Double circuits in bicircular matroids

Santiago Guzmán-Pro†,‡ and Winfried Hochstättler†,‡

1Facultad de Ciencias, Universidad Nacional Autónoma de México, C.P. 04510, Ciudad Universitaria, México
2FernUniversität in Hagen, Fakultät für Mathematik und Informatik, 58084 Hagen

Abstract

The first non-trivial case of Hadwiger’s conjecture for oriented matroids reads as follows. If \(O\) is an \(M(K_4)\)-minor free oriented matroid, then \(O\) has a nowhere-3-coflow, i.e., it is 3-colourable in the sense of Hochstättler-Nešetřil. The class of gammoids is a class of \(M(K_4)\)-free orientable matroids and it is the minimal minor-closed class that contains all transversal matroids. Towards proving the previous statement for the class of gammoids, Goddyn, Hochstättler, and Neudauer conjectured that every gammoid has a positive coline (equivalently, a positive double circuit), which implies that all orientations of gammoids are 3-colourable. This conjecture stems from their proof that every cobicircular matroid has a positive double circuit. In this brief note we disprove Goddyn, Hochstättler, and Neudauer’s conjecture by exhibiting a large class of bicircular matroids that do not contain positive double circuits.

1 Introduction

Hadwiger’s Conjecture is a well-known and long open conjecture regarding proper graph colourings. It states that for every positive integer \(k\) if a graph \(G\) contains no \(K_{k+1}\)-minor, then \(G\) is \(k\)-colourable. This conjecture has been proven true for \(k \leq 5\) [9], and remains open for larger integers.

The notion of proper graph colourings is extended to oriented matroids in different ways. In particular, Hochstättler and Nešetřil [5] propose to define the chromatic number of an oriented matroid in terms of nowhere-zero coflows (NZ coflows). It turns out that Hadwiger’s conjecture can be generalized to this context; but in this scenario, the first non-trivial case remains open and it reads as follows.

Conjecture 1. Every (loopless) \(M(K_4)\)-minor free oriented matroid has a nowhere-zero 3-coflow.

Towards proving Conjecture[1] Goddyn, Hochstättler and Neudauer, introduce the class of Generalized Series Parallel (GSP) oriented matroids and show that every GSP oriented matroid has a NZ 3-coflow [4]. It might be too much to hope for, but if every \(M(K_4)\)-free oriented matroid is GSP, then Conjecture[1] follows directly. In any case, this raises the fundamental problem of determining when a class \(C\) of oriented matroids is a subclass of GSP oriented matroids. To this end and in the same work, the previously mentioned authors show that if \(C'\) is a class of orientable matroids closed under minors such that every member of \(C'\) has a positive coline, then the class \(C\) of all orientations of matroids in \(C'\) is a class of GSP oriented matroids. Finally, they show that every bicircular matroid has a positive coline, so, if \(O\) is an oriented bicircular matroid, then \(O\) is GSP and thus it has a NZ 3-coflow.

Bicircular matroids are transversal matroids, and the smallest class closed under minors that contains transversal matroids is the class of gammoids. In turn, the class gammoids is a class of \(M(K_4)\)-free orientable matroids, so Goddyn, Hochstättler and Neudauer pose the following conjecture.

---

*The content of this note is included and extended in: arXiv:2209.06591
†sanguzpro@ciencias.unam.mx
‡winfried.hochstaettler@fernuni-hagen.de
Conjecture 2. Every simple gammoid of rank at least two has a positive coline.

In this work, we disprove this conjecture by exhibiting a large class of cobicircular matroids that do not have positive colines.

The rest of this work is organized as follows. In Section 2 we introduce all concepts needed to state Conjecture 2. In Section 3 we introduce bicircular matroids and prove the necessary results to exhibit a class of counterexamples to the previously mentioned conjecture. Finally, in Section 4 we notice that a simple observation made in Section 3 relates to known results about representability of bicircular matroids, and we pose some questions that arise from this relation.

2 Preliminaries

As previously mentioned, Conjecture 2 is motivated by showing that all orientations of gammoids are GSP, and thus exhibiting a large class of $M(K_4)$-free oriented matroids that admit a NZ 3-coflow. Nonetheless, we can state and disprove Conjecture 2 without introducing GSP and thus exhibiting a large class of matroids.

Conjecture 2. In Section 3, we introduce bicircular matroids and prove the necessary results to exhibit a class of matroids. These facts motivate Goddyn, Hochstättler, and Neudauer to conjecture that every orientation of a gammoid is GSP. Furthermore, they conjecture that the following statement is true.

Proposition 4. Let $C$ be minor closed class of orientable matroids. If every simple matroid in $C$ has a positive coline, then every orientation of a matroid in $C$ is GSP.

Using this proposition, Goddyn, Hochstättler, and Neudauer show that every orientation of a bicircular matroid is GSP. Every bicircular matroid is a transversal matroid, and the class of gammoids is the smallest dually and minor closed class that contains transversal matroids. These facts motivate Goddyn, Hochstättler, and Neudauer to conjecture that every orientation of a gammoid is GSP. Furthermore, they conjecture that the following statement is true.

Conjecture 2. Every simple gammoid of rank at least two has a positive coline.

Circuits are the dual complements of hyperplanes. That is, if $H$ is a hyperplane of a matroid $M$ then $E(M) \setminus H$ is a circuit of $M^*$. A double circuit of a matroid $M$ is a set $D$ such that $r(D) = |D| - 2$ and for every element $d \in D$ the rank of $D - d$ does not decrease, i.e. $r(D - d) = |D| - 2 = r(D)$. Dress and Lovász show that if $D$ is a double circuit then $D'$ has a partition $(D_1, \ldots, D_k)$ such that the circuits of $D'$ are $D \setminus D_i$ for $i \in \{1, \ldots, k\}$. We call this partition the circuit partition of $D'$ and say that the degree of $D'$ is $k$.

Observation 3. Let $M$ be a matroid and $D \subseteq E(M)$ a double circuit of $M$. If $D$ is a degree $k$ double circuit, then $M[D]$ is a series extension of $U_{k-2,k}$.

Proof. With out loss of generality suppose that $D = E(M)$, and let $(D_1, \ldots, D_k)$ be the circuit partition of $D$. If $|D_i| = 1$ for every $i \in \{1, \ldots, k\}$, then $M \cong U_{k-2,k}$. Now, the claim follows by a straightforward induction over the difference $|D| - k$.

Similar to how circuits are the dual complements of copoints, double circuits are the complements of colines. Moreover, the copoint partitions and circuit partitions relate as follows.

Observation 4. Let $M$ be a matroid, $L \subseteq E(M)$ and $(H_1, \ldots, H_k)$ a partition of $E(M) \setminus L$. Then, $L$ is a coline of $M$ with copoint partition $(H_1, \ldots, H_k)$ if and only if $E(M) \setminus L$ is a double circuit of $M^*$ with circuit partition $(H_1, \ldots, H_k)$. 
A positive double circuit $D$ is a double circuit with more singular than multiple classes in its circuit partition. By Observation 4, a matroid $M$ has a positive double circuit if and only if $M^*$ has a positive coline. Since gammoids are closed under duality, the following conjecture is equivalent to Conjecture 2.

Conjecture 5. Every cosimple gammoid of corank at least two has a positive double circuit.

3 Bicircular matroids and double circuits

Every bicircular matroid is a transversal matroid [7], and so, every bicircular matroid is a gammoid. In this section, we disprove Conjecture 2 by showing that its dual statement, Conjecture 5, does not hold for bicircular matroids. To do so, we begin by briefly introducing the class of bicircular matroids.

A standard reference for graph theory is [1]. In particular, given a graph $G$ and a subset of edges $I$ we denote by $G[I]$ the subgraph of $G$ induced by $I$. That is, $G[I]$ is the subgraph of $G$ with edge set $I$ and no isolated vertices.

Let $G$ be a (not necessarily simple) graph with vertex set $V$ and edge set $E$. The bicircular matroid of $G$ is the matroid $B(G)$ with base set $E$ whose independent sets are the edge sets $I \subseteq E$ such that $G[I]$ contains at most one cycle in every connected component. Equivalently, the circuits of $B(G)$ are the edge sets of subgraphs which are subdivisions of one of the graphs: two loops on the same vertex, two loops joined by an edge, or three parallel edges joining a pair of vertices.

Matthews [7] noticed that there are only a few uniform bicircular matroids.

Theorem 6. The uniform bicircular matroids are precisely the following:

- $U_{1,n}, U_{2,n}, U_{n,n}$, $(n \geq 0)$;
- $U_{n-1,n}$ $(n \geq 1)$;
- $U_{3,5}, U_{3,6}$ and $U_{4,6}$.

Recall that if a matroid $M$ has a double circuit of degree $k$ then $M$ contains a $U_{k-2,k}$ minor (Observation 3).

Corollary 7. The degree of a double circuit in a bicircular matroid is at most 6.

Suppose that a graph $G$ is obtained by subdividing edges of a graph $H_G$ with minimum degree 3. It is not hard to notice that $H_G$ is unique up to isomorphism. The subdivision classes of $G$ are the sets of edges that correspond to a series of subdivisions of an edge in $H_G$. An unsubdivided edge of $G$ is an edge of $H$ that is an edge of $H_G$. Clearly, if $G$ has no leaves an edge $xy$ is an unsubdivided edge of $G$ if and only if $d_G(x), d_G(y) \geq 3$.

Lemma 8. Let $G$ be a graph and $D \subseteq E$ a double circuit of $B(G)$. Then $G[D]$ has no leaves and contains at most 4 vertices $x_1, x_2, x_3$ and $x_4$ of degree greater than or equal to 3. Moreover, every subdivision class of $G[D]$ belongs to the same class of the circuit partition of $D$.

Proof. Since every element of $D$ belongs to a circuit of $D$, then $D$ has no coloops so $G[D]$ has no leaves. Let $V'$ be the vertex set of $G[D]$ and $r'$ the rank of $D$, so $r' = |V'|$. Since $D$ has no leaves then $d(x) \geq 2$ for every $e \in V'$. Let $t$ be the number of vertices in $V'$ with degree at least 3 in $G[D]$. By the handshaking lemma, $2|D| \geq 3t + 2(r' - t)$. Since $D$ is a double circuit then $r' = |D| - 2$, and thus $2(r' + 2) \geq 3t + 2(r' - t)$, so $t \leq 4$. Finally, let $S$ be a subdivision class of $G[D]$. Then $S$ is the edge set of a path $P$ path such that the internal vertices of $P$ have degree 2 in $G[D]$. Thus, every cycle in $G[D]$ that contains an edge of $P$ contains all edges of $P$. Hence, every circuit of $B(G)$ in $D$ that contains an edge in $P$ contains all of them, so $E(P)$ is contained in some circuit class of $D$.

Given a double circuit $D$ of a bicircular matroid $B(G)$, a distinguished vertex of $D$ is a vertex of degree at least 3 in $G[D]$. Since $G[D]$ has no leaves, the subdivision classes of $G[D]$ correspond to paths that contain distinguished vertices (only) as endpoints. In particular, unsubdivided edges of $G[D]$ are edges incident in distinguished vertices.
Theorem 9. Let $G$ be a graph. If $\text{girth}(G) \geq 5$ then $B(G)$ has no positive double circuits.

Proof. We proceed by contrapositive. Suppose that $D$ is a positive double circuit in $B(G)$ of degree $k$ with circuit partition $(D_1, \ldots, D_k)$ where $k \leq 6$ by Corollary If $k \in \{1, 2, 3, 4\}$ without loss of generality assume that $(D_1, \ldots, D_{k-1})$ are singular circuit classes of $D$. Since the union $D'$ of these classes is $D \setminus D_k$, then this union is a circuit of $D$. Thus, $G[D']$ contains two cycles of $G$, so $\text{girth}(G) \leq |D'| \leq 3$.

Now suppose that $k \in \{5, 6\}$ and $(D_1, \ldots, D_k)$ has at least 4 simple classes $\{e_1, \ldots, e_4\}$. By the moreover statement of Lemma the edges $e_1, e_2, e_3$ and $e_4$ must be unsubdivided edges of $G[D]$. Thus, the endpoints of these edges are distinguished vertices of $G[D]$, which by the same lemma there are at most 4 of these vertices. Putting all of this together we conclude that $D'$ is a set of four edges such that $G[D']$ has at most 4 vertices. Therefore, $G[D']$ contains at least one cycle of $G$, and so $\text{girth}(G) \leq |D'| \leq 4$.

The only remaining case is when the degree of $D$ is 5 and it has 3 singular classes. In this case there are 3 unsubdivided edges $e_1, e_2, e_3$ of $G[D]$. We claim that $\{e_1, e_2, e_3\}$ contains a cycle of $G$, and thus $\text{girth}(G) \leq 3$. Anticipating a contradiction, suppose that $\{e_1, e_2, e_3\}$ does not contain a cycle of $G$. This implies that $D$ has four distinguished $x_1, x_2, x_3$ and $x_4$. Notice that if we contract a subdivided edge $e$ of $G[D]$ we obtain a double circuit $D'$ of $B(G)/e$ with the same degree as $D$. Inductively, we end up with a double circuit $D_0$ and a graph $H$ with edge set $D_0$ such that $\{e_1, e_2, e_3\} \subseteq D_0$ and $V(H) = \{x_1, x_2, x_3, x_4\}$. Moreover, $\{e_1, e_2, e_3\}$ spans a tree of $H$. On the other hand, each edge $e_i$ for $i \in \{1, 2, 3\}$ belongs to a singular class of the circuit partition of $D_0$. Since the rank of $D_0$ is 4 then $D_0$ contains at most six edges. Also, the circuit partition of $D_0$ has 5 classes, so there must be an edge $e_4 \in D_0 \setminus \{e_1, e_2, e_3\}$ that belongs to a singular class. Notice that that $\{e_1, e_2, e_3, e_4\}$ contains at most one cycle of $H$ since $\{e_1, e_2, e_3\}$ spans a tree of $H$. On the other hand, $D_0 \setminus \{e_1, e_2, e_3, e_4\}$ is a class of $D_0$, so $\{e_1, e_2, e_3, e_4\}$ is a circuit of $D_0$, i.e. $\{e_1, e_2, e_3, e_4\}$ contains two cycles of $H$. We arrive at this contradiction by assuming that $\{e_1, e_2, e_3\}$ does not contain a cycle of $G$, thus $\text{girth}(G) \leq 3$, and the theorem follows. \[\square\]

This statement yields a large class $C$ of bicircular matroids with no positive double circuits. For instance, if $G$ is the dodecahedron graph and $P$ the Petersen graph (Figure then $B(G)$ and $B(P)$ are cosimple bicircular matroids of corank at least two that do not have positive double circuits. Dually, $B(G)^*$ and $B(P)^*$ are simple matroids of rank at least two that do not have positive colines.

Recall that every transversal matroid is a gammoid, and since bicircular matroids are transversal matroids, Theorem yields a class of cosimple gammoids of corank at least two that do not have positive double circuits.

Corollary 10. Not every cosimple gammoid of corank at least two has a positive double circuit. Equivalently, not every simple gammoid of rank at least two has a positive coline.
4 Conclusions

A simple observation used to prove Theorem 9 shows that the degree of double circuits in bicircular matroids is bounded above by 6. This raises the natural problem of describing the classes of matroids obtained by considering bicircular matroids whose double circuits have degree at most \( k \) where \( k \in \{3, 4, 5\} \). This question has been answered from another perspective for \( k = 3 \): the positive double circuits in a bicircular matroid \( B \) have degree at most 3 if and only if \( B \) is a binary bicircular matroid [7]. If we also restrict the degree of colines, for the case \( k = 4 \) we recover ternary bicircular matroids [10]. For the case \( k = 5 \), considering bicircular matroids with double circuits and colines of degree at most \( k \), we do not recover representation over \( GF(4) \) since \( P_6 \) has no positive double circuits nor colines of degree greater than or equal to 5, but is a bicircular matroid not representable over \( GF(4) \) [2]. We are interested in knowing if there is a meaningful description of bicircular matroids whose double circuits have bounded degree.

**Problem.** Provide a meaningful description of bicircular matroids where every double circuit has degree at most \( k \) for \( k \in \{4, 5\} \).

The motivation of this work was to settle Conjecture 2; we showed that it does not hold even for cobicircular matroids. Nonetheless, there are GSP oriented matroid, whose underlying matroid does not have a positive coline, for instance \( P_7 \) [4]. So, it still makes sense to ask the following question.

**Question.** Is every oriented cobicircular matroid GSP?

Acknowledgements

This work was carried out during a visit of the first author at FernUniversität in Hagen, supported by DAAD grant 57552339.

References

[1] J.A. Bondy and U.S.R Murty, Graph Theory, Springer, Berlin, 2008.
[2] D. Chun, T. Moss, D. Slilaty, X. Zhou, Bicircular Matroids representable over \( GF(4) \) and \( GF(5) \), Discrete Mathematics 339(9) (2016) 2239–2248.
[3] A. Dress and L. Lovász, On some combinatorial properties of algebraic matroids, Combinatorica 7(1) (1987) 39–48.
[4] L. Goddyn, W. Hochstättler, N. Neudauer, Bicircular matroids are 3-colourable, Discrete Mathematics 339(5) (2016) 1425–1429.
[5] W. Hochstättler and J. Nešetřil, Antisymmetric flows in matroids, European Journal of Combinatorics 27(7) (2006) 1129–1134.
[6] A.W. Ingleton, Gammoids and Transversal Matroids, Journal of Combinatorial Theory (B) 15 (1973) 51–68.
[7] L.R. Matthews, Bicircular Matroids, Quarterly Journal of Mathematics (2) 28(110) (1997) 213–227.
[8] J.G. Oxley, Matroid Theory, The Clarendon Press Oxford University Press, New York (1992).
[9] N. Robertson, P. Seymour, R. Thomas, Hadwiger’s conjecture for \( K_6 \)-free graphs, Combinatorica 13 (1993) 279–361.
[10] V. Sivaraman, Bicircular signed-graphic matroids, Discrete Mathematics 328 (2014) 1–4.