CLUSTER SCATTERING DIAGRAMS OF ACYCLIC AFFINE TYPE

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Abstract. We give an explicit construction of the cluster scattering diagram for any acyclic exchange matrix of affine type. We show that the corresponding cluster scattering fan coincides both with the mutation fan and with a fan constructed in the almost-positive roots model.

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Cluster scattering diagrams were introduced by Gross, Hacking, Keel, and Kontsevich \[11\] ("GHKK") to bring the powerful machinery of scattering diagrams to bear on the study of cluster algebras. In the process, GHKK proved (or corrected and proved) longstanding structural conjectures about cluster algebras. Since then, cluster scattering diagrams have become an important tool in the theory.

The key result of GHKK gives a recursive construction of the cluster scattering diagram, but no general, concrete description. A primary output of a cluster scattering diagram is a collection of theta functions, which, in essence, is a superset of the cluster monomials. However, the definition of a theta function, while combinatorial, requires listing all possible “broken lines” with a certain starting slope and endpoint. This can be prohibitively complicated in high dimension and/or when the cluster scattering diagram has infinitely many walls. Furthermore, computing structure constants for the multiplication of theta functions involves an even more complicated enumeration of broken lines. For all of these reasons, there is a need for combinatorial models of cluster scattering diagrams, to make the machinery of scattering diagrams concrete in specific cases.

For cluster algebras of acyclic finite type, the construction of cluster scattering diagrams is easy, by combining existing results on cluster algebras and results of GHKK. (See \[26\], Remark 4.8.) As an alternative, \[26\], Theorem 4.3] gives a tidy construction of cluster scattering diagrams of acyclic finite type using the machinery of shards \[17\], \[19\] and sortable elements \[21\], \[30\].

We will say that an exchange matrix \(B\) is of affine type if it is mutation-equivalent to an acyclic exchange matrix \(B'\) whose underlying Cartan matrix is of affine type in the usual sense. As a consequence of \[37\], Theorem 3.5], if \(B\) is of rank \(\geq 3\), then it is of affine type if and only if the associated cluster algebra has linear growth but is not finite. (For a less restrictive definition of affine type, with different motivations, see \[7\].)

In this paper, we construct cluster scattering diagrams of acyclic affine type with principal coefficients, again using the machinery of shards and sortable elements. We draw on results of \[32\] that construct the affine \(g\)-vector fan as the doubled Cambrian fan and results of \[34\] that establish an almost-positive roots model in affine type. In a future paper in preparation \[35\], we will use the results of this paper to compute theta functions and study their structure constants.

In this paper, we also show that, in the acyclic affine case, the cluster scattering fan (the fan cut out by the walls of the cluster scattering diagram \[27\]) coincides with two other fans: a fan called the mutation fan \[25\] that exists for any exchange matrix \(B\), and the generalized associahedron fan, constructed in \[34\] for \(B\) of acyclic affine type. Since the generalized associahedron fan is constructed as an explicit clique complex of its rays, the coincidence of these three fans provides an explicit construction of the cluster scattering fan and mutation fan in terms of rays.

The coincidence of the cluster scattering fan and mutation fan in acyclic affine type leads easily to a more general result: The cluster scattering fan and the mutation fan coincide in the affine case, even without the assumption of acyclicity.

Mutation fans were defined in \[25\] in order to construct universal geometric cluster algebras. In another future paper, we plan to use the fact that the mutation fan and the generalized associahedron fan coincide to construct universal geometric cluster algebras of affine type, proving a conjecture from \[25\].
The proofs in this paper draw heavily on three sets of tools: the sortable elements, Cambrian lattices/fans and doubled Cambrian fans of \([20, 21, 29, 30, 32]\), the shards/lattice theory of \([17, 19, 20, 22, 24]\); and the almost positive roots/generalized associahedra of \([4, 8, 9, 15, 33, 34]\). Some parts of the proofs use new results about these tools (most often generalizing something that was known in finite type). Beyond the basic definitions, tools from the theory of scattering diagrams do not play a role in the proofs. This is to be expected, because there are no general tools to explicitly construct cluster scattering diagrams. Instead, we must rely on combinatorial models.

Remark 1.1. Ours is not the only effort to understand cluster scattering diagrams using different tools. For example, Bridgeland \([3]\) has a general construction of cluster scattering diagrams in terms of motivic Hall algebras. In addition, Hanson, Igusa, Kim, and Todorov \([12]\) worked out the local structure of a cluster scattering fan of affine type near its limiting ray. Working in the representation-theoretic language of semi-invariant pictures, they described this local structure in terms of standard wall-and-chamber structures of Nakayama algebras.

2. Main results

We now describe our main results in more detail, leaving careful definitions until Section 3.

2.1. The cluster scattering diagram. We give several characterizations of the cluster scattering diagram in acyclic affine type, all having to do with a cutting relation on hyperplanes orthogonal to roots, and most phrased in terms of shards.

An exchange matrix \(B\) determines a Cartan matrix \(A\), a root system \(\Phi \subset V\), and a Coxeter group \(W\) as described in Section 3.1. An acyclic \(B\) also determines a Coxeter element \(c\) of \(W\) and, in this case, \(A\) and \(c\) together amount to the same information as \(B\). (Unfortunately, over the years, these equivalent ways of encoding the same data led to an ill-assorted bunch of notations; in order not to confuse readers already familiar with the topic we resist the urge to rationalize these notations here.) When \(B\) is acyclic and of affine type, \(A\), \(\Phi\), and \(W\) are of affine type in the usual sense.

We write \(\text{Scat}^T(B)\) for the (transposed) cluster scattering diagram for \(B\) with principal coefficients. (The transpose, relative to the conventions of \([11]\) is necessary to keep our conventions in line with \([10]\).) This is a collection of walls \((\mathfrak{d}, f_\mathfrak{d})\), where \(\mathfrak{d}\) is a codimension-1 rational cone in \(V^*\) and \(f_\mathfrak{d}\) is a formal power series in variables \(\hat{y}\) indexed by the simple roots. Specifically, if \(\mathfrak{d} \subset \beta^+\) with \(\beta\) a positive primitive vector, then \(f_\mathfrak{d}\) is a univariate formal power series evaluated on \(y^\beta\), the monomial whose exponents are given by the simple-root coordinates of \(\beta\).

Remark 2.1. Here we construct \(\text{Scat}^T(B)\) with principal coefficients. By \([27\text{ Proposition 2.6}]\), for arbitrary coefficients (chosen so that the extended exchange matrix has full rank), the scattering fan can then be obtained from the principal-coefficients scattering diagram by making a substitution in all scattering terms. The substitution amounts to replacing the monomials \(\hat{y}_i\) that appear here with the more general monomials \(\hat{y}_i\) defined in \([10\text{ (7.11)}]\).

Following constructions for finite type, in Section 3.3 we will define a cutting relation on hyperplanes orthogonal to roots (including the hyperplane orthogonal
to the imaginary root $\delta$). Each hyperplane is cut along its intersection with other hyperplanes into finitely many pieces, whose closures are called **shards**. The shards that intersect the Tits cone are in bijection with the join-irreducible elements of the weak order on $W$. Given a join-irreducible element $j$, we write $\text{Sh}(j)$ for the corresponding shard and let $f_j$ be $1 + \hat{y}^\beta$, where $\beta$ is the positive root orthogonal to $\text{Sh}(j)$.

We now introduce a distinguished wall called the **imaginary wall** ($\varnothing_\infty, f_\infty$). It is supported on the hyperplane $\delta^\perp$ that forms the boundary of the Tits cone and it is defined in terms of a skew-symmetric bilinear form $\omega_c$ that depends on $B$. We set $\Phi_{\text{fin}}^+=\{\beta \in \Phi_{\text{fin}} : \omega_c(\beta, \delta) > 0\}$, where $\Phi_{\text{fin}}$ denotes the underlying finite root system, and define $\varnothing_\infty = \{x \in \partial \text{Tits}(A) : \langle x, \beta \rangle \leq 0, \forall \beta \in \Phi_{\text{fin}}^+\}$. We also define $f_\infty$ to be

$$
(2.1) \quad f_\infty = \begin{cases} 
(1 - \hat{y}^\delta)^{-2} & \text{if } W \text{ is not of type } A_{2k}^{(2)}, \\
(1 + \hat{y}^\delta) \cdot (1 - \hat{y}^\delta)^{-2} & \text{if } W \text{ is of type } A_{2k}^{(2)}.
\end{cases}
$$

The $c$-sortable elements are certain elements of $W$ that, in finite type, play a role in Coxeter-Catalan combinatorics. We write $\text{J Irr}_c$ for the set of join-irreducible $c$-sortable elements.

Finally, define the scattering diagram $\text{DCScat}(A, c)$ as

$$
(2.2) \quad \{(\text{Sh}(j), f_j) : j \in \text{J Irr}_c(W)\} \cup \{(-\text{Sh}(j), f_j) : j \in \text{J Irr}_c^{-1}(W)\} \cup \{\varnothing_\infty, f_\infty\}.
$$

In general, the definition of scattering diagrams allows multiple copies of the same wall, but we don’t allow them in $\text{DCScat}(A, c)$. Thus $(2.2)$ should be understood in the ordinary sense of set union, rather than multiset union. There is some overlap between the first two sets of walls, but simple reasoning based on Proposition 5.17 shows that the overlap consists of a finite set of walls.

Our first main result is the following theorem.

**Theorem 2.2.** If $B$ is an acyclic exchange matrix of affine type and $A$ and $c$ are the corresponding Cartan matrix and Coxeter element, then $\text{Scat}^T(B)$ is $\text{DCScat}(A, c)$.

In [34], we combinatorially defined a set $\Phi_c$ of **almost positive Schur roots** and showed that it consists of the denominator vectors of cluster variables associated to $B$ together with the imaginary root $\delta$. In Proposition 5.24 we characterize the normal vectors to walls appearing in $(2.2)$ as precisely the positive roots in $\Phi_c$, thus obtaining the following corollary of Theorem 2.2.

**Theorem 2.3.** If $B$ is an acyclic exchange matrix of affine type, then the set of positive normals to walls of $\text{Scat}^T(B)$ is $\Phi_c \cap \Phi^+$.

Inspired by Theorem 2.3, we can also define the cluster scattering diagram directly in terms of $\Phi_c$, the cutting relation, and the form $\omega_c$. Given a pair $\beta, \gamma$ of positive roots, we say $\gamma$ **cuts** $\beta$ if $\beta^\perp$ is cut along its intersection with $\gamma^\perp$ in the construction of shards. Write $\text{cut}(\beta)$ for the set of positive roots $\gamma$ that cut $\beta$. This is always a finite set. Define $f_\beta$ to be

$$
(2.3) \quad \{x \in V^* : \langle x, \beta \rangle = 0 \text{ and } \langle x, \gamma \rangle \leq 0, \forall \gamma \in \text{cut}(\beta) \text{ with } \omega_c(\gamma, \beta) > 0\}.
$$

If $\beta$ is real, define $f_\beta$ to be $1 + \hat{y}^\beta$. The unique imaginary root in $\Phi_c$ is $\delta$, and we define $f_\delta = f_\infty$ as in $(2.1)$.

**Theorem 2.4.** If $B$ is an acyclic exchange matrix of affine type with corresponding Coxeter element $c$ and root system $\Phi$, then $\text{Scat}^T(B)$ is $\{(\varnothing_\beta, f_\beta) : \beta \in \Phi_c \cap \Phi^+\}$. 
The construction of cluster scattering diagrams in [11] defines a wall with normal vector $\beta$ to be **outgoing** if (in our notation) the vector $\omega_c(\cdot, \beta)$ is not in the wall. We define a cone to be **gregarious** if $-\omega_c(\cdot, \beta)$ is in the relative interior of the cone.

**Theorem 2.5.** Suppose $B$ is acyclic of affine type, associated to a root system $\Phi$ and a Coxeter element $c$. Then $\text{Scat}^T(B)$ can be constructed entirely of gregarious walls, with exactly one wall in each hyperplane $\beta^\perp$, where $\beta$ runs over all positive roots in $\Phi_c$, including the imaginary root $\delta$.

Indeed, we can add detail to Theorem 2.5 by describing walls in terms of shards and $\omega_c$.

**Theorem 2.6.** Suppose $B$ is acyclic of affine type, associated to a root system $\Phi$ and a Coxeter element $c$. Then the walls of $\text{Scat}^T(B)$ are

- the unique shard in $\beta^\perp$ containing $-\omega_c(\cdot, \beta)$ for each real root $\beta \in \Phi^\text{re}_c$ and
- the union of all shards in $\delta^\perp$ containing $-\omega_c(\cdot, \delta)$,

with $f_\beta$ and $f_\delta$ as in Theorem 2.4.

In general, the cluster scattering diagram is defined up to a notion of equivalence. We emphasize, however, that the descriptions of $\text{Scat}^T(B)$ in Theorems 2.2, 2.4 and 2.6 are identical, not merely equivalent.

The arguments of this paper simplify to recover a finite-type result [26, Proposition 4.10], and in fact, augment the finite-type result by giving explicit inequalities for the walls. See Theorem 6.15.

The finite-type analogs of Theorems 2.5 and 2.6 are [26, Proposition 4.12] and [26, Corollary 4.13]. Preliminary results in ongoing work with Greg Muller and Shira Viel suggest that similar results hold for cluster scattering diagrams associated to marked surfaces. Similar results also hold trivially in rank 2. Since all of these examples are special in that they involve mutation-finite exchange matrices, it seems premature to make a general conjecture. However, it seems appropriate to at least pose some questions. (Rather than $\omega_c$, we use the analogous form $\omega$ that can be defined directly from $B$, because we are not restricting the questions to acyclic $B$.)

**Question 2.7.** For a general exchange matrix $B$, can the (transposed) cluster scattering diagram for $B$ be constructed with at most one wall in each hyperplane, and with that wall being gregarious?

**Question 2.8.** For a general exchange matrix $B$, are all of the walls orthogonal to (real or imaginary) roots? For a (real or imaginary) root $\beta$, is the wall orthogonal to a root $\beta$ the union of all shards in $\beta^\perp$ that contain $-\omega(\cdot, \beta)$?

### 2.2. Three fans

Any consistent scattering diagram defines a complete (and often infinite) fan [27, Theorem 3.1]. We write $\text{ScatFan}^T(B)$ for the scattering fan associated to $\text{Scat}^T(B)$.

The mutation fan $\mathcal{F}_B$ associated to an exchange matrix $B$ encodes the piecewise-linear geometry of matrix mutation and is central to the notion of universal geometric cluster algebras.

In [34], in acyclic finite or affine type, we defined a complete fan $\mathcal{F}_c(\Phi)$ whose rays are spanned by the roots $\Phi_c$, and whose cones are defined by a certain compatibility relation on $\Phi_c$. This fan contains the fan of denominator vectors of cluster variables as a subfan. There is a piecewise-linear homeomorphism $\nu_c$ that takes
denominator vectors to $g$-vectors. This map is linear on every cone of $\text{Fan}_c(\Phi)$, so $\nu_c(\text{Fan}_c(\Phi))$ is a fan that contains the $g$-vector fan as a subfan, but also subdivides the space outside the $g$-vector fan.

The following is our main theorem about these three fans.

**Theorem 2.9.** Suppose that $B$ is an acyclic exchange matrix of affine type, let $\Phi$ be the associated root system, and let $c$ be the associated Coxeter element. Then $\text{ScatFan}_T(B)$, $\mathcal{F}_{B^T}$, and $\nu_c(\text{Fan}_c(\Phi))$ all coincide.

We will use Theorem 2.9 to prove the following more general theorem about affine scattering fans and mutation fans without the assumption of acyclicity.

**Theorem 2.10.** If $B$ is an exchange matrix of affine type, then the scattering fan $\text{ScatFan}_T(B)$ and the mutation fan $\mathcal{F}_{B^T}$ coincide.

The scattering fan and the mutation fan do not always coincide, but there is a refinement relation in general, as described in the following theorem [27, Theorem 4.10].

**Theorem 2.11.** $\text{ScatFan}_T(B)$ refines $\mathcal{F}_{B^T}$ for any exchange matrix $B$.

The following conjecture is [27, Conjecture 4.11].

**Conjecture 2.12.** Given an exchange matrix $B$, the scattering fan $\text{ScatFan}_T(B)$ coincides with the mutation fan $\mathcal{F}_{B^T}$ if and only if either

(i) $B$ is $2 \times 2$ and of finite or affine type or

(ii) $B$ is $n \times n$ for $n > 2$ and of finite mutation type.

Since an exchange matrix of affine type is in particular of finite mutation type, Theorem 2.10 provides evidence for Conjecture 2.12.

### 3. Definitions and background

In this section, we give the detailed definitions necessary to understand statements of the main results. (However, we postpone the definition of the mutation fan until Section 7.3 because it plays no role in the paper until then.)

**3.1. Root systems and Coxeter groups.** We assume the basic background on Root systems and Coxeter groups. Standard references include [1, 2, 13]. Much of the background can be found, with the notational conventions matching this paper, in [33, 34] and, with slightly different conventions, in [31, 32].

The initial data for a cluster scattering diagram or cluster algebra with principal coefficients is an $n \times n$ **exchange matrix**: a skew-symmetrizable integer matrix $B = [b_{ij}]$. We make a particular choice of the skew-symmetrizing constants (the $d_i$ such that $d_i b_{ij} = -d_j b_{ji}$ for all $i, j \in \{1, \ldots, n\}$). Specifically, we choose the $d_i$ so that $d_i^{-1}$ is an integer for each $i$ and $\gcd(d_i^{-1}; i \in \{1, \ldots, n\}) = 1$.

The exchange matrix $B$ determines a Cartan matrix $A = [a_{ij}]$ by $a_{ii} = 2$ for $i \in \{1, \ldots, n\}$ and $a_{ij} = -b_{ij}$ for $i, j \in \{1, \ldots, n\}$ with $i \neq j$. The matrix $A$ is symmetrizable, specifically with $d_i a_{ij} = d_j a_{ji}$ for all $i, j \in \{1, \ldots, n\}$. We make what is essentially the usual construction of roots, weights, co-roots, etc. associated to $A$. (We depart from the standard conventions in Lie theory in that we put roots and co-roots in the same space and put weights and co-weights in the dual space. This departure makes sense in our context as explained in [26, Remark 2.1] or [34, Remark 2.1].) Let $V$ be an $n$-dimensional real vector space with a basis $\alpha_1, \ldots, \alpha_n$. 
The $\alpha_i$ are the **simple roots**. The **simple co-roots** are $\alpha_i^\vee = d_i^{-1} \alpha_i$. Because of how we chose the constants $d_i$, the co-root lattice $Q^\vee = \text{Span}_\mathbb{Z}(\alpha_1^\vee, \ldots, \alpha_n^\vee)$ is a finite-index sublattice of the root lattice $Q = \text{Span}_\mathbb{R}(\alpha_1, \ldots, \alpha_n)$. The Cartan matrix defines a symmetric bilinear form $K$ on $V$ by $K(\alpha_i^\vee, \alpha_j) = a_{ij}$.

Let $V^*$ be the dual space of $V$, with a basis $\rho_1, \ldots, \rho_n$ defined by $\langle \rho_i, \alpha_j^\vee \rangle = \delta_{ij}$, where $\langle \cdot, \cdot \rangle : V^* \times V \to \mathbb{R}$ is the usual pairing and $\delta_{ij}$ is the Kronecker delta function. The $\rho_i$ are the **fundamental weights**. The **fundamental co-weights** are the basis $\rho_1^\vee, \ldots, \rho_n^\vee$ for $V^*$ defined by $\langle \rho_i^\vee, \alpha_j \rangle = \delta_{ij}$.

We write $s_i$ for the linear map on $V$ defined by $s_i(v) = v - K(\alpha_i^\vee, \alpha_i)\alpha_i$. The set $S = \{s_i : i = 1, \ldots, n\}$ of **simple reflections** generates the associated Coxeter group $W$. We will sometimes use $S$ as an alternative indexing set (rather than $\{1, \ldots, n\}$), writing $\alpha_i$ for $\alpha_i$ and $\rho_s$ for $\rho_i$ when $s = s_i$.

The set $\Phi^\vee$ of **real roots** associated to $A$ is the set $\{w\alpha_i : w \in W, i = 1, \ldots, n\}$. There are also **imaginary roots** that arise in Lie theory. We will not need the general definition of imaginary roots. We will only need imaginary roots associated to a Cartan matrix of affine type, and we describe these specifically in Section 3.3.

We write $\Phi$ for the **root system** associated to $A$: the set of (real and imaginary) roots. The notation $\Phi^+$ stands for the set $\Phi \cap \text{Span}_{\mathbb{R}_+}(\alpha_1, \ldots, \alpha_n)$ of positive roots.

The **inversion set** of $w \in W$ is the set $\text{inv}(w) = \{\beta \in \Phi^+ : w(\beta) \in -\Phi^+\}$. We use the symbol $\leq$ for the **weak order** on $W$, defined by $v \leq w$ if and only if $\text{inv}(v) \subseteq \text{inv}(w)$, and the symbol $\prec$ for cover relations in the weak order. These cover relations are $w \prec ws$ for $w \in W$ and $s \in S$ such that $\ell(w) < \ell(ws)$, where $\ell$ is the usual length function with respect to $S$. The weak order is a meet-semilattice, and is a lattice if and only if $W$ is finite. An element $j \in W$ is **join-irreducible** if and only if it covers a unique element.

We write $T$ for the set $\{ws^{-1} : w \in W, s \in \mathcal{S}\}$ of **reflections** in $W$. We write $\beta \mapsto t_\beta$ for the bijection between real positive roots and $T$, where $t_\beta$ is the map $v \mapsto v - K(\beta^\vee, v)\beta$ on $V$. Given $t \in T$ we write $\beta_t$ for the corresponding positive real root.

The **cover reflections** of $w \in W$ are the reflections $t$ such that $tw \prec w$. Thus there is exactly one cover reflection $t = ws^{-1}$ for each cover relation of the form $ws \prec w$. The root $\beta_t$ is the unique element of $\text{inv}(w) \setminus \text{inv}(ws)$. We write $\text{cov}(w)$ for the set of cover reflections of $w$.

If $I \subseteq S$, then the **parabolic subgroup** $W_I$ is the subgroup of $W$ generated by $I$. For each $w \in W$ and $I \subseteq S$, there is a unique element $w_I \in W_I$ such that $\text{inv}(w_I) = \text{inv}(W) \cap \Phi_I$. In particular, we will need the case where $I$ is obtained by removing one element from $S$, so for each $s \in S$, we define $\langle s \rangle$ to mean $S \setminus \{s\}$.

### 3.2. Coxeter elements and sortable elements

The sign information lost in passing from $B$ to $A$ is precisely an orientation of the Dynkin diagram of $A$. If this orientation is acyclic, then we say that $B$ is **acyclic**. In that case, the lost sign information is equivalent to a choice of Coxeter element $c$ of $W_I$, namely the product of the simple reflections $S$ ordered so that $s_i$ precedes $s_j$ if $b_{ij} > 0$. We will often assume that the simple roots/reflections have been numbered such that this Coxeter element is $s_1 \cdots s_n$, but there may be additional reduced words for this same Coxeter element. Given a Coxeter element $c$ and given $s \in S$, we say $s$ is **initial** in $c$ if there exists a reduced word for $c$ having $s$ as its first letter. Similarly, $s$ is **final** in $c$ if there is a reduced word for $c$ having $s$ as its last letter.

If $s$ is initial or final in $c$, then $scs$ is also a Coxeter element. If $s$ is initial in $c$, then
se is a Coxeter element of $W_\lambda$, and if $s$ is final in $c$, then $cs$ is a Coxeter element of $W_\lambda$.

Closely related to the Coxeter element $c$ are a skew-symmetric bilinear form $\omega_c$ and another form $E_c$, defined by

$$\omega_c(\alpha_i, \alpha_j) = \begin{cases} a_{ij} & \text{if } i > j, \\ 0 & \text{if } i = j, \text{ or } E_c(\alpha_i, \alpha_j) = \begin{cases} a_{ij} & \text{if } i > j, \\ 1 & \text{if } i = j, \text{ or } \\ -a_{ij} & \text{if } i < j. \end{cases} \end{cases}$$

We follow [34] in defining the piecewise linear homeomorphism $\nu_c : V \rightarrow V^*$. (Cf. [31] Section 5.3.) For a vector $\beta \in V$, write $I = \{ i : (\rho_i^\vee, \beta) < 0 \}$ and define $\beta_+ = \sum_{i \in I} (\rho_i^\vee, \beta)$. Then

$$\nu_c(\beta) = -\sum_{i \in I} (\rho_i^\vee, \beta)\rho_i - \sum_{i \in I} E_c(\alpha_i^\vee, \beta_+)\rho_i.$$ 

In particular, if $\beta$ is in the nonnegative linear span of $\Pi$, then $\nu_c(\beta) = -\sum_{i=1}^n E_c(\alpha_i^\vee, \beta)\rho_i = -E_c(\cdot, \beta)$.

We now give a definition of $c$-sortable elements, together with a distinguished reduced word for each $c$-sortable element, called its $c$-sorting word. The $c$-sortable elements are usually defined (e.g. in [21, 30]) by characterizing their $c$-sorting words, but here we are content to define both concepts by a recursion. As a base case for the recursion, the identity element is $c$-sortable (and has empty $c$-sorting word) for any $c$ in any $W$. Let $v \in W$ and suppose $s$ is initial in $c$. If $s \leq v$ (or equivalently if $\ell(sv) < \ell(v)$), then $v$ is $c$-sortable if and only if $sv$ is $scs$-sortable. In this case, the $c$-sorting word for $v$ is $s$ followed by the $scs$-sorting word for $sv$. If $s \not\leq v$, then $v$ is $c$ sortable if and only if $v \in W_{(s)}$ and is $sc$-sortable. In this case, the $c$-sorting word for $v$ equals the $sc$-sorting word for $w$.

If we write a reduced word $s_1 \cdots s_n$ for $c$, then the $c$-sorting word for a $c$-sortable element $v$ consists of $k$ copies of $s_1 \cdots s_n$ with $k \geq 0$ followed by the $c$-sorting word for a $c$-sortable element $v'$ contained in a standard parabolic subgroup.

### 3.3. Transposed cluster scattering diagrams

We follow [29] Section 2] in constructing principal-coefficients cluster scattering diagrams in the context of root systems. In particular we leave out some extra dimensions that occur in scattering diagrams in [11], and we take a global transpose relative to [11]. (The purpose of the transpose is to match the conventions of [10].)

We write $P = \text{Span}_Z(\rho_1, \ldots, \rho_n)$ for the weight lattice in $V^*$ and denote by $Q^+ = \{ \beta \in Q \setminus \{ 0 \} : (\rho_i^\vee, \beta) \geq 0, i = 1, \ldots, n \}$ the positive part of the root lattice. Let $\omega : V \times V \rightarrow \mathbb{R}$ be the bilinear form given by $\omega(\alpha_i^\vee, \alpha_j) = b_{ij}$. When $B$ is acyclic, $\omega$ coincides with $\omega_c$.

We take $x_1, \ldots, x_n$ and $y_1, \ldots, y_n$ to be indeterminates and for $i$ from 1 to $n$, we define $y_i = y_i x_1^{-b_{1i}} \cdots x_n^{-b_{ni}}$. Given $\lambda = \sum_{i=1}^n a_i \rho_i \in P$ and $\beta = \sum_{i=1}^n c_i \alpha_i \in Q$, define $x^\lambda y^\beta$ to be $x_1^{a_1} \cdots x_n^{a_n} y_1^{b_{11}} \cdots y_n^{b_{nn}}$. Taking $k$ to be a field of characteristic zero, we work in the ring $k[x_1^{\pm 1}, \ldots, x_n^{\pm 1}, [y_1, \ldots, y_n]]$ of formal power series in the $y_i$ with coefficients Laurent polynomials in the $x_i$. Let $m$ be the ideal consisting of series with constant term zero. Let $k[[y]]$ be the subring consisting of formal power series in the $y_i$ with coefficients in $k$. 

A wall is a pair \((\mathfrak{d}, f_\mathfrak{d})\), where \(\mathfrak{d}\) is a codimension-1 subset of \(V^*\) and \(f_\mathfrak{d}\) is in \(k[[\hat{y}]]\), subject to these requirements:

(i) \(\mathfrak{d}\) is contained in \(\beta^\perp\) for some \(\beta \in Q^+\) and is defined by inequalities of the form \((\cdot, \phi) \leq 0\) for \(\phi \in Q\). Importantly, we assume that \(\beta\) has been chosen to be primitive in \(Q\), meaning that if \(a\beta \in Q\) for some positive rational \(a\), then \(a\) is an integer.

(ii) \(f_\mathfrak{d}\) is a univariate power series evaluated at \(\hat{y}\beta\) for this primitive \(\beta\).

When we wish to identify the primitive vector \(\beta\) associated to a wall, we will write \(f_\mathfrak{d}(\hat{y}\beta)\) for \(f_\mathfrak{d}\). Two walls are parallel if they have the same \(\beta\). We will refer to \(f_\mathfrak{d}\) as the scattering term on the wall \((\mathfrak{d}, f_\mathfrak{d})\).

A scattering diagram is a collection \(\mathcal{D}\) of walls that may be infinite but must satisfy the following finiteness condition: For all \(k \geq 1\), the set of walls \((\mathfrak{d}, f_\mathfrak{d}) \in \mathcal{D}\) with \(f_\mathfrak{d} \neq 1\) modulo \(m^{k+1}\) is finite. The point of this finiteness condition is that for any given \(k\), we can make computations in the scattering diagram on monomials of total degree \(\leq k\) by considering only finitely many walls, and the full computations can be made as limits of formal power series. Note that since each scattering term \(f_\mathfrak{d}(\hat{y}\beta)\) is a formal power series in \(\hat{y}\beta\), a sufficient condition implying the finiteness condition is that there be only finitely many walls in each hyperplane \(\beta^\perp\).

A generic path for a scattering diagram \(\mathcal{D}\) is a piecewise differentiable path \(\gamma: [0, 1) \to V^*\) that does not pass through the relative boundary of any wall or the intersection of any two non-parallel walls, has endpoints not contained in any wall, and only intersects walls by crossing them transversely. Given a generic path \(\gamma\) and a wall \((\mathfrak{d}, f_\mathfrak{d})\) of \(\mathcal{D}\) with \(\gamma(t) \in \mathfrak{d}\) for some \(t \in (0, 1)\), the wall-crossing automorphism \(p_{\gamma, \mathfrak{d}}\) is given by

\[
\begin{align*}
\text{(3.1)} & & p_{\gamma, \mathfrak{d}}(x^\lambda) & = x^\lambda f_\mathfrak{d}^{(\lambda, \pm \beta^\vee)}, \\
\text{(3.2)} & & p_{\gamma, \mathfrak{d}}(\hat{y}^\phi) & = \hat{y}^\phi f_\mathfrak{d}^{(\pm \beta^\vee, \phi)},
\end{align*}
\]

where \(\beta^\vee\) is the normal vector to \(\mathfrak{d}\) that is contained in \(Q^+\) and is primitive in \(Q^\vee\), taking \(+\beta^\vee\) if \(\gamma\) crosses \(\mathfrak{d}\) against the direction of \(\beta^\vee\) (as \(t\) increases) or \(-\beta^\vee\) if \(\gamma\) crosses \(\mathfrak{d}\) in the direction of \(\beta^\vee\).

The path-ordered product \(p_{\gamma, \mathcal{D}}\) for a generic path \(\gamma\) is essentially the composition of all of the wall-crossing automorphisms for walls crossed by \(\gamma\). This composition makes sense even when the path crosses infinitely many walls; indeed the finiteness condition on \(\mathcal{D}\) allows us to compute the composition modulo \(m^{k+1}\) as a finite composition for each \(k \geq 1\), and \(p_{\gamma, \mathcal{D}}\) is the limit (in the sense of formal power series) of these finite compositions as \(k \to \infty\).

A scattering diagram \(\mathcal{D}\) is consistent if \(p_{\gamma, \mathcal{D}}\) depends only on the endpoints \(\gamma(0)\) and \(\gamma(1)\). Two scattering diagrams \(\mathcal{D}\) and \(\mathcal{D}'\) are equivalent if and only if \(p_{\gamma, \mathcal{D}} = p_{\gamma, \mathcal{D}'}\) whenever \(\gamma\) is generic for both \(\mathcal{D}\) and \(\mathcal{D}'\).

A wall \((\mathfrak{d}, f_\mathfrak{d})\) with associated primitive normal \(\beta\) is incoming if the vector \(\omega(\cdot, \beta)\) is in \(\mathfrak{d}\). Otherwise, \((\mathfrak{d}, f_\mathfrak{d})\) is outgoing. The crucial result of GHKK is [11 Theorem 1.12], which says that a consistent scattering diagram can be obtained by starting with walls \(\{(\alpha_i^-, 1 + \hat{y}_i) : i = 1, \ldots, n\}\) and appending outgoing walls, and that such a scattering diagram is unique up to equivalence. This is the transposed cluster scattering diagram with principal coefficients, which we denote by \(\text{Scat}^T(B)\).
A subset $C$ of $V^*$ is a **closed convex cone** if it is closed under addition and closed under nonnegative scaling and also closed in the usual sense. A **face** of $C$ is a cone $F$ that is a subset of $C$ such that if $x, y \in C$ and $F$ intersects the line segment $[x, y]$ at a point other than $x$ or $y$, then the whole segment $[x, y]$ is in $F$. A **facet** of $C$ is a face of $C$ with dimension one less than the dimension of $C$. A collection $\mathcal{F}$ of closed convex cones is a **fan** if it satisfies the following two conditions: if $C \in \mathcal{F}$ then every face of $C$ is in $\mathcal{F}$; and if $C, D \in \mathcal{F}$ then $C \cap D$ is a face of $C$ and a face of $D$. The notation $|\mathcal{F}|$ stands for the union of the cones in $\mathcal{F}$. A fan $\mathcal{F}$ in $V^*$ is called **complete** if $|\mathcal{F}|$ is all of $V^*$.

We now define a complete fan associated to a consistent scattering diagram. (See [27] for details.) Essentially, this is the fan “cut out by the walls” of the scattering diagram, but some care must be taken to make that notion precise.

Suppose $\mathcal{D}$ is a scattering diagram and $\beta \in Q^+$. The **rampart** of $\mathcal{D}$ associated to $\beta$ is the union of the walls of $\mathcal{D}$ that are contained in $\beta^\perp$. (In acyclic affine type, we will construct the transposed cluster scattering diagram with at most one wall in each hyperplane. As a consequence, each rampart is a single wall, so for the acyclic affine case, it is safe to replace “rampart” with “wall” throughout this definition.) Given a point $p \in V^*$, let $\text{Ram}_\mathcal{D}(p)$ stand for the set of ramparts of $\mathcal{D}$ containing $p$.

The **support** $\text{supp}(\mathcal{D})$ of a scattering diagram $\mathcal{D}$ is the union of the walls of $\mathcal{D}$. Every consistent scattering diagram is equivalent to a scattering diagram $\mathcal{D}$ with (uniquely) **minimal support**, meaning that $\text{supp}(\mathcal{D}) \subseteq \text{supp}(\mathcal{D}')$ for all $\mathcal{D}'$ equivalent to $\mathcal{D}$.

Now suppose $\mathcal{D}$ is consistent and has minimal support. We define an equivalence relation on $V^*$ by declaring that $p$ is $\mathcal{D}$-**equivalent** to $q$ if and only if there exists a path $\gamma$ with endpoints $p$ and $q$ on which $\text{Ram}_\mathcal{D}(\cdot)$ is constant. A $\mathcal{D}$-**class** is a $\mathcal{D}$-equivalence class. The closure of a $\mathcal{D}$-class is called a $\mathcal{D}$-**cone**. Indeed, every $\mathcal{D}$-cone $C$ is the closure of a unique $\mathcal{D}$-class, and this class contains the relative **interior** $\text{relint}(C)$ of $C$. Each $\mathcal{D}$-cone is a closed convex cone, and the collection $\text{Fan}(\mathcal{D})$ of all $\mathcal{D}$-cones and their faces is a complete fan in $V^*$. We write $\text{ScatFan}^T(B)$ for the **transposed scattering fan** $\text{Fan}(\text{Scat}^T(B))$.

### 3.4. Shards

We now discuss a notion, for $\beta \in \Phi$, of cutting the hyperplane $\beta^\perp$ into shards, generalizing some constructions from finite Coxeter groups and hyperplane arrangements. More specifically, we define shards and prove that the shards that intersect the Tits cone correspond to join-irreducible elements.

A **rank-2 subsystem** $\Phi'$ of $\Phi$ is a rank-2 root system that is the intersection of $\Phi$ with a plane. The **canonical roots** of $\Phi'$ are a canonical pair of simple roots for $\Phi'$, namely the unique pair of roots containing $\Phi' \cap \Phi^+$ in its nonnegative span. There are two distinct positive roots in $\Phi$ are contained in a unique rank-2 subsystem of $\Phi$, namely the intersection of $\Phi$ with the span of the two roots.

Given $\beta, \gamma \in \Phi^+$, say that $\gamma$ **cuts** $\beta$ if $\gamma$ is a canonical root of the rank-2 subsystem $\Phi'$ containing them, but $\beta$ not a canonical root in $\Phi'$. Write $\text{cut}(\beta)$ for the set of roots $\gamma$ such that $\gamma$ cuts $\beta$. For each $\beta \in \Phi^+$, consider the set $\beta^\perp \setminus \bigcup_{\gamma \in \text{cut}(\beta)} \gamma^\perp$. The closures of the connected components of this set are the **shards** in $\beta^\perp$. Less formally, we “cut” each $\beta^\perp$ along its intersections with all hyperplanes $\gamma^\perp$ with $\gamma \in \text{cut}(\beta)$, and the resulting pieces are the shards.

We define $D$ to be the closed convex cone $\bigcap_{i=1}^n \{x \in V^* : \langle x, \alpha_i \rangle \geq 0\}$ in $V^*$. The Coxeter group $W$ acts on $V^*$ by the action dual to its action on $V$. On the basis
of fundamental weights,

\[ s_k(\rho_j) = \begin{cases} 
\rho_j & \text{if } j \neq k, \\
\rho_k - \sum_{i=1}^{n} a_{ik}\rho_i & \text{if } j = k.
\end{cases} \tag{3.3} \]

The map \( w \mapsto wD \) is an injective map to a set of cones with disjoint interiors. The union of these cones is a convex cone called the \textit{Tits cone} \( \text{Tits}(A) \).

Each shard \( \Sigma \) is a cone defined by inequalities of the form \( \langle x, \beta \rangle \leq 0 \) for \( \beta \in \Psi \), so in particular if \( \Sigma \) intersects \( \text{Tits}(A) \) in codimension 1, then \( \Sigma \cap \text{Tits}(A) \) is a union of codimension-1 faces of cones \( wD \) for \( w \in W \). Call \( w \) an \textit{upper element} of \( \Sigma \) if \( wD \cap \Sigma \) has codimension 1 in \( V^* \) and \( \text{inv}(w) \) contains the positive root \( \beta \) such that \( \Sigma \in \beta^\perp \). We write \( \text{Upper}(\Sigma) \) for the set of upper elements of \( \Sigma \), and we consider \( \text{Upper}(\Sigma) \) as a partially ordered set, with the order induced by the weak order on \( W \).

We now establish a bijection between shards intersecting \( \text{Tits}(A) \) and join-irreducible elements of \( W \). Given a join-irreducible element \( j \), we write \( j_* \) for the unique element covered by \( j \) and define \( \text{Sh}(j) \) to be the unique shard containing \( jD \cap j_*D \). Versions of the following proposition for finite hyperplane arrangements can be found as [18, Proposition 2.2] and [19, Proposition 3.5].

**Proposition 3.1.** Given a shard \( \Sigma \) intersecting \( \text{Tits}(A) \) in codimension 1, the poset \( \text{Upper}(\Sigma) \) has a unique minimal element \( j_*(\Sigma) \), which is also the unique element of \( \text{Upper}(\Sigma) \) that is join-irreducible in \( W \). The maps \( \Sigma \mapsto j_*(\Sigma) \) and \( j \mapsto \text{Sh}(j) \) are inverse bijections.

To prove Proposition 3.1, we need a special property of the weak order. As explained in the paragraph after [30, Theorem 8.1], every interval in the weak order on \( W \) is a finite semidistributive lattice, meaning that the following conditions hold for all \( x, y, z \in W \) such that \( x \vee y \vee z \) exists: If \( x \vee y = x \vee z \) then \( x \vee y = x \lor (y \land z) \), and if \( x \land y = x \land z \) then \( x \land y = x \land (y \lor z) \).

We also need the following fact about rank-2 subsystems of \( \Psi \) and inversion sets of elements of \( W \), which is well known. (See, for example, [30, Lemma 2.17].)

**Lemma 3.2.** Suppose \( w \in W \) and \( \Psi' \) is a rank-2 subsystem of \( \Psi \) with canonical roots \( \beta \) and \( \gamma \).

1. If \( \beta, \gamma \in \text{inv}(w) \) then \( \Psi' \subseteq \text{inv}(w) \).
2. If \( \phi \in (\Psi' \setminus \{\beta, \gamma\}) \) and \( \phi \in \text{inv}(w) \), then \( \text{inv}(w) \cap \{\beta, \gamma\} \neq \emptyset \).
3. If \( \phi, \psi \in \Psi' \cap \text{inv}(w) \) and \( \chi \in \Psi \) is a positive linear combination of \( \phi \) and \( \psi \), then \( \chi \in \text{inv}(w) \).

**Proof of Proposition 3.1.** We first check that the poset \( \text{Upper}(\Sigma) \) is connected. Given \( v \) and \( w \) in \( \text{Upper}(\Sigma) \), choose a point \( p \in vD \cap \Sigma \) and a point \( q \in wD \cap \Sigma \). Choose generically, so that the line segment \( \overline{pq} \) does not pass through any intersections of hyperplanes in \( \{\beta^\perp : \beta \in \Psi^* \} \) such that the intersection is of codimension \( > 2 \) in \( V^* \). The entire line segment is in \( \Sigma \cap \text{Tits}(A) \) because this is a convex set. Since \( \text{inv}(v) \) and \( \text{inv}(w) \) are both finite, their symmetric difference is also finite, and therefore we move along \( \overline{pq} \) from \( p \) to \( q \) while passing through finitely many hyperplanes. Every time \( \overline{pq} \) passes from a cone \( xD \cap \Sigma \) to an adjacent cone \( yD \cap \Sigma \), with \( x, y \in \text{Upper}(\Sigma) \), since it does not leave \( \Sigma \), it crosses through the hyperplanes in a finite rank-2 subsystem \( \Psi' \) in which \( \beta \) is a canonical root. We see that either
inv(x) \cap \Phi' = \{\beta\} and \text{inv}(w) \cap \Phi' = \Phi'$, or vice versa. No other hyperplanes separate $xD$ from $yD$, and we see that $x \leq y$ or vice versa. We take finitely many such steps in passing from $v$ to $w$, and we conclude that $\text{Upper}(\Sigma)$ is connected.

Suppose $\Sigma$ intersects $\text{Tits}(A)$ in codimension 1. In particular, $\Sigma$ is contained in $\beta_\perp$ for some reflection $t$. Also, $\text{Upper}(\Sigma)$ is nonempty, so it has at least one minimal element. (Every nonempty subset of $W$ has a minimal element in the weak order: Take an element with minimal length.)

We check that an element of $\text{Upper}(\Sigma)$ is minimal in $\text{Upper}(\Sigma)$ if and only if it is join-irreducible in $W$. If $j$ is join-irreducible in $W$, then every element strictly below $j$ is weakly below $j_\ast = tj$, which has $\beta_\perp \not\in \text{inv}(tj)$. On the other hand, if $j \in \text{Upper}(\Sigma)$ is not join-irreducible, then let $t'$ be a reflection such that $j \geq t'j$ and $t' \neq t$. Let $\Phi'$ be the rank-2 subsystem containing $\beta_\perp$ and $\beta_\perp$. Since both $\text{inv}(j) \setminus \{\beta_\perp\}$ and $\text{inv}(j) \setminus \{\beta_\perp\}$ are inversion sets of elements of $W$, Lemma 3.2 implies that both $\beta_\perp$ and $\beta_\perp$ are canonical roots in $\Phi'$ and that $\text{inv}(j) \cap \Phi' = \Phi'$. We also see that there is an element $w$ of $\text{Upper}(\Sigma)$ with $\text{inv}(w) = (\text{inv}(j) \setminus \Phi') \cup \{\beta_\perp\}$, so $j$ is not minimal in $\text{Upper}(\Sigma)$.

Let $j$ be a minimal element of $\text{Upper}(\Sigma)$. As we have checked, $j$ is join-irreducible. If $j$ is not the unique minimal element, then since $\text{Upper}(\Sigma)$ is connected, there is a path in $\text{Upper}(\Sigma)$ from $j$ to another minimal element. Let $x$ be the first element along that path that is not $\geq j$, and let $y$ be the element immediately before $x$ (i.e. closer to $j$) on the path. Thus $x \leq y$ and $j \leq y$ but $j \not\leq x$. We have $j \geq tj$, $x > tx$, and $y > ty$, and in each case the difference in inversion sets is $\{\beta_\perp\}$. Thus $j \vee ty = x \vee ty = y$, but $j \wedge x < j$, and thus $j \wedge x \leq j_\ast = tj$. We have $(j \wedge x) \vee ty = ty$, contradicting semidistributivity in the interval below $y$. Thus $j$ is the unique minimal element.

Now $j = ji(\text{Sh}(j))$ because $j \in \text{Upper}(\text{Sh}(j))$. Since $ji(\Sigma)$ is minimal in $\text{Upper}(\Sigma)$, the intersection $ji(\Sigma)\Lambda \cap (ji(\Sigma))\Lambda$ is contained in $\Sigma$, so $\Sigma = \text{Sh}(ji(\Sigma))$. \qed

3.5. The affine almost-positive roots model. We now describe the affine version of the almost-positive roots model. The finite version is in [33, 34], and more details on the affine version can be found in [33, 34].

The Cartan matrix $A$ (and the associated root system $\Phi$ and Coxeter group $W$) are said to be of affine type if $A$ is positive semidefinite but all of its principal minors are positive. More details on root systems of affine type are in [14] Chapter 4. If $A$ is of affine type, then there is a unique positive imaginary root $\delta$ such that the imaginary roots in $\Phi$ are \{ $k\delta : k \in \mathbb{Z} \setminus \{0\}$ \}.

We depart from tradition of taking $A$ to be an $(n+1) \times (n+1)$ matrix indexed by $\{ 0, \ldots, n \}$. Instead, we choose an index $\text{aff} \in \{ 1, \ldots, n \}$ and call $\alpha_{\text{aff}}$ the affine simple root. We write $S_{\text{fin}} = S \setminus \{ s_{\text{aff}} \}$ and write $W_{\text{fin}}$ for the standard parabolic subgroup generated by $S_{\text{fin}}$. The choice of the index $\text{aff}$ is made in such a way that $W$ is a semidirect product of $W_{\text{fin}}$ with the lattice generated by $\{ \alpha_i^\vee : i \neq \text{aff} \}$. (In the traditional indexing, the affine simple root is indexed by 0.) We write $V_{\text{fin}}$ for the linear span of $\{ \alpha_i : i \neq \text{aff} \}$.

The fixed space in $V$ of a Coxeter element $c = s_1 \cdots s_n$ is spanned by $\delta$. We define $\gamma_c$ to be the unique vector in $V_{\text{fin}}$ that is a generalized 1-eigenvector associated to $\delta$ (meaning that $c\gamma_c = \delta + \gamma_c$).

The set $U^c = \{ v \in V : K(\gamma_c, v) = 0 \}$ is a hyperplane in $V$. We write $Y_{\text{fin}}^c$ for the set $\Phi \cap U^c \cap V_{\text{fin}}$. This is a finite root system of rank $n-2$, each of whose components are of type $A$. We define $\Lambda_{\text{fin}}^c$ to be the union of the $c$-orbits of the
positive roots in $\Upsilon_{\text{fin}}$ and we define $\Lambda_c$ to be $\Lambda_c^{re} \cup \{\delta\}$. Every root contained in $U^c$ has a finite $c$-orbit, so $\Lambda_c$ is a finite set.

The set of roots that participate in the affine almost-positive roots model is the set $\Phi^c = -\Pi \cup (\Phi^+ \setminus U^c) \cup \Lambda_c$. The set $\Phi^c = \Phi^c \setminus \{\delta\}$ of real roots in $\Phi^c$ is the set of $d$-vectors of cluster variables in the cluster algebra associated to $B$. (See [30, Theorem 2.7].)

We consider several permutations of $\Phi^c$ defined in [34] following the finite-type definitions in [8, 9, 15]. For each $s \in S$ that is initial or final in $c$, define an involution $\sigma_s : \Phi^c \to \Phi^c$ by

$$\sigma_s(\alpha) = \begin{cases} 
\alpha & \text{if } \alpha \in -\Pi \setminus \{-\alpha_s\} \\
s(\alpha) & \text{otherwise.}
\end{cases}$$

For $c = s_1 \cdots s_n$, define $\tau_c : \Phi^c \to \Phi^c$ by $\tau_c = \sigma_{s_1} \cdots \sigma_{s_n}$.

We now prepare to define the $c$-compatibility degree, which associates an integer to each pair of roots in $\Phi^c$. We will need additional notation related to roots contained in $U^c$.

The set $\Upsilon^c = \Phi^c \cap U^c$ is essentially a direct sum of root systems of affine type, with one component (of affine type $A$) for each component of $\Upsilon_{\text{fin}}$. However, in each component, the isotropic direction is $\mathbb{R}\delta$. Since the spans of these components intersect (in $\mathbb{R}\delta$), $\Upsilon^c$ is not a reducible root system in the usual sense, but is a root system in a sense considered in [5] and [6]. (See [30, Theorem 2.7].) We write $\Xi^c$ for the unique minimal set of roots whose nonnegative span is $\Upsilon^c \cap \Phi^+$. These should be thought of as the simple roots of $\Upsilon^c$. The Dynkin diagram for $\Upsilon^c$ consists of cycles and the Coxeter element $c$ of $W$ acts as a rotation on each cycle, moving each root in $\Xi^c$ to an adjacent root in the cycle.

Each real root $\beta \in \Upsilon^c$ has a unique expression as a linear combination of vectors in $\Xi^c$. We write $\text{Supp}_{\Xi}(\beta)$ for the set of vectors appearing in this expression with nonzero coefficient.

Given $\alpha \in \Upsilon^c$ and $\beta_j \in \Xi^c$, say $\beta_j$ is adjacent to $\alpha$ if $\beta_j$ is in $\text{Supp}_{\Xi}(c\alpha) \cup \text{Supp}_{\Xi}(c^{-1}\alpha)$ but not in $\text{Supp}_{\Xi}(\alpha)$. Given $\alpha, \beta \in \Lambda_c^{re}$, define $\text{adj}_{\alpha}(\beta)$ to be the number of roots in $\text{Supp}_{\Xi}(\beta)$ that are adjacent to $\alpha$. Furthermore, define

$$(\alpha \circ \beta)_c := \begin{cases} 
-1 & \text{if } \alpha = \beta, \\
0 & \text{if } \text{Supp}_{\Xi}(\alpha) \subset \text{Supp}_{\Xi}(\beta) \text{ or } \text{Supp}_{\Xi}(\beta) \subset \text{Supp}_{\Xi}(\alpha), \\
\text{adj}_{\alpha}(\beta) & \text{otherwise.}
\end{cases}$$

The $c$-compatibility degree is the unique function $\Phi^c \times \Phi^c \to \mathbb{Z}$, given by $(\alpha, \beta) \mapsto (\alpha \| \beta)_c$ satisfying the following conditions:

$$(3.4) \quad (-\alpha_i \| \beta)_c = (\rho_i^\vee, \beta) \text{ for } \alpha_i \text{ simple and } \beta \in \Phi^c,$$

$$(3.5) \quad (\beta \| -\alpha_i)_c = (\rho_i, \beta^\vee) \text{ for } \alpha_i \text{ simple and } \beta \in \Phi^c,$$

$$(3.6) \quad (\alpha \| \beta)_c = (\alpha \circ \beta)_c \text{ if } \alpha, \beta \in \Lambda_c^{re},$$

$$(3.7) \quad (\delta \| \alpha)_c = (\alpha \| \delta)_c = 0 \text{ if } \alpha \in \Lambda_c, \text{ and}$$

$$(3.8) \quad (\alpha \| \beta)_c = (\tau_c \alpha \| \tau_c \beta)_c.$$ 

Two roots $\alpha$ and $\beta$ in $\Phi^c$ are $c$-compatible if and only if $(\alpha \| \beta)_c = 0$. Although the $c$-compatibility degree is not symmetric, as a consequence of [34, Proposition 4.12], $c$-compatibility is a symmetric relation.
A \( c \)-cluster is a maximal set of pairwise \( c \)-compatible roots in \( \Phi_c \). An imaginary \( c \)-cluster is a \( c \)-cluster containing \( \delta \), and a real \( c \)-cluster is a \( c \)-cluster not containing \( \delta \).

Given a set \( C \) of pairwise \( c \)-compatible roots in \( \Phi_c \), let \( \text{Cone}(C) \) denote the nonnegative real span of \( C \). The affine \( c \)-cluster fan or affine generalized associahedron \( \text{Fan}_c(\Phi) \) consists of the cones \( \text{Cone}(C) \) for all sets \( C \) of pairwise \( c \)-compatible roots in \( \Phi_c \).

If \( \delta \in C \), then \( \text{Cone}(C) \) is an imaginary cone. Otherwise \( \text{Cone}(C) \) is a real cone. The subfan of \( \text{Fan}_c(\Phi) \) consisting of real cones is the real affine \( c \)-cluster fan \( \text{Fan}_{re}^c(\Phi) \).

We are interested in the image of \( \text{Fan}_c(\Phi) \) under the piecewise linear map \( \nu_c \) defined in Section 3.2. The following theorem is essentially [34, Theorem 1.1(1)], although it also incorporates the immediate observation that \( \nu_c \) is linear on the union of the imaginary cones in \( \text{Fan}_c(\Phi) \) (because these cones are all in the non-negative span of the simple roots).

**Theorem 3.3.** Suppose that \( B \) is an acyclic exchange matrix of affine type, let \( \Phi \) be the associated root system, and let \( c \) be the associated Coxeter element. The piecewise-linear homeomorphism \( \nu_c \) acts linearly on each cone of \( \text{Fan}_c(\Phi) \) and thus defines a complete fan \( \nu_c(\text{Fan}_c(\Phi)) \). The isomorphism \( \nu_c \) from \( \text{Fan}_c(\Phi) \) to \( \nu_c(\text{Fan}_c(\Phi)) \) restricts to an isomorphism from \( \text{Fan}_{re}^c(\Phi) \) to the \( g \)-vector fan.

(The \( g \)-vector fan associated to \( B \) is defined from the cluster variables in a principal-coefficients cluster algebra. We will not need the detailed definition here, but to avoid confusion, we point out that the \( g \)-vector is defined to be an integer vector, but here we interpret vectors in \( V^* \) as \( g \)-vectors by taking fundamental-weight coordinates.)

## 4. Beginning the cluster scattering diagram proofs

In this section, we begin to prove Theorem 2.2. The proof will take several sections.

Since the notion of \( \text{Scat}^T(B) \) is defined only up to equivalence, to show that \( \text{Scat}^T(B) \) “is” \( \text{DCScat}(A,c) \), we will show that \( \text{DCScat}(A,c) \) satisfies the finiteness condition, show that it contains the walls \( \{ (\alpha_i^+, 1 + \hat{y}_i) : i = 1, \ldots, n \} \), show that every other wall is outgoing, and show that \( \text{DCScat}(A,c) \) is consistent. Since every wall in \( \text{DCScat}(A,c) \) has a nontrivial scattering term, and since no two walls are parallel, it is easy to see that \( \text{DCScat}(A,c) \) has minimal support as well.

The finiteness condition on \( \text{DCScat}(A,c) \) is supplied by the following proposition, which is a combination of [30, Theorem 8.3] and the case \( k = 1 \) of [21, Theorem 6.1]. (See also [29, Corollary 3.9].)

**Proposition 4.1.** Suppose \( W \) is a Coxeter group and \( c \) is a Coxeter element of \( W \). Then for any reflection \( t \), there is at most one \( c \)-sortable join-irreducible element whose unique cover reflection is \( t \). If \( W \) is finite, then there is exactly one.

Immediately from Proposition 4.1 we can conclude that at most two walls of \( \text{DCScat}(A,c) \) are in the same hyperplane, which is already enough for the finiteness condition. (We will see later that there is at most one wall in each hyperplane. There would be two walls when there exist a join-irreducible \( c \)-sortable element \( j \) and a \( c^{-1} \)-sortable element \( j' \) such that \( \text{Sh}(j) \) and \( -\text{Sh}(j') \) are different shards in the same hyperplane. We will rule out that possibility in Proposition 6.8.)
Every wall $(\alpha^\perp, 1 + \tilde{y}^{\alpha^\vee})$ is present because each $s \in S$ is c-sortable and join-irreducible, and $\text{Sh}(s) = \alpha^\perp$ because cut$(\alpha_s) = \emptyset$.

Thus to prove Theorem 2.2, it remains to verify that every other wall is outgoing, and $\text{DCScat}(A, c)$ is consistent. We prove the assertion about outgoing walls in three propositions, one for each of the kinds of walls in $\text{DCScat}(A, c)$, namely walls associated to c-sortable join-irreducible elements, walls associated to $c^{-1}$-sortable join-irreducible elements, and the wall $(\emptyset_\infty, f_\infty)$. We state and prove the two propositions having to do with sortable elements in this section:

**Proposition 4.2.** If $j \in \text{JIrr}_c(W) \setminus S$, then $(\text{Sh}(j), f_j)$ is outgoing and gregarious.

**Proposition 4.3.** If $j \in \text{JIrr}_{c^{-1}}(W) \setminus S$, then $(-\text{Sh}(j), f_j)$ is outgoing and gregarious.

To prove these propositions, we need to quote and prove a few additional background results.

The following lemma is [21, Lemma 1.1]. It implies in particular that a join-irreducible element is in $W_I$ if and only if its unique cover reflection is.

**Lemma 4.4.** For any $I \subseteq S$, an element $w \in W$ is in $W_I$ if and only if every cover reflection of $w$ is in $W_I$.

Given a Coxeter element $c$ and a subset $I \subseteq S$, the **restriction** of $c$ to $W_I$ is the Coxeter element $c'$ of $W_I$ obtained by deleting the letters $S \setminus I$ from a reduced word for $c$. (Recall that for any $w \in W$ and $I \subseteq S$, we defined an element $w_I \in W_I$. In general $c_I$ is not equal to the restriction $c'$ of $c$ to $W_I$. For example, if $S = \{s_1, s_2\}$ with $c = s_1s_2 \neq s_2s_1$ and $I = \{s_2\}$, then $c' = s_2$ but $c_I$ is the identity.) We write $V_I$ for the subspace Span $\{\alpha_i : i \in I\}$ of $V$. The following lemma is immediate from the definition.

**Lemma 4.5.** Let $I \subseteq S$ and let $c'$ be the restriction of $c$ to $W_I$. Then $\omega_c$ restricted to $V_I$ is $\omega_{c'}$.

The **inversion sequence** of a word $a_1 \cdots a_k$ (in the alphabet $S$) is the sequence $(a_1 \cdots a_{i-1} \alpha_a : i = 1, \ldots, k)$. If $a_1 \cdots a_k$ is a reduced word for $w$, then the $k$ roots in the inversion sequence of $a_1 \cdots a_k$ are distinct and are precisely the roots in inv$(w)$. If $s \in S$ and $s < w$ in the weak order, then by writing a reduced word $a_1 \cdots a_k$ for $w$ with $a_1 = s$, we see that inv$(w) = s \cdot $ inv$(sw) \cup \{\alpha_s\}$.

The following is one direction of [30, Proposition 3.11]

**Proposition 4.6.** Let $v$ be a $c$-sortable element with $c$-sorting word $a_1 \cdots a_k$ and let $\beta_1, \ldots, \beta_k$ be the inversion sequence of $a_1 \cdots a_k$. Then $\omega_v(\beta_i, \beta_j) \geq 0$ for all $i < j$, and furthermore if $\omega_v(\beta_i, \beta_j) = 0$ then $K(\beta_i, \beta_j) = 0$.

The following lemma is [30, Lemma 3.8].

**Lemma 4.7.** If $s$ is initial or final in $c$, then $\omega_v(\beta, \beta') = \omega_{\text{scs}}(s\beta, s\beta')$ for all roots $\beta$ and $\beta'$.

We prove the following lemma for general Coxeter groups. The finite-type version is [19, Lemma 3.9].

**Lemma 4.8.** Suppose $j$ is join-irreducible in $W$ and $\text{cov}(j) = \{t\}$. Then $\text{Sh}(j) = \beta_t^\perp$ if and only if $j \in S$. 
Proof. One direction is easy because \( \text{cut}(\alpha_s) = \emptyset \) for all \( s \in S \). On the other hand, suppose \( j \notin S \), so that \( \ell(j) > 1 \). Given a reduced word \( a_1 \cdots a_k \) for \( j \), let \( m \) be minimal such that the word \( a_m \cdots a_k \) contains only the letters \( a_{k-1} \) and \( a_k \). Furthermore, choose the reduced word \( a_1 \cdots a_k \) to minimize \( m \). The roots \( a_1 \cdots a_r \alpha_{a_r} \) for \( r = m, \ldots, k \) are all contained in the same rank-2 subsystem \( \Phi' \), with canonical roots \( a_1 \cdots a_{m-1} \alpha_{a_{m-1}} \) and \( a_1 \cdots a_{m-1} \alpha_{a_m} \). One of these two is \( a_1 \cdots a_{m-1} \alpha_{a_m} \). If the other is \( a_1 \cdots a_{k-1} \alpha_{a_k} \), then using a single braid move, we can obtain a new reduced word ending in \( a_{k-1} \). Then \( j > ja_k \) and \( j > ja_{k-1} \), contradicting the fact that \( j \) is join-irreducible. We conclude that \( a_1 \cdots a_{m-1} \alpha_{a_m} \) cuts \( a_1 \cdots a_{k-1} \alpha_{a_k} \). But the latter is \( \beta_t \), so \( \text{Sh}(j) \neq \beta_t^\perp \). 

Throughout the section, it will be useful to set a convention on being above or below a hyperplane. Given a root \( \beta \in \Phi \) and a set \( U \subseteq V^* \), we say that \( U \) is above the hyperplane \( \beta^\perp \) if every point in \( U \) is either separated from \( D \) by \( \beta^\perp \) or is on \( \beta^\perp \). We say that \( U \) is below \( \beta^\perp \) if every point in \( U \) is either on \( \beta^\perp \) or on the same side of \( \beta^\perp \) as \( D \).

A version of the following proposition for finite simplicial hyperplane arrangements is [19, Lemma 3.7]. The proposition shows in particular that each shard is defined by a finite list of inequalities.

**Proposition 4.9.** If \( j \) is join-irreducible in \( W \) and \( \text{cov}(j) = \{t\} \), then

\[
\text{Sh}(j) = \{x \in V^* : \langle x, \beta_t \rangle = 0 \text{ and } \langle x, \gamma \rangle \leq 0 \text{ for all } \gamma \in \text{cut}(\beta_t) \cap \text{inv}(j)\}.
\]

Proof. For any root \( \gamma \in \text{cut}(\beta_t) \), there is another root \( \gamma' \in \text{cut}(\beta_t) \) such that \( \gamma, \gamma' \), and \( \beta_t \) are all in the same rank-2 subsystem \( \Phi' \) and the canonical roots of \( \Phi' \) are \( \gamma \) and \( \gamma' \). Up to swapping \( \gamma \) and \( \gamma' \), all of \( \text{Sh}(j) \) is above \( \gamma^\perp \) and below \( \langle \gamma', \gamma \rangle^\perp \). Thus since \( j \in \text{Upper}(\text{Sh}(j)) \), also \( jD \) is above \( \gamma^\perp \), or in other words \( \gamma \in \text{inv}(j) \). Thus \( \text{Sh}(j) \) is defined as the set of points in \( \beta_t^\perp \) above all hyperplanes \( \gamma^\perp \) with \( \gamma \in \text{cut}(\beta_t) \cap \text{inv}(j) \). 

We will now relate the shards associated to \( W \) with the shards associated to its parabolic subgroups. Given \( I \subseteq S \), we write \( V_I \) for the subspace \( \text{Span} \{\alpha_i : i \in I\} \) of \( V \), as before, and write \( V_I^* \) for its dual. The notation \( \text{Proj}_I \) stands for the surjection from \( V^* \) onto \( V_I^* \) that is dual to the inclusion of \( V_I \) into \( V \). We define \( D_I = \bigcap_{i \in I} \{x \in V_I^* : \langle x, \alpha_i \rangle \geq 0\} \), which equals \( \text{Proj}_I(D) \). For \( j \) a join-irreducible element of \( W_I \), we write \( \text{Sh}_I(j) \) for the shard in \( V_I^* \) containing \( jD_I \cap j, D_I \).

The **parabolic subsystem** \( \Phi_I \) is \( \Phi \cap V_I \). The restriction of the cutting relation on \( \Phi \) to \( \Phi_I \) coincides with the cutting relation intrinsically defined on \( \Phi_I \). Moreover, it is easily verified that when \( \beta \in \Phi_I \), the set \( \text{cut}(\beta) \) is also contained in \( \Phi_I \). (See, for example, [19, Lemma 6.6] for a finite-type argument that generalizes easily.) It is also well known that for \( w \in W_I \), the inversion set \( \text{inv}(w) \) is the same set of roots of \( \Phi_I \) whether it is computed in \( W \) or in \( W_I \). Thus for \( j \) a join-irreducible element of \( W_I \) with \( \text{cov}(j) = \{t\} \),

\[
\text{Sh}_I(j) = \{x \in V_I^* : \langle x, \beta_t \rangle = 0 \text{ and } \langle x, \gamma \rangle \leq 0 \text{ for all } \gamma \in \text{cut}(\beta_t) \cap \text{inv}(j)\}.
\]

Comparing this expression for \( \text{Sh}_I(j) \) with Proposition 4.9, we have the following lemma.

**Lemma 4.10.** If \( j \) is join-irreducible and lies in \( W_I \), then \( \text{Sh}(j) = \text{Proj}_I^{-1}\text{Sh}_I(j) \).

If \( s \) is initial in \( c \) and \( j \) is a \( c \)-sortable join-irreducible element with \( s \nleq j \), then \( j \) is in \( W(s) \), and in particular we have the following consequence of Lemma 4.10.
Proposition 4.11. Suppose $j$ is a c-sortable join-irreducible element and $s$ is initial in $c$ with $s \not< j$. Then $\text{Sh}(j) = \text{Proj}_{\langle s \rangle}^{-1}\text{Sh}(\langle s \rangle)(j)$.

We will also need the following fact relating $\text{Sh}(j)$ to $\text{Sh}(sj)$ when $s < j$.

**Proposition 4.12.** Suppose $j$ is join-irreducible with $\text{cov}(j) = \{t\}$ and let $s \in S$ have $s < j$. Then $(s \cdot \text{Sh}(sj)) \supseteq \text{Sh}(j)$ and

$$(s \cdot \text{Sh}(sj)) \cap \{x \in V^*: \langle x, \alpha_s \rangle \leq 0\} = \text{Sh}(j) \cap \{x \in V^*: \langle x, \alpha_s \rangle \leq 0\}.$$

**Proof.** Since $\pm \alpha_s$ are the only roots that change sign under the action of $s$, for any rank-2 subsystem $\Phi'$ that does not contain $\alpha_s$, the map $\beta \mapsto s\beta$ preserves the cutting relation in $\Phi'$. Thus in particular, for some root $\gamma$, if the rank-2 subsystem $\Phi'$ containing $\beta_\gamma$ and $\gamma$ has $\alpha_s \not\in \Phi'$, then $\gamma$ cuts $\beta_\gamma$ if and only if $s\gamma$ cuts $s\beta_\gamma$. Thus $s \cdot \text{cut}(s\beta_\gamma)$ and $\text{cut}(\beta_\gamma)$ agree except possibly that one or the other may contain $\alpha_s$.

Applying $s$ to $\text{Sh}(sj)$ yields a piece of the hyperplane $\beta_\gamma^+$. Proposition 4.9 says that $\text{Sh}(j)$ is defined by the inequalities $\langle x, \gamma \rangle \leq 0$ for all $\gamma \in \text{cut}(\beta_\gamma) \cap \text{inv}(j)$. Since $s < j$, $\text{inv}(sj) = s \cdot (\text{inv}(j) \setminus \{\alpha_s\})$. Again by Proposition 4.9 and by the relationship between $\text{inv}(sj)$ and $\text{inv}(j)$ and between $\text{cut}(\beta_\gamma)$ and $\text{cut}(s\beta_\gamma)$, we see that the inequalities defining $\text{Sh}(sj)$ are obtained from the inequalities defining $\text{Sh}(j)$ as follows: Throw away the inequality $\langle x, \alpha_s \rangle \leq 0$ if it is present, and then replace each inequality $\langle x, \gamma \rangle \leq 0$ with $\langle x, s\gamma \rangle \leq 0$. We see that $s \cdot \text{Sh}(sj)$ is defined by a weakly smaller set of inequalities than $\text{Sh}(j)$, but that intersecting both $s \cdot \text{Sh}(sj)$ and $\text{Sh}(j)$ with $\{x \in V^*: \langle x, \alpha_s \rangle \leq 0\}$ yields the same set. \hfill $\square$

**Proof of Proposition 4.12.** Suppose $\text{cov}(j) = \{t\}$. By Lemma 4.8 the cone $\text{Sh}(j)$ is not all of $\beta_\gamma^+$, so that it is at most half of a hyperplane. Thus the cone $\text{Sh}(j)$ cannot have both $\omega_c(\cdot, \beta_\gamma) \in \text{Sh}(j)$ and $-\omega_c(\cdot, \beta_\gamma) \in \text{relint}(\text{Sh}(j))$, so it is enough to prove that $-\omega_c(\cdot, \beta_\gamma) \in \text{relint}(\text{Sh}(j))$. We do so by induction on $\ell(j)$ and the rank of $W$. Let $s$ be initial in $c$.

If $s \not< j$, then $j \in W(\alpha)$ and, by induction on rank, $-\omega_{sc}(\cdot, \beta_\gamma) \in \text{relint}(\text{Sh}(\langle s \rangle)(j))$. By Lemma 4.3 and Proposition 4.11, we will see that $-\omega_c(\cdot, \beta_\gamma) \in \text{relint}(\text{Sh}(j))$.

If $s < j$ and $sj = r$ for some $r \in S$, then $j = sr$, and since $j$ is join-irreducible, $r$ and $s$ do not commute. Since $t = sr$, also $s$ and $t$ do not commute, so that $K(\alpha_s, \beta_\gamma) \neq 0$. But $sr$ is the c-sorting word for $j$ and $\alpha_s, \beta_\gamma$ is the inversion sequence of that word, so Proposition 4.6 implies that $-\omega_c(\alpha_s, \beta_\gamma) < 0$. Also $\text{cut}(\beta_\gamma) \cap \text{inv}(j) = \{\alpha_s\}$, so $\text{Sh}(j) = \{x \in \beta_\gamma^+: \langle x, \alpha_s \rangle \leq 0\}$ by Proposition 4.9. We see that $-\omega_c(\cdot, \beta_\gamma) \in \text{relint}(\text{Sh}(j))$.

If $s < j$ and $\ell(sj) > 1$, then by induction on $\ell(j)$, we have $-\omega_{sc}(\cdot, \beta_\gamma) \in \text{relint}(\text{Sh}(sj))$. Lemma 4.7 implies that $-\omega_c(\cdot, \beta_\gamma) \in \text{relint}(s \cdot \text{Sh}(sj))$. Since $s$ is initial in $c$, by definition of $\omega_c$, we have $-\omega_c(\alpha_s, \beta_\gamma) \leq 0$, with equality if and only if $t$ is contained in the parabolic subgroup generated by the elements of $S$ that commute with $s$. However, that parabolic subgroup is reducible, with a component $W'$ generated by the singleton $\{s\}$ and one or more other components. If $t$ is in the parabolic subgroup, then it is supported in only one of the components. Since $j \not< s$, we see that $t$ is in a component other than $W'$. But then $j$ is also in that component by Lemma 4.4, contradicting the assumption that $s < j$. We conclude that $-\omega_c(\alpha_s, \beta_\gamma) < 0$, so that $-\omega_c(\cdot, \beta_\gamma) \in \text{relint}(s \cdot \text{Sh}(sj)) \cap \{x \in V^*: \langle x, \alpha_s \rangle \leq 0\}$. By Proposition 4.12, $-\omega_c(\cdot, \beta_\gamma) \in \text{relint}(\text{Sh}(j))$. \hfill $\square$

**Proof of Proposition 4.12.** Again, suppose $\text{cov}(j) = \{t\}$. By Proposition 4.12 with $c^{-1}$ replacing $c$, we see that $-\omega_{c^{-1}}(\cdot, \beta_\gamma)$ is in the relative interior of $\text{Sh}(j)$ and...
that $\omega_{c^{-1}}(\cdot, \beta)$ is not in $\text{Sh}(j)$. But $\omega_{c^{-1}}(\cdot, \beta) = -\omega_{c}(\cdot, \beta)$, so $(-\text{Sh}(j), f_j)$ is outgoing and gregarious.

5. The doubled Cambrian fan

To complete the proof of Theorem 2.2 and the other results on affine cluster scattering diagrams, we need the affine-type version of the doubled Cambrian fan. This fan coincides with the $g$-vector fan (see Theorem 3.8 below) and thus constitutes much of the cluster scattering fan. Indeed, we will see below in Theorem 5.10 that the affine doubled Cambrian fan covers all of $V^*$ except for a certain codimension-1 cone, which is precisely the wall $\delta_\infty$ of $\text{DCScat}(A, c)$.

5.1. The doubled Cambrian fan in general. We now describe the Cambrian fan and doubled Cambrian fan construction from [29, 30, 31, 32]. The ambient space for both fans is $V^*$.

We associate a set $C_c(v)$ to each $c$-sortable element $v$, by setting $C_c(v)$ to be $\{\alpha_1, \ldots, \alpha_n\}$ (the set of simple roots) when $v$ is the identity element, and otherwise, for $s$ initial in $c$, setting

$$C_c(v) = \begin{cases} C_{sc}(v) \cup \{\alpha_s\} & \text{if } v \not\geq s, \text{ or} \\ sC_{sc}(sv) & \text{if } v \geq s. \end{cases}$$

Here $C_{sc}(v)$ is the analogous set of roots, with $v$ considered as an element of $W_s$.

For each $c$-sortable element $v$, let $\text{Cone}_c(v) = \{x \in V^*: \langle x, \beta \rangle \geq 0, \forall \beta \in C_c(v)\}$. This is a simplicial cone with inward-facing normal vectors $C_c(v)$. The $c$-Cambrian fan $F_c$ is the set of cones $\text{Cone}_c(v)$ where $v$ runs over all $c$-sortable elements. The doubled $c$-Cambrian fan $DF_c = F_c \cup (-F_{c^{-1}})$ is the simplicial fan consisting of all cones of $F_c$ and all antipodal images of cones in $F_{c^{-1}}$. (This is a fan by [32 Theorem 3.24].) When $W$ is of finite type, $DF_c$ is a complete fan and coincides with $F_c$. When $W$ is of affine type, $DF_c$ is not complete but, as we will see below, its support is all of $V^*$ except for a codimension-1 cone. When $W$ is infinite but not affine, the support of $DF_c$ is not dense in $V^*$.

Given a set of $c$-sortable elements of $W$, if the set has a join in the weak order, then that join is also $c$-sortable [30 Theorem 7.1]. As a consequence, for any $w \in W$, there is a unique maximal $c$-sortable element that is below $w$ in the weak order. We call this element $\pi_1^c(w)$. Thus $w$ is $c$-sortable if and only if $\pi_1^c(w) = w$, and otherwise, $\pi_1^c(w) < w$. The following theorem, which is [30 Theorem 6.3], shows that the geometry of the $c$-Cambrian fan is closely tied to the combinatorics of $c$-sortable elements. It also shows that the support of $F_c$ contains $\text{Tits}(A)$.

**Theorem 5.1.** Let $v$ be $c$-sortable. Then $\pi_1^c(w) = v$ if and only if $wD \subseteq \text{Cone}_c(v)$.

The following is [30 Proposition 6.13].

**Proposition 5.2.** Let $I \subseteq S$ and let $c'$ be the restriction of $c$ to $W_I$. Then $\pi_1^{c'}(w_I) = \pi_1^c(w_I)$ for any $w \in W$.

The following is an immediate consequence of [32 Lemma 3.17].

**Proposition 5.3.** For any $s \in S$, every cone in $DF_c$ is either above $\alpha_s^+$ or below $\alpha_s^-$. Specifically, if $v$ is $c$-sortable, then $\text{Cone}_c(v)$ is above $\alpha_s^+$ if and only if $v \geq s$, and if $u$ is $c^{-1}$-sortable, then $-\text{Cone}_{c^{-1}}(u)$ is above $\alpha_s^+$ if and only if $u \not\geq s$. 


The following proposition is a restatement of [30, Proposition 5.2]. (The proposition in [30] is a statement about \(C_c(v)\), but here we restate it as an assertion about \(\text{Cone}_c(v)\).)

**Proposition 5.4.** Let \(v\) be \(c\)-sortable. Then \(t \in \text{cov}(v)\) if and only if \(\text{Cone}_c(v)\) is above \(\beta_t^+\) and \(\text{Cone}_c(v) \cap \beta_t^+\) is a facet of \(\text{Cone}_c(v)\).

When \(s\) is initial in \(c\), the element \(s\text{cs}s\) is another Coxeter element for \(W\) and \(sc\) is a Coxeter element of \(W_{(s)}\). We quote a result that relates \(DF_c\) to \(DF_{scs}\) and \(DF_{sc}\) (the latter being constructed in \(V_{(s)}^s\)). For each \(s\) in \(S\), let \(DF_{c(s)}\) denote the set of cones in \(DF_c\) that are above \(\alpha_s^+\) and let \(DF_{c(s)}^1\) denote the set of cones in \(DF_c\) below \(\alpha_s^+\). As before, for any \(I \subseteq S\), let \(\text{Proj}_I\) be the surjection from \(V^s\) onto \((V_I)^s\) that is dual to the inclusion of \(V_I\) into \(V\). The following is [32, Proposition 3.28].

**Proposition 5.5.** Let \(s\) be initial in \(c\). Then

1. \(DF_c = DF_c^{(s)} \cup DF_{c(s)}^1\).
2. \(DF_c^{(s)} = s(DF_{sc}^1)\).
3. The map \(F \mapsto \text{Proj}_{(s)}^{-1}(F) \cap \{x \in V^s : \langle x, \alpha_s \rangle \geq 0\}\) is a bijection from the set of maximal cones of \(DF_{sc}\) to the set of maximal cones of \(DF_c^{(s)}\).
4. The map \(F \mapsto \text{Proj}_{(s)}^{-1}(F) \cap \{x \in V^s : \langle x, \alpha_s \rangle \leq 0\}\) is a bijection from the set of maximal cones of \(DF_{sc}\) to the set of maximal cones of \(DF_c^{(s)}\).

For each \(I \subseteq S\), write \(c'\) for the restriction of \(c\) to \(W_I\) and write \(F_{c'}\) for the \(c'\)-Cambrian fan defined in \(V_I^s\). We now quote and prove some results that relate the \(c\)-Cambrian fan to the \(c'\)-Cambrian fan. The following is [30, Proposition 9.4].

**Proposition 5.6.** Let \(I \subseteq S\) and let \(c'\) be the restriction of \(c\) to \(W_I\). Then \(\text{Proj}_{(s)}^{-1}(F_{c'}) \cap \text{Tits}(A)\) is a coarsening of \(F_{c} \cap \text{Tits}(A)\).

We prove the following proposition.

**Proposition 5.7.** Let \(I \subseteq S\) and let \(c'\) be the restriction of \(c\) to \(W_I\). If \(F\) is a codimension-1 face of \(F_c\) contained in \(\beta_t^+\) with \(t \in W_I\), then \(F \cap \text{Tits}(A)\) is contained in \(\text{Proj}_{(s)}^{-1}(F') \cap \text{Tits}(A)\) for some face \(F'\) of \(F_{c'}\) of codimension 1 in \(V_I^s\).

For the proof, we will need some more background. Recall that \(D_I = \text{Proj}_I(D)\). For each \(w \in W\), we have \(\text{Proj}_I(wD) \subseteq w_I(D_I)\) and

\[
(5.1) \quad \bigcup_{x \in W, x_I = w_I} xD = \text{Proj}^{-1}_I(w_ID_I) \cap \text{Tits}(A).
\]

Let \(A_I\) be the Cartan matrix obtained by deleting from \(A\) the rows and columns indexed by \(S \setminus I\), and let \(\Phi_I\) be the corresponding sub root system of \(\Phi\). As a consequence of (5.1), we have \(\text{Tits}(A) \subseteq \text{Proj}_I^{-1}(\text{Tits}(A_I))\).

**Proof of Proposition 5.7.** The face \(F\) is \(\text{Cone}_c(v) \cap \beta_t^+\) for a \(c\)-sortable element \(v\) and \(t \in \text{cov}(v)\). Thus by Theorem 5.1,

\[
F \cap \text{Tits}(A) = \beta_t^+ \cap \bigcup_{w \in W, \pi_I^s(w) = v} wD.
\]
By Proposition 5.2, \( v_I \) is \( c' \)-sortable and if \( \pi_i^c(w) = v \) then \( \pi_i^{c'}(w_I) = v_I \). Thus
\[
F \cap \Tits(A) \subseteq \beta^c_I \cap \bigcup_{w \in W} wD = \beta^c_I \cap \bigcup_{x \in W_I} \bigcup_{w \in W, w_I = x} wD.
\]

For any \( x \in W_I \), (5.1) says \( \bigcup_{w \in W} wD = \Proj^{-1}_I(xD_I) \cap \Tits(A) \). Thus \( F \cap \Tits(A) \) is contained in
\[
\beta^c_I \cap \bigcup_{x \in W_I} \Proj^{-1}_I(xD_I) \cap \Tits(A) = \beta^c_I \cap \Proj^{-1}_I(\Cone_{c'}(v_I)) \cap \Tits(A),
\]
because \( \Tits(A) \subseteq \Proj^{-1}_I(\Tits(A_I)) \). Since \( t \) is a cover reflection of \( w \), we have \( \inv(tw) = \inv(w) \setminus \{t\} \), and since \( t \in W_I \), we have \( \inv((tw)_I) = \inv(w_I) \setminus \{t\} \), and thus \( t \in \cov(v_I) \). Therefore, \( \Cone_{c'}(v_I) \) has a facet \( F' \) of codimension 1 in \( V^*_I \) contained in \( \beta^c_I \subseteq V^*_I \). Now \( \Proj^{-1}_I \) applied to \( \beta^c_I \subseteq V^*_I \) is \( \beta^c_I \subseteq V^* \). Thus \( \beta^c_I \cap \Proj^{-1}_I(\Cone_{c'}(v_I)) \cap \Tits(A) = \Proj^{-1}_I(F') \cap \Tits(A) \).

5.2. The affine doubled Cambrian fan. When \( A \) is of affine type, \( \Tits(A) \) is the union of the open halfspace \( \{ x \in V^* : \langle x, \delta \rangle > 0 \} \) with the zero vector in \( V^* \).

The fact that \( \Tits(A) \) is essentially a halfspace in affine type, together with the fact that \( \mathcal{F}_c \) covers \( \Tits(A) \) in general, means that the doubled Cambrian fan is a particularly useful model in affine type. We quote some results from [32] on \( \mathcal{DF}_c \) in the affine case, beginning with [32, Corollary 1.3]:

**Theorem 5.8.** If \( B \) is acyclic and of affine type, then \( \mathcal{DF}_c \) coincides with the \( g \)-vector fan associated to \( B \).

Recall that \( V^*_{fin} \) is the subspace of \( V \) spanned by \( \{ \alpha_i : i \neq \text{aff} \} \). We identify the dual space \( V^*_{fin} \) with \( \partial \Tits(A) = \delta^c \) by the inclusion that is dual to the projection map from \( V \) to \( V^*_{fin} \) with kernel \( \mathbb{R}\delta \). (This projection is not the same as \( \Proj(\text{stat}) \). See [32, p. 1455].) The **Coxeter fan** in \( V^*_{fin} \) is the fan defined by the hyperplanes \( \beta^c_I \) for \( \beta \in \Phi_{fin} \). By this identification, we obtain a fan in \( \delta^c \). This fan coincides with the fan defined by the intersections with \( \delta^c \) of the hyperplanes \( \beta^c_I \) for \( \beta \in \Phi_{fin} \).

We reuse the term Coxeter fan for this decomposition of \( \delta^c \). Every real root \( \beta \) in the affine root system \( \Phi \) is a scalar multiple of \( \phi + k\delta \) for some \( \phi \in \Phi_{fin} \) and \( k \in \mathbb{Z} \). For \( x \in \delta^c \), we have \( \langle x, \phi + k\delta \rangle = \langle x, \phi \rangle \), so the Coxeter fan in \( \delta^c \) is the fan defined by the intersections with \( \delta^c \) of all hyperplanes \( \beta^c_I \) for \( \beta \in \Phi_{fin} \), not just for \( \beta \in \Phi_{fin} \).

We define \( x_c \) to be \( -\omega_c(\cdot, \delta) \in V^* \). Since \( \omega_c \) is skew-symmetric, \( x_c \) is in \( \delta^c \subseteq V^* \), which we identified with \( V^*_{fin} \).

**Lemma 5.9.** The vector \( \nu_c(\delta) \in V^* \) is equal to \( \frac{1}{2}x_c \).

**Proof.** Since \( \delta \) is positive, \( \nu_c(\delta) = -E_c(\cdot, \delta) \). Thus for any \( \gamma \in V \), we have \( \langle \nu_c(\delta), \gamma \rangle = -E_c(\gamma, \delta) \). Since \( K(\gamma, \delta) = 0 \), we see that \( \omega_c(\gamma, \delta) \) equals
\[
\omega_c(\gamma, \delta) = K(\gamma, \delta) = (E_c(\gamma, \delta) - E_c(\gamma, \delta)) + (E_c(\gamma, \delta) + E_c(\gamma, \delta)) = 2E_c(\gamma, \delta). \quad \Box
\]

Recall that \( \Phi^c_{fin} \neq 0 \) is the set \( \{ \beta \in \Phi_{fin} : \omega_c(\beta, \delta) > 0 \} \). We write \( |\mathcal{DF}_c| \) for the union of the cones of \( \mathcal{DF}_c \). The following is [32, Corollary 4.9].
Theorem 5.10. If $A$ is of affine type, then $V^* \setminus |DF_c|$ is a nonempty open cone in $\partial \mathrm{Tits}(A)$, and thus is of codimension 1 in $V^*$. This cone is

$$V^* \setminus |DF_c| = \bigcap_{\beta \in \Phi_{\omega}^+} \{ x \in \partial \mathrm{Tits}(A) : \langle x, \beta \rangle < 0 \}.$$ 

Also, $V^* \setminus |DF_c|$ is the union of the relative interiors of those cones in Coxeter fan in $\delta^\perp$ that contain $x_c$.

Theorem 5.10 says that the imaginary wall $\partial_\infty$ coincides with the closure of $V^* \setminus |DF_c|$, which is the boundary of $|DF_c|$. We can combine results of [34] to describe $\partial_\infty$ in another way. According to [34, Proposition 6.13], the union of the cones $\mathrm{Cone}(C)$ for imaginary clusters $C$ is a cone whose extreme rays are spanned by $\Xi_c$. By Theorem 3.3, $\nu_c$ takes that cone to the closure of the complement of the $g$-vector fan, which according to Theorem 5.8 is the closure of $V^* \setminus |DF_c|$. We record this fact as the following theorem.

Theorem 5.11. The extreme rays of the cone $\partial_\infty$ are spanned by the vectors $\Xi_c$.

The following theorems are (part of) [32, Corollary 4.18] and (all of) [32, Corollary 4.19].

Theorem 5.12. Suppose $A$ is of affine type and let $F$ be a face of $DF_c$ of dimension $n - 2$. Then the maximal cones of $DF_c$ containing $F$ form either a finite cycle of adjacent cones or an infinite path of adjacent cones. If the relative interior of $F$ is in the interior of $|DF_c|$, then they form a cycle. Otherwise they form a path.

Theorem 5.13. If $A$ is of affine type, then every $(n - 2)$-dimensional face of $DF_c$ contained in $\partial \mathrm{Tits}(A)$ is in $\partial_\infty$.

The following easy consequence of Theorem 5.10 will also be important.

Lemma 5.14. Every cone of the Coxeter fan in $\delta^\perp$ that is not contained in $\partial_\infty$ is contained in some face of $DF_c$.

Proof. Theorem 5.10 implies that every maximal cone $F$ in the Coxeter fan in $\delta^\perp$ is either in $|DF_c|$ or has lower-dimensional intersection with $|DF_c|$. Since all cones in $DF_c$ are defined by real roots and because the Coxeter fan is the fan defined in $\delta^\perp$ by all of the reflecting hyperplanes of $W$, if $F$ is contained in $|DF_c|$ then it is contained in a unique maximal cone of $DF_c$. Thus every cone of the Coxeter fan, if it is not contained in $\partial_\infty$, is contained in the intersection of one or more maximal cones of $DF_c$. $\square$

As part of [32, Theorem 1.1], it is proved that the dual graph (i.e. adjacency graph of maximal cones) of $DF_c$ forms what is called a reflection framework. We will not need full details on reflection frameworks here, but in particular, the dual graph satisfies the “Reflection condition”. A weak form of this condition can be rephrased as the following property of $DF_c$.

Proposition 5.15. Suppose $C$ and $C'$ are adjacent maximal cones of $DF_c$ with shared facet $F$ and let $\beta = \pm \beta_t$ be the inward-facing normal to $C$ at the facet $F$. Suppose $G \neq F$ is a facet of $C$ with inward-facing normal $\gamma$, let $G' \neq F$ be the facet of $C'$ with $G \cap F = G' \cap F$, and let $\gamma'$ be the inward-facing normal of $C'$ at the facet $G'$. Then

$$\gamma' = \begin{cases} \gamma & \text{if } \omega_c(\beta_t, \gamma) \geq 0, \\ t\gamma & \text{if } \omega_c(\beta_t, \gamma) < 0. \end{cases}$$
In affine type, we can make the following statement analogous to Proposition 5.7.
(The following proposition applies to $\mathcal{DF}_c$ rather than $\mathcal{F}_c$ and also avoids the intersections with $\text{Tits}(A)$.)

Proposition 5.16. Suppose $W$ is of affine type. Let $I \subset S$ and let $c'$ be the restriction of $c$ to $W_I$.

1. Every cone in $\mathcal{DF}_c$ is contained in a cone in $(\text{Proj}_I)^{-1}(\mathcal{F}_{c'})$.
2. If $F$ is a codimension-1 face of $\mathcal{DF}_c$ contained in $\beta^+_t$ with $t \in W_I$, then $F$ is contained in $(\text{Proj}_I)^{-1}(F')$ for some face $F'$ of $\mathcal{F}_{c'}$ of codimension 1 in $V^*_I$.

Proof. Since $I \subset S$ and $W$ is of affine type, $W_I$ is finite, so the doubled $c'$-Cambrian fan coincides with $\mathcal{F}_{c'}$ and with $-\mathcal{F}_{(c')^{-1}}$.

Let $F$ be a cone in $\mathcal{DF}_c$. Proposition 5.6 implies that $F \cap \text{Tits}(A)$ is contained in a cone $F'$ of $(\text{Proj}_I)^{-1}(\mathcal{F}_{c'})$ and that $F \cap (-\text{Tits}(A))$ is contained in a cone $F''$ of $(\text{Proj}_I)^{-1}(-\mathcal{F}_{(c')^{-1}}) = (\text{Proj}_I)^{-1}(\mathcal{F}_{c'})$. Since $I \subset S$, there exists $s \in S \setminus I$, and $(\text{Proj}_I)^{-1}(F')$ contains the line spanned by $\rho_s$ and in particular, it crosses $\partial \text{Tits}(A)$. Thus we can take $(\text{Proj}_I)^{-1}(F') = (\text{Proj}_I)^{-1}(F'')$, so that $F' = F''$.

If $F$ is a codimension-1 face of $\mathcal{DF}_c$ contained in $\beta^+_t$ with $t \in W_I$, then Proposition 5.7 says that we can take $F'$ to have codimension 1 in $V^*_I$. \hfill \qed

We now prove several propositions connecting the roots $\Phi_c$, the join-irreducible sortable elements, and the Cambrian fans/doubled Cambrian fans in the affine case.

Proposition 5.17. Let $\Phi$ be a root system of affine type and let $c$ be a Coxeter element of the corresponding Weyl group. Given a hyperplane $H$ in $V^*$, the following are equivalent.

(i) $H = \beta^+_t$ for a real root $\beta = c^k \gamma$ such that $k \geq 0$ and $\gamma$ is contained in a proper parabolic subsystem of $\Phi$.
(ii) There exist adjacent cones of the Cambrian fan $\mathcal{F}_c$ whose shared facet is in $H$.
(iii) There exists a $c$-sortable join-irreducible element $j$ with $\text{cov}(j) = \{t\}$ and $H = \beta^+_t$.

The element $j$ in (iii) is unique if it exists.

In the proof of Proposition 5.17 we will use the following three propositions. The first is an immediate consequence of [30, Theorem 8.1] and [30, Proposition 8.2], the second is [30, Proposition 2.30], and the third is an immediate consequence of [30, Proposition 5.3].

Proposition 5.18. If $v$ is a $c$-sortable element and $t \in \text{cov}(v)$, then there exists a $c$-sortable join-irreducible element $j$ with $\text{cov}(j) = \{t\}$.

Proposition 5.19. Suppose $I \subset S$ and let $c'$ be the restriction of $c$ to $W_I$. Then an element $v \in W_I$ is $c$-sortable if and only if it is $c'$-sortable.

Proposition 5.20. If $j$ is a $c$-sortable join-irreducible element, if $s$ is final in $c$, and if $s \preceq j$, then $s = j$.

Proof of Proposition 5.17. We will prove (i) $\iff$ (iii) and (ii) $\iff$ (iii). The uniqueness of $j$ follows from Proposition 4.1.

First, suppose (ii) and let $t'$ be such that $\gamma = \beta_{t'}$. Since $\gamma$ is in a proper subsystem of $\Phi$, $t'$ is in a proper parabolic subgroup $W_I$ of $W$. Write $c'$ for the restriction of $c$ to $W_I$. Since $W$ is affine, $W_I$ is finite. By the finite case of Proposition 4.1 there exists
a \( c' \)-sortable join-irreducible element \( j' \) with \( \text{cov}(j') = \{ t' \} \), and Proposition 5.19 says that \( j' \) is also a \( c \)-sortable element of \( W \).

We will prove the following claim: If there exists a \( c \)-sortable join-irreducible \( j' \) with \( \text{cov}(j') = \{ t' \} \) then there exists a \( c \)-sortable join-irreducible \( j'' \) with \( \text{cov}(j'') = \{ ct'c^{-1} \} \). For an easier induction, we prove a stronger claim: If there exists a \( c \)-sortable join-irreducible \( j' \) with \( \text{cov}(j') = \{ t' \} \) and if \( s \) is final in \( c \), then there exists an \( scs \)-sortable join-irreducible \( j'' \) with \( \text{cov}(j'') = \{ st's \} \). (The claim about \( ct'c^{-1} \) follows by \( n \) applications of the stronger claim.)

If \( s \) is final in \( c \) and \( s \leq j' \), then Proposition 5.20 says that \( s = j' \), so \( j' \) is also \( scs \)-sortable. In this case, \( t' = s \), so we can take \( j'' = j' = s \), and \( \text{cov}(j'') = \{ s \} \) as desired. If \( s \not\leq j' \), then \( s \leq sj' \), which is \( scs \)-sortable and has \( st's \in \text{cov}(sj') \). Now Proposition 5.18 says that there exists an \( scs \)-sortable join-irreducible element \( j'' \) with \( \text{cov}(j'') = \{ st's \} \) as desired.

We have proved the claim. Applying the claim \( k \) times, we conclude that there exists a join-irreducible element \( j \) with \( \text{cov}(j) = \{ c^k t'c^{-k} \} \). Now \( \beta = c^k \gamma = \beta_{c^k t'c^{-k}} \), and we have established (iii).

Conversely, suppose (iii). We will prove (i) by showing that \( \beta_t = c^k \gamma \) for some \( \gamma \) contained in a proper subsystem of \( \Phi \). Writing a reduced word \( s_1 \cdots s_n \) for \( c \), recall that the \( c \)-sorting word for \( j \) consists of \( k \) copies of \( s_1 \cdots s_n \) with \( k \geq 0 \) followed by the \( c \)-sorting word for a \( c \)-sortable element \( j' \) contained in a standard parabolic subgroup. If \( j' \) is the identity, then \( t = c^k s_n c^{-k} \) and we are done, with \( \gamma = -\alpha_\nu \). If \( j' \) is not the identity, then \( j' \) is join-irreducible. Let \( t' \) be its unique cover reflection.

Now, assume (iii). Then there is a \( c \)-sortable element \( v \) such that \( H \) defines a facet of \( \text{Cone}_c(v) \) and \( \text{Cone}_c(v) \) is above \( H \). Thus \( H = \beta_t \) for some reflection \( t \), and Proposition 5.4 says that \( t \in \text{cov}(v) \). Now Proposition 5.18 says that there is a \( c \)-sortable join-irreducible element \( j \) with \( \text{cov}(j) = \{ t \} \). Conversely, assume (iii). Theorem 5.1 and the \( c \)-sortability of \( j \) imply that \( jD \) and \( j,D \) are in distinct cones of \( \mathcal{F}_c \), and Theorem 5.1 implies that these cones are adjacent, and are \( \text{Cone}_c(j) \) and \( \text{Cone}_c(\pi_1^c(j)) \). \( \square \)

The following proposition is part of [32, Proposition 4.4].

**Proposition 5.21.** No codimension-1 face of \( \mathcal{DF}_c \) is contained in \( \delta^\perp \).

As an easy consequence, we obtain the following proposition.

**Proposition 5.22.** Every pair of adjacent maximal cones in \( \mathcal{DF}_c \) is either a pair of adjacent maximal cones in \( \mathcal{F}_c \) or a pair of adjacent maximal cones in \( \mathcal{F}_{c-1} \), or both.

**Proof:** Proposition 5.21 implies that no shared facet of maximal cones in \( \mathcal{DF}_c \) is contained in \( \delta^\perp \), and as a consequence, given a pair of adjacent maximal cones in \( \mathcal{DF}_c \), either both cones intersect the interior of Tits(\( A \)) or both cones intersect the interior of \( -\text{Tits}(A) \). Since the support of \( \mathcal{F}_c \) contains Tits(\( A \)) and the support of \( \mathcal{F}_{c-1} \) contains \( -\text{Tits}(A) \), the proposition follows. \( \square \)

Recall that \( \Phi^e_c \) is the set \( -\Pi \cup (\Phi^+ \setminus U') \cup \Phi^e_c = \Phi_c \setminus \{ \delta \} \) of real roots in \( \Phi_c \). The following proposition is a rephrasing of [34, Proposition 3.13(4)].
Proposition 5.23. The set of positive real roots in $\Phi_c$ and their negations is the set of all roots $c^k\gamma$ such that $\gamma$ is in a proper subsystem of $\Phi$ and $k \in \mathbb{Z}$.

Proposition 5.24. Let $\Phi$ be a root system of affine type and let $c$ be a Coxeter element of the corresponding Weyl group. Given a hyperplane $H$ in $V^*$, the following are equivalent.

(i) There exists a positive root $\beta \in \Phi^+_c$ such that $H = \beta^\perp$.

(ii) There exist adjacent maximal cones in the doubled Cambrian fan $DF_c$ whose shared facet is in $H$.

(iii) Either there exist adjacent maximal cones in the Cambrian fan $F_c$ whose shared facet is in $H$ or there exist adjacent maximal cones in the opposite Cambrian fan $-F_{c-1}$ whose shared facet is in $H$, or both.

(iv) Either there is a $c$-sortable join-irreducible element $j$ with $\text{cov}(j) = \{t\}$ and $H = \beta_t^\perp$ or there is a $c^{-1}$-sortable join-irreducible element $j'$ with $\text{cov}(j') = \{t\}$ and $H = \beta_t^\perp$, or both.

In (iv), $j$ is unique if it exists and $j'$ is unique if it exists.

Proof. The equivalence of (i) and (iii) follows from Propositions 5.17 and 5.23. Proposition 5.22 implies that (ii) and (iii) are equivalent. The equivalence of (iii) and (iv) follows from Proposition 5.17. The uniqueness of $j$ and $j'$ in (iv) also follows from Proposition 5.17. □

Remark 5.25. Some of the equivalences in Proposition 5.24 can also be obtained, in the skew-symmetric case, by combining the main result of [16] with results of [32].

6. Completing the cluster scattering diagram proofs

We now complete the proof of Theorem 2.2 and the other results on affine cluster scattering fans. To begin, there is one more wall that must be shown to be outgoing and gregarious.

Proposition 6.1. The wall $(d_\infty, f_\infty)$ is outgoing and gregarious.

Proof. By Theorem 5.10, $d_\infty$ is the union of the cones in the Coxeter fan in $\delta^\perp$ that contain $-\omega_c(\cdot, \delta)$. Thus the wall $(d_\infty, f_\infty)$ is gregarious. Since $\Phi_{\text{fin}}$ has rank at least 1, no cone of the Coxeter fan contains both $\omega_c(\cdot, \delta)$ and $-\omega_c(\cdot, \delta)$. We conclude that $(d_\infty, f_\infty)$ is also outgoing. □

The last step in the proof of Theorem 2.2 which will take up most of Section 6 is to prove the following proposition:

Proposition 6.2. $\text{DCScat}(A, c)$ is consistent.

6.1. Rank-2 subsystems of affine type. At the heart of the proof of Proposition 6.2 is a reduction to the finite or affine rank-2 case. We need to differentiate between type $A^{(2)}_{2k}$ and all other types. (Accordingly, the definition of $\text{DCScat}(A, c)$ differentiates between these two possibilities.) The following lemma tells which affine rank-2 case we need to consider for type $A^{(2)}_{2k}$ and for all other types.

Lemma 6.3. Suppose $\beta \in \Phi^+_{\text{fin}}$ and suppose $d_\infty \cap \beta^\perp$ is a facet of $d_\infty$. If $\Phi$ is not of type $A^{(2)}_{2k}$, then the rank-2 subsystem $\Phi'$ containing $\beta$ and $\delta$ is of type $A^{(1)}_1$. If $\Phi$ is of type $A^{(2)}_{2k}$, then $\Phi'$ is of type $A^{(2)}_2$.
Proof. The restriction of $A$ to the span of $\beta$ and $\delta$ is positive semidefinite but not positive definite and not zero. Since the restriction is of rank 2, it is of affine type, and therefore $\Phi^\prime$ is of type $A_1^{(1)}$ or $A_2^{(2)}$. If $\Phi^\prime$ is of type $A_2^{(2)}$, then it contains a pair of roots whose ratio of squared lengths is 2. But only $A_{2k}^{(2)}$ contains roots with that ratio of squared lengths, and the first assertion of the lemma follows.

Suppose $\Phi$ is of type $A_{2k}^{(2)}$. We will show that for every $\beta$ such that $\mathcal{D}_\infty \cap \beta^\perp$ is a facet of $\mathcal{D}_\infty$, the rank-2 subsystem $\Phi^\prime$ containing $\beta$ and $\delta$ is of type $A_2^{(2)}$. If $k = 1$, then $\Phi^\prime = \Phi$, so assume $k > 1$. We first claim that it is enough to consider a single choice of $c$. Proposition 5.5.3 implies that, for $s$ initial in $c$, the wall $\mathcal{D}_\infty$ is above the hyperplane $\alpha_i^\perp$. Thus by Proposition 5.5.2, the imaginary wall, as defined for $scs$, is related to the imaginary wall, as defined for $c$, by the action of $s$. Since the action of $s$ preserves the lengths of roots, it also preserves the type of $\Phi^\prime$. The Dynkin diagram of type $A_{2k}^{(2)}$ is a tree (in fact a path), so all Coxeter elements are related by sequences of moves of the form $c \leftrightarrow scs$ for $s$ initial or final in $c$.

Having proved the claim, we choose a Coxeter element $c = s_1 \cdots s_n$ with each $s_i$ and $s_{i+1}$ not commuting, taking $s_1$ to be the longest root. Thus, $\alpha_i$ is the shortest root and $\text{aff} = n$. Then $\delta = \alpha_1 + 2 \sum_{i=2}^n \alpha_i$ and

$$ B = \begin{bmatrix} 0 & 1 & \cdots & 1 \\ -2 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ -1 & -1 & \cdots & -2 \end{bmatrix}. $$

Thus in the basis of fundamental weights, $x_c = -\omega_c(\cdot, \delta)$ is the transpose of

$$ \begin{bmatrix} 0 & 1 & \cdots & 1 \\ -2 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ -1 & -1 & \cdots & -2 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ \vdots \\ 2 \end{bmatrix} = \begin{bmatrix} -2 \\ 0 \\ \vdots \\ 0 \end{bmatrix}. $$

Recall that the dual space $V_{\text{fin}}^\ast$ for the finite root system is identified with the boundary of the Tits cone by the inclusion dual to the projection map $\pi : V \to V_{\text{fin}}$ with kernel $\mathbb{R}\delta$. Thus $x_c = \lambda \circ \pi$ for some unique $\lambda \in V_{\text{fin}}^\ast$. Since $\pi$ is the identity on $V_{\text{fin}}$, we can determine $\lambda$ by how $x_c$ acts on simple roots in $\Phi_{\text{fin}}$. We see that $\lambda = -2\rho_1$. (Recall that $\rho_1$ is the fundamental weight corresponding to $\alpha_1^\perp$.)

Theorem 5.10 says that $\mathcal{D}_\infty$ is the union of all cones in the Coxeter fan in $\delta^\perp$ that contain $x_c$. One of these cones is spanned by (the inclusion of) $\rho_1, \ldots, \rho_{n-1}$. The normal to the facet opposite $\rho_1$ in this cone is $\alpha_1$, which is a long root. We see that $\alpha_1$ defines a facet of the imaginary wall $\mathcal{D}_\infty$. The subsystem $\Phi^\prime$ containing $\alpha_1$ and $\delta$ has canonical roots $\alpha_1$ and $\gamma = \sum_{i=2}^n \alpha_i$. We compute that $\gamma = s_2 s_3 \cdots s_{n-1} \alpha_n$. In particular, since $\alpha_n$ is a short root, so is $\gamma$. Now, $\Phi^\prime$ is of affine type because it contains $\delta$, and it is of type $A_2^{(2)}$ because it contains real roots of two different lengths.

We have proved that $\Phi^\prime$ is of type $A_2^{(2)}$ when $\beta = \alpha_1$. Since $\mathcal{D}_\infty$ is the union of all maximal cones of the Coxeter arrangement containing $x_c = -2\rho_1$ and since all of these maximal cones are related by the action of $(W_{\text{fin}}(s_1))$, in particular, all of the root subsystems $\Phi^\prime$ contemplated in the second assertion of the lemma are related by the action of $W$. Thus for any $\beta$ as in the lemma, $\Phi^\prime$ is of type $A_2^{(2)}$. \qed
The rank-2 scattering diagrams of affine type are known. The following theorem is [36] Section 6] in the skew-symmetric case and [26] Theorem 3.4] in the non-skew-symmetric case.

**Theorem 6.4.** The function on the limiting wall of \( \text{Scat}^T \left( \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \right) \) is \( \frac{1}{(1 - y_1 y_2)^2} \).

The function on the limiting wall of \( \text{Scat}^T \left( \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \right) \) is \( \frac{1 + y_1 y_2^2}{(1 - y_1 y_2^2)^2} \).

6.2. Sortable elements and shards. The proof of Theorem 2.2 will also need some facts about shards associated to c-sortable join-irreducible elements. We begin by quoting a fundamental fact about c-sortable elements.

We say that \( w \in W \) is c-aligned with respect to a rank-2 subsystem \( \Phi' \) if one of the following cases holds (where \( \beta \) and \( \gamma \) are the canonical roots of \( \Phi' \)):

(i) \( \omega_c(\beta, \gamma) = 0 \) and \( \text{inv}(w) \cap \Phi' = \emptyset \) or \( \{ \beta \} \) or \( \{ \beta, \gamma \} \).

(ii) \( \omega_c(\beta, \gamma) > 0 \) and either \( \gamma \not\in \text{inv}(w) \) or \( \text{inv}(w) \cap \Phi' = \{ \gamma \} \) or \( \text{inv}(w) \supseteq \Phi' \).

(iii) \( \omega_c(\beta, \gamma) < 0 \) and either \( \beta \not\in \text{inv}(w) \) or \( \text{inv}(w) \cap \Phi' = \{ \beta \} \) or \( \text{inv}(w) \supseteq \Phi' \).

The following is a weaker version of [30] Theorem 4.3.

**Theorem 6.5.** An element \( w \in W \) is c-sortable if and only if \( w \) is c-aligned with respect to every generalized rank-2 parabolic sub root system of \( \Phi \).

**Lemma 6.6.** Suppose \( j \) is a c-sortable join-irreducible element with \( \text{cov}(j) = \{ t \} \). For \( \gamma \in \text{cut}(\beta_t) \), the following are equivalent.

(i) \( \gamma \in \text{inv}(j) \).

(ii) \( \omega_c(\gamma, \beta_t) > 0 \).

(iii) \( \omega_c(\gamma, \gamma') > 0 \), where \( \gamma' \) is the other canonical root in the rank-2 subsystem containing \( \gamma \) and \( \beta_t \).

**Proof.** Write \( a_1 \cdots a_k \) for the c-sorting word for \( j \) and let \( \beta_1, \ldots, \beta_k \) be the inversion sequence of \( a_1 \cdots a_k \). Crucially, since \( j \) is join-irreducible, \( \beta_i = \beta_k \).

Since \( \gamma \in \text{cut}(\beta_t) \), we know that \( \gamma \) is a canonical root in the rank-2 subsystem \( \Phi' \) containing \( \gamma \) and \( \beta_t \) and that \( \beta_t \) is not a canonical root in \( \Phi' \). Thus, if \( \gamma' \) is the other canonical root in \( \Phi' \), then \( \beta_t \) is a positive linear combination of \( \gamma \) and \( \gamma' \). Since \( \omega_c \) is skew-symmetric, we see that (i) and (iii) are equivalent.

If \( \omega_c(\gamma, \gamma') > 0 \), then since \( \beta_t \) is a positive combination of \( \gamma \) and \( \gamma' \), we see that \( \omega_c(\beta_t, \gamma') > 0 \) and thus \( \omega_c(\gamma', \beta_t) < 0 \). Since \( \beta_t = \beta_k \), Proposition 4.6 implies that \( \gamma' \) is not in the inversion sequence \( \beta_1, \ldots, \beta_k \). But then Lemma 3.2 implies that \( \gamma \not\in \text{inv}(j) \). We see that (iii) implies (i).

Suppose \( \gamma \in \text{inv}(j) \). Then \( \gamma = \beta_i \) for some \( i \leq k \). Since \( \gamma \in \text{cut}(\beta_t) \), \( i < k \) and \( \gamma \) is a canonical root in the rank-2 subsystem \( \Phi' \) containing \( \gamma \) and \( \beta_t \). Moreover, \( \beta_t \) is not a canonical root in \( \Phi' \). In particular, \( \Phi' \) has at least 3 roots, and thus, if \( \gamma' \) is the other canonical root in \( \Phi' \), we have \( K(\gamma, \gamma') \neq 0 \). Proposition 4.6 says that \( \omega_c(\gamma, \beta_t) \geq 0 \). If \( \omega_c(\gamma, \beta_t) > 0 \), then we have established (i). If \( \omega_c(\gamma, \beta_t) = 0 \), then Proposition 4.6 also says that \( K(\gamma, \beta_t) = 0 \). Since \( K(\gamma, \gamma') \neq 0 \), there exists a root \( \gamma'' \) in the positive linear span of \( \gamma \) and \( \beta_t \) with \( K(\gamma, \gamma'') \neq 0 \). Lemma 3.2 implies that \( \gamma'' \) appears between \( \gamma \) and \( \beta_t \) in the inversion sequence \( \beta_1, \ldots, \beta_k \), so Proposition 4.6 implies that \( \omega_c(\gamma, \gamma'') > 0 \). Since \( \gamma'' \) is a positive combination of \( \gamma \) and \( \gamma' \), we conclude that \( \omega_c(\gamma, \gamma') > 0 \), and we have established (iii). We have shown that (i) implies (ii) or (iii). □
As an immediate consequence of Proposition 4.9 and Lemma 6.6 we have the following description of shards for $c$-sortable join-irreducible elements.

**Proposition 6.7.** Suppose $c$ is a Coxeter element of $W$ and let $t$ be a reflection in $W$. If $j$ is a $c$-sortable join-irreducible element of $W$ and $\text{cov}(j) = \{t\}$, then $\text{Sh}(j)$ is

$$\{x \in V^*: \langle x, \beta_t \rangle = 0 \text{ and } \langle x, \gamma \rangle \leq 0 \text{ for all } \gamma \in \text{cut}(\beta_t) \text{ with } \omega_c(\gamma, \beta_t) > 0\}.$$ 

As another consequence of Lemma 6.6, we have the following relationship between shards for $c$-sortable and $c^{-1}$-sortable join-irreducible elements.

**Proposition 6.8.** Suppose $c$ is a Coxeter element of $W$ and let $t$ be a reflection in $W$. If there exist both a $c$-sortable join-irreducible element $j$ with $\text{cov}(j) = \{t\}$ and a $c^{-1}$-sortable join-irreducible element $j'$ with $\text{cov}(j') = \{t\}$, then $\text{Sh}(j') = -\text{Sh}(j)$.

**Proof.** Consider the inequalities of Proposition 6.7 for both $\text{Sh}(j)$ and $\text{Sh}(j')$. For any $\gamma \in \text{cut}(\beta_t)$, take $\gamma'$ as in Lemma 6.6. Since $\omega_{c^{-1}}(\gamma, \gamma') = -\omega_c(\gamma, \gamma')$, we see that the inequalities for $\text{Sh}(j')$ are obtained from the inequalities for $\text{Sh}(j)$ by replacing each $\gamma$ by $\gamma'$. Since $\gamma$ and $\gamma'$ are the canonical roots in a rank-2 subsystem in which $\beta_t$ is not a canonical root, for $x \in \beta_t^\perp$, the inequality $\langle x, \gamma' \rangle \leq 0$ is equivalent to $\langle x, \gamma \rangle \geq 0$. We conclude that $\text{Sh}(j') = -\text{Sh}(j)$.

**Proposition 6.9.** Suppose $W$ is of finite or affine type. Suppose $j$ is join-irreducible and $c$-sortable and $\text{Sh}(j)$ intersects the interior of $-\text{Tits}(A)$. Then there exists a $c^{-1}$-sortable element $j'$ such that $\text{Sh}(j) = -\text{Sh}(j')$.

**Proof.** Let $t$ be such that $\text{cov}(j) = \{t\}$. By Proposition 6.8, it is enough to show that there exists a $c^{-1}$-sortable element $j'$ with $\text{cov}(j') = t$. Then by Proposition 5.18 it is enough to show that there exists a $c^{-1}$-sortable element $v$ with $t \in \text{cov}(v)$. Let $s$ be initial in $c$.

- If $s \nleq j$, then $j \in W(s)$, and thus $\beta_t$ is in a proper subsystem of $\Phi$, so Proposition 5.17 implies that there exists a $c^{-1}$-sortable join-irreducible element $j'$ with $\text{cov}(j') = \{t\}$.

- If $s = j$, then $t = s$, and $j' = s$ is a join-irreducible $c^{-1}$-sortable element with $\text{cov}(j') = \{t\}$.

- If $s < j$, then $sj$ is a join-irreducible $\text{scs}$-sortable element with $\text{cov}(sj) = \{st\}j$ and $st \neq s$. By Proposition 4.12 we have $\text{Sh}(sj) \supseteq (s \cdot \text{Sh}(j))$, and since the action of $s$ preserves $-\text{Tits}(A)$, we see that $\text{Sh}(sj)$ intersects the interior of $-\text{Tits}(A)$. By induction on $t(j)$, there exists an $(\text{scs})^{-1}$-sortable element $u$ with $st \in \text{cov}(u)$.

We will show, more strongly, that there is an $(\text{scs})^{-1}$-sortable element $u$ with $st \in \text{cov}(u)$ and also $s \leq u$. Since there exists an $(\text{scs})^{-1}$-sortable element with $st$ as a cover reflection, Proposition 5.18 says that there exists a join-irreducible $(\text{scs})^{-1}$-sortable element $j''$ with $\text{cov}(j'') = \{st\}$. If $s \nleq j''$, then $j'' \in W(s)$.

We will show that the join $s \lor j''$ exists. Since $\text{Sh}(j)$ intersects the interior of $-\text{Tits}(A)$, there exists an element $w \in W$ such that $w(-D)$ is separated from $-D$ by the hyperplane $\beta_{st}^\perp$ and intersects $\text{Sh}(j)$ in a facet of $w(-D)$. Thus $wD$ is separated from $D$ by $\beta_{st}^\perp$ and intersects $-\text{Sh}(j)$ in a facet of $wD$. In other words, $w$ is an upper element of the shard $-\text{Sh}(j)$. Since $\text{Sh}(sj) \supseteq (s \cdot \text{Sh}(j))$, we see that $sw$ is an upper element of the shard $-\text{Sh}(sj)$. Since $j''$ is the unique minimal upper element of $-\text{Sh}(sj)$, we have $sw \geq j''$. Also $sw \geq s$, because $w \nleq s$. So $sw$ is an upper bound for $j''$ and $s$. Thus $s \lor j''$ exists.
Since \( j'' \) and \( s \) are both \((scs)^{-1}\)-sortable, \( s \vee j'' \) is also \((scs)^{-1}\)-sortable. Now [23 Lemma 2.8] says that if \( x \in W(s) \), then \( \operatorname{cov}(x \vee s) = \operatorname{cov}(x) \cup \{s\} \). Thus \( u = s \vee j'' \) is the desired \((scs)^{-1}\)-sortable element with \( sts \in \operatorname{cov}(u) \) and also \( s \leq u \).

Now \( s \) is final in \( scs \) and thus initial in \((scs)^{-1}\). Since \( u \) is \((scs)^{-1}\)-sortable and \( s \leq u \), also \( su \) is \( c^{-1}\)-sortable. Because \( sts \in \operatorname{cov}(u) \), there exists \( r \in S \) such that \( (sts)u = ur < u \). Since \( sts \neq s \), we have \( s \leq ur \), so \( s(ur) \leq su \). That is, \( t(su) = (su)r \leq su \), and thus \( t \in \operatorname{cov}(su) \).

We have constructed a \( c^{-1}\)- sortable element having \( t \) as a cover reflection. 

The following proposition relates the shards associated to \( c \)- and \( c^{-1}\)- sortable elements to the fan \( \nu_c(\operatorname{Fan}_c(\Phi)) \) in affine type. Once we have proved our main results, we will be able to improve on this proposition (as Corollary 7.20).

**Proposition 6.10.** Let \( \Phi \) be a root system of affine type and let \( c \) be a Coxeter element of the corresponding Weyl group.

1. If \( j \) is a \( c \)- sortable join-irreducible element with \( \operatorname{cov}(j) = \{t\} \), then every \((n-1)\)-dimensional face of \( \nu_c(\operatorname{Fan}_c(\Phi)) \) contained in \( \beta_1^+ \) is contained in \( \operatorname{Sh}(j) \) and \( \operatorname{Sh}(j) \cap (\operatorname{Tits}(A) \cup (\operatorname{Tits}(A))) \) is covered by \((n-1)\)- dimensional faces of \( \nu_c(\operatorname{Fan}_c(\Phi)) \).

2. If \( j' \) is a \( c^{-1}\)- sortable join-irreducible element with \( \operatorname{cov}(j') = \{t'\} \), then every \((n-1)\)-dimensional face of \( \nu_c(\operatorname{Fan}_c(\Phi)) \) contained in \( \beta_1^+ \) is contained in \( -\operatorname{Sh}(j') \) and \( -\operatorname{Sh}(j') \cap (\operatorname{Tits}(A) \cup (\operatorname{Tits}(A))) \) is covered by \((n-1)\)-dimensional faces of \( \nu_c(\operatorname{Fan}_c(\Phi)) \).

For the proof of Proposition 6.10 we will use the following less detailed version of [30 Proposition 5.3].

**Proposition 6.11.** Let \( s \) be final in \( c \) and let \( v \) be \( c \)- sortable with \( v \geq s \). Then \( v = s \vee v(s) \) and \( C_c(v) = \{ -\alpha_s \} \cup C_c(v(s)) \).

We will also use the following proposition to show that the two assertions of Proposition 6.10 are equivalent.

**Proposition 6.12.** \( \nu_c(\operatorname{Fan}_{c^{-1}}(\Phi)) = -\nu_c(\operatorname{Fan}_c(\Phi)) \).

**Proof.** [34 Proposition 3.2] says that \( \Phi_c = \Phi_{c^{-1}} \). Furthermore, [34 Proposition 4.10] says that \( c \)- compatibility and \( c^{-1}\)-compatibility coincide. As an immediate consequence, \( \operatorname{Fan}_c(\Phi) = \operatorname{Fan}_{c^{-1}}(\Phi) \). Now [34] implies that the map \( \tau_c \) on \( \Phi_c \) induces a piecewise linear automorphism of \( \operatorname{Fan}_c(\Phi) \), and [34 Lemma 9.3] says that \( \tau_c \) coincides with the map \( \beta \mapsto \nu_{c^{-1}}(\nu_c(\beta)) \).

**Proof of Proposition 6.11.** By Proposition 6.12 it is enough to prove Assertion 1. Assertion 2 then follows by replacing \( c \) with \( c^{-1} \).

Suppose \( j \) is a \( c \)- sortable join-irreducible element and \( \operatorname{cov}(j) = \{t\} \). Recall from Theorem 5.3 that \( \nu_c(\operatorname{Fan}_c(\Phi)) \) equals \( \operatorname{DF}_c \) and from Theorem 5.10 that the closure of the complement of \( |DF_c| \) is \( \delta_\infty \). Since \( \delta_\infty \subseteq \delta^+ = \partial(\operatorname{Tits}(A)) \), we can replace \( \nu_c(\operatorname{Fan}_c(\Phi)) \) by \( \operatorname{DF}_c \) throughout Assertion 1.

Let \( F \) be an \((n-1)\)-dimensional face of \( \operatorname{DF}_c \) contained in \( \beta_1^+ \). We will prove by induction on \( \ell(j) \) and on the rank of \( W \) that \( F \subseteq \operatorname{Sh}(j) \). Let \( s \) be initial in \( c \).

If \( s \not\leq j \), then \( j \) is an \( scs \)- sortable join-irreducible element of \( W(s) \). Thus also \( t \in W(s) \). By Proposition 5.16 \( F \) is contained in \( \operatorname{Proj}(s))^{-1}(F') \) for some face \( F' \)
of $\mathcal{F}_{sc}$ of dimension $n - 2$ in $V^*_s$. By induction on rank, $F'$ is contained in $\text{Sh}(j)^{(s)}$. Now Proposition 4.11 implies that $F$ is contained in $\text{Sh}(j)$.

If $s = j$, then $t = s$. Also, $\text{Sh}(j) = \beta^+_t$ by Lemma 4.8. Thus $F \subseteq \text{Sh}(j)$.

If $s < j$, then $j \notin W_s$, so $t \notin W_s$. We will show that $F$ is above $\alpha^+_s$. If $F$ is a face of $\mathcal{F}_c$ then it is $\text{Cone}_c(v) \cap \beta^+_t$ for some $c$-sortable element $v$ with $t \in \text{cov}(v)$. In particular, $v$ is not in $W_s$, and thus $v \geq s$. By Proposition 5.3, $\text{Cone}_c(v)$ is above $\alpha^+_s$, so $F$ is above $\alpha^+_s$. If $F$ is not a face of $\mathcal{F}_c$, then it is a face of $-\mathcal{F}_{c-1}$. In this case, by Proposition 5.3, if $F$ is not above $\alpha^+_s$ then it is $-\text{Cone}_{c-1}(u) \cap \beta^+_t$ for some $c^{-1}$-sortable element $u$ with $u \geq s$. Then Proposition 6.11 implies that every element of $C_{c-1}$ is either $-\alpha_s$ or is a root in $\Phi(s)$. But then $\beta_t \notin \pm C_{c-1}(u)$, contradicting the fact that $-\text{Cone}_{c-1}(u)$ has a facet defined by $\beta^+_t$. We conclude that $F$ is above $\alpha^+_s$ in either case.

Observe that since $s < j$, the element $sj$ is join-irreducible and has $\text{cov}(sj) = \{s, t, \ldots, j\}$. Thus by Proposition 4.12 and induction on $l(j)$, $sF \subseteq \text{Sh}(sj)$. Thus, by Proposition 4.12, since $F$ is above $\alpha^+_s$, $F$ is contained in $\text{Sh}(j)$.

To complete the proof of Assertion 1 we show that $\text{Sh}(j) \cap \text{Tits}(A) \cap (\text{inv}(x))$ is covered by $(n - 1)$-dimensional faces of $\mathcal{D}_{\mathcal{F}_c}$. Suppose $w$ and $x$ are elements of $W$ such that $wD \cap xD$ has codimension 1 in $V^*$. We must show that if $wD \cap xD$ is contained in $\text{Sh}(j)$, then there exists a codimension-1 cone of $\mathcal{F}_c$ containing $wD \cap xD$. We must also show that if $-(wD \cap xD)$ is contained in $\text{Sh}(j)$, then there exists a codimension-1 cone of $-\mathcal{F}_{c-1}$ containing $-(wD \cap xD)$.

First, suppose $wD \cap xD$ is contained in $\text{Sh}(j)$. Without loss of generality, $w \geq x$ in the weak order. By Proposition 3.1, $w \geq j$ in the weak order. It is also immediate that $x \geq j$, since $\beta_t \notin \text{inv}(x)$. Since $\pi^+_t(w)$ is the unique largest $c$-sortable element below $w$, and similarly for $\pi^+_t(x)$, we conclude that $\pi^+_t(w) \neq \pi^+_t(x)$. Thus $wD$ and $xD$ are in different cones of $\mathcal{F}_c$. These cones intersect in a codimension-1 face of $\mathcal{F}_c$ containing $wD \cap xD$.

Now suppose that $-(wD \cap xD)$ is contained in $\text{Sh}(j)$. In particular, $\text{Sh}(j)$ intersects the interior of $-\text{Tits}(A)$, so Proposition 6.9 says that there is a $c^{-1}$-sortable join-irreducible element $j'$ with $\text{Sh}(j) = -\text{Sh}(j')$. Thus $-(wD \cap xD)$ is contained in $-\text{Sh}(j')$. By the case proved above (replacing $c$ by $c^{-1}$), there exists a codimension-1 face of $-\mathcal{F}_{c-1}$ containing $-(wD \cap xD)$ and we are done. \qed

6.3. Consistency. In this section, we prove Proposition 6.2. To reduce the number of cases, we will use the following basic fact about $\text{Scat}^T(B)$. (See, for example, [26, Proposition 2.4].)

**Proposition 6.13.** For any exchange matrix $B$,

$$\text{Scat}^T(-B) = \{(d, f_\delta((y')^\beta)) : (d, f_\delta((y')^\beta)) \in \text{Scat}^T(B)\},$$

where as usual $y_i = y_i x^b_{1i} \ldots x^b_{ni}$ while $\tilde{y}_i = x^{-1}_{1i} \ldots x^{-1}_{ni}$.

Given a $(n - 2)$-dimensional intersection $F$ of walls of the scattering diagram $D\text{Scat}(A, c)$, we use the phrase *small loop about $F$* to denote a limit of closed curves about $F$, each passing exactly once through the relative interior of every wall containing $F$ and, in the limit, intersecting no other wall. Such a limiting loop can be obtained by starting with a closed curve about $F$ close to a point in the relative interior of $F$ and performing a dilation that limits to that relative interior point. A small loop about $F$ has a well defined path-ordered product, because for any $k \geq 1$, once the loop is dilated small enough, its path-ordered product relative
to DCScat\(q(A, c)\) does not change. The path-ordered product also does not depend on the precise choice of a small loop (except up to reversing direction).

Proof of Proposition 6.2. The proof proceeds by checking consistency locally, as we now explain. Let \(F\) be an \((n - 2)\)-dimensional intersection of walls of DCScat\((A, c)\). It is enough to check that for \(\gamma\) a small loop about \(F\), the path-ordered product \(p_\gamma\) is trivial. As pointed out in the proof of [24] Proposition 6.7, the finite-type analog of this theorem, passing from the rank-2 case to a loop about \(F\) is a simple change of variables. Specifically, the normal vectors to walls containing \(F\) are all contained in the 2-dimensional plane (in \(V\)) orthogonal to \(F\). Thus these normal vectors are roots in a rank-2 subsystem \(\Phi'\) of \(\Phi\).

By [24] Proposition 2.2, which rephrases [23] Proposition 2.5 in the conventions of this paper, to check consistency, it is enough to compute path-ordered products applied to monomials \(x^\lambda\) for vectors \(\lambda \in P\). However, for a small loop about \(F\), inspection of the wall-crossing automorphisms as described in (3.1) and (3.2) reveals that we can ignore the component of \(\lambda\) that is in the linear span of \(F\) (because that component is orthogonal to \(\Phi'\)), and that we only ever consider the restriction of \(\omega_c\) to the plane in \(V\) orthogonal to \(F\). Thus it is enough in every case to verify that the rank-2 scattering diagram induced on a plane complementary to \(F\) is consistent, relative to the restriction of \(\omega_c\).

We will consider three cases for how \(F\) is situated relative to \(\pm\text{Tits}(A)\): Either \(F \cap \text{Tits}(A)\) is \((n - 2)\)-dimensional, and/or \(F \cap (-\text{Tits}(A))\) is \((n - 2)\)-dimensional, or \(F\) is contained in the boundary of \(\text{Tits}(A)\). In light of Proposition 6.13, applying the antipodal map and replacing \(c\) by \(c^{-1}\) turns the case where \(F \cap (-\text{Tits}(A))\) is \((n - 2)\)-dimensional into the case where \(F \cap (\text{Tits}(A))\) is \((n - 2)\)-dimensional, so we only argue one of these cases.

The case where \(F \cap \text{Tits}(A)\) is \((n - 2)\)-dimensional is analogous to the finite-type case treated in [24] Proposition 4.7, and the argument given there extends without alteration to the affine case (because, although \(\mathcal{DF}_c\) is infinite, the set of cones of \(\mathcal{DF}_c\) containing \(F\) is finite).

More specifically, if \(F \cap \text{Tits}(A)\) is \((n - 2)\)-dimensional, then \(F\) contains an \((n - 2)\)-dimensional face \(F'\) of \(\mathcal{F}_c\). Theorem 5.12 implies that a small loop \(\gamma\) about \(F'\) intersects finitely many \((n - 1)\)-dimensional faces of \(\mathcal{F}_c\) (or equivalently of \(\mathcal{DF}_c\)). By Propositions 5.17 and 6.10 these intersections with faces of \(\mathcal{DF}_c\) are precisely the intersections with the walls of DCScat\((A, c)\). [30] Theorem 9.8 describes the star of \(F\) in \(\mathcal{DF}_c\) as a finite rank-2 Cambrian fan. The rank-2 exchange matrix associated to this fan is the matrix describing the restriction of \(\omega_c\) in the basis of canonical roots of \(\Phi'\). (See [32] Theorem 2.3(1)).) Thus the consistency of a small loop about \(F\) is equivalent to the consistency of rank-2 Cambrian scattering diagrams of finite type, just as in the proof of [24] Proposition 4.7).

We have completed the argument when \(F \cap \text{Tits}(A)\) and/or \(F \cap (-\text{Tits}(A))\) is \((n - 2)\)-dimensional. It remains to consider the case where \(F\) is in \(\partial\text{Tits}(A)\). Since each wall of DCScat\((A, c)\) is defined by hyperplanes orthogonal to roots, in this case \(F\) is a finite union of faces of the Coxeter fan in \(\delta^2\) that are \((n - 2)\)-dimensional. For simplicity, we will check consistency for a small loop about every \((n - 2)\)-dimensional face \(G\) of the Coxeter fan.

Let \(\Phi'\) be the rank-2 subsystem of \(\Phi\) consisting of roots that are orthogonal to \(G\). Write \(\beta\) and \(\gamma\) for the canonical roots of \(\Phi'\). Without loss of generality, we can take \(\beta \in \Phi_\text{fin}\). To see why, take any real root in \(\Phi'\). This root is a positive scalar multiple
of $\phi + k\delta$ for some $\phi \in \Phi_{\text{fin}}$ and $k \in \mathbb{Z}$. We take $\beta$ to be whichever of $\phi$ or $-\phi$ is positive. Since $\delta$ is orthogonal to $G$, so is $\beta$, and thus $\beta \in \Phi' \cap \Phi_{\text{fin}}$. Every root in $\Phi'$ is a linear combination of $\beta$ and $\delta$, and every positive root in $\Phi' \setminus \Phi_{\text{fin}}$ has a positive coefficient of $\alpha_{\text{aff}}$ in its expansion in the basis of simple roots. We conclude that $\beta$ is not a nonnegative linear combination of other positive roots in $\Phi'$. In other words, $\beta$ is a canonical root of $\Phi'$.

If $\omega_c(\beta, \delta) = 0$, then $\omega_c(\beta, \gamma) = 0$. In this case, Theorem 6.9 implies that there are at most two $c$-sortable join-irreducible elements whose unique cover reflection corresponds to a root in $\Phi'$, and each such root is a canonical root in $\Phi'$. Thus each associated shard is not cut along $\delta^\perp$, so if it contains $G$, it intersects $-\text{Tits}(A)$. Similarly, there are at most two $c^{-1}$-sortable join-irreducible elements whose unique cover reflection corresponds to a root in $\Phi'$, and each is a canonical root in $\Phi'$. Now Proposition 6.9 implies that any wall of $\text{DCScat}(A, c)$ that contains $G$ and is orthogonal to a real root must contain $G$ in its relative interior. But also, if $\mathfrak{d}_{\infty}$ contains $G$, then $G$ is in the relative interior of $\mathfrak{d}_{\infty}$ by Theorem 5.10 because $\omega_c(\beta, \delta) = 0$. We see that a small loop about $G$ crosses 1, 2, or 3 walls twice each, once in each direction. Since the restriction of $\omega_c$ to the span of $\Phi'$ is zero, all wall crossings commute and thus the path-ordered product about the loop is trivial. If $\omega_c(\beta, \delta) < 0$, then $G$ is not in the relative interior of $\mathfrak{d}_{\infty}$. To see why, recall that $\mathfrak{d}_{\infty}$ is the closure of the cone $V^* \setminus |DF_c|$ described in Theorem 5.10. The root $-\beta$ is in $\Phi_{\text{fin}}^{\ominus}$ because $\omega_c(\beta, \delta) < 0$, so $\mathfrak{d}_{\infty}$ is contained in $\{x \in \partial\text{Tits}(A) : \langle x, \beta \rangle \geq 0\}$, while $G$ is contained in $\beta^\perp$. Suppose $G$ is not contained in $\mathfrak{d}_{\infty}$. Then Lemma 5.14 says that $G$ is contained in some face $G'$ of $DF_c$, and we may as well take $G'$ minimal, so that $G$ is not in the relative boundary of $G'$. If $G'$ is of $(n - 2)$-dimensional, then since $G$ is of $(n - 2)$-dimensional and in $\partial\text{Tits}(A)$, also $G'$ is in $\partial\text{Tits}(A)$. But now Theorem 5.13 says that $G'$ is in $\mathfrak{d}_{\infty}$. This contradicts our supposition that $G$ is not in $\mathfrak{d}_{\infty}$, and we conclude that $G'$ has dimension $> n - 2$. If $G'$ is of $n$-dimensional, then since $G$ is not in the boundary of $G'$, there is a loop around $G$ that is contained in the interior of $G'$. Thus Proposition 6.10 implies that a small loop around $G$ intersects no walls of $\text{DCScat}(A, c)$, so that consistency about the loop is trivial. If $G'$ is of $(n - 1)$-dimensional, then Propositions 5.24, 6.9, and 6.10 combine to imply that a small loop about $G$ intersects only a single wall of $\text{DCScat}(A, c)$, so that consistency is again trivial.

Continuing in the case where $\omega_c(\beta, \delta) < 0$, it remains to consider the possibility that $G$ is in $\mathfrak{d}_{\infty}$ (specifically in the proper face $\mathfrak{d}_{\infty} \cap \beta^\perp$ of $\mathfrak{d}_{\infty}$). Since $\beta$ and $\gamma$ are canonical in the rank-two subarrangement $\Phi'$, the hyperplanes $\beta^\perp$ and $\gamma^\perp$ cut all other hyperplanes containing $G$ (the hyperplanes orthogonal to roots in $\Phi' \setminus \{\beta, \gamma\}$). The rank-two subarrangement $\Phi'$ is infinite because it contains an infinite collection of roots that are scalings of roots $\beta + k\delta$ for integers $k$. Thus Theorem 6.9 says that if $j$ is a $c$-sortable join-irreducible element with $\text{cov}(j) = \{t\}$ such that $\beta_t$ is a non-canonical root in $\Phi'$, then $\beta \not\in \text{inv}(j)$. Thus for such $j$, the shard $\text{Sh}(j)$ is below $\beta^\perp$. Similarly, if $j$ is a $c^{-1}$-sortable join-irreducible element with $\text{cov}(j) = \{t\}$ such that $\beta_t$ is a non-canonical root in $\Phi'$, then $\text{Sh}(j)$ is below $\gamma^\perp$ and thus $-\text{Sh}(j)$ is below $\beta^\perp$. We have already seen that $\mathfrak{d}_{\infty}$ is below $\beta^\perp$. Thus the hyperplanes orthogonal to roots in $\Phi' \setminus \{\beta, \gamma\}$ contain no walls of $\text{DCScat}(A, c)$ on the side of $G$ opposite $\mathfrak{d}_{\infty}$. Since $G$ is in the intersection of two or more walls, and since $DF_c$ is a fan, we conclude that both the wall contained in $\beta^\perp$ and the wall contained
in \( \gamma^\perp \) contain \( G \). Both of these walls extend above and below \( G \) into \( \text{Tits}(A) \) and \( -\text{Tits}(A) \).

Now Theorem 5.12 says that the set of maximal cones of \( \mathcal{DF}_c \) containing \( G \) forms a doubly infinite sequence, with each maximal cone adjacent to the cones before and after it. Propositions 5.17 and 6.10 imply that the intersection of each adjacent pair in the sequence is contained in a wall of \( \text{DCScat}(A,c) \), defining a doubly infinite sequence of walls containing \( G \). Since \( G \) is contained in the wall in \( \beta^\perp \) and in the wall in \( \gamma^\perp \), there is a maximal cone of \( \mathcal{DF}_c \) containing \( G \) and having \( \beta \) and \( \gamma \) as inward-facing normals. The normal vectors to all of the walls in the doubly infinite sequence are thus determined from \( \beta \) and \( \gamma \) by Proposition 5.15 and the restriction of \( \omega_i \) to \( G^\perp \) (the span of \( \Phi' \)). In both directions, these normal vectors limit to \( \pm \delta \), and the imaginary wall \( \partial_{\infty} \) also contains \( G \).

Now we can check consistency about \( G \) by considering the two-dimensional doubled Cambrian fan constructed in the span of \( \Phi' \), using the restriction of \( \omega_c \). That is, take \( H' = \begin{bmatrix} 0 & \omega_c(\beta^\gamma, \gamma) \\ -|\omega_c(\beta^\gamma, \gamma)| & 0 \end{bmatrix} \), take \( A' = \begin{bmatrix} 2 & -|\omega_c(\beta^\gamma, \gamma)| \\ 0 & 2 \end{bmatrix} \), and take \( c' = s_\gamma s_\beta \). We obtain the same sequence of normal vectors from Proposition 5.15 as we obtained inside \( \Phi \). We insert one additional limiting ray to obtain the cluster scattering diagram. This additional ray corresponds to \( \partial_{\infty} \). Theorem 6.4 and Lemma 6.3 imply that, with the scattering terms induced by \( \text{DCScat}(A,c) \), this rank-2 scattering diagram is the cluster scattering diagram, and in particular consistent. The consistency of the rank-2 cluster scattering diagram implies that the path-ordered product on a small loop about \( G \) is trivial.

If \( \omega_c(\beta, \delta) > 0 \), then replacing \( c \) with \( c^{-1} \) in the argument for the case \( \omega_c(\beta, \delta) < 0 \), we again conclude that the path-ordered product for a small loop around \( G \) is trivial.

This completes the proof of Theorems 2.2, 2.5, and 2.6. By Proposition 5.24 and the fact that \( \delta \) (the normal vector to \( \partial_{\infty} \)) is the unique imaginary root in \( \Phi_c \), Theorem 2.3 follows. We now prove our last result on affine cluster scattering diagrams.

**Proof of Theorem 2.4.** We argue that \( \{(\partial_\beta, f_\beta) : \beta \in \Phi_c \cap \Phi^+ \} \) is precisely equal to \( \text{DCScat}(A,c) \), which is \( \text{Scat}^T(B) \) by Theorem 2.2. Indeed, Proposition 6.7 says that each shard \( \text{Sh}(j) \) for \( j \) a \( c \)-sortable join-irreducible element is \( \partial_{\beta_i} \), where \( \text{cov}(j) = \{t\} \). Proposition 6.7 also implies that for \( j \) a \( c^{-1} \)-sortable join-irreducible element with \( \text{cov}(j) = \{t\} \), the wall \( -\text{Sh}(j) \) is

\[ \{ x \in V^* : (x, \beta_i) = 0 \text{ and } (x, \gamma) \geq 0 \text{ for all } \gamma \in \text{cut}(\beta_i) \text{ with } \omega_{c^{-1}}(\gamma, \beta_i) > 0 \} \]

But for each \( \gamma \in \text{cut}(\beta_i) \), there exists \( \gamma' \in \text{cut}(\beta_i) \) such that the signs of \( \omega_{c^{-1}}(\gamma', \beta) \) and \( \omega_{c^{-1}}(\gamma, \beta_i) \) are strictly opposite and, for \( x \in \beta_i^\perp \), the signs of \( (x, \gamma) \) and \( (x, \gamma') \) are weakly opposite. Thus \( -\text{Sh}(j) \) equals \( \partial_{\beta_i} \).

We have seen that every wall of \( \text{DCScat}(A,c) \) is of the form \( (\partial_\beta, f_\beta) \) for \( \beta \in \Phi_c \cap \Phi^+ \). Theorem 2.3 implies that every wall \( (\partial_\beta, f_\beta) \) is in \( \text{DCScat}(A,c) \). □

### 6.4. The finite-type construction

Although the focus of this paper is affine type, we pause to mention what this paper adds to the finite-type results already established in [26, Section 4].

When \( \Phi \) is of finite type, the doubled Cambrian fan is still a fan, but coincides with the Cambrian fan because \( \text{Tits}(A) \) is all of \( V^* \) and \( F_c = -F_{c^{-1}} \) in this case. Thus the proof we give for Proposition 6.10 can be simplified to give a proof of
the following result. (This result is a consequence of the earlier results, but seems not to have been explicitly stated before. It is, however, precisely equivalent to [24, Proposition 8.15], with some translation required to see the equivalence.)

**Proposition 6.14.** Let $\Phi$ be a root system of finite type and let $c$ be a Coxeter element of the corresponding Weyl group. If $j$ is a $c$-sortable join-irreducible element with $\text{cov}(j) = \{t\}$, then $\text{Sh}(j)$ is the union of all faces of $F_c$ contained in $\beta_t^\perp$.

Our proof of Theorem 2.2 simplifies to give an alternate proof of the following result, which is [26, Corollary 4.10].

**Theorem 6.15.** If $B$ is an acyclic exchange matrix of finite type, then $\text{Scat}^T(B)$ is \{$(\text{Sh}(j), f_j) : j \in \text{JIrr}_c(W)$\}.

Indeed, the proof given here is an extension of the proof given in [26]. But this paper augments the result because Proposition 6.7 provides explicit inequalities defining the walls $\text{Sh}(j)$ of $\text{Scat}^T(B)$.

Finally, our proof of Proposition 1.2 is valid for completely general Coxeter groups, without the assumption of affine type. In particular, it fills in the details of the finite-type version [26, Proposition 4.12], whose proof was only lightly sketched in [26].

7. Three fans

In this section, we prove Theorems 2.9 and 2.10 about coincidences between scattering fans, mutation fans, and generalized associahedron fans in affine type.

7.1. More on the generalized associahedron fan. In preparation for the proof of Theorem 2.9, we will prove the following two facts about the generalized associahedron fan.

**Proposition 7.1.** Let $\Phi$ be a root system of affine type, let $c$ be a Coxeter element, let $C$ be an imaginary $c$-cluster, and let $C' = C \setminus \{\delta\}$. Then there exists a bi-infinite sequence $(C_i : i \in \mathbb{Z})$ of distinct real $c$-clusters $C_i = C' \cup \{\beta_i, \beta'_i\}$ with

\[
\lim_{i \to \infty} R_{>0} \beta_i = \lim_{i \to \infty} R_{>0} \beta'_i = \lim_{i \to -\infty} R_{>0} \beta_i = \lim_{i \to -\infty} R_{>0} \beta'_i = R_{>0} \delta.
\]

Furthermore, for large enough $i$, the roots $\beta_i$ and $\beta'_i$ are both on one side of $U_c$, while $\beta_{-i}$ and $\beta'_{-i}$ are both on the opposite side of $U_c$.

Recall that $\Lambda_c^\tau$ is the set of real roots in $\Phi_c$ that are in finite $\tau_c$-orbits.

**Proposition 7.2.** Suppose $\Phi$ is of affine type and $c$ is a Coxeter element. If $C$ is a collection of $n-2$ pairwise $c$-compatible roots in $\Phi_c$ with $\delta \in C$, then there exists a root $\beta$ in $\Lambda_c^\tau$ with $\nu_c(C) \subset \beta^\perp$.

For the proofs of these propositions, we quote and prove some additional background. The following four propositions are parts of [34, Proposition 5.14]. In this subsection, we assume $\Phi$ to be of affine type.

**Proposition 7.3.** If $C$ is a real $c$-cluster, then $C$ consists of $n$ linearly independent roots.

**Proposition 7.4.** If $C$ is a real $c$-cluster, then $C$ contains at least 2 roots in the $\tau_c$-orbits of negative simple roots.
Proposition 7.5. If $C$ is an imaginary $c$-cluster, then $C$ consists of $n-1$ linearly independent roots.

Proposition 7.6. If $C$ is an imaginary $c$-cluster, then $C \setminus \{\delta\}$ consists of roots in $\Lambda^e_{\mathfrak{c}}$.

The following theorem is [33] Theorem 5.5.

Theorem 7.7. A set $C \subseteq \Phi_{\mathfrak{c}}$ is a real $c$-cluster if and only if it is a maximal set of pairwise $c$-compatible roots in $\Phi^e_{\mathfrak{c}}$.

Theorem 7.7 is not obvious, because conceivably there might exist a maximal set of pairwise $c$-compatible roots in $\Phi^e_{\mathfrak{c}}$, each of which is $c$-compatible with $\delta$. Such a set would not be a $c$-cluster.

For $j = 1, \ldots, n$, define $\psi^c_{e,j} = s_1 \cdots s_{j-1} \alpha_j$ and $\psi^c_{e+1,j} = s_n \cdots s_{j+1} \alpha_j$. Further, define $\overline{\psi}^e_c = \{\psi^c_{e,j} : j = 1, \ldots, n\}$ and $\overline{\psi}^c = \{\psi^c_{e+1,j} : j = 1, \ldots, n\}$. Recall that $\gamma_c$ is the unique vector in $V_{\text{fin}}$ with $c\gamma_c = \delta + \gamma_c$ and that $U^c$ is the hyperplane $\{v \in V : K(\gamma_c, v) = 0\}$ in $V$. The set $\overline{\psi}^e_c \cup \overline{\psi}^c$ has 2n distinct elements, the $c$-orbits of these elements are disjoint, and the union of these orbits is $\Phi^+ \setminus U^c$. The following is [33] Proposition 4.3.

Proposition 7.8. The $c$-orbits of roots in $\overline{\psi}^e_c$ are separated from the $c$-orbits of roots in $\overline{\psi}^c$ by the hyperplane $U^c$. Specifically, $K(\gamma_c, \beta) > 0$ for $\beta \in c^m \overline{\psi}^e_c$ and $m \in \mathbb{Z}$, while $K(\gamma_c, \beta) < 0$ for $\beta \in c^m \overline{\psi}^c$ and $m \in \mathbb{Z}$.

We now prove the first result of this section.

Proof of Proposition 7.7. Proposition 7.6 says that $C' \subseteq \Lambda^e_{\mathfrak{c}}$. Since $\Lambda_{\mathfrak{c}}$ is finite and fixed (as a set) by the action of $c$, there is some integer $m > 0$ such that $c^m$ fixes $C'$ pointwise. Since $c$ fixes $\delta$ also, by Proposition 7.5 we see that $c^m$ fixes the span of $C$, which is $U^c$.

Proposition 7.3 and Theorem 7.7 imply that there exist real roots $\beta$ and $\beta'$ such that $C' \cup \{\beta, \beta'\}$ is a $c$-cluster. Neither $\beta$ nor $\beta'$ is in $\Lambda^e_{\mathfrak{c}}$. (If, say, $\beta$ is in $\Lambda^e_{\mathfrak{c}}$, then by [33] Proposition 4.3, $C' \cup \{\beta\}$ is a set of pairwise $c$-compatible roots, contradicting the fact that $C'$ is a $c$-cluster.)

Let $\beta_i = \tau^m_c(\beta)$, $\beta'_i = \tau^m_c(\beta')$, and $C_i = \tau^m_c(C' \cup \{\beta, \beta'\}) = C' \cup \{\beta_i, \beta'_i\}$ for each $i \in \mathbb{Z}$. By (3.8), each $C_i$ is a $c$-cluster. By Proposition 7.4, we can reindex the sequence so that $\beta_0 = \psi^c_{e,j}$, $\beta_0 = -\alpha_j$ and $\beta_{-1} = \psi^c_{e+1,j}$ for some $j \in \{1, \ldots, n\}$.

The subspaces $U^c$ and $V_{\text{fin}}$ are distinct hyperplanes, because $\delta$ is in $U^c$ but not in $V_{\text{fin}}$. Thus $U^c \cap V_{\text{fin}}$ has codimension 2. Since $\gamma_c \notin U^c$, we can write $\beta_1 = a\gamma_c + b\delta + x$ with $x \in U^c \cap V_{\text{fin}}$. Since $\beta_1 \notin U^c$, also $a \neq 0$. Now $c$ fixes $\delta$ and sends $\gamma_c$ to $\gamma_c - \delta$. Also $c^m$ fixes $x$. Thus for $i \geq 0$, $\tau^m_c$ sends $\beta_i$ to $\beta_{i+1} = c^m\beta_i = a\gamma_c + (b + ami)\delta + x$. Proposition 7.8 implies that $a > 0$, so $\lim_{i \to \infty} \mathbb{R}_{\geq 0}\beta_i = \mathbb{R}_{\geq 0}\delta$. Similarly, we can write $\beta_{-1} = a'\gamma_c + b'\delta + x'$ with $a' < 0$ and compute $\beta_i = a'\gamma_c + (b' + ami)\delta + x'$ for $i \leq -1$, and conclude that $\lim_{i \to -\infty} \mathbb{R}_{\geq 0}\beta_i = \mathbb{R}_{\geq 0}\delta$. We obtain the remaining statements about limits by the symmetry between $\beta$ and $\beta'$.

Since $\beta$ and $\beta'$ are not in finite $\tau_c$-orbits, there exist simple roots $\alpha$ and $\alpha'$ and integers $k$ and $k'$ such that $\beta = \tau^k_c(-\alpha)$ and $\beta' = \tau^{k'}_c(-\alpha')$. Thus for large enough $i$, both $\beta_i$ and $\beta'_i$ are both on one side of $U^c$, by Proposition 7.8 and similarly $\beta_{-i}$ and $\beta'_{-i}$ are on the other side. \qed
Recall from Section 3.5 that $\Upsilon^c = \Phi \cap U^c$ and that $\Xi^c$ is, in essence, the set of simple roots of $\Upsilon^c$. Each root $\beta \in \Upsilon^c \setminus \{\delta\}$ has a unique expression as a linear combination of vectors in $\Xi^c$ and $\text{Supp}_\Xi(\beta)$ is the set of vectors appearing in this expression with nonzero coefficient. Recall also that the Dynkin diagram for $\Upsilon^c$ consists of cycles and that $\Xi$ acts as a rotation on each cycle, moving each root in $\Xi^c$ to an adjacent root in the cycle.

Two roots $\alpha, \beta \in \Lambda^c_\text{re}$ are \textit{nested} if $\text{Supp}_{\Xi}(\alpha) \subset \text{Supp}_{\Xi}(\beta)$ or $\text{Supp}_{\Xi}(\beta) \subset \text{Supp}_{\Xi}(\alpha)$. The roots are \textit{spaced} if $\text{Supp}_{\Xi}(c^{-1}\alpha) \cup \text{Supp}_{\Xi}(\alpha) \cup \text{Supp}_{\Xi}(c\alpha)$ is disjoint from $\text{Supp}_{\Xi}(\beta)$. In particular, roots from distinct components of $\Upsilon^c$ are spaced.

The following is Proposition 7.10.

\textbf{Proposition 7.10.} Suppose $\alpha, \beta \in \Lambda^c_\text{re}$. Then $E_\Upsilon(\alpha^\vee, \beta^\vee)$ is the number of roots in $\text{Supp}_{\Xi}(\alpha) \cap \text{Supp}_{\Xi}(\beta)$ minus the number of roots $\beta_i \in \text{Supp}_{\Xi}(\alpha)$ with $c\beta_i \in \text{Supp}_{\Xi}(\beta)$.

The following is Proposition 7.11.

\textbf{Proposition 7.11.} If $\beta$ is a root contained in $U^c$, then $E_\Upsilon(\beta, \delta) = 0$.

The following proposition accomplishes most of the work for the proof of Proposition 7.2.

\textbf{Proposition 7.12.} If $C$ is a collection of $n-2$ pairwise $c$-compatible roots in $\Lambda^c$ with $\delta \in C$, then there exists a root $\beta$ in $\Lambda^c_\text{re}$ with $E_\Upsilon(\beta, \beta') = 0$ for all $\beta' \in C$.

\textbf{Proof.} If $\beta \in \Lambda^c_\text{re}$, then $E_\Upsilon(\beta, \delta) = 0$ by Proposition 7.11. Thus we need to show that for any set $C'$ of $n-3$ pairwise $c$-compatible roots in $\Lambda^c_\text{re}$, there exists a root $\beta$ in $\Lambda^c_\text{re}$ with $E_\Upsilon(\beta, \beta') = 0$ for all $\beta' \in C'$. In light of Proposition 7.10, we need to find a root $\beta$ so that $|\text{Supp}_{\Xi}(\beta) \cap \text{Supp}_{\Xi}(\beta')| = |\text{Supp}_{\Xi}(\beta) \cap \text{Supp}_{\Xi}(c\beta')|$ for all $\beta' \in C'$. We will use Proposition 7.9 implicitly throughout the proof.

By Propositions 7.5 and 7.6, we can find a root $\alpha \in \Lambda^c_\text{re}$ such that $C' \cup \{\alpha, \delta\}$ is a $c$-cluster. Let $\Lambda'$ be the component of $\Upsilon^c$ containing $\alpha$. Then the restriction of $C' \cup \{\alpha\}$ to $\Lambda'$ is a maximal collection of pairwise $c$-compatible roots in $\Lambda^c_\text{re} \cap \Lambda'$.

The Coxeter diagram of $\Lambda'$ consists of a cycle with no edge labels, so that $\Lambda'$ is a root system of affine type $A^{(1)}$. The support $\text{Supp}_{\Xi}$ of a root in $\Lambda^c_\text{re}$ is an induced path in the cycle. We represent roots in $\Lambda^c_\text{re} \cap \Lambda'$ pictorially by circling their supports. When two supports are nested, we will draw the larger support around the smaller support. We will draw the cycle so that the action of $c$ is a counterclockwise rotation moving each vertex to an adjacent vertex.

The \textit{counterclockwise endpoint} of the root $\alpha$ is the element of $\text{Supp}_{\Xi}(\alpha)$ that is taken outside of $\text{Supp}_{\Xi}(\alpha)$ by counterclockwise rotation. The \textit{clockwise endpoint} of $\alpha$ is defined analogously. First, consider the case where no root $\beta' \in C'$ with $\text{Supp}_{\Xi}(\beta') \supset \text{Supp}_{\Xi}(\alpha)$ has the same clockwise endpoint as $\alpha$. A representative drawing of this case is shown in Fig. 1. In the figure, $\text{Supp}_{\Xi}(\alpha)$ is shown with a dotted line to distinguish it from the supports of the roots in $C'$, which are shown with solid lines. Some roots in $C'$ are not pictured. We now explain how to construct $\beta$. There is a unique vertex $v$ of the cycle that is in the support of $\alpha$ but is not in the support of any root $\beta'$ with $\text{Supp}_{\Xi}(\beta') \subset \text{Supp}_{\Xi}(\alpha)$. Let $\beta$ be the root
whose clockwise endpoint is the same as the clockwise endpoint of \( \alpha \), and whose counterclockwise endpoint is \( v \). In the figure, \( \text{Supp}_\Xi(\beta) \) is shown with a dashed line. We see that \( |\text{Supp}_\Xi(\beta) \cap \text{Supp}_\Xi(\beta')| = |\text{Supp}_\Xi(\beta) \cap \text{Supp}_\Xi(c\beta')| \) for all \( \beta' \in C' \).

Next, consider the case where some root \( \beta' \in C' \) with \( \text{Supp}_\Xi(\beta') \supset \text{Supp}_\Xi(\alpha) \) has the same clockwise endpoint as \( \alpha \). Choose \( \beta'' \) to have minimal support among such roots. Again, there is a unique vertex \( v \) of the cycle that is in the support of \( \alpha \) but is not in the support of any root \( \beta' \) with \( \text{Supp}_\Xi(\beta') \subseteq \text{Supp}_\Xi(\alpha) \). Let \( \alpha' \) be the root whose clockwise vertex is the vertex adjacent to and counterclockwise of \( v \), and whose counterclockwise vertex is the counterclockwise vertex of \( \beta'' \). Figure 2 shows two representative cases of the construction of \( \alpha' \). Again, we have circled the support of \( \alpha \) with a dotted line. The support of \( \beta'' \) is the largest circled support, and is shown with a solid line. The support of \( \alpha' \) is circled with a dashed line. Again, some of the roots \( \beta' \in C' \) are not shown. We replace \( \alpha \) by \( \alpha' \), and note that \( \alpha' \) is \( c \)-compatible with every \( \beta' \in C \). With this replacement, we fall into the previous case, so we construct \( \beta' \) as described above.

We now show how Proposition 7.2 follows from Proposition 7.12.
Proof of Proposition 7.12. Proposition 7.12 says that there is a positive root $\beta'$ in $\Lambda_c^+$ with $E_c(\beta, \beta') = 0$ for all $\beta' \in C$. Proposition 7.6 implies that every root in $C$ is in $\Lambda_c$. Since $\Lambda_c$ consists entirely of positive roots, each $\beta' \in C$ is positive, so the condition $E_c(\beta, \beta') = 0$ is equivalent to $\langle \nu_c(\beta'), \beta \rangle = 0$. \hfill $\square$

7.2. Affine scattering fans and generalized associahedron fans. We now prove the part of Theorem 2.9 that relates ScatFan$(T)$ with $\nu_c(\text{Fan}_c(\Phi))$.

Theorem 7.13. If $B$ is an acyclic exchange matrix of affine type, corresponding to a root system $\Phi$ and a Coxeter element $c$, then ScatFan$(T)$ and $\nu_c(\text{Fan}_c(\Phi))$ coincide.

We now prepare to prove Theorem 7.13.

Proposition 7.14. Every $(n - 1)$-dimensional face of $\nu_c(\text{Fan}_c(\Phi))$ is contained in a wall of DCScat$(A, c)$.

Proof. Suppose $F$ is an $(n - 1)$-dimensional face of $\nu_c(\text{Fan}_c(\Phi))$. If $F$ is contained in $\Phi_\infty$, then we are done, and if not, then $F$ is a face of $DF_c$ by Theorems 3.3, 5.8 and 5.10. Writing $t$ for the reflection such that $F \subseteq \beta_t^+$, condition (ii) of Proposition 5.24 holds. Therefore by condition (iv) of Proposition 5.24, either there is a $c$-sortable join-irreducible element $j$ with $\text{cov}(j) = \{t\}$ or there is a $c^{-1}$-sortable join-irreducible element $j'$ with $\text{cov}(j') = \{t\}$, or both. Now Proposition 6.10 says that $F$ is contained in $\text{Sh}(j)$ or $F$ is contained in $-\text{Sh}(j')$, or both. \hfill $\square$

The following lemma is [30, Lemma 3.3].

Lemma 7.15. If $s$ is initial or final in $c$, then $E_c(\beta, \beta') = E_{scs}(s\beta, s\beta')$ for all $\beta$ and $\beta'$ in $V$.

Lemma 7.16. For $s$ initial in $c$, suppose $\beta \in \Phi^+$. If $\beta \neq \alpha_s$, then $\nu_{scs}(s\beta) = s\nu_c(\beta)$.

Proof. Since $\beta \neq \alpha_s$, the root $s\beta$ is positive, so by Lemma 7.15, for any $\gamma \in V$,

$$\langle \nu_{scs}(s\beta), \gamma \rangle = -E_{scs}(\gamma, s\beta) = -E_c(s\gamma, \beta) = \langle \nu_c(\beta), s\gamma \rangle = \langle s\nu_c(\beta), \gamma \rangle.$$

\hfill $\square$

Lemma 7.17. Suppose $j$ is a $c$-sortable join-irreducible element with $\text{cov}(j) = \{t\}$. If $\gamma$ is in the nonnegative linear span of $\Pi$ and $\nu_c(\gamma) \in \beta_t^+$, then $\nu_c(\gamma) \in \text{Sh}(j)$.

Proof. We argue by induction on $\ell(j)$ and on the rank of $W$. Let $s$ be initial in $c$.

If $s = j$, then $t = s$, and Lemma 4.8 says that $\text{Sh}(j) = \beta_t^+$. If $s < j$, then $sj$ is join-irreducible with $\text{cov}(j) = \{sts\}$. Thus by Lemma 7.16, $\nu_{scs}(s\gamma) \in (s\beta_t)^+ = \beta_t^+$. By induction on $\ell(j)$, $\nu_{scs}(s\gamma) \in \text{Sh}(sj)$, or in other words, again by Lemma 7.16, $s\nu_c(\gamma) \in \text{Sh}(sj)$, so $\nu_c(\gamma) \in s \cdot \text{Sh}(sj)$. Since $\gamma$ is a nonnegative combination of $\Pi$, $\langle \nu_c(\gamma), \alpha_s \rangle = -E_c(\alpha_s, \gamma)$, which is nonpositive (by the definition of $E_c$) because $s$ is initial in $c$. Now Proposition 4.12 implies that $\nu_c(\gamma) \in \text{Sh}(j)$.

If $s \notin j$, then $j$ is an $sc$-sortable element of $W_{\langle s \rangle}$. Thus $\beta_t \in \Phi_{\langle s \rangle}$, and $\gamma$ is a multiple of $\alpha_s$ plus a nonnegative combination $\gamma'$ of $\Pi \setminus \{\alpha_s\}$. By induction on rank, $\nu_{sc}(\gamma') \in \text{Sh}(s\langle j \rangle)$. By Proposition 4.11, we can complete the proof by showing that $\nu_{sc}(\gamma') = \text{Proj}_{s\langle j \rangle}(\nu_c(\gamma))$. For any $z \in V_{\langle s \rangle} \subset V$, by definition (Proj)$(\nu_{s\langle j \rangle})(\nu_c(\gamma)), z) = \langle \nu_c(\gamma), z \rangle$. Since $\gamma$ is a nonnegative combination of $\Pi$, this is $-E_c(z, \gamma)$. Because $z \in V_{\langle s \rangle}$ and $s$ is initial in $c$, we see that $-E_c(z, \gamma) = -E_{sc}(z, \gamma')$. Since $\gamma'$ is a nonnegative combination of simple roots, $-E_{sc}(z, \gamma') = \langle \nu_{sc}(\gamma'), z \rangle$, and we conclude that $\nu_{sc}(\gamma') = \text{Proj}_{s\langle j \rangle}(\nu_c(\gamma))$, as desired. \hfill $\square$
Lemma 7.18. If $j$ is a $c$-sortable join-irreducible element with $\text{cov}(j) = \{t\}$, then $\text{Sh}(j) \cap \mathfrak{d}_\infty = \beta^j_\bot \cap \mathfrak{d}_\infty$. If $j$ is a $c^{-1}$-sortable join-irreducible element with $\text{cov}(j) = \{t\}$, then $-\text{Sh}(j) \cap \mathfrak{d}_\infty = \beta^j_\bot \cap \mathfrak{d}_\infty$.

Proof. The first assertion follows from Lemma 7.17 and the fact that the extreme rays of $\mathfrak{d}_\infty$ are spanned by the vectors $\nu_\gamma(\delta)$ for $\gamma \in \Xi^e$. The second assertion follows from the first because passing from $c$ to $c^{-1}$ changes $\mathfrak{d}_\infty$ by the action of the antipodal map. \qed

Proposition 7.19. Every $(n-2)$-dimensional face of $\nu_c(\Phi)$ that is contained in $\mathfrak{d}_\infty$ is also contained in another wall of $\text{DCScat}(A,c)$.

Proof. Suppose $F$ is an $(n-2)$-dimensional face of $\nu_c(\Phi)$ that is contained in $\mathfrak{d}_\infty$. If $F$ does not contain $\nu_\gamma(\delta)$, then $F$ is a face of $\nu_c(\Phi)$. Theorem 7.7 implies that $F$ is not a maximal face of $\nu_c(\Phi)$, so $F$ is contained in some $(n-1)$-dimensional face $F'$ of $\nu_c(\Phi)$. Proposition 7.14 says that $F'$ is contained in some wall of $\text{DCScat}(A,c)$. But Proposition 5.21 says that $F'$ is not contained in $\mathfrak{d}^1$, so that wall is not $\mathfrak{d}_\infty$.

If $F$ contains $\nu_\gamma(\delta)$, then Proposition 7.2 says that $F$ is contained in $\beta^j_\bot$ for some $\beta \in \Lambda^c_\infty$. Since $\Lambda^c_\infty \subset \Phi^c_\infty$, Proposition 5.24 implies that, for $\beta = \beta_\gamma$, either there is a $c$-sortable join-irreducible element $j$ with $\text{cov}(j) = \{t\}$ or there is a $c^{-1}$-sortable join-irreducible element $j'$ with $\text{cov}(j') = \{t\}$, or both. In either case, Lemma 7.18 implies that the corresponding wall of $\text{DCScat}(A,c)$ contains $F$. \qed

Proof of Theorem 7.13. Throughout the proof, we use Theorem 2.2, which implies that $\text{ScatFan}^\infty(B)$ is the set of $\text{DCScat}(A,c)$-cones and their faces. By construction and by Proposition 6.8, the ramparts of $\text{DCScat}(A,c)$ are precisely its walls. Thus points $p$ and $q$ are in the same $\text{DCScat}(A,c)$-class if and only if there is a path $\gamma$ from $p$ to $q$ such that the function taking a point to the set of all walls containing it is constant on $\gamma$. Recall that the $\text{DCScat}(A,c)$-cones are the closures of $\text{DCScat}(A,c)$-classes.

We first show that every cone of $\nu_c(\Phi)$ is contained in a $\text{DCScat}(A,c)$-cone. We will make several successive reductions/rephrasings: It is enough to show that every maximal cone of $\nu_c(\Phi)$ is contained in a $\text{DCScat}(A,c)$-cone. Since $F$ is the closure of $\text{relint}(F)$ and since the closure of a $\text{DCScat}(A,c)$-cone is a $\text{DCScat}(A,c)$-cone, it is enough to show that $\text{relint}(F)$ is in a $\text{DCScat}(A,c)$-class. Finally, it is enough to show that any wall that intersects $\text{relint}(F)$ actually contains $\text{relint}(F)$, because if so, any two points $p,q \in \text{relint}(F)$ are connected by a path inside $\text{relint}(F)$ that does not enter or leave any wall.

A maximal cone $F$ of $\nu_c(\Phi)$ is $\nu_c(\text{Cone}(C))$ for some $c$-cluster $C$. We break into two cases, based on whether $C$ is real or imaginary $c$-cluster.

First suppose $C$ is real. Theorems 3.3 and 5.8 combine to say that $F$ is a maximal cone of $DF_c$. In this case, we need to prove that no wall of $\text{DCScat}(A,c)$ intersects the interior of $F$. Suppose to the contrary that some wall of $\text{DCScat}(A,c)$ intersects the interior of $F$. Since $\mathfrak{d}_\infty$ is in the boundary of $|DF_c|$, it does not intersect the interior of $F$, so without loss of generality (up to replacing $c$ by $c^{-1}$ and applying the antipodal map), there is a $c$-sortable join-irreducible element $j$ such that $\text{Sh}(j)$ intersects the interior of $F$. However, $\text{Sh}(j)$ is normal to some real root and thus not contained in $\mathfrak{d}^1$, so the intersection of $\text{Sh}(j)$ with the interior of $F$ contains a point $p$ in $\text{Tits}(A) \cup (-\text{Tits}(A))$. However, Proposition 6.10.1 then says that $p$ is
contained in an \((n - 1)\)-dimensional face of \(\nu_c(\text{Fan}_c(\Phi))\). This contradiction shows that no wall of DCScat\((A, c)\) intersects the interior of \(F\).

Now suppose \(C\) is imaginary. Then \(F\) is contained in \(\mathcal{d}_\infty\), which is the closure of \(V^* \setminus |D F_c|\). We will show that the relative interior of \(F\) is disjoint from all walls of DCScat\((A, c)\) other than \(\mathcal{d}_\infty\).

Suppose for the sake of contradiction that some point \(p\) in the relative interior of \(F\) is contained in a wall \(\mathcal{d}\) of DCScat\((A, c)\) with \(\mathcal{d} \neq \mathcal{d}_\infty\). Since \(\mathcal{d}\) is a codimension-1 cone not contained in the hyperplane containing \(\mathcal{d}_\infty\), there exist a point \(q \notin \mathcal{d}_\infty\) such that the line segment \(\overline{pq}\) is contained in \(\mathcal{d}\).

Consider the bi-infinite sequence \(\ldots, C_{-1}, C_0, C_1, \ldots\) of distinct real \(c\)-clusters constructed in Proposition \(7.1\).

The sequences

\[
\nu_c(\text{Cone}(C_1)), \nu_c(\text{Cone}(C_2)), \ldots \text{ and } \nu_c(\text{Cone}(C_{-1})), \nu_c(\text{Cone}(C_{-2})), \ldots
\]

converge to \(F\) from opposite sides of the hyperplane containing \(F\). Thus there exists some \(i \in \mathbb{Z}\) such that \(\overline{pq}\) intersects the interior of \(\nu_c(\text{Cone}(C_i))\). Therefore \(\mathcal{d}\) intersects the interior of \(\nu_c(\text{Cone}(C_i))\). But \(\nu_c(\text{Cone}(C_i))\) is a full-dimensional cone of \(\nu_c(\text{Fan}_c(\Phi))\), so we have already shown that no wall of DCScat\((A, c)\) intersects the interior of \(\nu_c(\text{Cone}(C_i))\). This contradiction proves that the relative interior of \(F\) is disjoint from all walls of DCScat\((A, c)\) other than \(\mathcal{d}_\infty\).

We complete the proof by showing that every DCScat\((A, c)\)-cone is contained in some cone of \(\nu_c(\text{Fan}_c(\Phi))\). Since each DCScat\((A, c)\)-cone is the closure of a DCScat\((A, c)\)-class, it is enough to show that each DCScat\((A, c)\)-class is contained in some cone of \(\nu_c(\text{Fan}_c(\Phi))\).

Let \(\mathcal{C}\) be a DCScat\((A, c)\)-class. By definition, any two points of \(\mathcal{C}\) are connected by a path which never enters or leaves a wall of DCScat\((A, c)\). Since \(\mathcal{d}_\infty\) (the closure of \(V^* \setminus |D F_c|\)) is a wall of DCScat\((A, c)\), the DCScat\((A, c)\)-class \(\mathcal{C}\) is either contained in \(|D F_c|\) or contained in \(\mathcal{d}_\infty\).

First, suppose \(\mathcal{C}\) is contained in \(|D F_c|\). Proposition \(7.14\) says that every \((n - 1)\)-dimensional face of \(D F_c\) is contained in some wall of DCScat\((A, c)\). A point \(p \in \mathcal{C}\) is in the relative interior of some face \(F\) of \(D F_c\). If some \(q \in \mathcal{C}\) is not also in \(\text{relint}(F)\), then every path from \(p\) to \(q\) enters or leaves a wall of DCScat\((A, c)\); since \(D F_c\) is a simplicial fan, leaving \(\text{relint}(F)\) to a larger face leaves some \((n - 1)\)-dimensional face, and thus leaves the wall containing that face. Leaving \(\text{relint}(F)\) to a smaller face enters some \((n - 1)\)-dimensional face, and thus enters the wall containing that face. We conclude that all of \(\mathcal{C}\) is in the relative interior of \(F\) of \(D F_c\), which is, by Theorems \(3.3\) and \(5.8\), a face of \(\nu_c(\text{Fan}_c(\Phi))\).

Now suppose \(\mathcal{C}\) is contained in \(\mathcal{d}_\infty\). The faces of \(\nu_c(\text{Fan}_c(\Phi))\) that are contained in \(\mathcal{d}_\infty\) constitute a fan whose maximal faces have dimension \(n - 1\). This fan is simplicial as a fan in \(\mathcal{d}_\perp\). Proposition \(7.19\) says that every \((n - 2)\)-dimensional face of the fan in \(\mathcal{d}_\infty\) is contained in some wall of DCScat\((A, c)\) in addition to \(\mathcal{d}_\infty\). Arguing exactly as in the previous case, we see that all of \(\mathcal{C}\) is contained in the relative interior of a face of \(\nu_c(\text{Fan}_c(\Phi))\) contained in \(\mathcal{d}_\infty\).

As a consequence of Theorem \(7.13\), we have a strengthening of Proposition \(6.10\) (Part of the following, specifically Corollary \(7.20.3\) was already known by Theorems \(3.3\) and \(5.10\)).

**Corollary 7.20.** Let \(\Phi\) be a root system of affine type and let \(c\) be a Coxeter element of the corresponding Weyl group.
1. If \( j \) is a c-sortable join-irreducible element with \( \text{cov}(j) = \{ t \} \), then \( \text{Sh}(j) \) is the union of all faces of \( \nu_c(\text{Fan}_c(\Phi)) \) contained in \( \beta_*^\perp \).
2. If \( j \) is a \( c^{-1} \)-sortable join-irreducible element with \( \text{cov}(j) = \{ t \} \), then \(-\text{Sh}(j)\) is the union of all faces of \( \nu_c(\text{Fan}_c(\Phi)) \) contained in \( \beta_*^\perp \).
3. \( \delta_\infty \) is the union of all faces of \( \nu_c(\text{Fan}_c(\Phi)) \) contained in \( \delta^\perp \).

Proof. By definition, each rampart \( R \) of any scattering diagram \( D \) is a union of \( D \)-cones, and if \( R \) is closed, it is immediate that \( R \) is the union of all \( D \)-cones it contains. By the definition of a rampart, \( R \) is also the union of all \( D \)-cones contained in the hyperplane containing \( R \).

Since the ramparts of \( DCScat(A, c) \) are precisely its walls, each rampart is closed. All three assertions now follow. \( \square \)

7.3. Affine mutation fans. It now remains only to show that \( \text{ScatFan}^\beta(B) \) and \( \nu_c(\text{Fan}_c(\Phi)) \) coincide with the mutation fan \( \mathcal{F}_{B^\tau} \). We begin by reviewing the definition of the mutation fan.

Let \( B \) be an exchange matrix and let \( \tilde{B} \) be an extended exchange matrix: a matrix with \( n \) columns, whose first \( n \) rows are \( B \), with an additional \( \ell \) rows (with \( \ell \geq 0 \)) consisting of real entries. For each \( k = 1, \ldots, n \), we define the mutation \( \mu_k(\tilde{B}) \) of \( \tilde{B} \) at \( k \) to be the matrix \( \tilde{B}' = [b'_{ij}] \) given by

\[
(b'_{ij}) = \begin{cases} 
-b_{ij} & \text{if } i = k \text{ or } j = k; \\
 b_{ij} + \text{sgn}(b_{kj}) \max(b_{ik}b_{kj}, 0) & \text{otherwise.}
\end{cases}
\]

Here, \( \text{sgn}(x) \) is the sign \((-1, 0, 1)\) of \( x \).

If \( k = k_1, \ldots, k_l \) is a sequence of indices in \( \{1, \ldots, n\} \), then \( \mu_k \) means the composition \( \mu_{k_l} \circ \mu_{k_{l-1}} \circ \cdots \circ \mu_{k_1} \). Given an exchange matrix \( B \) and a sequence \( k \), the mutation map is a map \( \eta^B_k : V^* \rightarrow V^* \) defined as follows: Starting with \( x = \sum_{i=1}^{n} a_i \rho_i \in V^* \), let \( \tilde{B} \) the extended exchange matrix obtained by adjoining a single row \((a_i : i = 1, \ldots, n)\) below \( B \). Writing \((a'_i : i = 1, \ldots, n)\) for the last row of \( \mu_k(\tilde{B}) \), we define \( \eta^B_k(x) \) to be \( \sum_{i=1}^{n} a'_i \rho_i \in V^* \). Each mutation map is a piecewise linear homeomorphism from \( V^* \) to itself.

Given \( x = \sum_{i=1}^{n} a_i \rho_i \in V^* \), write \( \text{sgn}(x) \) for \((\text{sgn}(a_1), \ldots, \text{sgn}(a_n)) \) in \((-1, 0, 1)^n\). We define a notion of \( B \)-equivalence for vectors in \( x_1, x_2 \in V^* \). Say \( x_1 \equiv^B x_2 \) if and only if \( \text{sgn}(\eta^B_k(x_1)) = \text{sgn}(\eta^B_k(x_2)) \) for every sequence \( k \) of indices. We call the \( \equiv^B \)-equivalence classes \( B \)-classes and we call the closures of \( B \)-classes \( B \)-cones.

The mutation fan \( \mathcal{F}_B \) is the set of \( B \)-cones and their faces. This is a complete fan by [25, Theorem 5.13].

In light of Theorems 2.11 and 7.13, we can complete the proof of Theorem 2.9 by proving the following proposition.

**Proposition 7.21.** If \( B \) is an acyclic exchange matrix of affine type, corresponding to a root system \( \Phi \) and a Coxeter element \( c \), then \( \mathcal{F}_{B^\tau} \) refines \( \nu_c(\text{Fan}_c(\Phi)) \).

To prove Proposition 7.21, we quote and prove some additional background results, mostly about the mutation fan. We begin with the following less-detailed version of [25, Theorem 8.7].

**Theorem 7.22.** For any exchange matrix \( B \), the \( g \)-vector fan associated to \( B \) is a subfan of the mutation fan \( \mathcal{F}_{B^\tau} \).

The following theorem is [28, Theorem 3.8], which is obtained by combining results of other papers as explained in [28].
Theorem 7.23. If $B$ is an acyclic exchange matrix of finite type and $c$ is the associated Coxeter element, then the mutation fan $\mathcal{F}_{B^T}$ coincides with the Cambrian fan $\mathcal{F}_c$.

Define $B_\ell$ to be the matrix obtained from $B$ by deleting row $\ell$ and column $\ell$. We retain the original indexing of $B_\ell$, so its rows and columns are indexed by $\{1, \ldots, n\} \setminus \{\ell\}$. We also realize the mutation maps for $B_\ell$, the $B_\ell$-classes and $B_\ell$-cones, and the mutation fan $\mathcal{F}_{B_\ell}$, in the subspace $V^*_{B_\ell}$ (the dual space to $V_\ell = \text{Span}\{\alpha_i : i \neq \ell\}$. As before, $\text{Proj}_{B_\ell} : V^* \to V^*_{B_\ell}$ is the projection dual to the inclusion of $V_\ell$ into $V$. Define $\text{Proj}_{B_\ell}^{-1}(\mathcal{F}_{B_\ell})$ to be the fan with cones $\text{Proj}_{B_\ell}^{-1}(C) = \{x \in V^* : \text{Proj}_{B_\ell}(x) \in C\}$, where $C$ ranges over cones of $\mathcal{F}_{B_\ell}$.

Proposition 7.24. For $\ell \in \{1, \ldots, n\}$, the fan $\mathcal{F}_B$ refines the fan $\text{Proj}_{B_\ell}^{-1}(\mathcal{F}_{B_\ell})$.

Proof. The proposition is equivalent to the following assertion: For $\ell \in \{1, \ldots, n\}$, if $x_1 \equiv_B x_2$ then $\text{Proj}_{B_\ell}(x_1) \equiv_{B_\ell} \text{Proj}_{B_\ell}(x_2)$.

Suppose $x_1 \equiv_B x_2$ and let $k$ be a sequence of indices in $\{1, \ldots, n\} \setminus \{\ell\}$. Because $\ell$ doesn’t occur in $k$, the mutation maps used to compute $\eta_k^B$ commute with deleting row $\ell$ and column $\ell$. Thus $\text{sgn} \circ \eta_k^{B_\ell} \circ \text{Proj}_{B_\ell} = \text{sgn} \circ \text{Proj}_{B_\ell} \circ \eta_k^B$. We conclude that $x_1 \equiv_B x_2$ implies $\text{Proj}_{B_\ell}(x_1) \equiv_{B_\ell} \text{Proj}_{B_\ell}(x_2)$. \hfill $\square$

The following is [25, Proposition 5.3].

Proposition 7.25. For any exchange matrix $B$, and every sequence $k$ of indices, the mutation map $\eta_k^B$ is linear on every $B$-cone.

The following lemma is easy.

Lemma 7.26. If $B$ is an acyclic exchange matrix and $c = s_1 \cdots s_n$ is the corresponding Coxeter element, then $\mu_{12\cdots n}(B) = B$.

The following lemma is immediate by comparing the definition of the mutation map with the action of simple reflections on fundamental weights [33].

Lemma 7.27. Suppose $B$ is acyclic and the Coxeter element associated to $B$ is $c = s_1 \cdots s_n$. Let $x = \sum_{i=1}^n a_i \rho_i \in V^*$. If $a_1 \geq 0$, then $\nu_1^{B^T}(x) = -a_1 \rho_1 + \sum_{i=2}^n a_i \rho_i$. If $a_1 \leq 0$, then $\eta_1^{B^T}(x) = s_1 x$.

The map $\sigma_s : \Phi_c \to \Phi_{scs}$ was defined in Section 3.3 for $s$ initial or final in $c$.

Proposition 7.28. If $c = s_1 \cdots s_n$, then the maps $\nu_{s_1 \cdots s_1} \circ \sigma_s$ and $\nu_1^{B^T} \circ \nu_c$ from $\Phi_c$ to $V^*$ coincide.

Proof. We use Lemma 7.27 throughout. Suppose $\beta \in \Phi_c$.

If $\beta = -\alpha_\ell$ for some $\ell \neq 1$, then $\nu_{s_1 \cdots s_1}(\sigma_s(\beta)) = \nu_{s_1 \cdots s_1}(-\alpha_\ell) = \rho_\ell$ and $\eta_1^{B^T}(\nu_c(\beta)) = \eta_1^{B^T}(\rho_\ell) = \rho_\ell$.

If $\beta = -\alpha_1$, then $\nu_{s_1 \cdots s_1}(\sigma_s(\beta)) = \nu_{s_1 \cdots s_1}(\alpha_1) = -\sum_{i=1}^n E_{s_1 \cdots s_1}(a_i^\vee, \alpha_1) \rho_i$. Since $s_1$ is final in $s_1 \cdots s_1$, this is $-\rho_1$. Also, $\eta_1^{B^T}(\nu_c(\beta)) = \eta_1^{B^T}(\rho_1) = -\rho_1$.

If $\beta = \alpha_1$, then $\nu_{s_1 \cdots s_1}(\sigma_s(\beta)) = \nu_{s_1 \cdots s_1}(-\alpha_1) = \rho_1$. Also,

$$\nu_c(\beta) = -\sum_{i=1}^n E_c(a_i^\vee, \alpha_1) = -\rho_1 - \sum_{i \neq 1} a_i \rho_i = s_1 \rho_1,$$

where $E_c(a_i^\vee, \alpha_1)$ is the entry of $E_c$ of $\beta = \alpha_1$.
that

\[ \eta^{B_T}_1(\nu_c(\beta)) = \rho_1. \]

On positive roots \( \beta \neq \alpha_1 \), the map \( \sigma_{s_1} \) agrees with \( s_1 \), so \( \nu_{s_1 c s_1}(\sigma_{s_1}(\beta)) = s_1 \nu_c(\beta) \) by Lemma 7.16. Since \( \nu_c(\beta) \) has a nonpositive \( \rho_1 \)-coefficient in the basis of fundamental weights, \( \eta^{B_T}_1(\nu_c(\beta)) = s_1 \nu_c(\beta) \).

The next result is [25] Proposition 7.3.

**Proposition 7.29.** For any exchange matrix \( B \) and any sequence \( k \) of indices, the mutation map \( \eta^{B_T}_k \) is an isomorphism from \( \mathcal{F}_{B^T} \) to \( \mathcal{F}_{\mu_k(B^T)} \).

The following result is [27, Corollary 4.4].

**Theorem 7.30.** For any exchange matrix \( B \) and any sequence \( k \) of indices, the mutation map \( \eta^{B_T}_k \) is an isomorphism from ScatFan\(^T\)(\( B \)) to ScatFan\(^T\)(\( \mu_k(B) \)).

**Proposition 7.31.** Suppose \( B \) is an acyclic exchange matrix and the Coxeter element associated to \( B \) is \( c = s_1 \cdots s_n \). Then the piecewise linear map \( \eta^{B_T}_{12\cdots n} \)

1. has \( \eta^{B_T}_{12\cdots n} \circ \nu_c = \nu_c \circ \tau_c \) as maps on \( \Phi_c \),
2. is an automorphism of \( \mathcal{F}_B \),
3. is an automorphism of ScatFan\(^T\)(\( B \)), and
4. fixes \( \mathcal{d}_\infty \) as a set and agrees with \( c \) on \( \mathcal{d}_\infty \).

**Proof.** We first prove (1) by checking that \( \eta^{B_T}_{12\cdots n}(\nu_c)(\Phi_c) \). Proposition 7.28 says that \( \eta^{B_T}_1 = \nu_{s_1 c s_1} \circ \sigma_{s_1} \circ \nu_{c_1}^{-1} \). Since \( s_2 \) is initial in \( s_1 c s_1 \), which is the Coxeter element associated to \( \mu_1(B) \), and since \( c \) commutes with \( \nu_c \), we can write a similar expression for \( \eta^{B_T}_{12\cdots n} \). Composing, we see that \( \eta^{B_T}_{21} = \nu_{s_2 s_1 c s_1} \circ \sigma_{s_2} \sigma_{s_1} \circ \nu_{c_1}^{-1} \). Continuing in this manner, we see that \( \eta^{B_T}_{n(n-1)\cdots 1} = \nu_c \tau_c^{-1} \nu_{c_1}^{-1} \nu_{c_2}^{-1} \cdots \nu_{c_{n-1}}^{-1} \). Inverting and appealing to Lemma 7.26, we see that \( \eta^{B_T}_{12\cdots n} = \nu_c \circ \tau_c \circ \nu_{c_1}^{-1} \) as desired. Taking \( k = 12\cdots n \) in Proposition 7.29 and applying Lemma 7.26, we obtain Assertion 2. Assertion 3 follows from Theorem 7.30. Proposition 7.31 follows in the same way.

Now, \( \langle \alpha_c, \alpha_1^\vee \rangle = -\omega_c(\alpha_1^\vee, \delta) < 0 \) by the definition of \( \omega_c \), because \( \delta \) is a positive combination of all of the simple roots, and because there exists some \( j > 1 \) with \( a_{1j} \neq 0 \). Thus every cone of the Coxeter fan in \( \delta^L \) that contains \( x_c \) is in the halfspace \( \{ x \in V^\ast : \langle x, \alpha_1^\vee \rangle \leq 0 \} \). Thus by Theorem 5.10, every point \( \sum_{i=1}^n a_i \rho_i \in \mathcal{d}_\infty \) satisfies \( a_1 \leq 0 \), so Lemma 7.27 says that \( \eta^{B_T}_1 \) acts as \( s_1 \) on \( \mathcal{d}_\infty \). By Theorem 7.30, \( \eta^{B_T}_1 \) takes \( \mathcal{d}_\infty \) to the analogous imaginary wall relative to \( \mu_1(B^T) \). Thus we can apply Lemma 7.27 again to see that \( \eta^{B_T}_{12\cdots n} \) acts as \( s_2 \) on that imaginary wall, and continue until we show that \( \eta^{B_T}_{n(n-1)\cdots 1} \) acts as \( c^{-1} \) on \( \mathcal{d}_\infty \) and fixes \( \mathcal{d}_\infty \) as a set. Assertion 4 follows by taking inverses and appealing to Lemma 7.26.

We are now prepared to prove Proposition 7.21.

**Proof of Proposition 7.21** We need to show that if two points are in the same cone of \( \mathcal{F}_{B^T} \) then they are in the same cone of \( \nu_c(\text{Fan}_c(\Phi)) \). Since \( \mathcal{F}_{B^T} \) and \( \nu_c(\text{Fan}_c(\Phi)) \) both contain the \( g \)-vector fan as a subfan (Theorems 3.3 and 7.22), and since the \( g \)-vector fan covers the entire ambient space except for the relative interior of \( \mathcal{d}_\infty \) (Theorem 5.10), it remains only to prove the case where the two points are in \( \mathcal{d}_\infty \).

We will prove the contrapositive: Suppose two points \( p, q \in \mathcal{d}_\infty \) are not in the same cone of \( \nu_c(\text{Fan}_c(\Phi)) \). We will show that they are not in the same cone of \( \mathcal{F}_{B^T} \).
All maximal cones of \( \nu_c(F_{\infty}(\Phi)) \) contained in \( S_{\infty} \) are simplicial and of dimension \((n - 1)\). Thus since \( p \) and \( q \) are not in the same cone of \( \nu_c(F_{\infty}(\Phi)) \), there exists a face \( F \) of \( \nu_c(F_{\infty}(\Phi)) \) of dimension \( n - 2 \), contained in \( S_{\infty} \) such that \( p \) and \( q \) are not in the linear span of \( F \) but the line segment \( pq \) intersects \( F \). The face \( F \) is the nonnegative span of \( \nu_c(C) \) for a set \( C \) of \( n - 2 \) roots in \( \Phi_c \). If \( \delta \notin C \), then \( F \) is on the relative boundary of \( S_{\infty} \), so that the intersection of \( S_{\infty} \) with the linear span of \( F \) is a face of \( S_{\infty} \). But this is impossible since \( p \) and \( q \) are on opposite sides of the linear span of \( F \). Thus \( \delta \in C \), so Proposition 7.2 \( \Phi \) says that there exists \( \beta \in \Lambda_c^\alpha \) such that \( F \subseteq \beta \). 

By the definition of \( \Lambda_c^\alpha \) (see Section 3.5), there exists \( k \in \mathbb{Z} \) such that \( c^k \beta \) is in \( \Phi_{\text{fin}} \). We have \( c^k F \subseteq (c^k \beta)^\perp \). By Proposition 7.31.4, \( \nu_j^B(\delta(12...n)) \nu_j^B F = c^k F \). In light of Proposition 7.31.2(3) and Theorem 7.13 we may as well assume that \( k = 0 \), so that \( \beta \in \Phi_{\text{fin}} \). Let \( t \) be the reflection such that \( \beta = \beta_t \). 

Since \( \beta \in \Lambda_c^\alpha \subseteq \Phi_c^\alpha \), Proposition 5.24 says that either there is a \( c \)-sortable join-irreducible element \( j \) with \( \text{cov}(j) = \{t\} \) or there is a \( c^{-1} \)-sortable join-irreducible element \( j' \) with \( \text{cov}(j') = \{t\} \), or both. (Indeed, it must be “both”, but there is no need to argue that here.) will assume that \( j \) is a \( c \)-sortable join-irreducible element with \( \text{cov}(j) = \{t\} \) and \( \beta = \beta_t \); the other case is similar. Since the line segment \( pq \) intersects \( F \), it also intersects \( \beta \), but neither \( p \) nor \( q \) is in \( \beta \). Thus Lemma 7.18 says that \( pq \) intersects \( \text{Sh}(j) \). By Lemma 4.4 since \( \beta \in \Phi_{\text{fin}} \), we see that \( j \in W_{\text{fin}} \), so \( \text{Sh}(j) = \text{Proj}_{S_{\text{fin}}}^1 \text{Sh}_{S_{\text{fin}}}(j) \) by Lemma 4.10. Thus the segment connecting \( \text{Proj}_{S_{\text{fin}}}^1(p) \) to \( \text{Proj}_{S_{\text{fin}}}^1(q) \) intersects \( \text{Sh}_{S_{\text{fin}}}(j) \), but neither \( \text{Proj}_{S_{\text{fin}}}^1(p) \) nor \( \text{Proj}_{S_{\text{fin}}}^1(q) \) is in the linear span of \( \text{Sh}_{S_{\text{fin}}}(j) \).

Let \( B_{\text{fin}} \) be the exchange matrix obtained by deleting from \( B \) the row and column indexed by the affine index \( \text{aff} \). This is of finite type, so combining Theorem 7.23 with Proposition 6.14, \( \text{Sh}_{S_{\text{fin}}}(j) \) is the union of all faces of the mutation fan \( F_{(B_{\text{fin}})^T} \) contained in the hyperplane in \( V_{\text{fin}}^* \) orthogonal to \( \beta \). We see that \( \text{Proj}_{S_{\text{fin}}}^1(p) \) and \( \text{Proj}_{S_{\text{fin}}}^1(q) \) are not in the same cone of \( F_{(B_{\text{fin}})^T} \). Proposition 7.24 now implies that \( p \) and \( q \) are not in the same cone of the mutation fan \( F_{B_{\text{fin}}}^T \).

This completes the proof of Theorem 2.9. We conclude with the following proof.

**Proof of Theorem 2.10** An exchange matrix is of affine type if and only if it is mutation-equivalent to an acyclic exchange matrix of affine type. Theorem 2.9 is the special case of Theorem 2.10 where \( B \) is acyclic. Thus Proposition 7.29 and Theorem 7.30 combine to prove Theorem 2.10 for all exchange matrices of affine type. 

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