Koszul Property and Poincaré Series
of Matrix Bialgebras of Type $A_n$  

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Abstract

Bialgebras, defined by means of Yang-Baxter operators which verify the Hecke equation, are considered. It is shown that they are Koszul algebras. Their Poincaré series are calculated via the Poincaré series of the corresponding quantum planes.

The theory of deformations of function algebras on algebraic groups was invented by Manin [12], Faddeev, Reshetikhin, Takhtajan [7]. The idea is to consider deformations of coordinate spaces, called quantum planes and then consider the “symmetries” of these quantum planes. For example, the quantum group $GL_q(2)$ is the “symmetry group” of the quantum planes $A_{q|2}$ and $A_{q|2}$. 

Quantum planes are represented by quadratic algebras, which should be considered as the function algebras on them. Manin considered two quadratic algebras, defined on a finite dimensional vector space $V$ and its dual $V^*$. And in the simplest case, when the commutativity rules are controled by only one element $q$ from the ground field, he obtained the standard deformation of $GL_q(n)$.

Manin’s construction was generalized by Takeuchi [11] to the case of orthogonal and symplectic groups. Sudbery [15] and Mukhin [6] generalized the construction to the case of an arbitrary family of quadratic algebras defined only on $V$. In fact in Manin’s construction as well as in the classical case, the R-matrix is symmetric, so we can identify $V$ and $V^*$.

One should mention some earlier works of Lyubashenko [10] and Gurevich [8], where the authors quantized the “basic symmetry”, i.e. quantized the category, where we are working in.

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Thus being given a family of quadratic algebras, we want to consider their “symmetries”. In the language of algebraic geometry, these “symmetries” should be represented by a bialgebra (or a Hopf algebra if we need only “invertible symmetries”), which universally coacts on the quadratic algebras. It is shown that such a bialgebra (resp. Hopf algebra) always exists.

In this work we consider bialgebras, which are determined as above by only two quadratic algebras, such that the R-matrix obeys the Yang-Baxter equation and Hecke equation. We call such bialgebras matrix bialgebras of type $A_n$. We will show that these bialgebras are Koszul algebras and also consider their Poincaré series.

1 Quadratic algebras and matrix bialgebras

1.1 Quadratic algebras

Let $k$ be a fixed algebraically closed field of characteristic zero, $V$ be a $k$-linear vector space of finite dimension $d$. A quadratic algebra (QA) $A$ over $V$ is defined to be a factor algebra of the tensor algebra over $V$ by an ideal, generated by a set $R(A)$ of quadratic elements, i.e. elements of $V \otimes V$. $A$ inherits the grade of $T(V) : A = \oplus_n A_n$. The Poincaré series of $A$ is by definition the following formal series:

$$P_A(t) := \sum_{n=0}^{\infty} \dim_k A_n t^n.$$

The reader is referred to [12] for the definition of the Koszul complex (of second kind) of $A$. $A$ is said to have the Koszul property or to be a Koszul algebra if this Koszul complex is exact [12]. The following lemma is due to Backelin [1].

**Lemma 1.1** $A$ is a Koszul algebra iff the lattice generated by $R(A)$ in $V^{\otimes n}$ is distributive for all $n \geq 2$.

A lattice on $V^{\otimes n}$ is a set of subspaces of $V^{\otimes n}$, which is closed under $+$ and $\cap$. The lattice generated by $R(A)$ is the one generated by $R^n_i(A), i = 1, \cdots, n - 1$, where

$$R^n_i(A) := V^{\otimes i-1} \otimes <R(A)>_k \otimes V^{\otimes n-i-1}.$$

The lattice $L$ is distributive if and only if for all $u, v, w \in L$

$$u \cap (v + w) = (u \cap v) + (u \cap w),$$
$$u + (v \cap w) = (u + v) \cap (u + w).$$

Note that the two equations are in fact equivalent.
Lemma 1.2 Let $X$ be a vector space and $\mathcal{L}$ be a lattice on $X$. Let $X = \bigoplus_{i \in I} X_i$ be a decomposition of $X$ into its subspaces, such that for all $u \in \mathcal{L}$

$$u = \bigoplus_{i \in I} u \cap X_i.$$ 

Then $\mathcal{L}$ is distributive if and only if for all $i \in I$

$$\mathcal{L} \cap X_i := \{u \cap X_i | u \in \mathcal{L}\}$$

is a distributive lattice on $X_i$.

Proof. Let $u_i := u \cap X_i$. We have for $u, v \in \mathcal{L}$

$$u + v = \bigoplus (u + v)_i \supset \bigoplus (u_i + v_i),$$

$$u \subset \bigoplus (u_i + v_i), v \subset \bigoplus (u_i + v_i).$$

Hence $(u + v)_i = u_i + v_i$. Whence the assertion follows.

### 1.2 Matrix bialgebras

Assume we are given a family of quadratic algebras over a vector space, we want to study the algebra $E$, which universally coacts on this family [12, 13, 15]. It is shown that $E$ exists and is a quadratic algebra, and from the universal property it follows that $E$ is a bialgebra. We call such bialgebras matrix bialgebras.

We consider the following construction. Let $R : V \otimes V \rightarrow V \otimes V$ be a diagonalizable operator. Let $R = \sum_{i=1}^{k} c_i P_i$ be the spectral decomposition of $R$. Let $A_i$ be a QA with $R(A_i) = \text{Im} P_i$. Then the matrix bialgebra $E$, determined by the family $\{A_i | i = 1, 2, \cdots, k\}$, is the factor algebra of the tensor algebra over $E_1 := V^* \otimes V$ by the ideal, generated by

$$R(E) = \text{Im}(\mathcal{R} - 1),$$

where \(\mathcal{R} := s_{23}(R^{*-1} \otimes R)s_{23} : (V^* \otimes V)^{\otimes 2} \rightarrow (V^* \otimes V)^{\otimes 2}\), $s_{23}$ denotes the operator which interchanges elements in 2-nd and 3-rd places of the tensor product. Later on we shall however identify $\mathcal{R}$ with $R^{*-1} \otimes R$, acting on $V^{\otimes 2} \otimes V^{\otimes 2}$. The matrix $R$ is some times called an R-matrix. The construction is called Yang-Baxter if $R$ obeys the Yang-Baxter equation [6, 7].

We are interested in the case when $R$ obeys the Hecke equation

$$(R + 1)(R - q) = 0$$

with $q \neq 0$. $R$ is then called a Hecke operator and $E$ is called a matrix bialgebra of type $A_{d-1}$, i.e. $E$ is considered as a deformation of the function algebra on the
semigroup of $d \times d$ matrices. Thus we have two QA’s. We denote them by $\Lambda$ and $S$, where

$$R(\Lambda) = \text{Im}(R + 1), \quad R(S) = \text{Im}(R - q).$$

This construction is motivated by the symmetric and antisymmetric tensor algebras over $V$ together with the function algebra on $\text{End}(V)$ coacting upon them.

### 1.3 Standard deformations

The most interesting example of the construction described in Section 1.2 which has many applications in theoretical physics is the case, when $R$ is Drinfeld-Jimbo’s R-matrix. We restrict ourselves to the case of R-matrix of series $A_{d-1}$, which gives the standard deformation of $GL(d)$. This deformation plays a crucial role in my work. On some fixed basis $x_1, x_2, \ldots, x_d$ the operator $R^q$ is given by

$$R^q := q \sum_{i=1}^{d} e_i^i \otimes e_i^i + \sqrt{q} \sum_{i,j \neq j} e_i^i \otimes e_j^j + (q - 1) \sum_{i<j} e_i^i \otimes e_j^j, \quad (1)$$

where $e_i^j$ is the operator $e_i^j : V \rightarrow V$, $x_k e_i^j = \delta_k^i x_i$. $R$ obeys the Yang-Baxter and the Hecke equations. The two QA’s defined by $R$ are

$$S^q \cong k < x_1, \ldots, x_d \rangle / (x_i x_j = \sqrt{q} x_j x_i, i < j),$$

and $\Lambda^q \cong k < x_1, \ldots, x_d \rangle / (x_i^2 = 1, x_i x_j = -\sqrt{q} x_j x_i, i \leq j)$.

The matrix bialgebra $E$ is defined by:

$$z_i^k z_i^l - \sqrt{q} z_i^l z_i^k = 0, k < l,$$

$$z_i^k z_j^l - \sqrt{q} z_j^k z_i^l = 0, i < j,$$

$$z_j^k z_i^l - z_i^l z_j^k = 0, i < j, k < l,$$

$$z_i^k z_j^l + (\sqrt{q}^{-1} - \sqrt{q}) z_i^l z_j^k + z_i^j z_i^k = 0,$$

$$i < j, k < l.$$

It is shown that $S^q, \Lambda^q$ and $E^q$ have PBW bases [12]. By a theorem of Priddy [14], $S^q, \Lambda^q$ and $E^q$ are then Koszul algebras.

### 1.4 Hecke algebras

The reader is referred to [3] for a beautiful description of symmetric groups and Hecke algebras. We recall some important facts. The length of an element $w$ of the symmetric group $\mathfrak{S}_n$ is equal to

$$l(w) := \#\{ (i,j) | i < j \land iw > jw \}.$$
The basic transpositions \( v_i = (i, i + 1), i = 1, 2, \ldots, n - 1 \) generate \( S_n \). An element \( w \in S_n \) can be expressed as a product of \( l(w) \) basic transpositions. Let \( B \) denote the set of the basic transpositions. The elements \( v_i, i = 1, 2, \ldots, n - 2 \) generate a subgroup of \( S_n \), isomorphic to \( S_{n-1} \). By means of this isomorphism we will consider \( S_{n-1} \) as a subgroup of \( S_n \). The following lemma will be needed in the next section.

**Lemma 1.3** For every element \( v \) in \( S_n \setminus S_{n-1} \) there exist elements \( w \) and \( w' \) in \( S_{n-1} \) and \( i, j, 1 \leq i, j \leq n - 1 \) such that

\[
v = v_i v_{i+1} \cdots v_{n-1} w,
v = w' v_{n-1} v_{n-2} \cdots v_j
\]

for some \( 1 \leq i, j \leq n - 1 \).

**Proof.** For \( w \) in \( S_{n-1} \) and \( 1 \leq i \leq n - 1 \) we have

\[
l(v_i \cdots v_{n-1} w) = l(w) + n - i
\]

hence for \( w \neq w' \) in \( S_{n-1} \) and \( 1 \leq i, j \leq n - 1 \)

\[
wv_i \cdots v_{n-1} w \neq v_j \cdots v_{n-1} w'.
\]

Whence the first equation of the lemma follows, the second equation is proved analogously.

The Hecke algebra \( \mathcal{H}_n = \mathcal{H}_{q,n} \) as a \( k \)-space has the basis \( T_w, w \in S_n \) with the multiplication subject to:

1. \( T_1 = 1_{\mathcal{H}_n} \).
2. \( T_w T_v = T_{wv} \) if \( l(wv) = l(w) + l(v) \).
3. \( T_v^2 = q + (q - 1)T_v \) for \( v = (i, i + 1), i = 1, 2, \ldots, n - 1 \).

We shall always assume that \([n]_q! \neq 0 \) and \( q \neq 0 \), therefore \( \mathcal{H}_n \) is semisimple. If this is the case, \( \mathcal{H}_n \) is isomorphic to a direct product of matrix rings over \( k \) \([3, 4]\). Put

\[
x_n = \frac{1}{[n]_q!} \sum_{w \in S_n} T_w,\]
\[
y_n = \frac{1}{[-n]_q!} \sum_{w \in S_n} (-q)^{l(w)} T_w.
\]
Then \(x_n, y_n\) are idempotents and
\[
T_w x_n = x_n T_w = q^{l(w)} x_n \quad (2)
\]
\[
T_w y_n = y_n T_w = (-1)^{l(w)} y_n \quad (3)
\]
(cf. [3, 4]). According to Lemma 1.3 we have
\[
[n]_q x_n = x_{n-1} (1 + T_{v_{n-1}} + T_{v_{n-1}} T_{v_{n-2}} + \cdots + T_{v_{n-1}} \cdots T_{v_1}), \quad (4)
\]
where \(v_k\) denotes the basic transposition: \(v_k = (k, k + 1)\).

Let us consider the representation \(\rho\) of the Hecke algebra \(H_n\) on \(\text{End}_k(V \otimes^n)\) induced by a Hecke operator \(R\):
\[
\rho : H_n \rightarrow \text{End}_k(V \otimes^n),
\]
\[
\rho(T_{v_i}) : x \mapsto x R^n_i, \quad i = 1, 2, \cdots, n-1,
\]
where \(R^n_i := \text{id}^{n-1} \otimes R \otimes \text{id}^{n-i-1} : V \otimes^n \rightarrow V \otimes^n\). Let \(A\) and \(S\) be quadratic algebras with
\[
R(A) = \text{Im}(R + 1), \quad R(S) = \text{Im}(R - q).
\]
Then \(A_n \cong \text{Im} \rho(y_n)\), \(S_n \cong \text{Im} \rho(x_n)\) \([5]\), hence
\[
s_n := \dim_k S = \chi(x_n), \quad \lambda_n := \dim_k A_n = \chi(y_n) \quad (5)
\]
where \(\chi\) is the character of \(\rho\). Thus the character \(\chi\) determines the Poincaré series of \(A\) and \(S\). In Section 2.1 I will show that the Poincaré series of \(A\) determines \(\chi\).

2 Matrix bialgebras of type \(A_n\)

In this section we define the Schur algebra of a matrix bialgebra of type \(A_n\), which will be called R-Schur algebra. We show the double centralizer theorem for it. Using R-Schur algebra one can calculate the Poincaré series of the matrix bialgebra. This will be done in 2.1. In 2.2 we show that the matrix bialgebra is a Koszul algebra. Using this fact we give in 2.3 the formula for calculating the Poincaré series of the matrix bialgebra via the ones of the quantum planes.

2.1 R-Schur algebras and the Poincaré series of \(E\)

Let \(E\) be the matrix bialgebra defined as in the first section by a Hecke operator \(R\). Every homogeneous component \(E_n^*\) of \(E\) is a finite dimensional coalgebra, hence its dual \(E_n^*\) is an algebra. The R-Schur algebra \(S_d\) on \(V\) is defined to be
\[
S_d = \bigoplus_{n=0}^\infty S_{d,n}, \quad S_{d,n} := E_n^*,
\]
where $d$ is the dimension of $V$. Let $\theta_n$ be the isomorphism

$$\theta_n : V^{* \otimes n} \otimes V^{\otimes n} \rightarrow E_1^{\otimes n}$$

$$\theta_n(x_1 \otimes \cdots \otimes x_n \otimes y_1 \otimes \cdots \otimes y_n) = (x_1 \otimes y_1 \otimes \cdots \otimes x_n \otimes y_n).$$

Let us consider the natural action of $E_n^*$ on $V^{\otimes n}$ by

$$\text{ev}_{V^{\otimes n}} \circ \theta_n^* : E_n^* \otimes V^{\otimes n} \rightarrow V^{\otimes n} \otimes V^{* \otimes n} \otimes V^{\otimes n} \rightarrow V^{\otimes n}.$$ 

The Hecke algebra $H_n$ acts on $V^{\otimes n}$ by the representation $\rho$ in Section 1.4.

**Theorem 2.1** The natural action of $E_n^*$ on $V^{\otimes n}$ and the action $\rho$ of $H_n$ on $V^{\otimes n}$, induced by $R$, are centralizers of each other in $\text{End}_k(V)$.

**Proof.** Using $\theta_n^*$ one can identify $E_n^*$ with a subspace of $V^{\otimes n} \otimes V^{* \otimes n}$. Then $E_n^*$ acts on $V^{\otimes n}$ as follows. For $f = \sum_i f_i \otimes f_i^* \in E_n^*, x \in V^{\otimes n}$,

$$(\sum_i f_i \otimes f_i^*)x = \sum_i f_i < f_i^* | x >.$$

The Hecke algebra acts on $V^{\otimes n}$ from the right with

$$xT_{v_k} = xR_k, \ v_k \in B.$$ 

Since $E_n^*$ is invariant under $R_i, k = 1, 2, \cdots, n - 1$, we have for $f \in E_n^*

f = f R_{k}^{-1} = \sum_i (f_i \otimes f_i^*)(R_k^{-1} \otimes R_k) = \sum_i f_i R_k^{-1} \otimes f_i^* R_k^*.$$

Hence for $k = 1, 2, \cdots, n - 1,$

$$f(xT_{v_k}) = \sum_i f_i < f_i^* | x R_k > = \sum_i f_i < f_i^* R_k^* | x > = \sum_i f_i R_k^{-1} < f_i^* R_k^* | x > R_k = \sum_i f_i < f_i^* | x > R_k = (fx)T_{v_k}.$$

Thus we obtain the embedding $E_n^* \hookrightarrow \text{End}_{H_n}(V^{\otimes n}).$

Let now $g$ be an element of $\text{End}_{H_n}(V^{\otimes n}) \hookrightarrow \text{End}_k V^{\otimes n}$. Using $\theta$ one can consider $g$ as an element of $V^{\otimes n} \otimes V^{* \otimes n}$. $g = \sum_i g_i \otimes g_i^*$. Then $g$ acts on $V^{\otimes n}$ as follows.

$$gx = \sum_i g_i < g_i^* | x > .$$

Since $g \in \text{End}_{H_n}(V^{\otimes n})$, one has

$$g(xT_{v_k}) = (gx)T_{v_k}, \ \forall v_k \in B, x \in V^{\otimes n}.$$
Hence for \( k = 1, \ldots, n - 1, \)
\[
\sum_i g_i < g_i^* | x > = \sum_i g_i < g_i^* | R_k, \forall x \in V^\otimes n,
\]
therefore
\[
g^R = \sum_i (g_i \otimes g_i^*)(R_k^{-1} \otimes R_k^*) = \sum_i g_i \otimes g_i^*.
\]
Thus \( E^*_n \cong \text{End}_{H_n}(V^\otimes n) \) as subalgebras of the algebra \( \text{End}_k V^\otimes n \).

Since \( H_n \) is semisimple, \( E^*_n \cong \text{End}_{H_n}(V^\otimes n) \) is also semisimple and by the density theorem (see for example [2] Chapter 10) the homomorphism
\[
H_n \rightarrow \text{End}_{E^*_n} V^\otimes n
\]
is surjective, that completes the proof.

Let \( A_1, A_2, \ldots, A_m \) be simple subalgebras of \( H_n \). Since the \( k \)-dimension of \( A_i \) is independent of the field \( k \), it is a matrix ring over \( k \). Let \( a_i \) be the unit element of \( A_i \), then \( \sum_i a_i = 1_{H_n} \). We have
\[
V^\otimes n = \bigoplus_{i=1}^m V^\otimes n a_i = \bigoplus_{i=1}^m V^\otimes n A_i.
\]
Hence
\[
E^*_n = \text{End}_{H_n}(V^\otimes n A_1) \times \cdots \times \text{End}_{H_n}(V^\otimes n A_m).
\]
\( \dim_k(V^\otimes n A_i) \) is equal to \( \chi(a_i) \), where \( a_i \) is the unit element in \( A_i \). Let \( r_i \) be the \( k \)-dimension of the simple \( H_n \)-module, which corresponds to \( A_i \). Then
\[
\text{End}_{H_n}(V^\otimes n A_i) = M(\chi(a_i)/r_i),
\]
hence
\[
\dim_k E^*_n = \sum_i (\chi(a_i)/r_i)^2. \quad (6)
\]
Thus, for calculating the dimension of \( E^*_n \), it is sufficient to calculate the character \( \chi \) of \( \rho \). We show that this can be calculated via the Poincaré series of \( S \). We shall do it in three steps.

Let \( C_n := \{ c_i = v_1 v_2 \cdots v_{i-1} | v_i = (i, i+1), i = 1, 2, \ldots, n \} \) be a subset of \( S_n \).

In the first step we show the following lemma.

**Lemma 2.2** For all \( v \in S_n \), \( \chi(T_v) \) is a polynomial on \( \chi(T_{c_i}) \) with coefficients being polynomials in \( q \).
Proof. We use induction. Consider the sequence

$$\mathcal{S}_1 \subset \mathcal{S}_2 \subset \cdots \subset \mathcal{S}_n$$

where $\mathcal{S}_i$ is generated by $v_1, v_2, \ldots, v_{i-1}$ and assume that the assertion holds for elements of $\mathcal{S}_{n-1}$. Let $v \notin \mathcal{S}_{n-1}$. According to Lemma 1.3 one has

$$T_v = T_{v_1} T_{v_2} \cdots T_{v_{n-1}} T_{v'}, \ v' \in \mathcal{S}_{n-1}.$$ 

Hence

$$\text{tr} R_v = \text{tr}(R_{v_1} \cdots R_{v_{n-1}} R_{v'}) = \sum_{w \in \mathcal{S}_{n-1}} k_w \text{tr}(R_{v_{n-1}} R_w) = \sum_{w \in \mathcal{S}_{n-1} \setminus \mathcal{S}_{n-2}} k_w \text{tr}(R_{v_{n-1}} R_w) + \sum_{w \in \mathcal{S}_{n-2}} k_w d^{-n} \text{tr}(R_{v_{n-1}} R_w)$$

where $k_w$ are polynomials in $q$. Since $v_i$ and $\mathcal{S}_{n-2}$ commute, the terms in the latter sum of the last part of the above equation are polynomials in $\text{tr} c_i, i = 1, \ldots, n - 1$. Thus it is sufficient to show that $\text{tr}(R_{v_{n-1}} R_w)$ are polynomials in $\text{tr} c_i, i = 1, \ldots, n - 1$ for $w \in \mathcal{S}_{n-1} \setminus \mathcal{S}_{n-2}$. There exists an element $w' \in \mathcal{S}_{n-2}$ such that

$$T_w = T_{v_k} T_{v_{k+1}} \cdots T_{v_{n-1}} T_{w'}.$$ 

Proceed the above process once again, so that we can restrict ourselves to showing that $\text{tr}(R_{v_{n-1}} R_{v_{n-2}} R_u), u \in \mathcal{S}_{n-2} \setminus \mathcal{S}_{n-3}$ are polynomials in $\text{tr} c_i, i = 1, \ldots, n$. After $n - 2$ times we are led to the element $\text{tr}(R_{v_{n-1}} R_{v_{n-2}} \cdots R_{v_1})$. We have

$$T_{v_{n-1}} \cdots T_{v_1} = c_n,$$ 

which concludes the proof. \[\blacksquare\]

According to the equations (4.3) one has

$$[n]_q \chi(x_n) = \text{tr} \rho((x_{n-1}(1 + T_{v_{n-1}} + T_{v_{n-1}} T_{v_{n-2}} + \cdots + T_{v_{n-1}} T_{v_1}))) = \text{tr} \rho((1 + T_{v_{n-1}} + T_{v_{n-1}} T_{v_{n-2}} + \cdots + T_{v_{n-1}} T_{v_1}) x_{n-1}) = \text{tr} \rho(x_{n-1}) + [n - 1]_q \text{tr} \rho(T_{v_{n-1}} x_{n-1}) = \chi(x_{n-1}) + \text{tr} \rho(T_{v_{n-1}} x_{n-2}(1 + T_{v_{n-2}} + \cdots + T_{v_{n-2}} T_{v_1})) = \chi(x_{n-1}) + \text{tr} \rho(T_{v_{n-1}} x_{n-2}) + [n - 2]_q \text{tr} \rho(T_{v_{n-1}} T_{v_{n-2}} x_{n-1}) = \chi(x_{n-1}) + d^{-n} \chi(T_{c_2}) \chi(x_{n-2}) + [n - 2]_q \text{tr} \rho(T_{v_{n-1}} T_{v_{n-2}} x_{n-2}) = \cdots = \chi(x_{n-1}) + d^{-n} \chi(T_{c_2}) \chi(x_{n-2}) + \cdots + d^{-n} \chi(T_{c_{n-1}}) \chi(x_1) + \chi(T_{c_n}),$$

therefore

$$[n]_q \chi(x_n) = \chi(x_{n-1}) + d^{-n} \chi(T_{c_2}) \chi(x_{n-2}) + \cdots + d^{-n} \chi(T_{c_{n-1}}) \chi(x_1) + \chi(T_{c_n}). \quad (7)$$

That means $\chi(T_{c_i}), c_i \in C_n$ are uniquely determined by $\chi(x_1), \chi(x_2), \ldots, \chi(x_n)$. Consider $c_k$ and $x_k$ as elements of $\mathcal{H}_k$, thus we have $\chi(x_k) = s_k$. Let denote
The equation (7) is then
\[
[q]^n s_n = s_n - 1 p_0 + p_1 s_n - 2 + \cdots + p_n - 1 s_0
\] (8)
or \[ P_S(t) \cdot P(t) = \sum_{n=1}^\infty [n] q^n t^n. \] Thus we have shown the following theorem.

**Proposition 2.3** The character \( \chi \) of the representation \( \rho \) can be determined by rational functions in \( s_1, s_2, \ldots, s_n \) and \( q \).

From this proposition and the equation (3) the theorem follows immediately.

**Theorem 2.4** Assume that \( q \neq 0 \), \([n]_q \neq 0 \) then \( \dim_k E_n \) is a rational function on \( s_1, s_2, \ldots, s_n \) and \( q \).

### 2.2 The Koszul property

The idea of proving that \( S, \Lambda \) and \( E \) are Koszul algebras is based on using Lemmas 1.1 and 1.2. We decompose the homogeneous components of those algebras into simple \( H_n \)-modules and \( H_{q-1,n} \times H_n \)-modules and prove the assertion (in terms of lattices) for each module. The assertion for these modules is again provided by the Koszul property of the standard deformation and the mentioned above Lemmas.

Let us denote by \( \Lambda(d, n) \) the set of all compositions \( \lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k \cdots) \) of \( n \) with \( \lambda_a = 0 \) if \( a > d \) and \( \mathcal{P}(n) \) be the set of all partitions of \( n \). Then the set of simple \( H_n \)-modules can be indexed by \( \mathcal{P}(n) \). Let \( \mathcal{L}_\mu, \mu \in \mathcal{P}(n) \) denote the set of all simple \( H_n \)-modules. Then we have for some subset \( K \) of \( \mathcal{P}(n) \)
\[
V^\otimes n = \bigoplus_{k \in K} c_k \mathcal{L}_k, \tag{9}
\]
where the integer coefficient \( c_k \) denotes that \( \mathcal{L}_k \) appears \( c_k \) times in the decomposition. By considering the action of \( H_{q-1,n} \) on \( V^* \otimes n \) via \( R^{*-1} \) we get the similar equation
\[
V^* \otimes n = \bigoplus_{j \in J} c'_j \mathcal{L}'_j
\]
with \( \mathcal{L}'_j \) denoting simple modules of \( H_{q-1,n} \). Hence
\[
V^* \otimes n \otimes V^\otimes n = \bigoplus_{(j,k) \in J \times K} c_k c'_j \mathcal{L}'_j \otimes \mathcal{L}_k
\]
is the decomposition of $V^{* \otimes n} \otimes V^{\otimes n}$ as $\mathcal{H}_{q-1,n} \times \mathcal{H}_n$-module. We also have

$$\text{Im}(R_i + 1) = \bigoplus_{k \in K} c_k(\text{Im}(R_i + 1)|_{\mathcal{L}_k}),$$

$$\text{Im}(R_i - q) = \bigoplus_{k \in K} c_k(\text{Im}(R_i - q)|_{\mathcal{L}_k}),$$

$$\text{Im}(\mathcal{R}_i - 1) = \bigoplus_{(j,k) \in J \times K} c_k c'_j(\text{Im}(\mathcal{R}_i - 1)|_{\mathcal{L}_j^{c} \otimes \mathcal{L}_k}).$$

(10)

(11)

(12)

We remark that the action of $\mathcal{H}_n$ on $\mathcal{L}_\mu$ does not depend on $R$ but only on $\mu$.

Let us consider Drinfeld-Jimbo’s matrix $R^q$. From a theorem of Priddy ([14], Theorem 5.2) it follows that $A^q, S^q, E^q$ have the Koszul property. To every composition $\lambda$ there corresponds a partition $\overline{\lambda}$ obtained from $\lambda$ by reordering its elements. The equation (9) for $R^q$ is

$$V^{\otimes n} \cong \bigoplus_{\lambda \in \Lambda(d,n)} \mathcal{L}_\overline{\lambda}$$

(13)

([4], Proposition 5.1). If $d > n$, $\mathcal{P}(n) \subset \Lambda(d,n)$, hence all simple modules of $\mathcal{H}_n$ appear in this decomposition. According to Lemma 1.2 $\text{Im}(R_i + 1)|_{\mathcal{L}_k}, i = 1, 2, \ldots, n - 1$ generate a distributive lattice on $\mathcal{L}_k$. The same fact is true for $\text{Im}(R_i - q)|_{\mathcal{L}_k}, i = 1, 2, \ldots, n - 1$ and $\text{Im}(\mathcal{R}_i - 1)|_{\mathcal{L}_j^{c} \otimes \mathcal{L}_k}, i = 1, 2, \ldots, n - 1$. Using Lemma 1.2 again we obtain the following theorem.

**Theorem 2.5** Assume that $q \neq 0, [n]_q! \neq 0$ then the algebras $\Lambda, S$ and $E$ are Koszul algebras.

### 2.3 The Poincaré series of $E_n$

In this section we will give a formula to calculate the dimension of $E_n$ by the dimensions of $S_i, i = 1, 2, \ldots, n$. Since $E$ is a Koszul algebra it is enough to calculate the dimension of $B_n$, where $B_n := \cap_{i=1}^{n-1} \text{Im}(\mathcal{R}_i^{n-1})$ [12]. We will assume that $q$ is transcendent over $\mathbb{Q}$.

We introduce the following operator $P$: for $S$ being a Yang-Baxter operator on $V \otimes V$, $P_n(S)$ operates on $V^{\otimes n}$:

$$P_1(S) = \text{id}_V,$$

$$P_n(S) = [n]_q^{-1}(P_{n-1}(S) \otimes \text{id})(\text{id} + S_{n-1} + \cdots + S_{n-1}S_{n-2} \cdots S_{1}).$$

(14)

Let $\Phi_n = P_n(-\mathcal{R})$ and $\Phi_n^q := P_n(-R^q)$, $R^q$ is the Drinfeld-Jimbo’s Matrix [4].

For $x = x(q)$ a polynomial on $q$, one considers the following operation $(-)_i$:

$$(-)_i : x = x(q) \mapsto (x)_i := x(1).$$
As we remarked in the previous section, the representation $\rho$ of $H_n$, induced by $R$ on $V^\otimes n$, restricted on a simple modules $L_\mu$ does not depend on $R$ any more. The same is true for the representation of $H_n \otimes H_q - 1, n$ on $E_1^\otimes n$. For an element $x \in H_n$ one can then define $(\text{tr} \rho(x)|_{L_\mu})_t$ and $(\text{tr} \rho(x))_t := \sum_{\mu \in I}(\text{tr} \rho(x)|_{L_\mu})_t$.

Let $P_{n\lambda\mu} := P_n(-\mathcal{R})|_{L_\lambda' \otimes L_\mu}$, then $P_{n\lambda\mu}$ does not depend on $R$.

**Theorem 2.6** Assume that $q$ is transcendent over $\mathbb{Q}$ then

$$\text{Im} \Phi_n = \bigcap_{i=1}^{n-1} \text{Im}(\mathcal{R}_i^n - 1) = B_n,$$

$$\dim_k \text{Im} \Phi_n = (\text{tr} \Phi_n)_t.$$  \hspace{1cm} (15)

**Proof.** The inclusion

$$\text{Im} \Phi_n \subset \bigcap \text{Im}(\mathcal{R}_i^n - 1)$$

holds for all Yang-Baxter matrix $R$. It remains to show the inverse inclusion. We first show it for the R-matrix from (1). For $R = R^q$ with $q$ transcendent on $\mathbb{Q}$, $E^q$ has the correct dimension:

$$\dim_k E^q_n = \binom{d^2 + n - 1}{n}.$$  \hspace{1cm} (16)

Since $E$ is a Koszul algebra, we have

$$\dim_k \bigcap_i \text{Im}(\mathcal{R}_i^n - 1) = \binom{d^2}{n}.$$  \hspace{1cm} (17)

Thus

$$\dim_k \text{Im} \Phi^q_n \leq \binom{d^2}{n}.$$  \hspace{1cm} (18)

On the other hand, if $q = 1$, $\Phi^1_n$ is a projector, hence its eigenvalues are 1 and 0, and one has

$$\dim_k \text{Im} \Phi^1_n = \text{tr} \Phi^1_n = \binom{d^2}{n}.$$  \hspace{1cm} (19)

Rank of an operator is not less then the number of its non-zero eigenvalues, hence for $q$ transcendent the following holds

$$\dim_k \text{Im} \Phi^q_n \geq (\text{tr} \Phi^q_n)_t = \binom{d^2}{n}.$$  \hspace{1cm} (20)

Thus we have

$$\dim_k \text{Im} \Phi^q_n = (\text{tr} \Phi^q_n)_t = \binom{d^2}{n}.$$  \hspace{1cm} (21)
and
\[ \text{Im}\Phi^q_n = \bigcap_{i=1}^{n-1} \text{Im}(R^q_{i} - 1). \] (19)

The formulas (18) and (19) hold for all $n$ and $q$ transcendent. If $d > n$, then all simple $\mathcal{H}_n$ modules appear in the decomposition (13). Hence
\[ \text{Im}\Phi^q_n |_{\mathcal{L}_\mu} = \bigcap_{i=1}^{n-1} \text{Im}(R^q_{i} - 1) |_{\mathcal{L}_\mu}, \]
\[ \dim_k \text{Im}\Phi^q_n |_{\mathcal{L}_\mu} = (\text{tr}\Phi^q_n |_{\mathcal{L}_\mu}). \]

Using the Equations (9)–(12) we obtain the assertion for an arbitrary Hecke operator $R$. \hfill \blacksquare

Let $p_k := (\text{tr}R_{c_{k+1}})_t$, $k \geq 1$ where $c_k = (1, 2, \cdots, k) \in \mathcal{H}_k$, $p_0 := 1$. Set
\[ P(t) = \sum_{n=0}^{\infty} p_n t^n \quad \text{and} \quad P_2(t) = \sum_{n=0}^{\infty} p^2_n t^n. \]

**Theorem 2.7** Assume that $q$ is transcendent over $\mathbb{Q}$ then
\[ P_E(t) = \exp\left(\int_0^t P_2(t)\right), \] (20)
where $p_n$ can be calculated by the formula
\[ P(t) = P'_S(t) \cdot P_S(t)^{-1}. \] (21)

**Proof.** Let
\[ b_n := \dim_k \bigcap_{i=1}^{n-1} \text{Im}(R^q_{i} - 1) = (\text{tr}\Phi_n)_t. \]

By definition
\[ (\text{tr}R_{c_{k+1}})_t = p^2_k, \quad k \geq 0. \]

Since $(\text{tr}(R_v R_w))_t = (\text{tr}R_{vw})_t$ for all $v, w \in \mathcal{S}_n$ we have
\[ n(\text{tr}\Phi_n)_t = (\text{tr}\Phi_{n-1})_t - p^2_1(\text{tr}\Phi_{n-2})_t + \cdots + (-1)^{n-1} p^2_{n-1} \]
or
\[ nb_n = \sum_{k=0}^{n-1} (-1)^k p^2_k b_{n-k-1}. \]
Whence the equation (20) follows. On the other hand, from (8) it follows
\[ ns_n = \sum_{k=0}^{n-1} p_k s_{n-k-1}. \]
Whence the equation (21) follows.

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