SPACES OF GRAPHS, BOUNDARY GROUPOIDS AND THE COARSE
BAUM-CONNES CONJECTURE

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Abstract. We present a new approach to studying expander sequences with large girth, providing new geometric interpretations of previously known results of Willett and Yu. This geometric approach works though defining a new conjecture that captures the geometric structure of such sequences of finite graphs, which we call the coarse boundary conjecture. This conjecture provides a unifying approach to studying the coarse assembly map associated to expander sequences and provides new elementary proofs of many of the classical results present in the literature.

1. Introduction and Outline

We present a new approach to study the coarse Baum-Connes conjecture for spaces of graphs of bounded geometry, specifically providing new insight into counterexamples from [Hig99, HLS02, WY12a]. Recall that the coarse Baum-Connes conjecture states that a certain assembly map:

$$\mu_{\text{red}} : K_*^{\text{red}}(X) \longrightarrow K_*(C^*(X))$$

is an isomorphism for $X$ a uniformly discrete bounded geometry metric space. This conjecture is a geometric interpretation of the well-known Baum-Connes conjecture [BCH94]. For finitely generated groups the Cayley graph coming from some finite generating set will be a uniformly discrete space with bounded geometry, and a positive result for the coarse Baum-Connes conjecture in such situations has strong implications such as the Strong Novikov conjecture [Ros86], or the existence of metrics with positive scalar curvature.

The Baum-Connes conjecture can be developed in other directions, particularly into the realm of topological groupoids [Tu00], which arise naturally in topology and noncommutative geometry to study objects such as foliations on manifolds or group actions on topological spaces. It is a well known result from [STY02] that the above statement of the coarse Baum-Connes conjecture can be replaced with a conjecture with coefficients for some groupoid $G(X)$ that we can associate to any uniformly discrete bounded geometry metric space.

In this context, the coarse Baum-Connes conjecture states that the map:

$$\mu_r : K_*^{\text{top}}(G(X), \ell^\infty(X,K)) \to K_*(\ell^\infty(X,K) \rtimes_r G(X))$$

is an isomorphism. In the paper [HLS02], counterexamples were constructed for the coarse groupoid when the space was a expander graph [Lab10]. This class of spaces has subsequently been well studied with respect to the coarse Baum-Connes conjecture [Hig99, GY08, CTY08, GTY11, OY09, WY12a, WY12b]. The key property associated to expander graphs is the ability to construct ghost projections in the Roe algebra. These are infinite rank
projections that cannot be approximated well by operators of very small propagation as they have essentially no local information.

In the paper [HLS02] the method for constructing counterexamples to the conjecture for groupoids utilised a “short exact sequence” built from reductions of a groupoid -the main idea was that this sequence, whilst always exact for the maximal $C^*$-algebras of these groupoids may fail to be exact when we consider the left regular representation. The spotlight in [HLS02] was on the original groupoid, not the reductions, whilst in this paper we are going to consider specifically the groupoids that arise from reductions to boundaries of the unit space.

The object of this paper to give elementary proofs of many of these results concerning box spaces of residually finite groups as well as proving a strengthened geometric version of the main results of Willett and Yu in [WY12a, WY12b]. We obtain this framework by revisiting the ideas of HLS02, STY02 and by introducing a boundary conjecture associated to a uniformly discrete bounded geometry metric space $X$:

**Conjecture 1.** (Boundary Coarse Baum-Connes conjecture) Let $X$ be a uniformly discrete bounded geometry metric space and let $G(X)$ be the associated coarse groupoid on $X$. Then:

$$\mu_{X,\text{bdry}}: K^*_{\text{top}}(G(X)|_{\partial\beta X}, l^\infty(X,K)/C_0(X,K)) \to K^*((l^\infty(X,K)/C_0(X,K)) \rtimes_r G(X)|_{\partial\beta X})$$

is an isomorphism.

The intuition behind this conjecture is to quotient out by the ghost operators in the Roe algebra and then consider the K-theory of what remains. We formalise this idea in section 2.

Now consider a space $X = \bigsqcup X_i$ constructed from a sequence of finite graphs $\{X_i\}$. The boundary coarse Baum-Connes conjecture for $X$ in this instance allows us, via the Five Lemma and [STY02, Lemma 9], to conclude the coarse Novikov conjecture for $X$. We illustrate how this setting can be adapted to provide new proofs of results in WY12a concerning large girth expanders as well as the result of CTY11 concerning box spaces of linear groups. The maximal conjecture is also considered thereby allowing us to get a new proof a Theorem of GWY08, CTWY08, OGY09. This relationship makes it desirable to understand for which sequences this boundary conjecture is an isomorphism and to that end we prove:

**Theorem 2.** Conjecture 1 is true for the following classes of spaces:

(1) Sequences of spaces that uniformly uniformly embed into Hilbert Space.

(2) Sequences of spaces that have large girth and uniformly bounded vertex degree.

(3) Generalised Box Spaces associated to finitely generated groups with the Strong Baum-Connes Property.

This theorem allows us to recover all known information about spaces of the form $X = \bigsqcup X_i$ for sequences of finite graphs $\{X_i\}$. Remarkably this includes some examples of groups with property (T), whose box spaces have geometric property (T) introduced by Willett and Yu in WY12b. We recap now all the major themes.
1.1. Groupoids.

**Definition 1.3.** A groupoid is a set $\mathcal{G}$ equipped with the following information:

1. A subset $\mathcal{G}(0)$ consisting of the objects of $\mathcal{G}$, denote the inclusion map by $i : \mathcal{G}(0) \hookrightarrow \mathcal{G}$.
2. Two maps, $r$ and $s : \mathcal{G} \rightarrow \mathcal{G}(0)$ such that $r \circ i = s \circ i = Id$
3. An involution map $-1 : \mathcal{G} \rightarrow \mathcal{G}$ such that $s(g) = r(g^{-1})$
4. A partial product $\mathcal{G}(2) \rightarrow \mathcal{G}$ denoted $(g, h) \mapsto gh$, with $\mathcal{G}(2) = \{(g, h) \in \mathcal{G} \times \mathcal{G} | s(g) = r(h)\} \subseteq \mathcal{G} \times \mathcal{G}$ being the set of composable pairs.

Moreover we ask the following:

- The product is associative where it is defined in the sense that for any pairs: $(g, h), (h, k) \in \mathcal{G}(2)$ we have $(gh)k$ and $g(hk)$ defined and equal.
- For all $g \in \mathcal{G}$ we have $r(g)g = gs(g) = g$. 

A groupoid is principal if $(r, s) : \mathcal{G} \rightarrow \mathcal{G}(0) \times \mathcal{G}(0)$ is injective and transitive if $(r, s)$ is surjective. A groupoid $\mathcal{G}$ is a topological groupoid if both $\mathcal{G}$ and $\mathcal{G}(0)$ are topological spaces, and the maps $r, s, -1$ and the composition are all continuous. A Hausdorff, locally compact topological groupoid $\mathcal{G}$ is proper if $(r, s)$ is a proper map and étale or $r$-discrete if the map $r$ is a local homeomorphism. When $\mathcal{G}$ is étale, $s$ and the product are also local homeomorphisms, and $\mathcal{G}(0)$ is an open subset of $\mathcal{G}$.

**Definition 1.4.** Let $\mathcal{G}$ be a groupoid and let $x, y \in \mathcal{G}(0)$ and $A, B \subset \mathcal{G}(0)$. Set:

1. $\mathcal{G}_x = s^{-1}(x)$
2. $\mathcal{G}_y = r^{-1}(y)$
3. $\mathcal{G}_{xz} = \mathcal{G}_y \cap \mathcal{G}_x$

Denote by $\mathcal{G}|_A$ the subgroupoid $\mathcal{G}_A$, called the reduction of $\mathcal{G}$ to $A$. In particular it is worth noting that the groupoids $\mathcal{G}|_x$ are in fact groups, and we say that for a given $x \in \mathcal{G}(0)$ that the group $\mathcal{G}_x$ is the isotropy group at $x$.

**Definition 1.5.** Let $\mathcal{G}$ act on $Z$ (or $Z$ is a $\mathcal{G}$-space) if there is a continuous, open map $r : Z \rightarrow \mathcal{G}(0)$ and a continuous map $(\gamma, z) \mapsto \gamma.z$ from $\mathcal{G} \ast Z := \{(\gamma, z) \in \mathcal{G} \times Z | s_{\mathcal{G}}(\gamma) = r(z)\}$ to $Z$ such that $r_Z(z).z = z$ for all $z$ and $(\eta_\gamma).z = \eta.(\gamma.z)$ for all $\gamma, \eta \in \mathcal{G}(2)$ with $s_{\mathcal{G}}(\gamma) = r_Z(z)$.

When it is clear we drop the subscripts on each map. Right actions are dealt with similarly, replacing each incidence of $r_Z$ with $s_Z$.

**Definition 1.6.** Let $\mathcal{G}$ act on $Z$. The action is said to be free if $\gamma.z = z$ implies that $\gamma = r_Z(z)$.

We end this section with some useful examples.
Example 1.7. Let $X$ be a topological $\Gamma$-space. Then the transformation groupoid associated to this action is given by the data $X \times G \rightrightarrows X$ with $s(x, g) = x$ and $r(x, g) = g.x$. We denote this by $X \rtimes G$. A basis $\{U_i\}$ for the topology of $X$ lifts to a basis for the topology of $X \rtimes G$, given by sets $[U_i, g] := \{(u, g)|u \in U_i\}$.

Example 1.8. The construction in the example above can be generalized to actions of étale groupoids. We are concerned with the topology here: Given an étale groupoid $G$ and a $G$-space $X$ as well as a with a basis $\{U_i\}$ for $G(0)$. We can pull this basis back to a basis for $X \rtimes G$ given by $[r^{-1}_z(U_i), \gamma]$, where $U_i \subseteq s(\gamma)$.

1.2. Groupoid $C^*$-algebras. Let $G$ be a locally compact Hausdorff, étale groupoid - this reduces the problem from studying integrals over a Haar system to sums, as étale allows us to choose counting measure for our Haar system [Pat99, Page 46]. The main difference in constructing a $C^*$-algebra from $G$ over a topological group $G$ is the observation that everything need be fibered over the unit space $G(0)$, so any completions of $C_c(G)$ we consider will need to be in over fields of Hilbert spaces (or their spaces of continuous sections). As we will be working with $C_c(G)$ we need to know the product and the adjoint [Exe08]. For every $f, g \in C_c(G)$:

$$(f * g)(\gamma) = \sum_{(\sigma, \tau) \in G^{(2)}} f(\sigma)g(\tau)$$

$$f^*(\gamma) = \overline{f(\gamma^{-1})}$$

We can put a natural pre-Hilbert $C_0(G(0))$-module structure on this function algebra by defining the inner product:

$$\langle \zeta, \eta \rangle = \langle \zeta^* * \eta \rangle |_{G(0)}$$

We observe that for any function $f \in C_0(G(0))$ we can define a right action on $C_c(G)$ by: $(\eta.f)(\gamma) = \eta(\gamma)f(s(\gamma))$. We can then complete this as a Hilbert module, and we denote this by $L^2(G)$. The algebra $C_c(G)$ represents naturally on this algebra using the representation: $\lambda(f)(\eta) = f * \eta$.

It is well known that any Hilbert $C_0(G(0))$-module $M$ is the space of sections of a continuous field of Hilbert spaces $\{M_x\}_{x \in G(0)}$, with any bounded adjointable operator $T$ on $M$ decomposing as a strongly *-continuous field $(T_x)_{x \in G(0)}$ with, $\|T\| = \sup_{x \in G(0)} \|T_x\|$. We use this to get easier access to the norm by explicitly constructing each $M_x$. To do this, we construct an inner product for each $x \in G(0)$:

$$\langle \zeta, \eta \rangle_x = (\zeta^* * \eta)(x).$$

This defines an inner product on $C_c(G_x)$, which we can use to complete. We denote this completion by $L^2(G_x)$. This gives us the natural field of Hilbert spaces we were looking for, namely: $\{L^2(G_x)\}_{x \in G(0)}$. It also gives us a natural representation of $C_c(G)$ given by $\lambda_x(f)(\eta) = (f * \eta)(x)$. Hence we can conclude that $\|f\| = \sup_{x \in G(0)} \|\lambda(f)\| = \|\lambda(f)\|$. From this we can complete $C_c(G)$ in either the norm on $L^2(G)$ or the family of norms $\{L^2(G_x)\}_{x \in G(0)}$, getting the same completion, denoted by $C^*_r(G)$. The proof of the result outlined above can be found in [KS02, Theorem 2.3].

The following Proposition allows for density arguments; this is one of the major virtues to working over such a field rather than directly with the module structure on $L^2(G)$.
Proposition 1.9. [KS02, Corollary 2.4] Let $D$ be a dense subset of $\mathcal{G}^{(0)}$. Then the norm $\|f\| = \sup_{x \in D}\{\|\lambda_x(f)\|\}$

1.3. The Coarse Groupoid of a Metric Space. Let $X$ be a uniformly discrete bounded geometry (sometimes denoted uniformly locally finite) metric space. We want to define a groupoid with property that it captures the coarse information associated to $X$. To do this effectively we need to define the what we mean by a “coarse structure” associated to a metric.

Definition 1.10. Let $X$ be a set and let $\mathcal{E}$ be a collection of subsets of $X \times X$. If $\mathcal{E}$ has the following properties:

1. $\mathcal{E}$ is closed under finite unions;
2. $\mathcal{E}$ is closed under taking subsets;
3. $\mathcal{E}$ is closed under the induced product and inverse that comes from the groupoid product on $X \times X$.
4. $\mathcal{E}$ contains the diagonal

Then we say $\mathcal{E}$ is a coarse structure on $X$ and we call the elements of $\mathcal{E}$ entourages. If in addition $\mathcal{E}$ contains all finite subsets then we say that $\mathcal{E}$ is weakly connected.

For a given family of subsets $\mathcal{S}$ of $X \times X$ be can consider the smallest coarse structure that contains $\mathcal{S}$. This is the coarse structure generated by $\mathcal{S}$. We can use this to give some examples of coarse structures.

Example 1.11. Let $X$ be a metric space. Then consider the collection $\mathcal{S}$ given by the $R$-neighbourhoods of the diagonal in $X \times X$; that is, for every $R > 0$ the set:

$$\Delta_R = \{(x, y) \in X \times X | d(x, y) \leq R\}$$

Then let $\mathcal{E}$ be the coarse structure generated by $\mathcal{S}$. This is called the metric coarse structure on $X$. It is a uniformly locally finite proper coarse structure that is weakly connected when $X$ is a uniformly discrete bounded geometry (proper) metric space.

Example 1.12. Let $G$ be a group and let $X$ be a right $G$-set. Define:

$$\Delta_g = \{(x, x.g) | x \in X\}$$

We call the coarse structure generated by the family $\mathcal{S} := \{\Delta_g | g \in G\}$ the group action coarse structure on $X$. If $X$ is not a connected space; consider for example a space of graphs $X = \bigsqcup X_i$, the group action coarse structure need not be weakly connected.

Definition 1.13. Let $X$ be a coarse space with a coarse structure $\mathcal{E}$ and consider $\mathcal{S}$ a family of subsets of $\mathcal{E}$. We say that $\mathcal{E}$ is generated by $\mathcal{S}$ if every entourage $E \in \mathcal{E}$ is contained in a finite union of subsets of $\mathcal{S}$.

In the situation that $X$ admits a transitive $G$-action by translations, the group action coarse structure generates the metric coarse structure.

To build a groupoid from the metric coarse structure on $X$ we consider extensions of the pair product on $X \times X$. The most natural way to do this is by making use of the entourages arising from the metric. The approach to this problem is through the following Lemma:
**Lemma 1.14.** [Roe03, Corollary 10.18] Let $X$ be a uniformly discrete bounded geometry metric space and let $E$ be any entourage. Then the inclusion $E \to X \times X$ extends to an injective homeomorphism $\overline{E} \to \beta X \times \beta X$, where $\overline{E}$ denotes the closure of $E$ in $\beta(X \times X)$.

Now we can make the definition of the coarse groupoid $G(X)$:

**Theorem 1.15.** ([Roe03, Theorem 10.20]) Let $X$ be a coarse space with uniformly locally finite, weakly connected coarse structure $E$. Define $G(X) := \bigcup_{E \in E} E$. Then $G(X)$ is a locally compact, étale groupoid with the induced product, inverse and topology from $\beta X \times \beta X$.

As we are considering the metric coarse structure we can reduce this to considering only generators:

$$G(X) := \bigcup_{R > 0} \Delta_R$$

The following is an integral part of [STY02] and proofs of the quoted results can be found in [Roe03].

**Theorem 1.16.** Let $X$ be a uniformly discrete bounded geometry metric space. Then following hold:

1. $G(X)$ is an étale locally compact Hausdorff principal topological groupoid with unit space $G(X)^{(0)} = \beta X$. [Roe03 Theorem 10.20][STY02 Proposition 3.2];
2. $C^*_\text{r}(G(X))$ is isomorphic to the uniform Roe algebra $C^*_\text{u}(X)$. [Roe03 Proposition 10.29];
3. The coarse Baum-Connes conjecture for $X$ is equivalent to the Baum-Connes conjecture for $G(X)$ with coefficients in $\ell^\infty(X, K)$. [STY02 Lemma 4.7].

So this lets us appeal to the theory of groupoids to conclude coarse information about a given metric space $X$. In fact, this is precisely the strategy of [HLS02] when it comes to dealing with counterexamples to the coarse Baum-Connes conjecture.

### 1.4. Expanders, Asymptotic Coverings and Ghost operators.

**Definition 1.17.** Let $\{X_i\}$ be a sequence of finite graphs. Then $X = \sqcup_{i \in \mathbb{N}} X_i$ equipped with the metric such that $d_X(X_i, X_j) \to \infty$ as $i + j \to \infty$ and $d_X|_{X_i} = d_{X_i}$ is called the space of graphs associated with the sequence $\{X_i\}$.

In this paper we will be considering only sequences that grow in size, that is $|X_i| \to \infty$ as $i \to \infty$.

We denote the girth of a finite graph $X$ by $\text{girth}(X)$, by which we mean the length of the shortest simple cycle of the graph. We say a sequence of graphs has large girth if $\text{girth}(X_i) \to \infty$ as $i \to \infty$. Another way to think of large girth sequences is that they are the only sequences for which the universal covering sequence $\{\tilde{X}_i, p_i\}$ is asymptotically faithful, that is that for any $R > 0$ there exists an $n \in \mathbb{N}$ such that for all $i > n$ we have $x_i \in \tilde{X}_i$ such that $B_R(x_i) \cong B_R(p_i(x_i))$.

As we are going to be concerning ourselves with counterexamples to the coarse Baum-Connes conjecture, we need to consider a class of sequences known as expanders. For a sequence of
graphs \( \{X_i\} \) we measure the connectedness of each of the finite graphs \( X_i \) in our sequence using a weighted graph laplacian \( \Delta_i \), a bounded linear operator on \( \ell^2(V(X_i)) \) defined pointwise by:

\[
(\Delta_i f)(x) = f(x) - \sum_{d(x,y) = 1} \frac{f(y)}{\sqrt{\deg(x)\deg(y)}}
\]

If each \( X_i \) were a regular graph then this would reduce to the traditional graph laplacian; in the above equation we are weighting by the degree of each vertex. For the sequence, being an expander is then a condition on the spectral properties of the laplacians \( \Delta_i \).

**Definition 1.18.** Let \( \{X_i\} \) be a sequence of finite graphs and let \( X \) be the associated space of graphs. Then the space \( X \) (or the sequence \( \{X_i\} \)) is an expander if:

1. There exists \( k \in \mathbb{N} \) such that all the vertices of each \( X_i \) have degree at most \( k \).
2. \( |X_i| \to \infty \) as \( i \to \infty \).
3. There exists \( c > 0 \) such that \( \text{spectrum}(\Delta_i) \subseteq \{0\} \cup [c, 1] \) for all \( i \).

Expanders have many practical applications as well as providing us with an avenue to build counterexamples to the coarse Baum-Connes conjecture. In particular large girth sequences are integral for the construction of so called Gromov monster groups, which are finitely generated randomly constructed groups that contain a coarsely embedded expander [Gro03, AD08]. Such groups provide counterexamples to the Baum-Connes conjecture with coefficients, a motivation for their construction.

**Remark 1.19.** Each Laplacian \( \Delta_i \) has propagation 1, so we can form the product in the (algebraic) Roe algebra:

\[
\Delta := \prod_i (\Delta_i \otimes q) \in C^*_u(X) \otimes \mathcal{K} \subset C^*X
\]

For an expander \( X \) we have \( \text{spectrum}(\Delta) \subseteq \{0\} \cup [c, 1] \) for some \( c > 0 \). So we can consider the limit:

\[
p = \lim_{t \to \infty} e^{-t\Delta}
\]

The spectral gap in \( \text{spectrum}(\Delta) \) tells us that this limit converges in the strong operator topology and is a projection associated to the \( 0 \in \text{spectrum}(\Delta) \). \( p \) is the kernel of \( \Delta \), and is one method of producing the projection onto the constant functions on each \( X_i \). In fact this projection breaks up as:

\[
p = \prod_i p^{(i)}
\]

But the key point is that it is an element of \( C^*X \) (in fact an element of \( C[X] \)), and each \( p^{(i)} \) is the projection onto the constant functions in \( \ell^2(X_i) \).

The following notion is due to Guoliang Yu (unpublished):

**Definition 1.20.** An operator \( T \in C^*X \) is a ghost operator if \( \forall \epsilon > 0 \) there exists a bounded subset \( B \subset X \times X \) such that the norm: \( \|T_{xy}\| \leq \epsilon \) for all \( (x,y) \in (X \times X) \setminus B \).

The projection \( p \) built from the Laplacian \( \Delta \) is a ghost in \( C^*X \) [WY12a, Example 5.3]. It has a very definite global presence as it is a non-compact projection but locally it is invisible by the ghost property. For more details of this can be found in [WY12a, Section 5].
2. Counterexamples to the coarse Baum-Connes conjecture and Boundary Groupoids

Throughout this section let $G$ be an étale, Hausdorff locally compact topological groupoid. These are essentially unnecessary assumptions but we will only require groupoids in this class to fit with the coarse picture we are interested in. We outline the main concept introduced in [HLS02].

**Definition 2.1.** A subset of $F \subseteq G^{(0)}$ is said to be *saturated* if for every element of $\gamma \in G$ with $s(\gamma) \in F$ we have $r(\gamma) \in F$. For such a subset we can form subgroupoid of $G$, denoted by $G_F$ which has unit space $F$ and $G_F^{(2)} = \{ \gamma \in G | s(\gamma) \text{ and } r(\gamma) \in F \}$.

We will be considering closed saturated subsets $F$. We remark also that the compliment $F^c$ is an open saturated set. The motivation from our point of view for considering closed saturated subsets is the decomposition:

$$G = G_{F^c} \sqcup G_F$$

This lets us construct maps on the $*$-algebras of compactly supported functions associated with $G, G_F$ and $G_{F^c}$:

$$0 \to C_c(G_{F^c}) \to C_c(G) \to C_c(G_F) \to 0.$$ 

Where the quotient map $C_c(G) \to C_c(G_F)$ is given by restriction and the inclusion $C_c(G_{F^c}) \to C_c(G)$ is given by extension. By the functorial properties of the maximal $C^*$-norm this extends to the maximal groupoid $C^*$-algebras:

$$0 \to C^*_{\text{max}}(G_{F^c}) \to C^*_{\text{max}}(G) \to C^*_{\text{max}}(G_F) \to 0.$$ 

On the other hand this may fail to be an exact sequence when we complete in the norm that arises from any specific representation, for example the left regular representation $\lambda_G$; this can be detected at the level of K-theory, as discussed in [HLS02], by considering the sequence:

$$K_0(C^*_r(G_U)) \to K_0(C^*_r(G)) \to K_0(C^*_r(G_F))$$

This was used in [HLS02] to construct multiple different types of counterexample to the Baum-Connes conjecture for groupoids - each of which invokes the following Lemma:

**Lemma 2.2.** ([HLS02, Lemma 1]) Assume the sequence [1] is not exact at its middle term.

1. If the Baum-Connes map $K^{\text{top}}_0(G_F) \to K_0(C^*_r(G_F))$ is injective then the Baum-Connes map $K^{\text{top}}_0(G) \to K_0(C^*_r(G))$ fails to be surjective.

2. If the map $K_0(C^*_{\text{max}}(G_F)) \to K_0(C^*_r(G_F))$ is injective then the map $K_0(C^*_{\text{max}}(G)) \to K_0(C^*_r(G))$ fails to be surjective and a fortiori the Baum-Connes map $K^{\text{top}}_0(G) \to K_0(C^*_r(G))$ fails to be surjective.

We observe that whilst the sequence:

$$0 \to C^*_r(G_{F^c}) \xrightarrow{\alpha} C^*_r(G) \xrightarrow{q} C^*_r(G_F) \to 0$$


may fail to be exact in the middle term the maps $\alpha$ and $q$ both exist and we can see that the map $q$ is also surjective by considering the following diagram.

$$
\begin{array}{ccc}
C^*_\text{max}(G) & \longrightarrow & C^*_\text{max}(G_F) \\
\downarrow & & \downarrow \\
C^*_*(G) & \longrightarrow & C^*_*(G_F)
\end{array}
$$

It is also clear that the image of $\alpha$ is contained in the kernel of $q$, whence we can make the sequence exact artificially by replacing $C^*_*(G_F)$ by the ideal $I := \ker(q)$. We can then define a new assembly map in the first term to be the composition of the original assembly map $\mu_{F^c}$ and the K-theory map induced by inclusion $i_* : K_*(C^*_*(G_F)) \to K_*(I)$. Then in terms of assembly maps this gives us a new commutative diagram:

$$
\begin{array}{cccccccc}
& & K_1(C^*_*(G_F)) & \longrightarrow & K_0(I) & \longrightarrow & K_0(C^*_*(G)) & \longrightarrow & K_0(C^*_*(G_F)) & \longrightarrow & K_1(I) & \longrightarrow & \\
& & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
& & K_1^{\text{top}}(G_F) & \longrightarrow & K_0^{\text{top}}(I) & \longrightarrow & K_0^{\text{top}}(G) & \longrightarrow & K_0^{\text{top}}(G_F) & \longrightarrow & K_1^{\text{top}}(I) & \longrightarrow & \\
\end{array}
$$

where the rows here are exact. As in [HLS02] we would now choose suitable groupoids $G$ and subsets $F$ of the unit space $G^{(0)}$ that allow us to use the above sequence to analyse the Baum-Connes conjecture for the groupoid $G$. We have in mind the situation that $G = G(X)$, the coarse groupoid associated to some uniformly discrete bounded geometry metric space $X$.

2.1. The Coarse Groupoid Conjecture. Let $X$ be a uniformly discrete bounded geometry metric space. From what was described above we can associate to each closed saturated subset $F$ of the unit space space $\beta X$ a long exact sequence in K-theory. We consider the obvious closed saturated subset: $\partial \beta X \subset G^{(0)}$. This gives us the following commutative diagram (omitting coefficients):

$$
\begin{array}{cccccccc}
K_1(C^*_*(G(X)|_{\partial \beta X}) & \longrightarrow & K_0(I) & \longrightarrow & K_0(C^*_*(G(X))) & \longrightarrow & K_0(C^*_*(G(X)|_{\partial \beta X}) & \longrightarrow & K_1(I) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
K_1^{\text{top}}(G(X)|_{\partial \beta X}) & \longrightarrow & K_0^{\text{top}}(X \times X) & \longrightarrow & K_0^{\text{top}}(G(X)) & \longrightarrow & K_0^{\text{top}}(G(X)|_{\partial \beta X}) & \longrightarrow & K_1^{\text{top}}(X \times X) \\
\end{array}
$$

We can now properly formulate the boundary conjecture (replacing the coefficients):

**Conjecture 1** (Boundary Coarse Baum-Connes Conjecture). Let $X$ be a uniformly discrete bounded geometry metric space. Then the assembly map:

$$
\mu_{\partial \beta X} : K^*_*(G(X)|_{\partial \beta X}, l^\infty(X, K)/C_0(X, K)) \to K_*(l^\infty(X, K)/C_0(X, K)) \rtimes_r G(X)|_{\partial \beta X}
$$

is an isomorphism.

As mentioned in the introduction the intuitive view of this is supposed to be “quotient by the ghost ideal and consider the K-theory”. We confirm the technical approach meets the intuitive one; the kernel $I$ is precisely the ghost ideal $I_G$. In order to prove this we need some technology of [HLS02]:

**Lemma 2.3.** [HLS02] Lemma 9) If an étale topological groupoid $\mathcal{G}$ acts on a $C^*$-algebra $A$, then the map $C_c(\mathcal{G}, A) \to C_0(\mathcal{G}, A)$ extends to an injection (functorial in $A$) from $A \rtimes_r \mathcal{G}$ to $C_0(\mathcal{G}, A)$. \qed
Proposition 2.7. Let the map:

The downward maps being injective implies that the kernel is precisely the kernel of induced

behaviour of the inclusion

composition of \( \mu \). Recall that the assembly map

topology we can replace compact by bounded. So:

\[ C_0(\mathcal{G}, A) \rightarrow C_0(\mathcal{G}, B) \]

Remark 2.5. The map provided is not a \(*\)-homomorphism as it takes convolution in \( A \rtimes_r \mathcal{G} \)
to pointwise multiplication in \( C_0(\mathcal{G}, A) \). However it suffices for applications as the map is
continuous.

Proposition 2.6. Let \( X \) be a uniformly discrete bounded geometry metric space. Then the
kernel of the map

is the ghost ideal \( I_G \).

Proof. Lemma 2.3 implies that the following diagram commutes:

The downward maps being injective implies that the kernel is precisely the kernel of induced map:

\[ q : C^* X = \ell^\infty(X, \mathcal{K}) \rtimes_r G(X) \rightarrow C_0(G(X), \ell^\infty(X, \mathcal{K})/C_0(X, \mathcal{K})). \]

We can compute this kernel:

\[ I = \{ f \in C^* X | i(q(f)) = 0 \} \]

\[ = \{ f \in C^* X | q'(i'(f)) = 0 \} \]

\[ = \{ f | q'(f) \in C_0(G(X), C_0(X, \mathcal{K})) \} \]

\[ = \{ f | \forall \epsilon > 0 \exists K \subset X \times X \text{ compact} : |f_{xy}| \leq \epsilon \forall (x, y) \in X \times X \setminus K \}. \]

As \( X \) is uniformly discrete with bounded geometry and \( X \times X \) is equipped with the product
topology we can replace compact by bounded. So:

\[ I = \{ f | \forall \epsilon > 0 \exists K \subset X \times X \text{ bounded} : |f_{xy}| \leq \epsilon \forall (x, y) \in X \times X \setminus K \} = I_G. \]

Recall that the assembly map \( \mu_{I_G} \) associated to the open saturated subset \( X \) is given by the
composition of \( \mu_{X \times X} : K^\text{top}_{**}(X \times X) \rightarrow K_*(\mathcal{K}(\ell^2(X))) \) with the inclusion \( i_* : K_*(\mathcal{K}(\ell^2(X)) \rightarrow K_*(I_G) \). So to understand how the assembly map \( \mu_{I_G} \) behaves, it is enough to consider the behaviour of the inclusion \( i_* \) as the map \( \mu_{X \times X} \) is an isomorphism.

Proposition 2.7. Let \( \{ X_i \}_{i \in \mathbb{N}} \) be an expander sequence. Then:

\[ K^\text{top}_{**}(X \times X) \cong K_*(\mathcal{K}(\ell^2(X))) \rightarrow K_*(I_G) \]

is not surjective but is injective.
Proof. The proof is an adaption of a well known argument and relies on considering the algebra $C^*X_\infty$, which is the Roe algebra of the space of graphs with the disjoint union 'metric', and the following commutative diagram:

\[
\begin{array}{c}
\bigoplus_{i \in \mathbb{N}} \mathcal{K}(\ell^2(X_i)) \\
\oplus_{i \in \mathbb{N}} C^*X_i \\
\prod_{i \in \mathbb{N}} C^*X_i
\end{array}
\xrightarrow{q} \begin{array}{c}
C^*X_\infty \\
\bigoplus_{i \in \mathbb{N}} C^*X_i \\
\prod_{i \in \mathbb{N}} C^*(X_i)
\end{array}
\]

We remark that we can conclude $C^*X = C^*X_\infty + \mathcal{K}$. Let $I_{G,\infty} = I_G \cap C^*X_\infty$. Then by the second isomorphism theorem:

\[
\frac{I_G}{\mathcal{K}(\ell^2(X))} \cong \frac{I_{G,\infty}}{\bigoplus_{i \in \mathbb{N}} \mathcal{K}(\ell^2(X_i))}
\]

We define the map $d$ to be the compositions of $q$ and the inclusion $i$. The map $d$ induces a map on $K$-theory that is used to detect non-triviality of certain classes of projections - namely those associated to expander sequences [WY12a, Section 6]. In particular we observe that $d$ restricts to a map:

\[
d|_{G} : I_{G,\infty} \rightarrow \prod_{i \in \mathbb{N}} C^*(X_i) \bigoplus_{i \in \mathbb{N}} C^*X_i
\]

In particular: if $d|_{G,*}([p]) \neq 0$ then $q_*([p]) \neq 0$. Let $p$ be the ghost projection associated to Laplacian $\Delta$ of the $X_i$s, defined from the Laplacians $\Delta_i$ in remark 1.19. This lies in $C^*X$ as $X$ is an expander, and is clearly an element of $C^*X_\infty$ as it is defined piecewise. $p$ evalutates to a non-trivial class under $d_{G,*}$ so we know that $K_0(\prod_{i \in \mathbb{N}} C^*X_i) \neq 0$. From here we see that $K_0(\mathcal{K}) \rightarrow K_0(I_G)$ is not surjective and so $\mu_{I_G}$ is not surjective either.

To see injectivity it suffices to show that $K_1(I_G) \rightarrow K_0(\mathcal{K})$ is the zero map. Consider the following diagram:

\[
\begin{array}{c}
\bigoplus_{i \in \mathbb{N}} M_i \\
\oplus_{i \in \mathbb{N}} M_i^\mathcal{C} \\
\prod_{i \in \mathbb{N}} M_i
\end{array}
\xrightarrow{\mathcal{K}} \begin{array}{c}
I_{G,\infty} \\
I_{G,\infty} \\
I_{G,\infty}
\end{array}
\xrightarrow{0} \begin{array}{c}
C^*X_\infty \\
C^*X_\infty \\
C^*X_\infty
\end{array}
\xrightarrow{\prod_{i \in \mathbb{N}} M_i} \begin{array}{c}
I_{G,\infty} \\
I_{G,\infty} \\
I_{G,\infty}
\end{array}
\]
The boundary coarse Baum-Connes conjecture has applications to the coarse Baum-Connes conjecture for spaces of graphs:

**Proposition 2.8.** If $X$ satisfies the boundary coarse Baum-Connes conjecture then the following hold:

1. The coarse Baum-Connes assembly map for $X$ is injective.
2. If $X$ is an expander then the coarse Baum-Connes assembly map for $X$ fails to be surjective.

**Proof.** Consider the long exact sequences:

$$
\begin{align*}
K_1(C^*_r(G(X)|_{\partial X}) &\longrightarrow K_0(I_G) \rightarrow K_0(C^*_r(G(X))) \rightarrow K_0(C^*_r(G(X)|_{\partial X})) \rightarrow K_1(I_G) \\
K^\text{top}_1(G(X)|_{\partial X}) &\rightarrow K^\text{top}_0(X \times X) \rightarrow K^\text{top}_0(G(X)) \rightarrow K^\text{top}_0(G(X)|_{\partial X}) \rightarrow K^\text{top}_1(X \times X)
\end{align*}
$$

We remark here that $f_1$ is not surjective and that $f_1$ and $f_3$ are injective by Proposition 2.7.

The assumptions and these remarks, coupled with the Five Lemma, conclude the proof. □

In fact, the previous result can be improved by considering how the K-theory of the compact operators sits inside the K-theory of the Roe algebra.

**Proposition 2.9.** $K_0(K) \hookrightarrow K_0(C^*_X)$.

**Proof.** As in the proof of Proposition 2.7 we can just work with the subalgebras $\bigoplus_i M_{n_i}$ and $C^*X_\infty$. We observe that we have a similar diagram:

$$
\begin{array}{c}
\bigoplus_{i \in \mathbb{N}} M_i \\
\bigoplus_{i \in \mathbb{N}} M_i
\end{array} \longrightarrow C^*X_\infty \longrightarrow \prod_{i \in \mathbb{N}} M_i
\begin{array}{c}
\bigoplus_{i \in \mathbb{N}} M_i \\
\bigoplus_{i \in \mathbb{N}} M_i
\end{array}
$$

As the long bottom arrow is certainly injective on K-theory we can see that the arrow into $C^*X_\infty$ is also injective on K-theory. From this we can deduce, using a similar argument to Proposition 2.7 that this map actually induces an injection on K-theory between the compact operators $K$ and $C^*X$. □

Combining this proposition with the fact that the assembly map $\mu_{IG}$ factors through the standard assembly map $\mu_{X \times X}$ we can conclude:

**Proposition 2.10.** Let $X$ be a space of graphs for some sequence $\{X_i\}$ and assume $\mu_{\text{bdry}}$ to be injective. Then:

1. The coarse Baum-Connes assembly map $\mu$ is injective.
2. If $X$ is an expander then the coarse Baum-Connes assembly map fails to be surjective.
Proof. Consider the diagram:

\[
\begin{array}{cccc}
\prod_{i \in \mathbb{N}} K_0(C^*_i X_i) & \xrightarrow{d_s} & K_1(C^*_r(G(X)|_{\partial \beta X}) & \xrightarrow{\mu} \\downarrow \\
\bigoplus_{i \in \mathbb{N}} K_0(C^*_i X_i) & & K_0(I_G) & \xrightarrow{0} \\
\end{array}
\]

We prove (1) by considering an element \( x \in K_0^{top}(G(X)) \) such that \( \mu(x) = 0 \). Then \( x \) maps to \( 0 \in K_0^{top}(G(X)|_{\partial \beta X}) \) and so comes from an element \( y \in \mathbb{Z} \). Each square commutes hence \( y \) maps to \( 0 \in K_0(C^*_r(G(X))) \). As the composition up and left (as indicated in the diagram) is injective by Proposition 2.9, we know that \( y \in \mathbb{Z} \) is in fact \( 0 \in \mathbb{Z} \). Hence \( x = 0 \).

To see (2) we use Lemma 2.2 part 1 or directly: Take any non-compact ghost projection \( p \in K_0(I_G) \), which does not lie in the image of \( \mathbb{Z} \). Push this element to \( q \in K_0(C^*_r(G(X))) \). Assume for a contradiction that \( \mu \) is surjective. Then there is an element \( x \in \mathbb{Z} \) that maps to \( q \), and so \( d_s(q) = 0 \), as the image of compact operators lies in the kernel of the map \( d_s \). However, we know that \( d_s(q) = d_s(p) \) is certainly non-zero. \( \square \)

2.2. Tools to Prove the Boundary Conjecture. In general it is very hard to compute what happens in the boundary as our geometric intuition breaks down when considering non-principal ultrafilters. However we can salvage something when we have more information about the global geometry, for instance by requiring the space in question to admit a group action.

It is not always the case however that a group action is sufficient to capture all of the large scale information, for example a group \( \Gamma \) acting on a space of graphs \( X = \sqcup X_i \) so that the action preserves the decompositon, that is: \( \Gamma(X_i) = X_i \) for all \( i \in \mathbb{N} \). In this context the coarse structure generated by the action will not be weakly connected, but the metric coarse structure is weakly connected and so cannot be generated by the group action. To that end we introduce a definition:

**Definition 2.11.** Let \( \mathcal{E} \) be a coarse structure on \( X \) and let \( S \) be a family of subsets \( \mathcal{E} \). We say that \( S \) generates \( \mathcal{E} \) at infinity if for all \( E \in \mathcal{E} \):

\[
E \subseteq \bigcup_{k=1}^{n} S_k \cup F
\]

Where each \( S_k \in S \) and \( F \) is a finite subset of \( X \times X \).

**Remark 2.12.** The above definition is equivalent to asking that \( \overline{E} \setminus E \subseteq \bigcup_{k=1}^{n} \overline{S_k} \setminus S_k \), where the closure takes place in \( \beta X \times \beta X \) - as in Section 1.3.

Recall that \( \Delta_g := \{(x,x.g)|x \in X\} \) is the \( g \)-diagonal in \( X \).

**Proposition 2.13.** Let \( X \) be uniformly discrete bounded geometry metric space and let \( \Gamma \) be a finitely generated discrete group. If \( \Gamma \) acts on \( X \) such that the induced action on \( \beta X \) is free on \( \partial \beta X \) and the action generates the metric coarse structure at infinity. Then \( G(X)|_{\partial \beta X} \cong \partial \beta X \rtimes \Gamma \).
Proof. Recall that under the hypothesis that the $\Gamma$ action generates the metric coarse structure at infinity we know that $G(X) = (\bigcup_g \Delta_g) \cup (X \times X)$.

We consider a map from transformation groupoid $\beta X \rtimes \Gamma$ to $G(X)$. Observe that $\Delta_g$ is the bijective image of the set $\{(x, g) | x \in X\}$. We extend this map to the respective closures, giving a map for each $g$ from the set $\{(\omega, g) | \omega \in \beta X\}$ to $\Delta_g$. Let $\{x_\lambda\}$ be a net of elements in $X$ that converge to $\omega$. Then clearly each pair $(x_\lambda, g)$ is mapped to $(x_\lambda, x_\lambda.g)$ under the identification. Consider the element of the closure $\gamma_g = \lim_{\lambda}(x_\lambda, x_\lambda.g) \in \Delta_g$ and define the extension of the bijection to be the map: $(\omega, g) \mapsto \gamma_g$.

This map is well-defined as $G(X)$ is principal and so $\gamma_g$ is completely determined by its source and range - in particular any other net $y_\lambda$ that converges to $\omega$ gives rise to the same element $\gamma_g$. We can then extend this over the entire groupoid $\beta X \rtimes \Gamma$ elementwise, where it is certainly continuous but not in general injective or surjective (injectivity would require a free action and surjectivity a transitive one).

The map is a groupoid homomorphism because $G(X)$ is principal; this follows as principality implies the following diagram commutes:

\[
\begin{array}{ccc}
\beta X \rtimes \Gamma & \longrightarrow & G(X) \\
\downarrow (r,s) & & \downarrow (r,s) \\
\beta X \times \beta X & \rightarrow & \beta X \times \beta X
\end{array}
\]

We now restrict this map to the boundary $\partial \beta X$. We prove that the coronas $\Delta_g \setminus \Delta_g$ are disjoint: let $\gamma \in \Delta_g \setminus \Delta_g$ and $\Delta_h \setminus \Delta_h$ for $g \neq h$. Then we have that $s(\gamma) = \omega, r(\gamma) = \omega.g = \omega.h$, hence we have a fixed point on the boundary, which is a contradiction. Consider the restricted diagram:

\[
\begin{array}{ccc}
\partial \beta X \rtimes \Gamma & \longrightarrow & G(\partial \beta X) \\
\downarrow (r,s) & & \downarrow (r,s) \\
\partial \beta X \times \partial \beta X & \rightarrow & \partial \beta X \times \partial \beta X.
\end{array}
\]

As the coronas are disjoint, for any $\gamma \in G(X)$ it is possible to find a unique $g$ and $\omega$ such that $s(\gamma) = \omega, r(\gamma) = \omega.g$. It then follows that the pair $(\omega, g)$ map onto $\gamma$. Hence the map is a surjection. To see injectivity, we appeal again to freeness of the action. The action being free implies that the groupoid $\partial \beta X \rtimes \Gamma$ is principal. From a brief consideration of the diagram above injectivity follows.

It remains to consider the topology. For each $g \in \Gamma$, the map from $\partial \beta X \rtimes \Gamma$ to $G(\partial \beta X)$ on the piece $[\partial \beta X, g]$ is a homeomorphism as $\Delta_g \setminus \Delta_g$ can be identified with $\partial \beta X$, induced by the source map or range map. Hence the map takes clopen sets to clopen sets; these form a basis for the topology of $G(\partial \beta X)$ and so the map is a global homeomorphism.

This proposition provides a collection of examples of sequences that we can deal with, and will in general be the conduit we want to pass through to verify the conjecture in the presence of a group action.

**Example 2.14.** (Box spaces) Let $\Gamma$ be a finitely generated residually finite group and let $\{N_i\}$ be a family of finite index normal subgroups such that $N_i \leq N_{i+1}, \bigcap_{i \in \mathbb{N}} N_i = 1$ and
a fixed generating set $S$. Then the sequence of groups $\{\Gamma/N_i\}$ with generating sets $\pi_i(S)$ admits an action via quotient maps. Let $\Box \Gamma = \bigcup_{i \in \mathbb{N}} \Gamma_{N_i}$, equipped with a metric that restricts to the metric induced from the generating sets $\pi_i(S)$ for each $i$, and has the property that $d(\Gamma_{N_i}, \Gamma_{N_j}) \to \infty$ as $i + j \to \infty$. This is called a box space for $\Gamma$. The Stone-Čech boundary admits a free action of the group (either we can see this via the profinite completion $\hat{\Gamma}$, or via building sets as we will see in the following section) and the metric structure is generated at infinity by the quotients maps and the right action of the group via these maps. So we can conclude the boundary groupoid $G(\Box \Gamma)|_{\partial \beta \Box \Gamma}$ is homeomorphic to $\partial \beta \Box \Gamma \rtimes \Gamma$.

**Example 2.15.** In fact, we can weaken this to any finitely generated residually finite group by considering Scheirer quotients by finite index subgroups. We call these generalised box spaces (reserving box space for a sequence of quotients by normal subgroups).

This gives us some examples of situations where we can immediately verify the conjecture. We remark that a group is said to have the Strong Baum-Connes property if it satisfies the Baum-Connes conjecture with arbitrary coefficients. In particular this includes all amenable, a-T-menable [HK97] and, by remarkable recent results of Lafforgue [Laf12], groups that act properly isometrically on weakly geodesic strongly hyperbolic metric spaces.

**Theorem 2.16.** The boundary coarse Baum-Connes conjecture holds for sequences of graphs that are generalised box spaces of residually finite discrete groups that have the Strong Baum-Connes property.

This covers in particular certain expanding sequences that come from property (T) groups or property ($\tau$) with respect to the corresponding family of finite index subgroups.

Explicitly this behaviour occurs for the sequence $\{SL_2(\mathbb{Z}/p^n\mathbb{Z})\}_{n \in \mathbb{N}}$; coming from congruence quotients in $SL_2(\mathbb{Z})$. In fact, this example motivates [OOY09] - this sequence of finite graphs has small girth as $SL_2(\mathbb{Z}) \cong C_4 \ast C_2 \ast C_6$ implies that the group has cycles of length 4 and 6 in its Cayley graph. However this also acts as an upper bound on cycle length - it otherwise looks like a tree as it is a virtually free group. In particular the space of graphs for any family like the one above is coarsely equivalent to one of large girth.

Proposition 2.10 on the other hand tells us that the coarse Novikov conjecture holds in much more generality than this.

**Theorem 2.17.** $\mu_{dry}$ from the boundary coarse Baum-Connes conjecture is injective for the generalised box spaces associated to all residually finite uniformly embeddable groups.

**Proof.** We can use Proposition 2.13 to decompose our groupoid $G(X)|_{\partial \beta X}$ as $\partial \beta X \rtimes \Gamma$ for any generalised box space $X$ of $\Gamma$. As $\Gamma$ is uniformly embeddable we can conclude that the conjecture for $G(X)|_{\partial \beta X}$ is injective as it is equivalent to a conjecture with coefficients for $\Gamma$, using [STY02, Theorem 6.1]. Proposition 2.10 then allows us to conclude that the coarse Novikov conjecture holds for $X$. \square

**Corollary 2.18.** Let $\Gamma$ be a residually finite uniformly embeddable group and let $X$ be a generalised box space of $\Gamma$. Then the following hold:

1. The coarse Novikov conjecture holds for $X$. 

(2) If $X$ is an expander, then the assembly map $\mu$ fails to be surjective.

Proof. Proof follows from Theorem 2.17 and Proposition 2.10. □

This includes property (T) groups such as $SL_3(\mathbb{Z})$, and hence tells us something for small girth expanders. Using the recent results of Sako on the relationship between property A and the operator norm localization property for uniformly discrete bounded geometry spaces [Sak] we get a simpler proof of [CTWY08, Theorem 7.1]. Given that any countable subgroup of $GL(n, K)$ is exact for any field $K$ [GHW05] we can also conclude [GTY11, Theorem 5.3].

2.3. Some Remarks on the Max Conjecture.

**Proposition 2.19.** Let $X$ be the space of graphs arising from a sequence of finite graphs $\{X_i\}$. Then

(1) the maximal coarse Baum-Connes assembly map is an isomorphism if and only if the maximal Boundary coarse Baum-Connes map is an isomorphism.

(2) the maximal coarse assembly map is injective if and only if the maximal boundary assembly map is injective.

Proof. As before we consider a diagram, this time of maximal algebras:

$$
\begin{array}{c}
K_1(C^*(G(X)|_{\partial X})) \longrightarrow K_0(K) \longrightarrow K_0(C^*(G(X))) \Rightarrow K_0(C^*(G(X)|_{\partial X})) \longrightarrow K_1(K) \\
\mu_{\partial X} \downarrow \quad \downarrow 2\|4 \\
K_1^{top}(G(X)|_{\partial X}) \longrightarrow K_0^{top}(X \times X) \longrightarrow K_0^{top}(G(X)) \longrightarrow K_0^{top}(G(X)|_{\partial X}) \longrightarrow K_1^{top}(X \times X)
\end{array}
$$

Both parts follow from a diagram chase. □

**Theorem 2.20.** The maximal coarse Baum-Connes assembly map is an isomorphism for a space of graphs whose boundary groupoid decomposes as $\partial X \rtimes \Gamma$ where $\Gamma$ is a finitely generated discrete group with the Haagerup property.

Proof. The Haagerup property provides us with the Strong Baum-Connes property for the maximal Baum-Connes conjecture for $\Gamma$ with coefficients. It then follows from Proposition 2.19. □

This result captures completely [OYY09, Corollary 4.18], but the proof is very much more elementary.

It is obvious that for any space for which the coarse Novikov conjecture holds that the maximal coarse Novikov conjecture also holds. Hence we can see:

**Theorem 2.21.** Let $X$ be a generalised box space of a residually finite uniformly embeddable group $\Gamma$. Then the maximal coarse boundary assembly map for $X$ is injective. □

This result is related to the content of [GWY08, Theorem 5.1].
3. Main Theorem

The aim of the remainder of the paper is to prove the following result:

**Theorem 3.1.** The boundary coarse Baum-Connes conjecture holds for spaces of graphs with large girth and uniformly bounded vertex degree.

In the special case that each graph \( X_i \) in the sequence is \( 2k \)-regular we will show that the associated space of graphs \( X \) admits an action of the free group \( F_k \) that is free at infinity. Proposition 2.13 then implies that the boundary Baum-Connes conjecture for \( X \) is then a special case of the Baum-Connes conjecture with coefficients for the free group \( F_k \).

For the general case, we will only be able to construct a partial action of \( F_k \) for some finite \( k \). In this case we will show that the boundary groupoid \( \mathcal{G}(X)|_{\partial X} \) is a transformation groupoid \( \partial X \rtimes \mathcal{G}_X \) for some étale groupoid \( \mathcal{G}_X \). As before, this converts the boundary Baum-Connes conjecture into a special case of Baum-Connes with coefficients for the groupoid \( \mathcal{G}_X \). Lastly, we construct a continuous, proper groupoid homomorphism from \( \mathcal{G}_X \) to \( F_k \), whence the Haagerup property transfers from \( F_k \) to the groupoid \( \mathcal{G}_X \), allowing us to conclude the boundary Baum-Connes conjecture in this case.

We recall Pedersen’s Lemma \([\text{Køn90, Theorem 7, Chapter XI}]\):

**Lemma 3.2.** Let \( X \) be a finite graph. If \( 2k \) edges go into any vertex then all the edges of \( X \) can be divided into \( k \) classes such that two edges from the same class go into any vertex. \( \square \)

**Definition 3.3.** Let \( X \) be a finite \( 2k \)-regular graph. A \( k \)-orientation is a choice of edge orientation and labelling in letters \( a_1, \ldots, a_k \) that are compatible in the sense that precisely one edge oriented into and out of a vertex is labelled \( a_i \) for all \( i \).

Pedersen’s Lemma 3.2 provides us an avenue to construct \( k \)-orientations. We record this in a Lemma below.

**Lemma 3.4.** Every finite \( 2k \)-regular graph \( X \) can be \( k \)-oriented. Such a graph admits an action of the free group of rank \( k \) on the right by translations.

*Proof.* The fact that any finite \( 2k \)-regular graph can be \( k \)-oriented follows from Lemma 3.2 that tells us we can partition our edge set into \( k \)-pieces such that every vertex is incident on exactly two edges in each of the \( k \) partitioning sets. We can then orient this by mapping to a \( k \)-leafed bouquet and this covering map induces an inclusion of finite index of the group of deck transformations \( \tilde{G}(X) \) into \( \pi_1(\bigvee_{i=1}^k S^1) = F_k \), whose action is on the left by isometries on the Cayley graph of \( F_k \). So we allow \( F_k \) to act on itself on the right - this commutes with the left action and hence passes to the left quotient by \( \tilde{G}(X) \) giving us an action by translations on \( \tilde{G}(X) \setminus F_k = X \). \( \square \)

**Remark 3.5.** For the \( 4 \)-regular case we can get this result by appealing to the Eulerian tour that exists for all \( 2k \)-regular graphs; we label around any such tour using the letters \( a \) and \( b \) alternatively. This provides us almost with what we want as from this we then re-label so that every vertex has the orientation of a ball of radius 1 in the free group \([\text{Hat02, pg 57}]\). It is not easy to produce a labelling of an Euler tour that is compatible with the necessary labelling we are after for any \( k \) larger than two however.
Remark 3.6. Let $X$ be a finite $2k$-regular graph. A $k$-orientation can be thought of as providing a recipe to understand the action provided by the free group. Every vertex has entering (and leaving) precisely one edge labelled in the $a_1, \ldots, a_k$ and every undirected, not necessarily simple, path in the finite graph is now labelled in the letters $a_1, \ldots, a_k$ and has some assigned orientations. To describe the action, take a word in the free group with reduced form $w = \prod_i a_i^{e_i}$ and let $v \in V(X)$. Then we can apply $w$ to $v$ simply by following a walk along the letters $a_i$ that make up $w$. In addition, if we choose two vertices $x, y$ connected by some path, that path is now labelled and oriented and by reading the labels from this path we will attain an element of the free group that takes us from $x$ to $y$.

Lemma 3.7. Let $\{X_i\}$ be a sequence of finite connected graphs that have $|X_i| \to \infty$ as $i \to \infty$ and are $2k$-regular and let $X$ be the associated space of graphs. Then the action of $F_k$ generates the metric coarse structure on $X$ at infinity.

Proof. Let $(x, y) \in \Delta_R$ with $x, y \in X_i$ for some $i$. Then they are joined by a path that as a consequence of a $k$-orientation is labelled in the generators of $F_k$ and has assigned orientations. This provides us enough information to read the action of $F_k$, whence there is a $w \in F_k$ that takes $x$ to $y$. This implies $(x, y) \in \Delta_w$. Let $F$ be the finitely many pairs $(x, y) \in \Delta_R$ that come from distinct $X_i$s. Then $\Delta_R$ decomposes as:

$$\Delta_R = F \cup \bigcup_{|w| \leq R} \Delta_w$$

The intuition for this action at infinity can be gathered from the ultralimit of the sequence in the following way: If sequences of points in each $X_i$, when viewed as subsets of $X$, are fixed then the action is not free. An asymptotically faithful covering sequence tells us that no sequence is fixed.

Lemma 3.8. The action on $X$ of $F_k$ extends to $\beta X$ and is free on $\partial \beta X$.

Proof. Firstly, the action is continuous as we are acting on a discrete space $X$, hence it extends to a continuous action on the Stone-Čech compactification $\beta X$. We now deal with the second part of the claim.

Let $g \in F_k$ and for each $i$ fix a basepoint $x_i \in X_i$. As the graph is finite there exists an $n_i$ such that $g^{n_i}$ translates $x_i$ to itself. We assume that there is only a single orbit for the purposes of the following argument as the case of multiple orbits is similar. This gives us an action of $\Z/n_i\Z$ on $X_i$ for each $i$. As the girth of the $X_i$ tends to infinity as $i$ does, we know that the $n_i$ here also tends to infinity as $i$ does, otherwise there would be a cycle for each $i$ that was of bounded length. This is not enough to prove freeness however, we require an argument involving arbitrary subsets of $X$.

We observe also that there are two cases as for any $\omega \in \partial \beta X$ as we know that the pieces:

$$X_{\text{even}} = \bigsqcup_{n_i \equiv 0 \mod 2} X_i$$

$$X_{\text{odd}} = \bigsqcup_{n_i \equiv 1 \mod 2} X_i$$
are mutually complimentary and union to the entire of $X$, hence $\omega$ picks either $X_{\text{even}}$ or $X_{\text{odd}}$.

For the even case break the space into two complimentary pieces in the following way:

$$A_{i,0} := \{x_i g^n | n \equiv 0 \ mod \ 2\}$$
$$A_{i,1} := \{x_i g^n | n \equiv 1 \ mod \ 2\}$$

and let $A_j = \bigcup_{i | n_i \in \{\text{even}\}} A_{i,j}$. We assume for a contradiction that $\omega = \omega . g$ and then observe that $g$ permutes $A_0$ to $A_1$, so if, without loss of generality, $A_0 \in \omega$ we can deduce that $A_0 . g = A_1 = A_0^g \in \omega$, which is a contradiction.

The odd case is similar only we break each $X_i$ represented into three pieces:

$$B_{i,0} := \{x_i g^n | n \equiv 0 \ mod \ 2 \text{ and } n \neq n_i - 1\}$$
$$B_{i,1} := \{x_i g^n | n \equiv 1 \ mod \ 2\}$$
$$B_{i,2} := \{x_i^{n_i - 1}\}$$

$B_{i,2}$ is necessary here as the action of $g$ sends that point to $B_{i,0}$, which would otherwise have been a map from $B_{i,0}$ to itself. We build the corresponding $B_j = \bigcup_{i | n_i \in \{\text{odd}\}} B_{i,j}$. Again let $\omega . g = \omega$ and observe that $B_j \in \omega$ for some $j$. Acting by $g$ gives: $B_j . g \in \omega . g$, hence $B_j . g \in \omega$. Considering $j \mod 3$: $B_j . g \subset B_{j+1} \cup B_{j+2} = B_j^c$ which again gives a contradiction. □

**Theorem 3.9.** The boundary coarse Baum-Connes conjecture holds for spaces of graphs of large girth and $2k$-regularity.

**Proof.** Lemmas 3.7 and 3.8 combine with Proposition 2.13 to give us that $G(X) |_{\partial \beta X}$ is isomorphic to $\partial \beta X \rtimes F_k$. The proof follows using either a Pismner-Voiculescu argument or appealing to the Strong Baum-Connes property for $F_k$. □

**Corollary 3.10.** ([WY12a, Theorem 1.5]) For sequences of large girth and vertex degree $2k$ we have that the Coarse Baum-Connes assembly map is injective and if the sequence forms an expander then it also fails to be surjective.

**Proof.** Combine Proposition 2.8 and Theorem 3.9. □

### 3.1. Some Finite Graph Theory

The main idea in the previous Theorem was that we could utilise Pedersen’s Lemma to build an action of the free group on a large girth sequence. To adjust the results to a situation in which the vertex degree is odd everywhere we use some finite graph theory, this time we make use of 1-factors.

**Definition 3.11.** Let $X$ be a connected finite graph. A **1-factor** is a spanning subgraph $M$ such that for every vertex $v \in V(M) = V(X)$ we have $\text{deg}(v) = 1$.

Graph factorisation is well studied [AK88] and hence using some more of this theory we can arrive at an analogue of the Theorem 3.9. The issue with a direct analogue is that the universal cover of a $2k + 1$-regular graph is a $2k + 1$-regular infinite tree, which is not automatically a Cayley graph of a free group. However there is still technology to deal with this. The following is [Sum74, Corollary 2].

**Proposition 3.12.** Let $X$ be a finite connected graph with $|X| = 2n$ and no induced subgraphs isomorphic to $K_{1,3}$ then $X$ has a 1-factor. □
The removal of a 1-factor from a finite graph of uniform odd vertex degree \(2k+1\) gives a new finite graph that is a disjoint union of finitely many connected components that are each \(2k\) regular.

**Proposition 3.13.** Let \(X\) be a \(2k+1\)-regular finite graph that has a 1-factor. Then there is an action of \(F_k \ast C_2\) on \(X\).

**Proof.** Consider the 1-factor \(M \subset X\). Then consider the graph \(X'\) with the same vertex set as \(X\) but with the edges of \(M\) removed; this is a finite disjoint union of \(2k\)-regular induced subgraphs that we can now label and act on using Lemma 3.4. Now add back the edges of \(M\) but with no orientation. We observe that the edges of \(M\) can be thought of as ways to reflect in the graph. Hence we attain an action of \(F_k \ast C_2\) by combining the obvious actions of both the factors. \(\square\)

We can use Propositions 2.13 and 3.13 to prove the following:

**Theorem 3.14.** Let \(\{X_i\}\) be a sequence of finite graphs that are \(2k+1\)-regular and cofinitely many contain no induced \(K_{1,3}\)'s and let \(X\) be the associated space of graphs. Then conjecture \(I\) holds for \(X\).

**Proof.** We argue as we did in the \(2k\)-regular case. As the \(2k+1\)-regular infinite tree forms an asymptotically faithful covering sequence for the \(X_i\)'s we can conclude that our boundary groupoid: \(G(X)|_{\partial\beta X}\) is homeomorphic to \(\partial\beta X \rtimes (F_k \ast C_2)\). Now we can conclude the proof using either the strong Baum-Connes property or using an elementary argument in K-theory, which relies on the results of [Lan83] on free products and a Pimsner-Voiculescu argument. \(\square\)

However this is not very satisfying as there are many finite \(2k+1\)-regular graphs with edge chromatic number \(2(k+1)\) which do not immediately admit 1-factors. We would like this result to hold in much more generality than sequences of regular graphs; we are interested in sequences with only a uniform upper bound on their regularity in order to reach the most general results of [WY12a]. To tackle this we need a more flexible way to allow the free group to affect our finite graphs. We proceed via the notion of a \emph{partial action}.

### 3.2. The General Strategy via Graph Colourings.

As noted above we want to consider partial actions of a group \(\Gamma\) on a space \(X\). This means that the elements of \(\Gamma\) give rise to partial bijections of \(X\), i.e. bijections between subsets of \(X\). These partial bijections are algebraically encoded within an inverse semigroup, and the partial action of \(\Gamma\) is a \emph{dual prehomomorphism} from \(\Gamma\) into that semigroup, see section 3.5.

We begin by considering the more general result that accompanies Pedersens Lemma. The following is [Kön90, Theorem 6, Chapter XI]:

**Lemma 3.15.** Let \(X\) be a finite graph. If at most \(2k\) edges go into any vertex then all the edges of \(X\) can be divided into \(k\) classes such that at most two edges from the same class go into any vertex. \(\square\)

We want to use this to label any sequence of graphs that have uniformly bounded degree, which without loss of generality can be chosen to be an even uniform upper bound. We call such a labelling a \emph{partial k-orientation} and we say such graphs are \emph{partially or almost}
**k-oriented.** From the point of view of building a group action Lemma 3.15 is completely useless, however if we are willing to work with a reasonable generalisation of a group action Lemma 3.15 provides us ample information. When considering the space of graphs $X$ of a sequence of finite graphs $\{X_i\}$ the strategy is as follows:

1. Construct from a partial $k$-orientation a collection of *partial bijections* of each finite graph $X_i$. These will have disjoint support, whence they can be "added" together when we pass to the space of graphs - giving us, for each group element, a partial bijection on $X$. A natural thing to do then is ask how such things can be composed; they generate a submonoid of the *symmetric inverse monoid* over $X$.

2. Applying the work of Exel in [Exe08] (or Paterson [Pat99]) to this inverse monoid we can associate a groupoid over $X$. Combining this with an augmentation of Proposition 2.13 we can get a description of the boundary groupoid for the space $X$.

3. We utilise properties of the inverse monoid to prove that this groupoid has the Haagerup property. This in turn provides us with the Baum-Connes conjecture being an isomorphism with any coefficients for this groupoid. We use this to conclude the boundary conjecture is an isomorphism for $X$.

The remainder of this section is making these ideas precise.

We remark that we can always assume that the $2k$ here is minimal; there is a smallest even integer that bounds above the degree of all graphs in the sequence. This in particular stops us from doing something unnatural like embedding the 4-regular tree into a 6-regular tree.

### 3.3. An Interlude into Inverse Semigroup Theory.

**Definition 3.16.** Let $S$ be a semigroup. We say $S$ is *inverse* if there exists a unary operation $*: S \to S$ satisfying the following identities:

1. $(s^*)^* = s$
2. $ss^*s = s$ and $s^*ss^* = s^*$ for all $s \in S$
3. $ef = fe$ for all idempotents $e, f \in S$

Recall that a semigroup with a unit element is called a *monoid*, and it makes sense to talk about inverse monoids in the obvious way. A very fundamental example is the *symmetric inverse monoid* on any set $X$; consider the collection of all partial bijections of $X$ to itself, giving them the natural composition law associated to functions - that is find the largest possible domain on which the composition makes sense, shown below in Figure 1.

Explicitly:

$$f_2 \circ f_1 : f_1^{-1}(im(f_1) \cap dom(f_2)) \to f_2(im(f_1) \cap dom(f_2)).$$

When $X$ is a metric space we will be considering a *generalised (or partial) translation*. We denote this by $I_b(X)$.

**Definition 3.17.** Let $S$ be an inverse monoid. We denote by $E(S)$ the semilattice of idempotents (just by $E$ if the context is clear). This is a meet semilattice, where the meet is given...
by the product of $S$ restricted to $E$. In this situation, we can use the following partial order:

$$e \leq f \iff ef = e$$

This order can be extended naturally to the entire of $S$: $s \leq t$ if there exists $e \in E(S)$ such that $s = et$. In terms of partial bijections this order corresponds to restricting an element to a subset of its domain. We make use of this order later.

We remark that for a metric space $X$ every idempotent element in $I(X)$ moves elements no distance, and hence $E(I(X)) = E(I_b(X))$. An inverse submonoid with this property is often called full.

We want to consider quotient structures of an inverse monoid, and unlike in group theory where we have the concept of a normal subgroup our subsemigroups will not in general possess enough information. In general quotients are given by equivalence relations and in order to get an inverse monoid from the equivalence classes it is enough to impose a closure on the relation. This is the idea of a congruence on $S$.

**Definition 3.18.** An equivalence relation $\sim$ on $S$ is called a congruence if for every $u, v, s, t \in S$ such that $s \sim t$, we know that $su \sim tu$ and $vs \sim vt$. This allows us to equip the quotient $\frac{S}{\sim}$ with a product, making it into an inverse monoid.

One such example of this arises from an ideal in $S$.

**Definition 3.19.** Let $I$ be a subset of $S$. $I$ is an ideal of $S$ if $SI \cup IS \subset I$.

From an ideal we can get a quotient - at the cost of a zero element, that is an element $0$ such that $0s = s0 = 0$ for all $s \in S$.

**Definition 3.20.** Let $S$ be an inverse monoid and let $I$ be an ideal of $S$. Then we can define $\frac{S}{T}$ to be the quotient of the set $S$ by the congruence: $x \sim y$ if $x = y$ or $x$ and $y$ are elements of $I$. 

**Figure 1.** The multiplication of partial bijections
Another specific congruence we will be interested in is called the minimum group congruence on \( S \). This congruence, denoted by \( \sigma \), is given by:

\[
s \sigma t \iff (\exists e \in E)es = et
\]

This congruence is *idempotent pure*, that is for \( e \in E(S) \) and \( s \in S \), \( e \sim s \) implies \( s \in E \) - philosophically every idempotent is only related to other idempotents. This collects all idempotents into classes when we quotient out by that congruence. In general the minimum group congruence is the smallest idempotent pure congruence on \( S \).

**Definition 3.21.** An inverse monoid \( S \) is called *\( E \)-unitary* if for all \( e \in E \) and \( s \in S \) if \( e \leq s \) then \( s \in E \). \( S \) is *\( F \)-inverse* if each \( \sigma \) class has a maximal element in the order on \( S \). We denote these maximal elements by \( \text{Max}(S) \).

For an \( F \)-inverse monoid the minimum group congruence is given by considering all the maximal elements with a new product:

\[
(\forall s, t \in \text{Max}(S))s \ast t = u \text{ for } !u \in \text{Max}(S) \text{ with } st \leq u.
\]

If \( S \) has a zero element then \( \hat{\sigma} \) is the trivial group; The ideas above help us only so far in analysing properties of inverse monoids without zeroes, and in general the inverse monoids we are considering in this paper will have a zero element. However we can still make similar definitions:

**Definition 3.22.** We say \( S \) is *0-\( E \)-unitary* if \( \forall e \in E \setminus 0, s \in S \) \( e \leq s \) implies \( s \in E \). We say it is *0-\( F \)-inverse* if there exists a subset \( T \subset S \) such that for every \( s \in S \) there exists a unique \( t \in T \) such that \( s \leq t \) and if \( s \leq u \) then \( u \leq t \).

As mentioned before, the minimum group congruence on such monoids will return the trivial group. However by working in a category with a more relaxed type of morphism we can still build useful maps to groups. We develop this in section 3.5.

### 3.4. Groupoids from Inverse Monoids

We take an inverse monoid \( S \) and produce a universal groupoid \( G_{\hat{E}} \). One way to do this involves studying the actions of \( S \) on its semilattice \( E \). Working with semilattices, being generalisations of Boolean algebras, we still have access to a version of Stone duality; there exists many compactifications of \( E \), built from its order structure, that extends the natural conjugation action of \( S \). To any representation of \( S \) by partial bijections on a space \( X \) we get a corresponding representation on Hilbert space of the groupoid \( G_{\hat{E}} \). We can then use this to build a compactification of \( X \) that will allow us to reduce \( G_{\hat{E}} \), capturing the representation theory on \( X \).

We outline the steps in the construction.

1. Build an action of \( S \) on \( E \).
2. Build a dual space to \( E \), which is compact and Hausdorff. This is a *Stone dual* to \( E \). Show this admits an action of \( S \).
3. Build the groupoid \( G_{\hat{E}} \) from this data.

**Definition 3.23.** (1) Let \( D_e = \{ f \in E | f \leq e \} \). For \( ss^* \in E \), we can define a map \( \rho_s(ss^*) = s^*s \), extending to \( D_{ss^*} \) by \( \rho_s(e) = s^*es \). This defines a partial bijection on \( E \) from \( D_{ss^*} \) to \( D_{s^*s} \).
We consider a subspace of $2^E$ given by the functions $\phi$ such that $\phi(0) = 0$ and $\phi(ef) = \phi(e)\phi(f)$. This step is a generalisation of Stone duality [Law10]. We can topologise this as a subspace of $2^E$, where it is closed. This makes it compact Hausdorff, with a base of topology given by $\hat{D}_e = \{\phi \in \hat{E}|\phi(e) = 1\}$. This admits a dual action induced from the action of $S$ on $E$. This is given by the pointwise equation for every $\phi \in \hat{D}_s$:

$$\hat{\rho}_s(\phi)(e) = \phi(\rho_s(e)) = \phi(s^*es)$$

The use of $\hat{D}_e$ to denote these sets is not a coincidence, as we have the following map $D_e \to \hat{D}_e$:

$$e \mapsto \phi_e, \phi_e(f) = 1 \text{ if } e \leq f \text{ and } 0 \text{ otherwise}.$$

Remark 3.24. These character maps $\phi : E \to \{0, 1\}$ have an alternative interpretation, they can be considered as filters on $E$. A filter on $E$ is given a set $F \subset E$ with the following properties:

- for all $e, f \in F$ we have that $e \wedge f = ef \in F$
- for $e \in F$ with $e \leq f$ we have that $f \in F$ and
- $0 \notin F$

the relationship between characters and filters can be summarised as: To each character $\psi$ there is a filter:

$$F_\psi = \{e \in E|\psi(e) = 1\}.$$

And every filter $F$ provides a character by considering $\chi_F$, its characteristic function.

We take the set $S \times \hat{E}$, topologise it as a product and consider subset $\Omega := \{(s, \phi)|\phi \in D_{s^*s}\}$ in the subspace topology. We then quotient this space by the relation:

$$(s, \phi) \sim (t, \phi') \iff \phi = \phi' \text{ and } (\exists e \in E) \text{ with } \phi \in D_e \text{ such that } es = et$$

We can give the quotient $\hat{G}_E$ a groupoid structure with the product set, unit space and range and source maps:

$$\hat{G}_E^{(2)} := \{([s, \phi], [t, \phi'])|\phi = \hat{\rho}_t(\phi')\}$$

$$\hat{G}_E^{(0)} := \{[e, \phi]|e \in E\} \cong \hat{E}$$

$$s([t, \phi]) = [t^*t, \phi], r([t, \phi]) = [tt^*, \phi],$$

and product and inverse:

$$[s, \phi][t, \phi'] = [st, \phi'] \text{ if } ([s, \phi], [t, \phi']) \in \hat{G}_E^{(2)}, [s, \phi]^{-1} = [s^*, \hat{\rho}_s(\phi)]$$

For all the details of the above, we refer to [Exe08, Section 4]. We call this groupoid the universal groupoid associated to $S$. We collect some information about this groupoid from [Exe08, Pat99] in Theorem 3.25

**Theorem 3.25.** Let $S$ be a countable 0-E-unitary inverse monoid, $E$ its semilattice of idempotents and $\hat{G}_E$ its universal groupoid. Then the following hold for $\hat{G}_E$:

- $\hat{E}$ is a compact, Hausdorff and second countable space.
- $\hat{G}_E$ is a Hausdorff groupoid.
• Every unitary representation of $S$ on Hilbert space gives rise to a covariant representation of $\hat{G}_E$ and vice versa.

• We have $C^*_r(S) \cong C^*_r(\hat{G}_E)$.

Proof. The first point is a consequence of the fact that $E$ is countable, in this situation we know precisely that $2^E$ is metrizable, hence as a closed subset we know that $\hat{E}$ is second countable. It is compact and Hausdorff as it is a closed subset of a compact, Hausdorff space. The second point follows from [Exe08, Corollary 10.9], the third point is [Exe08, Corollary 10.16] and the fourth point follows from [Pat99], but a more elementary proof is given in [KS02]. □

We make use of the following technical property that arises from the presence of maximal elements:

**Claim 3.26.** Let $S$ be 0-F-inverse. Then every element $[s, \phi] \in G\hat{E}$ has a representative $[t, \phi]$ where $t$ is a maximal element.

Proof. Take $t = t_s$ the unique maximal element above $s$. Then we know $s = t_s s^* s$ and $s^* s \leq t_s^* t_s$. The second equation tells us that $t_s^* t_s \in F_\phi$ as filters are upwardly closed, thus $(t_s, \phi)$ is a valid element. Now to see $[t_s, \phi] = [s, \phi]$ we need to find an $e \in E$ such that $e \in F_\phi$ and $se = t_s e$. Take $e = s^* s$ and then use the first equation to see that $s(s^* s) = t_s(s^* s)$. □

Using Claim 3.26 we can forget the non-maximal elements in the monoid $S$ when working with $G\hat{E}$.

3.5. **(Dual) Prehomomorphisms and General Partial Actions.**

**Definition 3.27.** Let $\rho: S \to T$ be a map between inverse semigroups. This map is called a prehomomorphism if for every $s, t \in S$, $\rho(st) \leq \rho(s)\rho(t)$ and a dual prehomomorphism if for every $s, t \in S$ $\rho(s)\rho(t) \leq \rho(st)$.

We recall that a congruence is said to be idempotent pure for any $e \in E(S)$, $s \in S$ we have that $s$ is related to $e$ implies that $s \in E(S)$. We extend this definition to general maps in the following way.

**Definition 3.28.** A (dual) prehomomorphism $\rho$ is called idempotent pure if $\rho(s)^2 = \rho(s)$ implies $s \in E$.

In addition we call a map $S \to T$ 0-restricted if the preimage of 0 $\in T$ is 0 $\in S$.

**Definition 3.29.** Let $S$ be a 0-E-unitary inverse monoid. We say $S$ is strongly 0-E-unitary if there exists an idempotent pure, 0-restricted prehomomorphism, $\Phi$ to a group $G$ with a zero element adjoined, that is: $\Phi : S \to G^0$. We say it is strongly 0-F-inverse if it is 0-F-inverse and strongly 0-E-unitary. This is equivalent to the fact that the preimage of each group element under $\Phi$ contains a maximum element.
Remark 3.30. In this case a prehomorphism being idempotent pure is equivalent to having the preimage of an idempotent be idempotent as the semilattice of idempotents of a group with zero element adjoined is the two element Boolean algebra.

This class of inverse monoids is particularly important; the idempotent pure, 0-restricted prehomomorphism onto a group (with 0) can be thought of as a generalisation of the minimum group congruence in the larger category of inverse monoids with prehomomorphisms. We will utilise this technology later to regain some of the information from a group when we cannot quotient out in any meaningful way due to the presence of a zero element.

Example 3.31. In [BR84, LMS06] the authors introduce an inverse monoid that is universal for dual prehomomorphisms from a general inverse semigroup. In the context of a group $G$ this is called the prefix expansion; its elements are given by pairs: $(X, g)$ for \( \{1, g\} \subset X \), where $X$ is a finite subset of $G$. The set of such $(X, g)$ is then equipped with a product and inverse:

\[
(X, g)(Y, h) = (X \cup gY, gh), \quad (X, g)^{-1} = (g^{-1}X, g^{-1})
\]

This has maximal group homomorphic image $G$, and it has the universal property that it is the largest such inverse monoid. We denote this by $G^{Pr}$. The partial order on $G^{Pr}$ can be described by reverse inclusion, induced from reverse inclusion on finite subsets of $G$. It is $F$-inverse, with maximal elements: \( \{(\{1, g\}, g) : g \in G\} \). We make use of the prefix expansion later.

Definition 3.32. Let $G$ be a finitely generated discrete group and let $X$ be a (locally compact Hausdorff) topological space. A partial action of $G$ on $X$ is a dual prehomomorphism $\theta$ of $G$ in the symmetric inverse monoid $I(X)$ that has the following properties:

1. The domain $D_{\theta_g} \theta_g$ is an open set for every $g$.
2. $\theta_g$ is a continuous map.
3. The union $\bigcup_{g \in G} D_{\theta_g} \theta_g$ is $X$.

Given this data we can generate an inverse monoid $S$ using the set of $\theta_g$. This would then give a representation of $S$ into $I(X)$. If the space $X$ is a coarse space, then it makes sense to ask if each $\theta_g$ is a close to the identity. In this case, we would get a representation into the bounded symmetric inverse monoid $I_b(X)$. We call such a $\theta$ a bounded partial action of $G$.

Example 3.33. If we consider a subspace $X$ of a finitely generated group $\Gamma$ we can always equip $X$ with a partial action of $\Gamma$ in an obvious way; we restrict each element of $\Gamma$ to $X$. This gives us a partial representation when we consider $X$ with the standard subspace topology coming from the metric on $\Gamma$. This truncation provides a very nice example of a dual prehomomorphism of a group that will give rise to a bounded partial action.

Let $X = \sqcup X_i$ be a space of graphs admitting a bounded partial action of a discrete group $G$. We remark that in this setting partial bijections in the group can have the following form:

\[
\theta_g = \theta_g^0 \sqcup \bigcup_{i > i_0} \theta_g^i.
\]
Where $i_0$ is the first $i$ for which the distances between the $X_i$s is greater than the upper bound of the distance moved by $\theta_g$, and the $\theta_g^0$ are componentwise partial bijections of the $X_i$. We collect all the additional pieces that act only between the first $i_0$ terms into $\theta_g^0$, which could be the empty translation. We remark now that it is possible that there are partial bijections $\theta_g$ that could have finite support, that is only finitely many terms that are non-empty after $i_0$. To avoid this, we observe the following:

**Proposition 3.34.** Let $S = \langle \theta_g | g \in G \rangle$ and let $I_{fin} = \{ \theta_g | \text{supp}(\theta_g) \text{ is finite} \}$. Then $I_{fin}$ is an ideal and the Rees quotient $S_{inf} = S/ I_{fin}$ is an inverse monoid with 0.

*Proof.* To be an ideal, it is enough to show that $I_{fin}S \subset I_{fin}$, $SI_{fin} \subset I_{fin}$. Using the description of the multiplication of partial bijections from section 3.3 it is clear that either combination $si$ or $is$ yields an element of finite support. Now we can form the Rees quotient, getting an inverse monoid with a zero - the zero element being the equivalence class of elements with finite support. □

We want to utilise a partial action to construct a groupoid, so we apply the general construction outlined in section 3.3 to get an improved version of Proposition 2.13. As we are interested in specialising to a partial action that will somehow generate the boundary coarse groupoid we would like to know that we can get information about the metric coarse structure from the partial action when $X$ is a metric space. However in general it is too much to expect that our partial action generates the metric coarse structure completely. To understand this we need to define the length of a partial bijection:

**Definition 3.35.** The length of each $\theta_g$ is defined to be:

$$|\theta_g| = \sup \{ d(x, \theta_g(x)) : x \in \text{Dom}(\theta_g) \}.$$  

**Definition 3.36.** Recall that we say a bounded partial action $\theta$ generates the metric coarse structure at infinity if for all $R > 0$ there exists $S > 0$ such that $\Delta_R \setminus \Delta_R \subseteq \bigcup_{|\theta_g| < S} \Delta_{\theta_g} \setminus \Delta_{\theta_g}$. We say it finitely generates the metric coarse structure if the number of $\theta_g$ required for each $R$ is finite.

**Remark 3.37.** Recall a groupoid $G$ is said to be principal if the map $(s, r) : G \to G^{(0)} \times G^{(0)}$ is injective.

**Proposition 3.38.** Let $\{X_i\}$ be a sequence of finite graphs and let $X$ be the corresponding space of graphs. If $\theta : G \to \mathcal{I}(X)$ is a bounded partial action of $G$ on $X$ such that the induced action on $\beta X$ is free on $\partial \beta X$, the inverse monoid $S_{inf}$ is 0-F-inverse with maximal element set $\{ \theta_g | g \in G \}$ and the partial action finitely generates the metric coarse structure at infinity then there is a second countable, étale topological groupoid $G\tilde{X}$ such that $G(X)|_{\partial \beta X} \cong \partial \beta X \times G\tilde{X}$.

*Proof.* Observe now that the finite $\theta_g$ play no role in the action on the boundary. Thus we do not need to consider them during the following construction of a groupoid, we can just work with the strongly 0-F-inverse monoid $S_{inf}$. To build the groupoid $G\tilde{X}$ we first build a space $\tilde{X}$, which is done by considering the partial action and inverse monoid it generates inside $\mathcal{I}(X)$, and then using this space to construct a reduction of the universal groupoid $G\hat{E}(S_{inf})$.

We build the unit space $\tilde{X}$ of the groupoid using some results from [Exe08, Proposition 10.6, Theorem 10.16]. The first leads us to consider the representation of the inverse monoid $S_{inf}$...
on $\ell^2(X)$ induced by $\theta$ to get a representation $\pi_\theta : S \to \mathcal{B}(\ell^2(X))$. We can complete the semigroup ring in this representation to get an algebra $C^*_\nu S$, which has a unital commutative subalgebra $C^*_\nu E$. \cite[Proposition 10.6]{Exe08} then tells us that the spectrum of this algebra, which we will denote by $\hat{X}$, is a subspace of $\hat{E}$ that is closed and invariant under the action of $S$ on which the representation $\pi_\theta$ is supported (see \cite[Section 10]{Exe08} for more details).

As the space $\hat{X}$ is closed and invariant we can reduce the universal groupoid $\mathcal{G}_E$ for $S_{in}$ to $\hat{X}$. This we denote by $\mathcal{G}_{\hat{X}}$. \cite[Theorem 10.16]{Exe08} implies that with this representation in hand we have the isomorphism: $C^*_\nu(\mathcal{G}_{\hat{X}}) \cong C^*_\nu(S)$.

This gives us a groupoid, $\mathcal{G}_{\hat{X}}$. To make this act on $\beta X$ we make use of the assumption that $\theta_g$ is a bounded partial bijection for each $g \in G$ and again of the representation $\pi_\theta$. Each $\theta_g$ bounded implies that our algebra $C^*_\nu(\mathcal{G}_{\hat{X}})$ is a subalgebra of $C^*_\nu(G(X)) \cong C^*_\nu X$. We now remark that the representation $\pi_X$, when restricted to $C^*E$ assigns each idempotent a projection in $C^*_\nu X$, that is $C^*_\nu(E) = \pi_X(C^*E) \subset \ell^\infty(X)$. Taking the spectra associated to this inclusion then gives us a map:

$$r_{\beta X} : \beta X \to \hat{X}$$

which is continuous. In particular as both $\beta X$ and $\hat{X}$ are compact Hausdorff spaces, this map is closed (and open) and hence a quotient. We make use of this to define an action on $\beta X$. By Claim \ref{claim} we have that each element of our groupoid $\mathcal{G}_{\hat{X}}$ can be represented by a pair $[\theta_g, \phi]$, for some $\phi \in \hat{X}$. Observe also that as $X$ is discrete so are all of its subspaces, hence the maps $\theta_g$ are continuous (open) for each $g \in G$. These then extend to $\beta X$, and so coupled with the map $r_{\beta X}$ we can act by:

$$[\theta_g, \phi].\omega = \theta_g(\omega)$$

for all $\omega \in D_{\theta_g}\theta_g$ with $r_{\beta}(\omega) = \phi$. We see that $r_{\beta}(\omega).\omega = [\theta_{e}, r_{\beta}(\omega)].\omega = \omega$. And for all $([\theta_g, \phi], [\theta_h, \phi^{'}, \phi']) \in G^{(2)}_{\hat{X}}$ with $\phi^{'}, r_{\beta}(\omega)$ we have:

$$[\theta_g, \theta_h, \phi^{'}, \phi].\omega = \theta_g \theta_h(\omega) = \theta_g([\theta_h, \phi^{'}, \phi].\omega) = [\theta_g, \phi].([\theta_h, \phi^{'}, \phi].\omega)$$

as $r_{\beta}([\theta_h, \phi^{'}, \phi].\omega) = \theta_h(\phi^{'}) = \phi$.

It remains to prove the isomorphism of topological groupoids: $G(X)|_{\partial \beta X} \cong \partial \beta X \times \mathcal{G}_{\hat{X}}$. We follow the scheme of Proposition \ref{prop} and build a map from $\beta X \times \mathcal{G}_{\hat{X}}$ to $G(X)$. Recall that as the partial action of $G$ generates the metric coarse structure at infinity $G(X) = (\bigcup_g \Delta_{\theta_g}) \cup (X \times X)$. We observe that each $\Delta_{\theta_g}$ maps bijectively onto the domain of $\theta_g$, a subset of $X$.

This map extends to the closure of the domain precisely as in Proposition \ref{prop}, where here we map the pair $(\omega, [\theta_g, \phi])$ to the element $\gamma_g, \phi$ that is the limit $\lim_{\lambda}(x_\lambda, \theta_g(x_\lambda))$ for some net $\{x_\lambda\}$ that converges to $\omega$ (and also to $\phi$). This map is well defined as the groupoid $G(X)$ is principal, and it fits into the following commutative diagram:

$$\begin{CD}
\beta X \times \Gamma @>>> G(X) \\
@V{(r,s)}VV @VV{(r,s)}V \\
\beta X \times \beta X
\end{CD}$$

Again by principality, we can deduce that the covering map is a groupoid homomorphism.
We now restrict this map to the boundary $\partial \beta X$. As we know that the group action generates the metric coarse structure at infinity and that the partial action of the group $G$ is free on the boundary. Using these facts we can see that:

1. $\partial \beta X \rtimes \hat{G} \simeq \hat{X}$ is principal.
2. $G(X)|_{\partial \beta X} = \bigcup_{g} \Delta_{\theta_{g}} \setminus \Delta_{\theta_{g}}$.

From both (1) and (2) we can further deduce that the covering map is a bijection on the boundary. Both groupoids are also étale and so each component $\Delta_{\theta_{g}} \setminus \Delta_{\theta_{g}}$ is mapped homeomorphically onto its image and is therefore clopen. It follows then that we get the desired isomorphism $\partial \beta X \rtimes \hat{G} \simeq G(X)|_{\partial \beta X}$ of topological groupoids.

We are interested in understanding the analytic properties of the groupoid $\hat{G}$. In particular we are interested in showing that the groupoid has the Haagerup property, that is admits a proper affine isometric action on a field of Hilbert spaces. From results of Tu in [Tu99] this enough to conclude the Baum-Connes assembly map is an isomorphism for all coefficients for this groupoid. To do this we study the inverse monoid $S$ associated to the partial action $\theta$.

**Proposition 3.39.** Let $S = \langle \theta_{g} \mid g \in G \rangle$, where $\theta : G \to S$ is a dual prehomomorphism. If $S$ is 0-F-inverse with $\text{Max}(S) = \{\theta_{g} \mid g \in G\}$ and if $\theta_{g} \neq 0$ and $\theta_{g} \notin E(S)$ for $g \neq e$ then $S$ is strongly 0-F-inverse.

**Proof.** We build a map $\Phi$ back onto $G^{0}$. Let $m : S \to \text{Max}(S)$ be the map that sends each $s$ to the maximal element $m(s)$ above $s$ and consider the following diagram:

$$
\begin{array}{ccc}
G & \xrightarrow{\theta} & S \\
\text{Bar} & \Phi & \text{Bar} \\
G^{pr} & \xleftarrow{\sigma} & G^{0}
\end{array}
$$

where $G^{pr}$ is the prefix expansion of $G$. Define the map $\Phi : S \to G^{0}$ by:

$$
\Phi(s) = \sigma(m(\bar{\theta}^{-1}(m(s)))), \Phi(0) = 0
$$

For each maximal element the preimage under $\text{Bar}$ is well defined as the map $\theta_{g}$ has the property that $\theta_{g} = \theta_{h} \Rightarrow g = h$ precisely when $\theta_{g} \neq 0 \in S$. Given the preimage is a subset of the F-inverse monoid $G^{pr}$ we know that the maximal element in the preimage is the element $(\{1, g\}, g)$ for each $g \in G$, from where we can conclude that the map $\sigma$ takes this onto $g \in G$.

We now prove it is a prehomomorphism. Let $\theta_{g}, \theta_{h} \in S$, then:

$$
\begin{align*}
\Phi(\theta_{g}) &= \sigma(m(\bar{\theta}^{-1}(\theta_{g}))) = \sigma(\{1, g\}, g) = g \\
\Phi(\theta_{h}) &= \sigma(m(\bar{\theta}^{-1}(\theta_{h}))) = \sigma(\{1, h\}, h) = h \\
\Phi(\theta_{gh}) &= \sigma(m(\bar{\theta}^{-1}(\theta_{gh}))) = \sigma(\{1, gh\}, gh) = gh
\end{align*}
$$

Hence whenever $\theta_{g}, \theta_{h}$ and $\theta_{gh}$ are defined we know that $\Phi(\theta_{g} \theta_{h}) = \Phi(\theta_{g}) \Phi(\theta_{h})$. They fail to be defined if:

1. If $\theta_{gh} = 0$ in $S$ but $\theta_{g}$ and $\theta_{h} \neq 0$ in $S$, then $0 = \Phi(\theta_{g} \theta_{h}) \leq \Phi(\theta_{g}) \Phi(\theta_{h})$
(2) If (without loss of generality) \( \theta_g = 0 \) then 0 = \( \Phi(0, \theta_h) = 0, \Phi(\theta_h) = 0 \)

So prove that the inverse monoid \( S \) is strongly 0-F-inverse it is enough to prove then that the map \( \Phi \) is idempotent pure, and without loss of generality it is enough to consider maps of only the maximal elements - as the dual prehomomorphism property implies that in studying any word that is non-zero we will be less than some \( \theta_g \) for some \( g \in G \).

So consider the map \( \Phi \) applied to a \( \theta_g \):

\[
\Phi(\theta_g) = \sigma(m(\overline{\theta}^{-1}(\theta_g))) = \sigma(\{1, g\}, g) = g
\]

Now assume that \( \Phi(\theta_g) = e_G \). Then it follows that \( \sigma(m(\overline{\theta}^{-1}(\theta_g))) = e_G \). As \( \sigma \) is idempotent pure, it follows then that \( m(\overline{\theta}^{-1}(\theta_g)) = 1 \), hence for any preimage \( t \in \theta^{-1}(\theta_g) \) we know that \( t \leq 1 \), and by the property of being 0-E-unitary it then follows that \( t \in E(G^{pr}) \). Mapping this back onto \( \theta_g \) we can conclude that \( \theta_g \) is idempotent, but by assumption this only occurs if \( g = e \).

\[\Box\]

**Proposition 3.40.** Let \( S = \langle \theta_g | g \in G \rangle \) be a strongly 0-F-inverse monoid with maximal elements \( \text{Max}(S) = \{\theta_g : g \in G\} \), where \( \theta : G \to S \) is a dual prehomomorphism. Then the groupoid \( \hat{G}_E \) is Hausdorff, second countable with compact unit space. Also it admits a continuous proper groupoid homomorphism onto the group \( G \).

**Proof.** We record the topological facts about this groupoid here for reference.

Using the map \( \Phi \) we construct a map \( \rho : \hat{G}_E \to G \) as follows:

\[
\rho([m, \phi]) = \Phi(m)
\]

A simple check proves this is a groupoid homomorphism. This map sends units to units as the map \( \Phi \) is idempotent pure. We prove continuity by considering preimage of an open set in \( G \):

\[
\rho^{-1}(U) = \bigcup_{g \in U} [\theta_g, \hat{D}\theta_g\theta_g]
\]

This is certainly open as each \( [\theta_g, \hat{D}\theta_g\theta_g] \) are elements of the basis of topology of \( \hat{G}_E \). We check it is proper by observing that for groups \( G \) compact sets are finite, and they have preimage:

\[
\rho^{-1}(F) = \bigcup_{g \in F} [\theta_g, \hat{D}\theta_g\theta_g], |F| < \infty
\]

This is certainly compact as these are open and closed sets in the basis of topology for the groupoid \( \hat{G}_E \).

\[\Box\]

As \( \hat{G}_X \subseteq \hat{G}_E \), we also get a continuous proper groupoid homomorphism from \( \hat{G}_X \) onto a group.

A remark that comes from considering the work of Lawson in [Law98] is that we can consider the category of inverse monoids with prehomomorphisms equivalent to the category of ordered groupoids with groupoid homomorphisms, so it is reasonable to consider such maps when we want to understand the structure of the universal groupoid associated to \( S \).

We recall a special case of [Tu99, Lemme 3.12].
Lemma 3.41. Let $G$ and $H$ be locally compact, Hausdorff, étale topological groupoids and let $\varphi : G \to H$ be a continuous proper groupoid homomorphism. If $H$ has the Haagerup property then so does $G$. □

This lets us conclude the following:

Corollary 3.42. Let $\theta$ be a partial action of $G$ on $X$ such that all the conditions of Proposition 3.38 are satisfied and such that the inverse monoid $S_{\inf}$ is strongly 0-F-inverse. If $G$ has the Haagerup property then so does $G^X$. □

3.6. Partial Actions on Sequences of Graphs. Let $\{X_i\}$ be a sequence of finite graphs with degree $\leq 2k$ and large girth.

Lemma 3.43. Such a sequence can be almost $k$-oriented and this defines a bounded partial action of $F_k$ on $X$.

Proof. We work on just the $X_i$. Using Lemma 3.15, we partition the edges $E(X_i)$ into at most $k$ sets $E_j$ such that every vertex appears in at most 2 edges from each subset. Pick a generating set $S = \{a_j|j \in \{1,\ldots,k\}\}$ for $F_k$ and assign them to the edge sets $E_j$, and label the edges that appear in each $E_j$ by the corresponding generator. Pedersen’s Lemma ensures that no more than 2 edges at each point have the same label. This defines a map from the edges to the wedge $\bigvee_{j=1}^k S^1$. Choose an orientation of each circle and pull this back to the finite graph $X_i$ - this provides the partial $k$-orientation. Now define for each generator the partial bijection $\theta_{a_j}^i$ that maps any vertex appearing as the source of any edge in $E_j$ to the range of that edge. i.e:

$$\theta_{a_j}^i(v) := \begin{cases} r(e) & \text{if } \exists e \in E_j : s(e) = v \\ \text{undefined otherwise} & \end{cases}$$

For $g = a_{i_1}^e \ldots a_{m}^e$ we define $\theta_g^i$ as the product $\theta_{a_{i_1}} \ldots \theta_{a_{m}}$; i.e $\theta_g^i$ moves vertices along any path that is labelled by the word $g$ in the graph $X_i$. We observe that for $i \neq i'$ the domain $D_{\theta_{a_j}^i} \cap D_{\theta_{a_j}^{i'}}$ is empty hence we can add these partial bijections in $I(X)$ to form $\theta_g = \cup \theta_g^i$. It is a remark that as the topology of $X$ is discrete these maps are all continuous and open.

It is clear that as each $X_i$ is connected that the partial bijections have the property that $\cup_g D_{\theta_{a_j}^i} = X$. Lastly, this map is a dual prehomomorphism as for each $g, h \in G$ we have that $\theta_{gh} = \theta_g \theta_h$ precisely when both $\theta_h$ and $\theta_{gh}$ are defined and moreover if $\theta_g \theta_h$ is defined then so is $\theta_{gh}$. Hence this collection forms a partial action of $G$ on $X$. We also remark that as each bijection is given translation along a labelling in the free group it is clear that these move elements only a bounded distance and are therefore elements of $I_b(X)$. □

Remark 3.44. If we consider the proof of the above Lemma then it would seem that for every $\theta_g, \theta_h$ we have that $\theta_g \theta_h = \theta_{gh}$ However this might not be the case because of cancellation that occurs in the group but not in the partial bijections.
We would want to show that the partial action generates the metric coarse structure at infinity, we recall the length of a partial bijection:

**Definition 3.45.** The length of each $\theta_g$ is defined to be:

$$|\theta_g| = \sup \{d(x, \theta_g(x)) : x \in \text{Dom}(\theta_g)\}.$$  

**Remark 3.46.** As we have a concrete description of each $\theta_g$, given on each $X_i$, we can see that the length on each $X_i$ is given by:

$$|\theta_i^g| = \max \{|p| : p \in \{ \text{paths in } X_i \text{ labelled by } g \} \}.$$  

Then $|\theta_g| = \sup_i |\theta_i^g|$.

In this situation we require that the partial action contains plenty of infinitely supported elements.

**Proposition 3.47.** Let $\theta : F_k \to I(X)$ be the dual prehomomorphism corresponding to the bounded partial actions on each $X_i$. Then for each $R > 0$ there exist finitely many infinite $\theta_g$ with $|\theta_g| = |g| < R$.

**Proof.** In the general case we know the following for each $R > 0$ and $i \in \mathbb{N}$: $|\theta_i^g| \leq |g| \leq R$. From Lemma 3.43 we know that the partial action is defined by moving along paths inside each individual $X_i$. So for each $R$ we count the number of words in $F_k$ with length less than $R$; this is finite (consider the Cayley graph, which has bounded geometry). Now we observe that on the other hand there are infinitely many simple paths of length less than $R$, thus we must repeat some labellings infinitely many times. These labellings will be contained in words in $F_k$ of length less than $R$ hence when we take the supremum we observe that $|\theta_g| = |g| < R$. □

**Corollary 3.48.** The bounded partial action $\theta$ of $F_k$ on $X$ finitely generates the metric coarse structure at infinity, that is the set $\Delta_R \setminus \Delta_R = \bigcup_{|g| < R} \Delta_{\theta_g} \setminus \Delta_{\theta_g}$ where the index set is finite.

**Proof.** We proceed by decomposing $\Delta_R$ as we did in the proof of Proposition 2.13.

$$\Delta_R = (\bigcup_{|\theta_g| < R} \Delta_{\theta_g}) \cup F_R$$

Where $F_R$ is the finitely many elements of $\Delta_R$ who move between components. We now consider the following decomposition of the set $A := \{\theta_g||\theta_g| < R\}$ into:

$$A_\infty = \{\theta_g||\theta_g| < R \text{ and } |\text{supp}(\theta_g)| = \infty\}$$

$$A_{\text{fin}} = \{\theta_g||\theta_g| < R \text{ and } |\text{supp}(\theta_g)| < \infty\}$$

The first of these is in bijection with the words in $F_k$ that have $|g| < R$ and define an infinite $\theta_g$ from Proposition 3.47. Then:

$$\Delta_R = (\bigcup_{g \in A_\infty} \Delta_{\theta_g}) \cup (\bigcup_{g \in A_{\text{fin}}} \Delta_{\theta_g}) \cup F_R$$

We complete the proof by observing that for each $\theta_g \in A_{\text{fin}}$ the set $\Delta_{\theta_g}$ is finite. Therefore:

$$\Delta_R \setminus \Delta_R = \bigcup_{g \in A_\infty} \Delta_{\theta_g} \setminus \Delta_{\theta_g} = \bigcup_{|g| < R} \Delta_{\theta_g} \setminus \Delta_{\theta_g}$$

□
As in the case of uniform regularity we also need to see that the action is free. Ideally we would argue as if we were in the group case; for each \( g \in F_k \) choose a point in each \( X_i \) and consider its orbits as was implemented in Lemma 3.8. However, this argument does not work as we are faced with the problem that for a partial action the concept of orbit is not well-defined, or in particular it may not always be possible to apply an element \( \theta(g) \) twice to things within its domain.

**Lemma 3.49.** The bounded partial action extends to \( \beta X \) and is a free partial action on the boundary \( \partial \beta X \).

**Proof.** The partial action is by continuous maps on each element of the sequence. By the universal property of the Stone-Čech compactification we can then extend. So as with the case with a genuine action the technicality lies in proving the action is free on the boundary. The proof breaks into 3 cases relying as always on the “ultra” property. We break our domain \( D^{\partial_\omega\theta_g} := \cup_{i \in \mathbb{N}} D^{i \partial_\omega\theta_g} \) up into three (possibly infinite) pieces, based on how the action by \( g \) behaves. The first piece, index denoted \( I_0 \), consists of all the \( D^{i \partial_\omega\theta_g} \) such that \( D^{i \partial_\omega\theta_g} \cap D^{0 \partial_\omega\theta_g} \) is empty; the second piece, indexed by \( I_1 \), consists of all the \( D^{i \partial_\omega\theta_g} \) such that \( D^{0 \partial_\omega\theta_g} \cap D^{i \partial_\omega\theta_g} \) is not empty but \( D^{i \partial_\omega\theta_g} \) is not contained in \( D^{i \partial_\omega\theta_g} \); and the third, denoted by \( I_2 \) is all the \( D^{i \partial_\omega\theta_g} \) such that \( D^{i \partial_\omega\theta_g} = D^{i \partial_\omega\theta_g} \). These are illustrated in Figure 3.6.

![Figure 2. The three cases for Lemma 3.49](image)

We have that \( D^{\partial_\omega\theta_g} \) is:

\[
D^{\partial_\omega\theta_g} = (\bigcup_{i \in I_0} D^{i \partial_\omega\theta_g}) \cup (\bigcup_{i \in I_1} D^{0 \partial_\omega\theta_g}) \cup (\bigcup_{i \in I_2} D^{i \partial_\omega\theta_g})
\]

So any non-principal ultrafilter \( \omega \in \partial \beta X \) will pick between one of the three pieces, so it is enough to show that the action must be free given any choice.

In case \( i \) we have that \( \omega \) picks \( \bigcup_{i \in I_0} D^{i \partial_\omega\theta_g} \), hence \( \theta(g)(\omega) \) picks \( \bigcup_{i \in I_0} D^{i \partial_\omega\theta_g} \), which is contained in the compliment of \( \bigcup_{i \in I_0} D^{i \partial_\omega\theta_g} \) by construction. Whence \( \theta(g)(\omega) \neq \omega \) in this case.

In case \( iii \) every element of the domain \( \bigcup_{i \in I_2} D^{i \partial_\omega\theta_g} \) is also in the image \( \bigcup_{i \in I_2} D^{i \partial_\omega\theta_g} \), whence we can conclude that it is possible to apply any power of \( \theta(g) \) to this set. This allows us now to run the argument present in the proof of Lemma 3.8.

This leaves case \( ii \). Let \( A_{0,i} := D^{i \partial_\omega\theta_g} \setminus D^{i \partial_\omega\theta_g} \cap D^{0 \partial_\omega\theta_g} \) and \( A_1 = D^{i \partial_\omega\theta_g} \cap D^{0 \partial_\omega\theta_g} \). Then set \( A_j = \bigcup_{i \in I_j} A_{j,i} \) and let \( \omega \in \partial \beta X \). If \( A_0 \in \omega \) then \( \theta(g)(A_0) \subset A_0 \), which implies that \( \theta(g)(A_0) \not\in \theta(g)(\omega) \).
It remains to deal with $A_1 \in \omega$. Assume for a contradiction that $\theta(g)(\omega) = \omega$. Then $\theta(g)(A_1) \cap A_1 \in \omega$, and we can apply $\theta(g)$ again - denote by $A_1^m = A_1 \cap \theta(g)(A_1) \cap \ldots \cap \theta(g)^m(A_1)$. For each $i \in \mathcal{I}_2$ there exists a power $m_i$ of $\theta(g)$ such that this intersection stabilises in $X_i$, that is, $A_{1,i}^m = A_{1,i}^{m+1}$.

Let $\mathcal{A} = \bigcap_{m \in \mathbb{N}} A_1^m$. This set has the property that $\theta(g)(\mathcal{A}) = \mathcal{A}$, and we call this the stable core of the partial action. In this set the concept of orbit is well defined, and hence the argument from Lemma 3.3 will work without change, replacing the entire space by $\mathcal{A}$. The technicality here is that $\mathcal{A}$ may not be an element of $\omega$. It has an infinite compliment $\mathcal{A}^c$, so ultrafilters may fight over this. In particular we can remark that by the assumption that $\omega$ is fixed by $g$ every finite intersection $\bigcap_{k=0}^{k=n} A_1^m = A_1^k$ is an element of $\omega$.

So the last case to consider is that $\mathcal{A}^c \in \omega$. It suffices to work in $A_1$, so let $B_1 = \mathcal{A}^c \cup A_1$ and then define $B_i = \theta_g(B_{i-1}) \cap B_1$. It follows from the construction of $\mathcal{A}$ that every element $x \in B_1$ has an associated smallest natural number $n_x$ such that $\theta_{g,x}(x) \notin B_{n_x}$. It is clear from the definition that $B_{i+1} \subset B_i$ for every $i$. Lastly, define $B_{i+1}^{-1} := \theta_g^{-1}(B_{i+1}) \subset B_i$. From this we consider the decomposition of $B_i$ into two disjoint infinite pieces:

- $B_{i,even}^{+1} := \{x \in B_i^{-1} | n_x \equiv 0 \text{ mod } 2\}$
- $B_{i,odd}^{+1} := \{x \in B_i^{-1} | n_x \equiv 1 \text{ mod } 2\}$

$\omega$ must choose precisely one of these two pieces for each $i$. Assume without loss that $B_{i,even} \in \omega$. It is clear that $B_{i,even}^{-1} \cap B_{i,even} = B_{i,even}^{-1} \in \omega$ is sent, by $\theta_g$, to $B_{i,odd}^{-1}$ and so $B_{i,odd}^{-1} \in \theta_g(\omega)$. From the assumption that $\theta_g(\omega) = \omega$, we can conclude that $B_{i,odd}^{-1} \in \omega$. As ultrafilters are upwardly closed, we know also that $B_{i,odd} \in \omega$, which is a contradiction.

This freeness gives us a tool to understand the structure of $S_{\inf}$.

**Lemma 3.50.** Let $\{X_i\}$ be a sequence of graphs and let $G$ be a group which acts partially on each $X_i$. If $G$ fixes any sequence in $\{X_i\}$ then the partial action is not free on $\partial \beta X$.

**Proof.** Let $\theta_g$ denote the disjoint union of the $\theta_g^i$ arising from the partial action of $G$ on each $X_i$. To prove this it is enough to show that there is a single $\omega \in \partial \beta X$ that is fixed by the action of some $g \in G$. The hypothesis that $G$ fixes a sequence gives us $x := \{x_n\}_I$ with $I$ infinite and $\theta_g(\{x_n\}) = \{\theta_g^n(x_n)\}_I = \{x_n\}_I$.

Now consider an ultrafilter $\omega \in \partial \beta X$ that picks $x$. Then this ultrafilter $\omega$ is an element of $D_{\partial \beta X}$ as $x \subset D_{\partial \beta X}$. Now for any $A \in \omega$ and consider the intersection $A \cap x$. This is fixed by the action of $g$, as it is a subset of $x$. Hence we have: $\theta_g(A \cap x) \in \theta_g(\omega)$ for every $A \in \omega$. As $\theta_g(\omega)$ is an ultrafilter $A \in \theta_g(\omega)$, so in particular $\omega \subseteq \theta_g(\omega)$, whence $\theta_g(\omega) = \omega$. \qed

Recall that the inverse monoid $S_{\inf}$ is represented geometrically by partial bijections on $I(X)$. This representation gives us access to the geometry of $X$, which we can utilise, in addition to Lemma 3.50, to understand the structure of $S_{\inf}$.

**Lemma 3.51.** Consider the inverse monoid $S_{\inf}$ as a submonoid of $I(X)$. Then the following hold:

1. $S_{\inf}$ has the property that ($g \neq e_G$ and $\theta_g \neq 0$) implies $\theta_g$ is not an idempotent;
2. $S_{\inf}$ is 0-E-unitary;
(3) \( S_{\inf} \) has maximal element set \( \{ \theta_g : g \in F_k \} \).

Proof.

(1) We prove that no non-zero \( \theta_g \) are idempotent. To do this we pass to the induced action on \( \beta X \). We observe that if \( \theta_g \) is idempotent on \( X \) then it extends to an idempotent on \( \beta X \), hence on the boundary \( \partial \beta X \). \( \theta_g \) is non-zero implies that there is a non-principal ultrafilter \( \omega \) in the domain \( D_{\theta_g} \). The result then follows from the observation that \( \theta_g \circ \theta_g(\omega) = \theta_g(\omega) \) implies that \( \theta_g \) must now fix the ultrafilter \( \theta_g(\omega) \), which by Lemma \ref{sec:3.49} cannot happen.

(2) For 0-E-unitary it is enough to prove that \( f \leq \theta_g \) implies \( \theta_g \in E(S) \). Again, we extend the action to \( \beta X \). We observe that if \( \theta_g \) contains an idempotent, then we can build a sequence of elements of \( x_i \in f \cap D_{\theta_g} \cap X \) such that \( \theta_g \) fixes the sequence, and hence fixes any ultrafilter \( \omega \) that picks this sequence by Lemma \ref{sec:3.50}. This is a contradiction, from where we deduce that the only situation for which \( f \leq \theta_g \) is precisely when \( g = e_G \) hence trivially \( e \leq \theta_g \) implies \( \theta_g \in E(S) \). For the general case, we remark that by the above statement coupled with the dual prehomomorphism property shows that \( f \leq s \) implies \( s \leq \theta_e G \), hence is an idempotent.

(3) We construct the maximal elements. Observe that using the dual prehomomorphism it is clear that every non-zero word \( s \in S \) lives below a non-zero \( \theta_g \). So it is enough to prove that for \( \theta_g, \theta_h \neq 0, \theta_g \leq \theta_h \Rightarrow \theta_g = \theta_h \). Let \( \theta_g \leq \theta_h \). This translates to \( \theta_h \theta_g^* \theta_g = \theta_g \), hence for all \( x \in \hat{D}_{\theta_g} \cap X \) that \( \theta_g(x) = \theta_g(x) \). Hence \( \theta_g^* \theta_g \in E(S) \). From here we see that \( \theta_g^* \theta_g \leq \theta_e \). From (2) we can deduce: \( \theta_g^* \theta_h \leq \theta_g^{-1} \theta_h \) implies \( \theta_g^{-1} \theta_h \in E(S_{\inf}) \).

By (1) this implies \( \theta_g^{-1} \theta_h = \theta_e \), and this happens if and only if \( g^{-1} h = e \), i.e \( g = h \).

Appealing to the machinery we developed earlier in Propostions \ref{sec:3.39} and \ref{sec:3.40} we get the following corollary immediately.

**Corollary 3.52.** The inverse monoid \( S_{\inf} \) is strongly 0-F-inverse.

We now have enough tools to prove the general version of Theorems \ref{sec:3.9} and \ref{sec:3.14}.

**Theorem 3.53.** Let \( \{ X_i \} \) be a sequence of finite graphs of large girth and vertex degree uniformly bounded above by \( 2k \) and let \( X \) be the corresponding space of graphs. Then the boundary coarse Baum-Connes conjecture holds for \( X \).

Proof. Combining the results in the previous section we know that a free group of rank \( k \) acts partially freely on the boundary \( \partial \beta X \) in such a way as to give us a second countable locally compact Hausdorff étale topological groupoid \( G_X \). This groupoid implements an isomorphism \( G(X)|_{\partial \beta X} \cong \partial \beta X \rtimes G_X \). We also know using Corollary \ref{sec:3.52} that the inverse monoid generated by the infinite support elements \( S_{\inf} \) is a strongly 0-F-inverse monoid, admitting a 0-restricted idempotent pure prehomomorphism onto \( F_k \). Hence the groupoid \( G_X \) admits a proper continuous groupoid homomorphism onto \( F_k \), and so has the Haagerup property by Corollary \ref{sec:3.42}.
We can now conclude the Theorem by remarking that the isomorphism of groupoids $G(X)|_{\partial \beta X} \cong \partial \beta X \rtimes \mathcal{G}_X$ turns the Baum-Connes conjecture for $G(X)|_{\partial \beta X}$ into a specific case of the Baum-Connes conjecture for $\mathcal{G}_X$. As $\mathcal{G}_X$ has the Haagerup property we can conclude that the Baum-Connes assembly map with any coefficients is an isomorphism $\cite{Tu00}$ and so the assembly map required for the boundary conjecture is also an isomorphism. □

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