ON THE FIELD OF DEFINITION OF A CUBIC RATIONAL FUNCTION AND ITS CRITICAL POINTS

XANDER F ABER AND BIANCA THOMPSON

Abstract. Using essentially only algebra, we give a proof that a cubic rational function over $\mathbb{C}$ with real critical points is equivalent to a real rational function. We also show that the natural generalization to $\mathbb{Q}_p$ fails for all $p$.

1. Introduction

Let $K$ be a field of characteristic zero with algebraic closure $\bar{K}$. We say that two rational functions $f, g \in \bar{K}(z)$ are equivalent if there is a fractional linear transformation $\sigma \in \bar{K}(z)$ such that $f = \sigma \circ g$. Viewing $f$ and $g$ as endomorphisms of the projective line, we see that they are equivalent if they differ by a change of coordinate on the target. Note that equivalent rational functions have the same critical points. This is the equivalence relation used in the study of dessins d'enfants, as opposed to the equivalence used for dynamical systems.

Theorem 1.1 (Eremenko/Gabrielov). If $f \in \mathbb{C}(z)$ is a rational function with real critical points, then $f$ is equivalent to a rational function with real coefficients.

By relating equivalence classes of rational functions with special Schubert cycles, Goldberg [5] showed that there are at most 
$$\rho(d) := \frac{1}{d} \binom{2d-2}{d-1}$$
equivalence classes of degree $d$ rational functions with a given set of critical points. Eremenko and Gabrielov [1, 2] used topological, combinatorial, and complex analytic techniques to construct exactly $\rho(d)$ real rational functions with a given set of real critical points, which proves the theorem.

But the correspondence between a rational function and its critical points is purely algebraic, via roots of the derivative. This raises the question of whether a truly elementary proof of the above result exists — one that does

Key words and phrases. critical points, cubic rational functions, field of definition, B. and M. Shapiro conjecture.
not use any analysis or topology. We give such a proof for cubic functions in this note.

For a field $K$ and a nonconstant rational function $\phi \in K(z)$, we say that $K$ is $\phi$-perfect if the map $\phi : \mathbb{P}^1(K) \to \mathbb{P}^1(K)$ is surjective. For example, if $K$ has characteristic $p$ and $\phi(z) = z^p$, then $K$ is $\phi$-perfect if and only if it is a perfect field in the usual sense.

**Theorem 1.2.** Let $K$ be a field of characteristic zero with algebraic closure $\overline{K}$. The following statements are equivalent:

1. Any cubic rational function $f \in \overline{K}(z)$ with $K$-rational critical points is equivalent to a rational function in $K(z)$.
2. $K$ is $\phi$-perfect for the function $\phi(z) = \frac{-z^2 + 2z}{2z + 3}$.
3. $(2y - 1)^2 + 3$ is a square in $K$ for every $y \in K$.

Theorem 1.2 will be proved in §2.

**Corollary 1.3.** If $f \in \mathbb{C}(z)$ is a cubic rational function with real critical points, then $f$ is equivalent to a real rational function.

**Proof.** Evidently $(2y - 1)^2 + 3$ is a square in $\mathbb{R}$ for every $y \in \mathbb{R}$. □

**Remark 1.4.** For a quadratic function $f$, the correspondence between the field of definition of $f$ and the field of definition of its critical points is trivial. Direct computation shows that a function with critical points $c_1 \in \mathbb{P}^1(K)$ and $c_2 \in \mathbb{P}^1(K) \setminus \{\infty\}$ is equivalent to

$$
\begin{cases}
(z - c_2)^2 & \text{if } c_1 = \infty \\
\left(\frac{z-c_1}{z-c_2}\right)^2 & \text{if } c_1 \neq \infty.
\end{cases}
$$

For what other fields of interest $K$ does the corollary on equivalence of rational functions continue to hold? Said another way, which fields $K$ are $\phi$-perfect for $\phi(z) = -\frac{z^2 + 2z}{2z + 3}$?

For one source of (non-)examples, we look at non-Archimedean completions of the rational numbers.

**Proposition 1.5.** Set $\phi(z) = -\frac{z^2 + 2z}{2z + 3}$. The field $\mathbb{Q}_p$ is not $\phi$-perfect for any prime $p$.

This proposition will be proved in §3. Using Theorem 1.2, we obtain the following as a consequence:

**Corollary 1.6.** Let $p$ be a prime. There exists a cubic rational function $f \in \mathbb{Q}_p(z)$ that has $\mathbb{Q}_p$-rational critical points, but that is not equivalent to a rational function with coefficients in $\mathbb{Q}_p$. 
Remark 1.7. In fact, for fixed $p$ one can use the argument from the proof to show that there are infinitely many pairwise inequivalent cubic rational functions $f \in \bar{\mathbb{Q}}_p(z)$ satisfying the statement of the corollary.

The authors would like to thank Sebastian Dorn for pointing out an error in Proposition 1.5 in the published version of this paper [4]. This issue was addressed in an official corrigendum [3].

2. Proof of the theorem

We begin with a normal form for cubic functions. For $u \in \bar{K} \setminus \{-1, -2\}$, define

$$f_u(z) = \frac{z^2(z + u)}{2(u + 3)z - (u + 2)}.$$  \hfill (2.1)

(We exclude $u = -1, -2$ because otherwise a root of the numerator and the denominator collide, and $f_u$ degenerates to a quadratic.) This function has the property that it fixes $0, 1,$ and $\infty$, and each of these three points is critical.

Lemma 2.1. A cubic rational function that is critical at $0, 1,$ and $\infty$ is equivalent to a unique $f_u$, and the fourth critical point is $\phi(u) = \frac{-u^2 + 2u}{2u + 3}$.

Proof. Write $f$ for a cubic function that is critical at $0, 1,$ and $\infty$. By a change of coordinate on the target, we may assume that $0, 1,$ and $\infty$ are all fixed points. (The critical values of a cubic are distinct for local degree reasons.) Thus, $f$ is of the form

$$f(z) = \frac{z^3 + uz^2}{vz + (u - v + 1)}$$

for some $u, v \in \bar{K}$. The Wronskian has the form

$$z(2vz^2 + (uv + 3(u - v + 1))z + 2u(u - v + 1)).$$

Substituting $z = 1$ kills this expression since $1$ is a critical point. Solving the resulting equation for $v$ yields $v = 2u + 3$. Hence $f$ is equivalent to (2.1), as desired.

For the uniqueness statement, suppose that $f_u$ is equivalent to $f_v$ for some $u, v$. Then there is a fractional linear $\sigma \in \bar{K}(z)$ such that $f_u = \sigma \circ f_v$. But $f_u$ and $f_v$ both fix $0, 1,$ and $\infty$, so that $\sigma$ does as well. This means $\sigma(z) = z$, and $u = v$.

The fourth critical point of $f_u$ may be found by factoring the derivative. \hfill $\Box$

Remark 2.2. Note that taking $u = 0, -3,$ or $-3/2$ gives a double critical point at $0, 1,$ or $\infty$, respectively.
Proposition 2.3. If $f_u \in \overline{K}(z)$ is equivalent to a rational function with $K$-coefficients, then $u \in K$.

Proof. Let $\sigma \in \overline{K}(z)$ be a fractional linear map such that $\sigma \circ f_u$ has coefficients in $K$. The images of 0, 1, and $\infty$ under $\sigma \circ f_u$ all lie in $\mathbb{P}^1(K)$. We may therefore apply a further fractional linear transformation $\tau$ with $K$-coefficients so that $\tau \circ \sigma \circ f_u$ fixes 0, 1, and $\infty$. That is, $\tau \circ \sigma \circ f_u = f_v$ for some $v$. Since $\tau$ and $\sigma \circ f_u$ have $K$-coefficients, we know that $v \in K$. By uniqueness in the lemma, we conclude that $u = v$. □

Proof of Theorem 1.2. To prove the implication (1) $\Rightarrow$ (2), we take $y \in K$ and attempt to solve the equation $\phi(u) = y$ with $u \in K$. If $y = \infty$, then we may take $u = -3/2$. Otherwise, choose $u \in \overline{K}$ such that $\phi(u) = y$. Then the function $f_u$ has $K$-rational critical points $\{0, 1, \infty, y\}$. By (1), $f_u$ is equivalent to a rational function with $K$-coefficients. The above proposition implies that $u \in K$.

To prove (2) $\Rightarrow$ (1), we start with a rational function $f \in \overline{K}(z)$ with $K$-rational critical points. If $f$ has only two critical points, then each must have multiplicity 2 (by the Riemann-Hurwitz formula). Without loss, we assume they are 0 and $\infty$, and that 0, $\infty$ are fixed by $f$, so that $f(z) = az^2$ for some $a \in \overline{K}$. Evidently $a^{-1}f$ has coefficients in $K$.

Now suppose that $f$ has at least three distinct critical points. Without loss, we may assume that 0, 1, and $\infty$ are among them. In particular, by the lemma we see that $f$ is equivalent to $f_u$ for some $u \in \overline{K}$. The remaining critical point is $\phi(u) \in \overline{K}$. By (2), both solutions of $\phi(z) = \phi(u)$ lie in $\mathbb{P}^1(K)$, so that $u \in K$. That is, $f$ is equivalent to a rational function with $K$-coefficients.

To prove (2) $\iff$ (3), choose $y \in K$ and consider

$$\phi(z) = -\frac{z^2 + 2z}{2z + 3} = y.$$ 

Rearranging, we get a quadratic in $z$ with discriminant

$$(2y + 2)^2 - 4 \cdot 3y = (2y - 1)^2 + 3.$$ 

Thus, we can solve for $z \in K$ if and only if $(2y - 1)^2 + 3$ is a square in $K$. □

3. $p$-ADIC FIELDS

Our proof of Proposition 1.5 is split into the subcases $p = 2$, $p = 3$, and $p > 3$. We want to show that $\mathbb{Q}_p$ is not $\phi$-perfect for $\phi(z) = -\frac{z^2 + 2z}{2z + 3}$. This amounts to determining whether $\phi(z) = y$ has a solution in $\mathbb{P}^1(\mathbb{Q}_p)$ for $y \in \mathbb{Q}_p$. Rearranging gives the quadratic equation $z^2 + 2(1 + y)z + 3y = 0$, which has discriminant

$$\Delta = 4(y^2 - y + 1) = (2y - 1)^2 + 3. \quad (3.1)$$
Determining if $Q_p$ is $\phi$-perfect now amounts to determining whether $\Delta$ is a square in $Q_p$ for every $y \in Q_p$.

For $p = 2$, set $y = \frac{1}{2} + t$ with $t \in Z_2$. Then (3.1) becomes
\[
\Delta = 4t^2 + 3 \equiv 3 \pmod{4},
\]
which is not a square in $Q_2$. Hence $\phi(z) = \frac{1}{2} + t$ has no solution, and $Q_2$ is not $\phi$-perfect. (It is worth noting that what we have really proved is that the image of $P_1(Q_2)$ under $\phi$ is disjoint from the set $\frac{1}{2} + Z_2$.)

For $p = 3$, we set $y = 2 + 3t$ with $t \in Z_3$. Then $\text{ord}_3(\Delta) = 1$, which means $\Delta$ cannot be a square in $Q_3$.

Finally, we treat the case $p > 3$. The resultant of $\phi(z) = -z^2 + 2z + 3$ is $-3$, so this rational function may be reduced modulo $p$ to yield a quadratic function $\tilde{\phi} \in F_p(z)$. Note that $\tilde{\phi}(0) = \tilde{\phi}(-2)$, so that $\tilde{\phi}$ fails to be injective on $P^1(F_p)$. As $P^1(F_p)$ is a finite set, $\tilde{\phi}$ also fails to be surjective. Choose $\tilde{y} \in F_p$ such that $\tilde{\phi}(z) = \tilde{y}$ has no solution in $F_p$, and choose a lift $y \in Z_p$ such that $y \equiv \tilde{y} \pmod{p}$. It follows that $\phi(z) = y$ has no solution in $Z_p$. It remains to show that $\phi(z) = y$ has no solution in $Q_p \setminus Z_p$. If $\phi(x) = y$ with $|x|_p > 1$, then
\[
|\phi(x)|_p = |x|_p \cdot \left| \frac{1 + 2/x}{2 + 3/x} \right|_p = |x|_p > 1,
\]
which contradicts $y \in Z_p$. Hence $\phi(z) = y$ has no solution in $P^1(Q_p)$, and we have proved that $Q_p$ is not $\phi$-perfect.

4. Further thoughts

A general rational function of degree $d > 2$ has $2d + 1$ free parameters (coefficients) and $2d - 2$ critical points. Imposing the condition that $0, 1, \infty$ are fixed and critical reduces this to $2d - 5$ free parameters. If we fix a set of $K$-rational critical points and look at the Wronskian, then the $2d - 5$ free coefficients for the function must satisfy $2d - 5$ quadratic equations in $2d - 5$ variables over $K$. In the case $d = 3$, in which $2d - 5 = 1$, we were able to explicitly solve for the remaining critical point as an explicit function of the free parameter. Is it possible to solve for the critical points as explicit functions of the parameters for $d > 3$?

Bézout’s theorem gives an upper bound of $2^{2d-5}$ solutions for a general system of $2d - 5$ conics, while Goldberg [5] bounds the number of distinct solutions by the smaller quantity
\[
\frac{1}{d} \binom{2d-2}{d-1} \approx \frac{8}{\sqrt{\pi d^{3/2}}} 2^{2d-5}.
\]
This suggests a substantial amount of extra structure in our system of equations, which may make it possible to give elementary proofs of the theorem of Eremenko/Gabrielov in degree $d$ for other small $d > 3$. 

ON THE FIELD OF DEFINITION OF A CUBIC RATIONAL FUNCTION AND ITS CRITICAL POINTS
REFERENCES

[1] A. Eremenko and A. Gabrielov. Rational functions with real critical points and the B. and M. Shapiro conjecture in real enumerative geometry. *Ann. of Math. (2)*, 155(1):105–129, 2002.

[2] Alexandre Eremenko and Andrei Gabrielov. An elementary proof of the B. and M. Shapiro conjecture for rational functions. In *Notions of positivity and the geometry of polynomials. Dedicated to the memory of Julius Borcea*, pages 167–178. Basel: Birkhäuser, 2011.

[3] Xander Faber and Bianca Thompson. Corrigendum to “On the field of definition of a cubic rational function and its critical points” [J. Number Theory 167 (2016) 1–6]. *J. Number Theory*, 169:439–440, 2016.

[4] Xander Faber and Bianca Thompson. On the field of definition of a cubic rational function and its critical points. *J. Number Theory*, 167:1–6, 2016.

[5] Lisa R. Goldberg. Catalan numbers and branched coverings by the Riemann sphere. *Adv. Math.*, 85(2):129–144, 1991.

CENTER FOR COMPUTING SCIENCES, INSTITUTE FOR DEFENSE ANALYSES, BOWIE, MD

*Email address:* awfaber@super.org

WESTMINSTER COLLEGE, SALT LAKE CITY, UT

*Email address:* bthompson@westminstercollege.edu