THE HESS–APPELROT SYSTEM. III. SPLITTING OF SEPARATRICES AND CHAOS

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(Communicated by Amadeu Delshams)

Abstract. We consider a special situation of the Hess–Appelrot case of the Euler–Poisson system which describes the dynamics of a rigid body about a fixed point. One has an equilibrium point of saddle type with coinciding stable and unstable invariant 2-dimensional separatrices. We show rigorously that, after a suitable perturbation of the Hess–Appelrot case, the separatrix connection is split such that only finite number of 1-dimensional homoclinic trajectories remain and that such situation leads to a chaotic dynamics with positive entropy and to the non-existence of any additional first integral.

1. Introduction. A convenient way to study movements of a rigid body about a fixed point in presence of a constant gravitational force is to use the Euler–Poisson system

\[ \dot{M} = M \times \Omega - \Gamma \times K, \quad \dot{\Gamma} = \Gamma \times \Omega. \]  

(1.1)

The vectors \( M = (M_1, M_2, M_3) \), \( \Omega = (\Omega_1, \Omega_2, \Omega_3) \), \( K = mg(X_1, X_2, X_3) \) and \( \Gamma = (\Gamma_1, \Gamma_2, \Gamma_3) \), in the coordinate frame associated with the moving body, denote respectively: the angular momentum, the angular velocity, the gravity force \((mg)\) times the position of the center of mass of the body with respect to the fixed point and the unit vector along the gravity force. It is assumed that the inertia matrix is in the diagonal form,

\[ I = \text{diag}(I_1, I_2, I_3) \]  

(1.2)

(compare [4], [13], [28]); thus we have the relation

\[ M = I\Omega = (I_1\Omega_1, I_2\Omega_2, I_3\Omega_3). \]
System (1.1) is a Hamiltonian system with respect to the following Poisson structure:

\[
\{M_1, M_2 \} = M_3, \ldots, \{M_1, \Gamma_2 \} = -\{M_2, \Gamma_1 \} = \Gamma_3, \ldots, \{\Gamma_1, \Gamma_j \} = 0,
\]

where the dots indicate cyclic permutations of the indices (see [6]). This Poisson structure is degenerate, because the functions \(|\Gamma|^2\) (sometimes called the geometric integral and assumed equal to 1) and \(\langle M, \Gamma \rangle\) (the vertical component of the angular momentum, called also the areas integral\(^1\)) commute with all polynomials in \(M\) and \(\Gamma\). They are called the Casimir functions and they define the symplectic leaves

\[
\mathcal{Y}_c = \left\{ |\Gamma|^2 = 1, \langle M, \Gamma \rangle = c \right\},
\]

where the restricted Poisson structure is non-degenerate (i.e., symplectic). The Euler–Poisson vector field restricted to a symplectic leaf is Hamiltonian, \(X_H\) (in the usual sense \(X_H(F) = \{F, H\}\)), with the total energy

\[
H = \langle M, \Omega \rangle / 2 - \langle \Gamma, K \rangle
\]

as the Hamilton function.

Recall that there are only three cases when there exists an additional first integral (and the vector field \(X_H\) is completely integrable in the Liouville–Arnold sense, see [4], [27], and [20]):

- when \(K = 0\) (the Euler–Poincot case);
- when \(I_1 = I_2\) and the vector \(K = (0,0,K_3)\) is parallel to \(OX_3\) (the Lagrange case);
- when \(I_1 = I_2 = 2I_3\) and \(K_3 = 0\) (the Kovalevskaya case).

In these cases the motion takes place on 2-dimensional tori and typically such motion is quasi-periodic.

The Hess–Appelrot case (or the HA case) is defined by the following conditions:

\[
K_2 = 0, \quad K_3 \sqrt{I_1(I_2 - I_3)} = K_3 \sqrt{I_3(I_1 - I_2)},
\]

where \(0 < I_3 < I_2 < I_1\) and \(K_1, K_3 > 0\) (see [2,14]). This case is characterized by the fact that the following Hess surface

\[
S = \{ \langle M, K \rangle = 0 \}
\]

(i.e., a surface in each 3–dimensional variety defined by common levels of the first integrals (1.4)–(1.5)) is invariant for \(X_H\).

In the literature (see [12,13]) it is claimed that the Euler–Poisson system is partially integrable in this case. In [18] it was explained that in general the vector field \(V_H|_S\) does not admit nontrivial first integrals on the surface \(S\).

In [18] it was also proved that the Hess surface is a 2–dimensional torus (generally) and that the dynamics in it is of four types: hyperbolic (with two periodic solutions), parabolic (with one periodic solution), elliptic irrational (with all trajectories dense in the torus) or elliptic rational (with all trajectories periodic with the same period). Only in the last case there exists a continuous and non-trivial first integral, which is non-algebraic in general (see also Theorem 2 below).

In the papers [18,19] we were interested in two related questions. Firstly, we asked under which conditions the dynamics of the vector field \(X_H\) near the invariant surface \(S\) is normally hyperbolic. Secondly, assuming the normal hyperbolicity, we

\(^1\)This terminology comes from the second Kepler’s law in the celestial mechanics: the area spanned by the moving radius vector grows uniformly with time.
asked what occurs when one changes slightly the parameters of the Euler–Poisson system such that we fall out from the HA case. One expects that the invariant surface survives, it will be only slightly perturbed, but the new vector field restricted to this invariant surface will be not integrable and some limit cycles will appear.

What is the number of these limit cycles?

In general these problems are quite hard, so we restricted our analysis to the case when the invariant surface \( S \) is close to a degenerate torus, i.e., to so-called critical circle (when the radius \( r \) of one of the generating circles of the torus tends to 0). The dynamics on this torus is either quasi-periodic or \( p:q \) resonant. The main result of that paper states that the normal hyperbolicity (as \( r \to 0 \)) takes place only in the \( 1:q \) resonant case. Moreover, this normal hyperbolicity takes place not at the whole surface \( S \), one must delete two phase curves. Next, we studied perturbations of the Hess–Appelrot system in the \( 1:q \) resonance case within the Euler–Poisson class. In [19] we obtained approximate equations for the perturbed invariant surface \( S_\varepsilon \) and the vector field restricted to \( S_\varepsilon \). The problem of limit cycles in \( S_\varepsilon \) was reduced to the problem of zeroes of some highly nonstandard Melnikov integral. We estimated that number using analytic methods.

The aim of this paper is to analyze the dynamics in another limit situation of the HA case, when the Hess torus degenerates (topologically) to a \( 2 - \)dimensional sphere with two points identified (one of the meridians degenerates to a point \( O \)). We will call it the separatrix connection subcase. Here the vector field \( X_H \) has the singular point \( O \) of saddle type, with double eigenvalues \( \pm \lambda \), but without Jordan cells. The singular Hess surface forms a connection of the invariant manifolds (stable separatrix \( W^s \) and unstable separatrix \( W^u \)) of the point \( O, S = W^s = W^u \). A generic perturbation of this case depends on three small parameters: \( \varepsilon_1 \) and \( \varepsilon_2 \) responsible for destroying the Hess–Appelrot conditions and \( \varepsilon_0 \) proportional to the areas constant (which vanishes in the separatrix connection situation).

We denote \( \varepsilon = (\varepsilon_0, \varepsilon_1, \varepsilon_2) \), \( V_\varepsilon = X_{H_\varepsilon} \) (on \( Y_{\varepsilon_0} \)) with the singular point \( O_\varepsilon \) and invariant manifolds \( W^s = W^s_\varepsilon \) and \( W^u = W^u_\varepsilon \). For a local transverse section \( \Delta = \Delta_\varepsilon \) to trajectories of \( V_\varepsilon \) in \( W^s_\varepsilon \) one can define a Poincaré map

\[
T = T_\varepsilon : \Delta \mapsto \Delta,
\]

i.e., defined on a suitable domain.

The main result of the paper is the following

**Theorem 1.** For \( \varepsilon \) from an open and nonempty cone in \( \mathbb{R}^3 \) with vertex at \( \varepsilon = 0 \) there exists a compact set \( \Lambda \subset \Delta \) with the following properties:

- \( \Lambda \) is invariant and hyperbolic for \( T \),
- \( T|_\Lambda : \Lambda \mapsto \Lambda \) is topologically conjugate to the Bernoulli shift on \( N \geq 2 \) symbols.

This implies, in particular, that the Euler–Poisson system near the separatrix connection subcase of the Hess–Appelrot case, i.e., for \( \varepsilon \) from the above subset, does not admit any additional first integral depending analytically on the coordinates.

The invariant subset \( \Lambda \) from Theorem 1 is obtained via a Smale type horseshoe construction. The intersections of the section \( \Delta \) with the level hypersurfaces of the Hamilton function \( \Delta \cap \{ H = E \} \) are \( 2 - \)dimensional surfaces. They contain one or more rectangles \( P_j \) (their number depends on additional conditions) whose images \( T(P_j) \) intersect \( P_k \) along \( N \) thin strips. Such situation generates an invariant set with hyperbolic properties.
Let us present an idea of the proof of this statement. Firstly, one decomposes the Poincaré map (1.8) as follows:

\[ T = Q \circ R, \]

where \( Q = Q_\varepsilon : \Sigma \rightarrow \Delta \) is a correspondence map (of Dulac type) between sections \( \Sigma = \Sigma_\varepsilon \) and \( \Delta = \Delta_\varepsilon \) and \( R = R_\varepsilon : \Delta \rightarrow \Sigma \). Thus \( Q \) is defined by phase curves of \( V_\varepsilon \) passing near the critical point \( O_\varepsilon \) and \( R \) is defined by trajectories passing near the non-singular part of the Hess surface.

The map \( R \) measures the splitting of separatrices. Its essential feature is defined by so-called Melnikov function

\[ I_\varepsilon(q) = \varepsilon_0 I_{0}\varepsilon(q) + \varepsilon_1 I_{1}\varepsilon(q) + \varepsilon_2 I_{2}\varepsilon(q), \]

where \( I_j(q) \) are expressed by some contour integrals depending on the ‘direction’ \( q \) of a corresponding 1-dimensional homoclinic connection. The functions \( I_j \) turn out functionally independent, which implies that the function (1.10) can have two zeroes \( q_1(\varepsilon) \) and \( q_2(\varepsilon) \) which vary independently.

The map \( Q_0 \) is trivial (in a sense), but for \( \varepsilon \neq 0 \) the map \( Q_\varepsilon \) exhibits hyperbolic properties (like ones observed in the papers [8] of R. Devaney and [23] by D. Turaev and L. Shilnikov). The invariant set \( \Lambda \) lies near the intersection of the above separatrices with \( \Delta \).

The above problem of splitting of separatrices was considered by S. Dovbysh in [11]. It should be underlined that the problem is highly difficult. On the other hand, the Dovbysh’ work is very short and rather hard to follow. Main ideals and used tools are presented correctly, but many details of calculations are skipped and in several cases they turn out incorrect. Therefore we have decided to carry out and independent analysis which we hope is rigorous.

Let us say some words about other works concerning the problems of splitting of separatrices and of non-integrability in the rigid body dynamics. Those works concern perturbations of the Lagrange and Euler–Poincot cases.

Note that the Lagrange case \((K_1 = K_2 = 0 \text{ and } I_1 = I_2)\) lies in the ‘boundary’ of the HA stratum (compare Eqs. (1.6)). On the other hand, Eqs. (1.6) are linear with respect to \( K \); therefore, for small \(|K|\) (small gravitational force) there exist Hess–Appelrot perturbations of the Euler–Poincot case \((K = 0)\).

In [11] Dovbysh does not claim to prove results about chaotic dynamics (called quasi-random motion) for the general HA case. He studies the splitting of separatrices, but, when describing the quasi-random motion, he limits himself to the Lagrange case (and extends results of his previous paper [10]). In Remarks 3, 4, 5 and 9 we present those statements from [10] in greater detail.

There exists one situation where the splitting of separatrices leads to a chaotic dynamics quite explicitly. The Hamilton function is of the form \( H = H_0(p,q) + \mu H_1(p,q,\varphi) \) (i.e., with \( 1\frac{1}{2} \) degrees of freedom), where \( \mu \) (here and below in this section) is a small parameter, \( H_1 \) is periodic in the ‘time’ \( \varphi \) and the unperturbed vector field \( X_{H_0} \) has a separatrix loop (or a homoclinic cycle) in \( \mathbb{R}^2 \). Then the monodromy map, i.e., the evolution generated by \( X_H \) after the period, is 2-dimensional and one can exhibit a transverse intersection of corresponding separatrices for fixed points of the monodromy map. Some Hamiltonian systems with two degrees of freedom can be reduced to such situation (when restricting to a level of the first integral corresponding to the cyclic coordinate \( \varphi \)). This is the case of a perturbation of the
Euler–Poincaré case considered by Dovbysh \[9\], but not in the Euler–Poisson version. There one has the standard Hamiltonian on \(T^* SO(3)\), with so-called Andoyer–Deprit coordinates (which are related with the Euler angles, see \[16\]).

In the Euler–Poincaré case the Euler–Poisson equations have the following first integrals: \(|M|^2, 2H = \langle M, I^{-1} M \rangle\), \(|\Gamma|^2 = 1\) and \(\langle M, \Gamma \rangle\). In the non-symmetric case \(I_3 < I_2 < I_1\) on each invariant ellipsoid \(E_E = \{ M : \langle M, I^{-1} M \rangle = 2E \} \subset \mathbb{R}^3\)
the phase curves lie in intersections of \(E_E\) with the spheres \(\{|M|^2 = x\}\). There are two unstable equilibrium points \(M_\pm = (0, \pm \sqrt{2EI_3}, 0)\) with four heteroclinic connections along two circles \(C_1\) and \(C_2\). Assuming \(\langle M, \Gamma \rangle = c \neq 0\) we find that the equilibrium points \(M_\pm\) correspond to two circles \(D_\pm \subset \mathbb{R}^3 \times \mathbb{R}^3\) and the circles \(C_{1,2}\) correspond to two tori \(S_1\) and \(S_2\). Therefore we have heteroclinic connections between two hyperbolic periodic solutions. It was proved in \[16\] that, after perturbation of the Euler–Poincaré case, the separatrices of the limit cycles \(D_+\) and \(D_-\) (slightly deformed) become split in general. The non-splitting occurs only when the perturbation is of the Hess–Appelrot type; more precisely, one of the tori \(S_{1,2}\) becomes the Hess surface and the other is split (see also \[5\]).

In \[10,11\] an analogous result was proved for perturbation of the Lagrange case. We discuss this in Remark 5 in Section 5.

The following approach (which generalizes some result of Poincaré) is often used in proofs of the non-integrability results in the rigid body dynamics; see \[15–17\] (where the Andoyer–Deprit coordinates are used) and \[22\] (perturbation of the Lagrange case using the Euler angles and associated with them momenta). One deals with a perturbation of a completely integrable Hamiltonian system in the action–angle variables \(I = (I_1, I_2)\) and \(\varphi = (\varphi_1, \varphi_2)\). Then \(H = H_0(I) + \mu H_1(I, \varphi)\), with the Fourier expansion \(H_1 = \sum_n \mu H_n(I) \exp i \langle n, \varphi \rangle, n = (n_1, n_2) \in \mathbb{Z}^2, \langle n, \varphi \rangle = n_1 \varphi_1 + n_2 \varphi_2\). If the frequencies \(\omega_{1,2}(I) = \partial H_0 / \partial I_{1,2}\) satisfy some resonant relation \(\langle n, \omega(I) \rangle = 0\) on a so-called crucial set of \(I\)'s and \(h_n(I) \neq 0\) there, then the vector field \(X_H\) does not admit additional first integrals.

In \[16,26\] another idea was raised up. One has \(H = H_0(p, q, I) + \mu H_1(p, q, I, \varphi)\) where \(p\) is the momentum associated with the coordinate \(q\) and \(I = (\varphi)\) (action, angle). For \(\mu = 0\) we have a separatrix connection \(C\) between two equilibrium points \((p_1(I), q_1(I))\) and \((p_1(I), q_2(I))\). Then an obstacle to the existence of an additional first integral lies in a ramification (branching) of the action function \(I\) along a closed contour in a complex prolongation of the curve \(C\).

Finally, in \[27\] S. Ziglin used the monodromy group of a (linear) normal variation equation along a fixed algebraic solution of a Hamiltonian system to detect new obstacles to its integrability. Later his ideas were simplified in a way that the monodromy group was replaced with the differential Galois group; in the integrable case the identity component of the latter group should be abelian (see also \[20\]).

The plan of the paper is following. In Section 2 we recall from \[18\] some facts about the Hess–Appelrot case. In Section 3 we analyze the situation near the equilibrium point \(O\). In Section 4 we discuss periodic solutions near the separatrix connection in the HA case. In Section 5 we describe the Melnikov integrals from Eq. (1.10). In Section 6 we recall the ideas of Devaney and Turaev–Shilnikov (with our small contribution). In Section 7 we summarize the obtained statements and complete the proof of Theorem 1.

2. The Hess–Appelrot case. In this section we assume the Hess–Appelrot conditions (1.6).
2.1. The dynamics on the Hess surface. Introduce the following notations (compare [18]):

\[ x := |M|^2, \quad y := \langle M, K \times \Gamma \rangle, \quad z := \langle M, K \rangle, \quad \varepsilon_0 = \langle M, \Gamma \rangle. \]  

(2.1)

Therefore the Hess surface takes the form

\[ S = \{ z = 0 \} \]  

(see Eq. (1.7)) and the value of the areas integral has changed notation from \( c \) in Eq. (1.4) to \( \varepsilon_0 \). It follows directly from Eqs. (1.1) that

\[ \dot{x} = 2y. \]  

(2.2)

The following formula demonstrates that the Hess surface is really invariant:

\[ \dot{z} = f \cdot Mz, \quad f = \left( \frac{1}{I_3} - \frac{1}{I_2} \right) \frac{K_1}{K_3} > 0. \]  

(2.3)

It turns out that the kinetic energy is proportional to \(|M|^2\) on the Hess surface:

\[ \frac{\langle M, \Omega \rangle}{\langle M, M \rangle} |S = \frac{1}{I_2}. \]  

(2.4)

(In the proof one considers the following linear system on the \( M_j \)'s: \( \langle M, I^{-1}M \rangle = \lambda x, \ |M|^2 = x \) and \( K_1^2M_1 - K_3^2M_3 = 0, \) with degenerate matrix.)

Next, one expresses the vector \( \Gamma \) (on \( S \)) in the orthogonal frame \( M, K, M \times K \) (using Eq. (2.4)):

\[ \Gamma = \frac{\varepsilon_0}{x} M + \frac{x/2I_2 - E}{|K|^2} K + \frac{y}{|K|^2} M \times K \]  

(where \( E \) is the value of the Hamilton function). Then the condition \(|\Gamma|^2 = 1\) gives the following equation:

\[ y^2 = -x(x/2I_2 - E)^2 + |K|^2 x - (\varepsilon_0 |K|)^2 = R(x). \]  

(2.5)

The latter equation defines an elliptic curve in the \( x, y \) plane.

**Remark 1.** Note firstly that Eq. (2.5) is not uniquely solvable with respect to the energy value \( E \). Therefore two such curves for two different values of the energy (and with the same other constants) can intersect one another. The same is true about the value \( \varepsilon_0 \) of the areas integral.
Secondly, in [18] bifurcations of these curves, with $E$ and $\varepsilon_0$ as the parameters (denoted differently in [18]), were thoroughly analyzed.

In general, an elliptic curve can have one or two connected components. In the second case one of them is compact, an oval which can degenerate to a point. It turns out that physically realizable are only ovals, denoted $\gamma = \gamma_{E,\varepsilon_0}$, in the right half-plane. This follows from the following restrictions:

$$x \geq 0, \quad (\varepsilon_0 |K|)^2 \geq 0, \quad E \geq -|K|.$$  

In Figure 1 we present the ovals $\gamma_{E,\varepsilon_0}$, which fill corresponding domains $D_E$ (as $\varepsilon_0$ varies). The boundary of $D_E$ is formed by the curve $\gamma_{E,0}$.

On $S$ we have $M_3 = -(K_1/K_3) M_1$ and we can write

$$x = (|K|/K_3)^2 M_1^2 + M_2^2.$$  

So, it natural to use polar type coordinates $\sqrt{x}, \psi$ such that

$M_1 = (K_3/|K|) \sqrt{x} \cos \psi, \quad M_2 = \sqrt{x} \sin \psi.$  

The map

$$(x, y, \psi) : S \mapsto \mathbb{R}^2 \times S$$

realizes a diffeomorphism between $S$ and its image which is a 2-torus, i.e., when the oval $\gamma_{E,\varepsilon_0}$ is nondegenerate.

Following [13,21] we introduce the variable

$$u = \tan(\psi/2)$$

which takes values in $\mathbb{R}P^1 = \mathbb{R} \cup \infty$. Then one arrives at the following Hess–Appelrot system:

$$\begin{align*}
\dot{x} &= 2y, \\
\dot{y}^2 &= R(x), \\
\dot{u} &= A(x) + B(x)u^2,
\end{align*}$$  

where

$$A = b/x + a\sqrt{x}, \quad B = b/x - a\sqrt{x},$$  

$$a = K_3^2 \left( \frac{1}{3} - \frac{1}{2} \right), \quad b = \frac{1}{2} \varepsilon_0 |K|.$$  

The first two of the algebro–differential equations (2.7) are solved by means of the Weierstrass $P$–function

$$x = P(t)$$

which is inverse to the incomplete elliptic integral $\frac{1}{2} \int^x \pm ds/\sqrt{R(s)}$ and is periodic with the period $T = \frac{1}{2} \oint \frac{2}{y} \text{d}y$ (a complete elliptic integral along the oval $\gamma_{E,\varepsilon_0}$).

Substituting this solution to the third of Eqs. (2.7) we arrive at a Riccati equation with periodic coefficients. Important is the monodromy map $M : \mathbb{R}P^1 \mapsto \mathbb{R}P^1$, defined by the solutions over the time interval $[0, T]$. The monodromy map is fractional linear, $M(u) = (\alpha u + \beta) / (\gamma u + \delta)$, and, depending on the matrix

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in SL(2, \mathbb{R}).$$

we have different dynamics on $S$. We recall the corresponding result from [18].

**Theorem 2.** There are four types of the dynamics of the vector field $X_H$ on $S$:

- **hyperbolic,** with only two periodic trajectories which are hyperbolic and have period $T$;
- **parabolic,** with only one periodic (non-hyperbolic) trajectory, also of period $T$;
- **elliptic irrational,** with all trajectories dense in $S$ (quasi-periodic motion);
• elliptic rational, with all trajectories periodic with period $qT \in T\mathbb{Z}$.

**Remark 2.** From Eqs. (2.8) it follows that, when $a = 0$ (e.g., in the Lagrange subcase $K_1 = 0$ of the HA case) or when $b = 0$ (i.e., $\varepsilon_0 = 0$), the variables in the Riccati equation become separable. If $a = 0$ then we have the elliptic dynamics. If $b = 0$ then we have two $T-$periodic solutions defined by $u(t) \equiv 1$ and by $u(t) \equiv -1$ and the dynamics is hyperbolic (this was observed also by Yu. Varkhalev and G. Gorr in [24]).

Note also that in the Lagrange subcase one has $f = 0$ in Eq. (2.3).

2.2. The separatrix connection. In the case

$$E = |K|, \quad \varepsilon_0 = 0$$  \hfill (2.9)

the elliptic curve (2.5) becomes rational and the Riccati equation has separated variables. This is the **separatrix connection subcase of the Hess–Appelrot case**, which we study in our paper (and which was considered also by Dovbysh in [11]).

Let us compute the corresponding solutions. The elliptic curve takes the form

$$y^2 = x^2 (|K| - x/4I_2)/I_2$$

and we define its rational parametrization as follows:

$$x = 4I_2 (|K| - v^2), \quad y = 4\sqrt{I_2}v (|K| - v^2).$$  \hfill (2.10)

Then the first of Eq. (2.7) takes the form

$$\dot{v} = (v^2 - |K|)/\sqrt{I_2},$$  \hfill (2.11)

and we put the initial condition $v(0) = 0$ (i.e., $x(0) = 4I_2 |K|$ and $y(0) = 0$). The standard integration gives

$$x = \frac{c^2}{\cosh^2 \mu t}, \quad c^2 = 4I_2 |K|, \quad \mu = \sqrt{|K|/I_2}.$$  \hfill (2.12)

The Riccati equation from (2.7) becomes (compare Remark 2)

$$\dot{u} = -\frac{ac}{\cosh \mu t} (u^2 - 1).$$

Its integration gives

$$\frac{u - 1}{u + 1} = q \left( \frac{e^{-\mu t} + i}{e^{-\mu t} - 1} \right)^{i\nu}, \quad q = \frac{u(-\infty) - 1}{u(-\infty) + 1}, \quad \nu = \frac{ac}{\mu}.$$  \hfill (2.13)

This implies the following.

**Proposition 1.** The monodromy map corresponding to the evolution over the time interval $[-\infty, \infty]$ in the chart $q = (u - 1)/(u + 1)$ takes the form

$$q \mapsto e^{-\nu \pi} q.$$

It is hyperbolic with the fixed points $u = 1$ (attractor) and $u = -1$ (repeller).

Finally, the fact that $u = \tan(\psi/2)$ implies

$$\frac{u - 1}{u + 1} = \tan \chi, \quad \chi = \frac{\psi}{2} - \frac{\pi}{4}.$$  \hfill (2.14)
and hence
\[
\tan \chi = q \left( \frac{e^{-\alpha_i \delta_i}}{e^{-\pi/4}} \right)^{iv},
\]
\[
\cos \psi = -2 \tan \chi / (1 + \tan^2 \chi),
\]
\[
\sin \psi = (1 - \tan^2 \chi) / (1 + \tan^2 \chi).
\] (2.14)

3. Near the unstable equilibrium point. In this section we do not stick to the HA case. Therefore we introduce the following small parameters
\[
\varepsilon_1 = \frac{K_2}{|K|}, \quad \varepsilon_2 = \beta^2 \left( \frac{1}{I_3} - \frac{1}{I_2} \right) - \alpha^2 \left( \frac{1}{I_2} - \frac{1}{I_1} \right),
\] (3.1)
where
\[
\alpha = K_3 / |K|, \quad \beta = K_1 / |K|.
\]
The conditions \(\varepsilon_1 = \varepsilon_2 = 0\) are equivalent to the HA conditions (1.6).

We consider the equilibrium point \(O\) corresponding to the singularity of the Hess surface for \(E = |K|\) and \(\varepsilon_0 = 0\) (see Eqs. (2.9)). From Fig. 1 we see that \(x = y = 0\), i.e., \(M = 0\) and \(H(O) = -(K, \Gamma)\). So, \(K = -|K|\ \Gamma\), i.e., the body is in the upper vertical position:
\[
O : M = 0, \quad \Gamma = -K / |K|.
\]
It is natural to choose the (moving) coordinate system with \(K / |K|\) being the third versor (the same was done in [11]). Introduce a matrix \(C\) and its inverse by the formulas:
\[
C = \begin{pmatrix} \alpha & -\varepsilon_1 & \beta \\ 0 & 1 & \varepsilon_1 \\ -\beta & -\varepsilon_1 & \alpha \end{pmatrix}, \quad C^{-1} \approx C^\top \approx \begin{pmatrix} \alpha & 0 & -\beta \\ -\varepsilon_1 & 1 & -\varepsilon_1 \alpha \\ \beta & \varepsilon_1 & \alpha \end{pmatrix};
\] (3.2)
\(C\) is orthogonal modulo \(O(3,1)\). It is clear that \(C(0,0,|K|)^\top = K\). The inverse inertia matrix in the new coordinates takes the form
\[
J := C\Gamma^{-1}C^{-1} = \begin{pmatrix} J_2 + \varepsilon_2 & \varepsilon_1 L & J_0 \\ \varepsilon_1 L & J_2 & \varepsilon_1 L_1 \\ J_0 & \varepsilon_1 L_1 & J_3 \end{pmatrix},
\]
where
\[
J_2 = \frac{1}{I_2}, \quad J_0 = \alpha \beta \left( \frac{1}{I_3} - \frac{1}{I_1} \right), \quad J_3 = \alpha^2 \frac{1}{I_3} + \beta^2 \frac{1}{I_1},
\]
\[
L = \beta \left( \frac{1}{I_3} - \frac{1}{I_2} \right), \quad L_1 = \alpha \left( \frac{1}{I_3} - \frac{1}{I_2} \right).
\]
Below we denote
\[
J := J_2, \quad K := |K|.
\] (3.3)
Denote also
\[
(\gamma_1, \gamma_2, \gamma_3)^\top = C^{-1}\Gamma^\top, \quad (m_1, m_2, m_3)^\top = C^{-1}M^\top.
\] (3.4)
We have
\[
z = \langle M, K \rangle = m_3 |K| = m_3 K
\] (3.5)
and the Hess surface (for \(\varepsilon_1 = \varepsilon_2 = 0\)) becomes \(S = \{m_3 = 0\}\). The fact that the matrix \(J\) restricted to the Hess surface is \(J_2 \cdot Id\) reflects the Zhukovskii’s interpretation of the second of the HA conditions (1.6) (see [25]).
Near the equilibrium point \(\gamma_{1,2}\) and \(m_{1,2}\) are small. Thus
\[
(\gamma_1, \gamma_2, \gamma_3) = (\gamma_1, \gamma_2, -1 + \gamma_1^2 / 2 + \gamma_2^2 / 2 + \ldots),
\]
\[
(m_1, m_2, m_3) = (m_1, m_2, z / K).
\] (3.6)
We denote
\[ \gamma = (\gamma_1, \gamma_2), \quad m = (m_1, m_2); \] (3.7)
they are local coordinates in the symplectic leaf \( \mathcal{Y}_{\varepsilon_0} = \{ |\Gamma|^2 = 1, \langle M, \Gamma \rangle = \varepsilon_0 \} \).
Here the areas integral equals \( \varepsilon_0 = m_1 \gamma_1 + m_2 \gamma_2 - z/K + \ldots = \langle m, \gamma \rangle - z/K + \ldots \)
and hence
\[ m_3 = z/K = -\varepsilon_0 + \langle m, \gamma \rangle + \ldots \] (3.8)

The total energy equals
\[ H = K + (J/2)|m|^2 - (K/2)|\gamma|^2 - \varepsilon_0 J_0 m_1 + \varepsilon_1 L m_1 m_2 + (\varepsilon_2/2) m_2^2 + \ldots \]
\( = K + ((J + \varepsilon_2/2) \tilde{m}_1^2 + (J/2) m_2^2 - (K/2)|\gamma|^2 + \varepsilon_1 L \tilde{m}_1 m_2 + \ldots, \)
(3.9)
where
\[ \tilde{m}_1 = m_1 - m_0, \quad m_0^2 = \varepsilon_0 J_0/J, \]
and the dots denote higher order terms with respect to \( |\gamma|, |m| \) and \( |\varepsilon| \) (where \( \varepsilon = (\varepsilon_0, \varepsilon_1, \varepsilon_2) \)).
The corresponding differential equations are deduced for the Poisson brackets (1.3). For example, \( \dot{m}_1 = \{m_1, H\} = \{m_1, m_2\} \partial H/\partial m_2 + \{m_1, \gamma_2\} \partial H/\partial \gamma_2 \)
\( = (-\varepsilon_0 + \ldots)(J m_2 + \varepsilon_1 L m_1 + \ldots) + (-1 + \ldots)(-K \gamma_2 + \ldots), \)
etc. We get the following system
\[ \begin{align*}
\dot{m}_1 &= K \gamma_2 - J \varepsilon_0 m_2 + \ldots \\
\dot{\gamma}_2 &= (J + \varepsilon_2) m_1 + L \varepsilon_1 m_2 - J_0 \varepsilon_0 + \ldots \\
\dot{m}_2 &= -K \gamma_1 + J \varepsilon_0 m_1 + \ldots \\
\dot{\gamma}_1 &= -J m_2 - L \varepsilon_1 m_1 + \ldots
\end{align*} \] (3.10)
If \( \varepsilon = 0 \) then the equilibrium point is \( O : m = \gamma = 0 \). For \( \varepsilon \neq 0 \) the equilibrium point is moved to
\[ O_\varepsilon : m \approx (m_0^0, 0), \quad \gamma \approx 0, \]
where the linearization matrix of system (3.10) is
\[ A = \begin{pmatrix} 0 & K & -J \varepsilon_0 & 0 \\ J + \varepsilon_2 & 0 & L \varepsilon_1 & 0 \\ J \varepsilon_0 & 0 & 0 & -K \\ -L \varepsilon_1 & 0 & -J & 0 \end{pmatrix}. \] (3.12)
Introduce the variables
\[ \begin{align*}
v_1 &= \sqrt{J} \dot{m}_1 - \sqrt{K} \gamma_2, & v_2 &= \sqrt{J} m_2 + \sqrt{K} \gamma_1, \\
w_1 &= \sqrt{J} \dot{m}_1 + \sqrt{K} \gamma_2, & w_2 &= \sqrt{J} m_2 - \sqrt{K} \gamma_1, \\
v &= v_1 + i v_2, & w &= w_1 + i w_2.
\end{align*} \] (3.13)
Then
\[ m_1 + i m_2 = \frac{1}{2 \sqrt{J}} (v + w), \quad \gamma_1 + i \gamma_2 = \frac{i}{2 \sqrt{K}} (w - v) \]
and
\[ \{v_1, w_1\} \approx -2 \sqrt{JK}, \quad \{v_1, v_2\} = \{v_1, w_2\} = \{w_1, v_2\} = \{w_1, w_2\} \approx -J \varepsilon_0. \]
We have
\[ (H - K)|_{\varepsilon=0} = \frac{1}{2} Re v \bar{w} + \ldots, \] (3.14)
Moreover,
\[
\begin{align*}
\dot{v}_1 &= -\kappa v_1 - \varepsilon_2 (v_1 + w_1) - (\varepsilon_0 + \varepsilon_1) (v_2 + w_2), \\
\dot{v}_2 &= -\kappa v_2 + (\varepsilon_0 - \varepsilon_1) (v_1 + w_1), \\
\dot{w}_1 &= \kappa w_1 + \varepsilon_2 (v_1 + w_1) - (\varepsilon_0 - \varepsilon_1) (v_2 + w_2), \\
\dot{w}_2 &= \kappa w_2 + (\varepsilon_0 + \varepsilon_1) (v_1 + w_1)
\end{align*}
\] (3.15)

(plus higher order terms), where
\[
\kappa = \sqrt{K/J}, \quad \varepsilon_0 = \frac{1}{2} J \varepsilon_0, \quad \varepsilon_1 = \frac{1}{2} \sqrt{K/J} \ell_1, \quad \ell_2 = \frac{1}{2} \sqrt{K/J} \ell_2.
\]

The solutions to Eqs. (3.15) for \( \varepsilon = \varepsilon_0 = 0 \) are very simple
\[
v(t) = e^{-\kappa t} v(0) + \ldots, \quad w(t) = e^{\kappa t} w(0) + \ldots.
\] (3.16)

Of course, the local invariant manifolds (stable and unstable separatrices) of the equilibrium point for \( \varepsilon = 0 \) are
\[
W^s = W^s_0 = \{w = 0\}, \quad W^u = W^u_0 = \{v = 0\},
\] (3.17)

where we modify the variables (3.13) for \( \varepsilon = 0 \) in order that the latter conditions are satisfied. (In Section 7 we discuss the solutions to Eqs. (3.15) in the general case \( \varepsilon \neq 0 \).)

Let us consider sections to trajectories in \( W^s \) and in \( W^u \) as follows:
\[
\Sigma = \Sigma_0 = \{|v| = r, |w| < \mu\}, \quad \Delta = \Delta_0 = \{|v| < \mu, |w| = r\},
\] (3.18)

where \( r > 0 \) and \( \mu > 0 \) are small constants such that \( \mu < r \). They are of the form \( S^1 \times \mathbb{D} \) (circle times disc) and are parametrized by two angles and a radius:
\[
(\varphi, \theta, \rho) \in S^1 \times S^1 \times [0, \mu) \quad \text{such that} \quad (v, w) = (re^{i\varphi}, \rho e^{i\theta}) \in \Sigma \quad \text{and} \quad (v, w) = (\sigma e^{i\varphi}, \rho e^{i\theta}) \in \Delta \quad \text{(see Eqs. (3.13))}.
\]

The correspondence map \( Q : Q_0 : \Sigma \longrightarrow \Delta \) (of Dulac type), defined by trajectories near \( O \) is calculated using Eqs. (3.16). As \( \rho \to 0 \) we get
\[
Q : (\varphi, \theta, \rho) \mapsto (\theta, \varphi, \rho) = (\theta, \varphi, \rho) + O(\rho).
\] (3.19)

On the other hand, on the hypersurface \( \{H = K\} \) we have
\[
|\theta - \varphi| \approx \pi/2
\] (3.20)

(because \( \text{Re} \rho \bar{w} = |v| |w| \cos (\varphi - \theta) \)). In fact, the intersections of the hypersurface \( \{H = K\} \) with the sections \( \Sigma \) and \( \Delta \) form bundles over \( S^1 \) with segments of length \( 2\mu \) as fibers (but these bundles are topologically trivial). But on the hypersurfaces \( \{H = E\} \cap \Sigma, E \neq K \), property (3.20) definitely fails to hold.

Consider now the variables (2.1). We have
\[
\begin{align*}
x|_{\varepsilon=0} &\approx |m|^2 = \frac{1}{4J} |v + w|^2, \\
y|_{\varepsilon=0} &\approx K (m_2 \gamma_1 - m_1 \gamma_2) = \frac{1}{4} \sqrt{K/J} \left(|v|^2 - |w|^2\right), \\
z|_{\varepsilon=0} &\approx K \langle m, \gamma \rangle = \frac{1}{2} \sqrt{K/J} \text{Im} \rho \bar{w}.
\end{align*}
\] (3.21)

By Eq. (3.14) the Hess surface \( \{H = K, z = 0\} = \{v \bar{w} + \ldots = 0\} \) has two local components which coincide with the local invariant manifolds. For \( v = 0 \) we have \( x \approx |w|^2 / 4J \) and \( y \approx -\sqrt{K/J} |w|^2 / 4 \) and for \( w = 0 \) we have \( x \approx |v|^2 / 4J \) and \( y \approx \sqrt{K/J} |v|^2 / 4 \). On the other hand, Eq. (2.5) for the Hess surface in the \( x, y \)
variables becomes \( y^2 \approx (E/I_2) x^2 = KJx^2 \); so, the formulas agree. Note also that on the hypersurface \( \{ z = 0 \} \) we have
\[
\theta - \varphi \approx 0 \text{ or } \theta - \varphi \approx \pi. \tag{3.22}
\]

For \( \varepsilon = 0 \), instead of the representation (3.19) of the correspondence map \( Q \), it is sometimes suitable to express it in terms of the energy \( E \) and value of the Hess function \( z = \frac{1}{2}\sqrt{K/J} |v| |w| \sin (\varphi - \theta) \). We find that
\[
Q : (z, E) \mapsto (z, E) + \ldots. \tag{3.23}
\]

Applying the matrix (3.2) to the vector \( (m_1, m_2, m_3)^\top \) we get
\[
(M_1, M_2, M_3)^\top = C (m_1, m_2, \ldots)^\top \approx (\alpha m_1, m_2, -\beta m_1)^\top,
\]
where
\[
\alpha = K_3/K, \quad \beta = K_1/K,
\]
which, together with \( M_1 = (K_3/K) \sqrt{x} \cos \psi \) and Eqs. (3.13), gives
\[
\sqrt{x} e^{i\psi} = \left(1/2\sqrt{J}\right) (|v| e^{i\varphi} + |w| e^{i\theta}). \tag{3.24}
\]

Consider again the Dulac type map \( Q \), but restricted to the hypersurface \( \{ z = 0 \} \) (the intersection of the this hypersurface with the sections \( \Sigma \) and \( \Delta \) also form segment bundles over a circle). By Eq. (3.22) we have either \( \theta \approx \phi \) (and \( \phi \approx \vartheta \)) or \( \theta \approx -\phi \) (and \( \phi \approx -\theta \)). This implies the following.

**Lemma 1.** On \( W^s \) (resp., on \( W^u \)) the angle \( \psi \) equals the angle \( \varphi = \arg v \) (resp., the angle \( \vartheta = \arg w \)).

Moreover the Dulac type map \( Q \) restricted to the Hess hypersurface takes one of the two forms
\[
Q|_{z=0} : (\varphi, \rho) \mapsto (\vartheta, \sigma) \approx (\pm \varphi, \rho).
\]

4. **Long-periodic solutions close to the separatrix in the HA case.** In this section we consider the dynamics of the Euler–Poisson system near the singular Hess surface \( S \) in the HA case. Therefore, we assume that
\[
\varepsilon_1 = \varepsilon_2 = 0, \quad |\varepsilon_0| << 1, \tag{4.1}
\]
thus \( \varepsilon_0 \) can be nonzero but small. We look for periodic solutions located near \( S \); in fact, in \( S \).
Theorem 3. Assume that we are near the separatrix connection subcase of the Hess–Appelrot case (4.1); more precisely, \( |\varepsilon_0| \) and \( |E - K| \) are small and we assume additionally that
\[
|\varepsilon_0|^{1/2} < |E - K| < |\varepsilon_0|^{1/4}
\]
when \( \varepsilon_0 \neq 0 \).

Then the dynamics in the Hess surface is hyperbolic and hence there are only two periodic solutions with the period \( T \to \infty \) as \( (E, \varepsilon_0) \to (K, 0) \); in the hypersurface \( \{H = E\} \) these closed phase curves are of saddle type (with 2–dimensional stable and unstable invariant manifolds).

Proof. Consider firstly the situation with \( \varepsilon_0 = 0 \) and the motion on the Hess surface. By Remark 1, for \( E \neq K \) (but \( E \approx K \)) the Hess surface is a 2–torus. Consider the Poincaré map restricted to the Hess surface, \( T|_\Sigma : \Delta \cap \Sigma \mapsto \Delta \cap \Sigma \), where \( \Delta \cap \Sigma \simeq S^1 \simeq \mathbb{RP}^1 \). We have \( T|_\Sigma = \mathcal{Q}|_\Sigma \circ \mathcal{R}|_\Sigma \) where the map \( \mathcal{R}|_\Sigma : \Delta \cap \Sigma \mapsto \Delta \cap \Sigma \) is fractional linear and \( \mathcal{Q}|_\Sigma : \Sigma \cap \Sigma \mapsto \Delta \cap \Sigma \) was described in the previous section. (In Figure 2 we present the corresponding scheme). The map \( T|_\Sigma \) is also fractional linear.

In suitable projective coordinates \( u \) on \( \Sigma \cap \Sigma \) and \( u_1 \) in \( \Delta \cap \Sigma \) we have either \( u \mapsto u_1 \approx u \) or \( u \mapsto -u_1 \approx -u \) for \( \mathcal{Q}|_\Sigma \). The corresponding map should be fractional linear, which excludes the second possibility (as the matrix corresponding to \( u \mapsto -u \) has negative determinant).

On the other hand, the map \( \mathcal{R}|_\Sigma \) approximates (as \( E \mapsto K \) and \( r \mapsto 0 \)) the monodromy map described in Proposition 1. It follows that the map \( T|_\Sigma \) is hyperbolic with only two periodic solutions: near \( u = 1 \) (or \( \psi = \pi/2 \)) and near \( u = -1 \) (or \( \psi = -\pi/2 \)). Their period, the integral \( T = \frac{1}{2} \oint dx/y \) over the oval \( \gamma_{0.E} \), tends to infinity as the oval approaches the singular oval of the degenerate elliptic curve.

Consider now the case \( \varepsilon_0 \neq 0 \) (but \( \varepsilon_1 = \varepsilon_2 = 0 \)). From the analysis done in Section 7.2 below it follows that the analogous map \( \mathcal{Q}|_\Sigma : \Sigma \cap \Sigma \mapsto \Delta \cap \Sigma \) is close to the map \( \mathcal{Q}|_\Sigma \), i.e., to the map \( u \mapsto u_1 = u \). Therefore the above arguments work in the general case.

Finally, by Eq. (2.3) for \( f \neq 0 \), the above two periodic solutions considered in the hypersurface \( \{H = E\} \) are of saddle type, with 2–dimensional stable and unstable invariant manifolds (one of them is the Hess surface).

Remark 3. Dovbysh in [11, Section 3] studies periodic solutions near singular \( \Sigma \) in the HA case for \( \varepsilon_0 = 0 \). He divides the periodic orbits by the period of the corresponding periodic points of the Poincaré map \( T = T_\varepsilon \); if the minimal period is \( n \) then he calls such periodic solution as \( n \)–detour periodic solution. We recall his results (with the notations adopted to ours).

- When \( z = 0 \) and \( E > K \) the map \( T \) (i.e., on \( \Delta \cap \Sigma \cap \{H = E\} \)) has two fixed points of hyperbolic type (i.e., like in Theorem 1).
- When \( z = 0 \) and \( E < K \) the map \( T \) does not have periodic points of odd period and the entire circumference \( z = 0 \) consists of fixed points of \( T \circ T \).
- When \( z \neq 0 \) (i.e., outside the Hess surface) and \( E \neq K \) the map \( T \) has two fixed points. Those points are described by concrete formulas and the traces of the linear part of \( T \) are evaluated. In particular, depending on a value taken by some additional quantity, the corresponding periodic solutions are either hyperbolic or elliptic.
5. **Melnikov functions.** The separatrix connection is destroyed after perturbation, i.e., when $\varepsilon = (\varepsilon_0, \varepsilon_1, \varepsilon_2) \neq 0$.

Recall that we have the singular point $O_\varepsilon$ (defined in Eq. (3.11)) which lies in $\mathcal{Y}_{\varepsilon_0} \cap \{H = E_\varepsilon\}$, $E_\varepsilon = H(O_\varepsilon)$. For $\varepsilon = 0$ we have a family $\{\delta_q\}_{q \in \mathbb{R}^1}$ of 1-dimensional phase curves, called the **homoclinic trajectories**, which approach $O_\varepsilon$ as the time tends to $-\infty$ and to $+\infty$; they are defined in Eqs. (2.12)–(2.13), where

$$q = (u(-\infty) - 1) / (u(-\infty) + 1) = \tan \chi(-\infty), \quad \chi(-\infty) = \frac{1}{2} \psi(-\infty) - \frac{\pi}{2},$$

numerates these homoclinic curves. We ask which of these homoclinic trajectories $\delta_{q_j}$ survive the perturbation. More precisely, there should exist homoclinic trajectories $\delta_{q_j, \varepsilon}$ of the vector field $V_\varepsilon$ such that $\delta_{q_j, \varepsilon} \to \delta_{q_j}$ as $\varepsilon \to 0$.

Of course, $H$ is constant on $\delta_{q_j, \varepsilon}$, and we should have

$$z|_{\delta_{q_j, \varepsilon}} \to z(O_\varepsilon) = -\varepsilon_0 |K| = -\varepsilon_0 K$$

as a point in $\delta_{q_j, \varepsilon}$ tends to $O_\varepsilon$ (see also Eq. (3.8)). Therefore, the measure of splitting of separatrices along a trajectory $\delta(t)$ is

$$z(\infty) - z(-\infty) = \int_{-\infty}^{\infty} \dot{z}(t) \, dt,$$

where $z(t) = z \circ \delta(t)$.

We have the following perturbed version of Eq. (2.3):

$$\dot{z} = M_2 z + \varepsilon_1 \left( \tilde{F}_1 M_1 M_3 + F_2 M_2^2 \right) + \varepsilon_2 GM_1 M_2,$$

where $\tilde{F}_1 > 0$, $F_2 > 0$ and $G > 0$ are constants depending on the parameters $K_i$ and $I_j$ in the Euler–Poisson system (see also [19, Eqs. (2.1)–(2.3)]). When $\varepsilon_0 = 0$ we have

$$z(t) = \int_{-\infty}^{t} \left\{ \varepsilon_1 \left( \tilde{F}_1 M_1(s) M_3(s) + F_2 M_2^2(s) \right) + \varepsilon_2 GM_1(s) M_2(s) \right\} \, ds.$$

But, when $\varepsilon_1 = \varepsilon_2 = 0$ the solution to Eq. (5.1) is following:

$$z(t) = z(-\infty) \exp f \int_{-\infty}^{t} M_2(s) \, ds, \quad z(-\infty) = -\varepsilon_0 K.$$
(see also Eqs. (2.14)).

**Remark 4.** In [11, Section 4] an approximation of $z(\infty) - z(-\infty)$, linear in parameters of perturbations, is derived. It takes the form $I = \alpha I_\alpha + \beta I_\beta + j I_j$, where the parameters $\alpha, \beta$ are analogous to $\varepsilon_1, \varepsilon_2$ and $j$ is proportional to $\varepsilon_0$. The functions $\varepsilon I_\alpha$ and $I_\beta$ are integrals similar to $I_1$ and $I_2$. Also the function $I_j$ takes a similar integral form (without exponent of a contour integral), definitely different from the form our function $I_0$ takes.

Next, recall that for $f = 0$ we are in the Lagrange subcase (see Remark 2). Therefore, in perturbation of that case, we can treat $f$ as a new parameter, $f = \varepsilon_3$. Here one obtains the so-called **second order Melnikov function**

$$I_{\varepsilon}(q) = \varepsilon_1 I_1(q) + \varepsilon_2 I_2(q) + \varepsilon_0 \varepsilon_3 I_3(q), \quad I_3 = K \int M_2(t)dt. \quad (5.6)$$

In [11] such limit formula is not derived (due to the different formula for $I_j$) and in [10] only the terms with $\alpha$ and $\beta$ are taken into account (under the assumption $\varepsilon_0 = 0$).

Let us rewrite the above integrals in more transparent form. The function $\xi(t)$ from Eq. (5.5) takes values in the unit circle; we have

$$\xi(t) = e^{i\tau}, \quad 0 < \tau < \pi,$$

where $\tau \to 0$ for $t \to -\infty$ and $\tau \to \pi$ for $t \to \infty$.

We have

$$\xi(t)^{i\nu} = e^{-\nu \tau},$$
$$\cosh \mu t = 1/\sin \tau,$$
$$dt = d\tau/\mu \sin \tau.$$

This gives

$$\int M_2 dt = \frac{c}{\mu} \int \frac{1 - q^2 e^{-2\nu \tau}}{1 + q^2 e^{-2\nu \tau}} d\tau,$$
$$\int M_2^2 dt = \frac{4K^2\varepsilon^2}{K^2\mu} q^2 \int \sin \tau \frac{e^{-2\nu \tau}}{(1 + q^2 e^{-2\nu \tau})^2} d\tau,$$
$$\int M_2^3 dt = \frac{c^2}{\mu} \int \sin \tau \frac{(1 - q^2 e^{-2\nu \tau})^2}{(1 + q^2 e^{-2\nu \tau})^2} d\tau,$$
$$\int M_1 M_3 dt = -\frac{2K_3\varepsilon^2}{K \mu} q \int \sin \tau \frac{e^{-\nu \tau}}{(1 + q^2 e^{-2\nu \tau})^2} d\tau.$$

We expand them at $q = 0$ and at $q = \infty$.

We have

$$\int M_2 = \frac{c}{\mu} \int_0^\pi (1 - q^2 e^{-2\nu \tau} + \ldots) d\tau = \frac{c\pi}{\mu} - \frac{c}{\mu \nu} (1 - e^{-2\nu \pi}) q^2 + \ldots, \quad q \to 0,$$
$$\int M_2 = -\int_0^\pi (1 + \ldots) d\tau = -\frac{c\pi}{\mu} + \ldots, \quad q \to \infty.$$

Next,

$$I_1 = \tilde{F}_1 \mathcal{J}_1(q) + \tilde{F}_2 \mathcal{J}_2(q),$$
where $\hat{F}_{1,2}$ are constants proportional to $F_{1,2}$ and
\[ J_1(q) : = q^2 \int_0^\pi \sin \tau \frac{e^{-2\nu \tau}}{(1 + q^2 e^{-2\nu \tau})^2} d\tau, \]
\[ J_2(q) : = \int_0^\pi \sin \tau \left(1 - q^2 e^{-2\nu \tau}\right)^2 d\tau. \]

Therefore, as $q \to 0$ we have
\[ I_0 = A_0 + B_0 q^2 + \ldots, \]
\[ I_1 = C_0 + D_0 q^2 + \ldots, \]
\[ I_2 = E_0 q + \ldots, \]
\[ B_0 = -\frac{cK}{\mu} e^{f/c_0 / \mu} (1 - e^{-2\nu^2}), \]
\[ D_0 = \left(\hat{F}_1 - 4 \hat{F}_2\right) \left(1 + e^{-2\nu^2}\right) / (4\nu^2 + 1), \]
\[ A_0 = K (e^{f/c_0 / \mu} - 1), \]
\[ C_0 = 2 \hat{F}_2, \]
\[ E_0 = -\frac{2gK_e}{K^2 \mu}(1 + e^{-\nu^2}) / (\nu^2 + 1), \]
and as $q \to \infty$ we have
\[ I_0 = A_\infty + B_\infty q^{-2} + \ldots, \]
\[ I_1 = C_\infty + D_\infty q^{-2} + \ldots, \]
\[ I_2 = E_\infty q^{-1} + \ldots, \]
\[ A_\infty = K (e^{-f/c_0 / \mu} - 1), \]
\[ C_\infty = 2 \hat{F}_2, \]
\[ E_\infty = -\frac{2gK_e}{K^2 \mu}(1 + e^{\nu^2}) / (\nu^2 + 1). \]

**Proposition 2.** The functions $I_k(q)$, $k = 0, 1, 2$, are linearly independent (in the space of functions on $\mathbb{RP}^1$). This implies, in particular, that for any two distinct points $q_1$ and $q_2$ there exists $\varepsilon \neq 0$ (but with small $|\varepsilon|$) such that the function $I_\varepsilon(\cdot)$ has isolated zeroes at these points.

**Proof.** We see that the functions $I_0(q)$ and $I_1(q)$ are even and $I_2(q)$ is odd. Therefore it is enough to demonstrate the independence of $I_0$ and $I_1$. To this aim we use the expansions (5.8)–(5.9).

The corresponding coefficients, before $q^0$ and $q^2$ as $q \to 0$ and before $q^0$ as $q \to \infty$, in expansions of $I_0$ and $I_1$ form the matrix
\[
\begin{pmatrix}
A_0 & B_0 & A_\infty \\
C_0 & D_0 & C_\infty
\end{pmatrix}.
\]

If $\hat{F}_2 \neq 0$ then the first and the third columns form a non-degenerate matrix; otherwise, the matrix composed of the first two columns is nondegenerate for $\hat{F}_1 \neq 0$.

The second statement of the proposition standard; anyway, we provide its short proof. Let $q_1 \neq q_2$ be given points in $\mathbb{RP}^1$. Consider the determinants $D_{jk} = \begin{vmatrix} I_j(q_1) & I_j(q_2) \\ I_k(q_1) & I_k(q_2) \end{vmatrix}$. If some of them vanishes, say $D_{01} = 0$, then a suitable nonzero combination of the functions $I_0$ and $I_1$ vanishes at $q_{1,2}$. Otherwise, we fix $\varepsilon_2 \neq 0$ and solve the system $I_\varepsilon(q_1) = I_\varepsilon(q_2) = 0$ for $\varepsilon_0$ and $\varepsilon_1$. Since the functions $I_j$ are analytic and linearly independent, the corresponding their combinations have only isolated zeroes.

**Remark 5.** Interesting is the question of an upper bound for the number of solutions to Eq. (5.2). We do not know the answer, but in the Lagrange case the situation is quite clear.\(^3\)

\(^3\)It seems possible to bound the number of zeroes of the Melnikov function by prolonging it analytically to the complex domain and applying the argument principle.
Firstly, by Remark 3 the corresponding Melnikov function takes the form (5.6). Next, by Eqs. (2.8) the two constants $a$ and $b$ in the Riccati equation for $u$ are equal to zero. Therefore $\nu = 0$ in Eq. (2.13) and $u \equiv \text{const}$. It follows that the analogue of Eq. (5.2) is equivalent to

$$\mu_1 q^2 (1-q^2) + \mu_2 (1-q^2)^2 + \mu_3 q(1-q^2) + \mu_4 (1-q^4) = 0,$$

where $\mu_j$ are proportional to $\varepsilon_1$ or $\varepsilon_2$ or $\varepsilon_0 \varepsilon_3$.

Therefore, in perturbations of the Lagrange case, the number of isolated homoclinic trajectories is bound by $4$.

Moreover, since the functions $\mathcal{I}_1$, $\mathcal{I}_2$ and $\mathcal{I}_3$ in Eq. (5.6) are independent, the non-splitting situation $\mathcal{I}_2(q) \equiv 0$ occurs when either (i) $\varepsilon_1 = \varepsilon_2 = \varepsilon_0 = 0$ (the HA case), or (ii) $\varepsilon_1 = \varepsilon_2 = \varepsilon_3 = 0$ (the Lagrange case, see Remark 2). This demonstrates that a Dovbysh’s statement about the non-splitting of separatrices from [10,11] is correct, but the proof is different; in fact, in [10] only the case $\varepsilon_0 = 0$ was considered.

6. **Smale type horseshoes.** R. Devaney and T. Turaev with L. Shilnikov proved the existence of a chaotic dynamics in Hamiltonian systems with two degrees of freedom. Here we present those results with sketches of their proofs. In the next section we apply them to exhibit an analogous phenomenon in the HA case.

6.1. **A homoclinic connection of a saddle–focus.** Devaney [8] proved the following result. \footnote{4This theorem is proved in the first of his papers in [8]. The second paper contains a version with a heteroclinic connection.}

**Theorem 4.** (Devaney) Suppose that a Hamiltonian system with two degrees of freedom has the following properties:

- it has a singular point $O$ with eigenvalues $\pm (\alpha \pm i\omega) = \pm \lambda, \pm \alpha, \alpha, \omega > 0$;
- it has an isolated homoclinic trajectory $\delta$ with endpoints at $O$.

Then for any local section $\Delta$ transverse to $\delta$ and for any integer $N > 0$ there exists a compact set $\Lambda_N \subset \Delta$ invariant and hyperbolic for the Poincaré map, on which the latter map is topologically conjugate to the Bernoulli shift on $N$ symbols.

**Proof.** In the proof one uses local coordinates $x_1$, $x_2$, $y_1$, $y_2$ (with standard Poisson brackets), $x = x_1 + ix_2$, $y = y_1 + iy_2$, such that the local stable and unstable separatrices of $O$ are $W^s = \{y = 0\}$ and $W^u = \{x = 0\}$. Then the Hamilton function takes the form

$$H = -\alpha (x_1 y_1 + x_2 y_2) + \omega (x_2 y_1 - x_1 y_2) + \ldots = -\text{Re} \lambda xy + \ldots \quad (6.1)$$

We choose sections like in Eq. (3.18), i.e., $\Sigma = \{x = re^{i\varphi}, y = re^{i\theta} : r \text{ fixed}\}$ and $\Delta = \{x = re^{i\varphi}, y = re^{i\theta} : r \text{ fixed}\}$. The hypersurface $\{H = 0\}$ restricted to $\Sigma$ and $\Delta$ forces the restrictions $\theta \approx \varphi - \arg \lambda - \pi/2$ and $\phi \approx \vartheta + \arg \lambda + \pi/2$ (other possibility is with reverted signs before $\pi/2$). Thus the annulus $\Sigma \cap \{H = 0\}$ is parametrized by $(\rho, \varphi)$ and $\Delta \cap \{H = 0\}$ by $(\sigma, \vartheta)$. Since $x(t) = x(0)e^{-\alpha t}e^{-i\omega t} + \ldots$ and $y(t) = y(0)e^{\alpha t}e^{i\omega t} + \ldots$, the Dulac type map $\mathcal{Q} : \Sigma \cap \{H = 0\} \rightarrow \Delta \cap \{H = 0\}$ takes the form

$$\rho, \varphi \mapsto (\sigma, \vartheta) \approx \left( \rho, \varphi + \kappa + \frac{\omega}{\alpha} \ln (r/\rho) \right), \quad \kappa = -\arg \lambda + \frac{\pi}{2} \quad (6.2)$$

We see that the segments $\{\varphi = \text{const}\} \subset \Sigma \cap \{H = 0\}$ are sent to infinite curves spiralling into the circle $\{\rho = 0\}$ in the annulus $\Delta \cap \{H = 0\}$.
Let \((\rho, \varphi) = (0, \varphi_0)\) be the point of intersection of the homoclinic trajectory \(\delta\) with \(\Sigma\). We take a rectangle \(R \subset \Sigma \cap \{H = 0\}\) in the form
\[
R = \{\mu_N \leq \rho \leq \mu, \ \varphi_0 - \epsilon_N \leq \varphi \leq \varphi_0 + \epsilon_N\},
\]
where the small constants are defined below. Let \(R : P = R^{-1}(R) \mapsto R\) be the map defined by trajectories near the connection \(\delta\); here \(P\) is a curvilinear quadrangle. We choose the constants \(\mu, \mu_N\) and \(\epsilon_N\) such that the set \(Q(R)\) intersects the set \(P\) along \(N\) disjoint and almost horizontal strips \(P_1, \ldots, P_N\).

For the Poincaré map \(T = Q \circ R\) we have \(T(P) \cap P = P_1 \cup \cdots \cup P_N\); this is exactly like in the Smale horseshoe map. We have
\[
\Lambda_N = \bigcap_{n \in \mathbb{Z}} T^n(P).
\]

\[
6.2. \textbf{Homoclinic connections of a general saddle.} \text{ Turaev and Shilnikov} [23] \text{ considered Hamiltonian system with the following expansion of the Hamilton function at the critical point } O: \text{ }
\]
\[
H = -\lambda_1 x_1 y_1 - \lambda_2 x_2 y_2 + \ldots, \quad 0 < \lambda_1 < \lambda_2,
\]
with standard Poisson brackets. Therefore the eigenvalues are \(\pm \lambda_1, \pm \lambda_2\) (below we assume
\[
\lambda_1 = 1 < \lambda = \lambda_2
\]
and the local separatrices are \(W^s = \{y_1 = y_2 = 0\}\) and \(W^u = \{x_1 = x_2 = 0\}\). In \(W^u\) (resp., in \(W^s\)) the phase curves are of the form \(x_2 \approx \text{const} \cdot |x_1|^\lambda\) (resp., \(y_2 \approx \text{const} \cdot |y_1|^\lambda\)); therefore they are accumulated along the direction \(Ox_1\) (resp., \(Oy_1\)).

Assume that all homoclinic trajectories are grouped as follows:
- \(m_1\) trajectories begin in the sector \(\{y_1 > 0\} \subset W^u\) and end in the sector \(\{x_1 > 0\} \subset W^s\),
- \(m_2\) trajectories begin in the sector \(\{y_1 < 0\}\) and end in the sector \(\{x_1 > 0\}\),
- \(m_3\) trajectories begin in the sector \(\{y_1 < 0\}\) and end in the sector \(\{x_1 < 0\}\),
- \(m_4\) trajectories begin in the sector \(\{y_1 > 0\}\) and end in the sector \(\{x_1 < 0\}\).

Below, by Bernoulli shift on \(m = 0\) symbols we mean an empty set (without dynamics) and Bernoulli shift on \(m = 1\) symbols we mean a single fixed point for trivial dynamics.

\textbf{Theorem 5.} \text{ In this situation:}
- \text{ in each surface } \Delta \cap \{H = h\}, \ h < 0 \ (\text{and } |h| \text{ small}), \text{ there exists two distinct compact hyperbolic invariant subset } \Lambda_1 \text{ and } \Lambda_3 \text{ where the restricted Poincaré maps are topologically conjugated with a Bernoulli shifts on } m_1 \text{ and } m_3 \text{ symbols respectively,}
- \text{ in each surface } \Delta \cap \{H = h\}, \ 0 < h << 1 \text{ there exists two distinct compact hyperbolic invariant subset } \Lambda_2 \text{ and } \Lambda_4 \text{ where the restricted Poincaré maps are topologically conjugated with a Bernoulli shifts on } m_2 \text{ and } m_4 \text{ symbols respectively.}

Moreover, the sets \(\Xi_1, \ \Xi_3\) (resp., \(\Xi_2, \ \Xi_4\)) formed by phase curves of the Hamiltonian vector field \(X_H\) passing through the points of \(\Lambda_1, \ \Lambda_3\) (resp., \(\Lambda_2, \ \Lambda_4\)) are separated and constitute the whole invariant subset for \(X_H\) in the hypersurfaces.
\{H = h\} for \(h < 0\) (resp., for \(h > 0\)) localized in a neighborhood of the union of the homoclinic curves.

Proof. Let \(\delta_1, \ldots, \delta_m\) be the homoclinic trajectories in \(\{y_1 > 0\}\) and in \(\{x_1 > 0\}\).

We choose the local sections in different way than before; we take

\[
\Sigma_+ = \{x_1 = r > 0\}, \quad \Delta_+ = \{y_1 = r > 0\},
\]

where \(r > 0\) is fixed. Therefore \(\Sigma_+\) is parametrized by \((x_2, y_2, y_1)\) and \(\Delta_+\) by \((Y_2, X_2, X_1) = (y_2, x_2, y_1)|_{\Delta_+}\).

\[
\Sigma_+ \cap \mathcal{W}^s = \{x_1 = r, y_1 = y_2 = 0\}\]

is parametrized by \(x_2\) and \(\Delta_+ \cap \mathcal{W}^u = \{Y_1 = r, X_1 = X_2 = 0\}\) is parametrized by \(Y_2\). The homoclinic trajectories intersect \(\Sigma_+ \cap \mathcal{W}^s\) at points corresponding to \(x_2^{(1)}, \ldots, x_2^{(m_1)}\) and \(\Delta_+ \cap \mathcal{W}^u\) at points corresponding to \(Y_2^{(1)}, \ldots, Y_2^{(m_1)}\).

From the formulas for solutions

\[
x_1(t) \approx x_1(0)e^{-t}, \quad x_2(t) \approx x_2(0)e^{-\lambda t}, \quad y_1(t) \approx y_1(0)e^{\lambda t}, \quad y_2(t) \approx y_2(0)e^{\lambda t}
\]

we find the Dulac type map \(Q\):

\[
X_1 \approx y_1, \quad X_2 \approx x_2(y_1/r)^{\lambda}, \quad Y_2 \approx y_2(y_1/r)^{-\lambda}.
\]

Now we restrict everything to the levels of the Hamilton function \(\{H = h\}, h < 0\) (but with small \(|h|\)). From Eq. (6.4) we get

\[
\Sigma_+ \cap \{H = h\} = \{y_1 \approx (|h| - \lambda x_2 y_2)/r\}
\]

(parametrized by \(x_2\) and \(y_2\)) and \(\Delta_+ \cap \{H = h\}\) becomes analogously parametrized by \(X_2\) and \(Y_2\). Thus

\[
(y_1/r)^{\lambda} = : \Phi(y_2) \approx ((|h| - \lambda x_2 y_2)/r^2)^{\lambda}
\]

and the restricted map is following:

\[
(x_2, y_2) \mapsto (X_2, Y_2) = (x_2 \Phi(y_2), y_2/\Phi(y_2)).
\]

Suppose \(x_2^{(j)} > 0\) and \(Y_2^{(j)} > 0\) for some \(j\). Then the segment \(\{x_2 = x_2^{(j)}, -|h| \leq y_2 \leq c|h|\}, \quad c = 1/\lambda x_2^{(j)}\), is sent to an infinite half-line which is close to the half-line \(\{Y_2 \geq c'/|h|^\lambda, X_2 = 0\}\) for some \(c' > 0\); we have \(Y_2 \approx Cy_2/(c|h| - y_2)^\lambda\)

\[
X_2 \approx D(c|h| - y_2)^{-\lambda}
\]

for some constants \(C, D > 0\).

Now we take the rectangles

\[
R_j = \{x_2^{(j)} - \mu \leq x_2 \leq x_2^{(j)} + \mu, -c_j |h| \leq y_2 \leq c_j |h|\}, \quad j = 1, \ldots, m_1,
\]

where \(c_j = \max \{1, 1/\lambda x_2^{(j)} - \mu\}\) and \(\mu\) is sufficiently small. We have \(m_1\) preimages of these rectangles \(P_j = R^{-1}(R_j)\), which are curvilinear quadrangles near the points \((Y_2, X_2) = (y_2^{(j)}, 0)\).

(The map (6.7) preserves the natural orientations and hence also the restricted map \(R^{-1}\) preserves them (because the Hamiltonian flow preserves the Liouvillian measure). This implies that the sets \(R^{-1} \cap \{y_2 > 0\}\) lie almost completely in \(\{X_2 > 0\}\).)

Each set \(P_j\) intersects \(Q(\bigcup R_k)\) along \(m_1\) thin strips. We obtain \(m_1\) horseshoes which generate the invariant set

\[
\Lambda_1 = \bigcap_{n \in \mathbb{Z}} \mathcal{T}^n (P_1 \cup \cdots \cup P_{m_1}) \subset \Delta_+.
\]
By considering the sections
\[ \Sigma_+ = \{ x_1 = -r < 0 \}, \quad \Delta_+ = \{ y_1 = -r < 0 \} \]
we analogously construct the invariant subset \( \Lambda_3 \subset \Delta_- \) associated with the \( m_3 \) homoclinic trajectories in \( \{ y_1 < 0 \} \) and in \( \{ x_1 < 0 \} \). Here Eq. (6.6) takes the form
\[ \Sigma_- \cap \{ H = h \} = \{ y_1 = - (|h| - \lambda x_2 y_2)/r \}, \quad h < 0. \]
Since the sections \( \Delta_+ \) and \( \Delta_- \) are dynamically disjoint, the suspensions of the sets \( \Lambda_1 \) and \( \Lambda_3 \) are disjoint and support independent dynamics.

With \( m_2 \) homoclinic trajectories in \( \{ y_1 < 0 \} \) and \( \{ x_1 > 0 \} \) we associate the Dulac type map between the sections \( \Delta_+ \) and \( \Sigma_- \). Here we take \( 0 < h << 1 \), Eq. (6.6) takes the form \( \Delta_+ \cap \{ H = h \} = \{ y_1 \approx - (h + \lambda x_2 y_2)/r \} \) and \( \Phi(y_2) = ((h + \lambda x_2 y_2)/r^2)^\lambda \) in Eq. (6.7). In the above way we construct the set \( \Lambda_2 \subset \Delta_- \).

Finally, with the \( m_4 \) homoclinic trajectories from the last group we consider the correspondence map between the sections \( \Sigma_- \) and \( \Delta_+ \) and construct \( \Lambda_4 \subset \Delta_+ \).

There arises a natural question about independence of dynamics associated with homoclinic trajectories from the first and second groups (for example). Here we should consider Dulac type maps \( \Sigma_+ \rightarrow \Delta_+ \) and \( \Sigma_- \rightarrow \Delta_- \). By restricting the Hamiltonian to \( \{ x_2 = y_2 = 0 \} \), i.e., \( H = -x_1 y_1 + \ldots \), we guess that \( h < 0 \) for the first and \( h > 0 \) for the second, i.e., a contradiction.

Let us get this contradiction in another way. So we consider the restriction of the map \( \Sigma_+ \rightarrow \Delta_- \) to \( \{ H = h \} \) for \( h < 0 \). Eq. (6.6) becomes \( \Sigma_+ \cap \{ H = h \} = \{ y_1 \approx - (\lambda x_2 y_2 - |h|)/r \} \). But, for \( x_2 > 0 \) (for example) we get the condition \( y_2 > |h|/\lambda x_2 \). So, the eventual invariant subset should lie in \( \{ c_1 |h| < |y_2| < c_2 |h| \} \).

Therefore Eqs. (6.7) imply \( |X_2| < O(|h|) \) and \( \Phi(y_2) \approx -h \). The image in \( \Delta_- \) of that set under the map \( \mathbb{R}^{-1} \) would lie in \( \{ |X_2| > \text{const} \cdot |h| \} \). Even without the upper bound \( |y_2| < c_2 |h| \) one arrives at a contradiction.

This completes the proof of Theorem 5. \( \square \)

Remark 6. In [23] the corresponding result is more general. Namely, the number \( n \) of degrees of freedom is arbitrary, only one assumes the restriction \( 0 < \lambda_1 < \text{Re} \lambda_j, \quad j = 2, \ldots, n \). As above, the local separatrices are divided into strong ones, like \( W^{uu} \approx \{ x_1 = 0 \} \) and \( W^{us} = \{ y_1 = 0 \} \), and two open components of their complements. Also the homoclinic trajectories are divided into analogous four groups.

Turaev and Shilnikov obtain a topological Markov chain with the transition matrix of one of two forms:
\[
\begin{pmatrix}
  m_1 & m_2 \\
  m_3 & m_4
\end{pmatrix}, \quad \begin{pmatrix}
  m_2 & m_1 \\
  m_4 & m_3
\end{pmatrix}.
\]

They consider also situations with countable number of homoclinic trajectories and with heteroclinic connections between two distinct singular points.

Unfortunately, they have not published proofs of their statements.

6.3. Homoclinic connections of a degenerate saddle. Assume the following form of a Hamilton function near the singular point:
\[ H = -x_1 y_1 - x_2 y_2 + \varepsilon x_2 y_1 + \ldots, \] (6.9)
where \( \varepsilon > 0 \) is a small constant. The differential equations are following:
\[
\dot{x}_1 = -x_1 + \varepsilon x_2 + \ldots, \quad \dot{x}_2 = -x_2 + \ldots, \quad \dot{y}_1 = y_1 + \ldots, \quad \dot{y}_2 = y_2 - \varepsilon y_1 + \ldots;
\]
thus we have \( 1 : 1 \) resonance and two \( 2 \)-dimensional Jordan cells. The phase curves in the stable separatrix \( W^s = \{ y_1 = y_2 = 0 \} \) are accumulated along the \( Ox_1 \) axis.
and in \( W^u = \{ x_1 = x_2 = 0 \} \) along the \( Oy_2 \) axis; since \( \varepsilon \) is small, this accumulation is ‘weak’.

Consider the sections
\[
\Sigma_+ = \{ x_2 = r \}, \quad \Delta_+ = \{ y_1 = r \},
\]
(6.10)
r > 0, parametrized by \((x_1, y_1, y_2)\) and by \((Y_2, X_1, X_2)\) respectively. Their intersections with the levels of the Hamilton function are parametrized by \((x_1, y_1)\) and by \((X_2, Y_2)\) respectively; we have
\[
\Sigma_+ \cap \{ H = h \} = \{ y_2 \approx -(h + x_1 y_1) / r + \varepsilon y_1 \}
\]
and \( \Delta_+ \cap \{ H = h \} = \{ X_1 \approx -(h + X_2 Y_2) / r + \varepsilon X_2 \} \).

From the solutions
\[
x_1(t) \approx e^{-t} (x_1(0) + \varepsilon x_2(0) t), \quad x_2(t) \approx e^{-t} x_2(0),
\]
\[
y_1(t) \approx e^{t} y_1(0), \quad y_2(t) \approx e^{t} (y_2(0) - \varepsilon y_1(0) t),
\]
we get the following expressions for the Dulac type correspondence map \( Q|_{\{H=h\}} \)
\[
X_2 \approx \frac{1}{r} x_1 y_1, \quad Y_2 \approx -\frac{h}{y_1} - x_1 - \varepsilon r^2 \ln (r/y_1); \quad (6.11)
\]
(we have \( t \approx \ln (r/y_1), \ y_1 > 0 \)). Below we choose \( h < 0 \), but with small \(|h|\).

The function \( y_1 \mapsto \Psi(y_1) = |h|/y_1 - \varepsilon r^2 \ln (r/y_1) \) tends to \(+\infty\) as \( y_1 \to 0^+ \), it has a minimum at \( y_1 = |h|/\varepsilon r^2 \) with the value \( \varepsilon r^2 \left( 1 + \ln \left( |h|/\varepsilon r^2 \right) \right) \) (which tends to \(-\infty\) as \(|h| \to 0\)) and tends to \(+\infty\) as \( y_1 \to \infty \).

Let \( x_1^{(j)} \) and \( Y_2^{(j)}, \ j = 1, \ldots, m \), be coordinates corresponding to the intersection points of homoclinic trajectories \( \delta_1, \ldots, \delta_m \) with \( \Sigma_+ \) and with \( \Delta_+ \) respectively. The first of Eqs. (6.11), applied to the homoclinic connections, i.e., \( Y_2^{(j)} \approx \Psi(y_1) - x_1^{(j)} \), is solved for \( y_1 \), with \( y_1^{(j)} > 0 \) and small. Then \( X_2 \) from the second of Eqs. (6.11) will be also small.

Take the rectangles
\[
R_j = \left\{ x_1^{(j)} - \mu \leq x_1 \leq x_1^{(j)} + \mu, \ \mu \leq y_1 \leq \nu \right\} \subset \Sigma_+ \cap \{ H = h \}
\]
for small constants \( \mu \) and \( \nu \). We get their preimages \( P_j = R^{-1}(R_j) \) near \((Y_2, X_2) = (Y_2^{(j)}, 0)\). The sets \( Q(R_j) \) intersect each set \( P_i \) along \( m \) disjoint thin horizontal strips.

As before, we get the compact invariant hyperbolic set \( \Lambda \) defined in Eq. (6.8) and the following generalization of Theorems 4 and 5.

**Theorem 6.** Assume that a Hamiltonian systems has an equilibrium point \( O \) like in Eq. (6.7) and \( m \geq 2 \) homoclinic connections which begin in \( \{ y_1 > 0 \} \subset W^u \) and end in \( \{ x_2 > 0 \} \subset W^s \). Then for each \( h < 0 \), with sufficiently small \(|h|\), the Poincaré map restricted to \( \Sigma \cap \{ H = h \} \) has a compact hyperbolic invariant subset \( \Lambda \) where the restricted Poincaré map is topologically conjugated with a Bernoulli shift on \( m \) symbols.

An analogous statement holds for groups of homoclinic trajectories which begin and/or end in sections defined by different signs of \( y_1 \) and of \( x_2 \).
7. Hyperbolic properties near the equilibrium point. In this section we continue our analysis of the dynamics near the equilibrium point $O_\varepsilon$ which we began in Section 3 (essentially for $\varepsilon = 0$). Now we assume $\varepsilon \neq 0$. Firstly, we prove a result in the spirit of the previous section, i.e., we complete the proof of Theorem 1 from Introduction. Next, we demonstrate some estimates needed in the proof of Theorem 3 from Section 4.

7.1. Proof of Theorem 1. Of course, this proof consists of precise analysis of the maps $Q = Q_\varepsilon$ and $R = R_\varepsilon$. We begin with the Dulac type map.

Consider the differential system (3.15). We should make a change $(v, w) \mapsto (\tilde{v}, \tilde{w})$ such that the corresponding separatrices become $W^s = \{\tilde{w} = 0\}$ and $W^u = \{\tilde{v} = 0\}$. This change is following:

\[
\begin{align*}
    v_1 &= \tilde{v}_1 - \frac{1}{2\kappa} \tilde{\varepsilon}_2 \tilde{w}_1 - \frac{1}{2\kappa} (\tilde{\varepsilon}_0 + \tilde{\varepsilon}_1) \tilde{w}_2 + \ldots, \\
    v_2 &= \tilde{v}_2 + \frac{1}{2\kappa} (\tilde{\varepsilon}_0 - \tilde{\varepsilon}_1) \tilde{w}_1 + \ldots, \\
    w_1 &= \tilde{w}_1 - \frac{1}{2\kappa} \tilde{\varepsilon}_2 \tilde{v}_1 + \frac{1}{2\kappa} (\tilde{\varepsilon}_0 - \tilde{\varepsilon}_1) \tilde{v}_2 + \ldots, \\
    w_2 &= \tilde{w}_2 - \frac{1}{2\kappa} (\tilde{\varepsilon}_0 + \tilde{\varepsilon}_1) \tilde{v}_1 + \ldots,
\end{align*}
\]

where the dots denote higher order terms with respect to $\tilde{v}_i, \tilde{w}_j$ and $\tilde{\varepsilon}_k$. We arrive at two pairs of equations for $(\tilde{v}_1, \tilde{v}_2)$ and $(\tilde{w}_1, \tilde{w}_2)$ such that the matrices of their linear parts are

\[
B^s = \begin{pmatrix} -\kappa - \tilde{\varepsilon}_2 & -\tilde{\varepsilon}_0 - \tilde{\varepsilon}_1 \\ \tilde{\varepsilon}_0 - \tilde{\varepsilon}_1 & -\kappa \end{pmatrix}, \quad B^u = -(B^s)^T.
\]

Their characteristic polynomials are

\[
P_{B^s,u}(x) = (x + \kappa)^2 \pm \tilde{\varepsilon}_2 (x + \kappa) + \tilde{\varepsilon}_0^2 - \tilde{\varepsilon}_1^2,
\]

with the discriminant

\[
D(\varepsilon) = \tilde{\varepsilon}_2^2 + 4\tilde{\varepsilon}_1^2 - 4\tilde{\varepsilon}_0^2.
\]

Following the analysis of the previous section we call the situation with $D(\varepsilon) < 0$ as the Devaney case and the situation with $D(\varepsilon) > 0$ as the Turaev–Shilnikov case. In fact, these cases are ‘weak’ in the sense that the difference between eigenvalues with positive real parts are small, $|D(\varepsilon)|^{1/2} \leq 2 |\varepsilon|$. Of course, the case $D(\varepsilon) = 0$ is non-generic and we shall assume that

\[
c_1 \cdot |\varepsilon| < |D(\varepsilon)|^{1/2}
\]

with some fixed constant $c_1 > 0$.

Let us now focus our attention on the map $R$. Recall that its principal feature is the collection of homoclinic solutions which are governed by the Melnikov functions analyzed in Section 5.

By Proposition 2 we can ensure that there are generic situations with two homoclinic solutions; moreover, their initial ‘directions’ $q_1$ and $q_2$ can be chosen arbitrarily.

However, in the Turaev–Shilnikov case, in order to apply Theorem 5, we should localize those homoclinic solutions in a special way. Their initial directions and final directions should lie in the same ‘halves’ of the local separatrices. These ‘halves’ are defined by the eigen-directions of the above matrices $B^s$ and $B^u$. Therefore, we encounter the problem of compatibility of the behavior of zeroes of the Melnikov
function and the diagonalization procedure of our matrices. In Remark 7 below we discuss this topics in greater detail.

Fortunately, on the Devaney side the situation is definitely simpler. From the expansion formulas from the proof of Proposition 2 we find that

\[ \mathcal{I}_0(q) \to K \left( e^{\epsilon\pi/\mu} - 1 \right) > 0 \quad \text{as} \quad q \to 0, \]

\[ \mathcal{I}_0(q) \to K \left( e^{-\epsilon\pi/\mu} - 1 \right) < 0 \quad \text{as} \quad q \to \infty. \]

Moreover, \( d\mathcal{I}_0/dq \) is less than 0 for \( q > 0 \). Therefore the function \( \mathcal{I}_0(q) \) has a zero \( q_0 > 0 \). Also for \( \epsilon_1 < |\epsilon_0| \) and \( \epsilon_2 < |\epsilon_0| \) the whole Melnikov function \( \mathcal{M}(q) \) has a zero \( q = q_0(\epsilon) \).

Next, by Eq. (7.2) for

\[ |\epsilon_1,\epsilon_2| < c_2 \cdot |\epsilon_0| \]

we get \( D(\epsilon) < 0 \), i.e., the Devaney case, provided the positive constant \( c_2 \) is sufficiently small. In Theorem 4 we have \( \lambda = \alpha + i\omega \) with \( \alpha = -\kappa \) is an analytic closed curve in \( c_1(\epsilon) \), \( \omega \approx \sqrt{D(\epsilon)} = O(|\epsilon|) \).

In order to complete the proof of Theorem 1 we need some estimates. We should do two things:

- a Birkhoff type normalization of the differential system near the singular point \( O \), and

- a choice of the ‘distance’ \( r \) of the sections \( \Sigma \) and \( \Delta \) form this singular point.

Above we have preformed a partial reduction of system (3.15); namely, we have removed form the Hamiltonian the non-resonant quadratic terms \( v_1^2, v_1v_2, v_2^2, w_1^2, w_2 \) and \( w_3^2 \). Also cubic monomials are non-resonant and can be killed. In the quartic part there remain the following terms (resonant for \( \epsilon = 0 \)): \( v_1^2w_2^2, v_1w_2v_3^2, v_1v_2w_1^2, v_1v_2v_3w_2 \).

We want that the latter terms be small in comparison with the quadratic terms \( \tilde{e}_iv_jw_k \) when \( |v|, |w| \leq r \). For this it is enough to choose

\[ r = |\epsilon|^{3/4} \]  

and impose the following modification of condition (7.4):

\[ c_3 |\epsilon_0| < |\epsilon_1| + |\epsilon_2| < c_4 |\epsilon_0| \]  

with constants \( c_3, c_4 \) such that the above zero \( q_0(\epsilon) \) of \( \mathcal{M}(q) \) is simple. Then the quantity \( \frac{\epsilon}{\ln r} \sim O(|\epsilon| \ln 1/|\epsilon|) \) appearing in Eq. (6.2) is small.

Under these assumptions also the Melnikov function approximation of the map \( R : \Delta \to \Sigma \) is correct: the discrepancy between \( z|\Delta - z|\Sigma \) (with \( z = z(t) \) defined by a trajectory \( \delta_{g_0(\epsilon)}(t) \) of the vector field \( V_\epsilon \)) and \( \mathcal{M}(q) \) is small, tends to 0 as \( |\epsilon| \to 0 \).

Also the differences between the initial angle \( \varphi_0(\epsilon) \) on \( \Delta \cap W^u = \{ \hat{v} = re^{i\varphi}, \hat{w} = 0 \} \) (resp., final angle \( \varphi_0(\epsilon) \) on \( \Sigma \cap W^s = \{ \hat{v} = 0, \hat{w} = re^{i\varphi}\} \) and the corresponding angles for the homoclinic connection \( \delta_{g_0(\epsilon)} \) are small. To show this one should integrate approximately the Riccati equation in Eqs. (2.7) and use \( |\epsilon|/r \to 0 \) as \( \epsilon \to 0 \).

Repeating the proof of the Devaney theorem we find an invariant set \( \Lambda = \Lambda_N \subset \Delta \) on which the Poincaré map is topologically conjugated with the Bernoulli shift on \( N \) symbols, without restrictions on \( N \).

This implies the first part of Theorem 1.

Recall that this theorem contains a statement about the non-existence of a first integral, independent on the known integrals and regular with respect to the phase
variables. But this property is standard, when we have the quasi-random dynamics. For example, there exist infinite number of periodic trajectories with arbitrarily long period. We refer the reader to [5] and [16] for more precise arguments.

At this point we can finish the proof of the main theorem of this paper.

**Remark 7.** Above we have exhibited the quasi-random motion only in the Devaney domain \( \{ \varepsilon : D(\varepsilon) < 0 \} \). Here we present a possibility of a chaotic dynamics in the Turaev–Shilnikov domain \( \{ \varepsilon : D(\varepsilon) > 0 \} \).

It is possible to describe the situations when the Melnikov function \( I_\varepsilon(q) \) has double zero at \( q = 0 \) and at \( q = \infty \). From Eq. (5.7) it follows that

\[
I_\varepsilon(q) = A_0 \tilde{\varepsilon}_0 + C_0 \tilde{\varepsilon}_1 + E_0 \tilde{\varepsilon}_2 \cdot q + (B_0 \tilde{\varepsilon}_0 + D_0 \tilde{\varepsilon}_1) q^2 + \ldots
\]

as \( q \to 0 \). Therefore for

\[
\tilde{\varepsilon}_1 = -(A_0/C_0) \tilde{\varepsilon}_0, \quad \tilde{\varepsilon}_2 = 0 \quad (7.7)
\]

the function \( I_\varepsilon(q) \) has double zero at \( q = 0 \). Analogously, from Eq. (5.8) we find that for

\[
\tilde{\varepsilon}_1 = -(A_\infty/C_\infty) \tilde{\varepsilon}_0, \quad \tilde{\varepsilon}_2 = 0 \quad (7.8)
\]

this function has double zero at \( q = \infty \).

Next, after a suitable change of the parameters \( \tilde{\varepsilon}_j \), we obtain a situation when the function \( I_\varepsilon(q) \) has two or more (depending on the coefficients before \( q^2 \) and \( q^{-2} \)) zeroes \( q_1, q_2, \ldots \) localized in \( \{ 0 < q < \infty \} \) or in \( \{ -\infty < q < 0 \} \) near \( q = 0 \) or near \( q = +\infty \) or near \( q = -\infty \).

Recall that \( q = \frac{\varepsilon - \pi/2}{2} = \tan \frac{\theta - \pi/2}{2} \) is the initial ‘direction’ of a homoclinic trajectory in \( W^u \) (see Section 2.2 and Eq. (3.24)). Hence the values \( q = 0 \) and \( q = \infty \) correspond to \( \theta = \frac{\pi}{2} \) and \( \theta = -\frac{\pi}{2} \) respectively; since \( w_1 + iw_2 = |w| e^{i\theta} \), these cases correspond to \( \{ w_1 = 0 \} \).

By Proposition 1 the final ‘directions’ are

\[
g'_j = e^{-\pi\nu} q_j, \quad j = 1, 2, \ldots
\]

If \( q_j \)’s are of the same sign then also \( q'_j \)’s are of the same sign.

Let us now analyze the question of relation between the ‘directions’ \( q_j \) and \( q'_j \) and the eigen-directions of the linear parts of the Hamiltonian vector field in the stable and unstable invariant manifolds.

Firstly, we should be in the Turaev–Shilnikov domain. Under conditions (7.7) we have \( D(\varepsilon) = 4 (\tilde{\varepsilon}_1^2 - \tilde{\varepsilon}_0^2) = 4 \left( (A_0/C_0)^2 - 1 \right) \tilde{\varepsilon}_0^2 \) and under condition (7.8) we have \( D(\varepsilon) = 4 \left( (A_\infty/C_\infty)^2 - 1 \right) \tilde{\varepsilon}_0^2 \). Since \( C_0 = C_\infty = 2F_2 > 0 \) and \( A_{0,\infty} = K \left( e^{i\pi/\mu} - 1 \right) \), we obtain the following necessary condition:

\[
e^{i\pi/\mu} - 1 > 2F_2/K, \quad (7.9)
\]

By Eq. (7.1) the linear part of the Hamiltonian vector field on \( W^s \) under the condition (7.7) or (7.8) is

\[
B^u = \left( \begin{array}{c}
\kappa \\
\tilde{\varepsilon}_0 + \tilde{\varepsilon}_1 \\
\tilde{\varepsilon}_1 - \tilde{\varepsilon}_0
\end{array} \right).
\]

If \( D(\varepsilon) > 0 \), i.e., \( \tilde{\varepsilon}_1^2 > \tilde{\varepsilon}_0^2 \), then none of the eigen-directions is \( \{ \tilde{w}_1 = 0 \} \) (the matrix is not triangular). This means that none of these eigen-directions corresponds to \( q = 0 \) or to \( q = \infty \).

Summing up this remark we conclude the following.
If condition (7.9) is satisfied then for $\varepsilon$ from an open cone in $\{ D(\varepsilon) > 0 \}$ the dynamics of the vector field $X_H$ is chaotic on some level hypersurfaces $\{ H = E \}$ for $E > H(0_x)$ or for $E < H(0_x)$.

**Remark 8.** The above proof of Theorem 1 suggests that in the HA case with nonzero but small value of the areas integral there exist simultaneously two qualitatively different sorts of dynamics. One is regular on the Hess surface, i.e., hyperbolic in the sense of Theorem 2, and the other is chaotic. But the second one takes place on the energy level $\{ H = H(0_x) \}$ containing the saddle-focus type equilibrium point and the first one holds on the energy level separated form this singularity. We have not expected such phenomenon.

Also interesting is the question of bifurcations as $E$ varies from $E_1$, $|\varepsilon_0|^{1/2} < |E_1 - K| < |\varepsilon_0|^{1/4}$ to $E_2 = H(0_x)$.

**Remark 9.** The Dovbysh papers [10,11] do not contain such detailed study of the vector field $V_x$ near $0_x$. One cannot find there systems like (3.10) and (3.15) with arbitrary parameters $\varepsilon_i$. There we find only the case with $\varepsilon = 0$.

In [10] a variety of possible scenarios giving the chaotic dynamics, based on the Turaev–Shilnikov result, is presented.

### 7.2. The Dulac type map on the Hess surface.

Here we complete one missing point in the proof of Theorem 3.

We have the HA case $\varepsilon_1 = \varepsilon_2 = 0$, but with $\varepsilon_0 \neq 0$ (and small). Moreover, we consider the dynamics on the Hess surface (which is a smooth torus). We need to show that the correspondence map

$$Q|_S : \Sigma \cap S \longmapsto \Delta \cap S$$

is close to a trivial map (the identity in some coordinates).

From the previous section analysis we know that we are in the Devaney case. But from Eqs. (3.8) and (3.11) we know that the equilibrium point $0_x : (m, \gamma) = (m_0^1, 0, 0, 0)$, $n_0^1 \approx \varepsilon_0 J/J_1$ lies outside the Hess surface $S = \{ z = 0 \}$ at a distance $\sim \text{const} \cdot |\varepsilon_0|$.

We have then $\text{Im} w \sim \text{const} \cdot |\varepsilon_0|$ (see also Eqs. (3.21)). On the other hand, by the assumption (4.2), i.e., $|H - H(0_x)| = |H - K + O(\varepsilon_0^2)| \gtrsim |\varepsilon_0|^{1/2}$ we find that $|\text{Re} w| > |\text{Im} w|$ (see also Eq. (3.14)). Therefore

$$|\arg v\tilde{w}| < \text{const} \cdot |\varepsilon_0|^{1/2}$$

(is small) and hence in $\Sigma \cap S$ we have $|\theta - \varphi| \sim \text{const} \cdot |\varepsilon_0|^{1/2}$ and in $\Delta \cap S$ we have $|\theta - \phi| \sim \text{const} \cdot |\varepsilon_0|^{1/2}$.

Then, after choosing

$$r = |\varepsilon_0|^{3/4}$$

(as above), we get that the quantity

$$(\omega/\alpha) \ln (r/\rho),$$

which is responsible for change of the angle in Eq. (6.2), is small. Indeed, we have $|\omega| \sim |\varepsilon_0|$, $r \rho \sim \varepsilon_0^2$ (thus $\rho \sim |\varepsilon_0|^{1/4}$) and then $|\omega \ln (r/\rho)| \sim |\varepsilon_0| \ln (1/|\varepsilon_0|)$.

This implies that the map (7.10) takes the form $\varphi \longmapsto \varphi + O(|\varepsilon_0|^{1/2})$ (or $u \longmapsto u + O(|\varepsilon_0|^{1/2})$ where $u = \tan \varphi/2$). It completes the missing element in the proof of Theorem 3.
Acknowledgments. We would like to thank the referees who pointed out some mistakes (in formulas, statements and references) in the previous version of our work.

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Received December 2016; revised November 2017.

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