GHOST CHARACTERS AND CHARACTER VARIETIES OF 2-FOLD BRANCHED COVERS

FUMIKAZU NAGASATO AND SHINNOSUKE SUZUKI

Abstract. It is known that for any knot $K$ every (meridionally) trace-free $\text{SL}_2(\mathbb{C})$-representation of the knot group $G(K)$ gives an $\text{SL}_2(\mathbb{C})$-representation of the fundamental group $\pi_1(\Sigma_2K)$ of the 2-fold branched covering $\Sigma_2K$ of the 3-sphere branched along $K$. In this paper, we show by using a notion called a ghost character of a knot that for the $(4,5)$-torus knot $T_{4,5}$ the fundamental group $\pi_1(\Sigma_2T_{4,5})$ has an $\text{SL}_2(\mathbb{C})$-representation which cannot be given by any trace-free $\text{SL}_2(\mathbb{C})$-representation of $G(T_{4,5})$.

1. Background

In [10], we discovered several important properties of the trace-free slice of a knot $K$, denoted by $S_0(K)$, which is the cross-section of the character variety $X(K)$ of the knot group $G(K)$ cut by the hypersurface defined by meridionally trace-free condition $\text{tr}(\rho(\mu_K)) = 0$, where $\mu_K$ is a meridian of $K$. (For more details, refer to Subsection 2.2. See also [5, 10] etc.) One of the most remarkable properties of $S_0(K)$ is a mechanism such that $S_0(K)$ gives a large subset of the character variety $X(\Sigma_2K)$ of the fundamental group $\pi_1(\Sigma_2K)$ of the 2-fold branched cover $\Sigma_2K$ of 3-sphere $\mathbb{S}^3$ branched along the knot $K$. We denote the mechanism by the map $\hat{\Phi} : S_0(K) \to X(\Sigma_2K)$ as in [10]. The map $\hat{\Phi}$ is a very powerful tool to construct $\text{SL}_2(\mathbb{C})$-representations of $\pi_1(\Sigma_2K)$ from trace-free $\text{SL}_2(\mathbb{C})$-representations of $G(K)$\footnote{Such an $\text{SL}_2(\mathbb{C})$-representation of $\pi_1(\Sigma_2K)$ is called a $\tau$-equivalent representation. See [10].} since for any case where we have calculated $X(\Sigma_2K)$ the map $\hat{\Phi}$ is surjective. For example, $\hat{\Phi}$ is surjective for any 2-bridge knots and any pretzel knots. (Refer to [10] Theorem 1 and Lemma 23. See also [6, Theorem 1.3].) Moreover, we have shown in [9] Theorem 4.9 (1)] that for any 3-bridge knots the map $\hat{\Phi}$ is surjective. In this perspective, we have the following natural question:

Is the map $\hat{\Phi}$ surjective for any knot?

In this paper, we will give a negative answer to this question. More precisely, we will show the following.

Theorem 1.1. For the $(4,5)$-torus knot $T_{4,5}$, the map $\hat{\Phi}$ is not surjective. Namely, there exists an $\text{SL}_2(\mathbb{C})$-representation of $\pi_1(\Sigma_2T_{4,5})$ which cannot be given by any trace-free representations of $G(T_{4,5})$.

The key to this result is a ghost character of a knot, introduced in [8] (see also [9] Definition 4.7]). In general, it is not easy to find such a representation since basically we have to calculate all elements of both $S_0(K)$ and $X(\Sigma_2K)$ to compare them.
However, a ghost character let us know about an existence of such a representation relatively easy. In the following sections, we will explain the reason by reviewing briefly the trace-free slice $S_0(K)$ and the ghost characters. Then we will show that the $(4,5)$-torus knot $T_{4,5}$ has a ghost character and it is given by an $\text{SL}_2(\mathbb{C})$-representation of $\pi_1(\Sigma_2 T_{4,5})$, which essentially concludes the desired result. (See Theorem 3.2)

2. Ghost characters of a knot

A ghost character of a knot is defined via the characters of trace-free $\text{SL}_2(\mathbb{C})$-representations of the knot group. We first review them through the character varieties and its trace-free slice.

2.1. A quick review of the character variety $X(G)$ of a finitely presented group $G$. Let $G$ be a finitely presented group generated by $n$ elements $g_1, \cdots, g_n$. For a representation $\rho : G \to \text{SL}_2(\mathbb{C})$, we consider the character $\chi_\rho$ of $\rho$, which is the function on $G$ defined by $\chi_\rho(g) = \text{tr}(\rho(g))$ ($\forall g \in G$). We denote by $\mathcal{X}(G)$ the set of characters of $\text{SL}_2(\mathbb{C})$-representations of $G$. By $[\mathfrak{3}]$, the $\text{SL}_2(\mathbb{C})$-trace identity

$$\text{tr}(AB) = \text{tr}(A)\text{tr}(B) - \text{tr}(A B^{-1}) \quad (A, B \in \text{SL}_2(\mathbb{C}))$$

shows that the trace function $t_g(\rho) = \text{tr}(\rho(g))$ for an unspecified representation $\rho : G \to \text{SL}_2(\mathbb{C})$ and any $g \in G$ is expressed by a polynomial in

$$t_{g_i}(\rho) \quad (1 \leq i \leq n),$$
$$t_{g_i g_j}(\rho) \quad (1 \leq i < j \leq n),$$
$$t_{g_i g_j g_k}(\rho) \quad (1 \leq i < j < k \leq n).$$

It is known that the image of $\mathcal{X}(G)$ under the map $t$

$$t : \mathcal{X}(G) \to \mathbb{C}^{n+\binom{n}{2}+\binom{n}{3}}, \ t(\chi_\rho) = (t_{g_1}(\chi_\rho); t_{g_i g_j}(\chi_\rho); t_{g_i g_j g_k}(\chi_\rho)),$$

where we extend the trace function $t_g$ to the characters by $t_g(\chi_\rho) = t_g(\rho)$, is a closed algebraic set (refer to $[\Pi]$). That closed algebraic set is called the character variety of $G$ and denoted by $X(G)$. The character varieties has been calculated for many cases, however it is hard to determine the defining polynomial of them in general.

2.2. Trace-free slice $S_0(K)$ of the character variety $X(K)$. Let $G(K)$ be the knot group of $K$ and $\mu_K$ a meridian of $K$. A representation $\rho : G(K) \to \text{SL}_2(\mathbb{C})$ is said to be trace-free (traceless) if $\text{tr}(\rho(\mu_K)) = 0$ holds. We call its character $\chi_\rho$ a trace-free character. The set of trace-free characters gives a subset of $\mathcal{X}(G(K))$, denoted by $\mathfrak{S}_0(K)$:

$$\mathfrak{S}_0(K) = \{\chi_\rho \in \mathcal{X}(K) \mid \chi_\rho(\mu_K) = 0\}.$$

Again, by $[\Pi]$, $\mathfrak{S}_0(K)$ can be realized as an algebraic subset $S_0(K)$ of the character variety $X(K) = X(G(K))$ through the map $t$. By definition, the subset $S_0(K)$ can be thought of as the cross-section of $X(K)$ cut by the hyperplane $t_{\mu_K}(\chi_\rho) = 0$, so $S_0(K)$ is also a closed algebraic set. We call this algebraic set $S_0(K)$ the trace-free slice of $X(K)$ (or of $K$, simply). For a Wirtinger presentation

$$G(K) = \langle m_1, \cdots, m_n \mid r_1, \cdots, r_n \rangle,$$
since \( t_{\mu_K}(\chi_\rho) = 0 \) means \( t_{m_i}(\chi_\rho) = 0 \) for any \( 1 \leq i \leq n \), we have
\[
S_0(K) = t(\mathcal{S}_0(K)) \cong \left\{ (t_{m_i m_j}(\chi_\rho); t_{m_i m_j m_k}(\chi_\rho)) \in \mathbb{C}^n_{(2)} + \binom{n}{3} \middle| \chi_\rho \in \mathcal{S}_0(K) \right\}.
\]
Based on this description, the following theorem gives us a powerful tool to calculate the trace-free slice \( S_0(K) \). For a diagram \( D \) of a knot, \((i,j,k)\ (j < k)\) is called a Wirtinger triple if \( i \)th, \( j \)th and \( k \)th arcs \( a_i, a_j, a_k \) of \( D \) meet at a crossing such that \( a_i \) is the overarc and \( a_j, a_k \) are the underarcs.

**Theorem 2.1** ([8], cf. Theorems 3.1 and 3.2 in [3]). Let \( G(K) = \langle m_1, \ldots, m_n \mid r_1, \ldots, r_n \rangle \) be a Wirtinger presentation. Then \( S_0(K) \) is isomorphic to the following algebraic set in \( \mathbb{C}^n_{(2)} + \binom{n}{3} \):
\[
S_0(K) \cong \left\{ (x_{12}, \ldots, x_{nn-1}; x_{123}, \ldots, x_{n-2,n-1,n}) \in \mathbb{C}^n_{(2)} + \binom{n}{3} \middle| (F2), (H), (R) \right\},
\]
where \( (F2), (H) \) and \( (R) \) are the equations defined as follows:

**\( (F2) \):** the fundamental relations
\[
x_{ak} = x_{ij} x_{ai} - x_{aj} \quad (1 \leq a \leq n, \ (i,j,k) : \text{any Wirtinger triple}),
\]

**\( (H) \):** the hexagon relations
\[
x_{i_1 i_2 i_3} \cdot x_{j_1 j_2 j_3} = \frac{1}{2} \begin{vmatrix}
    x_{i_1 i_1} & x_{i_1 j_1} & x_{i_1 j_3} \\
    x_{i_2 j_1} & x_{i_2 j_2} & x_{i_2 j_3} \\
    x_{i_3 j_1} & x_{i_3 j_2} & x_{i_3 j_3}
\end{vmatrix} \quad (1 \leq i_1 < i_2 < i_3 \leq n, \ 1 \leq j_1 < j_2 < j_3 \leq n),
\]

**\( (R) \):** the rectangle relations
\[
\begin{vmatrix}
    2 & x_{12} & x_{1a} & x_{1b} \\
    x_{21} & 2 & x_{2a} & x_{2b} \\
    x_{a1} & x_{a2} & 2 & x_{ab} \\
    x_{b1} & x_{b2} & x_{ba} & 2
\end{vmatrix} = 0 \quad (3 \leq a < b \leq n),
\]

with \( x_{ii} = 2 \), \( x_{ji} = x_{ij} \) and \( x_{ij}^a x_{ij}^b x_{ij}^c = \text{sign}(\sigma) x_{i_1 i_2 i_3} \) for an element \( \sigma \) of the symmetric group \( G_3 \) of degree 3.

Note that the coordinates \( x_{ij} \) and \( x_{ijk} \) correspond to \(-t_{m_i m_j}(\chi_\rho)\) and \(-t_{m_i m_j m_k}(\chi_\rho)\) respectively for an unspecified trace-free representation \( \rho : G(K) \to \text{SL}_2(\mathbb{C}) \).

### 2.3. Ghost characters of a knot
Suppose that the knot group \( G(K) \) has a Wirtinger presentation:
\[
G(K) = \langle m_1, \ldots, m_n \mid r_1, \ldots, r_n \rangle.
\]
Then we define an algebraic set \( F_2(K) \) in \( \mathbb{C}^n_{(2)} \) by the fundamental relations \( (F2) \) in Theorem 2.1. Namely,
\[
F_2(K) = \left\{ (x_{12}, \ldots, x_{n-1,n}) \in \mathbb{C}^n_{(2)} \middle| x_{ak} = x_{ij} x_{ai} - x_{aj} \text{ for any } 1 \leq a \leq n \right\}.
\]
We remark that the coordinate ring of \( F_2(K) \) is isomorphic to the nilradical quotient of the complexified degree 0 abelian knot contact homology. (See Proposition 4.2 in
This was originally found in [7].) So $F_2(K)$ is an invariant of knots, justifying the notation as $F_2(K)$.

Some observations show that for many knots any point of $F_2(K)$ “lift” to $S_0(K)$, that is, any point of $F_2(K)$ satisfies (H) and (R). However, this would not be true for any knot. So we define the following.

**Definition 2.2** (Ghost characters of a knot [8, 9]). A point $(x_{ij}) ∈ F_2(K)$ which does not satisfy one of (H) and (R) is called a ghost character of $K$.

As shown in [9] Theorem 4.8, a knot with bridge index less than 4 does not have a ghost character. A computer experiment shows that there exist 4-bridge knots which have ghost characters. For example, any $(4, q)$-torus knot $T_{4,q}$ ($q ≥ 5$, odd), whose bridge index is 4, seems to have ghost characters. Here we will demonstrate the calculations for $T_{4,5}$.

Let $D$ be the diagram of a $(4, 5)$-torus knot shown in Figure 2.1. Set the meridians $m_1, \cdots, m_{15}$ as in Figure 2.1 for a Wirtinger presentation of $G(T_{4,5})$.

![Diagram of $T_{4,5}$](image)

**Figure 2.1.** A diagram $D$ of $T_{4,5}$ and the meridians $m_1, \cdots, m_{15}$ for the Wirtinger presentation associated with $D$.

In this setting, the algebraic set $F_2(T_{4,5})$ is given by

$$F_2(T_{4,5}) = \left\{ (x_{12}, \cdots, x_{14,15}) ∈ \mathbb{C}^{15}_{12} \left| \begin{array}{c} \forall a \leq 15 : \quad x_{ak} = x_{ij}x_{ai} - x_{aj} \\ \text{and any Wirtinger triple } (i, j, k) \end{array} \right. \right\}$$

We notice that a knot $K$ in braid position has a nice elimination of the fundamental relations (F2) in general. We demonstrate this for the current case $T_{4,5}$. At first, we have the following fundamental relations for $1 ≤ a ≤ 15$:

- $x_{a15} = x_{8,11}x_{a11} - x_{a8}$, $x_{a14} = x_{11,13}x_{a11} - x_{a13}$
- $x_{a13} = x_{5,8}x_{a8} - x_{a5}$, $x_{a12} = x_{8,10}x_{a8} - x_{a10}$
- $x_{a11} = x_{8,9}x_{a9} - x_{a9}$, $x_{a10} = x_{1,5}x_{a5} - x_{a1}$
- $x_{a9} = x_{5,7}x_{a5} - x_{a7}$, $x_{a8} = x_{5,6}x_{a5} - x_{a6}$
- $x_{a7} = x_{1,4}x_{a1} - x_{a4}$, $x_{a6} = x_{1,3}x_{a1} - x_{a3}$
- $x_{a5} = x_{1,2}x_{a1} - x_{a2}$
- $x_{a4} = x_{11,12}x_{a11} - x_{a12}$, $x_{a3} = x_{4,11}x_{a4} - x_{a11}$
- $x_{a2} = x_{4,15}x_{a4} - x_{a15}$, $x_{a1} = x_{4,14}x_{a4} - x_{a14}$

Note that the last 4 types of (F2) are described for the triple $(i, j, k)$ with $j > k$ (using the symmetry on $j$ and $k$) for a technical reason on the elimination process of (F2). Moreover, for an efficient elimination we add the following equations given...
by the fundamental relations:
\[ x_{12} = x_{4,14}x_{24} - x_{2,14} = x_{4,14}x_{4,15} - (x_{4,15}x_{4,14} - x_{14,15}) = x_{14,15}, \]
\[ x_{13} = x_{4,11}x_{14} - x_{1,11} = x_{4,11}x_{4,14} - (x_{4,14}x_{4,11} - x_{11,14}) = x_{11,14}, \]
\[ x_{14} = x_{4,14} = x_{11,12}x_{11,14} - x_{12,14}, \]
\[ x_{23} = x_{4,11}x_{24} - x_{2,11} = x_{4,11}x_{4,15} - (x_{4,15}x_{4,11} - x_{11,15}) = x_{11,15}, \]
\[ x_{24} = x_{4,15} = x_{11,12}x_{11,15} - x_{12,15}, \]
\[ x_{34} = x_{4,11} = x_{11,12}. \]

We start with the elimination of \( x_{a15} \) by applying \( x_{a15} = x_{8,11}x_{a11} - x_{a8} \) to the right hand sides of the other fundamental relations and the added relations. We continue this elimination from \( x_{a14} \) to \( x_{a5} \). Then \( x_{a15}, \ldots, x_{a5} \) (\( 1 \leq a \leq 15 \)) are described by polynomials in the polynomial ring \( R = \mathbb{C}[x_{12}, x_{13}, x_{14}, x_{23}, x_{24}, x_{34}] \).

As observed in the proof of [9, Theorem 4.8] the above elimination process can be understood more topologically. For example, a fundamental relations can be seen as sliding the corresponding loop at the corresponding crossing of the diagram \( D \) (this time from the left to the right) and resolving “the winding part” by the trace-free Kauffman bracket skein relation:

\[ \text{knot} \]

More precisely, we first identify \( x_{ak} \) with a loop freely homotopic to \( m_amm_k \). We next think of the loop \( x_{ak} \) as the union \( c_a \cup c_k \) of two arcs \( c_a \) and \( c_k \) corresponding to \( m_a \) and \( m_k \) respectively. Then slide \( c_k \) along the \( k \)th arc of \( D \) with fixing \( c_a \) and resolve the winding part (the part of the right side of \( D \) by the following trace-free Kauffman bracket skein relation:

\[ x_{ak} \]

Each of the resulting loops is freely homotopic to \( m_a m_i \) or \( m_a m_j \) or \( m_i m_j \). Regarding the disjoint union of loops as the product of their identified monomials, we obtain the fundamental relation \( x_{ak} = x_{ij}x_{ai} - x_{aj} \). We continue this topological operation until the loop under consideration is described by a polynomial in \( R \).

On this topological elimination process, as observed in the proof of [9, Theorem 4.8], the resulting polynomial can be also obtained by sliding the arcs to the right side of \( D \) first and resolving the winding parts along the way “associated with the fundamental relations” second. (Note that the order of slidings and resolvings are different from the original process.) In this case, there might be several ways to resolve the winding parts and thus several polynomial expressions as the elimination, however, the fundamental relations indicate a unique way to resolve the winding parts. (For example, mark every winding part with a natural number in order when
it appears and continue this operation until the sliding is finished, then resolve the marked winding parts in order. If there exists a mark which is not a winding part any more in the final position, then we erase the mark and skip the resolving.) By this we mean the way associated with the fundamental relations. This understanding is applied below. Here we describe the topological elimination process case by case:

(1) for $x_{ij}$ ($1 \leq i \leq j \leq 4$) in the original fundamental relations: slide the arc $c_i$ (resp. $c_j$) from the left side to the right side of $D$, meanwhile we fix the arc $c_j$ (resp. $c_i$), and resolve the winding parts along the way associated with the fundamental relations (the resulting polynomial in $R$ is denoted by $g_i(x_{ij})$ (resp. $g_j(x_{ij})$)),

(2) for $x_{ij}$ ($1 \leq i < j \leq 4$) in the added relations: just slide every loop from the left side to the right side of $D$ (the resulting loop does not have winding parts for the current diagram $D$),

(3) for the others: slide every loop all the way to the right side of $D$ and resolve the winding parts along the way associated with the fundamental relations.

By the above argument, we can define a biregular map (an isomorphic projection) $i : F_2(T_{4,5}) \to \text{Im}(i) \subset \mathbb{C}^2$, 

$$(x_{12}, \ldots, x_{14,15}) \mapsto (x_{12}, x_{13}, x_{14}, x_{23}, x_{24}, x_{34}).$$

Then the resulting equations $x_{ij} = g_i(x_{ij})$, $x_{ij} = g_j(x_{ij})$ in Process (1) turns out to be the defining polynomials of $\text{Im}(i)$. Hence we obtain

$$F_2(T_{4,5}) \cong \left\{ (x_{12}, \ldots, x_{34}) \in \mathbb{C}^2 \mid x_{ij} = g_i(x_{ij}), x_{ij} = g_j(x_{ij}) \ (1 \leq i \leq j \leq 4) \right\}.$$ 

We notice that the above presentation of $F_2(T_{4,5})$ can be naturally generalized for a knot $K$ in m-braid position:

$$F_2(K) \cong \left\{ (x_{12}, \ldots, x_{m-1,m}) \in \mathbb{C}^2 \mid x_{ij} = g_i(x_{ij}), x_{ij} = g_j(x_{ij}) \ (1 \leq i \leq j \leq m) \right\}.$$ 

Now, we can eliminate $x_{14}, x_{23}, x_{24}, x_{34}$ by $x_{12} = x_{23} = x_{34} = x_{14}$, $x_{13} = x_{24}$. The above topological process shows that substituting $x_{12} = x_{23}$, $x_{23} = x_{34}$, $x_{34} = x_{14}$, $x_{14} = x_{12}$, $x_{13} = x_{24}$, $x_{24} = x_{13}$ in $x_{ij} = g_i(x_{ij})$ and $x_{ij} = g_j(x_{ij})$ give $x_{i+1,j+1} = g_{i+1}(x_{i+1,j+1})$ and $x_{i+1,j+1} = g_{j+1}(x_{i+1,j+1})$, where every index shifts cyclically from 1 to 4, that is, if $i$ (resp. $j$) is 4, then $i+1$ (resp. $j+1$) means 1. Hence the relations $x_{ij} = g_i(x_{ij})$, $x_{ij} = g_j(x_{ij})$ for $(i,j) = (1,4), (2,3), (3,4)$ are reduced to $x_{12} = g_1(x_{12})$, $x_{12} = g_2(x_{12})$, and the relations $x_{ij} = g_i(x_{ij})$, $x_{ij} = g_j(x_{ij})$ for $(i,j) = (2,4)$ are reduced to $x_{13} = g_1(x_{13})$, $x_{13} = g_3(x_{13})$ by this substitution. Moreover, $x_{ii} = g_i(x_{ii})$ ($2 \leq i \leq 4$) is reduced to $x_{11} = g_1(x_{11})$. Therefore, $F_2(T_{4,5})$ is consequently isomorphic to

$$F_2(T_{4,5}) \cong \left\{ (x_{12}, x_{13}) \in \mathbb{C}^2 \mid x_{1j} = \tilde{g}_1(x_{1j}), x_{1j} = \tilde{g}_j(x_{1j}) \ (2 \leq j \leq 3) \mid x_{11} = \tilde{g}_1(x_{11}) \right\},$$

where $\tilde{g}_i(x_{ij})$ (resp. $\tilde{g}_j(x_{ij})$) denotes the polynomial given by substituting $x_{14} = x_{12}, x_{23} = x_{12}, x_{24} = x_{13}, x_{34} = x_{12}$ in $g_i(x_{ij})$ (resp. $g_j(x_{ij})$). By computer, we obtain
the following descriptions of the defining polynomials of $F_2(T_{4,5})$. Let $a = x_{12}, b = x_{13}$.

\[
x_{11} = 2 = \tilde{g}_1(x_{11}) = a^5 - 4a^3b + 3a^2b^2 - 2ab - 3a,
\]
\[
a = \tilde{g}_1(x_{12}) = a^6 - 4a^4b + 2a^4 + 3a^2b^2 + a^2b - 5a^2 - b^2 + 2,
\]
\[
a = \tilde{g}_2(x_{12}) = a^4b - a^4 - 3a^2b^2 + 4a^2b + b^3 - 3b,
\]
\[
b = \tilde{g}_1(x_{13}) = a^5b - a^5 - 4a^3b^2 + 6a^3b + 3ab^2 - a^3 - 3ab^2 - 5ab + 3a,
\]
\[
b = \tilde{g}_3(x_{13}) = a^5 - 3a^3b + a^3 + ab^2 + 2ab - 3a.
\]

Solving these equations, we obtain 6 points $(x_{12}, x_{13}) = (2, 2), (-1, 1), (\text{Root}(z^2 - 3z + 1), -2 + 2\text{Root}(z^2 - 3z + 1)), (\text{Root}(z^2 + z - 1), 2)$, giving the algebraic set $F_2(T_{4,5})$.

Here we focus on $(x_{12}, x_{13}) = (-1, 1)$. This gives us

$$(x_{12}, x_{13}, x_{14}, x_{23}, x_{24}, x_{34}) = (-1, 1, -1, -1, 1, -1).$$

One can easily check that this does not satisfy one of the rectangle relations (R):

\[
\begin{vmatrix}
2 & x_{12} & x_{13} & x_{14} \\
 x_{21} & 2 & x_{23} & x_{24} \\
 x_{31} & x_{32} & 2 & x_{34} \\
 x_{41} & x_{42} & x_{43} & 2
\end{vmatrix}
= 0.
\]

So we have the following.

**Proposition 2.3.** With the above setting, the point $(x_{12}, x_{13}) = (-1, 1) \in F_2(T_{4,5})$ is a ghost character of $T_{4,5}$.

In fact, we can show that $T_{4,5}$ has only this ghost character. This will not be necessary here for the proof of Theorem 1.1 and so will be discussed in another paper in a general setting.

3. Proof of Theorem 1.1

We will show Theorem 1.1 by using ghost characters. First, we will review the polynomial map $\hat{\Phi} : S_0(K) \to X(\Sigma_2K)$ in a general setting.

3.1. The map $\hat{\Phi}$ as a polynomial map. First, we review briefly the map $\hat{\Phi} : \mathcal{S}_0(K) \to \mathfrak{X}(\Sigma_2K)$ constructed in [10]. Let $p : C_2K \to E_K$ be the 2-fold cyclic covering of the knot exterior $E_K$, $\mu_2$ a meridian of $C_2K$. In this setting, $\pi_1(\Sigma_2K)$ is isomorphic to $\pi_1(C_2K)/\langle \langle \mu_2 \rangle \rangle$, where $\langle \langle \mu_2 \rangle \rangle$ denotes the normal closure of $\langle \mu_2 \rangle$. Here, the map $p$ naturally induces an injection $p_* : \pi_1(C_2K) \to G(K)$, so $\pi_1(C_2K)$ is isomorphic to $\text{Im}(p_*)$. Since $p_*\mu_2$ is $\mu_2^2$ for a meridian $\mu_K$ of $K$, $\pi_1(\Sigma_2K)$ can be considered as the quotient $\text{Im}(p_*)/\langle \langle \mu_2^2 \rangle \rangle$. Then the map $\hat{\Phi} : \mathcal{S}_0(K) \to \mathfrak{X}(\Sigma_2K)$ is defined as follows: for $\chi_\rho \in \mathcal{S}_0(K)$ and $g \in \pi_1(\Sigma_2K)$,

\[
\hat{\Phi}(\chi_\rho)(g) = (\sqrt{-1})^{\alpha(p_*(g))}\chi_\rho(p_*(g)),
\]

where $\alpha : G(K) \to H_1(E_K) = \langle \mu_K \rangle \cong \mathbb{Z}$ denotes the abelianization. For more details, refer to [10].

Now, we describe the map $\hat{\Phi}$ as a polynomial map $\hat{\Phi} : S_0(K) \to X(\Sigma_2K)$ using the following theorem.
Theorem 3.1 ([2], cf. [4]). For the knot group \( G(K) = \langle m_1, \ldots, m_n \mid r_1, \ldots, r_{n-1} \rangle \)
generated by \( n \) meridians \( m_1, \ldots, m_n \), we have
\[
\pi_1(\Sigma_2 K) \cong \langle m_i m_j \mid w(r_j), w(m_i r_j m_i^{-1}) \ (1 \leq j \leq n-1) \rangle,
\]
where \( w(r_j) \) (resp. \( w(m_i r_j m_i^{-1}) \)) is the word given by interpreting \( r_j \) (resp. \( m_i r_j m_i^{-1} \))
with the generators \( m_i \)'s.

Basically, a Fox’s theorem in [2] shows that for the presentation of \( G(K) \) in Theorem 3.1,
\[
\text{Im}(p_s) \cong \langle m_i m_j \mid w(r_j), w(m_i r_j m_i^{-1}) \ (1 \leq j \leq n-1) \rangle
\]
by a coset decomposition \( G(K) = \text{Im}(p_s) \cup \text{Im}(p_s) m_1 \).
Then the quotient by \( \langle m_1^2 \rangle \) gives Theorem 3.1.
(For more details, refer to [2]. See also [4].)

By this presentation, the character variety \( X(\Sigma_2 K) \) of \( \pi_1(\Sigma_2 K) \) has the following
description through the map \( t \) applied in Section 1.
Set the following for simplicity:
\[
y_a = t_{m_1 m_a}(\chi_\rho),
y_{ab} = t_{(m_1 m_a)(m_1 m_b)}(\chi_\rho),
y_{abc} = t_{(m_1 m_a)(m_1 m_b)(m_1 m_c)}(\chi_\rho),
\]
for an unspecified representation \( \rho : \pi_1(\Sigma_2 K) \to \text{SL}_2(\mathbb{C}) \).
Then as seen in Section 1 by [3] we obtain
\[
X(\Sigma_2 K) = \left\{ (y_a; y_{bc}; y_{def}) \in \mathbb{C}^{n-1+\binom{n-1}{2}+\binom{n-1}{3}} \left| \begin{array}{c}
\chi_\rho \in X(\Sigma_2 K) \\
2 \leq a \leq n \\
2 \leq b < c \leq n \\
2 \leq d < e < f \leq n
\end{array} \right. \right\}
\]
Note that by the \( \text{SL}_2(\mathbb{C}) \)-trace identity we have
\[
y_{ab} = y_a y_b - t_{m_a m_b}(\chi_\rho),
y_{abc} = (y_a y_b - t_{m_a m_b}(\chi_\rho)) y_c - t_{(m_a m_b)(m_a m_c)}(\chi_\rho).
\]
Hence resetting
\[
z_{ab} = t_{(m_a m_b)}(\chi_\rho), \quad z_{abcd} = t_{(m_a m_b)(m_b m_c)}(\chi_\rho),
\]
we can change the above coordinates of \( X(\Sigma_2 K) \) to
\[
X(\Sigma_2 K) = \left\{ (z_{ab}; z_{1cde}) \in \mathbb{C}^{\binom{n}{2}+\binom{n-1}{3}} \left| \begin{array}{c}
\chi_\rho \in X(\Sigma_2 K) \\
1 \leq a < b \leq n \\
2 \leq c < d < e \leq n
\end{array} \right. \right\}
\]
Using this expression and (1), we can describe the map \( \widehat{\Phi} \) as the following polynomial map.
For any trace-free character \( \chi_\rho = (x_{ij}; x_{ijk}) \in S_0(K) \),
\[
(2) \quad \widehat{\Phi}(x_{ij}; x_{ijk}) = \left( t_{m_a m_b}(\widehat{\Phi}(\chi_\rho)); t_{(m_1 m_c)(m_d m_e)}(\widehat{\Phi}(\chi_\rho)) \right)
\]
\[
= \left( x_{ij}; \frac{1}{2} (x_{1c} x_{de} + x_{1e} x_{cd} - x_{1d} x_{ce}) \right),
\]
since for any trace-free character \( \chi_\rho \in S_0(K) \) we have
\[
t_{m_a m_b}(\Phi(\chi_\rho)) = t_{m_a m_b}((\sqrt{-1})^2 \chi_\rho) = t_{m_a m_b}(-\chi_\rho) = -t_{m_a m_b}(\chi_\rho),
\]
be given by any trace-free representations of the knot group $G$ (cf. Theorem 1.1) surjective. Namely, there exists an $\SL_X$ precisely the map from $\rho$ construct a representation $T$ index more than 3. The target answer the question in Section 1 negatively, we need to focus on knots with bridge $23$, see also [6, Theorem 1.3] and any 3-bridge knots ([9, Theorem 4.9 (1)]). So, to that the map $\hat{\Phi}$ contradiction.

3.3. Ghost characters and surjectivity of $\hat{\Phi}$. In the previous research [9], we have shown that a ghost character satisfying a certain condition (see below) is an obstruction for the map $\hat{\Phi}$ to be surjective.

**Theorem 3.2** (Theorem 4.9 (2) in [9]). For a knot $K$ with a Wirtinger presentation $G(K) = \langle m_1, \ldots, m_n \mid r_1, \ldots, r_n \rangle$, the map $\hat{\Phi}$ is not surjective if there exists a representation $\rho : \Sigma_2(K) \to \SL_2(\mathbb{C})$ such that for the elements $m_im_j$ ($1 \leq i < j \leq n$) in $\pi_1(\Sigma_2K)$, the point $(t_{m_im_j}(\chi_\rho)) \in F_2(K)$ gives a ghost character of $K$.

We review the proof of Theorem 3.2 briefly.

*Proof.* We first note that for any representation $\rho : \Sigma_2(K) \to \SL_2(\mathbb{C})$, $(t_{m_im_j}(\chi_\rho))$ gives a point in $F_2(K)$. Indeed, for any Wirtinger triple $(i, j, k)$ and any $1 \leq a \leq n$, $m_am_k = (m_am_i)(m_jm_i)$ and $m_a^2 = 1$ hold in $\pi_1(\Sigma_2K) \cong \text{Im}(p_*)/\langle \langle m_1^2 \rangle \rangle$, and so for any character $\chi_\rho \in X(\Sigma_2K)$, we have

$$t_{m_am_k}(\chi_\rho) = t_{m_am_i}(\chi_\rho)t_{m_jm_i}(\chi_\rho) - t_{m_am_j}(\chi_\rho).$$

This shows that $t_{m_im_j}(\rho)$ ($1 \leq i < j \leq n$) satisfy the fundamental relations (F2) for $G(K)$ and thus $(t_{m_im_j}(\chi_\rho))$ gives a point of $F_2(K)$ for any representation $\rho : \Sigma_2(K) \to \SL_2(\mathbb{C})$. This naturally defines a map from $X(\Sigma_2K)$ to $F_2(K)$. (See the map $h^*$ defined in [9, Subsection 4.3]. The dual map has been considered in [12].)

Now, suppose that there exists a representation $\rho : \Sigma_2(K) \to \SL_2(\mathbb{C})$ such that $(t_{m_im_j}(\chi_\rho)) \in F_2(K)$ gives a ghost character of $K$. Then the character $\chi_\rho$ must be outside $\text{Im}(\hat{\Phi})$, since if it is not, the preimage $\hat{\Phi}^{-1}(\chi_\rho)$ is not empty and its projection to $F_2(K)$, which is exactly $(t_{m_im_j}(\chi_\rho))$ by [2], is not a ghost character, a contradiction. \qed

Since there does not exist a ghost character for any 2-bridge and 3-bridge knots by [9, Theorem 4.8], we cannot apply Theorem 3.2 for them. Moreover, we have shown that the map $\hat{\Phi}$ is surjective for any 2-bridge knots ([10] Theorem 1 and Lemma 23, see also [6, Theorem 1.3]) and any 3-bridge knots ([9, Theorem 4.9 (1)]). So, to answer the question in Section 1 negatively, we need to focus on knots with bridge index more than 3. The target $T_{4,5}$ is one of them, whose bridge index is 4.

3.3. Proof of Theorem 1.1. As shown in Subsection 2.3 the $(4,5)$-torus knot $T_{4,5}$ has a ghost character $(x_12, x_13) = (-1, 1) \in F_2(T_{4,5})$. So, by Theorem 3.2 (more precisely the map from $X(\Sigma_2K)$ to $F_2(K)$ in the proof of Theorem 3.2), if we can construct a representation $\rho : \pi_1(\Sigma_2T_{4,5}) \to \SL_2(\mathbb{C})$ satisfying

$$t_{m_im_2}(\chi_\rho) = -1, \ t_{m_im_3}(\chi_\rho) = 1,$$

then $\hat{\Phi}$ is not surjective. In fact, we can do it. Therefore we obtain Theorem 1.1

**Theorem 3.3** (cf. Theorem 1.1). For the $(4,5)$-torus knot $T_{4,5}$, the map $\hat{\Phi}$ is not surjective. Namely, there exists an $\SL_2(\mathbb{C})$-representation of $\pi_1(\Sigma_2T_{4,5})$ which cannot be given by any trace-free representations of the knot group $G(T_{4,5})$. 

$$t_{m_im_2}(\chi_\rho) = t_{m_im_2}(\chi_\rho) t_{m_im_3}(\chi_\rho) - t_{m_im_2}(\chi_\rho) t_{m_im_3}(\chi_\rho).$$
Proof. First, we need to calculate $\pi_1(\Sigma_2T_{4,5})$ by using Theorem 3.1.

Lemma 3.4 (cf. Theorem 3.1). For the Wirtinger presentation (without one relator) $G(T_{4,5}) = \langle m_1, \ldots, m_{15} | r_1, \ldots, r_{14} \rangle$ associated with the diagram $D$ in Figure 2.1 let $x = m_1 m_2$, $y = m_1 m_3$, $z = m_1 m_4$. Then we have

$$\pi_1(\Sigma_2T_{4,5}) \cong \langle x, y, z | w_i \ (1 \leq i \leq 6) \rangle,$$

where $w_i$ denotes the following relators:

$$w_1 = \zeta^{-1} x^{-1} y z^{-1} x z^{-1} y x^{-1} z^{-1},$$
$$w_2 = \zeta^{-1} x^{-1} y z^{-1} y x^{-1} z^{-1} x,$$
$$w_3 = \zeta^{-1} x^{-1} y z^{-1} y x^{-1} z^{-1} y,$$
$$w_4 = xy^{-1} x z^{-1} y x z,$$
$$w_5 = xy^{-1} y z^{-1} x z x^{-1},$$
$$w_6 = xy^{-1} y z^{-1} x y.$$

We demonstrate how to calculate $\pi_1(\Sigma_2T_{4,5})$ in Lemma 3.4. First, by the relators $r_1, \ldots, r_{14}$ of $G(T_{4,5})$, we have

$$m_5 = m_1 m_2 m_1^{-1},$$
$$m_6 = m_1 m_3 m_1^{-1},$$
$$m_7 = m_1 m_4 m_1^{-1},$$
$$m_8 = m_5 m_6 m_5^{-1} = m_1 m_2 m_3 m_2^{-1} m_1^{-1},$$
$$m_9 = m_5 m_7 m_5^{-1} = m_1 m_2 m_4 m_2^{-1} m_1^{-1},$$
$$m_{10} = m_5 m_1 m_5^{-1} = m_1 m_2 m_1 m_1^{-1},$$
$$m_{11} = m_8 m_9 m_8^{-1} = m_1 m_2 m_3 m_3^{-1} m_2^1 m_1^{-1},$$
$$m_{12} = m_8 m_{10} m_8^{-1} = m_1 m_2 m_3 m_3 m_1^{-1} m_2^{-1} m_1^{-1},$$
$$m_{13} = m_8 m_5 m_8^{-1} = m_1 m_2 m_3 m_3 m_2^{-1} m_1^{-1},$$
$$m_{14} = m_{11} m_{13} m_{11}^{-1} = m_1 m_2 m_3 m_4 m_2^{-1} m_3^{-1} m_2^{-1} m_1^{-1},$$
$$m_{15} = m_{11} m_{14} m_{11}^{-1} = m_1 m_2 m_3 m_4 m_3^{-1} m_2^{-1} m_1^{-1},$$
$$m_1 = m_4 m_{14} m_4^{-1} = m_1 m_2 m_3 m_4 m_2^{-1} m_3^{-1} m_2^{-1} m_1^{-1} m_4^{-1},$$
$$m_2 = m_4 m_{15} m_4^{-1} = m_1 m_2 m_3 m_4 m_3^{-1} m_2^{-1} m_1^{-1} m_4^{-1},$$
$$m_3 = m_4 m_{11} m_4^{-1} = m_1 m_2 m_3 m_4 m_3^{-1} m_2^{-1} m_1^{-1} m_4^{-1}.$$

By the Tietze transformations, the first 11 relations show that $G(T_{4,5})$ is generated by $m_1, m_2, m_3, m_4$ and then the set of relators of $G(T_{4,5})$ is generated normally by the last 3 relations. So we obtain

$$G(T_{4,5}) = \langle m_1, m_2, m_3, m_4 \mid w_1, w_2, w_3 \rangle,$$

where $w_1, w_2$ and $w_3$ are the following words:

$$w_1 = m_4 m_1 m_2 m_3 m_4 m_2^{-1} m_3^{-1} m_2^{-1} m_1^{-1} m_4^{-1} m_1^{-1},$$
$$w_2 = m_4 m_1 m_2 m_3 m_4 m_3^{-1} m_2^{-1} m_1^{-1} m_4^{-1} m_2^{-1},$$
$$w_3 = m_4 m_1 m_2 m_3 m_4 m_3^{-1} m_2^{-1} m_1^{-1} m_4^{-1} m_3^{-1}.$$
Consequently, by Theorem 3.1 we see that
\[ \pi_1(\Sigma_2 T_{4,5}) \cong \langle m_1 m_2, m_1 m_3, m_1 m_4 \mid w_i (1 \leq i \leq 6) \rangle, \]
where \( w_4 = m_1 w_1 m_1^{-1}, w_5 = m_1 w_2 m_1^{-1}, w_6 = m_1 w_3 m_1^{-1}. \) Note that the relators \( w_j \) \((1 \leq j \leq 6)\) should be words in \( m_1 m_i \) \((1 \leq i \leq 4)\). For simplicity, let \( x = m_1 m_2, y = m_1 m_3, z = m_1 m_4. \) Then we have
\[
\begin{align*}
w_1 &= z^{-1} x^{-1} y z^{-1} x z^{-1} y x^{-1} z^{-1}, \\
w_2 &= z^{-1} x^{-1} y z^{-1} y z^{-1} x, \\
w_3 &= z^{-1} x^{-1} y z^{-1} y x^{-1} z^{-1} y, \\
w_4 &= z x y z^{-1} x z^{-1} y, \\
w_5 &= z x y^{-1} z y^{-1} x z^{-1}, \\
w_6 &= z x y^{-1} z y^{-1} x z^{-1}.
\end{align*}
\]

This shows Lemma 3.4.

To complete the proof of Theorems 1.1 and 3.3 we show that there exists a representation \( \rho : \pi_1(\Sigma_2 T_{4,5}) \to \text{SL}_2(\mathbb{C}) \) satisfying \( \text{tr}(\rho(m_1 m_2)) = \text{tr}(\rho(m_1 m_4)) = -1 \) and \( \text{tr}(\rho(m_1 m_3)) = 1. \) To find such a representation, since \( \text{tr}(\rho(m_1 m_2)) \neq 2, \) we can assume up to conjugation that
\[
(\rho(m_1 m_2), \rho(m_1 m_3), \rho(m_1 m_4)) = \left( \begin{pmatrix} a & 0 \\
0 & a^{-1} \end{pmatrix}, \begin{pmatrix} b & c \\
d & e \end{pmatrix}, \begin{pmatrix} f & g \\
h & i \end{pmatrix} \right) \in \text{SL}_2(\mathbb{C})^3.
\]

Then we can check that there exists an \( \text{SL}_2(\mathbb{C}) \)-representation \( \rho \) satisfying \( t_{m_1 m_2}(\rho) = -1, \) \( t_{m_1 m_3}(\rho) = 1 \) and \( t_{m_1 m_4}(\rho) = -1. \) For example, we found the following representation \( \rho : \pi_1(\Sigma_2 T_{4,5}) \to \text{SL}_2(\mathbb{C}). \) Let \( i := \sqrt{-1}. \)
\[
(\rho(m_1 m_2), \rho(m_1 m_3), \rho(m_1 m_4)) = \left( \begin{pmatrix} e^{\frac{2}{3} \pi i} & 0 \\
0 & e^{-\frac{4}{3} \pi i} \end{pmatrix}, \begin{pmatrix} -\frac{1}{\sqrt{3}} e^{\frac{\pi}{3} i} & -\frac{2}{3} \pi i \\
\frac{1}{\sqrt{3}} e^{-\frac{\pi}{3} i} & 1 \end{pmatrix}, \begin{pmatrix} \frac{1}{\sqrt{3}} e^{\frac{\pi}{3} i} & \frac{1+2\alpha}{3} e^{-\frac{\pi}{3} i} \\
\frac{1}{\sqrt{3}} e^{-\frac{\pi}{3} i} & -\frac{1+2\alpha}{3} e^{-\frac{\pi}{3} i} \end{pmatrix} \right),
\]
where \( \alpha \) is a root of \( 2\alpha^2 + \alpha + 2 = 0. \) One can easily check that
\[
\begin{align*}
\text{tr}(\rho(m_1 m_2)) &= 2 \cos \left( \frac{2}{3} \pi \right) = -1, \\
\text{tr}(\rho(m_1 m_3)) &= -\frac{i}{\sqrt{3}} \cdot 2i \sin \left( \frac{\pi}{3} \right) = 1, \\
\text{tr}(\rho(m_1 m_4)) &= \frac{i}{\sqrt{3}} \cdot 2i \sin \left( \frac{\pi}{3} \right) = -1.
\end{align*}
\]

By Theorem 3.2 this shows that \( \rho \) cannot be given by any trace-free representations of \( G(T_{4,5}) \), completing the proof. \( \square \)
Additionally, we found the following representations giving the remaining points of $F_2(T_{4,5})$.

$$\rho(m_1m_2) = \begin{pmatrix} e^{\frac{2\pi i}{5}} & 0 \\ 0 & e^{-\frac{2\pi i}{5}} \end{pmatrix} \Rightarrow \text{tr}(\rho(m_1m_2)) = 2 \cos \left(\frac{2\pi k}{5}\right) = 2 \text{ or } -\frac{1}{2} \sqrt{5},$$

$$\rho(m_1m_3) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \Rightarrow \text{tr}(\rho(m_1m_3)) = 2,$$

$$\rho(m_1m_4) = \begin{pmatrix} e^{\frac{2\pi i}{5}} & 0 \\ 0 & e^{-\frac{2\pi i}{5}} \end{pmatrix} \Rightarrow \text{tr}(\rho(m_1m_4)) = 2 \cos \left(\frac{2\pi k}{5}\right) = 2 \text{ or } -\frac{1}{2} \sqrt{5},$$

where $k = 0, 1, 2, 3, 4$.

$$\rho(m_1m_2) = \begin{pmatrix} \frac{3\pm\sqrt{5}+\beta}{4} & 0 \\ 0 & \frac{3\pm\sqrt{5}-\beta}{4} \end{pmatrix} \Rightarrow \text{tr}(\rho(m_1m_2)) = \frac{3\pm\sqrt{5}}{2},$$

$$\rho(m_1m_3) = \begin{pmatrix} \frac{6(1\pm\sqrt{5})+26\beta^3}{32} & 1 \\ 1 & \frac{6(1\pm\sqrt{5})-26\beta^3}{32} \end{pmatrix} \Rightarrow \text{tr}(\rho(m_1m_3)) = 1 \pm \sqrt{5},$$

$$\rho(m_1m_4) = \begin{pmatrix} \frac{33(3\pm\sqrt{5})-7\beta^3}{32} & 1 \\ 1 & \frac{33(3\pm\sqrt{5})+7\beta^3}{32} \end{pmatrix} \Rightarrow \text{tr}(\rho(m_1m_4)) = \frac{3\pm\sqrt{5}}{2},$$

where $\beta = \sqrt{-2 \pm 6\sqrt{5}}$. These representations show that $X(\Sigma_2T_{4,5})$ gives all points in $F_2(T_{4,5})$.

**Acknowledgement**

The authors would like to thank Yoshikazu Yamaguchi for useful comments on $\tau$-equivalent representations and ghost characters. The first author has been partially supported by JSPS KAKENHI for Young Scientists (B) Grant Number 26800046.

**References**

[1] M. Culler and P. Shalen: *Varieties of group presentations and splittings of 3-manifolds*, Ann. of Math. 117 (1983), 109–146.

[2] R. Fox: *Free differential calculus III, subgroups*, Ann. of Math. 64 (1956).

[3] F. González-Acuña and J.M. Montesinos: *On the character variety of group representations in SL(2, C) and PSL(2, C)*, Math. Z., 214 (1993), 627–652.

[4] S. Kinoshita: *Isoukikagaku-nyuemon* (in Japanese), Baifukan, Tokyo, 2000.

[5] F. Nagasato: *On a behavior of a slice of the SL$_2$($\mathbb{C}$)-character variety of a knot group under the connected sum*, Topology Appl. 157 (2010), 182-187.

[6] F. Nagasato: *Finiteness of a section of the SL(2, C)-character variety of knot groups*, Kobe J. Math., 24 (2007), 125–136.

[7] F. Nagasato: *Algebraic varieties via a filtration of the KBSM and knot contact homology*, preprint.

[8] F. Nagasato: *On the trace-free characters*, RIMS Kokyuroku “Representation spaces, twisted topological invariants and geometric structures of 3-manifolds”, 1836 (2013), 110-123.

[9] F. Nagasato: *Trace-free characters and abelian knot contact homology I*, preprint.

[10] F. Nagasato and Y. Yamaguchi: *On the geometry of the slice of trace-free SL$_2$(C)-characters of a knot group*, Math. Ann. 354 (2012), 967–1002.

[11] L. Ng: *Knot and braid invariants from contact homology I*, Geom. Topol. 9 (2005), 247–297.

[12] L. Ng: *Knot and braid invariants from contact homology II*, Geom. Topol. 9 (2005), 1603-1637.
DEPARTMENT OF MATHEMATICS, MEIJO UNIVERSITY, TEMPAKU, NAGOYA 468-8502, JAPAN
E-mail address: fukky@meijo-u.ac.jp

GRADUATE SCHOOL OF MATHEMATICS, MEIJO UNIVERSITY, TEMPAKU, NAGOYA 468-8502, JAPAN
E-mail address: 163429002@ccalumni.meijo-u.ac.jp