Kaluza-Klein solitons reexamined

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Abstract

In (4 + 1) gravity the assumption that the five-dimensional metric is independent of the fifth coordinate authorizes the extra dimension to be either spacelike or timelike. As a consequence of this, the time coordinate and the extra coordinate are interchangeable, which in turn allows the conception of different scenarios in 4D from a single solution in 5D. In this paper, we make a thorough investigation of all possible 4D scenarios, associated with this interchange, for the well-known Kramer-Gross-Perry-Davidson-Owen set of solutions. We show that there are three families of solutions with very distinct geometrical and physical properties. They correspond to different sets of values of the parameters which characterize the solutions in 5D. The solutions of physical interest are identified on the basis of physical requirements on the induced-matter in 4D. We find that only one family satisfies these requirements; the other two violate the positivity of mass-energy density. The “physical” solutions possess a lightlike singularity which coincides with the horizon. The Schwarzschild black string solution as well as the zero moment dipole solution of Gross and Perry are obtained in different limits. These are analyzed in the context of Lake’s geometrical approach. We demonstrate that the parameters of the solutions in 5D are not free, as previously considered. Instead, they are totally determined by measurements in 4D. Namely, by the surface gravitational potential of the astrophysical phenomena, like the Sun or other stars, modeled in Kaluza-Klein theory. This is an important result which may help in observations for an experimental/observational test of the theory.

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1 Introduction

In four-dimensional general relativity, Birkhoff’s theorem establishes that the Schwarzschild metric is the only solution of the field equation $R_{\mu\nu} = 0$, with spherical symmetry. In more than four dimensions, in Kaluza-Klein theories, this theorem is no longer valid: there are a number of solutions to the field equations $R_{AB} = 0$, with spherical three-space.

However, a milder version of Birkhoff’s theorem is true in Kaluza-Klein. Namely, there is only one family of spherically symmetric exact solutions of the field equations $R_{AB} = 0$ which are asymptotically flat, static and independent of the “extra” coordinates. In five-dimensions, in the form given by Davidson and Owen, they are described by the line element

$$dS^2 = \left(\frac{a - 1}{a + 1}\right)^{2\sigma_k} dt^2 - \frac{1}{a^4 r^4 (a - 1)^{2\sigma(k-1)+1}} [dr^2 + r^2 d\Omega^2] - \left(\frac{a + 1}{a - 1}\right)^{2\sigma} dy^2,$$

where $d\Omega^2 = (d\theta^2 + \sin^2 \theta d\phi^2)$; $(t, r, \theta, \phi)$ are the usual coordinates for a spacetime with spherically symmetric spatial sections; $y$ denotes the extra coordinate; $a$ is a constant with dimensions of $L^{-1}$, and $\sigma$ along with $k$ are parameters that obey the constraint

$$\sigma^2 (k^2 - k + 1) = 1.$$ 

The above set of solutions has been rediscovered in different forms by Kramer, Gross and Perry, and, although in another context, by Millward. A particular case, in curvature coordinates, was given by Chatterjee and more recently by Millward. They form a subset of the “generalized Weyl solutions” of Emparan and Reall and are widely studied in the literature from different physical approaches. In particular they play a central role in the discussion of many important observational problems, which include the classical tests of relativity, as well as the geodesic precession of a gyroscope and possible departures from the equivalence principle.

We note that in the extra coordinate is spacelike. However, this is not a requirement of the field equations. Indeed, a closer examination of the field equations $R_{AB} = 0$, for solutions which are independent of the “extra” coordinates, reveals that the large extra dimension $y$ can be either spacelike or timelike. Thus, for generality, instead of we should consider

$$dS^2 = \left(\frac{a - 1}{a + 1}\right)^{2\sigma_k} dt^2 - \frac{1}{a^4 r^4 (a - 1)^{2\sigma(k-1)+1}} [dr^2 + r^2 d\Omega^2] \pm \left(\frac{a + 1}{a - 1}\right)^{2\sigma} dy^2.$$  

In this paper we study a number of aspects that arise from the fact that by interchanging the roles of $t$ and $y$ in , and keeping the freedom for the signature of the extra dimension, we generate the line element

$$dS^2 = \left(\frac{a + 1}{a - 1}\right)^{2\sigma} dt^2 - \frac{1}{a^4 r^4 (a - 1)^{2\sigma(k-1)+1}} [dr^2 + r^2 d\Omega^2] \pm \left(\frac{a - 1}{a + 1}\right)^{2\sigma_k} dy^2,$$

which also satisfies the 5D field equations in vacuum. The issue is that metrics and seem to produce different interpretations on four-dimensional spacetime sections orthogonal to the extra dimension. In particular, the 5D analogue of the 4D Schwarzschild metric in isotropic coordinates

$$dS^2_{Schw} = \left(\frac{1 - M/2r}{1 + M/2r}\right)^2 dt^2 - \left(1 + \frac{M}{2r}\right)^4 [dr^2 + r^2 d\Omega^2] \pm dy^2,$$

is recovered, for the same central mass $M = 2/a$, in different limits. Namely, $k \to \infty$ and $\sigma k \to 1$ for , while $k = 0$ and $\sigma = -1$ for .

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1. Our conventions are: $c = G = 1$; Greek indices run over four-dimensions $\mu, \nu = 0, 1, 2, 3$; upper case Latin letters symbolize Kaluza-Klein indices $A, B = 0, 1, 2, 3, 4, \ldots$. The signature of the metric is $(1, -1, -1, -1, \pm 1)$.
2. The inclusion of spin changes this simple picture: Emparan and Reall have shown that, apart from the black hole solutions of Myers and Perry, there are five-dimensional rotating black ring solutions, with the same values of mass and spin.
3. This metric is known as the Schwarzschild black string.
We note that in principle
\[ -\infty < k < +\infty, \]
and, as a consequence of the constraint \( \sigma \), \( \sigma^2 \) has a maximum, namely \( \sigma^2 = 4/3 \), at \( k = 1/2 \). Therefore,
\[ -\frac{2}{\sqrt{3}} \leq \sigma \leq \frac{2}{\sqrt{3}}. \]

The first goal of this work is to elucidate the link between the four-dimensional interpretation of metrics \( \mathcal{M} \) and \( \mathcal{N} \) (on hypersurfaces \( y = \text{constant} \)) and their parameters \( k \) and \( \sigma \). In this regard a number of questions arise. For example, what is the “appropriate” range of the parameters \( k \) and \( \sigma \)? What is the physical meaning of \( k \)? In this paper we discuss these questions in the context of the induced-matter interpretation. We recall that, in this context the curvature in 5D induces effective matter in 4D, and the metric \( \mathcal{M} \) can be interpreted as describing extended spherical objects called solitons.

In section 2, in order to identify the solutions of physical interest we impose physical requirements on the induced-matter. We find that these conditions demand \( k > 0 \) and \( \sigma > 0 \) for metric \( \mathcal{M} \), while \( k > 0 \) and \( \sigma < 0 \) for metric \( \mathcal{N} \). We show that, although the parameters \( \sigma \) and \( k \) are not independent, the transformation from metric \( \mathcal{M} \) to \( \mathcal{N} \), and vice-versa, corresponds to the simultaneous change \( k \rightarrow 1/k \), \( \sigma \rightarrow -\sigma \), or
\[ (\sigma k) \leftrightarrow -\sigma, \]
which leaves \( \sigma(k-1) \) and \( \sigma^2 k \) invariant.

In a recent paper, Lake \[11\] examines the properties of the Kramer-Gross-Perry-Davidson-Owen solutions in a purely geometrical way. The solutions are classified on the basis of the Weyl invariant, the geometrical mass and the character of the singularity. Therefore his results in 5D hold for any physical approach in 4D.

The second goal of this paper is to find out how the physics in 4D is subordinated to the general geometrical properties in 5D. Since the properties of the effective-matter depend on \( k \) and \( \sigma \), we need to relate these parameters to those of Lake \[11\]. In section 3, we provide a complete analysis of the metrics \( \mathcal{M} \) and \( \mathcal{N} \) in the context of the geometrical approach. Our analysis is very similar, but not identical, to Lake’s and leads to somewhat distinct results in 4D. For example, metrics \( \mathcal{M} \) and \( \mathcal{N} \) require \( ar \geq 1 \), which in terms of the coordinate \( h \) used by Lake corresponds to \( h \in (1, \infty) \); the region \( h \in (0, 1) \) considered in \[11\] is excluded here because it is not asymptotically flat\[4\]. We will see that this “lack” of symmetry results in two families of solutions with very different physical properties in 4D.

The third goal here is to understand the physical meaning of the parameter \( k \). In section 4, we demonstrate that \( k \) is completely determined by the degree of compactification of the soliton. Thus, \( k \) is neither a universal constant nor a free parameter, but varies from soliton to soliton. This feature has been overlooked in our previous work \[12\] and other subsequent studies \[9\], \[13\], \[14\].

In the Appendix we examine the more general case where the metric in 4D is taken to be conformal to the metric induced from 5D on four-dimensional hypersurfaces orthogonal to the extra dimension. We find the same kind of solutions as in the induced-matter approach, but with a different parameterization.

### 2 The induced-matter approach

The aim of this section is to compare and contrast the four-dimensional interpretation of metrics \( \mathcal{M} \) and \( \mathcal{N} \). In order to facilitate the presentation, let us restate some concepts that are essential in our discussion.

In five-dimensional models, our spacetime is identified with some 4D hypersurface \( y = \text{constant} \), which is orthogonal to the extra dimension. Therefore, for a given line element in 5D
\[ ds^2 = g_{\mu\nu}(x^\rho, y)dx^\mu dx^\nu + e\Phi^2(x^\rho, y)dy^2, \quad \epsilon = \pm 1, \]
the corresponding metric in 4D is just \( g_{\mu\nu} \). Such an identification is a standard\[5\] technique in the induced-matter approach as well as in brane-world models. However, it is worth to mention the approach where the metric in 4D is conformal to the metric induced on \( y = \text{constant} \) hypersurfaces. We will examine this approach in the Appendix.

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\[4\] As a consequence, the geometrical symmetry between quadrant 2 for \( h \in (1, \infty) \) and quadrant 1 for \( h \in (0, 1) \) is broken.

\[5\] For an alternative approach, which reproduces our 4D spacetime on a dynamical hypersurface, see Refs. \[15\] and \[16\].
In 4D, the effective energy-momentum tensor $T_{\mu\nu}$ is obtained from the 4 + 1 dimensional reduction of the field equations in 5D. In terms of the metric, it is given by

$$8\pi T_{\alpha\beta} = -\frac{\epsilon}{2\Phi^2} \left[ \frac{\Phi^{*\alpha\beta}}{\Phi} - g^{*\alpha\beta} + g^{\lambda\mu} g_{\alpha\lambda} g_{\beta\mu} \frac{1}{2} g^{\mu\nu} g_{\mu\nu} g_{\alpha\beta} + \frac{1}{4} g_{\alpha\beta} \left( g^{\nu\eta} g_{\mu\nu} + (g^{\mu\nu} g_{\mu\nu})^2 \right) \right] + \frac{\Phi_{=\alpha\beta}}{\Phi},$$  

(10)

where $f = \partial f / \partial y$. For the case where the 5D metric is independent of $y$, the effective matter is not affected by the signature of the extra dimension and $T_{\mu\nu}$ reduces to

$$8\pi T_{\mu\nu} = \frac{\Phi_{\mu\nu}}{\Phi},$$  

(11)

with $g^{\mu\nu} \Phi_{\mu\nu} = 0$, which follows from $R_{44} = 0$. What this means is that, in this case the effective matter in 4D is radiation like.

In the case under consideration the 5D metric has the form

$$dS^2 = e^{\nu(r)} dt^2 - e^{\lambda(r)} [dr^2 + r^2 d\Omega^2] + \epsilon \Phi^2(r) dy^2.$$  

(12)

Then, from $g^{\mu\nu} \Phi_{\mu\nu} = 0$, it follows that

$$\Phi'' = -\Phi' \left( \frac{\nu' + \lambda'}{2} + \frac{2}{r} \right),$$  

(13)

where primes denote derivatives with respect to $r$. Using this expression, the explicit form of the induced energy-momentum tensor (11) can be written as

$$8\pi T^0_0 = -\frac{e^{-\lambda} \Phi' \nu'}{2\Phi},$$  

$$8\pi T^1_1 = e^{-\lambda} \left( \frac{\nu'}{2} + \frac{\lambda'}{r} + \frac{2}{r^2} \right) \frac{\Phi'}{\Phi},$$  

$$8\pi T^2_2 = 8\pi T^3_3 = -e^{-\lambda} \left( \frac{1}{r} + \frac{\lambda}{2} \right) \frac{\Phi'}{\Phi}.$$  

(14)

### 2.1 Physical radius

For metrics (3) and (4) the “physical” radius $R$ of the sphere with coordinate radius $r$ is given by

$$R(r) = \frac{(ar + 1)\sigma(k-1)+1}{a^2r(a^2 - 1)\sigma(k-1)-1}.$$  

(15)

We note that for $(k > 0, \sigma > 0), (k > 0, \sigma < 0), (k < 0, \sigma > 0)$, and $(k = 0, \sigma = 1)$, the center of the sphere $R = 0$ corresponds to $ar = 1$ and $R$ increases monotonically with the increase or $r$, i.e., $(dR/dr) > 0$.

However, there is no origin if we choose either $(k < 0, \sigma < 0)$ or $(k = 0, \sigma = -1)$. Indeed, for this choice $(dR/dr)$ changes sign at

$$ar_{\text{min}} = -\sqrt{k^2 + |k| + 1} + \frac{|k| + 1 + \sqrt{|k|}}{\sqrt{k^2 + |k| + 1}} + 1.$$  

(16)

It is sometimes called “black” or “Weyl” radiation, because in this case $T_{\mu\nu} = -\epsilon E_{\mu\nu}$, where $E_{\mu\nu}$ represents the spacetime projection of the five-dimensional Weyl tensor, which is traceless.
Thus, for \((k < 0, \sigma < 0)\), \(R \to \infty\) as \(ar \to 1\) and \(ar \to \infty\). The radius \(R\) has a minimum at the value of \(ar\) given by (16). For \((k = 0, \sigma = -1)\), \(R \to \infty\) as \(ar \to 0\) and \(ar \to \infty\). We note that \(R\) is not well defined\(^7\) for \(ar < 1\). Therefore, in what follows we will consider \(ar \geq 1\) everywhere.

Sign of \(k\): From (14) we obtain

\[
8\pi T_0^0 = \frac{4a^6\sigma^2 k r^4}{(ar + 1)^4 (ar - 1)^2} \left(\frac{ar - 1}{ar + 1}\right)^{2\sigma(k-1)}.
\]  

(17)

for metrics (3) and (4). We note that this is invariant under transformation (8). Consequently, the positivity of mass-energy density requires \(k > 0\) for both metrics, which in turn assures that \(R = 0\) at \(ar = 1\) and \(dR/dr > 0\) everywhere.

### 2.2 Gravitational mass

In 4D, the gravitational mass inside a 3D volume \(V_3\) is given by the Tolman-Whittaker formula, viz.,

\[
M_g(r) = \int (T_0^0 - T_1^1 - T_2^2 - T_3^3)\sqrt{-g} dV_3.
\]  

(18)

Using (14) we obtain

\[
M_g(r) = -\frac{1}{2} \int_{1/a}^r e^{(\nu + \lambda)/2} \frac{\psi'}{\Phi} r^2 dr.
\]  

(19)

For the metric (3), after straightforward calculation we get

\[
M_g(r) = \frac{(2\sigma k)}{a} \left(\frac{ar - 1}{ar + 1}\right)^\sigma,
\]  

(20)

while for the metric (4)

\[
\bar{M}_g(r) = \frac{(-2\sigma)}{a} \left(\frac{ar + 1}{ar - 1}\right)^{\sigma k}.
\]  

(21)

Clearly, the interchange \(\sigma k \leftrightarrow -\sigma\) transforms \(M_g(r) \leftrightarrow \bar{M}_g(r)\).

Positivity of gravitational mass: Sign of \(\sigma\). Since \(a\) is related to the Schwarzschild mass we take \(a > 0\) everywhere. Therefore, the positivity of the gravitational mass \(M_g\), for metric (3) requires \(\sigma > 0\), i.e.,

\[
\sigma = +\frac{1}{\sqrt{k^2 - k + 1}}.
\]  

(22)

On the other hand, for the metric (4) the positivity of \(\bar{M}_g\) requires \(\sigma < 0\), i.e.,

\[
\bar{\sigma} = -\frac{1}{\sqrt{k^2 - k + 1}}.
\]  

(23)

In summary, the gravitational mass becomes

\[
M_g(r) = \frac{2k}{a\sqrt{k^2 - k + 1}} \left(\frac{ar - 1}{ar + 1}\right)^{1/\sqrt{k^2 - k + 1}},
\]  

(24)

\(^7\)One could think that for \(\sigma(k - 1) - 1 = 2n\), where \(n\) is some integer number, one could properly define \(R\), for \(ar < 1\), as \(R = (ar + 1)^{2(n+1)}/a^2 r(1 - ar)^{2n}\). In this case \(\sigma(k - 1) = 2n + 1\) and substituting into (2) we obtain a quadratic equation for \(k\), namely, \(4(n^2 + n)k^2 - (4n^2 + 4n - 1)k + 4(n^2 + n) = 0\). However, for an arbitrary \(n\) this equation has no real solution. There are only two real solutions: for \(n = -1\) and \(n = 0\). They correspond to the special cases \((k = 0, \sigma = 1)\) and \((k = 0, \sigma = -1)\) considered above.

\(^8\)In what follows quantities, as \(M_g, \sigma\) and others, corresponding to metric (4) will be denoted with a bar over them, i.e., \(\bar{M}_g, \bar{\sigma}, \text{etc.}\)
and

\[ M_g(r) = \frac{2}{a\sqrt{k^2 - k + 1}} \left( \frac{ar - 1}{ar + 1} \right)^{k/\sqrt{k^2 - k + 1}}, \]  

for (3) and (4), respectively.

### 2.3 Possible scenarios in 4D

Thus, in the full range of \( k \) and \( \sigma \), there are solutions with distinct physical properties. Namely, the original Davidson-Owen family of solutions (3) contain four different scenarios. These are:

1. \((k < 0, \sigma > 0) \leftrightarrow (\rho < 0, M_g < 0)\),
2. \((k < 0, \sigma < 0) \leftrightarrow (\rho < 0, M_g > 0)\),
3. \((k > 0, \sigma < 0) \leftrightarrow (\rho > 0, M_g < 0)\),
4. \((k > 0, \sigma > 0) \leftrightarrow (\rho > 0, M_g > 0)\).  

(26)

Under the transformation \( t \to y \) we get the metric (4), which allows the following scenarios:

1. \((k < 0, \sigma > 0) \leftrightarrow (\rho < 0, M_g < 0)\),
2. \((k < 0, \sigma < 0) \leftrightarrow (\rho < 0, M_g > 0)\),
3. \((k > 0, \sigma < 0) \leftrightarrow (\rho > 0, M_g > 0)\),
4. \((k > 0, \sigma > 0) \leftrightarrow (\rho > 0, M_g < 0)\).  

(27)

All these families, except for 2 and 2, have a center at \( ar = 1 \), where \( M_g = 0 \), as well as \( dR/dr > 0 \) everywhere. Solutions 2 and 2 have no center and \( M_g \to \infty \) for \( ar \to 1 \). In summary:

1. The physical properties of the first two solutions are invariant under the transformation \( t \leftrightarrow y \), i.e., 1 \( \leftrightarrow \) \( \bar{1} \) and 2 \( \leftrightarrow \) \( \bar{2} \).
2. The other two solutions show interchange symmetry, i.e., 3 \( \leftrightarrow \) \( \bar{4} \) and 4 \( \leftrightarrow \) \( \bar{3} \).
3. Solutions with \( \rho > 0 \) and \( M_g < 0 \), after the transformation \( t \leftrightarrow y \) become \( \rho > 0 \) and \( M_g > 0 \). Namely, 3 \( \to \) \( \bar{3} \) and 4 \( \to \) \( \bar{4} \).

### 2.4 Effective matter for solutions 4 and \( \bar{3} \)

These satisfy the requirements on the induced-matter in 4D. In Appendix A, we show that \( T_1^1 \neq T_2^2 \) is a general feature of solitons in theories where the metric in 4D is conformal to the metric induced induced on \( y = \) constant hypersurfaces. Therefore, the generic approach is to describe the soliton matter as an anisotropic fluid with an effective energy-momentum tensor of the form:

\[ T_{\mu\nu} = (\rho + p_\perp)u_\mu u_\nu - p_\perp g_{\mu\nu} + (p_r - p_\perp)\chi_\mu \chi_\nu, \]  

(28)

where \( u^\mu \) is the four-velocity; \( \chi^\mu \) is a unit spacelike vector orthogonal to \( u^\mu \); \( \rho \) is the energy density; \( p_r \) is the pressure in the direction of \( \chi_\mu \), and \( p_\perp \) is the pressure on the two-space orthogonal to \( \chi_\mu \). If we choose \( u^\mu = \delta_0^\mu e^{-\nu/2} \) and \( \chi^\mu = \delta_1^\mu e^{-\lambda/2} \), then \( T_0^0 = \rho, T_1^1 = -p_r \) and \( T_2^2 = -p_\perp \). Consequently, the equation of state becomes

\[ \rho = p_r + 2p_\perp, \]  

(29)

which shows that the matter has the nature of radiation.

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9 The identification used here for the distinct solutions is similar to the one used by Lake [11].
10 Under certain conditions, a single anisotropic fluid can be modeled as a multicomponent fluid [18]-[20].
Effective matter for solution 4: Collecting results, the evaluation of the induced-matter quantities (14) for the solution 4 in (26) yields

\[ 8\pi \rho = \frac{4a^6k^4}{(k^2 - k + 1)(ar + 1)^4(ar - 1)^4} \left( \frac{ar - 1}{ar + 1} \right)^{2(k-1)/\sqrt{k^2 - k + 1}}, \]  

(30)

\[ 8\pi p_r = \frac{4a^5r^3[ar(2 - k) + (a^2r^2 + 1)\sqrt{k^2 - k + 1}]}{(k^2 - k + 1)(ar + 1)^4(ar - 1)^4} \left( \frac{ar - 1}{ar + 1} \right)^{2(k-1)/\sqrt{k^2 - k + 1}}, \]  

(31)

\[ 8\pi p_\perp = \frac{2a^5r^3[2ar(k - 1) - (a^2r^2 + 1)\sqrt{k^2 - k + 1}]}{(k^2 - k + 1)(ar + 1)^4(ar - 1)^4} \left( \frac{ar - 1}{ar + 1} \right)^{2(k-1)/\sqrt{k^2 - k + 1}}. \]  

(32)

We note that \( \rho = p_r = p_\perp = 0 \) for \( k \to \infty \), as expected.

Effective matter for solution 3: Similarly, for the solution \( \bar{3} \) in (27) we find

\[ 8\pi \bar{\rho} = \frac{4a^6k^4}{(k^2 - k + 1)(ar + 1)^4(ar - 1)^4} \left( \frac{ar - 1}{ar + 1} \right)^{2(1-k)/\sqrt{k^2 - k + 1}}, \]  

(33)

\[ 8\pi \bar{p}_r = \frac{4ka^5r^3[ar(2k - 1) + (a^2r^2 + 1)\sqrt{k^2 - k + 1}]}{(k^2 - k + 1)(ar + 1)^4(ar - 1)^4} \left( \frac{ar - 1}{ar + 1} \right)^{2(1-k)/\sqrt{k^2 - k + 1}}, \]  

(34)

\[ 8\pi \bar{p}_\perp = \frac{2ka^5r^3[2ar(1 - k) - (a^2r^2 + 1)\sqrt{k^2 - k + 1}]}{(k^2 - k + 1)(ar + 1)^4(ar - 1)^4} \left( \frac{ar - 1}{ar + 1} \right)^{2(1-k)/\sqrt{k^2 - k + 1}}. \]  

(35)

Here \( \bar{\rho} = \bar{p}_r = \bar{p}_\perp = 0 \) for \( k = 0 \).

Clearly,

\[ (\rho, \ p_r, \ p_\perp) \longleftrightarrow (\bar{\rho}, \ \bar{p}_r, \ \bar{p}_\perp) \]  

(36)

for

\[ k \longleftrightarrow 1/k. \]  

(37)

Both distributions are identical for \( k = 1 \), but for any other \( k \) they are very different\(^{11}\).

Finally, we mention that for possible astrophysical applications of solitons, it is crucial to note that Kaluza-Klein solitons are more massive than the Schwarzschild one. Indeed, we find

\[ \frac{2}{a} \leq M_\rho(\infty) \leq \frac{4}{a\sqrt{3}}. \]  

(38)

This is an interesting result which advocates for solitons as candidates for dark matter \(^{9}\).

\(^{11}\)For \( k = 1 \) the solution takes a particular simple form. It was rediscovered by Chatterjee \(^{6}\).
3 The geometrical approach

In a recent paper, Lake examined the properties of the Kramer-Gross-Perry-Davidson-Owen solutions in a purely geometrical way [11]. He classified the solutions on the basis of the Weyl invariant, the nakedness and geometrical mass of their associated singularities. The natural question to ask is how the properties of induced matter in 4D are related, or subordinated, to the geometrical ones in 5D. In order to facilitate the discussion, in this section we use the codification of the solutions used by Lake.

3.1 Lake’s parameterization

In Lake’s work the solutions are described in terms of the parameters $\alpha$ and $\delta$, in such a way that the Davidson-Owen line element [3] is recovered by changing $\delta \to -2\sigma k$ and $\alpha \to 2\sigma$. Under the transformation

$$\sigma = \frac{\alpha}{2}, \quad k = -\frac{\delta}{\alpha},$$

the constraint [2] becomes

$$\alpha^2 + \delta^2 + \alpha\delta = 4,$$

which in the $(\alpha, \delta)$ plane represents an ellipse [13]. Conversely, setting $\alpha = 2\sigma$ and $\delta = -2\sigma k$ we recover [2].

As we have discussed in section 2.1, the physical radius in Davidson-Owen solutions, in the coordinates of [3], is well defined in the region $ar \geq 1$ only. This corresponds to $h \in (1, \infty)$ in Lake’s notation [14]. In this region there are three “regular” solutions which, in the attached figure, correspond to quadrants 1, 3 and 4. For these solutions $R = 0$ and $M_g = 0$ at $ar = 1$, besides $dR/dr > 0$. Quadrant 2 solutions are singular in the sense that there is no origin and $M_g \to \infty$ in the limit $ar \to 1$, which now corresponds to $R \to \infty$. Also, there are four “exceptional” solutions, namely $a = (2, 0), b = (0, 2), c = (-2, 0)$ and $d = (0, -2)$.

Regular solutions in quadrants 1 and 4 have positive $\sigma$, viz., $\sigma = +1/\sqrt{k^2 - 1}$. So, in our approach, they are described by the original Davidson-Owen line element [4].

In quadrants 1 and 4 the parameter $k$ increases clockwise, along the ellipse, from $k = -\infty$ at the exceptional solution $b$, to $k = 0$ at $a$ and $k = +\infty$ at $d$. Thus, $k < 0$ in quadrant 1 and $k > 0$ in quadrant 4. From [17] it follows that in quadrant 1 the energy condition $\rho > 0$ is violated. Meanwhile, in quadrant 4 the effective matter distribution, which is given by [30-32], satisfies the physical conditions $\rho > 0$, $M_g > 0$ and possesses an origin $R = 0$ at $ar = 1$.

The line element corresponding to the exceptional solution $a = (2, 0)$ is obtained from the metric [3] for $k = 0$ and $\sigma = 1$,

$$dS^2_{a(k=0, \sigma=1)} = dt^2 - \left(1 - \frac{1}{ar}\right)^4 [dr^2 + r^2 d\Omega^2] \pm \left(\frac{1 + 1/\ar}{1 - 1/\ar}\right)^2 dy^2.$$  

(41)

Quadrant 2 singular solutions and quadrant 3 regular solutions have negative $\sigma$, viz., $\sigma = -1/\sqrt{k^2 - 1}$. So, in our approach, they are described by the line element [4]. In these quadrants, the parameter $k$ also increases clockwise, along the ellipse. It goes from $k = -\infty$ at $d$, to $k = 0$ at $c$ and $k = +\infty$ at $b$. Thus, $k < 0$ in quadrant 2 but $k > 0$ in quadrant 3. Therefore, $\rho < 0$ in 2 but in 3 the effective matter distribution, which is given by [33-35], satisfies the physical conditions $\rho > 0$, $M_g > 0$ and possesses an origin $R = 0$ at $ar = 1$.

The line element corresponding to the exceptional solution $c = (-2, 0)$ is obtained from the metric [4] for $k = 0$ and $\sigma = -1$. Namely,

$$dS^2_{c(k=0, \sigma=-1)} = \left(\frac{1 - 1/\ar}{1 + 1/\ar}\right)^2 dt^2 - \left(\frac{1 + 1/\ar}{1 - 1/\ar}\right)^4 [dr^2 + r^2 d\Omega^2] \pm dy^2.$$  

(42)

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12The gravitational mass defined in [13] is not equivalent to the geometrical mass, which is defined via the sectional curvature of the two-sphere [11].

13A similar parameterization was considered by Lim, Overduin and Wesson [21].

14The region $h \in (0, 1)$ is excluded from our discussion because those solutions are not asymptotically flat. As a consequence, we loose the symmetry between quadrant 2 for $h \in (1, \infty)$ and quadrant 1 for $h \in (0, 1)$. But the symmetry between solutions in quadrants 3 and 4 is not affected. See Table 1 in Ref. [11].
which is the 5D analogue of the 4D Schwarzschild metric in isotropic coordinates with $a = 2/M$.

The point $b = (0,2)$ is attained from quadrant 1 (say $b_1$ from metric (3)), for $\sigma = 0, k = -\infty$ and from quadrant 3 (say $b_3$ from metric (3)), for $\sigma = 0, k = +\infty$. Therefore, there are two limiting metrics

$$dS^2_{b_1} = dS^2_{b(k=-\infty,\sigma=0)} = \left(\frac{1+1/ar}{1-1/ar}\right)^2 dt^2 - \left(1 - \frac{1}{ar}\right)^4 [dr^2 + r^2d\Omega^2] \pm dy^2. \quad (43)$$

$$dS^2_{b_3} = dS^2_{b(k=+\infty,\sigma=0)} = dt^2 - \left(1 - \frac{1}{ar}\right)^4 [dr^2 + r^2d\Omega^2] \pm \left(\frac{1+1/ar}{1-1/ar}\right)^2 dy^2. \quad (44)$$

The point $d = (0,-2)$ is attained from quadrant 4 (say $d_4$ from metric (3)), for $\sigma = 0, k = +\infty$ and from quadrant 2 (say $d_2$ from metric (3)) for $\sigma = 0, k = -\infty$, viz.,

$$dS^2_{d_2} = dS^2_{d(k=-\infty,\sigma=0)} = dt^2 - \left(1 - \frac{1}{ar}\right)^4 [dr^2 + r^2d\Omega^2] \pm \left(1 - \frac{1}{ar}\right)[1 + 1/ar]^2 dy^2. \quad (45)$$

$$dS^2_{d_4} = dS^2_{d(k=+\infty,\sigma=0)} = (1 - 1/ar)^2 dt^2 - (1 + 1/ar)^4 [dr^2 + r^2d\Omega^2] \pm dy^2. \quad (46)$$

Clearly, by changing $t \leftrightarrow y$ we convert $dS^2_b \leftrightarrow dS^2_b$ ($b_1 \leftrightarrow b_3$) and $dS^2_d \leftrightarrow dS^2_d$ ($d_2 \leftrightarrow d_4$). No such connection exists between solutions $a$ and $c$. The metrics (13) and (15) represent the Schwarzschild black string in 5D with $M = -2/a$ and $M = 2/a$, respectively. On the other hand, solutions (11), (14) and (16) represent the zero dipole moment soliton of Gross and Perry [3].

The top line $\alpha + \delta = 2/\sqrt{3}$ connects the solution $e = (4/\sqrt{3},-2/\sqrt{3})$, for which $k = 1/2$ and $\sigma = 2/\sqrt{3}$, with the solution $\bar{e} = (-2/\sqrt{3},4/\sqrt{3})$, for which $k = 2$ and $\sigma = -1/\sqrt{3}$.

The bottom line $\alpha + \delta = -2/\sqrt{3}$ connects the solution $f = (2/\sqrt{3},-4/\sqrt{3})$, for which $k = 2$ and $\sigma = 1/\sqrt{3}$, with the solution $\bar{f} = (-4/\sqrt{3},2/\sqrt{3})$, for which $k = 1/2$ and $\sigma = -2/\sqrt{3}$.

Solutions of quadrant 2 have been interpreted as describing wormholes by Agnese et al [13]. They possess positive gravitational mass, but violate the weak energy condition $\rho > 0$. The solution given recently by Millward is located in quadrant 1 and corresponds to the particular choice $k = -1, \sigma = 1/\sqrt{3}$ or $\alpha = \delta = 2/\sqrt{3}$ (see [11]). Therefore, it exhibits negative gravitational mass.

### 3.2 Interchange symmetry and physical equivalence

The ellipse (11) is invariant under the change $\alpha \to -\alpha, \delta \to -\delta$, which is equivalent to rotating the ellipse in $180^\circ$, in any direction. In terms of the Davidson-Owen parameters, this corresponds to the transformation

$$\sigma = -\frac{\alpha}{2}, \quad k = -\frac{\delta}{\alpha}. \quad (47)$$

It should be noted that setting $(\alpha = -2\sigma, \delta = 2k\sigma)$ in Lake’s solution, we recover the line element (3) instead of (3).

From (39) and (47), it follows that the interchange $(\alpha, \delta) \leftrightarrow (-\alpha, -\delta)$ is analogous to the choice of positive or negative $\sigma$, keeping the same $k$, as in (22) and (23). Consequently, if we use (47) instead of (39), then we obtain the quadrant interchanges $1 \leftrightarrow 2$ and $3 \leftrightarrow 4$ (say $\bar{e} \leftrightarrow f$ and $\bar{f} \leftrightarrow e$) along with the point interchanges $a \leftrightarrow c$, i.e., $dS^2_b \leftrightarrow dS^2_d$ and $dS^2_d \leftrightarrow dS^2_b$.

Regarding solutions $b$ and $d$ the invariance $\alpha \to -\alpha, \delta \to -\delta$ corresponds to $dS^2_b \leftrightarrow dS^2_d$ ($b_1 \leftrightarrow d_2$) and $dS^2_d \leftrightarrow dS^2_b$ ($b_3 \leftrightarrow d_4$).

However, it should be emphasized that this geometrical invariance is not accompanied by a “physical” equivalence. For example, it transforms the black string (12) into the zero dipole moment soliton (11). In the induced-matter approach, the effective energy-momentum tensors corresponding to $\bar{e}$ and $f$ (as well as $\bar{f}$ and $e$) are totally different.
3.3 Singularities and $t \leftrightarrow y$

We notice that the invariance under $\alpha \leftrightarrow \delta$ is equivalent to (8). This symmetry is not a consequence of any rotation in the $(\alpha, \delta)$ plane, but is a consequence of the interchange $t \leftrightarrow y$, which is allowed by the freedom of the signature of the extra dimension in Ricci flat $5D$ manifolds with spatial spherical symmetry and no-dependence of the extra dimension.

The singularity at $R = 0$ ($ar = 1$), for solutions in quadrants 3 and 4, corresponds to a lightlike singularity. The same kind of naked singularities, where the horizon coincides with the singularity, are found in black hole solutions ($R_{AB} = 0$) in other dimensions, for example in $d = 11$ supergravity [22].

Solutions c and b3 correspond, respectively, to the Schwarzschild solution with a spacelike singularity and the zero dipole moment soliton with timelike singularity. Thus, in quadrant 3 as $k$ goes from zero to infinity, the singularity changes from spacelike to lightlike and then to timelike. Similarly, in quadrant 4 as $k$ goes from zero to infinity, the singularity changes from timelike at $a$ to lightlike and then to spacelike at $d_4$.

4 Degree of compactification

The soliton matter is distributed in the form of centrally concentrated clouds, without a solid surface. However, the matter density decreases as $\rho \sim 1/a^2 r^4$ indicating that the matter is heavily concentrated near the origin. Therefore, it is always possible to define a sphere where most of the total mass is contained.

Let us define $r_\xi$, which represents the coordinate radius of the sphere containing the $\xi$-th part of the total gravitational matter of the soliton ($0 \leq \xi \leq 1$).

The total gravitational mass for the soliton described by metric (3) is $M_\xi(\infty) = 2k/(a\sqrt{k^2 - k + 1})$, which is obtained from (24) in the limit $ar \gg 1$. Therefore, we find

$$r_\xi(k) = \frac{1}{a} \left( \frac{1 + \xi \sqrt{k^2 - k + 1}}{1 - \xi \sqrt{k^2 - k + 1}} \right).$$

The corresponding physical radius $R_\xi = r_\xi e^{M(\infty)/2}$ is

$$R_\xi(k) = \frac{4}{a(1 - \xi^2 \sqrt{k^2 - k + 1})^\xi(1 - \sqrt{k^2 - k + 1}).}$$

We now define the “surface” gravitational potential $\phi$ as

$$\phi_\xi = \frac{M_\xi}{R_\xi},$$

which for the case under consideration becomes

$$\phi_\xi(k) = \frac{1}{2} \frac{k}{\sqrt{k^2 - k + 1}} \left( 1 - \xi^2 \sqrt{k^2 - k + 1} \right)^\xi(1 - \sqrt{k^2 - k + 1}).$$

This function has two important features. Namely, it is independent of parameter $a$, and is a monotonic function of $k$. Therefore, it gives a one-to-one connection between the surface gravitational potential $\phi$ and the soliton parameter $k$, which allows us to calculate this parameter for different astrophysical phenomena.

4.1 Evaluation of $k$

In order to study observational implications of extra dimensions, and test possible deviations from general relativity, the Sun and other stars are modeled as Kaluza-Klein solitons [23], [24]. Let us apply the above formulae to evaluate $k$ for the solar system.

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15 A rotation in 90° would be $\alpha \rightarrow \delta, \delta \rightarrow -\alpha$
In this case, $\phi$ can be taken as the surface gravitational potential of the Sun, which is $\phi_\odot = 0.212 \times 10^{-5}$. Although the Sun is a gaseous sphere, without a sharp boundary, most of its mass is contained within the photosphere of (mean) radius $R_\odot = 0.696 \times 10^6 km$. For the sake of argument, let us say that $99.9999\%$ of the total mass is enclosed there\footnote{Here we use $M_\odot = 1.9891 \times 10^{30} kg$ and that the mean diameter of the Sun is about $1.392 \times 10^9 km$. Also, since $c = G = 1$, it follows that $1kg = 7.41 \times 10^{-31} km$ and $M_\odot = 1.474 km$.}, which corresponds to $\xi = 0.999999$. Substituting these values into \footnote{This is within the accuracy allowed by the standard model for the interior of the Sun (no rotation, no diffusion) by Pierre Demarque and David Guenther in \cite{25}. For this selection, the remaining $10^{-8} M_\odot$ part is distributed above the photosphere with very low density. If we use \cite{26}, we find that $90\%$ of this part is located in a layer, whose thickness is $9$ times the solar radius, which is about $(1/10)$-th of the mean distance between the Sun and Mercury.} we obtain

$$k_\odot = 2.12.$$  

In order to see whether this number makes sense we use it in \footnote{If we model more compact astrophysical objects, like neutron stars, as Kaluza-Klein solitons, we find that $k$ is significantly larger. For example, for the value of $\xi$ taken above and the surface potentials}

$$\phi = (10^{-3}, \ 10^{-2}, \ 10^{-1}, \ 2 \times 10^{-1}, \ 3 \times 10^{-1}, \ 4 \times 10^{-1}), \quad (53)$$

from \footnote{The corresponding models in $4D$ scenarios, belonging to the well-known Kramers-Gross-Perry-Davidson-Owen family of solutions. In sections 2 and 3 we have provided a detailed study of the metrics \cite{4} and \cite{4} within the context of the induced-matter and the geometrical approaches.}

\footnote{The solutions of quadrants 1 and 2 are symmetrical in $5D$, and duplicate each other, iff we include in the discussion the region $0 < ar < 1 \ (h \in (0, 1)$ in Lake’s notation \cite{11}). However, when we restrict ourselves to solutions that are asymptotically flat, as we do here, then we exclude the region $0 < ar < 1$, which breaks the symmetry between them. The corresponding models in $4D$ bear the imprint of this lack of symmetry. Namely, the physical radius of quadrant} we obtain

$$k \approx (10^3, \ 10^4, \ 1.1 \times 10^5, \ 2.5 \times 10^5, \ 4.5 \times 10^5, \ 8 \times 10^5). \quad (54)$$

Clearly, $k \to \infty$ for $\phi \to 1/2$, which corresponds to the Schwarzschild black hole. A similar result can be obtained from metric \footnote{This is within the accuracy allowed by the standard model for the interior of the Sun (no rotation, no diffusion) by Pierre Demarque and David Guenther in \cite{25}. For this selection, the remaining $10^{-8} M_\odot$ part is distributed above the photosphere with very low density. If we use \cite{26}, we find that $90\%$ of this part is located in a layer, whose thickness is $9$ times the solar radius, which is about $(1/10)$-th of the mean distance between the Sun and Mercury.} with the corresponding change \footnote{This is within the accuracy allowed by the standard model for the interior of the Sun (no rotation, no diffusion) by Pierre Demarque and David Guenther in \cite{25}. For this selection, the remaining $10^{-8} M_\odot$ part is distributed above the photosphere with very low density. If we use \cite{26}, we find that $90\%$ of this part is located in a layer, whose thickness is $9$ times the solar radius, which is about $(1/10)$-th of the mean distance between the Sun and Mercury.}.

The main conclusion from this section is that the Kaluza-Klein potential of the astrophysical phenomena, like the Sun or other stars, modeled in Kaluza-Klein theory.

## 5 Summary and concluding remarks

When the metric coefficients are independent of $y$, the extra dimension can be either spacelike or timelike, without affecting the effective matter distribution \footnote{This is within the accuracy allowed by the standard model for the interior of the Sun (no rotation, no diffusion) by Pierre Demarque and David Guenther in \cite{25}. For this selection, the remaining $10^{-8} M_\odot$ part is distributed above the photosphere with very low density. If we use \cite{26}, we find that $90\%$ of this part is located in a layer, whose thickness is $9$ times the solar radius, which is about $(1/10)$-th of the mean distance between the Sun and Mercury.}. This is true for both, static and non-static metrics. In practice this means that $t$ and $y$ are interchangeable, which in turn allows us to generate different scenarios in $4D$.

The crucial question is, how do we recover our 4-dimensional world in higher-dimensional theories?. This is an unsettled question yet, but the popular wisdom is that we do this by going onto a hypersurface $y = constant$. With this assumption, the information about the fifth dimension is consolidated in the nonlocal stresses induced in $4D$ from the Weyl curvature in $5D$. As a consequence, even in the absence of matter, the exterior of a spherical star, like our sun, is not in general an empty Schwarzschild spacetime. We expect that these stresses, contributing to the effective pressure, will influence the conditions at the surface of a star \footnote{This is within the accuracy allowed by the standard model for the interior of the Sun (no rotation, no diffusion) by Pierre Demarque and David Guenther in \cite{25}. For this selection, the remaining $10^{-8} M_\odot$ part is distributed above the photosphere with very low density. If we use \cite{26}, we find that $90\%$ of this part is located in a layer, whose thickness is $9$ times the solar radius, which is about $(1/10)$-th of the mean distance between the Sun and Mercury.}.

In this work we have discussed in detail the relationship between the apparently different $4D$ scenarios, belonging to the well-known Kramers-Gross-Perry-Davidson-Owen family of solutions. In sections 2 and 3 we have provided a detailed study of the metrics \footnote{This is within the accuracy allowed by the standard model for the interior of the Sun (no rotation, no diffusion) by Pierre Demarque and David Guenther in \cite{25}. For this selection, the remaining $10^{-8} M_\odot$ part is distributed above the photosphere with very low density. If we use \cite{26}, we find that $90\%$ of this part is located in a layer, whose thickness is $9$ times the solar radius, which is about $(1/10)$-th of the mean distance between the Sun and Mercury.} and \footnote{This is within the accuracy allowed by the standard model for the interior of the Sun (no rotation, no diffusion) by Pierre Demarque and David Guenther in \cite{25}. For this selection, the remaining $10^{-8} M_\odot$ part is distributed above the photosphere with very low density. If we use \cite{26}, we find that $90\%$ of this part is located in a layer, whose thickness is $9$ times the solar radius, which is about $(1/10)$-th of the mean distance between the Sun and Mercury.} within the context of the induced-matter and the geometrical approaches. The solutions of quadrants 3 and 4 share the same geometrical properties in $5D$ (see Table 1 in Ref. \cite{11}). We have found that their four-dimensional counterparts are equivalent modulo transformation $k \leftrightarrow 1/k$: the corresponding matter quantities transform as $(\rho, \ p_r, \ p_\perp) \leftrightarrow (\bar{\rho}, \ \bar{p}_r, \ \bar{p}_\perp)$ for $k \leftrightarrow 1/k$. Besides, as $k$ increases from zero to infinity, the singularity of quadrants 3 solutions changes as: (spacelike $\rightarrow$ lightlight $\rightarrow$ timelike). For the same range of $k$, the singularity of quadrant 4 solutions changes as: (timelike $\rightarrow$ lightlight $\rightarrow$ spacelike). The solutions of quadrants 1 and 2 are symmetrical in $5D$, and duplicate each other, iff we include in the discussion the region $0 < ar < 1 \ (h \in (0, 1)$ in Lake’s notation \cite{11}). However, when we restrict ourselves to solutions that are asymptotically flat, as we do here, then we exclude the region $0 < ar < 1$, which breaks the symmetry between them. The corresponding models in $4D$ bear the imprint of this lack of symmetry. Namely, the physical radius of quadrant
solutions 2 has a positive minimum at the value of $r$ given by (16), while quadrant solutions 2 have an origin $R = 0$ at $ar = 1$ and $dR/dr > 0$ everywhere. In the context of induced-matter approach, this lack of symmetry yields solutions with positive gravitational mass for $(k < 0, \sigma < 0)$, and negative gravitational mass for $(k < 0, \sigma > 0)$.

In summary, the main conclusions from this paper are the following:

1. In the range $ar \geq 1$ (or $h \in (1, \infty)$ in Lake’s notation), the solutions in quadrants 1, 2 and 3 (or 4) are all distinct from each other; they have different physical and geometrical properties. They epitomize the character and nature of the effective four-dimensional picture in the conformal approach (A-1), where we find the same kind of solutions as in the induced-matter approach, but with a different parameterization.

2. Results concerning the signs of $k$ and $\sigma$, and their relations to Lake’s parameters allow us to compare and contrast both approaches. In the geometrical approach, based on the nature of the singularities at the origin, in [11] it is concluded that the physical solutions are those of quadrant 1 and 2. However, in the induced matter approach the solutions of physical interest are 3 and 4. Since the black string and the zero dipole moment solitons are unstable to any metric perturbation, one would expect them to decay into quadrant solutions 3 or 4, because the weak energy condition is violated in 1 and 2.

3. The singularity at $r = a$, in the physical region 3 (or 4), is lightlike and coincides with the horizon.

4. Solutions with $\rho > 0$ and $M_g < 0$, after the transformation $t \leftrightarrow y$ become identical to those with positive $\rho$ and $M_g$. This seems to be a general feature of solutions admitting a timelike extra dimension.

5. The basic characteristic of the effective soliton matter is that it behaves like an anisotropic fluid, which can be described by the energy-momentum tensor (28). This is true in the induced-matter approach, as well as in the conformal approach (A-1). This is a general result which, as far as we know, has never been unveiled in the literature.

6. The five-dimensional parameter $k$ can be evaluated by means of measurements in 4D, namely by the “surface” gravitational potential of the soliton. This is an important result which may help in observations for the experimental/observational test of the 5D theory.

Appendix A: The conformal approach

The prediction of physical effects from extra dimensions, which can be measured in experiments or observations, requires a “correct” identification of the physical or observable spacetime from the multidimensional one. Unfortunately, this is not an easy task [10].

Besides the approach discussed in section 2, we would like to briefly discuss the approach where the effective metric in 4D, say $g_{\mu\nu}^{eff}$, is conformal to the metric induced from 5D on four-dimensional hypersurfaces orthogonal to the extra dimension. The factorization $g_{\mu\nu}^{eff} = \Phi g_{\mu\nu}$ has been considered as a standard Kaluza-Klein technique in 5D theories with a compact extra dimension [18] [11] [28]. Here, for generality, we will examine

$$g_{\mu\nu}^{eff} = \Phi^N g_{\mu\nu},$$

where $N$ is some constant. For $N = 1$ we have the original Davidson-Owen interpretation, and for $N = 0$ the induced-matter interpretation discussed here.

For future references we provide the non-vanishing components of the effective energy-momentum tensor. For the metric [11] these are

$$8\pi T_0^0 = \frac{a^6 r^4 \sigma^2 [4k - N^2 - 4N(k - 1)]}{(ar + 1)^4} \left(\frac{ar - 1}{ar + 1}\right)^{2\sigma(k - 1 + N/2)},$$

18 A similar technique has been used in more than 5-dimensions [27]
\[ 8\pi T^1_1 = -\frac{4a^3\rho^2 \{(1-N)[ar(2-k) + (a^2r^2 + 1)/\sigma] + (3/4)N^2ar\}}{(ar + 1)^4(ar - 1)^4} \left(\frac{ar - 1}{ar + 1}\right)^{2\sigma(k-1+N/2)}, \] (A-3)

\[ 8\pi T^2_2 = 8\pi T^3_3 = -\frac{2a^5r^3\rho^2 \{2ar[k(1-N) - 1] - (1-N)(a^2r^2 + 1)/\sigma) + (1/2)N^2ar\}}{(ar + 1)^4(ar - 1)^4} \left(\frac{ar - 1}{ar + 1}\right)^{2\sigma(k-1+N/2)}, \] (A-4)

**Anisotropic pressures:** From the above we get

\[ 8\pi(T^2_2 - T^1_1) = -\frac{2a^5r^3\rho^2 \{[4k(1-N) + 4N - 6 - N^2]ar + 3(N-1)(a^2r^2 + 1)/\sigma\}}{(ar + 1)^4(ar - 1)^4} \left(\frac{ar - 1}{ar + 1}\right)^{2\sigma(k-1+N/2)}, \] (A-5)

which shows that \( T^2_2 \neq T^1_1 \) for any value of \( N \). Thus, the effective matter behaves as an anisotropic fluid, which can be described by the energy-momentum tensor \( \mathbf{T} \), not only in the induced-matter approach. In fact, we see that this is a general feature of the conformal approach \( \mathbf{A-1} \). This is an interesting result which, as far as we know, has never been exposed in the literature.

**Gravitational mass:** It can be verified that the gravitational mass, given by the Tolman-Whittaker formula \( \mathbf{18} \), becomes

\[ M_g(r) = \frac{(2k-N)\sigma}{a} \left(\frac{ar - 1}{ar + 1}\right)^{\sigma(1-N)}. \] (A-6)

Here, we require \((2k-N)\sigma > 0\) and \(\sigma(1-N) > 0\) in order to ensure the positivity of \(M_g\), as well as, the condition \(M_g = 0\) at \(ar = 1\), respectively.

**Factorization with \( N = 0 \):** This case corresponds to the induced-matter and braneworld approaches where the metric of the physical spacetime is identified with the metric induced in 4D. In this case the above expressions reduce to the ones in section 2.

**Factorization with \( N \neq 1 \):** This is the general case. Now the condition \( \rho > 0 \) demands \( k > [N(N-4)/4(1-N)] \) for \( N < 1 \), and \( k < [N(N-4)/4(1-N)] \) for \( N > 1 \). Once again, there are four different families of solutions. They are equivalent to those in \( \mathbf{26} \) for \( N = 0 \), but are displaced along the ellipse. For example, for \( N = 2 \) the physical solutions \( (\rho > 0; M_g > 0) \), with a center at \( ar = 1 \) where \( M_g = 0 \) move from quadrant 4 to the region encompassing quadrant 2 and the lower half (bellow the dotted line) of quadrant 3, i.e., \((k < 1, \sigma < 0)\). The wormhole solutions of Agnese et al \( \mathbf{13} \) are now parameterized by \((k > 1, \sigma > 0)\). So they move from quadrant 2 to the lower half (bellow the dotted line) of quadrant 4.

**Factorization with \( N = 1 \):** This factorization was considered by Davidson and Owen \( \mathbf{1} \). In this case the expressions for the effective energy-momentum tensor are specially simple. However, as far as we know, they are unreachable in the literature. Therefore, in view of their importance, we provide them here.

For the five-dimensional metric \( \mathbf{1} \), the effective spacetime is described by the line element

\[ ds^2 = \left(\frac{ar - 1}{ar + 1}\right)^{2\varepsilon} dt^2 - \frac{1}{a^4r^4(ar - 1)^{2(\varepsilon+1)}} [dr^2 + r^2 d\Omega^2], \] (A-7)

where \( \varepsilon = \sigma(k-1/2) \).

In this case

\[ 8\pi\rho = \frac{4a^6r^4(1-\varepsilon^2)}{(ar + 1)^4(ar - 1)^4} \left(\frac{ar - 1}{ar + 1}\right)^{2\varepsilon}. \] (A-8)
The equations of state for the anisotropic matter are
\[ p_r = \rho, \quad p_\perp = -\rho. \tag{A-9} \]
If we introduce the concept of average pressure, \( < p > \), as
\[ < p >= -\frac{1}{3}(T^1_1 + T^2_2 + T^3_3), \tag{A-10} \]
then, the effective matter satisfies the equation of state\(^{19}\)
\[ \rho = -3 < p >. \tag{A-11} \]

The gravitational mass for \( N = 1 \) is constant throughout the space, viz.,
\[ M_g(r) = \frac{2\varepsilon}{a}. \tag{A-12} \]
The Schwarzschild limit corresponds to \( \varepsilon = 1 \), for which we get \( M_g = 2/a \), as expected. We note that the only conditions on the “four-dimensional” parameter \( \varepsilon \) come from the positivity of \( \rho \) and \( M_g \), which require
\[ 0 \leq \varepsilon \leq 1. \tag{A-13} \]
Once we know its value, we can reconstruct the five-dimensional quantities \( \sigma \) and \( k \) as follows
\[ \sigma = \pm \frac{2\sqrt{1 - \varepsilon^2}}{\sqrt{3}}, \quad k = \frac{1}{2} \left( 1 \pm \frac{\sqrt{3}\varepsilon}{\sqrt{1 - \varepsilon^2}} \right) \tag{A-14} \]
Finally, we would like to mention that under the transformation \( t \leftrightarrow y \), the above expressions should be changed according to (A-11), i.e., \( \sigma k \leftrightarrow -\sigma \) everywhere.

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\(^{19}\)In the induced-matter approach, from (29) it follows that \( \rho = 3 < p > \). However, we should emphasize that this and (A-11) are just formal expressions, because the physical interpretation of an average pressure, in different “directions”, is not clear.
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