From Spinning Primaries to Permutation Orbifolds

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ABSTRACT

We carry out a systematic study of primary operators in the conformal field theory of a free Weyl fermion. Using $SO(4,2)$ characters we develop counting formulas for primaries constructed using a fixed number of fermion fields. By specializing to particular classes of primaries, we derive very explicit formulas for the generating functions for the number of primaries in these classes. We present a duality map between primary operators in the fermion field theory and polynomial functions. This allows us to construct the primaries that were counted. Next we show that these classes of primary fields correspond to polynomial functions on certain permutation orbifolds. These orbifolds have palindromic Hilbert series.

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1 Introduction

The remarkable success of the conformal bootstrap\cite{1,2,3,4} suggests that algebraic structures present in conformal field theory (CFT) can profitably be exploited to extract highly nontrivial information about the CFT. In the papers \cite{5,6} a systematic approach towards manifesting and exploiting some of these algebraic structures was outlined. The key result is that the algebraic structure of CFT defines a two dimensional topological field theory (TFT2) with $SO(4,2)$ invariance. Crossing symmetry is expressed as associativity of the algebra of local CFT operators. A basic observation which is at the heart of this result, is that the free four dimensional CFT of a scalar field can be formulated as an infinite dimensional associative algebra. This algebra admits a decomposition into linear representations of $SO(4,2)$, and is equipped with a non-degenerate bilinear product. A concrete application of these ideas has enabled a systematic study of primaries in bosonic free field theories in four dimensions, for scalar, vector and matrix models\cite{7,8}. For closely related ideas see \cite{9}.

We know from the AdS/CFT correspondence\cite{10,11,12} that strongly coupled CFTs have a dual holographic gravitational description. The combinatorics of the matrix model Feynman diagrams plays an important role in this holography. In this setting the TFT2
structure also appears as a powerful organizing structure, explicating algebraic structures that were not previously appreciated\cite{13,14,15,16}. Thus, it seems that the TFT2 idea is rich enough to incorporate the algebraic structure emerging both from the conformal symmetry, and from the color combinatorics.

In this paper we extend the study of \cite{7,8} by carrying out a systematic study of primaries in free fermion field theories in four dimensions. In section 2 we obtain formulae for the counting of primary fields constructed from \(n\) copies of the fundamental fermion, using the characters of representations of \(so(4,2)\). For a beautiful discussion of these characters, see \cite{17}. By specializing to particular classes of primaries, we can make the counting formulae very explicit. These special classes of primaries obey extremality conditions stated using relations between the charges under the Cartan subgroup of \(SO(4,2)\)). The construction of primary fields is then mapped to a problem of determining multi-variable polynomials subject to a system of algebraic and differential constraints. This relies on a function space realization of the conformal algebra, which is explained in section 3. We give concrete examples of polynomials obeying the constraints and the associated primary operators. Finally, in the last section we verify that the Hilbert series for the counting of extremal primaries are palindromic. The palindromy property of Hilbert series is indicative that the ring being enumerated is Calabi-Yau. It it interesting that palindromic Hilbert series also arise for moduli spaces of supersymmetric vacua of gauge theories, as found in \cite{18,19}.

2 Counting Primaries

We enumerate the \(SO(4,2)\) irreducible representations appearing among the composite fields made out of \(n = 2, 3, \cdots\) copies of a free chiral fermion field. The fermions are Grassman fields, so there is a sign change when two fields are swapped. Consequently, we should be taking the antisymmetric product of the \(SO(4,2)\) representations. Enumerating the primaries entails decomposing, the antisymmetrized tensor product \(\text{Asym}^n(W_+^n)\) into irreducible representations, where \(W_+^n = D_{\frac{1}{2},0}^n\) in the notation of \cite{17}. After obtaining a general formula in terms of an infinite product, we specialize to primaries that obey extremality conditions, that relate their dimension to their spin. For these primaries using results from \cite{22}, we find simple explicit formulas for the counting.

2.1 Generalities

The basic formula we use in this section states

\[
\det(1 + tM) = \sum_{n=0}^{\infty} t^n \chi_{(1^n)}(M)
\]  

(2.1)
where $\chi_{(1^n)}(M)$ is the trace over the antisymmetrized product of $n$ copies of $M$. From formula (3.44) of [17] we know the character of a left handed Weyl fermion is

$$\chi_{W_+}(s, x, y) = s^2 \chi_{1/2}(x) - s \chi_{1/2}(y)$$

$$= s^2 \sum_{q=0}^{\infty} s^q \chi_{2q+1/2}(x) \chi_{q+1/2}(y)$$

$$= \text{Tr}_{W_+}(M)$$

(2.2)

with $M = s^{D_3}x^{j_3}y^{j_3}$. It is straightforward to verify that

$$\text{det}(1 + tM) = \prod_{q=0}^{\infty} \prod_{a=-\frac{1}{2}}^{\frac{1}{2}} \prod_{b=-\frac{1}{2}}^{\frac{1}{2}} (1 + ts^{2q+1}x^a y^b)$$

(2.3)

Applying (2.1) we find the generating function of the characters of the antisymmetrized tensor products of the free Weyl fermion representation

$$Z(t, s, x, y) = \prod_{q=0}^{\infty} \prod_{a=-\frac{1}{2}}^{\frac{1}{2}} \prod_{b=-\frac{1}{2}}^{\frac{1}{2}} (1 + ts^{2q+q}x^a y^b) = \sum_{n=0}^{\infty} t^n \chi_{(1^n)}(s, x, y)$$

(2.4)

By expanding $Z(t, s, x, y)$ as a series in $t$ we can easily read off the character of the antisymmetrized tensor products of $n$ copies of the free Weyl fermion representation

$$\chi_{(1^n)}(s, x, y) = \sum_{[\Delta, j_L, j_R]} N_{[\Delta, j_L, j_R]} \chi_{[\Delta, j_L, j_R]}(s, x, y)$$

(2.5)

The coefficients $N_{[\Delta, j_L, j_R]}$ count how many times irrep $A_{[\Delta, j_L, j_R]}$ (in the notation of [17]) appears in $\text{Asym}_n(W_+)$. Hence, $N_{[\Delta, j_L, j_R]}$ are non-negative integers. The case that $n = 2$ is subtle because some of the irreducible representations appearing in the above decomposition are short. We will consider $n = 2$ separately in detail below. For $n \geq 3$ we have [17]

$$\chi_{[\Delta, j_1, j_2]}(s, x, y) = \frac{s^\Delta \chi_{j_1}(x) \chi_{j_2}(y)}{(1 - s \sqrt{xy})(1 - s \sqrt{x})(1 - s \sqrt{y})(1 - \frac{s}{\sqrt{xy}})}$$

(2.6)

It is useful to define

$$Z_n(s, x, y) = \sum_{[\Delta, j_1, j_2]} N_{[\Delta, j_1, j_2]} s^\Delta \chi_{j_1}(x) \chi_{j_2}(y)$$

(2.7)

so that

$$Z_n(s, x, y) = (1 - s \sqrt{xy})(1 - s \sqrt{x})(1 - s \sqrt{y})(1 - \frac{s}{\sqrt{xy}}) \chi_{(1^n)}(s, x, y)$$

(2.8)
The right hand side of (2.7) is a sum of (products of) SU(2) characters. Following [23], it can be simplified by using the orthogonality of SU(2) characters. The result is most easily stated in terms of the generating function

\[ G_n(s, x, y) = \left[ (1 - \frac{1}{x})(1 - \frac{1}{y})Z_n(s, x, y) \right] \geq \sum_{\Delta, j_1, j_2} N[\Delta, j_1, j_2] s^\Delta x^{j_1} y^{j_2} \] (2.9)

The subscript \( \geq \) is an instruction to keep only non negative powers of \( x \) and \( y \).

It is easy to check that this agrees with standard character computations. For example, the expansion

\[ G_3(s, x, y) = s^{\frac{11}{4}} x \sqrt{y} + s^{\frac{13}{4}} x^2 \frac{y}{2} + s^{\frac{15}{4}} x^2 y + s^{\frac{17}{4}} x^3 \frac{y}{2} + s^{\frac{19}{4}} x^2 y^2 + s^{\frac{17}{4}} x^2 y^2 \]

\[ + s^{\frac{15}{4}} x^2 y^2 + s^{\frac{19}{4}} x^2 y^2 + s^{\frac{17}{4}} x^2 y^2 + \frac{1}{2} s^{\frac{19}{4}} x^3 y^2 + s^{\frac{17}{4}} x^2 y^2 + \ldots \] (2.10)

can be reproduced using characters. The relevant Schur polynomial for this case is calculated as follows

\[ \chi_{(1^3)}(s, x, y) = \frac{1}{6} \left[ (\chi_L(s, x, y))^3 - 3\chi_L(s^2, x^2, y^2)\chi_L(s, x, y) + 2\chi_L(s^3, x^3, y^3) \right] \] (2.11)

Using Mathematica, we find the following terms

\[ \chi_{(1^3)}(s, x, y) = A_{\left(\frac{1}{4}, \frac{1}{4}, 1 \frac{1}{4}\right)} + A_{\left(\frac{1}{2}, \frac{3}{2}, 0 \frac{1}{2}\right)} + A_{\left(1 \frac{3}{4}, \frac{1}{4}, 1 \frac{1}{4}\right)} + A_{\left(\frac{1}{2}, \frac{3}{2}, 2 \frac{1}{2}\right)} \]

\[ + A_{\left(\frac{1}{2}, \frac{1}{2}, 3 \frac{1}{2}\right)} + A_{\left(\frac{1}{2}, \frac{5}{2}, 2 \frac{1}{2}\right)} + A_{\left(\frac{1}{2}, \frac{5}{2}, 4 \frac{1}{2}\right)} + A_{\left(\frac{3}{2}, \frac{1}{2}, 2 \frac{1}{2}\right)} + A_{\left(\frac{3}{2}, \frac{1}{2}, 3 \frac{1}{2}\right)} + A_{\left(\frac{3}{2}, \frac{1}{2}, 4 \frac{1}{2}\right)} \]

\[ + A_{\left(\frac{3}{2}, \frac{1}{2}, 3 \frac{1}{2}\right)} + \ldots \] (2.12)

in complete agreement with (2.11).

The case that \( n = 2 \) is complicated by the fact that representations that include null states appear in the decomposition. The condition for a short multiplet [23] is \( \Delta = f(j_1) + f(j_2) \) with \( f(j) = 0 \) if \( j = 0 \) or \( f(j) = j + 1 \) if \( j > 0 \). For \( n = 2 \) the decomposition includes a primary with \( \Delta = 3 \) and \( j_1 = j_2 = 0 \) which is not short, as well as primaries with \( \Delta = 2j \) \( j_1 = (2j-1)/2 \) and \( j_2 = (2j-3)/2 \) which are short representations and hence have null states. These null states (and their descendants) must be removed. These short representations arise because their primary operators are conserved higher spin currents

\[ \partial_{\mu} J^{\mu_{j_1} \cdots \mu_{j_2}} = 0 \] (2.13)
The subtraction of null states is achieved by removing the $\Delta = 3$ primary that does not need to be subtracted, dividing by $1 - s/\sqrt{xy}$ which removes the null descendents and then putting the original primary back in. In the end we have

$$G_2(s, x, y) = \left[ (1 - \frac{1}{x})(1 - \frac{1}{y}) (Z_2(s, x, y) - s^3) \right] \frac{1}{1 - \frac{s}{\sqrt{xy}}} + s^3 \geq s^3$$

This is indeed the correct result $[25]$.

### 2.2 Leading Twist Primaries

By restricting to well defined classes of primaries, we can significantly simplify the counting formulas of the previous section. The biggest simplification comes from focusing on the leading twist primaries, which have quantum numbers $[\Delta, j_1, j_2] = [\frac{n(n+2)}{2}, q, \frac{n(n+1)}{4} + \frac{q}{2}, \frac{n(n-1)}{4} + \frac{q}{2}]$. Each such primary operator comes in a complete spin multiplet of $(\frac{n(n+1)}{2} + q + 1)(\frac{n(n-1)}{2} + q + 1)$ operators. Choosing the operator with highest spin corresponds to studying primaries constructed using a single component $P_z$ of the momentum four vector operator. To count the leading twist primaries we can count this highest spin operator in each multiplet. The corresponding generating function is $G_{\text{max}}^n(s, x, y)$. This generating function is obtained after a simple modification of the results of the previous section. First, we replace $\chi_{\text{Asym}}(s, x, y)$ with a new function $\chi_{\text{max}}^n(s, x, y)$, by keeping only the highest spin state from each multiplet in the product

$$\prod_{q=0}^{\infty} (1 + ts^{\frac{n}{2}+q}x^{\frac{n}{2}} + s^{\frac{n}{2}}y^{\frac{n}{2}}) = \sum_{n=0}^{\infty} t^n \chi_{\text{max}}^n(s, x, y)$$

(2.15)

The leading twist primaries are constructed using a single component of the momentum, that raises left and right spin maximally. Consequently in (2.5) we replace

$$(1 - s\sqrt{xy})(1 - s\sqrt{\frac{x}{y}})(1 - s\sqrt{\frac{y}{x}})(1 - s\sqrt{\frac{s}{\sqrt{xy}}}) \rightarrow (1 - s\sqrt{xy})$$

(2.16)

Finally, for each spin multiplet we keep only 1 state so there is no longer any need to replace the multiplet of spin states by a single state when we count. The final result is

$$G_{\text{max}}^n(s, x, y) = (1 - s\sqrt{xy})\chi_{\text{max}}^n(s, x, y)$$

$$= \sum_{\Delta, j_1, j_2} N_{[\Delta, j_1, j_2]} s^{\Delta \cdot j_1} y^{j_2}$$

(2.17)

where $N_{[\Delta, j_1, j_2]}$ is the number of leading twist primaries of dimension $\Delta$ and spin $(j_1, j_2)$. For the leading twist primaries, once $n$ and the dimension of the operator is specified, the
spin of the primary is fixed. Consequently, we need not track the $x$ and $y$ dependence. This leads to the formula

$$\sum_{n=0}^{\infty} t^n G_n^\text{max}(s) = (1 - s) \prod_{q=0}^{\infty} (1 + ts^0)^q \equiv (1 - s)F(t, s) \quad (2.18)$$

We can obtain explicit expressions for $G_n^\text{max}(s)$ by developing $F(t, s)$ in a Taylor series. Define

$$f_q(t, s) = \frac{\partial^q}{\partial t^q} \log F(t, s) \quad (2.19)$$

Straight forward computation gives

$$f_q(t, s) = \sum_{k=0}^{\infty} (-1)^{q+1}(q - 1)!s^{\frac{3k+1}{2}}q^k + \cdots + \frac{s^{3n}}{1 - s^{k+\frac{3}{2}}}q^k \equiv (1 - s)^{f_q(t, s)} \quad (2.20)$$

so that, after reinstating $x$ and $y$, we have

$$f_k(0, s, x, y) = (k - 1)!(-1)^{k-1} \frac{s^{3k}x^\frac{k}{2}}{1 - s^k x^\frac{k}{2}y^\frac{k}{2}} \quad (2.21)$$

Explicit expressions for $G_n^\text{max}$ are now easily obtained. For example

$$G_3^\text{max}(s, x, y) = \frac{1}{3!} (1 - s\sqrt{xy}) \frac{\partial^3 F}{\partial t^3} \bigg|_{t=0} = \frac{1}{3!} (1 - s\sqrt{xy})(f_3 + 3f_1 f_2 + f_1^2)$$

$$= \frac{s^{15/2}x^3y^{3/2}}{(1 - s^2xy)(1 - s^3x^2y^2)} \quad (2.22)$$

Similarly

$$G_4^\text{max}(s, x, y) = \frac{s^{12}x^5y^3}{(1 - s^2xy)(1 - s^3x^2y^2)(1 - s^4x^2y^2)} \quad (2.23)$$

It is possible to obtain a general closed formula for $G_n^\text{max}(s)$. To make the argument as transparent as possible, again set $x = 1 = y$. Evaluate the derivative

$$\frac{\partial^n F}{\partial t^n} = \sum_{n_1, \ldots, n_q, k_1, \ldots, k_q} (n_1 k_1 + \cdots + n_q k_q)! \frac{n_1! \cdots n_q!(k_1)! \cdots (k_q)!}{k_1! \cdots k_q!} f_{k_1}^{n_1} \cdots f_{k_q}^{n_q} \delta_{n, n_1 k_1 + \cdots + n_q k_q} F \quad (2.24)$$

and use the formulas for the $f_k$’s to find

$$\frac{\partial^n F}{\partial t^n} \bigg|_{t=0} = \sum_{n_1, \ldots, n_q, k_1, \ldots, k_q} (-1)^{n-\sum n_i} n_i! s^{3k_1}_i \frac{s^{3k_1}}{1 - s^{k_1}} \cdots \frac{s^{3k_q}}{1 - s^{k_q}} \delta_{n, n_1 k_1 + \cdots + n_q k_q} \quad (2.25)$$

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Notice that this is a sum over conjugacy classes of $S_n$. The conjugacy class collects permutations with $n_q$ $k_q$-cycles. This interpretation follows because the coefficient

$$\frac{n!}{n_1! \cdots n_q! k_1^{n_1} \cdots k_q^{n_q}}$$

is the order of the conjugacy class. Each conjugacy class is weighted by the factor $(-1)^{\sum_i n_i}$ which is the signature of the permutation with $n_q$ $k_q$-cycles. There is a factor of $\frac{\text{det} V_{1n}}{1 - s}$ for each $k$-cycle in the permutation. The lowest weight discrete series irrep of $SL(2)$, built on a ground state with dimension $\frac{3}{2}$ has character

$$\chi_1(s) = \text{Tr}_{V_1}(s^{L_0}) = \frac{s^{\frac{3}{2}}}{1 - s}$$

Denote this irrep by $W_1$. It then follows that $(P_{[1n]}$ projects onto the antisymmetric irrep i.e. a single column of $n$ boxes)

$$\frac{1}{n!} \left. \frac{\partial^n F}{\partial t^n} \right|_{t=0} = \text{Tr}_{W_1}(P_{[1n]} s^{L_0}) = \frac{s^{\frac{3}{2}(n+1)}}{(1 - s)(1 - s^2)(1 - s^3) \cdots (1 - s^n)}$$

where the last equality follows from eqn (49) of [22], where these $SL(2)$ sector primaries were studied in the language of oscillators. We now easily find

$$G_n^{\text{max}}(s, x, y) = (s \sqrt{xy})^{\frac{3n-1}{2}} (s^{\frac{3}{2}} \sqrt{x})^n \prod_{k=2}^{n} \frac{1}{1 - (s \sqrt{xy})^k}$$

### 2.3 Extremal Primaries

We now consider the class of primaries with charges

$$\Delta = \frac{3n}{2} + q ; \quad J_3^L = \frac{n}{2} + \frac{q}{2}$$

The charge $J_3^R$, which is part of $SU(2)_R$, is not constrained. These primaries fill out complete multiplets of $SU(2)_R$. They are constructed using two components of the momentum four vector operator which are complex linear combinations of the (hermitian) $P_{\mu}$. Introduce a generating function $G_n^{z,w}(s, x, y)$, given by

$$G_n^{z,w}(s, x, y) = \left[ (1 - \frac{1}{y}) Z_n^{z,w}(s, x, y) \right]_+$$

where $Z_n(s, x, y)$ is defined by

$$Z_n^{z,w}(s, x, y) = (1 - s \sqrt{xy})(1 - s \sqrt{x/y}) \chi_n(s, x, y)$$
\[
\sum_{n=0}^{\infty} t^n \chi_n(s, x, y) = \prod_{q=0}^{\frac{3}{2}} \prod_{b=-\frac{1}{2}}^{\frac{3}{2}} (1 + ts^{\frac{3}{2} + q} x^{\frac{2}{3} + b} y^{\frac{1}{3} + b}) \equiv F^{(2)}(t, s, x, y) \quad (2.33)
\]

It is again possible to derive closed expressions for the generating functions \(Z_n^{z,w}(s, x, y)\) and \(G_n^{z,w}(s, x, y)\). Introduce the functions

\[
f_k(t, s, x, y) \equiv \frac{\partial^{k-1}}{\partial t^{k-1}} \log F^{(2)} = (-1)^{k-1}(k-1)! \sum_{q=0}^{\frac{3}{2}} \sum_{m=-\frac{1}{2}}^{q} \frac{s^{3q + \frac{3}{2} x^q} t^{\frac{3}{2} q^q} x^{\frac{1}{3} q^q} y^{\frac{2}{3} q^q}}{(1 + ts^{\frac{3}{2} + q} x^{\frac{2}{3} + b} y^{\frac{1}{3} + b})^k} \quad (2.34)
\]

It is simple to establish that

\[
f_k(0, s, x, y) = (-1)^{k-1}(k-1)! \frac{s^{\frac{3k}{2} x^\frac{k}{2}}}{(1 - s^{\frac{k}{3} x^\frac{k}{3} y^\frac{k}{3}})(1 - s^{\frac{k}{3} x^\frac{k}{3} y^\frac{k}{3}})} \quad (2.35)
\]

Exactly as above we have

\[
\frac{\partial^m F^{(2)}}{\partial t^n} \bigg|_{t=0} = \sum_{n_1, \ldots, n_q k_1, \ldots, k_q} \frac{(n_1 k_1 + \cdots + n_q k_q)!}{n_1! \cdots n_q! (k_1)! \cdots (k_q)!} f_{k_1}^{n_1} \cdots f_{k_q}^{n_q} \delta_{n_1 k_1, \ldots, n_q k_q} \quad (2.36)
\]

Inserting the formulas for the \(f_k\)'s expressions for the \(Z_n(s, x, y)\) now follows from (2.32). To extract spin multiplets, we need to compute

\[
G_n^{z,w}(z, w) = \left[ Z_n(s, x, y) \left( 1 - \frac{1}{y} \right) \right]_z = \frac{1}{2\pi i} \oint_C dz \frac{(1 - \frac{1}{y}) Z_n(s, x, z^2)}{z - \sqrt{y}} \quad (2.37)
\]

As an example, the generating functions counting the extremal primaries constructed from 3 fields are given by

\[
Z_3^{z,w}(s, x, y) = s^{\frac{13}{2} x^\frac{5}{2} y^\frac{3}{2}} \frac{y^{\frac{3}{2} + s^2 x y^\frac{3}{2} + s \sqrt{x} (1 + y)(1 + y^2)}}{(1 - s^2 x y)(1 - s^3 x^\frac{3}{2} y^\frac{3}{2})(1 - s^2 x y^\frac{3}{2})(1 - s^3 x^\frac{3}{2} y^\frac{3}{2})} \quad (2.38)
\]

\[
G_3^{z,w}(s, x, y) = s^{\frac{13}{2} x^\frac{5}{2} (1 + s \sqrt{x} y^\frac{3}{2})} \frac{1}{(1 - s^4 x^2)(1 - s^2 x y)(1 - s^3 x^\frac{3}{2} y^\frac{3}{2})}
\]

\[
= s^{\frac{13}{2} x^\frac{5}{2}} + s^{\frac{15}{2} x^3 y^\frac{3}{2}} + s^{\frac{17}{2} x^2 y^3} + s^{\frac{19}{2} x^4 y^\frac{3}{2}} + s^{\frac{21}{2} x^3 y^3} + \cdots \quad (2.39)
\]
3 Construction

In this section we will explain how the counting of the previous section can be used to derive concrete formulas for the construction of the primary operators in the free fermion CFT. For the leading twist counting this is manifest. For the counting of extremal primaries, we will argue that our formulas can naturally be phrased as counting the multiplicities of symmetric group representations. The quantities being counted are then easily constructed using projectors onto these representations. In this analysis, a polynomial representation of $SO(4, 2)$ will play an important role. This representation is described in the next subsection, after which we describe the construction of leading twist primaries and then extremal primaries.

3.1 Polynomial rep

We use the following representation of $SO(4, 2)$

\[
K_\mu = \frac{\partial}{\partial x^\mu}
\]

\[
D = (x \cdot \frac{\partial}{\partial x} - \frac{3}{2})
\]

\[
M_{\mu\nu} = x_\mu \frac{\partial}{\partial x^\nu} - x_\nu \frac{\partial}{\partial x^\mu} + M_{\mu\nu}
\]

\[
P_\mu = (x^2 \frac{\partial}{\partial x^\mu} - 2x_\mu x \cdot \frac{\partial}{\partial x} + 3x_\mu + 2x^\nu M_{\mu\nu})
\]

In the formula above we should replace $M_{\mu\nu}$ by the relevant matrix representing the spin part of the conformal group. In Minkowski spacetime we have (the two possibilities correspond to taking either a left handed $(\frac{1}{2},0)$ or a right handed $(0,\frac{1}{2})$ spinor)

\[
\mathcal{M}^{\mu\nu} = \sigma^{\mu\nu} \quad \text{or} \quad \bar{\sigma}^{\mu\nu}
\]

where

\[
(\sigma^{\mu\nu})_{\alpha}^{\beta} = \frac{1}{4} (\sigma^\mu \bar{\sigma}^\nu - \sigma^\nu \bar{\sigma}^\mu)_{\alpha}^{\beta}
\]

\[
(\bar{\sigma}^{\mu\nu})_{\dot{\alpha}}^{\dot{\beta}} = \frac{1}{4} (\bar{\sigma}^\mu \sigma^\nu - \bar{\sigma}^\nu \sigma^\mu)_{\dot{\alpha}}^{\dot{\beta}}
\]

and

\[
\sigma^{\mu}_{\alpha\dot{\beta}} = (1, \bar{\sigma}) \quad \bar{\sigma}^{\mu\dot{\alpha}} = (1, -\sigma)
\]
In Euclidean space we have

\[ M_{\mu\nu} = \sigma_{\mu\nu} \equiv \frac{1}{4} (\sigma^\mu \bar{\sigma}^\nu - \sigma^\nu \bar{\sigma}^\mu) \] (3.9)

or

\[ M_{\mu\nu} = \bar{\sigma}_{\mu\nu} \equiv \frac{1}{4} (\bar{\sigma}^\mu \sigma^\nu - \bar{\sigma}^\nu \sigma^\mu) \] (3.10)

where now

\[ \sigma^\mu = (-i\vec{\sigma}, 1) \quad \bar{\sigma}^\mu = (i\vec{\sigma}, 1) \] (3.11)

The generators in Minkowski space close the algebra

\[ [M_{\rho\sigma}, M_{\phi\theta}] = \eta_{\theta\rho} M_{\phi\sigma} + \eta_{\phi\sigma} M_{\theta\rho} - \eta_{\rho\sigma} M_{\theta\phi} \]

\[ [P_{\mu}, P_{\nu}] = 0 = [K_{\mu}, K_{\nu}] \quad [P_{\beta}, K_{\alpha}] = 2\eta_{\alpha\beta} D - 2M_{\alpha\beta} \]

\[ [M_{\beta\rho}, K_{\alpha}] = \eta_{\alpha\rho} K_{\beta} - \eta_{\alpha\beta} K_{\rho} \quad [M_{\beta\rho}, P_{\alpha}] = \eta_{\alpha\rho} P_{\beta} - \eta_{\alpha\beta} P_{\rho} \]

\[ [D, P_{\mu}] = P_{\mu} \quad [D, K_{\mu}] = -K_{\mu} \quad [D, M_{\mu\nu}] = 0 \] (3.12)

The Euclidean generators obey the same algebra with \( \eta_{\mu\nu} \) replaced with \( \delta_{\mu\nu} \).

States in this representation correspond to polynomials in the spacetime coordinates \( x_\mu \) times a constant spinor \( \zeta_\alpha \), which transforms in the \((\frac{1}{2}, 0)\) if we study the theory of a left handed fermion, or in the \((0, \frac{1}{2})\) if we study a right handed fermion. The 2×2 matrix \( M_{\mu\nu} \) acts on this constant spinor. Further, \( \zeta_\alpha \) is Grassman valued to account for the fact that the fermions are anticommuting fields. Concretely, each operator corresponds to a state (by the state operator correspondence) and each state corresponds to a polynomial times the spinor (thanks to the representation we have just described)

\[ x_{\mu_1} \cdots x_{\mu_k} \zeta_\alpha \] (3.13)

To deal with operators constructed from a product of \( n \) copies of the basic fermion field, we consider a “multiparticle system”. When we move to the multiparticle system, we have polynomials on the \( n \) particle coordinates \( x'^I \), times the \( n \) particle spinor, obtained by taking the tensor product of \( n \) copies of \( \zeta_\alpha \)

\[ (\zeta \otimes \zeta \otimes \cdots \otimes \zeta)_{\alpha_1, \alpha_2, \ldots, \alpha_n} \] (3.14)

To write the generator of the conformal group, for the multiparticle system, we need the matrices

\[ M_{\mu\nu}^{(I)} = 1 \otimes \cdots \otimes 1 \otimes M_{\mu\nu} \otimes 1 \otimes \cdots \otimes 1 \] (3.15)
where the matrix $M_{\mu\nu}$ on the right hand side is the $2 \times 2$ matrix we introduced above and it appears as the $I$th factor on the right hand side. In total $M^{(I)}_{\mu\nu}$ has $n$ factors. The $n$-particle representation of $SO(4,2)$ includes

$$K_\mu = \sum_{I=1}^{n} \frac{\partial}{\partial \vec{x}_I^I}$$

(3.16)

$$P_\mu = \sum_{I=1}^{n} ((x_I^\rho x_I^\nu \frac{\partial}{\partial x_I^\mu} - 2x_I^\mu x_I^I \frac{\partial}{\partial x_I^I} + 3x_I^I + 2x_I^I \cdot 2x_I^I M^{(I)}_{\mu\nu})$$

(3.17)

The representations introduced above all have null states. This is to be expected, since the dimension of the free fermion field saturates the unitarity bound. For the $(\frac{1}{2}, 0)$ field in Minkowski spacetime, for example, the null state is exhibited by verifying that

$$\bar{\sigma}^\mu P_\mu \zeta = 0$$

(3.18)

for any choice of $\zeta$.

Let us now spell out the conditions that the polynomial $P_O$ corresponding to an operator $O$ must obey if the operator $O$ is a primary operator. The general polynomial $P_O$ will have spinor indices (it is constructed from a tensor product of copies of $\zeta$) as well as four vector indices inherited from the spacetime coordinates. There are three conditions that must be imposed: Primaries are annihilated by the special conformal generator $K_\mu$

$$[K_\mu, O] = 0$$

(3.19)

This implies that the corresponding polynomial is translation invariant

$$\sum_{I=1}^{n} \frac{\partial}{\partial \vec{x}_I^I} P_O = 0$$

(3.20)

Secondly, the equation of motion must be obeyed by each fermionic field. Finally, we require that the polynomials are in the antisymmetric representation of $S_n$. Since the $\zeta$s are Grassman variables, we must impose this condition if we are to get a non-zero primary upon translating back to the language of the fermion field theory.

The above set of constraints on the polynomials corresponding to primaries is not yet very useful. To obtain a more manageable set of constraints, we will motivate replacing the constraint coming from the equation of motion with a constraint that simply requires that each polynomial is holomorphic. Our first observation is that the operator $\bar{\sigma}^\mu P_\mu$, known as the Cauchy-Fueter operator, has been used to define regular functions of a quarternionic variable. This theory of regular functions is well developed[20]. An important result, is Fueter’s Theorem[21], which gives a method for constructing Cauchy-Fueter regular functions in terms of holomorphic functions. In view of Fueter’s theorem, we will replace the equation of
motion constraint with the constraint that the polynomials are holomorphic. Thus, in the end we search for translation invariant, holomorphic polynomials that are in the antisymmetric representation of $S_n$. We will manage to test that the counting of these polynomials matches the counting of primaries in complete generality, and for a number of examples, we will construct the primary corresponding to a given polynomial and explicitly verify that it is annihilated by $K_\mu$.

### 3.2 Leading Twist

The leading twist primaries are given by polynomials in a single complex variable $z^I$, $I = 1, 2, \ldots, n$. Any such polynomial is automatically holomorphic, so we need not worry about the equation of motion constraint. To solve the translation invariance condition, we work with the hook variables $Z^a$, $a = 1, 2, \ldots, n - 1$ defined by

$$Z^a = \frac{1}{\sqrt{a(a+1)}} (z^{(1)} + z^{(2)} + \cdots + z^{(a)} - az^{(a+1)})$$

(3.21)

Our problem is now reduced to constructing antisymmetric polynomials from the hook variables. By construction, it is clear that the degree $k$ polynomials belong to a subspace of $V_H \otimes_k H$ of $S_n$. We can characterize the antisymmetric subspace, that we want to extract, using representation theory. Towards this end, consider the following decomposition in terms of $S_n \times S_k$ irreps

$$V_H^{\otimes k} = \bigoplus_{\Lambda_1 \vdash n, \Lambda_2 \vdash k} V^{(S_n)}_{\Lambda_1} \otimes V^{(S_k)}_{\Lambda_2} \otimes V^{\text{Com}(S_n \times S_k)}_{\Lambda_1, \Lambda_2}$$

(3.22)

In the above expression, $\text{Com}(S_n \times S_k)$ is the algebra of linear operators on $V_H^{\otimes k}$ that commute with $S_n \times S_k$. This decomposition has been studied in detail in [22]. The $Z$ variables are commuting so that we need to consider the case that $\Lambda_2 = [k]$ the symmetric representation given by a Young diagram with a single row of $k$ boxes. The resulting multiplicity is given by the coefficient of $q^k$ in

$$Z_{SH}(q; \Lambda_1) = (1 - q) q^{\sum_i c_i(c_i-1)/2} \prod_b \frac{1}{(1 - q^{h_b})} = \sum_k q^k Z^k_{SH}(\Lambda_1)$$

(3.23)

Here $c_i$ is the length of the $i$'th column in $\Lambda_1$, $b$ runs over boxes in the Young diagram $\Lambda_1$ and $h_b$ is the hook length of the box $b$. Evaluating this formula for the antisymmetric representations, for which $\Lambda_1$ is a single column, gives [22]

$$q^n(n-1) \over (1 - q^2) \cdots (1 - q^n)$$

(3.24)
After accounting for the dimension of \( n \) elementary fermion fields and reinstating \( x \) and \( y \), this is in complete agreement with (2.29).

Now that we have verified that the number of translation invariant, holomorphic polynomials in the antisymmetric representation of \( S_n \) agrees with the counting of leading twist primaries, we can move on to a construction formulas for these primaries. Indeed, the relevant polynomials are given by acting with a projector onto the antisymmetric representation, on the hook variables. This polynomial multiplies an anticommuting tensor product of Grassman valued constant spinors. The projector from the tensor product of \( k \) copies of the hook onto the antisymmetric representation of \( S_n \) is

\[
P(1^n) = \frac{1}{n!} \sum_{\sigma \in S_n} \text{sgn}(\sigma) \Gamma_k(\sigma)
\]

(3.25)

where \( \text{sgn}(\sigma) \) is the signature of permutation \( \sigma \). When acting on a product of variables, say \( Z^{a_1} Z^{a_2} \cdots Z^{a_k} \) we have

\[
\Gamma_k(\sigma) = \Gamma_{(n-1,1)}(\sigma) \otimes \cdots \otimes \Gamma_{(n-1,1)}(\sigma)
\]

(3.26)

where on the right hand side we take a tensor product (the usual Kronecker product) of \( k \) copies of the matrices of the hook representation of \( S_n \), labeled by a Young diagram with \( n-1 \) boxes in the first row and 1 box in the second row. Our construction formula is

\[
\frac{1}{n!} \sum_{\sigma \in S_n} \text{sgn}(\sigma) \Gamma_k(\sigma)_{a_1 a_2 \cdots a_k, b_1 b_2 \cdots b_k} Z^{b_1} Z^{b_2} \cdots Z^{b_k}(\zeta_1 \otimes \zeta_2 \cdots \otimes \zeta_n)_{a_1 \cdots a_n}
\]

(3.27)

The above formula produces an expression of the form \( \sum_i \hat{n}_i P_i(Z) \) where \( \hat{n}_i \) are unit vectors inside the carrier space of \( V_H^{\otimes k} \) and \( P_i(Z) \) are the polynomials that correspond to primary operators. To translate polynomials into momenta, the formula [7]

\[
z^k \leftrightarrow \frac{(-1)^k P_k}{2^k k!}
\]

(3.28)

is very useful. We will now gives some examples of polynomials obtained from formula (3.27). We will also translate these polynomials into primary operators.

If we consider \( n = 2 \) fields, there is a single hook variable given by \( Z = z_1 - z_2 \). To find a polynomial that is antisymmetric under swapping \( 1 \leftrightarrow 2 \), we must raise \( Z \) to an odd power. Thus, we predict that primaries for the fermion fields correspond to the polynomials

\[
(z_1 - z_2)^{2s+1} = \sum_{k=0}^{2s+1} \frac{(2s + 1)!}{k!(2s - k + 1)!} (-1)^k z_1^{2s-k+1} z_2^k
\]

(3.29)

Translating the polynomial variables into momenta we find the following primary

\[
|\psi\rangle = \sum_{k=0}^{2s+1} \frac{(-1)^k}{((2s - k + 1)k!)^2} P^k(\frac{3}{2}, \frac{1}{2}, 0) \otimes P^{2s-k+1}(\frac{3}{2}, \frac{1}{2}, 0)
\]

(3.30)
where, because our fields are fermions, we have
\[
\langle \frac{3}{2}, \frac{1}{2}, 0 \rangle_1 \otimes | \frac{3}{2}, \frac{1}{2}, 0 \rangle_2 = -\langle \frac{3}{2}, \frac{1}{2}, 0 \rangle_2 \otimes | \frac{3}{2}, \frac{1}{2}, 0 \rangle_1 \tag{3.31}
\]
Thus, our expression for the fermionic primaries built from two fields are
\[
\sum_{k=0}^{2s+1} \frac{(-1)^k}{((2s-k+1)!)^2} (\partial_1 + i\partial_2)^k \psi(x)(\partial_1 + i\partial_2)^{2s-k+1} \psi(x) \tag{3.32}
\]
which exactly matches the form of the higher spin currents\cite{26,27}.

For \( n = 3 \) fields it is easy to see that
\[
(z_1 - z_2)(z_1 - z_3)(z_2 - z_3) \tag{3.33}
\]
is holomorphic, translation invariant and in the antisymmetric representation of \( S_3 \). The corresponding primary operator can be simplified to
\[
\psi(x)(\partial_1 + i\partial_2)\psi(x)(\partial_1 + i\partial_2)^2 \psi(x) \tag{3.34}
\]
It is not difficult to see that this operator is indeed annihilated by \( K_\mu \), as discussed in Appendix A.

### 3.3 Extremal Primaries

In this section we will consider the construction of extremal primaries, which correspond to polynomials in two holomorphic coordinates, \( z \) and \( w \). We will characterize these polynomials by two degrees, one for \( Z \) and one for \( W \). Polynomials of degree \( k \) in \( Z \) and of degree \( l \) in \( W \) belong to a subspace of \( V^\otimes k_H \otimes V^\otimes l_H \) of \( S_n \). The relevant decompositions in terms of \( S_n \times S_k \) irreducible representations are
\[
\begin{align*}
V^\otimes k_H &= \bigoplus_{\Lambda_1 + \Lambda_2 = k} V^{(S_n)}_{\Lambda_1} \otimes V^{(S_k)}_{\Lambda_2} \otimes V^{\text{Com}}_{\Lambda_1, \Lambda_2} \\
V^\otimes l_H &= \bigoplus_{\Lambda_3 + \Lambda_4 = l} V^{(S_l)}_{\Lambda_3} \otimes V^{(S_l)}_{\Lambda_4} \otimes V^{\text{Com}}_{\Lambda_3, \Lambda_4}
\end{align*}
\tag{3.35}
\]
The tensor product \( V^\otimes k_H \otimes V^\otimes l_H \) is a representation of
\[
\mathbb{C}(S_n) \otimes \mathbb{C}(S_k) \otimes \mathbb{C}(S_l) \otimes \mathbb{C}(S_l) \tag{3.36}
\]
The \( Z \) and \( W \) variables are commuting so that \( \Lambda_2 \otimes \Lambda_4 = [k] \otimes [l] \) is the trivial representation of \( S_k \times S_l \). The multiplicity with which a given \( S_n \times S_k \) irrep \( (\Lambda_1, \Lambda_2) \) appears is given by the dimension of the irreducible representation of the commutants \( \text{Com}(S_n \times S_l) \) in \( V^\otimes k_H \).

Recall that since our polynomials multiply a product of anticommuting Grassman spinors, we want to project to states in \( V^\otimes k_H \otimes V^\otimes l_H \) which are in the totally antisymmetric irreducible
Thus, for the number of primaries constructed from $z_i, w_i$ we find that the number of $S_k \times S_l$ invariants and $S_n$ antisymmetric representations is

$$\sum_{\Lambda \vdash n} \text{Mult}(\Lambda^T, [k]; S_n \times S_k) \ \text{Mult}(\Lambda_1, [l]; S_n \times S_l)$$

(3.37)

Thus, for the number of primaries constructed from $z_i, w_i$ we get

$$\sum_{\Lambda \vdash n} Z^z_{SH}(\Lambda) Z^w_{SH}(\Lambda^T)$$

(3.38)

The above integer gives the number of primaries in the free fermion CFT, of weight $\frac{3n}{2} + k + l$, with spin $(J^l_3, J^R_3) = (\frac{k+l+n}{2}, \frac{k-l}{2})$. The generating function $Z^{z,w}_n(s, x, y)$ which encodes all $k, l$ is given by

$$Z^{z,w}_n(s, x, y) = s^{\frac{3n}{2}} x^\frac{n}{2} \sum_{\Lambda \vdash n} Z_{SH}(s\sqrt{xy}, \Lambda) Z_{SH}(s\sqrt{x}, \Lambda^T)$$

(3.39)

where $\Lambda$ is a partition of $n$ and we can use the formula (3.23). It is straight forwards to check, for example, that

$$Z^{z,w}_n(s, x, y) = s^\frac{3n}{2} x^\frac{n}{2} \left( Z_{SH}(s\sqrt{xy}, \begin{array}{c} 1 \\ \end{array} \begin{array}{c} 1 \\ \end{array}) Z_{SH}(s\sqrt{x}, \begin{array}{c} 2 \\ \end{array} \begin{array}{c} 2 \\ \end{array}) + Z_{SH}(s\sqrt{xy}, \begin{array}{c} 2 \\ \end{array} \begin{array}{c} 2 \\ \end{array}) Z_{SH}(s\sqrt{x}, \begin{array}{c} 1 \\ \end{array} \begin{array}{c} 1 \\ \end{array}) \right)$$

(3.40)

reproduces (2.38).

For $n = 3$ fields, it is easy to see that the polynomials

$$w_3(z_2 - z_1) + w_2(z_1 - z_3) + w_1(z_3 - z_2)$$

(3.41)

and

$$2w_1w_2z_1^2 - w_2^2 z_1^2 - 2w_1w_3z_1^2 + w_2^2 z_2^2 - 2w_1^2 z_1 z_2 + 2w_2^2 z_1 z_2 + 4w_1w_3z_1 z_2 - 4w_2w_3z_1z_2$$

$$+ 6w_1^2 z_2^2 - 2w_1w_2z_2^2 + 2w_2w_3z_2^2 - w_3^2 z_3^2 + 2w_1^2 z_1 z_3 - 4w_1w_2z_1z_3 + 4w_2w_3z_1z_3 - 2w_3^2 z_1 z_3$$

$$+ 4w_1w_2z_2z_3 - 2w_2^2 z_2 z_3 - 4w_1w_3z_2z_3 + 2w_3^2 z_2 z_3 - w_1^2 z_3^2 + w_2^2 z_3^2 + 2w_1w_3z_3^2 - 2w_2w_3z_3^2$$

(3.42)

are holomorphic, translation invariant and in the antisymmetric representation of $S_3$. To translate these polynomials into primary operators, we use the dictionary (see the appendix)

$$z^k w^l \leftrightarrow \frac{(-1)^{k+l} P_k^l P_w^l}{2^{k+l}(k + l)!}$$

(3.43)
where we have set \( P_z = P_1 - iP_2 \) and \( P_w = P_3 - iP_4 \). After a little work we finally obtain the following two primary operators

\[
\psi_1 = \psi(0)P_z\psi(0)P_w\psi(0)
\]

and

\[
\psi_2 = \frac{1}{3}P_wP_z^2\psi(0)P_w\psi(0)\psi(0) + \frac{1}{3}P_z\psi(0)P_w^2P_z\psi(0)\psi(0) \\
+ \frac{1}{4}P_w^2\psi(0)P_z^2\psi(0)\psi(0) + 2P_wP_z\psi(0)P_w\psi(0)
\]

(3.45)

In the appendix we verify that these operators are annihilated by the special conformal transformations.

### 4 Geometry

In this section we comment on the permutation orbifolds relevant for the combinatorics of the fermion primaries. The leading twist primaries are holomorphic polynomials in \( n \) complex variables. We mod out by translations and restrict to the antisymmetric representation of \( S_n \), so that the leading twist primaries correspond to holomorphic polynomial functions on

\[
(C)^n/(C \times S_n)
\]

(4.1)

A very similar argument shows that extremal primaries correspond to holomorphic polynomial functions on

\[
(C)^{2n}/(C^2 \times S_n)
\]

(4.2)

We will now argue that the Hilbert series of the fermionic primaries are counted by palindromic Hilbert series, suggesting that they are Calabi-Yau. We leave a more detailed study of these issues for the future. A palindromic Hilbert series obeys

\[
Z^{z,w}_n(q_1^{-1}, q_2^{-1}) = (q_1q_2)^{n-1}Z^{z,w}_n(q_1, q_2)
\]

(4.3)

Our Hilbert series \( Z^{z,w}_n(q_1, q_2) \) enjoy this transformation property. To demonstrate this, our starting point is the formula

\[
Z^{z,w}_n(q_1, q_2) = s^{2n}x^{\frac{1}{2}} \sum_{\Lambda \vdash n} Z_{SH}(q_1, \Lambda)Z_{SH}(q_2, \Lambda^T)
\]

(4.4)

where we have introduced the variables \( q_1 = s\sqrt{x/y}, q_2 = s\sqrt{x/y} \). This has the property \( Z^{z,w}_n(q_1, q_2) = Z^{z,w}_n(q_2, q_1) \). This follows because exchange of \( q_1, q_2 \) amounts to the inversion of \( y \), and by using the identity [7]

\[
Z_{SH}(q^{-1}, \Lambda) = (-q)^{n-1}Z_{SH}(q, \Lambda^T)
\]

(4.5)
Using this result

\[
Z_n^{z,w}(q_1^{-1}, q_2^{-1}) = s^n(q_1 q_2)^{n-1} \sum_{\Lambda_n} Z_{SH}(q_1, \Lambda^T) Z_{SH}(q_2, \Lambda)
\]

\[
= s^n(q_1 q_2)^{n-1} \sum_{\Lambda_n} Z_{SH}(q_1, \Lambda) Z_{SH}(q_2, \Lambda^T)
\]

\[
= (q_1 q_2)^{n-1} Z_n^{z,w}(q_1, q_2)
\]

(4.6)

The results of section (4.3) of [7] now imply that the Hilbert series \(G_n^{z,w}(s, x, y)\) also exhibit the palindromy property.

5 Summary and Outlook

Previous studies [7] have explained how to map the algebraic problem of constructing primary fields in the quantum field theory of a free scalar field \(\phi\) in four dimensions to one of finding polynomial functions on \((\mathbb{R}^4)_n\) that are harmonic, translation invariant and which are in the trivial representation of \(S_n\). In this article, we have extended this construction to describe primary fields in the free quantum field theory of a single Weyl fermion. Concrete results achieved with this new point of view include a counting formula for the complete set primary fields, explicit counting formulas (Hilbert series) for counting special classes of primaries, as well as detailed construction formulas for these primary operators. We have also established the palindromy of the Hilbert series.

One weak point in our analysis, that warrants further study, is the treatment of the constraint coming from the equation of motion. Motivated by results for Cauchy-Fueter regular functions, we simply stated that we will consider holomorphic polynomials. This has been verified explicitly, by checking that this leads to the correct number of primaries and further that when the polynomials are translated back into the operator language, that we do indeed obtain operators annihilated by \(K_\mu\). It would however be nice to perform a detailed analysis of the equation of motion constraint, which has to be carried out before the complete class of primaries can be treated.

Immediate generalizations of the current work include studies of CFTs which include gauge fields. The free limit of QCD and supersymmetric theories would be good starting points. Indeed, early constructions of primary fields in the \(SL(2)\) sector (leading twist primaries) were performed in the context of deep inelastic scattering in QCD (see for example the review [28]). Do the holomorphic primaries considered here have QCD applications? Explicit enumeration and construction of superconformal primary fields in \(\mathcal{N} = 4\) SYM will give a better understanding of the dual \(AdS_5 \times S^5\) background. Finally, another natural direction is to consider correlators involving the extremal primary fields and the determination of anomalous dimensions for these fields at the fixed point of the Gross-Neveu model in \(2 + \epsilon\) dimensions, using the techniques of [29, 30, 31, 32, 33].
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A Primaries examples

In this Appendix we will collect a few details on the translation from polynomials to primary operators and then test, for a few examples, that the primaries obtained are indeed annihilated by $K_\mu$.

A.1 Dictionary

We first show that the appropriate way to translate between polynomials and operators is given by (3.43). We again make use of the (Euclidean) representation

$$P_\mu = x^2 \partial_\mu - 2x_\mu x \cdot \partial + 3x_\mu + 2x_\nu M_{\mu\nu}$$

We consider a polynomial in $P_z = P_1 - iP_2 = \epsilon_z \cdot P$ and $P_w = P_3 - iP_4 = \epsilon_w \cdot P$ acting on the constant spinor $\zeta$. The $\epsilon$’s obey the following identities

$$\epsilon_z \cdot \epsilon_z = 0 = \epsilon_w \cdot \epsilon_w = \epsilon_z \cdot \epsilon_w$$

$$\epsilon_z \cdot x = z \quad ; \quad \epsilon_w \cdot x = w.$$  \hfill (A.2)

The translation we work out in this Appendix holds for the leading twist and extremal primaries. Recall that in this case we have fixed the left spin to a maximal value, corresponding to choosing the spinor $\zeta$ with spin up. Useful formulas to bear in mind are

$$\epsilon_z \cdot \partial \ z^m w^n \zeta = (m\epsilon_z \cdot \epsilon_z z^{m-1} w^n + n\epsilon_z \cdot \epsilon_z z^m w^{n-1})\zeta = 0$$

$$\epsilon_w \cdot \partial \ z^m w^n \zeta \quad = 0$$

$$(\epsilon_z)_\mu (3x_\mu + 2x_\nu M_{\mu\nu})\zeta = 2z\zeta$$

$$\quad (\epsilon_w)_\mu (3x_\mu + 2x_\nu M_{\mu\nu})\zeta = 2w\zeta$$  \hfill (A.3)

In the last formula above we have made use of the fact that $\zeta$ has maximal left spin. It is now straightforward to verify that

$$P_w^k P_z^l \psi(0) \leftrightarrow (-2)^{k+l}(k+l)!w^k z^l \zeta$$  \hfill (A.4)
A.2  \( n = 2 \) Example

We first study the operators given by (3.30). Introduce \( K_z = K_1 + iK_2 \). It is straightforward to verify that \([K_z, \bar{P}_z] = 0\) and

\[
[D, \bar{P}_z] = \bar{P}_z \\
[D, K_z] = -K \\
[K_z, \bar{P}_z] = -4D + 4iM_{21} \\
[M_{21}, K_z] = -iK_z \\
[M_{21}, \bar{P}_z] = i\bar{P}_z.
\] (A.5)

It follows that \( K_z \) annihilates (3.30). Using the above algebra we easily find

\[
K_z P^m \left| \frac{3}{2}, \frac{1}{2}, 0 \right> = -4P^{m-1}_z (m^2 - m(1 - D + iM_{21})) \left| \frac{3}{2}, \frac{1}{2}, 0 \right>
\] (A.7)

Consequently the action of \( K_z \) on the state (3.30) yields

\[
K_z |\psi\rangle = -4 \sum_{k=0}^{2s+1} \frac{(-1)^k k^2}{((2s - k + 1)!k!)^2} P^{k-1}_z \left| \frac{3}{2}, \frac{1}{2}, 0 \right> \otimes P^{2s-k+1}_z \left| \frac{3}{2}, \frac{1}{2}, 0 \right>
\]

\[
= 0
\] (A.8)

Next, consider

\[
K_3 P^m_z |0\rangle = -mP^{m-1}_z (M_{31} - iM_{32}) |0\rangle \\
K_4 P^m_z |0\rangle = -mP^{m-1}_z (M_{41} - iM_{42}) |0\rangle.
\] (A.9)

The operators \( M_{31} - iM_{32} \) and \( M_{41} - iM_{42} \) are raising operators for the right spin. Since the state \( |\frac{3}{2}, \frac{1}{2}, 0\rangle \) has vanishing right spin, we have

\[
iM_{21} \left| \frac{3}{2}, \frac{1}{2}, 0 \right> = iM_{34} \left| \frac{3}{2}, \frac{1}{2}, 0 \right> = \frac{1}{2} \left| \frac{3}{2}, \frac{1}{2}, 0 \right>
\]

\[
(M_{31} - iM_{32}) \left| \frac{3}{2}, \frac{1}{2}, 0 \right> = 0 = (M_{41} - iM_{42}) \left| \frac{3}{2}, \frac{1}{2}, 0 \right>.
\] (A.10)

It now follows that \( K_3 \) and \( K_4 \) annihilate (3.30), completing the demonstration that (3.30) is indeed a primary operator.
A.3 \( n = 3 \) Examples

We will show that the operators (3.44) and (3.45) are annihilated by the special conformal generators. Define \( K_w = K_3 + iK_4 \). It is straightforward to evaluate

\[
[K_w, P_z] = 2(M_{31} - iM_{32} + i(M_{41} - iM_{42})) \equiv 4M_{wz}
\]
\[
[M_{wz}, P_w] = -2P_z
\]
\[
[M_{wz}, P_z] = 0
\] (A.11)

and

\[
[K_z, P_w] = -2(M_{31} + iM_{32} + i(M_{41} + iM_{42})) \equiv 4M_{zw}
\]
\[
[M_{zw}, P_z] = -2P_w
\]
\[
[M_{zw}, P_w] = 0
\] (A.12)

To interpret these commutators, note that \( P_z \) has spin \((\frac{1}{2}, \frac{1}{2})\) and \( P_w \) has spin \((\frac{1}{2}, -\frac{1}{2})\). Thus, \( M_{wz} \) and \( M_{zw} \) are raising/lowering operators of the right spin. Since our fermion field has vanishing right spin it is clear that

\[
M_{zw}[\frac{3}{2}, \frac{1}{2}, 0] = M_{wz}[\frac{3}{2}, \frac{1}{2}, 0] = 0
\] (A.13)

which implies the identities

\[
K_z P^m_w P^m_z \frac{3}{2}, \frac{1}{2}, 0 = -4(nm + m^2) P^m_w P^{m-1}_z \frac{3}{2}, \frac{1}{2}, 0
\]
\[
K_w P^m_w P^m_z \frac{3}{2}, \frac{1}{2}, 0 = -4(nm + n^2) P^m_w P^{n-1}_z \frac{3}{2}, \frac{1}{2}, 0
\] (A.14)

It now follows that

\[
K_z \psi_1 = -4\psi(0)\psi(0) P_w \psi(0) = 0
\]
\[
K_w \psi_1 = -4\psi(0) P_z \psi(0) \psi(0) = 0
\] (A.15)

where we used the Grassman statistics of the field. For the action on \( \psi_2 \) we find

\[
-\frac{1}{4} K_z \psi_2 = \frac{1}{3} (6) P_w P_z \psi(0) P_w \psi(0) \psi(0) + \frac{1}{3} (3) P_z \psi(0) P^2_w \psi(0) \psi(0)
\]
\[
+ \frac{1}{4} (4) P^2_w \psi(0) P_z \psi(0) \psi(0) + 2 P_w P_z \psi(0) \psi(0) P_w \psi(0) = 0
\]
\[
-\frac{1}{4} K_w \psi_2 = \frac{1}{3} (3) P^2_z \psi(0) P_w \psi(0) \psi(0) + \frac{1}{3} (6) P_z \psi(0) P_w P_z \psi(0) \psi(0)
\]
\[
+ \frac{1}{4} (4) P_w \psi(0) P^2_z \psi(0) \psi(0) + 2 P_w P_z \psi(0) P_z \psi(0) \psi(0) = 0
\] (A.16)

again after using the Grassman nature of the field. This completes the demonstration that \( \psi_1 \) and \( \psi_2 \) are primary operators.
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