We define an entropy for a quantum field theory by combining quantum fluctuations, scaling and the maximum entropy concept. This entropy has different behavior in asymptotically free and non-asymptotically free theories. We find that the transition between the two regimes (from the asymptotically free to the non-asymptotically free) takes place via a continuous phase transition. For asymptotically free theories there exist regimes where the “temperatures” are negative. In asymptotically free theories there exist maser–like states mostly in the infrared; furthermore, as one goes into the ultraviolet and more matter states contribute to quantum processes, the quantum field system can shed entropy and cause the formation of thermodynamically stable entropy–ordered states. It is shown how the known heavier quarks can be thus described.
Quantum fluctuations are an unavoidable feature of any quantum theory, and quantum field theory in particular. They modify physical quantities, which become scale dependent objects, in a way which reflects the quantum nature of the underlying virtual cloud. At a given scale, where quantum fluctuations are active, there are contributions from virtual processes with an arbitrary (but smaller than this scale) impact parameter. In particular, they change the interaction energy which now has an extra, “built–in”, indeterminacy, manifesting as a deviation from its classical form. These corrections are taken into account in quantum field theory via the renormalization process, and their scale dependence is quantitatively described by the renormalization group equations (RGEs) satisfied by the corresponding $n$–point functions \[1\].

Another general feature of fluctuating systems is that one can describe many interesting properties of their collective behavior by associating with the system some probability distribution. From this probability distribution one can calculate, e.g., the entropy associated with it and obtain information on structural properties of the system.

We will introduce such a density for a quantum field system in its static limit. For this, we take advantage of the connection between potential theory and probability theory \[2\] where, for a potential that satisfies a Poisson equation, one can interpret the associated density as a probability density.

The starting point for our purposes is the quantum corrected, static limit of the interaction energy, which satisfies a Poisson equation, and that we can use to write down the probability density which will allow us to study some order-disorder properties of the quantum system.
At short distances the leading term in the static interaction energy between elementary “charges” interacting in the quantum vacuum,

and for massless mediating quanta, can be written at one loop, as

\[
V(r) \cong \frac{C}{4\pi g_0^2} r^{-1+\sigma}
\]  

(1)

where, \(\sigma\) is related to the beta function for the coupling \(g_0\) by \(\sigma = +2\beta_0 g_0^2\), and \(\mu dg_0^2/d\mu = -\beta_0 g_0^4\), with \(\mu\) the momentum scale.

Because of the Poisson equation satisfied by \(V(r)\), away from \(r = 0\), there exists a density \(\rho(r) \cong r^{-3+\sigma}\) which we interpret as a probability density (in the classical sense of probability) for the distribution of the virtual cloud that surrounds the elementary charges in the quantum-mechanical vacuum. Normalization of this charge density is possible if one introduces an IR-cut-off \(R_0\) in the case of asymptotically free (AF) theories and an UV-cut off \(r_0\) for non-asymptotically free (NAF) theories. One gets

\[
\rho(r) = A r^{-3+\sigma}
\]  

(2)

where \(A = \frac{\sigma}{4\pi} R_0^{-\sigma}\) when \(\sigma > 0\) and

\(A = \frac{-\sigma}{4\pi} r_0^{+\sigma}\) if the theory is NAF.

The fine grained (Gibbs’) entropy for \(\rho\) can be computed directly from the definition \(S = -\sum_i p_i \ln p_i\), where \(p_i\) is a probability density (\(\rho \equiv p_i\)). Alternatively, one may subject the system to the constraint \(\langle f \rangle = \sum_i p_i f(x_i)\), and perform a variational calculation where one is (formally) led to

\[1\] It is interesting to mention that this probability density is of the type associated with Lotka’s or Zipf’s laws, well known as typical examples of Pareto–Levy distributions.
\begin{equation}
p_i = \frac{e^{-\beta f(x_i)}}{Z(\beta)}, \quad Z(\beta) = \sum_i \exp [-\beta f(x_i)],
\end{equation}

and the entropy can be recast as

\begin{equation}
S = \log Z(\beta) + \beta \sum_i f(x_i) \frac{\exp [-\beta f(x_i)]}{Z(\beta)}
\equiv \log Z(\beta) + \beta U
\end{equation}

From this equation, we see that $U$ plays the rôle of an internal

energy, whereas $1/\beta$ can be identified with the “temperature” $T$ of the virtual cloud.\footnote{From now on, we will drop the quotes around “temperature”. The reader is kindly asked to mentally replace them each time she/he sees the word temperature in this paper. We will adopt “natural” units and take the equivalent of Boltzmann’s constant equal to 1. This $\beta$ should not be confused with $\beta_0$, the RG–quantity introduced above.}

Using these formulae on the density of Eq. (2), we perform at once the following identifications:

\begin{equation}
\beta = 3 - \sigma; \quad f(r) = \log r/R_0; \quad Z(\beta) = \frac{4\pi}{\sigma \Theta} R_0^3
\end{equation}

for AF–theories, and for NAF–theories

\begin{equation}
\beta = 3 + |\sigma|; \quad f(r) = \log r/r_0; \quad Z(\beta) = \frac{4\pi}{|\sigma| \Theta} r_0^3
\end{equation}

Here $\Theta$ plays a rôle similar to the “entropy constant” which is fixed in quantum statistical mechanics by using Nernst theorem. In principle, $\Theta$ is an arbitrary quantity with dimension of length to the cube.

We see at once

that $\sigma$ is related to the inverse temperature introduced in the variational calculation. The temperature ranges are as follows: $0 \leq T \leq 1/3$ for NAF–theories; in AF–theories, $T$
must be $\geq 1/3$ or $0 \geq T$, so that, in principle, negative temperature regimes are allowed in these theories. In Figure 1, we show the entropy as a function of $T$ for both AF– and NAF–theories.

The energy $U$ is given (for both NAF– and AF–theories) by

$$U = \frac{T}{1 - 3T} \quad (7)$$

With these identifications, the entropy obtained from Eq.(4) coincides with the fine–grained entropy calculated from the probability density of Eq.(2) [3].

The specific heat at constant volume $C_V$ (again for both types of theories) is given by

$$C_V = \frac{\partial U}{\partial T} \bigg|_V = \frac{1}{(1 - 3T)^2} \quad (8)$$

The specific heat blows–up (cf. Figure 2) when $T = 1/3$. A phase transition takes place in the non–abelian and static quantum system when $\sigma = 0$, that is when the system goes from the non–abelian to the abelian phase. As is seen from (8), all derivatives of $U$ diverge at $T = 1/3$, and the transition is a continuous phase transition. For an AF–theory the transition occurs as the size of the probed system decreases and, due to the decoupling theorem [3], more matter degrees of freedom start contributing to the beta–function, which has the possibility of changing sign and become negative.

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3This seems to also indicate the following: at the classical limit ($\hbar \to 0$) $\sigma$, which is proportional to $\hbar$, goes to zero; thus a phase transition of this kind can take place when a quantum field theory (both AF and NAF) goes into its classical regime.

4This change of phases in a quantum field theory is unrelated to the magnetostatic classification of the vacuum based on renormalization group behavior given by Pagels and Tomboulis in Reference [6]. The phase transition described here requires a change in the sign of $\sigma$ or that $\sigma$ be zero; the former is possible only for AF–theories, whereas the latter is also true for NAF–theories in their transition to the classical regime, as already pointed out.
We now analyze some features specific to AF–systems. In order to do this we introduce the variable $x \equiv -1/T$, which has the property that colder temperatures are mapped to $-\infty$, hotter to $+\infty$ and is useful in the description of negative temperature systems \cite{7}. For AF–regimes $x$ ranges between $x = -3^+$ and $x = +\infty$ (cf. Figure 3). In Figure 4, we plot $S$, $U$ and $C_V$ as functions of $x$; we see that $C_V$ has a minimum and vanishes at $x = 0$ ($\sigma = +3$) while the entropy has a maximum there.

The entropy of the system at first increases as we go from $x = -3^+$ to $x = 0^-$ (which corresponds to a positive increase in $\sigma$) to then decrease from $x = 0^+$ to $+\infty$. Since $\sigma$ becomes more and more positive as we go up in the size of the system we probe (less matter degrees of freedom contributing to $\beta_0$), we see that the entropy at first increases from $-\infty$ at $x = -3$ to its maximum at $x = 0$, to then decrease with increasing system size: thus when $\sigma = 3$ there is some form of “condensation” (or population inversion) that sets in.

The specific heat decreases very abruptly after the phase transition at $T = 1/3$, (accompanied by an equally abrupt increase in the entropy) and becomes zero at $\sigma = 3$. For $\sigma > 3$, the system is in a totally new state, where the specific heat increases with $\sigma$ and tends asymptotically to +1, as the size of the probed system is increased; that is, as we try to “raise the temperature” the system absorbs “energy” (up to a maximum of $U(\sigma = \infty) - U(\sigma = 3) = +1/3$) coming to a state where a large increase in negative temperature (for $\sigma >> 3$) increases the internal energy very little and yet the system absorbs it (the energy goes asymptotically to zero as $C_V \to +1$). Since this is accompanied by a decrease in entropy, one is tempted to interpret this situation as an indication that quantum field theory favors the existence of organized and complex states when $\sigma$ becomes $> 3$. This observation is prompted by the notion that a system sheds entropy to increase its level of organization \cite{8} or complexity \cite{9}. Unfortunately, for quantum chromodynamics

\footnote{As may be seen from Equations (1) and (2), this happens when $\sigma \equiv$ dimensionality of the physical space.}
(QCD), this range is well inside the non-perturbative regime where one has to question the validity of the perturbative calculations that led to Eq.(2), but it is present in the perturbative regime of other gauge theories.

Similar considerations apply when $\sigma$ is less than 3 and as it approaches zero, during the infrared–to–ultraviolet transition. In principle, within this range, the entropy decreases with $\sigma$, and the quantum system has the potential for sustaining the formation of organized states at a lower $\sigma$ by shedding entropy as the size of the probed region is decreased when going from the infrared to the ultraviolet. At each value of $\sigma$, there is a unique value of $\Theta$ that makes the entropy equal to zero, and leads to thermodynamical equilibrium

$$\left(\frac{\partial A}{\partial T}\right)_V = -S(\sigma, \Theta, R_0) = 0,$$

where $A$ is the free energy. Writing

$$\Theta = \bar{r}^3 \exp a,$$

where $a$ is an unspecified constant and $\bar{r}$ the maximum system size that can be (adiabatically) supported for this value of $\sigma$, the above condition can only be satisfied for a pair of values $(\sigma, R_0/\bar{r})$ once $a$ has been fixed at, say, $\bar{r} = R_0$. It is very illustrative to look at this situation from the point of view of order–disorder. For $\sigma < 3$, when we probe the system at shorter and shorter distances, the system “cools”, and the entropy decreases (cf. Figs. 1 and 4); in other words, going into the UV is accompanied by a drop in the temperature, which is

a “disorder–decreasing” process, reflected in QCD by a decrease

in the entropy. But in quantum field theory, as we decrease the size

of the system being probed, more momentum transfer is available, new

particle thresholds are crossed and more degrees of freedom become

active in the quantum system; the presence of additional degrees of freedom at shorter distances raises the number of states accessible to the system, a “disorder–increasing” process which makes the entropy grow. The simplest possibility for the net entropy is to assume that these

two competing effects compensate each other, and that the process of

probing the shorter distances is isentropic.

The isentropic nature of this process implies that the entropy
\[ S^{(\sigma > 0)} = 1 - \frac{3}{\sigma} - \ln \frac{\sigma}{4\pi} + 3 \ln \frac{R_0}{r} - a \]

stays constant, and the excess entropy goes into organizing and supporting new states. In QCD, where \( \beta_0 \) decreases each time the threshold for a new flavor is crossed, this insinuates the existence of a relationship between the different quark states, in particular, their masses must be related according to Eq. (9) with \( \Delta S = 0 \) between flavors. This relationship is shown in Figure 5. We compare the known experimental heavier quark masses (i.e., all except up and down) and strong interaction coupling constant, with the result of assuming \( \Delta S = 0 \) between quark states. The experimental data can be described by adjusting one single free parameter, the zero of the entropy: the resulting correlation between theory and experiment is better than .99. In this figure we have fitted \( \Delta S = 0 \), by adjusting \( a \) of Eq. (9), to the known experimental data \([10]\): the product is that the theoretical curve is never more than 8% away from the central experimental values.

The nature of the organized states for \( \sigma > 3 \) and \( \sigma < 3 \) is quite different: while the states with \( \sigma > 3 \) are thermodynamically unstable, the others are thermodynamically stable states. This follows from Equation (8) and the second derivative of the free energy with respect to the temperature,

\[ \left. \frac{\partial^2 A}{\partial T^2} \right|_V = - \left. \frac{\partial S}{\partial T} \right|_V = - \frac{C_V}{T} \]

In thermodynamically stable states this quantity must be negative \([11]\). We see that the states in the region of negative temperatures are un–stable because they correspond to a free energy which is a convex function of the temperature; the converse is true for the “entropic states” in the region of positive temperatures.

In summary, quantum field theories, through quantum fluctuations, as described by the renormalization group, and in the leading approximation, have charge densities which can be interpreted as

probability densities of the class associated with Pareto–Levy \([4]\) distributions. This
leads to a generalized notion of entropy and “temperature” in quantum field theory. For AF–theories there exist regimes of negative temperatures. At the transition from the AF–regime to the NAF–regime, there is a continuous phase transition. In AF–theories and when $\sigma \geq 3$ there is the possibility that there exist organized, highly complex, albeit thermodynamically unstable, maser–like states. However, in the regime where $\sigma \leq 3$, AF–theories support the existence of thermodynamically–stable–states whose order is provided by entropic effects; these states become patent as the quantum system goes from the infrared into the ultraviolet and the system entropy falls. Finally, the ideas discussed here and the results found are very general, and they can find application for example in grand unification physics, the quantum theory of gravity, multiparticle physics, astrophysics, the very early universe and those realms of physics where quantum field theory and/or the renormalization group are applicable, including fractal dynamics, chaos and complexity theory.

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FIGURES

FIG. 1. The entropy of asymptotically free and non-asymptotically free theories. The zeroes of the entropies have been (arbitrarily) chosen so as to display the curves in a position where their features can be easily appreciated.

FIG. 2. $C_V$ and $\sigma$ plotted as a function of the “temperature”. The quantity $\sigma$ increases with the size of the region of space being probed.

FIG. 3. The “temperature” $T$, and the variable $x$ plotted against the renormalization group quantity $\sigma$. The quantity $\sigma$ increases with the size of the region of space being probed.

FIG. 4. Entropy, $C_V$ and internal energy as functions of the variable $x = -1/T$. The variable $x$ increases with the size of the region of space being probed. The zero of the entropy has been (arbitrarily) chosen so as to display the curves in a position where their different features can be easily appreciated.

FIG. 5. The condition $\Delta S = 0$ for quarks in QCD. The experimental data have been fitted setting $S^{(\sigma>0)}(\sigma)$ of Equation (9) equal to zero, with (cf. Eq. (9)) $a = -0.721151$. The experimental values are given in Table 1. The Pearson’s correlation coefficient between experimental data and theory is greater than 0.99. In the inset we plot the residues for this fit. The reduced $\chi^2$ is 0.1120. All known quark masses are fitted with an error of less than 8%.
TABLE I. Experimental values used in fit for the quark masses.\textsuperscript{a}

| Flavor/State | $<3\ln(m_f/2m_{\text{strange}})>$ | $<\sigma_{\text{exp}}>\cdot\alpha_{\text{strong}}(M_Z) = 0.116 \pm 0.005$. The scale for $2m_{\text{strange}}$ was taken at 0.6 GeV; the data on quark masses is from Ref. \cite{10}. In addition to the quark masses, we have also included the masses of the corresponding $q\bar{q}$–states. | Residue     |
|--------------|----------------------------------|----------------|
| $b\bar{b}$   | 8.440232                         | 0.206233       | -0.27553 |
| $b$          | 6.360791                         | 0.240759       | -0.42370 |
| $c\bar{c}$   | 4.828314                         | 0.302973       | 0.37270  |
| $c$          | 2.748872                         | 0.384270       | 0.15042  |
| $s\bar{s}$   | 0.0                              | 0.670211       | 0.17611  |
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