The First Law for Rotating NUTs

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We address a long-standing problem of describing the thermodynamics of rotating Taub–NUT solutions. The obtained first law is of full cohomogeneity and allows for asymmetric distributions of Misner strings as well as their potential variable strengths—encoded in the gravitational Misner charges. Notably, the angular momentum is no longer given by the Noether charge over the sphere at infinity and picks up non-trivial contributions from Misner strings.

The Lorentzian Taub–NUT spacetime is one of the simplest yet one of the most puzzling vacuum solutions of general relativity. The spacetime possesses a Schwarzschild-like horizon that, as a consequence of the NUT charge, is accompanied by rotating string-like singularities on the north/south pole axes, known as Misner strings.

The Misner strings can be exiled from the spacetime at the cost of introducing a periodic time coordinate [1]. However, while this approach is customary it is not strictly necessary: as suggested by Bonnor [2], see also [3], the Misner strings can instead be interpreted as due to ‘singular’ sources of angular momentum, and the geometry from this perspective is less pathological than one might expect. For example, the spacetime is geodesically complete and while closed time-like curves exist near the axes, there are no causal pathologies for geodesic observers [4–7]. This raises a possibility that the NUT charge may actually be relevant for astrophysics. In fact, already in 1997 astrophysicists probed the possibility of detecting the NUT parameter by microlensing [8, 9], while its impact on the black hole shadow has been investigated in [10, 11]. Ongoing and forthcoming tests of general relativity in the strong field regime, e.g. the Event Horizon Telescope [12], may reveal interesting signatures of these spacetimes or at the very least provide tighter constraints.

Notwithstanding their potential pathologies, solutions with NUT charge have been the source of deep physical insights, for example in the context of black hole thermodynamics. The thermodynamics of these solutions has been predominantly studied in the Euclidean case and in the absence of Misner strings, e.g. [14, 21]. The most striking result to come from these studies is that, in equilibrium, the entropy of Taub–NUT solutions is not simply one quarter the horizon area, providing a counter-example to the ‘area law’.

A full understanding of the properties—in particular the first law—of the Lorentzian Taub–NUT solutions with Misner strings present is much subtler. In fact, to the best of our knowledge, until recently no consistent first law for these solutions has been presented. It was argued in [22, 23] that an additional relationship between the horizon radius and the NUT charge must be obeyed in order for the first law to hold. In [24], a Smarr relation was derived for the Lorentzian solution, but no first law consistent with the Smarr formula was obtained. The situation was remedied for the non-rotating solution in [25, 26], where it was found that these issues can be ameliorated through a judicious choice of a potential and conjugate charge for the NUT parameter.

The purpose of this note is to address the first law for Taub–NUT solutions with rotation. The Euclidean setup was considered in [28], where it was found that regularity (equilibrium) of the Euclidean sector effectively leads to a ‘discrete’ version of the first law. Here, focusing on the Lorentzian case and picking up the threads on [29], our first law not only allows for an arbitrary distribution of Misner strings, it also accounts for their variable strengths. The Misner gravitational charges that appear in this law are obtained by a (rotating-case) generalization of the Komar-like integrals introduced in [29] and the results are cross-checked by the Euclidean action calculation. One of the key results is the realization that Misner strings provide non-trivial contributions to the angular momentum of the spacetime, which is no longer simply given by the Noether charge over the sphere at infinity.

Let us begin our exploration by introducing the rotat-
ing Taub–NUT spacetime. The solution reads

\[
ds^2 = \frac{\Delta}{\Sigma} \left[ dt + (2n \cos \theta + 2s - a \sin^2 \theta) d\phi \right]^2 + \frac{\Sigma}{\Delta} dr^2 + \frac{\sin^2 \theta}{\Sigma} \left[ adt - (r^2 + a^2 + n^2 - 2as) d\phi \right]^2 + \Sigma d\theta^2 ,
\]

\[
\Delta = r^2 + a^2 - 2amr - n^2 ,
\]

\[
\Sigma = r^2 + (n + a \cos \theta)^2 ,
\]

where \( n, m, \) and \( a \) are the standard NUT, mass, and rotation parameters, respectively. There is yet another dimensionful physical parameter \( s \), which governs the overall distribution and strength of the Misner string singularities. In particular, when \( s = +a \), there is only one Misner string and it is located on the north pole (\( \cos \theta = +1 \)) axis, while the south pole (\( \cos \theta = -1 \)) axis is completely regular. The choice \( s = -n \) corresponds to the opposite situation, while for \( s = 0 \) the two Misner strings are ‘symmetrically distributed’ and both axes are ‘equally singular’.

The parameter \( s \) can be formally eliminated by performing the ‘large coordinate transformation’ \( [3] \)

\[
t \to t - 2s \phi ,
\]

effectively threading the spacetime with an ‘overall Misner string’ so that the symmetric distribution of Misner strings is restored. In general, changing the value of \( s \) corresponds to a physical transformation of the system that simultaneously affects the strengths of both Misner strings. Such ‘variations of strings strengths’ are captured by the first law of thermodynamics presented below. As shown in \( [5] \) for non-rotating Taub–NUTs, the spacetime remains geodesically complete for any value of \( s \), but the absence of closed timelike and null geodesics requires \(|s/n| \leq 1 \).

The spacetime is stationary and axisymmetric, corresponding to the \( k = \partial_t \) and \( \eta = \partial_\phi \) Killing vectors, and admits a number of Killing horizons. The ‘black hole horizon’ is located at the largest root \( r_+ \) of \( \Delta(r_+) = 0 \). It is generated by a (properly normalized at infinity) Killing vector

\[
\xi = k + \Omega \eta ,
\]

where \( \Omega \) is the angular velocity of the “Zero Angular Momentum Observer” (ZAMO) evaluated at \( r_+ \):

\[
\Omega = -\frac{g_{\theta \phi}}{g_{\phi \phi}} \bigg|_{r=r_+} = -\frac{a}{r_+^2 + a^2 + n^2 - 2as} .
\]

By standard arguments, it can be assigned the following entropy and temperature:

\[
S = \frac{\text{Area}}{4} = \pi (r_+^2 + a^2 + n^2 - 2as) ,
\]

\[
T = \frac{\Delta'(r_+)}{4\pi (r_+^2 + a^2 + n^2 - 2as)} .
\]

Note that, contrary to the non-rotating case \( [29] \), both quantities explicitly depend on the parameter \( s \).

As in the non-rotating case \( [30] \), when Misner strings are present, the north and south pole axes are also Killing horizons, generated by the following Killing vectors \( [2] \)

\[
\xi_\pm = k - \frac{1}{2(s \pm n)\eta} .
\]

The corresponding surface gravities \( \kappa_\pm \) can be obtained from the standard formula \( \kappa^2 = \frac{1}{4} (\nabla_\alpha \xi_\beta)(\nabla^\alpha \xi^\beta) \). Similar to \( [29] \) we associate with them the corresponding Misner potentials

\[
\psi_\pm = \frac{\kappa_\pm}{4\pi} = \frac{1}{8\pi(n \pm s)} .
\]

With these geometric quantities in hand, let us now turn towards thermodynamics. Following \( [25, 29] \) we identify the temperature of the spacetime with the temperature of the black hole horizon, \( \{r_+, a, n, s\} \), and the ‘entropy’ of the spacetime with the entropy of the black hole horizon, \( \{r_+, a, n, s\} \), and seek other thermodynamic charges so that the following generalized first law:

\[
\delta M = T \delta S + \Omega \delta J + \psi_+ \delta N_+ + \psi_- \delta N_-, \quad (9)
\]

together with the corresponding Smarr relation

\[
M = 2(TS + \Omega J + \psi_+ N_+ + \psi_- N_-) \quad (10)
\]

are satisfied. Obviously, such a first law is of full cohomogeneity: it has 4 terms on the r.h.s. which correspond to the variation of the 4 physical parameters of the solution: \( \{r_+, a, n, s\} \).

To find the thermodynamic charges \( M, J, \) and \( N_\pm \), let us generalize the Komar-like prescription developed in \( [29] \) to include rotation. The idea is to re-derive the Smarr formula, accounting properly for the new NUT-related boundaries of Misner tubes that surround Misner strings, and identify the ensuing Komar integrals with the thermodynamic charges.

As any Killing vector, the generator of the black hole horizon obeys the following integrability condition:

\[
\nabla_a \nabla^b \xi^a = -R^b_{\ a} \xi^a = 0 , \quad (11)
\]

with the last equality valid in vacuum spacetimes. Using the differential form language and integrating over the 3-dimensional \( (t = \text{const.}) \) hypersurface \( \Sigma \) we thus have

\[
0 = \int_{\Sigma} d* d\xi = \int_{\partial \Sigma} * d\xi , \quad (12)
\]

The spacetime boundary \( \partial \Sigma \) now consists of the two Misner string tubes \( T_+ \) and \( T_- \) positioned at \( \theta = \epsilon \) and \( \theta = \pi - \epsilon \).
The first two terms are standard and simply yield the thermodynamic quantities:

\[ \partial \Sigma = T_+ + S_\infty - T_- - H , \]  

see [29] for more details.

We split the integral (12) into the following parts:

\[ 0 = \int_{S_\infty} *dk - \int_{H} *d\xi + \int_{T_+} *dk - \int_{T_-} *dk + \Omega \left( \int_{S_\infty} *d\eta + \int_{T_+} *d\eta - \int_{T_-} *d\eta \right) . \]

The first two terms are standard and simply yield the mass \( M \) and the entropy \( S \):

\[ M = -\frac{1}{8\pi} \int_{S_\infty} *dk , \]
\[ S = -\frac{1}{16\pi T} \int_{H} *d\xi . \]

The next two terms were already studied in [29], and define the gravitational Misner charges:

\[ N_\pm \equiv \pm \frac{1}{16\pi\psi_\pm} \int_{T_\pm} *dk . \]

Finally, we define the total angular momentum \( J \) as

\[ J = \frac{1}{16\pi} \left( \int_{S_\infty} *d\eta + \int_{T_+} *d\eta - \int_{T_-} *d\eta \right) , \]

which obviously accounts for contributions of Misner strings. By construction, the defined quantities obey the generalized Smarr relation (10).

It may at first seem strange to consider additional string contributions in the definition of the angular momentum. Let us note, however, that regarding the string itself a source of angular momentum has precedence in the literature, e.g., [2, 31, 32]. Moreover, it is a somewhat remarkable fact that, individually, each of the three integrals in the definition of \( J \) above is divergent, yet their sum produces a finite result.

Evaluating these integrals, we find the following thermodynamic quantities:

\[ M = m , \]
\[ N_\pm = -\frac{2\pi(s \pm n)^2(n \pm a)}{r_+} , \]
\[ J = \frac{(r_+^2 + a^2 + n^2 - 2as)(a - s)}{2r_+} . \]

It is now easy to verify that these thermodynamic quantities, together with \( S \) given by (3), \( T \) [3], and \( \psi \)'s [3], satisfy the first law (9), providing a non-trivial consistency check of our results.

Thus, we have shown (for the first time ever) how to formulate consistent thermodynamics of rotating Taub–NUT solutions. The corresponding first law [9] is of full cohomogeneity and is valid for an arbitrary distribution of Misner strings. It also accounts for both: independent variations of the NUT parameter \( n \) and independent variations of the Misner strength parameter \( s \). As such, it may describe processes such as a capture of a rotating string by a black hole, or an axisymmetric merger of two Taub–NUT solutions.

To cross check our results, let us turn towards the Euclidean action and the corresponding derivation of the thermodynamic quantities. The action for our metric (11) can be obtained from the corresponding AdS action, see appendix, in the limit of the vanishing cosmological constant. It simply reads

\[ G = \frac{m}{2} . \]

Calculated in the grandcanonical ensemble, it is a function of the corresponding quantities:

\[ G = G(T, \psi_+, \psi_-, \Omega) , \]

or more explicitly

\[ G = \frac{\pi T}{16\Omega \psi_+ \psi_-} - \frac{\pi^2}{2\Omega^2} \frac{\psi_+ - \psi_-}{2} + \frac{\sqrt{(4\pi \psi_+ + \Omega)(4\pi \psi_- - \Omega)(4\pi^2 T^2 + \Omega^2)}}{16\Omega^2} . \]

It is then easy to verify that the thermodynamic quantities obtained above by the Komar integration can be recovered as

\[ S = -\frac{\partial G}{\partial T} , \quad N_\pm = -\frac{\partial G}{\partial \psi_\pm} , \quad J = -\frac{\partial G}{\partial \Omega} . \]

At the same time, the relation

\[ G = M - TS - \Omega J - \psi_+ N_+ - \psi_- N_- \]

is equivalent to the Smarr formula (10).

Let us also note the following two interesting facts about the total angular momentum \( J \). First, referring to the AdS case in the appendix again, one can employ the conformal method and calculate the asymptotic charge corresponding to the \( \eta \) Killing vector, to find in the asymptotically flat limit

\[ J_0 \equiv Q(\eta) = m(a - 3s) . \]

This means that the total angular momentum \( J \) actually differs from the asymptotic charge \( J_0 \) by the following Misner string contribution:

\[ J_s = J - J_0 = \frac{a(s^2 + n^2) + s(r_+^2 - 2n^2)}{r_+} . \]

In the special case where the strings are symmetrically distributed, then it is possible to compute finite values for \( J_s \) and \( J_0 \) directly via the Komar integration—those values agree with the \( s = 0 \) case above. Second, it is easy
to see that as long as $s \neq 0$, $J$ remains non-trivial also in the limit of zero rotation, $a \to 0$. However, there is no corresponding term in the first law, as $\Omega$ in this limit automatically vanishes.

Let us conclude here with a few points for discussion. First, we emphasize that the first law we present should be interpreted as a first law of black hole mechanics. Whether or not this first law corresponds, in the Lorentzian case, to a true thermodynamic first law for an equilibrium system depends on whether the potentials $\psi_{\pm}$ correspond to temperatures. If in fact they do, then the situation is somewhat analogous to de Sitter black holes, where one has multiple different temperatures and hence, in general, a non-equilibrium configuration. From this interpretation, regularity of the Euclidean sector—as considered in [28]—is tantamount to enforcing equality of the temperatures and, therefore, equilibrium. However, unlike the case for black holes and cosmological horizons [33], there as yet exists no independent confirmation of whether or not $\psi_{\pm}$ corresponds to a genuine temperature.

Our second point concerns the interpretation of $N_{\pm}$ as a Noether charge ‘entropy’, provided that $\psi_{\pm}$ corresponds to temperatures. In the non-rotating case considered in [29], provided the strings are symmetrically distributed, the Misner charges $N_{\pm}$ coincide with the Noether charge entropy [19] one would associate to the Killing horizons generated by $\xi_{\pm}$, properly normalizing by the respective surface gravities. However, this no longer appears to be the case in the presence of rotation, even in the case of symmetric string distribution. The reason for this is that the terms $N_{\pm}$ appearing in the first law arise from the integration of $k = \partial_t$ over the tubes surrounding the Misner strings, while the Noether charge entropy should correspond to the integral of $\xi_{\pm}$ over the respective tube, normalized by surface gravity. It is somewhat interesting, then, that the first law for the black hole involves a variation of only a ‘portion’ of the Misner string entropy.

Lastly, let us note that a natural next step consists in extending the ideas presented here to the asymptotically flat solution [11] studied in the main text, we consider the rotating AdS Taub–NUT solution [33], calculate its action by using the standard counterterms, and take the asymptotically flat limit.

The rotating AdS Taub–NUT solution reads

$$ds^2 = -\frac{\Delta}{\Sigma} [d\tau + (2n \cos \theta - a \sin^2 \theta + 2s) \frac{d\phi}{K}]^2 + \frac{P}{\Sigma} [adr - (r^2 + a^2 + n^2 - 2as) \frac{d\phi}{K}]^2 + \frac{\Delta}{\Sigma} dr^2 + \frac{\Sigma}{P} \sin^2 \theta d\theta^2 ,$$

(A2)

where $\Sigma = r^2 + (n + a \cos \theta)^2$, (A3)

$$\frac{P}{\sin^2 \theta} = 1 - \frac{4an \cos \theta - a^2 \cos^2 \theta}{l^2} ,$$

(A4)

$$\Delta = a^2 - n^2 - 2mr + r^2 + \frac{3(a^2 - n^2)n^2 + (a^2 + 6n^2)r^2 + r^4}{l^2} .$$

Here, we have included the Misner string strength parameter $s$, $K$ is a constant to regulate the conical singularities on the axes, and $l$ is the AdS length.

The Euclidean action is calculated with the usual counter terms [16]:

$$I = \frac{1}{16\pi} \int_M d^3 x \sqrt{g} (R + \frac{6}{l^2})$$

$$+ \frac{1}{8\pi} \int_{\partial M} d^2 x \sqrt{\kappa} \left[ K - \frac{2}{l} - \frac{l}{2} \mathcal{R} (h) \right] ,$$

(A6)

where $\kappa$ and $\mathcal{R} (h)$ are the extrinsic curvature and Ricci scalar of the boundary respectively. It is associated with a free energy, $\mathcal{G} = I/\beta$, where $\beta$ is the periodicity of the Euclidean time coordinate. This gives

$$G_{\text{AdS}} = \frac{m}{2K} - \frac{r_+ (a^2 + r_+^2 + 3n^2)}{2Kl^2} .$$

(A7)

The action [22] in the main text is then obtained by taking the asymptotically flat limit, $l \to \infty$, and setting $K = 1$ (upon which the axes become regular for the corresponding choice of $s$).

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**Appendix A: Euclidean action**

To find the Euclidean action for the asymptotically flat solution studied in the main text, we consider the rotating AdS Taub–NUT solution, calculate its action by using the standard counterterms, and take the asymptotically flat limit.

The rotating AdS Taub–NUT solution reads

$$ds^2 = -\frac{\Delta}{\Sigma} [d\tau + (2n \cos \theta - a \sin^2 \theta + 2s) \frac{d\phi}{K}]^2 + \frac{P}{\Sigma} [adr - (r^2 + a^2 + n^2 - 2as) \frac{d\phi}{K}]^2 + \frac{\Delta}{\Sigma} dr^2 + \frac{\Sigma}{P} \sin^2 \theta d\theta^2 ,$$

(A1)

where $\frac{P}{\sin^2 \theta} = 1 - \frac{4an \cos \theta - a^2 \cos^2 \theta}{l^2} , $ (A3)

$$\Delta = a^2 - n^2 - 2mr + r^2 + \frac{3(a^2 - n^2)n^2 + (a^2 + 6n^2)r^2 + r^4}{l^2} .$$

Here, we have included the Misner string strength parameter $s$, $K$ is a constant to regulate the conical singularities on the axes, and $l$ is the AdS length.

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$$G_{\text{AdS}} = \frac{m}{2K} - \frac{r_+ (a^2 + r_+^2 + 3n^2)}{2Kl^2} .$$

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The action [22] in the main text is then obtained by taking the asymptotically flat limit, $l \to \infty$, and setting $K = 1$ (upon which the axes become regular for the corresponding choice of $s$).
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