A new stability criterion for systems of two first-order linear ordinary differential equations

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Abstract. The Riccati equation method is used to establish a new stability criteria for systems of two first-order linear ordinary differential equations. Two examples are presented in which the obtained result is compared with the results obtained by the Lyapunov and Bogdanov methods, by a method involving estimates of solutions in the Lozinskii logarithmic norms and by the freezing method.

Key words: Riccati equation, linear systems of ordinary differential equations, Lyapunov stability, asymptotic stability, regular, normal and extremal solutions of Riccati equations.

1. Introduction. Let \( a(t), b(t), c(t) \) and \( d(t) \) be complex-valued locally integrable and locally bounded functions on \([t_0, +\infty)\). Consider the linear system

\[
\begin{align*}
\phi' &= a(t)\phi + b(t)\psi, \\
\psi' &= c(t)\phi + d(t)\psi, \quad t \geq t_0.
\end{align*}
\]

By a solution of this system we mean an ordered pair \((\phi(t), \psi(t))\) of absolutely continuous functions \(\phi(t)\) and \(\psi(t)\), satisfying (1.1) almost everywhere on \([t_0, +\infty)\).

Definition 1.1. The system (1.1) is called Lyapunov (asymptotically) stable if its all solutions are bounded on \([t_0, +\infty)\) (vanish at \(+\infty)\).

Study of the question of stability of the system (1.1), in general, of linear systems of ordinary differential equations, is an important problem of qualitative theory of differential equations. Being of interest not only in theory but also for applications it is the subject of numerous investigations (see e. g., [1 - 12]). There exist many methods of estimation of solutions of linear systems of ordinary differential equations allowing to describe (to detect) wide classes of stable and (or) unstable systems of ordinary differential equations. Among them the main ones include the Lyapunov’s, Bogdanov’s, Lozinski’s estimate...
methods and the freezing method (see [4], pp. 40-98, 132-145). The fundamental method of Lyapunov characteristic exponents allows to describe the asymptotic growth of solutions of linear systems of ordinary differential equations via these exponents and therefore carrying out the stability behavior of solutions of the system. However the application of this method has some difficulties, arising in the calculation process of Lyapunov characteristic exponents. There exist also other estimation methods for special classes of linear systems of ordinary differential equations (see e. g., [5 - 10]), allowing to describe wide classes of stable and (or) unstable linear systems of ordinary differential equations. However these, indicated above and other methods cannot completely describe the stable and unstable linear systems of ordinary differential equations (in terms of their coefficients).

In this paper we use the Riccati equation method to establish a new stability criterion for the system (1.1). By two examples we compare the obtained result with the results obtained by the Lyapunov and Bogdanov methods, by a method involving estimates of the linear equation
\[ y' + f(t)y^2 + g(t)y + h(t) = 0, \quad t \geq t_0. \quad (2.1) \]
\[ y' + f_1(t)y^2 + g_1(t)y + h_1(t) = 0, \quad t \geq t_0, \quad (2.2) \]
j = 1, 2, and the differential inequalities
\[ \eta' + f(t)\eta^2 + g(t)\eta + h(t) \geq 0, \quad t \geq t_0. \quad (2.3) \]
\[ \eta' + f_1(t)\eta^2 + g_1(t)\eta + h_1(t) \geq 0, \quad t \geq t_0. \quad (2.4) \]

By a solution of Eq. (2.1) (Ineq. (2.3)) on an interval \([t_1, t_2]\) \((t_0 \leq t_1 < t_2 \leq +\infty)\) we mean an absolutely continuous function \(y(t) (\eta(t))\) on \([t_1, t_2]\), satisfying (2.1) (2.3) almost everywhere on \([t_1, t_2]\). Note that for \(f(t) \geq 0 (f_1(t) \geq 0)\), \(t \geq t_0\) every solution of the linear equation \(\eta' + g(t)\eta + h(t) = 0 (\eta' + g_1(t)\eta + h_1(t) = 0)\) on \([t_1, +\infty)\) is also a solution of the inequality (2.3) (2.4). Therefore if \(f(t) \geq 0 (f_1(t) \geq 0), \ t \geq t_0, \) then the inequality (2.3) (2.4) has a solution on \([t_0, +\infty)\), satisfying any initial condition.

**Theorem 2.1.** Let \(y_1(t)\) be a solution of Eq. (2.2) on \([t_0, +\infty)\), and let \(\eta_0(t), \eta_1(t)\) be solutions of Ineq. (2.3) and Ineq. (2.4) respectively with \(\eta_k(t_1) \geq y_0(t_1), \ k = 0, 1.\) Let \(f(t) \geq 0,\)

\[ \lambda - y_1(t) + \int_{t_1}^{t} \exp \left\{ \int_{t_1}^{\tau} [f(\xi) (\eta_0(\xi) + \eta_1(\xi)) + g(\xi)] \right\} \times \]
for some \( \lambda \in [y_1(t_1), \eta_0(t_1)] \). Then for each \( y(0) \geq y_1(t_0) \) Eq. (2.1) has the solution \( y_0(t) \) on \([t_0, +\infty)\), satisfying the initial condition \( y_0(t_1) = y(0) \), moreover \( y_0(t) \geq y_1(t), \; t \geq t_0 \).

**Definition 2.1.** A solution of Eq. (2.1) is called \( t_1 \)-regular if it exists on \([t_1, +\infty)\).

**Definition 2.2.** A \( t_1 \)-regular solution \( y_0(t) \) of Eq. (2.1) is called \( t_1 \)-normal if there exists a \( \delta \)-neighborhood \( U_\delta(y_0(t_1)) \equiv (y_0(t_1) - \delta, y_0(t_1) + \delta) \) (\( \delta > 0 \)) of \( y_0(t_1) \) such that every solution \( y(t) \) of Eq. (2.1) with \( y(t_1) \in U_\delta(y_0(t_1)) \) is also \( t_1 \)-regular, otherwise \( y(t) \) is called \( t_1 \)-extremal.

Let \( y(t) \) be a \( t_1 \)-regular solution of Eq. (2.1). We can interpret \( y(t) \) as a solution of the linear equation

\[
y' + G(t)y + h(t) = 0, \quad t \geq t_1,
\]

where \( G(t) \equiv f(t)y(t) + g(t), \; t \geq t_1 \). Then by Cauchy formula we hve

\[
y(t) \equiv \exp \left\{ - \int_{t_1}^t [f(\tau)y(\tau) + g(\tau)]d\tau \right\} \left[ y(t_1) - \int_{t_1}^t \exp \left\{ \int_{t_1}^\tau g(s)ds \right\} h(\tau)\phi_0(\tau)d\tau \right], \quad t \geq t_1,
\]

where \( \phi_0(t) \equiv \exp \left\{ \int_{t_1}^t f(\tau)y(\tau)d\tau \right\} \), \( t \geq t_1 \). From here it follows

\[
y(t)\phi_0(t) = y(t_1) \exp \left\{ - \int_{t_1}^t g(\tau)d\tau \right\} - \int_{t_1}^t \exp \left\{ - \int_{t_1}^\tau g(s)ds \right\} h(\tau)\phi_0(\tau)d\tau, \quad t \geq t_1. \quad (2.5)
\]

**Lemma 2.1.** Let \( y(t) \) be a \( t_1 \)-regular solution of Eq. (2.1) and let \( f(t) \geq 0, \; t \geq t_1 \). Then

\[
\int_{t_1}^t f(\tau)y(\tau)d\tau \leq y(t_1) \int_{t_1}^t f(\tau) \exp \left\{ - \int_{t_1}^\tau g(s)ds \right\} d\tau - \int_{t_1}^t f(\tau) \int_{t_1}^\tau \exp \left\{ - \int_{t_1}^\xi g(s)ds \right\} h(\xi)d\xi,
\]

\( t \geq t_1 \).

Proof. By analogy of the proof of Lemma 2.2 from [11].

**Lemma 2.2.** Let the following conditions be satisfied.

\[
\int_{t_0}^{+\infty} g(\tau)d\tau = +\infty, \quad \int_{t_0}^t \exp \left\{ - \int_{t_0}^\tau g(s)ds \right\} |h(\tau)|d\tau \text{ is bounded on } [t_0, +\infty).
\]
Then for every continuous function \( \phi(t) \), vanishing at \(+\infty\), the relation
\[
\lim_{t \to +\infty} \int_{t_0}^{t} \exp \left\{ - \int_{\tau}^{t} g(s) d(s) \right\} |h(\tau)| \phi(\tau) d\tau = 0
\]
is valid.

Proof. By analogy of the proof of Lemma 2.5 from [11].
Along with the system (1.1) consider the following one
\[
\begin{cases}
\phi' = \text{Re} a(t) \phi + |b(t)| \psi, \\
\psi' = |c(t)| \phi + \text{Re} d(t) \psi, \quad t \geq t_0.
\end{cases}
\]
(2.6)

Lemma 2.3. If the system (2.6) is Lyapunov (asymptotically) stable, then the system (1.1) is also Lyapunov (asymptotically) stable.

Proof. Let \((\phi(t), \psi(t))\) be a solution of the system (1.1). We can interpret \(\phi(t)\) as a solution of the linear equation
\[
\phi' = a(t) \phi + L(t), \quad t \geq t_0,
\]
where \(L(t) \equiv b(t) \psi(t), \quad t \geq t_0\). Then by Cauchy formula we have
\[
\phi(t) = \exp \left\{ \int_{t_0}^{t} a(\tau) d\tau \right\} \left[ \phi(t_0) + \int_{t_0}^{t} \exp \left\{ - \int_{\tau}^{t} a(s) ds \right\} L(\tau) d\tau \right], \quad t \geq t_0
\]
or
\[
\phi(t) = \exp \left\{ \int_{t_0}^{t} a(\tau) d\tau \right\} \left[ \phi(t_0) + \int_{t_0}^{t} \exp \left\{ - \int_{\tau}^{t} a(s) ds \right\} b(\tau) \psi(\tau) d\tau \right], \quad t \geq t_0.
\]
(2.7)

By analogy for \(\psi(t)\) we can derive the equality
\[
\psi(t) = \exp \left\{ \int_{t_0}^{t} d(\tau) d\tau \right\} \left[ \psi(t_0) + \int_{t_0}^{t} \exp \left\{ - \int_{\tau}^{t} d(s) ds \right\} c(\tau) \phi(\tau) d\tau \right], \quad t \geq t_0.
\]
(2.8)

Substitute in place of \(\psi(t)\) the right hand part of the last equality into (2.7). After some simplifications we obtain
\[
\phi(t) = F(t) + \int_{t_0}^{t} K(t, \xi) \phi(\xi) d\xi, \quad t \geq t_0.
\]
(2.9)
where

\[ F(t) \equiv \phi(t_0) \exp \left\{ \int_{t_0}^{t} a(\tau) d\tau \right\} + \psi(t_0) \int_{t_0}^{t} \exp \left\{ \int_{\tau}^{t} d(s) ds + \int_{\tau}^{t} a(s) ds \right\} b(\tau) d\tau, \quad t \geq t_0, \]

\[ K(t, \xi) \equiv c(\xi) \exp \left\{ - \int_{t_0}^{\xi} d(s) ds \right\} \int_{t}^{\xi} \exp \left\{ \int_{s}^{t} a(s) ds \right\} b(\tau) d\tau, \quad t \geq \xi \geq t_0. \]

By analogy substituting in place of \( \phi(t) \) the right hand part of the equality (2.7) into (2.8) we arrive at the equality

\[ \psi(t) = G(t) + \int_{t_0}^{t} L(t, \xi) \psi(\xi) d\xi, \quad t \geq t_0, \quad (2.10) \]

where

\[ G(t) \equiv \psi(t_0) \exp \left\{ \int_{t_0}^{t} d(\tau) d\tau \right\} + \phi(t_0) \int_{t_0}^{t} \exp \left\{ \int_{\tau}^{t} a(s) ds + \int_{\tau}^{t} d(s) ds \right\} c(\tau) d\tau, \quad t \geq t_0, \]

\[ L(t, \xi) \equiv b(\xi) \exp \left\{ - \int_{t_0}^{\xi} a(s) ds \right\} \int_{t}^{\xi} \exp \left\{ \int_{s}^{t} d(s) ds \right\} c(\tau) d\tau, \quad t \geq \xi \geq t_0. \]

Let \( (\phi_0(t), \psi_0(t)) \) be a solution of the system (2.6). By (2.9) and (2.10) we have respectively

\[ \phi_0(t) = F_0(t) + \int_{t_0}^{t} K_0(t, \xi) \phi_0(\xi) d\xi, \quad t \geq t_0, \quad (2.11) \]

\[ \psi_0(t) = G_0(t) + \int_{t_0}^{t} L_0(t, \xi) \psi_0(\xi) d\xi, \quad t \geq t_0, \quad (2.112) \]

where

\[ F_0(t) \equiv \phi(t_0) \exp \left\{ \int_{t_0}^{t} \text{Re} a(\tau) d\tau \right\} + \psi(t_0) \int_{t_0}^{t} \exp \left\{ \int_{\tau}^{t} \text{Re} d(s) ds + \int_{\tau}^{t} \text{Re} a(s) ds \right\} b(\tau) d\tau, \]
\( t \geq t_0, \quad K_0(t, \xi) \equiv |c(\xi)| \exp \left\{ - \int_{t_0}^{\xi} \text{Re} a(s) ds \right\} \int_{t_0}^{t} \exp \left\{ \int_{\tau}^{t} \text{Re} a(s) ds \right\} |b(\tau)| d\tau, \quad t \geq \xi \geq t_0, \)

\[ G_0(t) \equiv \psi_0(t_0) \exp \left\{ \int_{t_0}^{t} \text{Re} a(\tau) d\tau \right\} + \phi_0(t_0) \int_{t_0}^{t} \exp \left\{ \int_{\tau}^{t} \text{Re} a(s) ds + \int_{\tau}^{t} \text{Re} d(s) ds \right\} |c(\tau)| d\tau, \]

\( t \geq t_0, \quad L_0(t, \xi) \equiv |b(\xi)| \exp \left\{ - \int_{t_0}^{\xi} \text{Re} a(s) ds \right\} \int_{t_0}^{t} \exp \left\{ \int_{\tau}^{t} \text{Re} d(s) ds \right\} |c(\tau)| d\tau, \quad t \geq \xi \geq t_0. \)

By (2.9) and (2.10) we can represent \( \phi(t) \) and \( \psi(t) \) respectively via the following series expansion

\[ \phi(t) = F(t) + \int_{t_0}^{t} K(t, \xi) F(\xi) d\xi + \int_{t_0}^{t} K(t, \xi) d\xi \int_{t_0}^{\xi} K(\xi, \zeta) F(\zeta) d\zeta + ..., \quad t \geq t_0, \quad (2.13) \]

\[ \psi(t) = G(t) + \int_{t_0}^{t} L(t, \xi) G(\xi) d\xi + \int_{t_0}^{t} L(t, \xi) d\xi \int_{t_0}^{\xi} L(\xi, \zeta) G(\zeta) d\zeta + ..., \quad t \geq t_0, \quad (2.14) \]

Assume \( \phi_0(t_0) = |\phi(t_0)|, \quad \psi_0(t_0) = |\psi(t_0)|. \) Then by (2.11) and (2.12) from (2.13) and (2.14) we obtain respectively:

\[ |\phi(t)| \leq F_0(t) + \int_{t_0}^{t} K_0(t, \xi) F_0(\xi) d\xi + \int_{t_0}^{t} K_0(t, \xi) d\xi \int_{t_0}^{\xi} K_0(\xi, \zeta) F_0(\zeta) d\zeta + ... = \phi_0(t), \quad (2.15) \]

\( t \geq t_0, \)

\[ |\psi(t)| \leq G_0(t) + \int_{t_0}^{t} L_0(t, \xi) G_0(\xi) d\xi + \int_{t_0}^{t} L_0(t, \xi) d\xi \int_{t_0}^{\xi} L_0(\xi, \zeta) G_0(\zeta) d\zeta + ... = \psi_0(t), \quad (2.16) \]

\( t \geq t_0. \) Assume the system (2.6) is Lyapunov (asymptotically) stable Then the estimates (2.15) and (2.16) imply that the system (1.1) is also Lyapunov (asymptotically) stable. The lemma is proved.

**3. Main result.** Set \( E(t) \equiv \text{Re} a(t) - \text{Re} d(t), \quad t \geq t_0. \)
**Theorem 3.1.** Let the following conditions be satisfied:

1) \( \sup_{t \geq t_0} \int_{t_0}^{t} \exp \left\{ \int_{\tau}^{t} Re a(s) ds \right\} |c(\tau)| d\tau < +\infty; \)

2) \( \sup_{t \geq t_0} \int_{t_0}^{+\infty} \left[ Re a(\tau) + |b(\tau)| \int_{\tau}^{t} \exp \left\{ - \int_{\xi}^{\tau} E(s) ds \right\} |c(\xi)| d\xi \right] d\tau < +\infty; \)

\[
(2') \int_{t_0}^{+\infty} E(\tau) d\tau = +\infty, \quad \lim_{t \to +\infty} \int_{t_0}^{t} \left[ Re a(\tau) + |b(\tau)| \int_{\tau}^{t} \exp \left\{ - \int_{\xi}^{\tau} E(s) ds \right\} |c(\xi)| d\xi \right] d\tau = -\infty.
\]

Then the system (1.1) is Lyapunov (asymptotically) stable.

Proof. Consider the Riccati equations

\[ y' + |b(t)| y^2 + E(t) y - |c(t)| = 0, \quad t \geq t_0, \tag{3.1} \]

\[ y' + |b(t)| y^2 + E(t) y = 0, \quad t \geq t_0. \]

Applying Theorem 2.1 to these equations we conclude that for every \( t_1 \geq t_0 \) and \( \gamma \geq 0 \) Eq. (3.1) has a solution \( y(t) \) on \( [t_1, +\infty) \) with \( y(t_1) = \gamma \) and

\[ y(t) \geq 0, \quad t \geq t_1. \tag{3.2} \]

All solutions \( y(t) \) of Eq. (3.1), existing on \([t_0, +\infty)\), are connected with solutions \((\phi(t), \psi(t))\) of the system (2.6) by relations (see [11]):

\[
\phi(t) = \phi(t_0) \exp \left\{ \int_{t_0}^{t} \left[ |b(\tau)y(\tau) + Re a(\tau)| \right] d\tau \right\}, \quad \phi(t_0) \neq 0, \quad \psi(t) = y(t)\phi(t), \quad t \geq t_0. \tag{3.3}
\]

Let \( y_0(t) \) be the solution of Eq. (3.1) with \( y_0(t_0) = 0 \). By already proven \( y_0(t) \) exists on \([t_0, +\infty)\) and is non-negative. Set \( \phi_0(t) = \exp \left\{ \int_{t_0}^{t} |b(\tau)y_0(\tau)| d\tau \right\}, \quad t \geq t_0 \). By (2.5) we have

\[
y_0(t)\phi_0(t) = \int_{t_0}^{t} \exp \left\{ - \int_{\tau}^{t} E(s) ds \right\} |c(\tau)| \phi_0(\tau) d\tau, \quad t \geq t_0. \tag{3.4}
\]

Let \((\phi(t), \psi(t))\) be the solution of the system (2.6) with \( \phi(t_0) = 1, \quad \psi(t_0) = 0 \). Then by (3.3) we have

\[
\phi(t) = \exp \left\{ \int_{t_0}^{t} \left[ |b(\tau)y_0(\tau) + Re a(\tau)| \right] d\tau \right\}, \quad \psi(t) = y_0(t)\phi(t), \quad t \geq t_0. \tag{3.5}
\]
This together with (3.4) implies
\[
\psi(t) = \int_{t_0}^{t} \exp\left\{ -\int_{\tau}^{t} \text{Re} d(s) ds \right\} |c(\tau)| \phi(\tau) d\tau, \quad t \geq t_0. \tag{3.6}
\]

In virtue of Lemma 2.1 from the first equality of (3.5) it follows
\[
0 < \phi(t) \leq \exp\left\{ \int_{t_0}^{t} \left[ \text{Re} a(\tau) + |b(\tau)| \int_{\tau}^{t} \exp\left\{ -\int_{\xi}^{\tau} E(s) ds \right\} |c(\xi)| d\xi \right] d\tau \right\}, \quad t \geq t_0. \tag{3.7}
\]

Show that \((\phi(t), \psi(t))\) is bounded (vanish at \(+\infty\)). The condition 2) (the condition 2') together with (3.6) implies that \(\phi(t)\) is bounded (vanish at \(+\infty\)). From here and from (3.6) (by Lemma 2.2 from here and from (3.6)) it follows that \(\psi(t)\) is also bounded (vanish at \(+\infty\)). Since \(c(t) \not\equiv 0\) there exists \(t_1 \geq t_0\) such that \(y(t) > 0, \ t \geq t_1\). Hence by the second equality of (3.5)
\[
\psi(t) > 0, \quad t \geq t_1. \tag{3.8}
\]

Let \((\phi_1(t), \psi_1(t))\) be a solution of the system (2.6) such that \(\phi_1(t_1) > 0, \ \psi_1(t_1) > 0\) and
det \(\begin{pmatrix} \phi(t_1) & \psi(t_1) \\ \phi_1(t_1) & \psi_1(t_1) \end{pmatrix} \neq 0\). Then \((\phi(t), \psi(t))\) and \((\phi_1(t), \psi_1(t))\) are linearly independent. Taking into account (3.5) and (3.8) we have
\[
\frac{\psi(t_1)}{\phi(t_1)} > 0, \quad \frac{\psi_1(t_1)}{\phi_1(t_1)} > 0. \tag{3.9}
\]

Let \(y_1(t)\) be the solution of Eq. (3.1) with \(y_1(t_1) = \frac{\psi(t_1)}{\phi(t_1)}\). Then by the second inequality of (3.9) and by the already proven \(y_1(t)\) is \(t_1\)-normal. By the first inequality of (3.9) and by the already proven \(y(t)\) is also \(t_1\)-normal. Therefore (see [14])
\[
M \equiv \sup_{t \geq t_1} \left| \int_{t_1}^{t} |b(\tau)|(y(\tau) - y_1(\tau)) d\tau \right| < +\infty. \tag{3.10}
\]

By (3.3) we have
\[
\phi_1(t) = \phi_1(t_1) \exp\left\{ \int_{t_1}^{t} \left[ |b(\tau)||y_1(\tau) + \text{Re} a(\tau) | \right] d\tau \right\} = \frac{\phi_1(t_1)}{\phi(t_1)} \exp\left\{ \int_{t_1}^{t} \left[ |b(\tau)||y(\tau) + \text{Re} a(\tau) | \right] d\tau \right\} \times
\]
This together with (3.10) implies

\[ 0 < \phi(t) < \frac{\phi(t_1)}{\phi(t_1)} \exp\{M\} \phi(t), \quad t \geq t_1. \]

Therefore \( \phi(t) \) is bounded (vanish at \( +\infty \)). Hence since \( (\phi(t), \psi(t)) \) and \( (\phi_1(t), \psi_1(t)) \) are linearly independent to complete the proof of the theorem it is enough to show that \( \psi_1(t) \) is bounded (vanish at \( +\infty \)). Let \( z(t) \) and \( z_1(t) \) be the solutions of the Riccati equation

\[ z' + |c(t)z^2 - E(t)z - |b(t)|| = 0, \quad t \geq t_1 \]

with \( z(t_1) = \frac{\phi(t_1)}{\psi(t_1)} > 0 \), \( z_1(t_1) = \frac{\phi_1(t_1)}{\psi_1(t_1)} > 0 \). Then by already proven \( z(t) \) and \( z_1(t) \) are \( t_1 \)-normal, and therefore (see [14])

\[ M_1 \equiv \sup_{t \geq t_1} \left| \int_{t_1}^{t} |c(\tau)(z(\tau) - z_1(\tau))d\tau \right| < +\infty. \quad (3.11) \]

By (3.3) we have

\[ \psi_1(t) = \psi_1(t_1) \exp\left\{ \int_{t_1}^{t} [||c(\tau)||z_1(\tau) + Red(\tau)]d\tau \right\} = \]

\[ = \frac{\psi_1(t_1)}{\psi(t_1)} \psi_1(t_1) \exp\left\{ \int_{t_1}^{t} [||c(\tau)||z(\tau) + Red(\tau)]d\tau \right\} \exp\left\{ \int_{t_1}^{t} [c(\tau)(z_1(\tau) - z(\tau))d\tau \right\}, \quad t \geq t_1. \]

This together with (3.11) implies that

\[ 0 < \psi_1(t) \leq \frac{\psi_1(t_1)}{\psi(t_1)} \exp\{M_1\} \psi(t), \quad t \geq t_1. \]

Hence \( \psi_1(t) \) is bounded (vanish at \( +\infty \)). The theorem is proved.

Let \( p(t), \ q(t) \) and \( r(t) \) be complex-valued continuous functions on \([t_0, +\infty)\) and let \( p(t) \neq 0, \ t \geq t_0 \). Consider the second order linear ordinary differential equation

\[ (p(t)\phi')' + q(t)\phi' + r(t)\phi = 0, \quad t \geq t_0. \quad (3.12) \]
Introducing a new variable $\psi = p(t)\phi'$ we reduce this equation to the system

$$
\begin{cases}
\phi' = \frac{1}{p(t)}\psi, \\
\psi' = -r(t)\phi - \frac{q(t)}{p(t)}\psi, \\
t \geq t_0.
\end{cases}
$$

(3.13)

**Definition 3.1.** Eq. (3.12) is called Lyapunov (asymptotically) stable if the system (3.13) is Lyapunov (asymptotically) stable.

From Theorem 3.1 we immediately get

**Corollary 3.1.** Let the functions

$$
I_1(t) \equiv \int_{t_0}^{t} \exp\left\{-\int_{\tau}^{t} \frac{q(s)}{p(s)} ds\right\} r(\tau)d\tau,
$$

$$
I_2(t) \equiv \sup_{t \geq t_0} \int_{0}^{t} \frac{d\tau}{|p(\tau)|} \int_{t_0}^{\tau} \exp\left\{-\int_{\xi}^{\tau} \frac{q(s)}{p(s)} ds\right\} |r(\xi)| d\xi, \quad t \geq t_0
$$

be bounded. Then Eq. (3.12) is Lyapunov stable.

**Remark 3.1.** In the case $p(t) > 0, r(t) \leq 0$ and $q(t)$ is real-valued the condition of Corollary 3.1 is also necessary for Lyapunov stable of Eq. (3.12) (see [12]). In this sense the conditions 1) and 2) of Theorem 3.1 are sharp.

**Remark 3.2.** It is not difficult to verify that in the case $p(t) > 0, r(t) \leq 0$ and $q(t)$ is real-valued the condition 2') of Theorem 3.1 for the system (3.13) is not satisfiable. On the other hand using Theorem 2.1 to the pair of equations

$$
y' + \frac{1}{p(t)}y^2 + \frac{q(t)}{p(t)}y + r(t) = 0, \quad t \geq t_0,
$$

$$
y' + \frac{1}{p(t)}y^2 + \frac{q(t)}{p(t)}y = 0, \quad t \geq t_0
$$

one can easily show that in this case Eq. (3.12) cannot be asymptotically stable (it has a positive and non decreasing solution). In this sense the condition 2') of Theorem 3.1 is sharp.

**Remark 3.3.** Theorem 3.1 is an improvement of results (Theorem 3.1 and Theorem 3.2) of the work [11].
Example 3.1. Consider the system

\[
\begin{cases}
\phi' = \nu(t)\phi + \frac{\mu(t)}{\ln^2 t} \psi, \\
\psi' = \mu(t)\phi + (\nu(t) - 1)\psi, \quad t \geq e,
\end{cases}
\]

where \(\nu(t)\) and \(\mu(t)\) are some real-valued continuous functions on \([e, +\infty)\) and \(\mu(t)\) is bounded on \([e, +\infty)\). Assume \(\int_{e}^{t} \nu(\tau) d\tau\) is upper bounded on \([e, +\infty)\) (\(\int_{e}^{+\infty} \nu(\tau) d\tau = -\infty\) and \(\int_{e}^{+\infty} (\varepsilon - \nu(\tau)) d\tau\) is upper bounded on \([e, +\infty)\) for some \(\varepsilon \in (0, 1)\)). Then it is not difficult to verify that the conditions 1) and 2) (2')) of Theorem 3.1 for the system (3.14) are satisfied. Therefore under the indicated restrictions the system (3.14) is Lyapunov (asymptotically) stable. Since at least one of the integrals \(\int_{e}^{+\infty} |\nu(\tau)| d\tau, \int_{e}^{+\infty} |\nu(\tau) - 1| d\tau\) diverges to \(+\infty\) the application of the estimates of Lyapunov and Yu. S. Bogdanov ([4], p. 133) to the system (3.14) gives no result. Let \(A(t)\) be the matrix of the coefficients of the system (3.14). Then

\[
\gamma_{\pm}(t) \equiv \frac{2\nu(t) - 1 \pm \sqrt{1 + \frac{4\mu^2(t)}{\ln^2 t}}}{2}, \quad t \geq e
\]

are its eigenvalues. Therefore if \(\sup_{t \geq e} \nu(t) \geq 0\), then the application of the freezing method ([4], p. 139, Theorem 4.6.4) to the system (3.14) gives no result. Let us now discuss the applicability of estimates of solutions via logarithmic norms \(\gamma_{I}(t)\), \(\gamma_{II}(t)\) and \(\gamma_{III}(t)\) of S. M. Lozinski ([4], pp. 135, 136). From the Lozinski’s theorem ([4], p. 137) it follows that if one of the integrals \(\int_{e}^{t} \gamma_{i}(\tau) d\tau, \quad i = I, II, III\) is upper bounded then the corresponding linear system is Lyapunov stable. For the system (3.14) we have

\[
\gamma_{I}(t) \geq \nu(t) + |\mu(t)|, \quad t \geq e.
\]

Therefore if \(\sup_{t \geq e} \int_{e}^{t} (\nu(\tau) + |\mu(\tau)|) d\tau = +\infty\), then the application of \(\gamma_{I}(t)\) to the system (3.14) gives no result. If \(|\mu(t)| \geq \frac{e}{e-1}, \quad t \geq e\) then the logarithmic norm \(\gamma_{II}(t)\) of the system (3.14) satisfies to the inequality

\[
\gamma_{II}(t) \geq 1 + \nu(t), \quad t \geq e.
\]
Hence if \( \sup_{t \geq e} \int (1 + \nu(\tau))d\tau = +\infty \), then the application of \( \gamma_{III}(t) \) to the system (3.14) gives no result. Finally the logarithmic norm \( \gamma_{III}(t) \) for the system (3.14) is

\[
\gamma_{III}(t) = \frac{2\nu(t) - 1 + \sqrt{1 + 4\mu^2(t)(1 + \frac{1}{\ln^2 t})}}{2}, \quad t \geq e.
\]

Therefore if \( \int e^\infty \mu^2(\tau)d\tau = +\infty \) and \( \int e^t \nu(\tau)d\tau \) is bounded from below then \( \int e^t \gamma_{III}(\tau)d\tau = +\infty \) and, hence the application of \( \gamma_{III}(t) \) to the system (3.14) gives also no result. Thus if \( \int e^t \nu(\tau)d\tau \) is bounded and \( |\mu(t)| \geq \frac{e}{e-t} \), \( t \geq e \) then none of the logarithmic norms \( \gamma_I(t), \gamma_{II}(t) \) and \( \gamma_{III}(t) \) is applicable to the system (3.14).

Example 3.2. Consider the system

\[
\begin{align*}
\phi' &= (\lambda - C \sin t)\phi + \mu_1 \psi, \\
\psi' &= \mu_2 \phi + \lambda_2 \psi, \quad t \geq 0,
\end{align*}
\]

(3.15)

where \( \lambda_k, \mu_k, \quad k = 1, 2, \quad C \) are some real constants, \( \mu_k > 0, \quad k = 1, 2, \quad C > 0 \). It is not difficult to verify that under the restrictions

\[
\lambda_k < 0, \quad k = 1, 2, \quad \lambda_1 - \lambda_2 > 0, \quad \lambda_1 + \frac{\mu_1 \mu_2}{\lambda_1 - \lambda_2} \leq 0 \quad (0)
\]

(3.16)

the conditions 1) and 2) (2') of Theorem 3.1 for the system (3.15) are satisfied. Therefore under these restrictions the system (3.15) is Lyapunov (asymptotically) stable. Since \( \int_0^\infty |\lambda_1 - C \sin t|dt = +\infty \) the application of estimates of Lyapunov and Yu. S. Bogdanov gives no result. Let \( A_1(t) \) be the matrix of coefficients of (3.15). Then

\[
\gamma(t) = \frac{\lambda_1 - \lambda_2 - C \sin t + \sqrt{(\lambda_2 - \lambda_1 + C \sin t)^2 + 4\mu_1 \mu_2}}{2}, \quad t \geq 0
\]

is its greatest eigenvalue. Hence, if \( C \geq |\lambda_1 + \lambda_2| \) then \( \sup_{t \geq 0} \gamma(t) \geq 0 \), and in this case the freezing method is not applicable to the system (3.15). For (3.15) we have the following logarithmic norms:

\[
\gamma_{II}(t) = \max\{\lambda_1 + \mu_1 - C \sin t, \lambda_2 + \mu_2\};
\]
\[ \gamma_{II}(t) = \max\{\lambda_1 + \mu_2 - C \sin t, \lambda_2 + \mu_1\}; \]
\[ \gamma_{III}(t) = \frac{\lambda_1 + \lambda_2 - C \sin t + \sqrt{(\lambda_2 - \lambda_1 + C \sin t)^2 + (\mu_1 + \mu_2)^2}}{2}, \quad t \geq 0. \]

The set of parameters \( \mu_k, \lambda_k \) for which at least one of norms \( \gamma_I(t), \gamma_{II}(t) \) is applicable to (3.15) does not include the set, defined by (3.16). For example for \( \lambda_1 = -1, \lambda_2 = -\frac{3}{2}, \mu_1 = 2, \mu_2 = \frac{1}{5} \) the application of \( \gamma_I(t) \) and \( \gamma_{II}(t) \) to (3.15) gives no result, whereas for these values of \( \lambda_k, \mu_k, k = 1, 2 \) the conditions (3.16) are satisfied. It is not difficult to verify that for all enough large (with respect to \( \lambda_k, \mu_k, k = 1, 2 \)) the equality \( \int_{0}^{+\infty} \gamma_{III}(t) dt = +\infty \) is satisfied. Therefore for all enough large \( C \) the application of \( \gamma_{III}(t) \) to the system (3.15) gives no result.

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