Rigid divisors on surfaces

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Abstract. We study effective divisors $D$ on surfaces with $H^0(\mathcal{O}_D) = k$ and $H^1(\mathcal{O}_D) = H^0(\mathcal{O}_D(D)) = 0$. We give a numerical criterion for such divisors, following a general investigation of negativity, rigidity and connectivity properties. Examples include exceptional loci of rational singularities, and spherelike divisors.

Keywords: negative divisors, rigid divisors, divisors on surfaces, spherelike sheaves.

Introduction

An important facet of algebraic geometry is the rich theory of positive divisors on varieties. The starting point are ample divisors, a notion that can be characterized geometrically, cohomologically and numerically. Moreover, there are various generalizations, leading to a mature tenet of positivity; see [1]. We point out that the theory began with Zariski’s work on surfaces.

In this article we conduct a systematic study of negativity properties. We restrict ourselves to surfaces but feel that a numerical approach going beyond intersection numbers should yield results in higher dimensions. Some classical negativity notions for surfaces are negative definiteness (appearing in Artin’s contraction criterion) and Ramanujan’s 1-connectedness.

Specifically, we consider effective divisors $D$ on a smooth algebraic surface $X$ which are well-connected ($H^0(\mathcal{O}_D) = k$) and rigid as sheaves on $X$ (that is, $H^1(\mathcal{O}_D) = H^0(\mathcal{O}_D(D)) = 0$). We then call $D$ a $(-n)$-divisor as necessarily $D^2 = -n < 0$. This turns out to be a good notion which, for example, has a nice numerical characterization.

Such divisors always consist of negative rational curves. We do not discuss negative curves on a fixed surface; for this and the Bounded Negativity Conjecture, see [2].

We have two principal motivations for this work. Geometrically, rigid and negative definite divisors arise as exceptional loci of rational singularities. In particular, $(-1)$-divisors come from blowing up smooth points, and numerical fundamental cycles of ADE singularities are $(-2)$-divisors. More generally, such cycles of rational singularities are always $(-n)$-divisors; see Proposition 6.8.

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Homologically, the structure sheaf $\mathcal{O}_D$ of a $(-2)$-divisor is a 2-spherelike object in the sense of [3], that is,

$$\text{Hom}(\mathcal{O}_D, \mathcal{O}_D) = \text{Ext}^2(\mathcal{O}_D, \mathcal{O}_D) = \mathbb{k} \quad \text{and} \quad \text{Ext}^1(\mathcal{O}_D, \mathcal{O}_D) = 0.$$ 

This was our initial motivation for studying these divisors. In previous work we associated with every such $D$ a natural maximal subcategory of $\mathcal{D}^b(X)$ in which $\mathcal{O}_D$ becomes a 2-Calabi–Yau object. As a homological detour, we compute these spherical subcategories and the related asphericities for some examples in §5.

We start out in §1 with a systematic study of divisors centred around negativity, rigidity and connectivity. Here we list non-standard notions in a very terse fashion; for more elaborate definitions and comments, see the referenced subsections. Let $D$ be an effective divisor. All $C_i$ occurring below are reduced irreducible curves.

**Negativity (§ 1.1).** $D$ is negatively closed if $A^2 < 0$ for all divisors $A$ with $0 \prec A \preceq D$. It is negative definite if $kD$ is negatively closed for all $k \geq 1$. It is negatively filtered if $D = C_1 + \cdots + C_m$, where $C_i.(C_i + \cdots + C_m) < 0$ for all $i = 1, \ldots, m$.

**Rigidity (§ 1.2).** $D$ is rigid as a subscheme if $H^0(\mathcal{O}_D(D)) = 0$. It is Jacobi rigid if $H^1(\mathcal{O}_D) = 0$. It is rigid if $D$ is rigid as a subscheme and Jacobi rigid, that is, $\mathcal{O}_D$ is an infinitesimally rigid sheaf on $X$.

**Connectivity (§ 1.3).** $D$ is well-connected if $H^0(\mathcal{O}_D) = \mathbb{k}$. It is 1-connected if $A.(D - A) \geq 1$ for all divisors $A$ with $0 \prec A \prec D$. It is 1-decomposable if $D = C_1 + \cdots + C_m$, where $C_i.(C_i+1 + \cdots + C_m) = 1$ for all $i = 1, \ldots, m - 1$.

\[
\begin{array}{cccccc}
\text{negative}^c_n & \implies & \text{negatively}^c_n \text{ closed} & \implies & \text{negatively}^n \text{ filtered} & \implies \\
\text{definite} & & & & \text{rigid as a}^c \text{ subscheme} \\
\end{array}
\]

\[
\begin{array}{c}
\text{well-connected} \\
& \implies \text{rigid}^c \\
& \implies \text{Jacobi}^c \text{ rigid} \\
& \implies \text{rational}^c \text{ forest} \\
\end{array}
\]

\[
\begin{array}{c}
\text{1-decomposable}^n \\
\implies \text{1-connected}^n \\
\end{array}
\]

\[
\begin{array}{c}
\text{1-connected}^n \\
\implies \text{well-connected} \\
\implies \text{connected} \\
\end{array}
\]

Our results about these properties are most concisely summed up in the scheme above. The ornaments $c$, $n$ indicate that a property is closed under subdivisors, numerical respectively; see Definition 2.12. By Proposition 2.16, all these properties are birationally invariant.

The crucial new notion is that of 1-decomposability because it yields a combinatorial grip on the conditions $H^0(\mathcal{O}_D) = \mathbb{k}$ and $H^1(\mathcal{O}_D) = 0$. One of our main results is the following characterization; see Theorem 3.1.

**Theorem A.** An effective divisor is well-connected and rigid if and only if is rational, 1-decomposable and negatively filtered.
Let $D$ be a well-connected rigid divisor, that is, a $(-n)$-divisor with $n = -D^2$. We are interested in simplifying $D$. Such a simplification can be

birational: if there is a contraction $\pi: X \to X'$ such that $D' = \pi(D)$ is a $(-n)$-divisor on $X'$ with $D = \pi^*D'$; for this to be possible, there has to be a $(−1)$-curve $E \subset X$ with $E.D = 0$;

homological: while $(-1)$-curves induce contractions (that is, categorical decompositions of the derived category), a $(-2)$-curve $C$ gives rise to an autoequivalence, the spherical twist $T_{O_C}$; if $C \preceq D$ with $C.D = −1$, then $T_{O_C}(-O_D) = O_{D-C}$, and we say that $C$ can be twisted off $D$;

numerical: a sum $D = D_1 + \cdots + D_m$ with $(-n_i)$-divisors $D_i$ such that $D_i.(D_{i+1} + \cdots + D_m) = 1$ and $-2 \geq -n_i \geq -n$ for all $i$, is called a curvelike decomposition; see Definition 4.13.

There is always a decomposition in the numerical sense with strong properties. The following result is Theorem 4.14.

**Theorem B.** When $n \geq 2$, any $(-n)$-divisor has a curvelike decomposition all of whose summands are pullbacks of either curves $C$ or chains of the form $(−m−1−2\cdots−2−1)$, where $C^2, -m \in \{-2, \ldots, -n\}$.

An alternative definition of a $(-n)$-divisor $D$ is that $H^\bullet(O_D) = H^\bullet(O_C)$ and $H^\bullet(O_D(D)) = H^\bullet(O_C(C))$, where $C$ is a $(-n)$-curve, that is, a smooth rational curve with $C^2 = -n$; see Lemma 1.6. In Proposition 4.20, we show that likewise

$H^\bullet(O_D(K_X)) = H^\bullet(O_C(K_X))$ and $H^\bullet(O_D(D + K_X)) = H^\bullet(O_C(C + K_X))$.

We say that $D$ is essentially a $(-n)$-curve if $D$ can be obtained from a $(-n)$-curve through blow-ups and spherical twists; see Definition 4.9. At the other extreme, we call $D$ a minimal $(-n)$-divisor if neither operation is possible. For general reasons, certain low-degree divisors are always essentially curves; see Proposition 4.19, Corollary 4.12 and Proposition 7.3 respectively.

**Theorem C.** A divisor which is exceptional, or spherical, or reduced and spherelike is essentially a $(-n)$-curve for some $n > 0$.

Moreover, the building blocks, that is, minimal $(-n)$-divisors, can be enumerated. For instance, there are five minimal $(-2)$-divisors on five curves; see Example 7.2:
The question remains as to whether such graphs can be realized as dual intersection graphs of effective divisors. In Proposition 7.6 we give a sufficient condition which is enough to deal with the examples above. This seems to be a subtle problem and it would be interesting to study it in greater depth.

Conventions. We work over an algebraically closed field $k$. Throughout this article, we consider a smooth algebraic surface, usually denoted by $X$, and effective projective divisors on it. For such divisors, the intersection product is well defined; see, for example, [4], Example 2.4.9. The canonical divisor on $X$ is denoted by $K_X$, or sometimes just by $K$. The structure sheaf of the surface is denoted by $\mathcal{O}$ instead of $\mathcal{O}_X$. Curves are always irreducible and reduced. By a $(-n)$-curve we mean a smooth rational curve on $X$ of self-intersection number $-n$.

Hom and Ext$^i$ refer to homomorphism and extension spaces on the surface unless we explicitly specify another variety. We occasionally abbreviate dimensions:

$$\text{hom}(M, N) := \dim \text{Hom}(M, N) \quad \text{and} \quad h^i(M) := \dim H^i(M).$$

The Hom-complex of two sheaves $M, N$ is

$$\text{Hom}^\bullet(M, N) := \bigoplus_i \text{Ext}^i(M, N)[-i].$$

It is a complex of vector spaces with zero differentials.

We sometimes use $M' \hookrightarrow M \twoheadrightarrow M''$ as a shorthand for a short exact sequence. Distinguished triangles in $D^b(X)$ are shown as $M' \to M \to M''$, omitting the connecting morphism $M'' \to M'[1]$, and just called ‘triangles’. We do not embellish the symbol for derived functors, for example, by writing $f_* : D^b(X) \to D^b(Y)$ for a proper scheme morphism $f : X \to Y$.

We depict divisors consisting of rational curves by their dual intersection graphs: \(-3\) denotes a reduced $(-1)$-curve, \(-3\) a double $(-3)$-curve, \(-2\)-\(-2\) two $(-2)$-curves intersecting at one point, and \(-3\)-\(-3\) two $(-3)$-curves intersecting at two points.

§ 1. Properties of divisors: negativity, rigidity, connectivity

Let $X$ be a smooth algebraic (that is, quasiprojective) surface defined over an algebraically closed field $k$. A curve on $X$ will always mean an irreducible reduced effective divisor, not necessarily smooth. Therefore, given an effective divisor $D$ on $X$, we will speak of its curve decomposition $D = \sum_i c_i C_i$, where the $C_i$ are pairwise distinct curves and all the $c_i$ are positive.

In this section, we distinguish typographically between recalling an established notion and introducing a new one.

Intersection numbers and the Euler characteristic. The intersection number of a divisor can be computed cohomologically using the Euler characteristic. Recall that for coherent sheaves $\mathcal{F}, \mathcal{G}$ on $X$ with proper support, their Euler characteristic is given by $\chi(\mathcal{F}, \mathcal{G}) = \sum_i (-1)^i \dim \text{Ext}^i(\mathcal{F}, \mathcal{G})$. 
Lemma 1.1. Let $A$ and $B$ be effective divisors on $X$. Then
\[ A \cdot B = \chi(O_A) + \chi(O_B) - \chi(O_{A+B}) = -\chi(O_A, O_B) \]
and, in particular, $A^2 = -\chi(O_A, O_A)$.

Proof. We start with the well-known formula (see, for example, [5], Exercise V.1.1)
\[ A \cdot B = \chi(O) - \chi(O(-A)) - \chi(O(-B)) + \chi(O(-A-B)), \]
replacing $X$ by an auxiliary compactification if necessary. This statement holds for all (projective) divisors, not just effective ones. The first equation follows immediately, using sequences of ideal sheaves. For the second, we compute
\[
\chi(O_A, O_B) = \chi(O, O_B) - \chi(O(-A), O_B) = \chi(O_B) - \chi(O_B(A)) = (\chi(O) - \chi(O(-B))) - (\chi(O(A)) - \chi(O(A - B))) = -A \cdot B. \]

For an effective divisor $D$ on $X$, we have the Riemann–Roch formula for the Euler characteristic of its structure sheaf:
\[ \chi(O_D) = -\frac{1}{2}(D^2 + D.K_X). \]

Lemma 1.2 (decomposition sequence). If $A + B$ is the sum of two effective divisors, then we have the following short exact sequence of sheaves on $X$:
\[ 0 \rightarrow O_A(-B) \rightarrow O_{A+B} \rightarrow O_B \rightarrow 0. \]

Proof. We include the proof just for the homological fun of it. Consider the following commutative diagram of ideal sheaf sequences:
\[
\begin{array}{cccccc}
0 & \rightarrow & O(-A - B) & \rightarrow & O & \rightarrow & O_{A+B} & \rightarrow & 0 \\
\downarrow \iota & & \downarrow \text{id} & & \downarrow \pi & & & & \\
0 & \rightarrow & O(-B) & \rightarrow & O & \rightarrow & O_B & \rightarrow & 0,
\end{array}
\]
where $\iota$ is the inclusion and $\pi$ the restriction. The snake lemma tells us that $\pi$ is surjective with kernel isomorphic to $\text{coker} \, \iota = O_A(-B)$. \(\square\)

1.1. Negativity properties of divisors. The negativity of a divisor $D$ can be measured in several ways:
- $D$ is negative if $D^2 < 0$;
- $D$ is negative definite if $A^2 < 0$ for all $A \neq 0$ with $\text{supp}(A) \subseteq \text{supp}(D)$;
- $D$ is negatively closed if $A^2 < 0$ for all $A$ with $0 < A \preceq D$;
- $D$ is negatively filtered if $D = C_1 + \cdots + C_m$ with curves $C_i$ such that $C_i(C_i + \cdots + C_m) < 0$ for all $i = 1, \ldots, m$.

In words, $D$ is negatively closed if all effective subdivisors are negative. By contrast, $D$ is negative definite if every divisor supported on (a subset of) $D$ is negative, that is, the intersection matrix of the curve components of $D$ is negative definite; in particular, $D$ is then negatively closed. Note that negative definiteness depends only on the curve configuration underlying $D$. 

**Remark 1.3.** To explain negative filtrations, let \( C \prec D \) be a curve with \( C.D < 0 \), and set \( D' := D - C \). The cohomology of the decomposition sequence \( \mathcal{O}_{D'}(D') \hookrightarrow \mathcal{O}_D(D) \twoheadrightarrow \mathcal{O}_C(D) \) gives \( H^0(\mathcal{O}_{D'}(D')) \cong H^0(\mathcal{O}_D(D)) \) since \( H^0(\mathcal{O}_C(D)) = 0 \) by the assumption that \( C.D < 0 \).

Therefore, applying this argument inductively to a negative filtration \( D = C_1 + \cdots + C_m \), we find that \( H^0(\mathcal{O}_D(D)) = \cdots = H^0(\mathcal{O}_{C_m}(C_m)) = 0 \), where the last equation uses the fact that \( C_m^2 < 0 \) from the final step of the filtration. Hence \( D \) is rigid as a subscheme in the parlance of § 1.2.

**Lemma 1.4.** Every negatively closed divisor \( D \) is negatively filtered.

**Proof.** More generally, we show that every curve \( C \) in a negatively closed divisor \( D \) with \( C.D < 0 \) can be extended to a negative filtration of \( D \).

First we note that every negatively closed divisor \( D \) contains a curve \( C_1 := C \) with \( C_1.D < 0 \). Otherwise, writing \( D = \sum n_i C_i \) as a sum of curves yields a contradiction: \( D^2 = \sum n_i C_i.D > 0 \). We now proceed by induction with \( D - C \), which is negatively closed as well. \( \square \)

**Example 1.5.** Let

\[
D = 2E + A + A' + B = -3 \quad -3 \quad -2.
\]

Then \( A, A', E, B, E \) is a negative filtration of \( D \). However, \( D^2 = -4 - 3 - 3 - 2 + 2(2 + 2 + 2) = 0 \), so that \( D \) is not negatively closed.

Example 6.16 describes a negatively closed divisor which is not negative definite.

1.2. **Rigidity properties of divisors.** An effective divisor \( D \) is said to be

- **rigid** (as a sheaf) if the first-order infinitesimal deformations of its structure sheaf \( \mathcal{O}_D \), as a sheaf on \( X \), are trivial, that is,

\[
\text{Ext}^1_X(\mathcal{O}_D, \mathcal{O}_D) = 0;
\]

- **rigid as a subscheme** if all first-order infinitesimal deformations of \( D \) as a closed subscheme of \( X \) are trivial. As these infinitesimal deformations are classified by global sections of the normal bundle of \( D \) in \( X \), this condition amounts to saying that \( H^0(\mathcal{O}_D(D)) = 0 \);

- **Jacobi rigid** if there are no first-order infinitesimal deformations of \( \mathcal{O}_D \) as a line bundle of degree zero on \( D \); this amounts to saying that \( H^1(\mathcal{O}_D) = 0 \), which is the tangent space of the Jacobian \( \text{Pic}^0(D) \) at \( [\mathcal{O}_D] \);

- **rational** if all curves in \( D \) are smooth rational curves.

**Lemma 1.6.** For every effective divisor \( D \) on \( X \), there are canonical isomorphisms of vector spaces

\[
\text{Hom}(\mathcal{O}_D, \mathcal{O}_D) = H^0(\mathcal{O}_D),
\]

\[
\text{Ext}^1(\mathcal{O}_D, \mathcal{O}_D) = H^0(\mathcal{O}_D(D)) \oplus H^1(\mathcal{O}_D),
\]

\[
\text{Ext}^2(\mathcal{O}_D, \mathcal{O}_D) = H^1(\mathcal{O}_D(D)).
\]

In particular, \( D \) is rigid \( \iff \) \( D \) is rigid as a subscheme and Jacobi rigid.
Proof. Let \( i : D \hookrightarrow X \) denote inclusion. We use the adjunction \( i^* \dashv i_* \) of exact functors between the derived categories \( D^b(D) \) and \( D^b(X) \). Note that \( i_* \) coincides with the underived pushforward of sheaves, in contrast to \( i^* \). Note that there is an isomorphism
\[
\mathcal{O}_D = i_* \mathcal{O}_D \cong [\mathcal{O}(-D) \rightarrow \mathcal{O}]
\]
in \( D^b(X) \) and, therefore, a decomposition
\[
i^*i_* \mathcal{O}_D = \mathcal{O}_D \oplus \mathcal{O}_D(-D)[1].
\]

We now compute that
\[
\text{Hom}^\bullet_X(\mathcal{O}_D, \mathcal{O}_D) = \text{Hom}^\bullet_X(i_* \mathcal{O}_D, i_* \mathcal{O}_D) = \text{Hom}^\bullet_D(i^* i_* \mathcal{O}_D, \mathcal{O}_D)
\]
\[
= \text{Hom}^\bullet_D(\mathcal{O}_D \oplus \mathcal{O}_D(-D)[1], \mathcal{O}_D)
\]
\[
= \text{Hom}^\bullet_D(\mathcal{O}_D, \mathcal{O}_D) \oplus \text{Hom}^\bullet_D(\mathcal{O}_D(-D)[1], \mathcal{O}_D)
\]
\[
= H^\bullet(\mathcal{O}_D) \oplus H^\bullet(\mathcal{O}_D(D))[1].
\]

\[\square\]

**Example 1.7.** We give a standard example of a curve which is rigid as a subscheme but not Jacobi rigid. Let \( C \subset \mathbb{P}^2 \) be a smooth cubic; in particular, \( C^2 = 9 \). Blowing up \( \mathbb{P}^2 \) at eleven points lying on \( C \), we can write the total transform of \( C \) as \( D = \overline{C} + E_1 + \cdots + E_{11} \), where \( \overline{C} \) is the strict transform of \( C \). As the intersection pairing is preserved, we have \( C^2 = \overline{C}^2 + 20 = \mathcal{O}_D(-D)[1] \). It is rigid as a subscheme but \( \mathcal{O}_C \) deforms in view of \( H^1(\mathcal{O}_C) = \mathbb{k} \). By Riemann–Roch, \( \overline{C}.K_X = 2 \).

**1.3. Connectivity properties of divisors.** Let \( D \) be an effective divisor.

- \( D \) is **connected** if its support is connected.
- \( D \) is **well-connected** if \( H^0(\mathcal{O}_D) = \text{Hom}(\mathcal{O}_D, \mathcal{O}_D) = \mathbb{k} \).
- \( D \) is (**numerically**) 1-connected if \( A.(D - A) \geq 1 \) for every divisor \( A \) with \( 0 \prec A \prec D \).

A well-connected divisor is obviously connected. For reduced divisors, the three notions are equivalent.

**Example 1.8.** Let \( E \) be a \((-1)\)-curve, that is, a smooth rational curve with \( E^2 = -1 \). The decomposition sequence for \( 2E \) easily gives \( H^0(\mathcal{O}_{2E}) = \mathbb{k}^3 \).

The following result of Ramanujan is classical.

**Lemma 1.9** (see, for example, [6], Lemma 3.11). *All 1-connected divisors are well-connected.*

**Example 1.10.** Consider the divisor
\[
D = \begin{array}{ccc}
-3 & \rightarrow & -2 \\
\rightarrow & \rightarrow & \rightarrow \\
-1 & \rightarrow & -2
\end{array},
\]
consisting of a triangle of rational curves, each with multiplicity two. This divisor is not 1-connected: \( D^2_{\text{red}} = -1 - 2 - 3 + 2 \cdot 3 = 0 \). Using the decomposition sequence for \( D = D_{\text{red}} + D_{\text{red}} \), one can check that \( D \) is well-connected.
Question 1.11. Is any well-connected divisor whose dual intersection graph is a tree automatically 1-connected?

This is about the converse of 1-connected \(\implies\) well-connected. Example 1.10 is a counterexample, but it is not a tree.

We consider two more conditions related to connectivity.

- \(D\) is a tree if the dual intersection graph of \(D\) is a tree and all intersections of irreducible curves in \(D\) are transversal.
- \(D\) is 1-decomposable if it can be written as \(D = C_1 + \cdots + C_m\), where each \(C_i\) is an irreducible curve and \(C_i, (C_{i+1} + \cdots + C_m) = 1\) for all \(i = 1, \ldots, m - 1\).

Remark 1.12. To explain the definition of 1-decompositions, let \(C \prec D\) be a rational curve with \(C.D' = 1\), where \(D' := D - C\). The cohomology of the decomposition sequence \(\mathcal{O}_C(-D') \hookrightarrow \mathcal{O}_D \to \mathcal{O}_{D'}\) gives \(H^*(\mathcal{O}_D) \cong H^*(\mathcal{O}_{D'})\) since \(H^*(\mathcal{O}_C(-D')) = 0\) by the assumption that \(C.D' = 1\).

Therefore, by applying this argument inductively, we see that a 1-decomposition of a rational divisor \(D = C_1 + \cdots + C_m\) yields \(H^*(\mathcal{O}_D) = \cdots = H^*(\mathcal{O}_{C_m}) = \mathbb{k}\).

Regarding the definition of trees, many sources include the condition that all curves are smooth and rational. For our purposes it is better to distinguish between these properties (for example, in the next paragraph).

Any divisor that is a reduced tree of curves has a 1-decomposition, by inductively pruning the leaves. In Lemma 2.8 we show that all 1-decomposable divisors are trees and we conjecture that they are also 1-connected.

1.4. \((-n)\)-divisors. In this article we are interested in well-connected rigid divisors. More concretely, for an effective divisor \(D\) this means that \(H^0(\mathcal{O}_D) = \mathbb{k}\) and \(\text{Ext}^1(\mathcal{O}_D, \mathcal{O}_D) = 0\). We recall that

\[
\text{Ext}^1(\mathcal{O}_D, \mathcal{O}_D) = H^0(\mathcal{O}_D(D)) \oplus H^1(\mathcal{O}_D).
\]

Remark 1.13. Thus, \(\mathcal{O}_D \in \mathcal{D}^b(X)\) is an object of a triangulated category with \(\text{Hom}^0(\mathcal{O}_D, \mathcal{O}_D) = \mathbb{k}\) and \(\text{Hom}^1(\mathcal{O}_D, \mathcal{O}_D) = 0\). In topological parlance, for example in the category of spectra, one might call \(\mathcal{O}_D\) a ‘simply connected’ object. We prefer the terminology ‘well-connected rigid’ to stress the rigidity of \(D\).

It is often convenient to specify \(\text{ext}^2(\mathcal{O}_D, \mathcal{O}_D)\). For well-connected rigid divisors \(D\) we have \(h^1(\mathcal{O}_D(D)) = \text{ext}^2(\mathcal{O}_D, \mathcal{O}_D) = \chi(\mathcal{O}_D, \mathcal{O}_D) - 1 = -D^2 - 1\). Thus,

- \(D\) is a \((-n)\)-divisor if \(D\) is well-connected and rigid with \(D^2 = -n < 0\).

By Lemma 1.6, \(D\) is a \((-n)\)-divisor if and only if \(H^1(\mathcal{O}_D(D)) = \mathbb{k}^n-1\), \(H^0(\mathcal{O}_D) = \mathbb{k}\) and \(H^1(\mathcal{O}_D) = H^0(\mathcal{O}_D(D)) = 0\). Assuming the other three provisions, \(H^1(\mathcal{O}_D(D)) = \mathbb{k}^{n-1}\) can be replaced by \(D^2 = -n\).

We mention three important special cases of this notion.

- \(D\) is exceptional if \(D\) is a \((-1)\)-divisor.
- \(D\) is spherelike if \(D\) is a \((-2)\)-divisor.
- \(D\) is spherical if it is spherelike and invariant under twisting by the canonical bundle of \(X\), that is,

\[
\mathcal{O}(K_X)|_D \cong \mathcal{O}_D.
\]
Saying that $D$ is exceptional, spherelike, or spherical, is just a short way of saying that the sheaf $\mathcal{O}_D \in D^b(X)$ is an exceptional object ([7], §8.3), or a 2-spherelike object ([3]), or a 2-spherical object ([7], §8.1) respectively.

The condition of being well-connected is crucial in the definition of a $(-n)$-divisor. The following remark spells out what happens when it is dropped in the first non-trivial case ($n = 2$).

Remark 1.14. Let $D$ be a rigid divisor with $D^2 = -2$, that is, we drop the condition of well-connectedness of $D$ from the definition of a spherelike divisor. Since $\chi(\mathcal{O}_D, \mathcal{O}_D) = -D^2 = 2$, there are only two options.

- $\text{hom}(\mathcal{O}_D, \mathcal{O}_D) = 1$, that is, $D$ is spherelike as defined above;
- $\text{hom}(\mathcal{O}_D, \mathcal{O}_D) = 2$, that is, $\mathcal{O}_D \in D^b(X)$ is a 0-spherelike object; see [3].

Because $k$ is algebraically closed, the 0-spherelike case in turn leads to either $\text{End}(\mathcal{O}_D) = k[x]/x^2$, or $\text{End}(\mathcal{O}_D) = k \times k$. Examples of these two types are $(\mathcal{O}_1 \mathcal{O}_2)$ and the disconnected divisor $(\mathcal{O}_1 \mathcal{O}_2)$, respectively. The first of these shows that a merely connected and rigid divisor is not necessarily a $(-n)$-divisor; one has to ask for well-connectedness.

§2. Relations between the properties of divisors

2.1. Properties of rigid divisors.

Lemma 2.1. Every rigid divisor is negative.

Proof. We have

$$-D^2 = \chi(\mathcal{O}_D, \mathcal{O}_D) = \text{hom}(\mathcal{O}_D, \mathcal{O}_D) + \text{ext}^2(\mathcal{O}_D, \mathcal{O}_D) > 0$$

with $\text{Ext}^1(\mathcal{O}_D, \mathcal{O}_D) = 0$ because $\mathcal{O}_D$ is rigid. \(\Box\)

Lemma 2.2. Any effective subdivisor of a rigid divisor is rigid.

This also holds with ‘rigid’ replaced by ‘Jacobi rigid’ or ‘rigid as a subscheme’.

Proof. Let $D = A + B$ be a decomposition of $D$ into two effective divisors.

We first assume that $D$ is Jacobi rigid, that is, $H^1(\mathcal{O}_D) = 0$. The long exact cohomology sequence of the decomposition sequence $\mathcal{O}_B(-A) \rightarrow \mathcal{O}_D \rightarrow \mathcal{O}_A$ contains a surjection $H^1(\mathcal{O}_D) \twoheadrightarrow H^1(\mathcal{O}_A)$. Thus, $H^1(\mathcal{O}_A) = 0$ and $A$ is also Jacobi rigid.

We now assume that $D$ is rigid as a subscheme, that is, $H^0(\mathcal{O}_D(D)) = 0$. Tensoring the other decomposition sequence $\mathcal{O}_A(-B) \hookrightarrow \mathcal{O}_D \rightarrow \mathcal{O}_B$ with $\mathcal{O}(D)$ yields an injection $H^0(\mathcal{O}_A(A)) \hookrightarrow H^0(\mathcal{O}_D(D))$. So $H^0(\mathcal{O}_A(A)) = 0$, and hence $A$ is rigid as a subscheme in $X$.

By Lemma 1.6, both cases together imply that $A$ is rigid when $D$ is. \(\Box\)

Combining these two lemmas, we obtain the following statement.

Corollary 2.3. Every rigid divisor is negatively closed.

Example 2.4. The sum of two rigid divisors need not be rigid. Indeed, let $C$ be a curve with $C^2 = 0$, for example, the fibre of a ruled surface $X$. Then $C$ obviously deforms, at least locally. Let $\tilde{X}$ be the blow-up of $X$ at any point of $C$. We denote
the total transform of $C$ by $\tilde{C} = (1-1)$. Then $\tilde{C}^2 = 0$ and $\tilde{C}$ is not rigid since $\text{Ext}_X(\mathcal{O}_{\tilde{C}}, \mathcal{O}_{\tilde{C}}) = \text{Ext}_X^1(\mathcal{O}_C, \mathcal{O}_C) \neq 0$.

**Lemma 2.5.** Every connected, Jacobi rigid divisor is a rational tree.

*Proof.* Let $D = \sum_i c_i C_i$ be the curve decomposition of $D$, and assume that $D$ is connected and Jacobi rigid. By Lemma 2.2, each curve $C_i$ is Jacobi rigid. Thus,

$$p_a(C_i) = 1 - \chi(\mathcal{O}_{C_i}) = 1 - h^0(\mathcal{O}_{C_i}) + h^1(\mathcal{O}_{C_i}) = 1 - 1 + 0 = 0,$$

that is, $C_i$ has arithmetic genus zero and is smooth and rational; see [8], §II.11.

The divisor $D_{\text{red}} = \sum_i C_i$ of $D$ is reduced by definition, connected by hypothesis, and has arithmetic genus $1 - \chi(\mathcal{O}_{D_{\text{red}}}) = 1 - 1 + 0 = 0$ (note that reduced and connected implies that $H^0(\mathcal{O}_{D_{\text{red}}}) = k$). Hence, again by [8], §II.11, this divisor is a tree of smooth rational curves. $\square$

**Example 2.6.** Let $D = \overline{\overline{2}}\overline{\overline{-3}}$ be a $(-2)$-curve and a $(-3)$-curve intersecting transversally at two points. Then $D$ is easily seen to be negative definite. But $D$ is not a tree and, in particular, not Jacobi rigid.

**Example 2.7.** Consider

$$D = 2Z + 2A + 2B + 2C = \begin{array}{ccc}
-1 & \cdot & \cdot \\
\cdot & -3 & \cdot \\
\cdot & \cdot & -3
\end{array},$$

where all curves are rational with $A^2 = B^2 = C^2 = -3$ and the central curve $Z^2 = -1$. An easy computation yields that $D$ is negatively closed. Put $D' := D_{\text{red}}$. The decomposition sequence for $D = D' + D'$ shows that $H^1(\mathcal{O}_D) = H^1(\mathcal{O}_{D'}(-D'))$ because $\mathcal{O}_{D'}$ is rigid and well-connected. The decomposition sequence for $D' = (A + B + C) + Z$, twisted by $-D'$, is

$$\mathcal{O}_{A+B+C}(-Z - D') \hookrightarrow \mathcal{O}_{D'}(-D') \twoheadrightarrow \mathcal{O}_Z(-D').$$

Its cohomology sequence ends with $H^1(\mathcal{O}_{D'}(-D')) \twoheadrightarrow H^1(\mathcal{O}_Z(-D')) = k$ because $Z.(-D') = Z.(-A - B - C - Z) = -2$. Therefore we have $H^1(\mathcal{O}_D) = H^1(\mathcal{O}_{D'}(-D')) \neq 0$. Hence $D$ is a negatively closed divisor which is a rational tree but not Jacobi rigid.

See also Example 6.15 for a negative definite divisor that is a tree but not Jacobi rigid.

**2.2. Properties of 1-decomposable divisors.** The results in this subsection are not used in the numerical criterion for $(-n)$-divisors (Theorem 3.1), and the reader may jump ahead to § 2.3.

**Lemma 2.8.** Every 1-decomposable divisor is a tree.

*Proof.* Note that 1-decomposable divisors are automatically connected, so we only need to show that there are no cycles. Let $D = C_1 + \cdots + C_m$ be a 1-decomposition of $D$. For a contradiction, assume that $C_{i_1}, \ldots, C_{i_k}$ is a cycle, where $i_1 < i_2 < \cdots < i_k$. Without loss of generality, the curve $C_{i_1}$ does not occur among the $C_j$ with $j > i_1$. The inequality $1 = C_{i_1}.(C_{i_1+1} + C_{i_1+2} + \cdots + C_m) \geq C_{i_1}.(C_{i_2} + C_{i_3} + \cdots + C_{i_k}) \geq 2$ is the desired contradiction. $\square$
Lemma 2.9. If $C$ is a multiple curve in a 1-decomposable divisor $D$, then $C^2 \leq 0$ with the single exception when $D = 2C$ and $C^2 = 1$.

Proof. Given a 1-decomposition $D = C_1 + \cdots + C_m$, we put $D' := C_{i+1} + \cdots + C_m$ with $i$ minimal such that $C_i = C$. By hypothesis, $C \preceq D'$.

Denote by $w \geq 2$ the multiplicity of $C$ in $D$, so that $C$ has multiplicity $w-1$ in $D'$. Next, denote by $v := C.(D'-(w-1)C)$ the valency of $C$ in $D'$, that is, the number of curves in $D'$ (with multiplicities) intersecting $C$. Then $1 = C.D' = (w-1)C^2 + v$ by the definition of 1-decomposition, whence $C^2 = (1-v)/(w-1)$. Thus, either $C^2 \leq 0$, or $v = 0$.

However, if $v = 0$, then $C^2 = 1$ and $w = 2$. As $D'$ is 1-decomposable and hence connected, $v = 0$ implies $D' = C$. This enforces $D = 2C$, lest we get a contradiction

$$1 = C_{i-1}.(C_1 + \cdots + C_m) = C_{i-1}.(C + D') = C_{i-1}.2C. \quad \square$$

Conjecture 2.10. Every 1-decomposable divisor is 1-connected.

Remark 2.11. This conjecture could more generally be seen as a purely combinatorial statement about finite weighted graphs. Note that there are graphs which cannot be realized on any surface as a divisor. In particular, Example 7.5 is a reduced tree (hence 1-decomposable). For dual graphs of rational divisors, it follows from Lemmas 3.7, 3.5 and 3.9. Hence the conjecture can be reduced to showing that if $D$ is a 1-decomposable graph on $X$, then there is a 1-decomposable rational graph $D'$ on some surface $X'$.

2.3. On properties of divisors. The multitude of properties of divisors can be overwhelming. In this short subsection we attempt to categorize them.

Definition 2.12. A property $(P)$ of effective divisors is said to be

- closed if, whenever $0 \prec D' \prec D$ and $D$ satisfies $(P)$, $D'$ also satisfies $(P)$;
- open if, whenever $D \prec D''$, supp$(D) = $ supp$(D'')$ and $D$ satisfies $(P)$, $D''$ also satisfies $(P)$;
- numerical if, whenever $D = c_1C_1 + \cdots + c_mC_m$ is the curve decomposition of a divisor satisfying $(P)$ and $D' = c'_1C'_1 + \cdots + c'_mC'_m$ with $C_i.C_j = C'_i.C'_j$ for all $i, j$, $D'$ also satisfies $(P)$;
- birational if it is preserved under contractions and blow-ups, that is, for every blow-up $\pi: X' \to X$ of a point on $X$, a divisor $D \subset X$ satisfies $(P)$ if and only if $\pi^*D \subset X'$ satisfies $(P)$;
- homological if it depends only on the graded vector space Ext$^\bullet_X(\mathcal{O}_D, \mathcal{O}_D)$.

Remark 2.13. The topological terminology comes from viewing the set of effective divisors on $X$ as a partially ordered set under $\preceq$ and giving it the Alexandrov topology. The notion ‘open’ makes sense only when, given a divisor $D$, we restrict ourselves to the subspace of divisors supported on $D$.

Example 2.14. The properties of being rigid, Jacobi rigid and rigid as a subscheme are closed (by Lemma 2.2), and the property of being negative definite is closed by definition.

A property is open and closed if it depends only on the reduced divisor, that is, on the underlying curve configuration. Examples are negative definiteness (by definition) and rigidity on exceptional loci of rational singularities.
The properties of being well-connected, rigid, a \((-n)\)-divisor, or spherelike are homological. Every homological property is birational.

Remark 2.15. A property \((P)\) is numerical if it depends only on the intersection matrix and the coefficients of the curves making up \(D\) (up to permutation). This is different from requiring that \((P)\) depends only on the numerical class \([D] \in \text{NS}(X)\) because this matrix and these coefficients do not determine the classes \([C]\) of curves in \(D\). Equivalently, \((P)\) can be checked on the weighted (by multiplicities) dual intersection graph of \(D\). By Theorem 3.1, the property of being a \((-n)\)-divisor is numerical when restricted to rational divisors.

See Remark 6.5 for a naturally occurring property which is numerical but not birational. By contrast, the following proposition shows that all the properties introduced in §1 are birational.

Proposition 2.16. The following properties of effective divisors on surfaces are birational: negative definite, negatively closed, negatively filtered, \((-n)\)-divisor, rigid, rigid as a subscheme, Jacobi rigid, 1-decomposable, 1-connected, well-connected, connected, rational.

Proof. We can assume that \(\pi : \tilde{X} \to X\) is the blow-up at a single point, \(D\) is an effective divisor on \(X\), and \(\tilde{D} = \pi^*D\) is its total transform on \(\tilde{X}\). It suffices to show that a property holds for \(D\) if and only if it holds for \(\pi^*D\).

The claim is obvious for the properties of being connected and rational.

For homological properties (rigidity, well-connectedness, and the property of being a \((-n)\)-divisor), we use the fact that \(\pi^*\) is fully faithful:

\[
\text{Ext}^i(\mathcal{O}_D, \mathcal{O}_D) = \text{Ext}^i(\pi^*\mathcal{O}_D, \pi^*\mathcal{O}_D) = \text{Ext}^i(\mathcal{O}_{\tilde{D}}, \mathcal{O}_{\tilde{D}})
\]

for all \(i\). Since \(\pi^*\mathcal{O}_X = \mathcal{O}_{\tilde{X}}\), we also find that the property of being Jacobi rigid is birational. By Lemma 1.6, the rigidity as a subscheme is birational too.

Now let \(D = C_1 + \cdots + C_m\) be a 1-decomposition of \(D\). Write \(C'_i\) for the strict transform of \(C_i\). We get a 1-decomposition of \(\tilde{D}\) by replacing each \(C_i\) by its total transform \(\pi^*C_i\), where the exceptional divisor comes first. More explicitly, if the blown-up point is not on \(C_i\), then \(\pi^*C_i = C'_i\) is unchanged; if the point is on \(C_i\), then take \(\pi^*C_i = E + C'_i\). This is again a 1-decomposition since \(E^2 = -1\) and \(C'^2_i = C_i^2 - 1\). Conversely, a 1-decomposition \(\tilde{D} = C'_1 + \cdots + C'_m\), can always be rearranged in such a way that \(E\) is in front of a curve \(C'_i\) with \(C'_iE = 1\) (since the intersection of \(E\) with curves untouched by the blow-down is always zero). Hence we can reverse the above process by replacing each \(C'_i\) by \(C_i\) and dropping all the \(E_s\).

A similar procedure applies to negative filtrations.

Regarding negatively closed divisors, we use some standard facts about intersection products on \(X\) and \(\tilde{X}\). For any divisors \(A, B\) on \(X\) we have \(\pi^*A.\pi^*B = A.B\). This shows immediately that if \(\tilde{D}\) is negatively closed, then so is \(\tilde{D}\). Conversely, for divisors \(B\) on \(X\) and \(\tilde{A}\) on \(\tilde{X}\) we have \(B.\pi_*\tilde{A} = \pi^*B.\tilde{A}\). If \(D\) is negatively closed and \(0 \leq \tilde{A} \leq \tilde{D}\), then we use this fact with \(B := \pi_*\tilde{A} \leq D\) and obtain that

\[
0 > (\pi_*\tilde{A})^2 = (\pi^*\pi_*\tilde{A}).\tilde{A} = (\tilde{A} + (\tilde{A}.E)E).\tilde{A} = \tilde{A}^2 + (\tilde{A}.E)^2 \geq \tilde{A}^2,
\]

showing that \(\tilde{D}\) is negatively closed.
A divisor $D$ is negative definite if all its multiples $kD$ with $k > 0$ are negatively closed. Hence the last paragraph also proves that the property of being negative definite is birational.

Finally, for 1-connectedness, one can argue as for negative closedness. If $\tilde{D}$ is 1-connected, then so is $D$:

$$A.(D - A) = \pi^* A.\pi^*(D - A) = \tilde{A}.(\tilde{D} - \tilde{A}) \geq 1.$$ 

Conversely, if $D$ is 1-connected and $0 \leq \tilde{A} \leq \tilde{D}$, then

$$1 \leq \pi^* \tilde{A}.(D - \pi^* \tilde{A}) = \pi^* \tilde{A} .\pi^*(\tilde{D} - \tilde{A}) = \pi^* \pi_* \tilde{A}.(\tilde{D} - \tilde{A}) = (\tilde{A} + (\tilde{A}.E)E).(\tilde{D} - \tilde{A}) = \tilde{A}.(\tilde{D} - \tilde{A}) - (\tilde{A}.E)^2,$$

using the fact that $\tilde{D}.E = \pi^* D.E = 0$. Hence,

$$\tilde{A}.(\tilde{D} - \tilde{A}) \geq 1 + (\tilde{A}.E)^2 \geq 1. \quad \square$$

§ 3. Numerical characterization of $(-n)$-divisors

In this section we prove a characterization of well-connected rigid divisors in terms of intersection numbers. This is useful because it turns a homologically defined notion into something much easier to test in practice. The criterion is about the existence of particularly nice curve decompositions. In examples, this is the easiest way to check that a divisor is a $(-n)$-divisor.

For the convenience of the reader, we recall the notions appearing in the theorem. Let $D = C_1 + \cdots + C_m$ be a curve decomposition of an effective divisor. The sequence of curves $C_1, \ldots, C_m$ is called

- a 1-decomposition if $C_i.(C_{i+1} + \cdots + C_m) = 1$ for all $i = 1, \ldots, m - 1$;
- a negative filtration if $C_i.(C_i + \cdots + C_m) < 0$ for all $i = 1, \ldots, m$.

Theorem 3.1. The following conditions on an effective divisor $D$ are equivalent:

1. $D$ is well-connected and rigid or, equivalently, $D$ is a $(-n)$-divisor for some $n > 0$;
2. $D$ is rational, 1-decomposable and negatively filtered.

Corollary 3.2. An effective rational divisor $D$ is a $(-n)$-divisor if and only if $D$ is 1-decomposable and negatively filtered with $D.K_X = n - 2$.

Proof. This follows from Riemann–Roch:

$$1 = \chi(O_D) = -\frac{1}{2}(D^2 + D.K). \quad \square$$

Remark 3.3. This corollary works well when computing spherelike examples, where we can check whether $D.K_X = 0$, which is easier to calculate than $D^2$.

The conditions ‘$D$ is rational’ and ‘$D.K = n - 2$’ are not numerical in the sense of Definition 2.12. This is unavoidable in view of the existence of negative non-rational curves, as in Example 1.7.

There cannot be a purely numerical characterization of (non-rational) rigid divisors, at least not with the properties in § 2.3. In particular, Example 2.7 contains a rational negatively closed tree which is a non-rigid divisor.
Example 3.4. The divisor

\[ D = B + 2C + C' + E = \begin{array}{c}
\text{2} \\
\text{2} \\
\text{3}
\end{array}, \]

where \( B^2 = -3 \), \( E^2 = -1 \) and \( C^2 = C'^2 = -2 \), is spherelike (hence it is a \((-2)\)-divisor) by Corollary 3.2: the sequence \( C, B, C', C, E \) is both a 1-decomposition and a negative filtration, and the conditions \( D.K = 0 \) or \( D^2 = -2 \) are easy to check.

The proof of Theorem 3.1 uses a series of lemmas.

Lemma 3.5. Every Jacobi rigid well-connected divisor is 1-connected.

Proof. Suppose that \( 0 \prec A \prec D \) and \( B := D - A \). We need to show that \( A.B \geq 1 \). Since \( D \) is Jacobi rigid, so are \( A \) and \( B \) by Lemma 2.2. Thus \( H^1(\mathcal{O}_A) = H^1(\mathcal{O}_B) = 0 \) and, in particular, \( \chi(\mathcal{O}_A) \geq 1 \) and \( \chi(\mathcal{O}_B) \geq 1 \). As to \( D \), we know that \( H^0(\mathcal{O}_D) = k \) and \( H^1(\mathcal{O}_D) = 0 \), whence \( \chi(\mathcal{O}_D) = 1 \). By Lemma 1.1,

\[ A.B = \chi(\mathcal{O}_A) + \chi(\mathcal{O}_B) - \chi(\mathcal{O}_D) \geq 1 + 1 - 1 = 1. \]

\[ \square \]

Lemma 3.6. Every negatively filtered divisor is rigid as a subscheme.

Proof. By Remark 1.3, if \( D = C_1 + \cdots + C_m \) is a negative filtration, then \( H^0(\mathcal{O}_D(D)) = \cdots = H^0(\mathcal{O}_{C_m}(C_m)) = 0 \) (the last equality holds because \( C_m^2 < 0 \)). \( \square \)

The following lemma contains a weakened version of Conjecture 2.10.

Lemma 3.7. Every 1-decomposable rational divisor \( D \) is well-connected and Jacobi rigid, that is, \( H^0(\mathcal{O}_D) = k \) and \( H^1(\mathcal{O}_D) = 0 \).

Proof. This follows from Remark 1.12. Indeed, if \( D = C_1 + \cdots + C_m \) is a 1-decomposition, then \( H^\bullet(\mathcal{O}_D) = \cdots = H^\bullet(\mathcal{O}_{C_m}) = k \). \( \square \)

Lemma 3.8. Let \( D = C_1 + \cdots + C_m \) be a curve decomposition of a rational divisor. Then \( -\sum_i C_i^2 = D.K_X + 2m \).

Proof. By Riemann–Roch for \( C_i \cong \mathbb{P}^1 \), we have \( C_i^2 + 2 = -C_i.K_X \). Adding these up, we get \( \sum C_i^2 + 2m = -D.K_X. \) \( \square \)

Lemma 3.9. Every Jacobi rigid 1-connected divisor is 1-decomposable.

Proof. By Lemma 2.5, every Jacobi rigid divisor is rational.

Given a 1-connected divisor \( D \), we have \( C.(D - C) \geq 1 \) for all curves \( C \) in \( D \). We claim that equality holds for some \( C \). Indeed, assuming that \( C.(D - C) \geq 2 \) throughout, we get the following inequality by summing over all curves \( C \prec D \) and using Lemma 3.8 and Riemann–Roch:

\[ 2m \leq \sum_{C \prec D} C.(D - C) = \sum_{C \prec D} C.D - \sum_{C \prec D} C^2 = D^2 + D.K + 2m = -2\chi(\mathcal{O}_D) + 2m. \]

This achieves the desired contradiction because \( \chi(\mathcal{O}_D) = h^0(\mathcal{O}_D) > 0 \) in view of the Jacobi rigidity of \( D \). Hence there is a curve \( C_1 \) such that \( C_1.(D - C_1) = 1 \).
For the induction step, observe that Jacobi rigidity passes from $D$ to $D' = D - C_1$ by Lemma 2.2. The only ingredient missing from the above argument is that $D'$ stays 1-connected. To see this, assume that the divisor $D = D' + C_1$ is 1-connected with $C_1.D' = 1$, and let $D' = A + (D' - A)$ with $0 < A < D'$. Then we can write $D' + C_1$ as a sum of effective subdivisions in two ways: $D' + C_1 = (A + C_1) + (D' - A) = A + (D' - A + C_1)$. Thus,

$$1 \leq (A + C_1).(D' - A) = A.(D' - A) - C_1.A + C_1.D',
1 \leq A.(D' - A + C_1) = A.(D' - A) + C_1.A.$$  

We deduce that $1/2 \leq A.(D' - A)$. Hence $D'$ is 1-connected. □

Remark 3.10. The proof of this lemma also shows that every partial 1-decomposition $D = C_1 + \cdots + C_l + D'$ of a Jacobi rigid 1-connected divisor can be completed to a 1-decomposition.

Proof of Theorem 3.1. (1) $\Rightarrow$ (2). Being a $(-n)$-divisor for some $n > 0$, the divisor $D$ is well-connected and rigid by definition. Thus $D$ is 1-connected by Lemma 3.5 and rational by Lemma 2.5. Then $D$ has a 1-decomposition by Lemma 3.9. As $D$ is rigid, it is negatively filtered by Corollary 2.3 and Lemma 1.4.

(2) $\Rightarrow$ (1). $D$ is rigid and well-connected by Lemma 3.6 and Lemma 3.7. □

§ 4. Minimal $(-n)$-divisors

4.1. Modifying $(-n)$-divisors: blow-up and blow-down. Consider a blow-up $\pi: \tilde{X} \to X$ at a point $P \in X$, with exceptional $(-1)$-curve $E$, and let $D$ be a $(-n)$-divisor on $X$ for some $n > 0$. As $\pi^*:\mathcal{D}^b(X) \to \mathcal{D}^b(\tilde{X})$ is fully faithful, $\text{Hom}^*(\pi^*\mathcal{O}_D, \pi^*\mathcal{O}_D) = \text{Hom}^*(\mathcal{O}_D, \mathcal{O}_D)$. Hence $\pi^*D$ stays a $(-n)$-divisor.

If $P \notin \text{supp}(D)$, then $\pi^*D$ is the same curve configuration as $D$. Otherwise, denote by $C_1, \ldots, C_k$ the curves in $D$ containing $P$, and let $n_1, \ldots, n_k$ be their multiplicities in $D$, respectively. Then the exceptional curve $E$ occurs in $\pi^*D$ with multiplicity $n_1 + \cdots + n_k$, and the self-intersection numbers of the strict transforms are $C_i^2 = C_i^2 - 1$ since $C_i$ is smooth and contains $P$. For later reference, we record the contraction criterion in the following proposition. In this situation, we say that the $(-n)$-divisor can be blown down.

Proposition 4.1. Let $D$ be a $(-n)$-divisor on $X$, $E \prec D$ a $(-1)$-curve with $E.D = 0$, and $\pi: X \to X'$ the contraction of $E$. Then $\pi(D)$ is a $(-n)$-divisor on $X'$.

4.2. Modifying $(-n)$-divisors: twisting spherical divisors. In some degree analogous to the contraction of $(-1)$-curves in a $(-n)$-divisor, there is a construction for $(-2)$-curves, using spherical twists. We give the definitions first, and then proceed to explain why they make sense.

Definition 4.2. A spherical component of a $(-n)$-divisor $D$ is a spherical sub-divisor $A \prec D$ such that $D - A$ is a $(-n)$-divisor.

A is a spherical component of $D$ if $A$ is a spherical divisor and a spherical component of $D$. In this case we say that $D$ is obtained by twisting $A$ on to $D - A$ and $D - A$ is obtained by twisting $A$ off $D$. 
There is a simple way of checking whether a given \((-2)\)-curve is a spherelike component (and hence a spherical component).

Lemma 4.3. If \(D\) is a \((-n)\)-divisor and \(C\) is a \((-m)\)-curve with \(m \geq 2\) and \(D.C = 1\), then \(D + C\) is a \((-n - m + 2)\)-divisor.

In particular, if \(C\) is a \((-2)\)-curve, then \(D + C\) stays a \((-n)\)-divisor.

Proof. This follows from the decomposition sequences

\[
\mathcal{O}(D) \hookrightarrow \mathcal{O}_{D+C} \rightarrow \mathcal{O}_D \quad \text{and} \quad \mathcal{O}_D(D) \hookrightarrow \mathcal{O}_{D+C}(D + C) \rightarrow \mathcal{O}_C(D + C)
\]

by taking cohomology and using the equalities \(C\cdot(D) = -1\) and \(C\cdot(D + C) = 1 - m \leq -1\) respectively. \(\Box\)

We turn to some easy and general properties of spherelike components.

Lemma 4.4. Let \(A < D\) be a spherelike component. Then

1. \(\text{Hom}(\mathcal{O}(D - A), \mathcal{O}_A(A - D)) = \mathbb{k}[-1]\).
2. \(\text{Hom}^\bullet(\mathcal{O}(D - A), \mathcal{O}_A(A - D)) = \mathbb{k}[-1]\).

Proof. (1) Write \(B := D - A\). Then \(-n = D^2 = (A + B)^2 = A^2 + 2A.B + B^2 = -2 + 2A.B - n\) since \(A\) is a spherelike component, and hence \(A.B = 1\). Next, the decomposition sequence yields that \(\text{Hom}^\bullet(\mathcal{O}(A - B)) \rightarrow \text{Hom}^\bullet(\mathcal{O}_D) \rightarrow \text{Hom}^\bullet(\mathcal{O}_B)\), and because \(\text{Hom}^\bullet(\mathcal{O}_D) \cong \text{Hom}^\bullet(\mathcal{O}_B) \cong \mathbb{k}\), the second map is an isomorphism induced by restriction of global sections. Hence \(\text{Hom}^\bullet(\mathcal{O}(A - D)) = 0\).

(2) We turn to \(\text{Hom}^\bullet(\mathcal{O}_B, \mathcal{O}_A(-B))\). For this, apply \(\text{Hom}^\bullet(\cdot, \mathcal{O}_A(-B))\) to \(\mathcal{O}(-B) \hookrightarrow \mathcal{O} \hookrightarrow \mathcal{O}_B\) and get

\[
\text{Hom}^\bullet(\mathcal{O}_A) \leftarrow \text{Hom}^\bullet(\mathcal{O}_A(-B)) \leftarrow \text{Hom}^\bullet(\mathcal{O}_B, \mathcal{O}_A(-B)).
\]

The left-hand term is isomorphic to \(\mathbb{k}\), and the middle term vanishes by (1), so the right-hand term is isomorphic to \(\mathbb{k}[-1]\), that is,

\[
\text{Ext}^1(\mathcal{O}_B, \mathcal{O}_A(-B)) = \mathbb{k} \quad \text{and} \quad \text{Hom}(\mathcal{O}_B, \mathcal{O}_A(-B)) = \text{Ext}^2(\mathcal{O}_B, \mathcal{O}_A(-B)) = 0. \quad \Box
\]

We justify the terminology in Definition 4.2. For every \(F \in \mathcal{D}^b(X)\), the twist functor \(\mathcal{T}_F: \mathcal{D}^b(X) \rightarrow \mathcal{D}^b(X)\) is defined on objects \(A\) by the triangles \(\text{Hom}^\bullet(F, A) \otimes F \rightarrow A \rightarrow \mathcal{T}_F(A)\), that is, \(\mathcal{T}_F(A)\) is the cone of the canonical evaluation morphism; see §2.1 in [3] for how these cones become functorial in an appropriate setting, for example, for \(\mathcal{D}^b(X)\). By Proposition 4.6, spherical components can be twisted off \((-n)\)-divisors, leaving smaller \((-n)\)-divisors.

Proposition 4.5 ([7], §8.1, [3], Lemma 3.1). Given two divisors \(D, D'\) with \(D\) effective, the functor \(\mathcal{T}_{\mathcal{O}_D(D')}\) is an autoequivalence of \(\mathcal{D}^b(X)\) if and only if \(D\) is spherical. In this case, \(\mathcal{T}_{\mathcal{O}_D(D')}\) is called a spherical twist functor.

Proposition 4.6. Let \(A < D\) be a spherical component. Then

1. \(\text{Hom}(\mathcal{O}(A - D), \mathcal{O}_D) = \mathbb{k}\) and
2. \(\mathcal{T}_{\mathcal{O}_A(A - D)}(\mathcal{O}_D) \cong \mathcal{O}_{D - A}\).
Proof. We apply $\text{Hom}^\bullet(\cdot, \mathcal{O}_A)$ to $\mathcal{O}(-A) \hookrightarrow \mathcal{O}(B) \twoheadrightarrow \mathcal{O}_D(B)$, where $B = D - A$, obtaining the triangle

$$H^\bullet(\mathcal{O}_A(A)) \leftarrow H^\bullet(\mathcal{O}_A(-B)) \leftarrow \text{Hom}^\bullet(\mathcal{O}_D, \mathcal{O}_A(-B)).$$

The middle term vanishes by Lemma 4.4, (1). Now $A$ is a spherical divisor, that is, $H^\bullet(\mathcal{O}_A(A)) = \mathbb{k}[-1]$ and $\mathcal{O}_A(K) \cong \mathcal{O}_A$, and Serre duality yields (1):

$$\mathbb{k} \cong H^\bullet(\mathcal{O}_A(A))[1] \cong \text{Hom}^\bullet(\mathcal{O}_D, \mathcal{O}_A(-B))[2] \cong \text{Hom}^\bullet(\mathcal{O}_A(-B), \mathcal{O}_D)^*.$$

Now (2) follows since $T_{\mathcal{O}_A(-B)}(\mathcal{O}_D)$ is defined by the triangle

$$\text{Hom}^\bullet(\mathcal{O}_A(-B), \mathcal{O}_D) \otimes \mathcal{O}_A(-B) \rightarrow \mathcal{O}_D \rightarrow T_{\mathcal{O}_A(-B)}(\mathcal{O}_D),$$

which reduces to the short exact sequence $\mathcal{O}_A(-B) \hookrightarrow \mathcal{O}_D \twoheadrightarrow \mathcal{O}_B$. □

**Corollary 4.7.** A well-connected rigid divisor $D$ is spherical if and only if it consists entirely of $(-2)$-curves.

**Proof.** ($\Rightarrow$) Assume that $D$ is spherical and let $C \leq D$ be any curve. The restriction $\mathcal{O}_D \rightarrow \mathcal{O}_C$ induces

$$\mathcal{O}_D = \mathcal{O}_D \otimes \omega_X \rightarrow \mathcal{O}_C \otimes \omega_X = \mathcal{O}_C(C.K_X).$$

Hence $C.K_X \geq 0$ and, by summing,

$$0 \leq \sum_{C \leq D} C.K_X = D.K_X = 0$$

by Lemma 3.8. Therefore $C.K_X = 0$ and each $C \leq D$ is a $(-2)$-curve.

($\Leftarrow$) Let $D$ be well-connected and rigid, consisting of $(-2)$-curves. The claim holds if $D$ is a single curve. We proceed inductively. By Theorem 3.1, $D$ is 1-decomposable; let $C \leq D$ be a curve with $C.D = 1$ and assume that $D - C$ is spherical. By Lemma 4.3, $D$ is spherelike. The decomposition sequence for $D$, tensored with $\mathcal{O}(K_X)$, is $\mathcal{O}_C(C - D) \hookrightarrow \mathcal{O}_D(K_X) \twoheadrightarrow \mathcal{O}_{D-C}$, as both $\mathcal{O}_{D-C}$ and $\mathcal{O}_C(C - D)$ are invariant by induction. By Lemma 4.4, (2), $\text{Ext}^1(\mathcal{O}_{D-C}, \mathcal{O}_C(C - D)) \cong \mathbb{k}$, and thus $\mathcal{O}_D(K_X) \cong \mathcal{O}_D$. □

**Remark 4.8.** Consider an abstract setting: if $S_1, \ldots, S_n$ are $d$-Calabi–Yau objects ($d$-CY objects) in a triangulated category $\mathcal{D}$, that is, $\text{Hom}(S_i, \cdot) \cong \text{Hom}(\cdot, S_i[d])^*$, will an object generated by $S_1, \ldots, S_n$ be $d$-CY as well?

Corollary 4.7 provides a very partial positive answer for a tree of $(-2)$-curves $D = C_1 + \cdots + C_n$. Then $\mathcal{O}_D \in \mathcal{D}^b(X)$ is a 2-spherical object and is generated by the 2-CY objects $\mathcal{O}_{C_1}, \mathcal{O}_{C_2}(-1), \ldots, \mathcal{O}_{C_n}(-1)$.

It is a separate question whether the subcategory $\mathcal{D}^b_D(X)$ of objects supported on $D$ is a 2-CY category, that is, its Serre functor is the shift by 2. This holds if $D$ is of ADE type; see Corollary 6.9.
4.3. Curvelike decompositions.

Definition 4.9. A \((-n)\)-divisor \(D\) is said to be minimal if no \((-1)\)-curves can be contracted from \(D\) and no \((-2)\)-curves can be twisted off \(D\), that is, if
- \(D.C \neq 0\) for all \((-1)\)-curves \(C\) in \(D\), and
- \(D.C \neq -1\) for all \((-2)\)-curves \(C\) in \(D\).

Remark 4.10. Any negative curve is a minimal \((-n)\)-divisor for \(n > 1\). If a \((-n)\)-divisor \(D\) can be obtained from a \((-n)\)-curve by blow-ups or spherical twists, then \(D\) is called essentially a \((-n)\)-curve.

Lemma 4.11. A \((-n)\)-divisor is minimal if and only if no \((-1)\)-curve in \(D\) can be contracted and no spherical component can be twisted off.

Proof. The direction (\(\Leftarrow\)) is clear. For the converse, let \(D\) be a \((-n)\)-divisor having a spherical component \(A\) but no contractible \((-1)\)-curves. We need to show that \(D\) is not a minimal \((-n)\)-divisor, that is, some \((-2)\)-curve can be twisted off.

Assume the opposite: \(D.C \neq -1\) for all \((-2)\)-curves \(C < D\). Then, as \(D\) is 1-connected, we have \((D - C).C \geq 2\) or, equivalently, \(D.C \geq 0\). In particular, by Corollary 4.7, the spherical component \(A\) consists entirely of \((-2)\)-curves, whence \(A.D \geq 0\). This is absurd as \(A.D = A^2 + A.(D - A) = -1\). □

The above proof (with \(D = A\)) works for the following statement (note that every spherical divisor consists only of \((-2)\)-curves).

Corollary 4.12. Every spherical divisor is essentially a \((-2)\)-curve.

Definition 4.13. Let \(D\) be a \((-n)\)-divisor for some \(n > 1\). A curvelike decomposition of \(D\) is a sum \(D = D_1 + \cdots + D_m\) of effective subdivisors such that all the \(D_i\) are \((-n_i)\)-divisors with \(-2 \geq -n_i \geq -n\) for all \(i\) and \(D_.(D_{i+1} + \cdots + D_m) = 1\) for \(i < m\).

To describe curvelike decompositions, we coin a term for the divisors obtained by successively blowing up a \((-n)\)-curve:

\[\underbrace{-n} - \underbrace{\cdots - \underbrace{-2} - \underbrace{-1}}\]

is called a simple \((-n)\)-chain.

We assume that such a simple \((-n)\)-chain always contains the \((-n - 1)\)-curve but not necessarily any \((-2)\)-curve.

Theorem 4.14. Let \(D\) be a \((-n)\)-divisor on \(X\) which is not the pullback of a \((-n)\)-curve. If \(n \geq 2\), then there is a contraction \(\pi: X \rightarrow Y\) to a smooth surface \(Y\) with \(D = \pi^*D'\) and a 1-decomposition \(D' = C_1 + \cdots + C_l + \cdots + C_m\) inducing a curvelike decomposition \(D' = A + B\), where \(A = C_1 + \cdots + C_l\) and
- either \(A\) is a \((-k)\)-curve with \(-2 \geq -k \geq -n\) and \(B\) is a \((-n + 2 - k)\)-divisor,
- or \(A\) is a simple \((-n)\)-chain with \(C_1^2 = -n - 1\) and \(C_2^2 = -1\) and \(B\) is a \((-2)\)-divisor.

Remark 4.15. The divisor in Example 5.11 admits only those curvelike decompositions \(D = A + B\) where both \(A\) and \(B\) are chains. Therefore the second case in the theorem above is necessary.
This theorem yields the curvelike decomposition $D = A_1 + \cdots + A_k$ in Theorem B by applying it iteratively.

**Corollary 4.16.** Every minimal $(-n)$-divisor which is not a curve has a non-trivial curvelike decomposition.

**Remark 4.17.** Let $D = \pi^*A + \pi^*B$ be a curvelike decomposition obtained from Theorem 4.14 with $A$ a simple $(-n)$-chain. In particular, there is a $(-1)$-curve $C$ in $A$. As $A$ is built from a 1-decomposition, $C.D = C.A + C.B = 0 + 1 = 1$. In particular, $D$ is not the pullback of a divisor from the surface where $C$ is contracted.

Before turning to the proof of Theorem 4.14, we need the following lemma.

**Lemma 4.18.** Let $D$ be a negatively closed divisor with a 1-decomposition $D = C_1 + \cdots + C_m$. Suppose that for some $l \leq m$, all the $A_i = C_1 + \cdots + C_i$ are reduced trees, where $i = 1, \ldots, l$. Then

$$A_i^2 + 1 = A_i.D, \quad A_i(D - A_i) = 1 \quad \text{and} \quad -1 \geq A_i^2 \geq D^2 - 1.$$  

**Proof.** We add the 1-decomposition pieces $1 = C_k.(D - C_1 - \cdots - C_k)$ up to $i \leq l$:

$$i = \sum_{k=1}^{i} C_k.(D - C_1 - \cdots - C_k) = \sum_{k} C_k.D - \sum_{k' < k} C_{k'}.C_k - \sum_{k} C_k^2$$

$$= A_i.D - A_i^2 + \sum_{k' < k} C_{k'}.C_k = A_i.D - A_i^2 + i - 1,$$

where the final sum expands to $i - 1$ because $A_i$ is a reduced tree with $i$ vertices. This yields the first formula. The second is obtained by plugging $D = (D - A_i) + A_i$ into the first. From this, we get

$$0 > (D - A_i)^2 = D.(D - A_i) - A_i.(D - A_i) = D^2 - D.A_i - 1 = D^2 - A_i^2 - 2$$

as $D$ is negatively closed. For the same reason, also $A_i^2 < 0$ and, combining these two, the last statement follows. \(\Box\)

In Definition 4.13 and Theorem 4.14, we exclude $(-1)$-divisors since these have no non-trivial curvelike decompositions by the following proposition.

**Proposition 4.19.** Every $(-1)$-divisor is essentially a $(-1)$-curve.

**Proof.** Let $D = C_1 + \cdots + C_m$ be a 1-decomposition of a $(-1)$-divisor. Then, by Lemma 4.18, $C_1$ is either a $(-1)$-curve or a $(-2)$-curve. As $C_1.(D - C_1) = 1$, this curve can be blown down or twisted off respectively. \(\Box\)

**Proof of Theorem 4.14.** Without loss of generality, we may assume that there are no $(-1)$-curves $C$ in $D$ that can be contracted, that is, with $D.C = 0$. Let $D = C_1 + \cdots + C_m$ be a 1-decomposition of the $(-n)$-divisor $D$. We set $A_i := C_1 + \cdots + C_i$ and $B_i := D - A_i$, so that $C_i.B_i = 1$. By Lemma 4.18, $-1 \geq C_i^2 = A_i^2 \geq D^2 - 1 = -n - 1$.

There are three cases:

- the case $C_1^2 = -1$ is excluded since it could be contracted (as $D.C_1 = 0$);
- $-2 \geq C_1^2 \geq -n$, and $D = C_1 + (D - C_1)$ forms a curvelike decomposition: $D - C_1$ is well-connected and rigid by Lemmas 2.2 and 3.7, so we are done;
- $C_1^2 = -n - 1$, and then $A_1.B_1 = 1$ and $B_1^2 = -1$. 

So we only have to deal with the last case, where a simple \((-n\))-chain will be constructed inductively. By Remark 3.10, we may assume in what follows that there are no \((-2\))-curves \(C\) in \(D\) with \(C.(D - C) = 1\). Otherwise we could start the 1-decomposition of \(D\) with such a curve (which could be twisted off \(D\)).

To make the induction step, assume that for some \(i\) we have

- \(C_i^2 = -n - 1\) and \(C_2^2 = \cdots = C_i^2 = -2\), so \(A_i^2 = -n - 1\);
- \(B_i^2 = -1\) and \(A_iB_i = 1\).

In particular, we find that \(i + 1 < m\). Indeed, when \(i + 1 = m\) we have \(B_i = C_m\) and, since \(D.C_m = 0\), we would obtain a contractible \((-1\))-curve \(C_m\) in \(D\), whose existence was excluded above.

First note that by the assumption of the negative closedness of \(D\) we get
\[
0 > (B_i - C_{i+1})^2 = B_i.(B_i - C_{i+1}) - C_{i+1}.(B_i - C_{i+1})
= B_i^2 - B_i.C_{i+1} - 1 = -2 - B_i.C_{i+1},
\]
also using the 1-decomposition and the fact that \(B_i^2 = -1\). It follows that \(B_i.C_{i+1} \geq -1\). Combining this with the equality \(1 = C_{i+1}.(B_i - C_{i+1})\) from the 1-decomposition, we get \(C_{i+1}^2 \in \{-1, -2\}\).

Next we claim that \(A_i.C_{i+1} \in \{0, 1\}\). To see this, note that by the 1-decomposition and the induction hypothesis
\[
A_{i+1}.B_{i+1} = (A_i + C_{i+1}).(B_i - C_{i+1})
= A_i.B_i - A_i.C_{i+1} + C_{i+1}.B_{i+1} = 2 - A_i.C_{i+1}. \tag{*}
\]
As \(D\) is 1-connected by Lemma 3.5, we get \(1 \leq A_{i+1}.B_{i+1} = 2 - A_i.C_{i+1}\), whence \(A_i.C_{i+1} \leq 1\). On the other hand, as \(A_i = (D - C_{i+1}) - B_{i+1}\), we obtain that
\[
A_i.C_{i+1} = (D - C_{i+1}).C_{i+1} - B_{i+1}.C_{i+1} \geq 1 - 1 = 0
\]
since \(D\) is 1-connected.

Case 1. \(A_i.C_{i+1} = 0\). Then we compute that
\[
D.C_{i+1} = B_i.C_{i+1} = B_{i+1}.C_{i+1} + C_{i+1}^2 = 1 + C_{i+1}^2.
\]
If \(C_{i+1}^2 = -1\), then \(C_{i+1}.D = 0\), but if \(C_{i+1}^2 = -2\), then \(C_{i+1}.(D - C_{i+1}) = 1\). Both cases are impossible since \(D\) is minimal by assumption.

Case 2. \(A_i.C_{i+1} = 1\). Recall that \(i + 1 < m\). Therefore we distinguish two subcases.

Subcase 2.1. \(C_{i+1}^2 = -1\). Here \(A_{i+1}^2 = -n\) and \(A_{i+1}.B_{i+1} = 1\) by (*). Hence we get a curvelike decomposition \(D = A_{i+1} + B_{i+1}\).

Subcase 2.2. \(C_{i+1}^2 = -2\). Then \(A_{i+1}^2 = -n - 1\), \(A_{i+1}.B_{i+1} = 1\) and, therefore, \(B_{i+1}^2 = -1\). Since \(A_{i+1}\) does not contain \((-1\))-curves either, we can repeat the whole argument with \(i \mapsto i + 1\).

So eventually we will end up in Subcase 2.1 with \(A_l = C_1 + \cdots + C_l\), where
\[
C_1^2 = -n - 1, \quad C_2^2 = \cdots = C_{l-1}^2 = -2, \quad C_l^2 = -1, \quad \text{and}
(C_1 + \cdots + C_l).C_{i+1} = 1 \quad \text{for} \quad 1 \leq i < l.
\]
We will construct a subdivisor \(A\) of \(A_l\) which is the desired simple \((-n\))-chain. By the second property, there is a unique \(i_1\) such that \(C_{i_1}.C_{i_1} = 1\). Likewise, as
Let Proposition 4.20. cohomology groups of

4.4. Cohomology of rigid.

by Lemma 4.18. Finally, by Lemmas 3.7 and 2.2,

By Remark 3.10, A

The first summand is 0 by minimality while the second is 1 by the 1-decomposition. The same holds for the remaining intersection numbers. Namely, when

The first summand is 0 by minimality while the second is 1 by the 1-decomposition. The same holds for the remaining intersection numbers. Namely, when

We claim that in the triangle

By Remark 3.10, A comes from a 1-decomposition of D. In particular, A.(D − A) = 1 by Lemma 4.18. Finally, by Lemmas 3.7 and 2.2, D − A is well-connected and rigid. □

4.4. Cohomology of (−n)-divisors. This and the following subsections provide applications of Theorem 4.14. Here we again justify our terminology: various cohomology groups of (−n)-divisors coincide with those of (−n)-curves.

Proposition 4.20. Let D be a (−n)-divisor. Then

\[ \text{Hom}^\bullet(\mathcal{O}_D, \mathcal{O}) \cong \mathbb{k}^{n-1}[-2], \quad H^\bullet(\mathcal{O}_D(K)) \cong \mathbb{k}^{n-1}, \]

\[ \text{Hom}^\bullet(\mathcal{O}_D(D), \mathcal{O}) \cong \mathbb{k}[-1], \quad H^\bullet(\mathcal{O}_D(D + K)) \cong \mathbb{k}[-1]. \]

Proof. The claims about \( H^\bullet(\mathcal{O}_D(K)) \) and \( H^\bullet(\mathcal{O}_D(D + K)) \) follow from their left-hand counterparts by Serre duality. Moreover, the statement about \( \text{Hom}^\bullet(\mathcal{O}_D, \mathcal{O}) \) follows from that about \( \text{Hom}^\bullet(\mathcal{O}_D(D), \mathcal{O}) \). To see this, apply \( \text{Hom}^\bullet(\mathcal{O}_D(D), \cdot) \) to \( \mathcal{O} \hookrightarrow \mathcal{O}(D) \twoheadrightarrow \mathcal{O}_D(D) \) and obtain the triangle

\[ \text{Hom}^\bullet(\mathcal{O}_D(D), \mathcal{O}) \rightarrow \text{Hom}^\bullet(\mathcal{O}_D, \mathcal{O}) \rightarrow \text{Hom}^\bullet(\mathcal{O}_D, \mathcal{O}_D). \]

By definition of D as a (−n)-divisor, \( \text{Hom}^\bullet(\mathcal{O}_D, \mathcal{O}_D) \cong \mathbb{k} \oplus \mathbb{k}^{n-1}[-2]. \) Hence

\[ \text{Hom}^\bullet(\mathcal{O}_D(D), \mathcal{O}) = \mathbb{k}[-1] \iff \text{Hom}^\bullet(\mathcal{O}_D, \mathcal{O}) = \mathbb{k}^{n-1}[-2]. \]

To prove that \( \text{Hom}^\bullet(\mathcal{O}_D(D), \mathcal{O}) \cong \mathbb{k}[-1], \) we may assume that D is not the pullback of a smaller (−n)-divisor. By Theorem 4.14 there is a decomposition

\[ D = A + B, \]

where either A is a (−m)-curve with \(-2 \geq m \geq n\), or

\[ A = \begin{array}{c}
\begin{array}{cccc}
2 & \cdots & 2 & -1
\end{array}
\end{array}. \]

We claim that in the triangle

\[ \text{Hom}^\bullet(\mathcal{O}_B(B), \mathcal{O}) \leftarrow \text{Hom}^\bullet(\mathcal{O}_D(D), \mathcal{O}) \leftarrow \text{Hom}^\bullet(\mathcal{O}_A(D), \mathcal{O}) \]
the right-hand term vanishes, so that the remaining isomorphism allows the reduction to a divisor $D'$ which is the pullback of a $(-n')$-curve $C$. But then

$$\text{Hom}^\bullet(\mathcal{O}_D(D), \mathcal{O}) \cong \text{Hom}^\bullet(\mathcal{O}_{D'}(D'), \mathcal{O}) \cong H^\bullet(\mathcal{O}_C(C + K))[-2] \cong \mathbb{k}[-1]$$

using Serre duality and the equality $C.(C + K) = -2$. It remains to show the vanishing of $\text{Hom}^\bullet(\mathcal{O}_A(D), \mathcal{O})$ or, by Serre duality, of $H^\bullet(\mathcal{O}_A(D + K))$. If $A$ is just a curve, then the vanishing follows since $A.(D + K) = A^2 + A.B - A^2 - 2 = -1$.

Otherwise, let $E = C_l$ be the single $(-1)$-curve in $A$ and let $\pi: X \to Y$ be the contraction of $E$. Then $A.E = 0$ and, therefore, $A = \pi^*A'$ for some shorter chain $A' = \langle -n - 1 \rangle - \langle 2 \rangle - \cdots - \langle 2 \rangle - \langle 1 \rangle$. Moreover, we have $B.E = 1$ since it came from a $1$-decomposition of $D$, that is, $D.E = 1$. Therefore $D = \pi^*D' - E$. Finally, $K_X = \pi^*K_Y + E$. Putting all this together, we find that $\mathcal{O}_A(D + K) = \pi^*\mathcal{O}_A'(D' + K)$.

Let $E'$ be the $(-1)$-curve in $A'$ whose strict transform becomes the last $(-2)$-curve $C = C_{l-1}$ in $A$. We again compute that

$$E'.D' = \pi^*E'.\pi^*D' = (C + E).(D + E) = C.D + 1 = C_l.(C_1 + \cdots + C_{l-1}) + C_l^2 + C_l.(C_{l+1} + \cdots + C_m) + 1 = 1$$

since the first and third summands are equal to $1$. This enables us to proceed inductively until we have contracted $A$ to a single $(-n)$-curve. □

4.5. Simple chains as spherelike components. The following result complements Proposition 4.6.

**Proposition 4.21.** Let $D = C_1 + \cdots + C_m$ be a $1$-decomposition of a $(-n)$-divisor such that the divisor $A = C_1 + \cdots + C_l = \langle -3 \rangle - \langle 2 \rangle - \cdots - \langle 2 \rangle - \langle 1 \rangle$ is a spherelike component for some $l$, $2 \leq l < m$. Then

$$\text{Hom}^\bullet(\mathcal{O}_A(A - D), \mathcal{O}_D) = \mathbb{k} \oplus \mathbb{k}[-1] \oplus \mathbb{k}[-2].$$

In particular, given a spherelike non-spherical component $A$ of $D$ as in the statement, the twist functor $T_{\mathcal{O}_A(A - D)}$ is not an autoequivalence, and $T_{\mathcal{O}_A(A - D)}(\mathcal{O}_D) \neq \mathcal{O}_{D - A}$. Roughly speaking, $A$ cannot be twisted off $D$.

Before the proof, we need a lemma extending Proposition 4.6, (1).

**Lemma 4.22.** Let $D$ be a $(-n)$-divisor with a spherelike component $A$, and let $B = D - A$. Then

$$\text{Hom}^\bullet(\mathcal{O}_A(-B), \mathcal{O}_D) \cong \mathbb{k} \oplus H^\bullet(\mathcal{O}_A(K - B))[-2] \cong \begin{cases} \text{either } \mathbb{k}, \\ \text{or } \mathbb{k} \oplus \mathbb{k}[-1] \oplus \mathbb{k}[-2]. \end{cases}$$

**Proof.** Applying $\text{Hom}^\bullet(\mathcal{O}_A(-B), \cdot)$ to the ideal sheaf sequence of $D$ yields the triangle

$$\text{Hom}^\bullet(\mathcal{O}_A(A), \mathcal{O}) \to \text{Hom}^\bullet(\mathcal{O}_A(-B), \mathcal{O}) \to \text{Hom}^\bullet(\mathcal{O}_A(-B), \mathcal{O}_D).$$

By Proposition 4.20, the term on the left is isomorphic to $\mathbb{k}[-1]$. The middle term is isomorphic to $H^\bullet(\mathcal{O}_A(K - B))[-2]$ by Serre duality. Combining these facts yields the first claimed equivalence since $\text{Hom}(\mathcal{O}_A(-B), \mathcal{O}_D) \neq 0$. \

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To prove the second, apply $\text{Hom}^\bullet(\cdot, O_D)$ to the decomposition sequence for $A + B$:

$$\text{Hom}^\bullet(O_A(-B), O_D) \leftarrow \text{Hom}^\bullet(O_D, O_D) \leftarrow \text{Hom}^\bullet(O_B, O_D).$$

Therefore $k \cong \text{Ext}^2(O_D, O_D) \rightarrow \text{Ext}^2(O_A(-B), O_D)$. This completes the proof since $\chi(O_A(K - B)) = \chi(O_A) + A.K - A.B = 1 + 0 - 1 = 0$. □

Proof of Proposition 4.21. By Lemma 4.22, it suffices to show that $H^0(O_A(K - B)) \neq 0$. Fix a 1-decomposition $D = C_1 + \cdots + C_m$ and put $D_i := C_1 + \cdots + C_i$. Then $A = D_l$. Because we are assuming $D = A + B$ to be as in Theorem 4.14, we have $O_A(K - B) = O_{D_l}(K - D + D_l)$. We now proceed by induction on $D_i$ until we reach $i = l$.

Step $i = 1$. Since $C_1$ is a $(-3)$-curve, we have $C_1.K = 1$ and, by the 1-decomposition $C_1.(D - C_1) = 1$,

$$H^\bullet(O_{C_1}(K - D + C_1)) \cong H^\bullet(O_{C_1}) \cong k.$$

Step $i, 1 < i < l$. We use the decomposition sequence for $D_{i-1} + C_i$, tensored with $O(K - D + D_i)$:

$$H^\bullet(O_{D_{i-1}}(K - D + D_{i-1})) \rightarrow H^\bullet(O_{D_i}(K - D + D_i)) \rightarrow H^\bullet(O_{C_i}(K - D + D_i)).$$

The term on the left is isomorphic to $k$ by the induction hypothesis. The term on the right vanishes since $C_i.(C_{i+1} + \cdots + C_m) = 1$ from the 1-decomposition and the fact that $C_i.K = 0$ because $C_i$ is a $(-2)$-curve. Hence the middle term is isomorphic to $k$.

Step $i = l$. When $i = l$, we look at the same triangle of cohomology spaces as in the previous step. The term on the left is still isomorphic to $k$. But $C_l.K = -1$ since $C_l$ is a $(-1)$-curve, so the term on the right is isomorphic to $O_{P^1}(-2) \cong k[-1]$. It follows that the middle term is

$$H^\bullet(O_A(K - B)) = H^\bullet(O_{D_l}(K - D + D_l)) = k \oplus k[-1],$$

as claimed. □

§ 5. Spherelike divisors

Exceptional divisors, that is, $(-1)$-divisors, can always be dismantled to $(-1)$-curves by contractions and spherical twists; see Proposition 4.19. In this section, we look at spherelike divisors, that is, $(-2)$-divisors.

We start with a simple observation. The numerical criterion in Corollary 3.2 yields an easy way of proving that a divisor $D$ is spherelike: specify a 1-decomposition and a negative filtration and test whether $D.K_X = 0$. By Lemma 3.8, $D.K_X = 0$ if and only if the average of the self-intersection numbers of all the curve components is $-2$.

We recall some central notions from [3], §4. Let $D$ be a spherelike divisor and $D'$ any other divisor. Then, by Serre duality,

$$\text{Hom}(O_D(D'), O_D(D' + K_X)) \cong \text{Ext}^2(O_D, O_D)^* \cong k.$$
Therefore we have a non-zero map, unique up to scalars, which can be completed to the asphericity triangle
\[ \mathcal{O}_D(D') \to \mathcal{O}_D(D' + K_X) \to Q_{\mathcal{O}_D(D')} \]
The last term \( Q_{\mathcal{O}_D(D')} \) of this triangle is called the asphericity of \( \mathcal{O}_D(D') \).

**Definition 5.1.** Let \( D \) be a spherelike divisor and \( D' \) an arbitrary divisor. The spherical subcategory of the spherelike object \( \mathcal{O}_D(D') \) is
\[ \mathcal{D}^b(X)_{\mathcal{O}_D(D')} := \{ M \in \mathcal{D}^b(X) \mid \text{Hom}^\bullet(M, Q_{\mathcal{O}_D(D')}) = 0 \} \]

**Proposition 5.2** ([3], Theorems 4.4 and 4.6). Let \( D \) and \( D' \) be divisors on \( X \) with \( D \) spherelike. Then \( \mathcal{D}^b(X)_{\mathcal{O}_D(D')} \) is the unique maximal full triangulated subcategory of \( \mathcal{D}^b(X) \) containing \( \mathcal{O}_D(D') \) as a spherical object.

**5.1. Spherical subcategories and blow-ups.** We first look at how the spherical subcategory changes under blow-ups. For every blow-up \( \pi: X' \to X \) of a surface \( X \) at a single point with exceptional divisor \( E \), we have the semi-orthogonal decompositions (see, for example, [7], Proposition 11.8)
\[ \mathcal{D}^b(X') = \langle \mathcal{O}_E(E), \pi^* \mathcal{D}^b(X) \rangle = \langle \pi^* \mathcal{D}^b(X), \mathcal{O}_E \rangle \]

**Proposition 5.3.** Let \( \pi: X' \to X \) be the blow-up of a surface \( X \) at a point \( P \) with exceptional divisor \( E \). For every spherelike divisor \( D \) on \( X \), the asphericities of \( D \) and \( D' = \pi^* D \) fit into the triangle
\[ \pi^* \mathcal{O}_D \to \mathcal{O}_{D'} \to \mathcal{O}_{D'} \otimes \mathcal{O}_E(E) \]
satisfying the following dichotomy for the derived tensor product \( T = \mathcal{O}_{D'} \otimes \mathcal{O}_E(E) \):
\[ P \notin \text{supp}(D) \implies T = 0 \quad \text{and} \quad \mathcal{D}^b(X')_{\mathcal{O}_{D'}} = \langle \mathcal{O}_E(E), \pi^* \mathcal{D}^b(X)_{\mathcal{O}_D} \rangle; \]
\[ P \in \text{supp}(D) \implies T = \mathcal{O}_E(E) \oplus \mathcal{O}_E(E)[1] \quad \text{and} \quad \pi^* \mathcal{D}^b(X)_{\mathcal{O}_D} \subseteq \mathcal{D}^b(X')_{\mathcal{O}_{D'}}. \]

**Proof.** We compare the pullback of the asphericity triangle of \( \mathcal{O}_D \) with that of \( \mathcal{O}_{D'} \) in the following diagram, where we set \( K = K_X, K' = K_{X'}, Q = Q_{\mathcal{O}_D}, Q' = Q_{\mathcal{O}_{D'}} \), and use the equalities \( \pi^* K = K' - E \) and \( E.K' = E^2 \):
\[ \mathcal{O}_{D'} \to \mathcal{O}_{D'}(K' - E) \to \pi^* Q. \]
\[ \mathcal{O}_{D'}(K') \]
\[ \mathcal{O}_{D'} \otimes \mathcal{O}_E(E) \]
\[ \mathcal{O}_{D'} \otimes \mathcal{O}_E(E) \]
\[ Q' \]

We now show that (**) commutes and hence induces the desired triangle by the octahedral axiom. For this, we first note that
\[ \mathcal{O}_{D'}(K' - E) \to \mathcal{O}_{D'}(K') \to \mathcal{O}_{D'} \otimes \mathcal{O}_E(E) \]
is the ideal sheaf sequence of \( E \) tensored with \( \mathcal{O}_{D'}(K') \). Tensoring the ideal sheaf sequence of \( D' \) with \( \mathcal{O}_E(E) \) and using the equation \( E.D' = 0 \), we obtain the triangle
\[ \mathcal{O}_E(E) \to \mathcal{O}_D \otimes \mathcal{O}_E(E). \]
Depending on the position of $P$, $\alpha$ is either an isomorphism or zero. Hence,

$$O_D \otimes O_E(E) \cong \begin{cases} O_E(E) \oplus O_E(E)[1], & P \in \text{supp}(D), \\ 0, & P \notin \text{supp}(D). \end{cases}$$

In both cases, applying $\text{Hom}(O_D, \cdot)$ to $(D')$ yields an isomorphism

$$\text{Hom}^\bullet(O_D, O_D'(K' - E)) \sim \text{Hom}^\bullet(O_D', O_D'(K'))$$

since $\text{Hom}^\bullet(O_D', O_E(E)) = \text{Hom}^\bullet(O_D, \pi_*O_E(E)) = 0$. Thus (***) commutes.

It remains to show that $\pi^*D^b(X)_{O_D} \subset D^b(X')_{O_D'}$ if $P \in \text{supp}(D)$. To do this, take any $A \in D^b(X)_{O_D}$, that is, with $\text{Hom}^\bullet(A, Q) = 0$. Then, applying $\text{Hom}^\bullet(\pi^*A, \cdot)$ to the triangle connecting the asphericities in $(Q)$, we find that

$$\text{Hom}^\bullet(A, Q) \rightarrow \text{Hom}^\bullet(\pi^*A, Q') \rightarrow \text{Hom}^\bullet(\pi^*A, O_E(E) \oplus O_E(E)[1]).$$

The term on the left vanishes by assumption, and the term on the right vanishes because $\pi_*O_E(E) = 0$. Hence $\text{Hom}^\bullet(\pi^*A, Q') = 0$, so $\pi^*A \in D^b(X')_{O_D'}$. $\square$

**Example 5.4** (compare with Example 5.6 in [3]). Let $Y$ be a surface containing a $(-2)$-curve $C$ and let $\psi: X \rightarrow Y$ be the blow-up at a point $P$. Then the spherical subcategory of $\psi^*O_C$ is

$$D^b(X)_{\psi^*O_C} = \begin{cases} \psi^*D^b(Y), & P \in C, \\ D^b(X), & P \notin C. \end{cases}$$

The following proposition shows that the spherical subcategory also keeps track of more complicated blow-up situations. We only treat the next case after Example 5.4. A general statement seems possible, but would have to take care of the combinatorial structure of iterated blow-ups.

**Proposition 5.5.** Let $\pi: X' \rightarrow X$ be the blow-up of a surface $X$ at a point $P$, with exceptional divisor $E'$. Let $D$ be a divisor on $X$ of type $\begin{array}{c} \circ \cr \circ \end{array}$. We put $D' = \pi^*D$ and consider the surface $Y$ obtained by contracting the $(-1)$-curve in $D$, and the composite $\pi': X' \rightarrow X \rightarrow Y$. Then one of the following three cases occurs, each distinguished by the spherical subcategory of $O_{D'}$:

1. $D'$ is of type $\begin{array}{c} \circ \cr \circ \end{array}$ if $P \notin D$; then $D^b(X')_{O_{D'}} = \psi'^*D^b(Y')$, where $\psi': X' \rightarrow Y'$ is the contraction of the $(-1)$-curve in $D'$;

2. $D'$ is of type $\begin{array}{c} \circ \cr \circ \end{array}$ if $P$ lies on the $(-3)$-curve but not on the $(-1)$-curve, then $D^b(X')_{O_{D'}} = \pi'^*D^b(Y)$;

3. $D'$ is of type $\begin{array}{c} \circ \cr \circ \end{array}$ or $\begin{array}{c} \circ \cr \circ \end{array}$ if $P$ lies on the $(-1)$-curve; then $D^b(X')_{O_{D'}} = (O_C(-1)) \oplus \pi'^*D^b(Y)$, where $C$ is the $(-2)$-curve in $D'$.

Let $M \in D^b(X')$. In the statement and proof of Proposition 5.5, we write $\langle M \rangle$ for the smallest triangulated full subcategory of $D^b(X')$ which contains $M$ and is closed under taking direct summands.
**Proof.** Note that $X$ is the blow-up $\psi: X \to Y$ of a surface $Y$ at a point $Q$ lying on a $(-2)$-curve with exceptional divisor $E$ and total transform $D$; this is the setting of Example 5.4. Hence Case (1) becomes a direct application of Example 5.4: if $\psi': X' \to Y'$ is the contraction of the $(-1)$-curve in $D'$, then we can write $\mathcal{O}_{D'}$ as $(\psi')^*\mathcal{O}_C$, where $C$ is a $(-2)$-curve on $Y'$, whence $\mathcal{D}^b(X')\mathcal{O}_{D'} = \psi'^*\mathcal{D}^b(Y')$.

In the other cases, Proposition 5.3 produces the following triangle of asphericities:

$$\pi^*\mathcal{O}_E(E) \oplus \pi^*\mathcal{O}_E(E)[1] \to Q\mathcal{O}_{D'} \to \mathcal{O}_{E'}(E') \oplus \mathcal{O}_{E'}(E')[1]. \tag{Q}$$

In Case (2), the composite $\pi': X' \xrightarrow{\pi} X \xrightarrow{\psi} Y$ is obtained by blowing up $Q$ and $\psi(P)$ in either order and, in particular, $\text{Hom}^*(\mathcal{O}_{E'}(E'), \pi^*\mathcal{O}_E(E)) = 0$ since $\text{supp}(E') \cap \text{supp}(\pi^*E) = \emptyset$. Hence $Q\mathcal{O}_{D'}$ is the direct sum of the outer terms of the triangle (Q), and the equalities

$$\mathcal{D}^b(X')\mathcal{O}_{D'} = \perp (\pi^*\mathcal{O}_E(E) \oplus \mathcal{O}_{E'}(E')) = \pi'^*\mathcal{D}^b(Y)$$

are obtained using the relation

$$\mathcal{D}^b(X') = (\mathcal{O}_{E'}(E'), \pi^*\mathcal{O}_E(E), \pi'^*\mathcal{D}^b(Y)).$$

In Case (3), we take a closer look at the degree-increasing morphism in the triangle (Q). One can check that $\text{Hom}^*(\mathcal{O}_{E'}(E'), \pi^*\mathcal{O}_E(E)) \cong \mathbb{k} \oplus \mathbb{k}[-1]$, so the following arrows are possible:

$$\mathcal{O}_{E'}(E')[1] \xrightarrow{e_2} \pi^*\mathcal{O}_E(E)[2] \oplus \mathcal{O}_{E'}(E') \xrightarrow{e_1} \pi^*\mathcal{O}_E(E)[1].$$

Consider the curve $C := \pi^*E - E'$. Then $C^2 = -2$, $C.E' = 1$, $C.\pi^*E = -1$.

**Step 1.** $\mathcal{O}_C(\pi^*E) \in \mathcal{D}^b(X')\mathcal{O}_{D'} = \perp Q\mathcal{O}_{D'}$.

Note that $\mathcal{O}_C(\pi^*E) \in (\mathcal{O}_{E'}(E'), \pi^*\mathcal{O}_E(E))$ from a decomposition sequence. Now $\text{Hom}^*(\mathcal{O}_C(\pi^*E), \mathcal{O}_{D'}) = 0$ implies that $\text{Hom}^*(\mathcal{O}_C(\pi^*E), Q\mathcal{O}_{D'}) = 0$ by Theorem 4.7 in [3], and thus it suffices to show that the former vanishes. Consider the decomposition sequence $\mathcal{O}_C(-E') \hookrightarrow \mathcal{O}_{\pi^*E} \to \mathcal{O}_{E'}$ and observe that $C.(-E') = C.\pi^*E$, whence $\mathcal{O}_C(-E') \cong \mathcal{O}_C(\pi^*E)$. Applying $\text{Hom}^*(\cdot, \mathcal{O}_{D'})$ gives

$$\text{Hom}^*(\mathcal{O}_C(\pi^*E), \mathcal{O}_{D'}) \hookrightarrow \text{Hom}^*(\mathcal{O}_{\pi^*E}, \mathcal{O}_{D'}) \hookrightarrow \text{Hom}^*(\mathcal{O}_{E'}, \mathcal{O}_{D'}).$$

Using the equality $\mathcal{D}^b(X') = (\pi'^*\mathcal{D}^b(Y), \pi^*\mathcal{O}_E, \mathcal{O}_{E'})$ or a direct calculation, we see that the middle and right terms are zero. Hence the left term also vanishes, as required. Thus the spherical subcategory of $\mathcal{O}_{D'}$ contains the spherical object $\mathcal{O}_C(\pi^*E)$.

**Step 2.** $e_1 \neq 0$ and $e_2 \neq 0$.

Assume the opposite. Then the asphericities have a direct summand $\mathcal{O}_{E'}(E')$ if $e_1 = 0$, or $\pi^*\mathcal{O}_E(E)[1]$ if $e_2 = 0$. We observe that:

- $\text{Hom}^*(\mathcal{O}_C(\pi^*E), \mathcal{O}_{E'}(E'))) \neq 0$ because of the extension corresponding to the decomposition sequence $\mathcal{O}_{E'}(E') \hookrightarrow \pi^*\mathcal{O}_E(E) \to \mathcal{O}_C(\pi^*E)$;
Tensoring the decomposition sequence
\[ \pi_3. \]

In particular, we would get that \( O_C(\pi^*E) \notin \perp Q_{O_{D'}} \), contradicting Step 1.

Step 3. \( D^b(X')_{O_{D'}} = \langle O_{E'}(E'), \pi^*O_E(E) \rangle \cap \perp O_F(F), \pi^*D^b(Y) \rangle. \)

This will follow from an explicit formula for \( Q_{O_{D'}} \). Consider the divisor \( F = \pi^*E + E' = C + 2E' \) of type \( \begin{pmatrix} 2 & -1 \\ -2 & 1 \end{pmatrix} \). Tensoring the decomposition sequence of \( F \) with \( O(F) \) yields that \( \pi^*O_E(E) \leftrightarrow O_F(F) \rightarrow O_{E'}(E') \). As the arrows \( e_i \) are non-zero, one can check the quasi-isomorphisms
\[ Q_{O_{D'}} \cong \begin{cases} Q_s := O_F(F) \oplus O_F(F)[1] & \text{if } \alpha = 0, \\ Q_n := \text{cone}(O_F(F) \overset{t}{\rightarrow} O_F(F)) & \text{if } \alpha \neq 0, \end{cases} \]

with \( t \in \text{Hom}^\bullet(O_F(F), O_F(F)) = \text{Hom}^\bullet(O_F, O_F) \cong k[t]/t^2 \) and \( \deg(t) = 0 \).

Even though these two possibilities differ, their left orthogonals inside \( \langle O_{E'}(E') \rangle \), \( \pi^*O_E(E) \) coincide. Indeed, to prove the inclusion \( \perp Q_s \subseteq \perp Q_n \), take any \( A \in \perp Q_s \), that is, \( \text{Hom}^\bullet(A, O_F(F)) = 0 \). This forces \( \text{Hom}^\bullet(A, \cdot) \) to the triangle \( O_F(F) \rightarrow O_F(F) \rightarrow Q_n \).

On the other hand, take any \( B \in \perp Q_n \) and suppose that \( 0 \neq \beta \in \text{Hom}^\bullet(B, O_F(F)) \). Applying \( \text{Hom}^\bullet(B, \cdot) \) to the triangle defining \( Q_n \), we get an isomorphism
\[ t^*: \text{Hom}^\bullet(B, O_F(F)) \rightarrow \text{Hom}^\bullet(B, O_F(F)). \]

Now \( t^*(\beta) = t\beta \neq 0 \), but \( t^*(\beta) = t^2\beta = 0 \), a contradiction.

Step 4. We will prove \( \langle O_{E'}(E'), \pi^*O_E(E) \rangle \cap \perp O_F(F) = \langle O_C(-1) \rangle \) via tilting.

Consider the triangulated category
\[ \mathcal{C} = \langle \mathcal{E}_0, \mathcal{E}_1 \rangle := \langle O_{E'}(E'), \pi^*O_E(E) \rangle. \]

The generators possess the following properties:

(a) \( \mathcal{E}_0, \mathcal{E}_1 \) are exceptional;
(b) \( \text{Hom}^\bullet(\mathcal{E}_1, \mathcal{E}_0) = 0; \)
(c) \( \text{Hom}^\bullet(\mathcal{E}_0, \mathcal{E}_1) = k \oplus k[-1]; \)
(d) the non-zero morphisms \( \mathcal{E}_0 \rightarrow \mathcal{E}_1 \) are injective.

Results in \([10]\) apply to exceptional sequences of this kind. Specifically, Proposition 1.7 in \([10]\) states that \( \mathcal{C} \cong D^b(\text{End}(T)\text{-mod}) \), where \( T \) is the iterated universal extension of the exceptional sequence.

\( O_F(F) \) is the unique non-trivial extension of \( \pi^*O_E(E) \) by \( O_{E'}(E') \), whence the tilting object is \( T = \pi^*O_E(E) \oplus O_F(F) \) and \( A := \text{End}(T) \) is the quiver algebra
\[ \begin{array}{c}
1 \\
\alpha \\
\beta \\
2
\end{array} \]

with the relation \( \beta\alpha = 0 \). Under the equivalence \( \mathcal{C} \overset{\sim}{\rightarrow} D^b(A\text{-mod}) \), the objects \( \pi^*O_E(E) \) and \( O_F(F) \) become the projective modules \( Ae_1 \) and \( Ae_2 \) respectively,
where $e_i$ is the idempotent corresponding to the vertex $i$. Moreover, $\mathcal{O}_C(-1)$ is sent to the simple module $S_1$ associated with the vertex 1. One can check that the following relation holds for every finite-dimensional quiver algebra $B$ and for the idempotent $e_i$ corresponding to a vertex $i$:

\[ Be_i^\perp = \langle S_j \mid j \neq i \rangle. \]

In particular, $Ae_2^\perp = \langle S_1 \rangle$. Going back to $C$, we obtain that $\mathcal{O}_F(F)^\perp = \langle \mathcal{O}_C(-1) \rangle$. As $\mathcal{O}_C(-1)$ is a spherical object and, in particular, a Calabi–Yau object, we conclude that $\perp \mathcal{O}_F(F) = \langle \mathcal{O}_C(-1) \rangle$. A similar argument shows that $\mathcal{O}_C(-1)$ is the two-sided orthogonal to $\pi^* \mathcal{D}^b(Y)$, completing the proof. □

Remark 5.6. The algebra $A$ in Step 4 of the proof appears in two well-known series. It is the derived-discrete algebra $\Lambda(1,2,0)$ (see [11], [12]) and it is the Auslander algebra of $k[t]/t^2$ (see [10], [14]).

Remark 5.7. One can check that the asphericities distinguish between $\frac{3}{3}$ and $\frac{4}{3}$ in Case (3) of Proposition 5.5. Namely,

\[ Q_{\mathcal{O}_{D'}} = \begin{cases} \mathcal{O}_F(F) \oplus \mathcal{O}_F(F)[1] & \text{if } D' = \frac{4}{3}, \\ \text{cone}(\mathcal{O}_F(F) \stackrel{\tau}{\rightarrow} \mathcal{O}_F(F)) & \text{if } D' = \frac{3}{3}. \end{cases} \]

Moreover, in Case (3) of Proposition 5.5 we have an orthogonal decomposition $\mathcal{D}^b(X')_{\mathcal{O}_D} = \langle \mathcal{O}_C(-1) \rangle \oplus \pi^* \mathcal{D}^b(Y)$. Also, $\langle \mathcal{O}_C(-1) \rangle$ is neither weakly admissible in $\mathcal{D}^b(X')$ (that is, its inclusion does not admit any adjoint), nor equivalent to $\mathcal{D}^b(Z)$ for any smooth projective surface $Z$. As a consequence, the same applies to $\mathcal{D}^b(X')_{\mathcal{O}_D'}$.

5.2. Spherical subcategories and decompositions. If $D$ is a spherelike divisor with a curvelike decomposition $D = A + B$ (Definition 4.13), then $A$ and $B$ are spherelike divisors themselves, and we emphasize this by calling $D = A + B$ a spherelike decomposition.

The remainder of this section is devoted to the behaviour of the spherical subcategory under spherelike decompositions. Proposition 4.6 together with Lemma 2.2 in [13] yields the following lemma.

Lemma 5.8. Let $D$ be a spherelike divisor. If $A \prec D$ is a spherical component, then $\mathcal{D}^b(X)_{\mathcal{O}_D} = T_{\mathcal{O}_A(-B)}^{-1}(\mathcal{D}^b(X)_{\mathcal{O}_B})$, where $B = D - A$.

Proposition 5.9. Let $D = A + B$ be a spherelike decomposition. Then the asphericity of $\mathcal{O}_D$ occurs in a triangle $R_A \rightarrow Q_{\mathcal{O}_D} \rightarrow R_B$, where

\[ R_A = \begin{cases} Q_{\mathcal{O}_A(-B)} & \text{if } H^0(\mathcal{O}_{B}(D)) = 0, \\ \mathcal{O}_A(K - B) \oplus \mathcal{O}_A(-B)[1] & \text{if } H^0(\mathcal{O}_{B}(D)) = k, \end{cases} \]

\[ R_B = \begin{cases} Q_{\mathcal{O}_B} & \text{if } H^0(\mathcal{O}_{A}(K - B)) = 0, \\ \mathcal{O}_B(K) \oplus \mathcal{O}_B[1] & \text{if } H^0(\mathcal{O}_{A}(K - B)) = k. \end{cases} \]
Proof. Note that the above possibilities for $R_A$ and $R_B$ exhaust all cases. For $R_B$ this follows from Lemma 4.22. For $\mathcal{O}_B(D)$ one can see this by considering the triangle $H^\bullet(\mathcal{O}_A(A)) \to H^\bullet(\mathcal{O}_D(D)) \to H^\bullet(\mathcal{O}_B(D))$ from the decomposition $D = A + B$ and using the relations $H^\bullet(\mathcal{O}_A(A)) \cong \mathbb{k}[-1]$ and $H^0(\mathcal{O}_D(D)) = 0$. We recall that $Q_{\mathcal{O}_D}$ is computed as the cone of the canonical map $\mathcal{O}_D \xrightarrow{\omega} \mathcal{O}_D(K)$. Using the decomposition sequence for $D = A + B$ and its $\mathcal{O}(K)$-twist, we obtain two triangles linked by $\omega$:

$$
\begin{align*}
\mathcal{O}_A(-B) & \xrightarrow{\iota} \mathcal{O}_D \xrightarrow{\pi} \mathcal{O}_B \\
\mathcal{O}_A(K - B) & \xrightarrow{\iota'} \mathcal{O}_D(K) \xrightarrow{\pi'} \mathcal{O}_B(K).
\end{align*}
$$

We already know that $\text{Hom}(\mathcal{O}_A(-B), \mathcal{O}_B(K)) = \text{Ext}^2(\mathcal{O}_B, \mathcal{O}_A(-B))^* = 0$ by Lemma 4.4, (2) and Serre duality. In particular, $\pi' \omega \iota = 0$ and this implies that $\omega$ extends to a map of triangles. Indeed, since $\text{Ext}^{-1}(\mathcal{O}_A(-B), \mathcal{O}_B(K)) = 0$, we see that $\omega$ determines the resulting morphisms $\alpha$ and $\beta$ uniquely. Taking the cones $R_A = \text{cone}(\alpha)$ and $R_B = \text{cone}(\beta)$, we get a triangle $R_A \to Q_{\mathcal{O}_D} \to R_B$. However, there are various cases depending on whether $\alpha \neq 0$ or $\beta \neq 0$.

If $\alpha \neq 0$, then this morphism is a multiple of $\omega_{\mathcal{O}_A(-B)}$ and, therefore, its cone is $R_A = Q_{\mathcal{O}_A(-B)} = Q_{\mathcal{O}_A}(-B)$. On the other hand, if $\alpha = 0$, then the triangle splits and $R_A$ is the direct sum given in the statement of the proposition. The same argument applies to $\beta$ and $R_B$. We now look at $R_A$:

$$
\begin{align*}
\alpha = 0 & \iff \iota' \alpha = \omega \iota = 0 \tag{1} \\
& \iff \text{Hom}(\mathcal{O}_B, \mathcal{O}_D(K)) \neq 0 \tag{2} \\
& \iff H^1(\mathcal{O}_B(D)) \neq 0 \tag{3} \\
& \iff H^0(\mathcal{O}_B(D)) = \mathbb{k}. \tag{4}
\end{align*}
$$

As to (1), $\iota'$ is injective and the left-hand square of (*** commutes.

As to (2), apply $\text{Hom}(\cdot, \mathcal{O}_D(K))$ to the top triangle of (*** to get

$$
0 \to \text{Hom}(\mathcal{O}_B, \mathcal{O}_D(K)) \to \text{Hom}(\mathcal{O}_D, \mathcal{O}_D(K)) \xrightarrow{\iota^*} \text{Hom}(\mathcal{O}_A(-B), \mathcal{O}_D(K)).
$$

If $\text{Hom}(\mathcal{O}_B, \mathcal{O}_D(K)) = 0$, then $\iota^*$ is injective, mapping $\omega \mapsto \omega \iota = \iota' \alpha \neq 0$. On the other hand, if $\text{Hom}(\mathcal{O}_B, \mathcal{O}_D(K)) \neq 0$, then $\iota^* = 0$ because we have

$$
\text{Hom}(\mathcal{O}_D, \mathcal{O}_D(K)) = \text{Ext}^2(\mathcal{O}_D, \mathcal{O}_D)^* = \mathbb{k}.
$$

As to (3), applying $\text{Hom}(\cdot, \mathcal{O}_B)$ to $\mathcal{O}(-D) \xleftarrow{\iota} \mathcal{O} \xrightarrow{\pi} \mathcal{O}_D$ yields the following segment of the long exact sequence, proving that $H^1(\mathcal{O}_B(D)) \cong \text{Hom}(\mathcal{O}_B, \mathcal{O}_D(K))$:

$$
\begin{align*}
\text{Ext}^1(\mathcal{O}, \mathcal{O}_B) & \xrightarrow{\pi} \text{Ext}^1(\mathcal{O}(-D), \mathcal{O}_B) \xrightarrow{\iota^*} \text{Ext}^2(\mathcal{O}_D, \mathcal{O}_B) \xrightarrow{\pi^*} \text{Ext}^2(\mathcal{O}, \mathcal{O}_B) \\
0 = H^1(\mathcal{O}_B) & \xrightarrow{\iota} H^1(\mathcal{O}_B(D)) \xrightarrow{\pi} \text{Hom}(\mathcal{O}_B, \mathcal{O}_D(K))^* \xrightarrow{\iota^*} H^2(\mathcal{O}_B) = 0.
\end{align*}
$$
As to (4), by Riemann–Roch,
\[ \chi(\mathcal{O}_B(D)) = \frac{1}{2}(B^2 + B.K) + B.D = 1 - 1 = 0, \]
whence
\[ H^0(\mathcal{O}_B(D)) \cong H^1(\mathcal{O}_B(D)) \cong \text{Hom}(\mathcal{O}_B, \mathcal{O}_D(K)). \]

We now look at the other cone \( R_B \):
\[ \beta = 0 \iff \beta \pi = \pi' \omega = 0 \quad (1') \]
\[ \iff \text{Hom}(\mathcal{O}_D, \mathcal{O}_A(K - B)) \neq 0 \quad (2') \]
\[ \iff H^0(\mathcal{O}_A(K - B)) = k. \quad (3') \]

As to (1’), \( \pi \) is surjective and the right-hand square of (*** ) commutes. As to (2’), apply \( \text{Hom}(\mathcal{O}_D, \cdot) \) to the bottom triangle of (*** ) and get
\[ 0 \to \text{Hom}(\mathcal{O}_D, \mathcal{O}_A(K - B)) \to \text{Hom}(\mathcal{O}_D, \mathcal{O}_D(K)) \xrightarrow{\pi_*} \text{Hom}(\mathcal{O}_D, \mathcal{O}_B(K)). \]

Note that \( \pi_* \) is injective \( \iff \text{Hom}(\mathcal{O}_D, \mathcal{O}_A(K - B)) = 0 \) since
\[ \text{Hom}(\mathcal{O}_D, \mathcal{O}_D(K)) = k. \]

As to (3’), this is Serre duality applied to Lemma 4.22. □

Remark 5.10. Let \( D \) be spherelike and consider its asphericity triangle \( \mathcal{O}_D \to \mathcal{O}_D(K) \to Q \). Taking the cohomology and combining it with Proposition 4.20 yields that \( H^\bullet(Q) = \text{Hom}^\bullet(\mathcal{O}, Q) = 0 \), that is, \( \mathcal{O}_X \in D^b(X)_{\mathcal{O}_D} \).

Example 5.11. Let
\[ D = 2B + C + C' + E + E' = \begin{array}{c} -3 \\ -2 \\ -1 \\ -1 \end{array} \]
be such that \( B^2 = -3, C^2 = C'^2 = -2, E^2 = E'^2 = -1 \) and \( D \) is rational.

Then \( D \) is spherelike by Corollary 3.2: a negative filtration is \( B, C, C', B, E, E' \), a 1-decomposition is \( B, C, C', E, E', B \) (it is crucial only that \( B \) comes first and last or next-to-last), and the equality \( D.K = 0 \) is clear.

The algorithm proving Theorem 4.14 produces a spherelike decomposition from a given 1-decomposition. The 1-decomposition \( B, C, C', E, E', B \) yields that \( D = (B + C + C' + E) + (B + E') \). Starting with the 1-decomposition \( B, C, E, C', E', B \), we obtain that \( D = (B + C + E) + (B + C' + E') \).

Because \( E.D = E'.D = 1 \neq 0 \), neither \((-1)\)-curve can be contracted to yield a smaller spherelike divisor; see § 4.1. Similarly, the fact that \( C.D = C'.D = 0 \neq 1 \) means that neither of the \((-2)\)-curves can be twisted off \( D \); see § 4.2. Therefore, \( D \) is a minimally spherelike divisor.

As to the asphericity of \( D \), we employ the criterion in Proposition 5.9 with the spherelike decomposition \( D = A + A' \), where \( A = B + C + C' + E \) and \( A' = B + E' \).
So we have to compute $H^0(\mathcal{O}_{A'}(D))$ and $H^0(\mathcal{O}_A(K - A'))$. For the former, the $\mathcal{O}(D)$-twisted decomposition sequence for $A' = B + E'$,

$$
\mathcal{O}_{E'}(D - B) \to \mathcal{O}_{A'}(D) \to \mathcal{O}_B(D)
$$

yields that $H^0(\mathcal{O}_{A'}(D)) = \mathbb{k}$. Thus the map

$$
\alpha: \mathcal{O}_A(-A') \to \mathcal{O}_A(K - A')
$$

is zero and, therefore, $R_A = \mathcal{O}_A(-A')[1] \oplus \mathcal{O}_A(K - A')$. In order to calculate $H^0(\mathcal{O}_A(K - A'))$, we use the decomposition sequence for $A = (C + C' + E) + B$ and note that $C + C' + E$ is a disjoint union:

$$
\mathcal{O}_C(-1) \oplus \mathcal{O}_{C'}(-1) \oplus \mathcal{O}_E(-1) \to \mathcal{O}_D \to \mathcal{O}_B,
$$

and twist it by $\mathcal{O}(K - A')$:

$$
\mathcal{O}_C(-2) \oplus \mathcal{O}_{C'}(-2) \oplus \mathcal{O}_E \to \mathcal{O}_D(K - A') \to \mathcal{O}_B,
$$

using that facts that $K.C = 0, K.E = -1, K.B = 1$ and $A'.C = A'.E = A'.B = 1$. We find that $0 \neq H^0(\mathcal{O}_D(K - A'))$ and, therefore, $\beta: \mathcal{O}_{A'} \to \mathcal{O}_{A'}(K)$ is also zero, forcing $R_{A'} = \mathcal{O}_{A'}[1] \oplus \mathcal{O}_{A'}(K)$. The asphericity of $\mathcal{O}_D$ thus sits in the triangle

$$
\mathcal{O}_A(-A')[1] \oplus \mathcal{O}_A(K - A') \to Q_D \to \mathcal{O}_A'[1] \oplus \mathcal{O}_{A'}(K).
$$

We take the cohomology exact sequence of this triangle:

$$
0 \to \mathcal{O}_A(-A') \to h^{-1}(Q_D) \to \mathcal{O}_{A'} \to \mathcal{O}_A(K - A') \to h^0(Q_D) \to \mathcal{O}_{A'}(K) \to 0.
$$

The map $\omega$ really is induced from $\omega: \mathcal{O}_D \to \mathcal{O}_D(K)$, again because $\alpha = 0$ and $\beta = 0$. The only common component of $A$ and $A'$ is the $(-3)$-curve $B$, so the map $\omega$ must be non-trivial there. We have $B.(K - A') = B.K - B.(B + E') = 1 - (-3 + 1) = 3$. Now $\text{hom}(\mathcal{O}_B, \mathcal{O}_B(3)) = 4$, but $\omega$ has to vanish on the intersections of $B$ with other curves in $A (C.(K - A') = C'.(K - A') = -2, E.(K - A') = 0$, so no poles are allowed). There is a unique map $\mathcal{O}_B \to \mathcal{O}_B(3)$ with these three prescribed
zeros. Splitting the above long exact sequence into short exact sequences, we get

(a) \[ \mathcal{O}_A(-A') \xrightarrow{\cdot} h^{-1}(Q_D) \xrightarrow{\cdot} \mathcal{O}_{E'}(-B), \]
(b) \[ \mathcal{O}_{E'}(-B) \xrightarrow{\cdot} \mathcal{O}_{A'} \xrightarrow{\cdot} \mathcal{O}_B, \]
(x) \[ \mathcal{O}_B(-C - C' - E) \xrightarrow{\cdot} \mathcal{O}_A \xrightarrow{\cdot} \mathcal{O}_{C+C'+E}, \]
(c) \[ \mathcal{O}_B \xrightarrow{\cdot} \mathcal{O}_A(K - A') \xrightarrow{\cdot} \mathcal{O}_{C+C'+E}(K - A'), \]
(d) \[ \mathcal{O}_{C+C'+E}(K - A') \xrightarrow{\cdot} h^0(Q_D) \xrightarrow{\cdot} \mathcal{O}_{A'}(K). \]

Here (a)–(b)–(c)–(d) splice together to give the long exact sequence, (b) and (x) are decomposition sequences, (c) is (x) twisted by \( K - A' \), using the fact that \( B.(A' - K) = -3 \) when \( \mathcal{O}_B(-C - C' - E) = \mathcal{O}_B(-3) = \mathcal{O}_B(A' - K) \). The last term of (c) is

\[ \mathcal{O}_{C+E+E'}(K - A') = (\mathcal{O}_C + \mathcal{O}_E + \mathcal{O}_{E'})(K - A') = \mathcal{O}_C(-1) + \mathcal{O}_E(-2) + \mathcal{O}_{E'}(-2). \]

The two further twisted decomposition sequences

\[ \mathcal{O}_A(-A') \xrightarrow{\cdot} \mathcal{O}_{B+C+C'+E+E'}(-B) \xrightarrow{\cdot} \mathcal{O}_{E'}(-B), \]
\[ \mathcal{O}_{C+C'+E}(K - A') \xrightarrow{\cdot} \mathcal{O}_{B+C+C'+E+E'}(K) \xrightarrow{\cdot} \mathcal{O}_{A'}(K) \]

show that

\[ h^{-1}(Q_D) = \mathcal{O}_{B+C+C'+E+E'}(-B), \]
\[ h^0(Q_D) = \mathcal{O}_{B+C+C'+E+E'}(K). \]

The degrees of these line bundles on \( B + C + C' + E + E' \) differ. At this point, it seems hard to compute the spherical subcategory \( \mathcal{D}^b(X)_{\mathcal{O}_D} = \frac{1}{2}Q_D. \) We do know that \( \mathcal{O}_D, \mathcal{O}_X \in \mathcal{D}^b(X)_{\mathcal{O}_D} \) and \( \frac{1}{2}h^{-1}(Q_D) \cap \frac{1}{2}h^0(Q_D) \subseteq \mathcal{D}^b(X)_{\mathcal{O}_D} \).

§ 6. Negative definite divisors and rational singularities

We recall a few facts about surface singularities, as can be found in [15], [9], [8], [16]. For modern proofs of the contraction results, see [6], § A.7 and § 4.15f.

Definition 6.1. A normal surface \( Y \) is called a rational singularity if there is a resolution of singularities \( \pi : X \rightarrow Y \) such that \( \pi_* \mathcal{O}_X = \mathcal{O}_Y \), that is, \( \pi \) has connected and acyclic fibres (\( R^i \pi_* \mathcal{O}_X = 0 \) for \( i > 0 \)).

Proposition 6.2 ([15], Theorem 2.3). A reduced connected divisor \( D \) on a given algebraic surface \( X \) can be contracted to a point \( P \) on an algebraic surface \( Y \) with \( \chi(Y) = \chi(X) \) if and only if \( D \) is negative definite and all the effective divisors supported on \( D \) are Jacobi rigid.

If this holds, then \( P \) is a rational singularity.

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Remark 6.3. The original phrasing of the second condition had \( \chi(\mathcal{O}_{D'}) \geq 1 \) for all \( D' \) supported on \( D \) instead of \( H^1(\mathcal{O}_{D'}) = 0 \) (Jacobi rigid). We claim that \( H^0(\mathcal{O}_{D'}(D')) \) also vanishes for such divisors. To see this, note that such divisors \( D' \) are also negative definite and, in particular, can be negatively filtered, whence \( H^0(\mathcal{O}_{D'}(D')) = 0 \) by Lemma 3.6. Hence any effective divisor supported on a configuration yielding a rational surface singularity is automatically rigid.

There is a related result stating that \( D \) is contractible in the analytic category if and only if \( D \) is negative definite; see [17].

Definition 6.4. Let \( \pi: X \to Y \) be a resolution of a normal surface singularity \( P \in Y \). So the exceptional locus \( \pi^{-1}(P) = \bigcup_i C_i \) is a union of projective curves. The \textit{numerical (fundamental) cycle} is the minimal divisor \( Z_{\text{num}} = \sum z_i C_i \) such that \( z_i > 0 \) and \( Z_{\text{num}}.C_i \leq 0 \) for all \( i \).

Remark 6.5. The condition defining numerical cycles makes sense for arbitrary divisors: \( D \) is said to be \textit{anti-nef} if \( C.D \leq 0 \) for all curves \( C \preceq D \). This property is obviously numerical in the sense of Definition 2.12. One can check that the property is birational because \( D \) allows contraction of a \((-1)\)-curve \( E \prec D \) if and only if \( E.D = 0 \).

Now consider the stronger variant \( C.D < 0 \) for all curves \( C \preceq D \). It is also numerical but certainly not birational: every reduced \((-1)\)-curve satisfies this condition, but its blow-up \( \frac{1}{2}(-1) \) does not. We remark that this property forces \( D \) to be negative definite. Indeed, let \( M \) be the intersection matrix of \( D_{\text{red}} \). Then the intersection matrix of \( D \) is \( TMT \), where \( T \) is the diagonal matrix of curve coefficients of \( D \). Because \( C.D < 0 \), all the column sums of \( TMT \) are negative, that is, \( -TMT \) is a symmetric strictly diagonal dominant matrix with positive diagonal entries, whence it is positive definite ([18], Corollary 1.22).

Remark 6.6. It is possible that the dual graph of a resolution of a normal surface singularity is not a tree. A singularity is said to be \textit{arborescent} if some resolution produces a tree (this property then holds for any good resolution); see [19], §4. Rational singularities have this property; see, for example, Proposition 6.8.

Definition 6.7. A rational singularity \( Y \) is called an \textit{ADE-singularity} if there is a crepant resolution of singularities \( \pi: X \to Y \), that is, \( K_X = \pi^*K_Y \).

This definition is anachronistic but the most convenient for us. It is well known that there are many characterizations of such singularities, leading to many equivalent notions such as rational double points and simple surface singularities, and they are often named after du Val or Klein; see [20].

Proposition 6.8. If \( X \to Y \) is a resolution of a rational singularity, then \( Z_{\text{num}} \) is a \((-n)\)-divisor for some \( n > 0 \). In particular, \( Z_{\text{num}} \) is spherelike if and only if \( Y \) is an ADE-singularity.

Proof. As noted in Remark 6.3, any effective divisor supported on the exceptional locus is rigid. In particular, this holds for \( Z_{\text{num}} \). Moreover, \( \chi(\mathcal{O}_{Z_{\text{num}}}) = 1 \) by Proposition 4.12 in [6], whence \( Z_{\text{num}} \) is well-connected.

If \( Y \) is an ADE-singularity, then \( Z_{\text{num}}^2 = -2 \), that is, \( Z_{\text{num}} \) is spherelike. The reverse implication holds by Theorem 3.31 in [16].
Corollary 6.9. Let \( X \to Y \) be a crepant resolution of an ADE-singularity. Then \( Z_{\text{num}} \) is spherical. Moreover, the subcategory \( \mathcal{D}^b_{Z_{\text{num}}}(X) \) of objects set-theoretically supported on \( Z_{\text{num}} \) is a 2-Calabi–Yau category.

Proof. We already know that \( Z_{\text{num}} \) is spherelike. The singular surface \( Y \) is Gorenstein, and the resolution is crepant. Hence there is an open subset \( U \subset X \) containing the exceptional locus and satisfying \( \omega_X|_U \cong \mathcal{O}_U \). In particular, \( \omega_X|_{Z_{\text{num}}} \cong \mathcal{O}_{Z_{\text{num}}} \). This shows that \( Z_{\text{num}} \) is spherical and also that \( M \otimes \omega_X \cong M \) for any \( M \) supported on \( Z_{\text{num}} \). Hence \( \mathcal{D}^b_{Z_{\text{num}}}(X) \) has Serre functor \( - \otimes \omega_X[2] = [2] \) and, therefore, it is a 2-Calabi–Yau category.

Example 6.10. The divisor

\[
\begin{array}{c}
-2 \\
\downarrow \\
-2 \quad -2 \quad -2
\end{array}
\]

is the numerical cycle of a minimal resolution of a \( D_4 \)-singularity. It is a non-reduced spherical divisor. See [8], p.96, for a complete list of the numerical cycles for ADE-singularities.

Proposition 6.11. Let \( D \) be a negative definite \((-n)\)-divisor that can be contracted. Then \( D \preceq Z_{\text{num}} \). In particular, if \( D \) contracts to a rational singularity, then \( Z_{\text{num}} \) is the maximal well-connected and rigid divisor with support \( \text{supp} \, D \).

The proof uses Laufer’s algorithm for computing the numerical cycle, and one can regard a 1-decomposition as a special case of it.

Proof. By Proposition 4.1 in [21], the numerical cycle can be computed recursively. Start with \( Z_0 := C \) for some curve \( C \prec D \). Given \( Z_i \), compute \( Z_i.C' \) for all curves \( C' \prec D \). If \( Z_i.C' > 0 \) for some \( C' \), set \( Z_{i+1} := Z_i + C' \). Otherwise \( Z_i.C' \leq 0 \) for all \( C' \), and then \( Z_{\text{num}} = Z_i \).

Using a 1-decomposition of \( D = C_1 + \cdots + C_m \), this algorithm yields that \( Z_i = C_{m-i} + \cdots + C_m \). In particular, \( D = Z_{m-1} \preceq Z_{\text{num}} \). □

Example 6.12. Let \( T \) be a tree of \((-2)\)-curves. As a reduced divisor, \( T \) is spherical by Corollary 4.12. If \( T \) forms an ADE-graph, then there is a unique maximal spherical divisor on \( T \), the numerical cycle \( Z_{\text{num}} \) by Definition 6.4.

The reverse implication also holds: if \( T \) is not an ADE-graph, then there is no maximal spherical divisor. For example, consider the following spherical divisors on a \( D_4 \)-configuration of \((-2)\)-curves:

\[
D_1 = \begin{array}{c}
-2 \\
\downarrow \\
-2 \quad -2
\end{array} \quad \text{and} \quad D_2 = \begin{array}{c}
-2 \\
\downarrow \\
-2 \quad -2 \quad -2
\end{array}.
\]

The smallest divisor \( D \) with \( D_1 \preceq D \) and \( D_2 \preceq D \) has \( D^2 = 0 \). In particular, no divisor containing \( D_1 \) and \( D_2 \) can be rigid since it contains \( D \) as a subdivisor.

Proposition 6.13. A negative definite spherelike divisor can be contracted either to a smooth point or to an ADE-singularity, and in the latter case it is the pullback of a spherical divisor.
Proof. If a spherelike divisor $D$ contains no $(-1)$-curves, then it has to be a configuration of $(-2)$-curves since the average of the self-intersection numbers of all curves in $D$ is $-2$ by Lemma 3.8. Then $D$ is spherical by Corollary 4.7. So $D$ is negative definite, and hence an ADE-configuration of $(-2)$-curves; see, for example, [20], §3. It is well known that such a configuration can be contracted to an ADE-singularity, which in turn gives $D \leq Z_{\text{num}}$ by Proposition 6.11.

If $D$ is not spherical, it contains a $(-1)$-curve $E$. As $D$ is 1-connected, we have $(D - E).E \geq 1$, whence $D.E \geq 0$. On the other hand, $D$ is negative definite, so $-1 \geq (D + E)^2 = -2 + 2D.E - 1$ and hence $D.E \leq 1$.

If $D.E = 0$, contract $E$ and start again with a smaller spherelike divisor.

If $D.E = 1$, then $(D + E)^2 = -1$. By the negative definiteness, $D + E$ is negatively filtered, whence $H^0(\mathcal{O}_{D+E}(D + E)) = 0$ by Lemma 3.6. On the other hand, the decomposition sequence yields a triangle $H^\bullet(\mathcal{O}_E(-D)) \to H^\bullet(\mathcal{O}_{D+E}) \to H^\bullet(\mathcal{O}_D)$. As $D.E = 1$, we get $H^\bullet(\mathcal{O}_{D+E}) = k$. Altogether, we find that $D + E$ is a $(-1)$-divisor and, moreover, can be contracted to a smaller $(-1)$-divisor $D'$ since $(D + E).E = 0$. Starting the proof again with this $D'$ shows that there has to be a $(-1)$-curve which can be contracted, inductively yielding a smooth point.

Otherwise we have $D.E = 0$ throughout, so that $D$ eventually becomes the pullback of a spherical divisor. □

Remark 6.14. With this proof one can also show that every negative definite $(-1)$-divisor can be iteratively contracted to a smooth point.

We end this section with two more examples, first of a divisor which contracts to an elliptic singularity, and then a spherelike divisor which is not negative definite.

Example 6.15 ([22], Example 4.20). Consider a minimal resolution of the surface singularity $\{x^3 + y^3 + z^4 = 0\} \subset k^3$, which is a minimal elliptic singularity, and in particular not rational. Its numerical cycle $Z_{\text{num}}$ is

![Diagram of Example 6.15](image)

The reduced divisor $D = (Z_{\text{num}})_{\text{red}}$ is a negative definite $(-3)$-divisor which can be twisted off to a single $(-3)$-curve. By contrast, an easy computation shows that $\chi(\mathcal{O}_{Z_{\text{num}}}) = 0$. Note that $Z_{\text{num}}$ is not 1-decomposable since $(Z_{\text{num}} - C).C = 2$ for all curves $C \prec Z_{\text{num}}$. Moreover, $Z_{\text{num}}$ is a negative definite divisor that is not Jacobi rigid.

Example 6.16. Let

$$D = D_n := B + E + C_1 + \cdots + C_n = -3 - 2 \cdots -2,$$

where $B^2 = -3$, $E^2 = -1$ and $C_i^2 = -2$. Then, clearly, $D.K = 0$ and $D$ is reduced, so pruning leaves yields both a 1-decomposition and a negative filtration.
Alternatively, one can first twist off the \((-2)\)-curves, and then blow down the remaining \((-1)\)-curve. In particular, \(D\) is essentially a \((-2)\)-curve.

We remark that \(D_n\) is not contractible to a rational singularity when \(n > 1\). This holds even in the analytic category since the equality \((B + 3E + C_1 + C_2)^2 = 2\) shows that \(D_n\) is not negative definite. The divisor \(D_1\) is not the pullback of any divisor, but contracts to a smooth point.

§ 7. Classification of minimal \((-n)\)-divisors

7.1. A graph-theoretical algorithm. \((-n)\)-divisors have a discrete, or combinatorial, flavour. More precisely, if we fix the self-intersection number \(-n\) and the topological type as a graph \(T\), then there are only finitely many building blocks, that is, minimal \((-n)\)-divisors. Here we are dealing only with exhausting these graphs algorithmically. There remains the question as to which of these graphs actually occur as dual intersection graphs of effective divisors (which are then necessarily \((-n)\)-divisors), and this is taken up in § 7.2.

Formally speaking, we consider weighted graphs with multiplicities below. Nonetheless, we will speak of ‘curves’ instead of ‘vertices’ and ‘self-intersection number’ instead of ‘weight’.

(1) Start with all curves of multiplicity 1 and unknown self-intersection.

(2) Form a list of partially defined divisors on \(T\) (some curves may not yet have an assigned self-intersection number), increasing in each step the multiplicity of one curve \(kC \mapsto (k + 1)C\) in such a way that the 1-decomposability condition is met. If \(k = 1\), this fixes \(C^2\).

(3) For the remaining entries, fill in unassigned self-intersection numbers in all possible ways admitting a negative filtration and satisfying \(D.K = n - 2\) or, equivalently, \(D^2 = -n\).

(4) Remove divisors having a \((-1)\)-curve which can be contracted or a \((-2)\)-curve which can be twisted off.

Graphs surviving the final step possess a 1-decomposition and a negative filtration and have self-intersection number \(-n\). Hence, if they are dual graphs of divisors, these are \((-n)\)-divisors by Corollary 3.2. The resulting list then needs to be condensed because it will contain multiple incarnations of the same divisor. Moreover, after Step (2), the resulting list will be infinite in general. Nonetheless, any \((-n)\)-divisor on a given graph will eventually be covered by this algorithm. The algorithm becomes more efficient if the following intermediate checks are also carried out during and after Step (2).

(2′) Remove an entry from the list if there is a subdivisor violating the property of being negatively closed, for example, \((-1)\) or \((-1)\) or \((-2)\).

(2″) Remove the divisors having a \((-1)\)-curve which can be contracted or a \((-2)\)-curve which can be twisted off.

By Proposition 4.19, \((-1)\)-divisors can always be worked down to \((-1)\)-curves. We now study spherelike divisors.

Example 7.1. If \(T\) is a tree with two, three or four vertices, then no spherelike divisor on \(T\) is minimal. We show this in the case when \(T\) is a four-chain; the
reasoning in the other cases is similar. The list after Step (2), cleaned up using (2'), is of the form

\[ -?, -1, -?, -?, \quad -?, -1, -2, -?, \quad -?, -1, -2, -? \]

together with the reduced chain \( -?, -?, -?, -? \), which cannot be a minimally spherelike divisor by Proposition 7.3. The \(-1\)-curves in the first and third divisors can be contracted. The \(-2\)-curve in the second divisor can be twisted off.

**Example 7.2.** We list all minimally spherelike divisors on five curves:

\[
\begin{align*}
-1 & -3 -1 -3 \quad -2 -3 -1 -3 -2 \\
-3 & -1 -3 -1 -3 \quad -2 -3 -1 -2 -3
\end{align*}
\]

Their existence as divisors (not just graphs) follows from Proposition 7.6 and can also be easily checked by hand.

**Proposition 7.3.** Every reduced spherelike divisor is essentially a \((-2)\)-curve.

**Proof.** We will show that \( D \) has a leaf (that is, a curve component intersecting only one other curve) of self-intersection \(-1\) or \(-2\): such a curve can be blown down or twisted off, obtaining a smaller divisor with the same properties.

For a contradiction, assume that \( D \) is a reduced negatively closed tree with \( C^2 \leq -3 \) for all its leaves \( C \). Write \( L \) for the subdivisor consisting of all, say \( l \), leaves, and let \( I := D - L \) be the complement of all inner curves.

Note that \( I \neq 0 \) unless \( D \) is either a single curve or two curves intersecting each other transversally at a point. Neither of these cases is possible under the assumption. Now the fact that \( I \neq 0 \) implies four things. First, \( I.L = l \) since \( D \) is reduced and each leaf intersects exactly one inner curve, with multiplicity 1. Second, \( L^2 \leq -3l \) since \( L \) is a disjoint union of \( l \) curves \( C \) with \( C^2 \leq -3 \). Third, \( I^2 < 0 \) since \( D \) is negatively closed. Fourth, \( l \geq 2 \). This contradicts the equality \( D^2 = -2 \):

\[
D^2 = (I + L)^2 = I^2 + 2I.L + L^2 = I^2 + 2l + L^2 \\
\leq I^2 + 2l - 3l = I^2 - l < -l. \quad \square
\]

**Remark 7.4.** This proof shows a bit more: if \( D \) is a reduced \((-3)\)-divisor that is not a chain, then it is essentially a \((-3)\)-curve. The provision is necessary, an example is \( -3 -1 -3 \).

### 7.2. From graphs to divisors.

The previous subsection contains a list of the weighted graphs which can occur as the dual intersection graphs of divisors with prescribed properties. However, it is a subtle problem to decide which of them can actually be realized by divisors.
Example 7.5. The following graph cannot be realized on any surface:

```
-2 2 2 2 1 3 3 1 2 2 2 2
```

Indeed, suppose the contrary and let $D$ be a rational reduced divisor with the dual intersection graph above. One can easily check that $D$ is 1-decomposable (since it is reduced) and negatively filtered. Moreover, $D^2 = -2$, so $D$ would be a spherelike divisor.

But if we iteratively blow down the $(-1)$-curves next to the two middle curves, then we will end up after five steps with $C_1 + C_2 = \begin{array}{c} 2 \\ -2 \end{array}$. Such a configuration cannot exist on any surface because it has $C_1(C_1 - 2C_2) = 0$, but $(C_1 - 2C_2)^2 > 0$ contrary to the Hodge index theorem ([5], Theorem V.1.9).

Many graphs can be realized, however. For the next proposition, we take a weighted tree $T$ with vertices denoted by $C$ and weights by $C^2$. Write $v(C)$ for the valency of a vertex $C$, that is, the number of vertices adjacent to $C$. For each vertex $C$, we introduce a local quantity $\sigma(C)$, which measures the excess positivity, and a global quantity $b(T)$, which counts the number of bad vertices:

\[
\sigma(C) := C^2 + v(C),
\]
\[
b(T) := \# \{ C \mid C^2 > -v(C) \} = \# \{ C \mid \sigma(C) \geq 1 \}.
\]

Moreover, the distance $d(C, C')$ between two vertices is the number of edges in the shortest path connecting them.

Proposition 7.6. Let $T$ be a finite weighted tree with badness $b = b(T)$ and bad vertices $C_1, \ldots, C_b$. There exists a reduced divisor on a rational surface with dual intersection graph $T$ if $b \leq 1$ or one of the following conditions holds:

1. $b = 2$, $d(C_1, C_2) = 1$ and $\sigma(C_1) = 1$, or $\sigma(C_2) = 1$, or $\sigma(C_1) = \sigma(C_2) = 2$;
2. $b = 2$, $d(C_1, C_2) = 2$ and $\sigma(C_1) = \sigma(C_2) = 1$;
3. $b = 3$ and $C_1, C_2, C_3$ form a 3-chain with $\sigma(C_1) = \sigma(C_2) = \sigma(C_3) = 1$.

Remark 7.7. Minimal examples in case (1) of the proposition are $\begin{array}{c} 0 \\ * \end{array}$ and $\begin{array}{c} 1 \\ -1 \end{array}$, and minimal examples in cases (2) and (3) are $\begin{array}{c} 0 \\ -2 \\ 0 \end{array}$ and $\begin{array}{c} 0 \\ -1 \\ 0 \end{array}$. Note that the first of these two can be obtained from the second by blowing up. Moreover, the last chain can be obtained by blowing up the intersection point of $\begin{array}{c} 1 \\ -1 \end{array}$, ignoring multiplicities.

Proof. It suffices to realize these trees as divisors $D$ which satisfy $C^2 = -v(C)$, that is, $\sigma(C) = 0$, for all but the bad vertices: any other weighted tree with smaller prescribed self-intersection numbers can be obtained by blowing up appropriate interior points on the curves of $D$.

Assume that $b = 0$. Then all leaves have weight $-1$. Thus we can contract each leaf in the numerical sense, that is, remove it and increase the weight of its neighbour by 1. Moreover, the condition also guarantees that this process can be iterated, stopping at a single vertex of weight 0. We can reverse this process on any surface with a 0-curve, for example, on $\mathbb{P}^1 \times \mathbb{P}^1$. 
Assume that $b = 1$. We apply the same procedure, but now we end up with the single bad vertex of weight $m = \sigma(C_1) > 0$. Let $F_m := \mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(m))$ be the Hirzebruch surface containing a smooth rational curve $L$ with $L^2 = m$. The process can again be reversed, starting with $L$ and ending with a tree $D$ of rational curves whose dual intersection graph is $T$.

Assume that $b = 2$. We can numerically contract all vertices except for the two bad vertices $C_1, C_2$ and the path between them, obtaining $x \overline{\overline{2}} \cdots \overline{\overline{2}} y$, where $x := \sigma(C_1) - 1$ and $y := \sigma(C_2) - 1$. If $C_1$ and $C_2$ are adjacent, that is, $d(C_1, C_2) = 1$, then each of the three cases can be realized on some Hirzebruch surface: $\overline{0 \overline{m}}$ on $F_m$, and $\overline{1 \overline{1}}$ on $\mathbb{P}^2$. The chain $\overline{0 \overline{2 \overline{1}} 0}$ is a double blow up of $\overline{1 \overline{1}}$.

Assume that $b = 3$. After contracting, we arrive at $\overline{0 \overline{1 \overline{1}} 0}$, already mentioned in the remark above. □

**Question 7.8.** Which graphs can be realized as dual graphs of divisors?

In a related vein: given a rational rigid (or negatively closed) divisor $D$ on some surface, can it be realized on a rational surface?

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