REGULARITY OF EXTREMAL SOLUTIONS OF SEMILINEAR ELLIPTIC PROBLEMS WITH NON-CONVEX NONLINEARITIES ON GENERAL DOMAINS

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Abstract. We consider the semilinear elliptic equation $-\Delta u = \lambda f(u)$ in a smooth bounded domain $\Omega$ of $\mathbb{R}^n$ with Dirichlet boundary condition, where $f$ is a $C^1$ positive and nondecreasing function in $[0, \infty)$ such that $\lim_{t \to \infty} f(t) t = \infty$. When $\Omega$ is an arbitrary domain and $f$ is not necessarily convex, the boundedness of the extremal solution $u^*$ is known only for $n = 2$, established by X. Cabrè [5]. In this paper, we prove this for higher dimensions depending on the nonlinearity $f$. In particular, we prove that if

$$\frac{1}{2} < \beta_- := \liminf_{t \to \infty} \frac{f'(t)F(t)}{f(t)^2} \leq \beta_+ := \limsup_{t \to \infty} \frac{f'(t)F(t)}{f(t)^2} < \infty,$$

where $F(t) = \int_0^t f(s) \, ds$, then $u^* \in L^\infty(\Omega)$, for $n \leq 6$. Also, if $\beta_- = \beta_+ > \frac{1}{2}$ or $\frac{1}{2} < \beta_- \leq \beta_+ < \frac{3}{4}$, then $u^* \in L^\infty(\Omega)$, for $n \leq 9$. Moreover, under the sole condition that $\beta_- > \frac{1}{2}$ we have $u^* \in H^{1,0}_0(\Omega)$ for $n \geq 1$. The same is true if for some $\epsilon > 0$ we have

$$\frac{f''(t)}{f(t)} \geq 1 + \frac{1}{(\ln t)^2 - \epsilon} \quad \text{for large } t,$$

which improves a similar result by Brezis and Vázquez [1].

1. Introduction. In this article, we consider the semilinear Dirichlet problem

$$\begin{cases}
-\Delta u = \lambda f(u) & x \in \Omega, \\
u > 0 & x \in \Omega, \\
u = 0 & x \in \partial \Omega,
\end{cases}$$

where $\Omega \subseteq \mathbb{R}^n$ is a smooth bounded domain, $n \geq 1$, $\lambda > 0$ is a real parameter, and the nonlinearity $f : [0, \infty] \to \mathbb{R}$ satisfies

$(H)$ $f$ is $C^1$, nondecreasing, $f(0) > 0$ and $\lim_{s \to \infty} f(s) s = \infty$.

By a weak solution of (1) we mean a nonnegative function $u \in L^1(\Omega)$ so that $f(u) \in L^1(\Omega) = L^1(\Omega, \delta(x) dx)$, $\delta(x) = \text{dist}(x, \partial \Omega)$ and

$$\int_\Omega (-\Delta \varphi) u = \int_\Omega \lambda f(u) \varphi,$$

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holds for any \( \varphi \in C^2(\Omega) \), \( \varphi = 0 \) on \( \partial \Omega \) (see Brezis et al. [3]).

It is well known (3 [11, 13]) that there exists a finite positive extremal parameter \( \lambda^* \) such that for any \( 0 < \lambda < \lambda^* \), problem (1) has a minimal classical solution \( u_\lambda \in C^2(\Omega) \), while no solution exists even in the weak sense for \( \lambda \geq \lambda^* \). The function \( \lambda \to u_\lambda \) is increasing and the increasing pointwise limit \( u^*(x) = \lim_{\lambda \to \lambda^*} u_\lambda(x) \) is a weak solution of (1) for \( \lambda = \lambda^* \) which is called the extremal solution. If \( \lambda < \lambda^* \) the solution \( u_\lambda \) is obtained by the implicit function theorem and it is semi-stable in the sense that the first Dirichlet eigenvalue of the linearized problem at \( u_\lambda \), 

\[ -\Delta - \lambda f'(u_\lambda), \] 

is nonnegative for all \( \lambda \in (0, \lambda_*) \). That is,

\[ \int_{\Omega} |\nabla \varphi|^2 - \lambda f(u_\lambda)' \varphi^2 \geq 0, \quad \varphi \in H_0^1(\Omega). \]  

(2)

The regularity and properties of the extremal solution of problem (1) have been studied extensively in the literature, in particular after Brezis and Vázquez raised some open problems in [4], see [1-18, 21, 22], and it is shown that it depends strongly on the dimension \( n \), domain \( \Omega \) and nonlinearity \( f \). When \( f(u) = e^u \) and \( \Omega \) is an arbitrary smooth bounded domain, it is well known that \( u^* \in L^\infty(\Omega) \) if \( n \leq 9 \) (see [11, 13]), while \( u^*(x) = -2\log |x| \) and \( \lambda^* = 2n - 4 \) if \( n \geq 10 \) and \( \Omega \) is the unit ball of \( \mathbb{R}^n \) (see [14]). A similar phenomenon happens when \( f(u) = (1 + u)^p \) with \( p > 1 \) (see [4]). For the general convex nonlinearity \( f \), Nedev in [16] proved that \( u^* \in L^\infty(\Omega) \) for \( n = 2, 3 \) in any domain \( \Omega \). When \( 2 \leq n \leq 4 \) the best known result was established by Cabré [5] who showed that \( u^* \in L^\infty(\Omega) \) for arbitrary nonlinearity \( f \) satisfying (H) if in addition \( \Omega \) is convex. In [21], Villegas proved the same result replacing the condition that \( \Omega \) is convex with \( f \) is convex. Cabré and Capella [8] proved that \( u^* \in L^\infty(\Omega) \) if \( n \leq 9 \) and \( \Omega = B_1 \). Also, in [8], Cabré and Ros-Oton showed that \( u^* \in L^\infty(\Omega) \) if \( n \leq 7 \) and \( \Omega \) is a convex domain of double revolution (see [8] for the definition).

By imposing extra assumptions on the convex nonlinearity \( f \) satisfying (H) much more is known, see [10]. Let \( f \) be convex and define

\[ \tau_- := \lim \inf_{t \to \infty} \frac{f(t)f''(t)}{f'(t)^2} \leq \tau_+ := \lim \sup_{t \to \infty} \frac{f(t)f''(t)}{f'(t)^2}. \]

Crandall and Rabinowitz [11] proved \( u^* \in L^\infty(\Omega) \) when \( 0 < \tau_- \leq \tau_+ < 2 + \tau_- + \sqrt{\tau_-} \) and \( n < 4 + 2\tau_- + 4\sqrt{\tau_-} \). This result was improved by Ye and Zhou in [22] and Sanchón in [18] establishing that \( u^* \in L^\infty(\Omega) \) when \( \tau_- > 0 \) and \( n < 6 + 4\sqrt{\tau_-} \). In [18] Sanchón proved that \( u^* \in L^\infty \) whenever \( \tau_- = \tau_+ \geq 0 \) and \( n \leq 9 \). Recently Cabré, Sanchón and Spruck [10] proved that if \( \tau_+ < 1 \) (without assuming \( \tau_- > 0 \)) and \( n < 2 + \frac{4}{\tau_+} \) then \( u \in L^\infty \), and if \( \tau_+ = 1 \) and \( n < 6 \) then \( u^* \in L^\infty \). These results are improved by the author in [1] as follows

\[ \text{if } 0 < \tau_+ < \infty \text{ and } n < \max \{ 2 + \frac{4}{\tau_+} + \frac{4}{\sqrt{\tau_+}}, 4 + \frac{2}{\tau_+} + \frac{4}{\sqrt{\tau_+}} \} \text{ then } u^* \in L^\infty(\Omega). \]

In particular, if \( \tau_+ < \frac{2}{9 - 2\sqrt{14}} \cong 1.318 \) and \( n < 10 \) then \( u^* \in L^\infty(\Omega) \).

It is still an open problem to prove that the extremal solution of the problem (1), in a bounded smooth domain with \( f \) satisfying (H) and convex, is always bounded if the dimension \( n \leq 9 \).

The case when \( f \) is not convex and \( \Omega \) is an arbitrary domain, is more challenging and there is nothing much in the literature on the boundedness of the extremal solution. In this case, the best known result is due to Cabré [5] who showed that \( u^* \in L^\infty(\Omega) \) for arbitrary \( f \) and \( \Omega \) in dimension \( n = 2 \). To the best of our knowledge, all
other known results in higher dimensions assume extra conditions on the geometry of the domain $\Omega$.

In this work we consider problem (1) for the case when $\Omega$ is an arbitrary smooth bounded domain and prove the boundedness of extremal solutions in higher dimensions under some extra assumptions on $f$ (but not the convexity property).

Let $f$ satisfy (H) and define

$$
\beta_- := \liminf_{t \to \infty} \frac{f'(t)F(t)}{f(t)^2} \leq \beta_+ := \limsup_{t \to \infty} \frac{f'(t)F(t)}{f(t)^2},
$$

(3)

where $F(t) := \int_0^t f(s)ds$, for $t \geq 0$.

The main results of this paper are as follows.

**Theorem 1.1.** Let $f$ (not necessarily convex) satisfy (H) with $\frac{1}{2} < \beta_- \leq \beta_+ < \infty$ and $\Omega$ be an arbitrary bounded smooth domain. Let $u^*$ be the extremal solution of problem (1). Then $u^* \in L^\infty(\Omega)$ for

$$
n < 4 + 4 \left( \frac{2\beta_+ - 1}{2\beta_+} + \sqrt{\frac{2\beta_- - 1}{\beta_+}} \right).
$$

(4)

Furthermore, if $\beta_- < 1$ then $u^* \in L^\infty(\Omega)$ for

$$
n < 6 + \frac{4}{2\beta_+ - 1} \left( 1 - \beta_+ + \sqrt{\beta_+ (2\beta_- - 1)} \right).
$$

(5)

As consequences, by the assumption $\frac{1}{2} < \beta_- \leq \beta_+ < \infty$, we have:

(a) If $n \leq 6$, then $u^* \in L^\infty(\Omega)$.
(b) If $\beta_- = \beta_+ \lor \beta_+ < \frac{7}{10}$, then $u^* \in L^\infty(\Omega)$ for $n \leq 9$.

It is worth mentioning here that, for a convex nonlinearity $f$ we always have $\beta_+ \geq \beta_- \geq \frac{1}{2}$. Indeed in this case $f'$ is a nondecreasing function, hence we have

$$
f'(t)F(t) = f'(t) \int_0^t f(s)ds \geq \int_0^t f'(s)f(s)ds = \frac{f(t)^2}{2} - \frac{f(0)^2}{2},
$$

now the fact that $f(t) \to \infty$ as $t \to \infty$ gives $\beta_- \geq \frac{1}{2}$.

Also for general nonlinearities $f$ (not necessarily convex) that satisfy only (H) we always have $\beta_+ \geq \frac{1}{2}$. To see this, by contradiction assume that $0 \leq \beta_+ < \frac{1}{2}$ and take a $\beta \in (\beta_+, \frac{1}{2})$. Then from the definition of $\beta_+$ there exists $T > 0$ such that $f'(t)F(t) \leq \beta f(t)^2$ for $t \geq T$, or equivalently, $\frac{d}{dt}(\frac{f(t)}{F(t)^\beta}) \leq 0$ for $t \geq T$. Thus,

$$
\frac{f(t)}{F(t)^\beta} \leq C := \frac{f(T)}{F(T)^\beta},
$$

for $t \geq T$, and by integration we get $F(t) \leq (C_1 t + C_2)^{\frac{1}{\beta}}$ for all $t > T$ and some constants $C_1, C_2$. But, from the superlinearity of $f$ we have $\lim_{t \to \infty} \frac{F(t)}{f(t)^\beta} = \infty$, hence we must have $\frac{1}{1-\beta} > 2$ or equivalently $\beta > \frac{1}{2}$ which is a contradiction. Hence, we always have $\beta_+ \geq \frac{1}{2}$.

Also, it is not hard to see that we always have $\beta_- \leq 1$. Indeed, if $\beta_- < 1$, take a $\beta \in (1, \beta_-)$ then by the definition of $\beta_-$ there exists $T > 0$ such that $\frac{f'(t)}{F(t)^\beta} \geq \beta \frac{f(t)}{F(t)^\beta}$ for $t \geq T$. Now integrating twice the later inequality and using the fact that $f(t) \to \infty$ as $t \to \infty$ lead us to a contradiction.

**Example 1.1.** Consider problem (1) in an arbitrary bounded smooth domain $\Omega$ with $f(u) = u^2 + 3u + 3 \cos u + 4$. It is easy to see that $f$ satisfies (H), but is not convex (even at infinity). Indeed, we have $f''(u) = 2 - 3 \cos u$, which is negative for all $u$ such that $\cos u > \frac{2}{3}$ (so none of the previous results apply). However, by a
simple computation we have $1 > \beta_- = \beta_+ = \frac{3}{2} > \frac{1}{2}$, hence by Theorem 1.1 we get $u^* \in L^\infty(\Omega)$ for $n \leq 15$.

Example 1.2. As an another example, consider problem \([1]\) with $f(u) = e^u(3 + 2\cos u)$ and arbitrary bounded smooth domain $\Omega$. Then, $f$ satisfies $(H)$, but is not convex. Indeed, we have $f''(u) = e^u(3 - 4\sin u)$, so $\liminf_{u \to \infty} f''(u) = -\infty$.

However, we have, after some simplification,

$$\frac{f'(t)(F(t) + 4)}{f(t)^2} = \frac{(3 + 2\cos u - \sin u)(3 + \sin u + \cos u)}{(3 + 2\cos u)^2} := \beta(u),$$

which is a periodic function with period $2\pi$, hence (as computed by Mathematica),

$$\beta_- = \min_{[0,2\pi]} \beta(u) \approx 0.786244 \text{ and } \beta_+ = \max_{[0,2\pi]} \beta(u) \approx 2.08846,$$

where we used also that $\lim_{t \to \infty} \frac{4f'(t)}{f(t)^2} = 0$. Now, using Theorem 1.1 we get $u^* \in L^\infty(\Omega)$ for $n \leq 9$.

Now consider the well-known convex nonlinearities $f(t) = e^t$ or $(1 + t)^p$, $p > 1$. When $f(t) = e^t$ we have $\beta_- = \beta_+ = 1$ then from Theorem 1.1 we get $u^* \in L^\infty(\Omega)$ for $n \leq 9$. Also, for $f(t) = (1 + t)^p$, $p > 1$ we have $\beta_- = \beta_+ = \frac{p}{p+1} < 1$, hence from Theorem 1.1 we get

$$u \in L^\infty(\Omega) \text{ for } n < 2\left(1 + \frac{2p}{p-1} + 2\sqrt{\frac{p}{p-1}}\right).$$

The above results are well-known in the literature \([11, 22, 18]\).

Also, notice that when $f$ is convex then the condition $\tau_- > 0$ (that is assumed in most of the previous known result) easily implies that $\beta_- > \frac{1}{2}$. Indeed, from the definition of $\tau_-$, the condition $\tau_- > 0$ yields that for every $0 < \tau < \tau_-$, there exists $T = T(\tau) > 0$ such that the function $\frac{t\tau f''(t)}{f(t)^2}$ is increasing in $[T, \infty)$, hence

$$f'(t)F(t) = f(t)^\tau \frac{f'(t)}{f(t)^\tau} \int_0^t f(s)ds \geq f(t)^\tau \int_t^T f'(s)\frac{f(s)}{f(s)^\tau} f(s)ds = \frac{f(t)^{2-\tau}}{2-\tau} - \frac{f(T)^{2-\tau}}{2-\tau},$$

now the facts that $f(t) \to \infty$ as $t \to \infty$ and $\tau > \tau_-$ was arbitrary give $\beta_\geq \frac{1}{2-\tau} > \frac{1}{2}$.

To get the regularity of the extremal solution in low dimensions or proving that it is in the energy class (i.e., $u^* \in H_0^1(\Omega)$) we can weaken the assumptions as follows. Notice that the $H_0^1$ regularity of $u^*$ is also an open problem raised by Brezis and Vázquez \([4]\). As a general result, Nedev \([17]\) proved, in an unpublished preprint, that $u^* \in H_0^1(\Omega)$ when $\Omega$ is convex and $f$ satisfies $(H)$, see also \([9]\). When $f$ is convex and $\Omega$ is arbitrary the best result is due to Villegas \([21]\) who proved that $u^* \in H_0^1(\Omega)$ if $n \leq 6$. Brezis and Vázquez in \([4]\) showed that under the extra condition that $f$ is at least of polynomial growth at infinity, namely

$$\liminf_{t \to \infty} \frac{tf'(t)}{f(t)} > 1,$$

then the extremal solution lies in the energy class. This condition is weakened in the first part of the following theorem.

Theorem 1.2. Let $f$ (not necessarily convex) satisfy $(H)$ and $\Omega$ be an arbitrary bounded smooth domain in $\mathbb{R}^n$. Let $u^*$ be the extremal solution of problem \([1]\). Then
(i) if for some $\epsilon > 0$ there exists $T_\epsilon$ such that
\[ tf'(t) \geq 1 + \frac{1}{(\ln t)^{2-\epsilon}}, \quad t > T_\epsilon, \] (6)
then $u^* \in H^1_0(\Omega)$. The same is true if for some $\epsilon > 0$ there exists $T_\epsilon$ such that
\[ \frac{f'(t)F(t)}{f(t)^2} \geq \frac{1}{2} + \frac{\epsilon t}{f(t)}, \quad t > T_\epsilon. \] (7)

In particular, this is true if $\beta_- > \frac{1}{2}$.

(ii) If for some $0 < \delta \leq 1$, $\lim_{t \to \infty} \frac{f(t)}{t^{2-\delta}} = \infty$, and there exists $t_0 > 0$ such that
\[ \frac{f'(t)F(t)}{f(t)^2} \geq \frac{1}{2} + \frac{1}{t^{2-\delta}}, \quad t > t_0 \] (8)
then $u^* \in L^\infty(\Omega)$ for $n < 5$. In particular, this is true if $\beta_- > \frac{1}{2}$.

2. Preliminaries and auxiliary results. To prove the main results we need the following simple technical lemma based on inequality (2), which is used frequently in the literature, for example [11, 10, 16, 22]. It is also proved in [1] for the general semilinear elliptic equation $-Lu = \lambda f(u)$ with zero Dirichlet boundary condition, but for the convenience of the reader we sketch a proof here for the case $L = \Delta$.

**Lemma 2.1.** Let $u_\lambda$ be the minimal solution of [1] and $g : [0, \infty] \to [0, \infty]$ be a $C^1$ function with $g(0) = 0$ and satisfy
\[ H(t) := g(t)^2 f'(t) - G(t) f(t) \geq 0, \text{ for } t \text{ sufficiently large,} \] (9)
where $G(t) := \int_0^t g'(s)^2 ds$. Then $||H(u_\lambda)||_{L^1(\Omega)} \leq C$, where $C$ is a constant independent of $\lambda$.

**Proof.** Let $u_\lambda \in C^2(\Omega)$ be the minimal classical solution of [1] where $0 < \lambda < \lambda^*$, and take $\varphi = g(u_\lambda)$ in the semi-stability condition [2]. Then we get
\[ \int_\Omega g'(u_\lambda)^2 |\nabla u_\lambda|^2 \, dx - \int_\Omega \lambda f'(u_\lambda) g(u_\lambda)^2 \, dx \geq 0. \] (10)
By using the Green’s formula one can show that
\[ \int_\Omega g'(u_\lambda)^2 |\nabla u_\lambda|^2 \, dx = \int_\Omega \lambda G(u_\lambda) f(u_\lambda) \, dx. \] (11)
Using [11] in (10) we obtain
\[ \int_\Omega H(u_\lambda) \, dx \leq 0. \] (12)
Now from (9) there exists $t_0 > 0$ so that $H(t) \geq 0$ for $t \geq t_0$, thus using (12) we obtain
\[ \int_\Omega |H(u_\lambda)| \, dx = \int_{u_\lambda \leq t_0} |H(u_\lambda)| \, dx + \int_{u_\lambda > t_0} H(u_\lambda) \, dx \]
\[ \leq \int_{u_\lambda \leq t_0} (|H(u_\lambda)| - H(u_\lambda)) \, dx \leq C_0 |\Omega|, \]
where $|\Omega|$ denotes the Lebesgue measure of $\Omega$ and $C_0 := \sup_{t \in [0, t_0]} (|H(t)| - H(t))$. Now, since $C_0$ is independent of $\lambda$ we get the desired result.

The following consequence of the above lemma is essential in the proof of the main results.
Proposition 1. Let \( u_\lambda \) be the minimal solution of \((1)\) and \( \xi : [0, \infty) \to [0, \infty) \) be a \( C^1 \) function such that for some \( t_0 > 0 \) we have \( \xi(t) \leq \frac{f'(t)}{f(t)} \), \( \xi'(t) + \xi(t)^2 \geq 0 \) for \( t \geq t_0 \), and \( \frac{E(t)}{f(t)} \to \infty \) as \( t \to \infty \) where \( E(t) \) is a \( C^1 \) function on \([0, \infty)\) such that

\[
E(t) = f(t) \left( \frac{f'(t)}{f(t)} - \xi(t) \right) e^{2 \int_0^t \xi(s) + \sqrt{\xi'(s) + \xi(s)^2} \, ds}, \quad t \geq t_0.
\]

(13)

Then \( ||E(u_\lambda)||_{L^1(\Omega)} \leq C \), where \( C \) is a constant independent of \( \lambda \).

Proof. Let \( g : [0, \infty) \to [0, \infty) \) be a \( C^1 \) function with \( g(0) = 0 \) and

\[
g(t) = e^{\int_0^t \xi(s) + \sqrt{\xi'(s) + \xi(s)^2} \, ds}, \quad t \geq t_0.
\]

Also, let \( G(t) = \int_0^t g'(s)^2 \, ds \) as in lemma \(2.1\) Then using the equality

\[
g'(t) = (\xi(t) + \sqrt{\xi'(t) + \xi(t)^2})g(t) \quad \text{for} \quad t \geq t_0,
\]

we compute

\[
\frac{d}{dt} \left( \xi(t)g(t)^2 - G(t) \right) = \xi'(t)g(t)^2 + 2\xi(t)g(t)g'(t) - g'(t)^2
\]

\[
= g(t)^2 \left( \xi'(t) + 2\xi(t)^2 + 2\xi(t)\sqrt{\xi'(t) + \xi(t)^2} - g'(t)^2
\]

\[
= g(t)^2 \left( \xi(s) + \sqrt{\xi'(t) + \xi(t)^2} \right)^2 - g'(t)^2
\]

\[
= g'(t)^2 - g'(t)^2 = 0, \quad \text{for} \quad t \geq t_0,
\]

implies that

\[
G(t) = \xi(t)g(t)^2 + C_0, \quad \text{where} \quad C_0 := G(t_0) - \xi(t_0).
\]

(14)

Now using \(14\), for \( t \geq t_0 \) we have

\[
H(t) := g(t)^2f'(t) - G(t)f(t) = g(t)^2f'(t) - g(t)\xi(t)f(t) - C_0f(t) = E(t) - C_0f(t),
\]

which is positive for large \( t \geq t_0 \) (by the assumption), hence by Lemma \(2.1\) we get \( ||H(u_\lambda)||_{L^1(\Omega)} \leq C_1 \), where \( C_1 \) is a constant independent of \( \lambda \). However, again by the assumption we have \( 0 < E(t) < 2H(t) \) for large \( t \) that also gives \( ||E(u_\lambda)||_{L^1(\Omega)} \leq C_2 \), where \( C_2 \) is a constant independent of \( \lambda \), which is the desired result.

To prove Theorem \(1.2\) in the next section, we also need the following rather standard result. For a simple proof see \([1]\).

Proposition 2. Let \( f \) satisfy \((H)\) and \( u_\lambda \) be the minimal solution of problem \((1)\). If there exists a positive constant \( C \) independent of \( \lambda \) such that

\[
||u_\lambda||_{L^1(\Omega)} \leq C \quad \text{and} \quad ||\frac{f'(u_\lambda)\alpha}{u_\lambda^\alpha}\||_{L^1(\Omega)} \leq C, \quad \text{for some} \quad 0 \leq \sigma \leq \alpha,
\]

where \( \tilde{f}(u) = f(u) - f(0) \) and \( \alpha \geq 1 \), then

\[
||u_\lambda||_{L^{\infty}(\Omega)} \leq \tilde{C} \quad \text{for} \quad n < 2\alpha,
\]

where \( \tilde{C} \) is a positive constant independent of \( \lambda \).
3. Proof of the main results.

Proof of Theorem 1.1. By the assumptions we have $\frac{1}{2} < \beta_- \leq \beta_+ < \infty$. Take $\frac{1}{2} < \beta_1 < \beta_2 < \beta_-$ and $\beta_3 \in (\beta_+, \infty)$, then by the definition of $\beta_-, \beta_+$ (see (3)) there exists a $t_0 > 0$ such that

$$\beta_1 < \beta_2 < \frac{f'(t)F(t)}{f(t)^2} < \beta_3, \text{ for } t \geq t_0. \tag{15}$$

Now let $\xi : [0, \infty] \to [0, \infty]$ be a $C^1$ function such that $\xi(t) = \beta_1 \frac{f(t)}{f(t)^2}$ for $t \geq t_0$, where $F(t) := \int_0^t f(s)ds$. Then from (15) we have $\xi(t) \leq \frac{f(t)}{f(t)^2} \beta_1$ and

$$\xi'(t) + \xi(t)^2 = \beta_1 \left( \frac{f'(t)}{F(t)} - (1 - \beta_1) \frac{f(t)^2}{F(t)^2} \right) \geq (2\beta_1 - 1) \frac{f'(t)}{F(t)} \geq \frac{2\beta_1 - 1}{\beta_3} \frac{f'(t)}{f(t)^2}, \tag{16}$$

for $t \geq t_0$. Notice that, in (16) we used the fact that $\beta_1 < 1$ (remember that we always have $\beta_- \leq 1$). Also, from (15) we have

$$\frac{f'(t)}{f(t)} - \xi(t) \geq (\beta_2 - \beta_1) \frac{f(t)}{F(t)}, \text{ for } t \geq t_0.$$

Now let the function $E(t)$ be given as in (13) in Proposition 1. By the later inequality, (15), (16) and the fact that $\int_0^t \xi(s)ds = \beta_1 (\ln F(t) - \ln F(t_0))$, we have

$$E(t) = f(t) \left( \frac{f'(t)}{f(t)} - \xi(t) \right) e^{\int_0^t \xi(s) + \sqrt{\xi'(s)^2 + \xi(s)^2}ds} \geq CF(t)^{2\beta_1 - 1} f(t)^{2 + 2\sqrt{\gamma_1^-}} \tag{17}$$

for $t \geq t_0$, where $C$ is a positive constant depends only on $f$. Now, writing the last inequality in (15) as $\frac{f'(t)}{f(t)} < \beta_3 \frac{f(t)}{f(t)^2}$ for $t > 0$, then integration from $t_0$ to $t$ gives

$$F(t) \geq C f(t)^{\frac{1}{\beta_3}} \text{ for } t \geq t_0. \tag{18}$$

Using (18) in (17) we arrive at

$$E(t) \geq f(t)^\gamma, \text{ where } \gamma := 2 + \frac{2\beta_1 - 1}{\beta_3} + 2 \sqrt{\frac{2\beta_1 - 1}{\beta_3}}, \text{ for } t \geq t_0. \tag{19}$$

And, since $\frac{E(t)}{f(t)} \rightarrow \infty \text{ as } t \rightarrow \infty$, from Proposition 1 we get $\|E(u^*)\|_{L^1(\Omega)} \leq C$ and then from (19), $\|f(u^*)\|_{L^1(\Omega)} \leq C$, where $C$ is a constant independent of $\lambda$. Now the standard elliptic regularity theory gives $u^* \in L^\infty(\Omega)$ for $n < 2\gamma$, and since $\beta_1$ and $\beta_3$ were arbitrary in the intervals $(\frac{1}{2}, \beta_-)$ and $(\beta_+, \infty)$, respectively, thus

$$u^* \in L^\infty(\Omega) \text{ for } n < 4 + 4 \left( \frac{2\beta_+ - 1}{2\beta_+} + \sqrt{\frac{2\beta_- - 1}{\beta_+}} \right) := \gamma_1, \tag{20}$$

that proves the first part.

Now assume that $\beta_+ < 1$, then we can also assume that $\beta_3 < 1$. From (18) we have $f(t)F(t)^{\beta_3} \leq C_1$, for $t \geq t_0$, and integration from $t_0$ to $t$ gives $F(t) \leq C_2 t^{\frac{1}{\beta_3}}$, $t \geq t_1$, for some $t_1 \geq t_0$. This together (18) implies that $f(t) \leq C_3 t^{\frac{1}{\beta_3}}$, for $t \geq t_1$, that also yields, for some $t_2 \geq t_1$, $f(t)^{\gamma_1} \geq C_4 f(t)^{\gamma_2}$, for $t \geq t_2$, $\gamma_2 := \frac{\beta_3}{2\beta_3 - 1} \gamma_1.$
where $\gamma_1$ is given in \[20\]. Hence, $\|\frac{f(u_{\lambda})}{u_{\lambda}}\|_{L^1(\Omega)} \leq C$, where $\int f(t) = f(t) - f(0)$ and $C$ is a constant independent of $\lambda$. Now, Proposition \[2\] gives $u^* \in L^\infty(\Omega)$ for $n < 2\gamma_2$, that proves the second part.

To prove part (a), note that in the case $\beta_+ \geq 1$, it is easy to see that the right hand side of \[4\] is larger than 6 and when $\beta_+ < 1$ we can use \[5\].

To show part (b), note that (from \[5\]) if $6 + \frac{4(1-\beta_+)}{2\beta_+ - 1} > 9$, which is equivalent to $\beta_+ < \frac{7}{10}$, then $u^* \in L^\infty(\Omega)$ for $n < 9$.

Now assume that $\beta_- = \beta_+$ then we have $\beta_+ \leq 1$ (as we always have $\beta_- \leq 1$). Also, from the later part we can consider only the case $\beta_+ \geq \frac{7}{10}$. If $\beta_+ = 1$ then from \[4\] we have $u^* \in L^\infty(\Omega)$ for $n < 10$. Also, if $\frac{7}{10} \leq \beta_+ < 1$ we can use \[6\]. We need to show that the right hand side of \[5\] is larger than 9. In the case $\frac{7}{10} \leq \beta_+ < 1$ this is equivalent to $68\beta_+^2 + 49 < 124\beta_+$, which obviously holds for $\beta_+ \in [\frac{7}{10}, 1)$. \[21\]

**Proof of Theorem \[1.2\]** First assume that \[6\] holds. Let $\xi : [0, \infty) \rightarrow [0, \infty]$ be a $C^1$ function such that

$$\xi(t) = \frac{1}{t} + \frac{1}{2t(\ln t)^2\epsilon},$$

for $t \geq T_\epsilon$. Then we have

$$\xi'(t) + \xi(t)^2 = \frac{1}{2t^2(\ln t)^2\epsilon^2} \left(1 + \frac{1}{2(\ln t)^2\epsilon^2} - \frac{2 - \epsilon}{\ln t}\right) \geq \frac{1}{4t^2(\ln t)^2\epsilon^2},$$

for $t \geq T_\epsilon$ and sufficiently large. Hence, we have

$$2 \int_{T_\epsilon}^{t} (\xi(s) + \sqrt{\xi'(s)} + \xi(s)^2) ds \geq 2 \ln t + \frac{2}{\epsilon}(\ln t)^{\frac{1}{2}} - C_\epsilon,$$

for $t \geq T_\epsilon$ and sufficiently large, where $C_\epsilon$ is a constant depends on $\epsilon$. This together \[6\] give us

$$E(t) = f(t) \left(\frac{f'(t)}{f(t)} - \xi(t)\right) e^{2 \int_{T_\epsilon}^{t} (\xi(s)+\sqrt{\xi'(s)}+\xi(s)^2) ds} \geq C \frac{tf(t)}{(\ln t)^2\epsilon^2} e^{\frac{2}{\epsilon}(\ln t)^{\frac{1}{2}}} \geq C tf(t),$$

for $t > T_\epsilon$ and sufficiently large, where $C$ is a positive constant depends only on $f$. Thus, from Proposition \[1\] we get $\|u_{\lambda}f(u_{\lambda})\|_{L^1(\Omega)} \leq C$, where $C$ is a constant independent of $\lambda$. Multiplying \[1\] by $u_{\lambda}$ we get

$$\int_{\Omega} |\nabla u_{\lambda}|^2 dx = \lambda \int_{\Omega} u_{\lambda} f(u_{\lambda}) dx \leq \lambda^* C,$$

which leads to $\int_{\Omega} |\nabla u^*|^2 dx \leq \lambda^* C$. Hence, $u^* \in H^1_0(\Omega)$.

Now assume \[7\] holds and let $\xi : [0, \infty) \rightarrow [0, \infty]$ be a $C^1$ function such that

$$\xi(t) = \frac{1}{2} \frac{f(t)}{F(t)}, \quad \text{for } t \geq T_\epsilon.$$ \[21\]

Then from \[7\] we have

$$\frac{f'(t)}{f(t)} - \xi(t) = \frac{f(t)}{F(t)} \left(\frac{f'(t)F(t)}{f(t)^2} - \frac{1}{2}\right) \geq \epsilon \frac{t}{F(t)}, \quad \text{for } t \geq T_\epsilon,$$

and

$$\xi'(t) + \xi(t)^2 = \frac{1}{2} \left(\frac{f'(t)F(t)}{f(t)^2} - \frac{1}{2}\right) \frac{f(t)^2}{F(t)^2} > 0 \quad \text{for } t \geq T_\epsilon.$$ \[23\]
From \((22)\) and \((23)\) and the fact that \(2 \int_{t_0}^{t} \xi(s) ds = \ln F(t) - \ln F(T)\) for \(t > T\), we get
\[
E(t) = f(t) \left( \frac{f'(t)}{f(t)} - \xi(t) \right) e^{2 \int_{t_0}^{t} (\xi(s) + \sqrt{\xi(s) + \xi(s)^2}) ds} \geq C t f(t), \quad t > T_0
\]
where \(C\) is a positive constant depends only on \(f\). Thus, using Proposition 4 and similar as above we get \(u^* \in H^1_0(\Omega)\), that completes the proof of part (i).

To prove part (ii), Let \(\xi(t)\) be as defined in \([21]\). Then, using \((8)\) and similar to the proof of part (i) we can show that \(E(t) \geq C t \frac{f(t)}{r^{2+\sigma}}\) for \(t > t_0\). Hence, by the assumption we get \(E(t)/f(t) \to \infty\) as \(t \to \infty\) then from Proposition 1 we find
\[
\frac{\dot{f}(u_\lambda)^2}{\lambda^{2-\delta}} \in L^\infty(\Omega).
\]
(24)

Now we proceed similar to the proof of Theorem 1.1 in \([21]\). From Proposition 2.3 in \([21]\), there exits a universal constant \(C_1\) independent of \(f, \Omega\) and \(\lambda\) such that
\[
\|u_\lambda\|_{L^\infty(\Omega)} \leq C_1 \|\nabla u_\lambda\|_{L^4(\Omega)}.
\]
(25)

Also, from the continuous inclusion \(W^{2,2}(\Omega) \subseteq W^{1,4}(\Omega)\) and elliptic regularity theory (see \([2]\)), there exits a constant \(C_2 = C_2(\Omega)\) such that
\[
\|u_\lambda\|_{W^{1,4}(\Omega)} \leq C_2 \|u_\lambda\|_{W^{2,2}(\Omega)} \quad \text{and} \quad \|u_\lambda\|_{W^{2,2}(\Omega)} \leq C_2 \|\lambda f(u_\lambda)\|_{L^2(\Omega)}.
\]
(26)

Now from \((24), (25)\) and \((26)\) we have (in the following inequalities various constants will be denoted by \(C\))
\[
\|u_\lambda\|_{L^\infty(\Omega)} \leq C \|\nabla u_\lambda\|_{L^4(\Omega)}
\leq C \|u_\lambda\|_{W^{1,4}(\Omega)}
\leq C \|u_\lambda\|_{W^{2,2}(\Omega)}
\leq C \|\lambda f(u_\lambda)\|_{L^2(\Omega)}
\leq C \lambda^\ast \left( \int_{u_\lambda \leq 1} f(u_\lambda)^2 dx + \int_{u_\lambda > 1} f(u_\lambda)^2 dx \right)^{\frac{1}{2}}
\leq C \left( f(1)^2 |\Omega| + \int_{u_\lambda > 1} \frac{f(u_\lambda)^2}{u_\lambda^{2-\delta}} dx \right)^{\frac{1}{2}}
\leq C \left( f(1)^2 |\Omega| + \tilde{C} \|u_\lambda\|_{L^\infty(\Omega)}^{2-\delta} \right)^{\frac{1}{2}},
\]
for every \(\lambda \in (0, \lambda_\ast)\). Hence, we must have
\[
\|u_\lambda\|_{L^\infty(\Omega)} \leq A + B \|u_\lambda\|_{L^\infty(\Omega)}^{2-\delta}, \quad \text{for every} \lambda \in (0, \lambda_\ast),
\]
where \(A\) and \(B\) are positive constants independent of \(\lambda\). This implies that \(\|u_\lambda\|_{L^\infty(\Omega)} \leq C\) with \(C\) independent of \(\lambda\), now letting \(\lambda \to \lambda_\ast\) gives \(u^* \in L^\infty(\Omega)\).

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