On the equilibrium positions of the planetary mechanism at the location of the center of mass of the rod between the axes of rotation

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Abstract. In this paper, the positions of stable and unstable equilibrium of the mechanism moving in the vertical plane are determined. The mechanism consists of a rod, load, counterweight and satellite, rolling without slipping on a fixed cylindrical surface. The potential energy function of the mechanism is examined. Only those variants of the mechanism design for which the ratio of the fixed surface radius to the satellite radius is a rational number are considered. The dependence of the number of equilibrium positions of the mechanism on the radius of the satellite and the distance of the load to the center of the satellite is determined by numerical analysis; and the periods of change in the potential energy of the mechanism for different values of the satellite radius are determined.

1. Introduction
Dynamic calculation of mechanical systems involves the determination of internal force factors that arise in its elements during operation. Sometimes it is necessary to consider the resonant operating modes of the system with frequencies close to its own frequencies. Therefore, great attention should be given to the calculation of possible movements of the mechanism and the natural frequencies of its elements [1-3].

In this paper we consider the mechanism of the planetary type, which can be an integral part of the planetary gearboxes, automotive differentials, on-Board planetary gears of heavy vehicles, agricultural machinery [4]. Oscillatory movements of the mechanism are investigated using the analysis of potential energy. A thorough analysis of the extremes of potential energy is carried out, which make it possible to judge the stability and instability of the equilibrium positions of a mechanical system [5-6]. The aim of the work is to study and demonstrate the peculiarities of changes in the potential energy of a mechanism when some of its characteristic parameters change.

2. Description of the mechanism and its mathematical model
A mechanism of planetary type is located in the vertical plane and contains a satellite 1 of radius $r$ and of mass $M_1=0.5$ kg (Figure 1). The satellite has the ability to ride without slipping on a fixed cylindrical surface 6 of radius $R=0.65$ m. The axis $A$ of the satellite is fastened with a rod 2 of length $L=0.8$ m and of mass $M_2=0.5$ kg. The clutch is fixed in a point $O$ and can rotate around the motionless horizontal axis passing around this point. At assembly of the mechanism or at change of radius of the satellite 1 the rod 2 slides along the clutch, and then attached to it with a screw.
A weightless bar 5 with a load 3 is rigidly attached to the satellite 1. The load 3 of mass \( M_3 = 0.4 \text{ kg} \) can be fixed on the bar 5 at a distance of \( AB = l_3 = 0.2 \text{ m} \). The counterweight 4 of mass \( M_4 = 0 \text{ kg} \) is on the rod 2 at a distance \( OD = l_4 = 0.15 \text{ m} \). The center of mass of the rod 2 is in its section \( C \).

To study the potential energy of the mechanism, write down her expression.

The mechanism under consideration has one degree of freedom. As a generalized coordinate, we take the angle \( \varphi \) of deviation of the rod 2 from the vertical. Figure 2 shows the mechanism at the current moment of time. The superscript indicates * the position of points \( A,N,B,C \) when \( \varphi = 0 \).

Since the lengths of the arcs \( \cup PN \) and \( \cup PN^* \) are equal, there is \( r \cdot \vartheta = R \cdot \varphi \), and the angle of rotation of the satellite 1 is

\[
\varphi = \theta \cdot \frac{R - r}{r} - \varphi.
\]

The active forces acting on the mechanism are the forces of gravity. Therefore the potential energies of the uniform gravity field of the satellite 1, rod 2, load 3 and counterweight 4 have the form:

\[
\begin{align*}
1 &= M_1 \cdot g \cdot (R - r) \cdot (1 - \cos \varphi), \\
2 &= M_2 \cdot g \cdot (R - r - L/2) \cdot (1 - \cos \varphi), \\
3 &= M_3 \cdot g \cdot \left[ (R - r) \cdot (1 - \cos \varphi) + l_3 \left( 1 - \cos \left( \frac{R - r}{r} \varphi \right) \right) \right], \\
4 &= -M_4 \cdot g \cdot l_4 \cdot (1 - \cos \varphi).
\end{align*}
\]

And the potential energy of the whole mechanism is

\[
\Phi = g \cdot \left[ (M_1 + M_2 + M_3) \cdot (R - r) - M_2 \cdot l_2/2 - M_4 \cdot l_4 \right] \cdot (1 - \cos \varphi) + M_3 \cdot g \cdot l_3 \left( 1 - \cos \left( \frac{R - r}{r} \varphi \right) \right),
\]

where \( g \) – is the acceleration of gravity.

It follows that the function \( \Phi(\varphi) \) is harmonic when \( l_3 = 0 \) (the load 3 is in the center of the satellite 1) and when \( R = 2r \).

Let us consider the change of the potential energy of the mechanism in which the centers of mass of the rod 2 and the satellite 1 are on the same side of the fixed hinge, that is, \( r < 0.25 \text{ m} \). In addition, in
this paper we limit ourselves to studying of the potential energy function only for rational numbers of the relation \( R/r \).

First, we analyze the change of this function depending on \( l_3 \). Assuming \( r=0.2 \) m and without changing other characteristics of the mechanism, except \( l_3 \), we have

\[
\phi = g \left[ 0.22 \left( 1 - \cos \phi \right) + 0.4 \cdot l_3 \left( 1 - \cos (2.2 \phi) \right) \right].
\]  

As follows from (1), at \( l_3=0 \) (load mass \( M_3 \) in the center of the satellite), the period of this function is \( T=2\pi \) rad, and at \( l_3 \neq 0 \) – \( T=8\pi \) rad (four turns of the rod 2).

Figure 3 shows the graph of the potential energy \( \Pi(\phi) \) at \( l_3=0.1 \) m. It is seen that at this value of \( l_3 \), the mechanical system is four times in the positions of stable (local minima \( \Pi(\phi) \)) and unstable (local maxima \( \Pi(\phi) \)) equilibrium for one period of potential energy change. If \( l_3=0.15 \) m, then there is a local maximum on the graph \( \Pi(\phi) \) (figure 4) at \( \phi=4\pi \) rad, and to the left (\( \phi=(4\pi - 4\pi/9)=11,1701 \) (rad)) and to the right (\( \phi=(4\pi+4\pi/9)=13,9626 \) (rad)) of it – two local minimum. The mechanical system will visit the positions of stable and unstable equilibrium for five times in one period of change \( \Pi(\phi) \) at this value of \( l_3 \).

From the analysis on the extremum of function (1) it follows that the transformation of a stable equilibrium position into an unstable (bifurcation) occurs at \( l_3=0.1086 \) m.

Figure 5 shows the graph \( \Pi(\phi) \) in the case of placing a load 3 of mass \( M_3 \) on the satellite rim \((l_3=r=0.2 \) m). It is seen that the number of equilibrium positions of the mechanical system has not changed compared with the previous case considered. The graph \( \Pi(\phi) \) (Figure 5) shows that with increasing distance \( l_3 \) to a value of 0.2025 m (load 3 outside the satellite) at \( \phi_1=2,2543 \) rad and \( \phi_2=22,8784 \) rad, two new local minimum begin to form at the same time (bifurcation) – new positions of stable equilibrium mechanical system.

The graph of the function \( \Pi(\phi) \) at \( l_3=0.23 \) m is shown in Figure 6. It can be seen that, with such a value of \( l_3 \), the mechanical system will visit the positions of stable and unstable equilibrium seven times in four turns of the rod \((0<\phi<8\pi \) rad). This situation persists with a further increase of \( l_3 \) up to the value \( l_3=0.265 \) m, when at \( \phi_1=7,8557 \) rad and \( \phi_2=17,2777 \) rad, two more local minimum appear on the graph \( \Pi(\phi) \) (bifurcation).
Now the mechanical system will be nine times in the positions of stable and unstable equilibrium for four turns of the rod. With a further increase in \( l_3 \), the number of equilibrium positions does not change, only extreme values of the potential energy \( \Pi(\phi) \) change (Figure 7). With a significant increase in the length \( l_3 \), the second term in expression (1) becomes predominant, the period of change of potential energy tends to \( 8\pi/9 \) rad, which is confirmed by the graph \( \Pi(\phi) \) with \( l_3=25r=5 \) m (Figure 8).

Now we will study the dependence of the potential energy of the mechanism on the satellite radius, taking \( l_3=0.2 \) m and not changing its other characteristics. Then

\[
(\phi, r) = g \cdot \left( 0.44 - 1.1 \cdot r \right) \cdot \left( 1 - \cos \phi \right) + 0.08 \left[ 1 - \cos \left( \frac{0.65}{r} \right) \phi \right].
\]

We restrict the study to cases when \( l_3>r \). Graphic dependences of function (2) with \( r=0.05 \) m; 0.1 m; 0.13 m and 0.1625 m are shown in figures 9,10,11 and 12 respectively. It can be seen that they represent the addition of two harmonics with periods of \( 2\pi \) rad and \( \pi/6 \) rad (Figure 9), \( 4\pi \) rad and \( 4\pi/11 \) rad (Figure 10), \( 2\pi \) rad and \( \pi/2 \) rad (Figure 11), \( 6\pi \) rad and \( 2\pi \) rad (Figure 12).

For other values of \( r \), the graph \( \Pi(\phi) \), as a rule, has a much more complex form, for example, such as with \( r=0.17 \) m (Figure 13).
Table 1 shows the dependences of the values of the periods $T$ of the function (2) on the radius $r$ of the satellite, as well as the number $n$ of positions of stable (unstable) equilibrium, in which there is a mechanism for one period of change of this function.

| $r$, m | $T$, rad | $n$ | $r$, m | $T$, rad | $n$ |
|--------|----------|-----|--------|----------|-----|
| 0.05   | $2\pi$   | 12  | 0.15   | $30\pi$  |     |
| 0.1    | $4\pi$   | 11  | 0.16   | $32\pi$  | 35  |
| 0.11   | $22\pi$  |     | 0.1625 | $2\pi$   | 1   |
| 0.12   | $24\pi$  | 53  | 0.17   | $34\pi$  | 31  |
| 0.13   | $2\pi$   | 4   | 0.18   | $36\pi$  | 30  |
| 0.14   | $28\pi$  |     | 0.19   | $38\pi$  | 28  |
| 0.145  | $58\pi$  |     | 0.2    | $8\pi$   | 5   |

From table № 1, it follows that a change in only the satellite radius in the range of $0.05 \leq r \leq 0.2$ (m) leads to an irregular change in the period of potential energy $2\pi \leq T \leq 58\pi$ (rad) and the number of equilibrium positions $1 \leq n \leq 53$.

The study of the motion of the mechanism in a nonlinear formulation shows a very high sensitivity of potential energy to changes in its geometric characteristics.

3. Conclusion

For a real planetary mechanism, a nonlinear expression of its potential energy is derived; shows the dependence of potential energy on the geometric and mass characteristics of the mechanism. It is confirmed that with rational values of the characteristics of the mechanism, the potential energy function is periodic; a numerical analysis of the nonlinear potential energy function made it possible to identify for all the cases considered all the provisions of stable and unstable equilibrium. It is shown that the period of change of potential energy and the number of equilibrium positions of the
mechanism substantially depend on the geometric dimensions of its elements. For the considered mechanism, it is established that as the load moves away from the center of the satellite, the number of equilibrium positions of the mechanism during the period of change of potential energy increases and may be four, five, seven and nine.

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