ON $\sigma$-CONVEX SUBSETS IN SPACES OF SCATTEREDLY CONTINUOUS FUNCTIONS

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We prove that for any topological space $X$ of countable tightness, each $\sigma$-convex subspace $F$ of the space $SC_p(X)$ of scatteredly continuous real-valued functions on $X$ has network weight $nw(F) \leq nw(X)$. This implies that for a metrizable separable space $X$, each compact convex subset in the function space $SC_p(X)$ is metrizable. Another corollary says that two Tychonoff spaces $X,Y$ with countable tightness and topologically isomorphic linear topological spaces $SC_p(X)$ and $SC_p(Y)$ have the same network weight $nw(X) = nw(Y)$. Also we prove that each zero-dimensional separable Rosenthal compact space is homeomorphic to a compact subset of the function space $SC_p(\omega^\omega)$ over the space $\omega^\omega$ of irrationals.

This paper was motivated by the problem of studying the linear-topological structure of the space $SC_p(X)$ of scatteredly continuous real-valued functions on a topological space $X$, addressed in [1, 2].

A function $f : X \to Y$ between two topological spaces is called scatteredly continuous if for each non-empty subspace $A \subset X$ the restriction $f|A : A \to Y$ has a point of continuity. Scatteredly continuous functions were introduced in [3] (as almost continuous functions) and studied

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in details in [4], [5] and [6]. If a topological space \( Y \) is regular, then the scattered continuity of a function \( f : X \to Y \) is equivalent to the weak discontinuity of \( f \); see [3], [5, 4.4]. We recall that a function \( f : X \to Y \) is weakly discontinuous if each subspace \( A \subset X \) contains an open dense subspace \( U \subset A \) such that the restriction \( f|U : U \to Y \) is continuous.

For a topological space \( X \) by \( SC_p(X) \subset \mathbb{R}^X \) we denote the linear space of all scatteredly continuous (equivalently, weakly discontinuous) functions on \( X \), endowed with the topology of pointwise convergence. It is clear that the space \( SC_p(X) \) contains the linear subspace \( C_p(X) \) of all continuous real-valued functions on \( X \). Topological properties of the function spaces \( C_p(X) \) were intensively studied by topologists, see [7]. In particular, they studied the interplay between topological invariants of topological space \( X \) and its function space \( C_p(X) \).

Let us recall [8, 9] that for a topological space \( X \) its

- \textit{weight} \( w(X) \) is the smallest cardinality of a base of the topology of \( X \);
- \textit{network weight} \( w(X) \) is the smallest cardinality of a network of the topology of \( X \);
- \textit{tightness} \( t(X) \) is the smallest infinite cardinal \( \kappa \) such that for each subset \( A \subset X \) and a point \( a \in \overset{\circ}{A} \) in its closure there is a subset \( B \subset A \) of cardinality \( |B| \leq \kappa \) such that \( a \in B \);
- \textit{Lindelöf number} \( l(X) \) is the smallest infinite cardinal \( \kappa \) such that each open cover of \( X \) has a subcover of cardinality \( \leq \kappa \);
- \textit{hereditary Lindelöf number} \( hl(X) = \sup \{ l(Z) : Z \subset X \} \);
- \textit{density} \( d(X) \) if the smallest cardinality of a dense subset of \( X \);
- \textit{the hereditary density} \( hd(X) = \sup \{ d(Z) : Z \subset X \} \);
- \textit{spread} \( s(X) = \sup \{ |D| : D \text{ is a discrete subspace of } X \} \).

By [7, §I.1], for each Tychonoff space \( X \) the function space \( C_p(X) \) has weight \( w(C_p(X)) = |X| \) and network weight \( nw(SC_p(X)) = nw(X) \). For the function space \( SC_p(X) \) the situation is a bit different.
Proposition 1. For any $T_1$-space $X$ we have
\[ s(SC_p(X)) = nw(SC_p(X)) = w(SC_p(X)) = |X|. \]

Proof. It is clear that $s(SC_p(X)) \leq nw(SC_p(X)) \leq w(SC_p(X)) \leq w(\mathbb{R}^X) = |X|$. To see that $|X| \leq s(SC_p(X))$, observe that for each point $a \in X$ the characteristic function
\[ \delta_a : X \to \mathbb{R} = \begin{cases} 1, & \text{if } x = a \\ 0, & \text{otherwise} \end{cases} \]
of the singleton $\{a\}$ is scatteredly continuous, and the subspace $D = \{\delta_a : a \in X\} \subset SC_p(X)$ has cardinality $|X|$ and is discrete in $SC_p(X)$. \hfill \Box

The deviation of a subset $\mathcal{F} \subset SC_p(X)$ from being a subset of $C_p(X)$ can be measured with help of the cardinal number $\text{dec}(\mathcal{F})$ called the decomposition number of $\mathcal{F}$. It is defined as the smallest cardinality $|\mathcal{C}|$ of a cover $\mathcal{C}$ of $X$ such that for each $C \in \mathcal{C}$ and $f \in \mathcal{F}$ the restriction $f|C$ is continuous. If the function family $\mathcal{F}$ consists of a single function $f$, then the decomposition number $\text{dec}(\mathcal{F}) = \text{dec}\{f\}$ coincides with the decomposition number $\text{dec}(f)$ of the function $f$, studied in [10]. It is clear that $\text{dec}(C_p(X)) = 1$.

Proposition 2. For a $T_1$ topological space $X$ the decomposition number $\text{dec}(SC_p(X))$ is equal to the decomposition number $\text{dec}(D)$ of the subset $D = \{\delta_a : a \in X\} \subset SC_p(X)$ and is equal to the smallest cardinality $d\text{dec}(X)$ of a cover of $X$ by discrete subspaces.

Proof. It is clear that $\text{dec}(D) \leq \text{dec}(SC_p(X)) \leq d\text{dec}(X)$. To prove that $\text{dec}(D) \geq d\text{dec}(X)$, take a cover $\mathcal{C}$ of $X$ of cardinality $|\mathcal{C}| = \text{dec}(D)$ such that for each $C \in \mathcal{C}$ and each characteristic function $\delta_a \in D$ the restriction $\delta_a|C$ is continuous. We claim that each space $C \in \mathcal{C}$ is discrete. Assuming conversely that $C$ contains a non-isolated point $c \in C$, observe that for the characteristic function $\delta_c$ of the singleton $\{c\}$ the restriction $\delta_c|C$ is not continuous. But this contradicts the choice of the cover $\mathcal{C}$. Therefore the cover $\mathcal{C}$ consists of discrete subspaces of $X$ and $d\text{dec}(X) \leq |\mathcal{C}| = \text{dec}(D)$. \hfill \Box
In contrast to the whole function space \( \text{SC}_p(X) \) which has large decomposition number \( \text{dec}(\text{SC}_p(X)) \), its \( \sigma \)-convex subsets have decomposition numbers bounded from above by the hereditary Lindelöf number of \( X \).

Following [11] and [12], we define a subset \( C \) of a linear topological space \( L \) to be \( \sigma \)-convex if for any sequence of points \( (x_n)_{n \in \omega} \) in \( C \) and any sequence of positive real numbers \( (t_n)_{n \in \omega} \) with \( \sum_{n=0}^{\infty} t_n = 1 \) the series \( \sum_{n=0}^{\infty} t_n x_n \) converges to some point \( c \in C \). It is easy to see that each compact convex subset \( K \subset L \) is \( \sigma \)-convex. On the other hand, each \( \sigma \)-convex subset of a linear topological space \( L \) is necessarily convex and bounded in \( L \).

The main result of this paper is the following:

**Theorem 1.** For any topological space \( X \) of countable tightness, each \( \sigma \)-convex subset \( F \subset \text{SC}_p(X) \) has decomposition number \( \text{dec}(F) \leq \text{hl}(X) \).

This theorem will be proved in Section 3. Now we derive some simple corollaries of this theorem.

**Corollary 1.** For any topological space \( X \) of countable tightness, each \( \sigma \)-convex subset \( F \subset \text{SC}_p(X) \) has network weight \( \text{nw}(F) \leq \text{nw}(X) \). Moreover,

\[
\text{nw}(X) = \max \{ \text{nw}(F) : F \text{ is a } \sigma \text{-convex subset of } \text{SC}_p(X) \}
\]

provided the space \( X \) is Tychonoff.

**Proof.** By Theorem 1, each \( \sigma \)-convex subset \( F \subset \text{SC}_p(X) \) has decomposition number \( \text{dec}(F) \leq \text{hl}(X) \). Consequently, we can find a disjoint cover \( \mathcal{C} \) of \( X \) of cardinality \( |\mathcal{C}| = \text{dec}(F) \leq \text{hl}(X) \) such that for each \( C \in \mathcal{C} \) and \( f \in F \) the restriction \( f[C] \) is continuous.

Let \( Z = \bigoplus \mathcal{C} = \{(x,C) \in X \times C : x \in C \} \subset X \times \mathcal{C} \) be the topological sum of the family \( \mathcal{C} \), and \( \pi : Z \to X, \pi : (x,C) \mapsto x \), be the natural projection of \( Z \) onto \( X \). Since the cover \( \mathcal{C} \) is disjoint, the map \( \pi : Z \to X \) is bijective and hence induces a topological isomorphism \( \pi^* : \mathbb{R}^X \to \mathbb{R}^Z \), \( \pi^* : f \mapsto f \circ \pi \). The choice of the cover \( \mathcal{C} \) guarantees that \( \pi^*(F) \subset C_p(Z) \).

By (the proof of) Theorem I.1.3 of [7], \( \text{nw}(C_p(Z)) \leq \text{nw}(Z) \) and hence \( \text{nw}(F) = \text{nw}(\pi^*(F)) \leq \text{nw}(C_p(Z)) \leq \text{nw}(Z) \leq |\mathcal{C}| \cdot \text{nw}(X) \leq \text{hl}(X) \cdot \text{nw}(X) = \text{nw}(X) \).
If the space $X$ is Tychonoff, then the “closed unit ball”

$$B = \{ f \in C_p(X) : \sup_{x \in X} |f(x)| \leq 1 \} \subset C_p(X)$$

is $\sigma$-convex and has network weight $nw(B) = nw(X)$ according to Theorem I.1.3 of [7]. So,

$$nw(X) = \max \{ nw(F) : F \text{ is a } \sigma\text{-convex subset of } SC_p(X) \}.$$  

\[ \square \]

In the same way we can derive some bounds on the weight of compact convex subsets in function spaces $SC_p(X)$.

**Corollary 2.** For any topological space $X$ of countable tightness, each compact convex subset $K \subset SC_p(X)$ has weight $w(K) \leq \max\{hl(X), hd(X)\}$. Moreover,

$$hl(X) \leq \sup \{ w(K) : K \text{ is a compact convex subset of } SC_p(X) \} \leq \max\{hl(X), hd(X)\}.$$  

**Proof.** Given a compact convex subset $K \subset SC_p(X)$, use Theorem 1 to find a disjoint cover $C$ of $X$ of cardinality $|C| = \text{dec}(K) \leq hl(X)$ such that for each $C \in C$ and $f \in K$ the restriction $f|C$ is continuous. Let $Z = \bigoplus C$ and $\pi : \bigoplus C \to X$ be the natural projection, which induces a linear topological isomorphism $\pi^* : \mathbb{R}^X \to \mathbb{R}^Z$, $\pi^* : f \mapsto f \circ \pi$, with $\pi^*(K) \subset C_p(Z)$. It follows that the topological sum $Z = \bigoplus C$ has density $d(Z) \leq \sum_{C \in C} d(C) \leq |C| \cdot hd(X) \leq \max\{hl(X), hd(X)\}$, and so we can fix a dense subset $D \subset Z$ of cardinality $|D| = d(Z) \leq \max\{hl(X), hd(X)\}$. Since the restriction operator $R : C_p(Z) \to C_p(D)$, $R : f \mapsto f|D$, is injective and continuous, we conclude that

$$w(K) = w(\pi^*(K)) = w(R \circ \pi^*(K)) \leq w(\mathbb{R}^D) = |D| \cdot \mathcal{N}_0 \leq \max\{hl(X), hd(X)\}.$$  

Next, we show that $hl(X) \leq \tau$ where

$$\tau = \sup \{ w(K) : K \text{ is a compact convex subset of } SC_p(X) \}.$$
Assuming conversely that $hl(X) > \tau$ and using the equality $hl(X) = \sup\{|Z| : Z \subset X \text{ is scattered}\}$ established in [9], we can find a scattered subspace $Z \subset X$ of cardinality $|Z| > \tau$. It is easy to check that each function $f : X \to [0, 1]$ with $f(X \setminus Z) \subset \{0\}$ is scatteredly continuous, which implies that the subset

$$K_Z = \{f \in SC_p(X) : f(Z) \subset [0, 1], \ f(X \setminus Z) \subset \{0\}\}$$

is compact, convex and homeomorphic to the Tychonoff cube $[0, 1]^Z$. Then $\tau \geq w(K_Z) = w([0, 1]^Z) = |Z| > \tau$ and this is a desired contradiction that completes the proof.

Corollaries 1 or 2 imply:

**Corollary 3.** For a metrizable separable space $X$, each compact convex subspace $K \subset SC_p(X)$ is metrizable.

Finally, let us observe that Corollary 1 implies:

**Corollary 4.** If for Tychonoff spaces $X, Y$ with countable tightness the linear topological spaces $SC_p(X)$ and $SC_p(Y)$ are topologically isomorphic, then $nw(X) = nw(Y)$.

1 Weakly discontinuous families of functions

In this section we shall generalize the notions of scattered continuity and weak discontinuity to function families.

A family of functions $\mathcal{F} \subset Y^X$ from a topological space $X$ to a topological space $Y$ is called

- **scatteredly continuous** if each non-empty subset $A \subset X$ contains a point $a \in A$ at which each function $f|A : A \to Y$, $f \in \mathcal{F}$ is continuous;

- **weakly discontinuous** if each subset $A \subset X$ contains an open dense subspace $U \subset A$ such that each function $f|U : U \to Y$, $f \in \mathcal{F}$ is continuous.
The following simple characterization can be derived from the corresponding definitions and Theorem 4.4 of [5] (saying that each scatteredly continuous function with values in a regular topological space is weakly discontinuous).

**Proposition 3.** A function family $\mathcal{F} \subset Y^X$ is scatteredly continuous (resp. weakly discontinuous) if and only if so is the function $\Delta \mathcal{F} : X \to Y^F$, $\Delta \mathcal{F} : x \mapsto (f(x))_{f \in \mathcal{F}}$. Consequently, for a regular topological space $Y$, a function family $\mathcal{F} \subset Y^X$ is scatteredly continuous if and only if it is weakly discontinuous.

Propositions 4.7 and 4.8 [5] imply that each weakly discontinuous function $f : X \to Y$ has decomposition number $\text{dec}(f) \leq \text{hl}(X)$. This fact combined with Proposition 3 yields:

**Corollary 5.** For any topological spaces $X, Y$, each weakly discontinuous function family $\mathcal{F} \subset Y^X$ has decomposition number $\text{dec}(\mathcal{F}) \leq \text{hl}(X)$.

### 2 Weak discontinuity of $\sigma$-convex sets in function spaces

For a topological space $X$ by $\text{SC}_p^*(X)$ we denote the space of all bounded scatteredly continuous real-valued functions on $X$. It is a subspace of the function space $\text{SC}_p(X) \subset \mathbb{R}^X$. Each function $f \in \text{SC}_p^*(X)$ has finite norm $\|f\| = \sup_{x \in X} |f(x)|$.

**Theorem 2.** For any topological space $X$ with countable tightness, each $\sigma$-convex subset $\mathcal{F} \subset \text{SC}_p^*(X)$ is weakly discontinuous.

**Proof.** By Proposition 3, the weak discontinuity of the function family $\mathcal{F}$ is equivalent to the scattered continuity of the function $\Delta \mathcal{F} : X \to \mathbb{R}^F$, $\Delta \mathcal{F} : x \mapsto (f(x))_{f \in \mathcal{F}}$. Since the space $X$ has countable tightness, the scattered continuity of $\Delta \mathcal{F}$ will follow from Proposition 2.3 of [5] as soon as we check that for each countable subset $Q = \{x_n\}_{n=1}^{\infty} \subset X$ the restriction $\Delta \mathcal{F}|_Q : Q \to \mathbb{R}^F$ has a continuity point. Assuming the converse, for each point $x_n \in Q$ we can choose a function $f_n \in \mathcal{F}$ such that the restriction $f_n|_Q$ is discontinuous at $x_n$. 
Observe that a function \( f : Q \to \mathbb{R} \) is discontinuous at a point \( q \in Q \) if and only if it has strictly positive oscillation
\[
\text{osc}_q(f) = \inf_{O_q} \sup \{|f(x) - f(y)| : x, y \in O_q \}
\]
at the point \( q \). In this definition the infimum is taken over all neighborhoods \( O_q \) of \( q \) in \( Q \).

We shall inductively construct a sequence \((t_n)_{n=1}^{\infty}\) of positive real numbers such that for every \( n \in \mathbb{N} \) the following conditions are satisfied:

1) \( t_1 \leq \frac{1}{2}, \ t_{n+1} \leq \frac{1}{2} t_n, \) and \( t_{n+1} \cdot \|f_{n+1}\| \leq \frac{1}{2} t_n \cdot \|f_n\| \);

2) the function \( s_n = \sum_{k=1}^{n} t_k f_k \) restricted to \( Q \) is discontinuous at \( x_n \),

3) \( t_{n+1} \cdot \|f_{n+1}\| \leq \frac{1}{8} \text{osc}_{x_n}(s_n|Q) \).

We start the inductive construction letting \( t_1 = 1/2 \). Then the function \( s_1|Q = t_1 \cdot f_1|Q \) is discontinuous at \( x_1 \) by the choice of the function \( f_1 \). Now assume that for some \( n \in \mathbb{N} \) positive numbers \( t_1, \ldots, t_n \) has been chosen so that the function \( s_n = \sum_{k=1}^{n} t_k f_k \) restricted to \( Q \) is discontinuous at \( x_n \).

Choose any positive number \( \tilde{t}_{n+1} \) such that
\[
\tilde{t}_{n+1} \leq \frac{1}{2} t_n, \ \tilde{t}_{n+1} \cdot \|f_{n+1}\| \leq \frac{1}{2} t_n \cdot \|f_n\| \quad \text{and} \quad \tilde{t}_{n+1} \cdot \|f_{n+1}\| \leq \frac{1}{8} \text{osc}_{x_n}(s_n|Q),
\]
and consider the function \( \tilde{s}_{n+1} = s_n + \tilde{t}_{n+1} f_{n+1} \). If the restriction of this function to \( Q \) is discontinuous at the point \( x_{n+1} \), then put \( t_{n+1} = \tilde{t}_{n+1} \) and finish the inductive step. If \( \tilde{s}_{n+1}|Q \) is continuous at \( x_{n+1} \), then put \( t_{n+1} = \frac{1}{2} \tilde{t}_{n+1} \) and observe that the restriction of the function
\[
s_{n+1} = \sum_{k=1}^{n+1} t_k f_k = s_n + \frac{1}{2} t_{n+1} f_{n+1} = \tilde{s}_{n+1} - \frac{1}{2} \tilde{t}_{n+1} f_{n+1}
\]
to \( Q \) is discontinuous at \( x_{n+1} \). This completes the inductive construction.

The condition (1) guarantees that \( \sum_{n=1}^{\infty} t_n \leq 1 \) and hence the number \( t_0 = 1 - \sum_{n=1}^{\infty} t_n \) is non-negative. Now take any function \( f_0 \in \mathcal{F} \) and consider
the function
\[ s = \sum_{n=0}^{\infty} t_n f_n \]
which is well-defined and belongs to \( \mathcal{F} \) by the \( \sigma \)-convexity of \( \mathcal{F} \).

The functions \( f_0, s \in \mathcal{F} \subset SC_p(X) \) are weakly discontinuous and hence for some open dense subset \( U \subset Q \) the restrictions \( s|U \) and \( f_0|U \) are continuous. Pick any point \( x_n \in U \). Observe that
\[ s = t_0 f_0 + s_n + \sum_{k=n+1}^{\infty} t_k f_k \]
and hence
\[ s_n = s - t_0 f_0 - \sum_{k=n+1}^{\infty} t_k f_k = s - t_0 f_0 - u_n, \]
where \( u_n = \sum_{k=n+1}^{\infty} t_k f_k \). The conditions (1) and (3) of the inductive construction guarantee that the function \( u_n \) has norm
\[ \|u_n\| \leq \sum_{k=n+1}^{\infty} t_k \|f_k\| \leq 2t_{n+1}\|f_{n+1}\| \leq \frac{1}{4} \text{osc}_{x_n}(s_n|Q). \]
Since \( s_n = s - t_0 f_0 - u_n \), the triangle inequality implies that
\[ 0 < \text{osc}_{x_n}(s_n|Q) \leq \text{osc}_{x_n}(s|Q) + \text{osc}_{x_n}(t_0 f_0|Q) + \text{osc}_{x_n}(u_n) \leq 0 + 0 + 2\|u_n\| \leq \frac{1}{2} \text{osc}_{x_n}(s_n|Q) \]
which is a desired contradiction, which shows that the restriction \( \Delta \mathcal{F}|Q \) has a point of continuity and the family \( \mathcal{F} \) is weakly discontinuous.

3 Proof of Theorem 1

Let \( X \) be a topological space with countable tightness and \( \mathcal{F} \) be a \( \sigma \)-convex subset in the function space \( SC_p(X) \). The \( \sigma \)-convexity of \( \mathcal{F} \) implies that for each point \( x \in X \) the subset \( \{f(x) : f \in \mathcal{F}\} \subset \mathbb{R} \) is bounded (in the opposite case we could find sequences \( (f_n)_{n \in \omega} \in \mathcal{F}^\omega \) and \( (t_n)_{n \in \omega} \in [0, 1]^\omega \)
with \( \sum_{n=0}^{\infty} t_n = 1 \) such that the series \( \sum_{n=1}^{\infty} t_n f_n(x) \) is divergent. Then \( X = \bigcup_{n=1}^{\infty} X_n \) where \( X_n = \{ x \in X : n \leq \sup_{f \in F} |f(x)| < n + 1 \} \) for \( n \in \omega \).

It follows that for every \( n \in \omega \) the family \( \mathcal{F}|X_n = \{ f|X_n : f \in \mathcal{F} \} \) is a \( \sigma \)-convex subset of the function space \( SC_p^*(X_n) \). By Theorem 2, the function family \( \mathcal{F}|X_n \) is weakly discontinuous and by Corollary 5, \( \text{dec}(\mathcal{F}|X_n) \leq hl(X_n) \). Then \( \text{dec}(\mathcal{F}) \leq \sum_{n=0}^{\infty} \text{dec}(\mathcal{F}|X_n) \leq \sum_{n=0}^{\infty} hl(X_n) \leq hl(X) \).

4 Some Open Problems

The presence of the condition of countable tightness in Theorem 1 and its corollaries suggests the following open problem.

**Problem 1.** Is it true \( w(\mathcal{K}) \leq nw(X) \) for each topological space \( X \) and each compact convex subset \( \mathcal{K} \subset SC_p(X) \)?

By Theorem 2, for each topological space \( X \) of countable tightness, each compact convex subset \( \mathcal{K} \subset SC_p^*(X) \) is weakly discontinuous.

**Problem 2.** For which topological spaces \( X \) each compact convex subset \( \mathcal{K} \subset SC_p(X) \) is weakly discontinuous?

According to Corollary 3, each compact convex subset \( \mathcal{K} \subset SC_p(\omega^\omega) \) is metrizable.

**Problem 3.** Is a compact subset \( \mathcal{K} \subset SC_p(\omega^\omega) \) metrizable if \( \mathcal{K} \) is homeomorphic to a compact convex subset of \( \mathbb{R}^f \)?

Let us recall that a topological space \( K \) is *Rosenthal compact* if \( K \) is homeomorphic to a compact subspace of the space \( \mathcal{B}_1(X) \subset \mathbb{R}^X \) of functions of the first Baire class on a Polish space \( X \). In this definition the space \( X \) can be assumed to be equal to the space \( \omega^\omega \) of irrationals.

**Problem 4.** Is each Rosenthal compact space homeomorphic to a compact subset of the function space \( SC_p(\omega^\omega) \)?

This problem has affirmative solution in the realm of zero-dimensional separable Rosenthal compacta.
Theorem 3. Each zero-dimensional separable Rosenthal compact space $K$ is homeomorphic to a compact subset of the function space $SC_p(\omega^\omega)$.

Proof. Let $D \subseteq K$ be a countable dense subset in $K$. Let $A = C_D(K, 2)$ be the space of continuous functions $f : K \to 2 = \{0, 1\}$ endowed with the smallest topology making the restriction operator $R : C_D(K, 2) \to 2^D$, $R : f \mapsto f|D$, continuous. By the characterization of separable Rosenthal compacta [13], the space $A$ is analytic, i.e., $A$ is the image of the Polish space $X = \omega^\omega$ under a continuous map $\pi : X \to A$. Now consider the map $\delta : K \to 2^A$, $\delta : x \mapsto (f(x))_{f \in A}$. This map is continuous and injective by the zero-dimensionality of $K$. The map $\pi : X \to A$ induces a homeomorphism $\pi^* : 2^A \to 2^X$, $\pi^* : f \mapsto f \circ \pi$. Then $\pi^* \circ \delta : K \to 2^X$ is a topological embedding.

We claim that $\pi^* \circ \delta(K) \subseteq SC_p(X) \cap 2^X$. Given a point $x \in K$, we need to check that the function $\pi^* \circ \delta(x) \in 2^X$ is scatteredly continuous. It will be convenient to denote the function $\delta(x) \in 2^A$ by $\delta_x$. This function assigns to each $f \in A = C_D(K)$ the number $\delta_x(f) = f(x) \in 2$.

By [14, 15], the Rosenthal compact space $K$ is Fréchet-Urysohn, so there is a sequence $(x_n)_{n \in \omega} \in D^\omega$ with $\lim_{n \to \infty} x_n = x$. Then the function $\delta_x : A \to 2$, $\delta_x : f \mapsto f(x)$, is the pointwise limit of the continuous functions $\delta_{x_n}$, which implies that $\delta_x$ is a function of the first Baire class on $A$ and $\delta_x \circ \pi : X \to 2$ is a function of the first Baire class on the Polish space $X$. Since this function has discrete range, it is scatteredly continuous by Theorem 8.1 of [5]. Consequently, $\pi^* \circ \delta(x) \in SC_p(X)$ and $K$ is homeomorphic to the compact subset $\pi^* \circ \delta(K) \subset SC_p(X)$.

A particularly interesting instance of Problem 4 concerns non-metrizable convex Rosenthal compacta. One of the simplest spaces of this sort is the Helly space. We recall that the Helly space is the subspace of $B_1(I)$ consisting of all non-decreasing functions $f : I \to I$ of the unit interval $I = [0, 1]$.

Problem 5. Is the Helly space homeomorphic to a compact subset of the function space $SC_p(\omega^\omega)$?
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ПРО $\sigma$-ОПУКЛІ ПІДПРОСТОРИ ПРОСТОРУ РОЗРІДЖЕНО НЕПЕРЕРВНИХ ФУНКЦІЙ

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Доведено, що для довільного топологічного простору $X$ зліченної тісноти кожний $\sigma$-опуклий підпростір $\mathcal{F}$ простору $SC_p(X)$ розріджено неперервних дійснозначних функцій на $X$ має сіткову вагу $nw(\mathcal{F}) \leq nw(X)$. З цього випливає, що для довільного метризованого сепарабельного простору $X$ кожний компактний опуклий підпростір в $SC_p(X)$ є метризованим, а також, що тихонівські простори $X$ і $Y$ зі зліченною тіснотою мають однакову сіткову вагу $nw(X) = nw(Y)$, якщо лінійні топологічні простори $SC_p(X)$ і $SC_p(Y)$ топологічно ізоморфні. Також доведено, що кожний нуль-вимірний сепарабельний компакт Розенталя вкладається у простір розріджено неперервних функцій $SC_p(\omega)$ над польським простором $\omega$. 