Fluctuations for stationary $q$-TASEP

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We consider the $q$-totally asymmetric simple exclusion process ($q$-TASEP) in the stationary regime and study the fluctuation of the position of a particle. We first observe that the problem can be studied as a limiting case of an $N$-particle $q$-TASEP with a random initial condition and with particle dependent hopping rate. Then we explain how this $N$-particle $q$-TASEP can be encoded in a dynamics on a two-sided Gelfand-Tsetlin cone described by a two-sided $q$-Whittaker process and present a Fredholm determinant formula for the $q$-Laplace transform of the position of a particle. Two main ingredients in its derivation is the Ramanujan’s bilateral summation formula and the Cauchy determinant identity for the theta function with an extra parameter. Based on this we establish that the position of a particle obeys the universal stationary KPZ distribution (the Baik-Rains distribution) in the long time limit.

1 Introduction

Large time fluctuations of certain class of interface growth exhibit universal critical behaviors, characterized by the Kardar-Parisi-Zhang (KPZ) universality class. For systems in the one dimensional KPZ class, the fluctuations of the height of an interface at a position grow as $O(t^{1/3})$ for large time $t$. The exponent $1/3$ are observed for example in various simulation models such as the ballistic deposition model and the Eden model [9]. Theoretically it was predicted by an application of a dynamical version of the renormalization group method [35, 51] to the celebrated KPZ equation,

$$\partial_t h = \frac{1}{2} \partial^2_x h + \frac{1}{2} (\partial_x h)^2 + \eta,$$

where $h = h(x,t)$ represents the height at position $x \in \mathbb{R}$ and at time $t \geq 0$ and $\eta = \eta(x,t)$ is the space-time Gaussian white noise with mean zero and covariance $\langle \eta(x,t)\eta(x',t') \rangle = \delta(x-x')\delta(t-t')$. Later the exponent was also confirmed by exactly solvable models.

One of the standard models in this one dimensional KPZ class is the totally asymmetric exclusion process (TASEP), in which each particle on one dimensional lattice hops to the right neighboring site with rate one as long as the target site is empty. By the interpretation of the occupied site (resp. empty site) as upward (resp. downward) slope, the TASEP can be mapped to a surface growth model, called the single step model. For TASEP the $1/3$ exponent was found exactly by looking at the spectral gap of the generator for the process on a ring [55]. For TASEP with step initial condition, in which all sites are occupied (resp. empty) to the left (resp. right) of the origin, this exponent was proved and even the limiting distribution of the height fluctuation was determined in [45]. In this case, the limiting distribution turned out to be the Tracy-Widom (TW) distribution of GUE (Gaussian unitary ensemble) type, which describes the largest eigenvalue distribution of the GUE random matrix ensemble in the limit of large matrix dimension [80].

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The limiting distributions, though universal in the sense that the same distribution appears in many other models and are even observed in experiments \([18, 19]\), turned out to have certain boundary/initial condition dependence. For instance for the flat interface, the limiting distribution is given by the TW ansatz. In the stationary measure of \(q\)-TAZRP was confirmed in \([64]\) by using Bethe ansatz. In the stationary measure of \(q\)-TASEP, all the gaps among the particles are independent and each gap is distributed as the Poisson random variable \([2, 5]\) with a parameter \(\alpha \in [0, 1]\) by which the average density of particles in the system can be controlled. In this article, we study the \(q\)-TASEP with a random initial condition,

\[
-1 - X_1(0) = Y_1, \quad X_{i-1}(0) - X_i(0) - 1 = Y_i \quad \text{for} \quad i = 2, 3, \ldots, N,
\]

where \(Y_1, Y_2, \ldots\) are independent \(q\)-Poisson random variables with parameter \(\alpha/a_i\) with \(0 \leq \alpha/a_i < 1, 1 \leq i \leq N\). This does not really correspond to our stationary \(q\)-TASEP in two respects. One is that the initial position of the first particle is taken to be random, not fixed at the origin. This can be handled as follows. Let us denote by \(X_N^{(0)}(t)\) the position of the \(i\)th particle at time \(t\) when the first particle is at the origin initially, i.e., the initial condition now reads

\[
X_1^{(0)}(0) = 0, \quad X_{i-1}^{(0)}(0) - X_i^{(0)}(0) - 1 = Y_i \quad \text{for} \quad i = 2, 3, \ldots, N,
\]

where \(Y_1, Y_2, \ldots\) are independent \(q\)-Poisson random variables with parameter \(\alpha/a_i\) with \(0 \leq \alpha/a_i < 1, 2 \leq i \leq N\). Then the difference between the two cases comes only from the randomness of \(Y_1\) and a simple relation \(X_i(t) = X_i^{(0)}(t) - Y_1, 1 \leq i \leq N\) holds. Since the distribution of \(Y_1\) is independent of \(X_i^{(0)}(t)\) and \(Y_1\) is just \(q\)-Poisson distributed, one can study \(X_i^{(0)}(t)\) by a combination of information on \(X_i(t)\) and \(Y_1\). The other difference of \([12]\) from the original problem we want to study is that the hopping rates depend
on particles. The stationary case can be accessed by specializing to $a_1 = a, a_i = 1, 2 \leq i \leq N$ and then taking the $a \rightarrow \alpha$ limit. In the $a \rightarrow \alpha$ limit, the distribution of $Y_1$ becomes singular (since the parameter of the $q$-Poisson distribution becomes one) but this difficulty can be handled by analytic continuation from $\alpha < a$ a case.

The fact that the largest eigenvalue of certain random matrix ensembles and the height of an interface growth share the same limiting distribution is staggering. It is still a challenging problem to understand the connection in full generality, but some part of it may be understood by a common determinantal structure which appear in the studies of a few concrete models. For GUE, the joint eigenvalue distribution is written as the square of a product of differences \[\text{[22, 78]}\]. As is well-known in linear algebra, a product of differences can be written as a Vandermonde determinant. Hence GUE is associated with a measure in the form of a product of two determinants. Once this structure is found, one can show that all correlation functions are written as determinants in terms of the same correlation kernel \[\text{[52]}\]. A determinantal point process is a point process whose correlation functions can be written as determinants in terms of the same correlation kernel \[\text{[12, 75]}\]. Hence GUE is an example of a determinantal point process.

The TASEP with step initial condition is, via the RSK mapping, associated with Schur measure, which is in the form of a product of two Schur functions \[\text{[15]}\]. It is well known that a Schur function can be written as a single determinant (Jacobi-Trudi identity) \[\text{[29, 74]}\]. Hence TASEP with step initial condition is associated with a measure in the form of a product of two determinants and the above machinery of the determinantal processes can be applied. Basically the same determinantal structure can be utilized to study the stationary case \[\text{[31, 40]}\]. For the flat case, a modified signed determinantal structure exists \[\text{[21, 69]}\].

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Situation is different for the KPZ equation, the OY polymer, $q$-TASEP, ASEP and so on. There the final results are written as Fredholm determinants after long calculations but not as a consequence of an underlying determinantal process (see the remark at the end of this introduction for recent related developments). In \[\text{[15]}\] the $q$-TASEP appears as a special case of the Macdonald process, which can be written as a product of two Macdonald polynomials \[\text{[59]}\]. For the Macdonald polynomials, a single determinat formula has not been found but the authors of \[\text{[15]}\] could use other properties of the Macdonald polynomials to study $q$-TASEP. There is another approach using duality \[\text{[18, 25, 41, 74]}\]. In either method, one first finds a multiple integral formula for the $q$-deformed moments of the form \[\text{[34, 61]}\] and sees that its certain generating function can be rewritten as a Fredholm determinant.

For the OY polymer, which is a finite temperature semi-discrete directed polymer model, some determinantal structures have been discussed. In \[\text{[62]}\] a connection to a GUE random matrix with a source is found though the resulting formula was not suited for asymptotics. In \[\text{[17]}\], this formula was transformed to the one in \[\text{[15]}\] for which the asymptotic analysis is possible. Another determinantal structure was also discussed in \[\text{[41]}\]. But all these discussions have been made after the application of the Bump-Stade identity \[\text{[22, 78]}\] which allows us to rewrite a certain quantity of the original model in an integral from with a better product structure in the integrand in a magical way.

In this paper we analyze the $N$-particle $q$-TASEP for the initial condition \[\text{[1.2]}\]. First we show that the process can be described as a marginal of a dynamics on an enlarged state space, which is a two-sided version of the Gelfand-Tsetlin cone $\mathbb{G}_N$. Here "two-sided" means that each component of the element takes a value from $\mathbb{Z}$ rather than $\mathbb{N}$. The element of $\mathbb{G}_N$ is in the form, $\lambda_N = (\lambda^{(1)}, \ldots, \lambda^{(N)})$ where $\lambda^{(i)} = (\lambda^{(i)}_1, \ldots, \lambda^{(i)}_i), 1 \leq i \leq N$ is a signature (two-sided partition), meaning $\lambda^{(i)}_j \in \mathbb{Z}$, with the interlacing condition $\lambda^{(i+1)}_j \leq \lambda^{(i)}_j \leq \lambda^{(i+1)}_j, 1 \leq j \leq i \leq N - 1$. We consider a Markov dynamics on $\mathbb{G}_N$, s.t. that probability $\mathbb{P}(\Delta_N(t) = \lambda)$ is given by a measure in the form,

$$P_t(\Delta_N) := \prod_{j=1}^{N} P_{\lambda^{(j)}/\lambda^{(j-1)}(a_j)} \cdot Q_{\lambda^{(N)}}(\alpha, t) \cdot \prod(a; \alpha, t). \quad (1.4)$$
Here $P_{\lambda/(\mu)}(a)$ is the $q$-Whittaker function of one variable, with the label $\lambda$ being a signature. $Q_{\lambda}(\alpha, t)$ is a generalization of a version of the Macdonald polynomial $Q_{\lambda}(x)$ to our situation with a finite density of particles to the left. (Note that our case is not included as the non-negative specialization treated in [15] because $\lambda$ is not restricted to partitions but is associated with a positive measure.) $\Pi(a; \alpha, t)$ is the normalization. We call this the two-sided $q$-Whittaker process. We will see that the dynamics of the $N$ particle $q$-TASEP with (1.2) is encoded as a marginal dynamics of $\lambda(j)(t), 1 \leq j \leq N$ in this process as $X_j(t) + j = \lambda(j)(t), 1 \leq j \leq N$.

To study the position of the $N$th particle in our $q$-TASEP, one can instead focus on the marginal for $\lambda(N)(t)$, which turns out to be written as

$$P[\lambda^{(N)}(t) = \lambda] = \frac{1}{\Pi(a; \alpha, t)} P_{\lambda}(a_1, \ldots, a_N) Q_{\lambda}(\alpha, t).$$

Here $P_{\lambda}(a_1, \ldots, a_N)$ is the two-sided $q$-Whittaker function and we call this marginal measure the two-sided $q$-Whittaker measure. We will calculate the $q$-Laplace transform, $\left\langle \frac{1}{(q^{\lambda(N)}; q)^{\infty}} \right\rangle$ with respect to this measure. Here $(a; q)^{\infty} = \prod_{i=1}^{\infty} (1 - aq^i)$ is the $q$-Pochhammer symbol. A difficulty to study a random initial condition for $q$-TASEP is that the $q$-moment $\langle q^{\lambda(N)} \rangle$ diverges for large $k$ [18]. So one can not expand the above $q$-Laplace transform as the generating function of these $q$-moments using the $q$-binomial theorem (4.2). Instead we compute the $q$-Laplace transform directly. To do so, we rewrite the Cauchy identity for Macdonald polynomials by separating the factor only on $N$th row of a Young diagram, see (B.8), and for the remaining summation of $\lambda_N$ over $\mathbb{Z}$, we use the Ramanujan’s bilateral summation formula, see (1.1).

Then, after some calculation, we notice a determinantal structure for our quantity in terms of the theta function. We can use the Cauchy determinant identity and that we use the Cauchy identity at the elliptic level with an extra parameter [50]. Novelty in our calculation is that we do not rely on the Bump-Stade identity. Some part of the procedure to find this Fredholm determinant formula is basically the same as that in [17] for the Log-Gamma (and OY) polymer case in which the trigonometric Cauchy determinant was used after the Bump-State identity. Novelty in our calculation is that we do not rely on the Bump-Stade identity and that we use the Cauchy identity at the elliptic level with an extra parameter [50].

Then we consider the stationary case by specializing to $a_1 = a, a_i = 1, i \geq 2$ and taking the limit $a \to \alpha$. The formula for the $q$-Laplace transform for the stationary $q$-TASEP is given in (5.52). We also perform the asymptotic analysis to our expression and establish that the limiting distribution is given by the Baik-Rains distribution, $F_0$. This is our second main result in the paper.
Theorem 1.2. For the stationary $q$-TASEP, with the parameter $\alpha = q^\theta$, $\theta > 0$ determining the average density through (2.6), we have, for $\forall s \in \mathbb{R}$,

$$
\lim_{N \to \infty} P(X_N^{(0)}(\kappa N) > (\eta - 1)N - \gamma N^{1/3}s) = F_0(s)
$$

(1.8)

where $\kappa = \kappa(\theta, q)$, $\eta = \eta(\theta, q)$, $\gamma = \gamma(\theta, q)$ are given by (2.21), (2.22), (2.26). $F_0$ is a special case of the Baik-Rains distribution (5.99).

This is a special ($\omega = 0$) case of Theorem 5.13. The coefficients $\kappa, \eta, \gamma$ in the theorem can be understood from the KPZ scaling theory [33, 65, 77]. In the main text we will study the case where $\alpha$ is also rescaled and obtain a parameter family of limiting distribution $F_\omega$ [8, 31, 43]. Our result is only for the one point distribution. As a conjecture it is expected that joint distributions of scaled height would be described by the process identified in the studies of TASEP [7]. The scaled two point correlation function would also show universal behaviors. Such analysis have been done for the PNG model, TASEP and the KPZ equation [31, 43, 67].

In the proof of the theorem we show that some part of the formula tends to the GUE Tracy-Widom distribution. In Appendix C we provide two lemmas to establish this limit when the kernel is written in a specific form. This would be useful for other situations as well.

Our approach can be applied to other models. As an example, we discuss two limiting cases, the OY polymer and the KPZ equation, in a companion paper [39]. For the OY model, the analysis for the case with multi parameters was already studied in [16]. In [39], we will provide a formula for the real stationary case. One can also consider the limit to the stationary KPZ equation.

The paper is organized as follows.

In section 2, we explain basic properties of $q$-TASEP and the corresponding $q$-boson TAZRP. We discuss their stationary measures, average density and current, hydrodynamic description, Burke theorem, and the KPZ scaling theory.

In section 3, we introduce a two-sided version of the Gelfand-Tsetlin (GT) cone, $q$-Whittaker function and $q$-Whittaker process. For the Schur case this type of generalization was already studied by [14]. The $q$-version was also mentioned in [27, 60]. Next we see that the dynamics of the $q$-TASEP is encoded in the marginal $\lambda^{(j)}_N$, $1 \leq j \leq N$ on one edge of GT cone. For the step case, corresponding relation is stated in [15] as a limit from discrete time dynamics but we show this more directly for the continuous time model. At the end of the section we discuss the two-sided $q$-Whittaker measure, as the marginal for $\lambda^{(N)}_N$ of the two-sided $q$-Whittaker process. In particular an expression for the distribution of $\lambda^{(N)}_N$ is given, which is essential for the subsequent discussions.

In section 4, we present a Fredholm determinant expression for the $q$-Laplace transform for the $N$ particle position. We will use in an essential way the Ramanujan’s bilateral summation formula and the Cauchy determinant identity for theta function.

In section 5, we first discuss in more detail the relationships between the two cases in which the initial position of the first particle is random and is fixed to be at the origin. Then we present a formula for the position of the particle for the stationary measure. Finally we perform asymptotic analysis and establish the Baik-Rains distribution for the position of the particle.

In section 6, we discuss the $q \to 0$ limit corresponding to the usual TASEP case studied in [31, 43].

In Appendix A, some $q$ notation and functions are summarized. In Appendix B, we explain and discuss a few basic properties of the ordinary (not two-sided) $q$-Whittaker function and process. In Appendix C, we discuss the Tracy-Widom limit when the kernel is written in a specific form. In Appendix D, we discuss the inverse of the $q$-Laplace transform.

Remark. After the main contents of this paper have been obtained and while the manuscript is prepared, a few papers in a similar direction have appeared. In [2], the authors consider ”generalized step Bernoulli
initial data” and discuss the BBP transition. In [1], the stationary ASEP is studied. In [63], the authors study some generalized initial condition for discrete $q$-TASEP. They are based on a new interesting observation on the relationship between certain observables of the Macdonald measure and the Schur measure [13]. Their approach also gives a better understanding on the determinantal structures of the models. Our approach is somewhat different and it would be certainly an interesting question to have a better understanding about the relationship between their methods and ours.

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2 Stationary $q$-TASEP and $q$-TAZRP

The $q$-TASEP on $\mathbb{Z}$, which was introduced rather informally in Introduction, is constructed in a standard manner [57] (cf [15] Sec 3.3.3), based on its generator,

$$L_{q\text{-TASEP}}f(\eta) = \sum_{x \in \mathbb{Z}} (1 - q^{g_x(\eta)}) \eta(x) \{f(\eta^{x,x+1}) - f(\eta)\}$$

(2.1)

where $\eta \in \{0, 1\}^\mathbb{Z}$ is a configuration of particles in $q$-TASEP, $g_x(\eta) + 1$ is the distance between $x$ and the first particle to the right in configuration $\eta$, $\eta^{x,x+1}$ is the configuration in which a particle at $x$ (if it exists) moves to $x + 1$, and $f$ is a local function on $\{0, 1\}^\mathbb{Z}$. The dynamics of the "gaps", $\xi_k = x_k - x_{k+1} - 1$, where $x_k$ is the position of the $k$th particle (labelled from right to left, with the 0th particle starting at the position which is positive and closest to the origin), is a version of zero range process whose generator is given by

$$L_{q\text{-TAZRP}}f(\xi) = \sum_{k \in \mathbb{Z}} c(\xi_k) f(\xi^{k,k+1}) - f(\xi)$$

(2.2)

with

$$c(l) = 1 - q^l, \quad l \in \mathbb{N}$$

(2.3)

where $\xi \in \mathbb{N}^\mathbb{Z}$ is a configuration of particles in a zero range process, and $f$ is a local function on $\mathbb{N}^\mathbb{Z}$. See Fig. 1. It is known that, when the function $c$ in (2.2) satisfies the condition,

$$\sup_l |c(l + 1) - c(l)| < \infty,$$

(2.4)

the process can be constructed (for a large class of initial conditions). The function $c$ (2.3) corresponding to our $q$-TAZRP certainly satisfies this condition when $0 < q < 1$. Because particles hop only in one direction and the (adjoint) generator is written in terms of $q$-boson [73], we call this the $q$-boson TAZRP (or simply $q$-TAZRP). We first explain some basic facts for this $q$-TAZRP.

2.1 $q$-TAZRP

For a zero range process (with certain mild conditions), the stationary measures (which are translationally invariant and extremal) are known [5]. For our $q$-TAZRP, the stationary measure is given by the one in which the number of particles at sites are independent and at each site the number is distributed by the $q$-Poisson distribution,

$$P[\xi_x = n] = (\alpha; q)_\infty \frac{\alpha^n}{(q; q)_n}, \quad n \in \mathbb{N},$$

(2.5)
with \( \alpha \in [0, 1) \) a parameter. Here \((a; q)_n, (a; q)_\infty\) are the \(q\)-Pochhammer symbols, see Appendix A. One observes that this is normalized due to the \(q\)-binomial theorem (A.2) and that it becomes singular as \( \alpha \uparrow 1 \). We call this the \(q\)-Poisson distribution because if we scale \( \alpha \) to \((1 - q)\alpha\) and take the \(q\rightarrow 1\) limit, this tends to the Poisson distribution (In [15], it was called the \(q\)-geometric distribution since it tends to the geometric distribution when \(q = 0\)). We sometimes write (2.5) as \(\xi \sim q\text{-Po}(\alpha)\). The average density for a given \(\alpha\) is

\[
\rho(\alpha) = \langle \xi_x \rangle = (\alpha; q)_\infty \sum_{n=0}^{\infty} \frac{n a^n}{(q; q)_\infty} = (\alpha; q)_\infty \alpha \frac{d}{d\alpha} (\alpha; q)_\infty = \alpha \sum_{k=0}^{\infty} q^k \left( 1 - \alpha q^k \right),
\]

(2.6)

where \(\langle \cdots \rangle\) means the expectation value. We denote by \(\alpha(\rho)\) the inverse function. The average current (to the right) is

\[
j(\rho) = \langle 1 - q^{\xi_x} \rangle = \alpha(\rho).
\]

(2.7)

Let us consider a quantity \(h(k, t)\) defined by

\[
h(k, t) = N_0(t) + \begin{cases} -\sum_{l=1}^{k} \xi_l(t), & k \geq 1, \\
0, & k = 0, \\
+\sum_{l=k+1}^{\infty} \xi_l(t), & k < 0, \end{cases}
\]

(2.8)

where \(N_0(t)\) is the number of particles which hop from 0 to 1 between the time period [0, \(t\)]. This can be interpreted as a height function. Since \(h(k, t) - h(k-1, t) = -\xi_k, k \in \mathbb{Z}\), the average density \(\rho\) is the (minus) average slope and the average current \(j\) is the average growth speed in the surface picture. See Fig. 2.

Macroscopically the evolution of the density profile \(\rho(x, t)\) (at a macroscopic space-time position \(x\) and \(t\)) is expected to be described by the hydrodynamic equation,

\[
\frac{\partial}{\partial t} \rho(x, t) + \frac{\partial}{\partial x} j(\rho(x, t)) = 0.
\]

(2.9)

See for instance [29, 37, 54, 76] for more information. In the stationary case, the macroscopic density is just a constant \(\rho(x, t) = \rho\). In the surface picture, macroscopic height profile is \(h(x, t) = j(\rho)t - px\). We will be interested in the fluctuation around the macroscopic shape. Since we are considering the stationary case, the nontrivial fluctuation is expected to be observed near the characteristic line from the origin \(x = j'(\rho)t\), along which the effect of fixing the height initially (i.e. \(h(0, 0) = 0\)) is propagated (cf e.g. [26, 29]).
A totally asymmetric zero range process can be considered as a system of queues in tandem. In the correspondence each site \( k \) is a service counter and the number of particles \( \xi_k \) is the number of customers waiting for a service at the counter at site \( k \). The service at each site is provided with rate \( 1 - q^k \). Queues are in tandem, which means that once a customer at a site \( k \) gets a service he/she moves to the next queue at site \( k + 1 \). \( N_0(t) \) represents the number of customers who get the service at site 0 and move to the site 1 between time period \([0, t]\). This is called the output process (at site 0) in the queue language. Likewise the input process for each service counter is the number of customers who join its queue between time period \([0, t]\). In the stationary situation, the number of customers at each site is distributed as the \( q \)-Poisson distribution (2.5). For this situation, we have

**Proposition 2.1.** For the queues in tandem corresponding to the stationary \( q \)-TAZRP with parameter \( \alpha \), the output process at the origin is the Poisson process with rate \( \alpha \).

**Proof.** This is basically due to the Burke theorem [23], which says that when a queue having an exponential waiting time for a service and the input process is the Poisson process, then the output process is also the Poisson process with the same rate.

The case of infinite queues in tandem corresponding to ordinary TASEP (\( q = 0 \)) was proved in [30]. The arguments there can be applied to our case (\( q \neq 0 \)) with \( \xi_x \) dependent hopping rate as well. One first notices that the process is reversible with respect to the stationary measure with parameter \( \alpha \). The output process at site 0 of the original process is in distribution the same as the input process at site 0 in the reversed process, which is by construction a Poisson process with rate calculated as the ratio of the stationary probability of having \( n \) and \( n + 1 \) particles at site 1 times the particle hopping rate when \( n + 1 \) particles are at site 1, i.e.,

\[
\frac{\alpha^{n+1}}{(q;q)_n} \frac{(q;q)_n}{\alpha^n} (1 - q^{n+1}) = \alpha.
\]

This proposition implies that, when we are only interested in the right half of the TAZRP, one can replace the whole left half by the Poisson input process at site 1 with rate \( \alpha \). We can suppose that there are infinite particles at the origin which acts as a source of particles with rate \( \alpha \), see the bottom-right figure in Fig. [1].

Next we explain the KPZ scaling theory [65][77] for our stationary \( q \)-TASEP. According to this conjectural theory, for a given density \( \rho \), it is expected that the scaled height around the characteristic
\[ \xi_t = (h(k, t) - (j(\rho)t - \rho k))/c(\rho)t^{1/3}, \text{ with } k = j'(\rho)t, \] (2.10)

exhibits universal behaviors, and the coefficient \( c(\rho) \) can be determined by the knowledge of the average current \( j(\rho) \) and the growth rate of the variance of height in space direction in the stationary measure,

\[ A(\rho) = \lim_{k \to \infty} \frac{1}{k} \langle (h(k, 0) - h(0, 0))^2 \rangle c, \] (2.11)

where the bracket is taken with respect to the stationary measure with a density \( \rho \) and \( C \) means the cumulant. Then one should take

\[ c(\rho) = -\left(-\frac{1}{2}j''(\rho)A(\rho)^2\right)^{1/3}. \] (2.12)

In our case since \( h(k, 0) = -\sum_{l=1}^{k} \xi_l(0), k > 0 \) is a sum of \( k \) independent \( q \)-Poisson variables with parameter \( \alpha \), \( A(\rho) \) can be easily calculated as

\[ A(\rho) = \sum_{k=0}^{\infty} \frac{\alpha(\rho) q^k}{(1 - \alpha q^k)^2}. \] (2.13)

From (2.7), we have

\[ j'(\rho) = \alpha'(\rho) = \frac{1}{\rho'(\alpha)}, \] (2.14)

\[ j''(\rho) = \alpha'(\rho) \frac{d}{d\rho} \left( \frac{1}{\rho'(\alpha)} \right) = -\frac{\rho''(\alpha)}{(\rho'(\alpha))^3}, \] (2.15)

where by (2.6)

\[ \rho'(\alpha) = \sum_{k=0}^{\infty} \frac{q^k}{(1 - \alpha q^k)^2}, \quad \rho''(\alpha) = 2 \sum_{k=0}^{\infty} \frac{q^{2k}}{(1 - \alpha q^k)^3}. \] (2.16)

Substituting (2.15) with (2.10) and (2.13) (note \( A(\rho) = \alpha \rho'(\alpha) \)) into (2.12), the constant \( c(\rho) \) (or \( c(\rho)^3 \)) in our case is

\[ c(\rho)^3 = -\frac{\alpha^2 \rho''(\alpha)}{2 \rho'(\alpha)} = -\frac{\sum_{k=0}^{\infty} \frac{\alpha^2 q^{2k}}{(1 - \alpha q^k)^2}}{\sum_{k=0}^{\infty} \frac{q^{2k}}{(1 - \alpha q^k)^3}} \] (2.17)

where \( \alpha \) on rhs should be understood as \( \alpha(\rho) \) defined below (2.6).

### 2.2 \( q \)-TASEP

Next we discuss corresponding and similar statements for \( q \)-TASEP. As already explained in the Introduction, in the stationary measure with parameter \( \alpha \), all the gaps among the particles are independent and each gap is distributed as the \( q \)-Poisson random variable (2.5). First, since the average distance between two consecutive particles is \( 1 + \langle \text{gap} \rangle \), the average density for a given \( \alpha \) is

\[ \rho_0(\alpha) = \frac{1}{1 + \alpha \sum_{k=0}^{\infty} \frac{q^k}{1 - \alpha q^k}} = \frac{1}{1 + \rho(\alpha)}. \] (2.18)

The inverse function is denoted by \( \rho_0(\rho_0) \). For \( q \)-TASEP, the average current for a given density \( \rho \) (or for a given \( \alpha \) through \( \rho = \rho_0(\alpha) \)) is given by

\[ j_0(\rho)|_{\rho = \rho_0(\alpha)} = \alpha \rho_0(\alpha). \] (2.19)
This is understood by noting that the average speed of each particle is the same as the average hopping probability at each site, which is equal to the average current \( \frac{2}{\eta} \) for \( q \)\-TAZRP and that the average current is given by this times the average density \( \rho \).

Next we explain how one can reduce the problem of the fluctuation of the position of a particle in stationary regime, with the conditioning that there is a particle at the origin initially, to the one of the \( N \)-particle \( q \)-TASEP with the initial condition (1.2). First, for \( q \)-TASEP, the proposition 2.1 implies that, in the stationary measure with a parameter \( \alpha \), the marginal dynamics of each particle is a Poisson random walk with rate \( \alpha \). Therefore, if one is only interested in particles starting from the left of a specific particle, one can replace the whole dynamics of particles to the right of that particles by postulating that the particle is performing the Poisson random walk with parameter \( \alpha \), see the bottom-left figure of Fig. 1. Because of the conditioning of the presence of a particle at the origin at \( t = 0 \), we put this particle at the origin initially. Next since in TASEP particles can not give any influence on the particles ahead, when we are interested in the fluctuation of the position of a particle we can study the fluctuation of the position of a particle by considering \( N \)-TASEP, but the current functions of \( q \)-TASEP, and hence also their characteristic lines, are different.) We will study the fluctuation of the position of the \( N \)-th particle when a nontrivial fluctuation is expected, that is when it comes around the characteristic line. For a given density \( \rho \) (or for a given \( \alpha \) through \( \rho = \rho_0(\alpha) \)),

\[
J_0' |_{\rho=\rho_0(\alpha)} = \alpha + \alpha_0' (\rho_0) \rho_0 (\alpha) = \alpha + \rho_0 (\alpha) / \rho_0 (\alpha) \tag{2.20}
\]

determines the characteristic line. If we parametrize \( \alpha = q^\theta, 0 < \theta < \infty \), for a given large \( N \), the time and the position that the \( N \)-th particle is around this characteristic line are given by \( t = \kappa N, x = (\eta - 1)N \) with

\[
\kappa = \frac{\Phi_q (\theta)}{(\log q)^2 q^\theta} = \sum_{n=0}^{\infty} \frac{q^n}{(1 - q^{\theta + n})^2} \tag{2.21}
\]

\[
\eta = \frac{\Phi_q' (\theta)}{(\log q)^2} - \frac{\Phi_q (\theta)}{\log q} - \frac{\log (1 - q)}{\log q} = \sum_{n=0}^{\infty} \frac{q^{2\theta + 2n}}{(1 - q^{\theta + n})^2}. \tag{2.22}
\]
where we used the $q$-digamma function $\Phi_q(z) = \partial_z \log \Gamma_q(z)$, see Appendix A. This is confirmed by checking that the average distance traveled by the particle is equal to the speed times time,

$$\alpha \times \kappa N = (\eta - 1)N + \frac{1}{\rho} N, \quad (2.23)$$

and that $t = \kappa N, x = (\eta - 1)N$ in fact is on the characteristic line, i.e.

$$j'_{0}(\rho) \kappa N = (\eta - 1)N \quad (2.24)$$

with $j'_{0}(\rho)$ given by (2.20).

We are interested in the fluctuation of the particle position around this macroscopic value. If one applies again the KPZ scaling theory [33, 77] to our $q$-TASEP (here again the reasonings are the same as for $q$-TAZRP but note the differences of the current functions and the stationary measures), it is expected we should scale the position as

$$\frac{X^{(0)}_N(\kappa: N) - (\eta - 1)N}{\gamma N^{1/3}} \quad (2.25)$$

where

$$\gamma = -\frac{1}{\log q} \left( \frac{\Phi'_q(\theta) \log q - \Phi'_q'(\theta)}{2} \right)^{1/3} = \left( \sum_{n=0}^{\infty} \frac{q^{2\theta+2n}}{(1-q^{\theta+n})^3} \right)^{1/3}. \quad (2.26)$$

In fact one can check that this scaling for the Nth particle position of $q$-TASEP is equivalent to (2.10) for $q$-TAZRP. First note $h(N, t) = X^{(0)}_N(t) + N$. By changing the large parameter from $t$ to $N$ by $N = j'(\rho)t$ and observing

$$\frac{1}{j'(\rho)} = \kappa, \quad \frac{j(\rho)}{j'(\rho)} - \rho = \eta, \quad -c(\rho)^3/j'(\rho) = \gamma^3, \quad (2.27)$$

one sees that the rhs of (2.10) is written as (2.25). The fact that the coefficients $\kappa, \eta, \gamma$ are consistent with the KPZ scaling theory was already confirmed in [33] (though with different character sets $\kappa, f, \chi$) but a remark here is that the calculations of the coefficients are somewhat easier in $q$-TAZRP language because the stationary measure of $q$-TAZRP is written in a product form.

So far our discussions have been for a fixed $\alpha$, but it is useful to also consider the scaling of $\alpha$ as

$$\alpha = q^\theta \left( 1 + \frac{\omega}{\gamma N^{1/3}} \right), \quad (2.28)$$

for a fixed $\theta > 0$. This scaling limit will be discussed in subsection 5.3.

3 Two-sided $q$-Whittaker process and $q$-TASEP

3.1 Definitions

For the case of the step initial condition, the dynamics of particles in $q$-TASEP can be described in terms of partitions, which are $n$-tuples of non-increasing non-negative integers [15]. To study the $q$-TASEP for a random initial condition, it is useful to consider $n$-tuples of non-increasing integers, each of which can take a value from $\mathbb{Z}$,

$$S_n := \{ \lambda = (\lambda_1, \cdots, \lambda_n) \in \mathbb{Z}^n | \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n \}. \quad (3.1)$$
Note that a signature $\lambda \in S_n$ becomes an ordinary partition when $\lambda_n \geq 0$. The set $S_n$ can be seen as a discrete version of the Weyl chamber but here we call an element $\lambda \in S_n$ a signature of length $n$. In this section we discuss this type of generalizations of partitions, $q$-Whittaker functions and so on. The Schur ($q = 0$) case is discussed in [14]. Some definitions and properties of the ordinary partitions and the $q$-Whittaker functions will be explained and discussed in Appendix B.

We also consider a set of $N(N + 1)/2$-tuples of integers with interlacing conditions,

$$
G_N := \{(\lambda^{(1)}, \lambda^{(2)}, \ldots, \lambda^{(N)}), \lambda^{(n)} \in S_n, 1 \leq n \leq N | \lambda^{(m+1)}_\ell \leq \lambda^{(m)}_\ell \leq \lambda^{(m+1)}_{\ell+1}, 1 \leq \ell \leq m \leq N - 1\}.
$$

(3.2)

See Fig. 3. Note that an element of $\Delta_N \in G_N$ can also be regarded as a point in $\mathbb{Z}^{N(N+1)/2}$, with the above interlacing conditions. We call $G_N, N \in \mathbb{N}$ the Gelfand-Tsetlin (GT) cone for signatures. Recall that the ordinary GT cone $G^{(0)}$ is given by (B.1), in which the sum is over the partitions rather than the signatures. The (ordinary) $q$-Whittaker process, introduced in [15], is defined as a process on $G^{(0)}_N$ and a marginal of the process gives the $q$-TASEP with the step initial condition. As a generalization, we will see in the following that a marginal of a process on our generalized GT cone $G_N$ gives the $q$-TASEP with the initial condition (1.2) with (2.5).

First we introduce the (skew) $q$-Whittaker function labeled by signatures.

**Definition 3.1.** Let $\lambda \in S_n, \mu \in S_{n-1}$ be two signatures of length $n$ and $n - 1$ respectively and $a$ an indeterminate. The skew $q$-Whittaker function (with 1 variable) is defined as

$$
P_{\lambda/\mu} (a) = \prod_{i=1}^{n} a^{\lambda_i} \cdot \prod_{i=1}^{n-1} \frac{a^{-\mu_i(q; q)_{\lambda_i-\lambda_{i+1}}}}{(q; q)_{\lambda_i-\mu_i(q; q)_{\mu_i-\lambda_{i+1}}}}.
$$

(3.3)

Using this, for a signature $\lambda \in S_N$ and $N$ indeterminates $a = (a_1, \ldots, a_N)$, we define the $q$-Whittaker function with $N$ variables as

$$
P_{\lambda} (a) = \sum_{\lambda^{(k)}_1 \leq \lambda^{(k)}_2 \leq \ldots \leq \lambda^{(k)}_{N-1}} \prod_{j=1}^{N} P_{\lambda^{(j)}/\lambda^{(j-1)}} (a_j).
$$

(3.4)

Here the sum is over the Gelfand-Tsetlin cone $G_N$ with the condition $\lambda^{(N)} = \lambda$.

Note that the notation $a$ is used for both 1 variable and $N$ variable cases. In this paper the skew functions always have only one variable. When $\lambda$ and $\mu$ are ordinary partitions, these definitions reduce to the
combinatorial definitions of the skew $q$-Whittaker function (with one variable) (3.2) and the $q$-Whittaker function (3.3) respectively [15,59].

Next we define another function labeled by a signature.

**Definition 3.2.** For a signature $\lambda$ of length $N$ (3.1), $t > 0$ and $\alpha = (\alpha_1, \ldots, \alpha_N) \in [0, 1)^N$, we define

$$Q_{\lambda}(\alpha, t) = \prod_{i=1}^{N-1} (q^{\lambda_i - \lambda_{i+1} + 1}; q)_\infty \int_{\mathbb{T}^N} \prod_{i=1}^{N} \frac{dz_i}{z_i} \cdot P_{\lambda}(1/z) \Pi(z; \alpha, t) m_N^q(z),$$

where $z = (z_1, \ldots, z_N)$,

$$m_N^q(z) = \frac{1}{(2\pi i)^N N!} \prod_{1 \leq i < j \leq N} (z_i/z_j; q)_\infty (z_j/z_i; q)_\infty$$

is the $q$-Sklyanin measure,

$$\Pi(z; \alpha, t) = \prod_{i,j=1}^{N} \frac{1}{(\alpha_i/z_j; q)_\infty} \cdot \prod_{j=1}^{N} e^{z_j t},$$

and $1/z$ in $P_{\lambda}$ is a shorthand notation for $(1/z_1, \ldots, 1/z_N)$.

When $\alpha_j = 0, j = 1, \ldots, N$, this function $Q_{\lambda}(\alpha, t)$ becomes zero unless $\lambda \in \mathbb{G}^0_N$ and on $\mathbb{G}^0_N$ reduces to a representation of the $q$-Whittaker function using the torus scalar product with the Plancherel specialization, see (B.15) with the remark below it. The case with nonzero $\alpha_j$ is not included as a special case of the nonnegative specialization defined for the ordinary $q$-Whittaker functions and treated in Sec.2.2.1 of [15]. But one can see that (3.5) is in fact nonnegative. The notation $\alpha$ here is for $N$ variables, but when we discuss the $q$-TASEP we consider the case where only one of them is nonzero, whose value is denoted by $\alpha \in [0, 1)$ corresponding to the parameter in the last section.

Let us discuss a few properties of the $q$-Whittaker functions labeled by signatures. When two signatures $\lambda, \nu \in S_n$ are related by

$$\nu_j = \lambda_j + k, \quad k \in \mathbb{Z}, \quad 1 \leq j \leq n,$$

let us call $\nu$ to be the shift of $\lambda$ by $k$. Suppose that $\nu, \tau \in S_n, \tau \in S_{n-1}$ are the shifts of $\lambda \in S_n, \mu \in S_{n-1}$ respectively by the same $k \in \mathbb{Z}$. Then the skew $q$-Whittaker function (3.3) of them are related by

$$P_{\lambda/\mu}(a) = a^{-k} P_{\nu/\tau}(a).$$

Accordingly, when $\nu \in S_N$ is the shift of $\lambda \in S_N$ by $k \in \mathbb{Z}$, the $q$-Whittaker function (3.4) satisfies

$$P_{\lambda}(a) = \prod_{\ell=1}^{N} a^{-k}_{\ell} \cdot P_{\nu}(a).$$

For a given signature $\lambda = (\lambda_1, \ldots, \lambda_n)$ of length $n$, one can define another signature of the same length, which we call the negation of $\lambda$ and denote by $-\lambda$, by

$$-\lambda = (-\lambda_n, -\lambda_{n-1}, \ldots, -\lambda_1).$$

Also, for a given element $\lambda_N = (\lambda^{(1)}, \ldots, \lambda^{(N)})$ in the Gelfand-Tsetlin cone $\mathbb{G}_N$, we define another element $-\lambda_N \in \mathbb{G}_N$ by

$$-\lambda_N = (-\lambda^{(1)}, \ldots, -\lambda^{(N)}).$$

The (skew) $q$-Whittaker functions introduced above has simple symmetry properties with respect to this negation.
Lemma 3.3. Let \( \lambda \in S_n, \mu \in S_{n-1} \) be two signatures of length \( n \) and \( n-1 \) respectively. We have

\[
P_{\lambda/\mu}(a) = P_{(-\lambda)/(-\mu)}(1/a).
\] (3.13)

For a signature \( \lambda \) of length \( N \) and \( a = (a_1, \ldots, a_N), \alpha = (\alpha_1, \ldots, \alpha_N), \) we have

\[
P_\lambda(a) = P_{-\lambda}(1/a),
\] (3.14)

\[
Q_\lambda(\alpha, 0) = \begin{cases} Q_{-\lambda}(\alpha), & \text{when } \lambda^{(N)} \leq \cdots \leq \lambda_1^{(N)} \leq 0, \\ 0, & \text{otherwise}, \end{cases}
\] (3.15)

where \( Q_\lambda(x), \lambda \in \mathcal{P}_N \) is defined in (B.3). For the \( q \)-Sklyanin measure (3.6) there is a related symmetry,

\[
m_q(z) = m_q(1/z).
\] (3.16)

Proof. The relations (3.13), (3.14) and (3.16) are obvious from the definitions (3.3), (3.4) and (3.6).

For (3.15), we first note that \( \Pi(e^{-i\theta_j}; \alpha, 0) = \prod_{i,j=1}^N \frac{1}{\alpha_i e^{i\theta_j q N}} \) is a function of \( \theta_1, \ldots, \theta_N \) and is periodic with the period \( 2\pi \) for each \( \theta_j \). Since \( 0 \leq \alpha_i < 1, 1 \leq i \leq N \) it is expanded uniquely as the Fourier series with some coefficients \( d_m \),

\[
\Pi \left( e^{-i\theta_j}; \alpha, 0 \right) = \sum_{m \in \mathbb{Z}^N} d_m \prod_{j=1}^N e^{im_j \theta_j}.
\] (3.17)

In addition, noting \( \Pi(e^{-i\theta_j}; \alpha, 0) \) is symmetric in \( \theta_1, \ldots, \theta_N \), we further write

\[
\Pi \left( e^{-i\theta_j}; \alpha, 0 \right) = \sum_{m \in \mathcal{S}_N} \tilde{d}_m \prod_{\sigma \in \mathcal{P}_N} \prod_{j=1}^N e^{im_{i\sigma(j)} \theta_j} = \sum_{m \in \mathcal{S}_N} \tilde{d}_m e^{im_N \sum_{j=1}^N \theta_j} \prod_{\sigma \in \mathcal{P}_N} \prod_{j=1}^N e^{i(m_{i\sigma(j)} - m_N) \theta_j},
\] (3.18)

where \( \mathcal{P}_N \) denotes the set of all permutations of \( \{1, \ldots, N\} \). Since \( \sum_{\sigma \in \mathcal{P}_N} \prod_{j=1}^N e^{i(m_{i\sigma(j)} - m_N) \theta_j} \) is symmetric in \( \theta_1, \ldots, \theta_N \) and \( m_{i\sigma(j)} - m_N \geq 0 \), we see that it is uniquely expanded by the \( q \)-Whittaker function with some coefficients \( \tilde{\gamma}_\nu \) as

\[
\sum_{\sigma \in \mathcal{S}_N} \prod_{j=1}^N e^{i(m_{i\sigma(j)} - m_N) \theta_j} = \sum_{\nu \in \mathcal{P}_N} \tilde{\gamma}_\nu \nu_\nu(e^{i\theta}).
\] (3.19)

Combining (3.18) with (3.19) and noting (3.10), we see that \( \Pi \left( e^{-i\theta_j}; \alpha, 0 \right) \) can be expanded by \( P_{\lambda}(e^{i\theta}) \) with \( \lambda \in \mathcal{S}_N \),

\[
\Pi \left( e^{-i\theta_j}; \alpha, 0 \right) = \sum_{m \in \mathcal{S}_N} \tilde{d}_m e^{im_N \sum_{j=1}^N \theta_j} \sum_{\nu \in \mathcal{P}_N} \tilde{\gamma}_\nu \nu_\nu(e^{i\theta}) = \sum_{\lambda \in \mathcal{S}_N} \gamma_\lambda P_{\lambda}(e^{i\theta}),
\] (3.20)

with some coefficient \( \gamma_\lambda \).

Next we discuss the orthogonality of \( P_{\lambda}(z) \). When \( \lambda \) is a partition, it is well-known that \( P_{\lambda}(z) \)'s are orthogonal with respect to the torus scalar product (B.3), see (B.11). But in fact we can see that the same orthogonality relation holds even for our generalized case, in which \( \lambda \) is a signature. Note that the relation (B.12) and thus the first equality in (B.13) hold also for the signatures. Furthermore, one sees in (B.12), \( m \in \mathbb{Z}_{\geq 0}^{N-1} \) even in the case \( \mu \in \mathcal{S}_N \). Thus we can apply (B.14) to the first relation in (B.13) and find that the orthogonal relation (B.11) holds also for the case \( \mu, \lambda \in \mathcal{S}_N \).
Using this, we find that the coefficient $\gamma_\lambda$ in (3.20) can be expressed as

$$\gamma_\lambda = \prod_{i=1}^{N-1} (q^{\lambda_i - \lambda_{i+1} + 1}; q)_\infty \left( \prod (e^{-i\theta}; \alpha, 0), P_\lambda(e^{i\theta}) \right)^t$$

$$= \prod_{i=1}^{N-1} (q^{\lambda_i - \lambda_{i+1} + 1}; q)_\infty \int_{[0, 2\pi]^N} \prod_{j=1}^N d\theta_j \cdot P_\lambda(e^{-i\theta}) \Pi(e^{-i\theta}; \alpha, 0) m_N^q(e^{i\theta}).$$

(3.21)

By (3.14), (3.16) and (3.5), this is written as

$$\gamma_\lambda = \prod_{i=1}^{N-1} (q^{\lambda_i - \lambda_{i+1} + 1}; q)_\infty \int_{[0, 2\pi]^N} \prod_{j=1}^N d\theta_j \cdot P_{-\lambda}(e^{-i\theta}) \Pi(e^{i\theta}; \alpha, 0) m_N^q(e^{i\theta}) = Q_{-\lambda}(\alpha, 0).$$

(3.22)

On the other hand, using the Cauchy identity (B.6) in (3.21), we find

$$\gamma_\lambda = \begin{cases} Q_\lambda(\alpha), & \lambda \text{ is a partition,} \\ 0, & \text{otherwise.} \end{cases}$$

(3.23)

Comparing these two expressions (3.22) and (3.23), we get (3.15).

3.2 Two-sided $q$-Whittaker process

Using Definitions 3.1 and 3.2 we introduce a measure on $\mathbb{G}_N$.

**Definition 3.4.** For $\Delta_N \in \mathbb{G}_N$, we define

$$P_t(\Delta_N) := \frac{\prod_{j=1}^N P_{\lambda(j)/\lambda(j-1)}(a_j) \cdot Q^{(N)}(\alpha, t)}{\Pi(a; \alpha, t)}.$$ 

(3.24)

We call this the two-sided $q$-Whittaker process. At this point it is not obvious that the rhs of (3.24) indeed gives a probability measure but we will see it is indeed the case in the next subsection.

When $\alpha_j = 0$ for all $j = 1, 2, \cdots, N$, this reduces to the ascending $q$-Whittaker process with the Plancherel specialization [15], for which it has been shown that the marginal with respect to $(\lambda^{(1)}_1, \cdots, \lambda^{(N)}_N)$ describes the probability density function of the particle positions in the $q$-TASEP with step initial condition at time $t$.

We will show that the $P_t(\Delta_N)$ is a pdf for a dynamics of interacting random walkers on $\mathbb{G}_N$ at time $t$. Furthermore we show that its marginal density of $(\lambda^{(1)}_1, \cdots, \lambda^{(N)}_N)$ in the special case where only one of $\alpha_j$ remains finite and all the other ones are zero, describes the $q$-TASEP with the initial condition (1.2) defined in the previous section.

First we show that at $t = 0$ the measure (3.24) is the pure $\alpha$ specialization [15] for $-\lambda$. Substituting (3.13) and (3.15) into (3.24), we have

**Proposition 3.5.** At $t = 0$, $P_0(\Delta_N)$ has a support on $-\mathbb{G}^{(0)}_N$ (B.1) and on this support it is written as the $q$-Whittaker process with pure $\alpha$ specialization, i.e., with $\mu_N = -\Delta_N$ defined by (3.12),

$$P_0(\Delta_N) = \begin{cases} \prod_{j=1}^N P_{\mu(j)/\mu(j-1)}(1/a_j) Q^{(N)}(\alpha) \Pi(a; \alpha, 0), & \mu_N \in \mathbb{G}^{(0)}_N, \\ 0, & \mu_N \notin \mathbb{G}^{(0)}_N, \end{cases}$$

(3.25)

where $P_{\mu(j)/\mu(j-1)}$ is the skew $q$-Whittaker function (B.2) and $Q^{(N)}(\alpha)$ is given by (B.3). In particular $P_0(\Delta_N)$ is a pdf on $-\mathbb{G}^{(0)}_N$. 

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The q-TASEP with the initial condition (12) corresponds to the case in which one of \( \alpha_j = \alpha, 1 \leq j \leq N \) and \( \alpha_k = 0, k \neq j \). But since \( Q^{(N)}_\mu(\alpha) \) is a symmetric function, the value of \( P_0(\Lambda_N) \) in (3.25) does not depend on which \( \alpha_j \) is nonzero. The Proposition 3.5 restricted to this case is the following:

**Proposition 3.6.** In the case where one of \( \alpha_j = \alpha \) and \( \alpha_k = 0, k \neq j, 1 \leq j, k \leq N \), we have, for \( \Lambda_N \in \mathbb{R}_+^N \),

\[
P_0(\Lambda_N) = \prod_{j=1}^N \left( \frac{\alpha_j}{a_j}; q \right)_\infty \frac{(\alpha/a_j)^{\lambda_j(j-1)-\lambda_j(j)}}{(q; q)_{\lambda_j(j-1)-\lambda_j(j)}} \cdot \prod_{k=1}^{N-1} \prod_{j=1}^N \delta_{\lambda_j(k), 0}, \tag{3.26}
\]

This vanishes unless \( \lambda_j(k) = 0, 1 \leq j < k \leq N \) and \( \lambda^{(N)}_N \leq \lambda^{(N-1)}_N \leq \cdots \leq \lambda^{(1)}_1 \leq 0 \). Focusing on \( \lambda^{(j)}_j, 1 \leq j \leq N \) and comparing with (1.2), we see that the marginal density at time \( t = 0 \) is exactly the initial condition (12) for the q-TASEP.

**Proof.** As we mentioned above, (3.25) is independent of which \( \alpha_j \) remains finite. We adopt the choice \( j = N \), i.e. \( \alpha_1 = \cdots = \alpha_{N-1} = 0, \alpha_N = \alpha \), where we can give the simplest proof.

First we consider the factor

\[
Q^{(N)}(\alpha, t = 0) = Q^{(N)}_\mu(0, \cdots, 0, \alpha) = \frac{P^{(N)}_\mu(0, \cdots, 0, \alpha)}{(q; q) \prod_{j=1}^{N-1} (q; q)_{\mu(j)-\mu(j+1)}}, \tag{3.27}
\]

where in the first equality we used (3.15) and in the second the definition (B.5) with (B.4). By the definition (B.3), \( P^{(N)}_\mu(0, \cdots, 0, \alpha) \) is given as

\[
P^{(N)}_\mu(0, \cdots, 0, \alpha) = \sum_{\mu^{(1)}, \cdots, \mu^{(N-1)}} \prod_{j=1}^{N-1} P^{(j-j-1)}_\mu(0) \cdot P^{(N-1)}_\mu(\alpha) \\
= \sum_{\mu^{(1)}, \cdots, \mu^{(N-1)}} \prod_{j=1}^{N-1} \delta_{\mu(j), 0} \prod_{i=1}^N \alpha^{\mu(i)} \prod_{i=1}^{N-1} \frac{(q; q)_{\mu(i)-\mu(i+1)}}{(q; q)_{\mu(i)}(q; q)_{-\mu(i+1)}} \\
= \prod_{i=1}^N \alpha^{\mu(i)} \prod_{j=1}^{N-1} \frac{(q; q)_{\mu(j)-\mu(j+1)}}{(q; q)_{\mu(j)}(q; q)_{-\mu(j+1)}} = \alpha^{\mu(1)} \prod_{j=1}^{N-1} \delta_{\mu(j+1), 0}, \tag{3.28}
\]

where we wrote the summation in (B.3) as \( \sum_{\mu^{(1)}, \cdots, \mu^{(N-1)}} \) and in the second equality we used the fact \( P^{(j-j-1)}_\mu(0) = \delta_{\mu(j), \mu(j-1)} \) while in the last equality we used another fact: for \( n \in \mathbb{N}, 1/(q; q)_n = (q^{n+1}; q)_\infty/(q; q)_\infty = \delta_{n, 0} \). Thus from (3.27) and (3.28), we see that

\[
Q^{(N)}_\mu(0, \cdots, 0, \alpha) = \frac{\alpha^{\mu(1)}(q; q)_{\mu(1)} \prod_{j=1}^{N-1} \delta_{\mu(j+1), 0}}{(q; q)_{\mu(1)}} \tag{3.29}
\]

and that \( P_0(\Lambda_N) \) can be written as

\[
P_{t=0}(\Lambda_N) = \prod_{j=1}^N \left( \frac{\alpha_j}{a_j}; q \right)_\infty \prod_{j=1}^{N-1} P^{(j-j-1)}_\mu(0) \cdot P^{(N-1)}_\mu \left( \frac{1}{a_N} \right) \cdot P^{(N-1)}_\mu \left( \frac{1}{a_N} \alpha^{\mu(1)}(q; q)_{\mu(1)} \prod_{j=1}^{N-1} \delta_{\mu(j+1), 0}. \tag{3.30}
\]

\[16\]
Here we find that the last factor in this equation can be calculated as

\[
P_{\mu^{(N)}}/\mu^{(N-1)} \left( \frac{1}{a_N} \right) \left( \frac{\alpha \mu^{(N)}_1}{(q; q) \mu^{(N)}_1} \right) \prod_{j=1}^{N-1} \delta_{\mu^{(N)}_j, 0} \\
= \left( \frac{1}{a_N} \right)^{\mu^{(N)}_1 - \mu^{(N-1)}_1} \left( \frac{(q; q) \mu^{(N)}_1}{(q; q) \mu^{(N-1)}_1 (q; q) \mu^{(N)}_1} \right) \prod_{j=2}^{N-1} \delta_{\mu^{(N)}_j, 0} \\
= \left( \frac{(q; q) \mu^{(N)}_1}{(q; q) \mu^{(N-1)}_1} \right)^{N-2} \prod_{j=1}^{N-2} \delta_{\mu^{(N)}_{j+1}, 0} \cdot \prod_{j=1}^{N-2} \delta_{\mu^{(N)}_{j+1}, 0}. \tag{3.31}
\]

Substituting this into (3.30), we get

\[
P_{t=0}(\Delta_N) = \prod_{j=1}^{N} \left( \frac{a_j}{a_N} - \frac{\alpha}{N} \right) \cdot \left( \frac{\alpha}{a_N} \right)^{\mu^{(N)}_1 - \mu^{(N-1)}_1} \prod_{j=1}^{N-1} \delta_{\mu^{(N)}_j, 0} \cdot \prod_{j=1}^{N-2} \delta_{\mu^{(N)}_{j+1}, 0} \\
\times P_{\mu^{(N-1)}}/\mu^{(N-2)} \left( \frac{1}{a_{N-1}} \right) \left( \frac{(q; q) \mu^{(N-1)}_1}{(q; q) \mu^{(N-1)}_1} \right)^{N-2} \prod_{j=1}^{N-2} \delta_{\mu^{(N-1)}_{j+1}, 0}. \tag{3.32}
\]

Note that the last factor in this equation is the same as the one in (3.30) with \( N \) replaced by \( N - 1 \) and (3.31) holds for any \( N = 1, 2, \ldots \). Applying (3.31) with \( N - 1, N - 2, \ldots, 1 \) repeatedly to (3.32) and remembering \( \mu_N = -\Delta_N \), we obtain (3.26).

Next we consider a time evolution property of \( P_t(\Delta_N) \) (3.24). For this purpose, for a \( \Delta_N \in \mathbb{G}_N \), we define \( \ell_{j,k} = \ell_{j,k}(\Delta_N) \) and \( m_{j,k} = m_{j,k}(\Delta_N) \) by the relations,

\[
\lambda_j^{(k-\ell_{j,k})} \ldots = \lambda_j^{(k-1)} = \lambda_j^{(k+1)} = \ldots = \lambda_j^{(k+m_{j,k})} < \lambda_j^{(k+m_{j,k}+1)}. \tag{3.33}
\]

Namely, for a given \( \Delta_N \), \( \ell_{j,k} \) (resp. \( m_{j,k} \)) denotes the number of particles on the GT cone which share the position and the lower indices with \( \lambda_j^{(k)} \) and whose upper indices are less (resp. bigger) than \( \lambda_j^{(k)} \). We also define \( \lambda_N^{k\pm} \) as

\[
\begin{align*}
(\lambda_N^{k+})_b^{(a)} &= \begin{cases} 
\lambda_b^{(a)} + 1, & a \in \{k, k+1, \ldots, k+m_{j,k}\} \text{ and } b = j, \\
\lambda_b^{(a)}, & \text{otherwise},
\end{cases} \\
(\lambda_N^{k-})_b^{(a)} &= \begin{cases} 
\lambda_b^{(a)} - 1, & a \in \{k - \ell_{j,k}, k - \ell_{j,k} + 1, \ldots, k\} \text{ and } b = j, \\
\lambda_b^{(a)}, & \text{otherwise}.
\end{cases}
\end{align*}
\tag{3.34, 3.35}
\]

In other words, \( \lambda_N^{k+} \) (resp. \( \lambda_N^{k-} \)) is the configuration of particles in \( \mathbb{G}_N \) in which the whole \( m_{j,k}+1 \) (resp. \( \ell_{j,k}+1 \)) particles in \( \Delta_N \) which share the positions with \( \lambda_j^{(k)} \) and whose lower indices are \( j \) and whose upper one is larger (resp. smaller) than \( k \) move one step to the right (resp. left). See Fig. 4.

Then we have the following Kolmogorov forward equation.

**Theorem 3.7.** For \( \Delta_N \in \mathbb{G}_N \), we have

\[
\frac{dP_t(\Delta_N)}{dt} = \sum_{\mu_N \in \mathbb{G}_N} P_t(\mu_N)L(\mu_N, \Delta_N) \tag{3.36}
\]
where the generator $L$ is expressed as
\[
L(\mu_N, \lambda_N) = \sum_{1 \leq j \leq k \leq N} r_{jk} \left( \delta_{\lambda_N - \mu_N^{jk+k}} - \delta_{\lambda_N - \mu_N^j} \right).
\] (3.37)

Here the rate $r_{jk} = r_{jk}(\mu_j^{(k-1)}, \mu_{j+1}^{(k-1)}, \mu_j^{(k)}, \mu_{j+1}^{(k)})$ is given by
\[
r_{jk} = a_k \frac{(1 - q^{\mu_j^{(k-1)} - \mu_j^{(k)}})(1 - q^{\mu_j^{(k)} - \mu_{j+1}^{(k+1)}})}{1 - q^{\mu_j^{(k)} - \mu_{j+1}^{(k-1)}}},
\] (3.38)
with the convention $\mu_0^{(j)} = \infty$, $\mu_{j+1}^{(j)} = -\infty$.

We find $L$ in (3.37) indeed satisfies the conditions of a generator of a Markov chain, see for instance the chapter 2 of \[58\]. It is easy to see $L(\mu_N, \lambda_N) \geq 0$ for $\mu_N \neq \lambda_N$ since $r_{jk} \geq 0$ for $\mu_N \in G_N$. One also finds $\sum_{\lambda_N} L(\mu_N, \lambda_N) = 0$, by noting that if $\mu_N \in G_N$, $r_{jk} = 0 \Leftrightarrow \mu_j^{(k)} = \mu_j^{(k+1)} \Leftrightarrow \mu_N^{j+k} \notin G_N$. (3.39)

This theorem tells us that $P_t(\lambda_N)$ represents a (discrete) probability density of an interacting particle system on $G_N$ (3.2) described by the following rules: Suppose for $j = 1, \cdots, k$, $k = 1, \cdots, N$ each particle labeled $(j, k)$ has the position $\lambda_j^{(k)}$ at time $t$ where $(\lambda^{(1)}, \cdots, \lambda^{(N)}) \in G_N$. Then the particle $(j, k)$ hops from $\lambda_j^{(k)}$ to $\lambda_j^{(k+1)}$ with rate $r_{jk}$. Furthermore when the particle $(j, k)$ hops, the other particles labeled $(j, k+1), \cdots, (j, k+m)$ are pushed to proceed forward one step, to keep the interlacing condition of the GT cone. Note that from (3.39), $r_{jk} = 0$ when $\lambda_j^{(k)} = \lambda_{j+1}^{(k-1)}$, which means that the particle $(j, k)$ is blocked by the particle $(j-1, k-1)$ when they have the same position, again to maintain the interlacing condition of the GT cone. In addition, one notices that the rules for $\lambda_j^{(k)}(t), 1 \leq j \leq N$ are independent from the rest and each $\lambda_j^{(k)}$ hops to the right with rate $a_j(1 - q^{\lambda_{j+1}^{(k)} - \lambda_j^{(k)}})$ unless $\lambda_j^{(k-1)} = \lambda_j^{(k)}$.

The above rules of the dynamics are the same as the $q$-Whittaker process with Plancherel specialization introduced in \[15\]. In particular, if we set $X_j(t) = \lambda_j^{(j)}(t) - j + 1$, the rules of the dynamics of $X_j(t)$ are exactly the ones of $q$-TASEP. In \[15\] this dynamics is shown to appear as a limiting case of a discrete time dynamics of the Macdonald process for the step initial condition (the last part of section 2.3 in \[15\]). See also Proposition 3.3.5 for a little reformulated proof for the discrete dynamics. Actually the generator (3.37) can be obtained by the expansion of the transition matrix $G_M(\mu_N, \lambda_N)$ of the discrete time case (3.23) as,
\[
G_M(\mu_N, \lambda_N) = \delta_{\mu_N, \lambda_N} + \epsilon L(\mu_N, \lambda_N) + O(\epsilon^2)
\] (3.40)
where in (B.23) we set $M = t\epsilon$ and expand $G_M(\mu,\lambda,\nu)$ in powers of $\epsilon$.

In the rest of this subsection, we give a proof of Theorem 3.7. Namely we will show that $P_t(\lambda)$ satisfies the Kolmogorov forward equation (3.36) directly, without detouring through a discrete time dynamics, for our random initial condition, using a few properties of the $q$-Whittaker functions. Below we will also use notation $P(\mu,\lambda; a) = P_{\lambda/\mu}(a), P(\lambda; a) = P_\lambda(a), Q(\lambda; \alpha, t) = Q_\lambda(\alpha, t)$ for the (skew) $q$-Whittaker functions to avoid complicated subscripts. We also use a notation $\lambda \pm 1_j$ to denote the Young diagram with one box added to (or removed from) $\lambda$. For the proof, we prepare three lemmas.

**Lemma 3.8.** The rate $r_{jk}$ (3.38) can be expressed as

$$r_{jk} = (1 - q^{\mu_j^{(k)} - \mu_j^{(k)}}) \frac{P(\mu^{(k)},\mu^{(k)} + 1_j; a_k)}{P(\mu^{(k)},\mu^{(k)}; a_k)}.$$  (3.41)

**Proof.** It follows immediately from the relation for the $q$-Whittaker functions,

$$P(\mu^{(k)},\mu^{(k)} + 1_j; a_k) = a_k \frac{1 - q^{\mu_j^{(k)} - \mu_j^{(k+1)}}}{1 - q^{\mu_j^{(k)} - \mu_j^{(k+1)}}} \frac{1 - q^{\mu_j^{(k)} - \mu_j^{(k+1)}}}{1 - q^{\mu_j^{(k)} - \mu_j^{(k+1)}}} P(\mu^{(k)},\mu^{(k)}; a_k),$$  (3.42)

which is obtained by the definition (B.2).

The next lemma is a direct consequence of the properties (B.18), (B.19) of the $q$-Whittaker functions.

**Lemma 3.9.** For $\lambda \in S_k$ and $\mu \in S_{k-1}$, $k = 1, 2, \ldots$, we have

$$\sum_{l=1}^k (1 - q^{\lambda_l - \lambda_l}) P(\mu,\lambda + 1_l; a_k)$$

$$= a_k P(\mu,\lambda; a_k) + \sum_{l=1}^{k-1} (1 - q^{\mu_l - \mu_l + 1}) P(\mu - 1_l,\lambda; a_k),$$  (3.43)

$$\frac{\partial}{\partial t} Q(\lambda; \alpha, t) = \sum_{l=1}^k (1 - q^{\lambda_l - \lambda_l + 1}) Q(\lambda - 1_l; \alpha, t)$$  (3.44)

with the convention $\lambda_0 = +\infty$.

The following special case of (3.43) will be useful in the following.

**Corollary 3.10.** For $\lambda \in S_k$ and $\mu \in S_{k-1}$, suppose $\mu = \lambda + 1_j$ holds for some $j$, $1 \leq j \leq k - 1$. Then we have

$$(1 - q^{\lambda_j - \lambda_j}) P(\mu,\lambda + 1_j; a_k) = (1 - q^{\mu_j - \mu_j + 1}) P(\mu - 1_j,\lambda; a_k).$$  (3.45)

**Proof.** Recall the skew $q$-Whittaker function $P(\mu,\lambda; a) = P_{\lambda/\mu}(a)$ vanishes if $\lambda/\mu$ is not a skew diagram. With the condition $\mu = \lambda + 1_j$, only the $j$th terms on both sides (3.43) remain.

**Proof of Lemma 3.9.** Remembering (3.39) and (3.10), and noting $\nu$ in (3.8) becomes an ordinary partition with positive integer values for sufficiently large $k$, one sees it is enough to show (3.43) and (3.44) for ordinary partitions.

First (3.43) comes from a property (B.18) of the $q$-Whittaker functions. Noting $Q_{\lambda/\mu}(r)$ with one variable (B.20) can be expanded as

$$Q_{\lambda/\mu}(r) = \delta_{\lambda,\mu} + r \sum_{j=1}^{\ell(\mu)} (1 - q^{\mu_j - \mu_j}) \delta_{\lambda,\mu+1_j} + O(r^2)$$  (3.46)

with
for small $r$ and comparing the coefficients of the $O(r^1)$ terms in both sides of (3.18) with $a = a_k$ and $\lambda, \mu$ exchanged, we obtain (3.43).

Next, for (3.44), we use the relation (1.19). In a similar way to the previous case, using (3.46) and the representation with the torus scalar product (cf (B.15) with (B.16)),

$$Q^{(3)}_{\lambda}(r_1, \cdots, r_M) = \prod_{i=1}^{k-1} (q^{\lambda_i - \lambda_{i+1} + 1}; q)_\infty \int_{\Gamma_k} \prod_{j=1}^k \frac{dz_j}{z_j} \cdot P\left(\lambda; 1/z\right) \prod_{i=1}^M \prod_{j=1}^k (1 + r_iz_j) m_N^q(z),$$

we compare the coefficients of $O(r^1)$ term in both hand sides of (1.19), which leads to the relation for $P_\lambda(z)$,

$$\sum_{j=1}^k (1 - q^{\lambda_j - 1}) P(\lambda - 1j; 1/z) = \sum_{j=1}^k z_j P(\lambda; 1/z).$$

(3.48)

Differentiating (3.2) this respect to $t$ and using (3.48), we get (3.44).

The last lemma provides a decomposition of the generator $L(\mu_N, \Delta_N)$ (3.37).

**Lemma 3.11.** The generator $L$ in (3.37) can be written as

$$L(\mu_N, \Delta_N) = \sum_{r=1}^N A_r(\mu_N, \Delta_N) + B(\mu_N, \Delta_N),$$

where

$$A_r(\mu_N, \Delta_N) = \sum_{1 \leq j \leq k \leq r - 1} (1 - q^{\mu_{j+1} - \mu_j}) \prod_{m=1}^{r-1} P(\lambda^{(m-1)}, \lambda^{(m)}; a_m) \prod_{m=k}^{r} P(\mu^{(m-1)}, \mu^{(m)}; a_m) \delta_{mjk, r-1-k} \delta_{\Delta_N, q^{k+}}$$

$$- \sum_{j=1}^{r-1} (1 - q^{\mu_{j+1} - \mu_j}) \frac{P(\mu^{(r-1)} - 1j; \mu^{(r)}; a_r)}{P(\mu^{(r-1)}, \mu^{(r)}; a_r)} \delta_{\Delta_N, q^{j+}} - a_r \delta_{\Delta_N, q^{j+}},$$

$$B(\mu_N, \Delta_N) = \sum_{1 \leq j \leq k \leq N} (1 - q^{\mu_{j+1} - \mu_j}) \prod_{m=1}^{r} P(\lambda^{(m-1)}, \lambda^{(m)}; a_m) \prod_{m=1}^{r} P(\mu^{(m-1)}, \mu^{(m)}; a_m) \delta_{mjk, r-1-k} \delta_{\Delta_N, q^{j+}}.$$

**Proof.** By (3.41), $L(\mu_N, \Delta_N)$ (3.37) is written as

$$L(\mu_N, \Delta_N) = \sum_{1 \leq j \leq k \leq N} (1 - q^{\mu_{j+1} - \mu_j}) \frac{P(\mu^{(k-1)}, \mu^{(k)} + 1j; a_k)}{P(\mu^{(k-1)}, \mu^{(k)}; a_k)} \left(\delta_{\Delta_N, q^{j+}} - \delta_{\Delta_N, q^{j+}}\right).$$

(3.52)

Using $1 = \sum_{r=k}^N \delta_{mjk, r-k}$, we further rewrite $L(\Delta_N, \mu_N)$ (3.52) as

$$L(\mu_N, \Delta_N) = \sum_{r=1}^N \sum_{1 \leq j \leq k \leq r} (1 - q^{\mu_{j+1} - \mu_j}) \frac{P(\mu^{(k-1)}, \mu^{(k)} + 1j; a_k)}{P(\mu^{(k-1)}, \mu^{(k)}; a_k)} \left(\delta_{mjk, r-k} \delta_{\Delta_N, q^{j+}} - \delta_{\Delta_N, q^{j+}}\right),$$

(3.53)

from which we find a decomposition of the generator (3.37) of the form (3.49) with $A_r, B$ replaced by

$$\tilde{A}_r(\mu_N, \Delta_N) = \sum_{1 \leq j \leq k \leq r-1} (1 - q^{\mu_{j+1} - \mu_j}) \frac{P(\mu^{(k-1)}, \mu^{(k)} + 1j; a_k)}{P(\mu^{(k-1)}, \mu^{(k)}; a_k)} \delta_{mjk, r-1-k} \delta_{\Delta_N, q^{j+}}$$

$$- \sum_{j=1}^r (1 - q^{\mu_{j+1} - \mu_j}) \frac{P(\mu^{(r-1)} - 1j; \mu^{(r)}; a_r)}{P(\mu^{(r-1)}, \mu^{(r)}; a_r)} \delta_{\Delta_N, q^{j+}},$$

(3.54)

$$\tilde{B}(\mu_N, \Delta_N) = \sum_{1 \leq j \leq k \leq N} (1 - q^{\mu_{j+1} - \mu_j}) \frac{P(\mu^{(k-1)}, \mu^{(k)} + 1j; a_k)}{P(\mu^{(k-1)}, \mu^{(k)}; a_k)} \delta_{mjk, r-1-k} \delta_{\Delta_N, q^{j+}}.$$

(3.55)
In fact in (3.53), the \((r+1)\)th summand, \(0 \leq r \leq N-1\), of the first term and the \(r\)th summand of the second term after the sum over \(k\) is taken gives (3.54) whereas the \(N\)-th summand of the first term gives \(\hat{B}\). Combining these two, we get the decomposition (3.49) with (3.54), (3.55). Below we show \(\hat{A}_r = A_r, \hat{B} = B\).

First we see \(\hat{A}_r = A_r\). Setting \(\mu = \mu^{(m-1)} + 1_j\) and \(\lambda = \mu^{(m)}\) for \(m = k+1, \cdots, r-1\) in (3.43), we see

\[
1 = \frac{1 - q^{(m-1)}_{\mu(m)} - \mu_j^{(m)}}{1 - q^{(m-1)}_{\mu(m)} - \mu_j^{(m)}} \frac{P(\mu^{(m-1)} + 1_j, \mu^{(m)} + 1_j, a_m)}{P(\mu^{(m-1)}, \mu^{(m)}, a_m)}
\]

where we used the fact that \(\mu_j^{(k)} = \cdots = \mu_j^{(r)}\) (since \(m_{jk} = r - k\) and

\[
P(\mu^{(m-1)} + 1_j, \mu^{(m)} + 1_i, a_m) = P(\mu^{(m-1)} + 1_j - 1_i, \mu^{(m)} + 1_i, a_m) = P(\mu^{(m-1)} + 1_j, \mu^{(m)} + 1_i, a_m) = 0
\]

for \(m \neq j\). Using (3.56) in the first term of (3.54) and (3.43) with \(\mu = \mu^{(r-1)}\) and \(\lambda = \mu^{(r)}\) in the second term, we have

\[
\hat{A}_r(\mu, \lambda) = \sum_{1 \leq j \leq k \leq r-1} (1 - q^{(k-1)}_{\mu(k)} - \mu_j^{(k)}) \frac{P(\mu^{(k-1)} + 1_j, a_k)}{P(\mu^{(k-1)}, \mu^{(k)}, a_k)}
\]

\[
\times \prod_{m=k+1}^{r-1} \frac{1 - q^{(m)}_{\mu(m)} - \mu_j^{(m)}}{1 - q^{(m)}_{\mu(m)} - \mu_j^{(m)}} \frac{P(\mu^{(m-1)} + 1_j, \mu^{(m)} + 1_j, a_m)}{P(\mu^{(m-1)}, \mu^{(m)}, a_m)} \delta_{m,j} \delta_{r-1-k} \delta_{\mu^{(k)+}}
\]

\[
- \sum_{j=1}^{r-1} (1 - q^{(r-1)}_{\mu(r-1)} - \mu_j^{(r-1)} + 1) \frac{P(\mu^{(r-1)} - 1_j, \mu^{(r)} + 1_i, a_r)}{P(\mu^{(r-1)}, \mu^{(r)}, a_r)} \delta_{\mu^{(r)+}} - a_r \delta_{\mu^{(r)+}}.
\]

Using

\[
(1 - q^{(k)}_{\mu(k)} - \mu_j^{(k)}) \prod_{m=k+1}^{r-1} \frac{1 - q^{(m)}_{\mu(m)} - \mu_j^{(m)}}{1 - q^{(m)}_{\mu(m)} - \mu_j^{(m)}} = 1 - q^{(r-1)}_{\mu(r-1)} - \mu_j^{(r-1)},
\]

\[
P(\mu^{(k-1)}, \mu^{(k)} + 1_j, a_k) \prod_{m=k+1}^{r-1} \frac{P(\mu^{(m-1)} + 1_j, \mu^{(m)} + 1_j, a_m)}{P(\mu^{(m-1)}, \mu^{(m)}, a_m)} = \prod_{m=k}^{r-1} P(\lambda^{(m-1)}, \mu^{(m)}, a_m)
\]

where \(\lambda^{(k-1)} = \mu^{(k-1)}, \lambda^{(m)} = \mu^{(m)} + 1_j\) for \(m = k, \cdots, r-1\), it is easy to see \(\hat{A}_r = A_r\).

Note that \(\hat{B}\) is equal to the first term of \(A_r\) with \(r = N + 1\). Thus we find \(\hat{B}\) is rewritten as (3.51) in a similar way to the first term of (3.50).

As we mentioned below Theorem 3.7, the generator \(L(\mu, \lambda)\) (3.37) appears in the continuous time limit of the transition matrix \(G_M(\mu, \lambda)\), see (3.40). The decomposition in the proposition is related to the one in (3.24) for the discrete-time dynamics, which is introduced for the proof of Proposition 3.5. Applying the small \(\epsilon\) expansion to the representation (3.24), one would see \(A_r\) (resp. \(B\)) in (3.49) corresponds to \(A_r^{(\beta)}\) (resp. \(B^{(\beta)}\)).

In addition, note that \(A_r^{(\beta)}\) can be written in two ways as

\[
A_r^{(\beta)}(r+1) = \sum_{\lambda^{(r-1)}} \frac{Q^{(\beta)}_{\lambda^{(r-1)}}(rM+1)}{P_{\lambda^{(r-1)}}(a)P^{(\beta)}_{\lambda^{(r-1)}}(rM+1)} = \frac{Q^{(\beta)}_{\lambda^{(r-1)}}(rM+1)}{(1 + a_r rM+1) \sum_p P^{(\beta)}_{\lambda^{(r-1)}}(a)Q^{(\beta)}_{\lambda^{(r-1)}}(rM+1)}.
\]

(3.61)
The first one is the definition itself (3.25) and the second one is obtained by applying (3.18) to \( \Delta(\lambda^{(j-1)}, \mu^{(j)}) \) in (3.25). The two expressions for \( A_r \) and \( B \) in the decomposition (3.53) correspond to this difference. The expressions \( \tilde{A}_r \) (3.51) and \( \tilde{B} \) (3.55) (resp. the expressions \( A_r \) (3.50) and \( B \) (3.54)) are associated with the first one (resp. second one) in (3.61). Here we gave a proof of the equivalence between the two expressions without passing through the discrete time dynamics. For the proof of Theorem 3.7, the expressions in the proposition are more useful.

**Proof of Theorem 3.7.** First from the definition (3.24) with (3.5) one observes

\[
\frac{\partial}{\partial t} P_l(\Delta_N) = -\sum_{r=1}^{N} a_r \cdot P_l(\Delta_N) + \prod_{l=1}^{N} \frac{P(\lambda^{(l-1)}, \lambda^{(l)}; a_l)}{\Pi(a; \alpha, t)} \frac{\partial}{\partial t} Q(\lambda^{(N)}; \alpha, t). \tag{3.62}
\]

Hence, for proving the relation, it is sufficient to show

\[
\sum_{\mu_N} P_l(\mu_N) A_r(\mu_N, \Delta_N) = -a_r P_l(\Delta_N), \tag{3.63}
\]

\[
\sum_{\mu_N} P_l(\mu_N) B(\mu_N, \Delta_N) = \prod_{l=1}^{N} \frac{P(\lambda^{(l-1)}, \lambda^{(l)}; a_l)}{\Pi(a; \alpha, t)} \frac{\partial}{\partial t} Q(\lambda^{(N)}; \alpha, t), \tag{3.64}
\]

where \( A_r(\Delta_N, \mu_N) \) and \( B(\mu_N, \Delta_N) \) are given as the second expressions (3.50), and (3.51) respectively. We use Lemma 3.11.

First we prove (3.63). We see that the part of \( \sum_{\mu_N} P_l(\mu_N) A_r(\mu_N, \Delta_N) \) coming from the first term of (3.50) can be written as

\[
\sum_{\mu_N} P_l(\mu_N) \sum_{1 \leq j \leq k \leq r-1} \left( 1 - q^{(r-1)}_{\mu_j} \right) \prod_{m=k}^{r-1} \frac{P(\lambda^{(m-1)}, \lambda^{(m)}; a_m)}{P(\mu^{(m-1)}, \mu^{(m)}; a_m)} \delta_{mjk}(\mu_N, \lambda_N) \cdot \frac{\partial}{\partial t} Q(\lambda^{(N)}; \alpha, t). \tag{3.65}
\]

Here we used the definition (3.24) and the fact that each factor of \( \prod_{l=1}^{N} P(\mu^{(l-1)}, \mu^{(l)}; a_l) \) from \( P_l(\mu_N) \) can be replaced by \( P(\lambda^{(l-1)}, \lambda^{(l)}; a_l) \) except \( l = r \) since \( \mu^{(l)} = \lambda^{(l)}, 1 \leq l \leq k-1, r+1 \leq l \leq N \) while the factor for \( l = r \) changes to \( P(\lambda^{(r-1)} - 1, \lambda^{(r)}; a_r) \) since \( \mu^{(r-1)} = \lambda^{(r-1)} - 1 \). We also rewrite the Kronecker’s deltas as

\[
\delta_{mjk}(\mu_N, \lambda_N) \cdot \frac{\partial}{\partial t} Q(\lambda^{(N)}; \alpha, t) = \delta_{j, r-1}(\Delta_N) \cdot \frac{\partial}{\partial t} Q(\lambda^{(N)}; \alpha, t). \tag{3.66}
\]

which follows from the definitions (3.33)-(3.35) (see also Fig. 1), and use \( \sum_{k=1}^{r-1} \delta_{j, r-1}(\Delta_N) \cdot \frac{\partial}{\partial t} Q(\lambda^{(N)}; \alpha, t) = \sum_{\mu_N} \delta_{\mu_N, \Delta_N} = 1. \)

Clearly the remaining parts of \( \sum_{\mu_N} P_l(\mu_N) A_r(\mu_N, \Delta_N) \), which come from the second and the last
terms of (3.50) become
\[- \sum_{\mu_N} P_t(\mu_N) \sum_{j=1}^{r-1} (1 - q^j_{j-1} - \mu^{(r-1)}_j + 1) \frac{P(\mu^{(r-1)} - 1_j, \mu^{(r)}; a_r)}{P(\mu^{(r-1)}, \mu^{(r)}; a_r)} \delta_{\Delta_N, \mu_N},\]
\[= - \sum_{j=1}^{r-1} (1 - q^{(r-1)}_{j-1} - \lambda^{(r)}_j) \prod_{l=1}^{N} P(\lambda^{(l-1)}, \lambda^{(l)}; a_l) \cdot P(\lambda^{(r-1)} - 1_j, \lambda^{(r)}; a_r) \frac{Q(\lambda^{(N)}; \alpha, t)}{\Pi(a; \alpha, t)} \]  
(3.67)
\[- \sum_{\mu_N} P_t(\mu_N)(a_r \delta_{\Delta_N, \mu_N}) = - a_r P_t(\Delta_N). \]  
(3.68)

Noting (3.65) and (3.67) cancel, only (3.68) remains and we obtain (3.63).

At last we prove (3.64). Note that $B(\mu_N, \Delta_N)$ (3.51) corresponds to the first term of $A_r(\mu_N, \Delta_N)$ (3.50) with $r = N + 1$. Thus as in (3.65), we get
\[\sum_{\mu_N} P_t(\mu_N)B(\mu_N, \Delta_N) = \sum_{j=1}^{N} (1 - q^{(N-1)}_{j-1} - \lambda^{(N)}_j) \prod_{l=1}^{N} P(\lambda^{(l-1)}, \lambda^{(l)}; a_l) \cdot \frac{Q(\lambda^{(N)} - 1_j; \alpha, t)}{\Pi(a; \alpha, t)}. \]  
(3.69)

Applying (3.14), we arrive at (3.64).

3.3 Marginal distributions

We will be interested in the distribution of $\lambda^{(N)}_N(t)$. To study it, it is useful to consider the marginal distribution of the two-sided $q$-Whittaker process with respect to the particles on the top line $\lambda^{(N)}$ ($= (\lambda^{(N)}_1, \cdots, \lambda^{(N)}_N)$) in the GT cone. By recalling the definition of the $q$-Whittaker function (3.4), one easily finds

**Proposition 3.12.** For $\lambda \in S_N$, 
\[\mathbb{P}[\lambda^{(N)}(t) = \lambda] = \frac{P_a(\alpha)Q(\alpha, t)}{\Pi(a; \alpha, t)}. \]  
(3.70)

We call this the two-sided $q$-Whittaker measure. Since this is the probability measure, the summation over $\lambda$ gives a version of the Cauchy identity for our case.

**Corollary 3.13.**
\[\sum_{\lambda} P_{\lambda}(a)Q_{\lambda}(\alpha, t) = \Pi(a; \alpha, t). \]  
(3.71)

When studying the distribution of $\lambda^{(N)}_N(t)$, the following representation will be useful.

**Proposition 3.14.** For $l \in \mathbb{Z}$,
\[\mathbb{P}[\lambda^{(N)}_N = l] = (q, q)_{N-1}^{l} \int_{\mathbb{T}^N} \prod_{j=1}^{N} \frac{dz_j}{z_j} \cdot \left(\frac{A}{Z}\right)^l m^q_N(z) \frac{\Pi(z; \alpha, t)}{\Pi(a; \alpha, t)} \cdot \frac{(A/Z; q)_{l}}{\prod_{i,j=1}^{N} (a_i/z_j; q)_{\infty}}. \]  
(3.72)

where $A = \prod_{i=1}^{N} a_i$ and $Z = \prod_{i=1}^{N} z_i$. 

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Proof of Proposition 3.14. Using (3.5) and (B.7), the two-sided $q$-Whittaker measure (3.70) is written as

$$
\mathbb{P}[\lambda^{(N)}(t) = \lambda] = \int_{T^N} \prod_{j=1}^{N} \frac{dz_j}{z_j} \cdot \left( \frac{A}{Z} \right)^{\lambda^{(N)}(z)} m_N^{(q)}(z) \prod_{\alpha; \alpha, t}^{(q; q)_{\infty}} \prod_{k=1}^{N-1} \frac{(q; q)_{\infty}}{(q; q)_{t_j}} \cdot R_{\ell}(a) R_{\ell}(1/z).
$$

Using the Cauchy identity in the form (B.8), we find

$$
\mathbb{P}[\lambda^{(N)}(t) = \lambda] = \sum_{\lambda \in S_N \text{ s.t. } \lambda_N = l} \mathbb{P}[\lambda^{(N)}(t) = \lambda] = \int_{T^N} \prod_{j=1}^{N} \frac{dz_j}{z_j} \cdot \left( \frac{A}{Z} \right)^{\lambda^{(N)}(z)} m_N^{(q)}(z) \prod_{\alpha; \alpha, t}^{(q; q)_{\infty}} \prod_{k=1}^{N-1} \frac{(q; q)_{\infty}}{(q; q)_{t_j}} \cdot R_{\ell}(a) R_{\ell}(1/z).
$$

$$
= (q; q)_{\infty}^{N-1} \int_{T^N} \prod_{j=1}^{N} \frac{dz_j}{z_j} \cdot \left( \frac{A}{Z} \right)^{l} m_N^{(q)}(z) \prod_{\alpha; \alpha, t}^{(q; q)_{\infty}} \prod_{k=1}^{N-1} \frac{(A/Z; q)_{\infty}}{(q; q)_{t_j} (a_i/\lambda_j; q)_{\infty}}.
$$

\square

4 Fredholm determinant formula for the $q$-Laplace transform

In this section, we derive a Fredholm determinant formula for the $q$-Laplace transform of the position of the $N$-th particle in $q$-TASEP with the initial condition (1.2). When taking the sum over the position of the particle and getting a determinantal expression, we need two essential ingredients. One is the Ramanujan’s summation formula and the other is the Cauchy determinant formula for the theta function. The former played an important role when studying the limit from ASEP to the KPZ equation in [71] but does not seem to have been utilized in the analysis of related models since then. Some part of the calculations in this section can be regarded as a generalization of those in [17].

Let us first recall the Ramanujan’s summation formula (A.7).

Theorem 4.1. For $|q| < 1, |b/a| < |z| < 1$, $a \notin q^n, n \in \mathbb{Z}$,

$$
\sum_{n \in \mathbb{Z}} \frac{(bq^n; q)_{\infty}}{(aq^n; q)_{\infty}} z^n = \frac{(az; q)_{\infty} (q/z; q)_{\infty} (a; q)_{\infty} (b; q)_{\infty}}{(a; q)_{\infty} (q; q)_{\infty} (z; q)_{\infty} (b; q)_{\infty}}.
$$

(4.1)

Next we introduce a modified Jacobi theta function ([36] p303),

$$
\theta(z) = (z; q)_{\infty} (q/z; q)_{\infty}, \quad |q| < 1, z \neq 0,
$$

(4.2)

which is related to the ordinary theta function as

$$
\theta_1(x, e^{\pi i \tau}) = \frac{1}{e^{-ix + \pi i \tau/4}} (q; q)_{\infty} \theta(e^{2ix}; q), \quad q = e^{2\pi i \tau}, \text{Im} \tau > 0, x \in \mathbb{C}.
$$

(4.3)

This function has been playing important role in various places. In the following we also use

$$
\tilde{\theta}(z) = \frac{1}{\sqrt{z}} \theta(z),
$$

(4.4)

which has a nicer symmetry property, $\tilde{\theta}(1/z) = \tilde{\theta}(z)$ where the square root is $\sqrt{z} = e^{\frac{1}{2} \log z}$ with the standard branch cut of logarithm.
Let \([x]\) be a nonzero holomorphic function which satisfies \([-x] = -[x]\) and the Riemann relation,

\[
[x + y][x - y][u + v][u - v] = [x + u][x - u][y + v][y - v] - [x + v][x - v][y + u][y - u].
\]

(4.5)

It is known that \([x]\) satisfying the above two relations is necessarily in the form \(e^{ax^2 + bf(x)}\) where \(f(x)\) is either \(f(x) = x\), \(f(x) = \sin \pi x\) or \(f(x) = \sigma(x)\) (cf [54] p415 20-53 ex.4 and p461 ex.38, [50]). Here the Weierstrass sigma function \(\sigma(x) = \sigma(x; \omega_1, \omega_2)\), with the half periods \(\omega_1, \omega_2\), can be written in terms of the ordinary theta function as (cf [84] p473 21-43)

\[
\sigma(x; \omega_1, \omega_2) = \frac{2\omega_1}{\pi \theta_1^{(1)}} \exp \left( -\frac{\pi^2 x^2 \theta_1^{(3)}}{24 \omega_1^2 \theta_1^{(1)}} \right) \theta_1 \left( \frac{\pi x}{2 \omega_1}, e^{i \frac{\pi x}{\omega_1}} \right)
\]

(4.6)

where \(\theta_1^{(n)} = \frac{d^n}{dx^n} \theta_1(x, q)|_{x=0}, n \in \mathbb{N}\). Combining (4.3), (4.6), one sees that our theta function \(\tilde{\theta}(q^*)\) is written in the form \(e^{ax^2 + b \sigma(x)}\) and hence an example of \([x]\).

**Theorem 4.2.** (cf [50]) For \([x]\) as above, the following Cauchy determinant formula holds,

\[
\frac{[\nu + B - C] \prod_{1 \leq i, j \leq N} [b_i - b_j] [c_i - c_j]}{[\nu] \prod_{i,j=1}^N [b_i - c_j]} = \text{det} \left( \frac{[\nu + b_i - c_j]}{[\nu] [b_i - c_j]} \right)_{1 \leq i, j \leq N}
\]

where \(\nu\) is a parameter, \(b_i, c_i, 1 \leq i \leq N\) are \(2N\) complex variables and \(B = \sum_{i=1}^N b_i, C = \sum_{i=1}^N c_i\).

Note that in a certain limit (\(\nu \to \infty\) for \(f(x) = x\), \(\nu \to i \infty\) for \(f(x) = \sin \pi x\) and for \(f(x) = \sigma(x)\) using \(\sigma(x + 2\omega_2) = -e^{2\eta_2(x+\omega_2)}\sigma(x), \eta_2 = -\frac{\pi^2 \omega_2 \theta_1^{(3)}}{12 \omega_1^2 \theta_1^{(1)}} - \frac{\pi i}{2 \omega_1}\), cf [84] p448-412), the factors including \(\nu\) on both sides of the identity cancel, leading to more familiar form of the Cauchy identity. This form of Cauchy identity for the theta function has already appeared in several contexts in random matrix theory (e.g. [34] 5.6.3), con-colliding diffusions [52-53] and so on. But in our discussions below, the formula with the extra parameter \(\nu\) plays an essential role.

Now we come back to the \(q\)-TASEP. By using the above two theorems, we will obtain

**Theorem 4.3.** For the two-sided \(q\)-Whittaker measure (3.70) with \(0 \leq \alpha_i < \alpha_j \leq 1, 1 \leq i, j \leq N\) and with \(\zeta \neq q^n, n \in \mathbb{Z}\),

\[
\langle \frac{1}{(\zeta q^n; q)_\infty} \rangle = \text{det} (1 - fK)_{\ell^2(\mathbb{Z})}
\]

(4.7)

where \(\langle \cdots \rangle\) means the average and the kernel of the Fredholm determinant on rhs is given by

\[
f(n) = \frac{1}{1 - q^n/\zeta},
\]

(4.8)

\[
K(n, m) = \sum_{l=0}^{N-1} \phi_1(m) \psi_1(n),
\]

(4.9)

\[
\phi_1(n) = \tau(n) \int_D dv \frac{e^{-vt}}{v^{n+1}} \prod_{j=1, a_j \neq 0}^l \frac{v - \alpha_j}{v - a_j} \prod_{k=1}^N \frac{(q\alpha_k/v; q)_\infty}{(qv/a_k; q)_\infty},
\]

(4.10)

\[
\psi_1(n) = \frac{a_{l+1} - a_l + 1}{\tau(n)} \int_{C_{\tau}} dz \frac{e^{zt}z^{n+1}}{z - a_{l+1}} \prod_{j=1, a_j \neq 0}^l \frac{z - \alpha_j}{z - a_j} \prod_{k=1}^N \frac{(qz/a_k; q)_\infty}{(q\alpha_k/z; q)_\infty}.
\]

(4.11)
Here the contour \( D \) is around \( \{ a_i, 1 \leq i \leq N \} \) and the contour \( C_r \) is around \( \{ 0, \alpha_i q^j, 1 \leq i \leq N, j \in \mathbb{N} \} \). See Fig. 5, \( \tau(n) \) is
\[
\tau(n) = \begin{cases} 
    b^n, & n \geq 0, \\
    c^n, & n < 0.
\end{cases}
\] (4.12)

where \( b, c \) are taken to satisfy \( 0 < \max \alpha_i < b < \min a_i \leq 1 < c \).

By the result in the previous section, we immediately have

**Corollary 4.4.** For \( q \)-TASEP with the initial condition \( (1.2) \) with \( 0 \leq \alpha < a_j \leq 1, 1 \leq j \leq N \) and with \( \zeta \neq q^n, n \in \mathbb{Z} \), the \( q \)-Laplace transform \( \langle q^{-T} X_N(t) + N; q \rangle^{\infty} \) for the \( N \)th particle position is written as the Fredholm determinant on rhs of (4.7) with (4.10), (4.11) specialized to \( \alpha_j = \alpha, \alpha_k = 0, k \neq j, 1 \leq j, k \leq N \).

In the rest of this section we will prove Theorem 4.3. First we discuss a few properties of the functions \( \phi_l(n), \psi_l(n) \) and the well-definedness of the Fredholm determinant on rhs.

**Proposition 4.5.** The functions \( \phi_l(n) \) and \( \psi_l(n) \) are biorthonormal, i.e.,
\[
\sum_{n \in \mathbb{Z}} \phi_l(n) \psi_m(n) = \delta_{l,m}.
\] (4.13)

**Proof.** This is easy to show by using their contour integral expressions.

**Lemma 4.6.** Fix \( t > 0, N \in \mathbb{Z}_{>0}, l \in \mathbb{N} \) and assume \( 0 < \max \alpha_i < b < \min a_i \leq 1 < c \). When \( a_i \)'s and \( \alpha_i \)'s are all different, the functions (4.10) and (4.11) have the following simple bounds:
\[
|\phi_l(n)| \leq \begin{cases} 
    C_{1+} b^n \left( \frac{1}{a_{\min}} \right)^n, & \text{for } n \geq 0, \\
    C_{1-} c^n \left( \frac{1}{a_{\max}} \right)^n, & \text{for } n \leq -1,
\end{cases}
\] (4.14)
\[
|\psi_l(n)| \leq \begin{cases} 
    C_{2+} a_{\max}^n / b^n, & \text{for } n \geq 0, \\
    C_{2-} d^n / c^n, & \text{for } n \leq -1,
\end{cases}
\] (4.15)

Here \( C_{1\pm}, C_{2\pm} > 0 \) are some constants (wrt \( n \)) and \( d \) is an arbitrary constant which is larger than 1. When some of the \( a_i \)'s or \( \alpha_i \)'s are the same, \( 1 / a_{\min} \) and \( a_{\max} \) (resp. \( 1 / a_{\max} \)) above should be replaced by a slightly larger (resp. smaller) value. Hence \( \phi_l \) and \( \psi_l \) are in \( \ell^2(\mathbb{Z}) \).

**Proof.** If we denote by \( \hat{\phi}_l, \hat{\psi}_l \) the rhs's of (4.10), (4.11) without the \( \tau \) factors, what we should show are (4.14), (4.15) without the factors in terms of \( b, c \). For \( \hat{\phi}_l \), when all \( a_i \)'s are different, by taking the poles in (4.10), it is easy to see,
\[
\hat{\phi}_l(n) = \sum_{i=1}^{N} C_i a_i^{-n},
\] (4.16)
with \( C_i, 1 \leq i \leq N \), some constants. When some of \( a_i \)'s are the same, polynomials in \( n \) appear in the coefficients but if one does the replacement stated in the Lemma, the following discussions would still be valid. Large \( n \) behaviors of \( \hat{\phi}_l(n) \) is dominated by the largest or smallest of \( a_i \)'s depending on the sign of \( n \). Multiplying \( \tau(n) \), we have (4.11).

Next we consider \( \psi_l(n) \). For positive \( n \), one first shrinks the contour in (5.21) to the one with radius which is slightly larger than \( \max \alpha_i \), which can be done without passing a singularity. We see the first case of (4.15). For negative \( n \), one can enlarge the contour as much as one wishes. If one takes the contour with radius \( d > 1 \), we have the bound in the second case of (4.15) after division by \( \tau(n) \).

By taking \( d > c \), one sees that the new \( \phi_l, \psi_l \) go to zero exponentially fast and hence they are in \( l^2(\mathbb{Z}) \).

**Proposition 4.7.** On rhs of (4.7) the operator \( fK \) is a trace-class operator and hence the Fredholm determinant is well-defined.

**Proof.** If one regards the kernel of \( fK \) as a product of two operators with kernels, \( \sqrt{T} \phi_l(m), \sqrt{T} \psi_l(n) \), the above Lemma [1.6] tells us that each operator is a Hilbert-Schmidt operator (Note that the sum over \( l \) is finite). Hence \( fK \) is a trace-class operator on \( l^2(\mathbb{Z}) \) (Th VI.22 in [68], see also Prop 2.4 in [47]). It is well-known that the Fredholm expansion of a trace-class kernel converges, giving the well-defined Fredholm determinant.

Now that a well-definedness of rhs of (4.7) has been checked, the remaining is to show how the formula (4.7) is derived.

**Proof of Theorem 4.3.** In this proof, products such as \( \prod_i \), run for \( i = 1, \ldots, N \) and determinants are \( N \) dimensionless unless otherwise stated. By Proposition 3.14 we have, with \( \{a_i\}_{1 \leq i \leq N}, \{\alpha_i\}_{1 \leq i \leq N} \) where \( |a_i|, |\alpha_i| < 1 \),

\[
\left\langle \frac{1}{(\zeta q^{\lambda^N}; q)_{\infty}} \right\rangle = \sum_{l \in \mathbb{Z}} \frac{1}{(\zeta q^l; q)_{\infty}} \int_{\mathbb{T}^N} \prod_{i=1}^{N} \frac{dz_i}{z_i} \left( \frac{A}{Z} \right)^l m_N(z) \frac{\Pi(z; \alpha, t)}{\Pi(a; \alpha, t)} \frac{q^{\lambda^N}(z)}{\prod_{i,j}(a_i/z_j; q)_{\infty}}, \tag{4.17}
\]

where \( A = \prod_{i=1}^{N} a_i, Z = \prod_{i=1}^{N} z_i \). The summation over \( l \) can be performed by using the Ramanujan’s formula with \( a = \zeta, b = 0, z = A/Z \),

\[
\sum_{l \in \mathbb{Z}} \frac{1}{(\zeta q^l; q)_{\infty}} \left( \frac{A}{Z} \right)^l = \frac{(\zeta A; q)_{\infty}(\zeta^2 Z; q)_{\infty}(q; q)_{\infty}}{(\zeta; q)_{\infty}(\zeta^2; q)_{\infty}(q; q)_{\infty}} = \frac{\theta(\zeta A)(q; q)_{\infty}}{\theta(\zeta)(\zeta^2; q)_{\infty}}. \tag{4.18}
\]

We find

\[
\left\langle \frac{1}{(\zeta q^{\lambda^N}; q)_{\infty}} \right\rangle = \frac{(q; q)_{\infty}^N}{N!} \int_{\mathbb{T}^N} \prod_{i=1}^{N} \frac{dz_i}{2\pi i z_i} \frac{\theta(A)}{\theta(\zeta)} \frac{\prod_{i,j}(z_i/z_j; q)_{\infty} \Pi(z; \alpha, t)}{\prod_{i,j}(a_i/z_j; q)_{\infty} \Pi(a; \alpha, t)}. \tag{4.19}
\]

Next we rewrite the two double products over \( i, j \) to apply the Cauchy determinant formula. Noting a simple fact

\[
(z/w; q)_{\infty}(w/z; q)_{\infty} = (1 - w/z)\theta(z/w), \tag{4.20}
\]

we have

\[
\prod_{i \neq j}(z_i/z_j; q)_{\infty} = \prod_{i < j}(1 - z_j/z_i) \prod_{i < j} \theta(z_i/z_j), \tag{4.21}
\]

\[
\prod_{i,j}(a_i/z_j; q)_{\infty} \prod_{i,j}(z_i/a_j; q)_{\infty} = \prod_{i,j}(1 - z_j/a_i) \prod_{i < j} \theta(a_i/z_j), \tag{4.22}
\]

\[
\prod_{i \neq j}(a_i/a_j; q)_{\infty} = \prod_{i < j}(1 - a_j/a_i) \prod_{i < j} \theta(a_i/a_j). \tag{4.23}
\]
and hence
\[ \frac{\prod_{i \neq j} (z_i/z_j; q)_{\infty}}{\prod_{i,j} (a_i/z_j; q)_{\infty}} = \frac{\prod_{i<j} (1 - z_j/z_i) \prod_{i,j} (z_i/a_j; q)_{\infty} \prod_{i<j} (1 - a_j/a_i) \prod_{i<j} (a_i/a_j)_{\infty}}{\prod_{i,j} (1 - z_j/a_i) \prod_{i,j} (a_i/z_j; q)_{\infty}} = \frac{\prod_{i \geq j} (a_i - a_j) \prod_{i \neq j} (z_i - z_j) \prod_{i \geq j} a_i \prod_{i,j} (a_i/z_j)_{\infty} \prod_{i \neq j} (a_i/a_j; q)_{\infty}}{\prod_{i \geq j} z_i \prod_{i,j} (a_i/z_j) \prod_{i \neq j} (a_i/a_j; q)_{\infty}}. \]

(4.24)

By rewriting
\[ \prod_{i \geq j} a_i \prod_{i < j} z_i = \prod_{i,j} \sqrt{a_i a_j z_j/a_i} \sqrt{Z/A} \]

(4.25)

and using the definition of \( \tilde{\theta}(z) \) \[ (4.4) \], we see
\[ \left\langle \frac{1}{{(\zeta q^{\lambda N}; q)_{\infty}}} \right\rangle = \frac{(q; q)^N}{N!} \int_{T^N} \prod_{i=1}^N \frac{dz_i}{z_i} \prod_{i < j}(a_i - a_j) \prod_{i < j}(z_i - z_j) \prod_{i < j} \tilde{\theta}(a_i/a_j) \prod_{i < j} \tilde{\theta}(z_i/z_j) \times \frac{\tilde{\theta}(z)}{\theta(\zeta)} \prod_i a_i \prod_{k \neq i} g(z_i/a_k; q)_{\infty} g(z_i; \alpha, t) \]

(4.26)

where
\[ g(z; \alpha, t) = \frac{e^{zt}}{\prod_{j} (a_j/z; q)_\infty}. \]

(4.27)

By using the Cauchy determinant formula for theta function here, we find
\[ \left\langle \frac{1}{{(\zeta q^{\lambda N}; q)_{\infty}}} \right\rangle = \frac{1}{N!} \int_{T^N} \prod_{i=1}^N \frac{dz_i}{z_i} \det \left( \frac{a_i}{a_i - z_j} \right) \det \left( \frac{\tilde{\theta}(\zeta a_i/z_j)}{\theta(\zeta) \tilde{\theta}(a_i/z_j)} \right) \prod_i \prod_{k \neq i} g(z_i/a_k; q)_{\infty} g(z_i; \alpha, t) \]

(4.28)

Note that the three \( \tilde{\theta} \)'s in the above determinant can be replaced by \( \theta \) because the square root factors in \[ (4.4) \] cancel in this combination. Using the Cauchy-Binet identity (or Andreief identity),
\[ \frac{1}{N!} \int \det(f_i(x))_{1 \leq i,j \leq N} \det(g_i(x))_{1 \leq i,j \leq N} \prod_{i=1}^N \frac{d\mu(x_i)}{z_i} = \det \left( \int f_i(x) g_j(x) d\mu(x) \right)_{1 \leq i,j \leq N}, \]

(4.29)

which holds for rather general \( f_i, g_i, \mu \) as long as the integrals in the formula make sense, we find
\[ \left\langle \frac{1}{{(\zeta q^{\lambda N}; q)_{\infty}}} \right\rangle = \det \left( \int \frac{dz}{z} \frac{a_i}{a_i - z} \theta(\zeta a_i/z) \frac{(q; q)_{\infty} \prod_k (z/a_k; q)_{\infty} g(z; \alpha, t)}{\prod_{k \neq j} (a_j/a_k; q)_{\infty} g(a_j; \alpha, t)} \right). \]

(4.30)

By making the contour smaller and taking the pole at \( z = a_i \),
\[ \left\langle \frac{1}{{(\zeta q^{\lambda N}; q)_{\infty}}} \right\rangle = \det \left( \int_{C_r} \frac{dz}{z} \frac{a_i}{a_i - z} \theta(\zeta a_i/z) \frac{(q; q)_{\infty} \prod_k (z/a_k; q)_{\infty} g(z; \alpha, t)}{\prod_{k \neq j} (a_j/a_k; q)_{\infty} g(a_j; \alpha, t)} \right). \]

(4.31)

Here using the Ramanujan's formula again with \( a = 1/\zeta, b = q/\zeta, z \to z/a_j \),
\[ \sum_{n \in \mathbb{Z}} \frac{1}{1 - q^n/\zeta} \left( \frac{z}{a_j} \right)^n = \frac{(\frac{\phi(z)}{a_j})_{\infty} (\frac{\theta(\zeta)}{z}; q)_{\infty} (q; q)_{\infty}}{(1/\zeta; q)_{\infty} (q\zeta; q)_{\infty} (z/a_j; q)_{\infty} (qa_j/z; q)_{\infty}} = \frac{\theta(\frac{\phi(z)}{a_j})}{\theta(1/\zeta) \theta(z/a_j)} (q; q)_{\infty}. \]

(4.32)
we see
\[
\left\langle \frac{1}{(\zeta q^\lambda N; q)_\infty} \right\rangle = \det \left( \delta_{ij} - \sum_{n \in \mathbb{Z}} \frac{1}{1 - q^n / \zeta} \int_{C_r} dz \frac{a_i}{z - a_i - z^n \prod_k (z/a_k; q)_\infty g(z; \alpha, t)} \right)
= \det(\delta_{ij} - \sum_{n \in \mathbb{Z}} A(i, n)B(n, j))_{1 \leq i, j \leq N}
\]
(4.33)

with
\[
A(i, n) = \sqrt{f(n)} \int_{C_r} \frac{dz}{z - a_i - z^n \prod_k (z/a_k; q)_\infty g(z; \alpha, t)},
\]
(4.34)
\[
B(n, j) = \frac{\sqrt{f(n)}}{a_j^n (q; q)_\infty \prod_{k \neq j} (a_j/a_k; q)_\infty g(a_j; \alpha, t)}.
\]
(4.35)

In the last determinant we state explicitly that this is for an \( N \)-dimensional matrix. Here use \( \det(1 - AB)_{1 \leq i, j \leq N} = \det(1 - BA)_{L^2(\mathbb{Z})} \), which holds for arbitrary Hilbert-Schmidt operators \( A, B \). The fact that \( A, B \) in our case (4.31), (4.35) are indeed Hilbert-Schmidt operators can be shown in the same way as Lemma 4.6 for \( \phi, \psi \). We see that the kernel (or the matrix element of the infinite matrix) is given by
\[
(BA)(m, n) = \sum_{i=1}^{N} B(m, i)A(i, n)
= \sum_{i=1}^{N} a_i^m (q; q)_\infty \prod_{k \neq i} (a_i/a_k; q)_\infty g(a_i; \alpha, t) \frac{1}{1 - q^n / \zeta} \int_{C_r} dz \frac{a_i}{z - a_i - z^n \prod_k (z/a_k; q)_\infty g(z; \alpha, t)}
= \frac{-1}{1 - q^n / \zeta} \int_D dv \int_{C_r} \frac{dz}{z - v - z^m} \frac{z^n \prod_k (z/a_k; q)_\infty g(z; \alpha, t)}{\prod_k (v/a_k; q)_\infty g(v; \alpha, t)},
\]
(4.36)

where the contour \( D \) is around \( \{a_i, 1 \leq i \leq N\} \). Rewriting the last product as
\[
\frac{\prod_k (z/a_k; q)_\infty g(z; \alpha, t)}{\prod_k (v/a_k; q)_\infty g(v; \alpha, t)} = \frac{\prod_k (qz/a_k; q)_\infty (q \alpha_k; q)_\infty e^{zt} (z - a_k)(v - a_k)}{\prod_k (qv/a_k; q)_\infty (q \alpha_k; q)_\infty e^{vt} (v - a_k)(z - a_k)}
\]
(4.37)

we have
\[
(BA)(m, n) = \frac{1}{1 - q^n / \zeta} \int_D dv \int_{C_r} \frac{dz z^n + N e^{zt} \prod_k (qz/a_k; q)_\infty (q \alpha_k; q)_\infty}{z v^n + N e^{vt} \prod_k (qv/a_k; q)_\infty (q \alpha_k; q)_\infty} \times \left( \frac{1}{z - v} \prod_{k} \frac{(z - a_k)(v - \alpha_k)}{(v - a_k)(z - \alpha_k)} - 1 \right)
\]
(4.38)

where \( -1 \) is inserted since it does not create singularities which change the value of the integral. By a simple relation,
\[
\frac{1}{z - v} \prod_{k} \frac{(z - a_k)(v - \alpha_k)}{(v - a_k)(z - \alpha_k)} - 1 = \sum_{l=0}^{N-1} \frac{a_{l+1} - a_l + 1}{(z - \alpha_{l+1})(v - a_l)} \prod_{j=1}^{l} \frac{(z - a_j)(v - \alpha_j)}{(z - \alpha_j)(v - a_j)},
\]
(4.39)

we finally arrive at our desired formula.

This completes the proof of Theorem 4.3.
5 Stationary case

5.1 $q$-TASEP with two parameters

In the last section, we found a formula for the two-sided $q$-Whittaker measure \([3.70]\) which corresponds to the $q$-TASEP with the initial condition \([1.2]\) when $\alpha_j = \alpha$ for one $j$ and $\alpha_k = 0, k \neq j$. Using this result, in this section, we will study the $q$-TASEP for the stationary situation with parameter $a$ under the conditioning that there is a particle at the origin at $t = 0$. The conditioning can be taken into account by considering the initial condition \([1.3]\). The stationary situation would be realized if we specialize the parameters as

$$\alpha_j = \alpha, \quad \alpha_k = 0, k \neq j, \quad a_1 = a, a_2 = \ldots = a_N = 1, \quad 0 < a, \alpha < 1,$$

and take the limit $a \to \alpha$. In this section we suppose $a, \alpha \in \mathbb{R}$ rather than $a, \alpha \in \mathbb{R}^N$ as in previous sections. Note that the $q$-TASEP with the initial condition \([1.2]\) is well-defined only when $\alpha < a$, since the initial position of the first particle $X_1(0)$ obeys $q\text{Po}(\alpha/a)$ \([2.5]\) which becomes singular when $a \to \alpha$.

As in Introduction let us denote by $X_N(t)$ (resp. $X_N^{(0)}(t)$) the $N$th particle position of $q$-TASEP with the initial condition \([1.2]\) (resp. \([1.3]\)) with \([5.1]\). We define

$$G(\zeta) = \langle \frac{1}{(\zeta q^{X_N(t)} + N; q)_\infty} \rangle = \sum_{l \in \mathbb{Z}} \frac{1}{(\zeta q; q)_\infty} \mathbb{P}[X_N(t) = N + l],$$

$$G_0(\zeta) = \langle \frac{1}{(\zeta q^{X_N^{(0)}(t)} + N - 1; q)_\infty} \rangle = \sum_{l \in \mathbb{Z}} \frac{1}{(\zeta q; q)_\infty} \mathbb{P}[X_N^{(0)}(t) = N - 1 = l].$$

It is easy to see that they are finite for $\zeta \notin q^n, n \in \mathbb{Z}$. Moreover when $\zeta \notin \mathbb{R}_{\geq 0} := \{x \in \mathbb{R} | x \geq 0\}$, this decays fast as $r \to \infty$ for $\zeta = re^{i\theta}, 0 < \theta < 2\pi$. Next we discuss a relation between $G(\zeta)$ and $G_0(\zeta)$ in the case $\alpha < a$. Later we perform analytic continuation to a region including $a = \alpha$. Since $X_N(t) = X_N^{(0)}(t) - \chi - 1$, where $\chi \sim q\text{Po}(\alpha/a)$ and $X_N^{(0)}$ and $\chi$ are independent with each other, one sees

$$G(\zeta) = (\alpha/a; q)_\infty \sum_{m=0}^{\infty} \frac{(\alpha/a)^m}{(q; q)_m} G_0(\zeta q^{-m}), \quad \zeta \notin \mathbb{R}_{\geq 0}.$$

The inversion formula can also be obtained.

**Lemma 5.1.** For $\zeta \notin \mathbb{R}_{\geq 0}$,

$$G_0(\zeta) = \frac{1}{(\alpha/a; q)_\infty} \sum_{k=0}^{\infty} \frac{(-1)^k q^{k(k-1)/2} (\alpha/a)^k}{(q; q)_k} G(\zeta q^{-k}).$$

**Proof.** Substituting \([5.4]\) into rhs of \([5.5]\), one has

$$\sum_{k,m=0}^{\infty} \frac{(-1)^k q^{k(k-1)/2} (\alpha/a)^k + m}{(q; q)_k (q; q)_m} G_0(\zeta q^{-(k+m)}) = \sum_{\ell=0}^{\infty} G_0(\zeta q^{-\ell}) \left( \frac{\alpha}{a} \right) \sum_{k=0}^{\ell} \frac{(-1)^k q^{k(k-1)/2}}{(q; q)_k (q; q)_{\ell-k}}.$$

Thus it is sufficient to prove

$$\sum_{k=0}^{\ell} \frac{(-1)^k q^{k(k-1)/2}}{(q; q)_k (q; q)_{\ell-k}} = \delta_{\ell,0},$$

\(30\)
and this is a direct consequence of a version of the $q$-binomial theorem (A.4),

$$
\sum_{k=0}^{\ell} \frac{(-1)^k q^{k(k-1)/2} (q; q)_k^\ell}{(q; q)_k^\ell} x^k = (1 - x)(1 - xq) \cdots (1 - xq^{\ell-1}).
\tag{5.8}
$$

Setting $x = 1$ in (5.8), we obtain (5.7).

By using the $q$-shift operator $T_q$ defined by $T_q f(z) = f(qz)$ and the $q$-binomial theorem (A.2), (5.4) is rewritten as

$$
G(\zeta) = \frac{(\alpha/a; q)_\infty}{(\alpha T_q^{-1}/a; q)_\infty} G_0(\zeta).
\tag{5.9}
$$

Formally this can be inverted as

$$
G_0(\zeta) = \frac{(\alpha T_q^{-1}/a; q)_\infty}{(\alpha/a; q)_\infty} G(\zeta),
\tag{5.10}
$$

from which one readily obtains (5.5) by another $q$-binomial formula (A.3),

$$
\sum_{n=0}^{\infty} \frac{(-1)^n q^{n(n-1)/2}}{(q; q)_n} z^n = (z; q)_\infty.
\tag{5.11}
$$

So far our discussions have been for the $q$-Laplace transform. Though this is enough for considering the long time limit of the distribution as we will see in section 5.3, we here give a formula for the distribution for finite time $t$. Let us consider the generating functions,

$$
Q_0(x) = \sum_{l \in \mathbb{Z}} \mathbb{P}(X_N^{(0)}(t) + N - 1 \geq l)x^l,
\tag{5.12}
$$

$$
Q(x) = \sum_{l \in \mathbb{Z}} \mathbb{P}(X_N(t) + N \geq l)x^l.
\tag{5.13}
$$

Noting $X_N(t) = X_N^{(0)}(t) - \chi - 1$, where $X_N^{(0)}$ and $\chi$ are independent and $\chi \sim q\text{Po}(\alpha/a)$, one has

$$
Q(x) = \sum_{l \in \mathbb{Z}} \left( \sum_{m=0}^{\infty} \mathbb{P}(X_N^{(0)}(t) \geq l + m + 1) \mathbb{P}(\chi = m) \right) x^l = \sum_{n \in \mathbb{Z}} \mathbb{P}(X_N^{(0)}(t) \geq n + 1) x^n \sum_{m=0}^{\infty} \mathbb{P}(\chi = m) x^{-m} = Q_0(x) \frac{(\alpha/a; q)_\infty}{(\alpha/ax; q)_\infty}.
\tag{5.14}
$$

Applying (D.18), we have the following.

**Proposition 5.2.** The distribution of $X_N^{(0)}(t)$ can be written in terms of $G(z)$, the $q$-Laplace transform of $X_N(t)$, as follows.

$$
\mathbb{P}(X_N^{(0)}(t) \geq n - N + 1) = \frac{1}{2\pi i} \int_{C_0} \frac{dx}{x^n} \frac{(\alpha/ax; q)_\infty (q; q)_\infty}{(x; q)_\infty} \sum_{k \in \mathbb{Z}} (qx)^k R_k(G).
\tag{5.15}
$$

where $C_0$ is a small contour around the origin and $R_k(G)$ means the residue at the pole $z = q^{-k}$ of the function $G$. 
5.2 Stationary limit

From the discussions in the last subsection, we can study the $q$-TASEP in the stationary regime using (5.5) from the result for the $q$-TASEP with the initial condition (1.2) with (5.1) and then taking the limit $a \to \alpha$. Here note that, as we have seen in section 3, the information about the distribution of the position of the $N$th particle $X_N(t)$ is contained in the two-sided $q$-Whittaker measure and the $q$-Whittaker functions $P_3(a), Q_3(\alpha, t)$ are symmetric functions. Therefore instead of (5.1) we can and will set the parameters in the formulas to

$$\alpha_1 = \cdots = \alpha_{N-1} = 0, \alpha_N = \alpha, \quad a_1 = \cdots = a_{N-1} = 1, a_N = a, \quad 0 < \alpha < a < 1,$$

and take the limit $a \to \alpha$.

To discuss the stationary limit, it is useful to rewrite the kernel (4.9) in the form,

$$K = K_{a,\alpha} + (a - \alpha)B_1 \otimes B_2,$$

where

$$K_{a,\alpha}(m, n) = \sum_{l=0}^{N-2} \phi_l(m)\psi_l(n).$$

Here the functions $\phi_n, \psi_n, B_1, B_2$ in the kernel are given by

$$\phi_l(n) = \tau(n) \int_D \frac{dv}{2\pi i} \frac{e^{-vt}}{v^n+1} \left( \frac{1}{v-1} \right)^{l+1} \frac{1}{(qv;q)_\infty^{N-1}} \frac{1}{(qv/a;q)_\infty} \frac{(q\alpha/v;q)_\infty}{(qv/a;q)_\infty},$$

$$B_1(n) := \phi_{N-1}(n) = \tau(n) \int_D \frac{dv}{2\pi i} \frac{e^{-vt}}{v^n+1} \left( \frac{1}{v-1} \right)^{N-1} \frac{1}{v-a(qv;q)_\infty^{N-1}} \frac{1}{(qv/a;q)_\infty} \frac{(q\alpha/v;q)_\infty}{(qv/a;q)_\infty},$$

$$\psi_l(n) = \frac{1}{\tau(n)} \int_{C_r} \frac{dz}{2\pi i} \frac{e^{zt}}{z^n-Nl-1}(z-1)^l(qz;q)_\infty^{N-1} \frac{(qz/a;q)_\infty}{(qz;q)_\infty} \frac{1}{(qz/a;q)_\infty},$$

$$B_2(n) := \psi_{N-1}(n) = \frac{1}{\tau(n)} \int_{C_r} \frac{dz}{2\pi i} \frac{e^{zt}}{z-a} \frac{z^n}{z-\alpha}(z-1)^{N-1} \frac{(qz/a;q)_\infty}{(qz;q)_\infty} \frac{1}{(qz/a;q)_\infty},$$

where $l = 0, 1, \cdots, N-2$, the contour $C_r$ is around 0 and $\alpha q^j, j \in \mathbb{N}$, and the contour $D$ is around 1 and $a$. $\tau(n)$ is defined in (4.12). The bounds in Lemma 4.6 still hold but note that since the integrand of $\psi_l, 0 \leq l \leq N-2$ does not have a pole at $\alpha$, the bound for this function can now be replaced by

$$|\psi_l(n)| \leq (\alpha q/b)^n, \quad n \geq 0, \quad 0 \leq l \leq N-2.$$

Let us set

$$A = fK_{a,\alpha},$$

where $f$ is an operator with kernel $f(m, n) = f(n)\delta_{m,n}$. By the same arguments as in Proposition 4.7, the operator $A$ is a trace-class operator on $\ell^2(\mathbb{Z})$. Because of the conditions in Lemma 4.6 and (5.23), this is so in the region $0 < \alpha q < b < a < 1$. The operator $fK$ is also a trace-class operator on $\ell^2(\mathbb{Z})$ but it is so in the smaller region $0 < \alpha < b < a < 1$ (see Lemma 4.6) due to the extra term in terms of $B_1, B_2$ in the kernel.

Furthermore we have

**Lemma 5.3.** For $\zeta < 0$ and $0 < q\alpha < b < a < 1$, we have

$$||A|| < 1,$$

where $|| \cdot ||$ denotes the operator norm in $\ell^2(\mathbb{Z})$. This implies that $1 - A$ is invertible.
Substituting (5.31)–(5.34) into (5.29), we write to be arbitrary as in Lemma 4.6. The bound for \( \|f\| \leq 1 \) now observe that \( f \) can be bounded as

\[
f \leq \frac{1}{2}(f_1 + f_2),
\]

(5.26)

where \( \|f_1\| \leq 1 \) and \( f_2 \) is a step function. For example one can take \( f_1(n) = f(n + n_0), f_2(n) = 1_{a+n_0 > 0} \) for large enough \( n_0 \in \mathbb{N} \). By the argument in Appendix B.3 of [31] (also Lemma D.3 of [16]), we know \( \|K_{a,\alpha} f_2\| < 1 \). Hence we have \( \|A\| \leq \frac{1}{2} \|f_1 + f_2\| K_{a,\alpha} \| + \|f_2 K_{a,\alpha}\| < 1 \). \( \square \)

From Lemma 5.3 we see that in the parameter region \( 0 < \alpha q < a < 1 \), \( G(\zeta) \) can be rewritten as

\[
G(\zeta) = \det(1 - A - (a - \alpha)f B_1 \otimes B_2)_{\zeta(\mathbb{Z})} = (a - \alpha) \det(1 - A) \left( \frac{1}{a - \alpha} - \sum_{n \in \mathbb{Z}} (\rho f B_1)(n) B_2(n) \right) = (a - \alpha) \det(1 - A) \left( L_{a,\alpha} - \sum_{n \in \mathbb{Z}} (A f B_1)(n) B_2(n) \right),
\]

(5.27)

where

\[
\rho(m, n) = (1 - A)^{-1}(m, n),
\]

(5.28)

\[
L_{a,\alpha} = \frac{1}{a - \alpha} - \sum_{n \in \mathbb{Z}} f(n) B_1(n) B_2(n).
\]

(5.29)

To perform the analytic continuation, we separate the contributions from the poles in \( B_1, B_2 \) as

\[
B_1(n) = B_1^{(1)}(n) + B_1^{(2)}(n), \quad B_2(n) = B_2^{(1)}(n) + B_2^{(2)}(n),
\]

(5.30)

where \( B_1^{(1)}(n) \) (resp. \( B_1^{(2)}(n) \)) denotes the residue at \( v = a \) (resp. \( z = \alpha \)) in (5.20) (resp. (5.22)) and \( B_1^{(2)}(n) \) (resp. \( B_2^{(2)}(n) \)) denotes the remaining contribution:

\[
B_1^{(1)}(n) = \frac{(-1)^{N-1} \tau(n) e^{-at}(qa/a; q)^\infty}{a^{n+1}(a; q)_{\infty}^{n-1}(q; q)_{\infty}},
\]

(5.31)

\[
B_1^{(2)}(n) = \tau(n) \int_{D_1} dv \frac{e^{-vt}}{2\pi i} v^{n+1} \left( \frac{1}{v - 1} \right)^{N-1} \frac{1}{v - a} (qv/a; q)_{\infty}^{N-1} (q/v; q)_{\infty}^{N-1} (qa/v; q)_{\infty},
\]

(5.32)

\[
B_2^{(1)}(n) = \frac{(-1)^{N-1} \alpha^{n+1} e^{at}(a; q)_{\infty}^{n-1}(qa/a; q)_{\infty}}{\tau(n)(q; q)_{\infty}},
\]

(5.33)

\[
B_2^{(2)}(n) = \frac{1}{\tau(n)} \int_{q \alpha < z < \alpha q} \frac{dz}{2\pi i} e^{zt} z^{n+1} = \frac{1}{\tau(n)} \int_{q \alpha < z < \alpha q} \frac{dz}{2\pi i} e^{zt} \left( z - \alpha \right)^{N-1}(qz; q)_{\infty}^{N-1} (qz/a; q)_{\infty} (aq/z; q)_{\infty},
\]

(5.34)

where \( D_1 \) denotes the contour enclosing 1 only. Note that, to separate the pole contribution in \( B_2 \), we should take the radius of the contour in (5.34) to be smaller than \( \alpha \) and one can not take \( d \) in (1.15) to be arbitrary as in Lemma 4.6. The bound for \( B_2^{(2)}(n) \) reads \( |B_2^{(2)}(n)| \leq (\alpha q/b)^n, n \geq 0, (\alpha/c)^n, n < 0. \) Substituting (5.31)–(5.34) into (5.29), we write

\[
L_{a,\alpha} = \frac{1}{a - \alpha} - \sum_{n \in \mathbb{Z}} f(n) B_1^{(1)}(n) B_2^{(1)}(n) - \sum_{i,j \in \{1, 2\}} \sum_{n \in \mathbb{Z}} f(n) B_1^{(i)}(n) B_2^{(j)}(n).
\]

(5.35)
From the above remark, for the last sum for \( i = 1, j = 2 \) on rhs to make sense we should take \( \alpha > aq \). Combining this with the condition in Lemma 5.3 we will consider the problem in the region

\[
0 < qa < \alpha < 1, 0 < q\alpha < a < 1.
\]

(5.36)

For the moment we should put the condition \( a \neq \alpha \) because of the apparent singularity at \( a = \alpha \) in (5.35). Below we remove this limitation \( a \neq \alpha \) by analytic continuation.

For the first two terms in (5.35), we have

\[
\frac{1}{a - \alpha} - \sum_{n \in \mathbb{Z}} f(n)B_1^{(1)}(n)B_2^{(1)}(n)
\]

\[
= \frac{1}{a - \alpha} - \frac{e^{(a-\alpha)t}}{a} \left( \frac{(qa/a;q)_\infty}{(q;q)_\infty} \right)^2 \left( \frac{(\alpha;q)_\infty}{(a;q)_\infty} \right)^N \sum_{n \in \mathbb{Z}} (\alpha/a)^n 1 - q^n/\zeta.
\]

(5.37)

Now we recall the Ramanujan’s summation formula (4.1). Setting \( a = 1/\zeta, b = q/\zeta, z = \alpha/a \), we have, for \( \zeta \neq q^n, n \in \mathbb{Z}, \)

\[
\sum_{n \in \mathbb{Z}} (\alpha/a)^n 1 - q^n/\zeta = \frac{(\alpha/a\zeta;q)_\infty(q;\zeta)_\infty^2(\alpha\zeta q/\alpha;q)_\infty}{(\alpha/a;q)_\infty(1/\zeta;q)_\infty(qa/\alpha;q)_\infty(q\zeta;q)_\infty}.
\]

(5.38)

Note that, the assumption in (4.1), \( |b/a| < |z| \), gives \( qa < \alpha \) in our case which is satisfied in our discussions. Substituting this into (5.37), we arrive at the following expression of \( L_{\alpha,a} \) (5.29):

\[
L_{\alpha,a} = \frac{1}{a - \alpha} \left( 1 - \frac{(\alpha/a\zeta;q)_\infty(a\zeta q/\alpha;q)_\infty(\alpha q/a;q)_\infty}{(1/\zeta;q)_\infty(q\zeta;q)_\infty(qa/\alpha;q)_\infty} \left( \frac{(\alpha;q)_\infty}{(a;q)_\infty} \right)^N e^{(a-\alpha)t} \right)
\]

\[- \sum_{\substack{i,j \in \{1,2\} \atop (i,j) \neq (1,1)}} \sum_{n \in \mathbb{Z}} f(n)B_1^{(i)}(n)B_2^{(j)}(n).
\]

(5.39)

Combining (5.27) with (5.39), we have

\[
G(\zeta) = (a - \alpha) \det(1 - A)
\]

\[
\times \left( \frac{1}{a - \alpha} \left( 1 - \frac{(\alpha/a\zeta;q)_\infty(a\zeta q/\alpha;q)_\infty(\alpha q/a;q)_\infty}{(1/\zeta;q)_\infty(q\zeta;q)_\infty(qa/\alpha;q)_\infty} \left( \frac{(\alpha;q)_\infty}{(a;q)_\infty} \right)^N e^{(a-\alpha)t} \right)
\]

\[- \sum_{\substack{i,j \in \{1,2\} \atop (i,j) \neq (1,1)}} \sum_{n \in \mathbb{Z}} f(n)B_1^{(i)}(n)B_2^{(j)}(n) - \sum_{n \in \mathbb{Z}} (ApfB_1)(n)B_2(n) \right).
\]

(5.40)

We have the following.

**Lemma 5.4.** \( G(\zeta) \) (5.40) is analytic in the region (5.36).

**Remark.** By "analytic for \( \alpha, a \) in (5.36)" we actually mean \( G(\zeta) \) is an analytic function in both \( \alpha \) and \( a \) in some complex domains containing the region (5.36). From the explicit formulas we have obtained so far, our main concern about a possible non-analyticity is at \( a = \alpha \).

**Proof.** We divide (5.40) into several parts and consider the analyticity of each part.

(i) The factor \( (a - \alpha) \det(1 - A) \) is analytic for \( 0 < q\alpha < a < 1 \) since, as mentioned before Lemma 5.3, \( A \) is a trace-class operator on \( \ell^2(\mathbb{Z}) \) in this region.
(ii) Next we consider the second line in (5.40). Note that the term
\[
\frac{(\alpha/a\zeta; q)_{\infty}(a\zeta q/\alpha; q)_{\infty}(aq/\alpha; q)_{\infty}}{(1/\zeta; q)_{\infty}(q\zeta; q)_{\infty}(qa/\alpha; q)_{\infty}} \left( \frac{(\alpha; q)_{\infty}}{(a; q)_{\infty}} \right)^{N-1} e^{(\alpha-a)t}
\]  
(5.41)
is analytic when \( qa/\alpha < 1 \), which is included in (5.36). Moreover, (5.41) is of order \( 1 + O(a - \alpha) \) when \( a \) and \( \alpha \) are close. Thus the second line in (5.40) is analytic in the region (5.36).

(iii) We consider the first term in the third line of (5.40), i.e.,
\[
\sum_{i,j \in \{1,2\}} \sum_{n \in \mathbb{Z}} f(n)B_1^{(i)}(n)B_2^{(j)}(n).
\]  
(5.42)
First for the case \( i = 1, j = 2 \), examining the large \(|n|\) behaviors of (5.31)-(5.34) and recalling the bounded for \( B_2^{(2)} \) mentioned below (5.31), one sees that the sum over \( n \) converges when \( aq/\alpha < 1 \) for \( n \to +\infty \) and when \( aq/\alpha < 1 \) for \( n \to -\infty \). One easily finds that both conditions are satisfied in (5.36). Thus (5.42) is analytic for (5.36).

For the cases with \( i = 2, j = 1 \) and \( i = j = 2 \), we take the sum over \( j = 1,2 \) and consider \( \sum_{n \in \mathbb{Z}} f(n)B_1^{(2)}B_2 \). Since the contour \( D_1 \) in (5.32) is only around \( 1 \), \( B_1^{(2)}(n) \) is \( \tau(n) \) times an \( \ell \)th order polynomial in \( n \). On the other hand, for \( B_2(n) \), one has the bounds (4.15) from Lemma 4.6. Combining these, one sees that the sum over \( n \) is finite whenever \( \alpha < 1 \).

(iv) At last we consider the second term in the third line of (5.40), i.e.,
\[
\sum_{n \in \mathbb{Z}} (Apf B_1(n))B_2(n).
\]  
(5.43)
By using the definition of \( A \) (5.21), we rewrite it as
\[
\sum_{l=0}^{N-2} \sum_{n \in \mathbb{Z}} f(n) \phi_l(n)B_2(n) \sum_{n_1,n_2 \in \mathbb{Z}} \psi_l(n_1)\rho(n_1,n_2)f(n_2)B_1(n_2).
\]  
(5.44)
For the first factor \( \sum_{n \in \mathbb{Z}} f(n) \phi_l(n)B_2(n) \), since the contour \( D \) goes around \( 1 \) in (5.19), \( \phi_l(n) \) is an \( \ell \)th order polynomial in \( n \) and one can apply the same argument as for (iii) above and sees that the summation over \( n \) converge whenever \( \alpha < 1 \). We also find that from (5.23) and Lemma 4.6 \( \psi_l \in \ell^2(\mathbb{Z}) \) and \( fB_1 \in \ell^2(\mathbb{Z}) \) when \( qa < b < a \). Recalling \( \rho = (1-A)^{-1} \) is a bounded operator by Lemma 5.3 one sees that the second factor in (5.44) is also analytic in \( qa < b < a \).

Therefore combining the above (i)-(iv), we see that (5.43) is analytic (at least) in the region (5.36).

Next we consider the stationary limit \( a \to \alpha \). If we restore the dependence on \( a, \alpha \) in \( B_j^{(i)} \), \( 1 \leq i, j \leq 2 \), (5.31)-(5.34) as \( B_j^{(i)}(n,a;\alpha), 1 \leq i, j \leq 2 \), it is easy to see that one can simply take the limit \( a \to \alpha \) of them: we set \( B_j^{(i)}(n;\alpha) = \lim_{a \to \alpha} B_j^{(i)}(n,a;\alpha), 1 \leq i, j \leq 2 \). For \( L_{a,\alpha} \) in (5.29) we have

**Lemma 5.5.**

\[
L := \lim_{a \to \alpha} L_{a,\alpha} = \frac{1}{\alpha} \sum_{n=0}^{\infty} \left( \frac{q^n/\zeta}{1 - q^n/\zeta} - \frac{\zeta q^{n+1}}{1 - \zeta q^{n+1}} + \frac{2q^{n+1}}{1 - q^{n+1}} + \frac{(N-1)aq^n}{1 - \alpha q^n} \right) + t
\]
\[
- \sum_{i,j=1,2} \sum_{n \in \mathbb{Z}} f(n)B_1^{(i)}(n;\alpha)B_2^{(j)}(n;\alpha).
\]  
(5.45)
Proof. Using

\[(1 + c)x; q)_\infty = \prod_{n=0}^{\infty} (1 - (1 + c)xq^n) = (x; q)_\infty - c(x; q)_\infty \sum_{n=0}^{\infty} \frac{xq^n}{1 - xq^n} + O(c^2) \]  

(5.46)

we find that when \(a\) and \(\alpha\) are close each factor of (5.41) can be expanded as

\[
\frac{(\alpha/a\zeta; q)_\infty}{(1/\zeta; q)_\infty} = 1 - \frac{\alpha - a}{a} \sum_{n=0}^{\infty} \frac{q^n/\zeta}{1 - q^n/\zeta} + O((a - \alpha)^2),
\]

(5.47)

\[
\frac{(aq\zeta/\alpha; q)_\infty}{(q\zeta; q)_\infty} = 1 - \frac{\alpha - a}{\alpha} \sum_{n=0}^{\infty} \frac{\zeta q^{n+1}}{1 - \zeta q^{n+1}} + O((a - \alpha)^2),
\]

(5.48)

\[
\frac{(aq/a; q)_\infty}{(a\zeta; q)_\infty} = 1 - \left(\frac{\alpha - a}{a} + \frac{\alpha - a}{\alpha}\right) \sum_{n=0}^{\infty} \frac{q^{n+1}}{1 - q^{n+1}} + O((a - \alpha)^2),
\]

(5.49)

\[
\frac{(\alpha; q)_\infty}{(a; q)_\infty} = 1 - \alpha - a \sum_{n=0}^{\infty} \frac{aq^n}{1 - aq^n} + O((a - \alpha)^2).
\]

(5.50)

Thus from (5.47)–(5.50), we see that (5.41) is written as

\[
\frac{1}{a - \alpha} - \sum_{n \in \mathbb{Z}} f(n) B_1^{(1)}(n) B_2^{(1)}(n) = -\frac{1}{a} \sum_{n=0}^{\infty} \frac{q^n/\zeta}{1 - q^n/\zeta} + \frac{1}{\alpha} \sum_{n=0}^{\infty} \frac{\zeta q^{n+1}}{1 - \zeta q^{n+1}} - \frac{\alpha + a}{\alpha} \sum_{n=0}^{\infty} \frac{q^{n+1}}{1 - q^{n+1}} - \frac{N - 1}{a} \sum_{n=0}^{\infty} \frac{aq^n}{1 - aq^n} + t.
\]

(5.51)

From (5.26) and (5.51) and using the fact that the term \(\sum_{n \in \mathbb{Z}} B_1^{(i)} B_2^{(j)}\) remain finite in the stationary limit except \(i = j = 1\) by the same arguments as for (iii) in Lemma 5.4, we arrive at (5.45). \(\square\)

Using Lemma 5.10, 5.4 and 5.3, we have an expression of \(G_0(\zeta)\) in the stationary limit.

**Proposition 5.6.** For \(q\)-TASEP with parameters (5.1) with \(a = \alpha\), we have, for \(\zeta \not\in \mathbb{R}_{\geq 0}\),

\[
\left\langle \frac{1}{(\zeta q^{-k}; q)_\infty} \right\rangle = G_0(\zeta) = \frac{\alpha}{(q; q)_\infty} \sum_{k=0}^{\infty} \left(-1\right)^k q^{k(k+1)/2} \left(V_N(\zeta q^{-k}) - V_N(\zeta q^{-k-1})\right)
\]

(5.52)

where

\[
V_N(\zeta) = \text{det}(1 - A) \left( L - \sum_{n \in \mathbb{Z}} (\rho A f B_1)(n; \alpha) B_2(n; \alpha) \right).
\]

(5.53)

Here \(A\) is given by (5.24), (5.18), (4.8) with \(a = \alpha\), \(L\) by (5.45), \(f(n)\) by (4.8), \(B\) by (5.30)–(5.34) with \(a = \alpha\) and \(\rho\) by (5.28) with \(a = \alpha\). Note that \(A\) and \(f\) also have \(\zeta\) dependence.

**Proof.** First we consider the region (5.36). Note that in this region \(G(\zeta)\) is well-defined due to Lemma 5.4. Substituting \(G(\zeta q^{-k}) = q^k G(\zeta q^{-k}) + (1 - q^k) G(\zeta q^{-k})\) into (5.51), we have

\[
G_0(\zeta) = \frac{1}{1 - a/a} \sum_{k=0}^{\infty} \left(-1\right)^k q^{k(k-1)/2} \frac{(\alpha q/a)^k}{(q; q)_k} \left(G(\zeta q^{-k}) - \frac{\alpha}{a} G(\zeta q^{-k-1})\right)
\]

(5.54)
Substituting (5.27) into (5.54), we get

\[ G_0(\zeta) = \frac{a}{(\alpha q/a; q)_\infty} \sum_{k=0}^{\infty} (-1)^k q^{k(k-1)/2} (\alpha q/a)^k \]
\[ \times \sum_{m=0}^{1} \left( -\frac{\alpha}{a} \right)^m \det(1 - A) \left( \sum_{n \in \mathbb{Z}} f(n)(\rho AB_1(n)B_2(n)) \right) \bigg|_{\zeta \to \zeta q^{-k-m}}. \]  

(5.55)

We take the stationary limit \( a \to \alpha \) in this expression using Lemma 5.5. To show the convergence of the last term, we notice that the resolvent converges in the operator norm in \( \ell^2(\mathbb{R}) \),

\[ \lim_{a \to \alpha} ||(1 - K_{a,a})^{-1} - (1 - K_{a,a})^{-1}|| = 0, \]

(5.56)

which follows from the convergence of the kernel as in Lem 7.6 of [16]. Therefore we arrive at (5.52).

By taking \( a \to \alpha \) limit in Proposition 5.57, one gets a formula for the distribution of the particle position.

**Proposition 5.7.** For the stationary \( q \)-TASEP with parameter \( \alpha \), the distribution of \( X_N^{(0)}(t) \) is written as

\[ \mathbb{P}(X_N^{(0)}(t) \geq n - N + 1) = \frac{1}{2\pi i} \int_{C_0} \frac{dx}{x^n} \frac{(q;x)_\infty}{(x;q)_\infty} \sum_{k \in \mathbb{Z}} (qx)^k R_k(\hat{G}). \]  

(5.57)

where \( C_0 \) is a small contour around the origin and \( R_k(\hat{G}) \) means the residue at the pole \( z = q^{-k} \) of the function \( \hat{G}(\zeta) = \lim_{a \to \alpha} G(\zeta)/(a - \alpha) \).

### 5.3 Long time limit

For the step initial condition, the long time limit of the \( q \)-TASEP was already discussed in [10, 33]. In this subsection we discuss the large time limit for the stationary case with the scaling discussed in section 2 based on the formulas obtained in the previous subsection. We consider the scaling (2.25) with (2.21), (2.22), (2.26), (2.28) and

\[ \zeta = -q^{-\eta N + \gamma N^{1/3}s}, \]

(5.58)

in Proposition 5.6. We will show that rhs of (5.52) tends to the Baik-Rains distribution in the long time limit. By the same reasoning as for the step initial condition in sec. 5 of [33], for which the GUE Tracy-Widom distribution appears, this implies that the limiting distribution of the particle position, \( \lim_{N \to \infty} \mathbb{P}(X_N^{(0)}(\kappa N) > (\eta - 1)N - \gamma N^{1/3}s) \), is also the Baik-Rains distribution. Hereafter we focus on the limiting behavior of rhs of (5.52). First we show that \( \det(1 - A) \) tends to the GUE Tracy-Widom distribution. In Appendix C, we will provide rather general lemmas to establish the GUE Tracy-Widom limit for a kernel of a specific form and we apply them to our case. Set

\[ C_{N,n,l} = \frac{v_c^{l-N-n+1}}{(v_c - 1)^{l+1}} \frac{1}{(v_c; q)_\infty^{N-1}}, \quad v_c = q^\theta, \]

(5.59)

and define

\[ \tilde{\phi}_l(n) = \frac{\gamma N^{1/3}}{C_{N,n,l}} \phi_l(n), \]

(5.60)

\[ \tilde{\psi}_l(n) = \gamma N^{1/3}(1 - v_c)C_{N,n,l}\psi_l(n). \]

(5.61)

The reason of considering the factor \( C_{N,n,l} \) becomes clear in the following lemma.
Lemma 5.8. The functions $\tilde{\phi}_l, \tilde{\psi}_l, 0 \leq l \leq N - 2$ satisfy the assumptions of Lemma C.1 with $(x, a, \gamma, c) \rightarrow (n, -\eta, \gamma, (1 - v_c)\gamma)$.

Proof. First we show that the assumption (a) in Lemma C.1 is satisfied. We provide a proof for only (5.60) since the one for (5.61) is similar. Let us write $\phi_l(n)$ (5.19) as

$$\phi_l(n) = \frac{(-1)^N}{2\pi i} \int_D e^{-Ng(v)} \left( \frac{v}{v-1} \right)^{-\gamma(v_c-1)N^{1/3}\lambda} \left( \frac{qv; q}{a; q} \right)_\infty \left( \frac{qv/a; q}{v-1} \right)_\infty,$$

(5.62)

where

$$g(v) = \kappa v - \eta \log v + \log(v; q)_\infty,$$

(5.63)

and set the scaling

$$n = -\eta N + \gamma N^{1/3}\xi, \quad l = N + \gamma(v_c - 1)N^{1/3}\lambda.$$

(5.64)

We find that $v_c = q^\theta$ is a solution of the saddle point equation,

$$g'(v_c) = \kappa - \frac{\eta}{v_c} - \sum_{n=0}^{\infty} \frac{q^n}{1 - q^n v_c} = 0.$$

(5.65)

This can be checked by simple calculations,

$$g'(v_c) = \sum_{n=0}^{\infty} \frac{q^n - q^{n+2} - (1 - q^{n+2})q^n}{(1 - q^{n+2})^2} = 0.$$

(5.66)

Furthermore the second and the third order derivatives of $g(v)$ can be calculated as

$$g''(v_c) = 0, \quad g'''(v_c) = -2v_c^{-3}\gamma^3,$$

(5.67)

where $\gamma$ is given by (2.26). Thus $g(v)$ is expanded as

$$g(v) = g(v_c) - \frac{\gamma^3}{v_c^3} \frac{(v - v_c)^3}{3} + O((v - v_c)^4).$$

(5.68)

If we take the contour $D$ to be the circle around 1 which passes through $v = v_c$, we see that the main contribution comes from around the saddle point $v = v_c$. Hence we focus on the contribution of the integral in (5.62) around the saddle point $v_c$. Scaling $v$ as

$$v = v_c \left( 1 - \frac{\bar{v}}{\gamma N^{1/3}} \right),$$

(5.69)

we find that each factor in (5.62) behaves asymptotically as

$$-Ng(v) \sim -Ng(v_c) - \frac{\bar{v}^3}{3},$$

(5.70)

$$\frac{1}{v^{\gamma N^{1/3}\xi}} \sim \frac{e^{\bar{v}\xi}}{v_c^{\gamma N^{1/3}\xi}},$$

(5.71)

$$\left( \frac{v}{v-1} \right)^{\gamma(v_c-1)N^{1/3}\lambda} \sim \left( \frac{v_c}{v_c-1} \right)^{\gamma(v_c-1)N^{1/3}\lambda} e^{\bar{v}\lambda},$$

(5.72)

$$\left( \frac{qv; q}{a; q} \right)_\infty \left( \frac{qv/a; q}{v-1} \right)_\infty \sim \left( \frac{qv_c; q}{a; q} \right)_\infty \left( \frac{qv_c/a; q}{v_c-1} \right)_\infty,$$

(5.73)
with the errors of order $O(N^{-1/3})$. Here we also applied the scaling for $\alpha$ given in (2.28). From (5.70)-(5.73), we see that in the scaling limit,
\[
\lim_{N \to \infty} \tilde{\phi}_t(n) = \lim_{N \to \infty} \frac{\phi_t(n)}{C_{N,n,t}} = \int_{i\mathbb{R}} \frac{d\xi}{2\pi} e^{-\frac{\xi^2}{2} + \theta(1 + \lambda)} = \text{Ai}(\xi + \lambda). \tag{5.74}
\]
This is uniform for $\xi, \lambda$ in a bounded domain.

Next we show the assumptions (b),(c) in the Lemma (5.1). Basic idea is taken from section 5 of [45]. We first observe $g(v) - g(v_c)$ can be written as
\[
g(v) - g(v_c) = -\gamma^3 \frac{(v - v_c)^3}{3} + \gamma^3 \frac{v - v_c}{v_c} \tilde{g}(v), \tag{5.75}
\]
where $\tilde{g}(v)$ satisfies $|\tilde{g}(v)| \leq 1/(3\delta_0)$ if $|v - v_c|/v_c < \delta_0$ for some $\delta_0 > 0$. By taking the contour $D$ to be a circle around 1 which passes $v_c(1 + \delta)$, one sees that integrand is bounded by the value at $v_c(1 + \delta)(< 1)$ and hence one has
\[
|\phi_n(x)| \leq Ce^{N(g(v_c) - g(v_c(1 + \delta))) - (y_1 + y_2(1 - v_c))\delta}. \tag{5.76}
\]
Then we see, due to the property of $\tilde{g}$ above and since $y_1 = \gamma N^{1/3}\xi, y_2 = (1 - v_c)\gamma N^{1/3}\lambda$,
\[
|\phi_n(x)| \leq Ce^{2\gamma N^{2/3}\delta^2 - (\xi + \lambda)\gamma N^{1/3}\delta}. \tag{5.77}
\]
Choose $\delta = \sqrt{\xi + \lambda}/(\gamma N^{1/3})$ if $\xi + \lambda < \gamma^2 N^{2/3}\delta_0^2$ and $\delta = \delta_0$ if $\xi + \lambda \geq \gamma^2 N^{2/3}\delta_0^2$. Then we have
\[
|\phi_n(x)| \leq c \exp(-\frac{1}{4} \min(\sqrt{\xi + \lambda} \gamma N^{1/3}\delta_0)(\xi + \lambda)). \tag{5.78}
\]
From this it is easy to see that the assumptions (b),(c) are satisfied for the function $\phi_n$. The same argument applies to the function $\psi_n$. □

**Lemma 5.9.** The kernel (5.24) and the function $f$ (4.3) with (5.58) and (5.64) satisfy the assumptions of Lemma (5.2).

**Proof.** (i)(ii) follows from Lemma (5.1) and Lemma (5.8) with (5.60) and (5.61). For (iii), one sees in the scaling limit (5.58) and (5.64),
\[
\lim_{N \to \infty} f(n) = \lim_{N \to \infty} \frac{1}{1 + q\gamma N^{1/3}(\xi - \delta)} = 1_{\geq s}(\xi) \tag{5.79}
\]
where $1_{\geq s}(\xi) = 1$(resp.0) for $\xi \geq s$ (resp. $\xi < s$). In fact it is easy to see that this limit holds as a convergence in $L^1(\mathbb{R})$ norm. The first condition of (iv) is included in Lemma (5.3) and the second one can also be checked in a similar manner. □

Combining Lemma (5.2) and Lemma (5.9) we have

**Lemma 5.10.**
\[
\lim_{N \to \infty} \det(1 - A) = F_2(s). \tag{5.80}
\]

Next we consider the remaining factors in (5.32). First we consider the asymptotics of $B_1(n)$ and $B_2(n)$. Set
\[
D_{N,n} = -\frac{(v_c; q)_\infty \gamma e^{-N\theta(v_c)}}{v_c^{n+1}} \tag{5.81}
\]
and
\[
B_\omega^{(1)}(\xi) = e^{\omega^{1/3} - \omega\xi}, \quad B_\omega^{(2)}(\xi) = -\int_0^\infty dz e^{\omega z} \text{Ai}(\xi + z), \tag{5.82}
\]
\[
B_\omega(\xi) = B_\omega^{(1)}(\xi) + B_\omega^{(2)}(\xi). \tag{5.83}
\]
Lemma 5.11. With the scaling (2.28), (5.64), we have
\[
\lim_{N \to \infty} B_1^{(i)}(n)/D_{N,n} = B_\omega^{(i)}(\xi), \quad i = 1, 2,
\]
(5.84)
\[
\lim_{N \to \infty} v_c D_{N,n} B_2^{(i)}(n) = B_\omega^{(i)}(\xi), \quad i = 1, 2.
\]
(5.85)

Proof. Since (5.85) can be shown in a similar way to (5.84), we provide the proof for only (5.84). To prove (5.84) with \(v_i\), where
\[
\text{Lemma 5.11.}
\]
\[
\text{Considering the scaling behaviors (5.70)-(5.73) and (5.87), we obtain}
\]
\[
\lim_{N \to \infty} v_c D_{N,n} B_2^{(i)}(n) = B_\omega^{(i)}(\xi), \quad i = 1, 2.
\]
(5.87)
we get (5.84) with \(i = 1\).

To prove (5.84) with \(i = 2\), we rewrite (5.32) as
\[
B_1^{(2)}(n) = \int_{D_1} \frac{dv}{2\pi iv} e^{-Ng(v)} \frac{v - 1}{v - a} (qv/a;q)_\infty (v/a;q)_\infty.
\]
(5.88)
\[
\text{Considering the scaling behaviors (5.70)-(5.73) and (5.87), we obtain}
\]
\[
B_1^{(2)}(n) \sim D_{N,n} \int_{\mathbb{R} + \eta} \frac{d\bar{v}}{2\pi} e^{-\frac{a^3}{3v} + \bar{v}(\xi + \lambda)}.
\]
(5.89)
Here \(\eta < -\omega\). This comes from the fact that in (5.32), \(a < |v|\) is satisfied since the contour \(D_1\) in (5.32) encloses only 1(\(> a\)). (5.84) with \(i = 2\) follows immediately from (5.89).

One can see that the convergence in the above lemma is uniform in a bounded domain and the remainder becomes small as \(N \to \infty\) as in the case of \(\phi_n, \psi_n\) in Lemma 5.8. Thus we find that in the limit (2.28), the ingredients in \(V_N(\xi)\) (5.53) besides \(\det(1 - A)\) goes to
\[
\lim_{N \to \infty} v_c \frac{\gamma}{\gamma N^{1/3}} \sum_{i,j=1,2} \sum_{n \in \mathbb{Z}} \sum_{(i,j) 
eq (1,1)} f(n) B_1^{(i)}(n; \alpha) B_2^{(j)}(n; \alpha) = \sum_{i,j=1}^{2} \int_{s}^{\infty} d\xi B_\omega^{(i)}(\xi) B_\omega^{(j)}(\xi),
\]
(5.90)
\[
\lim_{N \to \infty} v_c \frac{\gamma}{\gamma N^{1/3}} \sum_{n \in \mathbb{Z}} f(n)(\rho AB_1)(n; \alpha) B_2(n; \alpha) = \int_{s}^{\infty} d\xi (\rho A B_\omega)(\xi) B_\omega(\xi),
\]
(5.91)
where \(B_\omega(\xi)\) is defined in (5.83). \(A\) is the operator which has the kernel \(K(\xi, \zeta)1_{\geq s}\) with (C.1) and \(\rho_A = (1 - A)^{-1}\).

Furthermore, we have the following asymptotic behavior for the remaining part in (5.55):

Lemma 5.12. Under the scaling limit (2.28), we have
\[
\lim_{N \to \infty} v_c \frac{\gamma}{\gamma N^{1/3}} \left[ t - \sum_{n=0}^{\infty} \left( \frac{q^n/\zeta}{1 - q^n/\zeta} - \frac{\zeta q^{n+1}}{1 - \zeta q^{n+1}} + \frac{2q^{n+1}}{1 - q^{n+1}} + \frac{(N - 1)aq^n}{1 - aq^n} \right) \right] = s - \omega^2
\]
(5.92)
where \(v_c = q^\theta\) and \(\gamma\) is defined by (2.20) respectively.
Proof. Note that since $\eta > 0$,

$$\lim_{N \to \infty} \frac{1}{\alpha} \sum_{n=0}^{\infty} \frac{q^n / \zeta}{1 - q^n / \zeta} = \lim_{N \to \infty} \frac{1}{\alpha} \sum_{n=0}^{\infty} \frac{q^{n+\eta N - \gamma N^{1/3} s}}{1 + q^{n+\eta N - \gamma N^{1/3} s}} = 0$$ \hspace{1cm} (5.93)$$

and clearly one sees

$$\lim_{N \to \infty} \frac{v_c}{\gamma N^{1/3}} \sum_{n=0}^{\infty} \frac{2q^{n+1}}{1 - q^{n+1}} = 0.$$ \hspace{1cm} (5.94)$$

Thus nontrivial contributions come from the second and forth terms in the summation in (5.92). For the second term, we would like to show

$$\sum_{n=0}^{\infty} \frac{\zeta q^{n+1}}{1 - \zeta q^{n+1}} = -\sum_{n=0}^{\infty} \frac{q^{n+1-\eta N + \gamma N^{1/3} s}}{1 + q^{n+1-\eta N + \gamma N^{1/3} s}} = -\eta N + \gamma N^{1/3} s + O(N^0).$$ \hspace{1cm} (5.95)$$

Notice that, for a special case $\zeta = -q^{-m-1}, m \in \mathbb{N}$, minus the second expression is

$$\sum_{n=0}^{\infty} \frac{1}{1 + q^{m-n}} = \sum_{n=0}^{2m} \frac{1}{1 + q^{m-n}} + \sum_{n=2m+1}^{\infty} \frac{1}{1 + q^{m-n}} = m + 2 + \sum_{n=2m+1}^{\infty} \frac{1}{1 + q^{m-n}}$$ \hspace{1cm} (5.96)$$

and (5.95) holds. For general $\zeta = -q^{-\mu}, \mu \in \mathbb{R}$, the value of the second term is between those for $\zeta = -q^{-m}$ and $\zeta = -q^{-m-1}$ where $m = [\mu]$ is the largest integer which is smaller than or equal to $\mu$. Since the difference of these two values are of $O(m^0)$ for large $m$, one sees that (5.95) holds generally. Considering also the scaling for $\alpha$ in (2.28), we get

$$\frac{1}{\alpha} \sum_{n=0}^{\infty} \frac{\zeta q^{n+1}}{1 - \zeta q^{n+1}} = -\eta \frac{v_c}{v_c \gamma} N + \frac{\eta \omega}{v_c \gamma^2} N^{2/3} + \frac{\gamma^{3/2} \omega^2}{v_c \gamma^2} N^{1/3} + O(N^0).$$ \hspace{1cm} (5.97)$$

For the forth term in the sum in (5.92), expanding in $\omega$, we have

$$\frac{-N - 1}{\alpha} \sum_{n=0}^{\infty} \frac{\alpha q^n}{1 - \alpha q^n} = -\sum_{n=0}^{\infty} \frac{q^n}{1 - v_c q^n} N - \omega \sum_{n=0}^{\infty} \frac{v_c q^2 n}{1 - v_c q^n} N^{2/3} - \frac{\omega^2}{\gamma^2} \sum_{n=0}^{\infty} \frac{v_c^2 q^3 n}{1 - v_c q^n} N^{1/3} + O(N^0)$$

$$= \left( -\kappa + \frac{\eta}{v_c} \right) N - \frac{\eta \omega}{v_c \gamma} N^{2/3} + \left( \frac{\eta \omega^2}{v_c \gamma^2} - \frac{\gamma \omega^2}{v_c} \right) N^{1/3} + O(N^0),$$ \hspace{1cm} (5.98)$$

where in the last step we used (2.21),(2.22),(2.26). Considering (5.97) and (5.98) with $t = \kappa N$, we obtain (5.92).

Using Lemmas 5.6,5.12 we arrive at the following result:

**Theorem 5.13.** For the stationary q-TASEP, when the parameter $\alpha$ determining the average density through (2.7) scales as $\alpha = q^\theta (1 + \omega/\gamma N^{1/3})$, $\theta > 0, \omega \in \mathbb{R}$, we have, for $\forall s \in \mathbb{R}$,

$$\lim_{N \to \infty} \mathbb{P}(X_N(0) (\kappa N) > (\eta - 1) N - \gamma N^{1/3} s) = F_\omega(s) := \frac{\partial}{\partial s} \nu_\omega(s),$$ \hspace{1cm} (5.99)$$
where $\kappa, \eta, \gamma$ are given by (2.21), (2.22), (2.26). $F_\omega(s)$ is a distribution function with $\nu_\omega(s)$ expressed as

$$\nu_\omega(s) = F_2(s) \left( s - \omega^2 - \sum_{i,j=1 \atop (i,j)\neq (1,1)}^2 \int_s^\infty d\xi \mathcal{B}^{(i)}_\omega(\xi) \mathcal{B}^{(j)}_\omega(\xi) - \int_s^\infty d\xi (\rho_A A \mathcal{B}_\omega)(\xi) \mathcal{B}_{-\omega}(\xi) \right)$$  \hspace{1cm} (5.100)

where $F_2$ is the GUE Tracy-Widom distribution, $A, \rho_A$ are given below (5.91) and $\mathcal{B}_\omega, \mathcal{B}^{(i)}_\omega, i = 1, 2$ are defined in (5.32), (5.33).

**Proof.** As we have mentioned at the beginning of this subsection, it is enough to show that rhs of (5.52) goes to $F_\omega(s)$ as $t \to \infty$. For taking the limit on rhs, we want to consider

$$\lim_{N \to \infty} \frac{\alpha}{(q;q)_\infty} \sum_{k=0}^\infty \frac{(-1)^k q^{k(k+1)/2}}{(q;q)_k} \left( V_N(\zeta q^{-k}) - V_N(\zeta q^{-k-1}) \right).$$  \hspace{1cm} (5.101)

Combining Proposition 5.6, Lemmas 5.11, 5.12 and (5.90), (5.91), we see, for large $N$,

$$V_N(\zeta) = \nu_\omega(s) + O(N^{-1/3}).$$  \hspace{1cm} (5.102)

From this it is easy to see, for a given $k$,

$$\lim_{N \to \infty} \alpha(V_N(\zeta q^{-k}) - V_N(\zeta q^{-k-1})) = \frac{\partial}{\partial s} \nu_\omega(s).$$  \hspace{1cm} (5.103)

On the other hand, we see that det$(1 - A)$ in (5.52) and hence $V_N(\zeta q^{-k})$ goes to zero as $k \to \infty$ for $|\zeta| \to \infty$ for $\zeta \notin \mathbb{R}_{\geq 0}$ uniformly in $N$. Thus we find

$$\lim_{N \to \infty} \frac{\alpha}{(q;q)_\infty} \sum_{k=0}^\infty \frac{(-1)^k q^{k(k+1)/2}}{(q;q)_k} \left( V_N(\zeta q^{-k}) - V_N(\zeta q^{-k-1}) \right) = \frac{1}{(q;q)_\infty} \sum_{k=0}^\infty \frac{(-1)^k q^{k(k+1)/2}}{(q;q)_k} \frac{\partial}{\partial s} \nu_\omega(s)$$  \hspace{1cm} (5.104)

Using the relation (a special case of (A.3))

$$\sum_{k=0}^\infty \frac{(-1)^k q^{k(k-1)/2}q^k}{(q;q)_k} = (q;q)_\infty,$$  \hspace{1cm} (5.105)

we arrive at our desired expression. \hfill \square

**Remark.** The distribution function $F_\omega$ was first introduced in [8]. The representation in (5.99) with (5.100) was given in [43]. In [7, 31], somewhat different representation was discussed. In the context of the $q$-TASEP, this comes from the difference in the decomposition of the third factor in (5.27) for the $q$-TASEP. We decompose the factor in (5.27) as

$$\frac{1}{a - \alpha} - \sum_{n \in \mathbb{Z}} (\rho f B_1)(n) B_2(n)$$

$$= \frac{1}{a - \alpha} - \sum_{n \in \mathbb{Z}} f(n) B_1^{(1)}(n) B_2^{(1)}(n) - \sum_{i,j \in \{1,2\} \atop (i,j) \neq (1,1)} \sum_{n \in \mathbb{Z}} f(n) B_1^{(i)}(n) B_2^{(j)}(n) - \sum_{n \in \mathbb{Z}} (A \rho f B_1)(n) B_2(n).$$  \hspace{1cm} (5.106)
On the other hand, in the case of \([7, 31]\), the corresponding decomposition is
\[
\frac{1}{a - \alpha} - \sum_{n \in \mathbb{Z}} (\rho f B_1(n) B_2(n)) = \frac{1}{a - \alpha} - \sum_{n \in \mathbb{Z}} f(n) B_1^{(1)}(n) B_2(n) - \sum_{n \in \mathbb{Z}} ((\rho f B_1^{(1)}(n)) + (\rho f B_1^{(2)}(n))) B_2(n).
\]
(5.107)

Considering in the \(N \to \infty\) limit with \(a \to \alpha\), each function goes to
\[
f(n) \to 1_{\geq s}(\xi), \quad B_1^{(1)}(n) \to e^{\omega^3/3 + \omega \xi}, \quad B_1^{(2)}(n) \to e^{-\omega^3/3 + \omega \xi},
\]
\[
B_1^{(2)}(n) \to -\int_0^\infty d\lambda e^{\omega \lambda} \text{Ai}(\xi + \lambda), \quad B_2^{(1)}(n) \to -\int_0^\infty d\lambda e^{-\omega \lambda} \text{Ai}(\xi + \lambda),
\]
(5.108)
we easily find the representation in \([7, 31]\).

6 TASEP

In this section, we consider the stationary TASEP by taking \(q \to 0\) limit in the previous sections. The stationary TASEP was already studied and the limiting distribution was established in \([31]\). But there the relation between the stationary TASEP and the Schur process is not exact microscopically. Our approach provides a formula for the position of a tagged particle in the stationary case which is true even for finite time. Note that in the limit of reflecting Brownian motions, such formulas have been discussed \([32]\).

When \(q \to 0\), our skew and ordinary \(q\)-Whittaker functions \([3, 31]\) and \([3, 33]\) become a two-sided version of the skew and the ordinary Schur functions \([14]\). We denote them as \(s_\lambda/\mu(a)\) and \(s_\lambda(a_1, \cdots, a_N)\) respectively. On the other hand, the \(q \to 0\) limit of \(Q_\lambda(\alpha, t)\) in \((3.5)\) becomes
\[
s_\lambda(\alpha; t) := \lim_{q \to 0} Q_\lambda(\alpha; t) = \int_{\mathbb{T}^N} \prod_{i=1}^N \frac{dz_i}{z_i} \cdot s_\lambda(1/z) \Pi^0(z; \alpha, t) m_\lambda^0(z),
\]
(6.1)
where
\[
m_\lambda^0(z) = \frac{1}{(2\pi i)^N N!} \prod_{1 \leq i < j \leq N} \left(1 - \frac{z_i}{z_j}\right) \left(1 - \frac{z_j}{z_i}\right), \quad \Pi^0(z; \alpha, t) = \prod_{i,j=1}^N \frac{1}{1 - \alpha_i/z_j} \prod_{j=1}^N e^{z_j t}.
\]
(6.2)

As a special case of the remark below Definition \([3, 32]\) we find that \(s_\lambda(0; t)\) becomes the Schur function with the Plancherel specialization on the ordinary Gelfand-Tsetlin cone \(G_N^{(0)}\) \([14]\) while \(s_\lambda(\alpha; 0) = s_{-\lambda}(\alpha), \quad -\lambda \in G_N^{(0)}\). Hence we see that the \(q \to 0\) limit of \(P_t(\lambda_N)\) \((1.5)\), which is written as
\[
P_t^0(\lambda_N) = \lim_{q \to 0} P_t(\lambda_N) = \prod_{j=1}^N s_\lambda(j/j-1) (a_j) \cdot s_{\lambda(N)}(\alpha; t)
\]
(6.3)
is a type of the two-sided Schur process introduced by \([14]\). In addition, as \(q \to 0\), the generator \(L(\mu_N, \lambda_N)\) \((3.37)\) becomes a simple form
\[
L^0(\mu_N, \lambda_N) = \lim_{q \to 0} L(\mu_N, \lambda_N) = \sum_{1 \leq j \leq k \leq N} a_k 1_{\mu^k \in G_N(\mu_N)} \left(\delta_{\lambda_N, \mu^k} - \delta_{\lambda_N, \mu^k}\right).
\]
(6.4)
The dynamics defined by the generator \(L^0(\mu_N, \lambda_N)\) \((6.4)\) has been introduced in \([14, 19]\). In particular, the marginal \((\lambda_1^{(1)}, \lambda_2^{(2)}, \cdots, \lambda_N^{(N)})\) describes (the ordinary) TASEP where the rate of the \(k\)th particle is \(a_k\). (So we call the \(q \to 0\) limit the TASEP limit).

We readily obtain the TASEP limit of Theorem \([4, 3]\)
**Corollary 6.1.** For the two-sided Schur process \([6.3]\), we have
\[
\mathbb{P}\left(\lambda_N^{(N)} \geq m\right) = \det \left(1 - K_0\right)_{L^2(\mathbb{Z} \geq m + 1)}, \quad m \in \mathbb{Z},
\]
where the kernel \(K_0(m,n)\) is written as
\[
K_0(m,n) = \sum_{l=0}^{N-1} \phi_l^{(0)}(m)\psi_l^{(0)}(n)
\]
for \(l = 0, 1, \ldots, N - 2\). The contour \(C_r\) is around 0 and \(\alpha_j, 1 \leq j \leq N\), and the contour \(D\) is around 1 and \(a_j, 1 \leq j \leq N\).

**Proof.** In \([4.7]\) we set
\[
\zeta = -(1 - q)q^{-m+\delta}
\]
where \(m \in \mathbb{Z}_{\geq 0}\) and take the \(\delta \downarrow 0\) limit. Noting that under \([6.9]\)
\[
\lim_{\delta \downarrow 0} \lim_{q \to 0} \frac{1}{\zeta^{q^{2}}q_{\infty}} = 1_{\geq 0}(x - m)
\]
where \(1_{\geq 0}(x) = 1\) for \(x \geq 0\) and 0 for \(x < 0\), we find that the rhs of \([4.7]\) goes to
\[
\lim_{q \to 0} \left(1_{\zeta^{q^{2}}q_{\infty}}\right) = \mathbb{P}\left(\lambda_N^{(N)} \geq m\right).
\]
On the other hand we can also easily take the \(q \to 0\) limit of the rhs of \([4.7]\). As in \([6.10]\), under \([6.9]\) we have
\[
\lim_{q \to 0} \frac{1}{1 - q^{n}/\zeta} = 1_{\geq 0}(n - m + 1).
\]
In addition, \(\phi_l^{(0)}(n)\) \([6.7]\) and \(\psi_l^{(0)}(n)\) \([6.8]\) are easily obtained by taking \(q \to 0\) limit of \(\phi_l(n)\) \([4.10]\) and \(\psi_l(n)\) \([4.11]\) respectively. From these, we see that the rhs of \([4.7]\) goes to that of \([6.5]\). 

The relation \([6.5]\) can be interpreted as that for the \(X_N(t)\), the position of \(N\)th particle of TASEP since \(\lambda_N^{(N)}(t) = X_N(t) + N - 1\). One finds
\[
\mathbb{P}\left(\lambda_N^{(N)} \geq m\right) = \mathbb{P}_{\text{TASEP}}\left(X_N(t) \geq m - N + 1\right),
\]
where in the rhs \(\mathbb{P}_{\text{TASEP}}\) represents the probability measure with respect to both the TASEP dynamics and initial configurations. For general parameters \(a_j\), \(\alpha_j\), \(j = 1, \ldots, N\), the pdf of the initial configurations is given as the marginal density of \([6.3]\) on \(\lambda_1^{(1)}, \lambda_2^{(2)}, \ldots, \lambda_N^{(N)}\). For more special case where one of \(\alpha_j\) remain finite (and it is set to be \(\alpha\)) while all the others are zero, the initial configuration becomes the half stationary initial data, i.e. the gap between \(j - 1\)th and \(j\)th particle obeys the geometric distribution, \(X_j(0) - X_{j-1}(0) \sim \text{Ge}(\alpha/a_j), 1 \leq j \leq N\) with \(X_0(0) = 0\).
Corollary 6.2. For TASEP with the parameters $a, \alpha$ described above, $P_{TASEP}(X_N(t)\geq m-N+1)$ is written as the Fredholm determinant in rhs of (6.5) where in the functions $\phi_l^{(0)}, \psi_l^{(0)}$ the parameters are specialized as described above.

When we further specialize the parameters as $\alpha_1 = \cdots = \alpha_N = 0$, $a_1 = \cdots = a_N = 1$, the initial configuration reduces to the step initial condition, $X_j(0) = -j$. In this case, the kernel $K_0(m,n)$ of the Fredholm determinant can be written as

$$K_0(m,n) = \sum_{l=0}^{N-1} \frac{t^{2l-n-N+1} e^{-t}}{l!(l-n-N+1)!} C_l(l-n-N; t) C_l(l-n-N+1; t) \quad (6.14)$$

Here $C_l(x;t)$ is the $l$th order Charlier polynomial, and the orthogonal relation is given by

$$\sum_{x=0}^{\infty} \frac{t^x e^{-t}}{x!} C_m(x;t) C_n(y;t) = t^{-n} n! \delta_{m,n}. \quad (6.15)$$

This expression follows from the relation: in the case $\alpha_1 = \cdots = \alpha_N = 0$, $a_1 = \cdots = a_N = 1$,

$$\phi_l^{(0)}(n) = \int_D dv \frac{e^{-vt}}{v^{n+l} (v-1)^{l+1}} = \frac{t^l e^{-t}}{l!} C_l(l-n-N; t), \quad (6.16)$$

$$\psi_l^{(0)}(n) = \int_{C_r} dz e^{zt} z^{n+1-l-1} (z-1)^l = \frac{t^l e^{-t}}{(l-n-N+1)!} C_l(l-n-N+1; t). \quad (6.17)$$

Though these functions are not exactly the same as the ones in [45][46], one could perform the asymptotics with this kernel and establish the GUE Tracy-Widom distribution.

Next we take the TASEP limit of Proposition 5.6 for the stationary $q$-TASEP. Let us set

$$B_1^{(1)}(n) = \lim_{q \to 0} B_1^{(1)}(n) = \frac{e^{-\alpha t}}{\alpha^{n+1} (\alpha - 1)^{N-1}}, \quad (6.18)$$

$$B_1^{(2)}(n) = \lim_{q \to 0} B_1^{(2)}(n) = \int_{D_1} \frac{dv}{2\pi i} \frac{e^{-vt}}{v^{n+1} (v-1)} \left( \frac{1}{v-1} \right)^{N-1} \frac{1}{v-\alpha}, \quad (6.19)$$

$$B_2^{(1)}(n) = \lim_{q \to 0} B_2^{(1)}(n) = \alpha^{n+1} e^{\alpha t} (\alpha - 1)^{N-1}, \quad (6.20)$$

$$B_2^{(2)}(n) = \lim_{q \to 0} B_2^{(2)}(n) = \oint_{|z|<\alpha} \frac{dz}{2\pi i z} \frac{e^{zt} z^{n+1}}{z-\alpha} (z-1)^{N-1}, \quad (6.21)$$

where $B_j^{(i)}(n), i, j = 1, 2$ are defined by (5.31)–(5.34) respectively and we set $B_{10}(n) := B_1^{(1)}(n) + B_1^{(2)}(n)$ and $B_{20}(n) := B_2^{(1)}(n) + B_2^{(2)}(n)$.

We also define the kernel $A_0(m,n)$ as the $q \to 0$ limit of $A(m,n)$ (5.24) with (6.9),

$$A_0(m,n) = \lim_{q \to 0} A(n_1, n_2) = 1_{\geq 0}(n_1 - m + 1) \sum_{l=0}^{N-2} \phi_l^{(0)}(n_1) \psi_l^{(0)}(n_2), \quad (6.22)$$

where $\phi_l^{(0)}(n_1)$ and $\psi_l^{(0)}(n_2)$ are given by (5.7) and (6.8) respectively.

Then taking the $q \to 0$ limit of (5.5) with (6.9), we have the following.

Corollary 6.3. Let $X_N^{(0)}(t)$ be the position of the $N$th particle in the stationary TASEP, with the conditioning that there is a particle at the origin initially, with parameter $\alpha$ at time $t$. We have

$$P(X_N^{(0)}(t) \geq m-N) = \alpha (V_0(m) - V_0(m+1)), \quad (6.23)$$
where for \( m \in \mathbb{Z} \)

\[
V_0(m) = \det (1 - A_0) \left( -\frac{1}{\alpha}(m - 1) + \frac{(N - 1)\alpha}{1 - \alpha} + t \right.
- \sum_{i,j=1, i,j \neq (1,1)} \sum_{n \in \mathbb{Z}} 1_{n \geq 0}(n + m - 1) B^{(i)}_{10}(n; \alpha) B^{(i)}_{20}(n; \alpha)
- \sum_{n \in \mathbb{Z}} 1_{n \geq 0}(n + m - 1)(\rho A_0 A_0 B_{10})(a; \alpha) B_{10}(n; \alpha) \bigg)
\]

(6.24)

and \( \rho A_0 = (1 - A_0)^{-1} \).

**Proof.** We can show that \( G_0(\zeta) \) with \( a = \alpha \) goes to the lhs of (6.23) in the same way as (6.11). For the rhs, we note under (6.9)

\[
\lim_{\delta \downarrow 0} \lim_{q \to 0} \frac{q^n / \zeta}{1 - q^n / \zeta} = \lim_{\delta \downarrow 0} \frac{-q^{n+m-\delta}/(1-q)}{1 + q^{n+m-\delta}/(1-q)} = \begin{cases} 0, & n \geq -m + 1, \\ -1, & n \leq -m \end{cases}
\]

(6.25)

\[
\lim_{\delta \downarrow 0} \lim_{q \to 0} \frac{-\zeta q^{n+1}}{1 - \zeta q^{n+1}} = \lim_{\delta \downarrow 0} \frac{(1-q)q^{n+m+1+\delta}}{1 + (1-q)q^{n-m+\delta}} = \begin{cases} 0, & n \geq m - 1, \\ -1, & n \leq m - 2 \end{cases}.
\]

(6.26)

Combining these, we have the simple relation

\[
\lim_{\delta \downarrow 0} \lim_{q \to 0} \left( \frac{q^n / \zeta}{1 - q^n / \zeta} - \frac{\zeta q^{n+1}}{1 - \zeta q^{n+1}} \right) = m - 1.
\]

(6.27)

From (6.18)–(6.29) with the simple fact

\[
\lim_{q \to 0} \left( 2 \frac{q^n+1}{1 - q^{n+1}} + \frac{(N - 1)\alpha q^n}{1 - \alpha q^n} \right) = \frac{(N - 1)\alpha}{1 - \alpha},
\]

(6.28)

we see

\[
\lim_{q \to 0} V(\zeta) = V_0(m).
\]

(6.29)

Furthermore noting in the rhs of (5.52), only the term with \( k = 0 \) contributes to the \( q \to 0 \) limit, we find that the rhs goes to the one in (6.23). \( \square \)

By considering the \( q = 0 \) case of our analysis for \( q \)-TASEP in the previous section, one can prove that the fluctuations of a tagged particle in stationary TASEP is described by the Baik-Rains distribution.

So far the fluctuation properties of the stationary TASEP have been studied in [7,31]. There the authors introduced a last passage percolation (LPP) model whose last passage time approximates a particle position of the stationary TASEP. The LPP model has a nice property that the distribution function of the last passage time is represented in the language of the Schur measure. However it has been known that there is a discrepancy between the last passage time and the position of the stationary TASEP for finite time \( t \), although Proposition 3 in [31] shows that the discrepancy vanishes in the large time limit. Note that our formula (6.23) provides the exact distribution function of the particle position since we directly deal with the particle position of the \((q-)\)TASEP without using the approximations above.
A Some $q$-functions and $q$-formulas

In this appendix, we summarize a few $q$-notations, $q$-functions and $q$-formulas. The first is the $q$-Pochhammer symbol, or the $q$-

\[(a; q)_\infty = \prod_{n=0}^\infty (1 - aq^n), \quad (a; q)_n = \frac{(a; q)_\infty}{(aq^n; q)_\infty} \tag{A.1}\]

The $q$-binomial theorem will be useful in various places in the discussions,

\[\sum_{n=0}^\infty (a; q)_n z^n = \frac{(a z; q)_\infty}{(z; q)_\infty}, \quad |z| < 1. \tag{A.2}\]

In particular the $a = 0$ case appears in many applications. Another $q$-binomial formula reads (see e.g. Cor.10.2.2.b in [6])

\[\sum_{n=0}^\infty (-1)^n q^{n(n-1)/2} (q; q)_n z^n = (z; q)_\infty. \tag{A.3}\]

There is yet another version of the $q$-binomial theorem (see e.g. Cor.10.2.2.c in [6]),

\[\sum_{k=0}^\ell (-1)^k q^{k(k-1)/2} (q; q)_k (q^\ell; q)_{\ell-k} x^k = (1 - x)(1 - xq) \cdots (1 - xq^{\ell-1}). \tag{A.4}\]

The $q$-exponential function, denoted as $e_q(z)$ is defined to be

\[e_q(z) := \frac{1}{((1 - q)z; q)_\infty} = \sum_{n=0}^\infty \frac{(1 - q)_n}{(q; q)_n} z^n. \tag{A.5}\]

The second equality is by the above $q$-binomial theorem (A.2). From the series expansion expression, it is easy to see that this tends to the usual exponential function in the $q \to 1$ limit.

The $q$-Gamma function $\Gamma_q(x)$ is defined by

\[\Gamma_q(x) = (1 - q)^{1-x} (q; q)_\infty \frac{(q^x; q)_\infty}{(q^x; q)_\infty}. \tag{A.6}\]

The $q$-digamma function is defined by $\Phi_q(z) = \frac{d}{dz} \log \Gamma_q(z)$. In the $q \to 1$ limit, they tends to the usual $\Gamma$ function and the digamma function respectively.

Ramanujan’s summation formula (cf [6] p502, [36] p138) is a two-sided generalization of the above $q$-binomial theorem (A.2). For $|q| < 1, |b/a| < |z| < 1$,

\[\sum_{n \in \mathbb{Z}} (a; q)_n z^n = \frac{(az; q)_\infty (\frac{a}{z}; q)_\infty (q; q)_\infty (b; q)_\infty}{(z; q)_\infty (\frac{a}{z}; q)_\infty (b; q)_\infty (\frac{b}{a}; q)_\infty}. \tag{A.7}\]

B $q$-Whittaker functions and $q$-Whittaker process

In this appendix, we introduce the ordinary (skew) $q$-Whittaker functions labeled by partitions and the $q$-Whittaker process and then discuss some of their properties. Most of them are standard [59] but some of them are reformulated for the applications in the main text.

First a partition of length $n \in \mathbb{N}$ is an $n$-tuple $\lambda = (\lambda_1, \ldots, \lambda_n)$ with $\lambda_j \in \mathbb{Z}+, 1 \leq j \leq n$ s.t. $\lambda_1 \geq \ldots \geq \lambda_n$. The set of all partitions of length $n$ is denoted by $\mathcal{P}_n$. A partition $\lambda$ can also be
represented (and identified) as a Young diagram with \( n \) rows of length \( \lambda_1, \ldots, \lambda_n \). For two partitions \( \lambda \in \mathcal{P}_n, \mu \in \mathcal{P}_m \) s.t. \( m \leq n \) and \( \lambda_i - \mu_i \geq 0, 1 \leq i \leq n \) (with the understanding \( \mu_i \equiv 0, m < i \leq n \)), a pair \((\lambda, \mu)\) is called a skew diagram and is denoted by \( \lambda/\mu \). The transpose \( \lambda' \) is the partition of length \( \lambda_1 \) defined as \( \lambda'_i = \# \{ j \in \mathbb{Z}_+ | \lambda_j \geq i \} \), \( 1 \leq i \leq \lambda_1 \).

The Gelfand-Tsetlin (GT) cone for partitions, denoted by \( \mathcal{G}_N^{(0)} \), is defined as

\[
\mathcal{G}_N^{(0)} := \{ (\lambda^{(1)}, \lambda^{(2)}, \ldots, \lambda^{(N)}) \in \mathcal{P}_N, 1 \leq n \leq N | 0 \leq \lambda_{i+1}^{(m+1)} \leq \lambda_{i}^{(m)} \leq \lambda_{i}^{(m+1)}, 1 \leq \ell \leq m \leq N - 1 \}. \tag{B.1}
\]

An element \( \lambda_N \in \mathcal{G}_N^{(0)} \) is called a Gelfand-Tsetlin pattern. Next we define the (skew) \( q \)-Whittaker functions.

**Definition B.1.** Let \( \lambda \in \mathcal{P}_n, \mu \in \mathcal{P}_{n-1} \) be two partitions of order \( n \) and \( n-1 \) respectively and \( a \) an indeterminate. The skew \( q \)-Whittaker function (with 1 variable) is defined as (cf. \[59\] VI.7, (7.14) and Ex. 2.)

\[
P_{\lambda/\mu}(a) = \prod_{i=1}^{n} a^\lambda_i \cdot \prod_{i=1}^{n-1} \frac{a^{-\mu_i}(q;q)_{\lambda_i-\lambda_{i+1}}}{(q;q)_{\lambda_i-\mu_i}(q;q)_{\mu_i-\lambda_{i+1}}}. \tag{B.2}
\]

Using this, for a partition \( \lambda \in \mathcal{P}_N \) and \( a = (a_1, \ldots, a_N) \) being \( N \) indeterminates, we define the \( q \)-Whittaker function with \( N \) variables as (cf. \[59\] VI.7, (7.9'))

\[
P_{\lambda}(a) = \sum_{\lambda_i^{(k)} \leq a_j \leq \lambda_i^{(k+1)} \leq \lambda_i^{(k+1)}} \prod_{j=1}^{N} P_{\lambda^{(j)}/\lambda^{(j-1)}}(a_j). \tag{B.3}
\]

Here the sum is over the Gelfand-Tsetlin cone \( \mathcal{G}_N^{(0)} \) with the condition \( \lambda^{(N)} = \lambda \) and \( \lambda^{(0)} = \phi \).

It is known that the \( q \)-Whittaker function \( P_{\lambda}(a) \) forms a basis of \( \Lambda_N \), the space of \( N \)-variable symmetric polynomials with coefficients being rational functions in \( q \) (cf. \[59\] VI). There is an inner product \( \langle , \rangle \) in this space for which \( P_{\lambda} \)'s are orthogonal (\[59\] VI (6.19)):

\[
\langle P_{\lambda}, P_{\mu} \rangle = (q;q)_{\lambda_N} \prod_{i=1}^{N-1} (q;q)_{\lambda_i-\lambda_{i+1}} \delta_{\lambda,\mu}. \tag{B.4}
\]

Using this, we also introduce \( Q_{\lambda}(x) \).

**Definition B.2.** For \( \lambda \in \mathcal{P}_N \) and for \( x = (x_1, \ldots, x_N) \), we define (\[59\] VI (4.11), (4.12))

\[
Q_{\lambda}(x) = \frac{P_{\lambda}(x)}{\langle P_{\lambda}, P_{\lambda} \rangle}, \tag{B.5}
\]

Note that \( P_{\lambda} \) and \( Q_{\lambda} \) are orthonormal: \( \langle P_{\lambda}, Q_{\mu} \rangle = \delta_{\lambda,\mu} \).

If we take a sum over all partitions of length \( N \) of a product of \( P_{\lambda} \) and \( Q_{\lambda} \), the following Cauchy identity for the \( q \)-Whittaker functions holds (\[59\] VI(4.13),(2.5)),

\[
\sum_{\lambda \in \mathcal{P}_N} P_{\lambda}(x)Q_{\lambda}(y) = \prod_{i,j=1}^{N} \frac{1}{(x_iy_j;q)_\infty} =: \Pi(x; y). \tag{B.6}
\]
Here we rewrite this for applications in the main text. First one notices that one can express $P_\lambda(x)$ in a form,

$$ P_\lambda(x) = X^{\lambda N} R_\ell(x), \quad (B.7) $$

where $X = x_1 \cdots x_N$ and $\ell = (\ell_1, \ldots, \ell_{N-1})$ with $\ell_j = \lambda_j - \lambda_{j+1}, 1 \leq j \leq N - 1$ and $R_\ell \in \Lambda_N$. This is seen as follows. Since $(\lambda^{(1)}, \ldots, \lambda^{(N)})$ in $(B.3)$ is an element of $\mathbb{G}_N$, the order of each term in the sum in $(B.3)$ is bigger than or equal to $\lambda^{(N)}$ and one can factor out this lowest order common factor. Then a partition $\lambda$ is uniquely determined by $\lambda^{(N)}$ and $\ell_j, j = 1, \ldots, N - 1$ and the coefficient of each term depends only on $\ell_j, j = 1, \ldots, N - 1$. Then we have $(B.7)$. We do not write explicitly the form of the function $R_\ell(x)$ since it is not necessary in the following discussion. In terms of $R_\ell$, the Cauchy identity $(B.6)$ is equivalent to

$$ \sum_{\ell_1, \ldots, \ell_{N-1} = 0} R_\ell(x) R_\ell(y) \prod_{j=1}^{N-1} \frac{1}{(q; q)_{\ell_j}} = \frac{(X Y'; q)_\infty}{\prod_{i,j=1}^N (x; y; q)_\infty} \quad (B.8) $$

with $Y = y_1 \cdots y_N$. In fact if one substitutes $(B.7)$ into lhs of $(B.6)$ and uses $(B.8)$ and the $q$-binomial theorem $(A.2)$ with $a = 0$, we get rhs of $(B.6)$.

There is another inner product $\langle \cdot, \cdot \rangle'$ in $\Lambda_N$ called the torus scaler product: For the $N$-variable functions, $f(z), g(z) \in \Lambda_N$, the torus scalar product is defined by $(59)$ VI (9.10)

$$ \langle f, g \rangle'_N = \int_{\mathbb{T}^N} \prod_{j=1}^N \frac{dz_j}{z_j} \cdot f(z) \overline{g(z)} m^q_N(z), \quad (B.9) $$

where

$$ m^q_N(z) = \frac{1}{(2\pi i)^N N!} \prod_{1 \leq i < j \leq N} (z_i/z_j; q)_\infty (z_j/z_i; q)_\infty \quad (B.10) $$

is the $q$-Sklyanin measure. The $q$-Whittaker functions of $N$ variables are known to satisfy the following orthogonality relations with respect to this inner product $(59)$.

$$ \langle P_\lambda, P_\mu \rangle'_N = \frac{1}{(q^{\lambda_i-\lambda_{i+1}}; q)_\infty} \cdot \delta_{\lambda, \mu}. \quad (B.11) $$

Here we rewrite the orthogonality relation $(B.11)$. From the definition $(B.3)$, we find that $P_\lambda(e^{i\theta}) = P_\lambda(e^{i\theta_1}, \ldots, e^{i\theta_N})$ can be expressed as

$$ P_\lambda(e^{i\theta}) = e^{iN\tilde{\lambda} \tilde{\vartheta}} \tilde{P}_\ell(\tilde{\theta}), \quad (B.12) $$

where $\tilde{\theta} = (\theta_1 + \cdots + \theta_N)/N$ and $\tilde{\lambda} = (\lambda_1 + \cdots + \lambda_N)/N$ represent the barycentric coordinates while relative coordinates are denoted by $\tilde{\vartheta} = (\tilde{\theta}_1, \ldots, \tilde{\theta}_{N-1})$ and $\ell = (\ell_1, \ldots, \ell_{N-1})$ with $\tilde{\theta}_j = \theta_j - \theta_{j+1}$ and $\ell_j = \lambda_j - \lambda_{j+1}, 1 \leq j \leq N - 1$. Considering this we see that $(B.11)$ can be rewritten as, with $m_j = \mu_j - \mu_{j+1}, 1 \leq j \leq N - 1$

$$ \langle P_\lambda, P_\mu \rangle'_N = \int_0^{2\pi} d\tilde{\theta} e^{iN\tilde{\lambda} \tilde{\vartheta}} \int_{(-\pi, \pi)^{N-1}} d\tilde{\vartheta} \tilde{P}_\ell(\tilde{\theta}) \tilde{P}_m(-\tilde{\theta}) \prod_{1 \leq j < k \leq N-1} (e^{i(\theta_j - \cdots + \theta_k)}; q)_\infty^2 \Bigg|_{\tilde{\theta} = \hat{\theta}} \prod_{1 \leq j < k \leq N-1} \left| (e^{i(\theta_j - \cdots + \theta_k)}; q)_\infty \right|^2 \quad (B.13) $$

where

$$ \langle P_\lambda, P_\mu \rangle'_N = \frac{1}{(q^{\ell_i+1}; q)_\infty} \cdot \delta_{\lambda, \mu}. $$
Hence for the difference variables \( \ell, m \in \mathbb{Z}_{\geq 0}^{N-1} \), the orthogonality relation \((B.11)\) can be restated as

\[
\int_{(-\pi, \pi)^{N-1}} d\vec{\theta} \mathcal{P}_l(\vec{\theta}) \mathcal{P}_m(-\vec{\theta}) \prod_{1 \leq j < k \leq N-1} (e^{i(\theta_j + \cdots + \theta_k)}; q)_\infty)^2 = \prod_{i=1}^N \frac{1}{(q^{1+}; q)_\infty} \cdot \delta_{l,m}. \tag{B.14}
\]

One finds a representation of \( Q_\lambda \) using torus scalar product,

**Lemma B.3.** For \( y \in \mathbb{R}^N, \lambda \in \mathcal{P}_N \),

\[
Q_\lambda (y) = \frac{1}{\langle P_\lambda, P_\lambda \rangle_N} \langle (\Pi(\cdot, y), P_\lambda(\cdot))' \rangle_N
= \prod_{i=1}^{N-1} (q^{\lambda_i-\lambda_{i+1}+1}; q)_\infty \int_{\mathbb{T}^N} \prod_{i=1}^N \frac{dz_i}{z_i} \cdot P_\lambda (1/z) \Pi (z; y) m_N^q (z), \tag{B.15}
\]

where \( 1/z \) in \( P_\lambda \) is a shorthand notation for \( (1/z_1, \cdots, 1/z_N) \).

**Proof.** It immediately follows from the Cauchy identity \((B.6)\) and the orthogonality \((B.11)\). \( \square \)

We can consider a generalization of the function \( Q_\lambda \) by modifying \( \Pi(z; y) \) in \((B.15)\) to

\[
\Pi(z; \{\alpha, \beta, \gamma\}) = \prod_{i=1}^{N} \prod_{j=1}^{M} \frac{1 + z_i \beta_j}{(z_i \alpha_j; q)_\infty} e^{\gamma z_i}, \tag{B.16}
\]

with \( \alpha_j, \beta_j, \gamma \geq 0 \) for \( j = 1, \cdots, M \). We call this \( Q_\lambda (\{\alpha, \beta, \gamma\}) \). The case where only some of \( \alpha_j \)'s (resp. \( \beta_j \)'s or \( \gamma \)) are positive is called the \( \alpha \)-specialization (resp. \( \beta \)-specialization or Plancherel specialization) in \([15]\). Note that the \((B.16)\) for the \( \alpha \)-specialization is nothing but \( \Pi(z; \alpha) \) in \((B.6)\). The (ascending) \( q \)-Whittaker process corresponding to this generalization is defined by

\[
P(\lambda_N) := \prod_{j=1}^N P_{\lambda_j/\lambda_j-1} (a_j \cdot Q_{\lambda(N)} (\{\alpha, \beta, \gamma\})). \tag{B.17}
\]

For instance we define \( Q^{(\beta)}_\lambda (y), y \in \mathbb{R}^M, M \in \mathbb{N} \) by \((B.15)\) with \( \Pi(z; y) \) replaced by \( \Pi_M^{(\beta)} (z; y) := \Pi(z; \alpha = 0, \beta = y, \gamma = 0) = \prod_{i=1}^N \prod_{j=1}^M (1 + z_i y_j) \). We state two properties of the \( q \)-Whittaker functions with this \( \beta \)-specialization, which will be also useful to discuss our \( q \)-TASEP with a random initial condition (cf. \([59]\) VI 7).

\[
\sum_{\kappa} P_{\kappa/\lambda} (a) Q^{(\beta)}_{\kappa/\mu} (r) = \sum_{\tau} P_{\mu/\tau} (a) Q^{(\beta)}_{\lambda/\tau} (r) \cdot (1 + ra), \tag{B.18}
\]

\[
\sum_{\mu} Q^{(\beta)}_{\lambda/\mu} (r_{M+1}) Q^{(\beta)}_{\mu} (\{r\}_M) = Q^{(\beta)}_{\lambda} (\{r\}_{M+1}), \tag{B.19}
\]

where in \((B.19)\), \( Q^{(\beta)}_{\lambda/\mu} (r) \) with one variable is defined as

\[
Q^{(\beta)}_{\lambda/\mu} (r) := \prod_{i \geq 1, \lambda_i = \mu_i, \lambda_{i+1} = \mu_{i+1} + 1} (1 - q^{\mu_i - \mu_{i+1} + \mu_{i+1} + 1}) r_{i}^{|\lambda| - |\mu|} \tag{B.20}
\]

and we introduced a notation \( \{r\}_M = (r_1, \cdots, r_M) \) to consider a change of \( M \) below.

In the rest of this appendix we consider some properties of the \( q \)-Whittaker process corresponding to this pure \( \beta \) specialization.

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Definition B.4. For \( M \in \mathbb{N} \),
\[
P_M(\Delta_N) = \frac{1}{\prod_{j=1}^N P_{\lambda(j)/\lambda(j-1)}(a_j)} \prod_{j=1}^N P_{\lambda(j)/\lambda(j-1)}(a_j) \cdot Q_{\lambda(N)}^B(\{r\}_M). \tag{B.21}
\]

As mentioned above, the positivity of \( (B.21) \) is known. This process describes a distribution function at time \( M \) of a discrete time Markov process on \( G_N^{(0)} \) described by the following Kolmogorov forward equation.

**Proposition B.5.**
\[
P_{M+1}(\Delta_N) = \sum_{\mu_N} P_M(\mu_N)G_M(\mu_N, \Delta_N) \tag{B.22}
\]
where the transition matrix \( G_M(\mu_N, \Delta_N) \) is
\[
G_M(\mu_N, \Delta_N) = \prod_{j=1}^N \frac{P_{\lambda(j)/\lambda(j-1)}(a_j)Q_{\lambda(j)/\mu(j)}^B(r_{M+1})}{\Delta(\lambda(j-1), \mu(j))}, \tag{B.23}
\]
with \( \Delta(\lambda, \mu) = \sum_{k} P_{k/\lambda}(a)Q_{k/\mu}^B(r_M) \).

**Remark.** It is easy to see that each factor in \( (B.23) \) is a transition probability matrix from the definition of \( \Delta(\lambda, \mu) \).

**Proof.** We rewrite \( G_M(\mu_N, \Delta_N) \) as
\[
G_M(\mu_N, \Delta_N) = \prod_{r=1}^N A_r^B(r_{M+1}) \cdot B(r_{M+1}) \prod_{j=1}^N P_{\lambda(j)/\lambda(j-1)}(a_j), \tag{B.24}
\]
where
\[
A_r^B(r_{M+1}) = \frac{Q_{\lambda(r-1)/\mu(r-1)}^B(r_{M+1})}{\Delta(\lambda(r-1), \mu(r))}, \quad B(r_{M+1}) = Q_{\lambda(N)/\mu(N)}^B(r_{M+1}). \tag{B.25}
\]

Using \( (B.18) \) and \( (B.19) \), we have
\[
\sum_{\mu(N)} P_{\mu(r)/\mu(r-1)}(a_r)A_r^B(r_{M+1}) = \frac{1}{1 + a_r r_{M+1}}, \quad 1 \leq r \leq N, \tag{B.26}
\]
\[
\sum_{\mu(N)} Q_{\lambda(N)/\mu(N)}^B(\{r\}_M)B(r_{M+1}) = Q_{\lambda(N)}^B(\{r\}_{M+1}). \tag{B.27}
\]

Thus we have
\[
\sum_{\mu_N} P_M(\mu_N)G_M(\mu_N, \Delta_N)
\]
\[
= \frac{1}{\prod_{r=1}^N P_{r/\lambda}(a_r)} \prod_{r=1}^N \left( \sum_{\mu(r-1)} P_{\mu(r)/\mu(r-1)}(a_r)A_r^B(r_{M+1}) \right)
\]
\[
\times \sum_{\mu(N)} Q_{\lambda(N)/\mu(N)}^B(r_{M+1})B(r_{M+1}) \cdot \prod_{j=1}^N P_{\lambda(j)/\lambda(j-1)}(a_j)
\]
\[
= \frac{1}{\prod_{r=1}^N P_{r/\lambda}(a_r)} \prod_{r=1}^N \frac{1}{1 + a_r r_{M+1}} \cdot Q_{\lambda(N)}^B(\{r\}_{M+1}) \cdot \prod_{j=1}^N P_{\lambda(j)/\lambda(j-1)}(a_j) = P_{M+1}(\Delta_N). \tag{B.28}
\]
C Two lemmas regarding the Airy kernel and the GUE Tracy-Widom limit

Here we provide two lemmas for establishing the GUE Tracy-Widom limit when a kernel of a specific form is given. The Hermite kernel for the GUE is a simplest example. For GUE, one can also use a bound due to Ledoux, which holds for GUE and simplifies the proof for the case (cf [45,56]), but in other applications such a bound is not available. Our lemmas do not rely on such extra information but focus only on properties of the kernel. The essential part of the arguments are given in [45] and [20] but we reformulate and generalize them in a way which would be suited for various applications. The Airy kernel $K(\xi, \zeta)$ is defined as

$$K(\xi, \zeta) = \int_0^\infty d\lambda \text{Ai}(\xi + \lambda)\text{Ai}(\zeta + \lambda).$$

(C.1)

Our discussions below are given in a continuous setting but can be also applied to a discrete setting as well.

Lemma C.1. Suppose that we have a kernel of the form,

$$\mathcal{K}_N^{(0)}(x, y) = \sum_{n=0}^{N-1} \varphi_n(x)\psi_n(y),$$

(C.2)

where $\varphi_n, \psi_n, n \in \mathbb{N}$ are complex functions on $\mathbb{R}$. Assume that the functions $\varphi_n(x)$ satisfy the followings for some $a \in \mathbb{R}, \gamma > 0, c > 0$.

(a) For $\forall M > 0$ and $\forall L > 0$,

$$\lim_{N \to \infty} \varphi_{N-cN^{1/3}\lambda}(aN + \gamma N^{1/3}\xi) = \text{Ai}(\xi + \lambda),$$

(C.3)

uniformly for $|\xi| < L$ and for $\lambda \in [0, M]$.

(b) For $\forall L > 0, \forall \epsilon > 0$ and $N$ large enough,

$$\varphi_{N-cN^{1/3}\lambda}(aN + \gamma N^{1/3}\xi) \leq c_0 e^{-\xi-\lambda},$$

(C.4)

for $\xi, \lambda$ satisfying $\lambda > 0, L \leq |\xi + \lambda| \leq \epsilon N^{2/3}$ and for some constant $c_0$.

(c) For $\forall \epsilon > 0$ and $N$ large enough,

$$\varphi_{N-cN^{1/3}\lambda}(aN + \gamma N^{1/3}\xi) \leq c_1 e^{-\xi-\lambda},$$

(C.5)

for $\xi, \lambda$ satisfying $\lambda > 0, |\xi + \lambda| > \epsilon N^{2/3}$ and for some constant $c_1$.

The functions $\psi_n$ are also assumed to satisfy the same conditions (a),(b),(c) with the same parameters $a \in \mathbb{R}, \gamma > 0, c > 0$. Define the rescaled kernel,

$$\mathcal{K}_N(\xi, \zeta) = c N^{1/3} \mathcal{K}_N^{(0)}(aN + \gamma N^{1/3}\xi, aN + \gamma N^{1/3}\zeta).$$

(C.6)

Then we have

(i) For $\forall L > 0$,

$$\lim_{N \to \infty} \mathcal{K}_N(\xi, \zeta) = \mathcal{K}(\xi, \zeta), \quad \text{uniformly (in } N\text{) on } [-L, L]^2.$$ (C.7)

(ii) For $\forall L > 0$,

$$|\mathcal{K}_N(\xi, \zeta)| \leq c_1 e^{-\max(0, \xi) - \max(0, \zeta)}, \quad \text{uniformly (in } N\text{) for } \xi, \zeta \geq -L$$ (C.8)

for some constant $c_1$. 52
Proof. First we divide the sum over \(n\) in (C.2) as \(\sum_{n=0}^{N-1} = \sum_{n=0}^{[N-cN^{1/3}M]} + \sum_{n=[N-cN^{1/3}M]+1}^{N-1}\), where \([x]\) is the maximum integer which is less than or equal to \(x \in \mathbb{R}\). For the second sum, we have, due to the uniform convergence in (a),

\[
\lim_{N \to \infty} cN^{1/3} \sum_{n=N-cN^{1/3}M}^{N-1} \varphi_n(aN + cN^{1/3} \xi) \psi_n(aN + cN^{1/3} \zeta) = \int_{0}^{M} d\lambda \text{Ai}(\xi + \lambda) \text{Ai}(\zeta + \lambda) \tag{C.9}
\]

for \(\xi, \zeta \in [-L, L]\) for \(\forall L > 0\). The first sum is, due to (c),(d), bounded by \(\int_{M}^{\infty} e^{-2\lambda} d\lambda = e^{-2M/2}\). By taking the \(M \to \infty\) limit, we have (C.7) in (i). The bound in (ii) easily follows from the uniform convergence in (a) (Note \(\text{Ai}(\xi)\) is bounded by a constant as \(\xi \to -\infty\) and by \(e^{-\xi}\) as \(\xi \to \infty\)) combined with the bounds in (b),(c). \(\square\)

Lemma C.2. Suppose that a (sequence of ) kernel \(K_N : \mathbb{R} \times \mathbb{R} \to \mathbb{C}, N \in \mathbb{N}\) satisfies

(i) For \(\forall L > 0\),

\[
\lim_{N \to \infty} K_N(\xi, \zeta) = K(\xi, \zeta), \quad \text{uniformly in } [-L, L]^2. \tag{C.10}
\]

(ii) For \(\forall L > 0\),

\[
|K_N(\xi, \zeta)| \leq c_1 e^{-\max(0, \xi) - \max(0, \zeta)}, \quad \text{uniformly (in } N \text{) for } \xi, \zeta \geq -L, \tag{C.11}
\]

for some constant \(c_1\).

In addition suppose that a (sequence of ) function \(f_N : \mathbb{R} \to \mathbb{R}, N \in \mathbb{N}\) satisfy

(iii) The functions \(f_N, N \in \mathbb{N}\) are uniformly bounded and converges to \(1_{(s, \infty)}\) for \(s \in \mathbb{R}\) in \(L^1(\mathbb{R})\) norm,

\[
\lim_{N \to \infty} \int_{\mathbb{R}} |f_N(\xi) - 1_{(s, \infty)}(\xi)| d\xi = 0. \tag{C.12}
\]

(iv) For \(\forall L(> |s|), \sqrt{|f_N(\xi)f_N(\zeta)|}\) satisfies

\[
\lim_{N \to \infty} \sqrt{|f_N(\xi)f_N(\zeta)|} = e^{s+\xi} \tag{C.13}
\]

and

\[
\lim_{N \to \infty} \sqrt{|f_N(\xi)f_N(\zeta)|} = 0 \tag{C.14}
\]

uniformly (in } N\) for \(\xi, \zeta\) satisfying \(\xi \leq -L \) or \(\zeta \leq -L\).

Then for \(\forall s \in \mathbb{R}, \lim_{N \to \infty} \det(1 - f_NK_N)L^2(\mathbb{R}) = F_2(s)\). \(\tag{C.15}\)

Remark. The condition (i),(ii) are the same as (i)(ii) in Proposition C.1. For \(f_N = 1_{(s, \infty)}, \forall N \in \mathbb{N}\), (iii)(iv) are trivially satisfied. Hence the kernel \(K_N\) from Proposition C.1 with this special \(f_N\) automatically satisfies the above four conditions. This special case appears in many applications, for example in GUE and TASEP.

Proof. In this proof \(c_i, i = 1, 2, \ldots\) are some constants. We first prove the \(f_N = 1_{(s, \infty)}, \forall N \in \mathbb{N}\) case. By the remark above, we will show that for a kernel satisfying (i)(ii) in Proposition C.1 we have

\[
\lim_{N \to \infty} \det(1 - K_N)L^2(s, \infty) = F_2(s). \tag{C.16}
\]

The general case will be treated by considering the difference to this special case later in the proof. For a given \(s \in \mathbb{R}\), take \(L(> |s|)\). By the Hadamard’s inequality,

\[
|\det A| \leq \prod_{i=1}^{n} (\sum_{j=1}^{n} |a_{i,j}|^2)^{1/2}, \tag{C.17}
\]
which holds for general \( n \times n \) matrix \([38]\), and (ii), we have for \( \xi, \zeta \geq -L \),

\[
\left| \det(K_N(\xi_i, \xi_j))_{1 \leq i, j \leq k} \right| \leq \prod_{i=1}^{k} \left( \sum_{j=1}^{k} |K_N(\xi_i, \xi_j)|^2 \right)^{1/2} \\
\leq \prod_{i=1}^{k} \left( \sum_{j=1}^{k} c_i^2 (e^{-\max(0, \xi_i) - \max(0, \xi_j)})^2 \right)^{1/2} \\
\leq c_1^k k^{k/2} \prod_{i=1}^{k} e^{-\max(0, \xi_i)},
\]

(C.18)

and hence

\[
\left| \int_{(s, \infty)^k} \det(K_N(\xi_i, \xi_j))_{1 \leq i, j \leq k} \prod_{i=1}^{k} d\xi_i \right| \leq c_1^k k^{k/2} \prod_{i=1}^{k} \int_{(s, \infty)} e^{-\max(0, \xi_i)} d\xi_i \leq c_2^k k^{k/2}.
\]

(C.19)

Take \( \epsilon > 0 \). By (C.19), the Fredholm expansion,

\[
\det(1 - K_N)_{L^2(s, \infty)} = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \int_{(s, \infty)^k} \det(K_N(\xi_i, \xi_j))_{1 \leq i, j \leq k} \prod_{i=1}^{k} d\xi_i,
\]

(C.20)

converges and there exists \( l \in \mathbb{N} \) s.t.

\[
\left| \det(1 - K_N)_{L^2(s, \infty)} - \sum_{k=0}^{l} \frac{(-1)^k}{k!} \int_{(s, \infty)^k} \det(K_N(\xi_i, \xi_j))_{1 \leq i, j \leq k} \prod_{i=1}^{k} d\xi_i \right| \leq \sum_{k=l+1}^{\infty} \frac{c_1^k k^{k/2}}{k!} \leq \frac{\epsilon}{6}.
\]

(C.21)

By (ii), we have

\[
\left| \left( \int_{(s, \infty)^k} - \int_{(s, L)^k} \right) \det(K_N(\xi_i, \xi_j))_{1 \leq i, j \leq k} \prod_{i=1}^{k} d\xi_i \right| \\
\leq \int_{(s, \infty)^k} \sum_{j=1}^{k} \int_{\xi_j > L} c_i^k k^{k/2} \prod_{i=1}^{k} e^{-\max(0, \xi_i)} \prod_{i=1}^{k} d\xi_i \\
= c_1^k k^{k/2+1} \int_{L}^{\infty} e^{-\max(0, \xi_j)} d\xi_j \left( \int_{s}^{\infty} e^{-\max(0, \xi)} d\xi \right)^{k-1},
\]

(C.22)

where in the second inequality we used (C.18). Hence, for \( L \) large enough,

\[
\sum_{k=0}^{l} \frac{(-1)^k}{k!} \left( \int_{(s, \infty)^k} - \int_{(s, L)^k} \right) \det(K_N(\xi_i, \xi_j))_{1 \leq i, j \leq k} \prod_{i=1}^{k} d\xi_i \\
\leq \sum_{k=0}^{\infty} \frac{c_1^k k^{k/2+1}}{k!} \int_{L}^{\infty} e^{-\max(0, \xi_j)} d\xi_j \leq c_2 \int_{L}^{\infty} e^{-\max(0, \xi)} d\xi_j \leq \frac{\epsilon}{6}.
\]

(C.23)
uniformly in $N$. Combining (C.21) and (C.23), we have

$$\left| \det(1 - K_N)_{L^2(s,\infty)} - \sum_{k=0}^{l} \frac{(-1)^k}{k!} \int_{(s,L)^k} \det(K_N(\xi_i, \xi_j))_{1 \leq i,j \leq k} \prod_{i=1}^{k} d\xi_i \right| \leq \frac{\epsilon}{3}. \quad (C.24)$$

By the uniform convergence (i), for $N$ large enough,

$$\left| \sum_{k=0}^{l} \frac{(-1)^k}{k!} \int_{(s,L)^k} \det(K_N(\xi_i, \xi_j))_{1 \leq i,j \leq k} \prod_{i=1}^{k} d\xi_i \right| - \sum_{k=0}^{l} \frac{(-1)^k}{k!} \int_{(s,L)^k} \det(K(\xi_i, \xi_j))_{1 \leq i,j \leq k} \prod_{i=1}^{k} d\xi_i \right| \leq \frac{\epsilon}{3}. \quad (C.25)$$

By the same argument as to get (C.24), we have

$$\left| \det(1 - K)_{L^2(s,\infty)} - \sum_{k=0}^{l} \frac{(-1)^k}{k!} \int_{(s,L)^k} \det(K(\xi_i, \xi_j))_{1 \leq i,j \leq k} \prod_{i=1}^{k} d\xi_i \right| \leq \frac{\epsilon}{3}. \quad (C.26)$$

Combining (C.24), (C.25), (C.26), we have

$$\left| \det(1 - K_N)_{L^2(s,\infty)} - \det(1 - K)_{L^2(s,\infty)} \right| \leq \epsilon. \quad (C.27)$$

This completes the proof for the $f_N = 1_{(s,\infty)}$, $\forall N \in \mathbb{N}$ case.

Next we consider the general $f_N$ case. It is enough to show

$$\left| \int_{\mathbb{R}^k} \det(K_N(\xi_i, \xi_j))_{1 \leq i,j \leq k} \prod_{i=1}^{k} f_N(\xi_i) d\xi_i - \int_{(s,\infty)^k} \det(K_N(\xi_i, \xi_j))_{1 \leq i,j \leq k} \prod_{i=1}^{k} d\xi_i \right| \leq c^k k^{k/2+1}\epsilon \quad (C.28)$$

for large enough $N$. In the second term on lhs, the $\infty$ as the upper limit of the integrals can be replaced by a large $L$ with an error of the form $c_2 k^{k/2+1}\epsilon$ as in (C.22). Similarly, in the first term on lhs of (C.28), the $\infty$ as the upper limit of the integrals can be replaced by $L$ with an error of the form $c_2 k^{k/2+1}\epsilon$ and the $-\infty$ as the lower limit by $-L$ with an error of the form $c_2 k^{k/2+1}\epsilon$ due to (iv). Combining these, we can replace the integrals in (C.28) within $(-L, L]$ for large $L$ with an error $c_2 k^{k/2+1}\epsilon$. By a similar argument as to get (C.22), we have

$$\left| \int_{(-L, L]^k} \det(K_N(\xi_i, \xi_j))_{1 \leq i,j \leq k} \prod_{i=1}^{k} f_N(\xi_i) d\xi_i - \int_{(s,L)^k} \det(K_N(\xi_i, \xi_j))_{1 \leq i,j \leq k} \prod_{i=1}^{k} d\xi_i \right| \leq c_3 k^{k/2+1}\epsilon \quad (C.29)$$

Using (C.18) and (iii), one observes that the integral over $\xi_i, i \neq j$ in the first sum is finite. By using (iii), we see that the integral over $\xi_j$ in the first sum is bounded for large $N$ by $c_4 k^{k/2+1}\epsilon$. Similarly for the second term in (C.29), one first sees $\prod_{i=1}^{k} f_N(\xi_i) - 1 \leq c_5 \sum_{i=1}^{k} |f_N(\xi_i) - 1|$ since $f_N$ is bounded. Then using (iii) and (C.18) one sees that it is bounded by $c_6 k^{k/2+1}\epsilon$ for large $N$. Combining all of the above, we get (C.28). Taking the sum over $k$ and taking $N \to \infty$ limit, we find

$$\lim_{N \to \infty} \det(1 - f_N K_N)_{L^2(\mathbb{R})} = \lim_{N \to \infty} \det(1 - K_N)_{L^2(s,\infty)}. \quad (C.30)$$

This completes the proof for general $f_N$ case. \qed
D Inverse $q$-Laplace transforms

In this appendix we discuss inversion formulas of the $q$-Laplace transform. In Appendix B of [39], we discussed a few inversion formulas for the ordinary Laplace transform, which is summarized as follows. First the Laplace transform is defined as

$$\hat{\varphi}(u) = \int_0^\infty e^{-ux} \varphi(x) dx, \quad u \in \mathbb{C}. \quad (D.1)$$

(In [39] we used the notation $\tilde{\varphi}$. Here we use $\hat{\varphi}$ to keep the former for the transformed one, see below.) When $\varphi(x)$ is a (probability) distribution function on $(0, \infty)$, the Laplace transform is analytic for $\text{Re} \, u > 0$. The usual inversion formula is

$$\varphi(x) = \frac{1}{2\pi i} \int_{\delta+i\mathbb{R}} du e^{ux} \hat{\varphi}(u), \quad x > 0, \quad (D.2)$$

where $\delta$ should be taken so that the singularities of $\hat{\varphi}$ are to the left of the integration contour. If $\varphi$ is associated with a random variable $X$, we have $G(u) := \langle e^{-ux} \rangle = u\hat{\varphi}(u)$. The formula can be restated for the random variable $Y = \log X$. For instance for the distribution $F(y) = \mathbb{P}[Y \leq y] = \mathbb{P}[X \leq e^y] = \varphi(e^y)$, (D.1) is rewritten as

$$\hat{\varphi}(u) = \int_{\mathbb{R}} e^{-ue^y+y} F(y) dy =: \tilde{F}(u). \quad (D.3)$$

For discussing the distribution of the O’Connell-Yor polymer model, the following inversion formula was useful.

**Proposition D.1.** For a random variable $Y$, set $G(u) = \langle e^{-ue^Y} \rangle$. The distribution function of $Y$ is recovered from $G(u)$ as

$$F(y) = \frac{1}{2\pi i} \int_{\delta+i\mathbb{R}} d\xi \frac{e^{y\xi}}{\Gamma(\xi+1)} \int_0^\infty u^{\xi-1} G(u) du, \quad (D.4)$$

where $\delta > 0$. The corresponding density function $f(y) = F'(y)$, if it exists, is given by

$$f(y) = \frac{1}{2\pi i} \int_{\delta+i\mathbb{R}} d\xi \frac{e^{y\xi}}{\Gamma(\xi)} \int_0^\infty u^{\xi-1} G(u) du. \quad (D.5)$$

The formulas discussed in this appendix are $q$-analogues of the above.

Suppose we have a function $f(n), n \in \mathbb{Z}$ and denote by $\tilde{f}_q(z)$ its $q$-Laplace transform,

$$\tilde{f}_q(z) := \sum_{n \in \mathbb{Z}} \frac{f(n)}{z^q n; q}_\infty. \quad (D.6)$$

In this appendix, we mainly consider the case where $f(n), n \in \mathbb{Z}$ is a discrete probability density function, i.e., $f(n) \geq 0, \sum_{n \in \mathbb{Z}} f(n) = 1$, for which the $q$-Laplace transform converges and analytic except $z \neq q^n, n \in \mathbb{Z}$. By using the fact that the $q$-exponential function tends to the usual exponential function in a $q \to 1$ limit (see the comment after (A.5)), one sees that this formula goes to (D.1) as

$$\tilde{f}_q(-(1-q)u) = \sum_{n \in \mathbb{Z}} \frac{f(n)}{-(1-q)q^n u; q}_\infty \to \int_{\mathbb{R}} dy e^{y-ue^y} f(y) = \tilde{f}(u), \quad (D.7)$$

where in the limit ($\to$) we set $n = y/\log q$ and took $q \to 1$. An inverse formula for the $q$-Laplace transform is given by

$$f(n) = \int_{\gamma} \frac{dz}{2\pi i} z^n(q^{n+1}z; q)_\infty \tilde{f}_q(z), \quad (D.8)$$
where $\gamma$ is a contour enclosing $\mathbb{R}_+$ clockwise, see Fig. 6. Because of the factor $(q^{n+1}z; q)_\infty$, the poles at $z = q^{-k}, k = n+1, n+2, \ldots$ vanish and the contour can be taken to be around the poles at $z = 1, 2, \ldots, n$, but the infinite contour in Fig. 6 has an advantage that it can be used for any $n$.

In a certain $q \to 1$ limit, this, with the change of variable $z = -(1-q)u$, tends to \((D.2)\) as

$$f(n) = q^n(1-q) \int_{\gamma} \frac{du}{2\pi i} (-1-q)u^{n+1}f(u) \to \int_{\delta + \mathbb{R}} \frac{du}{2\pi i} e^{u\xi} f(u).$$  \((D.9)\)

There is also an inversion formula corresponding to \((D.5)\).

**Proposition D.2.** When $f(n), n \in \mathbb{Z}$ is a discrete probability density function,

$$f(n) = - \int_{C_0} \frac{dx}{2\pi i x^n+1} (q;x)_\infty \sum_{k \in \mathbb{Z}} (qx)^k R_k(\tilde{f}_q).$$ \((D.10)\)

Here $R_k(\tilde{f}_q)$ is the residue of $\tilde{f}_q(z)$ at the pole $z = q^{-k}$ and $C_0$ is a small contour around the origin.

**Proof.** It is equivalent to showing

$$\sum_{n \in \mathbb{Z}} f(n)x^n = - \frac{(q;x)_\infty}{(qx;x)_\infty} \sum_{k \in \mathbb{Z}} (qx)^k R_k(\tilde{f}_q),$$ \((D.11)\)

for $x = e^{i\theta}, \theta \in (-\pi, \pi]$. In fact considering the contour integral around the origin of both sides of \((D.11)\) divided by $x^{n+1}$, we get \((D.10)\). By \((D.8)\), this is further equivalent to showing

$$- \sum_{n \in \mathbb{Z}} x^n \int_{\gamma} \frac{dz}{2\pi i} \tilde{f}_q(z)q^n(q^{n+1}z; q)_\infty = \frac{(q;x)_\infty}{(qx;x)_\infty} \sum_{k \in \mathbb{Z}} (qx)^k R_k(\tilde{f}_q).$$ \((D.12)\)

Taking the poles at $z = q^{-k}, k \in \mathbb{Z}$, one can rewrite lhs of \((D.12)\) as

$$\sum_{k \in \mathbb{Z}} R_k(\tilde{f}_q) \sum_{n \in \mathbb{Z}} (qx)^n(q^{n+k+1}; q)_\infty = \sum_{k \in \mathbb{Z}} (qx)^k R_k(\tilde{f}_q) \sum_{n \in \mathbb{Z}} (qx)^{n-k}(q^{n-k+1}; q)_\infty.$$ \((D.13)\)

The last sum in \((D.13)\) can be taken owing to the $q$-binomial theorem \((A.2)\),

$$\sum_{n \in \mathbb{Z}} (qx)^{n-k}(q^{n-k+1}; q)_\infty = \sum_{n=k}^{\infty} (qx)^{n-k}(q^{n-k+1}; q)_\infty = \frac{(q;x)_\infty}{(qx;x)_\infty}.$$ \((D.14)\)

Substituting this into \((D.13)\), we get \((D.12)\). The last sum in \((D.10)\) formally seems to be a consequence of taking poles at $z = q^{-k}, k \in \mathbb{Z}$ of the integral, $\int_{\gamma} \frac{dz}{2\pi iz} \tilde{f}_q(z)$ (though this is not true since $z^2$ has a cut along $\mathbb{R}_+$). Setting $x = q^{-\xi}$ and taking the $q \to 1$ limit of this and recalling the factor $(q;x)_\infty/(qx;x)_\infty$ can be written in terms of the $q$-Gamma function \((A.6)\) which tends to the Gamma function as $q \to 1$, one would observe that \((D.10)\) may be regarded as a $q$-analogue of \((D.5)\).

Figure 6: The contour $\gamma$.  

As a corollary of proposition D.1, we get a formula for the distribution function \( F(n) = P[X \leq n], n \in \mathbb{N} \). It reads

\[
F(n) = -\int_{C_0} \frac{dx}{2\pi i x^{n+1}} \frac{(q; q)_{\infty}}{(x; q)_{\infty}} \sum_{k \in \mathbb{Z}} (qx)^k R_k(\tilde{f}_q).
\]  
(D.15)

For the density function, \( f(n) := F(n) - F(n - 1) \), this gives (D.10). (D.15) is an analogue of (D.4). If we introduce the generating function \( Q(x) = \sum_{n \in \mathbb{Z}} F(n)x^n \), clearly

\[
F(n) = \int_{C_0} \frac{dx}{2\pi i x^{n+1}} Q(x)
\]  
(D.16)

and (D.15) is equivalent to

\[
Q(x) = \frac{(q; q)_{\infty}}{(x; q)_{\infty}} \sum_{k \in \mathbb{Z}} (qx)^k R_k(\tilde{f}_q).
\]  
(D.17)

Suppose further that a random variable \( X \) having the density \( f(n), n \in \mathbb{Z} \) is written as \( X = X_0 + \chi \) where \( X_0 \) and \( \chi \) are independent random variables on \( \mathbb{Z} \). By the independence, the generating functions of them, \( Q(x) = \sum_{n \in \mathbb{Z}} F(n)x^n \), \( g(x) = \sum_{n \in \mathbb{Z}} P[\chi = n]x^n \), are related as \( Q(x) = g(x)Q_0(x) \). Combining this and (D.16), (D.17), we find a formula for the distribution of \( X_0 \),

\[
P[X_0 \leq n] = \int_{C_0} \frac{dx}{2\pi i x^{n+1}} \frac{(q; q)_{\infty}}{(x; q)_{\infty}} g(x) \sum_{k \in \mathbb{Z}} (qx)^k R_k(\tilde{f}_q).
\]  
(D.18)

Noting \( \tilde{f}_q(\zeta) = \langle \frac{1}{\langle \zeta q^{\chi}; q \rangle} \rangle \), this is a formula which gives the distribution function of \( X_0 \) in terms of the \( q \)-Laplace transform of \( X \).

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