Spinor and Isospinor Structure of Relativistic Particle Propagators

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Abstract

Representations by means of path integrals are used to find spinor and isospinor structure of relativistic particle propagators in external fields. For Dirac propagator in an external electromagnetic field all grassmannian integrations are performed and a general result is presented via a bosonic path integral. The spinor structure of the integrand is given explicitly by its decomposition in the independent $\gamma$-matrix structures. Similar technique is used to get the isospinor structure of the scalar particle propagator in an external non-Abelian field.
Propagators of relativistic particles in external fields (electromagnetic, non-Abelian or gravitational) contain important information about quantum behavior of these particles. Moreover, if such propagators are known in arbitrary external field, one can find exact one-particle Green’s functions in the corresponding quantum field theory, taking functional integrals over the external field. Dirac propagator in an external electromagnetic field distinguishes from one of a scalar particle by a complicated spinor structure, and its explicit form was unknown in general case until present time. This problem attracted attention of researchers already for a long time. Feynman, who had written first a path integral for probability amplitude in nonrelativistic quantum mechanics [1] and then a path integral for a scalar particle propagator [2], had also attempted to derive a representation for Dirac propagator via a bosonic path integral [3]. After Berezin had introduced the integral over grassmannian variables, it turned out to be natural to present Dirac propagator via both bosonic and grassmannian path integrals. Such representations have been discussed in the literature for a long time in different contexts [4–14]. Nevertheless, attempts to write Dirac propagator via a bosonic path integral only where continued. So, Polyakov [15] assumed that the propagator of free Dirac electron in \( D = 3 \) Euclidean space-time can be presented by means of a bosonic path integral, similar to a scalar particle, modified by a so called spinor factor. This idea was developed in [16] to write spinor factor for Dirac fermions, interacting with a non-Abelian gauge field in \( D \) dimensional Euclidean space-time. In that representation the spinor factor itself was presented via some additional bosonic path integrals. One ought to say that sometimes it is possible to find Dirac propagator in special configurations of electromagnetic field, thus, in these particular cases its spinor structure can be described explicitly. In fact, there are only few configurations of external field, where that can be done: constant homogeneous electromagnetic field [17], electromagnetic plane wave [18], crossed electric and magnetic fields [19,20], and combination of constant homogeneous electromagnetic field with plane wave field [21].

In this paper we consider a representation of Dirac propagator in arbitrary electromagnetic field as a path integral over bosonic and grassmannian variables and demonstrate that
all Grassmannian integrations can be performed so that a result can only be presented via a bosonic path integral over coordinates; the integrand of this path integral differs from the corresponding expression in scalar case by a spin factor, which spinor structure is given explicitly. Similar technique can be used to get the isospinor structure of the scalar particle propagator in an external non-Abelian field.

The propagator of a spinning particle in an external electromagnetic field $A_\mu(x)$ is the causal Green’s function $S^c(x,y)$ of Dirac equation in this field,

$$[\gamma^\mu (i\partial_\mu - gA_\mu(x)) - m] S^c(x,y) = -\delta^4(x-y),$$

where $x = (x^\mu)$, $[\gamma^\mu, \gamma^n]_+ = 2\eta^{\mu n}$, $\eta^{\mu\nu} = \text{diag}(1-1-1-1)$, $\mu, \nu = 0,3$.

Consider a lagrangian form of the path integral representation (see [13]) for, transformed by $\gamma^5 = \gamma^0\gamma^1\gamma^2\gamma^3$ function $\tilde{S}^c(x,y) = S^c(x,y)\gamma^5$,

$$\tilde{S}^c = \tilde{S}^c(x_{\text{out}}, x_{\text{in}}) = \exp \left\{ i\gamma^n \frac{\partial}{\partial \theta^n} \right\} \int_0^\infty d\epsilon_0 \int d\chi_0 \int_{x_0}^{x_{\text{out}}} D\chi \int_{x_{\text{in}}}^{x_{\text{out}}} Dx$$

$$\times \int D\pi_e \int D\pi_\chi \int_{\psi(0)+\psi(1)=\theta} \mathcal{D}\psi M(e) \exp \left\{ i \int_0^1 \left[ -\frac{\dot{x}^2}{2\epsilon} - \frac{e}{2} m^2 - g\dot{x}A(x) + iegF_{\mu\nu}(x)\psi^\mu\psi^\nu 

+ i \left( \frac{\dot{x}_\mu v^{\mu}}{e} - m\psi^5 \right) \chi - i\psi_n\dot{\psi}^n + \pi_e \dot{\epsilon} + \pi_\chi \dot{\chi} \right] d\tau + \psi_n(1)\psi_n(0) \right\} \right|_{\theta=0},$$

where $[\gamma^m, \gamma^n]_+ = 2\eta^{mn}$, $m,n = 0,3,5$, $\eta^{mn} = \text{diag}(1-1-1-1)$; $\theta^n$ are auxiliary Grassmannian (odd) variables, anticommuting by definition with the $\gamma$-matrices; $x^\mu(\tau)$, $e(\tau)$, $\pi_e(\tau)$ are bosonic trajectories of integration; $\psi^n(\tau)$, $\chi(\tau)$, $\pi_\chi(\tau)$ are odd trajectories of integration; and boundary conditions $x(0) = x_{\text{in}}$, $x(1) = x_{\text{out}}$, $e(0) = e_0$, $\psi^n(0) + \psi^n(1) = \theta^n$, $\chi(0) = \chi_0$ take place;

$$M(e) = \int Dp \exp \left\{ \frac{i}{2} \int_0^1 e^2 d\tau \right\}, \mathcal{D}\psi = D\psi \left[ \int_{\psi(0)+\psi(1)=0} D\psi \exp \left\{ \int_0^1 \psi_n\dot{\psi}^n d\tau \right\} \right]^{-1}. \quad (3)$$

We are going to demonstrate that the propagator (2) can be only expressed through a bosonic path integral over coordinates $x$. To this end one needs to fulfil several functional integrations. First, one can integrate over $\pi_e$ and $\pi_\chi$, and then use arisen $\delta$-functions to remove the functional integration over $e$ and $\chi,$
\[
\tilde{S}^c = -\exp\left\{i\gamma^\mu \frac{\partial}{\partial \theta^\mu}\right\} \int_0^\infty d\tau_0 \int_{x_0}^{x_{out}} D\tau \left(\frac{\dot{x}_\mu \psi^\mu}{\epsilon_0} - m \psi^5\right) d\tau
\]

\[
\times \exp\left\{i \int_0^1 \left[-\frac{\dot{x}^2}{2 \epsilon_0} - \frac{e_0}{2} m^2 - g \dot{x} A(x) + i e_0 F_{\mu \nu}(x) \psi^\mu \psi^\nu - i \psi_n \dot{\psi}_n \right] d\tau + \psi_n(1) \psi_n(0)\right\}\bigg|_{\theta=0}.
\]

Then, it is convenient to replace the integration over \(\psi\) by one over related odd velocities \(\omega\),

\[
\psi(\tau) = \frac{1}{2} \int_0^1 \varepsilon(\tau - \tau') \omega(\tau') d\tau' + \frac{1}{2} \theta, \quad \omega(\tau) = \dot{\psi}(\tau), \quad \varepsilon(\tau) = \text{sign} \tau.
\]

There are not more any restrictions on \(\omega\); because of (3) the boundary conditions for \(\psi\) are obeyed automatically. The corresponding Jacobian does not depend on variables and cancels with the same one from the measure (3). Thus,

\[
\tilde{S}^c = -\frac{1}{2} \exp\left\{i\gamma^\mu \frac{\partial}{\partial \theta^\mu}\right\} \int_0^\infty d\tau_0 \int_{x_0}^{x_{out}} D\tau \left[\frac{\dot{x}_\mu (\epsilon \omega^\mu + \theta^\mu)}{\epsilon_0} - m (\epsilon \omega^5 + \theta^5)\right]
\]

\[
\times \exp\left\{i \left[-\frac{\dot{x}^2}{2 \epsilon_0} - \frac{e_0}{2} m^2 - g \dot{x} A(x) - i e_0 g \frac{2}{4} (\omega^\mu \epsilon - \theta^\mu) F_{\mu \nu}(x) (\epsilon \omega^\nu + \theta^\nu) + \frac{i}{2} \omega_n \epsilon \omega_n\right]\right\}\bigg|_{\theta=0},
\]

where the measure \(D\omega\) is

\[
D\omega = D\omega \left[\int D\omega \exp\left\{-\frac{1}{2} \omega^5 \epsilon \omega_n\right\}\right]^{-1}.
\]

One can prove, that for a function \(f(\theta)\) in the Grassmann algebra, the following identity holds

\[
\exp\left\{i\gamma^\mu \frac{\partial}{\partial \theta^\mu}\right\} f(\theta)\bigg|_{\theta=0} = f\left(\frac{\partial}{\partial \zeta}\right) \exp \left\{i \zeta_n \gamma^n\right\}\bigg|_{\zeta=0}
\]

\[
= \sum_{k=0}^4 \sum_{n_1 \cdots n_k} f_{n_1 \cdots n_k} \frac{\partial}{\partial \zeta_{n_1}} \cdots \frac{\partial}{\partial \zeta_{n_k}} \sum_{l=0}^4 \frac{i^l}{l!} (\zeta_n \gamma^n)^l\bigg|_{\zeta=0},
\]

where \(\zeta_n\) are some odd variables. Using (5), we get

\[
\tilde{S}^c = -\frac{1}{2} \int_0^\infty d\tau_0 \int_{x_0}^{x_{out}} D\tau M(e_0) \left[\frac{\dot{x}_\mu \psi^\mu}{\epsilon_0} - m \psi^5\right]
\]

\[
\times \exp\left\{i \left[-\frac{\dot{x}^2}{2 \epsilon_0} - \frac{e_0}{2} m^2 - g \dot{x} A(x) + i e_0 g \frac{2}{4} F_{\mu \nu}(x) \frac{\partial}{\partial \zeta_{\mu}} \frac{\partial}{\partial \zeta_{\nu}}\right]\right\}
\]

\[
\times R\left[x, \rho, \frac{\partial}{\partial \zeta}\right] \exp \left\{i \zeta_\mu \gamma^\mu\right\}\bigg|_{\rho=0, \zeta=0},
\]

\[\text{footnote}{1}\text{Here and further, we are using condensed notations, e.g. } \omega \epsilon = \int_0^1 d\tau d\tau' \omega(\tau) \varepsilon(\tau - \tau') \omega(\tau') \text{ and so on.}\]
with
\[
R\left[ x, \rho, \frac{\partial_t}{\partial \zeta} \right] = \int D\omega \exp \left\{ -\frac{1}{2}\omega^n T_{nk}(x|g)\omega^k + I_n\omega^n \right\},
\]
(7)
\[
I_\mu = \rho_\mu - \frac{e_0 g}{2} \frac{\partial_t}{\partial \zeta} F_{\mu\nu}(x) \varepsilon, \; I_5 = \rho_5,
\]
\[
T_{nk}(x|g) = \begin{pmatrix} \Lambda_{\mu\nu}(x|g) & 0 \\ 0 & -\varepsilon \end{pmatrix}, \; \Lambda_{\mu\nu}(x|g) = \eta_{\mu\nu} \varepsilon - \frac{e_0}{2} \varepsilon g F_{\mu\nu}(x) \varepsilon.
\]
(8)
where \( \rho_n(\tau) \) are odd sources for \( \omega^n(\tau) \). Integral in (II) is gaussian one. It can be easily done [22], remembering its original definition [13],
\[
R\left[ x, \rho, \frac{\partial_t}{\partial \zeta} \right] = \left[ \frac{\text{Det}(T(x|g))}{\text{Det}(T(x|0))} \right]^{1/2} \exp \left\{ -\frac{1}{2} I_n \left[ T^{-1}(x|g) \right]^{nk} I_k \right\},
\]
(9)
The ratio \( \text{Det}T(x|g)/\text{Det}T(x|0) \) in (II) can be replaced by \( \text{Det}\Lambda(x|g)/\text{Det}\Lambda(x|0) \), due to the structure (III) of the matrix \( T(x|g) \), and the latter can be presented in a convenient form, which allows one to avoid problems with calculations of determinants of matrices with continuous indices,
\[
\frac{\text{Det}\Lambda(x|g)}{\text{Det}\Lambda(x|0)} = \exp \left\{ -e_0 \int_0^g dg' \text{Tr} \mathcal{G}(x|g') F(x) \right\}, \; \mathcal{G}^{\mu\nu}(x|g) = \frac{1}{2} \varepsilon \left[ \Lambda^{-1}(x|g) \right]^{\mu\nu} \varepsilon.
\]
(10)
Substituting (II) into (III), and performing functional differentiations with respect to \( \rho_\mu \), we get
\[
\tilde{S}^c = -\frac{1}{2} \int_0^\infty \frac{d\omega}{\omega} \int_{x_{\text{in}}}^{x_{\text{out}}} Dx M(e_0) \left[ \frac{\dot{x}^\mu}{e_0} K_{\mu\nu}(x) \frac{\partial_t}{\partial \zeta} - im\gamma^5 \right] \exp \left\{ i \left[ -\frac{\dot{x}^2}{2e_0} - \frac{e_0}{2} m^2 - g \dot{x} A(x) + i \frac{e_0}{2} \int_0^g dg' \text{Tr} \mathcal{G}(x|g') F(x) + i e_0 g F(x) K(x) \frac{\partial_t}{\partial \zeta} \right] \right\} \exp \left\{ i \zeta \gamma^\mu \right\} \bigg|_{\zeta=0},
\]
(11)
The differentiation over \( \zeta \) in (II) can be fulfilled explicitly, using eq. (5). Thus, finally
\[
\tilde{S}^c = \frac{i}{2} \int_0^\infty \frac{d\omega}{\omega} \int_{x_{\text{in}}}^{x_{\text{out}}} Dx M(e_0) \Phi(x, e_0) \exp \left\{ i \left[ -\frac{\dot{x}^2}{2e_0} - \frac{e_0}{2} m^2 - g \dot{x} A(x) \right] \right\},
\]
(12)
\[
\Phi(x, e_0) = \left[ m + (2e_0)^{-1} \dot{x} K(x) (1 - 2g F(x) K(x)) \gamma + i m e_0 g \frac{e_0}{4} (F(x) K(x))_{\mu\nu} \sigma^{\mu\nu} \right.
\]
\[
+ i \frac{g}{4} (\dot{x} K(x) \gamma) (F(x) K(x))_{\mu\nu} \sigma^{\mu\nu} + \frac{e_0 g^2}{16} (F(x) K(x))^*_{\mu\nu} (F(x) K(x))^{\mu\nu} \gamma^5 \right]
\]
\[
\times \exp \left\{ -\frac{e_0}{2} \int_0^g dg' \text{Tr} \mathcal{G}(x|g') F(x) \right\},
\]
(13)
where \( \sigma^{\mu\nu} = \frac{i}{2} [\gamma^\mu, \gamma^\nu] \), \((F(x)K(x))_{\mu\nu} = \frac{1}{2} \epsilon_{\mu\nu\rho\delta} (F(x)K(x))^\rho_\delta \), and \( \epsilon_{\mu\nu\rho\delta} \) is Levi-Civita symbol.

The eq.(12) gives a representation for Dirac propagator as a path integral over bosonic trajectories of a functional, which spinor structure is found explicitly, namely, its decomposition in all independent \( \gamma \)-structures is given. The functional \( \Phi(x, e_0) \) can be called spin factor, and namely it distinguishes Dirac propagator from the scalar one. One needs to stress that spin factor is gauge invariant, because of its dependence of \( F_{\mu\nu}(x) \) only.

In the same manner one can describe the isospinor structure of relativistic particle propagators. Here we restrict ourselves with a consideration of a scalar particle propagator in an external electromagnetic \( A_\mu(x) \) and non-Abelian \( B_\mu(x) \) fields. Such a propagator is the causal Green’s function \( D^c(x,y) \) of the Klein-Gordon equation in the fields,

\[
\left[ (i\partial - gA(x) - B^a(x)T_a)^2 - m^2 \right] D^c(x,y) = -\delta^4(x-y),
\]

where \( T^a \) are generators of a corresponding group. Choosing for simplicity \( SU(2) \) as the group, we have \( T_a = \frac{1}{2}\sigma_a \), where \( \sigma_a \) are Pauli matrices. The propagator \( D^c \) can be presented via bosonic and grassmannian path integrals [10,12],

\[
D^c = D^c(x_{out}, x_{in}) = \frac{i}{2} \exp \left\{ i \sigma_a^\nu \frac{\partial}{\partial \theta^a} \right\} \int_0^\infty d\epsilon_0 \int_{\epsilon_0}^\infty d\epsilon \int_{x_{in}}^{x_{out}} D\pi \int D\epsilon \int_\Phi \delta_{\Phi(0),\Phi(1)} \mathcal{D}\phi \times M(\epsilon) \exp \left\{ i \left[ \frac{\pi^2}{2e} - \frac{\epsilon}{2} m^2 - g\pi A(x) - \pi B^a(x) T_a - i\phi_a \frac{\partial}{\partial \pi} + \pi_\epsilon \right] + \phi_a(1) \phi_a(0) \right\} |_{\theta=0},
\]

where \( \theta_a \) are auxiliary odd variables, anticommuting by definition with the \( \sigma \)-matrices; \( \phi_a(\tau) \) are odd trajectories of integration and \( T_a = -i\epsilon_{abc} \phi_b \phi_c \). All grassmannian integrals can be done similar to the spinning particle case and final result presented in the form

\[
D^c = \frac{i}{2} \int_0^\infty d\epsilon_0 \int_{x_{in}}^{x_{out}} D\pi M(\epsilon_0) \Phi(x) \exp \left\{ i \left[ -\frac{\pi^2}{2e_0} - \frac{\epsilon_0}{2} m^2 - g\pi A(x) \right] \right\},
\]

\[
\Phi(x) = \left[ 1 + Tr R(x)G(x|1)R(x) - \frac{i}{2} L_{ab}(x) \epsilon_{abc} T_c \right] \exp \left\{ -\frac{1}{2} \int_0^1 d\lambda \ Tr G(x|\lambda) R(x) \right\},
\]

\[
G(x|\lambda) = \frac{1}{2} \epsilon Q^{-1}(x|\lambda) \epsilon , \ G(x|\lambda) = \epsilon I - \frac{\lambda}{2} \epsilon R(x) \epsilon, \ R_{ab}(x) = \dot{x} B^c(x) \epsilon_{cab},
\]

\[
L_{ab}(x) = R_{ab}(x) - [R(x)G(x|1)R(x)]_{ab};
\]
where $I$ is unit matrix in the group space. The isospinor factor in (17) is presented by its decomposition in the generators $T_a$ of the $SU(2)$ group. Explicit description of the spinor and isospinor structure of Dirac propagator in both Abelian and non-Abelian external fields is more complicated problem which, nevertheless, can be solved in the frame of the same approach.

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