Multiplicity of closed characteristics on $P$-symmetric compact convex hypersurfaces in $\mathbb{R}^{2n}$

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Abstract

There is a long standing conjecture that there are at least $n$ closed characteristics for any compact convex hypersurface $\Sigma$ in $\mathbb{R}^{2n}$, and the symmetric case, i.e. $\Sigma = -\Sigma$, has already been proved by C. Liu, Y. Long and C. Zhu in [Math. Ann., 323(2002), pp. 201-215]. In this paper, we extend the result in that paper to the $P$-symmetric case $\Sigma = P\Sigma$ for a certain class of orthogonal symplectic matrix $P$, and prove that there are at least $\lceil \frac{3n}{4} \rceil$ closed characteristics on $\Sigma$ for any positive integer $n$, where $\lceil a \rceil := \sup\{l \in \mathbb{Z}, l \leq a\}$. To obtain our result, the key problem is to estimate (3.5) which the method is based on the methods for symmetric case due to C. Liu, Y. Long and C. Zhu [15]. By using the properties of Maslov index and Maslov-type index for a certain kind of iteration of symplectic paths, we provide the new estimations (4.6-4.8), which are not considered in other papers.

Keywords: Compact convex hypersurfaces, $P$-symmetric closed characteristics, Iteration theory, Maslov-type index, Hamiltonian system.

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1. Introduction

This paper deal with the multiplicity of closed characteristics on $P$-symmetric compact convex hypersurfaces in $\mathbb{R}^{2n}$. For each $C^2$-compact convex hypersurface $\Sigma \in \mathbb{R}^{2n}$ surrounding 0, we will consider the following problem:

\[
\begin{aligned}
\dot{y}(t) &= JN_\Sigma(y(t)), \quad y(t) \in \Sigma, \quad \forall t \in \mathbb{R}, \\
y(\tau) &= y(0), \quad \tau > 0,
\end{aligned}
\]  

(1.1)

with the standard symplectic matrix $J$, i.e. \[
\begin{pmatrix}
0 & -I_n \\
I_n & 0
\end{pmatrix},
\] and the outward normal unit vector $N_\Sigma(x)$ of $x \in \Sigma$. A solution $(\tau, y)$ of (1.1) is called a closed characteristic on $\Sigma$, and we call it a prime closed characteristic if $\tau$ is the minimal period. If two closed characteristics $(\tau, y)$ and $(\sigma, z)$ are not completely overlapping, then they are called geometrically distinct. For the set of all prime closed characteristics and all geometrically distinct ones

\[
[(\tau, y)] = \{(\sigma, z) \in J(\Sigma)| y(\mathbb{R}) = z(\mathbb{R})\},
\]

we denote them by $J(\Sigma)$ and $\hat{J}(\Sigma)$, respectively.

In the last century, this problem attracted the attentions of many mathematicians. The milestone of this problem was made by P.H. Rabinowitz and A. Weinstein in 1978 [19, 25, who...
proved that
\[ \# \hat{J}(\Sigma, \alpha) \geq 1, \forall \Sigma \in \mathcal{H}(2n). \]

The notation $\mathcal{H}(2n)$ denotes the collection of all hypersurfaces as considered in (1.1). After this, I. Ekeland, H. Hofer, L. Lassoued and A. Szulkin \cite{6, 7, 21} provide a stronger result that
\[ \# \hat{J}(\Sigma) \geq 2, \forall \Sigma \in \mathcal{H}(2n), n \geq 2. \]

This result was improved greatly by Y. Long and C. Zhu \cite{17}. They showed that
\[ \# \hat{J}(\Sigma) \geq \left\lfloor \frac{n}{2} \right\rfloor + 1, \forall \Sigma \in \mathcal{H}(2n). \]

Later, the case $n = 3, 4$ for this conjecture were proved in \cite{26} and \cite{23}, respectively.

Besides, there are also plenty of results for special compact convex hypersurfaces. In \cite{15}, C. Liu, Y. Long and C. Zhu gave the first surprising result that
\[ \# \hat{J}(\Sigma) \geq n, \forall \Sigma = -\Sigma. \]

This is the only result that proves the conjecture in high dimension. After that, Y. Dong and Y. Long \cite{4} studied the $\mathcal{P}$-symmetric case for
\[ \mathcal{P} = \text{diag}(-I_{n-\kappa}, I_{\kappa}, -I_{n-\kappa}, I_{\kappa}), \kappa \in \{1, \cdots, n\}. \]

Under certain assumptions about $\Sigma$, it holds that
\[ \# \hat{J}(\Sigma) \geq n - 2\kappa, \Sigma = \mathcal{P}\Sigma. \]

The $P$-symmetric case was studied by D. Zhang in \cite{27}. He considered the symplectic and orthogonal matrix $P$ with satisfying $P^r = I_{2n}$ for some integer $r > 1$, and proved there are at least two geometrically distinct closed characteristics $(\tau_j, x_j)$ satisfying that
\[ x_j(t + \frac{\tau_j}{r}) = Px_j(t), t \in \mathbb{R}, j = 1, 2. \]

For other kinds of special hypersurfaces such as brake symmetric, pinched or star-shaped hypersurfaces, we refer to \cite{1, 3, 10, 11, 12, 14, 18, 20, 22, 24} and it’s references.

However there are still no results about the total number of closed characteristics on $P$-symmetric ones yet. Therefore, in this paper, we will focus on the $P$-symmetric case. Let $O(2n)$ be the orthogonal group and let the symplectic group be
\[ \text{Sp}(2n) = \{ M \in GL(2n, \mathbb{R}) | M^T J M = J \}. \]

For $P \in O(2n) \cap \text{Sp}(2n)$ and $\Sigma \in \mathcal{H}_P(2n) := \{ \Sigma \in \mathcal{H}(2n) | x, Px \in \Sigma, \forall x \in \Sigma \}$, we call $\Sigma P$-symmetric, and we also call a closed characteristic $(\tau, x)$ symmetric if $(\tau, x)$ belongs to the set
\[ \mathcal{J}_P(\Sigma) := \{(\tau, x) \in \mathcal{J}(\Sigma) | x(\mathbb{R}) = Px(\mathbb{R})\}. \]

Then we prove the following main results.

**Theorem 1.1.** Assume that $P \in O(2n) \cap \text{Sp}(2n)$ and $\Sigma \in \mathcal{H}_P(2n)$. If $P$ is similar to the matrix
\[ R(-\theta)^{\left\lfloor \frac{n}{2} \right\rfloor} \circ R(\theta)^{\left\lfloor \frac{n}{2} \right\rfloor} \]
with $\circ$ defined as \cite{4, 2}, then
\[ \# \hat{J}(\Sigma) \geq \frac{3n}{4}, \quad (1.3) \]
where \( R(\theta) = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix} \), \( e^{im\theta} = 1 \) for some integer \( m > 1 \) and \( \Theta \notin \mathbb{Z} \).

Actually, from Theorem 1.6 we can also obtained that \( \#\hat{J}(\Sigma) \geq n \) when \( m \) is even, i.e. the result in [15]. Besides, according to Proposition 3.1(3) below, we find out for any symmetric
prime closed characteristic \((\tau, x)\), there exist \( l \in \{1, \cdots, m-1\} \) such that \( x(t) = P x(t + \frac{l}{m}) \). However \( l \) is indeterminate. If we fix \( l \), the following result holds.

**Theorem 1.2.** Suppose that \( P \in O(2n) \cap Sp(2n) \) and integer \( m > 1 \). If the following conditions hold:

(i) \( P \) is similar to the matrix \( R(-\theta)^m \) with \( \theta \in (0, \pi] \) and \( e^{im\theta} = 1 \),

(ii) \( x(t) = P x(t + \frac{l}{m}) \) for any \((\tau, x) \in J_P(\Sigma) \) and \( t \in \mathbb{R} \),

then

\[ \#\hat{J}(\Sigma) \geq n. \]

**Remark 1.3.** If \( P \) is not orthogonal, by using Theorem 1.1 in [16] instead of Theorem 1.1 in [2], These results still hold. In this paper, we only focus on the orthogonal case.

An outline of this paper is as follows. In Section 2, the Maslov-type index and Maslov index are briefly introduced, and then we list some properties of splitting numbers. In Section 3, firstly, we provide the properties of symmetric closed characteristics, which have been considered by X. Hu and S. Sun in [9]. Then we transfer the multiplicity problem into the estimation (3.5) by using the approach in [15] with a small modification, i.e. Proposition 3.2. In Section 4, we provide some new estimations (4.6-4.8) and prove the main results.

2. Known properties

In this section, we first introduce the Maslov-type index and Maslov index briefly. Then we will list some useful properties of Splitting numbers.

Let \( P_r(2n) \) be the collection of symplectic paths in \( C([0, \tau], Sp(2n)) \) starting from \( I_{2n} \) and

\[ Sp(2n)_\omega = \{ M \in Sp(2n) | \det(M - \omega I_{2n}) = 0 \}, \quad Sp(2n)^*_\omega = Sp(2n) \setminus Sp(2n)_\omega \]

for any \( n \in \mathbb{N} \), \( \omega \in U := \{ z \in \mathbb{C}, |z| = 1 \} \) and \( \tau > 0 \).

Consider any symplectic path \( \gamma \in P_r(2n) \). The Maslov-type index \( (i_\omega(\gamma), \nu_\omega(\gamma)) \) of \( \gamma \) are defined by the intersection number of symplectic path \( e^{-\epsilon J} \gamma * \zeta \) on \( Sp(2n)^*_\omega \) and \( \dim_{\mathbb{C}} \ker_{\mathbb{C}}(\gamma(\tau) - \omega I_{2n}) \), respectively. Where \( \zeta \) is a path in \( Sp(2n)^*_\omega \) and \( \epsilon \) is small enough.

Similarly, the Maslov index \( \mu(Gr(Q^T), Gr(\gamma)) \) of \( \gamma \) with respect to \( Q \in Sp(2n) \) is defined by the intersection number of Lagrangian path \( e^{-\epsilon J}Gr(\gamma) \) on the Maslov cycle of the graph \( Gr(Q^T) := \{(x, Q^T x) | x \in \mathbb{R}^{2n}\} \) in Lagrangian Grassmannian \( \text{Lag}(2n) \). The transformation \( \hat{J} \) correspond to a symplectic structure of \( \mathbb{R}^{4n} \) in which the graph \( Gr(M) \) become a Lagrangian subspace for any \( M \in Sp(2n) \). All the details of this two definitions can be found in [13], [2] and [9].

The relation between this two indices was given by Lemma 4.6 of [9] as follows.

**Lemma 2.1.** Let \( P \in O(2n) \cap Sp(2n) \), we have

\[ \mu(Gr(\omega I), Gr(\gamma(t))) = \begin{cases} i_1(\gamma) + n, & \omega = 1 \\ i_\omega(\gamma), & \omega \in U \setminus \{1\}, \end{cases} \quad (2.1) \]

and

\[ \mu(Gr(\omega P^T), Gr(\gamma(t))) = \mu(Gr(\omega), Gr(P(\gamma(t)))) = i_\omega(\gamma * \xi) - i_\omega(\xi) \quad (2.2) \]
In addition, the splitting numbers of Proposition 2.2. For splitting numbers, we have

\[ S^+(\tau) = \lim_{\epsilon \to \pm 0} (i\exp(\sqrt{-1} \tau)\omega(\gamma) - i\omega(\gamma)). \tag{2.4} \]

For splitting numbers, we have

**Proposition 2.2.** Let \( M \in \text{Sp}(2n), \omega \in \mathbb{U}, \theta \in (0, \pi) \) and \( 0 \leq \theta \leq \theta \leq 2\pi, \sigma(M) \) is the spectral set of \( M \). Then there hold

\[
\begin{align*}
S^+_M(\omega) &\geq 0, \forall \omega \in \mathbb{U}, S^+_M(\omega) = 0, \omega \notin \sigma(M), \tag{2.5} \\
S^+_M(\omega) &\leq \dim \ker(M - \omega I)^{2n}, \tag{2.6} \\
P_\omega(M) - S^+_M(\omega) &\leq Q_\omega(M) - S^-_M(\omega) \geq 0, \tag{2.7} \\
S^+_M(\omega) + S^-_M(\omega) &\leq \dim \ker(M - \omega I)^{2n}, \omega \in \sigma(M), \tag{2.8} \\
(S^+_{I_2(1)}, S^-_{I_2(1)}) &=(1, 1), (S^+_{-I_2(-1)}, S^-_{-I_2(-1)})=(1, 1), \tag{2.9} \\
&S^+_{R(\theta)}(e^{\sqrt{-1} \theta}), S^-_{R(\theta)}(e^{\sqrt{-1} \theta})=(0, 1), \tag{2.10} \\
(S^+_{R(-\theta)}(e^{-\sqrt{-1} \theta}), S^-_{R(-\theta)}(e^{-\sqrt{-1} \theta}))=(1, 0), \tag{2.11} \\
0 \leq \nu_\omega(M) - S^-_M(\omega) &\leq P_\omega(M), 0 \leq \nu_\omega(M) - S^+_M(\omega) \leq Q_\omega(M), \tag{2.12} \\
(P_\omega(M), Q_\omega(M)) &=(Q^-_M, P^-_M), \tag{2.13} \\
\frac{1}{2} \sum_{\omega \in \mathbb{U}} \nu_\omega(M) &\leq \sum_{\omega \in \mathbb{U}} P_\omega(M) = \sum_{\omega \in \mathbb{U}} Q_\omega(M) \leq n. \tag{2.14} \\
i_{\sqrt{-1} \theta}(\gamma) &\leq i_1(\gamma) + \sum_{0<\theta<\hat{\theta}} S^+_M(e^{\sqrt{-1} \theta}) - \sum_{0<\theta<\hat{\theta}} S^-_M(e^{\sqrt{-1} \theta}). \tag{2.15} \\
\end{align*}
\]

**Proof.** (2.3), (2.4), follow from Lemma 9.1.6 and Lemma 9.1.9 in [13]. (2.5), (2.6), follow from Lemma 1.8.14, Theorem 9.1.7 in [13] and the definition of Krein type numbers. (2.7), (2.8), follow from Proposition 9.1.11 in [13]. (2.9), (2.10), (2.11), follow from Lemma 1.3.8 in [13], (2.12), (2.13), (2.14), (2.15), follows from the definition of Krein type numbers and (2.13), (2.14), (2.15), follows from Proposition 9.1.11 in [13]. \( \square \)

**Remark 2.3.** For any \( P \in O(2n) \cap \text{Sp}(2n) \), we also have

\[
\begin{align*}
S^+_P(\omega) + S^-_P(\omega) &\leq \nu_\omega(P), \forall \omega \in \mathbb{U}, \tag{2.16} \\
\sum_{\omega \in \mathbb{U}} S^+_P(\omega) &\leq \sum_{\omega \in \mathbb{U}} S^-_P(\omega) = n. \tag{2.17}
\end{align*}
\]
Since $P$ is unitarily diagonalizable, this remark is followed by (2.5-2.11).

3. Special properties of $P$-symmetric hypersurfaces

In this section, we first provide some useful properties of closed characteristics. Then based on the method in [15], we conclude that the multiplication problem is equivalent to the estimation (3.5).

Let $j_{\Sigma}: \mathbb{R}^{2n} \to \mathbb{R}$ be a gauge function of $\Sigma$ defined by

$$j_{\Sigma}(0) = 0, \; j_{\Sigma}(x) = \inf\{\lambda > 0, x_{\lambda} \in C\}, \; x \neq 0.$$  

Fix a constant $\alpha \in (1, 2)$ and define the Hamiltonian function $H_\alpha : \mathbb{R}^{2n} \to [0, \infty)$ by

$$H_\alpha(x) = j_{\Sigma}(x)^{\alpha}, \forall x \in \mathbb{R}^{2n}.$$ 

Note that $H_\alpha \in C^1(\mathbb{R}^{2n}, \mathbb{R}) \cap C^2(\mathbb{R}^{2n} \setminus \{0\}, \mathbb{R})$ is convex and $\Sigma = H_\alpha^{-1}(1)$. Since the gradient of $H_\alpha$ on $\Sigma$ is normal and nonzero, then the problem (1.1) is equivalent to the following fixed-energy problem

$$\begin{cases}
H_\alpha(x) = 1, \\
\dot{x} = JH_\alpha'(x), \\
x(0) = x(\tau).
\end{cases} \tag{3.1}$$

It’s well known that the solutions of (3.1) and (1.1) are in one to one correspondence with each other. It does not depend on the particular choice of $\alpha$.

For any $(\tau, x) \in J(\Sigma)$, we denote by $\gamma_{x} \in \mathcal{P}_{\tau}(2n)$ the fundamental solution of the linear system

$$\dot{y}(t) = JH_\alpha''(x(t))y(t),$$

which is the linearization of system (3.1). We also call $\gamma_{x}$ the associated symplectic path of $(\tau, x)$. Consider $P \in \text{Sp}(2n)$ and $\Sigma \in \mathcal{H}_{P}(2n)$, it implies that $H_\alpha(x) = H_\alpha(Px), \forall x \in \mathbb{R}^{2n}$. Then we have

$$H_\alpha'(x) = P^T H_\alpha'(Px), H_\alpha''(x) = P^T H_\alpha''(Px)P. \tag{3.2}$$

Denote by $(l, k)$ the greatest common divisor of $l, k \in \mathbb{N}$. $l$ and $k$ are relatively prime integers if $(l, k) = 1$. Then we have following properties, which have been considered by X. Hu and S. Sun in [9].

**Proposition 3.1.** Let $(\tau, x) \in J(\Sigma)$ be a prime closed characteristic, and let $\gamma_{x}$ be the associated symplectic path of $(\tau, x)$, then we have

1. $(\tau, Px) \in J(\Sigma)$ is also a prime closed characteristic.
2. $\gamma_{Px}(t) = P\gamma_{x}(t)P^{-1}$, where $\gamma_{Px}$ is the associated symplectic path of $(\tau, Px)$.
3. If there exist $a, b \in [0, \tau) (a < b)$ and a smallest integer $k > 0$ such that $x(a) = Px(b)$ and $x(t) = P^k x(t), \forall t \in [0, \tau)$, then $(\tau, x)$ is symmetric and

$$x(t) = Px(t + \frac{l\tau}{k}),$$

where $(l, k) = 1$.
4. If $(\tau, x)$ is symmetric, then $\gamma_{x}(t + \frac{l\tau}{k}) = P^{-1}\gamma_{x}(t)P\gamma_{x}(\frac{l\tau}{k})$. Especially,

$$\gamma_{x}(l\tau) = P^{-k}(P\gamma_{x}(\frac{l\tau}{k}))^k.$$
Proof. (1) Since \((\tau, x) \in J(\Sigma)\), then \((\tau, x)\) solves the system (3.1), i.e.
\[
\dot{x} = JH'_\alpha(x), x(0) = x(\tau).
\] (3.3)
\(\Sigma \in H_P(2n)\) implies that \(P x \in \Sigma\) and \(P x(0) = P x(\tau)\). By (3.2), we obtain
\[
P \dot{x} = P JH'_\alpha(x) = P J P^T H'_\alpha(P x) = JH'_\alpha(P x).
\]
Then (1) follows.

(2) Combining (3.2) and the assumption that \(P \in \text{Sp}(2n)\), we have
\[
\dot{\gamma}_x(t) = JH''_\alpha(x(t)) \gamma_x(t)
\] (3.4)
implies that
\[
P \dot{\gamma}_x(t) = JH''_\alpha(P x(t)) P \gamma_x(t).
\]
Since the fundamental solution starts from \(I_{2n}\), we get (2).

(3) By condition \(x(a) = P x(b)\) and the uniqueness of solution of (3.3), we have
\[
x(t) = P x(t + b - a) = P^2 x(t + 2(b - a)) = \cdots = P^k x(t + k(b - a)).
\]
Since \(x(t) = P^k x(t)\) with the smallest integer \(k > 0\), there exist \(l < k\) such that \(k(b - a) = l \tau\), i.e. \(x(t) = P x(t + \frac{l \tau}{k})\), and \(l, k\) are relatively prime integers. Otherwise, let \(l = rl_1, k = rk_1, r > 1\), we obtain \(k_1 < k\) such that
\[
x(t) = P x(t + \frac{l_1 \tau}{k_1}) = \cdots = P^{k_1} x(t),
\]
which is a contradiction.

(4) Replacing \(t\) of (3.4) by \(t + \frac{l \tau}{k}\), by (3.2) and (3) above, we get that
\[
P \dot{\gamma}_x(t + \frac{l \tau}{k}) = P JH''_\alpha(P^{-1} x(t)) \gamma_x(t + \frac{l \tau}{k}) = P J P^T H''_\alpha(x(t)) P \gamma_x(t + \frac{l \tau}{k}) = JH''_\alpha(x(t)) P \gamma_x(t + \frac{l \tau}{k}).
\]
Since the fundamental solution starts from \(I_{2n}\) again, we have
\[
P \gamma(t + \frac{l \tau}{k}) \gamma^{-1}(\frac{l \tau}{k}) P^{-1} = \gamma(t),
\]
then (4) follows. \(\square\)

By using the approach of in [15] with a small modification, the following Proposition holds.

Proposition 3.2. Let \(P \in O(2n) \cap \text{Sp}(2n), \Sigma \in H_P(2n)\) and assume \# \(\hat{J}(\Sigma) < +\infty\). If there exists an integer \(n_1 > 0\) such that for any symmetric closed characteristic \([\tau, x] \in \hat{J}(\Sigma)\),
\[
i(\gamma_x) + 2S^+_{\tau_x(\gamma)}(1) - \nu(\gamma_x) \geq n_1,
\] (3.5)
Then
\[
\# J(\Sigma) \geq \left\lfloor \frac{n_1 + n_2}{2} \right\rfloor.
\]
Proof. By the assumptions before, we can denote \( \mathcal{J}(\Sigma) \) by

\[
\{((\tau_1, x_1)), \cdots, ((\tau_p, x_p))\} \cup \bigcup_{i=1}^{q} \{((\tau_{p+i}, x_{p+i})), ((\tau_{p+i}, P x_{p+i})), \cdots, ((\tau_{p+i}, P^{k_i} x_{p+i}))\}. \tag{3.6}
\]

Where \( \{((\tau_j, x_j))\}_{j=1}^{q} \) are geometrically distinct symmetric closed characteristics and

\[
\{((\tau_{p+i}, x_{p+i})), ((\tau_{p+i}, P x_{p+i})), \cdots, ((\tau_{p+i}, P^{k_i} x_{p+i}))\}_{i=1}^{q}
\]

are distinct sets of asymmetric ones. Let \( K \) be the number of asymmetric closed characteristics. Since \( k_i \geq 2 \) for any \( i = 1, \cdots, q \), which is followed by Proposition 3.1(3), then we have \( \# \mathcal{J}(\Sigma) = p + K < +\infty \), \( K = k_1 + \cdots + k_q \geq 2q \).

Let \( i_j^m = i_1(\gamma_{jx}^m), u_j^m = \nu_1(\gamma_{jx}^m), M_j = \gamma_{jx}(\tau), j \in \{1, \cdots, p + q\} \). We can apply the common index jump Theorem 11.2.1 in [13] to the associated symplectic paths

\[
\{((\tau_1, x_1)), \cdots, ((\tau_p, x_p)), (2\tau_{p+1}, x_{p+1}^2), \cdots, (2\tau_{p+q}, x_{p+q}^2)\}.
\]

Then we obtain infinite many \( (N, m_1, \cdots, m_{p+2q}) \in \mathbb{N}^{p+2q+1}, N > n \) s.t. \( \forall j \in \{1, \cdots, p + q\} \), we have

\[
i_{j}^{2m_j+1} = 2N + i_j + \nu_j^{2m_j-1} + 2N = (i_j + 2S_{M_j}^+ (1) - \nu_j), \tag{3.7}
\]

\[
i_{j}^{2m_j} \geq 2N - n, \nu_j^{2m_j} \leq 2N + n. \tag{3.8}
\]

For \( \forall j \in \{1, \cdots, q\} \), we have

\[
i_{p+j}^{4m_{p+j}+2} = 2N + i_{p+j}^2, \tag{3.9}
\]

\[
i_{p+j}^{4m_{p+j}+2} - 2 + \nu_{p+j}^{4m_{p+j}+2} = 2N - (i_{p+j}^2 + 2S_{M_{p+j}}^+ (1) - \nu_{p+j}), \tag{3.10}
\]

\[
i_{p+j}^{4m_{p+j}+2} + \nu_{p+j}^{4m_{p+j}+2} \geq 2N - n, \nu_{p+j}^{4m_{p+j}+2} \leq 2N + n. \tag{3.11}
\]

Claim 1: \( m_{p+j} = 2m_{p+q+j} \) for any \( j \in \{1, \cdots, q\} \).

In fact, using (3.8), Lemma 15.6.3 in [13], (3.10) in [13] we have

\[
i_{p+j}^{2m_{p+j}} \geq 2N - n \geq 2N - (i_{p+j}^2 + 2S_{M_{p+j}}^+ (1) - \nu_{p+j})
\]

\[
= i_{p+j}^{4m_{p+j}+2} - 2 + \nu_{p+j}^{4m_{p+j}+2} \geq i_{p+j}^{4m_{p+j}+2} - 2.
\]

Then using (3.8), (3.10), (15.1.18) and (15.3.6) in [13],

\[
i_{p+j}^{2m_{p+j}} < i_{p+j}^{2m_{p+j}+1} + \nu_{p+j}^{2m_{p+j}} \leq 2N + n \leq 2N + i_{p+j}^2 = i_{p+j}^{4m_{p+j}+2}.
\]

Thus, by Lemma (15.3.6) in [13], we get

\[
4m_{p+q+j} - 2 < 2m_{p+j} < 4m_{p+q+j} + 2 \Rightarrow m_{p+j} = 2m_{p+q+j}.
\]

The claim follows.

According to Lemma 15.3.5 in [13], we get an injection map \( \Psi : \mathbb{N} \to \mathcal{J}(\Sigma) \times \mathbb{N} \). Let

\[
\Psi(N - s + 1) := \{((\tau_{j(s)}, x_{j(s)})), m(s)) \), \( s \in \{1, \cdots, n\},
\]

such that

\[
i_{j(s)}^{m(s)} \leq 2N - 2s + n \leq i_{j(s)}^{m(s)} + \nu_{j(s)}^{m(s)} - 1. \tag{3.12}
\]

where \( j(s) \in \{1, \cdots, p + q\}, m(s) \in \mathbb{N} \). Then From (3.12), (15.1.18) in [13], (3.7), we deduce
that
\[ i_{j(s)}^m \leq 2N - 2s + n < 2N + n \leq 2N + i_{j(s)}^1 = i_{j(s)}^{2m_{j(s)}+1}. \] 
(3.13)

Let
\[ S_1 = \{ k \in \{1, \ldots, \lfloor \frac{n_1 + n}{2} \rfloor \}, 1 \leq j(k) \leq p \}, S_2 = \{ k \in \{1, \ldots, n\}, p + 1 \leq j(k) \leq p + q \}. \] 
(3.14)

**Claim 2:** \#\( S_1 \leq p \).

In fact, let \( k \in S_1 \), then \( 1 \leq j(k) \leq p \). By (3.12), (3.7) and the assumptions before, it follows that
\[ i_{j(k)}^m + \nu_{j(k)}^m - 1 \geq 2N - 2k + n \geq 2N + n - 2(\frac{n_1 + n}{2}) = 2N - n_1 \]
\[ \geq 2N - (i_{j(k)}^1 + 2S_{j(k)}^+) (1 - \nu_{j(k)}^1) = i_{j(k)}^{2m_{j(k)}+1} + \nu_{j(k)}^{2m_{j(k)}+1}. \]

According to (3.13), we conclude that
\[ 2m_{j(k)} - 1 < m(k) < 2m_{j(k)} + 1 \Rightarrow m(k) = 2m_{j(k)}. \]

Then \( \Psi(N - k + 1) = (\{\bar{\tau}_{j(k)}, x_{j(k)}\], 2m_{j(k)}\}. \) Since \( \Psi \) is injective, by (3.14), we have \#\( S_1 \leq p \).

**Claim 3:** \#\( S_2 \leq 2q \).

In fact, let \( k \in S_2 \), then \( p + 1 \leq j(k) \leq p + q \). From (3.12), Lemma 15.6.3 in [13], (3.10) and Claim 1, we obtain
\[ i_{j(k)}^m + \nu_{j(k)}^m - 1 \geq 2N - 2s + n \geq 2N - n \geq 2n - (i_{j(k)}^2 + 2S_{j(k)}^+ (1 - \nu_{j(k)}^2)) \]
\[ = i_{j(k)}^{2m_{j(k)}+2} + \nu_{j(k)}^{2m_{j(k)}+2} = j_{j(k)}^{2m_{j(k)}+2} + \nu_{j(k)}^{2m_{j(k)}+2}. \]

By (3.13), we have
\[ 2m_{j(k)} - 2 < m(k) < 2m_{j(k)} + 1 \Rightarrow m(k) \in \{2m_{j(k)} - 1, 2m_{j(k)}\}. \]

Since \( \Psi \) is injective again, this claim follows.

Finally, by Claim 2 and Claim 3, we have
\[ \#\hat{J}(\Sigma) = p + K \geq p + 2q \geq \#S_1 + \#S_2 \geq \frac{n_1 + n}{2}. \]

\[ \square \]

4. **Index iteration theory and the proof of the main theorem**

In this section, we use the \((P,m)\)-iteration of symplectic path in [4] and the iteration formulas from [9] and [13] to provide the estimation in Theorem 4.4. Then we will give the proof of main results.

Firstly, we will show some notations. Let \( \Omega(2n) := O(2n) \cap \text{Sp}(2n) \) and
\[ \omega_k := e^{\sqrt{-1} \theta_k} = e^{\frac{2k\pi}{m}}, \quad k = 0, \ldots, m - 1. \]  
(4.1)

\( \sigma(M) \) denotes the spectrum set of matrix \( M \). Then we define
\[ \Omega_m(2n) := \{ P \in \Omega(2n) | P^m = I_{2n} \} \text{ and } \sigma(P) = \{ \omega, \tilde{\omega}_k \}, k \in \{1, \ldots, \lfloor \frac{m}{2} \rfloor \}, \]
\[ \Omega_{m,k}(2n) := \{ P \in \Omega_m(2n) | \omega_k \in \sigma(P) \}, \quad k = 1, \ldots, \lfloor \frac{m}{2} \rfloor, \]
Lemma 4.3.

Note that \( P \) is odd).

Proof. For path \( \gamma \) be a symmetric positive defined matrix, and satisfies \( B(t + \tau) = B(t), \forall t \in [0, \tau] \). \( \gamma \in \mathcal{P}_r(2n) \) denotes the fundamental solution of \( \dot{y}(t) = J B(t)y(t) \), then

\[
\mu(\text{Gr}(\omega I_{2n}), \text{Gr}(\gamma)) = \nu_1(\gamma) + \sum_{0 < t < \tau} \nu_\omega(\gamma(t)).
\]

Lemma 4.3. Let \( B(t) \) be a symmetric positive defined matrix, and satisfies \( B(t + \tau) = B(t), \forall t \in [0, \tau] \). \( \gamma \in \mathcal{P}_r(2n) \) denotes the fundamental solution of \( \dot{y}(t) = J B(t)y(t) \), then

\[
B_P(t) := - J P \dot{\gamma}(t)(P\gamma(t))^{-1} = (P^{-1})^TB(t)P^{-1} = PB(t)P^T,
\]

which is also positive definite. Similar to the proof of Theorem I.4.6 in [5], there are at most finitely many times \( \{t_1, \cdots, t_k\} \) such that \( \omega \in \sigma(P\gamma(t_j)), j \in \{1, \cdots, k\} \). Thus this lemma follows by the same approach of Proposition 15.1.3 in [13].
Then we have following results.

**Theorem 4.4.** Let \( P \in \Omega_m(2n) \) with \( m \geq 2 \). \((\tau, x) \in J(\Sigma)\) is a prime closed characteristic and \( \gamma_x \) is the associated symplectic path of \((\tau, x)\). Assume that

\[
x(t) = Px(t + \frac{\tau}{m}), \forall t \in [0, \tau),
\]

then

\[
\mu(Gr(I_{2n}), Gr(\gamma_x)) + 2S^+_{\gamma_x(\tau)}(1) - \nu(\gamma_x) \geq 2n - S^-_P(\omega), \ \omega \in \sigma(P) \cap U^+.
\]

(4.6)

Especially, if \( P \in \tilde{\Omega}_m(2n) \), then

\[
\mu(Gr(I_{2n}), Gr(\gamma_x)) + 2S^+_{\gamma_x(\tau)}(1) - \nu(\gamma_x) \geq \frac{3n}{2},
\]

(4.7)

else if \( P \in \tilde{\Omega}_m^{-1}(2n) \) or \( \tilde{\Omega}_m^{-1}(2n) \), then

\[
\mu(Gr(I_{2n}), Gr(\gamma_x)) + 2S^+_{\gamma_x(\tau)}(1) - \nu(\gamma_x) \geq \frac{3n}{2} + 1 \text{ or } \frac{3n}{2}.
\]

(4.8)

**Proof.** Denote by \( \omega = e^{\sqrt{-1} \theta} \) and \( \omega_k \) as [1.1]. Without loss of generality, we assume \( P \in \Omega_{m, k}(2n) \), i.e. \( \{\omega_k, \omega_{m-k}\} = \sigma(P) \) where \( k \in \{1, \ldots, \lfloor \frac{m}{2} \rfloor\} \). Let \( M = \gamma(\frac{\tau}{m}), \gamma_x(t) = \gamma_x(t), \forall t \in [0, \frac{\tau}{m}] \). According to Proposition 4.1 and Definition 4.1 we have \( \gamma_x = \hat{\gamma}_x^{m, P} \).

(1) When \( m > 2 \), we know that \( k - 1, k, m - k \) are all different. In view of Proposition 1.2, 2.9, 2.10, 2.12, 2.14, 2.16, Lemma 1.3 we deduce that

\[
\mu(Gr(I_{2n}), Gr(\gamma_x)) + 2S^+_{\gamma_x(\tau)}(1) - \nu(\gamma_x) \geq 3n - \left( \sum_{\theta_{k-1} < \theta < \theta_k} S^+_{\gamma_x(\tau)}(\omega) - \sum_{\theta_{k-1} < \theta < \theta_k} S^-_{\omega}(\omega) \right) + \left( \sum_{\theta_{k-1} < \theta < \theta_k} S^+_{\gamma_x(\tau)}(\omega) - \sum_{\theta_{k-1} < \theta < \theta_k} S^-_{\omega}(\omega) \right) + \left( \sum_{\theta_{k-1} < \theta < \theta_k} S^+_{\gamma_x(\tau)}(\omega) - \sum_{\theta_{k-1} < \theta < \theta_k} S^-_{\omega}(\omega) \right) = \frac{3n}{2}.
\]

(4.9)

By Remark 2.3, 2.10, 2.11, \( P \in \tilde{\Omega}_{m, k}(2n) \) implies that \( n \) is even and

\[
S^\pm_{\omega}(\omega_{m-k}) = \frac{n}{2}.
\]
Thus we get (4.7). Similarly, $P \in \tilde{\Omega}^1_{m,k}(2n)$ or $P \in \tilde{\Omega}^{-1}_{m,k}(2n)$ implies that $n$ is odd and

$$S_P^-(\omega_k) = \left\lfloor \frac{n}{2} \right\rfloor \text{ or } \left\lfloor \frac{n}{2} \right\rfloor + 1.$$ 

Then (4.8) follows.

Remark 4.5. If $m = 2$, the calculation in proof of Theorem 4.4(1) essentially coincides with the proof of C. Liu, Y. Long and C. Zhu in [13].

Then we conclude the following Theorem.

Theorem 4.6. Let $P \in \Omega_m(2n)$ with an integer $m \geq 2$, $\sigma(P) = \{\omega, \bar{\omega}\}$ and $\Sigma \in H_P(2n)$. It holds that

$$\# \hat{J}(\Sigma) \geq \left\lfloor \frac{n_1 + n}{2} \right\rfloor = n + \left\lfloor -\max\{S_P^-(\omega), S_P^-(\bar{\omega})\} \right\rfloor,$$

(4.10)

Especially, if $m$ is even, then

$$\# \hat{J}(\Sigma) \geq n.$$ 

(4.11)

Proof. We prove this result by following steps.

Step 1: Assume that $m$ is not prime. Denote by $m = m_1p$ and $P_1 = P^{m_1}$, where $p$ is a prime factor. Then we have $\Sigma = P_1^p\Sigma$, and it is sufficient to consider $m$ as a prime number.

Step 2: Let $m$ be a prime. Assume that $P \in \Omega_{m,k}(2n)$ and $(\tau, x) \in J_P(\Sigma)$ is a prime closed characteristic. By Proposition 3.1(3), there exist an integer $l \in \{1, \cdots, m-1\}$ such that

$$x(t) = Px(t + \frac{l\tau}{m}), \ \forall t \in \mathbb{R}.$$ 

Then we can choose $r \in \{1, \cdots, m-1\}$ such that

$$x(t) = P^r x(t + \frac{\tau}{m}), \ \forall t \in \mathbb{R}.$$ 

Assume that $\# \hat{J}(\Sigma) < +\infty$. If $\omega_k^r \in \{\omega_1, \cdots, \omega_{\lfloor\frac{m}{2}\rfloor}\}$, then by Theorem 4.4(1), (2.1) and Remark 4.5 we obtain that

$$i_1(\gamma_x) + 2S_{\gamma_x(\tau)}^+(1) - \nu_1(\gamma_x) \geq n - S_{P^r}(\omega_k^r) \geq n - \max\{S_P^-(\omega_k), S_P^-(\omega_{m-k})\},$$

(4.12)

else if $\omega_{m-k}^r \in \{\omega_1, \cdots, \omega_{\lfloor\frac{m}{2}\rfloor}\}$, (4.12) follows as well. Thus we have

$$n_1 = n - \max\{S_P^-(\omega_k), S_P^-(\omega_{m-k})\}.$$ 

Then by Proposition 3.2 (4.10) follows. However, if $m = 2$, it follows from (4.9) that $n_1 = n$. Then combine with Step 1, this theorem holds.

Remark 4.7. Since $P$ is an orthogonal symplectic matrix, which implies that $P$ is unitarily diagonalizable, then $P \in \Omega_m(2n)$ if and only if $P$ is similar to the following matrix

$$R(-\theta)^{\circ S_P^+(\omega)} \circ R(\theta)^{\circ S_P^-(\omega)},$$

where $R(\theta) = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix}$, $e^{im\theta} = 1$ and $\frac{\theta}{2\pi} \notin \mathbb{Z}$.

Then we prove Theorem 1.1 as follows.
Proof. If \( n \) is even, it follows from the assumption and Remark 4.7 that \( P \in \tilde{\Omega}_{m,k}(2n) \) for some \( k \in \{1, \cdots, \lfloor \frac{m}{2} \rfloor \} \). By Remark 2.3 and (4.10), we have \( n_1 = \frac{n}{2} \) and (1.3) holds. Similarly, if \( n \) is odd, \( P \in \tilde{\Omega}^\pm_{m,k}(2n) \) for some \( k \in \{1, \cdots, \lfloor \frac{m}{2} \rfloor \} \). By Remark 2.3 and (4.10) again, \( n_1 = \lfloor \frac{n}{2} \rfloor \) and then we deduce that \( \# \hat{J}(\Sigma) \geq \lfloor \frac{3n-1}{4} \rfloor \). Since \( n \) is odd, it holds that \( \lfloor \frac{3n}{4} \rfloor = \lfloor \frac{3n-1}{4} \rfloor \) and this result follows.

Proof of Theorem 1.2.

Proof. Let \( e^{it} = \omega \). From assumption (i), we know that \( S_P^- (\omega) = 0 \) and \( S_P^+ (\omega) = n \). According to (ii), (2.1) and (4.6), we have \( n_1 = n \). Then by Proposition 3.2 this result follows.

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