(\omega_1) + \frac{[m_n - m_{n-1}]^-(\omega_1)}{[m_n - m_{n-1}]^-(\omega_1) + [m_n - m_{n-1}]^+(\omega_1)}
\]

\[
 f_n(\omega_2) - \frac{[m_n - m_{n-1}]^-(\omega_2)}{[m_n - m_{n-1}]^-(\omega_2) + [m_n - m_{n-1}]^+(\omega_2)} \leq 1, \tag{135}
\]

where \( \Omega_n^- = \{\omega_1 \in \Omega, [m_n - m_{n-1}](\omega_1) \leq 0\}, \Omega_n^+ = \{\omega_2 \in \Omega, [m_n - m_{n-1}](\omega_2) > 0\} \).

Let us denote \( \xi_n(\omega) = [m_n - m_{n-1}](\omega) \). Then, the formula (135) is written in the form

\[
 f_n(\omega_1) + \frac{\xi_n^+(\omega_2)}{\xi_n^- (\omega_1) + \xi_n^+(\omega_2)} + \frac{\xi_n^-(\omega_1)}{\xi_n^- (\omega_1) + \xi_n^+(\omega_2)} f_n(\omega_2) \leq 1, \quad \omega_1 \in \Omega_n^-, \quad \omega_2 \in \Omega_n^+. \tag{136}
\]

From the inequalities (136), we obtain the inequalities

\[
 f_n(\omega_2) \leq 1 + \frac{1 - f_n(\omega_1)}{\xi_n^- (\omega_1)} \xi_n^+(\omega_2), \tag{137}
\]

\[
 \xi_n^-(\omega_1) > 0, \quad \xi_n^+(\omega_2) > 0, \quad \omega_1 \in \Omega_n^-, \quad \omega_2 \in \Omega_n^+. \tag{138}
\]

Two cases are possible: a) for all \( \omega_1 \in \Omega_n^-, f_n(\omega_1) \leq 1 \); b) there exists \( \omega_1 \in \Omega_n^- \) such that \( f_n(\omega_1) > 1 \). First, let us consider the case a).

Since the inequalities (137) are valid for every value \( \frac{1 - f_n(\omega_1)}{\xi_n^- (\omega_1)} \), as \( \xi_n^-(\omega_1) > 0 \), and \( f_n(\omega_1) \leq 1, \omega_1 \in \Omega_n^- \), then, if to denote

\[
 \alpha_n = \inf_{\{\omega_1, \xi_n^- (\omega_1) > 0\}} \frac{1 - f_n(\omega_1)}{\xi_n^- (\omega_1)}, \tag{139}
\]

we have \( 0 \leq \alpha_n < \infty \) and

\[
 f_n(\omega_2) \leq 1 + \alpha_n \xi_n^+(\omega_2), \quad \xi_n^+(\omega_2) > 0, \quad \omega_2 \in \Omega_n^+. \tag{140}
\]
From the definition of \( \alpha_n \), we obtain the inequalities
\[
f_n(\omega) \leq 1 - \alpha_n \xi_n^-(\omega), \quad \xi_n^-(\omega) > 0, \quad \omega_1 \in \Omega^-.
\]  \quad (141)

Now, if \( \xi_n^-(\omega_1) = 0 \) for some \( \omega_1 \in \Omega^- \), then in this case \( f_n(\omega_1) \leq 1 \). All these inequalities give the inequalities
\[
f_n(\omega) \leq 1 + \alpha_n \xi_n(\omega), \quad \omega \in \Omega^- \cup \Omega^+.
\]  \quad (142)

Consider the case b). From the inequality (137), we obtain the inequalities
\[
f_n(\omega_2) \leq 1 - \frac{1 - f_n(\omega_1)}{-\xi_n^-}(\omega_2),
\]  \quad (143)
\[
\xi_n^-(\omega_1) > 0, \quad \xi_n^+(\omega_2) > 0, \quad \omega_1 \in \Omega^-, \quad \omega_2 \in \Omega^+.
\]  \quad (144)

The inequalities (143) give the inequalities
\[
\frac{1 - f_n(\omega_1)}{-\xi_n^-}(\omega_1) \leq \inf_{\{\omega_2, \xi_n^+(\omega_2)>0\}} \frac{1}{\xi_n^+(\omega_2)} < \infty, \quad \xi_n^-(\omega_1) > 0, \quad \omega_1 \in \Omega^-.
\]  \quad (145)

Let us define \( \alpha_n = \sup_{\{\omega_1, \xi_n^-(\omega_1)>0\}} \frac{1 - f_n(\omega_1)}{-\xi_n^-}(\omega_1) < \infty \). Then, from (144) we obtain the inequalities
\[
f_n(\omega_2) \leq 1 - \alpha_n \xi_n^+(\omega_2), \quad \xi_n^+(\omega_2) > 0, \quad \omega_2 \in \Omega^+.
\]  \quad (146)

From the definition of \( \alpha_n \), we have the inequalities
\[
f_n(\omega_1) \leq 1 + \alpha_n \xi_n^-(\omega_1), \quad \xi_n^-(\omega_1) > 0, \quad \omega_1 \in \Omega^-.
\]  \quad (147)

The inequalities (146), (147) give the inequalities
\[
f_n(\omega) \leq 1 - \alpha_n \xi_n(\omega), \quad \omega \in \Omega^- \cup \Omega^+.
\]  \quad (148)

Theorem 10 is proved, since the set \( \Omega^- \cup \Omega^+ \) has the probability one.

**Theorem 11.** On the probability space \( \{\Omega, F, P\} \) with the filtration \( F_n \) on it, let \( M \) be a complete set of measures and let every \( \sigma \)-algebra \( F_n \), \( n = 1, \infty \), have an exhaustive decomposition. Then, every nonnegative super-martingale \( \{f_n, F_n\}_{n=0}^{\infty} \) is a local regular one.

**Proof.** Without loss of generality, we assume that \( f_n \geq d_0 > 0 \). From the last fact, we obtain
\[
E^Q^n \frac{f_n}{f_{n-1}} \leq 1, \quad Q^n \in M_n, \quad n = 1, \infty.
\]  \quad (149)

The inequalities (149) and Theorems 11, 10 prove Theorem 11.

**Theorem 12.** On the probability space \( \{\Omega, F\} \) with the filtration \( F_n \) on it, let \( M \) be a complete set of measures and let every \( \sigma \)-algebra \( F_n \), \( n = 1, \infty \), have an exhaustive decomposition. Then, every bounded from below super-martingale \( \{f_n, F_n\}_{n=0}^{\infty} \) is a local regular one.

**Proof.** Since the super-martingale \( \{f_n, F_n\}_{n=0}^{\infty} \) is bounded from below, then there exists a real number \( C_0 \) such that \( f_n + C_0 > 0 \). If to consider the super-martingale \( \{f_n + C_0, F_n\}_{n=0}^{\infty} \), then all conditions of Theorem 11 are true. Theorem 12 is proved.
5 Description of local regular super-martingales relative to a regular set of measures.

In this section, we give the description of local regular super-martingales.

**Theorem 13.** On the measurable space \( \{\Omega, \mathcal{F}\} \) with the filtration \( \mathcal{F}_n \) on it, let \( M \) be a regular set of measures. If \( \{f_m, \mathcal{F}_m\}_{m=0}^{\infty} \) is an adapted random process, satisfying the conditions

\[
f_m \leq f_{m-1}, \quad E^P|f_m| < \infty, \quad P \in M \quad m = 1, \infty, \quad \xi \in A_0,
\]

then the random process

\[
\{f_mE^P\{\xi|\mathcal{F}_m\}, \mathcal{F}_m\}_{m=0}^{\infty}, \quad P \in M,
\]

is a local regular super-martingale relative to the regular set of measures \( M \).

**Proof.** Due to Theorem 3, the random process \( \{E^P\{\xi|\mathcal{F}_m\}, \mathcal{F}_m\}_{m=0}^{\infty} \) is a martingale relative to the regular set of measures \( M \). Therefore,

\[
f_{m-1}E^P\{\xi|\mathcal{F}_{m-1}\} - E^P\{f_mE^P\{\xi|\mathcal{F}_m\}|\mathcal{F}_{m-1}\} =
\]

\[
E^P\{(f_{m-1}-f_m)E^P\{\xi|\mathcal{F}_m\}|\mathcal{F}_{m-1}\}, \quad m = 1, \infty.
\]

So, if to put \( \bar{g}_m^0 = (f_{m-1} - f_m)E^P\{\xi|\mathcal{F}_m\}, \quad m = 1, \infty, \) then \( \bar{g}_m^0 \geq 0, \) it is \( \mathcal{F}_m \)-measurable and \( E^P\bar{g}_m^0 \leq E^P\xi(|f_{m-1}| + |f_m|) < \infty. \) Due to Theorem 1, we obtain the proof of Theorem 13. \( \square \)

**Corollary 1.** If \( f_m = \alpha, \quad m = 1, \infty, \quad \alpha \in R^1, \quad \xi \in A_0, \) then \( \{\alpha E^P\{\xi|\mathcal{F}_m\}, \mathcal{F}_m\}_{m=0}^{\infty} \) is a local regular martingale. Assume that \( \xi = 1, \) then \( \{f_m, \mathcal{F}_m\}_{m=0}^{\infty} \) is a local regular super-martingale relative to the regular set of measures \( M \).

Denote \( F_0 \) the set of adapted processes

\[
F_0 = \{f = \{f_m\}_{m=0}^{\infty}, \quad P(|f_m| < \infty) = 1, \quad P \in M, \quad f_m \leq f_{m-1}\}.
\]

For every \( \xi \in A_0, \) let us introduce the set of adapted processes

\[
L_\xi =
\]

\[
\{\bar{f} = \{f_mE^P\{\xi|\mathcal{F}_m\}\}_{m=0}^{\infty}, \quad \{f_m\}_{m=0}^{\infty} \in F_0, \quad E^P\xi|f_m| < \infty, \quad P \in M\},
\]

and

\[
V = \bigcup_{\xi \in A_0} L_\xi.
\]

**Corollary 2.** Every random process from the set \( K, \) where

\[
K = \left\{ \sum_{i=1}^{m} C_i\bar{f}_i, \quad \bar{f}_i \in V, \quad C_i \geq 0, \quad i = 1, m, \quad m = 1, \infty \right\},
\]

is a local regular super-martingale relative to the regular set of measures \( M \) on the measurable space \( \{\Omega, \mathcal{F}\} \) with the filtration \( \mathcal{F}_n \) on it.
Proof. The proof is evident. □

**Theorem 14.** On the measurable space \( \{ \Omega, \mathcal{F} \} \) with the filtration \( \mathcal{F}_n \) on it, let \( M \) be a regular set of measures. Suppose that \( \{ f_m, \mathcal{F}_m \}_{m=0}^{\infty} \) is a nonnegative uniformly integrable super-martingale relative to the set of measures \( M \), then the necessary and sufficient conditions for it to be a local regular one is belonging it to the set \( K \).

Proof. The necessity. It is evident that if \( \{ f_m, \mathcal{F}_m \}_{m=0}^{\infty} \) belongs to \( K \), then it is a local regular super-martingale.

The sufficiency. Suppose that \( \{ f_m, \mathcal{F}_m \}_{m=0}^{\infty} \) is a nonnegative uniformly integrable local regular super-martingale. Then, there exists a nonnegative adapted process \( \{ g_m \}_{m=1}^{\infty} \), \( E^P g_m^0 < \infty, m = 1, \infty \), and a martingale \( \{ M_m, \mathcal{F}_m \}_{m=0}^{\infty} \), such that

\[
f_m = M_m - \sum_{i=1}^{m} g_i^0, \quad m = 0, \infty.
\]

(157)

Then, \( M_m \geq 0, m = 0, \infty \), \( E^P M_m < \infty, P \in M \). Since \( 0 < E^P M_m = f_0 < \infty \), we have \( E^P \sum_{i=1}^{m} g_i^0 < f_0 \). Let us put \( g_\infty = \lim_{m \to \infty} \sum_{i=1}^{m} g_i^0 \). Using the uniform integrability of \( \{ f_m, \mathcal{F}_m \}_{m=0}^{\infty} \), we can pass to the limit in the equality

\[
E^P(f_m + \sum_{i=1}^{m} g_i^0) = f_0, \quad P \in M,
\]

(158)

as \( m \to \infty \). Passing to the limit in the last equality, as \( m \to \infty \), we obtain

\[
E^P(f_\infty + g_\infty) = f_0, \quad P \in M.
\]

(159)

Introduce into consideration a random value \( \xi = \frac{f_\infty + g_\infty}{f_0} \). Then, \( E^P \xi = 1, P \in M \). From here, we obtain that \( \xi \in A_0 \) and

\[
M_m = f_0 E^P \{ \xi | \mathcal{F}_m \}, \quad m = 0, \infty.
\]

(160)

Let us put \( \bar{f}_m = - \sum_{i=1}^{m} g_i^0 \). It is easy to see that the adapted random process \( \bar{f}_2 = \{ \bar{f}_m^2, \mathcal{F}_m \}_{m=0}^{\infty} \) belongs to \( F_0 \). Therefore, for the super-martingale \( f = \{ f_m, \mathcal{F}_m \}_{m=0}^{\infty} \) the representation

\[
f = \bar{f}_1 + \bar{f}_2,
\]

is valid, where \( \bar{f}_1 = \{ f_0 E^P \{ \xi | \mathcal{F}_m \}, \mathcal{F}_m \}_{m=0}^{\infty} \) belongs to \( L_\xi \) with \( \xi = \frac{f_\infty + g_\infty}{f_0} \) and \( f_m^1 = f_0, \quad m = 0, \infty \). The same is valid for \( \bar{f}_2 \) with \( \xi = 1 \). This implies that \( f \) belongs to the set \( K \). Theorem 15 is proved. □

**Theorem 15.** On the measurable space \( \{ \Omega, \mathcal{F} \} \) with the filtration \( \mathcal{F}_n \) on it, let \( M \) be a regular set of measures. Suppose that the super-martingale \( \{ f_m, \mathcal{F}_m \}_{m=0}^{\infty} \) relative to the set of measures \( M \) satisfy the conditions

\[
|f_m| \leq C \xi_0, \quad m = 0, \infty, \quad \xi_0 \in A_0, \quad 0 < C < \infty,
\]

(161)

then the necessary and sufficient conditions for it to be a local regular one is belonging it to the set \( K \).
Proof. The necessity is evident.
The sufficiency. Suppose that \( \{f_m, \mathcal{F}_m\}_{m=0}^{\infty} \) is a local regular super-martingale. Then, there exists a nonnegative adapted random process \( \{g^0_m\}_{m=1}^{\infty}, E^P \bar{g}^0_m < \infty, m = 1, \infty \), and a martingale \( \{M_m\}_{m=0}^{\infty}, E^P |M_m| < \infty, m = 1, \infty, P \in M \), such that
\[
f_m = M_m - \sum_{i=1}^{m} \bar{g}^0_i, \quad m = 0, \infty. \tag{162}
\]
The inequalities \( f_m + C_0 \xi_0 \geq 0, m = 1, \infty \), give the inequalities
\[
f_m + C E^P \{ \xi_0 | \mathcal{F}_m \} \geq 0, \quad m = 0, \infty. \tag{163}
\]
From the inequalities \( \{161\} \), it follows that the super-martingale \( \{f_m, \mathcal{F}_m\}_{m=0}^{\infty} \) is a uniformly integrable one relative to the regular set of measures \( M \). The martingale \( \{E^P \{ \xi_0 | \mathcal{F}_m \}, \mathcal{F}_m\}_{m=0}^{\infty} \) relative to the regular set of measures \( M \) is also uniformly integrable one.

Then, \( M_m + C E^P \{ \xi_0 | \mathcal{F}_m \} \geq 0, m = 0, \infty \). Since \( 0 < E^P [M_m + C E^P \{ \xi_0 | \mathcal{F}_m \}] = f_0 + C < \infty \), we have \( E^P \sum_{i=1}^{m} \bar{g}^0_i < f_0 + C \). Let us put \( g_\infty = \lim_{m \to \infty} \sum_{i=1}^{m} \bar{g}^0_i \). Using the uniform integrability of \( f_m \) and \( \sum_{i=1}^{m} \bar{g}^0_i \), we can pass to the limit in the equality
\[
E^P (f_m + \sum_{i=1}^{m} \bar{g}^0_i) = f_0, \quad P \in M, \tag{164}
\]
as \( m \to \infty \). Passing to the limit in the last equality, as \( m \to \infty \), we obtain
\[
E^P (f_\infty + g_\infty) = f_0, \quad P \in M. \tag{165}
\]
Introduce into consideration a random value \( \xi_1 = \frac{f_\infty + C \xi_0 + g_\infty}{f_0 + C} \geq 0 \). Then, \( E^P \xi_1 = 1, P \in M \). From here, we obtain that \( \xi_1 \in A_0 \) and for the super-martingale \( f = \{f_m, \mathcal{F}_m\}_{m=0}^{\infty} \) the representation
\[
f_m = f_m^0 E^P \{ \xi_0 | \mathcal{F}_m \} + f_m^1 E^P \{ \xi_1 | \mathcal{F}_m \} + f_m^2 E^P \{ \xi_2 | \mathcal{F}_m \}, \quad m = 0, \infty, \tag{166}
\]
is valid, where \( f_m^0 = -C, f_m^1 = f_0 + C, f_m^2 = - \sum_{i=1}^{m} \bar{g}^0_i, m = 0, \infty, \xi_2 = 1 \). From the last representation, it follows that the super-martingale \( f = \{f_m, \mathcal{F}_m\}_{m=0}^{\infty} \) belongs to the set \( K \). Theorem \( 15 \) is proved.

**Corollary 3.** Let \( f_N, N < \infty, \) be a \( \mathcal{F}_N \)-measurable integrable random value, \( \sup_{P \in M} E^P |f_N| < \infty \), and let there exist \( \alpha_0 \in R^1 \) such that
\[
-\alpha_0 M_N + f_N \leq 0, \quad \omega \in \Omega,
\]
where \( \{M_m, \mathcal{F}_m\}_{m=0}^{\infty} = \{E^P \{ \xi | \mathcal{F}_m \}, \mathcal{F}_m\}_{m=0}^{\infty}, \xi \in A_0 \). Then, a super-martingale \( \{f_m^0 + \bar{f}_m\}_{m=0}^{\infty} \) is a local regular one relative to the regular set of measures \( M \), where
\[
f_m^0 = \alpha_0 M_m, \tag{167}
\]
\[
\bar{f}_m = \begin{cases} 0, & m < N, \\ f_N - \alpha_0 M_N, & m \geq N. \end{cases} \tag{168}
\]
Proof. It is evident that $\bar{f}_{m-1} - \bar{f}_m \geq 0$, $m = \overline{0, \infty}$. Therefore, the super-martingale

$$f^0_m + \bar{f}_m = \begin{cases} \alpha_0 M_m, & m < N, \\ \bar{f}_N, & m = N, \\ \bar{f}_N - \alpha_0 M_N + \alpha_0 M_m, & m > N \end{cases}$$

(169)

is a local regular one relative to the regular set of measures $M$. Corollary 3 is proved. \qed

6 Optional decomposition for super-martingales relative to a complete set of measures.

In this section, we prove that the bounded super-martingales are local regular ones with respect to the complete set of measures.

6.1 Measurable space with a finite decomposition.

In this and the next subsections, we reformulate the results of the paper [1]. Let $\{\Omega, \mathcal{F}\}$ be a measurable space. We assume that the $\sigma$-algebra $\mathcal{F}$ is a certain finite algebra of subsets of the set $\Omega$. We give a new proof of the optional decomposition for super-martingales relative to the complete set of measures. This proof does not use topological arguments as in [17]. Let $\mathcal{F}_n \subset \mathcal{F}_{n+1} \subset \mathcal{F}$ be an increasing set of algebras, where $\mathcal{F}_0 = \{\emptyset, \Omega\}$, $\mathcal{F}_N = \mathcal{F}$. Denote $M$ the complete set of measures on the measurable space $\{\Omega, \mathcal{F}\}$. It is evident that every algebra $\mathcal{F}_n$ is generated by sets $A^n_i$, $i = 1, N$, such decompositions are exhaustive one. Let $m_n = E^P\{\xi_0|\mathcal{F}_n\}$, $P \in M$, $n = 1, N$, $\xi_0 \in A_0$. Then, for $m_n$ the representation

$$m_n = \sum_{i=1}^{N_n} m^n_i \chi_{A^n_i}(\omega), \quad n = 1, N,$$

(170)

is valid.

Lemma 10. Let $M$ be a complete set of measures on the measurable space $\{\Omega, \mathcal{F}\}$ with the filtration $\mathcal{F}_n$ on it. Then, for every non negative bounded $\mathcal{F}_n$-measurable random value $\xi_n = \sum_{i=1}^{N_n} C^n_i \chi_{A^n_i}$ there exists a real number $\alpha_n$ such that

$$f_n(\omega) = \frac{\sum_{i=1}^{N_n} C^n_i \chi_{A^n_i}}{\sup \sum_{P \in M_i} C^n_i P(A^n_i)} \leq 1 + \alpha_n(m_n - m_{n-1}), \quad n = 1, N.$$

(171)

Proof. The random value $f_n(\omega)$ satisfy all conditions of Theorems 9, 10. This proves Lemma 10. \qed

Theorem 16. Let $M$ be a complete set of measures on the measurable space $\{\Omega, \mathcal{F}\}$ with the filtration $\mathcal{F}_n$ on it. Then, every non negative super-martingale $\{f_m, \mathcal{F}_m\}_{m=0}^N$ relative to the set of measures $M$ is a local regular one.

Proof. Without loss of generality, we assume that $f_n > a > 0, n = 1, N$. Then, the random value $\frac{f_n}{f_{n-1}}$ satisfy conditions of Theorems 10, 11. Therefore, all conditions of Theorem 4 are satisfied. Theorem 16 is proved. \qed

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Theorem 17. Let $M$ be a complete set of measures on the measurable space $\{\Omega, \mathcal{F}\}$ with the filtration $\mathcal{F}_n$ on it. Then, every bounded super-martingale $\{f_m, \mathcal{F}_m\}_{m=0}^N$ relative to the set of measures $M$ is a local regular one.

Proof. From the boundedness of super-martingale $\{f_m, \mathcal{F}_m\}_{m=0}^N$, there exists a constant $C_0 > 0$ such that $\frac{3C_0}{2} > f_m + C_0$, $\omega \in \Omega$, $m = 0, N$. From this, it follows that the super-martingale $\{f_m + C_0, \mathcal{F}_m\}_{m=0}^N$ is a nonnegative one and satisfies the conditions

$$\frac{f_n + C_0}{f_{n-1} + C_0} \leq 3, \quad n = 1, N. \quad (172)$$

It implies that the conditions of Theorem 16 are satisfied. Theorem 17 is proved.

6.2 Measurable space with a countable decomposition.

In this subsection, we generalize the results of the previous subsection onto the measurable space $\{\Omega, \mathcal{F}\}$ with the countable decomposition.

Let $\mathcal{F}_n \subset \mathcal{F}_{n+1} \subset \mathcal{F}$ be a certain increasing set of $\sigma$-algebras, where $\mathcal{F}_0 = \{\emptyset, \Omega\}$. Suppose that the $\sigma$-algebra $\mathcal{F}_n$ is generated by the sets $A_i^n$, $i = 1, \infty$, $A_i^n \cap A_j^n = \emptyset$, $i \neq j$, $\bigcup_{i=1}^{\infty} A_i^n = \Omega$, $n = 1, \infty$. We assume that $\mathcal{F} = \sigma(\bigvee_{n=0}^{\infty} \mathcal{F}_n)$. Denote $M$ the complete set of measures on the measurable space $\{\Omega, \mathcal{F}\}$. Introduce into consideration the martingale $m_n = \mathbb{E}^P(\xi_0|\mathcal{F}_n)$, $P \in M$, $n = 1, \infty$, $\xi_0 \in A_0$. Then, for $m_n$ the representation

$$m_n = \sum_{i=1}^{\infty} m_i^n \chi_{A_i^n}(\omega), \quad n = 1, \infty, \quad (173)$$

is valid.

Lemma 11. Let $M$ be a complete set of measures on the measurable space $\{\Omega, \mathcal{F}\}$ with the filtration $\mathcal{F}_n$ on it. Then, for every non negative bounded $\mathcal{F}_n$-measurable random value $\xi_n = \sum_{i=1}^{\infty} C_i^n \chi_{A_i^n}$, there exists a real number $\alpha_n$ such that

$$f_n(\omega) = \frac{\sum_{i=1}^{\infty} C_i^n \chi_{A_i^n}}{\sup_{P \in M_n} \sum_{i=1}^{\infty} C_i^n P(A_i^n)} \leq 1 + \alpha_n(m_n - m_{n-1}), \quad n = 1, \infty. \quad (174)$$

Proof. Every $\sigma$-algebra $\mathcal{F}_n$, $n = 1, \infty$, has an exhaustive decomposition. The random value $f_n(\omega)$ satisfy all conditions of Theorems 9, 10. This proves Lemma 11.

Theorem 18. Let $M$ be a complete set of measures on the measurable space $\{\Omega, \mathcal{F}\}$ with the filtration $\mathcal{F}_n$ on it. Then, every non negative super-martingale $\{f_n, \mathcal{F}_n\}_{n=0}^\infty$ relative to the set of measures $M$ is a local regular one.

Proof. Without loss of generality, we assume that $f_n > a > 0, n = 1, N$. Then, the random value $\frac{f_n}{f_{n-1}}$ satisfy the conditions of Theorems 10, 11. Therefore, all conditions of Theorem 4 are satisfied. Theorem 18 is proved.
7 Local regularity of majorized super-martingales.

In this section, we give the elementary proof that a majorized super-martingale relative to a complete set of measures is a local regular one.

**Theorem 19.** On the measurable space \( \{ \Omega, \mathcal{F} \} \) with the filtration \( \mathcal{F}_n \) on it, let \( M \) be a complete set of measures. Then, every bounded super-martingale \( \{ f_n, \mathcal{F}_n \}^\infty_{n=0} \) relative to the set of measures \( M \) is a local regular one.

**Proof.** From Theorem [19] conditions, there exists a constant \( 0 < C < \infty \) such that \( |f_n| \leq C \), \( n = 1, \infty \). Consider the super-martingale \( \{ f_n + C, \mathcal{F}_n \}^\infty_{n=0} \). Then, \( 0 \leq f_n + C \leq 2C \). Due to Theorem [18] for the super-martingale \( \{ f_n + C, \mathcal{F}_n \}^\infty_{n=0} \) the local regularity is true. So, the same statement is valid for the super-martingale \( \{ f_n, \mathcal{F}_n \}^\infty_{n=0} \). Theorem [19] is proved.

The next Theorem is analogously proved as Theorem [19].

**Theorem 20.** On the measurable space \( \{ \Omega, \mathcal{F} \} \) with the filtration \( \mathcal{F}_n \) on it, let \( M \) be a complete set of measures. Then, a super-martingale \( \{ f_n, \mathcal{F}_n \}^\infty_{n=0} \) relative to the set of measures \( M \), satisfying the conditions

\[
|f_n| \leq C_1 \xi_0, \quad f_n + C_1 \xi_0 \leq C_2, \quad n = 1, \infty, \quad \xi_0 \in A_0,
\]

(175)

for certain constants \( 0 < C_1, C_2 < \infty \), is a local regular one.

8 Discrete geometric Brownian motion.

In this section, we construct for the discrete evolution of risk assets the set of equivalent martingale measures and give a new formula for the fair price of super-hedge. Let \( \Omega_1 = R^1 \), \( \mathcal{F}_0 = B(R^1) \), where \( R^1 \) is a real axis, \( B(R^1) \) is a Borel \( \sigma \)-algebra of \( R^1 \). Let us put \( \Omega_i = \Omega_1 \), \( \mathcal{F}_i \) is \( \mathcal{F}_0 \), \( i = 1, \infty \), and let us construct the infinite direct product of the measurable spaces \( \{ \Omega_i, \mathcal{F}_i \} \), \( i = 1, \infty \). Denote \( \Omega = \prod_{i=1}^{\infty} \Omega_i \). Under the \( \sigma \)-algebra \( \mathcal{F} \) on \( \Omega \), we understand the minimal \( \sigma \)-algebra generated by sets \( \prod_{i=1}^{\infty} G_i \), \( G_i \in \mathcal{F}_i \), where in the last product only the finite set of \( G_i \) do not equal \( \Omega_i \). On the measurable space \( \{ \Omega, \mathcal{F} \} \), under the filtration \( \mathcal{F}_n \) we understand the minimal \( \sigma \)-algebra, generated by sets \( \prod_{i=1}^{\infty} G_i \), \( G_i \in \mathcal{F}_i \), where \( G_i = \Omega_i \) for \( i > n \). Suppose that the points \( t_0 = 0, t_1, t_2, \ldots, t_n, \ldots \), belongs to \( R^1 \) with \( \Delta t = t_i - t_{i-1} \) not depending on the index \( i \). Let us consider the probability space \( \{ \Omega, \mathcal{F}, P \} \), where \( P = \prod_{i=1}^{\infty} P_i \), \( P_i = P_{1, \infty} \), \( i = 1, \infty \),

\[
P_{1,\infty}(A) = \frac{1}{(2\pi\Delta t)^{1/2}} \int_A e^{-\frac{x^2}{2\Delta t}} dy, \quad A \in \mathcal{F}_1.
\]

(176)

Define on the set \( t_0 = 0, t_1, t_2, \ldots, t_n, \ldots \), the discrete Brownian motion. We say that the random process \( w(t_i), i = 0, \infty \), is a discrete Brownian motion, if on \( \{ \Omega, \mathcal{F} \} \) the joint distribution function is given by the formula

\[
P_0(w(t_i) \in A_i, \ldots, w(t_{ik}) \in A_{ik}) =
\]

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Further, we consider the discount evolutions of the risk assets by the sequence of random values $\zeta_t$. It is convenient to present the Brownian motion in the form

$$E\{w(t)\} = \int e^{-\frac{y^2}{2\Delta t}} \times \ldots \times e^{-\frac{[y_{ik} - y_{ik-1}]^2}{2\Delta t}} dy_{i1} \ldots dy_{ik}, \quad A_{ik} \in F_{i_k},$$

where $D = \left[2\pi\right]^{k/2}[\Delta t_1 \times \ldots \times \Delta t_{k}]^{1/2}$, $\Delta t_i = t_i - t_{i-1}$.

So defined above the random process $w(t_i)$ on the set $t_0, t_1, t_2, \ldots, t_n, \ldots$, with $w(0) = 0$, is a homogeneous one relative to the displacement on $k\Delta t$, where $k \geq 1$, and is a natural number, with the independent increments, the zero expectation and the correlation function $E_{i_k}w(t) = \min\{t, t_k\}$.

We assume that the evolution of non risk asset is given by the formula $B_n = e^{rt_n}, \ n = 0, \infty$, where $r$ is an interest rate. Let us consider on $\{\Omega, F, P\}$ two cases of evolutions of risk assets given by the laws

$$\bar{S}_n = S_0e^{\sigma w(t_n)}, \quad (178)$$

$$S_n = S_0e^{(\mu - \frac{\sigma^2}{2})t_n + \sigma w(t_n)}. \quad (179)$$

Further, we consider the discount evolutions of the risk assets

$$S_n = \frac{\bar{S}_n}{B_n} = S_0e^{\sigma w(t_n) - rt_n}, \quad (180)$$

$$S_n = \frac{\bar{S}_n}{B_n} = S_0e^{(\mu - \frac{\sigma^2}{2} - r)t_n + \sigma w(t_n)}. \quad (181)$$

It is convenient to present these evolutions in the form

$$S_n = (1 + \rho_n)S_{n-1}, \quad n = 1, \infty, \quad (182)$$

with $\rho_n = e^{\sigma w(t_n) - w(t_{n-1}) - rt_n} - 1$, $\rho_n = e^{(\mu - \frac{\sigma^2}{2} - r)t_n + \sigma w(t_n) - w(t_{n-1})} - 1$, correspondingly.

On the probability space $\{\Omega, F, P\}$ with the filtration $F_n$ on it, for further investigations it is convenient to present the Brownian motion in equivalent form. We present the Brownian motion by the sequence of random values $\zeta_n = \sum_{i=1}^{n} y_i, y_i \in \Omega_{i_k}, \ n = 1, \infty$, with the joint distribution functions

$$P(\zeta_1 \in A_{i_1}, \ldots, \zeta_k \in A_{i_k}) =$$

$$\frac{1}{D} \int e^{-\frac{y_1^2}{2\Delta t}} \times \ldots \times e^{-\frac{[y_{ik} - y_{ik-1}]^2}{2\Delta t}} dy_{i1} \ldots dy_{ik}, \quad A_{ik} \in F_{i_k},$$

where $D = \left[2\pi\right]^{k/2}[\Delta t_1 \times \ldots \times \Delta t_{k}]^{1/2}$.

Then, the discount evolutions of the risk assets we can rewrite in the form

$$S_n = S_0e^{\sigma \zeta_n - nr\Delta t}, \quad (184)$$

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$$S_n = S_0 e^{(\mu - \frac{\sigma^2}{2} - r)\Delta t + \sigma \zeta_n}.$$  

(185)

It is convenient to present these discount evolutions in the form

$$S_n = (1 + \rho_n(y_n))S_{n-1}, \quad n = 1, \infty,$$

(186)

with \( \rho_n(y_n) = e^{\sigma y_n - r \Delta t} - 1 = \rho_1(y_n), \rho_n(y_n) = e^{(\mu - \frac{\sigma^2}{2} - r)\Delta t + \sigma y_n} - 1 = \rho_1(y_n) \), correspondingly.

On the measurable space \( \{\Omega^N, \mathcal{F}^N\} \) with the filtration \( \mathcal{F}_n, n = 0, N \), on it, where

\[
\Omega^N = \prod_{i=1}^{N} \Omega_i^0, \quad \mathcal{F}^N = \prod_{i=1}^{N} \mathcal{F}_i^0,
\]

let us introduce into consideration the set of measures \( M^N \).

A measure \( Q \) belongs to \( M^N \), if \( Q = \prod_{i=1}^{N} Q_i \), where \( Q_i \in M_i^0 \) and for every \( \bar{Q} \in M_1^0 \) the representation

\[
\bar{Q}(A) = \int_{\Omega_1^-\Omega_1^+} \int \chi_A(y_1)y_2)\frac{\rho_1^-(y_1)}{\rho_1^-(y_1) + \rho_1^+(y_2)}d\mu(y_1, y_2) + \\
\int_{\Omega_1^-\Omega_1^+} \int \chi_A(y_2)y_1)\frac{\rho_1^-(y_1)}{\rho_1^-(y_1) + \rho_1^+(y_2)}d\mu(y_1, y_2), \quad A \in \mathcal{F}_1^0,
\]

(187)

\( \Omega_1^- = \{y \in R^1, \rho_1(y) \leq 0\} = \{y \in R^1, y \leq \frac{r\Delta t}{\sigma}\} \),

\( \Omega_1^+ = \{y \in R^1, \rho_1(y) > 0\} = \{y \in R^1, y > \frac{r\Delta t}{\sigma}\} \),

is valid, where \( \rho_1(y) = \rho_1^+(y) - \rho_1^-(y), \rho_1(y) = e^{\sigma y - r \Delta t} - 1, \mu = P^- \times P^+ \),

\[
P^-(A) = \frac{1}{[2\pi \Delta t]^{1/2}} \int_A e^{-\frac{y^2}{2\Delta t}}dy, \quad A \in B(\Omega_1^-),
\]

\[
P^+(A) = \frac{1}{[2\pi \Delta t]^{1/2}} \int_A e^{-\frac{y^2}{2\Delta t}}dy, \quad A \in B(\Omega_1^+).
\]

On the measurable space \( \{\Omega_1^- \times \Omega_1^+, B(\Omega_1^-) \times B(\Omega_1^+)\} \), the random value \( \alpha(y_1, y_2) \) satisfy the conditions:

\[
\mu(\{(y_1, y_2) \in \Omega_1^- \times \Omega_1^+, \alpha(y_1, y_2) > 0\}) = P(\Omega_1^+)P(\Omega_1^-),
\]

(188)

\[
\int_{\Omega_1^-\Omega_1^+} \int \alpha(y_1, y_2)\frac{\rho_1^-(y_1)\rho_1^+(y_2)}{\rho_1^-(y_1) + \rho_1^+(y_2)}d\mu(y_1, y_2) < \infty,
\]

(189)

\[
\int_{\Omega_1^-\Omega_1^+} \int \alpha(y_1, y_2)d\mu(y_1, y_2) = 1.
\]

(190)
Every bounded random value \( \alpha(y_1, y_2) > 0, (y_1, y_2) \in R^- \times R^+ \), satisfy the conditions (188) - (190), if \( \sigma < \frac{1}{2 \Delta t} \), since \( E^P \mathbb{E} |\rho_1(y)| < \infty \). It means that the set of equivalent martingale measures \( M^N \) for the discount evolution \( S_n = S_0 e^{\frac{\sigma y_n}{\Delta t}} \) of the risk asset contains more then one martingale measure. In this case, the financial market is an incomplete one.

Denote \( M_0^N = M_c^N \) the convex linear span of the set of measures \( M^N \). On the measurable space \( \{ \Omega^N, \mathcal{F}^N \} \) with the filtration \( \mathcal{F}_n, n = 0, N \), on it, in correspondence with Theorem 8, the set of measures \( M_0^N \) is a regular set of measures with the random variable \( \xi_0 = \prod_{i=1}^{N} (1 + \rho_i(y_i)) \), since the random value \( \eta_1 = \rho_1(y_1) \), figuring in Theorem 8, is an integrable one relative to the measure \( P \) and, therefore, \( E^P \xi_0 = 1, Q \in M_0^N \).

**Theorem 21.** On the measurable space \( \{ \Omega^N, \mathcal{F}^N \} \) with the filtration \( \mathcal{F}_n, n = 0, N \), on it, let the discount risk asset evolution is given by the formula \( S_n = S_0 e^{\frac{\sigma y_n}{\Delta t}} \). For the payment function \( f(S_N) \), satisfying the condition \( \sup_{Q \in M_0^N} E^Q f(S_N) < \infty \), the fair price of super-hedge is given by the formula

\[
\sup_{Q \in M_0^N} E^Q f(S_N) = \sum_{y_i \leq d, y_i > d, i=1, N} \sum_{i=1, ..., N=1} f \left( S_0 \prod_{s=1}^{N} (1 + \rho(y_s)) \right) \times \frac{1}{\prod_{s=1}^{N} \left| e^{\sigma(d+y_s^{i+1})} - e^{\sigma(d+y_s^1)} \right|^2},
\]

where we put \( d = -\frac{r \Delta t}{\sigma}, y_s^0 = y_s^1 \).

**Proof.** The Borel \( \sigma \)-algebra \( B(R^1) \) is generated by the exhaustive decomposition, since it has the countable set of intervals with the rational number ends that generate \( B(R^1) \). Therefore, the filtration \( \mathcal{F}_n, n = 1, N \), has the exhaustive decomposition, due to Remark 11 after Theorem 8, the set of measures \( \mu_{(y_1, y_2^i)} \), where

\[
\mu_{(y_1, y_2^i)}(A) = \chi_A(y_1) \frac{\rho_1^+(y_1^2)}{\rho_1^-(y_1^2) + \rho_1^+(y_1^2)} + \chi_A(y_2^i) \frac{\rho_1^-(y_1^i)}{\rho_1^-(y_1^i) + \rho_1^+(y_1^i)}. \tag{192}
\]

forms the extreme points of the convex set of measures \( M_0^N \). The formula (191) is obtained by integration relative to the measure \( \mu_{(y_1, y_2^i)} \) of the random value \( f(S_N) \) and taking the sup on the set of all extreme points. This prove the Theorem 21.

Now, let us consider the case, as \( \rho_n(y_n) = e^{(\mu - \frac{\sigma^2}{2} - r) \Delta t + \sigma y_n} - 1 = \rho_1(y_n) \).
On the measurable space \( \{ \Omega^N, \mathcal{F}^N \} \) with the filtration \( \mathcal{F}_n, n = 1, N \), on it, where \( \Omega^N = \prod_{i=1}^N \Omega_i^0 \), \( \mathcal{F}^N = \prod_{i=1}^N \mathcal{F}_i^0 \), we introduce into consideration the set of measures \( M^N \). A measure \( Q \) belongs to \( M^N \), if \( Q = \prod_{i=1}^N Q_i \), \( Q_i \in M_i^0 \). For every \( Q \in M_i^0 \) the representation

\[
\bar{Q}(A) = \int \int \chi_A(y_1) \alpha(y_1, y_2) \frac{\rho_1^+(y_2)}{\rho_1^-(y_1) + \rho_1^+(y_2)} d\mu(y_1, y_2) + \\
\int \int \chi_A(y_2) \alpha(y_1, y_2) \frac{\rho_1^-(y_1)}{\rho_1^-(y_1) + \rho_1^+(y_2)} d\mu(y_1, y_2), \quad A \in \mathcal{F}_1^0,
\]

is valid, where \( \rho_1(y) = \rho_1^+(y) - \rho_1^-(y) \), \( \rho_1(y) = e^{(\mu - \frac{\sigma^2}{2} - r) \Delta t + \sigma y} - 1 \), \( \mu = \bar{P} \times \bar{P}^+ \),

\[
P^- = \lfloor 2\pi \Delta t \rfloor^{1/2} \int_A e^{-\frac{y^2}{2\Delta t}} dy, \quad A \in B(\Omega_1^-),
\]

\[
P^+ = \lfloor 2\pi \Delta t \rfloor^{1/2} \int_A e^{-\frac{y^2}{2\Delta t}} dy, \quad A \in B(\Omega_1^+).
\]

On the measurable space \( \{ \Omega_1^- \times \Omega_1^+, B(\Omega_1^-) \times B(\Omega_1^+) \} \), the random value \( \alpha(y_1, y_2) \) satisfy the conditions

\[
\mu(\{(y_1, y_2) \in \Omega_1^- \times \Omega_1^+, \alpha(y_1, y_2) > 0\}) = P(\Omega_1^+)P(\Omega_1^-),
\]

\[
\int \int \alpha(y_1, y_2) \frac{\rho_1^-(y_1) \rho_1^+(y_2)}{\rho_1^-(y_1) + \rho_1^+(y_2)} d\mu(y_1, y_2) < \infty,
\]

\[
\int \int \alpha(y_1, y_2) d\mu(y_1, y_2) = 1,
\]

for every bounded \( \alpha(y_1, y_2) > 0 \), if \( \sigma < \frac{1}{2\sqrt{\Delta t}} \), since \( E^{P_i}\rho_i(y) \| \rho_i(y) \| < \infty \). Denote \( M_0^N = M_c^N \) the convex linear span of the set of measures \( M^N \). On the measurable space \( \{ \Omega^N, \mathcal{F}^N \} \) with the filtration \( \mathcal{F}_n, n = 0, N \), on it, in correspondence with Theorem 8 the set of measures \( M_0^N \) is a regular set of measures with the random variable \( \xi_0 = \prod_{i=1}^N (1 + \rho_i(y_i)) \),

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since the random value $\eta_1 = \rho_1(y_1)$, figuring in Theorem 8, is an integrable one relative to the measure $P_1^0$ and, therefore, $E^Q \xi_0 = 1, Q \in M^N_0$. It means that the set of equivalent martingale measures $M^N_0$ for the discount evolution $S_n = S_0 e^{(\mu - \frac{\sigma^2}{2} - r)\Delta t + \sigma \xi_n}$ of the risk asset contains more then one martingale measure. In this case, the financial market is an incomplete one.

**Theorem 22.** On the measurable space $\{\Omega^N, \mathcal{F}^N\}$ with the filtration $\mathcal{F}_n, n = 0, N$, on it, let the discount risk asset evolution is given by the formula $S_n = S_0 e^{(\mu - \frac{\sigma^2}{2} - r)\Delta t + \sigma \xi_n}$ for $\sigma < \frac{\sigma^2}{2\Delta t}$. For the payment function $f(S_N)$, satisfying the condition $\sup_{Q \in M^N_0} E^Q f(S_N) < \infty$, the fair price of super-hedge is giving by the formula

$$\sup_{Q \in M^N_0} E^Q f(S_N) =$$

$$\sup_{y_1^1 \leq -d, y_2^1 > -d, i=1,\ldots,N} \sum_{i=1}^{2} f \left( S_0 \prod_{s=1}^{N} (1 + \rho(\xi_s^i)) \right) \times$$

$$\prod_{s=1}^{N} \frac{|e^{\sigma(d+y_s^3)} - 1|}{|e^{\sigma(d+y_s^1)} - e^{\sigma(d+y_s^2)}|},$$

(197)

where we put $d = \frac{(\mu - \frac{\sigma^2}{2} - r)\Delta t}{\sigma}, y_3^s = y_1^s$.

The proof of Theorem 22 is the same as the proof of Theorem 21.

### 9 Conclusions.

In the paper, we generalize the results of the paper [1]. Section 2 contains the definition of local regular super-martingales. Theorem 1 gives the necessary and sufficient conditions of the local regularity of a super-martingale. In spite of its simplicity, the Theorem 1 appeared very useful for the description of the local regular super-martingales.

Section 3 contains the important Definition 3 of the set of equivalent measures consistent with the filtration. In Lemma 3, we give an example of the set of equivalent measures consistent with the filtration. Theorem 2 contains the sufficient conditions under that there exists a nonnegative super-martingale on a measurable space with the set of measures consistent with the filtration. In Theorem 3, the sufficient conditions are founded which guarantee the existence on a measurable space a regular martingale.

Lemma 4 gives the sufficient conditions of the existence of a set of measures consistent with the filtration.

Lemma 5 contains the description of the set of measures being equivalent to a given measure and satisfying the condition: mathematical expectation of a given random value relative to every such a measure equals zero. In Lemma 6 we obtain the representation for the set of measures being equivalent to a given measure and satisfying the condition: the conditional expectation of a given random value relative to every of which equals zero. At last, Theorem 4 gives the necessary and sufficient conditions of the local regularity of a nonnegative super-martingale.

In Section 4, in Lemma 7, we investigate the closure of the set of considered set of measure in the case of the countable space of elementary events. It is proved that in metrics (85) the closure of the set of considered set of measures contains the set of measures (86).
Further, we introduce the notion of the exhaustive decomposition of a measurable space. Using this notion, in Lemma 5, we describe the closure of the considered set of measures relative to the pointwise convergence of measures and the closure of expectation values relative to this set of measures.

Theorem 5 is a consequence of Lemma 5 and contains the description of the set of measures, being equivalent to the given measure, expectations relative to which are equal one. Theorem 6 states the necessary and sufficient conditions when the set of measures \((107)\) is consistent with filtration. In Theorem 7 we give the necessary and sufficient conditions of the consistency with the filtration of the set of measures \((107)\).

Theorem 7 states the necessary and sufficient conditions of the consistency with the filtration of the set of measure \((107)\). Using Lemma 5 in Lemma 9, we construct an example of the set of measures consistent with the filtration. In Theorem 8 we describe completely the local regular set of measures.

In Definition 6 we introduce a fundamental notion of the completeness of the regular set of measures.

Using Lemma 7 and 8, Theorem 9 states that the expectations of the integrable random values relative to the contraction of the complete set of measures on the \(\sigma\)-algebras of filtration contains the points \((132)\).

Theorem 10 states that for every nonnegative \(F_n\) measurable random value, mathematical expectation for which relative to every martingale measure is bounded by 1, the inequality \((134)\) is true.

In Theorem 11 it is proved that every nonnegative super-martingale relative to the regular set of measures is a local regular one. The same statement, as in Theorem 11, it is proved in Theorem 12 in the case, as a super-martingale is bounded from below.

Section 5 contains the description of the local regular super-martingales. Using Theorem 1, we prove Theorem 13 giving the possibility to describe the local regular super-martingales. Further, we introduce a class \(K\) of the local regular super-martingales relative to a regular set of measures. Theorem 14 states that every nonnegative uniformly integrable super-martingale relative to a regular set of measures belong to the class \(K\). The next Theorem 15 states that all super-martingales that are majorized by elements from the set \(A_0\) is also belong to the class \(K\). At last, in corollary 3 we give an example of the local regular super-martingale playing important role in the definition of the fair price of the contingent claim \([1]\).

Section 6 contains an application of the results obtained above. To make this helps us Theorem 4 giving the necessary and sufficient conditions of the local regularity of the nonnegative super-martingales. In subsection 6.1, we consider the applications of the results obtained in the case as \(\sigma\)-algebra on the set of elementary events is generated by the finite set of events. In this case, Lemma 10 states that inequality \((171)\) is true. Theorem 16 states that every nonnegative super-martingale is local regular one. The same statement is true, when a super-martingale is only bounded, as it is shown in Theorem 17. In subsection 6.2, we consider the measurable space with the countable decomposition. In Lemma 11 we obtain the inequality \((174)\). Theorem 18 states that every nonnegative super-martingale is a local regular one.

Section 7 contains two statements. The first statement is that every bounded super-martingale is a local regular one. It is contained in Theorem 19. The second statement is contained in Theorem 20. It declares that a majorized super-martingale is also a local regular one.

Section 8 contains the application of the results obtained above to calculation of the fair price of super-hedge, when the risk asset evolves by the discrete geometric Brownian motion. In this case, we describe the set of regular measures. We find the set of extreme points of the regular set of measures. It is proved that the fair price of the super-hedge is given by the formula \((197)\).
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