Research article

Complexity trees of the sequence of some nonahedral graphs generated by triangle

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ABSTRACT

Calculating the number of spanning trees of a graph is one of the widely studied graph problems since the Pioneer Gustav Kirchhoff (1847). In this work, using knowledge of difference equations we drive the explicit formulas for the number of spanning trees in the sequence of some Nonahedral (nine faced polyhedral) graphs generated by triangle using electrically equivalent transformations and rules of the weighted generating function. Finally, we evaluate the entropy of graphs in this manuscript with different studied graphs with an average degree being 4, 5 and 6.

1. Introduction

The trouble of counting spanning trees turns to be essential and more importantly, interesting. For instance, it has been shown, that if the graph represents an electrical community with each edge a unit resistor, the effective resistance of an edge is equal to the proportion of spanning trees that the edge is in. Also, the wide variety of spanning trees is used as an invariant for computing the entropy of certain networks related to physical processes. In addition, there are various applications of the wide variety of spanning trees within mathematics as well [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12].

A spanning tree of a connected graph is a subgraph that is a tree reaching all vertices. There exist numerous strategies for finding the number of spanning trees \( \tau(G) \) of a graph \( G \).

A classic technique called the matrix tree theorem, also called Kirchhoff’s matrix-tree theorem [13] which states that the number of nonidentical spanning trees of a graph \( G \) is same to any cofactor of its Laplacian matrix \( L = D - A \), in which \( D \) is the degree matrix and \( A \) is the adjacency matrix of the graph \( G \).

Another method to count this number is using Laplacian eigenvalues. Kelmans and Chelnoknov \cite{14} derived the following formula:

\[
\tau(G) = \frac{1}{\lambda_1} \prod_{i=1}^{p-1} \lambda_i, \tag{1.1}
\]

where \( G \) is a connected graph with \( p \) vertices and \( \lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_p = 0 \) are the eigenvalues of the Laplacian matrix \( L \).

One popular technique for finding the number of spanning trees, \( \tau(G) \), is the deletion-contraction method. This technique is a reliable method which lets into enumerate the number of spanning trees of a multigraph \( G \). This method makes use of the fact that

\[
\tau(G) = \tau(G - e) + \tau(G/e) \tag{1.2}
\]

where \( G - e \) denotes the graph obtained by deleting an arbitrary edge \( e \), and \( G/e \) denotes the graph obtained by contracting an arbitrary edge \( e \) \cite{15, 16}. For more results, see \cite{17, 18, 19, 20}.

2. Electrically equivalent transformations

An electrical network is an interconnection of electrical components (e.g. inductors, capacitors, batteries, resistors, switches, etc).

Kirchhoff’s motivation was studied of electrical networks: an edge-weighted graph can be regarded as an electrical network, where weights are the conductance of the respective edges. The effect conductance between two specific nodes \( u, v \) can be written as the quotient of (weighted) number of spanning trees and the (weighted) number of so called thickets, i.e., spanning forests with exactly two components and property that each of the components contains precisely one of the nodes \( u, v \) \cite{21, 22}. Next, we list the effect of some simple transformations on the number of spanning trees, suppose that \( G \) is an edge weighted

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2405-8440/© 2020 The Authors. Published by Elsevier Ltd. This is an open access article under the CC BY-NC-ND license (http://creativecommons.org/licenses/by-nc-nd/4.0/).
Theorem 1. The number of spanning trees in the sequence of the graph $G_n$, where $n \geq 1$, is given by

$$2^{-3} (97 + 17\sqrt{33})^2 \left( -1 + \left( \frac{1}{2} \left( 329 - 57\sqrt{33} \right) \right)^2 \right) \left( 19 - 3\sqrt{33} \right)^6 \left( 33 + \sqrt{33} \right) - (33 + \sqrt{33})^6 \left( 19 + 3\sqrt{33} \right)^6 \right)^2$$

$$3267 \left( 8(29 + 5\sqrt{33}) + 32 \left( 329 + 57\sqrt{33} \right)^{1-n} \right)^2$$

Proof: Let us be using the electrically equivalent transformation to transform $G_i$ to $G_i$. Figure 2 clarifies the transformation process from $G_2$ to $G_1 = K_3$.

By utilizing the properties that are given in section 2, the following the transformations are given:

$$r(G_i) = \frac{1}{27} r(G_2), \ r(G_i) = \frac{1}{27} r(G_1), \ r(G_i) = 9k_i r(G_i), \ r(G_i) = \frac{3}{9k_i + 8} r(G_i), \ r(G_i) = \frac{9k_i + 8}{72k_i} r(G_i)$$

Merging these seven transformations, we get

$$r(G_i) = 8(9k_i + 8)^3 r(G_i).$$

Moreover,

$$r(G_n) = \prod_{i=2}^{n} (9k_i + 8)^3 r(G_i) = 3 \times 8^{n-1} k_i^2 \left( \prod_{i=2}^{n} (9k_i + 8) \right)^2$$

where $k_i = \frac{11k_i + 8}{9k_i + 8}$, we have

$$k_{i-1} - 1 - \frac{1}{\sqrt{33}} \left( \frac{9k_i + 8}{9k_i + 8} \right) = \left( 19 - 3\sqrt{33} \right) \left( \frac{1}{k_i} \right) \frac{33}{2(9k_i + 8)}$$

Then by Eqs. (3.3) and (3.4), we get $h_{i-1} = \left( \frac{299\sqrt{33} - 32}{32} \right) h_i$ and $h_i = \left( \frac{299\sqrt{33} - 32}{32} \right)^{n-i} h_n$.

Thus $k_i = \frac{\left( \frac{299\sqrt{33} - 32}{32} \right)^{n-i} h_{i-1}}{h_i}$. Therefore,
Figure 2. The transformations from $G_2$ to $G_1$. 

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\[ k_i = \frac{\frac{2\sqrt{3} + \sqrt{33}}{32} \right)^{n-1}}{\left(\frac{2\sqrt{3} + \sqrt{33}}{32} \right)^{n-1} \xi_{n-1} - 1}. \tag{3.5} \]

Utilizing the expression \( k_{n-1} = \frac{11k_n + 8}{9k_n + 8} \) and indicating the coefficients of 11\( k_n \) + 8 and 9\( k_n \) + 8 as \( a_n \) and \( b_n \), we obtain

\[ \tau(G_n) = 3 \times 8^{-1} k_i^2 \left[ \left( \frac{561 + 97\sqrt{33}}{66} \right) \left( \frac{19 + 3\sqrt{33}}{2} \right)^{n-2} + \left( \frac{561 - 97\sqrt{33}}{66} \right) \left( \frac{19 - 3\sqrt{33}}{2} \right) \right]^2, \quad n \geq 2. \tag{3.11} \]

When \( n = 1 \), \( \tau(G_1) = 3 \) which verifies Eq. (3.11). Thus, the number of spanning trees in the sequence of the graph \( G_n \) is given by

\[ \tau(G_n) = 3 \times 8^{-1} k_i^2 \left[ \left( \frac{561 + 97\sqrt{33}}{66} \right) \left( \frac{19 + 3\sqrt{33}}{2} \right)^{n-2} + \left( \frac{561 - 97\sqrt{33}}{66} \right) \left( \frac{19 - 3\sqrt{33}}{2} \right) \right]^2, \quad n \geq 1. \tag{3.12} \]

\[ k_i = \frac{\frac{2\sqrt{3} + \sqrt{33}}{32} \right)^{n-1}}{\left(\frac{2\sqrt{3} + \sqrt{33}}{32} \right)^{n-1} \xi_{n-1} - 1}, \quad n \geq 1. \tag{3.13} \]

Putting Eq. (3.13) into Eq. (3.12), the result is obtained.

Consider the sequence of graphs \( H_1 = K_3, H_2, \ldots, H_n \) formed as illustrated in Figure 3.

According to this formation, the number of total vertices \( |V(H_n)| \) and edges \( |E(H_n)| \) are \( |V(H_n)| = 6n - 3, |E(H_n)| = 12n - 9, n = 1, 2, \ldots \) It is obvious that the average degree is convergently 46or a large \( n \).

**Theorem 2.** The number of spanning trees in the sequence of the graph \( H_n \), where \( n \geq 1 \), is given as

\[ \tau(H_n) = 9k_2\tau(H_2), \quad \tau(H_2) = \frac{1}{(3k_2 + 1)^2} \left( \sum_{i=1}^{n-2} \left( \frac{3k_2 + 1}{9k_2} \right)^i \right), \quad \tau(H_1) = \frac{1}{3k_2 + 1}. \]
Merging these nine transformations, we get

\[ \tau(H_s) = 2(18k_t + 5)^2 \tau(H_t). \]  

(3.14)

Moreover,

\[ \tau(H_s) = \prod_{i=1}^{n} 2(18k_t + 5) \tau(H_t) = 3 \times 2^{n-1} k_t \left( \prod_{i=1}^{n} (18k_t + 5) \right)^2 \]  

(3.15)

where \( k_{i-1} = \frac{11k_i+3}{18k_i+5} \) and \( n = 2, 3, \ldots, n \).

Its characteristic equation is \( 6x^2 - 2x - 1 = 0 \) which has two roots \( x_1 = \frac{1 + \sqrt{7}}{6} \) and \( x_2 = \frac{1 - \sqrt{7}}{6} \). Subtracting these two roots into both sides of \( k_{i-1} = \frac{11k_i+3}{18k_i+5} \), we get

\[ k_{i-1} = \frac{1 - \sqrt{7}}{6} \frac{11k_i + 3}{18k_i + 5} \Rightarrow 1 - \sqrt{7} = \left( 8 + 3\sqrt{7} \right) \frac{k_i - \frac{1 + \sqrt{7}}{6}}{18k_i + 5} \]  

(3.16)

\[ k_{i-1} = \frac{1 + \sqrt{7}}{6} \frac{11k_i + 3}{18k_i + 5} \Rightarrow 1 + \sqrt{7} = \left( 8 - 3\sqrt{7} \right) \frac{k_i - \frac{1 + \sqrt{7}}{6}}{18k_i + 5} \]  

(3.17)

Let \( h_i = \frac{\frac{1 + \sqrt{7}}{6} k_{i-1}}{18k_i + 5} \). Then by Eqs. (3.16) and (3.17), we get \( h_{i-1} = \frac{127 + 48\sqrt{7}}{18k_i + 5} h_i \).

Thus

\[ k_{i-1} = \frac{1 \pm \sqrt{7}}{6} \frac{11k_i + 3}{18k_i + 5} \Rightarrow \frac{1 \pm \sqrt{7}}{6} = \left( 8 \pm 3\sqrt{7} \right) \frac{k_i - \frac{1 + \sqrt{7}}{6}}{18k_i + 5} \]  

(3.18)

Utilizing the expression \( k_{i-1} = \frac{11k_i+3}{18k_i+5} \) and indicating the coefficients of \( 11k_i + 3 \) and \( 18k_i + 5 \) as \( a_i \) and \( b_i \), we obtain

\[ \tau(H_s) = 3 \times 2^{n-1} k_t \left[ \left( \frac{161 + 61 \sqrt{7}}{14} \right) \left( 8 + 3\sqrt{7} \right)^{n-2} + \left( \frac{161 - 61 \sqrt{7}}{14} \right) \left( 8 - 3\sqrt{7} \right)^{n-2} \right]^2, n \geq 2. \]  

(3.24)
Figure 4. The transformations from $H_2$ to $H_1$. 
When \( n = 1 \), \( \tau(H_1) = 3 \) which verifies Eq. (3.24). Thus, the number of spanning trees in the sequence of the graph \( H_n \) is given by

\[
\tau(H_n) = 3 \times 2^{n-1} k_1^n \left( \frac{161 + 61\sqrt{7}}{14} \right) \left( 8 + 3\sqrt{7} \right)^{-2} + \left( \frac{161 - 61\sqrt{7}}{14} \right) \left( 8 - 3\sqrt{7} \right)^{-2}, \quad n \geq 1.
\]  

(3.25)

where

\[
k_1 = \frac{\left( 127 + 48\sqrt{7} \right)^{n-1} \left( \frac{1+\sqrt{7}}{2} \right) - (1 - \sqrt{7})}{2\left( 127 + 48\sqrt{7} \right)^{n-1} \left( \frac{1+\sqrt{7}}{2} \right) - 6}.
\]  

(3.26)

Putting Eq. (3.26) into Eq. (3.25), we obtain the result.

Consider the sequence of graphs \( T_1 = K_3, T_2, ..., T_n \) as formed in Figure 5. According to this formation, the number of total vertices \( |V(T_n)| \) and edges \( |E(T_n)| = 6n - 3 \) for \( |E(T_n)| = 2 \) is shown in Table 3. It is obvious that the average degree is convergently 4 for a large \( n \).

**Theorem 3.** The number of spanning trees in the sequence of the graph \( T_n \), where \( n \geq 1 \), is given by

\[
\tau(T_n) = 9k_2^2 \tau(T_1), \tau(T_2) = \left( \frac{3k_2 + 2}{18k_2} \right)^3 \tau(T_1), \tau(T_3) = \left( \frac{9k_5 + 5}{18k_5 + 2} \right) \tau(T_2) \text{ and } \tau(T_1) = \tau(T_1)
\]

Merging these nine transformations, we get

\[
\tau(T_n) = 2(9k_2 + 5)^2 \tau(T_1)
\]  

(3.27)

Moreover,

\[
\tau(T_n) = \prod_{i=2}^{n} (2(9k_2 + 5)^2 \tau(T_1)) = 3 \times 2^{n-1} k_1^n \left( \frac{11k_2 + 6}{9k_2 + 5} \right)^2
\]  

(3.28)

where

\[
k_{i-1} = \frac{11k_2 + 6}{9k_2 + 5}, \quad i = 2, 3, ..., n.
\]

Its characteristic equation is \( 3x^2 - 2x - 2 = 0 \) which has two roots \( x_1 = \frac{-1 + \sqrt{7}}{3} \) and \( x_2 = \frac{-1 - \sqrt{7}}{3} \). Subtracting these two roots into both sides of \( k_{i-1} = \frac{11k_2 + 6}{9k_2 + 5} \), we have

\[
k_{i-1} = \frac{11k_2 + 6}{9k_2 + 5}.
\]

(3.29)

Let \( h_1 = \frac{1 - \sqrt{7}}{3}, h_2 = \frac{1 + \sqrt{7}}{3} \). Then by Eqs. (3.29) and (3.30), we have \( h_{i-1} = (127 + 48\sqrt{7})h_i \) and \( h_i = (127 + 48\sqrt{7})^{n-1}h_1 \).

Thus

\[
k_i = \frac{(127 + 48\sqrt{7})^{n-1} \left( \frac{1 + \sqrt{7}}{3} \right) + \left( \frac{1 - \sqrt{7}}{3} \right)}{(127 + 48\sqrt{7})^{n-1} \left( \frac{1 + \sqrt{7}}{3} \right) + 1}.
\]  

(3.31)

Utilizing the expression \( k_{i-1} = \frac{11k_2 + 6}{9k_2 + 5} \) and indicating the coefficients of \( 11k_2 + 6 \) and \( 9k_2 + 5 \) as \( a_r \) and \( b_r \), we obtain

\[
k_2 + 5 = a_0(11k_2 + 6) + b_0(9k_2 + 5),
\]

(3.32)

\[
k_{n-1} = a_1(11k_2 + 6) + b_1(9k_2 + 5)
\]

(3.33)

\[
k_{n-2} = a_2(11k_2 + 6) + b_2(9k_2 + 5)
\]

(3.34)

\[
k_{n-3} = a_3(11k_2 + 6) + b_3(9k_2 + 5)
\]

(3.35)

\[
k_{n-4} = a_4(11k_2 + 6) + b_4(9k_2 + 5)
\]

(3.36)

\[
k_{n-5} = a_5(11k_2 + 6) + b_5(9k_2 + 5)
\]

(3.37)

Thus, we get

\[
k_{i-1} = \frac{11k_2 + 6}{9k_2 + 5}.
\]

(3.29)
\[
\tau(T_n) = \left(3 \cdot 2^{n-1} k_i^2 \left[ a_{n-2}(11k_n + 6) + b_{n-2}(9k_n + 5) \right] \right)^2 
\] (3.34)

where \( a_0 = 0, b_0 = 1 \) and \( a_1 = 9, b_1 = 5 \). By the expression \( k_{n-1} = \frac{11k_n + 6}{9k_n + 5} \) and Eqs. (3.32) and (3.33), we obtain

\[
a_{n+1} = 16a_n - a_{n-1}; b_{n+1} = 16b_n - b_{n-1}
\] (3.35)

The characteristic equation of Eq. (3.35) is \( y^2 - 16y + 1 = 0 \) which has two roots \( y_1 = 8 + 3\sqrt{7} \) and \( y_2 = 8 - 3\sqrt{7} \). The general solution of Eq. (3.35) are \( a_i = \lambda_1 y_1^i + \lambda_2 y_2^i; b_i = \mu_1 y_1^i + \mu_2 y_2^i \).

Utilizing the initial conditions \( a_0 = 0, b_0 = 1 \) and \( a_1 = 9, b_1 = 5 \), yields

\[
a_i = \frac{3\sqrt{7}}{14} (8 + 3\sqrt{7})^i - \frac{3\sqrt{7}}{14} (8 - 3\sqrt{7})^i; b_i = \left(1 - \frac{\sqrt{7}}{14}\right)(8 + 3\sqrt{7})^i + \left(1 + \frac{\sqrt{7}}{14}\right)(8 - 3\sqrt{7})^i
\] (3.36)

If \( k_n = 1 \), yields \( T_n \) has no any electrically equivalent transformation. Substituting Eq. (3.36) into Eq. (3.34), we get

\[
\tau(T_n) = 3 \cdot 2^{n-1} k_i^2 \left( \frac{98 + 37\sqrt{7}}{14} (8 + 3\sqrt{7})^{n-2} + \frac{98 - 37\sqrt{7}}{14} (8 - 3\sqrt{7})^{n-2} \right)^2, n \geq 2.
\] (3.37)

When \( n = 1 \), \( \tau(T_1) = 3 \) which verifies Eq. (3.37). Thus, the number of spanning trees in the sequence of the graph \( H_n \) is given by

\[
\tau(T_i) = 3 \cdot 2^{n-1} k_i^2 \left( \frac{98 + 37\sqrt{7}}{14} (8 + 3\sqrt{7})^{n-2} + \frac{98 - 37\sqrt{7}}{14} (8 - 3\sqrt{7})^{n-2} \right)^2, n \geq 1.
\] (3.38)

Its characteristic equation is \( 9x^2 - 12x - 16 = 0 \) which has two roots \( x_1 = \frac{2+2\sqrt{5}}{3} \) and \( x_2 = \frac{2-2\sqrt{5}}{3} \). Subtracting these two roots into both sides of \( k_{i-1} = \frac{2k_i+16}{9k_i+8} \), we have

\[
k_{i-1} = \frac{2 - 2\sqrt{5}}{3} = \frac{20k_i + 16}{9k_i + 8} - \frac{2 - 2\sqrt{5}}{3} = 2\left(7 + 3\sqrt{5}\right) \frac{a_i - \frac{2-2\sqrt{5}}{3}}{9k_i + 8}
\] (3.42)

\[
k_{i-1} = \frac{2 + 2\sqrt{5}}{3} = \frac{20k_i + 16}{9k_i + 8} - \frac{2 + 2\sqrt{5}}{3} = 2\left(7 - 3\sqrt{5}\right) \frac{k_i - \frac{2+2\sqrt{5}}{3}}{9k_i + 8}
\] (3.43)

Proof: The electrically equivalent transformation to transform \( X_i \) to \( X_{i+1} \) is using Figure 8 clarifies the transformation process from \( X_2 \) to \( X_1 = K_3 \).

By utilizing the properties that are given in section 2, the following the transformations are given:

\[
\tau(X_i) = \frac{1}{27} \tau(X_1), \tau(X_1) = \tau(X_1), \tau(X_1) = 9k_2 \tau(X_1), \tau(X_1) = \left(\frac{3}{9k_2 + 8}\right) \tau(X_1), \tau(X_1), \tau(X_1) = \left(\frac{9k_2 + 8}{72k_1}\right) \tau(X_1) \text{ and } \tau(X_i) = \tau(X_i).
\]

Merging these seven transformations, we get

\[
\tau(X_i) = \frac{8(9k_2 + 8)^2 \tau(X_1)}{27}
\]

Moreover,

\[
\tau(X_i) = \prod_{i=2}^{n} (8(9k_2 + 8)^2 \tau(X_1)) = 3 \times 8^{n-1} k_i^2 \left(\prod_{i=2}^{n} (9k_2 + 8)\right)^2
\] (3.41)

where

\[
k_{i-1} = \frac{20k_i + 16}{9k_i + 8}, i = 2, 3, ..., n.
\]
Let \( h_t = \frac{h_{t-1} + \sqrt{h_{t-1}^2 + 4}}{2} \). Then by Eqs. (3.42) and (3.43), we get

\[
9k_{n-1} + 8 = \frac{a_1(20k_n + 16) + b_1(9k_n + 8)}{a_{n-1}(20k_n + 16) + b_{n-1}(9k_n + 8)}
\]

(3.45)

\[
9k_{n-2} + 8 = \frac{a_2(20k_n + 16) + b_2(9k_n + 8)}{a_{n-2}(20k_n + 16) + b_{n-2}(9k_n + 8)}
\]

(3.46)

Thus, we obtain

\[
\tau(X_n) = 3 \times 8^{n-1} k_1^2 \left[ \left( \frac{85 + 37\sqrt{5}}{10} \right) \left( 14 + 6\sqrt{5} \right)^{n-2} + \left( \frac{85 - 37\sqrt{5}}{10} \right) \left( 14 - 6\sqrt{5} \right)^{n-2} \right]^2, n \geq 2.
\]

(3.50)

Utilizing the initial conditions \( a_0 = 0, b_0 = 1 \) and \( a_1 = 9, b_1 = 8 \), by the expression \( k_{n-1} = \frac{20k_n + 16}{9k_n + 8} \) and Eqs. (3.45) and (3.46), we get

\[
a_{n+1} = 28b_n - 16a_n; b_{n+1} = 28b_n - 16b_{n-1}
\]

(3.48)

The characteristic equation of Eq. (3.48) is \( y^2 - 28y + 16 = 0 \) which has two roots \( y_1 = 14 + 6\sqrt{5} \) and \( y_2 = 14 - 6\sqrt{5} \). The general solution of Eq. (3.48) are

\[
a_t = a_t y_1^t + \beta y_2^t; b_t = \mu y_1^t + \nu y_2^t.
\]

Utilizing the initial conditions \( a_0 = 0, b_0 = 1 \) and \( a_1 = 9, b_1 = 8 \), yields

\[
a_t = \frac{3\sqrt{5}}{20} \left( 14 + 6\sqrt{5} \right)^t; b_t = \frac{3\sqrt{5}}{20} \left( 14 - 6\sqrt{5} \right)^t;
\]

(3.49)

If \( k_1 = 1 \), yields \( X_n \) has no any electrically equivalent transformation. Substituting Eq. (3.49) into Eq. (3.47), we get

\[
\tau(X_n) = 3 \times 8^{n-1} k_1^2 \left[ \left( \frac{85 + 37\sqrt{5}}{10} \right) \left( 14 + 6\sqrt{5} \right)^{n-2} + \left( \frac{85 - 37\sqrt{5}}{10} \right) \left( 14 - 6\sqrt{5} \right)^{n-2} \right]^2, n \geq 1.
\]

(3.51)
Figure 6. The transformations from $T_2$ to $T_1$. 
When \( n = 1 \), \( r(X_1) = 3 \) which verifies Eq. (3.50). Thus, the number of spanning trees in the sequence of the graph \( X_n \) is given by

\[
\tau(X_n) = \prod_{i=0}^{n-1} 2(18k_i + 11)^2 \tau(Y_i) = 3 \times 2^{n-1} k_1^2 \left[ \prod_{i=0}^{n-1} (18k_i + 11)^2 \right] \tag{3.54}
\]

where \( k_{i+1} = \frac{23k_i + 14}{18k_i + 11} \) and \( k_0 = 3 \). Subtracting these two roots into both sides of \( k_{i+1} \), we have

\[
k_{i+1} = \frac{1 - 2\sqrt{3}}{3} = \frac{23k_i + 14}{18k_i + 11} - \frac{1 - 2\sqrt{3}}{3} = \left( 17 + 12\sqrt{3} \right) \frac{k_i - \frac{1 + 2\sqrt{3}}{3}}{18k_i + 11} \tag{3.55}
\]

Moreover,

\[
\tau(Y_i) = \prod_{n=0}^{\infty} 2(18k_i + 11)^2 \tau(Y_i) = \frac{2^{n-5} \left( (10 + 7\sqrt{2}) (17 - 12\sqrt{2})^4 + (10 - 7\sqrt{2}) (17 + 12\sqrt{2})^4 \right)^3 \left( 1055 + 746\sqrt{2} - (11 + 8\sqrt{2})(577 + 408\sqrt{2}) \right)^3}{3 (577 + 408\sqrt{2} + (3 + 2\sqrt{2})(577 + 408\sqrt{2})^3} \text{ (3.57)}
\]

Proof: The electrically equivalent transformation to transform \( Y_i \) to \( Y_{i-1} \) is using. Figure 8 clarifies the transformation process from \( Y_2 \) to \( Y_1 = K_3 \). (see Figure 10).

By utilizing the properties that are given in section 2, the following the transformations are given:

\[
\tau(Y_i) = 9k_i \tau(Y_i), \tau(Y_i) = \left( \frac{1}{3k_i + 2} \right)^3 \tau(Y_i), \tau(Y_i) = \tau(Y_i), \tau(Y_i) = \left( \frac{4k_i + 27}{3k_i + 2} \right) \tau(Y_i), \tau(Y_i) = \left( \frac{3k_i + 2}{18k_i + 11} \right)^3 \tau(Y_i), \tau(Y_i) = \left( \frac{18k_i + 11}{45k_i + 27} \right) \tau(Y_i), \tau(Y_i) = \tau(Y_i)
\]

Merging these nine transformations, we obtain

\[
r(Y_2) = 2(18k_2 + 11)^2 \tau(Y_1). \tag{3.53}
\]
Figure 8. The transformations from $X_2$ to $X_1$. 
The characteristic equation of Eq. (3.61) is
\[ \tau(Y_n) = 3 \times 2^{n-1} k_1^2 \left( \frac{58 + 41 \sqrt{2}}{4} \right) \left( 17 + 12 \sqrt{2} \right)^{n-2} + \left( \frac{58 - 41 \sqrt{2}}{4} \right) \left( 17 - 12 \sqrt{2} \right)^{n-2} \] \( n \geq 2. \) (3.63)

When \( n = 1, \) \( \tau(Y_1) = 3 \) which verifies Eq. (3.63). Thus, the number of spanning trees in the sequence of the graph \( Y_n \) is given by

\[ \tau(Y_n) = 3 \times 2^{n-1} k_1^2 \left[ \frac{58 + 41 \sqrt{2}}{4} \right] \left( 17 + 12 \sqrt{2} \right)^{n-2} + \left( \frac{58 - 41 \sqrt{2}}{4} \right) \left( 17 - 12 \sqrt{2} \right)^{n-2} \] \( n \geq 1. \) (3.64)

Thus, we have
\[ \tau(Y_n) = 3 \times 2^{n-1} k_1^2 \left[ a_{n-2}(23k_n + 14) + b_{n-2}(18k_n + 11) \right] \] (3.60)

where \( a_0 = 0, b_0 = 1 \) and \( a_1 = 18, b_1 = 11. \) By the expression \( k_{n-1} = \frac{23k_n + 14}{18k_n + 11} \) and Eqs. (3.58) and (3.59), we have
\[ a_{n-1} = 34a_n - a_{n-2}, b_{n+1} = 34b_n - b_{n-1} \] (3.61)

The characteristic equation of Eq. (3.61) is \( y^2 - 34y + 1 = 0 \) which has two roots \( y_1 = 17 + 12 \sqrt{2} \) and \( y_2 = 17 - 12 \sqrt{2}. \) The general solution of Eq. (3.61) are \( a_n = \lambda_1^ny_1^n + \lambda_2^ny_2^n; b_n = \mu_1^ny_1^n + \mu_2^ny_2^n. \)

Utilizing the initial conditions \( a_0 = 0, b_0 = 1 \) and \( a_1 = 18, b_1 = 11, \) yields

\[ a_n = \frac{3\sqrt{2}}{8} \left( 17 + 12 \sqrt{2} \right) \left( 17 - 12 \sqrt{2} \right); b_n = \left( \frac{4 - \sqrt{8}}{8} \right) \left( 17 + 12 \sqrt{2} \right) + \left( \frac{4 + \sqrt{2}}{8} \right) \left( 17 - 12 \sqrt{2} \right). \] (3.62)

If \( k_n = 1, \) yields \( Y_n \) has no any electrically equivalent transformation. Substituting Eq. (3.62) into Eq. (3.60), we get

Putting Eq. (3.65) into Eq. (3.64), then the result is obtained. Consider the sequence of graphs \( Z_1 = K_3, Z_2, ..., Z_n, \) formed as illustrated in Figure 11.

According to this formation, the number of total vertices \( |V(Z_n)| \) and edges \( |E(Z_n)| \) are \( |V(Z_n)| = 6n - 3, |E(Z_n)| = 18n - 15, n = 1, 2, .... \) It is obvious that the average degree is convergently 6 for a large \( n. \)

**Theorem 6.** The number of spanning trees in the sequence of the graph \( Z_n, \) where \( n \geq 1, \) is given by \( G_{n-1}, \) where \( n \geq 1, \) is given by

\[ k_1 = \frac{577 + 408 \sqrt{2}}{577 + 408 \sqrt{2}} \left( \frac{577 + 408 \sqrt{2}}{3 + 2 \sqrt{2}} + 1 \right)^{n-1}. \] (3.65)

Proof: Let us be using the electrically equivalent transformation to form \( z_1 \) to \( z_{n-1}. \) Figure 2 clarifies the transformation process from \( z_2 \) to \( z_1 = K_3. \)
Figure 10. The transformations from $Y_2$ to $Y_1$. 
Proof: The electrically equivalent transformation to transform $Z_1$ to $Z_{i+1}$ is using. Figure 12 clarifies the transformation process from $Z_{i+1}$ to $Z_1 = K_3$.

By utilizing the properties that are given in section 2, the following transformations are given:

$$
\tau(Z_1) = 9k_2 \tau(Z_2), \quad \tau(Z_2) = \left( \frac{1}{3k_2 + 2} \right)^{\frac{3}{2}} \tau(Z_3), \quad \tau(Z_3) = \tau(Z_4), \quad \tau(Z_4) = \left( \frac{3k_2 + 2}{18k_2} \right)^{-1}\tau(Z_5),
$$

$$
= \left( \frac{3k_2 + 2}{21k_2 + 13} \right)^{\frac{3}{2}} \tau(Z_5), \quad \tau(Z_5) = \tau(Z_6), \quad \tau(Z_6) = \left( \frac{21k_2 + 13}{9(10k_2 + 6)} \right)^{\frac{3}{2}} \tau(Z_7), \quad \tau(Z_7) = \tau(Z_8)
$$

Merging these nine transformations, we obtain

$$
\tau(Z_8) = 4(21k_2 + 13)^{-\frac{3}{2}} \tau(Z_8).
$$

(3.66)

Moreover,

$$
\tau(Z_9) = \left( \frac{n}{4(21k_2 + 13)^{\frac{3}{2}}} \right)^{\frac{3}{2}} \tau(Z_9) = 3 \times 4^{n-1} k_2 \left[ \prod_{i=2}^{8} (21k_2 + 13) \right]^{-\frac{3}{2}} \tau(Z_{10}).
$$

(3.67)

where $k_{i-1} = \frac{34k_2 + 21}{2(21k_2 + 13)} i = 2, 3, ..., n$.

Its characteristic equation is $x^2 - x - 1 = 0$ which has two roots $x_1 = \frac{1 + \sqrt{5}}{2}$ and $x_2 = \frac{1 - \sqrt{5}}{2}$. Subtracting these two roots into both sides of $k_{i-1} = \frac{34k_2 + 21}{2(21k_2 + 13)}$, we have

$$
k_{i-1} - \frac{1 - \sqrt{5}}{2} = \frac{34k_2 + 21}{2(21k_2 + 13)} - \frac{1 - \sqrt{5}}{2} = \frac{a_i - \left( \frac{1 - \sqrt{5}}{2} \right)}{2(21k_2 + 13)}
$$

(3.68)

$$
k_{i-1} + \frac{1 + \sqrt{5}}{2} = \frac{34k_2 + 21}{2(21k_2 + 13)} + \frac{1 + \sqrt{5}}{2} = \frac{a_i + \left( \frac{1 + \sqrt{5}}{2} \right)}{2(21k_2 + 13)}
$$

(3.69)

Let $h_i = \frac{\sqrt{5} + 21\sqrt{5}}{2}$, Then by Eqs. (3.68) and (3.69), we have $h_{i-1} = \left( \frac{34k_2 + 21}{2(21k_2 + 13)} \right)^{\frac{3}{2}} h_i$ and $h_i = \left( \frac{34k_2 + 21}{2(21k_2 + 13)} \right)^{\frac{3}{2}} h_{i-1}$. Then by Eqs. (3.68) and (3.69), we have $h_{i-1} = \frac{34k_2 + 21}{2(21k_2 + 13)} h_i$ and $h_i = \left( \frac{34k_2 + 21}{2(21k_2 + 13)} \right)^{\frac{3}{2}} h_{i-1}$. Thus

$$
k_i = \left( \frac{34k_2 + 21}{2(21k_2 + 13)} \right)^{\frac{3}{2}} h_{i-1}
$$

Therefore,

$$
\tau(Z_8) = 3 \times 4^{n-1} k_2 \left[ \frac{85 + 38\sqrt{5}}{10} \left( \frac{47 + 21\sqrt{5}}{2} \right)^{-\frac{3}{2}} + \frac{85 - 38\sqrt{5}}{10} \left( \frac{47 - 21\sqrt{5}}{2} \right)^{-\frac{3}{2}} \right]^{-\frac{3}{2}} n \geq 2.
$$

(3.76)

Utilizing the expression $k_{n-1} = \frac{34k_2 + 21}{2(21k_2 + 13)}$ and indicating the coefficients of $34k_2 + 21$ and $21k_2 + 13$ as $a_i$ and $b_i$, we obtain

$$
21k_2 + 13 = a_i(34k_2 + 21) + b_i(21k_2 + 13),
$$

(3.71)

$$
21k_2 + 13 = a_i(34k_2 + 21) + b_i(21k_2 + 13) \quad a_i(34k_2 + 21) + b_i(21k_2 + 13)
$$

(3.72)

$$
21k_2 + 13 = a_i(34k_2 + 21) + b_i(21k_2 + 13) \quad a_i(34k_2 + 21) + b_i(21k_2 + 13)
$$

Thus, we get

$$
\tau(Z_8) = 3 \times 4^{n-1} k_2 \left[ a_{n-2}(34k_2 + 21) + b_{n-2}(21k_2 + 13) \right]^{-\frac{3}{2}}
$$

(3.73)

where $a_0 = 0, b_0 = 1$ and $a_1 = 21, b_1 = 13$. By the expression $k_{n-1} = \frac{34k_2 + 21}{2(21k_2 + 13)}$ and Eqs. (3.71) and (3.72), we have

$$
a_{n+1} = 47a_n - a_{n-1}, b_{n+1} = 47b_n - b_{n-1}
$$

(3.74)

The characteristic equation of Eq. (3.74) is $y^2 - 47y + 1 = 0$ which has two roots $y_1 = \frac{47 + \sqrt{5}}{2}$ and $y_2 = \frac{47 - \sqrt{5}}{2}$. The general solution of Eq. (3.73) are $a_i = \lambda_1 y_1^i + \lambda_2 y_2^i$, $b_i = \mu_1 y_1^i + \mu_2 y_2^i$.

Utilizing the initial conditions $a_0 = 0, b_0 = 1$ and $a_1 = 21, b_1 = 21$, yields

$$
a_i = \frac{\sqrt{5}}{5} \left( \frac{47 + 21\sqrt{5}}{2} \right)^i \left( \frac{47 - 21\sqrt{5}}{2} \right)^i b_i
$$

$$
= \left( \frac{47 + 21\sqrt{5}}{2} \right)^i \left( \frac{47 - 21\sqrt{5}}{2} \right)^i
$$

(3.75)

If $k_1 = 1$, yields $Z_8$ has no any electrically equivalent transformation. Substituting Eq. (3.75) into Eq. (3.73), we get
When \( n = 1 \), \( \tau(Z_1) = 3 \) which verifies Eq. (3.76). Thus, the number of spanning trees in the sequence of the graph \( Z_n \) is given by

\[
\tau(Z_n) = 3 \times 4^{n-1} k_1 \left[ \left( \frac{85 + 38\sqrt{5}}{10} \right)^{n-1} \left( \frac{47 + 21\sqrt{5}}{2} \right) + \left( \frac{85 - 38\sqrt{5}}{10} \right)^{n-1} \left( \frac{47 - 21\sqrt{5}}{2} \right) \right]^2, \quad n \geq 1.
\]  

(3.77)

Putting Eq. (3.77) into Eq. (3.78), hence the result is obtained.

4. Numerical results

Next tables illustrate some the values of the number of spanning trees in the graphs \( G_n, H_n, T_n, X_n, Y_n \) and \( Z_n \).

| \( n \) | \( \tau(G_n) \) | \( \tau(H_n) \) | \( \tau(T_n) \) |
|-------|-------------|-------------|-------------|
| 1     | 3           | 3           | 3           |
| 2     | 8664        | 1176        | 1734        |
| 3     | 22852800    | 596748      | 881292      |
| 4     | 60019201536 | 303141984   | 447690264   |
| 5     | 157597728780288 | 153993738288 | 227423130672 |
| 6     | 413814073710182400 | 78227606477184 | 115529159623776 |

| \( n \) | \( \tau(X_n) \) | \( \tau(Y_n) \) | \( \tau(Z_n) \) |
|-------|-------------|-------------|-------------|
| 1     | 3           | 3           | 3           |
| 2     | 31104       | 8214        | 36300       |
| 3     | 188940288   | 18960588    | 320498688   |
| 4     | 113639424000 | 43761009624 | 2829362006208 |
| 5     | 6833482751803392 | 101000334380592 | 24977602663502592 |
| 6     | 41091617468631220224 | 233108596706389344 | 22050223104361492352 |

5. Spanning tree entropy

After having explicit formulas for the number of spanning trees of the sequence of the six graphs \( G_n, H_n, T_n, X_n, Y_n \) and \( Z_n \), we can calculate its spanning tree entropy \( Z \) which is a finite number and a very interesting quantity characterizing the network structure, defined in [23, 24]: for a graph \( G \),

\[
Z(G) = \lim_{n \to \infty} \frac{\ln \tau(G)}{|V(G)|}.
\]

(5.1)

\[
Z(G_n) = \frac{1}{6} \left( \ln[2] + 2 \ln \left[ 19 + 3\sqrt{33} \right] \right) = 1.312187627,
\]

\[
Z(H_n) = \frac{1}{6} \left( \ln[2] - 2\ln \left[ 127 + 48\sqrt{7} \right] + 2\ln \left[ 2024 + 765\sqrt{7} \right] \right) = 1.038,
\]

\[
Z(T_n) = \frac{1}{6} \left( \ln[2] - 2\ln \left[ 127 + 48\sqrt{7} \right] + 2\ln \left[ 2024 + 765\sqrt{7} \right] \right) = 1.038,
\]

\[
Z(X_n) = \frac{1}{6} \left( \ln[8] + 2\ln \left[ 14 + 6\sqrt{5} \right] \right) = 1.45,
\]

\[
Z(Y_n) = \frac{1}{6} \left( \ln[2] + 2\ln \left[ 17 + 12\sqrt{2} \right] \right) = 1.291.
\]
In addition, the entropy of the graphs Gₜₙ and Tₑ of the same average degree 4 are equal and smaller than the entropy of the fractal scale-free lattice [25] which has the entropy 1. Also, the entropy of the graph Gₜₙ has the entropy 1.0445 of the same average degree 4.

6. Conclusions

In this paper, we have calculated the number of spanning trees in the sequences of some nonahedral (polyhedral graphs having nine vertices) graphs generated by Kₙ Using electrical equivalent transformations. The feature of this technique lies in the palsy of Brouwer et al. (2004) of spanning trees, ISCIT 2004, IEEE Int. Symposium. Info. Technol. 1 (2004) 601–604.

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Additional information

No additional information is available for this paper.

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