The resonant behaviour of the Faraday rotation in a medium with linear birefringence

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It is shown that the monochromatic optical wave propagating through the medium with linear birefringence in presence of a signal electromagnetic wave (whose wavelength is equal to the polarization beats length), displays Faraday rotation having the frequency of the signal wave and unsuppressed by linear birefringence. The effect is resonant with respect to the frequency of a signal wave. The "sharpness" of the resonance is defined by length of the birefringent medium.

I. INTRODUCTION

In the present paper we suggest the effect of resonant behaviour of the Faraday rotation of light propagating through a medium with linear birefringence. Due to the universality of the effect suggested the resonant frequency of the magnetic field has nothing to do with resonances of the susceptibility of the medium and completely defined by the value of linear birefringence of the medium in the transparency spectral region. In this section we describe the essence of the effect. For this reason we briefly remind some statements of the polarization optics. The optical wave electrical displacement vector in the medium runs over the ellipse whose plane is perpendicular to the direction of wave propagation. The parameters of this ellipse define the wave polarization. The case of ellipse squeezed to a line corresponds to linear polarization, the case of circle corresponds to circular polarization. The relative amplitudes and phases of the projections of this motion with respect to some fixed coordinate system completely define the polarization state. These projections can be described by Jones vector:

\[
|\xi > = \begin{pmatrix} \cos \phi \exp i\delta \\ \sin \phi \end{pmatrix}
\]

where \(\cos \phi\) and \(\sin \phi\) – relative amplitudes, and \(\delta\) – relative phase of the projections. For an arbitrary direction in anisotropic medium one can specify two linearly independent monochromatic waves – normal modes – propagating with no change of the polarization state. The normal modes Jones vectors \(|\xi_\pm >\) are the eigen vectors of the inverse permittivity tensor projected on the plane perpendicular to the wave propagation direction.
The corresponding eigenvalues define the normal modes refractive indexes \( n_{\pm} \) [1]. Below we consider the transparent anisotropic medium for which the matrix of this tensor is hermitian with its real symmetric part describing linear birefringence and imaginary antisymmetric part describing circular birefringence. For the problems of polarization optics not the refractive indexes of the normal modes themselves are of prime importance but their difference. Omitting some details which are not important for us we can introduce tensor \( \hat{\eta} \) whose eigenvalues reproduce the deviations of refractive indexes of normal modes from their main (or average) value (which we denote by \( n \)) and whose eigenvectors reproduce the Jones vectors of the normal modes. Real symmetric part of this tensor describe the linear birefringence and imaginary antisymmetric part describe the circular birefringence. In the coordinate system where the real part of this tensor is diagonal it can be always written in the following convenient form:

\[
2\hat{\eta} \equiv h_0 \hat{\sigma}_z + h_1 \hat{\sigma}_y
\]

where \( \hat{\sigma}_z \) and \( \hat{\sigma}_y \) – Pauli spin-matrices:

\[
\sigma_z \equiv \frac{1}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \sigma_y \equiv \frac{1}{2} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}
\]

In accordance with the above we call coefficient \( h_0 \) linear birefringence and coefficient \( h_1 \) we call gyration. The arbitrary polarized incident light can be always reproduced as a linear combination of the normal modes. Due to the fact that the propagation velocities of the normal modes differ from each other the polarization state will display the periodical changes along the propagation direction. The corresponding spatial period \( \Lambda_{\text{beats}} \) called the polarization beats length and can be calculated as:

\[
\Lambda_{\text{beats}} = \Lambda |n_- - n_+|
\]

Here \( \Lambda \) – is the vacuum wavelength of the incident light. For the case of the media with linear birefringence the normal modes are linearly polarized in two orthogonal directions and the polarization beats represent the periodical changes of ellipticity. For the case of circular birefringence (which appear for example in the isotropic medium affected by the magnetic field) the normal modes are the circularly polarized waves with different direction of rotation. In this case (if the incident wave is linearly polarized) the polarization beats represent the rotation of the polarization plane. Below we consider \( h_0, h_1 \ll n \equiv (n_+ + n_-)/2 \), which is almost always the case for the actual media. Now we are ready to pass to our effect. Suppose that the linearly polarized wave falls on the medium having length \( L \) with linear birefringence \( h_0 \). Suppose that this wave has one of eigen polarizations of the medium. This wave will pass through the medium with no change of polarization state. Now let us switch
on the weak longitudinal magnetic field which gives rise to gyration \( h_1 \ll h_0 \). If the linear birefringence is zero, the polarization plane would rotate by an angle \( \pi L h_1 / \Lambda \). Nevertheless it is known that if \( h_1 \ll h_0 \) (weak gyration) and \( L \gg \Lambda_{\text{beats}} = 2 \Lambda / h_0 \) (the medium length is greater than polarization beats spatial period) the above rotation is considerably suppressed [2]. This suppression can be removed as follows [2,3]. Suppose that the magnetic field is periodically varying in space and has the spatial period \( 2 \Lambda / h_0 \) i.e. equal to that of polarization beats. For example let gyration be the following function of coordinate: \( \sim h_1 \cos(\pi h_0 x / \Lambda) \). Then (under some additional conditions [3]) the polarization plane of the beam on exit of the medium rotate by an angle \( \pi L h_1 / \Lambda \), equal to that for the case of zero linear birefringence. It is wellknown effect of compensation of linear birefringence [2,3]. In this paper we suggest the effect of similar compensation by means of magnetic field which is constant in space (along the beam propagation direction) but varying in time. Let us introduce the incident plane wave as a sequence of pulses, each of which contain several spatial periods of the incident wave \( \Lambda \). One of these pulses is depicted on Fig.1 by a bold line. The temporal dependence of coordinate \( x(t) \) of an arbitrary pulse has the form:

\[
x(t) = x_0 + \frac{c}{n} t
\]

(5)

Here \( x_0 \) – is the pulse coordinate at \( t = 0 \), \( n \) – the main value of refractive index of the medium, \( c \) – the speed of light. Pulse can be identified by the value of \( x_0 \) and for this reason we denote the pulse whose coordinate is \( x_0 \) at \( t = 0 \) by a term ”pulse \( x_0 \)”. Now let us affect our medium by a time-dependent magnetic field directed along the \( x \)-axis. If we denote by \( \omega \) the frequency of this magnetic field, then this field gives rise to a gyration which can be written in the form:

\[
h = h_1 \cos(\omega t)
\]

(6)

It is easy to see that from the point of view of the pulse \( x_0 \) the gyration depends on \( x \). For this to see one should express \( t \) from (5) and substitute it in (6):

\[
h = h_1 \cos \left[ \omega n \frac{x - x_0}{c} \right]
\]

(7)

By changing the frequency of magnetic field \( \omega \) one can adjust the spatial period of gyration (7) to be equal to that of polarization beats related to the linear birefringence of the medium. This leads to the following relationship for this frequency:

\[
\Lambda_{\text{beats}} = \frac{2 \Lambda}{h_0} = \frac{2 \pi c}{n \omega} \equiv \frac{\lambda}{n}
\]

(8)

\[
\omega = \frac{\pi c h_0}{n \Lambda}
\]
Here λ is the vacuum wavelength of the electromagnetic wave having the frequency ω. So we see that the linear birefringence is compensated for the pulse x₀ and this pulse (under some additional conditions which we consider below) will display unsuppressed Faraday rotation \( \pi L h_1 / \Lambda \). This effect is resonant with respect to the deviation of frequency of the magnetic field from the value defined by (8). In the next section we present the quantitative calculation and show that the above resonance is accompanied by oscillations of the polarization plane with frequency ω.

II. CALCULATION

In this section we present a calculation of the above effect. It is convenient to perform calculations by means of the quasispin method - the method of polarization optics based on the formal similarity of the dynamics of quantum two-level system and the dynamics of polarization of light propagating in the anisotropic medium. Let us remind briefly the essence of this method. The polarization state is described by the vector of quasispin \( S \) whose components \( S_i \) expressed via the Pauli spin matrices \( \hat{\sigma}_i \) and Jones vector (1) as follows:

\[
S_i \equiv < \xi | \hat{\sigma}_i | \xi > \quad i = 1, 2, 3
\]  

While propagating through the medium the polarization of the beam can change and consequently the direction of quasispin \( S \) depends on \( x \)-coordinate along the propagation direction. This dependence can be described by the Bloch equation:

\[
\frac{dS}{dD} = [H, S]
\]  

The dimensionless coordinate \( D \) plays the role of "time" and can be expressed in terms of conventional coordinate \( x \) as: \( D \equiv 2 \pi x / \Lambda \). The effective "magnetic field" vector can be expressed via the linear birefringence and gyration as follows: \( H \equiv (0, h_1, h_0) \). The quantities \( h_1 \) and \( h_0 \) can depend on \( D \). Below we deal with the gyration having the sinusoidal dependence on \( D \). This dependence can be removed by passing to the rotating frame in the Bloch equation (10) [3]. Let us now turn to the problem we are interested in.

Consider the monochromatic beam having the vacuum wavelength \( \Lambda \) propagating in a medium with linear birefringence \( h_0 \) and with average refractive index \( n \) (Fig.1) We are interested in the dynamics of the polarization of this wave and we call this wave the optical wave. Let the other monochromatic wave (which will be hereinafter referred to as signal wave) be falling at the medium at an angle \( \beta \) so the vector of the magnetic field in this wave has the \( x \)- component \( \sim \cos \beta \). The magnetic field of the signal wave give rise to the gyration in the medium and the polarization of optical wave acquire the increment which we are interested in. Let \( \lambda \) be the vacuum wavelength of the signal wave, \( n_1 \) be the refractive
index of the environment, \( L \) be the medium length. Let the thickness of the medium be much smaller than \( \lambda \) so the magnetic field of the signal wave is homogeneous within our medium. The setup described corresponds to the case of the planar waveguide with the optical wave propagating inside it and irradiated by the signal wave falling at an angle \( \beta \) to the plane of the waveguide. It is easy to see that the distribution of gyration in the medium, produced by a signal wave, can be written as:

\[
h = h_1 \cos \left[ \omega t - x \frac{2\pi}{\lambda} n_1 \sin \beta \right]
\]

\[
h_1 \equiv 2H_0 V \cos \beta
\]

where \( V \) – the Verdet constant of the medium, \( H_0 \) – the amplitude of the magnetic field in the signal wave, \( \omega \) – the frequency of field oscillations in it. Now let us consider this distribution from the point of view of the pulse \( x_0 \) of the optical wave – in the same way it was done in the Introduction. For this reason express \( t \) from (5) and substitute it in (11). Passing to the dimensionless coordinate one can obtain that from the pulse \( x_0 \) point of view the following spatial distribution of the gyration take place:

\[
h(D) = h_1 \cos \left[ \Omega D + \phi_0 \right]
\]

\[
\Omega \equiv \frac{\Lambda}{\lambda} (n - n_1 \sin \beta), \quad \phi_0 \equiv -\frac{2\pi n}{\lambda} x_0
\]

If we denote the optical wave quasispin vector on the entrance of the medium by \( S_0 \), then the temporal behaviour of the quasispin of the pulse \( x_0 \) can be calculated from the Bloch equation (10) with the effective ”magnetic field” \( \tilde{H} = (0, h, h_0) \) and under the initial condition \( S(0) = S_0 \). The equation (10) should be solved within the interval \( D \in [0, 2\pi L/\Lambda] \).

The solution of this problem can be carried out in a conventional way and consists of the following steps:

1. Replace the effective ”magnetic field” oscillating in \( y \)-direction by the field rotating in \( xy \) plane and having the same \( y \)-component.

2. Pass to the frame rotating with ”frequency” \( \Omega \). We denote the matrix of corresponding coordinate transformation by \( \hat{R} \). The effective ”magnetic field” in the rotating \( R \)-frame has the form:

\[
\tilde{H} = (0, h_0 - \Omega, h_1).
\]

3. Perform the rotation around the \( x \)-axis of the \( R \)-frame by an angle:

\[
\theta = \arctan \frac{h_0 - \Omega}{h_1}.
\]

We denote the matrix of this transformation by \( \hat{T} \). In this \( TR \)-frame the effective ”magnetic field” has the components:

\[
\tilde{H} = (0, 0, \tilde{\Omega}),
\]
where
\[ \tilde{\Omega} \equiv \sqrt{h^2 + (h_0 - \Omega)^2} \]

The solution of Bloch equation in \( TR \)-frame represent the rotation around \( z \)-axis with "frequency" \( \tilde{\Omega} \). The corresponding matrix has the form:
\[
\hat{E} = \begin{pmatrix} \cos \tilde{\Omega}D & \sin \tilde{\Omega}D & 0 \\ -\sin \tilde{\Omega}D & \cos \tilde{\Omega}D & 0 \\ 0 & 0 & 1 \end{pmatrix},
\]
i.e., if \( \tilde{S}_0 = \hat{T}\hat{R}S_0 \) is the quasispin of light at \( D = 0 \) in \( TR \)-frame, then for the arbitrary \( D \), quasispin in \( TR \)-frame \( \tilde{S}(D) \) can be written as: \( \tilde{S}(D) = \hat{E}\tilde{S}_0 \). To obtain the final result one should pass back to the initial frame. Consequently, the quasispin dynamics can be calculated as:
\[
S(D) = \hat{R}^{-1}\hat{T}^{-1}\hat{E}\hat{T}\hat{R}S_0
\]

The explicit expressions for the above matrices are:
\[
\hat{R} = \begin{pmatrix} \cos(\phi_0 - \Omega D) & -\sin(\phi_0 - \Omega D) & 0 \\ \sin(\phi_0 - \Omega D) & \cos(\phi_0 - \Omega D) & 0 \\ 0 & 0 & 1 \end{pmatrix}
\]
\[
\hat{T} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \sin \theta & -\cos \theta \\ 0 & \cos \theta & \sin \theta \end{pmatrix}
\]

To calculate the quasispin on the exit of the medium one should set: \( D = 2\pi L/\Lambda \).

Remind now that we make our calculations for the polarization of the pulse \( x_0 \). This pulse goes out of the medium at \( t = n(L - x_0)/c \). Taking into account the expression (12) we obtain that for calculation of the temporal dynamics of the quasispin on the exit of the medium one should set
\[
\phi_0 = \omega t - \frac{2\pi n}{\lambda} L
\]
in formula (13). Let us consider the simple example with the incident optical wave having the eigen polarization of linear birefringent medium. In this case the initial quasispin on the entrance of the medium is directed along \( z \)-axis for an arbitrary pulse of the optical wave:
\[
S_0 = (0, 0, 1)
\]

For this initial condition let us perform all the matrix multiplications in (13) except the last one (\( \hat{R}^{-1} \)). For the quasispin on the exit of the medium we obtain:
\[
S(D) = \begin{pmatrix}
\cos(\phi_0 - \Omega D) & \sin(\phi_0 - \Omega D) & 0 \\
-\sin(\phi_0 - \Omega D) & \cos(\phi_0 - \Omega D) & 0 \\
0 & 0 & 1
\end{pmatrix} S_1
\]

(18)

\[
S_1 \equiv \begin{pmatrix}
-\cos \theta \sin(\tilde{\Omega} D) \\
\sin \theta \cos \theta [1 - \cos(\tilde{\Omega} D)] \\
\cos^2 \theta \cos(\tilde{\Omega} D) + \sin^2 \theta
\end{pmatrix}
\]

Note that \( S_1 \) has no temporal dependence. It is seen that the temporal behaviour of \( S(D) \)
represent the rotation of vector \( S_1 \) around \( z \)-axis with frequency \( \omega \) of the signal wave.

When \( h_0 = \Omega, \theta = 0 \) the polarization state of optical wave displays maximum change: the quasispin vector runs over the cone with an angle \( h_1 D = 2 \pi h_1 L/\Lambda \) what corresponds to the twice smaller oscillations of the polarization plane \( \pi h_1 L/\Lambda \), i.e. to the Faraday rotation of the optical wave unsuppressed by the linear birefringence. The direct calculation shows that the amplitude \( \phi \) of the Faraday rotation of the optical wave is:

\[
\phi = \frac{1}{2} \arcsin \sqrt{1 - S_{1z}^2}
\]

(19)

where

\[
S_{1z} = \frac{h_1^2}{h_1^2 + (h_0 - \Omega)^2} \left[ \cos \sqrt{h_1^2 + (h_0 - \Omega)^2 D} + \left( \frac{h_0 - \Omega}{h_1} \right)^2 \right]
\]

(20)

One can pass the above resonance not by changing frequency \( \omega \) but by changing \( \beta \) (12). The dependence of \( \phi \) on \( \beta \) (Fig.2) exhibit the resonant behaviour. The ”sharpness” of this resonance is \( \sim L/\Lambda_{beats} \).

[1] Landau
[2] Chetkin
[3] Zapasskii
[4] Zvezdin

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FIG. 1. The resonant behaviour of the Faraday rotation in a medium with linear birefringence. Horizontal rectangular – birefringent medium, sine function inside – optical wave. Bold fragment of the sine function – one of the pulses the optical wave consist of. Tilted arrow – the signal wave.
FIG. 2. The dependence of the Faraday rotation on an incident angle $\beta$ of the signal wave for the following values of the parameters: $n = n_1 = 2$, $h_0 = 0.02$, $\Lambda = 1\mu m$, $\lambda = 100\mu m$, $L = 3mm$. 